Universal eigenvarieties, trianguline Galois representations, and $p$-adic Langlands functoriality

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Abstract

Using the overconvergent cohomology modules introduced by Ash and Stevens, we construct eigenvarieties associated with reductive groups and establish some basic geometric properties of these spaces, building on work of Ash-Stevens, Urban, and others. We also formulate a precise modularity conjecture linking trianguline Galois representations with overconvergent cohomology classes. In the course of giving evidence for this conjecture, we establish several new instances of $p$-adic Langlands functoriality. Our main technical innovations are a family of universal coefficients spectral sequences for overconvergent cohomology and a generalization of Chenevier’s interpolation theorem.

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1 Introduction

1.1 Eigenvarieties and overconvergent cohomology

Since the pioneering works of Serre [Ser73], Katz [Kat73], and especially Hida [Hid86, Hid88, Hid94] and Coleman [Col96, Col97], p-adic families of modular forms have become a major topic in modern number theory. Aside from their intrinsic beauty, these families have found applications towards Iwasawa theory, the Bloch-Kato conjecture, modularity lifting theorems, and the local and global Langlands correspondences [BC04, BC09, BC11, BLGGT12, Ski09, Wil90]. One of the guiding examples in the field is Coleman and Mazur’s eigencurve [CM98, Buz07], a universal object parametrizing all overconvergent p-adic modular forms of fixed tame level and finite slope. Concurrently with their work, Stevens introduced his beautifully simple idea of overconvergent cohomology [Ste94], a group-cohomological avatar of overconvergent p-adic modular forms.

Ash and Stevens developed this idea much further in [AS08]: as conceived there, overconvergent cohomology works for any connected reductive \( \mathbb{Q} \)-group \( \mathbf{G} \) split at \( p \), and leads to natural candidates for quite general eigenvarieties. When the group \( \mathbf{G}^{\text{der}}(\mathbb{R}) \) possesses discrete series representations, Urban [Urb11] used overconvergent cohomology to construct eigenvarieties interpolating classical forms occurring with nonzero Euler-Poincaré multiplicities, and showed that his construction yields spaces which are equidimensional of the same dimension as weight space. In this article, we develop the theory of eigenvarieties for a connected reductive group over a number field \( F \), building on the ideas introduced in [AS08] and [Urb11], and we formulate precise conjectures relating these spaces with representations of the absolute Galois group \( \text{Gal}(\overline{F}/F) \).
In order to state our results, we first establish some notation. Fix a prime $p$ and a number field $F$ with ring of integers $\mathcal{O}_F$. For any place $v$ of $F$ we write $\mathcal{O}_v$ for the $v$-adic completion of $\mathcal{O}_F$ and $F_v$ for its fraction field, and we set $F_\infty = F \otimes_\mathbb{Q} \mathbb{R}$. Fix once and for all an algebraic closure $\overline{\mathbb{Q}}_p$ and an isomorphism $\iota : \mathbb{C} \cong \overline{\mathbb{Q}}_p$. Let $G$ be a connected reductive group over $\mathbb{Q}$ of the form $G = \text{Res}_{F/\mathbb{Q}} H$, with $H$ a connected reductive group over $F$ split over $F_v$ for each $v|p$. For any open compact subgroup $K_f \subset G(A_f) = H(A_{F,f})$, we have the associated locally symmetric space

$$Y(K_f) = G(\mathbb{Q}) \backslash G(A)/K_\infty \backslash K_f,$$

where $K_\infty$ denotes the identity component of a maximal compact-modulo-center subgroup $K_\infty \subset G(\mathbb{R}) = H(F_\infty)$.

For each place $v|p$, we set $H_v = H \times_{\text{Spec} F_v} \text{Spec} F_v$. Let $(B_v, T_v)$ be a split Borel pair in $H_v$. The group $H_v$ spreads out to a group scheme over $\text{Spec} \mathcal{O}_v$ (also denoted $H_v$) with reductive special fiber, and we assume that $B_v$ and $T_v$ are defined compatibly over $\mathcal{O}_v$ as well. Set $G = \prod_{v|p} \text{Res}_{\mathcal{O}_v/\mathcal{O}} H_v$, and define $B$ and $T$ analogously; we regard $G$, $B$ and $T$ as group schemes over $\text{Spec} \mathbb{Z}_p$, so e.g. $G(R) = \prod_{v|p} H_v(\mathcal{O}_v \otimes_{\mathbb{Z}_p} R)$ for $R$ any $\mathbb{Z}_p$-algebra and likewise for $B$ and $T$. Let $I$ be the Iwahori subgroup of $G(\mathbb{Z}_p)$ associated with $B$.

Let $\mathcal{W}$ be the rigid analytic space whose $\overline{\mathbb{Q}}_p$-points are given by

$$\mathcal{W}(\overline{\mathbb{Q}}_p) = \text{Hom}_{\text{cts}}(T(\mathbb{Z}_p), \overline{\mathbb{Q}}_p^\times).$$

We denote by $\lambda$ both a $\overline{\mathbb{Q}}_p$-point of $\mathcal{W}$ and the corresponding character of $T(\mathbb{Z}_p)$. Given any such character $\lambda : T(\mathbb{Z}_p) \to \overline{\mathbb{Q}}_p^\times$, the image of $\lambda$ generates a subfield $k_\lambda \subset \overline{\mathbb{Q}}_p$ finite over $\mathbb{Q}_p$. Following ideas of Stevens and Ash-Stevens [St94, AS08], we define a Fréchet $k_\lambda$-module $\mathcal{D}_\lambda$ of locally analytic distributions equipped with a continuous $k_\lambda$-linear left action of $I$. For any open compact subgroup $K^p \subset G(A^p_f)$, the quotient

$$(G(\mathbb{Q}) \backslash G(A)/K_\infty, K^p \times \mathcal{D}_\lambda)/I \to Y(K^p I)$$

defines a local system on $Y(K^p I)$ which we also denote by $\mathcal{D}_\lambda$. This local system is nontrivial if and only if $\lambda$ is trivial on $Z_G(\mathbb{Q}) \cap K^p I \subset T(\mathbb{Z}_p)$, and this condition cuts out a closed equidimensional rigid subspace $\mathcal{W}_{K^p} \subset \mathcal{W}$. The cohomology $H^*(Y(K^p I), \mathcal{D}_\lambda)$ is naturally a Hecke module, and when $\lambda$ is a $B$-dominant algebraic weight with associated algebraic representation $\mathcal{L}_\lambda$, there is a surjective $I$-equivariant map $i_\lambda : \mathcal{D}_\lambda \to \mathcal{L}_\lambda$ which induces a Hecke-equivariant and degree-preserving map

$$H^*(Y(K^p I), \mathcal{D}_\lambda) \to H^*(Y(K^p I), \mathcal{L}_\lambda).$$

The target, by Matsushima’s formula and its generalizations, is isomorphic as a Hecke module to a finite-dimensional space of classical automorphic forms; the source, on the other hand, is much larger. When $F = \mathbb{Q}$ and $G = \text{GL}_2$, Stevens proved that the finite-slope systems of Hecke eigenvalues appearing in $H^*(Y(K^p I), \mathcal{D}_\lambda)$ are exactly those appearing in a corresponding space of overconvergent modular forms. These cohomology modules make sense, however, even for groups without associated Shimura varieties, and in our opinion they are the “correct” surrogate for spaces of overconvergent $p$-adic modular forms.

To explain our results, we need to be precise about the Hecke algebras under consideration. After choosing a uniformizer $\pi_v$ of $\mathcal{O}_v$ for each $v|p$, the action of $I$ on $\mathcal{D}_\lambda$ extends canonically to an action of the monoid $\Delta \subset G(\mathbb{Q}_p)$ generated by $I$ and by the monoid

$$T^+ = \{t \in T(\mathbb{Q}_p) \mid t^{-1} B(\mathbb{Z}_p) t \subseteq B(\mathbb{Z}_p)\}.$$
The algebra $\mathcal{A}_p^+ = \mathcal{C}_c^\infty(I\Delta/I, \mathbb{Q}_p)$ is a commutative subalgebra of the Iwahori-Hecke algebra of $G$. Letting $U_t = [ht] \in \mathcal{A}_p^+$ be the Hecke operator associated with any $t \in T^+$, the map $t \mapsto U_t$ extends to an algebra isomorphism

$$\mathbb{Q}_p[T^+/T(\mathbb{Z}_p)] \xrightarrow{\sim} \mathcal{A}_p^+.$$ 

We define $U_t$ to be a controlling operator if $t \in T^+$ satisfies $\cap_{i \geq 1} t^{-1}B(\mathbb{Z}_p)t^i = \{1\}$.

Let $S(K^p)$ denote the finite set of places of $F$ where either $v|p$, or $\mathbb{H}/F_v$ is ramified, or $K^p = \mathbb{H}(F_v) \cap K^p$ is not a hyperspecial maximal compact subgroup of $\mathbb{H}(F_v)$. The main Hecke algebra of interest for us is

$$T(K^p) = \mathcal{A}_p^+ \otimes_{\mathbb{Q}_p'} \bigotimes_{v \in S(K^p)} \mathcal{C}_c^\infty(K^p_v \mathbb{H}(F_v)/K^p_v, \mathbb{Q}_p).$$

**Definition 1.1.1.** A finite-slope eigenpacket (of weight $\lambda$ and level $K^p$) is an algebra homomorphism $\phi : T(K^p) \rightarrow \overline{\mathbb{Q}_p}$ such that the space

$$\{ v \in H^*(Y(K^p)I, \mathcal{D}_\lambda) \otimes_{k_\lambda} \overline{\mathbb{Q}_p} | T \cdot v = \phi(T)v \forall T \in T(K^p) \}$$

is nonzero and such that $\phi(U_t) \neq 0$ for some controlling operator $U_t$.

This definition is independent of the specific choice of controlling operator, and has some consequences which are not entirely obvious: in particular, the set of finite-slope eigenpackets of given weight and level which satisfy $\psi_p(\phi(U_t)) \leq h$ is finite for any fixed $h$. The image of any finite-slope eigenpacket $\phi$ generates a subfield of $\overline{\mathbb{Q}_p}$ finite over $\mathbb{Q}_p$, and we denote by $k_\phi$ the compositum of this field with $k_\lambda$.

Our first main result is the existence of an eigenvariety parametrizing the finite-slope eigenpackets appearing in $H^*(Y(K^p)I, \mathcal{D}_\lambda)$ as $\lambda$ varies over $\mathcal{W}_{K^p}$.

**Theorem 1.1.2.** Notation and assumptions as above, there is a canonical separated rigid analytic space $\mathcal{X} = \mathcal{X}_{\mathcal{G}, K^p}$ equipped with a morphism $\mathcal{w} : \mathcal{X} \rightarrow \mathcal{W}_{K^p}$ and an algebra homomorphism $\phi : T(K^p) \rightarrow \mathcal{O}(\mathcal{X})$ such that:

i. The morphism $\mathcal{w}$ has discrete fibers and is locally finite in the domain.

ii. For any point $\lambda \in \mathcal{W}_{K^p}(\overline{\mathbb{Q}_p})$, there is a canonical bijection between points in the fiber $\mathcal{w}^{-1}(\lambda) \subset \mathcal{X}(\overline{\mathbb{Q}_p})$ and finite-slope eigenpackets of weight $\lambda$ and level $K^p$, realized by the map sending $x \in \mathcal{w}^{-1}(\lambda)$ to the algebra homomorphism

$$\phi_{\mathcal{X}, x} : T(K^p) \xrightarrow{\phi} \mathcal{O}(\mathcal{X}) \rightarrow k_x.$$ 

Writing $\phi \mapsto x(\phi)$ for the inverse of this map, we have $k_{\phi} = k_{x(\phi)}$.

iii. There are canonically defined sheaves of automorphic forms on $\mathcal{X}$ interpolating the modules $H^*(Y(K^p)I, \mathcal{D}_\lambda)$.

iv. For any controlling operator $U_t$, there is a canonical closed immersion $\mathcal{Z}_t \hookrightarrow \mathcal{W}_{K^p} \times \mathbb{A}^1$ and finite morphism $s : \mathcal{X} \rightarrow \mathcal{Z}_t$ such that $\mathcal{w}$ factors as the composite of $s$ with the morphism $\mathcal{Z}_t \rightarrow \mathcal{W}_{K^p} \times \mathbb{A}^1 \xrightarrow{pr_1} \mathcal{W}_{K^p}$.
One of the main results we shall prove in every relevant eigenvariety on the above list (and in particular, every eigenvariety known to the systematically comparing the resulting eigenvarieties in $X_{\lambda \mapsto \lambda}$ construction can be carried out, which admit no a priori are already defined in $M_{\lambda}$ occurring in $M$. Note that part iv. of this theorem characterizes the Grothendieck topology on $\mathcal{X}_{G,K^s}$. We construct $\mathcal{R}_{G,K^s}$ through a by-now-familiar process of gluing suitable affinoid local pieces. These affinoids are already defined in [AS08], but gluing them turns out to be a somewhat subtle affair. The main novelty of our analysis is not the gluing, however, but rather the construction of some universal coefficients spectral sequences which allow us to give a fairly soft analysis of the resulting spaces. We shall say more about both these points below.

In order to put this theorem in context, and to partially explain the title of this paper, we introduce a little formalism. Suppose we are given, for each $\lambda \in \mathcal{W}_{K^s}(\mathbb{Q}_p)$, a $k_\lambda \otimes \mathbf{T}(K^p)$-module $M_\lambda$ of “overconvergent modular forms of weight $\lambda$ and tame level $K^p$ on $G$.” If the assignment $\lambda \mapsto M_\lambda$ varies analytically with $\lambda$, then one has a chance of constructing an eigenvariety as in Theorem 1.1.2 whose fiber over any $\lambda$ parametrizes the finite-slope systems of Hecke eigenvalues occurring in $M_\lambda$. In practice there are several definitions of suitable families $\{M_\lambda\}_\lambda$ for which this construction can be carried out, which admit no a priori comparison. We shall take up the task of systematically comparing the resulting eigenvarieties in [Han14]. For now, let us make the following remarks:

- The union of irreducible components of $\mathcal{X}$ where $\phi_{\mathcal{X}}(U_t)$ is a $p$-adic unit admits a canonical integral model. The study of these components with their integral structures is usually known as “Hida theory”, following on the pioneering works of H. Hida [Hid86, Hid88, Hid95], and is best carried out from a formal-schematic perspective, as opposed to the rigid analytic setup of the present paper.

- When $G^{der}(R)$ is compact, the spaces $Y(K_f)$ are finite sets of points, and $H^*(Y(K_f), \mathcal{D}_\lambda) = H^0(Y(K_f), \mathcal{D}_\lambda)$ is a space of “algebraic overconvergent modular forms.” In this case, a number of authors have worked out special cases of the construction underlying Theorem 1.1.2 [BC09, Buz07, Buz04, Che04, Loe11, Tai12].

- If $G$ gives rise to Shimura varieties of PEL type, one might try to construct suitable $M_\lambda$’s using coherent cohomology. In the case $G = \text{GL}_2/\mathbb{Q}$, Coleman and Mazur initiated the entire theory of eigenvarieties in this way with the construction of their famous eigencurve [CM98], building on the results in [Col97, Col96]. Until recently, further coherent-cohomological constructions of $M_\lambda$ for general weights $\lambda$, e.g. [KL05, MT12], have relied (as did Coleman and Mazur) on tricks involving Eisenstein series or lifts of Hasse invariants, and have only given rise to one-dimensional families. However, canonical constructions of overconvergent modular forms of arbitrary $p$-adic weight on Shimura varieties have recently been discovered [AIS13, AIP13, Bra14, CHJ14] which do give rise to universal coherent-cohomological eigenvarieties.

- When $G^{der}(R)$ has a discrete series, there is a natural closed immersion from Urban’s eigenvariety into $\mathcal{X}_{G,K^s}$, and the image of this map is exactly the union of the $\dim \mathcal{W}_{K^p}$-dimensional irreducible components of $\mathcal{X}_{G,K^s}$.

- For general $G$, Emerton [Eme06] gave a construction of eigenvarieties in which $M_\lambda$ is (essentially) the weight-$\lambda$ subspace of the locally analytic Jacquet module of completed cohomology. We’ll carefully compare Emerton’s construction with the construction of the present paper in [Han14], generalizing the comparison given in [Loe11] in the case when $G^{der}(R)$ is compact.

One of the main results we shall prove in [Han14] is that for a given group $G$ and tame level $K^p$, every relevant eigenvariety on the above list (and in particular, every eigenvariety known to the
author) admits a natural closed immersion into \( \mathcal{X}_{G,K} \). In this light, it seems reasonable to regard the spaces \( \mathcal{X}_{G,K} \) as universal eigenvarieties, hence the title.

Let us outline the proof of Theorem 1.1.2. Again following [AS08], we define for any affinoid subdomain \( \Omega \subset \mathcal{H}_{K^p} \) a Fréchet \( \mathcal{O}(\Omega) \)-module \( \mathcal{D}_{\Omega} \) with continuous \( I \)-action, such that \( \mathcal{D}_{\Omega} \otimes_{\mathcal{O}(\Omega)} k_{\lambda} \cong \mathcal{D}_{\lambda} \) for all \( \lambda \in \Omega(\overline{\mathbb{Q}}_p) \). The cohomology modules \( H^*(Y(K^p I), \mathcal{D}_{\Omega}) \) don’t \textit{a priori} carry any natural topology, but after making some noncanonical choices, we obtain a chain complex \( C_{\Omega}^* \) of Fréchet modules such that \( H^*(C_{\Omega}^*) \cong H^*(Y(K^p I), \mathcal{D}_{\Omega}) \) together with an action of an operator \( \bar{U}_I \) on \( C_{\Omega}^* \) lifting the action of any given controlling operator \( U_I \) on cohomology. Using a fundamental result of Buzzard and the coherence of the assignment \( \Omega \rightsquigarrow C_{\Omega}^* \), we are then able to glue the finite-slope part of \( C_{\Omega}^* \) over varying \( \Omega \) into a complex of coherent sheaves on a certain noncanonical Fredholm hypersurface \( \mathcal{X} \subset \mathcal{H}_{K^p} \times \mathbb{A}^1 \). The cohomology sheaves \( \mathcal{M}^* \) of this complex are completely canonical, and Buzzard’s result gives an admissible covering of \( \mathcal{X} \) by affinoids \( \mathcal{D}_{\Omega, h} \), for suitably varying \( \Omega \subset \mathcal{H}_{K^p} \) and \( h \in \mathbb{Q}_{\geq 0} \), such that \( \mathcal{M}^* (\mathcal{D}_{\Omega, h}) \) is canonically isomorphic to the “slope-\( \leq h \) part” \( H^*(Y(K^p I), \mathcal{D}_{\Omega})_h \) of \( H^*(Y(K^p I), \mathcal{D}_{\Omega}) \). Recall that \( H^*(Y(K^p I), \mathcal{D}_{\Omega})_h \) - when it exists - is an \( \mathcal{O}(\Omega) \)-module-finite Hecke-stable direct summand of \( H^*(Y(K^p I), \mathcal{D}_{\Omega}) \) characterized as the maximal subspace of the latter on which (roughly) every eigenvalue of \( U_I \) has valuation \( \leq h \). Using this identification, we are able to glue the Hecke actions on \( H^*(Y(K^p I), \mathcal{D}_{\Omega})_h \) into an action on these cohomology sheaves, from which point we easily obtain \( \mathcal{X} \) by a simple “relative Spec”-type construction. We formalize this latter process in the definition of an eigenvariety datum (Definition 4.2.1).

Our main tool in analyzing this construction is family of spectral sequences which allows us to recover \( H^*(Y(K^p I), \mathcal{D}_\lambda) \) from \( H^*(Y(K^p I), \mathcal{D}_\Omega) \). For example, we prove the following result (our most general result in this direction is Theorem 3.3.1).

**Theorem 1.1.3.** There is a Hecke-equivariant second-quadrant spectral sequence

\[
E_2^{i,j} = \text{Tor}^{\mathcal{O}(\Omega)}_i \left( H^j(Y(K^p I), \mathcal{D}_{\Omega})_h, k_{\lambda} \right) \Rightarrow H^{i+j}(Y(K^p I), \mathcal{D}_{\lambda})_h
\]

for any \( \lambda \in \Omega(\overline{\mathbb{Q}}_p) \).

This sequence and its relatives turn out to be powerful tools for studying the geometry of eigenvarieties. To explain our results in this direction, recall that when \( G_{\text{der}}(\mathbb{R}) \) has a discrete series, standard limit multiplicity results yield an abundance of classical automorphic forms of essentially every arithmetic weight. One expects that correspondingly every irreducible component \( \mathcal{X} \) of the eigenvariety \( \mathcal{X} \) which contains a “suitably general” classical point has maximal dimension, namely \( \dim \mathcal{X} = \dim \mathcal{H}_{K^p} \). This numerical coincidence is characteristic of the groups for which \( G_{\text{der}}(\mathbb{R}) \) has a discrete series. More precisely, define the \textit{defect} and the \textit{amplitude} of \( G \), respectively, as the integers \( l(G) = \text{rank} G - \text{rank} K_{\infty} \) and \( q(G) = \frac{1}{2} \left( \text{dim}(G(\mathbb{R})/K_{\infty}) - l(G) \right) \). Note that \( l(G) \) is zero if and only if \( G_{\text{der}}(\mathbb{R}) \) has a discrete series, and that algebraic representations with regular highest weight contribute to \( (\mathfrak{g}, K_{\infty}) \)-cohomology exactly in the unbroken range of degrees \( [q(G), q(G) + l(G)] \). We say a point \( x \in \mathcal{X}(\overline{\mathbb{Q}}_p) \) is \textit{classical} if the weight \( \lambda_x \) factors as \( \lambda_x = \lambda_{\text{alg}} \cdot \varepsilon \) with \( \varepsilon \) finite-order and \( \lambda_{\text{alg}} \) \( B \)-dominant algebraic and the associated eigenpacket \( \phi_x \) matches the Hecke data of an algebraic automorphic representation \( \pi \) of \( G(\mathbb{A}_Q) \) such that \( \pi_{\infty} \) contributes to \( (\mathfrak{g}, K_{\infty}) \)-cohomology with coefficients in an irreducible algebraic representation of highest weight \( \lambda_{\text{alg}} \). A classical point is \textit{regular} if \( \lambda_{\text{alg}} \) is regular. The definitions of \textit{non-critical}, \textit{interior}, and \textit{strongly interior} points are slightly more subtle and we defer them until §3.2. The following conjecture is a special case of a conjecture of Urban (Conjecture 5.7.3 of [Urb11]).

\footnote{Here “rank” denotes the absolute rank, i.e. the dimension of any maximal torus, split or otherwise.}
Conjecture 1.1.4. Every irreducible component $\mathcal{X}_i$ of $\mathcal{H}_{G,K^p}$ containing a strongly interior, non-critical, regular classical point has dimension $\dim \mathcal{W}_{K^p} - l(G)$.

Using the spectral sequences, we verify Conjecture 1.1.4 in many cases. Our techniques yield new results even for groups with discrete series.

Theorem 1.1.5. If $l(G) \leq 1$, then Conjecture 1.1.4 is true, and if $l(G) \geq 1$, every irreducible component of $\mathcal{H}_{G,K^p}$ containing a strongly interior, non-critical, regular classical point has dimension at most $\dim \mathcal{W}_{K^p} - 1$.

In fact we prove slightly more, cf. Theorem 4.5.1. Some basic examples of groups with $l(G) = 1$ include $GL_3/Q$ and $Res_{F/Q}(H)$ where $F$ is a number field with exactly one complex embedding and $H$ is an $F$-inner form (possibly split) of $GL_2$.

It turns out that our techniques imply much more. Given any $x \in \mathcal{H}_{G,K^p}(\overline{Q}_p)$, set

$$l(x) = \sup \left\{ i \mid H^i(Y(K^p I), \mathcal{D}_x)_{\ker \phi_x} \neq 0 \right\} - \inf \left\{ i \mid H^i(Y(K^p I), \mathcal{D}_x)_{\ker \phi_x} \neq 0 \right\},$$

We shall see that interior, noncritical, regular classical points satisfy $l(x) = l(G)$. Upon reading an earlier version of this paper, James Newton discovered a proof of the following result, which is given here in Appendix B (cf. also the remark at the end of §4.5).

Theorem 1.1.6 (Newton). The dimension of any irreducible component $\mathcal{X}_i$ of $\mathcal{H}_{G,K^p}$ containing a given point $x$ satisfies $\dim \mathcal{X}_i \geq \dim \mathcal{W}_{K^p} - l(x)$. In particular, the lower bound of Conjecture 1.1.4 is true.

1.2 The conjectural connections with Galois representations

In this section we restrict our attention to the case $G = Res_{F/Q}(GL_n)$, choosing $B$ upper triangular and $T$ diagonal, so $T(Z_p) \cong \prod_{v|p} (O_F^\times)^n$ and $T(Q_p) \cong \prod_{v|p} (F_p^\times)^n$ with the obvious diagonal coordinates. For a given tame level $K^p$ we abbreviate $\mathcal{H}_{K^p} = \mathcal{H}_{G,K^p}$, and we write $T_{v,i}$ and $U_{v,i}$ for the usual Hecke operators at places $v \notin S(K^p)$ and $v|p$, respectively (cf. §4.6).

Let $G_F$ be the absolute Galois group of $F$, and let $\rho : G_F \to GL_n(\overline{Q}_p)$ be a continuous semisimple representation which is unramified almost everywhere. We say a tame level $K^p$ is admissible for $\rho$ if the set $S(K^p)$ contains the set of places where $\rho$ is ramified.

Definition 1.2.1. If $K^p$ is admissible for $\rho$ and $\phi : T(K^p) \to \overline{Q}_p$ is an algebra homomorphism, $\rho$ and $\phi$ are associated if the equality

$$\det(I_n - X \cdot \rho(\text{Frob}_v)) = \sum_{i=0}^{n} (-1)^i N^{\frac{i(i+1)}{2}} \phi(T_{v,i}) \lambda_x^i \text{ in } \overline{Q}_p[X]$$

holds for all $v \notin S(K^p)$.

This is a standard incarnation of the usual reciprocity between Galois representations and automorphic forms. Note that for any given $\phi$, there is at most one isomorphism class of continuous semisimple Galois representations associated with $\phi$, by the Brauer-Nesbitt theorem. Note also that if $\phi$ is associated with a Galois representation, the image of $\phi$ is necessarily contained in a finite extension of $Q_p$.

Conjecture 1.2.2. Given any point $x \in \mathcal{H}_{K^p}(\overline{Q}_p)$ with weight $\lambda_x$ and corresponding eigenpacket $\phi_x$, there is a continuous $n$-dimensional semisimple representation $\rho_x : G_F \to GL_n(\overline{Q}_p)$ with the following properties:

1. The tame level $K^p$ is admissible for $\rho_x$, and $\rho_x$ and $\phi_x$ are associated.
ii. The representation $\rho_x$ is odd: for any real infinite place $v$ with complex conjugation $c_v$, we have

$$\text{tr} \rho_x(c_v) = \begin{cases} 
\pm 1 & \text{if } n \text{ is odd} \\
0 & \text{if } n \text{ is even.}
\end{cases}$$

iii. For each place $v|p$, $\rho_x|G_{F_v}$ is trianguline, and the space of crystalline periods

$$D^+_{\text{crys}} \left( \wedge^i \rho_x|G_{F_v} \otimes (\lambda_{x,n} \cdots \lambda_{x,n+1-i} \cdot N^{\frac{i(i-1)}{2}} \otimes \chi_v) \right) \wedge_x = \phi_x(U_{v,i})$$

is nonzero for each $1 \leq i \leq n$, where $\chi_v : G_{F_v} \to \mathbb{Q}_v^\times$ is the Lubin-Tate character associated with our chosen uniformizer $\varpi_v$.

Part iii. of this conjecture is naturally inspired by a famous result of Kisin [Kis03] and its generalizations due to Bellaïche-Chenevier, Hellmann and others [BC09, Hel12]. When $F$ is totally real or CM, the existence of $\rho_x$ satisfying parts i. and ii. of this conjecture can be deduced from the recent work of Scholze, but part iii. seems quite difficult even for $\text{GL}_3/\mathbb{Q}$.

We would like to formulate a converse to this conjecture. More precisely, suppose we are given a continuous, absolutely irreducible, almost everywhere unramified representation $\rho : G_F \to \text{GL}_n(\mathbb{Q}_p)$ which is odd at all real infinite places. Let $N(\rho) \subset \mathcal{O}_F$ be the prime-to-$p$ Artin conductor of $\rho$.

We define

$$\mathcal{X}[\rho] = \{ x \in \mathcal{X}_{K_1(N(\rho))}(\mathbb{Q}_p) \mid \rho \text{ and } \phi_x \text{ are associated} \},$$

where $K_1(N)$ is the usual level subgroup appearing in the theory of new vectors for $\text{GL}_n$. What can we say about this set of points?

**Conjecture 1.2.3.** The set $\mathcal{X}[\rho]$ is nonempty if and only if $\rho$ is trianguline at all places dividing $p$.

To formulate a more quantitative statement, let $\mathcal{F} = \mathcal{F}_{n,F}$ denote the rigid space with $\mathcal{F}(L) = \text{Hom}_{\text{cts}}(T(\mathbb{Q}_p), L)$. We denote by $\delta$ both a point of $\mathcal{F}(\mathbb{Q}_p)$ and the associated character $\delta : T(\mathbb{Q}_p) \to \mathbb{Q}_p^\times$, and we identify any such character $\delta$ with an ordered $n$-tuple of continuous characters $\delta_i : \prod_{v|p} F_v^x \to \mathbb{Q}_p^\times$ in the natural way. Given any point $x \in \mathcal{X}_{K_1}(\mathbb{Q}_p)$, with associated weight $\lambda_x$, define $\delta_x \in \mathcal{F}(\mathbb{Q}_p)$ as follows:

i. $\delta_x(\varpi_v, \ldots, \varpi_v, 1, \ldots, 1) = \phi_{x,v}(U_{v,i})$, and

ii. $\delta_x(t) = \prod_{i=1}^n \lambda_{x,n+1-i}(t_i^{-1}N^{t_i-1}i)$ for $t = \text{diag}(t_1, \ldots, t_n) \in T(\mathbb{Z}_p)$.

Clearly there is a unique global character $\delta_{\mathcal{X}} : T(\mathbb{Q}_p) \to \mathcal{F}(\mathcal{X}[\rho])^\times$ specializing to $\delta_x$ at every point $x$. Let $\mathcal{F}[\rho] \subset \mathcal{F}(\mathbb{Q}_p)$ be the image of the map

$$\mathcal{X}[\rho] \to \mathcal{F}$$

$$x \mapsto \delta_x.$$
Define \( \mathcal{P}ar(\rho) \), the set of parameters of \( \rho \), as the set of characters \( \delta \in \mathcal{T}(\overline{\mathbb{Q}}_p) \) such that \( \delta|_{T_p(\overline{\mathbb{Q}}_p)} \) is the parameter of some triangulation of \( D_{\text{rig}}^1(\rho|_{G_{F_c}}) \) for all \( v|_{p} \). This set is clearly nonempty if and only if \( \rho|_{G_{F_c}} \) is trianguline for each \( v|_{p} \), and depends only on the possible triangulations of these local representations (we refer the reader to §6.1 for some background on trianguline representations).

**Conjecture 1.2.4.** If \( \delta \) is a parameter of \( \rho \), there is a unique point \( x = x(\rho, \delta) \in \mathcal{T}_{\overline{K_1(N(\rho))}}(\overline{\mathbb{Q}}_p) \) such that \( \rho \simeq \rho_x \) and \( \delta = \delta_x \). Equivalently, \( \mathcal{P}ar(\rho) \subset \mathcal{T}[\rho] \).

Our next conjecture gives a complete description of the set \( \mathcal{T}[\rho] \). To keep this introduction at a reasonable length, we refer the reader to §6.2 for two key definitions: briefly, given any character \( \chi \),

\[
\mathcal{T}[\rho] = \{ \chi \in \mathcal{T}(\overline{\mathbb{Q}}_p) : \chi \text{ is a twist of } \mathcal{T}[\rho] \text{ for some } \chi \}
\]

where \( \chi \) is the parameter of some triangulation of \( D_{\text{rig}}^1(\rho|_{G_{F_c}}) \) for all \( v|_{p} \). This set is clearly nonempty if and only if \( \rho|_{G_{F_c}} \) is trianguline for each \( v|_{p} \), and depends only on the possible triangulations of these local representations (we refer the reader to §6.1 for some background on trianguline representations).

