Solecki submeasures and densities on groups

Taras Banakh

Kielce-Lviv

Hejnice, 2013
Definition

A function $\mu : \mathcal{P}(X) \to [0, 1]$ on a power-set of a set $X$ is called:
- **monotone** if $\mu(A) \leq \mu(B)$ for any subsets $A \subset B$ of $X$;
- **subadditive** if $\mu(A \cup B) \leq \mu(A) + \mu(B)$ for any subsets $A, B \subset X$;
- **additive** if $\mu(A \cup B) = \mu(A) + \mu(B)$ for any disjoint subsets $A, B \subset X$;
- a **density** on $X$ if $\mu$ is monotone, $\mu(\emptyset) = 0$ and $\mu(X) = 1$;
- a **submeasure** if $\mu$ is a subadditive density on $X$;
- a **measure** if $\mu$ is an additive density on $X$.

So, all our measures are, in fact, finitely additive probability measures.
A function $\mu : \mathcal{P}(X) \to [0, 1]$ on a power-set of a set $X$ is called:

- **monotone** if $\mu(A) \leq \mu(B)$ for any subsets $A \subset B$ of $X$;
- **subadditive** if $\mu(A \cup B) \leq \mu(A) + \mu(B)$ for any subsets $A, B \subset X$;
- **additive** if $\mu(A \cup B) = \mu(A) + \mu(B)$ for any disjoint subsets $A, B \subset X$;
- a **density** on $X$ if $\mu$ is monotone, $\mu(\emptyset) = 0$ and $\mu(X) = 1$;
- a **submeasure** if $\mu$ is a subadditive density on $X$;
- a **measure** if $\mu$ is an additive density on $X$.

So, all our measures are, in fact, finitely additive probability measures.
A function $\mu : \mathcal{P}(X) \to [0, 1]$ on a power-set of a set $X$ is called:

- **monotone** if $\mu(A) \leq \mu(B)$ for any subsets $A \subset B$ of $X$;
- **subadditive** if $\mu(A \cup B) \leq \mu(A) + \mu(B)$ for any subsets $A, B \subset X$;
- **additive** if $\mu(A \cup B) = \mu(A) + \mu(B)$ for any disjoint subsets $A, B \subset X$;

- a **density** on $X$ if $\mu$ is monotone, $\mu(\emptyset) = 0$ and $\mu(X) = 1$;
- a **submeasure** if $\mu$ is a subadditive density on $X$;
- a **measure** if $\mu$ is an additive density on $X$.

So, all our measures are, in fact, finitely additive probability measures.
Definition

A function \( \mu : \mathcal{P}(X) \to [0, 1] \) on a power-set of a set \( X \) is called:

- **monotone** if \( \mu(A) \leq \mu(B) \) for any subsets \( A \subset B \) of \( X \);
- **subadditive** if \( \mu(A \cup B) \leq \mu(A) + \mu(B) \) for any subsets \( A, B \subset X \);
- **additive** if \( \mu(A \cup B) = \mu(A) + \mu(B) \) for any disjoint subsets \( A, B \subset X \);
- a **density** on \( X \) if \( \mu \) is monotone, \( \mu(\emptyset) = 0 \) and \( \mu(X) = 1 \);
- a **submeasure** if \( \mu \) is a subadditive density on \( X \);
- a **measure** if \( \mu \) is an additive density on \( X \).

So, all our measures are, in fact, finitely additive probability measures.
Densities and submeasures on sets

Definition

A function $\mu : \mathcal{P}(X) \to [0, 1]$ on a power-set of a set $X$ is called:

- **monotone** if $\mu(A) \leq \mu(B)$ for any subsets $A \subset B$ of $X$;
- **subadditive** if $\mu(A \cup B) \leq \mu(A) + \mu(B)$ for any subsets $A, B \subset X$;
- **additive** if $\mu(A \cup B) = \mu(A) + \mu(B)$ for any disjoint subsets $A, B \subset X$;
- a **density** on $X$ if $\mu$ is monotone, $\mu(\emptyset) = 0$ and $\mu(X) = 1$;
- a **submeasure** if $\mu$ is a subadditive density on $X$;
- a **measure** if $\mu$ is an additive density on $X$.

So, all our measures are, in fact, finitely additive probability measures.
Densities and submeasures on sets

**Definition**

A function $\mu : \mathcal{P}(X) \rightarrow [0, 1]$ on a power-set of a set $X$ is called:
- *monotone* if $\mu(A) \leq \mu(B)$ for any subsets $A \subset B$ of $X$;
- *subadditive* if $\mu(A \cup B) \leq \mu(A) + \mu(B)$ for any subsets $A, B \subset X$;
- *additive* if $\mu(A \cup B) = \mu(A) + \mu(B)$ for any disjoint subsets $A, B \subset X$;
- a *density* on $X$ if $\mu$ is monotone, $\mu(\emptyset) = 0$ and $\mu(X) = 1$;
- a *submeasure* if $\mu$ is a subadditive density on $X$;
- a *measure* if $\mu$ is an additive density on $X$.

So, all our measures are, in fact, finitely additive probability measures.
A function $\mu : \mathcal{P}(X) \rightarrow [0, 1]$ on a power-set of a set $X$ is called:

- **monotone** if $\mu(A) \leq \mu(B)$ for any subsets $A \subset B$ of $X$;
- **subadditive** if $\mu(A \cup B) \leq \mu(A) + \mu(B)$ for any subsets $A, B \subset X$;
- **additive** if $\mu(A \cup B) = \mu(A) + \mu(B)$ for any disjoint subsets $A, B \subset X$;
- a **density** on $X$ if $\mu$ is monotone, $\mu(\emptyset) = 0$ and $\mu(X) = 1$;
- a **submeasure** if $\mu$ is a subadditive density on $X$;
- a **measure** if $\mu$ is an additive density on $X$.

So, all our measures are, in fact, finitely additive probability measures.
A density $\mu : \mathcal{P}(G) \to [0, 1]$ on a group $G$ is called

- **left invariant** if $\mu(xA) = \mu(A)$ for all $x \in G$ and $A \subseteq G$;
- **right invariant** if $\mu(Ay) = \mu(A)$ for all $y \in G$ and $A \subseteq G$;
- **invariant** if $\mu(xAy) = \mu(A)$ for all $x, y \in G$ and $A \subseteq X$;
- **inversely invariant** if $\mu(A^{-1}) = \mu(A)$ for all $A \subseteq X$;
- **auto invariant** if $\mu(h(A)) = \mu(A)$ for any automorphism $h : G \to G$ and any subset $A \subseteq X$. 

Taras Banakh

Solecki submeasures and densities on groups
A density $\mu : \mathcal{P}(G) \to [0, 1]$ on a group $G$ is called

- **left invariant** if $\mu(xA) = \mu(A)$ for all $x \in G$ and $A \subset G$;
- **right invariant** if $\mu(Ay) = \mu(A)$ for all $y \in G$ and $A \subset G$;
- **invariant** if $\mu(xAy) = \mu(A)$ for all $x, y \in G$ and $A \subset X$;
- **inversely invariant** if $\mu(A^{-1}) = \mu(A)$ for all $A \subset X$;
- **auto invariant** if $\mu(h(A)) = \mu(A)$ for any automorphism $h : G \to G$ and any subset $A \subset X$. 
Definition

A density $\mu : \mathcal{P}(G) \to [0, 1]$ on a group $G$ is called

- **left invariant** if $\mu(xA) = \mu(A)$ for all $x \in G$ and $A \subset G$;
- **right invariant** if $\mu(Ay) = \mu(A)$ for all $y \in G$ and $A \subset G$;
- **invariant** if $\mu(xAy) = \mu(A)$ for all $x, y \in G$ and $A \subset X$;
- **inversely invariant** if $\mu(A^{-1}) = \mu(A)$ for all $A \subset X$;
- **auto invariant** if $\mu(h(A)) = \mu(A)$ for any automorphism $h : G \to G$ and any subset $A \subset X$. 
A density $\mu : \mathcal{P}(G) \to [0, 1]$ on a group $G$ is called

- **left invariant** if $\mu(xA) = \mu(A)$ for all $x \in G$ and $A \subset G$;
- **right invariant** if $\mu(Ay) = \mu(A)$ for all $y \in G$ and $A \subset G$;
- **invariant** if $\mu(xAy) = \mu(A)$ for all $x, y \in G$ and $A \subset X$;
- **inversely invariant** if $\mu(A^{-1}) = \mu(A)$ for all $A \subset X$;
- **auto invariant** if $\mu(h(A)) = \mu(A)$ for any automorphism $h : G \to G$ and any subset $A \subset X$. 
Invariant densities on groups

**Definition**

A density $\mu : \mathcal{P}(G) \to [0, 1]$ on a group $G$ is called

- **left invariant** if $\mu(xA) = \mu(A)$ for all $x \in G$ and $A \subset G$;
- **right invariant** if $\mu(Ay) = \mu(A)$ for all $y \in G$ and $A \subset G$;
- **invariant** if $\mu(xAy) = \mu(A)$ for all $x, y \in G$ and $A \subset X$;
- **inversely invariant** if $\mu(A^{-1}) = \mu(A)$ for all $A \subset X$;
- **auto invariant** if $\mu(h(A)) = \mu(A)$ for any automorphism $h : G \to G$ and any subset $A \subset X$. 

Taras Banakh

Solecki submeasures and densities on groups
A density $\mu : \mathcal{P}(G) \to [0, 1]$ on a group $G$ is called

- **left invariant** if $\mu(xA) = \mu(A)$ for all $x \in G$ and $A \subset G$;
- **right invariant** if $\mu(Ay) = \mu(A)$ for all $y \in G$ and $A \subset G$;
- **invariant** if $\mu(xAy) = \mu(A)$ for all $x, y \in G$ and $A \subset X$;
- **inversely invariant** if $\mu(A^{-1}) = \mu(A)$ for all $A \subset X$;
- **auto invariant** if $\mu(h(A)) = \mu(A)$ for any automorphism $h : G \to G$ and any subset $A \subset X$. 
Theorem (Haar, 1933)

Each compact topological group possesses a unique invariant probability $\sigma$-additive regular Borel measure $\lambda : \mathcal{B}(G) \to [0, 1]$ defined on the $\sigma$-algebra of Borel subsets of $G$.

The uniqueness of $\lambda$ implies that it is inversely and autoinvariant.

Problem

What about discrete groups? Do they have any canonical (sub)measures?
Theorem (Haar, 1933)

Each compact topological group possesses a unique invariant probability σ-additive regular Borel measure \( \lambda : \mathcal{B}(G) \to [0, 1] \) defined on the σ-algebra of Borel subsets of \( G \).

The uniqueness of \( \lambda \) implies that it is inversely and autoinvariant.

Problem

What about discrete groups? Do they have any canonical (sub)measures?
Theorem (Haar, 1933)

Each compact topological group possesses a unique invariant probability $\sigma$-additive regular Borel measure $\lambda : \mathcal{B}(G) \to [0, 1]$ defined on the $\sigma$-algebra of Borel subsets of $G$.

The uniqueness of $\lambda$ implies that it is inversely and autoinvariant.

Problem

What about discrete groups? Do they have any canonical (sub)measures?
Theorem (Banach, 1923)

*There exists an invariant measure on the group of integers $\mathbb{Z}$.*

Definition (von Neuman, 1929; Day, 1949)

A group $G$ is called amenable if it admits a left-invariant measure $\mu : \mathcal{P}(G) \to [0, 1]$.

Fact (Classics)

- Each abelian group is amenable;
- A non-commutative free group is not amenable.
Theorem (Banach, 1923)

There exists an invariant measure on the group of integers $\mathbb{Z}$.

Definition (von Neuman, 1929; Day, 1949)

A group $G$ is called amenable if it admits a left-invariant measure $\mu : \mathcal{P}(G) \to [0, 1]$.

Fact (Classics)

- Each abelian group is amenable;
- A non-commutative free group is not amenable.
Theorem (Banach, 1923)

There exists an invariant measure on the group of integers \( \mathbb{Z} \).

Definition (von Neuman, 1929; Day, 1949)

A group \( G \) is called **amenable** if it admits a left-invariant measure \( \mu : \mathcal{P}(G) \to [0, 1] \).

Fact (Classics)

- Each abelian group is amenable;
- A non-commutative free group is not amenable.
Theorem (Banach, 1923)

There exists an invariant measure on the group of integers \( \mathbb{Z} \).

Definition (von Neuman, 1929; Day, 1949)

A group \( G \) is called amenable if it admits a left-invariant measure \( \mu : \mathcal{P}(G) \to [0, 1] \).

Fact (Classics)

- Each abelian group is amenable;
- A non-commutative free group is not amenable.
Conclusion: There are groups admitting no invariant measure :( 

Problem
What about invariant submeasures?
Do they always exist on any group?

Yes! \( \mu(A) = \begin{cases} 0 & \text{if } A = \emptyset \\ 1 & \text{otherwise} \end{cases} \) But this is trivial :( 

Problem
Are there any canonical non-trivial and useful invariant submeasure on a group?

Yes!! 😊
Conclusion: There are groups admitting no invariant measure :( 

Problem: What about invariant submeasures?
Do they always exist on any group?

Yes!

\[ \mu(A) = \begin{cases} 0 & \text{if } A = \emptyset \\ 1 & \text{otherwise} \end{cases} \]

But this is trivial :( 

Problem: Are there any canonical non-trivial and useful invariant submeasure on a group?

Yes!! 😊
**Conclusion:** There are groups admitting no invariant measure :(

**Problem**

What about invariant submeasures?  
*Do they always exist on any group?*

Yes!  
\[ \mu(A) = \begin{cases} 0 & \text{if } A = \emptyset \\ 1 & \text{otherwise} \end{cases} \]

But this is trivial :(

**Problem**

Are there any canonical non-trivial and useful invariant submeasure on a group?  

