Bounds for the $p$-Angular Distance and Characterizations of Inner Product Spaces

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Abstract. Based on a suitable improvement of a triangle inequality, we derive new mutual bounds for $p$-angular distance $\alpha_p[x, y] = \|\|x\|^p - \|y\|^p\|\|x\|^{p-1}x - \|y\|^{p-1}y\|$, in a normed linear space $X$. We show that our estimates are more accurate than the previously known upper bounds established by Dragomir, Hile and Maligranda. Next, we give several characterizations of inner product spaces with regard to the $p$-angular distance. In particular, we prove that if $|p| \geq |q|$, $p \neq q$, then $X$ is an inner product space if and only if for every $x, y \in X \setminus \{0\}$,

$$\alpha_p[x, y] \geq \frac{\|x\|^p + \|y\|^p}{\|x\|^{q} + \|y\|^{q}} \alpha_q[x, y].$$

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1. Introduction

Throughout this paper, $(X, \|\cdot\|)$ stands for a real normed linear space. During decades, the problem of providing necessary and sufficient conditions for a normed space to be an inner product space has been studied by numerous authors. Some recent characterizations of inner product spaces can be found in [1–3, 5, 20, 22, 26] and the references therein.

One of the most celebrated characterizations of inner product spaces has been based on the so-called Dunkl–Williams inequality which provides an upper bound for an angular distance (or Clarkson distance, see [6]) between nonzero vectors $x, y \in X$, defined by

$$\alpha[x, y] = \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|.$$
Namely, Dunkl and Williams [12] proved that the inequality
\[ \alpha[x, y] \leq \frac{4 \|x - y\|}{\|x\| + \|y\|} \]
holds for all nonzero vectors \( x, y \) in a real normed linear space \( X \). The authors also showed that constant 4 can be replaced by 2 if \( X \) is an inner product space. In addition, Kirk and Smiley [16] proved that the inequality
\[ \alpha[x, y] \leq \frac{2 \|x - y\|}{\|x\| + \|y\|} \]  
(1)
serves as a characterization of an inner product space.

In 1993, Al-Rashed [2], generalized (1) by showing that if \( 0 < q \leq 1 \), then a normed linear space \( X = (X, \|\cdot\|) \) is an inner product space if and only if the inequality
\[ \alpha[x, y] \leq \frac{2^{\frac{1}{q}} \|x - y\|}{(||x||^q + ||y||^q)^{\frac{1}{q}}} \]  
(2)
holds for all nonzero vectors \( x, y \in X \). Moreover, in 2010, Dadipour and Moslehian [7], extended (2) by giving characterization of an inner product space expressed in terms of \( p \)-angular distance \( \alpha_p[x, y] \), defined by
\[ \alpha_p[x, y] = \|\|x\|^{p-1}x - \|y\|^{p-1}y\|, \quad p \in \mathbb{R}, \]
where \( x, y \) are nonzero vectors (note that \( \alpha_0[x, y] = \alpha[x, y] \)). Finally, Rooin et al. [24], extended the above results from [2,7] by giving a whole series of comparative relations for the \( p \)-angular distance which serve as characterizations of inner product spaces. We give here only a part of the main result from [24] that will be important in our study.

**Theorem A.** Let \( X \) be a normed linear space, \( \dim X \geq 3 \), and \( p, q \in \mathbb{R} \) with \( |p| \leq |q| \) and \( p \neq q \). Then the following statements are mutually equivalent:

(I) \( X \) is an inner product space.

(II) If \( 0 \leq \frac{p}{q} < 1 \), then for any \( x, y \neq 0 \),
\[ \alpha_p[x, y] \leq \frac{2\alpha_q[x, y]}{\|x\|^{q-p} + \|y\|^{q-p}}. \]

(III) For any \( x, y \neq 0 \),
\[ \frac{\alpha_p[x, y]}{\|x\|^\frac{q}{2} \|y\|^\frac{q}{2}} \leq \frac{\alpha_q[x, y]}{\|x\|^\frac{q}{2} \|y\|^\frac{q}{2}}. \]

(IV) If \( p \in \mathbb{R}\setminus\{0\} \), then for any \( x, y \neq 0 \),
\[ \alpha_{-p}[x, y] = \|x\|^{-p}||y||^{-p}\alpha_p[x, y]. \]

(V) If \( 0 \leq \frac{p}{q} < 1 \), then there exists \( r \in \mathbb{R} \) such that for any \( x, y \neq 0 \),
\[ \alpha_p[x, y] \leq \frac{2^{\frac{1}{r}}\alpha_q[x, y]}{(\|x\|^{r(q-p)} + \|y\|^{r(q-p)})^{\frac{1}{r}}}. \]
(VI) If $0 < \frac{p}{q} < 1$, then for any $x, y \neq 0$ with $\|x\| \neq \|y\|$, 
\[
\alpha_p[x, y] < \frac{\alpha_q[x, y]}{\min\{\|x\|^{q-p}, \|y\|^{q-p}\}}.
\]

Some interesting characterizations of inner product spaces connected with the concept of the $p$-angular distance can also be found in [4,9,17,23,25]. In particular, Rooin et al. [25] established a very nice characterization of an inner product space by giving an explicit formula for the $p$-angular distance. More precisely, they proved that if $p \neq 1$, then a normed space $X$ is an inner product space if and only if
\[
\alpha_p^2[x, y] = \|x\|^{p-1}\|y\|^{p-1}\|x - y\|^2 + (\|x\|^{p-1} - \|y\|^{p-1}) \left(\|x\|^{p+1} - \|y\|^{p+1}\right)
\]
holds for all nonzero elements $x, y \in X$.

In the last few decades, several authors have been interested in providing bounds for the $p$-angular distance in an arbitrary normed linear space. In 2006, Maligranda [19] established the following mutual bounds for angular distance $\alpha[x, y]$, based on improved form of the triangle inequality:
\[
\frac{\|x - y\| - \|x\| - \|y\|}{\min\{\|x\|, \|y\|\}} \leq \alpha[x, y] \leq \frac{\|x - y\| + \|x\| - \|y\|}{\max\{\|x\|, \|y\|\}}.
\]

The author also established the following upper bounds for the $p$-angular distance, dependent on index $p$:
\[
\alpha_p[x, y] \leq \begin{cases} 
  p \max\{\|x\|, \|y\|\}\|x - y\|, & p \geq 1, \\
  (2 - p)\|x - y\| \max\{\|x\|, \|y\|\}, & 0 \leq p < 1, \\
  (2 - p) \min\{\|x\|, \|y\|\}\|x - y\|, & p < 0.
\end{cases}
\]

We also recall the following upper bound for the $p$-angular distance stated by Hile [14]:
\[
\alpha_p[x, y] \leq \frac{\|y\|^p - \|x\|^p}{\|y\| - \|x\|}\|x - y\|, 
\]
for $p \geq 1$ and $x, y \in X$ with $\|x\| \neq \|y\|$.

In 2015, Dragomir [10] established the integral upper bounds for the $p$-angular distance, which resulted in a refinement of the Maligranda inequality (5), when $p \geq 2$. The author also showed that the Hile inequality (6) is more precise than the first inequality in (5), when $p \geq 2$. In 2018, Rooin et al. [25] generalized the above results from [10,14,19] in the sense that the above bounds for the $p$-angular distance were expressed in terms of the $q$-angular distance.

In addition, some mutual bounds for the $p$-angular distance based on the triangle inequality have been derived by Dragomir in [11], which will be discussed in the next section.

The main objective of the present paper is to establish some new bounds for the $p$-angular distance and to provide new characterizations of inner product spaces. After this introduction, in Sect. 2, we establish mutual bounds for the $p$-angular distance, based on a suitable refinement of the triangle inequality. In particular, we show that our estimates are more accurate than those in
(5), (6), and the corresponding ones in [11]. In Sect. 3 we provide some characterizations of inner product spaces which are not covered by Theorem A. First, we establish the inequality that seems familiar to the Hile inequality (6), although it shows significantly different behavior since it characterizes an inner product space. Finally, we give several characterizations of an inner product space which are established through the connection between mutual bounds for \(\alpha[x, y]\) given by (4), and the explicit formula for the \(p\)-angular distance (3), when \(p = 0\). It should be noted here that our results are mainly inspired by recent papers [24,25].

