Hamiltonian paths on directed grids

Hujter, Mihály and Kaszanyitzky, András
Budapest University of Technology and Economics
hujter@math.bme.hu; kaszi75@gmail.com

Abstract. Our studies are related to a special class of FASS-curves, which can be described in a node-rewriting Lindenmayer-system. These ortho-tile (or diagonal) type recursive curves inducing Hamiltonian paths. We define a special directed graph on a rectangular grid, and we enumerate all Hamiltonian paths on this graph. Our formulas are strongly related to both the Fibonacci numbers and the domino tilings of chessboards. The constructability of the regular 17-gon with straightedge and compass is also related.

Introduction. In 1877, G. Cantor proved for any positive integer $d$ that there exists a one-to-one point-to-point correspondence between a unit line segment and the entire $d$-dimensional space. In other words, the infinite number of points in a unit interval is the same cardinality as the infinite number of points in any finite-dimensional manifold. In 1890, G. Peano constructed a continuous mapping from the unit interval onto the unit square. This continuous curve that passes through every point of the unit square was the first example of space filling curves.

In 1891, D. Hilbert discovered another type of these recursive curves [Sa94]. These FASS-curves (space-Filling, self-Avoiding, Simple, self-Similar) can be described in a node-rewriting Lindenmayer-system [PL90,PLF91]. The ortho-tile type of FASS-curves can be represented only on a special directed grid graph. (See Figures 8.a and 8.b.) Their approximations are Hamiltonian paths between diagonally located points.

Definitions. Given fixed positive integers $p, q$, by a directed grid graph with an odd-even direction, or by $DGG_{p,q}$, in short, we mean a directed graph on the vertex set $\{1, 2, ..., p\} \times \{1, 2, ..., q\}$, as a subset of the $xy$ coordinate plane, with all possible arcs of the form $(x; y) \to (x+1; y)$ if $y$ is odd, and arcs of the form $(x; y) \to (x-1; y)$ if $y$ is even, and arcs of the form $(x; y) \to (x; y+1)$ if $x$ is odd, and arcs of the form $(x; y) \to (x; y-1)$ if $x$ is even. This graph has $pq$ vertices and $(p-1)q + p(q-1) = 2pq - p - q$ arcs. A Hamiltonian path is such a permutation $v_1 \to v_2 \to ... \to v_{pq}$ of all vertices for which $v_n \to v_{n+1}$ is an arc for each $n \in \{1, 2, ..., pq-1\}$. Let $h(p, q)$ denote the number of Hamiltonian paths in $DGG_{p,q}$. Furthermore, for a positive integer $r$, let $h_r(p, q)$ denote the number of those Hamiltonian paths of the form $v_1 \to v_2 \to ... \to v_{pq}$ for which $v_1 = (1; 1)$, $v_2 = (2; 1)$, ..., $v_r = (r; 1)$ hold but $v_{r+1} \neq (r+1; 1)$. For other definitions and for the history of the Hamiltonian paths we refer reader to [Ba06] and [We01].

Basic observations. We can make the following easy observations: For any positive integers $p, q$, we have $h(p, q) = h(q, p)$ and $h(p, 1) = h(1, q) = 1$. If $p$ is odd, then $h(p, 2) = h(2, p) = 1$. If both $p$ and $q$ are even, then $h(p, q) = 0$. If $pq$ is odd, then $h(p, q) > 0$.

We can also observe that for any Hamiltonian path $v_1 \to v_2 \to ... \to v_{pq}$ we have $v_1 = (1; 1)$. On the other hand, if both $p$ and $q$ are odd, then $v_{pq} = (p; q)$. If $p$ is odd but $q$ is even, then $v_{pq} = (1; q)$. Finally, if $p$ is even but $q$ is odd, then...
Theorem 1. There is a one-to-one correspondence between Hamiltonian paths and the domino tilings such that for each domino tiling the corresponding Hamiltonian path is the only one which bisects no domino of the tiling.

We will prove the theorem later. In Figures 3.a and 3.b we find the domino tilings that correspond to the Hamiltonian paths of Figure 2.a and 2.b., respectively.

