Experimental engineering of arbitrary qudit states with discrete-time quantum walks

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The capability to generate and manipulate quantum states in high-dimensional Hilbert spaces is a crucial step for the development of quantum technologies, from quantum communication to quantum computation. One-dimensional quantum walk dynamics represents a valid tool in the task of engineering arbitrary quantum states. Here we affirm such potential in a linear-optics platform that realizes discrete-time quantum walks in the orbital angular momentum degree of freedom of photons. Different classes of relevant qudit states in a six-dimensional space are prepared and measured, confirming the feasibility of the protocol. Our results represent a further investigation of quantum walk dynamics in photonics platforms, paving the way for the use of such a quantum state-engineering toolbox for a large range of applications.

Introduction — The preparation of high-dimensional quantum states is of great significance in quantum information science and technology. Compared to qubits, qudit states — describing quantum systems in d-dimensional spaces — enable stronger foundational tests of quantum mechanics [1–3] and better-performing applications in secure quantum communications [4–9], quantum emulation [10, 11], quantum error correction [12–14], fault-tolerant quantum computation [15–19], and quantum machine learning [20–22].

Protocols performed on systems living in large Hilbert spaces require great control in light of the number of parameters required to describe states and operations. Nonetheless, qudit states have been prepared successfully in various physical settings [11, 23–32]. Such schemes rely on ad hoc strategies whose dependence on the underpinning dynamics makes their translation across different physical platforms difficult.

A promising way to achieve a higher degree of platform-universality is the use of the rich dynamics offered by Quantum Walks (QWs) [33–35]. These can be thought of as the quantum counterparts of classical random walks and comprise — in their discrete version — a qudit, named walker, endowed with an internal two-dimensional degree of freedom dubbed coin. At every time step, the walker moves coherently to neighbouring sites on a lattice, conditionally to its coin state [36]. QWs have been successfully implemented [37] in systems as diverse as trapped atoms [38] and ions [39, 40], photonic circuits [41–50], and optical lattices [51]. An approach for state engineering based on their dynamics offers hope of being applicable in a variety of different systems, independently of the details of the physical implementation.

While the QW dynamics was previously shown to allow the engineering of specific walker’s states [52, 53], in Ref. [54] a scheme was proposed to use discrete-time QWs on a line to prepare arbitrary qudit states with high probability. This is achieved by enhancing the degree of control over the walk’s dynamics through the arrangement of suitable step-dependent coin operations, which affect the coin-walker quantum correlations by de facto steering the state of the walker towards the desired final state, and finally projecting in the coin space. This removes the correlations between walker and coin, thus producing a pure walker state with the desired features. In light of the large parameter space that characterizes the problem at hand, a systematic approach to the identification of the right set of coin operations and final projection is necessary.

In this paper, we use the scheme of Ref. [54] to demonstrate a state-engineering protocol based on the controlled dynamics generated by QWs. We use the orbital angular momentum (OAM) degree of freedom of single-photon states as a convenient embodiment of the walker [48, 55, 56]. OAM-based experiments offer the possibility to cover Hilbert spaces of large dimensions in light of the favourable (linear) scaling of the number of optical elements with the size of the walk. Moreover, the scheme allows for the full control of the coin operation that is key to the implementation of the walk. In order to demonstrate the versatility of our scheme, we focus on the interesting classes of cat-like states and spin-coherent states [57, 58]. Furthermore, we show experimentally the capability of engineering arbitrary states. The quality of the generated states and the feasibility of the experimental protocol that we have put in place, demonstrate the effectiveness of a hybrid platform for quantum state engineering. Such platform holds together a programmable quantum system, the photonic QW in the angular momentum, and classical optimization algorithms to effectively reach a given target.

Engineering quantum walks. — We consider a discrete-time QW with a two-dimensional coin with logical states labelled as \{\left\vert\downarrow\right\rangle_c, \left\vert\uparrow\right\rangle_c\}. The dynamics are made up of consecutive unitary steps. At step \(t\), a coin operator \(\hat{C}_t\) changes the coin state and is then followed by a shift operator \(\hat{S}_wc\), which moves the walker conditionally to the coin state. Such transformations are described by the operators

\[
\hat{C}_t = \begin{pmatrix}
e^{i\xi_t}\cos\theta_t & e^{i\zeta_t}\sin\theta_t \\
-e^{-i\zeta_t}\sin\theta_t & e^{-i\xi_t}\cos\theta_t
\end{pmatrix},
\]

