Perturbation Theory of KMS States

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Abstract. We extend the new perturbation formula of equilibrium states by Hastings to KMS states of general $W^*$-dynamical systems.

1. Introduction

Perturbation theory of equilibrium states is a basic problem in quantum statistical physics which has been studied from old time. For a finite system with Hamiltonian $H$, equilibrium state with inverse temperature $\beta$ is defined by the Gibbs state:

$$\frac{\text{Tr}(e^{-\beta H} A)}{\text{Tr}(e^{-\beta H})}.$$  \hfill (1)

If we perturb $H$ by $V$, $e^{-\beta H}$ is replaced by $e^{-\beta(H+V)}$. Using the Duhamel formula, $e^{-\beta(H+V)}$ can be obtained from $e^{-\beta H}$ by multiplying

$$E^\tau_{V} \left( \frac{i\beta}{2} \right) = \sum_{n\geq0} \left( -\frac{\beta}{2} \right)^n \int_{0\leq s_n \leq \cdots \leq s_1 \leq 1} ds_1 \cdots ds_n \tau_{i\beta s_n}(V) \cdots \tau_{i\beta s_1}(V)$$ \hfill (2)

and its adjoint from left and right, respectively. Here, $\tau$ is the dynamics given by $H$, i.e., $\tau_t(A) := e^{itH} A e^{-itH}$. Namely, we have

$$e^{-\beta(H+V)} = E^\tau_{V} \left( \frac{i\beta}{2} \right) e^{-\beta H} \left( E^\tau_{V} \left( \frac{i\beta}{2} \right) \right)^*.$$ \hfill (3)

In the thermodynamic limit, this $E^\tau_{V} \left( \frac{i\beta}{2} \right)$ can diverge. In spite of that, an analogous representation exists in infinite systems as well, thanks to Araki [1]. (See Theorem 9.) Corresponding to the possible divergence of $E^\tau_{V} \left( \frac{i\beta}{2} \right)$, the representation by Araki is written in terms of an unbounded operator.

In [5], Hastings introduced a new representation of $e^{-\beta(H+V)}$ with $E^\tau_{V} \left( \frac{i\beta}{2} \right)$ replaced by some bounded operator $O(V)$:

$$e^{-\beta(H+V)} = O(V) e^{-\beta H} (O(V))^*.$$ \hfill (4)
The main result of this paper is the extension of Hasting’s result to $W^*$-dynamical systems. For the notations and known facts about $W^*$-dynamical systems used in this paper, see “Appendix A” and Appendix B. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{M} \subset B(\mathcal{H})$ a von Neumann algebra. We denote the set of self-adjoint elements in $\mathcal{M}$ by $\mathcal{M}_{sa}$. The predual of $\mathcal{M}$ is denoted by $\mathcal{M}^*$. Let $\Omega \in \mathcal{H}$ be a cyclic and separating unit vector for $\mathcal{M}$. Define the functional $\omega \in \mathcal{M}^*$ by $\omega(A) = (\Omega, A\Omega)$ for all $A \in \mathcal{M}$. Let $\Delta$ be the modular operator and $J$ the modular conjugation associated with $(\mathcal{M}, \Omega)$. Let $\sigma$ be the modular automorphism group, and $\sigma^Q$ its perturbation by $Q \in \mathcal{M}_{sa}$. (We use the convention $\beta = -1$. Note that $\sigma^t$ corresponds to $\tau_{-\beta t}$ above.) Define Liouvillian $L$ of $\sigma$ by $L = \log \Delta$. (See “Appendix B.”) For $V \in \mathcal{M}_{sa}$, define $\Omega_V = e^{L+V/2}\Omega$ and the functional $\omega_V \in \mathcal{M}^*$ by $\omega_V(A) = (\Omega_V, A\Omega_V)$ for all $A \in \mathcal{M}$. It was proven in [1] that $\omega_V$ is a $(\sigma^V, -1)$-KMS state. The explicit expansion of $\omega_V$ is given, in terms of unbounded operator, the Liouvillian $L$. This corresponds to the expansion (3). (See Appendix Theorem 9(e).)

Our main theorem is an alternative representation of $\omega_V$, in terms of bounded operators, which corresponds to the expression by Hastings (4).

Define the real function $f$ on $\mathbb{R}$ by
\[
 f(t) = \begin{cases} 
 0 & (t = 0) \\
 \sum_{n=0}^{\infty} \frac{2}{2n+1} e^{-2\pi(n+\frac{1}{2})|t|} = -2 \log \left( \tanh \left( \frac{\pi|t|}{2} \right) \right) & (t \neq 0) 
\end{cases}
\] (5)
Then, $f \in L^1(\mathbb{R})$. Throughout this paper, we fix this $f$. Properties of $f$ are collected in “Appendix C.”

Define the maps $\Phi : \mathcal{M}_{sa} \times \mathbb{R} \to \mathcal{M}_{sa}$ by
\[
 \Phi(V; u) = \int_{\mathbb{R}} f(t) \sigma^u_t (V) \, dt,
\] (6)
for all $V \in \mathcal{M}_{sa}$ and $u \in \mathbb{R}$. The right-hand side integral is defined in the $\sigma$-weak topology [2]. Furthermore, $\Phi(V; u)$ is continuous in the norm topology with respect to $u \in \mathbb{R}$. (See Lemma 7.) Properties of $\Phi(V; u)$ we use in this paper are shown in Sect. 4. Define $\theta : \mathcal{M}_{sa} \to \mathcal{M}$ by
\[
 \theta(V) = \sum_{n=0}^{\infty} \int_{0}^{1} du_1 \int_{0}^{u_1} du_2 \cdots \int_{0}^{u_{n-1}} du_n \Phi(V; u_1) \cdots \Phi(V; u_n),
\] (7)
for all $V \in \mathcal{M}_{sa}$. The integration and summation are in the norm topology. Here is our main theorem.

**Theorem 1.** For any $V \in \mathcal{M}_{sa}$, we have
\[
 \omega_V(A) = (\Omega, \theta(V)^* A \theta(V) \Omega), \quad A \in \mathcal{M}.
\] (8)

**2. Proof of Theorem 1 for Analytic $V$**

In this section, we show Theorem 1 for analytic $V$. We will use facts in “Appendix B.” For $s \in \mathbb{R}$, define the vector $\Omega_s := \Omega_s V = e^{L+V/2}\Omega$ and the
functional \( \omega_s \in \mathcal{M} \), by \( \omega_s(A) := \omega_s V(A) = (\Omega_s, A\Omega_s) \) for all \( A \in \mathcal{M} \). Especially, we have \( \Omega_0 = \Omega \) and \( \omega = \omega \). For \( s \in \mathbb{R} \), the \( W^* \)-dynamics \( \sigma^{sV} \) satisfies \( \sigma^{sV}_t(A) := e^{it(L+sV)} A e^{-it(L+sV)} \) for all \( A \in \mathcal{M} \) and \( t \in \mathbb{R} \). Define \( \Delta_{\Omega_s} = e^{(L+sV-\sigma_jV)J} \) which is the modular operator of the pair \( (\mathcal{M}, \Omega_s) \). By Theorem 9(d), \( \Omega_s \) is continuous in the norm topology with respect to \( s \in \mathbb{R} \).

