A Discontinuous Differential Calculus in the Framework Colombeau’s Full Algebra

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Abstract

Starting from the Colombeau’s full generalized functions, the sharp topologies and the notion of generalized points, we introduce a new kind differential calculus (for functions between totally disconnected spaces). We study generalized point-values, Colombeau’s differential algebra, holomorphic and analytic functions. We show that the Embedding Theorem and the Open Mapping Theorem hold in this framework. Moreover, we study some applications in differential equations.

Key words and phrases— Colombeau’s full algebra, Nonlinear generalized functions, Pointvalues of a nonlinear generalized functions, Sharp topology and Differential equations.

Mathematics subject classification— 6F30; 46T20; 26E30; 30G06.

1 Introduction

The theory of Colombeau generalized functions appears in the 1980’s which the main aim was to define a product on the distribution space, see [10, 11, 12, 13] and [15] for more details. Colombeau constructed differential algebras $G(\Omega)$, $\Omega \subseteq \mathbb{R}^n$ containing the space of smooth functions defined on $\Omega$ as a subalgebra and the space of distributions on $\Omega$ as a subspace, i.e, it was constructed an associative, commutative differential algebra containing the space of distributions and hold the Leibniz rule for differential product of two distributions, where nowadays this algebra is known as Colombeau’s algebras. So far, Colombeau algebras are the only known differential algebras having all these properties enumerated above.

Fundamental investigations about the structure of these algebras containing the distributions space have been carried out by Rosinger, see [19, 20, 21] and [22] for more details. Moreover, with the results of [19, 20, 21] and [22] were constructed a general theory which characterized algebras of generalized functions containing the space of distributions, but was Colombeau that constructed differential algebras with good properties.

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(see \cite{11} and \cite{12}). The Colombeau algebras, simplified and full, rapidly developed during the last years and was found useful applications to linear and nonlinear partial differential equations, calculus of variations, mathematical physics problem, stochastic analysis and in differential geometry, where the theory of distributions have limitations of applications, because it is not nonlinear. In the presence of nonlinearity and the nonlinear generalized functions (Colombeau algebras), it produces new insight where the classical theory does not.

Our idea is to extend some results of \cite{6}, where the authors developed the discontinuous calculus for Colombeau’s simplified generalized functions. The definitions and topology of Colombeau’s full generalized numbers are far more complicated than the simplified version which makes an interesting object to introduce the differential calculus. It is important to say that Colombeau algebras is not just a regularization of functions, but an extension of the classical functions with the differential calculus developed here.

The starting point of an algebraic theory of the topological ring of Colombeau’s full generalized numbers was made in \cite{4}. This algebraic theory, together with the theory of the point values of a Colombeau’s full generalized functions that will be developed here, which is based with the constructions made in \cite{15}, are the base to introduce the differential calculus in the framework of Colombeau’s full generalized functions.

This paper is the continuation of a program whose aim is the development of a differential calculus in the Colombeau algebras setting. This program was effectively started in \cite{6} and \cite{1}, where the main point was to develop the differential calculus. Thus, the Colombeau’s simplified theory is a natural extension of classical calculus. We want to extend these studies for the Colombeau’s full theory, we want to show that the differential calculus in the framework of Colombeau’s full theory is a natural extension of classical calculus, too. As a future work, we want to study integration of generalized functions over membranes in the context of Colombeau’s full generalized numbers where it extends the ideas presented in \cite{1}.

This work is organized as follows: In the Section 2, we collect some basic definitions, results and notations to be used in the sequel of the paper and, as a rule, most of the proofs are omitted. In the Section 3 we present some results that are extensions of results obtained in \cite{6}, observing that the results obtained in \cite{4} was assumed that the support of the mollifiers are contained in the ball of center in $x_0$ and radio 1. In fact, we consider $\text{diam}(\text{supp}(\varphi)) = 1$ for $\varphi \in \mathcal{A}_0(\mathbb{R})$. In this case, we have that its support number, $d(\varphi) = \sup\{|x| : \varphi(x) \neq 0\} = \varepsilon$, see \cite{17} for more details. In the Section 4 we introduce the pointvalues in the framework Colombeau’s full algebras and we extend some results in \cite{15}. In the Section 5 we study the Colombeau differential full algebra over the image of $\mathcal{G}(\Omega)$ by the operator $\kappa$ that we define in this section. In the Section 6 we study the holomorphic and analytic generalized function and some applications in the framework Colombeau’s full generalized functions.

## 2 Definitions, results and notations

In this section we recall some basic definitions, results and notations that will be necessary to the development of this work. As a rule, the proofs will be omitted.

**Notation 1.**

a) $I := [0, 1]$, $\bar{I} := [0, 1]$ and $I_\eta := [0, \eta]$, $\forall \eta \in I$.

b) $A \setminus B := \{a \in A : a \notin B\}$.
c) $\mathbb{Q}$ denotes the field of rational numbers.

d) $\mathbb{K}$ denotes either the field of real or complex numbers, i.e., $\mathbb{R}$ or $\mathbb{C}$.

e) $\mathbb{K}^* := \mathbb{K} \setminus \{0\}$.

f) $\mathbb{N}$ and $\mathbb{Z}$ stand respectively for the set natural numbers and the set of integers. Moreover, $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$ and $\mathbb{Z}^* := \mathbb{Z} \setminus \{0\}$.

g) $\mathbb{R}^+ := \{x \in \mathbb{R} : x \geq 0\}$ and $\mathbb{R}^*_+ := \{x \in \mathbb{R} : x > 0\}$.

h) We denote $\mathbb{K}$ as the topological ring of Colombeau’s simplified generalized numbers, see [17].

i) $\mathcal{A}_0(\mathbb{R}) := \{\varphi \in \mathcal{D}(\mathbb{R}) : \int_0^\infty \varphi(x) \, dx = \frac{1}{2}, \varphi \text{ is even and } \varphi \equiv \text{ const. in } V_0\}$, where $V_0$ is a neighborhood of the origin. If $\varphi \in \mathcal{A}_0(\mathbb{R})$, then its support number is $d(\varphi) := \sup\{|x| : \varphi(x) \neq 0\}$, see [17].

j) $\mathcal{A}_q(\mathbb{R}) := \{\varphi \in \mathcal{A}_0(\mathbb{R}) : \int_0^\infty x^m \varphi(x) \, dx = 0, \text{ for } 1 \leq j, m \leq q, q \in \mathbb{N}\}$. If $\varphi \in \mathcal{A}_q(\mathbb{R})$, $q \in \mathbb{N}$, then for every $\varepsilon > 0$, $\varphi_\varepsilon(x) = \varepsilon^{-n} \varphi(\varepsilon x)$, $x \in \mathbb{R}^n$, belongs to $\mathcal{A}_q(\mathbb{R})$, see [17].

l) $\Gamma := \{\gamma : \mathbb{N} \to \mathbb{R}^+ : \gamma(n) < \gamma(n + 1), \forall n \in \mathbb{N} \text{ and } \lim_{n \to \infty} \gamma(n) = \infty\}$ is the set of the strict increasing sequences diverging to infinity when $n \to \infty$.

m) $\mathbb{K}$ denotes the topological ring of Colombeau’s full generalized numbers, see ([4], Definition 1.2)

n) For each $\Omega \subseteq \mathbb{R}^n$, denote $G_\alpha(\Omega)$ as the topological ring of Colombeau’s simplified generalized functions, see [4].

Let $\mathcal{E}(\Omega)$ be the ring (pointwise operations) of the functions $u : \mathcal{A}_0(\mathbb{K}) \times \Omega \to \mathbb{K}$ such that $u(\varphi, \cdot) = u_\varphi(\cdot) \in \mathcal{C}^\infty(\Omega)$ for each $\varphi \in \mathcal{A}_0(\mathbb{K})$. If $\alpha \in \mathbb{N}^n$ and $x \in \Omega$ we set $\partial^\alpha u_\varphi(x) := \partial^\alpha u_\varphi(\cdot)(x)$. Let $\mathcal{E}_M(\Omega)$ be the subring of $\mathcal{E}(\Omega)$ consisting of those functions satisfying the following “moderation” condition:

$M) \forall \alpha \in \mathbb{N}^n \exists N \in \mathbb{N} \text{ such that } \forall \varphi \in \mathcal{A}_\alpha(\mathbb{K}) \exists c = c(\varphi) > 0 \text{ and } \eta = \eta(\varphi) \in I$ verifying

$$\|\partial^\alpha u_\varphi(\cdot)\| \leq c\varepsilon^{-N}, \forall \varepsilon \in I_\eta.$$ 

We define an ideal $\mathcal{N}(\Omega)$ of $\mathcal{E}_M(\Omega)$ as the set of $u \in \mathcal{E}_M(\Omega)$ that satisfies the following “nullity” condition:

$N) \forall \alpha \in \mathbb{N}^n \exists N \in \mathbb{N} \text{ and } \gamma \in \Gamma \text{ such that } \forall q \geq N \text{ and } \forall \varphi \in \mathcal{A}_q(\mathbb{K}) \exists c = c(\varphi) > 0 \text{ and } \eta = \eta(\varphi) \in I$ verifying

$$\|\partial^\alpha u_\varphi(\cdot)\| \leq c\varepsilon^{\gamma(q)-N}, \forall \varepsilon \in I_\eta.$$ 

Note that $\mathcal{N}(\Omega)$ is a maximal differential ideal of $\mathcal{E}_M(\Omega)$. The Colombeau’s full generalized functions on $\Omega$ is defined by

$$\mathcal{G}(\Omega) := \mathcal{E}_M(\Omega)/\mathcal{N}(\Omega).$$

This definition appears in [3] for example. Now, in the case of $\Omega$, the topological sheaf of $\Omega \subset \mathbb{R}^n$ we have the Colombeau’s full generalized functions on $\Omega$, $\mathcal{G}(\Omega) = \mathcal{E}_M(\Omega)/\mathcal{N}(\Omega)$. 

