We discuss various aspects of the vortex state of a dilute superfluid atomic Fermi gas at \( T = 0 \). The energy of the vortex in a trapped gas is calculated and we provide an expression for the thermodynamic critical rotation frequency of the trap for its formation. Furthermore, we propose a method to detect the presence of a vortex by calculating the effect of its associated velocity field on the collective mode spectrum of the gas.

In Sec. \( \text{II} \), we calculate the energy of a vortex in a uniform gas, based on an approximate zero-temperature Ginzburg–Landau approach \( \text{[1]} \). We then calculate what this alternative method gives for the energy of the vortex, and compare the two results to show that they do not differ in any significant way.

\section{I. BASIC CONSIDERATIONS}

In the dilute ultracold limit the effective interaction between identical fermionic atoms vanishes due to the Pauli principle, and that between different ones can be well described by one parameter only, the \( s \)-wave scattering length \( a \). For a negative scattering length, the interaction is attractive and if the number of particles in the two internal states is the same the \( T = 0 \) ground state of the gas is a superfluid. The critical temperature \( T_c \) for the transition to such a superfluid state in a dilute gas was first determined for a uniform system by Gorkov and Melik-Barkhudarov \( \text{[10]} \), and using a more modern approach by Heiselberg \textit{et al.} \( \text{[11]} \). The predicted value is

\begin{equation}
\frac{\hbar k_B T_c}{\hbar} = \frac{\gamma}{\pi} \left( \frac{2}{\lambda} \right)^{7/3} \frac{\epsilon_F}{\epsilon_F} e^{-1/\lambda},
\end{equation}

where \( \lambda \) stands for \( 2kF|a|/\pi, \epsilon_F \) is the Fermi energy common to the two species of fermions, \( k_F \) the associated Fermi wavenumber, and \( \gamma \approx 1.781 \) is related to Euler’s constant \( C \) by \( \gamma = e^C \). The pairing gap \( \Delta \) at \( T = 0 \) is, as usual in BCS theory, related to the critical temperature by \( \Delta_0 = \pi \gamma^{-1} k_F T_c \) \( \text{[12,13]} \).

When applying this result to a gas trapped by a harmonic oscillator potential, as in the cases of experimental interest today, some requirements have to be met. The first one is, just as for the uniform case, that the density is everywhere so low that the gas is dilute, i.e. \( k_F(r) |a| \ll 1 \). We have introduced a local Fermi wavenumber \( k_F(r) \). This corresponds to using the Thomas-Fermi approximation, which is valid if \( \epsilon_F \gg \hbar \omega_T \), where \( \omega_T \) is the

\begin{equation}
\frac{\hbar k_B T_c}{\hbar} = \frac{\gamma}{\pi} \left( \frac{2}{\lambda} \right)^{7/3} \frac{\epsilon_F}{\epsilon_F} e^{-1/\lambda},
\end{equation}

where \( \lambda \) stands for \( 2kF|a|/\pi, \epsilon_F \) is the Fermi energy common to the two species of fermions, \( k_F \) the associated Fermi wavenumber, and \( \gamma \approx 1.781 \) is related to Euler’s constant \( C \) by \( \gamma = e^C \). The pairing gap \( \Delta \) at \( T = 0 \) is, as usual in BCS theory, related to the critical temperature by \( \Delta_0 = \pi \gamma^{-1} k_F T_c \) \( \text{[12,13]} \).

When applying this result to a gas trapped by a harmonic oscillator potential, as in the cases of experimental interest today, some requirements have to be met. The first one is, just as for the uniform case, that the density is everywhere so low that the gas is dilute, i.e. \( k_F(r) |a| \ll 1 \). We have introduced a local Fermi wavenumber \( k_F(r) \). This corresponds to using the Thomas-Fermi approximation, which is valid if \( \epsilon_F \gg \hbar \omega_T \), where \( \omega_T \) is the
frequency of the oscillator (which for the time being we assume to be isotropic). This condition is always satisfied if the particle number is sufficiently large, since for a harmonic potential \( \epsilon_F = (6N_\sigma)^{1/3}\hbar\omega_T \), with \( N_\sigma \) being the number of particles of one species.

Another condition for applicability of Eq. (1) is that \( k_F T_c \gg \hbar\omega_T \). When this latter condition is not satisfied, the shell structure of the harmonic oscillator is crucial when determining the superfluid properties of the gas, and Eq. (1) in general breaks down.