Yes!! 😊
Conclusion: There are groups admitting no invariant measure :(  

Problem

What about invariant submeasures?  
Do they always exist on any group?

Yes! \[ \mu(A) = \begin{cases} 0 & \text{if } A = \emptyset \\ 1 & \text{otherwise} \end{cases} \]  
But this is trivial :(  

Problem

Are there any canonical non-trivial and useful invariant submeasure on a group?  

Yes!! 😊
Conclusion: There are groups admitting no invariant measure :( 

Problem

What about invariant submeasures? Do they always exist on any group?

Yes! \( \mu(A) = \begin{cases} 0 & \text{if } A = \emptyset \\ 1 & \text{otherwise} \end{cases} \)

But this is trivial :(

Problem

Are there any canonical non-trivial and useful invariant submeasure on a group?

Yes!! 😊
Conclusion: There are groups admitting no invariant measure :(  

Problem  

What about invariant submeasures?  
Do they always exist on any group?  

Yes!  \[ \mu(A) = \begin{cases} 0 & \text{if } A = \emptyset \\ 1 & \text{otherwise} \end{cases} \] But this is trivial :(  

Problem  

Are there any canonical non-trivial and useful invariant submeasure on a group?  

Yes!! 😊  

Taras Banakh  
Solecki submeasures and densities on groups
Conclusion: There are groups admitting no invariant measure :(  

Problem

What about invariant submeasures?  
Do they always exist on any group?  

Yes! \( \mu(A) = \begin{cases} 0 & \text{if } A = \emptyset \\ 1 & \text{otherwise} \end{cases} \)  
But this is trivial :(  

Problem

Are there any canonical non-trivial and useful invariant submeasure on a group?  

Yes!! 😊
Each group $G$ possesses a canonical invariant submeasure $\sigma : \mathcal{P}(G) \to [0, 1]$ defined by

$$\sigma(A) = \inf_{F \in [G]^{<\omega}} \max_{x, y \in G} \frac{|F \cap xAy|}{|F|}.$$ 

This submeasure is inversely and auto invariant. The submeasure $\sigma$ was thoroughly studied by Solecki and because of that we decided to name it the Solecki submeasure.

**Example**

The subset $A = 2\mathbb{Z}$ in $\mathbb{Z}$ has Solecki submeasure $\sigma(A) = \frac{1}{2}$. 
Each group $G$ possesses a canonical invariant submeasure $\sigma : \mathcal{P}(G) \to [0, 1]$ defined by

$$\sigma(A) = \inf_{F \in [G]^{<\omega}} \max_{x, y \in G} \frac{|F \cap xAy|}{|F|}.$$  

This submeasure is inversely and auto invariant.

The submeasure $\sigma$ was thoroughly studied by Solecki and because of that we decided to name it the Solecki submeasure.

**Example**

The subset $A = 2\mathbb{Z}$ in $\mathbb{Z}$ has Solecki submeasure $\sigma(A) = \frac{1}{2}$.
Each group $G$ possesses a canonical invariant submeasure $\sigma : \mathcal{P}(G) \to [0, 1]$ defined by

$$\sigma(A) = \inf_{F \in [G]^{<\omega}, x, y \in G} \max \frac{|F \cap xAy|}{|F|}.$$ 

This submeasure is inversely and auto invariant. The submeasure $\sigma$ was thoroughly studied by Solecki and because of that we decided to name it the Solecki submeasure.

**Example**

The subset $A = 2\mathbb{Z}$ in $\mathbb{Z}$ has Solecki submeasure $\sigma(A) = \frac{1}{2}$. 
Each group $G$ possesses a canonical invariant submeasure $\sigma : \mathcal{P}(G) \to [0, 1]$ defined by

$$\sigma(A) = \inf_{F \in \mathcal{P}(G) < \omega} \max_{x,y \in G} \frac{|F \cap xAy|}{|F|}.$$ 

This submeasure is inversely and auto invariant.
The submeasure $\sigma$ was thoroughly studied by Solecki and because of that we decided to name it the **Solecki submeasure**.

**Example**
The subset $A = 2\mathbb{Z}$ in $\mathbb{Z}$ has Solecki submeasure $\sigma(A) = \frac{1}{2}$. 
The Solecki submeasure can be alternatively defined using finitely supported measures on $G$ instead of finite subsets of $G$.

A measure $\mu$ on a set $X$ is finitely supported if $\mu(F) = 1$ for some finite subset $F$. In this case it can be written as the convex combination $\mu = \sum_{i=1}^{n} \alpha_i \delta_{x_i}$ of Dirac measures.

By $P(X)$ we denote the set of all measures on a set $X$ and by $P_\omega(X)$ its subset consisting of finitely supported measures on $X$.

**Theorem (Solecki, 2005)**

Any subset $A$ of a group $G$ has Solecki submeasure

$$\sigma(A) = \inf_{\mu \in P_\omega(G)} \sup_{x,y \in G} \mu(xAy).$$

This theorem implies that $\sigma$ is subadditive.
The Solecki submeasure can be alternatively defined using finitely supported measures on $G$ instead of finite subsets of $G$.

A measure $\mu$ on a set $X$ is **finitely supported** if $\mu(F) = 1$ for some finite subset $F$. In this case it can be written as the convex combination $\mu = \sum_{i=1}^{n} \alpha_i \delta_{x_i}$ of Dirac measures.

By $P(X)$ we denote the set of all measures on a set $X$ and by $P_\omega(X)$ its subset consisting of finitely supported measures on $X$.

**Theorem (Solecki, 2005)**

Any subset $A$ of a group $G$ has Solecki submeasure

$$\sigma(A) = \inf_{\mu \in P_\omega(G)} \sup_{x,y \in G} \mu(xAy).$$

This theorem implies that $\sigma$ is subadditive.
The Solecki submeasure can be alternatively defined using finitely supported measures on $G$ instead of finite subsets of $G$.

A measure $\mu$ on a set $X$ is **finitely supported** if $\mu(F) = 1$ for some finite subset $F$. In this case it can be written as the convex combination $\mu = \sum_{i=1}^{n} \alpha_i \delta_{x_i}$ of Dirac measures.

By $P(X)$ we denote the set of all measures on a set $X$ and by $P_{\omega}(X)$ its subset consisting of finitely supported measures on $X$.

**Theorem (Solecki, 2005)**

Any subset $A$ of a group $G$ has Solecki submeasure

$$\sigma(A) = \inf_{\mu \in P_{\omega}(G)} \sup_{x,y \in G} \mu(xAy).$$

This theorem implies that $\sigma$ is subadditive.
The Solecki submeasure can be alternatively defined using finitely supported measures on $G$ instead of finite subsets of $G$.

A measure $\mu$ on a set $X$ is **finitely supported** if $\mu(F) = 1$ for some finite subset $F$. In this case it can be written as the convex combination $\mu = \sum_{i=1}^{n} \alpha_i \delta_{x_i}$ of Dirac measures.

By $P(X)$ we denote the set of all measures on a set $X$ and by $P_\omega(X)$ its subset consisting of finitely supported measures on $X$.

**Theorem (Solecki, 2005)**

*Any subset $A$ of a group $G$ has Solecki submeasure*

$$\sigma(A) = \inf_{\mu \in P_\omega(G)} \sup_{x,y \in G} \mu(x Ay).$$

This theorem implies that $\sigma$ is subadditive.
The Solecki submeasure can be alternatively defined using finitely supported measures on $G$ instead of finite subsets of $G$.

A measure $\mu$ on a set $X$ is **finitely supported** if $\mu(F) = 1$ for some finite subset $F$. In this case it can be written as the convex combination $\mu = \sum_{i=1}^{n} \alpha_i \delta_{x_i}$ of Dirac measures.

By $P(X)$ we denote the set of all measures on a set $X$ and by $P_\omega(X)$ its subset consisting of finitely supported measures on $X$.

**Theorem (Solecki, 2005)**

Any subset $A$ of a group $G$ has Solecki submeasure

$$\sigma(A) = \inf_{\mu \in P_\omega(G)} \sup_{x,y \in G} \mu(xAy).$$

This theorem implies that $\sigma$ is subadditive.
The Solecki submeasure can be alternatively defined using finitely supported measures on $G$ instead of finite subsets of $G$.

A measure $\mu$ on a set $X$ is **finitely supported** if $\mu(F) = 1$ for some finite subset $F$. In this case it can be written as the convex combination $\mu = \sum_{i=1}^{n} \alpha_i \delta_{x_i}$ of Dirac measures. By $P(X)$ we denote the set of all measures on a set $X$ and by $P_\omega(X)$ its subset consisting of finitely supported measures on $X$.

**Theorem (Solecki, 2005)**

Any subset $A$ of a group $G$ has Solecki submeasure

$$\sigma(A) = \inf_{\mu \in P_\omega(G)} \sup_{x,y \in G} \mu(xAy).$$

This theorem implies that $\sigma$ is subadditive.
Given any subsets $A, B \subset G$ we need to prove that

$$\sigma(A \cup B) \leq \sigma(A) + \sigma(B) + 2\varepsilon$$

for every $\varepsilon > 0$. Using the equivalent definition of the Solecki submeasures, find two finitely supported probability measures $\mu_A, \mu_B \in P_\omega(G)$ such that

$$\max_{x, y \in G} \mu_A(xAy) < \sigma(A) + \varepsilon \quad \text{and} \quad \max_{x, y \in G} \mu_B(xBy) < \sigma(B) + \varepsilon.$$ 

Write $\mu_A = \sum_i \alpha_i \delta_{a_i}$ and $\mu_B = \sum_j \beta_j \delta_{b_j}$ and consider the convolution measure

$$\mu = \mu_A * \mu_B = \sum_{i,j} \alpha_i \beta_j \delta_{a_i b_j}.$$
The subadditivity of the Solecki submeasure

Given any subsets $A, B \subset G$ we need to prove that

$$\sigma(A \cup B) \leq \sigma(A) + \sigma(B) + 2\varepsilon$$

for every $\varepsilon > 0$. Using the equivalent definition of the Solecki submeasures, find two finitely supported probability measures $\mu_A, \mu_B \in P_\omega(G)$ such that

$$\max_{x, y \in G} \mu_A(xAy) < \sigma(A) + \varepsilon \quad \text{and} \quad \max_{x, y \in G} \mu_B(xBy) < \sigma(B) + \varepsilon.$$

Write $\mu_A = \sum_i \alpha_i \delta_{a_i}$ and $\mu_B = \sum_j \beta_j \delta_{b_j}$ and consider the convolution measure

$$\mu = \mu_A * \mu_B = \sum_{i,j} \alpha_i \beta_j \delta_{a_i b_j}.$$
The subadditivity of the Solecki submeasure

Given any subsets $A, B \subset G$ we need to prove that

$$\sigma(A \cup B) \leq \sigma(A) + \sigma(B) + 2\varepsilon$$

for every $\varepsilon > 0$. Using the equivalent definition of the Solecki submeasures, find two finitely supported probability measures $\mu_A, \mu_B \in P_\omega(G)$ such that

$$\max_{x,y \in G} \mu_A(xAy) < \sigma(A) + \varepsilon \quad \text{and} \quad \max_{x,y \in G} \mu_B(xBy) < \sigma(B) + \varepsilon.$$

Write $\mu_A = \sum_i \alpha_i \delta_{a_i}$ and $\mu_B = \sum_j \beta_j \delta_{b_j}$ and consider the convolution measure

$$\mu = \mu_A * \mu_B = \sum_{i,j} \alpha_i \beta_j \delta_{a_i b_j}.$$
Given any subsets $A, B \subset G$ we need to prove that

$$\sigma(A \cup B) \leq \sigma(A) + \sigma(B) + 2\varepsilon$$

for every $\varepsilon > 0$. Using the equivalent definition of the Solecki submeasures, find two finitely supported probability measures $\mu_A, \mu_B \in P_\omega(G)$ such that

$$\max_{x,y \in G} \mu_A(xAy) < \sigma(A) + \varepsilon \quad \text{and} \quad \max_{x,y \in G} \mu_B(xBy) < \sigma(B) + \varepsilon.$$ 

Write $\mu_A = \sum_i \alpha_i \delta_{a_i}$ and $\mu_B = \sum_j \beta_j \delta_{b_j}$ and consider the convolution measure

$$\mu = \mu_A * \mu_B = \sum_{i,j} \alpha_i \beta_j \delta_{a_i b_j}.$$
Observe that for any \( x, y \in G \)

\[
\mu(xAy) = \sum_{i,j} \alpha_i \beta_j \delta_{a_i b_j}(xAy) = \sum_j \beta_j \sum_i \alpha_i \delta_{a_i}(xAyb_j^{-1}) = \\
= \sum_j \beta_j \mu_A(xAyb_j) < \sum_j \beta_j (\sigma(A) + \varepsilon) = \sigma(A) + \varepsilon
\]

and

\[
\mu(xBy) = \sum_{i,j} \alpha_i \beta_j \delta_{a_i b_j}(xBy) = \sum_i \alpha_i \sum_j \beta_j \delta_{b_j}(a_i^{-1}xBy) = \\
= \sum_i \alpha_i \mu_B(a_i^{-1}xBy) < \sum_i \alpha_i (\sigma(B) + \varepsilon) = \sigma(B) + \varepsilon.
\]

