Cobordisms with chronologies
and a generalisation of
the Khovanov complex

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Master’s thesis supervised by
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Krakow 2008
To my supervisor,

dr hab. Klaudiusz Wójcik,

for supervising my program of studies
as well as for his support and favour
during writing this thesis
To prof. Dror Bar-Natan
for his inestimable help and support during
my stay at the University of Toronto,
which bore fruits with these results
To my Parents
for making it possible to me to take
studies of my dreams and devoting
ungcountable many moments to my
education and breeding
Preface

The classical theory of knots deals with embeddings of circles into the Euclidean space $\mathbb{R}^3$. Allowing more circles as well as closed intervals we obtain links and tangles respectively. The internal structure of these objects depends only on the amount and type of components. Therefore, the embedding is the main object of research. It was G. W. Leibniz, who noticed in 1679 importance of embeddings, introducing the term *geometria situs*. First remarks on knots come from A. T. Vandermonde, whereas C. F. Gauss defined first invariants (see also [22]).

The end of 20th century is the time of a huge development of the theory due to papers of J. H. Conway and V. F. R. Jones [10], which introduced very strong polynomial invariants. Although the result of Conway is mostly a reformulation of a definition of the well-known before Alexander polynomial, the Jones polynomial $V_L(t)$ was a new invariant, given by three simple conditions:

1. $V_L(t)$ is a link invariant
2. $V_{S^1}(t) = 1$,
3. $t^{-1}V_{\chi}(t) - tV_{\chi}(t) = (t^{1/2} - t^{-1/2})V_{\chi}(t)$.

Several months later, L. Kauffman constructed in [12] the Jones polynomial by a bracket $\langle D \rangle$, which can be computed for non-oriented diagrams as a state-sum of polynomials defined for smoothed diagrams obtained from $D$ (see fig. 1). On one hand such a definition of the Jones polynomial resulted soon in proofs of almost hundred-years old Tait’s conjectures, on the other it gave impetus to the search for invariant defined in a similar way in quantum algebras.

The next step in understanding the Jones polynomial is the paper of M. Khovanov [14]. It contains a description of a graded homology groups $\mathcal{H}_{Kh}^*(L)$, with its Euler characteristic being the polynomial. In fact, it started the search for other homology theories, which Euler characteristics are other link invariants (such theories are called categorifications of these invariants). There are several reasons, why this hunting is worth to spend time on it: homology groups are often stronger invariants than their Euler characteristics, properties of those invariants may have a simpler explanation in terms of homology groups and finally they can be naturally extended over a larger class of objects, like tangles. One of the most recent results is the paper of P. Osváth, J. Rasmussen and Z. Szabó [20], where they constructed homology groups $\mathcal{H}_{ORS}^*(L)$, called odd Khovanov homology. They also categorify the Jones polynomial, but are far different from the one known before. In particular, there exist links having isomorphic homology groups of one type but not isomorphic of the other type.

![Figure 1. A diagram of a trefoil and some of its smoothed versions.](image)
The Khovanov’s construction starts with a commutative cube in the category of cobordisms, which is sent by a functor to a category of modules. In this category we can build a complex from this cube and compute its homology groups, which appear to be link invariants. A big step in understanding this construction is the paper of D. Bar-Natan [2], where he constructed the complex and proved its invariance in the category of cobordisms.\footnote{To be more precise, in the additive closure of cobordisms, i.e. the category extended by formal direct sums and formal sums of cobordisms (see the definition 3.1.6)} This approach gives a natural extension over tangles as well as cobordisms between tangles embedded in $\mathbb{R}^3 \times I$. Moreover, any functor $F: \mathbf{2Cob} \longrightarrow \mathbf{Mod}$ from the category of cobordisms into the category of modules, satisfying some additional conditions, induces an invariant complex and homology groups.

The approach described above does not work for odd homology groups, because the procedure $\mathcal{F}_{\text{ORS}}: \mathbf{2Cob} \longrightarrow \mathbf{Mod}$ is not a functor. Indeed, it is defined only up to a sign. This raises a question, if cobordisms can be enriched with some additional structure so that the construction of odd homology groups gets a functorial description. It would be great, if we could rewrite in this category the construction of Bar-Natan. The answer is positive: it suffices to equip cobordisms with projections onto the unit interval $\tau: M \longrightarrow I$ that have only non-degenerated critical points, all on different levels, and define orientations of these critical points (visualised with arrows in [20]). One of the most important properties of this new category is breaking symmetries. For example, contrary to the usual cobordisms, there is no associativity law:

\[
\begin{array}{c}
\begin{array}{c}
\text{1}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{1}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\text{1}
\end{array}
\end{array}
\] \neq \[
\begin{array}{c}
\begin{array}{c}
\text{1}
\end{array}
\end{array}
\]

A motivation for cobordisms is a topological quantum field theory (TQFT). Cobordisms stand for space-times, whereas manifolds are spaces. The projection $\tau: M \longrightarrow I$ can be seen as a description of the evolution of $\tau^{-1}(0)$ in time. Then each critical point denotes some special event, being a qualitative change of the space (i.e. a split into to spaces, a creation of a non-contractible loop, etc.). We can say that $\tau$ keeps a chronology of those events and therefore we call cobordisms with such projections chronological. This may result in a chronological topological quantum field theory.

In this paper, we describe the category of chronological cobordisms and construct a complex in this category, which is a tangle invariant. Then, applying appropriate functors, we can recover both usual and odd homology groups. Because the chronological cobordisms are strongly non-symmetric, we introduce changes of chronologies and connected to them relations – multiplication by some coefficients from $R$. As in the case of odd theory we obtain a cube that commutes up to invertible elements of $R$. Finally, we prove the cube can be fixed to be commutative in a canonical way and that the resulting complex is a tangle invariant.

The paper consists of four chapters. The first one is a brief introduction to the knot theory. It contains basic definitions and theorems, constructions of the Jones polynomial and the Kauffman bracket as well as basic facts on planar algebras, which generalises the algebra of tangles.

The chapter 2 is a description of the category of chronological cobordisms. It starts with definitions and basic facts from the theory of oriented cobordisms, which are followed by the definition and properties of chronologies. The section 4 contains a presentation of the category of two-dimensional chronological cobordisms in terms of generators and
relations. Next pages introduce changes of chronologies with explanation why the quotient category by these relations is non-trivial. This chapter ends with deliberation on cobordisms embedded in $\mathbb{R}^3$, which give a natural framework for the Khovanov cube and the whole construction.

A brief introduction to homological algebra is the main part of the chapter 3. We first define additive categories in which we can construct chain complexes and show that every category can be extended to an additive one. The sections 3 and 4 deal with special constructions in such categories: cubes and cube complexes. The are crucial from the view of the Khovanov complex and the proof of the its invariance.

The last chapter is devoted to the Khovanov complex. The construction is given in the beginning, then we prove its invariance. Next several properties are given: the behaviour of the complex under reversing orientations of some components of a link or mirroring the tangle. The section 4 contains examples of functors that can be used to compute homology groups. In particular, we define a functor $F_{XYZ}$, which generalises both $F_{Kh}$ and $F_{ORS}$. It leads to the notion of a chronological Frobenius algebra. The last section shows that this new functor categorifies the Jones polynomial.
Preface to the English version

This paper is a translation of my Master’s Thesis, originally written in Polish. The Thesis was defended in November 2008 and since that time several new things have been discovered: dotted chronological cobordisms, extension of the construction over cobordisms between tangles up to invertible elements. However, I tried to keep the translation as close to the original text as possible, making minor changes only if necessary. The chapters 1, 4 and most of 3 are faithful translations, modulo change of several symbols to avoid collisions. The second chapter has been changed in several places. In the section 2.2 the definition of equivalent chronological cobordisms is simplified and examples of isotopies of chronologies are included. The section 2.4 has been rewritten, due to gaps in the original proof of the classification theorem.
CHAPTER 1

A brief introduction to knots

This part contains basic definitions and facts on knots and links: existence of diagrams, Reidemester’s theorem as well as the constructions of the Kauffman bracket and the Jones polynomial. For a more comprehensive introduction the reader is referred to [4, 6] (with no results of Jones) or [7, 21, 23].

1. Basic definitions

Consider a standard oriented circle $S^1 = \{z \in \mathbb{C} | z\bar{z} = 1\}$. A disjoint sum of its $n$ copies will be denoted by $nS^1$.

**Definition 1.1.** A knot is a smooth embedding of a circle $S^1$ into the oriented space $\mathbb{R}^3$. An embedding of a disjoint sum of $n$ circles is called a link and the embedded circles are called components.

A knot is simply a link with one component. Some examples are provided in the figure [1].

**Remark 1.2.** One can also consider links in other space than $\mathbb{R}^3$, for instance thickened surfaces $\Sigma_g \times I$ or fibre bundles over surfaces. Non-trivial theories exist also for thin surfaces. For instance knots in a torus are in one-to-one correspondence with pairs of coprime numbers. (see [24], chapter 2).

![Figure 1. Examples of knots and links: the unknot 0_1, the trefoil 3_1, the Hopf link and the Whitehead link.](image)

Links with $n$ components form a space

\[
L^n = \{L: nS^1 \hookrightarrow \mathbb{R}^3 | L \text{ is a link} \} \subset C^\infty(nS^1, \mathbb{R}^3)
\]

with the open-compact Whitney topology. Denote by $\mathcal{L}$ the space of all links, being the disjoint sum of the spaces above. We will identify links lying in the same component of $\mathcal{L}$ according to the following definition.

**Definition 1.3.** Links $L_1$ and $L_2$ are regarded as equivalent, if there is a smooth path $\gamma: I \longrightarrow \mathcal{L}$ such that $\gamma(0) = L_1$ and $\gamma(1) = L_2$.

This definition is equivalent to the existence of a smooth isotopy $H: nS^1 \times I \longrightarrow \mathbb{R}^3$ from $L_1$ to $L_2$. It has a very geometrical meaning: links can be deformed so far as none

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1 The enumeration of knots used in this thesis comes from [24].
of its component is ,,torn” nor their parts intersect each other. Since now by links and knots we will mean their equivalence classes.

**Remark 1.4.** When forgetting orientation of circles, we get *non-oriented links* in opposite to *oriented links* defined above. The equivalence of oriented links descends to non-oriented if we identify links with different orientations.

**Remark 1.5.** One can also consider knots and links without the smoothness condition. However, with no additional restrictions it leads to a pathology called wild knots (fig. 2). To prevent from such situations a link is usually assumed to be equivalent to a sum of intervals (so called *PL-knots* or *combinatorial knots*). However, smooth knots are equivalent to combinatorial ones (see [4]).

![Figure 2. A wild knot](image)

We distinguish *trivial links* as those equivalent to an embedding of circles into a plane. They are denoted by \( nU \), where \( n \) stands for the number of components. In case \( n = 1 \) we write simply \( U \). The main problem of knot theory is to determine if two given links are equivalent or not, especially whether a given link is trivial (if it can be ,,untied”).

The are two basic operations on links:

- **mirroring**: \( L^* := S \circ L \), where \( S(x, y, z) = (x, y, -z) \) is the symmetry of \( \mathbb{R}^3 \) along the XY-plane,
- **reversion**: \(-L := L \circ (A, \ldots, A)\), where \( A(z) = \bar{z} \) is the symmetry of \( S^1 \) along the 0X-axis.

They commute and preserve the equivalence class of a link (compose the path \( \gamma \) in the definition of equivalence with an appropriate operation). \( L^* \) is called the mirror to \( L \) and \(-L \) the reversed link.

![Figure 3. Left- and right-handed trefoils are mirror knots. They are chiral and reversible.](image)

**Definition 1.6.** A link \( L \) is said to be *amphichiral* if \( L^* \sim L \) (otherwise it is called *chiral*) and *reversible* if \(-L \sim L \).
It is usually difficult to show chirality of a link. In particular, the fundamental group of the knot complement does not change after mirroring. In many cases the Jones polynomial defined in the section 5 is can be used. The figure eight knot $4_1$ is an example of an amphichiral knot, whereas the trefoil $3_1$ is chiral.

Even more challenging is to show non-reversibility, since most of known link invariants do not depend on orientations of link components. Among knots with up to eight crossings $8_{17}$ is the only non-reversible one. Another example is $K = 9_{32}$, producing four different oriented knots: $K, K^*, -K, -K^*$. We say $K$ is fully asymmetric.

![Figure 4](image1.png)

**Figure 4.** $8_{17}$ (in the middle) is non-reversible but equivalent to its mirror. $9_{32}$ (on the right) is fully asymmetric, whereas $4_1$ (on the left) is fully symmetric.

2. Link diagrams

Although links can be examined in a purely topological approach (using the fundamental group of their complements or covering spaces), one of the most efficient invariants arise from combinatorial approach based on diagrams of links. Consider a projection on a plain of a given link, satisfying the following conditions:

1. each point is an image of at most two points of the link
2. there are only finitely many double points
3. all intersections are transverse

Such a projection is called *regular*.

![Figure 5](image2.png)

**Figure 5.** Situations not allowed in regular projections

One can ask whether such projections exist. The Whitney’s embedding theorem implies a set of immerse projections is dense. Then transversality theorems gives a positive answer to the question.

**Theorem 2.1.** The subspace of regular projections of a given link is dense.

A regular projection does not fully describe a link — some information on double points (crossings) has to be included: which part of the link passes over (bridge) and which under (tunnel). It is noted by breaking the tunnel. A segment between two tunnels is called an arc.

**Definition 2.2.** A diagram of a link $L$ is a regular projection on a plain modified at each double point by breaking the part of the link that is closer to the plain.

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2 There is an analogous theorem for combinatorial knots, see [5, 21].
Let us distinguish two types of diagrams. We will see later, that links possessing such diagrams have very interesting properties.

**Definition 2.3.** A diagram \( D \) is *reduced* if each crossing meets four different regions. \( D \) is *alternating* if moving along any its component one goes alternately through tunnels and bridges. By an *alternating link* we mean a link with an alternating diagram.

Any link has a reduced diagram. Indeed, any not reduced crossing \( c \) can be removed by rotating part of the diagram. The first occurrence of a non-alternating knot is \( 8_{19} \).

![Figure 6. An example of an alternating non-reduced diagram.](image)

According to the theorem 2.1 each link has a diagram. On the other hand, a diagram encodes enough information to rebuild the link up to equivalence. Obviously, a link has a wide range of diagrams, but there exists a pretty elegant theorem classifying in some sense all of them.

**Theorem 2.4** (Reidemeister, 1927). Let \( L_1 \) and \( L_2 \) be links with diagrams \( D_1 \) and \( D_2 \) respectively. Then \( L_1 \) and \( L_2 \) are equivalent if and only if \( D_2 \) can be obtained from \( D_1 \) by applying isotopies of a plain and moves \( R_1 – R_3 \) (fig. 7).

![Figure 7. Reidemeister moves. Only one orientation for each case is shown. In the theorem 2.4 all possible orientations should be considered.](image)

Changes \( R_1 – R_3 \) are called *Reidemeister moves*. Elementary, though very technical proof of the above theorem can be found in [21, 23].

The number of crossings in a diagram is not a link invariant, because it the moves \( R_1 \) and \( R_2 \) does not preserve it. Define the *crossing number* \( c(L) \) of a link as the minimal number of crossings over all diagrams. Despite the simplicity of the definition, this number is very hard to compute. Any diagram bounds it from above, whereas homology groups defined in chapter 4 give estimation from below.

Theorem 2.4 gives no efficient method to check whether two diagrams represent the same link or not. For example, there exist diagrams of the unlink such that any move increases its number of crossings (so called demons, see fig. 8). Moreover, a single crossing may decide whether a link is trivial or not: by changing any crossing of the trefoil

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\(^3\) Those proofs are given for combinatorial knots. However, which some effort they can be translated into the language of differential topology.
to its mirror we get the unknot, although the trefoil itself is non-trivial. One can ask if any link can be untied by changing some crossings. The answer is positive: imagine a descending point moving over the given diagram. Change the crossings of this diagram such that the projection of the point is on a bridge at first pass through any crossing. The new diagram obtained in this way represents the unknot (fig. 9). In case of links repeat the procedure for each component.

Define the unknotting number $u(D)$ as the minimal number of crossings trivialising the diagram when changed into mirrors. It can be arbitrary large and is not a link invariant.

**Theorem 2.5** (K. Taniyama, 2008). A non-trivial link $L$ has a diagram with an arbitrary large unknotting number.

Analogously to the crossing number, define the unknotting number of a link $L$ as the minimum over all diagrams of $L$.

There are two types of crossings in oriented diagrams: positive and negative (fig. 10). The writhe number or the Tait number $w(D)$ of a diagram $D$ is the sum of signs over all crossings of $D$. It is changed by the first Reidemester move, but is preserved by others: the third move leaves all signs unchanged whereas the second creates or deletes two crossings with opposite signs.

Sings of crossings can be used to define another invariant. Let $L$ be a link, $L_1$ and $L_2$ its components. A linking number of $L_1$ and $L_2$ is the sum of signs over crossings of $L_1$ with $L_2$ divided by two:

$$\text{lk}(L_1, L_2) = \frac{1}{2} \sum_{c \in L_1 \cap L_2} \text{sgn}(c)$$
1. A brief introduction to knots

\[ \text{sgn} = +1 \quad \text{sgn} = -1 \]

**Figure 10.** Imagine components of a link are highways. Then the sign of the given crossing is determined by the vertical direction in turning right: positive, when moving up and negative when moving down.

In particular, one can compute the linking number of any component \( L_1 \) with the rest of a link \( L \setminus L_1 \). In opposite to the writhe number, the linking number is a link invariant, since move \( R_1 \) affects only one component.

3. The planar algebra of tangles

Links are compact sets. Hence, we can see them as embedded in a standard ball \( \mathbb{D}^3 \) instead of \( \mathbb{R}^3 \). This definition has the advantage that it can be extended over embeddings of intervals.

**Definition 3.1.** A *tangle* is a neat embedding\(^4\) of a disjoint sum of circles and closed intervals into a standard ball \( \mathbb{D}^3 \). Two tangles are called *equivalent*, if there exists an isotopy of the ball constant on the boundary, moving one tangle to the other.

**Figure 11.** A diagram of a tangle.

Analogously to links, a tangle has diagrams and two diagrams of a given tangle are related by Reidemeister moves. Denote by \( \mathcal{T}(B) \) the space of all tangles with endpoints in a set \( B \subset \partial \mathbb{D}^2 \).

Consider now a disk from the figure 12. One can put into its holes small tangles obtaining a bigger one. In this way we can construct from tangles any link. Obviously, instead of tangles one can put into holes other disks. This results in an algebraic structure, called the planar algebra of tangles. Now we will give a formal definition.

**Definition 3.2.** A *planar diagram* \( D \) with \( s \) inputs is a disk \( \mathbb{D}^2 \) missing smaller disks \( \mathbb{D}^2_i \) for \( i = 1, \ldots, s \), together with a neat embedding of disjoint circles and closed intervals. Say \( D \) is *oriented* if the embedded circles and intervals are oriented. Both oriented and non-oriented planar diagrams are considered up to planar isotopies constant on boundary of \( D \).

Denote by \( \mathcal{T}^0(B) \) the set of all tangle diagrams with the set of endpoints equal \( B \subset \mathbb{S}^1 \). Each planar diagram \( D \) induces a map

\[
\begin{align*}
D : \mathcal{T}^0(B_1) \times \cdots \times \mathcal{T}^0(B_s) & \longrightarrow \mathcal{T}^0(B) \\
(3.1)
\end{align*}
\]

\(^4\) An embedding of a manifold \( M \hookrightarrow N \) is called neat, if \( M \cap \partial N = \partial M \) and \( M \) is transverse to \( \partial N \).
for some sets $B, B_1, \ldots, B_s$. This gives us a structure in $\mathcal{T}^0$ called the planar algebra of tangle diagrams. In a similar way, oriented tangle diagrams $\mathcal{T}^0_+(B)$ with oriented planar diagrams create the oriented planar algebra $\mathcal{T}^0_+$. Due to locality of Reidemeister moves, both structures descend to tangles, resulting in planar algebras of non-oriented tangles $\mathcal{T}$ and oriented ones $\mathcal{T}_+$.

Among all planar diagrams we will distinguish the radial ones as those with one input in the middle, and only radial intervals. They induce identities in the above algebras.

Given a decomposition of a link into tangles and a planar diagram one may think how replacing any tangle by another one may affect the equivalence class of the link.

**Definition 3.3.** Let $D$ be a diagram of a link $L$ and $T$ its fragment being a tangle diagram with four ends in a corner of some square. A mutation of a link $L$ is a change given by rotating $T$ by $180^\circ$ along one of the following axes of the square: vertical, horizontal or perpendicular to the plane containing the diagram. The new link obtained by this change is called a mutant of $L$.

Many of known invariants cannot distinguish mutants, also the Jones polynomial defined later.

In next chapters we will find similar algebraic structures in other categories. Thus we will give now an abstract definition of a planar algebra (compare with [9]).

**Definition 3.4.** A planar algebra (oriented) $\mathcal{P}$ is a collection of sets $\mathcal{P}(B)$ defined for finite subsets $B \subset \mathbb{S}^1$ (oriented) together with an operator

\[(3.2) \quad D: \mathcal{P}(B_1) \times \cdots \times \mathcal{P}(B_s) \longrightarrow \mathcal{P}(B)\]

defined for each planar diagram (oriented) $D$, such that their composition is associative and radial diagrams correspond to identities.
1. A brief introduction to knots

Elements of \( \mathcal{P}(\emptyset) \) are \textit{closed} and operators taking values in this set — \textit{closure operators}. Denote by \( \mathcal{CPO} \) the set of all such operators. It splits into sets \( \mathcal{CPO}(B) \) consisting of operators with domain in \( \mathcal{P}(B) \). In general, let

\[
P(B_1, \ldots, B_s; B) := \{ D: \mathcal{P}(B_1) \times \cdots \times \mathcal{P}(B_s) \to \mathcal{P}(B) \}
\]

Then \( \mathcal{CPO}(B) = \mathcal{P}(B; \emptyset) \).

**Definition 3.5.** A \textit{morphism} of planar algebras \( \Phi: \mathcal{P}_1 \to \mathcal{P}_2 \) is a collection of maps \( \Phi_B: \mathcal{P}_1(B) \to \mathcal{P}_2(B) \) commuting with planar operators:

\[
D \circ (\Phi_{B_1}, \ldots, \Phi_{B_s}) = \Phi_B \circ D.
\]

Sending a tangle diagram to the tangle itself is an example of a morphism of planar algebras. Other examples will be given in the chapter 2.

4. The Kauffman bracket

Consider now non-oriented links. Each crossing has two resolutions — type 0 and type 1 (fig. 14). Unless it leads to confusion, denote by \( \times, \times' \) and \( \bigtimes \) diagrams which differ in a single crossing as is presented by the symbols.

\[
\begin{array}{ccc}
\times & \leftrightarrow & \times' \\
\bigtimes & \to & \bigtimes'
\end{array}
\]

**Figure 14.** A type of a resolution, similarly to a sign of a crossing, has a simple interpretation. Again, considering link components as highways, any resolution can be compared to a change of direction. Here a type of a resolution describe the level to be left (zero – lower, one – upper).

Define inductively a polynomial \( \langle D \rangle \) in variables \( A, B, d \) by the following equations:

(K1) \( \langle U \rangle = 1 \),

(K2) \( \langle U \sqcup D \rangle = d \langle D \rangle \),

(K3) \( \langle \bigtimes \rangle = A \langle \times \rangle + B \langle \times' \rangle \).

**Lemma 4.1.** Let \( B = A^{-1} \) and \( d = -(A^2 + A^{-2}) \). Then \( \langle D \rangle \) is an invariant under II and III Reidemeister moves as well as

\[
\begin{align*}
\langle \bigtimes \rangle &= -A^3 \langle \times \rangle \\
\langle \bigtimes' \rangle &= -A^{-3} \langle \times \rangle
\end{align*}
\]

**Proof.** To show (4.1) let us delete a crossing using (K3)

\[
\langle \bigtimes \rangle = A \langle \times \rangle + A^{-1} \langle \bigtimes \rangle = -A^3 \langle \times \rangle
\]

In a similar way we obtain the second equality. They imply invariance under II Reidemeister move:

\[
\langle \bigtimes \times \rangle = A \langle \bigtimes \times' \rangle + A^{-1} \langle \bigtimes \rangle = A^2 \langle \times \rangle + \langle \bigtimes \rangle - A^2 \langle \times' \rangle = \langle \bigtimes \rangle
\]

The proof of invariance under III Reidemeister move does not need relations among \( A, B \) and \( d \). It is derived directly from invariance under II move:

\[
\langle \bigtimes \rangle = A \langle \bigtimes \times \rangle + B \langle \bigtimes \times' \rangle = A \langle \bigtimes \times \rangle + B \langle \bigtimes \times' \rangle = \langle \bigtimes \rangle
\]

\( \square \)
The polynomial $\langle D \rangle$ in variable $A$ is called the Kauffman bracket. It was introduced by L. Kauffman in \cite{4.2}. As a consequence of \cite{K3} the Kauffman bracket of a link can be computed by summing up polynomials of trivial links. Define a Kauffman state $S$ of a diagram $D$ as a sequence of resolutions of all crossings:

\begin{equation}
S : \text{Cr}(D) \longrightarrow \{0, 1\}
\end{equation}

where $\text{Cr}(D)$ is the set of crossings of the diagram $D$. Denote by $S(D)$ the set of all states of $D$. Each state describes a collection $|S|$ of disjoint circles in a plane. Let $n_1(S), n_0(S)$ be the amounts of resolutions of type 1 and 0 accordingly and put $\tau(S) = n_1(S) - n_2(S)$.

\textbf{Theorem 4.2.} Let $D$ be a link diagram. Then:

\begin{equation}
\langle D \rangle = \sum_{S \in S(D)} (-1)^{|S|-1} A^{\tau(S)} (A^{-2} + A^2)^{|S|-1}.
\end{equation}

\textbf{Proof.} For trivial links the equality goes directly from \cite{K1} and \cite{K2}. Other cases are done by induction on the number of crossings, using relation \cite{K3}. Indeed, picking a crossing $c$ we have a bijection

$$S(\infty) = S_0(\infty) \cup S_1(\infty) \approx S(\infty) \cup S(\infty)$$

where $S_\alpha$ is the set of states satisfying $S(c) = \alpha$. Then

$$\langle \infty \rangle = A(\infty) + A^{-1} \langle \infty \rangle =$$

$$= \sum_{S \in S(\infty)} (-1)^{|S|-1} A^{\tau(S)+1} (A^{-2} + A^2)^{|S|-1} + \sum_{S \in S(\infty)} (-1)^{|S|-1} A^{\tau(S)-1} (A^{-2} + A^2)^{|S|-1} =$$

$$= \sum_{S \in S_0(\infty)} (-1)^{|S|-1} A^{\tau(S)} (A^{-2} + A^2)^{|S|-1} + \sum_{S \in S_1(\infty)} (-1)^{|S|-1} A^{\tau(S)} (A^{-2} + A^2)^{|S|-1} =$$

$$= \sum_{S \in S(\infty)} (-1)^{|S|-1} A^{\tau(S)} (A^{-2} + A^2)^{|S|-1}$$

\square

After appearance of M. Khovanov’s paper \cite{14}, O. Viro introduced in \cite{29} the notion of an enhanced Kauffman state, adding orientations to circles. The sum over these states has a simpler form — it is a sum of monomials.

\textbf{Definition 4.3.} An enhanced Kauffman state $S$ is a map which associates to each crossing a resolution and orientation to each circle in the smoothed diagram described by the resolutions.

The set of all enhanced states will be denoted by $ES(D)$. Let $d_+(S)$, $d_-(S)$ be the amounts of positively and negatively oriented circles accordingly and put $\sigma(S) = d_+(S) - d_-(S)$. Then

\begin{equation}
(A^{-2} + A^2)^{|S|} = \sum_{i=0}^{|S|} \binom{|S|}{i} A^{2(|S|-i)-2i} = \sum_{S'} A^{2\sigma(S')}
\end{equation}

where the last sum is taken over all enhanced states equal $S$, when we forget the orientations of circles. As a result we have the following statement.

---

\footnote{Formally, $\langle D \rangle$ is a Laurent polynomial, i.e. an element of a ring $\mathbb{Z}[A, A^{-1}]$.}
1. A brief introduction to knots

Theorem 4.4. Let $D$ be a link diagram. Then

\begin{equation}
\langle D \rangle = \sum_{S \in ES(D)} (-1)^{|S|-1} A^{r(S)+2sS}
\end{equation}

5. The Jones polynomial

Here we will define a polynomial that is an invariant of non-oriented links. Firstly, notice we have already defined for an oriented diagram $D$ two objects, which are preserved under II and III Reidemeister move:

- the Kauffman bracket $\langle D \rangle$ (when the orientation of $D$ is forgotten),
- the writhe number $w(D)$.