In a superfluid Fermi gas at zero temperature the size of the vortex core of a singly quantized vortex is given approximately by the BCS coherence length \( \xi_{BCS} = \hbar v_F/\pi\Delta_0 \), where \( v_F = \hbar k_F/m_\sigma \) is the Fermi velocity and \( m_\sigma \) the mass of a single atom. It is clear that in order for a vortex to appear at all, the BCS coherence length (size of the vortex core) at the center of the cloud has to be smaller than the size of the cloud itself, which in the Thomas-Fermi approximation is given by \( R_{TF} = (2\pi^2\hbar^2\sigma^2\omega_T^2)^{1/2} \). If this were not so, the superfluid properties of the system would be more like those of a nucleus (for which \( \xi \gg R \)) than those of a bulk superfluid.

Substituting the appropriate expressions one can immediately see that \( \xi_{BCS}/R_{TF} = \pi^{-1}\hbar\omega_T/\Delta_0 \). So that requiring \( \xi_{BCS} \ll R_{TF} \) corresponds to demanding that \( \Delta_0 \gg \pi^{-1}\hbar\omega_T \). This condition is automatically satisfied if \( k_F T_c \gg \hbar\omega_T \), but is not at all obviously realized in possible practical circumstances. Indeed, if we assume the validity of equation (1), and of the related value of \( \Delta_0 \), and use the expression \( \epsilon_F = (6N_\sigma)^{1/3}\hbar\omega_T \) for the Fermi energy, we obtain

\[
N_\sigma \gg \frac{(e/2)^7}{6\pi^3} e^{3/\lambda}
\]  

(2)

We may then immediately see that unless \( \lambda \), and therefore \( k_F a \), is sufficiently close to one, the exponential is very large and the condition in Eq. (1) is not satisfied, implying that the coherence length is much larger than the radius of the cloud and the rotation pattern very different from a vortex state. If however \( k_F a \) is too close to one (\( \sim 0.3 - 0.4 \) or more) the formula becomes unreliable because the gas is no more dilute and effects due to induced interactions, which strongly modify the value of \( \Delta_0 \) obtained in the dilute limit, must be taken into account. For the sake of the present work we shall not consider these effects; we study regions of densities in which Eq. (1) is reasonably reliable keeping \( k_F a \lesssim 0.4 \). There is then a region of applicability of Eq. (1) for a trapped gas which depends on the number of particles \( N_\sigma \) and the scattering length. In order to find this region we impose the equality in Eq. (2), and we plot in figure [1] the critical number of atoms \( N_{TF} \) for which \( \xi_{BCS}/R_{TF} = 1 \), as a function of \( k_F a \). Well above the curve we are in the regime where the local density approximation can be applied and a vortex may form, and below it the superfluid has a character more related to that of a nucleus. Since the value of \( k_F a \) can be simply increased by keeping the number of particles fixed and tightening the external trapping potential, we see that if \( N_\sigma \) is sufficiently large (\( \gtrsim 10^5 \)) these systems have the interesting possibility of going from one regime to the other.

In the remainder of this work we assume that we are in the upper region of fig. [1] and therefore that \( \xi_{BCS} \ll R_{TF} \). In this region a vortex forms in the cloud if it is stirred at an angular velocity greater than a critical one \( \omega_c \), which we shall calculate using a thermodynamic approach.

### II. VORTEX IN A UNIFORM GAS

Let us for the time being suppose that the vortex we want to describe is in a uniform gas. In particular we may take the system to be in a cylinder of radius \( R_c \gg \xi_{BCS} \).

Associated with the vortex there is a superfluid velocity flow which decreases with the distance from the vortex axis: \( v_F(\rho) = v_F(0) \rho/2m_\sigma \rho \), where \( \kappa \) is the number of quanta of circulation of the vortex. In a simple model this velocity field extends from \( \rho \sim \kappa \xi_{BCS} \) to \( \rho = R_c \). At distances shorter than \( \sim \kappa \xi_{BCS} \), the kinetic energy associated with the rotation becomes high enough to break the Cooper-pairs, and thus the fluid inside a cylinder of radius \( \sim \kappa \xi_{BCS} \) about the vortex axis can be thought of as being in a normal (non-superfluid) state. The energy per unit length associated with a vortex is then given by the sum of two contributions. One is the kinetic energy due to the flow:

\[
\mathcal{E}_{kin} = \int_0^{R_c} 2\pi\rho \ d\rho \ m_\sigma n_\sigma \left[ \frac{\kappa \hbar}{2m_\sigma} \right]^2
\]

\[
= \frac{\pi \kappa^2 \hbar^2 n_\sigma}{2m_\sigma} \ln \frac{R_c}{\kappa \xi_{BCS}}
\]  

