Analytical estimation of solution parameters for elliptic equations with variable coefficients in fluid dynamics

T I Lelekov and E T Lelekov
Reshetnev Siberian State University of Science and Technology, 31 Krasnoyarsky Rabochy Av., Krasnoyarsk, 660037, Russia

E-mail: tlelekov@mail.ru

Abstract. In this paper we consider multidimensional and two-dimensional elliptic partial differential equations of the second order with variable coefficients. They are often used in the oil and gas industry to calculate the inflow volume of liquids and gases from natural reservoirs to exciting structures. The analytical solution of that equations with variable values of the permeability coefficient is possible in rare cases, when the change in the formation permeability is expressed in the form of simple analytical dependencies over the area of fluid filtration. In this paper, we give estimated formulas for fluid inflow to structures under conditions of reservoir heterogeneity, which retain an analytical relationship with the permeability coefficient. The value of the proposed method lies in the ability to establish important qualitative regularities of an underground flow formation, based on simple formulas. The obtained expressions of the above theorems are consistent with the calculation formulas of underground fluid dynamics, and allows one to perform calculations of oil production in heterogeneous formations.

1. Introduction
Partial differential equations with variable coefficients are widely used in various branches of technology to describe various kinds of technological processes. One of the areas of use is the oil and gas industry, where it used for calculating the volume of inflow of liquids and gases from natural reservoirs to exciting structures. The calculation of fluid inflow to structures in a porous medium, in most cases is solved on the assumption of the reservoir homogeneity in terms of permeability. Under natural conditions, productive hydrocarbon reservoirs are rarely homogeneous. The permeability of the reservoirs most often changes over the filtration area, and in many cases it is required to determine the parameters of fluid movement in a heterogeneous medium. These tasks are basic in the exploitation of oil fields by setting the volume of oil production by wells, controlling the movement of the oil-bearing contour, determining the rate of productive formations watering.

In many cases, while calculations one resort to the averaging of the reservoir permeability over the area and take the filtration coefficient average value for the entire reservoir. If the non-homogeneity reaches significant values, then such a reduction will distort the true picture of filtration field and thus may give incorrect results of fluid inflow to the structures, and as a result will make it impossible to draw any conclusions about the rational operation of the structures.

To describe the true picture of fluid movement in heterogeneous formations, it is necessary to solve differential filtration equations for specific conditions of their deposition, namely, fluid filtration in heterogeneous porous formations in partial derivatives with variable values of coefficients [1, 2]. The
solution of such problems presents great mathematical difficulties. The analytical solution of partial differential equations with variable values of the permeability coefficient is possible in rare cases, when the change in the formation permeability is expressed in the form of simple analytical dependencies over the area of fluid filtration.

In this paper, we give estimated formulas for fluid inflow to structures under reservoir heterogeneity conditions. The value of the proposed method lies in the ability to establish important qualitative regularities of the formation of an underground flow based on simple formulas.

2. Multidimensional partial differential equations of elliptic type
Let $V$ be a domain in the space $\mathbb{R}^n$ whose boundary $S$ is a piecewise smooth closed surface. Consider a linear boundary value problem for elliptic equations

$$\nabla \cdot (k \nabla u) = 0, \ x \in V,$$

and one of the types of boundary conditions

$$\alpha u = f, \ \beta \frac{du}{dn} = g, \ \alpha u + \beta \frac{du}{dn} = h, \ x \in S.$$  \hspace{1cm} (2)

For this problem, we assume that

$$k \in C^1(V), \ f \in C(V), \ g \in C(V), \ h \in C(V), \ k \geq 0, \ x \in V,$$

and

$$\alpha \in C(V), \ \beta \in C(V), \ \alpha \geq 0, \ \beta \geq 0, \ x \in S.$$

Let us pose the problem of determining the function $u(x)$ and its derivative $\frac{du}{dn}$ on the boundary $S$ of the region $V$, satisfying equation (1) and boundary conditions (2).

