Extremal functions for a singular Hardy-Moser-Trudinger inequality

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Abstract In this paper, using the blow-up analysis, we prove a singular Hardy-Moser-Trudinger inequality, and find its extremal functions. Our results extend those of Wang and Ye (2012), Yang and Zhu (2016), Csató and Roy (2015) and Yang and Zhu (2017).

Keywords singular Trudinger-Moser inequality, extremal function, blow-up analysis

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1 Introduction and main results

Let \( \Omega \subset \mathbb{R}^2 \) be a smooth bounded domain, and \( W^{1,2}_0(\Omega) \) be the usual Sobolev space. The classical Moser-Trudinger inequality [18, 21] states that

\[
\sup_{u \in W^{1,2}_0(\Omega), \|\nabla u\|_2 \leq 1} \int_{\Omega} e^{\gamma u^2} \, dx < \infty,
\]

for any \( \gamma \leq 4\pi \). If \( \gamma > 4\pi \), then the supremum is infinite although the integrals in (1.1) are finite. In this sense, the inequality (1.1) is sharp. It plays an important role in geometric analysis and partial differential equations. Let \( \lambda_1(\Omega) \) be the first eigenvalue of the Laplacian operator with respect to the Dirichlet boundary condition. Adimurthi and Druet [1] proved that for any \( \alpha, 0 \leq \alpha < \lambda_1(\Omega) \),

\[
\sup_{u \in W^{1,2}_0(\Omega), \|\nabla u\|_2 \leq 1} \int_{\Omega} e^{4\pi u^2(1+\alpha \|u\|_2^2)} \, dx < \infty,
\]

whereas for any \( \alpha \geq \lambda_1(\Omega) \), the above supremum is infinity. The analogs of (1.2) were obtained on a compact Riemannian surface [23] and on a high dimensional Euclidean domain [24]. Clearly, the inequality (1.2) is stronger than the inequality (1.1). Tintarev [20] obtained an improvement of the Moser-Trudinger inequality as follows:

\[
\int_{\Omega} |\nabla u|^2 \, dx - \int_{\Omega} V(x)u^2 \, dx \leq 1, u \in C^\infty_0(\Omega) \int_{\Omega} e^{4\pi u^2} \, dx < \infty,
\]

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\int_{\Omega} |\nabla u|^2 \, dx - \int_{\Omega} V(x)u^2 \, dx \leq 1, u \in C^\infty_0(\Omega) \int_{\Omega} e^{4\pi u^2} \, dx < \infty,
\]
where $V(x) > 0$ is a specific class of potentials. Let us write for any $0 \leq \alpha < \lambda_1(\Omega)$,

$$
\|u\|_{1, \alpha} = \left( \int_{\Omega} |\nabla u|^2 \, dx - \alpha \int_{\Omega} u^2 \, dx \right)^{1/2}.
$$

(1.4)

Then a special case of (1.3) is the following:

$$
\sup_{u \in W^{1,2}_0(\Omega), \|u\|_{1, \alpha} \leq 1} \int_{\Omega} e^{4\pi u^2} \, dx < \infty,
$$

(1.5)

where $0 \leq \alpha < \lambda_1(\Omega)$. Note that the inequality (1.5) is stronger than (1.2). It was shown by Yang [25] that the supremum in (1.5) can be attained by some function $u_0 \in W^{1,2}_0(\Omega) \cap C^1(\Omega)$ with $\|u_0\|_{1, \alpha} = 1$.

For singular Moser-Trudinger inequalities, Adimurthi and Sandeep [2] proved that for any $\beta$, $0 \leq \beta < 1$, it holds that

$$
\sup_{u \in W^{1,2}_0(\Omega), \|u\|_{2, \beta} \leq 1} \int_{\Omega} \frac{e^{4\pi(1-\beta)u^2}}{|x|^{2\beta}} \, dx < \infty.
$$

(1.6)

An analog of (1.6) was established by Adimurthi and Yang [4] in the entire Euclidean space. The existence of the extremal function of (1.6) was proved by Csató and Roy [10]. Let $B \subset \mathbb{R}^2$ be the standard unit disc. Yuan and Zhu [28] proposed the following Adimurthi-Druet type inequality:

$$
\sup_{u \in W^{1,2}_0(B), \|u\|_{2, \beta} \leq 1} \int_{B} \frac{e^{4\pi(1-\beta)u^2(1+\|u\|^2_{\lambda, \beta})}}{|x|^{2\beta}} \, dx < \infty,
$$

(1.7)

where $0 \leq \beta < 1$, $\|u\|_{2, \beta} = (\int_{B} |x|^{-2\beta} u^2 \, dx)^{1/2}$, $0 \leq \alpha < \lambda_{1, \beta}(B)$ with

$$
\lambda_{1, \beta}(B) = \inf_{u \in W^{1,2}_0(B), \|u\|_{2, \beta} = 1} \int_{B} |\nabla u|^2 \, dx.
$$

Recently, (1.6) was generalized by Yang and Zhu [27] to the following: it holds that for any $\alpha$, $0 \leq \alpha \leq \lambda_1(\Omega)$,

$$
\sup_{u \in W^{1,2}_0(\Omega), \|u\|_{1, \alpha} \leq 1} \int_{\Omega} \frac{e^{4\pi(1-\beta)u^2}}{|x|^{2\beta}} \, dx < \infty,
$$

(1.8)

where $\|u\|_{1, \alpha}$ is defined as in (1.4). Moreover, the above supremum can be attained. A related result can be found in [14].

Another important inequality in analysis is the Hardy inequality, which says that

$$
\int_{B} |\nabla u|^2 \, dx - \int_{B} \frac{u^2}{(1 - |x|^2)^2} \, dx \geq 0, \quad \forall u \in W^{1,2}_0(B),
$$

(1.9)

where $B$ is the unit disc in $\mathbb{R}^2$. It was proved by Brezis and Marcus [6] that there exists a constant $C > 0$ such that

$$
\int_{B} |\nabla u|^2 \, dx - \int_{B} \frac{u^2}{(1 - |x|^2)^2} \, dx \geq C \int_{B} u^2 \, dx, \quad \forall u \in W^{1,2}_0(B).
$$

(1.10)

Hence

$$
\|u\|_\mathscr{H} = \left( \int_{B} |\nabla u|^2 \, dx - \int_{B} \frac{u^2}{(1 - |x|^2)^2} \, dx \right)^{1/2}
$$

defines a norm over $W^{1,2}_0(B)$. Denote by $\mathscr{H}$ the completion of $C_0^\infty(B)$ under the norm $\| \cdot \|_\mathscr{H}$. Then $\mathscr{H}$ is a Hilbert space. It was observed by Manchi and Sandeep [17] and Wang and Ye [22] that

$$
W^{1,2}_0(B) \subset \mathscr{H} \subset \cap_{p \geq 1} L^p(B).
$$

A Hardy-Moser-Trudinger inequality was first established by Wang and Ye [22]:

$$
\sup_{u \in \mathscr{H}, \|u\|_\mathscr{H} \leq 1} \int_{B} e^{4\pi u^2} \, dx < \infty.
$$

(1.11)
Moreover, the above supremum can be attained.

We slightly abuse some notations and write
\[
\lambda_1(\mathbb{B}) = \inf_{u \in \mathcal{H}, u \not\equiv 0} \frac{\|u\|_{\mathcal{H}}^2}{\|u\|_2^2},
\]
and \(\|u\|_{\mathcal{H}, \alpha} = (\|u\|_{\mathcal{H}} - \alpha \|u\|_2^2)^{1/2}\), where \(0 \leq \alpha < \lambda_1(\mathbb{B})\). In [26], Yang and Zhu improved the result of Wang and Ye [22] as below:
\[
\sup_{u \in \mathcal{H}, \|u\|_{\mathcal{H}, \alpha} \leq 1} \int_\mathbb{B} e^{4\pi u^2} \, dx < \infty. \tag{1.12}
\]
Moreover, the extremal function for the above supremum exists.