Note that in the lemmas below we use \( \sigma^{sV}_{-\frac{i\pi}{2}}(V) \) and \( E_{sV}(-iz) \) etc. The analyticity of \( V \) is used there.

**Lemma 2.** Let \( V \in \mathcal{M} \) be a \( \sigma \)-entire analytic and self-adjoint element. Then, \( \Omega_s \) is differentiable with respect to \( s \in \mathbb{R} \) in the norm topology, and we have

\[
\frac{d}{ds} \Omega_s = \frac{1}{2} \int_0^1 du \sigma^{sV}_{-\frac{i\pi}{2}}(V) \Omega_s.
\]

**Proof.** First we recall several equalities which can be shown using identity theorem: We have \( \Omega \in D \left( e^{z(L+sV)} \right) \) and

\[
e^{z(L+sV)} \Omega = E_{sV}(-iz) \Omega,
\]

for all \( z \in \mathbb{C} \) and \( s \in \mathbb{R} \).

We also have \( \Omega \in D \left( e^{z\frac{L+sV}{2}} e^{(1-z)\frac{L+sV}{2}} \right) \) for all \( z \in \mathbb{C} \) and \( s, \tilde{s} \in \mathbb{R} \),

\[
e^{z\frac{L+sV}{2}} e^{(1-z)\frac{L+sV}{2}} \Omega = E_{sV} \left( -\frac{iz}{2} \right) E_{\tilde{s}V} \left( \frac{i\tilde{z}}{2} \right)^* e^{\frac{L+sV}{2}} \Omega.
\]

The proofs are analogous to that of (15) and (16) that we omit them.

Since the right-hand side of (11) is analytic in norm, the left-hand side is analytic too. Therefore, it is differentiable with respect to \( z \) in norm. We claim that \( \Omega \in D \left( e^{z\frac{L+sV}{2}} V e^{(1-z)\frac{L+sV}{2}} \right) \) and the derivative is given by

\[
\frac{d}{dz} \left( e^{z\frac{L+sV}{2}} e^{(1-z)\frac{L+sV}{2}} \right) \Omega = \frac{s - \tilde{s}}{2} e^{z\frac{L+sV}{2}} V e^{(1-z)\frac{L+sV}{2}} \Omega
\]

for all \( z \in \mathbb{C} \) and \( s, \tilde{s} \in \mathbb{R} \).

To prove this, define the subspace \( D_s \subset \mathcal{H} \) for \( s \in \mathbb{R} \), by

\[
D_s := \bigcup_{N \in \mathbb{N}} E_{[-N,N]} \mathcal{H},
\]

where \( \{ E_\lambda \}_{\lambda \in \mathbb{R}} \) is the projection-valued measure such that \( \int_\mathbb{R} dE_\lambda \lambda = L + sV \). Then, \( D_s \) is a core for \( e^{z(L+sV)} \) for all \( z \in \mathbb{C} \).

For \( \xi \in D_s \), we have

\[
\frac{d}{dz} \left( e^{z\frac{L+sV}{2}} e^{(1-z)\frac{L+sV}{2}} \right) \xi = \frac{d}{dz} \left( e^{z\frac{L+sV}{2}} \xi, e^{(1-z)\frac{L+sV}{2}} \Omega \right)
\]

\[
= \left( \left( \frac{L + sV}{2} \right) e^{z\frac{L+sV}{2}} \xi, e^{(1-z)\frac{L+sV}{2}} \Omega \right)
\]

\[
+ \left( e^{\frac{z(L+sV)}{2}} \xi, \left( \frac{L + \tilde{s}V}{2} \right) e^{(1-z)\frac{L+sV}{2}} \Omega \right)
\]

\[
= \frac{s - \tilde{s}}{2} \left( e^{\frac{z(L+sV)}{2}} \xi, Ve^{(1-z)\frac{L+sV}{2}} \Omega \right).
\]
We claim
\[ \left( 1 - \frac{z}{L + 2sV} \right) L + \bar{s}V \Omega \in D \left( e^{z \left( \frac{L + 2sV}{2} \right)} \right) \] (15)
and
\[ e^{z \left( \frac{L + 2sV}{2} \right)} V e^{(1-z) \left( \frac{L + 2sV}{2} \right)} \Omega = E_{sV} \left( \frac{-iz}{2} \right) \sigma_{-\frac{z}{2}} \left( V \right) \left( E_{sV} \left( \frac{-i\bar{z}}{2} \right) \right)^* E_{sV} \left( \frac{-iz}{2} \right) \times \left( E_{\bar{s}V} \left( \frac{-i\bar{z}}{2} \right) \right)^* e^{z \left( \frac{L + 2sV}{2} \right)} \Omega. \] (16)

From (14) and (15), we obtain (12).

To prove (15) and (16), let \( \xi \) be an arbitrary element of \( D_s \) and consider the following two complex functions on \( \mathbb{C} \):
\[ z \mapsto \left( e^{z \left( \frac{L + 2sV}{2} \right)} \xi, V e^{(1-z) \left( \frac{L + 2sV}{2} \right)} \Omega \right) \] (17)
and
\[ z \mapsto \left( \xi, E_{sV} \left( \frac{-i\bar{z}}{2} \right) \sigma_{-\frac{z}{2}} \left( V \right) \left( E_{sV} \left( \frac{-i\bar{z}}{2} \right) \right)^* E_{sV} \left( \frac{-iz}{2} \right) \times \left( E_{\bar{s}V} \left( \frac{-i\bar{z}}{2} \right) \right)^* e^{z \left( \frac{L + 2sV}{2} \right)} \Omega \right). \] (18)

These two functions coincide on \( i\mathbb{R} \) since for \( t \in \mathbb{R} \) we have
\[ \left( e^{it \left( \frac{L + 2sV}{2} \right)} \xi, V e^{(1-it) \left( \frac{L + 2sV}{2} \right)} \Omega \right) = \left( \xi, e^{it \left( \frac{L + 2sV}{2} \right)} V e^{-it \left( \frac{L + 2sV}{2} \right)} e^{\frac{L + 2sV}{2}} \Omega \right) = \left( \xi, \sigma^{sV} \left( \frac{t}{2} \right) e^{it \left( \frac{L + 2sV}{2} \right)} e^{-it \left( \frac{t}{2} \right)} \right)^* \left( e^{it \left( \frac{L + 2sV}{2} \right)} e^{-it \left( \frac{t}{2} \right)} \right)^* e^{\frac{L + 2sV}{2}} \Omega. \] (19)

The second equality follows by (76), and the third equality follows due to (75) and (77). Since these two functions are analytic (recall (74) and (10)), they coincide on \( \mathbb{C} \). Since \( D_s \) is a core for \( e^{z \left( \frac{L + 2sV}{2} \right)} \), we obtain the claim (15) and (16). Hence, we have completed the proof of (12).

