A model problem for the initial-boundary value formulation of Einstein’s field equations

Oscar Reula
Facultad de Matemática, Astronomía y Física, Universidad Nacional de Córdoba,
Ciudad Universitaria, Medina Allende y Haya de la Torre, Córdoba 5000, Argentina

Olivier Sarbach
Department of Mathematics and Department of Physics and Astronomy,
Louisiana State University, 202 Nicholson Hall, Baton Rouge, Louisiana 70803-4001, USA and
Theoretical Astrophysics 130-33, California Institute of Technology, Pasadena, CA 91125, USA

In many numerical implementations of the Cauchy formulation of Einstein’s field equations one encounters artificial boundaries which raises the issue of specifying boundary conditions. Such conditions have to be chosen carefully. In particular, they should be compatible with the constraints, yield a well posed initial-boundary value formulation and incorporate some physically desirable properties like, for instance, minimizing reflections of gravitational radiation.

Motivated by the problem in General Relativity, we analyze a model problem, consisting of a formulation of Maxwell’s equations on a spatially compact region of spacetime with timelike boundaries. The form in which the equations are written is such that their structure is very similar to the Einstein-Christoffel symmetric hyperbolic formulations of Einstein’s field equations. For this model problem, we specify a family of Sommerfeld-type constraint-preserving boundary conditions and show that the resulting initial-boundary value formulations are well posed. We expect that these results can be generalized to the Einstein-Christoffel formulations of General Relativity, at least in the case of linearizations about a stationary background.

I. INTRODUCTION

For a number of years there has been an effort in the field of General Relativity to obtain a set of boundary conditions which are consistent with constraint propagation, physically reasonable and stable in the sense that they yield a well posed initial-boundary value formulation. Such a set is essential when integrating Einstein’s field equations on a domain with artificial timelike boundaries on a time scale of astrophysical relevance, as is the case in most numerical simulations of the binary black hole problem. Besides the success of [1] for the case of their frame formulation, and in spite of a lot of efforts [2, 3, 4, 5, 6, 7, 8, 9, 10], no boundary conditions which satisfy all the above properties have been found for the more commonly used tensorial formulations of the vector potential. In an attempt to understand this problem we consider in section II a toy model which in many aspects resembles some of the evolution systems currently employed for numerical evolution in General Relativity. This model is just Maxwell’s equations in a 3 + 1 decomposition, but written as a first order system in terms of the vector potential, the electric field, and all first order spatial derivatives of the vector potential. The allowance of all first order spatial derivatives of the vector potential as variables, and not just the antisymmetric, gauge-independent ones which describe the magnetic field, makes the situation similar to the one on the above mentioned systems, where all derivatives of the metric are indiscriminately promoted to evolution variables, regardless of their gauge dependence. Thus, the Maxwell system, which in gauge-independent variables (electric, and magnetic fields) has a nice symmetric hyperbolic initial value formulation as a set of evolution equations, here acquires most of the pathologies one encounters for the general relativistic systems. In particular, unphysical constraints appear, and the constraints propagate with nontrivial speeds at timelike boundaries. The indeterminacy of the evolution system under the addition of linear combination of constraints is used to construct a two-parameter family of evolution systems. For some values of the parameters the system is symmetric hyperbolic, for some others it is only strongly hyperbolic, while still for others it is neither, and so ill posed. This system has been used by a number of researchers in the past, in particular see [11].

The construction of boundary conditions for this toy model is easier than for the case of Einstein’s equations, and we can look at it in full generality. In particular, in the toy model, we know without ambiguities what the physically relevant boundary conditions are which lead to non-incoming radiation conditions, and so we can impose them, together with some conditions ensuring constraint propagation. As shown in section III for a range of parameters this set of boundary conditions satisfies the so called Kreiss condition [12, 13, 14] and so they yield a well posed problem for trivial initial data. But it turns out that, surprisingly, this set of conditions does not yield well posed problems in the traditional sense, that is when non-trivial initial data is allowed. This is shown by a counterexample consisting of a solution to the equations which does not satisfy the expected energy estimates. This counterexample also indicates the source of the problem which is, basically, the existence of static solutions which are represented by linearly in time
grown gauge potentials, and motivates a new gauge condition. By using this new condition, we are able to show well posedness in a Hilbert space that controls the $L^2$ norm of the main variables and the constraint variables. In section VI we derive a priori estimates. The main idea is to start with an estimate for the gauge-invariant quantities instead of estimating the norm given by the symmetrizer of the evolution system, which is gauge dependent. Based on these estimates we show existence of solutions in section VII. Since the new gauge condition is imposed by some elliptic equations, some elliptic theory is needed. For the proof of existence we employ the abstract theory of semigroups, for which the elliptic aspects of the problem are taken care of in a natural way. In section VIII we summarize our results and discuss some of their implications for the construction of boundary conditions for tensorial formulations of Einstein’s field equations. We have also provided an appendix with the details of the elliptic theory needed; the material there is standard. The main results and the counterexample that motivates the new gauge condition are stated in the next section.

II. MODEL PROBLEM AND MAIN RESULTS

In this section we present our model problem which consists of a system of evolution equations with constraints. We start with the case without boundaries and state in which sense the resulting initial-value problem is well posed. In particular, we introduce a Hilbert space $\mathcal{H}$ which controls the $L^2$ norm of the main and constraint variables. We then consider the presence of artificial boundaries, discuss boundary conditions and show by an explicit counterexample that the resulting initial-boundary value problem is not well posed in the expected space $\mathcal{H}$. This motivates a new gauge choice. Finally, we state the main result of this article which proves well posedness in $\mathcal{H}$ of the initial-boundary value problem for this new gauge choice.

Let $\Omega = \mathbb{R}^3$, and denote by $\nabla$ the covariant derivative with respect to the Eulerian metric $h = \delta_{ij} dx^i dx^j$ on $\mathbb{R}^3$ [32]. We are interested in the following evolution system on $\Omega$:

\begin{align}
\partial_t C &= \rho - \nabla^k E_k, \\
\partial_t A_i &= E_i + \nabla_i \phi, \\
\partial_t E_j &= \nabla^i W_{ij} - (1 + \alpha) \nabla^i W_{ij} + \alpha \nabla_j W + J_j, \\
\partial_t W_{ij} &= \nabla_i E_j + \frac{\beta}{2} h_{ij} \nabla^k E_k + \nabla_i \nabla_j \phi - \frac{\beta}{2} h_{ij} \rho,
\end{align}

where $\rho$ and $\phi$ are scalars on $\Omega$, $A_i$, $E_i$, and $J_i$ are one-forms on $\Omega$, and $W_{ij}$ a two-tensor on $\Omega$ with trace $W = h^{ij} W_{ij}$. $\alpha$ and $\beta$ are two parameters which determine the dynamics off the constraint hypersurface which is defined by

\begin{align}
C_i &\equiv \rho - \nabla^k E_k = 0, \\
C_{ij}^{(W)} &\equiv W_{ij} - \nabla_i A_j = 0.
\end{align}

Physically, the system [2334] is equivalent to Maxwell’s equations on Minkowski space, where $\phi$ represents the electrostatic potential, $A_i$ the vector potential, $E_i$ the electric field, $(B_i) = (W_{23} - W_{32}, W_{31} - W_{13}, W_{12} - W_{21})$ the magnetic field, and $\rho$ and $J_i$ represent the charge and current density, respectively, which obey the continuity equation [11]. Notice that for given current density, Eq. (1) can be integrated separately in order to obtain $\rho$, which can then be used as a source function in order to integrate the equations [2334]. However, we will find it more convenient to interpret $\rho$ as a field being evolved along with the fields $A_i$, $E_i$, $W_{ij}$ since in this case the constraint variable $C$ depends linearly on the evolution fields. We assume that the electrostatic potential and the current density are a priori given. The motivation for introducing the fields $W_{ij}$, which represent all the first order spatial derivatives of the vector potential, instead of using the magnetic field is to obtain a system of equations whose structure is similar to the one of the Einstein-Christoffel formulation of Einstein’s field equations [13, 15, 14, 16]. In the latter, one has evolution equations for the components of the three-metric $g_{ij}$, the extrinsic curvature $K_{ij}$, and some symbols $d_{kij}$ that are linear combinations of the Christoffel symbols. These fields are subject to the Hamiltonian and momentum constraints, and to the constraints $d_{kij} - d_{kij} = 0$. The structure of the equations is very similar to the ones in our model problem with the correspondence $g_{ij} \leftrightarrow A_i$, $K_{ij} \leftrightarrow E_i$, $d_{kij} \leftrightarrow W_{ij}$, $N^i \leftrightarrow \phi$, where $N^i$ is the shift vector field.

In the following, we impose the condition $\alpha \cdot \beta > 0$ which is a necessary and sufficient condition for the evolution system [1234] to be strongly hyperbolic which in turn yields well posedness of the associated Cauchy problem in $L^2(\Omega)$ [13]. The constraints’ propagation is described by the following evolution system

\begin{align}
\partial_t C_{ij}^{(W)} &= -\frac{\beta}{2} h_{ij} C, \\
\partial_t C &= -\alpha \nabla^k E_k, \\
\partial_t C_k &= -\beta \nabla_k C,
\end{align}
which is a consequence of the evolution system (1234). Here, we have introduced the new constraint variable
\[ C_k = 2H^{ij} \nabla_{[k} C_{ij]}^{(W)} = \nabla_k W - \nabla^i W_{ij} \] in order to obtain a first order evolution system. A simple energy estimate shows that the unique solution to Eqs. (789) with trivial initial data is trivial, and therefore, the constraints remain satisfied if satisfied initially. Summarizing, we have the following

**Theorem 1 (Well posedness of the Cauchy problem)** For \( \alpha \beta > 0 \) the constrained Cauchy problem associated with (1234), (67) on \( \Omega = \mathbb{R}^3 \) is well posed in the following sense: Let \( \tau > 0 \) be an arbitrary fixed constant with dimension of length, and write \( u(t) = (\tau \rho(t), \tau^{-1} A_i(t), E_i(t), W_{ij}(t)) \) (main variables), \( v(t) = (\tau^{-1} C_{ij}^{(W)}(t), C(t), C_k(t)) \) (constraint variables) and \( j(t) = (J_i(t), \tau \nabla_i J, \tau^{-1} \nabla_i \phi_i, \nabla_i \phi, \phi) \) (source functions). Given smooth source functions \( j(t) \in L^2(\Omega), t > 0 \) such that \( t \mapsto j(t) \) is continuous, and given smooth initial data \( u_0 \in L^2(\Omega) \), there exists a unique solution \( u(t) \in L^2(\Omega) \) of the evolution system (1234) with \( u(0) = u_0 \). Furthermore, the solution obeys the estimate

\[
\|u(t)\|_{L^2(\Omega)}^2 \leq a e^{b t/\tau} \left[ \|u_0\|_{L^2(\Omega)}^2 + \tau \int_0^t \|j(s)\|_{L^2(\Omega)}^2 ds \right],
\]

for some constants \( a, b \).

If the initial data is such that \( v_0 \in L^2(\Omega) \), then \( v(t) \in L^2(\Omega) \) and we also have the estimate

\[
\|v(t)\|_{L^2(\Omega)}^2 \leq c e^{d t/\tau} \|v_0\|_{L^2(\Omega)}^2,
\]

for some constants \( c, d \). In particular, this implies that initial data which satisfies the constraints initially automatically satisfies the constraints at later time.

**Remarks:**

1. The constant \( \tau \) is an artificial length scale that is introduced in order to avoid adding quantities which have different units. By choosing this scale to be arbitrarily large we can make the growth rate \( 1/\tau \) in the estimates as small as we like.

2. The precise sense in which there exists a solution \( u(t) \in L^2(\Omega) \) is in the sense of a strongly continuous semigroup in \( L^2(\Omega) \) whose generator is determined by the operator on the right-hand side of (1234).

3. In the following, we will replace the Hilbert space \( \{ u = (\tau \rho, \tau^{-1} A_i, E_i, W_{ij}) \in L^2(\Omega) \} \) by the Hilbert space

\[
\mathcal{H} = \{ u = (\tau \rho, \tau^{-1} A_i, E_i, W_{ij}) \in L^2(\Omega) : v = (\tau^{-1} C_{ij}^{(W)}, C, C_k) \in L^2(\Omega) \}
\]

with scalar product \( \langle u_1, u_2 \rangle_{\mathcal{H}} = \langle u_1, u_2 \rangle_{L^2(\Omega)} + \tau^2 \langle v_1, v_2 \rangle_{L^2(\Omega)} \). This Hilbert space controls the \( L^2 \) norm of the main variables \( and \) the constraint variables. In this case, we replace the estimate (10) by the estimate

\[
\|u(t)\|_{\mathcal{H}}^2 \leq a e^{b t/\tau} \left[ \|u_0\|_{\mathcal{H}}^2 + \tau \int_0^t \|j(s)\|_{L^2(\Omega)}^2 ds \right].
\]

The space \( \mathcal{H} \) will be important when boundaries are present, where we will be able to show well posedness in \( \mathcal{H} \) (see Theorem 2 below) but not in \( L^2 \). That is, we will be able to derive an estimate which involves the \( L^2 \) norms of the main and the constraint variables, but not the main variables alone.

The purpose of this article is to analyze the constrained evolution system (1234), (67) on a bounded domain \( \Omega \) of \( \mathbb{R}^3 \) with \( C^\infty \) boundary \( \partial \Omega \). In the following, \( n \) denotes the outward unit one-form to \( \partial \Omega \). We also introduce the projection operator \( H^i_j = \delta^i_j - n_i n^j \) on the tangent space of \( \partial \Omega \). For technical reasons, we assume that the extrinsic curvature of the boundary surface, defined by \( \kappa_{ij} = H^i_j H^j_k \nabla_k n_i \), is positive semi-definite at each point of the boundary.

