Finite-Temperature Supersymmetry: The Wess-Zumino Model

Kai Kratzert

Deutsches Elektronen-Synchrotron DESY
D–22603 Hamburg, Germany

March 31, 2003

Abstract

We investigate the breakdown of supersymmetry at finite temperature. While it has been proven that temperature always breaks supersymmetry, the nature of this breaking is less clear. On the one hand, a study of the Ward-Takahashi identities suggests a spontaneous breakdown of supersymmetry without the existence of a Goldstino, while on the other hand it has been shown that in any supersymmetric plasma there should exist a massless fermionic collective excitation, the phonino. Aim of this work is to unify these two approaches. For the Wess-Zumino model, it is shown that the phonino exists and contributes to the supersymmetric Ward-Takahashi identities in the right way displaying that supersymmetry is broken spontaneously with the phonino as the Goldstone fermion.

e-mail: kratzert@mail.desy.de
1 Introduction

Supersymmetry has become a central element in the extensions of the standard model of elementary particle physics that currently attract most attention. It is not only an intriguing mathematical concept with a distinct importance for theoretical physics as the only nontrivial extension of the Poincaré algebra in relativistic quantum field theory, but it has some very attractive properties also from the point of view of phenomenology.

On the other hand, our insight into the evolution of the universe has made an enormous progress over the last decades, and there has been a stimulating interchange of ideas between cosmology and quantum field theory. Of particular importance is the behaviour of symmetries in a hot plasma like it is believed to have existed shortly after the big bang, since phase transitions may have left their traces in the present state of the universe. For example, it is well-known that the spontaneously broken gauge symmetry of the standard model is restored at high temperature, a phenomenon that appears quite generally for global and gauge symmetries. So, if we believe that supersymmetry was realized in the hot early stages of the universe in one or the other way, it is indispensable to understand its behaviour at high temperatures.

Considering the fact that this is such a fundamental problem, it is astonishing how little has been done to solve it. Although it was realized that supersymmetry generally breaks down at finite temperature already in the earliest works on the subject \[1, 2\], the subsequent literature contains many contradictory statements, and our understanding of the mechanism of supersymmetry breaking at finite temperature is still unsatisfactory.

The current status can be summarized as follows. Only recently, it has been rigorously proven that supersymmetry is always broken at any finite temperature \[3\]. In fact, this is not surprising, as in thermal field theory the ground state is described by a statistical ensemble with different populations of bosons and fermions. Since the thermal ground state is responsible for the breakdown of supersymmetry, it has much in common with a spontaneous breaking. Thus, a natural question to ask is whether it is associated with the existence of a massless Goldstone fermion. The investigation of the supersymmetric Ward-Takahashi identities \[4, 5\] showed that there must be a zero-energy Goldstone mode. However, since the rest frame of the heat bath also breaks Lorentz invariance, this mode is not necessarily associated with a propagating Goldstone particle. Nevertheless, in \[6\] (clarifying earlier ideas in \[7\]) it has been shown from a complementary point of view that any model with thermally broken supersymmetry should contain a massless fermionic collective excitation, similar to the appearance of sound waves in a medium with spontaneously broken Lorentz invariance. The associated particle was baptized phonino.
because of its similarity to the phonon. For the simple case of the Wess-Zumino model, the existence of this phonino was proved.

In this work, we will again focus on the Wess-Zumino model. Though it appears relatively simple and has already been investigated in a number of papers, it reveals a very interesting structure. Our aim is to unify the different results in the literature and to obtain a complete picture of the breakdown of supersymmetry in this model. To this end, we will give an explicit proof of the existence of the phonino and investigate its contributions to the Ward-Takahashi identities of broken supersymmetry. It turns out that supersymmetry is indeed broken spontaneously by the heat bath with the phonino playing the role of the Goldstone particle. The results obtained for the Wess-Zumino model allow us to infer the behaviour of more general models.

The paper is organized as follows. After briefly reviewing the theoretical framework and the Wess-Zumino model in sections 2 and 3, the established knowledge about supersymmetry and one-loop behaviour of the Wess-Zumino model will be presented and partly extended for our purposes in sections 4 and 5. In section 6, we present a full calculation of the fermion propagator for low momenta and give an explicit proof of the existence of the phonino. Finally, section 7 is devoted to the investigation of the Ward-Takahashi identities.

2 Thermal field theory

We will work in the framework of finite temperature field theory which is the appropriate formalism for the description of quantum fields in thermal equilibrium. The thermal background will be described by the canonical ensemble with a density matrix

\[ \rho = Z^{-1} e^{-\beta H}. \]

Here, \( H \) is the Hamiltonian, and the inverse temperature \( \beta = (k_B T)^{-1} \) is chosen in units so that Boltzmann’s constant is unity. The partition function \( Z \) normalizes the density operator so that the expectation value of an observable \( O \) is given by

\[ \langle O \rangle_\beta = \text{tr} \rho O. \]

The basic effect of the presence of the thermal background on free quantum fields is a modification of the propagators. Since annihilation operators do not annihilate the thermal ground state, the propagator of a scalar field \( A \) gets an additional thermal contribution. It reads

\[ D(p) = \int d^4 x e^{i p(x-y)} \langle T A(x) A(y) \rangle_\beta = \frac{i}{p^2 - m^2 + i\epsilon} + 2\pi \delta(p^2 - m^2) n_B(p_0), \quad (1) \]
where
\[ n_B(p_0) = \frac{1}{e^{\beta|p_0|} - 1} \] (2)
is the Bose-Einstein distribution function. Analogously, the free fermion propagator is given by
\[ S(p) = \int d^4x e^{ip(x-y)} \langle T\Psi(x)\overline{\Psi}(y) \rangle_\beta = \frac{i(\phi + m)}{p^2 - m^2 + i\epsilon} - (\phi + m) 2\pi \delta(p^2 - m^2) n_F(p_0), \] (3)
with the Fermi-Dirac distribution function
\[ n_F(p_0) = \frac{1}{e^{\beta|p_0|} + 1}. \] (4)

Already at this point it becomes clear that supersymmetry has a hard time at finite temperature since the different distribution functions lead to quite different thermal contributions to the bosonic and fermionic propagators.

The treatment of an interacting theory requires much more effort. From the two formalisms that have been developed for this purpose, we will make use of the real-time formalism since it is suited for the direct perturbative calculation of thermal correlation functions in real time by standard Feynman diagram techniques. For consistency, this formalism requires a doubling of the degrees of freedom. Formally, one introduces for each field a ghost field with the same interaction as the original field, only of opposite sign. The thermal propagator of a scalar field is then given by a matrix of the form
\[ \tilde{D}(p) = M(p_0) \begin{pmatrix} \frac{i}{p^2 - m^2 + i\epsilon} & 0 \\ 0 & -\frac{i}{p^2 - m^2 - i\epsilon} \end{pmatrix} M(p_0), \] (5)

where
\[ M(p_0) = \begin{pmatrix} \cosh \theta(p_0) & \sinh \theta(p_0) \\ \sinh \theta(p_0) & \cosh \theta(p_0) \end{pmatrix} \quad \text{with} \quad \sinh^2 \theta(p_0) = n_B(p_0). \]

The 11-component of this matrix gives back the thermal propagator \( \Pi \). More general correlation functions can be calculated perturbatively by evaluating the same diagrams as in the vacuum theory, where external lines always correspond to physical 1-fields. It will later turn out that the ghost fields do not lead to relevant contributions in our calculations, but neverless must be considered in a consistent treatment.

The matrix structure \( [3] \) allows also the understanding of self energy corrections. Without going into any details (for which we refer to the standard literature...
as \[9\]), we note that the full propagator must have a similar structure, and the thermal self energy corrections only lead to the usual shift
\[ p^2 - m^2 \rightarrow p^2 - m^2 - \Pi_{\beta}(p), \]
while the overall structure \(\Pi\) of the thermal propagator remains unchanged. The real part of the thermal self energy function \(\Pi_{\beta}(p)\) is directly accessible to a perturbative calculation. It coincides with the real part of the self energy function with physical external lines, while the imaginary part requires minor corrections.

In case the self energy is real and small compared to the mass, it only induces a small, temperature-dependent mass shift. One should however note that there is now an ambiguity in defining the mass. Because of the breakdown of Lorentz invariance in the heat bath, the self energy \(\Pi_{\beta}(p)\) not only depends on \(p^2\) but also on \(\vec{p}\). Therefore, the value of the propagator at zero momentum or its pole for vanishing three-momentum lead to different notions of mass while they coincide at zero temperature. For reasons that will become clear later, we will adopt the definition of mass as the pole of the full propagator in the limit of vanishing three-momentum,
\[ m^2_\beta = m^2 + \Pi_{\beta}(m, 0). \] (6)

The fermionic case appears quite similar. Here, the thermal self energy leads to a temperature-dependent shift
\[ \phi - m \rightarrow \phi - m - \Sigma_{\beta}(p) \]
so that the thermal mass is given by
\[ m^2_\beta = \frac{m}{2} \text{tr}[(1 + \gamma^0)\Sigma_{\beta}(m, 0)], \] (7)
as long as the self energy is small. In general, the propagator can of course have a much more complicated structure and in particular involve an imaginary damping part.