(3)

and the other one is the loss in condensation energy about the vortex axis

\[
\mathcal{E}_{cond} \sim \pi \kappa^2 \xi_{BCS}^2 \epsilon_{cond}
\]

\[
\frac{\pi \kappa^2 \hbar^2 n_\sigma}{2m_\sigma} \frac{3}{\pi^2}
\]

(4)

where \( \epsilon_{cond} = 3\Delta_0^2 n_\sigma/4\epsilon_F \) is the condensation energy per unit volume due to the pairing \( \bar{\nu}^2 \), and the usual expression for \( \xi_{BCS} \) has been employed. Notice that we have here introduced the one-species particle density \( n_\sigma \), and we have supposed that this is a constant throughout the system, since contrary to the boson case it is not the particle density but only the pairing field that changes close to the vortex axis.

The total energy per unit length of a vortex is therefore given in the simple model by

\[
\mathcal{E}_v = \mathcal{E}_{kin} + \mathcal{E}_{cond} \approx \frac{\pi \kappa^2 \hbar^2 n_\sigma}{2m_\sigma} \ln \left( \frac{1.36 R_c}{\kappa \xi_{BCS}} \right)
\]

(5)
The important feature of this result is to point out that for large systems (i.e. for which \( R_c > \kappa \xi_{BCS} \)) the most relevant contribution is the logarithmic one arising from the kinetic integration. The value of the constant inside the logarithm will depend on the choice of the model used to describe the vortex. A more reliable value would be obtained from a numerical solution of the Bogoliubov-de Gennes equations, although it is unlikely that it will differ significantly from the one found here, since one expects it in any case to be of order one. As an example of what a different approach may yield, in App. A we show the result for the total energy of the vortex obtained using a zero-temperature Ginzburg-Landau model. As we shall see one obtains, as foreseen, the same expression as in Eq. (6), with coefficient 1.65 instead of 1.36 inside the logarithm.

For what follows we shall not need to know the precise value of this coefficient, which may be better determined in the future, and we shall therefore leave it unspecified and state our result as

\[
E_v \simeq \frac{\pi \kappa^2 \hbar^2 n_{\sigma}^2}{2m} \ln \left( \frac{D R_c}{\kappa \xi_{BCS}} \right)
\]

with the understanding that \( D \) is some constant of order one. From Eq. (6) it is already clear that one vortex with \( \kappa = \kappa \neq 1 \) has greater energy than \( \kappa \) vortices with \( \kappa = 1 \) since in any case we need to have \( \kappa \xi_{BCS} < R_z \). This implies that vortices with \( \kappa \neq 1 \) are unstable [17]. Therefore for the considerations that follow we shall set \( \kappa = 1 \).

With this solution, recalling that the thermodynamic critical velocity for formation of a first vortex is given by \( \omega_{c1} = E_v / L_v \) [9], and using the fact that the total angular momentum per unit length of the system with a vortex is \( L_v \simeq \hbar \pi R_z^2 n_{\sigma} \), corresponding to \( \hbar \) per Cooper pair, we can immediately state what the critical velocity is in a uniform system, which is of course a well known result

\[
\omega_{c1} = \frac{\hbar}{2m a R_c^2} \ln \left( \frac{D R_c}{\xi_{BCS}} \right).
\]

The result should be compared with the critical velocity found for a Bose-Einstein condensate. This is completely analogous if the mass of a single bosonic atom is replaced with that of a Cooper pair (\( 2m_a \)), and the boson coherence length by the BCS one.

Notice that since \( \xi_{BCS} \propto \Delta_0^{-1} \), from the measurement of \( \omega_{c1} \) one could in principle deduce the value of \( \Delta_0 \) if \( D \) is known. This possibility is usually lost however in a non-uniform system since several values of \( \Delta_0 \) are integrated over.

### III. VORTEX FORMATION IN TRAPPED GASES

In this section we calculate the energy of the vortex in a trapped gas, specializing to the trapping configurations used in a typical experiment.