We wish to specify boundary conditions on \( \partial \Omega \) which imply a well posed initial-boundary value problem in \( \mathcal{H} \). In particular, these conditions have to be specified such that the constraints are propagated which means that the constraints (5,6) should hold at later times if satisfied initially. We shall call boundary conditions which have this property *constraint preserving*. In order to specify such conditions we impose homogeneous maximally dissipative boundary conditions for the constraints’ propagation system (789), which means that we couple the in- to the outgoing characteristic fields with a smooth coupling function \( c \) on \( \partial \Omega \) with the property that \( |c| \leq 1 \):

\[
V^{(+)} = c V^{(-)},
\]

(14)
where

\[ V(+) = C - \frac{\sqrt{\alpha \beta}}{\beta} n^k C_k, \quad V(-) = C + \frac{\sqrt{\alpha \beta}}{\beta} n^k C_k, \]

are the in- and outgoing fields, respectively. A simple energy argument (see the next section) shows that the unique solution to Eqs. (7,8,9) with the boundary condition (14) and trivial initial data is trivial. Therefore, the boundary conditions (14) are constraint preserving. Furthermore, we impose the boundary conditions on the solution to Eqs. (7,8,9) with the boundary condition (14) and trivial initial data is trivial. Therefore, the boundary conditions (14) are constraint preserving. Furthermore, we impose the boundary conditions

\[ w_i^{(+)} = S_i^j w_j^{(-)} + g_i, \quad w_i^{(\pm)} = H_i^j E_j \pm n^k (W_{ki} - W_{ik}), \]

where \( S_i^j \) is a smooth matrix-valued function on \( \partial \Omega \) with the properties that \( S_i^j n^i = 0, S_i^j n_j = 0 \) and \( H^{ij} S_i^r S_j^s \leq H^{rs} \) if \( g_i = 0 \) and \( H^{ij} S_i^r S_j^s \leq \delta H^{rs} \) for some \( \delta < 1 \) otherwise. The function \( g_i \) is a smooth function on \( \partial \Omega \) with the property that \( n^i g_i = 0 \). Physically, the boundary condition (15) permits to control the normal component of the Poynting vector \( P_n = -n^i E^j (W_{ij} - W_{ji}) \): When \( g_i = 0 \), this condition makes sure that \( P_n \) is nonnegative, meaning that the total energy flux through the boundary is nonnegative. When \( S_i^j = 0 \) the data \( g_j \) permits to introduce a wave which travels in normal direction towards the boundary. With these boundary conditions one would expect to have an estimate, and the corresponding theorem of uniqueness and existence, of the following form:

\[
\|u(t)\|^2_{\mathcal{H}} \leq a e^{b t / \tau} \left[ \|u_0\|^2_{\mathcal{H}} + \int_0^t \|j(s)\|^2_{L^2(\Omega)} ds + \int_0^t \|g(s)\|^2_{L^2(\partial \Omega)} ds \right],
\]

\[
\|v(t)\|^2_{L^2(\Omega)} \leq a e^{b t / \tau} \|v_0\|^2_{L^2(\Omega)},
\]

for some constants \( a, b \). But this expectation is false, as shows the following counterexample.

### A. Ill posedness in \( L^2 \)

Suppose that the source functions \( \phi \) and \( J_j \) vanish identically, and consider the following family of solutions:

\[
\begin{align*}
\rho & = 0, \\
A_i & = t \nabla_i f, \\
E_i & = \nabla_i f, \\
W_{ij} & = t \nabla_i \nabla_j f,
\end{align*}
\]

(16)

where \( f \) is a smooth, time-independent, harmonic function \( (\nabla^2 f = 0) \). This family satisfies the constraints (10), the evolution equations (12,13,14), and the boundary condition (14). It has the initial data

\[
\rho|_{t=0} = 0, \quad A_i|_{t=0} = 0, \quad E_i|_{t=0} = \nabla_i f, \quad W_{ij}|_{t=0} = 0,
\]

and satisfies the boundary conditions (15) with boundary data given by

\[
g_i = w_i^{(+)} - S_i^j w_j^{(-)} = (H_i^j - S_i^j) \nabla_j f \bigg|_{x=0}.
\]

This family of solutions does not obey the desired estimate in \( \mathcal{H} \) since \( W_{ij} \) depends on second derivatives of \( f \) whereas \( \rho, A_i, E_i, W_{ij} \) depend only on first derivatives of \( f \). Physically, the solution (16) represents an electrostatic solution in a “bad” gauge corresponding to a boundary with charge density proportional to \( n^i \nabla_i f \). This gauge is “bad” in the sense that it is not adapted to electrostatic solutions since it requires the electric potential to be zero. As a consequence, the potential grows linearly in time. Notice that the solution (16) is trivial in the absence of boundaries since (if one requires the initial data to decay sufficiently fast at infinity) \( f \) has to be zero. This is compatible with the fact that in those cases one can show well posedness in \( L^2 \) since the evolution system is strongly hyperbolic (see Theorem 1).

### B. Main result

The above counterexample shows that the simple gauge condition \( \phi = 0 \) does not lead to a problem that is well posed in \( \mathcal{H} \). This counterexample motivates the following condition on \( \phi \):

\[
\nabla^k \nabla_k \phi = -\nabla^k E_k, \quad n^k \nabla_k \phi = -n^k E_k \quad \text{on} \quad \partial \Omega.
\]

(17)
This condition precludes the type of solution leading to the counterexample, and, as we will state shortly, allows for the formulation of a well-posed initial-boundary value problem. Notice that the gauge condition (17) guarantees that \( \partial_t(n^k A_k) = 0 \) on \( \partial \Omega \). In particular, this implies that the solution satisfies \( n^k A_k = 0 \) on \( \partial \Omega \) if the initial data satisfies this condition. In order to state the main result of this article, we introduce the Hilbert space

\[
\mathcal{H}' = \{ u \in \mathcal{H} : n^k A_k = 0 \text{ on } \partial \Omega \}. \tag{18}
\]

**Theorem 2 (Well posedness of the initial-boundary value problem)** Consider the constrained evolution system (1,2,3,4), (17), (14,15), where \( \phi \) is determined by the elliptic system (17). Assume that \( \alpha \beta > 0 \), and that the extrinsic curvature of the boundary surface, \( \kappa_{ij} \), is positive semi-definite at each point of the boundary. Assume that the matrix-valued function \( S_{ij} \) appearing in the boundary condition (15) has the form \( S_{ij} = sH_{ij} \) for some function \( s \in C^\infty(\partial \Omega) \), and assume that \( |c| < 1, |s| < 1 \) on \( \partial \Omega \).

Then, given smooth source functions \( \hat{j}(t) = (J_i(t), \nabla^j J_i(t)) \), \( t > 0 \), in \( L^2(\Omega) \), such that \( t \mapsto \hat{j}(t) \) is continuous, and given smooth initial data \( u_0 \in \mathcal{H}' \) and smooth boundary data \( g_i(t) \) in \( L^2(\partial \Omega) \) with appropriate compatibility conditions (see Eqs. (71,72) below) at \( \{ t = 0 \} \times \partial \Omega \), there exists a unique solution \( u(t) \in \mathcal{H}' \) of (1,2,3,4), (17), (14,15) with \( u(0) = u_0 \). Furthermore, the solution obeys the estimates

\[
\| u(t) \|_{\mathcal{H}}^2 \leq a e^{bt/\tau} \left[ \| u_0 \|_{\mathcal{H}}^2 + \tau \int_0^t \| \hat{j}(s) \|_{L^2(\Omega)}^2 ds + \int_0^t \| g(s) \|_{L^2(\partial \Omega)}^2 ds \right], \tag{19}
\]

\[
\| v(t) \|_{L^2(\Omega)}^2 \leq a e^{bt/\tau} \| v_0 \|_{L^2(\Omega)}^2. \tag{20}
\]

**Remarks:**

1. Again, the precise sense in which there exists a solution \( u(t) \in \mathcal{H}' \) is in the sense of the existence of a strongly continuous semigroup in \( \mathcal{H}' \). This is discussed in section V.

2. The assumptions on the nonnegativeness of \( \kappa_{ij} \) and on the special form of the coupling matrix \( S_{ij} \) can be weakened, but the proof is technically more complicated in those cases.

In the next section, we start with some preliminary investigations of the initial-boundary problem (1,2,3,4), (17), (14,15), which are valid in the “high-frequency limit”. That is, we consider the case where \( \Omega \) is a half plane and derive some necessary conditions for well-posedness, using Laplace and Fourier transformation techniques. A complete proof of Theorem 2 is given in the subsequent sections.

### III. CONSTRAINT-PRESERVING BOUNDARY CONDITIONS

In this section we construct a family of constraint-preserving boundary conditions for the case where \( \Omega = \{ (x, y, z) \in \mathbb{R}^3 : x > 0 \} \) is the half space. We apply Laplace-Fourier techniques in order to derive necessary conditions for well-posedness of the resulting initial-boundary value problem. We also derive a stronger condition, known as the Kreiss condition, which yields an \( L^2 \) estimate in the case of trivial initial data and trivial sources. However, the counterexample presented in the previous section shows that this result cannot be generalized to the case of non-trivial initial data when the temporal gauge \( \phi = 0 \) is adopted. This shows that the verification of the Kreiss condition for constraint-preserving boundary conditions does not necessarily yield well posedness in the space \( L^2 \). For simplicity we only consider the source-free case here, where \( \rho = 0, J_i = 0 \). We also neglect the evolution equations for \( A_i \) and \( C(W) \) in this section since they can be integrated separately once a solution of (81) and (82) has been obtained.

In order to construct constraint-preserving boundary conditions, we first find the characteristic speeds and fields with respect to the unit outward normal \( (n_i) = (-1, 0, 0) \). For the evolution system (81) these are

\[
\pm r, \quad v_x^{(\pm)} \equiv E_x \mp \frac{r}{\beta} W_x^A \tag{21}
\]

\[
\pm 1, \quad v_A^{(\pm)} \equiv E_A \mp (W_{xA} - (1 + \alpha)W_{Ax}) \tag{22}
\]

\[
0, \quad W_{Ax} \tag{23}
\]

\[
0, \quad \chi_{AB} \equiv W_{AB} - \frac{1}{2} \delta_{AB} W_C \tag{24}
\]

\[
0, \quad \kappa \equiv \beta W_{xx} - (1 + \beta/2)W_A^A \tag{25}
\]
where here and in the following, Capital indices refer to the directions $y$ and $z$ which are transversal to the boundary, $\delta_{AB} = 1$ if $A = B$ and zero otherwise, and $r = \sqrt{\alpha \beta} > 0$. For the constraints’ propagation system (31), the characteristic speeds and fields are

$$\pm r, \quad V^{(\pm)} \equiv C \pm \frac{r}{\beta} C_x$$

$$0, \quad C_A.$$  \hfill (26)

Defining the energy norm

$$\mathcal{E}_e = \frac{1}{2} \int_\Omega \left( C^2 + \frac{\alpha}{\beta} C^k C_k \right) d^3x,$$  \hfill (28)

taking a time derivative, using Eqs. (31) and integration by parts, we find

$$\frac{d}{dt} \mathcal{E}_e = -\alpha \int_\Omega \left( C \nabla^k C_k + C^k \nabla_k C \right) d^3x = \alpha \int_{x=0} CC_x dydz = \frac{r}{4} \int_{x=0} \left( (V^{(+)})^2 - (V^{(-)})^2 \right) dydz.$$ \hfill (29)

Therefore, if we impose the boundary condition

$$V^{(+)} = c V^{(-)}$$ \hfill (30)

with $|c| \leq 1$, it follows that the positive definite quantity $\mathcal{E}_e$ cannot increase; in particular a solution with initial data such that $C = 0$, $C_k = 0$ satisfies $C = 0$ and $C_k = 0$ at later times as well. Re-expressing condition (30) in terms of the variables of the main system yields

$$0 = r(V^{(+)} - c V^{(-)})$$

$$= \partial_t (v^{(+)}_x + c v^{(-)}_x) + \nabla A \left[ \frac{r}{\beta} (1 - c) E_A + (1 + c) F_{xA} - \frac{r}{\beta} (1 - c) \nabla A \phi \right],$$ \hfill (31)

where $F_{xA} = W_{xA} - W_{A} x$, and where we have used the main evolution equations (9) in order to trade normal derivatives ($\partial_x$) by derivatives that are tangential to the boundary ($\partial_t$, $\partial_A$). We can interpret this equation as an evolution equation for $v^{(+)}_x + c v^{(-)}_x$ at the boundary. Since the main evolution system has three ingoing modes, we need two more boundary conditions. One possibility is to require a maximally dissipative boundary condition on the transversal fields

$$v^{(+)}_A = S_A B v^{(-)}_B + g_A,$$ \hfill (32)

where $g_y$, $g_z$ are a priori specified functions on the boundary and where the $2 \times 2$ matrix $S = (S_A B)$ satisfies $S^1 S \leq 1$. From a physical point of view it is more convenient to require

$$E_A - F_{xA} = S_A B (E_B + F_{xB}) + g_A,$$ \hfill (33)

since this controls the radiation flux through the boundary, as discussed in the previous section. In order to explore both possibilities, we shall analyze the family of boundary conditions consisting of Eq. (30) and

$$[E_A - W_{xA} + (1 + a) W_{Ax}] = S_A B [E_B + W_{xB} - (1 + a) W_{Bx}] + g_A,$$ \hfill (34)

where the parameter $a$ is equal to either $\alpha$ or zero.

A. Particular cases

Here, we assume that the parameters $\alpha$ and $\beta$ satisfy $-2 < \alpha < 0, \beta < -2/3$ which makes sure that the evolution system (9) is symmetric hyperbolic. Suppose that $\phi$ satisfies $\nabla A \nabla A \phi = 0$ at the boundary. If we choose $c = -1$ and $S_A B = -S_A B$, the boundary conditions reduce to

$$v^{(+)}_x = v^{(-)}_x + g, \quad v^{(+)}_A = -v^{(-)}_A + g_A,$$ \hfill (35)

where, given $g_A$, the source function $g$ is obtained by integrating the equation $\partial_t g = -r \beta^{-1} \nabla A g_A$ at the boundary. Once $g$ is determined the resulting initial-boundary value problem is well posed since we specify maximally dissipative
boundary conditions for a symmetric hyperbolic system. Notice that $v^{(+)}_A + v^{(-)}_A = 2E_A$, so if $g_A = 0$ this means that the boundary is a conductor.

If we choose, instead, $c = 1$, $S_A^B = \delta_A^B$ and $a = \alpha$, we obtain

$$v^{(+)}_x = -v^{(-)}_x + g, \quad v^{(\pm)}_A = v^{(-)}_A + g_A,$$

where now $g$ is determined by the following evolution system at the boundary (given $g_A$ and $\nabla_x \phi$ at the boundary):

$$\partial_t g + 2\alpha \nabla^A W_{Ax} = \nabla^A g_A, \quad \partial_t W_{Ax} - \frac{1}{2} \nabla_A g = \nabla_A \nabla_x \phi.$$  \hfill (37)

Since $-2 < \alpha < 0$, this boundary evolution system is well posed, and it can be integrated separately to obtain the boundary function $g$ (and the zero speed field $W_{Ax}$). The functions $g$ and $g_A$ are then used as boundary data to integrate the bulk system, and the resulting initial-boundary value problem is well posed. However, in this case, the physical interpretation is less clear since the free data $g_A = -2F_{xA} + 2\alpha W_{Ax}$ is not gauge-invariant.

These two sets of well posed constraint-preserving boundary conditions have been generalized to the linearized Einstein-Christoffel formulation of Einstein’s equations \cite{1}. However, in order to construct a radiative-type boundary condition, as described in the previous section, we need to choose $S_A^B = 0$ and $a = 0$ in Eq. (34), and in this case the well posedness of the resulting system is more involved since the resulting boundary conditions are not in maximal dissipative form. We analyze this in the next subsection.

### B. Fourier-Laplace analysis

Since the equations and boundary conditions are linear and have constant coefficients, we can solve the initial-boundary value problem by performing a Laplace transformation in time and a Fourier transformation in the spatial directions which are tangential to the boundary. In other words, we are considering solutions of the form $f(x)e^{\omega t + i\omega_x Ax}$, where $s$ is a complex number with positive real part, $(\omega_A) = (\omega_\nu, \omega_v)$ is a real two-vector and $f \in L^2(\mathbb{R}_+)$. For given boundary data there must be a unique such solution for each $\text{Re}(s) > 0$ and $\omega_A$, otherwise the system admits modes that grow like $e^{\omega t}$ where $\text{Re}(s)$ can be arbitrarily large, and the system is ill posed \cite{13, 14}.

