RIGIDITY RESULTS ON $\rho$–EINSTEIN SOLITONS WITH ZERO SCALAR CURVATURE

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ABSTRACT. In this paper we show that a $\rho$-Einstein solitons conformal to a pseudo-Euclidean space, invariant under the action of the pseudo-orthogonal group with zero scalar curvature is steady and consequently flat. How application of the results obtained we present an explicit example for the question proposed by Kazdan in [17].

1. INTRODUCTION AND MAIN STATEMENTS

In this paper, we study two related problems. The first problem is on the existence of $\rho$–Einstein solitons with scalar curvature $K_\bar{g} = 0$. Besides that we present some rigity results.

The second problem consists in find all metrics that are conformal to the pseudo Euclidean metrics, with zero scalar curvature, which are invariant under the action of the pseudo-orthogonal group. This provides explicit solutions to Yamabe’s problem in the non-compact case. In the Riemannian case under some additional assumptions, all metrics obtained are complete. As application of this results we obtain a family of complete metrics in $\mathbb{R}^n \setminus \{0\}$ with scalar curvature positive, negative and zero, presenting an explicit example for a question proposed by Kazdan in [17].

In 1982, R. Hamilton introduced a nonlinear evolution equation for Riemannian metrics with the aim of finding canonical metrics on manifolds (see [1] or [16]). This evolution equation is known as the Ricci flow, and it has since been used widely and with great success, most notably in Perelman’s solution of the Poincaré conjecture. Furthermore, several convergence theorems have been established. One important aspect in the treatment of the Ricci flow is the study of Ricci solitons, which generate self-similar solutions to the flow and often arise as singularity models.

Given a semi-Riemannian manifold $(M^n, g)$, $n \geq 3$, we say that $(M, g)$ is a gradient Ricci soliton if there exists a differentiable function $h : M \rightarrow \mathbb{R}$ (called the potential function) such that

$$\text{Ric}_g + \text{Hess}_g(h) = \lambda g,$$

where $\text{Ric}_g$ is the Ricci tensor, $\text{Hess}_g(h)$ is the Hessian of $h$ with respect to the metric $g$, and $\lambda$ is a real number. We say that a gradient Ricci soliton is shrinking, steady, or expanding if $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively. Bryant [10] proved that there exists a complete, steady, gradient Ricci soliton that is spherically symmetric for any $n \geq 3$, which is known as Bryant’s soliton. In the bi-dimensional case an

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analogous nontrivial rotationally symmetric solution was obtained explicitly, and is known as the Hamilton cigar. Recently Cao-Chen [11] showed that any complete, steady, gradient Ricci soliton, locally conformally flat, up to homothety, is either flat or isometric to the Bryant’s soliton. The results obtained in [11] were extended to both flat gradient steady Ricci solitons (see [4]). Complete, conformally flat shrinking gradient solitons have been characterized as being quotients of $\mathbb{R}^n$, $\mathbb{S}^n$ or $\mathbb{R} \times \mathbb{S}^{n-1}$ (see [13]). In the case of steady gradient Ricci solitons, [2] provides all such solutions when the metric is conformal to an $n$-dimensional pseudo-Euclidean space and invariant under the action of an $(n-1)$-dimensional translation group.

Motivated by the notion of Ricci solitons on a semi-Riemannian manifold $(M^n, g)$, $n \geq 3$, it is natural to consider geometric flows of the following type:

\[
\frac{\partial}{\partial t} g(t) = -2(\text{Ric} - \rho R g)
\]

for $\rho \in \mathbb{R}$, $\rho \neq 0$, as in [3] and [6]. We call these the Ricci-Bourguignon flows. We notice that short time existence for the geometric flows described in (1.2) is provided in ([5]). Associated to the flows, we have the following notion of gradient $\rho$-Einstein solitons, which generate self-similar solutions:

**Definition 1.1.** Let $(M^n, g), n \geq 3$, be a Riemannian manifold and let $\rho \in \mathbb{R}, \rho \neq 0$. We say that $(M^n, g)$ is a gradient $\rho$-Einstein soliton if there exists a smooth function $h : M \rightarrow \mathbb{R}$, such that the metric $g$ satisfies the equations

\[
\text{Ric}_g + \text{Hess}_g h = \rho K_g g + \lambda g
\]

for some constant $\lambda \in \mathbb{R}$, where $K_g$ is the scalar curvature of the metric $g$.

A $\rho$-Einstein soliton is said to be shrinking, steady, or expanding if $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$, respectively. Furthermore, a $\rho$-Einstein solitons is said to be a gradient Einstein soliton, gradient traceless Ricci soliton, and gradient Schouten soliton if $\rho = \frac{1}{2}$, $\rho = \frac{1}{n}$, and $\rho = \frac{1}{2(n-1)}$, respectively.

The gradient $\rho$–Einstein solitons equation (1.3) links geometric information about the curvature of the manifold through the Ricci tensor and the geometry of the level sets of the potential function by means of their second fundamental form. Hence, classifying gradient $\rho$–Einstein solitons under some curvature conditions is a natural problem. The $\rho$–Einstein solitons were investigated by Catino and Mazzieri in [6], they obtained important rigidity results, proving that every compact gradient Einstein, Schouten, or traceless Ricci soliton is trivial. In addition, they proved that every complete gradient steady Schouten soliton is trivial, hence Ricci flat.

Gradient Ricci solitons with constant scalar curvature were investigated by Petersen and Wylie in [15], they proved that: If a non-steady gradient Ricci soliton has constant scalar curvature $K_g$, then it is bounded as $0 \leq K_g \leq n \lambda$ in the shrinking case, and $n \lambda \leq K_g \leq 0$ in the expanding case. Fernández-López and García-Río in [8] improved this result proving that: If an $n$–dimensional complete gradient Ricci soliton with constant scalar curvature $K_g$, then $K_g$ must be a multiple of $\lambda$.

In [9] the authors considered a $\rho$–Einstein solitons that are conformal to a pseudo-Euclidean space and invariant under the action of the pseudo-orthogonal group. They provide all the solutions for the gradient Schouten soliton case. Moreover, they proved that if a gradient Schouten soliton is both complete, conformal to a Euclidean metric, and rotationally symmetric, then it is isometric to $\mathbb{R} \times S^{n-1}$. 
In [12] the authors used the variational method to study the existence problem of metrics with constant scalar curvature on complete non-compact Riemannian manifolds. The assumptions of results are motivated from question in the work of Kazdan [17]. The question is that if M has complete metrics \( g_+ \) and \( g_- \) with positive (respectively negative) scalar curvature, is there one with zero scalar curvature? With several additional hypotheses the author provide an answer to the question posed by Kazdan, more details see [14].

