Particle-Antiparticle Asymmetry Due to Non-Renormalizable Effective Interactions

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Abstract

We consider a model for generating a particle-antiparticle asymmetry through out-of-equilibrium decays of a massive particle due to non-renormalizable, effective interactions.
1 Introduction

We study in this paper the generation of a fermion number ($F$) asymmetry in the early universe due to $F$-violations in non-renormalizable effective interactions. We are motivated by the possibility of generating the cosmological baryon number asymmetry through violation of global symmetries, such as global baryon or lepton numbers, by quantum gravity [1]. However, we will not restrict ourselves to this case. We consider the generic case of effective interactions due to physics at a scale $\Lambda \leq M_P$.

We have constructed a minimal toy model that, we expect, shows the main features of $F$-generation in generic models with $F$-violation only in non-renormalizable interactions. The toy model consists of fermions $b$ and $d$, and two scalars $\chi$ and $\sigma$. The $F$-asymmetry is generated through out-of-equilibrium decays of the massive $\chi$. All other fields are taken to be massless for simplicity. It is assumed that other non-specified interactions maintain equilibrium distributions for all particles except $\chi$.

The different scenarios of $F$-generation are classified by ranges of values of the following constants.

- $K \equiv (\Gamma_\chi / H)_{T=M}$, the “effectiveness of decay” parameter, and
- $K r_0 \simeq \Gamma^{NR}_\chi / H$, the “effectiveness of the non-renormalizable interactions,” where $r_0 \simeq \Gamma^{NR}_\chi / \Gamma_\chi$.

$K$ is given by the ratio of the decay rate $\Gamma_\chi$ of $\chi$ to the Hubble constant at the moment when the $\chi$ bosons are becoming non-relativistic, approximately when the temperature is equal to their mass, $M$. $r_0$ is the ratio of the $\chi$ decay width due to non-renormalizable decays through $F$-violating interactions, $\Gamma^{NR}_\chi$, to the total width, $\Gamma_\chi$. For the precise definition of $K$ and $r_0$ see (3.27) and (2.6).

The total $\chi$ decay width, $\Gamma_\chi$, is dominated by renormalizable interactions, whose coupling we call $g_1$. Thus $\Gamma_\chi(T \leq M) \simeq g_1^2 M / 8 \pi$ (see (3.12)). We call $g_2$ the coupling constant of the non-renormalizable $\chi$ interactions that provide $F$-violating $\chi$ decay, namely $\Gamma^{NR}_\chi(T \leq M) \simeq g_2^2 M^3 / \Lambda^2$. Taking $g_2$, the effective number of relativistic degrees of freedom appearing in the Hubble expansion rate $\mathcal{H}$, $\mathcal{H} \simeq \sqrt{g_2} T^2 / M_P$, from now on to be a reasonable number $g_2 \simeq 10^2$, we get for $K$ and $r_0$ (see (3.28) and (3.29)),

$$K \simeq 10^{-2} g_1^2 (M_P / M), \quad r_0 \simeq (g_2 M / g_1 \Lambda)^2. \quad (1.1)$$

The other independent parameters that completely define the resulting $F$-asymmetry are,
* $g_1$, the coupling of the renormalizable interactions that dominate the $\chi$ decay rate, and

* $\eta$ and $\xi$, two parameters related to $CP$-violation in decays and annihilation processes, respectively (see below).

We have chosen to keep these last three parameters constant at reasonable but arbitrary values, $g_1 = 10^{-1}$ and $\eta = \xi = 5 \times 10^{-4}$, in most of the cases presented below, to show the effects of the different ranges of the main parameters $K$ and $Kr_0$. In most cases the effects of changing $g_1$, $\eta$ and $\xi$ can be easily understood.

In order to be consistent with our Lagrangian, in which the effects of the physics at scale $\Lambda$ only remain in effective interactions, we have to consider energies below $\Lambda$, thus $T \leq \Lambda$. Since also $M \lesssim \Lambda$, from (1.1) we deduce that small values of $K$, $K \ll 1$, can only be obtained for $\Lambda$ not much lower than $M_P$, so that $M \simeq \Lambda \simeq M_P$ and $K$ becomes not much larger than $10^{-2}g_1^2$. Given that $r_0 \leq 1$ always, in the case $K \ll 1$ we necessarily have $Kr_0 \ll 1$. Values of $K$ not much smaller than 1 can be obtained for any value of $\Lambda$. As we will see, the main remaining ranges for which qualitatively different $F$-asymmetry generation patterns are obtained are

\begin{itemize}
    \item $Kr_0 \lesssim 10^{-2}$,
    \item $10^{-2} \lesssim Kr_0 \lesssim 10^7$,
    \item $10^7 \lesssim K$.
\end{itemize}

While the usual, renormalizable scenarios of $B$-violation can be classified with the sole parameter $K$ [3, 4, 5], that mainly regulates the departure of the $\chi$-abundance, $Y_\chi \equiv n_\chi/n_\gamma$ ($n$ stands for number density), with respect to its equilibrium abundance $Y_\chi^{eq}$, it is obvious that here $Kr_0$ is also important. This is so because only non-renormalizable interactions generate a net $F$-abundance, $Y_F$, whose “effectiveness” is given by $Kr_0 \simeq \Gamma_N^R/H$, as will become clear below.

\section{The Model}

The schematic model we consider in this paper belongs in the so-called standard scenario of out-of-equilibrium decays. As is well known, in such scenario three elements are necessary in order to generate dynamically an $F$-asymmetry [3, 4]. Namely, an $F$-violating interaction, violation of $C$
and CP symmetries, and a departure from thermal equilibrium. The last ingredient is provided by the expansion of the Universe, which we assume to be described by the Friedmann-Robertson-Walker cosmology, in the context of the hot big bang model [2, 8, 9]. The other two are discussed below.

2.1 Lagrangian

We include two different fermion species $b$ and $d$ to simplify calculations by avoiding the presence of cumbersome interference terms that are irrelevant for the classification of different $F$-generation scenarios, which is our main objective. Two different bosons $\chi$ and $\sigma$, with wisely chosen $F$-number, give rise to simple $F$-violating dimension five operators, containing $\chi\sigma$ and two fermions, while allowing for $F$-conserving renormalizable Yukawa couplings of $\chi$. This is a feature we want to preserve, namely, we expect that in generic models the decaying boson $\chi$ will have both renormalizable as well as non-renormalizable decays. Renormalizable Yukawa couplings of the second boson, $\sigma$, would only unnecessarily complicate the model, thus we do not include them. Therefore, trilinear couplings $\chi\chi\sigma$ are included in the scalar potential to define a non-zero $F$-number for $\sigma$, $F_\sigma = -2F_\chi$, while $F_\chi$ is defined in the Majorana type Yukawa coupling of $\chi$, $\bar{b}d\chi^\dagger$, to be $F_\chi = F_d + F_b$. By arbitrarily choosing $F_d = F_b = 1$ one obtains $F_\chi = 2$ and $F_\sigma = -4$ from the above couplings.

The complete Lagrangian of our toy model is, then,

$$\mathcal{L} = g_1(\bar{b}d\chi^\dagger + \text{h.c.}) + \frac{g_2}{\Lambda}(\bar{b}d\chi^\dagger\sigma + \text{h.c.}) + V(\chi, \sigma) \ ,$$

with

$$V(\chi, \sigma) = -M_\chi^2\chi^\dagger\chi + g_3(\chi\chi\sigma + \chi^\dagger\chi^\dagger\sigma^\dagger) + g_4\chi^\dagger\chi\sigma^\dagger\sigma \ .$$

Several comments are in order. We have assumed the existence of only one non-renormalizable $F$-violating term. It could be argued that, for example, quantum gravity could generate also $F$-violating renormalizable terms. Even though no strong argument can be given against this possibility, its effect would be so severe that any approximate conservation of a global number would be invalidated. Inclusion of other $F$-violating terms with dimensions $\geq 5$ would enormously complicate the model, and we would like to keep it as simple as possible. At any rate, dimension five operators will be dominant in most models. (Operators of dim-5 and 6 have been considered in the framework of SUSY GUTS in [10, 11].)
Even after these considerations the term
\[ \frac{g'_2}{\Lambda} (\bar{b}' d \chi \sigma + h.c.) \] (2.3)
should legitimately be included in (2.1). However, the effects of this term would be the same as the \( g_2 \) term in (2.1). In general \( g_2 \) and \( g'_2 \) will not be equal. If both, the \( g_2 \) and \( g'_2 \) terms, are of the same order of magnitude, their combined effect would be of the same order as that of just one of them. If, instead, one of them dominates, the other will again not change the order of magnitude of their combined effect. Thus, without loss of generality we assume \( g'_2 \ll g_2 \) and neglect the \( g'_2 \) term.

We will consider the masses of \( b, d, \) and \( \sigma \) to be negligibly small with respect to the mass of \( \chi, M \).

In spite of its simplicity, the model possesses several reaction channels, contributing to \( \chi \)-decay and inverse-decay, \( \chi \bar{\chi} \) annihilation, \( 2 \leftrightarrow 2 \) “point”-scatterings, due to contact interactions of the incoming and outgoing particles, and \( 2 \leftrightarrow 2 \) and \( 2 \leftrightarrow 3 \) scatterings. Two-to-three scatterings, in particular, give rise to a rather large number of diagrams.

The \( F \)-asymmetry is generated by decays and annihilations of \( \chi \) particles, while the other processes, inverse-decays and scatterings, partially or completely erase any asymmetry either of dynamical origin or due to an asymmetric initial condition. Decays and inverse-decays are the only relevant processes at small \( K, K \lesssim 1 \). As \( K \) becomes progressively larger than one, other terms in the evolution equations become important. Point scatterings, together with inverse-decays, are the most efficient damping processes, while \( 2 \leftrightarrow 3 \) scatterings are relevant only for extremely large values of \( K, K r_0 \gtrsim 10^7 \) with our choice of values of \( g_1, \eta \) and \( \xi \), simply because only then they dominate over inverse decays before all interactions other than decays go out of equilibrium.

### 2.2 CP violation

The interaction lagrangian (2.1) does not give rise to a violation of CP symmetry. In order to model CP violations, we should introduce more fields and couplings into (2.1). Since we want to keep our model as simple as possible, we shall instead resort to an explicit parametrization of the squared amplitudes for various processes, that respects CPT symmetry and unitarity but violates CP \([3, 4]\).
The branching ratios for $F$-violating decays of $\chi$ and $\bar{\chi}$ particles are defined as,

$$r \equiv \frac{\Gamma(\chi \to b\bar{d}\sigma)}{\Gamma_\chi}, \quad \bar{r} \equiv \frac{\Gamma(\bar{\chi} \to \bar{b}d\bar{\sigma})}{\Gamma_\chi}. \quad (2.4)$$

Notice that $\Gamma_{\bar{\chi}} = \Gamma_\chi$ due to $CPT$ invariance. When $CP$ is violated $r$ and $\bar{r}$ are different. We introduce the $CP$ violation parameter $\eta$,

$$\eta r_0 \equiv \frac{1}{2}(\bar{r} - r) \quad (2.5)$$

where

$$r_0 \equiv \frac{1}{2}(r + \bar{r}) \quad (2.6)$$

to describe $CP$ in decays and inverse decays.

Besides $\eta$, we have to consider the possibility of $CP$ violation in scatterings. There is no $CP$ violation in “point”-scatterings (i.e., $|\chi\bar{\sigma}\rangle \to |bd\rangle$, and crossed channels) and most of the $2 \leftrightarrow 3$ scatterings because of unitarity, as explained in appendix B. However, $CP$ violation is possible in $|\bar{b}\bar{b}\rangle \to |\bar{d}d\sigma\rangle$ and its back-process, as well as in $\chi\bar{\chi}$ annihilations. Moreover, the $CP$-violations in both processes are related to each other (see appendix B) so that we only need to define a single parameter $\xi$ as,

$$\xi \sigma(\chi\bar{\chi} \to \bar{b}b\sigma) \equiv \frac{1}{2} \left( \sigma(\chi\bar{\chi} \to \bar{b}b\bar{\sigma}) - \sigma(\chi\bar{\chi} \to b\bar{b}\sigma) \right) \quad (2.7)$$

Therefore, unitarity relations leave only two independent $CP$-violation parameters in our model, $\eta$ and $\xi$, that we assume to be independent of $x$.

