A PERTURBATION THEOREM FOR LINEAR HAMILTONIAN SYSTEMS WITH BOUNDED ORBITS

ROBERTA FABBRI
Università di Firenze
Dipartimento di Sistemi e Informatica
Via Santa Marta 3, 50139 Firenze, Italy

CARMEN NÚÑEZ
Universidad de Valladolid
Departamento de Matemática Aplicada, ETSII
Paseo del Cauce s/n, 47011 Valladolid, Spain

ANA M. SANZ
Universidad de Valladolid
Departamento de Análisis Matemático y Didáctica de la Matemática
Prado de la Magdalena s/n, 47005 Valladolid, Spain

(Communicated by Amadeu Delshams)

Abstract. This paper concerns the Sacker-Sell spectral decomposition of a one-parametric perturbation of a non-autonomous linear Hamiltonian system with bounded solutions. Conditions ensuring the continuous variation with respect to the parameter of the spectral intervals and subbundles are established. These conditions depend on the perturbation direction and are closely related to the topological structure of the flows induced by the initial system on the real and complex Lagrange bundles.

1. Introduction and preliminaries. Let us consider a recurrent linear Hamiltonian system, as well as the skew-product flow that it induces on the trivial bundle over the hull of the initial matrix. Assume that all the orbits of this flow are bounded. Results by Cameron [2] and Ellis and Johnson [4] ensure the existence of a change of variables which is continuous on the hull and takes such a system to a skew-symmetric form. Nevertheless, in spite of this strong simplification, the ergodic and topological structures of the corresponding flow over the Lagrange bundle are far away from being completely classified in the general dimension case, and the aspects of the dynamics which remain dark are rather numerous.

On the contrary, in the two-dimensional case, a complete classification of recurrent linear systems is known and, taking it as a starting point, different properties of the solutions have been described (see Novo and Obaya [12, 13] and references

2000 Mathematics Subject Classification. Primary 37B55, 34D08; Secondary 34C11, 37D20.
Key words and phrases. Linear Hamiltonian systems, Sacker-Sell spectral decomposition, perturbation theorems.
Partially supported by Junta de Castilla y León under project VA024/03B and by Ministerio de Ciencia y Tecnología under project BFM2002-03815.
therein). In particular, Núñez and Obaya [13] develop a one-parametric perturbation theory of these two-dimensional Hamiltonian systems, which is based on the connection between the ergodic and topological structures of the Lagrange (projective) bundle. This theory extends part of the classical perturbative results of Moser and Pöschel [10] (see also Eliasson [3]) for the quasi-periodic Schrödinger equation to a more general setting.

The main objective of this paper is to extend these two-dimensional properties to the general dimension case: we establish conditions on the perturbation directions of the initial (elliptic) family ensuring a) the exponential dichotomy (hyperbolicity) of the perturbed systems and b) the continuous variation of the Sacker and Sell spectral decomposition [16]. In addition, we analyze the limiting behavior of the corresponding subbundles as the perturbation parameter goes to zero, showing that the limits are uniform on the base and can be previously determined from the initial non-perturbed system. These results have a direct application to the study of the measurable and topological structures of the phase space of the systems considered: each perturbation direction determines a pair of Lagrange planes which are measurable (topological, if they are closed) components of the (unperturbed) limit flow. In this way, to understand their behavior contributes to understand the global ergodic and topological structures, in the line of the results of Novo and Obaya [12] and Arnold, Cong and Oseledets [1].

The results contained here are strongly based on those of Novo et al. [11], where one can find an ergodic and topological analysis of the qualitative behavior of the solutions of those Hamiltonian systems admitting a square-integrable change of variables taking them to a skew-symmetric form (a condition satisfied in the case considered here).

We conclude this section by explaining with some detail the setting of the problem and recalling some basic definitions and properties required in Section 2.

Let Ω be a compact metric space and \( \sigma : \mathbb{R} \times \Omega \to \Omega \), \( (t, \omega) \mapsto \omega \cdot t \) a minimal continuous flow. We consider the family of linear Hamiltonian systems

\[
\dot{z} = H(\omega \cdot t) z, \quad \omega \in \Omega, \quad (1.1)
\]

where \( H \) is a continuous real \( 2n \times 2n \) matrix-valued function belonging to the Lie-algebra of the infinitesimally symplectic \( 2n \times 2n \) matrices \( \text{sp}(n, \mathbb{R}) \); that is, \( JH + H^T J = 0 \), where \( J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix} \). Equivalently, \( H = \begin{bmatrix} H_1 & H_2 \\ H_3 & -H_1^T \end{bmatrix} \), where \( H_2 \) and \( H_3 \) are continuous symmetric \( n \times n \) matrix-valued functions.

This setting includes non-autonomous linear Hamiltonian systems with a wide class of coefficient functions: \( \Omega \) can be taken as the hull of a bounded and uniformly continuous matrix-valued function \( \hat{H}(t) \); i.e. as the closure on the compact-open topology of the set of time-translated functions \( \hat{H}_s(t) := \hat{H}(t + s) \). The flow \( \sigma \) is then given by translation, and is minimal for a large set of initial systems (commonly called recurrent), including quasi-periodic, almost-periodic (for which the base flow is, in addition, uniquely ergodic) and almost-automorphic potentials (see Shen and Yi [18]). It is well-known that this collective formulation has the important advantage of allowing us to identify the solutions with the orbits of a linear, skew-product continuous flow also in the non-autonomous case, as we explain in next paragraph. In addition, the connection between the systems of the family often causes the properties of the solutions of one of them to be transmitted (along
the trajectories of the base flow) to all the remaining ones; or, at least, to almost all, once an \((\Omega, \sigma)-\text{ergodic measure has been fixed.}\)

The mentioned skew-product flow is induced in a natural way by the family of systems \((1.1)\) on \(\Omega \times \mathbb{C}^{2n}\) and \(\Omega \times \mathbb{R}^{2n}\): if \(U(t, \omega)\) represents the fundamental matrix solution of equation \((1.1)\) for \(\omega \in \Omega\) with \(U(0, \omega) = I_{2n}\), the orbit or trajectory of \((\omega, z_0)\) is given by \(\{(\omega, t, U(t, \omega)z_0) | t \in \mathbb{R}\}\). In addition, the Hamiltonian character of the systems \((1.1)\) ensures that \(U(t, \omega)\) lies in the symplectic group \(Sp(n, \mathbb{R}) = \{G \in M_{2n}(\mathbb{R}) | G^T J G = J\}\). Consequently, the family of systems also induces a linear skew-product flow on the complex and real Lagrange bundles.

Recall that an \(n\)-dimensional vector subspace \(l\) of \(\mathbb{C}^{2n}(\mathbb{R}^{2n})\) is called a complex (real) Lagrange plane if \(x^T J y = 0\) for all \(x, y \in l\). One of these planes can be represented by a \(2n \times n\) complex (real) matrix \([F_1 F_2]\) of range \(n\) with \(F_1^T F_2 = F_2^T F_1\).