\textbf{2. More Accurate Bounds for the \(p\)-Angular Distance in a Normed Linear Space}

The main objective of this section is to establish some new mutual bounds for the \(p\)-angular distance in an arbitrary normed linear space, and to compare them with some previously known results. Our new estimates are based on a suitable improvement of the triangle inequality.

We start our discussion with the lower bounds that correspond to inequalities in (5). Namely, Rooin et al. [25] derived mutual bounds for angular distance \(\alpha_p[x, y]\), expressed in terms of angular distance \(\alpha_q[x, y]\), using a similar method as it has been done in [19]. Consequently, they obtained the lower bounds for the \(p\)-angular distance that correspond to inequalities in (5). However, the lower bounds for the \(p\)-angular distance can be established simply by transforming inequalities in (5). Therefore, for the reader’s convenience, we establish lower bounds that correspond to (5), with an alternative proof.

\textbf{Corollary 1.} (Rooin et al. [25]) Let \(X = (X, \|\cdot\|)\) be a normed linear space. Then the inequalities

\[
\alpha_p[x, y] \geq \begin{cases} 
\frac{p}{2p-1} \max^{p-1}\{\|x\|, \|y\|\}\|x - y\|, & p \geq 1, \\
\frac{p}{2p-1} \min^{p-1}\{\|x\|, \|y\|\}\|x - y\|, & 0 \leq p < 1,
\end{cases}
\]

hold for all nonzero vectors \(x, y \in X\).

An alternative proof. Let \(p \geq 1\). Rewriting the second inequality in (5) with \(\frac{1}{p}\) instead of \(p\), we obtain

\[
\alpha_{\frac{1}{p}}[x, y] \leq \frac{2p - 1}{p} \cdot \frac{\|x - y\|}{\max^{1-\frac{1}{p}}\{\|x\|, \|y\|\}}.
\]

Now, since

\[
\alpha_{\frac{1}{p}}[\|x\|^{p-1}x, \|y\|^{p-1}y] = \|\|x\|^{p(\frac{1}{p}-1)}\|x\|^{p-1}x - \|y\|^{p(\frac{1}{p}-1)}\|y\|^{p-1}y\| = \|x - y\|,
\]

substituting \(x' = \|x\|^{p-1}x\) and \(y' = \|y\|^{p-1}y\) instead of \(x\) and \(y\), respectively, in the last inequality, we obtain

\[
\|x - y\| \leq \frac{2p - 1}{p} \cdot \frac{\alpha_p[x, y]}{\max^{p-1}\{\|x\|, \|y\|\}}.
\]
which yields the first inequality in (7).

The second inequality in (7) holds trivially for \( p = 0 \). To prove that the second inequality in (7) holds for \( 0 < p < 1 \), we utilize the first inequality in (5) rewritten with \( \frac{1}{p} \) instead of \( p \), and with \( x' = \|x\|^{p-1}x \) and \( y' = \|y\|^{p-1}y \) instead of \( x \) and \( y \), respectively. Similarly to above, we obtain

\[
\|x - y\| \leq \frac{\max_{p}^{1-p}\{\|x\|, \|y\|\}}{p} \alpha_p[x, y],
\]

which yields the second inequality in (7). In the same way, the third inequality in (7) follows from the third inequality in (5) rewritten with parameter \( \frac{x}{p} \) and with vectors \( x' = \|x\|^{p-1}x \) and \( y' = \|y\|^{p-1}y \) instead of \( x \) and \( y \), respectively.

\[\square\]

Remark 1. By the same method as in the proof of Corollary 1, inequalities in (5) and (7) can be transformed into mutual bounds expressed in terms of \( \alpha_q[x, y] \), as it has been done in [25]. For example, consider the first inequality in (7) with \( \frac{p}{q} \geq 1 \) instead of \( p \):

\[
\alpha_{\frac{p}{q}}[x, y] \geq \frac{p}{2p - q} \max_{q}^{\frac{p}{q} - 1}\{\|x\|, \|y\|\}\|x - y\|.
\]

Now, with \( x'' = \|x\|^{2-1}x \) and \( y'' = \|y\|^{2-1}x \), instead of \( x \) and \( y \), respectively, the above inequality transforms to

\[
\alpha_p[x, y] \geq \frac{p}{2p - q} \max\{\|x\|^{2-p}q, \|y\|^{2-p}q\} \alpha_q[x, y],
\]

since \( \alpha_{p/q}[x'', y''] = \alpha_p[x, y] \). For more details about this class of inequalities the reader is referred to [25]. Obviously, the last inequality is equivalent to the first inequality in (7), so in this section, we will only consider inequalities in (5) and (7).

Now, our goal is to derive more accurate bounds for the \( p \)-angular distance than those in (5) and (7). We start with establishing a suitable refinement and reverse of the triangle inequality.

**Theorem 1.** Let \( X = (X, \| \cdot \|) \) be a normed linear space and let \( f : \mathbb{R}^+ \rightarrow \mathbb{R} \) be a nonnegative function. Then the inequalities

\[
\begin{align*}
&f(\|x\| \|x\| + f(\|y\| \|y\| - (\|x\| + \|y\| - \|x - y\|) \max\{f(\|x\|), f(\|y\|)\}) \\
&\leq \|f(\|x\|) x - f(\|y\|) y\| \\
&\leq f(\|x\|) \|x\| + f(\|y\|) \|y\| - (\|x\| + \|y\| - \|x - y\|) \min\{f(\|x\|), f(\|y\|)\}
\end{align*}
\]

(8)

hold for all nonzero vectors \( x, y \in X \).

**Proof.** Without loss of generality, we can suppose that \( f(\|x\|) \geq f(\|y\|) \). In this case, the left inequality in (8) reduces to

\[
f(\|y\|) \|y\| - f(\|x\|) \|y\| + f(\|x\|) x - f(\|y\|) y \leq \|f(\|x\|) x - f(\|y\|) y\|.
\]

Clearly, since \( f(\|x\|) \geq f(\|y\|) \), the last inequality is equivalent to

\[
\|f(\|x\|) x - f(\|y\|) y\| \leq \|f(\|x\|) x - f(\|y\|) y\| + \|f(\|y\|) y - f(\|x\|) y\|,
\]
Corollary 2. Let \( X = (X, \| \cdot \|) \) be a normed linear space and let \( p \geq 1 \). Then the inequalities
\[
\|x\|^p + \|y\|^p - (\|x\| + \|y\| - \|x - y\|) \max^{p-1}\{\|x\|, \|y\|\} \\
\leq \alpha_p[x, y] \\
\leq \|x\|^p + \|y\|^p - (\|x\| + \|y\| - \|x - y\|) \min^{p-1}\{\|x\|, \|y\|\}
\]
hold for all nonzero elements \( x, y \in X \). If \( p < 1 \), then the inequalities in (9) are reversed.

Proof. It follows from Theorem 1 by putting \( f(t) = t^{p-1} \) and noting that \( f \) is increasing (decreasing) for \( p \geq 1 \) (\( p < 1 \)).

Remark 2. If \( p = 0 \), then, taking into account obvious relations
\[
2 \min\{\|x\|, \|y\|\} = \|x\| + \|y\| - \|x\| - \|y\|, \\
2 \max\{\|x\|, \|y\|\} = \|x\| + \|y\| + \|x\| - \|y\|,
\]
Corollary 2 provides mutual bounds for angular distance \( \alpha[x, y] \), given by (4).