3 Boltzmann Equations

We assume that $b, d, \sigma$ and their antiparticles are kept in kinetic and chemical equilibrium with other particles through unspecified fast reactions. Thus we take the chemical potentials of each of these pairs of particle and antiparticle as always equal and opposite, $\mu_b = -\mu_{\bar{b}}, \mu_d = -\mu_{\bar{d}}$ and $\mu_\sigma = -\mu_{\bar{\sigma}}$. Moreover, we make the simplification of considering $\mu_b = \mu_d$, in view of the symmetry of the model under interchange of $b$ and $d$. We further assume that only the interactions in (2.1) are responsible for the evolution of these chemical potentials in the ranges of temperature where these interactions are dominant. Thus, we have,

$$f_b(p) = f_d(p) = e^{-(E-\mu_b)/T}, \quad f_\sigma(p) = e^{-(E-\mu_\sigma)/T}. \quad (3.1)$$
In the range of temperatures of interest for global-charge generation, the \( \chi \) and \( \bar{\chi} \) particles are going out of thermal equilibrium. This provides the out-of-equilibrium element needed for the generation of a particle asymmetry \[2, 8, 9\]. Thus, only for large \( T, T \gg M \), we can assume that \( f_\chi(p) \) will be close to the thermal equilibrium distribution \( f_\chi^{eq}(p) = e^{-E_\chi/T} \).

It is customary (e.g., \[3\]) to scale out the effect of the expansion of the universe by considering ratios of the number densities \( n_i \) to the number density of photons, \( Y_i = n_i/n_\gamma \). Because the interactions in (2.1) insure that
\[
\frac{d}{dt}(Y_\chi - Y_\bar{\chi}) = -\frac{d}{dt}(Y_b - Y_\bar{b}) ,
\]
as it will become clear below, we will need only to solve for \( Y_\chi, Y_\bar{\chi}, Y_\sigma \) and \( Y_\bar{\sigma} \). Actually, (3.2) implies that the net \( F \)-number abundance \( Y_F \) is just
\[
Y_F = 4(Y_\bar{\sigma} - Y_\sigma) \simeq (Y_\bar{\sigma} - Y_\sigma) .
\]
Since we are only interested in order-of-magnitude estimates, we will drop the factor of four in (3.3) from now on (which is equivalent to effectively taking \(-F_\bar{\sigma} = F_\sigma = 1\)). Thus, we will write below three coupled equations for \( Y_F \), and \( Y_- \), \( Y_+ \), defined as
\[
Y_\pm \equiv (Y_\chi \pm Y_\bar{\chi}) / 2 .
\]
This choice of densities is convenient because, as we will see, the contribution of \( Y_- \) to the evolution of \( Y_+ \) and \( Y_F \) is small, so that we will drop all terms containing \( Y_- \) and solve only two coupled equations.

Let us mention some useful relations between the scaled number densities \( Y_i \) and the chemical potentials \( \mu_i \). Because,
\[
n_\sigma = \frac{1}{(2\pi)^3} \int d^3p \ e^{-E/T+\mu_\sigma/T} = \frac{n_\gamma}{2} e^{+\mu_\sigma/T} ,
\]
we have,
\[
Y_F \simeq Y_\bar{\sigma} - Y_\sigma = (e^{-\mu_\sigma/T} - e^{\mu_\sigma/T})/2 .
\]
Since we are looking to produce small asymmetries, \(|Y_F| \simeq O(10^{-10})\), it is a good approximation to consider \( \mu_b, \mu_\sigma \ll T \). Expanding the exponentials we get \( Y_F \simeq -\mu_\sigma/T \) and, thus,
\[
e^{\pm\mu_\sigma/T} \simeq 1 \mp Y_F .
\]

\(^1\)We could have equally well used the entropy density \( s = g_*n_\gamma \) instead of \( n_\gamma \), since we consider the effective number of relativistic degrees of freedom \( g_* \) to be constant over the range of temperatures of interest to us.
Similarly, we obtain $e^\pm \mu_b/T = 1 \mp [(Y_b - Y_b)/2]$. Moreover, from (3.2) and the initial conditions (at high enough $T$) $Y_\pm = 0$, $Y_b - Y_b = 0$ we obtain $Y_\pm = Y_b - Y_b$ and, therefore,

$$e^\pm \mu_b/T \simeq 1 \mp Y_\pm . \tag{3.7}$$

### 3.1 Evolution Equation for $\chi$-Number Density

The processes that contribute to the generation of a net number of $\sigma$ and to its damping are $\chi$ decays ($D$) and inverse decays ($ID$), both renormalizable ($R$) and non-renormalizable ($NR$), “point”-scattering ($PS$) (namely scatterings due to contact interactions of the incoming and outgoing particles, such as $\chi\bar{\sigma} \rightarrow b\bar{d}\sigma$), $\bar{\chi}\chi$ renormalizable annihilations ($RA$) and non-renormalizable ones ($NRA$, such as $\chi\bar{\chi} \rightarrow b\bar{b}\sigma$) and their crossed channels ($NRCC$, such as $\chi\bar{\chi}\bar{\sigma} \rightarrow b\bar{b}$, etc). The Boltzmann equation for $n_\chi$ is of the form

$$\frac{dn_\chi}{dt} + 3n_\chi H = - \sum_\alpha \Theta_\alpha \tag{3.8}$$

where $\Theta_\alpha$ are the collision terms due to the processes just listed, so that $\alpha$ stands for $\alpha = D, ID, PS, RA, NRA, NRCC$. The full set of equations is given in appendix A.

We will now show how to deal with the $\Theta_\alpha$ terms by considering a few of them in detail. For example, $\Theta_{D+ID}$ contains the term

$$\Theta(\chi \leftrightarrow b_1d_2\sigma) = \int d\Pi_\chi d\Pi_b d\Pi_d d\Pi_\sigma \left[ f_\chi(p_\chi)|M(\chi \rightarrow bd\sigma)|^2 - f_b(p_b)f_d(p_d)f_\sigma(p_\sigma)|M(bd\sigma \rightarrow \chi)|^2 \right], \tag{3.9}$$

where $M(i \rightarrow j)$ is the Lorentz invariant amplitude for the process $|i\rangle$ going to $|j\rangle$, and $d\Pi_A \equiv (d^4p_A/(2\pi)^3)\delta(p_A^2 - m_A^2)$. By CPT invariance, we can replace $|M(bd\sigma \rightarrow \chi)|^2$ by $|M(\bar{\chi} \rightarrow b\bar{d}\sigma)|^2$, and momentum conservation implies $f_b(p_1)f_d(p_2)f_\sigma(p_\sigma) = \exp(2\mu_b/T) \exp(\mu_\sigma/T)f_\chi^eq(p_\chi)$, therefore

$$\Theta(\chi \leftrightarrow b_1d_2\sigma) = \int d\Pi_\chi d\Pi_b d\Pi_d d\Pi_\sigma \left[ f_\chi(p_\chi)|M(\chi \rightarrow bd\sigma)|^2 - e^{2\mu_b/T}e^{\mu_\sigma/T}f_\chi^eq(p_\chi)|M(\bar{\chi} \rightarrow b\bar{d}\sigma)|^2 \right] = \left[ n_\chi \Gamma - (1 - 2Y_\pm)(1 - Y_F)n_\chi^eq\Gamma \right] \langle \Gamma_\chi \rangle, \tag{3.10}$$
where $r$ and $\bar{r}$ are the $F$-violating branching ratios defined in section 2.2. To obtain the last line in (3.10), we have used (3.6) and (3.7), $n_\chi = \int f_\chi d^4p/(2\pi)^3$, and the thermal average of the $\chi$-decay width $\Gamma_\chi$,

$$
\langle \Gamma_\chi \rangle = \frac{1}{n_\chi} \int d\Pi_\chi \ f_\chi \Gamma_\chi = (\Gamma_\chi)_{\text{rest}} \frac{K_2(x)}{K_1(x)}.
$$

(3.11)

$K_1(x)$ and $K_2(x)$ are modified Bessel functions [12], $x \equiv M/T$, and $(\Gamma_\chi)_{\text{rest}}$ is the $\chi$-decay width in the rest frame $E_\chi = M$. We assume that $\Gamma_\chi$ is dominated by the renormalizable interactions,

$$
\Gamma_\chi = \frac{1}{2E_\chi} \int d\Pi_1 d\Pi_2 \ |M(\chi \rightarrow b_1 d_2)|^2 \approx \frac{g_1^2 M^2}{8\pi E_\chi}.
$$

(3.12)

Another term in the Boltzmann equations corresponds to the annihilation through renormalizable interactions $\chi \bar{\chi} \leftrightarrow b \bar{b}$ and its back-process,

$$
\Theta(\chi \bar{\chi} \leftrightarrow b \bar{b}) = \int d\Pi_\chi d\Pi_b d\Pi_{\bar{b}} \left[ f_\chi f_{\bar{\chi}} |M(\chi \bar{\chi} \rightarrow b \bar{b})|^2 - f_b f_{\bar{b}} |M(b \bar{b} \rightarrow \chi \bar{\chi})|^2 \right] = \int d\Pi_\chi d\Pi_b d\Pi_{\bar{b}} \left[ f_\chi f_{\bar{\chi}} - f_{\chi}^{eq} f_{\bar{\chi}}^{eq} \right] |M(\chi \bar{\chi} \rightarrow b \bar{b})|^2.
$$

(3.13)

Here we have used the equality $|M(b \bar{b} \rightarrow \chi \bar{\chi})|^2 = |M(\chi \bar{\chi} \rightarrow b \bar{b})|^2$, guaranteed by $CPT$ invariance. This allows us to write (3.13) in terms of the $\chi \bar{\chi}$-annihilation cross section. In fact, using that $f_{b} f_{\bar{b}} = f_\chi^{eq} f_{\bar{\chi}}^{eq}$, where $f_\chi^{eq}(p) = f_{\chi}^{eq}(p)$, and the definition of the thermal average of $v\sigma$,

$$
\langle v\sigma(\chi \bar{\chi} \rightarrow b \bar{b}) \rangle = \frac{1}{n_\chi n_{\bar{\chi}}} \int d\Pi_\chi d\Pi_{\bar{\chi}} d\Pi_b d\Pi_{\bar{b}} f_\chi f_{\bar{\chi}} |M(\chi \bar{\chi} \rightarrow b \bar{b})|^2,
$$

(3.14)

we get

$$
\Theta(\chi \bar{\chi} \leftrightarrow b \bar{b}) = \left( n_\chi n_{\bar{\chi}} - (n_\chi^{eq})^2 \right) \langle v\sigma(\chi \bar{\chi} \rightarrow b \bar{b}) \rangle.
$$

(3.15)

We can always relate amplitudes for $3 \rightarrow 2$ reactions to the corresponding amplitude for the $2 \rightarrow 3$ back-process through $CPT$ invariance. Therefore, proceeding as in this example, only cross-sections for processes with two particles in the initial state appear in the equations. In the same way $3 \rightarrow 1$ (or $2 \rightarrow 1$) processes are related to $1 \rightarrow 3$ (or $1 \rightarrow 2$) decays.
Using the evolution equation for $n_\chi$ analogous to (3.8) (see appendix A), and changing to the dimensionless variable $x \equiv M/T$ through the relation $dt = (xH)^{-1}dx$, we obtain,

$$
\frac{dY_+}{dx} = - \frac{\langle \Gamma_\chi \rangle}{xH} \left[ (Y_+ - Y^{eq}_+) \right]
+ \eta r_0 Y_+ Y^{eq}_+ - 2r_0 Y_+ Y^{eq}_-
- 24r_0 \frac{\langle \Gamma_\chi \rangle}{x^4H} \left[ Y_+ - Y^{eq}_+ + Y_+ Y_- \right]
- 96r_0 \frac{\Gamma_{\chi_{\text{rest}}}}{x^4H} \left[ 2(Y_+ - Y^{eq}_+) + (Y_+ - Y_+ Y^{eq}_+ + Y_-)Y_- \right]
- 2 \left[ Y^2_+ - (Y^{eq}_+)^2 - Y^2 \right] \frac{n_\gamma}{xH} \langle \sigma v (\chi \bar{\chi} \rightarrow b \bar{b}) \rangle
+ v \sigma' (\chi \bar{\chi} \rightarrow b b \sigma) + v \sigma' (\chi \bar{\chi} \rightarrow b b \sigma) + \text{c.ch.}.
$$

As we will see in section 4, the first line of (3.16) contains the dominant terms and we will always be able to neglect the others. The 1st and 2nd lines correspond to $D$ and $ID$, the 3rd and 4th to PS and the remaining lines to $RAN$, $NRAN$ and crossed channels (c.ch.) of the $NRAN$.