(1)
which accounts for the coin tossing, and \( \hat{S}_{w,c} = \sum_k |k - 1\rangle \langle k + 1| \pm |\uparrow\rangle \langle \uparrow| + \pm |\downarrow\rangle \langle \downarrow|\), which realizes the conditional motion of the walker. Here \( k \) is the lattice-site occupied by the walker and \( \{\theta_t, \xi_t, \zeta_t\} \) are parameters identifying a unitary transformation in two dimensions. The evolution through \( n \) steps of the QW is given by 

\[
\hat{U} = \prod_{t=1}^{n} \hat{S}_{w,c}\hat{C}_t.
\]

In Ref. [54] it was shown that it is always possible to find a set of coin operators \( \{\hat{C}_t\}_{t=1}^{n} \) that produce an arbitrary target state in the full coin-walker space. In addition, via suitable projection in the coin space, arbitrary walker states can also be obtained. The identification of the correct set of coin operators is enabled by a classical algorithm to maximize the fidelity between the final state of the walker, after projection of the coin, and the target \( (n + 1) \)-dimensional state.

To demonstrate the effectiveness of this approach for the state engineering of high-dimensional spaces, we here focus on classes of physically relevant states. First, we consider the synthesis of angular-momentum Schrödinger cat states [59], achieved by engineering coherent superpositions of extremal walker positions. The correspondence between the position space of the walker and an angular momentum of quantum number \( n/2 \), which will be illustrated and clarified later in this paper, makes the QW perfectly suited to synthesize this class of states. Schrödinger cat states play a crucial role in the investigations on foundations of quantum mechanics [60] and their generation is at the core of various quantum engineering protocols [57, 58, 61, 62]. The second class of states that we consider is spin-coherent states [63], which are the spin-like counterpart of coherent states of a quantum harmonic oscillator. Finally, in order to validate the flexibility of our approach, we demonstrate high-quality engineering of both balanced, and randomly sampled states.

Experimental apparatus — We have implemented a discrete-time QW with \( n = 5 \) steps, using the angular momentum states of light \( \{|m\rangle_w\} (m=\pm5,\pm3,\pm1) \) as the physical embodiment of the walker, while the logical states of the coin are encoded in circular-polarization states \( \{|R\rangle, |L\rangle\} \). We dub such degree of freedom as spin angular momentum (SAM) to mark the difference with OAM. Our experimental setup, which is shown schematically in Fig. 1 and follows Refs. [48, 49], allows for the full coin-walk evolution to take place in a single light beam, thus avoiding an exponential growth of optical paths as in previous interferometric implementations [48, 55, 56]. Arbitrary coin operators are achieved through a sequence of suitably arranged and oriented quarter-waveplates [64]. The shift operator \( \hat{S}_{w,c} \) is instead implemented using a Q-plate (QP) [65], an active device that uses an inhomogeneous birefringent medium to convert SAM into OAM and that can conditionally change the values of the OAM by a quantity \( 2q \) (here \( q \) is the topological charge of the
Figure 2. Experimental results for the engineering of angular momentum cat states. a) Representation on a Bloch-like ball of the four target states corresponding to the superposition of $| ± 5 \rangle$, which correspond to OAM states with maximum and minimum projection of the angular momentum along the quantization axis. b) Population of the OAM components after 5-step QW for the states $|\psi_i\rangle$ ($i = 1, 2, 3, 4$) in panel a). Odd-$m$ position states (bold lines on $x$-axis) should be the only ones involved in the state engineering. However, we report also the populations of even-$m$ position states (light-black numbers on $x$-axis) to illustrate possible imperfections at the generation and detection stages. The error bars associated with the experimental populations are shown by the transparent areas on top of each histogram. c)–d) Distributions of the probabilities $P_j = \langle B_j^{(j)} | \rho_{\text{exp}} | B_j^{(j)} \rangle$ ($j = 1, 2$) that the experimental walker state $\rho_{\text{exp}}$ is found to be one of the elements of the bases $B_j^{(j)} = \{|\psi_i\rangle, |\psi_{i+1}\rangle, |±4\rangle, |±3\rangle, |±2\rangle, |±1\rangle, |0\rangle\}$ with $p = 1$ for $j = 1$ and $p = 3$ for $j = 2$. All the error bars are due to Poissonian uncertainties, propagated through Monte Carlo methods. The state fidelities $F$ are calculated as described in the main text.