The column vectors form the basis of the Lagrange subspace; so two matrices \([F_1 F_2]\) and \([G_1 G_2]\) represent the same Lagrange plane if and only if there is a non-singular \(n \times n\) matrix \(P\) such that \(F_1 = G_1 P\) and \(F_2 = G_2 P\).

The set \(S_{\mathbb{C}}(n)\) of symmetric \(n \times n\) complex matrices parametrizes an open dense subset of the set of complex Lagrange planes, \(\mathcal{D} := \{[F_1^n M] | M \in S_{\mathbb{C}}(n)\}\). Taking these coordinates in \((1.1)\), we obtain the Riccati equations

\[
M' = -MH_2(\omega t) M - MH_1(\omega t) - H_1^T(\omega t) M + H_3(\omega t), \quad \omega \in \Omega. \tag{1.2}
\]

The flow on \(\Omega \times \mathcal{D}\) is then given by \((\omega, M_0) t = (\omega t, M(t, \omega, M_0))\), where \(M(t, \omega, M_0)\) is the solution of \((1.2)\) with initial data \(M(0, \omega, M_0) = M_0\).

The reader can find in \([1]\) many more details concerning the flow on the Lagrange bundles.

**Definition 1.1.** A family of (complex) Hamiltonian systems \((1.1)\) is said to have an exponential dichotomy (ED for short) over \(\Omega\) if there exist two positive constants \(C, \gamma\) and a splitting of the complex bundle into a Whitney sum of two invariant closed subbundles, \(\Omega \times \mathbb{C}^{2n} = L^+ \oplus L^-\), in such a way that

\[
\begin{align*}
(\text{i}) \quad \|U(t, \omega) z_0\| &\leq Ce^{-\gamma t}\|z_0\| \quad \text{for every } t \geq 0 \text{ and } (\omega, z_0) \in L^+, \\
(\text{ii}) \quad \|U(t, \omega) z_0\| &\leq Ce^{\gamma t}\|z_0\| \quad \text{for every } t \leq 0 \text{ and } (\omega, z_0) \in L^-.
\end{align*}
\]

The symplectic character of \(U(t, \omega)\) and the behavior of the solutions when \(|t| \to \infty\) provide an immediate proof of the fact that the sections of the stable and unstable subbundles,

\[
l^\pm(\omega) = L^\pm \cap (\{\omega\} \times \mathbb{C}^{2n}), \tag{1.3}
\]

are complex Lagrange planes; in particular the subbundles are always \(n\)-dimensional in the Hamiltonian case.

According to the results of \([15]\), the minimal character of the base flow ensures the equivalence of the ED of the whole family \((1.1)\) over \(\Omega\) and the ED (hyperbolicity) of a particular system. This result is fundamental to our purposes and, in particular, shows that the following definition is correct.

**Definition 1.2.** The Sacker-Sell spectrum of \((1.1), \Sigma(H)\), is the set of \(\lambda \in \mathbb{R}\) such that the family

\[
z' = (H(\omega t) - \lambda I_{2n}) z, \quad \omega \in \Omega
\]

does not have an ED over \(\Omega\).

In particular, the presence of an ED for the system \((1.1)\) can be characterized by \(0 \notin \Sigma(H)\). The Sacker-Sell spectral theorem \([16]\) asserts that \(\Sigma(H)\) is the union
of \( d \leq 2n \) non-overlapping closed intervals. In addition, there exist closed invariant subbundles \( Z_1, \ldots, Z_d \) such that \( \Omega \times \mathbb{C}^{2n} = Z_1 \oplus \cdots \oplus Z_d \) as Whitney sum. Each one of these subbundles is composed by those orbits corresponding to solutions whose characteristic exponents belong to each one of the spectral intervals. Recall that the four characteristic exponents of a pair \((\omega, z_0)\) are the values of the limits

\[
\limsup_{t \to \pm \infty} \frac{1}{t} \log \|U(t, \omega) z_0\|, \quad \liminf_{t \to \pm \infty} \frac{1}{t} \log \|U(t, \omega) z_0\|
\]

which are invariant along the trajectories.

In the case in which the four limits (1.4) agree, their value is one of the Lyapunov exponents of the system. It is known that there exist \( 2n \) Lyapunov exponents, equal or distinct, and that due to the Hamiltonian character of the system, \( n \) of them are the opposite of the other \( n: \pm \gamma_1, \ldots, \pm \gamma_n \).

A careful analysis of the relation between the spectral decomposition, the Lyapunov exponents and the characteristic exponents is made by Johnson, Palmer and Sell in [9]. A more specific analysis for the Hamiltonian case can be found in Johnson [7] and Novo et al. [11], as well as in the references therein.

Those spectral intervals and subbundles are the objects whose evolution we analyze in Section 2. As said before, we study one-parametric perturbations of (1.1) and establish conditions ensuring the uniform convergence of the spectral decomposition. This uniform convergence of subbundles means convergence of the fibers (i.e. convergence of linear spaces, equivalent to convergence of conveniently chosen bases), with the additional requirement of uniformity of the convergence with respect to the base space \( \Omega \).

2. Perturbation theorems. As said in the introduction, the analysis of the variation of the Sacker-Sell spectral decomposition carried out in this section is based on previous results concerning the ergodic and topological structures of the space of trajectories. We begin this section by summarizing the required properties.

From now on, we will assume that all the solutions of (1.1) are bounded. As said before, this condition ensures the existence of a continuous change of variables \( \tilde{z} = C(\omega \cdot t) z \) taking the initial family of systems (1.1) to

\[
\tilde{z}' = \tilde{H}(\omega \cdot t) \tilde{z}, \quad \omega \in \Omega,
\]

with \( \tilde{H} \) skew-symmetric (see [2] and [3]). Let \( m_0 \) be a \( \sigma \)-ergodic measure on \( \Omega \), and let \( \Gamma \) be a symmetric continuous matrix-valued function on \( \Omega \). This \( \Gamma \) will play the role of the perturbation direction we referred to in the introduction. The following result is proved in [11].

**Theorem 2.1.** The limits

\[
A \Gamma(\omega) = \lim_{t \to \infty} \frac{1}{2t} \int_{-t}^{t} U^T(s, \omega) \Gamma(\omega \cdot s) U(s, \omega) ds
\]

exist \( m_0 \)-a.e., and the symmetric matrix-valued function \( A \Gamma \) belongs to \( L^1(\Omega, m_0) \). In addition, the eigenvalues of the matrix-valued function \( J^{-1} A \Gamma(\omega) \) are constant \( m_0 \)-a.e. and the corresponding eigenspaces are invariant for the flow induced on \( \Omega \times \mathbb{C}^{2n} \) by (1.1).

