Optimal Ternary Cyclic Codes from Monomials

Cunsheng Ding and Tor Helleseth

Abstract
Cyclic codes are a subclass of linear codes and have applications in consumer electronics, data storage systems, and communication systems as they have efficient encoding and decoding algorithms. Perfect nonlinear monomials were employed to construct optimal ternary cyclic codes with parameters $[3^m - 1, 3^m - 1 - 2m, 4]$ by Carlet, Ding and Yuan in 2005. In this paper, almost perfect nonlinear monomials, and a number of other monomials over GF($3^m$) are used to construct optimal ternary cyclic codes with the same parameters. Nine open problems on such codes are also presented.

Index Terms
Almost perfect nonlinear functions, cyclic codes, monomials, perfect nonlinear functions, planar functions.

I. INTRODUCTION
Let $q$ be a power of a prime $p$. A linear $[n,k,d]$ code over GF($p$) is a $k$-dimensional subspace of GF($p$)$^n$ with minimum (Hamming) nonzero weight $d$. A linear $[n,k]$ code $C$ over the finite field GF($p$) is called cyclic if $(c_0,c_1,\ldots,c_{n-1}) \in C$ implies $(c_{n-1},c_0,c_1,\ldots,c_{n-2}) \in C$. Let gcd($n,p$) = 1. By identifying any vector $(c_0,c_1,\ldots,c_{n-1}) \in$ GF($p$)$^n$ with

$$c_0 + c_1x + c_2x^2 + \cdots + c_{n-1}x^{n-1} \in$GF($p$)$[x]/(x^n - 1),$$

any code $C$ of length $n$ over GF($p$) corresponds to a subset of GF($p$)$[x]/(x^n - 1)$. The linear code $C$ is cyclic if and only if the corresponding subset in GF($p$)$[x]/(x^n - 1)$ is an ideal of the ring GF($p$)$[x]/(x^n - 1)$. It is well known that every ideal of GF($p$)$[x]/(x^n - 1)$ is principal. Let $C = (g(x))$ be a cyclic code, where $g(x)$ is monic and has the least degree. Then $g(x)$ is called the generator polynomial and $h(x) = (x^n - 1)/g(x)$ is referred to as the parity-check polynomial of $C$.

The error correcting capability of cyclic codes may not be as good as some other linear codes in general. However, cyclic codes have wide applications in storage and communication systems because they have efficient encoding and decoding algorithms [10, 17, 25]. For example, Reed-Solomon codes have found important applications from deep-space communication to consumer electronics. They are prominently used in consumer electronics such as CDs, DVDs, Blu-ray Discs, in data transmission technologies such as DSL & WiMAX, in broadcast systems such as DVB and ATSC, and in computer applications such as RAID 6 systems.

Perfect nonlinear monomials were employed to construct optimal ternary cyclic codes with parameters $[3^m - 1, 3^m - 1 - 2m, 4]$ by Carlet, Ding and Yuan in [8]. In this paper, almost perfect nonlinear (APN) monomials and a number of classes of monomials over GF($3^m$) will be used to construct many classes of optimal ternary cyclic codes with the same parameters. Nine open problems on such codes are also presented.

II. PRELIMINARIES
In this section, we fix some basic notation for this paper and introduce almost perfect nonlinear and planar functions, and $q$-cyclotomic cosets that will be employed in subsequent sections.

C. Ding’s research was supported by The Hong Kong Research Grants Council, Proj. No. 600812. T. Helleseth’s research was supported by the Norwegian Research Council.

C. Ding is with the Department of Computer Science and Engineering, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong, China. Email: cding@ust.hk

T. Helleseth is with the Department of Informatics, The University of Bergen, N-5020 Bergen, Norway. Email: Tor.Helleseth@ii.uib.no
A. Some notation fixed throughout this paper

Throughout this paper, we adopt the following notation unless otherwise stated:

- \( p \) is an odd prime, and \( p = 3 \) for Section IV and subsequent sections.
- \( q = p^m \), where \( m \) is a positive integer.
- \( n = p^m - 1 \).
- \( \mathbb{Z}_n = \{0, 1, 2, \ldots, n - 1\} \) associated with the integer addition modulo \( n \) and integer multiplication modulo \( n \) operations.
- \( \alpha \) is a generator of \( \text{GF}(q)^* := \text{GF}(q) \setminus \{0\} \).
- \( m_a(x) \) is the minimal polynomial of \( a \in \text{GF}(q) \) over \( \text{GF}(p) \).
- \( \text{Tr}(x) \) is the trace function from \( \text{GF}(q) \) to \( \text{GF}(p) \).
- By the Database we mean the collection of the tables of best linear codes known maintained by Markus Grassl at [http://www.codetables.de/](http://www.codetables.de/).

B. The \( p \)-cyclotomic cosets modulo \( n = p^m - 1 \)

The \( p \)-cyclotomic coset modulo \( n \) containing \( j \) is defined by

\[
C_j = \{j, pj, p^2j, \ldots, p^{\ell_j - 1}j\} \subset \mathbb{Z}_n
\]

where \( \ell_j \) is the smallest positive integer such that \( p^{\ell_j}j \equiv j \pmod{n} \), and is called the size of \( C_j \). It is known that \( \ell_j \) divides \( m \). The smallest integer in \( C_j \) is called the coset leader of \( C_j \). Let \( \Gamma \) denote the set of all coset leaders. By definition, we have

\[
\bigcup_{j \in \Gamma} C_j = \mathbb{Z}_n.
\]

The following lemma is useful in the sequel.

**Lemma 2.1:** Let \( q = p^m \) and \( n = q - 1 \). For any \( 1 \leq e \leq n - 1 \) with \( \gcd(e, q - 1) = 2 \), the length \( \ell_e \) of the \( p \)-cyclotomic coset \( C_e \) is equal to \( m \).

