

CHEN–RUAN COHOMOLOGY OF SOME MODULI SPACES, II

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Abstract. Let $X$ be a compact connected Riemann surface of genus at least two. Let $r$ be a prime number and $ξ → X$ a holomorphic line bundle such that $r$ is not a divisor of $\deg(ξ)$. Let $M_ξ(r)$ denote the moduli space of stable vector bundles over $X$ of rank $r$ and determinant $ξ$. By $Γ$ we will denote the group of line bundles $L$ over $X$ such that $L^\otimes r$ is trivial. This group $Γ$ acts on $M_ξ(r)$ by the rule $(E, L) → E \otimes L$. We compute the Chen–Ruan cohomology of the corresponding orbifold.

1. Introduction

Let $X$ be a compact connected Riemann surface of genus $g$, with $g ≥ 2$. Let $M_η(2)$ denote the moduli space of stable vector bundles $E$ over $X$ of rank two with $\bigwedge^2 E = η$, where $η → X$ is a fixed holomorphic line bundle of degree one. This $M_η(2)$ is a smooth complex projective variety of dimension $3g - 3$. Let $Γ_2 = \text{Pic}^0(X)_2$ be the group of holomorphic line bundles $L$ over $X$ such that $L^\otimes 2$ is trivial. Such a line bundle $L$ defines an involution of $M_η(2)$ by sending any $E$ to $E \otimes L$. This defines a holomorphic action of $Γ_2$ on $M_η(2)$. In [3], we computed the Chen–Ruan cohomology algebra of the corresponding orbifold $M_η(2)/Γ_2$. (See [4], [5], [11], [1] for Chen–Ruan cohomology.) We note that $M_η(2)/Γ_2$ is the moduli space of topologically nontrivial stable $\text{PSL}(2, \mathbb{C})$–bundles over $X$.

Our aim here is to extend the above result to the moduli spaces of vector bundles of rank $r$ over $X$, where $r$ is any prime number.

Fix a holomorphic line bundle $ξ → X$ of degree $d$ such that $d$ is not a multiple of the fixed prime number $r$. Let $M_ξ(r)$ denote the moduli space of stable vector bundles $E → X$ of rank $r$ with $\bigwedge^r E = ξ$. This moduli space $M_ξ(r)$ is an irreducible smooth complex projective variety of dimension $(r^2 - 1)(g - 1)/2$. Let $Γ$ denote the group of holomorphic line bundles $L$ over $X$ such that $L^\otimes r$ is trivial. This group $Γ$ is isomorphic to $(\mathbb{Z}/r\mathbb{Z})^{\otimes 2g}$. Any line bundle $L ∈ Γ$ defines a holomorphic automorphism of $M_ξ(r)$ by sending any $E$ to $E \otimes L$. These automorphisms together define a holomorphic action of $Γ$ on $M_ξ(r)$.

The corresponding orbifold $M_ξ(r)/Γ$ is the moduli space of stable $\text{PSL}(r, \mathbb{C})$–bundles $F$ over $X$ such that the second Stiefel–Whitney class

$$w_2(F) ∈ H^2(X, \mathbb{Z}/r\mathbb{Z}) = \mathbb{Z}/r\mathbb{Z}$$

coincides with the image of $d$.

We compute the Chen–Ruan cohomology algebra of the orbifold $M_ξ(r)/Γ$.

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We note that the results of Section 2 and Section 3 are proved for all integers $r$. From Section 4 onwards we assume that $r$ is a prime number.

2. Group action and cohomology of fixed point sets

Let $X$ be a compact connected Riemann surface of genus $g$, with $g \geq 2$. Fix a holomorphic line bundle

$$\xi \to X.$$ 

Let $d \in \mathbb{Z}$ be the degree of $\xi$. Fix an integer $r \geq 2$ which is coprime to $d$. Let $\mathcal{M}_\xi(r)$ denote the moduli space of stable vector bundles $E \to X$ of rank $r$ and $\text{det} E := \bigwedge^r E = \xi$. This moduli space $\mathcal{M}_\xi(r)$ is an irreducible smooth complex projective variety of dimension $(r^2 - 1)(g - 1)$. The group of all holomorphic automorphisms of $\mathcal{M}_\xi(r)$ will be denoted by $\text{Aut}(\mathcal{M}_\xi(r))$; it is known to be a finite group.

Define

$$\Gamma := \{ L \in \text{Pic}^0(X) \mid L^{\otimes r} = \mathcal{O}_X \}.$$ 

It is a group under the operation of tensor product of line bundles. The order of $\Gamma$ is $r^{2g}$.

For any $L \in \Gamma$, let

$$\phi_L \in \text{Aut}(\mathcal{M}_\xi(r))$$

be the automorphism defined by $E \mapsto E \otimes L$. Let

$$\phi : \Gamma \to \text{Aut}(\mathcal{M}_\xi(r))$$

be the homomorphism defined by $L \mapsto \phi_L$. We will describe the fixed point set

$$\mathcal{M}_\xi(r)^L := \mathcal{M}_\xi(r)^{\phi_L} \subset \mathcal{M}_\xi(r)$$

of the automorphism $\phi_L$.

Take any nontrivial line bundle $L \in \Gamma \setminus \{\mathcal{O}_X\}$. Let $\ell$ denote the order of $L$. So $\ell$ is a divisor of $r$. Fix a nonzero holomorphic section

$$s : X \to L^{\otimes \ell}.$$ 

Define

$$Y_L := \{ z \in L \mid z^{\otimes \ell} \in \text{image}(s) \} \subset L.$$ 

Let

$$\gamma_L : Y_L \to X$$

be the restriction of the natural projection $L \to X$. Consider the action of the multiplicative group $\mathbb{C}^*$ on the total space of $L$. The action of the subgroup

$$\mu_\ell := \{ c \in \mathbb{C} \mid c^{\ell} = 1 \} \subset \mathbb{C}^*$$

preserves the complex curve $Y_L$ defined in Eq. (2.6). In fact, $Y_L$ is a principal $\mu_\ell$–bundle over $X$. Since any two nonzero holomorphic sections of $L^{\otimes \ell}$ differ by multiplication with a nonzero constant scalar, the isomorphism class of the principal $\mu_\ell$–bundle $Y_L \to X$ is independent of the choice of the section $s$. In particular, the isomorphism class of the complex curve $Y_L$ depends only on $L$.

Since the order of $L$ is exactly $\ell$, the curve $Y_L$ is irreducible.
Let $\mathcal{N}_L(d)$ denote the moduli space of stable vector bundle bundles $F \to Y_L$ of rank $r/\ell$ and degree $d$. We have a holomorphic submersion
\begin{equation}
(2.9) \quad \psi: \mathcal{N}_L(d) \to \text{Pic}^d(X)
\end{equation}
that sends any $F$ to $\mathcal{N}_L(d)\gamma_L^*F$.

Let $\text{Gal}(\gamma_L)$ be the Galois group of the covering $\gamma_L$ in Eq. (2.7); so $\text{Gal}(\gamma_L) = \mu_\ell$. For any $V \in \mathcal{N}_L(d)$, clearly
\[ \rho^*V \in \mathcal{N}_L(d) \]
for all $\rho \in \text{Gal}(\gamma_L)$. Therefore, $\text{Gal}(\gamma_L)$ acts on $\mathcal{N}_L(d)$; the action of $\rho \in \text{Gal}(\gamma_L)$ sends any $V$ to $\rho^*V$. We also note that $\psi(V) = \psi(\rho^*V)$, where $\psi$ is the projection in Eq. (2.9).

Therefore, the action of $\text{Gal}(\gamma_L)$ on $\mathcal{N}_L(d)$ preserves the subvariety
\begin{equation}
(2.10) \quad \psi^{-1}(\xi) \subset \mathcal{N}_L(d),
\end{equation}
where $\xi$ is the line bundle in Eq. (2.1).

**Lemma 2.1.** Take any nontrivial line bundle $L \in \Gamma$. The fixed point set $\mathcal{M}_\xi(r)^L$ (see Eq. (2.5)) is identified with the quotient variety $\psi^{-1}(\xi)/\text{Gal}(\gamma_L)$ (see Eq. (2.10)). The identification is defined by $F \mapsto \gamma_L^*F$.

**Proof.** Let $0_X \subset L$ be the image of the zero section of $L \to X$. Note that the pullback of $L$ to the complement $L \setminus \{0_X\}$ has a tautological trivialization. Since $Y_L \subset L \setminus \{0_X\}$, the holomorphic line bundle $\gamma_L^*L$ is canonically trivialized. This trivialization
\begin{equation}
(2.11) \quad \gamma_L^*L = \mathcal{O}_{Y_L}
\end{equation}
defines a holomorphic isomorphism
\begin{equation}
(2.12) \quad V \to V \otimes \mathcal{O}_{Y_L} = V \otimes \gamma_L^*L
\end{equation}
for any vector bundle $V \to Y_L$; the above isomorphism $V \to V \otimes \mathcal{O}_{Y_L}$ sends any $v$ to $v \otimes 1$. Using projection formula, the isomorphism in Eq. (2.12) gives an isomorphism
\begin{equation}
(2.13) \quad \gamma_L^*V \to \gamma_L^*(V \otimes (\gamma_L^*L)) = (\gamma_L^*V) \otimes L.
\end{equation}

Note that
\[ \gamma_L^*\gamma_L^*V = \bigoplus_{\rho \in \text{Gal}(\gamma_L)} \rho^*V. \]
Hence $\gamma_L^*\gamma_L^*V$ is semistable if $V$ is so. This implies that if $V$ is semistable, then the vector bundle $\gamma_L^*V$ is also semistable. Since $d = \text{degree}(\xi)$ is coprime to $r$, any semistable vector bundle over $X$ of rank $r$ and degree $d$ is stable. Therefore, $\gamma_L^*V \in \mathcal{M}_\xi(r)$ for each $V \in \psi^{-1}(\xi)$. In view of the isomorphism in Eq. (2.13) we conclude that
\[ \gamma_L^*V \in \mathcal{M}_\xi(r)^L \]
if $V \in \psi^{-1}(\xi)$.