Recall that \( U(t, \omega) \) represents the fundamental matrix solution of (1.1) with \( U(0, \omega) = I_{2n} \). In fact the existence \( m_0 \)-a.e. of the limit \( A \Gamma \in L^1(\Omega, m_0) \) can be guaranteed from the existence of a square-integrable (not necessarily continuous)
change of variables taking \((1,1)\) to skew-symmetric form. Conditions ensuring this existence, based on Kotani’s theory, are summarized in \([5]\). Results of \([11]\) and \([8]\) concerning the properties of differentiability of the rotation number and the evolution of the Weyl \(M\)-matrices (determined by the stable and unstable subbundles in the case of an ED) for the one-parametric perturbations of \((1,1)\) given by

\[
\mathbf{u}' = (H(\omega t) + \lambda J^{-1}(\omega t)) \mathbf{u}, \quad \omega \in \Omega,
\]

are strongly based on the existence and characteristics of these functions \(A_\Gamma\). In particular, if the perturbation \(\Gamma\) is \textit{positive definite} (i.e. if \(x^T \Gamma x > 0\) for all \(x \in \mathbb{R}^{2n} - \{0\}\)), the radial limit over the real axis of the Weyl \(M\)-matrices exists in the \(L^1(\Omega, m_0)\)-topology, and it is determined by the eigenspaces of the matrix \(J^{-1}A_\Gamma\).

Our results concerning the uniform variation with respect to \(\lambda\) of the Sacker-Sell spectral decomposition of these systems require, however, stronger hypotheses: we only consider perturbation directions \(\Gamma\) belonging to the following set.

**Definition 2.2.** Let \(\Gamma\) be a continuous \(2n \times 2n\) matrix-valued function on \(\Omega\). We say that \(\Gamma\) belongs to the set \(\mathcal{C}\) if the limit \((2.2)\) exists for every \(\omega \in \Omega\).

**Remarks 2.3.** 1. It can be shown (although the proof is not trivial) that, if the base flow is uniquely ergodic, there exists a perturbation direction \(\Gamma \in \mathcal{C}\) such that \(A_\Gamma\) exists everywhere and agrees with \(C^2 C\), where \(C\) is the continuous change of variables taking our system to skew-symmetric form whose existence is ensured by our hypotheses. In other words, the set \(\mathcal{C}\) is not empty. Recall that this base flow is known to be uniquely ergodic in the case that \(\Omega\) is the hull of an almost-periodic function. The reader is referred to \([14]\) for a more exhaustive analysis of the structure and properties of the set \(\mathcal{C}\) in the two-dimensional case: the conditions ensuring that it is not empty do not require ergodic uniqueness of the base flow; and there are cases in which it coincides with the whole set of continuous matrix-valued functions \(\Gamma\) on \(\Omega\).

2. A similar argument to the one used for \(q_\Gamma\) in \([14]\), Section 6, based on Theorem 3.1 of Furstenberg \([3]\), shows that, if \(\Gamma \in \mathcal{C}\), then the function \(A_\Gamma\) is continuous on \(\Omega\). Consequently, the invariant subbundles of \(\Omega \times \mathbb{C}^{2n}\) determined by the eigenspaces of \(J^{-1}A_\Gamma(\omega)\) are closed. We will refer to them as \textit{subbundles associated with} \(\Gamma\).

These facts will be fundamental to our purposes.

We will consider either complex perturbations for \(\Gamma \in \mathcal{C}\) with positive definite \(A_\Gamma\) or real perturbations for \(\Gamma \in \mathcal{C}\) such that \(J^{-1}A_\Gamma\) is non-singular and can be taken to real diagonal form. Both results are based on technical lemmas, non-trivial consequences of the Sacker-Sell perturbation theorem. To state the first one, we need to recall some basic algebraic facts.

**Remark 2.4.** Let \(D\) be a symmetric matrix. It is immediate to check that \(n\) of the eigenvalues of \(J^{-1}D\) are the opposite of the other \(n\). It is also possible to show that a symmetric matrix \(D\) is positive definite if and only if: a) the matrix \(J^{-1}D\) has purely imaginary eigenvalues and can be diagonalized; and b) the sums of eigenspaces of \(J^{-1}D\) corresponding to eigenvalues with either positive or negative imaginary parts are complex Lagrange planes which can be respectively represented as \(\frac{I_n}{N^+}\) and \(\frac{I_n}{N^-}\), with \(N^- = \overline{N^+}\) (complex conjugate) and \(\pm \Im N^\pm > 0\).

**Lemma 2.5.** Let \(D\) be a constant real positive definite (symmetric) \(2n \times 2n\) matrix. Given a continuous map \([0, \varepsilon_1] \to C(\Omega, M_\Omega(2n)), \varepsilon \mapsto T(\omega, \varepsilon)\) with \(T(\omega, 0) = 0,\)
we consider the families of linear systems
\[ z' = i\varepsilon J^{-1} (D + T(\omega, t, \varepsilon)) z, \quad \omega \in \Omega, \]  
for \( \varepsilon \in [0, \varepsilon_1] \). Let \( \pm i\mu_1, \ldots, \pm i\mu_s \) be the different eigenvalues of \( J^{-1} D \) with multiplicities \( m_1, \ldots, m_s \) respectively, ordered to get \( 0 < \mu_1 < \cdots < \mu_s \), and set
\[ \beta_0 = \frac{1}{2} \min (2\mu_1, \mu_2 - \mu_1, \ldots, \mu_s - \mu_{s-1}). \]

For each \( \beta \in (0, \beta_0) \) there exists \( \varepsilon(\beta) > 0 \) such that, if \( \varepsilon \in (0, \varepsilon(\beta)) \), then
(i) the Sacker-Sell spectrum of \((2.3)_{\varepsilon}\) is contained in the set
\[ \bigcup_{j=1}^{s} [\pm \varepsilon \mu_j - \varepsilon \beta, \pm \varepsilon \mu_j + \varepsilon \beta], \]
and each one of the \( 2s \) (disjoint) intervals of this union contains at least one spectral interval;
(ii) the sum of the spectral subbundles of \((2.3)_{\varepsilon}\) corresponding to the intervals contained in \([\pm \varepsilon \mu_j - \varepsilon \beta, \pm \varepsilon \mu_j + \varepsilon \beta]\) has dimension \( m_j \), varies continuously with \( \varepsilon \), and converges (uniformly on \( \Omega \)) as \( \varepsilon \to 0^+ \) to the constant subbundle given on each fiber by the eigenspace of \( J^{-1} D \) associated with \( \pm i\mu_j \);
(iii) if \( \tilde{\gamma}_{\varepsilon}^\pm(\omega) \) represents the sum of the Lyapunov exponents (equal or distinct) of \((2.3)_{\varepsilon}\) corresponding to the interval \([\pm \varepsilon \mu_j - \varepsilon \beta, \pm \varepsilon \mu_j + \varepsilon \beta]\), then
\[ \lim_{\varepsilon \to 0^+} \frac{\tilde{\gamma}_{\varepsilon}^\pm(\omega)}{\varepsilon} = \pm m_j \mu_j. \]

In particular, the families of systems corresponding to these values of \( \varepsilon \) admit an ED. For \( \varepsilon \) small enough, the complex Lagrange planes given by the stable and unstable subbundles can be represented by \( \left[ \begin{array}{c} I_n \\ M^\pm(\omega, \varepsilon) \end{array} \right] \), and the corresponding Weyl M-functions satisfy
\[ \lim_{\varepsilon \to 0^+} M^\pm(\omega, \varepsilon) = \mathbb{N}^\pm \]
uniformly on \( \Omega \), where \( \left[ \begin{array}{c} I_n \\ N^\pm \end{array} \right] \) represent respectively the complex Lagrange planes generated by the eigenvectors associated with the eigenvalues \( \pm i\mu_1, \ldots, \pm i\mu_s \) of \( J^{-1} D \).