Note that if \( \varphi \in D(\Omega) \), \( \int_\Omega \varphi(x) \, dx = 1 \) and \( \text{supp}(\varphi) \subseteq \overline{B}_1(0) \). Then, for all \( \varepsilon > 0 \), \( \varphi_\varepsilon(x) = \varepsilon^{-n} \varphi(\frac{x}{\varepsilon}) \), \( x \in \Omega \subseteq \mathbb{R}^n \) is such that \( \varphi_\varepsilon \in D(\Omega) \), \( \int_\Omega \varphi_\varepsilon(x) \, dx = 1 \) and \( \text{supp}(\varphi_\varepsilon) \subseteq \overline{B}_{\varepsilon}(0) \).

In [7], was defined an interesting subgroup of \( \text{Inv}(\mathbb{K}_n) \), i.e.,
\[
Q:= \{ \alpha_r : r \in \mathbb{R} \},
\]
where \( \alpha_r : I \to \mathbb{R}^*_+ \) defined by \( \alpha_r(\varepsilon) = \varepsilon^r \) with inverse \( \alpha_{-r} : I \to \mathbb{R}^*_+ \) given by \( \alpha_{-r}(\varepsilon) = \varepsilon^{-r} \). In [4], was defined its correspondent subgroup of \( \text{Inv}(\mathbb{K}) \),
\[
H:= \{ \hat{\alpha}_r : r \in \mathbb{R} \},
\]
where \( \hat{\alpha}_r : A_0(\mathbb{K}) \to \mathbb{R}^*_+ \) is given by \( \hat{\alpha}_r(\varphi) = (i(\varphi))^r \) \((i(\varphi) > 0) \) that is the diameter of the \( \text{supp}(\varphi) \), \( \varphi \in A_0(\mathbb{K}) \) with inverse \( \hat{\alpha}_{-r} : A_0(\mathbb{K}) \to \mathbb{R}^*_+ \) given by \( \hat{\alpha}_{-r}(\varphi) = (i(\varphi))^{-r} \). In particular,
\[
i(\varphi_\varepsilon) = \varepsilon i(\varphi), \ \forall \ \varepsilon > 0 \quad \text{and} \quad \hat{\alpha}_r(\varphi_\varepsilon) = \varepsilon^r (i(\varphi))^r = \alpha_r(\varepsilon)\hat{\alpha}_r(\varphi).
\]
Hence, if \( i(\varphi) \leq 1 \), then \( \hat{\alpha}_r(\varphi_\varepsilon) \leq \varepsilon^r = \alpha_r(\varepsilon), \ \forall \ r \in \mathbb{R} \) and if \( i(\varphi) = 1 \), then \( \hat{\alpha}_r(\varphi_\varepsilon) = \alpha_r(\varepsilon) = \varepsilon^r \), i.e.,
\[
H_{\varphi_\varepsilon}:= \{ \hat{\alpha}_r(\varphi_\varepsilon) | r \in \mathbb{R} \} \subseteq Q_\varepsilon:= \{ \alpha_r(\varepsilon) : r \in \mathbb{R} \} \quad \text{and} \quad H_{\varphi_\varepsilon} = Q_\varepsilon = \{ \varepsilon^r : r \in \mathbb{R} \}
\]
if and only if \( i(\varphi) = 1 \) for \( \varphi \in A_0(\mathbb{K}) \).

If \( \varphi \in A_0(\mathbb{K}) \), then its support number \( d(\varphi) \) is defined as in the Notation [1] item i). From [17], Remark 1.3) we shall suppose that \( i(\varphi) = 1 \), for \( \varphi \in A_0(\mathbb{K}) \). So from now on the support number of \( \varphi_\varepsilon \) is equal to \( \varepsilon \) and instead of \( d(\varphi_\varepsilon) \) we shall write \( \varepsilon \) only. This provides us an unique extraction of \( \varepsilon \) from \( \varphi_\varepsilon \) which is not the case in the original Colombeau theory.

Note that with above considerations we have that \( \hat{\alpha}_r \in \text{Inv}(\mathbb{K}), \ \forall \ r \in \mathbb{R} \), and for all \( \varphi \in A_0(\mathbb{K}), \ i(\varphi) = 1 \), there exists \( \eta = \eta(\varphi) \in ]0,1[ \) such that
\[
\lim_{r \to +\infty} \hat{\alpha}_r(\varphi_\varepsilon) = \lim_{r \to +\infty} (\varepsilon i(\varphi))^r = 0, \ \forall \ 0 < \varepsilon < \eta.
\]
In this case, we say that \( \lim_{r \to +\infty} \hat{\alpha}_r = 0 \). We shall use this, for example, in the proof of Lemma [7] in Section 3.

### 2.1 The sharp topology on Colombeau’s full generalized numbers: a review

In this subsection, we review some results and definitions about \( \mathbb{K} \), and we start with the following two definitions of [4] that are very important for the definition of the topology on \( \mathbb{K} \).

**Definition 2.** An element \( v \in \mathbb{K} \) is associated to zero, \( v \approx 0 \), if for some (hence for each) representative \((v(\varphi))_\varphi\) of \( v \) we have
\[
\exists \ p \in \mathbb{N} \text{ such that } \lim_{\varepsilon \downarrow 0} v(\varphi_\varepsilon) = 0, \ \forall \ \varphi \in A_p(\mathbb{K}).
\]
Two elements \( v_1, v_2 \in \mathbb{K} \) are associated, \( v_1 \approx v_2 \), if \( (v_1 - v_2) \approx 0 \). If there exists \( a \in \mathbb{K} \) with \( v \approx a \), then \( v \) is said to be associated with \( a \) and the latter is called the shadow of \( v \).
Definition 3. For a given $x \in \mathbb{K}$ we set $A(x) := \{ r \in \mathbb{R} : (\alpha - r, x) \approx 0 \}$ and define the valuation of $x$ as $V(x) = \sup(A(x))$.

For the relation of association “$\approx$” on $\mathbb{K}$ see ([4], Definition 1.3). It is easily seen that if $x \in \mathbb{K}$, then $r \in A(x) \iff \exists p \in \mathbb{N}$ such that $\lim_{\varepsilon \to 0} \varepsilon \cdot x(\varphi) = 0$, $\forall \varphi \in A_p(\mathbb{K})$ or equivalently $|x| \leq \alpha_r$, $\forall x \leq 1$ with $\varepsilon$ sufficiently small. From this, it easily follows that $D : \mathbb{K} \times \mathbb{K} \to \mathbb{R}_+$ defined by

$$D(x, y) := e^{-V(x-y)}$$

is an ultra-metric on $\mathbb{K}$ which is invariant under translations. The topology resulting from $D$ is so-called the sharp topology on $\mathbb{K}$ and it is denoted by $\tau_s$. Denote the norm of an element $x \in \mathbb{K}$ by $\|x\| := D(x, 0)$. Thus, we have the distance between two elements $x, y \in \mathbb{K}$ which is given by $D(x, y) := \|x - y\|$.

Now, we have the following result from [4].

Corollary 4. For given $x, y \in \mathbb{K}$, $r \in \mathbb{R}$, $s \in \mathbb{R}^*_+$ and $a, b \in \mathbb{K}$, we have:

i) $\|x + y\| \leq \max(\{\|x\|, \|y\|\})$ and $\|xy\| \leq \|x\|\|y\|$;

ii) $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$;

iii) $\|ax\| = \|x\|$, if $a \neq 0$;

iv) $(\alpha_s x) = e^{-r}\|x\|$ and $(\beta_s x) = s\|x\|$, where $\beta_s = \alpha_{-\log(s)}$;

v) $\|a\| = 1$, if $a \neq 0$;

vi) $\|a - b\| = 1 - \delta_{ab}$ (Kronecker’s $\delta$).

Now, we remind the following result of [4] which will be important to prove the Proposition [19] in the Section [4].

Lemma 5. i) $x \in B_1(0) \Leftrightarrow V(x) > 0$;

ii) If $x \in B_1(0)$, then $x \approx 0$ and $D(1, x) = 1$. Hence, $1 \notin B_1(0)$, $B_1(0) \cap B_1(1) = \emptyset$, $B_1(0) \supset B_1(0)$ and $B_1(0) \neq B_1(0)$.

The basic notation and some properties of the algebraic and the topological structure of $\mathbb{K}$ can be found in [4]. Let $\mathbb{K}^n := \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{K}, \forall i = 1, 2, \ldots n\}$ with the product topology, see [4] for more details.

If $x = (x_1, x_2, \ldots, x_n) \in \mathbb{K}^n$, we define $\|x\| := \max\{\|x_i\| : 1 \leq i \leq n\}$, where $\|x_i\|$ is defined as before, and frequently the subscript $n$ will be omitted from the notation.

If $r \in \mathbb{R}^*_+$ and $x_0 \in \mathbb{K}^n$, then

$$B_r(x_0) = \{x \in \mathbb{K}^n : \|x - x_0\| < r\}, B'_r(x_0) = \{x \in \mathbb{K}^n : \|x - x_0\| \leq r\}$$

and

$$S_r(x_0) = \{x \in \mathbb{K}^n : \|x - x_0\| = r\}$$

are the open ball of center in $x_0$ and ratio $r$, the closed ball of center in $x_0$ and ratio $r$ and the sphere of center in $x_0$ and ratio $r$, respectively.

Remark 6. It is convenient to point out that we easily extend for $\mathbb{K}^n$ the definitions of $B_r(x_0)$, $B'_r(x_0)$ and $S_r(x_0)$. 

5
3 Differential Calculus over the ring of Colombeau’s full generalized numbers

We begin with the following lemma that will be fundamental to introduce the concept of the differentiable functions in the framework of Colombeau’s full generalized numbers.

Lemma 7. Let \( U \subset \mathbb{K} \) be an open subset, \( f : U \to \mathbb{K} \) a function and \( x_0 \in U \). Then there exists at most one \( z_0 \in \mathbb{K} \), such that

\[
\lim_{x \to x_0} \frac{f(x) - f(x_0) - z_0(x - x_0)}{\beta_{\|x-x_0\|}} = 0,
\]

where \( \beta_{\|x-x_0\|} = \alpha_{-\log(\|x-x_0\|)} \) (as in Corollary \( \Box \) item iv), with \( s = \|x - x_0\| \).

