GLOBAL ANALYSIS OF STRONG SOLUTIONS FOR THE VISCOUS LIQUID-GAS TWO-PHASE FLOW MODEL IN A BOUNDED DOMAIN

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Abstract. In this paper, we investigate global existence and asymptotic behavior of strong solutions for the viscous liquid-gas two-phase flow model in a bounded domain with no-slip boundary. The global existence and uniqueness of strong solutions are obtained when the initial data is near its equilibrium in $H^2(\Omega)$. Furthermore, the exponential convergence rates of the pressure and velocity are also proved by delicate energy methods.

1. Introduction. Let $\Omega \subset \mathbb{R}^3$ be a smooth bounded domain, we are concerned with the following liquid-gas two-phase model for the motion of compressible viscous fluids:

\begin{equation}
\begin{aligned}
&m_t + \text{div}(mu) = 0, \\
&n_t + \text{div}(nu) = 0, \\
&(mu)_t + \text{div}(mu \otimes u) + \nabla P(m,n) = \text{div}T,
\end{aligned}
\end{equation}

with the initial and boundary conditions

\begin{equation}
\begin{aligned}
&m(x,0) = m_0(x), \quad n(x,0) = n_0(x), \\
&u(x,t)|_{\partial \Omega} = 0, \quad t \geq 0, \\
&\frac{1}{|\Omega|} \int_{\Omega} m_0(x)dx = \bar{m}_0 > 0, \\
&\frac{1}{|\Omega|} \int_{\Omega} n_0(x)dx = \bar{n}_0 \geq 0.
\end{aligned}
\end{equation}

Here the variables $m = \alpha_l \rho_l, n = \alpha_g \rho_g, u$ and $P = P(m,n)$ denote the the liquid mass, gas mass, the velocity of the liquid and gas, the common pressure for both
phases, respectively. The stress tensor is given by 
\[ T = \mu (\nabla u + \nabla u^T) + \lambda (\text{div}u)I, \]
where \( \mu \) and \( \lambda \) are the shear viscosity and the bulk viscosity coefficients of the fluid, respectively, which are assumed to satisfy the following physical condition:
\[ \mu > 0, \quad 3\lambda + 2\mu \geq 0. \]
\( \alpha_l, \alpha_g \in [0, 1] \) denote the liquid and gas volume fractions, satisfying the fundamental relation:
\[ \alpha_l + \alpha_g = 1. \]
The unknown variables \( \rho_l \) and \( \rho_g \) denote the liquid and gas densities, satisfying the equations of states
\[ \rho_l = \rho_{l,0} + \frac{P - P_{l,0}}{a_l^2}, \quad \rho_g = \frac{P}{a_g^2}, \]
where \( P_{l,0} \) and \( \rho_{l,0} \) are the reference pressure and density given as constants, \( a_l \) and \( a_g \) denote the sonic speeds of the liquid and the gas, respectively. Noticing (3) and (4), it is clear that the pressure satisfies
\[ P(m, n) = \kappa_1(-M(m, n) + \sqrt{M^2(m, n) + N(m, n)}) > 0, \]
where \( \kappa_1 = \frac{1}{2}a_l^2, \quad \kappa_2 = \rho_{l,0} - \frac{P_{l,0}}{a_l^2} > 0, \quad \kappa_3 = \frac{(a_g)^2}{a_l^2} \)
and
\[ M(m, n) = \kappa_2 - m - \kappa_3 n, \quad N(m, n) = 4\kappa_2\kappa_3 n. \]

Let us give some explanations about the above model. In fact, the system (1) is a simplified version of the general compressible liquid-gas two-phase flow model. So it is quite necessary to mention the recent work of Friis, Evje and Flåtten [12]. They have investigated the following full 2D compressible liquid-gas two-phase flow model, which is widely used within the petroleum industry to describe production and transport of oil and gas through long pipelines or wells:
\[
\begin{align*}
(\alpha_l \rho_l)_t + \text{div}(\alpha_l \rho_l u_l) &= 0, \\
(\alpha_g \rho_g)_t + \text{div}(\alpha_g \rho_g u_g) &= 0, \\
(\alpha_l \rho_l u_l)_t + \text{div}(\alpha_l \rho_l u_l \otimes u_l) + \alpha_l \nabla P(\alpha_l \rho_l, \alpha_g \rho_g) + \Delta P(\alpha_l \rho_l, \alpha_g \rho_g) \nabla \alpha_l &= Q_l + M_l, \\
(\alpha_g \rho_g u_g)_t + \text{div}(\alpha_g \rho_g u_g \otimes u_g) + \alpha_g \nabla P(\alpha_l \rho_l, \alpha_g \rho_g) + \Delta P(\alpha_l \rho_l, \alpha_g \rho_g) \nabla \alpha_g &= Q_g + M_g,
\end{align*}
\]
here \( u_l \) and \( u_g \) denote liquid velocity and gas velocity, \( \Delta P = P - P^i \) is the interface correction term, \( P^i \) is the pressure at the liquid-gas interface, \( M_g \) and \( M_l \) represent interfacial forces modeling interactions between the two phases and satisfy \( M_g + M_l = 0 \), and \( Q_g \) and \( Q_l \) represent external forces (friction and gravity), respectively. Of course, the system must be supplemented with equations of state for the liquid and gas phase. Here, we skip them, one can refer to [4] for more details. They gave some insight into the characteristic behavior of the above model by numerical methods.

From the mathematical point of view, the system (6) is not in conservation law form due to the pressure terms. Due to the super nonlinear terms \( \Delta P(\alpha_l \rho_l, \alpha_g \rho_g) \nabla \alpha_l \) and \( P(\alpha_l \rho_l, \alpha_g \rho_g) \nabla \alpha_g \) in the two separate momentum equations, it is very challenging to understand and analyze such a system. Especially, when the density function
may vanish or the phase takes vacuum states. In order to overcome these difficulties, we will deal with a simplified version of (8), which gives

\[
\begin{aligned}
&\left\{\begin{array}{l}
\partial_t (\alpha_l \rho_l) = 0, \\
\partial_t (\alpha_g \rho_g) = 0, \\
\partial_t (\alpha_l \rho_l u_l) + \partial_t (\alpha_g \rho_g u_g) = Q_l + Q_g.
\end{array}\right.
\end{aligned}
\]

We find the super nonlinear terms \(\Delta P(\alpha_l \rho_l, \alpha_g \rho_g)\nabla \alpha_l\) and \(P(\alpha_l \rho_l, \alpha_g \rho_g)\nabla \alpha_g\) no longer appear directly. By introducing the viscosity effects, the above system (7) can be rewritten as:

\[
\begin{aligned}
&\left\{\begin{array}{l}
\partial_t (\alpha_l \rho_l) = 0, \\
\partial_t (\alpha_g \rho_g) = 0, \\
\partial_t (\alpha_l \rho_l u_l) + \alpha_g \rho_g u_g = Q_l + Q_g.
\end{array}\right.
\end{aligned}
\]

where the external force \(f = -(Q_l + Q_g)\).

In this paper, we investigate some basic aspects of the above model. In other words, we will deal with a simplified version of (8). If we make the following simplifications:

- Due to the fact that the liquid phase is much heavier than the gas phase, typically to the order \(\frac{\rho_l}{\rho_g} \sim 10^{-3}\), we can neglect the gas phase in the mixture momentum equation;
- The effects of external force are ignored, i.e., \(f = 0\);
- A non-slip condition is assumed, i.e., \(u_l = u_g = u\).

Then by setting \(m = \alpha_l \rho_l\) and \(n = \alpha_g \rho_g\), it is easy to see that the system (8) is exactly the system (1). For more general two-fluid and the simplification (1) of the model, we can refer to [2, 9, 16, 24, 26] and references therein.

Due to the physical importance and mathematical challenges, a great deal of research has been devoted to many topics of the viscous liquid-gas two-phase flow model and related model. The 1D version of (1) with various initial and initial-boundary conditions has been investigated intensively during the past decades, both classical and weak solutions have been constructed, and long time behaviors of different solutions have been investigated. When the liquid is incompressible and the gas is polytropic, the existence, uniqueness, regularity, asymptotic behavior and decay rate estimates of \(\text{weak or classical) solutions to the free boundary problem have been studied in [5, 7, 9, 10, 11, 18, 22, 23]. If both of two fluids are compressible, Evje-Flåtten [6] obtained the global existence of weak solutions. As a generalization of the results in [6] to high dimensions, Yao et al. [24] proved the existence of the global solution to the 2D model when the initial energy is small and the initial density is bounded far away from the vacuum. Later, Guo et al. [13] studied the global existence of the strong solution to the Cauchy problem of the model (1) when the energy of the initial value is sufficiently small and the initial vacuum is allowed. Under the framework of Besov spaces, Hao-Li [14] obtained the existence and uniqueness of the global strong solution to the Cauchy problem of model (1), provided the initial value are close to a constant equilibrium state. The global existence and long time behavior for the Cauchy problem was investigated for the model (1) by Zhang-Zhu [28], where the global strong solution and optimal convergence rates were obtained. We refer to [15, 21, 25, 27] for the different types
of blow-up criteria for the local strong solution to the viscous liquid-gas two-phase flow model.