In finite temperature field theory, ultraviolet divergences can be absorbed by the same redefinition of the parameters as in the vacuum theory so that higher order vacuum contributions are small after renormalization. In this work, we will only be concerned with the additional thermal contributions.

3 The Wess-Zumino model

The Wess-Zumino model \[9\] is the simplest supersymmetric quantum field theory. It describes a single self-interacting chiral superfield
\[ \Phi = \phi + \sqrt{2} \theta \psi + \theta \theta F \]
whose component fields are a scalar field \( \phi \), a Weyl fermion \( \psi \) and an auxiliary scalar field \( F \). The Lagrangian reads

\[
\mathcal{L} = \Phi^\dagger \Phi + (W(\Phi))_{\theta\theta} + \text{h.c.}
\]

where the interaction is determined by the superpotential which we will take as

\[
W(\Phi) = \frac{m}{2}\Phi^2 + \frac{g}{3}\Phi^3.
\]

Without dwelling on the superspace formalism, we rewrite the Lagrangian in terms of the component fields. After eliminating the auxiliary field through its algebraic equation of motion, the on-shell Lagrangian reads

\[
\mathcal{L} = \frac{1}{2} \overline{\Psi} (i\partial - m) \Psi + \frac{1}{2} (\partial_\mu A \partial^\mu A - m^2 A^2) + \frac{1}{2} (\partial_\mu B \partial^\mu B - m^2 B^2)
\]
\[
- gm(A^2 + B^2) - \frac{1}{2} g^2 (A^2 + B^2)^2 - g \overline{\Psi} (A - i\gamma^5 B) \Psi.
\]

This Lagrangian describes a scalar field \( A \) and a pseudoscalar field \( B \), defined as the real components of the scalar field \( \phi \),

\[
A = \frac{1}{\sqrt{2}} (\phi + \phi^\dagger), \quad B = -\frac{i}{\sqrt{2}} (\phi - \phi^\dagger),
\]

in interaction with the Majorana fermion

\[
\Psi = \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix}.
\]

As a consequence of supersymmetry, all fields have equal mass \( m \), and all couplings are determined by the parameters \( g \) and \( m \). We assume the coupling \( g \) to be small, in order to allow a perturbative treatment.

The propagation of the three fields in a thermal background can be described by the usual real-time propagators for massive particles,

\[
D_{A/B}(q) = \frac{i}{q^2 - m^2 + i\epsilon} + 2\pi\delta(q^2 - m^2)n_B(q_0),
\]
\[
S(q) = (\not{q} + m) \left( \frac{i}{q^2 - m^2 + i\epsilon} - 2\pi\delta(q^2 - m^2)n_F(q_0) \right),
\]

where we will denote, in a diagrammatic language, \( A \)-bosons by dashed lines and \( B \)-bosons by wiggly lines, while solid lines stand for the fermion.
4 Supersymmetry

The action formed by the Lagrangian (9) is invariant under the on-shell supersymmetry transformations

\[\begin{align*}
\delta A &= \xi \Psi \\
\delta B &= \xi i \gamma^5 \Psi \\
\delta \Psi &= -(i\partial + m)(A + i\gamma^5 B)\xi - g(A + i\gamma^5 B)^2 \xi,
\end{align*}\]

where \(\xi\) is an infinitesimal fermionic transformation parameter. These continuous transformations are associated with a conserved current \(J^\mu\), the supercurrent. It is given by [10]

\[J^\mu = -(\partial + im)A\gamma^\mu \Psi + (\partial - im)i\gamma^5 B\gamma^\mu \Psi - ig(A + i\gamma^5 B)^2 \gamma^\mu \Psi.\]  

(14)

The supercharge

\[Q = \int d^3x J^0(x),\]

lets us express the above supersymmetry transformations as the commutator (for bosonic operators) or anticommutator (for fermionic ones) with the charge,

\[\delta \mathcal{O} = -i\xi [Q, \mathcal{O}]_\pm.\]

In terms of the supercharge, the supersymmetry algebra reads

\[\{Q, \overline{Q}\} = 2\gamma^\mu P_\mu,\]

(15)

where \(P_\mu\) is the energy-momentum operator. This relation with Poincaré symmetry displays why supersymmetry is necessarily broken in any thermal background. The nonvanishing energy density of the heat bath, \(\langle P_0\rangle_\beta \neq 0\), breaks Lorentz invariance spontaneously and, by relation (15), also supersymmetry breaks down.

4.1 Ward-Takahashi identities and Goldstone’s theorem

The existence of a conserved current is a highly nontrivial fact which leads to important relations between time-ordered correlation functions involving the symmetry current. The general form of these Ward-Takahashi identities reads

\[\partial_\mu \langle T J^\mu(x) \mathcal{O}_1(y_1) \cdots \mathcal{O}_n(y_n) \rangle = \sum_{i=1}^n \delta^{(4)}(x-y_i) \langle T \mathcal{O}_1(y_1) \cdots [Q, \mathcal{O}_i(y_i)]_\pm \cdots \mathcal{O}_n(y_n) \rangle.\]

(16)
It is important to note that these identities are basically operator identities, so that they are valid in the vacuum as well as in a thermal state, as it was established in [4]. Furthermore, they are valid even if supersymmetry is broken spontaneously. In the case of an explicit breaking, there is obviously no reason to expect their validity. Thus, the Ward-Takahashi identities provide a useful tool for the investigation of broken symmetries.

The spontaneous breakdown of supersymmetry is characterized by some fermionic operator $O$ transforming inhomogeneously,

$$\langle\{Q, O(y)\}\rangle \neq 0.$$ 

The corresponding Ward-Takahashi identity,

$$\partial^\mu \langle T J^\mu(x) O(y) \rangle = \delta^{(4)}(x-y) \langle\{Q, O(y)\}\rangle,$$

can be rewritten in momentum space. By defining

$$\Gamma^\mu_{J O}(k) = \int d^4x e^{ik(x-y)} \langle T J^\mu(x) O(y) \rangle,$$

one obtains

$$-ik_\mu \Gamma^\mu_{J O}(k) = \langle\{Q, O(y)\}\rangle \neq 0.$$ 

In order to satisfy this equation for all momenta $k$, the Fourier transformed correlation function $\Gamma^\mu_{J O}$ on the left hand side necessarily has a pole for $k = 0$. This is of course nothing but Goldstone’s theorem. In the vacuum, it follows from Lorentz invariance that there must be a pole for all lightlike momenta $k^2 = 0$, that is, there must exist a massless Goldstone particle. One cannot however draw this conclusion at finite temperature. In this case, Lorentz invariance is broken and there can in principle be an isolated pole for vanishing momentum without the need for a Goldstone particle. This observation lead the authors of [4, 5] to the conclusion that the thermal breakdown of supersymmetry is not associated with the existence of a Goldstone particle. A similar observation has been made for the related breakdown of Lorentz invariance [11]. In any case, the identification of a propagating Goldstone particle in a specific model requires some more effort.

For our case of the Wess-Zumino model, the simplest Ward-Takahashi identity one can consider is that for a single fermion operator,

$$\partial^\mu \langle T J^\mu(x) \Psi(y) \rangle = -i\delta^{(4)}(x-y)m \langle A \rangle.$$

On tree level, this identity is clearly fulfilled in a trivial way. The behaviour in the interacting theory is less trivial and will be discussed in section 7.
Secondly, we can establish the identity for the composite mode $A\Psi$. Inserting
the supersymmetry transformations (13) into the general formula (16), we obtain
\[
\partial_\mu<TJ_\mu(x)A(y)\Psi(z)\rangle_\beta = \delta^{(4)}(x-y) i<T\Psi(y)\Psi(z)\rangle_\beta
+ \delta^{(4)}(x-z)(\bar{\Psi}_y - im)(TA(y)A(z))_\beta + O(g).
\]

(18)

It is worth studying how this identity is satisfied at finite temperature. Let us
switch off the interaction for a while and consider the special case $y = z$. In
momentum space, the right hand side then reads
\[
\int d^4x e^{ik(x-y)} RHS = \int \frac{d^4q}{(2\pi)^4} (iS(q) + (-iq - im)D_A(q))
- 2\pi im \int \frac{d^4q}{(2\pi)^4} (n_F(q_0) + n_B(q_0))\delta(q^2 - m^2).
\]

(19)

As expected, the vacuum contributions from bosonic and fermionic propagators
cancel because of the equality of their masses. Thus, the right hand side is trivial at
zero temperature. At finite temperature, in contrast, the different thermal propa-
gators for boson and fermion no longer cancel but leave a nontrivial right hand side.
In the limits of nonrelativistic and relativistic temperatures, it is straightforward
to calculate,
\[
\int d^4x e^{ik(x-y)} RHS = \left\{ \begin{array}{ll}
-2ie^{-m/T} \left( \frac{Tm}{2\pi} \right)^{3/2} & T \ll m, \\
-imT^2 & T \gg m.
\end{array} \right.
\]

(20)

Thereby we have neglected any corrections suppressed by factors of order $T/m$
and $m/T$, respectively, which will always be done in the following, as long as they
do not become relevant.

