NONCOMMUTATIVE COHOMOLOGY AND ELECTROMAGNETISM ON $\mathbb{C}_q[SL_2]$ AT ROOTS OF UNITY

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Abstract We compute the noncommutative de Rham cohomology for the finite-dimensional quantum coordinate ring $\mathbb{C}_q[SL_2]$ at odd roots of unity and with its standard 4-dimensional differential structure. We find that $H^1$ and $H^3$ have three additional modes beyond the generic $q$-case where they are 1-dimensional, while $H^2$ has six additional modes. We solve the spin-0 and Maxwell theory on $\mathbb{C}_q[SL_2]$ including a complete picture of the self-dual and anti-self dual solutions and of Lorentz and temporal gauge fixing. The system behaves in fact like a noncompact space with self-propagating modes (i.e., in the absence of sources). We also solve with examples of ‘electric’ and ‘magnetic’ sources including the biinvariant element $\theta \in H^1$ which we find can be viewed as a source in the local (Minkowski) time-direction (i.e. a uniform electric charge density).

Keywords: noncommutative geometry, roots of unity, quantum groups, cohomology, electromagnetism, light

1 Introduction

By now there is a standard formulation of differential calculi or ‘exterior algebra’ of differential forms on quantum groups such as $\mathbb{C}_q[SL_2]$. The standard bicovariant ones correspond essentially to representations\(^3\), i.e. are labelled in this case by spin $j \in \frac{1}{2} \mathbb{Z}_+$ and have dimension $(2j+1)^2$ (there are also exotic twists of the standard ones which do not concern us). In our case the smallest nontrivial calculus is 4 dimensional and was already known since the earliest works\(^2\). The entire exterior algebra and exterior derivative are also known, and it is known that dimensions in each degree of forms and the resulting cohomology for generic $q$ are\(^3\)

$$\dim(\Omega) = 1 : 4 : 6 : 4 : 1, \quad H^0 = \mathbb{C}, \quad H^1 = \mathbb{C}, \quad H^2 = 0, \quad H^3 = \mathbb{C}, \quad H^4 = \mathbb{C}.$$
The nontrivial generator in degree 1 is the bi-invariant element $\theta$ that defines $d$ by graded-commutator. The further physics and geometry on such spaces has been mainly looked at for generic $q$, where (with some modifications such as a 1-dimensional extension) it follows broadly the line of the undeformed case.

What we show in the present purely computational paper is the existence of completely different and novel phenomena when, however, $q$ is an odd root of unity. This case is in many ways more relevant to both physics (e.g. in the Wess-Zumino-Witten model) and mathematics (e.g. the image of the quantum Frobenius map and because of known links to group theory in finite characteristic). We work with the reduced finite-dimensional quantum group, which is then a nonsemisimple Hopf algebra. This has the merit that all linear (and some nonlinear) aspects of the geometry can be fully computed. The model also contrasts markedly from the case of finite group algebras recently studied elsewhere[4]. For the differential calculus itself the theorem for factorisable quantum groups in [1] implies that these are classified by two-sided ideals in $u_q(sl_2)$. So the smallest nontrivial calculus is again the 4-dimensional one, which is the calculus that we use. Its structure is recalled briefly in Section 2.

We then find in Section 3 that there are additional elements of $H^i$ not present for generic $q$. In all cases that we have checked (namely 3,5,7’th roots) we find in fact that

$$H^i \cong \Lambda,$$

the space of right-invariant forms as a graded vector space, as well as an exact sequence for the $H^i$. The additional cohomology modes correspond to topological gauge fields with zero curvature in the Maxwell theory reflecting nontrivial topology created by the quotienting to the reduced quantum group. In Section 4 we use the family of ‘Killing form’ metrics in [3] and show how the requirement of $\star^2 = \text{id}$ for the Hodge-$\ast$ operator singles out a particular $q$-deformed Minkowski one (this applies for generic $q$). We then proceed to solve the Maxwell theory for $r = 3$ completely. Among interesting features, we find that for spin 0 the wave operator $\Box$ is not fully diagonalisable (this is due to the nonsemisimplicity), while on the other hand every solution of the sourceless Maxwell’s equations may be written as a sum of a self-dual and an antiself-dual solution. We are also able to completely analyse gauge fixing issues which are usually glossed over in gauge theories in physics; we find the novel result that not all solutions can be rendered in Lorentz gauge, nor all in temporal gauge, but that the two gauges between them ‘patch’ the moduli of solutions. We expect the phenomena found here by computation to hold for all odd roots.
2 Exterior algebra

Here we fix the algebras and exterior algebras in question in notation that we will use. In effect, in order to have reliable formulae for root of unity we carefully compute the (well-known) 4-D calculus from a modern crossed-module point of view. We let $q^2 \neq 1$. The quantum group $\mathcal{A} = \mathbb{C}_q[SL_2]$ has a matrix of generators $t^i{}_j = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with relations

$$ba = qab, \quad ca = qac, \quad db = qbd, \quad dc = qcd, \quad cb = bc, \quad da - ad = qbc, \quad ad - q^{-1}bc = 1,$$

where $\mu = 1 - q^{-2}$, and the matrix coalgebra structure. For its 4d calculus we take a basis $e_i^j = \begin{pmatrix} e_a & e_b \\ e_c & e_d \end{pmatrix}$, where $e_1^2 = e_b$, of the space $\Lambda^1$ of right-invariant differential 1-forms. This space $\Lambda^1$ is specified as a left $\mathbb{C}_q[SL_2]$-crossed module, namely with coaction and action

$$\Delta_L e_i^j = (St^k_i) t^j_i \otimes e_k^i, \quad a \triangleright \begin{pmatrix} e_a & e_b \\ e_c & e_d \end{pmatrix} = \begin{pmatrix} qe_a + \mu e_d & e_b \\ e_c & q^{-1}e_d \end{pmatrix},$$

$$b \triangleright \begin{pmatrix} e_a & e_b \\ e_c & e_d \end{pmatrix} = \begin{pmatrix} \mu e_c & q\mu e_d \\ 0 & 0 \end{pmatrix}, \quad c \triangleright \begin{pmatrix} e_a & e_b \\ e_c & e_d \end{pmatrix} = \begin{pmatrix} \mu e_b & 0 \\ q\mu e_d & 0 \end{pmatrix}, \quad d \triangleright \begin{pmatrix} e_a & e_b \\ e_c & e_d \end{pmatrix} = \begin{pmatrix} q^{-1}e_a & e_b \\ e_c & qe_d \end{pmatrix}. $$

