Generalized $k$-Center: Distinguishing Doubling and Highway Dimension

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Capacitated $k$-Center

Input

- graph $G = (V, E)$ with edge lengths $\ell: E \rightarrow \mathbb{R}^+$,
- integer $k$,
- capacities $c: V \rightarrow \mathbb{N}$.

Figure: C$k$C input with $k = 2$. 
Capacitated $k$-Center: Goal

Find $S \subseteq V$ and an assignment $\varphi : (V \setminus S) \rightarrow S$ such that

- $|S| \leq k$,
- for every $u \in S$, $|\varphi^{-1}(u)| \leq c(u)$, and
- $\max_{v \in V \setminus S} \text{dist}(v, \varphi(v))$ is minimal.

Figure: CKC solution for $k = 2$. 
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Figure: $C_{kC}$ solution for $k = 2$.

When $c(u) = |V|$ for every $u \in V \Rightarrow k$-CENTER.
Capacitated $k$-Center: Solution Prospects

**Capacitated $k$-Center** is NP-hard.

$\Rightarrow$ cannot solve exactly in polynomial time assuming $P \neq NP$. 

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c-approximation algorithm

Input $I$ → c-approximation algorithm → solution at most $c$ times worse than the optimum in time $O(poly(|I|))$
Capacitated \( k \)-Center: Solution Prospects

Capacitated \( k \)-Center is NP-hard.
⇒ cannot solve exactly in polynomial time assuming \( P \neq NP \).

Polynomial-time approximation scheme

\[ I \xrightarrow{\varepsilon > 0} \text{PTAS} \xrightarrow{\text{solution at most } (1 + \varepsilon) \text{ times worse than the optimum}} \in \mathcal{O}(\varepsilon (\text{poly}(|I|))) \]
Capacitated $k$-Center: Solution Prospects

c-approximation algorithm

An, Bhaskara, Chekuri, Gupta, Madan, Svensson. 2015
There is a 9-approximation algorithm for $C_{kn}$.

Cygan, Hajiaghayi, Khuller. 2012
There is no $(3 - \varepsilon)$-approximation algorithm for $C_{kn}$ unless $P = NP$. 
Capacitated $k$-Center: Solution Prospects

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There is no $(3 - \varepsilon)$-approximation algorithm for CkC unless $P = NP$.

Question

Are there settings where we can overcome this lower bound? Planar graphs, Euclidean spaces, real world, . . .
| **Special Settings?** |
|----------------------|
| **Doubling Dimension** $(\Delta)$ |
| **CAPACITATED $k$-CENTER** | generalizes the dimension of $\ell_q$ spaces |
| $k$-CENTER | $k^k / \varepsilon^{O(k\Delta)} \cdot \text{poly}(n)$ |
| Feldmann, Marx. 2020 |
| $k$-MEDIAN, $k$-MEANS, FACILITY LOCATION | $2^{(1/\varepsilon)^O(\Delta^2)} \cdot \text{poly}(n)$ |
| Cohen-Addad, Feldmann, Saulpic. 2021 |
| TSP, STEINER TREE | $\exp\{2^{O(\Delta)} \cdot (4\Delta \log n / \varepsilon)^\Delta\}$ |
| Talwar. 2004 |
### Special Settings?

| Capacitated k-Center | Doubling Dimension ($\Delta$) | Highway dimension ($h$) |
|----------------------|-------------------------------|------------------------|
| **k-Center**         | $k^k/\varepsilon^{O(k\Delta)} \cdot \text{poly}(n)$ | $f(k, h, \varepsilon) \cdot \text{poly}(n)^\dagger$ |
| **k-Median, k-Means, Facility Location** | | |
|                      | $2^{(1/\varepsilon)O(\Delta^2)} \cdot \text{poly}(n)$ | $n^{(2h/\varepsilon)^O(1)}$ |
| **TSP, Steiner Tree** | $\exp\{2^{O(\Delta)} \cdot (4\Delta \log n/\varepsilon)^\Delta\}$ | $\exp\left\{ \text{polylog}(n)^O(\log^2(h/\varepsilon)) \right\}$ |

$\dagger$: $f$: computable function

- **Doubling Dimension ($\Delta$)**: captures properties of transportation networks

- **Highway dimension ($h$)**: captures properties of transportation networks

- **Doubling Dimension ($\Delta$)**: $k^k/\varepsilon^{O(k\Delta)} \cdot \text{poly}(n)$
- **Highway dimension ($h$)**: $f(k, h, \varepsilon) \cdot \text{poly}(n)^\dagger$
Special Settings?