The left hand side of (18) gives
\[
LHS = \partial_\mu<T(-\bar{\Psi}_x - im)A(x)\gamma^\mu\Psi(x)A(y)\Psi(y)\rangle_\beta
= \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \int \frac{d^4q}{(2\pi)^4} (i\not{k} - i\not{q} - im)D_A(k - q)(-i\not{q})S(q).
\]

The loop integral gets two contributions from the thermal part of either of the
propagators. The knowledge that these have support on the mass shell simplifies
the algebra considerably, and one calculates

\[ \int d^4 x e^{ik(x-y)} LHS = \int \frac{d^4 q}{(2\pi)^4} \frac{i(k - q - m)k(k + m)}{(k - q)^2 - m^2 + i\epsilon} (-2\pi)n_F(q_0)\delta(q^2 - m^2) \]

\[ + \int \frac{d^4 q}{(2\pi)^4} \frac{i(q - m)k(k - q + m)}{(k - q)^2 - m^2 + i\epsilon} 2\pi n_B(q_0)\delta(q^2 - m^2) \]

\[ = -2\pi i m \int \frac{d^4 q}{(2\pi)^4} (n_F(q_0) + n_B(q_0))\delta(q^2 - m^2). \]

So, we are left with the same result as for the right hand side (19), and we have shown that the Ward-Takahashi identity (18) is fulfilled in a nontrivial way. In the vacuum, this would be possible only if there was a massless Goldstone particle generating the pole in the correlation function on the left hand side. At finite temperature, the situation is different. Although there is obviously no Goldstone fermion in the free, massive theory, nevertheless the identity is non-trivially satisfied.

The identity for the mode \( B\Psi \) behaves in the same way. One has

\[ \partial_\mu \langle T J^\mu(x) B(y) \bar{\Psi}(y) \rangle_\beta = \partial_\mu \langle T J^\mu(x) A(y) \bar{\Psi}(y) \rangle_\beta i\gamma^5. \quad (22) \]

Comparing the relative couplings of the two modes to the supercurrent, one can identify the linear combination

\[ \chi = A\Psi - i\gamma^5 B\Psi \quad (23) \]

as the Goldstone mode. This is not surprising once rewritten in terms of the original chiral superfield by means of equations (10) and (11),

\[ (A - i\gamma^5 B)\Psi = \left( \frac{\Phi^2}{\Phi^{12}} \right) \]

(24)

Thus, the Goldstone mode is the fermionic component field of the composite superfield \( \Phi^2 \) that determines the mass term in the superpotential (8). Note that the result (20) is nontrivial only because of this mass term.

It will turn out in section 7 that in the full interacting theory the Ward-Takahashi identities are satisfied in a somewhat different way. It will be shown that the Goldstone mode indeed corresponds to a propagating particle. Furthermore, the Goldstone mode \( \chi \) interacts with the fundamental fermion by means of the Yukawa interaction

\[ \mathcal{L}_Y = -g\bar{\Psi}(A - i\gamma^5 B)\Psi. \]

Thereby, a Goldstone pole will appear also in the fermion propagator which reflects itself in a nontrivial behaviour of the Ward-Takahashi identity (17).
So far, our discussion overlaps with earlier results from [4]. Let us go one step further and see what we can learn for the general case. Clearly, the way the first identity (17) is satisfied is highly dependent on the details of the model. The second identity (18) is more universal since it involves, to leading order, only the mass and behaves non-trivially already at the tree level. However, we would rather expect a nontrivial behaviour completely independent of masses and couplings since, for the reasons given above, the thermal breakdown of supersymmetry is a universal phenomenon. According to the supersymmetry algebra (15), it is tied to that of Lorentz invariance. Consequently, the most general Ward-Takahashi identity one should consider is that for the supercurrent itself. Its transformation law is given by

\[ \{ Q, J^\mu \} = 2 T^\mu\nu \gamma^\nu + \ldots, \]  

where the additional terms vanish when taken as the expectation value in a translation invariant state like the vacuum or a thermal equilibrium state [10]. This is basically the local form of (15) as the energy-momentum tensor \( T_{\mu\nu} \) is the current generating the translations \( P^\mu \). The corresponding Ward-Takahashi identity then reads

\[ \partial^\mu (T J^\mu(x), T^\nu(y)) = \delta^{(4)}(x - y) 2 \langle T^\mu \rangle \gamma_\mu \]  

and is completely independent of the model. The nonvanishing of the right hand side is characteristic to any thermal background, and the identity implies that a nontrivial Goldstone mode must appear in all operators contributing to the supercurrent. In this way, the identity (18) is rather a special case since, in the massive theory, the bilinear operator \( A\Psi \) is part of the supercurrent (14).

Let us verify the identity (26) for our case of the Wess-Zumino model. The right hand side involves the thermal expectation value of the energy-momentum tensor. In a thermal equilibrium state, it reads

\[ \langle T^{\mu\nu} \rangle_\beta = \text{diag}(\rho, p, p, p), \]

where the energy density \( \rho \) is given by

\[ \rho = \int \frac{d^3 q}{(2\pi)^3} \rho q \left( 2n_F(E_q) + 2n_B(E_q) \right) = \begin{cases} 4m e^{-m/T} \left( \frac{Tm}{2\pi} \right)^{3/2} & T \ll m, \\ \frac{\pi^2}{8} T^4 & T \gg m. \end{cases} \]

The pressure \( p \) is related with the energy density as

\[ p = \begin{cases} \frac{T}{m} \rho & T \ll m, \\ \frac{1}{3} \rho & T \gg m. \end{cases} \]

10
Hence, on the right hand side of the identity (26) we find, for \( T \ll m \),

\[
2\langle T^\nu\mu \rangle_\beta \gamma_\mu = 8\pi^2 e^{-m/T} \left( \frac{Tm}{2\pi} \right)^{3/2} \frac{\gamma_0}{Tm} \gamma^\gamma.
\]  

(27)

For \( T \gg m \), we have

\[
2\langle T^\nu\mu \rangle_\beta \gamma_\mu = \frac{\pi^2}{4} T^4 \left( \frac{\gamma_0}{\frac{1}{3} \gamma^\gamma} \right).
\]  

(28)

This must be compared with the left hand side of (26),

\[
\text{LHS} = \partial_\mu \langle T(-\partial_x - im)A(x)\gamma^\mu \Psi(x)\overline{\Psi}(y)\gamma^\nu(-\partial_y + im)A(y) \rangle_\beta
+ \partial_\mu \langle T(\partial_x - im)\gamma^5 B(x)\gamma^\mu \Psi(x)\overline{\Psi}(y)\gamma^\nu\gamma^5(\partial_y + im)B(y) \rangle_\beta.
\]

It can be evaluated in the same manner as equation (21). The same cancellations take place, and one obtains, after some algebra,

\[
\int d^4x e^{ik(x-y)} \text{LHS} = 4 \int \frac{d^4q}{(2\pi)^4} q^\nu q^\lambda 2\pi (n_F(q_0) + n_B(q_0)) \delta(q^2 - m^2).
\]

(29)

Calculating the remaining integral, one ends up with exactly the same result as for the right hand side, equations (27) and (28). Thus, the Ward-Takahashi identity (26) is shown to be fulfilled in a nontrivial way.

4.2 Supersymmetric sound

In view of the fact that the quantum field theoretical framework not necessarily requires a propagating Goldstone particle associated with the thermal breakdown of supersymmetry, it is even more interesting that the existence of a massless fermionic excitation can be deduced in a complementary setting. In [6] (see also [12]), a hydrodynamic approach to the supersymmetric plasma has lead to the prediction of a slow-moving collective excitation whose existence should be as general as that of sound waves.