Define a new polynomial for an oriented diagram $D$, evaluating $\langle D \rangle$ at $t^{-1/4}$ and multiplying it by $(-t)^{-3w(D)}$:

\begin{equation}
V_D(t) = (-t)^{-3w(D)} \langle D \rangle_{A=t^{1/4}}
\end{equation}

Proposition 5.1. The polynomial $V_D(t)$ defined by (5.1) is a link invariant and the following holds:

\begin{enumerate}
\item[(J1)] $V_U(t) = 1$,
\item[(J2)] $t^{-1} V_{\mathcal{X}}(t) - t V_{\mathcal{Y}}(t) = \left(t^{1/2} - t^{-1/2}\right) V_{\mathcal{Y}}(t)$.
\end{enumerate}

Proof. Invariance under II and III Reidemeister move is due to invariance of the writhe number and the Kauffman bracket. By lemma 4.1 I move also preserves $V_D$:

\[ V_{\mathcal{X}}(t) = (-A)^{-3w(\mathcal{X})} \langle \mathcal{X} \rangle = (-A)^{-3w(\mathcal{X})+1}(-A^3) \langle \mathcal{Y} \rangle = (-A)^{-3w(\mathcal{Y})} \langle \mathcal{Y} \rangle = V_{\mathcal{Y}}(t) \]

and similarly for the second loop.

The normalisation condition [J1] is satisfied by the definition of $V_D$. To show [J2] notice that [K3] implies:

\begin{align*}
A\langle \mathcal{X} \rangle &= A^2 \langle \mathcal{X} \rangle + \langle \mathcal{Y} \rangle \\
A^{-1}\langle \mathcal{X} \rangle &= \langle \mathcal{X} \rangle + A^{-2} \langle \mathcal{Y} \rangle \\
A\langle \mathcal{X} \rangle - A^{-1}\langle \mathcal{X} \rangle &= (A^2 - A^{-2}) \langle \mathcal{X} \rangle
\end{align*}

and the last equality can be written for an oriented link as:

\[ -A^4(-A)^{-3w(\mathcal{X})} \langle \mathcal{X} \rangle + A^{-4}(-A)^{-3w(\mathcal{X})} \langle \mathcal{X} \rangle = (A^2 - A^{-2})(-A)^{-3w(\mathcal{X})} \langle \mathcal{X} \rangle \]

The change of powers at the left hand side appears because of the difference in writhe numbers: diagrams at the left side have one more crossing than the one at the right side.

To end the proof put $t = A^{-4}$. \hfill $\square$

Due to the above theorem $V_L(t)$ is a Laurent polynomial in $t^{1/2}$ with integer coefficients, called the Jones polynomial of a link $L$. It was discovered before the Kauffman bracket and defined by axioms from proposition 5.1. The equality [J2] is called a skein relation and with the normalisation condition [J1] can be used to compute the polynomial without using Kauffman states.

Example 5.2. Consider the following three diagrams:
The first two of them represent the unknot, hence by the skein relation:

\[(t^{1/2} - t^{-1/2})V_\Sigma(t) = t^{-1}V_\Sigma(t) - tV_\Sigma(t) = t^{-1} - t\]

and as a result \(V_{2U}(t) = -t^{-1/2} - t^{1/2}\). By induction:

\[(5.3) \quad V_{nU}(t) = (-t^{1/2} - t^{-1/2})^{n-1}.\]

Obviously, it is the same as when computed from Kauffman states.

**Example 5.3.** We will compute now the polynomial for the left-handed trefoil using the computing tree from figure 15. Vertices 3₁ and X describe the equalities:

\[(5.4) \quad t^{-1}V_U(t) - tV_{3₁}(t) = (t^{1/2} - t^{-1/2})V_X(t)\]
\[(5.5) \quad t^{-1}V_{2U}(t) - tV_X(t) = (t^{1/2} - t^{-1/2})V_U(t)\]

Having already computed the polynomials of trivial links, we get:

\[V_{3₁}(t) = t^{-2}V_U(t) + (t^{-3/2} - t^{-1/2}) (t^{-2}V_{2U}(t) + (t^{-3/2} - t^{-1/2})V_U(t))\]
\[= - t^{-4} + t^{-3} + t^{-1}.\]

**Figure 15.** The computing tree for the trefoil.

**Remark 5.4.** The polynomial for the right-handed trefoil is

\[(5.6) \quad V_{3₁}^*(t) = -t^4 + t^3 + t\]

so the trefoil is chiral.

**Remark 5.5.** Reversing all link components does not affect signs of crossings. Hence writhe \(w(D)\) is preserved and

\[(5.7) \quad V_L(t) = V_{-L}(t)\]

Thus the Jones polynomial cannot distinguish a link from its reversion. In particular, it is well-defined for non-oriented knots.
The example 5.3 shows a general method how to compute the Jones polynomial using only the skein relation. Indeed, we can build a computing tree for any link. Recall that for each diagram $D$ there is a sequence of crossings $c_1, \ldots, c_k$, such that $D$ becomes trivial when all $c_i$’s are changed into mirrors (see fig. 9). Denote by:

- $D_i$ a diagram $D$ with changed $c_1, \ldots, c_i$ into mirrors,
- $D_i^0$ a diagram $D$ with changed $c_1, \ldots, c_{i-1}$ into mirrors and $c_i$ smoothed.

Build a tree according to the following rules:

1. $D$ is the root
2. vertices are given by $D_i, D_i^0$
3. branches go from $D_i$ to both $D_{i+1}$ and $D_{i+1}^0$ for $i < k$

This tree has $D_k$ and $D_i^0$’s in leaves. The first diagram is trivial and the rest have less crossings than $D$, so we can build inductively a computing tree for each of them. Eventually, we end with a tree having trivial links in leaves. Having computed polynomials for trivial links (example 5.2), we can calculate polynomials for vertices consecutively starting from leaves and ending in the root $D$. As a result we have

**Theorem 5.6.** The Jones polynomial is the unique invariant polynomial satisfying (J1) and (J2).

A spectacular application of the Jones polynomial was proving three conjectures, stated in the second half of 19th century by P. G. Tait, who tried to classify knots.

**Theorem 5.7 (Tait’s conjectures).** Let $L$ be an alternating link. Then

- $T1$: any reduced alternating diagram of $L$ has a minimal crossing number,
- $T2$: any two reduced alternating diagrams of $L$ have the same writhe number,
- $T3$: any reduced alternating diagram of $L$ can be obtained from another one by local flips (fig. 16).

**Figure 16.** Local moves classifying alternating diagrams.

First two statements have been proved independently by L. H. Kauffman [12], K. Murasugi [19] and M. B. Thistlethwaite [28] during two years after appearance of the paper of Jones. The third claim had been waiting to be proven till 1990’s, when papers of W. M. Menasco and M. B. Thistlethwaite appeared ([17, 18]).

There are non-trivial links with Jones polynomial equal 1 as well as different knots with the same Jones polynomial. The following theorem shows one way how to produce such pairs.

**Theorem 5.8.** Let $L_2$ be a link obtained from $L_1$ by a mutation. Then $V(L_1)$ and $V(L_2)$ are equal.
The proof can be found in [21]. Even more is shown there: any invariant defined by a skein relation is preserved by mutations. However, it is still unknown whether the Jones polynomial detects unknottedness or not.

\footnote{Such an invariant is said to be of Conway type.}
CHAPTER 2

Cobordisms with chronologies

The standard Khovanov complex lives in the category of oriented cobordisms, described with details in [13]. The notion of a chronology introduced in this chapter breaks some symmetries what results in a richer category still having a finite presentation. However, such a rigid structure is unnecessary to build a generalized complex. Thus we will weaken it by allowing some changes of chronologies keeping control over this process.

1. Oriented cobordisms

Let $M$ be an oriented manifold with a fixed orientation on its boundary $\partial M$. The input $M_{in}$ is the maximal boundary component with the orientation equal the induced from $M$, whereas the output $M_{out}$ is the maximal boundary component with the opposite orientation.

![Figure 1. Examples of cobordisms. Arrows denote inputs and outputs. They are usually omitted assuming the input is on the left-hand side of the cobordisms and the output on the right-hand side.](image)

The definition is independent of the choice of orientations in the following sense: reversing orientations of $M$ and $\partial M$ preserves both the input and the output of $M$.

**Definition 1.1.** An oriented cobordism between oriented $n$-manifolds $\Sigma_{in}$, $\Sigma_{out}$ is an oriented pair $(M, \partial M)$ of dimension $(n+1)$ along with diffeomorphisms $\varphi_{in}: \Sigma_{in} \rightarrow M_{in}$ and $\varphi_{out}: \Sigma_{out} \rightarrow M_{out}$ preserving the orientations. It is denoted by

$$\Sigma_{in} \rightarrow M \leftarrow \Sigma_{out} \quad \text{or} \quad M: \Sigma_{in} \Rightarrow \Sigma_{out}.$$

Diffeomorphic manifolds represent the same cobordism, if the diffeomorphism agrees with both input and output.

**Definition 1.2.** A diffeomorphism $\psi: M \rightarrow M'$ is called an equivalence of cobordisms $M: \Sigma_0 \Rightarrow \Sigma_1$ and $M': \Sigma_0 \Rightarrow \Sigma_1$, if the following diagram commutes:

$$\Sigma_0 \xrightarrow{\Sigma_0 \leftarrow \psi \rightarrow \Sigma_1} M \xrightarrow{M \leftarrow M'} \Sigma_1$$

Equivalent cobordisms will be identified.
2. Cobordisms and chronologies

Example 1.3. Let $\Sigma$ be an $n$-manifold. Consider the product $M = \Sigma \times I$ along with embeddings

$$\Sigma \approx \Sigma \times 0 \hookrightarrow \Sigma \times I \leftarrow \Sigma \times 1 \approx \Sigma$$

In this way we obtain a cobordism with $\Sigma$ being as both the input and the output. It is called the cylinder induced by $\Sigma$ and denoted by $C_\Sigma$ or simply $C$.

In general, an orientation preserving diffeomorphism $\varphi: \Sigma_1 \to \Sigma_0$ defines a cobordism

$$\Sigma_0 \xrightarrow{\text{id}} \Sigma_0 \times I \xleftarrow{\varphi} \Sigma_1$$

denoted by $C_\varphi$ and called the cylinder generated by $\varphi$. The cobordism $C_\Sigma$ will be called the identity cylinder on $\Sigma$.

$C_\varphi$ is not equivalent to the identity cylinder, unless $\varphi$ is isotopic to the identity.

Lemma 1.4. Cobordisms $C_{\varphi_1}, C_{\varphi_2}$ are equivalent if and only if the diffeomophisms $\varphi_1, \varphi_2$ are isotopic.

Proof. Let $\psi: C_{\varphi_1} \to C_{\varphi_2}$ be an equivalence of cobordisms. The desired isotopy is given by the following composition:

$$\Sigma_1 \times I \xrightarrow{\varphi_1 \times \text{id}} \Sigma_0 \times I \xrightarrow{\psi} \Sigma_0 \times I \xrightarrow{\pi_1} \Sigma_0$$

where $\pi_1$ is the projection on the first variable.

Conversely, an isotopy $H: \Sigma_1 \times I \to \Sigma_0$ between $\varphi_1$ and $\varphi_2$ induces an equivalence of cobordisms:

$$\begin{array}{c}
\Sigma_0 \\
\downarrow \text{id} \\
\Sigma_1 \times I \\
\downarrow (H, \pi_2) \\
\Sigma_0 \times I \\
\uparrow \text{id} \\
\Sigma_0 \\
\downarrow \text{id} \\
\Sigma_1 \\
\downarrow \varphi_2 \\
\Sigma_1 \\
\downarrow \varphi_1 \\
\Sigma_0 \\
\downarrow \varphi_1^{-1} \times \text{id} \\
\Sigma_0 \times I \\
\uparrow \varphi_0 \times \text{id} \\
\Sigma_1 \times I \\
\downarrow \text{id} \\
\Sigma_0 \\
\end{array}$$

where $\pi_2$ is the projection on the second variable. \qed

Example 1.5. Let $\Sigma_1$ and $\Sigma_2$ be $n$-manifolds. Take their disjoint sum

$$\Sigma_1 \sqcup \Sigma_2 = (\Sigma_1 \times 0) \cup (\Sigma_2 \times 1)$$
with the natural smooth structure. Exchanging its components defines a diffeomorphism

\[ s_{\Sigma_1,\Sigma_2} : \Sigma_1 \sqcup \Sigma_2 \longrightarrow \Sigma_2 \sqcup \Sigma_1 \]

In case \( \Sigma_1 = \Sigma_2 = \Sigma \) the diffeomorphism \( s_{\Sigma,\Sigma} \) is not isotopic to the identity, hence \( C_{s_{\Sigma,\Sigma}} \) is not an identity cylinder.

![Figure 3. A cobordism induced by a permutation of components of a disjoint sum is not equivalent to an identity cylinder.](image)

**Example 1.6.** Consider a family of diffeomorphisms of a two-dimensional torus:

\[ \varphi_{m,n}(z, w) = (zw^m, wz^n), \quad \gcd(m, n) = 1. \]

They send a curve \( \gamma(t) = (e^{2\pi it}, e^{2\pi it}) \) into non-homotopic curves, thus the diffeomorphisms are not isotopic. Hence all cobordisms \( C_{\varphi_{m,n}} \) are different.

A similar observation gives for an arbitrary manifold \( \Sigma \) a one-to-one correspondence

\[ \text{Diff}(\Sigma) / \sim \ni [\varphi] \mapsto C_\varphi \]

where \( \text{Diff}(\Sigma) / \sim \) is the group of isotopy classes of diffeomorphisms of \( \Sigma \).

We will now define basic operations on cobordisms.

1. The *reversion* of a cobordism \( M : \Sigma_0 \Rightarrow \Sigma_1 \) is the cobordism \( M^* : \Sigma_1 \Rightarrow \Sigma_0 \) given by reversing the orientations of \( M \) but preserving the orientation of \( \partial M \).

2. The *gluing* of cobordisms \( M_1 : \Sigma_0 \Rightarrow \Sigma_1 \) and \( M_2 : \Sigma_1 \Rightarrow \Sigma_2 \) is the cobordism \( M_1 \cup M_2 : \Sigma_0 \Rightarrow \Sigma_2 \) defined as the sum of manifolds \( M_1 \) and \( M_2 \) along the boundary \( \Sigma_1 \).

3. Say \( M_1 : \Sigma_0 \Rightarrow \Sigma_1 \) and \( M_2 : \Sigma_1 \Rightarrow \Sigma_2 \) is a *split* of \( M : \Sigma_0 \Rightarrow \Sigma_2 \), whenever \( M : \Sigma_0 \Rightarrow \Sigma_2 \) is a gluing of \( M_1 : \Sigma_0 \Rightarrow \Sigma_1 \) and \( M_2 : \Sigma_1 \Rightarrow \Sigma_2 \).

4. The *multiplication* of cobordisms \( M : \Sigma_0 \Rightarrow \Sigma_1 \) and \( M' : \Sigma_0' \Rightarrow \Sigma_1' \) is the disjoint sum:

\[ \Sigma_0 \sqcup \Sigma_0' \longrightarrow M \sqcup M' \leftarrow \Sigma_1 \sqcup \Sigma_1' \]

1 Details on the fundamental group and the mapping class group of a torus can be found in [24].
with induced input and output.

\[
\begin{array}{c}
\text{\includegraphics[width=0.5\textwidth]{cobordism.png}}
\end{array}
\]

The multiplication of \( n \) copies of a cobordism \( M \) is denoted by \( M^n \).

All the operations agree with the equivalence relation of cobordisms. For details look in [13].

**Lemma 1.7.** The operations defined above have the following properties:

1. \((M_1 M_2) M_3 = M_1 (M_2 M_3)\)
2. \(C_{\Sigma_0} M = M = MC_{\Sigma_1}\), where \( M : \Sigma_0 \Rightarrow \Sigma_1 \)
3. \(M_1 M_2 \sqcup N_1 N_2 = (M_1 \sqcup N_1)(M_2 \sqcup N_2)\)
4. \(C_{\Sigma} \sqcup C_{\Sigma'} = C_{\Sigma \sqcup \Sigma'}\)
5. \((M_1 M_2)^* = M_2^* M_1^*\)
6. \(C_{\Sigma}^* = C_{\Sigma'}^*\)
7. \(M^{**} = M\)
8. \((M \sqcup N)^* = M^* \sqcup N^*\)
9. if \(K_1, K_2\) is a split of \(M \sqcup N\), then there exist splits \(M = M_1 M_2, N = N_1 N_2\) such that \(K_i = M_i \sqcup N_i\)

**Proof.** All equalities except (9) follow directly from definitions of the operations and topological properties of cobordisms. For (9) take \(M_i = K_i \cap M\) and \(N_i = K_i \cap N\).  

Due to the first two points of the lemma, cobordisms form a category \(\text{Cob}\) as follows:

- objects are closed oriented manifolds
- morphisms are equivalence classes of oriented compact cobordisms
- the composition of morphisms is given by the gluing of cobordisms: \(M \circ N := NM\)
- the identity \(\text{id}_{\Sigma}\) is given by the cylinder \(C_{\Sigma}\)

Other points show \(\text{Cob}\) is a symmetric monoidal category.

**Definition 1.8.** A category \(\mathcal{C}\) is called **monoidal**, if there exists a functor \(\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}\) called **multiplication**, an object \(e \in \text{Ob}(\mathcal{C})\) called the **unit** and natural equivalences \(R_X : X \otimes e \rightarrow X, L_X : e \otimes X \rightarrow X, A_{XYZ} : X \otimes (Y \otimes Z) \rightarrow (X \otimes Y) \otimes Z\) such that the following diagrams commute:

\[
\begin{array}{c}
\text{\includegraphics[width=0.8\textwidth]{monoidal.png}}
\end{array}
\]

\(X \otimes (Y \otimes (Z \otimes W)) \xrightarrow{A} (X \otimes Y) \otimes (Z \otimes W) \xrightarrow{A} ((X \otimes Y) \otimes Z) \otimes W\)
Furthermore, if there exists a natural equivalence $S_{XY}: X \otimes Y \longrightarrow Y \otimes X$ such that $S_{XY} \circ S_{YX} = \text{id}$, $R_X = L_X \circ S_X$ and the following diagram commutes

$$
\begin{array}{ccl}
X \otimes (Y \otimes Z) & \xrightarrow{A} & (X \otimes Y) \otimes Z \\
\downarrow \text{id} \otimes S & & \downarrow S \\
X \otimes (Z \otimes Y) & \xrightarrow{A} & (X \otimes Z) \otimes Y
\end{array}
$$

the monoidal category $C$ is called symmetric.

A functor $F: C \longrightarrow D$ is called monoidal, if it preserves the multiplication:

$$
F \circ \otimes_C = \otimes_D \circ (F, F) \quad \text{and} \quad F e_C = e_D,
$$

and agrees with equivalences: $F \circ L_X^C = L_{F_X}$ and similarly for $R, A$. If the symmetry is also preserved, then $F$ is called a symmetric monoidal functor.

**Example 1.9.** The category $\text{Set}$ of sets possesses two monoidal structures given by the Cartesian product and by the disjoint sum. Both of them are symmetric.

**Example 1.10.** The category $\text{Mod}_R$ of $R$-modules is symmetric monoidal with the multiplication given by the tensor product. Another monoidal structure is given by the exterior product. This structure is symmetric with a skew-linear permutation.

**Corollary 1.11.** The category of cobordisms $\text{Cob}$ with the multiplication is symmetric monoidal. Moreover, the reversion is an contravariant functor which is inverse to itself.

**Proof.** Due to points (3) – (8) of the lemma 1.7, the multiplication of cobordisms is a functor with an identity $e = \emptyset$, whereas the reversion is an contravariant functor. Equivalences $A_{\Sigma\Sigma',\Sigma''}, L_{\Sigma}, R_{\Sigma}$ and $S_{\Sigma\Sigma'}$ are given by the standard diffeomorphisms $a_{\Sigma\Sigma',\Sigma''}, l_{\Sigma}, r_{\Sigma}, s_{\Sigma\Sigma'}$:

$$
\begin{align*}
A_{\Sigma\Sigma',\Sigma''}: (\Sigma \sqcup \Sigma') \sqcup \Sigma'' & \approx \Sigma \sqcup (\Sigma' \sqcup \Sigma'') \\
L_{\Sigma}: \emptyset \sqcup \Sigma & \Rightarrow \Sigma \\
r_{\Sigma}: \Sigma \sqcup \emptyset & \approx \Sigma \\
s_{\Sigma\Sigma'}: \Sigma \sqcup \Sigma' & \approx \Sigma' \sqcup \Sigma
\end{align*}
$$

Denote by $n\text{Cob}$ the subcategory of $(n-1)$-manifolds and $n$-cobordisms.

Hence,

$$
\text{Cob} = \bigcup_{n \in \mathbb{Z}_+} n\text{Cob}.
$$

and both the multiplication and the reversion preserve the decomposition.

**Example 1.12.** Hilbert spaces and linear operations with a tensor product form a symmetric monoidal category. A monoidal functor $F: n\text{Cob} \longrightarrow \text{Hilb}$ is called a topological quantum field theory (TQFT). Sometimes it is also assumed that reversion corresponds to conjugation: $F(M^*) = F(M)^*$.

Classification of surfaces gives a finite presentation of $2\text{Cob}$. We state the theorem below without proof, which can be found in [13].

---

2 The empty set $\emptyset$ is considered as a manifold of any dimension. It is both an object and a morphism of each category $n\text{Cob}$. 
Theorem 1.13. Any $(1+1)$-cobordism is generated under the composition and the multiplication by the following six cobordisms:

\begin{align*}
\text{cylinder} & \quad \text{merge} & \quad \text{birth} & \quad \text{split} & \quad \text{death} & \quad \text{permutation} \\
\end{align*}

Moreover, any two decompositions define equivalent cobordisms if and only if one can be obtained from the other by the following relations:

- **permutation group relations:**
  \[
  \begin{array}{c}
  \text{cylinder} \\
  \text{merge}
  \end{array}
  =
  \begin{array}{c}
  \text{cylinder} \\
  \text{merge}
  \end{array}
  \\
  \begin{array}{c}
  \text{cylinder} \\
  \text{split}
  \end{array}
  =
  \begin{array}{c}
  \text{cylinder} \\
  \text{split}
  \end{array}
  \\
  \begin{array}{c}
  \text{cylinder} \\
  \text{death}
  \end{array}
  =
  \begin{array}{c}
  \text{cylinder} \\
  \text{death}
  \end{array}
  \\
  \begin{array}{c}
  \text{cylinder} \\
  \text{permutation}
  \end{array}
  =
  \begin{array}{c}
  \text{cylinder} \\
  \text{permutation}
  \end{array}
  \\
  \begin{array}{c}
  \text{permutation}
  \end{array}
  =
  \begin{array}{c}
  \text{permutation}
  \end{array}
  \\
  \end{align*}

- **behaviour of a birth and a merge under a permutation:**
  \[
  \begin{array}{c}
  \text{cylinder} \\
  \text{birth}
  \end{array}
  =
  \begin{array}{c}
  \text{cylinder} \\
  \text{merge}
  \end{array}
  \\
  \begin{array}{c}
  \text{cylinder} \\
  \text{split}
  \end{array}
  =
  \begin{array}{c}
  \text{cylinder} \\
  \text{birth}
  \end{array}
  \\
  \begin{array}{c}
  \text{cylinder} \\
  \text{birth}
  \end{array}
  =
  \begin{array}{c}
  \text{cylinder} \\
  \text{split}
  \end{array}
  \\
  \begin{array}{c}
  \text{cylinder} \\
  \text{death}
  \end{array}
  =
  \begin{array}{c}
  \text{cylinder} \\
  \text{birth}
  \end{array}
  \\
  \begin{array}{c}
  \text{cylinder} \\
  \text{permutation}
  \end{array}
  =
  \begin{array}{c}
  \text{cylinder} \\
  \text{split}
  \end{array}
  \\
  \begin{array}{c}
  \text{cylinder} \\
  \text{permutation}
  \end{array}
  =
  \begin{array}{c}
  \text{cylinder} \\
  \text{death}
  \end{array}
  \\
  \end{align*}

- **behaviour of a merge and a split under a permutation:**
  \[
  \begin{align*}
  \text{merge} & \quad \text{split} \\
  \text{death}
  \end{align*}
  \\
  \begin{align*}
  \text{ merge} & \quad \text{split} \\
  \text{death}
  \end{align*}
  \\
  \begin{align*}
  \text{merge} & \quad \text{split} \\
  \text{death}
  \end{align*}
  \\
  \begin{align*}
  \text{merge} & \quad \text{split} \\
  \text{death}
  \end{align*}
  \\
  \begin{align*}
  \text{merge} & \quad \text{split} \\
  \text{death}
  \end{align*}
  \\
  \end{align*}

- **associativity and coassociativity laws:**
  \[
  \begin{align*}
  \text{merge} & \quad \text{split} \\
  \text{death}
  \end{align*}
  \\
  \begin{align*}
  \text{merge} & \quad \text{split} \\
  \text{death}
  \end{align*}
  \\
  \begin{align*}
  \text{merge} & \quad \text{split} \\
  \text{death}
  \end{align*}
  \\
  \begin{align*}
  \text{merge} & \quad \text{split} \\
  \text{death}
  \end{align*}
  \\
  \begin{align*}
  \text{merge} & \quad \text{split} \\
  \text{death}
  \end{align*}
  \\
  \end{align*}

- **commutativity and cocommutativity laws:**
  \[
  \begin{align*}
  \text{merge} & \quad \text{split} \\
  \text{death}
  \end{align*}
  \\
  \begin{align*}
  \text{merge} & \quad \text{split} \\
  \text{death}
  \end{align*}
  \\
  \begin{align*}
  \text{merge} & \quad \text{split} \\
  \text{death}
  \end{align*}
  \\
  \begin{align*}
  \text{merge} & \quad \text{split} \\
  \text{death}
  \end{align*}
  \\
  \begin{align*}
  \text{merge} & \quad \text{split} \\
  \text{death}
  \end{align*}
  \\
  \end{align*}

- **the unit and the counit laws:**
  \[
  \begin{align*}
  \text{merge} & \quad \text{split} \\
  \text{death}
  \end{align*}
  \\
  \begin{align*}
  \text{merge} & \quad \text{split} \\
  \text{death}
  \end{align*}
  \\
  \begin{align*}
  \text{merge} & \quad \text{split} \\
  \text{death}
  \end{align*}
  \\
  \begin{align*}
  \text{merge} & \quad \text{split} \\
  \text{death}
  \end{align*}
  \\
  \begin{align*}
  \text{merge} & \quad \text{split} \\
  \text{death}
  \end{align*}
  \\
  \end{align*}

- **the Frobenius law:**
  \[
  \begin{align*}
  \text{merge} & \quad \text{split} \\
  \text{death}
  \end{align*}
  \\
  \begin{align*}
  \text{merge} & \quad \text{split} \\
  \text{death}
  \end{align*}
  \\
  \begin{align*}
  \text{merge} & \quad \text{split} \\
  \text{death}
  \end{align*}
  \\
  \begin{align*}
  \text{merge} & \quad \text{split} \\
  \text{death}
  \end{align*}
  \\
  \begin{align*}
  \text{merge} & \quad \text{split} \\
  \text{death}
  \end{align*}
  \\
  \end{align*}

2. A chronology

Let $M$ be a cobordism and $\tau : M \to I$ its projection on the unit interval. It can be seen as a deformation of a space $\tau^{-1}(0)$ into $\tau^{-1}(1)$ in time. Critical points corresponds to the moments, when the modified space is not a manifold – some critical event occurs (i.e. in dimension two it can be a merge, a split, etc.). Our task in this chapter is to enrich cobordisms, so we can keep track on such critical events.
Definition 2.1. Let $M: \Sigma_0 \Rightarrow \Sigma_1$ be a cobordism. A chronology on $M$ is a function $\tau: M \rightarrow I$ such that

1. $\tau^{-1}(0) = M_{in}$
2. $\tau^{-1}(1) = M_{out}$
3. critical points of $\tau$ are non-degenerated
4. there is exactly one critical point for each critical level of $\tau$

A cobordism $M: \Sigma_0 \Rightarrow \Sigma_1$ with a chronology $\tau$ is denoted by $(M, \tau)$: $\Sigma_0 \Rightarrow \Sigma_1$ or $(M, \tau)$ and is called a cobordism with chronology or a chronological cobordism. We will write $M$, if the chronology is obvious from the context. Denote by $\text{Chron}(M)$ the space of all chronologies on $M$.

