On asymptotic effects of boundary perturbations in exponentially shaped Josephson junctions

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Received: date / Accepted: date

Abstract A parabolic integro differential operator $\mathcal{L}$, suitable to describe many phenomena in various physical fields, is considered. By means of equivalence between $\mathcal{L}$ and the third order equation describing the evolution inside an exponentially shaped Josephson junction (ESJJ), an asymptotic analysis for (ESJJ) is achieved, explicitly evaluating, boundary contributions related to the Dirichlet problem.

Keywords Superconductivity · Junctions · Laplace Transform · Initial-boundary problems for higher order parabolic equations

PACS 74.50.+r · 02.30.Jr

Mathematics Subject Classification (2010) 44A10 · 35A08 · 35K35 · 35E05

1 Introduction

Many equivalences among nonlinear operators and p.d.e. systems exist and an extensive bibliography is given. (see, f.i. [1,2,3]). Here, the semilinear equation, which characterizes exponentially shaped Josephson junctions in superconductivity (ESJJ) ([4,5,6,7,8] and references therein), is considered and equivalence with the following parabolic integro differential equation:

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\[ \mathcal{L} u = u_t - \varepsilon u_{xx} + au + b \int_{0}^{t} e^{-\beta(t-\tau)} u(x, \tau) d\tau = f(x, t, u) \]  

is proved. In this way, a priori estimates for (ESJJ) are obtained and, by means of a well known theorem on convolutions behavior, asymptotic effects of boundary perturbations are achieved for the solution of initial boundary value problem with Dirichlet conditions.

Operator \( \mathcal{L} \) defined in (1) can describe many linear and non linear physical phenomena and there is plenty of bibliography. In particular, when \( F = F(x, t, u) \), some non linear phenomena involve equation (1) both in superconductivity and biology and Dirichlet conditions in superconductivity refer to the phase boundary specifications, while in excitable systems occur when the pulse propagation in heart cells is studied. Besides, Dirichlet problem is also considered for stability analysis and asymptotic behavior of reaction-diffusion systems solutions, or in hyperbolic diffusion.

Previous analyses related to equation (1) have been developed assuming \( \varepsilon, a, b, \beta \) as positive constants. The fundamental solution \( K \) has been determined and many of its properties have been proved. Moreover, various boundary value problems have been considered as well as non linear integral equations have been determined, whose Green functions have numerous typical properties of the diffusion equation. Besides, the kernel \( e^{-\beta(t-\tau)} u(x, \tau) \) in (1) can be modified as physical situations demand, and this particular choice has been made taking into consideration superconductive and biological models. As a matter of fact, it is possible to prove that (1) can characterize reaction diffusion models, like the FitzHugh-Nagumo system, suitable to model the propagation of nerve impulses. Furthermore, the perturbed sine Gordon equation (PSGE) can be deduced and, as proved further, the evolution in a Josephson junction with nonuniform width can be characterized, too.

In superconductivity it is well known that the Josephson effect is modeled by the (PSGE) given by (see, f.i. [26]):

\[ \varepsilon u_{xxt} - u_{tt} + u_{xx} - \alpha u_t = \sin u - \gamma \]  

where \( u \) represents the difference between the phase of the wave functions related to the two superconductors of the junction, \( \gamma > 0 \) is a forcing term that is proportional to a bias current, the \( \alpha \)-term accounts for dissipation due to normal electrons crossing the junction, and the \( \varepsilon \)-term accounts for dissipation caused by normal electrons flowing parallel to the junction.

Besides, when the case of an exponentially shaped Josephson junction (ESJJ) is considered, denoting by \( \lambda \) a positive constant, the evolution of the phase inside the junction is described by the third order equation:

\[ \varepsilon u_{xxt} + u_{xx} - u_{tt} + \lambda u_{xt} + \lambda u_x - \alpha u_t = \sin u - \gamma \]  

where terms \( \lambda u_x \) and \( \lambda \varepsilon u_{xt} \) stand for the current as a consequence of the tapering and, more specifically, \( \lambda u_x \) represents a geometrical force leading
the fluxons from the wide edge to the narrow edge. Exponentially shaped Josephson junctions provide several advantages compared to rectangular ones. Among the others, we can mention the possibility of getting a voltage which is not chaotic anymore, but rather periodic. This allows to exclude some among the possible causes of large spectral width and to avoid the problem of trapped flux.

