Decompositions of some Specht modules I

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Abstract

We give a decomposition as a direct sum of indecomposable modules of several types of Specht modules in characteristic 2. These include the Specht modules labelled by hooks, whose decomposability was considered by Murphy, [14]. Since the main arguments are essentially no more difficult for Hecke algebras at parameter $q = -1$, we proceed in this level of generality.

1 Introduction

Let $K$ be a field and $r$ a positive integer. We write $\Sigma_r$ for the symmetric group of degree $r$. For each partition $\lambda$ of $r$ we have the Specht module $\text{Sp}(\lambda)$ and for each composition $\alpha$ of $r$ the permutation module $M(\alpha)$. The Specht module $\text{Sp}(\lambda)$ may be viewed as a submodule of $M(\lambda)$. James proved, [12, 13.17], that unless the characteristic of $K$ is 2 and $\lambda$ is 2-singular, the space of homomorphisms $\text{Hom}_{\Sigma_r}(\text{Sp}(\lambda), M(\lambda))$ is one dimensional. It follows that $\text{Sp}(\lambda)$ has one dimensional endomorphism algebra and in particular that $\text{Sp}(\lambda)$ is indecomposable (unless $K$ has characteristic 2 and $\lambda$ is 2-singular).

We now suppose $K$ has characteristic 2. Then, for $\lambda$ a 2-singular partition, the Specht module $\text{Sp}(\lambda)$ may certainly decompose but in general neither a criterion for decomposability nor the nature of a decomposition as a direct sum of indecomposable components, is known. The first example of such a module was discovered by James for the symmetric group $\Sigma_7$ and the Specht module $\text{Sp}(5,1,1)$, see [13]. Some years later, Murphy generalised James’ example and in [14] she gave a criterion for the decomposability of Specht modules labelled by hook partitions, i.e. partitions of the form $\lambda = (a,1^b)$. More recently, Dodge and Fayers found in [4] some new decomposable Specht modules for partitions of the form $\lambda = (a,3,1^b)$.

In the more general context of the Hecke algebras $H_q(r)$, Dipper and James showed in [3] that for $q \neq -1$ the corresponding Specht modules are
indecomposable. Recently Speyer generalised Murphy’s criterion regarding
the decomposability of Specht modules labeled by hook partitions for Hecke
algebras with \( q = -1 \), see [16].

We here obtain many new families of decomposable Specht modules for
Hecke algebras at parameter \( q = -1 \) and describe explicitly their indecom-
posable components. More precisely, we give a decomposition of the Specht
modules \( \text{Sp}(a, m-1, m-2, \ldots, 2, b) \), with \( a \geq m, b \geq 1 \) and \( a - m \) even
and \( b \) odd. Moreover, we show that the number of indecomposable sum-
mands in such decompositions is unbounded. We also point out that the
decomposition of \( \text{Sp}(a, m-1, m-2, \ldots, 2, b) \) lays the foundations for the
discovery of many other families of decomposable Specht modules. In fact,
using this approach we describe decompositions of Specht modules of the
form \( \text{Sp}(a, 3, b) \) which don’t appear in the list produced by Dodge and
Fayers. More results in this direction will appear in a follow up paper, [8].

Our method is to compare the situation with an analogous problem for
certain modules for the general linear groups and apply the Schur functor,
as in [7, Section 2.1].

Remark 1.1. We shall produce a decomposition of the Specht module \( \text{Sp}(\lambda) \),
for certain partitions \( \lambda \). In these cases \( \text{Sp}(\lambda) \) will actually be a Young module
(though not in general indecomposable). We note that in fact we already have
a supply of cases in which \( \text{Sp}(\lambda) \) is a Young module, namely when \( \lambda \) is, in
the terminology of [10], a Young partition, [10, Proposition 3.2 (i)].

However, in these cases the \( \text{Sp}(\lambda) \) is the indecomposable Young module
\( Y(\lambda) \). So these partitions are not of interest from the point of view of our
current investigation.

2 Preliminaries

2.1 Combinatorics

The standard reference for the polynomial representations of \( \text{GL}_n(K) \) is
the monograph [11]. Though we work in the quantised context this reference
is appropriate as the combinatorics is essentially the same and we adopt the
notation of [11] wherever convenient. Further details may also be found in
the monograph [7], which treats the quantised case.

We begin by introducing some of the associated combinatorics. By a part-
tion we mean an infinite sequence \( \lambda = (\lambda_1, \lambda_2, \ldots) \) of nonnegative integers
with \( \lambda_1 \geq \lambda_2 \geq \ldots \) and \( \lambda_j = 0 \) for \( j \) sufficiently large. If \( n \) is a positive
integer such that \( \lambda_j = 0 \) for \( j > n \) we identify \( \lambda \) with the finite sequence
\( (\lambda_1, \ldots, \lambda_n) \). The length \( l(\lambda) \) of a partition \( \lambda = (\lambda_1, \lambda_2, \ldots) \) is 0 if \( \lambda = 0 \)
and is the positive integer \( n \) such that \( \lambda_n \neq 0, \lambda_{n+1} = 0 \), if \( \lambda \neq 0 \). For
a partition \( \lambda \), we denote by \( \lambda' \) the transpose partition of \( \lambda \). We define the
degree of \( \lambda = (\lambda_1, \lambda_2, \ldots) \) by \( \deg(\lambda) = \lambda_1 + \lambda_2 + \cdots \).
We fix a positive integer $n$. We set $X(n) = \mathbb{Z}^n$. There is a natural partial order on $X(n)$. For $\lambda = (\lambda_1, \ldots, \lambda_n), \mu = (\mu_1, \ldots, \mu_n) \in X(n)$, we write $\lambda \leq \mu$ if $\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i$ for $i = 1, 2, \ldots, n - 1$ and $\lambda_1 + \cdots + \lambda_n = \mu_1 + \cdots + \mu_n$. We shall use the standard $\mathbb{Z}$-basis $e_1, \ldots, e_n$ of $X(n)$, where $e_i = (0, \ldots, 1, \ldots, 0)$ (with 1 in the $i$th position), for $1 \leq i \leq n$. We write $\omega_i$ for the element $\epsilon_1 + \cdots + \epsilon_i$ of $X(n)$, for $1 \leq i \leq n$.

We write $X^+(n)$ for the set of dominant $n$-tuples of integers, i.e., the set of elements $\lambda = (\lambda_1, \ldots, \lambda_n) \in X(n)$ such that $\lambda_1 \geq \cdots \geq \lambda_n$. We write $\Lambda(n)$ for the set of $n$-tuples of nonnegative integers and $\Lambda^+(n)$ for the set of partitions into at most $n$ parts, i.e., $\Lambda^+(n) = X^+(n) \cap \Lambda(n)$. We shall sometimes refer to elements of $\Lambda(n)$ as polynomial weights and to elements of $\Lambda^+(n)$ as polynomial dominant weights. For a nonnegative integer $r$ we write $\Lambda^+(n, r)$ for the set of partitions of $r$ into at most $n$ parts, i.e., the set of elements of $\Lambda^+(n)$ of degree $r$.

We fix a positive integer $l$. We write $X_1(n)$ for the set of $l$-restricted partitions into at most $n$ parts, i.e., the set of elements $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda^+(n)$ such that $0 \leq \lambda_1 - \lambda_2, \ldots, \lambda_{n-1} - \lambda_n, \lambda_n < l$.

A dominant weight $\lambda \in X^+(n)$ has a unique expression $\lambda = \lambda^0 + \tilde{\lambda}$ with $\lambda^0 \in X_1(n)$, $\tilde{\lambda} \in X^+(n)$, moreover if $\lambda \in \Lambda^+(n)$ then $\tilde{\lambda} \in \Lambda^+(n)$. We shall use this notation a great deal in what follows.

### 2.2 Rational Modules and Polynomial Modules

Let $K$ be a field. If $V, W$ are vector spaces over $K$, we write $V \otimes W$ for the tensor product $V \otimes_K W$. We shall be working with the representation theory of quantum groups over $K$. By the category of quantum groups over $K$ we understand the opposite category of the category of Hopf algebras over $K$. Less formally we shall use the expression “$G$ is a quantum group” to indicate that we have in mind a Hopf algebra over $K$ which we denote $K[G]$ and call the coordinate algebra of $G$. We say that $\phi : G \to H$ is a morphism of quantum groups over $K$ to indicate that we have in mind a morphism of Hopf algebras over $K$, from $K[H]$ to $K[G]$, denoted $\phi^*$ and called the co-morphism of $\phi$. We will say $H$ is a quantum subgroup of the quantum group $G$, over $K$, to indicate that $H$ is a quantum group with coordinate algebra $K[H] = K[G]/I$, for some Hopf ideal $I$ of $K[G]$, which we call the defining ideal of $H$. The inclusion morphism $i : H \to G$ is the morphism of quantum groups whose co-morphism $i^* : K[G] \to K[H] = K[G]/I$ is the natural map.