To do this, we first take a nonlinear transform as in [14, 28] so as to separate the two mass variables from each other, which enables us to decompose the original system into a homogeneous transport equation and a coupled hyperbolic-parabolic system. More precisely, let \( \bar{n}_0 \geq 0 \) and \( \bar{m}_0 > (1 - \text{sgn}\bar{n}_0)k_2 \), we introduce a new variable:

\[
s = \kappa_3 \left( \frac{n}{m} - \frac{\bar{n}_0}{\bar{m}_0} \right) . \tag{9}
\]

From (1)_{1,2}, the variable \( s \) satisfying the following homogeneous transport equation:

\[
s_t + u \cdot \nabla s = 0 , \tag{10}
\]

thus the estimates of the new variable \( s \) are expected to depend only on the mixed velocity.

To overcome the difficulties arising from the non-dissipation on \( m, n \), we will rewrite the system (1). The key idea here is that instead of the variables \((m, n, u)\), we study the system of the variables \((P, u, s)\). To see this, we rewrite the pressure in terms of the variables \((m, s)\) as follows

\[
P(m, s) = k_1 \left\{ \left( 1 + \kappa_3 \frac{\bar{n}_0}{\bar{m}_0} + s \right) m - k_2 \right. \left. + \sqrt{\left[ \left( 1 + \kappa_3 \frac{\bar{n}_0}{\bar{m}_0} + s \right) m - k_2 \right]^2 + 4k_2 \left( s + \kappa_3 \frac{\bar{n}_0}{\bar{m}_0} \right) m} \right\} . \tag{11}
\]

Then, due to (1), (10) and (11), the system (1) can be rewritten in terms of the variables \((P, u, s)\) as follows:

\[
\begin{cases}
P_t + u \cdot \nabla P + K(m, s)m\nabla u = 0 , \\
u m u + mu \cdot \nabla u + \nabla P = \mu \Delta u + (\mu + \lambda) \nabla \text{div} u , \\
s_t + u \cdot \nabla s = 0 ,
\end{cases} \tag{12}
\]

where

\[
K(m, s) = \kappa_1 \left\{ \left( 1 + \frac{\kappa_3 \bar{n}_0}{\bar{m}_0} + s \right) \\
+ \frac{\left[ \left( 1 + \kappa_3 \frac{\bar{n}_0}{\bar{m}_0} + s \right) m - \kappa_2 \left( 1 + \frac{\kappa_3 \bar{n}_0}{\bar{m}_0} + s \right) + 2\kappa_2 \left( \kappa_3 \frac{\bar{n}_0}{\bar{m}_0} + s \right) \right]}{\sqrt{\left[ \left( 1 + \kappa_3 \frac{\bar{n}_0}{\bar{m}_0} + s \right) m - \kappa_2 \right]^2 + 4\kappa_2 \left( \kappa_3 \frac{\bar{n}_0}{\bar{m}_0} + s \right) m}} \right\} . \tag{13}
\]

and \( m = m(P, s) \) can be obtained from (9) and (11) as follows:

\[
m = \frac{P^2 + 2\kappa_1 \kappa_2 P}{4\kappa_1^2 \kappa_2 (s + \frac{n \bar{n}_0}{m \bar{m}_0}) + 2\kappa_1 (s + 1 + \frac{n \bar{n}_0}{m \bar{m}_0})} . \tag{14}
\]

It is worth mentioning that the system (12) is a hyperbolic-parabolic system, where the dissipation comes from viscosity. In the present paper, we consider the initial boundary value problem for system (12), which is supplemented by the following initial and boundary conditions:

\[
\begin{cases}
(P, u, s)(x, 0) = (P_0, u_0, s_0)(x) , \\
u |_{\partial \Omega} = 0 , \\
1 |_{\Omega} \int_\Omega P_0 dx = P_0 ,
\end{cases} \tag{15}
\]

where $\bar{P}_0$ is a positive constant due to (5) and the fact that $\bar{n}_0 \geq 0$ and $\bar{m}_0 > (1 - \sgn \bar{n}_0)k_2$.

Before we state the main results, let us introduce some notations for the use throughout this paper. $C$ denotes some positive constant which may vary at different places. The norms in the Sobolev Spaces $H^m(\Omega)$ and $W^{m,q}(\Omega)$ are denoted respectively by $\| \cdot \|_m$ and $\| \cdot \|_{m,q}$ for $m \geq 0$ and $q \geq 1$. In particular, for $m = 0$ we will simply use $\| \cdot \|$ and $\| \cdot \|_{L^\infty}$. Finally,

$$\nabla = (\partial_1, \partial_2, \partial_3), \quad \partial_i = \partial_{x_i}, \ i = 1, 2, 3,$$

and for any integer $l \geq 0$, $\nabla^l f$ denotes all derivatives of order $l$ of the function $f$.

Now, we are in a position to state our main results:

**Theorem 1.1.** Let $\bar{n}_0 \geq 0$, $\bar{m}_0 > 1 - \sgn \bar{n}_0)k_2$ and assume that the initial boundary value $(P_0 - \bar{P}_0, u_0, s_0)$ satisfies the compatibility condition, i.e., $\partial_t^l u(0)|_{\partial \Omega} = 0, l = 0, 1$, where $\partial_t^l u(0)|_{\partial \Omega} = 0$ is the $l$th time derivative at $t = 0$ of any solution of the system (12)-(15), as calculated from (12) to yield an expression in terms of $P_0, u_0$ and $s_0$. Then there exists a constant $\varepsilon_0$ such that if

$$\|(P_0 - \bar{P}_0, u_0, s_0)\|_2 \leq \varepsilon_0, \quad (16)$$

the initial boundary value problem (12)-(15) admits a unique solution $(P, u, s)$ globally in time with $P > 0$, which satisfies

$$P - \bar{P}, s \in C^0([0, \infty); H^2(\Omega)) \cap C^1([0, \infty); H^1(\Omega)),
\quad u \in C^0([0, \infty); H^2(\Omega) \cap H^1_0(\Omega)) \cap C^1([0, \infty); L^2(\Omega)),$$

where

$$\bar{P}(t) = \frac{1}{|\Omega|} \int_\Omega P(x, t)dx. \quad (17)$$

Moreover, there exist two positive constant $C_0$ and $\eta_0$ such that for any $t \geq 0$, the following estimates hold

$$\|(P - \bar{P}, u)(t)\|_2 + \int_0^t (\|P(\tau) - \bar{P}(\tau)\|_2^2 + \|u(\tau)\|_2^3) d\tau \leq C_0\|(P_0 - \bar{P}_0, u_0)\|_2^2, \quad (18)$$

$$\|s(t)\|_2 \leq C_0\|s_0\|_2 \exp\{C_0\|(P_0 - \bar{P}_0, u_0)\|_2\}, \quad (19)$$

$$\|(P - \bar{P}, u)(t)\|_2 + \|\partial_t(P - \bar{P}, u, s)(t)\| \leq C_0\|(P_0 - \bar{P}_0, u_0)\|_2 \exp\{-\eta_0 t\}. \quad (20)$$

Finally, $\lim_{t \to \infty} \bar{P}(t)$ exists and let $\lim_{t \to \infty} \bar{P}(t) = \bar{P}$, the following convergence rate holds

$$\|\bar{P} - \bar{P}(t)\| \leq C_0\|(P_0 - \bar{P}, u_0, s_0)\|_2^2 \exp\{-\eta_0 t\}. \quad (21)$$

**Remark 1.** It is interesting to make a comparison between Theorem 1.1 and those of Zhang-Zhu [28]. It seems impossible to rewrite the equation (12) for $P$ in the conservative form owning to $K(m, s)m$ not only dependent on the variable $P$ but also dependent on the variable $s$. Thus we do not expect that the equilibrium $P$ of the pressure $P$ is a constant. It is worth mentioning that the convergence rates (20) and (21) are somewhat surprising, since the solution relaxes in the maximum norm to the constant background state at a rate of $(1 + t)^{-5/4}$ in the Cauchy problem case (see [28]).

