CAD Based Design Optimization of Four-bar Mechanisms: a coronaventilator case study

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Abstract

Design optimization of mechanisms is a promising research area as it results in more energy-efficient machines without compromising performance. However, machine builders do not actually use the design methods described in the literature as these algorithms require too much theoretical analysis. Moreover, the design synthesis approaches in the literature predominantly utilize heuristic optimizers leading to suboptimal local minima.

This research introduces a convenient optimization workflow allowing wide industrial adoption, while guaranteeing to reveal the global optimum. To guarantee that we find the global optimum, a mathematical expression of the constraints describing the feasible region of possible designs is of great importance. Therefore, kinematic analysis of the point-to-point (PTP) planar four-bar mechanism is discussed to obtain the static and dynamic constraints. Within the feasible region, objective value samples are generated through CAD multi-body software. These motion simulations determine the required torque to fulfill the movement for a certain combination of design parameters. Sparse interpo-
lation techniques allow minimizing the required amount of samples and thus CAD simulations. Moreover, this interpolation of simulation results enables the representation of the objective in a mathematical description without in-depth analytical design analysis by the machine designer. Subsequently, the mathematical expression of the objective allows global optimizers to find a global optimal design within the feasible design space. In a case study of a corona-ventilator mechanism with three design parameters (DP’s), 1870 CAD motion simulations from which only 618 are used to build a model allowed to reduce the RMS torque of the mechanism by 67%.

**Keywords:** Dimensional synthesis, Four-bar linkage, Optimization, Mechanical systems, Motion control

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1. Introduction

The energy consumption of industrial machinery is a topic of primary importance due to environmental and economic considerations [1]. The 45% share that electric motors have in the global electric consumption [2] supports the statement that any energy-saving method should be investigated thoroughly. The methodology proposed within this paper is applicable for all planar four-bar mechanisms with an imposed movement of the end-effector and/or output link BC (see Figure 1). The potential of this scope is indicated in [3, 4, 5], stating that four-bar linkages are extensively used in practical engineering applications. Moreover, reciprocating PTP machinery is progressively common within the industry [5].

The link lengths in a four-bar mechanism can differ while fulfilling the same task, being the PTP displacement of output link BC. Therefore, the geometry parameters depicted in Figure 1 can be considered as design parameters to enhance the mechanism. Design optimization of a Point-To-Point (PTP) mech-

Figure 1: The considered design parameters |OA|, |AB| and |BC| of a four-bar, in the present paper.
anism is one specific approach to reduce the energy consumption of electric machinery, as indicated in Figure 2. Awareness about the influence of machine components geometry on energy consumption has recently attracted attention [6, 7, 8]. Mechanism models [9, 10] replace the prototyping, allowing computational evaluation of multiple designs with limited costs.

A coronaventilator is used as a validation case within this study. This mechanism was constructed during the first wave of the covid-19 pandemic by a non-profit organization [11]. Having continuous (24/7) access to electricity is not obvious within low- and middle-income countries. Thus, having a mechanism that consumes a minimum of electric energy enabling the usage of batteries is highly relevant. Therefore, the objective of this study is to find the optimal design (being lengths $|OA|$, $|AB|$ and $|BC|$) leading to a minimal $T_{RMS}$ for a reciprocal four-bar mechanism. The method introduced in this paper relies on CAD software to sample the objective function through motion simulations. These simulation tools allow broad industrial applicability. Furthermore, sparse interpolation technique is implemented to avoid an infeasible computational burden of numerous CAD simulations. Moreover, the objective function is only to be considered in the feasible design space of the four-bar mechanism. Hence, constraints that limit the design space are highly relevant. State-of-the-art techniques generally use heuristic optimizers which cannot guarantee to find the global optimum [12]. However, the method described in this paper guarantees to reveal the global optimum.

1.1. Related work

In the literature the minimisation of the driving torque is done by establishing dynamic equations of the system to predict the dynamics. However, this makes the method inconvenient for machine builders. Moreover, [13, 14, 15] do
not define the feasible search domain nor include it in searching for the optimum result. The constraints that define the feasible design space are important as defects, giving infeasible designs, [16] frequently occur in the kinematic mechanism synthesis of a four-bar linkage. The optimization algorithms of [13, 14, 17] assure that the objective function has converged towards a minimum, yet it is generally not guaranteed that the designed linkage will be feasible. Therefore, the necessary constraints should be added so that the optimal solution can fulfill the movement without inconveniences. Using a constrained-global optimization algorithm requires a deterministic mathematical description of the constraints to find the global optimum. To the authors' knowledge, this has not been done yet in the literature [18].

Developing a four-bar mechanism that follows the desired output trajectory is a classic design problem that researchers extensively explore [19, 20, 21, 22, 23]. However, all methods above are not implementable in global optimizers as the algebraic expression (when provided) is only evaluated in discrete defined points $\left[ x_B(\theta), y_B(\theta) \right]_{i \in \mathbb{N}}$ on the coupler curve $B(\theta)$ (shown in Figure 1). Thus, these cannot deliver a deterministic mathematical description of the feasible design space, which is required.

1.2. Method

In general, it will be shown how CAD-based motion simulations combined with a sparse interpolation technique enable a global optimizer that guarantees revealing the global optimum and thereby outperform heuristic optimizers regarding energy savings.