Since $\gamma_L^*F = \gamma_L^*(\rho^*F)$, for all $\rho \in \text{Gal}(\gamma_L)$, we get a morphism
\begin{equation}
(2.14) \quad \tilde{\gamma}: \psi^{-1}(\xi)/\text{Gal}(\gamma_L) \to \mathcal{M}_\xi(r)^L
\end{equation}
defined by $V \mapsto \gamma_L^*V$. 

To construct the inverse of the map $\hat{\gamma}$, take any $E \in \mathcal{M}_\xi(r)^L$. Fix an isomorphism
\begin{equation}
(2.15) \quad \theta' : E \longrightarrow E \bigotimes L .
\end{equation}
Since $E$ is stable, it follows that $E$ is simple; this means that all automorphisms of $E$ are constant scalar multiplications. Therefore, any two isomorphisms between $E$ and $E \bigotimes L$ differ by multiplication with a constant scalar. Let
\begin{equation}
(2.16) \quad \theta \in H^0(X, \text{End}(E) \bigotimes L)
\end{equation}
be the section defined by $\theta'$ in Eq. (2.15).

Consider the pullback
\[ \gamma^*_L \theta \in H^0(Y_L, \gamma^*_L \text{End}(E) \bigotimes \gamma^*_L L) \]
of the section in Eq. (2.16). Using the canonical trivialization of the line bundle $\gamma^*_L L$ (see Eq. (2.11)), this section $\gamma^*_L \theta$ defines a holomorphic section
\begin{equation}
(2.17) \quad \theta_0 \in H^0(Y_L, \gamma^*_L \text{End}(E)) = H^0(Y_L, \text{End}(\gamma^*_L E)) .
\end{equation}
Since $Y_L$ is irreducible (this was noted earlier) it does not admit any nonconstant holomorphic functions, hence the characteristic polynomial of $\theta_0(x)$ is independent of the point $x \in Y_L$. Therefore, the set of eigenvalues of $\theta_0(x)$ does not change as $x$ moves over $Y_L$. Similarly, the multiplicity of each eigenvalue of $\theta_0(x)$ is also independent of $x \in Y_L$.

Therefore, for each eigenvalue $\lambda$ of $\theta_0(x)$, we have the associated generalized eigenbundle
\begin{equation}
(2.18) \quad \gamma^*_L E \supset E^\lambda \longrightarrow Y_L
\end{equation}
whose fiber over any $y \in Y_L$ is the generalized eigenspace of $\theta_0(y) \in \text{End}(\gamma^*_L E)_y$ for the eigenvalue $\lambda$.

Recall that $Y_L$ was constructed by fixing a section $s$ of $L \otimes \ell$ (see Eq. (2.6)); it was also noted that the isomorphism class of the covering $\gamma_L$ is independent of the choice of $s$. We choose $s$ such that the $\ell$-fold composition
\begin{equation}
(2.19) \quad (\theta')^\ell := \underbrace{\theta' \circ \cdots \circ \theta'}_{\ell\text{-times}} : E \longrightarrow E \bigotimes L \otimes \ell
\end{equation}
coincides with $\text{Id}_E \bigotimes s$, where $\theta'$ is the homomorphism in Eq. (2.15). Since the vector bundle $E$ is simple, there is exactly one such section $s$. In fact,
\[ s = \text{trace}((\theta')^\ell)/r .\]
We construct $Y_L$ using this $s$.

With this construction of $Y_L$, we have
\[ (\theta_0)^\ell := \underbrace{\theta_0 \circ \cdots \circ \theta_0}_{\ell\text{-times}} = \text{Id}_{\gamma^*_L E} ,\]
where $\theta_0$ is constructed in Eq. (2.17). Consequently, the set of eigenvalues of $\theta_0(x)$ is contained in $\mu_\ell$ (defined in Eq. (2.5)); we noted earlier that the set of eigenvalues of $\theta_0(x)$ along with their multiplicities are independent of $x \in Y_L$.

Since $Y_L$ is a principal $\mu_\ell$-bundle over $X$, the Galois group $\text{Gal}(\gamma_L)$ is identified with $\mu_\ell$. Note that $\text{Gal}(\gamma_L)$ has a natural action on the vector bundle $\gamma^*_L E$ which is a lift of
the action of \( \text{Gal}(\gamma_L) \) on \( Y_L \). Examining the construction of \( \theta_0 \) (see Eq. (2.17)) from \( \theta' \), it follows that the action of any

\[
\rho \in \text{Gal}(\gamma_L) = \mu_\ell
\]

on \( \gamma_L^*E \) takes the eigenbundle \( E^\lambda \) (see Eq. (2.18)) to the eigenbundle \( E^{\lambda \rho} \). This immediately implies that each element of \( \mu_\ell \) is an eigenvalue of \( \theta_0(x) \), and the multiplicities of the eigenvalues of \( \theta_0(x) \) coincide. Hence, the multiplicity of each eigenvalue of \( \theta_0(x) \) is \( r/\ell \).

Consider (2.20)

\[
E^1 \longrightarrow Y_L,
\]

which is the eigenbundle for the eigenvalue \( 1 \in \mu_\ell \). Define

\[
\tilde{E}^1 := \bigoplus_{\rho \in \text{Gal}(\gamma_L)} \rho^*E^1.
\]

There is a natural action of \( \text{Gal}(\gamma_L) = \mu_\ell \) on \( \tilde{E}^1 \). Since the action of any \( \rho \in \mu_\ell \) on \( \gamma_L^*E \) takes the eigenbundle \( E^\lambda \) to the eigenbundle \( E^{\lambda \rho} \), it follows immediately that we have a \( \text{Gal}(\gamma_L) \)-equivariant identification

(2.21)

\[
\gamma_L^*E = \tilde{E}^1 := \bigoplus_{\rho \in \text{Gal}(\gamma_L)} \rho^*E^1.
\]

In view of this \( \text{Gal}(\gamma_L) \)-equivariant isomorphism we conclude that the composition

\[
E \longrightarrow \gamma_L*\gamma_L^*E \longrightarrow \gamma_L*\tilde{E}^1 = \gamma_L*( \bigoplus_{\rho \in \text{Gal}(\gamma_L)} \rho^*E^1 ) \longrightarrow \gamma_L*E^1
\]

is an isomorphism; here

\[
E \longrightarrow \gamma_L*\gamma_L^*E
\]

is the natural homomorphism and \( \bigoplus_{\rho \in \text{Gal}(\gamma_L)} \rho^*E^1 \longrightarrow E^1 \) is the projection to the direct summand corresponding to \( \rho = 1 \).

Since \( E \) is stable, and \( \gamma_L*E^1 = E \), it follows that \( E^1 \) is stable. Indeed, if a subbundle \( F \subset E^1 \) violates the stability condition for \( E^1 \), then the subbundle

\[
\gamma_L*F \subset \gamma_L*E^1 = E
\]

violates the stability for \( E \). Let

(2.22)

\[
\Phi : \mathcal{M}_\xi(r)^L \longrightarrow \psi^{-1}(\xi)/\text{Gal}(\gamma_L)
\]

be the morphism that sends any \( E \) to \( E^1 \). Since \( E \) is isomorphic to \( \gamma_L*E^1 \), it follows that

\[
\hat{\gamma} \circ \Phi = \text{Id}_{\mathcal{M}_\xi(r)^L},
\]

where \( \hat{\gamma} \) is constructed in Eq. (2.14).

Therefore, to complete the proof of the lemma it suffices to show that for \( F, F' \in \psi^{-1}(\xi) \), if

(2.23)

\[
\gamma_L*F = \gamma_L*F',
\]

then

(2.24)

\[
F' = \tau^*F
\]

holds for some \( \tau \in \text{Gal}(\gamma_L) \).
Take \( F, F' \in \psi^{-1}(\xi) \) such that Eq. (2.23) holds. Note that

\[
\bigoplus_{\tau, \eta \in \text{Gal}(\gamma L)} \text{Hom}(\eta^* F, \tau^* F') = \bigoplus_{\tau \in \text{Gal}(\gamma L)} \text{Hom}(F, \tau^* F') \otimes = \text{Hom}(\gamma L^* \gamma L F, \gamma L^* \gamma L F').
\]

(2.25)

Since \( \gamma L F = \gamma L F' \),

\[
H^0(X, \text{Hom}(\gamma L F, \gamma L F')) \neq 0.
\]

Consequently,

\[
H^0(Y_L, \text{Hom}(\gamma L^* \gamma L F, \gamma L^* \gamma L F')) \neq 0.
\]

Therefore, from Eq. (2.25) we conclude that there is some \( \tau \in \text{Gal}(\gamma L) \) such that

\[
H^0(Y_L, \text{Hom}(F, \tau^* F')) \neq 0.
\]

Since both \( F \) and \( \tau^* F' \) are stable vector bundles with

\[
\frac{\text{degree}(F)}{\text{rank}(F)} = \frac{\text{degree}(\tau^* F')}{\text{rank}(\tau^* F')},
\]

from Eq. (2.26) we conclude that the vector bundle \( F \) is isomorphic to \( \tau^* F' \). In other words, Eq. (2.24) holds. This completes the proof of the lemma. \( \square \)

In Lemma 3.5 we will show that the action of \( \text{Gal}(\gamma L) \) on \( \psi^{-1}(\xi) \) is free.

