Quantization of $r - Z$-quasi-Poisson manifolds and related modified classical dynamical $r$-matrices

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Abstract

Let $X$ be a $C^\infty$-manifold and $\mathfrak{g}$ be a finite dimensional Lie algebra acting freely on $X$. Let $r \in \Lambda^2(\mathfrak{g})$ be such that $Z = [r, r] \in \Lambda^3(\mathfrak{g})^\mathfrak{g}$. In this paper we prove that every quasi-Poisson $(\mathfrak{g}, Z)$-manifold can be quantized. This is a generalization of the existence of a twist quantization of coboundary Lie bialgebras ([EH]) in the case $X = G$ (where $G$ is the simply connected Lie group corresponding to $\mathfrak{g}$). We deduce our result from a generalized formality theorem. In the case $Z = 0$, we get a new proof of the existence of (equivariant) formality theorem and so (equivariant) quantization of Poisson manifold (cf. [Ko, Do]). As a consequence of our results, we get quantization of modified classical dynamical $r$-matrices over abelian bases in the reductive case.

0. Introduction

Throughout this paper, the ground field will be $\mathbb{R}$. Let $\mathfrak{g}$ be a finite dimensional Lie algebra with a fixed element $r \in \Lambda^3(\mathfrak{g})$ such that $[r, r] = Z \in \Lambda^3(\mathfrak{g})^\mathfrak{g}$. In [AK, AKM], quasi-Poisson manifolds were introduced as a generalization of Poisson $\mathfrak{g}$-manifolds with Poisson bracket satisfying the Jacobi identity up to an invariant trivector corresponding to $Z$. More precisely:

**Definition 0.1.** A quasi-Poisson $(\mathfrak{g}, Z)$-manifold is a $\mathfrak{g}$-manifold $X$ with an invariant bivector $\pi$ such that the Schouten bracket $[\pi, \pi]_S$ equals $\gamma^{\otimes 3}(Z)$, where $\gamma : \mathfrak{g} \to \text{Vect}(X)$ is the action homomorphism.

The Schouten bracket will be described later. Thus the Poisson bracket $\{-, -\}$ associated to $\pi$ satisfies

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = m_0(\gamma^{\otimes 3}(Z)(f \otimes g \otimes h)),$$
where \( m_0 \) is the usual multiplication. In the framework of deformation quantization (see [BFFLS1, BFFLS2]), Enriquez and Etingof defined the quantization of quasi-Poisson manifolds in [EE1]: let \( h \) be a formal parameter and \( \Phi = 1 + \frac{h}{\theta} Z + O(h^3) \in (U(\mathfrak{g})^\otimes 3)_{\theta}[[h]] \) be an associator for \( \mathfrak{g} \) (Drinfeld proved in [Dr], Proposition 3.10, that such an associator always exists).

**Definition 0.2.** A quantization of \( X \) associated to \( \Phi \) is an invariant star-product \( * \) on \( X \), i.e. an invariant bidifferential operator on \( C^\omega(X) \), which satisfies \( f * g = fg + O(h) \) and the equation

\[
f * g - g * f = h\{f,g\} + O(h^2),
\]

and is associative in the tensor category of \( (U(\mathfrak{g})[[h]],\Phi) \)-modules. This means,

\[
m_\ast(m_\ast \otimes 1) = m_\ast(1 \otimes m_\ast)\gamma^{\otimes 3}(\Phi),
\]
on \( C^\omega(X)^{\otimes 3} \), where \( m_\ast(f \otimes g) = f * g \).

They also conjectured that such quantizations always exist when the action of \( \mathfrak{g} \) on the quasi-Poisson manifold \( X \) is free. Note that when the action is not free, Fronsdal ([Fr]) gave in 1978 a counter-example where such quantization is impossible even in the symplectic case. From now on, we will suppose that the manifold \( X \) is a \( G \)-bundle over a manifold \( M \), where \( G \) is the simply connected Lie group corresponding to \( \mathfrak{g} \). In the case \( G = \{e\} \), the conjecture is equivalent to the existence of star-products and was proved by Kontsevich ([Ko]). In the case \( Z = 0 \), the conjecture follows from the equivariant formality theorem of Dolgushev ([Do]).

In the general case, \( \gamma^{\otimes 3}(Z) \) commutes with all the left invariant polyvector fields in the following sense :

\[
[\gamma^{\otimes 3}(Z),X]_S = 0, \quad \text{for all invariant polyvector fields } X.
\]

Moreover, for \( \Phi \) an associator, \( \gamma^{\otimes 3}(\Phi) \) commutes with all the invariant differential operators in the following sense:

\[
[\gamma^{\otimes 3}(\Phi),C]_G = 0, \quad \text{for all invariant differential operator } C
\]

(\( C \) the Gerstenhaber bracket \( [-,-]_G \) will be described later in this paper). From now on, if \( s \in \Lambda^3(\mathfrak{g}) \), we will denote \( s \) instead of \( \gamma^{\otimes 3}(s) \) when no confusion is possible.

In this paper, we prove that there exists (a least) one associator for \( \mathfrak{g} \) such that Enriquez-Etingof’s conjecture is true:

**Theorem 0.3.** Let \( r \in \Lambda^3(\mathfrak{g}) \) such that \( [r,r] = Z \in \Lambda^3(\mathfrak{g})^\theta \). There exists \( \Phi = 1 + \frac{h^2}{3} Z + O(h^3) \in (U(\mathfrak{g})^\otimes 3)_{\theta}[[h]] \) and a deformation \( \mathfrak{g}_h \) of the Lie algebra \( \mathfrak{g} \) such that for every invariant bivector \( \pi \) satisfying \( [\pi,\pi]_S = \gamma^{\otimes 3}(Z) \), the quasi-Poisson manifold \( (X,\pi) \) admits a quantization associated to \( (\Phi,\mathfrak{g}_h) \) i.e. a multiplication associative in the tensor category of \( (U(\mathfrak{g}_h)[[h]],\Phi) \)-modules.

To prove this theorem, we will construct a formality between invariant polyvector and polydifferential operator as stated in Theorem 7.3. We first prove a local version of this theorem in the case \( X = \mathbb{R}^n \times \mathfrak{g} \). Using Fedosov’s resolutions we will be able to get a global version. We then get the wanted invariant star-product on the manifold \( X \) and classification of such deformations. We will then discuss the relation with quantization of modified classical dynamical \( r \)-matrices.
Remark 0.4. As a particular case, our results give a new proof of Kontsevich (and Dolgushin for equivariant) formality theorem. One can see this approach as related to Merkulov’s work (see [Me]) for quantization of Lie bialgebras. In our work the use of a graded version of Etingof-Kazhdan theorem was a crucial step to go from quantization of Lie bialgebra to quantization of Poisson manifolds.

