Velocity, Acceleration and Cosmic Distances in Cosmological Special Relativity

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Abstract

In this paper we present the fundamentals of the cosmological special relativity (CSR) [1-3] by discussing the dynamical concepts of velocity, acceleration and cosmic distances in spacevelocity. These concepts occur in CSR just as those of mass, linear momentum and energy appear in Einstein’s special relativity (ESR) of spacetime (see Chapter 7 of Ref. 1).
I. PRELIMINARIES

The most important result of Einstein’s special relativity is probably the relationship between mass and energy (see Chapter 7 of Ref. 1). How it happened that the mass became so critical in this theory? The answer is very simple. The theory involves the square of the speed of light, \( c^2 \). What physical quantity incorporates the square of velocity? It is the energy, \( mv^2/2 \). Hence it is the mass that goes with \( v^2 \) and \( c^2 \). Thus \( m \) is taken as an invariant under the Lorentz transformation. This becomes the rest mass \( m_0 \). The inertial mass \( m \) follows to depend on the velocity.

What is the comparable physical quantity in cosmological special relativity? Certainly not the mass. Here we have \( \tau^2 \), the square of the Hubble time constant. What physical quantity goes with the square of time? It is the acceleration \( \mathbf{a} \) because \( \mathbf{a}t^2/2 \) describes the distance a particle makes at time \( t \) when it is subject to acceleration \( \mathbf{a} \). Hence \( \mathbf{a} \) should be taken as the invariant quantity under the cosmological transformation. But then the acceleration depends on the cosmic time just as the mass depends on the velocity.

In this paper these concepts are explored.

II. VELOCITY AND ACCELERATION FOUR-VECTORS

We start our four-dimensional spacevelocity analysis by defining the velocity and acceleration (see Section 5.5 of Ref. 1 for the special relativistic treatment in spacetime).

The \emph{velocity} four-vector of a particle in spacevelocity is defined as the dimensionless quantity

\[
\mathbf{u}^\mu = \frac{dx^\mu}{ds},
\]

where \( \mu = 0, 1, 2, 3, x^\mu = (x^0, x^1, x^2, x^3) = (\tau v, x, y, z) \), and \( \tau \) is Hubble’s time in the limit of zero gravity, a universal constant whose value is \( 12.5 \times 10^9 \) years \([4,5]\). In flat spacevelocity one has for the line element,

\[
\tau^2 dv^2 - \left( dx^2 + dy^2 + dz^2 \right) = ds^2,
\]
thus
\[ \tau^2 \left( \frac{dv}{ds} \right)^2 \left( 1 - \frac{dv^2 + dy^2 + dz^2}{\tau^2 dv^2} \right) = 1. \] (3)

This gives
\[ \tau^2 \left( \frac{dv}{ds} \right)^2 \left( 1 - \frac{t^2}{\tau^2} \right) = 1, \] (4)

and therefore
\[ \frac{dv}{ds} = \frac{1}{\tau \sqrt{1 - \frac{t^2}{\tau^2}}}. \] (5)

The velocity four-vector in spacevelocity can thus be expressed as
\[ u^\mu = \frac{dx^\mu}{ds} = \frac{dx^\mu}{dv} \frac{dv}{ds} = \frac{1}{\tau \sqrt{1 - \frac{t^2}{\tau^2}}} \frac{dx^\mu}{dv}. \] (6)

The velocitylike component of \( u^\mu \) is therefore given by
\[ u^0 = \gamma, \] (7)

whereas its spatial components
\[ \mathbf{u} = u^k = (u^1, u^2, u^3) \] (8)
are given by
\[ u^k = \frac{\gamma}{\tau} \frac{dx^k}{dv}, \] (9)
where
\[ \gamma = \frac{1}{\sqrt{1 - \frac{t^2}{\tau^2}}}. \] (10)

It will be noted that, by Eq. (1),
\[ u_\alpha u^\alpha = 1, \] (11)

namely, the length of \( u^\mu \) is unity.

The acceleration four-vector of a particle in spacevelocity is defined by
\[ \frac{du^\mu}{ds} = \frac{d^2 x^\mu}{ds^2}. \] (12)
By differentiating Eq. (11) we find that the acceleration four-vector satisfies the orthogonality condition
\[ u^\alpha \frac{du^\alpha}{ds} = 0. \] (13)

The components of the acceleration four-vector of a particle in spacevelocity are, by Eqs. (5) and (7)–(9), then given by
\[ \frac{du^0}{ds} = \frac{\gamma d\gamma}{\tau dv}, \] (14)
\[ \frac{du^k}{ds} = \frac{\gamma d x^k}{\tau^2 dv} \left( \frac{d\gamma}{dv} \right), \] (15)
where \( \gamma \) is given by Eq. (10).

