Number of directions determined by a set in $\mathbb{F}_q^2$ and growth in $\text{Aff}({\mathbb{F}_q})$

Daniele Dona

Mathematisches Institut, Georg-August-Universität Göttingen
Bunsenstraße 3-5, 37073 Göttingen, Germany
daniele.dona@mathematik.uni-goettingen.de

Abstract. We prove that a set $A$ of at most $q$ non-collinear points in the finite plane $\mathbb{F}_q^2$ spans at least $\approx |A|^{\frac{1}{\sqrt{q}}}$ directions: this is based on a lower bound contained in [FST13], which we prove again together with a different upper bound than the one given therein. Then, following the procedure used in [RS18], we prove a new structural theorem about slowly growing sets in $\text{Aff}(\mathbb{F}_q)$ for any finite field $\mathbb{F}_q$, generalizing the analogous results in [Hel15] [Mur17] [RS18] over prime fields.

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1 Introduction

Among the many different problems related to the study of growth and expansion in finite groups, the study of the affine group over finite fields has occupied a particularly interesting place. The affine group

$$\text{Aff}(\mathbb{F}) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{F}^*, b \in \mathbb{F} \right\},$$

where $\mathbb{F}$ is a finite field, is one of the smallest interesting examples of an infinite family of finite groups on which questions of growth of sets $A \subseteq \text{Aff}(\mathbb{F})$ can yield nontrivial answers, and it has been used to showcase techniques applicable to more general situations, like the pivot argument; on the other hand, its shape makes its uniquely suitable to study the so-called sum-product phenomenon, related to growth of sets inside finite fields under both addition and multiplication. For both of these points of view, a remarkable example is provided in Helfgott’s survey [Hel15 §4.2].

Structural theorems about growth in $\text{Aff}(\mathbb{F}_p)$ ($p$ prime) have been produced in the last few years, describing in substance what a set $A$ with small growth must...
look like. Results like Helfgott’s [Hel15, Prop. 4.8] and Murphy’s [Mur17, Thm. 27] belong to a first generation of proofs that rely, one way or another, on sum-product estimates; they already accomplish the goal of characterizing quite well a slowly growing $A$: such a set must essentially either be a point stabilizer or be contained in a few vertical lines, which in addition get filled in finitely many steps if $|A| \gg p$.

Rudnev and Shkredov [RS18] have then quantitatively improved this classification in Aff($F_p$): the main attractiveness of their result, however, resides in the fact that, in their own words, “the improvement [they] gain is due [...] to avoiding any explicit ties with the sum-product phenomenon, which both proofs of Helfgott and Murphy relate to”, which makes their version of the characterization of slowly growing $A$ part of a new generation of efforts. What they rely on instead is a geometric theorem by Szönyi [Szö99, Thm. 5.2] that gives a good lower bound on the number of directions spanned by a set of non-collinear points in the plane $F_p^2$ for $p$ prime.

Following the approach by Rudnev and Shkredov, we first produce an analogous version of Szönyi’s result for the plane $F_q^2$, where $q$ is any prime power; then we use that estimate to prove a structural theorem on slowly growing sets in Aff($F_q$) (resembling the corresponding ones for Aff($F_p$) mentioned before), which to the best of our knowledge is the first of its kind.

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Throughout the paper, $p$ will always denote a prime and $q$ a power of $p$. Given a set $A$ inside the plane $F^2$, the set of directions spanned or determined by $A$ denotes the set

$$D = \left\{ \frac{b' - b}{a' - a} \mid (a, b), (a', b') \in A, (a, b) \neq (a', b') \right\} \subseteq F \cup \{\infty\},$$

where conventionally $\infty$ corresponds to the fraction with $a' - a = 0$. We make free use of the natural identification Aff($F$) $\leftrightarrow F^* \times F$ given by

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in \text{Aff}(F) \quad \longleftrightarrow \quad (a, b) \in F^* \times F,$$

so that we may refer to points, lines and directions even when speaking of the group Aff($F$); in particular, we call $\pi : \text{Aff}(F) \rightarrow F^*$ the map corresponding to the projection on the first component, so that the preimage of a point through this map is a vertical line. Aff($F$) acts also on $F$ as $(a, b) \cdot x = ax + b$, and we think of this action when we refer to $\text{Stab}(x)$ (which also looks like a line when seen in $F^2$); finally, $U$ denotes the unipotent subgroup corresponding to $\{1\} \times F$, again a vertical line.

As said before, one of the starting points of the new-style result for slowly growing sets in Aff($F_p$) is the following bound by Szönyi.

**Theorem 1.1.** Let $p$ be a prime, and let $A \subseteq F^2_p$ with $1 < |A| \leq p$. Then either $A$ is contained in a line or $A$ spans $\geq \frac{|A| + 3}{2}$ directions.
With that, Rudnev and Shkredov prove the following (see \cite{RS18} Thm. 5).