**Remark 2.** Our results of this paper are also right for the two-dimensional case. However, since $\|\nabla (P, u)(t)\|_1$ of the linear solution to system (12) decays only as $(1 + t)^{-1}$ in Cauchy problem (see [28]), which is not integrable, in particular making the strategy of [28] difficult to apply, construction of global existence and optimal
convergence rates for Cauchy problem of system (12) in the two-dimensional case is still an open problem.

**Remark 3.** Noting that $\bar{n}_0$ may be zero in Theorem 1.1, it is clear that our results still hold when the gas contains initial vacuum.

Now, we sketch the main idea of the proof and explain some the main difficulties and techniques involved in the process. First, due to the non-dissipation property for the variables $m$ and $n$, we rewrite the system (1) into (12) as in [28]. Then the dissipative variables $P$ and $u$ satisfy (12)$_1$-(12)$_2$ whose linear parts possess the same structure as that of the compressible isentropic Navier-Stokes equations, while the non-dissipative variable $s$ satisfies the homogeneous transport equation (12)$_3$. Thus in order to obtain a priori estimates to (12)-(15), we can apply the similar energy method as in [28] to the first two equations of (12) to obtain the uniform bound of $(P - \bar{P}, u)$ under the assumption that $\|(P - \bar{P}, u, s)\|_2$ is sufficiently small, see Lemmas 3.1-3.6 in Section 3. With these in hand, the variables $(P - \bar{P}, u)$ can be shown to converge exponentially to zero from the Poincaré's inequality and Gronwall's inequality. However, we should point out that since $\bar{P}(t)$ is not a constant, which makes the problem become much more difficult and needs us to develop some new energy estimates as in Lemmas 3.1-3.6. It is worth mentioning that the crucial part of the proof is to obtain a Lyapunov-type energy inequality (74). Then, the bound of $s$ will be derived by the exponential decay estimates on $(P - \bar{P}, u)$ and the Gronwall's inequality. Second, comparing to Cauchy problem [28], when establishing a priori estimates by the standard energy method, a new difficulty arises since the spatial derivatives are unknown on the boundary. To overcome this difficulty, we separate the energy estimates for the spatial derivatives into that over the region away from the boundary and near the boundary in spirit of Matsumura and Nishida [20]. In other words, we establish the energy estimates for the spatial derivatives by using cutoff functions and localizations of $\partial\Omega$. Although our proofs are in spirit of those for the Navier-Stokes equations, we should derive the new estimates due to the different dissipative effect and the special nonlinearity of (12). The main novelty of this paper is to obtain the global existence and exponential convergence rates of strong solutions in $H^2$ by getting the key uniform time-independent energy estimates on the solutions.

The plan of the rest of this article is as follows. In Section 2, we give some basic facts that will be used in this paper together with the local existence result. In Section 3, we do some careful a priori estimates for the strong solutions and then the global existence of the strong solutions is established by combining our a priori estimates and the local existence result.

### 2. Local existence and preliminaries.

In this section, we will recall some well-known facts and elementary inequalities that will be used frequently later. Let us begin with the local existence and uniqueness of the strong solution of problem (12)-(15). This does not rely much on the structure of the equations. Recently, Kagei-Kawashima [17] have proved the local $H^s$-solvability ($s \geq \lceil n/2 \rceil + 1$ being an integer) of the initial boundary problem for a general class of hyperbolic-parabolic system. In fact, we have the following local well-posedness theorem, which directly from the classical result in [17].

**Proposition 1.** *(Local existence).* Let $(P_0, u_0, s_0) \in H^2(\Omega)$ be such that

$$\inf_{x \in \Omega} \{P_0(x)\} > 0 \quad \text{and} \quad \partial^l_l u_0|_{\partial\Omega} = 0, l = 0, 1.$$
Then there exist positive numbers $T$ and $C$ such that problem (12)-(15) has a unique solution $(P, u, s) \in C([0, T]; H^2(\Omega))$ which satisfies
\[
\inf_{t \in [0, T], x \in \Omega} \{P(t, x)\} > 0, \quad P, s, u \in C([0, T]; H^1(\Omega)), \quad u \in L^2([0, T]; H^3(\Omega)), \quad u_t \in C([0, T]; L^2(\Omega)) \quad \text{and}
\]
\[
\|P(t)\|_2 + \|u(t)\|_2 + \|s(t)\|_2 \leq C(\|P_0\|_2 + \|u_0\|_2 + \|s_0\|_2).
\]

For later use we review some inequalities of Sobolev type (cf. [1, 29]), which read:

**Lemma 2.1.** Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^3$ and $f \in H^2(\Omega)$. Then, it holds:

(i) $\|f\|_{L^\infty} \leq C\|f\|_2$,

(ii) $\|f\|_{L^p} \leq C\|f\|_1, \quad 2 \leq p \leq 6,$

for some constant $C > 0$ depending only on $\Omega$.

Due to the slip boundary condition, the classical energy estimates can not be applied directly to spatial derivatives. We introduce the following lemma on the stationary Stokes equations to get the estimates on the tangential derivatives of both $P$ and $u$, cf. [20].

**Lemma 2.2.** Let $\Omega$ be any bounded domain in $\mathbb{R}^3$ with smooth boundary. Consider the problem
\[
\begin{cases}
-\mu \Delta u + \nabla P = g, \\
\text{div} u = f, \\
u|_{\partial \Omega} = 0,
\end{cases}
\]

where $f \in H^{k+1}(\Omega)$ and $g \in H^k$ ($k \geq 0$). Then the above problem has a solution $(P, u) \in H^{k+1} \times (H^{k+2} \cap H^0)$ which is unique modulo a constant of integration for $P$. Moreover, this solution satisfies
\[
\|u\|_{k+2}^2 + \|
abla P\|_k^2 \leq C\{\|f\|_{k+1}^2 + \|g\|_k^2\}.
\]  

3. The proof of global existence. In this section, we shall prove the global existence and large time behavior of the solution with small initial data (Theorem 1.1). The global existence of smooth solution of problem (12)-(15) can be established by the local existence theory, the uniformly a priori estimates, and the continuity argument. Thus it suffices for us to establish a priori estimate. Therefore, we assume a priori that
\[
\|(P - \bar{P}, u, s)(t)\|_2 + |\bar{P}(t) - \bar{P}_0| \leq \varepsilon \ll 1, \quad \text{for any} \quad t \geq 0.
\]  

Under the assumption (23), together with Sobolev’s inequality implies that
\[
\frac{1}{2} \bar{P}_0 \leq P(t) \leq 2\bar{P}_0, \quad \frac{1}{C} \leq m(t) \leq C \quad \text{for all} \quad t \geq 0.
\]  

This should be kept in mind in the rest of this paper.

In order to deduce the a priori estimate, in what follows, we will give some energy estimates in a few lemmas. First of all, we derive the lowest-order energy estimates.

**Lemma 3.1.** Under the conditions of Theorem 1.1 and (23), there exists a positive constant $C$ such that for any $t \geq 0$, it holds
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega m \|u\|^2 + \frac{(P - \bar{P})^2}{K} dx + \mu \int_\Omega |
abla u|^2 dx + (\mu + \lambda) \int_\Omega |\text{div} u|^2 dx \\
\leq C\varepsilon(\|\nabla P\|^2 + \|\nabla u\|^2),
\]

(25)
where
\[ \bar{K} = K(m(\bar{P}(t), 0), 0)m(\bar{P}(t), 0) > 0. \]  

**Proof.** A standard energy estimate for the equation (12) on \( u \) gives
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} m|u|^2 dx + \mu \int_{\Omega} |\nabla u|^2 dx + (\mu + \lambda) \int_{\Omega} |\text{div} u|^2 dx + \int_{\Omega} u \cdot \nabla P dx = 0. 
\]  

To get the estimate on \( P \), we shall deduce the equation of \( \bar{P}(t) \). By integrating (12)_1 over \( \Omega \) gives
\[
\bar{P}_t = \frac{1}{|\Omega|} \int_{\Omega} u \cdot \nabla(K(m, s)m - \bar{P}) dx. 
\]

Combining the above equality, (23) and (24), we obtain
\[
|\bar{P}_t| \leq C\|u\|(|\nabla P| + \|\nabla s\|), 
\]
where by (13) and (23), we have used the fact
\[
\|\nabla m\| \leq C(|\nabla P| + \|\nabla s\|).
\]