**Conjecture 1.2.5.** The set \( \mathcal{T}[\rho] \) consists exactly of those characters \( \eta \) such that for some \( \delta \in \mathcal{P}ar(\rho) \) we have \( \eta \in \mathcal{T}(\overline{\mathbb{Q}}_p) \) such that \( \delta \) is the parameter of some triangulation of \( D_{\text{rig}}^1(\rho|_{G_{F_c}}) \).

**Theorem 1.2.6.** Notation and assumptions as above, Conjecture 1.2.5 is true when \( n = 2 \), \( \rho \colon G_{\mathbb{Q}} \to \text{GL}_2(F) \) is absolutely irreducible, and \( \rho|_{G_{Q_p}} \) is not isomorphic to a twist of \( \chi_{\text{cyc}} \).

This result is due almost entirely to others, and the proof is simply a matter of assembling their results. More precisely, under the hypotheses of Theorem 1.2.6, Emerton [Eme11] proved that \( \rho \) is a twist of the Galois representation \( \rho_f \) associated with a finite-slope overconvergent cuspidal eigenform \( f \). The result then follows from work of Stevens and Bellaïche [Ste00, Bel12] showing that the Hecke data associated with overconvergent eigenforms appears in overconvergent cohomology. The situation is most interesting when the weight of \( f \) is an integer \( k \geq 2 \) and \( U_0 f = \alpha f \) with \( v_p(\alpha) > k - 1 \), in which case Conjecture 1.2.4 actually predicts the existence of the companion form of \( f \): a form \( g \) of weight \( 2 - k \) such that \( \rho_f \simeq \rho_g \otimes \chi_{\text{cyc}}^{1-k} \) [Col96]. We should remind the reader that Emerton’s magisterial work relies on the full force of the \( p \)-adic local Langlands correspondence for \( \text{GL}_2/\mathbb{Q} \), [BB10, Col10, Kis10], not to mention Khare and Wintenberger’s proof of Serre’s conjecture [KW09a, KW09b]. We should also note that in general we need a certain level-lowering result to deduce that \( f \) really does occur at the minimal level \( N(\rho_f) \) (cf. §6.3).

Surprisingly, we are able to offer some evidence for these conjectures beyond the cases where \( n = 2 \) or \( \rho \) is geometric. To state our results in this direction, let \( \rho_f \) be the Galois representation associated with a finite-slope cuspidal overconvergent eigenform \( f \), and let \( \delta_f = (\delta_{f, 1}, \delta_{f, 2}) \) be the canonical parameter of \( \rho_f \) (cf. §6.1). Suppose the residual representation \( \overline{\rho}_f \) satisfies the hypotheses of Theorem 1.2.6.

**Theorem 1.2.7.** Conjecture 1.2.4 is true for the representation \( \text{sym}^2 \rho_f \) and the parameter

\[
\text{sym}^2 \delta_f = (\delta_{f, 1}^2, \delta_{f, 1} \delta_{f, 2}, \delta_{f, 2}^2) \in \mathcal{P}ar(\text{sym}^2 \rho_f).
\]

Now choose a second eigenform \( g \) with Galois representation \( \rho_g \) and canonical parameter \( \delta_g \). Define
a map
\[ \mathcal{F}_2, \mathcal{I}_Q \times \mathcal{F}_2, \mathcal{I}_Q \to \mathcal{F}_4, \mathcal{I}_Q \]
by \( \delta \boxtimes \delta' = (\delta_1 \delta'_1, \delta_1 \delta'_2, \delta_2 \delta'_1, \delta_2 \delta'_2) \).

**Theorem 1.2.8.** Supposing \( f \) and \( g \) have tame level one, Conjecture 1.2.4 is true for the representation \( \rho = \rho_f \boxtimes \rho_g \) and the parameters \( \delta_f \boxtimes \delta_g \) and \( \delta_f \boxtimes \delta_g' \).

The assumption on the level is only for simplicity and can easily be removed. Note that even when \( f \) and \( g \) are refinements of classical level one eigenforms of distinct weights (and therefore \( \rho \) is crystalline with distinct Hodge-Tate weights at \( p \)), at least one of these two parameters is critical in the sense of Bellaïche-Chenevier, and the weight of the associated point is algebraic but not \( B \)-dominant! In this case, Theorem 1.2.8 asserts the existence of certain overconvergent companion forms on the split form of \( \text{GL}_4 / \mathbb{Q} \).

### 1.3 \( p \)-adic Langlands functoriality

Given an \( \mathbb{R} \)-nonsplit quaternion algebra \( D / \mathbb{Q} \) of discriminant \( d \), Buzzard [Buz04, Buz07] constructed a certain eigencurve \( \mathcal{C}_D \) using overconvergent algebraic modular forms on \( D \), and raised the question of whether there exists a closed immersion \( \iota_{4L} : \mathcal{C}_D \hookrightarrow \mathcal{C}(d) \) into the tame level \( d \) eigencurve interpolating the Jacquet-Langlands correspondence on classical points. Chenevier affirmatively answered this question in a beautiful paper [Che05] as a consequence of an abstract interpolation theorem.

In §5.1, inspired by Chenevier's results, we establish rather flexible interpolation theorems (cf. Theorems 5.1.2 and 5.1.6). As sample applications of these tools, we prove the following results.

**Theorem 1.3.1.** Let \( N \) be a squarefree integer and set \( \mathcal{X} = \mathcal{X}_{\text{GL}_2 / \mathbb{Q}, K_1 (N^2)} \). Let \( \mathcal{C}_0 (N) \) denote the cuspidal locus in the tame level \( N \) eigencurve \( \mathcal{C}(N) \). Then there is a finite morphism \( s : \mathcal{C}_0 (N) \to \mathcal{X} \) such that \( \rho_{s(x)} \simeq \text{sym}^2 \rho_x \) and \( \delta_{s(x)} = \text{sym}^2 \delta_x \) for all \( x \in \mathcal{C}_0 (N) (\overline{\mathbb{Q}}_p) \).

In fact we prove a much more precise result for arbitrary levels, taking into account the inertial behavior of \( \rho_x \) at all primes \( \ell \mid p \); this immediately implies Theorem 1.2.7.

**Theorem 1.3.2.** Set \( \mathcal{X} = \mathcal{X}_{\text{GL}_2 / \mathbb{Q}, K_1 (1)} \). Then there is a finite morphism \( t : \mathcal{C}_0 (1) \times \mathcal{C}_0 (1) \to \mathcal{X} \) such that \( \rho_{t(x,y)} \simeq \rho_x \boxtimes \rho_y \) and \( \delta_{t(x,y)} = \delta_x \boxtimes \delta_y \) for all \( (x,y) \in \mathcal{C}_0 (1) (\overline{\mathbb{Q}}_p) \times \mathcal{C}_0 (1) (\overline{\mathbb{Q}}_p) \).

This quickly implies Theorem 1.2.8. It’s tempting to proliferate \( p \)-adic functorialities (and, simultaneously, evidence towards Conjectures 1.2.4 and 1.2.5) by combining Theorem 5.1.6 with known classical functorialities.\(^4\) For example, we invite the reader to construct a symmetric eighth power map
\[ \mathcal{X}_{\text{GL}_2 / \mathbb{Q}, K_p} \to \mathcal{X}_{\text{GL}_3 / \mathbb{Q}, K_p} \]
(for compatible tame levels \( K^p, K^{(p)} \)) by applying Clozel and Thorne’s recent work [CT13].

### 1.4 Notation and terminology

Our notation and terminology is mostly standard. For \( p \) the prime with respect to which things are \( p \)-adic, we fix once and for all an algebraic closure \( \overline{\mathbb{Q}}_p \) and an isomorphism \( \iota : \overline{\mathbb{Q}}_p \to \mathbb{C} \). We denote by \( F \) (resp. \( K \)) a finite extension of \( \mathbb{Q} \) (resp. \( \overline{\mathbb{Q}}_p \)). Unless otherwise noted, \( L \) denotes a sufficiently large subfield of \( \overline{\mathbb{Q}}_p \) finite over \( \mathbb{Q}_p \), where the meaning of “sufficiently large” may change

\(^4\)Although in general there will be subtle issues involving \( L \)-packets; see [Lud13].
from one line to the next. If $F$ is a number field and $\rho : G_F \to \GL_n(L)$ is a Galois representation, and “blah” is an adjective from $p$-adic Hodge theory (crystalline, semistable, de Rham, Hodge-Tate, trianguline, etc.), we say “$\rho$ is blah” as shorthand for “$\rho|_{G_{F_v}}$ is blah for all places $v|p$”.

We normalize the reciprocity maps of local class field theory so uniformizers map to geometric Frobenii. If $\pi$ is an irreducible admissible representation of $\GL_n(K)$, we write $\rec(\pi)$ for the Frobenius-semisimple Weil-Deligne representation associated with $\pi$ via the local Langlands correspondence, normalized as in Harris and Taylor’s book. If $f = \sum_{n=1}^{\infty} a_f(n)q^n \in \mathcal{S}_k(\Gamma_1(N))$ is a classical newform, we write $\rho_{f,\ell}$ (or just $\rho_f$) for the two-dimensional semisimple $\Q_p$-linear representation of $G_\Q$ characterized by the equality $\text{tr}\rho_f(\text{Frob}_\ell) = a_f(\ell)$ for all $\ell \nmid Np$.

In nonarchimedean functional analysis and rigid analytic geometry we follow [BGR84]. If $M$ and $N$ are topological $\Q_p$-vector spaces, we write $\mathcal{L}(M,N)$ for the space of continuous $\Q_p$-linear maps between $M$ and $N$; if $M$ and $N$ are $\Q_p$-Banach spaces, the operator norm

$$|f| = \sup_{m \in M, |m|_p \leq 1} |f(m)|_N$$

makes $\mathcal{L}(M,N)$ into a Banach space. If $(A, |\cdot|_A)$ is a Banach space which furthermore is a commutative $\Q_p$-algebra whose multiplication map is (jointly) continuous, we say $A$ is a $\Q_p$-Banach algebra. An $A$-module $M$ which is also a Banach space is a Banach $A$-module if the structure map $A \times M \to M$ extends to a continuous map $A \hat{\otimes}_{\Q_p} M \to M$, or equivalently if the norm on $M$ satisfies $|am|_M \leq C|a|_A|m|_M$ for all $a \in A$ and $m \in M$ with some fixed constant $C$. For a topological ring $R$ and topological $R$-modules $M, N$, we write $\mathcal{L}_R(M,N)$ for the $R$-module of continuous $R$-linear maps $f : M \to N$. When $A$ is a Banach algebra and $M, N$ are Banach $A$-modules, we topologize $\mathcal{L}_A(M,N)$ via its natural Banach $A$-module structure. We write $\text{Ban}_A$ for the category whose objects are Banach $A$-modules and whose morphisms are elements of $\mathcal{L}_A(-,-)$. If $I$ is any set and $A$ is a Banach algebra, we write $c_I(A)$ for the module of sequences $a = (a_i)_{i \in I}$ with $|a_i|_A \to 0$; the norm $|a| = \sup_{i \in I} |a_i|_A$ gives $c_I(A)$ the structure of a Banach $A$-module. If $M$ is any Banach $A$-module, we say $M$ is orthonormalizable if $M$ is isomorphic to $c_I(A)$ for some $I$ (such modules are called “potentially orthonormalizable” in [Buz07]).

If $A$ is an affinoid algebra, then $\text{Sp}A$, the affinoid space associated with $A$, denotes the locally $G$-ringed space $(\text{Max}A, \mathcal{O}_A)$ where $\text{Max}A$ is the set of maximal ideals of $A$ endowed with the Tate topology and $\mathcal{O}_A$ is the extension of the assignment $U \mapsto A_U$, for affinoid subdomains $U \subset \text{Max}A$ with representing algebras $A_U$, to a structure sheaf on $\text{Max}A$. If $X$ is an affinoid space, we write $\mathcal{O}(X)$ for the coordinate ring of $X$, so $A \simeq \mathcal{O}(\text{Sp}A)$. If $A$ is reduced we equip $A$ with the canonical supremum norm. If $X$ is a rigid analytic space, we write $\mathcal{O}_X$ for the structure sheaf and $\mathcal{O}(X)$ for the ring of global sections of $\mathcal{O}_X$. Given a point $x \in X$, we write $\mathfrak{m}_x$ for the corresponding maximal ideal in $\mathcal{O}_X(U)$ for any admissible affinoid open $U \subset X$ containing $x$, and $k(x)$ for the residue field $\mathcal{O}_X(U)/\mathfrak{m}_x$. Given a point $x \in X(\Q_p)$, we write $k_x \subset \Q_p$ for the image of $k(x)$ under the associated embedding $k(x) \hookrightarrow \Q_p$. $\mathcal{O}_{X,x}$ denotes the local ring of $\mathcal{O}_X$ at $x$ in the Tate topology, and $\widehat{\mathcal{O}_{X,x}}$ denotes the $\mathfrak{m}_x$-adic completion of $\mathcal{O}_{X,x}$. A Zariski-dense subset $S$ of a rigid analytic space $X$ is a very Zariski-dense subset, or a Zariski-dense accumulation subset, if for any connected affinoid open $U \subset X$ either $U \cap S = \emptyset$ or $U \cap S$ is Zariski-dense in $U$.

In homological algebra our conventions follow [Wei94]. If $R$ is a ring, we write $\text{K}^+(R), ? \in \{+,-,b,\emptyset\}$ for the homotopy category of $?$-bounded $R$-module complexes and $\text{D}^+(R)$ for its derived category.
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2 Background

We maintain the notation of the introduction. Set \( X^* = \text{Hom}(T, G_m) \) and \( X_* = \text{Hom}(G_m, T) \), and let \( \Phi, \Phi^+ \) and \( \Phi^- \) be the sets of roots, positive roots, and negative roots respectively, for the Borel \( B \). We write \( X^*_+ \) for the cone of \( B \)-dominant weights; \( \rho \in X^* \otimes \mathbb{Z} \) denotes half the sum of the positive roots.

We write \( \overline{B} \) for the opposite Borel, \( N \) and \( \overline{N} \) for the unipotent radicals of \( B \) and \( \overline{B} \), and \( I \) for the Iwahori subgroup

\[
I = \{ g \in G(\mathbb{Z}_p) \text{ with } g \text{ mod } p \in B(\mathbb{Z}/p\mathbb{Z}) \}.
\]

For any integer \( s \geq 1 \), set \( \overline{B}^s = \{ b \in \overline{B}(\mathbb{Z}_p), b \equiv 1 \text{ in } G(\mathbb{Z}/p^s\mathbb{Z}) \} \), \( \overline{N}^s = \overline{N}(\mathbb{Z}_p) \cap \overline{B}^s \) and \( T^s = T(\mathbb{Z}_p) \cap \overline{B}^s \), so the Iwahori decomposition reads \( I = \overline{N}^s \cdot T(\mathbb{Z}_p) \cdot N(\mathbb{Z}_p) \). We also set

\[
I^s_0 = \{ g \in I, g \text{ mod } p^s \in \overline{B}(\mathbb{Z}/p^s\mathbb{Z}) \}
\]

and

\[
I^s_1 = \{ g \in I, g \text{ mod } p^s \in \overline{N}(\mathbb{Z}/p^s\mathbb{Z}) \}.
\]

Note that \( I^s_1 \) is normal in \( I^s_0 \), with quotient \( T(\mathbb{Z}/p^s\mathbb{Z}) \). Finally, we set \( I^s = I \cap \ker \{ G(\mathbb{Z}_p) \to G(\mathbb{Z}/p^s\mathbb{Z}) \} \).

We define semigroups in \( T(\mathbb{Q}_p) \) by

\[
T^+ = \left\{ t \in T(\mathbb{Q}_p), t \overline{N}^s t^{-1} \subseteq \overline{N}^s \right\}
\]
and

\[ T^{++} = \left\{ t \in T(Q_p), \bigcap_{i=1}^{\infty} t^i N^i t^{-i} = \{1\} \right\}. \]

A simple calculation shows that \( t \in T(Q_p) \) is contained in \( T^+ \) (resp. \( T^{++} \)) if and only if \( v_p(\alpha(t)) \leq 0 \) (resp. \( v_p(\alpha(t)) < 0 \)) for all \( \alpha \in \Phi^+ \). Using these semigroups, we define a semigroup of \( G(Q_p) \) by \( \Delta = IT^+I \). The Iwahori decomposition extends to \( \Delta \): any element \( g \in \Delta \) has a unique decomposition \( g = \overline{n}(g)t(g)u(g) \) with \( \overline{n} \in \overline{N} \), \( t \in T^+, u \in N(Z_p) \). Our chosen uniformizers of \( O_v, v | p \), induce a canonical group homomorphism \( \sigma : T(Q_p) \to T(Z_p) \) which splits the inclusion \( T(Z_p) \subset T(Q_p) \), and we set \( \Delta = T^+ \cap \ker \sigma \) and \( \Delta^+ = T^{++} \cap \ker \sigma \).

### 2.1 Symmetric spaces and Hecke operators

In this section we set up our conventions for the homology and cohomology of local systems on locally symmetric spaces. Following [AS08], we compute homology and cohomology using two different families of resolutions: some extremely large “adelic” resolutions which have the advantage of making the Hecke action transparent, and resolutions with good finiteness properties constructed from simplicial decompositions of the Borel-Serre compactifications of locally symmetric spaces.

**Resolutions and complexes**

Let \( G/Q \) be a connected reductive group with center \( Z_G \). Let \( G(R)^o \) denote the connected component of \( G(R) \) containing the identity element, with \( G(Q)^o = G(Q) \cap G(R)^o \). Fix a maximal compact-mod-center subgroup \( K_\infty \subset G(R) \) with \( K^o_\infty \) the connected component containing the identity. Given an open compact subgroup \( K_f \subset G(A_f) \), we define the locally symmetric space of level \( K_f \) by

\[ Y(K_f) = G(Q) \backslash G(A) / K_f K^o_\infty. \]

This is a possibly disconnected Riemannian orbifold. By strong approximation there is a finite set of elements \( \gamma(K_f) = \{ z_i, x_i \in G(A_f) \} \) with

\[ G(A) = \prod_{x_i \in \gamma(K_f)} G(Q)^o G(R)^o x_i K_f. \]

Defining \( Z(K_f) = Z_G(Q) \cap K_f \) and \( \tilde{\Gamma}(x_i) = G(Q)^o \cap x_i K_f x_i^{-1} \), we have a decomposition

\[ Y(K_f) = G(Q) \backslash G(A) / K_f K^o_\infty \simeq \prod_{x_i \in \gamma(K_f)} \Gamma(x_i) / D_\infty, \]

where \( D_\infty = G(R)^o / K^o_\infty \) is the symmetric space associated with \( G \) and \( \Gamma(x_i) \cong \tilde{\Gamma}(x_i) / Z(K_f) \) denotes the image of \( \Gamma(x_i) \) in the adjoint group. If \( N \) is any left \( K_f \)-module, the double quotient

\[ \overline{N} = G(Q) \backslash (D_\infty \times G(A_f) \times N) / K_f \]

naturally gives rise to a local system on \( Y(K_f) \), which is trivial unless \( Z(K_f) \) acts trivially on \( N \). Set \( D_A = D_\infty \times G(A_f) \), and let \( C_\bullet(D_A) \) denote the complex of singular chains on \( D_A \) endowed with
the natural bi-action of $G(Q) \times G(A_f)$. If $M$ and $N$ are right and left $K_f$-modules, respectively, we define the complexes of adelic chains and adelic cochains by

$$C_{ad}^\bullet(K_f, M) = C_\bullet(D_A) \otimes_{Z[G(Q) \times K_f]} M$$

and

$$C_{ad}^\bullet(K_f, N) = \text{Hom}_{Z[G(Q) \times K_f]}(C_\bullet(D_A), N),$$

and we define functors $H_\bullet(K_f, -)$ and $H^\bullet(K_f, -)$ as their cohomology.

**Proposition 2.1.1.** There is a canonical isomorphism

$$H^\bullet(Y(K_f), \tilde{N}) \simeq H^\bullet(K_f, N) = H^\bullet(C_{ad}^\bullet(K_f, N)).$$

**Proof.** Let $C_\bullet(D_{\infty})(x_i)$ denote the complex of singular chains on $D_{\infty}$, endowed with the natural left action of $\Gamma(x_i)$ induced from the left action of $G(Q)^\circ$ on $D_{\infty}$; since $D_{\infty}$ is contractible, this is a free resolution of $Z$ in the category of $Z[\Gamma(x_i)]$-modules. Let $N(x_i)$ denote the left $\Gamma(x_i)$-module whose underlying module is $N$ but with the action $\gamma \cdot x_i = x_i^{-1}\gamma x_i | n$. Note that the local system $\tilde{N}(x_i)$ obtained by restricting $\tilde{N}$ to the connected component $\Gamma(x_i) \setminus D_{\infty}$ of $Y(K_f)$ is simply the quotient $\Gamma(x_i) \setminus (D_{\infty} \times N(x_i))$. Setting

$$C_{\text{sing}}^\bullet(K_f, N) = \oplus_i \text{Hom}_{Z[\Gamma(x_i)]}(C_\bullet(D_{\infty})(x_i), N(x_i)),$$

the map $D_{\infty} \to (D_{\infty}, x_i) \subset D_A$ induces a morphism $x_i^* = \text{Hom}(C_\bullet(D_A), N) \to \text{Hom}(C_\bullet(D_{\infty}), N)$, which in turn induces an isomorphism

$$\oplus_i x_i^* : C_{ad}^\bullet(K_f, N) \to \oplus_i \text{Hom}_{\Gamma(x_i)}(C_\bullet(D_{\infty})(x_i), N(x_i)),$$

and passing to cohomology we have

$$H^\bullet(C_{ad}^\bullet(K_f, N)) \simeq \oplus_i H^\bullet(\Gamma(x_i) \setminus D_{\infty}, \tilde{N}(x_i))$$

$$\simeq H^\bullet(Y(K_f), \tilde{N})$$

as desired. $\square$

When $\Gamma(x_i)$ is torsion-free for each $x_i \in \gamma(K_f)$, we choose a finite resolution $F_\bullet(x_i) \to Z \to 0$ of $Z$ by free left $Z[\Gamma(x_i)]$-modules of finite rank as well as a homotopy equivalence $F_\bullet(x_i) \overset{f_i}{\to} C_\bullet(D_{\infty})(x_i)$. We shall refer to the resolution $F_\bullet(x_i)$ as a **Borel-Serre resolution**; the existence of such resolutions follows from taking a finite simplicial decomposition of the Borel-Serre compactification of $\Gamma(x_i) \setminus D_{\infty}$ [BS73]. Setting

$$C_\bullet(K_f, N) = \oplus_i F_\bullet(x_i) \otimes_{Z[\Gamma(x_i)]} M(x_i)$$

and

$$C^\bullet(K_f, N) = \oplus_i \text{Hom}_{Z[\Gamma(x_i)]}(F_\bullet(x_i), N(x_i)),$$

the maps $f_i, g_i$ induce homotopy equivalences

$$C_\bullet(K_f, M) \overset{f_i}{\Rightarrow} C_{ad}^\bullet(K_f, M)$$
and

\[ C^\bullet(K_f, N) \xrightarrow{g^*} C^\bullet(K_f, M). \]

We refer to the complexes \( C^\bullet(K_f, -) \) and \( C^\bullet(K_f', -) \) as Borel-Serre complexes, and we refer to these complexes together with a fixed set of homotopy equivalences \( \{ f_i, g_i \} \) as augmented Borel-Serre complexes. When the \( \Delta(x) \)'s are not torsion-free but \( M \) is uniquely divisible as a \( \mathbb{Z} \)-module, we may still define \( C^\bullet(K_f, M) \) in an ad hoc manner by taking the \( K_f/K'_f \)-coinvariants of \( C^\bullet(K'_f, M) \) for some sufficiently small normal subgroup \( K'_f \subset K_f \).

## Hecke operators

A Hecke pair consists of a monoid \( \Delta \subset G(\mathbb{A}_f) \) and a subgroup \( K_f \subset \Delta \) such that \( K_f \) and \( \delta K_f \delta^{-1} \) are commensurable for all \( \delta \in \Delta \). Given a Hecke pair and a commutative ring \( R \), we write \( T(\Delta, K_f)_R \) for the \( R \)-algebra generated by the double coset operators \( T_\delta = [K_f \delta K_f] \) under convolution.\(^5\)

Suppose \( M \) is a right \( R[\Delta] \)-module. The complex \( C^\bullet(D_A) \otimes_{\mathbb{Z}[G(\mathbb{Q})]} M \) receives a right \( \Delta \)-action via \( (\sigma \otimes m) | \delta = \sigma \delta \otimes m \delta \), and \( C^\bullet(K_f, M) \) is naturally identified with the \( K_f \)-coinvariants of this action. Given any double coset \( K_f \delta K_f = \bigsqcup_j \delta_j K_f \), the action defined on pure tensors by the formula

\[ (\sigma \otimes m) \cdot [K_f \delta K_f] = \sum_j (\sigma \otimes m) \delta_j\]

induces a well-defined algebra homomorphism

\[ \xi : T(\Delta, K_f)_R \rightarrow \text{End}_{\text{Ch}(R)}(C^\bullet(K_f, M)). \]

This action induces the usual Hecke action defined by correspondences on homology. Set \( \tilde{T} = g_* \circ \xi(T) \circ f_* \in \text{End}_{\text{Ch}(R)}(C^\bullet(K_f, M)) \). The map

\[ \tilde{\xi} : T(\Delta, K_f)_R \xrightarrow{T \mapsto \tilde{T}} \text{End}_{\text{Ch}(R)}(C^\bullet(K_f, M)) \rightarrow \text{End}_{K(R)}(C^\bullet(K_f, M)) \]

is a well-defined ring homomorphism, since \( g_* \circ \xi(T_1) \circ f_* \circ g_* \circ \xi(T_2) \circ f_* \) is homotopic to \( g_* \circ \xi(T_1 T_2) \circ f_* \).

Note that any individual lift \( \tilde{T} \) is well-defined in \( \text{End}_{\text{Ch}(R)}(C^\bullet(K_f, M)) \), but if \( T_1 \) and \( T_2 \) commute in the abstract Hecke algebra, \( \tilde{T}_1 \tilde{T}_2 \) and \( \tilde{T}_2 \tilde{T}_1 \) will typically only commute up to homotopy.

Likewise, if \( N \) is a left \( R[\Delta] \)-module, the complex \( \text{Hom}_{\mathbb{Z}[G(\mathbb{Q})]}(C^\bullet(D_A), N) \) receives a natural \( \Delta \)-action via the formula \( \delta \phi = \delta \cdot \phi(\sigma \delta) \), and \( C^\bullet(K_f, N) \) is naturally the \( K_f \)-invariants of this action. The formula

\[ [K_f \delta K_f] \cdot \phi = \sum_j \delta_j | \phi \]

yields an algebra homomorphism \( \xi : T(\Delta, K_f)_R \rightarrow \text{End}_R(C^\bullet(K_f, N)) \) which induces the usual Hecke action on cohomology, and \( f^* \circ \xi \circ g^* \) defines an algebra homomorphism \( T(\Delta, K_f)_R \rightarrow \text{End}_{K(R)}(C^\bullet(K_f, M)) \). It is extremely important for us that these Hecke actions are compatible with the duality isomorphism

\[ \text{Hom}_R(C^\bullet(K_f, M), P) \simeq C^\bullet(K_f, \text{Hom}_R(M, P)). \]

\(^5\)The ring structure on \( T(\Delta, K_f)_R \) is nicely explained in §3.1 of [Shi94].
where $P$ is any $R$-module.

We shall be mostly concerned with the following Hecke algebras. For $I$, $\Lambda$ and $\Delta$ as in the beginning of §2, set $A_p^+ = T(\Delta, I)_{Q_p}$. For any $t \in T^+$, the double coset operator $U_t = [ItI]$ defines an element of $A_p^+$, and the map $\Lambda \ni t \mapsto U_t \in A_p^+$ extends to a commutative ring isomorphism

$$Q_p[\Lambda] \xrightarrow{\sim} A_p^+ \quad \sum c_t t \mapsto \sum c_t U_t.$$

The operators $U_t = [ItI]$ are invertible in the full Iwahori-Hecke algebra $T(G(Q_p), I)_{Q_p}$ [IM65], and we define the Atkin-Lehner algebra $A_p$ as the commutative subalgebra of $T(G(Q_p), I)_{Q_p}$ generated by elements of the form $U_t$ and $U_t^{-1}$ for $t \in \Lambda$. There is a natural ring isomorphism $A_p \simeq Q_p[T(Q_p)/T(Z_p)]$, though note that $t \cdot T(Z_p)$ typically corresponds to the operator $U_t U_t^{-1}$ where $t_1, t_2 \in \Lambda$ are any elements with $t_1 t_2^{-1} \in t \cdot T(Z_p)$. A controlling operator is an element of $A_p$ of the form $U_t$ for $t \in \Lambda^+$.

Fix an open compact subgroup $K^p \subset G(A^p_p)$. We say $K^p$ is unramified at a place $v \mid p$ if $H/F_v$ is unramified and $K^p_v = K^p \cap H(F_v)$ is a hyperspecial maximal compact subgroup of $H(F_v)$, and we say $K^p$ is ramified otherwise. Let $S = S(K^p)$ denote the finite set of places where $K^p$ is ramified or $v | p$, and set $K^p_S = K^p \cap \prod_{v \in S} G(F_v)$, so $K^p$ admits a product decomposition $K^p = K^p_S \prod_{v \notin S} K^p_v$.

We mainly work with the (commutative) Hecke algebras

$$T^p(K^p) = \bigotimes_{v \notin S(K^p)} T(H(F_v), K^p_v)_{Q_p},$$

$$T(K^p) = A_p^+ \otimes T^p(K^p),$$

In words, $T(K^p)$ takes into account the prime-to-$p$ spherical Hecke operators together with certain Atkin-Lehner operators at $p$; we write $T_G(K^p)$ if we need to emphasize $G$. We also set $T_{\text{ram}}(K^p) = T(\prod_{v \in S, v | p} H(F_v), K^p_S)$.

### 2.2 Locally analytic modules

For each $s \geq 1$ fix an analytic isomorphism $\psi^s : Z_p^d \simeq \overline{N}^s$, $d = \dim N$.

**Definition.** If $R$ is any $Q_p$-Banach algebra and $s$ is a positive integer, the module $A(\overline{N}^1, R)^s$ of $s$-locally analytic $R$-valued functions on $\overline{N}^1$ is the $R$-module of continuous functions $f : \overline{N}^1 \to R$ such that

$$f(\psi^s(z_1, \ldots, z_d)) : Z_p^d \to R$$

is given by an element of the $d$-variable Tate algebra $T_{d,R} = R[z_1, \ldots, z_d]$ for any fixed $x \in \overline{N}^1$.

Letting $\| \cdot \|_{T_{d,R}}$ denote the canonical norm on the Tate algebra, the norm $\|f(\psi^s)\|_{T_{d,R}}$ depends only on the image of $x$ in $\overline{N}^1/\overline{N}^s$, and the formula

$$\|f\|_s = \sup_{x \in \overline{N}^s} \|f(\psi^s)\|_{T_{d,R}}$$

\[\text{as in the fixed ordering.} \]
defines a Banach $R$-module structure on $\mathbf{A}(\overline{N}^1, R)^s$, with respect to which the canonical inclusion $\mathbf{A}(\overline{N}^1, R)^s \subset \mathbf{A}(\overline{N}^1, R)^{s+1}$ is compact.

