ON 2-NILPOTENT MULTIPLIER OF LIE SUPERALGEBRAS

RUDRA NARAYAN PADHAN, NUPUR NANDI, AND K.C. PATI

1. Abstract

In this article we define the $c$-nilpotent multiplier of a finite dimensional Lie superalgebra. We characterize the structure of 2-nilpotent multiplier of finite dimensional nilpotent Lie superalgebras whose derived subalgebras have dimension at most one. Then we give an upper bound on the dimension of 2-nilpotent multiplier of any finite dimensional nilpotent Lie superalgebra. Moreover, we discuss the 2-capability of special as well as odd Heisenberg Lie superalgebras and abelian Lie superalgebras.

2. Introduction

Mathematics research comprises mainly classification of algebraic objects. Recently, several authors have considered $c$-nilpotent multiplier of Lie algebra. Salemkar et al. [18] developed a theory of $c$-nilpotent multiplier of Lie algebras. Let $0 \rightarrow R \rightarrow F \rightarrow L \rightarrow 0$ be a free presentation of a finite dimensional Lie algebra $L$ where $F$ is a free Lie algebra and $R$ is an ideal of $F$. Then for $c \geq 1$, the $c$-nilpotent multiplier of $L$ is defined as

\[
\mathcal{M}^{(c)}(L) = \frac{R \cap \gamma_{c+1}(F)}{\gamma_{c+1}(R, F)}
\]

where $\gamma_{c+1}(F)$ is the $(c+1)$-th term of the lower central series of $F$ and $\gamma_1(R, F) = R, \gamma_{c+1}(R, F) = [\gamma_c(R, F), F]$. This definition is analogous to the definition of the Baer-invariant of a group with respect to the variety of nilpotent groups of class at most $c$ [3]. For $c = 1$, the Schur multiplier $\mathcal{M}^{(1)}(L) = \mathcal{M}(L)$ has been studied by Batten [5].

The study on $c$-nilpotent multipliers of Lie algebras have seen several fruitful developments [2, 17, 15, 8, 1]. The main work on $c$-nilpotent multiplier of Lie algebras is to find an upper bound for the dimension of $c$-nilpotent multiplier and then classifying finite dimensional nilpotent Lie algebras under certain conditions. In [11], an upper bound for $\mathcal{M}^{(2)}(L)$ of finite dimensional Lie algebra $L$ has been investigated using the result of [12] and some characterization of 2-nilpotent multiplier of Heisenberg Lie algebras is given.

The aim of this paper is to generalize the notion of $c$-nilpotent multiplier of Lie algebras to the case of finite dimensional Lie superalgebras. Moreover, we give some upper bounds for $\mathcal{M}^{(2)}(L)$. In particular, we have found the dimension of 2-nilpotent multiplier of finite dimensional nilpotent Lie superalgebras whose derived subalgebras have dimension at most one. Then find an upper bound for $\mathcal{M}^{(2)}(L)$, i.e., for any nilpotent Lie superalgebra $L = L_0 \oplus L_1$ of dimension $(k \mid l)$ with \( \dim L^2 = (r \mid s), \ r + s \geq 1 \), then

\[
\dim \mathcal{M}^{(2)}(L) \leq \frac{1}{3}((k + l + 2r + 2s - 2)(k + l - r - s - 1) + 3(r + s - 1)) + 3.
\]

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In the last section, we have defined 2-capability of Lie superalgebra. Then discuss the 2-capability for the Heisenberg and abelian Lie superalgebras. Precisely, we have proved that $H(m,n)$ is 2-capable if and only if $m = 1, n = 0$. Also, the odd Heisenberg Lie superalgebras $H_m$ is 2-capable only for $m = 1$. Moreover, $A(m \mid n)$ is 2-capable if and only if $m + n \geq 2$.

3. Auxiliary Results

Throughout this article, for superdimension of a Lie superalgebra $L = L_\sigma \oplus L_\tau$ we simply write $\dim(L) = (m \mid n)$, where $\dim L_\sigma = m$ and $\dim L_\tau = n$. Non-zero elements of $L_\sigma \cup L_\tau$ are called homogeneous elements. For a homogeneous element $v \in L_\sigma$, with $\sigma \in \mathbb{Z}_2$ we set $|v| = \sigma$ is the degree of $v$. We denote an abelian Lie superalgebra of dimension $(m \mid n)$ by $A(m \mid n)$. A finite dimensional Lie superalgebra $L$ is said to be Heisenberg Lie superalgebra if $Z(L) = L'$ and $\dim Z(L) = 1$. A Heisenberg Lie superalgebra is a nilpotent Lie superalgebra of nilindex 2. Since the dimension of center of $L$ is one, by assuming the homogeneous generator of $Z(L)$ over an algebraically closed field, Heisenberg Lie superalgebra can be divided into two types, i.e., one with even center and another with odd center. Heisenberg Lie superalgebra with even center is known as special Heisenberg Lie superalgebra and we denote $H(m,n)$ as special Heisenberg Lie superalgebra of dimension $(2m + 1 \mid n)$. An odd Heisenberg Lie superalgebra is denoted by $H_m$ with dimension $(m \mid m + 1)$.[10][16]

Finally we adopt the convention that whenever the degree function appeared in a formula, the corresponding elements are supposed to be homogeneous. Now we list some useful results which we use in the next section.

Lemma 3.1. (See [13], Lemma 6.1) For any finite dimensional Lie superalgebra $L$, we have $Z^*(L) = Z^\langle\lambda\rangle(L)$.

Remark: $L$ is capable if and only if $Z^\langle\lambda\rangle(L) = 0$.

Lemma 3.2. (See [13], Theorem 6.3) $A(m \mid n)$ is capable if and only if $m = 0, n = 1$ and $m + n \geq 2$.

Lemma 3.3. (See [13], Theorem 6.4) $H(m \mid n)$ is capable if and only if $m = 1, n = 0$.

Lemma 3.4. (See [13], Theorem 6.7) $H_m$ is capable if and only if $m = 1$.

Lemma 3.5. (See [10], Proposition 3.4) Let $L$ be a nilpotent Lie superalgebra of dimension $(k \mid l)$ with $\dim L' = (r \mid s)$, where $r + s = 1$. If $r = 1, s = 0$ then $L \cong H(m,n) \oplus A(k - 2m - 1 \mid l - n)$ for $m + n \geq 1$. If $r = 0, s = 1$ then $L \cong H_m \oplus A(k - m \mid l - m - 1)$.

Let $X = X_0 \cup X_1$ be a totally ordered $\mathbb{Z}_2$-graded set. Let $\Gamma(X)$ be the groupoid of non-associative monomials in the alphabet $X$, $u \circ v = (u)(v)$ for $u, v \in \Gamma(X)$, and $S(X)$ be the free semigroup of associative words with the bracket removing homomorphism $- : \Gamma(X) \to S(X)$. For $u = x_1 \ldots x_n \in S(X)$, $x_i \in X$, we consider the word length $l_X(u) = n$, the multi degree $m(u)$, $|u| = \sum_{i=1}^n |x_i| \in \mathbb{Z}_2$. Now let $K$ be a commutative ring with 1, and let $A(X)$ and $F(X)$ be the free associative and non-associative $K$-algebras respectively. Let $A(X)_\sigma, F(X)_\sigma$ for $\sigma \in \mathbb{Z}_2$ be the $K$-linear spans of the subsets $S(X)_\sigma$ and $\Gamma(X)_\sigma$ respectively, $A(X)$ and $F(X)$ being the $\mathbb{Z}_2$-graded associative and non-associative algebras respectively. Let $I$ be the ideal generated by the homogeneous elements of the form $x \circ y - (1)^{|x||y|} y \circ x$ and $(x \circ y) \circ z - x \circ (y \circ z) - (1)^{|x||y|} y \circ (x \circ z)$, for $x, y \in \Gamma(X)$ then $L(X) = F(X)/I$, is the free Lie $K$-superalgebra (see [4]).

Suppose the set $S(X)$ is ordered lexicographically. A monomial $u \in \Gamma(X)$ is said to be regular if either $u \in X$ or;

1. $u = u_1 \circ u_2$ where $u_1, u_2$ are regular monomials with $\overline{u_1} > \overline{u_2}$,
2. $u = (u_1 \circ u_2) \circ u_3$ with $\overline{u_2} \leq \overline{u_3}$.
A monomial $u \in \Gamma(X)$ is said to be $s$-regular if either $u$ is a regular monomial or $u = (v)(v)$ with $v$ a regular monomial and $|v| = 1$. Then the set of all images of the $s$-regular monomials form a basis of the free Lie superalgebra $L(X)$. The analogue of Witt’s formula can be seen in Corollary 2.8 in [4].

**Theorem 3.6.** Let $X = X_0 \cup X_1$, $X_0 = \{x_1, \ldots, x_m\}$, $X_1 = \{y_1, \ldots, y_n\}$ be a totally ordered $\mathbb{Z}_2$-graded set and $L(X)$ be the free Lie superalgebra, $\mu(l)$ the Möbius function, and $W(\alpha_1, \ldots, \alpha_{m+n})$ the rank of the free module of elements of multi degree $\alpha = (\alpha_1, \ldots, \alpha_{m+n})$ in the free Lie algebra of rank $m + n$,

$$W(\alpha_1, \ldots, \alpha_{m+n}) = \frac{1}{|\alpha|} \sum_{e|\alpha} \mu(e) (|\alpha|/e)! (\alpha/e)!,$$

where $|\alpha| = \sum_{i=1}^{m+n} \alpha_i$. Let $SW(\alpha_1, \ldots, \alpha_{m+n})$ be the rank of the free module of elements of multi degree $\alpha = (\alpha_1, \ldots, \alpha_{m+n})$ in the free Lie superalgebra $L(X)$ of rank $m + n$. Then

$$SW(\alpha_1, \ldots, \alpha_{m+n}) = W(\alpha_1, \ldots, \alpha_{m+n}) + \beta W\left(\frac{\alpha_1}{2}, \ldots, \frac{\alpha_{m+n}}{2}\right),$$

where

$$\beta = \begin{cases} 0 & \text{if there exists an } i \text{ such that } \alpha_i \text{ is odd, or if } \frac{1}{2} \sum_{i=m+1}^{m+n} \alpha_i |\alpha_i| = 1, \\ 1 & \text{otherwise.} \end{cases}$$

**Corollary 3.7.** [14] Let $P$ and $M$ be two Lie superalgebras, then $F^n/F^{n+i}$ is an abelian Lie superalgebra with the basis of all $s$-regular monomials on $X$ of lengths $n, n+1, \ldots, n+i-1$ for all $1 \leq i \leq n$. In particular, $F^n/F^{n+1}$ is an abelian Lie superalgebra of dimension $\sum_{|\alpha|=n} SW(\alpha)$, where $F^n$ is the $n$-th term of the lower central series of $F$.