Performing the Laplace-Fourier transformation, and assuming trivial initial data and $\phi = 0$, we obtain the following system of ordinary differential equations

\begin{align*}
se_{Ax} &= \alpha \partial_x W_{Ax} + i\omega^A [W_{Ax} - (1 + \alpha)W_{Ax}], \quad (39) \\
sE_B &= \partial_x Q_B + i\omega^A W_{AB} - i(1 + \alpha)\omega^A W_{BA} + i\omega_B(W_{xx} + W^A), \quad (40) \\
sW_{xx} &= (1 + \beta/2) \partial_x E_x + \frac{i\beta}{2} \omega^A E_A, \quad (41) \\
sW_{xA} &= \partial_x E_A, \quad (42) \\
sW_{Ax} &= i\omega_A E_x, \quad (43) \\
sW_{AB} &= i\omega_A E_B + \frac{\beta}{2} \delta_{AB} \left(\partial_x E_x + i\omega^C E_C\right), \quad (44)
\end{align*}

where we have introduced $Q_A = W_{xA} - (1 + \alpha)W_{Ax}$. From this we can eliminate the variables with zero speed since their evolution equation becomes algebraic:

\begin{align*}
W_{Ax} &= \frac{i\omega_A}{s} E_x, \quad (45) \\
\chi_{AB} &= \frac{i}{s} \left[\omega_A E_B - \frac{1}{2} \delta_{AB} \omega^C E_C\right], \quad (46) \\
\kappa &= -\frac{i}{2s} (2 + 3\beta) \omega^A E_A. \quad (47)
\end{align*}

Suppose $|\omega| = \sqrt{\delta^{AB}\omega_A\omega_B} \neq 0$. Let $\omega_A = \omega_A/|\omega|$, and let $\tilde{\eta}_A$ be a unit two-vector which is orthogonal to $\omega_A$. Introduce the rescaled variables $\zeta = s/|\omega|$ and $\tilde{\zeta} = |\omega|x$, and define $E_\omega = E_A\tilde{\omega}^A$, $E_\eta = E_A\tilde{\eta}^A$, $Q_\omega = Q_A\tilde{\omega}^A$, $Q_\eta = Q_A\tilde{\eta}^A$. In terms of these variables, the evolution system decouples into the following two blocks

\begin{align*}
\partial_{\tilde{\zeta}} \left(\begin{array}{c}
E_\eta \\
Q_\eta
\end{array}\right) &= \left(\begin{array}{cc}
0 & \zeta + \frac{1}{\zeta} \\
\zeta + \frac{1}{\zeta} & 0
\end{array}\right) \left(\begin{array}{c}
E_\eta \\
Q_\eta
\end{array}\right), \quad (48)
\end{align*}
\[
\begin{align*}
\partial_\xi \begin{pmatrix} E_x \\ Q \\ E_\omega \\ Q_\omega \end{pmatrix} &= \begin{pmatrix} 0 & \frac{\zeta}{\zeta^2 + \alpha^2} & -i \frac{\alpha + \beta}{\zeta} & 0 \\
\frac{i(1 + \alpha)}{\zeta} & 0 & 0 & i(1 + \alpha) \\
0 & -i \frac{\alpha + \beta}{\zeta} & \zeta - \frac{\alpha + \beta + 1}{\zeta} & 0 \\
\end{pmatrix} \begin{pmatrix} E_x \\ Q \\ E_\omega \\ Q_\omega \end{pmatrix},
\end{align*}
\]

where \( Q = \alpha W^A \). The first matrix has the eigenvalues \( \pm \lambda \), where \( \lambda = \sqrt{\zeta^2 + 1} \). The second matrix has the eigenvalues \( \pm \lambda \) and \( \pm \mu \), where \( \mu = \sqrt{\zeta^2/(\alpha \beta) + 1} \) (the sign of the square root is chosen such that for \( \text{Re}(\zeta) > 0 \), \( \text{Re}(\lambda) > 0 \) and \( \text{Re}(\mu) > 0 \)). The solutions which are bounded as \( x \to \infty \) are given by

\[
\begin{align*}
\begin{pmatrix} E_\eta \\ Q_\eta \end{pmatrix} &= \sigma_0 \begin{pmatrix} -\zeta \\ \lambda \end{pmatrix} e^{-\lambda \xi}, \\
\begin{pmatrix} E_x \\ Q \\ E_\omega \\ Q_\omega \end{pmatrix} &= \sigma_1 \begin{pmatrix} \frac{\alpha}{\zeta} \\ -i \lambda \\ i(\zeta^2 - \alpha) \end{pmatrix} e^{-\lambda \xi} + \sigma_2 \begin{pmatrix} -\frac{\zeta - \alpha}{\zeta} \\ -\frac{i \alpha}{\zeta} \\ -\frac{i \alpha}{\zeta} \end{pmatrix} e^{-\mu \xi},
\end{align*}
\]

where \( \sigma_0, \sigma_1, \sigma_2 \) are integration constants. A necessary condition for the system to be well posed is that the boundary conditions must determine these constants uniquely. Using the relations and the previous notation the boundary conditions yield

\[
(1 - c)[i A E_\omega - \zeta Q] + (1 + c) \sqrt{\alpha \beta} \left( \zeta - \frac{\alpha}{\zeta} \right) E_x + i Q_\omega = 0,
\]

\[
(1 - d) E_\omega - (1 + d) \left( Q_\omega + i \frac{\alpha - a}{\zeta} E_x \right) = g_\omega,
\]

\[
(1 - d) E_\eta - (1 + d) Q_\eta = g_\eta,
\]

where for simplicity we have assumed that \( S_A^B = d \delta_A^B \) is diagonal. Plugging into this the general decaying solution we obtain the following equations

\[
[d_- \zeta - d_+ \lambda] \sigma_0 = g_\eta,
\]

\[
\begin{pmatrix}
\begin{pmatrix} d_- \zeta \lambda - d_+ (\zeta^2 - a) \\
0 \\
\end{pmatrix} & \begin{pmatrix} d_- \zeta + d_+ \mu \\
\lambda \end{pmatrix} \\
\end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2/\mu \end{pmatrix} = \begin{pmatrix} -i \zeta g_\omega \\ 0 \end{pmatrix},
\]

where \( c_\pm = c \pm 1 \), \( d_\pm = d \pm 1 \). In order to analyze in what cases these equations uniquely determine the constants \( \sigma_0, \sigma_1, \sigma_2 \) we use the following

**Lemma 1** Let \( P > 0 \), \( A, B \in \mathbb{R} \), \( (A, B) \neq (0, 0) \), and consider the function

\[
\psi : \{ \text{Re}(\zeta) > 0 \} \to \mathbb{C}, \psi(\zeta) = A \zeta - B \sqrt{\zeta^2 + P^2},
\]

where we choose the branch such that \( \text{Re}(\sqrt{\zeta^2 + P^2}) > 0 \) for \( \text{Re}(\zeta) > 0 \).

Then, \( \psi \) has zeroes if and only if \( A > B > 0 \) or \( A < B < 0 \). Furthermore, \( |\psi| \) is uniformly bounded away from zero if and only if \( A \cdot B < 0 \).

**Proof:** If \( B = 0 \) the Lemma is trivial. So let \( B \neq 0 \) and rescale \( \zeta, \psi \) and \( A \) such that \( \psi(\zeta) = A \zeta - \sqrt{\zeta^2 + 1} \). Suppose \( \psi(\zeta) = 0 \) for some \( \text{Re}(\zeta) > 0 \). Then it follows that \( (A^2 - 1)\zeta^2 = 1 \) and thus \( A^2 > 1 \). Therefore, \( \zeta = 1/\sqrt{A^2 - 1} \) and \( \sqrt{\zeta^2 + 1} = |A|/\sqrt{A^2 - 1} \). This satisfies \( \psi(\zeta) = 0 \) if and only if \( A > 1 \) is positive. This proves the first statement of the Lemma.

In order to prove the second assertion of the Lemma we restrict ourselves to the case \( A \leq 1 \) since otherwise \( \psi \) has zeroes. Notice first that for large \( |\zeta| \), \( \psi(\zeta) = (A - 1)\zeta + O(\zeta^{-1}) \), hence \( |\psi| \) is not bounded away from zero if \( A = 1 \). So let \( A < 1 \), and let \( \zeta_j \) be a sequence with \( \text{Re}(\zeta_j) > 0 \) such that \( \psi(\zeta_j) \to 0 \). This sequence is bounded for \( j \) large enough since otherwise \( A = 1 \). Therefore, there is a subsequence that converges to some \( \zeta^* \in \mathbb{C} \). We must have \( \text{Re}(\zeta^*) = 0 \).
since otherwise $\zeta^*$ is a zero of $\psi$. So we must determine in what cases zero lies on the boundary of the image of $\psi$ in $\mathbb{C}$. For $\zeta = i \cos \alpha$, $0 < \alpha < \pi$, we have $\sqrt{\zeta^2 + 1} = \sin \alpha$. Next, let $\gamma > 0$, $\varepsilon > 0$ and $\zeta = i \cosh \gamma + \varepsilon$. Then

$$\sqrt{\zeta^2 + 1} = \sqrt{-\sin^2 \gamma + 2i \cosh(\gamma) \varepsilon + \varepsilon^2 = +i \sinh \gamma + \coth(\gamma) \varepsilon + O(\varepsilon^2)},$$

where we have chosen the sign such that $\text{Re}(\sqrt{\zeta^2 + 1}) > 0$ for $\varepsilon$ small enough. Therefore, the boundary of the image of $\psi$ can be parametrized by

$$\psi(i \cos \alpha) = -\sin \alpha + i \lambda \cos \alpha, \quad 0 < \alpha < \pi,$$
$$\psi(\pm i \cos \gamma) = \pm i (\lambda \cosh \gamma - \sinh \gamma), \quad \gamma \geq 0.$$  
(57)  

(58)

We see that the boundary contains zero only if $0 \leq A \leq 1$. Therefore, $|\psi|$ is bounded away from zero if and only if $A < 0$.

We first apply the Lemma to the case $a = 0$ and $\alpha \beta > 0$. It follows that the $\sigma_0$, $\sigma_1$ and $\sigma_2$ are uniquely determined if and only if $-1 \leq c \leq -1 \leq d \leq 1$. Therefore, the conditions $-1 \leq c$, $|d| \leq 1$ are necessary for the well posedness of the initial-boundary value problem. This justifies the restriction on the matrix $S = (S^A_B)$ to satisfy $S^T S \leq 1$. Next, we impose the Kreiss deterministic condition \[12 \] \[13 \] \[14 \] which is stronger and requires that the constants $\sigma_0$ and $\sigma_1$ can be bounded by the boundary data with a bound that is independent on $\omega$ and $\zeta$. It follows from Lemma \[14 \] that this condition is satisfied if and only if $|d| < 1$. This immediately implies that the problem is well posed in $L^2$ if the initial data is trivial, since one can estimate the integration constants $\sigma_0$, $\sigma_1$ by the boundary data (see Lemma \[8.4.3 \] in \[13 \]) while $\sigma_2 = 0$ if $-1 \leq c$.

On the other hand, suppose that $d = 0$ and $a = \alpha$ which corresponds in setting to zero the transversal ingoing characteristic fields. The resulting initial-boundary value problem has been integrated in Ref. \[11 \] by numerical means for different values of the parameters $\alpha$ and $\beta$, subject to the restriction $\alpha \beta > 0$ (in the notation of \[11 \], $\alpha = -\gamma_1/2$, $\beta = -2\gamma_2$). In this case, $d_2 \zeta^2 - d_1 (\zeta^2 - a) = -((\lambda + \zeta)^2 - 2a - 1)/2$ and since the map $\zeta \mapsto (\lambda + \zeta)^2$ maps $\text{Re}(\zeta) > 0$ onto the outside of the unit disk minus the negative real axis, the system is ill posed if $2a + 1 > 1$, that is, if $\alpha > 0$. Indeed, the numerical results of Ref. \[11 \] exhibit instabilities in those cases.

## IV. ENERGY ESTIMATES

After the preliminary investigations in the previous section, we return to the case where $\Omega \in \mathbb{R}^3$ is a bounded domain with $C^\infty$ boundary $\partial \Omega$, and derive some a priori estimates in this section. For simplicity, we only consider the source-free case here, where $\rho = 0$, $J_i = 0$; the generalization to the inhomogeneous case is discussed at the end of next section. We assume that we are given a smooth function $u = (A_i, E_i, W_{ij})$ which satisfies the evolution equations \[2 \] \[3 \] \[4 \] and the boundary conditions \[14 \] \[15 \]. The goal of this section is to show that this solution satisfies the estimates of Theorem \[2 \]. These estimates are then used in the next section in order to prove existence with semigroup methods.

We start with estimates for the gauge invariant quantities $E_i$ and $F_{ij} = W_{ij} - W_{ji}$ (i.e. the electric and magnetic fields), and the constraint variables $C$, $C_k$, $C_{ij}^{(W)}$, and then estimate the gauge dependent quantities $A_i$ and the symmetric part of $W_{ij}$. It is the estimate for the latter quantities that requires a specific gauge condition, since the counterexample presented in section \[11 \] shows that the simple condition $\phi = 0$ does not allow to estimate the symmetric part of $W_{ij}$ in $L^2$. Assume first that $\alpha = \beta = 0$. In this case, the evolution system \[2 \] \[3 \] \[4 \] is only weakly hyperbolic. However, in this case, one can replace $W_{ij}$ by $F_{ij}$ by taking the antisymmetric part of Eq. \[4 \] and discarding its symmetric part. Consider the boundary conditions \[14 \]:

$$w_1^{(+)} = S_i^j w_j^{(-)} + g_i, \quad u_1^{(\pm)} = H_i^j E_j \leq n^k F_{ki},$$

where we recall that $S_i^j$ is a smooth matrix-valued function on $\partial \Omega$ with the properties that $S_i^j n^i = 0$, $S_i^j n_j = 0$ and $H^j S_i^j S_j^s \leq H^r s$ if $g_i = 0$ and $H^j S_i^j S_j^s \leq \delta H^r s$ for a $\delta < 1$ otherwise. The physical energy is defined by

$$E_{\text{phys}} = \frac{1}{2} \int_{\Omega} \left( E_j^i E_j + \frac{1}{2} F_{ij} F_{ij} \right) d^3 x,$$

where $n^i$ is a normal vector to $\partial \Omega$ with $n^i n_i = 0$. Taking a time derivative, using Eqs. \[14 \] and Gauss’ theorem, we obtain

$$\frac{d}{dt} E_{\text{phys}} = \int_{\Omega} \left( E_j^i \nabla_i F_{ij} + F_{ij} \nabla_i E_j \right) d^3 x = \int_{\partial \Omega} n^i E_j^i F_{ij} dS$$

$$= \frac{1}{2} \int_{\partial \Omega} H^j \left( w_1^{(+)} w_1^{(+)} - w_1^{(-)} w_1^{(-)} \right) dS,$$
where \( dS \) denotes the surface element on \( \partial \Omega \). If the boundary conditions are homogeneous \((g_i = 0)\) it follows immediately that the boundary integral is negative or zero, and we have an estimate for \( \mathcal{E}_{\text{phys}} \). If the boundary data is nonzero, we have the estimate

\[
H^{ij} \left( w_1^{(+)} w_j^{(+)} - w_1^{(-)} w_j^{(-)} \right) \leq -\varepsilon H^{ij} \left( w_1^{(+)} w_j^{(+)} + w_1^{(-)} w_j^{(-)} \right) + K H^{ij} g_i g_j ,
\]

for some constants \( \varepsilon = \varepsilon(\delta) > 0 \), \( K = K(\delta) > 0 \) from which we conclude

\[
\mathcal{E}_{\text{phys}}(t) \leq \mathcal{E}_{\text{phys}}(0) + a \int_0^t \| \mathbf{g}(s) \|_{L^2(\partial \Omega)}^2 ds
\]

for some constant \( a = a(\delta) > 0 \).

