Unconjugated Contact Forms

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Abstract
In this paper, we prove the existence of infinitely many number non-isomorphic contact structures on the torus $T^3$. Moreover, these structures are explicitly given by $\omega_n = \cos n\theta_1 \, d\theta_1 + \sin n\theta_1 \, d\theta_2 \; (n \in \mathbb{N})$.

Keywords: Contact structures; Reeb field; Poisson brackets

Introduction
In the acts of Colloquium of Brussels in 1958, Libermann [1] addressed the study of the automorphisms of the contact structures on a differentiable manifold $M$. She has proved that these automorphisms corresponds bijectively to functions on this manifold. This allows to transport the Lie algebra structure on the vector space $F(M)$ of the functions on $M$. We obtain, for given functions $f, g \in F(M)$, a Poisson bracket $\{f, g\}$ that depends of the contact form $\omega$. The study of the infinite dimensional Lie algebras obtained is far from be advanced. Thus, in 1973 Lichnerowicz [2] who hopes distinguish the contact structures by their Lie algebras, has given series of results that are all however of general character. Some works have appeared that have emphasis on the similarities of these algebras. In 1979, Lutz [3] has proved the existence of infinitely many non-isomorphic contact structures on the sphere $S^3$. In 1989, as reported by Lutz [3] himself, we have opened in our thesis [4] new perspectives in the other direction by studying the sub-algebras of finite dimension of these algebras. We know that if two contact structures $[\omega_1]$ and $[\omega_2]$ are isomorphic then their Lie algebras (of infinite dimension of course) $A([\omega_1])$ and $A([\omega_2])$ are also isomorphic.

Given an $n$-dimensional smooth manifold $M$, and a point $p \in M$, a contact element of $M$ with contact point $p$ is an $(n-1)$-dimensional linear subspace of the tangent space to $M$ at $p$: A contact contact element can be given by the zeros of a $1$-form on the tangent space to $M$ at $p$. However, if a contact element is given by the zeros of a $1$-form $\omega$, then it will also be given by the zeros of $\lambda \omega$ where $\lambda \neq 0$: thus $\{\lambda \omega: \lambda \neq 0\}$ are all give the same contact element. It follows that the space of all contact elements of $M$, denoted by $\mathfrak{X}(M)$, has proved series of results that are all however of general character. Some works have appeared that have emphasis on the similarities of these algebras. In 1979, Lutz [3] has proved the existence of infinitely many non-isomorphic contact structures on the sphere $S^3$. In 1989, as reported by Lutz [3] himself, we have opened in our thesis [4] new perspectives in the other direction by studying the sub-algebras of finite dimension of these algebras. We know that if two contact structures $[\omega_1]$ and $[\omega_2]$ are isomorphic then their Lie algebras (of infinite dimension of course) $A([\omega_1])$ and $A([\omega_2])$ are also isomorphic.

A contact structure on an odd dimensional manifold $M$, of dimension $2k + 1$, is a smooth distribution of contact elements, denoted by $\xi$, which is generic at each point. The genericity condition is that $\xi$ is non-integrable.

Assume that we have a smooth distribution of contact elements $\xi$ given locally by a differential $1$-form: i.e. a smooth section of the cotangent bundle. The non-integrability condition can be given explicitly as $\alpha \wedge (d\alpha)^k \neq 0$.

Notice that if $\xi$ is given by the differential $1$-form, then the same distribution is given locally by $\beta = \alpha$, where $\beta$ is a non-zero smooth function. If $\xi$ is co-orientable then is defined globally.

If is a contact form for a given contact structure, the Reeb vector field $R$ can be defined as the unique element of the kernel of $\alpha$ such that $\alpha(R) = 1$.

For more details, we can consult the previous studies [5-8].

The Main Result
The main result is contained in the following theorem:

Theorem 1
On the torus $T^3$ the contact structures defined by the contact forms $\omega_n = \cos n\theta_1 \, d\theta_1 + \sin n\theta_1 \, d\theta_2 \; (n \in \mathbb{N})$ are non-isomorphic.

To establish this result, we need the following lemma.

Lemma 2
Let $f$ a $C^\infty$-function on the torus $T^3$ and $R_n$ the Reeb field of $\omega_n$ defined by $R_n = \cos \theta_1 \, \partial_{\theta_1} + \sin \theta_1 \, \partial_{\theta_2}$.

If $R_n(f) = 0$, then $f$ depend only on $\theta_1$.

Proof: $R_n(f) = 0$ means that $f$ is constant along the integral curves of $R_n$ whose equations are:

$\frac{\partial \theta_1}{\partial t} = \cos \theta_1$

$\frac{\partial \theta_2}{\partial t} = \sin \theta_1$

$\frac{\partial \theta_3}{\partial t} = 0$

Thus, we have,

$\theta_1 = t \cos nk_1 + k_1$

$\theta_2 = t \sin nk_1 + k_2$

$\theta_3 = k_3$

where $k_1, k_2, k_3$ are real constants.

When $\tan k_1$ is irrational, the trajectories are dense on a torus $T^3$; so by continuity $f$ is constant on this torus. Hence, we get $\frac{\partial f}{\partial \theta_2} = \frac{\partial f}{\partial \theta_3} = 0$.