In the next section we will investigate the action of the isotropy subgroups on the tangent bundle for the action of \( \Gamma \) of \( M_\xi(r) \).

3. Action on the tangent bundle

Fix any \( L \in \Gamma \setminus \{O_X\} \). Take any

\[
E \in \mathcal{M}_\xi(r)^L
\]

(see Eq. (2.5)). Fix an isomorphism \( \theta' \) as in Eq. (2.15). Let \( s \) be the unique section of \( L \otimes \ell \) such that the homomorphism \( (\theta')^\ell \) in Eq. (2.19) coincides with \( \text{Id}_E \otimes s \). As we noted earlier, \( s = \text{trace}((\theta')^\ell)/r \). Let

\[
\gamma L : Y_L \longrightarrow X
\]

be the covering \( Y_L \) constructed as in Eq. (2.6) using this section \( s \). Recall that \( \text{Gal}(\gamma L) = \mu_\ell \), where \( \ell \) is the order of \( L \).

Let

\[
F := E^1 \longrightarrow Y_L
\]

be the vector bundle constructed in Eq. (2.20). The decomposition of \( \gamma L^* E \) in Eq. (2.21) yields the following decomposition of the pullback \( \gamma L^* \text{End}(E) = \text{End}(\gamma L^* E) \):

\[
\gamma L^* \text{End}(E) = \bigoplus_{t \in \text{Gal}(\gamma L)} \bigoplus_{u \in \text{Gal}(\gamma L)} \text{Hom}(u^* F, (ut)^* F) = \bigoplus_{t \in \text{Gal}(\gamma L)} \bigoplus_{u \in \text{Gal}(\gamma L)} (u^* F)^\vee \otimes (ut)^* F
\]

(3.2)

(see also Eq. (2.25)).

Note that for each \( t \in \text{Gal}(\gamma L) \), the vector bundle

\[
E_t := \bigoplus_{u \in \text{Gal}(\gamma L)} \text{Hom}(u^* F, (ut)^* F) \longrightarrow Y_L
\]

(3.3)
in Eq. (3.2) is left invariant by the natural action of \( \text{Gal}(\gamma_L) \) on the vector bundle \( \gamma_L \text{End}(E) \). Therefore, \( \mathcal{E}_t \) descends to \( X \). Let

\[(3.4) \quad \mathcal{F}_t \longrightarrow X \]

be the descent of \( \mathcal{E}_t \). So

\[(3.5) \quad \mathcal{F}_t = \gamma_L \text{Hom}(F, t^* F), \]

and

\[\gamma_L^* \mathcal{F}_t = \mathcal{E}_t.\]

The decomposition

\[\gamma_L^* \text{End}(E) = \bigoplus_{t \in \text{Gal}(\gamma_L)} \mathcal{E}_t \]

in Eq. (3.2) is preserved by the action of \( \text{Gal}(\gamma_L) \). Therefore, this decomposition descends to the following decomposition:

\[(3.6) \quad \text{End}(E) = \bigoplus_{t \in \text{Gal}(\gamma_L)} \mathcal{F}_t.\]

We will now describe the differential \( d\phi_L(E) \), where \( \phi_L \) is the automorphism in Eq. (2.3), and \( E \in \mathcal{M}_\xi(r)^L \).

Recall that we fixed an isomorphism \( \theta' : E \longrightarrow E \otimes L \) as in Eq. (2.15). This isomorphism \( \theta' \) induces an isomorphism of the endomorphism bundle \( \text{End}(E) \) with \( \text{End}(E \otimes L) \). Since

\[\text{End}(E) = E \otimes E^\vee = (E \otimes L) \otimes (E \otimes L)^\vee = \text{End}(E \otimes L),\]

the isomorphism of \( \text{End}(E) \) with \( \text{End}(E \otimes L) \) defined by \( \theta' \) gives an automorphism of \( \text{End}(E) \). Let

\[(3.7) \quad \hat{\theta} : \text{End}(E) \longrightarrow \text{End}(E) \]

be this automorphism constructed from \( \theta' \). Since any two isomorphisms between \( E \) and \( E \otimes L \) differ by a constant scalar, the automorphism \( \hat{\theta} \) is independent of the choice of \( \theta' \).

Let

\[\text{ad}(E) \subset \text{End}(E)\]

be the holomorphic subbundle of corank one given by the sheaf of trace zero endomorphisms. Clearly,

\[\hat{\theta}(\text{ad}(E)) \subset \text{ad}(E) .\]

We note that \( T_E \mathcal{M}_\xi(r) = H^1(X, \text{ad}(E)) \); here \( T \) denotes the holomorphic tangent bundle. Let

\[(3.8) \quad \overline{\theta} : H^1(X, \text{ad}(E)) \longrightarrow H^1(X, \text{ad}(E)) \]

be the automorphism induced by \( \hat{\theta} \).

From the construction of \( \phi_L \) it follows that the differential

\[(3.9) \quad d\phi_L(E) : T_E \mathcal{M}_\xi(r) \longrightarrow T_E \mathcal{M}_\xi(r) \]

coincides with \( \overline{\theta} \) constructed in Eq. (3.8).
In the proof of Lemma 2.1 we observed that the pullback to $Y_L$ of the isomorphism $\theta'$ coincides with the isomorphism

$$\gamma^*_L E \longrightarrow \gamma^*_L E \otimes \gamma^*_L L$$

obtained by tensoring with the tautological section of $\gamma^*_L L$ (see Eq. (2.11)). Consider the automorphism $\widehat{\theta}$ in Eq. (3.7) induced by $\theta'$. From the above description of $\theta'$ it follows immediately that $\widehat{\theta}$ acts on the subbundle $E_t$ (see Eq. (3.3)) as multiplication by $t \in \mu_\ell \subset \mathbb{C}^*$.

**Lemma 3.1.** Take any

$$t \in \mu_\ell \setminus \{1\}. \quad (3.10)$$

($\ell$ is the order of $L$; see Eq. (2.8) for $\mu_\ell$). Consider $F_t$ constructed in Eq. (3.4) (see also Eq. (3.6)). Then

$$F_t \subset \text{ad}(E). \quad (3.11)$$

**Proof.** We first note that $\dim H^0(X, \text{End}(E)) = 1$ because $E$ is stable. On the other hand,

$$\dim H^0(X, \gamma_L^* \text{Hom}(F, \text{Id}^* F)) \geq 1,$$

where $\text{Id} = 1 \in \text{Gal}(\gamma_L) = \mu_\ell$ is the identity element. Therefore, from Eq. (3.5) and Eq. (3.6),

$$H^0(X, F_t) = 0, \quad (3.12)$$

where $t$ is the element in Eq. (3.10).

**Remark 3.2.** Since the vector bundle $E$ is stable, it admits a unique Hermitian–Einstein connection. The connection on $\text{End}(E)$ induced by a Hermitian–Einstein connection on $E$ is also Hermitian–Einstein. Therefore, the vector bundle $\text{End}(E)$ is polystable of degree zero.

Continuing with the proof of Lemma 3.1 since $\text{End}(E)$ is polystable of degree zero, and $F_t$ is a direct summand of $\text{End}(E)$ (see Eq. (3.5) and Eq. (3.6)), it follows that $F_t$ is also polystable of degree zero. Consider the trace map

$$\text{End}(E) \supset F_t \xrightarrow{\text{trace}} O_X.$$

Since $F_t$ is polystable of degree zero, if the above trace homomorphism on $F_t$ is nonzero, then $O_X$ is a direct summand of $F_t$. If $O_X$ is a direct summand of $F_t$, then Eq. (3.12) contradicts. Hence the trace map on $F_t$ vanishes identically. This implies that Eq. (3.11) holds. This completes the proof of the lemma.

Consider the tautological trivialization of the line bundle $\gamma^*_L L$ (see Eq. (2.11)). The action of any element $t \in \text{Gal}(\gamma_L) = \mu_\ell$ takes the tautological section of $\gamma^*_L L$ to $t^{-1}$–times the tautological section. Using this it follows immediately that $t$ acts on $E_t$ in Eq. (3.3) as multiplication by $t$. Now from the construction of the automorphism $\widehat{\theta}$ in Eq. (3.7) it follows that $\widehat{\theta}$ acts on $F_t$ as multiplication by $t$. In view of this and Eq. (3.11), we conclude that for all $t \in \mu_\ell \setminus \{1\}$, the automorphism $\overline{\theta}$ in Eq. (3.11) acts on the subspace

$$H^1(X, F_t) \subset H^1(X, \text{ad}(E)) = T_E M_\xi(r) \quad (3.13)$$
as multiplication by \( t \).