The paper is organized as follows:
- In Section 1, we recall definitions of $L_\infty$-structures and formality morphisms.
- In Section 2, we give a graded version of quantization of Lie bialgebras: in particular, we get differential graded Etingof-Kazhdan quantization/dequantization functors.
- In Sections 3 and 4, we construct two useful functors between Lie and Gerstenhaber algebras “up to homotopy” and prove the existence of two resolutions for those algebras.
- In Section 5, we prove the existence of $L_\infty$-morphisms between DG Lie bialgebras and the Gerstenhaber algebra of their Etingof-Kazhdan quantization.
- In Section 6, we transpose the algebra structures into the category of $(U(\mathfrak{g})[[\hbar]], \Phi)$-modules. We define the graded Lie bialgebra $\tilde{\mathfrak{g}} = \mathbb{R} \oplus V[1] \oplus V^* \oplus \mathfrak{g}$, the direct sum of the Eisenberg Lie algebra $E = \mathbb{R} \oplus V[1] \oplus V^*$ and the Lie bialgebra $(\mathfrak{g}, [\cdot, -])$ which corresponds locally to the algebra of invariant poly-vectors. We prove the existence of the local wanted $L_\infty$-morphism.
- In Section 7, we show that this $L_\infty$-morphism can be globalized and prove our main theorem.
- In Section 8, we discuss relation between our quantization and quantization of modified classical dynamical r-matrices.

Notations
We use the standard notation for the coproduct-insertion maps: we say that an ordered set is a pair of a finite set $S$ and a bijection $\{1, \ldots, |S|\} \to S$. For $I_1, \ldots, I_m$ disjoint ordered subsets of $\{1, \ldots, n\}$, $(U, \Delta)$ a Hopf algebra and $a \in U^{\otimes m}$, we define

$$a^{I_1, \ldots, I_m} = \sigma_{I_1, \ldots, I_m} \circ (\Delta(1) \otimes \cdots \otimes \Delta(m)) (a),$$

with $\Delta^{(1)} = \text{id}$, $\Delta^{(2)} = \Delta$, $\Delta^{(n+1)} = (\text{id} \otimes \Delta \otimes \cdots \otimes \Delta) \circ \Delta^{(n)}$, and $\sigma_{I_1, \ldots, I_m}: U^\otimes \Sigma |I_i| \to U^\otimes n$ is the morphism corresponding to the map $\{1, \ldots, \Sigma |I_i|\} \to \{1, \ldots, n\}$ taking $(1, \ldots, |I_1|)$ to $I_1$, $(|I_1| + 1, \ldots, |I_1| + |I_2|)$ to $I_2$, etc. When $U$ is cocommutative, this definition depends only on the sets underlying $I_1, \ldots, I_m$.

Until the end of this paper, although we will often omit to mention it, we will always deal with graded structures.

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1 \(L_\infty\)-structures

1.1 Definitions

Let us recall definitions of \(L_\infty\)-algebras and \(L_\infty\)-morphisms. Let \(A\) be a graded vector space. We denote \(T_rA = T_r(A[-1])\) the free tensor algebra (without unit) of \(A\) which, equipped with the coshuffle coproduct, is a bialgebra. We also denote \(C(A) = S(A[-1])\) the free graded commutative algebra generated by \(A[-1]\), seen as a quotient of \(T_rA\). The coshuffle coproduct is still well defined on \(C(A)\) which becomes a cofree cocommutative coalgebra on \(A[-1]\). We also denote \(\Lambda A = S(A[1])\), the analogous graded commutative algebra generated by \(A[1]\) (in particular, for \(A_1, A_2 \in A, A_1A_2\) stands for the corresponding quotient of \(A_1[1] \otimes A_2[1]\) in \(\Lambda A\)). We will use the notations \(T^nA, \Lambda^nA\) and \(C^n(A)\) for the elements of degree \(n\).

**Definition 1.1.** A vector space \(A\) is endowed with a \(L_\infty\)-algebra (Lie algebra “up to homotopy”) structure if there are degree one linear maps \(d^1, \ldots, d^k: \Lambda^kA \to A[1]\) such that the associated coderivations (extended with respect to the cofree cocommutative structure on \(\Lambda A\)) \(d: \Lambda A \to \Lambda A, d_0 = 0\) where \(d\) is the coderivation

\[
d = d^1 + d^{1,1} + \cdots + d^{1,\ldots,1} + \cdots,
\]

In particular, a differential Lie algebra \((A, b, [-, -])\) is a \(L_\infty\)-algebra with structure maps \(d^1 = b[1], d^{1,1} = [-, -][1]\) and \(d^{k,\ldots,1}: \Lambda^kA \to A[1]\) are 0 for \(k \geq 3\). One can now define the generalization of Lie algebra morphisms:

**Definition 1.2.** A \(L_\infty\)-morphism between two \(L_\infty\)-algebras \((A_1, d_1)\) and \((A_2, d_2)\) is a morphism of codifferential cofree coalgebras, of degree 0,

\[
\varphi: (\Lambda A_1, d_1) \to (\Lambda A_2, d_2).
\]

In particular \(\varphi \circ d_1 = d_2 \circ \varphi\). As \(\varphi\) is a morphism of cofree cocommutative coalgebras, \(\varphi\) is determined by its image on the cogenerators, i.e., by its components: \(\varphi^1, \ldots, \varphi^{k,\ldots,1}: \Lambda^kA_1 \to A_2[1]\).

Let \(E\) be a graded vector space. Let us denote \(\mathcal{T}(E)\) the cofree tensor coalgebra of \(E\) with coproduct \(\Delta'\). Equipped with the shuffle product \(\bullet\) (defined on the cogenerators \(\mathcal{T}(E) \otimes \mathcal{T}(E) \to E\) as \(pr \otimes e + e \otimes pr\), where \(pr : \mathcal{T}(E) \to E\) is the projection and \(e\) is the counit), it is a bialgebra. Let \(\mathcal{T}_+(E)\) be the augmentation ideal. We denote \(\mathcal{T}''(E) = \mathcal{T}(E)/\mathcal{T}_+(E)\) the quotient by the shuffles. It has a graded cofree Lie coalgebra structure (with coproduct \(\delta = \Delta' - \Delta''\)). Then \(S(\mathcal{T}''(E)[1])\) has a structure of cofree coGerstenhaber algebra (i.e. equipped with cofree coLie and cofree cocommutative coproducts satisfying compatibility condition). We use the notation \(\mathcal{T}''(E)\) for the elements of degree \(n\).

**Remark 1.3.** One could also define \(G_\infty\)-structures. Most of the \(L_\infty\)-morphism constructed in this paper are also \(G_\infty\)-morphisms between corresponding \(G_\infty\)-structures. Definitions and extensions to \(G_\infty\)-structures can be found in [Ha].
2 Etingof-Kazhdan functors

2.1 QUE and QFSH algebras

We recall some facts from [Dr] (proofs and definitions can be found in [Gav]). Let us denote by QUE the category of quantized universal enveloping (QUE) algebras and by QFSH the category of quantized formal series Hopf (QFSH) algebras. Let us recall the definition of FSH and QFSH algebras:

**Definition 2.1.** A FSH algebra is a Hopf algebra of power series isomorphic as an algebra to $\mathbb{K}[\{u_i | i \in J\}]$ (for some set $J$).

There is an equivalence of categories between the category of FSH algebra and the category of Lie coalgebra (LC algebra), sending $\mathcal{O}_h$ to $\mathfrak{h} = \mathcal{O}_h + \mathcal{O}_h^2$ where $\mathcal{O}_h$ is the maximal ideal of $\mathcal{O}_h$.