Our aim is to extend (1.12) to a singular version of the Hardy-Moser-Trudinger inequality. Now, the main result of this paper can be stated as follows:

**Theorem 1.1.** Let \(0 \leq \beta < 1\) be fixed. Then for any \(\alpha, 0 \leq \alpha < \lambda_1(\mathbb{B})\), it holds that
\[
\sup_{u \in \mathcal{H}, \|u\|_{\mathcal{H}, \alpha} \leq 1} \int_\mathbb{B} e^{4\pi(1-\beta)u^2} \frac{dx}{|x|^{2\beta}} < \infty. \tag{1.13}
\]
Furthermore the supremum can be achieved by some function \(u_0 \in \mathcal{H}\) with \(\|u_0\|_{\mathcal{H}, \alpha} = 1\).

When \(\beta = 0\), the inequality (1.13) is reduced to (1.12). The existence of extremal functions for the Moser-Trudinger inequality originated in [7]. This result was generalized by Struwe [19], Flucher [12], Lin [16], Ding et al. [11], Adimurthi and Struwe [3], Li [15], Adimurthi and Druet [1], and so on. Compared with [26], there are difficulties caused by the term \(|x|^{-2\beta}\) in the process of the blow-up analysis. Here, we employ a classification theorem of Chen and Li [9] which was also used in [27], while another classification result [8] was also used in [26]. We derive an upper bound of the Hardy-Moser-Trudinger functionals by Onofri’s inequality (see [13, Theorem 1.1]), while an upper bound was obtained via the capacity estimate in [26]. The proof of Theorem 1.1 is composed of three steps. The first step is to reduce the problem on radially non-increasing functions and derive the associated Euler-Lagrange equation. The second step is the blow-up analysis. If the blow-up occurs, we analyze the asymptotic behavior of maximizing sequences near and away from the blow-up point. Then we estimate an upper bound of subcritical functionals. The final step is to construct test functions and get a contradiction with the upper bound derived in the previous step, which implies that the blow-up cannot occur. This completes the proof of Theorem 1.1.

## 2 Proof of main results

### 2.1 The singular subcritical functionals

In this subsection, we will prove the existence of the maximizers of subcritical functions. We recall Wang and Ye’s result [22] for our use later:

**Lemma 2.1** (See [22]). Let
\[
\mathcal{H}_0 = \{ u \in C^\infty_0(\mathbb{B}) : u(x) = u(r) \text{ with } r = |x|, u' \leq 0 \}
\]
and \(\mathcal{H}\) be the closure of \(\mathcal{H}_0\) in \(\mathcal{H}\). Then \(\mathcal{H}\) is embedded continuously in \(W_{loc}^{1,2}(\mathbb{B}) \cap C^{0,1}_{loc}(\mathbb{B}\{0}\})\).

Moreover, for any \(p \geq 1\), \(\mathcal{H}\) is embedded compactly in \(L^p(\mathbb{B})\).

Then we perform variation in \(\mathcal{H}\) instead of \(\mathcal{H}\) and get the following lemma.

**Lemma 2.2.** Assume \(0 \leq \alpha \leq \lambda_1(\mathbb{B})\). Then for any \(\epsilon, 0 < \epsilon < 1 - \beta\), there exists some \(u_\epsilon \in \mathcal{H} \cap C^\infty(\mathbb{B}\{0}\}) \cap C^0(\overline{\mathbb{B}})\) such that \(\|u_\epsilon\|_{\mathcal{H}, \alpha} = 1\) and
\[
\int_\mathbb{B} \frac{e^{4\pi(1-\beta-\epsilon)u^2}}{|x|^{2\beta}} \, dx = \sup_{u \in \mathcal{H}, \|u\|_{\mathcal{H}, \alpha} \leq 1} \int_\mathbb{B} \frac{e^{4\pi(1-\beta-\epsilon)u^2}}{|x|^{2\beta}} \, dx. \tag{2.1}
\]
Proof. First, we prove that
\begin{equation}
\sup_{u \in \mathcal{M}, \|u\|_{\mathcal{M}, \alpha} \leq 1} \int_{\mathcal{B}} \frac{e^{4\pi(1-\beta-\epsilon)u^2}}{|x|^{2\beta}} \, dx = \sup_{u \in \mathcal{M}, \|u\|_{\mathcal{M}, \alpha} \leq 1} \int_{\mathcal{B}} \frac{e^{4\pi(1-\beta-\epsilon)u^2}}{|x|^{2\beta}} \, dx,
\end{equation}
which reduces our problem on radially symmetric functions.

For any \(u \in C_0^\infty(\mathcal{B})\), denote by \(u^*\) the radially nonincreasing rearrangement of \(u\) with respect to the standard hyperbolic metric \(d\mu = \frac{1}{(1-|x|^2)} \, dx\). The argument in [5] leads to
\begin{align*}
\int_{\mathcal{B}} |\nabla u^*|^2 \, dx &\leq \int_{\mathcal{B}} |\nabla u|^2 \, dx, \\
\int_{\mathcal{B}} u^*^2 \, dx &\leq \int_{\mathcal{B}} u^2 \, dx
\end{align*}
and
\begin{align*}
\int_{\mathcal{B}} \frac{u^*^2}{(1-|x|^2)^2} \, dx &\leq \int_{\mathcal{B}} \frac{u^2}{(1-|x|^2)^2} \, dx.
\end{align*}
Thus, \(\|u\|_{\mathcal{M}, \alpha} \leq 1\) implies \(\|u^*\|_{\mathcal{M}, \alpha} \leq 1\). Using the Hardy-Littlewood inequality and noticing that the rearrangement of \(\frac{(1-|x|^2)^2}{|x|^{2\beta}}\) is just itself, we get
\begin{align*}
\int_{\mathcal{B}} \frac{e^{4\pi(1-\beta-\epsilon)u^2}}{|x|^{2\beta}} \, dx &\leq \int_{\mathcal{B}} \frac{e^{4\pi(1-\beta-\epsilon)u^2}}{|x|^{2\beta}} \, d\mu \\
&\leq \int_{\mathcal{B}} \frac{e^{4\pi(1-\beta-\epsilon)u^2}}{|x|^{2\beta}} \left(\frac{(1-|x|^2)^2}{|x|^{2\beta}}\right)^* \, d\mu \\
&= \int_{\mathcal{B}} \frac{e^{4\pi(1-\beta-\epsilon)u^2}}{|x|^{2\beta}} \, d\mu \\
&= \int_{\mathcal{B}} \frac{e^{4\pi(1-\beta-\epsilon)u^2}}{|x|^{2\beta}} \, dx.
\end{align*}
Thus,
\begin{align*}
\sup_{u \in C_0^\infty(\mathcal{B}), \|u\|_{\mathcal{M}, \alpha} \leq 1} \int_{\mathcal{B}} \frac{e^{4\pi(1-\beta-\epsilon)u^2}}{|x|^{2\beta}} \, dx &\leq \sup_{u \in \mathcal{M}, \|u\|_{\mathcal{M}, \alpha} \leq 1} \int_{\mathcal{B}} \frac{e^{4\pi(1-\beta-\epsilon)u^2}}{|x|^{2\beta}} \, dx.
\end{align*}
Combining the density of \(C_0^\infty(\mathcal{B})\) in \(\mathcal{M}\), we see that (2.2) holds.