**Theorem 1.2.** Let $p$ be a prime and let $A \subseteq \text{Aff}(F_p) \leftrightarrow F_p^2 \times F_p$ with $A = A^{-1}$ and $|A^2| = C|A|$. Then at least one of the following is true:

(a) $A \subseteq \text{Stab}(x)$ for some $x \in F_p$;

(b) when $1 < |A| \leq (1 + \varepsilon)p$ for some $0 < \varepsilon < 1$, we have $|\pi(A)| \leq 2C^4$;

(c) when $|A| > (1 + \varepsilon)p$ for some $0 < \varepsilon < 1$, we have $|\pi(A)| \ll \varepsilon \frac{1}{p} C^3|A|$, and in particular for $|A| > 4p$ we have $|\pi(A)| \leq \frac{2}{p} C^3|A|$ and $A^8 \supseteq U$.

Szönyi’s bound is part of a long history of applications of results about lacunary polynomials over finite fields to finite geometry: the reader interested in similar applications can check \cite{Sz99} and its bibliography.

Many results in this area can apply, with the appropriate modifications, to $F_q$ as well. In this case, however, bounds on the number of directions spanned by a set in the finite plane appear to be messier, and understandably so: unlike in the case of $F_p$, the number of directions determined by $A$ tends to congregate around values $\frac{|A|}{p}$ for powers $p^i|q$; this is due to the fact that there may exist sets with multiples of $p^i$ points on each line that are so well-structured that they sit in relatively few directions compared to the amount of points they have (see \cite{BBB+99} §5 for an example of this assertion when $|A| = q$).

The result we essentially use, on the number of directions spanned in $F_q^2$ by some set with $1 < |A| \leq q$, is due to Fancsali, Sziklai and Takáts \cite{FST13} Thm. 17: for the lower bound they found we give here a proof that is very similar to theirs, but we also prove a different upper bound that can be more or less advantageous than theirs depending on the situation (Theorem 2.2). Used directly, the lower bound can only give us $\frac{|A|}{q/p}$ directions; a tighter theorem, in the style of \cite{BBB+99} Thm. 1.1, would give not only $p^i|p^e = q$, but also $\ell e$ (and therefore the much better $\frac{|A|}{\sqrt{q}}$ for the number of directions): \cite{BBB+99} Thm. 1.1] however works only for $|A| = q$, and the lack of a complete set of $q$ points is crucial in worsening the condition on the denominator $p^i$ during the proof.

Nevertheless, it turns out that a simple observation can make us achieve the bound with $\sqrt{q}$ in the denominator: at its core, we use the fact that a set of points $A$ either sits on $\geq \sqrt{q}$ parallel lines or has a line with $\geq \frac{|A|}{\sqrt{q}}$ points on it. Our first main result then, playing the role of Szönyi’s bound in \cite{RS18}, is as follows.

**Theorem 1.3.** Let $q = p^e$ be a prime power, and let $A \subseteq F_q^2$ with $1 < |A| \leq q$. Then either $A$ is contained in a line or $A$ spans

(a) $\frac{|A|}{\sqrt{q}}$ directions for $e$ even,

(b) $\frac{|A|}{p^e/2^{e+i}}$ directions for $e$ odd.

Observe that the theorem is only a constant away from Szönyi’s bound when we use it for $q = p$; we add that actually the proof can be easily adjusted to yield
that bound exactly: we chose not to do so in order to get a cleaner statement, with case (1) valid for all $e$ odd.

Using Theorem 1.3 and following more or less the same proof as in [RS18], we obtain our second main result, generalizing Theorem 1.2 to any $\mathbb{F}_q$.

**Theorem 1.4.** Let $q = p^e$ be a prime power and let $A \subseteq \text{Aff}(\mathbb{F}_q) \leftrightarrow \mathbb{F}_q^* \times \mathbb{F}_q$ with $A = A^{-1}$ and $|A^3| = C|A|$. Then at least one of the following is true:

(a) $A \subseteq \text{Stab}(x)$ for some $x \in \mathbb{F}_q$;

(b) when $1 < |A| \leq q$ we have $|\pi(A)| < (p^{\frac{e}{2}} + 2)C^4$, while when $q < |A| < (3 + 2\sqrt{2})q$ we have $|\pi(A)| < (4 + 2\sqrt{2})C^4$;

(c) when $|A| \geq (3 + 2\sqrt{2})q$ we have $|\pi(A)| < \frac{2}{q}C^3|A|$ and $A^8 \supseteq U$.

The statement above looks remarkably similar to Theorem 1.2, and is qualitatively as strong a structural theorem as in the case of $\text{Aff}(\mathbb{F}_p)$.

Let us comment however on a small difference between the two. The case of a medium-sized $A$ (i.e. $1 < |A| \ll 1$) has been placed into alternative (1) by Rudnev and Shkredov and into alternative (1) by us, essentially losing the $A^k \supseteq U$ implication: this has been done because the subgroup $H$ of Kneser’s theorem can stifle the growth of $A$, in a way that Cauchy-Davenport could not (asking for $p$ large enough is innocuous there, but not here).