**Proof:** By definition, \( \ell_e \) is the smallest positive integer such that \( e(p^{\ell_e} - 1) \equiv 0 \pmod{p^m - 1} \). Hence \( \ell_e \) is the smallest positive integer such that \( (p^m - 1)|e(p^{\ell_e} - 1) \). Since \( \gcd(e, q - 1) = 2 \), \( \ell_e \) is the smallest positive integer such that \( \frac{p^m - 1}{2} |(p^{\ell_e} - 1) \). It then follows that \( \ell_e = m \). This completes the proof. \( \square \)

C. Perfect and almost perfect nonlinear functions on \( \text{GF}(q) \)

A function \( f : \text{GF}(q) \to \text{GF}(q) \) is called an almost perfect nonlinear (APN) if

\[
\max_{a \in \text{GF}(q)} \max_{b \in \text{GF}(q)} |\{x \in \text{GF}(q) : f(x + a) - f(x) = b\}| = 2,
\]

and is referred to as perfect nonlinear or planar if

\[
\max_{a \in \text{GF}(q)} \max_{b \in \text{GF}(q)} |\{x \in \text{GF}(q) : f(x + a) - f(x) = b\}| = 1.
\]

There is no perfect nonlinear (planar) function on \( \text{GF}(2^m) \). Known APN polynomials over \( \text{GF}(2^m) \) can be found in [1], [2], [3], [4], [5], [24], [18], [19], [15], [16], [20]. However, there are both planar and APN functions over \( \text{GF}(p^m) \) for odd primes \( p \). In the sequel, planar and APN monomials over \( \text{GF}(3^m) \) will be employed to construct optimal ternary cyclic codes. The results of this paper will be a nice demonstration of applications of APN functions in engineering.
III. THE CODES DEFINED BY PAIRS OF MONOMIALS

Let $p$ be a prime and let $q = p^m$, where $m$ is a positive integer. Let $m_{\alpha}(x)$ denote the minimal polynomial over $\text{GF}(p)$ of $\alpha^i$, where $\alpha$ is a generator of $\text{GF}(q)^*$. In this paper, we consider the cyclic code of length $n = q - 1$ over $\text{GF}(p)$ with generator polynomial $m_{\alpha}(x)m_{\alpha^i}(x)$, denoted by $C_{(1,e)}$, where $1 < e < q - 1$ and $e \notin C_1$. The condition that $e \notin C_1$ is to make sure that $m_{\alpha}(x)$ and $m_{\alpha^i}(x)$ are distinct. The dimension of $C_{(1,e)}$ is equal to $n - (m + \ell_e)$, where $\ell_e = |C_e|$ and $C_e$ is the $p$-cyclicotomic coset modulo $n$ containing $e$. The cyclic code $C_{(1,e)}$ is defined by the pair of monomials $x$ and $x^e$ over $\text{GF}(q)$.

When $p = 2$, it was proved in [6], [7], [23] that the binary code $C_{(1,e)}$ has parameters $[2^m - 1, 2^m - 1 - 2m, 5]$ if and only if $x^e$ is an APN monomial over $\text{GF}(2^m)$. When $p$ is odd and $x^e$ is a planar monomial over $\text{GF}(q)$, the codes $C_{(1,e)}$ were dealt with in [8], [26]. When $p > 3$ and $\ell_e = m$, the code $C_{(1,e)}$ has minimum distance 2 or 3 which may not be interesting. In this paper, we study the case that $p = 3$ only and prove that the ternary cyclic code $C_{(1,e)}$ is optimal for many classes of monomials $x^e$ over $\text{GF}(3^m)$.

IV. A BASIC THEOREM ABOUT THE TERNARY CYCLIC CODES $C_{(1,e)}$

From now on we consider only the code $C_{(1,e)}$ for $p = 3$. We are mostly interested in the case that $e \notin C_1$ and $\ell_e = |C_e| = m$. In this case, the dimension of the code is equal to $n - 2m = q - 1 - 2m$.

The following theorem is the fundamental result of this paper and will be used frequently in subsequent sections.

**Theorem 4.1:** Let $e \notin C_1$ and $\ell_e = |C_e| = m$. The cyclic code $C_{(1,e)}$ has parameters $[3^m - 1, 3^m - 1 - 2m, 4]$ if and only if the following conditions are satisfied:

C1: $e$ is even;
C2: the equation $(x + 1)^e + x^e + 1 = 0$ has the only solution $x = 1$ in $\text{GF}(q)^*$; and
C3: the equation $(x + 1)^e - x^e - 1 = 0$ has the only solution $x = 0$ in $\text{GF}(q)$.

**Proof:** Clearly, the minimum distance $d$ of the code $C_{(1,e)}$ cannot be 1. The code $C_{(1,e)}$ has a codeword of Hamming weight 2 if and only if there exist two elements $c_1$ and $c_2$ in $\text{GF}(3)^*$ and two distinct elements $x_1$ and $x_2$ in $\text{GF}(3^m)^*$ such that

$$\begin{cases} c_1x_1 + c_2x_2 = 0 \\ c_1x_1^e + c_2x_2^e = 0. \end{cases}$$

(1)

Note that $x_1 \neq x_2$. We have that $c_1 = c_2$. Hence (1) is equivalent to the following set of equations:

$$\begin{cases} x_1 + x_2 = 0 \\ x_1^e + x_2^e = 0. \end{cases}$$

It then follows that $C_{(1,e)}$ does not have a codeword of Hamming weight two if and only if $e$ is even.

The code $C_{(1,e)}$ has a codeword of Hamming weight 3 if and only if there exist three elements $c_1$, $c_2$ and $c_3$ in $\text{GF}(3)^*$ and three distinct elements $x_1$, $x_2$ and $x_3$ in $\text{GF}(3^m)^*$ such that

$$\begin{cases} c_1x_1 + c_2x_2 + c_3x_3 = 0 \\ c_1x_1^e + c_2x_2^e + c_3x_3^e = 0. \end{cases}$$

(2)

Due to symmetry it is sufficient to consider the following two cases.

**Case A, where $c_1 = c_2 = c_3 = 1$:** In this case, (2) becomes

$$\begin{cases} 1 + \frac{x_2}{x_1} + \frac{x_3}{x_1} = 0 \\ 1 + \left(\frac{x_2}{x_1}\right)^e + \left(\frac{x_3}{x_1}\right)^e = 0. \end{cases}$$

(3)

Putting $y_1 = x_2/x_1$ and $y_2 = x_3/x_1$. Then $y_i \notin \{0, 1\}$. Hence (3) has a desired solution if and only if the equation

$$(y + 1)^e + y^e + 1 = 0$$

has a solution $y \in \text{GF}(3^m) \setminus \{0, 1\}$. 

By definition, \( x \) has at least four distinct solutions, 1, \( x_1 \), \( x_2 \), \( x_3 \). Hence Condition C3 in Theorem 4.1 is met.

If Conditions C1, C2 and C3 are satisfied. This completes the proof of this theorem.

