On the Symmetric $K$-user Interference Channels with Limited Feedback

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Abstract

In this paper, we develop achievability schemes for symmetric $K$-user interference channels with a rate-limited feedback from each receiver to the corresponding transmitter. We study this problem under two different channel models: the linear deterministic model, and the Gaussian model. For the deterministic model, the proposed scheme achieves a symmetric rate that is the minimum of the symmetric capacity with infinite feedback, and the sum of the symmetric capacity without feedback and the symmetric amount of feedback. For the Gaussian interference channel, we use lattice codes to propose a transmission strategy that incorporates the techniques of Han-Kobayashi message splitting, interference decoding, and decode and forward. This strategy achieves a symmetric rate which is within a constant number of bits to the minimum of the upper bound on the symmetric capacity with infinite feedback, and the sum of the upper bound on the symmetric capacity without feedback and the amount of symmetric feedback. This constant is obtained as a function of the number of users, $K$. The symmetric achievable rate is used to characterize the achievable generalized degrees of freedom which exhibits a gradual increase from no feedback to perfect feedback in the presence of feedback links with limited capacity.

Index terms: $K$-user symmetric interference channel, rate-limited feedback, symmetric rate, achievability, generalized degrees of freedom.
I. INTRODUCTION

The interference channel (IC) has been studied in the literature since 1970's to understand performance limits of multiuser communication networks [1]. Although the exact characterization of the capacity region of a two-user Gaussian IC is still unknown, several inner and outer bounds have been obtained. These bounds have resulted in an approximate characterization of the capacity region, within one bit, in [2] and [3]. Such characterization includes outer bounds on the capacity region for the two-user Gaussian IC, as well as encoding/decoding strategies based on the Han-Kobayashi scheme [4], which performs close to the optimum. On the other hand, the $K$-user IC has been studied in [5, 6] for a symmetric scenario, where all direct links (from each transmitter to its respective receiver) have the same gain, and similarly, the gains of all cross (interfering) links are identical. For such a $K$-user symmetric IC, the number of symmetric generalized degrees of freedom (GDoF) is characterized in [5], and an approximate sum capacity is given in [6].

It is well known that feedback does not increase the capacity of point-to-point discrete memoryless channels [7]. However, feedback is beneficial in improving the capacity region of multi-user networks (see [8] and references therein). A number of works on ICs explore feedback strategies, where each receiver feeds back the channel output to its own transmitter [9–15]. Several coding schemes for the $K$-user Gaussian IC are developed in [15]. The effect of feedback on the capacity region of the two-user IC is studied in [9], where it is shown that feedback provides a multiplicative gain in the sum capacity at high signal-to-noise ratio (SNR), when the interference links are much stronger than the direct links. The capacity region of the two-user Gaussian IC with unlimited feedback is characterized within a 2 bit gap in [10]. The $K$-user symmetric IC with unlimited feedback is considered in [16], where the GDoF is characterized. A more realistic feedback model is one where the feedback links are rate-limited. The impact of rate-limited feedback is studied for a two-user Gaussian IC in [17], where it is shown that the maximum gain in the symmetric capacity with feedback is the amount of symmetric feedback.

In this paper, we study the impact of rate-limited feedback for a $K$-user IC. We first consider this problem for the linear deterministic model proposed in [18] as an approximation to the Gaussian model, and then treat the Gaussian model. For the Gaussian model, we develop an achievability scheme that employs the techniques of Han-Kobayashi message splitting, interference decoding and decode-and-forward. In order to effectively decode the interference, lattice codes are used such that the sum of signals can be decoded without decoding the individual signals. We also find the achievable symmetric GDoF with
rate-limited feedback. Except for the point where the SNR is equal to the interference-to-noise ratio (SNR = INR), the effect of interference from the other $K-1$ users is as if there were only one interferer in the network. This is analogous to the result of [5] and [16], where it is shown that for the cases of no feedback and unlimited feedback, respectively, the symmetric GDoF of the $K$-user IC is the same as that of a two-user IC.

In order to get the maximal benefit of feedback, we use an encoding scheme which combines two well-known interference management techniques, namely, interference alignment and interference decoding, which are also used in [16,17]. More precisely, the encoding at the transmitters is such that all the interfering signals are aligned at each receiver. However, a fundamental difference between our approach and the conventional interference alignment approach is that we need to decode interference to be able to remove it from the received signal, whereas the aligned interference is usually suppressed in conventional approaches. A challenge here, which makes the $K$-user problem fundamentally different from the two-user problem [17], is that the interference is a combination of multiple interfering messages instead of a single message as in the two-user case, and decoding all of them imposes strict bounds on the rate of the interfering messages. A key idea is that instead of decoding all the interfering messages individually, we will decode some combination of them that corrupts the intended message of interest. In the proposed scheme, the receiver decodes the sum of certain interfering signals and the intended signal, and sends it back to the transmitter. The transmitter, knowing the intended signal, can then decode the sum of interfering signals and transmits to the receiver in the next slot, to help the receiver to decode the intended signal. In order to decode the sum of certain intended/interfering signals, all transmitters employ a common structured lattice code [19] which has the property that the sum of different codewords is another codeword from the same codebook.

Our new scheme generalizes the prior works in [2,6,10,16,17,20] as follows. A two-user IC with rate-limited feedback is considered in [17], while this paper develops the achievability for a general $K$-user IC. A two-user IC without feedback is treated in [2], which is a special case of the $K$-user IC without feedback in [6,20]. A two-user IC with unlimited feedback is considered in [10], which is a special case of [16] where the $K$-user IC with unlimited feedback is treated. In this paper, we develop an achievability scheme for a $K$-user IC with limited feedback, which for the special cases of two-user, no feedback, and unlimited feedback, results in schemes that are different from those in [17], [6], and [16], respectively. Moreover, we provide a conjectured upper bound on the achievable symmetric rate, which corresponds to
the true upper bounds in [17], [2], and [16] for the cases of two-user, no feedback and unlimited feedback, respectively. We show that the symmetric rate of our proposed achievability scheme is within a constant gap to the conjectured upper bound. For the two-user case, the achievable symmetric rate in [17] is not within a constant gap to the upper bound for a certain interference region. Since the conjectured upper bound is the true upper bound for the two-user case, our achievable rate is within a constant gap to the upper bound for all interference regions.

For the achievability scheme for two-user IC in [17], the two transmitters have different and asymmetric encoding operations, and it cannot be generalized to arbitrary number of users. Also alignment of interfering signals and encoding them by a lattice code is not considered, because each receiver receives interference from only one transmitter. In our proposed achievability scheme, all the transmitters employ the same encoding operation and therefore all users are symmetric. Moreover, each receiver receives interference from the other \( K - 1 \) transmitters and we align and encode them using a lattice code so that the sum signal can be decoded. On the other hand, in the achievability scheme for the \( K \)-user IC with infinite feedback in [16], each receiver simply sends back all received signals to the corresponding transmitter whereas in our scheme each receiver sends back a lattice codeword (via the rate-limited feedback channel) with a strategy that is chosen depending on the interference regime. Finally the achievability scheme for the \( K \)-user IC with no feedback in [6] only performs alignment on the interfering signals and does not deal with feedback.

The remainder of this paper is organized as follows. Section II gives the symmetric achievable rate for the deterministic model, with some examples to illustrate the main ideas of the proposed achievability scheme. Section III gives our results for the Gaussian model, where the proposed achievability scheme is described and the achievable symmetric rate, a conjectured upper bound, and the achievable GDoF are given. Finally, Section IV concludes the paper.

II. DETERMINISTIC MODEL

A. System Model and Problem Formulation

We first consider the linear deterministic IC. This model was proposed in [18] to focus on signal interactions instead of the additive noise, and to obtain insights for the Gaussian model. In this model, there is a non-negative integer \( n_{kj} \) representing the channel gain from transmitter \( k \) to receiver \( j \), \( j, k \in \{1, \cdots, K\} \). We assume that \( n_{jk} = n \) for \( j = k \) and \( n_{jk} = m \) for \( j \neq k \). Also, define \( q \triangleq \max(m, n) \).
We write the channel input at transmitter \( k \) at time \( i \) as \( X_{k,i} = [X^1_{k,i}, X^2_{k,i}, \ldots, X^q_{k,i}] \in \mathbb{F}_2^q \), for \( k \in \{1, 2, \ldots, K\} \), such that \( X^1_{k,i} \) and \( X^q_{k,i} \) represent the most and the least significant bits of the transmitted signal, respectively. At each time \( i \), the received signal at the \( k \)th receiver is given by

\[
Y_{k,i} = D^{q-n}X_{k,i} + \sum_{j \neq k} D^{q-m}X_{j,i},
\]

where all the operations are performed modulo 2 and \( D \) is a \( q \times q \) shift matrix. We assume that there is a feedback channel from the \( k \)th receiver to the \( k \)th transmitter which is of capacity \( p \), and that \( 2p \) is an integer. The feedback is causal and hence at time \( i \) the signal received till time \( i - 1 \) is available at each receiver for encoding and feeding back to the corresponding transmitter.

For a deterministic IC, a symmetric rate \( R_{sym} \) is said to be achievable if there is a strategy such that all users can get a rate \( R_{sym} \). We further define \( \alpha \triangleq m/n \) and \( \beta \triangleq p/n \).

### B. Results for Linear Deterministic IC Model

In this section, we describe our proposed coding schemes for the \( K \)-user linear deterministic IC with rate-limited feedback. The following theorem gives our achievability result.

**Theorem 1.** For the \( K \)-user linear deterministic IC, the following symmetric rate is achievable:

\[
R_{sym} = \begin{cases} 
\min\{n - m + p, n - \frac{m}{2}\}, & \text{if } 0 \leq m \leq \frac{n}{2}, \\
\min\{m + p, n - \frac{m}{2}\}, & \text{if } \frac{n}{2} \leq m \leq \frac{2n}{3}, \\
n - \frac{m}{2}, & \text{if } \frac{2n}{3} \leq m < n, \\
\frac{n}{R}, & \text{if } m = n, \\
\frac{m}{2}, & \text{if } n < m \leq 2n, \\
\min\{n + p, \frac{m}{2}\}, & \text{if } 2n \leq m.
\end{cases}
\]

**Remark 1.** With infinite feedback, i.e., \( p = \infty \), according to Theorem 4 of [16], the symmetric capacity is

\[
C_{sym,\infty} = \begin{cases} 
n - \frac{m}{2}, & \text{if } 0 \leq m < n, \\
\frac{n}{R}, & \text{if } m = n, \\
\frac{m}{2}, & \text{if } n < m.
\end{cases}
\]
Corollary 1. With no feedback, i.e., \( p = 0 \), the symmetric capacity is

\[
C_{\text{sym,0}} = \begin{cases} 
  n - m, & \text{if } 0 \leq m \leq \frac{n}{2}, \\
  m, & \text{if } \frac{n}{2} \leq m \leq \frac{2n}{3}, \\
  n - \frac{m}{2}, & \text{if } \frac{2n}{3} \leq m < n, \\
  \frac{n}{K}, & \text{if } m = n, \\
  \frac{m}{2}, & \text{if } n < m \leq 2n, \\
  n, & \text{if } 2n \leq m.
\end{cases}
\]

(4)

Proof: The achievability follows from Theorem 1 for \( p = 0 \). The upper bound for \( \frac{2n}{3} \leq m \leq 2n \) follows from Remark 1; for \( 2n \leq m \) it is a simple cutset bound; and for \( 0 \leq m \leq \frac{2n}{3} \) it follows from (13) in [2].

Then, from (2)-(4) we have \( R_{\text{sym}} = \min\{C_{\text{sym,\infty}}, C_{\text{sym,0}} + p\} \). This result has been shown to be tight, i.e., \( R_{\text{sym}} \) is the symmetric capacity for \( K = 2 \) in [17], and we conjecture that \( R_{\text{sym}} \) is the symmetric capacity for a general \( K \).

Fig. 1 illustrates the (normalized) symmetric rate as a function of \( \alpha \), for different values of \( \beta = 0 \) (i.e., Corollary 1), \( \beta = 0.1 \), \( \beta = 0.2 \), and \( \beta = \infty \) (i.e., Remark 1).

The complete proof of Theorem 1 is given in Appendix A. In this section, we present several examples of the transmission schemes that achieve the symmetric rate as claimed in Theorem 1. For the range of \( \frac{2}{3} \leq \alpha \leq 2 \), we have \( C_{\text{sym,0}} = C_{\text{sym,\infty}} \). So, with limited feedback the symmetric capacity remains the same in this range. In the rest of this section, we will illustrate the proposed coding schemes for three
is ignored. The fourth output signal is sent to the transmitter over the feedback link, in order to be used that involves its intended symbol as well as interference. The equation received at the least significant bit symbols in its first channel use. Each receiver gets three interference-free symbols, and one more equation, that operates on a block of length 2. The basic idea can be seen from Fig. 2, where the coding scheme is demonstrated for \( K = 3, n = 5, m = 2, \) and \( p = 0.5. \)

1) Very Weak Interference Regime (\( \alpha \leq \frac{1}{2} \)): In the very weak interference regime, the goal is to achieve a symmetric rate of \( R_{sym} = \min\{n - m + p, n - \frac{m}{2}\} \) bits per user. We propose an encoding scheme that

...Fig. 2. Proposed coding scheme for the linear deterministic model in the very weak interference regime (\( \alpha \leq \frac{1}{2} \)), for \( K = 3, n = 5, m = 2 \) and \( p = 0.5. \)

As shown in Fig. 2, the proposed coding scheme is able to convey seven intended symbols from each transmitter to its respective receiver in two channel uses, i.e., \( 2R_{sym} = 7. \) Each transmitter sends four fresh symbols in its first channel use. Each receiver gets three interference-free symbols, and one more equation, that involves its intended symbol as well as interference. The equation received at the least significant bit is ignored. The fourth output signal is sent to the transmitter over the feedback link, in order to be used for the next transmission. In the second channel use, each transmitter forwards the interfering part of its...
received feedback on its top level. The next lower level should be empty and the three lowest levels will be used to transmit the remaining fresh symbols.

Now, consider the received signals at the first receiver in two channel uses. The information symbols intended for the first receiver are denoted by $a_{1,1}, \ldots, a_{1,7}$, respectively. The receiver receives eight linearly independent equations, involving nine variables, which seems to be unsolvable at the first glance. However, we do not need to decode all the symbols. Instead, we can solve the system of linear equations in $a_{1,1}, \ldots, a_{1,7}$ and $(a_{2,1}+a_{3,1})$ which can be solved to obtain the intended symbols. Similarly, the transmitted messages can be decoded at other receivers too. Hence, a symmetric rate of $\frac{7}{2}$ symbols/channel-use is achievable.

2) Weak Interference Regime ($\frac{1}{2} \leq \alpha \leq \frac{2}{3}$): In the weak interference regime, the goal is to achieve a symmetric rate of $R_{sym} = \min\{m+p, n-\frac{mp}{2}\}$ bits per user. We propose an encoding scheme that operates on a block of length 2. The basic idea can be seen from Fig. 3, where the coding scheme is demonstrated for $K=3$, $n=7$, $m=4$, and $p=0.5$.

As shown in Fig. 3, the proposed coding scheme is able to convey nine intended symbols from each transmitter to its respective receiver in two channel uses, i.e., $2R_{sym} = 9$. Each transmitter sends five fresh symbols in its first channel use: two symbols on the highest two levels, nothing on the next two levels, and three more on the lowest three levels. Each receiver gets four interference-free symbols, and one more equation, that involves their intended symbols as well as interference, which is sent to the transmitter over the feedback link, in order to be used for the next transmission. In the second channel use, each transmitter forwards the interfering part of its received feedback on its second top level. The highest level and the three lowest levels will be used to transmit the remaining fresh symbols and nothing is transmitted on the other two levels.

Now, consider the received signals at the first receiver in two channel uses. The information symbols intended for the first receiver are denoted by $a_{1,1}, \ldots, a_{1,9}$, respectively. We have twelve linearly independent equations, involving fifteen variables. Variables $a_{1,1}, a_{1,2}, a_{1,4}, a_{1,5}, a_{1,6}, a_{1,8}, a_{1,9}$ are received interference free and so does $(a_{2,2} + a_{3,2})$. Having $a_{1,2}$ and $(a_{2,2} + a_{3,2})$ decoded, $a_{1,3}$ and $a_{1,7}$ can then be decoded from equations $a_{1,3} + (a_{2,2} + a_{3,2})$ and $a_{1,7} + 2a_{1,2} + (a_{2,2} + a_{3,2})$. Also, $(a_{2,1} + a_{3,1})$ and $(a_{2,6} + a_{3,6})$ are not needed in the decoding. Similar decoding is performed at the other receivers. Hence, a per-user rate of $\frac{9}{2}$ symbols/channel-use is achievable.
transmitter to its respective receiver in two channel uses, i.e., fresh symbols in its first channel use on the three highest levels and nothing on the lower three levels.

Fig. 3. Proposed coding scheme for the linear deterministic model in the weak interference regime ($\frac{1}{2} \leq \alpha \leq \frac{2}{3}$), for $K = 3$, $n = 7$, $m = 4$ and $p = 0.5$.

3) Very Strong Interference Regime ($\alpha \geq 2$): In the very strong interference regime, the goal is to achieve a symmetric rate of $R_{\text{sym}} = \min\{n + p, m\}$ bits per user. We propose an encoding scheme that operates on a block of length 2. The basic idea can be seen from Fig. 4, where the coding scheme is demonstrated for $K = 3$, $n = 2$, $m = 6$, and $p = 0.5$.

