We derive basic properties of Triebel-Lizorkin-Lorentz spaces important in the treatment of PDE. For instance, we prove Triebel-Lizorkin-Lorentz spaces to be of class $\mathcal{H}T$, to have property $(\alpha)$, and to admit a multiplier result of Mikhlin type. By utilizing these properties we prove the Laplace and the Stokes operator to admit a bounded $H^\infty$-calculus. This is finally applied to derive local strong well-posedness for the Navier-Stokes equations on corresponding Triebel-Lizorkin-Lorentz ground spaces.

1 Introduction

The Triebel-Lizorkin-Lorentz spaces $F^{s,r}_{p,q}$, a unification of Triebel-Lizorkin spaces $F^s_{p,q}$ and Lorentz spaces $L_{p,r}$, were introduced by Yang, Cheng and Peng (see [23]) in 2005, where the possible parameters are $s \in \mathbb{R}$, $1 < p, q < \infty$ and $1 \leq r \leq \infty$. Implicitly the spaces $F^{s,r}_{p,q}$ already appear in the pertinent monograph of Triebel (see [20, Sec. 2.4.2]). By means of wavelet theory in [23] $F^{s,r}_{p,q}$ is proved to be a real interpolation scale of Triebel-Lizorkin spaces, which is very important and helpful, in particular for applications to PDE. In 2011, Xiang and Yan already considered Triebel-Lizorkin-Lorentz spaces in the context of partial differential equations and established the local well-posedness of a quasi-geostrophic equation (see [22]).

The scale $F^{s,r}_{p,q}$ contains many important function spaces: By setting $r = p$, we obtain the Bessel-potential spaces $H^s_p$ for $q = 2$ as well as the Sobolev-Slobodeckij spaces $W^s_p$ for $q = p$ in the case $s \notin \mathbb{Z}$ resp. $q = 2$ in the case $s \in \mathbb{Z}$. In particular, we obtain the Lebesgue spaces $L_p$ by setting $s = 0$ as well as the Lorentz spaces $L_{p,r} = F^{0,r}_{p,2}$.

It is therefore natural to ask, whether the scale of Triebel-Lizorkin-Lorentz spaces is suitable in the treatment of partial differential equations, since a corresponding outcome would yield results simultaneously in all spaces listed above.

The purpose of this paper is twofold. First we establish further fundamental properties of Triebel-Lizorkin-Lorentz spaces, such as of class $\mathcal{H}T$, property $(\alpha)$, useful equivalent norms, a Mikhlin type multiplier result, etc. (see Section 3). Second, we apply these properties in order to prove a bounded $H^\infty$-calculus for the Laplace and the Stokes operator (Section 4 and 5) which, in turn, will then be utilized to construct a maximal
strong solution \((u, \nabla p)\) of the Navier-Stokes equations

\[
\begin{aligned}
    \frac{d}{dt} u - \Delta u + \nabla p + (u \cdot \nabla) u &= f \quad \text{in} \ (0, T) \times \mathbb{R}^n, \\
    \text{div} \ u &= 0 \quad \text{in} \ (0, T) \times \mathbb{R}^n, \\
    u(0) &= u_0 \quad \text{in} \ \mathbb{R}^n
\end{aligned}
\]

in these spaces (Section 7). In fact, we prove the following result.

**Theorem 1.1.** Let \(n \in \mathbb{N}, n \geq 2, s > -1\) and let \(1 < p, q, r < \infty\) and \(1 < \eta < \infty\) such that \(\frac{s}{2p} + \frac{1}{\eta} < 1\). Then for every \(f \in L_{\eta}(0, \infty) \times (F_{p,q}^{s,r})^n\) and initial value \(u_0 \in (F_{p,q}^{s,r}, F_{p,q}^{s+2,r})^n\) with vanishing divergence, there is a maximal time \(T^* > 0\) such that the Navier-Stokes equations (NSE) \((f, u_0)\) have a unique maximal strong solution \((u, \nabla p)\) on \((0, T^*)\) satisfying

\[
u \in H^1_{\eta}((0, T), (F_{p,q}^s)^n) \cap L_\eta((0, T), (F_{p,q}^{s+2,r})^n),
\]

\[\nabla p \in L_\eta((0, T), (F_{p,q}^s)^n)\]

for every \(T \in (0, T^*)\). If additionally \(\frac{s}{2p} + \frac{2}{\eta} < 1\), then \(u\) is either a global solution or we have

\[
T^* < \infty \quad \text{and} \quad \limsup_{t \to T^*} \|u(t)\|_{(F_{p,q}^{s,r}, F_{p,q}^{s+2,r})^n_{1-1/\eta, \eta}} = \infty.
\]

**Remark 1.2.** The constraints on the parameters \(p, \eta\), especially the more restrictive one for the additional property that \(u\) is either a global solution or (1.1) holds, rely on the use of the multiplication result for \(F_{p,q}^{s,r}\)-spaces given in Lemma 6.4. They might be improved to the standard constraint in classical function spaces such as \(L^p\). This, however, requires optimal results on multiplication for \(F_{p,q}^{s,r}\)-spaces which by now are not available and would go beyond the scope of this note.

In order to prove Theorem 1.1 we consider the usual operatorial formulation relying on the use of Helmholtz projection and Stokes operator (see Theorem 7.1). Existence of the Helmholtz decomposition and maximal regularity for the Stokes operator in Triebel-Lizorkin-Lorentz spaces are proved in Section 5. The concept of maximal regularity is introduced in the next section. Starting from the pioneering works of Leray, Hopf, Fujita, Kato, Solonnikov, Giga, etc., local well-posedness for the Navier-Stokes equations in classical function spaces has been proved by a vast number of authors. We refrain from trying to give a complete list here. Instead we refer to the well-known monographs [19, 8] and the literature cited therein. For a comprehensive survey of results on the associated linear Stokes operator we also refer to [11]. As mentioned above, our approach has the advantage that it unifies many of the existing results on local well-posedness by the fact that Triebel-Lizorkin-Lorentz spaces include quite a number of classical function spaces. The approach to the Navier-Stokes equations given here is meant as a first step towards a theory in Triebel-Lizorkin-Lorentz spaces. Further investigations such as for instance well-posedness in critical Triebel-Lizorkin-Lorentz spaces are left as a future challenge.

This article is organized as follows. In Section 2 we fix notation and recall briefly basic notions and tools related to maximal regularity. In Section 3 we establish further properties of Triebel-Lizorkin-Lorentz spaces relevant for the handling of PDE. In Section 4 we prove a bounded \(H^\infty\)-calculus for the Laplacian and in Section 5 existence of the Helmholtz decomposition and a bounded \(H^{\infty,c}\)-calculus for the Stokes operator on
solenoidal subspaces. The same property is proved to hold for the time derivative operator in Section 6 where we also give embedding results important to handle nonlinearities. Theorem 1.1 is proved in Section 7. Finally, Appendix 8 collects basic facts on extension operators used in the sections before.

2 Basic notation and preliminary results

Generally $\| \cdot \|$ denotes the euclidean norm on $\mathbb{R}^n$ and the natural numbers $\mathbb{N}$ do not contain zero whereas $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For two equivalent norms $\| \cdot \|$ and $\| \cdot \|$ on a vector space $X$ we write $\| \cdot \| \sim \| \cdot \|$. Similarly, we use the notation $\| \cdot \| \lesssim \| \cdot \|$ if there is a constant $C > 0$ such that $\| \cdot \| \leq C \| \cdot \|$. Commonly $C$ denotes a generic positive constant. Especially during estimates we also use $C', C'', \ldots$ when the constant changes. The space of Schwartz functions is denoted by $\mathcal{S}(\mathbb{R}^n)$ and thus $\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distributions. The corresponding space of $X$-valued Schwartz functions (where $X$ is a Banach space) is $\mathcal{S}(\mathbb{R}^n, X)$ and we set $\mathcal{S}'(\mathbb{R}^n, X) = \mathcal{L}(\mathcal{S}(\mathbb{R}^n), X)$, that is, the space of continuous linear operators $T : \mathcal{S}(\mathbb{R}^n) \to X$. The space of Distributions is $\mathcal{D}'(\mathbb{R}^n)$. Nullspace resp. Range of a linear operator $T : \mathcal{D}(T) \subset X \to X$ are denoted by $\mathcal{N}(T)$ resp. $\mathcal{R}(T)$. The support of a function $f$ is denoted by $\text{spt}(f)$. For a Banach space $X$ and a measure space $(\Omega, \mathcal{A}, \mu)$ let $\mathcal{M}(\Omega, X)$ be the space of measurable (i.e., also separable-valued) functions $f : \Omega \to X$. The Lorentz space $L_{p,r}(X) = \mathcal{L}_{p,r}(\Omega, X) \subset \mathcal{M}(\Omega, X)$ with parameters $1 \leq p, r \leq \infty$ consists of those functions whose Lorentz quasinorm

$$
\|f\|_{L_{p,r}(\Omega, X)} = \begin{cases} 
\left( \int_0^\infty \left( t^{\frac{r}{p}} f^*(t) \right)^r \frac{dt}{t} \right)^{\frac{1}{r}}, & r < \infty \\
\sup_{t \geq 0} t^{\frac{1}{r}} f^*(t), & r = \infty
\end{cases}
$$

is finite, where $f^*(t) = \inf \{ \alpha \geq 0 : d_f(\alpha) \leq t \}, \quad t \geq 0$

is the decreasing rearrangement and

$$
d_f(\alpha) = \mu(\{|z| \leq \alpha, \|f(z)\| > \alpha\}), \quad \alpha \geq 0
$$

is the distribution function of $f \in \mathcal{M}(\Omega, X)$. Two functions in $L_{p,r}(\Omega, X)$ are considered equal, if they are equal on a null set (with respect to $\mu$).

For $1 < p_0, p_1, p < \infty, p_0 \neq p_1$, $1 \leq r_0, r_1, r \leq \infty$ and $0 < \theta < 1$ such that $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ we have

$$
(L_{p_0,r_0}(X), L_{p_1,r_1}(X))_{\theta,r} = L_{p,r}(X)
$$

(see [20, Rem. 1.18.6/4]) where $(\cdot, \cdot)_{\theta,r}$ denotes the real interpolation functor. In view of the identity $L_{p,p}(X) = L_p(X)$ the Lorentz spaces are identified as real interpolation spaces of the Lebesgue spaces $L_p(X)$. Note that $L_{p,r}(\Omega, X)$ is hence normable in the case $p > 1$. We denote the corresponding norm by $\| \cdot \|_{L_{p,r}(X)}$.

The space $l_q^s(X)$ (for $s \in \mathbb{R}$ and $1 \leq q < \infty$) consists of the sequences $a = (a_k)_{k \in \mathbb{N}_0} \subset X$ in a Banach space $X$ that fulfill

$$
\|a\|_{l_q^s(X)} = \left( \sum_{k \in \mathbb{N}_0} [2^{-ks} \|a_k\|^q]^\theta \right)^{\frac{1}{q}} < \infty.
$$

In the case $X = \mathbb{C}$ we write $l_q^s$ instead of $l_q^s(\mathbb{C})$. 

3
Real resp. complex interpolation of the Sobolev spaces

\[ W^k_p(\mathbb{R}^n, X) = \{ u \in L_p(\mathbb{R}^n, X) : \partial^\alpha u \in L_p(\mathbb{R}^n, X) \ \forall \alpha \in \mathbb{N}_0^m, |\alpha| \leq k \} \]

leads to Sobolev-Slobodeckij spaces

\[ W^s_p(\mathbb{R}^n, X) = \left( L_p(\mathbb{R}^n, X), W^k_p(\mathbb{R}^n, X) \right) \]

resp. Bessel-potential spaces

\[ H^s_p(\mathbb{R}^n, X) = \left( L_p(\mathbb{R}^n, X), W^k_p(\mathbb{R}^n, X) \right) \]

where \( k \in \mathbb{N}, 1 < p < \infty, 0 < s < k \) and \( X \) is a complex Banach space. In the following we assume that \( X \) is of class \( \mathcal{H}T \) (we give one possible definition for spaces of class \( \mathcal{H}T \) below). In the case \( s = m \in \mathbb{N}_0 \) the Bessel-potential spaces are given by the Sobolev spaces, so \( H^m_p(\mathbb{R}^n, X) = W^m_p(\mathbb{R}^n, X) \). For \( s \in \mathbb{R} \) and \( 1 < p < \infty \) we will also use the representation

\[ H^s_p(\mathbb{R}^n, X) = \left\{ u \in \mathcal{S}'(\mathbb{R}^n, X) : \mathcal{F}^{-1}(1 + |\xi|^2)^{s/2} \mathcal{F} u \in L_p(\mathbb{R}^n, X) \right\}, \]

where \( \| \mathcal{F}^{-1}(1 + |\xi|^2)^{2s/2} \mathcal{F} u \|_{L_p(\mathbb{R}^n, X)} \) is an equivalent norm in \( H^s_p(\mathbb{R}^n, X) \) and \( \mathcal{F} \) denotes the Fourier transform. Moreover, the continuous embeddings

\[ H^s_p(\mathbb{R}^n, X) \subset W^{s-\epsilon}_p(\mathbb{R}^n, X) \subset H^{s-2\epsilon}_p(\mathbb{R}^n, X) \]

hold for any \( \epsilon > 0 \). We refer to [12] and [21] for a detailed treatise of Bessel-potential and Sobolev-Slobodeckij spaces. In general, if \( \mathcal{F}(\mathbb{R}^n, X) \) is any normed function space (e.g. \( \mathcal{F} = H^s_p, W^s_p \) and \( \Omega \subset \mathbb{R}^n \) is any domain, we denote by \( \mathcal{F}(\Omega, X) \) the space of restrictions of functions \( u \in \mathcal{F}(\mathbb{R}^n, X) \) to \( \Omega \), equipped with the norm \( \| u \|_{\mathcal{F}(\Omega, X)} = \inf \{ \| v \|_{\mathcal{F}(\mathbb{R}^n, X)} : v \in \mathcal{F}(\mathbb{R}^n, X), v|_{\Omega} = u \} \).