The atoms are generally confined in a cylindrically symmetric harmonic potential of the form

\[
V_{ext}(r) = \frac{1}{2} m \omega_z^2 \left[ z^2 + \lambda_T^2 (x^2 + y^2) \right]
\]

and the density profile of the gas is, within the Thomas-Fermi approximation, given by

\[
n_{\sigma}(\rho, z) = n_{\sigma,0} \left( \frac{1 - \frac{\lambda_T^2 \rho^2 + z^2}{R_z^2}}{\left( \frac{h}{2m \rho} \right)^2} \right)^{3/2}.
\]

Here \( n_{\sigma,0} = n_{\sigma}(0,0) \) is the density at the center of the cloud and we have taken the profiles of the two species to be identical. The anisotropy of the trap is controlled by the coefficient \( \lambda_T \). The energy of a cloud with a vortex along the \( z \) axis can be calculated with the procedure devised by Lundh et al. [18]. One can divide the cloud in vertical slices of height \( dz \) and use the result [6] for a cylinder of radius \( \rho_1 \) such that \( \xi_{BCS} < \rho_1 < R_\perp = R_z / \lambda_T \), within which one can assume that the gas is approximately uniform. The energy per unit length associated with the vortex in a slice at \( z \) is then given by

\[
E_v(z) = \frac{\pi \hbar^2 n_{\sigma,0}(0, z)}{2m} \ln \left( \frac{D \rho_1}{\xi_{BCS}(z)} \right)
+ \int_{\rho_1}^{R_\perp(z)} 2\pi \rho \, d\rho \, \frac{m_a n_{\sigma}(\rho, z)}{2m_\rho} \left( \frac{\hbar}{2m_\rho} \right)^2,
\]

where \( R_\perp(z) = (1 - z^2/R_z^2)^{1/2} R_z / \lambda_T \) is the value of \( \rho \) up to which the cloud extends for a given \( z \), and \( n_{\sigma}(z,0) \) is the density on the \( z \)-axis at height \( z \). The second term in Eq. (10) gives the kinetic energy of the superfluid outside the cylinder of radius \( \rho_1 \).

With \( n_{\sigma}(\rho, z) \) given by [9] we then get

\[
E_v(z) = \frac{\pi \hbar^2 n_{\sigma,0}(0, z)}{2m} \ln \left( \frac{D \rho_1}{\xi_{BCS}(z)} \right)
+ \int_{\rho_1}^{R_\perp(z)} \left( 1 - \frac{\lambda_T^2 \rho^2 + z^2}{R_z^2} \right)^{3/2} \frac{d\rho}{\rho}.
\]

This result differs from the boson case in ref. [18] in the power 3/2 instead of 1 in the density distribution. Using the fact that

\[
\int (1 - x^2)^{3/2} \frac{dx}{x} = \sqrt{1 - x^2} + \ln \left( \frac{x}{1 + \sqrt{1 - x^2}} \right) + \frac{1}{3} (1 - x^2)^{3/2}
\]

and that unless \( z \) is very close to \( R_z \) one can assume \( \rho_1 \ll R_\perp(z) \), we finally obtain

\[
E_v(z) = \frac{\pi \hbar^2 n_{\sigma,0}(0, z)}{2m} \left( 1 - \frac{z^2}{R_z^2} \right)^{3/2} \ln \left( \frac{2}{e^{4/3}} \frac{D R_\perp(z)}{\xi_{BCS}(z)} \right).
\]

(13)
In order to proceed with the $z$ integration we need to know the explicit dependence of $\xi_{BCS}$ on $z$. In the dilute gas approximation where Eq. (11) is valid this is given by

$$
\xi_{BCS}(z) = \frac{2}{\pi} \left( \frac{\epsilon}{2} \right)^{7/3} \left( 1 - \frac{z^2}{R_z^2} \right)^{-1/2} \times \exp \left[ \frac{1}{\lambda_0} \left( 1 - \frac{z^2}{R_z^2} \right)^{-1/2} \right] k_{F,0}^2,
$$

(14)

where $k_{F,0} = (2m_\alpha \epsilon_F / \hbar^2)^{1/2}$ and $\lambda_0 = 2k_{F,0}|a|/\pi$ are the local Fermi wave-number and $\lambda$ respectively, evaluated at the center of the cloud. Inserting this value into Eq. (13), using the expression for $R_\perp(z)$ and integrating over $z$ we get after some cumbersome but straightforward calculations

$$
E_v = \frac{\pi \hbar^2 n_{\alpha,0} a^4}{2m_\alpha} 4 R_z \left[ \frac{9\pi}{32} \ln \left( \frac{2^{4/3} \pi D}{e^{5/2} \hbar \omega_{\perp}} \right) - \frac{1}{\lambda_0} \right] \tag{15}
$$