As shown in Fig. 4, the proposed coding scheme is able to convey five intended symbols from each transmitter to its respective receiver in two channel uses, i.e., $2R_{\text{sym}} = 5$. Each transmitter sends three fresh symbols in its first channel use on the three highest levels and nothing on the lower three levels.
Each receiver gets three interference equations, one empty level, and also two interference-free symbols in the lower two levels. The third output signal is sent to the transmitter over the feedback link, in order to be used for the next transmission. In the second channel use, each transmitter forwards its received feedback on its third top level. The next three lower levels are kept empty. The two highest levels are used to transmit the remaining fresh symbols.

Now, consider the received signals at the first receiver in two channel uses. The information symbols intended for the first receiver are denoted by $a_{1,1}, \ldots, a_{1,5}$, respectively. We have ten linearly independent
equations, involving fifteen variables. Variables $a_{1,1}, a_{1,2}, a_{1,4}, a_{1,5}$ are received interference free and so does $(a_{2,3} + a_{3,3})$. Having $(a_{2,3} + a_{3,3})$ decoded, $a_{1,3}$, can then be decoded from equation $2a_{1,3} + (a_{2,3} + a_{3,3})$. Similar decoding is performed at the other receivers. Hence, a per-user rate of $\frac{5}{2}$ symbols/channel-use is achievable.

III. GAUSSIAN INTERFERENCE CHANNEL

A. System Model and Problem Formulation

In this section, we describe the $K$-user symmetric Gaussian IC which consists of $K$ transmitters and $K$ receivers. Transmitter $i$ has a message $W_i$ that it wishes to send to receiver $i$. At time $t$, transmitter $i$ transmits a signal $X_i[t]$ over the channel with a power constraint $\text{tr}(\mathbb{E}(X_iX_i^\dagger)) \leq 1$ ($A^\dagger$ is the conjugate of $A$).

The received signal at receiver $i$ at time $t$ is denoted as $Y_i[t]$ and can be written as

$$Y_i[t] = \sqrt{\text{SNR}}X_i[t] + \sum_{j=1, j\neq i}^{K} \sqrt{\text{INR}}X_j[t] + Z_i[t],$$

where $Z_i[t] \sim \mathcal{CN}(0, 1)$ is i.i.d. complex Gaussian noise, SNR is the received signal-to-noise-ratio from transmitter $i$ to receiver $i$, and INR is the received interference-to-noise-ratio from transmitter $i$ to receiver $j$ for $i, j \in \{1, \ldots, K\}, i \neq j$. In other words, $\sqrt{\text{SNR}}$ is the power attenuation factor of the direct links and $\sqrt{\text{INR}}$ is the power attenuation factor of the interference links. Let $C_{FB}$ be the capacity of the feedback link from receiver $i$ to transmitter $i$, for all $i \in \{1, 2, \cdots, K\}$. We assume that the feedback channels are orthogonal to each other and they are also orthogonal to the data channels.

The encoding process at each node is causal, in the sense that the feedback signal transmitted from receiver $i$ at time $t$ is a function of whatever is received over the data channel up to time $(t - 1)$; and the transmitted signal by transmitter $i$ at time $t$ is a function of the message $W_i$ and the feedback received till time $t$. Each receiver decodes the message at $t = n$. If a message $W_i \in \{1, \ldots, 2^{nR}\}$ transmitted from transmitter $i$ is decoded at receiver $i$ for each $i \in \{1, \cdots, K\}$ with error probability $e_{i,n} = \Pr(\hat{W}_i \neq W_i) \to 0$ as $n \to \infty$, we say that the symmetric rate $R$ is achievable. We assume that $\text{SNR}, \text{INR} > 1$ and also define

$$\alpha = \frac{\log \text{INR}}{\log \text{SNR}}, \text{ and } \beta = \frac{C_{FB}}{\log \text{SNR}}.$$ (6)
B. Results of Gaussian IC Model

1) Overview: In this section, we will describe the achievability scheme for the symmetric $K$-user Gaussian IC with rate-limited feedback. This scheme will be shown to achieve a symmetric rate within a constant gap to a conjectured upper bound, which is the minimum of the symmetric rate upper bound with infinite feedback, and the sum of the symmetric rate upper bound without feedback and the amount of symmetric feedback.

The feedback helps decode the interference which can be useful for decoding the desired message. In addition, feedback helps to decode a part of the intended message that is conveyed from other transmitters through the feedback path. In a $K$-user IC, the receivers hear interference signals from multiple transmitters. Partial decoding of all interfering messages would dramatically decrease the maximum rate of the desired message. Thus, we decode the total interference from all the other users, without resolving the individual components of the interference. Our achievability strategy has two key features, namely, 1) interfering signals are aligned, and 2) the summation of interfering signals belong to a message set of proper size which can be decoded at each receiver. Here, the first property is satisfied since the network is symmetric (all interfering links have the same gain), and therefore, all interfering messages are received at the same power level. In order to satisfy the second property, we use a common lattice code for all transmitters, instead of random Gaussian codebooks. The structure of a lattice codebook and its closedness with respect to summation imply that the sum of aligned interfering codewords observed at each receiver is still a codeword from the same codebook. This allows us to perform decoding by searching over a single codebook, instead of the Cartesian product of all codebooks.

Lattice codes are a class of codes that can achieve the capacity of the Gaussian channel [21, 22], with lower complexity compared to the conventional random codes. A $T$-dimensional lattice $\Lambda$ is a subset of $T$-tuples with real elements, such that $x, y \in \Lambda$ implies $-x \in \Lambda$ and $x + y \in \Lambda$. For an arbitrary $x \in \mathbb{R}^T$, we define $[x \ mod \ \Lambda] = x - Q(x)$, where $Q(x) = \arg \min_{t \in \Lambda} ||x - t||$, is the closest lattice point to $x$. The Voronoi cell of $\Lambda$, denoted by $\nu$, is defined as $\nu = x \in \mathbb{R}^T : Q(x) = 0$. The Voronoi volume $V(\nu)$ and the second moment $\sigma^2(\Lambda)$ of the lattice are defined as $V(\nu) = \int_{\nu} dx$, and $\sigma^2(\Lambda) = \int_{\nu} ||x||^2 dx / T V(\nu)$, respectively. We further define the normalized second moment of $\Lambda$ as $G(\Lambda) = \frac{\sigma^2(\Lambda)}{V(\nu)^{2/T}} = \frac{\int_{\nu} ||x||^2 dx}{TV(\nu)^{1+1/T}}$. A sequence of lattices $\{\Lambda_T\}$ is called a good quantization code if $\lim_{T \to \infty} G(\Lambda_T) = \frac{1}{2\pi e}$. On the other hand, a sequence of lattices is known to be good for AWGN channel coding if $\lim_{T \to \infty} P_T[ z^T \in \nu_T] = 1$, where $z^T \sim \mathcal{N}(0, \sigma^2(\lambda_T))$ is random zero-mean Gaussian noise with proper variance. It is shown in [23] that
there exist sequences of lattices \( \{ \Lambda_T \} \) that are simultaneously good for quantization and AWGN channel coding.

2) Proposed Achievability Scheme: We will now describe our achievability strategies for a \( K \)-user symmetric Gaussian IC, which is inspired by the proposed schemes in Section II for the deterministic IC. We split the result into three regions, denoted as very weak interference where \( \alpha \leq 1/2 \), weak interference where \( 1/2 < \alpha \leq 2/3 \), and strong interference where \( \alpha \geq 2 \). We do not consider the case of \( 2/3 < \alpha < 2 \) since the upper bound for the symmetric capacity with perfect feedback in Theorem 3 of [16] and the lower bound for the symmetric capacity with no feedback in Theorem 1 of [6] are within a constant of \( \frac{1}{2} \log 9 + 16 + \frac{K-1}{2} + 3 \log K \) bits to each other for \( 2/3 < \alpha < 1 \), and are within a constant of \( \frac{1}{2} \log 6 + 6 + \frac{K-1}{2} + \log K \) bits to each other for \( 1 < \alpha < 2 \). We will use the notation \( X^{(a:b)} \triangleq X^{(a)} + X^{(a+1)} + \ldots + X^{(b)} \).

For the achievability scheme, we use a nested lattice code [24] which is generated using a good quantization lattice for shaping and a good channel coding lattice. We start with \( T \)-dimensional nested lattices \( \Lambda_c \subseteq \Lambda_f \), where \( \Lambda_c \) is a good quantization lattice with \( \sigma^2(\Lambda_c) = 1 \) and \( G(\Lambda_c) \approx 1/2\pi e \), and \( \Lambda_f \) is a good channel coding lattice. We construct a codebook \( C = \Lambda_f \cap \nu_c \), where \( \nu_c \) is the Voronoi cell of the lattice \( \Lambda_c \). We will use the following properties of lattice codes [16]:

1) Codebook \( C \) is a closed set with respect to summation under the “mod \( \Lambda_c \)” operation, i.e., if \( x_1, x_2 \in C \) are two codewords, then \( (x_1 + x_2) \mod \Lambda_c \in C \) is also a codeword.

2) Lattice code \( C \) can be used to reliably transmit up to rate \( R = \log(\text{SNR}) \) over a Gaussian channel modeled by \( Y = \sqrt{\text{SNR}}X + Z \) with \( \mathbb{E}[Z^2] = 1 \), while a more sophisticated scheme can achieve rate \( R = \log(1 + \text{SNR}) \).

The next result gives the symmetric achievable rate for the very weak interference regime (\( \alpha \leq 1/2 \)).
Theorem 2. For \( \alpha \leq 1/2 \), a symmetric rate of \( R_{sym} = \frac{R^{(1:4)}}{2} \) is achievable, for any \( R^{(1)}, \ldots, R^{(4)} \) satisfying

\[
\begin{align*}
R^{(1)} & \leq \log \left( \frac{\text{SNR}_{\mu}^{(1)}}{\text{SNR}_{\mu}^{(2:3)} + \text{SNR}_{\mu}^{(1:3)}(K-1) + \mu^{(1:3)}} \right), \\
R^{(2)} & \leq \log \left( \frac{\text{SNR}_{\mu}^{(2)}}{\text{SNR}_{\mu}^{(3)} + \text{SNR}_{\mu}^{(1:3)}(K-1) + \mu^{(1:3)}} \right), \\
R^{(3)} & \leq \log \left( \frac{\text{SNR}_{\mu}^{(3)}}{\text{SNR}_{\mu}^{(2:3)}(K-1) + \mu^{(1:3)}} \right), \\
R^{(4)} & \leq \log \left( \frac{\text{SNR}_{\mu}^{(4)}}{\text{SNR}_{\mu}^{(2:3)}(K-1) + (\mu^{(1)} + \mu^{(4)})} \right),
\end{align*}
\]

for any \( \mu^{(1)}, \mu^{(2)}, \mu^{(3)}, \mu^{(4)} \geq 0 \) such that \( \text{SNR}_{\mu}^{(3)} = \text{SNR}_{\mu}^{(1)} \).

Proof: Here, we will describe the achievability scheme only for the first user. Due to the symmetry of the scheme, the achievability for the other users is similar.

We take \( X_1 = \{ X_1^{(1)}, X_1^{(2)}, X_1^{(3)}, X_1^{(4)} \} \) as the signal that the first user transmits during two consecutive time slots. The encoded symbol \( X_1^{(i)} \) is unit power, and of rate \( R^{(i)} \) using the lattice codes, \( i \in \{1, 2, 3, 4\} \). The overall rate is thus \( R = \frac{R^{(1:4)}}{2} \). Let \( P^{(i)} \) be the power of \( X_1^{(i)} \) transmitted in the first round, and \( P^{(i)}' \) be the power of the \( X_1^{(i)} \) transmitted in the second round. The power allocations during the two rounds are chosen as

\[
\begin{align*}
P^{(1)} &= \frac{\mu^{(1)}}{\mu^{(1:3)}}, & P^{(2)} &= \frac{\mu^{(2)}}{\mu^{(1:3)}}, & P^{(3)} &= \frac{\mu^{(3)}}{\mu^{(1:3)}}, & P^{(4)} &= 0, \\
P^{(1)}' &= \frac{\mu^{(1)}}{\mu^{(1)} + \mu^{(4)}}, & P^{(2)}' &= 0, & P^{(3)}' &= 0, & P^{(4)}' &= \frac{\mu^{(4)}}{\mu^{(1)} + \mu^{(4)}}.
\end{align*}
\]

Transmission in the first time-slot: In the first time-slot, the \( j \)-th transmitter transmits \( \sum_{i=1}^{3} \sqrt{P^{(i)} X_j^{(i)}} \), for \( j \in \{1, \ldots, K\} \).

Decoding and feedback: The first receiver receives \( Y^{(1)}_1 = \sqrt{\text{SNR}} \sum_{i=1}^{3} \sqrt{P^{(i)} X_1^{(i)}} + \sqrt{\text{INR}} \sum_{j \neq 1} \sum_{i=1}^{3} \sqrt{P^{(i)} X_j^{(i)}} + Z^{(1)}_1 \). It decodes \( X_1^{(1)} \) by treating the rest of the signals as noise. The signal power is \( \text{SNR} P^{(1)} \) and the interference plus noise power is \( 1 + \text{SNR} P^{(2:3)} + \text{INR} P^{(1:3)}(K-1) \), and thus the decoding can be performed since (7) holds. After removing \( X_1^{(1)}, X_1^{(2)} \) is decoded by treating the rest as noise. Since
the signal power is $\text{SNR}P'(2)$ and the interference plus noise power is $1 + \text{SNR}P'(3) + \text{INR}P'(1:3)(K - 1)$, $X_1^{(2)}$ can be decoded since (8) holds.

The residual received signal after the contribution of $X_1^{(1)}$ and $X_1^{(2)}$ is removed is $\sqrt{\text{SNR}}\sqrt{P'(3)}X_1^{(3)} + \sqrt{\text{INR}}\sum_{j \neq 1}^3 \sqrt{P'(j)}X_j^{(1)} + Z_1^{(1)}$. Let $I = X_1^{(3)} + \sum_{j \neq 1} X_j^{(1)}$. Since $\text{SNR}\mu(3) = \text{SNR}\mu(1)$, we have the residual signal as $\sqrt{\text{SNR}}\sqrt{P'(3)}I + \sqrt{\text{INR}}\sum_{j \neq 1}^3 \sqrt{P'(j)}X_j^{(1)} + Z_1^{(1)}$. Note that $I$ is a lattice point, we can decode $I$ since (9) and (10) hold. After decoding $I$, $I$ is sent back to the transmitter. The maximum of the rates of the signals that $I$ is composed of is smaller than the feedback capacity by (11) and (12), and thus the message can be fed back.

**Transmission in the second time-slot:** The first transmitter receives $I = X_1^{(3)} + \sum_{j \neq 1} X_j^{(1)}$. Since it already knows $X_1^{(3)}$, and thus knows $\sum_{j \neq 1} X_j^{(1)}$, it sends $\sqrt{P'(1)}\sum_{j \neq 1} X_j^{(1)} + \sqrt{P'(4)}X_1^{(4)}$. In general, the $i^{th}$ transmitter sends $\sqrt{P'(1)}\sum_{j \neq i} X_j^{(1)} + \sqrt{P'(4)}X_i^{(4)}$, for $i \in \{1, \cdots, K\}$.

**Decoding:** The first receiver receives the signal $Y_1^{(2)} = \sqrt{\text{SNR}}\left(\frac{\sqrt{P'(1)}}{K-1}\sum_{j \neq 1} X_j^{(1)} + \sqrt{P'(4)}X_1^{(4)}\right) + \sqrt{\text{INR}}\sum_{j \neq 1} \left(\frac{\sqrt{P'(1)}}{K-1}\sum_{i \neq j} X_i^{(1)} + \sqrt{P'(4)}X_j^{(4)}\right) + Z_1^{(2)}$. Let $I_2 = \sum_{j \neq 1} X_j^{(1)}$. Since the receiver already knows $X_1^{(1)}$, the residual signal can be written as $\sqrt{\text{SNR}}\left(\frac{1}{K-1}\right) + \sqrt{\text{SNR}}\left(\frac{K-2}{K-1}\right)\sqrt{P'(1)}I_2 + \sqrt{\text{SNR}}\sqrt{P'(4)}X_1^{(4)} + \sqrt{\text{INR}}\sqrt{P'(4)}\sum_{j \neq 1} X_j^{(4)} + Z_1^{(2)}$. Thus, $I_2$ can be decoded by treating the rest of the terms as noise since (13) holds. Having decoded $I$ and $I_2$, $X_1^{(3)}$ can be decoded since it is the difference of the two. Having decoded $I_2$, the residual signal is $\sqrt{\text{SNR}}\sqrt{P'(4)}X_1^{(4)} + \sqrt{\text{INR}}\sqrt{P'(4)}\sum_{j \neq 1} X_j^{(4)} + Z_1^{(2)}$ from which $X_1^{(4)}$ can be decoded by treating $\sqrt{\text{INR}}\sqrt{P'(4)}\sum_{j \neq 1} X_j^{(4)} + Z_1^{(2)}$ as noise since (14) holds.

The next result gives the symmetric achievable rate for the weak interference regime ($1/2 < \alpha \leq 2/3$).