We studied the equation (1.3) in semi-Riemannian manifolds with scalar curvature constant. We consider gradient \( \rho \)-Einstein solitons conformal to a pseudo-Euclidean space, which are invariant under the action of the pseudo-orthogonal group. As a consequence of the Theorem 1.4, we obtain the zero (Theorem 1.7, Corollary 1.8 and Corollary 1.9).

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In the Proposition 1.1 we construct a family of complete metrics with zero scalar curvature. In the Corollary 1.12 we construct an explicit example for Kazdan’s question.

In what follows, we state our main results. We denote the second order derivative of \( \psi \) and \( h \) by \( \psi_{,i}x_j \) and \( h_{,i}x_j \), respectively, with respect to \( x_i x_j \).

**Theorem 1.2.** Let \((\mathbb{R}^n,g), n \geq 3\), be a pseudo-Euclidean space with coordinates \( x = (x_1,...,x_n) \) and metric components \( g_{ij} = \delta_{ij}\varepsilon_i \), \( 1 \leq i, j \leq n \), where \( \varepsilon_i = \pm 1 \). Consider a smooth function \( h : \mathbb{R}^n \rightarrow \mathbb{R} \). There exists a metric \( \tilde{g} = \frac{1}{\psi^2} g \) such that \((\mathbb{R}^n,\tilde{g})\) is a gradient \( \rho \)-Einstein soliton with \( h \) as a potential function if, and only if, the functions \( \psi \), and \( h \) satisfy

\[
(n - 2)\psi_{,i}x_j + \psi h_{,i}x_j + \psi_{,i}x_j h_{,j} + \psi_{,j}x_i h_{,i} = 0, \quad i \neq j,
\]

and

\[
\psi \left[ (n - 2) \psi_{,i}x_i + \psi h_{,i}x_i + 2\psi_{,i}x_i h_{,i} \right] + \varepsilon_i \sum_{k=1}^n \varepsilon_k \left[ (n - 1) \left( \rho \psi_{,k}^2 - 2\rho\psi_{,k}x_k - \psi_{,k}^2 \right) - \psi_{,k}x_k h_{,i} + \psi_{,k}x_k x_k \right] = \lambda \varepsilon_i , \quad i = j.
\]
Our objective is to determine solutions of the system (1.4), (1.5) of the form \( \psi(r) \) and \( h(r) \), where \( r = \sum_{k=1}^{n} \varepsilon_k x_k^2 \). The following theorem reduces the system of partial differential equations (1.4) and (1.5) into an system of ordinary differential equations.

**Corollary 1.3.** Let \((\mathbb{R}^n, g), n \geq 3\), be a pseudo-Euclidean space with coordinates \( x = (x_1, \ldots, x_n) \) and metric components \( g_{ij} = \delta_{ij} \varepsilon_i \), \( 1 \leq i, j \leq n \), where \( \varepsilon_i = \pm 1 \). Consider smooth functions \( \psi(r) \) and \( h(r) \) with \( r = \sum_{k=1}^{n} \varepsilon_k x_k^2 \). Then there exists a metric \( \bar{g} = \frac{1}{\psi(r)} g \) such that \((\mathbb{R}^n, \bar{g})\) is a gradient \( \rho \)-Einstein soliton with \( h \) as a potential function if, and only if, the functions \( \psi \) and \( h \) satisfy

\[
(n-2)\psi'' + \psi h'' + 2\psi'h' = 0, \tag{1.6}
\]

and

\[
2\psi \left[ (n-2)\psi' + \psi h' \right] + 2n\left[ 1 - 2(n-1)\rho \right] \psi\psi' \\
+ 4r \left\{ (n-1) \left[ (\rho n-1)(\psi')^2 - 2\rho \psi'' \right] - \psi\psi' + \psi'' \right\} = \lambda. \tag{1.7}
\]

The next we found all metrics that are conformal to the pseudo Euclidean metrics, with zero scalar curvature, which are invariant under the action of the pseudo-orthogonal group.

**Theorem 1.4.** Let \((\mathbb{R}^n, g)\) be a pseudo-Euclidean space, \( n \geq 3 \), with coordinates \( x = (x_1, \ldots, x_n) \) and \( g_{ij} = \delta_{ij} \varepsilon_i \), \( 1 \leq i, j \leq n \), where \( \varepsilon_i = \pm 1 \). Consider \( \bar{g} = \frac{1}{\psi(r)} g \) where \( r = \sum_{k=1}^{n} \varepsilon_k x_k^2 \). Then \( \bar{g} \) have scalar curvature \( K_{\bar{g}} = 0 \), if and only if

\[
\psi(r) = \frac{k_2 r}{\left( 1 + Ar^{\frac{n-2}{2}} \right)^{\frac{n}{n-2}}}, \tag{1.8}
\]

where \( A, k_2 \in \mathbb{R} \) with \( k_2 > 0 \). If \( A \geq 0 \) the metric \( \bar{g} \) is defined in \( \mathbb{R}^n \setminus \{0\} \). If \( A < 0 \) the set of singularity points of \( \bar{g} \) consist of the origin and a sphere \((n-1)-\)dimensional, with center at the origin and radius \( R = \sqrt{\left( \frac{1}{A} \right)^{\frac{2}{n-2}}} \).

**Remark 1.5.** If \((\mathbb{R}^n, g)\) is the Euclidean space, then we find in the Theorem 1.3 all metrics conformal to \( g \) and spherically symmetrical with zero scalar curvature. This provides explicit solutions to Yamabe’s problem in the non-compact case.

In \cite{7}, the authors showed that \( \{\mathbb{R}^n \setminus \{0\}, \tilde{g} = \frac{1}{r^2} g_0, \varphi(r) = \sqrt{r} \} \) is a complete Riemannian manifold and isometric at \( S^{n-1} \times \mathbb{R} \). As a consequence of Theorem 1.3 together with this fact, we obtain the following result:

**Corollary 1.6.** Let \((\mathbb{R}^n, g)\) be a Euclidean space, \( n \geq 3 \), with coordinates \( x = (x_1, \ldots, x_n) \) and \((g_0)_{ij} = \delta_{ij} \), \( 1 \leq i, j \leq n \). Consider \( \tilde{g} = \frac{1}{\varphi(r)^2} g_0 \) where \( r = \sum_{k=1}^{n} x_k^2 \). The metrics obtained in the Theorem 1.4 are complete whenever \( A > 0 \).