**Definition 2.2.** A QFSH algebra is a Hopf algebra $H$, which is a topologically free $\mathbb{K}[[\bar{\mathfrak{h}}]]$-module, such that $H_0 = H/\mathfrak{h}H$ is isomorphic to a FSH algebra.

Let us give an example of a FSH algebra, very important in this paper: let $V$ be a vector space and $\mathcal{C}TV$ (defined in the previous section) the cofree coalgebra, equipped with the shuffle product. Let us now complete $\mathcal{C}TV$. The algebra $\mathcal{C}TV$ is a graded algebra with $V$ being the set of elements of degree 1. Let us denote $\mathcal{M}_{\mathcal{C}TV}$ the set of elements of degree $\geq 1$. Finally, we denote $\hat{\mathcal{C}TV}$ the commutative cofree bialgebra, $\mathcal{M}_{\mathcal{C}TV}$-adic completion of $\mathcal{C}TV$.

**Proposition 2.3.** [Ha] $\hat{\mathcal{C}TV}$ is the FSH algebra $\mathcal{O}_{\mathcal{C}TV}$ associated with the Lie coalgebra $\mathcal{C}TV = \mathcal{C}T + V / (\mathcal{C}T + V)^2$, which is the cofree Lie coalgebra over $V$.

We have covariant functors QUE $\rightarrow$ QFSH, $U \mapsto U'$ and QFSH $\rightarrow$ QUE, $\mathcal{O} \mapsto \mathcal{O}^\vee$. These functors are also inverse to each other.

$U'$ is a subalgebra of $U$ defined as follows: for any ordered subset $\Sigma = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$ with $i_1 < \cdots < i_k$, define the morphism $j_\Sigma : U^{\otimes k} \rightarrow U^{\otimes n}$ by $j_\Sigma(a_1 \otimes \cdots \otimes a_k) := b_1 \otimes \cdots \otimes b_n$ with $b_i := 1$ if $i \notin \Sigma$ and $b_m := a_m$ for $1 \leq m \leq k$; then set $\Delta_\Sigma := j_\Sigma \circ \Delta^{(k)}$, $\Delta_\emptyset := \Delta^{(0)}$, and

$$\delta_\Sigma := \sum_{\Sigma \subseteq \Sigma} (-1)^{n-|\Sigma|} \Delta_\Sigma, \quad \delta_\emptyset := \varepsilon.$$

We shall also use the notation $\delta^{(n)} := \delta_{\{1,2,\ldots,n\}}$, $\delta^{(0)} := \delta_\emptyset$, and the useful formula

$$\delta^{(n)} = (id_U - \varepsilon)^{\otimes n} \circ \Delta^{(n)}.$$

Finally, we define

$$U' := \{ a \in U \mid \delta^{(n)}(a) \in h^n U^{\otimes n} \} \quad (\subseteq U)$$

and endow it with the induced topology.

On the other way, $\mathcal{O}^\vee$ is the $\mathfrak{h}$-adic completion of $\sum_{k \geq 0} h^{-k} \mathcal{M}^k \subset \mathcal{O}[1/h]$ (here $\mathcal{M} \subset \mathcal{O}$ is the maximal ideal).
2.2 The functor $DQ$

In [GH], a generalization of Etingof-Kazhdan theorem ([EK]) was proved in an appendix by Enriquez and Etingof:

**Theorem 2.4.** We have an equivalence of categories

$$DQ_{\Phi} : DGQUE \to DGLBA_h$$

from the category of differential graded quantized universal enveloping super-algebras to that of differential graded Lie super-bialgebras such that if $U \in Ob(DGQUE)$ and $a = DQ(U)$, then $U/hU = \mathbb{U}(a/ha)$, where $\mathbb{U}$ is the universal algebra functor, taking a differential graded Lie super-algebra to a differential graded super-Hopf algebra.

Here $\Phi$ is a Drinfeld associator. We will use any of these functors and denote it $DQ$.

3 Two functors

3.1 Functor $L-G$

Let $(\mathfrak{h}, \delta, d)$ be a differential Lie bialgebra. Let $C(\mathfrak{h}) = S(\mathfrak{h}[-1])$ be the free graded commutative algebra generated by $\mathfrak{h}$. Recall from the previous subsection that $C(\mathfrak{h})$ is also a cofree coalgebra and that coderivations $C(\mathfrak{h}) \to C(\mathfrak{h})$ are defined by their images in $\mathfrak{h}$. Thus, one easily checks that the coderivation $[-,-] : C(\mathfrak{h}) \to C(\mathfrak{h})$ extending the Lie bracket (with degree shifted by one) defines a Lie (even Gerstenhaber) algebra structure on $C(\mathfrak{h})$. Moreover, one can extend maps $d : \mathfrak{h} \to \mathfrak{h}$ and $\delta : \mathfrak{h} \to S^2(\mathfrak{h}[-1])$ on the free commutative algebra $C(\mathfrak{h})$ so that $(C(\mathfrak{h}), [-,-], \wedge, d + \delta)$ is a differential Gerstenhaber algebra. The differential $\delta$ is actually the Chevalley Eilenberg differential: the space $C(\mathfrak{h}) = S^*(\mathfrak{h}[-1])$ is isomorphic to the standard complex $(A^*(\mathfrak{h}))[\delta]$ and $\delta$ is simply the differential given by the underlying Lie coalgebra structure of $\mathfrak{h}$.

**Proposition 3.1.** [Ha] Any DGLA morphism $f : \mathfrak{h}_1 \to \mathfrak{h}_2$ can be extended into a DGLA (and even differential graded Gerstenhaber) morphism $C(f) : C(\mathfrak{h}_1) \to C(\mathfrak{h}_2)$ of free commutative algebras. This defines an exact functor $L-G$ from differential Lie bialgebras to differential Gerstenhaber algebras which sends $\mathfrak{h}$ to $C(\mathfrak{h})$. Quasi-isomorphisms $(\mathfrak{h}_1, d_1) \to (\mathfrak{h}_2, d_2)$ induce a quasi-isomorphisms $(C(\mathfrak{h}_1), d_1, \delta_1) \to (C(\mathfrak{h}_2), d_2, \delta_2)$.

3.2 Functor $L-G_{\infty}$

Consider now the category CFDLB of differential Lie bialgebras which are cofree as a Lie coalgebra. In other words we are interested in cofree Lie coalgebra $\mathcal{T}(E)$ on a graded vector space $E$ together with a differential $\ell$ and a cobracket $\ell$ on $\mathcal{T}(E)$ that makes it a differential Lie bialgebra. As $\mathcal{T}(E)$ is cofree, the differential is uniquely determined by its restriction to cogenerators $\ell^p : \mathcal{T}^p(E) \to E$. Similarly, the Lie bracket is uniquely determined by maps $L^{p_1,p_2} : \mathcal{T}^{p_1}(E) \wedge \mathcal{T}^{p_2}(E) \to E$.

**Proposition 3.2.** [Ha] Restriction map $\mathcal{T}^p(E) \to E$ defines an exact functor $L-G_{\infty}$ from CFDLB to the category of $G_{\infty}$ (and so Lie)-algebras.