**Proof.** We follow a standard argument based on the Sacker-Sell perturbation theorem. First we need to establish some notation. Consider the metric space \( \mathcal{M} \) of the uniformly continuous and bounded complex \( 2n \times 2n \) matrix-valued functions with the compact-open topology, and the real flow \( \sigma : \mathcal{M} \times \mathbb{R} \to \mathcal{M}, (L, t) \mapsto L_t \), where \( L_t(s) = L(t + s) \). Let \( U(t, L) \) be the fundamental matrix solution of
\[ w' = L(t) w \]  
with \( U(0, L) = I_{2n} \). The real skew-symmetric flow given on \( \mathcal{M} \times \mathbb{C}^{2n} \) by
\[ \tilde{\sigma} : \mathcal{M} \times \mathbb{C}^{2n} \times \mathbb{R} \to \mathcal{M} \times \mathbb{C}^{2n}, (L, w, t) \mapsto (L_t, U(t, L) w), \]
admits a restriction to each set \( \mathcal{K} \times \mathbb{C}^{2n} \), where \( \mathcal{K} \) is a compact \( \sigma \)-invariant subset of \( \mathcal{M} \). Denote by \( \Sigma(L) \) the Sacker-Sell spectrum of \((2.4)\), i.e. the set of \( \lambda \in \mathbb{R} \) such that \( w' = (L(t) - \lambda I_{2n}) w \) does not have an exponential dichotomy. Let
\[ \rho(L) = \mathbb{R} - \Sigma(L) \] is its resolvent. Denote also \( \Sigma(K) := \bigcup_{L \in K} \Sigma(L) \) and \( \rho(K) := \mathbb{R} - \Sigma(K) \cap \rho(L) \). For \( \lambda \in \rho(K) \), the sets
\[
S_\lambda(K) := \{(L, w) \in \mathbb{K} \times \mathbb{C}^2n \mid \| e^{-\lambda t} U(t, L)w \| \to 0 \text{ as } t \to \infty \},
\]
\[
U_\lambda(K) := \{(L, w) \in \mathbb{K} \times \mathbb{C}^2n \mid \| e^{-\lambda t} U(t, L)w \| \to 0 \text{ as } t \to -\infty \},
\]
are complementary \( \sigma \)-invariant closed subbundles of \( \mathbb{K} \times \mathbb{C}^2n \). We represent by \( S_\lambda(L) \) and \( U_\lambda(L) \) the sections of these subbundles for \( L \in \mathbb{K} \) (complex linear spaces). Recall that, according to the results of [10], given \( \lambda_1, \lambda_2 \in \rho(K) \) with \( \lambda_1 < \lambda_2 \), the following statements are equivalent:

a) there exists \( \mu \in (\lambda_1, \lambda_2) \cap \Sigma(K) \),

b) \( S_\lambda(K) \cap U_\lambda(K) \neq \mathbb{K} \times \{0\} \).

Furthermore, \( U_\lambda(K) \cap S_\lambda(K) \) is the sum of the spectral subbundles of \( \mathbb{K} \times \mathbb{C}^2n \) associated with the intervals of \( \Sigma(K) \) contained in \( (\lambda_1, \lambda_2) \).

Now we consider the family of systems (2.3). Taking \( w(t) = z(t/\varepsilon) \) we obtain
\[
w' = iJ^{-1}(D + T(\omega \cdot t/\varepsilon, \varepsilon))w, \quad \omega \in \Omega. \tag{2.5}_\varepsilon
\]
For each pair \((\omega, \varepsilon)\) we write \( T_{\omega,\varepsilon}(t) = T(\omega \cdot t/\varepsilon, \varepsilon) \) if \( \varepsilon > 0 \), and \( T_{\omega,0}(t) = 0 \). It is not difficult to check that our hypotheses guarantee that the map \( \Omega \times [0, \varepsilon_1] \to \mathcal{M}, \ (\omega, \varepsilon) \mapsto iJ^{-1}(D + T_{\omega,\varepsilon}) \) is continuous. Hence, if \( 0 \leq \varepsilon \leq \varepsilon_0 \leq \varepsilon_1 \), the sets
\[
\mathcal{K}(\varepsilon) := \{iJ^{-1}(D + T_{\omega,\varepsilon}) \mid \omega \in \Omega\},
\]
\[
\mathcal{K}_{\varepsilon_0} := \bigcup_{\varepsilon \in [\varepsilon_0, \varepsilon_1]} \mathcal{K}(\varepsilon) \cap \{iJ^{-1}(D + T_{\omega,\varepsilon}) \mid \omega \in \Omega, \varepsilon \in [0, \varepsilon_0]\}
\]
are connected compact invariant subsets of \( \mathcal{M} \). Besides, every neighborhood of \( \{iJ^{-1}D\} \) contains a set \( \mathcal{K}_{\varepsilon_0} \) for \( \varepsilon_0 > 0 \) small enough. It is clear that

c) the Sacker-Sell spectrum of the family of systems (2.3) coincides with \( \varepsilon \cdot \Sigma(\mathcal{K}(\varepsilon)) \);

d) the change of variables does not affect the spectral subbundles. More precisely, the section for \( iJ^{-1}(D + T_{\omega,\varepsilon}) \) of the spectral subbundle of \( \mathcal{K}(\varepsilon) \times \mathbb{C}^2n \) associated with the interval \([\lambda_1, \lambda_2]\) of its Sacker-Sell spectrum, coincides with the section for \( \omega \) of the spectral subbundle of \( \Omega \times \mathbb{C}^2n \) associated with the interval \([\varepsilon \lambda_1, \varepsilon \lambda_2]\) of the Sacker-Sell spectrum of (2.3).