In 2009, Dragomir [11] established the following two pairs of upper and lower bounds for the \( p \)-angular distance:
\[
\alpha_p[x, y] \leq \begin{cases} D_{\geq 1}(p), p \geq 1, \\ D_{< 1}(p), p < 1, \end{cases} \quad \text{and} \quad \alpha_p[x, y] \leq \begin{cases} S_{\geq 1}(p), p \geq 1, \\ S_{< 1}(p), p < 1, \end{cases}
\]
where
\[
D_{\geq 1}(p) = \|x - y\| \min^{p-1}\{\|x\|, \|y\|\} + \|x\|^{p-1} - \|y\|^{p-1} \max\{\|x\|, \|y\|\}, \\
D_{< 1}(p) = \|x - y\| \max^{p-1}\{\|x\|, \|y\|\} + \|x\|^{p-1} - \|y\|^{p-1} \max\{\|x\|, \|y\|\}, \\
S_{\geq 1}(p) = \|x - y\| \max^{p-1}\{\|x\|, \|y\|\} + \|x\|^{p-1} - \|y\|^{p-1} \min\{\|x\|, \|y\|\}, \\
S_{< 1}(p) = \|x - y\| \min^{p-1}\{\|x\|, \|y\|\} + \|x\|^{p-1} - \|y\|^{p-1} \min\{\|x\|, \|y\|\},
\]
and
\[
\alpha_p[x, y] \geq \begin{cases} d_{\geq 1}(p), p \geq 1, \\ d_{< 1}(p), p < 1, \end{cases} \quad \text{and} \quad \alpha_p[x, y] \geq \begin{cases} s_{\geq 1}(p), p \geq 1, \\ s_{< 1}(p), p < 1, \end{cases}
\]
where
\[d_{\geq 1}(p) = \|x - y\| \max^{p-1}\{\|x\|, \|y\|\} - \|x\|^{p-1} - \|y\|^{p-1} \max\{\|x\|, \|y\|\},\]
\[d_{< 1}(p) = \|x - y\| \min^{p-1}\{\|x\|, \|y\|\} - \|x\|^{p-1} - \|y\|^{p-1} \max\{\|x\|, \|y\|\},\]
\[s_{\geq 1}(p) = \|x - y\| \min^{p-1}\{\|x\|, \|y\|\} - \|x\|^{p-1} - \|y\|^{p-1} \min\{\|x\|, \|y\|\},\]
\[s_{< 1}(p) = \|x - y\| \max^{p-1}\{\|x\|, \|y\|\} - \|x\|^{p-1} - \|y\|^{p-1} \min\{\|x\|, \|y\|\}.\]

The estimates in (10) and (11) were also established as a consequence of certain variants of triangle inequalities, and are given here in a more suitable notation as in (10) and (11), the upper bounds given by (9) and its reverse will be denoted by \(K_{\geq 1}(p)\) and \(K_{< 1}(p)\), while the lower bounds will be denoted by \(s_{\geq 1}(p)\) and \(s_{< 1}(p)\). Here, the indices indicate the corresponding range of parameter \(p\). Note also that \(K_{\geq 1}(p) = k_{< 1}(p)\), as well as \(K_{< 1}(p) = k_{\geq 1}(p)\).

**Theorem 2.** Let \(x, y \in X\) be nonzero vectors. If \(p \geq 1\), then holds the relation
\[s_{\geq 1}(p) \leq d_{\geq 1}(p) \leq k_{\geq 1}(p) \leq \alpha_p[x, y] \leq K_{\geq 1}(p) = D_{\geq 1}(p) \leq S_{\geq 1}(p),\]
while for \(p < 1\) holds
\[s_{< 1}(p) \leq d_{< 1}(p) = k_{< 1}(p) \leq \alpha_p[x, y] \leq K_{< 1}(p) \leq D_{< 1}(p) \leq S_{< 1}(p).\]

**Proof.** Without loss of generality we assume that \(\|x\| \leq \|y\|\). Then it follows that
\[D_{\geq 1}(p) = K_{\geq 1}(p) = d_{< 1}(p) = k_{< 1}(p) = \|y\|^p - \|x\|^{p-1}\|y\| + \|x - y\||x||x|^{p-1}.\]

In addition, taking into account the triangle inequality and the fact that the function \(f(t) = t^\alpha\) is increasing (decreasing) for \(\alpha \geq 0\) (\(\alpha < 0\)), we obtain the following relations:
\[S_{\geq 1}(p) - D_{\geq 1}(p) = d_{\geq 1}(p) - s_{\geq 1}(p) = (\|y\|^{p-1} - \|x\|^{p-1})(\|x - y\| + \|x\| - \|y\|) \geq 0,\]
\[S_{< 1}(p) - D_{< 1}(p) = d_{< 1}(p) - s_{< 1}(p) = (\|x\|^{p-1} - \|y\|^{p-1})(\|x - y\| + \|x\| - \|y\|) \geq 0,\]
and
\[k_{\geq 1}(p) - D_{\geq 1}(p) = (\|y\| - \|x\|)(\|y\|^{p-1} - \|x\|^{p-1}) \geq 0,\]
\[D_{< 1}(p) - K_{< 1}(p) = (\|y\| - \|x\|)(\|x\|^{p-1} - \|y\|^{p-1}) \geq 0.\]

The proof is now complete. \(\square\)

**Remark 3.** Although our lower bounds \(k_{\geq 1}(p)\) and \(k_{< 1}(p)\) are better than the corresponding ones in (11), they are not always meaningful since they can take negative values. Obviously, their sign depends on parameter \(p\), as well as on vectors \(x, y \in X\). However, we are interested in finding values of \(p\) for which \(k_{\geq 1}(p)\) and \(k_{< 1}(p)\) are nonnegative, regardless of the choice of nonzero vectors \(x, y \in X\).

We first show that \(k_{\geq 1}(p) \geq 0\) for \(p \in [1, 2]\). Without loss of generality we suppose that \(\|x\| \leq \|y\|\). Then
\[k_{\geq 1}(p) = \|x\|^p - \|y\|^{p-1}\|x\| + \|y\|^{p-1}\|x - y\| = \|y\|^{p-1}\varphi(p),\]
where
\[ \varphi(p) = \|x\| \left( \frac{\|x\|}{\|y\|} \right)^{p-1} - \|x\| + \|x - y\| \]
is a decreasing function. Then utilizing the triangle inequality, we have
\[ \varphi(2) = \frac{\|x\|^2 - \|x\| \|y\| + \|x - y\| \|y\|}{\|y\|} \geq \frac{\|x\| (\|x\| - \|y\| + \|x - y\|)}{\|y\|} \geq 0. \]
This implies that \( k_{\geq 1}(p) \geq 0 \) for \( p \in [1, 2] \). It is easy to find vectors \( x, y \in X \) for which \( k_{\geq 1}(p) \) is negative, when \( p > 2 \). To see this, let \( \|x\| = n, \|y\| = n + 1 \) and \( \|x - y\| = 1 + \frac{1}{n}, n \in \mathbb{N} \). Then \( \varphi_n(p) = n \left( \frac{n}{n+1} \right)^{p-1} - n + 1 + \frac{1}{n} \), so by utilizing the L’Hospital rule it follows that \( \lim_{n \to \infty} \varphi_n(p) = 2 - p < 0 \), which proves our assertion.

Similarly, we show that \( k_{< 1}(p) \geq 0 \) for \( p \in [0, 1) \). Without loss of generality we suppose that \( \|x\| \leq \|y\| \). Then
\[ k_{< 1}(p) = \|y\|^p - \|x\|^{p-1} \|y\| + \|x\|^{p-1} \|x - y\| = \|x\|^{p-1} \psi(p), \]
where
\[ \psi(p) = \|y\| \left( \frac{\|y\|}{\|x\|} \right)^{p-1} - \|y\| + \|x - y\| \]
is an increasing function. We have, \( \psi(0) = \|x\| - \|y\| + \|x - y\| \geq 0 \), by the triangle inequality, which implies that \( k_{< 1}(p) \geq 0 \) for \( p \in [0, 1) \). Moreover, if \( x, y \in X \) are such that \( \|x\| = n, \|y\| = n + 1, \|x - y\| = 1 + \frac{1}{n}, n \in \mathbb{N} \), then, \( \psi_n(p) = (n + 1) \left( \frac{n+1}{n} \right)^{p-1} - n + 1 + \frac{1}{n} \) and \( \lim_{n \to \infty} \psi_n(p) = p \), by virtue of the L’Hospital rule. This shows that if \( p < 0 \), then \( k_{< 1}(p) \) can take negative values. In conclusion, the lower bounds given by Corollary 2 will be considered only for \( p \in [0, 2] \).

The upper bounds in (10) were roughly compared in [11] with the Maligranda upper bounds given by (5). In particular, it was shown that one of the estimates in (10) is better than (5) in the case when \( p \geq 1 \) (for more details, see [11]).

Now, by a more precise analysis, we will make comparison of our Corollary 2 with the Maligranda upper bounds in (5) and the lower bounds provided by Corollary 1. We will prove that the upper bounds in (9) and its reverse are always better than the corresponding estimates in (5). In addition, we will show that if \( 0 \leq p < 1 \), then the lower bound \( k_{< 1}(p) \) in the reverse of (9) is more precise than the corresponding one in (7). Following the notation as above, the upper bounds in (5) will be denoted by \( M_{\geq 1}(p) \), \( M_{[0,1]}(p) \), and \( M_{< 0}(p) \), respectively, while the lower bounds in (7) will be denoted by \( m_{\geq 1}(p) \), \( m_{[0,1]}(p) \), and \( m_{< 0}(p) \).