![Figure 4](image.png)

**Figure 4.** A chronology describes the order of critical events on a cobordism.

Theorem 2.2 (compare [8] theorem 6.1.2). Let $M: \Sigma_0 \Rightarrow \Sigma_1$ be an oriented cobordism. Then chronologies on $M$ form an open-dense subset in

$$C^\infty(M, M_{we}, M_{wy}; I; 0, 1) := \{ f \in C^\infty(M; I) | f(M_{we}) = 0 \land f(M_{wy}) = 1 \}$$

Due to the theorem, there exists a chronology for any oriented cobordism. If $\tau: M \rightarrow I$ is a chronology with critical points $p_1, \ldots, p_n$, then a diffeomorphism $\varphi: M \rightarrow M'$ induces on $M'$ a chronology $\tau' = \tau \circ \varphi^{-1}$ with critical points $\varphi(p_1), \ldots, \varphi(p_n)$. However, the critical levels are preserved, so the structure it gives is too rigid. We will soften it introducing a sort of deformations, which preserve the order of critical points.

Definition 2.3. An isotopy of chronologies is a smooth homotopy $H: M \times I \rightarrow I$ such that $H_t: M \rightarrow I$ is a chronology for each $t \in I$.

![Figure 5](image.png)

**Figure 5.** An example of a homotopy which is not an isotopy of chronologies.

Recall that a diffeotopy of a manifold $\Sigma$ is a smooth map $\Phi: \Sigma \times I \rightarrow \Sigma$ such that $\Phi_0 = \text{id}$ and $\Phi_t$ is a diffeomorphism for each $t \in I$.

Example 2.4. If $f_t: I \rightarrow I$ is a diffeotopy of an interval fixing the endpoints, we can define an isotopy $H^f$ of a chronology $\tau: M \rightarrow I$ by the following composition

$$H^f(p, t) = f_t(\tau(p))$$

It is called a reparametrization of $\tau$. 

---
Example 2.5. Let $\Phi: M \times I \rightarrow M$ be a diffeotopy of $M$. Then we can define an isotopy $H^\Phi$ of a chronology $\tau: M \rightarrow I$ as the composition
\[
(2.3) \quad H^\Phi(p,t) = \tau(\Phi_t^{-1}(p))
\]

The examples above do not exhaust all types of isotopies, because the first preserves levels in the sense that a level of $H_0$ is a level of $H_1$ and the second preserves critical values. Obviously one may take a composition of these two, but it is unknown whether all isotopies can be obtained in this way. The problem can be reduced to getting a smooth solution of some smooth family of linear equations with highest rank but non-invertible matrices.

Conjecture 2.6. Any isotopy $H$ of a chronology $\tau: M \rightarrow I$ is of the form
\[
H(\Phi_t(p), t) = f_t(\tau(p))
\]
for some $d$ $f_t: I \rightarrow I$ and $\Phi_t: M \rightarrow M$.

A partial result is given in the lemma 4.8.

A chronology $\tau \in \text{Chron}(M)$ induces a linear order on the set of critical points:
\[
(2.4) \quad p < q \iff \tau(p) < \tau(q)
\]
In general, a homotopy does not have to preserve neither the number of critical points nor their order (fig. 5). It is not the case of isotopies and yet an isotopy induces for each critical point a path on $M$, giving a natural isomorphism or ordered spaces for both chronologies.

Lemma 2.7. Let $H: M \times I \ni (p,t) \rightarrow H_t(p) \in I$ be an isotopy of chronologies on $M$. Denote by $p_1 < \cdots < p_n$ all critical points of $H_0$. Then there exist paths $\gamma_i: I \rightarrow M$ for $i = 1, \ldots, n$ such that $\gamma_i(0) = p_i$, $\gamma_i(t) \in \text{Crit}(H_t)$ and $\gamma_i(t) < \gamma_{i+1}(t)$ for each $t \in I$.

Proof. Critical points of a chronology are non-degenerated, so:
\[
(2.5) \quad \det \left( \frac{\partial^2 H}{\partial p^2} (p,t) \right) \neq 0, \quad \text{for } p \in \text{Crit}(H_t),
\]
and by the implicit function theorem there exists a unique smooth solution for $i = 1, \ldots, n$ to the equation:
\[
(2.6) \quad \begin{cases} 
\frac{\partial H}{\partial p} (\gamma_i(t), t) = 0 \\
\gamma_i(0) = p_i
\end{cases}
\]
defined for $t \in I$. Suppose for some $t \in I$ we have $H_t(\gamma_i(t)) \geq H_t(\gamma_j(t))$. Due to continuity, there is $t' \leq t$ such that $H_{t'}(\gamma_i(t')) = H_{t'}(\gamma_j(t'))$, so $\gamma_i(t') = \gamma_j(t')$ and by uniqueness of solutions $i = j$, what ends the proof.

Corollary 2.8. Let $\tau_1, \tau_2$ be isotopic chronologies on $M$. Then there exists a natural isomorphism $(\text{Crit}(\tau_1), <) \approx (\text{Crit}(\tau_2), <)$.

Example 2.9. Paths induced by the isotopy from example 2.5 are very easy to see. Indeed, it must be $\gamma_i(t) = \Phi_t(p_i)$, since critical values are preserved by diffeomorphisms.

Now we will define an equivalence relation on cobordisms with chronologies using isotopies.

Definition 2.10. An equivalence of cobordisms $(M, \tau): \Sigma_0 \Rightarrow \Sigma_1$ and $(M', \tau'): \Sigma'_0 \Rightarrow \Sigma'_1$ is an equivalence of oriented cobordisms $\psi: M \rightarrow M'$ such that $\tau'$ and $\tau \circ \psi^{-1}$ are isotopic.
All the operations defined for cobordisms can be lifted to the chronological ones. In case of multiplication some modification is necessary.

1. The reversion of \((M, \tau) : \Sigma_0 \Rightarrow \Sigma_1\) is the cobordism \((M^*, \tau^*) : \Sigma_1 \Rightarrow \Sigma_0\), where \(\tau^*(p) = \tau(1 - p)\).

\[\begin{array}{c}
\text{t} \\
\text{should lie in the result. First notice every regular value}
\end{array}\]

2. The gluing of cobordisms \((M_1, \tau_1) : \Sigma_0 \Rightarrow \Sigma_1\) and \((M_2, \tau_2) : \Sigma_1 \Rightarrow \Sigma_2\) is the cobordism \((M_1M_2, \tau_1 \cdot \tau_2) : \Sigma_0 \Rightarrow \Sigma_2\), where

\[
(\tau_1 \cdot \tau_2)(p) = \begin{cases} \frac{1}{2}\tau_1(p) & p \in M_1 \\ \frac{1}{2}(\tau_2(p) + 1) & p \in M_2 \end{cases}
\]

3. Say \((M_1, \tau_1) : \Sigma_0 \Rightarrow \Sigma_1\) and \((M_2, \tau_2) : \Sigma_1 \Rightarrow \Sigma_2\) is a split of \((M, \tau) : \Sigma_0 \Rightarrow \Sigma_2\), whenever \((M, \tau) : \Sigma_0 \Rightarrow \Sigma_2\) is the gluing of \((M_1, \tau_1)\) and \((M_2, \tau_2)\).

\[\begin{array}{c}
\text{t} \\
\text{values). Thus the multiplication has to shift both cobordisms: the fir} \\
\text{st to the left, and the second to the left.}
\end{array}\]

4. The multiplication of cobordisms \((M, \tau) : \Sigma_0 \Rightarrow \Sigma_1\) and \((M', \tau') : \Sigma_0' \Rightarrow \Sigma_1'\) is their disjoint sum with shifted chronologies:

\[
(M, \tau) \sqcup (M', \tau') := ((M \sqcup C_{\Sigma_1}) (C_{\Sigma_1} \sqcup M'), (\tau \cdot \pi) \sqcup (\pi \cdot \tau'))
\]

where \(\pi : C_{\Sigma} \longrightarrow I\) is a canonical chronology on a cylinder. The multiplication of \(n\) copies of a cobordism \(M\) is denoted by \(M^n\).

There is no natural chronology for the disjoint sum of cobordisms — the disjoint sum of two chronologies might not be a chronology (the cobordisms may have same critical values). Thus the multiplication has to shift both cobordisms: the first to the left, and the second to the right.\footnote{This can be seen as a left multiplication, in opposite to the right version, where the first cobordism is pushed to the right, and the second to the left.}

The multiplication can be generalized, by adding some information how critical points should lie in the result. First notice every regular value \(t \in I\) of a chronology \(\tau : M \longrightarrow I\) defines a split \(M = M_{[0,t]} \sqcup M_{[t,1]}\). Define for regular value \(a < b\) the cobordism \(M_{[a,b]} = \tau^{-1}([a, b])\) between manifolds \(\tau^{-1}(a)\) and \(\tau^{-1}(b)\) with a chronology given by restriction \(\tau|_{[a,b]}\). Next, denote by \(S_{m,n}\) the set of all zero-one sequences of length \(m + n\) with exactly \(n\) ones:

\[S_{m,n} = \left\{ s \in \{0,1\}^{m+n} \mid \sum_{i=1}^{n+m} s_i = n \right\}\]

The conjugation of a sequence \(s \in S_{m,n}\) is a sequence \(\bar{s} \in S_{n,m}\) such that \(\bar{s}_i = 1 - s_i\), whereas the multiplication of sequences \(s_1 \in S_{m_1,n_1}, s_2 \in S_{m_2,n_2}\) is defined as a concatenation:

\[
s_1s_2(i) = \begin{cases} s_1(i), & i \leq m_1 + n_1 \\ s_2(i - m_1), & i > m_1 + n_1 \end{cases}
\]
Let $\text{ChCob}_0^m$ be the set of chronological cobordisms with exactly $m$ critical points. A sequence $s \in S_{m,n}$ gives a ,,stretched” cobordism $(M, \tau) \in \text{ChCob}_0^m$ as follows:

1. Let $\text{Crit}(\tau) = \{ p_1 < \cdots < p_m \}$ and $\tau(p_i) = t_i$.
2. Take mid-points of chronology: $q_0 = 0, q_{m+1} = 1, q_i = \frac{1}{2}(t_i+1 + t_i)$ for $0 < i < m$.

Let $\Sigma_i = \tau^{-1}(q_i)$.

3. Build the cobordism $(M_s, \tau_s)$ due to the following rules:
   - if $s_j = 0$, append the cylinder $C_{\Sigma_i}$
   - if $s_j = 1$, append $M_{[q_i, q_{i+1}]}$, where $i = \sum_{k=1}^j s_k$.

Such a cobordism $(M_s, \tau_s)$ is obviously equivalent to $(M, \tau)$. The generalized multiplication of cobordisms $(M_1, \tau_1) \in \text{ChCob}_0^m$ and $(M_2, \tau_2) \in \text{ChCob}_0^n$ along $s \in S_{n,m}$ is the disjoint sum

\[
(M_1, \tau_1) \sqcup_s (M_2, \tau_2) = (M_1^s, \tau_1^s) \sqcup (M_2^s, \tau_2^s)
\]

with induced input and output.

![Figure 6. The generalized multiplication of two cobordisms along $s = (0, 1, 0)$.](image)

Reversion, gluing and multiplication satisfy analogous properties to lemma 1.7 except commutativity of multiplication and gluing. Therefore, chronological cobordisms form a category $\text{ChCob}_0$, but the multiplication is not a functor. However, when one of the arguments is a fixed cylinder, the multiplication is a functor in the other variable:

\[
\begin{align*}
MN \sqcup C &= (M \sqcup C)(N \sqcup C) \\
C \sqcup MN &= (C \sqcup M)(C \sqcup N)
\end{align*}
\]

**Definition 2.11.** Let $\mathbf{C}$ be a category. A chronological multiplication in $\mathbf{C}$ is a function $\boxtimes: \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$ being a half-functor: for each object $X$ the functions of one variable $(\cdot) \boxtimes X$ and $X \boxtimes (\cdot)$ are functors:

\[
\begin{align*}
(f \circ g) \boxtimes \text{id}_X &= (f \boxtimes \text{id}_X) \circ (g \boxtimes \text{id}_X) \\
\text{id}_X \boxtimes (f \circ g) &= (\text{id}_X \boxtimes f) \circ (\text{id}_X \boxtimes g)
\end{align*}
\]

with the property that $f \boxtimes g = (\text{id} \boxtimes g) \circ (f \boxtimes \text{id})$.

**Remark 2.12.** The following may be considered as a left multiplication due to the duality of the last equality in the definition. Then a right multiplication satisfies

\[
f \boxtimes g = (f \boxtimes \text{id}) \circ (\text{id} \boxtimes g).
\]

Replacing the word ,,multiplication” with ,,chronological multiplication” in the definition 1.8 and treating $L, R, A$ and $S$ as transformation of half-functors (i.e. when all
arguments except one are fixed) we obtain the definition of a chronological monoidal category and symmetric chronological monoidal category. In particular, any monoidal category is chronological monoidal.

**Corollary 2.13.** The category \( \text{ChCob}_0 \) is symmetric chronological monoidal. It splits into subcategories of \((n - 1)\)-manifolds with \(n\)-cobordisms

\[
\text{ChCob}_0 = \bigcup_{n \in \mathbb{Z}^+} n\text{ChCob}_0
\]

which are also symmetric chronological and monoidal. Moreover, the reversion is a symmetric chronological and monoidal contravariant functor.

### 3. Orientation of a critical point

Besides a chronology, we will enrich cobordisms with orientation of critical points, what will break commutativity laws. Notice that a chronology \( \tau \) induces on \( M \) a gradient flow \( \phi^\tau \) given by a vector field \( \chi^\tau = \nabla \tau \). Critical points of \( \tau \) are exactly the fixed points of \( \phi^\tau \).

Let \( p_0 \in M \) be a fixed point of \( \phi^\tau \) with Morse index \( \mu(p_0) \). As \( \tau \) is a Morse function, \( p_0 \) is isolated and hyperbolic. Choose its isolating neighbourhood \( U \) with orientation induced from \( M \). Let \( W^u_U \subset U \) be the local unstable manifold (which is diffeomorphic to \( \mathbb{R}^{\mu(p_0)} \)). For \( 0 < \mu(p_0) < n \) is has no natural orientation and we can choose it arbitrary.

\[ \begin{array}{ccc}
0 & \tau & 1 \\
W^u_U & & \\
\end{array} \]

**Figure 7.** A local unstable manifold for a gradient flow on a cobordism.

Let \( U, V \) be two isolating neighbourhoods of \( p_0 \). Then \( U \cap V \) is also an isolating neighbourhood and

\[
W^u_{U \cap V} = W^u_U \cap V = W^u_V \cap U.
\]

Consider two oriented local unstable manifolds \( W^u_U \) and \( W^u_V \) as equivalent, whenever their orientations agree on the intersection, i.e. both are equal on \( W^u_{U \cap V} \).

**Definition 3.1.** Let \( \phi \) be a gradient flow on \( M \) with a fixed point \( p_0 \). An orientation of a critical point \( p_0 \) is an equivalence class of oriented local unstable manifolds.

An isotopy of chronologies \( H: \tau_0 \simeq \tau_1 \) induces a homotopy of vector fields \( \nabla H: \chi_{\tau_0} \simeq \chi_{\tau_1} \), which defines a homotopy of flows \( \Phi: \phi^{\tau_0} \simeq \phi^{\tau_1} \), where each \( \Phi_t \) is a flow. It carries orientations of critical points of \( \tau \) to critical points of \( \tau' \). Called the orientation induced by \( H \).

**Definition 3.2.** Let \( H: M \times I \longrightarrow I \) be an isotopy of chronologies. Say the orientations of critical points \((M, H_0)\) and \((M, H_1)\) agree, if the orientation of points of \((M, H_1)\) is equal to the orientation induced by \( H \). Cobordisms \((M, \tau)\) and \((M', \tau')\) with oriented critical points are equivalent, if there exists an equivalence of chronological cobordisms \( \psi: M \longrightarrow M' \) preserving the orientation of critical points, i.e. the orientation of critical points on \( M' \) is equal to the induced one by \( \psi \).
Remark 3.3. All operations defined for chronological cobordisms lift to cobordisms with oriented critical points. Hence, there is a symmetric chronological monoidal category \( \text{ChCob} \) of cobordisms with oriented critical points.

Remark 3.4. An orientation of the local unstable manifold induces an orientation of the local stable manifold, as a complementary to the orientation of \( M \). In dimension two it corresponds to a rotation of an arrow pointing the chosen out-going trajectory by 90° clockwise.

Example 3.5. In the case of \((1+1)\)-cobordisms, a critical point with Morse index 0 or 2 has a natural orientation. For a point with index 1 an orientation corresponds to a choice of one of the two out-going trajectories and can be visualised by an arrow. The following two cobordisms are different, what shows that arrows are essential:

To show this it is sufficient to notice, that the positive trajectory (pointed by the arrow) in the left-hand side cobordism starts in the first (upper) circle, whereas in the right-hand side cobordism is starts in the second (lower) circle. Hence, they must be different, since an equivalence preserve both orientations and the order of circles on input.

Unless it is stated differently, all chronological cobordisms in this paper are considered to have oriented critical points.

4. A presentation of chronological \((1+1)\)-cobordisms

In the section \( \square \) we gave a finite presentation of the category \( \text{2Cob} \). The aim of this section is to give an analogous description of chronological \((1+1)\)-cobordisms. We will restrict to a full subcategory of \( \text{2ChCob} \), generated by a standard circle. There is no loss in generality, as for an arbitrary cobordism \((M, \tau): \Sigma_0 \Rightarrow \Sigma_1\) there are diffeomorphisms \( \varphi: \Sigma_0 \rightarrow nS^1 \) and \( \psi: \Sigma_1 \rightarrow mS^1 \) together with a cobordism \((M', \tau'): nS^1 \Rightarrow mS^1\) forming a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{C_{\varphi}} & nS^1 \\
M & \downarrow \quad & \downarrow \\
Y & \xleftarrow{C_{\psi}} & mS^1 \\
\end{array}
\]

so that \( M = C_{\varphi}M'C_{\psi}^{-1}. \)

**Theorem 4.1.** The category of chronological \((1+1)\)-cobordisms \((M, \tau): \Sigma_0 \Rightarrow \Sigma_1\) is generated under composition and multiplication by the following cobordisms:
2. Cobordisms and chronologies

In opposition to the theory of classical cobordisms, there is no associativity law. Due to anticommutativity laws we can reduce the number of generators taking only a positive split and a merge. Since now, if not stated differently, all critical points of index 1 have positive orientation.

The proof of the theorem 4.1 is divided into several steps. First we will show that the given set generates the category. Next we will show that any decomposition of a given cobordism can be reduced to be of a special type called a normal form. Finally, we will see that two normal forms are related by the relations listed above.

For the first step we will use the following results of the theory of cobordisms.

**Theorem 4.2** (cf. [8], theorem 6.2.2). Let \((M, \tau): \Sigma_0 \Rightarrow \Sigma_1\) be a cobordism with no critical points. Then there exists a diffeomorphism \(\psi: \Sigma_0 \times I \rightarrow M\) agreeing with \(\tau\), i.e. the following diagram commutes:

\[
\begin{array}{ccc}
\Sigma_0 \times I & \xrightarrow{\psi} & M \\
\downarrow{\pi} & & \downarrow{\tau} \\
I & \xrightarrow{} & I 
\end{array}
\]
2. Cobordisms and chronologies

**Theorem 4.3** (cf. [8], theorem 6.4.2). Let \((M, \tau): \Sigma_0 \Rightarrow \Sigma_1\) be a connected \((1 + 1)\)-chronological cobordism with exactly one critical point of index 1. Then \(M\) is diffeomorphic with a disk missing two smaller disks inside.

Directly from the theorem 4.2 follows the classification of cobordisms with no critical points.

**Corollary 4.4.** Let \((M, \tau): nS^1 \Rightarrow mS^1\) be a cobordism with no critical points. Then \(n = m\) and \((M, \tau)\) is a cylinder generated by a permutation of components of \(nS^1\).

**Proof.** Let \(\psi: nS^1 \times I \rightarrow M\) be a diffeomorphism given by the theorem 4.2. It induces an embedding \(\varphi: mS^1 \rightarrow nS^1 \times I\) and sets an equivalence of cobordisms \(M\) and \(C\varphi\), so \(n = m\). A diffeomorphism preserving an orientation of a circle is isotopic to identity, so by lemma 1.4 we may assume \(\varphi\) is a permutation of components of \(nS^1\). □

The second theorem gives us a list of generators of \(2\text{ChCob}\).

**Lemma 4.5.** Any chronological cobordism \((M, \tau): \Sigma_0 \Rightarrow \Sigma_1\) decomposes into generators listed in the theorem 4.1.

**Proof.** Use the induction on the number of critical points \(n\).

If \((M, \tau)\) has no critical points, due to corollary 4.4 it is generated be a permutation \(\sigma\). Taking its decomposition into transpositions \(\sigma = t_1 \cdots t_n, t_i = (i i + 1)\), we obtain

\[
M = C_{t_1} \cdots C_{t_n}
\]

where each \(C_{t_i}\) is a permutation of neighbouring circles.

Assume \(n \geq 1\) and \(t\) is a critical value of \(\tau\) at \(p\). Take \(\varepsilon > 0\) such that \(M_{[t-\varepsilon,t+\varepsilon]}\) has exactly one critical point. It gives us a decomposition

\[
M = M_{[0,t-\varepsilon]}M_{[t-\varepsilon,t+\varepsilon]}M_{[t+\varepsilon,1]}
\]

Let \(N\) be a connected component of \(M_{[t-\varepsilon,t+\varepsilon]}\) with the critical point \(p\). Then one of the following occurs:

- \(p\) is a local minimum and \(N\) is a birth
- \(p\) is a local maximum and \(N\) is a death
- \(p\) is a saddle and due to theorem 4.3 \(N\) is a merge or a split (according to the value of \(\tau\) on boundary components of \(N\))

The other components of \(M_{[t-\varepsilon,t+\varepsilon]}\) have no critical points, hence they are generated by a permutation and cylinders. The inductive hypothesis gives the decomposition of the other two terms in (4.2), what ends the proof. □

To prove the second part of the theorem, we will distinguish some special decompositions of cobordisms. Then we will show that an arbitrary decomposition can be reduced to a special one, using relations listed in the theorem.

**Definition 4.6.** Say \(M = P_1M_1P_2 \cdots P_nM_nP_{n+1}\) is in a normal form if

- each \(P_i\) is a permutation of circles
- each \(M_i\) decomposes as \(N_i \sqcup C_{S_i}^k\), where \(N_i\) is the unique component with a critical point
- if \(N_i\) is a split or a merge, the critical point has a positive orientation

**Lemma 4.7.** An arbitrary decomposition \(M = P_1M_1 \cdots P_nM_nP_{n+1}\) can be reduced to a normal form.
2. Cobordisms and chronologies

Proof. Notice that the relations from the theorem 4.1 implies $M = G ∪ C^n$ is equal to $P(G ∪ C^n)P'$, where $G$ is a generator and both $P$ and $P'$ are generated by permutations. We give below such a reduction for a split:

This was the easier part. The harder is to show that any two normal decompositions are related. To obtain this result we need three technical lemmas.

Lemma 4.8. Let $H: M × I → I$ be an isotopy of chronologies such that $H_0$ and $H_1$ has the same critical values. Then there exists an isotopy $f_t: I → I$ such that a map $(p, t) ↦ f_t(H(p, t))$ has fixed critical values for all $t ∈ I$.

Proof. Let $t_1, ..., t_n$ be the critical values of $H_0$ and set $t_0 = 0, t_{n+1} = 1$. Put

$$\varepsilon = \frac{1}{2} \min_{i=1, ..., n} \{t_{i+1} - t_i\}$$

and pick a smooth increasing function $h: I → I$ with all derivatives vanishing at 0 and 1 such that $h(0) = 0, h(1) = 1$. Using $h$, define a homotopy $g_t: I → I$ such that

$$g_t(H_t(γ_i(t))) = t_i - \varepsilon(H_t(γ_i(t)))$$

Then $f_t(x) = g_t(x) + \varepsilon t$ is the desired isotopy. □

Lemma 4.9. Let $(M, τ)$ and $(M', τ')$ be equivalent chronological cobordisms. Suppose $ϕ: M → M'$ is a diffeomorphism which agrees with chronologies and inputs of $M$ and $M'$, and preserves orientations of critical points. Then up to an isotopy $ϕ$ agrees also with outputs.

Proof. At first consider the case $τ$ and $τ'$ have no critical points. Without loss of generality we may assume that

- $M$ has the form $Σ_0 \xrightarrow{id} Σ_0 × I \xleftarrow{f} Σ_1$
- $M'$ has the form $Σ_0 \xrightarrow{id} Σ_0 × I \xleftarrow{g} Σ_1$

and $ϕ|_{Σ_0 × 0} = id$. Obviously, $ϕ|_{Σ_0 × 1} ∼ ϕ|_{Σ_0 × 0}$. Since $M$ and $M'$ are equivalent, $f ∼ g$ and the following triangle commutes up to an isotopy

what proves the hypothesis.

Suppose now that $τ$ and $τ'$ have critical points. Decompose $M$ and $M'$ into terms without critical points and terms with exactly one critical point. We may assume the latter are of the form $G ∪ C$, where $C$ is an identity cylinder and $G$ is a generator. Therefore, it remains to show the hypothesis for generators with one critical point. It holds trivially
for all except a split and in this case \( \varphi \) maps the positive output to the positive one and similar for the negative one, because it preserves orientations of critical points. \( \square \)

**Lemma 4.10.** Let \((M, \tau)\) and \((M', \tau')\) be two equivalent cobordisms with chronologies and suppose that \(M\) and \(M'\) are also equivalent. Then \(N \simeq N'\) as cobordisms with chronologies.

**Proof.** Without loss of generality we may assume \(MN = M'N'\) as cobordisms without chronologies and that \(\tau'\) has same critical values as \(\tau\) (since \(M\) and \(M'\) have equal numbers of critical points and the same holds for \(N\) and \(N'\), there is such a reparametrization of \(\tau'\) fixing \(\frac{1}{2}\)). Let \(H: \tau \simeq \tau'\) be an isotopy of chronologies. Applying a reparametrization from lemma 4.8 we may assume that critical values are fixed by \(H\) for all \(t \in I\).

Consider a set
\[
L = H^{-1}(1/2) = \{(p, t) \mid H(p, t) = 1/2\} \subset MN \times I
\]
Since \(\frac{1}{2}\) is a regular value of \(\tau\), so is of \(H\) and \(L\) is a cobordism from \(\tau^{-1}(1/2) = \Sigma_1\) to \(\tau'^{-1}(1/2) = \Sigma_2\). To show that \(L\) is a cylinder, consider a projection \(\pi: L \rightarrow I, \pi(p, t) = t\), which is a Morse function with no critical points.

\(L\) induces an isotopy \(\Sigma_1 \times I \rightarrow MN, (p, t) \mapsto H(p, t)\), which extends to a diffeotopy \(\Phi: MN \times I \rightarrow MN\). We may assume that \(\Phi\) is constant on boundary of \(MN\). To show that \(\Phi_{|N}: N \rightarrow N'\) is the desired equivalence we need to check that it agrees with the input and the output of \(N'\). The second holds trivially and the first is guaranteed by the lemma 4.9. Indeed, \(M\) is equivalent to \(M'\) and the lemma assures \(\Phi_1\) agrees with outputs of \(M\) and \(M'\), which are at the same time inputs of \(N\) and \(N'\). \(\square\)

Now we are ready to prove the last lemma, required for the proof of the main theorem in this section.

**Lemma 4.11.** Normal forms of equivalent cobordisms are equal up to the relations listed in the theorem 4.4.

**Proof.** Let \((M, \tau)\) and \((M', \tau')\) be equivalent cobordisms with normal forms
\[
M = P_1 M_1 \ldots P_n M_n P_{n+1}, \quad M' = P'_1 M'_1 \ldots P'_n M'_n P'_{n+1}
\]
For \(n = 0\) the lemma holds due to corollary 4.4.

