The Sylvester Equation and Kadomtsev–Petviashvili System

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Abstract: In this paper, we seek connections between the Sylvester equation and Kadomtsev–Petviashvili system. By introducing Sylvester equation $LM = rs^T$ together with an evolution equation set of $r$ and $s$, master function $S(i, j) = s^T K_i C(I + MC)^{-1} L_i r$ is used to construct the Kadomtsev–Petviashvili system, including the Kadomtsev–Petviashvili equation, modified Kadomtsev–Petviashvili equation and Schwarzian Kadomtsev–Petviashvili equation. The matrix $M$ provides $\tau$-function by $\tau = |I + MC|$. With the help of some recurrence relations, the reductions to the Korteweg–de Vries and Boussinesq systems are discussed.

Keywords: Sylvester equation; Kadomtsev–Petviashvili system; generalized Cauchy matrix approach

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1. Introduction

The Sylvester equation [1–3]

$$AM - MB = C,$$

also named the Rosenblum equation in control theory, has become a subject of considerable interest in recent decades. In (1), $A, B, C$ are known matrices and $M$ is an unknown matrix. The significance of Equation (1) comes from its important applications in many areas of applied mathematics, as well as in systems and control theory, signal processing, filtering, model reduction, image restoration, and so on. The Equation (1) includes as special cases several important linear equation problems, e.g., linear system $Ax = c$, multiple right-hand side linear system $AM = C$ and commuting matrices $XM - MX = 0$. So far, many efforts have been made to solve the Sylvester Equation (1) and to discuss its properties and applications. In [4], Bhatia and Rosenthal investigated lots of interesting and important theoretical results on this equation, including similarity, commutativity, hyperinvariant subspaces, spectral operators and differential equations, and so forth.

With the help of the Sylvester equation, many methods have been introduced to solve the integrable system, such as the operator method [5,6], the bidifferential calculus approach [7], a method based on Gel’fand-Levitan-Marchenko Equation [8], the direct linearization (DL) method [9,10] and the Cauchy matrix (CM) approach [11]. As a by-product of the DL method, the CM approach was first proposed by Nijhoff and his collaborators to investigate soliton solutions of the Adler–Bobenko–Suris lattice list except for the elliptic case of Q4. This scheme proves to be a powerful tool with the ability to provide integrable equations together with their soliton solutions, and amongst others, Lax representation, Bäcklund and Miura transformations. Subsequently, this method was extended to the elliptic case [12]. In [13], a generalized CM approach was developed to yield more types of exact solutions for the Adler–Bobenko–Suris lattice list beyond the soliton solutions. Up until now, the CM method has been used to investigate many discrete and semi-discrete integrable systems [14,15].
Besides the discrete and semi-discrete integrable systems, the CM structure is also valid in the study of continuous integrable systems. In [16], the links between Cauchy-type determinants and integrable systems were described, where the Sylvester equation with a Jordan-block type coefficient matrix was discussed emphatically. Xu et al. [17] took advantage of the CM approach to discuss the relations between the Sylvester equation

$$KM + MK = rs^T,$$

(2)

with $K, M \in \mathbb{C}_{N \times N}$, $r, s \in \mathbb{C}_{N \times 1}$, and some continuous integrable equations, including the KdV, modified KdV, Schwarzian KdV and sine-Gordon equations.

They showed that all these equations arose from the Sylvester Equation (2) and could be expressed by some discrete equations of master function $S^{(i,j)} = s^T K^i (I + M)^{-1} K^j r$, $(i, j \in \mathbb{Z})$ defined on certain points. Equation (2) corresponds to $A = -B$ and $C$ being of rank 1 in (1), where and whereafter $^T$ stands for the transpose of matrix and $I$ means the unit matrix. The master function $S^{(i,j)}$ possesses a symmetry property with respect to $i$ and $j$, namely, $S^{(i,j)} = S^{(j,i)}$.

Following a similar procedure, the CM approach for an AKNS system was considered in [18]. The relevant Sylvester equation reads

$$KM - MK = rs^T,$$

(3)

where $K, M, r$ and $s$ are block matrices in the form of

$$K = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix}, \quad M = \begin{pmatrix} 0 & M_1 \\ M_2 & 0 \end{pmatrix}, \quad r = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}, \quad s = \begin{pmatrix} 0 & s_1 \\ s_2 & 0 \end{pmatrix},$$

(4)

with $K_1 \in \mathbb{C}_{N_1 \times N_2}, M_1 \in \mathbb{C}_{N_1 \times N_2}, M_2 \in \mathbb{C}_{N_2 \times N_1}, r_i, s_i \in \mathbb{C}_{N_1 \times 1}, (i = 1, 2)$ and $N_1 + N_2 = 2N$. This corresponds to $A = B$ and $C$ being of rank 2 in (1).

In recent years, relations between the Sylvester equation and higher-dimensional integrable systems have drawn more and more attention. In [19], solutions to the Kadomtsev–Petviashvili (KP) equation were given in the form $w = p^T C^{-1} q$, and the Sylvester equation appeared in the solving procedure. Based on a Sylvester equation [20], some lattice KP-type equations were discussed, comprising lattice potential KP equation, lattice potential modified KP equation, lattice Schwarzian KP equation and lattice KP-type Nijhoff–Quispel–Capel equation [21]. In the recent paper [22], the DL was established for the discrete AKP, BKP and CKP equations, extending the earlier results on discrete KP equations of A-type [23]. Subsequently, continuous hierarchies for the AKP, BKP and CKP equations as well as their dimensional reductions were also discussed in terms of the DL framework. In reductions of the BKP and CKP hierarchies, the reduced measures admit antisymmetry and symmetry property, respectively [24]. It is worth noting that the Sylvester equation was also developed to study some interesting higher-dimensional physical models, such as a Yajima–Oikawa system, a Mel'nikov model and a self-dual Yang–Mills equation (see [25–27]). In [27], based on the Sylvester Equation (3), Li et al. investigated the master function $S^{(i,j)} = s^T K^i (I + M)^{-1} K^j r$ and revealed that this function has the symmetric property $S^{(i,j)} = -\sigma_2 S^{(j,i)} \sigma_2$, where $\sigma_2$ is a Pauli matrix.

In this paper, motivated by the CM approach and the understanding between the Sylvester equation and discrete KP system, we would like to adopt the CM approach to explore the links between the Sylvester equation and nonlinear KP system. A Sylvester equation will be introduced firstly, in terms of which some properties of the master function $S^{(i,j)}$ will be discussed. Next, we will impose the evolution relations on vectors $r$ and $s$, from which the evolution relations for master function $S^{(i,j)}$ will be obtained. Moreover, by setting values of indices $i$ and $j$, some KP-type equations, including the KP equation, modified KP equation and Schwarzian KP equation, will be constructed. Furthermore, the connection between $S^{(i,j)}$ and $\tau$-function will be also shown. In view of some special forms
of matrices $K$ and $L$, we will discuss the reduction in the KP system. As a consequence, the KdV and BSQ systems will be considered.

