Classification and quantification of entanglement through wedge product and geometry

Soumik Mahanti∗, Sagnik Dutta and Prasanta K Panigrahi

Department of Physical Sciences, Indian Institute of Science Education and Research Kolkata; Mohanpur, West Bengal 741246, India

*Author to whom any correspondence should be addressed.

E-mail: soumikmh1998@gmail.com, sagnikdutta17@gmail.com and pprasanta@iiserkol.ac.in

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Abstract

The wedge product of post-measurement vectors of a two-qubit state gives rise to a parallelogram, whose ‘area’ has been shown to be equivalent to the generalized 1-concurrence measure of entanglement. In multi-qudit systems, the wedge product of post-measurement vectors naturally leads to a higher dimensional parallelepiped which yields a modified faithful entanglement measure. Our new measure fine grains the entanglement monotone, wherein different entangled classes manifest with different geometries. We present a complete analysis of the bipartite qutrit case considering all possible geometric structures where three entanglement classes of pure bipartite qutrit states can be identified with different geometries of post-measurement vectors: three planar vectors; three mutually orthogonal vectors; and three vectors that are neither planar and not all of them are mutually orthogonal. It is further demonstrated that the geometric condition of area and volume maximization naturally leads to the maximization of entanglement. The wedge product approach uncovers an inherent geometry of entanglement and is found to be very useful for the characterization and quantification of entanglement in higher dimensional systems.

1. Introduction

Quantum entanglement has a fundamental importance in both applications of quantum information processing and the foundational understanding of quantum theory [1]. It is the most useful resource for various quantum communication protocols like quantum teleportation [2–6], dense coding [7–12], and secret sharing [13–15], which gives the quantum advantage over the classical communication protocols. Therefore, it is of practical interest to quantify this resource to estimate the efficiency of such protocols. Various aspects of quantum entanglement have been studied extensively [16]. However, the classification and quantification of entangled states are still not well understood in higher dimensional systems [17–22]. Entangled qudits show an advantage over entangled two-level systems in several communication and cryptographic protocols [23–25], exhibiting stronger nonlocality in maximally entangled states [26]. In short, the study of characterization and quantification of entanglement is a very rich area of research even today [27, 28], particularly from a geometric perspective, which can provide insight into the distributive nature of entanglement.

Concurrence, introduced by Hill and Wootters, is a faithful measure of entanglement that provides the necessary and sufficient conditions of separability for a pair of qubits [29]. Rungta et al extended the concurrence to bipartite pure states in $d_1 \otimes d_2$ systems [30]. Various geometry-based approaches have also been proposed to quantify entanglement [31–37]. Bhaskara and Panigrahi used the wedge product framework and Lagrange-Brahmagupta identity to provide for an entanglement measure based on geometry for pure states [33]. The Bhaskara–Panigrahi measure [33–35, 38] of entanglement across any bipartition of a multiparty state is the amount of total area formed by the post-measurement vectors of a subsystem. Therefore, if across any bipartition all the post-measurement vectors of a subsystem are parallel then the total area formed by them is zero; indicating the state is separable across that bipartition. A pure composite state is separable if it is separable across all possible bipartitions. This geometrical condition greatly simplifies the separability criteria in multi
Two qutrit systems have three entangled classes \( \mathcal{E} \). We introduce the Bhaskara-Panigrahi measure for entanglement through an example of two-qubit pure states. In any bipartition, wedge products between all post-measurement vectors construct a hypervolume of entanglement between any two arbitrary states of higher dimension. The geometric measure is generalized for pure multi-qudit states in section 6. The entanglement maximization conditions from geometry are also deduced for each scenario. Our study uncovers the rich geometry of entangled states and classes and provides a novel perspective to look at multi-party entanglement. Entangled qutrits are a better resource in many information theoretic tasks while entanglement classes also exhibit different resourcefulness under different tasks. For example, GHZ states have better teleportation fidelity than W states, whereas W states are more robust against particle loss. Our approach to use the geometrical criteria for separability and maximization are simpler and elegant in many cases, and easier to deduce. The case-by-case analysis for bipartite qutrit states establishes a deep connection between the ‘entanglement classes’ and the ‘geometry of post-measurement vectors,’ which can be readily extended to bipartite qudits to understand and visualize their entanglement classes. It will be interesting to see whether such an extension is feasible for entanglement families in multi-party settings. We organize the paper as follows. We give an overview of the Bhaskara-Panigrahi measure of entanglement and wedge product formalism for pure states in section 2. Our modified entanglement measure for two qutrit systems is presented in section 3, and we prove why it is a faithful measure of entanglement. In section 4, we discuss how entanglement is classified and what are the entangled classes for two qutrits. In section 5, we describe the geometric structures of pure two qutrit entangled states and their connection to the classification of entanglement. The entanglement maximization conditions from geometry are also deduced for each scenario. The geometric measure is generalized for pure multi-qudit states in section 6.