More than a half a century ago [Ka61] and [Te61] proved that the number of different domino tilings is exactly

\[ \prod_m \prod_k \left( 4 \cos^2 \frac{m \pi}{p} + 4 \cos^2 \frac{k \pi}{q} \right) \]

where the products are understood for all positive integers \( m \) and \( k \) such that \( 2m < p, 2k < q \). This formula is well-known as the Kasteleyn formula.
Example. We consider the case \( p = q = 5 \). Now
\[
4 \cos^2 \frac{\pi}{5} = \frac{3 + \sqrt{5}}{2} \quad 4 \cos^2 \frac{2\pi}{5} = \frac{3 - \sqrt{5}}{2}
\]
\[
\prod_{m \neq k} \left( 4 \cos^2 \frac{m\pi}{p} + 4 \cos^2 \frac{k\pi}{q} \right) = \left( 3 + \sqrt{5} \right) \cdot 3^2 \cdot \left( 3 - \sqrt{5} \right) = 36
\]
Therefore, by Theorem 1 we gain that \( h(5, 5) = 36 \).

Remark. The above mentioned Kasteleyn formula outputs 0 if both \( p \) and \( q \) are even. If \( \min\{p, q\} = 1 \), the meaning of the formula is \( 1 \). If \( \min\{p, q\} = 2 \) and \( |p - q| \in \{1, 3, 5, \ldots\} \), then the Kasteleyn formula outputs \( 1 \). Since \( 4 \cos^2 \frac{\pi}{3} = 1 \), if \( \min\{p, q\} = 3 \), then the Kasteleyn formula and our Theorem 1 produces the following two nice formulas for the Fibonacci numbers \( F_{2n} \) and \( F_{2n+1} \) for any positive integer \( n \).
\[
\prod_{k=1}^{n-1} \left( 1 + 4 \cos^2 \frac{k\pi}{2n} \right) = F_{2n} \quad \prod_{k=1}^{n} \left( 1 + 4 \cos^2 \frac{k\pi}{2n+1} \right) = F_{2n+1}
\]
For example for \( n = 8 \) and for \( x_k = 1 + 4 \cos^2 \frac{k\pi}{17} \), \( k = 1, 2, \ldots, 8 \), the latter formula produces \( x_1 x_2 \cdots x_8 = 1597 \). However, by the famous result of Gauss on the constructability of the regular 17-gon with straightedge and compass, we can derive a nice formula for each \( x_k \) separately. Namely, we can start from the well-known formula (see, e.g., [Gi07])
\[
16 \cos^2 \frac{2\pi}{17} = \sqrt{17} - 1 + \sqrt{34 - 2\sqrt{17}} + 2\sqrt{17 + 3\sqrt{17} - \sqrt{170 + 38\sqrt{17}}}
\]
and we can apply that
\[
1 + \cos \frac{2\pi}{17} = 2 \cos^2 \frac{\pi}{17} \quad 1 - \cos \frac{2\pi}{17} = 2 \sin^2 \frac{\pi}{17}
\]
This way we can express each \( x_k \) as \( \frac{1}{16} \) times an integer coefficient polynomial of \( v = \sqrt{34 - 2\sqrt{17}} \) and \( w = \sqrt{17 + 3\sqrt{17} - \sqrt{170 + 38\sqrt{17}}} \) because \( 64 \cos^2 \frac{\pi}{17} = 64 + (2 - v)v + 4w \). For example \( x_1 = 5 + \frac{(2-v)v}{16} + \frac{w}{4} \).

Table of \( h(p, q) \). By the Kasteleyn formula and by our Theorem 1 we can continue the above incomplete table as follows

| \( p, q \) | \( p = 4 \) | \( p = 5 \) | \( p = 6 \) | \( p = 7 \) | \( p = 8 \) | \( p = 9 \) |
|--------|----------|----------|----------|----------|----------|----------|
| \( h(p, 4) = \) | 0 | 11 | 0 | 41 | 0 | 153 |
| \( h(p, 5) = \) | 11 | 36 | 95 | 281 | 781 | 2245 |
| \( h(p, 6) = \) | 0 | 95 | 0 | 1183 | 0 | 14824 |
| \( h(p, 7) = \) | 41 | 281 | 1183 | 6728 | 31529 | 167089 |
| \( h(p, 8) = \) | 0 | 781 | 0 | 31529 | 0 | 1292697 |
| \( h(p, 9) = \) | 153 | 2245 | 14824 | 167089 | 1292697 | 12988816 |

Definition. Given a domino \( D \), there are six vertices at the perimeter of the domino; the canonical numbering of these vertices is \( D_1, D_2, \ldots, D_6 \) if all five \( D_j \rightarrow D_{j+1} \) are arcs in the graph, \( j = 1, 2, \ldots, 5 \). In Figure 4 we can see a canonical numbering. Observe, that each domino has exactly one canonical numbering. In this paper we always consider canonical numberings.
The proof of Theorem 1. The cases \( \min\{p, q\} \leq 2 \) are obvious. In the rest of the proof we assume that \( \min\{p, q\} \geq 3 \). The case when both \( p \) and \( q \) are even is also obvious. By symmetry we may assume that \( p \) is odd. We have at least one Hamiltonian path and we have at least one domino tiling.

