EXTENDED DOUBLING OF SELF-COMPLEMENTARY STRONGLY REGULAR GRAPHS AND AN ANALOGUE FOR DIGRAPHS

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Dedicated to the memory of Igor Faradžev

Abstract. In this paper, we construct symmetric association schemes of class 3 from self-complementary strongly regular graphs, and non-symmetric association schemes of class 3 from self-complementary non-symmetric association schemes of class 2.

1. Introduction

An association scheme of class 2 is said to be pseudo-cyclic if its two nontrivial multiplicities coincide. The first eigenmatrix of a symmetric pseudo-cyclic association scheme of class 2 is given by

\[
\begin{bmatrix}
1 & \frac{m-1}{2} & \frac{m-1}{2} \\
1 & -\frac{1+\sqrt{m}}{2} & -\frac{1-\sqrt{m}}{2} \\
1 & -\frac{1-\sqrt{m}}{2} & -\frac{1+\sqrt{m}}{2}
\end{bmatrix},
\]

where \( m \equiv 1 \pmod{4} \) (see [1, Remarks (2) after Theorem 4.4]). The graphs defined by nontrivial relations of such an association scheme are known as conference graphs. See [8, Part V, §6] for more information.

A non-symmetric association scheme of class 2 is necessarily pseudo-cyclic, and its first eigenmatrix is given by

\[
\begin{bmatrix}
1 & \frac{m-1}{2} & \frac{m-1}{2} \\
1 & -\frac{1+\sqrt{-m}}{2} & -\frac{1-\sqrt{-m}}{2} \\
1 & -\frac{1-\sqrt{-m}}{2} & -\frac{1+\sqrt{-m}}{2}
\end{bmatrix},
\]

where \( m \equiv 3 \pmod{4} \) (see, for example [11]). The digraphs defined by a nontrivial relation of such an association scheme are known as doubly regular tournaments, and the existence of such a digraph is equivalent to that of a skew Hadamard matrix. See [9] for details.

Let \( \mathcal{X} = (X, \{R_i\}_{i=0}^2) \) be a (not necessarily symmetric) association scheme of class 2 on \( m \) points. We call \( \mathcal{X} \) self-complementary if the (di)graphs \( (X, R_1) \) and \( (X, R_2) \) are isomorphic. It is easily seen that a self-complementary association scheme of class 2 is

Date: August 14, 2021.
2010 Mathematics Subject Classification. 05E30, 05B30.
Key words and phrases. association scheme, conference graph, skew Hadamard matrix.
This work was supported by JSPS KAKENHI grant number 20K03527.
pseudo-cyclic. If $m$ is an odd prime power, a self-complementary association scheme of class 2 can be constructed by taking $X$ to be the finite field of order $m$. There are self-complementary strongly regular graphs whose number of points are not prime powers. Indeed, Mathon [10] constructed a self-complementary symmetric association scheme of class 2 on 45 points. According to the database [4] (see also [5]), there are a large number of non-symmetric association schemes of class 2 in general. Table 1 gives the number of non-symmetric association schemes of class 2 2 on $m$ points, and that of self-complementary non-symmetric association schemes of class 2 on $m$ points. The latter numbers were determined with the help of MAGMA [2].

| # points | 3 | 7 | 11 | 15 | 19 | 23 | 27 | 31 |
|----------|---|---|----|----|----|----|----|----|
| # self-complementary | 1 | 1 | 1 | 0 | 2 | 1 | 26 | 5 |
| # non-sym. class 2 | 1 | 1 | 1 | 1 | 2 | 19 | 374 | 98300 |

Table 1. The number of self-complementary association schemes among non-symmetric association schemes of class 2

In this paper, we show that a self-complementary association scheme with first eigenmatrix (1) and (2), gives rise to an association scheme of class 3 with first eigenmatrix

$\begin{bmatrix}
1 & 1 & m & m \\
1 & -1 & \sqrt{m} & -\sqrt{m} \\
1 & -1 & -\sqrt{m} & \sqrt{m} \\
1 & 1 & -1 & -1
\end{bmatrix}$,

and

$\begin{bmatrix}
1 & 1 & m & m \\
1 & -1 & \sqrt{-m} & -\sqrt{-m} \\
1 & -1 & -\sqrt{-m} & \sqrt{-m} \\
1 & 1 & -1 & -1
\end{bmatrix}$,

respectively.