Next, we prove that for any \(\beta, 0 \leq \beta < 1\) and any \(\epsilon, 0 < \epsilon < 1 - \beta\), it holds that
\begin{equation}
\sup_{u \in \mathcal{M}, \|u\|_{\mathcal{M}, \alpha} \leq 1} \int_{\mathcal{B}} \frac{e^{4\pi(1-\beta-\epsilon)u^2}}{|x|^{2\beta}} \, dx < +\infty.
\end{equation}
We modify the proof of Theorem 3 in [22]. Let \(u \in \mathcal{M}_0\). Define
\begin{equation}
A_u(r) = \frac{1}{\pi r^2} \int_{B_r} \frac{u^2}{(1-|x|^2)^2} \, dx.
\end{equation}
We may assume \(u(0) > 1\); otherwise \(\int_{\mathcal{B}} \frac{e^{4\pi(1-\beta-\epsilon)u^2}}{|x|^{2\beta}} \, dx \leq \frac{\pi e^{4\pi(1-\beta-\epsilon)}}{1-\beta} \). Define \(r_1 = \inf\{r > 0 \mid u(r) \leq 1\} > 0\). By [22, Lemmas 3 and 4], we have for any \(r \leq r_1\),
\begin{align*}
\int_{B_r} |\nabla u|^2 \, dx &\leq 1 - \|u\|^2_{\mathcal{M}(\mathcal{B}_r^c)} + C \pi r^2 u(r)^2 \leq 1 - \|u\|^2_{\mathcal{M}(\mathcal{B}_r^c)} + \frac{C}{2} r \|u\|^2_{\mathcal{M}(\mathcal{B}_r^c)},
\end{align*}
where \(\mathcal{B}_r^c = \mathcal{B} \setminus B_r\), \(\| \cdot \|_{\mathcal{M}(\mathcal{B})}\) is the norm \(\| \cdot \|_{\mathcal{M}}\) on \(\mathcal{B}_r^c\) and \(C\) is a positive constant independent of \(u\). Hence for \(r_2 \in (0, r_1]\) small enough, independent of \(u\), it holds that \(\|\nabla u\|_{L^2(\mathcal{B}_r^c)} \leq 1\). Moreover, \(u(r_2)\) has an upper bound independent of \(u\). By the singular Moser-Trudinger inequality (1.6), we have
\begin{align*}
\int_{B_{r_2}} \frac{e^{4\pi(1-\beta)(u(r) - u(r_2))^2}}{|x|^{2\beta}} \, dx &\leq \int_{\mathcal{B}} \frac{e^{4\pi(1-\beta)(u(r) - u(r_2))^2}}{|x|^{2\beta}} \, dx < +\infty,
\end{align*}
where \([u(r) - u(r_2)]^+ = \max\{u(r) - u(r_2), 0\}\). Then we have for any \(r \leq r_2\),
\[
4\pi(1 - \beta - \epsilon)u(r)^2 \leq 4\pi(1 - \beta)[u(r) - u(r_2)]^2 + 8\pi(1 - \beta)u(r)u(r_2) - 4\pi\epsilon u^2(r) \\
\leq 4\pi(1 - \beta)[u(r) - u(r_2)]^2 + C_\epsilon,
\]
where \(C_\epsilon\) is a positive constant depending only on \(\epsilon\). Hence we get
\[
\int_B \frac{e^{4\pi(1-\beta-\epsilon)u^2}}{|x|^{2\beta}} \, dx = \int_{B_{r_2}} \frac{e^{4\pi(1-\beta-\epsilon)u^2}}{|x|^{2\beta}} \, dx + \int_{B_{\epsilon}^c} \frac{e^{4\pi(1-\beta-\epsilon)u^2}}{|x|^{2\beta}} \, dx \\
\leq \int_{B_{r_2}} \frac{e^{4\pi(1-\beta)(u-u(r_2))^2+C_\epsilon}}{|x|^{2\beta}} \, dx + \frac{\pi(1-r_2^{-2}\beta)}{1-\beta} e^{4\pi(1-\beta-\epsilon)u(r_2)^2} \\
< +\infty.
\]
One can see that (2.3) holds true.

We use a method of variation to prove (2.1). Choose a maximizing sequence \(u_j \in \mathcal{M}\) with \(\|u_j\|_{\mathcal{M}, \alpha} \leq 1\) such that
\[
\int_B |x|^{-2\beta} e^{4\pi(1-\beta-\epsilon)u_j^2} \, dx \to \sup_{u \in \mathcal{M}, \|u\|_{\mathcal{M}, \alpha} \leq 1} \int_B |x|^{-2\beta} e^{4\pi(1-\beta-\epsilon)u^2} \, dx
\]
as \(j \to \infty\). Noting \(0 \leq \alpha < \lambda_1(\mathbb{B})\), we obtain
\[
\left(1 - \frac{\alpha}{\lambda_1(\mathbb{B})}\right) \|u_j\|_{\mathcal{M}}^2 \leq 1.
\]
This implies that \(u_j\) is bounded in \(\mathcal{M}\). There exists some \(u_\epsilon \in \mathcal{M}\) such that up to a subsequence,
\[
u_j \to u_\epsilon \quad \text{weakly in} \quad \mathcal{M},
\]
\[
u_j \to u_\epsilon \quad \text{strongly in} \quad L^p(\mathbb{B}), \quad \forall \ p \geq 1,
\]
\[
u_j \to u_\epsilon \quad \text{a.e. in} \quad \mathbb{B}.
\]
Clearly, \(\|u_\epsilon\|_{\mathcal{M}, \alpha} \leq 1\) since \(\|u_j\|_{\mathcal{M}, \alpha} \leq 1\).

Because \(\langle u_j, u_\epsilon \rangle_{\mathcal{M}} \to \|u_\epsilon\|_{\mathcal{M}}^2\) and \(\|u_j\|_2 \to \|u_\epsilon\|_2\), we have
\[
\|u_j - u_\epsilon\|_{\mathcal{M}}^2 = \langle u_j - u_\epsilon, u_j - u_\epsilon \rangle_{\mathcal{M}} \\
= \|u_j\|_{\mathcal{M}}^2 + \|u_\epsilon\|_{\mathcal{M}}^2 - 2\langle u_j, u_\epsilon \rangle_{\mathcal{M}} \\
= \|u_j\|_{\mathcal{M}}^2 - \|u_\epsilon\|_{\mathcal{M}}^2 + o_j(1) \\
\leq 1 - \|u_\epsilon\|_{\mathcal{M}, \alpha}^2 + o_j(1).
\]
By Hölder’s inequality, we have for \(1 < p < 1/\beta\),
\[
\int_B |x|^{-2\beta p_q^*} e^{4\pi(1-\beta-\epsilon)p(1+\delta)u_j^2} \, dx \leq \int_B |x|^{-2\beta p_q^*} e^{4\pi(1-\beta-\epsilon)p(1+\delta)(u_j-u_\epsilon)^2} + 4\pi(1-\beta-\epsilon)p(1+\delta)u^2 \, dx \\
\leq \left( \int_B |x|^{-2\beta p_q^*} e^{4\pi(1-\beta-\epsilon)p(1+\delta)(u_j-u_\epsilon)^2} \, dx \right)^{1/q} \\
\times \left( \int_B \left| x \right|^{-2\beta p_q^*} e^{4\pi(1-\beta-\epsilon)p(1+\delta)u^2} \, dx \right)^{1/q'} \\
\leq \left( \int_B \left| x \right|^{-2\beta p_q^*} e^{4\pi(1-\beta-\epsilon)p(1+\delta)\left( u_j-u_\epsilon \right)^2} \, dx \right)^{1/q} \\
\times \left( \int_B \left| x \right|^{-2\beta p_q^*} \, dx \right)^{1/(q' \cdot s)} \\
\times \left( \int_B e^{4\pi(1-\beta-\epsilon)p q' s(1+\delta)u^2} \, dx \right)^{1/(q' \cdot s')} ,
\]
(2.6)
where $\delta > 0$, $1/q + 1/q' = 1$ and $1/s + 1/s' = 1$.

We choose $p, q, 1 + \delta, s$ sufficiently close to 1, and $\delta_1 < \epsilon$ such that

$$4\pi(1 - \beta - \epsilon)pq(1 + \delta)(u_j - u_{e})^2 < 4\pi(1 - \beta p - \delta_1)\|u_j - u_{e}\|_{J}^2, \quad \beta ps < 1.$$ 

Lemma 4 in [26] indicates that for any $\gamma > 0$ and any $u \in \mathcal{H}$, it holds that

$$\int_{\mathbb{B}} e^{\gamma u^2} \, dx < +\infty. \quad (2.7)$$

Combining (2.3), (2.6) and (2.7), we conclude that $|x|^{-2\beta} e^{4\pi(1-\beta-\epsilon)}u_j^2$ is bounded in $L^p(\mathbb{B})$ for some $p > 1$, which together with $|x|^{-2\beta} e^{4\pi(1-\beta-\epsilon)}u_j^2 \to |x|^{-2\beta} e^{4\pi(1-\beta-\epsilon)}u^2$ a.e. as $j \to \infty$ implies that

$$\lim_{j \to \infty} \int_{\mathbb{B}} |x|^{-2\beta} e^{4\pi(1-\beta-\epsilon)}u_j^2 \, dx = \int_{\mathbb{B}} |x|^{-2\beta} e^{4\pi(1-\beta-\epsilon)}u^2 \, dx. \quad (2.8)$$

We claim that $\|u_e\|_{\mathcal{H}, \alpha} = 1$. Otherwise, we have $\|u_e\|_{\mathcal{H}, \alpha} < 1$. Hence

$$\sup_{u \in \mathcal{H}, \|u\|_{\mathcal{H}, \alpha} \leq 1} \int_{\mathbb{B}} e^{4\pi(1-\beta-\epsilon)}u^2 \, dx < \int_{\mathbb{B}} |x|^{-2\beta} e^{4\pi(1-\beta-\epsilon)}u_e^2 \, dx \leq \sup_{u \in \mathcal{H}, \|u\|_{\mathcal{H}, \alpha} \leq 1} \int_{\mathbb{B}} e^{4\pi(1-\beta-\epsilon)}u^2 \, dx,$$

which leads to a contradiction.

