Stability of Self-Similar Spherical Accretion

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Abstract. Spherical accretion flows are simple enough for analytical study, by solution of the corresponding fluid dynamic equations. The solutions of stationary spherical flow are due to Bondi. The questions of the choice of a physical solution and of stability have been widely discussed. The answer to these questions is very dependent on the problem of boundary conditions, which vary according to whether the accretor is a compact object or a black hole. We introduce a particular, simple form of stationary spherical flow, namely, self-similar Bondi flow, as a case with physical interest in which analytic solutions for perturbations can be found. With suitable no matter-flux-perturbation boundary conditions, we will show that acoustic modes are stable in time and have no spatial instability at $r = 0$. Furthermore, their evolution eventually becomes ergodic-like and shows no trace of instability or of acquiring any remarkable pattern.

RELATIVISTIC POTENTIAL FLUID FLOW

We consider the adiabatic downfall of a perfect fluid onto a compact spherical body or a non-rotating black hole. In particular, we intend to analyse the stability of simple spherical, stationary flows (Bondi flows) [1]. General stability arguments have been given by Garlick [2] and Moncrief [3], but there are no exact solutions for perturbations, except in the WKB approximation. So we further restrict ourselves to self-similar flows, as a case amenable to analytic treatment and with physical interest.

We begin with a summary of the theory of relativistic potential fluid flow and its linear perturbations, introducing the sound metric. Then we proceed to the Newtonian limit, sufficient for our purposes. In this limit, we obtain the self-similar Bondi flows and the perturbation equations. These equations can be solved in terms of Bessel functions. We study their initial and boundary problems, and so we draw conclusions on their stability.

Let us consider the perfect fluid equations, namely, the energy-momentum tensor $T^\mu{}^\nu = (p + \rho)u^\mu u^\nu + pg^\mu{}^\nu$, $u^\mu u_\mu = -1$. and thermodynamic equations $h = (p + \rho)/n$, $dp = n(dh - T ds)$, where $n$ is the number density and $h$ is the enthalpy per particle (we have $u^\mu s_\mu = 0$). The equations of motion are the conservation equations $T^\mu{}^\nu ;_\nu = 0$, $(nu^\mu) ;_\mu = 0$. Let us further consider isentropic solutions and potential flow [3], such that $\omega^\mu{}^\nu = (hu_\mu)_;^\nu - (hu_\nu)_;^\mu$ fulfills $P_\alpha{}^\mu P_\beta{}^\nu \omega^\mu{}^\nu = 0$, where $P_\mu{}^\nu = g_\mu{}^\nu + u_\mu u_\nu$. Then we have $\omega^\mu{}^\nu = 0 \Rightarrow hu_\mu = \psi^\mu{};_\mu$ for some function $\psi$. The equations of motion become

$$\left(\frac{n}{h}\psi^\mu{};^\mu\right)_;^\mu = 0,$$

where $n$ is expressed in terms of $h$ by the equation of state: $n = \frac{\partial p}{\partial h}$, and $h^2 = \psi^\mu{};^\mu$. 

Linear Perturbations and Sound Metric

The linear perturbations of the equation for the scalar potential give the following scalar wave equation:

\[ \nabla^\mu \delta \psi_{,\mu} = 0, \]

where \( \nabla_\mu \) is the covariant derivative with respect to the sound metric

\[ G_{\mu\nu} = \frac{n}{h} c \left[ g_{\mu\nu} - \left( 1 - \frac{c_s^2}{c^2} \right) u_{\mu} u_{\nu} \right], \quad \frac{c_s^2}{c^2} = \frac{\partial p}{\partial \rho}{|_s}. \]

Therefore, causality in sound propagation is determined by \( g_{\mu\nu} \) (characteristics, etc), and the symmetries of \( G_{\mu\nu} \) are the ones common to \( g_{\mu\nu} \) and \( u^\mu \).