Proof. Let \( z_0, z_1 \) be elements of \( \mathbb{K} \) such that the limit in (3.1) is zero for both. Then

\[
\lim_{x \to x_0} \frac{(z_1 - z_0)(x - x_0)}{\beta_{\|x-x_0\|}} = 0.
\]

In particular, let \( x_n := x_0 + \alpha_n \). Then \( \alpha_n = x_n - x_0 \) and \( \beta_{\|x_n-x_0\|} = \alpha_{-\log(\|x_n-x_0\|)} = \alpha_n \). Thus, we have that

\[
0 = \lim_{x_n \to x_0} \frac{(z_1 - z_0)(x_n - x_0)}{\beta_{\|x_n-x_0\|}} = \lim_{n \to \infty} \frac{(z_1 - z_0)\alpha_n}{\alpha_n} = \lim_{n \to \infty} (z_1 - z_0) = z_1 - z_0.
\]

Hence, \( z_1 = z_0 \). \( \square \)

The Lemma 7 tell us that the following definition, which is the exact generalization of the Frechet derivative, is meaningful.

Definition 8. Given an open set \( U \subset \mathbb{K} \), \( f : U \to \mathbb{K} \) and \( x_0 \in U \) we shall say that \( f \) is differentiable in \( x_0 \) if there exists \( z_0 \in \mathbb{K} \), such that the limit in (3.1) is valid. In this case \( f \) is said to be differentiable in \( x_0 \), and we write \( D(f)(x_0) = z_0 \) and shall call \( z_0 \) the derivative of \( f \) in \( x_0 \). We shall say that \( f \) is differentiable if it is differentiable in each point of its domain.

Remark 9. a) The differentiability of \( f \) in \( x_0 \) is equivalent to the statement that

\[
\lim_{x \to x_0} \frac{\|T(x)\|}{\|x - x_0\|} = 0,
\]

where

\[
T(x) := f(x) - f(x_0) - D(f)(x_0)(x - x_0),
\]

because by Corollary \( \Box \) item iv), we have that

\[
\|\hat{\beta}_{\|x-x_0\|}\| = \|\alpha_{-\log(\|x-x_0\|)}\| = e^{\log(\|x-x_0\|)} = \|x - x_0\|.
\]

The choice of the limit that appears in (3.1) (instead of the limit in (3.2)) follows from the necessity to avoid additional difficulties in the proof of some properties. Moreover, by the fact that \( T(x) \in \mathbb{K} \) we have that it is natural to work, in the definition of the derivative, with a quotient of \( T(x) \) by an element of \( \mathbb{K} \) which is an "infinitesimal together with \( \|x - x_0\|". Since \( \mathbb{K} \) is not a field, this infinitesimal
must be an invertible element of \( \mathbb{K} \) and we get that the choice of \( \tilde{\beta}_{|x-x_0|} \) seems very natural since, \( \|\tilde{\beta}_r\| = r, \forall \ r \in \mathbb{R} \).

Note that the limit in (3.2) is not equivalent to

\[
\lim_{x \to x_0} \frac{T(x)}{\|x - x_0\|} = 0, \tag{3.4}
\]

because of Corollary 4 item iii), we have that

\[
\left\| \frac{T(x)}{\|x - x_0\|} \right\| = \left\| \frac{1}{\|x - x_0\|} T(x) \right\| = \|T(x)\|.
\]

Thus, if the differentiability of \( f \) in \( x_0 \) is defined by the limit in (3.4) with \( T(x) \) as in (3.3), then the continuity at \( x_0 \) would imply its differentiability at \( x_0 \). Hence, its derivative would be non-unique because of any element of \( \mathbb{K} \) would be derivative of \( f \) at \( x_0 \). So, this is not a good way to define differentiability in this context.

b) If \( f \) is differentiable in \( x_0 \), then we have that

\[
f(x) - f(x_0) = D(f)(x_0)(x - x_0) + E(x) \tag{3.5}
\]

with \( \lim_{x \to x_0} \frac{E(x)}{\tilde{\beta}_{|x-x_0|}} = 0 \). Moreover, \( D(f)(x_0) = \lim_{n \to \infty} \frac{f(x_0 + \alpha_n) - f(x_0)}{\alpha_n} \).

From now on we use Remark 9 item b) without further mention.

**Lemma 10.** Let \( U \subset \mathbb{K} \) be an open subset. If \( f : U \to \mathbb{K} \) is differentiable at \( x_0 \), then \( f \) is continuous at \( x_0 \).

**Proof.** Since

\[
\lim_{x \to x_0} \frac{E(x)}{\tilde{\beta}_{|x-x_0|}} = 0
\]

then for any \( \varepsilon > 0 \) there exists \( \delta > 0 \), such that \( \left\| \frac{E(x)}{\tilde{\beta}_{|x-x_0|}} \right\| < \varepsilon \) always that \( \|x - x_0\| < \delta \).

Note that

\[
\left\| \frac{E(x)}{\tilde{\beta}_{|x-x_0|}} \right\| = \left\| \frac{E(x)}{\|x - x_0\|} \right\| = \left\| \frac{1}{\|x - x_0\|} E(x) \right\| = \|E(x)\|
\]

where the last equality is due to Corollary 4 item iii). Thus, we have that \( \|E(x)\| < \varepsilon \) always that \( \|x - x_0\| < \delta \) which implies that \( \lim_{x \to x_0} E(x) = 0 \) and the result follows.

We now give an example of a non-constant function whose derivative vanishes everywhere. Hence a function that is not determined by its derivative. This example also shows that the "Mean Value Theorem" is false in general in our context, thus as in [6].
Example 11. Let

\[ f(x) = \begin{cases} \\
\dot{\beta}_{\|x\|^2}, & \text{if } x \in \mathbb{K}^* \\
0, & \text{if } x = 0.
\end{cases} \]

If \( x_0 \neq 0 \), then \( f \) is constant in the neighborhood \( S_{\|x_0\|} \) of \( x_0 \) and we have that \( D(f)(x_0) = 0 \) because of

\[
D(f)(x_0) = \lim_{n \to \infty} \frac{f(0 + \alpha_n) - f(0)}{\alpha_n} = \lim_{n \to \infty} \frac{f(\alpha_n)}{\alpha_n} = \lim_{n \to \infty} \frac{\alpha_{2n}}{\alpha_n} = \lim_{n \to \infty} \frac{\alpha_n}{\alpha_n} = 0.
\]

Now, it is convenient to point out that a function will be called almost constant if it has vanishing derivative.

Using the Remark 9 and the standard proofs of ordinary differential calculus we obtain the following result.

Proposition 12. Let \( U \subset \mathbb{K} \) be an open subset. If \( f, g : U \to \mathbb{K} \) be differentiable, then

a) \( fg \) is differentiable and \( D(fg) = D(f)g + fD(g) \);

b) If \( f(U) \) is contained in the domain of \( g \), then \( D(f \circ g) = (D(g) \circ f)D(f) \);

c) \( D(f \pm g) = D(f) \pm D(g) \) and \( D(cf) = cD(f) \), if \( c \) is constant;

d) if \( g(x) \in \text{Inv}(\mathbb{K}) \), \( \forall x \in U \) we have that

\[
D \left( \frac{f}{g} \right) = D(f)g - fD(g) \]

\[
g^2
\]

(3.6)

The Proposition 12 tells us that our notion of derivations satisfies the usual properties of the derivation of ordinary differential calculus.

Next, we introduce the differentiability in functions with more than one variable.

Definition 13. Let \( U \subset \mathbb{K}^n \) be an open subset, \( f : U \to \mathbb{K} \) a function, \( x = (x_1, x_2, \ldots, x_n), x_0 = (x_{01}, x_{02}, \ldots, x_{0n}) \in U \). Suppose that there exists an element \( a_i \in \mathbb{K} \), such that

\[
\lim_{h \to 0} \frac{f(x_1, x_2, \ldots, x_i + h, \ldots, x_n) - f(x_{01}, x_{02}, \ldots, x_{0i}, \ldots, x_{0n}) - a_i h}{\beta_{\|h\|}} = 0. \quad (3.6)
\]

Then we shall define \( \frac{\partial f}{\partial x_i}(x_0) := a_i \) and call it the partial derivative of \( f \) with respect to \( x_i \) in \( x_0 \). We shall say that \( f \) is differentiable in \( x_0 \) if there exists a vector \( a = (a_1, a_2, \ldots, a_n) \in \mathbb{K}^n \), such that

\[
\lim_{x \to x_0} \frac{f(x) - f(x_0) - (a_1, a_2, \ldots, a_n) \cdot (h_1, h_2, \ldots, h_n)}{\beta_{\|x-x_0\|}} = 0, \quad (3.7)
\]

where \( h_i = x_i - x_{0i}, \ i = 1, 2 \ldots, n \) are the components of the difference vector \( h \).
It is now standard to verify that all the known results of ordinary differential calculus hold also in our case. For example, if $f$ is differentiable in $x_0$, then it is continuous in $x_0$ and the $a_i$'s in the definition of $f$ being differentiable are exactly the partial derivatives in $x_0$. If $\mathbb{K} = \mathbb{R}$, the gradient of $f$ at $x_0$ is defined by the vector

$$\nabla(f)(x_0) := \left(\frac{\partial f}{\partial x_1}(x_0), \frac{\partial f}{\partial x_2}(x_0), \ldots, \frac{\partial f}{\partial x_n}(x_0)\right),$$

where $\frac{\partial f}{\partial x_i} = a_i$, for $i = 1, 2, \ldots, n$.

If $U$ is an open subset of $\mathbb{R}^n$ and $k \in \mathbb{N}$, we can define the set

$$\mathcal{C}^k(U; \mathbb{K}) := \{f : U \to \mathbb{K} | \partial^\alpha f \in \mathcal{C}(U; \mathbb{K}), \forall \alpha \in \mathbb{N}^n \text{ such that } 0 \leq |\alpha| \leq k\}$$

and $\mathcal{C}^\infty(U; \mathbb{K}) := \bigcap_{k \in \mathbb{N}} \mathcal{C}^k(U; \mathbb{K})$.

