The Fields of Ultra-Relativistic Gravitation

Barak Kol
Racah Institute of Physics, Hebrew University
Jerusalem 91904, Israel
barak_kol@phys.huji.ac.il

Abstract: In order to facilitate the study of weak ultra-relativistic (and Planckian) scattering we present an appropriate decomposition of the gravitational field together with its whole non-linear action. More generally the results apply to any reduction over a non-degenerate spacetime fiber.
1 Introduction

In a companion paper we discuss an effective field theory for weak ultra-relativistic scattering. The problem is to analyze the scattering of two light-like (or at least ultra-relativistic) point-like particles at a large impact parameter (see [1] and references therein). One finds that the physics localizes to the moment of passing in the longitudinal direction and interactions take place only in the transverse space.

When the two particles interact through the gravitational interaction, the components of the gravitational field split into several fields from the transverse perspective. In the literature (see for example [2]) the gravitational Einstein-Hilbert action is expanded perturbatively for weak gravitational fields. In this paper we shall define the field decomposition (2.2) and compute its whole non-linear action (2.3). Expanding it allows to readily read off all the gravitational bulk propagators and vertices for the perturbation theory. For example we reproduce a leading propagator term in (3.3).

In the related case of the post-Newtonian approximation (see [3, 4] for reviews, and [5] for an effective field theory approach) it is natural and useful to decompose the gravitational field through a temporal Kaluza-Klein reduction into Non-Relativistic Gravitational (NRG) fields [6, 7]. These consist of the Newtonian potential, the gravito-magnetic 3-vector and a spatial metric. The full, non-linear gravitational action for these fields was determined in [8].

Here we go further by allowing dimensions larger than one (and arbitrary in principle) for both (the longitudinal) fiber and (the transverse) base. Indeed, our main expression (2.3) includes novel terms which were not all present in the earlier cases. ¹

Computing the action by the standard method metric → Christoffel symbols → curvature tensor → action would be a very complicated analytical task, perhaps hopelessly so. Here we simplify the computation to manageable form with no computerized computation.

¹I was notified that an action whose mathematical form is essentially the same was given by Yoon [14]. While the mathematical context and tools there are essentially the same as here, namely a Kaluza-Klein reduction, the physical context and application is completely different and has no relation to ultra-relativistic gravitation. Instead [14] interprets 4d General Relativity as a 1+1 gauge theory. Technically, there the 1+1 space is the base while here it is a fiber, and here spacetime dimension is arbitrary. I thank S. Carlip for alerting me to [14].
by using a non-orthonormal frame within Cartan’s method, namely a hybrid method which incorporates both a non-trivial frame and a non-trivial metric as in [8].

2 Field decomposition and action

We work in the center of mass frame of a $d$ dimensional spacetime and we denote the longitudinal direction by $z$, and the transverse directions by $x^i$.

In the leading ultra-relativistic limit transverse gradients dominate over longitudinal ones

$$\partial_a \equiv \partial_z \ll \partial_i \equiv \partial_x \quad (2.1)$$

Therefore it is natural to perform a dimensional reduction à la Kaluza-Klein (KK) [9] of the metric over the light-cone coordinates $z^\pm$, a reduction which highlights the transformation properties (or tensor nature) with respect to gauge transformations which depend only on the transverse directions $x$.

The action is simply

$$S = \frac{1}{16\pi G} \int \sqrt{-G} \sqrt{g} d^{d-2} x \sqrt{dz} \cdot \left\{ - \langle K_{ab}[G]\rangle^2_{dW} + \frac{1}{4} \bar{F}^2 + \bar{R}[g] + \left\langle \frac{1}{2} \partial_a g_{ij} \right\rangle^2_{dW} - \bar{R}[G] \right\} \quad (2.3)$$

We proceed to define all the symbols and conventions. First

$$G := \det G_{ab}$$
$$g := \det g_{ij} \quad (2.4)$$
$$D_i = \partial_i + A_i^a \partial_a \quad (2.5)$$

The extrinsic curvature of the $1+1$ fiber is

$$K_{ab}[G] := -\frac{1}{2} \left( D_j G_{ab} + G_{c(a} \partial_{b)} A_c^j \right) = -\frac{1}{2} \left( \partial_b G_{ab} + \mathcal{L}_{A_i} G_{ab} \right) \quad (2.6)$$

\[2\text{In general the extrinsic curvature } K \text{ evaluated on two vector fields } X, Y \text{ which lie in a sub-manifold is defined by } K(X, Y) = (D_X Y)^\pm. \text{ This defines a symmetric tensor.}\]
where in the last expression $\mathcal{L}_{A_i}$ denotes the Lie derivative with respect to the longitudinal vector $A_i \equiv A_i^a$. The $\left(-\frac{1}{2}\right)$ prefactor was inserted to conform with the standard definition of the extrinsic curvature.