**Theorem 1.** For the boundary value problem of an elliptic equation with variable coefficients (1) with boundary conditions (2), the next two inequalities are true

$$\sum_{i=1}^{n} \left( \int_S u \cos(n\hat{x}_i) ds \right)^2 \cdot \frac{1}{\int_V k dv} \leq \int_S k u \frac{du}{dn} ds,$$  \hspace{1cm} (3)

$$\sum_{i=1}^{n} \left( \int_V k \frac{du}{dx_i} dv \right)^2 \cdot \frac{1}{\int_V k dv} \leq \int_S k u \frac{du}{dn} ds.$$  \hspace{1cm} (4)

**Proof.** Let's use the first Green's formula, which is written in the form

$$\int_V (\nabla \Phi \cdot \nabla \Psi + \Phi \Delta \Psi) dv = \int_S \Phi \frac{d\Psi}{dn} ds.$$  

Substitute here the expressions $\Phi = ku, \Psi = u$. In that case we have

$$\int_V (\nabla ku \cdot \nabla u + ku \Delta u) dv = \int_S ku \frac{du}{dn} ds.$$  

We write the integrand in the form

$$\nabla ku \cdot \nabla u + ku \Delta u = k(\nabla u)^2 + u \nabla k \cdot \nabla u + u \nabla (k \nabla u) - u \nabla k \cdot \nabla u = k(\nabla u)^2 + u \nabla (k \nabla u).$$

Then, the above integral will be equal to

$$\int_V (k(\nabla u)^2 + u \nabla (k \nabla u)) dv = \int_S ku \frac{du}{dn} ds.$$  

The condition $\nabla (k \nabla u) = 0$ of the theorem imply
\[ \int k(\nabla u)^2 dv = \int_S k u \frac{du}{dn} ds, \]

or

\[ \int k \sum_{i=1}^n (\frac{\partial u}{\partial x_i})^2 dv = \int_S k u \frac{du}{dn} ds. \]  

This equality can be written as

\[ \sum_{i=1}^n \int k \left( \frac{\partial u}{\partial x_i} \right)^2 dv = \int_S k u \frac{du}{dn} ds. \]  

To expand the left-hand side integral of this equality, we use the Cauchy-Schwarz inequalities

\[ |\int_V f(x)g(x) dx|^2 \leq \int_V |f(x)|^2 dx \int_V |g(x)|^2 dx. \]

Let's substitute into this inequality the expressions for the functions

\[ f_i(x) = \sqrt{k} \frac{\partial u}{\partial x_i}, \quad g(x) = \frac{1}{\sqrt{k}}, \]

and we will get

\[ \left| \int_V \frac{\partial u}{\partial x_i} dv \right|^2 \leq \int_V \left| \frac{\partial u}{\partial x_i} \right|^2 dv \cdot \int_V \frac{1}{k} dv. \]

Summation over \( i \) of the resulting inequality gives

\[ \sum_{i=1}^n \int_V k \left( \frac{\partial u}{\partial x_i} \right)^2 dv \geq \left( \sum_{i=1}^n \int_V \left| \frac{\partial u}{\partial x_i} \right|^2 dv \right)^2 \cdot \frac{1}{\int_V \frac{1}{k} dv}. \]

We transform this inequality using the Gauss-Ostrogradsky equation

\[ \int_V \left( \frac{\partial u}{\partial x_i} \right) dv = \int_S u \cos(\hat{n} x_i) ds. \]

and get

\[ \sum_{i=1}^n \int_V k \left( \frac{\partial u}{\partial x_i} \right)^2 dv \geq \left( \sum_{i=1}^n \int_S u \cos(\hat{n} x_i) ds \right)^2 \cdot \frac{1}{\int_V \frac{1}{k} dv}. \]

Substitution of this expression in (3) gives

\[ \sum_{i=1}^n \left( \int_S u \cos(\hat{n} x_i) ds \right)^2 \cdot \frac{1}{\int_V \frac{1}{k} dv} \leq \int_S k u \frac{du}{dn} ds, \]

Now substitute into the Cauchy-Schwarz inequality the next expressions for the functions

\[ f_i(x) = \sqrt{k} \frac{\partial u}{\partial x_i}, \quad g(x) = \sqrt{k}, \]

and we will get
\[ \left| \int_V \left( k \frac{\partial u}{\partial x_i} \right) dv \right|^2 \leq \int_V k \left| \frac{\partial u}{\partial x_i} \right|^2 dv \cdot \int_V k dv. \]

Summation over \( i \) and the corresponding transformation gives
\[ \sum_{i=1}^{n} \left( \int_V k \left( \frac{\partial u}{\partial x_i} \right)^2 dv \right) \geq \sum_{i=1}^{n} \left( \int_V \left( \frac{\partial u}{\partial x_i} \right)^2 dv \right) \cdot \frac{1}{\int_V k \ dv}. \]

Replacing the expression of the integral into this equation and summing over \( i \), we obtain
\[ \sum_{i=1}^{n} \left( \int_V k \left( \frac{\partial u}{\partial x_i} \right)^2 dv \right) \cdot \frac{1}{\int_V k dv} \leq \int_S \frac{\partial u}{\partial \hat{n}} ds. \]

The theorem is proved.