**Remark 14.** 1) Let $U \subset \mathbb{R}^n$ be an open subset and $f : U \to \mathbb{R}^m$ a function. We may write $f = (f_1, f_2, \ldots, f_m)$, where each $f_i : U \to \mathbb{R}$, $i = 1, 2, \ldots, m$ is the coordinated function of $f$. It is convenient to point out that the differentiability of $f$ at $x_0 \in U$ is equivalent to $f_i$ being differentiable at $x_0$, for all $i = 1, \ldots, m$, where $x_0 = (x_{01}, \ldots, x_{0n})$.

2) It is easy to see that $f$ is differentiable at $x_0$ if and only if there exists a $\mathbb{R}$-linear map $T : \mathbb{R}^n \to \mathbb{R}^m$, such that

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - T(x - x_0)}{\|x-x_0\|} = 0. \quad (3.9)$$

The map $T$ will be denoted by $D(f)(x_0)$ and

$$J = [T]_{m \times n} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \quad (3.10)$$

is the Jacobian matrix.

## 4 Pointvalues and generalized numbers

Within classical distribution theory a definition of pointvalues for distributions was introduced in [10] and see also [13]. However, this concept cannot be applied to arbitrary distributions at arbitrary points. Moreover, there is no way of characterizing distributions by their pointvalues in any way similar to classical functions. On the other hand, for elements of Colombeau algebras there is a very natural way of obtaining pointvalues by inserting points into representatives. The objects gained from such operation are sequences of numbers and then are not values in the field $\mathbb{K}$, but they are representatives of generalized numbers. Our first aim will be to gain an exact description of these objects. In this section we take some ideas of [15] to generalize some results about Colombeau’s simplified generalized function for Colombeau’s full generalized function.

It is convenient to point out at this point that $\mathbb{K}$ is embedded into $\overline{\mathbb{K}}$ via $c \mapsto \text{cl}[c(\varphi)]$ with $c(\varphi) = c$, $\forall \varphi \in \mathcal{A}_0(\mathbb{K})$ and $\overline{\mathbb{K}}$ is the natural home of pointvalues of elements of $\mathcal{G}(\Omega)$. Note that by an analogous proof to that given in [13] demonstrates that $\mathbb{K}_s$ can be a subring of $\mathbb{K}$. Moreover, $\mathbb{K}$ is the ring of constants in $\mathcal{G}(\Omega)$.

We begin this section with the following definition.
\begin{definition}
Let $U \in \mathcal{G}(\Omega)$ and $x \in \Omega$. The pointvalue of $U$ at $x$ is the element $cl([u_\varphi(x)])_\varphi$ of $\mathbb{K}$, where $u_\varphi$ is one of the representatives of $U$.
\end{definition}

\begin{proposition}
Let $\Omega$ be a connected open subset of $\mathbb{R}^n$ and $U \in \mathcal{G}(\Omega)$. Then $\nabla U \equiv 0$ if and only if $U \in \mathbb{K}$.
\end{proposition}

\begin{proof}
Evidently, $U \in \mathbb{K}$ implies $\nabla U \equiv 0$. Conversely, let $(\nabla u_\varphi)_\varphi \in \mathcal{N}(\Omega)^n$. Then, we assume that $\Omega$ is star-shaped and we have that there exists $p \in \mathbb{N}$, $\gamma \in \Gamma$ and $c = c_\varphi > 0$, such that
\[
|\nabla u_\varphi(x)| \leq c \varepsilon \gamma^{(q)-p}, \forall \varphi \in \mathcal{A}_q(\mathbb{K}), \ q \geq p
\]
and $\varepsilon$ sufficiently small. Hence,
\[
|u_\varphi(x) - u_\varphi(x_0)| = |(x - x_0) \int_0^1 \nabla u_\varphi(x - \sigma(x - x_0))d\sigma|
\]
\[
\leq |x - x_0| \int_0^1 |\nabla u_\varphi(x - \sigma(x - x_0))|d\sigma
\]
\[
\leq |x - x_0|c \varepsilon \gamma^{(q)-p}, \forall \varphi \in \mathcal{A}_q(\mathbb{K}), \ q \geq p
\]
for arbitrary $p$ and suitable $q$. Thus $((u_\varphi)_\varphi - (u_\varphi(x_0))_\varphi) \in \mathcal{N}(\mathbb{K})$. Now, if $\Omega$ is connected, then any point $x \in \Omega$ can be connected with some fixed $x_0 \in \Omega$ by a polygon and for an analogous argument to the one above we get the result.
\end{proof}

Let $\Omega$ be an open subset of $\mathbb{K}^n$ and $I_\eta$ as in Notation $[1]$ in the item $a)$. Define
\[
\Omega_M := \{ (x_\varphi) \in \Omega^{\mathcal{A}_0(\mathbb{K})} : \exists p \in \mathbb{N} \text{ s.t. } \forall \varphi \in \mathcal{A}_p(\mathbb{K}), \ \exists c = c_\varphi > 0, \text{ s.t. } |x_\varphi| \leq c \varepsilon^{-p}, \ \varepsilon \in I_\eta \}
\]
and we introduce the equivalence relation defined by
\[
(x_\varphi) \sim (y_\varphi) \iff \exists p \in \mathbb{N}, \gamma \in \Gamma, \ \exists c = c_\varphi > 0, \ \text{ s.t. } |x_\varphi - y_\varphi| \leq c \varepsilon \gamma^{(q)-p}, \ \varepsilon \in I_\eta
\]
\[\forall \varphi \in \mathcal{A}_q(\mathbb{K}), \ q \geq p.\]
Set $\tilde{\Omega} := \Omega_M / \sim$.

The set of compactly supported points is
\[
\tilde{\Omega}_c = \{ \tilde{x} \in \tilde{\Omega} : \exists \text{ repres. } (x_\varphi)_\varphi, \ \exists K \subset \subset \Omega, \ \exists p \in \mathbb{N} \text{ s.t. } x_\varphi \in K, \ \forall \varphi \in \mathcal{A}_p(\mathbb{K}), \ \forall \varepsilon \in I_\eta.\}
\]
It is clear that if the $\tilde{\Omega}_c$-property holds for one representative of $\tilde{x} \in \tilde{\Omega}$, then it holds for every representative. Also, for $\Omega = \mathbb{K}$, we have $\mathbb{K} = \mathbb{K}$. Thus, we have that the canonical identification $\mathbb{K}^n = \mathbb{K}^n = \mathbb{K}^n$. For $\mathbb{K}_c$ we write $\mathbb{K}_c$.

From considerations above, we have the following result.

\begin{proposition}
Let $U \in \mathcal{G}(\Omega)$ and $\tilde{x} \in \tilde{\Omega}_c$. Then the generalized pointvalue of $U$ at $\tilde{x} = cl([(x_\varphi)_\varphi]$ is $U(\tilde{x}) := cl([(u_\varphi(x_\varphi))_\varphi]$. Moreover, it is a well-defined element of $\mathbb{K}$.
\end{proposition}

\begin{proof}
If $\tilde{x} \in \tilde{\Omega}_c$, then there exists $K \subset \subset \Omega$, $p \in \mathbb{N}$, such that $x_\varphi \in K$, $\forall \varphi \in \mathcal{A}_p(\mathbb{K})$, $\forall \varepsilon \in I_\eta$. Since $U \in \mathcal{G}(\Omega)$ then we have that
\[
|u_\varphi(x_\varphi)| \leq \sup_{x \in K} |u_\varphi(x_\varphi)| \leq c \varepsilon^{-p}, \forall \varphi \in \mathcal{A}_p(\mathbb{K}), \forall \varepsilon \in I_\eta
\]
and
and for some $c = c_\varphi > 0$. Next we show that $\tilde{x} \sim \tilde{y} \Rightarrow U(\tilde{x}) \sim U(\tilde{y})$, i.e., $x_\varphi \sim y_\varphi \Rightarrow u_\varphi(x_\varphi) \sim u_\varphi(y_\varphi)$ and we need to prove that there exists $p' \in \mathbb{N}$, $\gamma \in \Gamma$ and $c' = c'_\varphi > 0$, such that

$$|u_\varphi(x_\varphi) - u_\varphi(y_\varphi)| \leq c' \varepsilon^{\gamma(q) - p'}, \forall \varphi \in \mathcal{A}_q(\mathbb{K}), q \geq p', \varepsilon \in I_\eta. \tag{4.1}$$

Note that

$$|u_\varphi(x_\varphi) - u_\varphi(y_\varphi)| = |(x_\varphi - y_\varphi) \int_0^1 \nabla u_\varphi(x_\varphi - \sigma(x_\varphi - y_\varphi)) d\sigma| \leq |x_\varphi - y_\varphi| \int_0^1 |\nabla u_\varphi(x_\varphi - \sigma(x_\varphi - y_\varphi))| d\sigma. \tag{4.2}$$

Since $x_\varphi \sim y_\varphi$ then there exists $p'' \in \mathbb{N}$, $\gamma' \in \Gamma$ and $c'' = c''_\varphi > 0$, such that

$$|x_\varphi - y_\varphi| \leq c'' \varepsilon^{\gamma'(q) - p''}, \forall \varphi \in \mathcal{A}_{q'}(\mathbb{K}), q' \geq p'', \varepsilon \in I_\eta. \tag{4.3}$$

By the fact that $x_\varphi - \sigma(x_\varphi - y_\varphi)$ remains within some compact subset of $\Omega$ for $\varepsilon \in I_\eta$ and $U \in \mathcal{G}(\Omega)$, we have that

$$|\nabla u_\varphi(x_\varphi - \sigma(x_\varphi - y_\varphi))| \leq \sup_{x \in K} |\nabla u_\varphi(x)| \leq c'''' \varepsilon^{\gamma''(q) - p''}, \forall \varphi \in \mathcal{A}_{q'''}(\mathbb{K}), \forall \varepsilon \in I_\eta \tag{4.4}$$

Replacing (4.3) and (4.4) in (4.2), we have that

$$|u_\varphi(x_\varphi) - u_\varphi(y_\varphi)| \leq (c'p'' + c''p'''(q') - p'' + \gamma(q')) \varepsilon^{\gamma(q) - p}, \forall \varphi \in \mathcal{A}_{q'}(\mathbb{K}), q' \geq (p'' + p'''), \varepsilon \in I_\eta,$$

where $p'' = p'' + p'''$, $\gamma(q) = \gamma'(q')$ and $c'' = c'''$. Thus the inequality in (4.1) holds which implies that $\tilde{x} \sim \tilde{y} \Rightarrow U(\tilde{x}) \sim U(\tilde{y})$. Next, if $(w_\varphi(x_\varphi))_\varphi \in \mathcal{N}(\Omega)$, then $w_\varphi(x_\varphi) \sim 0$, because of $\exists p \in \mathbb{N}, \gamma \in \Gamma$ and $c = c_\varphi > 0$, such that

$$|w_\varphi(x_\varphi)| \leq c \varepsilon^{\gamma(q) - p}, \varphi \in \mathcal{A}_q(\mathbb{K}), q \geq p, \varepsilon \in I_\eta$$

and we have that $x_\varphi$ stays within some compact subset of $\Omega$ for $\varepsilon \in I_\eta$. \hfill \Box

Next, we study when the elements in $\mathcal{G}(\Omega)$ are identically zero, i.e, we can characterize full generalized functions in $\tilde{\Omega}_c$ from their point values as in classical functions.

