Multivariate Subordination using Generalised Gamma Convolutions with Applications to V.G. Processes and Option Pricing*

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Abstract

We unify and extend a number of approaches related to constructing multivariate Variance-Gamma (V.G.) models for option pricing. An overarching model is derived by subordinating multivariate Brownian motion to a subordinator from the Thorin (1977) class of generalised Gamma convolution subordinators. A class of models due to Grigelionis (2007), which contains the well-known Madan-Seneta V.G. model, is of this type, but our multivariate generalization is considerably wider, allowing in particular for processes with infinite variation and a variety of dependencies between the underlying processes. Multivariate classes developed by Pérez-Abreu and Stelzer (2012) and Semeraro (2008) and Guillaume (2013) are also submodels. The new models are shown to be invariant under Esscher transforms, and quite

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explicit expressions for canonical measures (and transition densities in some cases) are obtained, which permit applications such as option pricing using PIDEs or tree based methodologies. We illustrate with best-of and worst-of European and American options on two assets.

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1 Introduction

Madan and Seneta [44] introduced the univariate “Variance Gamma” (V.G.) process as a model for a financial asset price process with a special view to more accurate option pricing on the asset, beyond the standard geometric Brownian motion (GBM) model. The V.G. model has proved to be outstandingly successful in this application, and is in common use by many financial institutions, as an alternative to the GBM model. Madan and Seneta extended the V.G. model [44] to a multi-asset version, again with a view to important applications in finance (“rainbow options”), by subordinating a multivariate Brownian motion with a single univariate Gamma process (also see [17, 18, 19, 58]). Modelling dependence between coordinates was incorporated by correlating the participating Brownian motions, and univariate Variance Gamma processes were obtained as the marginal processes.

Semeraro [57] generalized the multi-asset version of Madan and Seneta [44] to allow for multivariate subordination. This permits the dependence structure between asset prices to be modeled in a more flexible way. The economic intuition behind multivariate subordination is that each asset may have an idiosyncratic risk with its own activity time and a common risk factor, with a joint activity time for all assets. In specific cases it is possible to maintain V.G. processes for each single asset sub model, see [57] and related applications in Luciano and Semeraro [40], [41], [42], though this may be sacrificed for more flexible dependence modeling, as in Guillaume [24].
To summarize, a wide range of multi-asset models based on multivariate Gamma subordination of a Brownian motion has been proposed. However, there are still gaps in the literature concerning the characterization in general of the class of processes generated by Brownian motions subordinated by Gamma processes when the class is required to be stable under summation. Further, for this class theoretical results such as formulae for characteristic functions, Lévy measures and, when possible, transition densities, are needed for a comprehensive description of key properties. Additionally, the link between the real world measure and the pricing measure has been neglected in the literature to date.

The aim of the present paper is to contribute to filling these gaps by presenting a general class of $\mathbb{R}^d$-valued stochastic processes, constructed by subordinating multivariate Brownian motion with a subordinator drawn from a suitable class of multivariate subordinators. Our intention is to lay out a systematic formulation suitable for future development. For the new processes, we provide the formulae mentioned in the previous paragraph and link the real world and pricing measures by calculating Esscher transforms. To illustrate the practical possibilities, we show how the explicit formulae we derive can be used to price American and European multi-asset options. The most general class of subordinators we consider is Thorin’s \cite{61,62} class of generalised Gamma convolutions. We call it the GGC class of subordinators, and the process formed by subordinating Brownian motion in $\mathbb{R}^d$ with such a process we call a Variance Generalised Gamma convolution (VGG) process.

Grigelionis \cite{23} constructed such a VGG-class, which we called $VGG^{d,1}$ in the present paper. The $VGG^{d,1}$ class contains Madan-Seneta’s V.G. as a special case. Complementing Greliionis’ $VGG^{d,1}$ class, we introduce the $VGG^{d,d}$ class of Lévy processes. Our $VGG^{d,d}$ class includes a variety of previously derived models such as Semeraro’s $\alpha$-processes \cite{57} and Guillaume’s process \cite{25}. The general $VGG = VGG^{d,1} \cup VGG^{d,d}$ class extends the V.G. classes in a number of ways. In particular, the $VGG$ classes allow for infinite variation and heavy tails. Figure 1 depicts the connections between the various subordinated classes.

Our subordinated processes are, in particular, multivariate Lévy processes, and we obtain explicit expressions for their canonical measures and characteristic functions as well as transition densities in some special cases.

The $VGG$-class and its subclasses are shown to be invariant under Esscher transformation, so the risk-neutral distribution constructed as the Esscher transformation of a particular member is also in the $VGG$-class. Using those
concepts, we set up a market model and show how an option based on multiple assets may be priced. For illustration we restrict ourselves in this respect to a further subclass of the $VGG$-class which we term the $VM^d$-processes. These have the virtue of allowing a quite general dependency structure between the coordinate processes. As an example, we price best–of and worst–of European and American put options, using a tree-based algorithm.

The paper is organised as follows. Section 2 contains theory. In Subsections 2.1 and 2.2 we introduce $VGG$-classes and discuss existence of (exponential) moments and sample path behaviour. The remaining two subsections in Section 2 derive the Esscher transformation and introduce the subclass of $VM^d$-processes, and in Subsection 2.5 we compare our subordinator class with various others in the literature. Section 3 contains applications. Here the market model is introduced, risk-neutral valuation is discussed, and in Subsection 3.3 we price some cross-dependence sensitive options of both European and American types. Some illustrations of the kinds of dependencies the models allow is also given there. The concluding Section 4 gives an overview and summary of the advantages of our approach. Proofs of the results in Section 2 and some necessary methodological tools are relegated to Section 5, where polar decomposition of measures, subordination and a useful decomposition are briefly covered. The Appendix summarises some formulae concerning Bessel functions and formulae for transition densities for a subclass of the $VM^d$-class.

\[
\begin{align*}
VG & \rightarrow \text{Semeraro-\(\alpha\)} \rightarrow \text{Guillaume} & : & \text{VGG} \\
& \rightarrow \text{VGPAS}
\end{align*}
\]

\textbf{Figure 1: } \textit{Relations between multivariate V.G. classes. Madan-Seneta’s V.G.} \cite{44} \textit{occurs as marginals of Semeraro’s \(\alpha\)-process with inclusion in the univariate case; Semeraro’s} \cite{57} \textit{\(\alpha\)-class, Guillaume’s class} \cite{25}; \textit{VMT} \textit{\(d\) = Variance Matrix Gamma (finitely supported Thorin measures); VGPAS = Variance Gamma process based on Pérez-Abreu and Stelzer} \cite{52}; \textit{VGG-class based on Thorin’s class of GGC-subordinators. \(\rightarrow\) points in the direction of generalisation; \(\cdots\) indicates inclusion in special cases.)}
2 Theory

In this section, we reprise, in Subsection 2.1, the Madan-Seneta V.G. model and set out two major extensions: the Variance-Univariate GGC and the Variance-Multivariate GGC classes. This necessitates recalling, first, some basic facts about Gamma subordinators, and then outlining Thorin's GGC-class. Subsection 2.2 gives some results on the (exponential) moments and sample paths of the new processes, and Subsection 2.3 calculates their Escher transforms, stating the fact that both Variance GGC-classes remain invariant. Subsection 2.4 introduces the Variance-$\Gamma$ subclass on which we base the option pricing model in Section 3. Finally, Subsection 2.5 collects further properties of our subordinator class, including a comparison with those occuring in the literature.

2.1 Variance Generalised Gamma Convolutions (VGG)

Preliminaries. $\mathbb{R}^d$ is the $d$-dimensional Euclidean space $\mathbb{R}^d$; elements of $\mathbb{R}^d$ are column vectors $x = (x_1, \ldots, x_d)'$. Let $\langle x, y \rangle$ denote the Euclidean product, and set $\|x\|_2^2 := \langle x, \Sigma x \rangle$ for $x, y \in \mathbb{R}^d$ and $\Sigma \in \mathbb{R}^{d \times d}$. For $A \subseteq \mathbb{R}^d$ let $A_* = A \setminus \{0\}$. $1_A = 1\{A\}$ denotes indicator function. The Dirac measure with total mass in $x \in \mathbb{R}^d$ is $\delta_x$.

$X = (X_1, \ldots, X_d)' = (X(t))_{t \geq 0}$ is a $d$-dimensional Lévy process if $X$ has independent and stationary increments, $X(0) = 0$ and the sample paths $t \mapsto X(t) \in \mathbb{R}^d$ are càdlàg functions, i.e., are right-continuous with left limits.

The law of a Lévy process $X$ is determined by its characteristic function via $Ee^{i \langle \theta, X(t) \rangle} = \exp\{t \psi_X(\theta)\}$ with Lévy exponent, for $t \geq 0$, $\theta \in \mathbb{R}^d$,

$$\psi_X(\theta) = i \langle \gamma_X, \theta \rangle - \frac{1}{2} \|\theta\|_{\Sigma_X}^2 + \int_{\mathbb{R}_+^d} \left(e^{i \langle \theta, x \rangle} - 1 - i \langle \theta, x \rangle \mathbf{1}_{\|x\| \leq 1}\right) \Pi_X(dx). \tag{2.1}$$

Here $\gamma_X \in \mathbb{R}^d$, $\Sigma_X \in \mathbb{R}^{d \times d}$ is a symmetric and nonnegative matrix, $\Pi_X$ is a nonnegative Borel measure on $\mathbb{R}_+^d = \mathbb{R}^d \setminus \{0\}$ satisfying

$$\int_{\mathbb{R}_+^d} \|x\|^2 \wedge 1 \Pi_X(dx) < \infty, \tag{2.2}$$
and $\| \cdot \|$ is a given norm on $\mathbb{R}^d$. We write $X \sim L^d(\gamma_X, \Sigma_X, \Pi_X)$ whenever $X$ is a $d$-dimensional Lévy process with canonical triplet $(\gamma_X, \Sigma_X, \Pi_X)$.

Paths of $X$ are of (locally) finite variation ($FV^d$) whenever $\Sigma_X = 0$ and

$$\int_{0 < \|x\| \leq 1} \|x\| \Pi_X(dx) < \infty.$$  \hspace{1cm} (2.3)

In this case, we write $X \sim FV^d(D_X, \Pi_X)$ with $D_X$ denoting the drift of $X$:

$$D_X := \gamma_X - \int_{0 < \|x\| \leq 1} x \Pi_X(dx) \in \mathbb{R}^d.$$ 

A $d$-dimensional Lévy process $T$ with nondecreasing components is called a $d$-dimensional subordinator, possibly with drift $D_T$, written $T \sim S^d(D_T, \Pi_T)$. A general Lévy process $X \sim L^d(\gamma_X, \Sigma_X, \Pi_X)$ is a subordinator with drift $D_X$ if and only if $X \sim FV^d(D_X, \Pi_X)$ with $D_X \in [0, \infty)^d$ and $\Pi_X$ being concentrated on $[0, \infty)^d \setminus \{0\}$.

Finally, $B \sim BM^d(\mu, \Sigma) := L^d(\mu, \Sigma, 0)$ refers to a $d$-dimensional Brownian motion $B$ with $E[B(t)] = \mu t$ and covariance matrix $\text{Cov}(B(t)) = t\Sigma$. Brownian motions have continuous sample paths, but with infinite variation.

We write $X \overset{D}{=} Y$ and $X \sim Q$ whenever $\mathcal{L}(X) = \mathcal{L}(Y)$ and $\mathcal{L}(X) = Q$, respectively, where $\mathcal{L}(X)$ denotes the law of a random variable or stochastic process $X$. There is a correspondence between infinitely divisible distributions and Lévy processes $X$: for all $t \geq 0$ the law of $X(t)$, $P(X(t) \in dx)$, is infinitely divisible. Vice versa, any infinitely divisible Borel probability measure $Q$ on $\mathbb{R}^d$ determines uniquely the distribution of a Lévy process via $X(1) \sim Q$. This connection is used throughout the paper. For instance, we write $T \sim Q_S$ to indicate that $T$ is a subordinator with $T(1) \sim Q$.

See [1, 5, 10, 15, 39, 55] for basic properties of Lévy processes and their applications in finance.

**Subordination.** In [4] various kinds of subordination are introduced (see Subsection 5.2 for details). In the present paper, we will make use of two extreme cases: univariate and (strictly) multivariate subordination. Let $X = (X_1, \ldots, X_d)'$ be a $d$-dimensional Lévy process. $X$ serves as the subordinate.

Given a univariate subordinator $T$, independent of $X$, define a $d$-dimensional Lévy process, denoted $X \circ_{d,1} T$, by setting

$$(X \circ_{d,1} T)(t) := (X_1(T(t)), \ldots, X_d(T(t)))', \quad t \geq 0.$$ \hspace{1cm} (2.4)
In the sequel, we denote the law of $X \circ_d T$ by $\mathcal{L}(X) \circ_d \mathcal{L}(T)$. We refer to this type as univariate subordination (cf. Section 6 in [55]).

Suppose $X$ has independent components $X_1, \ldots, X_d$. Let $T = (T_1, \ldots, T_d)'$ be a $d$-dimensional subordinator, independent of $X$, and define a $d$-dimensional Lévy process by setting

$$X \circ_{d,d} T := (X_1 \circ_{1,1} T_1, \ldots, X_d \circ_{1,1} T_d)'.$$  \hspace{1cm} (2.5)

The law of $X \circ_{d,d} T$ is denoted by $\mathcal{L}(X) \circ_{d,d} \mathcal{L}(T)$.

**Remark 2.1.** When dealing with strictly multivariate subordination, we have to restrict the class of admissible subordinates $X$ to Lévy processes with independent components. This is necessary if we are to stay in the class of Lévy processes. For instance, let $B \sim BM_1(0,1)$ be a univariate standard BM. Then $X = (B, B)'$ is a Lévy process, but $t \mapsto (B(t), B(2t))'$ is not.

**Gamma subordinator.** Denote by $\Gamma(\alpha, \beta)$ a Gamma distribution with parameters $\alpha, \beta > 0$, i.e., a Borel probability measure having Lebesgue density

$$d\Gamma(\alpha, \beta) dx(x) = \mathbf{1}\{x > 0\} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \quad x \in \mathbb{R}.$$ \hspace{1cm} (2.6)

We write $G \sim \Gamma_S(\alpha, \beta)$ for a Gamma process $G = (G(t))_{t \geq 0}$ with parameters $\alpha, \beta > 0$, that is, $G$ is a univariate subordinator having marginal distributions $G(t) \sim \Gamma(\alpha t, \beta)$, $t > 0$. If $\alpha = \beta$ then $G$ is called a standard Gamma process: $G \sim \Gamma_S(\alpha) = \Gamma_S(\alpha, \alpha)$.

A Gamma process has zero drift, and its Lévy measure admits the following Lebesgue density (cf. p.16 & p.73 in [5]):

$$d\Pi_G (dr)(r) = \mathbf{1}\{r > 0\} \frac{\alpha}{\beta + \lambda} r^{-\alpha} e^{-\beta r}/r , \quad r \neq 0.$$ \hspace{1cm} (2.7)

Further, for $\lambda > -\beta$, $t > 0$, it follows from (2.7) that

$$E e^{-\lambda G(t)} = \left\{ \frac{\beta}{\beta + \lambda} \right\}^t \exp \left\{ -t \int_0^\infty (1 - e^{-\lambda r}) \frac{\alpha e^{-\beta r} dr}{r} \right\}.$$ \hspace{1cm} (2.8)

In (2.8), the first formula is well known, whereas the second identity follows from (2.7), also known as the Frullani integral (cf. [5], p.73). Note that

$$\int_{0<r<1} r^{1/2} \Pi_G (dr) < \infty.$$ \hspace{1cm} (2.9)
We collect some properties of the Gamma distribution into a lemma. In Part (a) we state the familiar scaling invariance of the Gamma distribution. Part (b) illustrates the fact that the class of Gamma distributions is not closed under convolutions (see Subsection 5.3 for proof).