Note that Eq. (15) predicts the energy cost of the vortex to be negative for small $k_{F,0}|a|$ and $\epsilon_F / \hbar \omega_{\perp}$ not too large. This is clearly an unphysical result reflecting the fact that in the limit of relatively few particles trapped and small $k_{F,0}|a|$, the condition $\xi_{BCS} \ll R_\perp$ is violated making Eq. (15) invalid. In the regime $\xi_{BCS} \ll R_\perp$, Eq. (15) yields positive vortex energies as expected. If we ignore the non-rotating particles at the core of the vortex, the total angular momentum of the vortex state is $L_v = N_\alpha \hbar$ and the critical rotation frequency $\omega_{c1} = E_v / L_v$ for the formation of a vortex in a trap given by Eq. (8) is

$$
\omega_{c1} = \omega_\perp \sqrt{\frac{16 \, \bar{l}_\perp^2}{3\pi R_z^2} \left[ \frac{9\pi}{32} \ln \left( \frac{2^{4/3} \pi D}{e^{5/2} \hbar \omega_{\perp}} \right) - \frac{1}{\lambda_0} \right]}
$$

(16)

with $l_\perp = \sqrt{\hbar / m_\alpha \omega_{\perp}}$ being the harmonic oscillator length in the radial direction. For realistic parameters of the gas, this critical frequency is small enough: choosing for $D$ the value obtained in App. A, taking $k_{F,0}|a| = 0.4$, $\omega_\perp = \omega_z = \omega_T$ (i.e. $\lambda_T = 1$) and $\epsilon_F = 200 \hbar \omega_T$ corresponding to an isotropic trap with $N_\alpha \sim 1.3 \times 10^9$, we obtain $\omega_{c1} \approx 0.0035 \omega_\perp$. The reason for the critical frequency being so small is that the angular momentum per atom is $\hbar / 2$ yielding $L_v = N_\alpha \hbar$ whereas the energy given by Eq. (15) only scales as $N_\alpha^{2/3}$. The Fermi pressure expands the cloud and reduces the density at the center of the trap. Since the energy of the vortex mainly comes from regions close to the vortex axis where the superfluid velocity is high, the energy of the vortex is correspondingly reduced.

**IV. OBSERVATION OF THE VORTEX**

Contrary to the situation for Bose-Einstein condensates, the presence of a vortex in the Fermi gas does not alter the density profile significantly [12]. One cannot therefore observe the vortex simply by looking at the density profile. It has been suggested to use the laser probing method of Ref. [20] to detect the local decrease of the pairing near the center of the vortex [8]. Here we examine a different method based on measuring the collective mode spectrum of the gas. In the case of no vortex present, excitations of the gas carrying equal and opposite angular momentum along the $z$-axis are degenerate in energy. The velocity field associated with a vortex aligned with this axis lifts the degeneracy since the rotational symmetry is removed; the velocity flow of the excitation is either parallel or anti-parallel to that of the vortex giving rise to an energy splitting of the modes [21, 22]. Since the collective mode frequencies of the gas can be measured with a fairly high precision, the possibility of detecting the presence of the vortex by its spectroscopic signatures is a promising method. Indeed, this method has proven to be very useful in the case of a vortex in a BEC [24]. The calculations will be carried out for an isotropic trap with $R_\perp = R_z = R_T$ and $\omega = \omega_z = \omega_T$ in the $\xi_{BCS} \ll R_T$ limit considered in this paper, the collective modes of the superfluid gas for $T = 0$ can be calculated using a hydrodynamic theory. The relevant continuity and superfluid velocity equations read [23]:

$$
\partial_t n(r, t) = -\nabla j_v(r, t) = -\frac{1}{m_\alpha} \nabla [m_\alpha \bar{v}_s^2 / 2 + \mu_F + V_{ext}]
$$

(17)

with $n_s(r, t), n_n(r, t),$ and $n(r, t) = n_s(r, t) + n_n(r, t)$ being the superfluid-, normal- and total density of the gas respectively. The total current is $j_v(r, t) = n_s(r, t) \bar{v}_v(r, t) + n_n(r, t) \bar{v}_n(r, t)$, $\bar{v}_v(r, t)$ is the superfluid velocity and $\bar{v}_n(r, t)$ the velocity of the normal fluid. For $\xi_{BCS} \ll R_T$, the extent of the vortex core is small compared to the size of the cloud, and the main effect of the vortex on the collective mode spectrum is the presence of the vortex velocity field $\bar{v}_v(r) = e_\phi \kappa \hbar / 2m_\alpha \rho$. In these considerations we keep the winding number $\kappa$ explicitly different from one for generality. Writing $n(r, t) = n_0(r) + \delta n(r, t)$ and $\bar{v}_v(r, t) = \bar{v}_v(r) + u(r, t)$, where $n_0(r)$ is the equilibrium density profile with the vortex alone (which we take to be coincident with the Thomas-Fermi one without vortex), and linearizing in $\delta n(r, t)$ and $u(r, t)$, Eq. (17) can be written as

$$
\left( \omega - \frac{\kappa \hbar m}{2m_\alpha \rho^2} \right)^2 m_\alpha n_0(r) \frac{\partial n_0(r)}{\partial P_0(r)} \Phi(r, t) = -\nabla \cdot [n_0(r) \nabla \Phi(r, t)].
$$