**Theorem 3.** If \( x, y \in X \) are nonzero vectors, then
\[
\alpha_p[x, y] \leq \begin{cases} 
K_{\geq 1}(p) \leq M_{\geq 1}(p), & p \geq 1, \\
K_{< 1}(p) \leq M_{[0,1]}(p), & 0 \leq p < 1, \\
K_{< 1}(p) \leq M_{< 0}(p), & p < 0. 
\end{cases} \tag{12}
\]
In addition, if $0 \leq p < 1$, then
\begin{equation}
    m_{[0,1)}(p) \leq k_{<1}(p) \leq \alpha_p[x,y].
\end{equation}

\textbf{Proof.} Without loss of generality we assume that $\|x\| \leq \|y\|$. We first prove that $K_{\geq 1}(p) \leq M_{\geq 1}(p)$, for $p \geq 1$. Taking into account our assumption, we have
\begin{align*}
    K_{\geq 1}(p) &= \|x\|^p - \|x\|^{p-1}\|y\| + \|x\|^{p-1}\|x - y\|, \\
    M_{\geq 1}(p) &= p\|y\|^{p-1}\|x - y\|,
\end{align*}
and we will show that $M_{\geq 1}(p) - K_{\geq 1}(p) \geq 0$, for $p \geq 1$. Since $M_{\geq 1}(p) - K_{\geq 1}(p) = \|y\|^{p-1}f(p)$, where
\begin{equation}
    f(p) = p\|x - y\| - \|y\| + (\|y\| - \|x - y\|) \left( \frac{\|x\|}{\|y\|} \right)^{p-1},
\end{equation}
it suffices to show that $f$ is nonnegative for $p \geq 1$. We consider two cases depending on whether $\|y\| - \|x - y\| > 0$ or $\|y\| - \|x - y\| \leq 0$. If $\|y\| - \|x - y\| \leq 0$, then $f$ is represented as a sum of two increasing functions. Consequently, $f$ is an increasing function which implies that $f(p) \geq f(1) = 0$, for $p \geq 1$. Otherwise, by taking a second derivative of $f$, we obtain
\begin{equation*}
    f''(p) = (\|y\| - \|x - y\|) \left( \frac{\|x\|}{\|y\|} \right)^{p-1} \log^2 \frac{\|x\|}{\|y\|} \geq 0.
\end{equation*}
Hence, if $\|y\| - \|x - y\| > 0$, then $f$ is convex on $\mathbb{R}$. Consequently, since $f(0) = \frac{\|y\|}{\|x\|} (\|y\| - \|x\| - \|x - y\|) \leq 0$, by the triangle inequality, and $f(1) = 0$, it follows that $f(p) \geq 0$, for all $p \geq 1$. In conclusion, $M_{\geq 1}(p) - K_{\geq 1}(p) \geq 0$, for all $p \geq 1$, as claimed.

Next we show that $K_{<1}(p) \leq M_{[0,1)}(p)$, for $0 \leq p < 1$. In this setting, we have
\begin{align*}
    K_{<1}(p) &= \|x\|^p - \|x\|\|y\|^{p-1} + \|y\|^{p-1}\|x - y\|, \\
    M_{[0,1)}(p) &= (2 - p)\|y\|^{p-1}\|x - y\|.
\end{align*}
Since $M_{[0,1)}(p) - K_{<1}(p) = \|y\|^{p-1}g(p)$, where
\begin{equation*}
    g(p) = (1 - p)\|x - y\| - \|x\| \left( \frac{\|x\|}{\|y\|} \right)^{p-1} + \|x\|,
\end{equation*}
it suffices to prove that $g(p) \geq 0$ for $0 \leq p < 1$. It should be noted here that $g$ is concave on $\mathbb{R}$ since
\begin{equation*}
    g''(p) = -\|x\| \left( \frac{\|x\|}{\|y\|} \right)^{p-1} \log^2 \frac{\|x\|}{\|y\|} \leq 0.
\end{equation*}
Consequently, since $g(0) = \|x - y\| + \|x\| - \|y\| \geq 0$, by the triangle inequality, and $g(1) = 0$, it follows that $g(p) \geq 0$, for all $0 \leq p < 1$.

The last step in connection with the upper bounds is to show that $M_{<0}(p) - K_{<1}(p) \geq 0$, for $p < 0$. Since
\begin{equation*}
    M_{<0}(p) = (2 - p)\|x\|^p \|x - y\|,
\end{equation*}
it follows that \( M_{<0}(p) - K_{<1}(p) = \frac{\|x\|^p}{\|y\|^p} h(p) \), where

\[
h(p) = (2 - p) \|x - y\| - \|y\| + (\|x\| - \|x - y\|) \left( \frac{\|y\|}{\|x\|} \right)^p.
\]

We will show that \( h \) is decreasing function for \( p < 0 \). We consider two cases depending on whether \( \|x\| - \|x - y\| < 0 \) or \( \|x\| - \|x - y\| \geq 0 \). If \( \|x\| - \|x - y\| < 0 \), then \( h \) is decreasing on \( \mathbb{R} \) since it is represented as a sum of two decreasing functions. Otherwise, if \( \|x\| - \|x - y\| \geq 0 \), then, by taking a derivative of \( h \), we obtain

\[
h'(p) = -\|x - y\| + (\|x\| - \|x - y\|) \left( \frac{\|y\|}{\|x\|} \right)^p \log \frac{\|y\|}{\|x\|}.
\]

Our intention is to show that \( h'(p) \leq 0 \), for \( p \leq 0 \). By the Lagrange mean value theorem, there exist \( \xi \in (\|x\|, \|y\|) \) such that \( \log \|y\| - \log \|x\| = \frac{\|y\| - \|x\|}{\xi} \), and consequently,

\[
\log \frac{\|y\|}{\|x\|} \leq \frac{\|y\| - \|x\|}{\|x\|} \leq \frac{\|y - x\|}{\|x\|}.
\]

Furthermore, since \( \left( \frac{\|y\|}{\|x\|} \right)^p \leq 1 \), for \( p \leq 0 \), it follows that

\[
h'(p) \leq -\frac{\|x - y\|^2}{\|x\|},
\]

so \( h \) is decreasing function for \( p \leq 0 \) in each case. On the other hand, since \( h(0) = \|x - y\| + \|x\| - \|y\| \geq 0 \), it follows that \( h(p) \geq h(0) = 0 \), for \( p < 0 \), which completes the proof of the third inequality in (12).

It remains to prove (13). In other words, we need to show that \( k_{<1}(p) - m_{(0,1)}(p) \geq 0 \), for \( 0 \leq p < 1 \). Since \( k_{<1}(p) = K_{\geq 1}(p) \) and \( m_{(0,1)}(p) = M_{\geq 1}(p) \), it follows from the first part of the proof that \( m_{(0,1)}(p) - k_{<1}(p) = \|y\|^{p-1} f(p) \), where \( f \) is defined by (14). We have already proved that \( f \) is a convex function such that \( f(0) \leq 0 \) and \( f(1) = 0 \). This implies that \( f(p) \leq 0 \), for all \( 0 \leq p < 1 \), and consequently, \( k_{<1}(p) - m_{(0,1)}(p) \geq 0 \), as claimed. The proof is now complete.

