Enhancing The Error Term of $|N_p(x) - \tau x|$ For Some Special Sequence

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Abstract. This paper concentrate on restricting the gap between the error term of $|N_p(x) - \tau x|$ and itself in Rieamnn hypothesis. Also, enhancing an algorithm in different technique to generate a primitive weird number from any sufficient large real x.
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1. Introduction
Prime numbers had studied firstly by ancient Greeks and so then many mathematicians improved the theories and gave a vast see of knowledge in number theory. The most known theorem in number theory called The Fundamental theorem of Arithmetic which states that any natural number greater than one is either prime or a multiplication of powers of primes. Beurling in 1937 defined the generalized prime system by positive increasing real sequence all of its terms greater than one and tend to infinity. The previous work by [2] on the arithmetical function $\psi_p(x)$ has been done to find weird number from any large enough real number $x$. Our concern is generating primitive weirds instead of weirds from any large enough real number $x$ by building an enough secure algorithm and defining a generalized prime system of primitive weird numbers in order to restrict the gap between the error term of $|N_p(x) - \tau x|$ and itself in Riemann hypothesis. This study concentrate on theoretical branch and leave the applicable side for the future work.

2. Important generalized counting function
Now we move our attention to define the known and important counting functions for any real number $a$, let $P_\ast = \{p_1\}_{i=1}^{\infty}$ be a sequence of reals satisfying Beurling and let $N_\ast$ be Beurling naturals :
1. $\pi_p(a) = \sum_{P_\ast < a} 1 \cdot p \in P_\ast$ 
2. $N_p(P) = \sum_{n < a} 1 \cdot p \in P_\ast$ 
3. $\Pi_p(a) = \sum_{k=1}^{\infty} \frac{1}{k} \pi_p(a^k)$ 
4. Chebyshev function $\psi_p(a) = \sum_{n < a} \Lambda(n) = \sum_{p^k < a} \log p$ 

The generalized Reimann zeta function defined as
\[ \zeta_p(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \text{ where } s = u + it. \] From several results in this subject it is obvious that there is a connection between \( \zeta_p(s) \) and \( N_p(x) \) as \( \zeta_p(s) = \int_{1-\gamma}^{\infty} x^{-s} \psi(x) dx \). The Milan transform tells us that \( N_p(s) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^s \zeta_p(s) dx \).

Actually it is impossible to know the sequence of g-primes from any given \( \pi_p \in S_0^1 \) \{all increasing functions with \( f(1) = 0 \}\). So it needs to deal with the given function approximately to find \( N_p(x) \in S_1^+ \) \{all increasing functions with \( f(1) = 1 \)\} and vise versa. Additionally, for this reason \( \zeta_p(s) \) must be involved.

3. **Enhancing the function \( \psi_p(x) = x + O(1) \) by [2]**

- Creating primitive weird number from sufficient large real \( x \)

The following definitions are needed for this work where the aim is creating a primitive weird number, so first Let s(a) be the sum of all proper devisors of natural a then Definition 1[2]. The natural \( a \) is abundant if \( a<s(a) \) and deficient if \( a>s(a) \).

Definition 2[2]. The natural \( a \) is Primitive abundant number if it is abundant number but all its proper devisors are deficient.

Definition 3[2]. Psedoperfect number is natural number that cannot be written as a sum of all or some of its proper devisors.

Definition 4.[2] If the number is abundant and not psedoperfect then it is weird number and it is primitive weird if it is primitive abundant and not psedoperfect.

It is worth mentioning that \( \Delta(a) \) refer to the deference between \( s(a) \) and a itself. If \( \Delta(a) > 1 \) then a is abundant, if the opposite then a is deficient.

Now let \( \psi_p = \lfloor x \rfloor - 1 \) for any large \( x \in \mathbb{R} \)

The function \( H(x) = \sum_{s=1}^{\infty} \mu(s) \psi_p(x^{1/s}) = \sum_{s=1}^{\infty} \mu(s) \lfloor x^{1/s} \rfloor - 1 \) will give us natural number from any real \( x \) using the work has been done by [2]. For example let \( x=24.5 \) then \( H(24.5) = 19 \).

In order to generate an abundant number from the natural \( H(x) \) we need to define the characteristic function \( \eta(x) \) as

\[ \eta(x) = \begin{cases} 1 & \text{if } s(H(x)) > H(x) \\ 0 & \text{if } s(H(x)) \leq H(x) \end{cases} \]

where \( s(H(x)) \) is the summation of all proper devisors of \( H(x) \) as we mentioned above.

Here the outcome of \( H_1(x) = \eta(x) H(x) \) must give us either zero or abundant number.