Assume \(n > 0\). The equivalence of cobordisms forces \(M_i\) and \(M'_i\) to be equal: the have the same number of inputs, outputs and a critical point of the same index. Suppose \(M_1 = G \sqcup C^n\), where \(G\) is a merge (other cases are proven in the same way). Denote by \(i, j, i', j'\) the input components of \(P_i\) and \(P'_i\) such that
\[
P_i(i) = P'_i(i') = 1, \quad P_i(j) = P'_i(j') = 2.
\]
Since \(M\) and \(M'\) are equivalent, \(i = i'\) and \(j = j'\) (components sent by \(P_i\) and \(P'_i\)) to inputs of \(G\) are determined uniquely, whereas the orientation of the critical point of \(G\) distinguishes the input circles). Therefore there is a permutation cobordism \(S_1\) such that
\[
P'_1 M_1 = P_1(C^2 \sqcup S_1)(G \sqcup C^{n-2}) = P_1(G \sqcup C^{n-2})(C \sqcup S_1) = P_1 M_1 S'_1
\]
and by the lemma 4.10
\[
P'_1 M_1 = P_1(C^2 \sqcup S_1)(G \sqcup C^{n-2}) = P_1(G \sqcup C^{n-2})(C \sqcup S_1) = P_1 M_1 S'_1
\]
and by the lemma 4.10
\[
P'_1 M_1 = P_1(C^2 \sqcup S_1)(G \sqcup C^{n-2}) = P_1(G \sqcup C^{n-2})(C \sqcup S_1) = P_1 M_1 S'_1
\]
The inductive hypothesis ends and the proof. \(\square\)

After proving all those lemmas we can connect them to obtain the presentation of \(2\text{ChCob}\).
Proof of the theorem 4.1. The first part is given by the lemma 4.5. For the second part, notice that the listed relations do not change the equivalence class of a cobordism. To see the list is complete, let $M$ and $M'$ be equivalent cobordisms with decompositions

$$M = P_1 M_1 \ldots P_n M_n P_{n+1}, \quad M' = P'_1 M'_1 \ldots P'_n M'_n P'_{n+1}.$$  

Due to the lemma 4.7 we may assume both decompositions are in normal forms. Now use the lemma 4.11 to end the proof. □

Remark 4.12. Taking into account the reversion of cobordisms, the set of generators can be reduced to a birth, a positive merge and a permutation. Then the set of relations can be restricted to those dealing with births, positive merges and permutations.

As an application of the theorem 4.1 we will introduce the 2-index of a chronological cobordism. Obviously, the number of critical points of a chosen type is fixed, so the following definition is independent on a decomposition of a cobordism.

Definition 4.13. Let $M$ be a chronological cobordism and take its decomposition. Denote by letters $m, b, s$ and $d$ the amounts of merges, births, splits and deaths respectively. A pair $\sigma(M) = (m - b, s - d)$ is called the 2-index of a cobordism $M$. If both numbers are equal to zero, we will write $\sigma(M) = 0$.

There is a simply correspondence between 2-indices of two cobordisms, their composition and multiplication. Furthermore, the 2-index of a cobordism imposes conditions on the number of inputs and outputs, especially for cobordisms of type zero.

Theorem 4.14. Let $M : n \mathbb{S}^1 \Rightarrow m \mathbb{S}^1$ and $N : m \mathbb{S}^1 \Rightarrow k \mathbb{S}^1$ be $(1 + 1)$-chronological cobordisms and put $\sigma(M) = (\alpha, \beta)$. Then

1. $\sigma(MN) = \sigma(M \sqcup N) = \sigma(M) + \sigma(N)$
2. $m - n = \beta - \alpha$
3. if $\sigma(M) = 0$, then $n = m \in \{0, 1\}$
4. $\chi(M) = -\alpha - \beta$
5. $g(M) = 1 + \frac{1}{2}(\alpha + \beta - m - n)$ if $M$ is connected, where $g(M)$ is the genus of $M$

Proof. The first point holds due to the definition of $\sigma$. Obviously (2) holds for generators of $2\text{ChCob}$ and other cases goes from (1). The point (3) is a special case of (2), and (4) is the formula for the Euler characteristic:

$$\chi(M) = \sum_{x \in M} (-1)^{\mu(x)} = -\alpha - \beta$$

where $\mu(x)$ is the Morse index of $x$. Using the relation between the Euler characteristic, the genus and the number of components of a given cobordism:

$$\chi(M) = 2 - 2g(M) - (n + m)$$

we obtain (5). □

5. A change of a chronology

An isotopy of chronologies preserves the order of critical points as well as their characters (i.e. merge is still a merge etc.). Now we will allow some changes of chronologies and introduce relations between two cobordisms differing by such a change.

Let $H : M \times I \longrightarrow I$ be a smooth homotopy. A critical moment of $H$ is $t \in I$ such that $H_t$ is not a chronology. Assume $t_0$ is an isolated critical moment of $H$ and one of the following occurs:
(CHCH1) $H_{t_0}$ has two critical points at some level and for a small $\varepsilon > 0$ the chronologies $H_{t_0-\varepsilon}$ and $H_{t_0+\varepsilon}$ are not isotopic.

(ChCH2) $H_{t_0}$ has a degenerated critical point and for a small $\varepsilon > 0$ $H_{t_0+\varepsilon}$ has two critical points more than $H_{t_0-\varepsilon}$.

(ChCH3) $H_{t_0}$ has a degenerated critical point and for a small $\varepsilon > 0$ $H_{t_0+\varepsilon}$ has two critical points less than $H_{t_0-\varepsilon}$.

Regard such changes to be respectively of type I, II and III.

Definition 5.1. Let $(M, \tau): \Sigma_0 \Rightarrow \Sigma_1$ be a chronological cobordism. A change of the chronology $\tau$ into $\tau'$ is a homotopy $H: M \times I \rightarrow I$ such that $H_0 = \tau$, $H_1 = \tau'$ and $H_t$ has finitely many critical moments $t_1, \ldots, t_n$, each of type I, II or III. If all critical moments of $H$ has the same type, say $H$ is of this type. A change $H$ with exactly one critical moment is called an elementary change of a chronology.

A change of a chronology $H$ from $\tau$ to $\tau'$ will be denoted by $H: \tau \rightsquigarrow \tau'$. If $\tau_i$ and $\tau'_i$ are isotopic for $i = 0, 1$, then regard changes $H: \tau_0 \rightsquigarrow \tau_1$ and $H': \tau'_0 \rightsquigarrow \tau'_1$ as equivalent. In particular, a change $H: \tau \rightsquigarrow \tau'$ is trivial, if $\tau$ and $\tau'$ are isotopic.

![Figure 8](image.png)

**Figure 8.** An elementary change of a chronology of type I. The middle state visualise the critical moment.

Remark 5.2. If $H_{t_0}$ has a degenerated critical point, then more than two critical points can be created. For instance, $H$ can create at the same time a split, a merge, a birth and a death. Thus in (CHCH2) and (CHCH3) the condition on the number of critical points is essential. In case of (CHCH1) the condition on non-isotopicity of chronologies guarantees that an elementary change of type I is non-trivial.

Given two changes of chronologies $H: \tau_0 \rightsquigarrow \tau_1$ and $H': \tau_1 \rightsquigarrow \tau_2$ define their *composition* $H \cdot H': \tau_0 \rightsquigarrow \tau_2$, as a change of a chronology given as follows:

$$H \cdot H'(p, t) = \begin{cases} H(p, 2t), & t \leq \frac{1}{2} \\ H'(p, 2t - 1), & t \geq \frac{1}{2} \end{cases}$$

and smoothed near $t = 1/2$ if necessary. Directly from the definition, this operation agrees with the equivalence relation of changes of chronologies and up to equivalence is associative, has neutral elements (trivial changes) and every change $H: \tau_0 \rightsquigarrow \tau_1$ has an inverse $H^{-1}: \tau_1 \rightsquigarrow \tau_0$.

**Remark 5.3.** Isotopy classes of chronologies on a cobordism $M$ with equivalence classes of changes of chronologies form a category $\text{Chron}(M)$. Notice, that every morphism is an isomorphism.

Elementary changes of chronologies affect the set of critical points in a way that can be easily described. Indeed, the techniques from the proof of the lemma 2.7 can be used to show the following three results.

---

4 Such a category is called a groupoid.
Lemma 5.4. Let $H: M \times I \to I$ be an elementary change of a chronology of type I and $p_1 < \cdots < p_n$ be all critical points of $H$. Then there exist $1 \leq i_0 \leq n$ and paths $\gamma_i: I \to M$ such that $\gamma_i(0) = p_0, \gamma_i(t) \in \text{Crit}(H_t)$ and $\gamma_i(t) < \gamma_j(t)$ for each $t \in I$ and $i < j, (i, j) \neq (i_0, i_0 + 1)$. Moreover, $\gamma_{i_0}(1) > \gamma_{i_0+1}(1)$.

Lemma 5.5. Let $H: M \times I \to I$ be an elementary change of a chronology of type II and $p_1 < \cdots < p_n$ be all critical points of $H$. Then there exist paths $\gamma_i: I \to M$ such that $\gamma_i(0) = p_0, \gamma_i(t) \in \text{Crit}(H_t)$ and $\gamma_i(t) < \gamma_{i+1}(t)$ for each $t \in I$.

Lemma 5.6. Let $H: M \times I \to I$ be an elementary change of a chronology of type III and $p_1 < \cdots < p_n$ be all critical points of $H$. Then there exist $1 \leq i_0 \leq n$ and paths $\gamma_i: I \to M$ for $i \neq i_0, i_0 + 1$ such that $\gamma_i(0) = p_0, \gamma_i(t) \in \text{Crit}(H_t)$ and $\gamma_i(t) < \gamma_{i+1}(t)$ for each $t \in I$ and $i < j$. Moreover, $\text{Crit}(H_{i_0}) = \{\gamma_i(1) \mid i \neq i_0, i_0 + 1\}$.

The lemmas guarantee that the description of an elementary change of a chronology given by a pair of critical points which are permuted (type I), created (type II) or deleted (type III) is unambiguous.

Remark 5.7. There exist nontrivial changes of chronologies preserving the cobordism. This is because cobordisms are considered up to equivalence which is stronger than just an isotopy of chronologies. The following pictures show two such non-trivial changes in dimension two, which preserve cobordisms up to the orientation of critical points (the normal form is given on the right-hand side).

(T1) \hspace{2cm} (T2)

The change (T1) preserves both orientations of the merge and the split, whereas (T2) changes orientation of one point. Furthermore, reversing the orientation of the merge or the split in the left-hand side cobordism, after applying the change, results in both cases in reversing the orientation of the other point.

In a change of a chronology which preserves the cobordism characters of permuted critical points must be changed. The Morse index, as a local property, is preserved (see the lemma 5.4), so in dimension two such a chronology has to permute a merge and a split lying in the same component of the cobordism. Depending on the beginning order of these points, there are exactly two such elementary changes:

1. a split of a circle in one point and then a merge in another point – (T1)
2. a merge of two circles in on point and then a split in another point – (T2)

Every change of a chronology is equivalent to a composition of homogeneous changes (i.e. in which each critical moment is of the same fixed type).

Theorem 5.8. For a given change of a chronology $H$ there exist changes $P, C, D$ of types I, II and III respectively such that $H \sim C \cdot P \cdot D$.

Proof. At first notice, that a creation of two points can be pulled back before any other change, i.e. for an elementary change $H$ and a change $H_c$ of type II there is a change
Indeed, let $H_c : (M, \tau) \rightsquigarrow (M', \tau')$ creates critical points $p_1$ and $p_2$ on a cylinder $M_{[t_0, t_1]}$. The one of the following occurs:

- $H$ is trivial on $M_{[t_0, t_1]}$. We may assume it is constant on this region and take as $H'$ a constant change on $M_{[t_0, t_1]}$ and equal to $H$ out of this region.
- $H$ carries some critical points through $M_{[t_0, t_1]}$. Then correct it to $H'$ by adding appropriate permutations.
- $H$ deletes points in $M_{[t_0, t_1]}$ and sends them beside this region. Then take as $H'$ an inverse change to the one described in the previous case.

Hence, we may assume $H = C \cdot H'$, where $H'$ has only critical moments of type I and III and $C$ has only critical moments of type II. In a similar way any change $H_d$ of type III can be pushed to the end of $H$, what ends the proof.

Corollary 5.9. If $C \cdot P \cdot D$ is a trivial change, then we may assume $D = C^{-1}$. In this situation $P$ is a trivial change.

Due to the lemma 5.4 a change $H$ of type I induces a permutation of critical points $\sigma_H$. Obviously $\sigma_H = \text{id}$ for a trivial change $H$. However, it is not the case of a change which preserves the cobordism, as it was shown in the remark 5.7. In dimension two even preserving types of critical points is not enough:

Changes of chronologies reduce the category $\mathbf{2ChCob}$ into $\mathbf{2Cob}$. Indeed, the relations of (co)commutativity, (co)associativity and (co)unity given in the theorem 1.13 can be described by changes of chronologies. We want to allow changes of chronologies but to avoid such a reduction. It can be done by colouring cobordisms and consider a change of a chronology as a change of colours.

Definition 5.10. Let $G$ be an Abelian group. A coloured chronological cobordism is a pair $(M, g)$, where $M$ is a chronological cobordism and $g \in G$. The colour of a composition or a multiplication is given by the multiplication of colours of both cobordisms:

\begin{align}
(M, g)(N, h) &:= (MN, gh) \\
(M, g) \sqcup (N, h) &:= (M \sqcup N, gh)
\end{align}

Remark 5.11. The multiplication in $G$ is associative, so is the operation (5.3) and we obtain a symmetric chronological monoidal category of coloured chronological cobordisms $G\mathbf{ChCob}$. 
2. Cobordisms and chronologies

For a change of a chronology $H: M \leadsto M'$ we want to introduce a relation
\[(5.4) \quad M' = r_H M, \quad r_H \in G\]
such that the coefficients $r_H$ agree with composition of changes:
\[(Ch1) \quad r_{H,H'} = r_H r_{H'}.
\]
Moreover, the quotient category should not be trivial. In particular, the following non-degeneracy condition ought to be satisfied, where $G_i$ are generators:
\[(Ch2) \quad g(G_1 \sqcup \cdots \sqcup G_n) = (G_1 \sqcup \cdots \sqcup G_n) \Rightarrow g = 1.
\]
Label a merge, a birth, a split and a death by $M, B, S, D$ respectively. Define coefficients $r_H$ for the following changes of chronologies:
- if $H$ permutes points $p < q$ labeled by $\alpha$ and $\beta$, and $H$ preserves types of these points, put $r_H = \lambda_{\alpha\beta}$
- if $H$ creates or deletes points $p < q$, put $r_H = 1$, provided that the birth or the death is on the positive side of the merge or the split (i.e. it is pointed by the arrow denoting the orientation of the critical point)$^5$

Every change of a chronology is invertible, thus due to (Ch2):
\[(5.5) \quad \lambda_{\alpha\beta} \lambda_{\beta\alpha} = 1.
\]

There are more relations for the coefficients $\lambda_{ab}$. A simple analysis of elementary changes gives the following necessary condition on the coefficients $\lambda_{\alpha\beta}$, such that the non-degeneracy condition holds.

**Theorem 5.12** (the change of a chronology condition). There exist elements $X, Y, Z \in G$ such that:
\[
\begin{align*}
\lambda_{MM} &= \lambda_{BB} = \lambda_{MB} = \lambda_{BM} = X \\
\lambda_{SS} &= \lambda_{DD} = \lambda_{SD} = \lambda_{DS} = Y \\
\lambda_{SM} &= \lambda_{DB} = \lambda_{MD} = \lambda_{BS} = Z \\
\lambda_{MS} &= \lambda_{BD} = \lambda_{DM} = \lambda_{SB} = Z^{-1}
\end{align*}
\]
and $X^2 = Y^2 = 1$.

**Proof.** Consider the following change of a chronology:
\[
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{Fig5_1.png}
\end{array}
\]
where $G_\alpha$ stands for a cobordism with exactly one critical point of type $\alpha$. Adding a creation of a birth at the beginning and deleting it at the end, we get:
\[(5.7) \quad \lambda_{Ma} \lambda_{Ba} G_\alpha = G_\alpha
\]
and from (Ch2) we have $\lambda_{Ba} = \lambda_{Ma}^{-1}$. In a similar way $\lambda_{Da} = \lambda_{Sa}^{-1}$. Due to (5.5) we have
\[(5.8) \quad \lambda_{Ma} = \lambda_{aB} \quad \text{and} \quad \lambda_{Sa} = \lambda_{aD}
\]

$^5$ There is no loss of generality, if one put $r_H = 1$ for these changes. Indeed, whenever these coefficients for creating points are equal $\mu_{ab}$, define the isomorphism of categories $F: G2\text{ChCob} \longrightarrow G2\text{ChCob}$ by multiplying a birth by $\mu_{BM}$ and a death by $\mu_{SD}$. In the target category both coefficients are equal 1.
When replacing $\alpha$ with $M, B, S, D$ we obtain the equalities
\begin{equation}
\begin{align*}
\lambda_{MM} &= \lambda_{BB} = \lambda_{MB} = \lambda_{BM} = X \\
\lambda_{SS} &= \lambda_{DD} = \lambda_{SD} = \lambda_{DS} = Y \\
\lambda_{SM} &= \lambda_{DB} = \lambda_{MD} = \lambda_{BS} = Z \\
\lambda_{MS} &= \lambda_{BD} = \lambda_{DM} = \lambda_{SB} = W
\end{align*}
\end{equation}

Finally, (5.5) implies $X^2 = Y^2 = ZW = 1$. \qed

Remark 5.13. The orientation of critical points were not used in the definition of the coefficients $\lambda_{\alpha\beta}$. In fact it plays no role, since there are changes of chronologies which reverse the orientation of a single critical point:

\begin{equation}
\begin{array}{c}
\quad \\
\quad
\end{array}
\end{equation}

and similarly for a split. Using such changes one can first set all orientations to be positive, then apply required changes and at the end restore the original orientation.

When calculating coefficients for changes (5.10) one gets $r_H = X$ for a merge and $r_H = Y$ for a split. Now we can complete the list of coefficients $r_H$, taking in account also the changes from the remark 5.7 as well as the other type of deletions and creations (with the opposite orientation). All of them are contained in the table 1.

Remark 5.14. The change of a chronology relations are defined locally, so they are compatible with operations on cobordisms. Therefore, there exists a quotient category $G2\text{ChCob}/_{XYZ}$ of cobordisms modulo changes of chronologies, which is symmetric chronological and monoidal. Reversion of a cobordism interchanges the role of $X$ and $Y$ (it gives the dual relations), so there is a contravariant chronological monoidal functor
\begin{equation}
\begin{array}{c}
\quad \\
\quad
\end{array}
\end{equation}

Every homomorphism $h: G \rightarrow G'$ induces a chronological monoidal functor
\begin{equation}
\begin{array}{c}
\quad \\
\quad
\end{array}
\end{equation}

In particular, these categories are isomorphic, provided $h$ is an isomorphism.

Instead of a group one can take a ring $R$ and pick the coefficients from the group of units $U(R)$. In the chapter 4 we will define a symmetric chronological monoidal functor $F_{XYZ}: R2\text{ChCob}/_{XYZ} \rightarrow \text{Mod}_R$, which maps every generator to a non-zero linear map between free modules, what gives that $r F_{XYZ}(G_1 \sqcup \cdots \sqcup G_n) = 0$ implies $r = 0$. In particular, for the group ring $R = \mathbb{Z}[G]$ we will get the following result.

Corollary 5.15 (the change of a chronology condition). The coefficients of changes of chronologies given in the table 1 satisfies the non-degeneracy condition (Ch2).\footnote{In (T1) both cobordisms $M_1$ and $M_2$ have a positive genus. Therefore, using the change from fig. 5.1 we get $XY M_1 = M_1$, so the coefficient is defined only up to the factor $XY$. The convention used in the table, in which changes (T1) with different orientations of critical points are distinguished, is due to the fact that when the orientation of a merge in $M_1$ is reversed, the split in $M_2$ gets a reversed orientation. Hence the change (T1) between cobordisms $M'_1$ and $M'_2$ with opposite orientations of the merge and the split can be described as a change from $M_1 = XM'_1$ to $M_2 = YM'_2$ with agreeing orientations. Different coefficients for these two types of (T1) will play a crucial role in proving the uniqueness of a coefficient for a change of a chronology in case of cobordisms embedded in $\mathbb{R}^3$ in the next paragraph.}
A description of a change $H$

Permutations preserving the types of the points

| $MM \sim MM$ | $MB \sim BM$ | $BB \sim BB$ |\hline
| $SS \sim SS$ | $SD \sim DS$ | $DD \sim DD$ |\hline
| $SM \sim MS$ | $MD \sim DM$ | $BS \sim SB$ | $DB \sim BD$ |

The coefficient $r_H$

| $X$ | $Y$ | $Z$ |

Permutations changing the types of the points

| $S^+M^+ \sim S^+M^+$ | $S^-M^- \sim S^-M^-$ |\hline
| $S^+M^- \sim S^-M^+$ | $M^-S^- \sim M^+S^-$ |\hline
| $M^+S^+ \sim M^-S^+$ | $M^-S^- \sim M^-S^+$ |\hline

Deleting and creating critical points

| $BM^+ \sim \emptyset$ |\hline
| $BM^- \sim \emptyset$ |\hline
| $S^+D \sim \emptyset$ |\hline
| $S^-D \sim \emptyset$ |

| 1 | $XY$ | 1 | $Y$ |

Table 1. The coefficients $r_H$ for elementary changes of chronologies. The table presents coefficients only for changes in one direction — for the opposite one take the inverse of the appropriate element. Signs “,” and “−” denote orientations of critical points if they are important. Elements $X, Y \in G$ has to be of order 2.

Let $M$ and $N$ be chronological cobordisms. Due to the theorem 5.12 a coefficient of a change of a chronology from $(M \sqcup C)(C \sqcup N)$ to $(C \sqcup N)(M \sqcup C)$ depends only on 2-indices of cobordisms $\sigma(M)$ and $\sigma(N)$. Indeed, by direct calculations

\[(5.13) \quad (M \sqcup C)(C \sqcup N) = X^{ac}Y^{bd}Z^{bc-ad}(C \sqcup N)(M \sqcup C)\]

where $\sigma(M) = (a, b)$ and $\sigma(N) = (c, d)$. Moreover, directly from the theorem 5.14 cobordisms with vanishing 2-index are central.

**Corollary 5.16.** Let $M: \Sigma_0 \Rightarrow \Sigma_1$ and $N: \Sigma_1 \Rightarrow \Sigma_0$ be chronological cobordisms and 2-indices of components of $M$ are zero. Then $MN = NM$.

**Proof.** If the 2-index of a connected cobordism $M'$ is zero, then there is a change of chronology between $M'$ and a cylinder. Hence, we have

\[(5.14) \quad MN = \lambda CN = \lambda NC = NM\]

\[\square\]

6. Cobordisms embedded in $\mathbb{R}^3$

This section deals (1+1)-cobordisms embedded into $\mathbb{R}^3$. Comparing to abstract cobordisms, the embedded ones have a richer structure, which can be used to introduce a chronology and equivalence of cobordisms in a more delicate way. In particular, there exists a planar algebra of cobordisms as well as for a certain type of changes of chronologies the coefficient depends only on the equivalence class of such a change.

**Definition 6.1.** Let $\Sigma_0$ and $\Sigma_1$ be compact one-dimensional submanifolds of a plane $\mathbb{R}^2$ with no boundary. An *embedded cobordism* in $\mathbb{R}^3$ from $\Sigma_0$ to $\Sigma_1$ is a surface $S \subset \mathbb{R}^2 \times I$ such that

\[(6.1) \quad S \cap (\mathbb{R}^2 \times \{1\}) = \Sigma_0 \quad \text{and} \quad S \cap (\mathbb{R}^2 \times \{0\}) = \Sigma_1.\]
2. Cobordisms and chronologies

We write $S: \Sigma_0 \Rightarrow \Sigma_1$.

We will consider the embedded cobordisms up to ambient isotopies, i.e. diffeotopies of $\mathbb{R} \times I$.

**Definition 6.2.** Regard two embedded cobordisms $S: \Sigma_0 \Rightarrow \Sigma_1$ and $Q: \Sigma_0 \Rightarrow \Sigma_1$ as **equivalent**, if there exists an ambient isotopy $\Psi_t$ of $S$ constant on the boundary of $\mathbb{R} \times I$ such that $\Psi_1(S) = Q$.

Embedded cobordisms form a category $\text{Cob}^3$ in a natural way: the composition $Q \circ S = SQ$ is given by placing $S$ on the top of $Q$ (see fig. 9). Moreover, there is a reversion defined as a symmetry along a plane $\mathbb{R}^2 \times \{\frac{1}{2}\}$.

The definition of an embedded cobordism can be extended over manifolds with boundary. Indeed, take an arbitrary cobordism $S: \Sigma_0 \Rightarrow \Sigma_1$ and cut it with a cylinder $\mathbb{D}^2 \times I$. Assume $S \cap (\partial \mathbb{D}^2 \times I) = B \times I$ for some finite $B \subset \partial \mathbb{D}^2$. Then $S' = S \cap (\mathbb{D}^2 \times I)$ is a surface between $\Sigma'_0$ and $\Sigma'_1$, where $\Sigma'_i = \Sigma_i \cap \mathbb{D}^2$. Elements of the set $B \times \{0, 1\}$ are called **corners** of the surface $S'$ and elements of $B$ are called **endpoints** of $\Sigma_i$.

**Definition 6.3.** A surface $S' \subset \mathbb{D}^2 \times I$ constructed above is called a **cobordisms with corners** from $\Sigma'_0 \subset \mathbb{D}^2$ to $\Sigma'_1 \subset \mathbb{D}^2$. Two cobordisms with corners $S$ and $Q$ are said to be **equivalent**, if there exists an ambient isotopy $\Psi_t$ of $S'$ constant on the boundary of $\mathbb{D}^2 \times I$ such that $\Psi_1(S) = Q$.

Directly from the definition, cobordisms with corners form a category, which extends the category $\text{Cob}^3$. We will denote it in the same way, putting $\text{Cob}^3(\emptyset)$ for cobordisms with no corners.

For a given cobordism $S: \Sigma_0 \Rightarrow \Sigma_1$ we have $\partial \Sigma_0 = \partial \Sigma_1$. Thus, there is a well defined subcategory $\text{Cob}^3(B)$ consisting of all submanifolds $\Sigma \subset \mathbb{D}^2$ with boundary $\partial \Sigma = B$ and cobordisms with corners in $B$. Hence, the category $\text{Cob}^3$ decomposes as

\[(6.2) \quad \text{Cob}^3 = \bigcup_{B \subset \partial \mathbb{D}^2} \text{Cob}^3(B)\]

Objects of $\text{Cob}^3(B)$ can be seen as diagrams of trivial tangles from $T^0(B)$ (see the section [13]), so they inherits a structure of a planar algebra. The structure lifts to cobordisms. Indeed, a planar diagram $D$ represents a three-dimensional curtain $D \times I$, which acts on cobordisms by putting them inside cylindrical holes (fig. 10). Clearly, each planar diagram $D$ lifts to a functor

\[(6.3) \quad D: \text{Cob}^3(B_1) \times \cdots \times \text{Cob}^3(B_s) \longrightarrow \text{Cob}^3(B)\]

which, for simplicity, will be denoted by the same letter. This defines a planar algebra structure on the category $\text{Cob}^3$. 

[Figure 9. The composition of embedded cobordisms in given by putting the second on the first.]
We will now add a chronology to our new framework. Notice there is a natural projection for a given cobordism \( S: \Sigma_0 \Rightarrow \Sigma_1 \):

\[
\pi: S \ni (x, y, z) \mapsto z \in I.
\]

**Definition 6.4.** A cobordism \( S: \Sigma_0 \Rightarrow \Sigma_1 \) is called *chronological*, if the projection \( \pi: S \Rightarrow I \) is a Morse function with at most one critical point at each level. An ambient isotopy \( \Psi_t \) of \( S \) is called *chronological*, if it is constant on the boundary of \( \mathbb{D}^2 \times I \) and \( \Psi_t(S) \) is a chronological cobordism for each \( t \in I \). Embedded chronological cobordisms \( S \) and \( Q \) are said to be *equivalent*, if there exists an ambient chronological isotopy \( \Psi_t \) of \( S \) such that \( \Psi_1(S) = Q \).

Like in the case of abstract cobordisms, the projection \( \pi \) induces a gradient flow on \( S \) and we can introduce orientations of critical points. The category of embedded chronological cobordisms with oriented critical points will be denoted by \( \text{ChCob}^3 \). In analogous to \( \text{Cob}^3 \) there exists a decomposition into subcategories

\[
\text{ChCob}^3 = \bigcup_{B \subset \partial \mathbb{D}^2} \text{ChCob}^3(B)
\]

**Remark 6.5.** A classification of \( \text{ChCob}^3 \) is at least as difficult as to classify knots (tubular neighbourhoods of non-equivalent knots give non-equivalent cobordisms). However, there is a forgetful functor

\[
\mathcal{U}: \text{ChCob}^3(\emptyset) \longrightarrow \text{2ChCob}
\]

assigning to an embedded cobordism its equivalence class in the sense of definition 2.10. Hence, any functor from the category \( \text{2ChCob} \) lifts to a functor from the category \( \text{ChCob}^3(\emptyset) \).