V. THE OPTIMAL CYCLIC CODES DEFINED BY PLANAR MONOMIALS \( x^e \) OVER \( GF(3^m) \)

We first prove the following lemma, where the first and the last conclusion should be known.

**Lemma 5.1:** If \( x^e \) is APN or planar over \( GF(3^m) \), then

- \( e \) must be even;
- \( \gcd(e, 3^m - 1) = 2 \);
- \( \ell_e = |C_e| = m \); and
- \( e \not\equiv 1 \).

**Proof:** If \( e \) is odd, then the equation \((x + 1)^e - x^e = 1\) will have three solutions \( x = 0 \) and \( x = \pm 1 \).

By definition \( x^e \) is not planar and APN. This proves the first conclusion.

Let \( x^e \) be planar or APN. Then \( e \) must be even. In this case, the equation \((x + 1)^e - x^e = 0\) has already one solution \( x = 1 \). If \( \gcd(e, 3^m - 1) > 2 \), then \( \gcd(e, 3^m - 1) \geq 4 \). In this case, the equation \( y^e = 1 \) has at least four distinct solutions, 1, \( y_1 = -1 \), \( y_2 \) and \( y_3 \). Define \( x_i = 1/(y_i - 1) \) for all \( i \in \{1, 2, 3\} \). Then \( x_1 \), \( x_2 \) and \( x_3 \) are three distinct solutions of the equation \((x + 1)^e - x^e = 0\). This is contrary to the assumption that \( x^e \) is planar or APN. Hence \( \gcd(e, 3^m - 1) = 2 \).

We now prove the third conclusion. Since \( \gcd(e, 3^m - 1) = 2 \), \( (3^m - 1)/2 \) must divide \( 3^\ell_e - 1 \). Hence \( \ell_e = m \).

Note that \( 3^i e - 1 \) is odd for any \( i \) and \( 3^m - 1 \) is even. We have \( 3^i e \not\equiv 1 \pmod{3^m - 1} \). Hence \( e \not\equiv 1 \).

The following theorem was proved by Carlet, Ding and Yuan in [8]. For completeness, we report it here and present a proof for it.

**Theorem 5.2:** (Carlet-Ding-Yuan [8]) If \( x^e \) is planar over \( GF(3^m) \), then \( C_{(1,e)} \) is an optimal ternary cyclic code with parameters \([3^m - 1, 3^m - 1 - 2m, 4] \).

**Proof:** Let \( x^e \) be planar. By Lemma 5.1 \( e \) is even. Notice that \( x = 0 \) is already a solution of the equation \((x + 1)^e - x^e = 1\). By the definition of planar functions, this equation cannot have other solutions. Hence Condition C3 in Theorem 4.1 is met.

We now prove that Condition C2 in Theorem 4.1 holds. Suppose on the contrary that \((-x - 1)^e + x^e + 1 = 0 \) for some \( x \in GF(q) \setminus \{1\} \). Then

\[ 1^e - x^e = x^e - (-x - 1)^e, \]

that is

\[ [x + (1 - x)]^e - x^e = [-x - 1 + (1 - x)]^e - (-x - 1)^e. \]

Note that \( x \neq -x - 1 \) as \( x \neq 1 \). This is contrary to the assumption that \( x^e \) is planar. Therefore Condition C2 in Theorem 4.1 is also satisfied.

By Lemma 5.1 and Lemma 2.1, the dimension of this code is equal to \( q - 1 - 2m \). The desired conclusions then follow from Theorem 4.1.
The following is a list of known planar monomials over $3^m$:

- $x^2$,
- $x^{3h+1}$, where $m / \gcd(m, h)$ is odd (12).
- $x^{(3h+1)/2}$, where $\gcd(m, h) = 1$ and $h$ is odd (11).

Each of these planar monomials gives a class of optimal ternary cyclic codes with parameters $[3^m - 1, 3^m - 1 - 2m, 4]$. More planar functions can be found in [11], [13], [27], [28].

**Example 5.3:** Let $(m, h) = (4, 3)$ and let $e = (3^h + 1)/2 = 14$. Let $\alpha$ be the generator of $\mathbb{GF}(3^m)^*$ with $
alpha^4 + 2\alpha^3 + 2 = 0$. Then $C_{(1,e)}$ is a ternary cyclic code with parameters $[80, 72, 4]$ and generator polynomial $x^8 + 2x^5 + x^3 + 2x^2 + 2$.

The dual of $C_{(1,e)}$ is a ternary cyclic code with parameters $[80, 8, 48]$ and weight enumerator

$$1 + 1320x^{48} + 2400x^{51} + 80x^{54} + 1920x^{57} + 480x^{60}.$$  

This dual code has the same parameters as the best known ternary linear code with parameters $[80, 8, 48]$ in the Database. The upper bound on the minimum distance $d$ of any ternary linear code of length 80 and dimension 8 is 49.

The following theorem is the partial inverse conclusion of Theorem 5.2 when $e$ is of a special form.

**Theorem 5.4:** Let $e = (3^h + 1)/2$, where $1 \leq h \leq m - 1$. Then the ternary cyclic code $C_{(1,e)}$ has parameters $[3^m - 1, 3^m - 1 - 2m, 4]$ if and only if $h$ is odd and $\gcd(m, h) = 1$, i.e., if and only if $x^e$ is planar over $\mathbb{GF}(3^m)$.

**Proof:** In the proof of Theorem 4.1 we see that $C_{(1,e)}$ has no codeword of Hamming weight 2 if and only if $e$ is even. Hence $h$ must be odd if $d = 4$.

Notice that multiplying the left sides of the two equations in Conditions C2 and C3 leads to

\[
((x + 1)^e + x^e + 1) ((x + 1)^e - x^e - 1) = (x + 1)^{2e} - (x^e + 1)^2
\]

\[
= (x + 1)^{3h+1} - (x^{(3h+1)/2} + 1)^2 = x \left(x^{(3h-1)/2} - 1\right)^2.
\]

It then follows that the equation

\[(x + 1)^{2e} - (x^e + 1)^2 = 0\]

does not have a solution $x \in \mathbb{GF}(q) \setminus \{0, \pm 1\}$ if and only if

\[\gcd((3^h - 1)/2, 3^m - 1) = 2,\]

which is equivalent to that $h$ is odd and $\gcd(h, m) = 1$.

When $h$ is odd and $\gcd(h, m) = 1$, we have $\gcd(e, 3^m - 1) = 2$. It then follows from Lemma 2.1 that $\ell_m = m$. Since $1 \notin C_e$, the dimension of this code is equal to $3^m - 1 - 2m$.