When \( \alpha \beta > 0 \), we obtain

\[
\frac{d}{dt} \mathcal{E}_{\text{phys}} = \int_{\Omega} \left( E^i \nabla ^i j C_j + \alpha C_j \right) d^3 x
\]

where we have used Schwarz' inequality and the inequality \((h_i)^2 \leq 3C_i^j C_j^i \) in the last step, and where we have set \( n_1 = 4\alpha/(3\beta) \) and \( r = \sqrt{\alpha \beta} \). On the other hand, using Schwarz' inequality again, we can estimate the norm

\[
\mathcal{E}_{\text{cons}} = \frac{1}{2} \int_{\Omega} \left( n_1 C^{(W)}_{\alpha \beta} + \tau^2 C^2 + \tau^2 \frac{\alpha}{\beta} C^{(W)} C \right) d^3 x.
\]

Here, \( \tau > 0 \) is a fixed constant with dimension of length and \( n_1 \) is a positive constant. We obtain

\[
\frac{d}{dt} \mathcal{E}_{\text{cons}} = -\int_{\Omega} \left( \frac{n_1 \beta}{2} h^{ij} C_j C_k + \alpha \tau^2 \left( C^{(W)}_{\alpha \beta} + \frac{C^2}{\beta} \right) \right) d^3 x
\]

which shows that we can estimate the gauge-invariant quantities \( E_j, F_{ij}, C, C_k, C_{ij}^{(W)} \). Notice that we can choose \( \tau \) arbitrary large, and thus we can make the exponential growth rate in the bound as small as we like. What is missing are estimates for the magnetic potential \( A_j \) and the symmetric part of \( W_{ij} \). Such estimates depend on the gauge
choice for \( \phi \). We have seen that the problem is not well posed in \( L^2 \) if we choose the temporal gauge \( \phi = 0 \). An \( L^2 \) estimate can be obtained if we impose the following gauge choice instead:

\[
\Delta \phi = -\nabla^k E_k, \quad n^k \nabla_k \phi = -n^k E_k \quad \text{on } \partial \Omega,
\]

where \( \Delta = \nabla^k \nabla_k \). The boundary condition on \( \phi \) implies that on \( \partial \Omega \), \( \partial_t (n^k A_k) = n^k E_k + n^k \nabla_k \phi = 0 \). This implies that \( n^k A_k|_{\partial \Omega} = 0 \) provided the initial data satisfies this condition. Next, we notice that we can estimate \( W \) since \( \partial_t W = -(1 + 3\beta/2)C + \Delta \phi = -3\beta C \). Since we control \( C_t^{(W)} \equiv W_{ij} - \nabla_i A_j \) we can also estimate \( \nabla^i A_i \). On the other hand, using Gauss’ theorem twice,

\[
\int_\Omega (\nabla_i A^i)(\nabla_j A^j) d^3x = \int_\Omega (\nabla_i A^i)(\nabla_j A^j) d^3x + \int_{\partial \Omega} n^i (A_i \nabla_j A^j - A^j \nabla_j A_i) dS = \int_\Omega (\nabla_i A^i)(\nabla_j A^j) d^3x + \int_{\partial \Omega} \kappa_{ij} A^i A^j dS,
\]

where in the last term, \( \kappa_{ij} = (h_{ik} - n_i n_k) \nabla^k n_j \) denotes the extrinsic curvature of the boundary \( \partial \Omega \), and where we have used the fact that \( A^i n_i = 0 \) on \( \partial \Omega \). Assuming that \( \kappa_{ij} \) is positive semi-definite at each point of \( \partial \Omega \)[1], we obtain the inequality

\[
\int_\Omega (2\nabla^i A^i \cdot \nabla_i A_j^i + \nabla_i A^i \cdot \nabla_j A^j) d^3x \geq \int_\Omega \nabla^i A^i \cdot \nabla_i A_j d^3x
\]

which allows us to estimate all spatial derivatives of \( A_j \); and in particular the symmetric part of \( W_{ij} \) since we control \( C_t^{(W)} \).

Finally, we show how to estimate \( A_i \): Because of Eq. \( 67 \) we have

\[
\int_\Omega (E^i + 2\nabla^i \phi)(E_i + 2\nabla_i \phi) d^3x = \int_\Omega \{ E^i E_i + 4E^i \nabla_i \phi + 4\nabla^i \phi \nabla_i \phi \} d^3x
\]

\[
= 4 \int_{\partial \Omega} \phi n^i (E_i + \nabla_i \phi) dS + \int_\Omega \{ E^i E_i - 4\phi (\nabla^i E_i + \Delta \phi) \} d^3x
\]

\[
= \int_\Omega E^i E_i d^3x,
\]

so we can estimate \( E_i + 2\nabla_i \phi \) and thus also \( \partial_t A_i = E_i + \nabla_i \phi \) since we have an estimate for \( E_i \). The net result is the a priori estimate

\[
\|u(t)\|^2_{L^2(\Omega)} + \tau^2 \|v(t)\|^2_{L^2(\Omega)} \leq a e^{bt/\tau} \left[ \|u(0)\|^2_{L^2(\Omega)} + \tau^2 \|v(0)\|^2_{L^2(\Omega)} + \int_0^t \|g(s)\|^2_{L^2(\partial \Omega)} d\tau \right],
\]

where \( a > 0, b > 0 \) are two constants and, \( u = (\tau^{-1} A_i, E_j, W_{ij}), v = (\tau^{-1} C_t^{(W)}), C, C_k \). This implies the estimate in Theorem \( 3 \) in the absence of source functions.

V. EXISTENCE

In this section, we prove well posedness for the initial-boundary value problem defined by the evolution equations \( 123 \), the gauge condition \( 17 \) and the constraint-preserving boundary conditions \( 1415 \). The idea is to represent the problem as an abstract Cauchy problem

\[
\frac{d}{dt} u(t) = A u(t), \quad u(0) = u_0 \in H,
\]

where \( A : D(A) \subset H \to H \) is a linear operator on an appropriate Hilbert space \( H \). This operator is basically given by the right-hand side of the evolution equations, and we will define its domain \( D(A) \) to be the space of smooth functions satisfying the boundary conditions with homogeneous boundary data. We then show that \( A \) has a unique extension \( \tilde{A} \) and that this extension generates a strongly continuous semigroup, which we write formally as \( P(t) = \exp(tA) \). Given initial data \( u_0 \) in the domain \( D(\tilde{A}) \) of the extension, the solution to the abstract Cauchy problem is given by \( u(t) = P(t)u_0 \). The semigroup properties imply that \( \|u(t)\| \leq a \exp(bt)\|u_0\| \) for some constants \( a > 0, b > 0 \) which are
independent of \(u_0\); thus the problem is well posed. In the following, we assume that the coupling functions \(c\) and \(S_{i,j}\) that appear in the boundary conditions are time-independent, lie in \(C^\infty(\partial \Omega)\) and satisfy \(|c| < 1, H^{ij}S_{i}S_{j} < H^{rs}\) at each point of \(\partial \Omega\).

The Hilbert space is motivated by the energy estimate in the previous section. Let

\[
H_A = \{ A \in L^2(\Omega, \mathbb{R}^3) : \text{The derivatives } \nabla_j A_i \text{ exist in the weak sense and belong to } L^2(\Omega), \text{ and on } \partial \Omega, n^i A_i = 0 \},
\]

\[
H_E = \{ E \in L^2(\Omega, \mathbb{R}^3) : \text{The derivative } \nabla^i E_i \text{ exists in the weak sense and belongs to } L^2(\Omega), \}.
\]

\[
H_W = \{ W \in L^2(\Omega, \mathbb{R}^3) : \text{The derivatives } \nabla_j W - \nabla^i W_{ji} \text{ exist in the weak sense and belong to } L^2(\Omega). \}.
\]

Here, \(C = -\nabla^i E_i\) exists in the weak sense and belongs to \(L^2(\Omega)\) means that \(C \in L^2(\Omega, \mathbb{R})\) has the property that

\[
\int_{\Omega} C \phi \, dx = \int_{\Omega} E^i \nabla_j \phi \, dx
\]

for all test functions \(\phi \in C_0^\infty(\Omega, \mathbb{R})\). Similar definitions apply for the weak derivatives \(\nabla_j A_i\) and \(\nabla_j W - \nabla^i W_{ji}\). We introduce the following scalar products on \(H_A, H_E\) and \(H_W\):

\[
(A, B)_A = \int_{\Omega} (\tau^{-2} A^i B_j + \nabla^i A^j \cdot \nabla E_j) \, d^3 x,
\]

\[
(E, F)_E = \int_{\Omega} (E^i F_j + \tau^2 \nabla^i E_j \cdot \nabla^j F_i) \, d^3 x,
\]

\[
(W, V)_W = \int_{\Omega} (W^{ij} V_{ij} + \tau^2 (\nabla^i W - \nabla^j W_{ij})(\nabla V - \nabla^k V_k)) \, d^3 x,
\]

where \(A, B \in H_A, E, F \in H_E,\) and \(W, V \in H_W\). Notice that the requirement \(n^i A_i = 0\) on \(\partial \Omega\) makes sense because of the trace theorems. The following results are standard.

**Lemma 2** The spaces \((H_A, \langle \cdot, \cdot \rangle_A), (H_E, \langle \cdot, \cdot \rangle_E)\) and \((H_W, \langle \cdot, \cdot \rangle_W)\) are Hilbert spaces.

**Lemma 3** Denote by \(C^\infty(\mathbb{R}^m, \mathbb{R}^m)\) the class of functions \(\mathbb{R}^m \rightarrow \mathbb{R}^m\) which are the restriction of a smooth function \(C^\infty(\mathbb{R}^3, \mathbb{R}^m)\) on \(\Omega\). Then, \(C^\infty(\Omega, \mathbb{R})\) is dense in \(L^2(\Omega, \mathbb{R})\).

\{ \mathcal{A} \in C^\infty(\Omega, \mathbb{R}^3) : n^i A_i = 0 \text{ on } \partial \Omega \} \text{ is dense in } H_A.

\(C^\infty(\Omega, \mathbb{R}^3)\) is dense in \(H_E\).

\(C^\infty(\Omega, \mathbb{R}^3)\) is dense in \(H_W\).

We define the total Hilbert space \(H = L^2(\Omega, \mathbb{R}) \times H_A \times H_E \times H_W\) with scalar product

\[
\langle u, \bar{u} \rangle_H = \int_{\Omega} \left( \tau^2 \rho^2 + \tau^{-2} A^i \bar{A}_i + E^i \bar{E}_i + 2 W^{ij} \bar{W}_{ij} + \frac{\alpha}{3} W \bar{W} + \frac{\alpha}{3} C^{ij} \bar{C}_{ij} + \tau^2 \bar{C} C + \frac{\alpha}{3} \bar{C} C \right) \, d^3 x,
\]

where \(u = (\rho, A_i, E_i, W_{ij})\), \(C^{ij} = W_{ij} - \nabla_i A_j\), \(C = \rho - \nabla^k E_k\), \(C_k = \nabla_k W - \nabla^j W_{kj}\). The estimate (69) implies that the norm induced by \(\langle \cdot, \cdot \rangle_H\) is equivalent to the one induced by the scalar product constructed from \((\cdot, \cdot)_{L^2}, \langle \cdot, \cdot \rangle_A, \langle \cdot, \cdot \rangle_E\) and \(\langle \cdot, \cdot \rangle_W\). Also, the Hilbert space \(H\) is topologically equivalent to the space \(H'\) defined in section III. Before we define the operator \(A\) on the Hilbert space \(H\), we need the following result from elliptic theory (see, for example [30]).

**Lemma 4** Let \(F \in C^\infty(\Omega)\), and \(g \in C^\infty(\partial \Omega)\). Then, the Neumann problem

\[
\Delta u = F \quad \text{on } \Omega,
\]

\[
n^k \nabla_k u = g \quad \text{on } \partial \Omega,
\]

has a solution \(u \in C^\infty(\Omega)\) if and only if

\[
\int_{\Omega} F \, d^3 x = \int_{\partial \Omega} g \, dS.
\]

The solution is unique up to an additive constant.
The linear operator

Proposition 3 (Well posedness in the homogeneous case) Theorem 3 (Well posedness in the homogeneous case) 

Proposition 1

Proposition 2

where \( \mathbf{a} = (a^j_i) = \mathbf{H}(\mathbf{H} - \mathbf{S})^{-1}(\mathbf{H} + \mathbf{S}) \) and \( b_0 = (1 + c)/(1 - c) \), and

\[
A \begin{pmatrix} \rho \\ A_j \\ E_j \\ W_{ij} \end{pmatrix} = \begin{pmatrix} 0 \\ E_j - \nabla_j \Delta^{-1} \nabla^i E_i \\ \nabla^i W_{ij} - (1 + \alpha) \nabla^i W_{ji} + \alpha \nabla_j W_{ij} \\ \nabla_i E_j + \frac{3}{2} h_{ij} \nabla^k E_k - \nabla_i \nabla_j \Delta^{-1} \nabla^k E_k - \frac{3}{2} h_{ij} \rho \end{pmatrix}
\]

(77)

The existence proof relies on the following three propositions, which imply by the Lumer-Phillips theorem [19] that \( A \) is closable and that its closure is the generator of a strongly continuous semigroup.

**Proposition 1** \( D(A) \) is dense in \( H \).

**Proposition 2** The operator \( A : D(A) \rightarrow H \) defined in (76), (77) is quasi-dissipative. That is, there is a constant \( b \) such that

\[
\Re \langle u, Au \rangle_H \leq \frac{b}{\tau} \langle u, u \rangle_H
\]

(78)

for all \( u \in D(A) \).

**Proposition 3** \((\lambda - A)(D(A))\) is dense in \( H \) for \( \lambda > 0 \) sufficiently large.

**Theorem 3** (Well posedness in the homogeneous case) The linear operator \( A : D(A) \rightarrow H \) is closable and its closure \( \bar{A} \) is the generator of a strongly continuous semigroup \( P(t) \) in \( H \). Given initial data \( u_0 \in D(\bar{A}) \), the map \( \mathbb{R}^+ \rightarrow D(\bar{A}) \), \( t \mapsto P(t)u_0 \) is strongly differentiable and satisfies

\[
\frac{d}{dt}u(t) = \bar{A}u(t), \quad t > 0,
\]

(79)

and so gives a solution to the constrained evolution system (76), (77) with \( J_j = 0 \) and homogeneous boundary data, \( g_i = 0 \). This solution obeys the estimate (78).