2. Bhaskara-Panigrahi measure of entanglement

We introduce the Bhaskara-Panigrahi measure for entanglement through an example of two-qubit pure states. A general two qubit pure state can be written as \( |\psi\rangle_{AB} = a|00\rangle_{AB} + b|01\rangle_{AB} + c|10\rangle_{AB} + d|11\rangle_{AB} \) where \( a, b, c, d \in C \), and \( |a|^2 + |b|^2 + |c|^2 + |d|^2 = 1 \). Hence, \( |\psi\rangle_{AB} = |0\rangle_A (a|0\rangle_B + b|1\rangle_B) + |1\rangle_A (c|0\rangle_B + d|1\rangle_B) \) are called the post-measurement vectors for the subsystem \( B \). The entanglement between the subsystems \( A \) and \( B \) is defined as the modulus of the wedge product of the post-measurement vectors as

\[
E = | \langle 0_A | \psi \rangle \wedge \langle 1_A | \psi \rangle | = |ad - bc|.
\]

Therefore, the condition for separability is \( \frac{a}{c} = \frac{b}{d} \) (or similarly, \( \frac{a}{c} = \frac{d}{b} \) if we consider post-measurement vectors for subsystem \( A \)). This implies, if the post-measurement vectors (of any subsystem) are parallel to each other, the state is separable. The faithful entanglement measure ‘Concurrence’ has a direct relation with this quantity

\[
C = 2 | \langle 0_A | \psi \rangle \wedge \langle 1_A | \psi \rangle | = 2 |ad - bc|.
\]

This definition of concurrence based on wedge products facilitates a geometric description for entanglement, as \( | \langle 0_A | \psi \rangle \wedge \langle 1_A | \psi \rangle | \) is the magnitude of the area spanned by the vectors \( \langle 0_A | \psi \rangle \) and \( \langle 1_A | \psi \rangle \) as shown in figure 1. Here, the vector \( |0\rangle_A \) is represented in the \( X \)-axis and \( |1\rangle_A \) is represented in the \( Y \)-axis. The post-measurement vectors are represented as brown arrows, and the blue-filled area is the area of the parallelogram constructed by the two vectors. In two-dimensional geometry, the wedge product corresponds to the area of the parallelogram (considering only the modulus). The concurrence, or the amount of entanglement is twice this area in value.
This geometric representation naturally reflects the condition for maximum entanglement, namely when the area of the parallelogram is maximum. This condition is satisfied when the post-measurement vectors are perpendicular and equal in magnitude, or the parallelogram is a square. For example, if one post-measurement vector is parallel to the X-axis \((0_B|\psi\rangle)\) and another is parallel to the Y-axis \((1_B|\psi\rangle)\) having equal length, the area will be maximum. The corresponding state is \(\frac{1}{\sqrt{2}} (|00\rangle_{AB} + |11\rangle_{AB})\), which we know is the maximally entangled two-qubit state.

This separability based on wedge product for multiparty system has been defined for each individual bipartition \([33]\). A composite state \(|\psi\rangle\) can be written about any bi-partition \(A|B\) as

\[
|\psi\rangle = \sum_{i=0}^{d-1} |i_A\rangle \otimes \langle i| \psi\rangle_B.
\]

Here \(|i\rangle\) is an orthonormal basis for subsystem \(A\). The state will be separable in that bi-partition if all the post-measurement vectors \(\langle i| \psi\rangle_B\) are parallel to each other. The generalized I-concurrence across the bi-partition given in terms of wedge product is found to be

\[
C_{AB}^2 = 4 \sum_{i<j} |\langle i| \psi\rangle \wedge \langle j| \psi\rangle|^2.
\]

Geometrically this implies the squared sum of the area of all parallelograms constructed by each pair of post-measurement vectors is the square of generalized I-concurrence. In the next sections, we introduce our modified entanglement measure which geometrically represents a higher dimensional parallelepiped with generalized volume elements and generalized area elements. This modification enables us to fine-grain the entanglement monotone and opens a new insight into the classification of pure entangled states in a novel geometric way.

### 3. Two qutrit entanglement measure

A pure two qutrit state can be written in the computational basis of Alice and Bob as

\[
|\psi\rangle = a|00\rangle + b|01\rangle + c|02\rangle + p|10\rangle + q|11\rangle + r|12\rangle + x|20\rangle + y|21\rangle + z|22\rangle.
\]

Where the co-efficients are complex numbers with the normalization condition \(|a|^2 + |b|^2 + |c|^2 + |p|^2 + |q|^2 + |r|^2 + |x|^2 + |y|^2 + |z|^2 = 1\). Rewriting the state in Alice’s computational basis, we get

\[
|\psi\rangle = |0_A\rangle |\eta_{0B}\rangle + |1_A\rangle |\eta_{1B}\rangle + |2_A\rangle |\eta_{2B}\rangle.
\]