Let $G$ be a quantum group over $K$. The category of left (resp. right) $G$-modules is the the category of right (resp. left) $K[G]$-comodules. We write Mod($G$) for the category of left $G$-modules and mod($G$) for the category of finite dimensional left $G$-modules. We shall also call a $G$-module a rational $G$-module (by analogy with the representation theory of algebraic groups). A $G$-module will mean a left $G$-module unless indicated otherwise. For a
finite dimensional $G$-module $V$ and a non-negative integer $d$ we write $V^\otimes d$ for the $d$-fold tensor product $V \otimes \cdots \otimes V$ and we write $V^{(d)}$ for the direct sum $V \oplus \cdots \oplus V$ of $d$ copies of $V$.

Let $V$ be a finite dimensional $G$-module with structure map $\tau : V \to V \otimes K[G]$. The coefficient space $\text{cf}(V)$ of $V$ is the subspace of $K[G]$ spanned by the “coefficient elements” $f_{ij}$, $1 \leq i, j \leq m$, defined with respect to a basis $v_1, \ldots, v_m$ of $V$, by the equations

$$\tau(v_i) = \sum_{j=1}^{m} v_j \otimes f_{ji}$$

for $1 \leq i \leq m$. The coefficient space $\text{cf}(V)$ is independent of the choice of basis and is a subcoalgebra of $K[G]$.

We fix a positive integer $n$. We shall be working with $G(n)$, the quantum general linear group of degree $n$, as in [7]. We fix a non-zero element $q$ of $K$. We have a $K$-bialgebra $A(n)$ given by generators $c_{ij}$, $1 \leq i, j \leq n$, subject to certain relations (depending on $q$) as in [7, 0.20]. The comultiplication map $\delta : A(n) \to A(n) \otimes A(n)$ satisfies $\delta(c_{ij}) = \sum_{r=1}^{n} c_{ir} \otimes c_{rj}$ and the augmentation map $\epsilon : A(n) \to K$ satisfies $\epsilon(c_{ij}) = \delta_{ij}$ (the Kronecker delta), for $1 \leq i, j \leq n$. The elements $c_{ij}$ will be called the coordinate elements and we define the determinant element

$$d_n = \sum_{\pi \in \text{Sym}(n)} \text{sgn}(\pi)c_{1,\pi(1)} \cdots c_{n,\pi(n)}.$$ 

Here $\text{sgn}(\pi)$ denotes the sign of the permutation $\pi$. We form the Ore localisation $A(n)_{d_n}$. The comultiplication map $A(n) \to A(n) \otimes A(n)$ and augmentation map $A(n) \to K$ extend uniquely to $K$-algebraic maps $A(n)_{d_n} \to A(n)_{d_n} \otimes A(n)_{d_n}$ and $A(n)_{d_n} \to K$, giving $A(n)_{d_n}$ the structure of a Hopf algebra. By the quantum general linear group $G(n)$ we mean the quantum group over $K$ with coordinate algebra $K[G(n)] = A(n)_{d_n}$.

We write $T(n)$ for the quantum subgroup of $G(n)$ with defining ideal generated by all $c_{ij}$ with $1 \leq i, j \leq n$, $i \neq j$. We write $B(n)$ for quantum subgroup of $G(n)$ with defining ideal generated by all $c_{ij}$ with $1 \leq i < j \leq n$. We call $T(n)$ a maximal torus and $B(n)$ a Borel subgroup of $G(n)$ (by analogy with the classical case).

We now assign to a finite dimension rational $T(n)$-module its formal character. We form the integral group ring $\mathbb{Z}X(n)$. This has $\mathbb{Z}$-basis of formal exponentials $e^\lambda$, which multiply according to the rule $e^\lambda e^\mu = e^{\lambda+\mu}$, $\lambda, \mu \in X(n)$. For $1 \leq i \leq n$ we define $\bar{c}_{ii} = c_{ii} + I_{T(n)} \in K[T(n)]$, where $I_{T(n)}$ is the defining ideal of the quantum subgroup $T(n)$ of $G(n)$. Note that $\bar{c}_{11} \ldots \bar{c}_{nn} = d_n + I_{T(n)}$, in particular each $\bar{c}_{ii}$ is invertible in $K[T(n)]$. For $\lambda = (\lambda_1, \ldots, \lambda_n) \in X(n)$ we define $\bar{e}^\lambda = \bar{c}_{11}^{\lambda_1} \cdots \bar{c}_{nn}^{\lambda_n}$. The elements $\bar{e}^\lambda$, $\lambda \in X(n)$, are group-like and form a $K$-basis of $K[T(n)]$. For
\( \lambda = (\lambda_1, \ldots, \lambda_n) \in X(n) \), we write \( K_\lambda \) for \( K \) regarded as a (one dimensional) \( T(n) \)-module with structure map \( \tau : K_\lambda \to K_\lambda \otimes K[T(n)] \) given by 
\[
\tau(v) = v \otimes e^\lambda, \quad v \in K_\lambda.
\]
For a finite dimensional rational \( T(n) \)-module \( V \) with structure map \( \tau : V \to V \otimes K[T(n)] \) and \( \lambda \in X(n) \) we have the weight space
\[
V^\lambda = \{ v \in V \mid \tau(v) = v \otimes e^\lambda \}.
\]
Moreover, we have the weight space decomposition \( V = \bigoplus_{\lambda \in X(n)} V^\lambda \). We say that \( \lambda \in X(n) \) is a weight of \( V \) if \( V^\lambda \neq 0 \). The dimension of a finite dimensional vector space \( V \) over \( K \) will be denoted by \( \dim V \). The character \( \chi V \) of a finite dimensional rational \( T(n) \)-module \( V \) is the element of \( \mathbb{Z}X(n) \) defined by \( \chi V = \sum_{\lambda \in X(n)} \dim V^\lambda e^\lambda \).

For each \( \lambda \in X^+(n) \) there is an irreducible rational \( G(n) \)-module \( L_n(\lambda) \) which has unique highest weight \( \lambda \), and \( \lambda \) occurs as a weight with multiplicity one. The modules \( L_n(\lambda), \lambda \in X^+(n), \) form a complete set of pairwise non-isomorphic irreducible rational \( G \)-modules. For a finite dimensional rational \( G(n) \)-module \( V \) and \( \lambda \in X^+(n) \) we write \( [V : L_n(\lambda)] \) for the multiplicity of \( L_n(\lambda) \) as a composition factor of \( V \).

We write \( D_n \) for the one dimensional \( G(n) \)-module corresponding to the determinant. Thus \( D_n \) has structure map \( \tau : D_n \to D_n \otimes K[G], \) given by \( \tau(v) = v \otimes d_n \), for \( v \in D_n \). We have \( D_n = L_n(\omega_n) = L_n(1, 1, \ldots, 1) \). We write \( E_n \) for the natural \( G(n) \)-module. Thus \( E_n \) has basis \( e_1, \ldots, e_n, \) and the structure map \( \tau : E_n \to E_n \otimes K[G(n)] \) is given by \( \tau(e_i) = \sum_{j=1}^n e_j \otimes c_{ij} \). We also have that \( E_n = L_n(1, 0, \ldots, 0) \).

A finite dimensional \( G(n) \)-module \( V \) is called polynomial if \( \text{cf}(V) \leq A(n) \). The modules \( L_n(\lambda), \lambda \in \Lambda^+(n), \) form a complete set of pairwise non-isomorphic irreducible polynomial \( G(n) \)-modules. We have a grading \( A(n) = \bigoplus_{r=0}^\infty A(n, r) \) in such a way that each \( c_{ij} \) has degree 1. Moreover each \( A(n, r) \) is a finite dimensional subcoalgebra of \( A(n) \). The dual algebra \( S(n, r) \) is known as the Schur algebra. A finite dimensional \( G(n) \)-module \( V \) is polynomial of degree \( r \) if \( \text{cf}(V) \leq A(n, r) \). We write \( \text{pol}(n) \) (resp. \( \text{pol}(n, r) \)) for the full subcategory of \( \text{mod}(G(n)) \) whose objects are the polynomial modules (resp. the modules which are polynomial of degree \( r \)).