Until the end of the paper, we will use the notations $TE$ for $T(E[-1])$ and $\mathcal{T}E$ for $\mathcal{T}(E[1])$.
4 Two resolutions

4.1 bialgebra structure on $cTT_+U$

Here, we will define a bialgebra structure on $cTT_+U$. One can construct a bialgebra structure on the space of Hochschild cochains of an algebra using the brace operations. In our case, we will firstly generalize the definition of brace operations for a general Hopf algebra. More precisely, let $(H, \Delta, \cdot)$ be a Hopf algebra (in our case $H$ will be the Etingof-Kazhdan quantization $U_h(a)$ of the Lie bialgebra $a$). We will define a brace structure on the cofree tensor coalgebra $cTT_+H$ of the free tensor algebra $T(H[−1])$ without unit. To distinguish the two tensor products, we denote $\otimes$ the tensor product on $T_+H$ and $\boxtimes$ the tensor product on $cTT_+H$.

**Definition 4.1.** We define brace operations on $cTT_+H$ by extending the following maps given on the cogenerators of the cofree algebra $cTT_+H$:

1. $B^0 = 0$,
2. $B^1 = b_{cH}$ (the coHochschild coboundary on $T_+H$),
3. $B^2 : \alpha \boxtimes \beta \mapsto \alpha \otimes \beta$,
4. $B^n = 0$ for $n > 2$,
5. $B^{0,1} = B^{1,0} = \text{id}$,
6. $B^{0,n} = B^{n,0} = 0$ for $n \geq 1$,
7. $B^{1,n} : (\alpha, \beta_1 \boxtimes \cdots \boxtimes \beta_n) \mapsto \sum_{0 \leq i_1 < \cdots < i_m \leq n \overline{i_1 + \cdots + i_m}} (-1)^e \alpha^{i_1+1-i_1+k_1, \cdots, i_m+1-i_m+k_m} \times 1_{i_1} \boxtimes \beta_1 \otimes 1_{i_2-1+i_2+k_1} \boxtimes \beta_2 \otimes \cdots \boxtimes \beta_n \otimes 1_{n-(k_m+k_n)}$,

where $k_s = |\beta_s|$, $n = |\alpha| + \sum_s k_s - m$ and $e = \sum_s (k_s - 1)i_s$.
8. $B^{k,l} = 0$ for $k > 1$.

Operations (2), (3) and (4) define a differential $d$ and (5), (6), (7) and (8) define a product $\ast$ deforming the shuffle product.

Note that, when $H = U(a)$, the enveloping algebra of a Lie algebra $a$, $T(H[−1])$ can be seen as the space of invariant polydifferential operators over the Lie group corresponding to $a$ and in that case, our definition coincides with usual braces operations.

We have:

**Theorem 4.2.** [Ha] The brace operations of Definition 4.1 define a differential bialgebra structure on the cofree tensor coalgebra $cTT_+H$, with product $\ast$ extending $\sum B^{p_1,p_2}$ and differential $d$ extending $\sum B^p$.

Let us now complete $cTT_+H$ as in section 2 with $V = T_+H$. We get a commutative cofree bialgebra $cTT_+H$, the $\mathcal{M}_{TT_+H}$-adic completion of $cTT_+H$ (where $\mathcal{M}_{TT_+H}$ is the maximal ideal of $cTT_+H$). Let us consider the free $\mathbb{K}[\![v]\!]$-module $cTT_+H[\![v]\!]$. One can now replace the operations $B^{p,q}$ of Definition 4.1 with $\mathbb{K}[\![v]\!]$-linear operations $v^{p+q-1}B^{p,q}$. Those operations are well defined on the completion $cTT_+H[\![v]\!]$ as this space is complete for the grading induced by the degree in $cTT_+H = cTV$ plus the $\hbar$-adic
valuation and because the operations we just defined are homogeneous for this grading. Thus we get a morphism of differential bialgebra

\[ I_v : \bar{cTT}H, \star, \Delta, d \rightarrow \bar{cTT}H[[v]][v^{-1}], \star_v, \Delta_v, d_v \]

where \(|x|\) is the degree in \(cT\). The morphism \(I_v\) extends to \(I_v : \bar{cTT}H[[v]], \star, \Delta, d \rightarrow \bar{cTT}H[[v]][v^{-1}], \star_v, \Delta_v, d_v\) which restricts to

\[ I'_v : \bar{cTT}H[[v]], \star, \Delta, d \rightarrow \bar{cTT}H[[v]][v^{-1}], \star_v, \Delta_v, d_v \]  \hspace{1cm} (4.3)

We have:

**Proposition 4.3.** [Ha] The algebra \((\bar{cTT}H[[v]], \star_v, \Delta_v, d_v)\) is a QFSH. The underlying differential Lie bialgebra structure on \(\bar{cTT}H\) is given by the Gerstenhaber bracket

\[ [\alpha, \beta]_G = B^{1,1}(\alpha, \beta) - (-1)^{|\alpha|-1(|\beta|-1)}B^{1,1}(\beta, \alpha) \]

and coHochschild differential

\[ b_{\text{coH}}(\alpha) = [1 \otimes 1, \alpha]_G. \]

for \(\alpha, \beta \in TH\) and then naturally extended on \(\bar{cTT}H\) using the cofree Lie cobracket.

**Remark 4.4.** Let now \(H\) be the QUE algebra \(U = U_h(\alpha)\). We have proved that \(T_+U\) can be equipped with a \(G_\infty\)-structure. Since the cofree Lie coalgebras are rigid, the differential Lie bialgebra corresponding to \(\bar{cTT}U[[v]]\) through Etingof-Kazhdan dequantization functor DQ is isomorphic to \(\bar{cTT}U[[v]]\) as a \(\mathbb{K}[[v]]\)-Lie coalgebra, and is therefore free.

### 4.2 A bialgebra quasi-isomorphism \(\phi_{\text{alg}} : U \rightarrow (\bar{cTT}_+U)^\vee\)

We have:

**Proposition 4.5.** Let \(U\) be a QUE algebra. One can define a bialgebra quasi-isomorphism \(\phi_{\text{alg}} : U \rightarrow \bar{cTT}_+U\) from the bialgebra \((U, \Delta_h, \times)\) to the bialgebra \((\bar{cTT}_+U, \Delta, \times)\) whose structure was described in the previous section.

Let \(U' \subset U\) (see section 2).

**Proposition 4.6.** [Ha] We have a bialgebra quasi-isomorphism \(\phi_{\text{alg}} : (U', \times) \rightarrow (\bar{cTT}_+U, \star_h)\) of QFSH algebra, where \((\bar{cTT}_+U, \star_h)\) is \((\bar{cTT}_+U[[v]], \star_v) / (v = 0)\) \(\bar{cTT}_+U[[v]][v]\) is the free \(\mathbb{K}[[h]]\)-module defined in the previous section: we der the operations \(B^{p,q}\) into \(v^{p+q-1}B^{p,q}\).