We will calculate the dimension of the subspace \( H^1(X, \mathcal{F}_t) \) in Eq. (3.13), where \( t \in \mu_\ell \setminus \{1\} \). Note that

\[
\text{rank}(\mathcal{F}_t) = \frac{r^2}{\ell} \quad \text{and} \quad \text{degree}(\mathcal{F}_t) = 0.
\]

Since \( H^0(X, \text{ad}(E)) = 0 \), from Eq. (3.11) we have

\[
H^0(X, \mathcal{F}_t) = 0.
\]

(3.14)

Therefore, the Riemann–Roch theorem says that

\[
\dim H^1(X, \mathcal{F}_t) = \frac{r^2 (g - 1)}{\ell}.
\]

Hence, we have proved the following lemma:

**Lemma 3.3.** Take any \( t \in \mu_\ell \setminus \{1\} \). Then \( \mathcal{F}_t \subset \text{ad}(E) \), and the automorphism \( \overline{\theta} \) in Eq. (3.8) acts on the subspace

\[
H^1(X, \mathcal{F}_t) \subset H^1(X, \text{ad}(E)) = T_E \mathcal{M}_\xi(r)
\]

as multiplication by \( t \). Also,

\[
\dim H^1(X, \mathcal{F}_t) = \frac{r^2 (g - 1)}{\ell}.
\]

Now we set \( t = 1 \in \mu_\ell = \text{Gal}(\gamma_L) \). The automorphism \( \overline{\theta} \) of End\((E)\) acts trivially on the subbundle \( \mathcal{F}_1 \subset \text{End}(E) \). Therefore, \( \overline{\theta} \) acts trivially on the subspace

\[
H^1(X, \mathcal{F}_1) \bigcap H^1(X, \text{ad}(E)) \subset H^1(X, \text{ad}(E)) = T_E \mathcal{M}_\xi(r).
\]

From Eq. (3.6) and Eq. (3.11),

\[
\text{ad}(E) = (\mathcal{F}_1 \bigcap \text{ad}(E)) \bigoplus \bigoplus_{\tau \in \mu_\ell \setminus \{1\}} \mathcal{F}_\tau.
\]

Consequently, the dimension of the subspace of \( H^1(X, \text{ad}(E)) \) on which \( \overline{\theta} \) in Eq. (3.8) acts as the identity map is

\[
\dim H^1(X, \text{ad}(E)) - \sum_{t \in \mu_\ell \setminus \{1\}} \dim H^1(X, \mathcal{F}_t)
\]

\[
= (r^2 - 1)(g - 1) - \frac{(\ell - 1)r^2(g - 1)}{\ell} = \frac{r^2(g - 1)}{\ell} - g + 1.
\]

Combining this with Lemma 3.3 we get the following proposition.

**Proposition 3.4.** The eigenvalues of the differential

\[
d\phi_L(E) : T_E \mathcal{M}_\xi(r) \longrightarrow T_E \mathcal{M}_\xi(r)
\]

are \( \mu_\ell \). For any \( t \in \mu_\ell \setminus \{1\} \), the multiplicity of the eigenvalue \( t \) is \( r^2(g - 1)/\ell \). The multiplicity of the eigenvalue \( 1 \) is \( 1 - g + r^2(g - 1)/\ell \).

**Lemma 3.5.** The action of \( \text{Gal}(\gamma_L) \) on \( \psi^{-1}(\xi) \) in Lemma 2.1 is free.
Proof. In Eq. (3.14) we saw that $H^0(X, \mathcal{F}_t) = 0$ for all $t \in \text{Gal}(\gamma_L) \setminus \{1\}$. Now, from Eq. (3.5) it follows that

$$H^0(X, \mathcal{F}_t) = H^0(Y_L, \text{Hom}(F, t^*F)) = 0$$

for all $t \in \text{Gal}(\gamma_L) \setminus \{1\}$. Since $F = E^1$ (see Eq. (3.1)), this implies that for each $t \in \text{Gal}(\gamma_L) \setminus \{1\}$, the vector bundle $E^1$ is not isomorphic to $t^*E^1$. This completes the proof of the lemma. \hfill \Box

4. Intersection of fixed point sets

Henceforth, we will always assume that $r$ is a prime number. As before, $d = \text{degree}(\xi)$ is assumed to be coprime to $r$.

We note that the group $\Gamma$ (see Eq. (2.2)) is a vector space over the field $\mathbb{Z}/r\mathbb{Z}$. For any $J, L \in \Gamma$, and any $n \in \mathbb{Z}/r\mathbb{Z}$,

$$J + L := J \otimes L \quad \text{and} \quad nL := L^\otimes n.$$ 

For a line bundle $L_0$ over $X$, by $L_0^\otimes 0$ we will denote the trivial line bundle $\mathcal{O}_X$.

Take two linearly independent elements

(4.1) \quad $J, L \in \Gamma$.

We note that the line bundles $J^\otimes i \otimes L^\otimes j, i, j \in [1, r-1]$, are all distinct and nontrivial. Take any

$$E \in \mathcal{M}_\xi(r)^J \bigcap \mathcal{M}_\xi(r)^L$$

(see Eq. (2.5)), where $J$ and $L$ are, as in Eq. (4.1), linearly independent. Fix isomorphisms

(4.2) \quad $\theta_1 : E \longrightarrow E \otimes J$ \quad and \quad $\theta_2 : E \longrightarrow E \otimes L$.

The isomorphism $\theta_1$ (respectively, $\theta_2$) gives an inclusion of the line bundle $J^\vee = J^\otimes (r-1)$ (respectively, $L^\vee = L^\otimes (r-1)$) in $\text{End}(E) = E \otimes E^\vee$. Note that since $r$ is a prime number, any $J^\otimes i$ (respectively, $L^\otimes i$) is a tensor power of $J^\vee$ (respectively, $L^\vee$). Using the associative algebra structure of the fibers of $\text{End}(E)$ defined by composition of homomorphisms, the above homomorphisms

$$J^\vee \longrightarrow \text{End}(E) \quad \text{and} \quad L^\vee \longrightarrow \text{End}(E)$$

give an inclusion of $(J^\vee)^\otimes i \otimes (L^\vee)^\otimes j$ in $\text{End}(E)$ for all $i, j \geq 0$. Therefore, we get an inclusion of $J^\otimes i \otimes L^\otimes j$ in $\text{End}(E)$ for all $i, j \geq 0$. The line bundle $J^\otimes 0 \otimes L^\otimes 0 = \mathcal{O}_X$ sits inside $\text{End}(E)$ as scalar multiplications.

Let

(4.3) \quad $\Theta : \bigoplus_{i,j=0}^{r-1} J^\otimes i \otimes L^\otimes j \longrightarrow \text{End}(E)$

be the homomorphism constructed as above.

Lemma 4.1. The homomorphism $\Theta$ in Eq. (4.3) is a holomorphic isomorphism of vector bundles.
Proof. Take any proper subset
\[ S \subset [0, r-1] \times [0, r-1], \]
and also take any \((i_0, j_0) \in [0, r-1] \times [0, r-1] \setminus S\) in the complement. Assume that the restriction of the homomorphism \(\Theta\) in Eq. (4.3) to
\[ W_S := \bigoplus_{(i,j) \in S} J^{\otimes i} \otimes L^{\otimes j} \]
is injective. Note that \(W_S\) is polystable of degree zero, and in Remark 3.2 we observed that the vector bundle \(\text{End}(E)\) is polystable. Since both \(W_S\) and \(\text{End}(E)\) are polystable vector bundles of degree zero, it follows that \(\Theta(W_S)\) is a subbundle of \(\text{End}(E)\). Furthermore, there is a holomorphic subbundle
\[ W'_S \subset \text{End}(E) \]
which is a direct summand of \(\Theta(W_S)\). In particular,
\[ \text{End}(E) = \Theta(W_S) \bigoplus W'_S. \tag{4.4} \]
We have
\[ H^0(X, \text{Hom}(J^{\otimes i_0} \otimes L^{\otimes j_0}, W_S)) = 0 \]
because \(J^{\otimes i_0} \otimes L^{\otimes j_0}\) is distinct from \(J^{\otimes i} \otimes L^{\otimes j}\) for all \((i, j) \in S\). Therefore, the projection of \(J^{\otimes i_0} \otimes L^{\otimes j_0}\) to the direct summand \(\Theta(W_S) \subset \text{End}(E)\) in Eq. (4.4) vanishes identically. This implies that
\[ \Theta(J^{\otimes i_0} \otimes L^{\otimes j_0}) \subset W'_S. \]
Hence \(\Theta\) makes \(W_S \bigoplus (J^{\otimes i_0} \otimes L^{\otimes j_0})\) a subbundle of \(\text{End}(E)\).

Now, using induction we conclude that \(\Theta\) in Eq. (4.3) is a pointwise injective homomorphism of vector bundles. Since
\[ r^2 = \text{rank}(\text{End}(E)) = \text{rank}(\bigoplus_{i,j=0}^{r-1} J^{\otimes i} \otimes L^{\otimes j}), \]
it follows that \(\Theta\) is an isomorphism of vector bundles. \(\square\)

Take two vector bundles \(E, F \in M_\xi(r)\). If the vector bundle \(\text{End}(E)\) is isomorphic to \(\text{End}(F)\), then at least one of the following two statements is valid:

1. There is a line bundle \(\zeta \longrightarrow X\) such that
\[ E = F \bigotimes \zeta. \tag{4.5} \]
2. There is a line bundle \(\zeta \longrightarrow X\) such that
\[ E = F^\vee \bigotimes \zeta. \tag{4.6} \]

Since \(\wedge^r E = \wedge^r F\), taking the \(r\)–th exterior power of both sides of Eq. (4.5) it follows that \(\zeta^{\otimes r} = \mathcal{O}_X\).

Similarly, since \(\wedge^r E = \wedge^r F = \xi\), taking the \(r\)–th exterior power of both sides of Eq. (4.6) it follows that \(\zeta^{\otimes r} = \xi^{\otimes 2}\). Recall that \(r\) is a prime number, and degree(\(\xi\)) is not a multiple of \(r\). Hence, if Eq. (4.6) holds, then \(r = 2\).
Therefore, Lemma 4.1 has the following corollary:

**Corollary 4.2.** Take two linearly independent elements \( J, L \in \Gamma \). If
\[
E, F \in M_\xi(r)^J \cap M_\xi(r)^L,
\]
then at least one of the following two statements is valid:

1. There is a line bundle \( \zeta \in \Gamma \) such that \( F \otimes \zeta \) is holomorphically isomorphic to \( E \).
2. There is a line bundle \( \zeta \in \Gamma \) such that \( F^\vee \otimes \zeta \) is holomorphically isomorphic to \( E \).

