On perfect $k$-matchings

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Abstract

In this paper, we generalize the notions of perfect matchings, perfect 2-matchings to perfect $k$-matchings and give a necessary and sufficient condition for existence of perfect $k$-matchings. For bipartite graphs, we show that this $k$-matching problem is equivalent to that matching question. Moreover, for regular graphs, we provide a sufficient condition of perfect $k$-matching in terms of edge connectivity.

Keywords: matching; 2-matching; k-matching.

1 Introduction

All graphs considered are multigraphs (with loops) and finite. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The number of vertices of a graph $G$ is called the order of $G$. Unless otherwise defined, we follow [3] for terminologies and definitions.

We denote the degree of vertex $v$ in $G$ by $d_G(v)$. For two subsets $S, T \subseteq V(G)$, let $e_G(S, T)$ denote the number of edges of $G$ joining $S$ to $T$. For a set $X$, we denote the cardinality of $X$ by $|X|$. A vertex of degree zero is called an isolated vertex. Let $Iso(G)$ denote the set of isolated vertices of $G$ and let $i(G) = Iso(G)$. Let $c_o(G)$ denote the number of odd components of $G$. Let $odd(G)$ denote the number of odd components with order at least three of $G$. For any subset $X$ of vertices of $G$, we define the neighbourhood of $X$ in $G$ to be the set of all vertices adjacent to vertices in $X$; this set is denoted by $N_G(X)$.

A matching $M$ of a graph $G$ is a subset of $E(G)$ such that any two edges of $M$ have no end-vertices in common. Let $k$ be a positive. A $k$-factor of a graph $G$ is a spanning subgraph $H$ of $G$ such that $d_H(x) = k$ for every $x \in V(G)$. A $\{K_2, C_{2t+1} \mid t \geq 1\}$-factor of

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a graph $G$ is a spanning subgraph of $G$ such that each of its components is isomorphic to one of $\{K_2, C_{2t+1} \mid t \geq 1\}$.

Let $f : \{0, 1, \ldots, k\} \to E(G)$ be an assignment such that the sum of weights of edges incident with any vertex is at most $k$, i.e., $\sum_{e \sim v} f(e) \leq k$ for any vertex $v \in V(G)$. A $k$-matching is a subgraph induced by the edges with weight among $1, \ldots, k$ such that $\sum_{e \sim v} f(e) \leq k$. The sum of all weights, i.e., $\sum_{e \in E(G)} f(e)$, is called size of a $k$-matching $f$. A $k$-matching is perfect if $\sum_{e \sim v} f(e) = k$ for every vertex $v \in V(G)$. Clearly, a $k$-matching is perfect if and only if its size is $k|V(G)|/2$. If $k = 1$, then a perfect $k$-matching is called a perfect matching. If $k = 2$, then a perfect $k$-matching is called a perfect 2-matching.

For perfect matching of bipartite graphs, Hall obtained the next result in terms of isolated vertices.

**Theorem 1.1 (Hall, [2])** Let $G = (X, Y)$ be a bipartite graph. Then $G$ has a perfect matching if and only if $|X| = |Y|$ and for any $S \subseteq X$, 

$$i(G - S) \leq |S|.$$ 

Tutte (1947) studied the perfect matching of general graphs and gave the sufficient and necessary condition.

**Theorem 1.2 (Tutte, [4])** A graph $G$ has a perfect matching if and only if for any $S \subseteq V(G)$, 

$$c_o(G - S) \leq |S|.$$ 

For perfect 2-matching, Tutte (1953) gave the following result.

**Theorem 1.3 (Tutte, [6])** Let $G$ be a connected graph. Then the following statements are equivalent:

1. $G$ has a perfect 2-matching;
2. $i(G - S) \leq |S|$ for all subsets $S \subseteq V(G)$;
3. $G$ has a $\{K_2, C_{2t+1} \mid t \geq 1\}$-factor.

In the proof, we need the following technical theorems.

**Theorem 1.4 (Tutte, [5])** Let $G$ be a graph and $k$ a positive integer. Then $G$ has a $k$-factor if and only if, for all $D, S \subseteq V(G)$ with $D \cap S = \emptyset$,

$$\delta_G(D, S) = k|D| - k|S| + \sum_{v \in S} d_{G-D}(v) - \tau_G(S, T) \geq 0,$$

where $\tau_G(S, T)$ is the weight of a minimum weight $k$-factor that covers $T$ and includes $S$ as a subset. Theorem 1.4 provides the necessary condition for the existence of a $k$-factor, which is equivalent to the Tutte’s condition for perfect 2-matching.