Finally, applying to \(\phi_{\text{alg}}\) the derived Drinfeld functor \((-)^\vee\), we get a bialgebra quasi-isomorphism \(\phi_{\text{alg}} : U \rightarrow (\bar{cTT}_+U)^\vee\).
4.3 A Lie bialgebra quasi-isomorphism $\phi_{\text{Lie}} : cTA \to cTC(cTA)$

Let $A$ be a vector space. Suppose now that the cofree Lie coalgebra $cTA$ has a structure $(cTA, \delta, [-, -], d)$ of a differential Lie bialgebra. Using the functor $L-G$ (see section 3), one gets a differential Gerstenhaber algebra $(C(cTA), [-, -], \wedge, d + \delta)$. One can extend the structure maps on the cofree Lie coalgebra $cTC(cTA)$ and one gets a differential cofree Lie bialgebra $(cTC(cTA), \delta', [-, -], d + \delta + \wedge)$ (we will set $d^1 = d + \delta$ and $d^2 = \wedge$).

**Proposition 4.7.** [Ha] Let $\phi_{\text{Lie}}$ be the composition map $\phi_{\text{Lie}} = cT i \circ \bar{\delta}$ of a map $\bar{\delta} : cTA \to cTC(cTA)$, $x \mapsto x + \sum_{k \geq 2} \bar{\delta}_k(x)$, where $\bar{\delta}_k$ is built using iterates of $\delta$, with $cT i : cTC(cTA) \to C(cTA)$ which is $cT$ of the inclusion $i: cTA[-1] \to C(cTA)$. Then $\phi_{\text{Lie}}$ is a differential Lie bialgebra quasi-isomorphism $\phi_{\text{Lie}} : cTA \to cTC(cTA)$.

5 $L_\infty$-morphism for Lie bialgebras

5.1 A Lie bialgebra quasi-isomorphism $\phi'_{\text{Lie}} : a \to cTT+U$

Let $(a, \delta_h)$ be a graded Lie bialgebra. We write $\delta_h = h \delta_1 + h^2 \delta_2 + \cdots$. Let $(U_h(a), \Delta_h)$ be the Etingof-Kazhdan canonical quantization of $(a, \delta_h)$. We denote $U = U_h(a)$ for short. In section 4, we proved the existence of a bialgebra structure on $cTT+U$ and a bialgebra quasi-isomorphism $\phi_{\text{alg}} : U \to (cTT+U)^\vee$. Thanks to Etingof-Kazhdan dequantization functor (see section 2), and the fact that $(cTT+U)^\vee$ is a QUE algebra quantizing $cTT+U$ (see section 4), we get a Lie bialgebra quasi-isomorphism $\phi'_{\text{Lie}} : a \to cTT+U$.

5.2 Inversion of formality morphisms

Let us recall Theorem 4.4 of Kontsevich ([Ko]):

**Theorem 5.1.** Let $g_1$ and $g_2$ be two $L_\infty$-algebras and $\mathcal{F}$ be a $L_\infty$-morphism from $g_1$ to $g_2$. Assume that $\mathcal{F}$ is a quasi-isomorphism. Then there exists an $L_\infty$-morphism from $g_2$ to $g_1$ inducing the inverse isomorphism between associated cohomology of complexes.

**Remark 5.2.** We know the existence of a similar $G_\infty$-version of this theorem. This result would imply the existence of corresponding $G_\infty$-morphisms.

5.3 $L_\infty$-morphism for Lie bialgebras

Let us summarize functors and quasi-isomorphisms constructed in the previous sections in the following diagram:
\[ C(\mathcal{C}(\mathcal{C}(TT+U) \mathcal{C}(TT+U)[1]) \mathcal{C}(TT+U) \mathcal{C}(TT+U)^\dagger \]
\[ \uparrow \varphi_{\text{Lie}} \quad \downarrow \varphi_{\text{Lie}} \quad \downarrow \varphi_{\text{Ger}} \quad \downarrow \varphi_{\text{Ger}} \quad \downarrow \varphi_{\text{alg}} \quad \downarrow \varphi_{\text{alg}} \]

\[ \mathcal{C}(TT+U) \quad T_+ U[1] \quad \mathcal{C}(a)[1] \quad a \quad U = U_\hbar(\hat{a}). \]

Thus, thanks to section 5.2, the composition \( \varphi: C(a) \to T_+ U \) of \( \varphi_{\text{Ger}} \) with the inverse of \( \varphi_{\text{Ger}} \) gives the wanted quasi-isomorphism.

**Theorem 5.3.** [Haj] the map \( \varphi: C(a) \to T_+ U \) is a \( L_\infty \)-quasi-isomorphism that maps \( v \in C(a) \) to \( \text{Alt}(v) \in T_+ U \mod \hbar. \)

### 5.4 \( L_\infty \)-morphism for \( X = \mathbb{R}^n \times \mathfrak{g} \)

We will now consider \( X = \mathbb{R}^n \times \mathfrak{g} \) and \( r \in \mathfrak{g} \wedge \mathfrak{g} \) such that \([r,r] = Z\). So \((\mathfrak{g}, [r,-])\) is a Lie bialgebra. Let us set \( V = \mathbb{R}\). From now on we will consider the graded Lie bialgebra \( \hat{\mathfrak{g}} = \mathbb{R} \oplus V[1] \oplus V^* \oplus \mathfrak{g} \), the direct sum of the Eisenberg Lie algebra \( E = \mathbb{R} \oplus V[1] \oplus V^* \) and the Lie bialgebra \((\mathfrak{g}, [r,-])\). We will now deduce our main result from:

**Proposition 5.4.** There exists a \( L_\infty \)-quasi-isomorphism \( \varphi_{\delta} \) between \((C(\hat{\mathfrak{g}}), [-,-], [r,-])\) and \((T, \underline{U}, [-,-], \mathfrak{h}, [1 \otimes 1, -\mathfrak{h}])\). Here \( \underline{U} \) is the Etingof-Kazhdan quantization of \( \hat{\mathfrak{g}} \) and so \( \underline{U} = U(E) \otimes U_\hbar(\hat{\mathfrak{g}}) \) where \( U_\hbar(\hat{\mathfrak{g}}) \) is the Etingof-Kazhdan quantization of \((\mathfrak{g}, [r,-])\). The bracket \([-,-]_\hbar \) denotes the Gerstenhaber bracket constructed in Section 4 corresponding to the coproduct of \( \underline{U} \).

