A survey of dimensionality reduction techniques based on random projection

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Abstract

Dimensionality reduction techniques play important roles in the analysis of big data. Traditional dimensionality reduction approaches, such as Principal Component Analysis (PCA) and Linear Discriminant Analysis (LDA), have been studied extensively in the past few decades. However, as the dimension of huge data increases, the computational cost of traditional dimensionality reduction approaches grows dramatically and becomes prohibitive. It has also triggered the development of Random Projection (RP) technique which maps high-dimensional data onto low-dimensional subspace within short time. However, RP generates transformation matrix without considering intrinsic structure of original data and usually leads to relatively high distortion. Therefore, in the past few years, some approaches based on RP have been proposed to address this problem. In this paper, we summarized these approaches in different applications to help practitioners to employ proper approaches in their specific applications. Also, we enumerated their benefits and limitations to provide further references for researchers to develop novel RP-based approaches.

Keywords: Random projection, Compressive sensing, Dimensionality reduction, High-dimensional data

1. Introduction

In machine learning and data mining scenarios, data usually has very high dimensionality [1]. For example, market basket data in the hypermarket is very high-dimensional, consisting of several thousand of merchanides. In text mining, each document is usually represented by a matrix whose dimensionality equals to the vocabulary size. In bioinformatics, gene expression profiles can also be considered as matrices with more than ten thousand of continuous values. High-dimensionality leads to burdensome computation and curse-of-dimensionality issues. Therefore, dimensionality reduction techniques have often been applied in machine learning tasks to solve curse-of-dimensionality problem [2]. During the past decades, traditional dimensionality reduction techniques such as PCA [3,4] and LDA [5] have been widely studied. However, as the data dimension increases, the computational cost of traditional dimensionality reduction approaches grows dramatically and becomes prohibitive. RP [6] is a simple and fast approach to reduce dimensionality, which projects the original high-dimensional matrix $X_{n \times d}$ onto a $k$-dimensional subspace using a random matrix $W_{d \times k}$. It can be formulated as following:

$$X_{n \times d}^{RP} = X_{n \times d}W_{d \times k}$$

The key idea of RP is based on the Johnson–Lindenstrauss lemma [7], which states that it is possible to project $n$ points in a space of arbitrarily high dimension onto an $O(\log n)$-dimensional space, such that the pairwise distances between points are approximately preserved.

Thus RP has attracted increasing attention in the past few years and has been employed in many machine learning scenarios, including classification [8,9,10], clustering [11,12,13], regression [14,15,16], manifold learning [17,18,19], and information retrieval [20,21]. Although RP is much less expensive in computational cost, it often fails to capture the task-related information because latent space is generated without considering the intrinsic structure of original data. Various strategies are proposed to overcome this issue and improve the performance of RP. These strategies are organized into four categories: feature extraction, dimensionality increase, ensemble, and optimization theory (Figure 1).

Feature extraction intends to construct informative and non-redundant features from a large set of data, which is the most commonly used strategy to improve the performance of RP. Schneider et al. [16] incorporated bag-of-words (BoW) model into RP to utilize consumer reviews for forecasting sales of products on Amazon. In order to gain higher performance in text classification tasks, Zhao et al. [10] developed two methods and achieved better balance between classification accuracy and computational complexity.

Increasing the dimensionality of the original feature space can sometimes improve the linear separability and the original features can be better represented in high-dimensional space. Ma et al. [22] firstly constructed an extremely high-dimensional feature space with rectangle filters. Then, RP was applied for dimensionality reduction. In bioinformatics, Alshamiri et al. [23] combined extreme learning machine (ELM) [24] with RP for data classification and clustering in gene expression datasets.

Several studies concentrated on ensembles of decision trees have been proposed, well known as Random Forest [25] and AdaBoost [26]. Schclar et al. [27] performed a voting scheme...
on ensemble classifiers that were constructed by applying multiple RP instances to the same dataset. Zhang et al. [28] and Yoshioka et al. [29] used a similar method for drug-target interaction prediction and dysarthric speech recognition, respectively.

Optimization theory methods treat the improvements of RP as an optimization problem. Drineas et al. [30] reduced the computational complexity of Lasso [31] with the help of RP.