Consider now the constant coefficients system
\[
w' = iJ^{-1}Dw. \tag{2.6}
\]
It is known that \( \Sigma(\mathcal{K}_0) = \Sigma(iJ^{-1}D) = \{-\mu_1, \ldots, -\mu_1, \mu_1, \ldots, \mu_s\} \), and the respective eigenspaces (with dimensions \( m_s, \ldots, m_1, m_1, \ldots, m_s \)) provide precisely the corresponding spectral subbundles of \( \mathcal{K}_0 \times \mathbb{C}^2n \). We take \( \beta \in (0, \beta_0) \), consider the neighborhood of \( \Sigma(iJ^{-1}D) \) given by
\[
V(\beta) := \bigcup_{j=1}^s [\pm \mu_j - \beta, \pm \mu_j + \beta],
\]
and choose \( \lambda_0 = 0, \lambda_1, \ldots, \lambda_s \) with
\[
\lambda_0 < \mu_1 - \beta, \quad \mu_j + \beta < \lambda_j < \mu_{j+1} - \beta \quad \text{for } j = 1, \ldots, s - 1, \quad \mu_s + \beta < \lambda_s.
\]
The perturbation theorem ([10], Theorem 6) ensures the existence of a constant \( \varepsilon(\beta) > 0 \) such that, if \( \varepsilon \in [0, \varepsilon(\beta)] \),
\[
\Sigma(\mathcal{K}(\varepsilon)) \subset V(\beta),
\]
and hence
\[ \Sigma(iJ^{-1}D) \subset \Sigma(K_{\varepsilon(\beta)}) \subset V(\beta). \]
Thus, \( \pm \lambda_j \in \rho(L) \) for every \( L \in K_{\varepsilon(\beta)}. \) Besides, \( S_{\pm \lambda_j}(L) \) and \( U_{\pm \lambda_j}(L) \) vary continuously with respect to \( L \in K_{\varepsilon(\beta)}, \) and therefore so do \( U_{\lambda_j}(L) \cap S_{\lambda_j+1}(L) \) and \( U_{-\lambda_j+1}(L) \cap S_{-\lambda_j}(L). \) In particular, this implies that the dimensions of all these vector subbundles are constant on \( K_{\varepsilon(\beta)}. \) Moreover, \( \dim(U_{\lambda_j}(L) \cap S_{\lambda_j+1}(L)) = \dim(U_{-\lambda_j+1}(L) \cap S_{-\lambda_j}(L)) = m_j, \) since this one is precisely the dimension of the spectral subbundle of \((2.6)\) associated with \( \pm \mu_j. \) In addition, the equivalence between the above properties a) and b), the fact that \( \dim S_{\lambda_j}(iJ^{-1}D) < \dim U_{\lambda_j}(iJ^{-1}D) \) (resp. \( \dim S_{-\lambda_j+1}(iJ^{-1}D) < \dim U_{-\lambda_j+1}(iJ^{-1}D) \)) for \( j = 0, \ldots, s - 1, \) and the invariance of these dimensions on \( K_{\varepsilon(\beta)}, \) guarantee that each one of the intervals of \( V(\beta) \) contains at least one interval of \( \Sigma(K(\varepsilon)) \) for every \( \varepsilon \in [0, \varepsilon(\beta)]. \)

This and equality c) above prove (i) for \( \varepsilon \in [0, \varepsilon(\beta)]. \) The continuous variation with respect to \( \varepsilon \in [0, \varepsilon(\beta)] \) of the sum of the spectral subbundles of \( K(\varepsilon) \times C^{\infty_n} \) corresponding to each interval of \( V(\beta) \) (which has dimension \( m_j \)) is deduced from the continuous variation on \( K_{\varepsilon(\beta)}; \) and this continuity guarantees the uniform convergence as \( \varepsilon \to 0 \) to the spectral subbundles of the limit system \((2.6). \) These facts and property d) provide the proof of (ii). Furthermore, \( \gamma_j^\infty(\varepsilon)/\varepsilon \) is not but the sum of the \( m_j \) Lyapunov exponents (equal or distinct) of \((2.5)_\varepsilon \) contained in the interval \([\pm \mu_j - \beta, \pm \mu_j + \beta], \) and this sum converges as \( \varepsilon \to 0 \) to the corresponding one for the limit system \((2.6); \) that is, to \( \pm \mu_j \mu_j. \) Hence (iii) is also proved.

Exponential dichotomy for systems \((2.3)\) with \( L \in K_{\varepsilon(\beta)} \) follows from the fact that \( \lambda_0 = 0 \notin V(\beta). \) According to (ii), the stable and unstable subbundles of \((2.3)_\varepsilon \) converge uniformly as \( \varepsilon \to 0 \) to the subbundles corresponding to \((2.6), \) that is, to the Lagrange planes \( \left[ \lambda_n^{2\infty_\varepsilon} \right] \) of the statement. Hence, for \( \varepsilon \) small enough, we can represent the stable and unstable subbundles (which are complex Lagrange planes) as \( \left[ \lambda_n^{2\infty(\omega;\varepsilon)} \right]. \) This completes the proof of the lemma.

As said before, our first result refers to the parametric variation of the perturbed family

\[ z' = \left( H(\omega\cdot t) + i\varepsilon J^{-1}\Gamma(\omega\cdot t) \right) z, \quad \omega \in \Omega, \quad (2.7) \]

with real \( \varepsilon, \) in the case in which \( \Gamma \) belongs to the set \( C \) and defines a positive definite finite limit \( A_\Gamma. \) (This last property holds not only for positive definite \( \Gamma \) but also for a wider class of perturbations: those satisfying an Atkinson definiteness condition; see [111].) We denote by \( \pm i\mu_{\Gamma,1}, \ldots, \pm i\mu_{\Gamma,s} \) the different eigenvalues of \( J^{-1}A_\Gamma \) with multiplicities \( m_{\Gamma,1}, \ldots, m_{\Gamma,s} \) respectively (see Theorem 2.1 and Remark 2.4), ordered to get \( \mu_{\Gamma,1} < \cdots < \mu_{\Gamma,s}, \) and set

\[ \beta_\Gamma := \frac{1}{2} \min \left( 2\mu_{\Gamma,1}, \mu_{\Gamma,2} - \mu_{\Gamma,1}, \ldots, \mu_{\Gamma,s} - \mu_{\Gamma,s-1} \right). \quad (2.8) \]

We also represent by \( \left[ \lambda_n^{2\infty_\varepsilon(\omega)} \right] \) and \( \left[ \lambda_n^{2\infty(\omega)} \right] \) the complex Lagrange planes respectively generated by the eigenvalues with positive and negative imaginary parts of \( J^{-1}A_\Gamma \) (see Remark 2.3), which are closed by continuity (see Remark 2.3). Then, as a consequence of the invariance of the eigenspaces, the symmetric \( n \times n \) matrix-valued functions \( N^\infty_\varepsilon \) are continuous solutions along the flow of the equation \((1.2). \) Besides,
Let us consider a perturbation corresponding to the interval \( \omega, i \varepsilon \) contained in the set \( \Omega \). In order to find a continuous change of variables taking the unperturbed system of \( (2.7) \) to a skew-symmetric family (2.1) (see Theorem 4.5 of [11]).

The continuous variation of the Sacker-Sell spectral decomposition is stated in the following theorem. (Clearly, a symmetric result can be stated for a negative definite limit \( A_R \).)