If Alice measures her subsystem in the computational basis, the state of Bob will be \(\frac{|\eta_{iB}\rangle}{|\eta_{i}\rangle} |\eta_{iA}\rangle\) corresponding to Alice’s measurement result \(i\). Hence we call the vectors \(|\eta_{iB}\rangle\) as post-measurement vectors (unnormalized) of...
operations emerged as local operations. The idea behind this is that distant parties can in
As entanglement appeared to be a very useful resource for quantum communication, the most important set of
4. Entanglement classes for bipartite qutrit systems
In terms of the continuous parameters
\[
E_G = 9|\det A|^2 + \left[ \frac{1}{2} (\| br - cq \|^2 + \| cp - ar \|^2 + \| az - bx \|^2 + \| ay - bz \|^2) \right].
\]
where \( A = \begin{pmatrix} a & b & c \\ p & q & r \\ x & y & z \end{pmatrix} \), and the \( \| . \| \) sign is used to represent the Euclidean norm of a column vector. Expanding
by terms, the full expression is the following
\[
E_G = 9[a(qz - ry) - b(pz - rx) + c(py - qx)]^2 \\
+ 2((br - cq)^2 + (cp - ar)^2 + (aq - bp)^2) \\
+ 2((bz - cy)^2 + (cx - az)^2 + (ay - bx)^2) \\
+ 2(qz - ry)^2 + (xz - pz)^2 + (py - qx)^2).
\]
Geometrically \( |\eta_0\rangle, |\eta_1\rangle, \) and \( |\eta_2\rangle \) are three vectors in \( C^3 \) Hilbert space. The first term of the equation (5)
geometrically represents the modulus square of the volume of the parallelepiped formed by \( |\eta_0\rangle, |\eta_1\rangle, \) and \( |\eta_2\rangle \).
The rest three terms are squares of all six surface areas of the 3-dimensional parallelepiped. The factor of 9 has
been put before the volume element to restrict the measure \( E_G \) to take a value between 0 and 1. It is 1 for the
maximally entangled states of the form \( \frac{1}{\sqrt{3}} [(00) + (|11\rangle + (22))] \), and 0 for separable states. Any measure of
entanglement is considered a ‘good’ measure if it follows the following three conditions.

- The measure must give a value of 0 for any separable state.
- Value of the measure should be local unitary invariant.
- Average entanglement value after local operation and classical communication (LOCC) must not increase.

We have already shown that \( E_G \) satisfies the first condition since parallelism of all vectors makes the wedge
product zero. Under local unitary operation on the second subsystem, the length and the angle between the post-
masurement vectors of that subsystem are unchanged. Hence the wedge products, and consequently \( E_G \) remain
invariant. Again, it is seen that the entanglement measure \( E_G \) is parity symmetric, which implies that its value is
independent of whether one takes the post-measurement vectors of subsystem 1 or 2. Hence, local unitary
operation on the first subsystem will also leave the value of \( E_G \) unchanged. Therefore, the second condition is also
satisfied. We only consider a single copy pure state scenario, for which local unitary invariance is the same as LOCC
invariance [16]. Hence, the entanglement measure \( E_G \) satisfies all the criteria for a faithful measure of entanglement.

4. Entanglement classes for bipartite qutrit systems
As entanglement appeared to be a very useful resource for quantum communication, the most important set of
operations emerged as local operations. The idea behind this is that distant parties can influence only their local
subsystem. Hence local operations and classical communication between distant parties are the only available
means for the manipulation of entangled states for communication purposes. Nielsen provided a significant result
regarding transformation criteria between two bipartite pure states under LOCC [40]. But exact transformation
under LOCC lacks continuity, and the Nielsen condition is not applicable for multiparty scenarios. The necessary and sufficient condition for LOCC transformation between two pure states is if they can be transformed under local unitary operation. W Dür et al. proposed another classification encapsulating qualitative features of entanglement in terms of stochastic local operation and classical communication (SLOCC) [41]. Two pure states are said to be equivalent if they can be transformed into each other by LOCC with some nonzero probability. Mathematically, if \( |\psi\rangle = A_1 \otimes \ldots \otimes A_n |\phi\rangle \), with \( A_i \) being reversible operators, then \( |\psi\rangle \) and \( |\phi\rangle \) are equivalent. Entangled states that cannot be transformed into each other by SLOCC belong to different entangled classes. Previous results in the search of entangled classes have shown that there are three inequivalent entangled classes for three-qubit pure states [39]. Bipartite pure states in \( d \otimes d \) dimension have \( d \) entangled classes of states [16]. Several works have been done regarding classification and characterization of entangled states in pure and mixed scenarios [42–47]. We present the classification of bipartite qutrit entangled states based on our geometric measure of entanglement. Our analysis considers pure states with a single copy available scenario.

5. Geometrical interpretation of entanglement classes

From equation (4), it is seen that there are three post-measurement vectors, belonging to \( \mathbb{C}^3 \). For geometrical representation in real space, \( |0\rangle \) is represented in the \( X \)-axis, \( |1\rangle \) in the \( Y \)-axis, and \( |2\rangle \) in the \( Z \)-axis. Post-measurement vector of the form \( a|0\rangle + b|1\rangle + c|2\rangle \) is represented as a point vector in \( \mathbb{R}^3 \). All the mathematical results presented here are valid for any complex number, but this \( 3 \)-dimensional real representation captures all the intricate geometries of the entangled states in bipartite qutrit systems. Below, three theorems are presented to show how the entanglement classes are related to the geometry.