As an consequence of the Theorem 1.4 we get the following rigidity results.
Theorem 1.7. Let \((\mathbb{R}^n, g)\) be a pseudo-Euclidean space, \(n \geq 3\), with coordinates \(x = (x_1, \cdots, x_n)\) and \(g_{ij} = \delta_{ij} \varepsilon_i\), \(1 \leq i, j \leq n\), where \(\varepsilon_i = \pm 1\). Consider \((\mathbb{R}^n, \bar{g})\), \(\bar{g} = \frac{1}{\psi^2} g\) a \(\rho\)-Einstein soliton with scalar curvature \(K_{\bar{g}} = 0\), where \(\psi(r)\) and \(h(r)\) are smooth functions, \(r = \sum_{k=1}^{n} \varepsilon_k x_k^2\) and \(h\) as a potential function. Then \(\lambda = 0\), that is \((\mathbb{R}^n, \bar{g})\) is steady.

Corollary 1.8. Let \((\mathbb{R}^n, g)\) be a pseudo-Euclidean space, \(n \geq 3\), with coordinates \(x = (x_1, \cdots, x_n)\) and \(g_{ij} = \delta_{ij} \varepsilon_i\), \(1 \leq i, j \leq n\), where \(\varepsilon_i = \pm 1\). Then \((\mathbb{R}^n, \bar{g})\), \(\bar{g} = \frac{1}{\psi^2} g\) is a \(\rho\)-Einstein soliton steady with scalar curvature \(K_{\bar{g}} = 0\), where \(\psi(r)\) and \(h(r)\) are smooth functions, \(r = \sum_{k=1}^{n} \varepsilon_k x_k^2\) and \(h\) as a potential function, if and only if, \((\mathbb{R}^n, \bar{g})\) is flat.

As a consequence of the previous results, we have the following result in the Riemannian case.

Corollary 1.9. Let \((M^n, \bar{g})\) be \(n \geq 3\) a \(\rho\)-Einstein soliton, Riemannian, locally conformally flat and rotationally symmetric with zero scalar curvature. Then \((M^n, \bar{g})\) is necessarily steady. Besides that \((M^n, \bar{g})\) is flat.

Remark 1.10. These results hold for \(\rho = 0\) and therefore they are extended to the Ricci solitons gradients, proving that a Ricci soliton gradient, conformal to the Euclidean space and spherically symmetrical with zero scalar curvature is necessarily steady and consequently flat.

Remark 1.11. As a consequence of the results obtained let’s make an application giving a positive answer for a question proposed by Kazdan in [17], as follows:

If \(M\) has complete metrics \(g_+\) and \(g_-\) with positive (respectively, negative) scalar curvature, is there one with zero scalar curvature? Kazdan showed in [18], that no, no case compacto the answer is "yes".

We built in \((\mathbb{R}^n \setminus \{0\})\) complete metrics with positive, negative and zero scalar curvature, respectively.

Proposition 1.1. Let \((\mathbb{R}^n, g)\) be a Euclidean space, \(n \geq 3\), with coordinates \(x = (x_1, \cdots, x_n)\) and \((g_0)_{ij} = \delta_{ij}\), \(1 \leq i, j \leq n\). Consider \(g = \frac{1}{\varphi(r)^2} g_0\) where \(r = \sum_{k=1}^{n} x_k^2\). If \(\varphi(r) = re^{-\left(1 + \frac{n-2}{n-2}\right)\frac{\varphi}{\psi}}\), then the metric \(g\) on \(\mathbb{R}^n\) is complete with scalar curvature negative given by

\[
K_g = h(r) \left[ (n-2) \left(1 + r \frac{\varphi}{\psi}\right)^{\frac{n-2}{n-2}} + 2(n-1) \frac{\varphi}{\psi} + 2(n-1) \right],
\]

where \(h(r) = \frac{4(n-1)e^{\frac{1}{2}}}{e^{\frac{1}{2}} \left(1 + \frac{n-2}{n-2}\right)^{\frac{n-2}{n-2}} \frac{\varphi}{\psi}}\).

In the next result we construct an explicit example in Riemannian manifolds for the question left by Kazdan [17].
Corollary 1.12. Note that \( \{ \mathbb{R}^n \setminus \{ 0 \}, \tilde{g} = \frac{1}{\varphi(r)} g_0, \varphi(r) = \sqrt{r} \} \) is a complete Riemannian manifold with scalar curvature positive and \( \{ \mathbb{R}^n \setminus \{ 0 \}, \tilde{g} = \frac{1}{\varphi_1^2} g_0, \varphi_1(r) = re^{-\left(1+r^{\frac{2}{m-2}}\right)^{-\frac{1}{2}}} \} \) is a complete Riemannian manifold with scalar curvature negative, exists a complete metric of scalar curvature zero.

2. Proofs of the main results

Proof. Proof of Theorem 1.2. It is well known (see, e.g., [2]) that if \( \tilde{g} = \frac{1}{\varphi^2} g \), then

\[
\text{Ric}_{\tilde{g}} = \frac{1}{\psi^2} \{(n - 2)\psi H_{\psi} + [\psi \Delta_{\psi} - (n - 1)|\nabla_{\psi}|^2]g\}
\]

and

\[
\tilde{R} = (n - 1) (2\psi \Delta_{\psi} - n|\nabla_{\psi}|^2).
\]

Hence, the equation

\[
\text{Ric}_{\tilde{g}} + H_{\psi}(h) = \rho \tilde{R}_{\tilde{g}} + \lambda \tilde{g},
\]

is equivalent to

\[
(2.1) \quad \frac{1}{\psi^2} \{(n - 2)\psi H_{\psi} + [\psi \Delta_{\psi} - (n - 1)|\nabla_{\psi}|^2]g\}
\]

and

\[
Hess_{\tilde{g}}(h)_{ij} = h_{,x_i x_j} - \sum_{k=1}^{n} \tilde{\Gamma}^k_{ij} h_{,x_k}
\]

where \( \tilde{\Gamma}^k_{ij} \) are the Christoffel symbols of the metric \( \tilde{g} \). For a distinct \( i, j, k \), we have

\[
\tilde{\Gamma}^k_{ij} = 0, \quad \tilde{\Gamma}^i_{ij} = -\frac{\psi_{,x_i}}{\psi}, \quad \tilde{\Gamma}^k_{ii} = \varepsilon_k \frac{\psi_{,x_k}}{\psi}, \quad \tilde{\Gamma}^i_{ii} = -\frac{\psi_{,x_i}}{\psi},
\]

therefore,

\[
(2.2) \quad Hess_{\tilde{g}}(h)_{ij} = h_{,x_i x_j} + \frac{\psi_{,x_i} h_{,x_j}}{\psi} + \frac{\psi_{,x_j} h_{,x_i}}{\psi} + \sum_{k=1}^{n} \varepsilon_k \frac{\psi_{,x_k} h_{,x_k}}{\psi}, \quad i \neq j.
\]