Mechanical design of systems is mainly done in Computer-Aided Design (CAD) software. These CAD models include all required information (i.e., volume, mass, friction, damping, joints, ...) to model the dynamics of a mechanism. This information is necessary to calculate the necessary torque of the mechanism through motion simulations. By driving the mechanism with the motion profile $\theta(t)$ at point $O$ (Figure 1), the location where the mechanism is driven in reality by a motor, the user can extract the necessary torque from the software (as in Figure 2) to fulfill the prescribed movement $\Delta \psi$ of the output link $BC$. Furthermore, within these motion simulations, the design parameters $|OA|$, $|AB|$ and $|BC|$ of the four-bar can be parameterized to simulate different designs. The objective value, to minimize by the optimizer is the RMS Torque ($T_{RMS}$) value, necessary to drive the mechanism fulfilling an imposed PTP motion ($\Delta \psi$). The literature states that minimizing the $T_{RMS}$ corresponds with reducing the energy losses in the system [5].

Hence, by calculating the RMS Torque based on CAD simulations as elucidated in Section 2 the objective value for a certain design (i.e., certain values for the three design parameters $|OA|$, $|AB|$ and $|BC|$) is obtained. The whole simulation process to obtain the objective value for different design parameter combinations ($|OA|$, $|AB|$ and $|BC|$) is automated. Constraints on the design parameter values are necessary to define an area containing feasible designs, as
discussed in Section 3, from which designs are selected to simulate their corresponding objective value ($T_{RMS}$). As one design evaluation can take on average 1 minute and 25 seconds, computational simulation time becomes a burden. Therefore, wise selection of the simulated designs within the feasible design space is essential. The brute force method requires an inconceivable number of $g^d$ motion simulations, with $g$ being the granularity of sampling and $d$ the number of design parameters. Even with state-of-the-art interpolation techniques, the construction of the objective function would require at least $(d + 1)n^d \log^2 \left(2d - 2(n)\right)$ samples [24], with $n$ the total number of terms in the mathematical description of the objective function. In the case of the coronaventilator this would mean 782,933 samples are required. Therefore, the selection of samples is performed with certain rules in order to use an innovative multidimensional sparse interpolation approach [25]. This novel interpolation technique, introduced in Section 4, allows obtaining the objective function with a sparse sampling method within the feasible design space. This reduces the number of required samples to 618, with an additional 1252 validation samples. Limiting the number of samples to construct the objective function is a major enabler for a global optimizer. As the interpolation limits the number of CAD motion simulations to a bare minimum. In this case the number of necessary samples is reduced from 10,000,000 to 1870.

2. CAD Motion Simulations

In kinematic analysis, linkage dimensions $|OA|$, $|AB|$, $|BC|$ and $|OC|$ are known and the resulting output motion $\psi(t)$ (and its derivatives) can be calculated. On the other hand, dimensional synthesis is regarded as the inverse, in which for a specific output motion $\psi(t)$ the feasible dimensions of the linkages are obtained [26]. This paper is based on the dimensional synthesis of a planar four-bar function generation [27]. As shown in Figure 3 the movement $\Delta \psi$ of output linkage BC caused by $\theta(t)$ is described by a starting angle $\psi_i$, and end angle $\psi_e$. In this paper, the machine designer only defined an output motion $\psi(t)$ which results in a reciprocal movement between the positions $\psi_i$ and $\psi_e$.

Figure 3: The output link BC requires a movement from $\psi_i$ to $\psi_e$, which is performed by moving $\theta$ over a design-specific angle.

The validation case is clarified to make all the following more tangible. This mechanism, shown in Figure 4, can ventilate a patient by pressing the indenter into the bag, which causes airflow towards the patient. Figure 4 presents
the CAD model of the coronaventilator and illustrates that the red beam, connected with the indentor (i.e., the end-effector), moves by rotating input link OA around point O. This is the point where an electric motor drives the mechanism. The red beam has two predefined angles: an angle $\delta_e$ that holds the mechanism in a position where the indentor touches the bag and an angle $\delta_i$ that corresponds to a position in which the air is compressed out of the bag. Figure 4 clearly shows that the mechanism is a four-bar linkage on which the method proposed in this paper can be applied.

![Figure 4: The used case within this research is a coronaventilator developed by Gear Up Medical vzw][11]. It can be seen that the mechanism is from a four-bar type.

However, the output motion of the coronaventilator is described by the angle $\delta$ (linked to the red beam), while the four-bar has an angle $\psi$ that is linked to the output link BC. A relation between these angles stated as

$$\psi = \text{asin} \left( \frac{\sin(\delta) \left( \frac{k}{\tan(\delta)} + \sqrt{b^2 - k^2} \right)}{b} \right)$$

(1)

allows a conversion from $\delta$ to $\psi$. The parameter $k$ in Equation (1) (Figure 4) is a constant value that changes neither in the optimization nor during the four-bar mechanism’s movement.

A CAD motion simulation can determine the necessary torque, to drive the mechanism at point O only if the required position profile $\theta(t)$, at that point O, is known. However, the user solely defines the required position profile of the end-effector, in this case $\delta(t)$. According to Equation (1) we obtain $\psi(t)$. The conversion of $\psi(t)$ to $\delta(t)$ depends on the values of the design parameters $|OA|$, $|AB|$ and $|BC|$. Therefore, each selected design is analysed by two motion simulations, as indicated in Figure 5. If the design is combined with the required output motion $\psi(t)$, the first kinematic motion simulation can extract the required motor position displacement $\theta(t)$. Subsequently, the motor motion profile $\theta(t)$ is used in the second motion simulation, from which the required driving torque is determined. This process with kinematic simulation and subsequent torque calculation extracts the objective value for predefined designs.
3. Design Parameter Constraints