**Theorem 2.6.** Let us consider a perturbation \( \Gamma \in C \) with positive definite \( A_R \), and define \( \beta_R \) by (2.8). For every \( \beta \in (0, \beta_R) \) there exists \( \varepsilon(\beta) > 0 \) such that, if \( \varepsilon \in (0, \varepsilon(\beta)) \), then

(i) the Sacker-Sell spectrum of \((2.7)_{\varepsilon}\) is contained in the set

\[
\bigcup_{j=1}^s [\pm \varepsilon \mu_{\Gamma, j} - \varepsilon \beta, \pm \varepsilon \mu_{\Gamma, j} + \varepsilon \beta],
\]

and each one of the \( 2s \) (disjoint) intervals of this union contains at least one spectral interval;

(ii) the sum of the spectral subbundles of \((2.7)_{\varepsilon}\) corresponding to the interval \([\pm \varepsilon \mu_{\Gamma, j} - \varepsilon \beta, \pm \varepsilon \mu_{\Gamma, j} + \varepsilon \beta]\) varies continuously with \( \varepsilon \), has dimension \( m_{\Gamma, j} \), and converges (uniformly on \( \Omega \)) as \( \varepsilon \to 0^+ \) to the subbundle associated with \( \Gamma \) corresponding to \( \pm \varepsilon \mu_{\Gamma, j} \);

(iii) if \( \tilde{\gamma}_{\Gamma, j}(\varepsilon) \) represents the sum of the Lyapunov exponents of \((2.7)_{\varepsilon}\) corresponding to the interval \([\pm \varepsilon \mu_{\Gamma, j} - \varepsilon \beta, \pm \varepsilon \mu_{\Gamma, j} + \varepsilon \beta]\), then

\[
\lim_{\varepsilon \to 0^+} \frac{\tilde{\gamma}_{\Gamma, j}(\varepsilon)}{\varepsilon} = \pm m_{\Gamma, j} \mu_{\Gamma, j}.
\]

In particular, the families of systems corresponding to these values of \( \varepsilon \) admit an ED. For \( \varepsilon \) small enough, the stable and unstable subbundles can be represented by

\[
\left[ I_n \atop M_{\Gamma}(\omega, i \varepsilon) \right], \text{ and the corresponding Weyl } M\text{-functions satisfy}
\]

\[
\lim_{\varepsilon \to 0^+} M_{\Gamma}^\pm(\omega, i \varepsilon) = N_{\Gamma}^\pm(\omega)
\]

uniformly on \( \Omega \).

**Proof.** We denote by \( U(\varepsilon)(t, \omega) \) the fundamental matrix solution of \((2.7)_{\varepsilon}\) satisfying \( U(\varepsilon)(0, \omega) = I_2n \). The subindex will be omitted for \( \varepsilon = 0 \).

The proof, which follows the same scheme as the one of Theorem 6.3 of [11], is divided in two steps. In the first one, we reformulate \((2.7)_{\varepsilon}\) in a new base \( (\Omega^1, \sigma^1) \), in order to find a continuous change of variables taking the unperturbed system of the family to \( \omega' = 0 \). To this end, we transform (1.1) to (2.1) by means of the change of variables \( \tilde{z} = C_\Gamma(\omega-t)z \). Then the compact metric space \( \Omega \times \mathcal{G} \), where

\[
\mathcal{G} := \left\{ \begin{bmatrix} \Phi \\ \Psi \end{bmatrix} \mid \Phi^T \Phi + \Psi^T \Psi = I_n, \Phi^T \Psi = \Psi^T \Phi \right\} \simeq Sp(n, \mathbb{R}) \cap SO(2n, \mathbb{R})
\]

is invariant for the continuous flow induced by (2.1) (see [11]); we choose a minimal subset \( \Omega^1 \), and represent by \( \sigma^1 \) the restriction to \( \Omega^1 \) of the flow, by \( \omega^1 = (\omega, [\varepsilon]) \) the elements of \( \Omega^1 \) and by \( \Pi : \Omega^1 \to \Omega, \omega^1 \mapsto \omega \) the projection over the first component.
Next we define \( \tilde{H}^1(\omega^1) = \tilde{H}(\Pi(\omega^1)) \) (see equation (2.11)) and consider the family of equations
\[
\tilde{\tilde{z}}' = \tilde{H}^1(\omega^1, t) \tilde{z}, \quad \omega^1 \in \Omega^1.
\]
As proved in [11], the function \( \tilde{V}^1(\omega^1, t) \), where
\[
\tilde{V}^1(\omega^1) = \begin{bmatrix} \Phi - \Psi \\ \Psi \end{bmatrix} \quad \text{for} \quad \omega^1 \in \left( \omega, \begin{bmatrix} \Phi \\ \Psi \end{bmatrix} \right),
\]
is a symplectic fundamental matrix solution of systems (2.10) evaluated along the trajectories of \((\Omega^1, \sigma^1)\). Consequently, the change of variables \( \tilde{z} = \tilde{V}^1(\omega^1, t) \mathbf{w} \) takes (2.10) to \( \mathbf{w}' = 0 \).

Setting \( C^1_2(\omega^1) = C^1_2(\Pi(\omega^1)) \) and \( H^1(\omega^1) = H(\Pi(\omega^1)) \) we observe that \( V^1(\omega^1) = (C^1_2)^{-1}(\omega^1) \tilde{V}^1(\omega^1) \) defines a fundamental matrix solution of
\[
\mathbf{z}' = H^1(\omega^1, t) \mathbf{z},
\]
also evaluated along the trajectories of \((\Omega^1, \sigma^1)\). Besides, \( V^1 \in L^2(\Omega^1, m^1_0) \). Clearly, the fundamental matrix solution \( U^1(t, \omega^1) \) of (2.11) with \( U^1(0, \omega^1) = I_{2n} \) (which is obviously given by \( U^1(t, \omega^1) = U(t, \Pi(\omega^1)) \)) is related to \( V^1 \) by
\[
U^1(t, \omega^1) = V^1(\omega^1, t) (V^1)^{-1}(\omega^1).
\]

Now we define \( H^1(\omega^1) = H(\Pi(\omega^1)) \) and \( \Gamma^1(\omega^1) = \Gamma(\Pi(\omega^1)) \) for \( \omega^1 \in \Omega^1 \), and consider the new extended family
\[
\mathbf{z}' = (H^1(\omega^1, t) + i\varepsilon J^{-1} \Gamma^1(\omega^1, t)) \mathbf{z}, \quad \omega^1 \in \Omega^1.
\]
Let \( U^1_\varepsilon(t, \omega^1) \) be the fundamental matrix solution of this system with \( U^1_\varepsilon(0, \omega) = I_{2n} \). It is obvious that \( U^1_\varepsilon(t, \omega^1) = U_\varepsilon(t, \Pi(\omega^1)) \) for every \( \omega^1 \in \Omega^1 \). Consequently, the Sacker-Sell spectrum and the Lyapunov exponents of (2.7) and (2.13) coincide, and the spectral decomposition of (2.13) can be obtained from that of (2.7) in a trivial way.