**Lemma 2.1.** Let \(c, \alpha, \beta, \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n > 0\). Let \(Z \sim \Gamma(\alpha, \beta)\) and \(Z_k \sim \Gamma(\alpha_k, \beta_k)\) for all \(1 \leq k \leq n\) be independent.

(a) \(cZ \sim \Gamma(\alpha, \beta/c)\); and

(b) (i) and (ii) are equivalent, where:

- (i) for all \(2 \leq k \leq n\), \(\beta_k = \beta_1\);
- (ii) there are \(a, b > 0\) such that \(\sum_{k=1}^{n} Z_k \sim \Gamma(a, b)\).

If (i) or (ii) is satisfied then \(b = \beta_1\) and \(a = \sum_{k=1}^{n} \alpha_k\).

**Madan-Seneta V.G. Process.** Madan and Seneta [44] (for extensive investigations and reviews cf. [17, 18, 19, 37, 38, 43, 58]) suggest subordinating Brownian motion with a Gamma process. For the parameters of this model we assume \(\mu \in \mathbb{R}^d\), \(b > 0\) and \(\Sigma \in \mathbb{R}^{d \times d}\), with \(\Sigma\) being symmetric and nonnegative definite.

Let \(B \sim BM^d(\mu, \Sigma)\) be a \(d\)-dimensional Brownian motion and \(G \sim \Gamma_S(b)\) be independent of \(B\). A Lévy process \(Y\) is a \(d\)-dimensional Variance Gamma \((VG^d)\) process with parameters \(b, \mu, \Sigma\) whenever \(Y \overset{D}{=} B \circ_{d,1} G\), which we write as

\[
Y \sim VG^d(b, \mu, \Sigma) := BM^d(\mu, \Sigma) \circ_{d,1} \Gamma_S(b, b). \tag{2.10}
\]

(a) Note that a V.G. process has zero drift and is of finite variation.

(b) The Laplace transformation of \(Y\) takes on an explicit form, straightforwardly derived from conditioning:

\[
E \exp\{-\langle \lambda, Y(t) \rangle\} = \int_{(0, \infty)} \exp \left\{ r \left( \frac{1}{2} \|\lambda\|_\Sigma^2 - \langle \mu, \lambda \rangle \right) \right\} \Gamma(tb, b)(dr)
\]

\[
= \left\{ \frac{b}{b + \langle \mu, \lambda \rangle - \frac{1}{2} \|\lambda\|_\Sigma^2} \right\}^{bt}, \tag{2.11}
\]

for \(t \geq 0\) and \(\lambda \in \mathbb{R}^d\) with \(\frac{1}{2} \|\lambda\|_\Sigma^2 - \langle \mu, \lambda \rangle < b\).

(c) If \(\Sigma\) is invertible, explicit formulae for the transition probability density
and the Lévy density \( f_Y(t) \) can be given for \( t > 0 \), as follows:

\[
f_Y(t)(y) = \frac{2^{(2-d)/2} b^t \exp\{\langle \Sigma^{-1} \mu, y \rangle \}}{\pi^{d/2} (\det \Sigma)^{1/2} \Gamma(bt)} \times \left( \frac{\|y\|_{\Sigma^{-1}}^2}{2b + \|\mu\|_{\Sigma^{-1}}^2} \right)^{(2bt-d)/4} \times K_{(2bt-d)/2} \left( \sqrt{(2b + \|\mu\|_{\Sigma^{-1}}^2) \|y\|_{\Sigma^{-1}}^2} \right), \quad y \in \mathbb{R}^d.
\]

Here \( K_{\nu} \) is the modified Bessel function of the second kind; see (A.1) in Appendix A.1. Further, still with \( \det \Sigma \neq 0 \), the canonical Lévy measure of \( Y \) is absolutely continuous with respect to Lebesgue measure and satisfies:

\[
\frac{d\Pi_Y}{dy}(y) = \frac{b^{2(2-d)/2} \exp\{\langle \Sigma^{-1} \mu, y \rangle \}}{\pi^{d/2} (\det \Sigma)^{1/2}} \left( \frac{\|y\|_{\Sigma^{-1}}^2}{\|\mu\|_{\Sigma^{-1}}^2} \right)^{d/4} \times \left( \frac{\|\mu\|_{\Sigma^{-1}}^2}{\|y\|_{\Sigma^{-1}}^2} + 2b \right) \times K_{d/2} \left( \sqrt{(2b + \|\mu\|_{\Sigma^{-1}}^2) \|y\|_{\Sigma^{-1}}^2} \right), \quad y \in \mathbb{R}^d.
\]

**Generalised Gamma Convolution Subordinator.** For our extension of the Madan-Seneta \( VG_d \)-class we use the subordinators corresponding to Thorin’s \([61, 62]\) class of generalised Gamma convolutions \((GGC)\). This is the smallest class of distributions that contains all Gamma distributions, but is closed under convolution and weak convergence (see \([7, 8, 23, 29, 56, 59]\); for multivariate extensions see \([3, 8, 52]\)). The class of GGC-distributions is a subclass of the self-decomposable distributions and, thus, infinitely divisible.

A \( d \)-dimensional \textit{Thorin measure} \( \mathcal{T} \) is a Borel measure on \([0, \infty)^d_*\) with

\[
\int_{(0, \infty)^d_*} (1 + \log^+ \|x\|) \wedge \left( \frac{1}{\|x\|} \right) \mathcal{T}(dx) < \infty.
\]

\((x = x^+ - x^-) \) denotes the decomposition of an extended real number \( x \in \bar{\mathbb{R}} \) into positive and negative part.)

A subordinator \( T \) is a \( GGC^d \)-subordinator with parameters \( a \) and \( \mathcal{T} \), in brief \( T \sim GGC^d(a, \mathcal{T}) \), when \( \mathcal{T} \) is a \( d \)-dimensional Thorin measure, \( a \in [0, \infty)^d \) and, for all \( t \geq 0, \lambda \in [0, \infty)^d \),

\[
- \log E \exp\{-\langle \lambda, T(t) \rangle\} = t \langle a, \lambda \rangle + t \int_{[0, \infty)^d} \log \left\{ \frac{\|x\|^2 + \langle \lambda, x \rangle}{\|x\|^2} \right\} \mathcal{T}(dx).
\]

The distribution of a Thorin subordinator is determined by \( a \) and \( \mathcal{T} \).
By Proposition 5.1, any Thorin measure $T$ admits a polar representation $T = \alpha \otimes_p K$. Here $\alpha$ is a Borel measure on $S^d_+ := \{ x \in [0, \infty)^d : \|x\| = 1 \}$ (the spherical component of $T$) and $K$ is a locally finite kernel from $S^d_+$ (the radial component of $T$).

The next lemma gives a formula for the Lévy measure of the corresponding subordinator (we omit the proof, but see [3] [proof of their Theorem F] and [52]).

**Lemma 2.2.** Let $T \sim GGC^d_S(a, \mathcal{T})$ with $\mathcal{T} = \alpha \otimes_p K$. Then $T \sim S^d(a, \Pi_T)$ where

$$\Pi_T = \int_{S^d_+} \int_{(0,\infty)} \delta_{rs} k(s, r) \frac{dr}{r} \alpha(ds),$$

$$k(s, r) = \int_{(0,\infty)} e^{-r\tau} K(s, d\tau), \quad r > 0, s \in S^d_+. \tag{2.16}$$

**Variance–Univariate GGC (VGG$^{d,1}$).** As a first extension of the $\text{VG}^d$-model, we review Grigelionis’ [23] class. Grigelionis used univariate subordination $\circ_{d,1}$ and subordinated Brownian motion with a univariate $GGC$-subordinator. For the parameters of his model we take $\mu \in \mathbb{R}^d$, $a > 0$ and $\Sigma \in \mathbb{R}^{d \times d}$, with $\Sigma$ being symmetric and nonnegative definite. Further, let $\mathcal{T}$ be a univariate Thorin measure. Let $B \sim BM^d(\mu, \Sigma)$ be a $d$-dimensional Brownian motion, $T \sim GGC^1_S(a, \mathcal{T})$, independent of $B$.

Given such $B$ and $T$, we call a Lévy process of the form $Y \overset{D}{=} B \circ_{d,1} T$ a $d$-dimensional Variance Univariate Generalised Gamma Convolution ($\text{VGG}^{d,1}$)-process with parameters $a, \mu, \Sigma, \mathcal{T}$. We write this as

$$Y \sim \text{VGG}^{d,1}(a, \mu, \Sigma, \mathcal{T}) := BM^d(\mu, \Sigma) \circ_{d,1} GGC^1_S(a, \mathcal{T}). \tag{2.18}$$

The next theorem gives the characteristic function and Lévy density. Part (a) is proved in Subsection 5.3. Part (b) occurs in [23] (see his Proposition 3). (Throughout log : $\mathbb{C}\setminus(-\infty, 0] \to \mathbb{C}$ denotes the principal branch of the logarithm.)

**Theorem 2.1.** Let $Y \sim \text{VGG}^{d,1}(a, \mu, \Sigma, \mathcal{T})$.

(a) For all $\theta \in \mathbb{R}^d$, $t \geq 0$,

$$E \exp\{i \langle \theta, Y(t) \rangle\} = \exp \left\{ at \left[ i \langle \mu, \theta \rangle - \frac{1}{2} \|\theta\|_\Sigma^2 \right] - t \int_{(0,\infty)} \log \left[ (\tau - i \langle \mu, \theta \rangle + \frac{1}{2} \|\theta\|_\Sigma^2) / \tau \right] \mathcal{T}(d\tau) \right\}. \tag{2.19}$$
(b) If \( \det \Sigma \neq 0 \) and \( T \neq 0 \) then \( \Pi_Y \) is absolutely continuous with respect to \( d\)-dimensional Lebesgue measure on \( \mathbb{R}^d \), where, for \( y \in \mathbb{R}^d \),

\[
\frac{d\Pi_Y}{dy}(y) = 2^{(2-d)/2} \pi^{-d/2} (\det \Sigma)^{-1/2} \|y\|_{\Sigma^{-1}}^{-d/2} \exp\{\langle \Sigma^{-1} \mu, y \rangle\} \times (2.20) \\
\times \int_{(0,\infty)} (2\tau + \|\mu\|_{\Sigma^{-1}}^2)^{d/4} K_{d/2} \left\{ \sqrt{2\tau + \|\mu\|_{\Sigma^{-1}}^2} \|y\|_{\Sigma^{-1}} \right\} T(d\tau).
\]

**Remark 2.2.** In both classes, \( VG^d \) and \( VGG_{G}^{d,1} \), we subordinate a Brownian motion with a single univariate subordinator. Thus the components of these processes must jump simultaneously. To allow the components to jump independently of each other we use multivariate subordination of Brownian motion. This motivates our next step, the introduction of our \( VGG^{d,d} \)-class. \( \square \)

**Variance–Multivariate GGC \( VGG^{d,d} \).** Next we give another modification of the \( VG^d \)-model which is constructed by multivariate subordination \( \circ_{d,d} \). This class contains Semeraro’s \( \alpha \)-processes \([57]\). For the parameters of this model we assume a \( d \)-dimensional Thorin measure \( T, \mu \in \mathbb{R}^d, a \in [0,\infty)^d \) and \( \Sigma \in \mathbb{R}^{d \times d} \), with \( \Sigma = \text{diag}(\sigma_1^2,\ldots,\sigma_d^2) \) being symmetric and nonnegative definite. (We impose on Brownian motion the requirement to have independent components, so as to stay in the class of Lévy processes, see Remark [2.1].)

Let \( B \sim BM^d(\mu,\Sigma) \) be a Brownian motion. Let \( T \sim GGC^d_S(a, T) \) be independent of \( B \). Given such \( B \) and \( T \), we call a Lévy process of the form

\[ Y \sim VGG^{d,d}(a, \mu, \Sigma, T) := BM^d(\mu,\Sigma) \circ_{d,d} GGC^d_S(a, T). \]

To state formulae for the characteristics of this process, it is convenient to introduce an outer \( \circ \)-product as

\[
y \circ z := (y_1z_1, y_2z_2, \ldots, y_dz_d)' \in \mathbb{R}^d,
\]

\[
\Sigma \circ z := \text{diag}(z_1,\ldots,z_d)\Sigma \in \mathbb{R}^{d \times d},
\]

for \( y = (y_1,\ldots,y_d)' \), \( z = (z_1,\ldots,z_d)' \) \( \in \mathbb{R}^d \) and \( \Sigma \in \mathbb{R}^{d \times d} \).

We can decompose \([0,\infty)^d = \bigcup_{\emptyset \neq I \subseteq \{1,\ldots,d\}} C_I \) into semi-cones \( C_I \subseteq \mathbb{R}^d \), where

\[
C_I := \left\{ \sum_{i \in I} x_i \mathbf{e}_i : x_i > 0 \right\}, \quad \emptyset \neq I \subseteq \{1,\ldots,d\},
\]
and \( e_i \) are the unit coordinate vectors. Let \( \# I \) be the cardinality of \( I \).

Finally, we need a family of reference measures (\( \ell \) denotes univariate Lebesgue measure): if \( T(C_I) = 0 \) then put \( \ell_I := 0 \); otherwise, if \( T(C_I) > 0 \) then define \( \ell_I := \bigotimes_{k=1}^{d} \ell_{I,k} \) as the product measure with the following factors

\[
el_{I,k} := \begin{cases} \ell, & \text{if } k \in I, \\ \delta_0, & \text{if } k \notin I, \end{cases} \quad 1 \leq k \leq d. \tag{2.24}
\]

The next theorem gives the characteristic function of \( Y \) and an expression for its Lévy measure. It is proved in Subsection 5.3.

**Theorem 2.2.** Let \( Y \sim VGG^d(a, \mu, \Sigma, T) \).

(a) For all \( \theta \in \mathbb{R}^d \), \( t \geq 0 \),

\[
E \exp \{i \langle \theta, Y(t) \rangle \} \quad \tag{2.25}
\]

\[
= \exp \left\{ t \langle \mu \odot a, \theta \rangle - \frac{1}{2} \| \theta \|_{\Sigma \circ a}^2 \right\}
\]

\[
- t \int_{[0, \infty)_+^d} \log \left[ \left( \| x \|^2 - i \langle \mu \odot x, \theta \rangle + \frac{1}{2} \| \theta \|_{{\Sigma \odot a}}^2 / \| x \|^2 \right) T(dx) \right].
\]

(b) Assume \( \det \Sigma \neq 0 \) and \( T \neq 0 \). Then \( \Pi_Y = \sum_{I \subseteq \{1, \ldots, d\}} \Pi_I \), where \( \Pi_I \) is absolutely continuous with respect to \( \ell_I \) on \( \mathbb{R}^d_+ \), for \( y \in \mathbb{R}^d_+ \), with density,

\[
\frac{d\Pi_I}{d\ell_I}(y) = 2^{(2-\# I)/2} \pi^{-\# I/2} \prod_{i \in I} \sigma_i^{-1} \exp \left\{ \sum_{i \in I} \frac{\mu_i y_i}{\sigma_i^2} \right\} \times \tag{2.26}
\]

\[
\int_{C_I} \frac{T(dx)}{\prod_{i \in I} x_i^{1/2}} \left\{ \frac{2\| x \|^2 + \langle \mu \odot x, \Sigma^{-1} \mu \rangle}{\sum_{i \in I} y_i^2 / (x_i \sigma_i^2)} \right\}^{\# I/4} \times
\]

\[
K_{\# I/2} \left\{ \left( \frac{2\| x \|^2 + \langle \mu \odot x, \Sigma^{-1} \mu \rangle}{\sum_{i \in I} y_i^2 / (x_i \sigma_i^2)} \right) \sum_{i \in I} y_i^2 / (x_i \sigma_i^2) \right\}^{1/2}
\]

**2.2 Moments and Sample Paths**

In Proposition 2.1, we provide conditions on the Thorin measure that can be used to check local integrability of \( \Pi_T \) and \( \Pi_Y \). We give the more refined result for the \( GGC \)-classes, and restrict our analysis of the \( VGG \)-class to a generic case. In particular, we see that both \( VGG \)-classes support pure jump processes with infinite variation and infinite moments (for a proof see Subsection 5.4).
Proposition 2.1. Let \( t > 0, k \in \{1, d\} \).
(a) If \( T \sim GGC^d_S(a, \mathcal{T}) \) then, for all \( 0 < q < 1 \),
\[
\int_{[0,1]^d} \|z\|^q \Pi_T(dz) < \infty \iff \int_{\|x\|>1} \mathcal{T}(dx)/\|x\|^q < \infty .
\]
(b) If \( Y \sim VGC^{d,k}(a, \mu, \Sigma, \mathcal{T}) \) with \( \det \Sigma \neq 0 \) then, for all \( 0 < q < 2 \),
\[
\int_{0<\|y\|\leq 1} \|y\|^q \Pi_Y(dy) < \infty \iff \int_{\|x\|>1} \mathcal{T}(dx)/\|x\|^{q/2} < \infty .
\]

Remark 2.3. To comply with [12], for instance, we show that the VGG-class support processes with infinite variation. Indeed, by (2.14), \( \mathcal{T}_\infty,\delta(dx) = 1\{x>1\} x^\delta dx \) is a univariate Thorin measure for all \( \delta < 0 \). To have (2.27), we must have \( 2 + 2\delta < q \). For instance, \( \mathcal{T}_\infty,\frac{-1}{2} \) is a valid Thorin measure, and the associated univariate \( VGG^{1,1}(0,0,1, \mathcal{T}_\infty,\frac{-1}{2}) \)-process has paths of infinite variation, because (2.3) is violated.