A non-symmetric association scheme of class 3 with first eigenmatrix (1) is a non-symmetric fission of a cocktail party graph, and self-dual. According to S. Y. Song [12] (5.3) Lemma], a self-dual non-symmetric fission of a complete multipartite graph has the first eigenmatrix

$\begin{bmatrix}
1 & k_1 & k_2 & k_2 \\
1 & -1 & \sqrt{-k_2} & -\sqrt{-k_2} \\
1 & -1 & -\sqrt{-k_2} & \sqrt{-k_2} \\
1 & k_1 & -\frac{k_1+1}{2} & -\frac{k_1+1}{2}
\end{bmatrix}$,

and our association scheme with the first eigenmatrix (1) is a special case where $k_1 = 1$.

I. A. Faradžev, M. H. Klin, and M. E. Muzichuk [3, Theorem 2.6.6] showed that there exists a non-symmetric association scheme of class 2 on $2m + 1$ points with first
eigenmatrix:

\[
\begin{bmatrix}
1 & m & m \\
1 & -1+\sqrt{-(2m+1)} & -1-\sqrt{2(2m+1)} \\
1 & -1-\sqrt{2(2m+1)} & -1+\sqrt{-(2m+1)}
\end{bmatrix},
\]

provided that there exists an association scheme with first eigenmatrix (4). This construction leads to the doubling of skew-Hadamard matrices (see [9, Theorem 14]). Since our association scheme with first eigenmatrix (4) has \(2^m + 2\) points, it seems reasonable to call our construction extended doubling. One may wonder if an association scheme with first eigenmatrix (4) may be related to skew-Hadamard matrices. However, by [7], such an association scheme does not contain a (complex) Hadamard matrix in its Bose–Mesner algebra. Also, it can be shown by [6, Lemma 7] that an association scheme with first eigenmatrix (3) does not contain a (complex) Hadamard matrix in its Bose–Mesner algebra.

The organization of this paper is as follows. In Section 2 we introduce necessary notation and give useful properties of pseudo-cyclic association schemes of class 2. In Section 3 we construct the adjacency matrices needed in our main theorem, and prove their multiplication formulas. In Section 4 we prove our main theorem.

2. Preliminaries

For fundamentals of the theory of association schemes, we refer the reader to [11]. Let \(\mathfrak{X} = (X, \{R_i\}_{i=0}^d)\) be a commutative association scheme of class \(d\) on \(n\) points. Let \(A_0, A_1, \ldots, A_d\) be the adjacency matrices of \(\mathfrak{X}\). The intersection numbers \(p_{i,j}^\ell\) are defined by

\[
A_i A_j = \sum_{\ell=0}^d p_{i,j}^\ell A_\ell,
\]

and the intersection matrices \(\{B_i\}_{i=0}^d\) are defined by \(B_i = p_{i,i}^\ell\). The linear span \(\mathcal{A} = \langle A_0, A_1, \ldots, A_d \rangle\) is called the Bose–Mesner algebra of \(\mathfrak{X}\), and it has primitive idempotents \(E_0 = \frac{1}{|X|} J, E_1, \ldots, E_d\). The first eigenmatrix \(P = (P_{i,j})_{0 \leq i,j \leq d}\) is defined by

\[
(A_0, A_1, \ldots, A_d) = (E_0, E_1, \ldots, E_d) P,
\]

and \(Q = |X| P^{-1}\) is called the second eigenmatrix of \(\mathfrak{X}\). Then we have

\[
|X| (E_0, E_1, \ldots, E_d) = (A_0, A_1, \ldots, A_d) Q.
\]

Let \(k_i\) \((i = 0, 1, \ldots, d)\) and \(m_i\) \((i = 0, 1, \ldots, d)\) be the valencies and the multiplicities of \(\mathfrak{X}\), respectively. Then the intersection numbers \(p_{i,j}^\ell\) are given by

\[
p_{i,j}^\ell = \frac{1}{nk_\ell} \sum_{\nu=0}^d m_{\nu} P_{\nu,i} P_{\nu,j} P_{\nu,\ell}
\]