For any symmetric tensor field we define a “deWitt” quadratic form (actually once applied to a differential of a metric and integrated over the manifold it becomes a metric on the space of metrics [10]), see [8] for its appearance in the NRG action

$$\langle h_{IJ}\rangle^2_{dW} := |h_{IJ}|^2 - h^2 \quad (2.7)$$

In particular

$$\langle K_{abi}(G)\rangle^2_{dW} = g^{ij}G^{ac}G^{bd} (K_{abi}K_{cdj} - K_{aci}K_{bdi}) \quad (2.8)$$

$$\left\langle \frac{1}{2} \partial_a g_{ij} \right\rangle^2_{dW} = \frac{1}{4} G^{ab}g^{ik}g^{jl} [\partial_a g_{ij} \partial_b g_{kl} - \partial_a g_{jk} \partial_b g_{il}] \quad (2.8)$$

The generalized (magnetic) field strength is defined by

$$F^a_{ij} := D_i A^a_j - D_j A^a_i = F^a_{ij} + A^b_i \partial_b A^a_j - A^b_j \partial_b A^a_i \quad (2.9)$$

and its square is given by

$$F^2 = G^{ab}g^{ik}g^{jl} F^a_{ij} F^b_{kl} \quad (2.10)$$

Finally $\tilde{R}[g]$ denotes the Ricci scalar of the transverse metric $g$ where the derivatives in its expression are replaced everywhere as follows $\partial_i \rightarrow D_i$. Borrowing notation from the Mathematica software this definition can be stated by

$$\tilde{R}[g] := R[g] / \partial_i \rightarrow D_i \quad (2.11)$$

**Derivation.** In order to compute the action we used a non-orthonormal frame within Cartan’s method, namely a hybrid method which incorporates both a non-trivial frame and a non-trivial metric as in [8]. This action generalizes an analogous result from the KK literature found by Aulakh and Sahdev [11], and the NRG (Non-Relativistic Gravitational) action [8].

**Tests.** We tested the Ultra-Relativistic Gravitational (URG) action (2.3) in several limits. For $d_Z = 1$ we reproduce the KK [11] and NRG actions. For $d_X = 1$ we reproduce the ADM [12] action (see for example [13]). As a Final test for $A^a_i = 0$ the action is symmetric with respect to the exchange $X \leftrightarrow Z$. The “stationary limit”, namely no $z$ dependence, is another interesting limit. In this limit the last two terms in the action (2.3) vanish,\(^3\) while the other terms simplify: $D_i \rightarrow \partial_i$, $\tilde{R} \rightarrow R$, $\tilde{F} \rightarrow F$ and $K_{abi} \rightarrow -(1/2)\partial_i G_{ab}$.

\(^3\)Recall that the Lie derivative by a vector field $V$ of a vector field $W$ is defined as $\mathcal{L}_V W^\mu := [V, W]^\mu := V^\nu \partial_\nu W^\mu - W^\nu \partial_\nu V^\mu$ where $[V, W]$ denotes the commutator of the two vector fields. When one extends this derivation to all tensors one finds that the Lie derivative of a co-vector $\omega$ is given by $\mathcal{L}_V \omega^\mu = V^\nu \partial_\nu \omega^\mu + \omega^\nu \partial_\nu V^\mu$. Similarly $\mathcal{L}_V G_{ab} = V^\nu \partial_\nu G_{ab} + G_{ac} \partial_c V^\nu + G_{bc} \partial_c V^\nu$.

\(^4\)Note that as usual the $1/4$ prefactor could have been avoided had we accompanied the anti-symmetrization in the definition of $\tilde{F}$ with a division by 2.

\(^5\)If a curved $Z$ fiber is allowed then some $R[G]$ would remain.
3 Perturbing around flat longitudinal spacetime

The dimensionally reduced action can be used to linearize the action around any prescribed product space-time \( X \times Z \). In our case the unperturbed space-time is flat and accordingly we may use

\[
G_{ab} = \eta_{ab} + H_{ab} \\
g_{ij} = \delta_{ij} + h_{ij}.
\]  

(3.1)

At leading ultra-relativistic order one source couple dominantly to \( H^{++} \) while the others couples to \( H^{--} \). For 2d metrics the deWitt metric simplifies

\[
\langle G_{ab} \rangle^2 dW = \frac{2}{-G} \left( dH^{++} dH^{--} - dH_0^2 \right)
\]

where

\[
H_0 := H^{+-} \\
-G = (1 + H_0)^2 - H^{++} H^{--}.
\]  

(3.2)

Substituting into (2.3) we find that the propagator for \( H^{++}, H^{--} \) is

\[
S \supset -\frac{1}{32\pi G} \int d^{d-2}x d^2z \nabla_\perp H^{++} \nabla_\perp H^{--}.
\]  

(3.3)

Longitudinal boosts are a global \( SO(1,1) \simeq \mathbb{R} \) symmetry of the action.\(^6\) Under this symmetry \( H^{++} \) has charge +2 while \( H^{--} \) has charge \((-2)\). This symmetry can be represented in the Feynman rules by representing the fields \( H^{++}, H^{--} \) by an oriented line and distinguishing them by its orientation, which represents the flow of charge.