When we try to define planar operators for embedded chronological cobordisms, we meet the same problem which arose in defining the multiplication of chronological cobordisms — using the naive definition we may get a cobordism which is not chronological. It can be overcome in the same way as before: a planar diagram \( D \) induces an operator

\[
D: \text{Mor}(\text{ChCob}^3(B_1)) \times \cdots \times \text{Mor}(\text{ChCob}^3(B_s)) \longrightarrow \text{Mor}(\text{ChCob}^3(B))
\]

which puts the \( i \)-th cobordism into the \( i \)-th cylindrical hole in \( D \times I \) in such a way, that all its critical points project onto \( (\frac{i}{s}, \frac{i}{s}) \). Hence, \( D(S_1, \ldots, S_s) \) has at first critical points of \( S_1 \), then critical points \( S_2 \), etc. However, functoriality is lost, as the following does not hold in general:

\[
D(S_1Q_1, \ldots, S_sQ_s) = D(S_1, \ldots, S_s)D(Q_1, \ldots, Q_s)
\]

But it is the case when \( S_i \) and \( Q_i \) are cylinders for all indices except one. For instance, operators given by diagrams with one input, especially closure operators, are functorial.
**Remark 6.6.** A functor $F: \text{ChCob}^3(\emptyset) \longrightarrow C$ might not extend naturally on the hole $\text{ChCob}^3$. But it can be extended over the category of sequences in $C$, by taking values of $F$ on all closure operators:

\[(6.9) \quad F|_{\text{ChCob}^3(B)}(S) := \{F(DS) \mid D \in \text{CPO}(B)\}\]

We have already defined in the section the change of a chronology relations in $G2\text{ChCob}$. It can be directly applied to $\text{ChCob}^3(\emptyset)$. However, some modifications are necessary for cobordisms with corners, due to the fact that a single saddle after applying to it a closure operator can become either a split or a merge.

Let $S \subset \mathbb{D}^2 \times I$ be a chronological cobordism and $\Psi$ its ambient isotopy, not necessary chronological. Call $t \in I$ a critical moment of $\Psi$, if $\Psi_t(S)$ is not a chronological cobordism.

**Definition 6.7.** An ambient isotopy $\Psi$ of $S$ constant on the boundary of $\mathbb{D}^2 \times I$ is called a change of a chronology, if it has finitely many critical moments $0 < t_1 < \cdots < t_k < 1$ and each $\Psi_{t_i}$ satisfies one of the conditions [(CHCH1) – (CHCH3)].

A change of a chronology from $S$ to $Q$ will be denoted by $\Psi: S \leadsto Q$. We will identify changes $\Psi: S \leadsto Q$ and $\Psi': S' \leadsto Q'$ if $S \simeq S'$ and $Q \simeq Q'$. Notice that changes (T1) and (T2) from the remark do not preserve embedded cobordisms.

Like in the paragraph we can define a composition of changes of chronologies, obtaining a category of embedded cobordisms and changes of chronologies $\text{Chron}$, which decomposes into subcategories $\text{Chron}(B)$ generated by cobordisms from $\text{ChCob}^3(B)$.

**Proposition 6.8.** The category $\text{Chron}$ has a structure of a planar algebra with functorial planar operators.

**Proof.** Let $D$ be a planar diagram with $s$ inputs and $\Psi^i: S_i \leadsto Q_i$ be changes of chronologies of appropriate types for $i = 1, \ldots, s$. Each $\Psi^i$ induces a change of a chronology

\[(6.10) \quad \Psi^i_D: D(Q_1, \ldots, Q_{i-1}, S_i, S_{i+1}, \ldots, S_s) \leadsto D(Q_1, \ldots, Q_{i-1}, Q_i, S_{i+1}, \ldots, S_s)\]

which is constant beyond the subset $\mathbb{D}^2 \times (\frac{t_i - 1}{s}, \frac{t_i}{s})$, on which is equal to $\Psi^i$. A change $D(\Psi^1, \ldots, \Psi^s)$ is defined as the composition of the induced changes:

\[(6.11) \quad D(\Psi^1, \ldots, \Psi^s) := \Psi^1_D \cdot \ldots \cdot \Psi^s_D.\]

Functoriality holds, because all $\Psi^i_D$ act on disjoint regions. \qed

The functor from the remark carries the change of a chronology relations from $G2\text{ChCob}$ to $\text{ChCob}^3(\emptyset)$ and further to $\text{ChCob}^3$, using the construction described in the remark. The details are provided below.

**Definition 6.9.** Let $G$ be an Abelian group. A coloured chronological embedded cobordism is a pair $(S, g)$ consisting of a chronological embedded cobordism $S$ with corners in $B$ and a function $g \in G^{\text{CPO}(B)}$ from the set of closure operators $\text{CPO}(B)$ into the group $G$. The colour of the composition is given by the multiplication in $G$:

\[(6.12) \quad (S, g)(Q, h) := (SQ, gh).\]

As before, we will usually write $gS$ instead of $(S, g)$.

Coloured cobordisms form a category denoted by $G\text{ChCob}^3$. The set of functions $G^X$ is an Abelian group for any set $X$, so this category is naturally equipped with a structure of a planar algebra:

\[(6.13) \quad D(g_1S_1, \ldots, g_sS_s) := g_1 \cdot \ldots \cdot g_sD(S_1, \ldots, S_s).\]
A change of a chronology $\Psi: S \leadsto Q$ of cobordisms in $\text{ChCob}^3(\emptyset)$ induces a change of a chronology of abstract cobordisms with some coefficient $r_\Psi$ defined as in the table 1. In the category $\text{ChCob}^3(B)$ the theorem \[ \ref{6.8} \] gives for a change of a chronology $\Psi: S \leadsto Q$ a function $r_\Psi: \mathcal{CPO}(B) \rightarrow G$ such that for every closure operator $D$ the following holds:

\[ DQ = r_\Psi(D)DS, \quad r_\Psi(D) = r_D. \]

It is called a coefficient of a change of a chronology of an embedded cobordism. The next proposition goes directly from the definition.

**Proposition 6.10.** Coefficients of changes of chronologies form a planar algebra, where a planar operator $D$ acts as follow:

\[ D(r_{\Psi_1}, \ldots, r_{\Psi_s}) := r_D(\Psi_1, \ldots, \Psi_s). \]

Moreover, the mapping $\Psi \mapsto r_\Psi$ is a morphism of planar algebras.

A priori the coefficient $r_\Psi$ may be different for two changes of a chronology $\Psi: S \leadsto Q$ and $\Psi': S' \leadsto Q$. An example is provided by the change \[ \ref{5.1} \] in the previous section. We will end this chapter showing that for some class of changes of chronologies the coefficient $r_\Psi$ is well-defined (i.e. depends only on $S$ and $Q$).

Let $D$ be a planar diagram with $k$ inputs. Then a permutation $\sigma \in S_k$ induces a planar diagram $D^\sigma$ which differs from $D$ only in the order of inputs: the $i$-th input of $D^\sigma$ is the $\sigma(i)$-th input of $D$. Let $S_1, \ldots, S_k$ be cobordisms such that $S_i$ fits into the $i$-th input of $D$. Directly from the definition, cobordisms $D(S_1, \ldots, S_k)$ and $D^\sigma(S_{\sigma(1)}, \ldots, S_{\sigma(k)})$ differ by some change of a chronology of type I, constant on the cylinder $D \times I$.

**Definition 6.11.** A change of a chronology induced by a planar diagram $D$ is called adapted to $D$ or simply a $D$-change.

Every elementary change of a chronology is adapted to some planar diagram $D$.

**Lemma 6.12.** Let $S$ be a cobordism with two critical points and $\Psi: S \leadsto Q$ be an elementary change of a chronology of type I. Then there exists a planar diagram $D$ with two inputs, such that $\Psi$ is a $D$-change.

**Proof.** Let $t_0$ be the unique critical moment of $\Psi$. Then there exists a unique critical level of $\Psi_{t_0}(S)$

\[ P = \Psi_{t_0}(S) \cap (\mathbb{D}^2 \times \{e\}). \]

It contains two critical points $x_1$ and $x_2$. Let $U_1$ and $U_2$ be their neighbourhoods such that $U_i \cap P$ is a point (if $\mu(x_i) \neq 1$) or a cross (if $\mu(x_i) = 1$). Then $D = P \setminus (U_1 \cup U_2)$ is a planar diagram with two inputs, inducing a change of a chronology equivalent to $\Psi$. \[ \square \]

Due to the above lemma we can narrow our interest to planar diagrams. All elementary changes of chronology of type I between saddles are listed in the figure \[ \ref{11} \]. Circles describe inputs, while saddles are shown as thick arcs — the output of a cobordism can be obtained by connecting circles with the boundary of bands attached along the arcs. As we need to put the arcs in order to have a well-defined chronology, permuting this order corresponds to the change of a chronology.

A composition of two changes adapted to the same planar diagram $D$ is also a $D$-change and there is a subcategory $\text{Chron}_D$ of cobordisms and $D$-changes. Moreover, each $D$-change $\Psi$ is given uniquely up to a composition with an isotopy of chronologies by...
2. Cobordisms and chronologies

Figure 11. Permutations of points of type $M$ and $S$ can be split into five groups $X$, $Y$, $Z$, $I$ and $R$. The names of the first three correspond to the coefficients they generate. Changes denoted by $I$ and $R$ are the changes (T1) from the remark 5.7, where in $I$ orientations of the merge and the split agrees but in $R$ they are opposite. Hence, we get coefficients 1 and $XY$. Thinner curves represents circles and thicker arcs describe saddles. If it matters, arrows describe orientations of critical points.

a permutation of critical points $\sigma_\Psi$ and a decomposition of $\sigma_\Psi$ into elementary transpositions (i.e. transpositions of the form $(i \ i+1)$) gives a decomposition of $\Psi$ into elementary changes.

Theorem 6.13. Let $\Psi: S \rightsquigarrow Q$ be a trivial change of a chronology adapted to some planar diagram $D$. Then $r_\Psi = 1$.

Proof. It is sufficient to show the theorem for $\text{ChCob}^3(\emptyset)$, since the uniqueness of coefficients for changes of chronologies of cobordisms with no corners implies the uniqueness in the general case.

We will show, that the coefficient $r_\Psi$ is independent on a decomposition of $\sigma_\Psi$. Denote by $\tau_i = (i \ i+1)$ an elementary transposition of elements $i$ and $(i+1)$. Any two decompositions of the same permutation are related by the following three relations:

(S1) \[ \tau_i \tau_j = \tau_j \tau_i, \quad |i - j| > 1, \]

(S2) \[ \tau_i^2 = 1, \]

(S3) \[ \tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}. \]

(S1) corresponds to commutativity of changes of chronologies acting on disjoint levels, whereas (S2) to the composition of a change with its inverse. It remains to show invariance under the last relation.
Let $D$ be a planar diagram with three inputs and pick three cobordisms $S_1, S_2, S_3$ with one critical point each, fitting into the inputs of $D$. Given a permutation $\sigma \in S_3$ write $S^\sigma = D^\sigma(S_{\sigma(1)}, S_{\sigma(2)}, S_{\sigma(3)})$. All these cobordisms form a hexagonal diagram:

(6.17)

where arrows represent changes of a chronologies with coefficients $r_i$. It suffices to show that for each diagram $D$ and surfaces $S_1, S_2, S_3$ the product of $r_i$'s is trivial:

(6.18) $r_1 \cdots r_6 = 1$.

The case of disconnected cobordisms is simple. Let the point $p_1$ lie on a different components than $p_2$ and $p_3$. Then in the diagram (6.17) we have equalities $r_3 = r_6^{-1}$ (a permutation of $p_2$ and $p_3$) and $r_1r_2 = r_5^{-1}r_4^{-1}$ (a permutation of $p_1$ with $p_2$ and $p_3$), what gives (6.18).

In the case of connected cobordisms all critical points are saddles. The figure 12 shows all possible situations. Likewise in figure 11 thinner curves represent inputs of cobordisms and thicker arcs describe saddles. Each diagram is equipped with sequences of six numbers, equal to amounts of elementary changes with coefficients respectively $X, Y, Z, Z^{-1}, 1$ and $XY$ which appear in the hexagon diagram (more than one for some diagrams, as different orientations of critical points may lead to different changes). For each such a sequence $(c_1, \ldots, c_6)$ we have $X^{c_1+c_6}Y^{c_2+c_6}Z^{c_3-c_4} = 1$, so the relation (S3) preserves in each case the coefficient of the change of a chronology $r_\Psi$. For instance, the first diagram leads to the following hexagon

where the diagram inside shows the chosen enumeration of inputs and orientations of critical points, whereas diagrams along arrows describe changes of chronologies. Notice, each of them is obtained by either forgetting one arrow (when the first two points are permuted) or applying a surgery along one arrow (when the second and the third point are permuted). In this example the coefficients are equal to $1, Z^{-1}, Y, Z, XY$ and $X$. □

**Corollary 6.14.** If changes $\Psi: S \rightsquigarrow Q$ and $\Psi': S \rightsquigarrow Q$ are adapted to the same planar diagram $D$, then $r_\Psi = r_{\Psi'}$. 

Figure 12. $D$-changes of three critical points. Each diagram corresponds to a hexagon of elementary changes. Numbers in brackets count the amounts of changes of type respectively $X, Y, Z, Z^{-1}, I$ and $R$ which appear in the hexagon. Different sequences correspond to different choices of orientations of thicker arcs.
CHAPTER 3

Elements of homological algebra

The object of interest of homological algebra are Abelian categories, for instance the category of modules over a commutative ring. Among its properties the following seem to be the most important: an additive structure of homorphisms and existence of direct sums, kernels, images and quotient modules. In this chapter we will introduce some machinery which can be used to carry over several constructions from the category of modules into an arbitrary category. This will allow us to construct the generalised Khovanov complex in the next chapter in the category of cobordisms and prove its invariance on that level.

1. Additive categories

Take an arbitrary category $\mathbf{C}$ and pick two of its objects $A$ and $B$.

**Definition 1.1.** An object $X$ together with morphisms $A \xleftarrow{\pi_A} X \xrightarrow{\pi_B} B$ is a product or a direct product of objects $A$ and $B$, if for any object $D$ and morphisms $A \xrightarrow{f} D \xleftarrow{g} B$ there exists a unique morphism $h: D \to X$ such that the following diagram commutes:

$$
\begin{array}{ccc}
D & \xrightarrow{g} & B \\
\downarrow{h} & & \downarrow{\pi_B} \\
A & \xleftarrow{\pi_A} & X
\end{array}
$$

The object $X$ is denoted by $A \times B$ and morphisms $\pi_A$ and $\pi_B$ are called projections. The unique morphism $h: D \to A \times B$ is denoted by $(f, g)$.

When we reserve the arrows, we obtain a dual construction.

**Definition 1.2.** An object $X$ together with morphisms $A \xrightarrow{i_A} X \xleftarrow{i_B} B$ is a coproduct or a direct sum of objects $A$ and $B$, if for any object $D$ and morphisms $A \xleftarrow{f} D \xrightarrow{g} B$ there exists a unique morphism $h: X \to D$ such that the following diagram commutes:

$$
\begin{array}{ccc}
D & \xleftarrow{f} & A \\
\downarrow{h} & & \downarrow{i_A} \\
X & \xrightarrow{i_B} & B
\end{array}
$$

The object $X$ is denoted by $A \oplus B$ and morphisms $i_A$ and $i_B$ are called embeddings. The unique morphism $h: A \oplus B \to D$ is denoted by $f + g$.

A product and a coproduct, if exist, are unique up to an isomorphism. Moreover, for morphisms $f: A \to C$ and $g: B \to D$ there are unique morphisms $f \times g: A \times B \to C \times D$.
and \( f \oplus g: A \oplus B \longrightarrow C \oplus D \) agreeing respectively with projections and embeddings. The properties below follow directly from the definitions of products and coproducts.

**Proposition 1.3.** Let \( A, B, C \) be objects of a category \( C \). Then there exist natural isomorphisms

- \( A \times B \cong B \times A \)
- \( A \times (B \times C) \cong (A \times B) \times C \)
- \( A \oplus B \cong B \oplus A \)
- \( A \oplus (B \oplus C) \cong (A \oplus B) \oplus C \)

as long as all the objects are well defined.

Naturality of the isomorphisms above means compliance with induced morphisms. For instance, when \( f: A \longrightarrow A' \) and \( g: B \longrightarrow B' \), the diagrams below commute

\[
\begin{array}{ccc}
A \times B & \cong & B \times A \\
\downarrow f \times g & & \downarrow g \times f \\
A' \times B' & \cong & B' \times A'
\end{array}
\quad \begin{array}{ccc}
A \oplus B & \cong & B \oplus A \\
\downarrow f \oplus g & & \downarrow g \oplus f \\
A' \oplus B' & \cong & B' \oplus A'
\end{array}
\]

and similarly in other cases.

**Example 1.4.** The category of sets \( \text{Set} \) has Cartesian products as products, and disjoint sums as coproducts. In the category of Abelian groups \( \text{Ab} \) Cartesian products also play the role of products, whereas coproducts are given by direct sums. In case of the full category of groups \( \text{Grp} \) products does not change, but coproducts are given by free products.

The definition of the product and the coproduct can be easily extended over any number of objects: the product of objects \( \{A_\lambda\}_{\lambda \in \Lambda} \) is the object \( X \) along with projections \( \{\pi_\lambda: X \longrightarrow A_\lambda\}_{\lambda \in \Lambda} \) such that given any object \( D \) with morphisms \( \{f_\lambda: X \longrightarrow A_\lambda\}_{\lambda \in \Lambda} \) there exists a unique morphism \( h: D \longrightarrow X \) making the following diagram commute for each \( \lambda \in \Lambda \):

\[
\begin{array}{ccc}
D & \downarrow h & \\
\downarrow f_\lambda & & \\
X & \pi_\lambda & A_\lambda
\end{array}
\]

The product is called finite if \( \Lambda \) is a finite set. In case \( \Lambda = \emptyset \) we obtain the terminal object \( T \) and there exists exactly one arrow to \( T \) from any object \( A \). We can define the general coproduct dually, obtaining for \( \Lambda = \emptyset \) the initial object \( I \).

**Proposition 1.5.** Let \( C \) be a category. If \( T \) and \( I \) are respectively the terminal and the initial object in \( C \), then

\[
(1) \ T \times A \cong A \times T \cong A \\
(2) \ I \oplus A \cong A \oplus I \cong A
\]

**Proof.** Isomorphisms are provided by the definitions of products and coproducts. We will show the case \( T \times A \cong A \) — other proofs are mostly the same.
Notice first, that $\pi_A: T \times A \to A$ is the only morphism, for which the following diagram commutes:

\[
\begin{array}{ccc}
T \times A & \xleftarrow{\pi_T} & T \\
\downarrow{\pi_A} & & \downarrow{\pi_A} \\
T & \xleftarrow{a} & A \\
\end{array}
\]

where $a: A \to T$ is the unique morphism to the terminal object $T$. Indeed, $a \circ \pi_A$ is a morphism from $T \times A$ to $T$, same as $\pi_T$. Since $T$ is terminal, these morphisms have to be equal. But the universal property of a product gives also a morphism $i_A: A \to T \times A$ which appears to be the inverse of $\pi_A$. Hence, $A \cong T \times A$.  

In the categories of Abelian groups $\text{Ab}$ and $R$-modules $\text{Mod}_R$ all finite products and coproducts exist. Moreover, a set of morphisms between any two objects is an Abelian group. This is a motivation for the following definition.

**Definition 1.6.** A category $\mathcal{C}$ is said to be **additive**, if

1. each set $\text{Mor}_{\mathcal{C}}(X,Y)$ is an Abelian group and composition of morphisms is additive from both sides:

   \[
   (f + g) \circ h = f \circ h + g \circ h, \quad h \circ (f + g) = h \circ f + h \circ g
   \]

2. finite products and coproducts exist

A category is called **preadditive** if the first condition holds but not the second.

A finite product of Abelian groups is also their coproduct. In fact, this holds in any additive category.

**Theorem 1.7.** Let $\mathcal{C}$ be a preadditive category and $A, B$ its objects. Then the product $A \times B$ exists if and only if there is the coproduct $A \oplus B$. Moreover, these two are equal and

\[
\begin{align*}
\pi_A i_A &= \text{id}_A, \quad \pi_B i_B = \text{id}_B, \quad \pi_A i_B = \pi_B i_A = 0, \quad i_A \pi_A + i_B \pi_B &= \text{id}_{A \times B}.
\end{align*}
\]

**Proof.** Assume the product $A \times B$ exists. Then there is a unique morphism $i_A$ given by the following diagram

\[
\begin{array}{ccc}
A & \xleftarrow{i_A} & A \times B \\
\downarrow{\pi_A} & & \downarrow{\pi_B} \\
A & \xleftarrow{i_A} & B
\end{array}
\]

and similarly we can define $i_B$. First fourth equalities in \((1.1)\) hold trivially. To show the last one consider the following diagram with $m = i_A \pi_A + i_B \pi_B$:

\[
\begin{array}{ccc}
A \times B & \xleftarrow{i_A} & A \\
\downarrow{\pi_A} & & \downarrow{\pi_B} \\
A & \xleftarrow{i_A} & B
\end{array}
\]

The diagram commutes, so $m = \text{id}$ and the triple $(A \times B, i_A, i_B)$ is a coproduct with the induced morphism $h = f \pi_A + g \pi_B$ for any $f$ and $g$. 
To end the proof we need to show that a terminal object $T$ is also an initial object. In a preadditive category there is a zero morphism $0: A \rightarrow B$ for any objects $A$ and $B$. In particular, $\text{id}_T = 0$. Hence, any morphism $f: T \rightarrow A$ is equal $f \circ \text{id}_T = 0$, so $T$ is initial.

In the same way one can prove that coproducts are products. □

**Corollary 1.8.** Categories $\text{Set}$ and $\text{Grp}$ cannot be extended to preadditive categories preserving both products and coproducts.

In a preadditive category the initial object, which is also terminal, is called the zero object $0$. Any morphism having it as a domain or a codomain has to be a zero morphism.

Now we will give a more general notion of an additive category.

**Definition 1.9.** Let $R$ be a commutative ring. Say a category $\mathcal{C}$ is $R$-preadditive, if a set of morphisms $\text{Mor}_{\mathcal{C}}(X,Y)$ is an $R$-module for any objects $X$ and $Y$. A category $\mathcal{C}$ is $R$-additive, if it is both additive and $R$-preadditive.

A preadditive category is $\mathbb{Z}$-preadditive. It can be extended to a $R$-preadditive one by tensoring it with $R$:

\[ \text{Mor}_{R\mathcal{C}}(X,Y) = \text{Mor}_{\mathcal{C}}(X,Y) \otimes R \]

**Remark 1.10.** A category $\mathcal{C}$ can be extended to an $R$-preadditive category $R\mathcal{C}$ in the following way:

- objects are preserved: $\text{Ob}(R\mathcal{C}) = \text{Ob}(\mathcal{C})$
- the set of morphisms $\text{Mor}_{R\mathcal{C}}(X,Y)$ is the free $R$-module generated by $\text{Mor}_{\mathcal{C}}(X,Y)$, i.e. it consists of formal finite sums
  \[ \sum_{i=1}^{n} r_i f_i : X \rightarrow Y \]
  where $r_i \in R$ and $f_i : X \rightarrow Y$ are morphisms in $\mathcal{C}$
- composition in $R\mathcal{C}$ is a bilinear extension of a composition in $\mathcal{C}$:
  \[ f \circ (rg + sh) = rf + sg \circ h \]
  \[ (rf + sg) \circ h = rf \circ h + s(g \circ h) \]

**Remark 1.11.** Any preadditive category $\mathcal{C}$ can be extended to an additive category $\text{Mat}(\mathcal{C})$ as follows:

- objects of $\text{Mat}(\mathcal{C})$ are formal direct sums $\bigoplus_{i=1}^{n} C_i$ of objects from $\mathcal{C}$
- a morphism $F: \bigoplus_{i=1}^{n} X_i \rightarrow \bigoplus_{j=1}^{m} Y_j$ is a matrix of morphisms
  \[ F = (F_{ji} : X_i \rightarrow Y_j) \]
- the addition of morphisms in $\text{Mat}(\mathcal{C})$ is defined as the addition of matrices
- the composition of morphisms is defined as the multiplication of matrices:
  \[ F_{ij} \circ G_{jk} = H_{ik}, \quad \text{where } H_{ik} = \sum_{j} F_{ij} \circ G_{jk} \]

Obviously, if $\mathcal{C}$ is $R$-preadditive, its extension to an additive category is $R$-additive. The category $\text{Mat}(\mathcal{C})$ is called the category of matrices over $\mathcal{C}$ or the additive closure of $\mathcal{C}$. Objects can be represented by finite sequences of objects from $\mathcal{C}$, while morphisms by bundles of morphisms from $\mathcal{C}$ (fig. 1). In this view, the component $(F \circ G)_{ik}$ is a sum of all paths from $X_k$ to $Z_i$. 
### Example 1.12
If $C$ is additive, $\text{Mat}(C)$ is equivalent to $C$. Indeed, by uniqueness of coproducts there is a natural isomorphism between a formal direct sum and the internal direct sum in the category $C$. Thus, the closure introduces no essential objects.

### Example 1.13
Let $R$ be a category consisting of a unique object being a commutative ring $R$ and $\text{Mor}(R) = \text{End}(R)$. Then $\text{Mat}(R)$ is the category of free modules over $R$. In particular, when $R = k$ is a field, we get the category $\text{Vect}_k$ of vector spaces over $k$.

A functor $F: C \rightarrow D$ between $R$-preadditive categories is called additive, if it is $R$-linear on morphisms, i.e. $F( rf + sg) = rFf + sFg$. Given a product $A \times B$, an additive functor $F$ induces morphisms

\[
FA \xrightarrow{F \pi_A} F(A \times B) \xrightarrow{F \pi_B} FB
\]

satisfying equations (1.1). Uniqueness of the product provides $F(A \times B) \cong FA \times FB$ and the following holds.

### Proposition 1.14
An additive functor $F: C \rightarrow D$ between preadditive categories preserves all products and coproducts.

We will end this section with the notion of gradation.

### Definition 1.15
An $R$-preadditive category $C$ has a gradation, if

1. for any objects $X, Y$ the set $\text{Mor}_C(X, Y)$ is a graded $R$-module with a distinguished subset $\tilde{\text{Mor}}_C(X, Y)$ consisting of functions called homogeneous such that $\text{id}_A$ is homogeneous for any object $A$.
2. there is a degree function $\text{deg}: \tilde{\text{Mor}}(C) \rightarrow \mathbb{Z}$ defined for homogeneous functions, agreeing with the composition, i.e. $\text{deg}(f \circ g) = \text{deg } f + \text{deg } g$.
3. there is an operation called gradation shift 

\[
\text{Ob}(C) \times \mathbb{Z} \ni (X, m) \mapsto X\{m\} \in \text{Ob}(C),
\]

preserving morphisms, i.e. $\text{Mor}_C(X\{m\}, Y\{n\}) = \text{Mor}_C(X, Y)$, but changing its gradings: a morphism $f \in \text{Mor}_C(X\{m\}, Y\{n\})$ has degree $\text{deg } f = d + n - m$, where $d$ is the degree of $f$ as an element of $\text{Mor}_C(X, Y)$.

### Example 1.16
Any additive category $C$ can be extended to a graded category $\Sigma C$ as follows:

1. objects are direct sums: $\bigoplus_{i \in \mathbb{Z}} X^i$
2. morphisms are the morphisms from $C$ and $f: X \rightarrow Y$ is homogeneous of degree $\text{deg } f = r$, if $f = \bigoplus_{i \in \mathbb{Z}} f^i$ is a direct sum of morphisms $f^i: X^i \rightarrow Y^{i+r}$
3. the gradation shift is defined by a shift of indices: $X\{m\} = \bigoplus_{i \in \mathbb{Z}} X^{i+m}$
Let $C$ be an arbitrary category. If there is a function $\text{deg}: \text{Mor}(C) \rightarrow \mathbb{Z}$ that is additive under compositions, then the category $C$ can be extended to a graded category $\tilde{C}$ such that $\text{deg}$ becomes a degree map. It is done by adding formal objects $X\{m\}$ for all $X \in \text{Ob}(C)$ and $m \in \mathbb{Z}$.

**Example 1.17.** Define the degree map in the category $\text{ChCob}^3(B)$ as an Euler characteristic corrected by half of the number of endpoints $B$:

$$\text{deg } M = \chi(M) - \frac{1}{2}|B|. \tag{1.4}$$

This function satisfies first two points of the definition 1.15. Hence, $\text{ChCob}^3$ can be extended to a graded category using the procedure described above.