**Theorem 3.** For $1/2 < \alpha \leq 2/3$, the symmetric rate of $R_{\text{sym}} = \frac{R^{(1:6)}}{2}$ is achievable, for any $R^{(1)}, \cdots, R^{(6)}$. 
satisfying

\[ R^{(1)} \leq \log \left( \frac{\text{SNR}_{\mu}^{(1)}}{\text{SNR}_{\mu}^{(2:4)} + \text{SNR}_{\mu}^{(1:4)}(K - 1) + \mu^{(1:4)}} \right), \]

\[ R^{(2)} \leq \log \left( \frac{\text{SNR}_{\mu}^{(2)}}{\text{SNR}_{\mu}^{(3:4)} + \text{SNR}_{\mu}^{(1:4)}(K - 1) + \mu^{(1:4)}} \right), \]

\[ R^{(1)} \leq \log \left( \frac{\text{SNR}_{\mu}^{(2)}}{\text{SNR}_{\mu}^{(3:4)} + \text{SNR}_{\mu}^{(1:4)}(K - 1) + \mu^{(1:4)}} \right), \]

\[ R^{(3)} \leq \log \left( \frac{\text{SNR}_{\mu}^{(3)}}{\text{SNR}_{\mu}^{(4)} + \text{SNR}_{\mu}^{(3:4)}(K - 1) + \mu^{(1:4)}} \right), \]

\[ R^{(2)} \leq \log \left( \frac{\text{SNR}_{\mu}^{(4)}}{\text{SNR}_{\mu}^{(3)} + \text{SNR}_{\mu}^{(3:4)}(K - 1) + \mu^{(1:4)}} \right), \]

\[ R^{(4)} \leq \log \left( \frac{\text{SNR}_{\mu}^{(5)}}{\text{SNR}_{\mu}^{(4)} + \text{SNR}_{\mu}^{(3:4)}(K - 1) + \mu^{(1:4)}} \right), \]

\[ R^{(2)} \leq 2C_{FB}, \]

\[ R^{(3)} \leq 2C_{FB}, \]

\[ R^{(5)} \leq \log \left( \frac{\text{SNR}_{\mu}^{(5)}}{\text{SNR}(\mu^{(2)} + \mu^{(6)}) + \text{SNR}_{\mu}^{(2)}(K - 1) + (\mu^{(2)} + \mu^{(6)}) - \text{SNR}_{\mu}^{(2)}(\mu^{(2)})} \right), \]

\[ R^{(2)} \leq \log \left( \frac{(\sqrt{\text{SNR}(\frac{1}{K-1}) + \sqrt{\text{SNR}_{\mu}^{(2)}(\frac{K-2}{K-1})} \mu^{(2)})}{\text{SNR}_{\mu}^{(6)} + \text{SNR}_{\mu}^{(5:6)}(K - 1) + (\mu^{(2)} + \mu^{(5:6)})}} \right), \]

\[ R^{(5)} \leq \log \left( \frac{\text{SNR}_{\mu}^{(5)}}{\text{SNR}_{\mu}^{(6)} + \text{SNR}_{\mu}^{(5:6)}(K - 1) + (\mu^{(2)} + \mu^{(5:6)})} \right), \]

\[ R^{(6)} \leq \log \left( \frac{\text{SNR}_{\mu}^{(6)}}{\text{SNR}_{\mu}^{(6)}(K - 1) + (\mu^{(2)} + \mu^{(5:6)})} \right), \]

for any \( \mu^{(1)}, \ldots, \mu^{(6)} \geq 0 \), where \( \text{SNR}_{\mu}^{(3)} = \text{SNR}_{\mu}^{(2)} \).

**Proof:** We take \( X_i = \{X_i^{(1)}, X_i^{(2)}, X_i^{(3)}, X_i^{(4)}, X_i^{(5)}, X_i^{(6)}\} \) as the encoded symbols that user 1 wants to transmit during two consecutive time slots. The encoded symbol \( X_i^{(i)} \) is unit power, and of rate \( R^{(i)} \) using the lattice codes, \( i \in \{1, 2, 3, 4, 5, 6\} \). The overall rate is thus \( R = \frac{R^{(i)}}{2} \). Let \( P^{(i)} \) be the power of \( X_i^{(i)} \) transmitted in the first round, and \( P^{(i)} \) be the power of the \( X_i^{(i)} \) transmitted in the second round. The power allocations during the two rounds are chosen as

\[
P^{(1)} = \frac{\mu^{(1)}}{\mu^{(1:4)}}, \quad P^{(2)} = \frac{\mu^{(2)}}{\mu^{(1:4)}}, \quad P^{(3)} = \frac{\mu^{(3)}}{\mu^{(1:4)}}, \quad P^{(4)} = \frac{\mu^{(4)}}{\mu^{(1:4)}}, \quad P^{(5)} = P^{(6)} = 0,
\]

\[
P^{(1)} = 0, \quad P^{(2)} = \frac{\mu^{(2)}}{\mu^{(2)} + \mu^{(5:6)}}, \quad P^{(3)} = P^{(4)} = 0, \quad P^{(5)} = \frac{\mu^{(5)}}{\mu^{(2)} + \mu^{(5:6)}}, \quad P^{(6)} = \frac{\mu^{(6)}}{\mu^{(2)} + \mu^{(5:6)}}.
\]

**Transmission in the first time-slot:** In the first time-slot, the \( j \)-th transmitter sends \( \sum_{i=1}^{4} \sqrt{P^{(i)}} X_{ij}^{(i)} \).

**Decoding and feedback:** The first receiver receives the signal \( Y_{1}^{(1)} = \sqrt{\text{SNR}} \sum_{i=1}^{4} \sqrt{P^{(i)}} X_{1}^{(i)} + \)
The receiver first decodes $X_1^{(1)}$ by treating other terms as noise. Due to the rate constraint (16), $X_1^{(1)}$ can be decoded. After cancelling the terms containing $X_1^{(1)}$, $X_1^{(2)}$ can further be decoded by treating the remaining terms as noise due to (17). After cancelling $X_1^{(1)}$ and $X_1^{(2)}$, the receiver decodes $\sum_{i \neq 1} X_i^{(1)}$ as a $(K-1)$-dimensional lattice point, by treating all other terms as noise. The signal power is $\text{INR} P^{(1)}$ and the interference plus noise power is $1 + \text{SNR} P^{(3:4)} + \text{INR}(K-1) P^{(2:4)}$, and decoding is possible due to (18). The residual signal is $\sqrt{\text{SNR}} \sum_{j \neq 1}^4 \sqrt{P^{(j)}} X_j^{(i)} + \sqrt{\text{INR}} \sum_{j \neq 1}^4 \sqrt{P^{(j)}} X_j^{(i)}$. Let $I = X_1^{(3)} + \sum_{i \neq 1} X_i^{(2)}$. Since $\text{SNR}_\alpha^{(3)} = \text{SNR}_\alpha^{(2)}$, we have that the residual signal is $\sqrt{\text{SNR}} \sqrt{P^{(3)}} I + \sqrt{\text{SNR}} \sqrt{P^{(4)}} X_1^{(4)} + \sqrt{\text{INR}} \sum_{j \neq 1}^4 \sum_{i = 3}^6 \sqrt{P^{(i)}} X_j^{(i)} + Z_1^{(1)}$. Thus, $I$ can be decoded by treating the other terms as noise due to (19) and (20). After cancelling $I$, $X_1^{(4)}$ can be decoded due to (21).

$I$ is fed back as a lattice point, and the rate constraint for the feedback channel is satisfied due to (22) and (23).

**Transmission in the second time-slot:** From the feedback $I$ and known $X_1^{(3)}$, the first transmitter obtains $\sum_{j \neq 1} X_j^{(2)}$. Using this, it transmits $\frac{P^{(2)}}{K-1} \sum_{j \neq 1} X_j^{(2)} + \sum_{i = 5}^6 P^{(i)} X_1^{(i)}$. In general, the $k^{th}$ transmitter sends $\frac{P^{(2)}}{K-1} \sum_{j \neq k} X_j^{(2)} + \sum_{i = 5}^6 P^{(i)} X_k^{(i)}$, $k \in \{1, \ldots, K\}$.

**Decoding:** The first receiver receives $Y_1^{(2)} = \sqrt{\text{SNR}} \left( \frac{P^{(2)}}{K-1} \sum_{j \neq 1} X_j^{(2)} + \sum_{i = 5}^6 \sqrt{P^{(i)}} X_1^{(i)} \right) + \sqrt{\text{SNR}} \sum_{j \neq 1} \left( \frac{P^{(2)}}{K-1} X_1^{(2)} + \sum_{i = 5}^6 \sqrt{P^{(i)}} X_j^{(i)} \right) + Z_1^{(2)}$. Based on this, the receiver cancels the terms $X_1^{(2)}$ and obtains $\sqrt{\text{SNR}} \left( \frac{P^{(2)}}{K-1} \sum_{j \neq 1} X_j^{(2)} + \sum_{i = 5}^6 \sqrt{P^{(i)}} X_1^{(i)} \right) + \sqrt{\text{INR}} \frac{P^{(2)}}{K-1} \sum_{j \neq 1} X_j^{(2)} + \sqrt{\text{INR}} \sum_{j \neq 1} \sum_{i = 5}^6 \sqrt{P^{(i)}} X_j^{(i)} + Z_1^{(2)}$. From this residual signal, $X_1^{(5)}$ can be decoded by treating the other terms as noise due to (24). Let $I_2 \triangleq \sum_{j \neq 1} X_j^{(2)}$. After cancelling $X_1^{(5)}$, the residual signal is $\left( \sqrt{\text{SNR}} \frac{P^{(2)}}{K-1} + \sqrt{\text{INR}} \frac{P^{(2)}(K-2)}{K-1} \right) I_2 + \sqrt{P^{(6)}} X_1^{(6)} + \sqrt{\text{INR}} \sum_{j \neq 1} \sum_{i = 5}^6 \sqrt{P^{(i)}} X_j^{(i)} + Z_1^{(2)}$. Based on this, $I_2$ can be decoded by treating the rest as noise due to (25). From $I$ and $I_2$, $X_1^{(3)}$ can be decoded since it is the difference of the two. Further, $\sum_{i \neq 1} X_i^{(5)}$ can be decoded after cancelling $I_2$ due to (26). After cancelling $\sum_{i \neq 1} X_i^{(5)}$, the residual signal is $\sqrt{P^{(6)}} X_1^{(6)} + \sqrt{\text{INR}} \sum_{j \neq 1} \sqrt{P^{(6)}} X_j^{(6)} + Z_1^{(2)}$. From this, $X_1^{(6)}$ can be decoded by treating $X_j^{(6)}$, $j \neq 1$ as noise due to (27).

The next result gives the symmetric achievable rate for the strong interference regime ($\alpha \geq 2$).
Theorem 4. For $\alpha \geq 2$, the symmetric rate of $R_{\text{sym}} = \frac{R^{(1:3)}}{2}$ is achievable, for any $R^{(1)}, \ldots, R^{(3)}$ satisfying

$$R^{(1)} \leq \log \left( \frac{\text{INR}\mu^{(1)}}{\text{SNR}\mu^{(1:2)} + \text{SNR}^{\alpha}\mu^{(2)}(K - 1) + \mu^{(1:2)}} \right),$$

(29)

$$R^{(2)} \leq \log \left( \frac{\text{INR}\mu^{(2)}}{\text{SNR}\mu^{(1:2)} + \mu^{(1:2)}} \right),$$

(30)

$$R^{(1)} \leq \log \left( \frac{\text{SNR}\mu^{(1)}}{\text{SNR}\mu^{(2)} + \mu^{(1:2)}} \right),$$

(31)

$$R^{(2)} \leq 2C_{FB},$$

(32)

$$R^{(3)} \leq \log \left( \frac{\text{SNR}^{\alpha}\mu^{(3)}}{\text{SNR}\mu^{(2)} + \text{SNR}^{\alpha}\mu^{(3)} + \mu^{(2:3)}} \right),$$

(33)

$$R^{(2)} \leq \log \left( \frac{\text{SNR}^{\alpha}\mu^{(2)}}{\text{SNR}\mu^{(2)} + \mu^{(2:3)}} \right),$$

(34)

$$R^{(3)} \leq \log \left( \frac{\text{SNR}\mu^{(3)}}{\mu^{(2:3)}} \right),$$

(35)

for any $\mu^{(1)}, \mu^{(2)}, \mu^{(3)} \geq 0$.

Proof: We take $X_1 = \{X_1^{(1)}, X_1^{(2)}, X_1^{(3)}\}$ as the encoded symbols that user 1 wants to transmit during two consecutive time slots. The encoded symbol $X_1^{(i)}$ is unit power, and of rate $R^{(i)}$ using the lattice codes, $i \in \{1, 2, 3\}$. The overall rate is thus $R = \frac{R^{(1:3)}}{2}$. Let $P^{(i)}$ be the power of $X_1^{(i)}$ transmitted in the first round, and $P''^{(i)}$ be the power of the $X_1^{(i)}$ transmitted the second round. The power allocations during the two rounds are chosen as

$$P^{(1)} = \frac{\mu^{(1)}}{\mu^{(1:2)}}, \quad P^{(2)} = \frac{\mu^{(2)}}{\mu^{(1:2)}}, \quad P^{(3)} = 0,$$

$$P''^{(1)} = 0, \quad P''^{(2)} = \frac{\mu^{(2)}}{\mu^{(2:3)}}, \quad P''^{(3)} = \frac{\mu^{(3)}}{\mu^{(2:3)}},$$

(36)

Transmission in the first time-slot: In the first time-slot, the $j^{th}$ transmitter sends $\sum_{i=1}^{2} \sqrt{P^{(i)}}X_j^{(i)}$, $j \in \{1, \ldots, K\}$.

Decoding and feedback: The first receiver receives the signal $Y_1^{(1)} = \sqrt{\text{SNR}} \sum_{i=1}^{2} \sqrt{P^{(i)}}X_1^{(i)} + \sqrt{\text{INR}} \sum_{j \neq 1} \sum_{i=1}^{2} \sqrt{P^{(i)}}X_j^{(i)} + Z_1^{(1)}$. Define $I_1 \triangleq \sum_{j \neq 1} X_j^{(1)}$ and also $I_2 \triangleq \sum_{j \neq 1} X_j^{(2)}$. Then, we can write $Y_1^{(1)} = \sqrt{\text{SNR}} \sum_{i=1}^{2} \sqrt{P^{(i)}}X_1^{(i)} + \sqrt{\text{INR}}(\sqrt{P^{(1)}}I_1 + \sqrt{P^{(2)}}I_2) + Z_1^{(1)}$. From this received signal, $I_1$ can be decoded by treating the other terms as noise due to (29). After cancelling $I_1$, $I_2$ can be decoded since (30) holds. After cancelling $I_2$, the residual signal is $\sqrt{\text{SNR}} \sum_{i=1}^{2} \sqrt{P^{(i)}}X_1^{(i)} + Z_1^{(1)}$ from which $X_1^{(1)}$ can be decoded by treating $X_1^{(2)}$ as noise due to (31).
The first receiver feeds back the lattice codeword $I_2$ to the first transmitter, and the rate constraint is satisfied due to (32).

**Transmission in the second time-slot:** The transmitter receives $I_2$ and transmits $\sqrt{\frac{P''}{K-1}} I_2 + \sqrt{P''} X_1^{(3)}$. In general, the $j^{th}$ transmitter transmits $\sqrt{\frac{P''}{K-1}} \sum_{i \neq j} X_i^{(2)} + \sqrt{P''} X_j^{(3)}$.

**Decoding:** The first receiver receives $Y_1^{(2)} = \sqrt{\text{SNR}} \left( \sqrt{\frac{P''}{K-1}} I_2 + \sqrt{P''} X_1^{(3)} \right) + \sqrt{\text{INR}} \sum_{j \neq 1} \left( \sqrt{\frac{P''}{K-1}} \sum_{i \neq j} X_i^{(2)} + \sqrt{P''} X_j^{(3)} \right) + Z_1^{(2)}$. Since the receiver knows $I_2$, it can be subtracted to get the residual signal $\sqrt{\text{INR}} \sqrt{P''} X_1^{(2)} + \sqrt{\text{SNR}} \sqrt{P''} X_1^{(3)} + \sqrt{\text{INR}} \sqrt{P''} \sum_{j \neq 1} X_j^{(3)} + Z_1^{(2)}$. From this, $\sum_{j \neq 1} X_j^{(3)}$ can be decoded, since (33) holds. Afterwards, the residual signal is $\sqrt{\text{INR}} \sqrt{P''} X_1^{(2)} + \sqrt{\text{SNR}} \sqrt{P''} X_1^{(3)} + Z_1^{(2)}$. Then, $X_1^{(2)}$ can be decoded by treating $X_1^{(3)}$ as noise due to (34). Finally, after cancelling $X_1^{(3)}$, $X_1^{(3)}$ can be decoded due to (35).

**Remark 2.** In the above Theorems 2-4, we used lattice codes for all signals $X^{(i)}$, which led to bounds of the form $R = \log(\text{SNR})$. We could improve the lower bounds by using Gaussian codebooks to code some of these signals as follows.

1) In Theorem 2, we could use the lattice code structure to code $X^{(1)}$ and $X^{(3)}$, but use Gaussian codes for $X^{(2)}$ and $X^{(4)}$, since $X^{(2)}$ and $X^{(4)}$ are never added. The coding, decoding, and feedback schemes would remain the same. This would lead to an improvement in terms of $R^{(2)}$ and $R^{(4)}$, as $\log(1 + \text{SNR})$ can be used rather than $\log(\text{SNR})$.

2) In Theorem 3, $X^{(4)}$ and $X^{(6)}$ can be chosen from a Gaussian codebook.

3) In Theorem 4, $X^{(3)}$ can be chosen from a Gaussian codebook.