In the proof, we need the following technical theorems.
where \( \tau_G(D, S) \) is the number of components \( C \) of \( G-(D\cup S) \) such that \( e_G(V(C), S)+k|C| \equiv 1 \pmod{2} \). Moreover, \( \delta_G(D, S) \equiv k|V(G)| \pmod{2} \).

## 2 Main Results

In this section, we gave a good characterization for perfect \( k \)-matchings.

**Theorem 2.1** Let \( k \geq 4 \) be even. Then \( G \) contains a perfect \( k \)-matching if and only if \( G \) contains a perfect 2-matching.

**Proof.** Suppose that \( G \) contains a perfect 2-matching. By Theorem 1.3, \( G \) contains a \( \{K_2, C_{2t+1}\} \)-factor \( H \). We assign every isolated edge of \( H \) with weight \( k \) and the rest edge with weight \( k/2 \). Then we obtain a perfect \( k \)-matching of \( G \).

Conversely, suppose \( G \) that contains a perfect \( k \)-matching \( H \). Then there exists a function \( f : V(G) \to \{0, 1, \ldots, k\} \) such that \( \sum_{v\sim e} f(e) = k \) for all \( v \in V(G) \). We claim \( i(G - S) \leq |S| \) for all \( S \subseteq V(G) \). Otherwise, assume that there exists \( S \subseteq V(G) \) such that \( i(G - S) > |S| \). Then we have

\[
ki(G - S) = \sum_{e \in E_G(Iso(G - S), S)} f(e) > k|S|,
\]

a contradiction. So by Theorem 1.3, \( G \) contains a perfect 2-matching. \( \square \)

**Corollary 2.2** Let \( k \geq 2 \) be even. Then a graph \( G \) contains a perfect \( k \)-matching if and only if \( i(G - S) \leq |S| \) for all \( S \subseteq V(G) \).

**Theorem 2.3** Let \( k \geq 1 \) be odd. Then \( G \) contains a perfect \( k \)-matching if and only if

\[
odd(G - S) + ki(G - S) \leq k|S| \quad \text{for all subsets } S \subseteq V(G).
\]

**Proof.** We first prove the necessity. Suppose that \( G \) has a perfect \( k \)-matching and there exists \( S \subseteq V(G) \) such that

\[
odd(G - S) + ki(G - S) > k|S|.
\]

Let \( f : E(G) \to \{0, 1, \ldots, k\} \) such that \( \sum_{e\sim v} f(e) = k \) for all \( v \in V(G) \). Let \( m = odd(G - S) \) and let \( C_1, \ldots, C_m \) denote the odd components of \( G - S \) with order at least three. Let \( W = C_1 \cup \cdots \cup C_m \). Since \( k \) is odd, by parity, every odd component with order at least
three can't contain a perfect \( k \)-matching. So \( \sum_{e \in E_G(V(C_i), S)} f(e) \geq 1 \) for \( i = 1, \ldots, m \). Then we have

\[
k|S| = \sum_{e \in S} \sum_{e \sim v} f(e) \geq \sum_{e \in E_G(V(W), S)} f(e) + \sum_{e \in E_G(Iso(G-S), S)} f(e)
\]

\[
\geq odd(G - S) + ki(G - S) > k|S|,
\]
a contradiction. So the result is followed.

We next prove the sufficiency. Let \( G^* \) be obtained from \( G \) by changing every edge of \( G \) into \( k \) parallel edges. Then \( G \) contains a perfect \( k \)-matching if and only if \( G^* \) contains a \( k \)-factor. Conversely, suppose that \( G \) contains no perfect \( k \)-matchings. Then \( G^* \) contains no \( k \)-factors. By Theorem 1.4 there exist two disjoint subset \( D, S \subseteq V(G^*) \) such that

\[
k|D| - k|S| + \sum_{x \in S} d_{G^* - D}(x) - \tau < 0,
\]

where \( \tau \) denote the number of components \( C \) of \( G^* - D - S \) such that \( k|V(C)| + e_{G^*}(V(C), S) \equiv 1 \) (mod 2). Let \( C_1, \ldots, C_\tau \) denote those components and \( W = \bigcup_{i=1}^\tau C_i \). By Theorem 1.2 we can suppose that \( k \geq 3 \).