Consider a relativistic perfect fluid with an energy-momentum tensor of the form

\[
\langle T^\mu\nu \rangle_\beta = \text{diag}(\rho, p, p, p).
\]

If the system is disturbed by a small, spacetime-dependent variation of, say, the temperature, \( \Delta T(x) \), the conservation of the energy-momentum tensor, \( \partial_\mu \langle \delta T^\mu\nu \rangle_\beta = 0 \), translates to a wave equation for the disturbance,

\[
\left( \frac{\partial \rho}{\partial T} \frac{\partial^2}{\partial T^2} - \frac{\partial p}{\partial T} \frac{\partial^2}{\partial \vec{x}^2} \right) \Delta T(x) = 0.
\]  

(29)
It describes the propagation of sound waves with the velocity

\[ v_S^2 = \frac{\partial p/\partial T}{\partial \rho/\partial T}. \]  (30)

Thus, as a consequence of the breakdown of Lorentz symmetry in the thermal bath, small perturbations in the energy-momentum tensor propagate as sound waves. Their quanta, the phonons, can be viewed as the Goldstone bosons associated with the spontaneous symmetry breaking.

Now, in the supersymmetric case, one can imagine the system undergoing a small, spacetime-dependent supersymmetry variation \( \xi(x) \). The conservation of the supercurrent, \( \partial_\mu \langle \delta J^\mu(x) \rangle_\beta = 0 \), translates to a wave equation for the transformation parameter,

\[ (\rho \gamma_0 \partial_0 + p \gamma_0 \partial_0) \xi(x) = 0, \]  (31)

where the components of the energy-momentum tensor enter through the transformation law (25). This Dirac equation describes a massless fermionic excitation propagating with the velocity

\[ v_{SS} = \frac{p}{\rho}. \]  (32)

Thus, in a system whose supersymmetry is broken by the thermal bath, there should exist ‘supersymmetric sound waves’. Their quanta, interpreted as the Goldstone fermions associated with the spontaneous symmetry breaking, are naturally called phoninos.

Both sound and supersymmetric sound have a very characteristic dispersion law given by

\[ v_S^2 = v_{SS} = \begin{cases} \frac{T}{m} & T \ll m \\ \frac{1}{3} & T \gg m \end{cases} \]  (33)

in the non-relativistic and relativistic limits.

Despite the similarity to sound waves, there is surely no classical picture of these fermionic waves. It should however be possible to verify their existence in the framework of thermal field theory. Just as sound waves appear as poles in the correlator \( \langle T_{\mu\nu}(x)T_{\rho\sigma}(y) \rangle_\beta \), supersymmetric sound must lead to poles in the correlation function \( \langle J_\mu(x)J_\nu(y) \rangle_\beta \). Since supersymmetric sound appears as a collective phenomenon, it is expected to appear in the quantum field theoretical framework only as a nonperturbative effect.

5 One-loop corrections

Though the thermal breakdown of supersymmetry is evident already at the tree level, a deeper understanding certainly requires the investigation of higher-order...
effects. The one-loop effects have already been investigated in earlier works, so that we can refer to [13, 6] for explicit calculations.

Firstly, the interaction with the heat bath leads to the appearance of a non-trivial expectation value of the fundamental scalar field. Its value is given by

$$\langle A \rangle_\beta = \begin{cases} -g\alpha m & T \ll m, \\ -\frac{gT^2}{2m} & T \gg m, \end{cases}$$

(34)

where we have introduced the small parameter

$$\alpha = 8m^{-3} \int \frac{d^3q}{(2\pi)^3} e^{-(m+\frac{q^2}{2m})/T} = e^{-m/T} \left( \frac{2T}{\pi m} \right)^{3/2}. \quad (35)$$

The nonzero value is another clear sign for the breakdown of supersymmetry. It must be stressed, however, that it is not a necessary condition like in the vacuum. In the massless Wess-Zumino model, for example, the expectation value vanishes because of chiral symmetry while supersymmetry is nevertheless broken for the general reasons given above.

Secondly, the interaction leads to a modification of the propagators. The dominant self energy corrections come from the proper one-loop diagrams and, with a contribution of the same order of magnitude, from the coupling to the thermal expectation value of the scalar field. For momenta $p = (m, 0)$, the relevant real parts of the thermal self energy functions $\Pi_i^\beta(p)$ have been calculated in [6]. Their result can easily be generalized to momenta on the mass shell, since none of the diagrams shows a dependence on the three-momentum $\vec{p}$, as long as $p^2 = m^2$. As a consequence, the self energy corrections only lead to a small displacement of the masses,

$$m_i^2 = m^2 + \Pi_i^\beta(p) \bigg|_{p^2=m^2}$$

without changing the dispersion relation

$$p_0^2 = \vec{p}^2 + m_i^2.$$  

Explicitly, the values of the effective thermal masses are given by

$$T \ll m : \begin{cases} m_A^2 = m^2 - \frac{20}{3}g^2\alpha m^2 \\ m_B^2 = m^2 \\ m_\Phi^2 = m^2 - 2g^2\alpha m^2 \end{cases} \quad T \gg m : \begin{cases} m_A^2 = m^2 - 2g^2T^2 \\ m_B^2 = m^2 \\ m_\Phi^2 = m^2 - g^2T^2 \end{cases} \quad (36)$$

As yet another evidence for the breakdown of supersymmetry, we find that the mass degeneracy of the three fields is lifted by the one-loop thermal self energy
corrections. As expected, the mass splitting vanishes in the limit of zero temperature when supersymmetry is restored.

In a model where supersymmetry is broken spontaneously already in the vacuum, the masses of the members of one supermultiplet are related by the mass formula

$$\sum J (-1)^{2J} (2J + 1) m_J^2 = 0.$$  

For the effective thermal masses $\Sigma$, we find that this relation is no longer valid, at least for low temperatures where $m_A^2 + m_B^2 \neq 2m_\Psi^2$. This raises the question whether the breakdown of supersymmetry should at all be considered spontaneous or rather explicit.

6 The phonino

In the previous section, we have seen that the influence of the heat bath on the fundamental fields is basically a small shift in their masses. According to the general arguments given in section 4, however, we suspect the existence of additional fermionic excitations. Hence, we should take a closer look at the full self energy function that determines the poles of the full fermion propagator. These are characterized by the condition

$$\det S^{-1}(p) = -i \det (\not p - m - \Sigma(p)) = 0.$$  

To leading order, the self energy $\Sigma$ gets contributions from the coupling to the expectation value of the scalar field $A$ as well as from the proper one-loop diagrams as drawn in figure 1. Altogether, one finds

$$-i \Sigma(p) = -2ig\langle A \rangle - 4g^2 \int \frac{d^4q}{(2\pi)^4} D(p - q) \left( S(q) + i\gamma^5 S(q)i\gamma^5 \right).$$  

We are particularly interested in the limit of small momenta. Thus, we set the...
three-momentum to zero and calculate

$$\Sigma_\beta(p_0, 0) = 2g\langle A\rangle_\beta + \frac{8g^2\gamma^0}{p_0} \int \frac{d^3q}{(2\pi)^3} \left( \frac{n_B(E_q)2E_q^2 - p_0^2}{E_q} + \frac{n_F(E_q)2E_q^2}{E_q} \right),$$

where $E_q = \sqrt{q^2 + m^2}$. One observes that the second part develops a pole for vanishing momentum. The reason is of course the degeneracy of bosonic and fermionic masses. In the limit $p \to 0$, the poles of both propagators in the loop coincide, leading to an anomalously big self energy.

In the relevant limits of low and high temperatures, the integration can easily be performed. For small momenta $p_0$, the result can be expressed in terms of $\langle A\rangle_\beta$ as

$$\Sigma_\beta(p_0, 0) = 2g\langle A\rangle_\beta - \frac{mg\langle A\rangle_\beta}{p_0}\gamma^0.$$ 

Inserting this result into condition (37) for the propagator poles, one obtains the dispersive equation

$$\left( p_0 + \frac{mg\langle A\rangle_\beta}{p_0} \right)^2 = (m + 2g\langle A\rangle_\beta)^2$$

with the solutions

$$|p_0| = m + g\langle A\rangle_\beta + O(g^2\langle A\rangle_\beta^2),$$

$$|p_0| = g\langle A\rangle_\beta + O(g^2\langle A\rangle_\beta^2).$$

The first solution reproduces the small shift in the fermion mass that was already calculated in the previous section. The existence of a second solution indicates another excitation with a tiny mass. In fact, such a pole is what we expect, albeit with exactly vanishing mass. It thus seems that our simple one-loop calculation is not sufficient.

As it will turn out in the following sections, a consistent calculation of the full propagator requires to take into account the full propagators in the internal lines, that is, to consider the differences in the effective masses of bosons and fermion due to the breakdown of supersymmetry. In this way, one must perform a nonperturbative, selfconsistent calculation in order to uncover the desired massless pole in the fermion propagator.

### 6.1 The case of high temperature

Let us now analyze the fermion self energy more carefully. At first, we restrict ourselves to the relativistic case. By the assumption that the mass shifts are
small compared to the mass, our discussion is then limited to a temperature range \( m \ll T \ll g^{-1}m \).