$\Omega^1 = \Lambda^1 \otimes \mathcal{A}$ is generated by these forms as a free right module over $\mathbb{C}_q[SL_2]$ while as a bimodule the left action is $h.e_a = (h^{(1)} \triangleright e_a)h^{(2)}$ for all $h \in \mathbb{C}_q[SL_2]$, etc. This comes out as

$$[\begin{pmatrix} c \\ d \end{pmatrix}, e_b] = \begin{pmatrix} a \\ b \end{pmatrix}, e_c] = \begin{pmatrix} a \\ b \end{pmatrix}, e_d]q^{-1} = \begin{pmatrix} c \\ d \end{pmatrix}, e_d]q = 0,$$

$$\begin{pmatrix} a \\ b \end{pmatrix}, e_b] = q\mu e_d \begin{pmatrix} c \\ d \end{pmatrix}, \quad [\begin{pmatrix} c \\ d \end{pmatrix}, e_c] = q\mu e_d \begin{pmatrix} a \\ b \end{pmatrix},$$

$$[\begin{pmatrix} c \\ d \end{pmatrix}, e_a]q^{-1} = \mu e_b \begin{pmatrix} a \\ b \end{pmatrix}, \quad [\begin{pmatrix} a \\ b \end{pmatrix}, e_a]q = \mu e_c \begin{pmatrix} c \\ d \end{pmatrix} + q\mu^2 e_d \begin{pmatrix} a \\ b \end{pmatrix},$$

where $[x, y]_q = xy - qyx.$

Also from the crossed module structure is the braiding $\Psi(e_a \otimes e_b) = e_a^{(1)} \triangleright e_b \otimes e_a^{(2)}$, etc., where $^{(1)}$ and $^{(2)}$ denote the outputs of $\Delta_L$. This comes out as

$$\Psi(e_a \otimes e_a) = e_a \otimes e_a - \mu(e_b \otimes e_c - e_c \otimes e_b) + q\mu^2 e_d \otimes (q e_a - q^{-1}e_d)$$

$$\Psi(e_b \otimes e_b) = e_b \otimes e_b$$

$$\Psi(e_c \otimes e_c) = e_c \otimes e_c$$

$$\Psi(e_d \otimes e_d) = e_d \otimes e_d$$

$$\Psi(e_a \otimes e_d) = e_d \otimes e_a$$
\[
\Psi(e_d \otimes e_a) = e_a \otimes e_d + \mu (e_b \otimes e_c - e_c \otimes e_b) - q \mu^2 e_d \otimes (q e_a - q^{-1} e_d)
\]
\[
\Psi(e_b \otimes e_c) = e_c \otimes e_b + q \mu e_d \otimes (q e_a - q^{-1} e_d)
\]
\[
\Psi(e_c \otimes e_b) = e_b \otimes e_c - q \mu e_d \otimes (q e_a - q^{-1} e_d)
\]
\[
\Psi(e_a \otimes e_b) = e_b \otimes e_a + q^2 \mu e_d \otimes e_b
\]
\[
\Psi(e_b \otimes e_a) = q^{-2} e_a \otimes e_b + \mu q^{-1} e_b \otimes (q e_a - q^{-1} e_d)
\]
\[
\Psi(e_a \otimes e_c) = e_c \otimes e_a - \mu e_d \otimes e_c
\]
\[
\Psi(e_c \otimes e_a) = q^2 e_a \otimes e_c - \mu q e_c \otimes (q e_a - q^{-1} e_d) + [2]_q^2 \mu^2 e_d \otimes e_c
\]
\[
\Psi(e_b \otimes e_d) = q^2 e_d \otimes e_b
\]
\[
\Psi(e_d \otimes e_b) = e_b \otimes e_d - q^2 \mu e_d \otimes e_b
\]
\[
\Psi(e_c \otimes e_d) = q^{-2} e_d \otimes e_c
\]
\[
\Psi(e_d \otimes e_c) = e_c \otimes e_d + \mu e_d \otimes e_c
\]

where \([n]_q = (1 - q^n)/(1 - q)\).

This extends as a bimodule map to an endomorphism of \(\Omega^1 \otimes_{\mathbb{C}_q[SL_2]} \Omega^1\). Following Woronowicz we then define \(\Omega^2 = \Omega^1 \otimes_M \Omega^1 / \ker(id - \Psi)\), etc. Equivalently, in a modern braided group approach which is computationally easier, \(\Omega\) is a free right \(\mathbb{C}_q[SL_2]\) module over the invariant exterior forms

\[
\Lambda = T\Lambda^1 / \oplus_n \ker A_n; \quad A_n = [n, -\Psi]! = (id \otimes A_{n-1})[n, -\Psi]
\]

\([n, -\Psi] = id - \Psi_{12} + \Psi_{12}\Psi_{23} + \ldots + (-1)^{n-1}\Psi_{12}\ldots\Psi_{n-1,n}\).

Here \([n, -\Psi]\) are the braided-integers induced by a braiding \(-\Psi\) and \(\Lambda\) is a braided group with additive coproduct \(\Delta e_a = e_a \otimes 1 + 1 \otimes e_a\), etc. The above relations ensure that it is dually paired with a similar braided group \(\Lambda^*\) and these together ensure Poincaré duality. In particular, \(A_2 = id - \Psi\) and hence the relations in degree 2, which are in fact all the relations for generic \(q\), come out as:

\[
< e_b, e_c, e_d > \quad \text{usual Grassmann algebra}, \quad e_a^2 = \mu e_b \wedge e_c, \quad e_a \wedge e_d + e_d \wedge e_a + \mu e_b \wedge e_c = 0,
\]
\[
e_a \wedge e_b + q^2 e_b \wedge e_a - \mu e_b \wedge e_d = 0, \quad e_c \wedge e_a + q^2 e_a \wedge e_c + \mu e_c \wedge e_d = 0
\]

Note that if we define the corresponding symmetric algebra by \(T\Lambda^1/\text{image}(id + \Psi)\) then we have q-Minkowski space in the braided-matrix form. The exterior algebra in this case has
a similar form to the above in terms of exact differentials, since both come from Meyer’s braiding for the additive braided group structure of q-Minkowski space.

Finally, the exterior derivative is

\[ \mathbf{d} = -[\theta, \cdot], \quad \theta = e_a + e_d \]

where we use the commutator on even degree and anticommutator on odd. Note that in our conventions \( \bar{\mathbf{d}} = [\cdot, \theta] \) is more natural but would be a right-derivation. The element \( \theta \) is closed but not exact and is biinvariant. Explicitly,

\[
\begin{align*}
\mathbf{d} \left( \begin{array}{c} a \\ b \end{array} \right) &= (q - 1)(e_a - q^{-1}(1 - \mu[2]q)e_d) \left( \begin{array}{c} a \\ b \end{array} \right) + \mu e_c \left( \begin{array}{c} c \\ d \end{array} \right), \\
\mathbf{d} \left( \begin{array}{c} c \\ d \end{array} \right) &= (q - 1)(e_d - q^{-1}e_a) \left( \begin{array}{c} c \\ d \end{array} \right) + \mu e_b \left( \begin{array}{c} a \\ b \end{array} \right)
\end{align*}
\]

\[ \mathbf{d} e_a = -\mu e_b \wedge e_c, \quad \mathbf{d} e_d = \mu e_b \wedge e_c, \quad \mathbf{d} e_b = -\mu (e_a \wedge e_b + q^{-2}e_b \wedge e_d), \quad \mathbf{d} e_c = \mu (q^2 e_a \wedge e_c + e_c \wedge e_d). \]