### 3.1. Non-abelian tensor product.

Recently, García-Martínez [7] introduced the notion of non-abelian tensor product of Lie superalgebras and exterior product of Lie superalgebras. Here we recall some of the known notations and results from [7].

Let $P$ and $M$ be two Lie superalgebras, then by an action of $P$ on $M$ we mean a $\mathbb{K}$-bilinear map of even grade,

$$P \times M \rightarrow M, \ (p, m) \mapsto p \cdot m,$$

such that

$$\begin{align*}
(1) \ [p, p']_M &= p(p' m) - (p')p|m| p'(p m), \\
(2) \ [p, [p', [p'', [p''', \ldots]]]]_M &= p([p', [p'', [p''', \ldots]]] m) - (-1)^{|p||p'|} [p', [p'', [p''', \ldots]]]'(p m).
\end{align*}$$
for all homogeneous $p, p' \in P$ and $m, m' \in M$. For any Lie superalgebra $M$, the Lie multiplication induces an action on itself via $^m m' = [m, m']$. The action of $P$ on $M$ is called trivial if $^p m = 0$ for all $p \in P$ and $m \in M$.

Given two Lie superalgebras $M$ and $P$ with action of $P$ on $M$, we define the semidirect product $M \rtimes P$ with underlying supermodule $M \oplus P$ endowed with the bracket given by $[(m, p), (m', p')] = ([m, m'] + ^p m - (-1)^{|m||p|}[^p m, [p, p']])$.

A crossed module of Lie superalgebras is a homomorphism of Lie superalgebras $\partial : M \longrightarrow P$ with an action of $P$ on $M$ satisfying

1. $\partial(^p m) = [p, \partial(m)]$,
2. $\partial(\partial(m)) = [\partial(m), \partial(m')]$,

for all $p \in P$ and $m, m' \in M$.

Let $M$ and $N$ be two Lie superalgebras with actions on each other. Let $X_{M,N}$ be the $\mathbb{Z}_2$-graded set of all symbols $m \otimes n$, where $m \in M_\mathbb{T} \cup M_\mathbb{T}$, $n \in N_\mathbb{T} \cup N_\mathbb{T}$ and the $\mathbb{Z}_2$-gradation is given by $|m \otimes n| = |m| + |n|$. The non-abelian tensor product of $M$ and $N$, denoted by $M \otimes N$, as the Lie superalgebra generated by $X_{M,N}$ and subject to the relations:

1. $\lambda (m \otimes n) = \lambda m \otimes n = m \otimes \lambda n$,
2. $(m + m') \otimes n = m \otimes n + m' \otimes n$, where $m, m'$ have the same grade, $m \otimes (n + n') = m \otimes n + m \otimes n'$, where $n, n'$ have the same grade,
3. $[m, m'] \otimes n = (m \otimes m') n - (-1)^{|m||m'|} (m' \otimes m) n$,
4. $m \otimes [n, n'] = (-1)^{|m||n|}(m' \otimes n) n - (-1)^{|m||n|}(m \otimes n) n'$,
5. $[m \otimes n, m' \otimes n'] = (-1)^{|m||n|}(m \otimes n') n - (-1)^{|m||n|}(m \otimes n) n'$,

for every $\lambda \in \mathbb{K}, m, m' \in M_\mathbb{T} \cup M_\mathbb{T}$ and $n, n' \in N_\mathbb{T} \cup N_\mathbb{T}$. If $M = M_0$ and $N = N_0$ then $M \otimes N$ is the non-abelian tensor product of Lie algebras introduced and studied [5].

Actions of Lie superalgebras $M$ and $N$ on each other are said to be compatible if

1. $[^m n] m' = (-1)^{|m||n|}[m, n] m'$,
2. $[^m n] m' = (-1)^{|m||n|}[m, m'] n$,

for all $m, m' \in M_\mathbb{T} \cup M_\mathbb{T}$ and $n, n' \in N_\mathbb{T} \cup N_\mathbb{T}$. For instance if $M$, $N$ are two graded ideals of a Lie superalgebra then the actions induced by the bracket are compatible.

Suppose $M$ and $N$ are Lie superalgebras acting compatibly with each other. There are two Lie superalgebra homomorphisms $\mu : M \otimes N \longrightarrow M$ and $\nu : M \otimes N \longrightarrow N$ [7]. As a result of this $[M, N]^N$ or $[N, M]^N$ is the submodule generated by $^m n$ and by the compatibility condition it is a graded ideal of $N$. Further with some given actions of both $M$ and $N$ on $M \otimes N$, the homomorphisms $\mu, \nu$ are crossed modules.

Now on wards, we consider all actions are compatible and we have the following well known results.

**Proposition 3.9.** [7] Proposition 3.5 Let $M$ and $N$ be Lie superalgebras acting on each other. Then the canonical map $M \otimes \mathbb{K} N \rightarrow M \otimes N$, $m \otimes n \mapsto m \otimes n$, is an even, surjective homomorphism of supermodules.

Further the result below tells us when the surjective homomorphism in Proposition 3.9 is an isomorphism.

**Proposition 3.10.** [7] Proposition 3.5 If the Lie superalgebras $M$ and $N$ act trivially on each other, then $M \otimes N$ is an abelian Lie superalgebra and there is an isomorphism of supermodules

$M \otimes N \cong M^{ab} \otimes \mathbb{K} N^{ab}$,

where $M^{ab} = M/[M, M]$ and $N^{ab} = N/[N, N]$. 
Proposition 3.14. \cite[Proposition 3.8]{7} Let \( m, n \) following straightforward result.

Corollary 3.11. \( A(m \mid n) \otimes A(r \mid s) \cong A(m \mid n) \otimes_{\mathbb{Z}} A(r \mid s) \cong A(mr + ns \mid ms + nr) \)

Lemma 3.12. \cite[Lemma 6.1]{7} Let \( M \square N \) be the submodule of \( M \otimes N \) generated by elements

\[(1) \ m \otimes n + (-1)^{|m||n|} m' \otimes n', \text{ where } \partial(m) = \partial'(n') \text{ and } \partial(m') = \partial'(n), \\
(2) \ m_0 \otimes n_0, \text{ where } \partial(m_0) = \partial'(n_0),
\]

with \( m, m' \in M_\mathbb{Z} \oplus M_\mathbb{T}, \ n, n' \in N_\mathbb{Z} \oplus N_\mathbb{T}, \ m_0 \in M_\mathbb{Z} \) and \( n_0 \in N_\mathbb{Z} \). Then \( M \square N \) is a graded central ideal of \( M \otimes N \).

Definition 3.13. Let \( P \) be a Lie superalgebra and \((M, \partial)\) and \((N, \partial')\) two crossed \( P \)-modules. Then the exterior product of \( M \) and \( N \) is denoted as \( M \wedge N \) and is defined as

\[
M \wedge N = \frac{M \otimes N}{M \square N}.
\]

Let \((M, \partial)\) and \((N, \partial')\) two crossed \( P \)-modules. Let \( L \) be a Lie superalgebra. An even bilinear map \( \rho : M \times N \to L \) is called Lie super exterior pairing if the following holds:

\[
\begin{align*}
(1) & \ \rho([m, m'], n) = \rho(m, m'n) - (-1)^{|m||n|}\rho(m', mn); \\
(2) & \ \rho([m, n], n') = (-1)^{|m||n|}\rho(n', m, n) - (-1)^{|m||n|}\rho(n, n', m); \\
(3) & \ \rho(m, [n, n']) = (-1)^{|m||n|}\rho(n', m, n); \\
(4) & \ \rho(m, [n, n']) \in 0 \text{ if } \partial(m) = \partial'(n') \text{ and } \partial(m') = \partial'(n); \\
(5) & \ \rho(m_0, n_0) = 0 \text{ if } \partial(m_0) = \partial'(n_0);
\end{align*}
\]

for every \( m, m' \in M_\mathbb{Z} \oplus M_\mathbb{T}, \ n, n' \in N_\mathbb{Z} \oplus N_\mathbb{T}, \ m_0 \in M_\mathbb{Z} \) and \( n_0 \in N_\mathbb{Z} \). A Lie super exterior pairing \( \rho : M \times N \to L \) is called universal whenever there is any other Lie super pairing \( \rho' : M \times N \to Q \), then there exists a unique Lie superalgebra homomorphism \( \tau : L \to Q \) such that \( \tau \rho = \rho' \).

Let us consider the category \( SLie^2_\mathbb{F} \). Let the Lie superalgebras \( M \) and \( N \), \( P \) and \( Q \) act compatibly on each other. Also \( \phi : M \to P \) and \( \psi : N \to Q \) be two Lie superalgebra homomorphisms which preserve the action, i.e.,

\[
\phi^m \psi^n = \phi^m \psi^n = \phi^m \psi^n.
\]

Then we have a homomorphism \( \phi \otimes \psi : M \wedge N \to P \wedge Q \) defined by \( m \wedge n \mapsto \phi(m) \wedge \psi(n) \).