**Theorem 18.** If $\Omega$ is an open subset of $\mathbb{R}^n$, then

$$U \equiv 0 \text{ in } \mathcal{G}(\Omega) \iff U(\tilde{x}) = 0 \text{ in } \mathbb{K}, \forall \tilde{x} \in \tilde{\Omega}_c.$$

**Proof.** ($\Rightarrow$) We suppose that $U \equiv 0$ in $\mathcal{G}(\Omega)$ and we show that $U(\tilde{x}) = 0$ in $\mathbb{K}$ for all $\tilde{x} \in \tilde{\Omega}_c$. In fact, let $u_\varphi, u_\varphi(x_\varphi)$, where $x_\varphi$ is a representative of $\tilde{x}$, $u_\varphi$ a representative of $U \in \mathcal{G}(\Omega)$ and $u_\varphi(x_\varphi)$ is a representative of $U(\tilde{x})$ in $\mathbb{K}$. Since $u_\varphi \in \mathcal{N}(\Omega)$ and $x_\varphi$ is a representative of $\tilde{x} \in \tilde{\Omega}_c$, then there exists $K \subset \subset \Omega \ni x_\varphi$, $p \in \mathbb{N}$, $c = c_\varphi > 0$, such that

$$|u_\varphi(x_\varphi)| \leq \sup_{x \in K} |u_\varphi(x)| \leq c \varepsilon^{\gamma(q) - p}, \forall \varphi \in \mathcal{A}_q(\mathbb{K}), q \geq p, \varepsilon \in I_\eta$$

for some $\gamma \in \Gamma$. Hence, $u_\varphi(x_\varphi) \in \mathcal{N}(\mathbb{K}), \forall x_\varphi$ representative of $\tilde{x} \in \tilde{\Omega}_c$, i.e. $U(\tilde{x}) = 0$ in $\mathbb{K}$, $\forall \tilde{x} \in \tilde{\Omega}_c$. 11
If $U \neq 0$ in $\mathcal{G}(\Omega)$, then there exists $K \subset \subset \Omega$, $\alpha \in \mathbb{N}^n$ such that $\forall \; p_1 \in \mathbb{N}, \; \forall \; \gamma \in \Gamma, \; \exists \; \varphi \in \mathcal{A}_{\eta} (\mathbb{K}), \; q_1 \geq p_1$ such that $\forall \; c_1 = c_1 \varphi \geq 0$, we have that
\[
\sup_{x \in K} |\partial^\alpha u_{\varphi_k}(x)| > c_1 \varepsilon^{\gamma(q_1) - p_1}, \; \forall \; \varepsilon \in I_\eta \tag{4.5}
\]
We choose $\alpha$ with the above property in such a way that $|\alpha|$ is minimal. Then, (4.5) yields the existence of sequences $\varepsilon_k \to 0$ and $x_{\varphi_k} \in K$ such that
\[
|\partial^\alpha u_{\varphi_k}(x_{\varphi_k})| > c_1 \varepsilon^{\gamma(q_1) - p_1}, \; \forall \; k \in \mathbb{N}. \tag{4.6}
\]
Let $\varepsilon > 0$ and we set $x_{\varphi} = x_{\varphi_k}$ for $\varepsilon_{k+1} < \varepsilon < \varepsilon_k, k \in \mathbb{N}$. Then, $(x_{\varphi})_\varphi \in \Omega_M$ and it has values in $K$. Consequently, $\tilde{x} = \text{cl}[(x_{\varphi})_\varphi]$ belongs to $\tilde{\Omega}_\varphi$. Also, from the equation (4.6), we have that $\partial^\alpha U(\tilde{x}) \neq 0$ in $\mathbb{K}$. Now, we have the following two cases:

$i)$ $\alpha = 0$;

$ii)$ $\alpha \neq 0$.

$i)$ If $\alpha = 0$, then $U(\tilde{x}) \neq 0$ in $\mathbb{K}$ and the result follows.

$ii)$ If $\alpha \neq 0$, we show that this leads to a contradiction. Indeed, since $|\alpha| = |(\alpha_1, \alpha_2, \ldots, \alpha_n)|$ was assumed to be minimal, then for any $\beta \in \mathbb{N}^n$ with $|\beta| = |\alpha| - 1$ and $L \subset \subset \Omega$, we have that there exists $p_2 \in \mathbb{N}, \gamma \in \Gamma$, such that for any $\varphi \in \mathcal{A}_{\eta^2} (\mathbb{K})$, $q_2 \geq p_2$ there exists $c_2 = c_2 \varphi > 0$, such that
\[
\sup_{x \in L} |\partial^\beta u_{\varphi_k}(x)| \leq c_2 \varepsilon^{\gamma(q_2) - p_2}, \; \forall \; \varepsilon \in I_\eta. \tag{4.7}
\]
Now, we may assume that $\alpha_1 \neq 0$ in $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. Let $\beta := (\alpha_1 - 1, \alpha_2, \ldots, \alpha_n)$, $\beta' = (\alpha_1 + 1, \alpha_2, \ldots, \alpha_n)$ and $x = (x_1, x')$ with $x' = (x_2, \ldots, x_n) \in \mathbb{R}^{n-1}$. Since $(u_{\varphi})_\varphi \in \mathcal{E}_M(\Omega)$, then we get that there exists $p_3 \in \mathbb{N}, c_3 = c_3 \varphi > 0$ and for any $\varphi \in \mathcal{A}_{\eta^3} (\mathbb{K})$ we have that
\[
\sup_{x \in L} |\partial^\beta u_{\varphi_k}(x)| \leq c_3 \varepsilon^{-p_3}, \; \forall \; \varepsilon \in I_\eta. \tag{4.8}
\]
Choose $L \subset \subset \Omega$ such that $K \subset L^0$, where $L^0$ denotes the interior of $L$. Then, for $k$ sufficiently large, we have that
\[
|\partial^\alpha u_{\varphi_k}(y_{1\varphi_k}, x_{2\varphi_k}, \ldots, x_{n\varphi_k})| = \left| \partial^\alpha u_{\varphi_k}(x_{\varphi_k}) + \int_{x_{\varphi_k}}^{y_{1\varphi_k}} \partial^\beta u_{\varphi_k} (\xi, x_{\varphi_k}) d\xi \right| \geq |\partial^\alpha u_{\varphi_k}(x_{\varphi_k})| - |y_{1\varphi_k} - x_{\varphi_k}| c_3 \varepsilon_k^{-p_3} \geq c_1 \varepsilon^{\gamma(q_1) - p_1} - |y_{1\varphi_k} - x_{\varphi_k}| c_3 \varepsilon_k^{-p_3}, \tag{4.9}
\]
because of the inequalities (4.6) and (4.8). For this $k$ (sufficiently large), we have that $(y_{1\varphi}, x_{2\varphi}, \ldots, x_{n\varphi})$ and $(x_{1\varphi}, x_{2\varphi}, \ldots, x_{n\varphi})$ belongs $K \subset \subset L^0 \subset L$ and we get that $(y_{1\varphi}, x_{2\varphi}, \ldots, x_{n\varphi}) \sim (x_{1\varphi}, x_{2\varphi}, \ldots, x_{n\varphi})$. Consequently, there exists $p_4 \in \mathbb{N}, c_4 = c_4 \varphi > 0$ with $q_3 \geq p_4, \gamma \in \Gamma$, such that for any $\varphi \in \mathcal{A}_{\eta^3} (\mathbb{K})$ we have that
\[
|y_{1\varphi_k} - x_{1\varphi_k}| \leq c_4 \varepsilon_k^{\gamma(q_1) - p_4}, \; \forall \; \varepsilon_k \in I_\eta. \tag{4.10}
\]
Replacing the inequality (4.10) in (4.9), observing the signal, we obtain that
\[
|\partial^\alpha u_{\varphi_k}(y_{1\varphi_k}, x_{2\varphi_k}, \ldots, x_{n\varphi_k})| \geq c_1 \varepsilon^{\gamma(q_1) - p_1} - c_4 \varepsilon_k^{\gamma(q_3) - p_4} c_3 \varepsilon_k^{-p_3}. \tag{4.11}
\]
Now, setting $\bar{x}_{\varphi_k} := (x_{\varphi_k} + c_4 \epsilon_k^{\gamma(q_3) - p_4}, x'_{\varphi_k})$, we have that

$$|\partial^3 u_{\varphi_k}(\bar{x}_{\varphi_k})| = |\partial^3 u_{\varphi_k}(x_{\varphi_k}) + \int_{x_{\varphi_k}} \partial^3 u_{\varphi_k}(\xi, x'_{\varphi_k}))d\xi|$$

$$\geq -c_2 \epsilon_k^{\gamma(q_3) - p_2} + |\bar{x}_{\varphi_k} - x_{\varphi_k}| c_4 \epsilon_k^{\gamma(q_1) - p_4},$$

(4.12)

because of the inequalities (4.6) and (4.7). Note that

$$|\bar{x}_{\varphi_k} - x_{\varphi_k}| = c_4 \epsilon_k^{\gamma(q_3) - p_4},$$

(4.13)

since $\bar{x}_{\varphi_k} - x_{\varphi_k} = (c_4 \epsilon_k^{\gamma(q_3) - p_4}, 0, \ldots, 0)$. Replacing, (4.13) in (4.12), we obtain

$$|\partial^3 u_{\varphi_k}(\bar{x}_{\varphi_k})| \geq -c_2 \epsilon_k^{\gamma(q_3) - p_2} + c_4 \epsilon_k^{\gamma(q_1) - p_4} c_4 \epsilon_k^{\gamma(q_1) - p_1}.$$  

(4.14)

Note that for $\gamma(q_2) - p_2$ large enough and $k > k_0$ we have that

$$\sup_{x \in \mathcal{L}} |\partial^3 u_{\varphi_k}(x)| \geq -c_2 \epsilon_k^{\gamma(q_3) - p_2} + c_4 \epsilon_k^{\gamma(q_1) - p_4} c_4 \epsilon_k^{\gamma(q_1) - p_1} \rightarrow 0,$$

this is a contradiction because of the inequality (4.14). Therefore, $\alpha = 0$ and the result follows.