Finally, we show (9). To prove this, first note that from (16) and the expansion of \( E_{sV}^\sigma(z) \) (79), we obtain the following continuity in the norm:
\[ \lim_{s \to \tilde{s}} e^{z \left( \frac{L + 2sV}{2} \right)} V e^{(1-z) \left( \frac{L + 2sV}{2} \right)} \Omega = e^{z \left( \frac{L + 2\tilde{s}V}{2} \right)} V e^{(1-z) \left( \frac{L + 2\tilde{s}V}{2} \right)} \Omega. \] (20)

For \( s, \tilde{s} \in \mathbb{R} \), since the map
\[ \mathbb{R} \ni u \mapsto e^{u \left( \frac{L + 2sV}{2} \right)} e^{(1-u) \left( \frac{L + 2\tilde{s}V}{2} \right)} \Omega \in \mathcal{H} \] (21)
is differentiable in the norm topology and satisfies
\[
\frac{d}{du} e^{u(L+sV)} e^{(1-u)(L+sV)} \Omega = \left( \frac{s - \tilde{s}}{2} \right) e^{u(L+sV)} V e^{(1-u)(L+sV)} \Omega, \tag{22}
\]
(from (12)), we have
\[
\Omega_s - \Omega_{\tilde{s}} = e^{L+sV} \Omega - e^{L+\tilde{s}V} \Omega
= \int_0^1 du \frac{d}{du} e^{u(L+sV)} e^{(1-u)(L+sV)} \Omega
= \frac{s - \tilde{s}}{2} \int_0^1 du e^{u(L+sV)} V e^{(1-u)(L+sV)} \Omega.
\tag{23}
\]
Therefore, from the continuity (20) and (16), using the Lebesgue’s dominated convergence theorem, \( \Omega_s \) is differentiable with respect to \( s \) in the norm topology and we have
\[
\frac{d}{ds} \Omega_s = \frac{1}{2} \int_0^1 du \left( \Omega_s, \sigma_{iu}^s V (V) A \Omega_s \right) \tag{24}
\]
\[
\text{for all } s \in \mathbb{R}.
\]

**Lemma 3.** Let \( V \in \mathcal{M} \) be a self-adjoint \( \sigma \)-entire analytic element, and \( A \in \mathcal{M} \). Then, \( \omega_s(A) \) is differentiable with respect to \( s \) and
\[
\frac{d}{ds} \omega_s(A) = \int_0^1 du \left( \Omega_s, \sigma_{iu}^s V (V) A \Omega_s \right) \tag{25}
\]
\[
\text{for all } s \in \mathbb{R}.
\]

**Proof.** By Lemma 2, \( \omega_s(A) \) is differentiable and we have
\[
\frac{d}{ds} \omega_s(A) = \frac{d}{ds} \left( \Omega_s, A \Omega_s \right)
= \left( \frac{1}{2} \int_0^1 du \sigma_{iu}^s V (V) \Omega_s, A \Omega_s \right) + \left( \Omega_s, A \left( \frac{1}{2} \int_0^1 du \sigma_{iu}^s V (V) \Omega_s \right) \right)
= \frac{1}{2} \int_0^1 du \left( \left( \Omega_s, \sigma_{iu}^s V (V) A \Omega_s \right) + \left( \Omega_s, A \sigma_{iu}^s V (V) \Omega_s \right) \right), \tag{26}
\]
We obtain
\[
\left( \Omega_s, A \sigma_{iu}^s V (V) \Omega_s \right) = \left( \Omega_s, \sigma_{i(1-\frac{1}{2})}^s V (V) A \Omega_s \right) \tag{27}
\]
since we have
\[
\left( \Omega_s, A \sigma_{iu}^s V (V) \Omega_s \right) = \left( A^* \Omega_s, \sigma_{iu}^s V (V) \Omega_s \right)
= \left( S_{\Omega_s} A \Omega_s, S_{\Omega_s} \sigma_{iu}^s V (V) \Omega_s \right) = \left( \Delta_{\Omega_s}^{\frac{1}{2}} \sigma_{iu}^s V (V) \Omega_s, \Delta_{\Omega_s}^{\frac{1}{2}} A \Omega_s \right)
\]
Therefore, we have
\[
\frac{d}{ds} \omega_s(A) = \frac{1}{2} \int_0^1 du \left( \Omega_s, \sigma^{sV}_{iu}(V)A\Omega_s \right) + \frac{1}{2} \int_0^1 du \left( \Omega_s, \sigma^{sV}_{i(1-\frac{V}{2})}(V)A\Omega_s \right)
\]
\[
= \int_0^1 du \left( \Omega_s, \sigma^{sV}_{iu}(V)A\Omega_s \right) + \int_0^1 du \left( \Omega_s, \sigma^{sV}_{i}(V)A\Omega_s \right)
\]
\[
= \int_0^1 du \left( \Omega_s, \sigma^{sV}_{iu}(V)A\Omega_s \right). \tag{29}
\]

Lemma 4. Let \( V \in \mathfrak{M} \) be a \( \sigma \)-entire analytic and self-adjoint element, and \( A \in \mathfrak{M} \). Let \( f \in L^1(\mathbb{R}) \) be the function defined in (5). Then, for all \( s \in \mathbb{R} \), we have
\[
\int_0^1 du \left( \Omega_s, \sigma^{sV}_{iu}(V)A\Omega_s \right) = (\Phi(V; s)\Omega_s, A\Omega_s) + (\Omega_s, A\Phi(V; s)\Omega_s), \tag{30}
\]
where
\[
\Phi(V; s) = \int_{\mathbb{R}} f(t) \sigma^{sV}_t(V) dt. \tag{31}
\]

Proof. Let \( G \) be a function defined on \( \mathbb{R}_{>0} \) by
\[
G(\lambda) = (1 + \lambda)^{-1} \int_0^1 du \lambda^u, \quad \lambda > 0. \tag{32}
\]
Define a function \( F \) on \( \mathbb{R} \) by
\[
F(x) = \begin{cases} 
\frac{e^x - 1}{e^x + 1} & (x \neq 0) \\
1/2 & (x = 0)
\end{cases}. \tag{33}
\]
It follows that \( F(x) = G(e^x) \) for all \( x \in \mathbb{R} \). Therefore, from (87) in “Appendix C,” we have
\[
G(e^x) = \int_{\mathbb{R}} f(t)e^{itx} dt, \tag{34}
\]
for all \( x \in \mathbb{R} \). We now show (30) for any \( \sigma \)-entire analytic element \( A \). We have
\[
\int_0^1 du \left( \Omega_s, \sigma^{sV}_{iu}(V)A\Omega_s \right) = \int_0^1 du \left( \sigma^{sV}_{iu}(V)\Omega_s, A\Omega_s \right)
\]
\[
= \left( \int_0^1 du \Delta^u_{\Omega_s} V\Omega_s, A\Omega_s \right) \tag{30}
\]
\begin{equation}
\left( \int_0^1 du \Delta_{\Omega_s}^u V \Omega_s, (1 + \Delta_{\Omega_s})^{-1}(1 + \Delta_{\Omega_s}) A\Omega_s \right)
= \left( (1 + \Delta_{\Omega_s})^{-1} \int_0^1 du \Delta_{\Omega_s}^u V \Omega_s, (1 + \Delta_{\Omega_s}) A\Omega_s \right).
\end{equation}

We have $A\Omega_s \in D(\Delta_{\Omega_s})$ due to the $\sigma$-entire analyticity of $A$.