### 3 Roots of unity and cohomology

We now study \( \mathcal{A} = \mathbb{C}_q[SL_2] \) reduced at \( q^r = 1 \) a primitive \( r \)'th root of unity by the additional relations

\[ c^r = b^r = 0, \quad a^r = d^r = 1, \]

which we suppose from now on. Here \( a^r, b^r, c^r, d^r \) generate an undeformed \( \mathbb{C}[SL_2] \) central sub-Hopf algebra of the original \( \mathbb{C}_q[SL_2] \). Note also that in the reduced case \( d = a^{-1}(1 + q^{-1}bc) \) is redundant and moreover the algebra becomes finite dimensional, with \( \dim(\mathcal{A}) = r^3 \). A basis of \( \mathcal{A} \) is \( \{a^m b^n c^k \} \) for \( 0 \leq m, n, k \leq r - 1 \). All kernel computations are done below for \( r = 3, 5, 7 \) for concreteness, but we expect identical results for all odd \( r \).

**Proposition 3.1** At least for \( r = 3, 5, 7 \), the exterior algebra for the reduced quantum group has the same dimensions as for generic \( q \) (namely 1:4:6:4:1) and is given entirely by relations in degree 2 (a quadratic algebra). Moreover, the exterior derivative descends to one over the reduced \( \mathbb{C}_q[SL_2] \).

**Proof** Since the reduced quantum group remains a Hopf algebra is b covariant calculi are still defined by quotient crossed modules of \( \ker \epsilon \). Our particular crossed module remains one with the same form of action and coaction, bimodule structure and braiding. Therefore it is only a matter of computing the explicit braided factorial matrices \([n, -\Psi]!\) for \( n = 2, 3, 4 \) and in particular the dimensions of their kernel, which we find to be the same provided \( r \) is odd (for
example \( r = 6 \) is different). Hence the algebra \( \Lambda \) is unchanged in this case. Since we have not discussed explicitly the projection from \( \ker \epsilon \) to \( \Omega_0 \) we also verify directly that \( d \) is consistent with the additional relations of the reduced quantum group. \( \diamond \).

Next, in order to compute cohomology we need \( d \) on a general element of \( A \). This is given by the Leibniz rule and the following:

**Lemma 3.2** For all invertible \( q^2 \neq 1 \),

\[
d(a^m b^n c^k) = e_a \cdot (q^{m+n-k} - 1) a^m b^n c^k + \mu e_b \cdot q^{n-k+1} \lfloor k \rfloor q^2 a^{m+1} b^n c^{k-1} \]

\[
+ \mu e_c \cdot q^{-n} \left( [m+n]q^2 a^{m-1} b^n c^{k+1} + q[n]q^2 a^{m-1} b^{n-1} c^k \right) \\
+ \mu^2 e_d \cdot q^{k-n+2} \left( [k+1]q^2 [m+n]q^2 a^{m+b^n} c^k + q[n]q^2 [k]q^2 a^{m+b^n-1} c^{k-1} \right) \\
+ e_d \cdot (q^{-n-m+k} - 1) a^{m+b^n} c^k
\]

**Proof** We first iterate the stated bimodule relations to obtain

\[
\begin{align*}
    u^n \cdot e_a &= q^n e_a \cdot u^n + q^{-1} [n] \mu e_c \cdot u^{n-1} v + q \mu^2 [n] e_d \cdot u^n \\
    v^n \cdot e_a &= q^{-n} e_a \cdot v^n + q [n] e_b \cdot v^{n-1} u \\
    u^n \cdot e_b &= e_b \cdot u^n + q \mu [n] e_d \cdot u^{n-1} v \\
    v^n \cdot e_b &= e_b \cdot v^n \\
    u^n \cdot e_c &= e_c \cdot u^n \\
    v^n \cdot e_c &= e_c \cdot v^n + q \mu [n] e_d \cdot v^{n-1} u \\
    u^n \cdot e_d &= q^n e_d \cdot u^n \\
    v^n \cdot e_d &= q^n e_d \cdot v^n
\end{align*}
\]

where \([n] = (q^n - q^{-n})/(q - q^{-1})\) and \( u = \begin{pmatrix} a \\ b \end{pmatrix}, v = \begin{pmatrix} c \\ d \end{pmatrix} \). Then by recurrence, one gets, for \( X = a^m b^n c^k \),

\[
\begin{align*}
    X \cdot e_a &= e_a \cdot q^{m+n-k} X + \mu e_b \cdot q^n [k] a^{m+1} b^n c^{k+1} \\
    &+ \mu e_c \cdot q^{n-k} \left( q^{m-1} [m] a^{m-1} b^n c^{k+1} + [n] a^{m-1} b^{n-1} c^k \right) \\
    &+ q \mu^2 e_d \cdot ( [k+1] [m+n] X + q^{-m} [n] [k] a^{m+b^n-1} c^{k-1} ) \\
    X \cdot e_d &= e_d \cdot q^{-n-m+k} X
\end{align*}
\]

We then compute \( dX = X \theta - \theta X \). \( \diamond \)
Finally, we choose an explicit basis for each degree of the exterior algebra. Here \( e_{abc} \equiv e_a \wedge e_b \wedge e_c \), etc. for our chosen basis elements

\[
\Lambda^2 = \{e_{ab}, e_{ac}, e_{ad}, e_{bc}, e_{bd}, e_{cd}\}, \quad \Lambda^3 = \{e_{abc}, e_{abd}, e_{acd}, e_{bcd}\}, \quad \Lambda^4 = \{e_{abcd}\}
\]

and we then use the above relations to explicitly define \( \wedge, d \) on right-invariant forms as a \( 16 \times 6 \) matrix and a \( 4 \times 6 \) matrix respectively. With these ingredients it is a matter of linear algebra to compute cohomology.

**Proposition 3.3** At least for \( r = 3, 5, 7 \) the noncommutative de Rham cohomology \( H^1 \) for the 4d calculus on the reduced quantum group \( \mathbb{C}_q[SL_2] \) is 4 dimensional with basis

\[
\theta = e_a + e_d, \quad h_1 = e_3a^{r-1}, \quad h_2 = e_4a^{r-1}b^{r-1}, \quad n = e_a + e_4a^{r-1}c
\]

**Proof** We write \( d_0 : \Omega^0 \to \Omega^1 \) as an \( r^3 \times 4r^3 \) matrix. We also compute the wedge product \( \Lambda^1 \otimes \Lambda^1 \to \Lambda^2 \) and exterior derivative \( d : \Lambda^1 \to \Lambda^2 \) as explained above. The Leibniz rule then allows us to define \( d_1 : \Omega^1 \to \Omega^2 \) from these ingredients as a \( 4r^3 \times 6r^3 \) matrix. We then compute the null spaces to find the dimension of the cohomology (for example the kernel of \( d_1 \) is 30-dimensional for \( r = 3 \) and 346-dimensional for \( r = 7 \), while the image of \( d_0 \) 26-dimensional and 342-dimensional respectively). We also verify \( d_1 d_0 = 0 \) as a programming check. Finally we chose 4 vectors in ker \( d_1 \) and not in the image of \( d_0 \) and verify that together with a basis of the image of \( d_0 \) they form a linearly independent set, i.e. their classes provide a basis of \( H^1 \). \( \diamondsuit \)

We find the same kind of phenomenon for the higher cohomology.