Consequently,

\[
\mu(x(A \cup B)y) \leq \mu(xAy) + \mu(xBy) < \sigma(A) + \sigma(B) + 2\varepsilon
\]

and

\[
\sigma(A \cup B) \leq \sup_{x,y \in G} \mu(x(A \cup B)y)) \leq \sigma(A) + \sigma(B) + 2\varepsilon.
\]
Observe that for any $x, y \in G$

$$
\mu(xAy) = \sum_{i,j} \alpha_i \beta_j \delta_{a_i b_j}(xAy) = \sum_j \beta_j \sum_i \alpha_i \delta_{a_i}(xAyb_j^{-1}) = \\
= \sum_j \beta_j \mu_A(xAyb_j) < \sum_j \beta_j(\sigma(A) + \varepsilon) = \sigma(A) + \varepsilon
$$

and

$$
\mu(xBy) = \sum_{i,j} \alpha_i \beta_j \delta_{a_i b_j}(xBy) = \sum_i \alpha_i \sum_j \beta_j \delta_{b_j}(a_i^{-1} xBy) = \\
= \sum_i \alpha_i \mu_B(a_i^{-1} xBy) < \sum_i \alpha_i(\sigma(B) + \varepsilon) = \sigma(B) + \varepsilon.
$$

Consequently,

$$
\mu(x(A \cup B)y) \leq \mu(xAy) + \mu(xBy) < \sigma(A) + \sigma(B) + 2\varepsilon
$$

and

$$
\sigma(A \cup B) \leq \sup_{x, y \in G} \mu(x(A \cup B)y)) \leq \sigma(A) + \sigma(B) + 2\varepsilon.
$$
Observe that for any \( x, y \in G \)

\[
\mu(xAy) = \sum_{i,j} \alpha_i \beta_j \delta_{a_i b_j}(xAy) = \sum_j \beta_j \sum_i \alpha_i \delta_{a_i}(xAyb_j^{-1}) =
\]

\[
= \sum_j \beta_j \mu_A(xAyb_j) < \sum_j \beta_j(\sigma(A) + \varepsilon) = \sigma(A) + \varepsilon
\]

and

\[
\mu(xBy) = \sum_{i,j} \alpha_i \beta_j \delta_{a_i b_j}(xBy) = \sum_i \alpha_i \sum_j \beta_j \delta_{b_j}(a_i^{-1}xBy) =
\]

\[
= \sum_i \alpha_i \mu_B(a_i^{-1}xBy) < \sum_i \alpha_i(\sigma(B) + \varepsilon) = \sigma(B) + \varepsilon.
\]

Consequently,

\[
\mu(x(A \cup B)y) \leq \mu(xAy) + \mu(xBy) < \sigma(A) + \sigma(B) + 2\varepsilon
\]

and

\[
\sigma(A \cup B) \leq \sup_{x,y \in G} \mu(x(A \cup B)y)) \leq \sigma(A) + \sigma(B) + 2\varepsilon.
\]
Observe that for any $x, y \in G$

$$\mu(xAy) = \sum_{i,j} \alpha_i \beta_j \delta_{a_i b_j}(xAy) = \sum_j \beta_j \sum_i \alpha_i \delta_{a_i}(xAyb_j^{-1}) =$$

$$= \sum_j \beta_j \mu_A(xAyb_j) < \sum_j \beta_j (\sigma(A) + \varepsilon) = \sigma(A) + \varepsilon$$

and

$$\mu(xBy) = \sum_{i,j} \alpha_i \beta_j \delta_{a_i b_j}(xBy) = \sum_i \alpha_i \sum_j \beta_j \delta_{b_j}(a_i^{-1}xBy) =$$

$$= \sum_i \alpha_i \mu_B(a_i^{-1}xBy) < \sum_i \alpha_i (\sigma(B) + \varepsilon) = \sigma(B) + \varepsilon.$$ 

Consequently,

$$\mu(x(A \cup B)y) \leq \mu(xAy) + \mu(xBy) < \sigma(A) + \sigma(B) + 2\varepsilon$$

and

$$\sigma(A \cup B) \leq \sup_{x,y \in G} \mu(x(A \cup B)y)) \leq \sigma(A) + \sigma(B) + 2\varepsilon.$$
The Solecki submeasure has natural left and right modifications called the left and right Solecki densities:

\[
\sigma^L(A) = \inf_{F \in [G]<\omega} \max_{x \in G} \frac{|F \cap xA|}{|F|}, \quad \sigma^R(A) = \inf_{F \in [G]<\omega} \max_{y \in G} \frac{|F \cap Ay|}{|F|}
\]

\[
\sigma_L(A) = \inf_{\mu \in P_\omega(G)} \max_{x \in X} \mu(xA), \quad \sigma_R(A) = \inf_{\mu \in P_\omega(G)} \max_{y \in X} \mu(Ay)
\]

It is clear that \( \sigma_L \leq \sigma^L \leq \sigma \geq \sigma^R \geq \sigma_R \).

If the group \( G \) is abelian, then \( \sigma_L = \sigma^L = \sigma = \sigma^R = \sigma_R \).

The densities \( \sigma_L, \sigma^L, \sigma_R, \sigma^R \) are (auto) invariant but not inversely invariant in general. However

\[
\sigma_L(A^{-1}) = \sigma_R(A) \quad \text{and} \quad \sigma^L(A^{-1}) = \sigma^R(A).
\]
The Solecki submeasure has natural left and right modifications called the left and right Solecki densities:

\[
\sigma_L(A) = \inf_{F \in [G]^{< \omega}} \max_{x \in G} \frac{|F \cap xA|}{|F|} \quad \sigma_R(A) = \inf_{F \in [G]^{< \omega}} \max_{y \in G} \frac{|F \cap Ay|}{|F|}
\]

\[
\sigma_L(A) = \inf_{\mu \in P_\omega(G)} \max_{x \in X} \mu(xA) \quad \sigma_R(A) = \inf_{\mu \in P_\omega(G)} \max_{y \in X} \mu(Ay)
\]

It is clear that \(\sigma_L \leq \sigma^L \leq \sigma \geq \sigma^R \geq \sigma_R\).

If the group \(G\) is abelian, then \(\sigma_L = \sigma^L = \sigma = \sigma^R = \sigma_R\).

The densities \(\sigma_L, \sigma^L, \sigma_R, \sigma^R\) are (auto) invariant but not inversely invariant in general. However

\[
\sigma_L(A^{-1}) = \sigma_R(A) \quad \text{and} \quad \sigma^L(A^{-1}) = \sigma^R(A).
\]
The Solecki submeasure has natural left and right modifications called the left and right Solecki densities:

\[
\sigma^L(A) = \inf_{F \in [G] < \omega} \max_{x \in G} \frac{|F \cap xA|}{|F|} \quad \sigma^R(A) = \inf_{F \in [G] < \omega} \max_{y \in G} \frac{|F \cap Ay|}{|F|}
\]
\[
\sigma_L(A) = \inf_{\mu \in P_\omega(G)} \max_{x \in X} \mu(xA) \quad \sigma_R(A) = \inf_{\mu \in P_\omega(G)} \max_{y \in X} \mu(Ay)
\]

It is clear that \( \sigma_L \leq \sigma^L \leq \sigma \geq \sigma^R \geq \sigma_R \).

If the group \( G \) is abelian, then \( \sigma_L = \sigma^L = \sigma = \sigma^R = \sigma_R \).

The densities \( \sigma_L, \sigma^L, \sigma_R, \sigma^R \) are (auto) invariant but not inversely invariant in general. However

\[
\sigma_L(A^{-1}) = \sigma_R(A) \quad \text{and} \quad \sigma^L(A^{-1}) = \sigma^R(A).
\]
The Solecki submeasure has natural left and right modifications called the left and right Solecki densities:

\[
\sigma^L(A) = \inf_{F \in [G]_{< \omega}} \max_{x \in G} \frac{|F \cap xA|}{|F|} \quad \quad \sigma^R(A) = \inf_{F \in [G]_{< \omega}} \max_{y \in G} \frac{|F \cap Ay|}{|F|}
\]

\[
\sigma_L(A) = \inf_{\mu \in P_\omega(G)} \max_{x \in X} \mu(xA) \quad \quad \sigma_R(A) = \inf_{\mu \in P_\omega(G)} \max_{y \in X} \mu(Ay)
\]

It is clear that \( \sigma_L \leq \sigma^L \leq \sigma \geq \sigma^R \geq \sigma_R \).

If the group \( G \) is abelian, then \( \sigma_L = \sigma^L = \sigma = \sigma^R = \sigma_R \).

The densities \( \sigma_L, \sigma^L, \sigma_R, \sigma^R \) are (auto) invariant but not inversely invariant in general. However

\[
\sigma_L(A^{-1}) = \sigma_R(A) \quad \text{and} \quad \sigma^L(A^{-1}) = \sigma^R(A).
\]
The Solecki submeasure has natural left and right modifications called the left and right Solecki densities:

\[
\sigma^L(A) = \inf_{F \in [G]^{< \omega}} \max_{x \in G} \frac{|F \cap xA|}{|F|} \quad \sigma^R(A) = \inf_{F \in [G]^{< \omega}} \max_{y \in G} \frac{|F \cap Ay|}{|F|}
\]

\[
\sigma_L(A) = \inf_{\mu \in P_\omega(G)} \max_{x \in X} \mu(xA) \quad \sigma_R(A) = \inf_{\mu \in P_\omega(G)} \max_{y \in X} \mu(Ay)
\]

It is clear that \(\sigma_L \leq \sigma^L \leq \sigma \geq \sigma^R \geq \sigma_R\).

If the group \(G\) is abelian, then \(\sigma_L = \sigma^L = \sigma = \sigma^R = \sigma_R\).

The densities \(\sigma_L, \sigma^L, \sigma_R, \sigma^R\) are (auto) invariant but not inversely invariant in general. However

\[
\sigma_L(A^{-1}) = \sigma_R(A) \quad \text{and} \quad \sigma^L(A^{-1}) = \sigma^R(A).
\]
The Solecki submeasure has natural left and right modifications called the left and right Solecki densities:

\[
\sigma_L(A) = \inf_{F \in [G] < \omega} \max_{x \in G} \frac{|F \cap xA|}{|F|}, \quad \sigma_R(A) = \inf_{F \in [G] < \omega} \max_{y \in G} \frac{|F \cap Ay|}{|F|},
\]

\[
\sigma_L(A) = \inf_{\mu \in P_\omega(G)} \max_{x \in X} \mu(xA), \quad \sigma_R(A) = \inf_{\mu \in P_\omega(G)} \max_{y \in X} \mu(Ay).
\]

It is clear that \(\sigma_L \leq \sigma^L \leq \sigma \geq \sigma^R \geq \sigma_R\).

If the group \(G\) is abelian, then \(\sigma_L = \sigma^L = \sigma = \sigma^R = \sigma_R\).

The densities \(\sigma_L, \sigma^L, \sigma_R, \sigma^R\) are (auto) invariant but not inversely invariant in general. However

\[
\sigma_L(A^{-1}) = \sigma_R(A) \quad \text{and} \quad \sigma^L(A^{-1}) = \sigma^R(A).
\]
The Solecki submeasure has natural left and right modifications called the left and right Solecki densities:

\[
\sigma^L(A) = \inf_{F \in [G]^{< \omega}} \max_{x \in G} \frac{|F \cap xA|}{|F|} \quad \sigma^R(A) = \inf_{F \in [G]^{< \omega}} \max_{y \in G} \frac{|F \cap Ay|}{|F|}
\]

\[
\sigma_L(A) = \inf_{\mu \in P_\omega(G)} \max_{x \in X} \mu(xA) \quad \sigma_R(A) = \inf_{\mu \in P_\omega(G)} \max_{y \in X} \mu(Ay)
\]

It is clear that \( \sigma_L \leq \sigma^L \leq \sigma \geq \sigma^R \geq \sigma_R \).

If the group \( G \) is abelian, then \( \sigma_L = \sigma^L = \sigma = \sigma^R = \sigma_R \).

The densities \( \sigma_L, \sigma^L, \sigma_R, \sigma^R \) are (auto) invariant but not inversely invariant in general. However

\[
\sigma_L(A^{-1}) = \sigma_R(A) \quad \text{and} \quad \sigma^L(A^{-1}) = \sigma^R(A).
\]
A group $G$ is called an **FC-group** if each $x \in G$ has finite conjugacy class $x^G = \{g x g^{-1} : g \in G\}$.

Abelian group $\implies$ FC-group $\implies$ Amenable group

**Theorem (Solecki, 2005)**

1. A group $G$ is an FC-group if and only if $\sigma_L = \sigma^L = \sigma = \sigma^R = \sigma_R$.
2. If $G$ is an amenable group, then $\sigma_L = \sigma^L$ and $\sigma_R = \sigma^R$ are subadditive.
3. If $G = F_2$ is a free group, then $\sigma_L \neq \sigma^L$ and $\sigma_R \neq \sigma^R$ and the densities $\sigma_L, \sigma^L, \sigma^R, \sigma_R$ are not subadditive.
A group $G$ is called an **FC-group** if each $x \in G$ has finite conjugacy class $x^G = \{gxg^{-1} : g \in G\}$.

A group $G$ is an FC-group if and only if $\sigma_L = \sigma^L = \sigma = \sigma^R = \sigma_R$.

If $G$ is an amenable group, then $\sigma_L = \sigma^L$ and $\sigma_R = \sigma^R$ are subadditive.