Similarly, by considering \( i = j \), we have

\[
(2.3) \quad Hess_{\tilde{g}}(h)_{ii} = h_{,x_i x_i} + \frac{2\psi_{,x_i} h_{,x_i}}{\psi} + \varepsilon_i \sum_{k=1}^{n} \varepsilon_k \frac{\psi_{,x_k} h_{,x_k}}{\psi}.
\]

However, we note that

\[
(2.4) \quad |\nabla_{\psi}|^2 = \sum_{k=1}^{n} \varepsilon_k \left( \frac{\partial \psi}{\partial x_k} \right)^2, \quad \Delta_{\psi} = \sum_{k=1}^{n} \varepsilon_k \psi_{,x_k x_k}, \quad Hess_{\psi}(\psi)_{ij} = \psi_{,x_i x_j}.
\]

If \( i \neq j \) in (2.1), we obtain

\[
(2.5) \quad (n - 2) \frac{Hess_{\psi}(\psi)_{ij}}{\psi} + Hess_{\tilde{g}}(h)_{ij} = 0.
\]
Substituting the expressions found in (2.2), and (2.4) into (2.5), we obtain

\[(n-2)\psi_{,i,x_i} + \psi_{,i,x_j} + \psi_{,i,x_j} + \psi_{,i,x_j}h_{,x_i} = 0, \quad i \neq j.\]

Similarly, if \(i = j\) in (2.4), we have

\[(2.6) \quad (n-2)\psi H_{\xi_{ii}}(\psi)_{,i} + \psi\Delta_g \psi \varepsilon_i - (n-1)|\nabla_g \psi|^2 \varepsilon_i + \psi^2 H_{\xi_{ii}}(h)_{,i} = 2(n-1)\rho \Delta_g \psi \varepsilon_i - n(n-1)\rho|\nabla_g \psi|^2 \varepsilon_i + \lambda \varepsilon_i.

Substituting the expressions found in (2.3), and (2.4) into (2.6), we obtain

\[\psi \left[(n-2) \psi_{,i,x_i} + \psi h_{,x_i,x_i} + 2\psi_{,i,x_i}\right]
+ \varepsilon_i \sum_{k=1}^n \varepsilon_k \left[(n-1) \left(\rho n \psi_{,x_k}^2 - 2\rho \psi \psi_{,x_k} - \psi_x^2 \right) - \psi \psi_{,x_k} h_{,x_k} + \psi \psi_{,x_k,x_k} \right] = \lambda \varepsilon_i.

This concludes the proof of Theorem 1.2. \qed

Proof. Proof of Corollary 1.3. Let \(\tilde{g} = \psi^{-2}g\) be a conformal metric of \(g\). We are assuming that \(\psi(r)\) and \(h(r)\) are functions of \(r\), where \(r = \sum_{k=1}^n \varepsilon_k x_k^2\). Hence, we have

\[
\psi_{,x_i} = 2\varepsilon_i \psi', \quad \psi_{,x_i,x_i} = 4x_i^2 \psi'' + 2\varepsilon_i \psi', \quad \psi_{,x_i,x_j} = 4\varepsilon_i \varepsilon_j x_i x_j \psi''
\]
and

\[
h_{,x_i} = 2\varepsilon_i h', \quad h_{,x_i,x_i} = 4x_i^2 h'' + 2\varepsilon_i h', \quad h_{,x_i,x_j} = 4\varepsilon_i \varepsilon_j x_i x_j h''.
\]

Substituting these expressions into (1.4), we obtain

\[
4\varepsilon_i \varepsilon_j \left[(n-2)\psi'' + \psi h'' + 2\psi' h' \right] x_i x_j = 0.
\]

Since there exist \(i \neq j\), such that \(x_i x_j \neq 0\), we have

\[
(n-2)\psi'' + \psi h'' + 2\psi' h' = 0.
\]

Similarly, considering the equation (1.3), we obtain

\[
4\psi \left[(n-2)\psi'' + \psi h'' + 2\psi' h' \right] x_i^2 + 2\psi \left[(n-2)\psi' + \psi h' \right] \varepsilon_i + 2\varepsilon_i n[1 - 2(n-1)\rho] \psi \psi' + 4\varepsilon_i \sum_{k=1}^n \varepsilon_k x_k^2 \left[(n-1) \left(\rho n - 1 \right) \left(\psi' \right)^2 - 2\rho \psi \psi'' \right] - \psi' h' + \psi h'' \varepsilon_i = \lambda \varepsilon_i.
\]

Note that \((n-2)\psi'' + \psi h'' + 2\psi' h' = 0\) and \(r = \sum_{k=1}^n \varepsilon_k x_k^2\). Therefore, we obtain

\[
2\psi \left[(n-2) \psi' + \psi h' \right] + 2n[1 - 2(n-1)\rho] \psi \psi' + 4r \left[(n-1) \left(\rho n - 1 \right) \left(\psi' \right)^2 - 2\rho \psi \psi'' \right] - \psi' h' + \psi'' h'' = \lambda.
\]
This concludes the proof of Corollary 1.3.

Proof. Proof of the Theorem 1.4 It is well known (see, e.g., [2] or [9]) that if \( \bar{g} = g \psi \), then

\[
K_{\bar{g}} = (n - 1) \left( 2\psi \Delta_g \psi - n|\nabla g \psi|^2 \right).
\]

How we are assuming that \( \psi(r) \) is a functions of \( r \), where \( r = \sum_{k=1}^{n} \varepsilon_k x_k^2 \), then we have that \( K_{\bar{g}} = 0 \) if, and only, if

\[
-n\psi\psi' - 2r\psi\psi'' + nr(\psi')^2 = 0,
\]

which is equivalent to

\[
-\frac{n}{2r} \psi' + n \left( \frac{\psi'}{\psi} \right)^2 - \frac{\psi''}{\psi} = 0.
\]

By equality \( \frac{\psi''}{\psi} = \left( \frac{\psi'}{\psi} \right)' + \left( \frac{\psi'}{\psi} \right)^2 \), follows that

\[
-\frac{n}{2r} \psi' + n \left( \frac{\psi'}{\psi} \right)^2 - \left( \frac{\psi'}{\psi} \right)' - \left( \frac{\psi'}{\psi} \right)^2 = 0.
\]

Taking \( y = \frac{\psi'}{\psi} \), the previous equation becomes

\[
(2.7) \quad y' = -\frac{n}{2r} y + \frac{n - 2}{2} y^2.
\]