3.1. Static constraints of a four-bar

The combinations of design parameters $|OA|$, $|AB|$ and $|BC|$, to consider in the workflow above, are chosen so that the designs are located within the feasible design space of the four-bar mechanism. To determine this region of feasible designs, the first step is looking for static constraints. This implies that only the designs which can be assembled for the maximal and minimal angle of the output link BC ($\psi_i$ and $\psi_e$) can be part of the feasible design space. An example of a design that cannot be assembled in $\psi_e$ due to the chosen values for DP’s $|OA|$, $|AB|$ and $|BC|$ is illustrated in Figure 6. This shows that the input link OA’ cannot be connected with the coupler link A”B. This assemblability define the first boundaries on the design space that is illustrated in Figure 7 (left) (only in 2D for illustrative purposes). These boundaries are obtained through a position analysis of the four-bar mechanism for both begin $\psi_i$- and end-position $\psi_e$. For analysis of the ventilator, the origin of the fixed frame is placed in joint O (the driver joint). Let $\theta$ be the angle of the input link OA measured relative to the x-axis and $\psi$ the angle of the output link BC relative to the x-axis, Figure 7 (right). A relation between the input angle $\theta$ and output
Figure 7: The static constraints (left), shown for 2 DP’s, limit the design space to the area of designs that reach the output link’s (BC) start- and end position ($\psi_i$ and $\psi_e$).

angle $\psi$ is obtained based on the length of the coupler link $|AB|$, which stays fixed during the mechanism’s movement. Therefore, the analysis can start with:

$$\begin{bmatrix} x_B(\psi) \\ y_B(\psi) \end{bmatrix} - \begin{bmatrix} x_A(\theta) \\ y_A(\theta) \end{bmatrix} = |AB|^2$$

where

$$x_A(\theta) = |OA| \cos(\theta) \quad x_B(\psi) = x_C + |BC| \cos(\psi)$$
$$y_A(\theta) = |OA| \sin(\theta) \quad y_B(\psi) = y_C + |BC| \sin(\psi).$$

By substitution of (3) in (2), the dependency of the input angle $\theta$ based on the output angle $\psi$ is noted as

$$\theta_{1,2}(\psi) = \arctan2(V(\psi),U(\psi)) \pm \arccos\left(\frac{W(\psi)}{\sqrt{U^2(\psi) + V^2(\psi)}}\right) + \pi$$

where

$$U(\psi) = -2x_C|OA| - 2|OA||BC| \cos(\psi)$$
$$V(\psi) = -2y_C|OA| - 2|OA||BC| \sin(\psi)$$
$$W(\psi) = x_C^2 + y_C^2 + |OA|^2 + |BC|^2 - |AB|^2 + 2\cos(\psi)x_C|BC| + 2\sin(\psi)y_C|BC|. $$

Equation (4) allows the derivation of the input angle $\theta$ from the output angle $\psi$. The latter is the imposed output motion defined by the $\Delta \psi$ range. However, the mechanism can be assembled in two ways for a single output angle $\psi$, resulting in two possible solutions for $\theta$ in Equation (4). This is a consequence of having the possibility to construct the four-bar, with a certain angle $\psi$, with output link BC on both sides of the diagonal OB, as shown in Figure 8. Both constructions, called the elbow-up OABC and elbow-down OA’BC, provide feasible solutions.
The method proposed in the present paper applies to both configurations, yet it is chosen to focus on the elbow-up OABC, as it is the most efficient one according to [30]. To obtain the corresponding \( \theta_1 \) which is smaller than \( \theta_2 \), the second term is subtracted from the first term in Equation (4).

Regardless of the elbow configuration, feasibility constraints due to the solvability of Equation (4) arise. A solution can be found if the argument of the \( \arccos \) is in the range \([-1,1]\). Thus, a solution solely exist when the inequality constraint:

\[
U^2(\psi) + V^2(\psi) - W^2(\psi) \geq 0 \quad (6)
\]

is satisfied. In this way, an inequality constraint for the two output angles \( \psi \) that bring point B farthest and closest to O must be established. Therefore, Equation (6) is evaluated for the maximal and minimal angle \( \psi_e \) and \( \psi_i \). This evaluates the assemblability in the positions \( \psi_i \) and \( \psi_e \).

\[
\left. U^2(\psi) + V^2(\psi) - W^2(\psi) \right|_{\psi=\psi_i,\psi_e} \geq 0 \quad (7)
\]

By fulfilling Equation (7), one can say that the designed mechanism is assemblable over its movement. This design lies than within the area formed by the blue lines, which means that the mechanism is assemblable in \( \psi_i \), and inside the area formed by red lines as it is assemblable in \( \psi_e \) (see Figure 7).

3.2. Dynamic constraints of a four-bar

The aforementioned static constraints in chapter 3.1 are not sufficient to exclude all infeasible designs. To ensure that the desired movement \( \psi(t) \) of the output linkage BC is feasible, all defects during the movement should also be excluded. The three types of defects that can occur during the motion of a four-bar linkage are branch, order and circuit defects. The broad review in [31] reveals that research about branch, order and circuit defect avoidance is of great significance in the field of linkage syntheses. With a branch defect, the mechanism cannot perform the desired movement continuously. More specifically, four-bar linkages can move in two different ways. In Figure 9, the input link OA moves between its extreme positions (\( \theta_{min} \) until \( \theta_{max} \)), resulting in a
change of the transmission angle $\zeta$ between 0 and $\pi$. The extreme input angle positions $\theta_{\text{min}}$ and $\theta_{\text{max}}$ corresponds with an angle $\zeta$ equal to respectively $\pi$ and 0. The movement is conducted by initiating the motion of the output link BC clockwise or counter-clockwise around C. The movement in each initial direction around C (clockwise or counter-clockwise) represents separate a branch. If the mechanism has to change branch while moving, a branch defect occurs for this linkage system design [32]. When a branch defect occurs, one can observe that the mechanism reaches the $\theta_{\text{min}}$ or $\theta_{\text{max}}$ position more than once during the movement. This results in a transmission angle $\zeta$ moving through 0 or $\pi$. Hence, when the mechanism moves through the positions $\zeta$ equal to 0 or $\pi$, a change in the direction of $\theta$ occurs.