### 2. Chain complexes

In the example 1.16 we have constructed a category $\Sigma C$ from any additive category $C$. Denote by $(C,d)$ a pair consisting of an object from $\Sigma C$ and a morphism $d: C \rightarrow C$ of degree $+1$.

**Definition 2.1.** Providing $d \circ d = 0$, a pair $(C,d)$ is called a **chain complex**, and a morphism $d$ is a differential of $C$. A complex $(C,d)$ is **bounded**, if $C_r = 0$ for sufficiently large and small $r$. A chain map $f: (C_1,d_1) \rightarrow (C_2,d_2)$ is a morphism $f: C_1 \rightarrow C_2$ which commutes with differentials, i.e. $f \circ d_1 = d_2 \circ f$.

Chain complexes with chain maps form a graded category. The subcategory consisting of all chain complexes and morphisms of degree 0 is denoted by $\text{Kom}(C)$. Both categories $C$ and $\Sigma C$ can be seen as its subcategories, when zero objects and differentials are added, i.e. an object $X \in \text{Ob } C$ can be seen as a complex

$$\cdots \rightarrow 0 \rightarrow X \rightarrow 0 \rightarrow \cdots \tag{2.1}$$

where the underlined term $X$ is in a degree 0.

**Remark 2.2.** If $C$ is graded, the category of complexes $\text{Kom}(C)$ gets another grading ($f$ is homogenous of degree $d$, if for every $r \in \mathbb{Z}$ $f^r$ is homogenous of degree $d$). Thus we obtain two gradations:

- a complex gradation (exterior): $C[i]_r := C^{r+i}$
- an induced gradation (interior): $C\{m\}_r := C^r\{m\}$

Since now, We will denote by $\text{deg}$ the degree connected to the induced gradation and differentials as assumed to have degree zero with respect to the induced grading (i.e. $\text{deg } d = 0$).

**Remark 2.3.** The category $\text{Kom}(C)$ is additive with natural products:

$$\text{Hom}(C_1, C_2) \oplus \text{Hom}(C_2, C_3) := \text{Hom}(C_1 \oplus C_2, C_3) \oplus \text{Hom}(C_2 \oplus C_3, C_4) \oplus \cdots$$

where $(C_1 \oplus C_2)^r = C_1^r \oplus C_2^r$, $(d_1 \oplus d_2)^r = d_1^r \oplus d_2^r$.

Notice that in $\text{Mod}_R$ each morphism $f: M \rightarrow M'$ has a kernel $\ker f = f^{-1}(0)$ and an image $\text{im } f = f(M)$, both being $R$-modules. The condition $d \circ d = 0$ implies that $\ker d \subset \text{im } d$, and we can create quotient modules

$$H^n(C) = \ker d^n / \text{im } d^{n-1} \tag{2.2}$$

The graded module $H^*(C)$ is called a **homology** of the complex $(C,d)$. $(C,d)$ is called a **exact sequence**, if $H^*(C) = 0$. 
Theorem 2.4. Take an exact sequence of complexes of $R$-modules:

\[ 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \]

Then there exists a long exact sequence of homologies:

\[ \cdots \longrightarrow H^i(A) \longrightarrow H^i(B) \longrightarrow H^i(C) \longrightarrow H^{i+1}(A) \longrightarrow \cdots \]

Definition 2.5. Let $f: (C_a, d_a) \longrightarrow (C_b, d_b)$ and $g: (C_a, d_a) \longrightarrow (C_b, d_b)$ be chain maps. A chain homotopy from $f$ to $g$ is a morphism $h: C_a \longrightarrow C_b[-1]$ such that

\[ f - g = hd + dh \]

In this case $f$ and $g$ are called homotopic and we write $f \sim_h g$.

Chain complexes $(C_a, d_a)$ and $(C_b, d_b)$ are called homotopic, if there exist morphisms $f: (C_a, d_a) \longrightarrow (C_b, d_b)$ and $g: (C_b, d_b) \longrightarrow (C_a, d_a)$ such that

\[ f \circ g \sim_h \text{id} \quad g \circ f \sim_h \text{id} \]

Morphisms $f$ and $g$ are called homotopy equivalences.

The homotopy equivalence relation agrees with compositions of morphisms. Therefore the quotient category $\text{Kom}_{/h}(C)$ is well defined. Moreover, directly from the definition, homotopic chain complexes have isomorphic homologies.

A special kind of homotopy equivalences are deformation retractions, which are the morphisms $g: (C_b, d_b) \longrightarrow (C_a, d_a)$ having a section $f: (C_a, d_a) \longrightarrow (C_b, d_b)$ being its homotopy inverse:

\[ g \circ f = \text{id}_{C_a} \quad f \circ g \sim_h \text{id} \]

If there exists a homotopy $h$ such that $h \circ f = 0$, then $g$ is called a strong deformation retraction, whereas $f$ is an inclusion in a strong deformation retract.

The next definition show how to construct a new chain complex from a chain map.

Definition 2.6. Let $f: (C_0, d_0) \longrightarrow (C_1, d_1)$ be a chain map. A cone of $f$ is a complex $(\text{cone}(f), \tilde{d})$ defined as follows:

\[ \text{cone}(f) = C_0 \oplus C_1[-1], \quad \tilde{d} = \begin{pmatrix} -d_0 & 0 \\ f & d_1[-1] \end{pmatrix} \]

A commutative square induces a morphism of cones. Indeed, directly from the definition 2.6 we have the following result.

Proposition 2.7. Let the following be a commutative diagram of complexes:

\[
\begin{array}{ccc}
C_{1a} & \overset{g_1}{\longrightarrow} & C_{1b} \\
\downarrow{f_a} & & \downarrow{f_b} \\
C_{2a} & \overset{g_2}{\longrightarrow} & C_{2b}
\end{array}
\]

Then the morphism $g = g_1 \oplus g_2[-1]: \text{cone}(f_a) \longrightarrow \text{cone}(f_b)$ is a chain map.
In particular, the following diagram (zero objects are omitted):

\[
\begin{array}{ccc}
  C_1 & \xrightarrow{id} & C_1 \\
  \downarrow f & & \downarrow f \\
  C_2 & \xrightarrow{id} & C_2
\end{array}
\]

induces morphisms \(i: C_2 \to \text{cone}(f)\) and \(\pi: \text{cone}(f) \to C_1\) which form a sequence:

\[
(2.9) \quad 0 \to C_2[-1] \xrightarrow{i} \text{cone}(f) \xrightarrow{\pi} C_1 \to 0
\]

In a category of \(R\)-modules the above sequence is exact and having an additive functor \(F: C \to \text{Mod}_R\) along with a morphism \(f: C \to D\) we get a long exact sequence of homologies

\[
(2.10) \quad \cdots \to H^i(FD) \to H^i(FC) \to H^i(\text{cone}(Ff)) \to H^{i+1}(FD) \to \cdots
\]

as due to the proposition 1.14 an additive functor preserves cones.

We will end this section with a theorem of invariance of cones under homotopies when composed with strong deformation retracts. In many cases using this theorem a given complex can be simplified a lot if an appropriate retract is known.

**Theorem 2.8.** Let the following be a commutative diagram

\[
\begin{array}{ccc}
  C_{0a} & \xrightarrow{f_0} & C_{0b} \\
  \downarrow F & & \downarrow F \\
  C_{1a} & \xrightarrow{g_1} & C_{1b}
\end{array}
\]

where \(f_0\) is an inclusion in a strong deformation retract and \(g_1\) is a strong deformation retraction. Then all the cones \(\text{cone}(F), \text{cone}(Ff_0), \text{cone}(g_1F)\) are homotopic.

**Proof.** For \(f_0\) there is a strong deformation retraction \(g_0: C_{0a} \to C_{0b}\) and a homotopy \(h_0: C_{0a} \to C_{0a}[-1]\) such that

\[
g_0f_0 = \text{id}, \quad \text{id} - f_0g_0 = dh_0 + h_0d, \quad h_0f_0 = 0
\]

Take the following morphisms:

\[
\begin{array}{l}
  \tilde{f}_0: \text{cone}(Ff_0) \to \text{cone}(F) \\
  \tilde{g}_0: \text{cone}(F) \to \text{cone}(Ff_0) \\
  \tilde{h}_0: \text{cone}(F)^* \to \text{cone}(F)^{*-1}
\end{array}
\]

\[
\begin{array}{c}
  \tilde{f}_0 = \begin{pmatrix} f_0 & 0 \\ 0 & \text{id} \end{pmatrix} \\
  \tilde{g}_0 = \begin{pmatrix} g_0 & 0 \\ Fh_0 & \text{id} \end{pmatrix} \\
  \tilde{h}_0 = \begin{pmatrix} -h_0 & 0 \\ 0 & \text{id} \end{pmatrix}
\end{array}
\]
They form a commutative diagram:

\[
\begin{array}{ccccccccc}
\text{cone}(Ff_0) : & \cdots & C'_{0b} & \oplus & C'_{1a}^{r-1} & \overset{d}{\rightarrow} & C'_{0b}^{r+1} & \oplus & C'_{1a}^r & \cdots \\
& & g_0 & f_0 & & & g_0^{-1} & f_0^{-1} & \\
\text{cone}(F) : & \cdots & C'_{0a} & \oplus & C'_{1a}^{r-1} & \overset{d}{\rightarrow} & C'_{0a}^{r+1} & \oplus & C'_{1a}^r & \cdots \\
\end{array}
\]

Moreover, \(g_0f_0 = \text{id}\) and \(\text{id} - f_0g_0 = d\tilde{h} + \tilde{h}_0d\). Hence, complexes \(\text{cone}(F)\) and \(\text{cone}(Ff_0)\) are homotopic. The other equivalence is shown in the same way. \(\square\)

3. Cubes

Let \(I^n\) be a standard unit \(n\)-cube in \(\mathbb{R}^n\). Its edges together with vertices form a directed graph. Vertices are labeled with zero-one sequences \(\xi = (\xi_1, \ldots, \xi_n)\) of length \(n\). Let \(|\xi| = \xi_1 + \cdots + \xi_n\). Replacing the \(i\)-th item with a star \(*\), we get a label of an edge \(\zeta = (\zeta_1, \ldots, *, \ldots, \zeta_n)\) going from \(\zeta(0) = (\zeta_1, \ldots, 0, \ldots, \zeta_n)\) to \(\zeta(1) = (\zeta_1, \ldots, 1, \ldots, \zeta_n)\).

**Definition 3.1.** A cube diagram of dimension \(n\) or an \(n\)-cube in a category \(\mathcal{C}\) is a mapping \(F: I^n \rightarrow \mathcal{C}\) which associates each vertex with an object of \(\mathcal{C}\) and each edge \(\zeta\) with a morphism \(F\zeta: F\zeta(0) \rightarrow F\zeta(1)\). A morphism \(\eta: F \rightarrow G\) of \(n\)-cubes is a collection of morphisms \(\{\eta_\xi: F_\xi \rightarrow G_\xi\}\).

All \(n\)-cubes in a given category \(\mathcal{C}\) form a category \(n\mathsf{Cub}(\mathcal{C})\) with an obvious composition. Denote by \(\mathsf{Cub}(\mathcal{C})\) the category of cubes of any dimension.

**Remark 3.2.** A morphism \(F_*: F_0 \rightarrow F_1\) of \(n\)-cubes induces an \((n+1)\)-cube \(F\) given as follows:

(3.1) \[F(\xi, i) = F_i(\xi)\]

(3.2) \[F(\xi, *) = (F_\xi)\]

Contrary, each \((n+1)\)-cube \(F\) produces a cube morphism \(F_*: F_0 \rightarrow F_1\), where \(F_i = F(\cdot, i)\). The above correspondence is clearly a bijection.

More generally, for every \((m+n)\)-cube \(F \in \mathsf{Cub}(\mathcal{C})\) we can construct an \(n\)-cube \(F^{(m)}(F) \in \mathsf{Cub}(m\mathsf{Cub}(\mathcal{C}))\) as follows:

(3.3) \[F^{(m)}(\xi) = F_\xi := F(\cdot, \xi)\]

(3.4) \[F^{(m)}(\zeta) = F_\zeta := F(\cdot, \zeta)\]

Hence, every \((m+n)\)-cube \(F\) can be seen as an \(n\)-cube \(F^{(m)}\) in a category of \(m\)-cubes. Contrary, each \(n\)-cube \(F\) in such a category describes an \((m+n)\)-cube in \(\mathcal{C}\).

Since now fix a commutative ring \(R\) and assume \(\mathcal{C}\) is \(R\)-additive. Denote by \(G = U(R)\) the group of invertible elements in \(R\). A projectivization of \(\mathcal{C}\) is the category \(\mathbb{P}\mathcal{C} = \mathcal{C}/G\), in which any two morphisms differing by an invertible element are identified. A projectivization of a cube \(F: I^n \rightarrow \mathcal{C}\) is defined as a composition of \(F\) with the canonical projection: \(\mathbb{P}F = \mathbb{P} \circ F: I^n \rightarrow \mathbb{P}\mathcal{C}\).
Definition 3.3. Choose any two dimensional face of a cube $F: I^n \rightarrow C$

\[
\begin{array}{ccc}
FA & \xrightarrow{Fa} & FB \\
F \downarrow & & \downarrow Fb \\
FC & \xrightarrow{Fd} & FD
\end{array}
\] (3.5)

This face is:

- **commutative**, if $Fb \circ Fa = Fd \circ Fc$
- **anticommutative**, if $Fb \circ Fa = -Fd \circ Fc$
- **projective**, if $Fb \circ Fa = \lambda Fd \circ Fc$ for some $\lambda \in G$

The cube $F$ is called **commutative**, **anticommutative** or **projective**, if all its faces are respectively commutative, anticommutative or projective.

It follows from the above definition that a cube $F: I^n \rightarrow C$ is projective if and only if its projectivization $\mathbb{P}F: I^n \rightarrow \mathbb{P}C$ is commutative. Projective cubes with equal projectivizations will be called $\mathbb{P}$-**equivalent**.

Definition 3.4. A cube morphism $\eta: F \rightarrow G$ is **commutative**, **anticommutative** or **projective**, if for each edge $\zeta$ the following square is respectively commutative, anticommutative or projective:

\[
\begin{array}{ccc}
F\xi & \xrightarrow{\eta} & G\xi \\
F\zeta & & \downarrow G\zeta \\
F\xi' & \xrightarrow{\eta'} & G\xi'
\end{array}
\] (3.6)

Two projective morphisms are called $\mathbb{P}$-**equivalent**, if their projectivizations are equal.

Both commutative and projective morphisms are closed under compositions. Hence we obtain three subcategories in $\textbf{Cub}(C)$: commutative cubes with commutative morphisms $\textbf{Cub}^c(C)$, anticommutative cubes with commutative morphisms $\textbf{Cub}^a(C)$ and projective cubes with projective morphisms $\textbf{Cub}^p(C)$.

Remark 3.5. In analogous to $\textbf{Cub}(C)$, each of the subcategories described above possesses a bijection between $(m+n)$-cubes and $n$-cubes in the category of $m$-cubes. In particular, in case $m = 1$ there is a bijection between $(n+1)$-cubes and morphisms of $n$-cubes, as every morphism is a 1-cube (commutative, anticommutative and projective at the same time). However, the morphism generated by an anticommutative $(n+1)$-cube $F$ is given by $\eta: -F_0 \rightarrow F_1$ (otherwise we will get an anticommutative one). Moreover, $\mathbb{P}$-equivalence is preserved for projective cubes.

Let $F$ be a projective $n$-cube. Denote its face from diagram (3.5) by $S$ and assume that directions of morphisms $Fa$ and $Fb$ agrees with the natural orientation of $S$. A cochain $\psi \in C^2(I^n; G)$ is **associated** to the cube $F$, if

\[
Fb \circ Fa = \psi(S)Fd \circ Fc
\] (3.7)

for every face $S$. 

Definition 3.6. Say a cube $F$ is a CC-cube or that is satisfies the cocycle condition, if there exists an associated cochain being a cocycle.

Notice that a cochain $\varphi \in C^1(I^n; G)$ defines a cube $\varphi_* F$ by multiplying edges of $F$ by the values of $\varphi$:

(3.8) $\varphi_* F(\xi) = F\xi$
(3.9) $\varphi_* F(\zeta) = \varphi(\zeta) F\zeta$

If $\psi_F$ is a cochain associated to $F$, the cochain $\psi_{\varphi_* F} = d\varphi \psi_F$ is associated to $\varphi_* F$. In particular, if $F$ is a CC-cube, so is $\varphi_* F$. Since $\mathbb{P}$-equivalent projective cubes differ only by a cochain, we have the following result.

Corollary 3.7. Let $F$ and $G$ be $\mathbb{P}$-equivalent projective cubes. Then $F$ is a CC-cube if and only if $G$ is a CC-cube.

Say a commutative cube $F: I^n \longrightarrow \mathbb{P}C$ is a CC-cube, if there exists (and due to the corollary every) its representative being a CC-cube.

A cochain $\varphi \in C^1(I^n; G)$ is a positive or a negative edge assignment of $F$, if $\varphi_* F$ is respectively a commutative or anticommutative cube. Directly from the definition of a differential we get the lemma below.

Lemma 3.8. An edge assignment $\varphi$ of a cube $F$ is positive (negative) if and only if $d\varphi = \psi$ (respectively: $d\varphi = -\psi$) for some associated cochain $\psi$.

In the above situation the edge assignment $\varphi$ is said to be of type $\psi$. Having two edge assignments $\varphi_1$ and $\varphi_2$ of the same type (positive or negative) the equality $d(\varphi_1 \varphi_2^{-1}) = 1$ is satisfied and the following holds.

Theorem 3.9. A cube $F$ has both a positive and a negative edge assignment if and only if $F$ is a CC-cube. Furthermore, two edge assignments (both positive or negative) of the same type induce isomorphic cubes (in the sense of commutative isomorphisms).

Proof. If $d\psi = 1$, then $\psi = d\varphi$ is a coboundary, since $H^2(I^n; G) = 0$. This shows the existence part (for negative assignments notice that $d(-\psi) = d\psi$). As $H^1(I^n; G) = 0$, having two edge assignments (both positive or negative) $\varphi_1$ and $\varphi_2$ there is a cochain $\eta \in C^0(I^n; G)$ such that $\varphi_1 = (d\eta) \varphi_2$. Thus there exists a commutative morphism $f: (\varphi_1)_* F \longrightarrow (\varphi_2)_* F$ given by:

$$f_\xi = \eta(\xi) id_\xi,$$

which is an isomorphism, since each $\eta(\xi) \in G$ is invertible. \hfill $\square$

Corollary 3.10. Up to cube isomorphisms, a CC-cube $F: I^n \longrightarrow \mathbb{P}C$ describes for each associated cocycle $\psi$ a unique commutative and a unique anticommutative cube in $C$.

According to the corollary $3.10$, each projective CC-cube is $\mathbb{P}$-equivalent to a commutative one. A morphism of projective cubes is called a CC-morphism, if it induces a CC-cube. Due to the previous observations, every CC-morphism is $\mathbb{P}$-equivalent to a morphism of commutative cubes. In fact, a stronger theorem holds.

\footnote{Notice we use multiplicative notation for the group $G$, so $d(-\psi) \neq -d\psi$. Instead, we have an equality $d\psi^{-1} = (d\psi)^{-1}$.}
3. Elements of homological algebra

**Theorem 3.11.** Let $F_0$ and $F_1$ be commutative $n$-cubes. Then for every $CC$-morphism $\eta: \mathbb{P}F_0 \to \mathbb{P}F_1$ there exists a representative $\tilde{\eta}: F_0 \to F_1$ that is a morphism of commutative cubes $F_0$ and $F_1$. Moreover, every two such representatives $\tilde{\eta}_1$ and $\tilde{\eta}_2$ differs by an invertible element, i.e. $\tilde{\eta}_1 = \lambda \tilde{\eta}_2$ for some $\lambda \in G$.

**Proof.** Let $\tilde{\eta}$ represent $\eta$. It induces an $(n+1)$-cube $F$, which is not commutative in general. We have to find a positive edge assignment for $F$, equals one on $I^n \times \partial I$.

Pick $\psi$ a cocycle associated to $F$. Then $\psi(S) = 1$ for each face $S$ in $F_0$ or $F_1$, so $\psi \in C^2(I^{n+1}, I^n \times \partial I; G)$ is a relative cochain. As the second relative homology group vanishes

$$H^2(I^{n+1}, I^n \times \partial I; G) = 0$$

there is a cochain $\varphi \in C^1(I^{n+1}, I^n \times \partial I; G)$ such that $d\varphi = \psi$. It is equal one on edges of both $F_0$ and $F_1$, so $\varphi \tilde{\eta}$ is a commutative morphism of cubes $F_0$ and $F_1$, representing $\eta$.

To show the second part notice that every two representatives $\tilde{\eta}_1$ and $\tilde{\eta}_2$ induces a cocycle $\varphi \in C^1(I^{n+1}, I^n \times \partial I; G)$ such that $\tilde{\eta}_2 = \varphi_\nu \tilde{\eta}_1$. Hence, for two edges $\xi, \xi'$ from $I \times 0$ to $I \times 1$ and a face $S$ connecting them we have

$$\varphi(\xi, *) \varphi(\xi', *)^{-1} = d\psi(S) = 1$$

what gives $\varphi(\xi, *) = \varphi(\xi', *)$. Connectedness of a cube provides $\varphi$ is constant, what ends the proof. \qed

**Corollary 3.12.** Let $F$ be an $(n+m)$-cube satisfying the cocycle condition such that each $m$-cube $F_\xi$ is commutative. Then there exists an edge assignment $\varphi$ of a cube $F$ such that $\varphi_\xi(F_\xi) = F_\xi$ for each edge $\xi \in I^n$. Moreover, if $\psi$ is a cochain associated to $F$ such that $\psi|_{F_\xi} = 1$ for each edge $\xi \in I^n$, we may assume that $\varphi$ is of type $\psi$.

**Proof.** The case $n = 0$ is trivial. Assume the hypothesis holds for $n = k$ and take a $(k+1+m)$-cube $F$. By induction hypothesis there exists edge assignments $\varphi_0$ and $\varphi_1$ of cubes $F_0$ and $F_1$. Applying the theorem [3.11] we find an edge assignment $\varphi'$ of a morphism $F_\xi: F_0 \to F_1$. The product $\varphi_0 \varphi_1 \varphi'$ is the desired edge assignment. \qed

**Corollary 3.13.** Let $F$ and $G$ be commutative cubes and let $\mathbb{P}\eta: \mathbb{P}F \to \mathbb{P}G$ be a $CC$-morphism. Then having a cocycle $\psi$ associated to $\eta$ there exists a unique commutative representative $\tilde{\eta}: F \to G$ such that $\tilde{\eta}(0, \ldots, 0) = \eta(0, \ldots, 0)$.

**Remark 3.14.** The above results also hold when the words ‘commutative’ are replaced with ‘anticommutative’.

Since $\mathbb{P}$-equivalence agrees with compositions, a composition of $CC$-morphisms is still a $CC$-morphism. Therefore, a category of projective cubes contains a subcategory of $CC$-cubes $\text{Cub}_{CC}(C)$.

4. Cube complexes

All categories $\text{Cub}(C)$, $\text{Cub}^e(C)$, $\text{Cub}^e(C)$, $\text{Cub}^e(C)$ and $\text{Cub}_{CC}(C)$ constructed in the previous section are $R$-additive. Indeed, the action of $R$ on morphisms is induced from the category $C$, whereas the coproduct of $F$ and $G$ is given as follows:

\begin{align}
(F \oplus G)(\xi) &:= F\xi \oplus G\xi \\
(F \oplus G)(\zeta) &:= F\zeta \oplus G\zeta
\end{align}

The zero element is the zero cube $0(\xi) := 0_C$. 
We will now pass to anticommutative cubes, as they can be used to produce chain complexes in an easy way.

**Definition 4.1.** Let $F$ be an anticommutative cube. A cube complex induced by $F$ is the complex $\text{Kom}(F) = (C_F, d_F)$ given as follows:

\begin{equation}
C_F^r := \bigoplus_{|\xi| = r} F\xi
\end{equation}

\begin{equation}
d_F^r|_{F\xi} := \sum_{\zeta: \xi \rightarrow \xi'} F\zeta
\end{equation}

The condition $d \circ d = 0$ holds due to anticommutativity of $F$. Notice that a commutative morphism $\eta: F \rightarrow G$ of anticommutative cubes induces in a natural way a chain map $\text{Kom}(\eta): C_F \rightarrow C_G$ and we get a functor $\text{Kom}$ from the category $\text{Cub}^a(C)$ to chain complexes $\text{Kom}(C)$. In particular, an anticommutative $(n+1)$-cube $F$ induces morphisms $\eta: - F_0 \rightarrow F_1$ and $\text{Kom}(\eta): \text{Kom}(- F_0) \rightarrow \text{Kom}(F_1)$. Directly from the definition of a cone we obtain the following

**Theorem 4.2.** The complex $\text{Kom}(F)$ is equal to the complex $\text{cone}(\text{Kom}(\eta))$.

The theorem 4.2 is the first step to compute cube complexes partially. Here, we can at first compute complexes $\text{Kom}(F_0)$ and $\text{Kom}(F_1)$, postponing computations of $\text{Kom}(F)$ to the next step. Now we will develop this approach.

At the beginning let us extend $\text{Kom}$ over categories of cubes of complexes, such that we will use the inner structure of complexes.

**Definition 4.3.** Let $F: I^n \rightarrow \text{Kom}(C)$ be an anticommutative cube. The extended cube complex of $F$ is the complex $\text{Kom}(F) = (C_F, d_F)$ defined as follows:

\begin{equation}
C_F := \bigoplus_{\xi \in I^n} (F\xi)[-|\xi|]
\end{equation}

\begin{equation}
d_F|_{(F\xi)[-|\xi|]} := d_F\xi[|\xi|] + \sum_{\zeta: \xi \rightarrow \xi'} (F\zeta)[-|\xi|]
\end{equation}
The construction is visualised in the figure 2. Definitions 4.1 and 4.3 agree with respect to the canonical embedding $C \rightarrow Kom(C)$. Moreover, treating $\eta: F_0 \rightarrow F_1$ as a 1-cube, we have the equality

$$cone(\eta) = Kom(\eta)$$

hence $Kom$ generalizes the notion of a cone. Define now the family of functors

$$Kom^m: (m+n) Cub(Kom(C)) \rightarrow n Cub(Kom(C))$$

which computes partial cube complexes as follows:

$$Kom^m(F)(\xi) := Kom(F_\xi)$$

$$Kom^m(F)(\zeta) := Kom(F_\zeta)$$

The above means that a vertex $\xi$ of a cube $Kom^m(F)$ contains a complex computed from the restricted $m$-cube $F_\xi$, what explain why the functor is called ,,partial''. Obviously, for any $n$-cube $F$ we have $Kom^n(F) = Kom(F)$. Moreover, direct calculation gives:

**Theorem 4.4.** Let $F$ be an anticommutative $k$-cube and $m + n \leq k$. Then

$$Kom^n(Kom^m(F)) = Kom^{n+m}(F)$$

Comparing equations (4.7) and (4.11) one can easily see that the theorem 4.4 generalises the theorem 4.2.

Perhaps the main strength of partial computations is that all $Kom^n$ preserve chain homotopies.

**Proposition 4.5.** Let $F$ and $G$ be anticommutative $n$-cubes in a category of complexes. If $\eta, \nu, h: F \rightarrow G$ are cube morphisms such that for each edge $\xi$

$$\eta_\xi - \nu_\xi = h_\xi d_{F_\xi} + d_{G_\xi} h_\xi$$

then $Kom(h)$ is a chain homotopy of induced morphisms $Kom(\eta)$ and $Kom(\nu)$.

The mapping $h$ in the proposition above is called a cube homotopy, whereas cubes $F$ and $G$ are said to be homotopic.

**Corollary 4.6.** Let $F$ and $G$ be anticommutative cubes in a category $C$ of dimensions respectively $(m_1+n)$ and $(m_2+n)$. Then if $n$-cubes $Kom^{m_1}(F)$ and $Kom^{m_2}(G)$ are homotopic, so are $Kom(F)$ and $Kom(G)$.