The prime in $\sigma'$ indicates that the contribution to the cross section of a real intermediate particle (i.e., an intermediate particle on mass-shell) has been removed. Scattering or annihilation processes involving a real intermediate particle are already taken into account by other terms in the Boltzmann equations, and must be subtracted to avoid double counting. For instance, production of a $\chi$ particle near the peak of the resonance through inverse decay, subsequently followed by its decay. This kind of time-ordered sequence of processes is described by the terms in the Boltzmann equation corresponding to each individual process. In our example these are inverse decay and decay.

The need for subtraction of pole contributions is mentioned in earlier papers on baryon-asymmetry generation [3, 4], but not described in detail in the literature until recently [3], after we had developed our own subtraction method. In order to subtract the contribution of the pole, we compute the Laurent expansion about $\Gamma_\chi = 0$ of $\sigma$, or of the thermal average $\langle \sigma v \rangle$ (since it may be easier to compute this average with the complete cross section).
We then subtract the term proportional to $\Gamma^{-1}_\chi$, the only negative power occurring in the expansion, and set $\Gamma_\chi = 0$. We thus identify the term of order $\Gamma_\chi^0$ as the virtual intermediate particle contribution to the cross-section in the narrow-width approximation. This procedure is easily seen to be equivalent to the methods proposed in [13], to leading order in the coupling constants. As pointed out there, virtual cross-sections defined this way can take negative values in the region around the pole. This negative values have no practical effects, as we have verified numerically, when the subtracted cross-section appears in a damping term in the evolution equations.

The case of $\chi\bar{\chi}$ annihilations, which are a source term and in which the intermediate particle is a stable fermion, is discussed in section 5.6.

### 3.2 Evolution Equation for $F$-Number Density

The evolution of $n_{\sigma}$ is due to processes that involve non-renormalizable interactions, the only ones that violate $F$-number in our model. These processes are non-renormalizable $\chi$ decays ($NRD$) and inverse decays ($NRID$), “point”-scatterings ($PS$), $\chi\bar{\chi}$ non-renormalizable annihilations ($NRAN$), their crossed-channels ($NRCC$) and $2 \leftrightarrow 3$ non-renormalizable scatterings of $b, d$ and $\sigma$ ($NRS$ such as $bd \rightarrow b\bar{d}\sigma$, $b\bar{d} \rightarrow b\bar{d}\sigma$ etc.).

The evolution equation for $n_{\sigma}$ is of the form,

$$\frac{dn_{\sigma}}{dt} + 3n_{\sigma}H = -\sum_\alpha \Theta_\alpha \quad (3.17)$$

where $\alpha = NRD, NRID, PS, NRAN, NRCC, NRS$. The full equations for $n_{\sigma}$ and $n_{\bar{\sigma}}$ are given in appendix A, and from them we obtain,

$$\frac{dY_F}{dx} = 2r_0 \frac{\langle \Gamma_\chi \rangle}{xH} \left[ \eta(Y_+ - Y_{eq}^+) - Y_F Y_{eq}^+ - (1 + 2Y_{eq}^+)Y_- \right]$$

$$-192r_0x^{-4}\frac{\Gamma_{\text{rest}}}{H} \left[ (Y_+ + Y_{eq}^+)Y_F + (2 + Y_+ + 3Y_{eq}^+)Y_- \right]$$

$$-48r_0x^{-3}\frac{\langle \Gamma_\chi \rangle}{H} \left[ Y_F Y_+ + (1 + 2Y_{eq}^+)Y_- \right]$$

$$+4\frac{n_{\sigma}}{xH} \left[ \left[ Y_+^2 - (Y_{eq}^+)^2 - Y_-^2 \right] \xi \langle v \sigma' (\chi\bar{\chi} \rightarrow b\bar{b}\sigma) \rangle \right.$$

$$+Y_F \left[ 2(Y_{eq}^+)^2 \langle v \sigma' (\chi\bar{\chi} \rightarrow b\bar{b}\sigma) + v \sigma' (\chi\bar{\chi} \rightarrow b\bar{b}\sigma) \rangle \right.$$

$$+\langle v \sigma' (bd \rightarrow b\bar{d}\sigma) + v \sigma' (b\bar{d} \rightarrow b\bar{d}\sigma) \rangle + \text{c.ch.} \right] \right]$$

$$+\text{more } 2 \leftrightarrow 3 \text{ c.ch.} \quad (3.18)$$
The 1st line corresponds to the non-renormalizable decays and inverse decays, and some terms from 2 ↔ 3 scatterings needed to obtain the correct minus sign for $Y^{eq}_+$, as explained below. The 2nd and 3rd lines correspond to PS processes. The rest of (3.18) corresponds to non-renormalizable annihilations and the remaining terms of 2 ↔ 3 scatterings (c.ch. stands for crossed channels).

If we take into account only decays and inverse decays and neglect other processes, the r.h.s. of (3.18) reduces to,

$$2\eta r_0 \frac{\langle \Gamma \chi \rangle}{x H} (Y_+ + Y^{eq}_+)$$  \hspace{1cm} (3.19)

Notice the plus sign in front of $Y^{eq}_+$. It is easy to see that a minus sign is necessary, because no $F$-number should be generated if $Y_+ = Y^{eq}_+$, i.e., if there is no departure from equilibrium. The correct sign is obtained once the 2 ↔ 3 scattering terms proportional to $|M'(b d \rightarrow b d \sigma)|^2 - |M'(\bar{b} \bar{d} \rightarrow \bar{b} \bar{d} \sigma)|^2$ are included. The unitarity relation,

$$\sum_i |M(i \rightarrow j)|^2 = \sum_i |M(\bar{i} \rightarrow \bar{j})|^2$$

applied to $|j\rangle = |bd\sigma\rangle$, relates scatterings and decays,

$$\int [d\Pi |M(\chi \rightarrow bd\sigma)|^2 + d\Pi_1 d\Pi_2 |M'(bd \rightarrow bd\sigma)|^2] = \int [d\Pi |M(\bar{\chi} \rightarrow bd\bar{\sigma})|^2 + d\Pi_1 d\Pi_2 |M'(\bar{b}d \rightarrow \bar{b}\bar{d}\sigma)|^2]$$  \hspace{1cm} (3.20)

Thus, the above-mentioned scattering terms are proportional to the non-renormalizable decay rates, namely

$$\int d\Pi_1 d\Pi_2 \left[ |M'(b_1 d_2 \rightarrow bd\sigma)|^2 - |M'(\bar{b}_1 \bar{d}_2 \rightarrow \bar{b}\bar{d}\bar{\sigma})|^2 \right] = \int d\Pi_s \left[ |M(\bar{\chi}_s \rightarrow \bar{b}\bar{d}\bar{\sigma})|^2 - M(\chi_s \rightarrow bd\sigma)|^2 \right]$$  \hspace{1cm} (3.21)

Therefore, the contribution of 2 ↔ 3 scatterings to the source terms on the r.h.s. of (3.18) is proportional to $Y^{eq}_+$ and is given by,

$$-\frac{2}{x H n_\gamma} (1 + 2Y_- Y_F) \int d\Pi_1 d\Pi_2 d\Pi_3 d\Pi_4 d\Pi_5 \ e^{-(E_1+E_2)/T} \left[ |M'(b_1 d_2 \rightarrow b_3 d_4 \sigma_5)|^2 - |M'(\bar{b}_1 \bar{d}_2 \rightarrow \bar{b}_3 \bar{d}_4 \bar{\sigma}_5)|^2 \right]$$
\[ \begin{align*}
&= -\frac{2}{xHn_\gamma} (1 + 2Y_+Y_F) \int d\Pi_1 d\Pi_2 d\Pi_3 d\Pi_4 d\Pi_5 \ e^{-(E_3 + E_4 + E_5)/T} \\
&\quad \int d\Pi_1 d\Pi_2 d\Pi_3 d\Pi_4 d\Pi_5 \ e^{-E_5/T} \\
&= -\frac{2}{xHn_\gamma} (1 + 2Y_+Y_F) \int d\Pi_1 d\Pi_2 d\Pi_3 d\Pi_4 d\Pi_5 \ e^{-(E_3 + E_4 + E_5)/T} \\
&\quad \int d\Pi_1 d\Pi_2 d\Pi_3 d\Pi_4 d\Pi_5 \ e^{-E_5/T} \\
\end{align*} \]

which together with (3.19) adds up to the source term on the first line of (3.18).

Notice that in (3.18), besides \( \eta \) that parametrizes the violation of \( CP \) in decays, there appears also the parameter \( \xi \) for the \( CP \)-violation in annihilations (see equations (2.5) and (2.7)).

### 3.3 Parameters and Initial Conditions

In order to solve numerically the coupled evolution equations for \( Y_+ \) and \( Y_F \) we impose two initial conditions at a small enough value of \( x = x_0 \), such that at \( x_0 \) it can be safely assumed that \( \chi \) and \( \tilde{\chi} \) are in equilibrium. Thus, we take

\[ Y_+(x_0) = Y_+^{eq}(x_0), \quad Y_F(x_0) = Y_F^{eq}, \]

(3.23)

where \( Y_F^{eq} \) is equal to zero only for \( F \)-symmetric initial conditions. Numerically, any \( Y_F^{eq} \ll Y_F(\infty) \approx 10^{-10} \) will be equivalent to zero, and in most cases we take \( Y_F^{eq} = 10^{-20} \) as our \( F \)-symmetric initial condition. We have also studied \( F \)-asymmetric initial conditions (section 3.1), assuming that some other \( F \)-generating processes have acted at earlier times. In this case, for large enough \( K\rho_0 \) there is an early erasure and subsequent generation of \( Y_F \) due to the processes we consider here.

Because our effective Lagrangian is not valid at energy scales larger than \( \Lambda \), we restrict our choice of \( x_0 \) to values where \( T < \Lambda \). At any rate, the solutions of the evolution equations are stable against variations in \( x_0 \), as long as \( x_0 \) is not too close to 1, \( x_0 \lesssim 0.1 \), so

\[ 0.1 \gtrsim x_0 \gtrsim \frac{M}{\Lambda} \simeq 30 \frac{g_1}{g_2} \sqrt{\rho_0}. \]

(3.24)

In the absence of any compelling reason to do otherwise, we take \( g_1/g_2 \sim 1 \).
The evolution of the $F$-asymmetry in our model is determined by five independent parameters (besides the initial conditions $x_0, Y_{F0}$). It is clear from the evolution equations (3.16) and (3.18) that the most suitable parameters to classify different scenarios for the production of $Y_F$ are: $K$, the “effectiveness of decay” parameter, $Kr_0$, the “effectiveness of non-renormalizable reactions” parameter, $g_1$ the coupling constant of renormalizable interactions, and $\eta$ and $\xi$, the only two independent CP-violation parameters. Therefore, we use

$$K, r_0, g_1, \eta \text{ and } \xi,$$

(3.25)
as independent parameters in the evolution equations. For our numerical solutions we choose reasonable but arbitrary values for three of them,

$$g_1 = 10^{-1}, \quad \xi = \eta \simeq 5 \times 10^{-4},$$

(3.26)

and examine different ranges of values of $K$ and $Kr_0$.

$K$ is the parameter whose value determines different baryon-asymmetry scenarios in out of equilibrium decay models with renormalizable interactions,

$$K \equiv \frac{(\Gamma_\chi)_{\text{rest}}}{H(x = 1)} = \frac{1}{\sqrt{g_\ast}} \left[ \frac{g_1^2}{8\pi} + \frac{g_2^2}{192(2\pi)^3} \left( \frac{M}{\Lambda} \right)^2 \right] \frac{M_P}{M}. \tag{3.27}$$

Here $(\Gamma_\chi)_{\text{rest}}$ is the decay rate of non relativistic $\chi$ particles, $\Gamma_\chi(T \geq M) = (\Gamma_\chi)_{\text{rest}}$ (it is the decay rate in the rest frame where $E_\chi = M$), $g_\ast$ is the effective number of relativistic degrees of freedom entering into the Hubble constant, that we take to be $g_\ast = 100$, and $M_P$ is the Planck mass. We assume that the decay width of $\chi$, $\Gamma_\chi$, is dominated by the renormalizable decays (see (3.12)), namely the first term in (3.27) is dominant,

$$K \simeq \frac{g_1^2}{\sqrt{g_\ast 8\pi}} \frac{M_P}{M}. \tag{3.28}$$

With this assumption on $\Gamma_\chi$, the parameter $r_0 \simeq \Gamma_\chi^{NR}/\Gamma_\chi$ defined in (2.6), is

$$r_0 \simeq \frac{1}{192\pi^2} \left( \frac{g_2 M}{g_1 \Lambda} \right)^2. \tag{3.29}$$

$K$ and $Kr_0$ determine the “effectiveness of the reactions” because all terms in $(dY_+/dx)$ in (3.16) are proportional to $K$ (even the last one, where $n_\gamma \langle v\sigma \rangle / H \sim Kg_1^2/x$) and all terms in $(dY_F/dx)$ in (3.18) are proportional to $Kr_0$ (even the last one, where $n_\gamma \langle v\sigma \rangle / H \sim Kr_0g_1^2/x$). As we will see, the final value of $Y_F$, $Y_F(x \rightarrow \infty)$, depends only on $r_0$ for $K \lesssim 1$ and on $K$ and $r_0$ for $K \gtrsim 10$. 