**Proposition 3.4** At least for \( r = 3, 5, 7 \) the cohomologies have the same dimensions as \( \Lambda \) in each degree. As representatives we have:

\[
H^2 : \quad m_1 = e_{bd}a^{r-1}, \quad m_2 = e_{ab}ac^{r-1}, \quad m_3 = e_{ac}a^{r-1}b^{r-1}, \quad m_4 = e_{cd}a^{r-1}b^{r-1},
\]

\[
m_5 = (e_{ac} - e_{cd})a^{r-1}c - e_{ad}, \quad m_6 = e_{bd}a^{r-1}c^{r-2} + q^4e_{cd}a^{r-1}b^{r-2}c^{r-1}.
\]

\[
H^3 : \quad \Theta = e_{bcd}b^{r-1}c^{r-1}, \quad h_1^* = e_{ab}ac^{r-1}, \quad h_2^* = e_{acd}a^{r-1}b^{r-1},
\]

\[
s = e_{abd}a^{r-1}c^{r-2} + q^4e_{acd}a^{r-1}b^{r-2}c^{r-1}, \quad H^4 : \quad e_{abcd}b^{r-1}c^{r-1}.
\]

**Proof** We proceed with respect to our basis above to compute the wedge products \( \Lambda^2 \otimes \Lambda^1 \to \Lambda^3 \) and \( \Lambda^1 \otimes \Lambda^2 \to \Lambda^3 \) as \( 24 \times 4 \) matrices. We also use the (graded) Leibniz rule to define \( d : \Lambda^2 \to \Lambda^3 \) using these projectors and the matrices already computed for \( H^1 \). Finally we combine these via
Leibniz with \( d_0 \) to obtain \( d_2 : \Omega^2 \rightarrow \Omega^3 \) as a \( 6r^3 \times 4r^3 \) matrix, and compute its kernel and the image of \( d_1 \) above (for example the kernel of \( d_2 \) is 84-dimensional for \( r = 3 \) and 1032-dimensional for \( r = 7 \), while the image of \( d_1 \) is 78-dimensional and 1026-dimensional respectively). Similarly we proceed to \( d_3 : \Omega^3 \rightarrow \Omega^4 \) (its kernel is 82-dimensional for \( r = 3 \) and 1030-dimensional for \( r = 7 \), while the image of \( d_2 \) is 78-dimensional and 1026-dimensional respectively). The image of \( d_3 \) has codimension 1, so \( H^4 \) is similarly 1-dimensional. We then chose representatives and check linear independence in the quotient spaces. Our notations for them will be relevant later. Note also that the kernel of \( d_0 \) is 1-dimensional and \( H^0 \) clearly has a basis given by 1.

Finally, we observe that the cohomology is itself a complex under the operation \( \theta \wedge \) since \( \theta \wedge \theta = 0 \) and \( d\theta = 0 \).

**Proposition 3.5** At least for \( r = 3, 5, 7 \), the sequence \( 0 \rightarrow H^0 \rightarrow H^1 \rightarrow H^2 \rightarrow H^3 \rightarrow H^4 \rightarrow 0 \) defined by \( \theta \wedge \) is exact.

**Proof** We let \( \theta_0 \) be \( \theta \) acting by multiplication in degree 0, etc. The image of \( \theta_0 \) is \( \theta \). Its complement in the basis shown has image

\[
\theta \wedge h_1 = m_2 - m_1, \quad \theta \wedge h_2 = m_3 - m_4, \quad \theta \wedge n = m_5
\]

which is a 3-dimensional subspace of \( H^2 \). Its complement has basis \( m_1 + m_2, m_3 + m_4, m_6 \). These map under \( \theta_2 \) up to normalisation to \( h_1^*, h_2^*, s \), which are three of the basis elements of \( H^3 \). Their complement \( \Theta \) maps under \( \theta_3 \) to the generator of \( H^4 \).

4 Wave equations and Hodge-*

Next we describe the Hodge * operator corresponding to the ‘Killing metric’ introduced in [5]. These are further geometric structures on the full \( \mathbb{C}q[SL_2] \) and after recalling them in the form that we need, we will then specialise to our reduced root of unity case. In our conventions the general metric is:

**Lemma 4.1** For all invertible \( q^2 \neq 1 \),

\[
\eta \equiv \eta^{ij} e_i \otimes e_j = e_b \otimes e_c + q^2 e_c \otimes e_b + \frac{(qe_a - e_d) \otimes (qe_a - e_d)}{[2]_q} + q(q - 1)e_a \otimes e_a + \lambda \theta \otimes \theta
\]

is nondegenerate for \( \lambda \neq q(1 - q)/[4]_q, \Delta_L \)-invariant and symmetric in the sense \( \wedge(\eta) = 0 \)

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Proof This is adapted from [5] and its properties then verified directly in our case.  

We next define the antisymmetrization tensor by
\[
\epsilon_{ijkl}\text{Top} = e_i \wedge e_j \wedge e_k \wedge e_l
\]
where Top = \(e_a \wedge e_b \wedge e_c \wedge e_d\) is \(\Delta_L\)-invariant and a basis of \(\Omega^4\). We can then define define
\[
\star(e_i) = d_1^{-1} \epsilon_{ijkl} \eta^{jm} \eta^{kn} \eta^{lp} e_p \wedge e_n \wedge e_m
\]
\[
\star(e_i \wedge e_j) = d_2^{-1} \epsilon_{ijkl} \eta^{kn} \eta^{ln} e_n \wedge e_m
\]
\[
\star(e_i \wedge e_j \wedge e_k) = d_3^{-1} \epsilon_{ijkl} \eta^{lm} e_m
\]
for some normalisations \(d_i\) to be chosen.

Note that all constructions here are \(\Delta_L\)-covariant, under which the space \(\Lambda^1\) is a direct sum
\[
\Lambda^1 = \text{sl}_2(q) \oplus \mathbb{C} \theta,
\]
and \(\star^2\) has these as eigenspaces. We now adjust \(\lambda\) so that the associated eigenvalues are the same.