Proposition 3.14. \cite[Proposition 3.8]{7} Given an exact sequence of Lie superalgebras

\[
(0, 0) \to (K, L) \xrightarrow{(i,j)} (M, N) \xrightarrow{(\phi, \psi)} (P, Q) \to (0, 0)
\]

there is an exact sequence of Lie superalgebras

\[
(K \wedge M) \times (M \wedge L) \xrightarrow{\alpha} M \wedge N \xrightarrow{\phi \otimes \psi} P \wedge Q \to 0.
\]

Any ideal \( M \) of the Lie superalgebra \( L \) act on \( L \) via Lie multiplication. Thus, we have the following straightforward result.

Lemma 3.15. Let \( M \) be an ideal of the Lie superalgebra \( L \) and the epimorphism \( \theta : M \wedge L \to [M, L] \) is defined by \( m \wedge l \mapsto [m, l] \). Then the Lie superalgebras \( L \) and \( M \wedge L \) act compatibly on each other as follows

\[
l' \wedge m = l' \wedge m + (-1)^{|l'||m|} m \wedge l', \quad m \wedge l = \theta(x)l,
\]

where \( x \in M \wedge L, l' \) and \( m \) are homogeneous elements of \( L \) and \( M \) respectively.
Now, following Lemma [3, 15] we can define the exterior product \((M \wedge L) \wedge L\). Thus inductively we can construct the exterior product
\[
M \wedge^{c+1} L = (\ldots ((M \wedge L) \wedge L) \cdots \wedge L), \; c \geq 1
\]
and for \(j \geq 1\) the epimorphism \(\theta_j : M \wedge^j L \to [M, j L]\) given by
\[
(m \wedge l_1 \wedge \cdots \wedge l_j) \mapsto [m, l_1, \ldots, l_j],
\]
where \([m, l_1, \ldots, l_j] = [\ldots [m, l_1], l_2, \ldots, l_j]\) and \(\theta_1 = \theta\). Similarly, if \(K\) is any other ideal of \(L\) with \([M, K] = 0\), then one can define the non-abelian exterior product \(M \wedge^{c+1} (L/K)\).

4. Bounds for \(\mathcal{M}^{(c)}(L)\)

Now we are ready to define \(c\)-nilpotent multiplier of a Lie superalgebra and obtain some of its bounds analogous to the case of Lie algebra.

Let \(L = L_0 \oplus L_1\) be a Lie superalgebra and \(F\) be the free Lie superalgebra such that \(0 \to R \to F \to L \to 0\) be a free presentation of \(L\). For \(c \geq 1\), we define the \(c\)-multiplier of \(L\) to be
\[
\mathcal{M}^{(c)}(L) = R \cap \gamma_{c+1}(F)/\gamma_{c+1}(R, F)
\]
where \(\gamma_{c+1}(F)\) is the \((c+1)\)-th term of the lower central series of \(F\) and \(\gamma_1(R, F) = R, \gamma_{c+1}(R, F) = [\gamma_c(R, F), F]\). In particular, if \(c = 1\), then \(\mathcal{M}^{(1)}(L) = \mathcal{M}(L)\) is Schur multiplier of Lie superalgebra \(L\) which has been studied in [10]. Following the definition, \(\mathcal{M}^{(0)}(L)\) is an abelian Lie superalgebra.

Let \(L = L_\pi \oplus L_\tau\) be a Lie superalgebra. Let us define the set \(Z_n(L) = \{x \in L \mid [x, y] \in Z_{n-1}(L) \forall y \in L\}\) where \(Z_0(L) = 0\). Then for each \(n\), \(Z_n(L)\) is a graded ideal of \(L\). In particular, \(Z_{n-1}(L)\) is a graded ideal of \(Z_n(L)\). If \(n = 1\), then we will get the center of \(L\). Furthermore, the upper central series of \(L\) is defined as
\[
\{0\} = Z_0(L) \subseteq Z_1(L) \subseteq \cdots \subseteq Z_n(L)
\]
where \(Z_1(L) = Z(L)\). One can verify that
\[
Z_n(L) = Z(L)^{l_{Z_n(L)}} \cdot \frac{Z_{n+1}(L)}{Z_n(L)}.
\]

Let \(K\) be a graded ideal of a Lie superalgebra \(L\). \(K\) is said to be \(n\)-central if \(K \subseteq Z_n(L)\). Evidently, \(\gamma_{n+1}(K, L) = 0\), then \(K \subseteq Z_n(L)\). Thus, if \(L\) is a nilpotent Lie superalgebra of nilindex \(n\), then \(Z_n(L) = L\).

**Proposition 4.1.** Let \(L = L_0 \oplus L_1\) be a Lie superalgebra with a free presentation \(0 \to R \to F \xrightarrow{\pi} L \to 0\), and \(M\) be a graded ideal of \(L\). Define \(M \cong N/R\) for some graded ideal \(N\) of \(F\), and \(T_i = \gamma_i(N, F)/\gamma_i(R, F)\) \((i \geq 1)\). Then

1. \(T_{i+1}\) is a homomorphic image of \(M \wedge^i L\).
2. If \(M\) is \(n\)-central and \(P = L/\gamma_{n+1}(L)\), then \(T_{i+1}\) is a homomorphic image of \(M \wedge^i P\).

**Proof.** (1) Let \(l\) and \(x\) be the homogeneous elements of \(L\) and \(\gamma_i(N, F)\) respectively. Let \(\pi^{-1}(l) = y\), then we have the following well-defined action of \(L\) and \(T_j\), acting on each other compatibly,
\[
l(x + \gamma_i(R, F)) = [y, x] + \gamma_i(R, F) \quad \text{and} \quad (x + \gamma_i(R, F))l = \pi([x, y])
\]
Since \(\gamma_i(R, F) \subseteq \ker \pi\), thus \(\varphi : T_i \to L\) is the induced map by \(\pi\). Moreover, \(\varphi\) is a crossed module with the action \([4.1]\). If \(\xi_i : T_i \times L \to T_{i+1}\) is defined by \(\xi_i(x + \gamma_i(R, F), l) = [x, y] + \gamma_{i+1}(R, F)\), then \(\xi_i\) is a Lie super exterior pairing. Thus from universal property, there exists a unique Lie superalgebra homomorphism \(\overline{\xi}_i : T_i \wedge L \to T_{i+1}\). By Proposition [3.12] \(\xi_i \wedge \text{Id}_L : (T_i \wedge L) \wedge L \to T_{i+1} \wedge L\) is an onto homomorphism, where \(\text{Id}_L : L \to L\). Now the proof follows using induction hypothesis.

(2) For any graded ideal \(I\) of \(F\), using induction it can be shown that \([I, \gamma_{i+1}(N, F)] \subseteq \gamma_{i+1}(I, F)\).

Now observe that \([\gamma_i(N, F), \gamma_{n+1}(F)] \subseteq \gamma_{i+n+1}(N, F) \subseteq \gamma_{i+1}(R, F)\). Thus from \([4.1]\) \(\gamma_{n+1}(F)\) acts trivially on \(T_i\) and the induce action of \(L/\gamma_{n+1}(L)\) and \(T_i\) on each other is compatibly. Thus from part (1) the result follows. \(\square\)
Corollary 4.2. By the assumption and notations in Proposition 4.1, the factor Lie superalgebra \( R \cap \gamma_{i+1}(N,F)/\gamma_{i+1}(R,F) \) is a homomorphic image of \( \ker(\theta_i) \).

Proof. From proposition 4.1, there exists surjective homomorphism \( \psi_i : M \wedge i L \to T_{i+1} \). Then the result follows from the following commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & \ker \theta_i \\
\psi_i & \downarrow & \theta_i \\
M \wedge i L & \longrightarrow & [M_i L] \\
\psi_i & \downarrow & \cong \\
0 & \longrightarrow & R \cap \gamma_{i+1}(N,F) \\
\gamma_{i+1}(R,F) & \longrightarrow & T_{i+1} \\
\gamma_{i+1}(N,F) & \longrightarrow & 0 \\
\end{array}
\]

where \( \psi_i \) is the restriction of \( \psi_i \) on \( \ker(\theta_i) \). \( \square \)

From 4.2 we have the exact sequences of \( c \)-nilpotent multipliers of Lie superalgebras.

Proposition 4.3. Let \( M \) be a graded ideal of a Lie superalgebra \( L = L_0 \oplus L_1 \). Then the following sequences are exact:

1. \( \ker(\theta_c) \to M^{(c)}(L) \to M^{(c)}(\frac{L}{M}) \to M^{(c)}(\gamma_{c+1}(L)_{\gamma_{c+1}(M,L)}) \to 0. \)

2. If \( M \) is \( n \)-central ideal with \( n \leq c \), \( M \wedge c \frac{L}{\gamma_{c+1}(M,L)} \to M^{(c)}(L) \to M \cap \gamma_{c+1}(L) \to 0. \)

Proof. Consider a free presentation of \( L \), i.e., \( 0 \to R \to F \to L \to 0 \). Then, there exists a graded ideal \( N \) of \( F \) such that \( M \cong N/R \). Evidently, the following inclusion maps

\[
R \cap \gamma_{c+1}(N,F) \subseteq R \cap \gamma_{c+1}(F) \subseteq N \cap \gamma_{c+1}(F) \subseteq (N \cap \gamma_{c+1}(F)) + R,
\]

induce the exact sequence of homomorphisms, i.e.,

\[
0 \to R \cap \gamma_{c+1}(N,F)_{\gamma_{c+1}(R,F)} \to R \cap \gamma_{c+1}(F)_{\gamma_{c+1}(R,F)} \to N \cap \gamma_{c+1}(F)_{\gamma_{c+1}(N,F)} \to (N \cap \gamma_{c+1}(F)) + R _{\gamma_{c+1}(N,F) + R} \to 0.
\]

If \( M \) is \( n \)-central with \( n \leq c \), then \( \gamma_{c+1}(N,F) \subseteq \gamma_{n+1}(N,F) \subseteq R \). Now, the result follows from Proposition 4.1 and Corollary 4.2 \( \square \)

Corollary 4.4. If \( M \) is \( n \)-central ideal of a Lie superalgebra \( L = L_0 \oplus L_1 \) with \( n \leq c \), then

\[
M \wedge c L \to M^{(c)}(L) \to M^{(c)}(\frac{L}{M}) \to M \cap \gamma_{c+1}(L)_{\gamma_{c+1}(M,L)} \to 0.
\]

Proof. Since \( M \) is \( n \)-central ideal, \( \ker(\theta_c) = M \wedge c L \). \( \square \)

Using the Proposition 4.3, the following inequalities can be obtained.