2 Statement of the problem

If \( T \) is an arbitrary positive constant and

\[
\Omega_T \equiv \{(x,t) : 0 \leq x \leq L ; \ 0 < t \leq T \},
\]

let us consider the following initial boundary value problem with Dirichlet boundary conditions:

\[
\begin{cases}
  u_t - \varepsilon u_{xx} + au + b \int_0^t e^{-\beta(t-\tau)} u(x,\tau) d\tau = F(x,t,u) \quad (x,t) \in \Omega_T \\
  u(x,0) = u_0(x) \quad x \in [0,L], \\
  u(0,t) = g_1(t) \quad u(L,t) = g_2(t) \quad 0 < t \leq T.
\end{cases}
\]  (4)

Denoting by \( K(x,t) \) the fundamental solution of the linear operator defined by (1), and considering \( \varepsilon, a, b, \beta \) as positive constants, it results [25]:

\[
K(r,t) = \frac{1}{2\sqrt{\pi \varepsilon}} \left[ e^{-\frac{r^2}{4\varepsilon} - at} - b \int_0^t e^{-\frac{r^2}{4\varepsilon} - ay} e^{-\beta(t-y)} J_1(2\sqrt{b(y(t-y))}) dy \right]
\]  (5)

where \( r = |x|/\sqrt{\varepsilon} \) and \( J_n(z) \) denotes the Bessel function of first kind and order \( n \). Moreover, the following theorem holds:

**Theorem 1** For all \( t > 0 \), the Laplace transform of \( K(x,t) \) with respect to \( t \) converges absolutely in the half-plane \( \Re s > \max(-a,-\beta) \) and it results:

\[
\hat{K}(r,s) = \int_0^\infty e^{-st} K(r,t) \ dt = \frac{e^{-r\sigma}}{2\sqrt{\varepsilon \sigma}}
\]  (6)

with \( \sigma^2 = s + a + \frac{b}{s+\beta} \).

Now, let us consider the following Laplace transforms with respect to \( t \):

\[
\hat{u}(x,s) = \int_0^\infty e^{-st} u(x,t) \ dt , \quad \hat{F}(x,s) = \int_0^\infty e^{-st} F[x,t,u(x,t)] \ dt ,
\]

and let \( \hat{g}_1(s), \hat{g}_2(s) \) be the \( L \) transforms of the data \( g_i(t) \) \((i = 1, 2)\).

Then the Laplace transform of the problem (4) is formally given by:
\[
\begin{aligned}
\left\{ \hat{u}_{xx} - \frac{\sigma^2}{\varepsilon} \hat{u} = -\frac{1}{\varepsilon} \left[ \hat{F}(x, s) + u_0(x) \right] \right. \\
\hat{u}(0, s) = \hat{g}_1(s) & \quad \hat{u}(L, s) = \hat{g}_2(s).
\end{aligned}
\] (7)

If one introduces the following theta function

\[
\hat{\theta}(y, \sigma) = \frac{1}{2} \sqrt{\varepsilon \sigma} \left\{ e^{-\frac{\varepsilon}{\sigma} (L - y)} + \sum_{n=1}^{\infty} \left[ e^{-\frac{\varepsilon}{\sigma} (2nL + y)} + e^{-\frac{\varepsilon}{\sigma} (2nL - y)} \right] \right\}
\] (8)

then, by (7) and (8) one deduces:

\[
\hat{u}(x, s) = \int_0^L \left[ \hat{\theta}(x + \xi, s) - \hat{\theta}(x - \xi, s) \right] u_0(\xi) + \hat{F}(\xi, s) \, d\xi - 2\varepsilon \hat{g}_1(s) \hat{\theta}_x(x, s) + 2\varepsilon \hat{g}_2(s) \hat{\theta}_x(L - x, s),
\] (9)

where

\[
\hat{\theta}_x(x, \sigma) = \frac{\sinh \left[ \sigma/\sqrt{\varepsilon} (x - L) \right]}{2 \varepsilon \sinh (\sigma/\sqrt{\varepsilon} L)}
\] (10)

3 Explicit solution

In order to obtain the inverse formula for (9), let us apply (6) to (8). Then, one deduces the following function which is similar to theta functions:

\[
\theta(x, t) = K(x, t) + \sum_{n=1}^{\infty} \left[ K(x + 2nL, t) + K(x - 2nL, t) \right] = \sum_{n=\infty}^{\infty} K(x + 2nL, t).
\] (11)

So that, denoting by

\[
G(x, \xi, t) = \theta \left( |x - \xi|, t \right) - \theta \left( x + \xi, t \right),
\]

when \( F = f(x, t) \), by (11) the explicit solution of the linear problem (4) is given by:

\[
u(x, t) = \int_0^L G(x, \xi, t) u_0(\xi) \, d\xi + \int_0^t d\tau \int_0^L G(x, \xi, t) f(\xi, \tau) \, d\xi \] (12)
On asymptotic effects of boundary perturbations in ESJJ

\[ -2\varepsilon \int_0^t \theta_x(x, t - \tau) g_1(\tau) \, d\tau + 2\varepsilon \int_0^t \theta_x(x - L, t - \tau) g_2(\tau) \, d\tau \]

So, owing to the basic properties of \( K(x, t) \), it is easy to deduce the following theorem:

**Theorem 2** When the linear source \( f(x, t) \) and the initial boundary data \( u_0(x), g_i (i = 1, 2) \) are continuous in \( \Omega_T \), then problem (4) admits a unique regular solution \( u(x, t) \) given by (12).