From the linear perturbation equation:

\[ \nabla_\nu \mathcal{T}^{\nu}_{\mu} = 0, \quad \mathcal{T}^{\nu}_{\mu} = \frac{1}{2} \left[ \delta \psi_{,\mu} \delta \psi_{,\nu} G^{\nu\kappa} - \frac{1}{2} \delta^{\nu}_{\mu} G_{,\sigma}^{\kappa} \delta \psi_{,\kappa} \delta \psi_{,\sigma} \right], \]

where \( \mathcal{T}^{\nu}_{\mu} \) is the energy-momentum tensor of scalar waves. We consider stationary flow \( \Rightarrow \) conserved energy:

\[ E = -2 \int d^3 x (- \det G)^{1/2} \mathcal{T}^{t}_{t} = \frac{1}{2} \int d^3 x (- \det G)^{1/2} \left[ - G^{tt} (\delta \psi_{,t})^2 + G^{ij} \delta \psi_{,i} \delta \psi_{,j} \right]. \]

In addition, we are interested in spherical flow \( \Rightarrow \) conserved angular momentum.

NEWTONIAN SELF-SIMILAR SPHERICAL FLOW

When \( r \gg R \geq GM/c^2 \) and \( |\mu| \ll 1 \), potential flow boils down to \( v = \nabla \psi \) and \[4\]

\[ \left( \frac{\partial}{\partial t} + v \cdot \nabla + \frac{c_s^2}{\rho} \nabla \left( \frac{\rho v}{c_s^2} \right) \right) \left( \frac{\partial}{\partial t} + v \cdot \nabla \right) \delta \psi = \frac{c_s^2}{\rho} \nabla \cdot (\rho v \delta \psi). \]

This is an equation of non-homogeneous wave propagation. The law of conservation of acoustic energy is simply \( \frac{\partial E}{\partial t} = \nabla \cdot \mathbf{W} \), where

\[ E = \frac{\rho}{2 c_s^2} \left[ (\partial_t \delta \psi)^2 + c_s^2 (\nabla \delta \psi)^2 - (v \cdot \nabla \delta \psi)^2 \right], \]

and the acoustic energy flux current is

\[ \mathbf{W} = \frac{\rho}{c_s^2} \partial_t \delta \psi \left[ v \partial_i \delta \psi - c_s^2 \nabla \delta \psi + v \cdot (v \cdot \nabla \delta \psi) \right] = - \partial_i \delta \psi \cdot \mathbf{j}, \]

with \( \mathbf{j} = \rho \mathbf{v} \) the matter flux current.
Basic flow: Bondi solutions

For stationary spherical flow we have the radial mass conservation and Euler equations
\[
\frac{d}{dr}(r^2 \rho v) = 0, \quad v \frac{dv}{dr} = -\frac{1}{\rho} \frac{dp}{dr} - \frac{GM}{r^2}.
\]
Integrating the first one, \(4\pi r^2 \rho v = \mathcal{K}_1\) (constant). We further use the polytropic equation of state \(p = \mathcal{K}_2 \rho^n\). We define the accretion radius \(\mathcal{R} = GM/(cs)^2\), and hence the non-dimensional variables
\[
x = \frac{r}{\mathcal{R}}, \quad y = \frac{\rho}{\rho_\infty}, \quad u = \frac{v}{(cs)_\infty}, \quad \lambda = \frac{\mathcal{K}_1}{4\pi \mathcal{R}^2 \rho_\infty (cs)_\infty}.
\]
Writing the equations of motion with these variables and solving for \(u\):
\[
\frac{\lambda^2}{2} x^{-4} y^{-2} + n(y^{1/n} - 1) = x^{-1},
\]
where the adiabatic index \(n = 1/(\gamma - 1)\).

Self-similar solutions

Assume \(x \ll 1 \Rightarrow \frac{\lambda^2}{2} x^{-3} y^{-2} + ny^{1/n} x = \frac{\lambda^2}{2} x^{-3} + 2n z^{-2n} + nz = 1\), where \(z = y^{1/n} x\). If \(n = 3/2\), we can solve for \(z(\lambda) \Rightarrow y = z(\lambda)^{3/2} x^{-3/2}, u = \lambda z(\lambda)^{-3/2} x^{-1/2}\) (power laws). Then, for \(n = 3/2 \iff \gamma = 5/3\),

\[
\rho(r) = \alpha r^{-3/2}, \quad v(r) = \beta r^{-1/2}, \quad c_s^2(r) = \mathcal{K}_2 \frac{5}{3} \alpha^{2/3} r^{-1} = \sigma^2 r^{-1},
\]
so the Mach number is given by \(\mathcal{M} = v(r)/c(r) = \beta / \sigma\) (constant).