We have
\begin{equation}
(1 + \Delta_{\Omega_s})^{-1} \int_0^1 du \Delta_{\Omega_s}^u V \Omega_s = G(\Delta_{\Omega_s}) V \Omega_s = \int_\mathbb{R} dt \Phi(t) \Delta_{\Omega_s}^u V \Delta_{\Omega_s}^{-u} \Omega_s = \Phi(V; s)\Omega_s.
\end{equation}

Therefore, we have
\begin{equation}
\int_0^1 du \left( \Omega_s, \sigma_{iu}^{sV}(V) A\Omega_s \right) = \left( (1 + \Delta_{\Omega_s})^{-1} \int_0^1 du \Delta_{\Omega_s}^u V \Omega_s, (1 + \Delta_{\Omega_s}) A\Omega_s \right)
= (\Phi(V; s)\Omega_s, (1 + \Delta_{\Omega_s}) A\Omega_s)
= (\Phi(V; s)\Omega_s, A\Omega_s) + (\Phi(V; s)\Omega_s, \Delta_{\Omega_s} A\Omega_s))
= (\Omega_s, \Phi(V; s) A\Omega_s) + (\Delta_{\Omega_s}^{\frac{1}{2}} \Phi(V; s)\Omega_s, \Delta_{\Omega_s}^{\frac{1}{2}} A\Omega_s)
= (\Omega_s, \Phi(V; s) A\Omega_s) + (J \Delta_{\Omega_s}^{\frac{1}{2}} A\Omega_s, J \Delta_{\Omega_s}^{\frac{1}{2}} \Phi(V; s)\Omega_s)
= (\Omega_s, \Phi(V; s) A\Omega_s) + (\Omega_s, A\Phi(V; s)\Omega_s). \tag{37}
\end{equation}

We repeatedly used the self-adjointness of $\Phi(V; s)$.

To extend (30) to general $A \in \mathcal{M}$, we just need to notice that both sides of (30) are continuous in $A$ with respect to the strong topology and recall that $\mathcal{M}_\sigma$ is strong-dense in $\mathcal{M}$. $\square$

Due to Lemma 7, we can define the map $\theta : \mathcal{M}_{sa} \times \mathbb{R} \to \mathcal{M}_{sa}$ by
\begin{equation}
\theta(V; s) = \sum_{n=0}^{\infty} \int_0^s du_1 \int_0^{u_1} du_2 \ldots \int_0^{u_{n-1}} du_n \Phi(V; u_1) \ldots \Phi(V; u_n). \tag{38}
\end{equation}

for all $V \in \mathcal{M}_{sa}$ and $s \in \mathbb{R}$.

We now prove the statement of Theorem 1 for $\sigma$-entire analytic $V$.

**Lemma 5.** Let $V \in \mathcal{M}$ be a $\sigma$-entire analytic and self-adjoint element. Then, we have
\begin{equation}
\omega_s(A) = \omega(\theta(V; s)^* A \theta(V; s)), \quad s \in \mathbb{R}, \quad A \in \mathcal{M}. \tag{39}
\end{equation}

**Proof.** For each $s \in \mathbb{R}$, we define the map $\mathcal{L}_s : \mathcal{M} \to \mathcal{M}$ by
\begin{equation}
\mathcal{L}_s(X) = \Phi(V; s) X + X \Phi(V; s), \quad X \in \mathcal{M}. \tag{40}
\end{equation}

By Lemma 3 and Lemma 4, for any $A \in \mathcal{M}$, we have
\begin{equation}
\frac{d}{ds} \omega_s(A) = \int_0^1 du \left( \Omega_s, \sigma_{iu}^{sV}(V) A\Omega_s \right)
= (\Phi(V; s)\Omega_s, A\Omega_s) + (\Omega_s, A\Phi(V; s)\Omega_s)
= \omega_s \circ \mathcal{L}_s(A). \tag{41}
\end{equation}
Next, define the state $\phi_s \in \mathfrak{M}_s$ by
\[
\phi_s(A) = \omega(\theta(V; s)^* A \theta(V; s)), \quad A \in \mathfrak{M}.
\] (42)

We now show that $\phi_s$ satisfies the same differential equation as $\omega_s$:
\[
\frac{d}{ds} \phi_s(A) = \phi_s \circ \mathcal{L}_s(A), \quad A \in \mathfrak{M}.
\] (43)

By Lemma 7, the map $\mathbb{R} \ni s \mapsto \Phi(V; s) \in \mathfrak{M}$ is continuous in the norm topology. Therefore, $\theta(V; s)$ is differentiable in the norm topology and we have
\[
\frac{d}{ds} \theta(V; s) = \Phi(V; s) \theta(V; s).
\] (44)

Since $\Phi(V; s)$ is self-adjoint, we obtain (43):
\[
\frac{d}{ds} \phi_s(A) = \omega(\theta(V; s)^* \Phi(V; s)^* A \theta(V; s)) + \omega(\theta(V; s)^* A \Phi(V; s) \theta(V; s))
= \omega(\theta(V; s)^* \mathcal{L}_s(A) \theta(V; s))
= \phi_s \circ \mathcal{L}_s(A).
\] (45)

As $\phi_s$ and $\omega_s$ satisfy the same differential equation and $\omega_0 = \phi_0$, we obtain $\omega_s = \phi_s$ for all $s \in \mathbb{R}$. $\Box$

3. Proof of Theorem 1

In order to extend the result to general $V$, we need the following continuity of $\theta(V; s)\Omega$.

**Lemma 6.** Let $\{V_m\}_{m \in \mathbb{N}} \subset \mathfrak{M}_{sa}$ be a sequence such that $V_m \to V \in \mathfrak{M}_{sa}$ strongly as $m \to \infty$. Then, we have $\|\theta(V_m; s)\Omega - \theta(V; s)\Omega\| \to 0$.