To end this first step, we take \( C^1_2(\omega^1) = C^1_2(\omega) \) and \( V^1_\varepsilon(\omega^1) = (C^1_2)^{-1}(\omega^1) \tilde{V}^1(\omega^1) \) and observe that the continuous and symplectic change of variables \( \mathbf{z} = V^1_\varepsilon(\omega^1, t) \mathbf{w} \) takes (2.13) to
\[
\mathbf{w}' = i\varepsilon J^{-1} W^1(\omega^1, t) \mathbf{w}, \quad \omega^1 \in \Omega^1,
\]
with \( W^1(\omega^1) = (V^1_\varepsilon)^T(\omega^1) \) and symmetric with \( \Gamma^1(\omega^1) \).

In the second part of the proof, as in [14], we follow a perturbation theory argument (see [10]) to transform (2.14) in a new family of linear systems satisfying the hypotheses of Lemma 2.5.

Note that, since \( U^1_0(t, \omega^1) = V^1_2(\omega^1, t) (V^1_2)^{-1}(\omega^1) \), we get
\[
W^1(\omega^1, t) = (V^1_2)^T(\omega^1) (U^1)^T(t, \omega) \Gamma^1(\omega^1, t) U^1(t, \omega) V^1_2(\omega^1).
\]

Consequently, the limit
\[
\lim_{t \to \infty} \frac{1}{2\varepsilon} \int_{-t}^{t} W^1(\omega^1, s) ds = (V^1_2)^T(\omega^1) A_T^1(\omega^1) V^1_2(\omega^1) = D_T(\omega)
\]
exists for every \( \omega^1 \in \Omega^1 \), since the fact that \( \Gamma \in C \) ensures that
\[
A_T^1(\omega) := \lim_{t \to \infty} \frac{1}{2\varepsilon} \int_{-t}^{t} U_T(\omega^1, s) \Gamma^1(\omega^1, s) U^1(\omega^1, s) ds = A_T(\Pi(\omega^1))
\]
exists for every \( \omega^1 \in \Omega^1 \). Moreover, the positive definite limit matrix \( D_T \) is constant on \( \Omega \). In particular, this implies that for every \( \sigma^1 \)-invariant measure \( \mu \) on \( \Omega^1 \),

\[
\int_{\Omega^1} (W_{T^1}(\omega^1) - D_T) \, d\mu = 0.
\]  

(2.15)

Note also that the eigenvalues of \( J^{-1} D_T \) agree with those of \( J^{-1} A_1(\omega^1) \), and the respective eigenspaces of both matrices are related by the symplectic matrix \( V^1_1(\omega^1) \).

Let \( (\kappa_j)_{j \in \mathbb{N}} \) be a decreasing sequence of positive real numbers with limit 0. The density of the set

\[
\{ f \in C(\Omega^1) \mid \text{there exists } g \in C(\Omega^1) \text{ with } g'(\omega^1) = f(\omega^1) \}
\]

(where \( g'(\omega^1) \) represents \( (d/dt)g(\omega^1 \cdot t)|_{t=0} \)) in

\[
\left\{ f \in C(\Omega^1) \mid \int_{\Omega^1} f \, d\mu = 0 \quad \text{for every } \sigma^1 \text{-invariant measure } \mu \text{ on } \Omega^1 \right\},
\]

(see Schwartzman [17], and property (2.13), allow us to choose a sequence of matrix functions \( (R_j)_{j \in \mathbb{N}} \) in \( C(\Omega^1, M_k(2n)) \), whose continuous derivatives along the flow satisfy

\[
\| R_j' - (W_{T^1} - D_T) \|_\infty < \kappa_j
\]

for every \( j \in \mathbb{N} \). (As usual, \( \| \cdot \|_\infty \) represents the supremum norm of a matrix-valued function on \( \Omega \), associated to a previously fixed matrix norm.) Let us denote \( r_j = \| R_j \|_\infty \). Given any constant \( r > 0 \) we can take a decreasing sequence \( (\varepsilon_j)_{j \in \mathbb{N}} \) of positive numbers with \( \lim_{j \to \infty} \varepsilon_j = 0 \) and \( \varepsilon_j (r_j + r_{j+1}) \leq r \) for every \( j \in \mathbb{N} \). We define

\[
R(\omega^1, \varepsilon) = \left( \frac{\varepsilon - \varepsilon_{j+1}}{\varepsilon_j - \varepsilon_{j+1}} R_j(\omega^1) + \frac{\varepsilon_j - \varepsilon}{\varepsilon_j - \varepsilon_{j+1}} R_{j+1}(\omega^1) \right) \quad \text{if } \varepsilon_j + 1 \leq \varepsilon \leq \varepsilon_j.
\]

Then \( R \) is a continuous function on \( \Omega^1 \times (0, \varepsilon_1] \). By choosing \( r \) small enough we can assure that

1) \( \det (I_{2n} + i\varepsilon J^{-1} R(\omega^1, \varepsilon)) \neq 0 \) for every \( \omega^1 \in \Omega^1 \);

2) the continuous linear change of variables \( w = (I_{2n} + i\varepsilon J^{-1} R(\omega^1, \varepsilon)) w^* \) takes

\[
(2.14)
\]

for \( \varepsilon \in (0, \varepsilon_1] \) to

\[
(w^*)' = i\varepsilon J^{-1} \left( D_T + W_{T^1}^*(\omega^1, t, \varepsilon) \right) w^*, \quad \omega^1 \in \Omega^1,
\]

(2.16)\( \varepsilon \)

with \( \| W_{T^1}^*(\omega^1, \varepsilon) \|_\infty \leq 2\kappa_1 \) if \( \varepsilon_{j+1} \leq \varepsilon \leq \varepsilon_j \); in particular, we can assure that \( \lim_{\varepsilon \to 0^+} W_{T^1}^*(\omega^1, \varepsilon) = 0 \) uniformly on \( \Omega \).

We define \( W^*_T(\omega^1, 0) = 0 \) and observe that (2.16)\( \varepsilon \) satisfies all the hypotheses of Lemma 2.3. Since the continuous transformation

\[
z = V^1_1(\omega^1 \cdot t) \left( I_{2n} + i\varepsilon J^{-1} R(\omega^1, t, \varepsilon) \right) \tilde{w}^*
\]

preserves the Sacker-Sell spectrum and the Lyapunov exponents of (2.16)\( \varepsilon \) and takes the spectral subbundles of this family to those of (2.13)\( \varepsilon \), in a continuous way, we obtain the desired conclusions for the extended family. Note that the constant subbundle determined by the eigenspaces of \( J^{-1} D_T \) for \( \pm i\mu_{T^1,j} \) is taken by the transformation \( z = V^1_1(\omega^1 \cdot t) \tilde{w}^* \) to the subbundle associated with \( \Gamma_j \) and \( \pm i\mu_{T^1,j} \). Our previous remark about the relation between the spectral decompositions of (2.13)\( \varepsilon \) and (2.7)\( \varepsilon \) completes the proof of (i), (ii) and (iii).