Given a tame level group $K^p \subset \mathbf{G}(\mathbb{A}_F^p)$, let $\overline{Z(K^p)}$ be the $p$-adic closure of $Z(K^p I)$ in $T(\mathbb{Z}_p)$. The weight space of level $K^p$ is the rigid space associated with the formal scheme $\text{Spf}(\mathbb{Z}_p[[T(\mathbb{Z}_p)/Z(K^p I)]]$ via Raynaud’s generic fiber functor (cf. §7 of [dJ95]). Given an admissible affinoid open $\Omega \subset \mathcal{W}$, we write $\chi : T(\mathbb{Z}_p) \to \mathcal{O}(\Omega)^\times$ for the unique character it determines. We define $s[\Omega]$ as the minimal integer such that $\chi_{\Omega \mid T^{s[\Omega]}}$ is analytic. For any integer $s \geq s[\Omega]$, we make the definition

$$\mathbf{A}^s_\Omega = \{ f : I \to \mathcal{O}(\Omega), f \text{ analytic on each } I^s - \text{coset}, f(gtn) = \chi_\Omega(t)f(g) \forall n \in N(\mathbb{Z}_p), t \in T(\mathbb{Z}_p), g \in I \} .$$

By the Iwahori decomposition, restricting an element $f \in \mathbf{A}^s_\Omega$ to $\overline{N}^1$ induces an isomorphism

$$\mathbf{A}^s_\Omega \simeq \mathbf{A}(\overline{N}^1, \mathcal{O}(\Omega))^s,$$

and we regard $\mathbf{A}^s_\Omega$ as a Banach $\mathcal{O}(\Omega)$-module via pulling back the Banach module structure on $\mathbf{A}(N(\mathbb{Z}_p), \mathcal{O}(\Omega))^s$ under this isomorphism. The rule $(f|\gamma)(g) = f(\gamma g)$ gives $\mathbf{A}^s_\Omega$ the structure of a continuous right $\mathcal{O}(\Omega)[I]$-module. More generally, the formula

$$\delta \ast (nB(\mathbb{Z}_p)) = \delta n\delta^{-1}\sigma(\delta)B(\mathbb{Z}_p), n \in \overline{N}^1 \simeq I/B(\mathbb{Z}_p) \text{ and } \delta \in T^+$$

yields a left action of $\Delta$ on $I/B(\mathbb{Z}_p)$ which extends the natural left translation action by $I$ (cf. §2.5 of [AS08]) and induces a right $\Delta$-action on $\mathbf{A}^s_\Omega$ which we denote by $f \ast \delta, f \in \mathbf{A}^s_\Omega$. For any $\delta \in T^{++}$, the operator $\delta \ast - \in \mathcal{L}_{\mathcal{O}(\Omega)}(\mathbf{A}^s_\Omega, \mathbf{A}^s_\Omega)$ factors through the inclusion $\mathbf{A}^{s-1}_\Omega \hookrightarrow \mathbf{A}^s_\Omega$, and so defines a compact operator on $\mathbf{A}^s_\Omega$. The Banach dual

$$\mathbf{D}^s_\Omega = \mathcal{L}_{\mathcal{O}(\Omega)}(\mathbf{A}^s_\Omega, \mathcal{O}(\Omega))$$

$$\simeq \mathcal{L}_{\mathcal{O}(\Omega)}(\mathbf{A}(\overline{N}^1, \mathcal{O}(\Omega))^s, \mathcal{O}(\Omega))$$

$$\simeq \mathcal{L}_{\mathbb{Q}_p}(\mathbf{A}(\overline{N}^1, \mathcal{O}(\mathbb{Q}_p))^s, \mathcal{O}(\Omega))$$

inherits a dual left action of $\Delta$, and the operator $- \ast \delta$ for $\delta \in T^{++}$ likewise factors through the inclusion $\mathbf{D}^{s-1}_\Omega \hookrightarrow \mathbf{D}^s_\Omega$.

We define the limit module

$$\mathcal{A}_\Omega = \lim_{s \to \infty} \mathbf{A}^s_\Omega,$$

where the direct limit is taken with respect to the natural compact, injective transition maps $\mathbf{A}^s_\Omega \to \mathbf{A}^{s+1}_\Omega$. Note that $\mathcal{A}_\Omega$ is topologically isomorphic to the module of $\mathcal{O}(\Omega)$-valued locally analytic functions on $\overline{N}^1$, equipped with the finest locally convex topology for which the natural maps $\mathbf{A}^s_\Omega \hookrightarrow \mathcal{A}_\Omega$ are continuous. The $\Delta$-actions on $\mathbf{A}^s_\Omega$ induce a continuous $\Delta$-action on $\mathcal{A}_\Omega$. Set

$$\mathcal{D}_\Omega = \{ \mu : \mathcal{A}_\Omega \to \mathcal{O}(\Omega), \mu \text{ is } \mathcal{O}(\Omega) - \text{linear and continuous} \} ,$$

and topologize $\mathcal{D}_\Omega$ via the coarsest locally convex topology for which the natural maps $\mathcal{D}_\Omega \to \mathbf{D}^s_\Omega$ are continuous. In particular, the canonical map

$$\mathcal{D}_\Omega \to \lim_{s \to \infty} \mathbf{D}^s_\Omega$$

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is a topological isomorphism of locally convex \( \mathcal{O}(\Omega) \)-modules, and \( \mathcal{D}_\Omega \) is compact and Fréchet. Note that the transition maps \( D^+_\Sigma \to D^+_{\Omega} \) are injective, so \( \mathcal{D}_\Omega = \cap_{\xi > 0} D^+_{\Omega} \).

Suppose \( \Sigma \subset \Omega \) is a Zariski closed subspace; by Corollary 9.5.2/8 of [BGR84], \( \Sigma \) arises from a surjection \( \mathcal{O}(\Omega) \to \mathcal{O}(\Sigma) \) with \( \mathcal{O}(\Sigma) \) an affinoid algebra. We make the definitions \( D^+_{\Sigma} = D^+_{\Omega} \otimes \mathcal{O}(\Sigma) \) and \( \mathcal{D}_\Sigma = \mathcal{D}_\Omega \otimes \mathcal{O}(\Sigma) \).

**Proposition 2.2.1.** There are canonical topological isomorphisms \( D^+_{\Sigma} \simeq \mathcal{L}_{\mathcal{O}(\Omega)}(A^1_{\Omega}, \mathcal{O}(\Sigma)) \) and \( \mathcal{D}_\Sigma \simeq \mathcal{L}_{\mathcal{O}(\Omega)}(A^1_{\Omega}, \mathcal{O}(\Sigma)) \).

**Proof.** Set \( a_{\Sigma} = \ker(\mathcal{O}(\Omega) \to \mathcal{O}(\Sigma)) \), so \( \mathcal{O}(\Sigma) \simeq \mathcal{O}(\Omega)/a_{\Sigma} \). The definitions immediately imply isomorphisms

\[
D^+_{\Sigma} \simeq \mathcal{L}_{\mathcal{O}(\Omega)}(A^1_{\Omega}, \mathcal{O}(\Sigma))/a_{\Sigma} \mathcal{L}_{\mathcal{O}(\Omega)}(A^1_{\Omega}, \mathcal{O}(\Omega))
\]

\[
\mathcal{D}_\Sigma \simeq \mathcal{L}_{\mathcal{O}(\Omega)}(A^1_{\Omega}, \mathcal{O}(\Sigma))/\mathcal{L}_{\mathcal{O}(\Omega)}(A^1_{\Omega}, a_{\Sigma}),
\]

so the first isomorphism will follow if we can verify that the sequence

\[
0 \to \mathcal{L}_{\mathcal{O}(\Omega)}(A^1_{\Omega}, a_{\Sigma}) \to \mathcal{L}_{\mathcal{O}(\Omega)}(A^1_{\Omega}, \mathcal{O}(\Omega)) \to \mathcal{L}_{\mathcal{O}(\Omega)}(A^1_{\Omega}, \mathcal{O}(\Sigma))
\]

is exact on the right. Given a \( \mathbb{Q}_p \)-Banach space \( E \), write \( b(E) \) for the Banach space of bounded sequences \( \{(e_i)_{i \in \mathbb{N}} : \sup_{i \in \mathbb{N}} |e_i|_E < \infty\} \). Choosing an orthonormal basis of \( \mathcal{O}(\Omega)^* \) gives rise to an isometry \( \mathcal{L}_{\mathcal{O}(\Omega)}(A^1_{\Omega}, E) \simeq b(E) \) for \( E \) any Banach \( \mathcal{O}(\Omega) \)-module. Thus we need to show the surjectivity of the reduction map \( b(\mathcal{O}(\Omega)) \to b(\mathcal{O}(\Sigma)) \). Choose a presentation \( \mathcal{O}(\Omega) = T_n/b_n \), so \( \mathcal{O}(\Sigma) = T_n/b_{\Sigma} \) with \( b_n \subset b_{\Sigma} \). Quite generally for any \( b \subset T_n \), the function

\[
f \in T_n/b \mapsto \|f\|_b = \inf_{f \in f+b} \|f\|_{T_n}
\]

defines a norm on \( T_n/b \). By Proposition 3.7.5/3 of [BGR84], there is a unique Banach algebra structure on any affinoid algebra. Hence for any sequence \( (f_i)_{i \in \mathbb{N}} \subset b(\mathcal{O}(\Sigma)) \), we may choose a bounded sequence of lifts \( (\tilde{f}_i)_{i \in \mathbb{N}} \subset b(T_n) \); reducing the latter sequence modulo \( b_n \), we are done.

Taking inverse limits in the sequence we just proved to be exact, the second isomorphism follows. □

Suppose \( \lambda \in X^+ \subset \mathcal{W} \) is a dominant weight for \( B \), with \( \mathcal{L}_\lambda \) the corresponding irreducible left \( G(\mathbb{Q}_p) \)-representation of highest weight \( \lambda \). We may realize \( \mathcal{L}_\lambda \) explicitly as

\[
\mathcal{L}_\lambda(L) = \{ f : G \to \mathcal{L} \text{ algebraic, } f(n'tg) = \lambda(t)f(g) \text{ for } n' \in \mathbb{N}, t \in T, g \in G \}
\]

with \( G \) acting by right translation. The function \( f_\lambda(g) \) defined by \( f_\lambda(n'tn) = \lambda(t) \) on the big cell extends uniquely to an algebraic function on \( G \), and is the highest weight vector in \( \mathcal{L}_\lambda \). For \( g \in G(\mathbb{Q}_p) \) and \( h \in I \), the function \( f_\lambda(gh) \) defines an element of \( \mathcal{L}_\lambda \otimes \mathcal{A}_\lambda \), and pairing it against \( \mu \in \mathcal{D}_\lambda \) defines a map \( i_\lambda : \mathcal{D}_\lambda \to \mathcal{L}_\lambda \) which we notate suggestively as

\[
i_\lambda(\mu)(g) = \int f_\lambda(gh)\mu(h).
\]

The map \( \mu \mapsto i_\lambda(\mu)(g) \) satisfies the following intertwining relation for \( \gamma \in I \):

\[
\gamma \cdot i_\lambda(\mu)(g) = i_\lambda(\mu)(g\gamma)
\]

\[
= \int f_\lambda(g\gamma n^\circ)\mu(n^\circ)
\]

\[
= \int f_\lambda(n^\circ g)(\gamma \cdot \mu)(n^\circ)
\]

\[
= i_\lambda(\gamma \cdot \mu)(g).
\]

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The case of GL\(_n/Q_p\)

We examine the case when \(G \cong \text{GL}_n/Q_p\). We choose \(B\) and \(\overline{B}\) as the upper and lower triangular Borel subgroups, respectively, and we identify \(T\) with diagonal matrices. The splitting \(\sigma\) is canonically induced from the homomorphism

\[
Q_p^\times \to \mathbb{Z}_p^\times, \quad x \mapsto p^{-v_p(x)}x.
\]

Since \(T(Z_p) \cong (Z_p^\times)^n\), we canonically identify a character \(\lambda : T(Z_p) \to \mathbb{R}^\times\) with the \(n\)-tuple of characters \((\lambda_1, \ldots, \lambda_n)\) where

\[
\lambda_i : Z_p^\times \to \mathbb{R}^\times, \quad x \mapsto \lambda \circ \text{diag}(1, \ldots, 1, x, 1, \ldots, 1).
\]

Dominant weights are identified with tuples of integers \((k_1, \ldots, k_n)\) with \(k_1 \geq k_2 \geq \cdots \geq k_n\), by associating to such a tuple the character with \(\lambda_i(x) = x^{k_i}\).

We want to explain how to “twist away” one dimension’s worth of weights in a canonical fashion.

For any \(\lambda \in \mathcal{W}\), a simple calculation shows that the \(\ast\)-action of \(\Delta\) on \(A^\lambda_s\) is given explicitly by the formula

\[
(f \ast \delta)(x) = \lambda(\sigma(t(\delta)) t(\delta)^{-1} t(\delta x)) f(\overline{t(\delta x)}), \quad \delta \in \Delta, \quad x \in \mathbb{N}^\lambda, \quad f \in A(\mathbb{N}^\lambda, k(\lambda))^s.
\]

Given \(1 \leq i \leq n\), let \(m_i(g)\) denote the determinant of the upper-left \(i\)-by-\(i\) block of \(g \in \text{GL}_n\). For any \(g \in \Delta\), a pleasant calculation left to the reader shows that

\[
t(g) = \text{diag}(m_1(g), m_1(g)^{-1} m_2(g), \ldots, m_i^{-1}(g) m_{i+1}(g), \ldots, m_{n-1}(g)^{-1} \det g).
\]

In particular, writing \(\lambda^0 = (\lambda_1 \lambda_n^{-1}, \lambda_2 \lambda_{n-1}^{-1}, \ldots, \lambda_{n-1} \lambda_n^{-1}, 1)\) yields a canonical isomorphism

\[
A^\lambda_s \cong A^{\lambda_0}_s \otimes \lambda_n(\det \cdot | \det |_p)
\]

of \(\Delta\)-modules, and likewise for \(D^\lambda_s\).

In the case of \(\text{GL}_2\), we can be even more explicit. Here \(\Delta\) is generated by the center of \(G(Q_p)\) and by the monoid

\[
\Sigma_0(p) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(Z_p), \quad c \in pZ_p, \quad a \in Z_p^\times, \quad ad - bc \neq 0 \right\}.
\]

Another simple calculation shows that the center of \(G(Q_p)\) acts on \(A^\lambda_s\) through the character \(z \mapsto \lambda(\sigma(z))\), while the monoid \(\Sigma_0(p)\) acts via

\[
(g \cdot f)(x) = (\lambda_1 \lambda_2^{-1})(a + bx) \lambda_2(\det g | \det g |_p) f \left( \frac{c + dx}{a + bx} \right), \quad f \in A(\mathbb{N}^\lambda, k)^s, \quad \begin{pmatrix} 1 \\ x \end{pmatrix} \in \mathbb{N}^\lambda.
\]

(almost) exactly as in [Ste94].
insomorphism for every multiplicative polynomial $Q$. Suppose $a) \leq$ slope $M$ such that $A$ (for short, "an
of the Newton polygon of $Q$. Let $M$ be an $A$-module equipped with an $A$-linear endomorphism $u : M \to M$ (for short, “an $A[u]$-module”). Fix a rational number $h \in \mathbb{Q}_{\geq 0}$. We say a polynomial $Q \in A[x]$ is multiplicative if the leading coefficient of $Q$ is a unit in $A$, and that $Q$ has slope $\leq h$ if every edge of the Newton polygon of $Q$ has slope $\leq h$. Write $Q^*(x) = x^{\deg Q}(1/x)$. An element $m \in M$ has slope $\leq h$ if there is a multiplicative polynomial $Q \in A[T]$ of slope $\leq h$ such that $Q^*(u) \cdot m = 0$. Let $M_{\leq h}$ be the set of elements of $M$ of slope $\leq h$; according to Proposition 4.6.2 of loc. cit., $M_{\leq h}$ is an $A$-submodule of $M$.

**Definition 2.3.1.** A slope-$\leq h$ decomposition of $M$ is an $A[u]$-module isomorphism

$$M \cong M_{\leq h} \oplus M_{>h}$$

such that $M_{\leq h}$ is a finitely generated $A$-module and the map $Q^*(u) : M_{>h} \to M_{>h}$ is an $A$-module isomorphism for every multiplicative polynomial $Q \in A[T]$ of slope $\leq h$.

The following proposition summarizes the fundamental results on slope decompositions.

**Proposition 2.3.2 (Ash-Stevens):**

a) Suppose $M$ and $N$ are both $A[u]$-modules with slope-$\leq h$ decompositions. If $\psi : M \to N$ is a morphism of $A[u]$-modules, then $\psi(M_{\leq h}) \subseteq N_{\leq h}$ and $\psi(M_{>h}) \subseteq N_{>h}$; in particular, a module can have at most one slope-$\leq h$ decomposition. Furthermore, ker $\psi$ and im$\psi$ inherit slope-$\leq h$ decompositions. Given a short exact sequence

$$0 \to M \to N \to L \to 0$$

of $A[u]$-modules, if two of the modules admit slope-$\leq h$ decompositions then so does the third.
b) If $C^\bullet$ is a complex of $A[u]$-modules, all with slope $\leq h$ decompositions, then

$$H^n(C^\bullet) \simeq H^n(C^\bullet_{\leq h}) \oplus H^n(C^\bullet_{> h})$$

is a slope $\leq h$ decomposition of $H^n(C^\bullet)$.

Proof. This is a rephrasing of (a specific case of) Proposition 4.1.2 of [AS08]. □

Suppose now that $A$ is a reduced affinoid algebra, $M$ is an orthonormalizable Banach $A$-module, and $u$ is a compact operator. Let

$$F(T) = \det(1 - uT)|M \in A[[T]]$$

denote the Fredholm determinant for the $u$-action on $M$. We say $F$ admits a slope $\leq h$ factorization if we can write $F(T) = Q(T) \cdot R(T)$ where $Q$ is a multiplicative polynomial of slope $\leq h$ and $R(T) \in A[[T]]$ is an entire power series of slope $> h$. Theorem 3.3 of [Buz07] guarantees that $F$ admits a slope $\leq h$ factorization if and only if $M$ admits a slope $\leq h$ decomposition. Furthermore, given a slope $\leq h$ factorization $F(T) = Q(T) \cdot R(T)$, we obtain the slope $\leq h$ decomposition of $M$ upon setting $M_{\leq h} = \{m \in M | Q^*(u) \cdot m = 0\}$, and $M_{> h}$ in this case is a finite flat $A$-module upon which $u$ acts invertibly.\footnote{Writing $Q^*(x) = a + x \cdot r(x)$ with $r \in A[x]$ and $a \in A^x$, $u^{-1}$ on $M_{\leq h}$ is given explicitly by $-a^{-1}r(u)$.} Combining this with Theorem 4.5.1 of [AS08] and Proposition 2.3.1, we deduce:

**Proposition 2.3.3.** If $C^\bullet$ is a bounded complex of orthonormalizable Banach $A[u]$-modules, and $u$ acts compactly on the total complex $\oplus C^i$, then for any $x \in \operatorname{Max}(A)$ and any $h \in Q_{\geq 0}$ there is an affinoid subdomain $\operatorname{Max}(A') \subset \operatorname{Max}(A)$ containing $x$ such that the complex $C^\bullet \hat{\otimes}_AA'$ of $A'[u]$-modules admits a slope $\leq h$ decomposition, and $(C^\bullet \hat{\otimes}_AA')_{\leq h}$ is a complex of finite flat $A'$-modules.

**Proposition 2.3.4.** If $M$ is an orthonormalizable Banach $A$-module with a slope $\leq h$ decomposition, and $A'$ is a Banach $A$-algebra, then $M \hat{\otimes}_AA'$ admits a slope $\leq h$ decomposition and in fact

$$(M \hat{\otimes}_AA')_{\leq h} \simeq M_{\leq h} \hat{\otimes}_AA'.$$

**Proposition 2.3.5.** If $N \in \operatorname{Ban}_A$ is finite and $M \in \operatorname{Ban}_A$ is an $A[u]$-module with a slope $\leq h$ decomposition, the $A[u]$-modules $M \hat{\otimes}_AA N$ and $\mathcal{L}_A(M, N)$ inherit slope $\leq h$ decompositions.

Proof. This is an immediate consequence of the $A$-linearity of the $u$-action and the fact that $\hat{\otimes}_A N$ and $\mathcal{L}_A(\cdot, N)$ commute with finite direct sums. □

If $A$ is a field and $M$ is either an orthonormalizable Banach $A$-module or the cohomology of a complex of such, then $M$ admits a slope $\leq h$ decomposition for every $h$, and if $h < h'$ there is a natural decomposition

$$M_{\leq h'} \simeq M_{\leq h} \oplus (M_{> h})_{\leq h'}$$

and in particular a projection $M_{\leq h'} \to M_{\leq h}$. We set $M^h = \lim_{h \to -h} M_{\leq h}$.

## 3 Overconvergent cohomology

Fix a connected, reductive group $G/\mathbb{Q}$ as in the introduction. For any tame level group $K^p \subset G(A_f^p)$, we abbreviate $H_*(K^p I, -)$ by $H_*(K^p, -)$, and likewise for cohomology.
3.1 Basic results

In this section we establish some foundational results on overconvergent cohomology. These results likely follow from the formalism introduced in Chapter 5 of [AS08], but we give different proofs. The key idea exploited here, namely the lifting of the $U_t$-action to the level of chain complexes, is due to Ash. We use freely the notations introduced in §2.1-§2.3.

Fix an augmented Borel-Serre complex $C_\ast(K^p, -) = C_\ast(K^p I_{\ast}, -)$. Fix an element $t \in \Lambda^+$, and let $\bar{U} = U_t$ denote the lifting of $U_t = [ItI]$ to an endomorphism of the complex $C_\ast(K^p, -)$ defined in §2.1. Given a connected admissible open affinoid subset $\Omega \subset \mathcal{W}_{K^p}$ and any integer $s \geq s[\Omega]$, the endomorphism $\bar{U}_t \in \mathrm{End}_{\mathcal{O}(\Omega)}(C_\ast(K^p, A^s_{\Omega}))$ is compact; let

$$F^s_{\Omega}(X) = \det(1 - X \bar{U}_t)|C_\ast(K^p, A^s_{\Omega}) \in \mathcal{O}(\Omega)[[X]]$$

denote its Fredholm determinant. We say $(U_t, \Omega, h)$ is a slope datum if $C_\ast(K^p, A^s_{\Omega})$ admits a slope-$h$ decomposition for the $U_t$ action for some $s \geq s[\Omega]$.

**Proposition 3.1.1.** The function $F^s_{\Omega}(X)$ is independent of $s$.

*Proof.* For any integer $s \geq s[\Omega]$ we write $C^s_\ast = C^s_\ast(K^p, A^s_{\Omega})$ for brevity. By construction, the operator $\bar{U}_t$ factors into compositions $\rho_s \circ \bar{U}_t$ and $\bar{U}_t \circ \rho_{s+1}$ where $\bar{U}_t : C^s_\ast \to C^{s-1}_\ast$ is continuous and $\rho_s : C^{s-1}_\ast \to C^s_\ast$ is compact. Now, considering the commutative diagram

\[
\begin{array}{ccc}
C^s_\ast & \xrightarrow{\rho_s} & C^{s-1}_\ast \\
\downarrow{\bar{U}_t} & & \downarrow{\bar{U}_t} \\
C^s_\ast & \xrightarrow{\rho_{s+1}} & C^{s-1}_\ast
\end{array}
\]

we calculate

$$\det(1 - X \bar{U}_t)|C^s_\ast = \det(1 - X \rho_s \circ \bar{U}_t)|C^s_\ast = \det(1 - X \bar{U}_t \circ \rho_s)|C^{s-1}_\ast = \det(1 - X \bar{U}_t)|C^{s-1}_\ast,$$

where the second line follows from Lemma 2.7 of [Buz07], so $F^s_{\Omega}(X) = F^{s-1}_{\Omega}(X)$ for all $s > s[\Omega]$. □

**Proposition 3.1.2.** The slope-$h$ subcomplex $C_\ast(K^p, A^s_{\Omega}) \leq h$, if it exists, is independent of $s$. If $\Omega'$ is an affinoid subdomain of $\Omega$, then the restriction map $A^s_{\Omega'} \to A^s_{\Omega'}$ induces a canonical isomorphism

$$C_\ast(K^p, A^s_{\Omega'}) \simeq C_\ast(K^p, A^s_{\Omega'}) \leq h$$

for any $s \geq s[\Omega]$.

*Proof.* Since $F^s_{\Omega}(X)$ is independent of $s$, we simply write $F^s_{\Omega}(X)$. Suppose we are given a slope-$h$ factorization $F^s_{\Omega}(X) = Q(X) \cdot R(X)$; by the remarks in §2.3, setting $C_\ast(K^p, A^s_{\Omega}) \leq h = \ker Q^\ast(\bar{U}_t)$ yields a slope-$h$ decomposition of $C_\ast(K^p, A^s_{\Omega})$ for any $s \geq s[\Omega]$. By Proposition 2.3.1, the injection $\rho_s : C_\ast(K^p, A^{s-1}_{\Omega}) \hookrightarrow C_\ast(K^p, A^s_{\Omega})$ gives rise to a canonical injection

$$\rho_s : C_\ast(K^p, A^{s-1}_{\Omega}) \leq h \hookrightarrow C_\ast(K^p, A^s_{\Omega}) \leq h$$

for any $s > s[\Omega]$. The operator $\bar{U}_t$ acts invertibly on $C_\ast(K^p, A^s_{\Omega}) \leq h$, and its image factors through $\rho_s$, so $\rho_s$ is surjective and hence bijective. This proves the first claim.
For the second claim, by Proposition 2.3.3 we have

\[ C_\bullet(K^p, A_{1h}^s) \leq h \otimes \mathcal{O}(\Omega) \mathcal{O}(\Omega') \cong (C_\bullet(K^p, A_{1h}^s) \otimes \mathcal{O}(\Omega) \mathcal{O}(\Omega')) \leq h \cong C_\bullet(K^p, A_{1h}^s) \leq h, \]

so the result now follows from the first claim. □

**Proposition 3.1.3.** Given a slope datum \((U_i, \Omega, h)\) and an affinoid subdomain \(\Omega' \subset \Omega\), there is a canonical isomorphism

\[ H_s(K^p, A_{1h}^s) \leq h \otimes \mathcal{O}(\Omega) \mathcal{O}(\Omega') \cong H_s(K^p, A_{1h}^s) \leq h \]

for any \(s \geq s[\Omega]\).

Proof. Since \(\mathcal{O}(\Omega')\) is \(\mathcal{O}(\Omega)\)-flat, the functor \(- \otimes \mathcal{O}(\Omega) \mathcal{O}(\Omega')\) commutes with taking the homology of any complex of \(\mathcal{O}(\Omega)\)-modules. Thus we calculate

\[ H_s(K^p, A_{1h}^s) \leq h \otimes \mathcal{O}(\Omega) \mathcal{O}(\Omega') \cong H_s(C_\bullet(K^p, A_{1h}^s) \leq h) \otimes \mathcal{O}(\Omega) \mathcal{O}(\Omega') \]
\[ \cong H_s(C_\bullet(K^p, A_{1h}^s) \leq h) \]
\[ \cong H_s(K^p, A_{1h}^s) \leq h, \]

where the third line follows from Proposition 2.3.4. □

**Proposition 3.1.4.** Given a slope datum \((U_i, \Omega, h)\), the complex \(C_\bullet(K^p, A_{1h})\) and the homology module \(H_s(K^p, A_{1h})\) admit slope-\(h\) decompositions, and there is an isomorphism

\[ H_s(K^p, A_{1h}) \leq h \cong H_s(K^p, A_{1h}^s) \leq h \]

for any \(s \geq s[\Omega]\). Furthermore, given an affinoid subdomain \(\Omega' \subset \Omega\), there is a canonical isomorphism

\[ H_s(K^p, A_{1h}) \leq h \otimes \mathcal{O}(\Omega) \mathcal{O}(\Omega') \cong H_s(K^p, A_{1h}^s) \leq h. \]

Proof. For any fixed \(s \geq s[\Omega]\), we calculate

\[ C_\bullet(K^p, A_{1h}) \cong \lim_{\to \delta} C_\bullet(K^p, A_{1h}^s) \]
\[ \cong \lim_{\to \delta} C_\bullet(K^p, A_{1h}^s) \leq h \oplus C_\bullet(K^p, A_{1h}^s) > h \]
\[ \cong C_\bullet(K^p, A_{1h}^s) \leq h \oplus \lim_{\to \delta} C_\bullet(K^p, A_{1h}^s) > h \]

with the third line following from Proposition 3.1.2. The two summands in the third line naturally form the components of a slope-\(h\) decomposition, so passing to homology yields the first sentence of the proposition, and the second sentence then follows immediately from Proposition 2.3.3. □

We’re now in a position to prove the subtler cohomology analogue of Proposition 3.1.4.

**Proposition 3.1.5.** Given a slope datum \((U_i, \Omega, h)\) and a Zariski-closed subspace \(\Sigma \subset \Omega\), the complex \(C^\bullet(K^p, \hat{\mathcal{D}}_\Sigma)\) and the cohomology module \(H^s(K^p, \hat{\mathcal{D}}_\Sigma)\) admit slope-\(h\) decompositions, and there is an isomorphism

\[ H^s(K^p, \hat{\mathcal{D}}_\Sigma) \leq h \cong H^s(K^p, \mathcal{D}_\Sigma) \leq h \]

for any \(s \geq s[\Omega]\).
for any $s \geq s[\Omega]$. Furthermore, given an affinoid subdomain $\Omega' \subset \Omega$, there are canonical isomorphisms

$$C^\bullet(K^p, \mathcal{D}_\Omega) \otimes_{\mathcal{O}(\Omega)} \mathcal{O}(\Omega') \simeq C^\bullet(K^p, \mathcal{D}_{\Omega'})$$

and

$$H^\bullet(K^p, \mathcal{D}_\Omega) \otimes_{\mathcal{O}(\Omega)} \mathcal{O}(\Omega') \simeq H^\bullet(K^p, \mathcal{D}_{\Omega'}) .$$

Proof. By a topological version of the duality stated in §2.1, we have a natural isomorphism

$$C^\bullet(K^p, D_\Sigma^s) = C^\bullet(K^p, \mathcal{L}_{\mathcal{O}(\Omega)}(A_{\Omega}^s, \mathcal{O}(\Sigma)))$$

$$\simeq \mathcal{L}_{\mathcal{O}(\Omega)}(C^\bullet(K^p, A_{\Omega}^s), \mathcal{O}(\Sigma))$$

for any $s \geq s[\Omega]$. By assumption, $C^\bullet(K^p, A_{\Omega}^s)$ admits a slope-$\leq h$ decomposition, so we calculate

$$C^\bullet(K^p, D_\Sigma^s) \simeq \mathcal{L}_{\mathcal{O}(\Omega)}(C^\bullet(K^p, A_{\Omega}^s), \mathcal{O}(\Sigma))$$

$$\simeq \mathcal{L}_{\mathcal{O}(\Omega)}(C^\bullet(K^p, A_{\Omega}^s)_{\leq h}, \mathcal{O}(\Sigma))$$

$$\oplus \mathcal{L}_{\mathcal{O}(\Omega)}(C^\bullet(K^p, A_{\Omega}^s)_{> h}, \mathcal{O}(\Sigma)) .$$

By Proposition 3.1.2, passing to the inverse limit over $s$ in this isomorphism yields a slope-$\leq h$ decomposition of $C^\bullet(K^p, D_\Sigma)$ together with a natural isomorphism

$$C^\bullet(K^p, D_\Sigma)_{\leq h} \simeq C^\bullet(K^p, D_\Sigma^s) \simeq \mathcal{L}_{\mathcal{O}(\Omega)}(C^\bullet(K^p, A_{\Omega}^s)_{\leq h}, \mathcal{O}(\Sigma))$$

for any $s \geq s[\Omega]$. This proves the first sentence of the proposition.

For the second sentence, we first note that since $C^\bullet(K^p, A_{\Omega}^s)_{\leq h}$ is a complex of finite $\mathcal{O}(\Omega)$-modules, the natural map

$$\mathcal{L}_{\mathcal{O}(\Omega)}(C^\bullet(K^p, A_{\Omega}^s)_{\leq h}, \mathcal{O}(\Omega)) \rightarrow \Hom_{\mathcal{O}(\Omega)}(C^\bullet(K^p, A_{\Omega}^s)_{\leq h}, \mathcal{O}(\Omega))$$

is an isomorphism by Lemma 2.2 of [Buz07]. Next, note that if $R$ is a commutative ring, $S$ is a flat $R$-algebra, and $M, N$ are $R$-modules with $M$ finitely presented, the natural map $\Hom_R(M, N) \otimes_R S \rightarrow \Hom_S(M \otimes_R S, N \otimes_R S)$ is an isomorphism. With these two facts in hand, we calculate as follows:

$$C^\bullet(K^p, \mathcal{D}_\Omega) \otimes_{\mathcal{O}(\Omega)} \mathcal{O}(\Omega') \simeq \Hom_{\mathcal{O}(\Omega)}(C^\bullet(K^p, A_{\Omega}^s)_{\leq h}, \mathcal{O}(\Omega)) \otimes_{\mathcal{O}(\Omega)} \mathcal{O}(\Omega')$$

$$\simeq \Hom_{\mathcal{O}(\Omega)}(C^\bullet(K^p, A_{\Omega}^s)_{\leq h} \otimes_{\mathcal{O}(\Omega)} \mathcal{O}(\Omega'), \mathcal{O}(\Omega'))$$

$$\simeq \Hom_{\mathcal{O}(\Omega)}(C^\bullet(K^p, A_{\Omega}^s)_{\leq h}, \mathcal{O}(\Omega'))$$

$$\simeq C^\bullet(K^p, \mathcal{D}_{\Omega'})_{\leq h} ,$$

where the third line follows from Proposition 2.3.3. Passing to cohomology, the result follows as in the proof of Proposition 3.1.3. □

3.2 Finite-slope eigenpackets and non-critical classes

In this section we explain two results which are fundamental in our analysis. First of all, we recall and summarize some of the work of Eichler, Shimura, Matsushima, Borel-Wallach, Franke, and Li-Schwermer on the cohomology of arithmetic groups. Next, we state a fundamental theorem of Ash-Stevens and Urban (Theorem 6.4.1 of [AS08], Proposition 4.3.10 of [Urb11]) relating overconvergent
cohomology classes of small slope with classical automorphic forms. The possibility of such a result was largely the original raison d’etre of overconvergent cohomology; in the case $G = \text{GL}_2/\mathbb{Q}$, Stevens proved this theorem in a famous preprint [Ste94].