For an arbitrary finite dimensional polynomial \( G(n) \)-module we may write \( V \) uniquely as a direct sum \( V = \bigoplus_{r=0}^\infty V(r) \) in such a way that \( V(r) \) is polynomial of degree \( r \), for \( r \geq 0 \). Let \( r \geq 0 \). The modules \( L_n(\lambda), \lambda \in \Lambda^+(n), \) form a complete set of pairwise non-isomorphic irreducible polynomial \( G(n) \)-modules which are polynomial of degree \( r \). We write \( \text{mod}(S) \) for the category of left modules for a finite dimensional \( K \)-algebra \( S \). The category \( \text{pol}(n, r) \) is naturally equivalent to the category \( \text{mod}(S(n, r)) \).

We shall also need modules induced from \( B(n) \) to \( G(n) \). (For details of the induction functor \( \text{Mod}(B(n)) \to \text{Mod}(G(n)) \) see, for example, [ ]). For \( \lambda \in X(n) \) there is a unique (up to isomorphism) one dimensional \( B(n) \)-module whose restriction to \( T(n) \) is \( K_\lambda \). We also denote this module by \( K_\lambda \).
The induced module \(\text{ind}_{B(n)}^G K_\lambda\) is non-zero if and only if \(\lambda \in X^+(n)\). For \(\lambda \in X^+(n)\) we set \(\nabla_n(\lambda) = \text{ind}_{B(n)}^G K_\lambda\). Then \(\nabla_n(\lambda)\) is finite dimensional and its character is given by Weyl’s character formula. In fact, for \(\lambda \in \Lambda^+(n)\) the character of \(\nabla_n(\lambda)\) is the Schur symmetric function corresponding to \(\lambda\). The \(G(n)\)-module socle of \(\nabla_n(\lambda)\) is \(L_n(\lambda)\). The module \(\nabla_n(\lambda)\) has unique highest weight \(\lambda\) and this weight occurs with multiplicity one.

A filtration \(0 = V_0 \leq V_1 \leq \cdots \leq V_r = V\) of a finite dimensional rational \(G(n)\)-module \(V\) is said to be good if for each \(1 \leq i \leq r\) the quotient \(V_i/V_{i-1}\) is either zero or isomorphic to \(\nabla_n(\lambda^i)\) for some \(\lambda^i \in X^+(n)\). For a rational \(G(n)\)-module \(V\) admitting a good filtration for each \(\lambda \in X^+(n)\), the multiplicity \(|\{1 \leq i \leq r \mid V_i/V_{i-1} \cong \nabla_n(\lambda)\}|\) is independent of the choice of the good filtration, and will be denoted \((V : \nabla_n(\lambda))\).

For \(\lambda, \mu \in X^+(n)\) we have \(\text{Ext}^1_{G(n)}(\nabla_n(\lambda), \nabla_n(\mu)) = 0\) unless \(\lambda > \mu\). Given Kempf’s Vanishing Theorem, \([7, \text{Theorem 3.4}]\), this follows exactly as in the classical case, e.g., \([5, \text{Lemma 3.2.1}]\), (or the original source \([2, \text{Corollary (3.2)}]\)). It follows that if \(V\) has a good filtration \(0 = V_0 \leq V_1 \leq \cdots \leq V_t = V\) with sections \(V_i/V_{i-1} \cong \nabla_n(\lambda_i)\), \(1 \leq i \leq t\), and \(\mu_1, \ldots, \mu_t\) is a reordering of the \(\lambda_1, \ldots, \lambda_t\) such that \(\mu_i < \mu_j\) implies that \(i < j\) then there is a good filtration \(0 = V'_0 < V'_1 < \cdots < V'_t = V\) with \(V'_i/V'_{i-1} \cong \nabla_n(\mu_i)\), for \(1 \leq i \leq t\).

Similarly it will be of great practical use to know that \(\text{Ext}^1_{G(n)}(\nabla_n(\lambda), \nabla_n(\mu)) = 0\) when \(\lambda\) and \(\mu\) belong to different blocks. Here the relationship with cores of partitions diagrams will be crucial for us. For a partition \(\lambda\) we denote by \([\lambda]\) the corresponding partition diagram (as in \([11]\)). The \(l\)-core of \([\lambda]\) is the diagram obtained by removing skew \(l\)-hooks, as in \([12]\). If \(\lambda, \mu \in \Lambda^+(n, r)\) and \([\lambda]\) and \([\mu]\) have different \(l\)-cores then the simple modules \(L_n(\lambda)\) and \(L_n(\mu)\) belong to different blocks and it follows in particular that \(\text{Ext}^i_{S(n, r)}(\nabla(\lambda), \nabla(\mu)) = 0\), for all \(i \geq 0\). A precise description of the blocks of the \(q\)-Schur algebras was found by Cox, see \([1, \text{Theorem 5.3}]\).

For a polynomial \(G(n)\)-module \(V\) and an \(l\)-core \(\gamma\) we mean by the expression, the component of \(V\) corresponding to \(\gamma\), the sum of all block components for blocks consisting of partitions with core \(\gamma\).

For \(\lambda \in \Lambda^+(n, r)\) we write \(I_n(\lambda)\) for the injective envelope of \(L_n(\lambda)\) in the category of polynomial modules. We have that \(I_n(\lambda)\) is a finite dimensional module which is polynomial of degree \(r\). Moreover, the module \(I_n(\lambda)\) has a good filtration and we have the reciprocity formula \((I_n(\lambda) : \nabla_n(\mu)) = [\nabla_n(\mu) : L_n(\lambda)]\) see e.g., \([6, \text{Section 4, (6)}]\).

### 2.3 The Frobenius Morphism

It will be important for us to make a comparison with the classical case \(q = 1\). In this case we will write \(\bar{G}(n)\) for \(G(n)\) and write \(x_{ij}\) for the coordinate element \(c_{ij}, 1 \leq i, j \leq n\). Also, we write \(\bar{L}(\lambda)\) for the \(\bar{G}\)-module
$L_n(\lambda), \lambda \in X^+(n)$, and write $\hat{E}_n$ for $E_n$.

We return to the general situation. If $q$ is not a root or unity, or if $K$ has characteristic 0 and $q = 1$ then all $G(n)$-modules are completely reducible, see e.g., [3] Section 4, (8)]. We therefore assume from now on that $q$ is a root of unity and that if $K$ has characteristic 0 then $q \neq 1$. Also, from now on, $l$ is the smallest positive integer such that $1 + q + \cdots + q^{l-1} = 0$.

Now we have a morphism of Hopf algebras $\theta : K[\hat{G}(n)] \to K[G(n)]$ given by $\theta(x_{ij}) = c_{ij}$, for $1 \leq i, j \leq n$. We write $F : G(n) \to \hat{G}(n)$ for the morphism of quantum groups such that $F^\sharp = \theta$. Given a $\hat{G}(n)$-module $V$ we write $V^F$ for the corresponding $G(n)$-module. Thus, $V^F$ as a vector space is $V$ and if the $\hat{G}(n)$-module $V$ has structure map $\tau : V \to V \otimes K[\hat{G}(n)]$ then $V^F$ has structure map $(\text{id}_V \otimes F) \circ \tau : V^F \to V^F \otimes K[G(n)]$, where $\text{id}_V : V \to V$ is the identity map on the vector space $V$.

For an element $\phi = \sum_{\xi \in X(n)} a_{\xi} e^\xi$ of $ZX(n)$ we write $\phi^F$ for the element $\sum_{\xi \in X(n)} a_{\xi} e^\xi$. Then, for a finite dimensional $\hat{G}(n)$-module $V$ we have $\text{ch} V^F = (\text{ch} V)^F$. Moreover, we have the following relationship between the irreducible modules for $G(n)$ and $\hat{G}(n)$, see [7] Section 3.2, (5)]

**Steinberg’s Tensor Product Theorem** For $\lambda^0 \in X_+(n)$ and $\bar{\lambda} \in X^+(n)$ we have

$$L_n(\lambda^0 + \bar{\lambda}) \cong L_n(\lambda^0) \otimes \hat{L}_n(\bar{\lambda})^F.$$ 

Usually we shall abbreviate the quantum groups $G(n)$, $B(n)$, $T(n)$ to $G$, $B$, $T$ and $\hat{G}(n)$ to $\hat{G}$. Likewise, we usually abbreviate the modules $L_n(\lambda)$, $\nabla_n(\lambda)$, $I_n(\lambda)$ and $\hat{L}_n(\lambda)$ to $L(\lambda)$, $\nabla(\lambda)$, $I(\lambda)$ and $\hat{L}(\lambda)$, for $\lambda \in \Lambda^+(n)$, and abbreviate the modules $E_n$ and $D_n$ to $E$ and $D$.