Note that equation \( (2.7) \) is an ordinary differential equation of Bernoulli. Therefore, you can determine all your solutions, whose general solution is given by

\[
(2.8) \quad y^{-1} = Ce^{F} - \frac{(n - 2)}{2} e^{F} \int e^{-F} dr, \quad \text{where} \quad F(r) = \frac{n}{2} \int \frac{1}{r} dr = \ln r^2, \quad C \text{ is an arbitrary constant (for more details see [19]). Thus}
\]

\[
y^{-1} = Cr^2 - \frac{(n - 2)}{2} r^2 \int r^{-2} dr
\]

equivalently,

\[
y^{-1} = \left( C - \frac{(n - 2)}{2} k_1 \right) r^2 + r,
\]

where \( k_1 \) is a real number. Implies that

\[
y^{-1} = Ar^2 + r,
\]

where \( A = C - \frac{(n - 2)}{2} k_1 \). It follow that

\[
y = \frac{r^2}{A + r^2}.
\]

how \( y = \frac{\psi'}{\psi} \), we get

\[
(2.9) \quad \psi(r) = \exp \left\{ \int \frac{r^2}{A + r^2} dr + \ln k_2 \right\}
\]
where \( k_2 \in \mathbb{R}_+^* \). Note that

\[
\int \frac{r^{-\frac{n}{2}}}{A + r^{\frac{n}{2}}} dr = \ln \left( A + r^{\frac{n}{2}} \right)^{\frac{2}{n-2}},
\]

combining the equations (2.9) and (2.10), the following that

\[
\psi(r) = k_2 B^{\frac{2}{n-2}},
\]

where \( B = A + r^{\frac{n}{2}} \). How \( n \geq 3 \) we obtain that

\[
\psi(r) = k_2 r \left( 1 + Ar^{\frac{n-2}{2}} \right)^{\frac{2}{n-2}}.
\]

**Proof.** Proof of the Corollary \[1.6\] If \( K_\bar{g} = 0 \), by Theorem \[1.4\] we get

\[
\psi(r) = k_2 \left( 1 + Ar^{\frac{n-2}{2}} \right)^{\frac{2}{n-2}}.
\]

We will show that \( \bar{g} = g_{0\psi}^2 \) is complete.

Consider the manifolds \( M = (\mathbb{R}^n \setminus \{0\}, \bar{g} = \frac{g_0}{\psi^2}) \), where \( \psi(r) = \frac{k_2 r}{(1 + Ar^{\frac{n-2}{2}})^{\frac{2}{n-2}}} \), and \( N = (\mathbb{R}^n \setminus \{0\}, g = \frac{g_0}{\phi^2}) \), where \( \phi(r) = \sqrt{r} \), and \( g_0 \) is a Euclidean metric. Note that

\[
|v|_\bar{g} = \frac{1}{\psi} |v|_{g_0} \quad \text{and} \quad |v|_g = \frac{1}{\varphi} |v|_{g_0},
\]

By other hand, we get

\[
|v|_\bar{g} = \frac{(1 + Ar^{\frac{n-2}{2}})^{\frac{2}{n-2}}}{k_2 r} |v|_{g_0} = \frac{(1 + Ar^{\frac{n-2}{2}})^{\frac{2}{n-2}}}{k_2} \frac{1}{\sqrt{r}} |v|_{g_0},
\]

thus,

\[
|v|_\bar{g} = f(r)|v|_g,
\]

where \( f(r) = \frac{(1 + Ar^{\frac{n-2}{2}})^{\frac{2}{n-2}}}{k_2 \sqrt{r}} \).

To find \( c > 0 \) such that \( |v|_\bar{g} \geq c|v|_g \), just solve the following problem

\[
\min_{r \in \mathbb{R}_+^*} f(r)
\]

The first derivative of \( f \) takes us

\[
f'(r) = \frac{r^\frac{1}{2} \frac{2-n}{2} \left( 1 + Ar^{\frac{n-2}{2}} \right)^{\frac{4-n}{2}} A \frac{n-2}{2} r^{\frac{n-2}{2}} - \frac{1}{2} r^{-\frac{1}{2}} \left( 1 + Ar^{\frac{n-2}{2}} \right)^{\frac{2}{n-2}}}{k_2 r},
\]

equivalently,

\[
f'(r) = \frac{(1 + Ar^{\frac{n-2}{2}})^{\frac{2}{n-2}}}{k_2 r} \left[ (1 + Ar^{\frac{n-2}{2}})^{-1} r^{\frac{n-2}{2}} - \frac{1}{2r^{\frac{1}{2}}} \right].
\]

Therefore,

\[
f'(r) = \frac{(1 + Ar^{\frac{n-2}{2}})^{\frac{4-n}{2}}}{k_2 r^\frac{1}{2}} \left( Ar^{\frac{n-2}{2}} - 1 \right).
\]
Given $f$ is a real function, we have that $r$ is a critical point if, and only if, $f'(r) = 0$. How $r > 0$, the minimum point candidate is given by

$$r = \frac{1}{A^{\frac{n-2}{2}}}.$$ 

Let’s calculate the second derivative of $f$ and evaluate at this point, this is,

$$f''(r) = \frac{1}{2k_2} \left[ \frac{4 - n}{n - 2} \left( 1 + Ar^{\frac{n-2}{2}} \right)^{\frac{6-2n}{n-2}} \left( \frac{n - 2}{2} r^{\frac{n-2}{2}} - \frac{2}{A^{\frac{n-2}{2}}} \right) \right] + \frac{1}{2k_2} \left[ \left( 1 + Ar^{\frac{n-2}{2}} \right)^{\frac{6-2n}{n-2}} \left( \frac{n - 5}{2} Ar^{\frac{n-2}{2}} + 3r^{\frac{n-2}{2}} \right) \right]$$

equivalently,

$$f''(r) = \frac{1}{2k_2} \left( 1 + Ar^{\frac{n-2}{2}} \right)^{\frac{6-2n}{n-2}} \left( \frac{4 - n}{2} A^2 r^{2-2n} - \frac{4-n}{2} \frac{r^{n-2}}{A^{\frac{n-2}{2}}} \right) + \frac{1}{2k_2} \left( 1 + Ar^{\frac{n-2}{2}} \right)^{\frac{6-2n}{n-2}} \left( \frac{n - 5}{2} Ar^{\frac{n-2}{2}} + 3A^{\frac{n-2}{2}} r^{\frac{n-2}{2}} \right)$$

implies that

$$f''(r) = \frac{1}{4k_2 r^{\frac{n-2}{2}}} \left( A^2 r^{n-2} + 6A r^{\frac{n-2}{2}} - 2An r^{\frac{n-2}{2}} - 3 \right).$$