Now, assume that \(\mathfrak{X}\) is a pseudo-cyclic association scheme of class 2 on \(m\) points. Then \(k_1 = k_2 = m_1 = m_2 = (m-1)/2\). If \(\mathfrak{X}\) is symmetric, then by (1) and (6) we
have

(7) \( A_1^2 = \frac{m-1}{2} A_0 + \frac{m-5}{4} A_1 + \frac{m-1}{4} A_2, \)
(8) \( A_2^2 = \frac{m-1}{2} A_0 + \frac{m-1}{4} A_1 + \frac{m-5}{4} A_2, \)
(9) \( A_1 A_2 = \frac{m-1}{4} (A_1 + A_2). \)

If \( \mathcal{X} \) is non-symmetric, then by (2) and (6) we have

(10) \( A_1^2 = m - 3 A_1 + \frac{m+1}{4} A_2, \)
(11) \( A_2^2 = m + 3 A_1 + \frac{m-1}{4} A_2, \)
(12) \( A_1 A_2 = \frac{m-1}{2} A_0 + \frac{m-3}{4} (A_1 + A_2). \)

**Lemma 1.** Let \( \mathcal{X} \) be a (not necessarily symmetric) pseudo-cyclic association scheme of class 2 on \( m \) points. Then we have the following:

(13) \( \epsilon J + A_1^2 + A_2^2 = \epsilon m A_0 + \frac{m-1}{2} (A_1 + A_2), \)
(14) \( (1 - \epsilon) J + 2 A_1 A_2 = (1 - \epsilon) m A_0 + \frac{m-1}{2} (A_1 + A_2), \)

where \( \epsilon = 1 \) if \( \mathcal{X} \) is symmetric, and \( \epsilon = 0 \) otherwise.

**Proof.** If \( \mathcal{X} \) is symmetric, then by (7) and (8) we have

\( A_1^2 + A_2^2 = (m-1) A_0 + \frac{m-3}{2} (A_1 + A_2). \)

If \( \mathcal{X} \) is non-symmetric, then by (10) and (11) we have

\( A_1^2 + A_2^2 = \frac{m-1}{2} (A_1 + A_2). \)

We can combine these to write

\( A_1^2 + A_2^2 = \epsilon (m-1) A_0 + \frac{m-1-2\epsilon}{2} (A_1 + A_2). \)

Since \( J = A_0 + A_1 + A_2 \), we have (13).

By (9) and (12) we have

\( A_1 A_2 = (1 - \epsilon) \frac{m-1}{2} A_0 + \frac{m-1-2(1-\epsilon)}{4} (A_1 + A_2). \)

Since \( J = A_0 + A_1 + A_2 \), we have (14). \( \square \)

**Lemma 2.** Let \( \mathcal{X} \) be a self-complementary association scheme of class 2 on \( m \) points, and \( S \) be a permutation matrix such that \( A_2 = S A_1 S^T \). Then \( A_2 = S^T A_1 S. \)
Proof. Since $S^T A_2 S = A_1$, we have

\[
S^T A_1 S = S^T (J - A_0 - A_2) S \\
= J - A_0 - A_1 \\
= A_2,
\]

as desired. \qed

3. Construction of adjacency matrices

Definition 3. Let $\mathfrak{X}$ be a (not necessarily symmetric) self-complementary association scheme of class 2 on $m$ points, and denote its adjacency matrices by $A_0, A_1, A_2$. Let $S$ be a permutation matrix such that $A_2 = S A_1 S^T$. Define

\begin{align*}
C_0 &= I_{2(m+1)}, \\
C_1 &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0^T & 0^T & 0_m & S^T \\ 0^T & 0^T & S & 0_m \end{bmatrix}, \\
C_2 &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ (1-\epsilon)1^T & (1-\epsilon)1^T & A_1 & S^T A_1 \\ (1-\epsilon)1^T & \epsilon1^T & SA_2 & A_2 \end{bmatrix}, \\
C_3 &= J - C_0 - C_1 - C_2,
\end{align*}

where $1$ is the all-one row vector of length $m$, and $\epsilon$ is as in Lemma 1. We call the set $\{C_0, C_1, C_2, C_3\}$ of matrices the extended doubling of $\mathfrak{X}$.