If $G = F_2$ is a free group, then $\sigma_L \neq \sigma^L$ and $\sigma_R \neq \sigma^R$ and the densities $\sigma_L, \sigma^L, \sigma^R, \sigma_R$ are not subadditive.
A group $G$ is called an **FC-group** if each $x \in G$ has finite conjugacy class $x^G = \{gxg^{-1} : g \in G\}$.

abelian group $\Rightarrow$ FC-group $\Rightarrow$ amenable group

**Theorem (Solecki, 2005)**

1. A group $G$ is an FC-group if and only if $\sigma_L = \sigma^L = \sigma = \sigma^R = \sigma_R$.

2. If $G$ is an amenable group, then $\sigma_L = \sigma^L$ and $\sigma_R = \sigma^R$ are subadditive.

3. If $G = F_2$ is a free group, then $\sigma_L \neq \sigma^L$ and $\sigma_R \neq \sigma^R$ and the densities $\sigma_L, \sigma^L, \sigma^R, \sigma_R$ are not subadditive.
A group $G$ is called an **FC-group** if each $x \in G$ has finite conjugacy class $x^G = \{gxg^{-1} : g \in G\}$.

abelian group $\Rightarrow$ FC-group $\Rightarrow$ amenable group

**Theorem (Solecki, 2005)**

1. A group $G$ is an FC-group if and only if $\sigma_L = \sigma^L = \sigma = \sigma^R = \sigma_R$.

2. If $G$ is an amenable group, then $\sigma_L = \sigma^L$ and $\sigma_R = \sigma^R$ are subadditive.

3. If $G = F_2$ is a free group, then $\sigma_L \neq \sigma^L$ and $\sigma_R \neq \sigma^R$ and the densities $\sigma_L, \sigma^L, \sigma^R, \sigma_R$ are not subadditive.
A group $G$ is called an **FC-group** if each $x \in G$ has finite conjugacy class $x^G = \{gxg^{-1} : g \in G\}$.

Abelian group $\Rightarrow$ FC-group $\Rightarrow$ Amenable group

**Theorem (Solecki, 2005)**

1. A group $G$ is an FC-group if and only if $\sigma_L = \sigma^L = \sigma = \sigma^R = \sigma_R$.
2. If $G$ is an amenable group, then $\sigma_L = \sigma^L$ and $\sigma_R = \sigma^R$ are subadditive.
3. If $G = F_2$ is a free group, then $\sigma_L \neq \sigma^L$ and $\sigma_R \neq \sigma^R$ and the densities $\sigma_L, \sigma^L, \sigma^R, \sigma_R$ are not subadditive.
A group $G$ is called an **FC-group** if each $x \in G$ has finite conjugacy class $x^G = \{gxg^{-1} : g \in G\}$.

**Theorem (Solecki, 2005)**

1. A group $G$ is an FC-group if and only if $\sigma_L = \sigma^L = \sigma = \sigma^R = \sigma_R$.
2. If $G$ is an amenable group, then $\sigma_L = \sigma^L$ and $\sigma_R = \sigma^R$ are subadditive.
3. If $G = \mathbb{F}_2$ is a free group, then $\sigma_L \neq \sigma^L$ and $\sigma_R \neq \sigma^R$ and the densities $\sigma_L, \sigma^L, \sigma^R, \sigma_R$ are not subadditive.
In the free group $F_2 = \langle a, b \rangle$ consider the set $A$ of irreducible words that start with $a$ or $a^{-1}$.

The set $A$ has right Solecki density $\sigma^R(A) = 0$ since for every set $F = \{b, b^2, \ldots, b^n\}$, $n \in \mathbb{N}$, we get $\sup_{y \in G} |F \cap Ay| \leq 1$ which implies $\sigma^R(A) \leq \sup_{y \in G} \frac{|F \cap Ay|}{|F|} \leq \frac{1}{n}$. By analogy we can prove that $\sigma^R(A) = 0$.

Then $\sigma^L(A^{-1}) = \sigma^R(A) = 0$ and $\sigma^L(B^{-1}) = \sigma^R(B) = 0$ and

$$F_2 = (A \cap A^{-1}) \cup (A \cap B^{-1}) \cup (B \cap A^{-1}) \cup (B \cap B^{-1})$$

is the union of four sets whose left and right Solecki densities are zero.
The Solecki densities are not subadditive on the free group $F_2$

In the free group $F_2 = \langle a, b \rangle$ consider the set $A$ of irreducible words that start with $a$ or $a^{-1}$.

The set $A$ has right Solecki density $\sigma^R(A) = 0$ since for every set $F = \{b, b^2, \ldots, b^n\}$, $n \in \mathbb{N}$, we get $\sup_{y \in G} |F \cap Ay| \leq 1$ which implies $\sigma^R(A) \leq \sup_{y \in G} \frac{|F \cap Ay|}{|F|} \leq \frac{1}{n}$. By analogy we can prove that $\sigma^R(A) = 0$.

Then $\sigma^L(A^{-1}) = \sigma^R(A) = 0$ and $\sigma^L(B^{-1}) = \sigma^R(B) = 0$ and

$$F_2 = (A \cap A^{-1}) \cup (A \cap B^{-1}) \cup (B \cap A^{-1}) \cup (B \cap B^{-1})$$

is the union of four sets whose left and right Solecki densities are zero.
The Solecki densities are not subadditive on the free group $F_2$

In the free group $F_2 = \langle a, b \rangle$ consider the set $A$ of irreducible words that start with $a$ or $a^{-1}$.

The set $A$ has right Solecki density $\sigma^R(A) = 0$ since for every set $F = \{b, b^2, \ldots, b^n\}$, $n \in \mathbb{N}$, we get $\sup_{y \in G} |F \cap Ay| \leq 1$ which implies $\sigma^R(A) \leq \sup_{y \in G} \frac{|F \cap Ay|}{|F|} \leq \frac{1}{n}$. By analogy we can prove that $\sigma^R(A) = 0$.

Then $\sigma^L(A^{-1}) = \sigma^R(A) = 0$ and $\sigma^L(B^{-1}) = \sigma^R(B) = 0$ and

$$F_2 = (A \cap A^{-1}) \cup (A \cap B^{-1}) \cup (B \cap A^{-1}) \cup (B \cap B^{-1})$$

is the union of four sets whose left and right Solecki densities are zero.
In the free group $F_2 = \langle a, b \rangle$ consider the set $A$ of irreducible words that start with $a$ or $a^{-1}$.

The set $A$ has right Solecki density $\sigma^R(A) = 0$ since for every set $F = \{b, b^2, \ldots, b^n\}$, $n \in \mathbb{N}$, we get $\sup_{y \in G} |F \cap Ay| \leq 1$ which implies $\sigma^R(A) \leq \sup_{y \in G} \frac{|F \cap Ay|}{|F|} \leq \frac{1}{n}$. By analogy we can prove that $\sigma^R(A) = 0$.

Then $\sigma^L(A^{-1}) = \sigma^R(A) = 0$ and $\sigma^L(B^{-1}) = \sigma^R(B) = 0$ and

$$F_2 = (A \cap A^{-1}) \cup (A \cap B^{-1}) \cup (B \cap A^{-1}) \cup (B \cap B^{-1})$$

is the union of four sets whose left and right Solecki densities are zero.
In the free group $F_2 = \langle a, b \rangle$ consider the set $A$ of irreducible words that start with $a$ or $a^{-1}$. The set $A$ has right Solecki density $\sigma^R(A) = 0$ since for every set $F = \{b, b^2, \ldots, b^n\}$, $n \in \mathbb{N}$, we get $\sup_{y \in G} |F \cap Ay| \leq 1$ which implies $\sigma^R(A) \leq \sup_{y \in G} \frac{|F \cap Ay|}{|F|} \leq \frac{1}{n}$. By analogy we can prove that $\sigma^R(A) = 0$. Then $\sigma^L(A^{-1}) = \sigma^R(A) = 0$ and $\sigma^L(B^{-1}) = \sigma^R(B) = 0$ and

$$F_2 = (A \cap A^{-1}) \cup (A \cap B^{-1}) \cup (B \cap A^{-1}) \cup (B \cap B^{-1})$$

is the union of four sets whose left and right Solecki densities are zero.
The Kelley intersection number $I(\mathcal{F})$ of a family $\mathcal{F}$ of subsets of a set $X$ is defined as

$$I(\mathcal{F}) = \inf_{F_1, \ldots, F_n \in \mathcal{F}} \sup_{x \in X} \frac{1}{n} \sum_{i=1}^{n} \chi_{F_i}(x).$$

**Theorem (B., 2012)**

For a subset $A$ of a group $G$ we get

$$\inf_{\mu \in P_\omega(G)} \sup_{y \in G} \mu(Ay) = \sigma_R(A) = I(\{xA\}_{x \in G}) = \sup_{\mu \in P(G)} \inf_{x \in G} \mu(xA).$$

Here $P(G)$ stands for the set of all (finitely additive probability) measures on $X$. 

Taras Banakh

Solecki submeasures and densities on groups
The **Kelley intersection number** $I(\mathcal{F})$ of a family $\mathcal{F}$ of subsets of a set $X$ is defined as

$$I(\mathcal{F}) = \inf_{F_1, \ldots, F_n \in \mathcal{F}} \sup_{x \in X} \frac{1}{n} \sum_{i=1}^{n} \chi_{F_i}(x).$$

**Theorem (B., 2012)**

For a subset $A$ of a group $G$ we get

$$\inf_{\mu \in P_\omega(G)} \sup_{y \in G} \mu(Ay) = \sigma_R(A) = I(\{xA\}_{x \in G}) = \sup_{\mu \in P(G)} \inf_{x \in G} \mu(xA).$$

Here $P(G)$ stands for the set of all (finitely additive probability) measures on $X$. 

**Taras Banakh**

Solecki submeasures and densities on groups
The upper Banach density $d^*(A)$ of a subset $A$ of an amenable group $G$ is defined as

$$d^*(A) = \sup_{\mu \in P_l(G)} \mu(A)$$

where $P_l(G)$ denotes the set of all left-invariant measures on $X$.

It is clear that the upper Banach density $d^* : \mathcal{P}(G) \to [0, 1]$ is a left-invariant submeasure on each amenable group $G$.

The Minimax Theorem describing the right Solecki density implies:

Corollary (B., 2013)

For any amenable group $G$ we get $\sigma_R = \sigma^R = d^*$. Consequently the right Solecki density on $G$ is subadditive.
The upper Banach density $d^*(A)$ of a subset $A$ of an amenable group $G$ is defined as

$$d^*(A) = \sup_{\mu \in P_l(G)} \mu(A)$$

where $P_l(G)$ denotes the set of all left-invariant measures on $X$.

It is clear that the upper Banach density $d^* : \mathcal{P}(G) \to [0, 1]$ is a left-invariant submeasure on each amenable group $G$.

The Minimax Theorem describing the right Solecki density implies:

**Corollary (B., 2013)**

For any amenable group $G$ we get $\sigma_R = \sigma^R = d^*$.

Consequently the right Solecki density on $G$ is subadditive.
The upper Banach density $d^*(A)$ of a subset $A$ of an amenable group $G$ is defined as

$$d^*(A) = \sup_{\mu \in P_l(G)} \mu(A)$$

where $P_l(G)$ denotes the set of all left-invariant measures on $X$.

It is clear that the upper Banach density $d^* : \mathcal{P}(G) \to [0, 1]$ is a left-invariant submeasure on each amenable group $G$. The Minimax Theorem describing the right Solecki density implies:

**Corollary (B., 2013)**

For any amenable group $G$ we get $\sigma_R = \sigma^R = d^*$. Consequently the right Solecki density on $G$ is subadditive.
Solecki-amenable groups

**Definition**

A group $G$ is **Solecki-amenable** if its Solecki density $\sigma_R$ is subadditive.

Amenable group $\Rightarrow$ Solecki-amenable

**Problem (Solecki, 2005)**

Is each Solecki-amenable group amenable?

**Theorem (B., 2012)**

For a group $G$ the following conditions are equivalent:

1. $G$ is amenable;
2. $G \times \mathbb{Z}$ is Solecki-amenable;
3. for every $n \in \mathbb{N}$ there is a finite group $F$ of cardinality $|F| \geq n$ such that the product $G \times F$ is a Solecki-amenable group;
4. $\sigma_R(f) + \sigma_R(1-f) \geq 1$ for any fuzzy set $f : G \rightarrow [0, 1]$. 
A group $G$ is **Solecki-amenable** if its Solecki density $\sigma_R$ is subadditive.

Amenable group $\implies$ Solecki-amenable

**Problem (Solecki, 2005)**

Is each Solecki-amenable group amenable?

**Theorem (B., 2012)**

For a group $G$ the following conditions are equivalent:

1. $G$ is amenable;
2. $G \times \mathbb{Z}$ is Solecki-amenable;
3. for every $n \in \mathbb{N}$ there is a finite group $F$ of cardinality $|F| \geq n$ such that the product $G \times F$ is a Solecki-amenable group;
4. $\sigma_R(f) + \sigma_R(1 - f) \geq 1$ for any fuzzy set $f : G \to [0,1]$. 

Taras Banakh

Solecki submeasures and densities on groups
Solecki-amenable groups

Definition
A group $G$ is **Solecki-amenable** if its Solecki density $\sigma_R$ is subadditive.

Amenable group $\Rightarrow$ Solecki-amenable

Problem (Solecki, 2005)

Is each Solecki-amenable group amenable?

Theorem (B., 2012)

For a group $G$ the following conditions are equivalent:

1. $G$ is amenable;
2. $G \times \mathbb{Z}$ is Solecki-amenable;
3. for every $n \in \mathbb{N}$ there is a finite group $F$ of cardinality $|F| \geq n$ such that the product $G \times F$ is a Solecki-amenable group;
4. $\sigma_R(f) + \sigma_R(1 - f) \geq 1$ for any fuzzy set $f : G \rightarrow [0, 1]$. 

Taras Banakh

Solecki submeasures and densities on groups
Solecki-amenable groups

Definition
A group $G$ is Solecki-amenable if its Solecki density $\sigma_R$ is subadditive.

Amenable group $\Rightarrow$ Solecki-amenable

Problem (Solecki, 2005)
Is each Solecki-amenable group amenable?