Corollary 4.5. Let \( M \) be a graded ideal of a finite dimensional Lie superalgebra \( L = L_0 \oplus L_1 \). Then

1. The Lie superalgebra \( M^{(c)}(L) \) is finite dimensional.
2. \( \dim(M^{(c)}(L)) \leq \dim(M^{(c)}(\frac{L}{M})) + \dim(M^{(c)}(\gamma_{c+1}(L)_{\gamma_{c+1}(M,L)})) \).
3. Following the notations of Proposition 4.1, \( \dim(M^{(c)}(L)) + \dim(M \cap \gamma_{c+1}(L)) = \dim(M^{(c)}(\frac{L}{M})) + \dim(M \cap \gamma_{c+1}(N,F)_{\gamma_{c+1}(R,F)}) \).
4. If \( M^{(c)}(L) = \{0\} \), then \( M^{(c)}(\frac{L}{M}) \cong M^{(c)}(\gamma_{c+1}(L)_{\gamma_{c+1}(M,L)}) \).

Proof. Since \( M \) is \( n \)-central ideal, \( \ker(\theta_c) = M \wedge c L \). \( \square \)
(5) If $M$ is $n$-central with $n \leq c$, then
\[
\dim(\mathcal{M}^{(c)}(L)) + \dim(M \cap \gamma_{c+1}(L)) \leq \dim(\mathcal{M}^{(c)}(\frac{L}{M})) + \dim \left( M \wedge^c \frac{L}{\gamma_{n+1}(L)} \right).
\]

Lemma 4.6. Let $L = L_0 \oplus L_1$ be a Lie superalgebra with a free presentation $0 \to R \to F \xrightarrow{\pi} L \to 0$, and $M$ be a graded ideal of $L$. Define $M \cong N/R$ for some graded ideal $N$ of $F$ and $T = L/M$. Then there exists a Lie superalgebra $P$ with a graded ideal $S$ such that the following hold:

1. $\gamma_{c+1}(L) \cap M \cong P/S$;
2. $S \cong \mathcal{M}^{(c)}(L)$;
3. $\mathcal{M}^{(c)}(T)$ is a homomorphic image of $P$.
4. $\gamma_{c+1} \cap M$ is a homomorphic image of $\mathcal{M}^{(c)}(T)$, whenever $M$ is a $c$-central ideal of $L$.

Proof. Now
\[
\gamma_{c+1}(L) \cap M = ((\gamma_{c+1}(F)) + R)/R \cap N/R
\] 
\[
= ((\gamma_{c+1}(F) \cap N) + R)/R
\]
\[
\cong (\gamma_{c+1}(F) \cap N)/(\gamma_{c+1}(F) \cap R)
\]
\[
\cong (\gamma_{c+1}(F) \cap N)/\gamma_{c+1}(R, F) \cong \frac{P}{S}
\]
Thus (1) and (2) follow by taking $P = (\gamma_{c+1}(F) \cap N)/\gamma_{c+1}(R, F)$, $S = (\gamma_{c+1}(F) \cap R)/\gamma_{c+1}(R, F) \cong \mathcal{M}^{(c)}(L)$. Furthermore,
\[
\mathcal{M}^{(c)}(T) \cong (\gamma_{c+1}(F) \cap N)/\gamma_{c+1}(N, F) \cong \frac{(\gamma_{c+1}(F) \cap N)/\gamma_{c+1}(R, F)}{\gamma_{c+1}(N, F)/\gamma_{c+1}(R, F)}
\]
Thus, $\mathcal{M}^{(c)}(T)$ is a homomorphic image of $P$. As $M$ is a $c$-central ideal of $L$, $\gamma_{c+1}(N, F) \subseteq \gamma_{c+1}(F) \cap R$. Thus,
\[
\gamma_{c+1}(L) \cap B = ((\gamma_{c+1}(F) + R)/R) \cap N/R
\]
\[
\cong (\gamma_{c+1}(F) \cap N)/(\gamma_{c+1}(F) \cap R)
\]
\[
\cong (\gamma_{c+1}(F) \cap N)/\gamma_{c+1}(N, F) \cong \frac{\gamma_{c+1}(F) \cap R}{\gamma_{c+1}(N, F)}
\]
which completes the proof.

The immediate consequence of the above lemma is as follow.

Corollary 4.7. Let $L = L_0 \oplus L_1$ be a Lie superalgebra with a graded ideal $M$ and $T = L/M$. Then
\[
\dim \mathcal{M}^{(c)}(T) \leq \dim \mathcal{M}^{(c)}(L) + \dim (\gamma_{c+1}(L) \cap M).
\]

Lemma 4.8. Let $L = L_0 \oplus L_1$ be a Lie superalgebra with a free presentation $0 \to R \to F \xrightarrow{\pi} L \to 0$, and $M$ be a graded ideal of $L$ with $M \subseteq Z(L)$. Define $M \cong N/R$ for some graded ideal $N$ of $F$ and $T = L/M \cong F/N$. Then $\gamma_{c+1}(N, F)/(\gamma_{c+1}(R, F) + \gamma_{c+1}(N))$ is a homomorphic image of $(T/T^2)^c \otimes M = T/T^2 \otimes \ldots \otimes T/T^2 \otimes M$, $c \geq 1$.

Proof. Define the map $\psi : T/T^2 \times \ldots \times T/T^2 \times M \rightarrow \gamma_{c+1}(N, F)/(\gamma_{c+1}(R, F) + \gamma_{c+1}(N))$ with
\[
\theta(f_1 + (F^2 + N), \ldots, f_c + (F^2 + N), x + R) := [x, f_1, \ldots, f_c] + (\gamma_{c+1}(R, F) + \gamma_{c+1}(N)),
\]
where \( f_1, f_2, \ldots, f_c \in F_0 \cup F_1 \); \( x \in N_0 \cup N_1 \). We will show that \( \theta \) is well-defined. Note that

\[
T/T^2 \cong F/(F^2 + N) \text{ and } M/M^2 \cong N/R.
\]

Suppose \((f_1 + F^2 + N, f_2 + F^2 + N, \ldots, f_c + F^2 + N, x + R) = (f'_1 + F^2 + N, f'_2 + F^2 + N, y + R), \) \( y \in N_0 \) for all homogeneous elements. Then \( f_i - f'_i \in F^2 + N \), for \( i = 1, 2, \ldots, c \) and \( x - y \in R_0 \cup R_1 \).

Which implies \( f_i - f'_i = g_i + s_i \), where \( g_i \) and \( s_i \) are homogeneous elements of \( F^2 \) and \( N \) respectively.

Again, \([x, f_1, f_2, \ldots, f_c] + \gamma_{c+1}(R, F) + \gamma_{c+1}(N) = [y, f'_1, f'_2, \ldots, f'_c] + \gamma_{c+1}(R, F) + \gamma_{c+1}(N)\) which implies \([x, f_1, f_2, \ldots, f_c] - [y, f'_1, f'_2, \ldots, f'_c] \in \gamma_{c+1}(R, F) + \gamma_{c+1}(N)\). By the assumption \( M \) is contained in \( Z(L) \), which implies \([N, F] \subseteq R\). Using Jacobi Identity, \([x, f_1, \ldots, f_c] - [x + r, f_1 + g_1 + s_1, \ldots, f_c + g_c + s_c] \in \gamma_{c+1}(R, F) + \gamma_{c+1}(N)\). Hence \( \theta \) is well-defined. Therefore, by universal property there exists a unique homomorphism

\[
\bar{\psi} : T/T^2 \times \cdots \times T/T^2 \to M \to \gamma_{c+1}(N, F)/\gamma_{c+1}(R, F) + \gamma_{c+1}(N)
\]

with \( \text{Im}(\bar{\psi}) = \gamma_{c+1}(N, F)/\gamma_{c+1}(R, F) + \gamma_{c+1}(N) \), as desired. \( \square \)

**Theorem 4.9.** Let \( L = L_0 \oplus L_1 \) be a finite dimensional Lie superalgebra with a free presentation 
\( 0 \to R \to F \to L \to 0 \), and \( M \) be a graded ideal of \( L \) with \( M \subseteq Z(L) \). Define \( T = L/M \), then

\[
\dim M(c)(L) + \dim(\gamma_{c+1}(L) \cap M) \leq \dim M(c)(T) + \dim M(c)(M) + \dim((T/T^2)^c \otimes M).
\]

**Proof.** There exists a graded ideal \( N \) of the free Lie superalgebra \( F \) such that \( M \cong N/R \), then \([N, F] \subseteq R\). Observe that

\[
\frac{\gamma_{c+1}(N, F)/\gamma_{c+1}(R, F)}{(\gamma_{c+1}(R, F) + \gamma_{c+1}(N))/\gamma_{c+1}(R, F)} \cong \frac{\gamma_{c+1}(N, F)}{\gamma_{c+1}(R, F) + \gamma_{c+1}(N)}.
\]

Now using Lemma 4.6, \( \dim M(c)(L) + \dim(\gamma_{c+1}(L) \cap M) = \dim M(c)(T) + \dim(\gamma_{c+1}(N, F)/\gamma_{c+1}(R, F)) \) and from Lemma 4.8 we have

\[
\dim M(c)(L) + \dim(\gamma_{c+1}(L) \cap M) = \dim M(c)(T) + \dim \left( \frac{\gamma_{c+1}(N, F)}{\gamma_{c+1}(R, F) + \gamma_{c+1}(N)} \right) + \dim \left( \frac{\gamma_{c+1}(R, F) + \gamma_{c+1}(N)}{\gamma_{c+1}(R, F)} \right)
\]

\[
= \dim M(c)(T) + \dim M(c)(M) - \dim \left( \frac{\gamma_{c+1}(N, F)}{\gamma_{c+1}(R, F) + \gamma_{c+1}(N)} \right) + \dim \left( \frac{\gamma_{c+1}(R, F) + \gamma_{c+1}(N)}{\gamma_{c+1}(R, F)} \right)
\]

\[
\leq \dim M(c)(T) + \dim M(c)(M) + \dim((T/T^2)^c \otimes M).
\]

\( \square \)

A Lie superalgebra \( H \) is said to be a generalized Heisenberg Lie superalgebra of rank \((r \mid s)\) if \( Z(H) = H^2 \) with \( \dim Z(H) = (r \mid s) \). As an application of the above theorem, we will find an upper bound for the c-nilpotent multiplier of generalized Heisenberg Lie superalgebra of rank \((r \mid s)\).