Furthermore, when the source term \( F = F(x, t, u) \) depends on the unknown function \( u(x, t) \), too, then problem (4) admits the following integral equation:

\[ u(x, t) = \int_0^L G(x, \xi, t)u_0(\xi)d\xi + \int_0^t d\tau \int_0^L G(x, \xi, t)F(\xi, \tau, u(x, \tau))d\xi - \]

\[ 2\varepsilon \int_0^t \theta_x(x, t - \tau) g_1(\tau) \, d\tau + 2\varepsilon \int_0^t \theta_x(x - L, t - \tau) g_2(\tau) \, d\tau \]

Indeed, if \( F = F(x, t, u(x, t)) \), let

\[ D = \{ (x, t, u) : (x, t) \in \Omega_T, \quad -\infty < u < \infty, \quad -\infty < p < \infty, \, \} \]

and let us assume the following **Hypotheses A**:

- The function \( F(x, t, u, p) \) is defined and continuous on \( D \) and it is bounded for all \( u \) and \( p \).
- For each \( k > 0 \) and for \( |u|, |p| < k \), the function \( F \) is Lipschitz continuous in \( x \) and \( t \) for each compact subset of \( D \).
- There exists a constant \( \beta_F \) such that:

\[ |F(x, t, u_1, p_1) - F(x, t, u_2, p_2)| \leq \beta_F \{ |u_1 - u_2| + |p_1 - p_2| \} \]

holds for all \( (u_i, p_i) i = 1, 2 \).

Moreover, let us assume \( \|z\| = \sup_{\Omega_T} |z(x, t)| \), and let \( B_T \) denote the Banach space

\[ B_T \equiv \{ z(x, t) : z \in C(\Omega_T), \|z\| < \infty \}. \] (14)

By means of standard methods related to integral equations and owing to properties of \( K \) and \( F \), it is easy to prove that the mapping defined by (13) is a contraction of \( B_T \) in \( B_T \) and so it admits an unique fixed point \( u(x, t) \in B_T \) [34,35]. Hence

**Theorem 3** When the data \( (u_0, g_1, g_2) \) are continuous functions, then the Dirichlet problem related to the non linear system (4), has a unique solution in the space of solutions which are regular in \( \Omega_T \).
4 Equivalence with (ESJJ)

Among many others equivalences, a significant one occurs between operator $\mathcal{L}$ and equation (3) which describes the evolution inside an exponentially shaped Josephson junction.

Indeed, let us assume

$$\beta = \frac{1}{\varepsilon}, \quad b = \beta^2 (1 - \alpha \varepsilon), \quad a \beta = \frac{\lambda^2}{4} - b$$

(15)

with

$$f = - \int_{0}^{\tau} e^{-\frac{1}{\varepsilon}(t-\tau)} f_1(x, \tau, u) \, d\tau,$$

(16)

From the integro differential equation (1) follows:

$$\varepsilon u_{xxt} - u_{tt} + u_{xx} - (\alpha + \varepsilon \frac{\lambda^2}{4}) u_t - \frac{\lambda^2}{4} u = f_1$$

(17)

Therefore, assuming $e^{\frac{x}{\varepsilon}} u = \bar{u}$, (17) turns into equation (3).

The diffusion effects due to the dissipative terms $\varepsilon (u_{xxt} - \lambda u_{xt})$ of (3) have already been investigated in [8]. Here, in sect 7 by means of this equivalence, the influences of data on the solution which is related to the Dirichlet problem, will be explicitly evaluated.