Let $\lambda \in X_+^*$ be a $B$-dominant algebraic weight, and let $K_f \subseteq G(\mathbb{A}_f)$ be any open compact subgroup. By fundamental work of Franke, the cohomology $H^*(Y(K_f), \mathcal{L}_\lambda)_C = H^*(Y(K_f), \mathcal{L}_\lambda) \otimes_{\mathbb{Q}_p, t} C$ admits an analytically defined splitting

$$H^*(Y(K_f), \mathcal{L}_\lambda)_C \simeq H^*_{\text{cusp}}(Y(K_f), \mathcal{L}_\lambda)_C \oplus H^*_{\text{Eis}}(Y(K_f), \mathcal{L}_\lambda)_C$$

into $T(G(\mathbb{A}_f), K_f)_C$-stable submodules, which we refer to as the cuspidal and Eisenstein cohomology, respectively. The cuspidal cohomology admits an exact description in terms of cuspidal group. By fundamental work of Franke, the cohomology $H^*(Y(K_f), \mathcal{L}_\lambda)_C$ is a regular weight, the natural inclusion of $H^*_{\text{cusp}}(Y(K_f), \mathcal{L}_\lambda)_C$ into $H^*_\text{cusp}(Y(K_f), \mathcal{L}_\lambda)$ is nonzero after localization at $\mathfrak{k}_\lambda$.

Let $\lambda$ be a regular weight, the natural inclusion of $H^*_{\text{cusp}}(Y(K_f), \mathcal{L}_\lambda)_C$ into $H^*_\text{cusp}(Y(K_f), \mathcal{L}_\lambda)$ is an isomorphism.

Note that if $\pi$ contributes nontrivially to the direct sum decomposition of the previous proposition, the central and infinitesimal characters of $\pi_\infty$ are necessarily inverse to those of $\mathcal{L}_\lambda$.

For any weight $\lambda \in W^f(K_p)$ and a given controlling operator $U_i$, we define $T_{\lambda,h}(K_p)$ as the subalgebra of $\text{End}_k_\lambda \left( H^*(K_p^\text{reg}, \mathcal{L}_\lambda) \right)$ generated by the image of $T(K_p^\text{reg}, \mathcal{L}_\lambda)$, and we set $T_{\lambda,h}(K_p) = \varinjlim_{\mathfrak{k}_\lambda} T_{\lambda,h}(K_p)$. The algebra $T_{\lambda,h}(K_p)$ is independent of the choice of controlling operator used in its definition.

**Proposition 3.2.1.** There is a canonical isomorphism

$$H^*_{\text{cusp}}(Y(K_f), \mathcal{L}_\lambda)_C \simeq \bigoplus_{\pi \in L^2_{\text{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}))} m(\pi)\pi^K_f \otimes H^*_\text{cusp}(g, K_\infty; \pi_\infty \otimes \mathcal{L}_\lambda)$$

of graded $T(G(\mathbb{A}_f), K_f)_C$-modules, where $m(\pi)$ denotes the multiplicity of $\pi$ in $L^2_{\text{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}))$. If $\lambda$ is a regular weight, the natural inclusion of $H^*_{\text{cusp}}(Y(K_f), \mathcal{L}_\lambda)_C$ into $H^*_\text{cusp}(Y(K_f), \mathcal{L}_\lambda)$ is an isomorphism.

Note that if $\pi$ contributes nontrivially to the direct sum decomposition of the previous proposition, the central and infinitesimal characters of $\pi_\infty$ are necessarily inverse to those of $\mathcal{L}_\lambda$.

For any weight $\lambda \in W^f(K_p)$ and a given controlling operator $U_i$, we define $T_{\lambda,h}(K_p)$ as the subalgebra of $\text{End}_k_\lambda \left( H^*(K_p^\text{reg}, \mathcal{L}_\lambda) \right)$ generated by the image of $T(K_p^\text{reg}, \mathcal{L}_\lambda)$, and we set $T_{\lambda,h}(K_p) = \varinjlim_{\mathfrak{k}_\lambda} T_{\lambda,h}(K_p)$. The algebra $T_{\lambda,h}(K_p)$ is independent of the choice of controlling operator used in its definition.

**Definition 3.2.2.** A finite-slope eigenpacket of weight $\lambda$ and level $K_p$ (or simply a finite-slope eigenpacket) is an algebra homomorphism $\phi : T_{\lambda}(K_p) \to \mathbb{Q}_p$.

It’s easy to check that this coincides with the definition given in the introduction. If $\phi$ is a finite-slope eigenpacket, we shall regard the contraction of $\ker \phi$ under the structure map $T(K_p) \to T_{\lambda}(K_p)$ as a maximal ideal in $T(K_p)$, which we also denote by $\ker \phi$. Note that $T_{\lambda}(K_p)$ is a countable direct product of zero-dimensional Artinian local rings, and the factors in this direct product are in natural bijection with the finite-slope eigenpackets.

A weight $\lambda$ is arithmetic if it factors as the product of a finite-order character $\varepsilon$ of $T(\mathbb{Z}_p)$ and an element $\lambda_{\text{alg}}$ of $X_+^*$; if $\lambda_{\text{alg}} \in X_+^*$ we say $\lambda$ is dominant arithmetic. If $\lambda = \lambda_{\text{alg}}\varepsilon$ is a dominant arithmetic weight, we are going to formulate some comparisons between $H^*(K_p, \mathcal{L}_\lambda)$ and $H^*(Y(K_p^\text{reg}), \mathcal{L}_{\lambda_{\text{alg}}})$. In order to do this, we need to twist the natural Hecke action on the latter module slightly. More precisely, if $M$ is any $A^+_p$-module and $\lambda \in X_+^*$, we define the $\star$-action in weight $\lambda$ by $U_i \star \lambda m = \lambda(t)^{-1}U_i m$ [AS08]. The map $i_{\lambda}$ defined in §2.2 induces a morphism

$$i_{\lambda} : H^*(K_p, \mathcal{L}_\lambda) \to H^*(K_p^\text{reg}, \mathcal{L}_{\lambda_{\text{alg}}})$$

for any $s \geq s[\varepsilon]$ which intertwines the standard action of $T(K_p)$ on the source with the $\star$-action on the target.

**Definition 3.2.3.** If $\lambda = \lambda_{\text{alg}}\varepsilon$ is an arithmetic weight and $s = s[\varepsilon]$, a finite-slope eigenpacket is classical if the module $H^*(Y(K_p^\text{reg}), \mathcal{L}_{\lambda_{\text{alg}}})$ is nonzero after localization at $\ker \phi$, and noncritical if the map

$$i_{\lambda} : H^*(K_p, \mathcal{L}_\lambda) \to H^*(K_p^\text{reg}, \mathcal{L}_{\lambda_{\text{alg}}})$$

is an isomorphism.
becomes an isomorphism after localization at ker \( \phi \). A classical eigenpacket is interior if \( H_2^p(K^p I^*_1, L^\alg_{\lambda^h}) \) vanishes after localization at ker \( \phi \), and strongly interior if \( H_2^p(K^p, \mathcal{D}_\lambda) \) vanishes after localization at ker \( \phi \) as well.

Next we formulate a result which generalizes Stevens’s control theorem [Ste94]. Given \( \lambda \in X^* \), we define an action of the Weyl group \( W \) by \( w \cdot \lambda = (\lambda + \rho)w - \rho \).

**Definition 3.2.4.** Fix a controlling operator \( U_i \), \( t \in \Lambda^+ \). Given an arithmetic weight \( \lambda = \lambda^\alg \varepsilon \), a rational number \( h \) is a small slope for \( \lambda \) if

\[
h < \inf_{w \in W \setminus \{id\}} v_p(w \cdot \lambda^\alg(t)) - v_p(\lambda^\alg(t)).
\]

**Theorem 3.2.5 (Ash-Stevens, Urban).** Fix an arithmetic weight \( \lambda = \lambda^\alg \varepsilon \) and a controlling operator \( U_i \). If \( h \) is a small slope for \( \lambda \), there is a natural isomorphism of Hecke modules

\[
H^s(K^p, \mathcal{D}_\lambda)_{\leq h} \cong H^s(Y(K^p I^*_1), L^\alg_{\lambda^h})_{T(\mathbb{Z}/p^r \mathbb{Z}) = \varepsilon}
\]

for any \( s \geq s[\varepsilon] \).

**Proof (sketch).** Suppose \( \lambda = \lambda^\alg \) for simplicity. Urban constructs a second quadrant spectral sequence

\[
E_1^{i,j} = \bigoplus_{w \in W, \ell(w) = -i} H^j(K^p, \mathcal{D}_{w, \lambda^h})_{T(\mathbb{Z}/p^r \mathbb{Z}) = \varepsilon} = H^{i+j}(K^p I, L^\alg_{\lambda^h})_{\leq h}
\]

which is equivariant for \( U_i \) if we twist the action as follows: \( U_i \) acts through the *-action in weight \( \lambda \) on the target and \( (\lambda(t)^{-1} w \cdot \lambda)(t) U_i \) acts on the \( w \)-summand of the \( E_1 \)-page. In particular, taking the slope \( \leq h \) part yields a spectral sequence

\[
E_1^{i,j} = \bigoplus_{w \in W, \ell(w) = -i} H^j(K^p, \mathcal{D}_{w, \lambda^h})_{\leq h} - v_p(w \cdot \lambda(t)) + v_p(\lambda(t)) = H^{i+j}(K^p I, L^\alg_{\lambda})_{\leq h}.
\]

But any element of \( \mathcal{A}^+ \) is contractive on \( H^j(K^p, \mathcal{D}_\lambda) \), so the \( w \)-summand of the \( E_1 \)-page is now empty for \( w \neq id \). □

### 3.3 The spectral sequences

In this section we introduce our main technical tool for analyzing overconvergent cohomology. Fix a choice of tame level \( K^p \) and an augmented Borel-Serre complex \( G_*(K^p, -) \).

**Theorem 3.3.1.** Fix a slope datum \( (U_i, \Omega, h) \), and let \( \Sigma \subseteq \Omega \) be an arbitrary rigid Zariski closed subspace. Then \( H^*(K^p, \mathcal{D}_\Sigma) \) admits a slope \( \leq h \) decomposition, and there is a convergent first quadrant spectral sequence

\[
E_2^{i,j} = \text{Ext}_i^\mathcal{O}(H_j(K^p, \mathcal{D}_\Omega)_{\leq h}, \mathcal{O}(\Sigma)) \Rightarrow H^{i+j}(K^p, \mathcal{D}_\Sigma)_{\leq h}.
\]

Furthermore, there is a convergent second quadrant spectral sequence

\[
E_2^{i,j} = \text{Tor}_i^\mathcal{O}(H^i(K^p, \mathcal{D}_\Omega)_{\leq h}, \mathcal{O}(\Sigma)) \Rightarrow H^{i+j}(K^p, \mathcal{D}_\Sigma)_{\leq h}.
\]

In addition, there are analogous spectral sequences relating Borel-Moore homology with compactly supported cohomology, and boundary homology with boundary cohomology, and there are morphisms between the spectral sequences compatible with the morphisms between these different cohomology
theories. Finally, the spectral sequences and the morphisms between them are equivariant for the natural Hecke actions on their $E_2$ pages and abutments; more succinctly, they are spectral sequences of $T(K^p)$-modules.

When no ambiguity is likely, we will refer to the two spectral sequences of Theorem 3.3.1 as “the Ext spectral sequence” and “the Tor spectral sequence.” The Hecke-equivariance of these spectral sequences is crucial for applications, since it allows one to localize the entire spectral sequence at any ideal in the Hecke algebra.

Proof of Theorem 3.3.1. By the isomorphisms proved in §3.1, it suffices to construct a Hecke-equivariant spectral sequence $\Ext^i_{\mathcal{O}(\Omega)}(H_j(K^p, A^s_{\Omega}) \leq h, \mathcal{O}(\Sigma)) \Rightarrow H^{i+j}(K^p, D^s_{\Sigma}) \leq h$ for some $s \geq s[\Omega]$.

Consider the hyperext group $\Ext^n_{\mathcal{O}(\Omega)}(C^d(K^p, A^s_{\Omega}), \mathcal{O}(\Sigma))$. Since $C^d(K^p, A^s_{\Omega})$ is a complex of $T(K^p)$-modules, this hyperext group is naturally a $T(K^p)$-module, and the hypercohomology spectral sequence

$$E^{i,j}_2 = \Ext^i_{\mathcal{O}(\Omega)}(H_j(K^p, A^s_{\Omega}) \leq h, \mathcal{O}(\Sigma)) \Rightarrow \Ext^{i+j}_{\mathcal{O}(\Omega)}(C^d(K^p, A^s_{\Omega}), \mathcal{O}(\Sigma))$$

is a spectral sequence of $T(K^p)$-modules. On the other hand, the quasi-isomorphism $C^d(K^p, A^s_{\Omega}) \simeq C_*^s(K^p, A^s_{\Omega})$ in $D^b(A(\Omega))$ together with the slope-$\leq h$ decomposition $C_*^s(K^p, A^s_{\Omega}) \simeq C_*^s(K^p, A^s_{\Omega}) \leq h \oplus C_*^s(K^p, A^s_{\Omega}) > h$ induces Hecke-stable slope-$\leq h$-decompositions of the abutment and of the entries on the $E_2$ page. By Proposition 3.2.2, the slope decomposition of the $E_2$ page induces slope decompositions of all entries on all higher pages of the spectral sequence. In other words, we may pass to the “slope-$\leq h$ part” of the hypercohomology spectral sequence in a Hecke-equivariant way, getting a spectral sequence

$$E^{i,j}_2 = \Ext^i_{\mathcal{O}(\Omega)}(H_j(K^p, A^s_{\Omega}) \leq h, \mathcal{O}(\Sigma)) \Rightarrow \Ext^{i+j}_{\mathcal{O}(\Omega)}(C^d(K^p, A^s_{\Omega}), \mathcal{O}(\Sigma)) \leq h$$

of $T(K^p)$-modules. But $\Ext^i_{\mathcal{O}(\Omega)}(H_j(K^p, A^s_{\Omega}) \leq h, \mathcal{O}(\Sigma)) \simeq \Ext^i_{\mathcal{O}(\Omega)}(H_j(K^p, A^s_{\Omega}) \leq h, \mathcal{O}(\Sigma))$, and

$$\Ext^n_{\mathcal{O}(\Omega)}(C^d(K^p, A^s_{\Omega}), \mathcal{O}(\Sigma)) \leq h \simeq \Ext^n_{\mathcal{O}(\Omega)}(C^s(K^p, A^s_{\Omega}) \leq h, \mathcal{O}(\Sigma)) \simeq H^n(\mathcal{O}(\Sigma)) \simeq H^n(K^p, D^s_{\Sigma}) \leq h,$$

where the third line follows from the projectivity of each $C_i(K^p, A^s_{\Omega}) \leq h$ and the fourth line follows from the proof of Proposition 3.1.5.

For the Tor spectral sequence, the isomorphism

$$C^s(K, D^s_{\Sigma}) \leq h \otimes_{\mathcal{O}(\Omega)} \mathcal{O}(\Sigma) \simeq C^s(K^p, D^s_{\Sigma}) \leq h$$

yields an isomorphism

$$C^s(K, D^s_{\Sigma}) \leq h \otimes_{\mathcal{O}(\Omega)} \mathcal{O}(\Sigma) \simeq C^s(K^p, D^s_{\Sigma}) \leq h$$

of $T(K^p)$-module complexes in $D^b(\mathcal{O}(\Omega))$, and the result follows analogously from the hypertor spectral sequence

$$\Tor^R_{-i}(H^j(C^s), N) \Rightarrow \Tor^R_{-i-j}(C^s, N).$$
Remark 3.3.2. If $(Ω, h)$ is a slope datum, $Σ_1$ is Zariski-closed in $Ω$, and $Σ_2$ is Zariski-closed in $Σ_1$, the transitivity of the derived tensor product yields an isomorphism
\[
C^∗(K^p, D_{Σ_2})_{≤ h} \cong C^∗(K^p, D_{Ω})_{≤ h} \otimes^L_{D_{Ω}} D(Σ_2)
\]
\[
\cong C^∗(K^p, D_{Ω})_{≤ h} \otimes^L_{D(Ω)} D(Σ_1) \otimes^L_{D(Σ_1)} D(Σ_2)
\]
\[
\cong C^∗(K^p, D_{Σ_1})_{≤ h} \otimes^L_{D(Σ_1)} D(Σ_2)
\]
which induces a relative version of the Tor spectral sequence, namely
\[
E^{i,j}_{2} = Tor_{D(Σ_1)}(H^j(K^p, D_{Σ_1})_{≤ h}, D(Σ_2)) \Rightarrow H^{i+j}(K^p, D_{Σ_2})_{≤ h}.
\]
This spectral sequence plays an important role in Newton’s proof of Theorem 1.1.6.

The boundary and Borel-Moore/compactly supported spectral sequences Notation as in §2.1, let $D_∞$ denote the Borel-Serre bordification of $D_∞$, and let $♭D_∞ = D_∞ \setminus D_∞$. Setting $DΑ = D_∞ \times G(A_f)$ and $♭DΑ = D_∞ \times G(A_f)$, the natural map $C^∗(D_Α) \to C^∗(DΑ)$ induces a functorial isomorphism $H_∗(K^f, M) \cong H_∗(C^∗(D_Α) \otimes^Z_{G(A_f)} M)$, so we may redefine $C^∗_ad(K^f, M)$ as $C^∗_ad(K^f, M) = C^∗(♭D_Α) \otimes^Z_{G(A_f)} M$. Setting $C^∗_ad(K^f, M) = C^∗(♭D_Α) \otimes^Z_{G(A_f)} M$, the natural inclusion induces a map $C^∗_ad(K^f, M) \to C^∗_ad(K^f, M)$, and we define $C^∗_ad(K^f, M)$ as the cone of this map. Not surprisingly, the homology of $C^∗_ad$ (resp. $C^∗_BM$) computes boundary (resp. Borel-Moore) homology. Choosing a triangulation of $Y(K^f)$ induces a triangulation on the boundary, and yields a complex $C^∗_ad(K^f, M)$ together with a map $C^∗_ad(K^f, M) \to C^∗_ad(K^f, M)$; defining $C^∗_BM$ as the cone of this map, these complexes all fit into a big diagram

\[
\begin{array}{ccc}
C^∗_ad(K^f, -) & \to & C^∗_BM(K^f, -) \\
\uparrow & & \uparrow \\
C^∗_ad(K^f, -) & \to & C^∗_BM(K^f, -)
\end{array}
\]

in which the rows are exact triangles functorial in $M$, and the vertical arrows are quasi-isomorphisms. We make analogous definitions of $C^∗_ad(K^f, M)$, etc.

The boundary and Borel-Moore/compactly supported sequences, and the morphisms between them, follow from “taking the slope-≤ h part” of the diagram

\[
\begin{array}{ccc}
\text{RHom}_{D(Ω)}(C^∗_ad(K^p, A^s_Ω), D(Σ)) & \to & C^∗_ad(K^p, \text{Hom}_{D(Ω)}(A^s_Ω, D(Σ))) \\
\text{RHom}_{D(Ω)}(C^∗_ad(K^p, A^s_Ω), D(Σ)) & \to & C^∗_ad(K^p, \text{Hom}_{D(Ω)}(A^s_Ω, D(Σ))) \\
\text{RHom}_{D(Ω)}(C^∗_ad(K^p, A^s_Ω), D(Σ)) & \to & C^∗_ad(K^p, \text{Hom}_{D(Ω)}(A^s_Ω, D(Σ)))
\end{array}
\]

in which the horizontal arrows are quasi-isomorphisms, the columns are exact triangles in $D^b(D(Ω))$, and the diagram commutes for the natural action of $T(K^p)$. 28
4 Eigenvarieties

In this section we begin to use global rigid analytic geometry in a more serious way; the main reference for this topic is [BGR84], and [Con08] is a nice survey of the main ideas. We shall repeatedly and tacitly use the fact that if $\Omega$ is an affinoid subdomain of an affinoid space $\Omega$, $\mathcal{O}(\Omega')$ is a flat $\mathcal{O}(\Omega)$-module; this is an easy consequence of the universal property of an affinoid subdomain together with the local criterion for flatness.

4.1 Fredholm hypersurfaces

Let $A$ be an affinoid integral domain. We say that such an $A$ is relatively factorial if for any $f = \sum_{n=0}^{\infty} a_n X^n \in A(X)$ with $a_0 = 1$, $(f)$ factors uniquely as a product of principal prime ideals $(f_i)$ where each $f_i$ may be chosen with constant term 1. A rigid analytic space $\mathcal{W}$ is relatively factorial if it has an admissible covering by relatively factorial affinoids. Throughout the remainder of this article, we reserve the letter $\mathcal{W}$ for a relatively factorial rigid analytic space.

**Definition 4.1.1.** A Fredholm series is a global section $f \in \mathcal{O}(\mathcal{W} \times A^1)$ such that under the map $\mathcal{O}(\mathcal{W} \times A^1) \rightarrow \mathcal{O}(\mathcal{W})$ induced by $j : \mathcal{W} \times \{0\} \rightarrow \mathcal{W} \times A^1$ we have $j^* f = 1$. A Fredholm hypersurface is a closed immersion $\mathcal{Z} \subset \mathcal{W} \times A^1$ such that the ideal sheaf of $\mathcal{Z}$ is generated by a Fredholm series $f$, in which case we write $\mathcal{Z} = \mathcal{Z}(f)$.

Note that the natural projection $\mathcal{W} \times A^1 \rightarrow \mathcal{W}$ induces a map $w : \mathcal{Z} \rightarrow \mathcal{W}$. Let $\mathcal{O}(\mathcal{W})\{\{X\}\}$ denote the subring of $\mathcal{O}(\mathcal{W})[[X]]$ consisting of series $\sum_{n=0}^{\infty} a_n X^n$ such that $|a_n|_{\Omega^n} \rightarrow 0$ as $n \rightarrow \infty$ for any affinoid $\Omega \subset \mathcal{W}$ and any $r \in \mathbb{R}_{>0}$. The natural injection $\mathcal{O}(\mathcal{W} \times A^1) \rightarrow \mathcal{O}(\mathcal{W})\{\{X\}\}$ identifies the monoid of Fredholm series with elements of $\mathcal{O}(\mathcal{W})\{\{X\}\}$ such that $a_0 = 1$. When $\mathcal{W}$ is relatively factorial, the ring $\mathcal{O}(\mathcal{W})\{\{X\}\}$ admits a good factorization theory, and we may speak of irreducible Fredholm series without ambiguity. We say a collection of distinct irreducible Fredholm series $\{f_i\}_{i \in I}$ is locally finite if $\mathcal{Z}(f_i) \cap U = \emptyset$ for all but finitely many $i \in I$ and any quasi-compact admissible open subset $U \subset \mathcal{W} \times A^1$.

**Proposition 4.1.2 (Coleman-Mazur, Conrad).** If $\mathcal{W}$ is relatively factorial, any Fredholm series $f$ admits a factorization $f = \prod_{i \in I} f_i^{n_i}$ as a product of irreducible Fredholm series with $n_i \geq 1$; any such factorization is unique up to reordering the terms, the collection $\{f_i\}_{i \in I}$ is locally finite, and the irreducible components of $\mathcal{Z}(f)$ are exactly the Fredholm hypersurfaces $\mathcal{Z}(f_i^{n_i})$. The nilreduction of $\mathcal{Z}(f)$ is $\mathcal{Z}(\prod_{i \in I} f_i)$.

**Proof.** See §1 of [CM98] and §4 of [Con99] (especially Theorems 4.2.2 and 4.3.2). □

**Proposition 4.1.3.** If $\mathcal{Z}$ is a Fredholm hypersurface, the image $w(\mathcal{Z})$ is Zariski-open in $\mathcal{W}$.

**Proof.** We may assume $\mathcal{Z} = \mathcal{Z}(f)$ with $f = 1 + \sum_{n=1}^{\infty} a_n X^n$ irreducible. By Lemma 1.3.2 of [CM98], the fiber of $\mathcal{Z}$ over $\lambda \in \mathcal{W}(\mathbb{Q}_p)$ is empty if and only if $a_n \in m_\lambda$ for all $n$, if and only if $\mathcal{I} = (a_1, a_2, a_3, \ldots) \subset m_\lambda$. The ideal $\mathcal{I}$ is naturally identified with the global sections of a coherent ideal sheaf, which cuts out a closed immersion $V(\mathcal{I}) \rightarrow \mathcal{W}$ in the usual way, and $w(V(\mathcal{I}))$ is the complement of $V(\mathcal{I})$. □

Given a Fredholm hypersurface $\mathcal{Z} = \mathcal{Z}(f)$, a rational number $h \in \mathbb{Q}$, and an affinoid $\Omega \subset \mathcal{W}$, we define $\mathcal{Z}_{\Omega,h} = \mathcal{O}(\mathcal{Z}) \langle p^h X \rangle / (f(X))$ regarded as an admissible affinoid open subset of $\mathcal{Z}$. The natural map $\mathcal{Z}_{\Omega,h} \rightarrow \Omega$ is flat but not necessarily finite, and we define an affinoid of the form $\mathcal{Z}_{\Omega,h}$ to be slope-adapted if $\mathcal{Z}_{\Omega,h} \rightarrow \Omega$ is a finite flat map. The affinoid $\mathcal{Z}_{\Omega,h}$ is slope-adapted if and only if $f_\Omega = f|_{\mathcal{Z}(\mathcal{Z})}$ admits a slope $\leq h$ factorization $Q(X) \cdot R(X)$, in which case $\mathcal{O}(\mathcal{Z}_{\Omega,h}) \cong \mathcal{O}(\mathcal{Z})[X]/(Q(X))$.  

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Proposition 4.1.4. For any Fredholm hypersurface $\mathcal{Z}$, the collection of slope-adapted affinoids forms an admissible cover of $\mathcal{Z}$.

Proof. See §4 of [Buz07].

4.2 Eigenvariety data

Definition 4.2.1. An eigenvariety datum is a tuple $\mathcal{D} = (\mathcal{W}, \mathcal{Z}, \mathcal{M}, \mathcal{T}, \psi)$ where $\mathcal{W}$ is a separated, reduced, equidimensional, relatively factorial rigid analytic space, $\mathcal{Z} \subset \mathcal{W} \times \mathbf{A}^1$ is a Fredholm hypersurface, $\mathcal{M}$ is a coherent analytic sheaf on $\mathcal{Z}$, $\mathcal{T}$ is a commutative $\mathbb{Q}_p$-algebra, and $\psi$ is a $\mathbb{Q}_p$-algebra homomorphism $\psi : \mathcal{T} \to \text{End}_{\mathcal{O}_z}(\mathcal{M})$.

In practice $\mathcal{T}$ will be a Hecke algebra, $\mathcal{Z}$ will be a “spectral variety” parametrizing the eigenvalues of some distinguished operator $U \in \mathcal{T}$ on some graded module $M^*$ of $p$-adic automorphic forms or on a complex whose cohomology yields $M^*$, and $\mathcal{M}$ will be the natural “spreading out” of $M^*$ to a coherent sheaf over $\mathcal{Z}$. We do not require that $\mathcal{M}$ be locally free on $\mathcal{Z}$.

Theorem 4.2.2. Given an eigenvariety datum $\mathcal{D}$, there is a rigid analytic space $\mathcal{X} = \mathcal{X}(\mathcal{D})$ together with a finite morphism $\pi : \mathcal{X} \to \mathcal{Z}$, a morphism $\psi : \mathcal{X} \to \mathcal{W}$, an algebra homomorphism $\phi_{\mathcal{X}} : \mathcal{T} \to \mathcal{O}(\mathcal{X})$, and a coherent sheaf $\mathcal{M}^1$ on $\mathcal{Z}$ together with a canonical isomorphism $\mathcal{M} \cong \pi_!\mathcal{M}^1$ compatible with the actions of $\mathcal{T}$ on $\mathcal{M}$ and $\mathcal{M}^1$ (via $\psi$ and $\phi_{\mathcal{X}}$, respectively). The points of $\mathcal{X}$ lying over $z \in \mathcal{Z}$ are in bijection with the generalized eigenspaces for the action of $\mathcal{T}$ on $\mathcal{M}(z)$.

Proof. Let $\mathcal{Cov} = \{\Omega_i\}_{i \in I}$ be an admissible affinoid cover of $\mathcal{Z}$; we abbreviate $\Omega_i \cap \Omega_j$ by $\Omega_{ij}$. For any $\Omega_i$ we let $\mathcal{T}_{\Omega_i}$ be the finite $\mathcal{O}(\Omega_i)$-subalgebra of $\text{End}_{\mathcal{O}(\Omega_i)}(\mathcal{M}(\Omega_i))$ generated by $\im \psi$, with structure map $\phi_{\Omega_i} : \mathcal{T} \to \mathcal{T}_{\Omega_i}$. Let $\mathcal{X}_{\Omega_i}$ be the affine rigid space $\text{Sp}\mathcal{T}_{\Omega_i}$, with $\pi : \mathcal{X}_{\Omega_i} \to \Omega_i$ the natural morphism. The canonical morphisms $\mathcal{T}_{\Omega_i} \otimes_{\mathcal{O}(\Omega_i)} \mathcal{O}(\Omega_{ij}) \to \mathcal{T}_{\Omega_{ij}}$ are isomorphisms, and so we may glue the affine rigid spaces $\mathcal{X}_{\Omega_i}$ together via their overlaps $\mathcal{X}_{\Omega_{ij}}$ into a rigid space $\mathcal{X}$ together with a finite map $\pi : \mathcal{X} \to \mathcal{Z}$. The $\mathcal{T}_{\Omega_i}$-module structure on $\mathcal{M}(\Omega_i)$ is compatible with the transition maps, and so these modules glue to a coherent sheaf $\mathcal{M}^1$. The structure maps $\phi_{\Omega_i}$ glue to a map $\phi : \mathcal{T} \to \mathcal{O}(\mathcal{X})$ which is easily seen to be an algebra homomorphism. The remainder of the theorem is straightforward from the construction. □

The space $\mathcal{X}$ is the eigenvariety associated with the given eigenvariety datum. For any point $x \in \mathcal{X}(\overline{\mathbb{Q}_p})$, we write $\phi_{\mathcal{X}, x} : \mathcal{T} \to k_x$ for the composite map

$$(\mathcal{O}(\mathcal{X}) \to \mathcal{O}_{\mathcal{X}, x} \to k_x) \circ \phi_{\mathcal{X}},$$

and we say $\phi_{\mathcal{X}, x}$ is the eigenpacket parametrized by the point $x$. If the map $x \mapsto \phi_{\mathcal{X}, x}$ determines a bijection of $\mathcal{X}(\overline{\mathbb{Q}_p})$ with a set of eigenpackets of a certain type, we write $\phi \mapsto x_\phi$ for the inverse map.