Order defects appear if certain points $\begin{bmatrix} x_B(\theta) \\ y_B(\theta) \end{bmatrix}^i{\ast}_{i \in \mathbb{N}}$ are not reached in a certain sequence or order [17]. Order defects are impossible in this study as a reciprocal mechanism is considered, which moves continuously (with a fixed motion profile $\pi(t)$) between the maximal and minimal angle $\psi_e$ and $\psi_i$. Figure 9 indicates that a circuit can exist out of two connected branches. Moreover, this design reveals that a mechanism can have an other circuit in which the mechanism reaches whole other positions $\begin{bmatrix} x_B(\theta) \\ y_B(\theta) \end{bmatrix}$, as shown in Figure 10. The maximum circuits a four-bar mechanism can have are limited to two. The mechanism can move in each circuit separately without the necessity of disconnecting any joints [33]. A circuit defect arises when the linkage mechanism must be disassembled and placed in the other circuit, shown in Figure 11 to complete the motion. To obtain a circuit defect, $\theta$ should become bigger or smaller than $\theta_{\text{max}}$ (with $\zeta=0$) or $\theta_{\text{min}}$ (with $\zeta=\pi$) respectively, to fulfill the desired movement of the output link BC ($\psi(t)$). A circuit defect has the same influence on $\theta$ as during a branch defect. In this paper, PTP movements with only a desired start- and endpoint are considered. The movement takes place through the actuation of one joint, point O. Therefore, the movement should stay within a single branch of a single circuit [34] (Figure 9 or 10).

To eliminate the possible circuit and branch defects, dynamic constraints
Figure 10: The second circuit of this specific four-bar design indicate that another circuit entails a complete different range. Nonetheless, the circuit is also constructed by two connected branches, with the same transition conditions for $\zeta$.

Figure 11: A designed linkage system that moves from one circuit to another must be disassembled, which is a circuit defect.

are created based on the consequence of a defect that changes the direction of the input angle $\theta$. The calculations of the motor angle are always chosen for the elbow-up OABC. However, by altering the circuit, the configuration becomes an elbow-down in which $\theta$ moves in the other direction. Therefore, one can exclude branch and circuit defects by guaranteeing monotonicity in the motor position profile $\theta(t)$. The dynamic constraint

$$\text{sign} \left( \dot{\theta}(\psi_i) \right) = \text{sign} \left( \dot{\theta}(\psi_e) \right)$$

will check if the first derivative of $\theta$, in the start- and end-position $\psi_i$ and $\psi_e$, does not alter its sign. Equation (8) is only applicable if the mechanism deals with an odd number of branch and or circuit defects while moving, as only then a change of sign is detected. Nonetheless, the method is still applicable when an even number of defects occur, because a defect results in very high required driving torques for each sign change. The interpolation in chapter 4 neglects these disproportional objective values. In that way, an even number of sign changes caused by an even number of defects will not affect the optimization. So, all the constraints together indicate the feasible design spaces, as shown in Figure 12. Within the feasible design spaces motion simulations, for certain samples, are performed to determine the objective value.

4. Multidimensional Sparse Interpolation

Determining the objective values, using CAD multi-body simulations, for the three-dimensional design problem, in a brute force way would lead to a tremen-
dous computational burden requiring 10,000,000 samples. Therefore, we rely on a sparse data fitting method to determine a mathematical model for the objective function $T_{RMS}(\|OA\|, |AB|, |BC|)$. While several interpolation methods are characterized by a trade-off between model accuracy and computing cost, sparse interpolation does not involve such compromise. The method introduced here uses a divide-and-conquer approach by splitting up the involved numerical linear algebra problems into smaller and hence better-conditioned independent sub-problems.

For this method, the objective value, within the feasible design space as defined in (7) and (8), is determined on $l$ distinct lines in 3D space that are all parallel to a chosen vector $\Delta = (\Delta_u, \Delta_v, \Delta_w)$. We let $\delta(i), i = 0, \ldots, l - 1$ indicate the 3D vector that the $i$-th parallel line is shifted over with respect to the line through the origin spanned by $\Delta$ for which we take $\delta(0) = 0$. Then the equidistant samples on these parallel lines, as depicted in Figure 13 left, are denoted by:

$$T_k^{(i)} := T_{RMS}(k\Delta + \delta(i)), \quad i = 0, \ldots, l - 1, \quad k = 0, \ldots, N_i - 1.$$  

Let us compactly denote the tuple of design variables $(|OA|, |BC|, |AB|)$ by

$$U = (u, v, w) := (|OA|, |BC|, |AB|)$$

and let $\langle \cdot, \cdot \rangle$ denote the standard inner product in 3D space. On each $i$-th parallel line the samples $T_k^{(i)}$ can be modeled by the sparse interpolant

$$T_{RMS,i}(U) = \sum_{j=1}^{n_i} \beta_j^{(i)} \exp \left( \langle \phi_j^{(i)}, U \rangle \right) \quad \text{(9)}$$

Figure 12: All constraints are shown for 2 DP’s (left) and 3 DP’s (right). The objects on the right are the feasible design spaces where the designs can perform the imposed reciprocal movement.
satisfying

\[
T_k^{(i)} = \sum_{j=1}^{n_i} \beta_j^{(i)} \exp \left( k(\phi_j^{(i)}, \Delta) \right), \quad i = 0, \ldots, l - 1, \quad k = 0, \ldots, N_i - 1.
\]

Note that the effect or influence of \( \delta^{(i)} \) is absorbed into the coefficients \( \beta_j^{(i)} \) in \( T_{RMS,i}(U) \) which models the behaviour of \( T_{RMS} \) on the \( i \)-th line.