In order to deal with operator-valued Fourier multipliers we employ the following concept. Let \( X, Y \) be complex Banach spaces. Let \( \mathcal{E}_P \) denote the set of families of random variables \( (\epsilon_i)_{i \in I} \) on a probability space \( P = (\Omega, \mathcal{A}, \mu) \) with values in \( \{ \pm 1 \} \), which are independent and symmetrically distributed. We refer to a family of continuous linear operators \( \mathcal{T} \subset \mathcal{L}(X, Y) \) as \( \mathcal{R}\text{-bounded} \) if there is a probability space \( P = (\Omega, \mathcal{A}, \mu) \) with \( \mathcal{E}_P \neq \emptyset, p \in [1, \infty) \) and a constant \( C > 0 \) such that for all \( N \in \mathbb{N}, (\epsilon_1, \ldots, \epsilon_N) \in \mathcal{E}_P, T_i \in \mathcal{T} \) and \( x_i \in X \) (for \( 1 \leq i \leq N \))

\[ \left\| \sum_{i=1}^N \epsilon_i T_i x_i \right\|_{L_p(\Omega, Y)} \leq C \left\| \sum_{i=1}^N \epsilon_i x_i \right\|_{L_p(\Omega, X)}. \tag{2.1} \]

In this case we call \( \mathcal{R}_p(\mathcal{T}) := \inf \{ C > 0 : (2.1) \text{ holds} \} \) the \( \mathcal{R}\text{-bound} \) or the \( \mathcal{R}_p\text{-bound} \). Note that \( \mathcal{R}\text{-boundedness} \) implies the boundedness of \( \mathcal{T} \subset \mathcal{L}(X, Y) \). If a family \( \mathcal{T} \subset \mathcal{L}(X, Y) \) is \( \mathcal{R}_p\text{-bounded} \) for a \( p \in [1, \infty) \) then it is also \( \mathcal{R}_q\text{-bounded} \) for any \( q \in (1, \infty) \). In this case (2.1) also holds for a (possibly different) constant \( C > 0 \) if we replace \( P \) by an arbitrary probability space \( Q \) with \( \mathcal{E}_Q \neq \emptyset \). Also note that, in view of Lebesgue’s dominated convergence theorem, it is sufficient to claim (2.1) for \( x_i \) in a dense subspace of \( X \). The following result is useful to extend boundedness to \( \mathcal{R}\text{-boundedness} \) in some concrete cases (see [3, Lemma 3.5]).
**Theorem 2.1** (Kahane’s contraction principle). Let $X$ be a Banach space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, $P = (\Omega, \mathcal{A}, \mu)$ a probability space and $1 \leq p < \infty$. Let $N \in \mathbb{N}$ and $a_j, b_j \in \mathbb{K}$ with $|a_j| \leq |b_j|$ for $j = 1, \ldots, N$. Then we have for all $x_1, \ldots, x_N \in X$ and $c_1, \ldots, c_N \in \mathcal{E}_P$
\[
\left\| \sum_{i=1}^{N} a_i c_i x_i \right\|_{L_p(\Omega, X)} \leq C_R \left\| \sum_{i=1}^{N} b_i c_i x_i \right\|_{L_p(\Omega, X)},
\]

where $C_R = 1$ and $C_C = 2$.

We call a linear and densely defined operator $A : \mathcal{D}(A) \subset X \rightarrow X$ pseudo-sectorial if its spectrum $\sigma(A)$ is contained in a closed sector $\Sigma_\varphi$ with angle $\varphi \in (0, \pi)$, where $\Sigma_\varphi = \{z \in \mathbb{C} \cup \{0\} : |\arg(z)| < \varphi\}$, and the family $\{\lambda(\lambda + A)^{-1} : \lambda \in \Sigma_\varphi\} \subset \mathcal{L}(X)$ is bounded. If $\{\lambda(\lambda + A)^{-1} : \lambda \in \Sigma_\varphi\} \subset \mathcal{L}(X)$ is even $\mathcal{R}$-bounded, $A$ is called pseudo-$\mathcal{R}$-sectorial. We omit the prefix "pseudo-" if the range $\mathcal{R}(A) \subset X$ is dense and so we get a sectorial resp. $\mathcal{R}$-sectorial operator. We denote the infimum over all $\varphi \in (0, \pi)$, such that the family $\{\lambda(\lambda + A)^{-1} : \lambda \in \Sigma_\varphi\} \subset \mathcal{L}(X)$ is bounded, by $\varphi_A$ (spectral angle) if $A$ is a (pseudo-)sectorial operator and likewise $\varphi_A^p$ is the infimum over all $\varphi \in (0, \pi)$ such that this family is $\mathcal{R}$-bounded if $A$ is a (pseudo)-$\mathcal{R}$-sectorial operator.

For a pseudo-sectorial operator $A$ and a fixed angle $\varphi < \varphi_A$ we will make use of the Dunford calculus
\[
f \mapsto f(A),
\]
which maps a function $f \in \mathcal{H}_0(\Sigma_\varphi) = \bigcup_{\alpha, \beta < 0} \mathcal{H}_{\alpha, \beta}(\Sigma_\varphi)$ to a bounded operator on $X$, as well as of its extension to $\mathcal{H}_p(\Sigma_\varphi) = \bigcup_{\alpha \in \mathbb{R}} \mathcal{H}_{\alpha}(\Sigma_\varphi)$, when $A$ is sectorial. Here $\mathcal{H}_{\alpha, \beta}(\Sigma_\varphi)$ is the space of holomorphic functions $f : \Sigma_\varphi \rightarrow \mathbb{C}$ such that
\[
\|f\|_{\alpha, \beta} = \sup_{|z|<1} |z^\alpha f(z)| + \sup_{|z|>1} |z^{-\beta} f(z)|
\]
is finite. We refer to [10] for a precise definition and treatise of this functional calculus. Note that for a function $f \in \mathcal{H}_{\alpha, \alpha}(\Sigma_\varphi)$ we get a bounded operator $f(A)$ in the case $\alpha < 0$ but in general only a closed operator on the domain $\mathcal{D}(f(A)) = \{f \in X : (g^k f)(A) x \in \mathcal{D}(A^k) \cap \mathcal{R}(A^k)\}$. Here $k > \alpha$ is a nonnegative integer and $g \in \mathcal{H}_0(\Sigma_\varphi)$ is the function $g(z) = \frac{1}{1+|z|^{1-\alpha}}$, which leads to a bijective mapping $g(A)^{-k} : \mathcal{D}(A^k) \cap \mathcal{R}(A^k) \rightarrow X$. A slight modification of this functional calculus leads to the following well-known characterization of sectorial operators (see [10] Prop. 3.4.4]).

**Proposition 2.2.** An operator $A : \mathcal{D}(A) \subset X \rightarrow X$ is pseudo-sectorial with angle $\varphi_A < \frac{\pi}{2}$ if and only if $-A$ is the generator of a bounded holomorphic strongly continuous semigroup.

The just introduced functional calculus is also used to describe an important property of an operator $A$. Let $A : \mathcal{D}(A) \subset X \rightarrow X$ be a sectorial operator. Then $A$ has a bounded $H^\varphi$-calculus in $X$ if for some $\varphi \in (\varphi_A, \pi)$ there is a constant $C_\varphi > 0$ such that for any $f \in \mathcal{H}_0(\Sigma_\varphi)$ we have
\[
\|f(A)\|_{\mathcal{L}(X)} \leq C_\varphi \|f\|_{L_\varphi(\Sigma_\varphi)}.
\]
In this case (2.2) also holds for all bounded holomorphic functions $f$ on $\Sigma_\varphi$. The infimum over all angles $\varphi \in (\varphi_A, \pi)$ such that (2.2) holds with a constant $C_\varphi > 0$, is called $H^\varphi$-angle and is denoted by $\varphi_A^\varphi$. Likewise we say that $A$ has an $\mathcal{R}$-bounded $H^\varphi$-calculus in $X$ if the set
\[
\{f(A) : f \in \mathcal{H}_0(\Sigma_\varphi), \|f\|_{L_\varphi(\Sigma_\varphi)} \leq 1\} \subset \mathcal{L}(X)
\]
is $\mathcal{R}$-bounded and the related $\mathcal{R}H^\infty$-angle is denoted by $\varphi_{A}^{\mathcal{R}\infty}$. If $A$ has a bounded $H^\infty$-calculus, then

$$\mathcal{D}(A^\alpha) = [X, \mathcal{D}(A)]_\alpha$$

holds for all $0 < \alpha < 1$ (see [16]), where the fractional power $A^\alpha : \mathcal{D}(A^\alpha) \subset X \to X$ is defined via the functional calculus above with the function $z \mapsto z^\alpha$.

We recall the following two assertions that frequently occur in the context of the $H^\infty$-calculus. They have been proved in [17] (see the proofs of Prop. 2.9 and Prop. 2.7, respectively).

**Lemma 2.3.**

(a) Let $0 < \varphi < \pi$. Then for all $\alpha \in \mathbb{N}_0$ there is a constant $C_{\alpha,\varphi} > 0$ so that for every holomorphic and bounded function $h : \Sigma_\varphi \to \mathbb{C}$ we have

$$\sup_{\xi \in \mathbb{R}^n \setminus \{0\}} |\xi|^\alpha |\mathcal{D}^\alpha h(|\xi|^2)| \leq C_{\alpha,\varphi} \|h\|_{L^\infty(\Sigma_\varphi)}.$$

(b) Let $\frac{\pi}{2} < \varphi < \pi$. Then for $k = 0, 1$ there is a constant $C_{\varphi} > 0$ so that for every $h \in \mathcal{H}_0(\Sigma_\varphi)$ we have

$$\sup_{\xi \in \mathbb{R}^n \setminus \{0\}} |\xi|^k |\mathcal{D}^k h(i\xi)| \leq C_{\varphi} \|h\|_{L^\infty(\Sigma_\varphi)}.$$

Next we give the definition of maximal regularity (due to [15]) for an operator $A : \mathcal{D}(A) \subset X \to X$, which is the generator of a bounded holomorphic strongly continuous semigroup $(S(t))_{t \geq 0}$ on a complex Banach space $X$. Therefore, we fix $1 < p < \infty$ and $0 < T \leq \infty$. $A$ has maximal $L_p$-regularity on $(0, T)$ if for all $f \in L_p((0, T), X)$ the solution

$$u(t) = \int_0^t S(t-s)f(s)ds$$

of the Cauchy problem

$$\begin{cases}
    u'(t) - Au(t) = f(t), & t \in (0, T) \\
    u(0) = 0
\end{cases}$$

(2.4)

is Fréchet differentiable a.e., takes its values in $\mathcal{D}(A)$ a.e. and $u', Au \in L_p((0, T), X)$. In this case we get

$$\|u'\|_{L_p((0,T),X)} + \|Au\|_{L_p((0,T),X)} \leq C\|f\|_{L_p((0,T),X)}$$

(2.5)

by application of the closed graph theorem. We write $A \in \text{MR}(X, C)$ if $A$ has maximal $L_p$-regularity for some (or equivalently for all) $1 < p < \infty$ on some $(0, T)$ so that (2.5) holds with a constant $C = C(T) > 0$. If $A \in \text{MR}(X, C)$ and $C$ doesn’t depend on $T$ (i.e., (2.5) holds for $T = \infty$) we write $A \in \text{MR}(X)$.

Now we take a look at the advantages of maximal regularity. Again for a complex Banach space $X$, let $A : \mathcal{D}(A) \subset X \to X$ be the generator of a bounded holomorphic strongly continuous semigroup. For $1 < p < \infty$ and $T \in (0, \infty]$ we set

$$E_T := H^1_p((0,T),X) \cap L_p((0,T), \mathcal{D}(A))$$
(the solution space of the related Cauchy problem) and
\[ F_T \times \mathbb{I} := L_p((0,T),X) \times \{ x = u(0) : u \in E_T \} \]
(the data space). Note that \( \mathbb{I} \) is a Banach space with the norm \( \|x\|_\mathbb{I} = \inf_{u(0)=x} \|u\|_{E_T} \), independent of \( T \) and we have
\[ \mathbb{I} = (X, \mathcal{D}(A))_{1 - \frac{1}{p}, p} \quad (2.6) \]
(see [17], Prop. 3.4.4). If \( A \) has maximal \( L_p \)-regularity on a finite interval \((0,T)\), then the solution operator
\[ \begin{align*}
L : E_T & \longrightarrow F_T \times \mathbb{I}, \\
\quad u & \longmapsto \left( \left( \frac{d}{dt} - A \right) u \right) \quad (2.7)
\end{align*} \]
is an isomorphism. This leads to the estimate
\[ \|u\|_{H^1((0,T),X) \cap L_p((0,T),\mathcal{D}(A))} \leq C(T)\left( \|f\|_{L_p((0,T),X)} + \|x\|_{\mathbb{I}} \right) \quad (2.8) \]
when \( u := L^{-1} \left( (\frac{d}{dt}) \right) \) is the solution for some \( \left( \frac{d}{dt} \right) \in F_T \times \mathbb{I} \).

**Lemma 2.4.** Let \( X \) be a Banach space, \( 1 < p < \infty \), \( 0 < T_0 < \infty \) and let \( A \in \text{MR}(X,C(T_0)) \). Then there exists a constant \( C' = C'(T_0) > 0 \) such that
\[ \|L^{-1} \left( (\frac{d}{dt}) \right) \|_{H^1((0,T),X) \cap L_p((0,T),\mathcal{D}(A))} \leq C'(T_0)\|f\|_{L_p((0,T),X)} \]
holds for all \( T \in (0,T_0] \) and for all \( f \in L_p((0,T),X) \), where \( L \) is the solution operator from \((2.7)\).

**Proof.** By the trivial extension of \( f \in L_p((0,T),X) \) to \((0,T_0)\) we get the estimate \((2.5)\) with a constant independent of \( T \in (0,T_0) \). Now the assertion follows from the fact that the Poincaré inequality \( \|u\|_{L_p((0,T),X)} \leq K\|u'\|_{L_p((0,T),X)} \) holds with a constant \( K > 0 \), which is independent of \( T \in (0,T_0] \) as well. \( \square \)

**Lemma 2.5.** Let \( 1 < p < \infty \) and \( T \in (0,\infty] \). Then, with the notation above, we have the continuous embedding
\[ E_T \subset BUC([0,T],\mathbb{I}) \]
(where \( BUC \), as usual, means bounded and uniformly continuous). Here the operator \( A : \mathcal{D}(A) \subset X \to X \) only needs to be closed and densely defined in a Banach space \( X \).