Although various approaches were developed to improve the performance of RP, there are still some issues need to be addressed. Practitioners also need some guidelines to select which strategy and approach to use in their application fields. Here we reviewed these approaches and summarized their benefits and limitations to provide references for further studies of RP-based methods.

2. Feature extraction approaches to improve the performance of RP

Feature extraction transforms the data in the high-dimensional space into a space with fewer dimensions, which is the most commonly used strategy to improve the performance of RP. It is often used before or after RP, which are called preprocessing or post-processing respectively (Figure 2). Most researchers prefer to use post-processing methods because feature extraction methods take less time in lower-dimensional subspace. Observe that RP rotates the original data to a random basis, which can be considered as a uniform sampling. Therefore, for approximately uniformly distributed data, it can be well represented after the usage of RP. The existing preprocessing and post-processing methods for RP in different application fields are listed in Table 1

In many machine learning and pattern recognition systems, feature extractors that transform raw data into feature vectors are required to be carefully designed, especially in specific application fields. So important are feature extractors that they directly affect the performance of the developed methods. There are two kinds of feature extraction methods: general methods, such as PCA and LDA, which can be applied to many fields, and application-specific methods that are designed for coping with a specific problem, like Nonsubsampled Pyramid (NSP) [39] and constrained energy minimization (CEM) [40].

2.1. General methods

Traditional dimensionality reduction techniques aim at seeking dimensions with maximum discriminative power, whereas RP performs well in fast finding low dimension space. It is natural that the two kinds of methods are combined to solve dimensionality reduction problem. So in the past few years, a great deal of researches based on the two techniques have been developed.

Xie et al. [9] incorporated RP into PCA, LDA, and feature selection (FS) [41] to classify gene expression profiles of breast cancer. Experimental results demonstrated that the classification accuracy of RP can be significantly improved by FS, especially in the small-n-large-P datasets [42]. However, in some cases, FS takes longer time with the increase of data dimension.

Zhao et al. [10] proposed Semi-Random Projection (SRP) method with the purpose of finding a discriminative subspace while having feasible computational load. Different from RP whose values of transformation matrix are assigned randomly, the weights for transformation vectors of SRP are obtained by LDA. SRP method (see Figure 3) includes three steps. First, the original data matrix $X \in \mathbb{R}^{n \times d}$ is mapped onto a subspace $\tilde{X}_i \in \mathbb{R}^{n \times k}$ using the randomly selected $k$ features. Next, the data with $k$ features is projected onto a single dimension $h_i$ using a transform vector $W \in \mathbb{R}^{k \times 1}$ learned from LDA. The above procedure is repeated $r$ times, and generates the following latent subspace $H \in \mathbb{R}^{n \times r}$:

$$H = [h_1| h_2| \ldots | h_r]$$

Experiments were performed on six datasets generated from the standard text 20 newsgroups corpus [43] for determining the top category of text. Experimental results indicated that the classification accuracy of SRP followed by PCA (SRP + PCA)
Figure 2: Preprocessing and post-processing methods for RP

Table 1: Existing preprocessing and post-processing methods for RP

| Preprocessing | Post-processing | Application                          | Ref. |
|---------------|-----------------|--------------------------------------|------|
| PCA / FS / None | PCA / LDA / FS / None | Tumor tissues classification         | [9]  |
| None          | LDA             | Text classification                   | [10] |
| BoW           | None            | Sale rank prediction                 | [16] |
| None          | BoW             | Texture images classification         | [32] |
| CNN           | RNN             | Text recognition in images           | [33] |
| FAST Corder Detector | None         | Image classification                 | [34] |
| NSP           | Mahalanobis distance | Target enhancing                   | [35] |
| None          | CEM             | Hyperspectral target detection       | [36] |
| None          | TCIMF           | Hyperspectral target detection       | [37] |
| None          | MUSIC           | Wideband spectrum sensing            | [38] |

Figure 3: Overview of SRP
increased by about 25% compared to that of RP, but it was still lower than that of PCA.