Figure 13: At the left, all the samples located at points \( k\Delta + \delta^{(i)} \). The convex hull, in blue, of the \( l \) lines covering the larger part of the design space, at the right.

The model for \( T_{RMS,i}(U) \) can be computed using any of the existing 1D exponential fitting methods, such as [35, 36, 37, 38]. The number of terms \( n_i \) in the sparse model can differ on each \( i \)-th line. The \( l \) individual models are only valid on their respective line spanned by \( \Delta \) and shifted over \( \delta^{(i)} \). Now we need to blend these individual sparse models into an overall sparse model, valid in the convex hull (blue area in Figure 13 right) of the \( l \) lines, which should cover the larger part of the region of interest. This requisite or demand actually dictates the more proper choices for \( \Delta \) and the \( \delta^{(i)}, i = 1, \ldots, l - 1 \).

In what follows, we consider every design parameter combination \( U \) in 3D space to lie on some line parallel with the one spanned by \( \Delta \), also if \( U \) is not an interpolation point. All the points on such a line take the form

\[
U + r\Delta, \quad r \in \mathbb{R}.
\]  
(10)

The normal plane through the origin and orthogonal to \( \Delta \) is given by the equation

\[
\Delta_u u + \Delta_v v + \Delta_w w = 0,
\]
or more compactly

\[
\langle \Delta, U \rangle = 0.
\]

The intersection point \( R \) of the normal plane with \((10)\) is thus given by

\[
\langle \Delta, R \rangle = 0, \quad R = U + r\Delta,
\]

13
or more explicitly,
\[ R = U - \frac{\langle \Delta, U \rangle}{||\Delta||^2} \Delta. \] (11)

Hence, the distance of \( U \) to this intersection point \( R \), expressed as a multiple of \( ||\Delta|| \), equals
\[ p(u, v, w) = \frac{\langle \Delta, U \rangle}{||\Delta||^2}, \quad U = (u, v, w) \]
and the points on the line given by (10) can be re-expressed as
\[ R + p(U)\Delta \] (12)

On each line through a point \( U \) parallel with \( \Delta \), the intersection point \( R \) with the normal plane, shown in Figure 14, is where \( p(R) = 0 \) on the line.

We therefore propose a blended 3D model of the following form, to represent the objective value \( T_{RMS} \) overall,
\[ T_{RMS}(u, v, w) \approx \sum_{i=0}^{l-1} \sum_{j=1}^{n_i} b_{ij}^{(i)}(u, v, w) \exp \left( p(u, v, w)\langle \phi_{ij}^{(i)}, \Delta \rangle \right), \] (13)
where the parameters \( \phi_{ij}^{(i)} \) and the value of \( p(u, v, w) \) are already determined and where furthermore the overall model continues to interpolate the values \( T_k^{(i)} \) in the sample points \( k\Delta + \delta^{(i)} \). The blended model (13) coincides with the 1D models (9) on each parallel line, and in between the lines the exponential terms fade in and out. Since
\[ p(k\Delta + \delta^{(i)}) = k + p(\delta^{(i)}) \]
this means
\[ T_{RMS}(k\Delta + \delta^{(i)}) = \sum_{j=1}^{n_i} b_{ij}^{(i)}(k\Delta + \delta^{(i)}) \exp \left( p(\delta^{(i)})\langle \phi_{ij}^{(i)}, \Delta \rangle \right) \exp \left( k\langle \phi_{ij}^{(i)}, \Delta \rangle \right). \] (14)

In other words, on each data line the model consists of only \( n_i \) terms, while in the convex hull of the parallel lines it consists of \( n_0 + \ldots + n_{l-1} \) terms. Remember that all of \( l \) and \( n_0, \ldots, n_{l-1} \) are small integer numbers.

From Equation (14) and Equation (9) we consequently find
\[ b_{ij}^{(i)}(k\Delta + \delta^{(i)}) = \beta_{ij}^{(i)} \exp \left( -p(\delta^{(i)})\langle \phi_{ij}^{(i)}, \Delta \rangle \right), \]
\[ k = 0, \ldots, N_i - 1, \quad i = 0, \ldots, l - 1, \quad j = 1, \ldots, n_i. \] (15)

Note that \( b_{ij}^{(i)}(U) \) remains constant along each line of the form \( R + p(U)\Delta \) and only varies with the projection \( R \) of that line onto the normal plane.