We will prove later that this solution also obeys the estimate (78).

### A. Proof of Proposition 1

The proof of Proposition 1 is based on Lemma 1 and

**Lemma 5** Let \( G \in C^\infty(\partial \Omega, \mathbb{R}) \), \( \mathbf{F} \in C^\infty(\partial \Omega, \mathbb{R}^3) \) with \( F_in_i = 0 \), and let \( \varepsilon > 0 \). There exists \( \mathbf{E} \in C^\infty(\bar{\Omega}, \mathbb{R}^3) \) such that \( \| \mathbf{E} \|_{C^\infty} < \varepsilon \) and such that on \( \partial \Omega \),

\[
H_j^i E_j = F_i, \quad \nabla^k E_k = G,
\]

where \( n_i \) denotes the unit outward normal to \( \Omega \), and \( H_j^i = \delta_j^i - n_i n_j \) is the projection operator on the tangent space of \( \partial \Omega \).

**Proof:** We construct \( \mathbf{E} \) in the following way: First, extend \( n_i \) to a neighborhood of \( \partial \Omega \) in \( \Omega \) by shooting geodesics through \( n_i \) at each point of \( \partial \Omega \) (such that \( n_i \nabla_i n_k = 0 \)). Denote by \( s \) the affine parameter which is such that \( \nabla_i s = n_i \) and \( s = 0 \) on \( \partial \Omega \). We consider a neighborhood \( U_{\delta} \) of \( \Omega \) which is spanned by \( s \in (-\delta, 0) \) for some \( \delta > 0 \). Next, extend \( G \) and \( F_i \) inside this neighborhood by solving the ordinary differential equations

\[
\nabla_n G + \kappa G = 0,
\]

\[
\nabla_n F_i + \kappa F_i = \kappa_{ji} F_j,
\]

where \( \kappa_{ij} = \nabla_i n_j \) is the extrinsic curvature of the surfaces \( s = c_{\text{onst}} \). Notice that the second equation implies that \( \nabla_n (n_i F_i) = -\kappa (n_i F_i) \), so \( n_i F_i = 0 \) on \( U_{\delta} \). Finally, let \( m > 2/\delta \) and define the function \( \psi_m \in C^\infty(\bar{\Omega}) \) by \( \psi_m = m^{-1} \psi (ms) \), where \( \psi \in C^\infty((0, \infty), \mathbb{R}) \) has the following properties:
(i) \( \psi(0) = 1, \psi(s) = 0 \) for all \( s \leq -2 \).

(ii) \( \psi'(0) = 1 \)

(iii) \( 0 \leq \psi'(s) \leq 1 \) for all \( s \leq 0 \).

Then, we define \( E_i = U_i + \varepsilon_{i\rho} \nabla^\rho V^s \), where \( U_i = \psi_m n_i G, \) \( V^s = -\psi_m \varepsilon^{i\rho\kappa} n_k F_i \) and \( \varepsilon_{i\rho\kappa} \) denotes the natural volume element on \( \Omega \). By construction, we have

\[
\nabla^i E_i = \nabla^i U_i = \nabla_n \psi_m \cdot G,
\]
\[
E_i = \psi_m n_i (G - \nabla^j F_j) + \nabla_n \psi_m \cdot F_i ,
\]
so \( E_i \) satisfies the boundary conditions. Furthermore,

\[
\| E \|_E^2 = \int_{U_s} \left[ \psi_m^2 (G - \nabla^j F_j)^2 + (\nabla_n \psi_m)^2 (F_i F_i + \tau^2 G^2) \right] d^3 x
\leq |\text{Vol}(U_{2/m})| \left[ \frac{1}{m^2} \int_{U_s} (G - \nabla^j F_j)^2 d^3 x + \int_{U_s} (F_i F_i + \tau^2 G^2) d^3 x \right].
\]

The right-hand side converges to zero as \( m \to \infty \), so the lemma follows.

Now let \( (\rho, A, E, W) \in H \), and let \( \varepsilon > 0 \) be arbitrarily small. According to Lemma 3 there exist \( \tilde{\rho}, \tilde{A}, \tilde{E}^{(1)}, \tilde{W} \in C^\infty(\overline{\Omega}) \) with \( \tilde{A}_n = 0 \) on \( \partial \Omega \) such that

\[
\| \rho - \tilde{\rho} \|_{L^2} < \frac{\varepsilon}{5}, \quad \| A - \tilde{A} \|_A < \frac{\varepsilon}{5}, \quad \| E - \tilde{E}^{(1)} \|_E < \frac{\varepsilon}{5}, \quad \| W - \tilde{W} \|_W < \frac{\varepsilon}{5}.
\]

Using Lemma 3 we can find \( \tilde{E}^{(2)} \in C^\infty(\overline{\Omega}) \) such that \( \| \tilde{E}^{(2)} \|_E < \varepsilon/5 \) and such that on \( \partial \Omega \),

\[
H_i^j \tilde{E}_j^{(2)} = -H_i^j \tilde{E}_j^{(1)} - a_i^j n_k (\tilde{W}_{kj} - \tilde{W}_{jk}),
\]
\[
\nabla^k \tilde{E}_k^{(2)} = -\nabla^k \tilde{E}_k^{(1)} - b_0 \frac{r}{\beta} n^i (\nabla_i \tilde{W} - \nabla^k \tilde{W}_{ik}) + \tilde{\rho}.
\]

Therefore, \( (\tilde{\rho}, \tilde{A}, \tilde{E}^{(1)} + \tilde{E}^{(2)}, \tilde{W}) \in D(A) \) and

\[
\| \rho - \tilde{\rho} \|_{L^2} + \| A - \tilde{A} \|_A + \| E - \tilde{E}^{(1)} - \tilde{E}^{(2)} \|_E + \| W - \tilde{W} \|_W < \varepsilon.
\]

This proves Proposition 1.

**B. Proof of Proposition 2**

Proposition 2 follows almost directly from the estimates in the previous section, so we only give the main steps here. Let \( u \in D(A) \). Using Gauss’ theorem and the estimate 17, we have, setting \( \phi = -\Delta^{-1} \nabla^k E_k \),

\[
\langle u, Au \rangle_H = \int_{\Omega} \left\{ \frac{1}{2\tau^2} A^i (E_j + 2 \nabla_j \phi) + \frac{1}{2\tau^2} A^i E_j + \alpha E^j C_j - \frac{\alpha}{2} WC - \frac{\alpha}{6} h^{ij} C^{(W)}_{ij} C \right\} d^3 x
\]
\[
+ \int_{\partial \Omega} \left\{ 2n^i E^j W_{ij} - \alpha \tau^2 n^i C C_j \right\} dS
\leq \frac{1}{2\tau} \int_{\Omega} \left\{ \frac{1}{\tau^2} A^i A_j + E^j E_j + r \left[ E^j E_j + \tau^2 \frac{\alpha}{\beta} C^j C_j + \frac{\alpha}{3\beta} W^2 + \frac{3\tau^2}{4} C^2 + \frac{\alpha}{3\beta} C^{(W)}_{ij} C^{(W)}_{ij} + \frac{\tau^2}{4} C^2 \right] \right\} d^3 x
\leq \frac{1}{2\tau} (1 + r) \langle u, u \rangle_H ,
\]

where \( r = \sqrt{\alpha \beta} \), and Proposition 2 follows with \( b = (1 + r)/2 \).
C. Proof of Proposition 3

Let \( v = (\sigma, B, F, V) \in C^\infty(\bar{\Omega}, \mathbb{R}^{16}) \) with \( n^iB_i = 0 \) on \( \partial\Omega \), and let \( \lambda > 0 \). We show that there exists \( u = (\rho, A, E, W) \in D(A) \) such that \((\lambda - A)u = v\); that is, such that

\[
\begin{align*}
\lambda\rho &= \sigma, \\
\lambda A_j &= E_j - \nabla_j \Delta^{-1} \nabla^k E_k + B_j, \\
\lambda E_j &= \nabla^i (W_{ij} - W_{ji}) + \alpha (\nabla_j W - \nabla^i W_{ji}) + F_j, \\
\lambda W_{ij} &= \nabla_i E_j + \frac{\beta}{2} h_{ij}(\nabla^k E_k - \rho) - \nabla_i \nabla_j \Delta^{-1} \nabla^k E_k + V_{ij}.
\end{align*}
\]

A consequence of this is the elliptic equation for \( E \)

\[
\lambda^2 E_j = \nabla^i \nabla_i E_j + (\alpha \beta - 1) \nabla_j \nabla^i E_i + S_j,
\]

where \( S_j = \nabla^i (V_{ij} - V_{ji}) + \alpha (\nabla_j V - \nabla^i V_{ji}) + \lambda F_j - \alpha \beta \lambda^{-1} \nabla_j \sigma \), with the boundary conditions on \( \partial\Omega \)

\[
\begin{align*}
\lambda H_i &= a_i^j E_j + a_i^k \nabla_k E_j - \nabla_j E_k = -a_i^j n^k (V_{kj} - V_{jk}), \\
\lambda \nabla^k E_k + b_0 r n^i \nabla_i \nabla^k E_k &= -b_0 \frac{r}{\beta} n^i (\nabla_i V - \nabla^j V_{ij}) - b_0 r \lambda^{-1} n^i \nabla_i \sigma + \sigma.
\end{align*}
\]

In appendix A it is proven in the case \( a_i^j = a_0 H_i^j \), where \( a_0 \in C^\infty(\partial\Omega) \) is a strictly positive function, that for sufficiently large \( \lambda > 0 \) there exists a unique solution \( E \in C^\infty(\bar{\Omega}) \) to this problem. Setting \( \rho = \lambda^{-1} \sigma \) and determining \( A_j \) and \( W_{ij} \) from Eq. (81) and Eq. (83), respectively, yields a solution \( u \in D(A) \) to \((\lambda - A)u = v\). Since \( \{(\sigma, B, F, V) \in C^\infty(\bar{\Omega}, \mathbb{R}^{16}) : n^iB_i = 0 \text{ on } \partial\Omega\} \) is dense in \( H \), proposition 3 follows.

D. Intertwining operators and the constraint hypersurface

Here we prove that the semigroup \( P(t) \) constructed in theorem 3 leaves the constraint manifold, defined by

\[
C = \{ u = (\rho, A_i, E_i, W_{ij}) \in H : W_{ij} = \nabla_i A_j, \nabla^i E_i = \rho \}
\]

invariant. In order to do so we introduce the Hilbert space \( G = \{ v = (C_{ij}^{(W)}, C, C_i) \in L^2(\Omega, \mathbb{R}^{13}) \} \) and the linear operator \( B : D(B) \subset G \rightarrow G \) which is defined by

\[
D(B) = \{ (C_{ij}^{(W)}, C, C_i) \in C^\infty(\bar{\Omega}, \mathbb{R}^{13}) : \text{On } \partial\Omega \text{ we have } C - b_0 \frac{r}{\beta} n^i C_i = 0 \},
\]

and

\[
B \begin{pmatrix} C_{ij}^{(W)} \\ C_j \\ C_k \end{pmatrix} = \begin{pmatrix} \frac{\alpha}{2} h_{ij} C \\ -\alpha \nabla^k C_k \\ -\beta \nabla C \end{pmatrix}.
\]

The operator \( B \) describes the evolution of the constraint variables, cf. Eqs. (78, 8). Similarly to before, one shows that \( D(B) \) is dense in \( G \), that \( B \) is closable and that its closure generates a strongly continuous semigroup \( Q(t) \) in \( G \). The Hilbert spaces \( H \) and \( G \) are related by the linear operator \( L : H \rightarrow G \) which is defined by

\[
L \begin{pmatrix} \rho \\ A_j \\ E_i \\ W_{ij} \end{pmatrix} = \begin{pmatrix} W_{ij} - \nabla_i A_j \\ \rho - \nabla^i E_i \\ \nabla_i W - \nabla^j W_{ij} \end{pmatrix}.
\]

Since \( L \) is continuous, the constraint hypersurface \( C = \{ u \in H : L(u) = 0 \} \) is a closed subset of \( H \). Furthermore, we have \( LD(A) \subset D(B) \), and we verify the intertwining relation

\[
LAu = BLu, \quad u \in D(A).
\]
Since $L$ is continuous and $\mathcal{A}$ and $\mathcal{B}$ are closable, it also follows that $LD(\tilde{\mathcal{A}}) \subset D(\tilde{\mathcal{B}})$ and that
\[ L\tilde{\mathcal{A}}u = \tilde{\mathcal{B}}Lu, \quad u \in D(\tilde{\mathcal{A}}). \tag{89} \]
This implies that for each $u \in D(\tilde{\mathcal{A}})$, the curve $v : \mathbb{R}_+ \to D(\tilde{\mathcal{B}})$, $t \mapsto LP(t)u$ is differentiable and satisfies
\[ \frac{d}{dt}v(t) = L\tilde{\mathcal{A}}P(t)u = \tilde{\mathcal{B}}LP(t)u = \tilde{\mathcal{B}}v(t), \quad v(0) = Lu. \]
Since $\tilde{\mathcal{B}}$ is the generator of the semigroup $Q(t)$, it follows that $LP(t)u = v(t) = Q(t)Lu$, $t > 0$. Therefore,
\[ LP(t) = Q(t)L, \quad t \geq 0, \tag{90} \]
since $D(\tilde{\mathcal{A}})$ is dense in $H$. Eq. \((90)\) implies that $P(t)$ leaves the constraint manifold $\mathcal{C}$ invariant.

E. Proof of Theorem 2

Finally, we discuss the case of nontrivial source functions and boundary data. We first reduce the problem to the case of homogeneous boundary conditions with nontrivial source functions and then use Duhamel’s principle to construct the solution of the resulting problem.