**Proof.** The case \( T = \infty \) follows essentially from the strong continuity of the translation semigroup. Then, by a standard extension and retraction argument one gets the case \( T < \infty \) as a consequence. See [3] Prop. 1.4.2] for details. \( \square \)

In the theory of partial differential equations, the notions of class \( \mathcal{H}T \) and property \( (\alpha) \) for Banach spaces turned out to be significant. A Banach space \( X \) is of class \( \mathcal{H}T \) if the Hilbert transform
\[ H : \mathcal{S}(\mathbb{R},X) \longrightarrow \mathcal{M}(\mathbb{R},X), \quad Hf(t) = \lim_{\epsilon \searrow 0} \int_{|s| > \epsilon} \frac{f(t-s)}{s} ds \]
has an extension $H \in \mathcal{L}(L_p(\mathbb{R}, X))$ for any (or equivalently for one) $1 < p < \infty$. A complex Banach space $X$ has property $(\alpha)$ if there exist $1 \leq p < \infty$, two probability spaces $P = (\Omega, \mathcal{A}, \mu), P' = (\Omega', \mathcal{A}', \mu')$ with $\mathcal{E}_p, \mathcal{E}_{p'} \neq \emptyset$ and a constant $\alpha > 0$ such that for all $N \in \mathbb{N}, \mathbf{x}_j \in X, a_{ij} \in \mathbb{C}, |a_{ij}| \leq 1 (i, j = 1, \ldots, N)$ and for all $(\varepsilon_1, \ldots, \varepsilon_N) \in \mathcal{E}_p, (\varepsilon_1', \ldots, \varepsilon_N') \in \mathcal{E}_{p'}$ we have

$$\left\| \sum_{i,j=1}^{N} \varepsilon_i \varepsilon_j' a_{ij} x_{ij} \right\|_{L_p(\Omega' \times \Omega, X)} \leq \alpha \left\| \sum_{i,j=1}^{N} \varepsilon_i \varepsilon_j' x_{ij} \right\|_{L_p(\Omega' \times \Omega, X)}. \tag{2.9}$$

A useful application of property $(\alpha)$ is the following one, which is a direct consequence of the Kalton-Weis theorem (see [17, Thm. 4.5.6]).

**Theorem 2.6.** Let $X$ be a Banach space with property $(\alpha)$ and let $A : D(A) \subset X \to X$ be an operator with a bounded $H^{\infty}$-calculus. Then $A$ has an $\mathcal{R}$-bounded $H^{\infty}$-calculus with $\varphi_A^R = \varphi_A^\infty$.

The following operator-valued version of Mikhlin’s theorem is important for our purposes as well as the subsequent characterization of maximal $L_p$-regularity. The results are due to Girardi and Weis (see [21] or [17, Thm. 4.3.9, Thm. 4.4.4]).

**Theorem 2.7.** Let $X, Y$ be complex Banach spaces of class $\mathcal{HT}$, which have property $(\alpha)$ and let $1 < p < \infty$. For $m_\lambda \in C^n(\mathbb{R}^n \setminus \{0\}, \mathcal{L}(X, Y))$, $\lambda \in \Lambda$ assume that $\kappa_\alpha := \mathcal{R}_p\{\xi^\alpha \partial^\alpha m_\lambda(\xi) : \xi \in \mathbb{R}^n \setminus \{0\}, \lambda \in \Lambda\} < \infty$ for each $\alpha \in \{0, 1\}^n$. Then the operator

$$\mathcal{F}^{-1} m_\lambda \mathcal{F} : \mathcal{S}(\mathbb{R}^n, X) \longrightarrow \mathcal{S}'(\mathbb{R}^n, Y)$$

has a unique extension $T_\lambda \in \mathcal{L}(L_p(\mathbb{R}^n, X), L_p(\mathbb{R}^n, Y))$ for every $\lambda \in \Lambda$ and we have

$$\mathcal{R}_p\{T_\lambda : \lambda \in \Lambda\} \subseteq C_{p,n} \sum_{\alpha \in \{0, 1\}^n} \kappa_\alpha =: C.$$

In particular, we have $\|\mathcal{F}^{-1} m_\lambda \mathcal{F} f\|_{L_p(Y)} \leq C\|f\|_{L_p(X)}$ for $f \in \mathcal{S}(\mathbb{R}^n, X)$ and $\lambda \in \Lambda$.

**Theorem 2.8.** Let $X$ be a Banach space of class $\mathcal{HT}, 1 < p < \infty$ and let $-A : \mathcal{D}(A) \subset X \to X$ be the generator of a bounded holomorphic strongly continuous semigroup. Then the following conditions are equivalent.

(i) $A$ has maximal $L_p$-regularity on $(0, \infty)$.

(ii) $A$ is pseudo-$\mathcal{R}$-sectorial with $\varphi_A^R < \frac{\pi}{2}$.

### 3 Properties of Triebel-Lizorkin-Lorentz spaces

For parameters $s \in \mathbb{R}, 1 < p, q < \infty, 1 \leq r \leq \infty$ we call

$$F^{s,r}_{p,q} := \left\{ u \in \mathcal{S}'(\mathbb{R}^n) : \|u\|_{F^{s,r}_{p,q}} < \infty \right\}$$

the **Triebel-Lizorkin-Lorentz space** (as defined in [23]). The norm is given by

$$\|u\|_{F^{s,r}_{p,q}} := \|\mathcal{F}^s * u\|_{L_p(l_q^r)}$$

where $(\mathcal{F}^k)_{k \in \mathbb{N}_0}$ is a dyadic decomposition defined as follows (cf. [20, Def. 2.3.1/2]).
**Definition 3.1.** Let $\Phi_N$ (for $N \in \mathbb{N}$) denote the set of systems of functions $(\varphi_k)_{k \in \mathbb{N}_0} \subset \mathcal{S}(\mathbb{R}^n)$ with the following properties.

- $\hat{\varphi}_k \geq 0$ for all $k \in \mathbb{N}_0$.
- $\text{spt}(\hat{\varphi}_k) \subset \{2^{k-N} \leq |x| \leq 2^{k+N}\}$ for $k \in \mathbb{N}$ and $\text{spt}(\hat{\varphi}_0) \subset \{|x| \leq 2^N\}$.
- There exist $D_1, D_2 > 0$ such that for all $\xi \in \mathbb{R}^n$
  
  $$D_1 \leq \sum_{k=0}^{\infty} \hat{\varphi}_k(\xi) \leq D_2. \quad (3.1)$$

- For any $\alpha \in \mathbb{N}_0^n$, there is $C_\alpha > 0$ such that for all $k \in \mathbb{N}_0$ and $\xi \in \mathbb{R}^n$
  
  $$|\xi|^\alpha|\hat{\varphi}_k(\xi)| \leq C_\alpha. \quad (3.2)$$

Additionally, we set $\Phi := \bigcup_{N \in \mathbb{N}} \Phi_N$ and call each family $(\hat{\varphi}_k)_{k \in \mathbb{N}_0}$ with $(\varphi_k)_{k \in \mathbb{N}_0} \in \Phi$ a **dyadic decomposition**.

Note that the constant $C_\alpha$ in (3.2) doesn’t depend on the index $k$ but on the selected $N \in \mathbb{N}$, that is, on the radius $2^N$ of the dyadic decomposition. Also note that the existence of $D_2$ in (3.1) can be deduced from (3.2) with $\alpha = 0$ and the properties of $\text{spt}(\hat{\varphi}_k)$. We will often use the following more specific dyadic decomposition.

**Example 3.2.** Let $\phi \in C^\infty(\mathbb{R}^n)$ be radial symmetric and $\text{spt}(\phi) \subset \{|x| \leq 1\}$, $\phi = 1$ on $\{|x| \leq \frac{1}{2}\}$ and $0 \leq \phi \leq 1$. We set $\hat{\psi}(\xi) := \phi(\frac{\xi}{2}) - \phi(\xi)$ and $\psi_k(\xi) := \hat{\psi}(2^{-k}\xi)$ for $\xi \in \mathbb{R}^n$ and $k \in \mathbb{Z}$. Now we set $\varphi_k := \psi_k$ for $k \geq 1$ and define $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)$ by

$$\hat{\varphi}_0(\xi) = \begin{cases} \sum_{j \leq 0} \hat{\psi}_j(\xi), & \text{if } \xi \neq 0 \\ 1, & \text{if } \xi = 0. \end{cases}$$

Then we get $(\varphi_k)_{k \in \mathbb{N}_0} \in \Phi_1$ with $\sum_{k \in \mathbb{N}_0} \varphi_k(\xi) = 1$ for all $\xi \in \mathbb{R}^n$. In addition, we have the following (easy to verify) properties:

(a) $\|\varphi_k\|_{L^1} = \|\psi\|_{L^1}$ for all $k \in \mathbb{N}$.

(b) $\sum_{k=1}^{N} \hat{\varphi}_k \xrightarrow{N \to \infty} 1$ locally uniformly on $\mathbb{R}^n$.

(c) $\sum_{j=0}^{N} \varphi_j \ast f \xrightarrow{N \to \infty} f$ in $\mathcal{S}(\mathbb{R}^n)$ for all $f \in \mathcal{S}(\mathbb{R}^n)$.

(d) $\sum_{j=0}^{N} \varphi_j \ast u \xrightarrow{N \to \infty} u$ in $\mathcal{S}'(\mathbb{R}^n)$ for all $u \in \mathcal{S}'(\mathbb{R}^n)$.

If we replace the Lorentz-norm $\| \cdot \|_{L_{p,r}(\mathbb{R}^n)}$ by $\| \cdot \|_{L_p(\mathbb{R}^n)}$ then we get the well-known Triebel-Lizorkin spaces $F_{p,q}^s$. More precisely we have $F_{p,q}^{s,p} = F_{p,q}^s$. One can find the following result as Remark 2.4.2/1 in [20].

**Proposition 3.3.** The Triebel-Lizorkin-Lorentz spaces are independent of the choice of the dyadic decomposition.

The following result is due to Yang, Cheng and Peng [23]. Their proof is based on wavelet theory. We notice that it is possible to derive the following interpolation property by $L^p$-interpolation and retraction and coretraction techniques as developed in [20], as well.
**Theorem 3.4.** For $s \in \mathbb{R}$, $1 < p_0, p_1, q < \infty$, $1 \leq r_0, r_1 \leq \infty$, $p_0 \neq p_1$ and $0 < \theta < 1$ such that $\frac{1}{p} = \frac{1}{p_0} + \frac{1}{p_1}$ we have

$$(F^{s,r_0}_{p_0,q}, F^{s,r_1}_{p_1,q})_{\theta,r} = F^{s,r}_{p,q}.$$  

**Lemma 3.5.** Let $s \in \mathbb{R}$, $1 < p, q < \infty$ and $1 \leq r \leq \infty$. Then we have the following continuous embeddings.

(i) $\mathcal{S}(\mathbb{R}^n) \subset F^{s,r}_{p,q} \subset \mathcal{S}'(\mathbb{R}^n)$ and the first embedding is dense.

(ii) $F^{s+\tau,r}_{p,q} \subset F^{s,r}_{p,q}$ for $\tau \geq 0$.

(iii) $F^{s,r}_{p,q} \subset L_{p,r}$ if $s > 0$.

**Proof.** (i) follows from the corresponding fact for Triebel-Lizorkin spaces since

$$\mathcal{S}(\mathbb{R}^n) \subset F^{s,r}_{p,q} \subset \mathcal{S}'(\mathbb{R}^n)$$

and since the intersection of an interpolation couple of Banach spaces is dense in their real interpolation space. (ii) is a consequence of $\mathcal{L}$ and the first embedding is dense.

In order to prove (iii) we use the dyadic decomposition $(\varphi_k)_{k \in \mathbb{N}_0}$ of Example 3.2. First we consider the estimate

$$\left\| \sum_{k=0}^{\infty} |\varphi_k \ast u| \right\|_{L_{p,r}} \leq C \left\| \left( \frac{1}{2^k} \right)_{k \in \mathbb{N}_0} \right\|_{L_{q,r}} \left\| u \right\|_{F^{s,r}_{p,q}}$$

that we get from H"older’s inequality with $1 + \frac{1}{q} = 1$. Since $s > 0$, the right-hand side is finite for $u \in F^{s,r}_{p,q}$. Applying Example 3.2, we have $u = \sum_{k=0}^{\infty} \varphi_k \ast u$ where the convergence is in $\mathcal{S}'(\mathbb{R}^n)$. Now (3.3) gives that the series even converges pointwise a.e. and thus $u$ is a measurable function. On the other hand (6.3) gives $\left\| u \right\|_{L_{p,r}} \leq C' \left\| u \right\|_{F^{s,r}_{p,q}}$.

**Proposition 3.6.** $F^{s,r}_{p,q}$ is of class $\mathcal{H}T$ for $s \in \mathbb{R}$ and $1 < p, q, r < \infty$.

**Proof.** We need to show that the Hilbert transform

$$H : \mathcal{S}(\mathbb{R}, F^{s,r}_{p,q}) \longrightarrow \mathcal{M}(\mathbb{R}, F^{s,r}_{p,q}), \quad H f(t) = \lim_{\epsilon \searrow 0} \int_{|s| > \epsilon} \frac{f(t - s)}{s} ds$$

has an extension $H \in \mathcal{L}(L_p(\mathbb{R}, F^{s,r}_{p,q}))$. For any $s \in \mathbb{R}$ and $1 < q < \infty$, Tonelli’s theorem implies that $L_q(\mathbb{R}, l^s_q)$ is a space of class $\mathcal{H}T$ and so is $L_p(\mathbb{R}, l^s_q)$ for arbitrary $1 < p < \infty$. Since the Triebel-Lizorkin space $F^{s}_{p,q}$ is a retract of $L_p(\mathbb{R}, l^s_q)$ we can transfer the $\mathcal{H}T$-Property to $F^{s}_{p,q}$ for any $s \in \mathbb{R}$ and $1 < p, q < \infty$.

Now for fixed parameters $s, p, q, r$ as in the assertion we can use Theorem 3.4 to complete the proof. As a direct conclusion of the interpolation property $L_r(\mathbb{R}, (X_0, X_1)_{\theta,r}) = (L_r(\mathbb{R}, X_0), L_r(\mathbb{R}, X_1))_{\theta,r}$, we get that for an interpolation couple of spaces of class $\mathcal{H}T$ $X_0, X_1$ the real interpolation space $(X_0, X_1)_{\theta,r}$ is also of class $\mathcal{H}T$. Thus $F^{s,r}_{p,q}$ is of class $\mathcal{H}T$.