If the second statement holds, then \( r = 2 \).

**Remark 4.3.** Take a line bundle \( L \in \Gamma \), and also take a vector bundle \( E \in M_\xi(r)^L \). Let
\[
f_0 : E \rightarrow E \otimes L
\]
be a holomorphic isomorphism.

- Take any \( \zeta \in \Gamma \). Then
\[
f_0 \otimes \text{Id}_\zeta : E \otimes L \otimes \zeta = E \otimes \zeta \otimes L
\]
is also an isomorphism. Hence \( E \otimes \zeta \in M_\xi(r)^L \). Therefore, if
\[
E \in M_\xi(r)^J \cap M_\xi(r)^L,
\]
and \( \zeta \in \Gamma \), then
\[
E \otimes \zeta \in M_\xi(r)^J \cap M_\xi(r)^L.
\]

- Consider the isomorphism
\[
(4.7) \quad f_0^\vee : E^\vee \otimes L^\vee = (E \otimes L)^\vee \rightarrow E^\vee.
\]
For any holomorphic line bundle \( \eta \rightarrow X \), tensoring the isomorphism in Eq. (4.7) by
\[
\text{Id}_{L \otimes \eta} : L \otimes \eta \rightarrow L \otimes \eta
\]
we get an isomorphism
\[
E^\vee \otimes \eta \rightarrow E^\vee \otimes L \otimes \eta = E^\vee \otimes \eta \otimes L.
\]
Therefore, if \( \wedge^r(E^\vee \otimes \eta) = \xi \), then \( E^\vee \otimes \eta \in M_\xi(r)^L \). Now using the first part of the remark we conclude the following: if
\[
E \in M_\xi(r)^J \cap M_\xi(r)^L,
\]
and \( \wedge^r(E^\vee \otimes \eta) = \xi \), then
\[
E^\vee \otimes \eta \otimes \zeta \in M_\xi(r)^J \cap M_\xi(r)^L
\]
for all \( \zeta \in \Gamma \). \( \Box \)
Remark 4.4. Assume that $r = 2$. Take any $E \in \mathcal{M}_\xi(2)$. Since $\bigwedge^2 E = \xi$, contracting both sides by $E^\vee$ it follows that $E = E^\vee \otimes \xi$. □

As before, take two linearly independent elements $J, L \in \Gamma$. Consider the coverings

$$\gamma_J : Y_J \to X \quad \text{and} \quad \gamma_L : Y_L \to X$$

constructed as in Eq. (2.7) from $J$ and $L$ respectively. Since the Galois groups of both $\gamma_J$ and $\gamma_L$ are $\mathbb{Z}/r\mathbb{Z}$, we get surjective homomorphisms

$$\rho_J : H_1(X, \mathbb{Z}) \to \mathbb{Z}/r\mathbb{Z} \quad \text{and} \quad \rho_L : H_1(X, \mathbb{Z}) \to \mathbb{Z}/r\mathbb{Z}. \tag{4.8}$$

Using $\rho_J$ and $\rho_L$, we will construct a homomorphism from $H_1(X, \mathbb{Z})$ to $\text{PGL}(r, \mathbb{C})$.

Let $D$ be the $r \times r$ diagonal matrix whose $(j,j)$–th entry is $\exp(2\pi\sqrt{-1}j/r)$. Let

$$\rho_1' : \mathbb{Z}/r\mathbb{Z} \to \text{PGL}(r, \mathbb{C}) \tag{4.10}$$

be the homomorphism that sends any $\tau$ to the image of $D^\tau$ in $\text{PGL}(r, \mathbb{C})$.

Let

$$R \in \text{GL}(r, \mathbb{C})$$

be the matrix defined by $R(e_i) = e_{i+1}$ for all $i \in [1, r-1]$ and $R(e_r) = e_1$, where $\{e_j\}_{j=1}^r$ is the standard basis of $\mathbb{C}^r$. Let

$$\rho_2' : \mathbb{Z}/r\mathbb{Z} \to \text{PGL}(r, \mathbb{C}) \tag{4.11}$$

be the homomorphism that sends any $\tau$ to the image of $R^\tau$ in $\text{PGL}(r, \mathbb{C})$. Note that the image of $R$, in $\text{PGL}(r, \mathbb{C})$, commutes with the image of the above defined diagonal matrix $D$. Consequently, the images of the two homomorphisms $\rho_1'$ and $\rho_2'$ commute.

Let

$$\Psi : H_1(X, \mathbb{Z}) \to \text{PGL}(r, \mathbb{C}) \tag{4.12}$$

be the homomorphism defined by $\gamma \mapsto \rho_1'(\rho_J(\gamma))\rho_2'(\rho_L(\gamma))$, where $\rho_J$ (respectively, $\rho_L$) is defined in Eq. (4.8) (respectively, Eq. (4.9)), and $\rho_1'$ (respectively, $\rho_2'$) is defined in Eq. (4.10) (respectively, Eq. (4.11)). Since the images of $\rho_1'$ and $\rho_2'$ commute, the map $\Psi$ is indeed a homomorphism.

The group $H_1(X, \mathbb{Z})$ is a quotient of the fundamental group of $X$, and $\text{PGL}(r, \mathbb{C})$ has the standard action on $\mathbb{C}P^{r-1}$. Hence any homomorphism from $H_1(X, \mathbb{Z})$ to $\text{PGL}(r, \mathbb{C})$ defines a flat projective bundle over $X$ of relative dimension $r - 1$. Let

$$\mathcal{P}_{J,L} \to X \tag{4.13}$$

be the flat projective bundle given by the homomorphism $\Psi$ in Eq. (4.12).

Let $\text{GL}(r, \mathbb{C})$ (respectively, $\text{PGL}(r, \mathbb{C})$) be the sheaf of locally constant functions on $X$ with values in $\text{GL}(r, \mathbb{C})$ (respectively, $\text{PGL}(r, \mathbb{C})$). We have the short exact sequence of sheaves

$$e \to \mathbb{Z}/r\mathbb{Z} \to \text{GL}(r, \mathbb{C}) \to \text{PGL}(r, \mathbb{C}) \to e \tag{4.14}$$
on $X$, where $\mathbb{Z}/r\mathbb{Z}$ is the sheaf of locally constant functions with values in $\mathbb{Z}/r\mathbb{Z}$; for notational convenience, we will denote the sheaf $\mathbb{Z}/r\mathbb{Z}$ by $\mathbb{Z}/r\mathbb{Z}$. Let

$$\chi : H^1(X, \underline{\text{PGL}(r, \mathbb{C})}) \rightarrow H^2(X, \mathbb{Z}/r\mathbb{Z}) = \mathbb{Z}/r\mathbb{Z}$$

be the homomorphism in the exact sequence of cohomologies associated to the short exact sequence in Eq. (4.14). The projective bundle $\mathcal{P}_{J,L}$ in Eq. (4.13) defines an element

$$c(\mathcal{P}_{J,L}) \in H^1(X, \underline{\text{PGL}(r, \mathbb{C})}).$$

Let

$$A_{J,L} := \chi(c(\mathcal{P}_{J,L})) \in H^2(X, \mathbb{Z}/r\mathbb{Z}) = \mathbb{Z}/r\mathbb{Z}$$

be the cohomology class, where $\chi$ is the homomorphism in Eq. (4.15).

A homomorphism $H_1(X, \mathbb{Z}) \rightarrow \mathbb{Z}/r\mathbb{Z}$ defines a cohomology class in $H^1(X, \mathbb{Z}/r\mathbb{Z})$.

Let

$$\overline{\rho}_J, \overline{\rho}_L \in H^1(X, \mathbb{Z}/r\mathbb{Z})$$

be the cohomology classes corresponding to the homomorphisms $\rho_J$ and $\rho_L$ constructed in Eq. (4.8) and Eq. (4.9) respectively. Let

$$\overline{\rho}_J \cup \overline{\rho}_L \in H^2(X, \mathbb{Z}/r\mathbb{Z})$$

be the cup product. It can be checked that

$$A_{J,L} = \overline{\rho}_J \cup \overline{\rho}_L \in H^2(X, \mathbb{Z}/r\mathbb{Z}),$$

where $A_{J,L}$ is constructed in Eq. (4.16).

Given any holomorphic projective bundle $\mathbb{P}$ over $X$, there is a holomorphic vector bundle $\mathcal{V} \rightarrow X$ such that $\mathbb{P}$ is isomorphic to the projective bundle over $X$ parametrizing the lines in the fibers of $\mathcal{V}$. Let

$$\mathcal{W} \rightarrow X$$

be a holomorphic vector bundle of rank $r$ such that the holomorphic projective bundle over $X$ parametrizing lines in the fibers of $\mathcal{W}$ is holomorphically isomorphic to the projective bundle $\mathcal{P}_{J,L}$ in Eq. (4.13).

**Proposition 4.5.** The vector bundle $\mathcal{W}$ in Eq. (4.19) is stable.

The image of $\text{deg}(\mathcal{W}) \in \mathbb{Z}$ in $\mathbb{Z}/r\mathbb{Z}$ coincides with $A_{J,L}$ in Eq. (4.16).

Also,

$$\text{End}(\mathcal{W}) = \bigoplus_{i,j=0}^{r-1} J^{\otimes i} \otimes L^{\otimes j}.$$  

In particular, $\mathcal{W}$ is isomorphic to both $\mathcal{W} \otimes J$ and $\mathcal{W} \otimes L$.