### 2.4 A truncation functor

Let $N, n$ be positive integers with $N \geq n$. We identify $G(n)$ with the quantum subgroup of $G(N)$ whose defining ideal is generated by all $c_{ii} - 1$, $n < i \leq N$, and all $c_{ij}$ with $1 \leq i \neq j \leq N$ and $i > n$ or $j > n$. We have an exact functor (the truncation functor) $d_{N,n} : \text{pol}(N) \to \text{pol}(n)$ taking $V \in \text{pol}(N)$ to the $G(n)$ submodule $\bigoplus_{\alpha \in \Lambda(n)} V^\alpha$ of $V$ and taking a morphism of polynomial modules $V \to V'$ to its restriction $d_{N,n}(V) \to d_{N,n}(V')$. For a discussion of this functor at the level of modules for Schur algebras in the classical case see [11] Section 6.5.

By [7] Section 4.2], the functor $d_{N,n}$ has the following properties:

(i) for polynomial $G(N)$-modules $X, Y$ we have $d_{N,n}(X \otimes Y) = d_{N,n}(X) \otimes d_{N,n}(Y)$;
(ii) for $\lambda \in \Lambda^+(N, r)$ and $X_\lambda = L_N(\lambda)$ or $\nabla_N(\lambda)$ then $d_{N,n}(X_\lambda) \neq 0$ if and only if $\lambda \in \Lambda^+(n, r)$;
(iii) for $\lambda \in \Lambda^+(n, r)$, $d_{N,n}(L_N(\lambda)) = L_n(\lambda)$ and $d_{N,n}(\nabla_N(\lambda)) = \nabla_n(\lambda)$. 


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Let \( \lambda \in \Lambda^+(N,r) \), for some \( r \geq 0 \), \( \alpha \in \Lambda(N,r) \) and \( \lambda_i = \alpha_i = 0 \), for \( n < i \leq N \). We identify \( \lambda \) and \( \alpha \) with elements of \( \Lambda^+(n,r) \) and \( \Lambda(n,r) \) in the obvious way. It follows that \( \dim L_N(\lambda)^\alpha = \dim L_n(\lambda)^\alpha \).

### 2.5 Connections with the Hecke algebras

We now record some connections with representations of Hecke algebra of type \( A \). We fix a positive integer \( r \). We write \( l(\pi) \) for the length of a permutation \( \pi \). The Hecke algebra \( \text{Hec}(r) \) is the \( K \)-algebra with basis \( T_w, w \in \text{Sym}(r) \), and multiplication satisfying

\[
T_w T_{w'} = T_{ww'}, \quad \text{if } l(ww') = l(w) + l(w'),
\]

\[
(T_s + 1)(T_s - q) = 0
\]

for \( w, w' \in \text{Sym}(r) \) and a basic transposition \( s \in \text{Sym}(r) \). For brevity we will denote the Hecke algebra \( \text{Hec}(r) \) by \( H(r) \).

For \( \lambda \) a partition of degree \( r \) we denote by \( \text{Sp}(\lambda) \) the corresponding (Dipper-James) Specht module and by \( Y(\lambda) \) the corresponding Young module. For \( \alpha \in \Lambda(n,r) \) we write \( M(\alpha) \) for the permutation module corresponding to \( \alpha \).

Let \( n \geq r \). We have the Schur functor \( f : \text{mod}(S(n,r)) \to \text{mod}(H(r)) \), see [7, Section 2.1]. By [7, Sections 4.4 and 4.5] we have that the functor \( f \) has the following properties:

(i) \( f \) is exact;
(ii) for \( \lambda \in \Lambda^+(n,r) \) we have \( f(\nabla(\lambda)) = \text{Sp}(\lambda) \);
(iii) for \( \lambda \in \Lambda^+(n,r) \) we have \( f(I(\lambda)) = Y(\lambda) \).

For a finite string of non-negative integers \( \alpha = (\alpha_1, \ldots, \alpha_m) \) we have the polynomial \( G(n) \)-modules

\[
S^\alpha E = S^{\alpha_1} E \otimes \cdots \otimes S^{\alpha_m} E
\]

and

\[
\wedge^\alpha E = \wedge^{\alpha_1} E \otimes \cdots \otimes \wedge^{\alpha_m} E.
\]

For \( \alpha \in \Lambda(n,r) \) we write \( H(\alpha) \) for the subalgebra \( H(\alpha_1) \otimes \cdots \otimes H(\alpha_n) \) of \( H(r) \). By [7, Section 2.1, (20)] we have that:

i) \( f(S^\alpha E) = H(r) \otimes_{H(\alpha)} K = M(\alpha) \);

ii) \( f(\wedge^\alpha E) = H(r) \otimes_{H(\alpha)} K_{\text{sgn}} \).

### 3 Some weight space multiplicities

Recall that our standard assumption is that \( q \) is a root of unity and that if the base filed \( K \) has characteristic 0 then \( q \neq 1 \). Also, \( l \) is the smallest positive integer such that \( 1 + q + \cdots + q^{l-1} = 0 \).
If $K$ has characteristic $p > 0$, the base $p$-expansion of a positive integer $r$ will be written $r = \sum_{i=0}^{\infty} p^{i}r_{i}$ or just $r = \sum_{i=0}^{N} p^{i}r_{i}$, if $r < p^{N+1}$.

**Definition 3.1.** Let $r, b$ be integers with $r \geq 0$. We shall say that the pair $(r, b)$ is $p$-special if

i) for $p = 0$ we have that $-r \leq b \leq r$ and $r - b$ is even;

ii) for $p > 0$ and $r = \sum_{i=0}^{\infty} p^{i}r_{i}$ be the $p$ expansion of $r$, there exists an expression $b = \sum_{i=0}^{\infty} p^{i}t_{i}$ with $-r_{i} \leq t_{i} \leq r_{i}$ and $r_{i} - t_{i}$ even for all $i \geq 0$.

**Definition 3.2.** Let $s, a$ be integers with $s \geq 0$. We write $s = s_{0} + l\bar{s}$ with $0 \leq s_{0} \leq l - 1$. We shall say that the pair $(s, a)$ is $(l, p)$-special if there exists an expression $a = a_{0} + l\bar{a}$, with $-s_{0} \leq a_{0} \leq s_{0}$, $s_{0} - a_{0}$ even and $(\bar{s}, \bar{a})$ $p$-special.

From Steinberg’s tensor product theorem for $G(2)$, we get the following, which in the case $l = 2$ is crucial for our analysis of a decomposition of some Specht modules.

**Lemma 3.3.** For a partition $(c, d) \in \Lambda^{+}(2, r)$ and for $(a, b) \in \Lambda(2, r)$, we have

$$\dim L(c, d)^{(a, b)} = \begin{cases} 1, & \text{if } (c - d, a - b) \text{ is } (l, p)\text{-special;} \\ 0, & \text{otherwise.} \end{cases}$$

For a positive integer $m$ we write $\delta_{m}$ for the partition $(m, m - 1, \ldots, 1)$ (of length $m$) and $\sigma_{m}$ for $(l - 1)\delta_{m}$.

**Remark 3.4.** Recall that if $\lambda \in \Lambda^{+}(n, r)$, $\alpha \in \Lambda(n, r)$ and $L(\lambda)^{\alpha} \neq 0$ then $\alpha \leq \lambda$. This implies that the length of $\alpha$ is at least the length of $\lambda$. If $\lambda$ and $\alpha$ have length $m$ then

$$\dim L(\lambda)^{\alpha} = \dim L_{m}(\lambda)^{\alpha} = \dim(D_{m} \otimes L(\lambda - \omega_{m}))^{\alpha} = \dim L(\lambda - \omega_{m})^{a - \omega_{m}}.$$

**Lemma 3.5.** For $m > 0$, let $a \geq m(l - 1)$ and $b \geq (m - 1)(l - 1)$. Let $n \geq m$ and $\mu \in \Lambda^{+}(n)$.

i) If $\mu_{m+1} = 0$ and $m > 2$ then

$$\dim L(\sigma_{m} + l\mu)^{(a, b, (m-2)(l-1), \ldots, 2(l-1), (l-1))}$$

$$= \dim L(\sigma_{m-1} + l\mu)^{(a-(l-1), b-(l-1), (m-3)(l-1), \ldots, l-1)}$$

and in fact $L(\sigma_{m} + l\mu)^{(a, b, (m-2)(l-1), \ldots, 2(l-1), (l-1))} = 0$ unless $\mu_{i} = 0$ for all $i > 2$.