Next, we rewrite (12)_1 in the linear form as following:
\[
\frac{(P - \bar{P})_t}{K} + \text{div} \left( \frac{\bar{P}_t + [K(m, s)m - \bar{K})]\text{div}u + u \cdot \nabla P}{K} \right) = 0. 
\]

Multiplying the above equality by \( P - \bar{P} \) and integrating over \( \Omega \) give
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \frac{(P - \bar{P})^2}{K} dx + \int_{\Omega} \text{div}(P - \bar{P}) dx 
\]
\[
= -\frac{1}{2} \int_{\Omega} \frac{(P - \bar{P})^2(K)\bar{P}_t dx + \int_{\Omega} \bar{P}_t + [K(m, s)m - \bar{K})\text{div}u + u \cdot \nabla P}{K} (P - \bar{P}) dx. 
\]

Adding the above equality to (27), we finally obtain
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} m|u|^2 + \frac{(P - \bar{P})^2}{K} dx + \mu \int_{\Omega} |\nabla u|^2 dx + (\mu + \lambda) \int_{\Omega} |\text{div} u|^2 dx 
\]
\[
= -\frac{1}{2} \int_{\Omega} \frac{(P - \bar{P})^2(K)\bar{P}_t dx + \int_{\Omega} \bar{P}_t + [K(m, s)m - \bar{K})\text{div}u + u \cdot \nabla P}{K} (P - \bar{P}) dx. 
\]

By using (23), (24), (29) , Lemma 2.1, Hölder’s inequality and Poincaré’s inequality, the right terms of the above equation can be estimated as following:
\[
\left| \int_{\Omega} \frac{(P - \bar{P})^2[K\bar{P}_t]}{K^2} dx \right| \leq C \left| \frac{(\bar{K})\bar{P}_t}{K^2} \right| \|P - \bar{P}\|^2 
\]
\[
\leq C\|u\|(|\nabla P| + |\nabla s|)\|P - \bar{P}\|^2 \leq C\varepsilon\|\nabla P\|^2, 
\]
\[
\left| \int_{\Omega} \frac{\bar{P}_t + u \cdot \nabla P}{K} (P - \bar{P}) dx \right| \leq C \frac{[P_1||P - \bar{P}|| + \|u\|_{L^6}\|\nabla P\||\|P - \bar{P}\|_{L^6})}{K} 
\]
\[
\leq C\|u\|(|\nabla P| + |\nabla s|)\|P - \bar{P}\| + \|\nabla u\|\|\nabla P\|^2 
\]
\[
\leq C\varepsilon(\|\nabla u\|^2 + \|\nabla P\|^2), 
\]
Next, in the following lemma we give the energy estimate of the time derivative for \((P, u)\).

**Lemma 3.2.** Under the conditions of Theorem 1.1 and (23), there exists a positive constant \(C\) such that

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} m|u|^2 + \frac{(P_t - \bar{P}_t)^2}{K} dx + \mu \int_{\Omega} |\nabla u_t|^2 dx + (\mu + \lambda) \int_{\Omega} |\text{div} u_t|^2 dx \\ \leq C \varepsilon (|\nabla u|^2 + |\nabla u_t|^2),
\]

for any \(t \geq 0\).

**Proof.** Differentiating (12) and (31) with respect to \(t\), we have

\[
\left\{ \begin{array}{l}
m u_{tt} + m u_t + (m u \cdot \nabla u)_t + \nabla P_t = \mu \Delta u_t + (\mu + \lambda) \nabla \text{div} u_t, \\
\frac{(P - \bar{P})_t}{K} - \frac{(P - \bar{P})_t}{K}(K) \frac{\mu}{P_t} \\
\quad + \text{div} u_t + \left[ \frac{\bar{P}_t + [K(m, s)m - \bar{K}] \text{div} u + u \cdot \nabla P}{K} \right]_t = 0.
\end{array} \right.
\]

Multiplying the above system by \(u_t\) and \((P - \bar{P})_t\) respectively, and then integrating them over \(\Omega\), we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} m|u|^2 + \frac{(P_t - \bar{P}_t)^2}{K} dx + \mu \int_{\Omega} |\nabla u_t|^2 dx + (\mu + \lambda) \int_{\Omega} |\text{div} u_t|^2 dx \\ = -\frac{1}{2} \int_{\Omega} m|u|^2 - \frac{(P_t - \bar{P}_t)^2}{K} \frac{\mu}{P_t} dx - \int_{\Omega} (m u \cdot \nabla u_t) u_t dx \\
\quad - \int_{\Omega} \left[ \frac{\bar{P}_t + [K(m, s)m - \bar{K}] \text{div} u + u \cdot \nabla P}{K} \right]_t (P - \bar{P})_t dx,
\]

where we have used the boundary condition: \(u_t|_{\partial \Omega} = 0\). For the first term on the right hand side of the above equality, we have from (23), (24), (29), (30), Lemma 2.1, Hölder’s inequality and Poincaré’s inequality that

\[
\left| \int_{\Omega} m|u|^2 - \frac{(P_t - \bar{P}_t)^2}{K^2} \frac{\mu}{P_t} dx \right| = \left| \int_{\Omega} \text{div}(mu)|u|^2 + \frac{(P_t - \bar{P}_t)^2}{K^2} \frac{\mu}{P_t} dx \right| \\
= \left| \int_{\Omega} 2mu \nabla u_t - \frac{(P_t - \bar{P}_t)^2}{K^2} \frac{\mu}{P_t} dx \right|
\]
where by (23) and (12), we have used the fact
\[\|P_t\|_{L^p} \leq C(\|\nabla u\|_{L^p} + \|u \cdot \nabla P\|_{L^p}) \leq C(\|\nabla u\|_{L^p} + \|u\|_{L^\infty} \|\nabla P\|_{L^p}) \leq C\|\nabla u\|_1, \tag{41}\]
for \(1 \leq p \leq 6\). Similarly, for the second term, we have
\[
\left| \int \Omega (mu \cdot \nabla u)_{t} dt \right| = \left| \int \Omega m u \cdot \nabla uu_{t} + mu_{t} \cdot \nabla uu_{t} + mu \cdot \nabla uu_{t} dt \right|
\leq C\|m\|_{L^\infty} \|u\|_{L^2} \|\nabla u\|_{L^2} \|\nabla uu_{t}\|_{L^2} + \|m\|_{L^\infty} \|u\|_{L^2} \|\nabla uu_{t}\|_{L^2} \|\nabla u_{t}\|_{L^2}
\leq C\|\nabla u\|^2 + \|\nabla u_{t}\|^2, \tag{42}\]
where by (14) and (23), we have used the fact
\[\|\nabla m\|_1 \leq C(\|\nabla P\| + \|\nabla s\|). \tag{43}\]
Next, together with (12)_3, (23), (28) and (41), we can obtain
\[
|\hat{P}_t| \leq C \left( \left| \int \Omega u_{t} \nabla (P - K(m,s)m) dt \right| + \left| \int \Omega \text{div}(P - K(m,s)m) dt \right| \right)
\leq C(\|u_t\| \|\nabla P\| + \|\nabla s\|) + \|\text{div}(P)\| \|\nabla u\| \|\nabla s\|)
\leq C\varepsilon(\|u_t\| + \|\nabla u\| + \|u\|). \tag{44}\]
Thus, for the last term, combining (41), (44) and (42) gives
\[
\left| \int \Omega \frac{\hat{P}_t + [K(m,s)m - \hat{K}] \text{div} u + u \cdot \nabla P}{K} (P - \bar{P}) dt \right|
\leq \left| \int \Omega \frac{\hat{P}_t + [Km] \rho \hat{P}_t + [Km]s_t - (\hat{K}) \rho \hat{P}_t}{K} \text{div} - [Km - \hat{K}] \text{div} u_t (P - \bar{P}) dt \right|
+ \left| \int \Omega \frac{\rho \nabla P + u \cdot \nabla u}{K} (P - \bar{P}) dt \right|
+ \left| \int \Omega \frac{(Km, s)m - \hat{K}] \text{div} u + u \cdot \nabla P}{K^2} (P - \bar{P}) dt \right|
\leq C(\|P_t\| + \|\hat{P}_t\| + \|\text{div} u\|_2 \|P_t\|_L^2) (\|P_t\|_{L^2} + \|s_t\|_{L^2})
+ |\hat{P}_t| \|\text{div} u\|_2 (\|P_t\|_{L^2} + \|s_t\|_{L^2} + |\hat{P}_t|)
constant $C$

Lemma 3.3. \( \chi \) estimates of solution into that over the region away from the boundary and near \( \Omega \) so as to deal with the boundary estimates of the solutions. More precisely, we will modify the standard technique developed in [20] that involves separating the estimates of solution into that over the region away from the boundary and near the boundary. Let \( \chi_0 \) be an arbitrary but fixed function in \( C_0^\infty(\Omega) \). Then, we have the following energy estimates on the region away from the boundary.