It is convenient to point out that the Proposition 17 and Theorem 18 say that the true domain of the full generalized functions are the sets $\Omega_c$. We now extend the definition of association presented (4), Definition 1.3) to $\mathbb{K}^n$: An element $x = (x_1, x_2, \ldots, x_n) \in \mathbb{K}^n$ is associated to $0 = (0, 0, \ldots, 0)$, $x \approx 0$ if and only if $x_i \approx 0, \forall i = 1, 2, \ldots, n$. We say that $x = (x_1, x_2, \ldots, x_n)$, and $y = (y_1, y_2, \ldots, y_n) \in \mathbb{K}^n$ are associated, denoted by $x \approx y$, if and only if $x_i$ and $y_i$ is associated in $\mathbb{K}$ for all $1 \leq i \leq n$. If there exists $a = (a_1, a_2, \ldots, a_n) \in \mathbb{K}^n$ with $a \approx x$, i.e., $a_i \approx x_i, \forall i = 1, 2, \ldots, n$, then $x \in \mathbb{K}^n$ is said to be associated with $a$ and $a$ is so-called the shadow of $x$.

Using the definition of the set $\mathbb{K}_{as} := \{z \in \mathbb{K} : \exists a \in \mathbb{K} \text{ such that } z \approx a\}$ defined in the Section 2 of (7), we can define the following subset of $\mathbb{K}$

$$\mathbb{K}_{as} := \{z \in \mathbb{K} : \exists a \in \mathbb{K} \text{ such that } z \approx a\},$$

so-called the shadow of $\mathbb{K}$, see (4), Definition 1.3). Note that this is the set of all elements $y \in \mathbb{K}$ such that there exists $p \in \mathbb{N}$ with

$$\lim_{x \rightarrow 0} \hat{y}(\varphi) = 0, \forall \varphi \in \mathcal{A}_p(\mathbb{K})$$

exists for some, and hence all representative $\hat{y}$ of $y$. It is easy to see that $\mathbb{K}_{as}$ is in fact a subalgebra of $\mathbb{K}$. Let $\alpha : \mathbb{K}_{as} \rightarrow \mathbb{K}$ be defined by $y \mapsto \alpha(y) := \lim_{x \rightarrow 0} \hat{y}(\varphi)$ for some $p \in \mathbb{N}, \forall \varphi \in \mathcal{A}_p(\mathbb{K})$. Then it is easy to see that $\alpha$ is a $\mathbb{K}$-algebra surjective homomorphism. We shall denote by $\mathbb{K}_0$ its kernel which is the subring of $\mathbb{K}$. of the elements associated to zero, i.e., for all $\hat{y}$ representative of $y$ there exists $p \in \mathbb{N}$, such that

$$\lim_{x \rightarrow 0} \hat{y}(\varphi) = 0, \forall \varphi \in \mathcal{A}_p(\mathbb{K}),$$

see (4), Definition 1.3). Thus we extend the (7), Proposition 2.15) in the next result which is an immediate consequence of Lemma 5 item ii) in the end of Section 2
Thus, if $f$ is a proposal of a new differential calculus.

Indeed, we shall extend the main results presented in [6] that contains a framework. Moreover, we show that the embedding theorem and the open mapping theorem hold. Indeed, we have that $x = \varepsilon + y$, i.e., there exists $p \in \mathbb{N}$ such that $\lim_{\varepsilon \to 0} (\hat{x} - \hat{y})(\varphi) = 0$, $\forall \varphi \in \mathcal{A}_p(\mathbb{K})$. 

Thus, if $K \subset \subset \Omega$ is such that $x_\varphi \in K$, then there exists a compact subset $L$ such that $K \subset L$ and for $\varepsilon$ sufficiently small. Thus, $y_\varphi \in L$ and we have that $y \in \tilde{\Omega}_c$. Note that $(x_\varphi)$ is a bounded sequence and we get that $\parallel x \parallel \leq 1$. We easily obtain that (ii) from (i). For (iii) we take $x \in \tilde{V}_c$ and let $K \subset \subset \Omega \setminus \{x_0\}$ with $x_\varphi \in K$ for $\varepsilon$ sufficiently small. Then there exists $r \in \mathbb{R}_+^\ast$ such that $|x - x_0| > r$ for $\varepsilon$ sufficiently small. Hence, $\|x - x_0\| \leq 1$, i.e., $x \in \tilde{\Omega}_c \setminus B_1(x_0)$. From (ii), we obtain that $\tilde{\Omega}_c \setminus B_1(x_0) \subset S_1(x_0)$.  

5 Colombeau’s full differential algebra $\kappa(\mathcal{G}(\Omega))$

In this section, we show that the Fundamental Theorem of Calculus holds in this framework. Moreover, we show that the embedding theorem and the open mapping theorem hold. Indeed, we shall extend the main results presented in [6] that contains a proposal of a new differential calculus.

From now on Colombeau’s full generalized functions, $\mathcal{G}^1(\tilde{\Omega}_c, \mathbb{K})$, will be endowed with the sharp topology, see [2, 3] and [9]. Moreover, let $\mathcal{C}^1(\tilde{\Omega}_c, \mathbb{K})$ with the topology of pointwise convergence, i.e., $f_n \rightarrow f$ if and only if $f_n(x) \rightarrow f(x)$, $\forall x \in \tilde{\Omega}_c$.

Now, we state the main result of this section which is a generalization of ([6], Theorem 4.1).

**Theorem 21 (Embedding Theorem).** Let $\Omega$ be an open subset of $\mathbb{R}^n$. The function $\kappa : \mathcal{G}(\Omega) \rightarrow \mathcal{C}^1(\tilde{\Omega}_c, \mathbb{K})$ defined by $\kappa(f)(x) = f(x)$ is an injective homomorphism of $\mathbb{K}$-algebras. Moreover, $\kappa$ is continuous and $\kappa \left( \frac{\partial f}{\partial x_i} \right) = \frac{\partial (\kappa(f))}{\partial x_i}$, $\forall f \in \mathcal{G}(\Omega)$ and $1 \leq i \leq n$.

**Proof.** We claim that $\kappa$ is a homomorphism of $\mathbb{K}$-algebras. In fact, if $f, g \in \mathcal{G}(\Omega)$ and $a \in \mathbb{K}$, then $\kappa(af)(x) = (af)(x) = af(x) = \alpha_k(f)(x), \forall x \in \tilde{\Omega}_c$ and we have that $\kappa(af) = \alpha_k(f)$. For each $f, g \in \mathcal{G}(\Omega)$ we have that $\kappa(f+g)(x) = (f+g)(x) = f(x)+g(x) = \kappa(f)(x) + \kappa(g)(x) = (\kappa(f)+\kappa(g))(x), \forall x \in \tilde{\Omega}_c$. and it follows that $\kappa(fg)(x) = (fg)(x) = f(x)g(x) = \kappa(f)(x)\kappa(g)(x) = (\kappa(f)\kappa(g))(x), \forall x \in \tilde{\Omega}_c$. Note that the injectivity of $\kappa$ is
immediately obtained from Theorem 18. Since $\kappa$ is a homomorphism, then we only need to prove the continuity at zero to get the continuity of $\kappa$. Let $f_n \xrightarrow{n \to \infty} 0$ in $G(\Omega)$, $x \in \Omega_c$ and $\hat{f}_n$, $(x(\varphi))_{\varphi}$ be representatives of these elements. Choose an exhaustion $(\Omega_m)_{m \in \mathbb{N}}$ of $\Omega$ and fix $m_0$ such that there exists $p \in \mathbb{N}$ with $x(\varphi_j) \in \Omega_{m_0}$, $\forall \varphi_j \in \mathcal{A}_p(\mathbb{K})$, $\forall \varepsilon \in I_n$. Then, $\kappa(f_n(x)) = f_n(x) \xrightarrow{n \to \infty} 0$ which implies that $\kappa(f_n(x)) \xrightarrow{n \to \infty} 0$ and, we have that $\kappa$ is continuous at zero. It remains to show that $\kappa(f)$ is differentiable at $x_0 \in \hat{\Omega}_c$ and prove this fact we firstly suppose that $n = 1$. Let $x_0 \in \hat{\Omega}_c$ whose support is contained in a compact subset to $K \subset \Omega$. We claim that $\kappa(f)$ is differentiable at $x_0$ and $D(\kappa(f))(x_0) = \kappa(f')(x_0) = f'(x_0)$ and this is equivalent to show that

$$\lim_{x \to x_0} \frac{\kappa(f)(x) - \kappa(f)(x_0) - D(\kappa(f))(x_0)(x-x_0)}{\|x-x_0\|} = 0,$$

where $x \in \hat{\Omega}_c$ with $\|x-x_0\| < 1$. In fact,

$$D(\kappa(f))(x_0) = \lim_{n \to \infty} \frac{\kappa(f)(x_0 + \alpha_n) - \kappa(f)(x_0)}{\alpha_n} = \lim_{n \to \infty} \frac{f(x_0 + \alpha_n) - f(x_0)}{\alpha_n} = f'(x_0) = \kappa(f')(x_0),$$

since $\kappa(f) = f \in \mathcal{C}^1(\hat{\Omega}_c; \mathbb{R})$. Replacing (5.2) in (5.1), and by the definition of $\kappa$, we have that (5.1) is equivalently to

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - \kappa(f')(x_0)(x-x_0)}{\|x-x_0\|} = 0,$$

and again by the Remark 9(b) the equality in (5.3) is true for $\kappa(f) = f$. Thus, we have the existence of partial derivatives. Now, to prove the differentiability in the general case it is enough to repeat, for $\mathbb{R}^n$ endowed with the product topology.