ii) We have

$$\dim L(\sigma_{m} + l\mu)^{(a, b, (m-2)(l-1), \ldots, 2(l-1), (l-1))}$$

$$= \dim L(\sigma_{1} + l\mu)^{(a-(m-1)(l-1), b-(m-1)(l-1))}.$$
(iii) If \( a + m \equiv 0 \) and \( b + m \equiv 1 \mod l \) and \( \mu = (c, d) \) then

\[
\dim L(\sigma_m + l\mu)^{(a, b, (m-2)(l-1), \ldots, 2(l-1), (l-1))}
\]

\[
= \begin{cases} 
1, & \text{if } (c - d, u - v) \text{ is } p\text{-special;} \\
0, & \text{otherwise.}
\end{cases}
\]

where \( a - m(l-1) = lu \) and \( b - (m-1)(l-1) = lv. \)

**Proof.** (i) If \( \mu_{m+1} = 0 \) then \( \sigma_m + l\mu \) and
\((a, b, (m-2)(l-1), \ldots, 2(l-1), (l-1))\) are partitions of length \( m \) with final term at least \( l - 1 \). Hence, from Remark 3.4, we obtain

\[
\dim L(\sigma_m + l\mu)^{(a, b, (m-2)(l-1), \ldots, 2(l-1), (l-1))}
\]

\[
= \dim L(\sigma_m + l\mu) - (l - 1)\omega_m^{(a, b, (m-2)(l-1), \ldots, 2(l-1), (l-1))}(l - 1)\omega_m
\]

\[
= \dim L(\sigma_{m-1} + l\mu)^{(a - (l-1), b - (l-1), (m-3)(l-1), \ldots, 2(l-1), (l-1))},
\]

This proves the first assertion.

Now if \( L(\sigma_m + l\mu)^{(a, b, (m-2)(l-1), \ldots, 2(l-1), (l-1))} \neq 0 \) then

\[
\sigma_m + l\mu \geq (a, b, (m-2)(l-1), \ldots, 2(l-1), (l-1))
\]

which gives \( \mu_{m+1} = 0 \). The second point follows now by induction on \( m \).

(ii) This follows by repeated application of (i).

(iii) We have \( L(\sigma_1 + l\mu) = L(\sigma_1) \otimes \hat{L}(\mu)^F \). We consider \( L(\sigma_1 + l\mu) \) as a \( G(2) \)-module. A weight of \( L(\sigma_1) \otimes \hat{L}(\mu)^F \) has the form \( \alpha + l\beta \), where

\[
\alpha \in \{(l-1, 0), (l-2, 1), \ldots, (0, l-1)\}
\]

and \( \beta \) is a weight of \( \hat{L}(\mu) \) and all non-zero weight spaces are one dimensional. Since \( a - (m-1)(l-1) \) is congruent to \( l - 1 \) (modulo \( l \)) the only solution for \( \alpha \) is \( (l-1, 0) \) and the dimension of the weight space

\[
L(\sigma_1 + l\mu)^{(a - (m-1)(l-1), b - (m-1)(l-1))}
\]

is equal to the number of weights \( \beta \) of \( \hat{L}(\mu) \) such that

\[
(l - 1, 0) + l\beta = (a - (m-1)(l-1), b - (m-1)(l-1))
\]

so this dimension is 1 if \( (a - m(l-1), b - (m-1)(l-1)) \) is a weight of \( \hat{L}(\mu)^F \), i.e., if \( (u, v) \) is a weight of \( \hat{L}(\mu) \), and 0 otherwise. Now the result follows from Lemma 3.3 (applied in the classical case \( q = 1 \)).

\[\square\]
4 Decompositions of some polynomial modules

The main technical point of this paper is an analysis, for \( l = 2 \), of a certain block component of a module of the form

\[
S^a E \otimes \Lambda^b E \otimes \Lambda^{m-2} E \otimes \cdots \otimes \Lambda^2 E \otimes E.
\]

Suppose that \( n \geq r \). Then, for \( \alpha \in \Lambda(n, r) \), the module \( S^\alpha E \) is injective and we have

\[
S^\alpha E = \bigoplus_{\lambda \in \Lambda^+(n, r)} I(\lambda)^{(d_{\lambda\alpha})}
\]

where \( d_{\lambda\alpha} = \dim L(\lambda)^\alpha \), [7, Section 2.1 (8)].

Applying the Schur functor to (1), for any \( \alpha \in \Lambda(n, r) \), we get

\[
M(\alpha) = \bigoplus_{\lambda \in \Lambda^+(n, r)} Y(\lambda)^{(d_{\lambda\alpha})}
\]

From Lemma 3.5 and equations (1) and (2) above we get the following two corollaries.

**Corollary 4.1.** For \( m \geq 2 \), let \( a \geq m(l-1) \) and \( b \geq (m-1)(l-1) \). Suppose that \( a + m \equiv 0 \) and \( b + m \equiv 1 \mod l \). The component of

\[
S^a E \otimes S^b E \otimes S^{(m-2)(l-1)} E \otimes \cdots \otimes S^{2(l-1)} E \otimes S^{l-1} E
\]

corresponding to the core \( \sigma_m \) is

\[
\bigoplus_{\mu} I(\sigma_m + l\mu)
\]

where the sum is over all partitions \( \mu = (c, d) \) such that, \( c + d = u + v \) and \( (c - d, u - v) \) is \( p \)-special, where \( a - m(l-1) = lu \) and \( b - (m-1)(l-1) = lv \).

Applying the Schur functor we then obtain the following.

**Corollary 4.2.** For \( m \geq 2 \), let \( a \geq m(l-1) \) and \( b \geq (m-1)(l-1) \). Suppose that \( a + m \equiv 0 \) and \( b + m \equiv 1 \mod l \). The block component

\[
M(a, b, (m - 2)(l - 1), \ldots, 2(l - 1), l - 1)
\]

corresponding to the core \( \sigma_m \) is

\[
\bigoplus_{\mu} Y(\sigma_m + l\mu)
\]

where the sum is over all partitions \( \mu = (c, d) \) such that, \( c + d = u + v \) and \( (c - d, u - v) \) is \( p \)-special, where \( a - m(l-1) = lu \) and \( b - (m-1)(l-1) = lv \).
5 Adapted partitions and symmetric polynomials

For $\lambda \in \Lambda^+(n)$ we write $s(\lambda)$ for the Schur symmetric function corresponding to $\lambda$. The elements $s(\lambda)$ of $\mathbb{Z}X(n)$, as $\lambda$ varies over partitions with at most $n$ parts, form a $\mathbb{Z}$-basis of the ring of symmetric functions $\mathbb{Z}[x_1, \ldots, x_n]^\Sigma_n$.

Let $\gamma \in \Lambda^+(n)$ be an $l$-core. For a polynomial $G(n)$-module $V$ we write $V(\gamma)$ for the component of $V$ corresponding to $\gamma$. For $\lambda \in \Lambda^+(n)$ we define $C^*_\gamma(s(\lambda))$ to be $s(\lambda)$ if the core of $\lambda$ is $\gamma$ and 0 otherwise. We extend $C^*_\gamma$ additively to an endomorphism of the ring of symmetric functions $\mathbb{Z}[x_1, \ldots, x_n]^\Sigma_n$.

Lemma 5.1. Let $\gamma \in \Lambda^+(n)$ be an $l$-core. For a finite dimensional polynomial module $V$ we have

$$\text{ch} V(\gamma) = C^*_\gamma(\text{ch} V).$$

Proof. Since both sides are additive on short exact sequences of $G(n)$-modules, it is enough to check for a set of polynomial modules that generate the Grothendieck group of finite dimensional polynomial $G(n)$-modules. Hence it is enough to check for $V = \nabla(\lambda)$, $\lambda \in \Lambda^+(n)$, and for these modules it is clear from the definition.

We now restrict attention to the case $l = 2$. We need to keep track of the part of a symmetric polynomial corresponding to the core $\sigma_m = (m, m-1, \ldots, 1)$, $m \geq 0$ (where $\sigma_0 = 0$). To this end we introduce the following notion.

Definition 5.2. Let $m$ be a non-negative integer and $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition. We say that $\lambda$ is $m$-adapted if $\lambda_i > m - i$, for all $i \geq 1$ with $\lambda_i > 0$.

We write $C_m$ for the additive endomorphism of the ring of symmetric functions in $n$ variables such that

$$C_m(s(\lambda)) = \begin{cases} s(\lambda), & \text{if } \lambda \text{ is } m\text{-adapted;} \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 5.3. For a symmetric polynomial $g$ (in $n$ variables) and $0 \leq r \leq n$ we have

$$C_{m+1}(gs(1^r)) = C_{m+1}(C_m(g)s(1^r)).$$

Proof. It is enough to check this for $g = s(\lambda)$ for a partition $\lambda$, with at most $n$ parts. By Pieri’s Formula, [15, 5.17], we have

$$s(\lambda)s(1^r) = \sum_{\mu} s(\mu)$$
where the sum is over all partitions \( \mu \) with at most \( n \) parts, whose Young diagram may be obtained from the Young diagram of \( \lambda \) by adding a box in each of \( r \) distinct rows. Hence we have \( \mu_i \leq \lambda_i + 1 \) for each \( \mu \) appearing in the above sum.