**Proof.** From the uniform boundedness principle, we have $c := \sup_m \|V_m\| < \infty$. By Lemma 7, $\Phi(V_m; s)$ converges to $\Phi(V; s)$ strongly. From this convergence and the uniform boundedness
\[
\sup_m \|\Phi(V_m; s)\| \leq \|f\|_{L^1}, \quad \sup_m \|V_m\| \leq c \|f\|_{L^1},
\] (46)

we have
\[
\lim_{m \to \infty} \|\Phi(V_m; u_1) \cdots \Phi(V_m; u_n) \Omega - \Phi(V; u_1) \cdots \Phi(V; u_n) \Omega\| = 0.
\] (47)

We also have
\[
\|\Phi(V_m; u_1) \cdots \Phi(V_m; u_n) \Omega - \Phi(V; u_1) \cdots \Phi(V; u_n) \Omega\| \leq (c^n + \|V\|^n) \|f\|_{L^1}^n,
\] (48)
for any $u_1, \ldots, u_n \in \mathbb{R}$ and $n \in \mathbb{N}$. From the definition of $\theta(V; s)$, we have
\[
\|\theta(V_m; s)\Omega - \theta(V; s)\Omega\|
\leq \sum_{n=0}^\infty \int_0^s du_1 \int_0^{u_1} du_2 \cdots \int_0^{u_{n-1}} du_n \|\Phi(V_m; u_1) \cdots \Phi(V_m; u_n)
- \Phi(V; u_1) \cdots \Phi(V; u_n)\| \Omega\|.
\] (49)
From (47) and (48), applying Lebesgue’s dominated convergence theorem, we obtain
\[
\lim_{m \to \infty} \| \theta(V_m; s) \Omega - \theta(V; s) \Omega \| = 0.
\] (51)

\[\square\]

**Proof of Theorem 1** Note that \( \theta(V) = \theta(V; 1) \). There exists a sequence \( \{V_m\} \subset \mathcal{M}_{sa} \) such that \( V_m \) is \( \sigma \)-entirely analytic for all \( m \in \mathbb{N} \) and that \( V_m \to V \) strongly. Let \( A \in \mathcal{M} \). By Lemma 6, we have
\[
(\Omega, \theta(V_m)^* A \theta(V_m) \Omega) \to (\Omega, \theta(V)^* A \theta(V) \Omega)
\] (52)
as \( m \to \infty \).

By Theorem 9(d), \( \omega_{V_m} \) converges to \( \omega_V \) in the norm topology. By Lemma 5, we have
\[
\omega_{V_m}(A) = (\Omega, \theta(V_m)^* A \theta(V_m))
\] (53)
for all \( m \in \mathbb{N} \). Therefore, we conclude that
\[
\omega_V(A) = (\Omega, \theta(V)^* A \theta(V) \Omega).
\] (54)
\[\square\]

### 4. Properties of \( \Phi(V; s) \)

In this section, we collect some properties of \( \Phi \) used in Sects. 2 and 3.

**Lemma 7.** (1) For all \( V \in \mathcal{M}_{sa} \) and \( s \in \mathbb{R} \), we have
\[
\| \Phi(V; s) \| \leq \| f \|_{L^1} \| V \|.
\] (55)

(2) For all \( V \in \mathcal{M}_{sa} \), the map defined by \( \mathbb{R} \ni s \mapsto \Phi(V; s) \in \mathcal{M} \) is norm-continuous.

(3) For any \( s \in \mathbb{R} \) and sequence \( V_N \) in \( \mathcal{M}_{sa} \) such that \( V_N \to V \in \mathcal{M}_{sa} \) in the strong operator topology, we have \( \Phi(V_N; s) \to \Phi(V; s) \) in the strong operator topology.

**Proof.** For each \( V \in \mathcal{M}_{sa} \), \( t \in \mathbb{R} \) and \( n = 0, 1, \ldots \), set
\[
T_n(V; t) := i^n \int_{0 \leq s_n \leq \cdots \leq s_1 \leq t} ds_1 \cdots ds_n (\sigma_{s_n}(V) \cdots \sigma_{s_1}(V)).
\] (56)

Here, the integral is defined in \( \sigma \)-weak topology. Note that the map \( \mathbb{R} \ni t \mapsto T_n(V; t) \in \mathcal{M} \) is norm-continuous. Furthermore, its norm is bounded
\[
\| T_n(V; t) \| \leq \frac{|t|^n \| V \|^n}{n!}.
\] (57)

We have
\[
E^\sigma_{sV}(t) = \sum_{n \geq 0} s^n T_n(V; t),
\] (58)
where the summation converges in norm. From (75), we have
\[
\sigma_t^s V(A) = E_s^\sigma(t) \sigma_t(A) E_s^\sigma(t)^* = \sum_{n \geq 0} \sum_{m \geq 0} s^{n+m} T_n(V; t) \sigma_t(A) (T_m(V; t))^*,
\]
where the summation in the last term converges in the norm topology.

For each \( \varepsilon > 0 \), we fix \( M_\varepsilon > 0 \) such that
\[
\int_{|t| \geq M_\varepsilon} dt \ |f(t)| < \varepsilon.
\]

Now, let us prove the claim of the lemma. The first statement of the lemma is trivial. Let us prove the second statement of the lemma. For all \( V \in \mathcal{M}_{sa}, \xi \in \mathcal{H}, s, \tilde{s} \in \mathbb{R} \) and \( \varepsilon > 0 \), using (57) and (60), we have
\[
\| (\Phi(V; \tilde{s}) - \Phi(V; s)) \xi \|
\leq \int_{|t| \leq M_\varepsilon} dt \ f(t) \left\| \left( \sigma_t^{\tilde{s}V} - \sigma_t^s V \right) \xi \right\|
+ \int_{|t| \geq M_\varepsilon} dt \ f(t) \left\| \left( \sigma_t^{\tilde{s}V} - \sigma_t^s V \right) \xi \right\|
\leq \int_{|t| \leq M_\varepsilon} dt \ f(t) \sum_{n \geq 0} \sum_{m \geq 0} |\tilde{s}^{n+m} - s^{n+m}| \left\| \left( T_n(V; t) \sigma_t(V) (T_m(V; t))^* \right) \xi \right\|
+ 2\varepsilon \| \xi \| \| V \|
\leq \left( \sum_{n \geq 0} \sum_{m \geq 0} |\tilde{s}^{n+m} - s^{n+m}| \| f \|_{L^1} \frac{|M_\varepsilon|^{n+m}}{n!m!} \| V \|^{n+m+1} + 2 \varepsilon \| V \| \right) \| \xi \|.
\]

Therefore, we have
\[
\| \Phi(V; \tilde{s}) - \Phi(V; s) \| \leq \sum_{n \geq 0} \sum_{m \geq 0} |\tilde{s}^{n+m} - s^{n+m}| \| f \|_{L^1} \frac{|M_\varepsilon|^{n+m} \| V \|^{n+m+1}}{n!m!} + 2 \varepsilon \| V \|,
\]
for any \( \varepsilon > 0, s, \tilde{s} \in \mathbb{R} \) and \( V \in \mathcal{M}_{sa} \). As \( \tilde{s} \to s \), the first term converges to 0 by Lebesgue’s dominated convergence theorem and we obtain
\[
\limsup_{\tilde{s} \to s} \| \Phi(V; \tilde{s}) - \Phi(V; s) \| \leq 2 \varepsilon \| V \|,
\]
for any \( \varepsilon > 0, s \in \mathbb{R} \) and \( V \in \mathcal{M}_{sa} \). Therefore, we have
\[
\lim_{\tilde{s} \to s} \| \Phi(V; \tilde{s}) - \Phi(V; s) \| = 0,
\]
for any \( s \in \mathbb{R} \) and \( V \in \mathcal{M}_{sa} \). This proves the second statement of lemma.