Next, as preparation for our analysis in Subsection 2.3, we provide conditions on the Thorin measure, ensuring finiteness of (exponential) moments for the associated VGG-model. We use the notation
\[
\mathcal{D}_Y = \left\{ \lambda \in \mathbb{R}^d : E \exp \langle \lambda, Y(t) \rangle < \infty \right\}
\]
\[
= \left\{ \lambda \in \mathbb{R}^d : \int_{\|y\|>1} \exp \langle \lambda, y \rangle \Pi_Y(dy) < \infty \right\}.
\]
\( \mathcal{D}_Y \) is a convex subset of \( \mathbb{R}^d \), containing the origin (see [55], p. 165). Further, we need to introduce
\[
\mathcal{O}_\lambda := \{ x \in [0, \infty)^d_\ast : \|x\|^2 > \langle \lambda, x \rangle \}, \quad \lambda \in \mathbb{R}^d .
\]

Proposition 2.2. Let \( p, t > 0, \lambda \in \mathbb{R}^d \). Assume \( T \sim GGC^d_S(a, \mathcal{T}) \). Then:
(a) \( E[\|T(t)\|^p] < \infty \iff \int_{[0,1]^d} \mathcal{T}(dx)/\|x\|^p < \infty . \)
(b) \( \lambda \in \mathcal{D}_T \iff \text{simultaneously, } \mathcal{T}([0, \infty)^d_\ast \setminus \mathcal{O}_\lambda) = 0 \) and
\[
\int_{\mathcal{O}_\lambda} \log^+ \frac{\|x\|^2 - \langle \lambda, x \rangle}{\|x\|} \mathcal{T}(dx) < \infty .
\]
The next proposition follows from Proposition 2.2 (see Subsection 5.4 for a proof). In Part (a) we restrict our analysis to cover a generic case, and it is left to the reader to explore other parameter choices.

**Proposition 2.3.** Let \( p, t > 0, \lambda \in \mathbb{R}^d, k \in \{1, d\} \). Let \( Y \sim VGG^{d,k}(a, \mu, \Sigma, T) \).

(a) If \( \mu = 0 \) and \( \det \Sigma > 0 \) then
\[
E[\|Y(t)\|^p] < \infty \iff \int_{[0,1]^d} T(dx)/\|x\|^{p/2} < \infty.
\] (2.31)

(b) Without restrictions on \((a, \mu, \Sigma, T)\): \( \lambda \in \mathcal{D}_Y \Leftrightarrow q_{\lambda,k} \in \mathcal{D}_T \), where
\[
q_{\lambda,k} = \begin{cases} 
\langle \lambda, \mu \rangle + \frac{1}{2} \|\lambda\|_{\Sigma}^2, & \text{if } k = 1, \\
\lambda \odot \mu + \frac{1}{2} \Sigma(\lambda \odot \lambda), & \text{if } k = d.
\end{cases}
\] (2.32)

**Remark 2.4.** (i) It has been suggested that log returns have an infinite fourth moment [28]. As an example consider \( Y^{(0,\delta)} \sim VGG^{1,1}(0, 0, 1, T_{0,\delta}) \), with \( \delta > -1 \), a valid Thorin measure by (2.14). (2.31) holds for \( Y = Y^{(0,\delta)} \), \( p, t > 0 \), if and only if \( 2\delta + 2 > p \). In particular, \( Y^{(0,1)} \) is a well defined \( VGG \)-process with infinite fourth moment.

(ii) We construct \( Y \sim VGG^{1,1}(0, 0, 1, T) \) without finite \( p \)-moments. Plainly,
\[
T(dx) := 1 \{0 < x < 1/2\} \frac{dx}{x \log(1/x)^3}
\]
defines a Thorin measure as (2.14) holds. On the other hand, (2.31) fails for any \( p, t > 0 \), with \( Y \sim VGG^{1,1}(0, 0, 1, T) \) being left without \( p \)-moments. \( \square \)

### 2.3 Esscher Transformation

Assume that \( Y \sim L^d(\gamma_Y, \Sigma_Y, \Pi_Y) \) is a Lévy process with respect to an underlying stochastic basis \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)\).

The **Esscher transform** on \( \mathcal{F}_t \) with respect to \( Y \) is given by
\[
\frac{dQ^Y_{\lambda,t}}{dP} = \frac{\exp \langle \lambda, Y(t) \rangle}{E_P \exp \langle \lambda, Y(t) \rangle}, \quad t \geq 0, \ \lambda \in \mathcal{D}_Y.
\] (2.33)

(Recall (2.28)). For \( t \geq 0 \) and \( \lambda \in \mathcal{D}_Y \) it is well-known that \( Q^Y_{\lambda,t} : \mathcal{F}_t \to [0, 1] \) defines a probability measure, equivalent to \( P : \mathcal{F}_t \to [0, 1] \). Besides this, \( \{Y(s) : 0 \leq s \leq t\} \) remains a Lévy process under the new measure \( Q^Y_{\lambda,t} \).
Next we show that both VGG-classes are invariant under Esscher transformations (for a proof see Subsection 5.5; recall (2.29) and (2.32)).

**Theorem 2.3.** Let \( t \geq 0, \ k \in \{1, d\} \). Assume \( Y \sim VGG^{d,k}(a, \mu, \Sigma, T) \). Assume \( \lambda \in \mathcal{D}_Y \). Let \( q = q_{\lambda,k} \) with \( q_{\lambda,k} \in \mathbb{R}^k \) as defined in (2.32).

Then we have \( q \in \mathcal{D}_T \) and

\[
\{Y(s) : 0 \leq s \leq t\} | Q_{\lambda,t} \sim VGG^{d,k}(a, \mu + \Sigma \lambda, \Sigma, T_\lambda), \quad t \geq 0,
\]

where, for all Borel sets \( A \subseteq [0, \infty)_k^k \),

\[
T_\lambda(A) = T \left( S_q^{-1}(A) \right).
\]

Here \( O_q \subseteq \mathbb{R}^k \) is as in (2.29), but with \( \lambda \) replaced by \( q \). Also, \( S_q : O_q \to [0, \infty)_k^k \) is a bijective transformation, defined by

\[
S_q(x) = \frac{\|x\|^2 - (q, x)}{\|x\|^2} x.
\]

### 2.4 VMT\(^d\)-Class

In this subsection we restrict ourselves to finitely supported Thorin measures and consider a corresponding subclass of \( VGG^{d,d} \).

**MT\(^d\)-Subordinator.** The parameters are as follows: let \( n \in \mathbb{N} = \{1, 2, \ldots\} \), \( b_* = (b_1, \ldots, b_n)' \in (0, \infty)^n \), \( M \in \mathbb{R}^{d \times n} \) having columns \( M_1, \ldots, M_n \in [0, \infty)^d_\star \). Let \( M = (m_{k,l})_{1 \leq k \leq d, 1 \leq l \leq n} \).

Let \( G_1 \sim \Gamma_S(b_1, b_1), \ldots, G_n \sim \Gamma_S(b_n, b_n) \) be independent standard Gamma processes, and set

\[
T \overset{\mathbb{P}}{=} M(G_1, \ldots, G_n)' = \sum_{l=1}^{n} G_l M_l.
\]

We call \( T \) a \( d \)-dimensional \( MT\)-subordinator with parameters \( n, b_*, M \), written as \( T \sim MT^{d}_S(n, b_*, M) \).

In Subsection 5.6 we show that \( MT^{d}\)-subordinators are \( GGC^{d}\)-subordinators, but having zero drift \( a = 0 \) and finitely supported Thorin measure.
Lemma 2.3. Let \( T \sim M \Gamma^d_S(n, b_*, M) \). Then \( T \sim S^d(0, \Pi_T) = GGC^d_S(0, T_T) \),
where, simultaneously,

\[
\Pi_T = \sum_{l=1}^{n} b_l \int_{(0, \infty)} \delta_{rM_l} \exp\{-b_l r\} \frac{dr}{r}, \quad (2.37)
\]

\[
T_T = \sum_{l=1}^{n} b_l \delta_{b_l M_l/\|M_l\|^2}, \quad (2.38)
\]

\[
D_T = \bigcap_{l=1}^{n} \{ \lambda \in \mathbb{R}^d : \langle M_l, \lambda \rangle < b_l \} \quad (2.39)
\]

and, for \( t \geq 0 \), \( \lambda \in D_T \),

\[
- \log E \exp \langle \lambda, T(t) \rangle = t \sum_{l=1}^{n} b_l \log \left\{ \frac{b_l - \langle M_l, \lambda \rangle}{b_l} \right\}. \quad (2.40)
\]

**Variance–MG (VMT^d).** With parameters \( b_* = (b_1, \ldots, b_n)' \in (0, \infty)^n \) and \( M \in \mathbb{R}^{d \times n} \) as set for an \( M \Gamma^d \)-subordinator, in addition, take \( \mu = (\mu_1, \ldots, \mu_d)' \in \mathbb{R}^d \) and a diagonal matrix \( \Sigma = \text{diag}(\sigma_1^2, \ldots, \sigma_d^2) \) with non-negative entries.

Whenever \( Y \overset{D}{=} B \circ_{d,d} T \), with \( B, T \) being independent and \( B \sim BM^d(\mu, \Sigma) \) being Brownian motion, while \( T \sim M \Gamma^d_S(n, b_*, M) \), we call \( Y \) a **Variance MT (VMT^d)**-process, written in the following as

\[
Y \sim VMT^d(n, b_*, M, \mu, \Sigma) := BM^d(\mu, \Sigma) \circ_{d,d} M \Gamma^d_S(n, b_*, M). \quad (2.41)
\]

For a generic case, where \( \text{det} \Sigma \neq 0 \), we give formulae for the canonical Lévy measure \( \Pi_Y \). To each column \( M_l \) we associate both a dimension \( 1 \leq d_l \leq d \) by

\[
d_l := \#\{ 1 \leq k \leq d : m_{k,l} > 0 \}, \quad 1 \leq l \leq n,
\]

and a \( \sigma \)-finite Borel measure \( M_l := \bigotimes_{k=1}^{d_l} M_{k,l} \) on \( \mathbb{R}^d \) as a product measure with the following factors

\[
M_{k,l} := \begin{cases} \ell, & \text{if } m_{k,l} > 0, \\ \delta_0, & \text{if } m_{k,l} = 0, \end{cases} \quad 1 \leq k \leq d, \quad 1 \leq l \leq n. \quad (2.42)
\]

For \( 1 \leq l \leq n \), we set

\[
\beta_l := 2b_l + \sum_{m_{k,l} \neq 0} m_{k,l} \mu_k^2 / \sigma_k^2 = 2b_l + \langle \mu \odot M_l, \Sigma^{-1} \mu \rangle,
\]

\[
\alpha_l := \left( \frac{(2^{d_l}/2 \pi)^{d_l/2} b_l \beta_l^{d_l/4}}{\prod_{m_{k,l} \neq 0} \sigma_k m_{k,l}^{1/2}} \right). \quad (2.43)
\]
The next theorem gives formulae for the Lévy measure and Laplace exponent of \( Y \), which has finite variation (recall (2.3)) and is invariant in form under Esscher transformations. It is proved in Subsection 5.6.

**Theorem 2.4.** Assume \( Y \sim VM\Gamma_\nu^d(n, b_*, M, \mu, \Sigma) \). Then:

(a) We have \( Y \sim VGG^d_\nu(0, \mu, \Sigma, T) \) with \( T = \sum_{l=1}^n b_l \delta_{b_l M_l/\|M_l\|^2} \).

(b) Always, \( Y \sim FV_\nu^d(0, \Pi_Y) \). When, in addition, \( \det(\Sigma) > 0 \), then for all Borel sets \( A \subseteq \mathbb{R}^d \),

\[
\Pi_Y(A) = \sum_{l=1}^n \alpha_{l,t} \int_A \frac{K_{d/2}}{\sum_{m, k, l \neq 0} y_k^2 / (\sigma_k^2 m_{k,l})} \exp \left\{ \sum_{m, k, l \neq 0} \mu_k y_k / \sigma_k^2 \right\} M_l(dy).
\]

(c) We have (recall (2.28))

\[
D_Y = \{ \lambda \in \mathbb{R}^d : \langle \mu \odot M_l, \lambda \rangle + \frac{1}{2} \| \lambda \|_{\Sigma \odot M_l}^2 < b_l, \quad 1 \leq l \leq n \} ,
\]

and, for \( t \geq 0 \) and \( \lambda \in D_Y \),

\[
- \log E e^{i \lambda, Y(t)} = t \sum_{l=1}^n b_l \log \left\{ \left( b_l - \langle \mu \odot M_l, \lambda \rangle - \frac{1}{2} \| \lambda \|_{\Sigma \odot M_l}^2 \right) / b_l \right\} ,
\]

and

\[
\{ Y(s) : 0 \leq s \leq t \} \mid Q_{\lambda,t}^Y \sim VM\Gamma_\nu^d(n, b_*, M_\lambda, \mu_\lambda, \Sigma).
\]

Here \( \mu_\lambda = \mu + \Sigma \lambda \), and \( M_\lambda \in [0, \infty)^{d \times n} \) has the following columns \( M_1^\lambda, \ldots, M_n^\lambda \):

\[
M_l^\lambda = \frac{b_l}{b_l - \langle \mu \odot M_l, \lambda \rangle - \frac{1}{2} \| \lambda \|_{\Sigma \odot M_l}^2} M_l, \quad 1 \leq l \leq n.
\]

**Remark 2.5.** Let \( Y \sim VM\Gamma_\nu^d(n, b_*, M, \mu, \Sigma) \). It follows from (2.44) that

\[
Y \overset{\mathbb{D}}{=} \sum_{l=1}^n Y_l
\]

where \( Y_1, \ldots, Y_n \) are independent with \( Y_l \sim VG^d_\nu(b_l, \mu \odot M_l, \Sigma \odot M_l) \) for \( 1 \leq l \leq n \). It is, thus, possible to construct \( VM\Gamma_\nu^d \)-processes by superimposing independent Madan-Seneta \( VG^d_\nu \)-processes.
In Subsection 2.5, we show that Semeraro’s $\alpha VG$-process [57] is a $VM\Gamma^d$-process. In particular, it is possible to write any $\alpha VG$-process as a superposition of suitable $VG^d$-processes. Wang [63] comes to similar conclusions, and constructs multivariate Lévy processes with $VG^1$-components by superimposing suitable $VG^d$-processes, just as in the right hand-side of (2.46). In general, $VM\Gamma^d$-processes do not have $VG^1$-components, but we return to this question in Subsection 2.5.