4 Evolution of $Y_+$

The evolution of $Y_+$ is mainly determined by the first term in (3.16), due to decays and inverse decays. We can then write,

$$\frac{dY_+}{dx} \simeq -Kx \frac{K_1(x)}{K_2(x)}(Y_+ - Y_{eq}^+). \quad (4.1)$$

In first approximation, we shall set $Y_F = 0, Y_- = 0$ in (3.16). This approximation will be justified below.

The 5th and 6th lines in (3.16), correspond to $\chi\bar{\chi}$ annihilations. Non-renormalizable annihilation terms are always much smaller than the renormalizable one, on the 5th line of (3.16), and can be neglected. This is due to the fact that $(g_2M/g_1\Lambda)^2 \ll 1$, and also to phase-space considerations. The cross-section for $\chi\bar{\chi} \to b\bar{b}$ can be expressed as,

$$\langle v\sigma(\chi\bar{\chi} \to b\bar{b}) \rangle = \frac{g_4^2}{M^2}f(x). \quad (4.2)$$

A fit to the function $f(x)$ is shown in figure 1. At small $x \ll 1$ this term is dominant. However, in this region $Y_+$ closely follows $Y_{eq}^+ = 1/2 - O(x^2)$, the difference being of order $Y_+ - Y_{eq}^+ \sim O(x^2)$. Thus, $Y_+$ remains approximately constant, independently of the detailed form of the r.h.s. of (3.14). A plot of $Y_+(x)$ for several values of $K$ in figure 2 shows the effect of the annihilations term which, as argued, is small.

For $x > \sim 1$, the annihilation rate is suppressed relative to decays, and the first term in (3.16) determines the evolution of $Y_+$. It is in this region, in which $\chi$ particles are becoming non-relativistic and $Y_{eq}^+$ varies rapidly, where large departures from equilibrium can occur. This is, then, the most relevant region for generating an asymmetry $Y_F$.

The ratio of the 5th line, to the 1st term, contains $(Y_+ + Y_{eq}^+) \lesssim 1$, and the factor $n_c\langle v\sigma\rangle/\langle T_\chi \rangle = 32g_4^2K_2(x)f(x)/K_1(x)x^3$, where $f(x)$, the fit function just mentioned, is $f(x) < 10^{-2}$. Thus, for values of $x$ not much smaller than 1, the 5th line of (3.16) is smaller than the 1st term by a factor $g_4^2K_2(x)/x^3K_1(x) \ll 1$.

We also have that the ratio of the 1st term to the one on the 3rd line of (3.16) is given by $x^2/24r_0 \gg 1$, for $x \gtrsim 1$. A similar argument holds for the term on the 4th line. Both terms come from “point”-scattering processes.

We have numerically checked that $Y_-$ is always at least one order of magnitude smaller than $Y_F$ and $Y_+$, so it can be safely ignored. This can be
Figure 1: Cross-section for renormalizable annihilations $\chi \overline{\chi} \rightarrow bb$ as a function of $x$.

explained by noticing that the source term in the evolution equation for $Y_-$ (see appendix A) is proportional to $r_0 Y_F Y_+$ and $r_0 \ll 1$.

We are then left with the term proportional to $\eta r_0 Y_F$ on the 2nd line in (B.10), stemming from non-renormalizable decays and inverse decays. This term would be non-negligible only for very large values of $Y_F$

$$Y_F \simeq \frac{1}{\eta r_0} \frac{(Y_+ - Y_+^{eq})}{Y_+^{eq}}, \tag{4.3}$$

that would never be produced through the mechanism discussed in this paper. These large values of $Y_F$ could only arise from large initial values $Y_{F0}$ of the $F$-asymmetry, if its erasure, as discussed below, is not efficient. However in this case the value of $Y_F$ never significantly departs from $Y_{F0}$ and the mechanism discussed here becomes irrelevant.

We mentioned above that we only need to find the density of $\chi$ particles, $Y_+$, produced at $x$ of order 1 and larger in order to estimate $Y_F$. Let us consider this statement in more detail. We shall treat separately the cases of $K \ll 1$ and $K \gtrsim 1$.

For $K \ll 1$, all $\chi$-number changing interactions are out of equilibrium at $x \simeq 1$. Thus $Y_+$ does not change with respect to $Y_+(x \simeq 1) \simeq Y_+^{eq}(x \simeq 1) \simeq 0.5$ until the $\chi$-decays occur, at $x \simeq x_{\text{Decay}}$. We define $x_{\text{Decay}}$ as the
value of $x$ at which $\chi$-decays enter thermal equilibrium, namely,

$$\langle \Gamma_\chi(x_{\text{Decay}}) \rangle = H(x_{\text{Decay}}) = H(x = 1)/x_{\text{Decay}}^2,$$

and using $\langle \Gamma_\chi(x_{\text{Decay}}) \rangle \simeq (\Gamma_\chi)_{\text{rest}}$ we get $x_{\text{Decay}} \simeq (\sqrt{K})^{-1}$. Since $dY_F/dx \sim Y_+ - Y_+^{eq}$, the overabundance of $\chi$ in this case is responsible for most of the $Y_F$ produced.

For $K \gtrsim 1$, instead, there is never a large overabundance of $\chi$, because the reactions that change $\chi$-number are in equilibrium at $x \simeq 1$. While $\chi$-decays are always in equilibrium, inverse decays and $\chi$-number generating scatterings go out of equilibrium at some point $x_f$ at which all remaining $\chi$-particles decay. Therefore, because $dY_F/dx \sim Y_+ - Y_+^{eq}$, the production of $Y_F$ happens steadily for both $x \lesssim 1$ and $x \gtrsim 1$ and, consequently, the production for $x$ of $O(1)$ and larger gives the right order of magnitude.

---

$^2$To clarify this argument assume the production rate is almost constant, i.e. $dY_F/dx \simeq C$ where $C$ is a constant, so the increment of $Y_F$ is proportional to the increment of $x$.
An approximate analytical solution of (4.1), for \( K \geq 1 \) and all \( x \) is given in appendix C. We will actually only use a simpler approximation to this solution, valid for \( K > \sim 10 \) and \( K^{-1/3} \ll x \),

\[
\frac{Y_+(x) - Y^\text{eq}_+(x)}{Y^\text{eq}_+} \simeq \frac{1}{Kx}
\]

5 Evolution of \( Y_F: \) Symmetric Initial Conditions.

In this section we give analytical and numerical estimates of the asymmetry \( Y_F \) produced in different regimes, characterized by small or large values of \( K \) and \( K \tau_0 \). We shall be more specific about these regimes below.

The evolution equation (3.18) has two source terms, corresponding to generation of \( Y_F \) by decays and annihilations of \( \chi \) particles, respectively, and their inverse processes. We shall consider them separately, deferring the treatment of annihilations until the end of this section since, as we shall see, they turn out to give a negligible contribution compared to decays.

5.1 Decays and Inverse Decays as a Source.

For \( K \lesssim 1 \), all \( \chi \)-number changing processes are out of equilibrium at \( x \gtrsim 1 \). \( Y_+ \) remains constant beyond \( x = 1 \),

\[
Y_+(x > 1) \simeq Y_+(x = 1) \simeq Y^\text{eq}_+(x = 1) = 0.5
\]

until \( \chi \) and \( \bar{\chi} \) decay. This happens when the decay rate finally equals the expansion rate of the Universe, \( \langle \Gamma_\chi(x_{\text{Decay}}) \rangle = H(x_{\text{Decay}}) \), at \( x_{\text{Decay}} \simeq K^{-1/2} \).

This overabundance of \( Y_+ \) for \( x \gtrsim 1 \) is responsible for most of the \( Y_F \) produced. In this case it is easy to estimate the final \( F \)-asymmetry, because each \( \chi\bar{\chi} \) pair that decays produces a net \( F \)-number

\[
F_\theta(\bar{r} - r) = F_\theta 2\eta \tau_0
\]

where the CP-violating parameter \( \eta \) was defined in (2.5) and (2.6). Thus, we obtain

\[
Y_F(\infty) \simeq F_\theta 2\eta \tau_0 Y_+(x_D) = \eta \tau_0
\]

5.1 Decays and Inverse Decays as a Source.
since \( Y_+(x_D) \simeq 1/2 \), and we have effectively set \( F_0 = 1 \) for simplicity (see (3.3)).

For \( K > 10 \), \( \chi \)-number changing reactions are in equilibrium at \( x > 1 \), therefore \( Y_+(x > 1) \) follows \( Y_+^{eq} \) closely. We have to consider different regimes characterized by the value of \( K r_0 \). As shown below, for \( \xi \simeq \eta \) and \( g_1^2 \leq 10^{-2} \) (corresponding to our choice of parameters in (3.26)), the annihilation term is negligible as a source for \( Y_F \) compared to the decay term for reasonable values of \( K \) and \( K r_0 \). Thus, taking into account only decays, we have the following results.

* For \( K r_0 < 10^{-2} \), we have \( Y_F(\infty) \sim r_0 \eta \). Notice that \( r_0 \lesssim 10^{-3} \) (see (3.29)), so this case includes in particular the case \( K < 1 \).

* For \( 10^{-2} < K r_0 \lesssim 10 \) we have damping by point scatterings,

\[
Y_F(\infty) \sim \frac{\eta [\ell \ln(10^2 K r_0)]^2}{10^2 K}.
\]

* For \( 10 \lesssim K r_0 \lesssim 10^7 \) inverse decays are the dominant damping process and we have,

\[
Y_F(\infty) \sim \frac{\eta}{[\ell \ln(10^2 K r_0)]/K}.
\]

* For very large values of \( K r_0 \), \( K r_0 \gtrsim 10^7 \), we obtain exponential damping of the form \( Y_F(\infty) \sim \exp\{-(36 \pi K r_0 g_1^2)^{1/6}\} \), due to \( 2 \leftrightarrow 3 \) scatterings.

In the following subsections we discuss these results in more detail.

### 5.2 Damping by point scatterings

We have seen that for \( K r_0 \lesssim 10^{-2} \) we can ignore damping terms. As \( K r_0 \) grows larger other terms in the equation for \( Y_F \) start being relevant. We shall take them into account in what follows, beginning with the term corresponding to point scattering processes. In all cases we evaluate \( Y_F \) by quadratures, applying the saddle-point approximation where appropriate.

We shall now assume \( 192 K r_0 \gtrsim 1 \). The evolution equation including only the source term due to decays and the damping term due to point scatterings reads,

\[
\frac{dY_F}{dx} = 2 \eta r_0 K \frac{K_1(x)}{K_2(x)} x \Delta(x) \left[ Y_+ + Y_+^{eq} \right] Y_F
\]

\[
- 192 K r_0 \frac{1}{x^2} \left( Y_+ + Y_+^{eq} \right) Y_F
\]

(5.2)
with,
\[ \Delta(x) \equiv Y_+(x) - Y_+^{eq}(x). \]

The expression for \( Y_F(\infty) \) can then be written as,
\[
Y_F(\infty) = \int_{x_0}^{\infty} \frac{K_1(u)}{K_2(u)} \left[ \Delta(u)e^u \right] \times \exp \left\{ -u - \int_{u}^{\infty} \frac{dz}{z^2} \frac{192Kr_0}{2} \left( Y_+(z) + Y_+^{eq}(z) \right) \right\} , \quad (5.3)
\]

Since we are considering decays as the source for \( Y_F \), we expect this integral to be dominated by the contribution of the region \( u > 1 \). For this reason we explicitly extracted the exponential dependence from \( \Delta \). Furthermore, we have \( K \sim 1/(192r_0) \), so we can use the leading-order approximate expressions,
\[
\Delta(u) \simeq \frac{Y_+^{eq}(u)}{Ku} ; \quad Y_+(u) + Y_+^{eq}(u) \simeq 2Y_+^{eq}(u) \quad (5.4)
\]
\[
\frac{K_1(u)}{K_2(u)} \simeq 1 ; \quad Y_+^{eq}(u) \simeq \sqrt{\frac{\pi}{2}} \frac{u^2}{4} e^{-u} , \quad (5.5)
\]
to evaluate \( Y_F(\infty) \).