Lemma 3.3. Under the conditions of Theorem 1.1 and (23), there exists a positive constant $C$ such that

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega m|\nabla u_0|_0^2 + \frac{|\nabla P_0|_0^2}{K} + |u_t|_0^2 + \frac{\int_\Omega |\nabla \delta u|_0^2 dx}{K} + (\mu + \lambda) \int_\Omega |\nabla \delta u|_0^2 dx \\
\leq C \epsilon (\|\nabla u_t\|_0^2 + \|\nabla u_t\|_0^2 + \|\nabla u\|_0^2) + C \|\nabla u(t)\|_0^2 + \|\nabla \delta u\|_0^2,
\]

for any $t \geq 0$.

Proof. As in Lemma 3.2, here, we only sketch the outline and shall omit the detailed calculations for simplicity. Differentiating (12) and (31) with respect to \( x_i \), multiplying the resulting equations by \( u_{x_i} \), \( P_{x_i} \), \( \chi_0 \) respectively, we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega m|u_{x_i}|_0^2 + \frac{|P_{x_i}|_0^2}{K} + |u_{x_i}|_0^2 + \frac{\int_\Omega |\nabla \delta u|_0^2 dx}{K} + (\mu + \lambda) \int_\Omega |\nabla \delta u|_0^2 dx \\
= \frac{1}{2} \int_\Omega m|u_{x_i}|_0^2 - \frac{|P_{x_i}|_0^2}{K} (K_{x_i} - \hat{K}_{x_i}) |\nabla u|_0^2 dx \\
- \int_\Omega \left[ P_{x_i} + [K(m, s) - \hat{K}] |\nabla u|_0^2 \right]_{x_i} \chi_0^2 dx \\
- (\mu + \lambda) \int_\Omega |\nabla \delta u|_0^2 dx \\
\leq C \epsilon (\|\nabla u_t\|_0^2 + \|\nabla u_t\|_0^2 + \|\nabla \delta u\|_0^2) + C \|\nabla u(t)\|_0^2 + \|\nabla \delta u\|_0^2,
\]

Substituting (40), (43), and (45) into (39) gives (41). The proof of Lemma 3.2 is completed. \( \square \)

Due to the no-slip boundary condition, the classical energy estimates can not be applied directly to spatial derivatives. To overcome the difficulty, we need to localize for any \( t \in [0, T] \) the classical energy estimates. More precisely, we will modify the standard technique developed in [20] that involves separating the estimates of solution into that over the region away from the boundary and near the boundary. Let \( \chi_0 \) be an arbitrary but fixed function in \( C_0^\infty(\Omega) \). Then, we have the following energy estimates on the region away from the boundary.

Lemma 3.3. Under the conditions of Theorem 1.1 and (23), there exists a positive constant $C$ such that

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega m|u_{t, \chi_0}|_0^2 + \frac{|P_{t, \chi_0}|_0^2}{K} + |u_{t, \chi_0}|_0^2 + \frac{\int_\Omega |\nabla \delta u_{t, \chi_0}|_0^2 dx}{K} + (\mu + \lambda) \int_\Omega |\nabla \delta u_{t, \chi_0}|_0^2 dx \\
\leq C \epsilon (\|\nabla u_{t, \chi_0}\|_0^2 + \|\nabla u_{t, \chi_0}\|_0^2 + \|\nabla P\|_0^2) + C \|\nabla u_{t, \chi_0}(t)\|_0^2 + \|\nabla \delta u_{t, \chi_0}\|_0^2,
\]

for any \( t \geq 0 \).

Proof. As in Lemma 3.2, here, we only sketch the outline and shall omit the detailed calculations for simplicity. Differentiating (12) and (31) with respect to \( x_i \), multiplying the resulting equations by \( u_{x_i, \chi_0} \), \( P_{x_i, \chi_0} \) respectively, we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega m|u_{x_i, \chi_0}|_0^2 + \frac{|P_{x_i, \chi_0}|_0^2}{K} + |u_{x_i, \chi_0}|_0^2 + \frac{\int_\Omega |\nabla \delta u_{x_i, \chi_0}|_0^2 dx}{K} + (\mu + \lambda) \int_\Omega |\nabla \delta u_{x_i, \chi_0}|_0^2 dx \\
= \frac{1}{2} \int_\Omega m|u_{x_i, \chi_0}|_0^2 - \frac{|P_{x_i, \chi_0}|_0^2}{K} (K_{x_i} - \hat{K}_{x_i}) |\nabla u|_0^2 dx \\
- \int_\Omega \left[ P_{x_i} + [K(m, s) - \hat{K}] |\nabla u|_0^2 \right]_{x_i} \chi_0^2 dx \\
- (\mu + \lambda) \int_\Omega |\nabla \delta u_{x_i, \chi_0}|_0^2 dx \\
\leq C \epsilon (\|\nabla u_{x_i, \chi_0}\|_0^2 + \|\nabla u_{x_i, \chi_0}\|_0^2 + \|\nabla \delta u\|_0^2) + C \|\nabla u_{x_i, \chi_0}(t)\|_0^2 + \|\nabla \delta u_{x_i, \chi_0}\|_0^2,
\]
which implies (46). Repeating the above procedure again for 2nd order spatial derivatives we get the following
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega m|u_{x_ix_j}x_0|^2 + \frac{|P_{x_ix_j}x_0|^2}{K} + \mu \int_\Omega |\nabla u_{x_ix_j}x_0|^2 + (\mu + \lambda) \int_\Omega |\text{div} u_{x_ix_j}x_0|^2
\]
\[
= \frac{1}{2} \int_\Omega m_i |u_{x_ix_j}x_0|^2 - \frac{|P_{x_ix_j}x_0|^2[K]_i P_i}{K^2} - \int_\Omega [m_{x_ix_j}u_t + m_{x_i}u_{tx_j} + m_{x_j}u_{tx_i}]
\]
\[
+ (m \cdot \nabla u)_{x_ix_j} |u_{x_ix_j}x_0|^2 - \int_\Omega \left[ \frac{\bar{P}_i + [K(m,s)m - \bar{K}] \text{div} u + u \cdot \nabla P}{K} \right] P_{x_ix_j}x_0^2
\]
\[
- \mu \int_\Omega u_{x_ix_j} \nabla u_{x_ix_j} \nabla \chi_0^2 dx - (\mu + \lambda) \int_\Omega \text{div} u_{x_ix_j} u_{x_ix_j} \nabla \chi_0^2 + \int_\Omega P_{x_ix_j} u_{x_ix_j} \nabla \chi_0^2
\]
\[
\leq C \varepsilon (\|\nabla u\|_2^2 + \|\nabla u_t\|^2 + \|\nabla P\|_2^2) + C\|\nabla^2 u\| (\|\nabla^3 u\|_0^2 + \|\nabla^2 P\|_0^2),
\]
which gives (47). The proof of Lemma 3.3 is completed. \(\square\)

Finally, let us establish the estimates near the boundary. Similar to that in [20], we need a more detailed argument using the trick of estimating the tangential derivatives and the normal derivatives separately. We choose a finite number of bounded open sets \(\{O_j\}_{j=1}^N\) in \(\mathbb{R}^3\), such that \(\partial \Omega \subset \bigcup_{j=1}^N O_j\). In each open set \(O_j\) we choose the local coordinates \(y = (y_1, y_2, y_3)\) as follows:

- The surface \(O_j \cap \partial \Omega\) is the image of a smooth vector function \(z^j(y_1, y_2) = (z^j_1, z^j_2, z^j_3)(y_1, y_2)\) (e.g., take the local geodesic polar coordinate), satisfying
  \[
  |z^j_1| = 1, z^j_1 \cdot z^j_2 = 0 \quad \text{and} \quad |z^j_2| = \delta > 0,
  \]
  where \(\delta\) is some positive constant independent of \(1 \leq j \leq N\).
- Any \(x = (x_1, x_2, x_3) \in O_j\) is represented by
  \[
  x_i := \Psi_i(y) = y_3 \Psi_i^j(z^j(y_1, y_2)) + z^j_i(y_1, y_2) \quad \text{for} \quad i = 1, 2, 3,
  \]
  where \(\Psi_i(y_1, y_2) = (\Psi_1^i, \Psi_2^i, \Psi_3^i)(z^j(y_1, y_2))\) represents the internal unit normal vector at the point \(z^j(y_1, y_2)\) of the surface \(\partial \Omega\).