2.5 Subclasses of $GGC$-Subordinators

In this subsection we review subordinator classes as they occur in the literature and relate them to our formulations. Our $GGC^d_S$ and $MT^d$-classes were introduced in Subsections 2.1 and 2.4. Various other classes, such as the ones introduced by Semeraro [57], Guillaume [25] and Pérez-Abreu and Stelzer [52], are related to them as shown in Figure 2. (Compare Figure 2 with Figure 1.)

In the univariate case, where $d = 1$, note that $\Gamma^1_S = \Gamma^1_s = \Gamma$. The Multivariate Gamma Subordinator. We reproduce the model in [52]. Let $S^d_+ := \{x \in [0, \infty)^d : \|x\| = 1\}$. Let $\beta : S^d_+ \to (0, \infty)$ be a Borel function, and $\alpha$ a finite Borel measure on $S^d_+$ such that

$$
\int_{S^d_+} \log \left\{ (1 + \beta(s))/\beta(s) \right\} \alpha(ds) < \infty. \tag{2.47}
$$

We refer to a $d$-dimensional subordinator $T$ as a $\Gamma^d$-subordinator with parameters $\alpha$ and $\beta$, written as $T \sim \Gamma^d_S(\alpha, \beta)$, whenever, for all $\lambda \in [0, \infty)^d$,

$$
-\log E e^{-\langle \lambda, T(t) \rangle} = \int_{S^d_+} \log \left\{ \left( \beta(s) + \langle \lambda, s \rangle \right)/\beta(s) \right\} \alpha(ds). \tag{2.48}
$$

In the univariate case, $d = 1$, we have $\Gamma^1_S(\alpha \delta_1, \beta) = \Gamma(\alpha, \beta(1))$. Also, note that $T \sim S^d(0, \Pi_T)$ with

$$
\Pi_T = \int_{S^d_+} \int_0^\infty \delta_{rs} e^{-\beta(s)r} \frac{dr}{r} \alpha(ds). \tag{2.49}
$$

The connection with our $GGC$-class is $\Gamma^d_S(\alpha, \beta) = GGC^d_S(0, \alpha \otimes_p \delta_{\beta(\cdot)})$ (see Lemma 2.2 for polar decomposition $\otimes_p$).
Let $T \sim \Gamma^d_S(\alpha, \beta), \lambda \in \mathbb{R}^d, q = q_\lambda \in \mathcal{D}_T$ as in (2.32). We get from Part (ii) of Proposition 2.2 that, simultaneously, $\alpha\{\beta(\cdot) \leq \langle q, \cdot \rangle\} = 0$ and (2.47) holds with $\beta$ replaced by $\beta_\lambda(\cdot) := \beta(\cdot) - \langle q, \cdot \rangle$. For the image of the Thorin measure in (2.34), observe that $(\alpha \otimes_p \delta_{\beta(\cdot)})_\lambda = \alpha \otimes_p \delta_{\beta_\lambda}$. Consequently, the associated $\Gamma^d_S$-class of subordinated Brownian motions is closed under the Esscher transformation in the interpretation of Theorem 2.3.

**Semeraro’s $\alpha$-Subordinator.** Semeraro [57] introduced another approach to multivariate Gamma subordinators (also see [40, 41, 42]). The parameters of this model are as follows: let $a, b \in (0, \infty), \alpha_* = (\alpha_1, \ldots, \alpha_d)' \in (0, \infty)^d$ such that, simultaneously, $b > a\alpha_k$ for all $1 \leq k \leq d$. Let $S_1, \ldots, S_{d+1}$ be independent such that

$$S_k \sim \Gamma_S\left(\frac{b}{\alpha_k} - a, \frac{b}{\alpha_k}\right), \quad 1 \leq k \leq d, \quad S_{d+1} \sim \Gamma_S(a, b).$$

We refer to $T$ as an $\alpha$-subordinator, in brief $T \sim \alpha \Gamma^d_S(a, b, \alpha_*)$, provided $T \overset{D}{=} (T_1, \ldots, T_d)'$ with

$$T_k := S_k + \alpha_k S_{d+1}. \quad (2.50)$$

Observe that an $\alpha$-subordinator $T$ admits standard Gamma marginal distributions: $T_k \sim \Gamma(b/\alpha_k)$. As a result, by subordinating Brownian motion with Semeraro’s $\alpha$-subordinators, it is possible to construct processes with V.G.-marginal distributions.

We give an alternative representation of $T$ in (2.50). Introduce parameters $b_* = (b_1, \ldots, b_{d+1})' \in (0, \infty)^{d+1}$ and independent standard Gamma subordinators $G_1, \ldots, G_{d+1}, G_k \sim \Gamma_S(b_k)$ for $1 \leq k \leq d+1$, by setting

$$b_k := b - a\alpha_k, \quad 1 \leq k \leq d, \quad b_{d+1} := a,$$

and, with $S_1, \ldots, S_{d+1}$ as above,

$$G_k := \frac{b}{b-a\alpha_k} S_k, \quad 1 \leq k \leq d, \quad G_{d+1} := \frac{b}{a} S_{d+1}.$$

For $T$ in (2.50) we conclude that $T \sim M\Gamma^d_S(d+1, b_*, M_{a,b,\alpha_*})$, where in our notation

$$M_{a,b,\alpha_*} := \left(\frac{1}{b} \text{diag} \left(b - a\alpha_1, \ldots, b - a\alpha_d\right), \frac{a}{b} \alpha_*\right) \in [0, \infty)^{d \times (d+1)}. \quad (2.51)$$
We show that the \( V\alpha \Gamma^d_S \) process is not closed under Esscher transform by considering the following bivariate example. Next, we show that the \( V\alpha \Gamma^d_S \)-class is not closed under Esscher transform. In Part (c) of Theorem 2.4, we have \( \lambda := (1,0)' \in \mathcal{D}_Y \) for \( \mu = (0,0)' \), \( \Sigma := \text{diag}(1,1) \) and

\[
T \sim \alpha \Gamma^d_S(1,2,(1,1)') = M\Gamma^d_S \left( 3, (1,1,1)', \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1/2 & 1/2 \end{pmatrix} \right),
\]

but also, recalling (2.45),

\[
M_{(1,0)'} = (M^{(1,0)'}_1, M^{(1,0)'}_2, M^{(1,0)'}_3) = \begin{pmatrix} 2/3 & 0 & 2/3 \\ 0 & 1/2 & 2/3 \end{pmatrix}.
\]

Plainly, \( 2/3 + 2/3 > 1 \); \( M_{(1,0)'} \) is not of the form we require in (2.51). The associated \( V\alpha \Gamma^d_S \) of subordinated Brownian motions is not closed under the Esscher transformation in the interpretation of Theorem 2.3.

**Guillaume’s Subordinator.** Guillaume [25] extends Semeraro’s \( \alpha \)-class as follows: let \( \alpha_* = (\alpha_1, \ldots, \alpha_d)' \), \( a_* = (a_1, \ldots, a_d)' \), \( \beta_* = (\beta_1, \ldots, \beta_d)' \in (0, \infty)^d \), \( c_1, c_2 > 0 \). Let \( S_1, \ldots, S_{d+1} \) be independent such that

\[
S_k \sim \Gamma_S(a_k, \beta_k), \quad 1 \leq k \leq d, \quad S_{d+1} \sim \Gamma_S(c_1, c_2).
\]

We refer to \( T \) as a \( \mathcal{G} \)-**subordinator**, in brief \( T \sim \mathcal{G}^d_S(\alpha_*, a_*, \beta_*, c_1, c_2) \), provided \( T \overset{D}{=} (T_1, \ldots, T_d)' \) with \( T_k := S_k + \alpha_k S_{d+1} \).

With \( S_1, \ldots, S_{d+1} \) as above, introduce independent standard Gamma subordinators \( G_1 \sim \Gamma_S(a_1), \ldots, G_d \sim \Gamma_S(a_d), G_{d+1} \sim \Gamma_S(c_1) \) by setting

\[
G_k := \frac{\beta_k}{\alpha_k} S_k, \quad 1 \leq k \leq d, \quad G_{d+1} := \frac{c_2}{c_1} S_{d+1}.
\]

We conclude that \( T \sim M\Gamma^d_S(d+1, b_*, M_{\alpha_*, a_*, \beta_*, c_1, c_2}) \), where in our notation, \( b_* = (a_1, \ldots, a_d, c_1)' \in (0, \infty)^{d+1} \) and

\[
M_{\alpha_*, a_*, \beta_*, c_1, c_2} := \left( \text{diag}(a_1/\beta_1, \ldots, a_d/\beta_d), (c_1/c_2) \alpha_* \right) \in [0, \infty)^{d \times (d+1)}.
\]

Further, observe that

\[
\{ \mathcal{G}^d_S(\alpha_*, a_*, \beta_*, c_1, c_2) : \alpha_*, a_*, \beta_* \in (0, \infty)^d, c_1, c_2 > 0 \} = \{ M\Gamma^d_S(d+1, b_*, \text{diag}(x_*'), y_*) : b_*, x_*, y_* \in (0, \infty)^d \}.
\]

20
By Part (c) of Theorem 2.4, the $VG^d_S$-class of subordinated Brownian motions is, thus, closed under the Esscher transformation in Theorem 2.3.

**$MT^d$-Class.** Already defined in (2.36), in Lemma 2.3 the $MT^d_S$-class was identified to be the subclass of $GGC^{MT^d}$-subordinators with drift $a = 0$, having finitely supported Thorin measures $T$. For example, Semeraro’s $\alpha$-process is of $MT^d_S$-class. By Part (b) of the next Proposition 2.4, contemplating (2.51) and (2.52) yields that both $\alpha$–subordinator and $G$-subordinator are $MT^d_S$-subordinators, concluding settlement of our diagram in Figure 2 (for a proof of the next result see Subsection 5.7).

**Proposition 2.4.** Let $T = (T_1, \ldots, T_d)' \sim MT^d_S(n, b, M)$.

(a) Then (i) $\iff$ (ii), where

(i) there are $p_k, q_k > 0$ such that $T_k \sim \Gamma_S(p_k, q_k)$;

(ii) there exists $1 \leq l_0 \leq n$ with $m_{k,l_0} > 0$ such that,

$$m_{k,l} > 0, \quad \text{for all } 1 \leq l \leq n \text{ with } m_{k,l} \neq 0,$$

we have $b_lm_{k,l_0} = b_{l_0}m_{k,l};$

(b) (i') $\iff$ (ii'), where

(i’) $T \sim \Gamma^d_S(\alpha, \beta)$ for some $\alpha, \beta$;

(ii’) for all $1 \leq k, l \leq n$, the following implication holds

$$\|M_l\|M_k = \|M_k\|M_l \Rightarrow \|M_l\|b_k = \|M_k\|b_l.$$

In addition, if one of (i) or (ii) holds then we have $q_k = b_{l_0}/m_{k,l_0}$ and $p_k = \sum_{m_{k,l} \neq 0} b_l$. Also, if one of (i') or (ii') is satisfied then $\alpha = \sum_{l=1}^n b_l\delta_{M_l/\|M_l\|}$ and $\beta(M_l/\|M_l\|) = b_l/\|M_l\|$ (1 $\leq l \leq n$).

**Remark 2.6.** Neither the $VG^d$ nor the $VMT^d$-classes support processes with infinite variation. Yet, extensions of the $VG^1$-class to univariate and multivariate $CGMY$-models [12, 40] comprise a range of possible sample path behaviour. This is in the spirit of our Proposition 2.1. By allowing subordinators $T$ to be from the larger $GGC$-class it is possible to have processes $B \circ T$ with infinite variation. It would be interesting to investigate whether the $CGMY$-model can be represented as $VGG$-processes. A multivariate special case occurs in [40]. We believe that this is possible; one could exploit results of [12, 30, 40, 45, 53, 54].

Investigations related to ours we have already mentioned are [25, 40]. Loosely connected to our paper are [24, 35, 47] who do not deal with subordinated processes; we also refer to [2] who give up-to-date discussion of multivariate Lévy processes in finance. □
3 Applications

In this section, we are primarily concerned with demonstrating how our \( V\Gamma^d \) subclass can be applied, in particular, to price multi-asset options. The \( V\Gamma^d \) subclass, as we showed, contains other popular models, such as the multivariate VG [44], the Semeraro \( \alpha VG \) [57], and the extended \( \alpha VG \) [25].

In Subsection 3.1 a market model using the \( V\Gamma^d \) process is introduced, and we give explicit formulae for the expected value of the \( k \)-dimensional log-price process and its covariance matrix, and for the expected value of the price process itself. This allows us to tabulate values of these quantities for a specific parameter set which we will use to illustrate the results. The corresponding densities are calculated using the formula for the characteristic function given in (2.25) of Theorem 2.2 and displayed in Figure 3. The parameters required to make the Esscher transform an equivalent martingale measure linking the real world and risk neutral dynamics are derived in Proposition 3.2 of Subsection 3.2. As an example, pricing of two kinds of two-asset options, specifically, European and American best-of and worst-of put options, can then be operationalised as we demonstrate in Subsection 3.3. The exact form of the Lévy measure as given in Theorem 2.4 (b) is an essential ingredient here.

3.1 A \( V\Gamma^d \)-Market Model

We employ the \( V\Gamma^d \)-process to model the log-prices of risky assets of a financial market. Potentially latent risk factors are described by a process

\[
\begin{align*}
\alpha \Gamma^d_S &\rightarrow G^d_S & &\rightarrow M\Gamma^d_S &\rightarrow GGC^d_S \nonumber \\
\Gamma^d_S &\rightarrow G^d_S & &\rightarrow M\Gamma^d_S &\rightarrow GGC^d_S 
\end{align*}
\]

Figure 2: An arrow points in the direction of generalisations of different subordinator classes, as described in the text. \( \ldots \) indicates inclusion in special cases.
where for the second coordinate the relationship is inverted. This effect
\( \rho_m \) for both coordinates are below \( \{ -1, 0, 1 \} \), and correlation of \( \rho = (0, 0) \) \( R \) is well-defined and \( \sim \). Accordingly, \( AY \) and \( R \) are not necessarily \( \Gamma^d \)-processes, but are of course Lévy processes. \( \square \)

Proposition 3.1. Let \( R \) be given by (3.1). Then:

(a) \( E R(t) = (m - q + \omega + A \sum_{i=1}^n \mu \odot M_i) t, \quad t \geq 0. \)

(b) \( \text{Cov}(R(t)) = A \left[ \sum_{i=1}^n \left( \frac{1}{b_i} (\mu \odot M_i) (\mu \odot M_i)' + \Sigma \odot M_i \right) \right] A' t, \quad t \geq 0. \)

(c) Assume \( \{ A^i : 1 \leq i \leq d \} \subseteq D_Y \), then

\[
\omega_i = - \sum_{i=1}^n b_i \log \left( \frac{b_i}{b_i - (\mu \odot M_i, A^i) - \frac{1}{2} \| A^i \|^2 \Sigma \odot M_i} \right),
\]

is well-defined and \( \mathbb{E} S_i(t) = S_i(0) e^{(m_i - q_i) t}, \) for \( t \geq 0 \) and \( 1 \leq i \leq k. \)

We investigate the distribution of \( R \) for parameters: \( n = 3, \) \( d = k = 2, \) \( m = (0.1, 0.1), q = (0, 0), b_* = (5, 5, 10)', M = (0.5, 0.5, 0.5; 0, 0.5, 0.5), \)
\( \mu = (-0.14, -0.25), \Sigma = \text{diag}(0.0144, 0.04) \) and \( A = (1, \rho \rho, 1)^0.5 \) with \( \rho \in \{-0.3, 0, 0.3\} \). Table states the expected value, volatility (square root of variance), and correlation of \( R(1) \), for \( \rho \in \{-0.3, 0, 0.3\} \). The expected values for both coordinates are below \( m = (0.1, 0.1) \) and are robust when varying \( \rho \). The expected value of the first coordinates becomes maximal for \( \rho = 0 \) whereas for the second coordinate the relationship is inverted. This effect
is determined by the term \( A \sum_{l=1}^{n} \mu \odot M_l \) in Proposition 3.1 (a). A similar behavior can be observed for the volatilities, however, here the roles of the coordinates are exchanged. Most notably, the correlation differs considerably from the dependence parameter \( \rho \). The main driver of this difference is the first component \( A \left[ \sum_{l=1}^{n} \frac{1}{b_l} (\mu \odot M_l)(\mu \odot M_l)' \right] A' \) in Proposition 3.1 (b). Depending on the sign of the entries of \( A\mu \) this term increases or decreases the correlation. For \( \rho \in \{-0.30, 0, 0.30\} \), \( A\mu \) has negative entries in both coordinates, consequently increasing the correlation above \( \rho \). This effect weakens when decreasing the dependence parameter \( \rho \).