Remains to determine the \( b_{ij}^{(i)}(u, v, w) \). These functions can be determined from the interpolation conditions given in Equation (15). A simple model for
$b_j^{(i)}(u, v, w)$ is a 2D polynomial interpolant $a_j^{(i)}(p, q)$, as we outline now. Let us denote the coordinates of the intersection point given in Equation (11) by $R = (r, s, t)$. The collection of points on a particular line perpendicular to the normal plane, say here through $R$, is entirely identified by the remaining two degrees of freedom that pinpoint the intersection point of such a line with the normal plane. Since every point $U = (u, v, w)$ on the line perpendicular to the normal plane and passing through $R$ satisfies the conditions

$$
\begin{align*}
\frac{u-r}{\Delta_u} &= \frac{v-s}{\Delta_v} = \frac{w-t}{\Delta_w},
\end{align*}
$$

we can take any two of the values

$$
\begin{align*}
\Delta_vu - \Delta_u v &= r\Delta_v - s\Delta_u \\
\Delta_wu - \Delta_u w &= r\Delta_w - t\Delta_u \\
\Delta_w v - \Delta_v w &= s\Delta_w - t\Delta_v
\end{align*}
$$

(16)

to characterize the full line. Over the whole of such a perpendicular line the right hand sides of Equation (16) are constant and independent of the points $U$ on the line. The right hand sides of Equation (16) are only determined by $\Delta$ and $R$. Say, for now, that we take the first two of (16), without any loss of generality: $p = \Delta_vu - \Delta_u v, q = \Delta_wu - \Delta_u w$. For the $l$ parallel lines on which the samples where collected, we find

$$
(p^{(i)}, q^{(i)}) = (\Delta_v\delta_u^{(i)} - \Delta_u \delta_v^{(i)}, \Delta_w\delta_u^{(i)} - \Delta_u \delta_w^{(i)}), \quad i = 0, \ldots, l - 1.
$$

Let us abbreviate the values in the right hand side of Equation (15) by $\alpha_j^{(i)}$ and replace $b_j^{(i)}(u, v, w)$ in Equation (13) by the more appropriate $a_j^{(i)}(p, q)$ since

Figure 14: The intersection point $R$ of the line $U + r\Delta$, parallel with $\Delta$ and through $U$, at the left. All $l = 7$ lines’ intersection points $(p^{(i)}, q^{(i)})$ with the normal plane at the right.
the interpolation conditions for \( \phi_j^{(i)}(u, v, w) \) hold for a whole line and vary only with the intersection point of such a line with the normal plane:

\[
T_{RMS}(u, v, w) \approx \sum_{i=0}^{l-1} \sum_{j=1}^{n_i} a_j^{(i)}(p, q) \exp \left( p(u, v, w) \langle \phi_j^{(i)}, \Delta \rangle \right).
\]

(17)

Finally, the 2D polynomial interpolant

\[
a_j^{(i)}(p, q) = \sum_{h,\ell} \tau_h^{(i,j)} T_h(p) T_\ell(q)
\]

where \( T_n(\cdot) \) denotes the well-known Chebyshev polynomial (of the first kind) of degree \( n \), is computed from the interpolation conditions

\[
a_j^{(i)}(p^{(m)}, q^{(m)}) = \begin{cases} 
\alpha_j^{(i)}, & m = i \\
0, & m \neq i,
\end{cases}
\]

\[i, m = 0, \ldots, l-1, \quad j = 0, \ldots, n_i.
\]

Figure 15: Region of interest delimited by (7) and (8) at the left and sampling locations on \( l = 7 \) parallel lines at the right in red.

We now apply the above to our four-bar problem. The region of interest for the design variables \(|OA|, |AB|, |BC|\) and restricted by the conditions (7) and (8) is shown in Figure 15 and the sampling performed in this region is shown in red in Figure 15 (right). We take \( l = 7 \) and \( \Delta = (0.000, 0.920, 0.503) \) to guarantee maximal coverage of the region of interest. Also, the whole domain is translated over \( -(31.000, 257.859, 72.705) \) to start sampling at the origin, in line with our description. In total only 618 samples are determined by the simulations explained in Section 2 which shape the objective function. We find that \( n_i = 5 \) for all \( i = 0, \ldots, 6 \), thus yielding \( 7 \times 5 \) terms in the global model \( T_{RMS}(|OA|, |AB|, |BC|) \). The coefficients \( a_j^{(i)}(p, q) \) are interpolated by a linear combination of the 7 bivariate Chebyshev polynomials \( T_m(p) T_n(q), 0 \leq m + n \leq 2 \) and \( T_2(p) T_1(q) + T_1(p) T_2(q) \). As a final step, we validate the blended model by collecting 1252 more simulation data on 10 other lines within the convex hull, along directions different from \( \Delta \). These evaluation directions are shown in purple in Figure 16 (left) and the result of this validation is shown in Figure 16 (right). In Figure 16 (right) the red and purple markers
depict the simulated data and the blue markers represent the value computed from the blended model (17). Each partial curve shows the function values of $T_{RMS}(u, v, w)$ restricted to one of the lines where samples were collected, either for interpolation (red) or validation (purple). The overall Root Mean Square Error (RMSE) equals 0.0281Nm, indicating a very good fit. When restricting our attention to $T_{RMS}$ values below 5 – reasonable to locate a minimum – the RMSE reduces to 0.0153Nm.

After this validation we look for a minimum of the modelled $T_{RMS}(u, v, w)$ (17) in the convex hull of the parallel lines shown in Figure 15 (right). This was fulfilled through a brute force evaluation of the objective function (17), in which 10,000,000 calculations were performed in 3 minutes. Thus, we can conclude that the most time consuming was the collection of all 1870 samples, in which generating a sample takes on average 1 minute and 25 seconds of simulation time. The model reaches a minimal value of 2.5989Nm at $U = (33.246, 266.088, 79.435)$. If we want to achieve the same result through brute force evaluation of the simulations, 10,000,000 simulations would be required.

5. Conclusion

This study proposes an industrially applicable approach that guarantees to reveal the global optimal design of a four-bar mechanism based on CAD motion simulations and sparse interpolation. The process of sampling the objective-value $T_{RMS}$ for a combination of design parameters $|OA|$, $|AB|$ and $|BC|$ is automated by means of CAD multi-body motion simulations. Subsequently, the constraints limiting the feasible design space are introduced based on the position analysis of the four-bar mechanism. This guarantees that all designs considered by the optimiser can be assembled, and no circuit or branch defect will occur during the mechanism’s movement.