The goal of this paper is to show that $\{C_0, C_1, C_2, C_3\}$ is the set of adjacency matrices of an association scheme of class 3.

By (15)–(18), we have

\begin{equation}
C_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ (1-\epsilon)1^T & \epsilon1^T & A_2 & S^T A_2 \end{bmatrix}.
\end{equation}

Remark 4. For comparison, we display the adjacency matrices of an association scheme with first eigenmatrix $[5]$ constructed in [3, Theorem 2.6.6]:

\[
\tilde{A}_0 = I_{2m+1}, \quad \tilde{A}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0^T & A_1 & A_2 + I \\ 1^T & A_2 & A_2 \end{bmatrix}, \quad \tilde{A}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1^T & A_2 & A_1 \\ 0^T & A_1 + I & A_1 \end{bmatrix},
\]

where $A_2^T = A_1$. 


We claim that, in the extended doubling of $X$, the choice of a permutation matrix $S$ is unimportant. Indeed, let $T$ be a permutation matrix such that $A_2 = TA_1T^T$. The extended doubling of $X$, where $S$ is replaced by $T$, is the set of matrices \( \{D_0, D_1, D_2, D_3\} \), where

\[
D_0 = I_{2(m+1)},
\]

\[
D_1 = \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & m & T^T & 0_m
\end{bmatrix},
\]

\[
D_2 = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
(1 - \epsilon)1^T & (1 - \epsilon)1^T & A_1 & T^TA_1 \\
\epsilon1^T & \epsilon1^T & TA_2 & A_2
\end{bmatrix},
\]

\[
D_3 = J - D_0 - D_1 - D_2.
\]

Define

\[
U = \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & m & T
\end{bmatrix}.
\]

Then we have $UC_0U^T = D_0$, $UC_1U^T = D_1$, and

\[
UC_2U^T = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
(1 - \epsilon)1^T & \epsilon1^T & SA_1S^T & A_1T^T \\
\epsilon1^T & (1 - \epsilon)1^T & TSA_2S^T & TA_2T^T
\end{bmatrix}
= D_3,
\]

(20)

and similarly $UC_3U^T = D_3$. Therefore, the extended doubling is uniquely defined up to simultaneous permutation of rows and columns.

**Proposition 5.** Let $\{C_0, C_1, C_2, C_3\}$ be the extended doubling of $X$. Then the (di)graphs defined by $C_2$ and $C_3$ are isomorphic.

**Proof.** Setting $S = T$ in the preceding discussion, we have $D_3 = C_3$, and hence $UC_2U^T = D_3 = C_3$. \( \square \)

The next lemma is necessary in the next section in order to establish our main result.
Lemma 6. We have the following:

\[(21)\]
\[C_1^2 = C_0,\]
\[(22)\]
\[C_1 C_2 = C_2 C_1 = C_3,\]
\[(23)\]
\[C_1 C_3 = C_3 C_1 = C_2,\]
\[(24)\]
\[C_2^2 = C_3^2 = mC_{1-\epsilon} + \frac{m-1}{2}(C_2 + C_3),\]
\[(25)\]
\[C_2 C_3 = C_3 C_2 = mC_\epsilon + \frac{m-1}{2}(C_2 + C_3).\]

Proof. We can easily see that (21) and (22) hold. Then (23) follows immediately from (21) and (22).

Since
\[(C_2^2)_{1,1} = (C_2^2)_{2,2} = \epsilon m,\]
\[(C_2^2)_{1,2} = (C_2^2)_{2,1} = (1-\epsilon)m,\]
\[(C_2^2)_{1,3} = (C_2^2)_{1,4} = (C_2^2)_{2,3} = (C_2^2)_{2,4} = \frac{m-1}{2}1,\]
\[(C_2^2)_{3,1} = (C_2^2)_{3,2} = (C_2^2)_{4,1} = (C_2^2)_{4,2} = \frac{m-1}{2}1^\top,\]
\[(C_2^2)_{3,3} = (C_2^2)_{4,4} = \epsilon J + A_1^2 + A_2^2\]
\[= \epsilon m A_0 + \frac{m-1}{2}(A_1 + A_2)\] (by Lemma 2),
\[(C_2^2)_{3,4} = S^\top((1-\epsilon)J + 2A_1 A_2)\]
\[= (1-\epsilon)m S^\top + \frac{m-1}{2}S^\top(A_1 + A_2)\] (by (13)),
\[(C_2^2)_{4,3} = S((1-\epsilon)J + 2A_1 A_2)\]
\[= (1-\epsilon)m S + \frac{m-1}{2}S(A_1 + A_2)\] (by (14)),
we have \(C_2^2 = mC_{1-\epsilon} + \frac{m-1}{2}(C_2 + C_3)\). Similarly, we have \(C_3^2 = mC_\epsilon + \frac{m-1}{2}(C_2 + C_3)\). Finally, (25) follows from (21)–(24). \(\square\)