To further improve the performance of SRP, Stacked Semi-Random Projection (SSRP) was developed. In the SSRP, SRP is stacked by feeding the output of previous SRP layer as the input of the subsequent SRP layer. Suppose the output of the k-th SRP layer is denoted by $\mathbf{H}_k$ with the dimensionality of $d^k$, and the first layer is initialized with the original data $\mathbf{H}^0 = \mathbf{X}$ with the dimensionality of $d^0$. The relationship between dimensionality of the k-th and the $(k - 1)$-th SRP layer can be calculated as:

$$d^k = \alpha [d^{k-1}]^\beta$$  \hspace{1cm} (3)

where $\alpha$ is a hyperparameter. The output of the k-th SRP layer can be determined as:

$$\mathbf{H}_k = \frac{1}{1 + \exp(-||\mathbf{W}_k||^2 \mathbf{H}_k^{-1})}$$  \hspace{1cm} (4)

where $\mathbf{W}_k$ is the transformation vector of the k-th SRP layer learned from regularized LDA $\mathbf{H}_k = \tilde{\mathbf{X}}$ of rank $k$. The overall process of SSRP is shown in Figure 4. The SSRP outperformed other methods, including PCA, sparse PCA $\mathbf{H}_k$, RP, RP + PCA, mSDA $\mathbf{H}_k$, and SRP + PCA on 5 of 6 datasets in classification accuracy. In addition, the computation cost of SSRP is much lower than that of PCA though it is 2 times higher than that of RP. Due to LDA is adopted, SRP and SSRP may easily overfit in the presence of labeling noise and are not applicable for non-linear problems $\mathbf{H}_k$.

BoW model is a simplified representation used in natural language processing and computer vision $\mathbf{H}_k$. The fact that the number of words in a BoW model usually reaches tens of millions, which explodes the dimensionality of predictor matrices. In the past few years, RP has been used to address this issue. Schneider et al. $\mathbf{H}_k$ proposed an attributes-based regression model in which historical data were used to forecast one-week-ahead rolling sales for tablet computers. Different from extant approaches $\mathbf{H}_k$, the authors took customer reviews into account and used BoW model to analyze feedback of products. However, a key challenge of BoW is that millions of words in the bag lead to infeasible computation. To tackle this problem, RP was adopted for reducing the dimension of BOW predictor matrices. Assuming that the BoW matrix is denoted by $\mathbf{BOW}_{\text{seed}}$, where $n$ and $d$ are the number of reviews and words, respectively, RP maps original space onto a subspace $\mathbf{BOW}_{\text{seed}}$ spanned by the $k$ features. Compared to baseline model without BoW, the proposed model with BoW $\mathbf{BOW}_{\text{seed}}$ had better predictive performance on the dataset named “Market Dynamics and User Generated Content about Tablet Computers” $\mathbf{BOW}_{\text{seed}}$.

In computer vision, the BoW model has been applied to texture research (including synthesis, classification, segmentation, compression, and shape from texture) $\mathbf{BOW}_{\text{seed}}$ by treating texture features as words. In BoW, texture images are statistically described as histograms over a dictionary of features. It has been proved that local texture feature descriptors learned from BoW were proved to be insensitive to local image perturbations such as rotation, affine changes and scale $\mathbf{BOW}_{\text{seed}}$. One limitation of existing methods without BoW $\mathbf{BOW}_{\text{seed}}$ is that the image patch features are not rotationally invariant. In order to address this limitation, Liu et al. $\mathbf{BOW}_{\text{seed}}$ proposed a robust and powerful texture classifier based on RP and BoW. Firstly, RP is used to extract a small set of random features from local image patches. Then, random features are embedded into a BoW model to carry out texture classification. Finally, learning and classification are performed in the compressed domain. By comparison, the proposed method made an improvement by 10.38% and 3.32% in classification accuracy compared to local binary pattern (LBP) $\mathbf{BOW}_{\text{seed}}$ and the combination of LBP and normalized Gabor filter (NGF) $\mathbf{BOW}_{\text{seed}}$. Furthermore, the proposed method took considerable less time and storage.

Traditional machine learning techniques are limited in their ability to handle natural data in their row form. Deep learning $\mathbf{BOW}_{\text{seed}}$ has performed remarkably well, leaving traditional machine learning in the dust. Wu et al. $\mathbf{BOW}_{\text{seed}}$ proposed a novel method based on RP and deep neural network (DNN) $\mathbf{BOW}_{\text{seed}}$ for text recognition in natural scene images. First of all, convolutional neural network (CNN) $\mathbf{BOW}_{\text{seed}}$ is adopted as a feature extractor to convert word images to a multi-layer CNN feature sequence with slicing windows. Then RP is used to map high-dimensional CNN features onto subspace with low dimensionality. Finally, recurrent neural network (RNN) $\mathbf{BOW}_{\text{seed}}$ is used for decoding the embedded RP-CNN features to recognize the text in the image. Multiple RNNs are ensembled by Recognizer Output Voting Error Reduction (ROVER) algorithm $\mathbf{BOW}_{\text{seed}}$ so as to further improve the recognition rate. The authors implied that the recognition rate of RP-CNN features, with 85% dimensionality reduction, is similar to the original high-dimensional ones. In spite of the fact that CNNs and RNNs dramatically improved the performance in recognition rate, CNNs and RNNs are difficult to train and require lots of time in selecting hyperparameters.