We could still use Alon’s bound [Alo86, (4.2)] on the number of lines in the projective plane as done in [RS18], since it holds for $\mathbb{F}_q$ as well: this would give for example $|\pi(A)| < \frac{2(\sqrt{5} + 1)}{(7 - 3\sqrt{5})q}C^3|A|$ for $|A| \geq \frac{\sqrt{5} - 1}{2}q$ (where the maximum of $\frac{e^2(1-e)}{2(1+e)}$ is located) and in general $|\pi(A)| \ll \frac{1}{q}C^3|A|$ for $|A| \geq (1 + \epsilon)q$; then, upon using Kneser’s theorem, one could either ask for $p$ large enough ($p > 100$ in the first case, say, and $p \gg 1$ in general) or classify separately the sets $A$ with large $H$ (which should be possible, because having large $H = \text{Stab}(A^2)$ is a rather restrictive condition to satisfy), and an additional conclusion $A^k \supseteq U$ for $k \ll 1$ would be reached. It would probably be interesting to explore more deeply these medium-sized sets: however, for the purpose of obtaining a structural result like Theorem 1.4 whose numerical details are of secondary relevance, we deemed to be simpler and just as effective to reduce that case to alternative (1), especially as the observation behind our ability to do so (Lemma 2.1) is very elementary.

## 2 Number of directions in $\mathbb{F}_q^2$

In the present section we prove bounds about the number of directions determined by sets of points in the plane $\mathbb{F}_q^2$, which lead eventually to Theorem 1.3.

Let us start with the following simple statement: it does not concern Theorem 1.3, but it will allow us in the next section to deal quickly with the sets $A$ whose size is slightly larger than $q$.

**Lemma 2.1.** Any set $A \subseteq \mathbb{F}_q^2$ with $|A| > q$ spans all $q + 1$ directions.
Proof. The result is immediate: by the pigeonhole principle, for any given direction, one of the \( q \) parallel lines in \( \mathbb{F}_q^2 \) following that direction has to contain at least two points of \( A \). \qed

As a complement to Lemma 2.1, the following theorem deals with the number of directions spanned by sets of size at most \( q \). As remarked before, a theorem of the same nature appears already in [FST13], and it is proved very similarly using the same techniques deriving from the study of lacunary polynomials.

Theorem 2.2. Let \( q = p^e \) be a prime power, let \( A \subseteq \mathbb{F}_q^2 \) with \( 1 < |A| \leq q \), and let \( D \) be the set of directions determined by \( A \). Then either \(|D| = 1 \) (and \( A \) is contained in a line), \(|D| = q + 1 \) (and \( A \) spans all directions) or there are two integers \( 0 \leq l_2 \leq l_1 < e \) such that

\[
|D| \geq |A| - 1 \frac{p^{l_2} + 1}{p^{l_1}} + 2,
\]

\[
|D| \leq q - |A| + \max \left\{ 1, \frac{|A| - 1 - (q - |A|)}{p^{l_1}} \max \{0, |A| + p^{l_1} - q - 1\} \right\}.
\]

A little notational comment: if \( l_1 = 0 \) we consider the upper bound trivial (but the lower bound becomes \( \frac{|A|+3}{2} \), which is quite strong, identical to Szönyi’s bound for \( \mathbb{F}_p \)).

Before we go to the proof, let us spend a few more words comparing this result with the one in [FST13]: their bounds are written as \( |A| - 1 + t + 2 \leq |D| \leq |A| - 1 + s - 1 \), for some appropriately defined \( s, t \). The lower bound is the same as the one presented here, as \( t \) and \( p^{l_2} \) are defined in the same way. The situation for the upper bound is more interesting: we have \( s \leq t = p^{l_2} \leq p^{l_1} \), because the authors define \( s \) looking at the multiplicities in \( H_y(x) \) alone (see the proof below for details) instead of the whole \( x^g + g_y(x) \), which also gives a stronger geometric meaning to their \( s \) than to our \( l_1 \); however, our upper bound tends to be stronger when \(|A| \) is fairly close to \( q \) and there is a gap between \( s \) and \( p^{l_1} \) (which can happen, as observed in [FST13]).

Proof. First of all, we can suppose \( \infty \in D \). If this were not true, we could take any \( d \in D \setminus \{0\} \) (\( D \) is nonempty for \(|A| > 1 \), and \( D = \{0\} \) concludes the theorem) and consider \( A' \) made of points \((a - db, b)\) for any \((a, b) \in A\), which implies also that \(|A'| = |A|\): such a set would span directions given by

\[
\frac{b' - b}{a' - db' - a + db} = \frac{1}{b' - b} - d,
\]

from which it is clear that the new set of directions \( D' \) is as large as \( D \), since equalities are preserved, and that moreover \( \infty \in D' \).