**Lemma 4.10.** Let \( A(m \mid n) \) be an abelian Lie superalgebra of dimension \((m \mid n)\). Then \( \dim(M(c)(L)) = \sum_{|\alpha| = c+1} SW(\alpha) \), where \( SW(\alpha_1, \ldots, \alpha_m, \ldots, \alpha_{m+1}, \ldots, \alpha_{m+n}) \) is the rank of the free module of elements of multi degree \( \alpha = (\alpha_1, \ldots, \alpha_{m+n}) \) in the free Lie superalgebra \( L(X) \) of rank \( m + n \).

**Proof.** Let \( X = X_0 \cup X_1 \), \( X_0 = \{x_1, \ldots, x_m\}, X_1 = \{y_1, \ldots, y_n\} \) be a \( \mathbb{Z}_2 \)-graded set and \( F = L(X) \) be the free Lie superalgebra over \( X \). Then \( 0 \to F^2 \to F \to A(m \mid n) \to 0 \) is a free presentation for \( A(m \mid n) \). Therefore, \( M(c)(L) \cong \gamma_c(F)/\gamma_{c+1}(F) \). Now the result follows from Theorem 3.8. \( \square \)
Corollary 4.11. Let $H$ be a generalized Heisenberg Lie superalgebra of dimension $(m|n)$ with rank $(r|s)$. Then

$$\dim M^{(c)}(H) \leq \sum_{|\alpha|=c+1} SW(\alpha) + \sum_{|\alpha'|=c+1} SW(\alpha') + (m + n - r - s)c(r + s),$$

where $SW(\alpha)$ (resp. $SW(\alpha')$) is the rank of the free module of elements of multi degree $\alpha = (\alpha_{1}, \ldots, \alpha_{m-r}, \alpha_{m-r+1}, \ldots, \alpha_{m-r+n-s})$ (resp. $\alpha' = (\alpha'_{1}, \ldots, \alpha'_{r}, \alpha'_{r+1}, \ldots, \alpha'_{r+s})$) in the free Lie superalgebra $L(X_{1})$ (resp. $L(X_{2})$) of rank $m - r + n - s$ (resp. $r + s$).

Proof. We conclude the result using Theorem 4.9. To use Theorem 4.9 take $M = Z(H)$. Then

$$\dim(\gamma_{c+1}(H) \cap Z(H)) = \begin{cases} (r|s) & \text{if } c = 1 \\ 0 & \text{if } c \geq 2 \end{cases}$$

Now $\dim \left( \frac{H}{Z(H)} \right) = (m - r|n - s)$. Then using Lemma 4.10

$$\dim M^{(c)}(H) \leq \dim M^{(c)}\left( \frac{H}{Z(H)} \right) + \dim M^{(c)}(Z(H)) + \dim \left( \left( \frac{H}{Z(H)} \right)^{c} \otimes Z(H) \right)$$

$$= \sum_{|\alpha|=c+1} SW(\alpha) + \sum_{|\alpha'|=c+1} SW(\alpha') + (m + n - r - s)c(r + s).$$

\[\square\]

5. 2-nilpotent Multiplier of Lie Superalgebra

In this section we define direct product of two Lie superalgebras and then study some of its properties relating to 2-nilpotent multiplier. In particular, we derive the dimension of $M^{(2)}(L)$, when $L$ is a nilpotent Lie superalgebra with dim $L^{2} = 1$.

Let $L_{1}$ and $L_{2}$ be two Lie superalgebras with the free presentation

$$0 \rightarrow R_{1} \rightarrow F_{1} \xrightarrow{\delta_{1}} L_{1} \rightarrow 0$$

and

$$0 \rightarrow R_{2} \rightarrow F_{2} \xrightarrow{\delta_{2}} L_{2} \rightarrow 0$$

of $L_{1}$ and $L_{2}$ respectively, where $F_{1}$ and $F_{2}$ are free Lie superalgebras on some $\mathbb{Z}_{2}$-graded set.

Definition 5.1. Let $L_{1}$, $L_{2}$ and $L$ be Lie superalgebras with $\rho_{i} : L_{i} \rightarrow L$ be Lie superalgebra homomorphism for $i = 1, 2$. Then $(L, (\rho_{1}, \rho_{2}))$ is called a free product of $L_{1}$ and $L_{2}$ if, for any Lie superalgebra $K$ with the Lie superalgebra homomorphism $\sigma_{i} : L_{i} \rightarrow K$, $i = 1, 2$, then there exists a unique Lie superalgebra homomorphism $\pi : L \rightarrow K$ such that $\pi \rho_{i} = \sigma_{i}$, $i = 1, 2$.

The free product of $L_{1}$ and $L_{2}$ is denoted by $L_{1} \ast L_{2}$.

Lemma 5.2. If $F_{1}$ and $F_{2}$ are free Lie superalgebras on the $\mathbb{Z}_{2}$-graded sets $X$ and $Y$ respectively, then $F_{1} \ast F_{2}$ is a free Lie superalgebra on the $\mathbb{Z}_{2}$-graded set $X \cup Y$. 
The following lemma can be easily concluded from Lemma 5.2.

Lemma 5.3. If \( F = F_1 \ast F_2 \), then
\[
F^3 = [F_2, F_1, F_1] + [F_2, F_1, F_2] + F_1^3 + F_2^3.
\]

If we define an ordering on the \( \mathbb{Z}_2 \)-graded set \( X \cup Y \) by imposing every element of \( X \) proceeds each element of \( Y \). Then we have the following result.

Theorem 5.4. Let \( F_1 \) and \( F_2 \) be two free Lie superalgebras freely generated by \( X \) and \( Y \), respectively, and \( F = F_1 \ast F_2 \). Then \( [F_2, F_1, F_1] + [F_2, F_1, F_2] \) \((\text{mod } F^4)\) is an abelian Lie superalgebra.

Proof. The proof follows from the Theorem 3.3.

Lemma 5.5. Let \( F = F_1 \ast F_2 \) be the free product of \( F_1 \) and \( F_2 \). Then
\[
0 \rightarrow R \rightarrow F \xrightarrow{\delta} L_1 \oplus L_2 \rightarrow 0
\]
is a free presentation for \( L_1 \oplus L_2 \) in which \( R = R_1 + R_2 + [F_1, F_2] \).

Proof. By definition, we have the Lie superalgebra homomorphism \( \delta : F \rightarrow L_1 \oplus L_2 \) such that \( \delta \rho_i = \pi_i \delta \) for \( i = 1, 2 \), where \( \pi_i \) is the projection map from \( L_i \) into \( L_1 \oplus L_2 \). We will show that \( \ker(\delta) = R_1 + R_2 + [F_1, F_2] \). To check this, we observe that \( \delta(R_i) = 0 \) and for any \( x \in F_1 \), \( y \in F_2 \), then \( [\delta(x) , \delta(y)] = 0 \). Thus \( R_1 + R_2 + [F_1, F_2] \subseteq \ker(\delta) \). Conversely, let \( x \in \ker(\delta) \), then we can express \( x \) as a finite linear combination of basic commutators of \( F \). Because \( \delta \rho_i = \pi_i \delta \); this, in turn, gives \( x \in R \), ending the proof.

Lemma 5.6. Let \( L_1 \) and \( L_2 \) be two Lie superalgebras, then
\[
\mathcal{M}^{(2)}(L_1 \oplus L_2) \cong \mathcal{M}^{(2)}(L_1) \oplus \mathcal{M}^{(2)}(L_2) \oplus K
\]
for some subalgebra \( K \) of \( \mathcal{M}^{(2)}(L_1 \oplus L_2) \).