device) according to transformations

$$
|L, m\rangle \xrightarrow{\text{QP}} \cos \frac{\delta}{2} |L, m\rangle + ie^{2i\alpha_0} \sin \frac{\delta}{2} |R, m + 2q\rangle,
$$

$$
|R, m\rangle \xrightarrow{\text{QP}} \cos \frac{\delta}{2} |R, m\rangle + ie^{-2i\alpha_0} \sin \frac{\delta}{2} |L, m - 2q\rangle.
$$

The additional phase $\alpha_0$ between the two polarizations is compensated by changing the orientations of the waveplates which implement the coin operator of the subsequent step.

Single-photon states are generated via a type-II, collinear spontaneous-parametric-down-conversion source [cf. Fig. 1]. The photons emitted by the source are separated with a polarizing beam splitter (PBS) and coupled to two single-mode fibers (SMF). One photon acts as the trigger signal, while the other one undergoes the QW evolution. After the propagation in the SMF and the first PBS, the initial state of the walker and coin is prepared in $|\psi_0\rangle_{wc} = |0\rangle_w \otimes |+\rangle_c$ with $|+\rangle_c = (|\uparrow\rangle_c + |\downarrow\rangle_c)/\sqrt{2}$. At the end of an $n$-step QW, the protocol involves a projection of the coin state onto $|+\rangle_c$. This is experimentally implemented by a final PBS. The OAM analysis is performed through a spatial-light modulator (SLM) followed by coupling into a single-mode fiber, which allows for the measurement of arbitrary superposition of OAM components with high accuracy [66, 67]. The quantum state fidelity between the actual state of the walker and the target $(n + 1)$-dimensional state is estimated by projecting the OAM state onto a basis that contains the given target state [cf. Fig. 1].

**Engineering cat-like states in high dimensions.**—Our investigation on the engineering of quantum states living in Hilbert spaces of large dimensions starts from coherent superpositions of two extremal lattice sites of the walker. The isomorphism of the OAM with an angular momentum of quantum number $n/2$ allows us to put in correspondence the position states of the walker on the lattice $|±5\rangle$ with angular momentum states with minimum and maximum projections onto the quantization axis $|±5/2\rangle$ (for simplicity of notation, we will use position states only). Such isomorphism makes a coherent superposition state such as $(|5\rangle + e^{i\varphi}|−5\rangle)/\sqrt{2}$ (with $\varphi$ a suitable phase) a faithful angular momentum Schrödinger cat state [59], thus benchmarking the performance of our experiment with a relevant class of states [52, 53, 62] that is also used in quantum sensing [68, 69].

In Fig. 2 we report the experimental results for the generation of four of such states, which are conveniently pictured as the states pointing towards the poles of a Bloch-like ball. Quantum coherence between the components of such states has been tested by changing their relative phase. The values of the state fidelity between the experimentally synthesized states and their respective target ones are reported in Fig. 2. Hereafter we compute fidelities by projecting the state on the orthonormal basis which includes the target qudit in the 6-dimensional subspace associated to our 5-step QW, generated by the OAM eigenstate $\{|m\rangle_w\}$ ($m = ±5, ±3, ±1$).

The second class of relevant states that we addressed are spin-coherent states (SCSs) [61]. These are the counterparts of coherent states of the harmonic oscillator for a particle with spin $s$ [61, 70–72]. SCSs are eigenstates $|s\rangle$ of the component of the total spin-momentum operator $\hat{\mathbf{S}}$ pointing along the direction identified by the polar spherical angles $\{\theta, \phi\}$ [61, 63, 71, 73] A decomposition of such states over the $\{|s_z\rangle\}$ basis of the projected spin along z-direction.

$$
|L, m\rangle \xrightarrow{\text{QP}} \cos \frac{\delta}{2} |L, m\rangle + ie^{2i\alpha_0} \sin \frac{\delta}{2} |R, m + 2q\rangle,
$$

$$
|R, m\rangle \xrightarrow{\text{QP}} \cos \frac{\delta}{2} |R, m\rangle + ie^{-2i\alpha_0} \sin \frac{\delta}{2} |L, m - 2q\rangle.
$$
Cat-like
Spin-coherent
Computational basis
Fourier basis
Random states