**Theorem 1.** Three planar vectors remain in the same plane after local unitary operation in any of the subsystems.

**Proof.** From equation (3), an arbitrary two qutrit state can be written as \( |\psi\rangle = |0\rangle_A |\eta_0\rangle_B + |1\rangle_A |\eta_1\rangle_B + |2\rangle_A |\eta_2\rangle_B \). Applying any local unitary in the second subsystem does not change the angle between the vectors \( |\eta_0\rangle_B \), \( |\eta_1\rangle_B \), and \( |\eta_2\rangle_B \). Hence, only the orientation of the parallelepiped can change but not the shape. Now, let Alice apply an arbitrary local unitary operation \( U \) in her subsystem. \( U(|0\rangle_A |\psi\rangle) = U(0|0\rangle_A + x_0|1\rangle_A + x_2|2\rangle_A) \), \( U(|1\rangle_A |\psi\rangle) = x_1|0\rangle_A + x_1|1\rangle_A + x_2|2\rangle_A \), and \( U(|2\rangle_A |\psi\rangle) = x_2|0\rangle_A + x_2|1\rangle_A + x_2|2\rangle_A \).

Therefore, \( (U \otimes I)|\psi\rangle = |0\rangle_A (x_{00}|0\rangle_B + x_{01}|1\rangle_B + x_{02}|2\rangle_B) + |1\rangle_A (x_{10}|0\rangle_B + x_{11}|1\rangle_B + x_{12}|2\rangle_B) + |2\rangle_A (x_{20}|0\rangle_B + x_{21}|1\rangle_B + x_{22}|2\rangle_B) \).

If the vectors lie in a plane, then without loss of generality, \( |\eta_2\rangle_B = m|\eta_0\rangle_B + n|\eta_1\rangle_B \). Therefore, \( |\chi_{0B}\rangle = (x_{00} + mx_{00})|\eta_0\rangle_B + (x_{01} + nx_{01})|\eta_1\rangle_B \), \( |\chi_{1B}\rangle = (x_{10} + mx_{10})|\eta_0\rangle_B + (x_{11} + nx_{11})|\eta_1\rangle_B \), and \( |\chi_{2B}\rangle = (x_{20} + mx_{20})|\eta_0\rangle_B + (x_{21} + nx_{21})|\eta_1\rangle_B \).

So, the post-measurement vectors \( |\chi_0\rangle_A \), \( |\chi_1\rangle_A \), and \( |\chi_2\rangle_A \) lie in the same plane as do \( |\eta_0\rangle_B \), \( |\eta_1\rangle_B \), and \( |\eta_2\rangle_B \). Therefore, the planar structure is retained after applying local unitary to either of the subsystems. \( \square \)

**Theorem 2.** Three orthogonal vectors remain orthogonal after local unitary operation in any of the subsystems.

**Proof.** As stated in the proof of theorem 1, only local unitary operation on the first subsystem can change the shape of the parallelepiped. If a local unitary operator \( U \) is applied on the first subsystem, the state transforms according to equation (8) as

\[
(U \otimes I)|\psi\rangle = |0\rangle_A (x_{00}|0\rangle_B + x_{01}|1\rangle_B + x_{02}|2\rangle_B) + |1\rangle_A (x_{10}|0\rangle_B + x_{11}|1\rangle_B + x_{12}|2\rangle_B) + |2\rangle_A (x_{20}|0\rangle_B + x_{21}|1\rangle_B + x_{22}|2\rangle_B) \]

Given the condition, \( \langle \eta_i|\eta_i\rangle_B = 0 \), for \( i \neq j \), we need to show that \( \langle \chi_i|\chi_j\rangle_B = 0 \), for \( i \neq j \).

\[
UU^+ = \begin{pmatrix}
0 & x_{00} & x_{01} & x_{02} \\
0 & x_{10} & x_{11} & x_{12} \\
x_{00} & x_{10} & x_{11} & x_{12} \\
x_{01} & x_{11} & x_{12} & x_{12}
\end{pmatrix} = I.
\]
Therefore, the off-diagonal terms are zero and we get the following conditions:
\[
\begin{align*}
\langle x_0^* | x_1 \rangle &= 0, \\
\langle x_0^* | x_2 \rangle &= 0, \\
\langle x_1^* | x_2 \rangle &= 0.
\end{align*}
\]
Hence it is proved that the vectors remain orthogonal after applying local unitary to either of the subsystems.

**Theorem 3.** Three mutually non-orthogonal vectors are local unitarily equivalent to one pair or two pairs of orthogonal vectors.