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Received May 29, 2018; Accepted July 21, 2018; Published July 30, 2018

Citation: Aggoun S (2018) Unconjugated Contact Forms. J Phys Math 9: 280. doi: 10.4172/2090-0902.1000280

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for $\theta_1, \theta_2$ arbitrary and $\theta_1$ in a dense subset of the circle. It follows that $f$ is constant with respect to $\theta_1$ and $\theta_2$.

This completes the proof of the lemma.

**Proof of the theorem:** It suffices to prove that the structures $[\omega_1]$ and $[\omega_2]$ are non-isomorphic.

From a study [1] we recall that the Poisson brackets associated to $[\omega_1]$ and $[\omega_2]$ are given respectively by:

$$\{ f, g \} = \begin{pmatrix} \frac{\partial f}{\partial \theta_1} - \frac{\partial g}{\partial \theta_1} + \frac{\partial f}{\partial \theta_3} \frac{\partial g}{\partial \theta_3} - \frac{\partial f}{\partial \theta_2} \frac{\partial g}{\partial \theta_2} \\ \frac{\partial f}{\partial \theta_2} - \frac{\partial g}{\partial \theta_2} + \frac{\partial f}{\partial \theta_3} \frac{\partial g}{\partial \theta_3} - \frac{\partial f}{\partial \theta_1} \frac{\partial g}{\partial \theta_1} \\ \frac{\partial f}{\partial \theta_3} - \frac{\partial g}{\partial \theta_3} + \frac{\partial f}{\partial \theta_1} \frac{\partial g}{\partial \theta_1} - \frac{\partial f}{\partial \theta_2} \frac{\partial g}{\partial \theta_2} \end{pmatrix} \cos \theta_3$$

$$+ \begin{pmatrix} \frac{\partial f}{\partial \theta_1} - \frac{\partial g}{\partial \theta_1} + \frac{\partial f}{\partial \theta_3} \frac{\partial g}{\partial \theta_3} - \frac{\partial f}{\partial \theta_2} \frac{\partial g}{\partial \theta_2} \\ \frac{\partial f}{\partial \theta_2} - \frac{\partial g}{\partial \theta_2} + \frac{\partial f}{\partial \theta_3} \frac{\partial g}{\partial \theta_3} - \frac{\partial f}{\partial \theta_1} \frac{\partial g}{\partial \theta_1} \\ \frac{\partial f}{\partial \theta_3} - \frac{\partial g}{\partial \theta_3} + \frac{\partial f}{\partial \theta_1} \frac{\partial g}{\partial \theta_1} - \frac{\partial f}{\partial \theta_2} \frac{\partial g}{\partial \theta_2} \end{pmatrix} \sin \theta_1$$

Thus we have $[\phi_1, \psi_1]=\omega_1$, $[\psi_1, \omega_1]=\phi_1$ and $[\omega_1, \phi_1]=-\psi_1$.

We obtain the two equations:

$$\frac{\partial u}{\partial \theta_1} \cos w + \frac{\partial v}{\partial \theta_1} \sin w = \lambda \cos 2\theta_1$$

$$\frac{\partial u}{\partial \theta_2} \cos w + \frac{\partial v}{\partial \theta_2} \sin w = \lambda \sin 2\theta_2$$

Suppose that $[\omega_1]$ and $[\omega_2]$ are isomorphic that is $F^*\omega_1=\lambda \omega_2$, where $\lambda$ is a function on $\mathbb{T}^3$ without zeros and $F$ be this diffeomorphism defined from $\mathbb{T}^3$ into $\mathbb{T}^3$ by:

$$F(\theta_1, \theta_2, \theta_3)=(u(\theta_1, \theta_2, \theta_3), v(\theta_1, \theta_2, \theta_3), w(\theta_1, \theta_2, \theta_3))$$

We obtain the following way:

$$u(\theta_1, \theta_2, \theta_3)=\theta_1 \alpha_1(\theta_3)+\theta_2 \beta_1(\theta_3)+\gamma_1(\theta_3),$$

$$v(\theta_1, \theta_2, \theta_3)=\theta_1 \alpha_2(\theta_3)+\theta_2 \beta_2(\theta_3)+\gamma_2(\theta_3),$$

$$w(\theta_1, \theta_2, \theta_3)=\theta_1 \alpha_3(\theta_3)+\theta_2 \beta_3(\theta_3)+\gamma_3(\theta_3).$$

If $\Phi(\theta_1, \theta_2, \theta_3)=\sin \theta_3, \Psi(\theta_1, \theta_2, \theta_3)=\cos \theta_1$ and $\Omega(\theta_1, \theta_2, \theta_3)=-\sin \theta_1$:

Then $[\omega_1]$ and $[\omega_2]$ are non-isomorphic. Consequently, there are infinitely many non-isomorphic contact structures $[\omega_n]$ on the torus $\mathbb{T}^3$ given by

$$[\omega_n]=\lambda_n \omega_1,$$

where the functions $\alpha_i, \beta_i$ and $\gamma_i$ take only integer values and subject to the condition $\alpha_1 \beta_2 + \alpha_2 \beta_1 = \pm 1$.

We return now to the eqns. (1) and (2); we obtain:

$$\alpha_1 \beta_2 \gamma_1 + \alpha_2 \beta_1 \gamma_2 = \pm 1.$$