Define \( n \geq 0 \) such that \(|A| = q - n\). First, define the Rédei polynomial

\[
H_y(x) = \prod_{i=1}^{q-n} (x + ya_i - b_i) \in \mathbb{F}_q[x, y],
\]

\[

\]
where the product is among all the \((\alpha_i, b_i) \in A\): it is a polynomial of degree \(q - n\) in two variables (some authors, like in \cite{BB1999+}, define it as a homogeneous polynomial in three variables, but by ensuring that \(\infty \in D\) we do not need to do so). The usefulness of such polynomial lies in the fact that two points of \(A\) sitting on the same line with slope \(y_0\) yield the same \(x + y_0 a - b\), so that a multiple root in \(H_{g_0}(x)\) reflects the presence of a line with multiple points, i.e. a secant of \(A\), and indicates that \(y_0 \in D\). We also define another function in two variables, 

\[
f_y(x) = \sum_{j=0}^{n} (-1)^j \sigma_j(\mathbb{F}_q \setminus \{ya_i - b_i | (\alpha_i, b_i) \in A\})x^n - j,
\]

where \(\sigma_j(S)\) is the \(j\)-th elementary symmetric polynomial of the elements in the set \(S\); \(f_y(x)\) is itself a polynomial in two variables (see \cite{Sz66} Thm. 4) for a recursive definition of \(f_y(x)\), in which the coefficient of \(x^{n-j}\) has \(y\)-degree \(j\): therefore we can write 

\[x^q + g_y(x) = H_y(x)f_y(x) \in \mathbb{F}_q[x,y],\]

where \(g_y(x)\) is a polynomial in two variables of \(x\)-degree \(\leq q - 1\).

Substituting \(y = y_0\) for some \(y_0 \notin D\), we observe that by definition the set \(\mathbb{F}_q \setminus \{ya_i - b_i | (\alpha_i, b_i) \in A\}\) has \(n\) elements and that \(f_{y_0}(x)\) is simply the product of the \(x-k_i\), for all the \(k_i \in \mathbb{F}_q\) not counted in \(H_{g_0}(x)\), so \(g_{y_0}(x) = -x\): this means that the coefficients of \(x^{q-1}, x^{q-2}, \ldots, x^{D}\) in \(g_{y_0}(x)\) are polynomials of degree \(\leq q - |D|\) in \(y\) that take value 0 for the \(|D| + 1\) values \(y_0 \in \mathbb{F}_q \setminus D\). Thus, these coefficients are the zero polynomial; in other words, the \(x\)-degree of \(g_{y_0}(x)\) is at most \(|D| - 1\).

Working with \(x, y\) has allowed us to give a bound on the degree of \(g_{y_0}(x)\). From now on, for the sake of simplicity we substitute one value \(y \in D \setminus \{\infty\}\) inside our polynomials and drop the index, and we will work with only one variable; this is possible unless \(D = \{\infty\}\), from which \(|D| = 1\) and \(A\) is contained in a vertical line (or a general line, if we got to \(\infty \in D\) by the linear transformation at the beginning of the proof).

Call \(l_2\) the largest integer for which \(g(x) \in \mathbb{F}_q[x^{p^2}]:\) by the fact that any \(x \mapsto x^q\) is an automorphism of \(\mathbb{F}_q\), we have \(g(x) = (g(x))^p\) for some \(g(x) \in F_q[x] \setminus F_q[x^p]\). Decompose \(x^q + g(x)\) into its irreducible factors, and call \(l_1\) the largest integer for which \(p^{l_1}\) divides the multiplicity of each linear factor (hence \(l_1 \geq l_2\)): \(l_1, l_2\) depend on our choice of \(y\), so for our definition we suppose that we have chosen a \(y\) that yields the smallest \(l_1\). We can write

\[x^{q/p^{l_2}} + \tilde{g}(x) = (R(x))^{p^{l_1-l_2}} N(x),\]

where \(R(x) \in F_q[x] \setminus F_q[x^p]\) is such that \((R(x))^{p^{l_1}}\) is the divisor of \(x^q + g(x)\) made of its linear factors (the fully reducible part of \(x^q + g(x)\)) and \(N(x) \in F_q[x] \setminus F_q[x^p]\) is such that \((N(x))^{p^{l_2}}\) is the divisor of \(x^q + g(x)\) made of its nonlinear factors. Note that \((N(x))^{p^{l_2}}\) must be a divisor of \(f(x)\). If \(l_1 = e\) then \(x^q + g(x) = (x + c)^q\) for some \(c \in F_q\), which means that all the points of \(A\) lie on a line of slope equal to the \(y\) we have fixed, contradicting \(\infty \in D\): hence \(l_2 \leq l_1 < e\).

Call \(R^*(x)\) the divisor of \(R(x)\) made of all the irreducible factors of \(R(x)\) counted without multiplicity: \(R^*(x)\) divides also \(x^q - x\) by definition, so it divides
for any choice of polynomials $x^q + g(x) - (x^q - x) = q(x) + x \neq 0$ ($y \in D$ prevents us from having $g(x) = -x$).