Let $J(t) \in C^\infty(\Omega, \mathbb{R}^3) \subset H_E$, $t > 0$, be a continuous curve in $H_E$, let $u_0 = (\rho^{(0)}, A^{(0)}, E^{(0)}, W^{(0)}) \in C^\infty(\Omega, \mathbb{R}^{16}) \cap H$ be smooth initial data, and let $g_t(t) \in C^\infty(\partial\Omega)$, $t > 0$, be a continuously differentiable curve in $L^2(\partial\Omega)$ with the property that $g_t(t)n^i = 0$ on $\partial\Omega$ for all $t > 0$. Assume that the initial and boundary data satisfy the zeroth order compatibility condition
\[ H_i \bar{J}_j E_{j}^{(0)} + a_i^j n^k(W_{k}^{(0)} - W_{jk}^{(0)}) = g_j, \tag{91} \]
\[ \nabla^k E_k^{(0)} - \rho^{(0)} + b_0 r_{ij} n^i(\nabla_i W^{(0)} - \nabla^j W_{ij}^{(0)}) = 0, \tag{92} \]
on $\{t = 0\} \times \partial\Omega$. Choose $F(t) \in C^\infty(\Omega, \mathbb{R}^3)$ such that $H_i \bar{J}_j F_{j}(t) = g_j(t)$ on $\partial\Omega$ and $\nabla^k F_k(t) = 0$ on $\Omega$ (see Lemma 5) for all $t > 0$. We can choose $F(t)$ such that it describes a continuously differentiable curve in $H^1(\Omega, \mathbb{R}^3)$. Define $v(t) = (0, 0, F_i(t), 0) \in H$, $S(t) = (\nabla^i J_i(t), 0, J_i(t), 0) \in H$ and $F(t) = \mathcal{A}v(t) - \dot{v}(t) + S(t)$, where $\mathcal{A}v(t)$ is defined by Eq. \((77)\). One can verify that $v(t), S(t), F(t), t > 0$ define continuous curves in the Hilbert space $H$. Next, consider the inhomogeneous Cauchy problem
\[ \frac{d}{dt}w = \mathcal{A}w + F, \tag{93} \]
\[ w(0) = w_0 \equiv u_0 - v(0). \tag{94} \]
The compatibility conditions \((91),(92)\) make sure that $w_0 \in D(\mathcal{A})$. We can solve this problem in an abstract way using Duhamel’s principle:
\[ w(t) = P(t)w_0 + \int_0^t P(t - s)F(s)ds, \tag{95} \]
where $P(t)$ is the semigroup constructed in theorem 3 and where the above integral is a $H$-valued Riemann integral. By construction, $u(t) = v(t) + w(t)$ satisfies the inhomogeneous initial-boundary value problem \((1)\), \((17)\), \((14)\) with $u(0) = u_0$:
\[ \frac{d}{dt}u = \mathcal{A}v + S + \mathcal{A}w + F = \mathcal{A}u + S, \tag{96} \]
and $u(t)$ satisfies the inhomogeneous boundary conditions. Furthermore, using the fact that $L v(t) = 0$ and $L S(t) = 0$ for all $t \geq 0$, and using the intertwining relation \((41)\), we find $Lu(t) = Q(t)Lu_0$, which shows that the constraints are propagated. This concludes the proof of Theorem 2.
VI. CONCLUSIONS

In this article we considered the initial-boundary value problem for a formulation of Maxwell’s equations which is structurally similar to the Einstein-Christoffel family of formulations of Einstein’s field equations. In particular, our problem includes constraints that propagate with nontrivial speeds at timelike boundaries and so need to be controlled by specifying constraint-preserving boundary conditions. These conditions are supplemented with boundary conditions that control the incoming electromagnetic radiation. We have first considered the case of a fixed, a priori specified electromagnetic potential \( \phi \) and shown that the resulting initial-boundary value problem is not well posed in the expected sense, even though the Kreiss condition is verified. This is shown by an explicit counterexample representing an electrostatic solution in a “bad” gauge. This gauge is “bad” in the sense that it gives rise to modes growing linearly in \( \omega t \), where \( \omega \) are the frequency components of the initial data. The counterexample also motivates how to deal with these modes by choosing a different gauge which eliminates them. In this new gauge, we are able to derive well posed initial-boundary value formulations; that is, we show that a solution exists in some appropriate Hilbert space, is unique, and depends continuously on the initial and boundary data.

One key idea in our proof of well posedness is to start with an estimate for the physical energy instead of estimating the norm defined by the symmetrizer of the main evolution system, and with an estimate that controls the constraint variables. This gives rise to estimates for the gauge-invariant quantities. Once these quantities are under control, one estimates the gauge-dependent quantities (in our case the components of the vector potential and their first order spatial derivatives). These a priori estimates then motivate the Hilbert space in which the solutions are shown to exist using methods from semigroup theory.

Unfortunately, the new gauge choice is of elliptic type, implying one has to solve a scalar elliptic equation at each time step of a numerical simulation. Although one can start an iterative method for solving it with the prior time value as seed, the procedure has some computational costs. Therefore, an interesting question is whether there exists other gauge choices leading to a well posed initial-boundary value problem which are easier to implement numerically.

The relevant two places where the gauge choice is important is in estimating the \( L^2 \) norm of the vector potential \( A_i \) and of its derivatives \( \nabla_i A_j \). For the first case, since a \( L^2 \) estimate for \( E_i \) is given from the estimates for the gauge-invariant quantities, all that is needed is that the gradient of \( \phi \) be bounded in \( L^2 \) too. For the second case, the vanishing of the normal component, \( A_n \), of \( A_i \) at the boundary is needed. In particular, this is true if \( A_n \big|_{\partial \Omega} = 0 \) initially and \( n^k (\nabla_k \phi + E_k) \big|_{\partial \Omega} = 0 \) for all times. So any gauge fixing satisfying these conditions should lead to a well posed system.

How much of the above discussion can be carried over to General Relativity? As an example, consider the Einstein-Christoffel formulations with an a priori given shift vector that is tangential to the boundary. For this system, one has to specify six boundary conditions. There are three conditions that are needed in order to preserve the constraints \( \mathcal{C} \). Then, there are two conditions that should control incoming gravitational radiation. At least in the weak field approximation, where linearizations about Minkowski space are considered, it should be possible to specify such conditions in terms of the Weyl tensor, since – in the weak field approximation – this tensor is gauge invariant. Finally, there is a remaining boundary condition that has to be used in order to control a gauge freedom. We expect that for the a priori given shift case a similar counterexample as the one we discussed here can be found and that the resulting initial-boundary value formulation will not be well posed in the expected sense. On the other hand, we also expect that this problem can be avoided by requiring an appropriate elliptic equation for the shift and that well posedness can be derived for the resulting problem, at least for the case of linearizations about a stationary background.

Finally, we briefly discuss a different toy model describing Maxwell’s equations which resembles more closely the BSSN-type of formulations of General Relativity (see [20, 21] for the original references and [10, 22, 23, 24] for an analysis of the mathematical structure of the equations). Instead of the variables \( W_{ij} \) we introduce an extra variable \( \Gamma \) along with the constraint \( C^\Gamma \equiv \Gamma - \nabla^k A_k = 0 \), and consider the mixed first order-second order system

\[
\begin{align*}
\partial_t A_i &= E_i + \nabla_i \phi, \\
\partial_t E_j &= \nabla_i A_j + \alpha \nabla_j \Gamma - (1 + \alpha) \nabla_j \nabla^i A_i, \\
\partial_t \Gamma &= (1 + \beta) \nabla^k E_k + \nabla^k \nabla_k \phi,
\end{align*}
\]

where \( \alpha, \beta \) are parameters subject to the condition \( \alpha \cdot \beta > 0 \). This model problem has been discussed by several authors [8, 22, 26, 27] in the past. The constraint variables \( C = -\nabla^k E_k, C^\Gamma \) propagate according to

\[
\begin{align*}
\partial_t C &= -\alpha \nabla^i \nabla_i C^\Gamma, \\
\partial_t C^\Gamma &= -\beta C.
\end{align*}
\]
In analogy to our previous model example, we consider the boundary conditions

\[ V^{(+)} = c \, V^{(-)} , \quad V^{(\pm)} \equiv C \mp \frac{\sqrt{\alpha \beta}}{\beta} n^k \nabla_k C' , \quad (102) \]

\[ w_i^{(+)} = S_i^j \, w_j^{(-)} , \quad w_i^{(\pm)} \equiv H_i^j E_j \pm n^k (\nabla_k A_i - \nabla_i A_k) , \quad (103) \]

where \( |c| \leq 1 \), and \( S_i^j \) is a smooth matrix-valued function on \( \partial \Omega \) with the properties that \( S_i^j n_i = 0 \), \( S_i^j n_j = 0 \) and \( H^{ij} S_i^r S_j^s \leq H^{rs} \). There is an analogous example to the one presented in section \[ IV \] which shows that it is not possible to prove well posedness in a space that controls the \( L^2 \) norm of the fields \( A_i \), \( E_i \), \( \Gamma \) and the first spatial derivatives of \( A_i \) when the gauge \( \phi = 0 \) is adopted. However, it is not difficult to see that one can adapt the estimates of section \[ IV \] when the elliptic gauge condition \[ III \] is imposed instead, and so one can proceed as in this article to show well posedness.

VII. ACKNOWLEDGEMENTS

It is a pleasure to thank G. Calabrese, L. Lehner, L. Lindblom, G. Nagy, M. Scheel, M. Tiglio and M. Tom for useful comments and discussions. OS particularly thanks H. Beyer for many illuminating discussions on the use of semigroup methods and for communicating his manuscript \[ 28 \]. We also thank the Caltech Visitors Program for the Numerical Simulation of Gravitational Wave Sources, the Klavi Institute for Theoretical Physics Visitor Program: Gravitational Interaction of Compact Objects, and the Isaac Newton Institute for Mathematical Sciences, Cambridge Visitor Program: Hyperbolic Models in Astrophysics and Cosmology for financial support. This work was supported in part by the Center for Computation \\& Technology at Louisiana State University, by grants nsf-phy0244355, nsf-phy034699, nsf-phy0312049, NSF-PHY-0204937, NSF-PHY-0307290, NASA-NAG5-13430, by funds from the Horace Hearne Jr. Laboratory for Theoretical Physics, and by CONICET; SECYT-UNC; and Agencia Córdoba Ciencia.

APPENDIX A: ELLIPTIC PROBLEM

Here we show existence of smooth solutions to the elliptic boundary value problem which arises in the proof of Proposition \[ 3 \]. We first simplify the problem and then apply standard methods to analyze the resulting boundary value problem.

Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^3 \) with \( C^\infty \) boundary \( \partial \Omega \). In the following, \( n = n_i dx^i \) denotes the outward unit co-normal to \( \partial \Omega \), and \( \nabla \) denotes the covariant derivative with respect to the Eulerian metric \( ds^2 = \delta_{ij} dx^i dx^j \) on \( \mathbb{R}^3 \). We also introduce the projection operator \( H_i^j = \delta_i^j - n_i n^j \) on the tangent space of \( \partial \Omega \), and the extrinsic curvature of the boundary surface, \( \kappa_{ij} = H_i^j H_j^k \nabla_k n_s \).

Let \( \lambda \) and \( r \) be strictly positive constants, and let \( c_0 \) and \( d \) be a positive \( C^\infty \) function and a matrix-valued \( C^\infty \) function, respectively, defined on the boundary of \( \Omega \), where \( d = (d_k^j) \) satisfies \( n^j d_j^k = 0 \), \( d_j^k n_k = 0 \). We are interested in the following boundary value problem:

\[ \lambda^2 E_j - \nabla^i \nabla_i E_j - (r^2 - 1) \nabla_j \nabla^i E_i = S_j \quad \text{on} \ \Omega , \quad (A1) \]

\[ n^j \nabla_j + \lambda c_0 \nabla^i E_i = G \quad \text{on} \ \partial \Omega , \quad (A2) \]

\[ n^j (\nabla_i E_j - \nabla_j E_i) + \lambda d_j^k E_k = g_k \quad \text{on} \ \partial \Omega , \quad (A3) \]

where \( S \in C^\infty (\bar{\Omega}, \mathbb{R}^3) \) and \( G \in C^\infty (\partial \Omega, \mathbb{R}) \), \( g \in C^\infty (\partial \Omega, \mathbb{R}^3) \) are given functions with the property that \( n^j g_j = 0 \). Notice that this has the form of the problem arising in Proposition \[ 3 \] with \( r = \sqrt{\alpha \beta} \),

\[ c_0 = \frac{1}{r \lambda} = \frac{1 - c}{r^2 + c^2} \]

\[ d = \left( \begin{array}{ccc} 1 - c & 0 & 0 \\ 0 & 1 - c \\ 0 & 0 & 1 - c \end{array} \right) \]

\[ d = a^{-1} - H(H + S)^{-1}(H - S) . \]

In this appendix, we prove

**Theorem 4** If \( c_0 \geq 0 \) and \( d = d_0 H \) where \( d_0 \) is a strictly positive function on \( \partial \Omega \), and if \( \lambda > 0 \) is sufficiently large, the boundary value problem \[ A1, A2, A3 \] possesses a unique solution \( E \in C^\infty (\Omega, \mathbb{R}^3) \).
We first simplify the problem and show that it is, basically, sufficient to consider the case where \( r = 1 \) and \( c_0 \) is replaced by an arbitrary positive \( C^\infty \) function \( \epsilon_0 \) on \( \partial \Omega \). This freedom in the choice of \( \epsilon_0 \) will be important in view of constructing coercive quadratic forms.

Let \( S \in C^\infty(\Omega, \mathbb{R}^3) \) and \( G \in C^\infty(\partial \Omega, \mathbb{R}) \), \( g \in C^\infty(\partial \Omega, \mathbb{R}^3) \) be given. A solution to the problem \( A1 \), \( A2 \), \( A3 \) can be obtained by performing the following steps:

1. Decompose \( S = S^T + \nabla J \) where \( S^T \in C^\infty(\Omega, \mathbb{R}^3) \), \( J \in C^\infty(\Omega, \mathbb{R}) \) satisfy \( \text{div} S^T = \nabla^i S^T_i = 0 \) and \( J\mid_{\partial \Omega} = 0 \) (this requires solving the Dirichlet problem \( \Delta J = \text{div} S \), \( J\mid_{\partial \Omega} = 0 \)).

2. Define \( \tilde{G} = \lambda^{-2} (r^2 G + n^i \nabla_i J) \big|_{\partial \Omega} \) and solve the problem

\[
(\lambda^2 - r^2 \nabla^i \nabla_i) \phi = J \quad \text{on } \Omega, \quad (A4)
\]

\[
(n^i \nabla_i + \lambda \epsilon_0) \phi = \tilde{G} \quad \text{on } \partial \Omega, \quad (A5)
\]

Using similar techniques as the one used below, it is not difficult to show that this problem has a unique solution \( \phi \in C^\infty(\Omega, \mathbb{R}) \) if \( r > 0 \), \( \epsilon_0 \geq 0 \).

3. Solve the boundary value problem

\[
\begin{align*}
\lambda^2 F_j - \nabla^i \nabla_i F_j &= S^T_j \quad \text{on } \Omega, \quad (A6) \\
(n^i \nabla_i + \lambda \epsilon_0) F_i &= 0 \quad \text{on } \partial \Omega, \quad (A7) \\
n^i (\nabla_i F_j - \nabla_j F_i) + \lambda d_j^k F_k &= g_j - \lambda d_j^k \nabla_k \phi \quad \text{on } \partial \Omega, \quad (A8)
\end{align*}
\]

where \( \epsilon_0 \) is an arbitrary positive and smooth function on \( \partial \Omega \). Notice that a solution \( F \in C^\infty(\Omega, \mathbb{R}^3) \) satisfies \( (\lambda^2 - r^2 \nabla^i \nabla_i) \text{div} F = 0 \), \( (n^i \nabla_i + \lambda \epsilon_0) \text{div} F = 0 \big|_{\partial \Omega} \), so \( \text{div} F = 0 \) according to the previous step.