Maintaining the notation of the previous theorem, we note the following useful tautology: if $\mathcal{Y} \hookrightarrow \mathcal{X}$ is a closed immersion with associated ideal sheaf $\mathcal{I}_\mathcal{Y} \subset \mathcal{O}_{\mathcal{X}}$, then $\mathcal{Y}$ can be interpreted as the eigenvariety associated with the eigenvariety datum

$$\mathcal{D}_\mathcal{Y} = (\mathcal{W}, \mathcal{Z}, \pi_*(\mathcal{M}^1 \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{Y}}/\mathcal{I}_\mathcal{Y}), \mathcal{T}, \phi_{\mathcal{X}} \mod \mathcal{I}_\mathcal{Y}).$$

4.3 Eigenvariety data from overconvergent cohomology

Fix $\mathbf{G}$, $\mathbb{K}^p$, a controlling operator $U_1$, and an augmented Borel-Serre complex $C_\bullet(K^p, -)$. For $\Omega \subset \mathcal{W}_{\mathbf{K}^p}$ an affinoid open, the Fredholm series $f_\Omega(X) = \text{det}(1 - U_1 X)|C_\bullet(K^p, \mathcal{A}_{\mathbf{K}^p}^\infty)$ is well-defined independently of $s \geq s[\Omega]$ by Proposition 3.1.1, and if $\Omega' \subset \Omega$ is open then $f_\Omega(X)|_{\Omega'} = f_{\Omega'}(X)$.  

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By Tate’s acyclicity theorem, there is a unique $f(X) \in \mathcal{O}(\Psi)\{\{X\}\}$ with $f(X)|_{\Omega} = f_{\Omega}(X)$ for all $\Omega$. Set $\mathcal{Z} = \mathcal{Z}_f$. If $\mathcal{Z}_{\Omega,h} \subset \mathcal{Z}$ is a slope-adapted affinoid, then $C^*(K^p, \mathcal{Z}_{\Omega})$ admits a slope-\(\leq h\) decomposition, with $C^*(K^p, \mathcal{Z}_{\Omega})_{\leq h} \cong \text{Hom}_{\mathcal{O}(\Omega)}(C_*(K^p, \mathcal{A}_{\Omega})_{\leq h}, \mathcal{O}(\Omega))$ for any $s \geq s[\Omega]$, and $C^*(K^p, \mathcal{Z}_{\Omega})_{\leq h}$ is naturally a (graded) module over $\mathcal{O}(\mathcal{Z}_{\Omega,h}) \cong \mathcal{O}(\Omega)[X]/(Q_{\Omega,h}(X))$ via the map $X \mapsto U_t^{-1}$; here $Q_{\Omega,h}(X)$ denotes the slope-\(\leq h\) factor of $f_{\Omega}$.

**Proposition 4.3.1.** There is a unique complex $\mathcal{X}^*$ of coherent analytic sheaves on $\mathcal{Z}$ such that $\mathcal{X}^*(\mathcal{Z}_{\Omega,h}) \cong C^*(K^p, \mathcal{Z}_{\Omega})_{\leq h}$ for any slope-adapted affinoid $\mathcal{Z}_{\Omega,h}$.

Proof. For $\mathcal{Z}_{\Omega,h}$ a slope-adapted affinoid, we simply set $\mathcal{X}^*(\mathcal{Z}_{\Omega,h}) = C^*(K^p, \mathcal{Z}_{\Omega})_{\leq h}$, with $\mathcal{X}^*(\mathcal{Z}_{\Omega,h})$ regarded as an $\mathcal{O}(\mathcal{Z}_{\Omega,h})$-module as in the previous paragraph. We are going to show that the formation of $\mathcal{X}^*(\mathcal{Z}_{\Omega,h})$ is compatible with overlaps of slope-adapted affinoids; since slope-adapted affinoids form a base for the ambient $G$-topology on $\mathcal{Z}$, this immediately implies that the $\mathcal{X}^*(\mathcal{Z}_{\Omega,h})$'s glue together into a sheaf over $\mathcal{Z}$.

If $\mathcal{Z}_{\Omega,h} \in \mathcal{O}v$ and $\Omega' \subset \Omega$ with $\Omega'$ connected, a calculation gives $\mathcal{O}(\mathcal{Z}_{\Omega',h}) \cong \mathcal{O}(\mathcal{Z}_{\Omega,h}) \otimes_{\mathcal{O}(\Omega)} \mathcal{O}(\Omega')$, so then $\mathcal{Z}_{\Omega',h} \in \mathcal{O}v$. Fix $\mathcal{Z}_{\Omega',h} \subset \mathcal{Z}_{\Omega,h} \in \mathcal{O}v$ with $\mathcal{Z}_{\Omega',h} \in \mathcal{O}v$; we necessarily have $\Omega' \subset \Omega$, and we may assume $h' \leq h$. Set $C_{\Omega,h} = C^*(K^p, \mathcal{Z}_{\Omega})_{\leq h}$. We now trace through the following sequence of canonical isomorphisms:

$$
C_{\Omega,h} \otimes_{\mathcal{O}(\mathcal{Z}_{\Omega,h})} \mathcal{O}(\mathcal{Z}_{\Omega',h}) \cong C_{\Omega,h} \otimes_{\mathcal{O}(\Omega)} \mathcal{O}(\mathcal{Z}_{\Omega'}) \cong C_{\Omega,h} \otimes_{\mathcal{O}(\mathcal{Z}_{\Omega,h})} \mathcal{O}(\mathcal{Z}_{\Omega',h}) \cong C_{\Omega'} \otimes_{\mathcal{O}(\mathcal{Z}_{\Omega',h})} \mathcal{O}(\mathcal{Z}_{\Omega',h}) \cong C_{\Omega'}.
$$

The fourth line here follows from Proposition 3.1.5. □

Taking the cohomology of $\mathcal{X}^*$ yields a graded sheaf $\mathcal{M}^*$ on $\mathcal{Z}$ together with canonical isomorphisms $\mathcal{M}^*(\mathcal{Z}_{\Omega,h}) \cong H^*(K^p, \mathcal{Z}_{\Omega})_{\leq h}$ for any slope-adapted affinoid $\mathcal{Z}_{\Omega,h}$. By Proposition 3.1.5 the natural maps $T(K^p) \to \text{End}_{\mathcal{O}(\mathcal{Z}_{\Omega,h})}(H^*(K^p, \mathcal{Z}_{\Omega})_{\leq h})$ glue together into a degree-preserving algebra homomorphism $\psi : T(K^p) \to \text{End}_{\mathcal{O}(\mathcal{Z})}(\mathcal{M}^*)$.

**Definition 4.3.2.** The eigenvariety $\mathcal{Y}_{G,K^p}$ is the eigenvariety associated with the eigenvariety datum $(\Psi_{K^p}, \mathcal{Z}, \mathcal{M}^*, T(K^p), \psi)$. For $n$ a given integer, $\mathcal{Y}_{G,K^p,n}$ is the eigenvariety associated with the eigenvariety datum $(\Psi_{K^p}, \mathcal{Z}, \mathcal{M}^*, T(K^p), \psi)$.

Note that $\mathcal{Z}$ is highly noncanonical, depending as it does on a choice of augmented Borel-Serre complex. However, $\mathcal{Y}_{G,K^p}$ is completely canonical and independent of this choice. Furthermore, setting $f = \text{ann}_{\mathcal{O}(\mathcal{Z})}\mathcal{M}^* \subset \mathcal{O}(\mathcal{Z})$, the closed immersion $\mathcal{Z}_f \hookrightarrow \mathcal{Z} \times \mathcal{A}^1$ cut out by $\mathcal{O}_{\Psi_{K^p}} \to \mathcal{O}_f = \mathcal{O}_f | \mathcal{Z}_f$ is independent of all choices, and is the “true” spectral variety for $U_t$. Note also that in practice, the eigenvarieties $\mathcal{Y}_{G,K^p}$ carry some extra structure which we don’t really exploit in this article: in particular, the sheaves $\mathcal{M}^*$ are sheaves of $T_{\text{ram}}(K^p)$-modules. Our first main result on the eigenvarieties $\mathcal{Y}_{G,K^p}$ is the following.

**Theorem 4.3.3.** The points $x \in \mathcal{Y}_{G,K^p}(\overline{\mathbb{Q}_p})$ lying over a given weight $\lambda \in \Psi_{K^p}(\overline{\mathbb{Q}_p})$ are in bijection with the finite-slope eigenpackets for $G$ of weight $\lambda$ and level $K^p$, and this bijection is realized by sending $x \in \mathcal{Y}(\overline{\mathbb{Q}_p})$ to the eigenpacket $\phi_{x,x}$. This theorem is due to Ash and Stevens (Theorem 6.2.1 of [AS08]), but the following proof is new.
Proof. Given a finite-slope eigenpacket \( \phi \) of weight \( \lambda \), fix a slope-adapted affinoid \( \mathcal{A}_{\Omega, h} \) with \( \lambda \in \Omega \) and \( h > \nu_p(\phi(U)) \), and let \( T_{\Omega, h} = T_{\Omega, h}(K^p) \) be the \( \mathcal{O}(\Omega) \)-subalgebra of \( \text{End}_{\mathcal{O}(\Omega)}(H^*(K^p, \mathcal{D}_{\Omega})_{\leq h}) \) generated by \( T(K^p) \otimes \mathbb{Q}_p \mathcal{O}(\Omega) \). Let \( \mathfrak{M} \) be the maximal ideal of \( T(K^p) \otimes \mathbb{Q}_p \mathcal{O}(\Omega) \) defined by

\[
\mathfrak{M} = (T \otimes 1 + 1 \otimes x), \quad T \in \ker \phi \quad \text{and} \quad x \in \mathfrak{m}_\lambda.
\]

After localizing the spectral sequence

\[
E_2^{i,j} = \text{Tor}_{-i}^{\mathcal{O}(\Omega)}(H^j(K^p, \mathcal{D}_{\Omega})_{\leq h}, k(\lambda)) \Rightarrow H^{i+j}(K^p, \mathcal{D}_{\Omega})_{\leq h}
\]

at \( \mathfrak{M} \), the abutment is nonzero by assumption, so the source must be nonzero as well. Therefore \( \mathfrak{M} \) determines a maximal ideal of \( T_{\Omega, h} \) lying over \( \mathfrak{m}_\lambda \), or equivalently a point \( x \in \mathcal{X}_{\Omega, h} \) with \( w(x) = \lambda \).

On the other hand, given a point \( x \in \mathcal{X}_{\Omega, h} \) with \( w(x) = \lambda \), let \( \mathfrak{M} = \mathfrak{M}_x \subset T_{\Omega, h} \) be the maximal ideal associated with \( x \), and let \( d \) be the largest degree for which \( H^j(K^p, \mathcal{D}_{\Omega})_{\leq h, \mathfrak{M}} \neq 0 \). Localizing the spectral sequence at \( \mathfrak{M} \), the entry \( E_2^{d,d} \) is nonzero and stable, so the spectral sequence induces an isomorphism

\[
0 \neq H^d(K^p, \mathcal{D}_{\Omega})_{\leq h, \mathfrak{M}} \otimes \mathcal{O}(\Omega) k(\lambda) \simeq H^d(K^p, \mathcal{D}_{\Omega})_{\leq h, \mathfrak{M}},
\]

and thus \( \mathfrak{M} \) induces a finite-slope eigenpacket in weight \( \lambda \) as desired. \( \Box \)

### 4.4 The support of overconvergent cohomology modules

As in the previous section, fix \( G, K^p \), and an augmented Borel-Serre complex \( C_\bullet(K^p, -) \). We are going to prove the following theorem.

**Theorem 4.4.1.** Fix a slope datum \((U, \Omega, h)\).

1. For any \( i \), \( H_i(K^p, \mathcal{A}_{\Omega})_{\leq h} \) is a faithful \( \mathcal{O}(\Omega) \)-module if and only if \( H^i(K^p, \mathcal{D}_{\Omega})_{\leq h} \) is faithful.
2. If the derived group of \( G \) is \( \mathbb{Q} \)-anisotropic, \( H_i(K^p, \mathcal{A}_{\Omega})_{\leq h} \) and \( H^i(K^p, \mathcal{D}_{\Omega})_{\leq h} \) are torsion \( \mathcal{O}(\Omega) \)-modules for all \( i \), unless \( G^{\text{der}}(R) \) has a discrete series, in which case they are torsion for all \( i \neq 1/2 \dim G(R)/K_\infty \).

Let \( R \) be a Noetherian ring, and let \( M \) be a finite \( R \)-module. We say \( M \) has full support if \( \text{Supp}(M) = \text{Spec}(R) \), and that \( M \) is torsion if \( \text{ann}(M) \neq 0 \). We shall repeatedly use the following basic result.

**Proposition 4.4.2.** If \( \text{Spec}(R) \) is reduced and irreducible, the following are equivalent:

1. \( M \) is faithful (i.e. \( \text{ann}(M) = 0 \)),
2. \( M \) has full support,
3. \( M \) has nonempty open support,
4. \( \text{Hom}_R(M, R) \neq 0 \),
5. \( M \otimes_R K \neq 0 \), \( K = \text{Frac}(R) \).

**Proof.** Since \( M \) is finite, \( \text{Supp}(M) \) is the underlying topological space of \( \text{Spec}(R/\text{ann}(M)) \), so i) obviously implies ii). If \( \text{Spec}(R/\text{ann}(M)) = \text{Spec}(R) \) as topological spaces, then \( \text{ann}(M) \subset \sqrt{(0)} = (0) \) since \( R \) is reduced, so ii) implies i). The set \( \text{Supp}(M) = \text{Spec}(R/\text{ann}(M)) \) is a priori closed; since \( \text{Spec}(R) \) is irreducible by assumption, the only nonempty simultaneously open and closed subset of \( \text{Spec}(R) \) is all of \( \text{Spec}(R) \), so ii) and iii) are equivalent. By finiteness, \( M \) has full support if and only if (0) is an associated prime of \( M \), if and only if there is an injection \( R \to M \); tensoring with \( K \) implies the equivalence of ii) and v). Finally, \( \text{Hom}_R(M, R) \otimes_R K \simeq \text{Hom}_K(M \otimes_R K, K) \), so \( M \otimes_R K \neq 0 \) if and only if \( \text{Hom}_R(M, R) \neq 0 \), whence iv) and v) are equivalent. \( \Box \)
Proof of Theorem 4.4.1.i. (I’m grateful to Jack Thorne for suggesting this proof.) Tensoring the Ext spectral sequence with $K(\Omega) = \text{Frac}(\mathcal{O}(\Omega))$, it degenerates to isomorphisms

$$\text{Hom}_{K(\Omega)}(H_i(K^p, \mathcal{A})_{\leq h} \otimes_{\mathcal{O}(\Omega)} K(\Omega), K(\Omega)) \simeq H^i(K^p, \mathcal{D}_{\Omega})_{\leq h} \otimes_{\mathcal{O}(\Omega)} K(\Omega),$$

so the claim is immediate from the preceding proposition. $\square$

Proof of Theorem 4.4.1.ii. We give the proof in two steps, with the first step naturally breaking into two cases. In the first step, we prove the result assuming $\Omega$ contains an arithmetic weight. In the second step, we eliminate this assumption via analytic continuation.

**Step One, Case One: G doesn’t have a discrete series.** Let $\mathcal{W}^{\text{ad}}$ be the rigid Zariski closure in $\mathcal{W}_{K^p}$ of the arithmetic weights whose algebraic parts are the highest weights of irreducible $G$-representations with nonvanishing $\mathcal{O}$-cohomology. A simple calculation using §II.6 of [BW00] shows that $\mathcal{W}^{\text{ad}}$ is the union of its countable set of irreducible components, each of dimension $< \text{dim} \mathcal{W}_{K^p}$. An arithmetic weight is non-self-dual if $\lambda \notin \mathcal{W}^{\text{ad}}$.

Now, by assumption $\Omega$ contains an arithmetic weight, so $\Omega$ automatically contains a Zariski dense set $\mathcal{N}_h \subset \Omega \times \Omega \cap \mathcal{W}^{\text{ad}}$ of non-self-dual arithmetic weights for which $h$ is a small slope. By Theorem 3.2.5, $H^*(K^p, \mathcal{D}_h)_{\leq h}$ vanishes identically for any $\lambda \in \mathcal{N}_h$. For any fixed $\lambda \in \mathcal{N}_h$, suppose $m_\lambda \in \text{Supp}_\Omega H^*(K^p, \mathcal{D}_h)_{\leq h}$; let $d$ be the largest integer with $m_\lambda \in \text{Supp}_\Omega H^d(K^p, \mathcal{D}_h)$. Taking $\Sigma = \lambda$ in the Tor spectral sequence gives

$$E_{ij}^2 = \text{Tor}_{i-1}^{\mathcal{O}(\Omega)}(H^j(K^p, \mathcal{D}_h)_{\leq h}, k(\lambda)) \Rightarrow H^{i+j}(K^p, \mathcal{D}_h)_{\leq h}.$$ 

The entry $E_{i,j}^{0,d} = H^d(K^p, \mathcal{D}_h)_{\leq h} \otimes_{\mathcal{O}(\Omega)} k(\lambda)$ is nonzero by Nakayama’s lemma, and is stable since every row of the $E_2$-page above the $d$th row vanishes by assumption. In particular, $E_{i,j}^{0,d}$ contributes a nonzero summand to the grading on $H^d(K^p, \mathcal{D}_h)_{\leq h}$ - but this module is zero, contradicting our assumption that $m_\lambda \in \text{Supp}_\Omega H^*(K^p, \mathcal{D}_h)$. Therefore, $H^*(K^p, \mathcal{D}_h)_{\leq h}$ does not have full support, so is not a faithful $\mathcal{O}(\Omega)$-module.

**Step One, Case Two: G has a discrete series.** The idea is the same as Case One, but with $\mathcal{N}_h$ replaced by $\mathcal{R}_h$, the set of arithmetic weights with regular algebraic part for which $h$ is a small slope. For these weights, Proposition 3.2.5 together with known results on $(g, K_\infty)$-cohomology (see e.g. Sections 4-5 of [LS04]) implies that $H^i(K^p, \mathcal{D}_h)_{\leq h}$, $\lambda \in \mathcal{R}_h$ vanishes for $i \neq d_G = \frac{1}{2}\text{lim}_{\rightarrow}(\mathbb{R})/K_\infty$. The Tor spectral sequence with $\Sigma = \lambda \in \mathcal{R}_h$ then shows that $\mathcal{R}_h$ doesn’t meet $\text{Supp}_\Omega H^i(K^p, \mathcal{D}_h)_{\leq h}$ for any $i > d_G$. On the other hand, the Ext spectral sequence with $\Sigma = \lambda \in \mathcal{R}_h$ then shows that $\mathcal{R}_h$ doesn’t meet $\text{Supp}_\Omega H^i(K^p, \mathcal{D}_h)_{\leq h}$ for any $i < d_G$, whence the Ext spectral sequence with $\Sigma = \Omega$ shows that $\mathcal{R}_h$ doesn’t meet $\text{Supp}_\Omega H^i(K^p, \mathcal{D}_h)_{\leq h}$ for any $i < d_G$. The result follows.

**Step Two.** We maintain the notation of §4.3. As in that subsection, $H^n(K^p, \mathcal{D}_h)_{\leq h}$ glues together over the slope-adapted affinoids $\mathcal{Z}_{\Omega, h} \subseteq \mathcal{Z}$ into a coherent $\mathcal{O}_\mathcal{Z}$-module sheaf $\mathcal{M}^n$, and in particular, the support of $\mathcal{M}^n$ is a closed analytic subset of $\mathcal{Z}$. Let $\pi : \mathcal{Z} \rightarrow \mathcal{W}$ denote the natural projection. For any $\mathcal{Z}_{\Omega, h} \in \mathcal{C}_{\text{ov}}$, we have

$$w_! \text{Supp}_{\mathcal{Z}_{\Omega, h}} \mathcal{M}^n(\mathcal{Z}_{\Omega, h}) = \text{Supp}_\Omega H^n(K^p, \mathcal{D}_h)_{\leq h}.$$ 

Suppose $\text{Supp}_\Omega H^n(K^p, \mathcal{D}_h)_{\leq h} = \Omega$ for some $\mathcal{Z}_{\Omega, h} \in \mathcal{C}_{\text{ov}}$. This implies that $\text{Supp}_{\mathcal{Z}_{\Omega, h}} \mathcal{M}^n(\mathcal{Z}_{\Omega, h})$ contains a closed subset of dimension equal to $\text{dim} \mathcal{Z}$, so contains an irreducible component of $\mathcal{Z}_{\Omega, h}$.
Since $\text{Supp}_\mathcal{X} \mathcal{M}^n$ is a priori closed, Corollary 2.2.6 of \cite{Con99} implies that $\text{Supp}_\mathcal{X} \mathcal{M}^n$ contains an entire irreducible component of $\mathcal{X}$, say $\mathcal{Z}_0$. By Proposition 4.1.3, the image of $\mathcal{Z}_0$ is Zariski-open in $\mathcal{Y}$, so we may choose an arithmetic weight $\lambda_0 \in w_* \mathcal{Z}_0$. For some sufficiently large $h_0$ and some affinoid $\Omega_0$ containing $\lambda_0$, $\mathcal{Z}_{\Omega_0,h_0}$ will contain $\mathcal{Z}_{\Omega_0,h_0} \cap \mathcal{Z}_0$ as a nonempty union of irreducible components, and the latter intersection will be finite flat over $\Omega_0$. Since $\mathcal{M}^n(\mathcal{Z}_{\Omega_0,h_0}) \simeq H^n(K^p, \mathcal{D}_{\Omega_0}) < h_0$, we deduce that $\text{Supp}_{\mathcal{Z}_{\Omega_0,h_0}} H^n(K^p, \mathcal{D}_{\Omega_0}) < h_0 = \Omega_0$, whence $H^n(K^p, \mathcal{D}_{\Omega_0}) < h_0$ is faithful, so by Step One $G^{\text{def}}(\mathcal{R})$ has a discrete series and $n = \frac{1}{2} \dim G(\mathcal{R}) / K_{\infty}$.

4.5 Eigenvarieties at noncritical interior points.

In this section we prove the following result; part i. of this theorem is a generalization of "Coleman families".

**Theorem 4.5.1.** Let $x = x(\phi) \in G_{G, K^p}(\mathbb{Q}_p)$ be a point associated with a classical, noncritical, interior eigenpacket $\phi$ such that $w(x)$ has regular algebraic part.

i. If $l(G) = 0$, every irreducible component of $\mathcal{X}$ containing $x$ has dimension equal to $\dim \mathcal{Y}_{K^p}$.

ii. If $l(G) \geq 1$ and $\phi$ is strongly interior, then every irreducible component of $\mathcal{X}_{G, K^p}$ containing $x$ has dimension $\leq \dim \mathcal{Y}_{K^p} - 1$, with equality if $l(G) = 1$.

**Proof.** By the basic properties of irreducible components together with the construction given in §4.2-4.3, it suffices to work locally over a fixed $\mathcal{Z}_{\Omega, h}$. Suppose $x \in \mathcal{Z}_{\Omega, h}$ is as in the theorem, with $\phi : \mathcal{T}_{\Omega, h} \to \mathbb{Q}_p$ the corresponding eigenpacket. Set $\mathfrak{M} = \ker \phi$, and let $\mathfrak{m} = \mathfrak{m}_\lambda$ be the contraction of $\mathfrak{M}$ to $\mathcal{O}(\Omega)$. Let $\mathcal{Y} \subset \mathcal{T}_{\Omega, h}$ be any minimal prime contained in $\mathfrak{M}$, and let $\mathfrak{p}$ be its contraction to a prime in $\mathcal{O}(\Omega)$. The ring $\mathcal{T}_{\Omega, h} / \mathfrak{p}$ is a finite integral extension of $\mathcal{O}(\Omega) / \mathfrak{p}$, so both rings have the same dimension. Note also that $\mathfrak{p}$ is an associated prime of $H^s(K^p, \mathcal{D}_{\Omega}) < h$.

**Proposition 4.5.2.** The largest degrees for which $\phi$ occurs in $H^s(K^p, \mathcal{D}_{\Omega}) < h$ and $H^s(K^p, \mathcal{D}_{\lambda}) < h$ coincide, and the smallest degrees for which $\phi$ occurs in $H^s(K^p, \mathcal{D}_{\lambda}) < h$ and $H^j(K^p, \mathcal{D}_{\lambda}) < h$ coincide. Finally, the smallest degree for which $\phi$ occurs in $H^s(K^p, \mathcal{D}_{\lambda}) < h$ is greater than or equal to the smallest degree for which $\phi$ occurs in $H^s(K^p, \mathcal{D}_{\lambda}) < h$.

**Proof.** For the first claim, localize the Tor spectral sequence at $\mathfrak{M}$, with $\Sigma = \lambda$. If $\phi$ occurs in $H^i(K^p, \mathcal{D}_{\lambda}) < h$ then it occurs in a subquotient of $\text{Tor}_j^{\mathcal{O}(\Omega)}(H^{i+j}(K^p, \mathcal{D}_{\Omega}) < h, k(\lambda))$ for some $j \geq 0$. On the other hand, if $d$ is the largest degree for which $\phi$ occurs in $H^d(K^p, \mathcal{D}_{\Omega}) < h$, the entry $E_2^{0,d}$ of the Tor spectral sequence is stable and nonzero after localizing at $\mathfrak{M}$, and it contributes to the grading on $H^d(K^p, \mathcal{D}_{\lambda}) < h, \mathfrak{M}$. The second and third claims follow from an analogous treatment of the Ext spectral sequence. □

First we treat the case where $l(G) = 0$, so $G^{\text{def}}(\mathcal{R})$ has a discrete series. By the noncriticality of $\phi$ together with the results recalled in §3.2, the only degree for which $\phi$ occurs in $H^j(K^p, \mathcal{D}_{\lambda}) < h$ is the middle degree $d = \frac{1}{2} \dim G(\mathcal{R}) / K_{\infty}$, so Proposition 4.5.2 implies that the only degree for which $\phi$ occurs in $H^s(K^p, \mathcal{D}_{\Omega}) < h$ is the middle degree as well. The Tor spectral sequence localized at $\mathfrak{M}$ now degenerates, and yields

$$\text{Tor}_i^{\mathcal{O}(\Omega)}(H^d(K^p, \mathcal{D}_{\Omega}) < h, \mathfrak{M}, k(\lambda)) = 0$$

for all $i \geq 1$.

so $H^d(K^p, \mathcal{D}_{\Omega}) < h, \mathfrak{M}$ is a free module over $\mathcal{O}(\Omega) / \mathfrak{m}$ by Proposition A.3. Since $\mathcal{O}(\Omega) / \mathfrak{m}$ is a domain and $\mathfrak{p}$ is (locally at $\mathfrak{m}$) an associated prime of a free module, $\mathfrak{p} = 0$ and thus $\dim \mathcal{T}_{\Omega, h} / \mathfrak{p} = \dim \mathcal{O}(\Omega) / \mathfrak{m} = \dim \mathcal{Y}$. 

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Now we turn to the case $l(G) \geq 1$. First we demonstrate the existence of an affinoid open \( \mathcal{V} \subset \mathcal{Z}_{\Omega,h} \) containing \( x \), and meeting every component of \( \mathcal{Z}_{\Omega,h} \) containing \( x \), such that every regular classical non-critical point in \( \mathcal{V} \) is cuspidal. By our assumptions, \( \phi \) does not occur in \( H^*_{\partial}(K^0, \mathcal{D}_\lambda)_{\leq h} \), so by the boundary spectral sequence \( \phi \) does not occur in \( H^*_{\partial}(K^0, \mathcal{D}_\lambda)_{\leq h} \). Since \( \text{Supp}_{\Omega,h} H^*_{\partial}(K^0, \mathcal{D}_\lambda)_{\leq h} \) is closed in \( \mathcal{Z}_{\Omega,h} \) and does not meet \( x \), the existence of a suitable \( \mathcal{V} \) now follows easily. Shrinking \( \Omega \) and \( \mathcal{V} \) as necessary, we may assume that \( \mathcal{O}(\mathcal{V}) \) is finite over \( \mathcal{O}(\Omega) \), and thus \( \mathcal{M}^*(\mathcal{V}) = H^*_{\partial}(K^0, \mathcal{D}_\lambda)_{\leq h} \otimes_{\mathcal{O}_{\Omega,h}} \mathcal{O}(\mathcal{V}) \) is finite over \( \mathcal{O}(\Omega) \) as well. Exactly as in the proof of Theorem 4.4.1, the Tor spectral sequence shows that \( \text{Supp}_{\Omega,h} H^*_{\partial}(K^0, \mathcal{D}_\lambda)_{\leq h} \). Taking \( \Sigma = \lambda \) in the Ext spectral sequence and localizing at \( \mathfrak{M} \), Proposition 4.5.2 yields

\[
H^d(K^0, \mathcal{D}_\lambda)_{\leq h, \mathfrak{M}} \simeq \text{Hom}_{\mathcal{O}(\Omega)_\mathfrak{M}}(H^d_{\partial}(K^0, \mathcal{D}_\lambda)_{\leq h, \mathfrak{M}}, \mathcal{O}(\Omega)_\mathfrak{M}).
\]

Since the left-hand term is a torsion \( \mathcal{O}(\Omega)_\mathfrak{M} \)-module, Proposition 4.4.2 implies that both modules vanish identically. Proposition 4.5.2 now shows that \( d + 1 \) is the only degree for which \( \phi \) occurs in \( H^*_{\partial}(K^0, \mathcal{D}_\lambda)_{\leq h} \). Taking \( \Sigma = \lambda \) in the Tor spectral sequence and localizing at \( \mathfrak{M} \), the only nonvanishing entries are \( E_{2d+1}^{i,d+1} \) and \( E_{2d+1}^{d+1,i} \). In particular, \( \text{Tor}_i^{\mathcal{O}(\Omega)}(H^{d+1}_{\partial}(K^0, \mathcal{D}_\lambda)_{\leq h, \mathfrak{M}}, \mathcal{O}(\Omega)/\mathfrak{M}) = 0 \) for all \( i \geq 2 \), so \( H^{d+1}_{\partial}(K^0, \mathcal{D}_\lambda)_{\leq h, \mathfrak{M}} \) has projective dimension at most one by Proposition A.3. Summarizing, we’ve shown that \( H^i(K^0, \mathcal{D}_\lambda)_{\leq h, \mathfrak{M}} \) vanishes in degrees \( \neq q(G) + 1 \), and that \( H^{q(G)+1}_{\partial}(K^0, \mathcal{D}_\lambda)_{\leq h, \mathfrak{M}} \) is a torsion \( \mathcal{O}(\Omega)_\mathfrak{M} \)-module of projective dimension one, so \( \text{ht} \mathfrak{p} = 1 \) by Proposition A.6.

We remark here that results towards Newton’s Theorem 1.1.6 were established by Stevens and Urban in unpublished work; however, in the notation of the present subsection, their proofs required an a priori assumption that the ideal \( \mathfrak{p} \) is generated by a regular sequence locally in \( \mathcal{O}(\Omega)_\mathfrak{M} \), which seems quite hard to check.

### 4.6 General linear groups

In this section we examine the special case when \( G \simeq \text{Res}_{F/ \mathbb{Q}} \mathbb{H} \) for some number field \( F \) and some \( F \)-inner form \( \mathbb{H} \) of \( \text{GL}_n/F \), introducing notation which will remain in effect throughout Chapters 5 and 6. In particular, we often work with a canonical family of tame level subgroups suggested by the theory of new vectors. More precisely, given an integral ideal \( n = \prod \mathfrak{p}^{e_v(n)} \subset \mathcal{O}_F \) with \( e_v = 0 \) if \( v \mid p \) or if \( H(F_v) \not\simeq \text{GL}_n(F_v) \), set

\[
K_1(n) = \prod_{v \text{ with } e_v(n) > 0} K_v(\varpi_v^{e_v(n)}) \prod_{v \text{ with } e_v(n) = 0} K_v
\]

where \( K_v(\varpi_v^{e_v}) \) denotes the open compact subgroup of \( \text{GL}_n(\mathcal{O}_v) \) consisting of matrices with lowest row congruent to \((0, \ldots, 0, 1) \mod \varpi_v^{e_v} \) and \( K_v \) denotes a fixed maximal compact subgroup of \( H(F_v) \).

The Hecke algebra \( \mathbf{T}(K_1(n)) \) then contains the usual operators \( T_{v,i} \) corresponding to the double cosets of the matrices \( \text{diag}(\varpi_{v,1}, \ldots, \varpi_{v,i}, 1, \ldots, 1) \) for \( 1 \leq i \leq n \) and \( v \) a place of \( F \) such that \( e_v(n) = 0 \) and \( H(F_v) \simeq \text{GL}_n(F_v) \). For a place \( v \mid p \) we write \( U_{v,i} \) for the element of \( \mathcal{A}_p^+ \) corresponding
to the double coset of \( \text{diag} \left( 1, \ldots, 1, \varpi_1, \ldots, \varpi_v \right) \), and we set \( u_{v,i} = U_{v,1}^{-1} U_{v,i} \in \mathcal{A}_p \). The operator \( U_p = \prod_{v \mid p} \prod_{i=1}^{n-1} U_{v,i} \) is a canonical controlling operator. If \( n = 2 \) we adopt the more classical notation, writing \( T_v = T_{v,1} \) and \( S_v = T_{v,2} \). We write \( T_\lambda(n) \) for the finite-slope Hecke algebra of weight \( \lambda \) and tame level \( K_1(n) \) as in §3.2, and we abbreviate \( \mathcal{I}_{\text{Res}_F/\text{GL}_n, K_1(n)} \) by \( \mathcal{I}_{\text{GL}_n/F, n} \).