5 Some properties on $\theta$ and $\theta_x$ functions

In order to obtain a priori estimates and asymptotic effects related to boundary perturbations, some properties of the fundamental solution $K$ and the theta function defined in (11) ought to be evaluated. In [11,23,24,25] some of these have already been proved and it results:

$$\int_{L}^{L} |\theta(x - \xi, t)| \, d\xi \leq (1 + \sqrt{b} \pi t) e^{-\omega t}, \quad \omega = \min\{a, \beta\}$$

(18)

$$\left| \int_{L}^{L} \theta(x - \xi, t) \, d\xi \right| \leq \sum_{n=-\infty}^{\infty} \int_{L}^{L} |K(|x - \xi + 2nL|, t)| \, d\xi =$$

$$= \sum_{n=-\infty}^{\infty} \int_{x+(2n-1)L}^{x+2nL} |K(y, t)| \, dy \leq \int_{R} |K(y, t)| \, dy.$$

(19)

Moreover, letting:
\[ K_1 \equiv \int_0^t e^{-\beta(t-\tau)} K(x, \tau) \, d\tau, \quad (20) \]

one has:

\[ \int_\mathbb{R} |K_1| \, d\xi \leq E(t); \quad \int_0^t d\tau \int_\mathbb{R} |K_1| \, d\xi \leq \beta_1 \quad (21) \]

being

\[ E(t) = \frac{e^{-\beta t} - e^{-\alpha t}}{a - \beta} > 0 \quad \text{and} \quad \beta_1 = (a \beta)^{-1}. \]

Furthermore, as for \( \theta_x \), from (5), it is well-rendered that the \( x \) derivative of the integral term vanishes for \( x \to 0 \), while the first term represents the derivative with respect to \( x \) of the fundamental solution related to the heat equation. Indeed, since (11), it results:

\[ \theta_r = K_r + \sum_{n=1}^{\infty} \left[ -\frac{x+2nL}{2t} K(x+2nL, t) - \frac{x-2nL}{2t} K(x-2nL, t) \right] \quad (22) \]

\[ = K_r + J(r, t) \]

and, by means of classic calculations,(see, f.i. [34]), one has:

\[ \lim_{x \to 0} J(r, t) = 0. \quad (23) \]

So, classic theorems assure that conditions (3) are surely satisfied. Moreover, being:

\[ \lim_{t \to \infty} \int_0^t \theta_x(x, \tau) \, d\tau = \lim_{s \to 0} \hat{\theta}_x(x, s) \quad \text{Re } s > \max \{-a, -\beta\} \quad (24) \]

from (10), it results:

\[ \lim_{t \to \infty} \int_0^t \theta_x(x, \tau) \, d\tau = \frac{1}{2 \epsilon} \frac{\sinh \sigma_0 (x-L)}{\sinh (\sigma_0 L)} \quad (25) \]

where \( \sigma_0 = \sqrt{\left( a + \frac{\alpha}{\beta} \right)^2} \).

Besides, for all \( x > \delta > 0 \) it results:

\[ \sum_{n=-\infty}^{\infty} e^{-\frac{(x+2nL)^2}{4\epsilon}} \leq t \frac{4\epsilon \pi^2}{4 \pi^2 \epsilon} \csc^2(\pi \delta/2L), \quad (26) \]

and one has

\[ |K_r(r, t)| \leq r \frac{e^{-x^2/4\epsilon}}{\sqrt{4\pi \epsilon t^3}} \left[ 1 + 4 \beta t^2 \right] e^{-rt}. \quad (27) \]

So, the following theorem can be proved:
Theorem 4 For all \(0 \leq x \leq L\), there exists a positive constant \(C\) depending only on constants \(a, \varepsilon, b, \beta\) such that:

\[
\int_0^\infty |\theta_x(x, \tau)| \, d\tau \leq C.
\] (28)

Moreover, it results:

\[
\lim_{t \to \infty} \theta_x(x, t) = 0,
\] (29)

6 Analysis of boundary contributions

By means of the following well known theorem (see f.i. [24] and references therein), boundary contributions related to the Dirichlet problem (4) can be explicitly evaluated.

Theorem 5 Let \(h(t)\) and \(\chi(t)\) be two continuous functions on \([0, \infty[\) satisfying the following hypotheses

\[
\exists \lim_{t \to \infty} h(t) = h(\infty), \quad \exists \lim_{t \to \infty} \chi(t) = \chi(\infty) \quad (30)
\]

\[
\dot{h}(t) \in L_1[0, \infty).
\] (31)

Then, it results:

\[
\lim_{t \to \infty} \int_0^t \chi(t - \tau) \dot{h}(\tau) \, d\tau = \chi(\infty) \cdot [h(\infty) - h(0)].
\] (32)

According to this, the following theorem holds:

Theorem 6 If data \(g_i\) \((i = 1, 2)\) are two continuous functions satisfying condition (30) then, one has:

\[
\lim_{t \to \infty} \int_0^t \theta_x(x, \tau) g_i(t - \tau) \, d\tau = g_{i, \infty} \cdot \frac{1}{2 \varepsilon} \cdot \frac{\sinh \sigma_0 (x - L)}{\sinh \sigma_0 L} \quad (33)
\]

being \(g_{i, \infty} = \lim_{t \to \infty} g_i\), \((i = 1, 2)\) and \(\sigma_0 = \sqrt{\left(a + \frac{b}{\beta}\right) \frac{1}{\varepsilon}}\).