Let $J$ be an open interval of $\mathbb{R}$, $f \in G(J)$ and $a, b \in J_c$. We define

$$\left( \int_a^b \kappa(f) \right)(\varphi) := \int_{a(\varphi)}^{b(\varphi)} f(\varphi, t) \, dt,$$

where $(a(\varphi)), (b(\varphi))$ and $f(\varphi, \cdot)$ are representatives of $a, b$ and $f$, respectively, the second integral is Riemann integral and $\int_a^b \kappa(f)$ is a well-defined element of $\mathbb{R}$. It is easy to see that

i) If $f, g \in G(J)$, $\lambda \in \mathbb{R}$ and $a, b, c \in J_c$, $a \leq c \leq b$, then

$$\int_a^b \kappa(f + \lambda g) = \int_a^b \kappa(f) + \lambda \int_a^b \kappa(g) \quad \text{and} \quad \int_a^c \kappa(f) = \int_a^b \kappa(f) + \int_c^b \kappa(f);$$

ii) If $a, b \in J$, then $\int_a^b \kappa(f) = \int_a^b f$, where the second integral is the integral of generalized functions (see, for example, [15]).
We show by using Theorem 21 that the Fundamental Theorem of Calculus holds in this framework, i.e., if $J$ is an open interval of $\mathbb{R}$, $a \in J$, $f \in G(J)$ and $F_x$ is the function defined on $J$ by $F_x = \int_a^x \kappa(f)$, then $F_x$ is a differentiable function and
\[
F'_x = D \left( \int_a^x \kappa(f) \right) = \kappa(f),
\]
that is, $F'_x = \kappa(f)$. In fact, let $d \in J$, $(a(\varphi))$ a $(f(\varphi, \cdot))$ be representatives of $a$ and $f$, respectively. Then
\[
F_x(\varphi) = \left( \int_a^x \kappa(f) \right) (\varphi)
= \int_{a(\varphi)}^{d(\varphi)} f(\varphi, t) \, dt
= \int_{a(\varphi)}^{y(\varphi)} f(\varphi, t) \, dt + \int_{y(\varphi)}^{d(\varphi)} f(\varphi, t) \, dt
= \int_{a(\varphi)}^{y(\varphi)} f(\varphi, t) \, dt + \kappa(G_y),
\]
where $G_y := \int_a^y f(\varphi, t) \, dt$ in $G(J)$ and $y = x(\varphi)$. Thus,
\[
F'_x = \left( \int_{a(\varphi)}^{y(\varphi)} f(\varphi, t) \, dt + \kappa(G_y) \right)' = (\kappa(G_y))' = \kappa(G'_y) = \kappa(f),
\]
because $G'_y = (\int_{a(\varphi)}^{y(\varphi)} f(\varphi, t) \, dt)' = f(\varphi, y)$ a representative of $f \in G(J)$. These results show the consistency of our proposal and allow one to use standard techniques of differential calculus.

**Definition 22.** Let $x \in \tilde{\mathbb{K}}$ and $\hat{x}$ one of its representatives. Then the function $|\hat{x}| : A_0(\mathbb{K}) \to \mathbb{R}_+$ defined by $|\hat{x}|(\varphi) = |\hat{x}(\varphi)|$ rises to an element $|x| \in \mathbb{R}_+$ which depends only on $x$ and it is called the module of $x$.

The next lemma was called in [6] as Generalized Cauchy-Schwarz inequality in the case of the simplified algebra. The same holds for the case of the full algebras as the next result shows whose proof is analogous to the classical one.

**Lemma 23.** Let $x, y \in \mathbb{R}^n$. Then $|\langle x | y \rangle| \leq [x]_2[y]_2$, where $[\cdot]_2 := (\sum_{i=1}^n | \cdot_i |^2)^{\frac{1}{2}}$.

**Proposition 24.** Let $\Omega$ be open subset of $\mathbb{R}^n$ and $f \in G(\Omega)$. The following assertions hold:

1) $\kappa(G(\Omega)) \subset \mathcal{C}(\tilde{\Omega}_c; \tilde{\mathbb{K}})$ and $\kappa(G(\Omega)) \neq \mathcal{C}(\tilde{\Omega}_c; \tilde{\mathbb{K}})$;

2) If $\Omega$ is connected, then for each $x, y \in \tilde{\Omega}_c$ there is $c \in \tilde{\Omega}_c$ such that
\[
\kappa(f)(x) - \kappa(f)(y) = (\nabla \kappa(f)(c)|x - y)
\]
and
\[
|\kappa(f)(x) - \kappa(f)(y)| \leq [\nabla \kappa(f)(c)]_2[x - y]_2;
\]
3) If \( \omega \) is convex and \( D(\kappa(f)) = 0 \), then \( f \) is a constant;

4) If \( K \subset \Omega \) is a compact subset, then \( \kappa(f)|_K \) is bounded. Moreover, if \( 0 \in \Omega \), then
\[
\kappa(f)|_K \cap B_1(0) \text{ is Lipschitz function, where}
\]
\[
\tilde{K} := \{ x \in \tilde{\Omega} \mid \exists \text{ a representative } (x(\varphi)) \text{ of } x \text{ such that } x(\varphi) \in K, \forall \varphi \in \mathcal{A}_0(\mathbb{K}) \};
\]

5) If \( n = 1, m \in \mathbb{N}^* \) and \( \Omega \) is convex, then given \( x, y \in \tilde{\Omega}_c \) there is \( z \in \tilde{\Omega}_c \) such that
\[
\kappa(f)(x) = \sum_{0 \leq j \leq m} \frac{\kappa(f^{(j)})(y)(x-y)^j}{j!} + \frac{\kappa(f^{m+1})(z)(x-y)^{m+1}}{(m+1)!}
\]
and there exist \( (x(\varphi)), (y(\varphi)) \) and \( (z(\varphi)) \), representatives of \( x, y \) and \( z \), respectively with \( x(\varphi) \leq z(\varphi) \leq y(\varphi) \);

6) If \( n = 1, m \in \mathbb{N}^* \) and \( x \in \tilde{\Omega}_c \), then
\[
\lim_{||x-y|| \to 0} \frac{1}{(\beta||x-y||)^n} \left( \kappa(f)(x) - \sum_{0 \leq j \leq m} \frac{\kappa(f^{(j)})(y)(x-y)^j}{j!} \right) = 0.
\]

Proof. The proof is a simple adaptation of ([4], Proposition 4.4). From Theorem 21 together with Example 11 we get 1. The second part of 2) follows from Lemma 23. Assertion 3) follows from 2). The other assertions are proved with similar arguments used in the proof of Theorem 21.

Proposition 25. Let \( f \in \mathcal{C}^\infty(\Omega) \subset \mathcal{G}(\Omega), U := \kappa(f)(\tilde{\Omega}_c), V := f(\Omega) \) and
\[
\tilde{V}_c := \{ x \in \mathbb{K} : \exists \text{ repres. } (x(\varphi))_\varphi, \exists p \in \mathbb{N} \text{ s.t. } x_{\varphi_p} \in K, \forall \varphi \in \mathcal{A}(\mathbb{K}), \forall \varepsilon \in I_\eta \}.
\]
Then the following statements hold:

a) \( U \subset \tilde{V}_c \) and \( \kappa(f) \) is a bounded function;

b) If \( f \) is an open mapping, then \( U = \tilde{V}_c \) and \( U \) is an open subset of \( \tilde{K}_c \).

Proof. Again, we have a little adaptation of the original proof of the corresponding result presented in [4].

a) It is immediate that \( U \subset \tilde{V}_c \) and hence, from Proposition 20 item i), \( \kappa(f) \) is a bounded function.

b) Let \( z \in \tilde{V}_c, (z(\varphi))_\varphi \) a representative of \( z \) and \( K \subset \subset V \) such that \( z_{\varphi_\varepsilon} \in K \) for \( \varepsilon \) sufficiently small and \( \forall \varphi \in \mathcal{A}_\varepsilon(\mathbb{K}) \). Then there exists an \( L \subset \subset \Omega \) such that \( K \subset f(L) \), since \( f \) is an open mapping. Hence there exists an element \( x \in \tilde{\Omega}_c \), whose support is contained in \( L \), such that \( f(x_{\varphi_\varepsilon}) = z_{\varphi_\varepsilon} \). From Proposition 20 item ii), \( U \) is open.

Notice that we do not really need that \( f \) is an open mapping, what we actually need is that there exists an exhaustion \( (\Omega_n)_{n \in \mathbb{N}} \) of relatively compact sets of \( \Omega \) such that \( (f(\Omega_n))_{n \in \mathbb{N}} \) is an exhaustion of \( \Im(f) \).

The following corollary is called the Open Mapping Theorem.

Corollary 26 (Open Mapping Theorem). Let \( f \in \mathcal{C}^\infty(\Omega) \) be an open mapping. Then for every open subset \( W \subset \Omega \) we have that \( \kappa(f)(\tilde{W}_c) \) is open.

Proposition 27. Let \( \Omega \) be connected, \( f \in \mathcal{G}(\Omega) \) and suppose that \( \Im(\kappa(f)) \) is a discret set. Then \( f \) is constant.