**Proof.** Consider the homomorphism $\Psi$ in Eq. (4.12). Its image is a finite subgroup of $\text{PGL}(r, \mathbb{C})$, and hence $\Psi(H_1(X, \mathbb{Z}))$ lies inside a maximal compact subgroup of $\text{PGL}(r, \mathbb{C})$. Also, the subgroup

$$\Psi(H_1(X, \mathbb{Z})) \subset \text{PGL}(r, \mathbb{C})$$
is irreducible in following sense. Consider the standard action of \( \mathrm{PGL}(r, \mathbb{C}) \) on the projective space \( \mathbb{CP}^{r-1} \) that parametrizes all lines in \( \mathbb{C}^r \). The action of the subgroup \( \Psi(H_1(X, \mathbb{Z})) \) leaves invariant no proper linear subspace of \( \mathbb{CP}^{r-1} \).

Since \( \Psi(H_1(X, \mathbb{Z})) \) is an irreducible subgroup of \( \mathrm{PGL}(r, \mathbb{C}) \) lying inside a maximal compact subgroup, it follows that the principal \( \mathrm{PGL}(r, \mathbb{C}) \)–bundle over \( X \) defined by the projective bundle \( \mathcal{P}_{J,L} \) (see Eq. (4.13)) is stable [10, p. 146, Theorem 7.1]. Consequently, the corresponding vector bundle \( \mathcal{W} \) in Eq. (4.19) is stable.

From the definition of \( A_{J,L} \) in Eq. (4.16) it follows that

\[
\text{degree}(\mathcal{W}) \equiv A_{J,L} \mod r.
\]

To construct the isomorphism in Eq. (4.20), consider the homomorphism

\[
h : \mathbb{Z}/r\mathbb{Z} \longrightarrow \mathbb{C}^*
\]
defined by \( n \mapsto \exp(2\pi \sqrt{-1}n/r) \). We note that \( h \circ \rho_J \) is a character of \( H_1(X, \mathbb{Z}) \), where \( \rho_J \) is the homomorphism in Eq. (4.8). Any character of \( H_1(X, \mathbb{Z}) \) defines a flat complex line bundle over \( X \) (since \( H_1(X, \mathbb{Z}) \) is a quotient the fundamental group of \( X \), a character of \( H_1(X, \mathbb{Z}) \) is also a character of the fundamental group, and hence any character of \( H_1(X, \mathbb{Z}) \) defines a flat complex line bundle over \( X \)). The holomorphic line bundle corresponding to the character \( h \circ \rho_J \) is itself. Similarly, the holomorphic line bundle over \( X \) corresponding to the character \( h \circ \rho_L \) of \( H_1(X, \mathbb{Z}) \), where \( \rho_L \) is the homomorphism in Eq. (4.9), is identified with \( L \).

Let \( \mathfrak{m}(J) \) (respectively, \( \mathfrak{m}(L) \)) be the one–dimensional complex \( H_1(X, \mathbb{Z}) \)–module defined by the character \( h \circ \rho_J \) (respectively, \( h \circ \rho_L \)) of \( H_1(X, \mathbb{Z}) \). The holomorphic line bundle over \( X \) associated to the \( H_1(X, \mathbb{Z}) \)–module \( \mathfrak{m}(J) \) (respectively, \( \mathfrak{m}(L) \)) coincides with the holomorphic line bundle corresponding to the character \( h \circ \rho_J \) (respectively, \( h \circ \rho_L \)) of \( H_1(X, \mathbb{Z}) \). Therefore, the holomorphic line bundle over \( X \) associated to the \( H_1(X, \mathbb{Z}) \)–module \( \mathfrak{m}(J) \) (respectively, \( \mathfrak{m}(L) \)) coincides with \( J \) (respectively, \( L \)).

On the other hand, consider the adjoint action of \( \mathrm{PGL}(r, \mathbb{C}) \) on the vector space \( \mathcal{M}(r, \mathbb{C}) \) of \( r \times r \)–matrices with entries in \( \mathbb{C} \). Using this action, and the homomorphism \( \Psi \) constructed in Eq. (4.12), the vector space \( \mathcal{M}(r, \mathbb{C}) \) becomes a \( H_1(X, \mathbb{Z}) \)–module. This \( H_1(X, \mathbb{Z}) \)–module \( \mathcal{M}(r, \mathbb{C}) \) has the following decomposition:

\[
\mathcal{M}(r, \mathbb{C}) = \bigoplus_{i=0}^{r-1} \bigoplus_{j=0}^{r-1} \mathfrak{m}(J)^{\otimes i} \otimes \mathfrak{m}(L)^{\otimes j} \tag{4.21}
\]

where \( \mathfrak{m}(J) \) and \( \mathfrak{m}(L) \) are the one–dimensional \( H_1(X, \mathbb{Z}) \)–modules defined above.

The holomorphic vector bundle over \( X \) associated to the above mentioned \( H_1(X, \mathbb{Z}) \)–module \( \mathcal{M}(r, \mathbb{C}) \) is identified with the vector bundle \( \text{End}(\mathcal{W}) \), where \( \mathcal{W} \) is the vector bundle in Eq. (4.19). On the other hand, we noted earlier that the holomorphic line bundle over \( X \) associated to the \( H_1(X, \mathbb{Z}) \)–module \( \mathfrak{m}(J) \) (respectively, \( \mathfrak{m}(L) \)) is \( J \) (respectively, \( L \)). Therefore, fixing an isomorphism as in Eq. (4.21) we obtain an isomorphism as in Eq. (4.20).

A nonzero holomorphic homomorphism

\[
p : \text{End}(\mathcal{W}) \longrightarrow J \tag{4.22}
\]
gives a nonzero holomorphic section of $\text{End}(W)^\vee \otimes J = \text{End}(W) \otimes J$. Hence $p$ gives a nonzero holomorphic homomorphism of vector bundles

$$W \to W \otimes J.$$ 

Since the vector bundle $W$ is stable, and $\deg(J) = 0$, any nonzero homomorphism $W \to W \otimes J$ must be an isomorphism.

The decomposition in Eq. (4.20) ensures that a nonzero homomorphism $p$ as in Eq. (4.22) exists. Hence we conclude that $W$ is isomorphic to $W \otimes J$. Similarly, the vector bundle $W \otimes L$ is isomorphic to $W$. This completes the proof of the proposition. □

Recall that $\mathcal{M}_\xi(r)^L$ is the fixed point set defined in Eq. (2.5).

**Lemma 4.6.** Take two linearly independent elements $J, L \in \Gamma$. Let $\overline{\rho}_J, \overline{\rho}_L \in H^1(X, \mathbb{Z}/r\mathbb{Z})$ be the corresponding cohomology classes constructed as in Eq. (4.17). Then

$$\mathcal{M}_\xi(r)^J \bigcap \mathcal{M}_\xi(r)^L = \emptyset$$

if and only if the cup product $\overline{\rho}_J \cup \overline{\rho}_L \in H^2(X, \mathbb{Z}/r\mathbb{Z})$ vanishes.

If $\mathcal{M}_\xi(r)^J \bigcap \mathcal{M}_\xi(r)^L \neq \emptyset$, then

$$\#(\mathcal{M}_\xi(r)^J \bigcap \mathcal{M}_\xi(r)^L) = r^{2g-2}.$$ 

**Proof.** First assume that

$$\overline{\rho}_J \cup \overline{\rho}_L \equiv d := \deg(\xi) \mod r.$$ 

Then from the second part of Proposition 4.5 we know that

$$\deg(W) = ar + d$$

for some integer $a$, where $W$ is the vector bundle in Eq. (4.19). Since $W$ is also stable (see the first part of Proposition 4.5), there is a holomorphic line bundle $L_0 \to X$ of degree $-a$ such that

$$W_0 := W \otimes L_0 \in \mathcal{M}_\xi(r).$$

The vector bundle $W \to X$ is isomorphic to both $W \otimes J$ and $W \otimes L$ (see Proposition 4.5). Hence $W_0$ is isomorphic to both $W_0 \otimes J$ and $W_0 \otimes L$. Indeed, for any isomorphism $f_0 : W \to W \otimes J$, the homomorphism

$$f_0 \otimes \text{Id}_{L_0} : W_0 := W \otimes L_0 \to W \otimes J \otimes L_0 = W \otimes L_0 \otimes J = W_0 \otimes J$$

is an isomorphism; similarly, $W_0$ is isomorphic to $W_0 \otimes L$. In other words, we have

$$W_0 \in \mathcal{M}_\xi(r)^J \bigcap \mathcal{M}_\xi(r)^L.$$ 

Now assume that

$$\delta := \overline{\rho}_J \cup \overline{\rho}_L \neq 0,$$ 

(4.26)
where $\rho_J$ and $\rho_L$ are as in Eq. (4.24). Fix a positive integer $n_0$ such that
\begin{equation}
(4.27) \quad d \equiv n_0 \delta \pmod{r}.
\end{equation}
We note that such an integer $n_0$ exists because $r$ is a prime number, $d \not\equiv 0 \pmod{r}$, and $\delta \neq 0$. Replace the line bundle $J$ by
\begin{equation}
(4.28) \quad J_0 = J^{n_0},
\end{equation}
and keep the line bundle $L$ unchanged. From Eq. (4.26) and Eq. (4.27),
\begin{equation}
(4.29) \quad \rho_{J_0} \cup \rho_L \equiv d \pmod{r},
\end{equation}
where $\rho_{J_0} \in H^1(X, \mathbb{Z}/r\mathbb{Z})$ is the cohomology class constructed as in Eq. (4.17) for the line bundle $J_0$ in Eq. (4.28).