Lemma 4.2 For all invertible \(q^2 \neq 1\) there exists precisely one value, \(\lambda = q(1 - q - q^2)/[2]_q\), such that \(\eta\) is invertible and \(\star^2 \propto \text{id}\) on \(\Lambda^1\). In this case we suppose \([3]_q \neq 0\) and set
\[
d_1 = 2q^2(1 - q + q^2)[3]_q, \quad d_2 = q^2[2]_q, \quad d_3 = q^2
\]
Then,
\[
\star e_a = -e_{abc} - \mu e_{bcd}, \quad \star e_b = -e_{abd}, \quad \star e_c = q^2 e_{acd}, \quad \star e_d = e_{bcd}
\]
\[
\star e_{ab} = -e_{ab} + 2\mu e_{ad}, \quad \star e_{ac} = e_{ac}, \quad \star e_{ad} = \frac{1}{2} \left(2e_{bc} - q^2 \mu e_{ad}\right)
\]
\[
\star e_{bc} = \frac{q^2}{[2]_q}(2e_{ad} + \mu e_{bc}), \quad \star e_{bd} = e_{bd}, \quad \star e_{cd} = -e_{cd}
\]
\[
\star e_{abc} = -e_a - \mu e_d, \quad \star e_{abd} = -e_b, \quad \star e_{acd} = q^2 e_c, \quad \star e_{bcd} = e_d
\]
and \(\star^2 = \text{id}\) on all degrees. The spaces of self-dual and antiselfdual 2-forms are each 3-dimensional. We define \(\star\) directly by these formulae for all invertible \(q^2 \neq \pm 1\).

Proof We first compute \(\epsilon\) as defined above. Its nonzero values are
\[
\epsilon_{1141} = -\epsilon_{1141} = -\epsilon_{1312} = \epsilon_{1411} = -\epsilon_{3121} = -\epsilon_{4111} = \epsilon_{4111} = -q^2 \epsilon_{1213} = q^2 \epsilon_{2131} = \mu
\]
\[
\epsilon_{1234} = -\epsilon_{1234} = -\epsilon_{1324} = \epsilon_{1423} = -\epsilon_{1432} = -q^2 \epsilon_{2134} = q^2 \epsilon_{2143} = \epsilon_{2314} = -\epsilon_{2341}
\]
\[
= -q^2 \epsilon_{2413} = \epsilon_{2431} = q^2 \epsilon_{3124} = -q^2 \epsilon_{3142} = -\epsilon_{3214} = \epsilon_{3241} = q^2 \epsilon_{3412} = -\epsilon_{3421} = -\epsilon_{4123}
\]
\[
= \epsilon_{4132} = q^2 \epsilon_{4213} = -\epsilon_{4231} = -q^2 \epsilon_{4312} = \epsilon_{4321} = 1.
\]
Using this, we define $\star$ (without normalisations) and compute $\star^2$ on $\Lambda^1$. We solve for $\lambda$ such that its two eigenvalues coincide. This has one solution which is such that $\eta$ is degenerate, and the one shown. We then find that $\star^2 \propto \text{id}$ in degree 2 also, and normalise $\star$ so that $\star^2 = \text{id}$ in all degrees. This only fixes the product $d_1 d_3$ but we chose these to reduce repeated factors in $\star$.

Also, it is clear by inspection that 

$$\Lambda^2_+ = \{e_{bd}, e_{ac}, e_{ad} + e_{bc}\}, \quad \Lambda^2_- = \{e_{cd}, e_{ab} - \mu e_{bd}, e_{ad} - q^{-2} e_{bc}\}.$$ 

⋄

Note that for the special value of $\lambda$ found in the proposition above, which we use from now on, we have 

$$\eta = e_b \otimes e_c + q^2 e_e \otimes e_b - q^2 (e_a \otimes e_d + e_d \otimes e_a + \mu e_d \otimes e_d) = e_b \otimes e_c + q^2 e_e \otimes e_b + q^2 (\frac{1 - q^2}{2} e_z \otimes e_z - \frac{q^4}{2} \theta \otimes \theta).$$

This is precisely (in some conventions) the metric of $q$-Minkowski space with $\theta$ the time direction. Likewise $\epsilon$ is basically that for exact differentials on $q$-Minkowski space in that context, see [6]. In our case however, the space is $SU_q(2)$ so there is no ‘time coordinate’. Instead, $\theta$ being a generator of $H^1$, we see that the ‘time direction’ is created by q-deformation of the differential calculus on $SU_2$ but is not exact, i.e. not $d$ of any time coordinate.

With these general-$q$ preliminaries, we specialise from now on to the reduced quantum group at the root given by $r = 3$. We obtain all specific formulae for this case, but expect similar features for all odd $r$ as discussed at the end. We actually obtain such results in the basis $\{a^m b^n c^k\}$ whereas the natural answers equally involve the variable $d = a^2 (1 + q^2 bc)$, to which we convert using the identities 

$$d^2 = a(b^2 c^2 - qbc + 1), \quad d^2 b = -q(ab^2 c - q^2 ab), \quad d^2 c = -q(abc^2 - q^2 ab)$$

$$db^2 = a^2 b^2, \quad dc^2 = a^2 c^2, \quad (bc - q) = q^2 (a^2 b^2 c^2 - q^2 a^2).$$

We say that a form is harmonic if it is closed and coclosed. The latter means in the kernel of $\delta = \star \circ d \circ \star$. Likewise, coexact means with respect to $\delta$, i.e. that the Hodge $\star$ of the form is exact.

**Proposition 4.3** At least for $r = 3$, the element $\theta$ is coexact. The element $\star \Theta$ is not closed. Moreover, $H^1, H^2$ have a basis of harmonic representatives, while the space of harmonic elements of $H^3$ is the 3-dimensional kernel of $\theta \wedge$. 

10
Proof For $H^1$ the first three representatives are already harmonic, while $n$ can be replaced by a harmonic 1-form

$$h_3 = qe_z - q^2 e_b d^2 b + e_c a^2 c.$$ 

One can also put $-q e_a$ for the first term since the difference is $\theta$ already in the basis. For $H^2$ the $m_1, \cdots, m_4$ are already harmonic since they are up to a linear combination self-dual or antiself-dual. They become part of our harmonic (anti)self-dual basis

$$h_1^+ = e_{bd} ac^r, \quad h_1^- = (e_{ab} - \mu_e b_d) a c^r, \quad h_2^+ = e_{ac} a^r b^r, \quad h_2^- = e_{cd} d^r b^r.$$ 

The remaining $m_5, m_6$ can be replaced by harmonic ones

$$h_3^+ = e_{bd} d^2 b + e_{ac} a^2 c - (e_{ad} + e_{bc}), \quad h_3^- = q e_{cd} d^2 b + (e_{ad} - q^{-2} e_{bc}).$$ 

which are respectively self-dual and antiself-dual. The facts on $\theta, \Theta$ can be directly verified. Finally, we take a basis of Harmonic 3-forms and eliminate all those that are exact. This leaves only three. Hence the dimension of the quotient is at most 3. On the other hand three harmonic 3-forms linearly independent in the quotient are provided by applying $\star$ to the above harmonic representatives of $H^1$. Up to coboundary and normalisation, this gives a basis by $\Theta, h_1^*, h_2^*$ as before and

$$h_3^* = e_{abcd} d^2 b + (e_{ac} a^2 c - (e_{ab} + e_{bc}).$$

We can also write $qe_{abc}$ for the last term here since the difference is a multiple of $\star \theta$ and this is exact.