Since the interaction with the heat bath modifies the propagators by a small, constant mass correction, we will approximate the full propagators by the one-loop resummed propagators

\[
S(q) = (\Slash{q} + m + \Sigma'(q)) \left( \frac{i}{q^2 - m_\Psi^2 + i\epsilon} - 2\pi\delta(q^2 - m_\Psi^2)n_F(q_0) \right)
\]

\[
\mathcal{D}_{A/B}(q) = \frac{i}{q^2 - m_{A/B}^2 + i\epsilon} + 2\pi\delta(q^2 - m_{A/B}^2)n_B(q_0)
\]  

with the effective thermal masses as given in equation (36). As one can check from equation (38), the one-loop self energy \( \Sigma_\beta \) splits into a constant part \( \Sigma_1 \) proportional to the unit matrix and a traceless part \( \Sigma_\gamma \) (which is an odd function of the momentum), so that one has

\[
\Sigma'(q) = \Sigma_1 - \Sigma_\gamma(q).
\]

With this approximation for the full propagators, we can now evaluate the fermion self energy (38). For small momenta \( k \), the product of both propagators in the loop can be approximated as

\[
\mathcal{D}_i(k - q)S(q) = -2\pi i (\Slash{q} + m + \Sigma'(q)) \delta(q^2 - m^2) \frac{n_B(q_0) + n_F(q_0)}{m_\Psi^2 - m_i^2 - 2qk},
\]

where we only keep corrections linear in \( k \) to small terms of order \( g^2 \). A general loop integral then reads, for sufficiently regular \( f \),

\[
\int \frac{d^4q}{(2\pi)^4} \mathcal{D}_i(k - q)S(q)f(q)
\]

\[
= -i \int \frac{d^3q}{(2\pi)^3} \sum_{q_0 = \pm E_q} \frac{n_B(E_q) + n_F(E_q)}{2E_q} \frac{\Slash{q} + m + \Sigma'(q)}{m_\Psi^2 - m_i^2 - 2qk} f(q).
\]

One finds that the small mass splitting between bosons and fermion regularizes the loop integral. So, in contrast to the one-loop calculation with tree-level masses, the integral no longer diverges in the limit \( k = 0 \) but becomes anomalously large because of the small mass difference appearing in the denominator. We assume that \( qk \ll m_\Psi^2 - m_i^2 \) with a typical loop momentum \( q \sim T \) which restricts ourselves to momenta \( k \ll g^2T \).
The explicit calculation of the full self energy then gives

\[ \Sigma_\beta(k) = -\frac{4g^2}{m_\Psi^2 - m_A^2}(m + \Sigma_1) \int \frac{d^3 q}{(2\pi)^3} \frac{n_B(E_q) + n_F(E_q)}{E_q} \]

\[ -\frac{4g^2}{m_\Psi^2 - m_B^2}(-m - \Sigma_3) \int \frac{d^3 q}{(2\pi)^3} \frac{n_B(E_q) + n_F(E_q)}{E_q} \]

\[ -8g^2 \left( k_0 \gamma_0 + \frac{1}{3} \vec{k} \vec{\gamma} \right) \left( \frac{1}{(m_\Psi^2 - m_A^2)^2} + \frac{1}{(m_\Psi^2 - m_B^2)^2} \right) \cdot \int \frac{d^3 q}{(2\pi)^3} E_q \left( n_B(E_q) + n_F(E_q) \right) \]

\[ + \Sigma_1. \]

One can now insert the mass splittings resulting from equation (36),

\[ m_\Psi^2 - m_A^2 = m_B^2 - m_\Psi^2 = g^2 T^2, \]

and perform the integration. One ends up with the simple result

\[ \Sigma_\beta(k) = -m - \frac{\pi^2}{g^2} \left( k_0 \gamma_0 + \frac{1}{3} \vec{k} \vec{\gamma} \right). \]

Remarkably, the result is exactly the negative Lagrangian mass so that, in our approximation, the full inverse fermion propagator reads

\[ iS^{-1}(k) = \not{k} - m - \Sigma_\beta(k) \approx \frac{\pi^2}{g^2} \left( k_0 \gamma_0 + \frac{1}{3} \vec{k} \vec{\gamma} \right). \quad (42) \]

Because of the cancellation of the mass term, we find that the full fermion propagator has an additional pole for vanishing momentum as well as on the dispersion curve

\[ k_0 = \pm \frac{1}{3} |\vec{k}|. \quad (43) \]

This displays the existence of a massless excitation propagating with velocity \( v = \frac{1}{3} \), just as one expects from the general arguments given in section 4.2. Consequently, we identify this pole in the fermion propagator as the phonino.

It should be added that this mechanism to cancel the fermion mass by loop corrections goes back to a widely unnoticed paper by Kapusta who performed a similar calculation for the somewhat simpler case of the massless Wess-Zumino model. However, the author attributed the existence of the soft fermion to chiral symmetry and not to the breakdown of supersymmetry.
6.2 The case of low temperature

The above calculation can in principle be translated also to the case of low temperature. It turns out, however, that this case requires a more thorough investigation. We have seen that contributions of the form (40) from two propagators of nearly degenerate mass and opposite momentum lead to an anomalously large value of the self energy. So, for consistency, one must take care of all diagrams of a similar structure that could give equally large contributions.

Take a look at the vertex corrections in figure 2. For vanishing external momentum and low temperature, the contribution from diagram b is of order

\[ g^3 m \int \frac{d^3 q}{(2\pi)^3} \frac{n_F(E_q) + n_B(E_q)}{E_q} \left( \frac{m}{m^2_\psi - m^2_A m^2} \right) \frac{1}{\sqrt{\frac{E_q}{m}}} \sim g. \]

Hence, it is as relevant as the tree-level coupling. For high temperature, in contrast, one finds

\[ g^3 m \int \frac{d^3 q}{(2\pi)^3} \frac{n_F(E_q) + n_B(E_q)}{E_q} \left( \frac{T}{m^2_\psi - m^2_A m^2} \right) \frac{1}{\sqrt{\frac{T}{T}}} \sim \frac{m}{T^3 g}, \]

so it does not give a relevant contribution because of the tree-level Yukawa coupling being proportional to \( m \ll T \). The same is true for diagrams with fermion exchange like in figure 2c. Thus, for high temperature there is no need to take these diagrams into account, while for low temperatures one must evaluate all relevant contributions to the vertex.

Obviously, many-loop diagrams play an equally important role, as long as each intermediate vertex or line is again followed by a pair of particles with a tiny mass difference, which leads to a cancellation of the small coupling constants and distribution functions that otherwise suppress any higher-order diagram. Altogether, one must sum up all ladder diagrams of this kind.

The standard way to perform such a resummation is by solving the corresponding Bethe-Salpeter equations. Their general form reads, diagrammatically,
Here, the shaded spot stands for the full proper (one-particle irreducible) vertex while the square denotes any proper (two-particle irreducible) scattering subdiagram, and a sum over all possible intermediate states is implied. The internal lines stand for full propagators.

In our specific case, the intermediate state in the Bethe-Salpeter equation is always a boson-fermion pair of nearly degenerate mass. The two-particle irreducible scattering amplitude gets contributions from both boson and fermion exchange. So, the Bethe-Salpeter equation for the vertex function involving the $A$ boson reads

$$\Psi_A = \frac{1}{AB} + \frac{1}{AC} + \frac{1}{AD} + \ldots + \frac{1}{AF} \tag{44}$$

A similar equation must hold for the $B$ boson vertex,

$$\Psi_B = \frac{1}{AG} + \frac{1}{AH} + \frac{1}{AI} + \frac{1}{AJ} + \frac{1}{AK} + \ldots + \frac{1}{AL} \tag{45}$$

Let us translate these diagrammatic equations into mathematical language. The amputated full proper vertex functions will be denoted by $\Gamma_A(k, p)$ and $\Gamma_B(k, p)$, respectively,

$$\equiv \Gamma_A(k, p), \quad \equiv \Gamma_B(k, p),$$

which we want to calculate for small momenta $k$. The thermal contributions to the Bethe-Salpeter equations come from loop momenta on the mass shell. Therefore, we consider only on-shell momenta $p^2 = m^2$.