**Corollary 3.7.** $F^{s,r}_{p,q}$ is reflexive for $s \in \mathbb{R}$ and $1 < p, q, r < \infty$ (due to [18]).
Corollary 3.7 could also be obtained in a direct way, regarding the following result which is a conclusion of the corresponding result for Triebel-Lizorkin spaces (see [20, Thm. 2.6.2]) and Theorem 3.4.

Proposition 3.8. The dual space to $F^{s,r}_{p,q}$ is given by $F^{-s,-r}_{p',q'}$ for $s \in \mathbb{R}$ and $1 < p, q, r < \infty$ where $1 < p', q', r' < \infty$ are given by $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$ and $\frac{1}{r} + \frac{1}{r'} = 1$.

Proposition 3.9. $F^{s,r}_{p,q}$ has property $(\alpha)$ for $s \in \mathbb{R}$ and $1 < p, q, r < \infty$.

Proof. The Triebel-Lizorkin spaces $F^{s}_{p,q}$ have property $(\alpha)$ since there exists a continuous embedding in $L_p(\mathbb{R}^n, l^q_s)$. This implies the assertion, since property $(\alpha)$ preserves under real interpolation. We refer to [20, Thm. 4.5].

Theorem 3.10 (Multiplier theorem for Triebel-Lizorkin-Lorentz spaces). Let $s \in \mathbb{R}$, $1 < p, q < \infty$ and $1 \leq r < \infty$. Let $(m_\lambda)_{\lambda \in \Lambda} \subset C^n(\mathbb{R}^n \setminus \{0\}, \mathbb{C})$ such that $C_\alpha := \sup_{\xi \in \mathbb{R}^n \setminus \{0\}, \lambda \in \Lambda} |\xi^\alpha \partial^\alpha m_\lambda(\xi)| < \infty$ for all $\alpha \in (0,1)^n$. Then for every $\lambda \in \Lambda$

$$\mathcal{F}^{-1} m_\lambda \mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$$

has a (unique) continuous extension $T_\lambda : F^{s,r}_{p,q} \to F^{s,r}_{p,q}$ such that

$$\|T_\lambda\|_{\mathcal{L}(F^{s,r}_{p,q})} \leq C \max_{\alpha \in [0,1]^n} C_\alpha,$$

where the constant $C > 0$ only depends on $n$ and the parameters $p, q, s, r$. Furthermore, $(T_\lambda)_{\lambda \in \Lambda} \subset \mathcal{L}(F^{s,r}_{p,q})$ is $\mathcal{R}$-bounded in the case $1 < r < \infty$.

Proof. We define $M_\lambda \in L_X(\mathbb{R}^n, \mathcal{L}(l^q_s))$ by setting $M_\lambda(\xi)x := (m_\lambda(\xi)x_k)_{k \in \mathbb{N}_0}$ for $\xi \in \mathbb{R}^n \setminus \{0\}$, $x = (x_k)_{k \in \mathbb{N}_0} \in l^q_s$ and $\lambda \in \Lambda$. By Kahane’s contraction principle we see that the assumption $C_\alpha < \infty$ implies the $\mathcal{R}$-boundedness of $\{\xi^\alpha \partial^\alpha M_\lambda(\xi) : \xi \in \mathbb{R}^n \setminus \{0\}, \lambda \in \Lambda\} \subset \mathcal{L}(l^q_s)$ and the $\mathcal{R}_\theta$-bound doesn’t exceed $2 \max_{\alpha \in [0,1]^n} C_\alpha$. Since $l^q_s$ is of class $\mathcal{HT}$ (note that $1 < q < \infty$) with property $(\alpha)$, Theorem 3.7 gives that $M_\lambda$ is a Fourier multiplier, i.e.,

$$\mathcal{F}^{-1} M_\lambda \mathcal{F} : \mathcal{S}(\mathbb{R}^n, l^q_s) \to \mathcal{S}'(\mathbb{R}^n, l^q_s)$$

has a (unique) continuous extension $S_\lambda : L_p(l^q_s) \to L_p(l^q_s)$ such that

$$\mathcal{R}_\theta(\{S_\lambda : \lambda \in \Lambda\}) \leq C \max_{\alpha \in [0,1]^n} C_\alpha =: K \tag{3.4}$$

for all $\lambda \in \Lambda$. From the identity

$$(\varphi_k \ast \mathcal{F}^{-1} m_\lambda \mathcal{F} f)_{k \in \mathbb{N}_0} = \mathcal{F}^{-1} M_\lambda \mathcal{F}(\varphi_k \ast f)_{k \in \mathbb{N}_0} \tag{3.5}$$

we get $\|\mathcal{F}^{-1} m_\lambda \mathcal{F} f\|_{L^p_{p,q}} \leq K \|f\|_{F^{s,r}_{p,q}}$ for $f \in \mathcal{S}(\mathbb{R}^n)$ and consequently we have a continuous extension $T_\lambda : F^{s}_{p,q} \to F^{s}_{p,q}$ of $\mathcal{F}^{-1} m_\lambda \mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$. Now (3.4) and (3.5) imply $\mathcal{R}_\theta(\{T_\lambda : \lambda \in \Lambda\}) \leq K$. Hence the assertion is proved in the case $p = r$.

In order to generalize the result, we select $1 < p_0 < p < p_1 < \infty$ and $0 < \theta < 1$ such that $\frac{1}{p} = \frac{1}{p_0} + \frac{\theta}{p_1}$ and get $F^{s,r}_{p,q} = (F^{s}_{p_0,q} \cdot F^{s}_{p_1,q})_{\theta,r}$. Thus for

$$T_\lambda : (F^{s}_{p_0,q} \cdot F^{s}_{p_1,q})_{\theta,r} \to (F^{s}_{p_0,q} \cdot F^{s}_{p_1,q})_{\theta,r}$$
we get the estimate \( |T_\\lambda|_{\mathcal{L}(F^{s,r}_{p,q})} \leq C' \max_{\alpha \in \{0,1\}^n} C_\alpha \) since the real interpolation method is exact of type \( \theta \), where \( C' > 0 \) is also a constant depending only on \( s, p, q, r \) and \( n \).

Since \( F^s_{p,q} \) is of class \( \mathcal{H}T \) for \( j = 0, 1 \) we get the \( \mathcal{R} \)-boundedness of

\[
T_\\lambda: (F^s_{p_0,q_0} F^s_{p_1,q_1})^\theta, r \rightarrow (F^s_{p_0,q_0} F^s_{p_1,q_1})^\theta, r
\]

for \( 1 < r < \infty \) as a consequence of the case \( p = r \) proved above (see [14, Thm. 3.19]).

**Proposition 3.11.** For \( s \in \mathbb{R}, 1 < p, q < \infty \) and \( 1 \leq r \leq \infty \) the following representations hold.

(i) \( F^{s+\sigma,r}_{p,q} = \{ u \in \mathcal{S}'(\mathbb{R}^n) : \mathcal{F}^{-1}(1 + |\xi|^2)^{\frac{\sigma}{2}} F u \in F^{s,r}_{p,q} \} \) for \( \sigma \in \mathbb{R} \).

(ii) \( F^{s+k,r}_{p,q} = \{ u \in \mathcal{S}'(\mathbb{R}^n) : \partial_\alpha u \in F^{s,r}_{p,q} \forall \alpha \in \mathbb{N}_0^n, |\alpha| \leq k \} \) for \( k \in \mathbb{N}_0 \).

(iii) \( F^{s+2m,r}_{p,q} = \{ u \in \mathcal{S}'(\mathbb{R}^n) : \Delta^j u \in F^{s,r}_{p,q} \forall j \in \mathbb{N}_0, j \leq m \} \) for \( m \in \mathbb{N}_0 \).

The corresponding norms are equivalent, where the norm of the space on the right-hand side is given by \( \| \mathcal{F}^{-1}(1 + |\xi|^2)^{\frac{\sigma}{2}} F u \|_{F^{s,r}_{p,q}} \) in (i), by \( \sum_{|\alpha| \leq k} \| \partial_\alpha u \|_{F^{s,r}_{p,q}} \) in (ii) and by \( \sum_{0 \leq j \leq m} \| \Delta^j u \|_{F^{s,r}_{p,q}} \) in (iii).

**Proof.** We consider the Bessel-potential operator \( B^\sigma u : = \mathcal{F}^{-1}(1 + |\xi|^2)^{\frac{\sigma}{2}} F u \) for \( u \in \mathcal{S}'(\mathbb{R}^n) \) and \( \sigma \in \mathbb{R} \). If we fix \( (\varphi_k)_{k \in \mathbb{N}_0} \in \Phi \) and \( \sigma \in \mathbb{R} \), then by setting \( \psi_k(\xi) = \frac{\varphi_k(\xi)}{(1 + |\xi|^2)^{\frac{\sigma}{2}}} \) we get \( (\psi_k)_{k \in \mathbb{N}_0} \in \Phi \) (see also the proof of [20, Thm. 2.3.4]). Hence

\[
\| u \|_{F^{s,r}_{p,q}} \sim \| B^\sigma u \|_{F^{s-\sigma,r}_{p,q}} \tag{3.6}
\]

and we get (i).

Now the special case \( \sigma = 2m \) in (3.6) leads to \( F^{s+2m,r}_{p,q} = \{ u \in \mathcal{S}'(\mathbb{R}^n) : (1 - \Delta)^m u \in F^{s,r}_{p,q} \} \) together with the equivalence \( \| u \|_{F^{s+2m,r}_{p,q}} \sim \| (1 - \Delta)^m u \|_{F^{s,r}_{p,q}} \). Hence for (iii) it remains to show \( \sum_{0 \leq j \leq m} \| \Delta^j u \|_{F^{s,r}_{p,q}} \leq C\| (1 - \Delta)^m u \|_{F^{s,r}_{p,q}} \) since the converse estimate is obvious. For this purpose we write

\[
(-\Delta)^j u = \mathcal{F}^{-1} \frac{|\xi|^{2j}}{(1 + |\xi|^2)^k} F(1 - \Delta)^k u.
\]

Now the associated symbol \( \frac{|\xi|^{2j}}{(1 + |\xi|^2)^k} \) fulfills the conditions of Theorem 3.10 and we get the assertion.

In order to verify (iii) we write

\[
\partial_\alpha u = i^{|\alpha|}\mathcal{F}^{-1} \frac{\xi^\alpha}{(1 + |\xi|)^{|\alpha|}} F B^{|\alpha|} u \quad \text{for} \quad |\alpha| \leq k \tag{3.7}
\]

and

\[
B^k u = \mathcal{F}^{-1} \sum_{|\alpha| \leq k} \frac{k!}{\alpha! (k - |\alpha|)!} \frac{\xi^\alpha}{(1 + |\xi|^2)^{\frac{k}{2}}} F u. \tag{3.8}
\]

Now using again Theorem 3.10 and (3.7) we get \( \| u \|_{F^{s+k,r}_{p,q}} \sim \sum_{|\alpha| \leq k} \| \partial_\alpha u \|_{F^{s,r}_{p,q}} \) where (3.8) gives the estimate \( \ll \) and (3.7) gives \( \gg \).
4 The Laplace operator in $F^{s,r}_{p,q}$

The Laplace operator in $F^{s,r}_{p,q}$ for $s \in \mathbb{R}$, $1 < p, q < \infty$ and $1 \leq r \leq \infty$ is defined as

$$A_L = A^{s,r}_{L,p,q} : \mathcal{D}(A_L) \subset F^{s,r}_{p,q} \to F^{s,r}_{p,q}: \ u \mapsto -\Delta u,$$

where the domain is $\mathcal{D}(A_L) = F^{s+2,r}_{p,q}$.

**Proposition 4.1.** $A_L$ is $\mathcal{R}$-sectorial with $\varphi_{A_L}^\mathcal{R} = 0$ for $s \in \mathbb{R}$ and $1 < p, q, r < \infty$ and we have $(A^{s,r}_{L,p,q})' = A^{s,r}_{L,p',q'}$.

**Proof.** Lemma 3.3(i) implies that $A_L$ is densely defined. For $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ we would like to have $\lambda \in \rho(-A_L)$ with

$$\left(\lambda + A_L\right)^{-1} = \mathcal{F}^{-1} \frac{1}{\lambda + |\xi|^2} \mathcal{F}. \quad (4.1)$$

Therefore, we consider the symbols $\frac{1}{\lambda + |\xi|^2}$ and $\frac{|\xi|^2}{\lambda + |\xi|^2}$, which are smooth and fulfill the conditions of Theorem 3.10. From the first symbol we get that (4.1) defines a bounded operator on $F^{s,r}_{p,q}$. From the second symbol and Proposition 3.11 we get that (4.1) has in fact values in $F^{s+2,r}_{p,q}$ and hence must be the inverse operator of $\lambda + A_L$.

To prove the claimed $\mathcal{R}$-boundedness of $\{\lambda(\lambda + A_L)^{-1} : \lambda \in \Sigma_\varphi\} \subset \mathcal{L}(F^{s,r}_{p,q})$, we need the uniform estimate

$$\sup_{\xi \in \mathbb{R}^n, \lambda \in \Sigma_\varphi} |\xi^\alpha \partial^\alpha m_\lambda(\xi)| < \infty$$

for all $\alpha \in \mathbb{N}_0^n$ and $\varphi < \pi$, where $m_\lambda(\xi) := \frac{\lambda}{\lambda + |\xi|^2}$. This is a consequence of Lemma 2.3(a), so we can apply Theorem 3.10. Summarizing, $A_L$ is pseudo-$\mathcal{R}$-sectorial with $\varphi_{A_L}^\mathcal{R} = 0$.

Let now initially $s > -2$. Then we obtain in an elementary way that $A_L$ is injective: For $u \in \mathcal{N}(A_L)$ we have $\text{spt}(\hat{u}) \subset \{0\}$ and thus $u$ is a polynomial (see e.g. [9, Cor. 2.4.2]). Lemma 3.5 gives that $F^{s+2,r}_{p,q} \subset L^{p,\infty}$ and it is not hard to show that $L^{p,\infty}$ doesn’t contain any nontrivial polynomials. Hence $u = 0$. Now we consider the decomposition $F^{s,r}_{p,q} = \mathcal{N}(A_L) \oplus \mathcal{R}(A_L)$, which is a consequence of the pseudo-$\mathcal{R}$-sectoriality proved above and of the reflexivity of $F^{s,r}_{p,q}$ obtained in Corollary 3.4 (see e.g. [10, Prop. 2.1.1]). The injectivity of $A_L$ then gives the density of $\mathcal{R}(A_L) \subset F^{s,r}_{p,q}$.