(18)

Here, $\omega$ is the frequency of the collective mode, $P_0(r)$ is the equilibrium pressure profile, and $m$ is the magnetic quantum number of the mode. The velocity field associated with the mode has been written as $u(r, t) = \nabla \Phi(r, t)$ with $\Phi(r, t) = \Phi(r, \theta) \exp[\i (n \phi - \omega t)]$. The term $\kappa \hbar m / 2m_\alpha \rho^2$ in Eq. (13) comes from the presence of the vortex velocity field $\bar{v}_v$. Without this term, Eq. (13) has been solved for a spherical symmetric trap by writing $\Phi_{n \ell m}(r) = \Phi_n(r) Y_{\ell m}(\theta, \phi)$ yielding the spectrum $\omega_{nl0} = 2\omega_T \sqrt{(n^2 + 2n + \ell^2 + 6l/4)/3}$ with $n = 0, 1, 2, \ldots$ [20].
From Eq. [19], we see that the frequency shift of a given mode induced by the vortex can be calculated perturbatively as

$$\omega_{nml}^2 - \omega_{n0}^2 = \frac{\kappa \hbar m \omega_{n0}}{m_a} \frac{\langle \Phi_{nml} | \rho^{-2} | \Phi_{nml} \rangle}{\langle \Phi_{nml} | \Phi_{nml} \rangle}. \quad (19)$$

Here $\langle \Phi_{nml} | f(r) | \Phi_{nml} \rangle$ denotes the spatial average $\int_0^{R_z} w(r) dr \int d^2 \Phi_{nml}(r) f(r)$ with the weight function $w(r) = r^2 (1 - r^2 / R_{TF}^2)^{1/2}$. This anomalous weight has to be introduced in place of simply $r^2$ because the operator in Eq. [18] without the perturbation is not Hermitian.

As pointed out in Ref. [23], the perturbative procedure works for $|m| \geq 2$; for $|m| < 2$ it predicts an unphysical $\rho \to 0$ divergence in the density fluctuation of the mode. With no vortex present, the lowest mode for a given angular momentum is the surface mode $\Phi_{n=0l=0}(r) \propto r^l$ with frequency $\sqrt{\kappa} \omega_l$. Recalling that $\rho = r \sin \theta$ and using the fact that $(4\pi)^{-1} \int d\Omega |Y_{lm}(\Omega)|^2 / \sin^2 \theta = (2l + 1)/2|m|$, the matrix elements in Eq. [19] can be calculated analytically for these surface modes and we obtain for the frequency shift:

$$\frac{\omega_{nml}^2 - \omega_{n0}^2}{\omega_{n0}^2} = \frac{\kappa(l + 2)}{2 \sqrt{6N_a}^{1/3}}. \quad (20)$$

with $|m| \geq 2$. As expected, the vortex splits the $2l + 1$ degenerate modes depending on the direction of the projection of their angular momentum on the $z$-axis. Not all the modes are split however since the splitting is independent of $|m|$ in analogy with the equivalent result for bosons [23]. Particularly important is the result $(\omega_{2,2}^2 - \omega_{2,-2}^2) / \omega_{n0}^2 = \pm \sqrt{2} \kappa / (6N_a)^{1/3}$ for the quadrupolar mode $l = 2, m = \pm 2$, since this mode is easily excited in trapped gases and has been already employed for a precise determination of the critical frequency for vortex nucleation in Bose gases.

The same result can be obtained following the sum rule approach of ref. [23]. From that the splitting is found to be given by $\omega_{2,2} - \omega_{2,-2} = 2 < l_z > / (m_a < \rho^2 >)$, with $< l_z >$ expectation value of the angular momentum along the $z$-axis per atom ($\kappa \hbar / 2$ in the case of a vortex), and $< \rho^2 >$ expectation value of $r^2 + y^2$ (equal to $R_{TF}^2 / 4$ for an isotropic cloud with a Thomas-Fermi density profile). From the latter result one can immediately see that the splitting of the modes of a Fermi superfluid is in general smaller compared with the BEC case, the reason being that given the same number of atoms the radius of a fermionic cloud is usually larger due to the Pauli repulsion and thus the expectation value of $r^2$ is also correspondingly larger, and the splitting reduced. For $2N_\sigma = 10^6$ particles trapped, the $m = \pm 2$ quadrupole modes are split by $\sim 1\%$. Although this is a rather small shift, it should be measurable assuming the same high spectroscopic precision demonstrated for BEC’s can be obtained for trapped Fermi gases [27].