Figure 3 illustrates the density of \( R \) for \( t \in \{0.01, 0.25\} \) when varying \( \rho \in \{-0.30, 0, 0.30\} \). For \( t = 0.01 \), the superposed processes \( A\mu \odot T \) dominate \( A \Sigma^{1/2} \hat{B} \circ_{d,d} T \), where \( T \sim M \Gamma_{S}^{d}(n, b_*, M) \) and \( \hat{B} \) is \( d \)-dimensional standard Brownian motion. For \( \rho = 0 \), most of the probability mass is located near the \( x \)- and \( y \)-axes. For \( \rho = 0.30 \), additionally mass appears around two straight lines in the first and third quadrants (positive dependence). For \( \rho = -0.30 \), additionally mass appears around two straight lines in the second and fourth quadrants (negative dependence). For \( t = 0.25 \), the density tends to normality with nearly elliptical level lines. Note, though, that for \( \rho = 0 \) the density is not symmetric but skewed towards the left and lower values.

Remark 3.2. A desirable property of a parametrisation of a multivariate distribution is to be able to distinguish between parameters describing marginal distributions, and parameters describing the dependence. For the \( V M \Gamma^{d} \), however, this is in general not possible. Each parameter appears in at least one marginal distribution. This is a consequence of the fact that the family of Gamma distributions is not stable under convolution, except for singular cases; see Lemma 2.1 (b). These are the cases analysed by [57]. See also [35].

| \( \rho \) | \( ER_1(1) \) | \( ER_2(1) \) | \( \text{Var}(R_1(1))^{1/2} \) | \( \text{Var}(R_2(1))^{1/2} \) | \( \text{Cor}(R_1(1), R_2(1)) \) |
|---|---|---|---|---|---|
| 0.30 | 0.0917 | 0.0782 | 0.1296 | 0.2104 | 0.3651 |
| 0.00 | 0.0921 | 0.0780 | 0.1260 | 0.2114 | 0.0329 |
| -0.30 | 0.0919 | 0.0785 | 0.1276 | 0.2092 | -0.3076 |

Table 1: Expected value, volatility and correlation of \( R(1) \) for \( A = (1, \rho; 1)^{0.5} \), \( \rho \in \{-0.30, 0, 0.30\} \), \( Y \sim V M \Gamma^{d}(n, b_*, M, \mu, \Sigma) \) with parameters \( n = 3, d = k = 2, m = (0.1, 0.1), q = (0, 0), b_* = (5, 5, 10)', \( M = (0.5, 0, 0.5; 0, 0.5, 0.5), \mu = (-0.14, -0.25), \Sigma = \text{diag}(0.0144, 0.04) \).
\[ t = 0.01 \]

\[ t = 0.25 \]

\[
\begin{align*}
\rho &= 0.30 \\
\rho &= 0.00 \\
\rho &= -0.30
\end{align*}
\]

\textbf{Figure 3:} Density level lines of \( R(t) = (m - q + \omega)t + AY(t) \) for \( t \in \{0.01, 0.25\} \), \( A = (1, \rho; \rho, 1)^{0.5} \), \( \rho \in \{-0.30, 0, 0.30\} \), \( Y \sim \text{VM} \Gamma^d(n, b_*, M, \mu, \Sigma) \) with parameters the same as for Table 1.

for correlating Lévy process and related applications. \( \Box \)

\section*{3.2 Risk-Neutral Valuation via Esscher Transform}

Option pricing requires a risk-neutral measure as the basis for risk-neutral valuation. In the general Lévy process setting, such a measure is not guaranteed to exist and further, if it exists it is in general not unique. But in Part (c) of Theorem 2.4 we showed that the \( \text{VM} \Gamma^d \)-class is invariant under an Esscher transformation, and here we follow common practice by adopting the Esscher transformation for identifying a risk-neutral measure, see [15, 20, 60].

For the processes \( R, X, Y \) in (3.1) and \( h \in \mathcal{D}_R = \mathcal{D}_X = \mathcal{D}_{AY} \) the Esscher transform is given by (see (2.33))

\[
\frac{dQ^R_{h,t}}{dP} = \frac{e^{(h,R(t))}}{E_P[e^{(h,R(t))}]} = \frac{e^{(h,X(t))}}{E_P[e^{(h,X(t))}]} = \frac{e^{(A'Y_h,Y(t))}}{E_P[e^{(A'Y_h,Y(t))}]}, \quad t \geq 0, \quad (3.2)
\]

such that, with \( h \in \mathcal{D}_R = \mathcal{D}_X = \mathcal{D}_{AY} \),

\[
\frac{dQ^R_{h,t}}{dP} = \frac{dQ^X_{h,t}}{dP} = \frac{dQ^Y_{A'Y_h,t}}{dP}, \quad \text{for} \ t \geq 0. \quad (3.3)
\]
By Part (c) of Theorem 2.4 as \( \mathcal{D}_R = \mathcal{D}_X = \mathcal{D}_{AY} \), we observe that

\[
\mathcal{D}_R = \left\{ h \in \mathbb{R}^k : \langle \mu \odot M_l, A'h \rangle + \frac{1}{2} \| A'h \|^2_{\Sigma \odot M_l} < b_l, \quad 1 \leq l \leq n \right\}.
\]

Also, by replacing \( \lambda \) with \( A'h \) in Theorem 2.4, it follows from (3.3) that

\[
\{Y(s) : 0 \leq s \leq t}\big| Q_{R,h,t} \sim VM \Gamma^d (n^h, b^h_*, M^h, \mu^h, \Sigma^h), \quad h \in \mathcal{D}_R, \quad t \geq 0,
\]

with \( n^h = n, b^h_* = b_*, \mu^h = \mu + \Sigma A^\top h, \Sigma^h = \Sigma \), and

\[
M^h_l = \frac{b_l}{b_l - \langle \mu \odot M_l, A'h \rangle - \frac{1}{2} \| A'h \|^2_{\Sigma \odot M_l}} M_l, \quad 1 \leq l \leq n.
\]

Next, we summarise risk-neutral pricing under the Esscher transform, as follows (see Subsection 5.8 for a proof):

**Proposition 3.2.** Assume \( h^* \in \mathbb{R}^k \) such that \( h^*, c_i + h^* \in \mathcal{D}_R = \mathcal{D}_X = \mathcal{D}_{AY}, 1 \leq i \leq k \). Then, for the market with price process \( S_0 = e^{Rt} \) and \( S_i = S_i(0) e^{R_i t} \) with \( S_i(0) \in \mathbb{R}^+, 1 \leq i \leq k \), the Esscher transform \( Q_{R,h^*} \) is an equivalent martingale measure with respect to the numeraire \( S_0 \): \( Q_{R,h^*}^{R^*} \sim P \) and \( e^{h_i} S_i / S_0 \) are \( Q_{R,h^*}^{R^*} \)-martingales, for \( 1 \leq i \leq k \) and \( T > 0 \) if, and only if,

\[
m_i - r = \Lambda_{AY(1)} (c_i) + \Lambda_{AY(1)} (h^*) - \Lambda_{AY(1)} (c_i + h^*), \quad \text{for } 1 \leq i \leq k, \quad (3.4)
\]

where \( \Lambda_X \) is the cumulant-generating function of an \( \mathbb{R}^d \)-valued random variable \( X \), i.e. \( \Lambda_X (u) = \log E e^{(u,X)} \), \( u \in \{ v \in \mathbb{R}^d : E e^{(v,X)} < \infty \} \).

**Remark 3.3.** The parameter \( h^* \) is called the **Esscher parameter.** For general exponential Lévy market models, Theorem 7.2.8 of [9] states that \( h^* \) is unique, provided the driving Lévy process does not degenerate under \( P \) in the sense of Definition 24.16 of [55]. An application of this result yields that market model (3.1) admits a unique \( h^* \), provided \( \text{rank}(A) \geq k, \text{rank}(M) \geq d \) and \( \det \Sigma > 0 \).

Next we set the interest rate to \( r = 0.05 \) and keep the remaining model parameters as in Subsection 3.1. The resulting Esscher parameter, the adjusted risk-neutral parameters and some basic statistics are provided in Table 2. The first row indicates the three different scenarios, i.e. \( \rho \in \{-0.30, 0, 0.30\} \). In the second row the Esscher parameter \( h^* \) is seen to be to be negative and
increasing in $\rho$. The third row gives the transformed parameter $\mu^h$ which tends to be lower than the original parameter under $P$ and is increasing in $\rho$ as well. The matrix distributing the Gamma subordinators to the coordinates $M^h$ is displayed in the fourth row. The elements are all greater than those of $M$ and the more negative the dependence parameter $\rho$ becomes the stronger is this effect. The resulting characteristics of the distribution are displayed in rows 5 to 8. These numbers can be compared to the numbers under $P$ in Table 1. The expected values of $R(1)$ under the Esscher martingale measure are lower than under $P$. The volatilities increase across the board by nearly 1%. For the correlation the same can be observed; an increase of about 1% is found when comparing the Esscher numbers to the original numbers under $P$. Summarising, volatilities and correlations increase when we change from $P$ to $Q^h$. Thus under the pricing measure $Q^h$ risk in the form of volatilities requires a higher risk premium than would be anticipated under $P$, e.g., when pricing a call or put option. Further, diversification effects are less pronounced under the pricing measure, e.g., requiring a higher premium for basket options.

3.3 Pricing Best-of and Worst of Put-Options

The financial market model presented above can capture a wide range of dependencies between different asset prices. As an illustration we price some cross-dependence sensitive options of both European and American styles.

|       | $\rho = 0.30$ | $\rho = 0.00$ | $\rho = -0.30$ |
|-------|---------------|---------------|---------------|
| $h^*$ | $(-2.5626, -0.5351)$ | $(-2.9662, -1.0410)$ | $(-3.8416, -1.8390)$ |
| $\mu^h$ | $(-0.1776, -0.2867)$ | $(-0.1827, -0.2916)$ | $(-0.1907, -0.2994)$ |
| $M^h$ | $(0.5217, 0)$ | $(0.5251, 0)$ | $(0.5309, 0)$ |
|       | $(0, 0.5126)$ | $(0, 0.5145)$ | $(0, 0.5176)$ |
|       | $(0.5171, 0.5171)$ | $(0.5198, 0.5198)$ | $(0.5241, 0.5241)$ |
| $E_{h^*} R(1)$ | $(0.0908, 0.0768)$ | $(0.0912, 0.0764)$ | $(0.0909, 0.0766)$ |
| $\text{Var}_{h^*}^{1/2} R_1(1)$ | 0.1365 | 0.1334 | 0.1359 |
| $\text{Var}_{h^*}^{1/2} R_2(1)$ | 0.2178 | 0.2195 | 0.2185 |
| $\text{Cor}_{h^*}(R_1, R_2)$ | 0.3751 | 0.0492 | -0.2864 |

Table 2: Esscher parameter and resulting basic statistics for $A = (1, \rho; \rho, 1)^{0.5}$, $\rho \in \{-0.30, 0, 0.30\}$, $r = 0.05$, $Y \sim VM^{d}(n, b, M, \mu, \Sigma)$ with parameters the same as for Table 1.
European options can be conveniently priced by Fourier methods \[13\]. Thus, we can draw on the results provided in Theorem 2.2 to compute European option prices. Pricing American options can be carried out by finite difference methods, discretising the respective pricing partial integro-differential equations, or by using tree-based methods. See \[26\] for a recent survey on numerical methods in exponential Lévy process models. Both methods require formulae for the Lévy measure that we provided in Theorem 2.4.

As an example we consider best/worst-of put options with respective early exercise values

\[
\chi_{\text{bop},k}(t) = \left( K - \max_{i=1}^{k} S_i(t) \right)^+, \quad \chi_{\text{wop},k}(t) = \left( K - \min_{i=1}^{k} S_i(t) \right)^+, \quad (3.5)
\]

for \(0 \leq t \leq T\), where \(T\) is the maturity date and \(K \in \mathbb{R}^+\) the exercise price.

The risk-neutral parameters are: \(n = 3\), \(d = k = 2\), \(b_\ast = (5, 5, 10)'\), \(M = (0.5, 0, 0.5; 0, 0.5, 0.5)\), \(\Sigma = \text{diag}(0.0144, 0.04)\), \(\mu = (-0.14, -0.25)\), \(m = (0.1, 0.1)\), \(q = (0, 0)\) and \(A = (1, \rho; \rho, 1)^{0.5}\) with \(\rho \in \{-0.3, 0, 0.3\}\). Note that we have set here \(r = 0.1\) in contrast to Subsection 3.2, resulting in \(h^\ast = 0\) and \(Q^{h^\ast} = P\). This allows us to interpret the option price dependencies on the parameter \(\rho\) without confounding this with effects of the Esscher transform on the option premium. To compute American option prices we use the tree approach as outlined in \[31\], \[32\], based on \[46\]. The European option prices are obtained as a byproduct of this procedure.

The recombining multinomial tree calculation we use has probability weights derived from the Lévy measure, as provided in Theorem 2.4. The option parameters are set to \(T = 0.25\) and \(K \in \{90, 95, 100, 105, 110\}\). The tree models the bivariate process \(Y = (Y_1, Y_2)'\) directly, with an exponential transform to obtain the price process. At each node of the tree the process branches on a regular rectangular \(127 \times 127\) grid. The minimum step sizes are \(4.92 \times 10^{-3}\) and \(8.37 \times 10^{-3}\) for \(Y_1\) and \(Y_2\) respectively. Prices are then obtained to an accuracy of three significant figures. The time increment is \(1.25 \times 10^{-3}\). Run times are reduced by truncating propagation of the tree in its spatial dimensions after one time increment. Allowing the tree to grow further does not affect the results. The results are presented in Table 3. As expected, put options prices are increasing in the exercise price \(K\). Also, the worst-of put option prices exceed the corresponding best-of put option prices, which is consistent with no-arbitrage. For out-of-the-money options, the early exercise premium is higher for the worst-of put compared to the
| \( \rho \) | \( K \) | Best-of put price | Worst-of put price |
|-----|-----|----------------|----------------|
|     |     | European  | American | European  | American |
| 0.3 | 90  | 0.04      | 0.05     | 0.75      | 0.81     |
| 0.3 | 95  | 0.18      | 0.24     | 1.76      | 1.90     |
| 0.3 | 100 | 0.71      | 1.06     | 3.74      | 4.03     |
| 0.3 | 105 | 2.17      | 5.00     | 7.00      | 7.49     |
| 0.3 | 110 | 4.98      | 10.00    | 11.32     | 11.98    |
| 0   | 90  | 0.01      | 0.02     | 0.76      | 0.82     |
| 0   | 95  | 0.09      | 0.13     | 1.83      | 1.98     |
| 0   | 100 | 0.44      | 0.77     | 3.96      | 4.27     |
| 0   | 105 | 1.63      | 5.00     | 7.48      | 7.96     |
| 0   | 110 | 4.27      | 10.00    | 12.01     | 12.62    |
| -0.3| 90  | 0.00      | 0.01     | 0.77      | 0.83     |
| -0.3| 95  | 0.03      | 0.06     | 1.85      | 2.01     |
| -0.3| 100 | 0.24      | 0.53     | 4.14      | 4.45     |
| -0.3| 105 | 1.19      | 5.00     | 7.94      | 8.42     |
| -0.3| 110 | 3.66      | 10.00    | 12.63     | 13.20    |

Table 3: Best-of and worst-of put option prices for \( T = 0.25 \), \( K \in \{90, 95, 100, 105, 110\} \), \( A = (1, \rho; \rho, 1)^{0.5} \), \( \rho \in \{-0.30, 0, 0.30\} \), \( r = 0.10 \), \( Y \sim \text{VMaG}(n, b_*, M, \mu, \Sigma) \) with parameters the same as for Table 1.
best-of put. The early exercise premium for at-the-money options is approximately similar in both cases. For in-the-money options, the early exercise premium is higher for the best-of put compared to the worst-of put. The dependence parameter $\rho$ affects the option prices as expected. The payoff of the best-of put increases the contingency that both price processes fall jointly, thus the option premium is increasing in $\rho$. The payoff of the worst-of put increases if at least one price process falls, thus the option premium is decreasing in $\rho$.