3. Two-dimensional partial differential equations of elliptic type

Consider a two-dimensional elliptic equation in a rectangular domain. Such equations are quite common in technical problems of filtration and heat transfer. When solving, a rectangular area is obtained by conformal transformation of the process flow area, in order to transform a complex area shape to a simpler rectangular one. An important feature of this course of solution is the preservation of the form of the original equation of the process \( \nabla \cdot (k \nabla u) = 0 \).

A two-dimensional problem with any boundary conditions is formulated with the equation
\[ \frac{\partial}{\partial \xi} (k(\xi, \eta) \frac{\partial u}{\partial \xi}) + \frac{\partial}{\partial \eta} (k(\xi, \eta) \frac{\partial u}{\partial \eta}) = 0, \]

where \( U \) is process function, \( k(\xi, \eta) \) is the intensity coefficient of the medium process, which varies over the domain of definition.

The solution of the equation \( U(\xi, \eta) \) is sought in a simply connected domain \( V \) with the boundary \( S \), consisting of four parts \( S_1 + S_2 + S_3 + S_4 \). The coefficient \( k(\xi, \eta) \) is non-constant and positive everywhere in the domain \( V \). \( U_1 \) and \( U_2 \) are the values of the function \( U(\xi, \eta) \) on the boundaries \( S_1 \) and \( S_2 \) of the area, whilst \( \frac{\partial u}{\partial \eta}\bigg|_{S_3} \) and \( \frac{\partial u}{\partial \eta}\bigg|_{S_4} \) are derivatives along the normal to the boundary contour on \( S_3 \) and \( S_4 \) boundary parts.

An important component of solving the posed problems is to determine the rate of change of process state variable (flow rate of liquid, volume of heat, etc.) \( Q \) in the domain \( V \), as well as the intensity parameter \( R = (U_2 - U_1)/Q \).

**Theorem 2.** For the boundary value problem \( \frac{\partial}{\partial \xi} (k(\xi, \eta) \frac{\partial u}{\partial \xi}) + \frac{\partial}{\partial \eta} (k(\xi, \eta) \frac{\partial u}{\partial \eta}) = 0 \) with boundary conditions \( U|_{S_1} = U_1, \ U|_{S_2} = U_2, \frac{\partial u}{\partial \eta}\bigg|_{S_3} = 0, \frac{\partial u}{\partial \eta}\bigg|_{S_4} = 0 \), the next relations are true
\[ Q \geq (U_1 - U_0) \int_V \frac{d\eta}{\int_V k(\xi, \eta) d\xi}, \quad Q \leq (U_1 - U_0) \int_V \frac{1}{\int_V k(\xi, \eta) d\xi}, \]

where \( Q = \int_{S_1} k(\xi, \eta) \frac{\partial u}{\partial \xi} d\eta = \int_{S_3} k(\xi, \eta) \frac{\partial u}{\partial \eta} d\eta. \)

**Proof.** For the proof we apply the Cauchy-Schwarz inequality, written in a rectangular region along one coordinate
\[
\left| \int f(\xi, \eta) g(\xi, \eta) d\xi \right|^2 \leq \int |f(\xi, \eta)|^2 d\xi \cdot \int |g(\xi, \eta)|^2 d\xi. \quad (7)
\]

Similarly, making in this expression the replacement
\[
f(\xi, \eta) = \sqrt{k(\xi, \eta) \frac{\partial U}{\partial \xi}}, \quad g(\xi, \eta) = \frac{1}{\sqrt{k(\xi, \eta)}}
\]
we get
\[
\int_S k(\xi, \eta) \left( \frac{\partial U}{\partial \xi} \right)^2 d\xi \geq \left( \int_S \frac{\partial U}{\partial \xi} d\xi \right)^2 \int_S \frac{1}{k(\xi, \eta)} d\xi.
\]

Because the
\[
\left( \int_S \frac{\partial U}{\partial \xi} d\xi \right)^2 = (U_2 - U_1)^2,
\]
where \(U_1\) and \(U_2\) are the values of the function \(U(\xi, \eta)\) on the area boundaries, then
\[
\int_S k(\xi, \eta) \left( \frac{\partial U}{\partial \xi} \right)^2 d\xi \geq (U_2 - U_1)^2 \int_S \frac{1}{k(\xi, \eta)} d\xi.
\]