If \( \lambda \) is not \( m \)-adapted then \( C_m(s(\lambda)) = 0 \). Moreover, in this case, we have \( \lambda_i \leq m - i \) for some \( i \), and so, for \( \mu \) appearing in the above sum we have \( \mu_i \leq \lambda_i + 1 \leq (m + 1) - i \). Hence, \( \mu \) is not \((m + 1)\)-adapted and \( C_{m+1}(s(\mu)) = 0 \). Therefore we get \( C_{m+1}(s(\lambda)s(1^r)) = 0 \).

Suppose now that \( \lambda \) is \( m \)-adapted. Thus we have

\[
C_{m+1}(C_m(s(\lambda)s(1^r))) = C_{m+1}(s(\lambda)s(1^r))
\]

and we are done.

\[\square\]

**Lemma 5.4.** Let \( m \geq 2 \), \( a \geq m \) and \( b \geq m - 1 \). Then we have, for all \( n \) sufficiently large,

\[
C_m(s(a)s(1^b)s(1^{m-2}) \ldots s(1)) \\
= s(a + 1, m - 1, \ldots, 2, 1^{b-m+1}) + s(a, m - 1, \ldots, 2, 1^{b-m+2}).
\]

*Proof.* We argue by induction on \( m \). First suppose \( m = 2 \). Then

\[
s(a)s(1^b) = s(a + 1, 1^{b-1}) + s(a, 1^b)
\]

by Pieri’s Formula. Both \((a+1, 1^{b-1})\) and \((a, 1^b)\) are \(2\)-adapted, so the result holds in this case.

Now suppose that \( m \geq 3 \) and the result holds for \( m - 1 \). By Lemma 5.3 and the induction hypothesis we have

\[
C_m((s(a)s(1^b)s(1^{m-3}) \ldots s(1))s(1^{m-2})) \\
= C_m(C_{m-1}(s(a)s(1^b)s(1^{m-3}) \ldots s(1))s(1^{m-2})) \\
= C_m(s(a + 1, m - 2, \ldots, 2, 1^{b-m+2})s(1^{m-2})) \\
+ C_m(s(a, m - 2, \ldots, 2, 1^{b-m+3})s(1^{m-2})).
\]

But now, again by Pieri’s Formula, for \( \lambda = (a + 1, m - 2, \ldots, 2, 1^{b-m+2}) \) we have

\[
s(\lambda)s(1^{m-2}) = \sum_{\mu} s(\mu)
\]

where the sum is over all partitions whose diagram is obtained by adding a box to \( m - 2 \) rows of the Young diagram of \( \lambda \). But such a diagram is not \( m \)-adapted unless the boxes are added to rows 2 up to \( m - 1 \) and in this case we have \( \mu = (a + 1, m - 1, \ldots, 2, 1^{b-m+1}) \). Hence we obtain

\[
C_m(s(a + 1, m - 2, \ldots, 2, 1^{b-m+2})s(1^{m-2})) = s(a + 1, m - 1, \ldots, 2, 1^{b-m+1})
\]
and similarly
\[ C_m(s(a, m - 2, \ldots, 2, 1^{b-m+3})s(1^{m-2})) = s(a, m - 1, \ldots, 2, 1^{b-m+2}), \]
so we are done.

We write \( C^*_m \) for \( C^*_\sigma_m \). We note that for a symmetric polynomial \( g \) contributions to \( C^*_m(g) \) can only come from Schur polynomials \( s(\lambda) \) with \( \lambda \) an \( m \)-adapted partition. So we have \( C^*_m(g) = C^*_m(C_m(g)) \), for a symmetric polynomial \( g \). Using Lemma 5.4 it is now easy to verify the following.

**Corollary 5.5.** Let \( m \geq 2, a \geq m \) and \( b \geq m - 1 \). Then we have, for all \( n \) sufficiently large, \( C^*_m(s(a)s(1^b)s(1^{m-2}) \ldots s(1)) \)

\[
\begin{cases}
  s(a + 1, m - 1, \ldots, 2, 1^{b-m+1}), & \text{if } a - m \text{ is odd and } b - m \text{ is even;} \\
  s(a, m - 1, \ldots, 2, 1^{b-m+2}), & \text{if } a - m \text{ is even } b - m \text{ is odd;} \\
  0, & \text{otherwise.}
\end{cases}
\]

Now the module
\[ S^n E \otimes \Lambda^b E \otimes \Lambda^{m-2} E \otimes \cdots \otimes \Lambda^2 E \otimes E \]
has a good filtration, e.g. by [6, Section 4, (3)]. Interpreting Corollary 5.5 in terms of \( G(n) \)-modules we obtain the following.

**Corollary 5.6.** Assume that \( m \geq 2 \). Let \( a \geq m \) and \( b \geq m - 1 \). For all \( n \) sufficiently large, the component of the module
\[ S^n E \otimes \Lambda^b E \otimes \Lambda^{m-2} E \otimes \cdots \otimes \Lambda^2 E \otimes E \]
corresponding to the core \( \sigma_m \) is:
\[
\begin{cases}
  \nabla(a + 1, m - 1, \ldots, 2, 1^{b-m+1}), & \text{if } a - m \text{ is odd and } b - m \text{ is even;} \\
  \nabla(a, m - 1, \ldots, 2, 1^{b-m+2}), & \text{if } a - m \text{ is even } b - m \text{ is odd;} \\
  0, & \text{otherwise.}
\end{cases}
\]

We specialise to the following.

**Corollary 5.7.** Assume that \( m \geq 2 \). Let \( a \geq m \) and \( b \geq m - 1 \) with \( a - m \) even and \( b - m \) odd. Then, for all \( n \) sufficiently large, the component of the module
\[ S^n E \otimes \Lambda^b E \otimes \Lambda^{m-2} E \otimes \cdots \otimes \Lambda^2 E \otimes E \]
corresponding to the core \( \sigma_m \) is
\[ \nabla(a, m - 1, \ldots, 2, 1^{b-m+2}). \]
6 Decomposable Specht modules

We continue to assume that \( q = -1 \) and so \( l = 2 \). Let \( n, r \geq 0 \) with \( n \geq r \).

Recall that for \( \alpha \in \Lambda(n, r) \) we write

\[
H(\alpha) = H(\alpha_1) \otimes \cdots \otimes H(\alpha_n)
\]

of \( H(r) \) and that we have

\[
f(S^\alpha E) = H(r) \otimes H(\alpha) K
\]

\[
f(\Lambda^\alpha E) = H(r) \otimes H(\alpha) K_{\text{sgn}}
\]

For finite strings of non-negative integers \( \alpha = (\alpha_1, \ldots, \alpha_a) \) and \( \beta = (\beta_1, \ldots, \beta_b) \) we write \( (\alpha | \beta) \) for the concatenation \( (\alpha_1, \ldots, \alpha_a, \beta_1, \ldots, \beta_b) \).

Assume that \( a + b \leq n \) and that \( \deg(\alpha) = r_1 \), \( \deg(\beta) = r_2 \) with \( r = r_1 + r_2 \).

Then it follows that

\[
f(S^\alpha E \otimes \Lambda^\beta E) = H(r) \otimes H(\alpha | \beta) (K \otimes K_{\text{sgn}}).
\]

But since \( q = -1 \) we have that \( K_{\text{sgn}} \cong K \), e.g. by [3, Lemma 3.1], and so

\[
f(S^\alpha E \otimes \Lambda^\beta E) = H(r) \otimes H(\alpha | \beta) K = M(\alpha | \beta).
\]

Applying the Schur functor to Corollary 5.7 we get immediately the following result.

**Corollary 6.1.** Assume that \( m \geq 2 \). Let \( a \geq m \) and \( b \geq m - 1 \) with \( a - m \) even and \( b - m \) odd. Then the block component of the module \( M(a, b, m - 2, \ldots, 2, 1) \) corresponding to the core \( \sigma_m \) is

\[
\text{Sp}(a, m - 1, \ldots, 2, 1^{b-m+2}).
\]

Comparing now Corollary 4.2 with Corollary 6.1 we get the main result of this paper.