4 Conclusion

The Thorin \cite{61, 62} generalized Gamma convolutions provide a very natural class of distributions on which to base our multivariate V.G. generalizations. As we showed, they facilitate construction of a very general class of subordinators and corresponding multivariate Lévy processes obtained as subordinated $d$-dimensional Brownian motions. Our new class complements \cite{23}, and contains a number of currently known versions of multivariate V.G. processes, and extends them significantly in a variety of important ways. Although rather technical in appearance, our approach is very much directed toward practical usage of the methodology. Explicit expressions for characteristic functions or Laplace transforms, and Lévy measures or densities, are derived and exhibited for all our processes. This permits easy programming of option pricing routines as we demonstrate by an example, focusing in particular on the pricing of American style options on a bivariate underlying; a thorny problem not often tackled in this context.

Some advantages of our approach can be noted:

- Our processes are invariant under Esscher transform, important for option pricing purposes.

- They may have support on $\mathbb{R}^d$ (whereas those of \cite{57} for example are based on finitely supported measures.)

- By use of the Thorin class, we obtain processes possibly with infinite variation or infinite moments. (Neither uni– nor multivariate Gamma subordinators $T$ can produce processes $X \circ T$ with infinite variation. See Remarks \ref{2.3} and \ref{2.4} for discussion of this.)
- They further satisfy a number of nice theoretical properties. The subordinator class is closed under convolution. And as indicated in Remark 2.5 and Proposition 5.2 below, there are a number of useful relationships which can be expressed by superpositions and decompositions. We hope to expand on these points elsewhere.

- Luciano and Semeraro (2010) extend the $\alpha VG$ model to a multivariate $CGMY$-model, for instance. It would be possible to extend their models, using our methods. We leave this as an interesting avenue of future research, but see Remark 2.6 at the end of Section 2.

5 Proofs

5.1 Polar Decomposition of Lévy & Thorin Measures

We modify a result of [3] (see Lemma 2.1 of [3], also see [23, 53, 54]).

For $\sigma$-finite measures $\mu, \nu$, $\mu \otimes \nu$ denotes the corresponding unique $\sigma$-finite product measure. The trace field of the Borel field $\mathcal{B}(\mathbb{R}^d)$ in $A \in \mathcal{B}(\mathbb{R}^d)$ is denoted by $\mathcal{B}(A)$. Let $\emptyset \neq B \subseteq \mathbb{R}^d$ be a Borel set. We say that a Borel measure $\mu$ is locally finite relative to $B$, provided $\mu(C) < \infty$ for all compact subsets $C \subseteq B$. Let $\| \cdot \|$ be a given norm on $\mathbb{R}^d$ with unit sphere $S^d := \{x \in \mathbb{R}^d : \|x\| = 1\}$. Let $\alpha$ be a finite Borel measure on $S^d$. Let $K : S^d \times \mathcal{B}((0, \infty)) \to [0, \infty]$ be a locally finite Borel transition kernel relative to $(0, \infty)$: simultaneously, $s \mapsto K(s, B)$ is Borel measurable; $B \mapsto K(s, B)$ is a Borel measure, locally finite relative to $(0, \infty)$.

It follows from Exercise 3.24, Chapter III of [27], for instance, that there exists a measure $\alpha \otimes K : \mathcal{B}(S^d) \otimes \mathcal{B}((0, \infty)) \to [0, \infty]$, locally finite relative to $S^d \times (0, \infty)$ and uniquely determined by

$$(\alpha \otimes K)(A \times B) = \int_A K(s, B) \alpha(ds), \quad A \in \mathcal{B}(S^d), \ B \in \mathcal{B}((0, \infty)).$$

Define $\alpha \otimes_p K : \mathcal{B}(\mathbb{R}^d_+) \to [0, \infty]$ as the image of $\alpha \otimes K$ under homeomorphism $S^d \times (0, \infty) \ni (s, r) \mapsto rs \in \mathbb{R}^d_+$. By construction, $\alpha \otimes_p K$ is a locally finite Borel measure relative to $\mathbb{R}^d_+$. For all nonnegative Borel functions $f$, we have the familiar

$$\int_{\mathbb{R}^d_+} f(x) \ (\alpha \otimes_p K)(dx) = \int_{S^d} \int_{(0, \infty)} f(rs) \ K(s, dr) \alpha(ds).$$
Next, we provide a polar decomposition of measures, also dealing with additional integrability conditions:

**Proposition 5.1.** [Polar Decomposition] Let \( w : (0, \infty) \to (0, \infty) \) be a continuous function with \( I := \int_{\mathbb{R}^d} w(\|x\|) \mu(dx) \in (0, \infty) \) for a Borel measure \( \mu \) on \( \mathbb{R}^d \). Then we have:

(a) \( \mu \) is locally finite relative to \( \mathbb{R}^d \) with \( \mu(\mathbb{R}^d) \in (0, \infty] \).

(b) There exists a pair \((\alpha, \beta)\) such that, simultaneously,

- (i) \( \alpha \) is a finite Borel measure on \( \mathbb{S}^d \);
- (ii) \( K : \mathbb{S}^d \times \mathcal{B}((0, \infty)) \) is a Borel kernel, locally finite relative to \((0, \infty)\);
- (iii) \( 0 < \int w(r) K(s, dr) < \infty \) for all \( s \in \mathbb{S}^d \);
- (iv) \( \mu = \alpha \otimes_\beta K \).

(c) If \((\alpha', K') \) is another pair, simultaneously satisfying (i)–(iv), then there exists a Borel function \( c : \mathbb{S}^d \to (0, \infty) \) such that \( \alpha(ds) = c(s)\alpha'(ds) \) and \( c(s)K(s, dr) = K'(s, dr) \).

**Proof of Proposition [5.1]** (a) As \( w \) is continuous, observe that

\[
i_n := \inf\{w(x) : 1/n \leq \|x\| \leq n\} > 0, \quad n \in \mathbb{N},
\]

and, thus, \( \mu(1/n \leq \|x\| \leq n) \leq \int_{\mathbb{R}^d} \int \mu(\|x\|) \mu(dx)/i_n < \infty \) for all \( n \in \mathbb{N} \). Thus, \( \mu \) is locally finite relative to \( \mathbb{R}^d \). It is obvious that \( \mu(\mathbb{R}^d) \in (0, \infty] \).

(b) Define a probability measure \( \mu^0 : \mathcal{B}(\mathbb{R}^d) \to [0, 1] \) by \( d\mu^0/d\mu(x) := w(\|x\|)/I, x \in \mathbb{R}^d \). Let \( X \) be a random vector with \( X \sim \mu^0 \). Let \( \alpha^0(ds) = P(X/\|X\| \in ds) \). Then there is a Markov kernel \( K^0 : \mathbb{S}^d \times \mathcal{B}((0, \infty)) \to [0, 1] \) such that \( P(\|X\| \in dr \mid X/\|X\|) = K^0(X/\|X\|, dr) \), almost surely (see [33], Theorem 5.3). Set \( \alpha = I\alpha^0 \) and \( K^1(s, dr) = K^0(s, dr)/w(r) \). Note that

\[
\int_{\mathbb{S}^d} \int_{(0, \infty)} w(r) K^1(s, dr) \alpha(ds) = I \int_{\mathbb{S}^d} \int_{(0, \infty)} \frac{w(r)}{w(r)} K^0(s, dr) \alpha^0(ds) = I.
\]

In particular, there exists \( S_0 \in \mathcal{B}(\mathbb{S}^d) \) such that, simultaneously, \( \alpha(S_0^c) = 0 \) and \( 0 < \int_{(0, \infty)} w(r) K^1(s, dr) < \infty \) for all \( s \in S_0 \). Set

\[
K(s, A) := 1_{S_0}(s) K^1(s, A) + 1_{S_0^c}(s) \delta_1(A), \quad s \in \mathbb{S}^d, A \in \mathcal{B}((0, \infty))
\]

Observe that \( 0 < \int_{(0, \infty)} w(r) K(s, dr) < \infty \) and \( K(s, C) < \infty \) for all \( s \in \mathbb{S}^d \) and compact \( C \subseteq (0, \infty) \). (The latter follows from the first by the same
argument as in Part (a.) It is clear that \((\alpha, \beta)\) satisfies (i)–(iv) of (b).

(c) Uniqueness follows as in \([3]\) by replacing \(\| \cdot \|^2 \wedge 1\) with a general \(w\). \(\square\)

**Remark 5.1.** By Proposition \([5.1]\), any Lévy measure \(\Pi\) admits a polar representation \(\Pi = \alpha \otimes_p K\) with \(w(r) = r^2 \wedge 1\) (also see \([3, 53, 54]\)). By (2.14), any Thorin measure \(\mathcal{T}\) admits a polar representation \(\mathcal{T} = \alpha \otimes_p K\) with \(w(r) = (1 + \log^- r) \wedge (1/r)\). \(\square\)

### 5.2 Subordination and Decomposition

Let \(L^{d, d}(\gamma_X, \Sigma_X, \Pi_X) \subseteq L^d(\gamma_X, \Sigma_X, \Pi_X)\) be the class of Lévy processes having independent components. Let \(L^{d, 1}(\gamma_X, \Sigma_X, \Pi_X) := L^d(\gamma_X, \Sigma_X, \Pi_X), d \in \mathbb{N}\).

Recall (2.22). For a Borel measure \(\nu\) on \(\mathbb{R}_+^d\) and \(z \in [0, \infty)^d\), we define a Borel measure \(\nu \circ z\) on \(\mathbb{R}_+^d\) where \((\nu \circ z)(A) := \sum_{l=1}^d z_l \nu(A \cap A_{d, l})\) for a Borel \(A \subseteq \mathbb{R}_+^d\). Here \(A_{1, 1} := \mathbb{R}\) and

\[
A_{d, l} := \{x = (x_1, \ldots, x_d)' \in \mathbb{R}^d : x_m = 0 \text{ for } m \neq l\}, \quad d \geq 2, \ 1 \leq l \leq d.
\]

Set \(\circ_{d, d} := \circ\). When \(z \in [0, \infty)^d\), \(y \in \mathbb{R}^d\), \(\Sigma \in \mathbb{R}^{d \times d}\) and \(\nu\) is a Borel measure on \(\mathbb{R}_+^d\), we set \(y \circ_{d, 1} z := z y, \Sigma \circ_{d, 1} z := z \Sigma\) and \(\nu \circ_{d, 1} z := z \nu\).

Recall (2.4) and (2.5). We collect some formulae for the associated canonical triplets of \(X \circ_{d, k} T\) (see Theorem 30.1 in \([55]\) for the univariate subordination; see Theorem 3.3 in \([4]\) for the multivariate subordination).

**Lemma 5.1.** Let \(k \in \{1, d\}\). Let \(X \sim L^{d, k}(\gamma_X, \Sigma_X, \Pi_X)\). Let \(T \sim S_k(D_T, \Pi_T)\) be independent of \(X\). Then we have:

(a) \(X \circ_{d, k} T \sim L^d(\gamma_{X \circ_{d, k} T}, \Sigma_{X \circ_{d, k} T}, \Pi_{X \circ_{d, k} T})\) with

\[
\begin{align*}
\gamma_{X \circ_{d, k} T} &= \gamma_X \circ_{d, k} D_T + \int_{[0, \infty)^k} \int_{\mathbb{R}^d} x P(X(s) \in dx) \Pi_T(ds), \\
\Sigma_{X \circ_{d, k} T} &= \Sigma_X \circ_{d, k} D_T, \\
\Pi_{X \circ_{d, k} T}(dx) &= (\Pi_X \circ_{d, k} D_T)(dx) + \int_{[0, \infty)^k} P(X(s) \in dx) \Pi_T(ds).
\end{align*}
\]

(b) For all \(t \geq 0\)

\[
P\{(X \circ_{d, k} T)(t) \in dx\} = \int_{[0, \infty)^k} P(X(s) \in dx) P(T(t) \in ds).
\]

(c) If, in addition, \(D_T = 0\) and \(\int_{[0, 1]^k} \|t\|^{1/2} d\Pi_T(t) < \infty\) then \(X \circ_{d, k} T \sim FV^d(0, \Pi_{X \circ_{d, k} T})\).
In Part (a) of Lemma 5.1 the dependence of $T$ enters into the formulae in a linear fashion via both $D_T$ and $\Pi_T$. As a result, if a process $X$ is independently subordinated by a superposition of independent subordinators then it can be written (in distribution) as the sum of independent processes:

**Proposition 5.2.** Let $n \geq 1$, $k \in \{1, d\}$ and $X \sim L^{d,k}(\gamma_X, \Sigma_X, \Pi_X)$.

Let $X, T_1, \ldots, T_n$ be independent with $T_l \sim S_k(D_{T_l}, \Pi_{T_l})$ for $1 \leq l \leq n$. Let $T := \sum_{l=1}^{n} T_k$ and $Y := X \circ T$. Then we have:

(a) $T \sim S_k(D_T, \Pi_T)$ with $D_T = \sum_{l=1}^{n} D_{T_l}$ and $\Pi_T = \sum_{l=1}^{n} \Pi_{T_l}$.

(b) $Y \sim L^d(\gamma_Y, \Sigma_Y, \Pi_Y)$ with $\gamma_Y = \sum_{l=1}^{n} \gamma_X \circ_{d,k} T_l$, $\Sigma_Y = \sum_{l=1}^{n} \Sigma_X \circ_{d,k} T_l$ and $\Pi_Y = \sum_{l=1}^{n} \Pi_X \circ_{d,k} T_l$.

(c) If $X_1, \ldots, X_n$ are independent copies of $X$, also being independent of $T_1, \ldots, T_n$, then $Y \overset{D}{=} \sum_{l=1}^{n} X_l \circ T_l$.

(d) If, in addition, both $\sum_{l=1}^{n} \int_{[0,1]} \|t\|^{1/2} d\Pi_{T_l}(t) < \infty$ and $\sum_{l=1}^{n} D_{T_l} = 0$, then $Y \sim FV^d(0, \Pi_Y)$ and $X \circ_{d,k} T_l \sim FV^d(0, \Pi_X \circ_{d,k} T_l)$ for all $1 \leq l \leq d$.

**Proof of Proposition 5.2.** (a) is well known, but can alternatively be deduced from the Laplace transformation.

(b) follows from Part (a), owing to Part (a) of Lemma 5.1.

(c) follows from Part (b).

(d) follows from Part (a) as an implication of Part (c) of Lemma 5.1.

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5.3 Proofs for Subsection 2.1

**Proof of Lemma 2.1.** (b) It suffices to show that (ii) $\Rightarrow$ (i) holds. Suppose that there exist $a, b > 0$ such that $\sum_{k=1}^{n} Z_k \sim \Gamma(a, b)$. As we have assumed that $Z_1, \ldots, Z_n$ are independent we get from (2.8) that

$$a \int_{0}^{\infty} (1 - e^{-\lambda r}) \exp\{-br\} \frac{dr}{r} = \int_{0}^{\infty} (1 - e^{-\lambda r}) \sum_{k=1}^{n} \alpha_k \exp\{-\beta_k r\} \frac{dr}{r},$$

for all $\lambda \geq 0$. As Laplace exponents determine Lévy measures on the positive real axis we must have $a = \sum_{k=1}^{n} \alpha_k e^{(b-\beta_k)r}$ for all $r > 0$, Lebesgue a.e. and, thus, for all $r > 0$, by continuity. From this we see that $b = \beta_k$ for all $1 \leq k \leq n$. (Alternatively, this follows from the Thorin representation.)