4. Then, it is easy to verify that \( E = F + \nabla \phi \) solves the boundary-value problem \( A1 \), \( A2 \), \( A3 \). This solution is unique, because given two solutions their difference \( E^\Delta \) satisfies \( (\lambda^2 - r^2 \nabla^i \nabla_i) \text{div} E^\Delta = 0 \), \( (n^i \nabla_i + \lambda \epsilon_0) \text{div} E^\Delta = 0 \big|_{\partial \Omega} \), so \( \text{div} E^\Delta = 0 \) by the second step, and so \( E^\Delta \) satisfies the homogeneous version of the problem in the third step for which uniqueness will be shown.

Therefore, it is sufficient to prove the statement of theorem 4 for the case where \( r = 1 \) and where \( \epsilon_0 \) is one particular positive and smooth function. Before we attack the problem, it turns out to be convenient to rewrite the boundary condition \( A2 \) in a different form. Using Eq. \( A1 \) with \( r = 1 \), we rewrite

\[
n^j \nabla_j \nabla_i E_i = n^j (n^i n^k + H^{ik}) \nabla_j \nabla_i E_k
\]

\[
= (\lambda^2 E_n - S_n) + H^{ik} \nabla_i (n^i \nabla_j E_k - n^j \nabla_k E_i) + \kappa^{ik} (\nabla_i E_k - \nabla_k E_i)
\]

\[
= (\lambda^2 E_n - S_n) + H^{ik} \nabla_i (g_k - \lambda d_k^j E^tg_j),
\]

where \( E_n = E_i n^i \), \( E^tg_i = H^{tg}_i E_k \), and where we have used the fact that \( \kappa_{ij} \) is symmetric. Combining this with \( \nabla^i E_j = n^i (n^j \nabla_j + \kappa) E_i + H^{ik} \nabla_j E^tg_k \) the boundary condition \( A2 \) can be rewritten in the form

\[
n^i (n^i \nabla_j + \kappa + \lambda \epsilon_0^{-1} E_i + (H^{ik} - \epsilon_0^{-1} d^{ik}) D_i E^tg_k - \epsilon_0^{-1} (D_i d^{ik}) E^tg_k) = G_0,
\]

where \( G_0 = (\lambda \epsilon_0^{-1})(G + S_n - D^j g_j) \) and where \( D \) denotes the covariant derivative with respect to the induced metric \( H_{ij} \) on \( \partial \Omega \). Finally, we combine this with the boundary condition \( A3 \) and obtain the following boundary value problem:

\[
\begin{align*}
\lambda^2 E_j - \nabla^i \nabla_i E_j &= S_j \quad \text{on } \Omega, \quad (A9) \\
n^j \nabla_i E_j - D_j E_n + n_j (H^{ik} - \epsilon_0^{-1} d^{ik}) D_i E^tg_k + \mu^{jk} E^tg_k + \eta n_j E_n - \epsilon_0^{-1} n_j (D_i d^{ik}) E^tg_k &= n_j G_0 + g_j \quad \text{on } \partial \Omega, \quad (A10)
\end{align*}
\]

where \( \mu_{jk} = \lambda d_{jk} + \kappa_{jk} \) and \( \eta = \lambda \epsilon_0^{-1} + \kappa \).

From now on, we specialize to the case where \( d_{kj} = d_0 H_{kj} \) where \( d_0 \) is a strictly positive and smooth function on \( \partial \Omega \). Although we believe that this restriction is not necessary, it simplifies the proof below. Choose \( \epsilon_0 = d_0 / 2 \) in which case the boundary condition \( A10 \) reduces to

\[
n^j \nabla_i E_j - D_j E_n - n_j D^s E^s_i + \mu^{jk} E^tg_k + \eta n_j E_n - 2 n_j d_0^{-1} (D^j d_0) E^tg_k = n_j G_0 + g_j \quad \text{on } \partial \Omega, \quad (A11)
\]

where \( \mu_{jk} = \lambda d_0 H_{jk} + \kappa_{jk} \) and \( \eta = 2 \lambda d_0^{-1} + \kappa \). Notice that the third term in the boundary condition \( A11 \) is minus the formal adjoint with respect to \( L^2(\partial \Omega) \) of the second term in \( A11 \). This and the positivity of \( \mu_{ij} \) and \( \eta \) will allow us to find the appropriate elliptic estimates and to prove
Theorem 5 Let $d_0 \in C^\infty(\partial \Omega)$ be strictly positive at each point of $\partial \Omega$. Denote by $\kappa_1 \leq \kappa_2$ the nonzero eigenvalues of $\kappa_{ij}$ at each point of $\partial \Omega$, and set $N = \sqrt{H^3 D_1 \log d_0 - D_j \log d_0}$. Then, if $\lambda$ is sufficiently large such that
\[ \lambda \geq \max \{0, \frac{1}{2} (N - (\kappa_1 + \kappa_2)d_0), N - \frac{\kappa_1}{d_0} \} \] (A12)
the boundary value problem [A12, A14] possesses a unique solution $E \in C^\infty(\bar{\Omega}, \mathbb{R}^3)$.

The proof uses standard methods [29, 30, 31]: We first recast the boundary value problem in weak form and show the existence and uniqueness of weak solutions. After that, we derive regularity results which show that this solution is, in fact, smooth.

1. Weak formulation

In the following, let $H^k(\Omega, \mathbb{R}^m), k \geq 0$, denote the Sobolev space of functions $\Omega \to \mathbb{R}^m$ whose partial derivatives of order $\leq k$ are in $L^2(\Omega)$. We abbreviate $W^k = H^k(\Omega, \mathbb{R}^3)$. Introduce the linear operators $L : W^{k+2} \to W^k$, $b : W^{k+2} \to H^{k+1/2}(\partial \Omega, \mathbb{R}^3)$ defined by
\[
L(E)_j = (\lambda^2 - \nabla^i \nabla_i) E_j, \\
b(E)_j = \eta E_j,
\]
where $\lambda$ is a bounded bilinear form on the Hilbert space $L^3(\Omega, \mathbb{R}^3)$.

We consider the boundary value problem $L(E)_j = S_j$, $b(E)_j = g_j$, where $S \in W^k$ and $g \in H^{k+1/2}(\partial \Omega, \mathbb{R}^3)$. Let $E, F \in C^\infty(\bar{\Omega}, \mathbb{R}^3)$. Using Gauss’ theorem we find
\[
\int_\Omega F^j L(E)_j d^3x = \int_\Omega (\lambda^2 E_j + \nabla^i F^j \cdot \nabla_i E_j) d^3x - \int_{\partial \Omega} F^j b(E)_j ds
\]
where $ds$ denotes the surface element on $\partial \Omega$ and where we have defined the bilinear form
\[
Q_\lambda(E, F) = \int_\Omega (\lambda^2 F^j E_j + \nabla^i F^j \cdot \nabla_i E_j) d^3x + \int_{\partial \Omega} [\mu E_i F_j + \eta E_n F_n - 2F_n(D^i \log d_0)E_i^{\delta g} - F_n D^i E_i^{\delta g} - F_i^{\delta g} D^i E_n] ds.
\]
Therefore,
\[
(F, L(E))_{L^2(\Omega)} + (F, b(E))_{L^2(\partial \Omega)} = Q_\lambda(E, F).
\] (A13)

Because of the trace theorems we can extend $Q_\lambda$ to $W^1 \times W^1$: If $E, F \in W^1$, they are also in $H^{1/2}(\partial \Omega)$, and (in the sense of distributions) $D_i E_n \in H^{-1/2}(\partial \Omega)$, which is the dual space of $H^{1/2}(\partial \Omega)$, so the integral $\int_{\partial \Omega} F_i^{\delta g} D^i E_n dS$ makes sense. Furthermore, there is a constant $C$ such that
\[
|Q_\lambda(E, F)| \leq C \|E\|_{W^1} \cdot \|F\|_{W^1} \text{ for all } E, F \in W^1.
\] (A14)

Thus $Q_\lambda$ is a bounded bilinear form on the Hilbert space $W^1$. Equation [A13], which holds for all $E \in W^2$, $F \in W^1$, implies

Lemma 6 Let $E \in W^2$, and suppose that $S \in W^0$, $g \in H^{1/2}(\partial \Omega, \mathbb{R}^3)$ are given. Then, $E$ is a (strong) solution of [A10, A11] if and only if
\[
(F, S)_{L^2(\Omega)} + (F, g)_{L^2(\partial \Omega)} = Q_\lambda(E, F).
\] (A15)

for all $F \in W^1$.

Proof: The “only if” part is clear. In order to show the “if” part assume that [A15] holds for all $F \in W^1$. In particular, it holds for all $F \in C_0^\infty(\Omega, \mathbb{R}^3)$. Since $C_0^\infty(\Omega, \mathbb{R}^3)$ is dense in $L^2(\Omega, \mathbb{R}^3)$ it follows, in view of Eq. [A13], that $L(E) = S$. Next, choose $F \in C^\infty(\bar{\Omega}, \mathbb{R}^3)$ with $F_j \neq 0$ at the boundary, which implies $b(E)_j = g_j$.

This motivates the following
Lemma 8 Suppose $\lambda > 0$, and let $S \in W^0$, $g \in H^{1/2}(\partial \Omega, \mathbb{R}^3)$ be given. We call $E \in W^1$ a weak solution of (A19), (A11) if equation (A11) holds for all $F \in W^1$.

The next lemma implies the existence of weak solutions:

Lemma 7 Suppose the assumptions of Theorem 4 are met. Then, $Q_\lambda$ is coercive. That is, there exists a constant $a_0 = a_0(\lambda) > 0$ such that

$$Q_\lambda(E, E) \geq a_0 \|E\|^2_{W^1}$$

for all $E \in W^1$.

**Proof:** Let $E \in C^\infty(\bar{\Omega}, \mathbb{R}^3)$. We have

$$Q_\lambda(E, E) = \int_\Omega (\lambda^2 E^j E_j + \nabla^i E^j \cdot \nabla_i E_j) \, d^3 x$$

$$+ \int_{\partial \Omega} [\mu^{ij} E_i E_j + \eta E^2_n - 2E_n (D^i \log d_0) E^i g - E_n D^i E^i g - E^i g D^i E_n] \, dS.$$  

The last two terms in the boundary integral form a total divergence, and so their contribution to the integral vanish. Therefore, setting $a_0 = \min \{ \lambda^2, 1 \}$, $N = \sqrt{H^{ij} D_i \log d_0 - D_j \log d_0}$ and using Schwarz’ inequality,

$$Q_\lambda(E, E) \geq a_0 \|E\|^2_{W^1} + \int_{\partial \Omega} [\mu^{ij} E_i E_j + \eta E^2_n - N \{ d_0 H^{ij} E_i E_j + d_0^{-1} E^2_n \}] \, dS$$

$$= a_0 \|E\|^2_{W^1} + \int_{\partial \Omega} [(\lambda - N) d_0 H^{ij} + \kappa^{ij} \} E_i E_j + [(2\lambda - N) d_0^{-1} + \kappa] E^2_n \] \, dS.$$  

$$\geq a_0 \|E\|^2_{W^1}.$$  

Since $C^\infty(\bar{\Omega}, \mathbb{R}^3)$ is dense in $W^1$ and since $Q_\lambda$ is bounded, the lemma follows. \qed

Corollary 1 (Existence and Uniqueness of weak solutions) Suppose the assumptions of Theorem 4 are met. Then, the problem (A19), (A11) possesses a unique weak solution.

**Proof:** Define the linear functional $J \in W^{1*}$ by

$$J(F) = (F, S)_{L^2(\Omega)} + (F, g)_{L^2(\partial \Omega)}, \quad F \in W^1.$$

According to the Lax-Milgram lemma there is a unique $E$ such that $Q_\lambda(E, .) = J$. \qed

For later use, we introduce the operator $L_\lambda : W^1 \to W^{1*}$ which is defined by

$$(L_\lambda E)(F) = Q_\lambda(E, F)$$

for $E, F \in W^1$. Since $|Q_\lambda(E, F)| \leq C \|E\|_{W^1} \cdot \|F\|_{W^1}$ it follows that $\|L_\lambda E\|_{W^{1*}} \leq C \|E\|_{W^1}$, so $L_\lambda$ is bounded. Furthermore, it is injective because of $(L_\lambda E)(F) = Q_\lambda(E, E) \geq a_0 \|E\|^2_{W^1}$ and onto because of the implications of the Lax-Milgram lemma. The weak form of (A19), (A11) is simply $L_\lambda E = J$.

The following estimate will be important:

Lemma 8 $L_\lambda : W^1 \to W^{1*}$ defines a linear bounded and bijective operator. For each $E \in W^1$ it satisfies

$$\|E\|_{W^1} \leq a_0^{-1} \|L_\lambda E\|_{W^{1*}}.$$  

\[(A17)\]

**Proof:** According to Lemma 7 we have, for $E \in W^1$,

$$a_0 \|E\|^2_{W^1} \leq Q_\lambda(E, E) = (L_\lambda E)(E) \leq \|L_\lambda E\|_{W^{1*}} \cdot \|E\|_{W^1}.$$  

\[\]
Theorem 6 (Regularity) Suppose the assumptions of Theorem 3 are satisfied. Let \( k \geq 0 \) and suppose that \( S \in W^k, \ g \in H^{k+1/2}(\partial \Omega, \mathbb{R}^3) \). Then, the unique weak solution \( E \) of the boundary value problem \( (A9), (A11) \) lies in \( W^{k+2} \). Furthermore, there is a constant \( C_k > 0 \) such that

\[
\|E\|_{W^{k+2}} \leq C_k (\|S\|_{W^k} + \|g\|_{H^{k+1/2}(\partial \Omega)} + \|E\|_{W^{k+1}}).
\]  

(A18)

Remark: For \( k = 0 \) in particular, the theorem shows that any weak solution lies in \( W^2 \) and so is a strong solution according to Lemma 6.

Remark: Since the solution is unique, one can show (see, for example, Lemma 8.38 in Ref. [31]) that the last term in the right-hand side of the estimate (A18) can be dropped. The resulting estimate then implies well posedness in the sense that the solution depends continuously on the data.

We break the proof of the theorem into several steps. First, we choose a finite covering of \( \Omega \) and show that the problem can be localized. Applying difference quotients to the estimates (A17), (A18) we then show interior regularity first. Next, we flatten the boundary by choosing an appropriate coordinate patch and map the problem to a half plane problem. Using difference quotients again, we first show differentiability in the directions tangential to the boundary, and then use the solution properties to show differentiability in the normal directions as well. We start with the case \( k = 0 \).