Let us consider a fixed Hamiltonian path \( H \) and a fixed domino \( D \). According to the canonical numbering let the vertices around \( D \) be \( D_i \), \( i = 1, 2, ..., 6 \). Observe that if \( D_5 \to D_2 \) is not an arc in \( H \), then \( D_5 \to D_6 \) must be an arc in \( H \), and \( D_1 \to D_6 \) cannot be an arc in \( H \). Given \( H \) and given \( k \) pairwise nonoverlapping dominoes, we gain \( 2k \) arcs such that none of them can be in \( H \). These \( 2k \) arcs are all distinct because the \( D_5 \to D_2 \) arcs are all inside the pairwise nonoverlapping dominoes, and each \( D_1 \to D_6 \) arc belongs to the domino situated in the right angle determined by the two arcs started at \( D_1 \). In a domino tiling there are \( (p - 1)(q - 1)/2 \) pairwise distinct dominoes. In summary, if \( H \) avoids each domino in a fixed domino tiling, then there are \( pq - 1 \) arcs in \( H \) and there are \( (p - 1)(q - 1) \) further arcs not in \( H \). However

\[
pq - 1 + (p - 1)(q - 1) = 2pq - q - p
\]

This is exactly the total number of arcs in the graph. Therefore, starting from the given domino tiling, some (but not necessary all) of the canonical numbering \( D_j \to D_{j+1} \) arcs of the domino’s perimeter form the Hamiltonian path. On the other hand, the \( (p - 1)(q - 1) \) arcs missing from the a Hamiltonian path determine two-by-one all dominoes of the domino tiling.

Now we make the above argument more explicit. As on a usual chessboard, we say that a unit square of our grid is black if the sum of bottom-left corner coordinates is even. (See Figure 5 as an illustration.) The other unit squares are called white. Clearly, each domino \( D \) consists of a black square and of a white square; the corners of the black square are \( D_1, D_2, D_5, D_6 \) according to the canonical ordering, and the white square’s corners are \( D_2, D_3, D_4, D_5 \). If the domino is an element of the domino tiling, than the arc between the domino’s squares must not be in the corresponding Hamiltonian path. These \( (p - 1)(q - 1)/2 \) pairwise distinct arcs of \( DGG_{p,q} \) are called domino-axis arcs. According to the canonical ordering of a domino \( D \), the domino-axis arc is the \( D_5 \to D_2 \) arc. On the other hand, if a Hamiltonian path avoids a domino \( D \), then the \( D_1 \to D_6 \) arc can not be in the Hamiltonian path, either. We call these arcs domino-black-end arcs. Clearly, each domino-axis arc corresponds to exactly one domino in a domino tiling, and each domino-black-end arc corresponds to exactly one domino in a domino tiling, too. The total number of domino-axis arcs and domino-black-end arcs is \( (p - 1)(q - 1) \).

The number of the remaining arcs is exactly the same as the number of the arcs in a Hamiltonian path. Therefore a domino tiling determines at most one Hamiltonian path which bisects no domino of the tiling.

We can observe that in case of a domino \( D \) of a domino tiling, the arcs \( D_1 \to D_2 \) and \( D_5 \to D_6 \) can be neither domino-axis arcs nor domino-black-end arcs. Therefore both arcs must be in the Hamiltonian path corresponding to the domino tiling.

Now let us consider such arcs of the grid graph which are at the perimeter of the grid and which are at the perimeter of a white square at the same time. One can easily compute that there are \((p - 1)/2\) such arcs at the bottom line, \((p - 1)/2\) such arcs at the top line, and there are other \( q - 1 \) such arcs at the vertical sides of the grid. We call such arcs as white perimeter arcs. Observe that each white perimeter arc must be in each Hamiltonian path. Obviously, a perimeter arc can
be neither a domino-axis nor a domino-black-end arc in case of any domino tiling. The total number of white perimeter arcs is $p + q - 2$. Since a Hamiltonian path contains $pq - 1$ arcs, there are exactly $pq - 1 - (p + q - 2) = (p - 1)(q - 1)$ arcs in any Hamiltonian path which are neither white perimeter arcs, nor domino-axis arcs nor domino-black-end arcs. Given a domino tiling, we call all arcs as domino-black-side arcs which are neither white perimeter arcs, nor domino-axis arcs nor domino-black-end arcs.