Thus, the above simple modification of choosing some of the signals from a Gaussian codebook can lead to a slight improvement in the achievable rates.

3) A Conjectured Upper Bound: According to Theorem 1 of [2], an upper bound on the symmetric capacity without feedback is given by

$$R_{\text{sym},0}^u = \min \{ \log(1 + \text{SNR}), \log(1 + \text{INR} + \frac{\text{SNR}}{1 + \text{INR}}) \}. \quad (37)$$

Moreover, according to Section VI of [16], an upper bound on the symmetric capacity with infinite feedback is given by

$$R_{\text{sym},\infty}^u = \frac{1}{2} \log(1 + \frac{\text{SNR}}{1 + \text{INR}}) + \frac{1}{2} \log(1 + \text{SNR} + \text{INR}) + \frac{K - 1}{2} + \log K. \quad (38)$$
We conjecture that the following upper bound holds for a $K$-user symmetric Gaussian IC with rate-limited feedback

\[ R^u_{\text{sym}} = \min \{ R^u_{\text{sym}, \infty}, R^u_{\text{sym}, 0} + C_{FB} \}. \]  

(39)

Note that the conjecture holds true for $K = 2$ as shown in [17].

The next result shows that the achievable symmetric rate given in the last section is within a constant number of bits to the conjectured upper bound $R^u_{\text{sym}}$ for a particular choice of the parameters for each interference regime.

**Theorem 5.** For the $K$-user symmetric Gaussian IC with rate-limited feedback with $\frac{\text{INR}}{\text{SNR}} \not\in \left(\frac{1}{2}, 2\right)$ and $\text{SNR}, \text{INR} \geq 1$, there is an achievability scheme that achieves a symmetric rate within a constant $L$ bits to $R^u_{\text{sym}}$, where

\[
L = \max \left\{ \frac{1}{2} \log \left( 2304(K - 1)^2 K^2 \left( K + \frac{1}{3} \right) \left( K + \frac{2}{3} \right)^2 (K + 2)^2 \left( K + \frac{11}{4} \right) \right), \log 3 + 16 + \log K^3 \right\} + \frac{K - 1}{2}.
\]

(40)

The detailed proof for this result is provided in Appendix B. The parameters $\mu^{(i)}$ of the achievability scheme that are chosen for this result are as follows.

Case 1 ($\alpha \leq \frac{1}{2}$): We take $\mu^{(1)} = \frac{1}{2\text{INR}} \min \{ 2^{-2C_{FB}}, \text{INR} - 1 \}$, $\mu^{(2)} = \frac{1}{\text{INR}} - \frac{1}{25\text{SNR}} \min \{ 2^{-2C_{FB}}, \text{INR} - 1 \}$, and $\mu^{(4)} = \frac{1}{\text{INR}}$ in Theorem 2.

Case 2 ($\frac{1}{2} \leq \alpha \leq \frac{2}{3}$): We take $\mu^{(4)} = \frac{1}{4\text{INR}} \max \{ 2^{-2C_{FB}}, \frac{\text{INR}^3}{\text{SNR}^2} \}$, $\mu^{(6)} = \mu^{(3)} = \frac{1}{3\text{INR}} - \frac{1}{4\text{INR}} \max \{ 2^{-2C_{FB}}, \frac{\text{INR}^3}{\text{SNR}^2} \}$, $\mu^{(1)} = 1 - \mu^{(2:4)}$, and $\mu^{(5)} = 1 - \mu^{(2)} - \mu^{(6)}$ in Theorem 3.

Case 3 ($2 \leq \alpha$): We take $\mu^{(2)} = \frac{\text{SNR}}{2\text{INR}} \min \{ 2^{-2C_{FB}}, \frac{\text{INR}^3}{\text{SNR}^2} \}$, and $\mu^{(1)} = \mu^{(3)} = 1 - \mu^{(2)}$ in Theorem 4.

**Remark 3.** An achievability scheme for a two-user ($K = 2$) symmetric Gaussian IC is given in [17], and it is claimed that the achievable rate is within a constant of 14.8 bits from the upper bound $R^u_{\text{sym}}$. However, there is an error in the calculation for the range of $\alpha \in [1/2, 2/3]$ and consequently the gap is actually not constant for this range. We will explain this non-constant gap in Remark 5. On the other hand, our proposed achievability scheme when specialized to $K = 2$, results in a symmetric rate that is within a constant of 21.085 bits to the symmetric rate upper bound, according to Theorem 5.

**Remark 4.** For the special cases of no feedback and infinite feedback, $R^u_{\text{sym}}$ in (39) becomes the true symmetric upper bounds given in [2] and [16], respectively. Furthermore, the achievability schemes in
[6] and [16] achieve symmetric rates within constant gaps of $9 + \log(K^2)$ and $\frac{1}{2} \log(16K^4(K + 1)) + \frac{K-1}{2}$ bits to the corresponding upper bounds, for no feedback and infinite feedback, respectively. Although these gaps are tighter, they are only for the two extreme cases.

4) Numerical Results: We now provide numerical results on symmetric rate of the $K$-user symmetric Gaussian IC with limited feedback. In Fig. 5, we consider three different values of $\alpha$ corresponding to the three interference regions - very weak, weak and strong interferences, and plot the symmetric rate as a function of SNR for $K = 3$. It is seen that the achievable symmetric rate increases with the feedback capacity.

![Graphs showing symmetric rate as a function of SNR for $K = 3$.](image)

(a) Very weak interference with $\alpha = \frac{1}{4}$.

(b) Weak interference with $\alpha = \frac{7}{12}$.

(c) Strong interference with $\alpha = \frac{5}{2}$.

Fig. 5. Achievable symmetric rate as a function of SNR for $K = 3$.

We next consider the special case of no feedback and compare the achievable rate of our scheme to that of the scheme in [6]. We let $C_{FB} = 0$, and consider some values of $\alpha$ corresponding to the different interference regimes. Note that in this case, the conjectured upper bound in (40) becomes the upper bound in [2]. The achievable rate of our proposed scheme and that of the scheme in [6] as well as the upper
bound, are plotted in Fig. 6 for $K = 3$. We note that the proposed achievable symmetric rate is better than that in [6] for the parameters considered in weak and strong interference regimes. Also although for the case of very weak interference the achievable rate in [6] is higher, the slope of our scheme is higher. We can compare the constant gaps between the upper and lower bounds given in Theorem 1 of [6] for $C_{FB} = 0$ and those given in Appendix B of this paper for general $C_{FB}$, for the parameters of Fig. 6: for the very weak, weak, and strong interference regimes, the gaps of [6] are 3 bits, 11 bits, and 2 bits, respectively; for our scheme with $C_{FB} = 0$, the gaps are 5.02 bits, 13.7 bits, and 4.45 bits, respectively, according to (55), (73), and (85), respectively.

![Graph](image_url)

(a) Very weak interference with $\alpha = \frac{1}{4}$.

(b) Weak interference with $\alpha = \frac{7}{12}$.

(c) Strong interference with $\alpha = \frac{5}{2}$.

Fig. 6. Comparison of our results with that in [6] and [2] for the case of no feedback and $K = 3$.

We next consider the special case of infinite feedback. In this case, the conjectured upper bound in (39) becomes the upper bound in [16]. In Fig. 7, we compare the achievable rate of the proposed scheme when $C_{FB} = \infty$ to that of the scheme in [16] for some values of $\alpha$ corresponding to the different interference regimes for $K = 3$. We note that the proposed achievable symmetric rate is better than the achievable...
rate in [16] for the parameters considered in strong and very weak interference regimes. For the weak interference regime, our achievability is better for high SNR as compared to that in [16] and the slope of our scheme is higher. We can also compare the constant rate gaps for \( C_{FB} = \infty \) in [16] and our constant gaps for general \( C_{FB} \). In particular, according to the proof of Theorem 1 (Section V and Section VI) of [16], for the very weak, weak, and strong interference regimes, the gaps are 7.17 bits, 7.17 bits, and 4.38 bits, respectively; and our corresponding constant gaps are 9.84 bits, 15.49 bits, and 7.62 bits, respectively, according to (56), (74), and (86), respectively.

Finally we consider the special case of two-user IC with limited feedback. We set \( K = 2 \), \( C_{FB} = 1 \). In Fig. 8, we compare our achievable symmetric rate with that in [17] in different interference regimes. In this case, the conjectured upper bound in (39) becomes the true upper bound in [17]. It is seen that our rate is better in the strong interference regime. And for the other two regions, our scheme has higher slopes and are better at high SNR. We can also compare the constant gaps between the upper and lower
bounds given in Appendix D of [17] for $K = 2$ and those given in Appendix B of this paper for general $K$, for the parameters of Fig. 8: for the very weak and strong interference regimes, the gaps of [17] are 9.6 bits and 7 bits, respectively; and our corresponding gaps are 5.27 bits and 4.6 bits, respectively. Hence our bounds are tighter in these two regimes. For the weak interference regime, i.e., $1/2 < \alpha < 2/3$ as noted in Remark 3 and Remark 5, the achievable rate in [17] is actually not within a constant gap to the upper bound; whereas our proposed achievability scheme achieves a symmetric rate that is within 21.085 bits to the upper bound.

![Graph](image1)

(a) Very weak interference with $\alpha = \frac{1}{4}$.

![Graph](image2)

(b) Weak interference with $\alpha = \frac{7}{12}$.

![Graph](image3)

(c) Strong interference with $\alpha = \frac{5}{2}$.

Fig. 8. Comparison of the proposed achievability scheme with that in [17] for the two-user case, $K = 2$, $C_{FB} = 1$.

5) Achievable Symmetric GDoF: The symmetric GDoF characterize the ratio of the symmetric capacity to $\log \text{SNR}$ as $\text{SNR}$ goes to infinity, i.e., $\text{GDoF} = \lim_{\text{SNR} \to \infty} \frac{C_{\text{sym}}}{\log \text{SNR}}$. Recall that $\alpha = \frac{\log \text{INR}}{\log \text{SNR}}$ and $\beta = \frac{C_{FB}}{\log \text{SNR}}$. We have the following result.
Theorem 6. The symmetric GDoF of a $K$-user symmetric Gaussian IC with rate-limited feedback satisfies

$$\text{GDoF}_{\text{sym}} \geq \begin{cases} 
\min\{1 - \alpha + \beta, 1 - \frac{\alpha}{2}\}, & \text{if } 0 \leq \alpha \leq \frac{1}{2}, \\
\min\{\alpha + \beta, 1 - \frac{\alpha}{2}\}, & \text{if } \frac{1}{2} \leq \alpha \leq \frac{2}{3}, \\
1 - \frac{\alpha}{2}, & \text{if } \frac{2}{3} \leq \alpha < 1, \\
\text{not well defined}, & \text{if } \alpha = 1, \\
\frac{\alpha}{2}, & \text{if } 1 < \alpha \leq 2, \\
\min\{1 + \beta, \frac{\alpha}{2}\}, & \text{if } 2 \leq \alpha. 
\end{cases}$$ (41)

Proof: Since the achievable symmetric rate is within a constant gap to $R_{u_{\text{sym}}}$ in (39), we can write

$$\text{GDoF}_{\text{sym}} \geq \frac{R_{u_{\text{sym}}}}{\log \text{SNR}} = \lim_{\text{SNR} \to \infty} \frac{R_{u_{\text{sym}}}}{\log \text{SNR}} = \min\{\text{GDoF}_{\text{sym},\infty}, \text{GDoF}_{\text{sym},0} + \beta\}$$

where $\text{GDoF}_{\text{sym},0}$ and $\text{GDoF}_{\text{sym},\infty}$ are given in Theorem 3.1 of [5] and Theorem 1 of [16], respectively.

We note that if we normalize (2) by $n$ and use the definitions of $\alpha = m/n$, $\beta = p/n$, then we obtain (41), except for $\alpha = 1$. There is a discussion on $\alpha = 1$ in [16]. Hence Fig. 1 describes the achievable symmetric GDoF of a $K$-user symmetric Gaussian IC as well.

Remark 5. Consider the achievability scheme for the two-user symmetric Gaussian IC in [17] for the case of $1/2 < \alpha < 2/3$. We set the feedback capacity as $C_{FB} = \log \left( \frac{\text{SNR}^2}{\text{INR}^2} - 1 \right)$. In this case, the GDoFs corresponding to the six terms in Eq.(55) in [17] under the power allocation given by Eq. (84) in [17] are $1 - \alpha, 0, 2\alpha - 1, 1 - \alpha, 0,$ and $2\alpha - 1$, respectively, with a sum of $2\alpha$. However, the sum GDoF of the achievability scheme which is the sum of these six terms, is claimed in Eq. (87) of [17] to be $2 - \alpha = 2\alpha + (2 - 3\alpha) > 2\alpha$ which is incorrect. Since in this range of $\alpha$, the upper bound on sum rate satisfies $\lim_{\text{SNR} \to \infty} \frac{2R_{u_{\text{sym}}}}{\log \text{SNR}} = 2 - \alpha$, the gap between the upper and lower bounds for high SNR is $(2 - 3\alpha)\log \text{SNR} + o(\log \text{SNR})$, i.e., it is unbounded.

IV. Conclusions

We have developed achievability schemes for symmetric $K$-user interference channels with rate-limited feedback, for both the linear deterministic model, and the Gaussian model. For the deterministic model, the achievable symmetric rate is the minimum of the symmetric capacity with infinite feedback, and the sum of the symmetric capacity without feedback and the amount of symmetric feedback. And for the Gaussian model, the achievable rate is within a constant gap to the minimum of the upper bound on the symmetric capacity with infinite feedback, and the sum of the upper bound of the symmetric capacity without feedback and the amount of symmetric feedback. For the Gaussian model, the proposed achievability
scheme employs lattice codes to perform Han-Kobayashi message splitting, interference-decoding, and decode-and-forward. Further, the achievable generalized degrees of freedom (GDoF) is characterized with rate-limited feedback. It is shown that the per-user GDoF does not depend on the number of users, so that it is the same as that of the two-user interference channel with rate-limited feedback.

We conjecture that the minimum of the upper bound of the symmetric capacity with infinite feedback, and the sum of the upper bound of the symmetric capacity without feedback and the amount of symmetric feedback is an upper bound for the symmetric capacity of the Gaussian IC with rate-limited feedback for any number of users $K$. This conjecture has been shown to hold for the $K$-user IC without feedback in [6], the $K$-user IC with infinite feedback in [16], and $K = 2$ in [17]. However, it remains open for general $K$ and $C_{FB}$.

APPENDIX A

PROOF OF THEOREM 1

In this section, we prove Theorem 1 by breaking the result into three regimes. We denote that $a_{i,j} \triangleq \sum_{k=1, k \neq i}^{K} a_{k,j}$

Lemma 1. For the $K$-user linear deterministic IC, a symmetric rate of $n \min \{1 - \alpha + \beta, 1 - \frac{\alpha}{2}\}$ is achievable for $0 \leq \alpha \leq \frac{1}{2}$.

Proof: Define $l \triangleq (m - 2p)^+$. For the $i$th transmitter, $i \in \{1, ..., K\}$, we transmit $a_{i,1}, ..., a_{i,n} - m$ in two transmission slots.

First Round:

1. Transmission: In the first round, the $i$th transmitter sends $a_{i,1}, ..., a_{i,n-l}$ on the highest $n-l$ transmission levels, respectively, and nothing on the lowest $l$ transmission levels.

2. Reception: Since $0 \leq \alpha \leq \frac{1}{2}$, the $i$th receiver receives $a_{i,1}, ..., a_{i,n-m}$ on the highest $n-m$ reception levels, respectively, and $a_{i,n-m+1} + a_{i,1}, ..., a_{i,n-l} + a_{i,m-l}$ on the next $m-l$ levels, respectively, and throws away whatever it receives on the last $l$ levels.

Feedback:

Receiver $i$ sends back $a_{i,n-m+1} + a_{i,1}, ..., a_{i,n-l} + a_{i,m-l}$ over the feedback channel to transmitter $i$ ($m-l$ bits). Since $0 \leq m-l \leq 2p$, the feedback rate is $p$ bits per channel use. With this feedback, transmitter $i$ decodes $a_{i,1}, ..., a_{i,m-l}$. Since the feedback does not increase the achievable rate in the statement of the Theorem beyond $p = m/2$, we only use $m/2$ bits of feedback if $p > m/2$. 
Second Round:

1. Transmission: In the second round, the $i^{th}$ transmitter sends $\overline{a_{i,1}}, ..., \overline{a_{i,m-l}}$ on the highest $m-l$ transmission levels, respectively, nothing on the next lower $l$ levels, and new bits of $a_{i,n-l+1}, ..., a_{i,2n-m-l}$ on the last $n-m$ levels, respectively.

2. Reception: The $i^{th}$ receiver receives $\overline{a_{i,1}}, ..., \overline{a_{i,m-l}}$ on the highest $m-l$ levels, nothing on the next $l$ levels, $a_{i,n-l+1}, ..., a_{i,2n-m-l}$ on the next $n-2m$ levels, $a_{i,2n-2m-l+1} + (K-2)\overline{a_{i,1}} + (K-1)a_{i,1}, ..., a_{i,2n-m-2} + (K-2)\overline{a_{i,m-l}} + (K-1)a_{i,m-l}$ on the next $m-l$ levels, and $a_{i,2n-m-2l+1}, ..., a_{i,2n-m-l}$ on the lowest $l$ levels.