The paper is organized as follows. In Section 2, we first set up the Sylvester equation. In addition, we introduce the master function $S^{(i,j)}$ and discuss some properties. In Section 3, for different values of $i$ and $j$, we construct KP-type equations. Section 4 is devoted to the discussion on the reduction of the KP system. Some conclusions are made in Section 5. Additionally, we have an Appendix A as a compensation of the paper.

2. The Sylvester Equation and Master Function

2.1. The Sylvester Equation

The solvability of the Sylvester Equation (1) can be summarized as the following Proposition [1].

**Proposition 1.** Let us denote the eigenvalue sets of matrices $A$ and $B$ by $\mathcal{E}(A)$ and $\mathcal{E}(B)$, respectively. For the known matrices $A, B$ and $C$, the Sylvester Equation (1) has a unique solution $M$ if and only if $\mathcal{E}(A) \cap \mathcal{E}(B) = \emptyset$.

With some more conditions on $\mathcal{E}(A)$ and $\mathcal{E}(B)$, solution $M$ of (1) can be expressed via series or integration [4] (see also Ref. [17]).

In the present paper, we consider a Sylvester equation in the form of

$$LM - MK = rs^t,$$  \hspace{1cm} (5)

where $L \in \mathbb{C}_{N \times N}$, $K \in \mathbb{C}_{N' \times N'}$, $M \in \mathbb{C}_{N \times N'}$, $r = (r_1, r_2, \ldots, r_N)^t$ and $s = (s_1, s_2, \ldots, s_{N'})^t$. Equation (5) corresponds to $C$ being of rank 1 in (1) and has unique solution $M$ when $\mathcal{E}(K) \cap \mathcal{E}(L) = \emptyset$. In the remaining portion of this section, we assume that $K$ and $L$ satisfy such conditions and $0 \notin \mathcal{E}(K) \cup \mathcal{E}(L)$, i.e., $K$ and $L$ are invertible constant matrices. In a discrete case, the Sylvester Equation (5) has been used to construct the lattice KP system and their various solutions [21], where $r$ and $s$ satisfy some discrete evolution equations.

2.2. Master Function $S^{(i,j)}$ and Some Properties

2.2.1. The Definition of $S^{(i,j)}$

By the Sylvester Equation (5), we now introduce master function

$$S^{(i,j)} = s^t K^t C (I + MC)^{-1} L^t r, \quad i, j \in \mathbb{Z},$$ \hspace{1cm} (6)

where $C \in \mathbb{C}_{N' \times N}$ is an arbitrary constant matrix, such that the product $MC$ is a square $N \times N$ matrix. The $S^{(i,j)}$ (6) first appeared in [20] to construct the lattice KP-type equations. In what follows, we will show that the master function $S^{(i,j)}$ can be also used to generate the nonlinear KP system. For convenience, we introduce an auxiliary vector function

$$u^{(i)} = (I + MC)^{-1} L^t r, \quad i \in \mathbb{Z}.$$ \hspace{1cm} (7)

Then $S^{(i,j)}$ defined in (6) can be simplified to

$$S^{(i,j)} = s^t K^t Cu^{(i)}, \quad i, j \in \mathbb{Z}.$$ \hspace{1cm} (8)

It is noteworthy that $S^{(i,j)}$ is not symmetric w.r.t. the interchange of the parameters $i$ and $j$, i.e., $S^{(i,j)} \neq S^{(j,i)}$, which is different from that in the KdV case [17].

In the following sections, we discuss some properties of the master function $S^{(i,j)}$. 

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2.2.2. Similarity Invariance of $S^{(i,j)}$

Now suppose that matrices $K_1$ and $L_1$ are similar to $K$ and $L$, respectively, under the transform matrices $T_1$ and $T_2$, i.e.,

$$K_1 = T_1 KT_1^{-1}, \quad L_1 = T_2 LT_2^{-1}. \quad (9)$$

We denote

$$M_1 = T_2 MT_1^{-1}, \quad C_1 = T_1 CT_2^{-1}, \quad r_1 = T_2 r, \quad s_{1} = s T_1^{-1}. \quad (10)$$

Then by direct substituting, the Sylvester Equation (5) and master function (6) yield

$$L_1 M_1 - M_1 K_1 = r_1 s_{1},$$

and

$$S^{(i,j)} = s_{1}^{j} K^{j} C (I + MC)^{-1} L^{j} r = s_{1}^{j} K^{j} C_1 (I + M_1 C_1)^{-1} K^{j} r_1.$$

It is shown that master function $S^{(i,j)}$ is invariant under the transformations (9) and (10).

2.2.3. Identities of $S^{(i,j)}$

With some special relations between matrices $L$ and $K$, one can derive several important equalities for the master function $S^{(i,j)}$, which will be used in Section 4. Here we suppose the orders of matrices $K$ and $L$ are the same, i.e., $N' = N$, and the constant matrix $C$ in (6) is the $N$th order unit matrix.

**Proposition 2.** Assuming $K = -L$, then for the master function $S^{(i,j)}$ with $M, L, r, s$ satisfying the Sylvester Equation (5), we have the following relation

$$S^{(i,j+2s)} = S^{(i+2s,j)} - \sum_{l=0}^{2s-1} (-1)^{l} S^{(2s-1-l,j)} S^{(l,j)}, \quad (s = 1, 2, \ldots),$$

$$S^{(i,j-2s)} = S^{(i-2s,j)} + \sum_{l=0}^{2s-1} (-1)^{l} S^{(1-l,j)} S^{(i-2s+l)}, \quad (s = 1, 2, \ldots).$$

In particular, when $s = 1$, we have

$$S^{(i,j+2)} = S^{(i+2,j)} - S^{(i,0)} S^{(1,j)} + S^{(i,1)} S^{(0,j)}, \quad (11)$$

$$S^{(i,j-2)} = S^{(i-2,j)} + S^{(i-2,j-1)} - S^{(i-1,j)} S^{(-2,j)}. \quad (12)$$

The proof of Proposition 2 can be referred to Ref. [17]. The relation (11) firstly appeared in a discrete case [11] and plays a crucial role in the construction of the KdV system [17]. It is worth noting that the master function $S^{(i,j)}$ in Proposition 2 is of form

$$S^{(i,j)} = (-1)^{s} L^{j} (I + M)^{-1} L^{j} r,$$

which has the symmetric property $S^{(i,j)} = S^{(j,i)}$ when $j - i$ is an even number and antisymmetric property $S^{(i,j)} = -S^{(j,i)}$ when $j - i$ is an odd number [13,17].