Some open questions.

- In 4d spacetime the transverse metric is 2d which may bring about additional simplifications.
- In the post-Newtonian case Weyl rescaling of the metric was employed, and actually was necessary even to reproduce the Newtonian potential. In 4d space the transverse space is 2d and Weyl rescaling is less effective, but it could possibly be of use at least in higher dimensions.

---

\(^6\)From the transverse point of view \( G_{ab} \) are three scalars and being a quadratic form the action is invariant under a similarity transformation \( G \rightarrow R^T G R \) for any \( R \in GL(2,\mathbb{R}) \). Moreover the vacuum (unperturbed solution) \( G_{ab} = \eta_{ab} \) is invariant under the \( SO(1,1) \simeq \mathbb{R} \) subgroup of 2d Lorentz transformations, which accordingly is a symmetry of the linearized action.
References

[1] G. ’t Hooft, “Graviton Dominance in Ultrahigh-Energy Scattering,” Phys. Lett. B198, 61-63 (1987).
D. Amati, M. Ciafaloni, G. Veneziano, “Towards an S-matrix description of gravitational collapse,” JHEP 0802, 049 (2008). [arXiv:0712.1209 [hep-th]].

[2] D. Amati, M. Ciafaloni, G. Veneziano, “Effective action and all order gravitational eikonal at Planckian energies,” Nucl. Phys. B403, 707-724 (1993).

[3] L. Blanchet, “Gravitational radiation from post-Newtonian sources and inspiralling compact binaries,” Living Rev. Rel. 5, 3 (2002), update: Living Rev. Rel. 9, 4 (2006) [arXiv:gr-qc/0202016].

[4] G. Schaefer, “Post-Newtonian methods: Analytic results on the binary problem,” arXiv:0910.2857 [gr-qc].

[5] W. D. Goldberger and I. Z. Rothstein, “An effective field theory of gravity for extended objects,” Phys. Rev. D 73, 104029 (2006) [arXiv:hep-th/0409156].

[6] B. Kol and M. Smolkin, “Classical Effective Field Theory and Caged Black Holes,” Phys. Rev. D 77, 064033 (2008) [arXiv:0712.2822 [hep-th]].

[7] B. Kol and M. Smolkin, “Non-Relativistic Gravitation: From Newton to Einstein and Back,” Class. Quant. Grav. 25, 145011 (2008) [arXiv:0712.4116 [hep-th]].

[8] B. Kol, M. Smolkin, “Einstein’s action in terms of Newtonian fields,” [arXiv:1009.1876 [hep-th]].

[9] T. Kaluza, “Zum Unitätsproblem in der Physik”. Sitzungsber. Preuss. Akad. Wiss. Berlin. (Math. Phys.) 1921: 966972. O. Klein, ”Quantentheorie und fünfdimensionale Relativitätstheorie”. Zeitschrift für Physik A Hadrons and Nuclei 37 (12): 895906.

[10] B. deWitt, “Quantum theory of gravity I: the canonical theory,” Phys. Rev. 160 1113 (1967), eq. (6.6).

[11] C. S. Aulakh and D. Sahdev, “The Infinite Dimensional Gauge Structure Of Kaluza-Klein Theories. 1. D = 1+4,” Phys. Lett. B 164, 293 (1985).

[12] R. L. Arnowitt, S. Deser and C. W. Misner, “The dynamics of general relativity,” in Gravitation: an introduction to current research, Louis Witten ed. (Wiley 1962), chapter 7, p 227 [arXiv:gr-qc/0405109].

[13] B. Kol, M. Levi, M. Smolkin, “Comparing space+time decompositions in the post-Newtonian limit,” [arXiv:1011.6024 [gr-qc]].

[14] J. H. Yoon, “4-dimensional Kaluza-Klein approach to general relativity in the (2,2)-splitting of spacetimes,” arXiv:gr-qc/9611050. “Kaluza-Klein Formalism of General Spacetimes,” Phys. Lett. B 451, 296 (1999) [arXiv:gr-qc/0003059].