By integration by parts we easily obtain $A^{s-r}_{L,p',q'} \subset (A^{s,r}_{L,p,q})'$. The fact that $1 \in \rho(A^{s,r}_{L,p,q})$ for all $s \in \mathbb{R}$ and $1 < p, q, r < \infty$ then gives $A^{s-r}_{L,p',q'} = (A^{s,r}_{L,p,q})'$. Since $(F^{s,r}_{p,q})' = F^{s-r}_{p',q'}$, the $\mathcal{R}$-sectoriality with $\varphi_{A_L}^\mathcal{R} = 0$ for $s \leq -2$ now follows by standard permanence properties for $\mathcal{R}$-sectorial operators.

**Remark 4.2.** The proof of Proposition 4.1 shows that for $r \in \{1, \infty\}$ we still have that $A_L$ is pseudo-sectorial with $\varphi_{A_L} = 0$ and, in the case $s > -2$, $A_L$ is injective.

**Proposition 4.3.** Let $s \in \mathbb{R}$ and $1 < p, q, r < \infty$. Then $A_L$ has an $\mathcal{R}$-bounded $H^\infty$-calculus with $\varphi_{A_L}^\mathcal{R} = 0$.

**Proof.** Thanks to Proposition 3.9 and Theorem 2.6 it is sufficient to prove that $A_L$ has a bounded $H^\infty$-calculus with $\varphi_{A_L}^\mathcal{R} = 0$. Let $\varphi \in (0, \pi)$ and $f \in \mathcal{H}_0(\Sigma_\varphi)$. $A_L$ is sectorial thanks to Proposition 4.1. Using Cauchy’s integral formula we get $f(A_L)u =$
\[ \mathcal{F}^{-1}f(\|\xi\|) \mathcal{F}u \text{ for all } u \in \mathcal{S}(\mathbb{R}^n). \] Now the symbol \( f(\|\xi\|) \) fulfills the condition of Theorem 3.10 (due to Lemma 2.3 (a)) so we have

\[ \|f(A_L)\|_{\mathcal{S}(F^{s,r}_{p,q})} \leq C_{\varphi}\|f\|_{L_{\infty}(\Sigma_{\varphi})}. \]

\[ \square \]

Note that Proposition 4.3 implies 4.1 if we only knew the sectoriality of \( A_L \). But, as the proof of Proposition 4.1 shows, \( \mathcal{R} \)-sectoriality can be obtained in a direct way at essentially the same cost.

Now we consider an alternative representation for Triebel-Lizorkin-Lorentz spaces. We would like to prove that \( F^{s+2\alpha,r}_{p,q} \) is the domain of \( (1 - \Delta)^{\alpha} \) in \( F^{s,r}_{p,q} \), where \( \alpha \in [0,1] \).

**Proposition 4.4.** Let \( s \in \mathbb{R} \), \( 1 < p, q < \infty \) and \( 1 \leq r \leq \infty \). Then

\[ \mathcal{A} : \mathcal{D}(\mathcal{A}) = \mathcal{D}(A_L) \subset F^{s,r}_{p,q} \rightarrow F^{s,r}_{p,q} \], \( u \rightarrow (1 - \Delta)u \)

is sectorial with angle \( \varphi_{\mathcal{A}} = 0 \) and for \( \alpha \in [0,1] \)

\[ \mathcal{D}(\mathcal{A}^\alpha) = \{ u \in F^{s,r}_{p,q} | \mathcal{F}^{-1}(1 + |\xi|^2)^\alpha \mathcal{F}u \in F^{s,r}_{p,q} \} = F^{s+2\alpha,r}_{p,q} \] \quad (4.2)

holds with equivalent norms, i.e., \( \|u\|_{\mathcal{D}(\mathcal{A}^\alpha)} \sim \|\mathcal{F}^{-1}(1 + |\xi|^2)^\alpha \mathcal{F}u\|_{F^{s,r}_{p,q}} \) for all \( u \in \mathcal{D}(\mathcal{A}^\alpha) \). Moreover, we have

\[ \mathcal{A}^\alpha u = \mathcal{F}^{-1}(1 + |\xi|^2)^\alpha \mathcal{F}u \] \quad (4.3)

for all \( u \in \mathcal{D}(\mathcal{A}^\alpha) \).

**Proof.** The second equality in (4.2) is Proposition 3.11 (i). The Laplace operator \( A_L \) is pseudo-sectorial with angle \( \varphi_{A_L} = 0 \) and so is \( \mathcal{A} \). Now \( -1 \in \rho(A_L) \), so \( \mathcal{A} \) is bijective and thus sectorial.

We now assume \( \alpha \in (0,1) \) since the cases \( \alpha = 0 \) and \( \alpha = 1 \) are obvious. We set \( g(z) := \frac{z}{(1 + z)^2} \) and \( h_\alpha(z) := z^\alpha \). Using Cauchy’s integral formula, we obtain

\[ (gh_\alpha)(\mathcal{A})f = \mathcal{F}^{-1}\frac{(1 + |\xi|^2)^\alpha + 1}{(2 + |\xi|^2)^2} \mathcal{F}f \] \quad (4.4)

for all \( f \in \mathcal{S}(\mathbb{R}^n) \). Theorem 3.10 gives that (4.4) even holds for all \( f \in F^{s,r}_{p,q} \). Now \( \mathcal{A} : \mathcal{D}(\mathcal{A}) \rightarrow F^{s,r}_{p,q} \) is bijective with \( \mathcal{A}^{-1}f = \mathcal{F}^{-1}\frac{1 + |\xi|^2}{1 + |\xi|^2} \mathcal{F}f \) for \( f \in F^{s,r}_{p,q} \) and thus we get

\[ g(\mathcal{A})^{-1}f = (1 + \mathcal{A}^2)^{-1}f = \mathcal{F}^{-1}\frac{(2 + |\xi|^2)^2}{1 + |\xi|^2} \mathcal{F}f \] \quad (4.5)

for all \( f \in \mathcal{D}(\mathcal{A}) \). Relations (4.4) and (4.5) yield (4.3) since \( \mathcal{A}^\alpha \) is given by \( g(\mathcal{A})^{-1}(gh_\alpha)(\mathcal{A}) \).

Now we verify (4.2) together with the equivalence of the norms. For this purpose let first \( u \in F^{s,r}_{p,q} \) so that \( \mathcal{F}^{-1}(1 + |\xi|^2)^\alpha \mathcal{F}u \in F^{s,r}_{p,q} \). Then, using (4.3) and Theorem 3.10 we get \((2 - \Delta)(gh_\alpha)(\mathcal{A})u \in F^{s,r}_{p,q} \). Consequently, we have \((gh_\alpha)(\mathcal{A})u \in \mathcal{D}(\mathcal{A}) \). Now, again using (4.3), we can also write \((gh_\alpha)(\mathcal{A})u = (1 - \Delta)v \), where \( v := \mathcal{F}^{-1}\frac{(1 + |\xi|^2)^\alpha}{(2 + |\xi|^2)^2} \mathcal{F}u \). Then Theorem 3.10 gives \( v \in \mathcal{D}(\mathcal{A}) \) and thus \((gh_\alpha)(\mathcal{A})u \in \mathcal{D}(\mathcal{A}) \). Summarizing, we obtain
Proposition 5.1. \( u \in \mathcal{D}(\mathcal{A}^\alpha) \). Hence we have \( \{ u \in F_{p,q}^{s,r} \mid \mathcal{F}^{-1}(1 + |\xi|^2)^\alpha \mathcal{F} u \in F_{p,q}^{s,r}\} \subset \mathcal{D}(\mathcal{A}^\alpha) \), so we can restrict ourselves to \( u \in \mathcal{D}(\mathcal{A}^\alpha) \) to show the equivalence

\[
\|u\|_{\mathcal{D}(\mathcal{A}^\alpha)} = \|u\|_{F_{p,q}^{s,r}} + \|\mathcal{A}^\alpha u\|_{F_{p,q}^{s,r}} \sim \|\mathcal{F}^{-1}(1 + |\xi|^2)^\alpha \mathcal{F} u\|_{F_{p,q}^{s,r}}. 
\] (4.6)

For \( u \in \mathcal{D}(\mathcal{A}^\alpha) \) we can apply \((4.3)\), so we directly get \( \geq \) in \((4.6)\). Applying Theorem \(3.10\) to the symbol \( \frac{1}{(1+|\xi|^2)^\alpha} \) and using \((4.3)\) again, we get the estimate \( \|u\|_{F_{p,q}^{s,r}} \leq C \|\mathcal{F}^{-1}(1 + |\xi|^2)^\alpha \mathcal{F} u\|_{F_{p,q}^{s,r}} \) and consequently the converse inequality in \((4.6)\). Hence we have proved the equivalence \((4.6)\) and this also shows \( \mathcal{D}(\mathcal{A}^\alpha) \subset \{ u \in F_{p,q}^{s,r} \mid \mathcal{F}^{-1}(1 + |\xi|^2)^\alpha \mathcal{F} u \in F_{p,q}^{s,r}\} \). \( \square \)

As a consequence of Propositions \(4.3\) and \(4.4\) and of \((2.3)\) we also get the following result on complex interpolation of Triebel-Lizorkin-Lorentz spaces.

Corollary 4.5. Let \(-\infty < s_0 \leq s_1 < \infty\) and \(1 < p, q, r < \infty\). Then for \( \eta \in (0,1) \) we have

\[
[F_{p,q}^{s_0,r}, F_{p,q}^{s_1,r}]_\eta = F_{p,q}^{(1-\eta)s_0 + \eta s_1,r}. 
\]

Proof. For \( s \in \mathbb{R} \) we get

\[
[F_{p,q}^{s,r}, F_{p,q}^{s+2k\theta,q}]_\eta = F_{p,q}^{s+2k\theta,q}. 
\] (4.7)
in the case \( k = 1, \theta = 1 \) from Propositions \(4.3\) and \(4.4\) Since for any \( \beta \geq 0 \) we can write \( \mathcal{A}^\beta = \mathcal{A}^m \mathcal{A}^\alpha \) for some \( m \in \mathbb{N}_0 \) and \( \alpha \in [0,1] \), \((4.7)\) holds for all \( k \in \mathbb{N}_0 \) and \( \theta = 1 \). Application of the reiteration theorem now gives \((4.7)\) for all \( \theta \in [0,1] \) and \( k \in \mathbb{N}_0 \). This proves the claim. \( \square \)

5 The Stokes operator in \( F_{p,q}^{s,r} \)

We first introduce the Helmholtz projection on \((F_{p,q}^{s,r})^n\). Again \( n \in \mathbb{N} \) is the dimension and \( s \in \mathbb{R} \), \( 1 < p, q, r < \infty \), \( 1 \leq r \leq \infty \). For \( u \in \mathcal{S}(\mathbb{R}^n) \) we set

\[
P u := \mathcal{F}^{-1} \left[ 1 - \frac{\xi_j^T}{|\xi|^2} \right] \mathcal{F} u = u - \left( \sum_{j=1}^n \mathcal{F}^{-1} \frac{\xi_j \xi_j}{|\xi|^2} \mathcal{F} u_j \right)_{1 \leq i \leq n}.
\]

From Theorem \(3.10\) we get the Helmholtz projection as the bounded extension \( P \in \mathcal{L}((F_{p,q}^{s,r})^n) \). The space of solenoidal functions is

\[
(F_{p,q}^{s,r})^n : = \{ u \in (F_{p,q}^{s,r})^n \mid \text{div} \, u = 0 \}
\]

and the space of gradient fields in \((F_{p,q}^{s,r})^n\) is

\[ \mathcal{G} := \{ \nabla p \mid p \in \mathcal{D}'(\mathbb{R}^n), \nabla p \in (F_{p,q}^{s,r})^n \}. \]

Furthermore, let \( C^\infty_c(\mathbb{R}^n) \) denote the smooth functions with compact support and vanishing divergence. Now we get the Helmholtz decomposition:

Proposition 5.1. Similar to the Definition of the space of gradient fields we set

\[
\mathcal{G}^* := \{ \nabla p \mid p \in \mathcal{D}'(\mathbb{R}^n), \nabla p \in (F_{p,q}^{s,r})^n \}.
\]
Let \( 1 < p, q, r < \infty \) and \( n \geq 2 \). If \( s > -2 \), we additionally admit \( r \in \{1, \infty\} \). Then range and nullspace of the Helmholtz projection are given by \( \mathcal{R}(P) = (F_{p,q}^{s,r})^n_\sigma \) and \( \mathcal{N}(P) = \mathcal{G} = \mathcal{G}^* \). In particular the Helmholtz decomposition

\[
(F_{p,q}^{s,r})^n = (F_{p,q}^{s,r})^n_\sigma \oplus \mathcal{G}
\]

holds.

**Proof.** We prove the claim in three steps and start with some general observations that we will make use of. First we remark that one gets the inclusion \( \mathcal{R}(P) \subset (F_{p,q}^{s,r})^n_\sigma \) by direct computation (and approximation). Second the injectivity of the Laplace operator (see Proposition \[4.1\] and Remark \[4.2\]) yields

\[
(F_{p,q}^{s,r})^n_\sigma \cap \mathcal{G} = \{0\}.
\]

Furthermore, de Rham’s theorem (see \[3\] and the references therein) gives that \( \mathcal{G} \) is a closed subspace of \( (F_{p,q}^{s,r})^n \).

**Step 1.** We show \( \mathcal{N}(P) \subset \mathcal{G}^* \subset \mathcal{G} \subset \mathcal{N}(P) \) (in the stated order). For a fixed \( \eta \in \mathcal{G}^* \) we have \( \eta = \xi (\mathcal{G}^* \cap \mathcal{G}) \) and \( \eta \in \mathcal{G} \). In particular the Helmholtz decomposition

\[
\mathcal{G}^* = \mathcal{G} \oplus \mathcal{N}(P)
\]

is valid since the converse inclusion is obvious. Besides, \( \mathcal{G}^* \cap \mathcal{G} \) is dense in \( (F_{p,q}^{s,r})^n \).

**Step 2.** We use the first step to show the inclusions \( \mathcal{N}(P) \subset \mathcal{G}^* \subset \mathcal{G} \subset \mathcal{N}(P) \) as a conclusion. The second inclusion \( \mathcal{G}^* \subset \mathcal{G} \) is valid since \( \mathcal{G} \) is closed. For the third inclusion we fix \( u \in \mathcal{G} \). Since we have already shown \( \mathcal{N}(P) \subset \mathcal{G}^* \subset \mathcal{G} \), we obtain \( Pu = u - (1 - P)u \in \mathcal{G} \). On the other hand, we have \( Pu \in \mathcal{G} \). Consequently, \( \mathcal{N}(P) \subset \mathcal{G} \).