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**Proof of Theorem 2.1.** (a) Let $Y \overset{D}{=} B \circ_{d,1} T \sim VGG^{d,1}(a, \mu, \Sigma, T)$ where $T, B$ are independent with $T \sim GCC^1(a, \mathcal{T})$ and $B \sim BM^d(\mu, \Sigma)$.
Observe that (2.15) extends to $\lambda \in \mathbb{C}$ with $\Re \lambda \geq 0$:

$$
E \exp\{-\lambda T(t)\} = \exp\{-ta\lambda - t \int_{(0,\infty)} \log[(x + \lambda)/x] \ T(dx)\}.
$$

(5.1)

This follows from Schwarz’s principle of reflection: the proof of Theorem 24.11 of [55] can be adapted to our situation.

Let $\theta \in \mathbb{R}^d$ and set $\lambda_{\theta} := \frac{1}{2} ||\theta||^2 - i \langle \mu, \theta \rangle$ such that $E[\exp(i \langle \theta, B_t \rangle)] = \exp(-t\lambda_{\theta})$. Now (2.19) follows from (5.1) via conditioning on $T(t)$:

$$
E[\exp(i \langle \theta, Y_t \rangle)] = E[e^{-T_t \lambda_{\theta}}] = \exp\{-ta\lambda_{\theta} - t \int_{(0,\infty)} \log[(x + \theta)/x] \ T(dx)\}.
$$

Here the right hand-side matches the formulae in (2.19).

(b) See [23] [see his Proposition 3.3].

Proof of Theorem 2.2.

(a) We omit the proof as it is similar to the proof of Part (a) of Theorem 2.1.

(b) Assume $\det \Sigma > 0$. Recall (2.23). We decompose $T$ into a superposition of independent subordinators $T = \sum_{I \subseteq \{1, \ldots, d\}} T_I$ where $T_\emptyset := aI$ and

$$
T_I := \sum_{0 < s \leq t} 1_{C_I}(\Delta T_s) \Delta T_s, \quad t \geq 0, \ \emptyset \neq I \subseteq \{1, \ldots, d\}.
$$

(5.2)

(Here $\Delta T(t) = T(t) - T(t-)$ for $t > 0$.) By Proposition 5.2, $Y \overset{D}{=} \sum_{I \subseteq \{1, \ldots, d\}} Y_I$.

Here $(Y_I)$ is a family of independent Lévy processes with $Y_\emptyset \overset{D}{=} B \circ_{d,d} (aI)$. For $I \neq \emptyset$ we have $Y \overset{D}{=} B \circ_{d,d} T_I \sim L^d(\gamma_I, 0, \Pi_Y^I)$ with $T_I \sim S^d(0, \Pi_Y^I)$ where, by (2.16), with $T = \alpha \otimes_p K$ and $k(s, r)$ as in (2.17),

$$
\Pi_Y^I = \int_{C_I \cap \mathbb{R}_+^d} \int_0^\infty \delta_{rs} \frac{k(s, r)}{r} \ dr \ \alpha(ds).
$$

From Proposition 5.2 we get $\Pi_Y = \sum_{\emptyset \neq I \subseteq \{1, \ldots, d\}} \Pi_Y^I$. It remains to show that $\Pi_I = \Pi_Y^I$ with $\Pi_I$ as in (2.26). To see this, let $\emptyset \neq I \subseteq \{1, \ldots, d\}$, and assume $T(C_I) > 0$ without loss of generality. (Otherwise, we have $\Pi_Y^I \equiv 0 = \Pi_I$.) In view of Lemma 5.1

$$
\Pi_Y^I(dx) = \int_{C_I} P\left(\mu \circ t + (\Sigma \circ t)^{1/2}Z \in dx\right) \Pi_Y^I(dt),
$$
where $Z$ is a $d$-dimensional standard normal vector.

As both $\det \Sigma > 0$ and $\Pi_I^T(C_I) > 0$, $\Pi_I^T$ must be absolutely continuous with respect to $\ell_I$, admitting the following density $h_I = d\Pi_I^T/d\ell_I$ where

$$h_I(y) = \int_{C_I \cap S^d} \int_0^\infty \int_0^\infty \exp \left\{ -r\tau - \frac{1}{2} \|y - r\mu \circ s\|_{I,rs}^2 \right\} \frac{1}{r(2\pi r)^{\# I/2}} \prod_{i \in I} \sigma_i s_i^{1/2} \, dr \, K(s, d\tau) \, \alpha(ds),$$

for $y \in \mathbb{R}^d$. Here we set $\|x\|_{I,rs}^2 := \sum_{i \in I} x_i^2 / (rs_i \sigma_i^2)$ for $r > 0$, $s \in C_I$, $x \in \mathbb{R}^d$.

Consequently, we get from (A.1) that, for $y \in \mathbb{R}^d$,

$$h_I(y) = 2^{(2-\# I)/2} \pi^{-\# I/2} \left\{ \prod_{i \in I} \sigma_i^{-1} \right\} \exp \left\{ \sum_{i \in I} \mu_i y_i / \sigma_i^2 \right\} \times \int_{C_I \cap S^d} \int_0^{(0,\infty)} \left[ \frac{2r + \sum_{i \in I} s_i \mu_i^2 / \sigma_i^2}{r^{1/2}} \right] \prod_{i \in I} s_i^{-1/2} \times K_{\# I/2} \left( \frac{2r + \sum_{i \in I} s_i \mu_i^2 / \sigma_i^2}{r^{1/2}} \right) \|y\|_{I,s} \, K(s, d\tau) \, \alpha(ds).$$

The proof of Part (b) is completed by noting that the right hand-side of (5.3) matches (2.26). \hfill \Box

### 5.4 Proofs for Subsection 2.2

**Proof of Proposition 2.1.**

(a) Let $t > 0$ and $0 < q < 1$. Pick $\varepsilon > 0$ such that, for all $\tau > 0$,

$$\varepsilon^2 \tau^{-q} \mathbf{1}_{\tau > 1} \leq \varepsilon \tau^{-q} \int_0^\tau r^{q-1} e^{-r} \, dr \leq 1 \wedge \tau^{-q}. \quad (5.4)$$

By Lemma 2.2 we get from Fubini’s theorem and simple substitution that

$$\int_{0 \leq \|z\| \leq 1} \|z\|^q \, \Pi_T(dx) = \int_{0,\infty}^\infty \|x\|^q \int_0^\infty r^{q-1} e^{-r} \, dr \, \mathcal{T}(dx). \quad (5.5)$$

In view of (2.14), it follows from (5.4) and (5.5) that $\int_{\|x\| > 1} \mathcal{T}(dx) / \|x\|^q$ is finite if and only if $\int_{[0,1]^d} \|z\|^q \, \Pi_T(dx)$ is, completing the proof of (a).

(b) follows from (a). \hfill \Box
Proof of Proposition 2.3. (a) Let \( p, t > 0 \). Pick \( \varepsilon > 0 \) such that, for all \( \tau > 0 \),

\[
\varepsilon^2 \tau^{-p} 1_{0 < \tau \leq 1} \leq \varepsilon \tau^{-p} \int_{\tau}^{\infty} r^{p-1} e^{-r} \, dr \leq 1_{0 < \tau \leq 1} \tau^{-p} + 1_{\tau > 1} e^{-\tau}. \tag{5.6}
\]

By Lemma 2.2, we get from Fubini’s theorem and simple substitution that

\[
\int_{\|z\| \geq 1} \|z\|^p \Pi_T(dz) = \int_{[0,\infty)} \int_{\|x\|} \int_{\infty}^\infty e^{r(\langle \mu, s \rangle - \varepsilon \|x\|)} d\mu \, K(s, d\tau) \alpha(ds). \tag{5.7}
\]

We get from (2.14), (5.6) and (5.7) that

\[
\int_{\|z\| \geq 1} \|z\|^p \Pi_T(dz) \text{ is finite if and only if } \int_{[0,\infty)} d\star T(d\mu) / \|\mu\|^p \text{ is, completing the proof of (a)}.
\]

(b) Recall (2.28). Let \( \lambda \in \mathbb{R}^d \). We get from Fubini’s theorem and (2.16) that

\[
\int_{\|x\| \geq 1} e^{\langle \mu, x \rangle} \Pi_T(dx) = \int_{S^d} \int_{[0,\infty)} \int_{\infty}^\infty e^{r(\langle \mu, s \rangle - \varepsilon \|x\|)} d\mu \, K(s, d\tau) \alpha(ds). \tag{5.8}
\]

Consequently, if \( T([0,\infty) d \setminus \Omega) > 0 \) then \( \lambda \notin \mathcal{D}_T \). For the remaining part, assume \( T([0,\infty) d \setminus \Omega) = 0 \), and choose \( \varepsilon > 0 \) such that, for all \( \tau > 0 \),

\[
\varepsilon^2 \log^{-}(\tau) \leq \varepsilon \int_{\tau}^{\infty} e^{-r} \frac{dr}{r} \leq \log^{-}(\tau) + e^{-\tau}. \tag{5.9}
\]

Note that

\[
\int_{\Omega} \exp\{(\langle \lambda, x \rangle - ||x||^2)/||x||\} T(dx) \leq \sup_{x \in S^d} e^{\langle \lambda, x \rangle} \times \int_{\Omega} e^{-\|x\|} T(dx). \tag{5.10}
\]

In (5.10) the right hand-side is finite in view of (2.14). The proof of Part (b) is easily completed by combining (5.8), (5.9) and (5.10). \( \square \)

Proof of Proposition 2.3. (a) follows directly from Part (a) of Proposition 2.2. (b) Let \( \lambda \in \mathbb{R}^d, t > 0 \). If \( k = 1 \) then \( E \exp \langle \lambda, B(t) \rangle = E \exp\{t q_{\lambda, 1}\} \) and, thus,

\[
E \exp \langle \lambda, Y(t) \rangle = E \exp \langle \lambda, B(T(t)) \rangle = E \exp\{q_{\lambda, 1} T(t)\}. \]

Otherwise, if \( k = d \), then we have

\[
E e^{\langle \lambda, (B_{0,d,d} T)(t) \rangle} = E e^{\langle \mu \circ T(t), \lambda \rangle + \frac{1}{2} ||\lambda||^2_{\mathbb{H}^{d,d} T(t)}} = E \exp \langle q_{\lambda, d}, T(t) \rangle. \tag{5.11}
\]

In either way, this completes the proof of Part (b). \( \square \)
5.5 Proofs for Subsection 2.3

Proof of Theorem 2.3. Let $k = d$, $t > 0$, $\lambda \in \mathcal{D}_Y$. Let $q := (q_1, \ldots, q_d)' = q_{\lambda,d} \in \mathbb{R}^d$ as in (2.32). As we assumed $\lambda \in \mathcal{D}_Y$ we must have $q \in \mathcal{D}_T$ by Part (b) of Proposition 2.3. Let $O_q$ as in (2.29), but with $\lambda$ replaced by $q$. Observe $T([0, \infty)^d \setminus O_q) = 0$, the latter by Part (b) of Proposition 2.2.

Let $T_\lambda$ be as defined in (2.34). We show that $T_\lambda$ is a Thorin measure. With $S_q$ as in (2.35), note that there is a constant $C \in (0, \infty)$ such that, for all $x \in O_q$ with $\|S_q(x)\| \geq 1$,

$$\|x\|/\|S_q(x)\| = 1 + \langle q, x \rangle / \|x\|^2 - \langle q, x \rangle \leq 1 + \langle q, x \rangle / \|x\| \leq C,$$

and, thus, by the transformation theorem,

$$\int_{[0, \infty)^d} (1 + \log^+ \|x\|) \wedge (1/\|x\|) \ T_\lambda(dx)$$

$$= \int_{O_q} (1 + \log^+ \|S_q(x)\|) \wedge (1/\|S_q(x)\|) \ T(dx)$$

$$\leq \int_{O_q} (1 + \log^+ \|S_q(x)\|) \wedge (C/\|x\|) \ T(dx).$$

In view of (2.14) and (2.30), the right hand-side is finite, and $T_\lambda$ is a Thorin measure, as desired.

Let $a = 0$. Adapting arguments from the proof of Theorem 25.17 of [55], for example, we get from (2.13) that, for $z \in \mathbb{C}^d$ with $\Re z_k \leq q_k$ for all $1 \leq k \leq d$,

$$E \exp \langle z, T(t) \rangle = \exp \left\{ -t \int_{O_q} \log \left( (\|x\|^2 - \langle z, x \rangle) / \|x\|^2 \right) \ T(dx) \right\}. \quad (5.12)$$

Note that $S_q(x)/\|S_q(x)\|^2 = x/(\|x\|^2 - \langle q, x \rangle)$ for all $x \in O_q$. Set $\mu_\lambda = \mu + \Sigma \lambda$. Extending (5.11) as well, we get from (5.12) that, still with $a = 0$,

$$E \exp \langle \lambda + i\theta, Y(t) \rangle / E \exp \langle \lambda, Y(t) \rangle$$

$$= \exp \left\{ -t \int_{O_q} \log \left( \|x\|^2 - \langle q, x \rangle - i \langle \mu_\lambda \odot x, \theta \rangle + \frac{1}{2} \|\theta\|^2_{\Sigma \odot x} \right) \ T(dx) \right\}$$

$$= \exp \left\{ -t \int_{O_q} \log \left( \|S_q(x)\|^2 - i \langle \mu_\lambda \odot S_q(x), \theta \rangle + \frac{1}{2} \|\theta\|^2_{\Sigma \odot S_q(x)} \right) \ T(dx) \right\}. \quad (5.12)$$
Next, apply the transformation theorem to $\mathcal{T}$ and $\mathcal{S}_q : \mathcal{O}_q \to [0, \infty)^d$ to see that the right hand-side of the last display matches (2.25), but with $a, \mu, \mathcal{T}$ replaced by $0, \mu, \lambda, \mathcal{T}$, respectively.

According to (2.25), if $a \neq 0$ it is possible to decompose $Y = B + Y_0$ into independent $B, Y_0$ where $B \sim BM^d(\mu \circ a, \Sigma \circ a)$ and $Y_0 \sim VGG^d, d(0, \mu, \lambda, \mathcal{T}, a)$. Using the independence, the proof is completed for $k = d$ by noting that

$$E \exp \langle \lambda + \theta_i, B(t) \rangle / E \exp \langle \lambda, B(t) \rangle = \exp(\langle \mu, \lambda, a \rangle - \frac{t}{2} \|	heta\|^2_{\Sigma \circ a}), \theta \in \mathbb{R}^d.$$  

The proof of the remaining case, where $k = 1$, is similar, but simpler. This completes the proof of the theorem. 

5.6 Proofs for Subsection 2.4

Proof of Lemma 2.3. By (2.15), (2.40) follows straight forwardly from (2.38). Part (a) of Proposition (5.2) is applicable to (2.36) giving $\Pi_T = \sum_{i=1}^n \Pi_{G_i, M_i}$, with a similar superposition and intersection in place for $\mathcal{T}_T$ and $\mathcal{D}_T$, respectively. It remains to verify the formulae in (2.37)-(2.39), but with $n = 1, M = M_1, b = b_1$ and $G = G_1 \sim \Gamma_S(b)$.

With $\Pi := b \int_{(0, \infty)} \delta_r M e^{-br} dr / r$ and $\lambda \in \mathbb{R}^d$ with $\langle \lambda, M \rangle < b$ calculate

$$\int_{\mathbb{R}^d} (1 - e^{-\langle \lambda, x \rangle}) \Pi(dx) = b \int_0^\infty (1 - e^{-r \langle \lambda, M \rangle}) e^{-br} \frac{dr}{r}. \quad (5.13)$$

On the other hand, $E \exp \langle \lambda, G(t) M \rangle = E \exp \{G(t) \langle \lambda, M \rangle\}$ in which we can substitute the characteristic exponents of $G$ using (2.8). By (5.13), the resulting expressions match those in (2.37)-(2.38). 