We shall need the following lemma later.

**Lemma 5.5:** Let $e = 3^h + 1$, where $0 \leq h \leq m - 1$. When $m$ is odd, $\ell_e = m$. When $m$ is even,

\[\ell_e = \begin{cases}
\frac{m}{2} & \text{if } h = \frac{m}{2} \\
\frac{m}{2} & \text{if } h \neq \frac{m}{2}.
\end{cases}\]  

(5)

**Proof:** First of all, we have

\[\gcd(3^j - 1, 3^m - 1) = 3^\gcd(j,m) - 1.\]  

(6)

It is also known that

\[\gcd(3^h + 1, 3^m - 1) = \begin{cases}
2 & \text{if } m / \gcd(m, h) \text{ is odd} \\
3^\gcd(h,m) + 1 & \text{if } m / \gcd(m, h) \text{ is even}.
\end{cases}\]  

(7)
When \( h \neq m/2 \), we have
\[
\gcd(3^h + 1, 3^m - 1) < 3^{m/2} + 1.
\]
Since \( 1 \leq j < m \), we have
\[
\gcd(3^j - 1, 3^m - 1) = 3^\gcd(j,m) - 1 \leq 3^{m/2} - 1.
\]
It then follows that
\[
gcd(3^h + 1, 3^m - 1) \gcd(3^j - 1, 3^m - 1) < 3^m - 1
\]
for all \( 1 \leq j < m \). Hence
\[
gcd(3^h + 1, 3^m - 1) \gcd(3^j - 1, 3^m - 1) \not\equiv 0 \pmod{3^m - 1}
\]
for all \( 1 \leq j < m \). Therefore, \( \ell_e = m \) if \( h \neq m/2 \).

When \( h = m/2 \), let \( j = m/2 \). Then \( (3^j - 1)(3^h + 1) \equiv 0 \pmod{3^m - 1} \). It then follows from the discussions above that \( \ell_e = m/2 \).

**Theorem 5.6:** Let \( e = 3^h + 1 \), where \( 0 \leq h \leq m - 1 \). Then the ternary cyclic code \( C_{(1,e)} \) has parameters \([3^m - 1, 3^m - 1 - 2m, 4]\) if and only if \( m/\gcd(m,h) \) is odd, i.e., if and only if \( x^e \) is planar over GF\( (3^m) \).

**Proof:** We first prove the necessity of the condition. Notice that
\[
gcd \left( x^{3^h - 1} + 1, x^{3^m - 1} - 1 \right) = 1.
\]
We have
\[
gcd \left( x^{3^h - 1} + 1, x^{3^m - 1} - 1 \right)
= \frac{\gcd \left( x^{2(3^h - 1)} - 1, x^{3^m - 1} - 1 \right)}{\gcd \left( x^{3^h - 1} - 1, x^{3^m - 1} - 1 \right)}
= \frac{x^{\gcd(2(3^h - 1),3^m - 1)} - 1}{x^{\gcd(3^h - 1,3^m - 1)} - 1}
= \frac{\gcd(3^h - 1,3^m - 1) \gcd \left( 2, \frac{3^m - 1}{\gcd(3^h - 1,3^m - 1)} \right) - 1}{x^{\gcd(h,m) - 1}}
\]
Suppose first that \( m/\gcd(m,h) \) is even. Then \( 3^{\gcd(h,m)} - 1 \) divides \((3^m - 1)/2\). Let \( \alpha \) be a generator of GF\( (3^m)^* \). Define then \( x = \alpha^{3^{\gcd(h,m)} - 1} \). Then \( x^{3^{\gcd(h,m)} - 1} + 1 = 0 \).

It is easily checked that
\[
(x + 1)^e - x^e - 1 = x \left( x^{3^h - 1} + 1 \right).
\]
Hence \( x = \alpha^{3^{\gcd(h,m)} - 1} \) is a solution of \((x + 1)^e - x^e - 1 = 0\). This means that Condition C3 in Theorem 4.1 is not met. So we have reached a contradiction. This proves the necessity of the condition in this theorem.

We now prove the sufficiency of this condition. When \( m/\gcd(m,h) \) is odd, it was proved in [11] that \( x^e \) is planar. It then follows from Theorem 5.2 that \( C_{(1,e)} \) has parameters \([3^m - 1, 3^m - 1 - 2m, 4]\). This completes the proof of this theorem.
VI. The optimal cyclic codes defined by APN monomials $x^e$ over GF(3$^m$)

In this section, we present seven classes of optimal cyclic codes defined by APN monomials $x^e$ over GF(3$^m$).

**Theorem 6.1**: The ternary cyclic code $C_{(1,e)}$ has parameters $[3^m - 1, 3^m - 1 - 2m, 4]$ if $x^e$ is APN over GF(3$^m$).

**Proof**: Let $x^e$ be APN over GF($q$), where $q = 3^m$. It then follows from Lemma 5.1 that gcd($e, 3^m - 1$) = 2. By Lemma 2.1, the dimension of this code is equal to $3^m - 1 - 2m$.

We now prove that Condition C2 of Theorem 4.1 is satisfied. To this end, we need to prove that there does not exist two distinct elements $y$ and $z$ in GF($q$) \ {0, 1} such that

$$\begin{cases} 1 + y + z = 0 \\ 1 + y^e + z^e = 0. \end{cases}$$

(8)

Suppose on the contrary that (8) has such a solution $(y, z)$. Then adding 1 to both sides of both equations in (8) yields

$$\begin{cases} y - 1 = 1 - z \\ y^e - 1 = 1 - z^e. \end{cases}$$

(9)

Define $a = 1 - y \neq 0$ and $b = 1 - y^e$. It then follows from (9) that

$$\begin{cases} (1 - a)^e - 1^e = -b \\ (z - a)^e - z^e = -b. \end{cases}$$

(10)

Adding $z$ and $z^e$ to the first and second equation of (10) gives

$$\begin{cases} 1 - z = z - y = -a \\ 1^e - z^e = z^e - y^e = -b. \end{cases}$$

(11)

It then follows that

$$(y - a)^e - y^e = -b.$$  

(12)

Hence the equation $(x - a)^e - x^e = -b$ has three distinct solutions 1, $y$ and $z$. This is contrary to the assumption that $x^e$ is APN. Note that $(x - a)^e - x^e = -b$ has at most two solutions $x \in$ GF($q$) for any $(a, b) \in$ GF($q$)$^* \times$ GF($q$) according to the definition of APN functions. This completes the proof of Condition C2 in Theorem 4.1.