**Proof.** Like in the previous cases, under the local unitary \( U \) on the first subsystem, the state takes the form of equation (9). Let \( \langle \psi_1 | \psi_2 \rangle = 1 \), \( \langle \psi_1 | \psi_3 \rangle = m \), \( \langle \psi_2 | \psi_3 \rangle = n \). This gives:
\[
\begin{align*}
\langle x_0 | x_1 \rangle &= x_0^*(2x_1 + n^*x_2) + x_0^*(x_0^* + mx_1) + x_0^*(nx_0 + m^*x_2), \\
\langle x_0 | x_2 \rangle &= x_0^*(x_1 + n^*x_2) + x_0^*(x_0^* + mx_2) + x_0^*(nx_0 + m^*x_1), \\
\langle x_1 | x_2 \rangle &= x_1^*(x_1 + n^*x_2) + x_1^*(x_0^* + mx_2) + x_1^*(nx_0 + m^*x_1).
\end{align*}
\]
Now, let \( l = m = 0 \), and \( n = 0 \). Then from equation (11), \( \langle x_0 | x_1 \rangle = n^*x_0^*x_2 + nx_{02}^*x_{10} \), \( \langle x_0 | x_2 \rangle = n^*x_0^*x_2 + nx_{02}^*x_{10} \), and \( \langle x_1 | x_2 \rangle = n^*x_0^*x_2 + nx_{02}^*x_{10} \). In general, all three inner products are nonzero unless \( n = 0 \). Hence, two pairs of orthogonal vectors can be local unitarily transformed into three mutually non-orthogonal vectors, and vice versa. Similarly, taking one orthogonal pair of vectors, it can be shown that local unitary operation can transform it into three mutually non-orthogonal vectors. Therefore, the proof is complete.

It is easily seen that the new measure of entanglement based on the wedge product of complex vectors constructs different geometry to different entangled states. Under local unitary, the geometry of the post-measurement vectors can not be transformed arbitrarily. For pure states, local unitary transformation is equivalent to LOCC transformation. Since any of the three geometries has zero probability to be transformed into another under LU, they are not connected by stochastic LOCC. Hence, they constitute different entangled classes. These theorems encompass all possible geometry that can arise from the 3d geometry of the parallelepiped formed by the post-measurement vectors. Our results show how simple 3d geometry can distinguish between the different classes of two qutrit entangled states.

The two qutrit pure states have nine basis elements. Taking a linear combination of two or more terms, we present a case-by-case analysis for all possible states and their geometric shapes. We carry on the analysis in a similar spirit as done by Pan et al [39]. The geometrical shapes, entanglement maximization conditions, and entangled types are presented in table formats.

### 5.1. Two term states

The nine terms in the computational basis of two qutrits can be listed as \( \begin{pmatrix} 00 & 01 & 02 \\ 10 & 11 & 12 \\ 20 & 21 & 22 \end{pmatrix} \). Picking a combination of any two terms from the same row or same column will not be entangled. Entangled states consisting of any two terms have the same geometric shape. For the example state in table 1, the post-measurement vectors are along the X and Y axis, and the shape is of a rectangle with the side lengths \( |a| \), and \( |q| \) respectively. The area is maximum, if each side is of equal length to make it a square. The entanglement maximization condition is thus \( |a| = |q| \).

### 5.2. Three terms

Picking three terms from the basis can be done in three inequivalent ways. Their geometric shape and entanglement types are shown in table 2. From the geometrical shape, the maximum entanglement value found for the planar shape is 0.5 but can be 1 for non-planar structures. These two structures give two different types of entanglement classes. The planar structure corresponds to the Type I entanglement class and the non-planar structure with three mutually orthogonal pairs is of the Type II entanglement class.

### 5.3. Four term states

Four terms can be picked in 5 inequivalent ways. Examples of each type of state and their properties are described in table 3. Among these four-term states, we find a new geometry of entangled state (third row of table 3), that was not present for three-term cases. Geometrically this state has a non-planar shape with at least one pair of vectors non-orthogonal. This type of entangled state is named Type III entanglement.
Table 1. Geometric structure and entanglement type of two-term states.

| Example State ($\psi$) | Max. Ent. Value | Figure (Example) | Geometry Structure | Ent. Class |
|------------------------|-----------------|------------------|-------------------|-----------|
| $a|00\rangle + q|11\rangle = |0\rangle_A (a|0\rangle_B + |1\rangle_A (q|1\rangle_B)$ | $E_{\text{Gme}} = \frac{1}{2}$ Condition: $|a| = |q|$ | Planar | No. of Orthogonal pair (Op): $Op = 1$ | Type I |

Diagram: 3D geometric representation of the state with axes labeled $X$, $Y$, and $Z$.
Table 2. Geometric structure and entanglement type of three-term states.

| Example State ($\psi$) | Max. Ent. Value | Figure (Example) | Geometry Structure | Ent. Class |
|------------------------|------------------|-----------------|-------------------|-----------|
| $a|00\rangle + q|11\rangle + z|22\rangle = |0\rangle_a(a|0\rangle + |1\rangle_a(q|1\rangle + |2\rangle_a(z|2\rangle)$ | $E_{\text{Ent}} = 1$ | Condition: $|a| = |q| = |z| = \frac{1}{\sqrt{3}}$ | Non-Planar | $\text{Op} = 3$ | Type II |
| $a|00\rangle + q|01\rangle + z|10\rangle = |0\rangle_a(a|0\rangle + b|1\rangle + |1\rangle_a(p|0\rangle$ | $E_{\text{Ent}} < \frac{1}{2}$ | Remarks: Maximizes as $a \rightarrow 0; |b|^2, |p|^2 \rightarrow \frac{1}{2}$ | Planar | $\text{Op} = 0$ | Type I |
| $a|00\rangle + q|11\rangle + z|22\rangle = |0\rangle_a(a|0\rangle + b|1\rangle_a + |2\rangle_a(z|0\rangle$ | $E_{\text{Ent}} = \frac{1}{2}$ | Condition: $|a|^2 + |b|^2 = |z|^2 = \frac{1}{2}$ | Planar | $\text{Op} = 1$ | Type I |
Table 3. Geometric structure and entanglement type of four-term states.