Low rank approximation plays a central role in data analysis. In mathematics, it is often desirable to find a good approximation of a given matrix with lower rank. Eigenvalue decomposition is a typical strategy, best known as Singular Value Decomposition (SVD) $\mathbf{BOW}_{\text{seed}}$, but it usually leads to heavy computation. RP is a simple technique and has been widely used for accelerating the computation for such approximations. Assume that the original matrix can be formulated as $\mathbf{X} \in \mathbb{R}^{n \times d}$. The quality of the approximation depends on how well the method captures the important part of $\mathbf{X}$. Martinsson $\mathbf{BOW}_{\text{seed}}$ proposed an approximation method based on QR decomposition $\mathbf{BOW}_{\text{seed}}$ and RP. First of all, a matrix $\mathbf{Y} \in \mathbb{R}^{n \times d}$ is formed according to $\mathbf{Y} = (XX^T)^{-1}XX$, where $p$ stands for the number of iterations, and $\mathbf{S} \in \mathbb{R}^{p \times d}$ follows Gaussian distribution $N(0, 1)$. Then, QR decomposition is applied: $\mathbf{Y} = QR$. The low rank approximation of matrix $\mathbf{X}$ can be represented as $\mathbf{X} = Q(Q^T\mathbf{X})$. The proposed method constructed a nearly optimal rank-k approximation with much lower time complexity. Alternatively, the sampling matrix $\mathbf{S}$ can also be sampled from Subsampled Randomized Hadamard Transform (SRHT) $\mathbf{BOW}_{\text{seed}}$, and the approximation method is known as structured RP $\mathbf{BOW}_{\text{seed}}$. Experimental
results revealed that the structured RP is faster but less accurate than the method proposed by Martinsson [66].

2.2. Application-specific methods

In some specific application fields, especially in image processing, specific feature extraction methods are often designed to obtain better experimental results. Arriaga et al. [34] proposed two RP-based methods: slide window RP (SWRP) and corner RP (CRP). In the SWRP, a corresponding location in the projected images is filled with a synthetic color. The synthetic color is generated from a random combination of colors in a window that slides over the original image (see Figure 5a). Motivated by the intuition that human visual systems are feature detectors for colors, lines, corners, and so on, the features from accelerated segment test (FAST) [70] corner detector is adopted in CRP to detect features in images. The projected image can be represented as a combination of multiple corner images (see Figure 5b). The authors indicated that SWRP performed better than CRP in classification accuracy. Compared to neural networks without SWRP, neural networks with SWRP have similar classification accuracy but much lower computational cost. However, the performance of SWRP may be serious affected for images consisting of similar colors since SWRP randomly combines colors in a sliding window.

By taking advantages of characteristics of RP that it is insensitive to noise in images, Qin et al. [35] proposed a method for suppressing clutters while enhancing targets in infrared images (see Figure 6). First, a signal decomposition algorithm named NSP [39] is adopted to decompose image into one low-frequency subband and multiple high-frequency subbands. After K-scale NSP decomposition, K+1 subbands including one low-frequent subband and K high-frequency subbands are produced. The high-frequency subbands mainly contain targets and little clutters background, so it is easier to extract target information by dealing with high-frequency subbands. In order to preserve target information as much as possible, a 3D image cube is constructed by concatenating all the high-frequency subbands. The cube model can be expressed as:

$$F_{K \times M} = (f_1, f_2, \ldots, f_\text{K})^T$$

where $f_k$ is the row vector representation of a subband and $M$ is the number of pixels of the $k$-th high-frequency subband. Then, RP is used to project 3D image cube $F_{K \times M}$ onto a low-dimensional subspace $Q_{K \times M}$ with the intention of reducing the spatial redundancy of the target and background information. Finally, Mahalanobis distance [21] is applied to remove background clutters in dimensionality-reduction high-frequency subbands obtained above. Compared to other state-of-art methods for background suppression of infrared small target images, the proposed model outperformed other methods including max-median [72], morphological (top-hat) [73], Phase Spectrum of Quaternion Fourier Transform (PQFT) [74], Variable Spectrum of Quaternion Fourier Transform (VSQFT) [75], and Wavelet Transform (WRX) [76] in signal-to-clutters ratio gain (SCRG) and background suppress factor (BSF) [77].