**Step 3.** It remains to prove \( \mathcal{R}(P) = (F_{p,q}^{s,r})^n_\sigma \). In view of what we have already seen we get \( (F_{p,q}^{s,r})^n = \mathcal{R}(P) \oplus \mathcal{N}(P) = \mathcal{R}(P) \oplus \mathcal{G} \subset (F_{p,q}^{s,r})^n_\sigma \). Now the last inclusion is an equality since the converse inclusion is obvious. Besides, \( \mathcal{R}(P) \) yields the directness of the sum, so \( \mathcal{R}(P) \oplus \mathcal{G} = (F_{p,q}^{s,r})^n_\sigma \) together with \( \mathcal{R}(P) \subset (F_{p,q}^{s,r})^n_\sigma \) gives \( \mathcal{R}(P) = (F_{p,q}^{s,r})^n_\sigma \).

**Remark 5.2.** The space \( (F_{p,q}^{s,r})^n_\sigma \) is of class \( \mathcal{HT} \) for \( 1 < p, q, r < \infty \) and \( s \in \mathbb{R} \). This is a consequence of Proposition \[3.3\].

Now we are able to define the** Stokes operator** as

\[
A_S = A_{S,F_{p,q}^{s,r}} : \mathcal{D}(A_S) \subset (F_{p,q}^{s,r})^n_\sigma \longrightarrow (F_{p,q}^{s,r})^n_\sigma, \quad u \longmapsto -P\Delta u
\]

on the domain \( \mathcal{D}(A_S) := (F_{p,q}^{s+2,r})^n_\sigma \).

**Proposition 5.3.** For \( s \in \mathbb{R} \) and \( 1 < p, q, r < \infty \) we have \( A_S = A_L|_{\mathcal{D}(A_S)} \). Besides, we have \( \rho(A_L) \subset \rho(A_S) \) with \( (\lambda - A_S)^{-1} = (\lambda - A_L)^{-1}|_{(F_{p,q}^{s,r})^n_\sigma} \) for all \( \lambda \in \rho(A_L) \).
Proof. For $u \in \mathcal{D}(A_S)$ we get $u \in \mathcal{A}'(1 - P)$ since $P$ is a projection and thus $Pu = u$. By Proposition 3.11 and the continuity of $\Delta : \mathcal{A}'(\mathbb{R}^n) \to \mathcal{A}'(\mathbb{R}^n)$ we get $P\Delta = \Delta P$ on $(F^{s+2,r}_{p,q})^n$. This shows $A_Su = A_Lu$ for $u \in \mathcal{D}(A_S)$.

Now let $\lambda \in \rho(A_L)$ and set $T_\lambda v := (\lambda - A_L)^{-1}v$ for $v \in (F^{s,r}_{p,q})^n$. Again we can use $P\Delta = \Delta P$ and get $PT_\lambda = T_\lambda$ by the injectivity of $\lambda - A_L$, i.e. $T_\lambda$ maps into $(F^{s,r}_{p,q})^n$. Consequently, $T_\lambda = (\lambda - A_S)^{-1}$ on $(F^{s,r}_{p,q})^n$.

**Proposition 4.5.** Let $s \in \mathbb{R}$ and $1 < p, q, r < \infty$. Then $A_S$ is $\mathcal{R}$-sectorial with $\varphi_{A_S} = 0$. Hence $A_S \in \text{MR}((F^{s,r}_{p,q})^n)$. Furthermore, we have $(A_S^{s,r}_{p,q})' = A_S^{s,r}_{p',q'}$.

**Proof.** Let $0 < \varphi < \pi$. Then we get the $\mathcal{R}$-boundedness of $\{\lambda (\lambda + A_S)^{-1} : \lambda \in \Sigma_{\varphi}\} \subset \mathcal{L}((F^{s,r}_{p,q})^n)$ as a direct consequence of Propositions 3.11 and 5.3. $(F^{s,r}_{p,q})_0^n$ is reflexive (see Corollary 5.7). Consequently, we get the density of $\mathcal{D}(A_S) \subset (F^{s,r}_{p,q})_0^n$ (see e.g. [10, Prop. 2.1.1]). The Laplace operator $A_L$ is injective and so is $A_S$. The remaining proof is thus completely analogous to the proof of Proposition 4.1.

### 6 The time derivative $\frac{d}{dt}$ and some embeddings

We take a look at the time derivative operator $\frac{d}{dt}$, more precisely at

$$
B : \mathcal{D}(B) = H^1_p(\mathbb{R}, X) \subset L_p(\mathbb{R}, X) \rightarrow L_p(\mathbb{R}, X), \quad u \mapsto (1 + \frac{d}{dt})u.
$$

**Proposition 6.1.** Let $1 < p < \infty$ and let $X$ be of class $\mathcal{H}T$ with property $(\alpha)$. Then $B$ is sectorial with angle $\varphi_B = \frac{\pi}{2}$ and we have

$$
\mathcal{D}(B^n) = H^n_p(\mathbb{R}, X)
$$

for $\alpha \in [0, 1]$. The related norms are equivalent. Furthermore, we have

$$
B^n u = \mathcal{F}^{-1}(1 + i\xi)^n \mathcal{F} u \quad \forall u \in \mathcal{D}(B^n).
$$

**Proposition 6.2.** Let $X$ be a Banach space of class $\mathcal{H}T$ with property $(\alpha)$ and $1 < p < \infty$. Then $B$ has an $\mathcal{R}$-bounded $H^\infty$-calculus in $L_p(\mathbb{R}, X)$ with $\varphi_B^{\mathcal{R},\infty} = \frac{\pi}{2}$.

We omit the proofs of Propositions 6.1 and 6.2. On the one hand this can be done very similar to the proofs of Propositions 4.1 and 4.3 respectively and on the other hand most of the assertions are already proved in [7].

**Lemma 6.3.** Let $s \in \mathbb{R}$, $1 < p, q < \infty$ and $1 \leq r \leq \infty$ such that $p > \frac{n}{2}$. Let $\delta > 0$ such that $\frac{\delta}{p} + \delta < 1$. Then we have the continuous embedding

$$
F^{s+2,\delta,r}_{p,q} \subset F^{s+1,r}_{2p,q}.
$$

**Proof.** We use an embedding theorem for Triebel-Lizorkin spaces and deduce the result via interpolation. For $\epsilon' := \frac{n}{2}(p - \frac{n}{2}) > 0$ we have $\frac{1}{2} > 1 + \frac{1}{\epsilon'}$. Now select $0 < \epsilon < \epsilon'$ such that $\frac{\delta}{p} > 1 + \frac{1}{\epsilon}$. Then we have $\frac{n}{2} + \frac{\epsilon}{2} < p$, so it’s possible to select parameters

$$
\text{max}\{1, \frac{n}{2} + \frac{\epsilon}{2}\} < p_0 < p < p_1 < \infty.
$$
Additionally, select \( \theta \in (0, 1) \) s.t. \( \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \). From [20, Thm. 2.8.1] we get \( F^{s,\frac{\theta}{p_0}}_{2p_0q} \subset F^{s-1+\delta}_{2p_1q} \) for \( j = 0, 1 \) and Theorem 5.3 gives
\[
F^{s+2-\delta,\frac{\theta}{p_0}}_{p,q} \subset \left( F^{s+2-\delta}_{2p_0q} \cdot F^{s+1}_{2p_1q} \right)_{\theta,r} = F^{s+1,\frac{\theta}{p_0}}_{2p_1q}.
\]

Lemma 6.4. Let \( s > 0, 1 < p, q < \infty \) and \( 1 \leq r \leq \infty \). Then the product \( \pi : F^{s,r}_{2p,q} \times F^{s,r}_{2p,q} \rightarrow F^{s,r}_{p,q} \) is continuous.

Proof. Again we make use of a corresponding fact for Triebel-Lizorkin spaces (which is included in the paper of J. Johnsen [13]) and extend this to Triebel-Lizorkin-Lorentz spaces via interpolation. We fix parameters \( 1 < p_0 < p < p_1 < \infty \). The product \( \pi : F^{s}_{2p,q} \times F^{s}_{2p,q} \rightarrow F^{s}_{p,q} \) is continuous for \( j = 0, 1 \) due to [13, Thm. 6.1] and so is \( \pi(\cdot, u) : F^{s}_{2p,q} \rightarrow F^{s}_{p,q} \) for each \( u \in F^{s}_{2p,q} \). Interpolation of the respective spaces leads to the continuity of \( \pi(\cdot, u) : F^{s,r}_{2p,q} \rightarrow F^{s,r}_{p,q} \) for any \( u \in F^{s}_{2p,q} \cup F^{s}_{p,q} \) and thus the whole product \( \pi : F^{s,r}_{2p,q} \times F^{s,r}_{2p,q} \rightarrow F^{s,r}_{p,q} \) is continuous for \( i = 0, 1 \).

Now by repeating an analogue argument with \( \pi(v, \cdot) : F^{s}_{p,q} \rightarrow F^{s,r}_{2p,q} \) we get the continuity of \( \pi : F^{s,r}_{2p,q} \times F^{s,r}_{2p,q} \rightarrow F^{s,r}_{p,q} \), where we made use of the (simpler) fact, that we get \( F^{s,r}_{p,q} \) by real interpolation with itself.

Consider any function space \( \mathcal{F} \) of time-dependent functions on some time interval \( (0, T) \) (or in other words on \([0, T]\) since we usually identify two functions differing on a null set). If \( \mathcal{F} \) contains the smooth functions with compact support on \((0, T)\) then we denote their closure in \( \mathcal{F} \) by \( \partial \mathcal{F} \). For the function spaces of time-dependent functions that appear in the sequel, \( \partial \mathcal{F} \) consists of those functions \( u \in \mathcal{F} \) with \( u(0) = 0 \) if the trace in time exists for \( \mathcal{F} \). Note that we usually have \( \partial \mathcal{F} = \mathcal{F} \) if the trace in time doesn’t exist (cf. [20, Thm. 4.3.2/(1)(a)]).

Lemma 6.5. Let \( s \in \mathbb{R}, 1 < p, q, r < \infty, 1 < \eta < \infty \) and \( \alpha \in [0, 1] \). Then for \( T \in (0, \infty) \) we have the continuous embeddings
\[
H^{1}_\eta(\mathbb{R}, F^{s,r}_{p,q}) \cap L^p_{\eta}(\mathbb{R}, F^{s+2,r}_{p,q}) \subset H^{\alpha}_\eta(\mathbb{R}, F^{s+2(1-\alpha),r}_{p,q})
\]
and
\[
H^{1}_{\eta}((0, T), F^{s,r}_{p,q}) \cap L^p_{\eta}((0, T), F^{s+2,r}_{p,q}) \subset H^{\alpha}_{\eta}((0, T), F^{s+2(1-\alpha),r}_{p,q}).
\]
For \( T \in (0, \infty) \) we also have the continuous embedding
\[
0H^{1}_{\eta}((0, T), F^{s,r}_{p,q}) \cap L^p_{\eta}((0, T), F^{s+2,r}_{p,q}) \subset 0H^{\alpha}_{\eta}((0, T), F^{s+2(1-\alpha),r}_{p,q})
\]
locally uniformly in time, i.e., for every \( T_0 > 0 \) there exists an embedding constant \( C > 0 \) for (6.5), which is independent of \( T \in (0, T_0] \).

Proof. Let \( \mathcal{A} = 1 - \Delta \) in \( F^{s,r}_{p,q} \) be the operator from Proposition 4.4 and \( B = 1 + \frac{\partial}{\partial t} \) in \( L^p_{\eta}(\mathbb{R}, F^{s,r}_{p,q}) \) the operator from Proposition 6.1. We have already seen that \( \mathcal{A} \) and \( B \) have a bounded \( H^{\infty} \)-calculus with \( \varphi^\mathcal{A}_T + \varphi^B_T < \pi \). Note that \( \mathcal{A} \) can be interpreted as an operator in \( L^p_{\eta}(\mathbb{R}, F^{s,r}_{p,q}) \) instead of \( F^{s,r}_{p,q} \) in a trivial way, where it still admits a bounded \( H^{\infty} \)-calculus with the same angle \( \varphi^\mathcal{A}_T = 0 \). Obviously \( \mathcal{A} \) and \( B \) are resolvent commuting operators. So all conditions of the mixed derivative theorem (in the version of [7, Lem. 4.1]) are fulfilled. This yields that
\[
\|\mathcal{A}^{1-\alpha}B^\alpha u\|_{L^p_{\eta}(\mathbb{R}, F^{s,r}_{p,q})} \leq C\|\mathcal{A}u + Bu\|_{L^p_{\eta}(\mathbb{R}, F^{s,r}_{p,q})}
\]
holds for all \( u \in \mathcal{D}(\mathcal{A}) \cap \mathcal{D}(B) \) and all \( \alpha \in [0, 1] \). Now we use Propositions \[ 6.1, 6.4 \] and \[ 6.11 \] and get for all \( u \in \mathcal{D}(\mathbb{R}, F^{\alpha+2}_p, r) \subset H^1_\eta(\mathbb{R}, F^{\alpha+2}_p) \cap L_\eta(\mathbb{R}, F^{\alpha+2}_p) \)

\[
\|u\|_{H^1_\eta(\mathbb{R}, F^{\alpha+2}_p)} \lesssim \|B^\alpha u\|_{L_\eta(\mathbb{R}, F^{\alpha+2}_p)} \lesssim \|B^\alpha u\|_{L_\eta(\mathcal{A}(\mathcal{A}^{-1}))} \\
\lesssim \|B^{1-\alpha} u\|_{L_\eta(\mathbb{R}, F^{\alpha}_p)} \lesssim \|Bu\|_{L_\eta(\mathbb{R}, F^{\alpha}_p)} \lesssim \|u\|_{H^1_\eta(\mathbb{R}, F^{\alpha}_p) \cap L_\eta(\mathbb{R}, F^{\alpha+2}_p)}.
\]

This proves \( (6.3) \).

We get \( (6.4) \) as a conclusion of \( (6.3) \) by suitable retraction and extension. More precisely we make use of \( (8.1) \), which yields an extension operator simultaneously on \( H^1_\eta((0, T), F^{\alpha}_p, r) \) and on \( L_\eta((0, T), F^{\alpha+2}_p) \).