Decoding:

Decoding by the $i^{th}$ receiver, $i \in \{1, ..., K\}$, is performed as follows. First, $a_{i,1}, ..., a_{i,n-m}$ are decoded from the highest $n-m$ levels of the first reception. Then, $\overline{a_{i,1}}, ..., \overline{a_{i,m-l}}$ are decoded from the highest $m-l$ levels of the second reception. Then, having $\overline{a_{i,1}}, ..., \overline{a_{i,m-l}}$, the receiver decodes $a_{i,n-m+1}, ..., a_{i,n-l}$ from $a_{i,n-m+1} + \overline{a_{i,1}}, ..., a_{i,n-l} + \overline{a_{i,m-l}}$ on the next $m-l$ levels of the first reception. Then, the receiver decodes $a_{i,n-l+1}, ..., a_{i,2n-2m-l}$ from the $(m+1)^{th}$ to $(n-m)^{th}$ highest levels of the second reception, respectively. Then, having $a_{i,1}, ..., a_{i,m-l}$, and $\overline{a_{i,1}}, ..., \overline{a_{i,m-l}}$, the receiver decodes $a_{i,2n-2m-l+1}, ..., a_{i,2n-m-2l}$ from $a_{i,2n-2m-l+1} + (K-2)\overline{a_{i,1}} + (K-1)a_{i,1}, ..., a_{i,2n-m-2} + (K-2)\overline{a_{i,m-l}} + (K-1)a_{i,m-l}$ on the next $m-l$ lower levels of the second reception. Finally, the receiver decodes $a_{i,2n-m-2l+1}, ..., a_{i,2n-m-l}$ from the lowest $l$ levels of the second reception.

Rate:

With the above strategy, each user transmits $2n-m-l$ bits in two uses of the channel which proves the lemma because

\[
\frac{1}{2}(2n-m-l) = \frac{1}{2}(2n - m - (m - 2p)^+) = \frac{1}{2} \min\{2n - m, 2n - 2m + 2p\} = \min\{n - \frac{1}{2}m, n - m + p\} = n \min\{1 - \frac{\alpha}{2}, 1 - \alpha + \beta\}. \tag{42}
\]

Lemma 2. For the $K$-user linear deterministic IC, a symmetric rate of $n \min\{\alpha + \beta, 1 - \frac{\alpha}{2}\}$ is achievable.
for $\frac{1}{2} \leq \alpha \leq \frac{2}{3}$.

**Proof:** Define $l' \triangleq (2n - 3m - 2p)^+$. For the $i^{th}$ transmitter, $i \in \{1, \ldots, K\}$, we transmit $a_{i,1}, \ldots, a_{i,2n-m-l'}$ in two transmission slots.

**First Round:**

1. Transmission: In the first round, the $i^{th}$ transmitter sends $a_{i,1}, \ldots, a_{i,n-m-l'}$ on the highest $n - m - l'$ transmission levels, nothing on the next lower $2m - n + l'$ levels, and $a_{i,n-m-l'+1}, \ldots, a_{i,2n-m-l'}$ on the lowest $n - m$ levels.

2. Reception: Since $\frac{1}{2} \leq \alpha \leq \frac{2}{3}$, the $i^{th}$ receiver receives $a_{i,1}, \ldots, a_{i,n-m-l'}$ on the highest $n - m - l'$ reception levels, nothing on the next lower $l'$ levels, $\overline{a_{i,1}}, \ldots, \overline{a_{i,2m-n}}$ on the next lower $2m-n$ levels, $a_{i,n-m-l'+1} + \overline{a_{i,2m-n+1}}, \ldots, a_{i,3n-4m-2l'} + \overline{a_{i,n-m-l'}}$ on the next $2n - 3m - l'$ levels, and $a_{i,3n-4m-2l'+1}, \ldots, a_{i,2n-m-l'}$ on the lowest $2m - n + l'$ levels.

**Feedback:**

Receiver $i$ sends back $a_{i,n-m-l'+1} + \overline{a_{i,2m-n+1}}, \ldots, a_{i,2n-m-l'} + \overline{a_{i,n-m-l'}}$ over the feedback channel to transmitter $i$ ($2n - 3m - l'$ bits). Since $0 \leq 2n - 3m - l' \leq 2p$, the feedback rate is $p$ bits per channel use. With this feedback, transmitter $i$ decodes $\overline{a_{i,2m-n+1}}, \ldots, \overline{a_{i,n-m-l'}}$.

**Second Round:**

1. Transmission: In the second round, the $i^{th}$ transmitter sends the new bits $a_{i,2n-2m-l'+1}, \ldots, a_{i,n-l'}$ on the highest $2m - n$ transmission levels, $\overline{a_{i,2m-n+1}}, \ldots, \overline{a_{i,n-m-l'}}$ on the next $2n - 3m - l'$ levels, nothing on the next lowest $2m - n + l'$ levels, and the new bits $a_{i,n-l'+1}, \ldots, a_{i,2n-m-l'}$ on the lowest $n - m$ levels.

2. Reception: In this round, the $i^{th}$ receiver receives $a_{i,2n-2m-l'+1}, \ldots, a_{i,n-l'}$ on the highest $2m - n$ reception levels, $\overline{a_{i,2m-n+1}}, \ldots, \overline{a_{i,n-m-l'}}$ on the next $2n - 3m - l'$ levels, nothing on the next lower $l'$ levels, $\overline{a_{i,2n-2m-l'+1}}, \ldots, \overline{a_{i,n-l'}}$ on the next lower $2m - n$ levels, $a_{i,n-l'+1} + (K - 2)\overline{a_{i,2m-n+1}} + (K - 1)a_{i,2m-n+1}, \ldots, a_{i,3n-3m-2l'} + (K - 2)\overline{a_{i,2n-3m-l'}} + (K - 1)a_{i,2n-3m-l'}$ on the next lower $2n - 3m - l'$ levels, and $a_{i,3n-3m-2l'+1}, \ldots, a_{i,2n-m-l'}$ on the lowest $2m - n + l'$ levels.

**Decoding:**

Decoding by the $i^{th}$ receiver, $i \in \{1, \ldots, K\}$, is performed as follows. First, $a_{i,1}, \ldots, a_{i,n-m-l'}$ are decoded from the highest $n - m - l'$ levels of the first reception. Then, $a_{i,3n-4m-2l'+1}, \ldots, a_{i,2n-2m-l'}$ are decoded from the lowest $2m - n + l'$ levels of the first reception. Further, $a_{i,2n-2m-l'+1}, \ldots, a_{i,n-l'}$ are decoded from the highest $2m - n$ levels of the second reception, and $\overline{a_{i,2m-n+1}}, \ldots, \overline{a_{i,n-m-l'}}$ are decoded from the next $2n - 3m - l'$ levels of the second reception. Moreover, $a_{i,3n-3m-2l'+1}, \ldots, a_{i,2n-m-l'}$ are decoded from the
lowest $2m - n + l'$ levels of the first transmission, respectively.

Then, having $a_{i,2m-n+1}, ..., a_{i,n-m+l'}$, the receiver decodes $a_{i,n-m-l'+1}, ..., a_{i,2n-3m-2l'}$ from $a_{i,n-m-l'+1} + a_{i,2m-n+1}, ..., a_{i,2n-3m-2l'} + a_{i,n-m-l'}$ in the first reception. Finally, having $a_{i,2m-n+1}, ..., a_{i,2n-3m-l'}$, the receiver decodes $a_{i,n-l'+1}, ..., a_{i,3n-3m-2l'}$ from $a_{i,n-l'+1} + (K-2)a_{i,2m-n+1} + (K-1)a_{i,2n-3m-l'}$ in the second reception.

Rate:
With the above strategy, each user transmits $2n - m - l'$ bits in two uses of the channel which proves the lemma because

$$\frac{1}{2}(2n - m - l') = \frac{1}{2}(2n - m - (2n - 3m - 2p)^+) = \frac{1}{2} \min\{2n - m, 2m + 2p\} = \min\{n - \frac{1}{2} m, m + p\} = n \min\{1 - \frac{\alpha}{2}, \alpha + \beta\}. \quad (43)$$

Lemma 3. For the $K$-user linear deterministic IC, a symmetric rate of $n \min\{1 + \beta, \frac{\alpha}{2}\}$ is achievable for $\alpha \geq 2$.

Proof: Define $l'' \triangleq (m - 2n - 2p)^+$. For the $i$th transmitter, $i \in \{1, ..., K\}$, we transmit $a_{i,1}, ..., a_{i,m-l''}$ in two transmission slots.

First Round:
1. Transmission: In the first round, the $i$th transmitter sends $a_{i,1}, ..., a_{i,m-n-l''}$ on the highest $m - n - l''$ transmission levels, respectively, and nothing on the lower $n + l''$ levels.
2. Reception: Since $\alpha \geq 2$, the $i$th receiver receives $a_{i,1}, ..., a_{i,m-n-l''}$ on the highest $m - l''$ reception levels, nothing on the next lower $l''$ levels, and $a_{i,1}, ..., a_{i,n}$ on the lowest $n$ levels.

Feedback:
Receiver $i$ sends back $a_{i,n+1}, ..., a_{i,m-n-l''}$ over the feedback channel to the $i$th transmitter $(m - 2n - l''$ bits). Since $0 \leq m - 2n - l'' \leq 2p$, the feedback rate is $p$ bits per channel use.

Second Round:
1. Transmission: In the second round, the $i$th transmitter sends new bits $a_{i,m-n-l''+1}, ..., a_{i,m-l''}$ on the
highest \( n \) transmission levels, \( a_{i,n+1}, \ldots, a_{i,m-n-l''} \) on the next \( m-2n-l'' \) levels, and nothing on the lower \( n+l'' \) levels.

2. Reception: The \( i^{th} \) receiver receives \( a_{i,n+1}, \ldots, a_{i,m-n-l''} \) on the highest \( n \) reception levels, \( (K-1)a_{i,n+1} + (K-2)a_{i,n+1}, \ldots, (K-1)a_{i,m-n-l''} + (K-2)a_{i,m-n-l''} \) on the next \( m-2n-l'' \) levels, nothing on the next lower \( l'' \) levels, and \( a_{i,m-n-l''+1}, \ldots, a_{i,m-l''} \) on the lowest \( n \) levels.

Decoding:

Decoding at the \( i^{th} \) receiver, \( i \in \{1, \ldots, K\} \) is performed as follows. First, \( a_{i,1}, \ldots, a_{i,n} \) are decoded from the lowest \( n \) levels of the first reception, \( a_{i,n+1}, \ldots, a_{i,m-n-l''} \) are decoded from the \((n+1)^{th}\) to \((m-n-l'')^{th}\) highest levels of the first reception, and \( a_{i,m-n-l''+1}, \ldots, a_{i,m-l''} \) are decoded from the lowest \( n \) levels of the second reception. Then, having \( a_{i,n+1}, \ldots, a_{i,m-n-l''} \), the receiver decodes \( a_{i,n+1}, \ldots, a_{i,m-n-l''} \) from \( (K-1)a_{i,n+1} + (K-2)a_{i,n+1}, \ldots, (K-1)a_{i,m-n-l''} + (K-2)a_{i,m-n-l''} \) in the second reception.

Rate:

With the above strategy, each user transmits \( m-l'' \) bits in two uses of the channel which proves the lemma because

\[
\frac{1}{2}(m-l'') = \frac{1}{2}(m-(m-2n-2p)^+) = \frac{1}{2}\min\{m,2n+2p\} = \min\{\frac{m}{2},n+p\} = n\min\{\frac{\alpha}{2},1+\beta\}. \tag{44}
\]

\[\blacksquare\]

APPENDIX B

PROOF OF THEOREM 5

We split the proof into three cases: \( \alpha \leq \frac{1}{2}, \frac{1}{2} < \alpha \leq \frac{2}{3}, \) and \( \alpha \geq 2. \)

Case 1 \((\alpha \leq \frac{1}{2})\): We use the following parameters in Theorem 2: \( \mu^{(1)} = \frac{1}{\ln\text{R}} \min\{2^{2C_{FB}}, \ln\text{R} - 1\}, \)
\( \mu^{(2)} = \frac{1}{\ln\text{R}} - \frac{1}{\ln\text{SNR}} \min\{2^{2C_{FB}}, \ln\text{R} - 1\}, \) and \( \mu^{(4)} = \frac{1}{\ln\text{R}}. \) We first lower bound the right-hand sides (RHS) of (7)-(14) as follows.
RHS of (7):

\[
\begin{align*}
\log \left( \frac{\text{SNR}_\mu^{\text{a}}}{\text{SNR}_\mu^{\text{b}} + \text{SNR}_\alpha^{\text{c}}(K - 1) + \mu^{\text{d}}(K - 1) + \mu^{\text{e}}(K - 1)} \right) \\
= & \log \left( \frac{\frac{1}{2} \text{SNR}^{1 - \alpha} \min\{2^{2CFB}, \text{INR} - 1\}}{\text{SNR}^{1 - \alpha} + (K - 1) + \frac{1}{2} \min\{2^{2CFB}, \text{INR} - 1\}(K - 1) + \mu^{\text{d}}(K - 1) + 1} \right) \\
\overset{(a)}{\geq} & \log \left( \frac{\frac{1}{2} \text{SNR}^{1 - \alpha} \min\{2^{2CFB}, \text{INR} - 1\}}{\text{SNR}^{1 - \alpha} + (K - 1) + \frac{1}{2} \min\{2^{2CFB}, \text{INR} - 1\}(K - 1) + 1} \right) \\
\overset{(b)}{\geq} & \log \left( \frac{\frac{1}{2} \text{SNR}^{1 - \alpha} \min\{2^{2CFB}, \text{INR} - 1\}}{\text{SNR}^{1 - \alpha} + (K - 1) + \frac{1}{2} (\text{INR} - 1)(K - 1) + 1} \right) \\
\overset{(c)}{\geq} & \log \left( \frac{\frac{1}{2} \text{SNR}^{1 - \alpha} \min\{2^{2CFB}, \text{INR} - 1\}}{\text{SNR}^{1 - \alpha} (K + 1) + \frac{1}{2}(K + 1)} \right) \\
= & \log \left( \frac{\frac{1}{2} \text{SNR}^{1 - \alpha} \min\{2^{2CFB}, \text{INR} - 1\}}{\text{SNR}^{1 - \alpha} + 1} \right) - \log(K + 1) \\
\overset{(d)}{\geq} & \log \left( \frac{\frac{1}{2} \text{SNR}^{1 - \alpha} \min\{2^{2CFB}, \text{INR} - 1\}}{2 \text{SNR}^{1 - \alpha}} \right) - \log(K + 1) \\
= & \log \left( \frac{\text{SNR}^{1 - \alpha} \min\{2^{2CFB}, \text{INR} - 1\}}{\text{SNR}^{1 - \alpha}} \right) - \log(4(K + 1)) \\
= & \log \left( \min\{2^{2CFB}, \text{INR} - 1\} \right) - \log(4(K + 1)),
\end{align*}
\]

(45)

where (a) follows since $\mu^{\text{c}} \leq 1$, (b) follows since $\min\{2^{2CFB}, \text{INR} - 1\} \leq \text{INR} - 1$, (c) follows since $\text{INR} \leq \text{SNR}^{1 - \alpha}$, and (d) follows since $1 \leq \text{SNR}^{1 - \alpha}$. 
RHS of (8):

\[
\log \left( \frac{\text{SNR}\mu^{(2)}}{\text{SNR}\mu^{(3)} + \text{SNR}^\alpha \mu^{(1:3)}(K - 1) + \mu^{(1:3)}} \right)
\]

\[
= \log \left( \frac{\text{SNR}^{1-\alpha} - \frac{1}{2} \min\{2\text{C}_{FB}, \text{INR} - 1\}}{\frac{1}{2} \min\{2\text{C}_{FB}, \text{INR} - 1\} + (K - 1) + \frac{1}{2} \min\{2\text{C}_{FB}, \text{INR} - 1\}(K - 1) + \mu^{(1:3)}} \right)
\] \quad \text{(a)}

\[
\geq \log \left( \frac{\text{SNR}^{1-\alpha} - \frac{1}{2} \min\{2\text{C}_{FB}, \text{INR} - 1\}}{\frac{1}{2} \min\{2\text{C}_{FB}, \text{INR} - 1\} + K + \frac{1}{2} \min\{2\text{C}_{FB}, \text{INR} - 1\}(K - 1)} \right)
\]

\[
= \log \left( \frac{\text{SNR}^{1-\alpha} - \frac{1}{2} \min\{2\text{C}_{FB}, \text{INR} - 1\}}{K + \frac{1}{2} \min\{2\text{C}_{FB}, \text{INR} - 1\}K} \right)
\]

\[
\geq \log \left( \frac{\frac{1}{2}(\text{SNR}^{1-\alpha} + 1)}{K + \frac{1}{2} \min\{2\text{C}_{FB}, \text{INR} - 1\}K} \right)
\] \quad \text{(b)}

\[
= \log \left( \frac{\frac{1}{2}(\text{SNR}^{1-\alpha} + 1)}{1 + \frac{1}{2} \min\{2\text{C}_{FB}, \text{INR} - 1\}} \right) - \log(K)
\]

\[
= \log \left( \frac{(\text{SNR}^{1-\alpha} + 1)}{2 + \min\{2\text{C}_{FB}, \text{INR} - 1\}} \right) - \log(K)
\]

\[
\geq \log \left( \frac{(\text{SNR}^{1-\alpha} + 1)}{3 \min\{2\text{C}_{FB}, \text{INR} - 1\}} \right) - \log(K)
\] \quad \text{(c)}

\[
= \log \left( 1 + \text{SNR}^{1-\alpha} \right) - \min\{2\text{C}_{FB}, \log(\text{INR} - 1)\} - \log(3K),
\] \quad (46)

where (a) follows since \(\mu^{(1:3)} \leq 1\), (b) follows since \(\min\{2\text{C}_{FB}, \text{INR} - 1\} \leq \text{INR} - 1\) and \(\text{INR} \leq \text{SNR}^{1-\alpha}\), and (c) follows since \(0 \leq \text{C}_{FB}\) and \(1 \leq \text{INR}\).