**Remark 4.** Note that the previous theorem gives no answer to the question about comparison of the lower bounds \( k_{\geq 1}(p) \) and \( m_{\geq 1}(p) \), when \( p \in [1, 2] \). Of course, if \( p = 1 \) these bounds coincide, otherwise they are not comparable. To see this, note that the difference \( k_{\geq 1}(p) - m_{\geq 1}(p) \), provided that \( \|x\| \leq \|y\| \), can be rewritten as \( k_{\geq 1}(p) - m_{\geq 1}(p) = \|y\|^{p-2} \xi(p) \), where

\[
\xi(p) = \|x\| \left( \frac{\|x\|}{\|y\|} \right)^{p-1} - \|x\| + \frac{p - 1}{2p - 1} \|x - y\|.
\]

Now, if \( \|x\| = \|y\| \), then \( \xi(p) \geq 0 \), for all \( p \in (1, 2] \). On the other hand, if \( \|x\| = n, \|y\| = n + 1, \|x - y\| = 1 + \frac{1}{n}, n \in \mathbb{N} \), then \( \xi_n(p) = n \left( \frac{n}{n+1} \right)^{p-1} - n + \frac{p - 1}{2p - 1} (1 + \frac{1}{n}) \), and consequently \( \lim_{n \to \infty} \xi_n(p) = -\frac{2(p-1)^2}{2p-1} < 0 \), for all \( p \in (1, 2] \). This means that for each \( p \in (1, 2] \), we can choose \( x, y \in X \) such that \( k_{\geq 1}(p) > m_{\geq 1}(p) \) or \( k_{\geq 1}(p) < m_{\geq 1}(p) \).
Although the lower bounds $k_{\geq 1}(p)$ and $k_{< 1}(p)$ can be negative when $p$ does not belong to interval $[0, 2]$, they are still not comparable with bounds in (7). For example, if $\|x\| = 1$, $\|y\| = 4$ and $\|x - y\| = 4$, then $m_{< 0}(-1) - k_{< 1}(-1) = 0.083 > 0$, while for $\|x\| = 1$, $\|y\| = 4$ and $\|x - y\| = 4.5$, we have $m_{< 0}(-1) - k_{< 1}(-1) = -0.375 < 0$.

We have already discussed that Dragomir [10], showed that the Hile inequality (6) is more precise than the first inequality in (5), when $p \geq 2$ (see also [25]). However, it turns out that our inequality in (9) is better than the Hile inequality (6), for all $p \geq 1$. For the reader’s convenience, the right-hand side of inequality (6) will be denoted by $H_{\geq 1}(p)$.

**Theorem 4.** If $x, y \in X$ are nonzero vectors and $p \geq 1$, then

$$\alpha_p[x, y] \leq K_{\geq 1}(p) \leq H_{\geq 1}(p).$$

**Proof.** Without loss of generality we assume that $\|x\| \leq \|y\|$. Then considering the difference $H_{\geq 1}(p) - K_{\geq 1}(p)$ and factoring the obtained expression, we have that

$$H_{\geq 1}(p) - K_{\geq 1}(p) = \frac{||y||^p - ||x||^p}{||y|| - ||x||} ||x - y|| - (||y||^p - ||x||^{p-1}||y|| + ||x||^{p-1}||x - y||)$$

$$= \frac{||y||^p ||x - y|| - ||y||^p + ||y|| ||x|| - ||x|| ||y||}{||y|| - ||x||}$$

$$+ \frac{||x||^{p-1}||y||^2 - ||x||^{p-1}||y|| ||x - y||}{||y|| - ||x||}$$

$$= \frac{||y||(||y||^{p-1} - ||x||^{p-1})(||x - y|| + ||x|| - ||y||)}{||y|| - ||x||}.$$}

Now, since $||y||^{p-1} - ||x||^{p-1} \geq 0$ and $||x - y|| + ||x|| - ||y|| \geq 0$, by the triangle inequality, it follows that $H_{\geq 1}(p) - K_{\geq 1}(p) \geq 0$, which ensures (15). \(\square\)

Theorem 1 can also be exploited in deriving mutual bounds for the skew $p$-angular distance. Recall that the skew $p$-angular distance between nonzero elements $x, y \in X$ (see [25]) is defined by

$$\beta_p[x, y] = \|\|y\|^{p-1}x - ||x||^{p-1}y\||,$$ 

$p \in \mathbb{R}$.

When $p = 0$, we set $\beta[x, y]$ for $\beta_p[x, y]$ and call it simply the skew angular distance between $x, y \in X$. It is easy to see that the $p$-angular distance and the skew $p$-angular distance are related by

$$\beta_p[x, y] = \|x\|^{p-1}||y||^{p-1}\alpha_{2-p}[x, y].$$

**Corollary 3.** Let $X = (X, \|\cdot\|)$ be a normed linear space and let $p \leq 1$. Then the inequalities

$$\|x\||y||^{p-1} + \|x||^{p-1}\|y\| - \frac{\|x\| + ||y|| - ||x - y||}{\min^{1-p}\{\|x\|, ||y||\}}$$

$$\leq \beta_p[x, y]$$

$$\leq \|x\||y||^{p-1} + \|x||^{p-1}\|y\| - \frac{\|x\| + ||y|| - ||x - y||}{\max^{1-p}\{\|x\|, ||y||\}}$$

(17)
hold for all nonzero elements $x, y \in X$. If $p > 1$, then the inequalities in (17) are reversed.

Proof. We utilize inequalities in (9) with $2 - p$ instead of $p$, and the above formula (16).

Remark 5. Taking into account Remark 3 and relation (16), it follows that the lower bounds in (17) are nonnegative for all $p \in [0, 2]$. In particular, if $p = 0$, then the inequalities in (17) reduce to the following mutual bounds for $\beta[x, y]$, obtained by Dehghan [9]:

$$\frac{\|x - y\|}{\min\{|\|x\|, |\|y\|\}} - \frac{|\|x\| - |\|y\||}{\max\{|\|x\|, |\|y\|\}} \leq \beta[x, y] \leq \frac{|\|x\| - |\|y\||}{\max\{|\|x\|, |\|y\|\}} + \frac{\|x-y\|}{\min\{|\|x\|, |\|y\|\}}.$$

To conclude this section, we give yet another pair of mutual bounds for the $p$-angular distance which relies on the estimates in (4). Namely, due to homogeneity of angular distance $\alpha[x, y]$, we obtain mutual bounds for $\alpha_p[x, y]$, expressed in terms of $\alpha[x, y]$.

Corollary 4. Let $X = (X, \| \cdot \|)$ be a normed linear space. If $p \geq 0$, then the inequalities

$$\max^p\{|\|x\|, |\|y\|\}\alpha[x, y] - |\|x\|^p - |\|y\|^p| \leq \alpha_p[x, y] \leq \min^p\{|\|x\|, |\|y\|\}\alpha[x, y] + |\|x\|^p - |\|y\|^p|$$

(18)

hold for all nonzero vectors $x, y \in X$. If $p < 0$, then

$$\min^p\{|\|x\|, |\|y\|\}\alpha[x, y] - |\|x\|^p - |\|y\|^p| \leq \alpha_p[x, y] \leq \max^p\{|\|x\|, |\|y\|\}\alpha[x, y] + |\|x\|^p - |\|y\|^p|.$$  

(19)

Proof. Rewriting (4) with $x' = \|x\|^{p-1}x$, $y' = \|y\|^{p-1}y$ instead of $x, y$, respectively, as well as noting that $\alpha[ax, by] = \alpha[x, y]$ when $ab > 0$, we obtain

$$\frac{\alpha_p[x, y] - |\|x\|^p - |\|y\|^p|}{\min\{|\|x\|^p, |\|y\|^p\}} \leq \alpha[x, y] \leq \frac{\alpha_p[x, y] + |\|x\|^p - |\|y\|^p|}{\max\{|\|x\|^p, |\|y\|^p\}}.$$

Consequently, the result follows from relations $\min\{|\|x\|^p, |\|y\|^p\} = \min^p\{|\|x\|, |\|y\|\}$, for $p \geq 0$, $\min\{|\|x\|^p, |\|y\|^p\} = \max^p\{|\|x\|, |\|y\|\}$, for $p < 0$, and the similar formulas for maximum. \qed

Remark 6. The lower bound in (18) is meaningful if the condition

$$\alpha[x, y] \geq \frac{|\|x\|^p - |\|y\|^p|}{\max^p\{|\|x\|, |\|y\|\}}$$

(20)

is valid. Obviously, (20) holds if $\alpha[x, y] \geq 1$. On the other hand, it is easy to show that the lower bound in (18) can take negative values when $\alpha[x, y] < 1$. For example, if $x, y \in X$ are such that $\alpha[x, y] < 1$, $|\|x\| = n$ and $|\|y\| = n^2$, $n \in \mathbb{N}$, then the lower bound in (18) is equal to $n^{2p}(\alpha[x, y] - 1 + n^{-p})$, which is evidently negative for $p > 0$ and sufficiently large $n$. The same conclusion can be drawn for inequality (19), we omit details here.
3. Bounds in Inner Product Spaces and Characterizations of Inner Product Spaces

Rooin et al. [24] proved that if $X$ is an inner product space and $|\frac{p}{q}| \geq 1$, then the inequality

$$\alpha_p[x, y] \leq \frac{\|y\|^p - \|x\|^p}{\|y\|^q - \|x\|^q} \alpha_q[x, y]$$  \hspace{1cm} (21)

holds for all $x, y \in X$ such that $\|x\| \neq \|y\|$. On the other hand, generalizing the integral bounds for the $p$-angular distance established by Dragomir [10], Rooin et al. [25] proved that if $\frac{p}{q} \geq 1$, then inequality (21) is valid in any normed linear space. By putting $q = 1$ in (21), we obtain the Hile inequality (6) provided that $p > 1$.