Note also that in this harmonic basis the action of $\theta \wedge$ in Proposition 3.5 is more symmetric. It clearly sends harmonic forms to harmonic forms. In fact we find $\theta \wedge h_1 = h_1^- - q^{-2} h_1^+$, $\theta \wedge h_2 = h_2^+ - h_2^-$ as before and $\theta \wedge h_3 = h_3^+ - q^2 h_3^-$. Their complement has basis $h_1^+ + q^2 h_1^-$, $h_2^+ + h_2^-$ as before, and $h_3^+ + q^2 h_3^-$. The image of these under $\theta \wedge$ is now a multiple of $h_1^*, h_2^*, h_3^*$, with complement $\Theta$. ⊢

This immediately implies that there is no Hodge decomposition theorem (into a direct sum of exact, coexact and harmonic forms in each degree), precisely because $\theta$ is a nonzero element that is both coexact and harmonic.

Leaving now cohomology, we consider general forms and ‘wave equations’. As well as the operator $d + \delta$ who’s kernel is the harmonic forms (given that they map into different degrees), we also have the Laplacian $\Box = \delta d + d \delta$. 11
\begin{table}[h]
\centering
\begin{tabular}{l|ccccc}
\hline
$r = 3$ & $\Omega^0$ & $\Omega^1$ & $\Omega^2$ & $\Omega^3$ & $\Omega^4$ \\
\hline
All & 27 & 108 & 162 & 108 & 27 \\
Closed & 1 & 30 & 84 & 82 & 27 \\
Exact & 0 & 26 & 78 & 78 & 26 \\
Harmonic & 1 & 16 & 30 & 16 & 1 \\
ker $\Box$ & 13 & 33 & 40 & 33 & 13 \\
\hline
\end{tabular}
\caption{Number of independent forms of various types in each degree, for $r = 3$.}
\end{table}

**Proposition 4.4** For $r = 3$ the dimensions over $\mathbb{C}$ of the spaces of Harmonic forms and the kernel of $\Box$ are shown in Table 1. Also for comparison we remind the dimensions of the closed and exact forms in each degree as found in Section 3. Coclosed and coexact are given by reversing the relevant rows. In particular, harmonic $\subset$ ker $\Box$ is strict.

**Proof** This is direct computation once the matrices for the various operators above have been found explicitly. \hfill $\diamond$

Next we look in detail at physical ‘wave equations’. For spin 0 or scaler fields, we find that $\Box$ is not fully diagonalisable. This is related to the nonsemisimplicity of the Hopf algebra.

**Proposition 4.5** For $r = 3$ a full set of 13 zero-modes of $\Box$ in spin zero are

\[ 1, a, b, c, d, ab^2, a^2b, db^2, d^2b, ac^2, a^2c, dc^2, d^2c. \]

In addition there are 9 ‘massive’ modes of eigenvalue $6(q + 1)$ given by

\[ a^2, b^2, c^2, d^2, ab, ac, db, dc, bc - 1. \]

**Proof** Elementary computation once $\Box$ is defined. Note the zero modes $ab^2c - q^2 ab$ and $a^2c$ already featuring in the construction of harmonic forms above. \hfill $\diamond$

Note that we do not consider ‘orthogonality’ since the correct ‘reality’ properties are not clear when $q$ is a root of unity. Instead we are guided at our algebraic level by simplicity of expressions. It is worth noting that there is, however, necessarily a translation-invariant ‘integral’ functional in the Hopf algebra sense.

Next we solve the ‘spin 1’ or 1-form system. Following the notations in physics, we say that a 1-form is in *Lorentz gauge* if it is coclosed. It is in *temporal gauge* if it can be written entirely in terms of $e_b, e_c, e_z$ (i.e. no $\theta$ component when taking these four as basis). By number of ‘modes’ we will mean only the dimensions of the relevant spaces or quotient spaces (the number of linearly independent vectors in any basis).
**Proposition 4.6** Let \( \text{Max} = \delta d \) be the Maxwell operator on \( \Omega^1 \). Then for \( r = 3 \): (i) \( \ker \text{Max} \) is 54-dimensional, hence up to gauge equivalence (i.e. modulo exact 1-forms) there are 28 ‘true’ spin 1 zero modes, of which exactly 4 have zero curvature \( dA \) (namely the harmonic basis of \( H^1 \)). (ii) If we ‘gauge fix’ to Lorentz gauge by looking among coclosed 1-forms \( A \) then there are 32 zero modes but only 20 true ones when taken modulo exact. (iii) Ditto for temporal gauge. (iv) Every zero mode is gauge equivalent to the sum of a zero mode in Lorentz gauge and one in temporal gauge (with 12 modes in both gauges up to equivalence.)

**Proof** We compute the dimension of the kernel of \( \text{Max} \) as 54. It contains the exact 1-forms, so subtracting 26 gives the true dimension ‘modulo gauge’. Much more work gives explicit bases of representatives of the various types of modes constructed as kernels of suitable linear maps. Here we use the same method as for the cohomology computations, namely we first eliminate all elements of the relevant kernel which are exact. The remainder could still be linearly dependent in the quotient. We then painstakingly chose enough representatives to give the required dimensions, i.e. checking that together with the image of \( d_0 \) they form a basis of the original kernel. Of course this process is not unique (we choose the simplest representatives where possible). Explicitly, they are as follows. 20 modes obeying the gauge fixing are the elements \( \{h_1, h_2, h_3\} \) of the Harmonic basis of \( H^1 \) above (with zero curvature) plus the 13 modes of the form

\[
A = \theta f, \quad \Box f = 0
\]

(i.e. induced by spin 0 solutions as given above), and 4 more coclosed modes which we have to specify. E.g. the vector space of coclosed modes which are also in temporal gauge is 19 dimensional, reducing to 7 true modes in the quotient, of which 3 are \( \{h_1, h_2, h_3\} \) already counted. The remaining four are:

\[
\begin{align*}
A_1 &= e_z d(bc - q) - e_y b(bc - q^2) + q e_c d^2 c \\
A_2 &= q e_y a b^2 - e_y a b^2 c + e_c de ac \\
A_3 &= q e_y b^2 c - q e_b a b^2 + e_c d^2 a \\
A_4 &= e_z b c^2 - q e_b a^2 d + e_c dc^2.
\end{align*}
\]

Finally, we must complete the basis with 8 modes which are not, however, coclosed. We find that the dimension of the space of solutions in temporal gauge is (like for Lorentz gauge) 32 dimensional, reducing to 20 true temporal-gauge modes in the quotient. We have already used
7 of them above and can choose 8 more from among the remainder, e.g.

\[ A_5 = e_b a^2 c, \quad A_6 = e_c d^2 b, \quad A_7 = q e_c b^2 c - e_z a b^2, \quad A_8 = q e_b + e_z a^2 c, \quad A_9 = q^2 e_b a b^2 + e_z d a^2 \]
\[ A_{10} = e_c a b c - q e_z a^2 b, \quad A_{11} = e_0 d b^2 + q^2 e_z d^2 b, \quad A_{12} = e_b a^2 - q e_z a^2 c. \]

We can also use \( A_6' = e_z d b^2 \) in place of \( A_6 \) with the same curvature up to normalisation. In this way we may ‘patch’ the moduli of solutions into Lorentz and temporal gauge, with some overlap. We could equally chose 20 temporal gauge modes and complete with 8 more in Lorentz but not temporal gauge if we preferred. This means that there are 12 true modes which can be viewed in either gauge by a gauge transformation (but only 7 which can be transformed to a solution in both gauges simultaneously as explained above).