Theorem (B., 2012)
For a group $G$ the following conditions are equivalent:
1. $G$ is amenable;
2. $G \times \mathbb{Z}$ is Solecki-amenable;
3. for every $n \in \mathbb{N}$ there is a finite group $F$ of cardinality $|F| \geq n$ such that the product $G \times F$ is a Solecki-amenable group;
4. $\sigma_R(f) + \sigma_R(1 - f) \geq 1$ for any fuzzy set $f : G \to [0, 1]$. 

Taras Banakh
Solecki submeasures and densities on groups
A subset $A$ of a group $G$ is called

- **Solecki null** if $\sigma(A) = 0$;
- **Solecki positive** if $\sigma(A) > 0$;
- **Solecki one** if $\sigma(A) = 1$.

Solecki one sets can be characterized as follows:

**Proposition**

A subset $A$ of a group $G$ is Solecki one if and only if for each finite subset $F \subset G$ there are points $x, y \in G$ such that $xFy \subset A$. 
A subset $A$ of a group $G$ is called

- **Solecki null** if $\sigma(A) = 0$;
- **Solecki positive** if $\sigma(A) > 0$;
- **Solecki one** if $\sigma(A) = 1$.

Solecki one sets can be characterized as follows:

**Proposition**

A subset $A$ of a group $G$ is Solecki one if and only if for each finite subset $F \subset G$ there are points $x, y \in G$ such that $xFy \subseteq A$. 
A subset $A$ of a group $G$ is called

- **Solecki null** if $\sigma(A) = 0$;
- **Solecki positive** if $\sigma(A) > 0$;
- **Solecki one** if $\sigma(A) = 1$.

Solecki one sets can be characterized as follows:

**Proposition**

A subset $A$ of a group $G$ is Solecki one if and only if for each finite subset $F \subseteq G$ there are points $x, y \in G$ such that $xFy \subseteq A$. 
A subset $A$ of a group $G$ is called

- **Solecki null** if $\sigma(A) = 0$;
- **Solecki positive** if $\sigma(A) > 0$;
- **Solecki one** if $\sigma(A) = 1$.

Solecki one sets can be characterized as follows:

**Proposition**

A subset $A$ of a group $G$ is Solecki one if and only if for each finite subset $F \subset G$ there are points $x, y \in G$ such that $xFy \subset A$. 
The subadditivity of the Solecki submeasure $\sigma$ implies that the Solecki null sets of a group $G$ form an invariant ideal $S_G$ on $G$.

**Problem**

*Given a group $G$, study the properties of the ideal $S_G$. In particular, calculate its cardinal characteristics*

\[
\text{add}(S_G) = \min\{|A| : A \subset S_G, \cup A \notin S_G\},
\]
\[
\text{cov}(S_G) = \min\{|A| : A \subset S_G, \cup A = \cup S_G\},
\]
\[
\text{non}(S_G) = \min\{|A| : A \subset G, A \notin S_G\},
\]
\[
\text{cof}(S_G) = \min\{|C| : C \subset S_G, \forall A \in S_G \exists C \in C \text{ with } A \subset C\}.
\]
The ideal of Solecki null sets

The subadditivity of the Solecki submeasure $\sigma$ implies that the Solecki null sets of a group $G$ form an invariant ideal $S_G$ on $G$.

**Problem**

*Given a group $G$, study the properties of the ideal $S_G$. In particular, calculate its cardinal characteristics*

$$\text{add}(S_G) = \min\{|A| : A \subset S_G, \cup A \notin S_G\},$$

$$\text{cov}(S_G) = \min\{|A| : A \subset S_G, \cup A = \cup S_G\},$$

$$\text{non}(S_G) = \min\{|A| : A \subset G, A \notin S_G\},$$

$$\text{cof}(S_G) = \min\{|C| : C \subset S_G, \forall A \in S_G \exists C \in C \text{ with } A \subset C\}.$$
The ideal of Solecki null sets

The subadditivity of the Solecki submeasure $\sigma$ implies that the Solecki null sets of a group $G$ form an invariant ideal $S_G$ on $G$.

**Problem**

Given a group $G$, study the properties of the ideal $S_G$.

In particular, calculate its cardinal characteristics

\[
\text{add}(S_G) = \min\{|A| : A \subset S_G, \cup A \notin S_G\},
\]
\[
\text{cov}(S_G) = \min\{|A| : A \subset S_G, \cup A = \cup S_G\},
\]
\[
\text{non}(S_G) = \min\{|A| : A \subset G, A \notin S_G\},
\]
\[
\text{cof}(S_G) = \min\{|C| : C \subset S_G, \forall A \in S_G \exists C \in C \text{ with } A \subset C\}.
\]
For each infinite group $G$ we get

\[
\begin{align*}
\text{non}(S_G) & \rightarrow \text{cof}(S_G) \rightarrow 2^{|G|} \\
\omega & \rightarrow \text{add}(S_G) \rightarrow \text{cov}(S_G) \rightarrow |G|
\end{align*}
\]

Example (Not exciting)
For any infinite countable group $G$

\[
\omega = \text{add}(S_G) = \text{non}(S_G) = \text{cov}(S_G) < \text{cof}(S_G).
\]
If $G$ is abelian, then $\omega = \text{add}(S_G) = \text{cov}(S_G)$ and $\text{non}(S_G) = |G|$.

Example (Exciting)
For any infinite cardinal $\kappa$ there is an amenable group $G$ such that $|G| = \kappa$ and $\omega = \text{add}(S_G) = \text{non}(S_G)$.
For each infinite group \( G \) we get

\[
\begin{align*}
\text{non}(S_G) & \rightarrow \text{cof}(S_G) \rightarrow 2|G| \\
\omega & \rightarrow \text{add}(S_G) \rightarrow \text{cov}(S_G) \rightarrow |G|
\end{align*}
\]

Example (Not exciting)

For any infinite countable group \( G \)

\[
\omega = \text{add}(S_G) = \text{non}(S_G) = \text{cov}(S_G) < \text{cof}(S_G).
\]

If \( G \) is abelian, then \( \omega = \text{add}(S_G) = \text{cov}(S_G) \) and \( \text{non}(S_G) = |G| \).

Example (Exciting)

For any infinite cardinal \( \kappa \) there is an amenable group \( G \) such that

\[
|G| = \kappa \text{ and } \omega = \text{add}(S_G) = \text{non}(S_G).
\]
For each infinite group $G$ we get

$$\begin{align*}
\text{non}(S_G) &\longrightarrow \text{cof}(S_G) \longrightarrow 2^{|G|} \\
\omega &\longrightarrow \text{add}(S_G) \longrightarrow \text{cov}(S_G) \longrightarrow |G|
\end{align*}$$

Example (Not exciting)
For any infinite countable group $G$

$$\omega = \text{add}(S_G) = \text{non}(S_G) = \text{cov}(S_G) < \text{cof}(S_G).$$

If $G$ is abelian, then $\omega = \text{add}(S_G) = \text{cov}(S_G)$ and $\text{non}(S_G) = |G|.$

Example (Exciting)
For any infinite cardinal $\kappa$ there is an amenable group $G$ such that $|G| = \kappa$ and $\omega = \text{add}(S_G) = \text{non}(S_G).$
For each infinite group $G$ we get

\[ \non(S_G) \rightarrow \cof(S_G) \rightarrow 2|G| \]

\[ \omega \rightarrow \add(S_G) \rightarrow \cov(S_G) \rightarrow |G| \]

**Example (Not exciting)**

For any infinite countable group $G$

$\omega = \add(S_G) = \non(S_G) = \cov(S_G) < \cof(S_G)$.

If $G$ is abelian, then $\omega = \add(S_G) = \cov(S_G)$ and $\non(S_G) = |G|$.

**Example (Exciting)**

For any infinite cardinal $\kappa$ there is an amenable group $G$ such that $|G| = \kappa$ and $\omega = \add(S_G) = \non(S_G)$. 
The exciting example

In the group $G = FS_\kappa$ of finitely supported permutations of the cardinal $\kappa$ consider the countable subgroup $H = FS_\omega$ consisting of all permutations $f : \kappa \rightarrow \kappa$ with finite support

$$\text{supp}(f) = \{x \in \kappa : f(x) \neq x\} \subset \omega.$$ 

It can be shown that $\sigma(H) = 1$. So, $H \not\in \mathcal{S}_G$ and

$$\omega \leq \text{add}(\mathcal{S}_G) \leq \text{non}(\mathcal{S}_G) \leq |H| = \omega.$$ 

Problem

*Calculate $\text{cov}(\mathcal{S}_G)$ for the group $FS_\kappa$.***
In the group $G = \text{FS}_\kappa$ of finitely supported permutations of the cardinal $\kappa$ consider the countable subgroup $H = \text{FS}_\omega$ consisting of all permutations $f : \kappa \to \kappa$ with finite support

$$\text{supp}(f) = \{x \in \kappa : f(x) \neq x\} \subset \omega.$$ 

It can be shown that $\sigma(H) = 1$. So, $H \notin \mathcal{S}_G$ and

$$\omega \leq \text{add}(\mathcal{S}_G) \leq \text{non}(\mathcal{S}_G) \leq |H| = \omega.$$ 

**Problem**

*Calculate $\text{cov}(\mathcal{S}_G)$ for the group $\text{FS}_\kappa$.***
In the group $G = FS_\kappa$ of finitely supported permutations of the cardinal $\kappa$ consider the countable subgroup $H = FS_\omega$ consisting of all permutations $f : \kappa \to \kappa$ with finite support

$$\text{supp}(f) = \{x \in \kappa : f(x) \neq x\} \subset \omega.$$ 

It can be shown that $\sigma(H) = 1$.

So, $H \notin S_G$ and

$$\omega \leq \text{add}(S_G) \leq \text{non}(S_G) \leq |H| = \omega.$$ 

Problem

*Calculate $\text{cov}(S_G)$ for the group $FS_\kappa$.***
The exciting example

In the group $G = FS_\kappa$ of finitely supported permutations of the cardinal $\kappa$ consider the countable subgroup $H = FS_\omega$ consisting of all permutations $f : \kappa \to \kappa$ with finite support

$$\text{supp}(f) = \{x \in \kappa : f(x) \neq x\} \subset \omega.$$ 

It can be shown that $\sigma(H) = 1$. So, $H \notin S_G$ and

$$\omega \leq \text{add}(S_G) \leq \text{non}(S_G) \leq |H| = \omega.$$ 

Problem

*Calculate $\text{cov}(S_G)$ for the group $FS_\kappa$.***
The exciting example

In the group $G = FS_\kappa$ of finitely supported permutations of the cardinal $\kappa$ consider the countable subgroup $H = FS_\omega$ consisting of all permutations $f : \kappa \to \kappa$ with finite support

$$\text{supp}(f) = \{x \in \kappa : f(x) \neq x\} \subset \omega.$$ 

It can be shown that $\sigma(H) = 1$. So, $H \notin S_G$ and

$$\omega \leq \text{add}(S_G) \leq \text{non}(S_G) \leq |H| = \omega.$$ 

Problem

*Calculate $\text{cov}(S_G)$ for the group $FS_\kappa$.***
Theorem

If a group \( G \) admits a homomorphism onto an infinite compact Hausdorff group, then \( \text{non}(S_G) \geq \text{cov}(\mathcal{E}) \).

Here \( \text{cov}(\mathcal{E}) \) denotes the smallest cardinality of a cover of an infinite compact metrizable group by closed Haar null subsets. This cardinal was thoroughly studied by Bartoszynski and Shelah.

Corollary

For any infinite cardinal \( \kappa \) the group \( FS_\kappa \) admits no homomorphism onto an infinite compact Hausdorff topological group.

So, the properties of the ideal \( S_G \) depends essentially on the topologizability properties of the group \( G \).
Theorem

If a group $G$ admits a homomorphism onto an infinite compact Hausdorff group, then $\text{non}(S_G) \geq \text{cov}(\mathcal{E})$.

Here $\text{cov}(\mathcal{E})$ denotes the smallest cardinality of a cover of an infinite compact metrizable group by closed Haar null subsets. This cardinal was thoroughly studied by Bartoszynski and Shelah.

Corollary

For any infinite cardinal $\kappa$ the group $FS_\kappa$ admits no homomorphism onto an infinite compact Hausdorff topological group.

So, the properties of the ideal $S_G$ depends essentially on the topologizability properties of the group $G$. 
Theorem

If a group $G$ admits a homomorphism onto an infinite compact Hausdorff group, then $\text{non}(S_G) \geq \text{cov}(E)$.

Here $\text{cov}(E)$ denotes the smallest cardinality of a cover of an infinite compact metrizable group by closed Haar null subsets. This cardinal was thoroughly studied by Bartoszynski and Shelah.

Corollary

For any infinite cardinal $\kappa$ the group $FS_\kappa$ admits no homomorphism onto an infinite compact Hausdorff topological group.

So, the properties of the ideal $S_G$ depends essentially on the topologizability properties of the group $G$. 
Let $G$ be a compact topological group and $\lambda$ be its Haar measure. For a subset $A \subset G$ let $\bar{A}$ be the closure of $A$ in $X$ and $A^\bullet$ (resp. $A^\circ$) be the largest open set $U \subset G$ such that $U \setminus A$ is meager in $G$ (resp. empty).

It is clear that $A^\circ$ is the interior of $A$ and $A^\circ \subset A^\bullet \subset \bar{A}$.

Example: Each dense $G_\delta$-set $A \subset G$ has $A^\bullet = G$.