If \( F = \mathbb{Q} \), we define \( \mathcal{W}^0_{K_F} \) as the subspace of \( \mathcal{W}_{K_F} \) parametrizing characters trivial on the one-parameter subgroup \( \text{diag}(1, \ldots, 1, t_n) \), and we set \( \mathcal{I}^0_{G,K_F} = \mathcal{I}_{G,K_F} \cap w^{-1}(\mathcal{W}^0_{K_F}) \). By the remarks in §2.2, \( \mathcal{I}^0_{G,K_F} \) is a disc bundle over \( \mathcal{I}_{G,K_F} \): for any point \( x \in \mathcal{I}_{G,K_F} \) with \( w(x) = (\lambda_1, \ldots, \lambda_n) \in \mathcal{W}_{K_F} \), there is a unique point \( x^0 \in \mathcal{I}^0_{G,K_F} \) with \( \lambda^0 = (\lambda_1 \lambda_n^{-1}, \ldots, \lambda_{n-1} \lambda^{-1}) \) such that \( \phi(x)(T_{\ell,i}) = \lambda_n(t) \phi(x^0)(T_{\ell,i}) \). Restricting attention to \( \mathcal{I}^0_{G,K_F} \) amounts to factoring out “wild twists”, and \( \mathcal{I}^0_{\text{GL}_2/\mathbb{Q}, K_F} \) is canonically isomorphic to the Coleman-Mazur-Buzzard eigencurve of tame level \( K_F \) (this is a theorem of Bellaïche [Bel12]).

5 \( p \)-adic Langlands functoriality

5.1 An interpolation theorem

**Definition 5.1.1.** Given an eigenvariety datum \( \mathcal{D} = (\mathcal{W}, \mathcal{U}, \mathcal{M}, T, \psi) \) with associated eigenvariety \( \mathcal{E} \), the core of \( \mathcal{E} \), denoted \( \mathcal{E}^\circ \), is the union of the dim \( \mathcal{W} \)-dimensional irreducible components of the reduction \( \mathcal{E}^\text{red} \), regarded as a closed subspace of \( \mathcal{E} \). An eigenvariety \( \mathcal{E} \) is unmixed if \( \mathcal{E}^\circ \simeq \mathcal{E} \).

Let \( \mathcal{Z}^\circ \) denote the subspace of points in \( \mathcal{Z} \) whose preimage in \( \mathcal{E} \) meets the core of \( \mathcal{E} \), with its reduced rigid subspace structure; \( \mathcal{Z}^\circ \) is naturally a union of irreducible components of \( \mathcal{E}^\text{red} \). We will see below that \( \mathcal{Z}^\circ \) really is an eigenvariety, in the sense of being associated with an eigenvariety datum.

Suppose we are given two eigenvariety data \( \mathcal{D}_i = (\mathcal{W}_i, \mathcal{Z}_i, \mathcal{M}_i, T, \psi_i) \) for \( i = 1, 2 \), together with a closed immersion \( j : \mathcal{W}_1 \hookrightarrow \mathcal{W}_2 \); we write \( j \) for the natural extension of \( j \) to a closed immersion \( j \times \text{id} : \mathcal{W}_1 \times \mathbb{A}^1 \hookrightarrow \mathcal{W}_2 \times \mathbb{A}^1 \). Given a point \( z \in \mathcal{Z}_1 \) and any \( T \in \mathcal{T} \), we write \( D_t(T, X)(z) \in k(z)[X] \) for the characteristic polynomial \( (1 - \psi(T)(X)) \cdot \mathcal{M}_1(z) \).

**Theorem 5.1.2.** Notation and assumptions as in the previous paragraph, suppose there is some very Zariski-dense set \( \mathcal{E}_1 \subset \mathcal{E}_1^\circ \) with \( j(\mathcal{E}_1) \subset \mathcal{E}_2 \) such that the polynomial \( D_t(T, X)(z) \) divides \( D_2(T, X)(j(z)) \) in \( k(z)[X] \) for all \( T \in \mathcal{T} \) and all \( z \in \mathcal{E}_1^\circ \). Then \( j \) induces a closed immersion \( \zeta : \mathcal{E}_1 \hookrightarrow \mathcal{E}_2 \), and there is a canonical closed immersion \( i : \mathcal{E}_1^\circ \hookrightarrow \mathcal{E}_2 \) such that the diagram

\[
\begin{array}{ccc}
\mathcal{E}_1^\circ & \xhookrightarrow{i} & \mathcal{E}_2 \\
\mathcal{W}_1 \downarrow \; \; \downarrow j & & \; \; \downarrow w_2 \\
\mathcal{W}_1 & \xhookrightarrow{w_1} & \mathcal{W}_2
\end{array}
\]
and

\[ \begin{array}{ccc}
T & \xrightarrow{\phi_1} & \mathcal{O}(X_1) \\
\downarrow \phi_2 & & \downarrow i^* \\
\mathcal{O}(X_2) & & 
\end{array} \]

commute.

Before proving this result, we establish three lemmas.

**Lemma 5.1.3.** Suppose \( A \) is an affinoid algebra, \( B \) is a module-finite \( A \)-algebra, and \( S \) is a Zariski-dense subset of \( \text{Max}A \). Then

\[ I = \bigcap_{m \in \text{Max}B \text{ with } m \text{ lying over } n} m \subset B \]

is contained in every minimal prime \( p \) of \( B \) with \( \dim B/p = \dim A \).

*Proof.* Translated into geometric language, this is the self-evident statement that the preimage of \( S \) under \( \pi : \text{Sp}B \to \text{Sp}A \) is Zariski-dense in any irreducible component of \( \text{Sp}B \) with \( \dim A \)-dimensional image in \( \text{Sp}A \). □

**Lemma 5.1.4.** Suppose \( A \to B \) are affinoid algebra, with \( \text{Max}B \) an affinoid subdomain of \( \text{Max}A \). Let \( A^o \) be the maximal reduced quotient of \( A \) which is equidimensional of dimension \( \dim A \). Then \( A^o \otimes_A B \) is the maximal reduced quotient of \( B \) which is equidimensional of dimension \( \dim A \) if \( \dim B = \dim A \), and is zero if \( \dim B < \dim A \).

*Proof.* Set \( d = \dim A \). The kernel of \( A \to A^o \) is the ideal \( I^o = \cap_{p \in \text{Spec}A, \text{coht}p = d} p \), so tensoring the sequence

\[ 0 \to I^o \to A \to A^o \to 0 \]

with \( B \) yields

\[ 0 \to I^o \otimes_A B \to B \to A^o \otimes_A B \to 0 \]

by the \( A \)-flatness of \( B \). It thus suffices to prove an isomorphism

\[ I^o \otimes_A B \simeq \bigcap_{p \in \text{Spec}B, \text{coht}p = d} p. \]

By the Jacobson property of affinoid algebras, we can rewrite \( I^o \) as follows:

\[ I^o = \bigcap_{\text{coht}p = d} p \]

\[ = \bigcap_{m \in \text{Max}(A) \text{ with } m \supset p \text{ and } \text{coht}p = d} m \]

\[ = \bigcap_{m \in \text{Max}(A) \text{ with } \text{ht}m = d} m. \]
Since $B$ is $A$-flat, we have $(I_1 \cap I_2) \otimes_A B = I_1B \cap I_2B$ for any ideals $I_i \subset A$, so

$$I^\circ \otimes_A B = \bigcap_{m \in \text{Max}(A) \text{ with } ht m = d} mB = \bigcap_{m \in \text{Max}(B) \text{ with } ht m = d} m = \bigcap_{p \in \text{Spec} B \cup \{0\}} p,$$

and we're done if $\dim B = \dim A$. But if $\dim B < \dim A$, then $mB = B$ for all $m \in \text{Max}(A)$ of height $d$, so $I^\circ \otimes_A B = B$ as desired. □

**Lemma 5.1.5.** Let $T$ be a commutative $\mathbb{Q}_p$-algebra, $L/\mathbb{Q}_p$ a finite extension, and $M_1, M_2$ a pair of $T \otimes_{\mathbb{Q}_p} L$-modules finite over $L$. For any $t \in T$ set $D_i(X,t) = \det(1 - Xt)|_{M_i} \in L[X]$. If $D_1(X,t)\mid D_2(X,t)$ in $L[X]$ for any $t \in T$, then the same divisibility holds for any $t \in T \otimes_{\mathbb{Q}_p} L$.

Proof. Given an arbitrary element $t = \sum_{i=1}^{n} t_i \otimes a_i \in T \otimes_{\mathbb{Q}_p} L$ with $t_i \in T$ and $a_i \in L$, we set

$$P_i(X, X_1, \ldots, X_n) = \det(1 - X(X_1t_1 + X_2t_2 + \cdots + X_nt_n))|_{M_i} \in L[X, X_1, \ldots, X_n],$$

so in particular $P_i(X, a_1, \ldots, a_n) = D_i(X, t)$. Consider the meromorphic quotient

$$Q(X, X_1, \ldots, X_n) = \frac{P_2(X, X_1, \ldots, X_n)}{P_1(X, X_1, \ldots, X_n)}.$$

Let $U \subset \text{Spec} L[X_1, \ldots, X_n][a^n]$ be a sufficiently small connected affinoid open neighborhood of the origin and let $C \gg 0$ be an integer such that on the domain $\{|X| \leq p^{-C}\} \times U$, $Q$ admits a convergent power series expansion

$$Q(X, X_1, \ldots, X_n) = \sum_{j=0}^{\infty} f_j(X_1, \ldots, X_n)X^j \in \mathcal{O}(U) \langle p^{-C}X \rangle.$$

By hypothesis, for any $(b_1, \ldots, b_n) \in \mathbb{Q}_p^n$ the specialization $Q(X, b_1, \ldots, b_n)$ is an element of $L[X]$, necessarily of degree at most $d = \dim L M_2$. In particular, for any $j > d$ the function $f_j$ vanishes on the Zariski-dense set $U \cap \mathbb{Q}_p^n$ and therefore vanishes identically. We now have the identity $Q(X, X_1, \ldots, X_n) = \sum_{0 \leq j \leq d} f_j(X_1, \ldots, X_n)X^j \in \mathcal{O}(U)[X]$. For some $N \gg 0$ we have $(p^N a_1, \ldots, p^N a_n) \in U$; specializing $Q$ at this point and making the change of variables $X \mapsto p^{-N}X$ yields

$$\frac{D_2(X,t)}{D_1(X,t)} = Q(X, a_1, \ldots, a_n) = Q(p^{-N}X, p^N a_1, \ldots, p^N a_n) = \sum_{0 \leq j \leq d} f_j(p^N a_1, \ldots, p^N a_n)p^{-Nj} X^j \in L[X].$$
Proof of Theorem 5.1.2. Set $d = \dim \mathcal{M}_1$. First, we establish the theorem in the special case when $\mathcal{M}_1 = \mathcal{M}_2$, $j = \text{id}$, $\mathfrak{Z}_1 \simeq \mathfrak{Z}_2$, and $\mathfrak{Z}_1^{\circ} = \mathfrak{Z}_1$; we refer to this as the narrow case. For brevity we write $\mathcal{W} = \mathcal{W}_1$ and $\mathfrak{Z} = \mathfrak{Z}_1$. As in §4.2, let $\mathcal{Cov} = \{ \Omega_i \}_{i \in I}$ be an admissible affinoid covering of $\mathfrak{Z}$. For any $\Omega \in \mathcal{Cov}$ and $i \in \{ 1, 2 \}$, let $\mathcal{T}_{\Omega, i}$ denote the $\mathcal{O}^\prime(\Omega)$-subalgebra of $\text{End}_{\mathcal{O}^\prime(\Omega)}(\mathcal{M}_i(\Omega))$ generated by the image of $\psi_i$, and let $I_{\Omega, i}$ be the kernel of the natural surjection

$$\phi_{\Omega, i} : \mathcal{T} \otimes_{\mathcal{Q}_p} \mathcal{O}^\prime(\Omega) \to \mathcal{T}_{\Omega, i}.$$ 

We are going to establish an inclusion $I_{\Omega, 2} \subseteq I_{\Omega, 1}$ for all $\Omega \in \mathcal{Cov}$. Granting this inclusion, let $\mathfrak{A}_\Omega \subseteq \mathcal{T}_{\Omega, 2}$ be the kernel of the induced surjection $\mathcal{T}_{\Omega, 2} \to \mathcal{T}_{\Omega, 1}$. If $\Omega' \subset \Omega$ is an affinoid subdomain, then applying $- \otimes_{\mathcal{Q}_p} \mathcal{O}^\prime(\Omega')$ to the sequence

$$0 \to I_{\Omega, i} \to \mathcal{T} \otimes_{\mathcal{Q}_p} \mathcal{O}^\prime(\Omega) \xrightarrow{\phi_{\Omega, i}} \mathcal{T}_{\Omega, i} \to 0$$

yields a canonical isomorphism $I_{\Omega, i} \otimes_{\mathcal{Q}_p} \mathcal{O}^\prime(\Omega') \cong I_{\Omega', i}$, so applying $- \otimes_{\mathcal{Q}_p} \mathcal{O}^\prime(\Omega')$ to the canonical isomorphism $\mathfrak{A}_\Omega \cong I_{\Omega, 2}/I_{\Omega, 1}$ yields an isomorphism $\mathfrak{A}_\Omega \otimes_{\mathcal{Q}_p} \mathcal{O}^\prime(\Omega') \cong \mathfrak{A}_{\Omega'}$. Therefore the assignments $\Omega \mapsto \mathfrak{A}_\Omega$ glue together into a coherent ideal subsheaf of the structure sheaf of $\mathfrak{Z}_2$. Cutting out $\mathfrak{A}_\Omega$ equivalently, the surjections $\mathcal{T}_{\Omega, 2} \to \mathcal{T}_{\Omega, 1}$ glue together over $\Omega \in \mathcal{Cov}$ into the desired closed immersion.

It remains to establish the inclusion $I_{\Omega, 2} \subseteq I_{\Omega, 1}$. Let $\mathfrak{Z}^{\text{reg}}$ be the maximal subset of $\mathfrak{Z}$ such that $\mathcal{O}_{\mathfrak{Z}, z}$ is regular for all $z \in \mathfrak{Z}^{\text{reg}}$ and the sheaves $\mathcal{M}_1$ and $\mathcal{M}_2$ are locally free after restriction to $\mathfrak{Z}^{\text{reg}}$, since $\mathfrak{Z}^{\text{reg}}$ is naturally the intersection

$$\text{Reg}(\mathfrak{Z}) \bigcap \left( \mathfrak{Z} \setminus \text{Supp} \oplus_{i=1}^d \text{Ext}_i^1(\mathcal{M}_1 \oplus \mathcal{M}_2, \mathcal{O}_\mathfrak{Z}) \right),$$

and $\text{Reg}(\mathfrak{Z})$ is Zariski-open by the excellence of affinoid algebras, $\mathfrak{Z}^{\text{reg}}$ is naturally a Zariski-open and Zariski-dense rigid subspace of $\mathfrak{Z}$. For any $T \in \mathcal{T}$, let $D_i(T, X) \in \mathcal{O}(\mathfrak{Z}^{\text{reg}})[X]$ be the determinant of $1 - \psi_i(T)X$ acting on $\mathcal{M}_i|_{\mathfrak{Z}^{\text{reg}}}$, defined in the usual way (this is why we need local freeness). For any $z \in \mathfrak{Z}^{\text{reg}}$, the image of $D_i(T, X)$ in the residue ring $k(z)[X]$ is simply $D_i(T, X)(z)$. By our hypotheses, the formal quotient

$$Q(T, X) = D_2(T, X)/D_1(T, X) = \sum_{n \geq 0} a_n X^n \in \mathcal{O}(\mathfrak{Z}^{\text{reg}})[[X]]$$

reduces for any $z \in \mathfrak{Z}^{\text{reg}} \cap \mathfrak{Z}^{\text{cl}}$ to an element of $k(z)[X]$ with degree bounded uniformly above as a function of $z$ on any given irreducible component of $\mathfrak{Z}^{\text{reg}}$. In particular, the restriction of $a_n$ to any given irreducible component of $\mathfrak{Z}^{\text{reg}}$ is contained in a Zariski-dense set of maximal ideals for sufficiently large $n$, and so is zero. Thus $D_1(T, X)(z)$ divides $D_2(T, X)(z)$ in $k(z)[X]$ for any $z \in \mathfrak{Z}^{\text{reg}}$ and any $T \in \mathcal{T}$. This extends by Lemma 5.1.5 to the same divisibility for any $T \in \mathcal{T} \otimes_{\mathcal{Q}_p} k(z)$.

Suppose now that $T \in \mathcal{T} \otimes_{\mathcal{Q}_p} \mathcal{O}(\Omega)$ is contained in $I_{\Omega, 2}$. Since $D_2(T, X)(z) = 1$ for any $z \in \Omega \cap \mathfrak{Z}^{\text{reg}}$, the deduction in the previous paragraph shows that $D_1(T, X)(z) = 1$ for any $z \in \Omega \cap \mathfrak{Z}^{\text{reg}}$. But then

$$\phi_{1}(T) \in \bigcap_{x \in \text{Sp} \mathcal{T}_{\Omega, 1}} \mathfrak{m}_x \subseteq \bigcap_{p \in \text{Sp} \mathcal{T}_{\Omega, 1}, \text{coht } p = d} p = 0,$$
where the middle inclusion follows from Lemma 5.1.3 and the rightmost equality follows since $T_{\Omega,1}$ is reduced and equidimensional of dimension $d$ by assumption. This establishes the narrow case.

It remains to establish the general case. By the hypotheses of the theorem, $j(\mathcal{Z}^c) \in j(\mathcal{Z}_1^c) \cap \mathcal{Z}_2$, so $j$ induces the closed immersion $\zeta : \mathcal{Z}_1^c \hookrightarrow \mathcal{Z}_2$ by the Zariski-density of $\mathcal{Z}^c$ in $\mathcal{Z}_1^c$. Let $\mathcal{Z}_2$ denote the fiber product $\mathcal{Z}_2 \times_{\mathcal{Z}_1} \mathcal{Z}_2$, note that $\mathcal{Z}_2^c$ is the eigenvariety associated with the eigenvariety datum $\mathcal{D}_2 = (\mathcal{W}_1, \mathcal{Z}_1^c, \xi^c, \mathcal{M}_2, \mathbf{T}, \xi^c \psi_2)$, and there is a canonical closed immersion $i' : \mathcal{Z}_2^c \hookrightarrow \mathcal{Z}_2$ by construction. For $\Omega \subseteq \mathcal{Z}_1$ an affinoid open we define an ideal $\mathcal{I}(\Omega) \subseteq T_{\Omega,1}$ by the rule

$$\mathcal{I}(\Omega) = \begin{cases} T_{\Omega,1} & \text{if } \dim T_{\Omega,1} < d \\ \bigcap_{x \in \text{Sp} T_{\Omega,1}, \text{ht} m_x = d} m_x & \text{if } \dim T_{\Omega,1} = d. \end{cases}$$

By Lemma 5.1.4 the ideals $\mathcal{I}(\Omega)$ glue into a coherent ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_{\mathcal{Z}_1}$. The support of $\mathcal{O}_{\mathcal{Z}_1}/\mathcal{I}$ in $\mathcal{Z}_1$ is exactly $\mathcal{Z}_1^c$, and in fact the closed immersion cut out by $\mathcal{I}$ is exactly the core of $\mathcal{Z}_1$. In particular, the core of $\mathcal{Z}_1$ is the eigenvariety associated with the (somewhat tautological) eigenvariety datum $\mathcal{D}_1 = (\mathcal{W}_1, \mathcal{Z}_1^c, \pi_* (\mathcal{M}_1 \otimes \mathcal{O}_{\mathcal{Z}_1}/\mathcal{I}), \mathbf{T}, \psi \bmod \mathcal{I})$. The narrow case of Theorem 5.1.2 applies to the pair of eigenvariety data $\mathcal{D}_1^c$ and $\mathcal{D}_2^c$, producing a closed immersion $\iota^* : \mathcal{Z}_1^c \hookrightarrow \mathcal{Z}_2^c$, and the general case follows upon setting $i = i' \circ i^\sigma$. \qed

From Theorem 5.1.2, it’s easy to deduce the following more flexible interpolation theorem.

**Theorem 5.1.6.** Suppose we are given two eigenvariety data $\mathcal{D}_i = (\mathcal{W}_i, \mathcal{Z}_i, \mathcal{M}_i, \mathbf{T}_i, \psi_i)$ for $i = 1, 2$, together with the following additional data:

1. A closed immersion $j : \mathcal{W}_1 \hookrightarrow \mathcal{W}_2$.
2. An algebra homomorphism $\sigma : \mathbf{T}_2 \rightarrow \mathbf{T}_1$.
3. A very Zariski-dense set $\mathcal{Z}_1^c \subset \mathcal{Z}_1^c$ with $j(\mathcal{Z}^c) \subset \mathcal{Z}_2$ such that $D_2(\sigma(T), X)(z)$ divides $D_2(T, X)(j(z))$ in $k(z)[X]$ for all $z \in \mathcal{Z}^c$ and all $T \in \mathbf{T}_2$.

Then there exists a morphism $i : \mathcal{Z}_1^c \rightarrow \mathcal{Z}_2^c$ such that the diagrams

$$\begin{array}{ccc} \mathcal{Z}_1^c & \xrightarrow{i} & \mathcal{Z}_2^c \\ \downarrow \psi_1 & & \downarrow \psi_2 \\ \mathcal{W}_1 & \xrightarrow{\sigma} & \mathcal{W}_2 \end{array}$$

and

$$\begin{array}{ccc} \mathcal{O}(\mathcal{Z}_2) & \xrightarrow{i^*} & \mathcal{O}(\mathcal{Z}_1^c) \\ \downarrow \phi_2 & & \downarrow \phi_1^* \\ \mathcal{T}_2 & \xrightarrow{\sigma} & \mathcal{T}_1 \end{array}$$

commute, and $i$ may be written as a composite $i_c \circ i_f$ where $i_f$ is a finite morphism and $i_c$ is a closed immersion.

**Proof.** Let $\mathcal{D}_1^c$ be the eigenvariety datum $(\mathcal{W}_1, \mathcal{Z}_1, \mathcal{M}_1, \mathbf{T}_2, \psi_1 \circ \sigma)$. Theorem 5.1.2 produces a closed immersion $i' : \mathcal{Z}(\mathcal{D}_1^c) \hookrightarrow \mathcal{Z}(\mathcal{D}_2)$. The inclusion $\text{im}(\psi_1 \circ \sigma)(\mathbf{T}_2) \subset \text{im}\psi_1(\mathbf{T}_1) \subset \text{End}_{\mathcal{O}(\Omega)}(\mathcal{M}_1(\Omega))$ induces a finite morphism $i_f : \mathcal{Z}(\mathcal{D}_1) \rightarrow \mathcal{Z}(\mathcal{D}_1^c)$. The ideal subsheaf of $\mathcal{O}(\mathcal{Z}(\mathcal{D}_1))$ cut out by the kernel of the composite $\mathcal{O}(\mathcal{Z}(\mathcal{D}_1)) \rightarrow \mathcal{O}(\mathcal{Z}(\mathcal{D}_1^c)) \rightarrow \mathcal{O}(\mathcal{Z}(\mathcal{D}_2))$ determines a closed intermorphism...
In order to apply the interpolation theorems of the previous section, we need a systematic way of including the Atkin-Lehner operators. The key is Theorem 3.2.5 together with Proposition 5.2.1 below.

Let \( \pi \) be an unramified generic irreducible representation of \( G = GL_n(Q_p) \) defined over \( L \), and let \( r : W_{Q_p} \to GL_n(L) \) be the unramified Weil-Deligne representation satisfying \( r \simeq \text{rec} (\pi \otimes | det |^{\frac{1}{2n}}) \). Let \( \varphi_1, \ldots, \varphi_n \) be any fixed ordering on the eigenvalues of \( r(\text{Frob}_p) \), and let \( \chi \), \( \sigma \in S_n \) be the character of \( A_p \) defined by \( A_p(u_i) = p^{1-i} \varphi_{\sigma(i)} \).

**Proposition 5.2.1.** For every \( \sigma \in S_n \), the module \( \pi^I \) contains a nonzero vector \( v_\sigma \) such that \( A_p \) acts on \( v_\sigma \) through the character \( \chi_\sigma \).

**Proof.** Assembling some results of Casselman (cf. §3.2.2 of [Tai12]), there is a natural isomorphism of \( \Lambda \)-modules

\[
\pi^I \simeq (\pi_p^T)^T(Z_p) \otimes \delta_p^{-1}
\]

5.2 Refinements of unramified representations

In order to apply the interpolation theorems of the previous section, we need a systematic way of producing sets \( \mathcal{F}^{cl} \) such that \( \mathcal{M}_1(z) \) consists entirely of classical automorphic forms for \( z \in \mathcal{F}^{cl} \). The key is Theorem 3.2.5 together with Proposition 5.2.1 below.

Let \( G \simeq GL_n(Q_p) \), with \( B \) the upper-triangular Borel and \( \overline{B} \) the lower-triangular Borel. In this case we may canonically parametrize \( L \)-valued characters of \( A_p \) and unramified characters of \( T(Q_p) \) by ordered \( n \)-tuples \( a = (a_1, \ldots, a_n) \in (L^X)^n \), the former via the map

\[
a \mapsto \chi_a(u_{p,i}) = a_{n+1-i}
\]

and the latter via the map

\[
a \mapsto \chi_a(t_1, \ldots, t_n) = \prod_{i=1}^{n} a_{p(t_i)}.
\]

Let \( \xi \) be the unramified Hecke algebras to include the Atkin-Lehner operators.

\[
\begin{array}{ccc}
\mathcal{F} \leftarrow \mathcal{F}(\mathcal{D}_1^\tau) & \xrightarrow{\iota} & \mathcal{F}(\mathcal{D}_1) \\
\iota_c' & & \iota_c'' \\
\mathcal{F} & \xrightarrow{\iota_c} & \mathcal{F}(\mathcal{D}_1^\tau) & \xrightarrow{\iota_c'} & \mathcal{F}(\mathcal{D}_1^\tau)
\end{array}
\]

and taking \( \iota_c = \iota_c' \iota_c'' \) concludes. \( \square \)

The template for applying these results is as follows. Consider a pair of connected, reductive groups \( G \) and \( H \) over \( Q \), together with an \( L \)-homomorphism \( L \sigma : L G \to L H \) which is known to induce a Langlands functoriality map. The \( L \)-homomorphism induces a morphism \( T_H \to T_G \) of unramified Hecke algebras in the usual way. When \( G \) and \( H \) are inner forms of each other, \( L \sigma \) is an isomorphism, and Theorem 5.1.2 (with \( \mathcal{M}_1 = \mathcal{M}_2 \) and \( \iota = \text{id} \)) gives rise to closed immersions of eigenvarieties interpolating correspondences of Jacquet-Langlands type and/or comparing different theories of overconvergent automorphic forms. In the general case, the homomorphism \( X^*(\overline{T}_H) \to X^*(\overline{T}_G) \) together with the natural identification \( T(A) \simeq X^*(\overline{T}) \otimes_Z A \) induces a homomorphism \( \tau : T_H(Z_p) \to T_G(Z_p) \), and \( \iota \) is given by sending a character \( \lambda \) of \( T_G \) to the character \( (\lambda \circ \tau) \cdot \xi_\sigma \) for some fixed character \( \xi_\sigma \) of \( T_H \) which may or may not be trivial. In this case, Theorem 5.1.6 then induces morphisms of eigenvarieties interpolating the functoriality associated with \( L \sigma \). In practice, one must carefully choose the character \( \xi_\sigma \), compatible tame levels for \( G \) and \( H \), and an extension of the map on unramified Hecke algebras to include the Atkin-Lehner operators.
where \( t \in \Lambda \) acts on the left-hand side by \( U_t \). By Theorem 4.2 of [BZ77] and Satake’s classification of unramified representations, we may write \( \pi \) as the full normalized induction

\[
\pi = \text{Ind}_{Z}^{G} \chi
\]

where \( \chi \) is the character of \( T \) associated with the tuple \( (p^{\frac{1}{n_a}} \varphi(\epsilon), \ldots, p^{\frac{1}{n_a}} \varphi(1)) \). By Frobenius reciprocity, there is an embedding of \( T \)-modules

\[
L(\chi_{\delta}) \hookrightarrow (\text{Ind}_{Z}^{G} \chi_{\varphi})_{\infty}.
\]

so \( L(\chi_{\frac{1}{\delta B}}) \hookrightarrow \pi' \), and \( \chi_{\sigma} = \chi_{\delta_{B}^{-1}} \) upon noting that \( \delta_{B}^{-1} \) corresponds to the tuple \( (p^{\frac{1}{n_a}}, \ldots, p^{\frac{1}{n_a}}) \).

\[\square\]

5.3 Some quaternionic eigencurves

Fix a squarefree positive integer \( \delta \geq 2 \), a positive integer \( N \) prime to \( \delta \), and a prime \( p \) with \( p \nmid N\delta \). Let \( D \) be the quaternion division algebra over \( \mathbb{Q} \) ramified at exactly the finite places dividing \( \delta \), and ramified or split over \( \mathbb{R} \) according to whether \( \delta \) has an odd or even number of distinct prime divisors. Let \( G \) be the inner form of \( \text{GL}_{2}/\mathbb{Q} \) associated with \( D \), and let \( \mathcal{X}_{D} \) be the eigenvariety \( \mathcal{X}^{\text{ncm}}_{G,K_{1}(N\delta)} \) as defined in §4.2 and §4.6. Let \( \mathcal{X} \) be the eigenvariety \( \mathcal{X}^{0}_{\text{GL}_{2}/\mathbb{Q},K_{1}(N\delta)} \). The eigenvarieties \( \mathcal{X} \) and \( \mathcal{X}_{D} \) are both unmixed of dimension one.

**Theorem 5.3.1.** There is a canonical closed immersion \( \iota_{\text{JL}} : \mathcal{X}_{D} \hookrightarrow \mathcal{X} \) interpolating the Jacquet-Langlands correspondence on non-critical classical points.

**Proof.** Let \( \mathcal{D}_{1} \) and \( \mathcal{D}_{2} \) be the eigenvariety data giving rise to \( \mathcal{X}_{D} \) and \( \mathcal{X} \) as in Definition 4.3.2, and let \( \mathcal{M}^{1} \) and \( \mathcal{M}^{1}_{D} \) be the sheaves of automorphic forms carried by \( \mathcal{X} \) and \( \mathcal{X}_{D} \), respectively. Let \( \mathcal{M}^{1}_{\text{nc}} \subset \mathcal{M}^{1}_{0}(\mathbb{Q}_{\mathbb{A}}^{1})/(\mathcal{W} \times \mathbf{A}^{1})(\mathbb{Q}_{p}) \) be the set of points \( z = (\lambda, \alpha^{-1}) \) with \( \lambda \in \mathcal{W}(\mathbb{Q}_{p}) \) of the form \( \lambda(x) = x^{k}, k \in \mathbb{Z}_{\geq 1} \), and with \( v_{p}(\alpha) < k + 1 \). For any \( z \in \mathcal{M}^{1}_{\text{nc}} \), Theorem 3.2.5 together with the classical Eichler-Shimura isomorphism induces isomorphisms of Hecke modules

\[
\mathcal{M}^{1}(z) \simeq (S_{k+2}(\Gamma_{1}(N\delta) \cap \Gamma_{0}(p)) \oplus M_{k+2}(\Gamma_{1}(N\delta) \cap \Gamma_{0}(p)))^{U_{p} \equiv \epsilon(\alpha)}
\]

and

\[
\mathcal{M}^{1}_{D}(z) \simeq (S_{k+2}^{0}(\Gamma_{1}(N) \cap \Gamma_{0}(p))^{\equiv \epsilon})^{U_{p} \equiv \epsilon(\alpha)}
\]

where \( \epsilon = 1 \) or 2 according to whether \( D \) is ramified or split over \( \mathbb{R} \). The set \( \mathcal{M}^{1}_{\text{nc}} \) forms a Zariski-dense accumulation subset of \( \mathcal{M}^{1}_{0} \), and \( D_{1}(T, X)(z) \) divides \( D_{2}(T, X)(z) \) in \( k(z)[X] \) for any \( z \in \mathcal{M}^{1}_{\text{nc}} \) by the classical Jacquet-Langlands correspondence. Theorem 5.1.2 now applies.