Finally we prove that Condition C3 is also satisfied. Suppose on the contrary that the equation $(x + 1)^e - x^e - 1 = 0$ has a solution $x \in$ GF($q$)$^*$. Then $x \neq \pm 1$. Whence $x^2 \neq 1$ and thus $x \neq x^{-1}$. Dividing both sides of the equation $(x + 1)^e - x^e - 1 = 0$ with $x^e$ yields $(x^{-1} + 1)^e - x^{-e} - 1 = 0$. Therefore $x^{-1}$ is also a solution of the equation $(x + 1)^e - x^e - 1 = 0$. So this equation has three distinct solutions 0, $x$ and $x^{-1}$. This is contrary to the assumption that $x^e$ is APN over GF($q$). Hence Condition C3 in Theorem 4.1 is indeed met.

The desired conclusions of this theorem then follow from Theorem 4.1.

The following is a summary of known APN monomials $x^e$ over GF(3$^m$). Each of them gives a class of optimal ternary cyclic codes with parameters $[3^m - 1, 3^m - 1 - 2m, 4]$.

- $e = 3^{m-1} - 1$, $m$ is odd.
- $e = 3^{(m+1)/2} - 1$, $m$ is odd.
- $e = 3^{m-3}/4$, $m \geq 5$ and $m$ is odd (14).
- $e = 3^{m/4} + 3^{m-1}/2$, $m \geq 3$ and $m$ is odd (14).
- $e = (3^{(m+1)/4} - 1)(3^{(m+1)/2} + 1)$, $m \equiv 3 \pmod{4}$ (28).
- The exponent $e$ is defined by

$$e = \begin{cases} 3^{(m+1)/2} - 1 \quad &\text{if } m \equiv 3 \pmod{4} \\ 3^{m+1/2} - 1 + 3^{m-1}/2 \quad &\text{if } m \equiv 1 \pmod{4}. \end{cases}$$
• The exponent $e$ is defined by

$$e = \begin{cases} 
\frac{3^{m+1}-1}{8} & \text{if } m \equiv 3 \pmod{4} \\
\frac{3^{m+1}-1}{8} + \frac{3^{m}-1}{2} & \text{if } m \equiv 1 \pmod{4}.
\end{cases}$$

There are other APN monomials over $\mathbb{GF}(p^m)$ for $p \geq 5$. The reader is referred to [14], [28] for details.

**Example 6.2:** Let $m = 3$ and let $e = 3^{(m+1)/2} - 1 = 8$. Let $\alpha$ be the generator of $\mathbb{GF}(3^m)$ with $\alpha^3 + 2\alpha + 1 = 0$. Then $C_{(1,e)}$ is a ternary cyclic code with parameters $[26,20,4]$ and generator polynomial

$$x^6 + 2x^5 + x^4 + x^3 + 2.$$ 

The dual of $C_{(1,e)}$ is a ternary cyclic code with parameters $[26,6,15]$ and weight enumerator

$$1 + 312x^{15} + 260x^{18} + 156x^{21}.$$ 

This code is also optimal, while the optimal ternary cyclic code with the same parameters in the Database is not known to be cyclic.

**VII. THE OPTIMAL CYCLIC CODES DEFINED BY OTHER MONOMIALS $x^e$ OVER $\mathbb{GF}(3^m)$**

In the previous sections, optimal ternary cyclic codes from planar and APN monomials over $\mathbb{GF}(3^m)$ were presented. In this section, we construct several classes of optimal ternary cyclic codes with parameters $[3^m - 1, 3^m - 1 - 2m, 4]$ using monomials over $\mathbb{GF}(3^m)$ that are neither planar nor APN.

We prove first the following theorem.

**Theorem 7.1:** Let $m = 3^h - 1/2$, where $2 \leq h \leq m - 1$. The ternary cyclic code $C_{(1,e)}$ has parameters $[3^m - 1, 3^m - 1 - 2m, 4]$ if and only if

(a) $m$ is odd;
(b) $h$ is even;
(c) $\gcd(h,m) = 1$; and
(d) $\gcd(h-1,m) = 1$.

**Proof:** The code $C_{(1,e)}$ does not have a codeword of Hamming weight two if and only if $e$ is even. Clearly $e$ is even if and only if $h$ is even.

Obviously, Conditions C2 and C3 in Theorem 4.1 are simultaneously satisfied if and only if the equation $(x + 1)^{2e} - (x^e + 1)^2 = 0$ does not have a solution $x \in \mathbb{GF}(q) \setminus \{0, \pm 1\}$.

Let $x \in \mathbb{GF}(q) \setminus \{0, \pm 1\}$. Note that

$$(x + 1)^{2e} - (x^e + 1)^2 = (x + 1)^{3h-1} - (x^{3h-1} - x^{(3h-1)/2} + 1)$$

$$= (x + 1)^{3h} - (x + 1)(x^{3h-1} - x^{(3h-1)/2} + 1)$$

$$= x + 1 \frac{x^{(3h-3)/2} - 1}{x^{(3h-1)/2} - 1}.$$ 

The equation $(x + 1)^{2e} - (x^e + 1)^2 = 0$ does not have a solution $x \in \mathbb{GF}(q) \setminus \{0, \pm 1\}$ if and only if

$$\gcd \left( \frac{x^{3h-1} - 1}{2}, 3^m - 1 \right) \in \{1, 2\} \quad (13)$$

and

$$\gcd \left( \frac{x^{h-1} - 1}{2}, 3^m - 1 \right) \in \{1, 2\}. \quad (14)$$
It is easily seen that for any $i \geq 1$
\[ \gcd \left( \frac{3^i - 1}{2}, 3^m - 1 \right) = \begin{cases} 3^{\gcd(i,m)} - 1 & \text{if } i \text{ is odd} \\ 3^{\gcd(i,m)} & \text{if } i \text{ is even}. \end{cases} \]

Conditions (a), (b), (13) and (14) together are equivalent to Conditions (a), (b), (c) and (d) together.
When all the conditions of this theorem are met, we have
\[ \gcd(e, 3^m - 1) = 2. \]
It then follows from Lemma 2.1 that $\ell_e = m$.
Note that $e = (3^h - 1)/2 = 3^{h-1} + 3^{h-2} + \cdots + 3 + 1$. Hence $e \not\equiv 3^i \pmod{3^m - 1}$ for any $1 \leq i \leq m - 1$. Thus $e \not\in C_1$.