These results show several key features of the electromagnetic theory. First and foremost, there are 24 ‘electromagnetic’ modes with nonzero curvature \( F = dA \) obeying the source-less Maxwell equation (i.e. forming a basis with the zero curvature ones). They are the analogue of the photon self-propagation modes in usual physics. I.e. ‘there is light’. The remaining 4 modes of zero curvature indicate nontrivial topology and the existence of the ‘Bohm-Aharanov’ effect. Finally, we see that usual gauge fixing to ‘Lorentz gauge’ (where \( \delta A = 0 \)) does not work: not all solutions obey the gauge fixing condition. Likewise for temporal gauge fixing. Such problems can potentially plague any nontrivial gauge theory but here in our concrete model we see how the moduli space can instead be ‘covered by patches’ built from Lorentz and temporal gauge. Note also that two representatives in Lorentz gauge can only differ by \( df \) with \( f \in \text{ker} \Box \) and in usual electromagnetism this would be forced to be zero by boundary conditions at infinity (so that there would be a unique representative fixed by the gauge condition); in our case we do not have any such natural conditions, i.e. the possibility of nontrivial ‘Gribov ambiguities’\(^{[9]}\). This would be relevant to the quantum electromagnetic theory if one tried to impose gauge fixing in the functional integral.

**Proposition 4.7** Of the 28 true zero modes of \( \text{Max} \) for \( r = 3 \), exactly 16 have self-dual curvature and 16 antiself-dual curvature. Every zero mode is gauge equivalent to the sum of a self-dual and an antiself-dual zero modes (with the four zero curvature modes in both classes).

**Proof** We compute dimensions as kernels of suitable maps. Thus the space of 1-forms with self-dual curvature is 42 dimensional, reducing to 16 true modes in the quotient. Similarly for antiself-dual. We next proceed to find reasonable representatives in the space of 1-forms
modulo exact ones forming a basis as in the computations above. To this end, we also note that the dimension of the space of 1-forms which have self-dual curvature and are coclosed is 20-dimensional, reducing to 8 true modes in the quotient, including the 4 of zero curvature (the harmonic basis of $H^1$) already given. This leads us to 8 of our basis of 16 forms given by \{\theta, h_1, h_2, h_3\} and

$$A_1 = e_a a, \quad A_2 = e_a b, \quad A_3 = e_d c, \quad A_4 = e_d d$$

say (these are equivalent up to normalisation and coboundaries to coclosed modes but not themselves coclosed). We complete the basis of 1-forms with self-dual curvature by

$$A_5 = (\mu e_d + e_a)a^2 b + e_c abc, \quad A_6 = (\mu e_d + e_a)ab^2 + q^2 e_a b^2 c$$

$$A_7 = e_d a^2 c^2 + q^2 e_b c^2, \quad A_8 = e_d d^2 c + q e_b d c$$

$$A_9 = e_d b^2, \quad A_{10} = e_d a c^2, \quad A_{11} = e_b - e_a a^2 c, \quad A_{12} = e_d a d^2 b - q e_b d c^2.$$

These are all chosen to have simple expressions for their self-dual curvatures, namely (with $e_+ \equiv e_{ad} + e_{bc}$, and up to normalisations),

$$F_1 = e_+ a + q^2 e_{ac} c, \quad F_2 = e_+ b + q^2 e_{ac} d, \quad F_4 = e_+ d + q e_{bd} b, \quad F_3 = e_+ c + q e_{bd} a$$

$$F_5 = e_{ac} a, \quad F_6 = e_{ac} b, \quad F_7 = e_{bd} c, \quad F_8 = e_{bd} d$$

$$F_9 = e_{ac} d^2 b - q e_+ d b^2, \quad F_{10} = e_+ a c^2 - q e_{bd} a^2 c$$

$$F_{11} = e_+ a^2 c - q^2 e_{ac} a c^2 - e_{bd}, \quad F_{12} = e_{ac} + q e_{bd} d b^2 - e_+ d^2 b.$$

These are exact and coclosed (hence harmonic) 2-forms. Note that this is possible because the Hodge decomposition again does not hold, here in degree 2. We can similarly find 12 antiself-dual forms completing the zero curvature ones to a basis of the antiself-dual moduli space. One then checks that these 12, the above 12 self-dual modes and the 4 zero curvature modes are linearly independent modulo exact forms. This decomposition also holds before working modulo exact forms, with the 30 closed forms as the intersection of the two 42-dimensional spaces.

It turns out that we can also ‘patch’ the moduli of solutions of the sourceless Maxwell equations into Lorentz gauge and self-dual ones. Here the self-dual modes $A_5, \ldots, A_{12}$ are beyond the reach of the Lorentz gauge fixing condition, being linearly independent modulo exact forms to the basis of the Lorentz gauge-fixed solutions in Proposition 4.6. Similarly for temporal gauge.
Corollary 4.8 At least for $r = 3$, (i) every zero mode of $\text{Max}$ is gauge equivalent to the sum of one of the form $\theta f$ where $\Box f = 0$ and a self-dual one (with the mode $\theta$ in both classes). (ii) every zero mode of $\text{max}$ is gauge equivalent to the sum of one in temporal gauge and a self-dual one (with 8 modes including 4 zero curvature ones in the overlap). Similarly using antiself-dual modes.

Proof (i) We check that the 16 modes $\theta f$, $\Box f = 0$ and $\{h_1, h_2, h_3\}$ in the proof of Proposition 4.6 are linearly independent modulo exact forms from the $A_1, \cdots, A_{12}$ self-dual modes in Proposition 4.7. Also note that if we want to have as much as possible of the basis in Lorentz gauge then we could equally well use the coclosed self-dual modes

$$
A_1' = e_a - e_b a^2 b + q e_c (bc - q)c + q e_z abc \\
A_2' = q e_a b - \theta b + q e_z b^2 c - e_b a b^2 + e_c d (bc - q) \\
A_3' = e_d c + q e_z b c^2 - q e_b abc + q e_a a^2 c^2 \\
A_4' = e_d d - e_z (d (bc - q) - q^2 c) - q e_c d^2 c + e_0 (b^2 c + qa).
$$

These are gauge equivalent (up to normalisation) to the $A_1, \cdots, A_4$ in Proposition 4.7, giving a full set of 20 coclosed modes and a basis along with the $A_5, \cdots, A_{12}$.