To prove the third statement, let \( V_N \in \mathcal{M}_{sa} \) be a sequence such that \( V_N \to V \in \mathcal{M}_{sa} \) in the strong operator topology. By the uniform boundedness principle, we have \( c := \sup_N \| V_N \| < \infty \). Fix arbitrary \( \varepsilon > 0, \xi \in \mathcal{H} \), and \( s \in \mathbb{R} \). We have
\[ \| (\Phi(V_N; s) - \Phi(V; s)) \varrho \| \]
\[ \leq \int_{|t| \leq M_\varepsilon} dt \ f(t) \left\| \left( \sigma^{sV_N}_t (V_N) - \sigma^{sV}_t (V) \right) \varrho \right\| + \int_{|t| \geq M_\varepsilon} dt \ f(t) \left\| \left( \sigma^{sV_N}_t (V_N) - \sigma^{sV}_t (V) \right) \varrho \right\| \]
\[ \leq \int_{|t| \leq M_\varepsilon} dt \ f(t) \sum_{n \geq 0} \sum_{m \geq 0} |s^{n+m}| \left\| \left( T_n(V_N; t) \sigma_t(V_N) \left( T_m(V_N; t) \right)^* - T_n(V; t) \sigma_t(V) \left( T_m(V; t) \right)^* \right) \varrho \right\| \]
\[ + \varepsilon \| \varrho \| \left( \| V \| + c \right), \quad (65) \]

for all \( N \in \mathbb{N} \). Note that
\[ \left\| \left( T_n(V_N; t) \sigma_t(V_N) \left( T_m(V_N; t) \right)^* - T_n(V; t) \sigma_t(V) \left( T_m(V; t) \right)^* \right) \varrho \right\| \]
\[ \leq \int_{0 \leq s_1 \leq \ldots \leq s_t \leq t} ds_1 \ldots ds_n \int_{0 \leq u_1 \leq \ldots \leq u_t \leq t} du_1 \ldots du_m \]
\[ \left\| \left( \sigma_{s_1}(V_N) \ldots \sigma_{s_t}(V_N) \sigma_{u_1}(V_N) \ldots \sigma_{u_m}(V_N) \right. \right. \]
\[ \quad - \left. \left. \sigma_{s_n}(V) \ldots \sigma_{s_{m+t}}(V) \sigma_{u_1}(V) \ldots \sigma_{u_m}(V) \right) \varrho \right\| . \quad (66) \]

From Lemma 8, the integrand on the last line converges to 0 as \( N \to \infty \), point wise. On the other hand, they are bounded by \( \| V \|^{n+m+1} + c^{n+m+1} \). Therefore, from Lebesgue’s dominated convergence theorem, we see that
\[ \lim_{N \to \infty} \left\| \left( T_n(V_N; t) \sigma_t(V_N) \left( T_m(V_N; t) \right)^* - T_n(V; t) \sigma_t(V) \left( T_m(V; t) \right)^* \right) \varrho \right\| = 0. \quad (67) \]

From (66), we also have
\[ \left\| \left( T_n(V_N; t) \sigma_t(V_N) \left( T_m(V_N; t) \right)^* - T_n(V; t) \sigma_t(V) \left( T_m(V; t) \right)^* \right) \varrho \right\| \]
\[ \leq \frac{M_\varepsilon^{n+m}}{n! m!} \left( \| V \|^{n+m+1} + c^{n+m+1} \right) \| \varrho \|, \quad (68) \]

for any \( |t| \leq M_\varepsilon \). From (67) and (68), applying the Lebesgue’s dominated convergence theorem, we have
\[ \lim_{N \to \infty} \int_{|t| \leq M_\varepsilon} dt \ f(t) \sum_{n \geq 0} \sum_{m \geq 0} |s^{n+m}| \left\| \left( T_n(V_N; t) \sigma_t(V_N) \left( T_m(V_N; t) \right)^* \right. \right. \]
\[ \quad - \left. \left. T_n(V; t) \sigma_t(V) \left( T_m(V; t) \right)^* \right) \varrho \right\| = 0. \quad (69) \]

Substituting this into (65), we obtain
\[ \limsup_{N \to \infty} \| (\Phi(V_N; s) - \Phi(V; s)) \varrho \| \leq \varepsilon \| \varrho \| \left( \| V \| + c \right), \quad (70) \]
for any $\varepsilon > 0$, $\xi \in \mathcal{H}$, and $s \in \mathbb{R}$. As $\varepsilon$ is arbitrary, we have
\begin{equation}
\lim_{N \to \infty} \| (\Phi(V_N; s) - \Phi(V; s)) \xi \| = 0,
\end{equation}
for any $\xi \in \mathcal{H}$, and $s \in \mathbb{R}$. This proves the third statement. \hfill \Box

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### Appendix A. $W^*$-Dynamical Systems

We collect known facts on $W^*$-dynamical systems. See [3] and [4] and references therein for more information. Let $\mathcal{H}$ be a Hilbert space and $\mathfrak{N} \subset B(\mathcal{H})$ a von Neumann algebra. We denote by $\mathfrak{N}_s$ the predual of $\mathfrak{N}$. We also use $\mathfrak{N}_{sa}$ the set of all self-adjoint elements in $\mathfrak{N}$. Let $\text{Aut}(\mathfrak{N})$ denote the group of all $*$-automorphisms of the von Neumann algebra $\mathfrak{N}$. A family $\{\tau_t\}_{t \in \mathbb{R}} \subset \text{Aut}(\mathfrak{N})$ is called $W^*$-dynamics if it is satisfied that for all $t, s \in \mathbb{R}$, $\tau_s \circ \tau_t = \tau_{s+t}$, $\tau_0 = 1$ and $\tau_t(A)$ converges to $A$ in the $\sigma$-weak topology as $t \to 0$. As $\tau_t$ is an automorphism, for any $A \in \mathfrak{N}$, $\mathbb{R} \ni t \mapsto \tau_t(A) \in \mathfrak{N}$ is continuous with respect to the $\sigma$-strong topology. The pair $(\mathfrak{N}, \tau)$ is called a $W^*$-dynamical system. For an entire $\tau$-analytic $A \in \mathfrak{N}$, we denote by $\tau_z(A)$, for $z \in \mathbb{C}$ the analytic continuation of $\mathbb{R} \ni t \mapsto \tau_t(A) \in \mathfrak{N}$. We denote the set of all $\tau$-entire analytic elements in $\mathfrak{N}$ by $\mathfrak{N}_\tau$. Then, $\mathfrak{N}_\tau$ is a $*$-subalgebra in $\mathfrak{N}$, which is dense in $\mathfrak{N}$ with respect to the strong topology. Let $\beta \in \mathbb{R}$. A normal state $\omega$ on $\mathfrak{N}$ is called a $(\tau, \beta)$-KMS state if for all $A \in \mathfrak{N}$ and $B \in \mathfrak{N}_\tau$,
\begin{equation}
\omega(A \tau_i \beta(B)) = \omega(BA).
\end{equation}

Any $(\tau, \beta)$-KMS state is $\tau$-invariant.