Efficient Parameterized Approximation Scheme

| Capacitated $k$-Center, $k$-Median, $k$-Means, Facility Location |
|---------------------------------------------------------------|
| $k$-Center                                                   |
| $k^k / \varepsilon^{O(k\Delta)} \cdot \text{poly}(n)$       |
| $f(k, h, \varepsilon) \circ \text{poly}(n)\dagger$           |
| Feldmann, Marx. 2020                                         |
| Becker, Klein, Saulpic. 2018                                 |

| Parameter $p \in \mathbb{N} \times \cdots \times \mathbb{N}$ of the input |

| TSP, Steiner Tree                                            |
|-------------------------------------------------------------|
| $\exp\{2^{O(\Delta)} \cdot (4\Delta \log n / \varepsilon)^{\Delta}\}$ |
| $\exp \left\{ \text{polylog}(n)^{O(\log^2(h/\varepsilon))} \right\}$ |
| Talwar. 2004                                                |
| Feldmann, Fung, Königemann, Post. 2018                      |

$\dagger$: $f$: computable function

Input $I$

$\varepsilon > 0$

EPAS

Solution at most $(1 + \varepsilon)$ times worse than the optimum

In time $O(f(p, \varepsilon) \cdot |I|^{\mathcal{O}(1)})$

where $f$ is a computable function.
## Special Settings?

| Setting                        | Doubling Dimension ($\Delta$)                                                                 | Highway Dimension ($h$)                                                                 |
|-------------------------------|---------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------|
| Capacitated $k$-Center        | $k^k / \varepsilon^{O(k\Delta)} \cdot \text{poly}(n)$                                      | $f(k, h, \varepsilon) \cdot \text{poly}(n)^{\dagger}$                                  |
|                               | **Theorem 2**                                                                               |                                                                                         |
| $k$-Center                    | $k^k / \varepsilon^{O(k\Delta)} \cdot \text{poly}(n)$                                      | $f(k, h, \varepsilon) \cdot \text{poly}(n)^{\dagger}$                                  |
|                               | [Feldmann, Marx, 2020](#FeldmannMarx2020)                                                  | [Becker, Klein, Saulpic, 2018](#BeckerKleinSaulpic2018)                                  |
| $k$-Median, $k$-Means, Facility Location | $2^{(1/\varepsilon)^{O(\Delta^2)}} \cdot \text{poly}(n)$                                 | $n^{2(h/\varepsilon)^{O(1)}}$                                                          |
|                               | [Cohen-Addad, Feldmann, Saulpic, 2021](#CohenAddadFeldmannSaulpic2021)                     | [Feldmann, Saulpic, 2021](#FeldmannSaulpic2021)                                         |
| TSP, Steiner Tree             | $\exp\{2^{O(\Delta)} \cdot (4\Delta \log n/\varepsilon)^{\Delta}\}$                      | $\exp\{\text{polylog}(n)^{O(\log^2(n/h/\varepsilon))}\}$                              |
|                               | [Talwar, 2004](#Talwar2004)                                                                | [Feldmann, Fung, Könemann, Post, 2018](#FeldmannFungKönenmannPost2018)                  |

$\dagger$: $f$: computable function
| Problem                      | Doubling Dimension ($\Delta$) | Highway dimension ($h$) |
|------------------------------|-------------------------------|-------------------------|
| **CAPACITATED $k$-CENTER**   | $k^k / \varepsilon^{O(k\Delta)} \cdot \text{poly}(n)$ | $\exists c > 1$: no $c$-approximation in $O(\varepsilon (f(k, h) \cdot \text{poly}(n)))^{\dagger, \S}$ |
| **$k$-CENTER**               | $k^k / \varepsilon^{O(k\Delta)} \cdot \text{poly}(n)$ | $f(k, h, \varepsilon) \cdot \text{poly}(n)^\dagger$ |
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$\dagger$: $f$: computable function  
$\S$: unless FPT = W[1]
Let $M = (X, \text{dist})$ be a metric space.

Figure: $B_r(u)$: Ball of radius $r$. 

Doubling Dimension
Doubling Dimension

Doubling dimension $\Delta(M)$: smallest $\Delta \in \mathbb{N}$ such that

$$\text{the ball } B_r(u) \text{ for every } u \in X \text{ and every } r \in \mathbb{R}^+ \text{ is contained in } \bigcup_{v \in V} B_{r/2}(v) \text{ for some } V \subseteq X \text{ with } |V| \leq 2^\Delta.$$
Doubling Dimension

Doubling dimension $\Delta(M)$: smallest $\Delta \in \mathbb{N}$ such that

- the ball $B_r(u)$ for every $u \in X$ and every $r \in \mathbb{R}^+$
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- is contained in $\bigcup_{v \in V} B_{r/2}(v)$ for some $V \subseteq X$ with $|V| \leq 2^\Delta$. 

![Diagram of balls and points illustrating the concept of doubling dimension]
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- is contained in $\bigcup_{v \in V} B_{r/2}(v)$ for some $V \subseteq X$ with $|V| \leq 2^\Delta$.

$\Rightarrow d$-dimensional $\ell_q$ metrics have doubling dimension $\mathcal{O}(d)$. 
Highway Dimension: Shortest Path Cover

- Let $G$ be an edge-weighted graph and fix a scale $r \in \mathbb{R}^+$. 
- Let $P_r$ be the set of paths of $G$ such that
  - they are a shortest path between their endpoints,
  - their length is more than $r$ and at most $2r$.