5.4 A symmetric square lifting

Let \( \mathcal{C}_{0}(N) \) be the cuspidal locus of the Coleman-Mazur-Buzzard eigencurve of tame level \( N \). Given a non-CM cuspidal modular eigenform \( f \in S_{k}(\Gamma_{1}(N)) \), Gelbart and Jacquet constructed a cuspidal automorphic representation \( \Pi(\text{sym}^{2} f) \) of \( \text{GL}_{3}/\mathbb{Q} \) characterized by the isomorphism

\[
\iota_{\text{WD}}(\text{sym}^{2} R_{f, \ell} \mid G_{Q}) \simeq \text{rec} (\Pi(\text{sym}^{2} f)_{\ell} \otimes | \det_{\ell}^{-1})
\]

for all primes \( \ell \). We are going to interpolate this map into a morphism \( s : \mathcal{C}_{0}^{\text{ncm}}(N) \rightarrow \mathcal{X} \), where \( \mathcal{C}_{0}^{\text{ncm}}(N) \) is the Zariski-closure inside \( \mathcal{C}_{0}(N) \) of the classical points associated with non-CM
eigenforms and $\mathcal{X}$ is an eigenvariety arising from overconvergent cohomology on $\text{GL}_3$. For our intended application to Galois representations, we need a more precise result.

**Definition.**

i. For a prime $\ell \neq p$, an inertial Weil-Deligne representation $\tau_\ell$ is a pair $(r_\ell, N_\ell)$ consisting of a continuous semisimple representation $r_\ell : I_{Q_p} \to \text{GL}_2(\overline{Q_p})$ and a nilpotent matrix $N_\ell \in M_2(\overline{Q_p})$ such that $r_\ell N_\ell = N_\ell r_\ell$.

ii. A global inertial representation is a formal tensor product $\tau = \bigotimes_{\ell \neq p} \tau_\ell$ of inertial Weil-Deligne representations such that $r_\ell(I_{Q_p}) = \{1\}$ and $N_\ell = 0$ for all but finitely many $\ell$.

A global inertial representation $\tau$ has a well-defined conductor $N(\tau)$, given formally as a product $\prod \ell^{f(\tau_\ell)}$ of local conductors.

**Definition.** The non-CM cuspidal eigencurve of inertial type $\tau$, denoted $\mathcal{C}^\text{ncm}_0(\tau)$, is the union of the irreducible components of $\mathcal{C}(N(\tau))$ which contain a Zariski-dense set $Z$ of non-CM classical cuspidal points. Under the hypotheses above, there is a morphism $\mathcal{D}_1 \to \mathcal{D}_2$ as the eigenpacket associated with $\phi_{f,\alpha}$.

Now take $H = \text{GL}_3/Q$ and $T_2 = T_{H}(K_1(N(\text{sym}^2 \tau)))$. Let $\mathcal{D}_2$ be the eigenvariety datum from Definition 4.3.2, with $\mathcal{X} = \mathcal{X}_{H,K_1(N(\text{sym}^2 \tau))}$ the associated eigenvariety.

**Theorem 5.4.1.** Under the hypotheses above, there is a morphism $s : \mathcal{C}^\text{ncm}_0(\tau) \to \mathcal{X}$ interpolating the symmetric square lift on classical points.

Let $j : \mathcal{X}_1 \to \mathcal{X}_2$ be the closed immersion sending a character $\lambda$ to the character $j(\lambda)(t_1, t_2, t_3) = \lambda(t_1^2 t_2)$. Let $\sigma : T_2 \to T_1$ be the map defined on generators by

$$
\begin{align*}
\sigma(T_{\ell,1}) &= T_{\ell}^2 - \ell S_{\ell}, \\
\sigma(T_{\ell,2}) &= T_{\ell}^2 S_{\ell} - \ell S_{\ell}^2, \\
\sigma(T_{\ell,3}) &= S_{\ell}^3, \\
\sigma(U_{p,1}) &= U_{p}^3, \\
\sigma(U_{p,2}) &= U_{p}^2 S_{p}, \\
\sigma(U_{p,3}) &= S_{p}^3.
\end{align*}
$$

**Lemma 5.4.2.** If $f \in S^{\text{ncm}}_{k+2}(\Gamma_1(N))$ has inertial type $\tau$ and nebentype $\varepsilon_f$, and $X^2 - a_f(p) X + p^{k+1} \varepsilon_f(p)$ has a root $\alpha$ with $v_p(\alpha) < \frac{k+1}{2}$, then $\Pi(\text{sym}^2 f)$ contributes to $H^*(K_1(N(\text{sym}^2 \tau)), \mathcal{L}_{(2k,k,0)})$. 43
and $H^*(K_1(N(\text{sym}^2\tau)), D_{(2k,k,0)})$ contains a nonzero vector $v$ such that every $T \in T_2$ acts on $v$ through the scalar $(\phi_{f,\alpha} \circ \sigma)(T)$.

Proof. Fix $f$ and $\alpha$ as in the lemma, and let $\lambda$ be the highest weight $(2k, k, 0)$. By the local Langlands correspondence, $\Pi(\text{sym}^2 f)$ has conductor exactly $N(\text{sym}^2\tau)$. Since $f$ is not CM, $\Pi(\text{sym}^2 f)$ is cuspidal. The Hecke module $\Pi(\text{sym}^2 f)|_{K_1(N(\text{sym}^2\tau))}$ occurs in $H^*(K_1(N(\text{sym}^2\tau)), L_\lambda)$ by the Gelbart-Jacquet lifting and the calculations in [Clo90]. At primes $\ell \nmid Np$, $\Pi(\text{sym}^2 f)_{\ell}$ is unramified, and $T_{\ell,i}$ acts on the unramified line via the scalar $(\phi_{f,\sigma}(T_{\ell,i})$. A simple calculation using Proposition 5.2.1 shows that $\Pi(\text{sym}^2 f)|_{K_1(N(\text{sym}^2\tau))}$ contains a vector on which $\mathcal{A}_p$ acts through the character associated with the tuple $(p^{-2}\beta^2, p^{-1}\alpha\beta, \alpha^2)$, so there is a vector $v'$ in the $\Pi(\text{sym}^2 f)$-isotypic component of $H^*(K_1(N(\text{sym}^2\tau)), L_\lambda)$ such that the $\pi$-action of $\mathcal{A}_p$ is given by the character associated with the tuple $(\alpha^{-2}e(f)^2, e(f)(p), \alpha^2)$. In particular, $U_p$ acts on $v'$ through the scalar $\alpha^4e(p)$. By Proposition 3.2.5, the integration map $i_\lambda : H^*(K_1(N(\text{sym}^2\tau)), L_\lambda) \to H^*(K_1(N(\text{sym}^2\tau)), L_\lambda)$ is an isomorphism on the subspace where $U_p$ acts with slope $< k + 1$, so $v = i_\lambda^{-1}(v')$ does the job. \qed

Now we take $\mathcal{X}^{\text{cl}}$ to be the set of points in $\mathcal{Z}_1(\mathcal{Q}_p)$ of the form $(\lambda, \alpha^{-1})$, where $\lambda$ is a character of the form $\lambda(x) = x^k, k \in \mathbb{Z}_{\geq 3}$ and $\alpha$ satisfies $\nu_p(\alpha) < \frac{k+1}{4}$. This is a Zariski-dense accumulation subset of $\mathcal{Z}_1$. By Coleman’s classicality criterion, there is a natural injection $\mathcal{M}_1(z) \rightarrow \mathcal{S}^\text{num} \Gamma_1(N) \cap \Gamma_0(p) U_{\ell} = l(\alpha)$ of $\mathcal{T}_1$-modules, so Theorem 5.1.6 now applies, with the divisibility hypothesis following from Lemma 5.4.2. We thus conclude. \end{proof}

It’s not hard to show that the image of $s$ is actually a union of irreducible components of $\mathcal{X}^0$.

5.5 A Rankin-Selberg lifting

Let $f$ and $g$ be a pair of level one holomorphic cuspidal eigenforms of weights $k_f + 2, k_g + 2$ with associated Galois representations $V_{f,i}$ and $V_{g,i}$. By a deep theorem of Ramakrishnan [Ram00], there is a unique isobaric automorphic representation $\Pi(f \otimes g)$ of $\text{GL}_4(\mathbb{A}_Q)$ characterized by the equality

$$\text{rec}_\ell \left( \Pi(f \otimes g) \otimes | \det |_{\ell}^{\frac{3}{2}} \right) \simeq \ell \text{WD}(V_{f,i} \otimes V_{g,i}, G_{\mathbb{Q}_\ell})$$

for all primes $\ell$. We are going to interpolate the map $(f, g) \rightarrow \Pi(f \otimes g)$ into a morphism of eigenvarieties $\mathcal{C}_0 \times \mathcal{C}_0 \rightarrow \mathcal{X}$, where $\mathcal{C}_0$ denotes the cuspidal locus of the Coleman-Mazur eigencurve and $\mathcal{X}$ denotes an eigenvariety associated with overconvergent cohomology on $\text{GL}_4/Q$.

Set $G = \text{GL}_2/Q$, $T_1 = T_G(K_1(1)) \otimes T_G(K_1(1))$, and $\mathfrak{H}_1 = \text{Hom}_{cts}(\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times, G_m)$; we regard an $A$-point of $\mathfrak{H}_1$ as a pair of characters $\lambda_1, \lambda_2 : \mathbb{Z}_p^\times \rightarrow A^\times$ in the obvious way. The product $\mathcal{C}_0 \times \mathcal{C}_0$ arises from an eigenvariety datum $D_1 = (\mathfrak{H}_1, \mathfrak{Z}_1, \mathfrak{M}_1, T_1, \psi_1)$, where $(\lambda_1, \lambda_2, \alpha^{-1}) \in \mathcal{X}_1(\mathcal{Q}_p)$ if and only if there exist cuspidal overconvergent eigenforms $f_1$ and $f_2$ of weights $\lambda_1$ and $\lambda_2$ such that $U_p f_1 \otimes U_p f_2 - \alpha$ annihilates $f_1 \otimes f_2$.

Set $H = \text{GL}_4/Q$, $T_2 = T_H(K_1(1))$, and let $D_2$ be the eigenvariety datum from Definition 4.3.2, with $\mathcal{X}$ the associated eigenvariety.

**Theorem 5.5.1.** Under the hypotheses above, there is a morphism $\mathfrak{t} : \mathcal{C}_0 \times \mathcal{C}_0 \rightarrow \mathcal{X}$ interpolating the Rankin-Selberg lift on classical points.

Let $j : \mathfrak{H}_1 \rightarrow \mathfrak{H}_2$ be the closed immersion defined by sending a character $\lambda \in \mathfrak{H}_1$ to the character $j(\lambda)(t) = (t_{12})^{-1} \lambda_1(t_{12}) \lambda_2(t_{12})$, $t = \text{diag}(t_1, t_2, t_3, t_4) \in T_H$. 

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Define a map $\sigma : T_2 \to T_1$ on generators by
\[
\begin{align*}
\sigma(T_{f,1}) &= T_f \otimes T_f, \\
\sigma(T_{f,2}) &= S_f \otimes T_f^2 + T_f^2 \otimes S_f - 2\ell S_f \otimes S_f, \\
\sigma(T_{f,3}) &= \ell^{-1} S_f T_f \otimes S_f T_f, \\
\sigma(T_{f,4}) &= \ell^{-2} S_f^2 \otimes S_f^2, \\
\sigma(U_{p,1}) &= U_p \otimes U_p, \\
\sigma(U_{p,2}) &= U_p^2 \otimes S_p, \\
\sigma(U_{p,3}) &= U_p S_p \otimes S_p, \\
\sigma(U_{p,4}) &= S_p^2 \otimes S_p^2.
\end{align*}
\]

Let $(f, g)$ be an ordered pair as above with $k_f - 1 > k_g > 0$. Set $\lambda = \lambda(f \otimes g) : (x_1, x_2) \mapsto x_2^{k_f} x_2^{k_g} \in \mathcal{X}_1$. Let $\alpha_f, \beta_f$ be the roots of the Hecke polynomial $X^2 - a_f(p)X + p^{k_f+1}$, and likewise for $g$.

**Lemma 5.5.2.** The module $\Pi(f \otimes g)^f_p$ contains a vector $v$ on which $A_p$ acts through the character associated with the tuple $(p^{-3}\beta_f \beta_g, p^{-2}\beta_f \alpha_g, p^{-1}\alpha_f \beta_g, \alpha_f \alpha_g)$; in particular, $U_p$ acts via the scalar $p^{k_g+k_f-1}\alpha_f^2 \alpha_g^2$.

*Proof.* This is a direct consequence of Proposition 5.2.1, together with the characterization of $\Pi(f \otimes g)$ given above. \(\square\)

**Lemma 5.5.3.** If $\alpha_f$ and $\alpha_g$ satisfy $v_p(\alpha_f^2 \alpha_g^2) < \min(k_f - k_g, k_g + 1)$, then $H^*(K(1), \mathcal{Z}_{j(\lambda)})$ contains a nonzero vector $v$ such that every $T \in T_2$ acts on $v$ through the scalar $(\phi_{f, \alpha_f} \otimes \phi_{g, \alpha_g})(\sigma(T))$.

*Proof.* Dominance of $j(\lambda)$ is obvious, so $\Pi(f \otimes g)$ is cohomological in the weight $j(\lambda)$. For primes $\ell \nmid p$, $\Pi(f \otimes g)^f_\ell$ is unramified and $T_{f, i}$ acts on the unramified line via the scalar $(\phi_{f, \alpha_f} \otimes \phi_{g, \alpha_g})(\sigma(T_{f, i}))$.

Next, recall that the $*$-action of $A_p$ on $\Pi(f \otimes g)^f_p$ is simply the usual action rescaled by $j(\lambda)(1, p, p^2, p^3)^{k_f-1}$, and $j(\lambda)$ corresponds to the highest weight $(k_f + k_g - 1, k_f - 1, k_g, 0)$, so $\lambda(1, p, p^2, p^3)^{k_f-1} = p^{1-2k_g-k_f}$. In particular, $\Pi(f \otimes g)^f_p$ contains a vector on which $U_p$ acts through the scalar $\alpha_f^2 \alpha_g^2$ by Lemma 5.5.2. Writing $\kappa = \min(k_f - k_g, k_g + 1)$, the integration map $i_{j(\lambda)}$ induces an isomorphism
\[
\begin{align*}
H^*(K(1), \mathcal{Z}_{j(\lambda)})_{< \kappa} &\cong H^*(K(1), \mathcal{Z}'_{j(\lambda)})_{< \kappa},
\end{align*}
\]
and the target contains a vector satisfying the claim of the theorem. \(\square\)

Finally, we take $\mathcal{Z}'_1$ to be the set of points in $\mathcal{Z}_1$ of the form $(\lambda_1, \lambda_2, \alpha^{-1})$ where $\lambda_1(x) = x^{k_1}$ and $\lambda_2(x) = x^{k_2}$ with $k_1 \in \mathbb{Z}$ and $0 < k_2 < k_1 - 1$, and $\alpha$ satisfies $\alpha < \min(k_1 - k_2, k_2 + 1)$. This is a Zariski-dense accumulation subset, and Theorem 5.1.6 applies with the divisibility hypothesis following from Lemma 5.5.3. This proves Theorem 5.5.1. \(\square\)

## 6 Galois representations

### 6.1 Background on trianguline representations and $(\varphi, \Gamma)$-modules

In this section we give some background and context on $(\varphi, \Gamma)$-modules and trianguline representations. Our primary references here are the articles [BC09, Ber11, KPX13]. Throughout this section, we let $K \subset \overline{\mathbb{Q}}_p$ denote a finite extension of $\mathbb{Q}_p$, with $K_0$ the maximal unramified subfield of $K$. Let $K'_0$ be the maximal unramified subfield of $K(\zeta_p)$. We identify $G_K \simeq \text{Gal}(\overline{\mathbb{Q}}_p/K)$ without
comment. Set $H_K = \text{Gal}(\mathbb{Q}_p/K(\zeta_{p^n}))$ and $\Gamma_K = \text{Gal}(K(\zeta_{p^n})/K) = G_K/H_K$; the cyclotomic character $\chi_{\text{cyc}} : \Gamma_{\mathbb{Q}_p} \rightarrow \mathbb{Z}_p^\times$ identifies $\Gamma_K$ with an open subgroup of $\mathbb{Z}_p^\times$.

Let $B_{\text{rig}}^\dagger$ and $\hat{B}_{\text{rig}}^\dagger$ be the topological $\mathbb{Q}_p$-algebras defined in [Ber02]. These rings are equipped with a continuous action of $G_{\mathbb{Q}_p}$ and a commuting operator $\varphi$. The Robba ring is the ring $\mathcal{R}_K = (\hat{B}_{\text{rig}}^\dagger)^{H_K}$, with its natural actions of $\varphi$ and $\Gamma_K$. There is an isomorphism

$$\mathcal{R}_K \simeq \{ f(\pi_K) \mid f(T) \in K_0[[T,T^{-1}]] \text{ with } f \text{ convergent on } r \leq |T| < 1 \text{ for some } r = r_f < 1 \}.$$ 

Here $\pi_K$ is a certain indeterminate arising from the theory of the field of norms; $\varphi$ and $\Gamma_K$ act on the coefficients of $f$ through the absolute Frobenius and the natural map $\Gamma_K \rightarrow \text{Gal}(K_0'/K_0)$, respectively, but the actions on $\pi_K$ are noncanonical in general. The topological ring $\mathcal{R}_K$ is naturally an LF-space, i.e. a strict inductive limit of Fréchet spaces: setting

$$\mathcal{R}_K^{r,s} \simeq \{ f(\pi_K) \mid f(T) \in K_0[[T,T^{-1}]] \text{ with } f \text{ convergent on } r \leq |T| \leq s \}$$

with its natural affinoid structure, $\mathcal{R}_K = \bigcup_{r<1} \cap_{s<1} \mathcal{R}_K^{r,s}$. In particular, for any affinoid algebra $A$, the completed tensor product $\mathcal{R}_K \hat{\otimes} A$ is well-defined, and the $\varphi$- and $\Gamma_K$-actions extend naturally.

**Definition 6.1.1.** $A(\varphi, \Gamma_K)$-module over $L$ is a finite free $\mathcal{R}_K \otimes L$-module $D$ equipped with commuting $L$-linear and $\mathcal{R}_K$-semilinear actions of $\varphi$ and $\Gamma_K$ such that $\mathcal{R}_K \cdot \varphi(D) = D$ and such that the map $\Gamma_K \rightarrow \text{End}(D)$ is continuous.

We write $\text{Mod}(\varphi, \Gamma_K)_L$ for the category of $(\varphi, \Gamma_K)$-modules over $L$. On the other hand, let $\text{Rep}(G_K)_L$ denote the category of pairs $(V, \rho)$ where $V$ is a finite-dimensional $L$-vector space and $\rho : G_K \rightarrow \text{End}_L(V)$ is a continuous group homomorphism. The significance of $(\varphi, \Gamma_K)$-modules arises from the following beautiful theorem of Berger, Cherbonnier-Colmez, and Fontaine.

**Theorem 6.1.2.** The functor

$$\text{Rep}(G_K)_L \rightarrow \text{Mod}(\varphi, \Gamma_K)_L$$

$$\rho \mapsto D_{\text{rig}}(\rho)$$

defined by

$$D_{\text{rig}}(\rho) = (\rho \otimes \mathbb{Q}_p \hat{B}_{\text{rig}}^\dagger)^{H_K}$$

is additive, tensor-exact and fully faithful, with a natural quasi-inverse given by

$$D \mapsto V_{\text{rig}}(D) = (D \otimes \mathcal{R}_K \hat{B}_{\text{rig}}^\dagger)^{\varphi=1}.$$ 

Historically, this is actually the third flavor of $(\varphi, \Gamma)$-module introduced. Fontaine initiated the theory [Fon90] by constructing a functor $\rho \mapsto D(\rho)$ to a similar category, but with the role of $\mathcal{R}_K$ played by the ring

$$B_K = \left( \lim_{\rightarrow} (O_{K_0}/p^n)[[\pi_K]][[\pi_K^{-1}]] \right)[\frac{1}{p}].$$

Cherbonnier and Colmez [CC98] then showed, in a difficult piece of work, that $D(\rho)$ arises via base change from a module $D(\rho)$ defined over the subring

$$B_K^\dagger = \{ f(\pi_K) \in B_K \mid f(T) \text{ convergent on } r(f) < |T| < 1 \text{ for some } r(f) < 1 \}.$$
Berger [Ber02] then defined $D^1_{\text{rig}}$ as the base change $D^1_{\text{rig}}(\rho) = D^1(\rho) \otimes B_K^1 \mathcal{R}_K$. The change from $B_K^1$ to $\mathcal{R}_K$ may look slight, but in fact has enormous consequences. For example, the ring $\mathcal{R}_K$ is intimately connected with the theory of $p$-adic differential equations, and Berger used this link to give the first proof of Fontaine’s $p$-adic monodromy conjecture. Berger also showed how to functorially recover the usual Fontaine modules $D_*(\rho)$ for $* \in \{\text{crys}, \text{st}, dR, \text{Sen}, \text{dif}\}$ from $D^1_{\text{rig}}(\rho)$; in particular, there is a natural equality

$$D_{\text{crys}}(\rho) = \left( D^1_{\text{rig}}(\rho) \right)^{\Gamma_K}$$

of filtered $\varphi$-modules over $K_0 \otimes L$, and a $(\varphi, \Gamma_K)$-equivariant inclusion

$$D_{\text{crys}}(\rho) \otimes_{K_0} \mathcal{R}_K[\varphi] \subseteq D^1_{\text{rig}}(\rho)$$

which is an isomorphism if and only if $\rho$ is crystalline. In retrospect we might say the Robba ring has “just the right amount of flexibility”: the ring $B_K^1$ is a field and so its abstract module theory has little structure, while the ring $B_K^1$ is too small to contain the ubiquitous $t$ of $p$-adic Hodge theory, its elements being bounded as analytic functions on their annuli of definition.

For our purposes, the following classification of rank one $(\varphi, \Gamma_K)$-modules is indispensable [KPX13, Nak09]:

**Theorem 6.1.3.** The rank one $(\varphi, \Gamma_K)$-modules over $L$ are naturally parametrized by the continuous characters $\delta : K^\times \to L^\times$. Writing $\mathcal{R}_K(\delta)$ for the module corresponding to a character $\delta$, there is a natural isomorphism

$$\mathcal{R}_K(\delta_1) \otimes_{\mathcal{R}_K \otimes L} \mathcal{R}_K(\delta_2) \simeq \mathcal{R}_K(\delta_1 \delta_2).$$

If $\delta(\varpi_K) \in \mathcal{O}_K^\times$ for some uniformizer of $K$, then

$$\mathcal{R}_K(\delta) \simeq D^1_{\text{rig}}(\text{rec}(\delta))$$

where $\text{rec}(\delta)$ is the unique continuous character of $G_K$ such that

$$\text{rec}(\delta) \circ \text{Art}_K = \delta.$$

One very significant point in the study of Galois representations via their $(\varphi, \Gamma)$-modules is that the latter may be highly reducible even when $\rho$ is irreducible. To explain this, recall that Kedlaya [Ked04] associated with any $\varphi$-module $D$ over $\mathcal{R}_K$ a finite set of slopes $s(D) \subset \mathbb{Q}$, and proved that $D$ admits a unique filtration $0 = D_0 \subset D_1 \subset \cdots D_j \subset D_{j+1} = D$ by $\varphi$-submodules such that each $s(D_{i+1}/D_i)$ is a singleton, say $s(D_{i+1}/D_i) = \{s_i(D)\}$, and such that $s_i(D) > s_{i-1}(D)$ for all $1 \leq i \leq j$. One says that each $D_{i+1}/D_i$ is pure of slope $s_i$. Berger [Ber02] then proved that the essential image of $D^1_{\text{rig}}$ consists exactly of those $(\varphi, \Gamma_K)$-modules whose underlying $\varphi$-module is pure of slope zero, or étale. However, Kedlaya’s slope filtration theorem does not preclude étale $(\varphi, \Gamma_K)$-modules from containing subobjects whose slopes are positive.

Let us now formalize the notion of a Galois representation whose $(\varphi, \Gamma_K)$-module is totally reducible. Given a continuous representation $\rho : G_K \to \text{GL}_n(L)$, an ordered $n$-tuple $\delta = (\delta_1, \ldots, \delta_n)$ of continuous characters $\delta_i : K^\times \to L^\times$ is a parameter of $\rho$ if $D^1_{\text{rig}}(\rho)$ admits a filtration

$$0 = \text{Fil}^0 \subset \text{Fil}^1 \subset \cdots \subset \text{Fil}^n = D^1_{\text{rig}}(\rho)$$

by $(\varphi, \Gamma_K)$-stable $\mathcal{R}_K \otimes L$-free direct summands such that $\text{Fil}^i/\text{Fil}^{i-1} \simeq \mathcal{R}_K(\delta_i)$ for all $1 \leq i \leq n$. Let $\mathcal{P}ar(\rho)$ denote the set of parameters of $\rho$.\footnote{In the global setup of §1.2, we have $\mathcal{P}ar(\rho) = \prod_{v \mid p} \mathcal{P}ar(\rho(G_{F_v}))$.} Note that a given representation may not admit
Definition 6.1.4. [Col08] A representation $\rho$ is trianguline if $\mathcal{P}\ar(\rho)$ is nonempty.

The most well-studied trianguline representations are the nearly ordinary representations, in which case the representation space of $\rho$ itself admits a $G_K$-stable full flag $0 = V^{(0)} \subset V^{(1)} \subset \cdots \subset V^{(n)} = V$ such that $\Fil^i = D^\dual_{\rig}(\rho^{(i)})$; a parameter $\delta$ corresponds to a nearly ordinary structure on $\rho$ if and only if $\delta_i(\varpi_K) \in O_K^{\dual}$ for $1 \leq i \leq n$. However, most trianguline representations are irreducible in the category of Galois representations.

Example 6.1.1: de Rham representations

Suppose $\rho : G_K \to \GL_n(\Q_p)$ is de Rham, and let $\WD(\rho) = (r, N)$ denote the associated Frobenius-semisimple Weil-Deligne representation.

Proposition 6.1.5. The following are equivalent:

i. $\rho$ is triangular.

ii. $\rho$ becomes semistable over an abelian extension of $K$.

iii. $r$ is a direct sum of characters.

iv. The local Langlands correspondent $\pi$ of $\WD(\rho)$ is a subquotient of a representation induced from a Borel subgroup of $\GL_n(K)$.

Proof (sketch). The equivalence of i. and ii. follows from Berger’s dictionary [Ber08] relating filtered $(\varphi, N, G_K)$-modules and $(\varphi, \Gamma_K)$-modules. The equivalence of ii. and iii. is an easy consequence of Fontaine’s construction of $\WD(\rho)$ in terms of $\calD_{pst}(\rho)$. The equivalence of iii. and iv. follows from Bernstein and Zelevinsky’s work [BZ77, Zel80], and in particular from the fact that $r$ determines the cuspidal support of $\pi$. $\square$

Example 6.1.2: Overconvergent modular forms

Throughout this example, and the remainder of §6, we write $f$ for an overconvergent cuspidal eigenform of finite slope and some tame level $N_f$, with associated Galois representation $\rho_f : G_\Q \to \GL_2(L)$. We define the weight of $f$ as the unique continuous character $w : \Z_p^\times \to L^\times$ such that $\det \rho_f|_{G_\Q} \simeq w(\chi_{\cyc}^{-1})\chi_{\cyc}^{-1}$. Set $k = 2 + \frac{\log w(1+p)}{\log(1+p)}$, so the Sen weights of $\rho_f$ are exactly $0$ and $k - 1$.

Let $\alpha_f$ be the $U_1$-eigenvalue of $f$. We recall a brilliant result of Kisin [Kis03]:

Theorem (Kisin). The space of crystalline periods $\calD^+_{cryst}(\rho_f|_{G_{\Q_p}})^{w=\alpha_f}$ is nonzero.

We now associate a canonical parameter $\delta_f$ with $f$. Before doing so, we partition the set of finite-slope overconvergent eigenforms $f$ into five types:

1a. $k \in \Z_{\geq 2}$ and $0 \leq v(\alpha_f) < k - 1$.

1b. $k \in \Z_{\geq 1}$, $v_p(\alpha_f) = k - 1$, and $\rho_f|_{G_{\Q_p}}$ is de Rham and indecomposable.

2. $k \in \Z_{\geq 1}$, $v_p(\alpha_f) = k - 1$, and $\rho_f|_{G_{\Q_p}}$ is a direct sum of characters.

3a. $k \in \Z_{\geq 1}$, $v_p(\alpha_f) \geq k - 1$, and $\rho_f|_{G_{\Q_p}}$ is not de Rham.

3b. $k \notin \Z_{\geq 1}$.

Forms of type 1 are always classical, while forms of type 3 are never classical.

If $f$ is of type 1 or 3b, then we define

$$\delta_f = (\mu_\alpha, \mu_{\alpha-1}w(x_0)^{-1}x_0^{-1})$$
where \( \eta = \det \rho_f(\text{Frob}_p) \); here for \( x \in \mathbb{Q}_p^\times \) we set \( x_0 = x|_x \) and \( \mu_\alpha(x) = \alpha^{wp}(x) \). If \( f \) is of type 2 or 3a, we define
\[
\delta_f = (\mu_{p^1-k}\alpha x_0^{-k}, \mu_{\alpha^{-1}}(x_0))^{-1}
\]
where \( w(x) = x^{k-2}\varepsilon(x) \) with \( \varepsilon \) of finite order. (For a proof that these really are parameters, see e.g. Proposition 5.2 of [Che08].) These are the unique parameters compatible with Kisin’s result, in the sense that the associated \( \text{Fil}^1 \subset D_{\text{rig}}(\rho_f|G_{\mathbb{Q}_p}) \) satisfies
\[
0 \neq D_{\text{cris}}(\text{Fil}^1) \subset D_{\text{cris}}(\rho_f|G_{\mathbb{Q}_p})^{\alpha = \alpha'}. \]

### 6.2 The general conjecture

Let \( A \) be a finite étale \( \mathbb{Q}_p \)-algebra, and fix an integer \( n \geq 1 \). Let \( \mathcal{T} = \mathcal{T}_{n,A} \) be the rigid analytic space parametrizing continuous characters of \((A^\times)^n\). Let \( \delta \in \mathcal{T}(\mathbb{Q}_p) \) be any character, so we may regard \( \delta \) as an ordered \( n \)-tuple of characters \( \delta_i : A^\times \to \mathbb{Q}_p^\times \) in the obvious way. Set \( \Sigma = \text{Hom}_{\text{alg}}(A, \mathbb{Q}_p) \). Given \( \delta = (\delta_1, \ldots, \delta_n) \) as above, there is some sufficiently small open subgroup \( U \subset A \) such that the linear form \((\partial_\delta)(a) = \log(\delta_i(\exp(a))) : U \to \mathbb{Q}_p^\times \) is well-defined, and we have
\[
(\partial_\delta)(a) = \sum_{\sigma \in \Sigma} k_{\sigma,i}(\delta)\sigma(a)
\]
for some uniquely determined constants \( k_{\sigma,i}(\delta) \in \mathbb{Q}_p^\times \). When \( A = K \) is a finite extension of \( \mathbb{Q}_p \) and \( \delta \) is a parameter of a trianguline representation \( \rho : G_K \to \text{GL}_n(L) \), the multiset \( \{-k_{\sigma,i}(\delta)\}_{1 \leq i \leq n} \) is exactly the set of \( \sigma \)-Sen weights of \( \rho \).

**Definition 6.2.1.** \( W(\delta)^\sigma < S_n \) is the group generated by the transpositions \((ij)\) such that \( k_{\sigma,i}(\delta) - k_{\sigma,j}(\delta) \in \mathbb{Z} \), and
\[
W(\delta) = \prod_\sigma W(\delta)^\sigma < S_n^\Sigma.
\]

The reflections in \( W(\delta) \) are those elements which are a transposition in exactly one \( W(\delta)^\sigma \) and trivial in \( W(\delta)^{\sigma'} \) for all \( \sigma' \neq \sigma \).