Besides the above case, we have the following two results, where parameter $\omega$ satisfies $\omega^2 + \omega + 1 = 0$. 

Proposition 3. Assuming $K = \omega L$, then for the master function $S^{(i,j)}$ with $M, L, r, s$ satisfying the Sylvester Equation (5), we have the following relation

$$S^{(i,j+3s)} = S^{(i+3s,j)} - \sum_{l=0}^{3s-1} S^{(3s-1-l,j)} S^{(i,l)}, \quad (s = 1, 2, \ldots).$$

(13)

In particular, when $s = 1$, we have

$$S^{(i,j+3)} = S^{(i+3,j)} - S^{(i,2)} S^{(0,j)} - S^{(i,1)} S^{(1,j)} - S^{(i,0)} S^{(2,j)}.$$  

(14)

Proposition 4. Assume $K = \text{Diag}(\omega K_1, \omega^2 K_2), \ L = \text{Diag}(K_1, K_2)$, where $K_i \in \mathbb{C}^{N_i \times N_i}$ with $N_1 + N_2 = N$. Then for the master function $S^{(i,j)}$ with $M, L, K, r, s$ satisfying the Sylvester Equation (5), we have the relations (13) and (14).

The proofs of Proposition 3 and Proposition 4 are similar to the one for Proposition 2, which are omitted here.

3. The KP System

3.1. Evolution of $M$

We suppose the following evolution equation set

$$r_x = Lr, \quad s_x = -K's, \quad (15)$$
$$r_y = -L^2 r, \quad s_y = (K^2)s, \quad (16)$$
$$r_t = 4L^3 r, \quad s_t = -4(K^3)s, \quad (17)$$

where $r, s$ and $M$ are functions of $(x, y, t)$ while $K$ and $L$ are non-trivial constant matrices.

We now discuss the dynamical properties of matrix $M$, i.e., the evolution relations of $M$ w.r.t. independent variables. The derivative of the Sylvester Equation (5) w.r.t. $x$ together with (15) yields

$$LM_x - M_x K = r_x s^T + rs_x = Lrs^T - rs^T K,$$

which gives rise to the relation

$$M_x = rs^T,$$  

(18)

in light of Proposition 1.

The $y$-derivative of Equation (5) leads to

$$LM_y - M_y K = r_y s^T + rs_y$$
$$= -L^2 rs^T + rs^T K^2$$
$$= L(-L^2 M + MK^2) - (-L^2 M + MK^2)K,$$

(19)

where in the last step the term $rs^T$ is replaced by $LM - MK$. Then we obtain

$$M_y = -L^2 M + MK^2,$$  

(20)

which can be rewritten as

$$M_y = -rs^T K - Lrs^T.$$

(21)
Analogous to the earlier analysis, we deduce that the time evolution of $M$ is of form
\[ M_t = 4(L^3 M - MK^3) = 4(L^2 rs^i + Lrs^i K + rs^i K^2). \] (22)

Equations (18), (21) and (22) encode all the information on the dynamics of the matrix $M$ w.r.t. the independent variables $x$, $y$ and $t$, in addition to (5), which can be thought as the defining property of $M$.

3.2. Evolution of $S^{(i,j)}$

To begin, we take into account the dynamical properties of the vector function $u^{(i)}$ defined by (7). It follows from (7) that equation
\[ (I + MC)u^{(i)} = L^i r, \] (23)
holds identically. Substituting (15) and (18) into the $x$-derivative of Equation (23) yields
\[ (I + MC)u_x^{(i)} = L^{i+1} r - rs^i C u^{(i)}, \]
which implies
\[ u_x^{(i)} = u^{(i+1)} - S^{(i,0)} u^{(0)}, \] (24)
where relation (8) has been used. After a similar analysis as aforementioned, we arrive at the evolution of $u^{(i)}$ in $y$, $t$-directions
\[ u_y^{(i)} = -u^{(i+2)} + S^{(i,0)} u^{(1)} + S^{(i,1)} u^{(0)}, \] (25)
\[ u_t^{(i)} = 4(u^{(i+3)} - S^{(i,2)} u^{(0)} - S^{(i,1)} u^{(1)} - S^{(i,0)} u^{(2)}). \] (26)

Multiplying (24), (25) and (26) from the left by the row vector $s^i K^j C$ and noting the Equations (15), (16), (17) and the connection (8) between $u^{(i)}$ and $S^{(i,j)}$, we have the evolution relations of $S^{(i,j)}$:
\[ S_t^{(i,j)} = S^{(i+1,j)} - S^{(i,j+1)} - S^{(i,0)} S^{(0,j)}, \] (27)
\[ S_x^{(i,j)} = -S^{(i+2,j)} + S^{(i,j+2)} + S^{(i,1)} S^{(0,j)} + S^{(i,0)} S^{(1,j)}, \] (28)
\[ S_y^{(i,j)} = 4(S^{(i+3,j)} - S^{(i,j+3)} - S^{(i,0)} S^{(2,j)} - S^{(i,1)} S^{(1,j)} - S^{(i,2)} S^{(0,j)}). \] (29)

Some higher-order derivatives of $S^{(i,j)}$ can be readily obtained w.r.t. independent variables from the above relations by iterate calculation. Here, we just present the expressions of $S_{xx}^{(i,j)}$, $S_{xy}^{(i,j)}$ and $S_{yy}^{(i,j)}$ as follows:
\[ S_{xx}^{(i,j)} = S^{(i+2,j)} + S^{(i,j+2)} - 2S^{(i+1,j+1)} - 2S^{(i+1,0)} S^{(0,j)} + 2S^{(i,0)} S^{(0,j+1)} - S^{(i,0)} S^{(1,j)} + S^{(i,1)} S^{(0,j)} + 2S^{(0,0)} S^{(i,0)} S^{(0,j)}; \] (30)
\[ S_{xxx}^{(i,j)} = S^{(i+3,j)} - S^{(i,j+3)} - 3S^{(i+2,j+1)} + 3S^{(i+1,j+2)} - 3S^{(i+2,0)}S^{(0,j)} - 3S^{(i,0)}S^{(0,j+1)} + 6S^{(i+1,0)}S^{(0,j+1)} - 3S^{(i+1,0)}S^{(1,j)} - 3S^{(i,0)}S^{(0,j)+1} + 6S^{(i+1,0)}S^{(0,j)} - 6S^{(i,0)}S^{(0,j)+1} + 2S^{(i,1)}S^{(1,j)} + 3S^{(i,0)}S^{(0,0)}S^{(1,j)} - 3S^{(i,0)}S^{(0,0)}S^{(0,j)} - 3S^{(i,1)}S^{(0,0)}S^{(0,j)} - 6S^{(i,0)}S^{(0,0)}S^{(0,j)} ,
\]
\[ S_{xy}^{(i,j)} = -S^{(i+3,j)} - S^{(i,j+3)} + S^{(i+2,j+1)} + S^{(i+1,j+2)} + S^{(i+2,j)} - S^{(i,0)}S^{(0,j) + 2} + S^{(i,j)}S^{(0,j) + 1} + S^{(i+1,0)}S^{(0,j) + 1} - S^{(i+1,0)}S^{(1,j)} + S^{(i,0)}S^{(0,0)}S^{(0,j) + 1} - S^{(i,0)}S^{(0,0)}S^{(0,j)} + S^{(i,0)}S^{(1,0)}S^{(0,j)} - S^{(i,0)}S^{(0,1)}S^{(0,j)} - S^{(i,0)}S^{(0,1)}S^{(0,j)} ,
\]