Now let’s evaluate the second derivative at the point $r = \frac{1}{A^{\frac{n-2}{2}}}$, that is,

$$f'' \left( \frac{1}{A^{\frac{n-2}{2}}} \right) = -\left( 1 + A \left( \frac{1}{A^{\frac{n-2}{2}}} \right)^n \right)^{\frac{6-2n}{n-2}} \left( A^2 \left( \frac{1}{A^{\frac{n-2}{2}}} \right)^n + 6A \left( \frac{1}{A^{\frac{n-2}{2}}} \right)^{2n-2} - 2An \left( \frac{1}{A^{\frac{n-2}{2}}} \right)^{n-2} - 3 \right),$$

equiv,

$$f'' \left( \frac{1}{A^{\frac{n-2}{2}}} \right) = -\frac{1}{4k_2} \left( 1 + 6 - 2n - 3 \right),$$

implies that

$$f'' \left( \frac{1}{A^{\frac{n-2}{2}}} \right) = -\frac{2^{\frac{6-2n}{2}}}{4k_2} \frac{1}{A^{\frac{n-2}{2}}} (1 + 6 - 2n - 3),$$

Therefore,

$$f'' \left( \frac{1}{A^{\frac{n-2}{2}}} \right) = -\frac{2^{\frac{6-2n}{2}}}{k_2} A^{\frac{n-2}{2}} (4 - 2n).$$

How $n \geq 3$, $A, k_2 \in \mathbb{R}_+$, we get $f'' \left( \frac{1}{A^{\frac{n-2}{2}}} \right) > 0$, consequently $r = \frac{1}{A^{\frac{n-2}{2}}}$ it’s a minimum point. Therefore,

$$f \left( \frac{1}{A^{\frac{n-2}{2}}} \right) = \frac{1}{k_2} \left( 1 + A \left( \frac{1}{A^{\frac{n-2}{2}}} \right)^n \right)^{\frac{6-2n}{2}} \left( \frac{1}{A^{\frac{n-2}{2}}} \right)^{\frac{n-2}{2}}$$

(2.13)
Consequently, $(2.15)$ 
\[ \psi_B = 2 \begin{pmatrix} n \end{pmatrix} + \begin{pmatrix} n \end{pmatrix} \]
where $B = A + r^{2-n}$. Consequently,
\[ (2.15) \quad \psi'(r) = k_2B^{2-n}r^{-2} \quad \text{and} \quad \psi''(r) = \frac{k_2n}{2} \left( B^{2-n}r^{-n} - B^{2-n}r^{-(n+2)} \right). \]

Replacing the expressions found in $(2.14)$ and $(2.15)$ in $(1.7)$, we have
\[ 2(n-1)(1-np)k_2^2B^{2-n}r^{-2} + 2(n-1)(n-1)k_2B^{2-n}r^{-n} 
+ 2(n-1)(n-1) \left( k_2B^{2-n}r^{-(n+2)} \right) \]
\[ + k_2^2B^{2-n} \left( B^{2-n} - 2B^{2-n}r^{2-n} \right) h' = \lambda \]
equivalently,
\[ 2(n-1)(1-np)k_2^2B^{2-n}r^{-2} + 2(n-1)(1-n) \left( k_2B^{2-n}r^{-n} \right) 
+ k_2^2B^{2-n}r^{-(n+2)} + 2(n-1)(n-1)k_2B^{2-n}r^{2-n} \]
\[ + k_2^2B^{2-n} \left( B^{2-n} - 2B^{2-n}r^{2-n} \right) h' = \lambda \]

How $B \neq 0$, we get
\[ 2(n-1)(1-np)B^{2-n}r^{-2} + (n-1) \left( k_2B^{2-n}r^{-n} \right) 
- 2(n-1)(1-n)B^{2-n}r^{2-n} + k_2^2B^{2-n} \left( B^{2-n} - 2B^{2-n}r^{2-n} \right) h' = \frac{\lambda}{2k_2B^{2-n}} \]

Consequently,
\[ (n-2) \left( B^{2-n}r^{-2} - B^{2-n}r^{2-n} \right) + \left( B^{2-n} - 2B^{2-n}r^{2-n} \right) h' = \frac{\lambda}{2k_2B^{2-n}} \]
equivalently,
\[ 2k_2^2B^{2-n}(n-2) \left( B^{2-n}r^{-2} - B^{2-n}r^{2-n} \right) + 2k_2^2B^{2-n} \left( B^{2-n} - 2B^{2-n}r^{2-n} \right) h' = \lambda. \]