In order to prove \( (6.5) \), we make use of the extension operator \( (8.4) \) in the case \( \beta = 1 \). For a fixed \( T_0 > 0 \) we get

\[
\|u\|_{H^1_\eta(\mathbb{R}, F^{\alpha+2}_p)} \lesssim \|E_{\mathcal{X}, 1} E_T u\|_{H^1_\eta(\mathbb{R}, F^{\alpha+2}_p)} \\
\lesssim C\|E_{\mathcal{X}, 1} E_T u\|_{H^1_\eta(\mathbb{R}, F^{\alpha+2}_p) \cap L_\eta(\mathbb{R}, F^{\alpha+2}_p)} \\
\leq C'\|u\|_{H^1_\eta(\mathbb{R}, F^{\alpha+2}_p) \cap L_\eta(\mathbb{R}, F^{\alpha+2}_p)}
\]

for all \( u \in 0H^1_\eta((0, T), F^{\alpha}_p, r) \cap L_\eta((0, T), F^{\alpha+2}_p) \) with a constant \( C' > 0 \), independent of \( T \in (0, T_0) \).

We will additionally need the following embeddings for Bessel-potential spaces on a time-interval.

**Lemma 6.6.** Let \( 1 < \eta < \infty \) and let \( X \) be a Banach space of class \( \mathcal{HT} \). Then for \( s > \frac{1}{2\eta} \) and \( T \in (0, \infty) \) we have the continuous embedding

\[
H^s_\eta((0, T), X) \subset L_2(0, T, X).
\]

For \( \alpha > \frac{1}{2\eta} \) and \( T_0 > 0 \) the continuous embedding

\[
0H^\alpha_\eta((0, T), X) \subset L_2(0, T, X)
\]

holds with an embedding constant \( C > 0 \), which is independent of \( T \in (0, T_0) \).

**Proof.** For \( (6.7) \) let first \( \alpha \in \left( \frac{1}{2\eta}, 1 \right] \). We select \( \epsilon > 0 \) such that \( \alpha - 2\epsilon > \frac{1}{\eta} \). The embedding constant of \( H^\alpha_\eta((0, T), X) \subset W^{\alpha-\epsilon}_\eta((0, T), X) \) doesn’t depend on \( T \in (0, \infty) \) and for the extension operator

\[
E_{\mathcal{X}, 1} E_T : 0W^{\alpha-\epsilon}_\eta((0, T), X) \longrightarrow 0W^{\alpha-\epsilon}_\eta(\mathbb{R}, X)
\]

from \( (8.4) \) there exists a continuity constant independent of \( T \in (0, T_0) \). Thus for \( u \in 0H^\alpha_\eta((0, T), X) \) we conclude

\[
\|u\|_{L_2(0, T, X)} \lesssim \|E_{\mathcal{X}, 1} E_T u\|_{L_2(0, T, X)} \lesssim C\|E_{\mathcal{X}, 1} E_T u\|_{H^2_\eta(\mathbb{R}, X)} \\
\leq C'\|E_{\mathcal{X}, 1} E_T u\|_{W^{\alpha-\epsilon}_\eta(\mathbb{R}, X)} \leq C''\|u\|_{W^{\alpha-\epsilon}_\eta((0, T), X)} \leq C'''\|u\|_{H^\alpha_\eta((0, T), X)}
\]

where \( C'' > 0 \) is a constant independent of \( T \in (0, T_0) \). Now let \( \alpha \in \left( \frac{1}{2\eta}, \frac{3}{4} \right] \). In this case we have \( 0H^\alpha_\eta((0, T), X) = H^\alpha_\eta((0, T), X) \), (see \[ 20 \] Thm. 4.3.2/1(a)) so we can make use of an extension argument as well, where we have the trivial extension available this time. The case \( \alpha > 1 \) is an obvious consequence.

Relation \( (6.0) \) is a well-known Sobolev embedding. It can be obtained by an analogous extension argument as above, where we make use of \( (8.2) \) instead of \( (8.1) \). For \( \mathbb{R} \) instead of \( (0, T) \) see e.g. \[ 4 \] Thm. 3.7.5].
7 The Navier-Stokes equations

We fix $s \in \mathbb{R}$, $1 < p, q, r < \infty$, $1 < \eta < \infty$ and $X_\sigma := (F_{p,q}^{s,r})^n$ with dimension $n \geq 2$. As above, $A_S$ is the Stokes operator in $X_\sigma$. The solution space for the Stokes equation is

$$E_T := H_\eta^1((0,T), X_\sigma) \cap L_\eta((0,T), \mathscr{D}(A_S)),$$

where $T \in (0, \infty]$. Next, as in Section 2 [89] we set

$$F_T := L_\eta((0,T), X_\sigma) \text{ and } \mathbb{I} := \{u_0 = u(0) : u \in E_T\}, \quad (7.1)$$

equipped with the norm $\|u_0\|_\mathbb{I} = \inf_{u(0)=u_0} \|u\|_{E_T}$, so $F_T \times \mathbb{I}$ is the data space with right-hand side functions $f \in F_T$ and initial values $u_0 \in \mathbb{I}$. Note that by [26], Proposition 5.1 and [20] Thm. 1.9.3/1] we obtain

$$\mathbb{I} = (X_\sigma, \mathscr{D}(A_S))_{1-1/n, \eta} = P \left( (F_{p,q}^{s,r}, F_{p,q}^{s+2,r})^n \right)_{1-1/n, \eta}.$$

The proof of the additional statement in Theorem 7.1 will essentially make use of the

$$\left( \frac{d}{dt} - A_S \right) u = \left( \frac{d}{dt} - A_S \right) u(0) \quad \text{in } (0,T) \times \mathbb{R}^n,$$

is an isomorphism when $T < \infty$, due to Proposition 5.4. The nonlinear term is

$$G(u) := -P(u \cdot \nabla)u = -P \text{div}(uuT), \quad u \in (F_{p,q}^{s,r})^n,$$

where $P \in \mathcal{L}((F_{p,q}^{s,r})^n)$ denotes the Helmholtz projection introduced in Section 5.

**Theorem 7.1.** Let $n \in \mathbb{N}$, $n \geq 2$, $s > -1$ and let $1 < p, q, r < \infty$ and $1 < \eta < \infty$ such that $\frac{s}{2p} + \frac{s+2}{\eta} < 1$. Then for all $(\tilde{f}, \tilde{u}_0) \in F_\infty \times \mathbb{I}$

$$(\text{PNSE})_{f,u_0} \left\{ \frac{d}{dt} u - \Delta u + P(u \cdot \nabla)u = f \quad \text{in } (0,T) \times \mathbb{R}^n, \right.$$ 

$$u(0) = u_0 \quad \text{in } \mathbb{R}^n$$

has a unique maximal strong solution $u : [0,T^*) \rightarrow \mathbb{I}$ with $T^* \in (0,\infty]$ and $u \in E_T$ for all $T \in (0,T^*)$. If additionally $\frac{s}{2p} + \frac{s+2}{\eta} < 1$, then $u$ is either a global solution or we have $T^* < \infty$ and $\limsup_{t \to T^*} \|u(t)\|_\mathbb{I} = \infty$.

We first convince ourselves that the two systems $(\text{NSE})_{f,u_0}$ and $(\text{PNSE})_{f,u_0}$ are equivalent. This particularly shows that Theorem 7.1 implies Theorem 1.1. Indeed, when $u$ is the solution of (PNSE)$_{f,u_0}$, given by Theorem 7.1, we get the solution $(u, \nabla p)$ of (NSE)$_{f,u_0}$ as claimed in Theorem 1.1 by setting $\nabla p = -(1-P)(u \cdot \nabla)u$. On the other hand, if $(u, \nabla p)$ is a solution of (NSE)$_{f,u_0}$, then $u$ solves (PNSE)$_{f,u_0}$ and consequently $\nabla p = -(1-P)(u \cdot \nabla)u$.

The proof of the additional statement in Theorem 7.1 will essentially make use of the following embedding for the space of initial values.

**Lemma 7.2.** Let $s \in \mathbb{R}$, $1 < p, q, r < \infty$ and $1 < \eta < \infty$ such that $\frac{s}{2p} + \frac{s+2}{\eta} < 1$. Then we have the continuous embedding

$$\mathbb{I} \subset (F_{2p,q}^{s+1,r})^n.$$
Proof. Select \( 0 < \epsilon < \min\{\eta - 1, \frac{n}{2p} + \frac{1}{\eta}\} \) and \( T \in (0, \infty) \). Then we have the continuous embeddings

\[
 E_T \subset H_{\eta}^{\frac{2\alpha+1}{\eta}}((0, T), (F_{p,q}^{s+2(1-\frac{1}{\eta})})^n) \subset C([0, T], (F_{p,q}^{s+2(1-\frac{1}{\eta})})^n),
\]

where the first embedding follows from Lemma 6.5 and the second one can be deduced from standard Sobolev embedding in the same way as in the proof of Lemma 6.3. Now, setting \( \delta := \frac{n}{2p} + \delta < 1 \) so Lemma 6.3 gives the continuous embedding

\[
 (F_{p,q}^{s+2(1-\frac{1}{\eta})})^n \subset (F_{2p,q}^{s+1,r})^n.
\]

This leads to \( \|u_0\|_{(F_{2p,q}^{s+1,r})^n} \leq C\|u\|_{E_T} \) for \( u_0 \in \mathbb{I} \) and any \( u \in E_T \) with \( u(0) = u_0 \) so the assertion is proved.

Proof of Theorem 7.1. Let \( (\tilde{f}, \tilde{g}) \in \mathbb{F}_{\infty} \times \mathbb{I} \). We start with the local existence and uniqueness, so we need to show that there is a unique solution \( u \in E_T \) for

\[
 Lu = \left( f + G(u) \right) u_0
\]

on some time interval. First of all we note that it’s possible to restrict ourselves to those solutions with \( u(0) = 0 \). In fact, by setting \( u^*: = L^{-1}(f_0) \), we can always consider \( \tilde{u} = u - u^* \in 0E_T \) for \( u \in E_T \), so for any \( T \in (0, \infty) \) the following assertions are equivalent:

(a) \( Lu = \left( f + G(u) \right) u_0 \) has a unique solution \( u \in E_T \).

(b) \( L\tilde{u} = \left( G(\tilde{u} + u^*) \right) \) has a unique solution \( \tilde{u} \in 0E_T \).

Before we are able to verify (b), it’s necessary to have the continuous embedding

\[
 G(E_T) \subset \mathbb{F}_T
\]

for \( T \in (0, \infty) \). For \( u \in E_T \), using Proposition 3.11, we have

\[
 \|G(u)\|_{\mathbb{F}_T} = \|P (u \cdot \nabla) u\|_{L_{\eta}(0,T), (F_{p,q}^{s+1})^n} \leq C\|\text{div}(uu^T)\|_{L_{\eta}(0,T), (F_{p,q}^{s+1})^n}) \leq C\|\text{div}(uu^T)\|_{L_{\eta}(0,T), (F_{p,q}^{s+1})^n}) \leq C\|u\|_{L_2(0,T), (F_{p,q}^{s+1})^n})^2,
\]

where we applied H{"o}lder’s inequality together with Lemma 6.3 (note that \( s + 1 > 0 \) is assumed) to get the last inequality. Now it remains to prove \( E_T \subset L_2((0, T), (F_{2p,q}^{s+1,r})^n) \), to get (7.3). Due to the condition \( \frac{n}{2p} + \frac{1}{\eta} < 1 \), we can select \( \delta > \frac{1}{\eta} \) such that \( \frac{n}{2p} + \delta < 1 \). Then we have \( F_{p,q}^{s+2-\delta,r} \subset F_{2p,q}^{s+1,r} \), according to Lemma 6.3. By setting \( \alpha := \frac{\delta}{2} \) we get the continuous embeddings

\[
 E_T \subset H_{\eta}^{\alpha}((0, T), (F_{p,q}^{s+2(1-\alpha), r})^n) \subset L_2((0, T), (F_{p,q}^{s+2(1-\alpha), r})^n) \subset L_2((0, T), (F_{2p,q}^{s+1,r})^n)
\]

where we used Lemma 6.3 for the first embedding, Lemma 6.6 for the second embedding and Lemma 6.3 for the last embedding. This yields (7.3).
In order to obtain (1), we define
\[ N : 0 \mathbb{E}_T \rightarrow \mathbb{F}_T \times \{0\}, \quad \bar{u} \mapsto L\bar{u} - \left( G(\bar{u} + u^*) \right) \tag{7.5} \]
for \( T \in (0, \infty) \). Because of (3.3) we know that \( N \) is well-defined, i.e., we have indeed \( N(\bar{u}) \in \mathbb{F}_T \times \{0\} \) for all \( \bar{u} \in \partial \mathbb{E}_T \). Furthermore, \( N \) is continuously Fréchet-differentiable, where
\[
DN(0)v = Lv - \left( DG(u^*)v \right) = Lv + \left( P\left( u^* \cdot \nabla \right)v + P(v \cdot \nabla)u^* \right) \quad \forall v \in \partial \mathbb{E}_T
\]
is the derivative at the zero point. Our aim is to verify that there exists a unique \( \bar{u} \in \partial \mathbb{E}_T \) such that \( N(\bar{u}) = 0 \) for small time intervals \( (0, T) \).