**III. ACCELERATION AND COSMIC DISTANCES**

Multiplying Eq. (4) by \( a_0^2 \tau^4 \), where \( a_0 \) is the ordinary three-vector acceleration as measured in the cosmic frame of reference (see Section 2.5 of Ref. 1) at cosmic time \( t = 0 \) (i.e. now). We obtain
\[ a_0^2 \tau^6 \left( \frac{dv}{ds} \right)^2 \left( 1 - \frac{t^2}{\tau^2} \right) = a_0^2 \tau^4. \] (16)

Using now Eq. (5) in Eq. (16) the latter then yields
\[ \left( 1 - \frac{t^2}{\tau^2} \right)^{-1} \left( a_0^2 \tau^4 - a_0^2 t^2 \tau^2 \right) = a_0^2 \tau^4. \] (17)

We now define the acceleration \( a \) at an arbitrary cosmic time \( t \) by
\[ a = \frac{a_0}{\sqrt{1 - \frac{t^2}{\tau^2}}}, \] (18)
then Eq. (17) will have the form
\[ a^2 \tau^4 - a^2 t^2 \tau^2 = a_0^2 \tau^4, \] (19)
or
\[ a^2 \tau^4 - \tau^2 v^2 = a_0^2 \tau^4, \] (20)
where \( \mathbf{v} = \mathbf{a} t \) is the ordinary three-dimensional velocity. Equation (20) is the analog to

\[
m^2 c^4 - c^2 \mathbf{p}^2 = m_0^2 c^4, \tag{21}
\]
in ESR, where \( \mathbf{p} = m \mathbf{v} \) is the linear momentum (see Section 7.2 of Ref. 1). Thus \( \mathbf{a} \) reduces to \( \mathbf{a}_0 \) when the cosmic time \( t = 0 \) (i.e. at present).

The comparable to Eq. (18) in ESR is, of course,

\[
m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{22}
\]
where \( m \) and \( m_0 \) are the inertial mass and the rest mass of the particle, with \( m \) reduces to \( m_0 \) at \( v = 0 \). If we multiply Eq. (22) by \( c^2 \) and expand both sides in \( v/c \), we obtain

\[
m c^2 = m_0 c^2 + \frac{m_0}{2} v^2 + \cdots \tag{23}
\]
Doing the same with Eq. (18) but multiplication by \( \tau^2 \), and expanding in \( t/\tau \), we obtain

\[
\mathbf{a} \tau^2 = \mathbf{a}_0 \tau^2 + \frac{\mathbf{a}_0}{2} t^2 + \cdots \tag{24}
\]
Equation (24) in CSR is of course the analog to Eq. (23) in ESR.

**IV. ENERGY IN ESR VERSUS COSMIC DISTANCE IN CSR**

While Eq. (23) yields

\[
E = E_0 + \frac{m_0}{2} v^2 + \cdots \tag{25}
\]
Eq. (24) gives

\[
S = S_0 + \frac{\mathbf{a}_0}{2} t^2 + \cdots \tag{26}
\]
In the above equations \( E \) is, of course, the energy of the particle, whereas \( S \) is the cosmic distance. \( E_0 = m_0 c^2 \) is the rest energy of the particle whereas \( m_0 v^2/2 \) is the Newtonian kinetic energy. What about the terms in Eq. (26)? The term \( \mathbf{a}_0 t^2/2 \) is, of course, the Newtonian distance the particle makes due to the acceleration \( \mathbf{a}_0 \), the term \( S_0 = \mathbf{a}_0 \tau^2 \) is unique to CSR, and it might be called the *intrinsic* cosmic distance of the particle.
Equation (20) can now be written as

$$S^2 - \tau^2 v^2 = S_0^2,$$  \hspace{1cm} (27)

in complete analogy to

$$E^2 - c^2 p^2 = E_0^2$$  \hspace{1cm} (28)

in ESR.