If an irreducible polynomial $k_1(x)$ divides another $k_2(x)$ with multiplicity $m$ then it divides $k_2'(x)$ with multiplicity $m - 1$, so

$$\frac{(R(x))^{p^{j-1}-1}}{R^*(x)}(x^{p^j} + \tilde{g}(x))' = \tilde{g}'(x) \neq 0,$$

where the last is true because $\tilde{g}(x) \notin \mathbb{F}_q[x^p]$. From the reasoning above, we obtain

$$x^q + g(x) = \left(R^*(x) \cdot \frac{(R(x))^{p^{j-1}-1}}{R^*(x)}\right)^{p^j} (N(x))^{p^j}(g(x) + x)^{p^j} (\tilde{g}'(x))^{p^j} f(x) \neq 0,$$

and therefore $q = \deg(x^q + g(x)) \leq p^j (\deg(g(x) + x) + \deg(\tilde{g}'(x)) + \deg f(x)$; we have already determined that $\deg(g(x) + x) \leq \deg g(x) \leq |D| - 1$, and similarly $\deg \tilde{g}'(x) \leq \frac{\deg g(x)}{p^j} - 1 \leq \frac{|D|-1}{p^j} - 1$, hence from the definition of $f(x)$ we get

$$q \leq p^j \left(|D| - 1 + \frac{|D|-1}{p^j} - 1\right) + n \implies |D| \geq \frac{q - n - 1}{p^j + 1} + 2,$$

settling the lower bound.

Let us focus now on the upper bound. Fix a point $(a, b) \in A$ and take a slope $y_0 \in \mathbb{F}_q$; the multiplicity of the linear factor $x + y_0a - b$ inside $H(x)$ determines how many points of $A$ sit on the line defined by $(a, b)$ and $y_0$. We know that the multiplicity of every linear factor in the whole $H(x)f(x)$ is a multiple of $p^j$ and that it is at least 1 for this particular linear factor, since $(a, b)$ sits on the line; however, we need a way to keep under control the number of false positives that come from the fully reducible part of $f(x)$ (inexistent “ghost points” that make us overcount the contribution of a single line to $A$, and thus undercount $|D|$). The way to go is to bound the number of lines passing through $(a, b)$ for which ghost points exist.

Let $f_y(x)$ be as in (2.1), call it for simplicity $f_y(x) = \sum_{j=0}^{n} \sigma_{y,j} x^{n-j}$ where the $\sigma_{y,j}$ are polynomials in $y$ of degree $j$. Assume that $|D| < q + 1$: then there will be a direction $y_0 \in \mathbb{F}_q \setminus D$, as $\infty \in D$. For this $y_0$, $H_{y_0}(x)f_{y_0}(x) = x^q - x$ and $x + y_0a - b$ has multiplicity 1 in it; moreover, it must come from our fixed point $(a, b)$, which means that it must divide $H_{y_0}(x)$ and be coprime with $f_{y_0}(x)$: this fact implies that the two-variable linear polynomial $x + y_0a - b$ cannot divide $f_y(x)$.

In other words, we cannot write

$$(x^{n-1} + \tau_{y,1}x^{n-2} + \ldots + \tau_{y,n-1})(x + y_0a - b) = x^n + \sigma_{y,1} x^{n-1} + \ldots + \sigma_{y,n} (2.2)$$

for any choice of polynomials $\tau_{y,i}$; however, defining

$$\tau_{y,i} = \sum_{j=0}^{i} (-1)^{j} (ya + b)^{j} \sigma_{y,i-j}$$
(here $\sigma_{y,0} = 1$) we can ensure that the equality (2.2) works at least at the level of
the coefficients of $x, x^2, \ldots, x^{n-1}$, which means that we must have
\[
\sum_{j=0}^{n} (-1)^j (ya + b)^j \sigma_{y,n-j} \neq 0,
\] (2.3)
so as to violate (2.2) for the free coefficient.

Every time $f_{y_i}(x)$ has a $x + y_i a - b$ factor (or, geometrically speaking, every
time the line determined by $(a, b)$ and $y_i$ has a ghost point), (2.2) is true for
$y = y_i$ though, and in particular the LHS of (2.3) is indeed 0: that expression is
a polynomial in $y$ of degree $n$, so if there were $n + 1$ lines with ghost points (2.3)
would not be true, contradicting the fact that $x + ya - b$ cannot divide $f_{y}(x)$ as
polynomials in two variables. Hence, at most $n$ non-vertical lines through $(a, b)$
have ghost points.

If $|D| - 1 \leq n$ the upper bound stated in the theorem is already true, so suppose
that the opposite holds: then there is a non-vertical line through $(a, b)$ whose slope
is in $D$ with no ghosts. We can transform $A$ at the beginning of the proof to
make that slope $\infty$, i.e. $(a, b)$ lies on a vertical secant of $A$.

Each non-vertical line through $(a, b)$ whose slope is in $D$ has a multiple of $p^{l_1}$
among true points of $A$ and ghost points ($l_1$ has been defined so as to make that
statement true for all slopes at the same time). On the ghost-free lines there are at
least $p^{l_1} - 1$ true points besides $(a, b)$, while on the ones with ghosts we can only
say that there are at least $\max\{0, p^{l_1} - 1 - n\}$ of them (as the $x$-degree of $f_{y}(x)$ is
$n$); finally, the vertical secant has at least $p^{l_1}$ points including $(a, b)$, as we made
sure that it had no ghosts before the transformation. Combining all of this with
the bound on the number of lines with ghosts, and counting all the points of $A$,
we get
\[
(|D| - 1 - n)(p^{l_1} - 1) + n \max\{0, p^{l_1} - 1 - n\} + p^{l_1} \leq q - n.
\]
As we remarked after the statement of the theorem, for $l_1 = 0$ there is no upper
bound. For $l_1 > 0$, the inequality above concludes the proof.