If the unconstrained design space would be considered by a brute force approach 10,000,000 objective value samples would be required. As each objective value sample requires a simulation of approximately 1 minute and 25 seconds this would be practically impossible and seriously hampering the identification of the global optimum. However, thanks to the mathematical description of
Table 1: Saving potential achieved with design optimization.

| Design          | $|OA|$ | $|AB|$ | $|BC|$ | $T_{rms}$ | $T_{max}$ | $T_{rms}$ savings | $T_{max}$ savings |
|-----------------|------|------|------|----------|----------|-------------------|-------------------|
| Original        | 53   | 65   | 282  | 7.91     | 13.26    | -                 | -                 |
| Local optimum   | 40.6 | 77.2 | 263.23| 4.19     | 6.30     | 47                | 52.5              |
| Global optimum  | 33.246| 79.435| 266.088| 2.60     | 4.35     | 67                | 67                |

the design space constraints, introduced in this paper, sparse interpolation can be applied. The innovative sparse interpolation technique, described and applied in this paper reduces the number of necessary simulations to only 618. This allowed to identify the global optimal design. As shown in Table 1, the method clearly outperforms the best result (local optimum) obtained through the HEEDS Sherpa heuristic optimizer 39. The global optimum is 38 % more efficient than the local optimum. Moreover, the global optimum also reduces the $T_{max}$ by 67 % compared to the original design, which means that the mechanism can operate with a smaller, and thus cheaper motor.
References

[1] D. Dornfeld, Green manufacturing: Fundamentals and applications, Vol. 9781441960, Springer US, 2013. doi:10.1007/978-1-4419-6016-0

[2] P. Waide, C. U. Brunner, Energy-Efficiency Policy Opportunities for Electric Motor-Driven Systems, Internationale energy agency na (na) (2011) 132. doi:10.1787/20792581

[3] K. Russell, R. S. Sodhi, On the design of slider-crank mechanisms. Part I: Multi-phase motion generation, Mechanism and Machine Theory 40 (3) (2005) 285–299. doi:10.1016/j.mechmachtheory.2004.07.009

[4] V. L. Reis, G. B. Daniel, K. L. Cavalca, Dynamic analysis of a lubricated planar slider-crank mechanism considering friction and Hertz contact effects, Mechanism and Machine Theory 74 (2014) 257–273. doi:10.1016/j.mechmachtheory.2013.11.009

[5] G. Berselli, F. Balugani, M. Pellicciari, M. Gadaleta, Energy-optimal motions for Servo-Systems: A comparison of spline interpolants and performance indexes using a CAD-based approach, Robotics and Computer-Integrated Manufacturing 40 (2016) 55–65. doi:10.1016/j.rcim.2016.01.003

[6] G. Carabin, E. Wehrle, R. Vidoni, A review on energy-saving optimization methods for robotic and automatic systems, Robotics 6 (4) (2017). doi:10.3390/robotics6040039

[7] T. Mashimo, T. Urakubo, T. Kanade, Singularity-based four-bar linkage mechanism for impulsive torque with high energy efficiency, Journal of Mechanisms and Robotics 7 (3) (2015) 1–8. doi:10.1115/1.4028930

[8] P. Sheppard, S. Rahimifard, Improving energy efficiency in manufacturing using peer benchmarking to influence machine design innovation, Clean Technologies and Environmental Policy 21 (6) (2019) 1213–1235. doi:10.1007/s10098-019-01701-4

[9] N. V. Oosterwyck, F. Vanbecelaere, M. Haemers, D. Ceulemans, K. Stockman, S. Derammelaere, CAD Enabled Trajectory optimization and Accurate Motion Control for Repetitive Tasks., in: IEEE International Conference on Control and Automation, ICCA, Vol. 2019-July, IEEE, 2019, pp. 387–392. doi:10.1109/ICCA.2019.8899728

[10] N. Van Oosterwyck, A. B. Yahya, A. Cuyt, S. Derammelaere, CAD based trajectory optimization of PTP motions using chebyshev polynomials, in: IEEE/ASME International Conference on Advanced Intelligent Mechatronics, AIM, Vol. 2020-July, 2020, pp. 403–408. doi:10.1109/AIM43001.2020.9158893
[11] J. Herregodts, S. Herregodts, Gear Up Medical (2019). URL https://www.coronaventilator.be/

[12] A. Piazzi, A. Visioli, Global minimum-time trajectory planning of mechanical manipulators using interval analysis, International Journal of Control 71 (4) (1998) 631–652. doi:10.1080/002071798221713.

[13] B. El-Kribi, A. Houidi, Z. Affi, L. Romdhane, Application of multi-objective genetic algorithms to the mechatronic design of a four bar system with continuous and discrete variables, Mechanism and Machine Theory 61 (2013) 68–83. doi:10.1016/j.j.mechmachtheory.2012.11.002.

[14] Z. Affi, B. El-kribi, L. Romdhane, Advanced mechatronic design using a multi-objective genetic algorithm optimization of a motor-driven four-bar system, Mechatronics 17 (2007) 489–500. doi:10.1016/j.mechatronics.2007.06.003.

[15] R. Rayner, M. N. Sahinkaya, B. Hicks, Combining inverse dynamics with traditional mechanism synthesis to improve the performance of high speed machinery, in: 2008 Proceedings of the ASME Dynamic Systems and Control Conference, DSCC 2008, 2009, pp. 393–400. doi:10.1115/dscd2008-2186.