### 6 Deformed structures and local \( L_\infty \)-morphism

#### 6.1 Deformed structures

Suppose we are given \( \Phi \in (U(\mathfrak{g}) \otimes \mathfrak{g})[[\hbar]] \) an associator. In particular, \( \gamma \otimes \Phi \) commutes with all the invariant differential operator. We have in fact:

\[ [C, \gamma \otimes \Phi]_G = 0 \text{ for all } C \in U(\hat{\mathfrak{g}})[[\hbar]]. \]

From now on, we will consider the tensor category of \((U(\mathfrak{g})[[\hbar]], \Phi)\)-modules (in which we want to construct an associative star-product). Let us define the “deformed” Gerstenhaber bracket as the Bracket defined in Section 4 but in the new tensor category. We get a new Lie algebra structure on \( U(\hat{\mathfrak{g}})[[\hbar]] \) given by the bracket \([-,-]_\Phi \) defined, for \( D, E \in U(\hat{\mathfrak{g}})[[\hbar]] \), by

\[ [D, E]_\Phi = \{ D[E]_\Phi = (-1)^{|E||D|} \{ E[D]_\Phi, \]

where for \( D \in U(\hat{\mathfrak{g}})^{\otimes d} \) and \( E \in U(\hat{\mathfrak{g}})^{\otimes e}, \)

\[ \{ D[E] \} = \sum_{i \geq 0} (\hbar-1)^i \Phi D^{i, i+1, i+1} \cdots E^{i+1, i+e}. \]

\( \Phi \) corresponds to the obvious change of parenthesis in the tensor category of \((U(\mathfrak{g})[[\hbar]], \Phi)\)-modules. For example, if \( A \) and \( B \) are two 2-cochains in \( U(\hat{\mathfrak{g}})[[\hbar]] \), one has

\[ \{ A, B \}_\Phi = A^{12, 1} B^{1, 2} - \Phi^{-1} A^{1, 2, 3} B^{2, 3}. \]
Remark 6.1. One could also define a deformed bialgebra structure on \((\hat{\mathcal{T}}T, U(\hat{g})[[\hbar]])\) and so using Etingof-Kazhdan quantization a \(G_\infty\)-structure on \(U(\hat{g})[[\hbar]]\) (proof can be copied from [Ta] or [GH]).

6.2 Twist quantization of coboundary Lie bialgebras

Let us recall results from [EH]:

**Theorem 6.2.** [Ha] Let \((a, [r, -])\) be a coboundary Lie bialgebra. There exists a coboundary quantization of it: \((U_h(a), \Delta_h, R_h)\).

Then, following [Dr], it was proved in [EH]:

**Theorem 6.3.** There exists a deformation \(a_\hbar\) of \(a\) in the category of topologically free \(\mathbb{R}[[\hbar]]\)-Lie algebras, \(J = 1 + hr/2 + O(h^2) \in U(\hbar a)\otimes^1 a_\hbar\) and \(\Phi_0 \in (U(\hbar a)\otimes^3 a_\hbar\jmath)\) such that the coboundary Hopf algebra \((U_h(a), \Delta_h, R_h)\) is isomorphism through \(J\) to the coboundary quasi-Hopf algebra \((U(\hbar a), \Delta_0, 1, \Phi_0)\) and we have, in \(U_h(a)\),

\[
J\Delta_0 J^{-1} = \Delta_h \text{ and } J^{1.2} J^{12.3} = J^{2.3} J^{1.23} \Phi_0.
\]

6.3 local \(L_\infty\)-morphism

Let us keep the notation of the previous section for \(a = \hat{g}\), the Lie algebra define in Section 5.4. Let us set \(F = J^{-1}\) and \(\Phi = \Phi_0\). We can now prove the existence of a \(L_\infty\)-morphism for our structures, in the local case:

**Theorem 6.4.** There exists a \(L_\infty\)-quasi-isomorphism \(\varphi_\infty\) between the differential Lie algebra \((\hat{S}(V) \otimes \Lambda(V^* \oplus g), [-, -, [r, -]])\) (corresponding to local invariant polynomials) and the Lie algebra \((\hat{T}(S(V)) (U(V^* \oplus g))[[\hbar]]), [-, -]_{\Phi}, [F, -]_{\Phi}\) (corresponding to invariant polydifferential operators).

**Proof.** Let us consider the Lie bialgebra \(\hat{g}\) defined in Section 5. Let \(\hat{U}\) be its Etingof-Kazhdan quantization. We know from Section 5 that there exists a \(L_\infty\)-quasi-isomorphism \(\Phi_h: (\hat{C}(\hat{g}), [-, -, [r, -]]) \rightarrow (\hat{S}(V) \otimes \Lambda(V^* \oplus g), [-, -], [r, -]) \rightarrow (\hat{T}(\hat{S}(V)) (U(V^* \oplus g))[[\hbar]]), [-, -]_{\Phi}, [F, -]_{\Phi}\).

Let now define \(\varphi_F: T_\hbar U_h(\hat{g}) \rightarrow T_\hbar U(\hat{g}_h)[[\hbar]]\) to be the map defined as follows: for \(x \in U_h(\hat{g})\),

\[
\varphi_F(x) = F^{12 \cdots n-1} \cdots F^{12} x_{F},
\]

in \(U(\hat{g}_h)\). It is clear that \(\varphi_F\) is an isomorphism of differential Lie algebras sending the bracket \([-,-]_{\Phi}\) to \([-,-]_{\Phi}\) and the differential \([1 \otimes 1, -]_{\hbar}\) to \([F, -]_{\Phi}\). Composing \(\varphi_h\) with \(\varphi_F\) we get a \(L_\infty\)-quasi-isomorphism:

\[
(\hat{S}(V) \otimes \Lambda(V^* \oplus g), [-, -], [r, -]) \rightarrow (T, U(\hat{g}_h), [-, -]_{\Phi}, [F, -]_{\Phi}).
\]

This gives the result as one can identify \((T, U(\hat{g}_h), [-, -]_{\Phi}, [F, -]_{\Phi})\) with \((T_{\hat{S}(V)} (U(V^* \oplus g))[[\hbar]]), [-, -]_{\Phi}, [F, -]_{\Phi}\) as differential Lie algebras.

**Remark 6.5.** Construction of the Lie algebras isomorphism \(\varphi_F\) can be generalized to differential Lie algebras between any two twist equivalent quasi-Hopf algebras \((H_1, \Delta_1, \Phi_1)\) and \((H_1, \Delta_1, \Phi_2)\) as far as the associators \(\Phi_1\) and \(\Phi_2\) are invariant (so that one can define corresponding Lie algebras on \(T_\hbar H_1\)).
7 Globalization and Proof of Theorem 0.3

7.1 Globalization

In this section $X$ is a principal $G$-bundle over a manifold $M$. We will use Kontsevich globalization procedure as described in [Do]. One can deduce global version of the local formality theorem (proved in Section 6) to a global one using Fedosov resolution as described in [Do]. The only things one has to check are the extra conditions that the $L_\infty$-quasi-isomorphism $\phi_{\text{loc}}$ has to fulfill:

1. The $L_\infty$-quasi-isomorphism $\phi_{\text{loc}}$ is equivariant with respect to linear transformations of coordinates.
2. $\phi(v_1, v_2) = 0$ for any formal vector fields $v_1$ and $v_2$.
3. If $n \geq 2$ and $v$ is a linear vector field in the coordinates on $\mathbb{R}^n$, then for any set of polyvector fields $\gamma_1, \ldots, \gamma_n$ we have $\phi(v, \gamma_1, \ldots, \Lambda \gamma_n) = 0$.