Our results show the viability of QW-based approaches to state engineering, reinforcing the idea that numerical optimization [54]. Our tests have been run in a photonic platform using OAM as the embodiment of a quantum walker. This allowed us to implement a five-step QW, without exponential overhead in the number of required optical paths and with full control on the preparation, coin-operation, and detection stages. We focused on significant instances of high-dimensional states to benchmark the effectiveness of the protocol, demonstrating its ability to synthesize high-quality cat-like states. Our results show the viability of QW-based approaches to state engineering, reinforcing the idea that numer-
ical optimization complementing quantum dynamics of a sufficient degree of complexity is effective for high-dimensional state engineering. Further improvements of our approach can be envisaged by identifying appropriate routines to optimize the state engineering process in the presence of actual experimental imperfections. To this end, machine learning algorithms can be a promising add-on to our numerical optimization approach to adapt the coin operators to a given experimental implementation.

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Supplementary material: Experimental engineering of arbitrary qudit states with discrete-time quantum walks

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I. SUMMARY OF THE STATES ENGINEERED

In the following table we report the summary of target states engineered during the experiment, with relative quantum state fidelities and generation probabilities. The latter is provided by the algorithm developed in Ref.[1] together by the coin operators needed in the engineering process. The expected fidelities for all states is 1. The protocol and the experimental platforms are tested firstly with trivial states, as the element of computational basis corresponding to the eigenstate of OAM operator \{ |m\rangle \} = \{ |\pm 5\rangle, |\pm 3\rangle, |\pm 1\rangle \}. Then, superposition of two OAM components up to more complex states with arbitrary no-zero amplitudes on OAM basis, such as spin-coherent states, the Fourier basis and random extracted states. Quantum state fidelities are calculated measuring target state on orthonormal basis which contains the state itself. They are constructed according to Gram-Schmidt orthogonalization, starting from an ensemble of linearly independent states composed by elements of the computational basis and the target state.

Let us clarify the notation used in Table I. For the Fourier basis the convention employed is the following: \( |\text{QFT}_k\rangle = \frac{1}{\sqrt{6}} \sum_{j=1}^{6} e^{\frac{2\pi i j k}{5}} |j\rangle \), where \{ |j\rangle \} stands for the logical basis that in our case corresponds to the OAM eigenstates \{ |m\rangle \}. The notation \( |r_k\rangle \) and \( |c_k\rangle \) refers to real and complex random states respectively. Amplitudes of real states have been sampled uniformly in the range \([-0.5, 0.5]\). In Table II we report the resulting amplitudes.

| Target State | Probability | \( F_{\text{exp}} \) | Target State | Probability | \( F_{\text{exp}} \) |
|--------------|-------------|----------------|--------------|-------------|----------------|
| \(-5\)       | 0.5         | 0.981 ± 0.007 | \( \text{QFT}_1 \) | 0.14        | 0.969 ± 0.007 |
| \(-3\)       | 0.5         | 0.982 ± 0.007 | \( \text{QFT}_2 \) | 0.17        | 0.923 ± 0.022 |
| \(-1\)       | 0.5         | 0.960 ± 0.007 | \( \text{QFT}_3 \) | 0.17        | 0.911 ± 0.011 |
| \(1\)        | 0.5         | 0.995 ± 0.007 | \( \text{QFT}_4 \) | 0.17        | 0.980 ± 0.011 |
| \(3\)        | 0.5         | 0.975 ± 0.007 | \( \text{QFT}_5 \) | 0.17        | 0.936 ± 0.011 |
| \(5\)        | 0.5         | 0.994 ± 0.001 | \( \text{QFT}_6 \) | 0.17        | 0.945 ± 0.007 |
| \(-5 + 5\)   | 0.5         | 0.995 ± 0.001 | \( r_1 \)     | 0.22        | 0.911 ± 0.011 |
| \(-5 - 5\)   | 0.5         | 0.947 ± 0.002 | \( r_2 \)     | 0.16        | 0.923 ± 0.012 |
| \(-5 + i5\)  | 0.5         | 0.969 ± 0.002 | \( r_3 \)     | 0.17        | 0.941 ± 0.004 |
| \(-5 - i5\)  | 0.5         | 0.936 ± 0.003 | \( r_4 \)     | 0.14        | 0.947 ± 0.015 |
| \(S_1\rangle = \sqrt{2/5} (|5\rangle + |-5\rangle)\) | 0.15 | 0.970 ± 0.002 | \( r_5\rangle | 0.19 | 0.950 ± 0.005 |
| \(S_2\rangle = \sqrt{2/5} (|5\rangle - |-5\rangle)\) | 0.15 | 0.961 ± 0.003 | \( c_1\rangle | 0.16 | 0.956 ± 0.004 |
| \(-1/2 (S_1 + S_2)\) | 0.15 | 0.932 ± 0.004 | \( c_2\rangle | 0.29 | 0.935 ± 0.006 |
| \(-1/2 (S_1 - S_2)\) | 0.15 | 0.942 ± 0.004 | \( c_3\rangle | 0.17 | 0.925 ± 0.008 |
| \(S_1 + iS_2\rangle | 0.23 | 0.974 ± 0.003 | \( c_4\rangle | 0.16 | 0.944 ± 0.008 |
| \(S_1 - iS_2\rangle | 0.23 | 0.964 ± 0.004 | \( c_5\rangle | 0.28 | 0.946 ± 0.004 |