All contributions to the Bethe-Salpeter equations have a very similar structure. Approximating the full propagators in the internal lines by the one-loop resummed propagators $\langle 39 \rangle$, we again find expressions of the form $\langle 41 \rangle$. For simplicity, we introduce

$$\Delta_A = m^2_{\Psi} - m^2_A = \frac{14}{3} g^2\alpha m^2, \quad \Delta_B = m^2_{\Psi} - m^2_B = -2g^2\alpha m^2. \tag{46}$$
Like in the case of high temperatures, we only consider momenta $k$ small compared to the mass differences and keep only the linear terms, while small corrections of order $\Delta_i/m$ or $k/m$ will be neglected. Furthermore, we can safely neglect contributions of order $\tilde{p}/m$ or $\tilde{q}/m$ (where $q$ is the momentum of the fermion in the loop), since $\tilde{p}$ and $\tilde{q}$ are thermal momenta of order $\sqrt{Tm}$ which, for low temperature, is negligible compared to the mass. The propagators of the exchanged particles are then essentially constant but depend on the relative sign of the zero components of outgoing and loop momenta.

Altogether, the Bethe-Salpeter equation for the $A$ boson vertex can be written as

$$\Gamma_A(k, p) = -2ig - 4g^2 \int \frac{d^3q}{(2\pi)^3} \frac{e^{-E_q/T}}{E_q} \cdot \left\{ \begin{array}{l} \left( \frac{\psi + 2m}{3m^2} + \frac{3m}{-m^2} \right) \frac{\dot{q} + m}{\Delta_A - 2qk} \Gamma_A(k, q) \bigg|_{q_0 = E_q \cdot \text{sign} \, p_0} \\ + \left( \frac{-\psi + 2m}{-m^2} + \frac{3m}{3m^2} \right) \frac{\dot{q} + m}{\Delta_A - 2qk} \Gamma_A(k, q) \bigg|_{q_0 = -E_q \cdot \text{sign} \, p_0} \\ - \left( \frac{-\psi + 2m}{3m^2} + \frac{m}{-m^2} \right) \frac{-\dot{q} + m}{\Delta_A - 2qk} i\gamma^5 \Gamma_B(k, q) \bigg|_{q_0 = E_q \cdot \text{sign} \, p_0} \\ - \left( \frac{-\psi + 2m}{-m^2} + \frac{m}{3m^2} \right) \frac{-\dot{q} + m}{\Delta_A - 2qk} i\gamma^5 \Gamma_B(k, q) \bigg|_{q_0 = -E_q \cdot \text{sign} \, p_0} \end{array} \right\}.$$  

(47)

For the $B$ boson vertex, one finds

$$i\gamma^5 \Gamma_B(k, p) = -2ig - 4g^2 \int \frac{d^3q}{(2\pi)^3} \frac{e^{-E_q/T}}{E_q} \cdot \left\{ \begin{array}{l} \left( \frac{\psi}{3m^2} + \frac{-m}{-m^2} \right) \frac{\dot{q} + m}{\Delta_A - 2qk} \Gamma_A(k, q) \bigg|_{q_0 = E_q \cdot \text{sign} \, p_0} \\ + \left( \frac{-\psi}{-m^2} + \frac{-m}{3m^2} \right) \frac{\dot{q} + m}{\Delta_A - 2qk} \Gamma_A(k, q) \bigg|_{q_0 = -E_q \cdot \text{sign} \, p_0} \\ - \left( \frac{-\psi}{3m^2} + \frac{m}{-m^2} \right) \frac{-\dot{q} + m}{\Delta_A - 2qk} i\gamma^5 \Gamma_B(k, q) \bigg|_{q_0 = E_q \cdot \text{sign} \, p_0} \\ - \left( \frac{-\psi}{-m^2} + \frac{m}{3m^2} \right) \frac{-\dot{q} + m}{\Delta_A - 2qk} i\gamma^5 \Gamma_B(k, q) \bigg|_{q_0 = -E_q \cdot \text{sign} \, p_0} \end{array} \right\}.$$  

(48)

These seemingly complex coupled integral equations can be simplified consid-
erably by introducing the functions

\[ \tilde{\Gamma}_A(k, p) = \frac{\not{p} + m}{\Delta_A - 2p k} \Gamma_A(k, p), \quad \tilde{\Gamma}_B(k, p) = -i\gamma^5 \frac{\not{p} + m}{\Delta_B - 2p k} \Gamma_B(k, p). \]

Rewriting the Bethe-Salpeter equations in terms of these functions, it turns out that the right hand sides involve only the symmetric parts

\[ V_{A,B}(k, p) = \left( \tilde{\Gamma}_{A,B}(k, p) + \tilde{\Gamma}_{A,B}(k, -p) \right) \bigg|_{p_0 = E_p}. \]  

(49)

By straightforward calculation, one can express the Bethe-Salpeter equations in terms of the functions \( V_{A,B} \) as

\[ V_A(k, p) = \left( \frac{2m}{\Delta_A} + \frac{4\not{p} k}{\Delta_A^2} \right) \left\{ -2ig + \frac{4g^2}{m^2} \int \frac{d^3q}{(2\pi)^3} e^{-E_q/T} \left( 2V_A(k, q) + \frac{2}{3} V_B(k, q) \right) \right\} \]

\[ V_B(k, p) = \left( -\frac{2m}{\Delta_B} + \frac{4\not{p} k}{\Delta_B^2} \right) \left\{ -2ig + \frac{4g^2}{m^2} \int \frac{d^3q}{(2\pi)^3} e^{-E_q/T} \left( -\frac{2}{3} V_A(k, q) + \frac{2}{3} V_B(k, q) \right) \right\} , \]  

(50)

where we have again made a linear approximation for small momenta \( k \).

For vanishing \( k \), one finds that the functions \( V_{A,B}(0, p) \) do not depend on the momentum \( p \) at all. The integration can then easily be performed by making use of the relations

\[ \frac{4g^2}{m^2} \int \frac{d^3q}{(2\pi)^3} e^{-E_q/T} = \frac{1}{2} g^2 \alpha m = \frac{3}{28} \Delta_A m = -\frac{1}{4} \Delta_B m, \]

which immediately follow from equations (35) and (46). One arrives at the system of linear equations

\[ V_A(0, p) = \frac{2m}{\Delta_A} \left\{ -2ig + \frac{3\Delta_A}{28} \left( 2V_A(0, p) + \frac{2}{3} V_B(0, p) \right) \right\} \]

\[ V_B(0, p) = -\frac{2m}{\Delta_B} \left\{ -2ig - \frac{\Delta_B}{4} \left( -\frac{2}{3} V_A(0, p) + \frac{2}{3} V_B(0, p) \right) \right\} \]  

(51)

which is easy to solve. The solution is given by

\[ V_A(0, p) = V_B(0, p) = \frac{2m}{\Delta_B} \cdot 2ig. \]  

(52)

It is illustrative to express this result in terms of the amputated vertex functions \( \Gamma_A \) and \( \Gamma_B \) we started with. One finds

\[ \Gamma_B(0, p) = -2ig \left( -i\gamma^5 \right), \quad \Gamma_A(0, p) = -2ig \left( -\frac{\Delta_A}{\Delta_B} \right) \].  

(53)
In other words, the Yukawa coupling to the \( B \) boson comes out of the whole resummation without a change while the coupling to the \( A \) boson is enhanced by a factor of
\[
\frac{\Delta_A}{\Delta_B} = \frac{7}{3}.
\]

Let us now calculate the momentum dependence of the vertex functions. To this end, we expand in \( k \),
\[
V_{A,B}(k, p) = V_{A,B}(0, p) + k_\mu C_{A,B}^\mu(p),
\]
which is substituted in the Bethe-Salpeter equations (50). The resulting integral equations for the functions \( C_{A,B}^\mu(p) \) can be simplified by replacing the known values of \( V_{A,B}(0, p) \) and integrating, so that one obtains
\[
k_\mu C_{A}^\mu(p) = -\frac{14}{3} g \frac{4 \phi p_{\mu} k_{\mu}}{\Delta_A^2} + \frac{8g^2}{m\Delta_A} \int \frac{d^3q}{(2\pi)^3} e^{-E_q/T} \left( 2k_\mu C_A^\mu(q) + \frac{2}{3} k_\mu C_B^\mu(q) \right),
\]
\[
k_\mu C_{B}^\mu(p) = -2ig \frac{4 \phi p_{\mu} k_{\mu}}{\Delta_B^2} - \frac{8g^2}{m\Delta_B} \int \frac{d^3q}{(2\pi)^3} e^{-E_q/T} \left( -\frac{2}{3} k_\mu C_A^\mu(q) + \frac{2}{3} k_\mu C_B^\mu(q) \right).
\]
(54)

At this point, it is no longer possible to neglect the momentum dependence since in particular the vector components \( \vec{C}_{A,B}^\mu(p) \) receive an important contribution rising quadratically with \( |\vec{p}| \). Yet, the system can be solved by multiplication by \( e^{-E_q/T} \), followed by integration. In this way, one finds a linear system with the solution
\[
\int \frac{d^3q}{(2\pi)^3} e^{-E_q/T} k_\mu C_{A}^\mu(q) = \int \frac{d^3q}{(2\pi)^3} e^{-E_q/T} k_\mu C_{B}^\mu(q) = \frac{im^2}{2g\Delta_B} \left( m k_0 \gamma_0 + T \vec{k} \vec{\gamma} \right).
\]
(55)

One can now proceed and calculate the functions \( C_{A,B}^\mu(q) \) by inserting this result into equations (55). We skip this last step since the result will not be needed in the following.