Proposition 7.1. The $L_\infty$-quasi-isomorphism $\phi_{\text{loc}}$ can be built so that it satisfies those three conditions

Proof. Let us recall that the map $\phi_{\text{loc}}$ was built from two differential Lie algebra morphism: $\phi_{\text{Ger}}: C(\tilde{\mathfrak{g}}) \rightarrow C(\mathcal{T}\mathcal{T}_u \bar{U})$ and $\phi_{\text{loc}}: C(\mathcal{T}_u \bar{U}) \rightarrow C(\mathcal{T}\mathcal{T}_u \bar{U})$. Recall also that $\phi_{\text{Ger}}$ was built as a resolution of $\mathcal{T}_u \bar{U}$. Now, instead of using Theorem 5.1 we will construct directly the composition of $\phi_{\text{Ger}}$ with the inverse of $\phi_{\text{loc}}$. More precisely, we will construct $\bar{\phi}_{\text{Ger}}$, a $L_\infty$-quasi-isomorphism deforming $\phi_{\text{Ger}}$ so that the image of $\bar{\phi}_{\text{Ger}}$ is contained in the space of cocycle of $C(\mathcal{T}\mathcal{T}_u \bar{U})$, and satisfy conditions of the theorem. This will then give the result. Let us now recall a useful lemma that can be found in [Do]:

Lemma 7.2. Let $\phi$ be a $L_\infty$-quasi-isomorphism between two DGLAs $(\mathfrak{g}_1, d_1)$ and $(\mathfrak{g}_2, d_2)$. Let $\phi^n: S^n(\mathfrak{g}_1[1]) \rightarrow \mathfrak{g}_2$ be the structure maps of $\phi$. Let $m \geq 1$. Then it is possible to construct a deformed $L_\infty$-quasi-isomorphism $\bar{\phi}$ satisfying

- $\bar{\phi}(\gamma_1, \ldots, \gamma_n) = \phi(\gamma_1, \ldots, \gamma_n)$, for $n < m$.
- $\bar{\phi}(\gamma_1, \ldots, \gamma_m) = \phi(\gamma_1, \ldots, \gamma_m) + d_2 V(\gamma_1, \ldots, \gamma_m) - \sum_{1 \leq i, j \leq m} (-1)^{i+j+\sum_{k=1}^j} V(\gamma_1, \ldots, d_1 \gamma_i, \ldots, \gamma_j, \ldots, \gamma_m)$, where $V: S^m(\mathfrak{g}_1[1]) \rightarrow \mathfrak{g}_2$ is an arbitrary polylinear map.

Moreover one has explicit computation of $\bar{\phi}$ from $\phi$ and $V$: let $D_1 = d_1 + d_1^{1,1}$ and $D_2 = d_2 + d_2^{1,2}$ be the structure maps of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ and $\Delta_1, \Delta_2$ the associated free comultiplications (see Section 1). Then, for $x \in C(\mathfrak{g}_1[1])$, $\bar{\phi}(x) = \phi(x) + D_2 V(x) + V(D_1 x)$, where $V$ is extended as follows:

$$\Delta_2 D_2 V(x) = \left( \phi \otimes V + V \otimes \phi + \frac{1}{2} (V \otimes D_2 V + D_2 V \otimes V) + \frac{1}{2} (V \otimes V D_1 + V D_1 \otimes V) \right) \Delta_1(x).$$

Let us denote by $\partial$ the differential in $C(\mathcal{T}\mathcal{T}_u \bar{U})$ and $\phi$ will be the map $\phi_{\text{Ger}}$. We know that the complex $(C(\mathcal{T}\mathcal{T}_u \bar{U}), \partial)$ is acyclic except for elements of $\mathcal{T}_u \bar{U}$. We will write $(x)_0$ for the component in $\mathcal{T}_u \bar{U}$ of an element $x \in C(\mathcal{T}\mathcal{T}_u \bar{U})$. So, as $\phi \phi^1(x) = 0$, there exists a linear map $V: C(\tilde{\mathfrak{g}}) \rightarrow C(\mathcal{T}\mathcal{T}_u \bar{U})$ such that for every element $v \in C(\tilde{\mathfrak{g}})$, $\phi^1(x) = (\phi^1(x))_0 + \partial V(x)$. Note that for degree reasons, when $x$ is a vector field, $(V(x))_0$
is a function and so can be chosen to be zero. Moreover, the map $\phi$ is equivariant with respect to change of coordinates (see [Ha2], Section 5.2). So we can assume that $V$ is also equivariant. So one can define $\tilde{\phi}$ as in the lemma and $\tilde{\phi}$ satisfies the first condition of the theorem. Moreover $(\tilde{\phi})_0$ clearly satisfy the second condition and the third is again a consequence of equivariance with respect to change of coordinates. Let us replace $\phi$ with $\tilde{\phi}$. We will now proceed by induction and suppose that the first condition of the theorem is true the structure maps of $\phi$, that the second and third conditions are true for their $(\alpha)_0$ parts, and are also true for the structure maps of $\phi^i = (\tilde{\phi})_0$ for $i \leq n$. Using the induction hypothesis and the fact that $\phi$ is a $L_\infty$-morphism, we get that $(\partial \phi^{n+1})_0 = 0$. So there exists $W$ such that $\phi^{n+1} = (\tilde{\phi}^{n+1})_0 + \partial W$. Again $W$ can be chosen equivariant. $(W)_0$ can be chosen to satisfy the last condition (the second is automatic for degree reason) as $(\phi^{n+1})_0$ satisfies it. Then again thanks to equivariance, one checks that $\tilde{\phi}$ obtained from $\phi$ and $W$ satisfies the hypothesis of the induction. This concludes the proof. \hfill \Box

7.2 Proof of Theorem 0.3

Let us summarize what we have done so far:

**Theorem 7.3.** Let $X$ be a principal $G$-bundle over a manifold $M$. Let $r \wedge^2 \mathfrak{h}$ such that $[r, r] = Z \in (\wedge^2 \mathfrak{g})^\mathfrak{h}$. There exists $\Phi = 1 + \frac{\mathcal{O}}{2} Z + O(h^3) \in (U(\mathfrak{g})^\otimes 3)^\mathfrak{h}$, $J = 1 + hr + O(h^2) \in U(\mathfrak{g})^\otimes 2 [i\hbar]$ such that $J_1, J_2, J_3 = J_2 J_1, J_3, J_1, J_2 J_3 \Phi$, a deformation $\mathfrak{g}_\Phi$ of the Lie algebra $\mathfrak{g}$ and $\Phi$ a $L_\infty$-quasi-isomorphism

$$\varphi : (T^{\text{inv}}_{\text{poly}}(M), [-, -], [r, -]) \to (D^{\text{inv}}_{\text{poly}}(M), [-, -], [J, -])\Phi,$$

where $T^{\text{inv}}_{\text{poly}}(M)$ and $D^{\text{inv}}_{\text{poly}}(M)$ are respectively the spaces of invariant polyvectors fields on $M$ or polydifferential operators on $M$ and $[-, -]_\Phi$ is the deformed Gerstenhaber bracket in the tensor category of $(U(\mathfrak{g}_\Phi)[[i\hbar]], \Phi)$-modules.