Proof. As mention in Lemma 5.2 \( F_1, F_2 \) and \( F \) are free Lie superalgebras. Thus we have a surjective map \( F \rightarrow F_1 \times F_2 \) and this map induces a surjective homomorphism
\[
\eta_1 : \frac{R \cap F^3}{[R, F, F]} \rightarrow \frac{R_1 \cap F_1^3}{[R_1, F_1, F_1]} \oplus \frac{R_2 \cap F_2^3}{[R_2, F_2, F_2]}.
\]

Let us define \( \eta_2 : \frac{R \cap F^3}{[R_1, F_1, F_1]} \oplus \frac{R \cap F^3}{[R_2, F_2, F_2]} \rightarrow \frac{R \cap F^3}{[R, F, F]} \) by
\[
\eta_2(x + [R_1, F_1, F_1], y + [R_2, F_2, F_2]) = x + y + [R, F, F].
\]

Since \( [R_1, F_1, F_1] + [R_2, F_2, F_2] \subseteq [[R, F], F] \), thus \( \eta_2 \) is a well-defined map and a homomorphism. Also, if \( (x + [R_1, F_1, F_1], y + [R_2, F_2, F_2]) \in \frac{R \cap F^3}{[R, F, F]} \) where \( x \in R_1 \cap F_1^3 \) and \( y \in R_2 \cap F_2^3 \), then
\[
\eta_2 \eta_1(x + y + [R, F, F]) = \eta_2(x + [R_1, F_1, F_1], y + [R_2, F_2, F_2]) = x + y + [R, F, F].
\]
That is \( \eta_2 \eta_1 = Id \). Thus, there exists a subalgebra \( K \) of \( \mathcal{M}^{(2)}(L_1 \oplus L_2) \) such that \( \mathcal{M}^{(2)}(L_1 \oplus L_2) \equiv \mathcal{M}^{(2)}(L_1) \oplus \mathcal{M}^{(2)}(L_2) \oplus K \).

The aim is to find the complete structure of \( K \) in order to derive the behavior of \( \mathcal{M}^{(2)}(L_1 \oplus L_2) \).

Lemma 5.7. Consider the surjective homomorphism defined in Lemma 5.6
\[
\eta_1 : \mathcal{M}^{(2)}(L_1 \oplus L_2) \rightarrow \mathcal{M}^{(2)}(L_1) \oplus \mathcal{M}^{(2)}(L_2).
\]
Then
\[
\ker \eta_1 \equiv [F_2, F_1, F_1] + [F_2, F_1, F_2] \pmod{[R, F, F]}.
\]
Proof. We can see that $[F_2, F_1, F_1] + [F_2, F_1, F_2] \pmod {R, F, F} \subset \ker \eta_1$. Now, suppose that $x + [R, F, F] \in \ker \eta_1$, then by Lemma 5.3 we can write $w = v_1 + v_2 + v_3$ where $v_1 \in [F_2, F_1, F_1] + [F_2, F_1, F_2], v_2 \in F_1^2$, and $v_3 \in F_3^2$. The way $\eta$ is defined, we get $v_2 \in [R_1, F_1, F_1]$ and $v_3 \in [R_2, F_2, F_2]$. If $x \equiv v_1 \pmod {R, F, F}$, as desired.

**Theorem 5.8.** With the above notations we have

$$[F_2, F_1, F_1] + [F_2, F_1, F_2] \equiv (L^{ab}_2 \otimes L^{ab}_1 \otimes L^{ab}_1) \oplus (L^{ab}_2 \otimes L^{ab}_1 \otimes L^{ab}_2) \pmod {R, F, F}.$$  

Proof. First we define $\psi_1: [F_2, F_1, F_1] + [F_2, F_1, F_2]/[R, F, F] \to (L^{ab}_2 \otimes L^{ab}_1 \otimes L^{ab}_1) \oplus (L^{ab}_2 \otimes L^{ab}_1 \otimes L^{ab}_2)$ by

$$\psi_1([x, y, z] + [R, F, F]) = \overline{x} \otimes \overline{y} \otimes \overline{z},$$

which gives a homomorphism. Conversely, if we define $\psi_2: (L^{ab}_2 \otimes L^{ab}_1 \otimes L^{ab}_1) \oplus (L^{ab}_2 \otimes L^{ab}_1 \otimes L^{ab}_2) \to [F_2, F_1, F_1] + [F_2, F_1, F_2]/[R, F, F]$ by

$$\overline{x} \otimes \overline{y} \otimes \overline{z} \mapsto [a, b, c] + [R, F, F]$$

which defines another homomorphism. Then $\psi_1 \psi_2 = Id = \psi_2 \psi_1$. □

If we know $M^{(2)}(L_1)$ and $M^{(2)}(L_2)$, then we can find $M^{(2)}(L_1 \oplus L_2)$ as follows.

**Theorem 5.9.** Let $L_1$ and $L_2$ be two Lie superalgebras, then

$$M^{(2)}(L_1 \oplus L_2) \cong M^{(2)}(L_1) \oplus M^{(2)}(L_2) \oplus (L^{ab}_2 \otimes L^{ab}_1 \otimes L^{ab}_1) \oplus (L^{ab}_2 \otimes L^{ab}_1 \otimes L^{ab}_2).$$

Proof. Acting Theorem 5.7 and Theorem 5.8 the desired conclusion follows. □

An useful result on the dimension of 2-nilpotent multiplier of an abelian Lie superalgebra of dimension $(m \mid n)$ is given by the even and odd dimension of 2-nilpotent multiplier.

**Corollary 5.10.** Let $A(m \mid n)$ be an abelian Lie superalgebra of dimension $(m \mid n)$. Then

$$M^2(A(m \mid n)) \cong A\left(\frac{1}{3}(m^3 + 3n^2m - m) \left| \frac{1}{3}(3m^2n + n^3 - n)\right.\right).$$

Proof. From Lemma 4.10 it is clear that $\dim(M^{(2)}(L)) = \sum_{|\alpha|=3} SW(\alpha)$. Let $\alpha = (\alpha_1, ..., \alpha_{m+n})$, then $\sum_{i=1}^{m+n} \alpha_i = 3$. Thus, from Theorem 3.6 at least one of $\alpha_i$ is odd, i.e., $\beta = 0$ which implies $\sum_{|\alpha|=3} SW(\alpha) = \sum_{|\alpha|=3} W(\alpha) = \frac{1}{3}[(m + n)^2 - (m + n)$. Now the required result follows from the Corollary 5.7 □

Using the above corollary, we can find the structure of 2-nilpotent multiplier of non-capable special Heisenberg Lie superalgebras as well as odd Heisenberg Lie superalgebras [10, 10].

**Theorem 5.11.** Let $H(m, n)$ be a non-capable special Heisenberg Lie superalgebra. Then

1. $M^{(2)}(H(m, n)) \cong A\left(\frac{1}{3}(8m^3 + 6n^2m - 2m) \left| \frac{1}{3}(n^3 + 12m^2n - n)\right.\right)$ if $m + n \geq 2$
2. $M^{(2)}(H(m, n)) \cong 0$ if $m = 0, n = 1$.

Proof. From Lemma 3.7 and Lemma 3.8 we have $Z^*(H(m, n)) = H(m, n)^2$ for $m + n \geq 2, m = 0, n = 1$. Now using Corollary 4.4

$$M^{(2)}(H(m, n)) \cong M^{(2)}(A(2m \mid n)).$$

Similarly, if $m = 0, n = 1$, then $M^{(2)}(H(0, 1)) \cong M^{(2)}(A(0 \mid 1))$. Now the result follows from Corollary 5.10 □
Theorem 5.12. Let $H(m)$ be a non-capable odd Heisenberg Lie superalgebra. Then

\[ \mathcal{M}^{(2)}(H(m)) \cong A \left( \frac{1}{3} (4m^3 - m) \right) \cong A \left( \frac{1}{3} (4m^3 - m) \right). \]

Proof. The proof follows from Corollary 5.10 and Lemma 3.4 \qed

Lemma 5.13 (see [11], Theorem 2.10). $\mathcal{M}^2(H(1,0)) \cong \mathcal{M}^2(H(1)) \cong A(5 \mid 0)$.

Now we will characterize $\mathcal{M}^2(L)$, where $L$ is a finite dimensional nilpotent Lie superalgebra whose derived subalgebra has dimension one. At first we consider nilpotent Lie superalgebras whose derived subalgebra have dimension $(1 \mid 0)$.

Lemma 5.14. Let $L$ be a nilpotent Lie superalgebra of dimension $(k \mid l)$ with $\dim L^2 = (1 \mid 0)$, then $\mathcal{M}^2(L)$ is isomorphic to either $A\left( \frac{1}{3} k(k-1)(k+1) + (k-1)(l^2 - k) \mid \frac{1}{3} l(l-1)(l+1) + l(k-1)^2 \right)$ or $A\left( \frac{1}{3} k(k-1)(k+1) + (k-1)(l^2 - k) + 1 \mid \frac{1}{3} l(l-1)(l+1) + l(k-1)^2 + 1 \right)$ or $A\left( \frac{1}{3} k(k-1)(k+1) + (k-1)(l^2 - k) + 3 \mid \frac{1}{3} l(l-1)(l+1) + l(k-1)^2 \right)$.

Proof. From Lemma 3.5, we know that $L \cong H(m,n) \oplus A(k-2m-1 \mid l-n)$. By Corollary 3.11, Theorem 5.9, Theorem 5.10, Theorem 5.11 and Lemma 5.13, we prove the theorem by considering case by case.