RHS of (9):

\[
\log \left( \frac{\text{SNR}^{(3)}}{\text{SNR}^{\alpha} \mu^{(2:3)}(K - 1) + \mu^{(1:3)}} \right)
\]

\[
= \log \left( \frac{\frac{1}{2} \min\{2\text{C}_{FB}, \text{INR} - 1\}}{(K - 1) + \mu^{(1:3)}} \right)
\]

\[
\geq \log \left( \frac{\frac{1}{2} \min\{2\text{C}_{FB}, \text{INR} - 1\}}{K} \right)
\] \quad (a)

\[
= \log \left( \min\{2\text{C}_{FB}, \text{INR} - 1\} \right) - \log(2K)
\]

\[
= \min\{2\text{C}_{FB}, \log(\text{INR} - 1)\} - \log(2K),
\] \quad (47)

where (a) follows since \(\mu^{(1:3)} \leq 1\).

RHS of (10) is equal to RHS of (9) since \(\text{SNR}^{(3)} = \text{SNR}^{\alpha} \mu^{(1)}\).
RHS of (13):

\[
\log \left( \frac{\sqrt{\text{SNR}} \left( \frac{1}{K-1} \right) + \sqrt{\text{SNR}^\alpha} \left( \frac{K-2}{K-1} \right) }{\text{SNR} \mu^{(4)} + \text{SNR}^\alpha \mu^{(4)}(K-1) + (\mu^{(1)} + \mu^{(4)})} \right)
\]
\[
\overset{(a)}{=} \log \left( \frac{\sqrt{\text{SNR}} \left( \frac{1}{K-1} \right)^2 \mu^{(1)} }{\text{SNR} \mu^{(4)} + \text{SNR}^\alpha \mu^{(4)}(K-1) + (\mu^{(1)} + \mu^{(4)})} \right)
\]
\[
\overset{(b)}{=} \log \left( \frac{\text{SNR} \left( \frac{1}{K-1} \right)^2 \mu^{(1)} }{\text{SNR} \mu^{(4)} + (K-1) + (\mu^{(1)} + \mu^{(4)})} \right)
\]
\[
\overset{(c)}{=} \log \left( \frac{\text{SNR} \left( \frac{1}{K-1} \right)^2 \mu^{(1)} }{\text{SNR} \mu^{(4)} + K} \right)
\]
\[
\overset{(d)}{=} \log \left( \frac{\frac{1}{2} \left( \frac{1}{K-1} \right)^2 \text{SNR}^{1-\alpha} \min \{2^{2C_{FB}}, \text{INR} - 1\} }{\text{SNR}^{1-\alpha} + K} \right)
\]
\[
\overset{(e)}{=} \log \left( \frac{\frac{1}{2} \left( \frac{1}{K-1} \right)^2 \text{SNR}^{1-\alpha} \min \{2^{2C_{FB}}, \text{INR} - 1\} }{(K+1)\text{SNR}^{1-\alpha}} \right)
\]
\[
\overset{(f)}{=} \log \left( \frac{\frac{1}{2} \left( \frac{1}{K-1} \right)^2 \min \{2^{2C_{FB}}, \text{INR} - 1\} }{(K+1)} \right) - \log(K+1)
\]
\[
\overset{(g)}{=} \log \left( \frac{\frac{1}{2} \left( \frac{1}{K-1} \right)^2 \min \{2^{2C_{FB}}, \text{INR} - 1\} }{(K+1)} \right) - \log(2)(K+1) - 2 \log(K-1)
\]
\[
\overset{(h)}{=} \min \{2C_{FB}, \log(\text{INR} - 1)\} - \log(2)(K+1) - 2 \log(K-1),
\]

where (a) follows since \( \sqrt{\text{SNR}} \left( \frac{1}{K-1} \right) \leq \sqrt{\text{SNR}} \left( \frac{1}{K-1} \right) + \sqrt{\text{SNR}^\alpha} \left( \frac{K-2}{K-1} \right) \), (b) follows since \( \mu^{(1)} + \mu^{(4)} \leq 1 \), and (c) follows since \( 1 \leq \text{SNR} \).

RHS of (14):

\[
\log \left( \frac{\text{SNR} \mu^{(4)} }{\text{SNR}^\alpha \mu^{(4)}(K-1) + (\mu^{(1)} + \mu^{(4)})} \right)
\]
\[
\overset{(a)}{=} \log \left( \frac{\text{SNR}^{1-\alpha} }{(K-1) + (\mu^{(1)} + \mu^{(4)})} \right)
\]
\[
\overset{(b)}{=} \log \left( \frac{\text{SNR}^{1-\alpha} }{K} \right)
\]
\[
\overset{(c)}{=} \log \left( \text{SNR}^{1-\alpha} \right) - \log(K),
\]

where (a) follows since \( \mu^{(1)} + \mu^{(4)} \leq 1 \).

Also we do not need (11) and (12) anymore, because we have tighter bounds for \( R^{(1)} \) and \( R^{(3)} \) in (47).
Thus, we find the achievable rate expressions can be reduced as follows:

\[ R^{(1)} \leq \min\{2C_{FB}, \log(\text{INR} - 1)\} - \log 2(K + 1) - 2 \log(K - 1) \quad (50) \]

\[ R^{(2)} \leq \log(1 + \text{SNR}^{1-a}) - \min\{2C_{FB}, \log(\text{INR} - 1)\} - \log(3K) \quad (51) \]

\[ R^{(3)} \leq \min\{2C_{FB}, \log(\text{INR} - 1)\} - \log(2K) \quad (52) \]

\[ R^{(4)} \leq \log(\text{SNR}^{1-a}) - \log(K). \quad (53) \]

Putting these bounds all together, we achieve \( \frac{R^{(1)} + R^{(2)} + R^{(3)} + R^{(4)}}{2} = \)

\[ \frac{1}{2} \log(1 + \text{SNR}^{1-a}) + \min\{C_{FB}, \frac{1}{2} \log(\text{INR} - 1)\} + \frac{1}{2} \log(\text{SNR}^{1-a}) - \frac{1}{2} \log(K + 1) \]

\[ - \log(K - 1) - \frac{3}{2} \log(K) - \frac{1}{2} \log(12). \quad (54) \]

Next we will bound the gap between (54) and the conjectured rate upper bound in (39). We split this into 2 regimes. The first is when \( C_{FB} \leq \frac{1}{2} \log(\text{INR} - 1) \), and the second is when \( C_{FB} > \frac{1}{2} \log(\text{INR} - 1) \). In
the first case, we find the distance between (54) and \( R_{sym,0}^{u} + C_{FB} \) to get

\[
R_{sym,0}^{u} + C_{FB} - \left( \frac{1}{2} \log (1 + \text{SNR}^{1-\alpha}) + \min \{ C_{FB}, \frac{1}{2} \log (\text{INR} - 1) \} \right) \left( \frac{1}{2} \log (\text{SNR}^{1-\alpha}) - \frac{1}{2} \log (K + 1) - \frac{3}{2} \log (K) - \frac{1}{2} \log (12) \right)
\]

\[
\leq \left( \log (1 + \text{INR} + \frac{\text{SNR}}{1 + \text{INR}}) + C_{FB} \right) - \left( \frac{1}{2} \log (1 + \text{SNR}^{1-\alpha}) + \min \{ C_{FB}, \frac{1}{2} \log (\text{INR} - 1) \} \right)
\]

\[
\frac{1}{2} \log (\text{SNR}^{1-\alpha}) - \frac{1}{2} \log (K + 1) - \frac{3}{2} \log (K) - \frac{1}{2} \log (12)
\]

\[
= \log (1 + 2^{\text{SNR}^{1-\alpha}}) - \frac{1}{2} \log (1 + \text{SNR}^{1-\alpha}) - \frac{1}{2} \log (\text{SNR}^{1-\alpha}) + \frac{1}{2} \log (K + 1) + \log (K - 1)
\]

\[
+ \frac{3}{2} \log (K) + \frac{1}{2} \log (12)
\]

\[
\leq \log (1 + 2^{2\text{SNR}^{1-\alpha}}) - \frac{1}{2} \log (1 + \text{SNR}^{1-\alpha}) - \frac{1}{2} \log (\text{SNR}^{1-\alpha}) + \frac{1}{2} \log (K + 1) + \log (K - 1)
\]

\[
+ \frac{3}{2} \log (K) + \frac{1}{2} \log (12)
\]

\[
= \log (1 + 2^{2\text{SNR}^{1-\alpha}}) - \frac{1}{2} \log (1 + \text{SNR}^{1-\alpha}) - \frac{1}{2} \log (\text{SNR}^{1-\alpha}) + \frac{1}{2} \log (K + 1) + \log (K - 1)
\]

\[
+ \frac{3}{2} \log (K) + \frac{1}{2} \log (12)
\]

\[
= \frac{1}{2} \log \left( \frac{4 (\text{SNR}^{1-\alpha})^2}{(1 + \text{SNR}^{1-\alpha}) (\text{SNR}^{1-\alpha})} \right) + \frac{1}{2} \log (K + 1) + \log (K - 1) + \frac{3}{2} \log (K) + \frac{1}{2} \log (12)
\]

\[
= \frac{1}{2} \log \left( \frac{(\frac{1}{2} + \text{SNR}^{1-\alpha})^2}{(1 + \text{SNR}^{1-\alpha}) (\text{SNR}^{1-\alpha})} \right) + \frac{1}{2} \log (K + 1) + \log (K - 1) + \frac{3}{2} \log (K) + \frac{1}{2} \log (48)
\]

\[
= \frac{1}{2} \log \left( 1 + \frac{1}{4 (1 + \text{SNR}^{1-\alpha})} \right) + \frac{1}{2} \log (K + 1) + \log (K - 1) + \frac{3}{2} \log (K) + \frac{1}{2} \log (48)
\]

\[
\leq \frac{1}{2} \log \left( 1 + \frac{1}{8} \right) + \frac{1}{2} \log (K + 1) + \log (K - 1) + \frac{3}{2} \log (K) + \frac{1}{2} \log (48)
\]

\[
= \frac{1}{2} \log (K + 1) + \log (K - 1) + \frac{3}{2} \log (K) + \frac{1}{2} \log (54).
\]

In the second case when \( C_{FB} > \frac{1}{2} \log (\text{INR} - 1) \), we find the distance between (54) and \( R_{sym,\infty}^{u} \) as follows.
Since \( \min\{C_{FB}, \frac{1}{2} \log(INR - 1)\} = \frac{1}{2} \log(INR - 1) \)

\[
R^n_{sym, \infty} - \left( \frac{1}{2} \log \left( 1 + \text{SNR}^{-1 - \alpha} \right) + \min\{C_{FB}, \frac{1}{2} \log(INR + 1)\} - \frac{1}{2} \log \left( \text{SNR}^{-1 - \alpha} \right) - \frac{1}{2} \log(K + 1) - \log(K - 1) - \frac{3}{2} \log(K) - \frac{1}{2} \log(12) \right)
\]

\[
= \left( \frac{1}{2} \log\left( 1 + \frac{\text{SNR}}{1 + \text{INR}} \right) + \frac{1}{2} \log\left( 1 + \text{SNR + INR} \right) + \frac{K - 1}{2} + \log K \right) - \left( \frac{1}{2} \log \left( 1 + \text{SNR}^{-1 - \alpha} \right) + \min\{C_{FB}, \frac{1}{2} \log(INR + 1)\} - \frac{1}{2} \log \left( \text{SNR}^{-1 - \alpha} \right) - \frac{1}{2} \log(K + 1) - \log(K - 1) - \frac{3}{2} \log(K) - \frac{1}{2} \log(12) \right)
\]

\[
= \frac{1}{2} \log\left( 1 + \frac{\text{SNR}}{1 + \text{INR}} \right) + \frac{1}{2} \log\left( 1 + \text{SNR + SNR}^\alpha \right) - \frac{1}{2} \log \left( 1 + \text{SNR}^{-1 - \alpha} \right) - \frac{1}{2} \log \left( \text{SNR}^\alpha + 1 \right) - \frac{1}{2} \log \left( \text{SNR}^{-1 - \alpha} \right) + \frac{K - 1}{2} + \log K + \frac{1}{2} \log(K + 1) + \log(K - 1) + \frac{3}{2} \log(K) + \frac{1}{2} \log(12)
\]

\[
\leq \frac{1}{2} \log\left( 1 + 2\text{SNR} \right) + \frac{1}{2} \log\left( 1 + 2\text{SNR} \right) - \frac{1}{2} \log \left( 1 + \text{SNR}^{-1 - \alpha} \right) - \frac{1}{2} \log \left( \text{SNR}^\alpha + 1 \right) - \frac{1}{2} \log \left( \text{SNR}^{-1 - \alpha} \right) + \frac{K - 1}{2} + \log K + \frac{1}{2} \log(K + 1) + \log(K - 1) + \frac{3}{2} \log(K) + \frac{1}{2} \log(12)
\]

\[
= \frac{1}{2} \log \left( \frac{(1 + 2\text{SNR})(1 + 2\text{SNR})}{(1 + \text{SNR}^\alpha)^2(1 + \text{SNR}^{-1 - \alpha})(\text{SNR}^{-1 - \alpha})}) \right) + \frac{K - 1}{2} + \log K + \frac{1}{2} \log(K + 1) + \log(K - 1) + \frac{3}{2} \log(K) + \frac{1}{2} \log(12)
\]

\[
= \frac{1}{2} \log \left( \frac{(3\text{SNR})(3\text{SNR})}{(\text{SNR}^\alpha)^2(\text{SNR}^{-1 - \alpha})^2} \right) + \frac{K - 1}{2} + \log K + \frac{1}{2} \log(K + 1) + \log(K - 1) + \frac{3}{2} \log(K) + \frac{1}{2} \log(12)
\]

\[
= \frac{1}{2} \log \left( \frac{9\text{SNR}^2}{\text{SNR}^2} \right) + \frac{K - 1}{2} + \log K + \frac{1}{2} \log(K + 1) + \log(K - 1) + \frac{3}{2} \log(K) + \frac{1}{2} \log(12)
\]

\[
= \frac{K - 1}{2} + \frac{1}{2} \log(K + 1) + \log(K - 1) + \frac{5}{2} \log(K) + \frac{1}{2} \log(108)
\]

\[
= \frac{K - 1}{2} + \frac{1}{2} \log(108(K - 1)(K^5(K + 1)) \right)
\]

From (55) and (56), we find that the achievable symmetric rate is within \( \frac{1}{2} \log(108(K - 1)(K^5(K + 1))) + \frac{K - 1}{2} \) bits to the conjectured upper bound in (39) when \( \alpha \leq \frac{1}{2} \).
Case 2 ($\frac{1}{2} \leq \alpha \leq \frac{2}{3}$): We use the following parameters in Theorem 3: $\mu^{(4)} = \frac{1}{4\text{INR}} \max\{2^{-2C_{FB}}, \frac{\text{INR}^3}{\text{SNR}^2}\}$, $\mu^{(6)} = \mu^{(3)} = \frac{1}{3\text{INR}} - \frac{1}{4\text{INR}} \max\{2^{-2C_{FB}}, \frac{\text{INR}^3}{\text{SNR}^2}\}$, $\mu^{(1)} = 1 - \mu^{(2,4)}$, and $\mu^{(5)} = 1 - \mu^{(2)} - \mu^{(6)}$. We first lower bound the RHS of (16)-(27) as follows.

RHS of (16):

$$
\log \left( \frac{\text{SNR}\mu^{(1)}}{\text{SNR}\mu^{(2,4)} + \text{SNR}^3\mu^{(1,4)}(K - 1) + 1} \right) \\
\geq \log \left( \frac{\frac{2}{3}\text{SNR}}{\frac{2}{3}\text{SNR}^{2-2\alpha} + \text{SNR}^3(K - 1) + 1} \right) \\
\geq \log \left( \frac{\frac{2}{3}\text{SNR}}{(K + \frac{2}{3})\text{SNR}^{2-2\alpha}} \right) \\
= \log \left( \frac{\text{SNR}^{2\alpha - 1}}{\frac{3}{2}(K + \frac{2}{3})} \right) \\
= \log (\text{SNR}^{2\alpha - 1}) - \log \left( \frac{3}{2} \left( K + \frac{2}{3} \right) \right),
$$

where (a) follows since $\mu^{(1,4)} = 1$, $\mu^{(1)} \geq \frac{2}{3}$, and $\mu^{(2,4)} \leq \frac{2}{3}\text{SNR}^{1-2\alpha}$, and (b) follows since $\text{SNR}^{2-2\alpha} \geq \text{SNR}^\alpha \geq 1$.