Suppose now that $(X, \pi, r, Z)$ is a quasi-$(r, Z)$-Poisson manifold. Set $\pi = \pi' + r$. Then $\pi$ is a Maurer-Cartan in the DGLA $(T^{\text{inv}}_{\text{poly}}(M), [-, -], [r, -])$. Thanks to Theorem 7.3, we know that those Maurer-Cartan elements, up to Gauge transform are in one to one correspondence with Maurer-Cartan elements of the DGLA $(D^{\text{inv}}_{\text{poly}}(M), [-, -], [J, -])\Phi$, up to Gauge transform. If $m'$ is such a Maurer-Cartan element, set $m = m' + J$, $m$ is a quantization of the quasi-$(r, Z)$-Poisson manifold $(X, \pi, r, Z)$. We have prove:

**Theorem 7.4.** Let $(X, \pi, r, Z)$ be a quasi-$(r, Z)$-Poisson manifold, quantization of $X$, up to equivalence, are in one to one correspondence with

$$\{ \pi_\hbar = h\pi + O(h^2) \text{ such that } [r, \pi_\hbar] + \frac{1}{2} [\pi_\hbar, \pi_\hbar] = 0 \}.$$

8 Quantization of modified dynamical Yang-Baxter $r$-matrices

We know that modified classical dynamical Yang-Baxter $r$-matrices provide examples of quasi-Poisson manifolds. Let us recall their definition: let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$. Let $\rho$ be a $\mathfrak{h}$-equivariant map $\rho : \mathfrak{h}^* \to \Lambda^2(\mathfrak{g})$, solution of the modified classical dynamical Yang-Baxter equation:

$$- \text{Alt}(d\rho) + \text{CYB}(\rho) = Z,$$
where
\[ \text{CYB}(\rho) = [\rho^{1,2}, \rho^{1,3}] + [\rho^{1,2}, \rho^{2,3}] + [\rho^{1,3}, \rho^{2,3}] \]
and
\[ \text{Alt}(d\rho) = \sum_i h_i^1 \frac{\partial \rho^{2,3}}{\partial \lambda_i} - \sum_i h_i^2 \frac{\partial \rho^{1,3}}{\partial \lambda_i} + \sum_i h_i^3 \frac{\partial \rho^{1,2}}{\partial \lambda_i}. \]

Using a quasi-Poisson generalization of a construction of Xu [Xu], Enriquez and Etingof [EE1] built a quasi-Poisson manifold \( X_{\rho} \) associated to \( \rho \) (for which the action of the corresponding group \( G \) is free). They then prove (following [Xu]) that any twist quantization \( J \) associated to an associator \( \Phi \) (i.e.

- \( J \in \text{Mer}(\mathfrak{g}^*, U(\mathfrak{g}) \otimes \mathfrak{g}^*) \), \( \mathfrak{g} \)-invariant, such that \( J(\lambda) = 1 + O(\hbar) \),
- \( J^{12,3}(\lambda) \ast J^{12}(\lambda + \hbar^3) = \Phi^{-1} J^{123}(\lambda) \ast J^{123}(\lambda) \),
- \( Z = \text{Alt} \left( \frac{\Phi - 1}{\hbar} \right) \mod \hbar \),
- \( \rho(\lambda) = \left( \frac{J(\lambda) - 1}{\hbar} \right) - \left( \frac{J(\lambda) - 1}{\hbar} \right)^{2,1} \mod \hbar \)

- gives rise to a quantization of the quasi-Poisson manifold \( X_{\rho} \). Our result provides us with a quantization of the manifolds \( X_{\rho} \) when \( Z \) satisfies our conditions but unfortunately, we don’t know whether this quantization provides us with a twist quantization of the modified dynamical \( r \)-matrix \( \rho \).

Let us write, according to [Xu], the Poisson bracket associated \( \pi_{\rho} \) to a dynamical \( r \)-matrix \( \rho \): in the decomposition of \( T^\text{inv}_{\text{poly}} = \wedge T \mathfrak{h}^* \otimes \mathfrak{g} \),

\[ \pi_{\rho} = \pi_{\mathfrak{h}^*} + \sum_i \frac{\partial}{\partial \lambda_i} \wedge h_i + \rho \in \wedge^2 T \mathfrak{h}^* \oplus T \mathfrak{h}^* \oplus \mathfrak{g} \oplus \wedge^2 \mathfrak{g}, \]

where \( h_i \) and \( \lambda_i \) are basis and dual basis of \( \mathfrak{h} \) and \( \pi_{\mathfrak{h}^*} \) is the Kostant-Kirilov-Souriau Poisson bracket (which we will denote \( \ast_{\mathfrak{h}^*} \)). Now, still following [Xu], a quantization \( \ast \) of \( X_{\rho} \) corresponds to a quantization of \( \rho \) is an only if: it satisfies the following conditions:

1. for any \( f, g \in C^\infty(\mathfrak{h}^*) \), \( f(\lambda) \ast g(\lambda) = f \ast_{\mathfrak{h}^*} g \).
2. for any \( f \in C^\infty(G) \) and \( g \in C^\infty(\mathfrak{h}^*) \), \( f(x) \ast g(\lambda) = f(x)g(\lambda) \).
3. for any \( f \in C^\infty(\mathfrak{h}^*) \) and \( g \in C^\infty(G) \), \( f(\lambda) \ast g(x) = \sum_{k \geq 0} \frac{\hbar}{k!} \frac{\partial^k f}{\partial \lambda_1 \cdots \partial \lambda_k} h_1 \cdots h_k g \).
4. for any \( f, g \in C^\infty(G) \), \( f(x) \ast g(x) = R(f, g) \), where \( R \) would be the quantization of \( \rho \).

Let us notice that our quantization of \( X_{\rho} \) will not satisfy those conditions as, for symmetry conditions, conditions 2 and 3 will not be fulfilled. So we will use a trick proved by Alekseev and Calaque ([AC]). Let us first recall their definition of strongly \( \mathfrak{g} \)-invariant quantization of quasi-Poisson manifold.

**Definition 8.1.** Let \( \ast \) be a quantization of a quasi-Poisson manifold \((X, \pi, Z)\). Suppose \( \mu : X \rightarrow \mathfrak{h}^* \) is a momentum map for which the map \( M = U(\mu^*) \circ \text{sym} : (\mathfrak{g}^*, [\hbar], \ast_{\text{PBW}}) \rightarrow (\mathfrak{o}_X([\hbar]), \ast) \) is an algebra morphism satisfying \([M(x), f] = \hbar \{\mu^* x, f\}\) for any \( f \in \mathfrak{o}_X \) and \( \mu(x) = 0 \), then we say that the quantization \( \ast \) is strongly \( \mathfrak{g} \)-invariant.

**Proposition 8.2.** [AC] Assume that \( \ast \) is a strongly \( \mathfrak{g} \)-invariant quantization of \( X_{\rho} \). Then there exists a gauge equivalent quantization \( \ast' \) of \( X_{\rho} \) such that it corresponds to a quantization of \( \rho \).
Thus we only need to prove that we can construct a strongly $g$-invariant quantization of $X_\rho$.

**Theorem 8.3.** Suppose that $\mathfrak{h}$ is an abelian subalgebra of $g$ and that there exists a decomposition $g = \mathfrak{h} \oplus m$ with $[\mathfrak{h}, m] \subseteq m$. Then the modified classical dynamical Yang-Baxter $r$-matrice $(r - Z)$ can be quantized.

**Proof.** Using Proposition 8.2, we need to prove that there exists a a strongly $g$-invariant quantization of the quasi-Poisson manifold $X_\rho$. To do so, we can copy the proof of Proposition 7.1 to get adapted product.

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