Instead, we proceed to the calculation of the full fermion propagator. With our knowledge of the full one-particle irreducible vertex functions, we can now evaluate the full proper self energy. In terms of full propagators and vertex functions, the main contribution from (38) translates to
\[
-i \Sigma_\beta(k) = -2ig \int \frac{d^4q}{(2\pi)^4} \mathcal{D}(k - q) \left( \Gamma_A(k, q) S(q) + \Gamma_B(k, q) S(q)i\gamma^5 \right),
\]
which, by making use of equations (41) and (49), is easily brought into the form
\[
\Sigma_\beta(k) = -2ig \int \frac{d^3q}{(2\pi)^3} \frac{e^{-E_q/T}}{E_q} (V_A(k, q) + V_B(k, q)).
\]
With our results for $V_{A,B}$ in (52) and (55), one calculates

$$
\Sigma_\beta(k) = -m - \frac{1}{g^2\alpha} \left( k_0 \gamma_0 + \frac{T}{m} \vec{k} \gamma \right). \tag{56}
$$

Thus, the self energy $\Sigma_\beta(0) = -m$ exactly cancels the Lagrangian mass term in the full inverse propagator which thereby becomes

$$
i S^{-1}(k) = k - m - \Sigma_\beta(k) \approx \frac{1}{g^2\alpha} \left( k_0 \gamma_0 + \frac{T}{m} \vec{k} \gamma \right). \tag{57}
$$

So we have finally found the additional massless pole in the full fermion propagator. It becomes singular for vanishing momentum as well as on the dispersion curve

$$
k_0 = \pm \frac{T}{m} |\vec{k}| \tag{58}
$$

which is nothing but the dispersion law predicted in section 4.2. Therefore, this pole can be identified with the phonino. Thus we have shown that the phonino pole in the fermion propagator is present at any moderate temperature, only the residue and dispersion law change with temperature. In the limit $T \to 0$, the residue vanishes and the Goldstone mode disappears, since supersymmetry is restored.

Due to our approximation, we have proven the existence of the phonino pole only for small momenta. We have assumed that $kq \ll \Delta_A$ with a typical thermal momentum $|\vec{q}| \sim \sqrt{Tm}$, so that the derivation is restricted to momenta $|\vec{k}| \ll \Delta_A/\sqrt{Tm}$. Interestingly, this is not only a technical point but of physical relevance. In the hydrodynamic picture, the existence of sound and supersymmetric sound waves is restricted to long wavelengths much greater than the mean free path so that there is always time to establish local thermodynamic equilibrium. At higher frequencies, the waves are strongly damped as it was shown in [7].

The fact that our calculation requires a resummation of both one-loop corrections to the propagators and higher-order vertex corrections shows that the existence of the phonino is really a nonperturbative phenomenon. This is consistent with the interpretation as a collective excitation according to the hydrodynamic explanation given in section 4.2. Furthermore, it explains why earlier calculations were not able to relate the Goldstone mode to a propagating excitation by simpler one-loop calculations [4, 5].

The existence of the low-temperature phonino pole and dispersion law was already shown in [6] by investigating the homogeneous Bethe-Salpeter equations for the full (i.e., not one-particle irreducible) vertex functions. Our explicit calculation of the full fermion propagator is however a new result which will be essential for the following discussion of the Ward-Takahashi identities.
7 Verification of the Ward-Takahashi identities

The appearance of the massless phonino derived in the previous section is apparently linked with the breakdown of supersymmetry. In order to obtain a complete picture and to prove that the phonino is indeed the Goldstone fermion of spontaneously broken supersymmetry, let us now take a closer look at the supersymmetric Ward-Takahashi identities and verify that the phonino contributes in exactly the way expected for a Goldstone particle.

In section 4, we have derived the identity
\[ \partial_\mu \langle T J^\mu(x) \bar{\psi}(y) \rangle_\beta = -i m \langle A \rangle_\beta \delta^{(4)}(x - y). \] (59)

At the one-loop level, the right hand side is nontrivial because of the nonvanishing thermal expectation value of the scalar field given in equation (34). Thus, the left hand side must get a contribution from a Goldstone mode. Surely, we expect the phonino to act this part, with the main contributions coming from the class of diagrams drawn in figure 3. Here, the encircled cross denotes the supercurrent (14). The shaded spot stands for the full vertex, and the thick lines mean full resummed thermal propagators.

The evaluation of these diagrams thus gives, to leading order,
\[ \langle T J^\mu(x) \bar{\psi}(y) \rangle_\beta = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \left\{ \int \frac{d^4q}{(2\pi)^4} (ik^\mu - i\gamma^\mu D_A(k - q)S(q)) \Gamma_A(k, q)S(k) \right. \\
\left. + \int \frac{d^4q}{(2\pi)^4} (-ik^\mu + i\gamma^\mu D_B(k - q)S(q)) \Gamma_B(k, q)S(k) \right\}. \] (60)

Going over to momentum space, we amputate the fermion propagator and define
\[ \Gamma^\mu_\beta(k) = \int d^4x e^{ik(x-y)} \langle T J^\mu(x) \bar{\psi}(y) \rangle_\beta S^{-1}(k) \]

Figure 3: Main contributions to the Ward-Takahashi identity (59)
so that the Ward-Takahashi identity (59) we want to verify can be written as
\[-i k_\mu \Gamma^\mu_j \bar{\Psi}(k) = -im \langle A \rangle_\beta S^{-1}(k).\] (61)

Now to the calculation of the left hand side. The loop integrals in (60) are again of the form (41) well-known by now, and the Dirac algebra is simplified by the fact that the loop-momentum is on-shell. Since we are mainly interested in the role of the Goldstone mode, we consider small momenta \(k\) and only keep the linear terms. We find
\[-i k_\mu \Gamma^\mu_j \bar{\Psi}(k) = \int \frac{d^3 q}{(2\pi)^3} \sum_{q_0 = \pm E_q} \left\{ (-2\hat{q} \cdot q k) \frac{n_B(E_q) + n_F(E_q)}{2E_q} \frac{\Gamma_A(0, q)}{m^2 - m_A^2} \right.\]
\[+ \left. 2\hat{q} \cdot q k \frac{n_B(E_q) + n_F(E_q)}{2E_q} \frac{i\gamma^5 \Gamma_B(0, q)}{m^2 - m_B^2} \right\}.\]

The integration gives, for nonrelativistic temperatures \(T \ll m\),
\[-i k_\mu \Gamma^\mu_j \bar{\Psi}(k) = \frac{m^2}{g} \left( k_0 \gamma_0 + \frac{T}{m} \vec{k} \gamma \right).\] (62)

Similarly, for the relativistic case \(T \gg m\), one finds
\[-i k_\mu \Gamma^\mu_j \bar{\Psi}(k) = \frac{\pi^2 T^2}{2g} \left( k_0 \gamma_0 + \frac{1}{3} \vec{k} \gamma \right).\] (63)

One immediately observes the similarity to the inverse phonino propagators obtained in equations (42) and (57), and indeed, one finds by comparison with equation (57) that the difference is just a factor of \(-im \langle A \rangle_\beta\). So, the Ward-Takahashi identity (61) is fulfilled through the contribution from the phonino, and we can conclude that the phonino is the Goldstone fermion associated with the spontaneous breakdown of supersymmetry in the thermal background.

Once we have convinced ourselves that the Ward-Takahashi identity is satisfied at finite temperature in the full resummed interacting theory, we can in turn use it to obtain more information about the full propagator. If one keeps also the linear terms in the calculation of \(\Gamma^\mu_j \bar{\Psi}(k)\), equation (60), one can deduce the nonlinear corrections to the phonino dispersion law. In agreement with [6], they are found to be small in the region where the phonino exists.

Let us now proceed to the second Ward-Takahashi identity, the one for the composite mode \(A\bar{\Psi}\) that was derived in equation (18). Its right hand side was evaluated already in equation (20) so that we can rewrite the identity in terms of \(\langle A \rangle_\beta\) as
\[\partial^\mu \langle T J^\mu(x)A(y)\bar{\Psi}(y)\rangle_\beta = i \frac{m^2}{4g} \langle A \rangle_\beta \delta^{(4)}(x - y).\] (64)
Again, the right hand side is nontrivial at nonvanishing temperature, so that the left hand side must get a contribution from the Goldstone mode.