| Example State \((\psi)\) | Max. Ent. Value | Figure (Example) | Geometry Structure | Ent. Class |
|--------------------------|-----------------|------------------|-------------------|------------|
| \(a|00\rangle + b|01\rangle + c|02\rangle + p|10\rangle = |0\rangle_s(a|0\rangle_a + b|1\rangle_a + c|2\rangle_a) + 1\rangle_s(p|0\rangle_a)\) | \(E_{\text{Ent}} < \frac{1}{2}\) Remarks: Maximizes as \(a \to 0\); \(|b|^2 + |c|^2\), \(|p|^2 \to \frac{1}{2}\) | Planar No. of Orthogonal pair\((\text{Op})\): \(\text{Op} = 0\) | Type I |
| \(a|00\rangle + b|01\rangle + p|10\rangle + r|12\rangle = |0\rangle_s(a|0\rangle_a + b|1\rangle_a) + |1\rangle_s(p|0\rangle_a + r|2\rangle_a)\) | \(E_{\text{Ent}} < \frac{1}{2}\) Remarks: Maximizes as \(p \to 0\); \(|a|^2 + |b|^2\), \(|r|^2 \to \frac{1}{2}\) | Planar No. of Orthogonal pair\((\text{Op})\): \(\text{Op} = 0\) | Type I |
| \(a|00\rangle + b|01\rangle + p|10\rangle + q|22\rangle = |0\rangle_s(a|0\rangle_a + b|1\rangle_a) + |1\rangle_s(p|0\rangle_a) + |2\rangle_s(q|2\rangle_a)\) | \(E_{\text{Ent}} < 1\) Remarks: Maximizes as \(a \to 0\); \(|b|^2\), \(|p|^2, |q|^2 \to \frac{1}{2}\) | Non-Planar No. of Orthogonal pair \((\text{Op})\): \(\text{Op} = 2\) | Type III |
| \(a|00\rangle + b|01\rangle + p|10\rangle + q|11\rangle = |0\rangle_s(a|0\rangle_a + b|1\rangle_a) + |1\rangle_s(p|0\rangle_a + q|1\rangle_a)\) | \(E_{\text{Ent}} = \frac{1}{2}\) Condition: \(|a|^2 + |b|^2 = |p|^2 + |q|^2 = \frac{1}{2}\); \(a^*p + b^*q = 0\) | Planar No. of Orthogonal pair\((\text{Op})\): \(\text{Op} = 0/1\). | Type I |
| Example State (|ψ⟩) | Max. Ent. Value | Figure (Example) | Geometry Structure | Ent. Class |
|----------------|----------------|-----------------|------------------|----------|
| $a|00⟩ + b|01⟩ + r|12⟩ + z|22⟩ = |0⟩, $Δ(a|0⟩_b + b|1⟩_a) + |1⟩, $Δ(r|2⟩_a) + |2⟩, $Δ(z|2⟩_a)$ | $E_{\text{Giv}} = \frac{1}{2}$ Condition: $|a|^2 + |b|^2 = |r|^2 + |z|^2 = \frac{1}{2}$ | Planar No. of Orthogonal pair (Op): $Op = 2$ | Type I |

Table 3. (Continued.)
Table 4. Geometric structure and entanglement type of five-term states.

| Example State (\(\psi\)) | Max. Ent. Value | Figure (Example) | Geometry Structure | Ent. Class |
|--------------------------|----------------|-----------------|--------------------|-----------|
| \[a(00) + b(01) + c(02) + p(10) + q(11) = |0\rangle, a(0) + b(1) + c(2) + p(1) + q(1)\] | \[E_{\text{Ent}} = \sum \text{Condition: } |a|^2 + |b|^2 + |c|^2 = |p|^2 + |q|^2 = \frac{1}{2}, \]
|                          | \[a^* p + b^* q = 0.\] |
|                          | \[E_{\text{Ent}} = 1 \text{ Condition: } |a|^2 + |b|^2 + |c|^2 = \frac{1}{2}, a^* p + b^* q = 0.\] |
|                          | \[E_{\text{Ent}} < 1 \text{ Remarks: Maximizes as } a \to 0; b|f|^2 + |c|^2, \]
|                          | \[|p|^2 + |x|^2 \to \frac{1}{2}.\] |
|                          | \[E_{\text{Ent}} < 1 \text{ Remarks: Maximizes as } a, \]
|                          | \[b \to 0; |c|^2, |p|^2, |y|^2 \to \frac{1}{2}.\] |

Planar: No. of Orthogonal pair \(\Omega_p\): \(\Omega_p = 0/1.\)

Planar/Non-Planar: No. of Orthogonal pair \(\Omega_p: \Omega_p = 2/3.\)