**Theorem 6.2.** Assume that \( m \geq 2 \). Let \( a \geq m \) and \( b \geq m - 1 \) with \( a - m \) even and \( b - m \) odd. Then we have

\[
\text{Sp}(a, m - 1, \ldots, 2, 1^{b-m+2}) = \bigoplus_{\mu} Y(\sigma_m + 2\mu)
\]

where the sum is over all partitions \( \mu = (c, d) \) such that \( c + d = u + v \) and \( (c - d, u - v) \) is \( p \)-special, where \( a - m = 2u \) and \( b - m + 1 = 2v \).

We give now an example of such a decomposition to point out that the number of indecomposable summands is unbounded.

**Example 6.3.** Assume that \( K \) has characteristic 2. By Theorem 6.2 and using a simple inductive argument it follows that for \( k \geq 1 \) we have

\[
\text{Sp}(2^k + 2, 1^{2^k-1}) = \bigoplus_{j=1}^k Y(2^k + 2^j, 2^k - 2^j + 1).
\]
7 Further Remarks

7.1 Hook partitions

As an immediate application of Theorem 6.2 for \( m = 2 \), one obtains a decomposition for the Specht module \( \text{Sp}(a, 1^b) \) where \( a \) is even and \( b \) is odd. We point out here that our method gives also a decomposition of \( \text{Sp}(a, 1^b) \) when \( a \) is odd and \( b \) is even. In fact, we have the following proposition.

**Proposition 7.1.1.** Let \( a, b \geq 1 \) and assume that \( a \) and \( b \) have different parity. Then we have the following decompositions

(i) For \( a \) even and \( b \) odd,

\[
\text{Sp}(a, 1^b) = \bigoplus_{\mu} Y(\sigma_2 + 2\mu),
\]

where the sum is over all partitions \( \mu = (c, d) \), such that \( c + d = u + v \) and \( (c - d, u - v) \) is \( p \)-special, where \( a - 2 = 2u \) and \( b - 1 = 2v \).

(ii) For \( a \) odd and \( b \) even,

\[
\text{Sp}(a, 1^b) = \bigoplus_{\mu} Y(\sigma_1 + 2\mu),
\]

where the sum is over all partitions \( \mu = (c, d) \), such that \( c + d = u + v \) and \( (c - d, u - v) \) is \( p \)-special, where \( a - 1 = 2u \) and \( b = 2v \).

**Proof.** Part (i) follows directly from Theorem 6.2 for \( m = 2 \). So we can move on to the second part. Let \( a \) be odd and \( b \) be even.

We consider the tensor product \( S^a E \otimes \bigwedge^b E \). This module decomposes as

\[
S^a E \otimes \bigwedge^b E = \nabla(a + 1, 1^{b-1}) \oplus \nabla(a, 1^b),
\]

(\( \dagger \))

where the partition \( (a + 1, 1^{b-1}) \) has 2-core \( \sigma_2 \) and the partition \( (a, 1^b) \) has 2-core \( \sigma_1 \).

Now, after applying the Schur functor to (\( \dagger \)) we get, as in Section 6, that

\[
M(a, b) = \text{Sp}(a + 1, 1^{b-1}) \oplus \text{Sp}(a, 1^b).
\]

Hence, the block component of \( M(a, b) \) corresponding to the core \( \sigma_1 \) is \( \text{Sp}(a, 1^b) \).

Moreover it is easy to see, as in Corollary 4.2, that the block component of \( M(a, b) \) corresponding to the core \( \sigma_1 \) is

\[
\bigoplus_{\mu} Y(\sigma_1 + 2\mu),
\]

where the sum is over all partitions \( \mu = (c, d) \), such that \( c + d = u + v \) and \( (c - d, u - v) \) is \( p \)-special, where \( a - 1 = 2u \) and \( b = 2v \). Comparing now these two expressions we get immediately the result.

\[ \square \]
7.2 More Decomposable Specht Modules

We point out here that the decomposition of the Specht modules described in Theorem 6.2, lays the foundations for the discovery of other families of decomposable Specht modules. We describe such a case here and we take our considerations further in [8]. For simplicity throughout this subsection we assume that $K$ has characteristic 2. The Hecke algebras analogues of the results of this section will appear in a much more general form in [8].

Since $K$ has characteristic 2 there is no need from now on to distinguish between the irreducible modules $L(\lambda)$ and $\dot{L}(\lambda)$. We will need the following lemma regarding the dimension of some weight spaces for certain irreducible $G(n)$-modules.

Lemma 7.2.1. Let $a, b \geq 2$ with $a \not\equiv 0 \mod 4$, and $b$ be odd. Let $\mu = (\mu_1, \mu_2)$ be a two part partition with expansion $\mu = \mu_1 + 2\bar{\mu}$. Then

$$\dim L(\sigma_2 + 2\mu)^{(a,b,2)} = \begin{cases} 2\dim L(\mu)^{(u+1,v)} + \dim L(2\bar{\mu})^{(u-2,v)}, & \text{if } \mu_0 = (2,1); \\ 2\dim L(\mu)^{(u+1,v)}, & \text{otherwise} \end{cases}$$

where $a = 2(u + 1)$ and $b = 2v + 1$.

Proof. The weights for the $G(3)$-module $L(\sigma_2)$ are

$$\{(2, 1, 0), (2, 0, 1), (0, 2, 1), (0, 1, 2), (1, 0, 2), (1, 2, 0), (1, 1, 1)\},$$

e.g. by considering the Schur function $s(2, 1, 0)$. Since $a$ is even and $b$ is odd the only weights in this list that can contribute to the $(a, b, 2)$ weight space of $L(\sigma_2 + 2\mu) = L(\sigma_2) \otimes L(\mu)^F$ are $(2, 1, 0)$ and $(0, 1, 2)$. In these cases we can have that $(a, b, 2) = (2, 1) + 2\rho$ and $(a, b, 2) = (0, 1, 2) + 2\xi$ for some weights $\rho$ and $\xi$ of $L(\mu)$. Therefore we get that

$$\dim L(\sigma_2 + 2\mu)^{(a,b,2)} = \dim L(2\mu)^{(a-2,b-1,2)} + \dim L(2\mu)^{(a,b-1)}.$$

First notice that if $\mu_0 = 0$ then by Steinberg’s tensor product theorem we get directly that $L(2\mu)^{(a-2,b-1,2)} = 0$. Moreover since $a \not\equiv 0 \mod 4$ we also have that $L(2\mu)^{(a,b-1)} = 0$ and so the result holds in this case.

Hence, we might assume that $\mu_0 \not= 0$ and so it has one of the forms $(1), (1, 1)$ or $(2, 1)$.

We write $a$ and $b$ in the form $a = 2(u + 1)$ and $b = 2v + 1$. Since $a \not\equiv 0 \mod 4$, then $u$ must be even. We have that

$$\dim L(2\mu)^{(a-2,b-1,2)} = \dim L(\mu)^{(u,v,1)} \quad \text{and} \quad \dim L(2\mu)^{(a,b-1)} = \dim L(\mu)^{(u+1,v)}.$$
We consider now all the possible remaining cases for \( \mu^0 \). Using Steinberg's tensor product theorem one easily obtains the following.

i) Let \( \mu^0 = (1) \).

Then \( \dim L(\mu^{(u,v,1)}) = \dim L(2\bar{\mu}^{(u,v)}) \).

Also \( \dim L(\mu^{(u+1,v)}) = \dim L(2\bar{\mu}^{(u,v)}) \), since \( u \) is even, and the result holds.

ii) Let \( \mu^0 = (1,1) \).

As above we have that \( \dim L(\mu^{(u,v,1)}) = \dim L(2\bar{\mu}^{(u,v-1)}) \), since \( u \) is even.

Moreover, \( \dim L(\mu^{(u+1,v)}) = \dim L(2\bar{\mu}^{(u,v-1)}) \) and so we get again the result.

iii) Finally let \( \mu^0 = (2,1) \).

Here we get that \( \dim L(\mu^{(u,v,1)}) = \dim L(2\bar{\mu}^{(u-2,v)}) + \dim L(2\bar{\mu}^{(u,v-2)}) \).

On the other hand we have that \( \dim L(\mu^{(u+1,v)}) = \dim L(2\bar{\mu}^{(u,v-2)}) \).