VI. CONCLUSION

In this paper we considered various aspects of the vortex state of a dilute superfluid Fermi gas at $T = 0$. For a trapped system, we found that a large number of particles and a not too small scattering length yields $\xi_{BCS} \ll R_{TF}$ and the vortex is well confined within the gas. We then used a simple model to calculate the energy of a vortex in a uniform medium. Subsequently, using the fact that the structure of the vortex near the rotation axis is essentially unaffected by the trapping potential we derived an expression for the vortex energy in a trap, and we employed this energy expression to calculate the thermodynamic critical rotation frequency for the formation of a vortex. Finally, we suggested a way of observing the presence of the vortex by calculating perturbatively its influence on the collective mode spectrum of the gas. In the Appendix we report an alternative, less naive, description of the vortex in a uniform medium and find a slightly different value for its energy compared the one obtained in Sec. [1].

VI. ACKNOWLEDGMENT

We would like to acknowledge valuable discussions with C. J. Pethick.

FIG. 1. Critical number of atoms per spin species for which $\xi_{BCS} / R_{TF} = 1$ in an isotropic trap. Well above the line the local density approximation, and thus Eq. [1], applies, below the line the system is intrinsically finite sized.

APPENDIX A: A GINZBURG–LANDAU DESCRIPTION OF THE VORTEX CORE

We here present a Ginzburg–Landau description of the vortex core in a uniform gas, and the consequent result...
for the total energy of a vortex in a cylindrical bucket of radius \( R_c \). As is well-known, Ginzburg–Landau theory is only valid for temperatures such that \( |T - T_c|/T_c \ll 1 \) but the following calculation can be used for a qualitative estimate at \( T = 0 \).

The extension of the Ginzburg-Landau theory to zero temperature for a uniform system can be done by imposing to the free energy [2]

\[
F_{GL} = \int d^3 r f_{GL}(r) = \int \left[ \frac{\hbar^2}{4m_a} \nabla^2 \psi^2 + A|\psi|^2 + B|\psi|^4 \right] d^3 r, \tag{A1}
\]

to be equal to the condensation energy density, which in a uniform system is given by \( \epsilon_{\text{cond}} = -3\Delta_0^2 n_a/4\epsilon_F \) [2]. \( \psi \) is here the well known Ginzburg-Landau order parameter. Upon minimization of (A1) with respect to \( \psi \) we obtain the Ginzburg-Landau equation

\[
-\frac{\hbar^2}{4m_a} \nabla^2 \psi + A\psi + B|\psi|^2 \psi = 0. \tag{A2}
\]

For a uniform system the solution is \( |\psi_0|^2 = -A/B \) and from the Ginzburg–Landau free energy then coincides with the condensation one \( f_{GL} = -3\Delta_0^2 n_a/4\epsilon_F \), we obtain \( A = -3\Delta_0^2/2\epsilon_F \) and \( B = -A/n_a \).

We now calculate the structure and the energy of the vortex. A vortex along the \( z \)-axis is described by writing the order parameter in cylindrical coordinates as \( \psi(r) = f(\rho)e^{i\sigma_0} \). Replacing this expression into Eq. (A2) we find

\[
-\frac{1}{x} \frac{d}{dx} \left( x \frac{d\chi}{dx} \right) + \frac{\kappa^2}{x^2} \chi + \chi^3 - \chi = 0, \tag{A3}
\]

where we introduced the dimensionless quantities \( \chi = f/|\psi_0| \) and \( x = \rho/\xi_{GL} \). We used the fact that \( f \) does not vary along the \( z \) direction if the system is uniform, and we defined the Ginzburg–Landau coherence length \( \xi_{GL}^2 = \hbar^2/4m_a A \), which implies \( \xi_{GL} = \hbar v_F/\sqrt{2\Delta_0} = 0.907\xi_{BCS} \). This equation has exactly the same form as the Gross-Pitaevskii equation for a vortex in a uniform boson cloud [17,28]. It can be solved numerically and the results for the lowest \( \kappa \) (\( \kappa = 1, 2, 3 \)) was first obtained by Ginzburg and Pitaevskii [17]. A very good approximate solution for \( \kappa = 1 \) can be obtained by a variational calculation yielding \( \chi = x/(2 + x^2)^{1/2} \) [29]. Using this solution in Eq. (A1), one finds that the energy cost associated with the vortex core is given by

\[
\mathcal{E}_v(z) = \pi \frac{\hbar^2}{2m_a} n_a \ln \left( \frac{e^{3/4} R_c}{\sqrt{2} \xi_{GL}} \right). \tag{A4}
\]