To leading order in \( K \) and \( u \) we then have,
\[
Y_F(\infty) = \sqrt{\frac{\pi}{2}} \frac{\eta r_0}{2} \int_{x_0}^{\infty} u^2 e^{u} \mathcal{E}(u) , \quad (5.6)
\]
where we defined the exponent \( \mathcal{E}(u) \) as,
\[
\mathcal{E}(u) = u + 96Kr_0 \sqrt{\frac{\pi}{2}} \Gamma \left( \frac{1}{2}, u \right) . \quad (5.7)
\]
Writing \( \mathcal{E} \) in terms of an incomplete Gamma function \[12\] will be useful below, when we consider more terms in the equation.

The exponent \( \mathcal{E}(u) \) is minimal at \( u = u_F \) given by the equation
\[
\sqrt{u_F} e^{u_F} = 96\sqrt{\frac{\pi}{2}} Kr_0 , \quad (5.8)
\]

The exponent \( \mathcal{E}(u_F) \) is minimal at \( u = u_F \) given by the equation
\[
\sqrt{u_F} e^{u_F} = 96\sqrt{\frac{\pi}{2}} Kr_0 , \quad (5.8)
\]
corresponding to the epoch of “freeze-out” of the damping process \[2\]. At the minimum we have,
\[
\mathcal{E}(u_F) \simeq u_F + 1 \quad (5.9)
\]
\[
\mathcal{E}''(u_F) = 1 + \frac{1}{2u_F} . \quad (5.10)
\]
We then obtain the expression,

\[ Y_F(\infty) = \frac{\sqrt{2\pi} \eta}{192e K} \frac{u_F^2}{\sqrt{1 + \frac{1}{2u_F}}}, \tag{5.11} \]

for the final asymmetry.

The freeze-out epoch \( u_F \) is an increasing function of \( K r_0 \). We can then roughly approximate,

\[ u_F \simeq \ell n \left( 96\sqrt{\frac{\pi}{2} K r_0} \right) \tag{5.12} \]

for large enough values of \( K r_0 \). In this way we arrive at an explicit form for \( Y_F \),

\[ Y_F(\infty) = \frac{\sqrt{2\pi} \eta}{192e K} \left[ \ell n \left( 96\sqrt{\frac{\pi}{2} K r_0} \right) \right]^2, \tag{5.13} \]

where we ignored the square root in the denominator of (5.11).

Notice that \( Y_F(\infty) \) in (5.13) depends on \( r_0 \) only through the combination \( K r_0 \), and that the dependence is very weak. This is unlike the “free decay” case where we had a linear dependence on \( r_0 \). The transition between the two regimes is, of course, not sharp. The linear dependence with \( r_0 \) becomes flatter as \( K r_0 \) grows, turning into the logarithmic form of (5.13) at about \( K r_0 \simeq 10^{-2} \).

The factor of \( K \) in the denominator represents the damping due to point scatterings. This suppression of the generated asymmetry as a power of \( K \) is similar to the damping obtained in renormalizable models \([2]\).

### 5.3 Damping by inverse decays

The next term we shall take into account corresponds to inverse decays. It is the term proportional to \( Y_F \) on the first line of (3.18). The evolution equation now reads,

\[
\frac{dY_F}{dx} = 2r_0 K \frac{K_1(x)}{K_2(x)} \left( \eta \Delta(x) - Y_F Y^{eq}_+ \right) - 192K r_0 \frac{1}{\pi^2} \left( Y_+ + Y^{eq}_+ \right) Y_F \tag{5.14}
\]

We shall proceed as in the previous section. \( Y_F(\infty) \) is given by (5.6), but now there is an extra term in the exponent,

\[
\mathcal{E}(u) = u + \frac{K r_0}{\pi} \left( 192 \Gamma \left( \frac{1}{2}, u \right) + \Gamma \left( \frac{7}{2}, u \right) \right). \tag{5.15}
\]
The equation for the freeze-out point \( u_F \) takes the form,
\[
\sqrt{\frac{\pi K r_0}{2}} \left( \frac{192}{\sqrt{u_F}} + u_F^{5/2} \right) e^{-u_F} = 1 .
\]  
(5.16)

When \( u_F^{5/2} < 192/\sqrt{u_F} \), corresponding to \( K r_0 \lesssim 6.5 \), we can neglect the second term in the parentheses in (5.16). In this case we recover the results from the previous section.

For \( K r_0 \gtrsim 6.5 \) we consider the approximate equation for \( u_F \) (compare with (5.8)),
\[
\sqrt{u_F} e^{u_F} = u_F^3 \sqrt{\frac{\pi K r_0}{2}} .
\]  
(5.17)

At \( u = u_F \) the exponent is minimal, taking the value,
\[
E(u_F) \approx u_F + 1
\]  
(5.18)
\[
E''(u) \approx 1 - \frac{5}{2} u_F^{-1} ,
\]  
(5.19)

where we used \( 5u_F^3 > 192 \), and (5.17).

Using the approximate saddle-point condition (5.17), we then obtain the expression for \( Y_F(\infty) \),
\[
Y_F(\infty) = \frac{\sqrt{2\pi}}{e} \frac{\eta}{K u_F} \frac{1}{\sqrt{1 - \frac{5}{2} u_F}} \quad \text{for} \quad K r_0 > 6.5 ,
\]  
(5.20)

to be compared with (5.11).

In order to obtain an explicit expression for \( Y_F \) we need a solution to (5.17). We cannot neglect the factor \( u_F^3 \) in that equation, because that would be inconsistent with our previous approximations. Instead, for moderate values of \( K r_0 \gtrsim 6.5 \) we can use the matching condition with the regime of the previous section and write
\[
u_F \approx \ell n \left( 96 \sqrt{\frac{\pi}{2}} K r_0 \right) ,
\]
to obtain,
\[
Y_F(\infty) = \frac{\sqrt{2\pi}}{e} \frac{\eta}{K \ell n \left( 96 \sqrt{\frac{\pi}{2}} K r_0 \right) } .
\]  
(5.21)

We see that the damping effects of inverse decays are stronger than those of point scatterings, and that they appear at a later epoch.
For larger values of $K r_0$, $K r_0 \gg 6.5$, we must use an iterated solution of (5.17),

$$u_F \simeq \ell n \left( \frac{1}{2} \sqrt{\pi} K r_0 \right) + 3 \ell n \left( \frac{1}{2} \sqrt{\pi} K r_0 \right)$$

and replace it in (5.20).

5.4 Point scatterings revisited

Point scattering processes give rise to another damping term in the evolution equation, given by the 3rd line of (3.18). Repeating the same analysis as in the previous sections, we are led to the following expression for the exponent,

$$E(u) = u + \frac{K r_0}{2} \sqrt{\frac{\pi}{2}} \left( 192 \Gamma \left( \frac{1}{2}, u \right) + \Gamma \left( \frac{7}{2}, u \right) + 24 \Gamma \left( \frac{3}{2}, u \right) \right). \quad (5.22)$$

The last term in $E$ is new. It never dominates the exponent, however, and only gives small corrections to $Y_F(\infty)$, not larger than 30%. We will, therefore, not take it into account, since we are interested only in order-of-magnitude estimates.

5.5 Damping by $2 \leftrightarrow 3$ scatterings

Having analyzed the effect of the damping terms corresponding to point scatterings and inverse decays, we are left with those related to $2 \leftrightarrow 3$ scatterings. The number of diagrams for this kind of processes is large, making the analysis quite intricate.

Since the simplicity of our model does not warrant a detailed treatment of this problem, we shall consider only two examples which we consider representative of the general situation. As we shall see, this processes turn out to be relevant only for very large values of $K r_0 \gtrsim 10^7$, which for fixed $r_0$ correspond to $K \gtrsim 10^{10}$.

The first reaction channel we shall consider is $\bar{b} \bar{\sigma} \rightarrow \bar{b} dd$ (see figure 3a). Its thermally averaged cross-section, for small and large values of $x$, has the asymptotic form,

$$\langle \nu \sigma \rangle \simeq \begin{cases} G^2 \frac{0.012}{x^4} & x \gtrsim 10 \\ G^2 \left( 2.83 \times 10^{-5} - 1.27 \times 10^{-4} \ell n(x^2) \right) & x \lesssim 0.1 \end{cases}, \quad (5.23)$$

where we used the notation $G^2 \equiv g_1^2 g_2^2 / \Lambda^2$ for brevity.
Adding the corresponding term to (5.14) we obtain,

\[
\frac{dY_F}{dx} = 2r_0K\frac{K_1(x)}{K_2(x)}\left(\eta\Delta(x) - Y_F Y^\text{eq}_+\right) \\
- 192Kr_0\frac{1}{x^2}\left(Y_+ + Y^\text{eq}_+\right)Y_F \\
- 2^{11}3\pi Kr_0\frac{g_1^2}{x^2}\frac{\langle v\sigma\rangle}{G^2}Y_F(x) .
\] (5.24)

For small values of \(x\), taking into account (5.23), the last term can be written as,

\[
2^{11}3\pi Kr_0\frac{g_1^2}{x^2}\left(2.83 \times 10^{-5} - 1.27 \times 10^{-4}\ell n(x^2)\right) Y_F(x) ,
\] (5.25)

and turns out to be negligible compared to the term on the second line of (5.24) for \(x > 10^{-10}\). The latter will, therefore, be the most relevant term in determining \(Y_F(\infty)\) for \(Kr_0 \lesssim 6.5\), as we have seen in previous sections since, for reasonable values of the parameters, \(x_0 \gg 10^{-10}\).

For very large values of \(Kr_0 \gg 6.5\), \(2 \leftrightarrow 3\) scatterings can be important. In this regime, we write \(Y_F(\infty)\) in the form (5.6), with the exponent given by

\[
\mathcal{E}(x) = x + Kr_0 \int_x^\infty dz \left[2zY^\text{eq}_+(z) + 2^{11}3\pi \frac{g_1^2}{z^2} \frac{\langle v\sigma\rangle}{G^2}\right]
\] (5.26)

\[
\mathcal{E}'(x) = 1 - Kr_0 \left[2xY^\text{eq}_+(x) + 2^{11}3\pi \frac{g_1^2}{x^2} \frac{\langle v\sigma\rangle}{G^2}\right]
\] (5.27)

The minimum of \(\mathcal{E}\) is found numerically to satisfy \(x_F > 10\), so we can
approximate the expression for $E'$ as,

$$E'(x) \simeq 1 - Kr_0 \left[ \frac{1}{2} \sqrt{\frac{\pi}{2}} x^{5/2} e^{-x} + 36\pi g_1^2 \frac{1}{x^6} \right] \tag{5.28}$$

Let us define $\tilde{x}$ such that for $x < \tilde{x}$ the first term in $E'$ is larger than the second. In this case the results of section 5.3 hold. We are interested now in the situation in which $x_F > \tilde{x}$, so that it is 2 $\leftrightarrow$ 3 scatterings that determine $Y_F(\infty)$.