A straightforward calculation shows that $u_e$ satisfies the following Euler-Lagrange equation:

$$\begin{cases}
-\Delta u_e - \frac{u_e}{(1 - |x|^2)^2} - \alpha u_e = \frac{1}{\lambda_e} |x|^{-2\beta} e^{4\pi(1-\beta-\epsilon)}u_e^2 & \text{in } \mathbb{B}, \\
u_e \geq 0 & \text{in } \mathbb{B}, \\
\lambda_e = \int_{\mathbb{B}} |x|^{-2\beta} u_e^2 e^{4\pi(1-\beta-\epsilon)}u_e^2 \, dx. & \text{in } \mathbb{B}.
\end{cases} \quad (2.9)$$

Applying standard elliptic estimates to (2.9), we get $u_e \in C^\infty(\mathbb{B}\setminus\{0\})$. Observing $u_e \in \mathcal{H}$, we have $u_e \in C^0(\mathbb{B})$.

\[\square\]

### 2.2 The blow-up analysis

We use the method of the blow-up analysis to describe the asymptotic behavior of the maximizers $u_e$ in Lemma 2.2. Note that $\|u_e\|_{\mathcal{H}, \alpha} = 1$, so $u_e$ is bounded in $\mathcal{H}$. Thus, there exists $u_0 \in \mathcal{H}$ such that up to a subsequence,

$$u_e \to u_0 \quad \text{weakly in } \mathcal{H},$$

$$u_e \to u_0 \quad \text{strongly in } L^p(\mathbb{B}), \quad \forall p \geq 1,$$

$$u_e \to u_0 \quad \text{a.e. in } \mathbb{B}.$$

It is clear that

$$\|u_0\|_{\mathcal{H}, \alpha} \leq \liminf_{\epsilon \to 0} \|u_\epsilon\|_{\mathcal{H}, \alpha} = 1.$$

Denote $c_e = u_e(0) = \max_{\mathbb{B}} u_e$. If $c_e$ is bounded, then the Lebesgue dominated convergence theorem yields that

$$\int_{\mathbb{B}} |x|^{-2\beta} e^{4\pi(1-\beta-\epsilon)}u_e^2 \, dx = \lim_{\epsilon \to 0} \int_{\mathbb{B}} |x|^{-2\beta} e^{4\pi(1-\beta-\epsilon)}u_e^2 \, dx = \sup_{u \in \mathcal{H}, \|u\|_{\mathcal{H}, \alpha} \leq 1} \int_{\mathbb{B}} e^{4\pi(1-\beta)}u^2 \, dx.$$

Thus, \( u_0 \) is the desired maximizer. Now, we assume that
\[
\epsilon \rightarrow +\infty \quad \text{as} \quad \epsilon \rightarrow 0.
\] (2.10)
The simple inequality \( e^t \leq 1 + te^t \) implies
\[
\int_{\mathbb{B}} |x|^{-2\beta}e^{4\pi(1-\beta-\epsilon)}u^2 \, dx \leq \int_{\mathbb{B}} |x|^{-2\beta} \, dx + 4\pi \epsilon.
\]
This together with (2.1) yields
\[
\liminf_{\epsilon \to 0} \lambda_\epsilon > 0.
\] (2.11)
Now, we claim that \( u_0 \equiv 0 \). Otherwise, \( \| u_0 \|_{\mathcal{H}, \alpha} > 0 \). Calculate
\[
\| u_\epsilon - u_0 \|^2_{\mathcal{H}} = \langle u_\epsilon - u_0, u_\epsilon - u_0 \rangle_{\mathcal{H}}
= \| u_\epsilon \|^2_{\mathcal{H}} + \| u_0 \|^2_{\mathcal{H}} - 2\langle u_\epsilon, u_0 \rangle_{\mathcal{H}}
= \| u_\epsilon \|^2_{\mathcal{H}, \alpha} - \| u_0 \|^2_{\mathcal{H}, \alpha} + o_\epsilon(1)
\leq 1 - \| u_0 \|^2_{\mathcal{H}, \alpha} + o_\epsilon(1).
\] (2.12)
By the similar estimates to those in (2.6), we get \(|x|^{-2\beta}e^{4\pi(1-\beta-\epsilon)}u^2 \) is bounded in \( L^p(\mathbb{B}) \) for some \( p > 1 \).
In view of (2.11), applying elliptic estimates to (2.9), we have that \( u_\epsilon \) is bounded in \( C^0_{\text{loc}}(\mathbb{B}) \). This contradicts (2.10).

Set
\[
r_\epsilon = \sqrt{\lambda_\epsilon c^{-1}_\epsilon e^{-2\beta(1-\beta-\epsilon)}c^2}.
\] (2.13)
Note that \( u_\epsilon \to 0 \) in \( L^q(\mathbb{B}) \) for any \( q \geq 1 \). Then, by Hölder’s inequality and (2.3), we have
\[
\lambda_\epsilon = \int_{\mathbb{B}} |x|^{-2\beta} u^2 \, dx \leq e^{4\pi c^2} \int |x|^{-2\beta} u^2 \, dx \leq Ce^{4\pi c^2}
\] (2.14)
for \( 0 < \delta < 1 - \beta \), where the constant \( C \) is independent of \( u_\epsilon \). This leads to
\[
r^2 \leq Cc^{-2}e^{-4\pi(1-\beta-\epsilon)c^2} \to 0, \quad \text{as} \quad \epsilon \to 0.
\] (2.15)
Let \( \Omega_\epsilon = \{ x \in \mathbb{R}^2 : r^{1/(1-\beta)}_\epsilon x \in \mathbb{B} \} \). We define two blow-up sequences
\[
\psi_\epsilon(x) = c^{-1}_\epsilon u_\epsilon(r^{1/(1-\beta)}_\epsilon x), \quad \varphi_\epsilon(x) = c_\epsilon(u_\epsilon(r^{1/(1-\beta)}_\epsilon x) - c_\epsilon).
\] (2.16)
Then \( \psi_\epsilon \) satisfies the following equation:
\[
-\Delta \psi_\epsilon = \frac{\psi_\epsilon r^{2/(1-\beta)}_\epsilon}{(1 - r^{2/(1-\beta)}_\epsilon |x|^2)^2} + \alpha r^{2/(1-\beta)}_\epsilon \psi_\epsilon + c^{-2}_\epsilon |x|^{-2\beta} \psi_\epsilon e^{4\pi(1-\beta-\epsilon)}(1+\psi_\epsilon)\psi_\epsilon \quad \text{in} \quad \Omega_\epsilon.
\] (2.17)
By (2.15), we have \( r_\epsilon \to 0 \), and hence \( \Omega_\epsilon \to \mathbb{R}^2 \). Using elliptic estimates to (2.17), we conclude that \( \psi_\epsilon \to \psi \) in \( C^1_{\text{loc}}(\mathbb{R}^2 \setminus \{0\}) \cap C^0_{\text{loc}}(\mathbb{R}^2) \), where \( \psi \) is a distributional harmonic function. Clearly, \( \psi(0) = \lim_{\epsilon \to 0} \psi_\epsilon(0) = 1 \). The Liouville theorem implies that \( \psi \equiv 1 \) on \( \mathbb{R}^2 \). Hence we have
\[
\psi_\epsilon \to 1 \quad \text{in} \quad C^1_{\text{loc}}(\mathbb{R}^2 \setminus \{0\}) \cap C^0_{\text{loc}}(\mathbb{R}^2).
\] (2.18)
A straightforward computation shows that
\[
-\Delta \varphi_\epsilon = \frac{\psi_\epsilon r^{2/(1-\beta)}_\epsilon}{(1 - r^{2/(1-\beta)}_\epsilon |x|^2)^2} + \alpha c^2_\epsilon r^{2/(1-\beta)}_\epsilon \psi_\epsilon + |x|^{-2\beta} \psi_\epsilon e^{4\pi(1-\beta-\epsilon)}(1+\psi_\epsilon)\varphi_\epsilon \quad \text{in} \quad \Omega_\epsilon.
\] (2.19)
In view of (2.18) and \( \varphi_\epsilon(x) \leq \varphi_\epsilon(0) = 0 \) for all \( x \in \Omega_\epsilon \), we have by the elliptic estimates that
\[
\varphi_\epsilon \to \varphi_0 \quad \text{in} \quad C^1_{\text{loc}}(\mathbb{R}^2 \setminus \{0\}) \cap C^0_{\text{loc}}(\mathbb{R}^2),
\] (2.20)
where \( \varphi_0 \) is a solution to
\[
- \Delta \varphi_0 = |x|^{-2 \beta} e^{8 \pi (1-\beta) \varphi_0} \quad \text{in} \quad \mathbb{R}^2 \setminus \{0\}.
\] (2.21)