Now that we have the lower bound provided by Theorem 2.2, we can proceed
with the proof of the first main theorem. We retain the same notation as in the
previous proof.

Proof of Thm. 1.3. Suppose that $|D| \neq 1, q + 1$ (otherwise the theorem is already
proved): fix a slope $y_0 \neq \infty$ and consider the polynomial $R^\ast(x)$ defined as in the
proof of Theorem 2.2. Let $\varepsilon > 0$ be small enough, and let $q' = p^{\varepsilon} - \varepsilon$ for $e$ even
and $q' = p^{\varepsilon - 1}$ for $e$ odd.

If the degree of $R^\ast(x)$ is $\leq q'$, the set $A$ must be contained in $\leq q'$ lines with
slope $y_0$, which means that one of them (call it $L$) will have to contain $\geq \frac{|A|}{q'}$ points
of $A$: since $A$ is not contained in one line there must be also a point of $A$ outside
$L$, and each secant laid between this point and a point of $A \cap L$ has a different
slope, so that $|D| \geq \frac{|A|}{q'}$: for $e$ even it means $|D| > \frac{|A|}{\sqrt{q'}}$, while for $e$ odd it means
$|D| \geq \frac{|A|}{p^{\varepsilon - 1}}$. 8
If $R^*(x)$ has degree $> q'$, then by the fact that $(R^*(x))^{p^e}$ divides $x^q + g(x)$ we must have $p^e \leq p^{q/2} < q/2$: regardless of whether $e$ is even or odd, $p^e \leq p^{q/2}$ since $l_2$ is an integer. Using the lower bound in Theorem 2.2 (which holds for our $A$), we have

$$|D| \geq \frac{|A| - 1}{p^{l_2} + 1} + 2 = \frac{|A|}{p^{l_2}} + 2 - \frac{|A|}{p^{l_2}(p^{l_2} + 1)} - \frac{1}{p^{l_2} + 1}.$$ 

For $e$ even, the bound above implies $|D| > \frac{|A|}{\sqrt{q}}$, while for $e$ odd we can obtain $|D| \geq \frac{|A|}{p^{l_2} + 1}$.

\[\square\]

### 3 Growth in $\text{Aff}(F_q)$

We move now to the proof of Theorem 1.4. We follow closely the proof of the analogous result in [RS18] for $F_p$: the only difference is that we use Theorem 1.3 instead of Szónyi’s bound, and that as we have already said we absorb the case of $A$ of medium size into alternative (b), without resorting to Alon’s bound to fall into (c).

We remind the reader of two well-known and by now classical results. First, an inequality, deducible in multiple ways from bounds by Ruzsa (see for instance [Ruz98]), states that for any group $G$ and any $A = A - 1 \subseteq G$ the equality $|A^3| = C|A|$ implies $|A^k| \leq C^{k-2}|A|$ for any $k \geq 4$. Second, Kneser’s theorem [Kne53] tells us that, given any abelian $G$ and any $A, B \subseteq G$, there is a proper subgroup $H$ with $|A + B| \geq \min\{|G|, |A| + |B| - |H|\}$.

Theorem 1.3 and Lemma 2.1 will take care of small and medium $|A|$, respectively. For $|A|$ large we will instead make use of the following bound, due to Vinh [Vin11, Thm. 3]: the statement therein says actually something weaker, but it is based on a well-known graph-theoretic result [AS16, Cor. 9.2.5] that lets us reformulate as follows (as [RS18] does for $F_p$).

**Proposition 3.1.** Let $q$ be a prime power, let $P$ be a set of points in $\mathbb{F}^2_q$ and let $L$ be a set of lines in $\mathbb{F}^2_q$; define $I(P, L)$ as the set of pairs $(p, l) \in P \times L$ s.t. $p \in l$. Then

$$|I(P, L)| - \frac{|P||L|}{q} \leq \sqrt{q|P||L|}.$$ 

Let us also give here separately a lemma that will provide the upper bounds on $\pi(A)$ in Theorem 1.4(b)-(c).