[16] A. Hernández, A. Muñoz, M. Utrizar, E. Amezua, Comprehensive approach for the dimensional synthesis of a four-bar linkage based on path assessment and reformulating the error function, Mechanism and Machine Theory 156 (2021). doi:10.1016/j.mechmachtheory.2020.104126.

[17] G. R. Gogate, S. B. Matekar, Optimum synthesis of motion generating four-bar mechanisms using alternate error functions, Mechanism and Machine Theory 54 (2012) 41–61. doi:10.1016/j.mechmachtheory.2012.03.007.

[18] Q. Shen, W. T. Lee, K. Russell, On adjustable planar four-bar motion generation with order, branch and circuit defect rectification, Journal of Mechanisms and Robotics 7 (3) (2015). doi:10.1115/1.4028828.

[19] J. A. Hrones, G. L. Nelson, Analysis of the four-bar linkage: its application to the synthesis of mechanisms, Technology Press of the Massachusetts Institute of Technology, 1978.

[20] A. Jaiswal, H. P. Jawale, Comparative study of four-bar hyperbolic function generation mechanism with four and five accuracy points, Archive of Applied Mechanics 87 (12) (2017) 2037–2054. doi:10.1007/s00419-017-1310-5.

[21] S. Bai, J. Angeles, Coupler-curve synthesis of four-bar linkages via a novel formulation, Mechanism and Machine Theory 94 (2015) 177–187. doi:10.1016/j.mechmachtheory.2015.08.010.
[22] X. Li, S. Wei, Q. Liao, Y. Zhang, A novel analytical method for four-bar path generation synthesis based on Fourier series, Mechanism and Machine Theory 144 (2020). doi:10.1016/j.mechmachtheory.2019.103671

[23] X. Li, S. Wei, Q. Liao, Y. Zhang, A novel analytical method for function generation synthesis of planar four-bar linkages, Mechanism and Machine Theory 101 (2016) 222–235. doi:10.1016/j.mechmachtheory.2016.03.013

[24] T. Sauer, Prony’s method in several variables: Symbolic solutions by universal interpolation, Journal of Symbolic Computation 84 (2018) 95–112. doi:10.1016/j.jsc.2017.03.006

[25] A. Cuyt, W. shin Lee, Multivariate exponential analysis from the minimal number of samples, Advances in Computational Mathematics 44 (4) (2018) 987–1002. doi:10.1007/s10444-017-9570-8

[26] W. T. Lee, K. Russell, Developments in quantitative dimensional synthesis (1970-present): four-bar motion generation, Inverse Problems in Science and Engineering 26 (1) (2018) 133–148. doi:10.1080/17415977.2017.1310858

[27] W. T. Lee, K. Russell, Developments in quantitative dimensional synthesis (1970–present): four-bar path and function generation, Inverse Problems in Science and Engineering 26 (9) (2018) 1280–1304. doi:10.1080/17415977.2017.1396328

[28] Simcenter 3D for motion simulation (2020). URL www.cardsplmsolutions.com/nl/producten/simcenter-3d

[29] J. M. McCarthy, Gim Song Soh, Geometric design of linkages, 2nd Edition, Springer US, 2010. doi:10.1007/978-1-4419-7892-9

[30] R. A. Srivatsan, S. Bandyopadhyay, On the position kinematic analysis of MaPaMan: A reconfigurable three-degrees-of-freedom spatial parallel manipulator, Mechanism and Machine Theory 62 (April) (2013) 150–165. doi:10.1016/j.mechmachtheory.2012.11.008

[31] S. S. Balli, S. Chaud, Defects in link mechanisms and solution rectification, Mechanism and Machine Theory 37 (9) (2002) 851–876. doi:10.1016/S0094-114X(02)00035-6

[32] R. Singh, H. Chaudhary, A. K. Singh, Defect-free optimal synthesis of crank-rocker linkage using nature-inspired optimization algorithms, Mechanism and Machine Theory 116 (2017) 105–122. doi:10.1016/j.mechmachtheory.2017.05.018

[33] T. R. Chase, J. A. Mirth, Circuits and branches of single-degree-of-freedom planar linkages, 21st Biennial Mechanism Conference (1993) 11–19.
[34] M. S. Feki, F. Chaari, M. S. Abbes, F. Viadero, A. Fdez. Del Rincon, M. Haddar, Identification of a Usable Six-Bar Linkage for Dimensional Synthesis, Mechanisms and Machine Science 7 (2013) 311–318. doi:10.1007/978-94-007-4902-3

[35] M. Briani, A. Cuyt, F. Knaepkens, W. shin Lee, VEXPA: Validated EXPo-nential Analysis through regular sub-sampling, Signal Processing 177 (Cmi) (2020). arXiv:1709.04281 doi:10.1016/j.sigpro.2020.107722

[36] R. Roy, T. Kailath, ESPRIT-estimation of signal parameters via rotational invariance techniques, IEEE Transactions on Acoustics, Speech, and Signal Processing 37 (7) (1989) 984–995. doi:10.1109/29.32276

[37] R. Schmidt, Multiple emitter location and signal parameter estimation, IEEE Transactions on Antennas and Propagation 34 (3) (1986) 276–280. doi:10.1109/TAP.1986.1143830

[38] W. M. Steedly, C. Ying, R. Moses, Statistical analysis of TLS-based prony techniques, Autom. 30 (1994) 115–129. doi:10.1016/0005-1098(94)90232-1

[39] Red Cedar Technology, SHERPA, An Efficient and Robust Optimization (Search) Algorithm (2014) 1–3.