Feng et al. [36] developed a new approach to detect hyperspectral target using the CEM method [40]. The objective of CEM is to design a finite impulse response (FIR) linear filter with $L$ filter coefficients, and the FIR filter can be represented using an $L$-dimensional vector $w = \{w_1, w_2, \ldots, w_L\}$. Let $r_i (i = 1, 2, \ldots, n)$ be an $L$-dimensional sample pixel vector. The optimized target detector $w$ can be given as:

$$w^* = \frac{R_{\text{LxL}}^{-1} \mathbf{d}}{d^T R_{\text{LxL}}^{-1} \mathbf{d}}$$

where $R_{\text{LxL}} = (1/n) \sum r_i r_i^T$ is the sample autocorrelation matrix of the target and $\mathbf{d}$ denotes the spectral signature of the target. In order to resolve the issue of "curse of dimensionality" in hyperspectral imagery, RP is incorporated. The noise suppression effect of RP is superior to that of maximum-noise fraction (MNF) [77] and PCA. Therefore, CEM method using dimensionality reduction RP (RP-CEM) outperformed MNF-CEM and PCA-CEM in both detection accuracy and computation time. The drawback of CEM is that it can only detect single target because CEM simply treats the undesired targets as interference and does not take fully use of the known information. Target-Constrained Interference-Minimized Filter (TCIMF) [78] was developed to tackle this issue. It assumes pixels of an image are made up of three separate signal sources: $D$ (desired targets), $U$ (undesired targets), and $I$ (interference), while CEM simply treats $U$ as a part of $I$. 

![Figure 4: Overview of SSRP](image-url)
Figure 5: A glance of SWRP and CRP

Let $D = \left[ d_1, d_2, \ldots, d_p \right]$ and $U = \left[ u_1, u_2, \ldots, u_q \right]$ denote the desired-target signature and undesired-target signature, respectively. The spectral signature of the target $d$ in Eq. 6 can be replaced with $[D, U]$. The optimal weight vector $w$ can be formulated as:

$$w^* = \frac{R_{D,U}^{-1} \left[ D \ U \right]}{[D \ U]^T R_{D,U}^{-1} [D \ U]} \left[ I_{p \times 1} \ 0_{q \times 1} \right] \label{7}$$

where $I_{p \times 1}$ is a $p \times 1$ column vector with ones in its components, which is used to constrain the desired targets in $D$. Similarly, $0_{q \times 1}$ is a $q \times 1$ column vector with zeros in all components, which is used to suppress the undesired targets in $U$. Analogous to Feng et al. [36], Du et al. [37] conducted target detection by TCIMF, where RP is used for dimensionality reduction. Not only did RP reduce computation complexity, but also improved target-detection accuracy by decision fusion across multiple RP instances. Experimental results demonstrated that the detection accuracy of RP-TCIMF with a single run of RP was a bit lower than that of TCIMF, but it could be further improved by multiple runs.

Subspace based spectrum estimation is often used for wide-band spectrum sensing. However, subspace based techniques which require eigen decomposition of original data are computationally expensive. To alleviate this issue, Majee et al. [38] applied low-rank approximation before MUltiple SIgnal Classification (MUSIC) [79] which is employed for wideband spectrum sensing. The low-rank approximation technique used in this work is based on Cholesky Factorization (CF) [80] and RP. Suppose $X \in \mathbb{R}^{n \times n}$ is an arbitrary square matrix. First of all, RP is performed as $\hat{X} = XW$, where $W \in \mathbb{R}^{n \times k}$ is a random projection matrix. Then $\Phi^T$ is filled with $k$ left singular values of $\hat{X}$, and thus $\hat{X}$ can be estimated as $\hat{X}^* = \Phi \Theta \Phi^T$. After that, CF is applied as $\tilde{X}^* = L L^T$, and $D$ can be calculated as $D = \tilde{X}^* \tilde{X} \Phi^T (L^T)^{-1}$. SVD is then applied as $D = U \Sigma V^T$, and $k$-rank approximate of $X$ can be obtained as $X^{LR} = U \Sigma^2 V^T$. The authors concluded that the proposed method had a marginal reduction in time complexity. In addition, the spectrum sensing performance of the proposed method was comparable or superior to that of MUSIC in terms of probability of alarm.
3. Increase dimensionality of original data to improve the performance of RP