**Theorem**

*Any subset $A$ of a compact topological group $G$ has*

$$\max\{\lambda_*(A), \lambda(A^\bullet)\} \leq \sigma(A) \leq \lambda(\bar{A}).$$

*Here $\lambda_*(A) = \sup\{\lambda(B) : B \subset A$ is a Borel subset in $X\}$ is the lower Haar density of $A$.***
Let $G$ be a compact topological group and $\lambda$ be its Haar measure. For a subset $A \subset G$ let $\bar{A}$ be the closure of $A$ in $X$ and $A^\bullet$ (resp. $A^\circ$) be the largest open set $U \subset G$ such that $U \setminus A$ is meager in $G$ (resp. empty).

It is clear that $A^\circ$ is the interior of $A$ and $A^\circ \subset A^\bullet \subset \bar{A}$.

Example: Each dense $G_\delta$-set $A \subset G$ has $A^\bullet = G$.

**Theorem**

Any subset $A$ of a compact topological group $G$ has

$$\max\{\lambda_*(A), \lambda(A^\bullet)\} \leq \sigma(A) \leq \lambda(\bar{A}).$$

Here $\lambda_*(A) = \sup\{\lambda(B) : B \subset A \text{ is a Borel subset in } X\}$ is the lower Haar density of $A$. 
Let $G$ be a compact topological group and $\lambda$ be its Haar measure. For a subset $A \subset G$ let $\bar{A}$ be the closure of $A$ in $X$ and $A^\bullet$ (resp. $A^\circ$) be the largest open set $U \subset G$ such that $U \setminus A$ is meager in $G$ (resp. empty).

It is clear that $A^\circ$ is the interior of $A$ and $A^\circ \subset A^\bullet \subset \bar{A}$.

**Example:** Each dense $G_\delta$-set $A \subset G$ has $A^\bullet = G$.

**Theorem**

Any subset $A$ of a compact topological group $G$ has

$$\max\{\lambda_*(A), \lambda(A^\bullet)\} \leq \sigma(A) \leq \lambda(\bar{A}).$$

Here $\lambda_*(A) = \sup\{\lambda(B) : B \subset A \text{ is a Borel subset in } X\}$ is the lower Haar density of $A$. 

Taras Banakh

Solecki submeasures and densities on groups
Let $G$ be a compact topological group and $\lambda$ be its Haar measure. For a subset $A \subset G$ let $\overline{A}$ be the closure of $A$ in $X$ and $A^\bullet$ (resp. $A^\circ$) be the largest open set $U \subset G$ such that $U \setminus A$ is meager in $G$ (resp. empty).

It is clear that $A^\circ$ is the interior of $A$ and $A^\circ \subset A^\bullet \subset \overline{A}$.

**Example:** Each dense $G_\delta$-set $A \subset G$ has $A^\bullet = G$.

**Theorem**

Any subset $A$ of a compact topological group $G$ has 
\[ \max\{\lambda^*(A), \lambda(A^\bullet)\} \leq \sigma(A) \leq \lambda(\overline{A}). \]

Here $\lambda^*(A) = \sup\{\lambda(B) : B \subset A \text{ is a Borel subset in } X\}$ is the lower Haar density of $A$. 
Let $G$ be a compact topological group and $\lambda$ be its Haar measure. For a subset $A \subset G$ let $\bar{A}$ be the closure of $A$ in $X$ and $A^\bullet$ (resp. $A^\circ$) be the largest open set $U \subset G$ such that $U \setminus A$ is meager in $G$ (resp. empty).

It is clear that $A^\circ$ is the interior of $A$ and $A^\circ \subset A^\bullet \subset \bar{A}$.

**Example:** Each dense $G_\delta$-set $A \subset G$ has $A^\bullet = G$.

**Theorem**

*Any subset $A$ of a compact topological group $G$ has*

$$\max\{\lambda_*(A), \lambda(A^\bullet)\} \leq \sigma(A) \leq \lambda(\bar{A}).$$

Here $\lambda_*(A) = \sup\{\lambda(B) : B \subset A \text{ is a Borel subset in } X\}$ is the lower Haar density of $A$. 

The Haar measure is determined by the Solecki submeasure.

**Corollary**

*Each closed subset $A$ of a compact topological group $G$ has $\sigma(A) = \lambda(A)$.*

This means that the Haar measure $\lambda$ is completely determined by the Solecki submeasure:

**Theorem**

*For a compact Hausdorff topological group $G$ its Haar measure is a unique regular $\sigma$-additive Borel measure $\lambda$ such that $\lambda(A) = \sigma(A)$ for each closed subset $A \subset G$.*

So, the Haar measure, being a topologo-algebraic object has more essential algebraic component than could be expected.
The Haar measure is determined by the Solecki submeasure

**Corollary**

*Each closed subset $A$ of a compact topological group $G$ has $\sigma(A) = \lambda(A)$.*

This means that the Haar measure $\lambda$ is completely determined by the Solecki submeasure:

**Theorem**

*For a compact Hausdorff topological group $G$ its Haar measure is a unique regular $\sigma$-additive Borel measure $\lambda$ such that $\lambda(A) = \sigma(A)$ for each closed subset $A \subset G$.*

So, the Haar measure, being a topologo-algebraic object has more essential algebraic component than could be expected.
Corollary

Each closed subset $A$ of a compact topological group $G$ has 
$\sigma(A) = \lambda(A)$. 

This means that the Haar measure $\lambda$ is completely determined by 
the Solecki submeasure:

Theorem

For a compact Hausdorff topological group $G$ its Haar measure is a 
unique regular $\sigma$-additive Borel measure $\lambda$ such that $\lambda(A) = \sigma(A)$ 
for each closed subset $A \subset G$.

So, the Haar measure, being a topologo-algebraic object has more 
essential algebraic component than could be expected.
Ramsey Applications of the Solecki submeasure
Van der Waerden and Gallai’s Theorem

**Theorem (Van der Waerden, 1927)**

For any partition $\mathbb{Z} = A_1 \cup \cdots \cup A_n$ of integers there is a cell $A_i$ of the partition containing arbitrarily long arithmetic progressions.

This theorem can be deduced from a more general:

**Theorem (Gallai, ≤ 1933)**

For any finite partition $G = A_1 \cup \cdots \cup A_n$ of the group $G = \mathbb{Z}^n$ there is a cell $A_i$ of the partition containing the homothetic copy of each finite set $F \subset G$. 

Taras Banakh  
Solecki submeasures and densities on groups
Van der Waerden and Gallai’s Theorem

**Theorem (Van der Waerden, 1927)**

For any partition \( \mathbb{Z} = A_1 \cup \cdots \cup A_n \) of integers there is a cell \( A_i \) of the partition containing arbitrarily long arithmetic progressions.

This theorem can be deduced from a more general:

**Theorem (Gallai, \( \leq 1933 \))**

For any finite partition \( G = A_1 \cup \cdots \cup A_n \) of the group \( G = \mathbb{Z}^n \) there is a cell \( A_i \) of the partition containing the homothetic copy of each finite set \( F \subset G \).
By a **homothetic copy** of a set $F$ in a group $G$ we understand the image $h(F)$ of $F$ under a polynomial map $h : G \to G$ of the form $h(x) = a_0 x a_1 \ldots a_{n-1} x a_n$ for some constants $c_0, \ldots, c_n \in G$.

If $n = 1$, then $h(x) = c_0 x c_1$ and we say that $h(F) = c_0 F c_1$ is a **translation copy** of the set $F$. 
By a homothetic copy of a set $F$ in a group $G$ we understand the image $h(F)$ of $F$ under a polynomial map $h : G \to G$ of the form $h(x) = a_0x a_1 \ldots a_{n-1}x a_n$ for some constants $c_0, \ldots, c_n \in G$.

If $n = 1$, then $h(x) = c_0xc_1$ and we say that $h(F) = c_0Fc_1$ is a translation copy of the set $F$. 
Theorem (B., 2012)

If a subset $A$ of a group $G$ is:
- **Solecki one**, then $A$ contains a *translation* copy of each finite subset $F \subset G$;
- **Solecki positive**, then $A$ contains a *homothetic* copy of each finite subset $F \subset G$.

This theorem combined with the subadditivity of the Solecki submeasure implies the following generalization of Gallai’s Theorem:

**Corollary**

For any finite partition $G = A_1 \cup \cdots \cup A_n$ of any group $G$ there is a cell $A_i$ of the partition containing a homothetic copy of each finite subset $F \subset G$. 
If a subset $A$ of a group $G$ is:
- **Solecki one**, then $A$ contains a *translation* copy of each finite subset $F \subset G$;
- **Solecki positive**, then $A$ contains a *homothetic* copy of each finite subset $F \subset G$.

This theorem combined with the subadditivity of the Solecki submeasure implies the following generalization of Gallai’s Theorem:

**Corollary**

For any finite partition $G = A_1 \cup \cdots \cup A_n$ of any group $G$ there is a cell $A_i$ of the partition containing a homothetic copy of each finite subset $F \subset G$. 

Taras Banakh

Solecki submeasures and densities on groups
Generalizing Van der Waerden and Gallai Theorem

Theorem (B., 2012)

If a subset $A$ of a group $G$ is:
- **Solecki one**, then $A$ contains a translation copy of each finite subset $F \subset G$;
- **Solecki positive**, then $A$ contains a homothetic copy of each finite subset $F \subset G$.

This theorem combined with the subadditivity of the Solecki submeasure implies the following generalization of Gallai’s Theorem:

**Corollary**

For any finite partition $G = A_1 \cup \cdots \cup A_n$ of any group $G$ there is a cell $A_i$ of the partition containing a homothetic copy of each finite subset $F \subset G$. 
Theorem (Green, Tao, 2008)

The set of prime numbers contains arbitrarily long arithmetic progressions.

Unfortunately, this theorem cannot be deduced from our result because of:

Proposition

The set of primes is Solecki null in the group $\mathbb{Z}$. 
Theorem (Green, Tao, 2008)

*The set of prime numbers contains arbitrarily long arithmetic progressions.*

Unfortunately, this theorem cannot be deduced from our result because of:

Proposition

*The set of primes is Solecki null in the group $\mathbb{Z}$.**
Steinhaus-Weil Theorem

**Theorem (Steinhaus-Weil)**

*For any measurable subset $A$ of positive Haar measure $\lambda(A)$ in a compact topological group $G$ the difference set $AA^{-1}$ is a neighborhood of zero in $G$.***

**Problem**

*Can the Haar measure in this theorem replaced with the Solecki submeasure $\sigma$ or the right Solecki density $\sigma^R$?*

**Answer**

*Partially Yes! (for the right Solecki density $\sigma^R$).*
Steinhaus-Weil Theorem

Theorem (Steinhaus-Weil)

For any measurable subset $A$ of positive Haar measure $\lambda(A)$ in a compact topological group $G$ the difference set $AA^{-1}$ is a neighborhood of zero in $G$.

Problem

Can the Haar measure in this theorem replaced with the Solecki submeasure $\sigma$ or the right Solecki density $\sigma^R$?

Answer

Partially Yes! (for the right Solecki density $\sigma^R$).
### Theorem (Steinhaus-Weil)

For any measurable subset $A$ of positive Haar measure $\lambda(A)$ in a compact topological group $G$ the difference set $AA^{-1}$ is a neighborhood of zero in $G$.

### Problem

Can the Haar measure in this theorem replaced with the Solecki submeasure $\sigma$ or the right Solecki density $\sigma^R$?

### Answer

Partially Yes! (for the right Solecki density $\sigma^R$).
A subset $A$ of a group $G$ is called
- right-Solecki null if $\sigma^R(A) = 0$;
- right-Solecki positive if $\sigma^R(A) > 0$;
- right-Solecki one if $\sigma^R(A) = 1$ (equivalently, if $\sigma_R(A) = 1$).

Right-Solecki one sets can be characterized as follows:

**Proposition**

A subset $A$ of a group $G$ is right-Solecki one iff for each finite subset $F \subset G$ there is a point $y \in G$ such that $Fy \subset A$. 

Tara Banakh

Solecki submeasures and densities on groups
A subset $A$ of a group $G$ is called
- **right-Solecki null** if $\sigma^R(A) = 0$;
- **right-Solecki positive** if $\sigma^R(A) > 0$;
- **right-Solecki one** if $\sigma^R(A) = 1$ (equivalently, if $\sigma_R(A) = 1$).

Right-Solecki one sets can be characterized as follows:

**Proposition**

A subset $A$ of a group $G$ is right-Solecki one iff for each finite subset $F \subset G$ there is a point $y \in G$ such that $Fy \subset A$. 
For a subset $A$ of a group $G$ the cardinal

\[ \text{pack}_L(A) = \sup \{|E| : E \subset G \ (xA)_{x \in E} \text{ is disjoint} \} \]

is called the **left packing index** of $A$;

\[ \text{cov}_L(A) = \min \{|E| : E \subset G, \ EA = G \} \]

is called the **left covering number** of $A$.

**Theorem**

\[ \text{cov}_L(AA^{-1}) \leq \text{pack}_L(A) \leq \frac{1}{\sigma^R(A)}. \]

**Corollary**

*If an (analytic) subset $A$ a Polish group $G$ is right-Solecki positive, then $AA^{-1}$ is not meager (and $AA^{-1}AA^{-1}$ is a neighborhood of the unit) in $G$.***
For a subset $A$ of a group $G$ the cardinal

- $\text{pack}_L(A) = \sup\{|E| : E \subset G \text{ (xA)}_{x \in E} \text{ is disjoint}\}$
  is called the left packing index of $A$;

- $\text{cov}_L(A) = \min\{|E| : E \subset G, EA = G\}$
  is called the left covering number of $A$.

**Theorem**

$$\text{cov}_L(AA^{-1}) \leq \text{pack}_L(A) \leq \frac{1}{\sigma^R(A)}.$$  

**Corollary**

If an (analytic) subset $A$ a Polish group $G$ is right-Solecki positive, then $AA^{-1}$ is not meager
(and $AA^{-1}AA^{-1}$ is a neighborhood of the unit) in $G$. 