**Case-1:** Suppose $m = 1, n = 0$. Then

\[ \mathcal{M}^2(L) \cong \mathcal{M}^2(H(1,0)) \oplus \mathcal{M}^2(A(k-3 \mid l)) \oplus \left( \frac{H(1,0)}{(H(1,0))^2} \otimes \frac{H(1,0)}{(H(1,0))^2} \right) \]

\[ \cong A(5 \mid 0) \oplus A\left( \frac{1}{3} ((k-3)^3 + 3l^2(k-3) - k + 3) \mid \frac{1}{3} (l^3 + 3(k-3)^2 l - l) \right) \oplus A(4k - 12 \mid 4l) \]

\[ \oplus A(2 \mid 0) \oplus A(1 \mid 0) \oplus A(2 \mid 0) \oplus A(2 \mid 0) \]

\[ \cong A\left( \frac{1}{3} k(k-1)(k+1) + (k-1)(l^2 - k) + 3 \mid \frac{1}{3} l(l-1)(l+1) + l(k-1)^2 \right) \]

**Case-2:** Suppose $m = 0, n = 1$, i.e., consider $L \cong H(0,1) \oplus A(k-1 \mid l-1)$. Then
\[ M^2(L) \cong M^2(H(0,1)) \oplus M^2(A(k-1|l-1)) \oplus \left( \frac{H(0,1)}{(H(0,1))^2} \otimes \frac{H(0,1)}{(H(0,1))^2} \right) \otimes \frac{A(k-1|l-1)}{(A(k-1|l-1))^2} \]
\[ \oplus \left( \frac{A(k-1|l-1)}{(A(k-1|l-1))^2} \otimes \frac{A(k-1|l-1)}{(A(k-1|l-1))^2} \right) \otimes \frac{H(0,1)}{(H(0,1))^2} \]
\[ \cong A\left( \frac{1}{3}(k-1)^3 + 3(k-1)^2(l-1) - (k-1) \right) \frac{1}{3}(3(k-1)^2(l-1)) + (l-1)^3 - (l-1) \]
\[ \oplus (A(0|1) \otimes Z) A(k-1|l-1) \oplus (A(k-1|l-1) \otimes Z) A(0|1) \]
\[ \cong A\left( \frac{1}{3}(k-1)^3 + 3(k-1)^2(k-1) - (k-1) \right) \frac{1}{3}((l-1)^3 + 3(k-1)^2(l-1)) - (l-1) \]
\[ \oplus A(k-1|l-1) \oplus A(2(k-1)(l-1))(k-1)^2 + (l-1)^2 \]
\[ \Rightarrow A\left( k(k-1)(k+1) + (k-1)(l^2 - k) + 1 \right) \frac{1}{3}((l-1)(l+1) + l(k-1)^2 + 1) \]

**Case-3:** Suppose \( m + n \geq 2 \), then
\[ L \cong H(m,n) \oplus A(k-2m-1|l-n) \]

Now
\[ M^2(L) \cong M^2(H(m,n)) \oplus M^2(A(k-2m-1|l-n)) \oplus \left( \frac{H(m,n)}{(H(m,n))^2} \otimes \frac{H(m,n)}{(H(m,n))^2} \right) \otimes \frac{A(k-2m-1|l-n)}{(A(k-2m-1|l-n))^2} \]
\[ \oplus \left( \frac{A(k-2m-1|l-n)}{(A(k-2m-1|l-n))^2} \otimes \frac{A(k-2m-1|l-n)}{(A(k-2m-1|l-n))^2} \right) \otimes \frac{H(m,n)}{(H(m,n))^2} \]
\[ \cong A\left( \frac{1}{3}(8m^3 + 6n^2m - 2m) \right) \frac{1}{3}(n^3 + 12m^2n - n) \]
\[ \oplus A\left( \frac{1}{3}(k-2m-1)^3 + (k-2m-1)(l-n)^2 - \frac{1}{3}(k-2m-1) \right) \frac{1}{3}((4m^2 + n^2)(k-2m-1) + 4mn(l-n)) \]
\[ \oplus A(2m(k-2m-1)^2 + 2m(l-n)^2 + 2n(k-2m-1)(l-n)|n(k-2m-1)^2 + n(l-n)^2 \]
\[ + 4m(k-2m-1)(l-n)) \]
\[ = A\left( \frac{1}{3}k(k-1)(k+1) + (k-1)(l^2 - k) + 1 \right) \frac{1}{3}((l-1)(l+1) + l(k-1)^2) \]

From the above, it should be noted that \( M^2(L) \) is isomorphic to an abelian Lie superalgebra of dimension either \( \frac{1}{3}(k+l)^3 - (k+l)^2 + \frac{2}{3}(k+l) + 3 \) or \( \frac{1}{3}(k+l)^3 - (k+l)^2 + \frac{2}{3}(k+l) + 2 \) or \( \frac{1}{3}(k+l)^3 - (k+l)^2 + \frac{2}{3}(k+l) \). Using free presentation of \( H_1 \), we find the 2-multiplier of \( H_1 \), which we give in the next theorem.

**Theorem 5.15.** \( M^2(H_1) \cong A(2|2) \).

**Proof.** At first we will find the free presentation of \( H_1 \). Let \( X = \{x_1\} \cup \{x_2\} \) be a \( \mathbb{Z}_2 \)-graded set. Let \( F \) be the free Lie superalgebra over \( X \). Since \( H_1 \) is of nilpotency class two, then \( F^3 \subseteq R \), thus put
\[ R = \langle [x_2, x_2] \rangle + F^3 \]
Then we can see that \(0 \to R \to F \to H_1 \to 0\) is a free presentation of \(H_1\). Therefore,

\[
\mathcal{M}^2(H_1) \cong \frac{R \cap F^3}{[\langle R, F \rangle, F]} \cong \frac{R \cap F^3 / F^5}{[\langle R, F \rangle, F] / F^5}.
\]

Using Theorem 3.6 and Theorem 3.8

\[
\dim R \cap F^3 / F^5 = \dim F^3 / F^5 = \sum_{|\alpha|=3} SW(\alpha) + \sum_{|\alpha|=4} SW(\alpha) = (1 \mid 1) + (2 \mid 2) = (3 \mid 3).
\]

Now \([\langle R, F \rangle, F] / F^5 = < \langle [x_2, x_2], x_1 \rangle, x_1 \rangle + F^5, \langle [x_2, x_2], x_1 \rangle, x_2 \rangle + F^5 >\). Thus, \(\dim \frac{R \cap F^3}{[\langle R, F \rangle, F]} = (3 \mid 3) - (1 \mid 1) = (2 \mid 2)\).

Now we are in a position to determine the 2-nilpotent multiplier of a Lie superalgebra \(L\) whose derived subalgebra has dimension \((0 \mid 1)\).

**Theorem 5.16.** Let \(L\) be a nilpotent Lie superalgebra of dimension \((k \mid l)\) with \(\dim L^2 = (0 \mid 1)\). Then \(\mathcal{M}^2(L)\) is isomorphic to \(A(\frac{1}{3}k(k+1)(k-1) + k(l-1)^2 + 1 \mid \frac{1}{3}l(l-1)(l+1) + (l-1)(k^2 - l) + 1)\) or \(A(\frac{1}{3}k(k+1)(k-1) + k(l-1)^2 \mid \frac{1}{3}l(l-1)(l+1) + (l-1)(k^2 - l))\).

**Proof.** Using Lemma 3.5 we can decompose \(L \cong H_m \oplus A(k-m \mid l-m - 1)\). We will consider separately for different values of \(m\). We conclude the following cases using Corollary 3.11 Corollary 5.10 Theorem 5.12 and Theorem 5.15

**Case-1:** Suppose \(m = 1\), then

\[
\mathcal{M}^2(L) \cong \mathcal{M}^2(H_1) \oplus \mathcal{M}^2(A(k-1 \mid l-2)) \oplus \left(\frac{H_1}{H_1^2} \otimes \frac{H_1}{H_1^2}\right) \otimes \frac{A(k-1 \mid l-2)}{(A(k-1 \mid l-2))^2}
\]

\[
\oplus \left(\frac{A(k-1 \mid l-2)}{(A(k-1 \mid l-2))^2} \otimes \frac{A(k-1 \mid l-2)}{(A(k-1 \mid l-2))^2}\right) \otimes \frac{H_1}{H_1^2}
\]

\[
\cong \mathcal{M}^2(H_1) \oplus \mathcal{M}^2(A(k-1 \mid l-2)) \oplus (A(1 \mid 1) \otimes A(1 \mid 1) \otimes A(k-1 \mid l-2)) \oplus (A(k-1 \mid l-2)) \otimes A(1 \mid 1)
\]

\[
\cong A(2 \mid 2) \oplus A\left(\frac{1}{3}(k-1)^3 + (k-1)(l-2)^2 - \frac{(k-1)}{3}(l-2)^3 + (k-1)^2(l-2) - \frac{(l-2)}{3}\right) \oplus A(2(k-1) + 2(l-2)|2(k-1) + 2(l-2))
\]

\[
\oplus A((k-1)^2 + (l-2)^2 + 2(k-1)(l-2))(k-1)^2 + (l-2)^2 + 2(k-1)(l-2))
\]

\[
\cong A\left(\frac{1}{3}k(k+1)(k-1) + k(l-1)^2 + 1 \mid \frac{1}{3}l(l-1)(l+1) + (l-1)(k^2 - l) + 1\right).
\]
Case-2: Suppose $m > 1$, then
\[
\mathcal{M}^2(L) \cong \mathcal{M}^2(H_m) \oplus \mathcal{M}^2(A(k-m|l-m-1)) \oplus \left( \frac{H_m}{H_m^2} \otimes \frac{H_m}{H_m^2} \right) \otimes A(k-m|l-m-1) \]
\[
\oplus \left( \frac{(A(k-m|l-m-1)^2 \otimes \frac{H_m}{H_m^2}}{A(k-m|l-m-1))^2 \otimes \frac{H_m}{H_m^2}} \right)
\cong \mathcal{M}^2(H_m) \oplus \mathcal{M}^2(A(k-m|l-m-1)) \oplus (A(m|m) \otimes \frac{H_m}{H_m^2}) \otimes A(k-m|l-m-1)
\oplus (A(k-m|l-m-1)) \otimes \frac{H_m}{H_m^2}
\cong A\left(\frac{1}{3}(4m^3 - m)\right) \oplus A\left(\frac{1}{3}(k-m)^3 + (k-m)(l-m-1)^2 - \frac{(k-m)}{3}(l-m-1)^3 + (k-m)^2(l-m-1) - \frac{(l-m-1)}{3}\right) \oplus A(2m^2(k-m)
\cong A\left(\frac{1}{3}(k+1)(k+1) + (k-l-1)^2 + \frac{1}{3}(l-1)(l+1) + (l-1)(k^2-l))\right)
\]

It is worth pointing out that from the Theorem 5.14 and Theorem 5.16, if $L$ is a nilpotent Lie superalgebra of dimension $(k \mid l)$ with $\dim L^2 = (r \mid s)$, $r + s = 1$, then $\mathcal{M}^2(L)$ is isomorphic to an abelian Lie superalgebra of dimension $\frac{1}{3}(k+l)^3 - (k+l)^2 + \frac{2}{3}(k+l)$ or $\frac{1}{3}(k+l)^3 - (k+l)^2 + \frac{2}{3}(k+l) + 3$ or $\frac{1}{3}(k+l)^3 - (k+l)^2 + \frac{2}{3}(k+l)^3 + 3$. Now we are able to find an upper bound on the dimension of 2-nilpotent multiplier of $L$.