In some studies, researchers acted in diametrically opposed ways. They firstly map the original dataset onto a higher dimensional feature space to better represent the original features. Then RP is applied to reduce dimensionality and save computational cost.

Ma et al. [22] proposed a robust method for face recognition, as illustrated in Figure 7. In the proposed method, multi-radius local binary pattern (MLBP) is firstly adopted to incorporate more structural facial feature. Then, a high-dimension multiscale and multi-radius LBP (MMLBP) space $P \in \mathbb{R}^{L \times m}$ is obtained by convolving rectangle filters. The rectangle filters of a given face image with width $w$ and height $h$ can be defined as follows:

$$p_i^{u_0,v_0} = \begin{cases} 1, & u_0 \leq u \leq w, v_0 \leq v \leq h \\ 0, & \text{otherwise} \end{cases} \quad (8)$$

where $u_0$ and $v_0$ represent the offset coordinates of the filter $p_i$ with width $w$ and height $h$, respectively. It can be found that there are approximately $L = m^2 = (wh)^2$ (i.e. the number of possible locations times that of possible scales) exhaustive rectangle filters for each face image. Generally, a large multiscale rectangle filter matrix $P$ can be obtained by stacking $p_i$ together, where $p_i \in \mathbb{R}^{1 \times m}$ can be formulated as a row vector whose dimensionality equals to $w \times h$. Since $L$ is usually up to $10^6 - 10^{10}$, sparse RP [81] is taken for dimensionality reduction. The subspace can be formulated as $M_{\text{scor}} = W_{\text{best}} P_{\text{scor}}$, where $W$ is a transformation matrix generated by RP. The proposed method not only archived higher recognition rate but also showed better robustness to corruption, occlusion, and disguise compared to Randomface [82] and Eigenfaces [83], even in low dimensional spaces.

There have been lots of works concentrating on visual tracking [84]-[86], where RP are favored by researchers because of its computational effective and data independent characteristics. Zhang et al. [85]-[86] proposed a tracking framework, named CT tracker, where rectangle filters following Eq. 8 were adopted to form a very high-dimensional multiscale image feature vector and RP was applied to compress samples of foreground targets and the background. The tracking window in the first frame is determined manually. To predict the target in the next frame, positive samples are sampled near the current target while negative samples are sampled far away from the current target. Both positive and negative samples are used to update the Bayes classifier. However, rectangle filters are sensitive to the presence of a few outliers in samples. To overcome this problem, a more stable model was proposed by Gao et al. [87], where rectangle filters were replaced by MSERs [88]. To robustly adapt the variations of target appearance, a least squares support vector machine (LS-SVM) [89] is employed. The authors stated that the proposed tracker outperformed the CT-based tracker [86] in terms of mean distance precision and mean frame rate (FPS).

Recent studies have revealed that ELM performs well in regression and classification problems [24]. Additionally, compared to traditional algorithms such as back-propagation (BP) and support vector machine (SVM), ELM provides not only good generalization performance, but also fast learning speed. Alshamiri et al. [23] performed RP in conjunction with ELM for low-dimensional data classification. The method consists of two phases. Firstly, the original data $X \in \mathbb{R}^{n \times d}$ is projected onto the subspace spanned by $L (L \gg d)$ using ELM. The linear separability of the data often increases by mapping the data onto a high-dimensional ELM feature space. After that, RP is applied to reduce the dimensionality of the ELM feature space. There is a slight improvement in classification accuracy of ELM-RP compared to that of ELM. Since ELM consists of a single hidden layer, it is hard to encode complex things and archive satisfying accuracy. Also, on small-n-large-p datasets, ELM is prone to overfit because of a lack of training samples.