Then for any fixed \( R > 0 \), we have
\[
\int_{B_R(0)} |x|^{-2 \beta} e^{8 \pi (1-\beta) \varphi_0} \, dx \leq \limsup_{\epsilon \to 0} \int_{B_R(0)} |x|^{-2 \beta} e^{4 \pi (1-\beta) - \epsilon} (1 + \psi_\epsilon) \varphi_0 \, dx
\]
\[
\leq \limsup_{\epsilon \to 0} \lambda_\epsilon^{-1} \int_{B_{Rr_\epsilon(1-\beta)(0)}} |y|^{-2 \beta} \epsilon \varphi_0 e^{4 \pi (1-\beta) - \epsilon} u^2(y) \, dy
\]
\[
\leq \limsup_{\epsilon \to 0} \lambda_\epsilon^{-1} \int_{B_{Rr_\epsilon(1-\beta)(0)}} |x|^{-2 \beta} \epsilon \varphi_0 e^{4 \pi (1-\beta) - \epsilon} u^2(y) \, dy
\]
\[
\leq 1.
\]
Therefore,
\[
\int_{\mathbb{R}^2} |x|^{-2 \beta} e^{8 \pi (1-\beta) \varphi_0} \, dx \leq 1.
\]

The classification result of Chen and Li [9, Theorem 3.1] leads to
\[
\varphi_0(x) = -\frac{1}{4 \pi (1-\beta)} \log \left( 1 + \frac{\pi}{1-\beta} |x|^{2(1-\beta)} \right).
\] (2.22)

It follows that
\[
\int_{\mathbb{R}^2} |x|^{-2 \beta} e^{8 \pi (1-\beta) \varphi_0} \, dx = 1.
\] (2.23)

We now analyze the behavior of \( u_\epsilon \) away from the zero. Let \( u_{\epsilon, \tau} = \min \{ u_\epsilon, \tau c_\epsilon \} \) for any \( \tau, 0 < \tau < 1 \). We have the following lemma.

**Lemma 2.3.** For any \( \tau, 0 < \tau < 1 \), it holds that \( \lim_{\epsilon \to 0} \| u_{\epsilon, \tau} \|_{\mathcal{H}^\alpha} = \tau \).

**Proof.** By the equation (2.9), the integration by parts yields
\[
\int_{\mathbb{R}^2} |\nabla u_{\epsilon, \tau}|^2 \, dx = \int_{\mathbb{R}^2} \nabla u_{\epsilon, \tau} \cdot \nabla u_\epsilon \, dx
\]
\[
= \int_{\mathbb{R}^2} \left( \frac{u_{\epsilon, \tau} u_\epsilon}{(1 - |x|^2)^2} + \alpha u_{\epsilon, \tau} u_\epsilon + \lambda_\epsilon^{-1} |x|^{-2 \beta} u_{\epsilon, \tau} u_\epsilon e^{4 \pi (1-\beta) - \epsilon} \right) \, dx.
\]
Thus,
\[
\| u_{\epsilon, \tau} \|^2_{\mathcal{H}^\alpha} = \int_{\mathbb{R}^2} \left( \nabla u_{\epsilon, \tau} \cdot \nabla u_\epsilon - \frac{u_{\epsilon, \tau}^2}{(1 - |x|^2)^2} - \alpha u_{\epsilon, \tau}^2 \right) \, dx
\]
\[
= \int_{\mathbb{R}^2} \left( \frac{u_{\epsilon, \tau} (u_\epsilon - u_{\epsilon, \tau})}{(1 - |x|^2)^2} + \alpha u_{\epsilon, \tau} (u_\epsilon - u_{\epsilon, \tau}) + \lambda_\epsilon^{-1} |x|^{-2 \beta} u_{\epsilon, \tau} u_\epsilon e^{4 \pi (1-\beta) - \epsilon} \right) \, dx
\]
\[
\geq \int_{\mathbb{R}^2} \left( \lambda_\epsilon^{-1} |x|^{-2 \beta} u_{\epsilon, \tau} u_\epsilon e^{4 \pi (1-\beta) - \epsilon} \right) \, dx
\]
\[
\geq \tau \int_{B_{Rr_\epsilon(1-\beta)}} \left( \lambda_\epsilon^{-1} |x|^{-2 \beta} c_\epsilon e^{4 \pi (1-\beta) - \epsilon} \right) \, dx
\]
\[
\geq \tau \int_{B_{R}(0)} (1 + o_\epsilon(1)) |x|^{-2 \beta} e^{4 \pi (1-\beta) \varphi_0} \, dx
\]
for any fixed \( R > 0 \). Letting \( \epsilon \to 0 \) first and then \( R \to \infty \), we get
\[
\liminf_{\epsilon \to 0} \| u_{\epsilon, \tau} \|^2_{\mathcal{H}^\alpha} \geq \tau.
\]

Noting \( |\nabla (u_\epsilon - \tau c_\epsilon)^+|^2 = |\nabla (u_\epsilon - \tau c_\epsilon)^+| \nabla u_\epsilon \) on \( B \) and \( (u_\epsilon - \tau c_\epsilon)^+ = (1 + o_\epsilon(1))(1 - \tau) c_\epsilon \) on \( B_{Rr_\epsilon(1-\beta)}(0) \), we have by the similar calculation that
\[
\liminf_{\epsilon \to 0} \| (u_\epsilon - \tau c_\epsilon)^+ \|^2_{\mathcal{H}^\alpha} \geq 1 - \tau.
\]
Since \( |\nabla u_\epsilon|^2 = |\nabla u_{\epsilon,\tau}|^2 + |\nabla (u_\epsilon - \tau c_\epsilon)^+|^2 \) and \( u_\epsilon \to 0 \) in \( L^p(\mathbb{B}) \) for any fixed \( p \geq 1 \), we have

\[
\lim_{\epsilon \to 0} (\|u_{\epsilon,\tau}\|^2_{\mathcal{H},\alpha} + \|u_\epsilon - \tau c_\epsilon\|^2_{\mathcal{H},\alpha}) = \lim_{\epsilon \to 0} \|u_\epsilon\|^2_{\mathcal{H},\alpha} = 1.
\]

Hence,

\[
\lim_{\epsilon \to 0} \|u_{\epsilon,\tau}\|^2_{\mathcal{H},\alpha} = \tau, \quad \lim_{\epsilon \to 0} \|u_\epsilon - \tau c_\epsilon\|^2_{\mathcal{H},\alpha} = 1 - \tau.
\]

This completes the proof.

\[\square\]