The desired conclusions in this theorem then follow from Theorem 4.1.
\[ \text{Example 7.2: Let } (m, h) = (5, 2) \text{ and let } e = (3^h - 1)/2 = 4. \text{ Let } \alpha \text{ be the generator of } \GF(3^m)^* \text{ with } \alpha^5 + 2\alpha + 1 = 0. \text{ Then } C_{(1,e)} \text{ is a ternary cyclic code with parameters } [242, 232, 4] \text{ and generator polynomial } x^{10} + 2x^9 + 2x^7 + x^5 + 2x^4 + x^3 + x^2 + x + 2. \text{ The dual of } C_{(1,e)} \text{ is a ternary cyclic code with parameters } [242, 10, 153] \text{ and weight enumerator } 1 + 21780x^{153} + 19844x^{162} + 17424x^{171}. \]
This code has the same parameters as the best known ternary linear code with parameters $[242, 10, 153]$ in the Database. The upper bound on the minimum distance $d$ of any ternary linear code of length 242 and dimension 10 is 155.

**Lemma 7.3:** Let $e = 2(3^h + 1)$, where $0 \leq h \leq m - 1$. When $m$ is odd, $\ell_e = m$. When $m$ is even,
\[ \ell_e = \begin{cases} \frac{m}{2} & \text{if } h = \frac{m}{2} \\ m & \text{if } h \neq \frac{m}{2}. \end{cases} \] (15)

**Proof:** The proof is similar to that of Lemma 5.5 and is omitted.

**Theorem 7.4:** Let $e = 2(1 + 3^h)$, where $0 \leq h \leq m - 1$, and let $m$ be even. Then the ternary cyclic code $C_{(1,e)}$ has parameters $[3^m - 1, k, 3]$, where
\[ k = \begin{cases} 3^m - 1 - 2m & \text{if } h \neq m/2 \\ 3^m - 1 - 3m/2 & \text{if } h = m/2. \end{cases} \]

**Proof:** It is easily seen that $e \not\in C_1$. The conclusions on the dimension then follow from Lemma 7.3. Since $e$ is even, the minimum distance $d \geq 3$. We now prove that $d = 3$. To this end, we prove that Condition C2 or C3 in Theorem 4.1 is not satisfied.

Since $m$ is even, $(3^m - 1)/2$ is even. Define $x = \alpha^{(3^m - 1)/4}$, where $\alpha$ is a generator of $\GF(3^m)^*$. Then $x^2 = \alpha^{(3^m - 1)/2} = -1$.

When $h$ is even, we have
\[
(x + 1)^{2(1+3^h)} - x^{2(1+3^h)} - 1
= (x^2 - x + 1)^{1+3^h} - (x^2)^{1+3^h} - 1
= (-x)^{1+3^h} - (-1)^{1+3^h} - 1
= 0.
\]

In this case, Condition C3 in Theorem 4.1 is not met, and hence $d = 3$.
When $h$ is odd, we have
\[
(x + 1)^{2(1+3^h)} + x^{2(1+3^h)} + 1 = (-1)^{(1+3^h)/2} + 2 = 0.
\]
In this case, Condition C3 in Theorem 4.1 is not met, and hence $d = 3$.

The code $C_{1,e}$ of Theorem 7.4 is almost optimal when $h \neq m/2$, and is optimal when $(m,h) = (2,1)$ and $(m,h) = (4,2)$.

**Open Problem 7.5:** Let $e = 2(1 + 3^h)$, where $0 \leq h \leq m - 1$, and let $m$ be odd. Does the ternary cyclic code $C_{1,e}$ have parameters $[3^m - 1, 3^m - 1 - 2m, 4]$?

When $3 \leq m \leq 13$, the answer to this question is positive and confirmed by Magma.

**Theorem 7.6:** Let $e = 3^h - 1$, where $1 \leq h \leq m - 1$. The ternary cyclic code $C_{1,e}$ has parameters $[3^m - 1, 3^m - 1 - 2m, 4]$ if and only if

(A) $\gcd(h, m) = 1$; and

(B) $\gcd(3^h - 2, 3^m - 1) = 1$.

**Proof:** Let Conditions (A) and (B) be met. We first prove that $\ell_e = m$. To this end, we prove that there is no $1 \leq j \leq m - 1$ such that $3^m - 1$ divides $(3^j - 1)(3^h - 1)$. Note that

$$\gcd(3^m - 1, (3^j - 1)(3^h - 1)) = \frac{\gcd(3^m - 1, 3^h - 1) \gcd(3^m - 1, 3^j - 1)}{\gcd(3^j - 1, 3^h - 1)} = \frac{2^{\gcd(m,f)} - 1}{3^{\gcd(j,h) - 1}} < 3^m - 1.$$ 

Hence $\ell_e = m$. Clearly, $e \not\in C_1$. Thus the dimension of the code is equal to $3^m - 1 - 2m$.

It is now time to prove that the minimum distance $d = 4$. Suppose on the contrary that the equation $(x + 1)^e + x^e + 1 = 0$ has a solution $x \in \text{GF}(3^m) \setminus \{0, \pm 1\}$. Then we have

$$(x + 1)^{3^h - 1} + x^{3^h - 1} + 1 = 0.$$ 

Multiplying both sides of this equation with $x + 1$ yields

$$x^{3^h - 1} - x^{3^h} + x - 1 = (1 - x)(x^{3^h - 1} - 1) = 0.$$ 

Whence $x^{3^h - 1} = 1$. It then follows from Condition (A) that $x^2 = 1$. This is contrary to the assumption that $x \neq \pm 1$. Therefore Condition C2 in Theorem 4.1 is met.

Now suppose on the contrary that the equation $(x + 1)^e - x^e - 1 = 0$ has a solution $x \in \text{GF}(3^m) \setminus \{0, \pm 1\}$. Then we have

$$(x + 1)^{3^h - 1} - x^{3^h - 1} - 1 = 0.$$ 

Multiplying both sides of this equation with $x(x + 1)$ gives

$$x^{3^h} + x^2 = 0.$$ 

Whence $x^{3^h - 2} + 1 = 0$ and $x^{2(3^h - 2)} = 1$. It then follows from Condition (B) that $x^2 = 1$. This is contrary to the assumption that $x \neq \pm 1$. Therefore Condition C3 in Theorem 4.1 is satisfied.