The value of $\tilde{x}$ is given by the equation,

$$\frac{1}{2} \sqrt{\frac{\pi}{2}} \tilde{x}^{5/2} e^{-\tilde{x}} = 36\pi g_1^2 \frac{1}{\tilde{x}^6} . \tag{5.29}$$

Thus, we have,

$$\tilde{x}(g_1) \simeq 27.7 + \ln \left( \frac{0.01}{g_1^2} \right) \quad (g_1 \lesssim 0.1) . \tag{5.30}$$

The minimum of $E$ will then be at

$$x_F \simeq (36\pi K r_0 g_1^2)^{1/6} , \tag{5.31}$$

as long as $Kr_0$ is large enough so that the consistency condition $x_F > \tilde{x}$ is satisfied. For $g_1 = 0.1$, we obtain

$$x_F > \tilde{x} \iff Kr_0 \gtrsim 4 \times 10^8 . \tag{5.32}$$

At $x = x_F$ we have,

$$E''(x_F) \simeq \frac{6}{x_F} . \tag{5.33}$$

The minimum of $E$ is therefore broad, and the steepest-descent approximation cannot be applied. However, in view of the preceding considerations, we expect $Y_F(\infty)$ to be exponentially damped in this regime,

$$Y_F(\infty) \sim e^{-(36\pi Kr_0 g_1^2)^{1/6}} \tag{5.34}$$

As another example of 2 $\leftrightarrow$ 3 scattering we consider the diagram in figure 3b. The last term in (5.24) must now be substituted by,

$$-\frac{2}{\pi^2} g_1^2 Kr_0 f(x) Y_F(x) , \tag{5.35}$$
where,

\[ f(x) \simeq \begin{cases} 
8x^2 + 2 + O(x^2) & x \ll 1 \\
\frac{21032}{x^6} + O(x^{-8}) & x \gg 1 
\end{cases} \]  \tag{5.36}

As in the previous case, for small \( x \) this term can be neglected compared to the point-scattering term on the second line of (5.24). For \( x \sim 1 \), cross-section (5.36) is very suppressed due to the subtraction of real-intermediate-particle contributions. Only when \( x \gtrsim 10 \) can this term be relevant.

The analysis follows the same lines as for the previous diagram, since the asymptotic dependence is \( x^{-6} \) in both cases. This is, in fact, a general result; since the limit \( x \gg 1 \) corresponds to low temperatures, \( i.e. \) low initial energies, the cross-section in this limit is essentially determined by the final three-body phase-space.

The result in this case is,

\[ x_F \simeq 4(Kr_0 g_1^2)^{1/6} \tag{5.37} \]

valid for \( Kr_0 \gtrsim 10^7 \). Thus, we expect this 2 \( \leftrightarrow \) 3 process to start being relevant before the previous one as \( Kr_0 \) grows, and to have a somewhat stronger damping effect.

| \( K \)   | \( r_0 \)  | \( x_0 \) | \( \chi\) decays | \( \chi\bar{\chi} \) annihilations |
|-----|-----|-----|----------------|------------------|
| 0.1  | \( 10^{-4} \) | 0.1  | \( 5.0 \times 10^{-8} \) | \( 3.9 \times 10^{-10} \) |
| 10   | \( 10^{-4} \) | 0.1  | \( 4.8 \times 10^{-8} \) | \( 4.5 \times 10^{-10} \) |
| 100  | \( 10^{-4} \) | 0.1  | \( 3.5 \times 10^{-8} \) | \( 1.9 \times 10^{-10} \) |
| 100  | \( 5 \times 10^{-6} \) | 0.03 | \( 2.4 \times 10^{-9} \) | \( 4.9 \times 10^{-11} \) |
| 1000 | \( 10^{-4} \) | 0.1  | \( 1.4 \times 10^{-8} \) | \( 8.0 \times 10^{-12} \) |
| 1000 | \( 5 \times 10^{-6} \) | 0.03 | \( 2.0 \times 10^{-9} \) | \( 1.9 \times 10^{-11} \) |

Table 1: Numerically obtained values of \( Y_F(\infty) \) for both, \( \chi \) decays/inverse decays and \( \chi\bar{\chi} \) annihilations as source terms. Fixed parameters are specified in (3.26).
5.6 Annihilations and their back processes

The analysis of the term on the 4th line of (3.18), due to $\chi\bar{\chi}$ annihilations and their back processes, as a source of $F$-number generation can be carried out along lines similar to those of the previous sections. We shall only quote numerical results here. As expected on physical grounds, and can be seen from Table 1, this term is quantitatively much less important than decays and inverse decays.

\begin{figure}[h]
\centering
\begin{subfigure}{0.5\textwidth}
\centering
\begin{align*}
\chi & \quad \bar{\chi} \quad \sigma \\
\bar{\chi} & \quad \chi
\end{align*}
\caption{(a) $\chi\bar{\chi}$ annihilation diagram.}
\end{subfigure}
\begin{subfigure}{0.5\textwidth}
\centering
\begin{align*}
\chi & \quad \sigma \\
\bar{\chi} & \quad \chi
\end{align*}
\caption{(b) $\chi$ decay followed by point scattering.}
\end{subfigure}
\caption{(a) $\chi\bar{\chi}$ annihilation diagram. (b) $\chi$ decay followed by point scattering.}
\end{figure}

A Feynman diagram for $\chi\bar{\chi}$ annihilation is shown in figure 4(a). The intermediate fermion can be on mass shell, which makes the tree-level cross-section singular. Taking $p_4^0$ as a variable (see fig. 4(a)), at fixed center-of-mass energy $\sqrt{s}$, the kinematical region where the intermediate particle can be on-shell is given by

$$
\frac{\sqrt{s} - \sqrt{s - 4M}}{4} \leq p_4^0 \leq \frac{\sqrt{s} + \sqrt{s - 4M}}{4}.
$$

(5.38)

Notice that this interval is completely contained within the kinematical domain $0 \leq p_4^0 \leq \sqrt{s}/2$.

The singularity of the amplitude at tree level corresponds to a space-time-ordered sequence of $\chi$ decay and point scattering, illustrated in figure 4(b), in which the intermediate particle propagates over macroscopic distances. The fact that singularities in the physical region represent ordered sequences of processes has been proved in general in [13]. Clearly, the decay width of $\chi$ has to be included in the propagator in order to obtain a finite cross-section.

The differential cross-section $d\sigma/dp_4^0$ for $\chi\bar{\chi}$ annihilation is shown in figure 5, for $\Gamma_\chi = 4 \times 10^{-4}M$ which corresponds to the values of parameters
Figure 5: $\chi \bar{\chi}$ annihilation differential cross section for $s = 5$, $\Gamma_\chi = 0.0004$ and $M = 1$ ($G \equiv g_1 g_2 / \Lambda$). The vertical lines show the boundaries of the kinematical domain of real-intermediate-particle exchange.

Figure 6: Solid line: same as previous figure. Dashed line: contribution of real particle exchange only, computed at $\Gamma_\chi = 0$. Dot-dashed line: contribution of virtual particle exchange only, at $\Gamma_\chi = 0$. 
given in (3.26). It is also shown in figure 3 together with its value at zero width, and the differential cross-section for exchange of a real intermediate particle, also computed at $\Gamma_{\chi} = 0$. The former is singular at the boundaries of interval (5.38), and the latter is zero outside that interval.

In order to estimate the $F$-number generated by this process, we set the subtracted differential cross-section $d\sigma'/dp^0_4$ (i.e., the virtual-particle exchange differential cross-section) to be equal to $d\sigma/dp^0_4$, with the above-mentioned value for $\Gamma_{\chi}$, outside the kinematical limits for real-intermediate-particle exchange and zero inside that interval.

The evolution equation for $Y_F$, with $\chi\bar{\chi}$ annihilations as the only source term, can then be numerically solved with the results shown in table 4. This gives an estimate of the importance of this term relative to the other source term. The $F$-number generated in this case is negligible compared to that arising from decays and inverse decays, except for very large values of $K$ and very small values of $r_0$. In this last case both the decays- and annihilations-generated $F$-numbers are small themselves, since the system never departs much from equilibrium.

6 Asymmetric initial conditions.

We consider now the case in which the initial value for the density $Y_F$ is different from zero. To be concrete, we take $Y_{F0} > 0$. It is clear from the foregoing analysis that the processes which will be relevant to the erasure of $Y_{F0}$ are point scatterings. Therefore, we have,

$$Y_F(\infty) = Y_{F0} \exp \left\{-192K r_0 \int_{x_0}^{\infty} \frac{dz}{z^2} \left( Y_+(z) + Y_{eq}^+(z) \right) \right\} + Y^\text{sym}_F(\infty),$$

(6.1)

where $Y^\text{sym}_F(\infty)$ refers to the value of $Y_F(\infty)$ obtained in the previous sections with $Y_{F0} = 0$.

We can easily evaluate the exponent as,

$$\int_{x_0}^{\infty} dz \frac{1}{z^2} \left( Y_+(z) + Y_{eq}^+(z) \right) \simeq \frac{1}{x_0}$$

(6.2)

to write,

$$Y_F(\infty) \simeq Y_{F0} \exp \left(-\frac{192K r_0}{x_0}\right) + Y^\text{sym}_F(\infty).$$

(6.3)

As mentioned in section 3.3, the value of $x_0$ cannot be chosen arbitrarily small, since lagrangian (2.1) is not applicable at energies higher than $O(\Lambda)$.  


More precisely, tree-level diagrams such as non-renormalizable decays and point scatterings violate the unitarity bound at an energy scale of order $\sim \Lambda/g_2$. The appearance of $g_2$ here should not be surprising, since $\Lambda$ enters $\mathcal{L}$ only in the combination $g_2/\Lambda$.

Thus, the minimal value of $x_0$ we can choose is,

$$x_0 \sim \frac{g_2 M}{\Lambda} \simeq 30 g_1 \sqrt{r_0}, \quad (6.4)$$

where we used (3.24) in the last approximate equality. With this value of $x_0$ we have, finally,

$$Y_F(\infty) \simeq Y_{F0} \exp \left(\frac{-4K \sqrt{r_0}}{g_1}\right) + Y_{F\text{sym}}(\infty). \quad (6.5)$$

Depending on the values of the parameters $K$, $r_0$, $g_1$ and $\eta$ (since $Y_{F\text{sym}}(\infty) \propto \eta$), and of $Y_{F0}$, one of the two terms will dominate. Only when the first one is much smaller than the second will this model display dynamical, initial-condition-independent generation of global charge $Y_F$. This is illustrated in figure 7 for $K = 30$ and 5. In the first case the initial condition is completely erased, and only partially in the second.
7 Final Remarks

We considered a schematic model for dynamical generation of a global charge. The interactions giving rise to the violation of $F$-number are given by an operator of dimension 5, whereas the renormalizable piece of the Lagrangian is $F$-symmetric. Our model is a special case of the standard out-of-equilibrium decays scenario [2, 8, 9], with the particularity that the heavy $\chi$ boson can decay through two channels only one of which, the one mediated by the effective interactions, violates conservation of $F$-number.

We introduced $CP$ violation through a parametrization of the matrix elements for $\chi$ decays and $\chi\bar{\chi}$ annihilations consistent with unitarity and $CPT$ (to the order considered here in the coupling constants) [3, 4]. We have, then, two independent $CP$ violation parameters, $\eta$ and $\xi$, which represent the net $F$-number generated in decay and annihilation reactions, respectively. All other possible violations of $CP$ are related to $\eta$ and $\xi$ through unitarity and $CPT$. We made one additional assumption, that $\xi$ is constant. Genuine $CP$ violation would require explicit modelling [2, 8], with the introduction of more fields and coupling constants. We preferred not to do so, to keep our model as simple as possible.

The model assumes the standard Friedmann-Robertson-Walker cosmology [2, 8, 9, 4], which enters the evolution equations explicitly through the Hubble parameter $H$. Furthermore, other species and interactions are assumed to exist, which do not change $F$-number, give a $g_\ast \sim O(100)$, and maintain light degrees of freedom in local thermal equilibrium. In order to establish the evolution equations and make them tractable we made two approximations that are known to be valid in more realistic models such as GUT-based baryogenesis models [3, 4, 5]. These are the neglect of degeneracy factors and the use of Maxwell-Boltzmann equilibrium distributions. We also neglected processes of order higher than the first in $g_1$ and $g_2$. In view of the negligibly small effects of $2 \rightarrow 3$ scatterings for all reasonable values of parameters, this approximation is justified.

The model has five parameters, which can be chosen as explained in section 3.3. We kept $g_1$, $\eta$ and $\xi$ fixed (see 3.26), and studied the final value of the global charge for broad ranges of values of $K$ and $r_0$. Since we set the coupling constant for non-renormalizable interactions $g_2 = 0.1 = g_1$, the condition $M < \Lambda$ leads to an upper bound for $r_0$, $r_0 \lesssim 5 \times 10^{-4}$.