Proof of Theorem 2.4. (a) follows from Lemma 2.3 and (2.41). By construction (or with reference to Part (a) of Proposition 2.1), $\int_{[0, 1]} \|x\|^{1/2} \Pi_T(dx)$ is finite for $M \Gamma^d$-subordinators $T$. Also, $M \Gamma^d$-subordinators $T$ have zero drift. Thus, $Y \sim FV(0, \Pi_Y)$ by Part (c) of Lemma 5.1. In view of Part (a) of the proposition, the remaining parts (b)-(c) follow from Theorems 2.2-2.3. 

5.7 Proofs for Subsection 2.5

Proof of Proposition 2.4. In view of (2.36), the $k$th component of $T$ can be decomposed into $n$ univariate Gamma subordinators. Thus, Part (a) follows
from Part (c) of Lemma \text{2.1}. To show (b), we restrict ourselves to show ‘(ii’)⇒(i)’, leaving ‘(i)’⇒(ii)’ to the reader.

Let $S \sim \Gamma_S(\alpha, \beta)$, $x \in [0, \infty)^d$ with Euclidean norm $\|x\|^2 := \langle x, x \rangle = 1$ and introduce a univariate subordinator $S^x$ by

$$S^x(t) := \sum_{0 < s \leq t} 1\{\alpha x : \alpha > 0\} (\Delta S(s)) \langle x, \Delta S(s) \rangle, \quad t \geq 0.$$ 

In view of (2.48), for $\theta \geq 0$, observe that

$$- \log E \exp\{-\theta S^x(1)\} = \alpha\{x/\|x\|\} \int_0^\infty (1 - e^{-\theta r/\|x\|}) e^{-r\beta(x/\|x\|)} \frac{dr}{r}.$$

Substituting $r = r'/\|x\|$ on the right of the last display, we get from (2.8) that either $S^x \sim \Gamma_S(\alpha\{x/\|x\|\}, \|x\|\beta(x/\|x\|))$ or $S^x = 0$.

Thus prepared, let $T \sim \Gamma_T^d(n, b, M)$. For $1 \leq k \leq n$ let $S^{M_k/\|M_k\|_2}$ be the univariate Gamma subordinator, associated to $M_k/\|M_k\|_2$. In view of (2.36), observe that

$$S^{M_k/\|M_k\|_2} \overset{D}{=} \sum_{\|M_l\| = \|M_k\|/\|M_l\|} G_I(M_k, M_l)/\|M_k\|_2.$$

Suppose $T \sim \Gamma_T^d(\alpha, \beta)$. Then $S^{M_k/\|M_k\|_2}$ must either be degenerate or a univariate Gamma subordinator. Consequently, by Part (c) of Lemma \text{2.1}, we must have $b_l\|M_k\|_2^2 = b_k(M_k, M_l)$ for $1 \leq l \leq n$ with $\|M_l\|_2 = \|M_k\|_2$. The latter is equivalent to (ii’), completing the proof of ‘(ii)’⇒(i)’.

\section*{5.8 Proofs for Section 3}

\textbf{Proof of Proposition 3.1.} By Remark \text{2.3} $Y$ has the same distribution as the sum of $n$ independent processes $Y_l$, i.e. $Y \overset{D}{=} \sum_{l=1}^n Y_l$, where $Y_l \sim VG^d(b_l, \mu \odot M_l, \Sigma \odot M_l)$ for $1 \leq l \leq n$.

Observe that $EY_l(t) = \mu \odot M_l$, $1 \leq l \leq n$, where the last step follows from the fact that each coordinate of $Y_l$ is VG since $\Sigma$ is a diagonal matrix, and from the expected value of univariate VG, see, e.g., (A4) in \text{43}. By linearity of the expectation part (a) follows.

We have $\text{Cov}(Y(t), Y(t)) = \sum_{l=1}^n \text{Cov}(Y_l(t), Y_l(t))$, since $Y_1, ..., Y_n$ are independent. We have that $Y_l \sim VG^d(b_l, \mu \odot M_l, \Sigma \odot M_l)$, and thus $Y_{1,i} \overset{D}{=}$
\[ \mu_i M_{t,i} G_l + \sigma_i M_i^{1/2} \dot{B}_{t,i} \circ 1,1 G_l, \ 1 \leq i \leq k, \] where \( \dot{B}_l = (\dot{B}_{l,1}, ..., \dot{B}_{l,k}) \) is a standard Brownian motion, \( 1 \leq l \leq n \). Observe that Cov \( \left( G_l(t), \dot{B}_{t,i}(G_l(t)) \right) = E \left[ (G_l(t) - t) \dot{B}_{t,i}(G_l(t)) \right] = E \left[ E \left[ (G_l(t) - t) \dot{B}_{t,i}(G_l(t)) \mid G_l(t) \right] \right] = 0, \ 1 \leq i \leq k, \ 1 \leq l \leq n. \) Conditioning on \( G_l(t) \) gives

\[
\text{Cov}(Y_{l,i}(t), Y_{l,j}(t)) = \mu_i M_{l,i} \mu_j M_{l,j} \text{Cov}(G_l(t), G_l(t)) + \sigma_i M_i^{1/2} \sigma_j M_j^{1/2} \text{Cov} \left( \dot{B}_{l,i}(G_l(t)), \dot{B}_{l,j}(G_l(t)) \right)
\]

\[
= \left( \frac{1}{b_t} (\mu \circ M)_i (\mu \circ M)_j + (\Sigma \circ M)_i 1_{i=j} \right) t,
\]

and the last equality follows from \( EG_l(t) = t \) and \( Var(G_l(t)) = \frac{1}{b_t} t, \ 1 \leq i, j \leq k, \ 1 \leq l \leq n. \) Since

\[
\text{Cov}(R(t), R(t)) = \text{Cov}(AY(t), AY(t)) = A \text{Cov}(Y(t), Y(t)) A'
\]

part (b) follows. Part (c) is a direct consequence of Theorem 2.4 (c). \( \square \)

**Proof of Proposition 3.2.** Let \( h \in D_{AY} \) such that \( h + \epsilon_i \in D_{AY}, \) for \( 1 \leq i \leq k. \) Then \( Q_h := Q^R_{h,T} \) is well-defined and \( E_{Q_h}[e^{q_t S_i(t)/S_0(t)}] < \infty, \) for \( 1 \leq i \leq k \) and \( 0 \leq t \leq T. \) Note that \( e^{q_t S_i(t)/S_0(t)} \) is the exponential of a Lévy process, under both \( P \) and \( Q^R_{h,t} \), and thus for \( 1 \leq i \leq k \) and \( 0 \leq t \leq T \) it is the case that

\[
E_{Q_h} \left[ e^{q_t S_i(t)/S_0(t)} \right] = e^{q_t S(t)/S_0(t)} \left( e^{q_t E_{Q_h} \left[ S_i(1)/S_0(0) \right]} \right)^{T-t}
\]

\[
= e^{q_t S(t)/S_0(t)} \left( e^{m_i + \omega_i - r} \frac{E_P \left[ e^{(h_t, AY(1))} e^{(h_t, R(1))} \right]}{E_P \left[ e^{(h_t, R(1))} \right]} \right)^{T-t}
\]

\[
= e^{q_t S(t)/S_0(t)} \left( e^{m_i + \omega_i - r} \frac{E_P \left[ e^{(h_t, AY(1))} \right]}{E_P \left[ e^{(h_t, AY(1))} \right]} \right)^{T-t}
\]

\[
= e^{q_t S(t)/S_0(t)} e^{(m_i + \omega_i - r + \Lambda_{AY}(\epsilon_i) + h)} (t) (T-t).
\]

Thus, \( e^{q_t S_i(t)/S_0(t)} \) is a \( Q_h \)-martingale if and only if \( h \) satisfies (3.4). \( \square \)
A Appendix

A.1 Modified Bessel Functions of the Second Kind

We recall the following identities regarding the modified Bessel function $K$ of the second kind, e.g. Eq (3.471)–9 and Eq (8.469)–3 in [21]:

$$2 \left( \frac{\delta}{\gamma} \right)^{\nu/2} K_{|\nu|} \left( 2 \sqrt{\delta \gamma} \right) = \int_0^\infty x^{\nu-1} e^{-\frac{\delta}{x} - \frac{\gamma}{x}} \, dx,$$  \hspace{1cm} (A.1) 

$$K_{1/2}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}.$$ \hspace{1cm} (A.2) 

for $z, \delta, \gamma > 0$, $\nu \in \mathbb{R}$. See also [1], Section 5.4.6 & Appendix 5.6, and the Appendix in [15].

A.2 Transition Densities

Let $b_\ast = (b_1, \ldots, b_{d+1})' \in (0, \infty)^{d+1}$ and $T \sim \Gamma_d(d+1, b_\ast, M)$. For this subsection, we assume that $M \in [0, \infty)^{d \times (d+1)}$ such that, simultaneously,

$$M = (m_{k,t})_{1 \leq k \leq d, 1 \leq t \leq d+1} = \left( \text{diag}(m_{1,1}, \ldots, m_{d,d}), M_{d+1} \right),$$

$$\prod_{k=1}^d m_{k,k} \times \prod_{k=1}^d m_{k,d+1} \neq 0. \hspace{1cm} (A.3)$$

With $t > 0$ define

$$C^*_t := C^*_t(b_\ast, M) := \left\{ \frac{t^{b_{d+1}}}{\Gamma(t_{d+1})} \right\} \left\{ \prod_{k=1}^d \frac{t_{b_k}}{\Gamma(t_k)m_{k,k}^{t_{b_k}}} \right\}, \hspace{1cm} (A.4)$$

$$\beta^* := \beta^*(b_\ast, M) := -b_{d+1} + \sum_{k=1}^d b_k m_{k,d+1} / m_{k,k}. \hspace{1cm} (A.5)$$

The proof of the next result follows from a similar analysis as in Section 48.3.1 in [36]. (We admit to being unable to provide substantial simplification of the integral in (A.6). However, using the results in [49], it is possible to expand the integral in terms of Lauricella functions.)
Lemma A.1. Let $t > 0$ and $T \sim \Gamma_{d}^S(d+1, b_*, M)$ with $M$ satisfying (A.3). Then $T(t)$ admits a Lebesgue density $f_T$: for $\tau = (\tau_1, \ldots, \tau_d)' \in \mathbb{R}^d$:

$$f_{T(t)}(\tau) = C^*_t \cdot 1_{(0, \infty)^d}(\tau) \exp \left\{ - \sum_{k=1}^d b_k \tau_k / m_{k,k} \right\} \times$$

$$\int_0^{\sum_{k=1}^d \tau_k / m_{k,d+1}} e^{b^* \cdot s} s^{b_{d+1}-1} \prod_{k=1}^d (\tau_k - m_{k,d+1}s)^{tb_k-1} \, ds. \quad (A.6)$$

With the help of (A.4) and (A.5) define

$$a_k := 1/(2m_{k,d+1}\sigma_k^2), \quad \hat{a}_k := m_{k,d+1} \left[ (b_k/m_{k,k}) + (\mu_k^2/(2\sigma_k^2)) \right],$$

$$c_k := 2b_k + m_{k,k}(\mu_k/\sigma_k)^2, \quad \hat{c}_k := \sqrt{c_k/(\sigma_k^2 m_{k,k})}.$$ 

Further, for $t > 0$, we set

$$C_t := C^*_t(b_*, M) \cdot 2^{-d/2} \pi^{-d/2} \left\{ \prod_{k=1}^d 1/\sigma_k \right\} \left\{ \prod_{k=1}^d m_{k,d+1}^{tb_k-1/2} \right\}$$

$$= \left\{ b_{d+1}^{tb_{d+1}}/(2^{d/2}\pi^{d/2} \Gamma(tb_{d+1})) \right\} \prod_{k=1}^d b_k^{tb_k} m_{k,d+1}^{tb_k-1/2} / (\sigma_k \Gamma(tb_k) m_{k,k}^{tb_k}) \right\},$$

$$c_{d+1} := 2b_{d+1} + \sum_{k=1}^d m_{k,d+1}\mu_k^2/\sigma_k^2,$$

$$D_t := 2\pi^{-d} c_{d+1}^{(d-2b_{d+1})/4} \left\{ \prod_{k=1}^{d+1} b_k^{tb_k} / \Gamma(b_k) \right\} \left\{ \prod_{k=1}^d c_k^{(1-2b_k t)/4} \right\} \times$$

$$\times \left\{ \prod_{k=1}^d \sigma_k^{-(3+2b_k t)/2} \right\} \left\{ \prod_{k=1}^d m_{k,k}^{-(1+2b_k t)/4} \right\} \left\{ \prod_{k=1}^d m_{k,k}^{(1-2b_k t)/4} \right\} \left\{ \prod_{k=1}^d m_{k,d+1}^{-1/2} \right\}.$$ 

Theorem A.1. Let $t > 0, Y \sim VM^d(d+1, b_*, M, \mu, \Sigma)$. If (A.3) holds then
\( Y(t) \) admits Lebesgue density \( f_{Y(t)} \): for \( y = (y_1, \ldots, y_d)' \in \mathbb{R}^d \),

\[
f_{Y(t)}(y) = C_t \exp \left\{ \sum_{k=1}^{d} \mu_k y_k / \sigma_k^2 \right\} \times
\int_{0}^{\infty} e^{bs} s^{\frac{d-2}{2} - 1} \sum_{k=1}^{d+1} b_k \prod_{k=1}^{d} \int_{0}^{1} \exp \left\{ -a_k u y_k^2 / s - \tilde{a}_k s \right\} \frac{1-u}{u} t^{b_k-1} du \, ds
\]

\[
= D_t \exp \left\{ \sum_{k=1}^{d} \mu_k y_k / \sigma_k^2 \right\} \times
\int_{\mathbb{R}^d} K_{[2b_d+1-t-d]/2} \left( \sqrt{c_{d+1} \sum_{k=1}^{d} z_k^2 / (\sigma_k^2 m_k, d+1)} \right) \times
\left\{ \prod_{k=1}^{d} K_{[2b_k t-1/2]} \left( \tilde{c}_k |y_k - \tilde{z}_k| \right) / |y_k - \tilde{z}_k| (1-2bt)/2 \right\} \, d(z_1, \ldots, z_d)
\]

**Proof of Theorem A.1.** Let \( \circ = \circ_{d,d}, \pi = \pi_{d,d} \). By Part (b) of Lemma 5.1, (A.7) follows from Lemma A.1 and Fubini’s theorem. Observe that \( Y \overset{D}{=} B \circ T \) with independent \( B \sim BM^d(\mu, \Sigma) \) and \( T \sim \Gamma_{S}^{d+1}(d+1, b_*, M) \). Next write \( T = T_1 + T_2 \) with \( T_1 \sim \Gamma_{S}^{d}(d, (b_1, \ldots, b_d)', \text{diag}(m_{1,1}, \ldots, m_{d,d})) \) and \( T_2 = \Gamma_{S,d}^{d+1}(1, (b_{d+1}, \ldots, b_{d,d+1}), \text{diag}(m_{1,1}, \ldots, m_{d,d,d+1})) \) with \( B, T_1, T_2 \) being independent. By Part (c) of Proposition 5.2, we have \( X \overset{D}{=} B \circ T \overset{D}{=} X_1 \circ B_2 + B_1 \circ T_2 \), where \( B_1, B_2 \) are independent copies of \( B \), also being independent of \( T_1 \) and \( T_2 \). Observe that the \( d \)-dimensional process \( B_1 \circ T_1 \) has independent components with the \( k \)-th component being a \( VG_{1}^{d}(b_k, \mu_k m_k, \sigma_k^2 m_k, k) \)-process \( (1 \leq k \leq d) \). Further, \( B_2 \circ T_2 \) is a \( VG_{d}^{d}(b_{d+1}, \mu \circ M_{d+1}, \Sigma \circ M_{d+1}) \)-process. The formula in (A.8) follows from (2.12) by convolution. \( \square \)

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