Note that $e$ is even. It then follows from Theorem 4.1 that $d = 4$.

**Example 7.7:** Let $(m,h) = (5,2)$ and let $e = 3^h - 1 = 8$. Let $\alpha$ be the generator of $\text{GF}(3^m)^*$ with $\alpha^5 + 2\alpha + 1 = 0$. Then $C_{1,e}$ is a ternary cyclic code with parameters $[242,232,4]$ and generator polynomial $x^{10} + 2x^9 + x^6 + x^5 + 2x^3 + x^2 + 2$.

The dual of $C_{1,e}$ is a ternary cyclic code with parameters $[242,10,147]$ and weight enumerator

\[
1 + 2420x^{147} + 4840x^{150} + 2420x^{153} + 7260x^{156} + 9680x^{159} + 10164x^{162} + 9680x^{165} + 2420x^{168} + 7260x^{171} + 2420x^{174} + 484x^{186}.
\]
Below we list a number of open problems on the ternary cyclic codes $C_{(1,e)}$.

**Open Problem 7.8:** Let $e = 2(3^{m-1} - 1)$. Does the ternary cyclic code $C_{(1,e)}$ have parameters $[3^m - 1, 3^m - 1 - 2^m, 4]$ if $m \geq 5$ and $m$ is prime?

When $m \in \{5, 7, 11, 13\}$, the answer to this question is positive and confirmed by Magma.

**Open Problem 7.9:** Let $e = (3^h + 5)/2$, where $1 \leq h \leq m - 1$. Is it true that the ternary cyclic code $C_{(1,e)}$ has parameters $[3^m - 1, 3^m - 1 - 2^m, 4]$ if

- $m$ is odd and $m \not\equiv 0 \pmod{3}$;
- $h$ is odd; and
- $\gcd(h, m) = 1$?

For all $5 \leq m \leq 15$, the answer to this question is positive and confirmed by Magma.

**Open Problem 7.10:** Let $e = (3^h - 5)/2$, where $2 \leq h \leq m - 1$. Let $m$ be odd and $m \not\equiv 0 \pmod{3}$. Is it true that the ternary cyclic code $C_{(1,e)}$ has parameters $[3^m - 1, 3^m - 1 - 2^m, 4]$ if $h$ is even?

For all $3 \leq m \leq 15$, the answer to this question is positive and confirmed by Magma.

**Open Problem 7.11:** Let $e = (3^h - 5)/2$, where $2 \leq h \leq m - 1$. Let $m$ be even. What are the conditions on $m$ and $h$ under which the ternary cyclic code $C_{(1,e)}$ has parameters $[3^m - 1, 3^m - 1 - 2^m, 4]$?

**Open Problem 7.12:** Let $e = 3^h + 5$, where $2 \leq h \leq m - 1$. Let $m$ be even. Is it true that the ternary cyclic code $C_{(1,e)}$ has parameters $[3^m - 1, 3^m - 1 - 2^m, 4]$ if one of the following conditions is met?

- $m \equiv 0 \pmod{4}$, $m \geq 4$ and $h = m/2$.
- $m \equiv 2 \pmod{4}$, $m \geq 6$ and $h = (m+2)/2$.

For $m \in \{6, 10, 14, 18\}$, the answer to this question is positive and confirmed by Magma.

**Open Problem 7.13:** Let $e = 3^h + 5$, where $0 \leq h \leq m - 1$. Let $m \geq 5$ and let $m$ be a prime. Is it true that the ternary cyclic code $C_{(1,e)}$ has parameters $[3^m - 1, 3^m - 1 - 2^m, 4]$ for all $h$ with $0 \leq h \leq m - 1$?

For $m \in \{3, 5, 7, 11, 13, 17\}$, the answer to this question is positive and confirmed by Magma.

**Open Problem 7.14:** Let $e = 3^h + 13$, where $3 \leq h \leq m - 1$. Let $m$ be an odd prime. What are the conditions on $h$ under which the ternary cyclic code $C_{(1,e)}$ has parameters $[3^m - 1, 3^m - 1 - 2^m, 4]$?

**Open Problem 7.15:** Let $e = (3^{m-1} - 1)/2 + 3^h + 1$, where $0 \leq h \leq m - 1$. What are the conditions on $m$ and $h$ under which the ternary cyclic code $C_{(1,e)}$ has parameters $[3^m - 1, 3^m - 1 - 2^m, 4]$?

**VIII. Conclusion remarks**

When $x^e$ is planar or APN over $GF(3^m)$, the code $C_{(1,e)}$ has parameters $[3^m - 1, 3^m - 1 - 2^m, 4]$ and is thus optimal. However, as demonstrated in this paper, the code may have the same parameters when $x^e$ is neither planar nor APN. It looks hard to completely characterize the ternary cyclic codes $C_{(1,e)}$ with parameters $[3^m - 1, 3^m - 1 - 2^m, 4]$. In this paper, we presented nine open problems on the codes $C_{(1,e)}$. It would be nice if some of these open problems could be settled. Finally, we inform the reader that there are other examples of monomials $x^e$ that define optimal cyclic codes $C_{(1,e)}$ according to our Magma experimental data.

**Acknowledgments**

The authors are grateful to the reviewers for their comments that improved the presentation and quality of this paper.

**References**

[1] T. Beth, C. Ding, “On almost perfect nonlinear permutations,” in: Advances in Cryptology–EUROCRYPT 93, Lecture Notes in Comput. Sci. 765, Springer-Verlag, New York, 1993, pp. 6576.

[2] L. Budaghyan, C. Carlet, “Classes of quadratic APN trinomials and hexanomials and related structures,” IEEE Trans. Inform. Theory, vol. 54, no. 5, pp. 2355–2357, 2008.

[3] L. Budaghyan, C. Carlet, G. Leander. “Two classes of quadratic APN binomials inequivalent to power functions,” IEEE Trans. Inform. Theory, vol. 54, no. 9, pp. 4218–4229, 2008.

[4] L. Budaghyan, C. Carlet, G. Leander, “Constructing new APN functions from known ones,” Finite Fields and Their Applications, vol. 15, no. 2, pp. 150–159, April 2009.
T. Helleseth, C. Rong, D. Sandberg, “New families of almost perfect nonlinear power mappings,” IEEE Trans. Inform. Theory, vol. 52, no. 3, pp. 1141–1152, March 2006.