(a) Metro and tram network in Prague city center.

(b) Czech railway network.
Highway Dimension: Shortest Path Cover

Let $G$ be an edge-weighted graph and fix a scale $r \in \mathbb{R}^+$. Let $\mathcal{P}_r$ be the set of paths of $G$ such that they are a shortest path between their endpoints, their length is more than $r$ and at most $2r$.

The shortest path cover $\text{SPC}_r(G)$ is a hitting set\(^1\) for $\mathcal{P}_r$.

\(^1\)For every $P \in \mathcal{P}_r$ we have $P \cap \text{SPC}_r(G) \neq \emptyset$. 
Highway Dimension

*highway dimension* of an edge-weighted graph $G$:

- smallest integer $h$ such that,
- for any scale $r \in \mathbb{R}^+$,
- there exists $H := \text{SPC}_r(G)$ so that,
- $|H \cap B_{2r}(u)| \leq h$ for every $u \in V(G)$. 
$k$-CENTER algorithm

$M = (X, d)$

Optimum solution of cost OPT.
**k-Center algorithm**

\[ M = (X, d) \]

\[ \forall x \in X \exists y \in Y : d(x, y) \leq \varepsilon \text{OPT}, \text{ and} \]

\[ \forall y_1 \neq y_2 \in Y : d(y_1, y_2) > \varepsilon \text{OPT}. \]

**Net**: \( Y \subseteq X \) such that

- \( \forall x \in X \exists y \in Y : d(x, y) \leq \varepsilon \text{OPT}, \) and
- \( \forall y_1 \neq y_2 \in Y : d(y_1, y_2) > \varepsilon \text{OPT}. \)

Optimum solution of cost \( \text{OPT}. \)
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\[ M = (X, d) \]

- We get a \((1 + \varepsilon)\)-approximate solution.

\[ \forall x \in X \exists y \in Y : d(x, y) \leq \varepsilon \text{OPT}, \text{ and} \]
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Net: \( Y \subseteq X \) such that

- Replace every optimum center by its nearest net point.

\[ \Rightarrow \text{We get a } (1 + \varepsilon)\text{-approximate solution.} \]
**k-Center algorithm**

\[ M = (X, d) \]

We get a \((1 + \varepsilon)\)-approximate solution.

Guess the \(k\)-tuple near the optimum centers to get an EPAS with parameters \(k\), \(\varepsilon\), and \(\Delta\).

\[ \forall x \in X \exists y \in Y : d(x, y) \leq \varepsilon \text{OPT}, \text{ and} \]
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Replace every optimum center by its nearest net point.

\[ \Rightarrow \text{We get a } (1 + \varepsilon)\text{-approximate solution}. \]
\[ \Rightarrow \text{It can be shown that } |Y| \leq k(1/\varepsilon)^O(\Delta). \]
\[ \Rightarrow \text{Guess the } k\text{-tuple near the optimum centers to get an EPAS with parameters } k, \varepsilon, \text{ and } \Delta. \]
CkC algorithm obstacles

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**CkC algorithm obstacles**

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⇒ We get a \((1 + \varepsilon)\)-approximate solution.

⇒ Guess the \(k\)-tuple near the optimum centers to get an EPAS with parameters \(k, \varepsilon, \Delta\).

∀\(x \in X\) ∃\(y \in Y\): \(d(x, y) \leq \varepsilon\) OPT, and

∀\(y_1 \neq y_2 \in Y\): \(d(y_1, y_2) > \varepsilon\) OPT.

Net: \(Y \subseteq X\) such that
- \(\forall x \in X \exists y \in Y: d(x, y) \leq \varepsilon\) OPT, and
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\textbf{CkC algorithm obstacles}

- \textit{Net}: \( Y \subseteq X \) such that
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## Conclusion

| Problem                        | Doubling Dimension ($\Delta$)                                                   | Highway dimension ($h$)                                                                 |
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| **Capacitated $k$-Center**    | $k^k / \varepsilon^{O(k\Delta)} \cdot \text{poly}(n)$                      | $\exists c > 1$: no $c$-approximation in $\mathcal{O}_\varepsilon (f(k, h) \cdot \text{poly}(n))^{\dagger, \S}$ |
| **$k$-Center**                | $k^k / \varepsilon^{O(k\Delta)} \cdot \text{poly}(n)$                      | $f(k, h, \varepsilon) \cdot \text{poly}(n)^{\dagger}$                             |
| **$k$-Median, $k$-Means, Facility Location** | $2^{(1/\varepsilon)^{O(\Delta^2)}} \cdot \text{poly}(n)$                   | $n^{(2h/\varepsilon)^{O(1)}}$                                                     |
| **TSP, Steiner Tree**         | $\exp\{2^{O(\Delta)} \cdot (4\Delta \log n / \varepsilon)^{\Delta}\}$      | $\exp\left\{\text{polylog}(n)^{O(\log^2(h/\varepsilon))}\right\}$               |

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**Thank you for your attention!**

Questions, comments, ...?