**Proof.** Pick two commutative cube morphisms

$$\eta: Kom^{m_1}(F) \rightarrow Kom^{m_2}(G)$$

$$\nu: Kom^{m_2}(G) \rightarrow Kom^{m_1}(F)$$

together with cube homotopies $h_F: \nu \eta \simeq id$ and $h_G: \eta \nu \simeq id$. Due to the proposition 4.5 the morphisms $h_F$ and $h_G$ induce chain homotopies

$$Kom(h_F): Kom(\nu) Kom(\eta) \simeq id$$

$$Kom(h_G): Kom(\eta) Kom(\nu) \simeq id$$

and due to the equation (4.11) we have homotopies of complexes:

$$Kom(F) = Kom(Kom^{m_1}(F)) \simeq Kom(Kom^{m_2}(G)) = Kom(G).$$
KHovanov complex

In this chapter we will construct the generalized Khovanov complex in the spirit of Bar-Natan. At first, we will define a cube and a complex in the additive closure of ChCob$^3$, then we will prove an invariance of the latter up to chain homotopies and some relations. Finally we will give examples of functors into Abelian categories, such that homology groups can be computed. All of them will categorify the Jones polynomial.

1. The construction of the complex

One picture is worth of thousand words, therefore we will describe the generalized Khovanov complex explaining the figure which shows the complex $[3,1]$ for the trefoil.

A knot. In the left top corner we can see a diagram $D$ of the trefoil with enumerated crossings. Minus signs stand for negative crossings. The caption $(n_+,n_-) = (0,3)$ means the diagram possesses three negative crossings and no positive ones. Moreover, each crossing is equipped with an arrow oriented in a way such that it connects the two arcs in the type 0 resolution. Notice there are two choices of the arrow for each crossing.

Vertices. The main part of the picture consists of smooth diagrams placed in vertices of a three-dimensional cube $I^3$. The diagram $D_\xi$ in a vertex $\xi = (\xi_1, \xi_2, \xi_3)$ is obtained from $D$ by replacing $i$-th crossing with its resolution of type $\xi_i$.

Edges. Every edge is directed to the diagram with more type 1 resolutions. Globally, the arrows give all possible paths from the left-most diagram (all resolutions of type 0) to the right-most diagram (all resolutions of type 1) such that in each step one resolution is changed.

Pick any edge $\zeta: \xi \rightarrow \xi'$ and let $U$ be a small neighbourhood of the crossing, which resolution is changed by this edge. This edge is labeled with a cobordism in $R^2 \times I$, being a cylinder $(D \setminus U) \times I$ with a saddle inserted over $U$. Orientation of the critical point is given by the arrow in the knot diagram in the left-top corner. An example is given in the left-bottom corner for the edge $(0,*,0)$.

Anticommutativity. Fix a commutative ring $R$ and units $X,Y,Z \in U(R)$ such that $X^2 = Y^2 = 1$. Consider the change of chronology relations in the $R$-preadditive closure of ChCob$^3$. Above we have a description of a projective 3-cube $I_0(D)$ in ChCob$^3(\emptyset)/XYZ$, since each face corresponds to some change of chronology. Coefficients next to cobordisms form a negative edge assignment $\varphi \in C^1(I^n, U(R))$. Such a modified cube will be denoted by $I(D, \varphi)$ and called the Khovanov cube. Since the isomorphism class of the cube is independent of the edge assignment, we can write also $I(D)$.

The complex. Due to the previous chapter, we have a complex $\text{Kom}(I(D))$ in the category of matrices $\text{Mat}(\text{ChCob}^3(\emptyset))$ given by summing the complex over diagonals $|\xi| = r$. It is visualised by vertical dotted arrows.

Gradation. The differential has degree $-1$ with respect to the internal gradation of cobordisms. Therefore, the last step is to fix the grading of $\text{Kom}(I(D))$ by shifting the
4. Khovanov complex

Figure 1. The generalised Khovanov complex for the trefoil.

$r$-th term by $r$. It is shown on the picture by figures in brackets. The complex defined above will be called the formal Khovanov bracket\footnote{This definition differs from the one given by D. Bar-Natan in \cite{2}, since it includes partially gradation but lacks of the horizontal shift. This is motivated by the interplay between the Kauffman bracket and the Jones polynomial and is more similar to the construction described in the earlier paper \cite{3}.} and denoted by $[D]_{\varphi}$.

Let us make a remark before describing the general situation. A planar diagram $D$ with $n$ inputs together with cobordisms $S_i: \Sigma_i \Rightarrow \Sigma_i'$ forms a projective $n$-cube $D^S$ given as follows:

\begin{align*}
D^S(\xi) &:= D(S_1^{\xi_1}, \ldots, S_n^{\xi_n}), \quad \text{where } S_i^1 = \Sigma_i, \quad S_i^0 = \Sigma_i' \\
D^S(\zeta) &:= D(S_1^{(\zeta_1)}, \ldots, S_n^{(\zeta_n)}), \quad \text{where } S_i^{(1)} = \Sigma_i \times I, \quad S_i^{(0)} = \Sigma_i' \times I, \quad S_i^{(\ast)} = S_i
\end{align*}

Call it a $D$-cube. There is a canonical associated cochain given by chronology change relations. Theorem 2.6.13 asserts the cochain is a cocycle, what is proven below.

**Proposition 1.1.** Any $D$-cube satisfies the cocycle condition.

**Proof.** Let $F$ be a $D$-cube obtained from a planar diagram $D$. We will show that the canonical associated cochain $\psi$ given by chronology change relations is a cocycle. To do this pick any 3-cube in $F$: 
and denote by \(M_{abc}\) the cobordism given by the path consisting of edges parallel to \(a\), \(b\) and \(c\) (in this order). Then

\[
M_{xyz} = \psi(S_t)M_{yxz} = \psi(S_f)\psi(S_t)M_{yxx} = \cdots = d\psi(C)M_{xyz}
\]

where \(S_t\) and \(S_f\) stand respectively for the top and front faces of \(C\). Since a coefficient of a change of a chronology does not depend on a presentation of a permutation as a composition of transpositions, \(d\psi(C) = 1\), what ends the proof. \(\square\)

We can now go back to the construction of the Khovanov complex. Let \(T \in \mathcal{T}^0(B)\) be a diagram with \(n\) crossings of a tangle equipped with arrows over crossings. As before, we can form an \(n\)-cube \(\mathcal{I}_0(T)\) of resolutions of \(T\) in \(\text{ChCob}^3(B)\). Each face is a change of a chronology of type 1, so the cube is projective. Moreover, \(\mathcal{I}_0(T)\) is a \(D\)-cube and due to the proposition \(\square\) it has a negative edge assignment \(\varphi\). Indeed, remove from \(T\) small neighbourhoods of its crossings to get a planar diagram \(D\) (fig. 2). Each crossing of \(T\) describes a saddle with one critical point. This saddles together with \(D\) form a \(D\)-cube being equal to \(\mathcal{I}_0(T)\).

![Figure 2](image)

**Figure 2.** Having a tangle diagram one creates a planar diagram by removing small neighbourhoods of crossings.

**Definition 1.2.** Let \(T\) be a tangle diagram with \(n\) crossings equipped with arrows. The formal Khovanov bracket of \(T\) is the complex \([T]_\varphi\) given by

\[
[T]_\varphi := \text{Kom}(\mathcal{I}(T, \varphi))^r\{r\}
\]

where \(\mathcal{I}(T, \varphi)\) is the \(D\)-cube induced by \(T\) with an edge assignment \(\varphi\). If \(n_+\) and \(n_-\) are respectively the numbers of positive and negative crossings in \(T\), define the generalized Khovanov complex \(\text{K}h(T, \varphi)\) as a shift of the formal bracket

\[
\text{K}h(T, \varphi) = [T]_\varphi [-n_-]\{n_+ - 2n_-\}.
\]

Directly from the definition the formal bracket \([T]_\varphi\) is defined for unoriented tangles, whereas the complex \(\text{K}h(T, \varphi)\) for oriented.

To construct the Khovanov complex we have enumerated crossings in the diagram and assigned an arrow for each crossing. The order of crossings is obviously irrelevant (a permutation of crossings induces an isomorphism of complexes). The independence of the choice of arrows comes from existence of appropriate edge assignments as shown below.
Lemma 1.3. Let $D_1, D_2$ be diagrams of a tangle $T$ with $n$ crossings, which differ only in orientations of arrows over crossings. Then for any edge assignment $\varphi_1$ for $D_1$ there exists an edge assignment $\varphi_2$ of $D_2$ such that $I(D_1, \varphi_1) = I(D_2, \varphi_2)$.

Proof. Without loss of generality we may assume $D_1$ and $D_2$ differ in an orientation of exactly one arrow, say over the $i$-th crossing. Reversing the arrow reverses orientations of critical points of cobordisms assigned to edges $\zeta$ with $\zeta_i = \ast$. Let $\psi_i$ be the canonical cocycle of the cube $I_0(D_i)$. For a negative edge assignment $\varphi_1$ for $I_0(D_1)$ define
\begin{equation}
\varphi_2(\zeta) = \begin{cases} 
\varphi_1(\zeta), & \zeta_i \neq \ast \\
\lambda_\zeta \varphi_1(\zeta), & \zeta_i = \ast 
\end{cases}
\end{equation}
where $\lambda_\zeta$ is the coefficient of reversing the orientation of the critical point of the cobordism assigned to $\zeta$. For a face $S$ with the boundary $\zeta_1 \zeta_2 \zeta_3^{-1} \zeta_4^{-1}$ we have
\begin{equation}
\psi_2(S) = \lambda_{\zeta_1} \lambda_{\zeta_2} \lambda_{\zeta_3}^{-1} \lambda_{\zeta_4}^{-1} \psi_1(S).
\end{equation}
Therefore $d\varphi_2(S) = \lambda_{\zeta_1} \lambda_{\zeta_2} \lambda_{\zeta_3}^{-1} \lambda_{\zeta_4}^{-1} d\varphi_1(S) = -\psi_2(S)$, so $\varphi_2$ is the desired edge assignment. \hfill \Box

Remark 1.4. The formal Khovanov bracket $[D]$ and the generalized Khovanov complex $\text{Kh}(D)$ are well defined up to an isomorphism for any tangle diagram.

The formal Khovanov bracket has properties similar to the ones of the Kauffman bracket. Directly from the construction we have the following.

Proposition 1.5. The formal Khovanov bracket satisfies the following equations:

\begin{align*}
\text{(Kh1)} \quad & [U] = (0 \rightarrow U \rightarrow 0) \\
\text{(Kh2)} \quad & [D \sqcup U] = [[D] \sqcup U] \\
\text{(Kh3)} \quad & [\times] = \text{cone}([\times] \xrightarrow{d} [\times])
\end{align*}

where in \text{(Kh2)} $[D] \sqcup U$ stands for the complex $[D]$ with a trivial tangle $U$ added to each item and the identity $C_U$ added to the differential.

Remark 1.6. The morphism $d: [[\times]] \xrightarrow{d} [\times]$ in \text{(Kh3)} is induced from the cube morphism given by the decomposition of $I(\times)$. It is a bunch of cobordisms
\begin{equation}
M_\xi: D_\times(\xi) \rightarrow D_\times(\xi)
\end{equation}
with one critical point each, given by the change of a resolution of the distinguished crossing.

Similarly to the case of the Kauffman bracket, properties \text{(Kh1),(Kh3)} determines the formal Khovanov bracket uniquely. We will come back to the interplay between the two brackets in the section 5.

2. The proof of invariance

The Khovanov complex $\text{Kh}(T)$ is not a tangle invariant. For example it depends on the number of crossings in a chosen diagram. To have an invariant construction, we will introduce three relations $S, T$ and $4Tu$. The theorem 2.4 says the complex in the quotient category is a tangle invariant up to chain homotopies.
The $S$ stands for a sphere and means that a sphere with two critical points is the zero object. Using the monoidal structure of cobordisms we see that any cobordism $M$ with a component being a sphere with exactly two critical points must be zero (the two critical points must be consecutive in $M$).

The $T$ stands for a torus and means that a torus is equal to $Z(X + Y)$. Again, by the monoidal structure, having any cobordism $M$ with a component being a torus with exactly four critical points which are consecutive in $M$ and agreeing orientations of the split and the merge, we can remove this component and multiply the rest by $Z(X + Y)$. However, to be consistent with change of chronology relations, one has to use a different coefficient if the points are not consecutive or the toroidal component has more than four critical points (apply an appropriate change of a chronology first).

Finally, $4Tu$ stands for four tubes. It is best described locally. Let $M$ be an identity cylinder over $S^1 \sqcup S^1$ with two components $M'$ and $M''$. If we cut one of them and close the holes with a birth and a death, we get two cobordisms $M_1$ and $M_2$. Construct $M_3$ by cutting both components and connecting together the upper remaining parts. Append deaths to the lower parts so that the higher death lies below the negative part of the split. $M_4$ is constructed dually. Then $4Tu$ says that $Z \cdot M_1 + Z \cdot M_2 = X \cdot M_3 + Y \cdot M_4$.

Notice that both $T$ and $4Tu$ preserve the 2-index of a cobordism. Indeed, the 2-index of a torus is zero and of the 2-index of each cobordism in $4Tu$ it is $(-1, -1)$. Therefore, the coefficient of the change of a chronology is well defined in the quotient category as well as the degree of a cobordism.

Let $\text{ChCob}^3_{/l}$ be the category of embedded cobordisms modulo $S, T, 4Tu$ and chronology change relations. By the above it is a well-defined graded $R$-preadditive category and we can construct the category of double graded complexes in a standard way. Denote by $\text{Kob}, \text{Kob}(\emptyset)$ and $\text{Kob}(B)$ the subcategories of cube complexes in $\text{Kom}(\text{Mat}(\text{ChCob}^3_{/l}))$, $\text{Kom}(\text{Mat}(\text{ChCob}^3_{/l}(\emptyset)))$ and $\text{Kom}(\text{Mat}(\text{ChCob}^3_{/l}(B)))$ respectively. The corresponding homotopy categories will be denoted by $\text{Kob}_{/h}, \text{Kob}_{/h}(\emptyset)$ and $\text{Kob}_{/h}(B)$.

Now we are ready to construct homotopy equivalences between complexes of tangles appearing in the definitions of the Reidemeister moves.

**Lemma 2.1.** The complex $\text{Kh}(\otimes)$ is a strong deformation retract of $\text{Kh}(\otimes)$.

**Proof.** We want to show that the first of the following complexes is a retract of the other:

\[
\begin{align*}
\text{Kh}(\otimes) : & \quad 0 \longrightarrow \otimes \longrightarrow 0 \\
\text{Kh}(\otimes) : & \quad 0 \longrightarrow \otimes \{1\} \xrightarrow{d} \otimes \{2\} \longrightarrow 0
\end{align*}
\]

where $d = [\otimes \{1\}]$. In both complexes we underlined the item in degree zero. Construct maps $F : \text{Kh}(\otimes) \longrightarrow \text{Kh}(\otimes)$ and $G : \text{Kh}(\otimes) \longrightarrow \text{Kh}(\otimes)$ as in the figure. For simplicity the arrows describing orientations of critical points are omitted – assume all are directed to the right. The differential $d$ has to be a merge for any closing operator, whereas $F^0$ has to be a split. Moreover, due to grading shifts, all morphisms have degree 0. Directly from
4. Khovanov complex

Figure 3. Invariance under the $R_1$ move

chronology change relations we have

$$dF^0 = XY - YZ = YZ - YZ = 0$$

so $F$ is a chain map. Showing $G$ is a chain map is trivial. We will prove now, that $F$ and $G$ are mutually inverse homotopy equivalences. The $T$ relation implies $GF = I$:

$$G^0 F^0 = YZ - XY = Y(X + Y) - XY = I$$

Due to $4Tu$ we have $F^0 G^0 - I = hd$:

$$0 = Z + Z - X - Y = YZ + XZ - XZ - Y = -XZ(G^0 F^0 - I - hd)$$

what together with $dh = -I$ (remove the birth) gives $FG - I = hd + dh$. Obviously $hF = 0$, so $\text{Kh}(\otimes)$ is a strong deformation retract of $\text{Kh}(\otimes)$. \qed

Lemma 2.2. The complex $\text{Kh}(\otimes)$ is a strong deformation retract of $\text{Kh}(\otimes)$.

Proof. This lemma is proven in the same way as the previous one. Consider the diagram in the figure where, as before, the omitted arrows for critical points are directed to the right. Notice that $F^0 = h^1 d_{i\bullet}$ and $G^0 = d_{\bullet 0} h^0$. Moreover, due to grading shifts, all morphisms have degree 0.

Equalities $dF = 0$ and $Gd = 0$ are either trivial or can be derived from chronology change relations, so both $F$ and $G$ are chain maps. The $S$ relation implies $GF = I$ and $hF = 0$. To end the proof it remains to show that $h$ is a chain homotopy between $FG$ and an identity. It is trivial in gradings $-1$ and 1. In the zero grading we have to check.
Figure 4. Invariance under the $R_2$ move

the matrix condition:

$$\begin{pmatrix} G^0 & F^0 \\ G^0 & I \end{pmatrix} - \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} h^1d_{\bullet \cdot} + d_0h^0 & h^1d_{\cdot \cdot} \\ d_{\bullet 0}h^0 & 0 \end{pmatrix}$$

The only non-trivial equality $F^0G^0 - I = h^1d_{\cdot 1} + d_0h^0$ can be derived from 4Tu:

$$0 = Z + Z - X - Y = XZ + XYZ - XZ - XYZ = XZ(-F^0G^0 + I + h^1d_{\bullet \cdot} + d_{\cdot \bullet}h^0)$$

When modifying the second term, we first used chronology change relations, then anti-commutativity of the lower square in $\square$ and finally the expressions of $F$ and $G$ in terms of $h$ and $d$. □

The case of the third move is the simplest one, although it deals with the largest complex. This is because it can be derived from the invariance under the second move, as it was in the case of the Kauffman bracket.

**Lemma 2.3.** The complexes $\text{Kh}(\otimes)$ and $\text{Kh}(\otimes)$ are chain homotopic.

**Proof.** First notice that the lemma can be shown at the level of formal Khovanov brackets. This is because for both tangles $\otimes$ and $\otimes$ the Khovanov complexes are the formal brackets with the same shifts.
Due to (Kh3) the complex $\llbracket \bigcirc \rrbracket$ is a cone of the chain morphism $\Psi = \llbracket \bigcirc \rrbracket: \llbracket \bigcirc \rrbracket \longrightarrow \llbracket \bigcirc \rrbracket$ given by the following four morphisms:

Now we can use the homotopy equivalence $F$ from the proof of the previous lemma. Since it is an embedding into a strong deformation retract, the theorem 3.2.8 says $\llbracket \bigcirc \rrbracket$ is chain homotopic to the cone of $\Psi_L = \Psi F$, which is presented in the figure 5. For the same argument $\llbracket \bigcirc \rrbracket$ is chain homotopic to $\Psi_R$. Since $\llbracket \bigcirc \rrbracket$ and $\llbracket \bigcirc \rrbracket$ are isotopic, $\Psi_L$ and $\Psi_R$ are isomorphic, what gives the invariance of the Khovanov complex under the third Reidemeister move. □

**Figure 5.** Cones for tangles describing the $R_3$ move

The next step is to show that the homotopy equivalences built above extend for Reidemeister moves applied to any tangle diagram. Before, let us make some observations on planar algebras.

Planar operators in $\textbf{ChCob}^3$ are not functors in general. The exceptions are operators with exactly one input, so that they can be naturally extended over the categories of cubes or complexes.

Let $D: T^0(B_1) \times T^0(B_2) \longrightarrow T^0(B)$ be a planar diagram and $T \in T^0(B_1)$ be a tangle diagram. Denote by $D^T_\xi$ the planar diagram obtained from $D$ by inserting into the first input the diagram $T$ smoothed with respect to $\xi$. For any diagram $T' \in T^0(B_2)$ with $m$ crossings we can define a cube

$$I^{DT}(T') := \textbf{Kom}^m I(D(T, T'))$$

which has in a vertex $\xi$ a complex computed for the tangle $D^T_\xi(T')$. Moreover, a morphism $f: \text{Kh}(T_1) \longrightarrow \text{Kh}(T_2)$ lifts to a morphism of cubes $f^{DT}: I^{DT}(T_1) \longrightarrow I^{DT}(T_2)$ given by

$$f^{DT}(\xi) = D^T_\xi(f)$$
Since each $D^T_\xi$ is a functor, the operation $(\cdot)^{DT}$ is functorial:

\[(fg)^{DT} = f^{DT}g^{DT}\]

We will use this observation in the proof of the invariance theorem.

**Theorem 2.4.** The Khovanov complex $Kh(T)$ is an invariant of a tangle $T \in T(B)$ in the category $\text{Kob}_{\#}(B)$ up to an isomorphism. In other words, Khovanov complexes computed for two diagrams of a given tangle $T$ are homotopic modulo relations $S, T, 4Tu$.

**Proof.** Let $T$ be a tangle diagram and $T'$ be obtained from $T$ by applying a Reidemeister move $R_i$. Pick a planar diagram $D$ with two inputs and tangle diagrams $T_1, T_2$ and $T'_2$ such that $T = D(T_1, T_2)$ and $T' = D(T_1, T'_2)$, where $T_2$ and $T'_2$ describe $R_i$.

Theorem 3.4.4 together with existence of edge assignments (see the corollary 3.3.12) gives

$$Kh(T) = \text{Kom} T^{DT_1}(T_2)$$

$$Kh(T') = \text{Kom} T^{DT_1}(T'_2)$$

and it remains to show that the maps induced by homotopy equivalences from the lemmas above are also homotopy equivalences.

**The move $R_1$.** Both $(F^0)^{DT}$ and $(G^0)^{DT}$ are $D$-morphisms and due to the proposition [14] there exist edge assignments for them (notice that $F^0$ is homogeneous). Furthermore, the uniqueness of the edge assignment assures that the equality $G^0 F^0 = I$ is preserved. By the definition, $h$ has the opposite sign to $d$ ($\sigma(h) = -\sigma(d)$). Hence $h^{DT}$ is a morphism of anticommutative cubes so it is a cube homotopy between $FG$ and the identity.

**The move $R_2$.** Same as before, the homotopy $h$ induces a cube homotopy $h^{DT}$. Moreover, directly from the definitions, $(F^0)^{DT}$ and $(G^0)^{DT}$ are commutative, since each of their components is either an identity or a composition of anticommutative morphisms:

$$(F^0)^{DT} = h^{DT}d^{DT}, \quad (G^0)^{DT} = d^{DT}h^{DT}$$

**The move $R_3$.** Since $(\cdot)^{DT}$ is functorial, $F^{DT}$ from the previous paragraph is still an embedding into a strong deformation retract and we can repeat the proof of the lemma [23] for any tangle. \(\square\)

3. Properties of the complex

In this section we will show basic properties of the Khovanov complex. All proofs are taken with minor modifications from [14].

Let $T$ be an oriented tangle with a diagram $D$ and pick its component $T_0$ with linking number $l = \text{lk}(T_0, T - T_0)$. Reversing the orientation of $T$ forms a new tangle $T'$ with a diagram $D'$ and $\text{lk}(T_0, T' - T_0) = -l$. Since $[D] = [D']$ and

$$n_+(D') = n_+(D) - 2l, \quad n_-(D') = n_-(D) + 2l,$$

we have

**Proposition 3.1.** Let $D$ and $D'$ be the diagrams given above. Then

$$Kh'(D') = Kh^{r+2l}(D)\{2l\}$$

Reversing the global orientation (i.e. of all component) preserves signs of crossings. Hence the complex does not depend on the global orientation.
Proposition 3.2. Let $T$ be a tangle. Denote by $-T$ the tangle $T$ with reversed orientation of all components. Then
\[(3.3) \text{Kh}(T) = \text{Kh}(-T)\]
Due to (Kh3) the complex $[\times]$ is a cone of $d: [\otimes] \longrightarrow [\vee]$. Therefore the sequence below is exact
\[(3.4) 0 \longrightarrow [\chi] \longrightarrow [\times] \longrightarrow [\otimes] \longrightarrow 0\]
and similar to the Jones polynomial, we have the following

Proposition 3.3. There is an exact sequence of complexes:
\[(3.5) 0 \longrightarrow \text{Kh}(\otimes)[2] \longrightarrow \text{Kh}(\times)[1][1] \longrightarrow \text{Kh}(\otimes)[2] \longrightarrow \text{Kh}(\otimes)[2] \longrightarrow 0\]

Proof. First write down the exact sequences (3.4) for diagrams $\otimes$ and $\times$:
\[
0 \longrightarrow [\chi][{-}1] \longrightarrow [\times] \longrightarrow [\otimes] \longrightarrow 0
\]
\[
0 \longrightarrow [\otimes][{-}1] \longrightarrow [\times] \longrightarrow [\chi] \longrightarrow 0
\]
We can combine them together and get an exact sequence
\[(3.6) 0 \longrightarrow [\times] \longrightarrow [\times][1] \longrightarrow [\otimes][2] \longrightarrow [\otimes][2] \longrightarrow 0\]
If we orient the diagrams and make the necessary grading shifts, we will get the desired sequence. □

Let $T^*$ be a mirror tangle to $T$. Recall that there is a contravariant functor
\[(3.7) *: \text{ChCob}^3_{/XYZ} \longrightarrow \text{ChCob}^3_{/YXZ}\]
It preserves all relations $S$, $T$ and $4Tu$, so it induces a functor between the categories of complexes. Directly from the construction of the Khovanov complex

Proposition 3.4. If $T$ a mirror tangle to $T^*$ then
\[(3.8) \text{Kh}_{XYZ}(T^*) = \text{Kh}_{YXZ}(T)^*\]

Proof. First consider the category $\text{ChCob}^3$ with chronology change relations. Then
\[\mathcal{I}_0(T^*) = \mathcal{I}_0(T)^*\]
Moreover, any edge assignment $\varphi$ for $\mathcal{I}_0(T)$ with chronology change relations given by $X, Y, Z$ induces a dual edge assignment $\varphi'$ for $\mathcal{I}_0(T)^*$ with the dual chronology change relations given by coefficients $Y, X, Z$, satisfying
\[\varphi'((\zeta^*)) = \varphi(\zeta),\]
where $\zeta^*$ is the edge dual to $\zeta$ (exchange zeros with ones). □

4. Examples of homology groups

The complex $\text{Kh}(T)$ is an invariant of a tangle $T$, but it is hard (if even possible) to make computations in $\text{ChCob}^3_{/n}$. For example, how to check that two complexes are chain homotopic or not? Therefore, it is more convenient to use a functor $\mathcal{F}: \text{ChCob}^3_{/n} \longrightarrow A$ into an Abelian category such as modules or vector spaces, where we can compute homology groups. Such a functor can be additively extended over $\text{Mat}^*(\text{ChCob}^3_{/n})$ and then to a functor $\mathcal{F}: \text{Kob} \longrightarrow \text{Kom}(A)$. In this way we obtain a complex $\mathcal{F} \text{Kh}(T)$ being a tangle invariant up to chain homotopies. Clearly, the isomorphism classes of homology groups $H^*(\mathcal{F} \text{Kh}(T))$ are tangle invariants.
If $A$ is graded and $\mathcal{F}$ preserves degrees of morphisms, $H^\bullet(\mathcal{F} \text{Kh}(T))$ is double graded with homological gradation the the one induced by $\mathcal{F}$. If $\mathcal{F}$ is defined only on $\text{ChCob}^3(\emptyset)$ we obtain a priori only invariants of links. However, it can be extended for tangles by the construction described in the remark 2.6.6. Notice, that the result does not depend on whether we first extend $\mathcal{F}$ over the whole $\text{ChCob}^3$ and then to the category of complexes or in the different order: first to complexes $\text{Kob}(\emptyset)$ and then over $\text{Kob}$.

**Definition 4.1.** Pick $(A, \otimes, e, L, R, A, S)$ be a symmetric monoidal subcategory of $R$-modules. A Frobenius algebra in $A$ is an $R$-module $A \in A$ together with operations

\[
\begin{align*}
\mu: A \otimes A &\longrightarrow A \\
\eta: R &\longrightarrow A \\
\varepsilon: A &\longrightarrow R
\end{align*}
\]

called multiplication, comultiplication, unit and counit, equipping $A$ with the structure of (co)associative (co)commutative and (co)unital algebra and coalgebra

\[
\begin{align*}
\mu \circ (\mu \otimes \text{id}) &= \mu \circ (\text{id} \otimes \mu) & \mu \circ S &= \mu & \mu \circ (\eta \otimes \text{id}) &= \mu \\
(\Delta \otimes \text{id}) \circ \Delta &= (\text{id} \otimes \Delta) \circ \Delta & S \circ \Delta &= \Delta & (\varepsilon \otimes \text{id}) \circ \Delta &= \Delta
\end{align*}
\]

satisfying the Frobenius equation:

\[
(\Delta \otimes \text{id}) \circ (\text{id} \otimes \mu) = \mu \circ \Delta = (\text{id} \otimes \Delta) \circ (\mu \otimes \text{id})
\]

**Remark 4.2.** A Frobenius algebra $(A, \mu, \Delta, \eta, \varepsilon)$ gives a monoidal functor $\mathcal{F}_A: \text{2Cob} \longrightarrow A$ as follows:

\[
\begin{align*}
\mathcal{F}_A(n \mathbb{S}^1) &= A^\otimes n \\
\mathcal{F}_A(\bigcirc) &= \mu \\
\mathcal{F}_A(\bigotimes) &= \eta \\
\mathcal{F}_A(\bigotimes) &= \varepsilon \\
\mathcal{F}_A(\bigotimes) &= S
\end{align*}
\]

Also the opposite holds: any monoidal functor $F: \text{2Cob} \longrightarrow A$ comes from a Frobenius algebra $(A, \mu, \Delta, \eta, \varepsilon)$, where the module $A$ is given by the value of $F$ on a circle and the operations are the values of $F$ on appropriate generators of $\text{2Cob}$.