Let \( I = \{n_i\} \in \mathbb{N} \) be the sequence of \( H_1(x) \) devisors and then \( s(n_i) \) is the sum of devisors of \( n_i \in I \). Now we will generate primitive abundant number from the abundant number \( H_1(x) \) by defining the characteristic function \( \eta_1(x) \) by

\[ \eta_1(x) = \begin{cases} 1 & \text{if } s(n_i) \leq n_i \\ 0 & \text{if } s(n_i) > n_i \end{cases} \]

And so then the outcome of the function

\( H_2(x) = \eta_1(x) H_1(x) \) is either zero or primitive abundant number.

To generate primitive weird from primitive abundant \( H_2(x) \) also by defining the characteristic function \( \eta_2(x) \) by

\[ \eta_2(x) = \begin{cases} 1 & \text{if } H_2(x) \neq \sum_{i \in I} n_j, n_j \in I \\ 0 & \text{if } H_2(x) = \sum_{i \in I} n_j, n_j \in I \end{cases} \]

The outcome of the function \( H_3(x) = \eta_2(x) H_2(x) \) is either zero or primitive weird number.
For more strength we defined \( \eta_3(x) \) as

\[
\eta_3(x) = \begin{cases} 
1 & \text{if } \mathcal{H}_3(x) = m \prod_{i=1}^{k} p_i, \Delta(m) < 1, p_i \text{ are primes} \\
0 & \text{if } \mathcal{H}_3(x) \neq m \prod_{i=1}^{k} p_i, \Delta(m) < 1, p_i \text{ are primes} 
\end{cases}
\]

So, \( \mathcal{H}_4(x) = \eta_3(x) \mathcal{H}_3(x) \) is either zero or primitive weird number of the form \( m \prod_{i=1}^{k} p_i \) where \( m \) is deficient and all \( p_i \) are primes.

For more generality, suppose that there is a sequence of primitive weird numbers satisfies Buehring conditions named \( W = \{w_n\}_{n=1}^{\infty} \) with the form \( w_n = m \prod_{i=1}^{k} p_i \) where \( m \) is deficient and all \( p_i \) are primes.

4. Enhancing bounds of the order of \( |N_p(x) - \tau x| \) for \( \psi_e(x) = x + O(1) \)

During the work had been done in 2015 (see[1]) showing that there is a Beurling system satisfies \( \psi_e = x + O(1) \) and

\[
N_r(x) = \tau x + O(x^{\log\log\log\log x}), \quad \forall c > 1 \quad \text{(*)}
\]

where \( k(x) = \frac{\log x}{\log\log\log x} \). The above system tells us that there is a gap between the error term of \( N_r(x) \) in(*) and the error term of \( N_r(x) \) in Reimann hypothesis. Moreover, equation (*) informed us that \( |N_r(x) - \tau x| \geq \delta(x^{-ck(x)}) \), \( \forall c > 1 \)

where \( k(x) = \frac{\log x}{\log\log\log x} \) as \( x \) goes to infinity. Whenever, if Reimann hypothesis exist then one can see how big the gap between the error term of \( |N_r(x) - \tau x| \) in (*) and it self in Reimann hypothesis.

For more details one can see [1]. In order to improve the above work, that is to restrict the gap as mentioned above it needs some hard work, so define a sequence \( W = \{w_n\}_{n=1}^{\infty} \) of primitive weird numbers of the form \( w_n = m \prod_{i=1}^{k} p_i \) where \( m \) is deficient and all \( p_i \) are primes. It is clear that

\[
(\log\log\log(m^{p^*}))^{\alpha} > \log\log\left( m \prod_{i=1}^{k} p_i \right)
\]

where \( p^* > \prod_{i=1}^{k} p_i, \quad \alpha > 1 \)

Hence

\[
\frac{\log\log\log(m^{p^*})}{(\log\log(m^{p^*}))^{\alpha}} \leq \frac{\log\log\log(m \prod_{i=1}^{k} p_i)}{\log\log(m \prod_{i=1}^{k} p_i)} \quad \text{where} \quad \alpha > 1 .
\]

Now let \( w = m \prod_{i=1}^{k} p_i \)

Then \( xe^{-c(\log m + \log p^*)/(\log\log\log(m^{p^*}))^{\alpha}} \geq xe^{-c\log\log\log\log w/\log\log w} \)

Name \( k^*(w) = (\log m + \log p^*)/(\log\log\log(m^{p^*})) \) and

\[
x e^{-ck^*(x)} \geq xe^{-ck(x)} \).
\]

It is beneficial to note that the gap between the error term of (*) and the error term in Reimann hypothesis could be restricted more if possible after a challenge work.
5. Conclusion
The manipulation have been done in this work for $x$ takes a primitive weird numbers as $x = m \prod_{i=1}^{k} p_i$. This showed that the gap between the error term of $|N_p(x) - \tau x|$ and itself in the Riemann hypothesis could be restricted more than in [1]. One can use a different sequence with some different details to give a better enhancing. We left rest for the future work.

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