In particular, the initial family of systems has an ED for \( \varepsilon > 0 \) small enough, and the corresponding subbundles converge uniformly on \( \Omega \) as \( \varepsilon \to 0 \) to the Lagrange planes \( V^1(\omega^1) \left[ \frac{I_n}{\lambda} \right] \), where \( \left[ \frac{I_n}{\lambda} \right] \) are the complex Lagrange planes generated by
Let us consider a perturbation corresponding to the interval \( J \) is not completely described. The topological structure has not been so far completely described. and Pöschel \[ }\]

Theorem 2.7. Let us consider a perturbation \( \Gamma \in \mathcal{C} \) such that
- the different eigenvalues of \( J^{-1}A_{\Gamma}(\omega) \) are ±\( \mu_{\Gamma,1}, \ldots, \pm \mu_{\Gamma,s} \in \mathbb{R} \setminus \{0\} \),
- \( J^{-1}A_{\Gamma}(\omega) \) can be diagonalized,

and define \( \beta \) by (2.8). For every \( \beta \in (0, \beta_{\Gamma}) \) there exists \( \varepsilon(\beta) > 0 \) such that, if \( \varepsilon \in (0, \varepsilon(\beta)) \), then
(i) the Sacker-Sell spectrum of (2.17) is contained in the set
\[
\bigcup_{j=1}^{s} \left[ \pm \varepsilon \mu_{\Gamma,j} - \varepsilon \beta, \pm \varepsilon \mu_{\Gamma,j} + \varepsilon \beta \right],
\]
and each one of the 2s (disjoint) intervals of this union contains at least one spectral interval;
(ii) the sum of the spectral subbundles of (2.17) corresponding to the interval \( [\pm \varepsilon \mu_{\Gamma,j} - \varepsilon \beta, \pm \varepsilon \mu_{\Gamma,j} + \varepsilon \beta] \) varies continuously with \( \varepsilon \), has dimension \( m_{\Gamma,j} \), and converges as \( \varepsilon \to 0^+ \) to the subbundle associated with \( \Gamma \) corresponding to \( \pm \mu_{\Gamma,j} \);
(iii) if \( \gamma_{\Gamma,j}^\varepsilon(\varepsilon) \) represents the sum of the Lyapunov exponents of (2.17) corresponding to the interval \( [\pm \varepsilon \mu_{\Gamma,j} - \varepsilon \beta, \pm \varepsilon \mu_{\Gamma,j} + \varepsilon \beta] \), then
\[
\lim_{\varepsilon \to 0^+} \frac{\gamma_{\Gamma,j}^\varepsilon(\varepsilon)}{\varepsilon} = \pm m_{\Gamma,j} \mu_{\Gamma,j}.
\]

The proof of this result follows the same scheme as the one of Theorem 2.6 and lies in a technical result similar to Lemma 2.5. Note that Theorem 2.7 is an extension to the ergodic Hamiltonian context of part of the results of Moser and Pöschel [10] in the two-dimensional case. It is also possible to show that the required hypotheses are equivalent to the appearance of two closed real Lagrange bundles \( F_1 \) and \( F_2 \) with \( \Omega \times \mathbb{R}^{2n} = F_1 \oplus F_2 \). The hypotheses assumed on the initial Hamiltonian system do not suffice to guarantee this property. A more exhaustive analysis is possible in the two-dimensional case: this occurrence is equivalent to the topological decomposition of the projective flow in a family of closed minimal sets, given by copies of the base flow (see [12] and [14]). But to obtain a similar characterization in the higher dimension case is far away from trivial, as far as the topological structure has not been so far completely described.
Acknowledgement. The authors thank Professor R. Obaya for his help and useful comments.

REFERENCES

[1] L. Arnold, N.D. Cong, V.I. Oseledets, Jordan normal form for linear cocycles, *Random Oper. Stoch. Equ.* 7 (4) (1999), 303–358.
[2] R.H. Cameron, Almost periodic properties of bounded solutions of linear differential equations with almost periodic coefficients, *J. Math. Phys.* 15 (1936), 73–81.
[3] L.H. Eliasson, Floquet solutions for the 1-dimensional quasi-periodic Schrödinger equation, *Comm. Math. Phys.* 146 (1992), 447–482.
[4] R. Ellis, R. Johnson, Topological dynamics and linear differential systems, *J. Differential Equations* 44 (1982), 21–39.
[5] R. Fabbri, R. Johnson, C. Núñez, Rotation number for non-autonomous linear Hamiltonian systems II: The Floquet coefficient, *Z. Angew. Math. Phys.* 54 (2003), 652–676.
[6] H. Furstenberg, Strict ergodicity and transformations of the torus, *Amer. J. Math.* 85 (1963), 573–601.
[7] R. Johnson, m-functions and Floquet exponents for linear differential systems, *Ann. Mat. Pura Appl.* 147 (1987), 211–248.
[8] R. Johnson, S. Novo, R. Obaya, Ergodic properties and Weyl M-functions for linear Hamiltonian systems, *Proc. Roy. Soc. Edinburgh* 130A (2000), 1045–1079.
[9] R. Johnson, K. Palmer, G.R. Sell, Ergodic theory of linear dynamical systems, *SIAM J. Math. Anal.* 18 (1987), 1–33.
[10] J. Moser, J. Pöschel, An extension of a result by Dinaburg and Sinai on quasi-periodic potentials, *Comment. Helv. Math.* 59 (1984), 39–85.
[11] S. Novo, C. Núñez, R. Obaya, Ergodic properties and rotation number for linear Hamiltonian systems, *J. Differential Equations* 148 (1) (1998), 148–185.
[12] S. Novo, R. Obaya, An ergodic classification of bidimensional linear systems, *J. Dynam. Differential Equations* 8 (3) (1996), 373–406.
[13] S. Novo, R. Obaya, On the dynamical behaviour of an almost periodic linear system with an ergodic 2-sheet studied by R. Johnson, *Israel J. Math.* 105 (1998), 235–249.
[14] C. Núñez, R. Obaya, Non-tangential limit of the Weyl m-functions for the ergodic Schrödinger equation, *J. Dynam. Differential Equations* 10 (2) (1998), 209–257.
[15] R.J. Sacker, G.R. Sell, Dichotomies and invariant splittings for linear differential systems I, *J. Differential Equations* 15 (1974), 429–458.
[16] R.J. Sacker, G.R. Sell, A spectral theory for linear differential systems, *J. Differential Equations* 27 (1978), 320–358.
[17] S. Schwartzman, Asymptotic cycles, *Ann. of Math.* 66 (1957), 270–284.
[18] W. Shen, Y. Yi, Almost Automorphic and Almost Periodic Dynamics in Skew-Product Semiflows, *Mem. Amer. Math. Soc.* 647, Amer. Math. Soc., Providence, 1998.

Received August 2004; revised February 2005.

E-mail address, R. Fabbi: fabbi@dsi.unifi.it
E-mail address, C. Núñez: carnun@wmatem.eis.uva.es
E-mail address, A.M. Sanz: anasan@wmatem.eis.uva.es