Finally, we consider the perturbation theory of the dynamics. Let $(\mathfrak{N}, \tau)$ be a $W^*$-dynamical system and $Q \in \mathfrak{N}_{sa}$. For $t \in \mathbb{R}$, we define $\tau_t^Q$ by
\begin{equation}
\tau_t^Q(A) = \sum_{n \geq 0} (it)^n \int_{0 \leq s_n \leq \cdots \leq s_1 \leq 1} ds_1 \cdots ds_n \left[ \tau_{s_n t}(Q), \cdots, [\tau_{s_1 t}(Q), \tau_t(A)] \cdots \right],
\end{equation}
for all $A \in \mathfrak{N}$. The integral is defined in the $\sigma$-weak topology, and the summation is in the norm topology.

Then, $\tau^Q$ is a $W^*$-dynamics. For all $t \in \mathbb{R}$, define $E_Q^*(t) \in \mathfrak{N}$ by
\begin{align}
E_Q^*(t) &= \sum_{n \geq 0} (it)^n \int_{0 \leq s_n \leq \cdots \leq s_1 \leq 1} ds_1 \cdots ds_n \tau_{s_n t}(Q) \cdots \tau_{s_1 t}(Q) \\
&= \sum_{n \geq 0} i^n \int_{0 \leq s_n \leq \cdots \leq s_1 \leq t} ds_1 \cdots ds_n \tau_{s_n}(Q) \cdots \tau_{s_1}(Q).
\end{align}

The integral is defined in the $\sigma$-weak topology and the summation is in the norm topology. This $E_Q^*(t)$ is a unitary element of $\mathfrak{N}$ and
\begin{equation}
\tau_t^Q(A) = E_Q^*(t) \tau_t(A) E_Q^*(t)^*.
\end{equation}
for all $t \in \mathbb{R}$ and $A \in \mathfrak{M}$. Suppose that there exists a self-adjoint operator $L$ on $\mathcal{H}$ satisfying $\tau_t(A) = e^{itL}Ae^{-itL}$ for all $t \in \mathbb{R}$ and $A \in \mathfrak{M}$. Then, for all $t \in \mathbb{R}$, $A \in \mathfrak{M}$ and $Q \in \mathfrak{M}_{sa}$,

$$\tau_t^Q(A) = e^{it(L+Q)}Ae^{-it(L+Q)}$$

and

$$E_t^\tau(Q) = e^{it(L+Q)}e^{-itL}.$$  

If $A, Q$ are entire analytic for $\tau$, then $A$ is $\tau^Q$-entire analytic and $\mathbb{R} \ni t \mapsto E_t^\tau(Q) \in \mathfrak{M}$ has an entire analytic continuation. We have the expansion

$$\tau_z^Q(A) = \sum_{n \geq 0} (iz)^n \int_{0 \leq s_n \leq \cdots \leq s_1 \leq 1} ds_1 \cdots ds_n \left[ \tau_{s_n z}(Q), \cdots, \tau_{s_1 z}(Q) \right] \tau_z(A) \cdots],$$

and

$$E_z^\tau(Q) = \sum_{n \geq 0} (iz)^n \int_{0 \leq s_n \leq \cdots \leq s_1 \leq 1} ds_1 \cdots ds_n \tau_{s_n z}(Q) \cdots \tau_{s_1 z}(Q)$$

for all $z \in \mathbb{C}$, respectively. The integration and the summation converge in the norm topology [4].

### Appendix B. Tomita–Takesaki Theory

We collect known facts Tomita–Takesaki theory. See [3] and [4] and references therein for more information. Let $\mathcal{H}$ be a Hilbert space, $\mathfrak{M} \subset B(\mathcal{H})$ a von Neumann algebra, and $\Omega \in \mathcal{H}$ a cyclic and separating vector for $\mathfrak{M}$. Define the operator $S_0^\Omega$ on $\mathcal{H}$ with the domain $\mathfrak{M}\Omega$ by

$$S_0^\Omega A\Omega = A^*\Omega, \quad A \in \mathfrak{M}.$$  

(80)

Then, $S_0^\Omega$ is anti-linear and closable. Let $S_\Omega$ be its closure. Let $S_\Omega = J_\Omega \Delta_{\Omega}^{1/2}$ be the polar decomposition of $S_\Omega$. The operators $\Delta_\Omega$ and $J_\Omega$ are called, respectively, the modular operator and the modular conjugation for $(\mathfrak{M}, \Omega)$. The modular operator $\Delta_\Omega$ is non-singular. The modular operator and the modular conjugation satisfy $J_\Omega \Delta_\Omega J_\Omega = \Delta_\Omega^{-1}$, $\Delta_{\Omega}^z \Omega = \Omega$ for all $z \in \mathbb{C}$ and $\log \Delta_{\Omega} \Omega = 0$. Furthermore, $J_\Omega$ interchange $\mathfrak{M}$ and $\mathfrak{M}'$, i.e., $J_\Omega \mathfrak{M} J_\Omega = \mathfrak{M}'$. From the fact that $\Delta_{\Omega}^{it}\mathfrak{M}\Delta_{\Omega}^{-it} = \mathfrak{M}$ all $t \in \mathbb{R}$, we can define a $*$-automorphism $\sigma_t$ on $\mathfrak{M}$ by $\sigma_t(A) = \Delta_{\Omega}^{it} A \Delta_{\Omega}^{-it}$, $A \in \mathfrak{M}$. This $W^*$-dynamics $\sigma$ is called the modular automorphism associated with $\Omega$. We set $L_\Omega = \log \Delta_\Omega$ and call it the Liouvillean of $\sigma$. One can check that the normalized state of $\omega$ is a $(\sigma, -1)$-KMS state.

The set $\mathfrak{M}_\sigma \Omega$ is a core for $S_\Omega$ and $\Delta_{\Omega}^{1/2}$ (Proposition 2.5.22 of [2]). We use the following lemma.

**Lemma 8.** Let $Q \in \mathfrak{M}$, $n \in \mathbb{N}$ and $t_1, \ldots, t_n \in \mathbb{R}$. Let $\{Q_m\}_{m \in \mathbb{N}} \subset \mathfrak{M}$ satisfy that $Q_m \to Q$ and $Q^*_m \to Q^*$ strongly as $m \to \infty$. Then,

$$\Delta_{\Omega}^{it_n}Q_m \cdots \Delta_{\Omega}^{it_2}Q_m\xi \to \Delta_{\Omega}^{it_n}Q \cdots \Delta_{\Omega}^{it_1}Q\xi$$

as $m \to \infty$, for any $\xi \in \mathcal{H}$. 


The following theorem is shown in [1].