We omit the subscript \(j\) in what follows for the simplicity of presentation. For \(k = 1, 2\), we define the unit vectors
\[
e_1 = z_1 \quad \text{and} \quad e_2 = \frac{z_2}{|z_2|}.
\]
Then Frenet-Serret’s formula gives that there exist smooth functions \((\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2)\) of \((y_1, y_2)\) satisfying
\[
\frac{\partial}{\partial y_1} \left( \begin{array}{c} e_1 \\ e_2 \\ \Psi \end{array} \right) = \left( \begin{array}{ccc} 0 & -\gamma_1 & -\alpha_1 \\ \gamma_1 & 0 & -\beta_1 \\ \alpha_1 & \beta_1 & 0 \end{array} \right) \left( \begin{array}{c} e_1 \\ e_2 \\ \Psi \end{array} \right),
\]
\[
\frac{\partial}{\partial y_2} \left( \begin{array}{c} e_1 \\ e_2 \\ \Psi \end{array} \right) = \left( \begin{array}{ccc} 0 & -\gamma_2 & -\alpha_2 \\ \gamma_2 & 0 & -\beta_2 \\ \alpha_2 & \beta_2 & 0 \end{array} \right) \left( \begin{array}{c} e_1 \\ e_2 \\ \Psi \end{array} \right),
\]
where \(e^i_m\) denote the \(i\)-th component of \(e_m\). An elementary calculation shows that the Jacobian \(J\) of the transform \((49)\) is
\[
J = \Psi_{y_1} \times \Psi_{y_2} \cdot \Psi = |z_2| + (\alpha_1|z_2| + \beta_2)y_3 + (\alpha_1\beta_2 - \beta_1\alpha_2)y_3\]
\[
= \frac{(\alpha_1|z_2| + \beta_2)y_3 + \alpha_2\beta_2 - \beta_1\alpha_2}{\alpha_1\beta_2 - \beta_1\alpha_2} y_3^2. \quad (50)
\]
By (50), we have the transform \((49)\) is regular by choosing \(y_3\) so small that \(J \geq \delta/2\) for some positive \(\delta\). Therefore, the inverse function of \(\Psi(y) := (\Psi_1, \Psi_2, \Psi_3)(y)\)
exists, and we denote it by \( y = \Psi^{-1}(x) \). Moreover \( (y_1, y_2, y_3, x) \) make sense and can be expressed by, using a straightforward calculation,

\[
\begin{align*}
\partial_{x_1} y_1 &= \frac{1}{f}(\Psi_{y_2} \times \Psi_{y_3})_i = \frac{1}{f}(A e_i^1 + B e_i^2) =: a_{1i}, \\
\partial_{x_2} y_2 &= \frac{1}{f}(\Psi_{y_3} \times \Psi_{y_1})_i = \frac{1}{f}(C e_i^1 + D e_i^2) =: a_{2i}, \\
\partial_{x_3} y_3 &= \frac{1}{f}(\Psi_{y_1} \times \Psi_{y_2})_i = \Omega_i =: a_{3i},
\end{align*}
\]

where \( A = \|z_{y_2}\| + \beta_2 y_3, B = -y_3 \alpha_2, C = -\beta_1 y_3, D = 1 + \alpha_1 y_3, \)

\[ J = AD - BC \geq \delta/2. \]

Obviously, (51) gives

\[
\sum_{i=1}^{3} a_{3i}^2 = |u|^2 = 1, \quad a_1 a_3 = a_2 a_3 = 0, \quad J^2 = (AC + BD)^2 - (A^2 + B^2)(C^2 + D^2)
\]

and

\[
\partial_{x_i} = a_{ki} \partial_{y_k},
\]

where we have used the Einstein convention of summing over repeated indices.

Thus, in each \( O_j \), the first two equations of (12) can be rewritten in the local coordinates \( (y_1, y_2, y_3) \) as follows:

\[
\begin{align*}
F^p &:= \frac{dP}{dt} + \tilde{K} \left[ (A e_1 + B e_2) \cdot u_{y_1} + (C e_1 + D e_2) \cdot u_{y_2} + J \mathcal{N} \cdot u_{y_3} \right] = g, \\
F^u &:= \mu u_t - \frac{\mu}{J^2} \left[ (A^2 + B^2) u_{y_1 y_1} + 2(A C + B D) u_{y_1 y_2} + (C^2 + D^2) u_{y_2 y_2} \right. \\
&\quad \left. + J^2 u_{y_3 y_3} \right] + \text{one order terms of } u + \frac{1}{J} (A e_1 + B e_2) \left[ \frac{\mu + \lambda}{K} \frac{dP}{dt} + P \right]_{y_1} \\
&\quad + \frac{1}{J} (C e_1 + D e_2) \left[ \frac{\mu + \lambda}{K} \frac{dP}{dt} + P \right]_{y_2} + \mathcal{N} \left[ \frac{\mu + \lambda}{K} \frac{dP}{dt} + P \right]_{y_3} = h,
\end{align*}
\]

where

\[
\begin{align*}
\frac{d}{dt} &= \partial_t + u \cdot \nabla \text{ denotes the material derivative,} \\
g &= -(K(m, s)m - \tilde{K}) \text{div} u, \\
h &= \mu u_t \cdot \nabla u + \frac{\mu}{K} \nabla g, \\
J^2 &= (AC + BD)^2 - (A^2 + B^2)(C^2 + D^2).
\end{align*}
\]

Let us denote the tangential derivatives by \( \partial = (\partial_{y_1}, \partial_{y_2}) \) and \( \chi_j \) be arbitrary but fixed function in \( C^\infty_0(\Omega_j) \). Obviously, \( \chi_j \partial^k u = 0 \) on \( \partial \Omega_j^{-1} \), where \( 0 \leq k \leq 2 \) and \( \Omega_j^{-1}(y) := \{ y | y = \Psi^{-1}(x), x \in \Omega_j \in \Omega_j \cap \Omega \} \). Estimating the tangential derivatives in the similar way as the above lemma, we have

**Lemma 3.4.** Under the conditions of Theorem 1.1 and (23), there exists a positive constant \( C \) such that

\[
\begin{align*}
\frac{d}{dt} \int_{\Omega_j^{-1}} m |\partial u \chi_j|^2 + \frac{|\partial P \chi_j|^2}{K} dy + \int_{\Omega_j^{-1}} |\partial \nabla u \chi_j|^2 dy &+ \int_{\Omega_j^{-1}} \left| \frac{\partial}{dt} \chi_j \right|^2 dy \leq C \varepsilon (\|\nabla u\|^2 + \|\nabla u\|^2 + \|\nabla P\|^2) + C \|\nabla u\|(\|\nabla u\| + \|\nabla P\|), \\
\frac{d}{dt} \int_{\Omega_j^{-1}} m |\partial^2 u \chi_j|^2 + \frac{|\partial^2 P \chi_j|^2}{K} dy &+ \int_{\Omega_j^{-1}} |\partial^2 \nabla u \chi_j|^2 dy + \int_{\Omega_j^{-1}} \left| \frac{\partial^2}{dt} \chi_j \right|^2 dy \leq C \varepsilon (\|\nabla u\|^2 + \|\nabla u\|^2 + \|\nabla P\|^2) + C \|\nabla^2 u\|(\|\nabla u\| + \|\nabla^2 P\|),
\end{align*}
\]
for any $t \geq 0$, $1 \leq j \leq N$.

Next, we turn to deduce the estimates of derivatives in the normal directions.