Observe that around any black square, any Hamiltonian path contains exactly two opposite arcs. Therefore from all black squares a Hamiltonian path contains exactly $(p - 1)(q - 1)$ arcs.

Let $A$ be the set of all arcs in the grid which are not white perimeter edges. We have that

$$|A| = 2pq - q - p - (p + q - 2) = 2(p - 1)(q - 1)$$

(In Figure 6 we find an illustration for $p = 4$, $q = 3$.) Each Hamiltonian path contains exactly half of the arcs in $A$.

Each domino tiling also takes exactly a quarter of the arcs of $A$ as domino-black-end arcs and exactly a quarter of the arcs as domino-axis arcs. And these three parts must be pairwise disjoint if and only if the Hamiltonian path bisects neither domino of the tiling.

We obtained that a domino tiling determines the corresponding Hamiltonian path. On the other hand, any Hamiltonian path determines for each black square that in a domino tiling the domino containing the black square is in vertical position or in horizontal position. This leaves one or two choices for the white neighbor. However, if there are two choices, then both are vertical or both are horizontal. In case of the bottom-left black square there is only one choice. (In Figures 7.a and 7.b we find illustrations for $p = 9$, $q = 9$.)

Given a Hamiltonian path $H$ we make a bipartite graph $B_H$ as follows: The black squares form one color class of the vertices in $B_H$ and the white squares form the other color class of the vertices. A black and a white square will be adjacent in $B_H$ if they are neighbors and the two squares form such a domino whose position is allowed in the previous sense. Observe that the Hamiltonian path allows no cycle in $B_H$. Therefore, there exists at most one perfect matching in the bipartite graph. Therefore a given Hamiltonian path $H$ allows at most one domino tiling whose dominoes are all avoided by $H$. This completes the proof of Theorem 1.

References

[Ba09] Bang-Jensen, J. and Gutin, G.: Digraphs: theory, algorithms and applications, Second edition, Springer, 2009.
[Gi07] Gindikin, S. (translated from Russian by Suchat, A.): Tales of mathematicians and physicists, Springer, 2007.
[Ka61] Kasteleyn, P. W.: The statistics of dimers on a lattice, Physica 27 (1961) 1209–1225.
[PL90] Prusinkiewicz, P. and Lindenmayer, A.: The algorithmic beauty of plants, Springer, 1990.
[PLF91] Prusinkiewicz, P., Lindenmayer, A., and Fracchia, F. D.: Synthesis of space-filling curves on the square grid. In: H.-O. Peitgen, J. M. Henrques & L. F. Penedo, eds.: Fractals in the fundamental and applied sciences, North-Holland (1991) 341–366.
[Sa94] Sagan, H., *Space-filling curves*, Springer, 1994.

[Te61] Temperley, H. N. V. and Fisher, M. E.: *Dimer problem in statistical mechanics—an exact result*, Phil. Mag. 68 (1961) 1061–1063.

[We01] West, D. B.: *Introduction to graph theory*, 2nd ed., Pearson Education, 2001.
Figures

Figure 1. The directed graph $DGG_{5,4}$

Figure 2.a. A Hamiltonian path

Figure 2.b. Another Hamiltonian path
Figure 3.a. A tiling corresponding to Fig. 2.a

Figure 3.b. A tiling corresponding to Fig. 2.b

Figure 4. A canonical numbering around a domino
Figure 5. The grid as a chessboard

Figure 6. Without the white perimeter arcs
Figure 7.a. The opposite arcs of the black squares and the white perimeter arcs form a Hamiltonian path covering a Euclidean disk.
Figure 7.b. A Hamiltonian path and the corresponding domino tiling.
Figure 8.a. Possible connecting edges between self similar parts in ortho-tile type, node-rewriting FASS-curves. (Grey squares grow out from the nodes of the path.)

Figure 8.b. Directed Grid Graph: $DGG_{3,3}$ and the 2 possible Hamiltonian paths on it.