**Theorem 9.** Let $Q \in \mathcal{M}_{sa}$. Then, $\Omega \in D\left(e^{L_\Omega + Q}\right)$.

Define the vectors $\Omega_Q := e^{L_\Omega + Q} \Omega$, and the functional $\omega_Q$ on $\mathcal{M}$ by

$$
\omega_Q(A) = (\Omega_Q, A\Omega_Q)
$$

(82)

for all $A \in \mathcal{M}$. Then, the following conditions are satisfied;

(a) $\Omega_Q$ is cyclic and separating for $\mathcal{M}$;
(b) The normalized state of $\omega_Q$ is a $(\sigma_Q, -1)$-KMS state;
(c) By (a), we can define the modular operator $\Delta_{\Omega_Q}$ and the modular conjugation $J_{\Omega_Q}$ for the pair $(\mathcal{M}, \Omega_Q)$. Then, $J_{\Omega} = J_{\Omega_Q}$,

$$
\log \Delta_{\Omega_Q} = L_\Omega + Q - J_{\Omega_Q}J_{\Omega_Q}
$$

(83)

and

$$
\sigma^Q_t(A) = \Delta_{\Omega_Q}^{it} A \Delta_{\Omega_Q}^{-it} = e^{it(L_\Omega + Q)} A e^{-it(L_\Omega + Q)};
$$

(84)

(d) Assume that a sequence $\{Q_n\}_{n \in \mathbb{N}} \subset \mathcal{M}_{sa}$ converges to $Q \in \mathcal{M}_{sa}$ strongly. Then, $\Omega_{Q_n} \to \Omega$ and $\omega_{Q_n} \to \omega_Q$ in the norm topology;

(e) For any $0 \leq s_n \leq \cdots \leq s_1 \leq 1$, $\Omega$ is in the domain of $e^{\frac{1}{2} s_n L_Q} \cdot e^{\frac{1}{2} (-s_n + s_{n-1}) L_Q} \cdots e^{\frac{1}{2} (-s_2 + s_1) L_Q}$ and $\Omega$ has an expansion

$$
\sum_{n \geq 0} \left(\frac{1}{2}\right)^n \int_{0 \leq s_n \leq \cdots \leq s_1 \leq 1} \cdots ds_n e^{\frac{1}{2} s_n L_Q} e^{\frac{1}{2} (-s_n + s_{n-1}) L_Q} \cdots e^{\frac{1}{2} (-s_2 + s_1) L_Q} \Omega.
$$

(85)

Due to (c) of this theorem, let us denote $J$ the modular conjugation if there is no danger of confusion.

### Appendix C. Properties of $f$

In this section, we consider some properties of $f$. Clearly, $f$ is in $L^1(\mathbb{R})$ by Fubini’s theorem. This $f$ and $F$ in (33) are Fourier transform of each other.

**Lemma 10.** For $f$ defined in (5) and $F$ defined in (33), we have

$$
\lim_{R \to \infty} \int_{-R}^{R} F(w)e^{-iwt}dw = f(t)
$$

(86)

for all $t \in \mathbb{R} \setminus \{0\}$ and

$$
F(x) = \int_{-\infty}^{\infty} f(t)e^{ixt}dt
$$

(87)

for all $x \in \mathbb{R}$.

**Proof.** In order to prove this, we consider the analytic continuation of $F$ (defined in (33)) to $\mathbb{C} \setminus (2\pi i (\mathbb{Z} + \frac{1}{2}))$:

$$
F(z) = \begin{cases} 
\frac{e^z - 1}{e^z + 1} & (z \in \mathbb{C} \setminus (2\pi i (\mathbb{Z} + \frac{1}{2}) \cup \{0\})) \\
\frac{1}{2} & (z = 0).
\end{cases}
$$

(88)
The claim (86) for \( t < 0 \) follows applying the residue theorem to \( F(z)e^{-izt} \) along the path in the following picture:

![Path Diagram]

Each pole \( z = 2\pi i \left( n + \frac{1}{2} \right) \) of \( F(z)e^{-itz} \) is a simple pole of \( F(z)e^{-itz} \), and

\[
\text{Res} \left( F(z)e^{-itz}; 2\pi i \left( n + \frac{1}{2} \right) \right) = \lim_{z \to 2\pi i \left( n + \frac{1}{2} \right)} \left( z - 2\pi i \left( n + \frac{1}{2} \right) \right) F(z)e^{-itz} = -\frac{i}{\pi \left( n + \frac{1}{2} \right)} e^{2\pi \left( n + \frac{1}{2} \right) t}. \tag{89}
\]

From the residue theorem, we obtain

\[
\int_{-R}^{R} \frac{dw}{w} F(w) e^{-iwt} = \int_{\gamma_1} F(z) e^{-itz} dz
\]

\[
= 2\pi i \sum_{k=0}^{n-1} \text{Res} \left( F(z)e^{-itz}; 2\pi i \left( k + \frac{1}{2} \right) \right) - \int_{\gamma_2} F(z) e^{-itz} dz - \int_{\gamma_3} F(z) e^{-itz} dz - \int_{\gamma_4} F(z) e^{-itz} dz. \tag{90}
\]

The integral along \( \gamma_3 \) goes to 0 as \( n \to \infty \). The integral along \( \gamma_2, \gamma_4 \) goes to 0 as \( R \to \infty \). Hence, we obtain (86) for all \( t < 0 \).

To show that (86) for \( t > 0 \), we note

\[
\int_{-R}^{R} F(w) e^{-iwt} dw = \int_{-R}^{R} \frac{F(w) e^{-iwt}}{F(-w)} dw = \int_{-R}^{R} F(w) e^{-iw(-t)} dw. \tag{91}
\]

The final term converges to \( f(-t) = f(t) \) as \( R \to \infty \) due to \( -t < 0 \). Hence, we obtain (86) for all \( t \neq 0 \).

Finally, we show (87). Note that \( F|_R \in L^2(\mathbb{R}) \). Let \( \hat{F} \in L^2(\mathbb{R}) \) be the Fourier transform of \( F|_R \). For \( R > 0 \), we define the function \( \phi_R \) on \( \mathbb{R} \) by

\[
\phi_R(t) = \int_{-R}^{R} F(x) e^{-ixt} dx. \tag{92}
\]
for all $t \in \mathbb{R}$. Then, we have $||\phi_R - \hat{F}||_2 \to 0$ as $R \to \infty$ by Theorem 9.13 of [6]. Therefore, there exists a sequence $\{n_k\}_{k \in \mathbb{N}}$ such that $\phi_{n_k}(t) \to \hat{F}(t)$ as $k \to \infty$ a.e. $t$. Since we have shown

$$\lim_{R \to \infty} \phi_R(t) = f(t) \quad (93)$$

for all $t \in \mathbb{R}\setminus\{0\}$, it follows that $f(t) = \hat{F}(t)$ a.e. $t$. Hence, $\hat{F} = f$ belongs to $L^1(\mathbb{R})$. By Theorem 9.14 of [6], we have

$$F(x) = \int_{-\infty}^{\infty} \hat{F}(t)e^{ixt}dt = \int_{-\infty}^{\infty} f(t)e^{ixt}dt \quad (94)$$

a.e. $x$. The left-hand side is continuous. The right-hand side is continuous too due to $f \in L^1(\mathbb{R})$. Therefore, the equality holds for all $x \in \mathbb{R}$. \hfill \Box

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