**Theorem 5.17.** Let $L = L_0 \oplus L_1$ be a nilpotent Lie superalgebra of dimension $(k \mid l)$ with $\dim L^2 = (r \mid s)$, $r + s = 1$. Then
\[
\dim \mathcal{M}^2(L) \leq \frac{1}{3}(k+l-r-s)[(k+l+2r+2s-2)(k+l-r-s-1) + 3(r+s-1)] + 3.
\]
Moreover, $\dim \mathcal{M}^2(L) \leq \frac{1}{3}(k+l)(k+l-1)(k+l-2) + 3$. Also, the equality holds in the last inequality if and only if $L \cong H(1, 0) \oplus A(k-3 \mid l)$.

**Proof.** We will use induction on $(r \mid s)$ to complete the proof. If $(r \mid s) = (1 \mid 0)$ or $(0 \mid 1)$, then by Theorem 5.14 and Theorem 5.16, the above result holds. Now assume that $\dim L^2 = (r \mid s)$ with $r \geq 1$ and $s \geq 1$. Then we get an one dimensional central ideal $W = W_0 \oplus W_1$. Suppose $\dim W = (1 \mid 0)$. As $W$ and $L/L^2$ act on each other trivially from Theorem 4.9, we have
\[
(W \otimes L/L^2) \otimes L/L^2 \cong (W \otimes \frac{L/L^2}{(L/L^2)^2}) \otimes \frac{L/L^2}{(L/L^2)^2}.
\]
Then using Theorem 4.9, $\dim \mathcal{M}^2(L) + \dim (W \cap L^3) \leq \dim \mathcal{M}^2(L/W) + \dim (W \otimes \frac{L/L^2}{(L/L^2)^2} \otimes \frac{L/L^2}{(L/L^2)^2})$. Since $\dim L/W = (k-1 \mid l)$ and $\dim (L^2/W) = (r-1 \mid s)$, then from induction hypothesis
\[
\dim \mathcal{M}^2(L/W) \leq \frac{1}{3}(k+1-r-s)[(k+1+2r+2s-5)(k+1-r-s-1) + 3(r+s-2)] + 3.
\]
Thus, $\dim \mathcal{M}^2(L) \leq \frac{1}{3}(k+l-r-s)[(k+l+2r+2s-5)(k+l-r-s-1) + 3(r+s-2)] + 3 + (k+l-r-s)^2 - 1$
\[
\leq \frac{1}{3}(k+l-r-s)[(k+l+2r+2s-5)(k+l-r-s-1) + 3(r+s-1)] + 3.
\]
By considering the odd dimension zero in the above theorem, we will get the result for Lie algebra (see [11], Theorem 2.14).

**Corollary 5.18.** Let $L$ be a nilpotent Lie algebra of dimension $k$ with $\dim L^2 = r$, $r \geq 1$. Then
\[
\dim M^2(L) \leq \frac{1}{3}(k-r)[(k+2r-2)(k-r-1) + 3(r-1)] + 3.
\]
Moreover, $\dim M^2(L) \leq \frac{1}{3}(k)(k-1)(k-2)+3$. Also, the equality holds in the last inequality if and only if $L \cong H(1) \oplus A(k-3)$.

An important consequence of the Theorem 5.17 is the following corollary.

**Corollary 5.19.** Let $L = L_0 \oplus L_1$ be a nilpotent Lie superalgebra of dimension $(k \mid l)$ with $k+l \geq 3$. If $\dim M^2(L) \leq \frac{1}{3}(k+l)(k+l+1)(k+l-1)$, then $L \cong A(k \mid l)$.

6. **2-CAPABILITY OF LIE SUPERRALGEBRA**

Let $L = L_0 \oplus L_1$ be a Lie superalgebra, then we can define the set
\[
Z_2(L) = \{x \in L \mid [x, L] \subseteq Z(L)\}.
\]
Thus $Z_2(L)$ is a subalgebra of $L$ with $Z(L) \subseteq Z_2(L)$.

**Definition 6.1.** A Lie superalgebra $L$ is said to be 2-capable if there exists a Lie superalgebra $H$ such that $L \cong H/Z_2(H)$.

A Lie superalgebra $L$ is capable [13] if there exists a Lie superalgebra $H$ such that $L \cong H/Z(H)$. Suppose $L$ is 2-capable, then
\[
L \cong H/Z_2(H) \cong \frac{H/Z(H)}{Z_2(H)/Z(H)} \cong \frac{H/Z(H)}{Z(H/Z(H))},
\]
i.e., $L$ is capable. Therefore, every 2-capable Lie superalgebra is capable. Further, we can analyze the 2-capability of a Lie superalgebra by looking at its even part.

**Theorem 6.2.** Let $L = L_0 \oplus L_1$ be a 2-capable Lie superalgebra with $L_0 \cap Z_2(L)$ is non-empty. Then $L_0$ is 2-capable Lie algebra.

**Proof.** Since $L$ is 2-capable, there exists a Lie superalgebra $H$ such that the map $\varphi : L \rightarrow H/Z_2(H)$ is an isomorphism. Define the map $\psi : L_0 \rightarrow H_0/H_0 \cap Z_2(H)$ by $\psi(x_0) = \varphi(x_0) + H_0 \cap Z_2(H)$. Now it is easy to check that $L_0 \cong H_0/H_0 \cap Z_2(H)$, and also $H_0 \cap Z_2(H)$ is non-empty as $L_0 \cap Z_2(L) \neq \emptyset$. Therefore, $L_0$ is 2-capable.

Let $L = L_0 \oplus L_1$ be a Lie superalgebra, then we define $Z^*_2(L)$ be the smallest graded ideal of $L$ such that $L/Z^*_2(L)$ is 2-capable. Thus, $Z^*(L/Z^*_2(L)) = 0$. In [19], c-capability of Lie algebra $L$ has been discussed and they have characterized $Z^*_c(L)$. Now we list some useful results on 2-capability of Lie superalgebra $L$. Evidently, the proof of the following results on $Z^*_2(L)$ are analogous to the case of Lie algebra [11, 13, 19].

**Proposition 6.3.** Let $L$ be a Lie superalgebra with the free presentation $0 \rightarrow R \rightarrow F \xrightarrow{\pi} L \rightarrow 0$, then $Z^*_c(L) = \pi(Z_c(F/\gamma_{c+1}[R, F]))$.

**Proposition 6.4.** A Lie superalgebra $L = L_0 \oplus L_1$ is 2-capable if and only if $Z^*_2(L) = 0$.

**Theorem 6.5.** Let $L$ be a Lie superalgebra with a graded ideal $K$ such that $K \subseteq Z^*_2(L)$. Then the natural Lie superalgebra homomorphism $M^2(L) \rightarrow M^2(L/K)$ is injective.

**Lemma 6.6.** (see [11], Theorem 3.3) $H(m)$ is 2-capable if and only if $m = 1$. 

Next theorems discusses the 2-capability of Heisenberg Lie superalgebras of even as well as odd center.

**Theorem 6.7.** $H(m, n)$ is 2-capable if and only if $m = 1$, $n = 0$.

**Proof.** As we have noticed, 2-capability of $L$ implies capability of $L$. Thus from Lemma 6.3 $H(m, n)$ is not 2-capable except $m = 1$, $n = 0$. Since $H(1, 0) \cong H(1)$. By Lemma 6.6 $H(1, 0)$ is 2-capable. This ends the proof. □

**Theorem 6.8.** $H_m$ is 2-capable if and only if $m = 1$.

**Proof.** Now from Lemma 3.3, $H_m$ is not 2-capable for $m \geq 2$. For $H_1$, let us take $K = Z(H_1)$. Then $H/Z(H_1) \cong A(1 \mid 1)$. Thus from Theorem 5.15 and Corollary 5.10 dim $M^2(H_1) = (2 \mid 2)$ and dim $M^2(A(1 \mid 1)) = (1 \mid 1)$. Now the result follows from Proposition 6.4 and Theorem 6.5 □

**Theorem 6.9.** $A(m \mid n)$ is 2-capable only for $m + n \geq 2$.

**Proof.** Let $K$ be any one dimensional subalgebra of $A(m \mid n)$ with dim $K = (r \mid s)$, $r + s = 1$. Then from Corollary 5.10
\[
\dim M^2(A(m\mid n)) = \left(\frac{1}{3}(m^3 + 3n^2m - m)\right) - \left(\frac{1}{3}(3m^2n + n^3 - n)\right),
\]
\[
\dim M^2(A(m-r\mid n-s)) = \left(\frac{1}{3}((m-r)^3 + 3(n-s)^2(m-r) - (m-r))\right) - \left(\frac{1}{3}(3((m-r)^2(n-s) + (n-s)^3 - (n-s))\right).
\]
Now from Theorem 6.5 $Z^*(A(m \mid n)) = 0$. Therefore, $A(m \mid n)$ is 2-capable, by acting Proposition 6.4 □

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Department of Mathematics, National Institute of Technology, Rourkela, Odisha-769028, India
*E-mail address*: rudra.padhan6@gmail.com

Department of Mathematics, National Institute of Technology, Rourkela, Odisha-769028, India
*E-mail address*: nupurnandi999@gmail.com

Department of Mathematics, National Institute of Technology, Rourkela, Odisha-769028, India
*E-mail address*: kcpati@nitrkl.ac.in