Linear Perturbations

We try the separation of variables in spherical coordinates: \(\delta \psi(r, \theta, \varphi, t) = R(r) Y_{lm}(\theta, \varphi) e^{-i\omega t}\). So we obtain the radial differential equation
\[
(\sigma^2 - \beta^2) r^2 R'' + \left(\frac{\sigma^2}{2} - \beta^2 + 2i\beta \omega r^{3/2}\right) r R' + \left[-l(l+1)\sigma^2 + \omega^2 r^2 + i\beta \omega r^{3/2}\right] R = 0.
\]
Note that it is not of Sturm-Liouville type. Its general solution is
\[
R(r) = r^{1/4} e^{-i\mu r^{3/2}} [C_1 J_{v}(\kappa r^{3/2}) + C_2 J_{-v}(\kappa r^{3/2})],
\]
where \( \mu = \frac{2\beta \omega}{3(\sigma^2 - \beta^2)} \), \( \kappa = \frac{2\beta \omega}{3(\sigma^2 - \beta^2)} \), \( \nu = \sqrt{\frac{1+16(l+1)}{6\sigma^2 - \beta^2}} \).

We need appropriate boundary conditions at two radii. In the self-similar case, we must use \( r_1 = 0 \) and \( r_2 \approx \mathcal{R} \to \infty \), but we keep the latter finite to have a discrete spectrum. Since \( W = -\partial_t \delta \psi \delta j \), holding the mass flow, as done in Ref. [5], is equivalent to holding the energy flow. So, using the variable \( z = \kappa r^{3/2} \), we impose

\[
 z^{-1/6} \left( C_1 [J_\nu(z) + 6zJ'_\nu(z)] + C_2 [J_{-\nu}(z) + 6zJ'_{-\nu}(z)] \right) = 0
\]

at \( r_1, r_2 \), and so we obtain the \( \kappa \)-spectrum. Remarkably, we have regularity at \( r_1 = 0 \), namely, the physical quantities \( \delta v_r/\nu, \delta v_{\theta, \phi}/\nu, \delta \rho/\rho \) stay finite.

**Evolution of radial eigen-functions**

Since our boundary problem is not of Sturm-Liouville type, we have no eigenfunction orthogonality. Let us turn to the first order equations for radial perturbations:

\[
 \frac{\partial \delta \rho}{\partial t} + \frac{1}{r^2} \partial_r \left[ r^2 (\delta \rho \nu + \rho \delta \nu) \right] = 0, \quad \frac{d(A \cdot x)}{dr} = i\omega x, \\
 \frac{\partial \delta \nu}{\partial t} + \partial_r \left( \frac{c^2}{\rho} \delta \rho + \nu \delta \nu \right) = 0 \quad \Leftrightarrow \quad A = \begin{pmatrix} v(r) & r^2 \rho(r) \\ \frac{c(r)^2}{r^2 \rho(r)} & v(r) \end{pmatrix},
\]

with \( x = (r^2 \delta \rho, \delta \nu) \). If \( y = A \cdot x \), \( U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), then we get the orthogonality relation

\[
 \int y_n^* \cdot U \cdot A^{-1} \cdot y_m dr \propto \delta_{nm}.
\]

Then we express the initial condition \( y(r) \) as a sum of orthogonal modes:

\[
 y(r) = \sum_{n=-\infty}^{\infty} c_n y_n(r) = c_0 y_0 + 2 \Re \left[ \sum_{n=1}^{\infty} c_n y_n(r) \right].
\]

Due to orthogonality, \( c_n = \langle y_n, y \rangle / \langle y_n, y_n \rangle \). The time evolution is given by \( c_n(t) = c_n e^{-i\omega_n t} \). The norm \( \langle y, y \rangle \) is invariant and, in fact, is proportional to the energy. The energy spectrum \( \{ |c_n|^2 \}_{n=0}^{\infty} \) is invariant but the generic evolution of the correlation between the phases is to decrease with time, leading to quasi-ergodicity and, therefore, (marginal) stability.

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