4. Main results

Theorem 7. Let \(\mathcal{C} = \{C_0, C_1, C_2, C_3\}\) be the extended doubling of a self-complementary association scheme \(\mathcal{X}\) of class 2.

(i) If \(\mathcal{X}\) is symmetric, then \(\mathcal{C}\) is the set of adjacency matrices of an association scheme with first eigenmatrix (3).

(ii) If \(\mathcal{X}\) is non-symmetric, then \(\mathcal{C}\) is the set of adjacency matrices of an association scheme with first eigenmatrix (4).

Proof. Suppose \(\mathcal{X}\) is an association scheme on \(m\) points. Let \(\mathcal{A} = \{C_0, C_1, C_2, C_3\}\) be the linear span of the extended doubling \(\mathcal{C}\). First we observe that \(\mathcal{A}\) is closed under
multiplication by Lemma 6. Thus \( A \) is the Bose–Mesner algebra of an association scheme \( \tilde{X} \) of class 3.

Secondly we compute the first eigenmatrix of \( \tilde{X} \). Let \( r = \sqrt{m} \), \( \epsilon = 1 \) if \( \tilde{X} \) is symmetric, and \( r = \sqrt{-m} \), \( \epsilon = 0 \) otherwise. By Lemma 6 the intersection matrices \( B_1, B_2, B_3 \) of \( \tilde{X} \) are given by

\[
B_1 = \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
\end{bmatrix},
\]

\[
B_2 = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
em & (1 - \epsilon)m & \frac{m-1}{2} & \frac{m-1}{2} \\
(1 - \epsilon)m & em & \frac{m-1}{2} & \frac{m-1}{2} \\
\end{bmatrix},
\]

\[
B_3 = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
(1 - \epsilon)m & em & \frac{m-1}{2} & \frac{m-1}{2} \\
em & (1 - \epsilon)m & \frac{m-1}{2} & \frac{m-1}{2} \\
\end{bmatrix}.
\]

Let

\[
E_0 = \frac{1}{2(m+1)}(C_0 + C_1 + C_2 + C_3),
\]

\[
E_1 = \frac{1}{4}(C_0 - C_1 + \frac{1}{r}(C_2 - C_3)),
\]

\[
E_2 = \frac{1}{4}(C_0 - C_1 - \frac{1}{r}(C_2 - C_3)),
\]

\[
E_3 = \frac{1}{2(m+1)}(m(C_0 + C_1) - (C_2 + C_3)).
\]

Then by (26)–(28) we have \( E_i E_j = \delta_{i,j} E_i \). By (29)–(32) the second eigenmatrix of \( \tilde{X} \) is given by

\[
\begin{bmatrix}
1 & \frac{m+1}{2} & \frac{m+1}{2} & m \\
1 & -\frac{m+1}{2} & -\frac{m+1}{2} & m \\
1 & \frac{m+1}{2r} & -\frac{m+1}{2r} & -1 \\
1 & -\frac{m+1}{2r} & \frac{m+1}{2r} & -1 \\
\end{bmatrix}.
\]

Then the first eigenmatrix of \( \tilde{X} \) is given by (3) if \( \tilde{X} \) is symmetric, and (4) otherwise. \( \square \)

**Remark 8.** In the database of [4], as12[10], as20[10], and as28[13] can be constructed by (i) in Theorem 7, and as08[6], as16[11], and as24[14] can be constructed by (ii) in Theorem 7.

**Acknowledgements.** We would like to thank Ferenc Szöllősi for providing us with electronic data of self-complementary strongly regular graphs on 45 points.
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