Remark 7. If $\frac{p}{q} \geq 1$, then inequality (21) can be derived directly from the Hile inequality (6). Namely, assuming that (6) holds, it follows that holds the inequality

$$\alpha_{\frac{p}{q}}[x', y'] \leq \frac{\|y\|^\frac{p}{q} - \|x'^{\frac{p}{q}}\|}{\|y\|^1 - \|x'^{\frac{p}{q}}\|} \|x' - y'\|,$$

where $x' = \|x\|^{q-1}x$ and $y' = \|y\|^{q-1}y$. Clearly, the last inequality reduces to (21) since $\alpha_{p/q}[x', y'] = \alpha_p[x, y]$ and $\|x' - y'\| = \alpha_q[x, y]$.

The extended Hile inequality (21) in an inner product space has been established as a consequence of a generalized Dunkl–Williams identity between the $p$-angular and the $q$-angular distances (for more details, see [24]). However, this can be achieved through a simpler identity that establishes a connection between $\alpha_p[x, y]$, $p \neq 0$, and $\alpha[x, y]$. Namely, if $p = 0$, the explicit formula for the $p$-angular distance, given by (3), reduces to

$$\alpha^2[x, y] = \frac{\|x - y\|^2}{\|x\| \|y\|}.$$  \hspace{1cm} (22)

Then considering (22) with $x' = \|x\|^{p-1}x$ and $y' = \|y\|^{p-1}y$ instead of $x$ and $y$, respectively, as well as taking into account homogeneity of $\alpha[x, y]$, we arrive at the following identity that was used in [17]:

$$\alpha^2_{p}[x, y] = (\|x\|^p - \|y\|^p)^2 + \|x\|^p \|y\|^p \alpha^2[x, y].$$  \hspace{1cm} (23)

Now, the quotient

$$\frac{\alpha^2_{p}[x, y]}{\alpha^2_{q}[x, y]} = \frac{(\|x\|^p - \|y\|^p)^2 + \|x\|^p \|y\|^p \alpha^2[x, y]}{(\|x\|^q - \|y\|^q)^2 + \|x\|^q \|y\|^q \alpha^2[x, y]}$$

can be rewritten as

$$\alpha^2_{q}[x, y] (\|x\|^p - \|y\|^p)^2 - \alpha^2_{p}[x, y] (\|x\|^q - \|y\|^q)^2$$
$$= \alpha^2[x, y] (\|x\|^q \|y\|^q \alpha^2_{p}[x, y] - \|x\|^p \|y\|^p \alpha^2_{q}[x, y]),$$

which implies that inequality (21) is valid if and only if holds the inequality

$$\alpha_p[x, y] \geq \|x\|^\frac{p-2}{2} \|y\|^\frac{p-2}{2} \alpha_q[x, y].$$  \hspace{1cm} (24)
Finally, by virtue of statement (III) in Theorem A, inequality (24) holds for $|p| \geq |q|$, which means that (21) holds for $\frac{p}{q} \geq 1$ in an inner product space.

Utilizing identity (23) once again, we will establish yet another inequality through equivalence with (24). We have already discussed that inequality (21) holds in an arbitrary normed linear space when $\frac{p}{q} \geq 1$. On the other hand, our next result shows significantly different behavior from (21) since we will obtain a characterization of an inner product space. To establish the corresponding result, we will keep in mind the next two theorems due to Lorch and Ficken.

**Theorem B.** (Lorch [18]) Let $X = (X, ||||)$ be a normed linear space. Then the following statements are mutually equivalent:

(i) For each $x, y \in X$ if $||x|| = ||y||$, then $||x + y|| \leq ||\mu x + \mu^{-1}y||$, for all $\mu \neq 0$.

(ii) For each $x, y \in X$ if $||x + y|| \leq ||\mu x + \mu^{-1}y||$, for all $\mu \neq 0$, then $||x|| = ||y||$.

(iii) $X = (X, ||||)$ is an inner product space.

**Theorem C.** (Ficken [13]) Let $X = (X, ||||)$ be a normed linear space. Then the following statements are mutually equivalent:

(i) For each $x, y \in X$ if $||x|| = ||y||$, then $||\lambda x + \mu y|| = ||\mu x + \lambda y||$, for all $\lambda, \mu > 0$.

(ii) For each $x, y \in X$ if $||x|| = ||y||$, then $||\mu x + \mu^{-1}y|| = ||\mu^{-1}x + \mu y||$, for all $\mu > 0$.

(iii) $X = (X, ||||)$ is an inner product space.

Now, we are in position to state and prove the main result of this section.

**Theorem 5.** Let $|p| \geq |q|$, $p \neq q$. Then a normed linear space $X = (X, ||||)$ is an inner product space if and only if the inequality

$$\alpha_p[x, y] \geq \frac{||x||^p + ||y||^p}{||x||^q + ||y||^q} \alpha_q[x, y]$$  \hspace{1cm} (25)

holds for all nonzero vectors $x, y \in X$.

**Proof.** Let $X$ be an inner product space. We will show equivalence of relations (24) and (25), by rewriting (23) in a more suitable form:

$$\alpha_p^2[x, y] = (||x||^p + ||y||^p)^2 + ||x||^p||y||^p(\alpha^2[x, y] - 4).$$

Therefore, we have

$$\frac{\alpha_p^2[x, y]}{\alpha_q^2[x, y]} = \frac{(||x||^p + ||y||^p)^2 + ||x||^p||y||^p(\alpha^2[x, y] - 4)}{(||x||^q + ||y||^q)^2 + ||x||^q||y||^q(\alpha^2[x, y] - 4)},$$

which can be rewritten as

$$\alpha_q^2[x, y] (||x||^p + ||y||^p)^2 - \alpha_p^2[x, y] (||x||^q + ||y||^q)^2 = (\alpha^2[x, y] - 4) (||x||^q||y||^q \alpha_q^2[x, y] - ||x||^p||y||^p \alpha_q^2[x, y]).$$

Now, since $\alpha[x, y] \leq 2$, the previous identity implies that inequalities (24) and (25) are equivalent for $\alpha[x, y] < 2$. It remains to prove (25) when $\alpha[x, y] = 2,$
which is trivial. Namely, if $\alpha[x, y] = 2$, then there exist $\lambda > 0$ such that $x + \lambda y = 0$. Then, $\alpha_p[x, y] = (1 + \lambda^p)\|x\|^p = \|x\|^p + \|y\|^p$, and we have equality sign in (25). In conclusion, inequality (25) holds whenever $|p| \geq |q|$.

Now, let $X$ be a normed linear space satisfying inequality (25). We prove that $X$ is an inner product space by considering three cases $|p| > |q| > 0$, $p = -q$, and $q = 0$ separately.