**Lemma 3.5.** Under the conditions of Theorem 1.1 and (23), there exists a positive constant $C$ such that

\[
\frac{d}{dt} \int_{\Omega_j^{-1}} |P_{y_j} \chi_j|^2 dy + \int_{\Omega_j^{-1}} \left| \left( \frac{dP}{dt} \right)_{y_j} \right|^2 dy \\
\leq C \left[ \|u(t)\|^2 + \|u\|^2 + \varepsilon(\|\nabla P\|^2_1 + \|\nabla u\|^2_1) + \int_{\Omega_j^{-1}} |\partial \nabla u \chi_j|^2 dy \right],
\]

\[\tag{56}\]

\[
\frac{d}{dt} \int_{\Omega_j^{-1}} |\partial^{k+1} P \chi_j|^2 dy + \int_{\Omega_j^{-1}} \left| \partial^{k} \left( \frac{dP}{dt} \right) \chi_j \right|^2 dy \\
\leq C \left[ \|\nabla u\|^2_1 + \|u\|^2_1 + \varepsilon(\|\nabla P\|^2_1 + \|\nabla^2 u\|^2_1) + \int_{\Omega_j^{-1}} |\partial^{k+1} \partial \nabla u \chi_j|^2 dy \right],
\]

\[\tag{57}\]

for any $t \geq 0$, $k + l = 1, k, l \geq 0$, $1 \leq j \leq N$.

**Proof.** First, using the equations $\partial_{y_j} (F^P - g) = 0$ and $\mathfrak{N}(F^u - h) = 0$, we obtain the following form:

\[
\left( \frac{dP}{dt} \right)_{y_j} + \frac{K}{\mathfrak{H}} \left[ (Ac_1 + Bc_2) \cdot u_{y_1 y_3} + (Cc_1 + Dc_2) \cdot u_{y_2 y_3} + J\mathfrak{N} \cdot u_{y_3 y_3} \right] \\
+ \text{one order terms of } u = g_{y_j},
\]

\[\tag{58}\]

\[
\mathfrak{N} \mu \nu_t - \frac{\mu}{\mathfrak{H}^2} ([A^2 + B^2] \mathfrak{N} u_{y_1 y_3} + 2(Ac + B\mathfrak{H}) \mathfrak{N} u_{y_1 y_2} + (C^2 + D^2) \mathfrak{N} u_{y_2 y_2} \\
+ J^2 \mathfrak{N} u_{y_3 y_3}) + \text{one order terms of } u + \left[ \frac{\mu + \lambda}{K} \frac{dP}{dt} \right]_{y_j} = \mathfrak{N} h,
\]

\[\tag{59}\]

To eliminate $u_{y_3 y_3}$ in (58), we take the summation $\frac{\mu}{K} \times (58) + (59)$ which gives

\[
2\mu + \lambda \left( \frac{dP}{dt} \right)_{y_j} + P_{y_j} = \frac{\mu}{\mathfrak{H}^2} ([A^2 + B^2] \mathfrak{N} u_{y_3 y_3} \\
+ 2(Ac + B\mathfrak{H}) \mathfrak{N} u_{y_1 y_2} + (C^2 + D^2) \mathfrak{N} u_{y_2 y_2}) \\
- \mathfrak{N} \mu u_t - \frac{\mu}{\mathfrak{H}^2} ([Ac_1 + Bc_2) \cdot u_{y_1 y_3} + (Cc_1 + Dc_2) \cdot u_{y_3 y_3} \\
+ \text{one order terms of } u + \mathfrak{N} h + \frac{\mu}{K} g = F.
\]

\[\tag{60}\]

Multiplying the above equation by $\chi_j^2 \left( \frac{dP}{dt} \right)_{y_j}$ and integrating on $\Omega_j^{-1}$, we have

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega_j^{-1}} |P_{y_j} \chi_j|^2 dy + \frac{2\mu + \lambda}{K} \int_{\Omega_j^{-1}} \left| \left( \frac{dP}{dt} \right)_{y_j} \chi_j \right|^2 dy \\
= \int_{\Omega_j^{-1}} -(u \cdot \nabla P)_{y_j} P_{y_j} \chi_j^2 + \left( \frac{dP}{dt} \right)_{y_j} F \chi_j^2 dy.
\]

\[\tag{61}\]
For the first term on the right hand of (61), we have from Lemma 2.1, Hölder’s inequality and Cauchy’s inequality that
\[
\left| \int_{\Omega_j} (u \cdot \nabla P)_{y_3} P_{y_3} \chi_j^2 dy \right| \leq \left\| \nabla u \right\|_1 \left\| \nabla\nabla P \right\|_1 \leq C \varepsilon \left\| \nabla P \right\|_1^2.
\]

Similarly, we also have
\[
\left| \int_{\Omega_j} \left( \frac{dP}{dt} \right)_{y_3} F \chi_j^2 dy \right| \leq \frac{2\mu + \lambda}{2K} \int_{\Omega_j} \left( \frac{dP}{dt} \right)_{y_3} \chi_j^2 dy + \left| \int_{\Omega_j} |F\chi_j|^2 dy \right|
\]
\[
\leq \frac{2\mu + \lambda}{2K} \int_{\Omega_j} \left( \frac{dP}{dt} \right)_{y_3} \chi_j^2 dy + + \int_{\Omega_j} \left| \partial\nabla u\chi_j \right|^2 dy + C(\left\| \nabla u \right\|^2 + \left\| u_t \right\|^2 + \varepsilon \left\| \nabla u \right\|^2_1).
\]

Substituting (62) and (63) into (61) yields
\[
\frac{d}{dt} \int_{\Omega_j} |P_{y_3} \chi_j|^2 dy + \frac{2\mu + \lambda}{K} \int_{\Omega_j} \left( \frac{dP}{dt} \right)_{y_3} \chi_j^2 dy 
\leq C \left[ \left\| \nabla u \right\|^2 + \left\| u_t \right\|^2 + \varepsilon (\left\| \nabla P \right\|^2 + \left\| \nabla u \right\|^2) + \int_{\Omega_j} \left| \partial\nabla u\chi_j \right|^2 dy \right],
\]

which implies (56).

Applying \( \partial^k \partial_{y_3}^l (k + l = 1) \) to (59), multiplying it by \( \chi_j^2 \partial^k \partial_{y_3}^{l+1} (dP/dt) \) and using similar arguments as in deducing (64), we also have
\[
\frac{d}{dt} \int_{\Omega_j} |\partial^k \partial_{y_3}^{l+1} P \chi_j|^2 dy + \frac{2\mu + \lambda}{K} \int_{\Omega_j} \left| \partial^k \partial_{y_3}^{l+1} \left( \frac{dP}{dt} \right) \chi_j \right|^2 dy 
\leq C \left[ \left\| \nabla u \right\|^2_1 + \left\| u_t \right\|^2 + \varepsilon (\left\| \nabla P \right\|^2_1 + \left\| \nabla^2 u \right\|^2_1) + \int_{\Omega_j} \left| \partial^{k+1} \partial_{y_3}^l \partial\nabla u\chi_j \right|^2 dy \right],
\]

which implies (57). The proof of Lemma 3.5 is completed.

Finally, we use Lemma 2.2 to deduce the estimates on the tangential derivatives of \((P, u)\).

**Lemma 3.6.** Under the conditions of Theorem 1.1 and (23), there exists a positive constant \(C\) such that
\[
\left\| \nabla^2 u \right\|^2 + \left\| \nabla P \right\|^2 \leq C \left( \left\| \frac{dP}{dt} \right\|^2_1 + \left\| u_t \right\|^2 + \left\| \nabla u \right\|^2 \left\| \nabla^2 u \right\|^2 \right),
\]

\[
\int_{\Omega_j} \left| \partial\nabla^2 u\chi_j \right|^2 dy + \int_{\Omega_j} \left| \partial\nabla P\chi_j \right|^2 dy 
\leq C \left( \left\| \nabla u \right\|^2 + \left\| u_t \right\|^2 + \left\| \nabla P \right\|^2 + \left\| \nabla P \right\| \left\| \frac{dP}{dt} \right\| + \left\| \nabla u \right\|^2 \left\| \nabla^3 u \right\|^2 + \left\| \nabla u \right\|^2 \left\| \nabla^3 u \right\|^2 \right)
\]

\[
+ C \int_{\Omega_j} \left| \partial\nabla \frac{dP}{dt} \chi_j \right|^2 dy,
\]
for any $t \geq 0$.

**Proof.** We rewrite the perturbed equations as the Stokes problem:

$$
\begin{cases}
\begin{align*}
\text{div} u &= -\frac{1}{K(m,s)m} \frac{dP}{dt}, \\
-\mu \Delta u + \nabla P &= (\mu + \lambda) \text{div} u - (mu_t + mu \cdot \nabla u),
\end{align*}
\end{cases}

(68)
\right.
\]

Thus, by applying Lemma 2.2 to (68), one can easily get (66).