A. Canteaut, P. Charpin, H. Dobbertin, “Weight divisibility of cyclic codes, highly nonlinear functions on \( F_{2^n} \), and crosscorrelation of maximum-length sequences,” SIAM, J. Discrete Math., vol. 13, no. 1, pp. 105–138, 2000.

C. Carlet, P. Charpin, V. Zinoviev, “Codes, bent functions and permutations suitable for DES-like cryptosystems,” Des. Codes Cryptogr., vol. 15, pp. 125–156, 1998.

C. Carlet, C. Ding and J. Yuan, “Linear codes from highly nonlinear functions and their secret sharing schemes,” IEEE Trans. Inform. Theory, vol. 51, no. 6, pp. 2089-2102, 2005.

P. Charpin, Open problems on cyclic codes, in: Handbook of Coding Theory, Part 1: Algebraic Coding, V. S. Pless, W. C. Huffman, and R. A. Brualdi, Eds. Amsterdam, The Netherlands: Elsevier, 1998, ch. 11.

R. T. Chien, “Cyclic decoding procedure for the Bose-Chaudhuri-Hocquenghem codes,” IEEE Trans. Inform. Theory, vol. 10, pp. 357–363, 1964.

R. S. Coulter, R. W. Matthews, “Planar functions and planes of LenzBarlotti class II,” Des. Codes Cryptogr., vol. 10, pp. 167–184, 1997.

P. Dembowski, T. G. Ostrom, “Planes of order \( n \) with collineation groups of order \( n^2 \),” Math. Z., vol. 193, pp. 239–258, 1968.

C. Ding, J. Yuan, “A family of skew Hadamard difference sets,” J. of Combinatorial Theory, Series A, vol. 113, pp. 1526–1535, 2006.

T. Helleseth, C. Rong, D. Sandberg, “New families of almost perfect nonlinear power mappings,” IEEE Trans. Inform. Theory, vol. 45, no. 2, pp. 475–485, 1999.

H. Dobbertin, “Almost perfect nonlinear power functions on \( GF(2^n) \): the Welch case,” IEEE Trans. Inform. Theory, vol. 45, pp. 1271–1275, 1999.

H. Dobbertin, “Almost perfect nonlinear power functions on \( GF(2^n) \): The Niho case,” Inform. and Comput., vol. 151, pp. 57–72, 1999.

G. D. Forney, “On decoding BCH codes,” IEEE Trans. Inform. Theory, vol. 11, no. 4, pp. 549–557, 1995.

R. Gold, “Maximal recursive sequences with 3-valued recursive crosscorrelation functions,” IEEE Trans. Inform. Theory, vol. 14, pp. 154–156, 1968.

T. Kasami, “The weight enumerators for several classes of subcodes of the second order binary Reed-Muller codes,” Inform. and Control, vol. 18, pp. 369–394, 1971.

H. D. L. Hollmann, Q. Xiang, “A proof of the Welch and Niho conjectures on cross-correlations of binary \( m \)-sequences,” Finite Fields and Their Applications, vol. 7, no. 2, pp. 253–286, April 2001.

W. C. Huffman and V. Pless, Fundamentals of Error-Correcting Codes, Cambridge University Press, Cambridge, 2003.

L. Liidl, and H. Niederreiter, Finite Fields, Cambridge University Press, Cambridge, 1997.

J. H. van Lint and R. M. Wilson, “On the minimum distance of cyclic codes,” vol. 32, pp. 23–40, 1986.

K. Nyberg, “Differentially uniform mappings for cryptography,” in: Advances in Cryptology–EUROCRYPT 93, Lecture Notes in Comput. Sci. 765, Springer-Verlag, New York, 1993, pp. 55–64.

E. Prange, “Some cyclic-error-correcting codes with simple decoding algorithms,” Air Force Cambridge Research Center-TN-58-156, Cambridge, Mass., April 1958.

J. Yuan, C. Carlet and C. Ding, “The weight distribution of a class of linear codes from perfect nonlinear functions,” IEEE Trans. Information Theory, vol. 52, no. 2, pp. 712–717, Feb. 2006.

Z. Zha, G. M. Kyureghyan, X. Wang, “Perfect nonlinear binomials and their semifields”, Finite Fields and Their Applications, vol 15, pp. 125–133, 2009.

Z. Zha, X. Wang, “Almost perfect nonlinear power functions in odd characteristic”, IEEE Trans. Inform. Theory, vol. 57, no. 7, pp. 4826–4832, 2011.

### Cunsheng Ding

M’98–SM’05 was born in 1962 in Shaanxi, China. He received the M.Sc. degree in 1988 from the Northwestern Telecommunications Engineering Institute, Xian, China; and the Ph.D. in 1997 from the University of Turku, Turku, Finland. From 1988 to 1992 he was a Lecturer of Mathematics at Xidian University, China. Before joining the Hong Kong University of Science and Technology in 2000, where he is currently Professor of Computer Science and Engineering, he was Assistant Professor of Computer Science at the National University of Singapore.

His research fields are cryptography and coding theory. He has coauthored four research monographs, and served as a guest editor or editor for ten journals. Dr. Ding co-received the State Natural Science Award of China in 1989.

### Tor Helleseth

M’89–SM’96–F’97 received the Cand. Real. and Dr. Philos. degrees in mathematics from the University of Bergen, Bergen, Norway, in 1971 and 1979, respectively.

From 1973–1980, he was a Research Assistant at the Department of Mathematics, University of Bergen. From 1981–1984, he was at the Chief Headquarters of Defense in Norway. Since 1984, he has been a Professor in the Department of Informatics, University of Bergen. During the academic years 1985–1987 and 1992–1993, he was on sabbatical leave at the University of Southern California, Los Angeles, and during 1979–1980, he was a Research Fellow at the Eindhoven University of Technology, Eindhoven, The Netherlands. His research interests include coding theory and cryptography.

Prof. Helleseth served as an Associate Editor for Coding Theory for the IEEE TRANSACTIONS ON INFORMATION THEORY from 1991 to 1993. He was Program Chairman for Eurocrypt 1993 and for the Information Theory Workshop in 1997 in Longyearbyen, Norway. He was a Program Co-Chairman for SETA04 in Seoul, Korea, and SETA06 in Beijing, China. He was also a Program Co-Chairman for the IEEE Information Theory Workshop in Solstrand, Norway in 2007. During 2007–2009 he served on the Board of Governors for the IEEE Information Theory Society. In 1997 he was elected an IEEE Fellow for his contributions to coding theory and cryptography. In 2004 he was elected a member of Det Norske Videnskaps-Akademiet.