The relation (27) implies that the following identities
\[ S^{(i,j+2)} = -S^{(i,j+1)} + S^{(i+1,j+1)} - S^{(0,j+1)}S^{(i,0)} ,\]
\[ S^{(i+2,j)} = S^{(i+1,0)}S^{(0,j)} + S^{(i,0)}S^{(0,1)}S^{(0,j)} ,\]
hold. Thus, the subtraction of (33) from (34) leads to
\[ S^{(i,j+2)} - S^{(i,j+2)} = \partial_x (S^{(i,j+1)} + S^{(i+1,j)}) + S^{(0,j+1)}S^{(i,0)} + S^{(0,j)}S^{(i+1,0)} .\]

Plugging (35) into (28) and utilizing
\[ S^{(0,j+1)} = -S^{(0,j)} + S^{(i,j)} - S^{(i,0)}S^{(0,j)} ,\]
\[ S^{(i+1,0)} = S^{(i,0)} + S^{(i,1)}S^{(i,0)} ,\]
we finally arrive at
\[ \partial_x^{-1} S^{(i,j)} = -(S^{(i,j+1)} + S^{(i+1,j)}) + \partial_x^{-1} (S^{(i,0)}S^{(0,j)} - S^{(i,0)}S^{(0,j)}) ,\]
\[ \partial_y^{-1} S^{(i,j)} = S^{(i+3,j)} - S^{(i,j+3)} + S^{(i+2,j+1)} - S^{(i+1,j+2)} - S^{(0,j)}S^{(i,1)} + S^{(i,0)}S^{(0,j)} - S^{(i,j)}S^{(i+1,0)} - S^{(i,0)}S^{(i+1,0)} + \partial_y\partial_x^{-1} (S^{(i,0)}S^{(0,j)} - S^{(i,0)}S^{(0,j)} ,\]

where \( \partial_x^{-1} = \frac{1}{2} (\frac{d}{dx} - \int_0^x ) \) and relation (39) is derived from (38) by taking \( y \)-derivative. All the relations in (27)–(39) can be viewed as semi-discrete equations when the parameters \( i \) and \( j \) are recognized as discrete independent variables.

### 3.3. The KP System

From the evolution relations (27)–(39), various KP-type equations can be constructed for special values of the parameters \( i \) and \( j \), including KP equation, modified KP equation and Schwarzian KP equation. Analogue to the discrete case [20,21], we will show that the master functions \( S^{(0,0)}, S^{(-1,0)}, S^{(0,-1)} \) and \( S^{(-1,-1)} \) will be the generating functions for the resulting equations.

#### 3.3.1. The KP Equation

To derive the KP equation, we take \( i = j = 0 \) in (27)–(39) and denote \( u = S^{(0,0)} \). In this case, some evolution relations in (27)–(39) give rise to
\[ u_x = -S^{(0,1)} + S^{(1,0)} - u^2, \quad (40) \]
\[ u_t = 4(S^{(3,0)} - S^{(0,3)} - uS^{(2,0)} - uS^{(0,2)} - S^{(1,0)}S^{(0,1)}), \quad (41) \]
\[ u_{xxx} = -S^{(0,3)} + S^{(3,0)} + 3(S^{(1,2)} - S^{(2,1)}) - 4u(S^{(2,0)} + S^{(0,2)}) + 8S^{(1,0)}S^{(0,1)} \]
\[ - 3(S^{(1,0)})^2 + 6uS^{(1,1)} + 12u^2(S^{(1,0)} - S^{(0,1)}) - 6u^4, \quad (42) \]
\[ \partial^{-1} u_{yy} = S^{(3,0)} - S^{(0,3)} + S^{(2,1)} - 2uS^{(1,1)} - S^{(0,1)^2} - S^{(1,0)^2}. \quad (43) \]

From (40)–(43), one can easily find the following potential KP equation:
\[ u_t - u_{xxx} - 6u_x^2 - 3\partial^{-1} u_{yy} = 0. \quad (44) \]

By transformation \( \omega = 2u_x \), Equation (44) is transformed to the KP equation
\[ \omega_t - \omega_{xxx} - 6\omega\omega_x - 3\partial^{-1}\omega_{yy} = 0, \quad (45) \]
which has solution
\[ \omega = 2(s^1C(I + MC)^{-1}r)_{xx} \quad (46) \]
where matrix \( M \) and vectors \( s, r \) satisfy systems (5), (15), (16) and (17).

3.3.2. The Modified KP Equation

To derive the modified KP equation, we consider the following two cases:
\[ i = 0 \text{ and } j = -1; \quad (47) \]
\[ i = -1 \text{ and } j = 0. \quad (48) \]

For case \( (47) \), a new variable \( v = S^{(0,-1)} + 1 \) is introduced. Then some evolution relations in (27)–(39) become
\[ v_x = S^{(1,-1)} - uv, \quad (49) \]
\[ v_t = 4(S^{(3,-1)} + vS^{(0,2)} - S^{(1,-1)}S^{(0,1)} - S^{(2,-1)}u), \quad (50) \]
\[ v_{xx} = vS^{(0,1)} + S^{(2,-1)} - 2vS^{(1,0)} + 2u^2v - uS^{(1,-1)}, \quad (51) \]
\[ v_{xxx} = S^{(3,-1)} - vS^{(0,2)} + 3vS^{(1,1)} - 3vS^{(2,0)} - 6uvS^{(0,1)} + 9uvS^{(1,0)} \]
\[ - 6u^3v - uS^{(2,-1)} - 3S^{(1,0)}S^{(1,-1)} + 2S^{(0,1)}S^{(1,-1)} + 3u^2S^{(1,-1)}, \quad (52) \]
\[ v_y = vS^{(0,1)} - S^{(2,-1)} + uS^{(1,-1)}, \quad (53) \]
\[ v_{xy} = -S^{(3,-1)} - vS^{(0,2)} + vS^{(1,1)} + vS^{(2,0)} - 2uvS^{(0,1)} - uvS^{(1,0)} \]
\[ + uS^{(2,-1)} + S^{(1,0)}S^{(1,-1)} - u^2S^{(1,-1)}, \quad (54) \]