We noted above that from Eq. (4.29) it follows that $M_\xi(r)_{J_0} \cap M_\xi(r)L \neq \emptyset$ (see Eq. (4.25)). Take any
\begin{equation}
(4.30) \quad V \in M_\xi(r)_{J_0} \cap M_\xi(r)L.
\end{equation}
Since $V$ is isomorphic to $V \otimes J_0$, it follows that $V$ is isomorphic to $V \otimes J_0^{\otimes n}$ for all $n$. Indeed, if
\[
f : V \rightarrow V \otimes J_0
\]
is an isomorphism, then the composition
\[
\begin{array}{c}
V \xrightarrow{f} V \otimes J_0 \xrightarrow{f \otimes \text{Id}_{J_0}} \cdots \xrightarrow{f \otimes \text{Id}_{J_0}} V \otimes J_0^{\otimes n}
\end{array}
\]
is an isomorphism for all $n \geq 1$.

Take a positive integer $m_0$ such that $m_0n_0 \equiv 1 \pmod{r}$, where $n_0$ is the integer in Eq. (4.27). Therefore, the line bundle $J_0^{\otimes m_0}$ is isomorphic to $J$ (see Eq. (4.28)). Since $V$ is isomorphic to $V \otimes J_0^{\otimes m_0} = V \otimes J$, we conclude that
\[
V \in M_\xi(r)J.
\]
Hence from Eq. (4.30),
\[
V \in M_\xi(r)^J \cap M_\xi(r)^L.
\]
Therefore, we have proved that
\[
M_\xi(r)^J \cap M_\xi(r)^L \neq \emptyset
\]
if $\overline{\rho}_J \cup \overline{\rho}_L \neq 0$.

To prove the converse, assume that
\begin{equation}
(4.31) \quad \overline{\rho}_J \cup \overline{\rho}_L = 0.
\end{equation}
If $E \rightarrow X$ is a holomorphic vector bundle such that
\[
\text{End}(E) = \bigoplus_{i,j=0}^{r-1} J^{\otimes i} \otimes L^{\otimes j},
\]
then from Eq. (4.31) it follows that degree$(E) \equiv 0 \text{ mod } r$. Since $d := \text{degree}(\xi)$ is coprime to $r$, this implies that
\[ \mathcal{M}_\xi(r)^J \cap \mathcal{M}_\xi(r)^L = \emptyset. \]

Therefore,
\[ \mathcal{M}_\xi(r)^J \cap \mathcal{M}_\xi(r)^L = \emptyset \]
if and only if $\bar{\rho}_J \cup \bar{\rho}_L = 0$.

To prove the last statement in the lemma, assume that
\[ \mathcal{M}_\xi(r)^J \cap \mathcal{M}_\xi(r)^L \neq \emptyset. \]
Fix a vector bundle $E \in \mathcal{M}_\xi(r)^J \cap \mathcal{M}_\xi(r)^L$. Then from Corollary 4.2 and Remark 4.3 we know that $\mathcal{M}_\xi(r)^J \cap \mathcal{M}_\xi(r)^L$ is the orbit of $E$ under the action of $\Gamma$ on $\mathcal{M}_\xi(r)$. The isotropy subgroup $\Gamma_E \subset \Gamma$ for $E$ is generated by $J$ and $L$. Now, Eq. (4.23) holds because $\#\Gamma = r^2 g$, and the order of the subgroup of $\Gamma$ generated by $J$ and $L$ is $r^2$. This completes the proof of the lemma.

\[ \square \]

5. The cohomologies

The cohomology groups $H^i(\mathcal{M}_\xi(r), \mathbb{Q})$, $i \geq 0$, are computed in [7], [2]. The cohomology algebra $\bigoplus_{i \geq 0} H^i(\mathcal{M}_\xi(r), \mathbb{Q})$ is computed by Kirwan in [8].

Consider the action of $\Gamma$ on $\bigoplus_{i \geq 0} H^i(\mathcal{M}_\xi(r), \mathbb{Q})$ given by the action of $\Gamma$ on $\mathcal{M}_\xi(r)$. It is known that this action is trivial [7, p. 220, Theorem 1]. Therefore, the cohomology algebra $\bigoplus_{i \geq 0} H^i(\mathcal{M}_\xi(r)/\Gamma, \mathbb{Q})$.

Take any nontrivial line bundle $L \in \Gamma \setminus \{O_X\}$. Since $r$ is a prime number, the order of $L$ is $r$. Let
\[ \gamma_L : Y_L \longrightarrow X \]
be the Galois covering of degree $r$ constructed as in Eq. (2.7). Let
\[ \text{Prym}_\xi(\gamma_L) \subset \text{Pic}^d(Y_L) \]
be the Prym variety parametrizing all line bundles $\eta \longrightarrow Y_L$ such that
\[ \det \gamma_L \eta := \bigwedge^r (\gamma_L \eta) = \xi. \]

Note that the Galois group $\text{Gal}(\gamma_L)$ acts on $\text{Prym}_\xi(\gamma_L)$. The action of $\tau \in \text{Gal}(\gamma_L)$ on $\text{Prym}_\xi(\gamma_L)$ sends any line bundle $\eta$ to $\tau \eta$. From Lemma 2.1 we know that
\[ \text{Prym}_\xi(\gamma_L) / \text{Gal}(\gamma_L) = \mathcal{M}_\xi(r)^L. \]

We also know that the action of $\text{Gal}(\gamma_L)$ on $\text{Prym}_\xi(\gamma_L)$ is free (see Lemma 3.3).

The group $\Gamma$ acts on $\text{Prym}_\xi(\gamma_L)$. The action of any $\zeta \in \Gamma$ is given by the map $\eta \mapsto \eta \otimes \gamma_L^* \zeta$; note that by the projection formula,
\[ \bigwedge^r \gamma_L^* (\eta \otimes \gamma_L^* \zeta) = (\bigwedge^r \gamma_L^* \eta) \otimes \zeta^{\otimes r} = \bigwedge^r \gamma_L^* \eta, \]
hence $\eta \otimes \gamma_L^* \zeta \in \text{Prym}_\xi(\gamma_L)$ if $\eta \in \text{Prym}_\xi(\gamma_L)$.

It is straightforward to check that the actions of $\Gamma$ and $\text{Gal}(\gamma_L)$ on $\text{Prym}_\xi(\gamma_L)$ commute.
For any \( i \geq 0 \), consider the action of \( \Gamma \) on \( H^1(\text{Prym}_{\xi}(\gamma_L), \mathbb{Q}) \) given by the above action of \( \Gamma \) on \( \text{Prym}_{\xi}(\gamma_L) \). Since the action of \( \Gamma \) on \( \text{Prym}_{\xi}(\gamma_L) \) is through translations, it follows immediately that this action of \( \Gamma \) on \( H^1(\text{Prym}_{\xi}(\gamma_L), \mathbb{Q}) \) is the trivial one.

We will now recall the topological model of a cyclic covering of \( X \) of degree \( r \).

The isomorphism classes of unramified cyclic coverings

\[
Y \longrightarrow X
\]
of degree \( r \) with \( Y \) connected are parametrized by the complement

\[
H^1(X, \mathbb{Z}/r\mathbb{Z})_0 := H^1(X, \mathbb{Z}/r\mathbb{Z}) \setminus \{0\},
\]
because the space of all the surjective homomorphisms

\[
\pi_1(X, x_0) \longrightarrow \mathbb{Z}/r\mathbb{Z}
\]
is parametrized by \( H^1(X, \mathbb{Z}/r\mathbb{Z})_0 \). Let \( \text{Diff}^+(X) \) denote the group of all orientation preserving diffeomorphisms of \( X \). This group \( \text{Diff}^+(X) \) has a natural action on \( H^1(X, \mathbb{Z}/r\mathbb{Z}) \). The action of \( \text{Diff}^+(X) \) on \( H^1(X, \mathbb{Z}/r\mathbb{Z})_0 \) can be shown to be transitive. To prove this, let \( \text{Aut}(H^1(X, \mathbb{Z})) \) denote the group of automorphisms of \( H^1(X, \mathbb{Z}) \) preserving the cup product; so \( \text{Aut}(H^1(X, \mathbb{Z})) \) can be identified with the symplectic group \( \text{Sp}(2g, \mathbb{Z}) \) after fixing a symplectic basis of \( H^1(X, \mathbb{Z}) \). It is known that the natural homomorphism

\[
\text{Diff}^+(X) \longrightarrow \text{Aut}(H^1(X, \mathbb{Z}))
\]
is surjective \cite[p. 114]{top}. On the other hand, it is easy to check that the natural action of \( \text{Aut}(H^1(X, \mathbb{Z})) \) on \( H^1(X, \mathbb{Z}/r\mathbb{Z})_0 \) is transitive. Hence, we conclude that the action of \( \text{Diff}^+(X) \) on \( H^1(X, \mathbb{Z}/r\mathbb{Z})_0 \) is transitive.

Therefore, given two unramified cyclic coverings

\[
\pi_1 : Y_1 \longrightarrow X \quad \text{and} \quad \pi_2 : Y_2 \longrightarrow X
\]
of degree \( r \) with both \( Y_1 \) and \( Y_2 \) connected, there is a diffeomorphism

\[
\varphi : X \longrightarrow X
\]
such that \( \varphi \) pulls back the covering \( \pi_2 \) to \( \pi_1 \).