RHS of (17):

$$
\log \left( \frac{\text{SNR}\mu^{(2)}}{\text{SNR}\mu^{(3,4)} + \text{SNR}^3\mu^{(1,4)}(K - 1) + 1} \right) \\
\geq \log \left( \frac{\frac{1}{12}\text{SNR}^{2-2\alpha}}{\frac{1}{3}\text{SNR}^{1-\alpha} + \text{SNR}^3(K - 1) + 1} \right) \\
\geq \log \left( \frac{\frac{1}{12}\text{SNR}^{2-2\alpha}}{\text{SNR}^\alpha(K + \frac{1}{3})} \right) \\
= \log \left( \frac{\text{SNR}^{2-3\alpha}}{12(K + \frac{1}{3})} \right) \\
= \log (\text{SNR}^{2-3\alpha}) - \log \left( 12 \left( K + \frac{1}{3} \right) \right),
$$

where (a) follows since $\mu^{(1,4)} = 1$, $\mu^{(1)} \geq \frac{2}{3}$, and $\mu^{(2,4)} \leq \frac{2}{3}\text{SNR}^{1-2\alpha}$, and (b) follows since $\text{SNR}^{2-2\alpha} \geq \text{SNR}^\alpha \geq 1$. 
RHS of (18):

\[
\log \left( \frac{\text{SNR}^\alpha \mu^{(1)}}{\text{SNR}^{\mu^{(3:4)}} \text{SNR}^\alpha \mu^{(2:4)} (K - 1) + 1} \right) \\
\overset{(a)}{\geq} \log \left( \frac{2/3 \text{SNR}^\alpha}{\frac{1}{3} \text{SNR}^{1-\alpha} + \frac{1}{3} \text{SNR}^{1-\alpha} (K - 1) + 1} \right) \\
= \log \left( \frac{2\text{SNR}^\alpha}{\text{SNR}^{1-\alpha} (K + 3)} \right) \\
\overset{(b)}{\geq} \log \left( \frac{\text{SNR}^\alpha}{\text{SNR}^{1-\alpha} (K + 3)} \right) \\
= \log \left( \frac{\text{SNR}^{2\alpha - 1}}{\text{SNR}^{1-\alpha}} \right) - \log \left( \frac{1}{2} (K + 3) \right) \\
= \log \left( \text{SNR}^{2\alpha - 1} \right) - \log \left( \frac{1}{2} (K + 3) \right),
\]

where (a) follows since \(\mu^{(1:4)} = 1\), \(\mu^{(1)} \geq \frac{7}{3}\), and \(\mu^{(3:4)} \leq \frac{1}{3} \text{SNR}^{-\alpha}\), and (b) follows since \(\text{SNR}^{1-\alpha} \geq 1\).

RHS of (19):

\[
\log \left( \frac{\text{SNR}^\alpha \mu^{(2)}}{\text{SNR}^{\mu^{(4)}} \text{SNR}^\alpha \mu^{(3:4)} (K - 1) + 1} \right) \\
\overset{(a)}{\geq} \log \left( \frac{1/2 \text{SNR}^{1-\alpha}}{\frac{1}{4} \text{SNR}^{1-\alpha} \max \{2^{-2C_{FB}}, \frac{\text{INR}^3}{\text{SNR}^2}\} + \frac{1}{3} (K - 1) + 1} \right) \\
\geq \log \left( \frac{\text{SNR}^{1-\alpha}}{3\text{SNR}^{1-\alpha} \max \{2^{-2C_{FB}}, \frac{\text{INR}^3}{\text{SNR}^2}\} + 4(K - 1) + 12} \right) \\
= \log \left( \frac{\text{SNR}^{1-\alpha}}{4(K + \frac{11}{4}) \text{SNR}^{1-\alpha} \max \{2^{-2C_{FB}}, \frac{\text{INR}^3}{\text{SNR}^2}\} } \right) \\
= \log \left( \frac{\text{SNR}^{1-\alpha}}{4(K + \frac{11}{4}) \text{SNR}^{1-\alpha} \max \{2^{-2C_{FB}}, \frac{\text{INR}^3}{\text{SNR}^2}\} } \right) \\
= \log \left( \frac{\min \{2^{2C_{FB}}, \text{SNR}^2 \}}{4 \left( K + \frac{11}{4} \right)} \right) \\
\geq \log \left( \min \{2^{2C_{FB}}, \text{SNR}^2 \} \right) - \log \left( 4 \left( K + \frac{11}{4} \right) \right) \\
= \min \{2^{C_{FB}}, \log \left( \text{SNR}^{2-3\alpha} \right) \} - \log \left( 4 \left( K + \frac{11}{4} \right) \right),
\]

where (a) follows since \(\mu^{(2)} \geq \frac{1}{12} \text{SNR}^{1-2\alpha}\), and \(\mu^{(3:4)} \leq \frac{1}{3} \text{SNR}^{-\alpha}\).

RHS of (20) is equal to RHS of (19) since \(\text{SNR}^{\mu^{(3)}} = \text{SNR}^\alpha \mu^{(2)}\).
RHS of (21):

\[
\begin{align*}
\log \left( \frac{\text{SNR}^{\mu(4)}}{\text{SNR}^{\alpha} \mu^{(3:4)}(K - 1) + 1} \right) \\
\overset{(a)}{\geq} \log \left( \frac{\frac{1}{3} \text{SNR}^{1-\alpha} \max \{2^{-2C_{FB}}, \frac{\text{INR}^3}{\text{SNR}^2} \}}{\frac{1}{3}(K - 1) + 1} \right) \\
= \log \left( \frac{\text{SNR}^{1-\alpha} \max \{2^{-2C_{FB}}, \frac{\text{INR}^3}{\text{SNR}^2} \}}{\frac{4}{3}(K + 2)} \right) \\
= \log (\text{SNR}^{1-\alpha}) + \log \left( \frac{\text{SNR}^{\alpha} \max \{2^{-2C_{FB}}, \frac{\text{INR}^3}{\text{SNR}^2} \}}{\frac{4}{3}(K + 2)} \right) \\
= \log (\text{SNR}^{1-\alpha}) - \log \left( \min \{2^{2C_{FB}}, \frac{\text{SNR}^2}{\text{INR}^3} \} \right) - \log \left( \frac{4}{3}(K + 2) \right) \\
= \log (\text{SNR}^{1-\alpha}) - \min \{2^{C_{FB}}, \log (\text{SNR}^{2-3\alpha}) \} - \log \left( \frac{4}{3}(K + 2) \right),
\end{align*}
\]

where (a) follows since \(\mu^{(3:4)} \leq \frac{1}{3} \text{SNR}^{-\alpha}\).

RHS of (24):

\[
\begin{align*}
\log \left( \frac{\text{SNR}^{\mu(5)}}{\text{SNR}(\mu^{(2)} + \mu^{(6)}) + \text{SNR}^{\alpha}(\mu^{(2)} + \mu^{(5:6)})(K - 1) + 1 - \text{SNR}^{\alpha} \mu^{(2)}} \right) \\
\overset{(a)}{\geq} \log \left( \frac{\frac{1}{3} \text{SNR}}{\text{SNR}(\mu^{(2)} + \mu^{(6)}) + \text{SNR}^{\alpha}(K - 1) + 1} \right) \\
\overset{(b)}{\geq} \log \left( \frac{\frac{1}{3} \text{SNR}}{\text{SNR}(\frac{2}{3} \text{SNR}^{1-2\alpha}) + \text{SNR}^{\alpha}(K - 1) + 1} \right) \\
\overset{(c)}{\geq} \log \left( \frac{\frac{1}{3} \text{SNR}}{(\text{SNR}^{2-2\alpha})(K + \frac{2}{3})} \right) \\
= \log (\text{SNR}^{2\alpha-1}) - \log \left( 3 \left( K + \frac{2}{3} \right) \right),
\end{align*}
\]

where (a) follows since \(\mu^{(2)} + \mu^{(5:6)} = 1\), and \(\mu^{(5)} \geq \frac{1}{3}\), (b) follows since \(\mu^{(2)} + \mu^{(6)} \leq \frac{2}{3} \text{SNR}^{1-2\alpha}\), and (c) follows since \(\text{SNR}^{2-2\alpha} \geq \text{SNR}^{\alpha} \geq 1\).
RHS of (25):

\[
\log \left( \frac{(\sqrt{\text{SNR}} \left( \frac{1}{K-1} \right) + \sqrt{\text{SNR}^\alpha \left( \frac{K-2}{K-1} \right)^2 \mu^{(2)}})}{\text{SNR} \mu^{(6)} + \text{SNR}^\alpha \mu^{(5:6)} (K - 1) + 1} \right)
\geq \log \left( \frac{\text{SNR} \left( \frac{1}{K-1} \right)^2 \mu^{(2)}}{\text{SNR} \mu^{(6)} + \text{SNR}^\alpha \mu^{(5:6)} (K - 1) + 1} \right)
\geq \log \left( \frac{\frac{1}{12} \text{SNR}^{2-2\alpha} \left( \frac{1}{K-1} \right)^2}{\frac{1}{3} \text{SNR}^{1-\alpha} + \text{SNR}^\alpha (K - 1) + 1} \right)
\geq \log \left( \frac{\frac{1}{12} \text{SNR}^{2-2\alpha} \left( \frac{1}{K-1} \right)^2}{\text{SNR}^\alpha (K + \frac{1}{3})} \right)
= \log \left( \text{SNR}^{2-3\alpha} \right) - \log \left( 12(K - 1)^2(K + \frac{1}{3}) \right),
\]

where (a) follows since \( \sqrt{\text{SNR}} \left( \frac{1}{K-1} \right) \leq \sqrt{\text{SNR}} \left( \frac{1}{K-1} \right) + \sqrt{\text{SNR}^\alpha \left( \frac{K-2}{K-1} \right)^2} \), (b) follows since \( \mu^{(5:6)} \leq 1 \), \( \mu^{(6)} \leq \frac{1}{3} \text{SNR}^{-\alpha} \), and \( \mu^{(2)} \geq \frac{1}{12} \text{SNR}^{1-2\alpha} \), and (c) follows since \( \text{SNR}^\alpha \geq \text{SNR}^{1-\alpha} \geq 1 \).

RHS of (26):

\[
\log \left( \frac{\text{SNR}^\alpha \mu^{(5)}}{\text{SNR} \mu^{(6)} + \text{SNR}^\alpha \mu^{(6)} (K - 1) + 1} \right)
\geq \log \left( \frac{\frac{1}{3} \text{SNR}^\alpha}{\frac{1}{3} \text{SNR}^{1-\alpha} + \frac{1}{3} (K - 1) + 1} \right)
= \log \left( \frac{\text{SNR}^\alpha}{\text{SNR}^{1-\alpha} + (K - 1) + 3} \right)
\geq \log \left( \frac{\text{SNR}^\alpha}{(K + 3) \text{SNR}^{1-\alpha}} \right)
= \log \left( \frac{\text{SNR}^\alpha}{\text{SNR}^{1-\alpha}} \right) - \log(K + 3)
= \log \left( \text{SNR}^{2\alpha-1} \right) - \log(K + 3),
\]

where (a) follows since \( \mu^{(6)} \leq \frac{1}{3} \text{SNR}^{-\alpha} \), and \( \mu^{(5)} \geq \frac{1}{3} \), and (b) follows since \( \text{SNR}^{1-\alpha} \geq 1 \).
RHS of (27):

\[
\log \left( \frac{\text{SNR}_\mu^{(6)}}{\text{SNR}^\alpha \mu^{(6)}(K - 1) + 1} \right) \\
= \log \left( \frac{\frac{1}{3} \text{SNR}^{1-\alpha} - \frac{1}{4} \text{SNR}^{1-\alpha} \max\left\{2^{-2C_{FB}}, \frac{\text{INR}^3}{\text{SNR}^2}\right\}}{\left(\frac{1}{3} - \frac{1}{4} \max\left\{2^{-2C_{FB}}, \frac{\text{INR}^3}{\text{SNR}^2}\right\}\right)(K - 1) + 1} \right) \\
\geq \log \left( \frac{\frac{1}{12} \text{SNR}^{1-\alpha}}{\frac{1}{3}(K - 1) + 1} \right) \\
= \log \left( \frac{\frac{1}{3} \text{SNR}^{1-\alpha}}{(K + 2)} \right) \\
= \log \left( \text{SNR}^{1-\alpha} \right) - \log(4(K + 2)).
\]

(65)

Also we do not need (22) and (23) anymore, because we have tighter bounds for \( R^{(2)} \) and \( R^{(3)} \) in (60).

Thus, we find the achievable rate expressions can be reduced as follows

\[
R^{(1)} \leq \log \left( \text{SNR}^{2\alpha - 1} \right) - \log \left( \frac{3}{2} \left( K + \frac{2}{3} \right) \right)
\]

(66)

\[
R^{(2)} \leq \min\{2C_{FB}, \log (\text{SNR}^{2-3\alpha})\} - \log \left( 12(K - 1)^2(K + \frac{1}{3}) \right)
\]

(67)

\[
R^{(3)} \leq \min\{2C_{FB}, \log (\text{SNR}^{2-3\alpha})\} - \log \left( 4 \left( K + \frac{11}{4} \right) \right)
\]

(68)

\[
R^{(4)} \leq \log \left( \text{SNR}^{1-\alpha} \right) - \min\{2C_{FB}, \log (\text{SNR}^{2-3\alpha})\} - \log \left( \frac{4}{3} (K + 2) \right)
\]

(69)

\[
R^{(5)} \leq \log \left( \text{SNR}^{2\alpha - 1} \right) - \log \left( 3 \left( K + \frac{2}{3} \right) \right)
\]

(70)

\[
R^{(6)} \leq \log \left( \text{SNR}^{1-\alpha} \right) - \log(4(K + 2)).
\]

(71)
Putting these bounds all together, we achieve the rate 
\[ \frac{R^{(1)} + R^{(2)} + R^{(3)} + R^{(4)} + R^{(5)} + R^{(6)}}{2} = \]
\[ \frac{1}{2} \left( \log (\text{SNR}^{2\alpha-1}) - \log \left( \frac{3}{2} \left( K + \frac{2}{3} \right) \right) + \right. \]
\[ \min \{ 2C_{FB}, \log (\text{SNR}^{2-3\alpha}) \} - \log \left( 12(K - 1)^2(K + \frac{1}{3}) \right) + \]
\[ \min \{ 2C_{FB}, \log (\text{SNR}^{2-3\alpha}) \} - \log \left( 4 \left( K + \frac{11}{4} \right) \right) + \]
\[ \log (\text{SNR}^{1-\alpha}) - \min \{ 2C_{FB}, \log (\text{SNR}^{2-3\alpha}) \} \right) - \log \left( \frac{4}{3} (K + 2) \right) + \]
\[ \log (\text{SNR}^{2\alpha-1}) - \log \left( 3 \left( K + \frac{2}{3} \right) \right) + \]
\[ \log (\text{SNR}^{1-\alpha}) - \log (4(K + 2)) \right) \]
\[ = \log (\text{SNR}^{2\alpha-1}) + \log (\text{SNR}^{1-\alpha}) + \min \{ C_{FB}, \frac{1}{2} \log (\text{SNR}^{2-3\alpha}) \} \]
\[ - \frac{1}{2} \log \left( 768 (K - 1)^2 \left( K + \frac{1}{3} \right) \left( K + \frac{2}{3} \right)^2 (K + 2)^2 \left( K + \frac{11}{4} \right) \right) \].  