After a straightforward computation, we obtain the equation
\[ v_t - v_{xxx} - 6u_x v_x + 6v_{xy} + 3v_{xy} = 0, \quad (55) \]
where \( u_x \) is given by (40) and
\[ u_y = -S^{(2,0)} + S^{(0,2)} + S^{(1,0)}u + uS^{(1,0)}. \quad (56) \]

Moreover, noting (51), (53) and (40), we also have
\[ v_{xx} = -2uv_x - v_y. \quad (57) \]
Substituting (57) into (55) and with a direct computation, we obtain the potential modified KP equation

\[ v_t - v_{xxx} + 3 \frac{v_x v_{xx}}{v} + 6v \partial^{-1} \left( \frac{v_x}{v} \cdot \frac{v_y}{v} \right) - 3v \partial^{-1} \left( \frac{v_{yy} v - v_y^2}{v^2} \right) = 0. \] (58)

By transformation \( \mu = \partial_x \ln v \), Equation (58) is transformed into a modified KP equation

\[ \mu_t - \mu_{xxx} + 6\mu^2 \mu_x + 6\mu_x \partial^{-1} \mu_y - 3\partial^{-1} \mu_{yy} = 0, \] (59)

which possesses the solution

\[ \mu = \partial_x \ln (1 + s^1 K^{-1} C (I + MC)^{-1} r), \] (60)

where matrices \( M, K \) and vectors \( s, r \) satisfy systems (5), (15), (16) and (17).

Noting that \( \omega = 2u_x \) and \( \mu = \partial_x \ln v \), relation (57) implies

\[ -\omega = \mu_x + \mu^2 + \partial^{-1} \mu_y, \] (61)

which is the Miura transformation between modified KP Equation (59) and KP equation (45).

For case (48), we consider a new variable

\[ w = 1 - s^{(-1,0)}, \] (62)

whose various derivatives can be derived directly from (27)–(39) with \( i = -1 \) and \( j = 0 \):

\[ w_x = S^{(-1,1)} - uw, \] (63)

\[ w_t = 4(S^{(-1,3)} - wS^{(2,0)} + S^{(-1,1)} S^{(1,0)} + S^{(-1,2)} u), \] (64)

\[ w_{xx} = -wS^{(1,0)} - S^{(-1,2)} + 2wS^{(0,1)} + 2u^2 w - uS^{(-1,1)}, \] (65)

\[ w_{xxx} = S^{(-1,3)} - wS^{(2,0)} - 3wS^{(0,2)} + 3wS^{(1,1)} + 6uwS^{(1,0)} - 9uwS^{(0,1)} \]
\[ - 6u^3 w + uS^{(-1,2)} + 3S^{(0,1)} S^{(-1,1)} - 2S^{(1,0)} S^{(-1,1)} + 3u^2 S^{(-1,1)}, \] (66)

\[ w_y = -S^{(-1,2)} - uS^{(-1,1)} + wS^{(1,0)}, \] (67)

\[ w_{xy} = S^{(-1,3)} - wS^{(0,2)} - wS^{(1,1)} + wS^{(2,0)} - 2uwS^{(1,0)} - uwS^{(0,1)} \]
\[ + uS^{(1,2)} + S^{(0,1)} S^{(-1,1)} + u^2 S^{(-1,1)}. \] (68)

A straightforward calculation yields equations

\[ w_t - w_{xxx} - 6u_x w_x - 6uw_y - 3w_{xy} = 0, \] (69)

\[ w_{xx} = -2uw_x + w_y, \] (70)

where \( u_x \) and \( u_y \) are defined by (40) and (56). Similarly, taking (70) into (69), we derive one more potential modified KP equation

\[ w_t - w_{xxx} + 3 \frac{w_x w_{xx}}{w} - 6w \partial^{-1} \left( \frac{w_x}{w} \cdot \frac{w_y}{w} \right) - 3w \partial^{-1} \left( \frac{w_{yy} w - w_y^2}{w^2} \right) = 0. \] (71)

In terms of transformation \( v = \partial_x \ln w \), we obtain the corresponding modified KP equation

\[ v_t - v_{xxx} + 6v^2 v_x - 6v_x \partial^{-1} v_y - 3\partial^{-1} v_{yy} = 0, \] (72)
whose solution is given by

\[ \nu = \partial_x \ln(1 - s^T C (I + MC)^{-1} L^{-1} r), \]  

(73)

where matrices \( M, L \) and vectors \( s, r \) satisfy systems (5), (15), (16) and (17). It is easy to know that \(-\mu\) also satisfies the modified KP Equation (72).

In the light of \( \omega = 2u_x \) and \( v = \partial_x \ln w \), relation (70) gives rise to the Miura transformation between the modified KP Equation (72) and KP Equation (45), i.e.,

\[ -\omega = \nu_x + \nu^2 - \partial^{-1} \nu_y. \]  

(74)

3.3.3. The Schwarzian KP Equation

Let us examine the equation related to function \( S(-1, -1) \). We introduce

\[ z = S(-1, -1) + x. \]  

(75)

Setting \( i = j = -1 \) in (27)–(39) yields the following expressions:

\[ z_x = vw, \]  

(76)

\[ z_{xx} = vS^{(-1,1)} + wS^{(1,-1)} - 2uvw = (vw)_x, \]  

(77)

\[ z_{xxx} = -vS^{(-1,2)} + wS^{(2,-1)} + 2S^{(-1,1)}S^{(1,-1)} - 3uvS^{(1,-1)} - 3uvS^{(-1,1)} \]

\[ -3uvwS^{(1,0)} + 3uvwS^{(0,1)} + 6u^2vw, \]  

(78)

\[ z_t = 4(-vS^{(-1,2)} + wS^{(2,1)} - S^{(-1,1)}S^{(1,-1)}), \]  

(79)

\[ z_y = S^{(-1,1)}v - S^{(1,-1)}w = w_z v - w v_x, \]  