Taras Banakh  
Solecki submeasures and densities on groups
For a subset $A$ of a group $G$ the cardinal

- $\text{pack}_L(A) = \sup \{|E| : E \subset G \ (xA)_{x \in E} \text{ is disjoint}\}$
  is called the left packing index of $A$;

- $\text{cov}_L(A) = \min \{|E| : E \subset G, \ EA = G\}$
  is called the left covering number of $A$.

**Theorem**

$$\text{cov}_L(AA^{-1}) \leq \text{pack}_L(A) \leq \frac{1}{\sigma^R(A)}.$$ 

**Corollary**

If an (analytic) subset $A$ a Polish group $G$ is right-Solecki positive, then $AA^{-1}$ is not meager
(and $AA^{-1}AA^{-1}$ is a neighborhood of the unit) in $G$. 

Taras Banakh
Solecki submeasures and densities on groups
For a subset $A$ of a group $G$ the cardinal
- \( \text{pack}_L(A) = \sup \{|E| : E \subset G \ (xA)_{x \in E} \text{ is disjoint} \} \)
is called the \text{left packing index} of $A$;
- \( \text{cov}_L(A) = \min \{|E| : E \subset G, \ EA = G \} \)
is called the \text{left covering number} of $A$.

**Theorem**

\[
\text{cov}_L(AA^{-1}) \leq \text{pack}_L(A) \leq \frac{1}{\sigma^R(A)}.
\]

**Corollary**

If an (analytic) subset $A$ of a Polish group $G$ is right-Solecki positive, then $AA^{-1}$ is not meager (and $AA^{-1}AA^{-1}$ is a neighborhood of the unit) in $G$. 
Protasov’s Problem

Problem (Protasov)

Let $G = A_1 \cup \cdots \cup A_n$ be a finite partition of a group $G$. Is $\text{cov}_L(A_iA_i^{-1}) \leq n$ for some $i$?

Theorem (Protasov-B., \leq 2003)

For any partition $G = A_1 \cup \cdots \cup A_n$ of a group $G$ there is $i \leq n$ such that $\text{cov}_L(A_iA_i^{-1}) \leq 2^{n-1}-1$. 
Problem (Protasov)

Let \( G = A_1 \cup \cdots \cup A_n \) be a finite partition of a group \( G \). Is \( \text{cov}_L(A_iA_i^{-1}) \leq n \) for some \( i \)?

Theorem (Protasov-B., ≤ 2003)

For any partition \( G = A_1 \cup \cdots \cup A_n \) of a group \( G \) there is \( i \leq n \) such that \( \text{cov}_L(A_iA_i^{-1}) \leq 2^{2^{n-1}} - 1 \).
A subset $A \subset G$ is called \textit{inner-invariant} if $\forall x \in G \ xAx^{-1} = A$.

\begin{tabular}{|l|
\hline
Theorem (B.-Protasov-Slobodianiuk, 2013) \\
\hline
Let $G = A_1 \cup \cdots \cup A_n$ be a partition of a group $G$. If $G$ is Solecki-amenable or all sets $A_i$ are inner-invariant, then $\cov_L(A_iA_i^{-1}) \leq n$ for some $i$. \\
\hline
\end{tabular}

\begin{tabular}{|l|
\hline
Proof. \\
\hline
If $G$ is Solecki-amenable, then the right Solecki submeasure $\sigma_R$ is subadditive and then $\sigma_R(A_i) \geq 1/n$ for some $i$ and hence

$$\cov_L(A_iA_i^{-1}) \leq \frac{1}{\sigma_R(A)} \leq \frac{1}{\sigma_R(A)} \leq n.$$ \\
\hline
If each set $A_i$ is inner-invariant, then $\sigma(A_i) \geq \frac{1}{n}$ for some $i$ by the subadditivity of the Solecki submeasure. The inner invariance of $A_i$ implies that $\sigma^R(A_i) = \sigma(A_i) \geq 1/n$ and $\cov_L(A_iA_i^{-1}) \leq \frac{1}{\sigma^R(A)} \leq n$. \\
\hline
\end{tabular}
A partial answer to Protasov’s Problem

A subset $A \subset G$ is called \textit{inner-invariant} if $\forall x \in G \ xA x^{-1} = A$.

**Theorem (B.-Protasov-Slobodianiuk, 2013)**

Let $G = A_1 \cup \cdots \cup A_n$ be a partition of a group $G$. If $G$ is Solecki-amenable or all sets $A_i$ are inner-invariant, then $\text{cov}_L(A_i A_i^{-1}) \leq n$ for some $i$.

**Proof.**

If $G$ is Solecki-amenable, then the right Solecki submeasure $\sigma_R$ is subadditive and then $\sigma_R(A_i) \geq 1/n$ for some $i$ and hence

$$\text{cov}_L(A_i A_i^{-1}) \leq \frac{1}{\sigma_R(A)} \leq \frac{1}{\sigma_R(A)} \leq n.$$  

If each set $A_i$ is inner-invariant, then $\sigma(A_i) \geq \frac{1}{n}$ for some $i$ by the subadditivity of the Solecki submeasure. The inner invariance of $A_i$ implies that $\sigma^R(A_i) = \sigma(A_i) \geq 1/n$ and $\text{cov}_L(A_i A_i^{-1}) \leq \frac{1}{\sigma^R(A)} \leq n$.  

Taras Banakh

Solecki submeasures and densities on groups
A partial answer to Protasov’s Problem

A subset $A \subset G$ is called inner-invariant if $\forall x \in G \ xAx^{-1} = A$.

Theorem (B.-Protasov-Slobodianiuk, 2013)

Let $G = A_1 \cup \cdots \cup A_n$ be a partition of a group $G$. If $G$ is Solecki-amenable or all sets $A_i$ are inner-invariant, then $\text{cov}_L(A_iA_i^{-1}) \leq n$ for some $i$.

Proof.

If $G$ is Solecki-amenable, then the right Solecki submeasure $\sigma_R$ is subadditive and then $\sigma_R(A_i) \geq 1/n$ for some $i$ and hence

$$\text{cov}_L(A_iA_i^{-1}) \leq \frac{1}{\sigma_R(A)} \leq \frac{1}{\sigma_R(A)} \leq n.$$ 

If each set $A_i$ is inner-invariant, then $\sigma(A_i) \geq \frac{1}{n}$ for some $i$ by the subadditivity of the Solecki submeasure. The inner invariance of $A_i$ implies that $\sigma_R(A_i) = \sigma(A_i) \geq 1/n$ and $\text{cov}_L(A_iA_i^{-1}) \leq \frac{1}{\sigma_R(A)} \leq n$. 

Taras Banakh

Solecki submeasures and densities on groups
A subset $A \subset G$ is called *inner-invariant* if $\forall x \in G \ xA x^{-1} = A$.

**Theorem (B.-Protasov-Slobodianiuk, 2013)**

Let $G = A_1 \cup \cdots \cup A_n$ be a partition of a group $G$. If $G$ is Solecki-amenable or all sets $A_i$ are inner-invariant, then $\text{cov}_L(A_i A_i^{-1}) \leq n$ for some $i$.

**Proof.**

If $G$ is Solecki-amenable, then the right Solecki submeasure $\sigma_R$ is subadditive and then $\sigma_R(A_i) \geq 1/n$ for some $i$ and hence

$$\text{cov}_L(A_i A_i^{-1}) \leq \frac{1}{\sigma_R(A)} \leq \frac{1}{\sigma_R(A)} \leq n.$$

If each set $A_i$ is inner-invariant, then $\sigma(A_i) \geq \frac{1}{n}$ for some $i$ by the subadditivity of the Solecki submeasure. The inner invariance of $A_i$ implies that $\sigma^R(A_i) = \sigma(A_i) \geq 1/n$ and $\text{cov}_L(A_i A_i^{-1}) \leq \frac{1}{\sigma^R(A)} \leq n$. \qed
A partial answer to Protasov’s Problem

A subset \( A \subset G \) is called \textit{inner-invariant} if \( \forall x \in G \ \ xAx^{-1} = A \).

**Theorem (B.-Protasov-Slobodianiuk, 2013)**

Let \( G = A_1 \cup \cdots \cup A_n \) be a partition of a group \( G \). If \( G \) is Solecki-amenable or all sets \( A_i \) are inner-invariant, then

\[
\text{cov}_L(A_iA_i^{-1}) \leq n \text{ for some } i.
\]

**Proof.**

If \( G \) is Solecki-amenable, then the right Solecki submeasure \( \sigma_R \) is subadditive and then \( \sigma_R(A_i) \geq 1/n \) for some \( i \) and hence

\[
\text{cov}_L(A_iA_i^{-1}) \leq \frac{1}{\sigma_R(A)} \leq \frac{1}{\sigma_R(A)} \leq n.
\]

If each set \( A_i \) is inner-invariant, then \( \sigma(A_i) \geq \frac{1}{n} \) for some \( i \) by the subadditivity of the Solecki submeasure. The inner invariance of \( A_i \) implies that \( \sigma_R(A_i) = \sigma(A_i) \geq 1/n \) and \( \text{cov}_L(A_iA_i^{-1}) \leq \frac{1}{\sigma_R(A)} \leq n \). 

\[\square\]
A subset $A \subset G$ is called *inner-invariant* if $\forall x \in G \ xAx^{-1} = A$.

**Theorem (B.-Protasov-Slobodianiuk, 2013)**

Let $G = A_1 \cup \cdots \cup A_n$ be a partition of a group $G$. If $G$ is Solecki-amenable or all sets $A_i$ are inner-invariant, then $\text{cov}_L(A_i A_i^{-1}) \leq n$ for some $i$.

**Proof.**

If $G$ is Solecki-amenable, then the right Solecki submeasure $\sigma_R$ is subadditive and then $\sigma_R(A_i) \geq 1/n$ for some $i$ and hence

$$\text{cov}_L(A_i A_i^{-1}) \leq \frac{1}{\sigma_R(A)} \leq \frac{1}{\sigma_R(A)} \leq n.$$  

If each set $A_i$ is inner-invariant, then $\sigma(A_i) \geq \frac{1}{n}$ for some $i$ by the subadditivity of the Solecki submeasure. The inner invariance of $A_i$ implies that $\sigma^R(A_i) = \sigma(A_i) \geq 1/n$ and $\text{cov}_L(A_i A_i^{-1}) \leq \frac{1}{\sigma^R(A)} \leq n$. 

Taras Banakh

Solecki submeasures and densities on groups
The **Bohr topology** on a group $G$ is the largest totally bounded group topology on $G$.

Equivalently, it can be defined as the smallest topology on $G$ in which every homomorphism $h : G \to K$ to a compact Hausdorff topological group $K$ is continuous.

In this case we can assume that $K = \prod_{n=1}^{\infty} O(n)$.

Elements of the Bohr topology are called **Bohr open subsets of $G$**.
The **Bohr topology** on a group $G$ is the largest totally bounded group topology on $G$.

Equivalently, it can be defined as the smallest topology on $G$ in which every homomorphism $h : G \to K$ to a compact Hausdorff topological group $K$ is continuous.

In this case we can assume that $K = \prod_{n=1}^{\infty} O(n)$.

Elements of the Bohr topology are called **Bohr open subsets** of $G$. 
The **Bohr topology** on a group $G$ is the largest totally bounded group topology on $G$.

Equivalently, it can be defined as the smallest topology on $G$ in which every homomorphism $h : G \to K$ to a compact Hausdorff topological group $K$ is continuous.

In this case we can assume that $K = \prod_{n=1}^{\infty} O(n)$.

Elements of the Bohr topology are called **Bohr open subsets of $G$**.
The Bohr topology on a group $G$ is the largest totally bounded group topology on $G$.

Equivalently, it can be defined as the smallest topology on $G$ in which every homomorphism $h : G \to K$ to a compact Hausdorff topological group $K$ is continuous.

In this case we can assume that $K = \prod_{n=1}^{\infty} O(n)$.

Elements of the Bohr topology are called **Bohr open subsets** of $G$. 

---

Taras Banakh

Solecki submeasures and densities on groups
Now we shall generalize results of Bogoliuboff, Følner, Cotlar, Ricabarra (1954), Ellis, Keynes (1972), Beiglböck, Bergelson, Fish (2010).

**Theorem (B., 2013)**

For each right-Solecki positive set $A$ in an amenable group $G$ there are a Bohr open neighborhood $U \subset G$ of the unit $1_G$ and a right-Solecki null subset $N \subset G$ such that $U \setminus N \subset AA^{-1}$.

**Corollary (B., 2013)**

For any right-Solecki positive set $A, B$ in an amenable group $G$ the set $B^{-1}AA^{-1}$ has non-empty interior and $AA^{-1}BB^{-1}$ is a neighborhood of the unit $1_G$ in the Bohr topology on the group $G$.

**Problem (Ellis)**

Is $AA^{-1}$ is Bohr neighborhood of the unit for each right-Solecki positive set $A$ in the group $G = \mathbb{Z}$?
Now we shall generalize results of Bogoliuboff, Følner, Cotlar, Ricabarра (1954), Ellis, Keynes (1972), Beiglbock, Bergelson, Fish (2010).

**Theorem (B., 2013)**

For each right-Solecki positive set $A$ in an amenable group $G$ there are a Bohr open neighborhood $U \subset G$ of the unit $1_G$ and a right-Solecki null subset $N \subset G$ such that $U \setminus N \subset AA^{-1}$.