**Lemma 2.4.** It holds that

\[
\limsup_{\epsilon \to 0} \int_{\mathbb{B}} |x|^{-2\beta} e^{4\pi(1-\beta-\epsilon)} u_\epsilon^2 dx \leq \frac{\pi}{1-\beta} + \limsup_{\epsilon \to 0} \frac{\lambda_\epsilon}{c_\epsilon^2}.
\]

**Proof.** For any \( \tau, 0 < \tau < 1 \), we have

\[
\int_{\mathbb{B}} |x|^{-2\beta} e^{4\pi(1-\beta-\epsilon)} u_\epsilon^2 dx = \int_{\{u_\epsilon > \tau c_\epsilon\}} |x|^{-2\beta} e^{4\pi(1-\beta-\epsilon)} u_\epsilon^2 dx + \int_{\{u_\epsilon \leq \tau c_\epsilon\}} |x|^{-2\beta} e^{4\pi(1-\beta-\epsilon)} u_\epsilon^2 dx
\]

\[
\leq \int_{\mathbb{B}} |x|^{-2\beta} e^{4\pi(1-\beta-\epsilon)} u_{\epsilon,\tau}^2 dx + \frac{\lambda_\epsilon}{\tau^2 c_\epsilon^2}.
\]

By Lemma 2.3, we have that \( |x|^{-2\beta} e^{4\pi(1-\beta-\epsilon)} u_{\epsilon,\tau}^2 \) is bounded in \( L^p(\mathbb{B}) \) for some \( p \geq 1 \). Therefore,

\[
\lim_{\epsilon \to 0} \int_{\mathbb{B}} |x|^{-2\beta} e^{4\pi(1-\beta-\epsilon)} u_{\epsilon,\tau}^2 dx = \int_{\mathbb{B}} |x|^{-2\beta} dx = \frac{\pi}{1-\beta}.
\]

By combining the above estimates and letting \( \epsilon \to 0 \) first, then \( \tau \to 1 \), we finish the proof.

\[\square\]

Similar to [22, 26], we prove the following lemma.

**Lemma 2.5.** For any \( \phi \in C^\infty(\mathbb{B}) \), it holds that

\[
\lim_{\epsilon \to 0} \int_{\mathbb{B}} \lambda_\epsilon^{-1} |x|^{-2\beta} c_\epsilon u_\epsilon e^{4\pi(1-\beta-\epsilon)} u_\epsilon^2 \phi dx = \phi(0).\]

**Proof.** Divide \( \mathbb{B} \) into three parts:

\[
\mathbb{B} = \left( \{u_\epsilon > \tau c_\epsilon\} \setminus B_{R_\epsilon^{1/(1-\beta)}}(0) \right) \cup \left( \{u_\epsilon \leq \tau c_\epsilon\} \setminus B_{R_\epsilon^{1/(1-\beta)}}(0) \right) \cup B_{R_\epsilon^{1/(1-\beta)}}(0),
\]

for some \( 0 < \tau < 1 \). Denote the integrals on the above three domains by \( I_1, I_2 \) and \( I_3 \), respectively. We estimate them one by one. In view of (2.18), (2.20) and (2.23), we have

\[
I_1 \leq \sup_{\mathbb{B}} |\phi| \int_{\{u_\epsilon > \tau c_\epsilon\} \setminus B_{R_\epsilon^{1/(1-\beta)}}(0)} \lambda_\epsilon^{-1} |x|^{-2\beta} c_\epsilon u_\epsilon e^{4\pi(1-\beta-\epsilon)} u_\epsilon^2 dx
\]

\[
\leq \frac{1}{\tau} \sup_{\mathbb{B}} |\phi| \left( 1 - \int_{B_{R_\epsilon^{1/(1-\beta)}}(0)} \lambda_\epsilon^{-1} |x|^{-2\beta} u_\epsilon^2 e^{4\pi(1-\beta-\epsilon)} u_\epsilon^2 dx \right)
\]

\[
\leq \frac{1}{\tau} \sup_{\mathbb{B}} |\phi| \left( 1 - \int_{B_R(0)} |x|^{-2\beta} e^{8\pi(1-\beta)} \phi_0 dx + o_\epsilon(R) \right),
\]

where \( o_\epsilon(R) \to 0 \) as \( \epsilon \to 0 \) for any fixed \( R > 0 \). Thus, \( I_1 \to 0 \) by letting \( \epsilon \to 0 \) first and then \( R \to +\infty \). Noting that \( |x|^{-2\beta} e^{4\pi(1-\beta-\epsilon)} u_{\epsilon,\tau}^2 \) is bounded in \( L^p(\mathbb{B}) \) for some \( p \geq 1 \), we have

\[
I_2 = \int_{\{u_\epsilon \leq \tau c_\epsilon\} \setminus B_{R_\epsilon^{1/(1-\beta)}}(0)} \lambda_\epsilon^{-1} |x|^{-2\beta} c_\epsilon u_\epsilon e^{4\pi(1-\beta-\epsilon)} u_\epsilon^2 \phi dx
\]

\[
\leq \sup_{\mathbb{B}} |\phi| \frac{c_\epsilon}{\lambda_\epsilon} \int_{\mathbb{B}} |x|^{-2\beta} u_\epsilon e^{4\pi(1-\beta-\epsilon)} u_{\epsilon,\tau}^2 \phi dx
\]

\[
\leq \sup_{\mathbb{B}} |\phi| \frac{c_\epsilon}{\lambda_\epsilon} \|u_\epsilon\|_{L^p(\mathbb{B})} \| |x|^{-2\beta} e^{4\pi(1-\beta-\epsilon)} u_{\epsilon,\tau}^2 \|_{L^p(\mathbb{B})},
\]

where \( \|u_\epsilon\|_{L^p(\mathbb{B})} \to 1 \) as \( \epsilon \to 0 \) for any fixed \( p \geq 1 \).
where $1/q + 1/p = 1$. Lemma 2.4 yields $\lambda_\epsilon/c_\epsilon \to +\infty$, and hence $c_\epsilon/\lambda_\epsilon \to 0$. We obtain $I_2 \to 0$ as $\epsilon \to 0$.

Next,

$$I_3 = \int_{\mathbb{R}^n \setminus B_{\lambda_\epsilon} \setminus B_{(1-\beta)(0)}} \lambda_\epsilon^{-1} |x|^{-2\beta} c_\epsilon u_\epsilon e^{4\pi(1-\beta-\epsilon)} u_\epsilon^2 \phi \, dx$$

$$= \phi(0)(1 + o_\epsilon(1)) \left( \int_{B_{\lambda_\epsilon}(0)} |x|^{-2\beta} e^{8\pi(1-\beta)\phi_0} \, dx + o_\epsilon(R) \right).$$

Letting $\epsilon \to 0$ first and then $R \to +\infty$, we have $I_3 \to \phi(0)$. Finally, we get $\lim_{\epsilon \to 0} (I_1 + I_2 + I_3) = \phi(0)$. \hfill \Box

Define the operator

$$L_\alpha = -\Delta - \frac{1}{(1 - |x|^2)^2} - \alpha.$$

We have the following lemma.

**Lemma 2.6.** It holds that $c_\epsilon u_\epsilon \to G$ weakly in $W^{1,p}_0(\mathbb{B})$ for any $1 < p < 2$, strongly in $L^q(\mathbb{B})$ for any $q \geq 1$ and in $C^0(\overline{\mathbb{B}})$ for any $0 < r < 1$, where $G$ is a Green function satisfying

$$L_\alpha(G) = \delta_0,$$  \hspace{1cm} (2.24)

where $\delta_0$ is the Dirac measure centered at 0.