Without loss of generality, among all such subsets, we choose subsets \( D \) and \( S \) such that \( S \) is minimal. We have \( S \neq \emptyset \), otherwise, \( k|D| < \tau \) and \( |V(C_i)| \) is odd for \( i = 1, \ldots, \tau \). So we have

\[
k|D| - ki(G - D) - odd(G - D) \leq k|D| - \tau < 0,
\]
a contradiction. Let \( M = G^* - D - S - V(W) \).

**Claim 1.** \( G[S] \) consists of isolated vertices.

Otherwise, let \( e = uv \in G[S] \). Let \( |N_G(v) \cap V(W)| = m \). Let \( D' = D \) and \( S' = S - v \). Let \( \tau' \) denote the number of components \( C \) of \( G^* - D' - S' \) such that \( k|V(C)| + e_{G^*}(V(C), S') \equiv 1 \) (mod 2). Then we have

\[
k|D'| - k|S'| + \sum_{x \in S'} d_{G^* - D'}(x) - \tau' \leq k|D| - k(|S| - 1) + \sum_{x \in S - v} d_{G^* - D}(x) - (\tau - m)
\]

\[
\leq k|D| - k|S| + k + \sum_{x \in S} d_{G^* - D}(x) - d_{G^* - D}(v) - (\tau - m)
\]

\[
\leq k|D| - k|S| + \sum_{x \in S} d_{G^* - D}(x) - k(m + 1) - \tau + m
\]

\[
\leq k|D| - k|S| + \sum_{x \in S} d_{G^* - D}(x) - \tau < 0,
\]

contradicting to the minimality of \( S \). This completes the claim.
With the similar proof of Claim 1, we obtain the following claim.

**Claim 2.** \( e_G(S, V(M)) = \emptyset. \)

**Claim 3.** \(|N_G(x) \cap V(W)| \leq 1\) for all \( x \in S. \)

Otherwise, suppose that there exists \( v \in S \) such that \( m = |N_G(v) \cap V(W)| \geq 2. \) Let \( D'' = D \) and \( S'' = S - v. \) Let \( \tau'' \) denote the number of components \( C \) of \( G^* - D'' - S'' \) such that \( k|V(C)| + e_{G^*}(V(C), S'') \equiv 1 \) (mod \( 2. \)) Then we have

\[
|D''| - |S''| + \sum_{x \in S''} d_{G^* - D''}(x) - \tau'' \leq |D| - |S| - 1 + \sum_{x \in S - v} d_{G^* - D}(x) - (\tau - m)
\]

\[
\leq |D| - |S| + k + \sum_{x \in S} d_{G^* - D}(x) - d_{G^* - D}(v) - (\tau - m)
\]

\[
\leq |D| - |S| + k + \sum_{x \in S} d_{G^* - D}(x) - km - \tau + m
\]

\[
= |D| - |S| + \sum_{x \in S} d_{G^* - D}(x) - \tau - (k - 1)(m - 1) + 1
\]

\[
\leq |D| - |S| + \sum_{x \in S} d_{G^* - D}(x) - \tau < 0,
\]

contradicting to the minimality of \( S. \) This completes the claim.

**Claim 4.** \( E_G(S, V(W)) = \emptyset. \)

Otherwise, by Claim 2, suppose that there exists an edge \( uv \in E_G(S, V(W)), \) where \( v \in S \) and \( u \in V(W). \) Let \( D''' = D \) and \( S''' = S - v. \) Let \( \tau''' \) denote the number of components \( C \) of \( G^* - D''' - S''' \) such that \( k|V(C)| + e_{G^*}(S''', V(C)) \equiv 1 \) (mod \( 2. \)). Without loss of generality, suppose that \( u \in C_1. \) By Claims 1, 2 and 3, then \( G^*[V(C_1) \cup \{v\}] \) is a component of \( G^* - D''' - S'''. \) Note that

\[
k|V(C_1) \cup \{v\}| + e_{G^*}(S''', V(C_1) \cup \{v\}) \equiv k|V(C_1)| + e_{G^*}(S, V(C_1)) \equiv 1 \text{ (mod } 2). \]