The diagrams contributing to the left hand side fall into two classes, drawn in figure 4. For the first class of diagrams, one must have control over the full one-particle irreducible four-point function denoted by a shaded box that could in principle have additional poles. However, in the limit of small momentum we are interested in, it can easily be shown that this is not the case. The relevant contributions to this four-point function can be resummed by means of the same Bethe-Salpeter equations that were set up for the calculation of the one-particle irreducible three-point functions in section 6.2. At this point, it is only important that this resummation does not introduce any new poles, in other words that the homogeneous Bethe-Salpeter equation does not have any nontrivial solutions. This can easily be seen from equation (51) which, as a homogeneous system, has only the trivial solution. Hence, this class of diagrams stays finite in the limit of low momentum.

Thus, the relevant contributions come from the second class of diagrams involving the phonino, schematically drawn in figure 4b. Here, the big smudge stands for the sum of all diagrams discussed above in connection with the first Ward-Takahashi identity. So, for the evaluation of this correlation function in the low-momentum limit, we can take advantage of our earlier results and write

\[
\Gamma_{JA\bar{\Psi}}(k) \equiv \int d^4 x e^{ik(x-y)} \langle TJ^\mu(x)A(y)\bar{\Psi}(y) \rangle \beta
\]

\[
= \Gamma_{\bar{\Psi}}(k)\mathcal{S}(k) \int \frac{d^4 q}{(2\pi)^4} \frac{1}{\mathcal{D}_A(k-q) \mathcal{S}(q)}
\]

\[
= \Gamma_{\bar{\Psi}}(k)\mathcal{S}(k) \left( -\frac{m^2}{4g} \right),
\]

where the numerical factor is the same for low and high temperatures.

We can now make use of the first Ward-Takahashi identity (61) and obtain

\[
-i k_\mu \Gamma_{JA\bar{\Psi}}(k) = i\frac{m^2}{4g} \langle A \rangle \beta
\]

(65)

Figure 4: Contributions to the Ward-Takahashi identity (61)
which is nothing but the Ward-Takahashi identity we intended to prove, written in momentum space.

Let us compare the ways the Ward-Takahashi identity is fulfilled for the free and interacting theories. In the free theory, the loop diagram in figure (without any interaction) is the only one that contributes to the left hand side of the identity. Because of the equality of the masses of both propagators, it becomes singular in the vanishing momentum limit, thereby saturating the Ward-Takahashi identity non-trivially without the need for a massless mode. In the interacting theory, in contrast, there appears a mass splitting between fermion and bosons, so that diagram stays finite even in the limit of vanishing momentum. Hence, there must be a massless Goldstone mode that contributes to the left hand side in order to saturate the Ward-Takahashi identity. Our calculation shows that the phonino pole in the fermion propagator exactly does this job.

Finally, the Ward-Takahashi identity involving the supercurrent,
\[
\partial_\mu \langle T J^\mu(x) \overline{\mathcal{J}}^\nu(y) \rangle_\beta = \delta^{(4)}(x-y) 2 \langle T^\nu \rangle_\beta \gamma_\mu,
\] (66)
can be verified in the same way. The dominant contributions to the left hand side come again from diagrams with a single fermion intermediate state. Hence, we can write, in momentum space,
\[
\Gamma_{\mu J J}(k) = \int d^4x e^{ik(x-y)} \langle T J^\mu(x) \overline{\mathcal{J}}^\nu(y) \rangle_\beta = \Gamma_{\mu}(k) \mathcal{S}(k) \Gamma_{J J}(k).
\]
By making use of the Ward-Takahashi identity, we find
\[
-i k_\mu \Gamma_{J J}(k) = -im \langle A \rangle_\beta \overline{\Gamma}_{J J}(k),
\]
which can now be calculated by using our earlier results on \(\Gamma_{J J}(k)\).

First, in the limit of low temperature, we make use of (62) and obtain the left hand side of the Ward-Takahashi identity as
\[
-i k_\mu \Gamma_{J J}(k) = -m^2 \langle A \rangle_\beta \frac{T m}{\Delta B} \left( \frac{\gamma_0}{T m} \right) = 8m e^{-m/T} \left( \frac{T m}{2\pi} \right)^{3/2} \left( \frac{\gamma_0}{T m} \right).
\]
For high temperatures, one calculates in the same way, using (63),
\[
-i k_\mu \Gamma_{J J}(k) = m^2 \langle A \rangle_\beta \frac{\pi^2}{g^2} \left( \frac{\gamma_0}{T m} \right) = \frac{\pi^2}{4} T^4 \left( \frac{\gamma_0}{T m} \right).
\]
Comparing with our earlier results on the energy-momentum tensor, equations (27) and (28), we find that this can be written as
\[
-i k_\mu \Gamma_{J J}(k) = 2 \langle T^\nu \rangle_\beta \gamma_\mu.
\] (67)
This is however nothing but the momentum space version of the identity we wanted to prove.
8 Conclusion

Our calculations have lead to a consistent picture of the behaviour of the Wess-Zumino model at finite temperature. At moderate temperatures, the interaction with the thermal background leads, on the one-loop level, to a small splitting of the effective masses, and the scalar field develops a nontrivial thermal expectation value. Both are signs, if not necessary conditions, for the breakdown of supersymmetry. As an additional feature, the phonino pole appears in the propagator of the fundamental fermion, as a full nonperturbative calculation shows. The reason is the continuous interaction with boson-fermion pairs of nearly degenerate mass. As a consequence, the same pole appears in all modes that couple to the supercurrent. This relation to the supercurrent indicates the role of the phonino as the Goldstone particle of spontaneously broken supersymmetry, which could be confirmed by the investigation of the Ward-Takahashi identities that are saturated by the contribution from the phonino in a nontrivial way.

However, our results are not restricted to the Wess-Zumino model. The general picture we have drawn and verified in this simple model, according to which the breakdown of supersymmetry is a consequence of the breakdown of Lorentz invariance, can immediately be translated to any supersymmetric model. Just as the existence of supersymmetric sound was derived in [6] in a model-independent way, the Ward-Takahashi identity for the supercurrent generically predicts the existence of a Goldstone pole in any mode that couples to the supercurrent, as long as the fields involved contribute to the nonvanishing energy density of the thermal bath. By the same mechanism that we explicitly studied for the Wess-Zumino model, this pole must be associated with a propagating particle whenever the interaction lifts the boson-fermion mass degeneracy that otherwise allows the saturation of the Ward-Takahashi identities without the need for a Goldstone particle.

Up to now, our analysis was restricted to chiral superfields and (in the vacuum) unbroken supersymmetry. In order to come closer to phenomenologically attractive models, it is necessary to understand the behaviour of gauge fields [15] as well as the interplay with other mechanisms of supersymmetry breaking.

Acknowledgements

I would like to thank Prof. Wilfried Buchmüller for his continuous guidance in this project. The opportunity to finish this work under financial support by DESY is gratefully acknowledged.
References

[1] A. Das and M. Kaku, Phys. Rev. D 18 (1978) 4540.

[2] L. Girardello, M. T. Grisaru and P. Salomonson, Nucl. Phys. B 178 (1981) 331.

[3] D. Buchholz and I. Ojima, Nucl. Phys. B 498 (1997) 228.

[4] D. Boyanovsky, Phys. Rev. D 29 (1984) 743;
H. Aoyama and D. Boyanovsky, Phys. Rev. D 30 (1984) 1356.

[5] H. Matsumoto, M. Nakahara, Y. Nakano and H. Umezawa, Phys. Rev. D 29 (1984) 2838;
H. Matsumoto, M. Nakahara, H. Umezawa and N. Yamamoto, Phys. Rev. D 33 (1986) 2851.

[6] V. V. Lebedev and A. V. Smilga, Nucl. Phys. B 318 (1989) 669.

[7] R. Gudmundsdottir and P. Salomonson, Nucl. Phys. B 285 (1987) 1.

[8] N. P. Landsman and C. G. van Weert, Phys. Rept. 145 (1987) 141.

[9] J. Wess and B. Zumino, Nucl. Phys. B 70 (1974) 39.

[10] M. F. Sohnius, Phys. Rept. 128 (1985) 39.

[11] H. Aoyama, Phys. Lett. B 171 (1986) 420;
I. Ojima, Lett. Math. Phys. 11 (1986) 73;
H. Matsumoto, H. Umezawa, N. Yamamoto and N. J. Papastamatiou, Phys. Rev. D 34 (1986) 3217.

[12] R. G. Leigh and R. Rattazzi, Phys. Lett. B 352 (1995) 20.

[13] S. Midorikawa, Prog. Theor. Phys. 73 (1985) 1245.

[14] J. I. Kapusta, Phys. Lett. B 118 (1982) 343 [Erratum-ibid. B 122 (1983) 486].

[15] K. Kratzert, in preparation.