Non-Planar: No. of Orthogonal pair \(\Omega_p: \Omega_p = 1.\)
Table 4. (Continued.)

| Example State ($|\psi\rangle$) | Max. Ent. Value | Figure (Example) | Geometry Structure | Ent. Class |
|-------------------------------|----------------|------------------|-------------------|------------|
| $a(00) + b(01) + p(10) + r(12) + y(21) = |0\rangle, a|a(0)\rangle_a + b|1\rangle_a + |1\rangle, a|p(0)\rangle_a + r|2\rangle_a + |2\rangle, a|y(1)\rangle_a$ | $E_{\text{max}} < 1$ Remarks: Maximizes as $b$, $p \to 0$; $|a|^2$, $|y|^2$, $|r|^2 \to \frac{1}{2}$ | Non-Planar No. of Orthogonal pair (Op): Op = 1 | Type III |
| $a(00) + b(01) + p(10) + r(12) + z(22) = |0\rangle, a|a(0)\rangle_a + b|1\rangle_a + |1\rangle, a|p(0)\rangle_a + r|2\rangle_a + |2\rangle, a|z(2)\rangle_a$ | $E_{\text{max}} < 1$ Remarks: Maximizes as $a$, $r \to 0$; $|b|^2$, $|p|^2$, $|z|^2 \to \frac{1}{2}$ | Non-Planar No. of Orthogonal pair (Op): Op = 1 | Type III |
### Table 5. Geometric structure and entanglement type of six-term states.

| Example State (|ψ⟩) | Max. Ent. Value | Figure (Example) | Geometry Structure | Ent. No. of Orthogonal pair| Ent. Class |
|----------------|-----------------|------------------|------------------|------------------------|-------------------|-----------|
| a(00) + b(01) + c(02) + p(10) + q(11) + r(12) = [0]|a⟩|a⟩|a⟩|a⟩ + [1]|a⟩|a⟩ + [2]|a⟩|a⟩ + [3]|a⟩|a⟩ + |4⟩|a⟩ + |5⟩|a⟩ + |6⟩|a⟩ | $E_{\text{Max}} = \frac{3}{2}$ Condition: |a|$^2$ + |b|$^2$ + |c|$^2$ = |p|$^2$ + |q|$^2$ + |r|$^2$ = \frac{3}{2}$. |
| a(00) + b(01) + c(02) + p(10) + q(11) + r(12) = [0]|a⟩|a⟩|a⟩|a⟩ + [1]|a⟩|a⟩ + [2]|a⟩|a⟩ + [3]|a⟩|a⟩ + |4⟩|a⟩ + |5⟩|a⟩ + |6⟩|a⟩ | $E_{\text{Max}} < 1$ Remarks: Maximizes as b, p, a → 0; |a|$^2$, |b|$^2$, |c|$^2$ → \frac{1}{7} |
| a(00) + b(01) + c(02) + p(10) + q(11) + r(12) = [0]|a⟩|a⟩|a⟩|a⟩ + [1]|a⟩|a⟩ + [2]|a⟩|a⟩ + [3]|a⟩|a⟩ + |4⟩|a⟩ + |5⟩|a⟩ + |6⟩|a⟩ | $E_{\text{Max}} < 1$ Remarks: Maximizes as a, b, p → 0; |a|$^2$, |b|$^2$, |c|$^2$ → \frac{1}{7} |
| a(00) + b(01) + c(02) + p(10) + q(11) + r(12) = [0]|a⟩|a⟩|a⟩|a⟩ + [1]|a⟩|a⟩ + [2]|a⟩|a⟩ + [3]|a⟩|a⟩ + |4⟩|a⟩ + |5⟩|a⟩ + |6⟩|a⟩ | $E_{\text{Max}} < 1$ Remarks: Maximizes as c → 0; |b|$^2$, |p|$^2$, |q|$^2$, |r|$^2$ → \frac{1}{7} \cdot a^2 + b^2 + c^2 = 0. |

Planar No. of Orthogonal pair (Op): Op=0/1.

Non-Planar No. of Orthogonal pair (Op): Op=0.

Non-Planar No. of Orthogonal pair (Op): Op=0/1.

Non-Planar No. of Orthogonal pair (Op): Op=1/2.
5.4. Five term states

Five terms can be picked up in five inequivalent ways from the nine elements of the product basis.

Among all states consisting of five terms, there is no new type of entanglement class. But there exists a special type of five-term state (second row of Table 4) which can belong to any one of Type I, II, or III entanglement classes depending on the value of the coefficients. That state shows Type I entanglement when \( |x| = \frac{|E|}{|q|} \) and Type III for \( a^*p + b^*q \neq 0 \).

5.5. Six term states

Six terms can be picked up in four inequivalent ways. Similar to five-term states, no new classes are found in six-term states also. But, here are some non-planer states with no mutually orthogonal post-measurement vectors (rows two and three of Table 5). These states belong to Type III entangled states.

States consisting of seven basis terms or more give neither new classes nor new geometric shapes different from the ones already listed.