Next, we prove (67). To do this, by operating $\chi_j \partial$ to Stokes equation (68) and together with (68), implies that the following Stokes problem:

$$
\begin{cases}
\begin{align*}
\text{div}(\chi_j \partial u) &= \chi_j \partial \left(-\frac{1}{K(m,s)m} \frac{dP}{dt}\right) + \nabla \chi_j \partial u, \\
-\mu \Delta(\chi_j \partial u) + \nabla(\chi_j \partial P) &= -2\mu \nabla \chi_j \nabla(\partial u) - \Delta \chi_j \partial u + \nabla \chi_j \partial P - (\mu + \lambda) \chi_j \nabla \chi_j \partial u,
\end{align*}
\end{cases}
\right.

(69)
\]

Thus applying Lemma 2.2 to (69) gives (67). The proof of Lemma 3.6 is completed.

Now, we are in a position to prove Theorem 1.1.

**Proof of Theorem 1.1.** We will do it by four steps.

**Step 1.** We first deal with the lower order derivatives for $(P,u)$. Let $D$ be a fixed but large positive constant. By summing up $D^2 \times ((25) + (37)) + D \times ((46) + (54)) + (56)$, there exists a function $H_1(P,u)$ which is equivalent to $\|u\|^2 + \|P - \bar{P}\|^2 + \|u_t\|^2 + \|P_t - \bar{P}_t\|^2 + \|\nabla P\|^2$ and satisfies

$$
\frac{d}{dt} \left\{ H_1 + \int_{\Omega} m|\nabla u\chi_0|^2dx + \sum_{j=1}^{N} \int_{\Omega_j^{-1}} m|\partial u\chi_0|^2dy \right\} + D(\|\nabla u\|^2 + \|\nabla u_t\|^2 + \|\nabla u\|^2)
\right.
\]

(70)

Substituting (66) into the above inequality, and using the fact $\frac{dP}{dt} = -K(m,s)m \text{div} u$, we obtain

$$
\frac{d}{dt} \left\{ H_1 + \int_{\Omega} m|\nabla u\chi_0|^2dx + \sum_{j=1}^{N} \int_{\Omega_j^{-1}} m|\partial u\chi_0|^2dy \right\} + \|\nabla u\|^2 + \|\nabla P\|^2 + \|\nabla u_t\|^2 + \left| \frac{dP}{dt} \right|^2
\right.
\]

(71)

$$
\leq C\varepsilon \|\nabla^2 P\|^2,
$$

(72)
since $D$ is large and $\varepsilon$ is small.

**Step 2.** Next, we turn to estimate the higher order derivatives for $(P, u)$. Taking $l = 0$ in (57) and summing up $D \times [2 \times (47) + (55)] + (57)$, we have

$$
\frac{d}{dt} \left\{ D \int_\Omega m|\nabla^2 u| \chi_0|^2 dx + D \sum_{j=1}^N \int_{\Omega_j} m|\partial^2 u \chi_j|^2 dx 
+ \left| \frac{\partial^2 P \chi_j}{K} \right|^2 dy + \int_{\Omega_j} |\partial \partial_y^3 P \chi_j|^2 dy \right\} + \int_\Omega |\nabla^3 u \chi_0|^2 dx 
+ \int_\Omega \left| \nabla^2 \frac{dP}{dt} \chi_0 \right|^2 \chi_0 dy 
\leq CD(\|\nabla u\|^2 + \|\nabla u_t\|^2) + CD \delta (\|\nabla P\|^2 + \|\nabla^2 u\|^2) 
+ CD\|\nabla^2 u\|(\|\nabla u\|_2 + \|\nabla^2 P\|).
$$

(72)

Then by taking $l = 1$ in (57) and substituting (67) into (57), we have

$$
\frac{d}{dt} \int_{\Omega_j} |\partial^2_y P \chi_j|^2 dy 
+ \int_{\Omega_j} \left| \frac{\partial^2 P}{\partial t} \chi_j \right|^2 dy 
\leq C \left( \|\nabla u\|^2 + \|u_t\|^2 + \|\nabla P\| \right) \left( \|\nabla P\| + \left| \nabla \frac{dP}{dt} \right| \right) + \delta \|\nabla P\|^2 + \|\nabla^2 u\|^2.
$$

(73)

Adding $D \times (72)$ to (73), there exists $H_2(P)$ which is equivalent to $\|\nabla^2 P\|^2$, such that

$$
\frac{d}{dt} \left\{ D^2 \int_\Omega m|\nabla^2 u \chi_0|^2 d + D^2 \sum_{j=1}^N \int_{\Omega_j} m|\partial^2 u \chi_j|^2 dy + H_2 \right\} 
+ \int_\Omega |\nabla^3 u \chi_0|^2 dx + \sum_{j=1}^N \int_{\Omega_j} |\partial^2 \nabla u \chi_j|^2 + \int_\Omega \left| \nabla^2 \frac{dP}{dt} \right|^2 \chi_0 dy 
\leq CD(\|\nabla u\|^2 + \|\nabla u_t\|^2 + \|\nabla P\|^2) 
+ CD^2 \delta (\|\nabla P\|^2 + \|\nabla^2 u\|^2) + CD^2 \|\nabla^2 u\|(\|\nabla u\|_2 + \|\nabla^2 P\|).
$$

(74)

Applying Lemma 2.2 to (68), we have

$$
\|\nabla^3 u\|^2 + \|\nabla^2 P\|^2 \leq C (\|\nabla u\|^2 + \|u_t(t)\|^2 + \|\nabla P\|^2 + \left| \nabla \frac{dP(t)}{dt} \right|_1^2 
+ \|\nabla u\|^2 + \|\nabla^2 u\|^2).
$$

(75)

**Step 3.** Now, we are able to establish the energy inequality of Gronwall-type. An application of the $L^p$-estimate of elliptic system to (12) gives

$$
\|\nabla^2 u\|^2 \leq C (\|u_t\|^2 + \|\nabla P\|^2 + \|\nabla u\|^2).
$$

(76)

Thus by summing up $D^4 \times (71) + D \times (74) + (75)$, there exists a function $H_3(P, u)$ which is equivalent to $\|P - \bar{P}\|^2 + \|u\|^2 + \|P - \bar{P}\|^2 + \|u_t\|^2$ such that

$$
\frac{dH_3}{dt} + CH_3 + C\|\nabla^3 u\|^2 \leq 0,
$$

(77)
where we have used the Poincaré’s inequality $\|P - \bar{P}\| \leq C\|\nabla P(t)\|$. Integrating the above inequality over $[0, t]$ gives (18). Using Gronwall’s inequality to (77), it is clear that there exist two positive constant $C_1$ and $\eta_1$ such that

$$H_3(P, u) \leq C_1H_3(P_0, u_0)e^{-\eta_1 t}.$$ 

which together with (12) implies (20).

**Step 4.** Finally, we prove (19) and (21). By symmetry and some tedious but straightforward calculation, one easily concludes the energy estimates on $s$ as following:

$$\frac{d}{dt}\|s\|_2^2 \leq C\|u\|_2\|s\|_2^2.$$ 

By Gronwall’s inequality again, we obtain

$$\|s\|_2^2 \leq \|s_0\|_2\exp\left\{C\int_0^t \|u(\tau)\|_2 d\tau\right\},$$

which together with (20) implies (19). To prove (21), we first show that $\lim_{t \to \infty} \bar{P}(t)$ exists. In fact, for any arbitrary positive constant $\varepsilon$, there exists a positive constant $T = \max\{1, \frac{\ln \eta_0}{\eta_0} - \frac{\eta_0}{C_0}\}$ such that for any $t_2 > t_1 > T$, it holds that

$$|\bar{P}(t_2) - \bar{P}(t_1)| = \left|\int_{t_1}^{t_2} \bar{P}_\tau d\tau\right| \leq C_0 \int_{t_1}^{t_2} e^{-\eta_0 \tau} d\tau \leq \frac{C_0}{\eta_0} e^{-\eta_0 t_1} < \varepsilon,$$

which implies that $\lim_{t \to \infty} \bar{P}(t)$ exists. Now, setting $\bar{P} = \lim_{t \to \infty} \bar{P}(t)$, and combining (29) and (30), we obtain

$$|\bar{P} - \bar{P}(t)| = \left|\int_t^\infty \bar{P}_\tau d\tau\right| \leq C \int_t^\infty \|u(\tau)\| (\|\nabla P(\tau)\| + \|\nabla s(\tau)\|) d\tau,$$

which together with (20) implies (21).

Thus, we have completed the proof of Theorem 1.1. \qed

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