This result is now identical with that for a Bose-Einstein condensate, with the mass of a single bosonic atom replaced by that of a Cooper pair \((2m_a)\), and the boson coherence length replaced by the Ginzburg-Landau one. Using \( \xi_{GL} = 0.907\xi_{BCS} \), we obtain \( \mathcal{E}_v = \pi \hbar^2 n_a \ln(D R_c/\xi_{BCS})/2m_a \) with \( D = 1.65 \).

[1] B. DeMarco, S. B. Papp, and D. S. Jin, cond-mat/0101445; B. DeMarco and D. S. Jin, Science 285, 1703 (1999).
[2] A. G. Truscott et al., Science 291, 2570 (2001).
[3] F. Schreck et al., cond-mat/0011291.
[4] K. M. O’Hara et al., Phys. Rev. Lett. 85, 2092 (2000).
[5] H. T. C. Stoof, M. Houbiers, C. A. Sackett, and R. G. Hulet, Phys. Rev. Lett. 76, 10 (1996).
[6] F. Weig and W. Zwerger, Europhys. Lett. 49, 282 (2000); W. Zhang, C. A. Sackett, and R.G. Hulet, Phys. Rev. A 60, 504 (1999); J. Ruostekoski, Phys. Rev. A 60, R1775 (1999); P. Törma and P. Zoller, Phys. Rev. Lett. 85, 487 (2000).
[7] See e.g. A. L. Fetter and A. A. Svidzinsky, J. Phys. Cond. Mat. 13, R135 (2001).
[8] M. Rodriguez, G.-S. Paraoanu, and P. Törma, cond-mat/0104491.
[9] R. I. Epstein and G. Baym, Ap. J. 328, 680 (1988).
[10] L. P. Gorkov and T. K. Melik-Barkhudarov, Sov. Phys. JETP 13, 1018 (1961).
[11] H. Heiselberg, C. J. Pethick, H. Smith, and L. Viverit, Phys. Rev. Lett. 85, 2418 (2000).
[12] P. G. de Gennes, Superconductivity of Metals and Alloys (Benjamin, New York, 1966).
[13] A. L. Fetter and J. D. Walecka, Quantum Theory of Many-Particle Systems (McGraw-Hill, New York, 1971).
[14] G. Bruun, Y. Castin, R. Dum, and K. Burnett, Eur. Phys. J. D 7, 433 (1999).
[15] R. Combescot, Phys. Rev. Lett. 83, 3766 (1999); H. Heiselberg, Phys. Rev. A 63, 043606 (2001).
[16] P. Nozieres and D. Pines, The Theory of Quantum Liquids (Benjamin, New York, 1966), vol II.
[17] V. L. Ginzburg and L. P. Pitaevskii, Sov. Phys. JETP 34, 858 (1958).
[18] E. Lundh, C. J. Pethick, and H. Smith, Phys. Rev. A 55, 2126 (1999).
[19] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. 71, 463 (1999).
[20] F. Zambelli and S. Stringari, Phys. Rev. Lett. 81, 1754 (1998).
[21] S. Sinha, Phys. Rev. A 55, 4325 (1997); R. J. Dodd, K. Burnett, M. Edwards, and C. W. Clark, Phys. Rev. A 56, 587 (1997).
[22] A. A. Svidzinsky and A. Fetter, Phys. Rev. A 58, 3168 (1998).
[23] F. Chevy, K. W. Madison, and J. Dalibard, Phys. Rev. Lett. 85, 2223 (2000).
[24] L. D. Landau and E. M. Lifshitz, Fluid Mechanics (Perg-
[26] G. M. Bruun and C. W. Clark, Phys. Rev. Lett. 83, 5418 (1999); M. A. Baranov and D. S. Petrov, Phys. Rev. A 62, 041601(R) (2000).

[27] D. S. Jin et al., Phys. Rev. Lett. 77, 420 (1996); D. M. Stamper-Kurn et al., Phys. Rev. Lett. 81, 500 (1998).

[28] C. J. Pethick and H. Smith, Bose-Einstein Condensation in Dilute Gases (to be published by Cambridge University Press).

[29] A. L. Fetter, In Lectures in Theoretical Physics, eds. K. T. Mahanthappa and W. E. Britten (Gordon and Breach, New York, 1969), Vol. XIB p. 351.