3. Localization

Suppose \( E \in W^1 \) is a weak solution, and let \( \chi \) be the restriction on \( \Omega \) of a function in \( C_0^\infty(\mathbb{R}^3) \). Then \( \chi \cdot F \in W^1 \) for each \( F \in W^1 \) and \( J(\chi F) = (\mathcal{L}_\lambda E)(\chi F) = (\mathcal{L}_\lambda (\chi E))(F) + R(E, F) \), where, using the Leibnitz rule and Gauss’ theorem, one finds

\[
R(E, F) = Q_\lambda (E, \chi F) - Q_\lambda (\chi E, F) = \int_\Omega \left[ \nabla^i E^j \cdot \nabla_i \chi \cdot F_j + \nabla_i (\nabla^i \chi \cdot E^j) \cdot F_j \right] \, dx + \int_{\partial \Omega} \left[ D^s \chi (F_n E_s^g + E_n F_s^g) - \nabla_n \chi \cdot E^j F_j \right] \, dS.
\]

Therefore, \( \mathcal{L}_\lambda (\chi E) = \tilde{J} \) where

\[
\tilde{J}(F) = (F, \tilde{S})_{L^2(\Omega)} + (F, \tilde{g})_{L^2(\partial \Omega)}
\]

with

\[
\tilde{S}_j = \chi S_j - 2 \nabla_i \chi \cdot \nabla^i E_j - (\nabla^i \nabla_i \chi) E_j,
\]

(A19)

\[
\tilde{g}_j = \chi g_j + \left[ \nabla_n \chi \cdot E_j - D_j \chi \cdot E_n - n_j D^s \chi \cdot E_s^g \right]_{\partial \Omega},
\]

(A20)

Since \( E \in W^1 \), we have \( \tilde{S} \in W^0 \) and \( \tilde{g} \in H^{1/2}(\partial \Omega, \mathbb{R}^3) \), and \( \tilde{S}, \tilde{g} \) have the same support as \( \chi \). Therefore, it is sufficient to assume that \( E \) and the data \( F, g \) are supported in some coordinate patch.

Finally, we notice that taking \( F \in C_0^\infty(\Omega, \mathbb{R}^3) \) in equation (A15) implies that

\[
(\lambda^2 - \nabla^i \nabla_i) E_j = S_j
\]

(A21)

in the sense of distributions.

4. Interior regularity

Assume first that \( E \) and \( S \) are supported in \( \Omega \) (and so the boundary data vanishes). Then, we can extend them to elements in \( H^1(\mathbb{R}^3, \mathbb{R}^3) \), and \( H^0(\mathbb{R}^3, \mathbb{R}^3) \), respectively. By rewriting \( (\lambda^2 - \nabla^i \nabla_i) E_j = S_j \) in the form \( (1 - \nabla^i \nabla_i) E_j = S_j + (1 - \lambda^2) E_j \), it follows immediately from the next lemma that \( E \in H^2(\mathbb{R}^3, \mathbb{R}^3) \).
Lemma 9 For each \( s \in \mathbb{R} \), the operator \( 1 - \nabla^3 \nabla_k \) induces an isometry of \( H^s(\mathbb{R}^3) \) into \( H^{s-2}(\mathbb{R}^3) \).

Proof: This follows immediately using Fourier transformation.

In view of the generalization to operators with variable coefficient and in preparation of the boundary regularity result, we give a different proof which is based on the use of difference quotients. So let \( E \in H^1(\mathbb{R}^3, \mathbb{R}^3) \) be a weak solution with compact support. The idea is to apply the estimate (A17) to the difference quotient \( D_h E \), defined by
\[
D_h^h E(x) = \frac{1}{h} (E(x + he_i) - E(x)),
\]
where \( h > 0 \) and \( e_i \) denote the unit standard vectors in \( \mathbb{R}^3 \). The partial derivative \( D_i E \) exists in the weak sense if and only if \( D_i^h E \) is bounded in \( h \) [31]. According to Eq. (A17), we have
\[
\|D^h_i E\|_{H^{1}} \leq a_0^{-1}\|\lambda D^h_i E\|_{H^{1}}.
\]
Now for any \( F \in H^1(\mathbb{R}^3, \mathbb{R}^3) \),
\[
\langle L\lambda D_i^h E, F \rangle = Q_\lambda(D_i^h E, F) = Q_\lambda(E, D_i^{-h} F) + \tilde{R}(E, F),
\]
where
\[
\tilde{R}(E, F) = Q_\lambda(D_i^h E, F) - Q_\lambda(E, D_i^{-h} F).
\]
Using the “Leibnitz rule” and “integration by parts” for difference quotients [31], it is not difficult to see that there is a constant \( C > 0 \) (which depends only on the metric coefficients and their derivatives) such that \( |\tilde{R}(E, F)| \leq C\|E\|_{H^1} \cdot \|F\|_{H^1} \). Furthermore, since \( E \) is a weak solution,
\[
Q_\lambda(E, D_i^{-h} F) = J(D_i^{-h} F)
\]
\[
\leq \|S\|_{L^2(\mathbb{R}^3)} \cdot \|D_i^{-h} F\|_{L^2(\mathbb{R}^3)}
\]
\[
\leq \|S\|_{L^2(\mathbb{R}^3)} \cdot \|F\|_{H^1(\mathbb{R}^3)}.
\]
Taking everything together, we obtain
\[
\|D_i^h E\|_{H^1(\mathbb{R}^3)} \leq a_0^{-1}(\|S\|_{L^2(\mathbb{R}^3)} + C\|E\|_{H^1(\mathbb{R}^3)}).
\]
Taking the limit \( h \to 0 \) we obtain \( D_i E \in H^1(\mathbb{R}^3, \mathbb{R}^3) \). Repeating the argument for each \( i = 1, 2, 3 \), it follows that \( E \in H^2(\mathbb{R}^3, \mathbb{R}^3) \) and that
\[
\|E\|_{H^2(\mathbb{R}^3)} \leq C_1(\|S\|_{L^2(\mathbb{R}^3)} + \|E\|_{H^1(\mathbb{R}^3)}),
\]
for some constant \( C_1 \).

5. Boundary regularity

Let \( x_0 \in \partial \Omega \) and suppose that \( E \) and the data \( S, g \) vanish for all \( |x - x_0| > \delta > 0 \). If \( \delta \) is sufficiently small, we can introduce new coordinates \( y \) such that the boundary is described by \( y = (y_1, y_2, y_3) = (0, y_2, y_3) \). That is, we can consider the problem on the half plane \( \mathbb{R}^3_+ \). In order to show regularity, we proceed as above, except that now we are only allowed to take the difference quotients \( D^h_A E \), \( A = 2, 3 \) in the directions tangent to the boundary. Also, instead of Eq. (A25), we have
\[
Q_\lambda(E, D^h_A F) = J(D^h_A F)
\]
\[
\leq \|S\|_{L^2(\mathbb{R}^3_+)} \cdot \|D^h_A F\|_{L^2(\mathbb{R}^3_+)} + \|g\|_{H^{1/2}(\mathbb{R}^3)} \cdot \|D^h_A F\|_{H^{-1/2}(\mathbb{R}^3)}
\]
\[
\leq \left(\|S\|_{L^2(\mathbb{R}^3_+)} + \|g\|_{H^{1/2}(\mathbb{R}^3)}\right) \|F\|_{H^1(\mathbb{R}^3_+)},
\]
where we have used the trace theorem in the last step. Therefore, it follows that \( D_A E \in H^1(\mathbb{R}^3_+, \mathbb{R}^3) \), \( A = 2, 3 \). In order to show that \( D_A E \in H^1(\mathbb{R}^3_+, \mathbb{R}^3) \) we remember that \( L(E) = S \) in the weak (distributional) sense which allows us to express \( (D_A)^2 E \) in terms of all other second (and lower order) derivatives of \( E \) for which we know that they are in \( L^2(\mathbb{R}^3_+) \). Therefore, we have \( E \in H^2(\mathbb{R}^3_+, \mathbb{R}^3) \), with an estimate
\[
\|E\|_{H^2(\mathbb{R}^3_+)} \leq C_1 \left(\|S\|_{L^2(\mathbb{R}^3_+)} + \|g\|_{H^{1/2}(\mathbb{R}^3)} + \|E\|_{H^1(\mathbb{R}^3_+)}\right).
\]
Taking a finite covering of \( \Omega \) with a corresponding partition of unity, and summing up the interior and boundary regularity results, it follows that \( E \in W^2, \) and the estimate (A18) holds for \( k = 0 \). This proves theorem (6) for \( k = 0 \).
6. Higher order regularity

Suppose \( \mathbf{E} \in W^{k+1}, k \geq 1 \), is a strong solution of \( L(\mathbf{E})_j = S_j, b(\mathbf{E})^j = g^j \), where \( \mathbf{S} \in W^k, g \in H^{k+1/2}(\partial \Omega, \mathbb{R}^3) \). Suppose also that we have the estimate

\[
\| \mathbf{E} \|_{W^{k+1}} \leq C_{k-1} \left( \| L(\mathbf{E}) \|_{W^{k-1}} + \| b_j(\mathbf{E}) \|_{H^{k-1/2}(\partial \Omega)} + \| \mathbf{E} \|_{W^k} \right)
\]

for all \( \mathbf{E} \in W^{k+1} \). We show this implies that \( \mathbf{E} \in W^{k+2} \). The statement of theorem 6 then follows by induction.

The idea of the proof is the same as for \( k = 0 \): We first localize the problem: Let \( \chi \) be the restriction on \( \Omega \) of a function in \( C_0^\infty(\mathbb{R}^3) \). Then \( \chi \cdot \mathbf{E} \in W^{k+1} \) satisfies the problem

\[
L(\chi \mathbf{E})_j = \dot{S}_j, b(\chi \mathbf{E})^j = \dot{g}^j \quad \text{where} \quad \mathbf{S} \in W^k \quad \text{and} \quad \dot{g} \in H^{k+1/2}(\partial \Omega, \mathbb{R}^3)
\]

are given by Eqs. (A19) and (A20), respectively. So we can assume that \( \mathbf{E} \) and the data are supported in a coordinate patch, and so it is sufficient to consider the problem on \( \mathbb{R}^3 \) or on the half plane \( \mathbb{R}^3_+ \).

We discuss only the case \( \mathbb{R}^3_+ \) here; interior higher regularity follows in the same way. We apply the estimate (A27) to \( D_A^h \mathbf{E} \), where \( A = 2, 3 \) denote tangential directions. Introducing the commutators

\[
R^h(\mathbf{E}) = L(D_A^h \mathbf{E}) - D_A^h L(\mathbf{E}),
\]

\[
r^h(\mathbf{E}) = b(D_A^h \mathbf{E}) - D_A^h b(\mathbf{E}),
\]

which satisfy the estimates (which follow from the “Leibnitz rule” for difference quotients)

\[
\| R^h(\mathbf{E}) \|_{H^{k-1}(\mathbb{R}^3_+)} \leq C \| \mathbf{E} \|_{H^{k+1}(\mathbb{R}^3_+)} ,
\]

\[
\| r^h(\mathbf{E}) \|_{H^{k-1/2}(\mathbb{R}^3_+)} \leq C \| \mathbf{E} \|_{H^{k+1/2}(\mathbb{R}^3_+)} ,
\]

where the constant \( C \) depends on bounds for the metric coefficients \( h_{ij} \) and their \( k + 2 \) order derivatives, we obtain

\[
\| D_A^h \mathbf{E} \|_{H^{k+1}(\mathbb{R}^3_+)} \leq \tilde{C}_k \left( \| D_A^h L(\mathbf{E}) \|_{H^{k-1}(\mathbb{R}^3_+)} + \| D_A^h b(\mathbf{E}) \|_{H^{k-1/2}(\mathbb{R}^3_+)} + \| \mathbf{E} \|_{H^{k+1}(\mathbb{R}^3_+)} \right)
\]

\[
\leq \tilde{C}_k \left( \| \mathbf{S} \|_{H^k(\mathbb{R}^3_+)} + \| g \|_{H^{k+1/2}(\mathbb{R}^3_+)} + \| \mathbf{E} \|_{H^{k+1}(\mathbb{R}^3_+)} \right) .
\]

Therefore, \( D_A^h \mathbf{E} \in H^{k+1}(\mathbb{R}^3_+, \mathbb{R}^3) \), \( A = 2, 3 \). Because \( \mathbf{E} \) satisfies the equation \( L(\mathbf{E}) = \mathbf{S} \) this also implies \( D_A^1 \mathbf{E} \in H^{k+2}(\mathbb{R}^3_+, \mathbb{R}^3) \) with a corresponding estimate.

[1] H. Friedrich and G. Nagy, Comm. Math. Phys. 201, 619 (1999).
[2] J.M. Stewart, Class. Quantum Grav. 15, 2865 (1998).
[3] G. Calabrese, L. Lehner, and M. Tiglio, Phys. Rev. D 65, 104031 (2002).
[4] B. Szilagyi, B. Schmidt, and J. Winicour, Phys. Rev. D 65, 064015 (2002).
[5] G. Calabrese and J. Winicour, Phys. Rev. D 68, 041501 (2003).
[6] G. Calabrese, J. Pullin, O. Reula, O. Sarbach and M. Tiglio, Comm. Math. Phys. 240, 377 (2003).
[7] G. Calabrese and O. Sarbach, J. Math. Phys. 44, 3888 (2003).
[8] A. Anderson and J.W. York, Jr., Phys. Rev. Lett. 82, 4384 (1999).
[9] O. Sarbach, B. Schmidt, and J. Winicour, Phys. Rev. D 70, 044031 (2004).
[10] J.M. Stewart, Class. Quantum Grav. 15, 2865 (1998).
[11] I. Lindblom, M.A. Scheel, L.E. Kidder, H.P. Pfeiffer, D. Shoemaker, and S.A. Teukolsky, Phys. Rev. D 69, 124025 (2004).
[12] H. Kreiss, Commun. Pure Appl. Math. 23, 277 (1970).
[13] H.O. Kreiss and J. Lorenz, “Initial-Boundary Value Problems and the Navier-Stokes Equations,” Academic Press, (1989).
[14] B. Gustafsson, H. Kreiss, and J. Oliger, “Time dependent problems and difference methods,” John Wiley & Sons, New York (1995).
[15] M. Shibata and T. Nakamura, Phys. Rev. D 52, 5428 (1995).
[16] T.W. Baumgarte and S.L. Shapiro, Phys. Rev. D 59, 024007 (1998).
[17] O. Sarbach, G. Calabrese, J. Pullin, and M. Tiglio, Phys. Rev. D 66, 064002 (2002).
Throughout this article, we use the Einstein summation convention and raise and lower indices by means of the metric $h$.

In order to see this more explicitly, assume that $\Omega = \mathbb{R}_+ \times S^1 \times S^1$ where the $y$ and $z$ directions are periodic with period 1, and consider the family of solutions parametrized by

$$f_n(x, y, z) = e^{-\omega_n(x+iy)},$$

where $\omega_n = 2\pi n$ and $n$ is a fixed integer. Clearly, $f_n$ is harmonic. Furthermore,

$$I \equiv \|E_i\|_{L^2(\Omega)}^2 = 2\omega_n^2 \int_0^\infty e^{-2\omega_n x} dx = \omega_n,$$  

$$B \equiv \|H_j \nabla_j f\|_{L^2([0,t] \times \partial \Omega)}^2 = \omega_n^2 \int_0^t 1 dt = t\omega_n^2.$$

However, we have

$$N \equiv \|W_{ij}(t)\|_{L^2(\Omega)}^2 = 4t^2 \omega_n^4 \int_0^\infty e^{-2\omega_n x} dx = 2t^2 \omega_n^3.$$

Therefore, for each fixed $t$, $N/I = 2t^2\omega_n^2 \to \infty$ and $N/B = 2t\omega_n \to \infty$ as $n \to \infty$ so the system is ill posed in $H$.

This condition can probably be weakened by using the trace theorems.