4. Ensemble of multiple RP instances to improve the performance of RP

Inspired by the fact that an ensemble strategy can significantly improve the performance of weak classifiers, several algorithms were proposed based on ensembles of decision trees, well known as Random Forest [25] and AdaBoost [26]. To alleviate computational burdens of high-dimensional datasets, RP is incorporated as a preprocessing step before AdaBoost [29]. Furthermore, ensembles of multiple RP instances helps RP to yield more stable results.

Schclar et al. [27] proposed an ensemble method based on RP and nearest-neighbor (NN) inducers [91]. First of all, $K$ random matrices are generated by RP. Then, $K$ training sets for ensemble classifiers are constructed by applying random matrices to the original dataset. After that, $K$ NN classifiers are trained and the final classification result is produced via a voting scheme. The proposed method appeared to be more accurate compared to the non-ensemble NN classifier.

Zhang et al. [28] proposed an RP ensemble method which is analogous to the work of Schclar’s et al. [27] for drug-target interaction prediction, where “PaDEL-Descriptor” [92] was adopted as a feature detector. The authors stated that the proposed drug-target interaction prediction method achieved an improvement in accuracy by 4.5%-8.2% compared to the work [93]. Similarly, Yoshioka et al. [29] also proposed RP ensemble method for dysarthric speech recognition, in which an automatic speech recognition (ASR) system was adopted as a feature detector. As compared to the PCA-based feature projection method, the method proposed in [29] made an improvement by 5.23% in recognition rate.

Gondara [94] also proposed an ensemble classifier using RP. Different from the method proposed by Schclar’s et al. [27] where the RP matrices are applied to the same feature space, in this method, the RP matrices are applied to random subsets of the original feature set. The authors demonstrated that the proposed method performed equally well or better than Random Forest and AdaBoost in classification accuracy.

So as to alleviate the high distortion in single run of clustering algorithm in feature spaces produced by RP, Fern et al. [11] investigated how RP could best be used for clustering.
and proposed a cluster ensemble approach based on expectation–maximization (EM) [95] and RP. In the proposed ensemble approach, EM generates a probabilistic model $\theta$ of a mixture of $k$ Gaussian distribution after each run of RP. The final clusters are aggregated by measuring similarity of clusters in different clustering results, where the similarity of two clusters $c_i$ and $c_j$ can be defined as:

$$sim(c_i, c_j) = \min_{p \in \mathcal{C}_i, q \in \mathcal{C}_j} P_{p,q}$$

where $P_{p,q}$ denotes the probability of data point $i$ and $j$ belonging to the same cluster and can be calculated as:

$$P_{p,q} = \sum_{l=1}^{k} P(l|\theta) P(l|i, j, \theta)$$

The authors mentioned that the proposed ensemble method (RP + EM) was more robust and produced better clusters compared to PCA + EM.

5. Use optimization theory to improve the performance of RP

The optimization problem of high-dimensional datasets is computationally expensive. RP is an effective method and has been widely used to reduce the computational cost for optimization problems.

Least squares approximation has been widely used to find an approximate solution to an overdetermined system that has no exact solution. The Least-Square problem, also known as Lasso [31], can be formulated as:

$$\beta_\ast = \arg\min_{\beta \in \mathbb{R}^d} \| Y - X\beta \|^2_2$$

where $X = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^{n \times d}$ and $Y = (y_1, y_2, \ldots, y_n)^T \in \mathbb{R}^n$ represent the data matrix and corresponding response vector, respectively. $\beta \in \mathbb{R}^d$ is a vector denoting model coefficients. So as to speed up calculation of least square problem, Drineas et al. [30] applied RP matrices to reduce the dimensionality of original feature space, which can be formulated as follows:

$$\beta_\ast = \arg\min_{\beta \in \mathbb{R}^d} \left\| WHD(Y - X\beta) \right\|^2_2$$

where $W$ is a data-independent random matrix, $H$ and $D$ are produced by SRHT [68].