**TABLE I.** Summary of the measured states with relative generation probabilities and experimental quantum state fidelities.

| State | Amplitudes |
|-------|------------|
| \( r_1 \) | (0.51, 0.27, 0.13, 0.10, 0.29, 0.75) |
| \( r_2 \) | (0.19, 0.40, 0.04, 0.53, 0.37, 0.62) |
| \( r_3 \) | (0.50, 0.74, 0.40, 0.16, 0.10, 0.006) |
| \( r_4 \) | (0.50, 0.47, 0.55, 0.31, 0.36, 0.04) |
| \( r_5 \) | (0.24, 0.12, 0.72, 0.16, 0.54, 0.30) |
| \( c_1 \) | (0.04 + 0.35i, 0.34 + 0.41i, 0.10 + 0.24i, 0.18 − 0.26i, 0.11 − 0.11i, −0.47 + 0.22i) |
| \( c_2 \) | (0.19 − 0.33i, −0.43 + 0.30i, −0.18 − 0.02i, −0.37 + 0.42i, −0.12 − 0.10i, 0.23 + 0.38i) |
| \( c_3 \) | (−0.19 − 0.30i, −0.02 + 0.39i, 0.30 − 0.15i, 0.25 − 0.22i, −0.13 + 0.42i, 0.24 + 0.48i) |
| \( c_4 \) | (0.06 + 0.07i, 0.30 − 0.37i, −0.23 + 0.08i, 0.11 − 0.13i, −0.22 + 0.57i, 0.07 − 0.54i) |
| \( c_5 \) | (0.07 + 0.14i, 0.48 − 0.34i, −0.41 + 0.18i, −0.41 − 0.09i, −0.10 + 0.32i, 0.32 + 0.18i) |

**TABLE II.** Amplitudes of random states.
II. CAT STATES BASED ON SPIN COHERENT STATES: PHASE-SPACE PICTURE

In the main manuscript we have introduced the decomposition of a spin coherent state (SCS) $|s, \theta, \phi\rangle$ over the basis of eigenstates of angular momentum \{$s_z$\}. This reads

$$|s, \theta, \phi\rangle = \sum_{s_z=-s}^{s} \sqrt{(2s)! \over (s+s_z)!(s-s_z)!} e^{-i\phi s_z} C_s^{s+s_z} S_\theta^{s-s_z} |s_z\rangle,$$

where the functions $C_\theta$ and $S_\theta$ have been defined in the main manuscript. We have also introduced the SCS-based Schrödinger cat states built as the following superpositions of orthogonal states $|S_1\rangle := |5/2, \pi/2, 0\rangle$ and $|S_2\rangle := |5/2, -\pi/2, 0\rangle$:

$$|\psi_1\rangle = {1 \over \sqrt{2}} (|S_1\rangle + |S_2\rangle), \quad |\psi_2\rangle = {1 \over \sqrt{2}} (|S_1\rangle - |S_2\rangle).$$

In this Section, we aim at providing a brief analysis of the features of such states, which are best analyzed in a suitably defined phase space \cite{2}. In particular, we shall be considering the analogous of the Husimi $Q$ function \cite{3} defined as

$$Q_j(\alpha, \beta) = |(5/2, \alpha, \beta|\psi_j\rangle|^2 \quad (j = 1, 2)$$

in the spherical polar space where the Cartesian coordinates $(x, y, z)$ are mapped into $x \rightarrow Q_j(\alpha, \beta) \sin \alpha \cos \beta$, $y \rightarrow Q_j(\alpha, \beta) \sin \alpha \sin \beta$ and $z \rightarrow Q_j(\alpha, \beta) \cos \alpha$. Despite the simplicity of its definition, $Q_j(\alpha, \beta)$ captures important information about the quantum interference between the orthogonal components of $|\psi_j\rangle$, which differentiate such states from the incoherent mixture of SCSs $(|S_1\rangle \langle S_1| \pm |S_2\rangle \langle S_2|)/2$.