As a first step to obtain this, we prove that \( DN(0) : \partial \mathbb{E}_T \rightarrow \mathbb{F}_T \times \{0\} \) is an isomorphism when \( T > 0 \) is small enough. Similarly to the verification of (3.3) we obtain for \( T > 0 \) and \( v \in \partial \mathbb{E}_T \) that
\[
\left\| \begin{pmatrix} DG(u^*)v \\ 0 \end{pmatrix} \right\|_{\mathbb{F}_T \times \{0\}} = \left\| P\text{div}(u^*v^T) + P\text{div}(v(u^*)^T) \right\|_{L_n((0,T),(F_{p,q}^{s+1,r})^n)} 
\leq C \left\| \text{div}(u^*v^T) + \text{div}(v(u^*)^T) \right\|_{L_n((0,T),(F_{p,q}^{s+1,r})^n)} 
\leq C' \left\| u^*v^T + v(u^*)^T \right\|_{L_n((0,T),(F_{p,q}^{s+1,r})^n \times n)} 
\leq C'' \left\| u^* \right\|_{L_{2q}((0,T),(F_{p,q}^{s+2,1-r})^n)} \left\| v \right\|_{L_{2q}((0,T),(F_{p,q}^{s+1,r})^n)}, \tag{7.6} \right.
\]
in view of Proposition (3.11). Lemma (6.4) and Hölder’s inequality, where the constant \( C'' > 0 \) is independent of \( T \in (0, \infty) \). Again let \( \delta > \frac{1}{q} \) such that \( \frac{n}{2p} + \delta < 1 \) and set \( \alpha := \frac{\delta}{2} \). Then we have \( \mathbb{F}_{p,q}^{s+2-\delta,r} \subset \mathbb{F}_{p,q}^{s+1,r} \) and, since \( \alpha > \frac{1}{2q} \), we obtain for any fixed \( T_0 > 0 \)
\[
0 \mathbb{E}_T \subset \mathbb{H}^\alpha((0,T), (\mathbb{F}_{p,q}^{s+2(1-\alpha),r})^n) \subset L_{2q}((0,T), (\mathbb{F}_{p,q}^{s+2(1-\alpha),r})^n) \subset L_{2q}((0,T), (\mathbb{F}_{p,q}^{s+1,r})^n) \tag{7.7},
\]
where the embeddings are continuous with an embedding constant independent of \( T \in (0, T_0] \), due to Lemmas (6.5) and (6.6). Hence we have in total
\[
\left\| \begin{pmatrix} DG(u^*)v \\ 0 \end{pmatrix} \right\|_{\mathbb{F}_T \times \{0\}} \leq C_1 \left\| u^* \right\|_{L_{2q}((0,T),(\mathbb{F}_{p,q}^{s+1,r})^n)} \left\| v \right\|_{\mathbb{E}_T} \tag{7.8},
\]
for all \( v \in \partial \mathbb{E}_T \) and for all \( T \in (0, T_0] \). Thanks to Lemma (2.4) there is also a constant \( C_2 > 0 \) such that \( \left\| L^{-1} \right\|_{L(\mathbb{E}_T \times \{0\}, \partial \mathbb{E}_T)} \leq C_2 \) for all \( T \in (0, T_0] \).

The size of the finite time interval \((0,T_0)\) was arbitrary up to this point. Proceeding from any finite \( T_0 > 0 \), we will shrink the interval \((0,T_0)\) in the following to get a unique local solution. The constants \( C_1 \) and \( C_2 \), found above, can be assumed to be fixed so they don’t change by shrinking \((0,T_0)\). First, let \((0,T_0)\) be small enough, so that
\[
\left\| u^* \right\|_{L_{2q}((0,T_0),(\mathbb{F}_{p,q}^{s+1,r})^n)} \leq \frac{1}{2C_1 C_2} \tag{7.9},
\]
holds. Then we obtain from (7.8) and (7.9)
\[
\left\| \begin{pmatrix} DG(u^*) \\ 0 \end{pmatrix} \right\|_{L(\mathbb{E}_T \times \{0\})} \leq \frac{1}{\left\| L^{-1} \right\|_{L(\mathbb{E}_T \times \{0\}, \partial \mathbb{E}_T)}}.
\]

for all \( T \in (0, T_0] \). Hence by the Neumann series we get that \( DN(0) : 0 \mathbb{E}_T \to \mathbb{F}_T \times \{0\} \) is an isomorphism for all \( T \in (0, T_0] \).

We apply the inverse function theorem (see e.g. [3] Thm. VII.7.3) and get open neighborhoods \( 0 \in U_T < 0 \mathbb{E}_T \) and \( N(0) \in V_T \subset \mathbb{F}_T \times \{0\} \) such that \( N : U_T \to V_T \) is bijective. Now we fix \( T \in (0, T_0] \) and define for \( 0 < T' < T \) a function \( F_{T'} : \mathbb{F}_T \) by

\[
F_{T'}(t) := \begin{cases} 
0, & \text{if } t \in (0, T') \\
G(u^*)(t), & \text{if } t \in [T', T].
\end{cases}
\]

Then we have

\[
\| \left( F_{T'} \right) - \left( G(u^*) \right) \|_{\mathbb{F}_T \times \{0\}} = \int_0^T \| F_{T'}(t) - G(u^*)(t) \|_{X_T} \, dt
\]

and thus \( \left( F_{T'} \right) \to 0 \) as \( T' \to 0 \). Since \( V_T \) is a neighborhood of \( N(0) \), this yields \( \left( F_{T'} \right) \in V_T \), if \( T' \in (0, T) \) is small enough and consequently for \( \bar{u} := N^{-1} \left( F_{T'} \right) \in U_T \) we have \( N(\bar{u}) = \left( F_{T'} \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \) on \( (0, T') \). Hence, by restriction of \( \bar{u} \) to \( (0, T') \), we get a solution \( \bar{u} \in 0 \mathbb{E}_{T'} \). Since \( N : U_T \to V_T \) is bijective, this solution is unique.

Having established the local existence and uniqueness of a solution for \((\text{PNSE})_{f,u_0}\), we now extend the solution to a maximal time interval \([0, T^*]\). First we note that uniqueness holds on any time interval: Considering two solutions \( u, v \in \mathbb{E}_T \) of \((\text{PNSE})_{f,u_0}\) on \([0, T]\) for some \( T \in (0, \infty) \), we know from the established local uniqueness that \( u = v \) holds on some \([0, T'] \subset [0, T)\). We assume that \( u \) and \( v \) do not coincide on \([0, T)\). Then Lemma 2.5 allows to apply a continuity argument, which provides some \( 0 < t_1 < t_2 < T \) so that \( u(t) = v(t) \) for all \( t \in [0, t_1] \) and \( u(t) \neq v(t) \) for all \( t \in (t_1, t_2) \). Now, setting \( u_1 := u(t_1) \) and \( f_1 := f(t_1 + \cdot) \), we can apply local uniqueness of the solution of \((\text{PNSE})_{f,u_1}\) and get \( u(t_1 + \cdot) = v(t_1 + \cdot) \) on some \([0, T''(t_1)]\), contradictory to \( u(t) \neq v(t) \) for all \( t \in (t_1, t_2) \).

In order to get a maximal time interval \([0, T^*]\) for the solution of \((\text{PNSE})_{f,u_0}\), we define

\[
M := \{(J_T, u_T) : T \in (0, \infty), \exists \text{ solution } u_T \in \mathbb{E}_T \text{ of } (\text{PNSE})_{f,u_0} \text{ on } J_T = [0, T)\},
\]

\[
J^* := \bigcup \{(J_T, u_T) : M \} =: [0, T^*)
\]

and \( u : [0, T^*) \to \mathbb{E}, u(t) := u_T(t) \) for \( t \in J_T \). Due to the uniqueness proved above, \( u \) is well defined and consequently the desired maximal solution.

Now let additionally \( \frac{m}{2(p+1)} + \frac{2}{p} < 1 \). We assume \( T^* < \infty \) and \( \limsup_{T \to T^*} \|u(t)\|_1 < \infty \) for the maximal solution \( u \). Then we have \( u \in BC([0, T^*], \mathbb{I}) \) (i.e., bounded and continuous). For \( T \in (0, T^*) \) and \( v \in \mathbb{E}_T \) we define the linear operator

\[
Bv := \left( P\text{div}(uw^T) \right).
\]

Then we have \((L + B)u = \left( \begin{array}{c} f_l \\ 0 \end{array} \right)\). As in [7,6] we get

\[
\|Bv\|_{\mathbb{E}_T \times \{0\}} \leq C \|u\|_{L_2(0,T), (F_{2p,q}^{0+1, r})^n} \|v\|_{L_2(0,T), (F_{2p,q}^{0+1, r})^n} \quad \forall v \in \mathbb{E}_T
\]

(7.10)
with a constant \(C > 0\) independent of \(T\). Concerning (7.11) and Lemma (2.12) we get

\[
\|Bv\|_{\mathcal{F}_T \times I} \leq C' \left( \int_0^T \|u(t)\|_{(P^\eta_{2\eta+1,t})}^{2\eta} \right)^{\frac{1}{2\eta}} \|v\|_{\mathcal{E}_T} \leq C'' T^{\frac{1}{2\eta}} \|u\|_{BC([0,T^*], \mathcal{E})} \|v\|_{\mathcal{E}_T}
\]

for all \(v \in \eta \mathcal{E}_T\) with a constant \(C'' > 0\) independent of \(T \in (0, T^*]\). Due to (7.11) we can also deduce \(B \in \mathcal{L}(\mathcal{E}_T, \mathcal{F}_T \times I)\) from (7.10). Furthermore, Lemma (2.14) gives a constant \(K > 0\), such that \(\|L^{-1}\|_{\mathcal{L}(\mathcal{F}_T \times \{0\}, \mathcal{E}_T)} \leq K\) holds for all \(T \in (0, T^*]\). Consequently, we obtain for sufficiently small \(T \in (0, T^*]\) that

\[
\|B\|_{\mathcal{L}(\mathcal{E}_T, \mathcal{F}_T \times \{0\})} < \frac{1}{\|L^{-1}\|_{\mathcal{L}(\mathcal{F}_T \times \{0\}, \mathcal{E}_T)}},
\]

which yields that \(L + B : \eta \mathcal{E}_T \to \mathcal{F}_T \times \{0\}\) is an isomorphism. More precisely, we need to choose

\[
T \leq \frac{1}{(2C''K \|u\|_{BC([0,T^*], \mathcal{E})})^{2\eta}}.
\]

(7.11)

Now, for \(T\) as in (7.11), we can select \(T_1 \in (0, T^*]\) and repeat the argument on \((T_1, T + T_1)\) instead of \((0, T]\). This yields that \(L + B : \eta \mathcal{E}_T \to \mathcal{F}_T \times \{0\}\) is an isomorphism, where \(\eta \mathcal{E}_{T_1, T + T_1}\) (resp. \(\mathcal{F}_{T_1, T + T_1}\)) consists of the translations of functions in \(\eta \mathcal{E}_T\) (resp. \(\mathcal{F}_T\)) by \(T_1\). We repeat this argument \(k\) times on the interval \((kT_1, T + kT_1) \cap (0, T^*]\) until we reach \(T + kT_1 \geq T^*\). Finally we have that \(L + B : \eta \mathcal{E}_{T^*} \to \mathcal{F}_{T^*} \times \{0\}\) is an isomorphism. Now it is not hard to deduce that

\[
L + B : \mathcal{E}_{T^*} \xrightarrow{\cong} \mathcal{F}_{T^*} \times I
\]

is an isomorphism: Continuity and injectivity are obvious while one gets the surjectivity by setting \(v^* := L^{-1}\left( \frac{0}{\eta} \right)\) and \(v := (L + B)^{-1}\left( \eta^{-\text{div}}(v^* u_T) \right) + v^* \in \mathcal{E}_{T^*}\) for \((\eta_0) \in \mathcal{F}_T \times I\).

As a consequence of (7.12) and Lemma (2.5) we can achieve

\[
u = (L + B)^{-1}\left( \begin{pmatrix} f \\ u_0 \end{pmatrix} \right) \in \mathcal{E}_{T^*} \subset BUC([0, T^*], \mathcal{I})
\]

and hence \(u(T^*) = \lim_{T \to T^*} u(t) \in I\). Application of the local existence and uniqueness now gives a solution of (PNSE)\(f, \cdot, u_{T^*}\) on some time interval \([0, T^*]\), which yields an extension of \(u\) to a solution of (PNSE)\(f, u_0\) on \([0, T^* + T^*]\), in contradiction to the maximality of \(u\).

\(\square\)

8 Appendix: Extension operators

Let \(1 < \eta < \infty\). For fixed \(m \in \mathbb{N}\) and \(T \in (0, \infty]\) there exists a mapping \(u \mapsto E_{T,m} u\) for functions \(u\) (defined on \((0, T]\) with values in any vector space) such that for all \(k \in \{0, 1, \ldots, m\}\) and any Banach space \(X\) we have an extension operator

\[
E_{T,m} : W^k_\eta((0, T], X) \longrightarrow W^k_\eta(\mathbb{R}, X).
\]

(8.1)

A precise proof can be found in [1] Thm. 4.26] for the case of scalar-valued functions, but the given proof can be directly transferred to the vector-valued case. \(E_{T,m}\) is the correction of

\[
R : W^k_\eta(\mathbb{R}, X) \longrightarrow W^k_\eta((0, T], X), \quad u \longmapsto u|_{(0, T)},
\]

24
so, by the interpolation $W^s_\eta(\mathbb{R}, X) = (L^q_\eta(\mathbb{R}, X), W^k_\eta(\mathbb{R}, X))_{\frac{s}{k}, \eta}$ for $0 < s < k$, we get

$$W^s_\eta((0, T), X) = (L^q_\eta((0, T), X), W^k_\eta((0, T), X))_{\frac{s}{k}, \eta}$$

and the extension operator

$$E_{T, m} : W^s_\eta((0, T), X) \rightarrow W^s_\eta(\mathbb{R}, X) \quad (8.2)$$

(see [20 Thm. 1.2.4]).

Now let $T \in (0, \infty)$, $1 < \eta < \infty$ and $X$ a Banach space. For a function $u$ defined on $(0, T)$ with values in any vector space we set

$$E_T u(\tau) := \begin{cases} u(\tau), & \text{if } 0 < \tau < T \\ u(2T - \tau), & \text{if } T \leq \tau < 2T \\ 0, & \text{if } 2T \leq \tau \end{cases}$$

(see also [16]). Then, due to [16 Prop. 6.1], this leads to an extension operator

$$E_T : \mathcal{D}(0W^\beta_\eta((0, T), X) \rightarrow \mathcal{D}(0W^\beta_\eta((0, \infty), X) \quad (8.3)$$

for $\beta \in \left(\frac{1}{p}, 1\right]$ such that for any fixed $T_0 \in (0, \infty)$ there is a constant $C = C(T_0)$ with $\|E_T\| \leq C$ for all $T \in (0, T_0]$. Now we use (8.2) in the case $T = \infty$ and $m = 1$ and get the extension operator

$$E_{x, 1} E_T : \mathcal{D}(0W^\beta_\eta((0, T), X) \rightarrow \mathcal{D}(0W^\beta_\eta(\mathbb{R}, X) \quad (8.4)$$

(for $\beta \in \left(\frac{1}{p}, 1\right]$), whose operator norms $\|E_{x, 1} E_T\|, T \in (0, T_0]$ are bounded above for a fixed $T_0 > 0$ as well. The structure of $E_T$ also gives that $\|E_T u\|_{L^q((0, \infty), X)} \leq 2 \|u\|_{L^q((0, T), X)}$.

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