**Case 1.** Let $|p| > |q| > 0$. From validity of inequality (25), it follows that holds the inequality

$$\alpha_p[x', y'] \geq \frac{\|x'||^p + \|y'||^p}{\|x'||^q + \|y'||^q} \alpha_q[x', y'],$$

where $x' = \|x\|^{\frac{1}{\lambda}} x$ and $y' = \|y\|^{\frac{1}{\lambda}} y$. Consequently, since $\alpha_p[x', y'] = \|x - y\|$ and $\alpha_q[x', y'] = \alpha_{q/p}[x, y]$, it follows validity of the inequality

$$\alpha_r[x, y] \leq \frac{\|x\|^r + \|y\|^r}{\|x\| + \|y\|} \|x - y\|,$$  \hspace{1cm} (26)

where $0 < |r| = \frac{2}{p} < 1$ and $x, y \neq 0$. Now, let $x, y \in X$ be such that $\|x\| = \|y\|$ and let $\mu \neq 0$. According to Theorem B, it suffices to prove that $\|x + y\| \leq \|\mu x + \mu^{-1} y\|$. If $\|x\| = \|y\| = 0$, this relation holds trivially. Otherwise, applying inequality (26) to $x'' = \mu^{-n} x$ and $y'' = -\mu^{-n} y$, where $\mu > 0$, instead of $x$ and $y$, respectively, we obtain

$$\alpha_r[x'', y''] \leq \frac{\|x''||^r + \|y''||^r}{\|x''|| + \|y''||} \|x'' - y''\|.$$

Moreover, since $\|x\| = \|y\|$, we have that

$$\alpha_r[x'', y''] = \left\| \mu r^{(r-1)} \mu^{-n} \|x\|^r x + \mu^{-r(n-1)} \mu^{-r} \|y\|^{r-1} y \right\|
= \|x\|^{r-1} \left\| \mu^{r+1} x + \mu^{-r+1} y \right\|,$$

so the above inequality reduces to

$$\left\| \mu^{r+1} x + \mu^{-r+1} y \right\| \leq \frac{\mu^{r+1} + \mu^{-r+1}}{\mu^r + \mu^{-r}} \left\| \mu^{r} x + \mu^{-r} y \right\|.  \hspace{1cm} (27)$$

Now, consider the function $f(x) = ax + a^{-x}$, where $a > 0$, $a \neq 1$. Clearly, $f$ is an even function. In addition, since $f'(x) = \log a(a^x - a^{-x})$, it follows that $f$ is increasing for $x > 0$, while it is decreasing for $x < 0$. Consequently, we have that $a^x + a^{-x} < a + a^{-1}$, for $|r| < 1$. Hence, by putting $a = \mu^r$ in the last inequality, we obtain $\mu^{r+1} + \mu^{-r+1} \leq \mu^r + \mu^{-r}$, so (27) implies that

$$\left\| \mu^{r+1} x + \mu^{-r+1} y \right\| \leq \left\| \mu^r x + \mu^{-r} y \right\|.$$

In other words, $a_n = \|\mu^r x + \mu^{-r} y\|$, $n \in \mathbb{N} \cup \{0\}$, is a decreasing sequence which implies that

$$\|x + y\| = \lim_{n \to \infty} a_n \leq a_0 = \|\mu x + \mu^{-1} y\|.$$
In addition, if $\mu$ is negative, then by putting $-\mu$ in the last inequality, we obtain

$$\|x + y\| \leq \|(\mu)x + (\mu)^{-1}y\| = \|\mu x + \mu^{-1}y\|.$$ 

In conclusion, $X$ is an inner product space by Theorem B.

**Case 2.** Let $p = -q$. Then, taking into account (26), it follows that the inequality

$$\|x\|^{-2} - \|y\|^{-2} = \alpha_{-1}[x, y] \leq \frac{\|x\|^{-1} + \|y\|^{-1}}{\|x\| + \|y\|} \|x - y\|$$

holds for all nonzero vectors $x, y \in X$. Clearly, the above relation is equivalent to

$$\left\| \frac{y}{\|x\|} x - \frac{x}{\|y\|} y \right\| \leq \|x - y\|. \quad (28)$$

Now, assume that $x, y \in X$ are such that $\|x\| = \|y\|$ and let $\mu > 0$. Applying inequality (28) to $\mu x$ and $-\mu^{-1}y$ instead of $x$ and $y$, respectively, we obtain the inequality $\|\mu^{-1}x + \mu y\| \leq \|\mu x + \mu^{-1}y\|$. In the same way, inequality (28) with $\mu^{-1}x$ and $-\mu y$ instead of $x$ and $y$, respectively, yields the inequality with the reversed sign, which implies that $\|\mu^{-1}x + \mu y\| = \|\mu x + \mu^{-1}y\|$. Therefore, Theorem C ensures that $X$ is an inner product space.

**Case 3.** It remains to consider the case of $q = 0$. In this setting, inequality (25) reduces to

$$\alpha[x, y] \leq \frac{2}{\|x\|^{p} + \|y\|^{p}} \alpha_{p}[x, y], \quad (29)$$

which is covered by Theorem A, statement (II), when dimension of $X$ is not less than 3. Alternatively, we can also exploit Theorem B which has no restriction on dimension of a space. Namely, let $x, y$ be nonzero vectors such that $\|x\| = \|y\|$ and let $\mu > 0$. Then applying inequality (29) to $\mu^{\frac{1}{p}}x$ and $-\mu^{-\frac{1}{p}}y$ instead of $x$ and $y$, respectively, as well as utilizing the arithmetic geometric mean inequality, we obtain

$$\|x + y\| \leq \frac{2}{\mu + \mu^{-1}} \|\mu x + \mu^{-1}y\| \leq \|\mu x + \mu^{-1}y\|.$$ 

Consequently, Theorem B implies that $X$ is an inner product space which completes the proof. \[\square\]

**Remark 8.** It should be noted here that the proof of Theorem A relies on the symmetry of Birkhoff–James orthogonality which is a characterization of inner product spaces when dimension of the corresponding space is at least three (for more details, see [8,15,24]). If we would replace an arbitrary real parameter $r$ in statement (V) of Theorem A with a nonnegative parameter $r$, then the statements (I)–(V) would be equivalent in any normed linear space, regardless of dimension. Namely, in that case, the corresponding equivalence could be established via Theorems B and C.
Remark 9. If \( q = 1 \), inequality (25) reduces to
\[
\alpha_p[x, y] \geq \frac{\|x\|^p + \|y\|^p}{\|x\| + \|y\|} \|x - y\|, \tag{30}
\]
where \(|p| \geq 1\). Although inequality (30) seems to be familiar to the Hile inequality (6), they show significantly different behavior. Namely, the Hile inequality holds in every normed linear space, while (30) characterizes an inner product space for \(|p| > 1\). On the other hand, if \( q = 0 \), inequality (25) becomes
\[
\alpha_p[x, y] \geq \frac{\|x\|^p + \|y\|^p}{2} \alpha[x, y],
\]
which coincides with the corresponding inequality in Theorem A.

Theorem A provides characterizations of inner product spaces expressed in terms of power means. Recall that a weighted power mean \( M_r(x_1, x_2, \ldots, x_n) \) is defined by
\[
M_r(x_1, x_2, \ldots, x_n) = \left( \sum_{k=1}^{n} p_k x_k^r \right)^{\frac{1}{r}},
\]
where \( \sum_{k=1}^{n} p_k = 1 \), \( p_k > 0 \), and \( (x_1, x_2, \ldots, x_n) \) is a positive \( n \)-tuple. The nonweighted means correspond to the setting \( p_k = 1/n \), \( k = 1, 2, \ldots, n \). Recall that for \( p = 1, 0, -1 \) we obtain respectively, the arithmetic, geometric and harmonic mean. In addition, \( M_{-\infty}(x_1, x_2, \ldots, x_n) = \min\{x_1, x_2, \ldots, x_n\} \) and \( M_{\infty}(x_1, x_2, \ldots, x_n) = \max\{x_1, x_2, \ldots, x_n\} \). The generalized mean inequality asserts that if \( r < s \), then
\[
M_r(x_1, x_2, \ldots, x_n) \leq M_s(x_1, x_2, \ldots, x_n). \tag{31}
\]
Inequality (31) is true for real values of \( r \) and \( s \), as well as for positive and negative infinity values. For more details about the means inequalities, the reader is referred to [21].

Taking into account the monotonicity property (31), Theorem A asserts that the inequality
\[
\alpha_p[x, y] \geq \left( \frac{\|x\|^{pr} + \|y\|^{pr}}{2} \right)^{\frac{1}{r}} \alpha[x, y] \tag{32}
\]
characterizes an inner product space for every \( r \leq 1 \). Recall, once again, that for \( r = 0 \) the above inequality yields the estimate \( \alpha_p[x, y] \geq \|x\|^\frac{p}{2} \|y\|^\frac{p}{2} \alpha[x, y] \), due to the well-known limit value \( \lim_{x \to 0} \left( \frac{a^p + b^p}{2} \right)^{\frac{1}{p}} = \sqrt{ab} \).