V. COSMIC DISTANCE-VELOCITY FOUR-VECTOR

We now define the cosmic distance-velocity four-vector. It is the analogous to the energy-momentum four-vector in ESR (see Section 7.4 of Ref. 1). It is defined by

$$v^\mu = a_0 \tau^2 u^\mu,$$  \hspace{1cm} (29)

where \( u^\mu \) has been defined in Section 2. We have

$$v^0 = a_0 \tau^2 u^0,$$  \hspace{1cm} (30a)

$$v^k = a_0 \tau^2 u^k,$$  \hspace{1cm} (30b)

where \( u^0 \) and \( u^k \) are given by Eqs. (7)–(10), with

$$v_0 = v^0, \quad v_k = -v^k.$$  \hspace{1cm} (31)

Accordingly we have

$$v^2 = v_0 \cdot v^0 + v_k \cdot v^k = a_0^2 \tau^4 \left( u_0 u^0 + u_k u^k \right)$$

$$= a_0^2 \tau^4 u_\alpha u^\alpha = a_0^2 \tau^4 = S_0^2.$$  \hspace{1cm} (32)

But, using Eqs. (7), (9) and (18),

$$v^0 = a_0 \tau^2 \gamma = a \tau^2 = S,$$  \hspace{1cm} (33a)

$$v^k = a_0 \tau \gamma \frac{dx^k}{dv} = a \tau \frac{dx^k}{dv} = a \tau \frac{dx^k}{dt} \frac{dt}{dv} = \tau v,$$  \hspace{1cm} (33b)
where \( \mathbf{v} \) is the three-dimensional velocity. Hence

\[
\mathbf{v}_\alpha \cdot \mathbf{v}^\alpha = \mathbf{v}_0 \cdot \mathbf{v}^0 + \mathbf{v}_k \cdot \mathbf{v}^k = a_0^2 \tau^4 \gamma^2 - a_0^2 \tau^2 \gamma^2 \left( \frac{d\tau_k}{d\tau} \right)^2
\]

\[
= a^2 \tau^4 - \tau^2 v^2 = S^2 - \tau^2 v^2,
\]

and accordingly, using (32),

\[
S^2 - \tau^2 v^2 = S_0^2,
\]

which is exactly Eq. (20) with \( S = a \tau^2 \) and \( S_0 = a_0 \tau^2 \). The above analysis also shows that Eq. (35) is covariant under spacevelocity cosmological transformation.

Equation (35) is the analog to

\[
E^2 - c^2 p^2 = E_0^2
\]

in ESR, where \( E \) and \( E_0 \) are the energy and rest energy, respectively, \( S \) and \( S_0 \) are the cosmic distances at cosmic time \( t \) and present time \( (t = 0) \), respectively.

Finally, in ESR when the rest mass is zero (like the photon), one then has

\[
E = cp,
\]

which is valid at the light cone (see Chapter 6 of Ref. 1). In the case of CSR one also has, when \( S_0 = 0 \),

\[
S = \tau v,
\]

now valid at the galaxy cone (see Section 2.14 of Ref. 1).

VI. CONCLUSIONS

In this paper it has been shown that the comparable quantity to the mass in Einstein’s special relativity is the ordinary acceleration three-vector in cosmological special relativity. They both have similar behavior, one with respect to \( v/c \) and the other with \( t/\tau \):

\[
m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}},
\]

7
and

\[ a = \frac{a_0}{\sqrt{1 - \frac{t^2}{\tau^2}}}, \quad (40) \]

Furthermore, the role of the energy in Einstein’s theory is being taken over by the cosmic distance,

\[ E = mc^2, \quad (41) \]

and

\[ S = a \tau^2. \quad (42) \]

Finally, the analog of the energy formula

\[ E^2 - c^2p^2 = E_0^2 \quad (43) \]

in ordinary special relativity is

\[ S^2 - \tau^2v^2 = S_0^2 \quad (44) \]

in cosmological special relativity.
REFERENCES

[1] M. Carmeli, *Cosmological Special Relativity: The Large Scale Structure of Space, Time and Velocity* (World Scientific, River Edge, N.J., 1997).

[2] M. Carmeli, Cosmological relativity: a special relativity for cosmology, *Foundations of Physics* **25**, 1029 (1995).

[3] M. Carmeli, Cosmological special relativity, *Foundations of Physics* **26**, 413 (1996).

[4] M. Carmeli and T. Kuzmenko, Value of the cosmological constant: theory versus experiment, in: *Proceedings of the 20th Texas Symposium on Relativistic Astrophysics*, held 10-15 December 2000, Austin, Texas, H. Martel and J.C. Wheeler, Editors (American Institute of Physics, 2001).

[5] M. Carmeli, Accelerating universe, cosmological constant and dark energy, to be published.