**Proof.** Note that

$$L_\alpha(c_\epsilon u_\epsilon) = \frac{1}{\lambda_\epsilon} |x|^{-2\beta} c_\epsilon u_\epsilon e^{4\pi(1-\beta-\epsilon)} u_\epsilon^2.$$  \hspace{1cm} (2.25)

Denote $f_\epsilon = \frac{1}{\lambda_\epsilon} |x|^{-2\beta} c_\epsilon u_\epsilon e^{4\pi(1-\beta-\epsilon)} u_\epsilon^2$. The rest of the proof is the same as in [26]. For completeness, we give the main steps. Let $\nu_\epsilon$ be a solution to

$$\begin{cases}
    L_\alpha \nu_\epsilon = f_\epsilon & \text{in } \mathbb{B}_{1/2}, \\
    \nu_\epsilon = 0 & \text{on } \partial \mathbb{B}_{1/2}.
\end{cases}$$  \hspace{1cm} (2.26)

Then for any $q, 1 < q < 2$, it holds that

$$\nu_\epsilon \rightharpoonup \nu_0 \text{ weakly in } W^{1,q}_0(\mathbb{B}_{1/2}).$$

Set $w_\epsilon = c_\epsilon u_\epsilon - \phi \nu_\epsilon$, where $\phi$ is a cut-off function in $C_0^\infty(\mathbb{B})$ with $0 \leq \phi \leq 1$, $\phi \equiv 1$ on $\mathbb{B}_{1/8}$ and $\phi \equiv 0$ outside $\mathbb{B}_{1/4}$. Then

$$w_\epsilon \rightharpoonup w_0 \text{ weakly in } \mathcal{H},$$  \hspace{1cm} (2.27)

and $G = \phi \nu_0 + w_0$. \hfill \Box

The Green function $G$ can be represented by

$$G = -\frac{1}{2\pi} \log r + A_0 + \Phi,$$  \hspace{1cm} (2.28)

where $A_0$ is a constant and $\Phi \in C^1_{\text{loc}}(\mathbb{B})$.

**2.3 Upper bound estimates**

In this subsection, we will use Iula and Mancini’s result [13] to derive the upper bound of the Hardy-Moser-Trudinger functionals.

**Lemma 2.7** (See [13]). Let $u_n \in W^{1,2}_0(\mathbb{B})$ be such that $\int_{\mathbb{B}} |\nabla u_n|^2 \, dx \leq 1$ and $u_n \rightharpoonup 0$ in $W^{1,2}_0(\mathbb{B})$, and then for any fixed $\beta$, $0 \leq \beta < 1$, we have

$$\sup_{n \to \infty} \int_{\mathbb{B}} \frac{e^{4\pi(1-\beta)u_n^2}}{|x|^{2\beta}} \, dx \leq \frac{\pi (1 + e)}{1 - \beta},$$  \hspace{1cm} (2.29)
We proceed as in [7, 22] and get the following lemma.

**Lemma 2.8.** It holds that

\[
\sup_{u \in \mathcal{H}, \|u\|_{\mathcal{H}, \alpha} \leq 1} \int_{B} \frac{e^{4\pi(1-\beta)u^2}}{|x|^{2\beta}} \, dx \leq \frac{\pi}{1-\beta} (1 + \alpha^{1+4\pi(1-\beta)\mathcal{A}_0}).
\]

**Proof.** In view of (2.20), we have

\[
\int_{B_{R+1/(1-\beta)}} |x|^{-2\beta} e^{4\pi(1-\beta-\epsilon)u^2} \, dx = \frac{\lambda_{\epsilon}}{c_{\epsilon}^2} \left( \int_{B_R} |x|^{-2\beta} e^{8\pi(1-\beta)\varphi_\epsilon \, dx} + o_\epsilon(R) \right).
\]

Therefore,

\[
\lim_{R \to +\infty} \limsup_{\epsilon \to 0} \int_{B_{R+1/(1-\beta)}} |x|^{-2\beta} e^{4\pi(1-\beta-\epsilon)u^2} \, dx = \limsup_{\epsilon \to 0} \frac{\lambda_{\epsilon}}{c_{\epsilon}^2}.
\]

Let \( \rho \in (0, 1) \). By Lemma 2.6, we have

\[
\lim_{\epsilon \to 0} c_{\epsilon} u_\epsilon(\rho) = G(\rho) \tag{2.30}
\]

and

\[
\lim_{\epsilon \to 0} \int_{B_\rho} \left( \frac{1}{(1 - |x|^2)^2} + \alpha \right) (c_{\epsilon} u_\epsilon)^2 \, dx = \int_{B_\rho} \left( \frac{1}{(1 - |x|^2)^2} + \alpha \right) G^2 \, dx =: E_1(\rho). \tag{2.31}
\]

By (2.25), we have

\[
\int_{B_\rho} \left( \frac{\partial^2 (c_\epsilon u_\epsilon)}{\partial \nu} (c_\epsilon u_\epsilon) \, d\sigma \right) = - c_\epsilon u_\epsilon(\rho) \int_{B_\rho} \Delta (c_\epsilon u_\epsilon) \, dx
\]

\[
= c_\epsilon u_\epsilon(\rho) \left( \int_{B_\rho} \frac{c_\epsilon u_\epsilon}{(1 - |x|^2)^2} \, dx + \int_{B_\rho} \alpha c_\epsilon u_\epsilon \, dx \right)
\]

\[
+ c_\epsilon u_\epsilon(\rho) \int_{B_\rho} \lambda_{\epsilon}^{-1} |x|^{-2\beta} c_\epsilon u_\epsilon e^{4\pi(1-\beta-\epsilon)u^2} \, dx
\]

\[
\to G(\rho) \left( \int_{B_\rho} \frac{G}{(1 - |x|^2)^2} \, dx + \int_{B_\rho} \alpha G \, dx + 1 \right) =: E_2(\rho).
\]

Hence,

\[
\int_{B_\rho} \nabla u_\epsilon^2 \, dx = 1 - \int_{B_\rho} \left( \frac{\nabla u_\epsilon^2}{(1 - |x|^2)^2} - \alpha u_\epsilon^2 \right) \, dx
\]

\[
+ \int_{B_\rho} \left( \frac{u_\epsilon^2}{(1 - |x|^2)^2} + \alpha u_\epsilon^2 \right) \, dx
\]

\[
= 1 - \frac{1}{c_{\epsilon}^2} [E_2(\rho) - E_1(\rho) + o_\epsilon(1)]. \tag{2.32}
\]
Let $F_\rho := E_2(\rho) - E_1(\rho)$. Then
\[
\int_{B_\rho} |\nabla u_\epsilon|^2 dx = 1 - \frac{F_\rho + o_\epsilon(1)}{c_\epsilon^2}.
\]

Let $u = [u_\epsilon - u_\epsilon(\rho)]^+$ and $s_\epsilon = u/\|\nabla u\|_{L^2(B_\rho)}$. Obviously, $s_\epsilon \in W^{1,2}_0(B_\rho)$, $\|\nabla s_\epsilon\|_2 = 1$ and $s_\epsilon \to 0$ in $W^{1,2}_0(B_\rho)$. By (2.18), we have that $c_\epsilon^{-1} s_\epsilon \to 1$ uniformly on $B_{R_\epsilon^{1/(1-\beta)}}$. Therefore, we have
\[
\begin{aligned}
\frac{u_\epsilon^2}{\epsilon} &\leq [s_\epsilon(x) + u_\epsilon(\rho)]^2 \|\nabla u_\epsilon\|_{L^2(B_\rho)}^2 \\
&= [s_\epsilon(x) + c_\epsilon^{-1} G(\rho) + o_\epsilon(c_\epsilon^{-1})]^2 [1 - c_\epsilon^{-2} F_\rho + o_\epsilon(c_\epsilon^{-2})] \\
&= s_\epsilon^2(x) + 2G(\rho) - F_\rho + o_\epsilon(1),
\end{aligned}
\]
where $o_\epsilon(1)$ goes to 0 uniformly in $B_{R_\epsilon^{1/(1-\beta)}}$. According to (2.29), we get
\[
\begin{aligned}
\limsup_{\epsilon \to 0} \int_{B_{R_\epsilon^{1/(1-\beta)}}} |x|^{-2\beta} e^{4\pi(1-\beta)\epsilon} u_\epsilon^2 dx &\leq \limsup_{\epsilon \to 0} \int_{B_{R_\epsilon^{1/(1-\beta)}}} |x|^{-2\beta} (e^{4\pi(1-\beta)\epsilon} s_\epsilon^2 - 1) dx \\
&\leq e^{4\pi(1-\beta)(2G(\rho) - F_\rho)} \limsup_{\epsilon \to 0} \int_{B_{R_\epsilon^{1/(1-\beta)}}} |x|^{-2\beta} (e^{4\pi(1-\beta)\epsilon}s_\epsilon^2 - 1) dx \\
&\leq e^{4\pi(1-\beta)(2G(\rho) - F_\rho)} \limsup_{\epsilon \to 0} \int_{B_\rho} |x|^{-2\beta} (e^{4\pi(1-\beta)\epsilon}s_\epsilon^2 - 1) dx \\
&\leq \pi(1 - \beta)^{-1} \rho^{2(1-\beta)} e^{1+4\pi(1-\beta)(2G(\rho) - F_\rho)}.
\end{aligned}
\]

In view of (2.28), we obtain
\[
\begin{aligned}
\pi(1 - \beta)^{-1} \rho^{2(1-\beta)} e^{1+4\pi(1-\beta)(2G(\rho) - F_\rho)} &= \pi(1 - \beta)^{-1} e^{1+4\pi(1-\beta)(\epsilon \log \rho + 2G(\rho) - F_\rho)} \\
&\to \pi(1 - \beta)^{-1} e^{1+4\pi(1-\beta)A_0},
\end{aligned}
\]
as $\rho \to 0$. Combining Lemma 2.4, we finish the proof.