Moreover, when data \(\varphi_i\) \((i = 1, 2)\) verify both condition (30) and condition (31), then it results:

\[
\lim_{t \to \infty} \int_0^t \theta_x(x, \tau) \varphi_i(t - \tau) \, d\tau = 0 \quad (i = 1, 2) \quad (34)
\]

Proof Let us assume, in (32), \(h = \int_0^t \theta_x(x, \tau) \, d\tau\) and \(\chi = g_i\), \((i = 1, 2)\). Then, (33) follows by (25) and (28).

Besides, when \(\chi = \theta_x(x, t)\) and \(h = g_i\), since (29), (34) is deduced.
7 Asymptotic behaviours related to (ESJJ)

According to the equivalence between (3) and the integro differential equation (1), previous results can be applied to (ESJJ).

Let us consider the following initial boundary value problem with Dirichlet conditions:

\[
\begin{cases}
\varepsilon u_{xxt} + u_{xx} - \varepsilon \lambda u_{x} - \lambda u - \alpha u_t = \sin u - \gamma (x, t) & (x, t) \in \Omega T \\
u(x, 0) = u_0(x) & x \in [0, L], \\
u_t(x, 0) = v_0(x) & x \in [0, L], \\
0 < t \leq T.
\end{cases}
\] (35)

When in (4) one assumes

\[
F(x, t, u) = e^{-\frac{1}{2}x} \left[ \int_0^t e^{-\frac{1}{2}(t-\tau)} \left( \sin \left( e^{x/2} u \right) - \gamma \right) d\tau - v_0(x) e^{-\frac{1}{2}t} \right].
\] (36)

and \(a, b, \beta, \lambda\) are given by (15), system (35) can be given the form (4) and hypotheses A assure that the following integro equation states:

\[
u(x, t) = \int_0^L G(x, \xi, t) e^{-\frac{1}{2}x} u_0(\xi) d\xi + \int_0^t d\tau \int_0^L G(x, \xi, t) F(\xi, \tau, u(x, \tau)) d\xi - 2 \varepsilon \int_0^t \theta_x(x, t-\tau) g_1(\tau) d\tau + 2 \varepsilon \int_0^t \theta_x(x-L, t-\tau) g_2(\tau) d\tau.
\] (37)

So, a priori estimates of the solution and asymptotic effects of boundary perturbations can be obtained. Indeed, letting

\[\|u_0\| = \sup_{0 \leq x \leq L} |u_0(x)|, \quad \|v_0\| = \sup_{0 \leq x \leq L} |v_0(x)|,\]

and

\[\|f\| = \sup_{\Omega T} |f(x, t, u)|,\]

with \(f\) defined by (15)2 (16), the following theorems hold:

**Theorem 7** When \(g_1 = g_2 = 0\), the solution \(u(x, t)\) related to (ESJJ) model, and determined in (13), satisfies the following estimate:

\[|u| \leq 2 \left( \|u_0\| + \|v_0\| + \|f\| \right) + \|E(t) + \beta_1 \|f\|.\] (38)

**Proof** The initial disturbances can be evaluated by means of (18), (19) and (21)1. Besides, since \(21)2\), the behaviour of the source term is determined.
Further, according to theorem (9), if \( u_0 = v_0 = 0 \) and \( F = 0 \), one has:

**Theorem 8** When \( t \) tends to infinity and data \( g_i \) \( (i = 1, 2) \) are two continuous functions satisfying condition (30), it results:

\[
    u = g_1,\infty \frac{\sinh \frac{\lambda}{2} (x - L)}{2 \epsilon \sinh \frac{\lambda L}{2}} + g_2,\infty \frac{\sinh \frac{\lambda}{2} (x - 2L)}{2 \epsilon \sinh \frac{\lambda L}{2}}
\]

(39)

Otherwise, in the hypotheses (30) (31), the effects determined by boundary disturbance vanish.

**Remarks.** When \( t \) tends to infinity, the effect due to the initial disturbances \( (u_0, v_0) \) is vanishing, while the effect of the nonlinear source is bounded for all \( t \). Furthermore, for large \( t \), the effects due to boundary disturbances \( g_1, g_2 \) are null or at least everywhere bounded.

**Acknowledgements** This paper has been performed under the auspices of G.N.F.M. of I.N.D.A.M.
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