(72)

Next we will bound the gap between (72) and the conjectured upper bound in (39). We split this into 2 regimes. The first is when \( 2C_{FB} \leq \log (\text{SNR}^{2-3\alpha}) \), and the second is when \( C_{FB} > \log (\text{SNR}^{2-3\alpha}) \). In the
first case, we find the distance between (72) and the bound $R_{sym,0}^u + C_{FB}$ as follows.

\[
R_{sym,0}^u + C_{FB} - \left( \log (\text{SNR}^{2\alpha - 1}) + \log \left( \frac{\text{SNR}^{1-\alpha}}{1+\text{INR}} \right) + \min \{C_{FB}, \frac{1}{2} \log (\text{SNR}^{2-3\alpha}) \} \right)
\]
\[
- \frac{1}{2} \log \left( 768 (K-1)^2 \left( K + \frac{1}{3} \right) \left( K + \frac{2}{3} \right)^2 (K+2) \left( K + \frac{11}{4} \right) \right)
\]
\[
\leq \left( \log(1 + \text{INR} + \frac{\text{SNR}}{1+\text{INR}}) + C_{FB} \right) -
\left( \log (\text{SNR}^{2\alpha - 1}) + \log \left( \frac{\text{SNR}^{1-\alpha}}{1+\text{INR}} \right) + \min \{C_{FB}, \frac{1}{2} \log (\text{SNR}^{2-3\alpha}) \} \right)
\]
\[
- \frac{1}{2} \log \left( 768 (K-1)^2 \left( K + \frac{1}{3} \right) \left( K + \frac{2}{3} \right)^2 (K+2) \left( K + \frac{11}{4} \right) \right)
\]
\[
= \left( \log(1 + \text{INR} + \frac{\text{SNR}}{1+\text{INR}}) + C_{FB} \right) - (\log (\text{SNR}^{2\alpha - 1}) + \log (\text{SNR}^{1-\alpha}) + C_{FB})
\]
\[
- \frac{1}{2} \log \left( 768 (K-1)^2 \left( K + \frac{1}{3} \right) \left( K + \frac{2}{3} \right)^2 (K+2) \left( K + \frac{11}{4} \right) \right)
\]
\[
= \log(1 + \text{INR} + \frac{\text{SNR}}{1+\text{INR}}) - \log (\text{SNR}^{2\alpha - 1}) - \log (\text{SNR}^{1-\alpha})
\]
\[
+ \frac{1}{2} \log \left( 768 (K-1)^2 \left( K + \frac{1}{3} \right) \left( K + \frac{2}{3} \right)^2 (K+2) \left( K + \frac{11}{4} \right) \right)
\]
\[
= \log \left( \frac{(1 + \text{INR} + \frac{\text{SNR}}{1+\text{INR}})}{(\text{SNR}^{2\alpha - 1})(\text{SNR}^{1-\alpha})} \right)
\]
\[
+ \frac{1}{2} \log \left( 768 (K-1)^2 \left( K + \frac{1}{3} \right) \left( K + \frac{2}{3} \right)^2 (K+2) \left( K + \frac{11}{4} \right) \right)
\]
\[
\leq \log \left( \frac{3\text{INR}}{(\text{SNR}^{2\alpha - 1})(\text{SNR}^{1-\alpha})} \right)
\]
\[
+ \frac{1}{2} \log \left( 768 (K-1)^2 \left( K + \frac{1}{3} \right) \left( K + \frac{2}{3} \right)^2 (K+2) \left( K + \frac{11}{4} \right) \right)
\]
\[
\leq \log 3 + \frac{1}{2} \log \left( 768 (K-1)^2 \left( K + \frac{1}{3} \right) \left( K + \frac{2}{3} \right)^2 (K+2) \left( K + \frac{11}{4} \right) \right)
\]
\[
= \frac{1}{2} \log \left( 6912 (K-1)^2 \left( K + \frac{1}{3} \right) \left( K + \frac{2}{3} \right)^2 (K+2) \left( K + \frac{11}{4} \right) \right).
\]
In the second case when \( C_{FB} > \log (\text{SNR}^{2-3\alpha}) \), we find the gap between (72) and \( R_{sym,\infty}^u \) as follows.

\[
R_{sym,\infty}^u = \left( \log (\text{SNR}^{2\alpha-1}) + \log (\text{SNR}^{1-\alpha}) + \min\{C_{FB}, \frac{1}{2}\log (\text{SNR}^{2-3\alpha})\} \right)
- \frac{1}{2} \log \left( 768 (K - 1)^2 \left( K + \frac{1}{3} \right) \left( K + \frac{2}{3} \right)^2 (K + 2)^2 \left( K + \frac{11}{4} \right) \right)

= \left( \frac{1}{2} \log (1 + \frac{\text{SNR}}{1 + \text{INR}}) + \frac{1}{2} \log (1 + \text{SNR} + \text{INR}) + \frac{K - 1}{2} + \log K \right) -
\left( \log (\text{SNR}^{2\alpha-1}) + \log (\text{SNR}^{1-\alpha}) + \min\{C_{FB}, \frac{1}{2}\log (\text{SNR}^{2-3\alpha})\} \right)
- \frac{1}{2} \log \left( 768 (K - 1)^2 \left( K + \frac{1}{3} \right) \left( K + \frac{2}{3} \right)^2 (K + 2)^2 \left( K + \frac{11}{4} \right) \right)

\leq \frac{1}{2} \log (1 + \text{SNR}^{1-\alpha}) + \frac{1}{2} \log (3\text{SNR}) + \frac{K - 1}{2} + \log K
- \log (\text{SNR}^{2\alpha-1}) - \log (\text{SNR}^{1-\alpha}) - \frac{1}{2} \log (\text{SNR}^{2-3\alpha})
+ \frac{1}{2} \log \left( 768 (K - 1)^2 \left( K + \frac{1}{3} \right) \left( K + \frac{2}{3} \right)^2 (K + 2)^2 \left( K + \frac{11}{4} \right) \right)

\leq \frac{1}{2} \log (3\text{SNR}) - \log (\text{SNR}^{2\alpha-1}) - \frac{1}{2} \log (\text{SNR}^{1-\alpha}) - \frac{1}{2} \log (\text{SNR}^{2-3\alpha})
+ \frac{1}{2} \log \left( 768 (K - 1)^2 K^2 \left( K + \frac{1}{3} \right) \left( K + \frac{2}{3} \right)^2 (K + 2)^2 \left( K + \frac{11}{4} \right) \right) + \frac{K - 1}{2}

= \frac{1}{2} \log \left( \frac{(3\text{SNR})}{(\text{SNR}^{2\alpha-1})^2 (\text{SNR}^{2-3\alpha}) (\text{SNR}^{1-\alpha})} \right) + \frac{K - 1}{2}
+ \frac{1}{2} \log \left( 768 (K - 1)^2 K^2 \left( K + \frac{1}{3} \right) \left( K + \frac{2}{3} \right)^2 (K + 2)^2 \left( K + \frac{11}{4} \right) \right)

= \frac{1}{2} \log 3 + \frac{1}{2} \log \left( 768 (K - 1)^2 K^2 \left( K + \frac{1}{3} \right) \left( K + \frac{2}{3} \right)^2 (K + 2)^2 \left( K + \frac{11}{4} \right) \right) + \frac{K - 1}{2}

= \frac{1}{2} \log \left( 2304 (K - 1)^2 K^2 \left( K + \frac{1}{3} \right) \left( K + \frac{2}{3} \right)^2 (K + 2)^2 \left( K + \frac{11}{4} \right) \right) + \frac{K - 1}{2}. \quad (74)

From (73) and (74), we find that the achievable symmetric rate is within \( \frac{1}{2} \log \left( 2304 (K - 1)^2 K^2 \left( K + \frac{1}{3} \right) \right) \left( K + \frac{2}{3} \right)^2 (K + 2)^2 \left( K + \frac{11}{4} \right) \) + \( \frac{K - 1}{2} \) bits to the conjectured upper bound (39) when \( \frac{1}{2} \leq \alpha \leq \frac{2}{3} \).

**Case 3** (\( \alpha \geq 2 \)): We use the following parameters in Theorem 4: \( \mu_1(\alpha) = \frac{\text{SNR}}{\text{INR}} \min\{2C_{FB}, \frac{\text{INR}}{\text{SNR}^{\alpha}}\} \), and
\(\mu^{(1)} = \mu^{(3)} = 1 - \mu^{(2)}\). We first lower bound the RHS of (29)-(35) as follows.

RHS of (29):

\[
\log \left( \frac{\text{INR} \mu^{(1)}}{\text{SNR} \mu^{(1:2)} + \text{SNR}^\alpha \mu^{(2)} (K - 1) + 1} \right)
\]

\[
\overset{(a)}{=} \log \left( \frac{\frac{1}{2} \text{INR}}{\text{SNR} + \frac{1}{2} \text{SNR} \min \{2^{2C_{FB}}, \frac{\text{INR}}{\text{SNR}^{2}} \} (K - 1) + 1} \right)
\]

\[
\overset{(b)}{=} \log \left( \frac{\frac{1}{2} \text{INR}}{\text{SNR} \min \{2^{2C_{FB}}, \frac{\text{INR}}{\text{SNR}^{2}} \} (K + 3)} \right)
\]

\[
= \log \left( \frac{\text{SNR}^{\alpha-1}}{\min \{2^{2C_{FB}}, \frac{\text{INR}}{\text{SNR}^{2}} \} (K + 3)} \right)
\]

\[
= \log \left( \text{SNR}^{\alpha-1} \max \left\{ 2^{-2C_{FB}}, \frac{\text{SNR}^{2}}{\text{INR}} \right\} \right) - \log(K + 3)
\]

\[
\geq \log \left( \text{SNR}^{\alpha-1} \left( \frac{\text{SNR}^{2}}{\text{INR}} \right) \right) - \log(K + 3)
\]

\[
= \log (\text{SNR}) - \log(K + 3),
\]

(75)

where (a) follows since \(\mu^{(1)} \geq \frac{1}{2}\), and \(\mu^{(1:2)} = 1\), (b) follows since \(\min \{2^{2C_{FB}}, \frac{\text{INR}}{\text{SNR}^{2}} \} \geq 1\), and \(\text{SNR} \geq 1\).

RHS of (30):

\[
\log \left( \frac{\text{INR} \mu^{(2)}}{\text{SNR} \mu^{(1:2)} + 1} \right)
\]

\[
= \log \left( \frac{\frac{1}{2} \text{SNR} \min \{2^{2C_{FB}}, \frac{\text{INR}}{\text{SNR}^{2}} \}}{\text{SNR} + 1} \right)
\]

\[
\overset{(a)}{=} \log \left( \frac{\frac{1}{2} \text{SNR} \min \{2^{2C_{FB}}, \frac{\text{INR}}{\text{SNR}^{2}} \}}{2 \text{SNR}} \right)
\]

\[
= \log \left( \frac{\text{SNR} \min \{2^{2C_{FB}}, \frac{\text{INR}}{\text{SNR}^{2}} \}}{\text{SNR}} \right) - \log(4)
\]

\[
= \log \left( \min \{2^{2C_{FB}}, \frac{\text{INR}}{\text{SNR}^{2}} \} \right) - \log(4)
\]

\[
= \min \{2C_{FB}, \log (\text{SNR}^{\alpha-2}) \} - \log(4),
\]

(76)

where (a) follows since \(\text{SNR} \geq 1\).
RHS of (31):

\[
\log \left( \frac{\text{SNR}_{\mu}^{(1)}}{\text{SNR}_{\mu}^{(2)} + 1} \right) \\
\geq \log \left( \frac{\frac{1}{2} \text{SNR}}{\frac{1}{2} + 1} \right) \\
= \log \left( \frac{\text{SNR}}{3} \right) \\
= \log (\text{SNR}) - \log(3),
\]

(77)

where (a) follows since \( \mu^{(1)} \geq \frac{1}{2} \).

RHS of (33):

\[
\log \left( \frac{\text{SNR}^{\alpha} \mu^{(3)}}{\text{SNR}^{\alpha} \mu^{(2)} + \text{SNR} \mu^{(2:3)} + 1} \right) \\
\geq \log \left( \frac{\frac{1}{2} \text{INR}}{\frac{1}{2} \text{SNR} \min\{2^{2C_{FB}}, \text{INR} \frac{\text{SNR}^{2}}{\text{SNR}^{2}}\} + \text{SNR} + 1} \right) \\
= \log \left( \frac{\text{SNR}^{\alpha - 1} \min\{2^{2C_{FB}}, \text{INR} \frac{\text{SNR}^{2}}{\text{SNR}^{2}}\}(5)}{\min\{2^{2C_{FB}}, \text{INR} \frac{\text{SNR}^{2}}{\text{SNR}^{2}}\}(5)} \right) \\
= \log \left( \frac{\text{SNR}^{\alpha - 1} \max\{2^{-2C_{FB}}, \text{SNR}^{2} \frac{\text{INR}^{2}}{\text{INR}}\}}{\min\{2^{2C_{FB}}, \text{INR} \frac{\text{SNR}^{2}}{\text{SNR}^{2}}\}(5)} \right) \\
= \log (\text{SNR}) - \log(5),
\]

(78)

where (a) follows since \( \mu^{(3)} \geq \frac{1}{2} \), and \( \mu^{(2:3)} = 1 \), (b) follows since \( \min\{2^{2C_{FB}}, \text{INR} \frac{\text{SNR}^{2}}{\text{SNR}^{2}}\} \geq 1 \), and \( \text{SNR} \geq 1 \).
RHS of (34):

\[
\log \left( \frac{\text{SNR}^{(2)} \mu^{(2)}}{\text{SNR}^{(3)} + 1} \right) \\
\geq (a) \log \left( \frac{\frac{1}{2} \text{SNR} \min\{2^{2C_{FB}}, \frac{\text{INR}}{\text{SNR}^{(2)}}\}}{\text{SNR} + 1} \right) \\
\geq (b) \log \left( \frac{\frac{1}{2} \text{SNR} \min\{2^{2C_{FB}}, \frac{\text{INR}}{\text{SNR}^{(2)}}\}}{2\text{SNR}} \right) \\
= \log \left( \frac{\text{SNR} \min\{2^{2C_{FB}}, \frac{\text{INR}}{\text{SNR}^{(2)}}\}}{\text{SNR}} \right) - \log(4) \\
= \log \left( \min\{2^{2C_{FB}}, \frac{\text{INR}}{\text{SNR}^{(2)}}\} \right) - \log(4) \\
= \min\{2C_{FB}, \log \left( \text{SNR}^{\alpha-2} \right) \} - \log(4),
\]

(79)

where (a) follows since \( \mu^{(3)} \leq 1 \), and \( \mu^{(2;3)} = 1 \), (b) follows since \( \text{SNR} \geq 1 \).

RHS of (35):

\[
\log \left( \frac{\text{SNR}^{(3)}}{1} \right) \\
\geq (a) \log \left( \frac{\text{SNR}^{1/2}}{1} \right) \\
= \log \left( \text{SNR} \right) - \log(2),
\]

(80)

where (a) follows since \( \mu^{(3)} \geq \frac{1}{2} \).

Thus, we find the achievable rate expressions can be reduced as follows:

\[
R^{(1)} \leq \log \left( \text{SNR} \right) - \log(K + 3) \quad (81)
\]

\[
R^{(2)} \leq \min\{2C_{FB}, \log \left( \text{SNR}^{\alpha-2} \right) \} - \log(4) \quad (82)
\]

\[
R^{(3)} \leq \log \left( \text{SNR} \right) - \log(5) \quad (83)
\]

Putting these bounds all together, we achieve \( R^{(1)} + R^{(2)} + R^{(3)} \):

\[
\frac{1}{2} \left( \log \left( \text{SNR} \right) - \log(K + 3) + \min\{2C_{FB}, \log \left( \text{SNR}^{\alpha-2} \right) \} - \log(4) \\
+ \log \left( \text{SNR} \right) - \log(5) \right) \\
= \log \left( \text{SNR} \right) + \min\{C_{FB}, \frac{1}{2} \log \left( \text{SNR}^{\alpha-2} \right) \} - \frac{1}{2} \log(20(K + 3)) .
\]

(84)

Next, will bound the gap between (84) and the conjectured upper bound (39). We split this into 2 regimes.
The first is when $2C_{FB} \leq \log (\text{SNR}^{\alpha-2})$, and the second is when $C_{FB} > \log (\text{SNR}^{\alpha-2})$. In the first case, we find the distance between \((84)\) and $R_{\text{sym},0}^u + C_{FB}$ as follows:

$$R_{\text{sym},0}^u + C_{FB} - \left( \log (\text{SNR}) + \min\{C_{FB}, \frac{1}{2} \log (\text{SNR}^{\alpha-2})\} - \frac{1}{2} \log (20(K + 3)) \right) \leq \log (1 + \text{SNR}) + C_{FB} - \left( \log (\text{SNR}) + \frac{1}{2} \log (\text{SNR}^{\alpha-2}) - \frac{1}{2} \log (20(K + 3)) \right)$$

$$= \log (2) + \frac{1}{2} \log (20(K + 3))$$

$$= \frac{1}{2} \log (80(K + 3)). \quad (85)$$

In the second case we find the gap between \((84)\) and $R_{\text{sym},\infty}^u$ as follows:

$$R_{\text{sym},\infty}^u - \left( \log (\text{SNR}) + \min\{C_{FB}, \frac{1}{2} \log (\text{SNR}^{\alpha-2})\} - \frac{1}{2} \log (20(K + 3)) \right)$$

$$= \left( \frac{1}{2} \log (1 + \frac{\text{SNR}}{1 + \text{INR}}) + \frac{1}{2} \log (1 + \text{SNR} + \text{INR}) + \frac{K - 1}{2} + \log K \right) - \left( \log (\text{SNR}) + \min\{C_{FB}, \frac{1}{2} \log (\text{SNR}^{\alpha-2})\} - \frac{1}{2} \log (20(K + 3)) \right)$$

$$= \frac{1}{2} \log (1 + \frac{\text{SNR}}{1 + \text{INR}}) + \frac{1}{2} \log (1 + \text{SNR} + \text{INR}) + \frac{K - 1}{2} + \log K$$

$$- \log (\text{SNR}) - \frac{1}{2} \log (\text{SNR}^{\alpha-2}) + \frac{1}{2} \log (20(K + 3))$$

$$= \frac{1}{2} \log \left( \frac{(1 + \text{SNR} + \text{INR})(1 + \text{SNR} + \text{INR})}{(\text{SNR}^{\alpha-2})(1 + \text{INR})(\text{SNR})^2} \right) + \frac{K - 1}{2} + \log K + \frac{1}{2} \log (20(K + 3))$$

$$\leq \frac{1}{2} \log \left( \frac{(3\text{SNR}^{\alpha})^2}{(\text{SNR}^{\alpha-2})(\text{INR})(\text{SNR})^2} \right) + \frac{K - 1}{2} + \log K + \frac{1}{2} \log (20(K + 3))$$

$$= \frac{1}{2} \log 9 + \frac{K - 1}{2} + \frac{1}{2} \log (20K^2(K + 3))$$

$$= \frac{K - 1}{2} + \frac{1}{2} \log (180K^2(K + 3)). \quad (86)$$

From (85) and (86), we find that the achievable symmetric rate is within $\frac{K - 1}{2} + \frac{1}{2} \log (180K^2(K + 3))$ bits to the conjectured upper bound in (39) when $\alpha \geq 2$.

Combining these three cases together with the gap of $\frac{1}{2} \log 9 + 16 + \frac{K - 1}{2} + 3 \log K$ bits when $2/3 < \alpha < 1$ regime, and the gap of $\frac{1}{2} \log 6 + 6 + \frac{K - 1}{2} + \log K$ bits when $1 < \alpha < 2$ (gap between the upper bound for the symmetric capacity with perfect feedback in Theorem 3 of [16] and the lower bound for the symmetric capacity with no feedback in Theorem 1 of [6]), we find that the achievable symmetric rate is within $L$
bits to the conjectured upper bound in (39), where \( L \) is given by (40).

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