One example of a cyclic covering of degree \( r \) is the following:

Let \( X_0 \) be a compact surface of genus one, and let \( X_1 \) be a compact surface of genus \( g - 1 \). Take an orientation preserving free action of \( \mathbb{Z}/r\mathbb{Z} \) on \( X_0 \). Let \( X'_0 \) be the complement of \( r \) open disks in \( X_0 \) such that \( X'_0 \) is preserved by the action of \( \mathbb{Z}/r\mathbb{Z} \) on \( X_0 \). Let \( X'_1 \) be the complement of a closed disk in \( X_1 \). Now attach \( r \) copies of \( X'_1 \) to \( X'_0 \) along the \( r \) boundary circles of \( X'_0 \). The resulting compact connected surface of genus \( r(g - 1) + 1 \) will be denoted by \( Y \). The action of \( \mathbb{Z}/r\mathbb{Z} \) on \( X'_0 \) and the permutation action of \( \mathbb{Z}/r\mathbb{Z} \) on the \( r \) copies of \( X'_1 \) together define an action of \( \mathbb{Z}/r\mathbb{Z} \) on \( Y \). This action is clearly free, and the quotient is of genus \( g \). Up to diffeomorphisms of \( Y/(\mathbb{Z}/r\mathbb{Z}) \), all connected unramified cyclic coverings of \( Y/(\mathbb{Z}/r\mathbb{Z}) \) of degree \( r \) coincide with the covering

\[
Y \longrightarrow Y/(\mathbb{Z}/r\mathbb{Z}) .
\]
In particular, the topological model of the covering \( \gamma_L \) in Eq. (2.7) is the covering in Eq. (5.3).

Using this model of \( \gamma_L \) it follows that \( \text{Prym}_{\xi}(\gamma_L) \) (defined in Eq. (5.1)) is topologically isomorphic to a real torus of dimension \( 2(r - 1)(g - 1) \). The action of the Galois group
Gal(γ_L) on H^1(Prym_ξ(γ_L), C) can also be calculated using the above topological model of the covering γ_L.

To calculate the action of Gal(γ_L) on H^1(Prym_ξ(γ_L), C), consider the group μ_r defined in Eq. (2.8), which is identified with Gal(γ_L). Let

\[ \hat{\mu}_r := \text{Hom}(\mu_r, \mathbb{C}^*) \]

be the group of characters of μ_r. It is a cyclic group of order r generated by the tautological character of μ_r defined by the inclusion of μ_r in C^*.

Each nontrivial element of \( \hat{\mu}_r \) is an eigen-character of μ_r for the action of Gal(γ_L) = μ_r on H^1(Prym_ξ(γ_L), C), and furthermore, the multiplicity of each eigen-character is 2(g - 1).

To prove the above assertion, consider the covering in Eq. (5.3). We noted earlier that it is the topological model of the covering γ_L. Let

(5.4) \[ A : H^1(X_1, \mathbb{C})^{\oplus r} \rightarrow H^1(X_1, \mathbb{C}) \]

be the homomorphism defined by

\[ (c_1, \cdots, c_r) \mapsto \sum_{j=1}^r c_j ; \]

the surface X_1 is the one used in the construction of the covering in Eq. (5.3). The complex vector space H^1(Prym_ξ(γ_L), C) is identified with the kernel of the homomorphism

\[ H^1(\text{Pic}^d(Y_L), \mathbb{C}) = H^1(Y_L, \mathbb{C}) \rightarrow H^1(X, \mathbb{C}) \mathbb{C} \]

that sends any \( c \in H^1(Y_L, \mathbb{C}) \) to the class in

\[ H^1(X, \mathbb{C}) \mathbb{C} = H^1(Y_L, \mathbb{C})^{\text{Gal}(\gamma_L)} \]

defined by \( \sum_{\tau \in \text{Gal}(\gamma_L)} \tau^* c. \) We have

\[ H^1(\text{Pic}^d(Y_L), \mathbb{C}) = H^1(X_1, \mathbb{C})^{\oplus r} \bigoplus H^1(X_0, \mathbb{C}) \]

(\( X_0 \) is the surface of genus one in the construction of the covering in Eq. (5.3)), and

\[ H^1(\text{Prym}_\xi(\gamma_L), \mathbb{C}) = \ker(A) \subset H^1(X_1, \mathbb{C})^{\oplus r} \subset H^1(\text{Pic}^d(Y_L), \mathbb{C}) , \]

where A is the homomorphism in Eq. (5.4). The action of

\[ \exp(2\pi \sqrt{-1}/r) \in \mu_r = \text{Gal}(\gamma_L) \]
on H^1(\text{Pic}^d(Y_L), \mathbb{C}) = H^1(X_1, \mathbb{C})^{\oplus r} \bigoplus H^1(X_0, \mathbb{C}) is given by the automorphism defined by

\[ (c_1, \cdots, c_r; d) \mapsto (c_2, \cdots, c_r, c_1; d) \in H^1(X_1, \mathbb{C})^{\oplus r} \bigoplus H^1(X_0, \mathbb{C}) . \]

Also, \( \dim H^1(X_1, \mathbb{C}) = 2(g - 1). \) It is now easy to see that each nontrivial character of μ_r is an eigen-character of multiplicity 2(g - 1) for the action of Gal(γ_L) = μ_r on H^1(Prym_ξ(γ_L), C).

The cohomology algebra \( \bigoplus_{i \geq 0} H^i(\text{Prym}_\xi(\gamma_L), \mathbb{C}) \) is identified with the exterior algebra \( \bigoplus_{i \geq 0} \wedge^i H^i(\text{Prym}_\xi(\gamma_L), \mathbb{C}) \). Therefore, from the above description of the action of Gal(γ_L) on H^1(Prym_ξ(γ_L), C) we obtain a description of the action of Gal(γ_L) on the cohomology algebra \( \bigoplus_{i \geq 0} H^i(\text{Prym}_\xi(\gamma_L), \mathbb{C}) \).
5.1. **The Chen–Ruan cohomology.** The case of $r = 2$ was already considered in \[3\]. Here we will assume that $r \geq 3$.

The $i$–th Chen–Ruan cohomology group is the degree shifted direct sum

\[ H^i_{CR}(\mathcal{M}_\xi(r)/\Gamma, \mathbb{Q}) = \bigoplus_{L \in \Gamma} H^{i-2(L)}(\mathcal{M}_\xi(r)^L/\Gamma, \mathbb{Q}) . \]

The degree shifting number $\iota(L)$ is obtained from Proposition \[3.4\]

\[ \iota(L) = \begin{cases} 0 & \text{if } L = \mathcal{O}_X \\ \frac{1}{2}(r^2 - r)(g - 1) & \text{otherwise.} \end{cases} \]

As in \[3\], we will denote $H^{*+2q}(\mathcal{M}_\xi(r)^L/\Gamma, \mathbb{Q})$ by $A^*(L)$. Then for $\alpha_1 \in A^p(L_1)$ and $\alpha_2 \in A^q(L_2)$, the Chen–Ruan product

\[ \alpha_1 \cup \alpha_2 \in A^{p+q}(L_1 \otimes L_2) \]

is defined via the relation

\[ \langle \alpha_1 \cup \alpha_2 , \alpha_3 \rangle = \int_{S/\Gamma} e_1^* \alpha_1 \wedge e_2^* \alpha_2 \wedge e_3^* \alpha_3 \wedge c_{\text{top}} \mathcal{F} \]

for all $\alpha_3 \in A^*(L_3)$ such that $L_1 \otimes L_2 \otimes L_3 = \mathcal{O}_X$, where $\langle , \rangle$ is the nondegenerate bilinear Poincaré pairing for Chen–Ruan cohomology (see \[4, 3 (6.20)\]),

\[ S := \mathcal{M}_\xi(r)^{L_1} \bigcap \mathcal{M}_\xi(r)^{L_2} , \]

and $e_i : S/\Gamma \longrightarrow \mathcal{M}_\xi(r)^{L_i}/\Gamma$ are the canonical inclusions, and $\mathcal{F}$ is a complex $\Gamma$–bundle over $S$, or equivalently, an orbifold vector bundle over $S/\Gamma$, of rank

\[ \text{rank}(\mathcal{F}) = \dim_{\mathbb{C}} S - \dim_{\mathbb{C}} \mathcal{M}_\xi(r) + \sum_{j=1}^3 \iota(L_j) . \]

From Lemma \[4.6\] it follows that $S$ is empty or zero dimensional if $L_1$ and $L_2$ are linearly independent. If $S$ is empty, then the corresponding Chen–Ruan products are automatically zero. Even otherwise, since $L_3$ is also nontrivial if $L_1$ and $L_2$ are linearly independent, using Eq. \[5.6\] we get that

\[ \text{rank}(\mathcal{F}) = \frac{1}{2}(r - 1)(r - 2)(g - 1) > 0 = \dim_{\mathbb{C}} S \]

(recall that $r \geq 3$). Hence, $c_{\text{top}} \mathcal{F} = 0$, and again all the corresponding Chen–Ruan products are zero.

If $L_1$ and $L_2$ are linearly dependent then we have the following three cases:

1. If $L_1$ and $L_2$ are both trivial then the Chen–Ruan products are just the usual products in the singular cohomology of $\mathcal{M}_\xi(r)/\Gamma$.

2. If all the three line bundles are nontrivial, meaning $L_i = L^{\otimes k_i}$ for some $L \neq \mathcal{O}_X$ and $1 \leq k_i \leq (r - 1)$, $i \in \{1, 2, 3\}$, then we have $S = \mathcal{M}_\xi(r)^L$, and

\[ \text{rank}(\mathcal{F}) = \frac{1}{2}(r^2 - r)(g - 1) > (r - 1)(g - 1) = \dim_{\mathbb{C}} S \]

(recall that $r \geq 3$). Hence $c_{\text{top}} \mathcal{F} = 0$, and all the corresponding Chen–Ruan products are zero.
(3) If exactly two of the three $L_i$’s are nontrivial, then we must have $L_{i_1} = L$, $L_{i_2} = L^\otimes(r-1)$ and $L_{i_3} = \mathcal{O}_X$ for some $L \neq \mathcal{O}_X$ and some permutation $\{i_1, i_2, i_3\}$ of $\{1, 2, 3\}$. The calculations of the Chen–Ruan products are analogous to the cases a), b) and c) in Section 6.1 of [3].

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