**Corollary (B., 2013)**

For any right-Solecki positive set $A, B$ in an amenable group $G$ the set $B^{-1}AA^{-1}$ has non-empty interior and $AA^{-1}BB^{-1}$ is a neighborhood of the unit $1_G$ in the Bohr topology on the group $G$.

**Problem (Ellis)**

Is $AA^{-1}$ is Bohr neighborhood of the unit for each right-Solecki positive set $A$ in the group $G = \mathbb{Z}$?
Now we shall generalize results of Bogoliuboff, Følner, Cotlar, Ricabarra (1954), Ellis, Keynes (1972), Beiglböck, Bergelson, Fish (2010).

**Theorem (B., 2013)**

For each right-Solecki positive set $A$ in an amenable group $G$ there are a Bohr open neighborhood $U \subset G$ of the unit $1_G$ and a right-Solecki null subset $N \subset G$ such that $U \setminus N \subset AA^{-1}$.

**Corollary (B., 2013)**

For any right-Solecki positive set $A, B$ in an amenable group $G$ the set $B^{-1}AA^{-1}$ has non-empty interior and $AA^{-1}BB^{-1}$ is a neighborhood of the unit $1_G$ in the Bohr topology on the group $G$.

**Problem (Ellis)**

Is $AA^{-1}$ is Bohr neighborhood of the unit for each right-Solecki positive set $A$ in the group $G = \mathbb{Z}$?
Now we shall generalize results of Bogoliuboff, Følner, Cotlar, Ricabarra (1954), Ellis, Keynes (1972), Beiglböck, Bergelson, Fish (2010).

**Theorem (B., 2013)**

For each right-Solecki positive set $A$ in an amenable group $G$ there are a Bohr open neighborhood $U \subset G$ of the unit $1_G$ and a right-Solecki null subset $N \subset G$ such that $U \setminus N \subset AA^{-1}$.

**Corollary (B., 2013)**

For any right-Solecki positive set $A, B$ in an amenable group $G$ the set $B^{-1}AA^{-1}$ has non-empty interior and $AA^{-1}BB^{-1}$ is a neighborhood of the unit $1_G$ in the Bohr topology on the group $G$.

**Problem (Ellis)**

Is $AA^{-1}$ is Bohr neighborhood of the unit for each right-Solecki positive set $A$ in the group $G = \mathbb{Z}$?
The following theorem generalizes results of Jin (2002) and Beiglböck, Bergelson, Fish (2010).

**Theorem (B., 2013)**

For any right-Solecki positive sets $A, B$ in an amenable group $G$ the sumset $AB$ contains the intersection $U \cap T$ for some non-empty Bohr open set $U$ and some right-Solecki one set $T \subset G$.

**Corollary (B., 2013)**

For any right-Solecki positive sets $A, B$ in an amenable group $G$ the set $ABB^{-1}A^{-1}$ is a neighborhood of the unit $1_G$ in the Bohr topology on $G$. 
The following theorem generalizes results of Jin (2002) and Beiglböck, Bergelson, Fish (2010).

**Theorem (B., 2013)**

For any right-Solecki positive sets $A, B$ in an amenable group $G$ the sumset $AB$ contains the intersection $U \cap T$ for some non-empty Bohr open set $U$ and some right-Solecki one set $T \subset G$.

**Corollary (B., 2013)**

For any right-Solecki positive sets $A, B$ in an amenable group $G$ the set $ABB^{-1}A^{-1}$ is a neighborhood of the unit $1_G$ in the Bohr topology on $G$.
The Bohr topology on $G$ will be called **trivial** if the only Bohr open subsets of $G$ are $\emptyset$ and $G$.

The Bohr topology on a group $G$ is trivial if and only if each homomorphism $h : G \to K$ to a compact Hausdorff topological group $K$ is constant.

Examples of groups with trivial Bohr topology are:

- the group $S_X$ of all permutations of an infinite set $X$;
- the group $A_X$ of all even finitely supported permutations of an infinite set $X$.

The group $A_X$ is locally finite and hence amenable.
The Bohr topology on $G$ will be called trivial if the only Bohr open subsets of $G$ are $\emptyset$ and $G$.

The Bohr topology on a group $G$ is trivial if and only if each homomorphism $h : G \to K$ to a compact Hausdorff topological group $K$ is constant.

Examples of groups with trivial Bohr topology are:

- the group $S_X$ of all permutations of an infinite set $X$;
- the group $A_X$ of all even finitely supported permutations of an infinite set $X$.

The group $A_X$ is locally finite and hence amenable.
The Bohr topology on $G$ will be called **trivial** if the only Bohr open subsets of $G$ are $\emptyset$ and $G$.

The Bohr topology on a group $G$ is trivial if and only if each homomorphism $h : G \to K$ to a compact Hausdorff topological group $K$ is constant.

Examples of groups with trivial Bohr topology are:

- the group $S_X$ of all permutations of an infinite set $X$;
- the group $A_X$ of all even finitely supported permutations of an infinite set $X$.

The group $A_X$ is locally finite and hence amenable.
Groups with trivial Bohr topology

The Bohr topology on \( G \) will be called **trivial** if the only Bohr open subsets of \( G \) are \( \emptyset \) and \( G \).

The Bohr topology on a group \( G \) is trivial if and only if each homomorphism \( h : G \to K \) to a compact Hausdorff topological group \( K \) is constant.

Examples of groups with trivial Bohr topology are:

- the group \( S_X \) of all permutations of an infinite set \( X \);
- the group \( A_X \) of all even finitely supported permutations of an infinite set \( X \).

The group \( A_X \) is locally finite and hence amenable.
The Bohr topology on $G$ will be called *trivial* if the only Bohr open subsets of $G$ are $\emptyset$ and $G$.

The Bohr topology on a group $G$ is trivial if and only if each homomorphism $h : G \to K$ to a compact Hausdorff topological group $K$ is constant.

Examples of groups with trivial Bohr topology are:

- the group $S_X$ of all permutations of an infinite set $X$;
- the group $A_X$ of all even finitely supported permutations of an infinite set $X$.

The group $A_X$ is locally finite and hence amenable.
Characterizing amenable groups with trivial Bohr topology

**Theorem**

If an amenable group $G$ has trivial Bohr topology, then for any right-Solecki positive sets $A, B \subseteq G$ we get

1. **$AB$ is right-Solecki one** and $G \setminus AA^{-1}$ is right-Solecki null;
2. $G = B^{-1}AA^{-1} = AA^{-1}A = ABB^{-1}A^{-1}$.

**Theorem**

An amenable group $G$ has trivial Bohr topology iff for every partition $G = A_1 \cup \cdots \cup A_n$ there is a cell $A_i$ with $A_iA_i^{-1}A_i = G$.

A group $G$ is **odd** if every element of $G$ has odd order.

**Theorem (B.-Nykyforchyn-Gavrylkiv, 2008)**

A group $G$ is odd iff for any partition $G = A_1 \cup A_2$ there is a cell $A_i$ of the partition such that $A_iA_i^{-1} = G$. 
Theorem

If an amenable group $G$ has trivial Bohr topology, then for any right-Solecki positive sets $A, B \subset G$ we get

1. $AB$ is right-Solecki one and $G \setminus AA^{-1}$ is right-Solecki null;
2. $G = B^{-1}AA^{-1} = AA^{-1}A = ABB^{-1}A^{-1}$.

Theorem

An amenable group $G$ has trivial Bohr topology iff for every partition $G = A_1 \cup \cdots \cup A_n$ there is a cell $A_i$ with $A_iA_i^{-1}A_i = G$.

A group $G$ is odd if every element of $G$ has odd order.

Theorem (B.-Nykyforchyn-Gavrylkiv, 2008)

A group $G$ is odd iff for any partition $G = A_1 \cup A_2$ there is a cell $A_i$ of the partition such that $A_iA_i^{-1} = G$. 
If an amenable group $G$ has trivial Bohr topology, then for any right-Solecki positive sets $A, B \subset G$ we get

1. $AB$ is right-Solecki one and $G \setminus AA^{-1}$ is right-Solecki null;
2. $G = B^{-1} AA^{-1} = AA^{-1} A = ABB^{-1} A^{-1}$.

An amenable group $G$ has trivial Bohr topology iff for every partition $G = A_1 \cup \cdots \cup A_n$ there is a cell $A_i$ with $A_iA_i^{-1}A_i = G$.

A group $G$ is odd if every element of $G$ has odd order.

A group $G$ is odd iff for any partition $G = A_1 \cup A_2$ there is a cell $A_i$ of the partition such that $A_iA_i^{-1} = G$. 
Characterizing amenable groups with trivial Bohr topology

**Theorem**

If an amenable group $G$ has trivial Bohr topology, then for any right-Solecki positive sets $A, B \subset G$ we get

1. $AB$ is right-Solecki one and $G \setminus AA^{-1}$ is right-Solecki null;
2. $G = B^{-1}AA^{-1} = AA^{-1}A = ABB^{-1}A^{-1}$.

**Theorem**

An amenable group $G$ has trivial Bohr topology iff for every partition $G = A_1 \cup \cdots \cup A_n$ there is a cell $A_i$ with $A_iA_i^{-1}A_i = G$.

A group $G$ is odd if every element of $G$ has odd order.

**Theorem (B.-Nykyforchyn-Gavrylkiv, 2008)**

A group $G$ is odd iff for any partition $G = A_1 \cup A_2$ there is a cell $A_i$ of the partition such that $A_iA_i^{-1} = G$. 
Characterizing amenable groups with trivial Bohr topology

Theorem

If an amenable group $G$ has trivial Bohr topology, then for any right-Solecki positive sets $A, B \subset G$ we get

1. $AB$ is right-Solecki one and $G \setminus AA^{-1}$ is right-Solecki null;
2. $G = B^{-1}AA^{-1} = AA^{-1}A = ABB^{-1}A^{-1}$.

Theorem

An amenable group $G$ has trivial Bohr topology iff for every partition $G = A_1 \cup \cdots \cup A_n$ there is a cell $A_i$ with $A_iA_i^{-1}A_i = G$.

A group $G$ is **odd** if every element of $G$ has odd order.

Theorem (B.-Nykyforchyn-Gavrylkiv, 2008)

A group $G$ is odd iff for any partition $G = A_1 \cup A_2$ there is a cell $A_i$ of the partition such that $A_iA_i^{-1} = G$. 
Theorem

If an amenable group $G$ has trivial Bohr topology, then for any right-Solecki positive sets $A, B \subseteq G$ we get

1. $AB$ is right-Solecki one and $G \setminus AA^{-1}$ is right-Solecki null;
2. $G = B^{-1}AA^{-1} = AA^{-1}A = ABB^{-1}A^{-1}$.

Theorem

An amenable group $G$ has trivial Bohr topology iff for every partition $G = A_1 \cup \cdots \cup A_n$ there is a cell $A_i$ with $A_iA_i^{-1}A_i = G$.

A group $G$ is odd if every element of $G$ has odd order.

Theorem (B.-Nykyforchyn-Gavrylkiv, 2008)

A group $G$ is odd iff for any partition $G = A_1 \cup A_2$ there is a cell $A_i$ of the partition such that $A_iA_i^{-1} = G$. 
Corollary

If a subset $A$ of an infinite alternating group $G = A_X$ is right-Solecki positive, then $AA^{-1}A = G$.

Problem

Is $AA^{-1}A = G$ for each (inner-invariant) right-Solecki positive set $A$ in an infinite permutation group $G = S_X$?

Applying some results of Bergman (2006) it is possible to prove:

Theorem (B., 2013)

For any inner-invariant Solecki positive subset $A$ of an infinite permutation group $G = S_X$ we get $(AA^{-1})^{18} = G$. 
Corollary

If a subset A of an infinite alternating group $G = A_X$ is right-Solecki positive, then $AA^{-1}A = G$.

Problem

Is $AA^{-1}A = G$ for each (inner-invariant) right-Solecki positive set $A$ in an infinite permutation group $G = S_X$?

Applying some results of Bergman (2006) it is possible to prove:

Theorem (B., 2013)

For any inner-invariant Solecki positive subset $A$ of an infinite permutation group $G = S_X$ we get $(AA^{-1})^{18} = G$. 
Corollary

If a subset $A$ of an infinite alternating group $G = A_{\infty}$ is right-Solecki positive, then $AA^{-1}A = G$.

Problem

Is $AA^{-1}A = G$ for each (inner-invariant) right-Solecki positive set $A$ in an infinite permutation group $G = S_\infty$?

Applying some results of Bergman (2006) it is possible to prove:

Theorem (B., 2013)

For any inner-invariant Solecki positive subset $A$ of an infinite permutation group $G = S_\infty$ we get $(AA^{-1})^{18} = G$. 

Taras Banakh

Solecki submeasures and densities on groups
Problem

Is there a group $G$ such that $\sigma(A) \in \{0, 1\}$ for any subset $A \subset G$?

Problem

Let $H$ be a meager analytic subgroup of a compact topological group $G$. Is $H$ Solecki null? (Yes, if $G$ is abelian).
Two Open Problems

Problem

Is there a group $G$ such that $\sigma(A) \in \{0, 1\}$ for any subset $A \subset G$?

Problem

Let $H$ be a meager analytic subgroup of a compact topological group $G$. Is $H$ Solecki null? (Yes, if $G$ is abelian).
Two Open Problems

Problem

Is there a group $G$ such that $\sigma(A) \in \{0, 1\}$ for any subset $A \subset G$?

Problem

Let $H$ be a meager analytic subgroup of a compact topological group $G$. Is $H$ Solecki null? (Yes, if $G$ is abelian).
T.Banakh, *Solecki submeasures and densities on groups*, preprint (arXiv:1211.0717).

* * *

Thanks!
T. Banakh, *Solecki submeasures and densities on groups*, preprint (arXiv:1211.0717).

* * *

Thanks!