Integration of the last inequality over another coordinate gives
\[
\iint_V k(\xi, \eta) \left( \frac{\partial U}{\partial \xi} \right)^2 d\xi d\eta \geq (U_2 - U_1)^2 \int_V \frac{d\eta}{k(\xi, \eta) d\xi}. \quad (8)
\]

Let us represent the expression (5) for the considered two-dimensional region in the corresponding coordinates as
\[
\int_V k \left( \left( \frac{\partial U}{\partial \xi} \right)^2 + \left( \frac{\partial U}{\partial \eta} \right)^2 \right) d\nu = \int_S k U \frac{\partial U}{\partial \eta} d\sigma.
\]

Since
\[
\left( \frac{\partial U}{\partial \eta} \right)^2 \geq 0, \quad \int_S k U \frac{\partial U}{\partial \eta} d\sigma = Q(U_2 - U_1),
\]
then
\[
\int_V k \left( \frac{\partial U}{\partial \xi} \right)^2 d\nu \leq Q(U_2 - U_1).
\]

Substitution of this expression into (8) gives
\[
Q \geq (U_2 - U_1) \int_V \frac{d\eta}{k(\xi, \eta) d\xi}.
\]

If we replace functions in expression (7), in the form
\[
f(\xi, \eta) = \sqrt{k(\xi, \eta) \frac{\partial U}{\partial \xi}}, \quad g(\xi, \eta) = \sqrt{k(\xi, \eta)}.
\]
then one can get
\[
\int_V k(\xi, \eta) \left( \frac{\partial U}{\partial \xi} \right)^2 d\eta \geq \left( \int_V k(\xi, \eta) \frac{\partial U}{\partial \xi} d\eta \right)^2 \frac{1}{\int_V k(\xi, \eta) d\eta}.
\]

Since
\[
\int_{s_1} k(\xi, \eta) \frac{\partial u}{\partial \xi} d\eta = Q,
\]
then integration over \(\xi\) gives
\[
\iint_V k(\xi, \eta) \left(\frac{\partial u}{\partial \xi}\right)^2 d\xi d\eta \geq Q^2 \int_V \frac{d\eta}{k(\xi, \eta) d\xi}.
\]
Expression (8) implies
\[
Q(U_2 - U_1) \geq Q^2 \int_V \frac{d\eta}{k(\xi, \eta) d\xi},
\]
and this implies the second statement of the theorem
\[
Q \leq (U_2 - U_1) \left(\int_V \frac{d\xi}{k(\xi, \eta) d\eta}\right)^{-1}.
\]
The theorem is proved.

4. Conclusion
The obtained expressions of the above theorems are in good agreement with the calculated formulas of underground fluid dynamics, for example, when \(k(\xi, \eta) = k(\xi),\) or \(k(\xi, \eta) = k(\eta).\) In these cases, for a reservoir of \(L \times M\) size and unit thickness, the estimate
\[
Q \geq (U_2 - U_1) \frac{M}{\int_V k(\xi) d\xi}, \quad Q \leq (U_2 - U_1) \frac{\int_V k(\eta) d\eta}{L}
\]
is performed for the fluid flow rate, which coincides with the formulas for calculating the movement of underground fluids in heterogeneous formations [7]. To calculate the case of more complex change of the equation coefficients, one should apply the estimated inequalities of the derived formulas in the theorems of this article.

References
[1] Xu F and Liu L 2016 Journal of Nonlinear Sciences and Applications 09 6371-81
[2] Silvestre L 2014 Regularity estimates for parabolic integro-differential equations and applications International Congress of Mathematicians-Seoul 2014 vol III ed. Sa K M (Seoul) 873-94
[3] Córdoba D, Gancedo F and Orive R 2005 Journal of Mathematical Physics 48 065206
[4] Xing J, Shenoi R A, Wilson P A and Xing J T 2004 Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences 460 1905-20
[5] Yu W and He Y 2014 Boundary Value Problems 2014 95
[6] Chemetov N 2014 Systems of Coupled Differential Equations Transport in bounded domains Ph.D. thesis FCUL / University of Lisbon
[7] Bear J 1988 Dynamics of Fluids in Porous Media (New York: Dover)