Remark 10. It was shown in [24] that inequality (32) is the best possible in the sense that there is no exponent \( r > 1 \) such that (32) is valid for all nonzero vectors \( x, y \in X \). This can also be nicely seen by considering any two linearly dependent vectors \( x, y \) such that \( x + \lambda y = 0, \lambda > 0 \). Then, \( \alpha_p[x, y] = \|x\|^p + \|y\|^p, \alpha[x, y] = 2 \), so in this case (32) reduces to
\[
\frac{\|x\|^p + \|y\|^p}{2} \geq \left( \frac{\|x\|^{pr} + \|y\|^{pr}}{2} \right)^{\frac{1}{r}},
\]
which is in contrast to (31).
According to Remark 10, inequality (32) does not hold for \( r > 1 \) in general. However, it is easy to describe pairs of vectors for which inequality (32) holds when \( r = 2 \). The corresponding result relies on identity (23).

**Corollary 5.** Let \( X = (X, \langle \cdot, \cdot \rangle) \) be an inner product space and let \( x, y \in X \) be nonzero vectors. If \( \alpha[x, y] \leq \sqrt{2} \), then holds the inequality

\[
\alpha_p[x, y] \geq \sqrt{\frac{\|x\|^{2p} + \|y\|^{2p}}{2}} \alpha[x, y],
\]

(33)

while for \( \alpha[x, y] > \sqrt{2} \), the sign of inequality (33) is reversed.

**Proof.** Since \( X \) is an inner product space, identity (23) is valid and it can be rewritten as

\[
\alpha_2[x, y] = \frac{\|x\|^{2p} + \|y\|^{2p}}{2} \alpha^2[x, y] + \frac{1}{2} \left( \|x\|^p - \|y\|^p \right)^2 (2 - \alpha^2[x, y]).
\]

Now, if \( \alpha[x, y] \leq \sqrt{2} \), then the second term on the right-hand side of the previous identity is nonnegative, which implies (33). In the same way, we conclude that for \( \alpha[x, y] > \sqrt{2} \) holds the reverse in (33). □

Our next intention is to establish connection between mutual bounds for angular distance \( \alpha[x, y] \) given by (4), and the explicit formula for \( \alpha[x, y] \) given by (22), which is a characterization of an inner product space. It is interesting that if (22) is valid, then the square of angular distance is equal to the product of mutual bounds in (4). This fact allows us to refine these bounds in an inner product space.

**Theorem 6.** Let \( X = (X, \langle \cdot, \cdot \rangle) \) be an inner product space and let \( r > 0 \). If \( x, y \in X \) are nonzero vectors with \( x \neq y \), then hold the inequalities

\[
\left( 1 + c^r[x, y] \right)^{-\frac{1}{r}} \frac{\|x - y\| - \|x\| - \|y\|}{\min\{\|x\|, \|y\|\}} \leq \alpha[x, y] \leq \left( 1 + c^r[x, y] \right)^{\frac{1}{r}} \frac{\|x - y\| + \|x\| - \|y\|}{\max\{\|x\|, \|y\|\}},
\]

(34)

where

\[
c[x, y] = \frac{\max\{\|x\|, \|y\|\}}{\min\{\|x\|, \|y\|\}} \cdot \frac{\|x - y\| - \|x\| - \|y\|}{\|x - y\| + \|x\| - \|y\|} < 1.
\]

**Proof.** Taking into account formula (22), we have that \( \alpha^2[x, y] = AB \), where

\[
A = \frac{\|x - y\| - \|x\| - \|y\|}{\min\{\|x\|, \|y\|\}} \quad \text{and} \quad B = \frac{\|x - y\| + \|x\| - \|y\|}{\max\{\|x\|, \|y\|\}}.
\]

This means that \( \alpha[x, y] \) is a geometric mean of \( A \) and \( B \). Consequently, due to the monotonicity property (31), it follows that the inequalities

\[
\left( \frac{A^{-r} + B^{-r}}{2} \right)^{-\frac{1}{r}} \leq \alpha[x, y] \leq \left( \frac{A^r + B^r}{2} \right)^{\frac{1}{r}}
\]

hold for every \( r > 0 \). Now, the result follows by noting that \( c[x, y] = \frac{A}{B} \). □
Remark 11. It follows from the construction that the inequalities in (34) provide refinements of mutual bounds in (4) for each exponent \( r > 0 \). More precisely, since \( \alpha[x, y] \) is represented as the geometric mean of bounds in (4), it follows by monotonicity property (31) that the upper bound in (34) is not greater than the upper bound in (4). In the same way, the lower bound in (34) is more accurate than the lower bound in (4). In particular, if \( r = 1 \), the inequalities in (34) reduce to the following form:

\[
\frac{2[\|x - y\|^2 - (\|x\| - \|y\|)^2]}{\|x - y\|(\|x\| + \|y\|) - (\|x\| - \|y\|)^2} \leq \alpha[x, y] \leq \frac{\|x - y\|(\|x\| + \|y\|) - (\|x\| - \|y\|)^2}{2\|x\||\|y\|}. \tag{35}
\]

The second inequality in (35) can be exploited in obtaining a parametric family of characterizing relations for an inner product space.

Corollary 6. Let \( X = (X, \|\cdot\|) \) be a normed linear space, let \( 0 < r \leq 1 \), and let \( c[r, x, y] \) be defined as in the statement of Theorem 6. Then, \( X \) is an inner product space if and only if the inequality

\[
\alpha[x, y] \leq \left( \frac{1 + c^r[x, y]}{2} \right)^{\frac{1}{r}} \frac{\|x - y\| + \|\|x\| - \|y\||}{\max\{|\|x\|, \|y\|\}} \tag{36}
\]

holds for all nonzero vectors \( x, y \in X \) with \( x \neq y \).

Proof. It suffices to prove that a normed linear space satisfying (36) is an inner product space. Taking into account Theorem 6, Remark 11, property (31) and the triangle inequality, we have

\[
\alpha[x, y] \leq \left( \frac{1 + c^r[x, y]}{2} \right)^{\frac{1}{r}} \frac{\|x - y\| + \|\|x\| - \|y\||}{\max\{|\|x\|, \|y\|\}} \\
\leq \frac{\|x - y\|(\|x\| + \|y\|) - (\|x\| - \|y\|)^2}{2\|x\||\|y\|} \\
= \frac{\|x - y\|(\|x\| + \|y\|)^2 - (\|x\| + \|y\|)(\|x\| - \|y\|)^2}{2\|x\||\|y\|} \\
\leq \frac{[(\|x\| + \|y\|)^2 - (\|x\| - \|y\|)^2]\|x - y\|}{2\|x\||\|y\|} = \frac{2\|x - y\|}{\|x\| + \|y\|}.
\]

Obviously, this proves our assertion due to characterization (1) given by Kirk and Smiley. \( \square \)

To conclude this paper, we give the lower bound for the \( p \)-angular distance expressed in terms of angular distance \( \alpha[x, y] \), which is also consequence of the right inequality in (35). The following lower bound for \( \alpha_p[x, y] \) also gives a characterization of inner product spaces.

Corollary 7. Let \( p \neq 0 \). Then a normed linear space \( X = (X, \|\cdot\|) \), \( \dim X \geq 3 \), is an inner product space if and only if the inequality

\[
\alpha_p[x, y] \geq \frac{2\|x\|^p\|y\|^p\alpha[x, y] + (\|x\|^p - \|y\|^p)^2}{\|x\|^p + \|y\|^p} \tag{37}
\]
holds for all nonzero vectors $x, y \in X$.

Proof. Let $X$ be an inner product space. Then rewriting the right inequality in (35) with $x' = \|x\|^{p-1}x$, $y' = \|y\|^{p-1}y$ instead of $x$, $y$, respectively, we obtain the inequality
\[
\alpha[x, y] \leq \frac{\alpha_p[x, y](\|x\|^p + \|y\|^p) - (\|x\|^p - \|y\|^p)^2}{2\|x\|^p\|y\|^p},
\]
due to homogeneity of $\alpha[x, y]$, that is, $\alpha[x', y'] = \alpha[x, y]$. Obviously, the last inequality is equivalent to (37).

On the other hand, assume that $X$ is a normed linear space satisfying inequality (37). Then we have
\[
\alpha_p[x, y] \geq \frac{2\|x\|^p\|y\|^p\alpha[x, y]}{\|x\|^p + \|y\|^p} = \left(\frac{\|x\|^{-p} + \|y\|^{-p}}{2}\right)^{-1} \alpha[x, y],
\]
so $X$ is an inner product space due to Theorem A, statement (V). \qed

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