2.4 The existence result

If $c_\epsilon$ is bounded, then our theorem holds true. If $c_\epsilon \to +\infty$, we will construct a sequence of functions $\phi_\epsilon(x) \in \mathcal{K}$ satisfying $\|\phi_\epsilon(x)\|_{\mathcal{K}, \alpha} \leq 1$ and
\[
\int_{B} |x|^{-2\beta} e^{4\pi(1-\beta)\phi_\epsilon} dx > \pi(1 - \beta)^{-1}(1 + e^{1+4\pi(1-\beta)A_0}).
\]
This leads to a contradiction. Then the proof of Theorem 1.1 is completed since $c_\epsilon$ must be bounded.

Let
\[
\phi_\epsilon(x) = \begin{cases}
  c + \frac{1}{c} \left( -\frac{1}{4\pi(1-\beta)} \log \left( 1 + \frac{\pi}{1-\beta} \frac{|x|^{2(1-\beta)}}{e^{2(1-\beta)}} \right) + b \right), & x \in B_{R}, \\
  \frac{G(x)}{c}, & x \in B \setminus B_{R},
\end{cases}
\]
where $R = (-\log \epsilon)^{1/(1-\beta)}$, and $b$ and $c$ are constants to be determined later. We require
\[
\begin{aligned}
c + \frac{1}{c} \left( -\frac{1}{4\pi(1-\beta)} \log \left( 1 + \frac{\pi}{1-\beta} R^{2(1-\beta)} \right) + b \right) &= \frac{1}{c} \left( -\frac{1}{2\pi} \log(R \epsilon) + A_0 + O(R \epsilon) \right),
\end{aligned}
\]
which gives
\[
\begin{aligned}
c^2 &= -\frac{1}{2\pi} \log \epsilon + A_0 - b + \frac{1}{4\pi(1-\beta)} \log \frac{\pi}{1-\beta} + O\left( \frac{1}{R^{2(1-\beta)}} \right).
\end{aligned}
\]
By (2.27), \( G = w_0 \) on \( \mathbb{B} \setminus \mathbb{B}_{1/2} \). Noting \( w_0 \in \mathcal{H} \) and \( \phi_\epsilon - w_0/c \in W^{1,2}_0(\mathbb{B}) \), we get \( \phi_\epsilon \in \mathcal{H} \). We have by integration by parts that

\[
\int_{\mathbb{B} \setminus \mathbb{B}_{R\epsilon}} \left( |\nabla \phi_\epsilon|^2 - \frac{\phi^2_\epsilon}{(1 - |x|^2)^2} - \alpha \phi^2_\epsilon \right) dx = \frac{1}{c^2} \int_{\partial \mathbb{B}_{R\epsilon}} G \frac{\partial G}{\partial \nu} d\sigma + \frac{1}{c^2} \int_{\mathbb{B} \setminus \mathbb{B}_{R\epsilon}} G \mathcal{L}_\alpha G dx
\]

\[
= \frac{1}{c^2} \left( - \frac{1}{2\pi} \log(R\epsilon) + A_0 + O(R\epsilon) \right). \tag{2.35}
\]

A direct calculation shows that

\[
\int_{\mathbb{B}_{R\epsilon}} |\nabla \phi_\epsilon|^2 dx = \frac{1}{4\pi(1 - \beta)c^2} \left( \log \frac{\pi}{1 - \beta} + \log R^{2-2\beta} - 1 + O\left( \frac{1}{R^{2-2\beta}} \right) \right). \tag{2.36}
\]

Refer to [27] for detailed calculations for (2.36) and the following (2.39). Then we have

\[
\|\phi_\epsilon\|_{\mathcal{H},\alpha}^2 \leq \int_{\mathbb{B} \setminus \mathbb{B}_{R\epsilon}} \left( |\nabla \phi_\epsilon|^2 - \frac{\phi^2_\epsilon}{(1 - |x|^2)^2} - \alpha \phi^2_\epsilon \right) dx + \int_{\mathbb{B}_{R\epsilon}} |\nabla \phi_\epsilon|^2 dx
\]

\[
= \frac{1}{c^2} \left( - \frac{1}{2\pi} \log \epsilon + A_0 + \frac{1}{4\pi(1 - \beta)} \log \frac{\pi}{1 - \beta} - \frac{1}{4\pi(1 - \beta)} + O\left( \frac{1}{R^{2-2\beta}} \right) \right). \tag{2.37}
\]

Letting the last term in the above inequality equal 1, we get

\[
c^2 = - \frac{1}{2\pi} \log \epsilon + A_0 + \frac{1}{4\pi(1 - \beta)} \log \frac{\pi}{1 - \beta} - \frac{1}{4\pi(1 - \beta)} + O\left( \frac{1}{R^{2-2\beta}} \right). \tag{2.38}
\]

Combining (2.35) and (2.37), we get

\[
b = \frac{1}{4\pi(1 - \beta)} + O\left( \frac{1}{R^{2-2\beta}} \right). \tag{2.39}
\]

We also have

\[
\int_{\mathbb{B}_{R\epsilon}} |x|^{-2\beta} e^{4\pi(1 - \beta)\phi^2} dx \geq \frac{\pi}{1 - \beta} e^{1 + 4\pi(1 - \beta)A_0} + O\left( \frac{1}{R^{2-2\beta}} \right). \tag{2.40}
\]

Next, we calculate

\[
\int_{\mathbb{B} \setminus \mathbb{B}_{R\epsilon}} |x|^{-2\beta} e^{4\pi(1 - \beta)\phi^2} dx \geq \int_{\mathbb{B} \setminus \mathbb{B}_{R\epsilon}} |x|^{-2\beta} (1 + 4\pi(1 - \beta)\phi^2) dx
\]

\[
= \int_{\mathbb{B} \setminus \mathbb{B}_{R\epsilon}} |x|^{-2\beta} dx + \frac{4\pi(1 - \beta)}{c^2} \int_{\mathbb{B} \setminus \mathbb{B}_{R\epsilon}} |x|^{-2\beta} G^2 dx
\]

\[
= \int_{\mathbb{B}} |x|^{-2\beta} dx - \int_{\mathbb{B}_{R\epsilon}} |x|^{-2\beta} dx + \frac{4\pi(1 - \beta)}{c^2} \int_{\mathbb{B}} |x|^{-2\beta} G^2 dx
\]

\[
- \frac{4\pi(1 - \beta)}{c^2} \int_{\mathbb{B}_{R\epsilon}} |x|^{-2\beta} G^2 dx
\]

\[
= \frac{\pi}{1 - \beta} + \frac{4\pi(1 - \beta)}{c^2} \int_{\mathbb{B}} |x|^{-2\beta} G^2 dx + O\left( \frac{1}{R^{2-2\beta}} \right).
\]

Noting \( O\left( \frac{1}{R^{2-2\beta}} \right) = o\left( \frac{1}{c^2} \right) \) and combining the above estimates, we get

\[
\int_{\mathbb{B}} |x|^{-2\beta} e^{4\pi(1 - \beta)\phi^2} dx \geq \frac{\pi}{1 - \beta} + 1 + 4\pi(1 - \beta)A_0,
\]

if \( \epsilon \) is sufficiently small.

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