For \( g \in W(\delta) \), we define \( g \cdot \delta \) by
\[
(g \cdot \delta)_i(x) = \delta_i(x) \prod_{\sigma} \sigma(x)^{k_{\sigma,i}(\delta) - k_{\sigma,j}(\delta)} \quad \forall 1 \leq i \leq n.
\]

It’s easy to check that this is a left action. Let \( W(\delta) \cdot \delta \) be the orbit of \( \delta \).

**Definition 6.2.2.** Given a character \( \eta \in W(\delta) \cdot \delta \) and a reflection \( g = (ij)_\sigma \in W(\delta) \) with \( i < j \), \( \eta \) precedes \( g \cdot \eta \) if \( k_{\sigma,i}(\eta) = k_{\sigma,j}(\eta) \in \mathbb{Z}_{\geq 0} \). We define a partial ordering on \( W(\delta) \cdot \delta \) by \( \eta \preceq \eta' \) if either \( \eta = \eta' \) or if there is a chain of characters \( \eta_0, \eta_1, \ldots, \eta_N \in W(\delta) \cdot \delta \) with \( \eta_0 = \eta \) and \( \eta_N = \eta' \) such that \( \eta_{i-1} \) precedes \( \eta_i \) for all \( 1 \leq i \leq N \).

For any \( \delta \in \mathcal{T}(\mathbb{Q}_p) \) we set
\[
\mathcal{T}[\delta] = \{ \eta \in W(\delta) \cdot \delta \mid \delta \preceq \eta \}.
\]

Now in the setting of Conjecture 1.2.5 we simply specialize all these notions to \( A = F \otimes_{\mathbb{Q}} \mathbb{Q}_p \), in which case
\[
\mathcal{T}_{n,F}(\mathbb{Q}_p) : = \mathcal{T}_{n,F \otimes_{\mathbb{Q}} \mathbb{Q}_p}(\mathbb{Q}_p) = \prod_{v|p} \mathcal{T}_{n,F_v}(\mathbb{Q}_p).
\]

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Let us give a slight reformulation of Conjecture 1.2.5. For $K/\mathbb{Q}_p$ finite and $\rho : G_K \to \text{GL}_n(\mathbb{Q}_p)$ a Galois representation, $\mathcal{P}_{\text{ar}}(\rho)$ is naturally a subset of $\mathcal{F}_{n,K}$, and we set (with $A = K$)

$$
\mathcal{F}[\rho] = \bigcup_{\delta \in \mathcal{P}_{\text{ar}}(\rho)} \mathcal{F}[\delta] 
\subset \mathcal{F}_{n,K}(\mathbb{Q}_p).
$$

Given a global Galois representation $\rho : G_F \to \text{GL}_n(\mathbb{Q}_p)$ as in §1.2, we define

$$
\mathcal{F}[\rho]^{\text{loc}} = \prod_{v \mid p} \mathcal{F}[\rho|_{G_{F_v}}] 
\subset \prod_{v \mid p} \mathcal{F}_{n,F_v}(\mathbb{Q}_p) 
= \mathcal{F}_{n,F}(\mathbb{Q}_p).
$$

Recall, on the other hand, the automorphically defined set $\mathcal{F}[\rho] \subset \mathcal{F}_{n,F}(\mathbb{Q}_p)$. The following conjecture is easily seen to be equivalent to Conjecture 1.2.5.

**Conjecture 6.2.3.** Notation and assumptions as in §1.2, we have

$$
\mathcal{F}[\rho] = \mathcal{F}[\rho]^{\text{loc}}.
$$

**Remarks.**

1. As we’ve already remarked, this conjecture is strongly analogous with Serre’s modularity conjecture and its generalizations, and the formulation just given highlights this analogy. Furthering the analogy, all of our partial results are of the form “for certain representations $\rho$ and certain elements $\eta \in \mathcal{F}[\rho]^{\text{loc}}$ we have $\eta \in \mathcal{F}[\rho]$”. On the other hand, there doesn’t seem to be any obvious mod $p$ structure analogous with the set of parameters of $\rho$ and the role they play in describing the total set $\mathcal{F}[\rho]^{\text{loc}}$.

2. This conjecture is rather nontrivial even when $\rho$ is classically automorphic. For concreteness, suppose $\rho : G_{\mathbb{Q}} \to \text{GL}_n(L)$ is crystalline with $n$ distinct Hodge-Tate weights. A refinement of $\rho$, in the terminology of [BC09], is a choice of an ordering $\alpha_\bullet = \{\alpha_1, \ldots, \alpha_n\}$ on the eigenvalues of the crystalline Frobenius $\varphi$ on $\text{D}_{\text{crys}}(\rho|_{G_{\mathbb{Q}_p}})$. Refinements are in bijection with complete $\varphi$-stable flags

$$
\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n = \text{D}_{\text{crys}}(\rho|_{G_{\mathbb{Q}_p}}) 
$$

of $L$-vector spaces by the association $\mathcal{F}_i = \ker \prod_{1 \leq j \leq i} (\varphi - \alpha_j)$. Any such flag, in turn, determines a triangulation of $\text{D}_{\text{rig}}(\rho|_{G_{\mathbb{Q}_p}})$ by setting $\text{Fil}^i = \mathcal{F}_i \otimes_{\mathbb{Q}_p} \mathcal{R}_{Q_i, \ell}[\frac{1}{l}] \cap \text{D}_{\text{rig}}(\rho|_{G_{\mathbb{Q}_p}})$, the intersection taking place in $\text{D}_{\text{crys}}(\rho|_{G_{\mathbb{Q}_p}}) \otimes_{\mathbb{Q}_p} \mathcal{R}_{Q_i, \ell}[\frac{1}{l}] \cong \text{D}_{\text{rig}}(\rho|_{G_{\mathbb{Q}_p}})[\frac{1}{l}]$. It’s easy to check that these associations determine bijections between these three sets (orderings on $\varphi$-eigenvalues, complete $\varphi$-stable flags, triangulations). Given a refinement $\alpha_\bullet$ with $\delta = (\delta_1, \ldots, \delta_n)$ the parameter of the associated triangulation, it transpires that $\delta_i(x) = \mu_{\alpha_i}(x)x^{-k_i}$, where $k_1, \ldots, k_n$ is some ordering on the Hodge-Tate weights of $\rho$. In particular, $W(\delta) = S_n$, and there’s a unique element $\delta^\dagger \in W(\delta) \cdot \delta$ which is maximal for the partial ordering $\leq$. Again following [BC09], we say the refinement (or triangulation) is noncritical if $k_1 < k_2 < \cdots < k_n$, and critical otherwise. It’s easy to see that the following conditions are equivalent:

i. The refinement $\alpha_\bullet$ is noncritical.

ii. The weight $\lambda$ of the putative point $x(\rho, \delta)$ is $B$-dominant.
iii. $\delta = \delta^{cl}$, or equivalently, $\mathcal{T}[\delta] = \{\delta\}$.

Now the weight $\lambda$ of the putative point $x^{cl}$ characterized by $\rho_x \cong \rho$ and $\delta = \delta^{cl}$ is $B$-dominant, and the subspace of $H^*(K_1(N), \mathcal{Z}_\lambda)$ predicted by Conjecture 1.2.5 is compatible with the Fontaine-Mazur-Langlands conjecture under the map

$$H^*(K_1(N), \mathcal{Z}_\lambda) \to H^*(K_1(N), \mathcal{Z}_\lambda)$$

(this calculation is sketched in [Han13]). Qualitatively, Conjecture 1.2.5 says the classical point $x^{cl}$ has “companion points” if and only if $\delta$ is critical, and $x^{cl}$ has more companion points the further $\delta$ is from being noncritical.

3. There is a purely local analogue of Conjecture 6.2.3, which seems quite interesting in its own right. To formulate this, fix a finite field $F$ of characteristic $p$ and an absolutely irreducible residual representation $\overline{\rho} : \mathrm{GL}_n(\mathbb{F})$. Let $\mathcal{T}_{\overline{\rho}}$ be the rigid generic fiber of the universal pseudodeformation space of $\overline{\rho}$, and let $X(\overline{\rho}) \subset \mathcal{B}_{\mathcal{T}_{\overline{\rho}}} \times \mathcal{T}_{n,K}$ be the finite slope space (Definition 2.10 of [HS13]).

**Conjecture 6.2.4.** For any point $x \in \mathcal{T}_{\overline{\rho}}(\overline{\mathbb{Q}}_p)$ with associated representation $\rho_x : G_K \to \mathrm{GL}_n(\mathbb{F})$, we have

$$\text{pr}_2(\text{pr}_1^{-1}(x) \cap X(\overline{\rho})) = \mathcal{T}[\rho_x]$$

as subsets of $\mathcal{T}_{n,K}(\overline{\mathbb{Q}}_p)$.

We also strongly believe that if there is a functor

$$\text{Irr}_n(G_K)/L \to \text{Ban}_{\text{adm}}(\text{GL}_n(K))/L$$

$$\rho \mapsto \Pi(\rho)$$

which deserves to be called the $p$-adic local Langlands correspondence, then (modulo details of normalization) the set $\mathcal{T}[\rho]$ is exactly the set of characters appearing in the locally analytic Jacquet module of $\Pi(\rho)$.

### 6.3 Evidence for two-dimensional Galois representations

In this section we prove Theorem 1.2.6. Given $f$ an overconvergent finite-slope eigenform, we write $N(\rho_f)$ for the prime-to-$p$ Artin conductor of $\rho_f$. Note that a priori the only relation between the tame level $N_f$ and the Artin conductor is the divisibility $N(\rho_f)|N_f$, since we don’t require that $f$ be a newform. However, we have the following result.

**Proposition 6.3.1 (Level-lowering).** If $\overline{\rho}$ satisfies the hypotheses of Theorem 1.2.6, then there exists a finite-slope eigenform $f_0$ of tame level $N_{f_0} = N(\rho_f)$ such that $\rho_{f_0} \cong \rho_f$.

**Proof.** This follows from Emerton’s $p$-adic version of Mazur’s principle (Theorem 6.2.4 of [Eme11]) together with the results in [Pau11]. $\square$

Now, let $\rho$ be as in Theorem 1.2.6. By Corollary 1.2.2 of [Eme11], there is some $f$ and some continuous character $\nu : G_\mathbb{Q} \to \overline{\mathbb{Q}}_p^{*}$ such that $\rho \cong \rho_f \otimes \nu$. We may choose $\nu$ and $f$ such that $\nu$ is unramified outside $p$ and $\infty$, and then by the previous proposition we may choose $f$ such that $N_f = N(\rho)$. Since Conjecture 1.2.5 is compatible with twisting by characters unramified outside $p\infty$, we may assume $\nu = 1$ and $\rho = \rho_f$. Let $w : \mathbb{Z}_p^{\times} \to L^{\times}$ be the weight of $f$ as in §6.1 above, and let $S_w(\Gamma_1(N))$ be the linear span of finite-slope overconvergent cusp forms of weight $w$ and level $N$. For any character $\lambda = (\lambda_1, \lambda_2)$ in the $\text{GL}_2/\mathbb{Q}$ weight space, work of Stevens [Ste00] and Bellaïche [Bel12] yields a noncanonical injection of semisimplified Hecke modules

$$\beta_\lambda : S_w^{cl}(\Gamma_1(N))^{ss} \otimes \lambda_2 \circ (\det | \det |) \hookrightarrow H^1(K_1(N), \mathcal{Z}_\lambda)^{ss}$$
where of course the right-hand side denotes overconvergent cohomology for \( \text{GL}_2 \).

In general we have \( |\mathcal{P}\text{ar}(\rho_f)| \leq 2 \). Suppose first that \( \delta_f \) is the unique parameter of \( \rho_f \). This is true if and only if either \( f \) is type 1 and Steinberg at \( p \), or type 3. If \( f \) is type 1 or type 3b, the point \( x = x(\rho_f, \delta_f) \) we seek is the point associated with the eigenspace of \( f \) under the map \( \beta_{(w,0)} \).

If \( f \) is type 3a, with weight \( w(x) = x^{k-2} \varepsilon \), then by deep work of Coleman, \( f \) has a companion form \( g \) [Col96], namely a form \( g \) of type 3b such that \( \rho_f \simeq \rho_g \otimes \chi_{\text{cyc}}^{1-k} \) and \( \alpha_f = p^{k-1} \alpha_g \). It’s easy to see that \( g \) has weight \( x^{-k} \varepsilon \), and the eigenspace of \( g \otimes 1 \) under the map \( \beta_{(x^{-1}x^{k-1})} \) corresponds to the point \( x(\rho_f, \delta_f) \) predicted by Conjecture 1.2.4.

Now suppose that \( |\mathcal{P}\text{ar}(\rho_f)| = 2 \), so \( f \) is type 1 or 2. The point \( x(\rho_f, \delta_f) \) is given exactly as in the type 1 or type 3b subcases of the previous case, respectively. There is a classical newform \( f \) \( \in S_k(\Gamma_1(Np^n)) \) for some \( n \geq 0 \) which is “partially unramified principal series” at \( p \), such that \( f \) is in the same generalized eigenspace as the refinement \( f_n \), of \( f \). Let \( \varepsilon_f \) be the \( p \)-part of the nebentypus of \( f \). The second parameter \( \delta' \) is characterized by the fact that \( f \otimes \varepsilon_f^{-1} \) admits a type 1 refinement \( h \) with parameter \( \delta' \otimes \varepsilon_f \), and \( x(\rho_f, \delta') \) corresponds to the eigenspace of \( h \) under the map \( \beta_{(w_\varepsilon, x_\varepsilon, 0)} \). □

### 6.4 Evidence for three- and four-dimensional Galois representations

In this section we prove Theorems 1.2.7 and 1.2.8.

**Proof of Theorem 1.2.7.** Notation as in the theorem, let us show that the character \( \text{sym}^2 \delta_f \) really is a parameter of \( \text{sym}^2 \rho_f \). For brevity set \( D_f = D_\text{rig}^+(\rho_f|G_{Q_p}) \) and \( D_{\text{sym}^2 f} = D_\text{rig}^+(\text{sym}^2 \rho_f|G_{Q_p}) \).

We may realize \( D_{\text{sym}^2 f} \) as the \( \mathcal{A} \)-span of symmetric tensors in \( D_f \otimes_{\mathcal{A}} D_f \). Suppose \( D_f \) has a triangulation \( 0 \rightarrow \mathcal{A}(\delta_1) \rightarrow D_f \rightarrow \mathcal{A}(\delta_2) \rightarrow 0 \).

Let \( v_1, v_2 \) be a basis for \( D_f \) with \( v_1 \) spanning \( \mathcal{A}(\delta_1) \). The filtration

\[
0 \subset \mathcal{A}(\delta_1^2) \subset \text{Span}_{\mathcal{A}}(v_1 \otimes v_1) \subset \text{Span}_{\mathcal{A}}(v_1 \otimes v_1, v_1 \otimes v_2 + v_2 \otimes v_1)
\]

then exhibits \( (\delta_1^2, \delta_1 \delta_2, \delta_2^2) \) as an element of \( \mathcal{P}\text{ar}(\text{sym}^2 \rho_f) \).

Since \( \text{sym}^2 \rho_f \) is assumed irreducible, \( \rho_f \) is neither reducible or dihedral up to twist. If \( f \) is of type 1 or 3b, \( f \) defines a point \( x_f = x(\rho_f, \delta_f) \in \mathcal{C}_0^\text{cm}(\tau) \) (with notation as in §5.4 and §6.3), and the point \( s(x_f) \in \mathcal{A}_{\text{GL}_2}(\mathbb{Q}, \mathcal{A}) \) is the point predicted by Conjecture 1.2.4. If \( f \) is of type 2 or 3a with companion form \( g \), we take a suitable twist of \( s(x_g) \). □

**Proof of Theorem 1.2.8.** Notation as in the theorem, we first demonstrate that the claimed characters are parameters. Suppose \( D_f = D(\rho_f|G_{Q_p}) \) and \( D_g = D(\rho_g|G_{Q_p}) \) admit triangulations

\[
0 \rightarrow \mathcal{A}(\delta_{f,1}) \rightarrow D_f \rightarrow \mathcal{A}(\delta_{f,2}) \rightarrow 0
\]

and likewise for \( D_g \). Let \( v_1, v_2 \) be a basis for \( D_f \) with \( v_1 \) generating \( \mathcal{A}(\delta_{f,1}) \), and let \( w_1, w_2 \) be an analogous basis for \( D_g \). The character \( \delta_f \otimes \delta_g \) is then the parameter associated with the triangulation

\[
0 \subset \mathcal{A}(\delta_{f,1}) \otimes \mathcal{A}(\delta_{g,1}) \subset \mathcal{A}(\delta_{f,1}) \otimes D_g \subset \text{Span}_{\mathcal{A}}(v_1 \otimes w_1, v_1 \otimes w_2, v_2 \otimes w_1) \subset D_f \otimes D_g,
\]

and \( \delta_g \otimes \delta_f \) corresponds to the triangulation

\[
0 \subset \mathcal{A}(\delta_{f,1}) \otimes \mathcal{A}(\delta_{g,1}) \subset \mathcal{A}(\delta_{g,1}) \otimes D_f \subset \text{Span}_{\mathcal{A}}(v_1 \otimes w_1, v_2 \otimes w_1, v_1 \otimes w_2) \subset D_f \otimes D_g,
\]

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Notation as in §5.5, if \( f, g \) are both of type 1 and/or type 3b then \( t(x_f, x_g) \) and \( t(x_g, x_f) \) (with \( x_f, x_g \) as in the previous proof) are exactly the points \( x(\rho_f \otimes \rho_g, \delta_f \boxtimes \delta_g) \) and \( x(\rho_f \otimes \rho_g, \delta_g \boxtimes \delta_f) \). The general case is similar: the points we seek are given by twisting the points obtained by applying \( t \) to the companion points of \( f \) and/or \( g \). □

A Some commutative algebra

In this appendix we collect some results relating the projective dimension of a module \( M \) and its localizations, the nonvanishing of certain \( \text{Tor} \) and \( \text{Ext} \) groups, and the heights of the associated primes of \( M \). We also briefly recall the definition of a perfect module, and explain their basic properties. These results are presumably well-known to experts, but they are not given in our basic reference [Mat89].

Throughout this subsection, \( R \) is a commutative Noetherian ring and \( M \) is a finite \( R \)-module. Our notations follow [Mat89], with one addition: we write \( \text{mSupp}(M) \) for the set of maximal ideals in \( \text{Supp}(M) \).

**Proposition A.1.** There is an equivalence

\[
\text{projdim}_R(M) \geq n \iff \text{Ext}^n_R(M, N) \neq 0 \text{ for some } N \in \text{Mod}_R.
\]

See e.g. p. 280 of [Mat89] for a proof.

**Proposition A.2.** The equality

\[
\text{projdim}_R(M) = \sup_{m \in \text{mSupp}(M)} \text{projdim}_{R_m}(M_m)
\]

holds.

**Proof.** Any projective resolution of \( M \) localizes to a projective resolution of \( M_m \), so \( \text{projdim}_{R_m}(M_m) \leq \text{projdim}_R(M) \) for all \( m \). On the other hand, if \( \text{projdim}_R(M) \geq n \), then \( \text{Ext}^n_R(M, N) \neq 0 \) for some \( N \), so \( \text{Ext}^n_R(M, N)_m \neq 0 \) for some \( m \); but \( \text{Ext}^n_R(M, N)_m \simeq \text{Ext}^n_R(M_m, N_m) \), so \( \text{projdim}_{R_m}(M_m) \geq n \) for some \( m \) by Proposition A.1. □

**Proposition A.3.** For \( M \) any finite \( R \)-module, the equality

\[
\text{projdim}_R(M) = \sup_{m \in \text{mSupp}(M)} \sup \{ i | \text{Tor}^R_i(M, R/m) \neq 0 \}
\]

holds. If furthermore \( \text{projdim}_R(M) < \infty \) then the equality

\[
\text{projdim}_R(M) = \sup \{ i | \text{Ext}^R_i(M, R) \neq 0 \}
\]

holds as well.

**Proof.** The module \( \text{Tor}^R_i(M, R/m) \) is a finite-dimensional \( R/m \)-vector space, so localization at \( m \) leaves it unchanged, yielding

\[
\text{Tor}^R_i(M, R/m) \simeq \text{Tor}^R_i(M, R/m)_m
\]

\[
\simeq \text{Tor}^R_i(M_m, R_m/m).
\]

Since the equality \( \text{projdim}_S(N) = \sup \{ i | \text{Tor}^R_i(N, S/m_S) \neq 0 \} \) holds for any local ring \( S \) and any finite \( S \)-module \( N \) (see e.g. Lemma 19.1.ii of [Mat89]), the first claim now follows from Proposition A.2.

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For the second claim, we first note that if \( S \) is a local ring and \( N \) is a finite \( S \)-module with \( \text{projdim}_S(N) < \infty \), then \( \text{projdim}_S(N) = \sup\{i|\Ext^i_S(N, S) \neq 0\} \) by Lemma 19.1.iii of [Mat89]. Hence by Proposition A.2 we have

\[
\text{projdim}_R(M) = \sup_{m \in \text{Supp}(M)} \sup \{i|\Ext^i_{R_m}(M_m, R_m) \neq 0\}
= \sup \{i|\Ext^i_R(M, R)_m \neq 0 \text{ for some } m\}
= \sup \{i|\Ext^i_R(M, R) \neq 0\},
\]

as desired. □

**Proposition A.4.** If \( R \) is a Cohen-Macaulay ring, \( M \) is a finite \( R \)-module of finite projective dimension, and \( p \) is an associated prime of \( M \), then \( \text{ht}p = \text{projdim}_{R_p}(M_p) \). In particular, \( \text{ht}p \leq \text{projdim}_R(M) \).

**Proof.** Supposing \( p \) is an associated prime of \( M \), there is an injection \( R/p \hookrightarrow M \); this localizes to an injection \( R_p/p \hookrightarrow M_p \), so \( \text{depth}_{R_p}(M_p) = 0 \). Now we compute

\[
\text{ht}p = \dim(R_p)
= \text{depth}_{R_p}(R_p) \text{ (by the CM assumption)}
= \text{depth}_{R_p}(M_p) + \text{projdim}_{R_p}(M_p) \text{ (by the Auslander – Buchsbaum formula)}
= \text{projdim}_{R_p}(M_p),
\]

whence the result. □

Now we single out an especially nice class of modules, which are equidimensional in essentially every sense of the word. Recall the *grade* of a module \( M \), written \( \text{grade}_R(M) \), is the \( \text{ann}(R)-\text{depth} \) of \( R \); by Theorems 16.6 and 16.7 of [Mat89],

\[
\text{grade}_R(M) = \inf \{i|\Ext^i_R(M, R) \neq 0\},
\]

so quite generally \( \text{grade}_R(M) \leq \text{projdim}_R(M) \).

**Definition A.5.** A finite \( R \)-module \( M \) is perfect if \( \text{grade}_R(M) = \text{projdim}_R(M) < \infty \).

**Proposition A.6.** Let \( R \) be a Noetherian ring, and let \( M \) be a perfect \( R \)-module, with \( \text{grade}_R(M) = \text{projdim}_R(M) = d \). Then for any \( p \in \text{Supp}(M) \) we have \( \text{grade}_{R_p}(M_p) = \text{projdim}_{R_p}(M_p) = d \). If furthermore \( R \) is Cohen-Macaulay, then \( M \) is Cohen-Macaulay as well, and every associated prime of \( M \) has height \( d \).

**Proof.** The grade of a module can only increase under localization (as evidenced by the Ext definition above), while the projective dimension can only decrease; on the other hand, \( \text{grade}_R(M) \leq \text{projdim}_R(M) \) for any finite module over any Noetherian ring. This proves the first claim.

For the second claim, Theorems 16.6 and 17.4.i of [Mat89] combine to yield the formula

\[
\dim(M_p) + \text{grade}_{R_p}(M_p) = \dim(R_p)
\]

for any \( p \in \text{Supp}(M) \). The Auslander-Buchsbaum formula reads

\[
\text{depth}_{R_p}(M_p) + \text{projdim}_{R_p}(M_p) = \text{depth}_{R_p}(R_p).
\]

But \( \dim(R_p) = \text{depth}_{R_p}(R_p) \) by the Cohen-Macaulay assumption, and \( \text{grade}_{R_p}(M_p) = \text{projdim}_{R_p}(M_p) \) by the first claim. Hence \( \text{depth}_{R_p}(M_p) = \dim(M_p) \) as desired. The assertion regarding associated primes is immediate from the first claim and Proposition A.4. □
B The dimension of irreducible components

by James Newton

In this appendix we use the results of the above article to give some additional evidence for Conjecture 1.1.4. In the notation and terminology of Section 1 above, we prove

Proposition B.1. Any irreducible component of \( \mathcal{H}_{G,K^p} \) containing a given point \( x \) has dimension at least \( \dim(\mathcal{H}_{K^p}) - l(x) \).

Note that Proposition 5.7.4 of [Urb11] implies that at least one of these components has dimension at least \( \dim(\mathcal{H}_{K^p}) - l(x) \). This is stated without proof in that reference, and is due to G. Stevens and E. Urban. We learned the idea of the proof of this result from E. Urban — in this appendix we adapt that idea and make essential use of Theorem 3.3.1 (in particular the 'Tor spectral sequence') to provide a fairly simple proof of Proposition B.1.

We place ourselves in the setting of Sections 1 and 4.3. In particular, \( G \) is a reductive group over \( \mathbb{Q} \). Fix an open compact subgroup \( K^p \subseteq G(A_F^p) \) and a slope datum \( (U_\ell, \Omega, h) \). Suppose that \( \mathfrak{M} \) is a maximal ideal of \( \mathfrak{T}_{\Omega,h}(K^p) \) corresponding to a point \( x \in \mathcal{X}_{G,K^p}(\mathbb{Q}_p) \). Denote by \( m \) the contraction of \( \mathfrak{M} \) to \( \mathcal{O}(\Omega) \). Let \( \mathcal{P} \) be a minimal prime of \( \mathfrak{T}_{\Omega,h}(K^p) \) contained in \( \mathfrak{M} \). Since \( H^*(K^p, \mathcal{P}) \) is a finite faithful \( \mathfrak{m} \)-dimension at least \( \mathfrak{m} \)-dimension of \( \mathfrak{m} \), \( \mathfrak{m} \) elements of \( \mathfrak{m} \) is non-zero.

By induction, it suffices to prove the following: let \( i \) be an integer satisfying \( 0 \leq i \leq d-1 \). Suppose

\[ H^{r-i}(K^p, \mathcal{X}_{\Sigma_i}) \leq h, \mathcal{P} \]

is a non-zero \( A_i \)-module, with \( \mathfrak{m} A_i \) an associated prime, and

\[ H^i(K^p, \mathcal{X}_{\Sigma_i}) \leq h, \mathcal{P} = 0 \]

We will show that \( d \leq l(x) \).

Denote by \( A_i \) the quotient \( \mathcal{O}(\Omega)_{2i}/(x_1, ..., x_i) \mathcal{O}(\Omega)_{2i} \) and denote by \( \Sigma_i \) the Zariski closed subspace of \( \Omega \) defined by the ideal \( (x_1, ..., x_i) \mathcal{O}(\Omega) \). The affinoids \( \Sigma_i \) may be non-reduced. Note that \( A_i = \mathcal{O}(\Sigma_i)_{2i} \) and \( \mathcal{O}(\Sigma_{i+1}) = \mathcal{O}(\Sigma_i)/x_{i+1} \mathcal{O}(\Sigma_i) \).

Lemma B.3. The space

\[ H^{r-d}(K^p, \mathcal{X}_{\Sigma_d}) \leq h, \mathcal{P} \]

is non-zero.

Proof. By induction, it suffices to prove the following: let \( i \) be an integer satisfying \( 0 \leq i \leq d-1 \). Suppose

\[ H^{r-i}(K^p, \mathcal{X}_{\Sigma_i}) \leq h, \mathcal{P} \]

is a non-zero \( A_i \)-module, with \( \mathfrak{m} A_i \) an associated prime, and

\[ H^i(K^p, \mathcal{X}_{\Sigma_i}) \leq h, \mathcal{P} = 0 \]
for every $t < r - i$. Then

$$H^{r-i-1}(K^p, \mathcal{D}_{\Sigma_i+1}) \leq h, \mathcal{P}$$

is a non-zero $A_{i+1}$-module, with $\wp A_{i+1}$ an associated prime, and

$$H^t(K^p, \mathcal{D}_{\Sigma_i+1}) \leq h, \mathcal{P} = 0$$

for every $t < r - i - 1$.

Note that the hypothesis of this claim holds for $i = 0$, by the minimality of $r$. Suppose the hypothesis is satisfied for $i$. It will suffice to show that

- $H^t(K^p, \mathcal{D}_{\Sigma_i+1}) \leq h, \mathcal{P} = 0$ for every $t < r - i - 1$
- there is an isomorphism of non-zero $A_i$-modules
  $$\iota : \text{Tor}^A_{1}(H^{r-i}(K^p, \mathcal{D}_{\Sigma_i}) \leq h, \mathcal{P}, A_i/x_{i+1}A_i) \cong H^{r-i-1}(K^p, \mathcal{D}_{\Sigma_i+1}) \leq h, \mathcal{P}.$$  

Indeed, the left hand side (which we denote by $T$) of the isomorphism $\iota$ is given by the $x_{i+1}$-torsion in $H^{r-i}(K^p, \mathcal{D}_{\Sigma_i}) \leq h, \mathcal{P}$, so a non-zero $A_i$-submodule of $H^{r-i}(K^p, \mathcal{D}_{\Sigma_i}) \leq h, \mathcal{P}$ with annihilator $\wp A_i$ immediately gives a non-zero $A_{i+1}$-submodule of $T$ with annihilator $\wp A_{i+1}$.

Both the claimed facts are shown by studying the localisation at $\mathcal{P}$ of the spectral sequence

$$E_2^{s,t} : \text{Tor}^{A_{\Sigma_i}}(H^{t}(K^p, \mathcal{D}_{\Sigma_i}) \leq h, A(\Sigma_{i+1})) \Rightarrow H^{s+t}(K^p, \mathcal{D}_{\Sigma_{i+1}}) \leq h$$

(cf. Remark 3.3.2). After localisation at $\mathcal{P}$, the spectral sequence degenerates at $E_2$. This is because we have a free resolution

$$0 \to A_i \xrightarrow{x_{i+1}} A_i \to A_{i+1} \to 0$$

of $A_{i+1}$ as an $A_i$-module (we use the fact that $x_{i+1}$ is not a zero-divisor in $A_i$), so $(E_2^{s,t})_{\mathcal{P}}$ vanishes whenever $s \notin \{-1,0\}$. Moreover, since

$$H^t(K^p, \mathcal{D}_{\Sigma_i}) \leq h, \mathcal{P} = 0$$

for every $t < r - i$, we know that $(E_2^{s,t})_{\mathcal{P}}$ vanishes for $t < r - i$. The existence of the isomorphism $\iota$ and the desired vanishing of $H^t(K^p, \mathcal{D}_{\Sigma_{i+1}}) \leq h, \mathcal{P}$ are therefore demonstrated by the spectral sequence, since the only non-zero term $(E_2^{s,t})_{\mathcal{P}}$ contributing to $(E_2^{-1,-1})_{\mathcal{P}}$ is given by $s = -1, t = r - i$, whilst $(E_2^{s,t})_{\mathcal{P}} = 0$ for all $(s,t)$ with $s + t < r - i - 1$. $\square$

**Corollary B.4.** We have an inequality $r - d \geq q$. Since $r \leq q + l$ we obtain $d \leq l$. In particular $\wp$ has height $\leq l$, so the irreducible component of $T_{\Omega,h}(K^p)$ corresponding to $\mathcal{P}$ has dimension $\geq \dim(\Omega) - l$.

**Proof.** It follows from Proposition 4.5.2 (with $\Omega$ replaced by $\Sigma_d$) that

$$H^t(K^p, \mathcal{D}_{\Sigma_d}) \leq h, \mathcal{P}$$

is zero for $t < q$. Our Lemma therefore implies that $r - d \geq q$. The conclusion on dimensions follows from the observation made in Section 4.5 that $T_{\Omega,h}(K^p)/\mathcal{P}$ has the same dimension as $\mathcal{O}(\Omega)/\wp$. $\square$

Proposition B.1 follows immediately from the Corollary. We have also shown that if $d = l$, then $r = q + l$.  

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