Note that $B^{2-n} - 2B^{2-n}r^{2-n} \neq 0$, otherwise $B = 2r^{2-n}$ and Consequently $B = 2A$. 
Thus,
\[ h'(r) = \frac{\lambda}{2k_2^2B^{2-n}} \left( B^{2-n} - 2B^{2-n}r^{2-n} \right) \]
Making
\[ \varphi(r) = \frac{\lambda}{2k_2^2B^{\frac{n}{2-n}}(B^{\frac{2}{2-n}} - 2B^{\frac{n}{2-n}}r^{\frac{2-n}{2-n}})}, \]
and \[ w(r) = (2 - n) \left( \frac{B^{\frac{2(n-1)}{2-n}}r^{-\frac{n}{2-n}} - B^{\frac{2(n-1)}{2-n}}} {B^{\frac{2}{2-n}} - 2B^{\frac{n}{2-n}}r^{\frac{2-n}{2-n}}} \right), \]
the first derivative of these equations leads us to
\[ \varphi'(r) = -\frac{\lambda}{2k_2^2} \frac{(nB^{-1}r^{-\frac{n}{2-n}} - (n + 2)B^{-2r^{1-n}})}{(B^{\frac{2}{2-n}} - 2B^{\frac{n}{2-n}}r^{\frac{2-n}{2-n}})^2}, \]
and
\[ w'(r) = \frac{(2 - n)}{2} \left( 3nB^{\frac{n}{2-n}}r^{-n} - 4(n - 1)B^{\frac{3n-2}{2-n}}r^{\frac{2-n}{2-n}} + 2(n - 2)B^{\frac{4(n-1)}{2-n}}r^{2(1-n)} - nB^{\frac{n+2}{2-n}}r^{\frac{n+2}{2-n}} \right) \]
Replacing the functions in the equation (1.10), that is,
\[ (n - 2)\psi'' + \psi h' + 2\psi'h = 0, \]
we obtain
\[ (n - 2)k_2^2 \left( B^{\frac{2(n-1)}{2-n}}r^{-n} - B^{\frac{n}{2-n}}r^{\frac{2-n}{2-n}} \right) \left( B^{\frac{n}{2-n}} - 4B^{\frac{2n}{2-n}}r^{\frac{2-n}{2-n}} + 4B^{\frac{n}{2-n}}r^{\frac{2-n}{2-n}} \right) \]
\[ + \frac{(2 - n)}{2} k_2 B^{\frac{n}{2-n}} \left( 3nB^{\frac{n}{2-n}}r^{-n} - 4(n - 1)B^{\frac{3n-2}{2-n}}r^{\frac{2-n}{2-n}} + 2(n - 2)B^{\frac{4(n-1)}{2-n}}r^{2(1-n)} - nB^{\frac{n+2}{2-n}}r^{\frac{n+2}{2-n}} \right) \]
\[ \cdot 2(n - 2)k_2 B^{\frac{n}{2-n}}r^{-\frac{n}{2-n}} \left( B^{\frac{n}{2-n}}r^{-\frac{n}{2-n}} - B^{\frac{2(n-1)}{2-n}}r^{1-n} \right) \left( B^{\frac{n}{2-n}} - 2B^{\frac{2(n-1)}{2-n}}r^{\frac{2-n}{2-n}} \right) \]
\[ - \frac{\lambda}{2k_2^2} \left( nB^{\frac{n}{2-n}}r^{-\frac{n}{2-n}} - (n + 2)B^{\frac{2(n-1)}{2-n}}r^{1-n} \right) = 0, \]
Which is equivalent to
\[ B^{\frac{2(n+1)}{2-n}}r^{-n} - 2B^{\frac{3n-2}{2-n}}r^{\frac{2-n}{2-n}} + B^{\frac{2(2n-1)}{2-n}}r^{2(1-n)} = \frac{\lambda}{(n - 2)^2k_2^2} \left( nB^{\frac{n}{2-n}}r^{-\frac{n}{2-n}} - (n + 2)B^{\frac{2(n-1)}{2-n}}r^{1-n} \right). \]
Note that \( nB^{\frac{n}{2-n}}r^{-\frac{n}{2-n}} - (n + 2)B^{\frac{2(n-1)}{2-n}}r^{1-n} \neq 0 \), because otherwise \( B = \frac{n+2}{n}r^{\frac{2-n}{2-n}} \).
On the other hand,
\[ B^{\frac{2(n+1)}{2-n}}r^{-n} - 2B^{\frac{3n-2}{2-n}}r^{\frac{2-n}{2-n}} + B^{\frac{2(2n-1)}{2-n}}r^{2(1-n)} = 0 \]
therefore,
\[ \left( B^{\frac{n+1}{2-n}}r^{-\frac{n}{2-n}} - B^{\frac{2n-1}{2-n}}r^{1-n} \right)^2 = 0, \]
and consequently \( B = r^{\frac{2-n}{2-n}}, \) but this is a contradiction, because \( B = \frac{n+2}{n}r^{\frac{2-n}{2-n}} \).
Therefore,
\[ \lambda = (n - 2)^2k_2^2 \frac{\left( B^{\frac{n+1}{2-n}}r^{-\frac{n}{2-n}} - B^{\frac{2n-1}{2-n}}r^{1-n} \right)^2} {nB^{\frac{n}{2-n}}r^{-\frac{n}{2-n}} - (n + 2)B^{\frac{2(n-1)}{2-n}}r^{1-n}}. \]
How \( \lambda \) is constant, we have that
Theorem 1.7 and Corollary 1.8 are satisfied. □

Proof of the Proposition 1.1 Consider the manifolds $\psi$ and $r$

A $\rho$-Einstein soliton with scalar curvature $K = 0$. Besides that, if, and only if,

$\frac{d}{dr}\left(\frac{B^{\frac{n+1}{2}} r^{-\frac{n}{2}} - B^{\frac{n-1}{2}} r^{1-n}}{nB^{\frac{n+1}{2}} r^{-\frac{n}{2}} - (n+2)B^{\frac{n-1}{2}} r^{1-n}}\right)^2 = 0.$

if, and only if,

$(4n-1)B^{\frac{n-2}{2}} r^{-2n} = (n+1)(n+2)B^{\frac{n-2}{2}} r^{-2n} - n^2B^{\frac{n-2}{2}} r^{-2n} + (n+2)nB^{\frac{n+1}{2}} r^{-2n} + (2-5n)nB^{\frac{n-2}{2}} r^{2-3n} - 2(2-5n)(n+2)B^{\frac{n-2}{2}} r^{2-3n} + (2n-1)nB^{\frac{n+1}{2}} r^{2-3n} - \frac{1}{2}(3n^2+2n-4)B^{\frac{n+1}{2}} r^{-2n} - (n+2)(n+1)B^{\frac{n+1}{2}} r^{2-3n} + (n^2+2n-2)B^{\frac{n+1}{2}} r^{2-3n} - \frac{n^2}{2}B^{\frac{n-1}{2}} r^{-2n} + (3n^2+2n-4) + \frac{n^2}{2}B^{\frac{n-1}{2}} r^{-2n} - 2(n^2+n-2)B^{\frac{n-1}{2}} r^{-2n} = 0$

if, and only if,

$\frac{d}{dr}\left(\frac{B^{\frac{n+1}{2}} r^{-\frac{n}{2}} - B^{\frac{n-1}{2}} r^{1-n}}{nB^{\frac{n+1}{2}} r^{-\frac{n}{2}} - (n+2)B^{\frac{n-1}{2}} r^{1-n}}\right)^2 = 0.$

We will prove that equation (2.20) is satisfied if, and only if, $A = 0$. For this, consider

If $f(r) = 0 \forall r > 0$ and $A \neq 0$, then its derivative also is zero. Since there is a single value of $r$ such that $f'(r) = 0$ given by $r = \left(\frac{4(n-1)}{n^2+1}\right)^{\frac{n}{2}},$ we get a contradiction.