**Lemma 3.2.** For any $g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in \text{Aff}(F_q) \setminus \{\text{Id}\}$, define the map

$$\varphi_g : \text{Aff}(F_q) \to \text{Aff}(F_q), \quad \varphi_g(h) = hgh^{-1}.$$ 

Then,

(a) any point in the image of $\varphi_g$ has as preimage a line of slope $\frac{b}{a-1}$;
(b) if \( A = A^{-1} \subseteq \text{Aff}(\mathbb{F}_q) \) and \( g \in A^k \) then \( |\pi(A)| \leq \frac{|A^{k+3}|}{|\varphi_g(A)|} \)

Proof. \( \square \) This is just an easy computation: as

\[
\left( \begin{array}{rr}
 r & s \\
 0 & 1
\end{array} \right) \left( \begin{array}{rr}
 a & b \\
 0 & 1
\end{array} \right) \left( \begin{array}{rr}
 r^{-1} & -r^{-1}s \\
 0 & 1
\end{array} \right) = \left( \begin{array}{rr}
 a & br + (1-a)s \\
 0 & 1
\end{array} \right),
\]

two elements are in the preimage of a single point if and only if \( br + (1-a)s = br' + (1-a)s' \), from which all pairs of elements with \( \frac{r-r'}{s-s'} = \frac{b}{a} \) must be sent to the same point by \( \varphi_g \) (we allow \( a = 1 \) and a slope equal to \( \infty \), but we avoid \( (a,b) = (1,0) \) since \( g \neq \text{Id} \)).

\( \square \) On one hand we have \( |A \varphi_g(A)g^{-1}| = |A \varphi_g(A)| \leq |A^{k+3}| \), while on the other hand any element of \( A \varphi_g(A)g^{-1} \) is of the form \( a_1a_2g_2^{-1}g^{-1} \in AU \): since

\[
\left( \begin{array}{rr}
 x & y \\
 0 & 1
\end{array} \right) \left( \begin{array}{rr}
 1 & z \\
 0 & 1
\end{array} \right) = \left( \begin{array}{rr}
 x & xz + y \\
 0 & 1
\end{array} \right),
\]

pairs in \( A \times U \) with either distinct \( x \) or with the same \( x, y \) but distinct \( z \) will all give different products in \( AU \); hence we can select one element of \( A \) for each value of \( x \) (therefore \( |\pi(A)| \) of them) and all the \( a_2g_2^{-1}g^{-1} (|\varphi_g(A)| \) of them, they are all multiplied by the same \( g^{-1} \) and obtain the other side of the bound, namely \( |A \varphi_g(A)g^{-1}| \geq |\pi(A)||\varphi_g(A)| \).

With these tools at our disposal, we can proceed with the proof.

Proof of Thm. \( \square \) Let us start with the case of \( A \) large: impose \( |A| \geq cq \) for a constant \( c > 1 \) to be chosen later.

We use the bound from Proposition \( \square \) with \( P = A \) and \( L = L(A) \) (the set of lines that are not determined by \( A \)), interpreted as a lower bound on the expression inside the absolute value, and combine it with the trivial observation that \( I(A, L(A)) \leq |L(A)| \) by the definition of \( L(A) \): this yields

\[
|L(A)| \leq q^2 \frac{c}{(c-1)^2} \quad \implies \quad |L(A)| \geq q + q^2 \left( 1 - \frac{c}{(c-1)^2} \right).
\]

If \( c \geq 1 + \frac{1+\sqrt{1-4-\frac{4}{q}}}{1+\frac{1}{p}} \) (or \( c \geq 3 + 2\sqrt{2} \), which is an upper bound for all primes \( p \)), by the pigeonhole principle there must exist \( \geq \frac{q}{2} \left( 1 + \frac{1}{p} \right) \) non-vertical lines of \( L(A) \) parallel to each other; call \( d \) the direction defined by such lines. Given any two elements of \( A \) sitting on one of these lines, we have

\[
g = \left( \begin{array}{rr}
 a_1 & b_1 \\
 0 & 1
\end{array} \right)^{-1} \left( \begin{array}{rr}
 a_2 & b_2 \\
 0 & 1
\end{array} \right) = \left( \begin{array}{rr}
 a_2a_1^{-1} & b_2a_1^{-1} - b_1a_1^{-1} \\
 0 & 1
\end{array} \right) = \left( \begin{array}{rr}
 a' & b' \\
 0 & 1
\end{array} \right),
\]

with \( \frac{a'}{a^{-1}} = \frac{b-a}{b_2-a_2} = d \), so by Lemma \( \square \) there are \( \geq \frac{q}{2} \left( 1 + \frac{1}{p} \right) > \frac{q}{2} \) elements in \( \varphi_g(A) \); by Lemma \( \square \) and Ruzsa’s inequality, this implies that \( |\pi(A)| < \frac{2|A|^3}{q} \leq \frac{q}{2} C^3 |A| \). Moreover, the unipotent subgroup \( U \) is isomorphic to \( \mathbb{F}_q \) as an
additive group, so that its largest proper subgroup is of size \( \frac{q}{p} \); therefore, since

\( \phi_g(A)g^{-1} \subseteq A^6 \cap U \) has \( |\phi_g(A)g^{-1}| \geq \frac{q}{2} \left( 1 + \frac{1}{p} \right) \), by Kneser’s theorem we must have

\[
A^6 \supseteq AqAg^{-1}A \supseteq (\phi_g(A)g^{-1})(\phi_g(A)g^{-1})^{-1} \supseteq U,
\]

and we fall into case 3 of the theorem.