Zhang et al. [96] proposed a method that accelerated linear regression with the help of low-rank matrix approximation implemented with RP and QR decomposition, as introduced in Section 2.1. Let $X$ be the low-rank approximation of the original feature set, and thus the coefficient vector $\beta$ can be calculated according to the following recursive formula:

$$\beta_{i+1} = \arg\min_{\beta \in \mathbb{R}^d} \left( -\frac{1}{n} (\beta - \beta_i)^T \tilde{X}(Y - \tilde{X}^T\beta_i) + \Phi_\eta \| \beta \|_1 + \frac{\lambda}{2} \| \beta - \beta_i \|^2_2 \right)$$

where $\lambda$ represents the regularization term. $\beta_0$ is set to $0$ and $\Phi_\eta = \min(\Phi_{\text{min}}, \Phi_{\text{max}})$ in which $\eta \in (0, 1)$ controls the shrinkage speed of $\Phi_\eta$. As compared to existing methods excluding RP, such as PGH [27] and ADG [98], the proposed method made a significant reduction by 98.90% and 98.55% in computation time. Also, the convergence rate of the proposed method was comparable to that of PGH, and much higher than that of ADG.

Zhang et al. [99] tried to solve an optimization problem by recovering from the dual solution of the low-dimensional optimization problem. Assume a training sample and the corresponding binary class assignment are denoted by $x_i \in \mathbb{R}^d$ and $y_i \in \{-1, +1\}$ ($i = 1, 2, \ldots, n$), respectively. In general, a linear classifier $\beta \in \mathbb{R}^d$ can be learned by solving a regularized optimization problem as following:

$$\beta_\ast = \arg\min_{\beta \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^{n} \ell(y_i \beta^T x_i) + \frac{\lambda}{2} \| \beta \|^2_2$$

where $\ell(\cdot)$ is a differentiable convex loss function, and $\lambda$ and $n$
denote the regularization term and the number of training examples, respectively. Using conjugate function, Eq. (14) can be turned into a dual problem:

$$\alpha_* = \text{argmax}_{\alpha \in \mathbb{R}^d} - \sum_{i=1}^{n} \ell_i(\alpha_i) - \frac{1}{2n}(\alpha \circ \gamma)^T \mathbf{X} \mathbf{X}(\alpha \circ \gamma)$$

(15)

where $\alpha_*$ represents the optimal solution in dual problem and $\ell_i(\cdot)$ is the convex conjugate of $\ell_i(\cdot)$. Let $\mathbf{W} \in \mathbb{R}^{d \times k}$ be an RP matrix following Gaussian distribution $\mathcal{N}(0, 1/k)$, where $k \ll d$. The primal problem can be solved in the low-dimensional space generated by RP, which can be formulated as following:

$$\gamma_* = \text{argmin}_{\gamma \in \mathbb{R}^d} \frac{1}{n} \sum_{i=1}^{n} \ell(y_i, \gamma^T \mathbf{W} x_i) + \frac{d}{2} \| \gamma \|_2^2$$

(16)

And the corresponding dual problem is

$$\delta_* = \text{argmax}_{\delta \in \mathbb{R}^d} - \sum_{i=1}^{n} \ell_i(\delta_i) - \frac{1}{2n}(\delta \circ \gamma)^T \mathbf{X}^T \mathbf{W}^T \mathbf{X}(\delta \circ \gamma)$$

(17)

The optimal primal solution $\beta_*$ can be computed as following when given the optimal dual solution $\delta_*$ of the low-dimensional optimization problem:

$$\beta_* = -\frac{1}{n} \mathbf{X}(\delta_* \circ \gamma)$$

(18)

Compared to the naive recover method where the primal solution was computed as $\beta = \mathbf{W} \gamma$ [100], the authors stated that the proposed method had much lower relative recovery error and less computational time.

6. Conclusions

RP is an efficient and powerful dimensionality reduction technique and has been developed and matured over the last 15 years. With the rapid increase of big data, RP has brought about tangible benefits to counteract the burdensome computation and has met the needs of real-time processing in some situations. Despite the fact that RP is computationally efficient, RP often introduces relatively high distortion. In order to solve this problem, various strategies have been proposed to improve the performance of the RP, which were summarized in this survey. Experimental results have proved that feature extraction methods including PCA, LDA, BoW and other application-specific methods can significantly improve the performance of RP. It has also been found that an ensemble of RP instances can improve the generalization ability of original features by bringing information diversity with randomness.

Although RP yields satisfying performance in many scenarios, such as text classification, face recognition, and video tracking. In some other complex tasks including advertisement recommendation, cancer classification based on high-throughput microarray data, however, RP may not well capture the intrinsic structure of the original datasets and its performance need to be further improved.

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