(ii) For temporal gauge we find that there is similarly a 20-dimensional space of forms which are both in temporal gauge and have self-dual curvature, reducing to 8 in the quotient. They include 4 of zero curvature (so $H^1$ has a basis of representatives in temporal gauge) and all 8 are in fact gauge equivalent to the above 8 modes that were self-dual and renderable in Lorentz gauge (the $\{\theta, h_1, h_2, h_3\}$ and $A_1, \cdots, A_4$ (or $A_1', \cdots, A_4'$ as just discussed). So the self-dual $A_5, \cdots, A_{12}$ in Proposition 4.7 again complete to a full set of self-dual forms. This time we find that the 12 temporal gauge modes $A_1, \cdots A_{12}$ in Proposition 4.6 then complete to full set of 28 zero modes of $\text{Max}$. $\diamond$

Finally, we give examples of some source $J$ and solve the full Maxwell equation $\delta F = J$. Recall that the element $\theta$ is coexact as one would need for any source $J$.

Proposition 4.9 For $r = 3$, a basis of valid sources (i.e. in the image of $\text{Max}$) in the direction of $\theta$ is provided by

$$
\theta \{1, a, b, c, d\}.
$$

In particular, the element $\theta$ is a valid source and has a gauge field (not uniquely determined since we can add any of the above zero modes) given in Lorentz gauge by

$$
A = -\frac{q^2}{12} \theta bc (1 + bc) - \frac{qh}{12} (e_a + e_c a^2 c).
$$
Its curvature is

\[ F = \frac{q}{4} e_{cd} - \frac{\mu}{12} \left( (e_{ab} - e_{bd})d^2b + q(e_{cd} - e_{ac})a^2c \right) \]

**Proof** We first compute the dimension of the subspace of the image of Max of the form \( \theta f \) as 5. This is found as the dimension of the image of Max minus that of the image of \( T \circ \text{Max} \) where \( T \) is the linear map whose kernel is spanned by \( \theta \) over \( \mathcal{A} \). We then solve explicitly for the example \( J = \theta \) in Lorentz gauge. Note that the second term in \( A \) is topological, being a multiple of the fourth basis element of \( H^1 \) in Section 3. It can be omitted so that \( A \) is itself \( \theta \) times a function (if we abandon the Lorentz gauge), without changing the curvature. \( \diamond \)

According to the physical picture mentioned above, \( \theta \) could be viewed as a Minkowski time direction. So there is a ‘current’ in the cotangent space of \( SU_q(2) \) in this direction (but no actual current flow as time is not a coordinate) generating this gauge field. In usual Maxwell theory such a current in the time direction corresponds to a static electric charge density. Accordingly, the source \( \theta \) can be viewed as a uniform charge density over the noncommutative \( S^3 \) leading to gauge field and (electric) curvature field as stated. There are of course many other sources, the dimension of the image of Max being 54 (for \( r = 3 \)).

**Proposition 4.10** For \( r = 3 \), the subspace of sources in the ‘spatial’ directions spanned by \( e_z, e_b, e_c \) is 40-dimensional. Those purely along each of the three directions have bases

\[ e_z, \quad e_b \{1, e^2, d^2, dc, dc^2, d^2c\}, \quad e_c \{1, a^2, b^2, ab, ab^2, a^2b\}. \]

In particular, the gauge fields and their curvatures

\[
\begin{align*}
A_1 &= \frac{q^2}{6} e_z, & F_1 &= \frac{q^2}{6} \mu e_{bc} \\
A_2 &= \frac{q^2}{6} e_b, & F_2 &= -\frac{\mu}{6} (q^2 e_{ab} + e_{bd}) \\
A_3 &= -\frac{1}{6} e_b d^2b, & F_3 &= -\frac{\mu}{6} (e_{bc} d^2b + (q^2 e_{ab} + e_{bd}) db^2)
\end{align*}
\]

are solutions for the sources \( e_z, e_b, e_c \) respectively.

**Proof** Here we compute the dimension of the subspace of spatial currents as the image of Max as the dimension of the image of Max minus that of the image of \( S \circ \text{Max} \) where \( S \) is the linear map whose kernel is the spatial directions. Similarly along each of the directions \( e_z, e_b, e_c \) separately we obtain dimensions 1,6,6 respectively. We then find the right number of independent modes. Finally, we solve for some of these in temporal gauge and exhibit the solutions for the three
$r = 5$ | $\Omega^0$ | $\Omega^1$ | $\Omega^2$ | $\Omega^3$ | $\Omega^4$
---|---|---|---|---|---
All | 125 | 500 | 750 | 500 | 125
Closed | 1 | 128 | 378 | 376 | 125
Exact | 0 | 124 | 372 | 372 | 124
Harmonic | 1 | 36 | 70 | 36 | 1

Table 2: Number of independent forms of various types in each degree, for $r = 5$.

Maxwell | $r = 3$ | $r = 5$
---|---|---
All zero modes | 28 (54) | 68 (192)
Coclosed | 20 (32) | 52 (84)
Temporal | 20 (32) | 52 (84)
Cocl. $\cap$ Temp. | 7 (19) | 19 (51)
self − dual | 16 (42) | 36 (160)
zero curv. | 4 (30) | 4 (128)
Cocl. $\cap$ s.d. | 8 (20) | 20 (52)
Temp. $\cap$ s.d. | 8 (20) | 20 (52)
$\theta f$ modes | 13 (13) | 33 (33)
All sources | 54 | 308
spatial sources | 40 | 216
$\theta f$ sources | 5 | 17

Table 3: Summary of electromagnetic theory for $r = 3, 5$. Number of independent solutions of the sourceless Maxwell equations modulo exact forms (in brackets before making the quotient). We also summarize the types of valid sources.

constant sources directions. Again, these solutions are not unique since we can add any of the above zero modes of Max. ◊

In classical electrodynamics spatial sources would correspond to currents inducing magnetic configurations. Another example is the source $e_b b^2$ having a solution with curvature proportional to $(e_{cd} + q e_{ac}) b^2$. From these various (and other) solutions we see that the natural electric and magnetic curvature directions under this time/space decomposition are spanned by

$$\Lambda^2_E = \{e_{ad}, e_{ab} - e_{bd}, e_{cd} - e_{ac}\}, \quad \Lambda^2_B = \{e_{bc}, q^2 e_{ab} + e_{bd}, e_{cd} + q e_{ac}\}$$

respectively.

It should be mentioned in conclusion that other odd roots appear to give similar features as the $r = 3$ case. The preliminary Table 2 summarizes the form dimensions for $r = 5$, after which Table 3 summarizes the Maxwell theory above and the corresponding numbers for $r = 5$. From these and further inspection we find the same qualitative features, e.g. all solutions can be
written as sums of self-dual and antiself-dual solutions with overlap given by the zero curvature modes (both modulo exact 1-forms and before taking the quotient); the temporal and Lorentz gauges patch the moduli space, etc. Also, $\theta, e_z, e_b, e_c$ are all valid sources of electric and magnetic type among others in the numbers shown. From the tables we note another novel feature that we expect for all odd $r$, namely a linear isomorphism between harmonic 1-forms and self-dual solutions modulo exact forms.

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