Therefore the equation (2.20) is satisfied if, and only if, $A = 0$. In this case $B = r^{\frac{2-n}{2}}$. Substituting $B = r^{\frac{2-n}{2}}$ in (2.18) we obtain that $\lambda = 0$. Therefore the proof is done. □

Proof. Proof of the Theorem 1.8 Follows by Theorem 1.7 that $(\mathbb{R}^n, \bar{g}), \bar{g} = \frac{1}{\psi^2}g$ is a $\rho$-Einstein soliton with scalar curvature $K_{\bar{g}} = 0$ is steady if, and only if, $A = 0$. Besides that $h(r)$ is constant. How $A = 0$ we obtain from Lemma 1.4 that $\psi(r) = k\bar{r}$. Follow of the 1.4 that $(\mathbb{R}^n, \bar{g})$ have sectional curvature zero. Therefore, we conclude que $(\mathbb{R}^n, \bar{g})$ is flat. The reciprocal is automatically satisfied. □

Proof. Proof of the Corollary 1.9 How $M^n, \bar{g}$ is locally conformally flat and rotationally symmetric, then locally the metric $\bar{g}$ is given by $\bar{g} = 1_{\psi^2}g$ where $\psi = \psi(r)$ and $r = \sum_{k=1}^n x_k^2$ and $g$ is the euclidean metric. Therefore the results obtained in Theorem 1.7 and Corollary 1.8 are satisfied. □
By other hand, we get
\[ |v|_g = \frac{e^{\left(1 + r \frac{n-2}{2}\right) \frac{n}{n-2}}}{r} |v|_{g_0} = \frac{e^{\left(1 + r \frac{n-2}{2}\right) \frac{n}{n-2}}}{\left(1 + r \frac{n-2}{2}\right) \frac{n}{k_2 r}} |v|_{g_0}. \]

Thus,
\[ |v|_g = h(r) |v|_{\bar{g}}, \]
where, \[ h(r) = \frac{e^{\left(1 + r \frac{n-2}{2}\right) \frac{n}{n-2}}}{\left(1 + r \frac{n-2}{2}\right) \frac{n}{k_2}}. \]

To find \( c_1 > 0 \) such that \( |v|_{\bar{g}} \geq c_1 |v|_g \), just solve the following problem
\[ \min_{r \in \mathbb{R}^+} h(r) \]

The first derivative of \( h \) takes us
\[ h'(r) = e^{\left(1 + r \frac{n-2}{2}\right) \frac{n}{n-2}} \left(1 + r \frac{n-2}{2}\right) \frac{n}{k_2} \left[ \left(1 + r \frac{n-2}{2}\right) \frac{n}{k_2} - 1 \right]. \]

How \( r > 0 \), we get \( h'(r) > 0, \forall r \), so the \( h \) function is strictly increasing. Therefore,

\[ (2.21) \quad \min_{r \in \mathbb{R}^+} h(r) = \lim_{r \to 0} \frac{e^{\left(1 + r \frac{n-2}{2}\right) \frac{n}{n-2}}}{\left(1 + r \frac{n-2}{2}\right) \frac{n}{k_2}} = e. \]

Just take \( c_1 = e \), we get \( h(r) \geq c_1, \forall r \). Thus \( |v|_{\bar{g}} \geq c |v|_g \), how \( M = \left( \mathbb{R}^n \setminus \{0\}, \bar{g} = \frac{g_0}{\bar{g}} \right) \) is complete, implies that \( N = \left( \mathbb{R}^n \setminus \{0\}, g = \frac{g_0}{\bar{g}} \right) \) is complete.

We will show that \( (\mathbb{R}^n, \bar{g}) \) has negative scalar curvature. Indeed, note that

\[ (2.22) \quad \varphi'(r) = \frac{1 - r \frac{n-2}{2} \left(1 + r \frac{n-2}{2}\right) \frac{4-n}{n}}{e^{\left(1 + r \frac{n-2}{2}\right) \frac{n}{n-2}}}, \]

and
\[ \varphi''(r) = -\frac{1}{2e^{\left(1 + r \frac{n-2}{2}\right) \frac{n}{n-2}}} \left[ (4 - n) r \frac{2-n}{2} \left(1 + r \frac{n-2}{2}\right) \frac{6-2n}{n-2} + (n - 2) r \frac{n-2}{2} \left(1 + r \frac{n-2}{2}\right) \frac{4-n}{n-2} \\
-2r \frac{n-2}{2} \left(1 + r \frac{n-2}{2}\right) \frac{4-n}{n-2} - 2r \frac{2-n}{2} \left(1 + r \frac{n-2}{2}\right) \frac{2(4-n)}{n-2} \right], \]

implies that
\[ (2.23) \quad \varphi''(r) = -\frac{\left[ n r \frac{n-2}{2} \left(1 + r \frac{n-2}{2}\right) \frac{4-n}{n-2} + (4 - n) r \frac{2-n}{2} \left(1 + r \frac{n-2}{2}\right) \frac{6-2n}{n-2} - 2r \frac{2-n}{2} \left(1 + r \frac{n-2}{2}\right) \frac{2(4-n)}{n-2} \right]}{2e^{\left(1 + r \frac{n-2}{2}\right) \frac{n}{n-2}}} \]
It is well known (see, e.g., [2] or [9]) that if $\bar{g} = g_{\psi^2}$, then
\begin{equation}
K_{\bar{g}} = 4r \left[ 2(n-1)\varphi'' - n(n-1)(\varphi')^2 \right] + 4(n-1)\varphi' \cdot \tag{2.24}
\end{equation}
Substituting the expressions found in (2.22) and (2.23) into (2.24), we have
\begin{equation}
K_{\bar{g}} = \frac{4(n-1)}{e^{\left(1+r\frac{n-2}{2}\right)\frac{n-2}{n-2}}} \left[ (2-n)r^{n-1} \left(1 + r\frac{n-2}{n-2}\right) \frac{2(n-3)}{n-2} + 2(1-n)r^\frac{2}{2} \left(1 + r\frac{n-2}{n-2}\right) \right]
\end{equation}
equivalently,
\begin{equation}
K_{\bar{g}} = \frac{4(n-1)}{e^{\left(1+r\frac{n-2}{2}\right)\frac{n-2}{n-2}}} \left[ (2-n)r^{n-1} \left(1 + r\frac{n-2}{n-2}\right) \frac{2(n-3)}{n-2} + 2(1-n)r^\frac{2}{2} \left(1 + r\frac{n-2}{n-2}\right) \right]
\end{equation}
implies that,
\begin{equation}
K_{\bar{g}} = \frac{4(n-1)}{e^{\left(1+r\frac{n-2}{2}\right)\frac{n-2}{n-2}}} \left[ (2-n)r^{n-1} \left(1 + r\frac{n-2}{n-2}\right) \frac{2(n-3)}{n-2} + 2(1-n)r^\frac{2}{2} \left(1 + r\frac{n-2}{n-2}\right) \right]
\end{equation}

Therefore,
\begin{equation}
K_{\bar{g}} = \frac{4(n-1)}{e^{\left(1+r\frac{n-2}{2}\right)\frac{n-2}{n-2}}} \left[ (n - 2) \left(1 + r\frac{n-2}{n-2}\right) \frac{2(n-2)}{n-2} + 2(n-1)r^\frac{2-n}{2} + (n+2) \right] r^{n-1}.
\end{equation}

Proof. Proof of Corollary 1.12 Follow immediately from Corollary 1.6.

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