When $R = \mathbb{Z}$ and $X = Y = Z = 1$, the category $\text{ChCob}^3$ reduces to embedded cobordisms $\text{Cob}^3$. Therefore, a Frobenius algebra $A$ gives a functor $\mathcal{F}_A: \text{ChCob}^3(\emptyset) \longrightarrow A$, and if it preserves $S, T$ and $4T_u$ relations, then it may be used to compute homology groups of the Khovanov complex.

**Example 4.3** (M. Khovanov, 1999). Let $A = \mathbb{Z}v_+ \oplus \mathbb{Z}v_-$ be a free graded module with two generators $v_+$ and $v_-$ in degrees respectively $+1$ and $-1$. Define the structure of a Frobenius algebra as below:

- **multiplication $\mu: A \otimes A \longrightarrow A$:**

\[
\begin{align*}
\mu(v_+ \otimes v_+) &= v_+ \\
\mu(v_- \otimes v_+) &= v_- \\
\mu(v_+ \otimes v_-) &= v_-
\end{align*}
\]

- **unit $\eta: \mathbb{Z} \longrightarrow A$:**

\[
\eta(1) = v_+
\]

- **comultiplication $\Delta: A \longrightarrow A \otimes A$:**

\[
\begin{align*}
\Delta(v_+) &= v_- \otimes v_+ + v_+ \otimes v_- \\
\Delta(v_-) &= v_- \otimes v_-
\end{align*}
\]
Definition 4.4. Let \((A, \boxtimes, e, L, R, A, S)\) be a symmetric chronological monoidal subcategory of \(R\)-modules. A chronological Frobenius algebra is an \(R\)-module \(A\) together with operations

\[
\mu: A \boxtimes A \longrightarrow A \quad \Delta: A \longrightarrow A \boxtimes A
\]

\[
\eta: R \longrightarrow A \quad \varepsilon: A \longrightarrow R
\]

called multiplication, comultiplication, unit and counit, satisfying chronology change relations with respect to invertible elements \(X, Y, Z \in R\):

\[
\begin{align*}
(4.5) \quad (\text{id}_A \boxtimes \mu) \circ (\mu \boxtimes \text{id}_{A^2A}) &= X(\mu \boxtimes \text{id}_A) \circ (\text{id}_{A^2A} \boxtimes \mu) \\
(4.6) \quad (\text{id}_A \boxtimes \eta) \circ (\mu \boxtimes \text{id}_R) &= X(\mu \boxtimes \text{id}_A) \circ (\text{id}_{A^2A} \boxtimes \eta) \\
(4.7) \quad (\text{id}_A \boxtimes \eta) \circ (\eta \boxtimes \text{id}_R) &= X(\eta \boxtimes \text{id}_A) \circ (\text{id}_R \boxtimes \eta)
\end{align*}
\]

\[
\begin{align*}
(4.8) \quad (\text{id}_{A^2A} \boxtimes \Delta) \circ (\Delta \boxtimes \text{id}_A) &= Y(\Delta \boxtimes \text{id}_{A^2A}) \circ (\text{id}_A \boxtimes \Delta) \\
(4.9) \quad (\text{id}_R \boxtimes \Delta) \circ (\varepsilon \boxtimes \text{id}_A) &= Y(\varepsilon \boxtimes \text{id}_{A^2A}) \circ (\text{id}_A \boxtimes \Delta) \\
(4.10) \quad (\text{id}_R \boxtimes \varepsilon) \circ (\varepsilon \boxtimes \text{id}_A) &= Y(\varepsilon \boxtimes \text{id}_R) \circ (\text{id}_A \boxtimes \varepsilon)
\end{align*}
\]

\[
\begin{align*}
(4.11) \quad (\text{id}_{A^2A} \boxtimes \mu) \circ (\Delta \boxtimes \text{id}_{A^2A}) &= Z(\Delta \boxtimes \text{id}_A) \circ (\text{id}_A \boxtimes \mu) \\
(4.12) \quad (\text{id}_R \boxtimes \mu) \circ (\varepsilon \boxtimes \text{id}_{A^2A}) &= Z(\varepsilon \boxtimes \text{id}_{A^2A}) \circ (\text{id}_A \boxtimes \mu) \\
(4.13) \quad (\text{id}_{A^2A} \boxtimes \eta) \circ (\Delta \boxtimes \text{id}_R) &= Z(\Delta \boxtimes \text{id}_A) \circ (\text{id}_A \boxtimes \eta) \\
(4.14) \quad (\text{id}_R \boxtimes \eta) \circ (\varepsilon \boxtimes \text{id}_R) &= Z(\varepsilon \boxtimes \text{id}_A) \circ (\text{id}_A \boxtimes \eta)
\end{align*}
\]

The degree of \(v_1 \otimes \cdots \otimes v_n\) is defined as the sum of degrees: \(\deg(v_1) + \cdots + \deg(v_n)\).

The most general homology groups given in [14] are defined over the ring of polynomials \(\mathbb{Z}[c]\). However, such a functor does not preserve the relation \(S\) nor \(4Tu\). Our example is the specialization to \(c = 0\).
and equipping $A$ with the structure of (co)associative (co)commutative and (co)unital algebra and coalgebra in the chronological sense:

\begin{align}
4.15 \quad & \mu \circ (\mu \boxtimes \text{id}_A) = X \mu \circ (\text{id}_A \boxtimes \mu) \quad \mu \circ S = X \mu \quad \mu \circ (\eta \boxtimes \text{id}_A) = \mu \\
4.16 \quad & (\Delta \boxtimes \text{id}_A) \circ \Delta = Y (\text{id}_A \boxtimes \Delta) \circ \Delta \quad S \circ \Delta = Y \Delta \quad (\varepsilon \boxtimes \text{id}_A) \circ \Delta = \Delta
\end{align}

satisfying the chronological Frobenius equation:

\begin{align}
4.17 \quad & (\Delta \boxtimes \text{id}_A) \circ (\text{id}_A \boxtimes \mu) = Z \mu \boxtimes \Delta = (\text{id}_A \boxtimes \Delta) \circ (\mu \boxtimes \text{id}_A) \circ \Delta = \Delta
\end{align}

This choice of axioms gives an analogous correspondence between chronological Frobenius algebras and symmetric chronological monoidal functors $\mathcal{F} : \text{2ChCob} \rightarrow A$ to the one described in the remark $4.2$. Indeed any such an algebra $(A, \mu, \Delta, \eta, \varepsilon)$ gives a functor $\mathcal{F}_A : \text{2ChCob} \rightarrow A$ given below:

\begin{align}
\mathcal{F}_A(n \mathcal{S}^1) & = A^{\otimes n} \\
\mathcal{F}_A \left( \begin{array}{c} 0 \cr 0 \end{array} \right) & = \mu \\
\mathcal{F}_A \left( \begin{array}{c} 0 \cr 1 \end{array} \right) & = \eta \\
\mathcal{F}_A \left( \begin{array}{c} 1 \cr 0 \end{array} \right) & = S
\end{align}

In the other direction, the algebra $(A, \mu, \Delta, \eta, \varepsilon)$ is given by the values of $\mathcal{F}$ on generators of $\text{2ChCob}$.

**Example 4.5** (P. Ozsváth, J. Rasmussen, Z. Szabó, 2007). Consider the category of exterior algebras of free modules over $\mathbb{Z}$. Define the chronological product to be the exterior product:

\begin{align}
4.18 \quad & (\Lambda^* \mathbb{Z}\langle v_1, \ldots, v_n \rangle) \boxtimes (\Lambda^* \mathbb{Z}\langle w_1, \ldots, w_m \rangle) := (\Lambda^* \mathbb{Z}\langle v_1, \ldots, v_n, w_1, \ldots, w_m \rangle)
\end{align}

with a permutation $S : \Lambda^* \mathbb{Z}\langle v_1, v_2 \rangle \rightarrow \Lambda^* \mathbb{Z}\langle v_1, v_2 \rangle$ defined on generators as follows

\begin{align}
4.19 \quad & S(v_1) = v_2 \\
& S(v_2) = v_1
\end{align}

Let $A = \Lambda^* \mathbb{Z}a_1$ be the exterior algebra of a free module with one generator. Then its $n$-th power $A^{\otimes n}$ is the exterior algebra on a free module with $n$-generators $a_1, \ldots, a_n$. The chronological Frobenius algebra on $A$ is given by the following operations:

- **multiplication** $\mu : A \wedge A \rightarrow A$ is given by identifying the two generators and taking the wedge product:

\begin{align}
\mu(a_1) & = a_1 \\
\mu(a_2) & = a_1 \\
\mu(1) & = 1 \\
\mu(a_1 \wedge a_2) & = 0
\end{align}

- **unit** $\eta : \mathbb{Z} \ni \lambda \rightarrow \lambda 1 \in A$ is the standard embedding

- **comultiplication** $\Delta : A \rightarrow A \wedge A$ is given by the wedge product with the difference of generators:

\begin{align}
\Delta(1) & = a_1 - a_2 \\
\Delta(a_1) & = a_1 \wedge a_2
\end{align}

- **counit** $\varepsilon : A \rightarrow \mathbb{Z}$ is the dual to the generator:

\begin{align}
\varepsilon(1) & = 0 \\
\varepsilon(a_1) & = 1
\end{align}

One may check that all the axioms of a chronological Frobenius algebra are satisfied for $X = Z = 1$ and $Y = -1$. Define the degree in $\Lambda^* \mathbb{Z}(a_1, \ldots, a_n)$ by

\begin{align}
4.20 \quad & \deg(a_1 \wedge \cdots \wedge a_k) = n - 2k
\end{align}
Then multiplication and comultiplication have degree $-1$, whereas unit and counit have degree $1$. The functor $\mathcal{F}_{ORS} : \mathcal{ZChCob}^{3/1,-1,1}(\emptyset) \rightarrow \text{Mod}_R$ obtained in this way was described for the first time in [20]. It preserves both the grading and the relations $S,T,4Tu$, so we can use it to compute odd link homology groups.

Both constructions presented above are the specific cases of a more general one presented below.

**Example 4.6.** Let $V = Rv_+ \oplus Rv_-$ be a free $R$-module on two generators $v_+$ and $v_-$ in degrees $+1$ and $-1$. Pick invertible elements $X, Y, Z$ in $R$ such that $X^2 = Y^2 = 1$ and define $S : V \otimes V \rightarrow V \otimes V$ as follows:

$$S(v_+ \otimes v_+) = Xv_+ \otimes v_+ \quad S(v_- \otimes v_+) = Zv_+ \otimes v_-$$
$$S(v_+ \otimes v_-) = Z^{-1}v_- \otimes v_+ \quad S(v_- \otimes v_-) = Yv_- \otimes v_-$$

Since $S^n_k = \text{id} \otimes (n-1) \otimes S \otimes \text{id} \otimes (k-n-2)$, $k = 1, \ldots, n-1$, satisfies the relations of permutation groups, it gives us a symmetry $S$ in a monoidal subcategory generated by modules $V^{\otimes n}$. It defines a chronological product $\boxtimes$ as below:

$$X \boxtimes Y := X \otimes Y$$
$$f \boxtimes \text{id}_Z := f \otimes \text{id}_Z \quad \text{id}_Z \boxtimes f := S_{YZ} \circ (f \otimes \text{id}_Z) \circ S_{ZX}$$

where $f : X \rightarrow Y$. Equip $V$ with a structure of a chronological Frobenius algebra by the following operations:

- **multiplication** $\mu : V \boxtimes V \rightarrow V$: $\mu(v_+ \otimes v_+) = v_+$, $\mu(v_- \otimes v_+) = XZv_-$, $\mu(v_+ \otimes v_-) = v_-$, $\mu(v_- \otimes v_-) = 0$

- **unit** $\eta : R \rightarrow V$: $\eta(1) = v_+$

- **comultiplication** $\Delta : V \rightarrow V \boxtimes V$: $\Delta(v_+) = v_- \otimes v_+ + YZv_+ \otimes v_-$, $\Delta(v_-) = v_- \otimes v_-$

- **counit** $\varepsilon : V \rightarrow R$: $\varepsilon(v_+) = 0$, $\varepsilon(v_-) = 1$

As before one may check that all the axioms of a chronological Frobenius algebra are satisfied, both multiplication and comultiplication have degree $-1$ and both unit and counit have degree $1$. Hence we have a functor $\mathcal{F}_{XYZ} : \mathcal{RChCob}^3(\emptyset)/XYZ \rightarrow \text{Mod}_R$ preserving the grading and one can check that it preserves also the relations $S,T,4Tu$. It generalizes both functors described above. Indeed $\mathcal{F}_{Kh} = \mathcal{F}_{1,1,1}$ for $R = \mathbb{Z}$ and for $\mathcal{F}_{ORS}$ notice first that if $R = \mathbb{Z}$ then there is an isomorphism $A \cong V$ given by

$$(4.21) \quad 1 \leftrightarrow v_+ \quad a_1 \leftrightarrow v_-$$

Put $X = Z = 1$ and $Y = -1$. Then under this isomorphism the permutation $S^A$ in the algebra $A$ corresponds to the permutation $S^V$ in $V$. The same holds for other operations and we have an isomorphism of chronological Frobenius algebras

$$(4.22) \quad (A, \mu, \Delta, \eta, \varepsilon, S, \wedge) \cong (V, \mu, \Delta, \eta, \varepsilon, S, \boxtimes)$$

so $\mathcal{F}_{ORS}$ is equivalent to $\mathcal{F}_{1,-1,1}$. 

Let $R_0 < R$ be a subring of $R$ generated by the coefficients $X, Y, Z$. Directly from the construction one can see that the Khovanov complex is built in $R_0\text{ChCob}_h^3$. Moreover, the functor $F_h$ from the remark 2.5.14 induced by a ring homomorphism $h: R \rightarrow R'$ agrees with relations $S, T, 4Tu$, hence it extends to a functor between categories of complexes $F_h: \text{Kob}_{XYZ} \rightarrow \text{Kob}_{h(X)h(Y)h(Z)}$. It is easy to check that $F_h(X)h(Y)h(Z) \circ F_h = F_h \circ F_{XYZ}$.

**Proposition 4.7.** Let $h: R \rightarrow R'$ be a ring homomorphism and $L, L'$ be links. If homology groups of the links computed for $F_{XYZ}$ are isomorphic, so are the ones computed for $F_{h(X)h(Y)h(Z)}$. In particular, if $h$ is an isomorphism, $F_h$ is an isomorphism of categories and both $F_{XYZ}$ and $F_{h(X)h(Y)h(Z)}$ carry the same amount of information.

**Remark 4.8.** Let $R_U = \mathbb{Z}[x, y, z, z^{-1}]/(x^2 = y^2 = 1)$ be a reduced ring of polynomials, and put $F_U = F_{xyz}$. Then for any functor $F_{XYZ}$ we have

$$F_{XYZ} \circ F_h = F_h \circ F_U$$

where $F_h$ is a functor given by the ring epimorphism $h: R_U \rightarrow R$ sending $x, y, z \in R_U$ respectively to $X, Y, Z \in R$.

The functor $F_U$ given above is the generalization of both $F_{Kh}$ and $F_{ORS}$. In particular, if the homology groups of two links computed for this functor are isomorphic, neither the standard Khovanov homology nor the odd version can distinguish the links.

Notice also that an isomorphism $h: R \rightarrow R'$ induces an isomorphism of chronological Frobenius algebras $F_{XYZ}(S^1)$ and $F_{h(X)h(Y)h(Z)}(S^1)$. For instance, homology groups computed for the functors

$$F_{\pm x, \pm y, \pm z}, \quad F_{\pm y, \pm x, \pm z}, \quad F_{\pm x, \pm y, \pm z^{-1}}, \quad F_{\pm y, \pm x, \pm z^{-1}}$$

over $R_U$ are all isomorphic.

There is also a natural choice for the functor $F$: the module of morphisms from some fixed object.

**Example 4.9.** Define the tautological functor $F_X: \text{ChCob}_h^3 \rightarrow \text{Mod}_R$ for a given object $X \in \text{ChCob}_h^3$ as follows:

$$F_X(Y) := \text{Mor}(X, Y) \quad F_X(S) := S \circ (\cdot)$$

It gives a chronological Frobenius algebra $A_X$ in an obvious way.

In the case of chronological cobordisms without change of chronology relations, the functor $F_X: \text{ChCob}^3(\emptyset) \rightarrow \text{Mod}_R$ is faithful and any $F: \text{ChCob}^3(\emptyset) \rightarrow A$ factors by it.

**Question 4.10.** Is the tautological functor $F_X$ faithful for a given object $X \in \text{ChCob}_h^3(\emptyset)$?

A positive answer to the question above for some object $X$ will imply $F_X$ is universal and if for some links $L, L'$ the homology groups $F_X Kh(L), F_X Kh(L')$ are equal, none homology groups described in this paper can distinguish $L$ form $L'$.

**Question 4.11.** If $F_X$ is a faithful functor, are the complexes $F_X Kh(L)$ and $F_X Kh(L')$ homotopic if and only if the complexes $Kh(L)$ and $Kh(L')$ are homotopic? Furthermore, does an isomorphism of homology groups $H^* F_X Kh(L) \cong H^* F_X Kh(L')$ give a chain homotopy between $Kh(L)$ and $Kh(L')$?
5. A categorification of the Jones polynomial

We will now show the connection between the Jones polynomial and homology groups given by $\mathcal{F}_{XYZ}$. Let us first recall basic facts about the Euler characteristic of a chain complex.

**Definition 5.1.** A graded rank of a graded $R$-module $M = \bigoplus_{i \in \mathbb{Z}} M_i$ is the polynomial

$$\dim_q M := \sum_{i \in \mathbb{Z}} q^i \dim M_i$$

where $\dim M_i$ stands for the rank of $M_i$. The Euler characteristic of a complex of graded $R$-modules $(C, d)$ is the alternating sum of graded dimensions of terms of $C$:

$$\chi_q(C) = \sum_{r \in \mathbb{Z}} (-1)^r \dim_q C^r$$

From the basic linear algebra we know the rank of a quotient module is the difference of ranks of the divided module and the divisor. Therefore

$$\dim_q(M/N) = \dim_q(M) - \dim_q(N).$$

**Corollary 5.2.** Let $(C, d)$ be a complex of graded $R$-modules. Then the homology groups $H^*(C)$ are also graded and

$$\chi_q(C) = \chi_q(H^*(C))$$

In particular the Euler characteristic is preserved by chain homotopies.

**Corollary 5.3.** Pick a finite exact sequence of graded complexes

$$0 \to C_r \to C_{r+1} \to \ldots \to C_s \to 0$$

Then the alternating sum of their Euler characteristics vanishes:

$$\sum_{i=r}^{s} (-1)^i \chi_q(C_i) = 0$$

After these short remarks we are ready to show how to recover the Jones polynomial from the Khovanov complex. For this denote by $J_D(q)$ the Euler characteristic of the Khovanov complex of a link diagram $D$ given by the functor $\mathcal{F}_{XYZ}$:

$$J_D(q) := \chi_q(\mathcal{F}_{XYZ} \text{Kh} D)$$

We will show it is the Jones polynomial up to normalisation.

**Theorem 5.4.** The polynomial $J_D(q)$ has the following properties:

- $(qJ1)$ $J_U(q) = q + q^{-1}$,
- $(qJ2)$ $q^{-2} J_{\mathbb{Z}}(q) - q^2 J_{\mathbb{Z}}(q) = (q^{-1} - q) J_{\mathbb{Z}}(q)$

Therefore we have an equality

$$V_L(t) = \frac{J_L(-t^{1/2})}{(-t^{1/2} - t^{-1/2})}$$

**Proof.** The point $(qJ1)$ follows from the definition of the complex, whereas $(qJ2)$ is a consequence of existence of the short exact sequence (3.5). The last equality is due to the uniqueness theorem for the Jones polynomial. \qed
Remark 5.5. The connection between the Jones polynomial and \( J_D(q) \) can also be obtained more directly. Let \( \langle D \rangle_q \) be the Euler characteristic of the formal Khovanov bracket \( \mathcal{F}_{XYZ}[D] \). From the construction of the bracket we can see directly the following properties:

\((qK1)\) \( \langle \emptyset \rangle_q = 1 \)
\((qK2)\) \( \langle U \sqcup D \rangle_q = (q + q^{-1}) \langle D \rangle_q \)
\((qK3)\) \( \langle X \rangle_q = \langle \infty \rangle_q - q \langle \infty \rangle_q \)

and obviously \( J_D(q) = (-1)^{n(D)}q^{w_D(D) - 2n_D(D)} \langle D \rangle_q \). The bracket \( \langle D \rangle_q \) has the same meaning for \( J_D(q) \) as the Kauffman bracket for the Jones polynomial. In particular it can be expressed as a state sum over smoothed diagrams:

\[
\langle D \rangle_q = \sum_{s \in S(D)} (-q)^{n(s)}(q + q^{-1})^{\lvert s \rvert}
\]

It is clear now that if we substitute \( q = -A^{-2} \) and \( t = A^{-1/4} \) we get

\[
(q + q^{-1})^{-1} J_D(q) = (-A)^{-3w(D)} \langle A \rangle = V_D(t)
\]

As a corollary from the theorem 5.4 we have at hand several properties of the Jones polynomial.

Proposition 5.6. Let \( L \) be any link. Then

1. \( J_{L^*}(q) = J_L(q^{-1}) \)
2. \( J_{-L}(q) = J_L(q) \)
3. \( J_{L'}(q) = q^2 J_L(q) \), where \( L' \) is obtained from \( L \) by reversing the orientation of its component \( L_0 \) with the linking number \( \text{lk}(L_0, L \setminus L_0) = l \).
CHAPTER 5

Odds and ends

The main goal of this paper was to find a generalisation of both construction given by M. Khovanov and P. Osváth, J. Rasmussen and Z. Szabó. We enriched the category of oriented cobordisms so that the second got a functorial description. Then we constructed a complex in this category and proved it was a tangle invariant. Thanks to this we found a common description for both homology theories.

One strange step in [20], which does not appear in the Khovanov’s construction, is looking for an edge assignment for the cube of resolutions. Here we explained the existence of the assignment by the fact that a coefficient of change of a chronology is independence of a decomposition of the change as a permutation of neighbouring critical points. In this way the problem of existence of an edge assignment is reduced to the problem of uniqueness of a chronology change coefficient, which seems to be more natural. However, the prove given by us is still based on checking several cases. Moreover, it is only a minor modification of the one given in [20].

Problem 1.7. Why a coefficient of a chronology change adapted to some planar diagram is well-defined? Is that true for a larger class of changes of chronologies? Is there a simpler proof of the theorem 2.6.13 which is not based on checking different cases?

The next problem is the lack of functoriality of planar operators in ChCob. This is a reason why we was unable to naturally define a planar algebra of complexes in Kob. In the case of classical cobordisms such a structure gives automatically the invariance of the complex for any tangle, provided the invariance of elementary tangles in the definitions of the Reidemeister moves. We overcame the problem by computing complexes partially. In fact, this proof gives a clue, how we can restrict cube morphisms to have a planar algebra.

We can define embedded cobordisms not only between tangles in a disk, but also in any compact two-dimensional submanifold of a plain. In particular, we have cobordisms between planar diagrams $M : D_1 \Rightarrow D_2$ and cubes in the category of planar diagrams and cobordisms between them. Every such a cobordism induces a mapping

$$(1.11) \quad M : \text{ChCob}^3(B_1) \times \cdots \times \text{ChCob}^3(B_s) \longrightarrow \text{ChCob}^3(B)$$

which acts on objects $\Sigma_1, \ldots, \Sigma_s$ by filling holes with cylinders $C_{\Sigma_1}, \ldots, C_{\Sigma_s}$, whereas for cobordisms $S_i : \Sigma_i \Rightarrow \Sigma_i'$ we have a diagram

$$(1.12) \quad \begin{array}{ccc}
D_1(\Sigma_1, \ldots, \Sigma_s) & \xrightarrow{D_1(S_1, \ldots, S_s)} & D_1(\Sigma'_1, \ldots, \Sigma'_s) \\
\downarrow{M(\Sigma_1, \ldots, \Sigma_s)} & & \downarrow{M(\Sigma'_1, \ldots, \Sigma'_s)} \\
D_2(\Sigma_1, \ldots, \Sigma_s) & \xrightarrow{D_2(S_1, \ldots, S_s)} & D_2(\Sigma'_1, \ldots, \Sigma'_s)
\end{array}$$

which commutes up to invertible elements of $R$. In case of classical cobordisms, $M$ is a natural transformation of functors $D_1$ and $D_2$. This situation is similar, if we treat $D_1$
and $D_2$ as functors of one variable (half-functors). Therefore, we have an induced action on the category of cube complexes.

**Definition 1.8.** Say a morphism of cube complexes $f: C \to D$ is regular, if for any $CC$-cube $I$ in $CPO(ChCob^3)$ the induced cube morphism $I(f)$ is a $CC$-cube.

It turns out that the category of cube complexes with regular morphisms has a natural structure of a planar algebra and we in this framework the proof of the theorem 4.2.4 is a bit shorter. However, one may ask if the category is natural in some sense or whether there is its simpler definition.

**Problem 1.9.** Is there a natural category with a structure of a planar algebra, containing cube complexes, in which we can proof the invariance of the Khovanov complex?

The problems described so far are technical and do not bring much to the main goal of the paper. The following two deal with possible constructions directly connected to homology groups.

All homology groups defined by functors $F_{XYZ}$ categorify the Jones polynomial. In [2] D. Bar-Natan showed how to recover the polynomial directly from the complex $Kh(T)$. Unfortunately, it cannot be repeat in the same way for chronological cobordisms, because we do not have the neck-cutting relation. In our case it has the form

\[(1.13) \quad Z(X + Y) = \text{\begin{comment}} \quad + \end{comment}\]

For usual cobordisms, the coefficient at the left-hand side is equal 2 and is invertible when we extend $Z$ by a fraction $\frac{1}{2}$. In our case we can repeat it only if $X \neq Y$. In particular, we cannot do this for odd theory. Moreover, the existence of $(X + Y)^{-1}$ implies $X = Y$. On the other hand, each functor $F_{XYZ}$ categorifies the Jones polynomial. This suggests we can obtain the polynomial directly from the complex $Kh(T)$.

Recall a trace in an $R$-additive category $C$ is an $R$-linear mapping $\text{Tr}: \text{End}(C) \to G$, where $\text{End}(C)$ is the class of endomorphisms of the category $C$ and $G$ is an Abelian group, satisfying the following condition:

\[(1.14) \quad \text{Tr}(FG) = \text{Tr}(GF) \]

for any two morphisms $F: X \to Y$ and $G: Y \to X$. Then we can define the dimension of an object $X$ as a trace of the identity $\dim(X) = \text{Tr}(\text{id}_X)$ and we have an Euler characteristic of a complex given in a usual way. In particular, we can take for $G$ the trace group

\[(1.15) \quad \Xi(C) = \text{End}(C)/\langle FG - GF \mid F: X \to Y, G: Y \to X \rangle \]

and the universal trace $\text{Tr}_*: \text{End}(C) \to \Xi$. It can be shown that any trace factorise by the universal one.

**Problem 1.10.** Show the connection between the universal trace $\text{Tr}_*$ in $ChCob^3$ and the Jones polynomial.

The operation $Kh$ which associates a complex in $Cob^3$ to a tangle induces chain maps between complexes for cobordisms between tangles. In particular, a cobordism $M$ between empty tangles (i.e. when $T_1 = T_2 = \emptyset$) is a knotted surface and $Kh(M)$ is a multiplication by a number. This gives invariants of surfaces.
In case of chronological cobordisms we can repeat the proof from [2] with minor modifications to show that the naive definition gives a chain map well-defined up to a global invertible element. We strongly believe that this the whole construction can be fixed to produce well-defined chain maps.

**Conjecture 1.11.** The map \( \text{Kh} \) extends functorially over cobordisms between tangles.
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