6. Generalization to higher dimension

In this section, we extend our entanglement measure for multi-party arbitrary dimensional systems. First, we present the entanglement measure for two qudit systems. Any two qudit systems of dimension \( d_1 \) and \( d_2 \), respectively can be represented as \( \psi = \sum_{ij=0}^{d_1-1} a_{ij} |i\rangle |j\rangle \) where \( a_{ij} \) are complex coefficients and \( i \) and \( j \) are corresponding basis elements. In terms of post-measurement vectors, it can be rewritten as \( \psi = \sum_{ij=0}^{d_1-1} |i\rangle |j\rangle |\gamma_i\rangle |\gamma_j\rangle \) where \( |\gamma_i\rangle = \sum_{p=0}^{d_1-1} a_{ip} |p\rangle \) are the post-measurement vectors. Hence, the entanglement measure for this bipartite qudit state is given by:

\[
E_G = \frac{1}{2} \sum_{i,j=0}^{d_1-1} \left| \gamma_i \right|^2 + \frac{1}{2} \sum_{i,j=0}^{d_1-1} \left| \gamma_j \right|^2 + \ldots + \left| \gamma_{d_1-1} \right|^2,
\]

where we define, \( \gamma |k\rangle = |0\rangle \wedge |1\rangle \ldots \wedge |p\rangle \) and \( \gamma |k\rangle = |0\rangle \ldots \wedge |l-1\rangle \wedge |l+1\rangle \wedge \ldots |p\rangle \).

Consider any \( n \)-qudit pure state \( \psi \) of arbitrary dimension. It can be represented as:

\[
|\psi\rangle = \sum_{i_1,i_2,\ldots,i_n=0}^{d_1-1} a_{i_1\ldots i_n} |i_1\rangle \ldots |i_n\rangle.
\]

Here, \( |i_j\rangle \) are the basis elements for qudit of dimension \( d_i \). Consider any bipartition of \((m|n-m)\) and we look into the post-measurement vectors in the \((n-m)\) subsystem.

Without loss of generality we relabel those \( m \) parties as \( \{1, 2, \ldots, m\} \) and the remaining parties as \( \{m+1, m+2, \ldots, n\} \). In terms of the post-measurement vector of those \( m \)-n parties, the state can be rewritten as \( \psi = \sum_{i_1\ldots i_n} |i_1\rangle \ldots |i_n\rangle |\alpha_{i_1\ldots i_n}\rangle \) where \( |\alpha_{i_1\ldots i_n}\rangle = \sum_{i_{m+1},i_{m+2},\ldots,i_n} a_{i_1\ldots i_n} |i_{m+1}\rangle |i_{m+2}\rangle \ldots |i_n\rangle \) is the post-measurement vector of dimension \( D = d_{m+1} \cdot d_{m+2} \ldots d_n \). We propose the entanglement measurement for such \( m \)-party bipartition as:

\[
E_G^n = \frac{1}{2} \sum_{i_1\ldots i_m=0}^{d_1-1} \left| \alpha_{i_1\ldots i_m}\right|^2 + \frac{1}{2} \sum_{i_1\ldots i_m=0}^{d_1-1} \left| \alpha_{i_{m+1}\ldots i_n}\right|^2 + \ldots + \frac{1}{2} \sum_{i_1\ldots i_m=0}^{d_1-1} \left| \alpha_{i_1\ldots i_n}\right|^2,
\]

Therefore, we define the total entanglement of the \( n \)-partite qudit state as:

\[
E_G = \sum_{m=1}^{n-1} C_m^n E_G^n,
\]

where \( C_m^n \) denotes all possible combinations of \( m \) items selected from \( n \) items.

7. Conclusion

Our entanglement measure for bipartite qutrit systems incorporates area and volume elements of the 3-dimensional representation of the complex parallelepiped constructed by the post-measurement vectors. This geometry entirely encapsulates the distinct structures of entangled classes of those states. We have shown that there are three non-transformable types of geometries, namely three planar vectors, three mutually orthogonal vectors, and three vectors neither mutually orthogonal nor planar. States with different types of geometries...
cannot be transformed into one another with nonzero probability under LOCC, hence they correspond to
different classes of entangled states. The geometric shapes along with the maximizing condition of entanglement
have been presented for all types of bipartite pure qutrit states. The maximum entanglement can be found for
the type II entangled class, with the maximum value being $\frac{1}{2}$. The planar structure, being type I entangled states have
entanglement of no more than $\frac{1}{2}$. Type III states have the maximum amount of entanglement strictly less than
unity. The entanglement maximization condition from geometry has been presented for each state which admits
simplified algebraic conditions. Finally, the measure has been generalized to arbitrary pure states and to
multipartiy scenarios. Our results show that entanglement has an inherent connection with geometry and our
entanglement measure based on wedge product formalism geometrically classifies different entangled classes as
well as simplifies the entanglement maximization criteria for pure states. It will be interesting to explore the
geometrical interpretation of entanglement families in multi-party settings.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

ORCID iDs

Soumik Mahanti @ https://orcid.org/0000-0002-0380-0324
Sagnik Dutta @ https://orcid.org/0000-0003-2083-9922
Prasanta K Panigrahi @ https://orcid.org/0000-0001-5812-0353

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