So \( \tau = \tau'''. \) Hence

\[
k|D'''| - |S'''| + \sum_{x \in S'''} d_{G^* - D'''}(x) - \tau''' = |D| - |S| + \sum_{x \in S - v} d_{G^* - D}(x) - \tau
\]

\[
\leq |D| - |S| + k + \sum_{x \in S} d_{G^* - D}(x) - d_{G^* - D}(v) - \tau
\]

\[
\leq |D| - |S| + \sum_{x \in S} d_{G^* - D}(x) - \tau < 0,
\]

contradicting to the minimality of \( S. \) This completes the claim.
Since \(e_{G^*}(V(C_i), S) + k|V(C_i)| \equiv 1 \pmod{2}\) and \(k\) is odd, by Claim 4, we have \(|V(C_i)| \equiv 1 \pmod{2}\) for \(i = 1, \ldots, \tau\). By Claims 1, 2, and 4, we have

\[
0 > k|D| - k|S| + \sum_{x \in S} d_{G^* - D}(x) - \tau \\
= k|D| - k|S| - \tau \\
\geq k|D| - ki(G - D) - \text{odd}(G - D).
\]

Hence we have \(k|D| < ki(G - D) + \text{odd}(G - D)\), a contradiction. We complete the proof. \(\square\)

**Theorem 2.4** Let \(G = (U, W)\) be a bipartite graph, where \(|U| = |W|\). Then \(G\) contains a perfect matching if and only if \(G\) contains a perfect \(k\)-matching.

**Proof.** Necessity is obvious. Now we prove the sufficiency. Suppose that \(G\) contains a perfect \(k\)-matching. Let \(f : E(G) \to \{0, 1, \ldots, k\}\) such that \(\sum_{v \sim e} f(e) = k\) for all \(v \in V(G)\).

Then for all independent set \(S\), we have

\[
k|S| = \sum_{v \in S} \sum_{v \sim e} f(e) \leq \sum_{e \in E_G(S, N(S))} f(e) \leq k|N(S)|.
\]

So we have \(i(G - S) \leq |S|\) for all \(S \subseteq U\). By Theorem 1.1, \(G\) contains a perfect matching. This completes the proof. \(\square\)

**Corollary 2.5** Let \(k \geq 1\) be an odd integer and \(G\) be an \(r\)-regular, \(\lambda\)-edge-connected graph. Suppose that

\[
\lambda = \begin{cases} 
\left\lceil \frac{r}{k} \right\rceil - 1 & \text{if } \left\lceil \frac{r}{k} \right\rceil \equiv r \pmod{2} \\
\left\lceil \frac{r}{k} \right\rceil & \text{if } \left\lceil \frac{r}{k} \right\rceil \not\equiv r \pmod{2}.
\end{cases}
\]

Then \(G\) contains a perfect \(k\)-matching.

**Proof.** Suppose that the result doesn’t hold. By Theorem 2.3, there exists a subset \(S \subseteq V(G)\) such that

\[
\text{odd}(G - S) + ki(G - S) > k|S|.
\]

Let \(m = \text{odd}(G - S)\). Let \(C_1, \ldots, C_m\) denote these odd components with order at least three of \(G - S\). Since \(r|C_i| - e_G(V(C_i), S) = \sum_{x \in V(C_i)} d_{G^*}(x)\) is even, so \(e_G(V(C_i), S) \equiv r\). Since \(G\) is an \(r\)-regular, \(\lambda\)-edge-connected graph, so if \(\left\lceil \frac{r}{k} \right\rceil \equiv r \pmod{2}\), then \(e_G(V(C_i), S) \geq \lambda + 1\) for \(i = 1, \ldots, m\). So we have

\[
r|S| \geq \left\lceil \frac{r}{k} \right\rceil \text{odd}(G - S) + ri(G - S).
\]
Hence,

\[ kr|S| \geq k\left\lceil \frac{r}{k} \right\rceil odd(G - S) + kri(G - S) \]
\[ \geq rodd(G - S) + kri(G - S) > kr|S|, \]

a contradiction. This completes the proof. \( \Box \)

**Corollary 2.6 (Bäbler, [1])** Let \( G \) be an \( r \)-regular, \((r-1)\)-edge-connected graph with even order. Then \( G \) contains a perfect matching.

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