Let us deal now with \( A \) of medium size: suppose \( q < |A| < cq \), so that by Lemma 2.1 every direction is determined by some pair of points of \( A \). Partition \( A^2 \setminus \{ \text{Id} \} \) into \( q + 1 \) subsets, collecting into each one of them elements having the same value for \( \frac{a}{n+1} \in \mathbb{F}_q \cup \{ \infty \} \). Every two distinct \( a_1, a_2 \in A \) yield an element \( a_1^{-1}a_2 \in A^2 \) that is located into the part corresponding to the slope of the line they define: by the pigeonhole principle there will be a part (identifiable with some \( d \in \mathbb{F}_q \cup \{ \infty \} \) with at most \( \frac{|A|^2 - 1}{q+1} \) elements, and therefore every line of \( L(A) \) in the direction \( d \) must have at most \( \frac{|A|^2 - 1}{q+1} + 1 \) elements of \( A \) on it, since \( a_1^{-1}a_i \neq a_1^{-1}a_j \) for \( a_i \neq a_j \). We have thus given a bound on the number of points of \( A \) sent to the same element by the map \( \phi_g \) for some \( g \in A^2 \) with \( \frac{a}{n+1} = d \), which translates to

\[
|\phi_g(A)| \geq \frac{(q + 1)|A|}{|A^2| + q} > \frac{|A|}{Cc + 1};
\]

proceeding as before, by Lemma 5.2 and Ruzsa’s inequality we conclude that \( |\pi(A)| < (Cc + 1)C^3 \leq (4 + 2\sqrt{2})C^4 \) and we reach case 1 of the theorem.

For \( A \) small (i.e. \( 1 < |A| \leq q \)) we repeat essentially what we did for \( A \) medium, but instead of \( |D| = q + 1 \) we use the bounds in Theorem 1.3 on the number of directions \( |D| \) spanned by \( A \). We obtain

\[
|\phi_g(A)| > \frac{|A|^2}{q'(|A^2| - 1) + |A|} > \frac{|A|}{Cc'q' + 1},
\]

where \( q' = \sqrt{7} \) for \( e \) even and \( q' = \frac{1}{2} \sqrt{2} + 1 \) for \( e \) odd, from which we get \( |\pi(A)| < (p^{1+1/2} + 2)C^4 \) and end up in case 3. Finally, we need to deal with the other alternative in Theorem 1.3 namely that \( A \) may be contained in one line: in other words, the elements of \( A \) are either all of the form \( (a, ad + b) \) for some \( b, d \in \mathbb{F}_q \), through the identification of \( \text{Aff}(\mathbb{F}_q) \) with \( \mathbb{F}^*_q \times \mathbb{F}_q \), or all contained in \( U \). In the latter alternative \( A \subseteq U \) implies \( |\pi(A)| = 1 \), yielding 3; in the former, since \( A = A^{-1} \) and \( (a, ad + b)^{-1} = (a^{-1}, -a^{-1}b - d) \), we must have \( b = -d \) and then \( A \subseteq \text{Stab}(-d) \).

\( \square \)

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References

[Alo86] N. Alon. Eigenvalues, geometric expanders, sorting in rounds and Ramsey theory. *Combinatorica*, 6(3):207–219, 1986.

[AS16] N. Alon and J. H. Spencer. *The probabilistic method*. Wiley Publishing, fourth edition, 2016.

[BBB+99] A. Blokhuis, S. Ball, A. E. Brouwer, L. Storme, and T. Szőnyi. On the number of slopes of the graph of a function defined on a finite field. *J. Combin. Theory Ser. A*, 86:187–196, 1999.

[FST13] S. L. Fancsali, P. Sziklai, and M. Takts. The number of directions determined by less than \( q \) points. *J. Algebraic Combin.*, 37:27–37, 2013.

[Hel15] H. A. Helfgott. Growth in groups: ideas and perspectives. *Bull. Amer. Math. Soc. (N.S.)*, 52(3):357–413, 2015.

[Kne53] M. Kneser. Abschätzung der asymptotischen Dichte von Summenmengen. *Math. Z.*, 58:459–484, 1953. In German.

[Mur17] B. Murphy. Upper and lower bounds for rich lines in grids. *arXiv:1709.10438*, 2017.

[RS18] M. Rudnev and I. D. Shkredov. On growth rate in \( SL_2(\mathbb{F}_p) \), the affine group and sum-product type implications. *arXiv:1812.01671*, 2018.

[Ruz96] I. Z. Ruzsa. Sums of finite sets. In D. V. Chudnovsky, G. V. Chudnovsky, and M. B. Nathanson, editors, *Number Theory: New York Seminar 1991-1995*, pages 281–293. Springer-Verlag, New York (USA), 1996.

[Sző96] T. Szőnyi. On the number of directions determined by a set of points in an affine Galois plane. *J. Combin. Theory Ser. A*, 74:141–146, 1996.

[Sző99] T. Szőnyi. Around Rédei’s theorem. *Discrete Math.*, 208/209:557–575, 1999.

[Vin11] L. A. Vinh. The Szemerédi-Trotter type theorem and the sum-product estimate in finite fields. *European J. Combin.*, 32:1177–1181, 2011.