The orthogonality of $|S_1\rangle$ and $|S_2\rangle$ allows one to cast $Q_j(\alpha, \beta)$ as

$$Q_j(\alpha, \beta) = \frac{1}{2} \left( |q_+(5/2, \alpha, \beta)|^2 + |q_-(5/2, \alpha, \beta)|^2 + \text{sign}_j 2 \text{Re}[q_+(5/2, \alpha, \beta)q_-(5/2, \alpha, \beta)] \right)$$

where $q_\pm(s, \alpha, \beta) = \langle s, \alpha, \beta| s, \pm \theta, 0\rangle$ and $\text{sign}_1 = -\text{sign}_2 = +1$. Such scalar products can be evaluated explicitly for any value of $s$ by using the decomposition in Eq. (3) to get

$$q_\pm(s, \alpha, \beta) = (\pm 1)^s \frac{\Gamma(2s+1)}{\Gamma(s+2)} S^s(\alpha) C^s(\alpha) S^s(\theta) C^s(\theta) \left[ {}_2F_1(1, -s; s+1; \mp e^{-i\beta} T(\alpha) T(\theta)) + {}_2F_1(1, -s; s+1; \mp e^{i\beta} T^{-1}(\alpha) T^{-1}(\theta)) - 1 \right],$$

where $T(\alpha) = S(\alpha)/C(\alpha) = \tan(\alpha/2)$, $\Gamma(a) = \Gamma(a)/\Gamma(a-1)$, and $\Gamma(d)$ is the ordinary Gamma function with argument $d$.

Using such expressions, we can compute $Q_j(\alpha, \beta)$ to investigate its features. However, looking at such function directly does not provide sufficient information for the discrimination of an incoherent mixture and a state such as $|\psi - 1, 2\rangle$. On the other hand, we find more informative to consider that $\frac{1}{2} \left( |q_+(5/2, \alpha, \beta)|^2 + |q_-(5/2, \alpha, \beta)|^2 \right)$ is precisely the spherical SCS-based $Q$ function for the incoherent state $(|S_1\rangle \langle S_1| \pm |S_2\rangle \langle S_2|)/2$. Let us call it $Q_{inc}(\alpha, \beta)$, so that

$$Q_j(\alpha, \beta) = Q_{inc}(\alpha, \beta) + \text{sign}_j \text{Re}[q_+(5/2, \alpha, \beta)q_-(5/2, \alpha, \beta)],$$

which pinpoints the contribution coming from the fixed-phase relation typical of a coherent superposition. We thus focus on state $|\psi_2\rangle$, which is the one that has been addressed in our experimental endeavors, and look at the term $-\text{Re}[q_+(5/2, \alpha, \beta)q_-^*(5/2, \alpha, \beta)]$, and represent it in the spherical polar plane defined above. Fig. 1 (a) shows the results of our calculations.

Such interference term exhibits 10 equally separated lobes, and is clearly displays both rotation and inversion symmetry. In fact, one can show that, for a generic value of $s$, the interference term in the corresponding $Q$ function exhibits 4s equally spaced lobes. It is worth mentioning that in Ref. [2] another figure of merit for the analysis of the effects of the interference term was adopted. More specifically, Ref. [2] studied the form of

$$\frac{Q_j(\alpha, \beta)}{Q_{inc,j}(\alpha, \beta)} = 1 + \text{sign}_j \frac{2 \text{Re}[q_+(5/2, \alpha, \beta)q_-^*(5/2, \alpha, \beta)]}{Q_{inc,j}(\alpha, \beta)},$$
which thus quantifies the effect of quantum coherence as the deviation of $Q_j(\alpha, \beta)$ from 1, whose representation in the chosen spherical polar space is a sphere of unit radius. When making use of such figure of merit, we find Fig. 1 (b), which shows a lobate behavior significantly different from the (incoherent) spherical trend.

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