(80)

where in (77) and (80), we have made use of the relations (49) and (63). By a forward calculation associated with (76) and (80), we have

\[ z_t - z_{xxx} + \frac{3}{2} \frac{z_{xx}^2 - z_y^2}{z_x} - 3z_x \partial^{-1} \left( \frac{z_y}{z_x} \right)_y \]

\[ = z_t - z_{xxx} + \frac{3}{2} \frac{z_{xx}^2 - z_y^2}{z_x} - 3z_x \left( \frac{w_y}{w} - \frac{v_y}{v} \right) \]

\[ = z_t - z_{xxx} + \frac{3}{2} \frac{z_{xx}^2 - z_y^2}{z_x} - 3(vw_y - v_y w) = 0, \]  

(81)

where \( v_y \) and \( w_y \) are defined by (53) and (67), respectively. It is worth pointing out that (81) is the Schwarzian KP equation [28], which has solution

\[ z = s^T K^{-1} C (I + MC)^{-1} L^{-1} r + x, \]  

(82)

where matrices \( M, K, L \) and vectors \( s, r \) satisfy system (5), (15), (16) and (17).

Due to the definitions of variables \( \mu \) and \( \nu \), relations (76), (77) and (80) provide Miura transformations between modified KP Equation (59) and Schwarzian KP Equation (81), respectively, as well as modified KP Equation (72) and Schwarzian KP Equation (81), given by

\[ \mu = \frac{z_{xx} - z_y}{2z_x}, \]  

(83)

\[ \nu = \frac{z_{xx} + z_y}{2z_x}. \]  

(84)
3.4. The τ-Function

To discuss the bilinear structure of the KP system, we introduce the τ-function

$$\tau = |I + MC|,$$  \hspace{1cm} (85)

for which the following result holds.

**Proposition 5.** For the scalar function $$S^{(i,j)}$$ defined in (6), where $$K, L, M, r, s$$ are formulated by the Sylvester Equation (5) and $$r, s$$ obey the evolutions (15)–(17), we have

$$S^{(i,j)} = \frac{g}{\tau},$$  \hspace{1cm} (86)

with some function $$g = -\begin{vmatrix} 0 & s^T K C \\ L^T r & I + MC \end{vmatrix}$$. Specifically, for $$S^{(0,0)}$$, we have

$$S^{(0,0)} = \frac{\tau_x}{\tau}.$$  \hspace{1cm} (87)

The proof is similar to the one given in [17]. The rational expressions (86) and (87) are always used in the bilinearization of integrable equations.

4. Reductions

As we all know, the KdV equation and BSQ equation can be derived from the KP equation by imposing dimensional reductions. In this section, the reduction of the resulting KP system given in Section 3.3 will be carried out by taking constraints on matrices $$K$$ and $$L$$ in system (5), (15), (16) and (17). As a consequence, the KdV system and BSQ system will be obtained. For this purpose, we take $$N' = N$$ in system (5), (15), (16) and (17), while $$C = I$$ in scalar function $$S^{(i,j)}$$.  

4.1. Reduction to the KdV System

The CM approach for the KdV system has been discussed in Ref. [17]. To derive the KdV system from the KP system by reduction, we suppose $$K = -L$$. Then systems (5), (15), (16) and (17) lead to the equation set

$$LM + ML = rs',$$  \hspace{1cm} (88)

$$r_x = lr, \hspace{0.5cm} s_x = L's,$$  \hspace{1cm} (89)

$$r_y = -L^2 r, \hspace{0.5cm} s_y = (L^T)^2 s,$$  \hspace{1cm} (90)

$$r_t = 4L^3 r, \hspace{0.5cm} s_t = 4(L^T)^3 s,$$  \hspace{1cm} (91)

and $$S^{(i,j)}$$ in (6) is of the form

$$S^{(i,j)} = (-1)^j s^T L^j (I + M)^{-1} L^i r.$$  \hspace{1cm} (92)

Comparing the equality (11) with (28), we have $$S^{(i,j)}_L = 0$$. Hence, the KdV system can be obtained immediately from the KP systems (45), (59), (72) and (81). We list these equations as follows.

**KdV equation:**

$$\omega_t - \omega_{xxx} - 6\omega \omega_x = 0,$$  \hspace{1cm} (93)

of which the solution is given by

$$\omega = 2S^{(0,0)}_x = 2(s^T (I + M)^{-1} r)_x.$$  \hspace{1cm} (94)
**Modified KdV equation:** It follows from the anti-symmetric property $S(-1,0) = -S(0,-1)$ that

$$\mu = v = \partial_x \ln(1 - S(-1,0)).$$

Thus, both modified KP Equations (59) and (72) are reduced to the same modified KdV equation

$$\mu_t - \mu_{xxx} + 6\mu^2\mu_x = 0,$$

of which the solution is given by

$$\mu = \partial_x \ln(1 - S(-1,0)) = \partial_x \ln(1 - s^i(I + M)^{-1} L^{-1} r).$$

A natural fact is that (61) and (74) are reduced to the Miura transformation between the modified KdV Equation (95) and KdV Equation (93), i.e.,

$$-\omega = \mu_x + \mu^2.$$ (97)

**Schwarzian KdV equation:**

$$z_t - z_{xxx} + 3 \frac{z_{xx}^2}{2 z_x} = 0,$$ (98)

of which the solution is given by

$$z = S(-1,-1) + x = -s^i L^{-1} (I + M)^{-1} L^{-1} r + x.$$ (99)

In this case, the Miura transformation between the modified KdV Equation (95) and Schwarzian KdV Equation (98) is described as

$$\mu = \frac{z_{xx}}{2z_x}.$$ (100)

In solutions (94), (96) and (99), $L$, $M$, $r$ and $s$ are determined by equation set (88)–(91).

### 4.2. Reduction to the BSQ System

Now the reduction to the BSQ system will be carried out due to different relations between $L$ and $K$. We consider the following two cases.

**Case 1:** $K = \omega L$, where $\omega$ is defined in Proposition 3. In this case, systems (5), (15), (16) and (17) become

\[
LM - \omega ML = rs^i,
\]

\[
r_x = Lr, \quad s_x = -\omega L s,
\]

\[
r_y = -L^2 r, \quad s_y = \omega^2 (L^t)^2 s, \quad (102)
\]

\[
r_t = 4L^3 r, \quad s_t = -4 (L^t)^3 s,
\]

and $S^{(i,j)}$ in (6) reads

\[
S^{(i,j)} = \omega^i s^i L^j (I + M)^{-1} L^t r.
\]

Noting the equality (14) and the expression of $S^{(i,j)}$ given in (29), we find $S^{(i,j)}_t = 0$. The KP systems (45), (59), (72) and (81) thereby are reduced to the BSQ system. We list these equations as follows, where $y$ can be viewed as a temporal variable.