Besides these five parameters, initial conditions $x_0, Y_{F_0}$ must be chosen. In the symmetric case $Y_{F_0} \ll Y_F(\infty)$ the final value of $Y_F(\infty)$ is insensitive to the previous value of $x_0$. We have used $x_0 \sim M/\Lambda$ in this case. For
asymmetric initial conditions the choice of $x_0$ is more delicate. Whether large initial asymmetries are completely erased or not depends on the value of $Y_{F0}$, $K$ and $r_0$, but also on $x_0$. Due to the nature of our model, we cannot approximate $x_0 \simeq 0$ as in other simple models [2], since some of the terms in the evolution equation (notably point scatterings) are singular at $x = 0$. Thus, we had to determine the minimal value $x_0$ can take. This is the value at which non-renormalizable processes such as point-scatterings start violating unitarity bounds, $x_0 \sim g_2 M/\Lambda$. For $x$ much less than this value, the description of $F$ violating interactions as effective operators ceases to be valid.

The case of symmetric initial conditions is the more interesting, since it displays purely dynamical generation of global charge. The appearance of an additional parameter $r_0$ in our model, besides $K$, which determines the effectiveness of damping, introduces some differences with respect to renormalizable models. As shown in section 3, damping is effective only when $Kr_0 \gtrsim 10^{-2}$, so even for relatively large values of $K$ we can have undamped generation of $F$-charge. The largest $Y_F(\infty)$ attainable in this case is $\eta r_0$.

Also as a result of having a branching ratio $r_0$, exponential damping due to $2 \rightarrow 3$ scatterings is effectively postponed to very large values of $K$, since $Kr_0$ itself must be large for these processes to be in thermal equilibrium. We mentioned in section 3 that the number of $2 \rightarrow 3$ scattering channels is rather large in our model, and considered two specific examples which showed that these processes are irrelevant for $K \lesssim 10^9$. (Notice, however, that they are important in deriving the Boltzmann equations, as explained in section 3).

An important class of processes not present in renormalizable models is two-body point scatterings. (These have been considered in the framework of SUSY GUTs in 10, 11.) These scatterings turn out to be the most efficient damping process at low $x$. Thus, they are crucial for erasing initial asymmetries. For symmetric initial conditions, $2 \rightarrow 2$ scatterings give algebraic damping ($\sim 1/K$) for $Kr_0 \gtrsim 10^{-2}$. Inverse decays are also important for $Kr_0 \gtrsim 7$. They are in equilibrium for $x \sim 1$, for large enough $Kr_0$, and also produce $1/K$ damping of the generated $Y_F$.

Besides $\chi$ decays and inverse decays, $\chi \bar{\chi}$ annihilations and their back processes are also a source of $F$-generation. Since $CP$ violation in this case is independent of that in decays, it is always possible to choose parameters in such a way that annihilations be the dominant source term (by setting $\xi \gg \eta$). In the absence of explicit modelling of $CP$ violation this situation
cannot be ruled out. We considered this possibility artificial, and used $\xi \sim \eta$ in our computations in order to be able to compare the relative importance of both source terms. As shown in section 5.6, the annihilations term is negligible compared to decays over a wide range of values of $K$ and $r_0$. When $K$ is large and $r_0$ small enough to block damping ($Kr_0 \lesssim 10^{-2}$), the $F$-number generated by annihilations is comparable to that generated by decays, typically one order of magnitude smaller.

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A Evolution equations

Here, we give the full evolution equations of \( n_\chi, n_\sigma, n_b \) and show that the \( F \)-number asymmetry only depends on \( Y_\sigma - Y_\bar{\sigma} \). We define

\[
\Theta(\alpha_1 \alpha_2 \cdots \leftrightarrow \beta_1 \beta_2 \cdots) \equiv \int d\Pi_{\alpha_1} d\Pi_{\alpha_2} \cdots d\Pi_{\beta_1} d\Pi_{\beta_2} \cdots \\
\times \left[ f_{\alpha_1} f_{\alpha_2} \cdots |M(\alpha_1 \alpha_2 \cdots \rightarrow \beta_1 \beta_2 \cdots)|^2 \\
- f_{\beta_1} f_{\beta_2} \cdots |M(\beta_1 \beta_2 \cdots \rightarrow \alpha_1 \alpha_2 \cdots)|^2 \right] \\
= -\Theta(\beta_1 \beta_2 \cdots \leftrightarrow \alpha_1 \alpha_2 \cdots),
\]

(A.1)

where \( \alpha_1 \alpha_2 \cdots, \beta_1 \beta_2 \cdots \) label different particles, and \( \Theta' \) is given by a similar expression where \( M' \) replaces \( M \) (namely the amplitude with on-mass shell propagators subtracted). The full \( n_\chi \) evolution equations is,

\[
\frac{dn_{\chi}}{dt} + Hn_{\chi} = -\Theta_{D+ID}(\chi \leftrightarrow bd) - \Theta_{NRD+NRID}(\chi \leftrightarrow bd\sigma) \\
- \Theta_{PS}(\bar{\sigma}\chi \leftrightarrow bd) - 2\Theta_{PS}(\bar{b}\chi \leftrightarrow d\sigma) - 2\Theta_{AN}(\chi \bar{\chi} \leftrightarrow \bar{b}\bar{b}) \\
- 2\Theta'_{NRAN+NRCC}(\chi \bar{\chi} \leftrightarrow \bar{b}\bar{b}\sigma) - 2\Theta'_{NRAN+NRCC}(\chi \bar{\chi} \leftrightarrow \bar{b}\bar{b}\sigma) \\
- 2\Theta_{NRCC}(\sigma \chi \bar{\chi} \leftrightarrow \bar{b}\bar{b}) - 2\Theta_{NRCC}(\sigma \chi \bar{\chi} \leftrightarrow \bar{b}\bar{b}) \\
- 2\Theta_{NRCC}(b\chi \bar{\chi} \leftrightarrow b\sigma) - 2\Theta_{NRCC}(b\chi \bar{\chi} \leftrightarrow b\sigma) \\
- 2\Theta_{NRCC}(\bar{b}\chi \bar{\chi} \leftrightarrow \bar{b}\sigma) - 2\Theta_{NRCC}(\bar{b}\chi \bar{\chi} \leftrightarrow \bar{b}\sigma),
\]

(A.2)

where the factor of 2 in \( PS \) and \( AN \) (\( R \) or \( NR \)) terms accounts for similar processes with \( b \) and \( d \) exchanged. The \( n_{\bar{\chi}} \) evolution equation is obtained by replacing in (A.2) all particles by their antiparticles.

The full \( n_b \) evolution equation is,

\[
\frac{dn_{b}}{dt} + Hn_{b} = +\Theta_{D+ID}(\chi \leftrightarrow bd) + \Theta_{NRD+NRID}(\chi \leftrightarrow bd\sigma) + \Theta_{AN}(\chi \bar{\chi} \rightarrow \bar{b}\bar{b}) \\
+ \Theta_{PS}(\bar{d}\chi \leftrightarrow b\sigma) - \Theta_{PS}(\bar{b}\chi \leftrightarrow d\sigma) + \Theta_{PS}(\bar{\sigma}\chi \leftrightarrow bd) \\
+ \Theta_{NRAN+NRCC}(\chi \bar{\chi} \leftrightarrow \bar{b}\bar{b}\sigma) + \Theta_{NRAN+NRCC}(\chi \bar{\chi} \leftrightarrow \bar{b}\bar{b}\sigma) \\
- \Theta_{NRCC}(\bar{b}\bar{b} \leftrightarrow \sigma\chi\bar{\chi}) - \Theta_{NRCC}(\bar{b}\bar{b} \leftrightarrow \sigma\chi\bar{\chi}) \\
+ \Theta_{NRCC}(\bar{b}\chi \leftrightarrow \bar{b}\chi) + \Theta_{NRCC}(\bar{b}\chi \leftrightarrow \bar{b}\chi) \\
+ \Theta_{NRCC}(\bar{\sigma}\chi \leftrightarrow \bar{b}\chi) + \Theta_{NRCC}(\bar{\sigma}\chi \leftrightarrow \bar{b}\chi) \\
- \Theta_{NRS}(\bar{b}\bar{b} \leftrightarrow \bar{d}\bar{d}\sigma) - \Theta_{NRS}(\bar{b}\bar{b} \leftrightarrow \bar{d}\bar{d}\sigma) + \Theta_{NRS}(\bar{d}\bar{d} \leftrightarrow \bar{b}\bar{b}\sigma) \\
+ \Theta_{NRS}(\bar{d}\bar{d} \leftrightarrow \bar{b}\bar{b}\sigma) + \Theta_{NRS}(\bar{d}\bar{d} \leftrightarrow \bar{b}\bar{b}d) + \Theta_{NRS}(\bar{d}\bar{d} \leftrightarrow \bar{b}\bar{b}d) \\
+ \Theta_{NRS}(\bar{d}\bar{d} \leftrightarrow \bar{b}\bar{b}d) + \Theta_{NRS}(\bar{d}\bar{d} \leftrightarrow \bar{b}\bar{b}d).
\]

(A.3)
Similar expressions hold for \( n_d \) (switching \( b \) and \( d \)), and for \( n_{\bar{b}}, n_{\bar{d}} \) (replacing the particles by their antiparticles. Using (A.2) and (A.3) one can show \( n_b - n_{\bar{b}} \) and \( n_\chi - n_{\bar{\chi}} \), evolve equally, i.e.

\[
\frac{d(n_b - n_{\bar{b}})}{dt} + H(n_b - n_{\bar{b}}) = \frac{d(n_d - n_{\bar{d}})}{dt} + H(n_d - n_{\bar{d}})
\]

\[
= +\Theta_{D+ID}(\chi \leftrightarrow bd) - \Theta_{D+ID}(\bar{\chi} \leftrightarrow \bar{b}\bar{d})
+\Theta_{NRD+NRID}(\chi \leftrightarrow b\sigma) - \Theta_{NRD+NRID}(\bar{\chi} \leftrightarrow \bar{b}\bar{d}\sigma)
+\Theta_{PS}(\bar{\sigma}\chi \leftrightarrow bd) - \Theta_{PS}(\bar{\sigma}\bar{\chi} \leftrightarrow \bar{b}\bar{d})
+2\Theta_{PS}(b\bar{\chi} \leftrightarrow d\sigma) - 2\Theta_{PS}(\bar{b}\bar{\chi} \leftrightarrow \bar{d}\sigma)
\]

\[
= - \left[ \frac{d}{dt}(n_\chi - n_{\bar{\chi}}) + H(n_\chi - n_{\bar{\chi}}) \right].
\]  

(A.4)

Hence

\[
\frac{d(Y_b - Y_{\bar{b}})}{dt} = - \frac{d(Y_\chi - Y_{\bar{\chi}})}{dt} = \frac{d(Y_d - Y_{\bar{d}})}{dt},
\]  

(A.5)

and since \( Y_F = 4(Y_\sigma - Y_{\bar{\sigma}}) + 2(Y_\chi - Y_{\bar{\chi}}) + (Y_b - Y_{\bar{b}}) + (Y_d - Y_{\bar{d}}), \) \( Y_F \) is given by \( Y_\sigma - Y_{\bar{\sigma}} \).

The full \( n_\sigma \) evolution equation is

\[
\frac{dn_\sigma}{dt} + Hn_\sigma = +\Theta_{NRD+NRID}(\chi \leftrightarrow b\sigma) + 2\Theta_{PS}(b\bar{\chi} \leftrightarrow d\sigma) + \Theta_{PS}(\bar{b}\bar{d} \leftrightarrow \bar{\sigma}\bar{\chi})
+2\Theta'_{NRA+NRC}(\chi \leftrightarrow \bar{b}\bar{\sigma}) + 2\Theta_{NRC}(b\chi \leftrightarrow b\sigma\chi)
+2\Theta_{NRC}(\bar{b}\chi \leftrightarrow \bar{b}\sigma\bar{\chi}) + \Theta_{NRCC}(b\sigma \leftrightarrow b\chi\bar{\sigma})
+2\Theta_{NRCC}(\bar{b}\chi \leftrightarrow \bar{b}\sigma\bar{\chi}) - \Theta_{NRC}(\sigma\chi \leftrightarrow b\chi\bar{\sigma})
-2\Theta_{NRCC}(\sigma\bar{\chi} \leftrightarrow b\bar{b}\sigma) + 2\Theta_{NRCC}(b\bar{b} \leftrightarrow \sigma\chi\bar{\sigma})
-2\Theta_{NRCC}(b\sigma \leftrightarrow b\chi\bar{\sigma}) - 2\Theta_{NRC}(b\sigma \leftrightarrow b\chi\bar{\sigma})
+\Theta'_{NRS}(b\bar{d} \leftrightarrow d\bar{\sigma}) + \Theta'_{NRS}(\bar{b}\bar{d} \leftrightarrow \bar{b}\bar{d}\sigma)
+2\Theta_{NRS}(b\bar{d} \leftrightarrow d\bar{\sigma}) + 2\Theta_{NRS}(b\bar{d} \leftrightarrow d\bar{d}\sigma)
-2\Theta_{NRS}(b\sigma \leftrightarrow b\bar{d}) - 2\Theta_{NRS}(b\sigma \leftrightarrow b\bar{d}\bar{d}).
\]  

(A.6)

Changing all particles by their antiparticles in (A.6) one obtains the equation for \( n_\sigma \).