This completes all the cases and proves the result.

\[ \Box \]

**Proposition 7.2.2.** Let \( a \geq 4, b \geq 3 \) with \( a \) even and \( a \not\equiv 0 \mod 4 \), and let \( b \) be odd. Then

\[ \text{Sp}(a,3,1^{b-1}) = \bigoplus_{\mu} Y(\sigma_2 + 2\mu) \oplus \bigoplus_{\rho} Y(\sigma_2 + 2\rho), \]

where the sums are over all partitions \( \mu \) and \( \rho \) such that,

\( \mu = (\mu_1, \mu_2, 1) \), \( \mu_1 + \mu_2 = u + v \) and \( (\mu_1 - \mu_2, u - v) \) is 2-special; and

\( \rho = (\rho_1, \rho_2) \) with \( \rho = \sigma_2 + 2\bar{\rho} \), \( 2(\bar{\rho}_1 + \bar{\rho}_2) = u + v - 2 \) and \( 2(\bar{\rho}_1 - \bar{\rho}_2), u - v - 2 \)

is 2-special, where \( a = 2(u + 1) \) and \( b = 2v + 1 \) for some \( u, v \geq 1 \).

**Proof.** We consider the \( G(n) \)-module \( S^a E \otimes \bigwedge^b E \otimes S^2 E \), for \( n \) sufficiently large. This has the following decomposition

\[ S^a E \otimes \bigwedge^b E \otimes S^2 E = (\nabla(a+1,1^{b-1}) \otimes S^2 E) \oplus (\nabla(a,1^b) \otimes S^2 E), \]

where the partition \( (a+1,1^{b-1}) \) has 2-core \( \sigma_1 \) and \( (a,1^b) \) has 2-core \( \sigma_2 \).

Now using Pieri's Formula we can easily see that the block component of \( \nabla(a+1,1^{b-1}) \otimes S^2 E \) corresponding to the core \( \sigma_2 \) is just \( \nabla(a+2,1^b) \).

Let \( V \) be the block component of the module \( \nabla(a,1^b) \otimes S^2 E \) corresponding to the core \( \sigma_2 \). Then \( V \) has a good filtration and again by Pieri's Formula we see that \( V \) actually fits in the short exact sequence

\[ 0 \rightarrow \nabla(a,3,1^{b-1}) \rightarrow V \rightarrow \nabla(a+2,1^b) \rightarrow 0. \]

By \([7]\) Section 4.2 (17), one has that

\[ \text{Ext}^1_{G(n)}(\nabla(a+2,1^b), \nabla(a,3,1^{b-1})) = \text{Ext}^1_{G(2)}(\nabla(a+1), \nabla(a-1,2)), \]

and since \( a \not\equiv 0 \mod 4 \), we have that \( \text{Ext}^1_{G(2)}(\nabla(a+1), \nabla(a-1,2)) = 0 \), see for e.g. \([9]\) Corollary 5.12. Therefore, \( V = \nabla(a,3,1^{b-1}) \oplus \nabla(a+2,1^b) \).

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Hence, the block component of $S^a E \otimes \wedge^b E \otimes S^2 E$ corresponding to the core $\sigma_2$ is

$$\nabla(a, 3, 1^{b-1}) \oplus \nabla(a + 2, 1^b) \oplus \nabla(a + 2, 1^b).$$

Now as in Sections 4 and 6, when we apply the Schur functor to the $G(n)$-module $S^a E \otimes \wedge^b E \otimes S^2 E$ and restrict attention to the block component of $M(a, b, 2)$ corresponding to the core $\sigma_2$, we get

$$\text{Sp}(a, 3, 1^{b-1}) \oplus \text{Sp}(a + 2, 1^b) \oplus \text{Sp}(a + 2, 1^b) = \bigoplus_{\mu} Y(\sigma_2 + 2\mu)(d_\mu),$$

where the sum is over all partitions $\mu = (\mu_1, \mu_2, \mu_3)$ with $d_\mu = \dim L(\sigma_2 + 2\mu)(a,b,2) \neq 0$.

Assume first that $\mu_3 \neq 0$. If $L(\sigma_2 + 2\mu)(a,b,2) \neq 0$, then $(a, b, 2) \leq \sigma_2 + 2\mu$ and so we must have $\mu_3 = 1$.

Therefore in this case

$$\dim L(\sigma_2 + 2\mu)(a,b,2) = \dim L(\sigma_2 + 2(\mu - \omega_3))(a-2,b-2) = \dim L(2(\mu - \omega_3))(a-4,b-3).$$

We write $a = 2(u + 1)$ and $b = 2v + 1$. Then

$$\dim L(2(\mu - \omega_3))(a-4,b-3) = \dim L(\mu_1 - 1, \mu_2 - 1)(a-1,v-1),$$

and by Lemma 3.3 this dimension is 1 if $(\mu_1 - \mu_2, u - v)$ is 2-special and 0 otherwise.

Assume now that $\mu_3 = 0$ and so $\mu = (\mu_1, \mu_2)$. Then Lemma 7.2.1 and Lemma 3.3 give the dimension of $L(\sigma_2 + 2(\mu_1, \mu_2))(a,b,2)$.

Putting these two points together we conclude that the direct sum

$$\bigoplus_{\mu} Y(\sigma_2 + 2\mu) \oplus \bigoplus_{\nu} Y(\sigma_2 + 2\nu)(2) \oplus \bigoplus_{\rho} Y(\sigma_2 + 2\rho),$$

(1)

where the sums are over all partitions $\mu, \nu$ and $\rho$ respectively such that,

$\mu = (\mu_1, \mu_2, 1), \mu_1 + \mu_2 = u + v$ and $(\mu_1 - \mu_2, u - v)$ is 2-special;

$\nu = (\nu_1, \nu_2), \nu_1 + \nu_2 = u + v + 1$ and $(\nu_1 - \nu_2, u - v + 1)$ is 2-special;

$\rho = (\rho_1, \rho_2)$ with $\rho = \sigma_2 + 2\tilde{\rho}, 2(\tilde{\rho_1} + \tilde{\rho_2}) = u + v - 2$ and $(2(\tilde{\rho_1} - \tilde{\rho_2}), u - v - 2)$ is 2-special.

On the other hand, by Theorem 6.2 we have

$$\text{Sp}(a + 2, 1^b) = \bigoplus_{\nu} Y(\sigma_2 + 2\nu),$$

(2)

where the sum is over all partitions $\nu = (\nu_1, \nu_2)$ such that $\nu_1 + \nu_2 = u + v + 1$ and $(\nu_1 - \nu_2, u - v + 1)$ is 2-special.

Comparing now the decompositions (1) and (2) we get immediately that
\[ \text{Sp}(a,3,1^{b-1}) = \bigoplus_{\mu} Y(\sigma_2 + 2\mu) \oplus \bigoplus_{\rho} Y(\sigma_2 + 2\rho), \]

where the sums run over all partitions \( \mu \) and \( \rho \) as described in the statement.

\[ \square \]

**Remark 7.2.3.** Proposition 7.2.2 gives many new decomposable Specht modules of the form \( \text{Sp}(a,3,1^b) \) which do not appear in the list of Dodge and Fayers, [4, Theorem 3.1 and Corollary 3.2]. For instance we have the following example.

**Example 7.2.4.** For \( a = 14 \) and \( b = 9 \) we have that

\[ \text{Sp}(14,3,1^8) = Y(14,9,2) \oplus Y(18,5,2) \oplus Y(14,11). \]

Of course we can now obtain new decomposable Specht modules by considering the linear duals of the modules appeared above. More precisely, by [12, Theorem 8.15] we have that in characteristic 2, \( \text{Sp}(\lambda)^* = \text{Sp}(\lambda') \), where \( \text{Sp}(\lambda)^* \) is the linear dual of \( \text{Sp}(\lambda) \) and \( \lambda' \) is the transpose of the partition \( \lambda \). Moreover, we have that the Young modules are self-dual modules.

Therefore by considering the linear duals of the modules appeared in Proposition 7.2.2 we get the following result.

**Corollary 7.2.5.** Let \( a \geq 4, b \geq 3 \) with \( a \) even and \( a \not\equiv 0 \mod 4 \), and \( b \) odd. Then

\[ \text{Sp}(b+1,2,2,1^{a-3}) = \bigoplus_{\mu} Y(\sigma_2 + 2\mu) \oplus \bigoplus_{\rho} Y(\sigma_2 + 2\rho), \]

where the sums are over all partitions \( \mu \) and \( \rho \) such that,

- \( \mu = (\mu_1, \mu_2, 1) \), \( \mu_1 + \mu_2 = u + v \) and \( (\mu_1 - \mu_2, u - v) \) is 2-special; and
- \( \rho = (\rho_1, \rho_2) \) with \( \rho = \sigma_2 + 2\tilde{\rho} \), \( 2(\tilde{\rho}_1 + \tilde{\rho}_2) = u + v - 2 \) and \( (2(\tilde{\rho}_1 - \tilde{\rho}_2), u - v - 2) \) is 2-special, where \( a = 2(u + 1) \) and \( b = 2v + 1 \) for some \( u, v \geq 1 \).

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