**BSQ equation:**

$$\omega_{xxx} + 6\omega\omega_x + 3\omega^{-1}\omega_{yy} = 0,$$ (105)
of which the solution is given by
\[ \omega = 2S_x^{(0,0)} = 2(s^t(I + M)^{-1}r)_x. \] (106)

**Modified BSQ equation:** The modified KP Equation (59) is reduced to the modified BSQ equation
\[ -\mu_{xxx} + 6\mu^2\mu_x + 6\mu_x\partial^{-1}\mu_y - 3\partial^{-1}\mu_{yy} = 0, \] (107)
of which the solution is given by
\[ \mu = \partial_x \ln(1 + S^{(0,-1)}) = \partial_x \ln(1 + \frac{1}{\omega} s^tL^{-1}(I + M)^{-1}r). \] (108)
In this case, (61) becomes the Miura transformation between modified BSQ Equation (107) and BSQ Equation (105).

Similarly, (72) is reduced to another modified BSQ equation
\[ -\nu_{xxx} + 6\nu^2\nu_x - 6\nu_x\partial^{-1}\nu_y - 3\partial^{-1}\nu_{yy} = 0, \] (109)
of which the solution is given by
\[ \nu = \partial_x \ln(1 - S^{(-1,0)}) = \partial_x \ln(1 - s^t(I + M)^{-1}L^{-1}r). \] (110)
Naturally, (74) turns into the Miura transformation between modified BSQ Equation (109) and BSQ Equation (105).

**Schwarzian BSQ equation:**
\[ -z_{xxxx} + \frac{3}{2}z_{xx}^2 - \frac{z_y^2}{z_x} - 3z_x\partial^{-1}\left(\frac{z_y}{z_x}\right)_y = 0, \] (111)
of which the solution is given by
\[ z = S^{(-1,-1)} + x = \frac{1}{\omega} s^tL^{-1}(I + M)^{-1}L^{-1}r + x. \] (112)

The Miura transformations between modified BSQ Equation (107) and Schwarzian BSQ Equation (111), respectively, modified BSQ Equation (109) and Schwarzian BSQ Equation (111), are in (83) and (84). In (106), (108), (110) and (112), L, M, r and s are determined by equation set (100)–(103).

**Case 2:** Supposing
\[ L = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix}, \quad K = \begin{pmatrix} \omega K_1 & 0 \\ 0 & \omega^2 K_2 \end{pmatrix}, \] (113)
where \( K_i \in \mathbb{C}_{N_1 \times N_1} \) with \( N_1 + N_2 = N \), then for the object (6) defined by systems (5), (15), (16) and (17), we have Proposition 4, which implies \( S^{(i,j)} = 0 \). Consequently, the KP systems (45), (59), (72) and (81) still, respectively, lead to BSQ Equation (105), modified BSQ Equations (107), (109) and Schwarzian BSQ Equation (111).

Although the reduced BSQ system of Case 1 is the same as the one of Case 2, the exact solutions for these two cases are totally different. Similar to discrete case [14,15], in Case 1 \( S^{(i,j)} \) only contains one kind of plane wave factor, while Case 2 \( S^{(i,j)} \) has two kinds of plane wave factors.
5. Conclusions

In the present paper, a more general Sylvester Equation (5) than the one in Ref. [17] is proposed to investigate the KP system. The Sylvester equation utilized here shares the same form as the discrete case [21]. By the Sylvester Equation (5), we define a master function $S(i, j)$, where an arbitrary $N' \times N$ constant matrix $C$ is introduced to guarantee the order of product $MC$. With some evolution relations of vectors $r$ and $s$, various evolution relations of $S(i, j)$ are presented. Furthermore, by some special values of $i$ and $j$, KP type equations, including a KP equation, modified KP equation and Schwarzian KP equation, are derived. In addition, the Miura transformations among these equations are also obtained. The procedure shown in present paper can be viewed as a continuous version of the CM approach in the discrete case [21]. By imposing some constraints on matrices $K$ and $L$, various equalities of $S(i, j)$ are proposed, by which the reductions of the KP system to KdV and BSQ systems are considered. The details for solving the Sylvester Equation (5) have been shown in [21], which are listed in the Appendix A.

When vectors $r$ and $s$ in systems (5), (15), (16) and (17) are replaced by matrices, then matrix KP system or noncommutative KP system can be constructed, which will be considered in the future. By means of a CM approach, elliptic soliton solutions to the lattice KdV system, ABS lattice and lattice KP system have been constructed by Nijhoff and his collaborators in recent papers [12,29]. It is of great interest to discuss the elliptic soliton solutions for continued integrable systems by utilizing this method, which will be one part of the ongoing research in this area.

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Appendix A. Exact Solutions to Systems (5), (15), (16) and (17)

Because of the invariance of $S(i, j)$ under any similar transformation of $K$ and $L$ (see Section 2.2.2), here we only need to consider the following canonical equation set (see [21]):

\begin{align}
\Gamma M - MA &= rs', \\
r_x &= \Gamma r, \quad s_x = -\Lambda' s, \\
r_y &= -\Gamma^2 r, \quad s_y = (\Lambda')^2 s, \\
r_t &= 4\Gamma^3 r, \quad s_t = -4(\Lambda')^3 s,
\end{align}

where $\Gamma$ and $\Lambda$ are, respectively, the Jordan canonical forms of $L$ and $K$. Corresponding to solvability condition $\mathcal{E}(K) \cap \mathcal{E}(L) = \emptyset$, hereafter we suppose $\mathcal{E}(\Gamma) \cap \mathcal{E}(\Lambda) = \emptyset$. Among the equation set (A1)–(A4), evolution Equations (A2), (A3) and (A4) are used to determine plane wave factor vectors $r$ and $s$, and the Sylvester Equation (A1) is used to define matrix $M$.