The evolution equation of \( Y_- \) is,

\[
\frac{dY_-}{dx} = -\frac{\langle T_\chi \rangle}{xH} \left[ Y_- (1 + 2Y_+^{eq}) + r_0 Y_F Y_+^{eq} (1 - 2r_0 Y_-) \right] - 24r_0 x^{-3}\frac{\langle T_\chi \rangle}{H} \left[ Y_- (1 + 2Y_+^{eq}) + Y_+ \right].
\]
\[
-96r_0x^{-4}\frac{\langle \Gamma \chi \rangle}{H} [Y_-(2 + Y_+ + 2Y_{eq}^\pm) + Y_F(Y_+ + Y_{eq}^\pm)] . \tag{A.7}
\]

## B \quad CP \ \text{violation}

All the processes we have considered in this paper are up to the order \((g_2/\Lambda)^2\). In this Appendix, we will use unitarity to check the possibility of \(CP\) violating differences \(|M(a \rightarrow b)|^2 - |M(b \rightarrow a)|^2\) up to the order \((g_2/\Lambda)^4\).

In order to have \(CP\) violation in a process \(|a\rangle \rightarrow |b\rangle\), there must be other final states besides \(|b\rangle\) for \(|a\rangle\) to go to and other initial state besides \(|a\rangle\) that can go to \(|b\rangle\). Namely, unitarity imposes the following relations,

\[
\sum_j |M(a \rightarrow j)|^2 = \sum_j |M(\bar{a} \rightarrow \bar{j})|^2 \tag{B.1}
\]
\[
\sum_i |M(i \rightarrow b)|^2 = \sum_i |M(\bar{i} \rightarrow \bar{b})|^2 \tag{B.2}
\]

where \(\bar{a}, \bar{b}, \bar{i}, \bar{j}\) denote the \(CP\) conjugates of \(a, b, i, j\) respectively. Therefore, if \(|\bar{b}\rangle\) is the unique state of \(|a\rangle\) or \(|\bar{a}\rangle\) the unique initial state for \(|\bar{b}\rangle\), then either (B.1) or (B.2) imply

\[
|M(a \rightarrow b)|^2 = |M(\bar{a} \rightarrow \bar{b})|^2,
\]
which is equivalent to the requirement of \(CP\) conservation in the process. Using this condition, we find most processes do not violate \(CP\) up to the order of \((g_2/\Lambda)^4\).

Let us look at some examples:

1. Consider \(|\bar{b}\chi\rangle\) as initial state. By (B.1), we get

\[
\int \left[ d\Pi_d d\Pi_\sigma |M(\bar{b}\chi \rightarrow d\sigma)|^2 + d\Pi_b d\Pi_\sigma d\Pi_\chi |M(\bar{b}\chi \rightarrow \bar{b}\sigma\chi)|^2 \\
+ d\Pi_b d\Pi_\sigma d\Pi_\chi |M(\bar{b}\chi \rightarrow \bar{b}\sigma\chi)|^2 \right]
\]

\[
= \int \left[ d\Pi_d d\Pi_\sigma |M(b\tilde{\chi} \rightarrow d\tilde{\sigma})|^2 + d\Pi_b d\Pi_\sigma d\Pi_\chi |M(b\tilde{\chi} \rightarrow b\tilde{\sigma}\tilde{\chi})|^2 \\
+ d\Pi_b d\Pi_\sigma d\Pi_\chi |M(b\tilde{\chi} \rightarrow b\tilde{\sigma}\tilde{\chi})|^2 \right] \tag{B.3}
\]

that may seem to allow \(CP\) violation. But with \(|\bar{b}\sigma\chi\rangle\) and \(|\bar{b}\tilde{\sigma}\tilde{\chi}\rangle\) as final states by (B.2) we get

\[
|M(\bar{b}\chi \rightarrow \bar{b}\sigma\chi)|^2 = |M(b\tilde{\chi} \rightarrow b\tilde{\sigma}\tilde{\chi})|^2, \tag{B.4}
\]
\[
|M(\bar{b}\chi \rightarrow \bar{b}\tilde{\sigma}\tilde{\chi})|^2 = |M(b\tilde{\chi} \rightarrow b\tilde{\sigma}\tilde{\chi})|^2 \tag{B.5}
\]
and using (B.3), we have
\[ |M(b\chi \to d\sigma)|^2 = |M(b\bar{\chi} \to d\bar{\sigma})|^2 \] (B.6)

Therefore there is no CP violation in any of the three processes in (B.3).

2. With the assumption of no asymmetry of b and d in all processes up to the order of \((g_2/\Lambda)^2\), i.e.
\[ |M(\bar{\sigma}\chi \to b\bar{\chi})|^2 = |M(\bar{\sigma}\chi \to d\bar{\chi})|^2 \] (B.7)

we find again no CP violation in processes that involve the state |\bar{\sigma}X\rangle. In this case, the relation (B.1) is
\[
\int \left[ d\Pi_b d\Pi_d |M(\bar{\sigma}\chi \to bd)|^2 + d\Pi_b d\Pi_d |M(\bar{\sigma}\chi \to bb\chi)|^2 \\
+ d\Pi_d d\Pi_d |M(\bar{\sigma}\chi \to dd\chi)|^2 \right] \\
= \int \left[ d\Pi_b d\Pi_d |M(\sigma\bar{\chi} \to \bar{b}d)|^2 + d\Pi_b d\Pi_d |M(\sigma\bar{\chi} \to \bar{b}b\bar{\chi})|^2 \\
+ d\Pi_d d\Pi_d |M(\sigma\bar{\chi} \to d\bar{d}\chi)|^2 \right]. \tag{B.8}
\]

Now, consider |bb\chi\rangle as the final state then (B.2) gives
\[ |M(\sigma\chi \to bb\chi)|^2 = |M(\sigma\chi \to b\bar{b}\chi)|^2 = |M(\sigma\bar{\chi} \to bb\bar{\chi})|^2 \] (B.9)

and with |\sigma\chi\rangle as initial state, (B.1) and (B.7) yield
\[ |M(\sigma\chi \to bb\chi)|^2 = |M(\sigma\chi \to dd\chi)|^2 = |M(\sigma\bar{\chi} \to bb\bar{\chi})|^2 = |M(\sigma\bar{\chi} \to dd\bar{\chi})|^2 \] (B.10)

Then, combining (B.9) and (B.10), one obtain
\[ |M(\bar{\sigma}\chi \to bb\chi)|^2 = |M(\bar{\sigma}\chi \to bb\bar{\chi})|^2, \tag{B.11} \]

and (B.11) together with (B.8) give
\[ |M(\bar{\sigma}\chi \to bd)|^2 = |M(\sigma\bar{\chi} \to \bar{b}d)|^2 \] (B.12)

To sum up, we find that only the processes |\chi\rangle \to |bd\rangle, |\chi\rangle \to |bd\sigma\rangle, |bd\rangle \to |bd\sigma\rangle, |bb\rangle \to |dd\sigma\rangle, |dd\rangle \to |bb\sigma\rangle, |\chi\bar{\chi}\rangle \to |dd\bar{\sigma}\rangle, |\chi\bar{\chi}\rangle \to |bb\bar{\sigma}\rangle, and their CP conjugates violate CP. We have related the CP violation in |\chi\rangle \to |bd\rangle and |\chi\rangle \to |bd\sigma\rangle and their CP conjugates by using the branching ratios \(r\) and \(\bar{r}\). The CP violation in |\chi\rangle \to |bd\sigma\rangle and the violation in
|bd⟩ → |bdσ⟩ are related in (3.20). With |bσ⟩ as the final state in (B.2), we obtain

\[
\int \left[ d\Pi_x d\Pi \chi |M'(\chi \bar{\chi} \to b\bar{b}\sigma)|^2 + d\Pi_d d\Pi_d |M(\bar{d}d \to b\bar{b}\sigma)|^2 \right] = \int \left[ d\Pi_x d\Pi \bar{\chi} |M'(\bar{\chi} \chi \to b\bar{b}\sigma)|^2 + d\Pi_d d\Pi_d |M(\bar{d}d \to b\bar{b}\sigma)|^2 \right],
\]

namely,

\[
\int d\Pi_x d\Pi \bar{\chi} \left[ |M'(\bar{\chi} \chi \to b\bar{b}\sigma)|^2 - |M'(\chi \bar{\chi} \to b\bar{b}\sigma)|^2 \right] = - \int d\Pi_d d\Pi_d \left[ |M(\bar{d}d \to b\bar{b}\sigma)|^2 - |M(\bar{d}d \to b\bar{b}\sigma)|^2 \right] = 2\xi \int d\Pi_x d\Pi \bar{\chi} |M(\chi \bar{\chi} \to b\bar{b}\sigma)|^2 \tag{B.13}
\]

where ξ is defined in (2.7). We have a similar equation for |d\bar{d}σ⟩ as final state. Note that in (B.14) we take |M(\chi \bar{\chi} \to d\bar{d}\sigma)|^2 = |M(\chi \bar{\chi} \to b\bar{b}\sigma)|^2, since we assume no asymmetry of b and d in all the processes upto order \((g_2/\Lambda)^2\).

\section{C Approximate solution for \(Y_{+}\)}

The equation for \(\Delta(x) = Y_{+} - Y_{+}^{eq}\), neglecting terms proportional to \(Y_B\) and \(Y_{-}\), is given by,

\[
\frac{d\Delta}{dx} = -Kg(x)\Delta(x) - \frac{dY_{+}^{eq}}{dx}, \tag{C.1}
\]

where we used the notation \(g(x) = xK_1(x)/K_2(x)\). Defining \(\Delta_K(x) = K\Delta(x/K)\), we get the equation,

\[
\frac{d\Delta_K(x)}{dx} = g\left(\frac{x}{K}\right) \Delta_K(x) - \frac{dY_{+}^{eq}}{dx} \left(\frac{x}{K}\right) \tag{C.2}
\]

with the initial condition \(\Delta_K(Kx_0) = K\Delta(x_0) = 0\).

For very large \(K\), we can expand the coefficient functions \(g(x/K)\) and \(dY_{+}^{eq}(x/K)/dx\) in these equations about \(x/K = 0\) (i.e. for \(K \gg x\)), solve the resulting equation for \(\Delta_K\), and transform back to \(\Delta\) to obtain,

\[
\Delta(x) = \frac{1}{8} \left( x^2 \text{ _1F_1}\left(\frac{2}{3}, \frac{5}{3}, \frac{Kx^3}{6}\right) - x_0^2 \text{ _1F_1}\left(\frac{2}{3}, \frac{5}{3}, \frac{Kx_0^3}{6}\right) \right) \times \exp(-Kx^3/6) \tag{C.3}
\]
where $_1F_1$ is a confluent hypergeometric function \[12\].

For $Kx^3 \gg 1$ (i.e. $K^{-1/3} \ll x \ll K$) we obtain,

$$\Delta(x) \approx \frac{1}{2} \frac{1}{Kx}$$  \hfill (C.4)

which coincides with the result (see \[4\], eqn. (6.29)),

$$\Delta(x) = \frac{Y_{eq}^+(x)}{Kx}$$  \hfill (C.5)

if we approximate, $Y_{eq}^+ \approx 1/2$, valid for $x < 1$. Notice that in our model $\chi$ is a scalar, so $Y_{eq}^+(x \ll 1) = 1/2$, instead of 1. This suggests that we take, for large $K$,

$$\Delta(x) = \frac{1}{4} Y_{eq}^+(x) \left( x^2 \ _1F_1\left( \frac{2}{3}, \frac{5}{3}, \frac{Kx^3}{6} \right) - x^2 \ _1F_1\left( \frac{5}{3}, \frac{Kx^3}{6} \right) \right) \times \exp(-Kx^3/6).$$  \hfill (C.6)

This expression is a very accurate representation of $\Delta$ for large values of $K$, which already at $K = 10$ departs by about 5% from the numerical solution of equation \[C.1\] at $x \simeq 1$, the region where the error is largest.