The solutions to the Sylvester Equation (A1) have been discussed systematically in the recent paper [21], where matrix $M$ was factorized into $M = FGH$ for $N \times N$ matrix.
Without showing the details, here we simply present some of the main results of solutions to the equation set \((A1)-(A4)\). We list some special matrices below:

- **Diagonal matrices:**
  \[
  \Gamma_0^\mathbb{N}(\{l_i\}_1^N) = \text{Diag}(l_1, l_2, \ldots, l_N),
  \]
  \[
  \Lambda_0^\mathbb{N}(\{k_j\}_1^{N'}) = \text{Diag}(k_1, k_2, \ldots, k_{N'}),
  \]

- **Jordan block matrices:**
  \[
  \Gamma_i^\mathbb{N}(a) = \begin{pmatrix}
  a & 0 & 0 & \cdots & 0 \\
  1 & a & 0 & \cdots & 0 \\
  0 & 1 & a & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & 1 \\
  \end{pmatrix}_{N \times N},
  \]
  \[
  \Lambda_i^\mathbb{N}(b) = \begin{pmatrix}
  b & 0 & 0 & \cdots & 0 \\
  1 & b & 0 & \cdots & 0 \\
  0 & 1 & b & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & 1 \\
  \end{pmatrix}_{N' \times N'},
  \]

- **Lower triangular Toeplitz matrix:** \([30,31]\)
  \[
  T^\mathbb{N}(\{a_i\}_1^N) = \begin{pmatrix}
  a_1 & 0 & 0 & \cdots & 0 \\
  a_2 & a_1 & 0 & \cdots & 0 \\
  a_3 & a_2 & a_1 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  a_N & a_{N-1} & a_{N-2} & \cdots & a_2 & a_1 \\
  \end{pmatrix}_{N \times N}.
  \]

- **Skew triangular Toeplitz matrix:**
  \[
  H^\mathbb{N}(\{b_j\}_1^{N'}) = \begin{pmatrix}
  b_1 & b_{N-1} & b_{N-2} & \cdots & b_{N'} \\
  b_2 & b_{N-1} & b_{N-2} & \cdots & 0 \\
  b_3 & b_{N-2} & b_{N-1} & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  b_{N'} & 0 & 0 & \cdots & 0 \\
  \end{pmatrix}_{N' \times N'}.
  \]

Meanwhile, the following expressions need to be considered:

**Exponential function:** \(\rho_i = e^{\xi_i}, \quad \xi_i = l_i x - l_i^2 y + 4l_i^3 t + \xi_{i}^{(0)}, \) with constants \(\xi_{i}^{(0)}\),
\[
(A11)
\]

**Exponential function:** \(\sigma_i = e^{\eta_i}, \quad \eta_i = -k_i x + k_i^2 y - 4k_i^3 t + \eta_{i}^{(0)}, \) with constants \(\eta_{i}^{(0)}\),
\[
(A12)
\]

**N, N' th-order vectors:** \(\tau = (r_1, r_2, \ldots, r_N)^T, \quad s = (s_1, s_2, \ldots, s_{N'})^T, \)
\[
(A13)
\]

**N \times N' matrix:** \(G_{\mathbb{N},0}^{\mathbb{N}}(\{l_i\}_1^N; \{k_j\}_1^{N'}) = (g_{i,j})_{N \times N'}, \quad g_{i,j} = \frac{1}{l_i - k_j}, \)
\[
(A14)
\]

**N_1 \times N'_2 matrix:** \(G_{\mathbb{N},0}^{\mathbb{N},2}(\{l_i\}_1^N; b) = (g_{i,j})_{N_1 \times N'_2}, \quad g_{i,j} = \left(\frac{1}{l_i - b}\right)^j, \)
\[
(A15)
\]

**N_2 \times N'_1 matrix:** \(G_{\mathbb{N},0}^{\mathbb{N},1}(a; \{k_j\}_1^{N'}) = (g_{i,j})_{N_2 \times N'_1}, \quad g_{i,j} = -\left(\frac{-1}{a - k_j}\right)^i, \)
\[
(A16)
\]
$N_1 \times N'_2$ matrix: $G_{[N_1 \times N'_2]}^{(i+j)}(a; b) = (g_{ij})_{N_1 \times N'_2}$, $g_{ij} = C_{i+j}^{-1} \frac{(-1)^{i+1}}{(a - b)^{i+j-1}}$.

(A17)

where

$$C_j = \frac{j!}{h(j-i)!} \quad (j \geq i).$$

**Theorem A1.** When $\Gamma$ and $\Lambda$ are taken as

$$\Gamma = \text{Diag}(T^{(N_i)}_1 \{ I_{[1]} \}, T^{(N_j)}_2 (I_{N_1} + 1), T^{(N_j)}_3 (I_{N_1} + 2), \ldots, T^{(N_j)}_i (I_{N_1} + (s-1))),$$

(A18)

$$\Lambda = \text{Diag}(\Lambda^{(N_i)}_0 (\{ k_j \}^N_1), \Lambda^{(N_j)}_0(k_{N'_1} + 1), \Lambda^{(N_j)}_0(k_{N'_1} + 2), \ldots, \Lambda^{(N_j)}_0(k_{N'_1} + (s-1))),$$

(A19)

with $\sum_{i=1}^s N_i = N$ and $\sum_{i=1}^s N'_i = N'$, then we have solutions

$$r = \begin{pmatrix}
  r_1^{(N_1)}(\{ I_{[1]} \}) \\
  r_2^{(N_1)}(I_{N_1} + 1) \\
  \vdots \\
  r_s^{(N_1)}(I_{N_1} + (s-1))
\end{pmatrix}, \quad s = \begin{pmatrix}
  s_1^{(N_1)}(\{ k_j \}^N_1) \\
  s_1^{(N_1)}(k_{N'_1} + 1) \\
  \vdots \\
  s_1^{(N_1)}(k_{N'_1} + (s-1))
\end{pmatrix},$$

(A20)

and $M = FGH$, where

$$F = \text{Diag}(T^{(N_i)}_1 \{ r_j \}^N_1), T^{(N_j)}_2 (I_{N_1} + 1), T^{(N_j)}_3 (I_{N_1} + 2), \ldots, T^{(N_j)}_i (I_{N_1} + (s-1))),$$

(A21)

$$H = \text{Diag}(\Lambda^{(N_i)}_0 (\{ s_j \}^N_1), H^{(N_j)}_0(k_{N'_1} + 1), H^{(N_j)}_0(k_{N'_1} + 2), \ldots, H^{(N_j)}_0(k_{N'_1} + (s-1))),$$

(A22)

and $G$ possesses block structure

$$G = (G_{ij})_{s \times s},$$

(A23)

with

$$G_{11} = G^{(N_1 \times N'_1)}_0 (\{ I_{[1]} \}; \{ k_j \}^N_1),$$

(A24)

$$G_{1j} = G^{(N_1 \times N'_1)} (\{ I_{[1]} \}; \{ k_j \}^N_1; k_{N'_1 + j-1}), \quad (1 < j \leq s),$$

(A25)

$$G_{11} = G^{(N_1 \times N'_1)}_0 (I_{N_1 + i - 1}; \{ k_j \}^N_1), \quad (1 < i \leq s),$$

(A26)

$$G_{11} = G^{(N_1 \times N'_1)}_0 (I_{N_1 + j - 1}; k_{N'_1 + j-1}), \quad (1 < i, j \leq s).$$

(A27)

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