Proof. The prove is same that appears in ([4] for Proposition 4.7).
6 Holomorphic and analytic generalized functions and applications

In this section we shall define the notions of holomorphic and analytic functions in the framework of Colombeau’s full algebra. Here $\mathbb{K}$ shall always stand for $\mathbb{C}$, $\Omega$ denotes a nonvoid open set of $\mathbb{C}$, $\mathcal{H}(\Omega) := \{ f \in C^1(\Omega; \mathbb{C}) : \overline{\partial} f = 0 \}$ and $\mathcal{H}G(\Omega) := \{ f \in G(\Omega; \mathbb{C}) : \overline{\partial} f = 0 \}$. It is obvious that $\overline{\mathbb{C}} = \mathbb{R} + i\mathbb{R}$, where $i^2 = -1$. Thus, we consider $\overline{\mathbb{C}}$ to be $\mathbb{R}$-isomorphic to $\mathbb{R}^2$. If $z = x + iy$, with $x, y \in \mathbb{R}$, then we define the following operators:

$$\frac{\partial}{\partial z} := \frac{1}{2}(\partial_x - i\partial_y), \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}(\partial_x + i\partial_y).$$

As in [6] our Embedding Theorem gives, in the obvious way, an Embedding Theorem in the Complex case. We also have an Open Mapping Theorem in this case.

**Theorem 28.** If $f \in \mathcal{H}$ is non-constant, then $(\kappa(f))(\hat{W}c)$ is an open subset for all sets $W \subset \Omega$.

**Proof.** This follows at once from Proposition 25. \hfill $\Box$

**Definition 29.** Let $f \in G(\Omega)$. We shall say that $f$ is sub-linear in $\Omega$ if there exists a representative $\hat{f}$ of $f$ with the following property: for all $x \in \hat{\Omega}_c$, there exists a representative $(x(\varphi))$ of $x$, $k \in \mathbb{R}$, a sequence $(\eta_n)_{n \in \mathbb{N}} \in I = [0, 1]$ and sequences $(C_n)_{n \in \mathbb{N}}$, $(p_n)_{n \in \mathbb{N}}$ in $\mathbb{R}$ such that $\lim_{n \to \infty} (p_n + \eta_n) = \infty$ and

$$|\hat{f}^{(n)}(\varphi_c, x(\varphi_c))| \leq C_n \varepsilon^{-p_n}, \quad \forall \varphi \in \mathcal{A}_{p_n}(\mathbb{K}), \forall \varepsilon \in I_{\eta_n} = [0, \eta_n[.$$

Note that the definition does not depend on the representative of $f$. It is immediate to verify that the set of all sublinear functions of $\Omega$ is a $\mathbb{K}$-subalgebra of $G(\Omega)$.

**Example 30.** If $f \in C^\infty(\Omega) \subset G(\Omega)$, then $f$ is sub-linear.

**Definition 31.** Let $U \subset \mathbb{K}$ be an open subset and $z_0 \in U$. We say that $f : U \to \mathbb{K}$ is analytic in $z_0$ if there exists a sequence $(a_n)_{n \in \mathbb{N}} \in \mathbb{K}$ and a series of the form $\sum_{n \geq 0} a_n(z - z_0)^n$ which converges in a neighborhood of $z_0$ such that $f(z) = \sum_{n \geq 0} a_n(z - z_0)^n$ in this neighborhood. Moreover, we say that $f$ is analytic if $f$ is analytic for all $z_0 \in U$ and we write $f \in \mathcal{A}G(U)$.

As in [6], in the proof of the results below we use the following obvious fact, which holds for general complete ultra-metric abelian groups $G$, but here we restrict our attention to the case $G = \mathbb{K}$: If $(b_n)_{n \in \mathbb{N}}$ is a sequence in $\mathbb{K}$, then $\sum b_n$ converges if and only if $\lim_{n \to \infty} \|b_n\| = 0$.

**Theorem 32.** Let $r > 0$, $z_0 \in \mathbb{K}$ and $f(z) = \sum_{n \geq 0} a_n(z - z_0)^n \in \mathcal{A}G(B_r(z_0))$. Then $f$ is differentiable and $f'(z) = \sum_{n \geq 1} na_n(z - z_0)^{n-1}$. 

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Proof. Let \( r > 0 \), \( z \in B_r(z_0) \) and \( s \in \mathbb{R}^*_+ \) such that \( \| z - z_0 \| < s < r \). Since \( \sum_{n \geq 0} a_n(\hat{\beta}_s)^n \), where \( \hat{\beta}_s = \hat{\alpha} - \log(s) \), converges we have, for \( w \in B_s(z_0) \subset B_r(z_0) \), that

\[
\lim_{n \to \infty} \|na_n(w - z_0)^{n-1}\| = \lim_{n \to \infty} \|a_n(w - z_0)^{n-1}\| \\
\leq \lim_{n \to \infty} \|a_n\| (w - z_0)^{-n+1} \\
\leq \lim_{n \to \infty} \|a_n\| s^{n-1} \\
\leq s^{-1} \lim_{n \to \infty} \|a_n(\hat{\beta}_s)^n\| \\
= 0
\]

which implies that \( \sum_{n \geq 1} na_n(w - z_0)^{n-1} \) converges uniformly on \( B_s(z_0) \). Hence, if

\[
g(w) := \frac{f(w) - f(z) - (w - z) \sum_{n \geq 1} na_n(z - z_0)^{n-1}}{\hat{\beta} \|w - z\|}, \quad \forall \ w \in B_s(z_0) \setminus \{z\},
\]

then

\[
\lim_{w \to z} g(w) = \lim_{w \to z} \lim_{m \to \infty} \sum_{1 \leq n \leq m} a_n \left( (w - z_0)^n - (z - z_0)^n - n(z - z_0)^{n-1}(w - z) \right) / \hat{\beta} \|w - z\| \\
= \lim_{m \to \infty} \sum_{1 \leq n \leq m} 0 \\
= 0.
\]

Thus \( f'(z) = \sum_{n \geq 1} na_n(z - z_0)^{n-1} \). \( \square \)

It is convenient to point out that in the last result we used the following fact: the sequence \( \{\psi_m\}_{m \in \mathbb{N}} \), where \( \psi_m(w) := \sum_{1 \leq n \leq m} a_n \frac{(w - z_0)^n - (z - z_0)^n - n(z - z_0)^{n-1}(w - z)}{\hat{\beta} \|w - z\|} \) uniformly converges on \( B_s(z_0) \setminus \{z\} \) to \( g \), and the change of the order of the limits in the second equality follows from a classical result presented in \((23), \text{Theorem 7.11}\), which clearly holds for function with values in \( \mathbb{K} \), because it is complete.

**Corollary 33.** Let \( r > 0 \), \( z_0 \in \mathbb{K} \) and \( f(z) = \sum_{n \geq 0} a_n(z - z_0)^n \) for \( z \in B_r(z_0) \). Then \( f \in \mathcal{G}^\infty(B_r(z_0) ; \mathbb{K}) \) and for \( k \in \mathbb{N}^* \) one has \( f^{(k)}(z) = \sum_{k \geq 0} n(n-1) \ldots (n-k+1)a_n(z - z_0)^{n-k} \).

In particular, \( k!a_k = f^{(k)}(z_0) \).

**Theorem 34.** Let \( f \in \mathcal{G}(\Omega) \). The following assertions hold.

i) If \( \kappa(f) \) is analytic, then \( f \) is sub-linear;

ii) If \( f \in \mathcal{H}\mathcal{G}(\Omega) \) and \( f \) is sub-linear, then \( \kappa(f) \) is analytic and for all \( z_0 \in \tilde{\Omega}_e \) there exists \( r \in ]0, 1[ \) such that for all \( z \in B_r(z_0) \) one has \( \kappa(f)(z) = \sum_{n \geq 0} \frac{\kappa(f)(n)(z_0)}{n!} (z - z_0)^n \).

Moreover, this series converges uniformly in \( B_r(z_0) \) and \( \frac{\partial}{\partial z} (\kappa(f)) = 0 \).
Proof. i) Let \( z_0 \in \tilde{\Omega}_\varepsilon \) and suppose that \( \kappa(f)(z) = \sum_{n \geq 0} a_n (z - z_0)^n \), \( \forall z \in B_R(z_0) \) with \( R \in \mathbb{R}^*_+ \), where by Corollary 33, we have that \( n! a_n = \kappa(f)^{(n)}(z_0) \). Let \( r \in ]0,1[ \) be such that \( \|\dot{\alpha}_r\| = e^{-r} < R \) and define \( z = z_0 + \dot{\alpha}_r \). Since \( \|z - z_0\| = \|\dot{\alpha}_r\| = e^{-r} < R \) one has that the series
\[
\sum_{n \geq 0} a_n (z - z_0)^n = \sum_{n \geq 0} (\dot{\alpha}_r)^n
\]
converges and from Corollary \( \square \) item iv), we have that
\[
0 = \lim_{n \to \infty} \|a_n (\dot{\alpha}_r)^n\| = \lim_{n \to \infty} e^{-(V(a_n) + nr)}.
\]
Let \( (f_\varphi(\cdot))_\varphi \) be a representative of \( f \) and \( (z_{0,\varphi})_\varphi \) one of \( z_0 \). Take \( k = r \), \( c_n = 1 \) and
\[
p_n = \begin{cases}
V(a_n) - 1, & \text{if } V(a_n) \in \mathbb{R};
\end{cases}
\]
\[
n, & \text{if } V(a_n) = \infty \text{ (i.e., } a_n \in \mathcal{N}(\mathbb{R})).\]

Then \( \lim_{n \to \infty} (p_n + nk) = \infty \) and it follows that
\[
|f^{(n)}_{\varphi_\nu}\left(z_{0,\varphi_\nu}\right)| = n!|a_n| \leq c_n \varepsilon^{p_n}
\]
for \( \varepsilon \) sufficiently small.

ii) Let \((K_\nu)_\nu\) be an exhaustive sequence of compact subsets of \( \Omega \) such that \( \tilde{K}_\nu \) is a \( \mathcal{C}_c^\infty \)-strictly pseudoconvex domain for each \( \nu \in \mathbb{N} \), see (133, Corollaries 1.5.6 and 1.5.11). From (5, Theorem 2) there exists a representative \((f_\nu(\cdot))_\varphi\) of \( f \) such that \( f_\nu(\cdot) \in \mathcal{H}(\tilde{K}_\nu) \) for \( \varepsilon \in I \) and \( \nu \in \mathbb{N} \). Take \( z_0 \in \tilde{\Omega}_\varepsilon \). Since \( f \) is sublinear then there exists \((\eta_n)_{n \in \mathbb{N}}\) a sequence in \( I, k \in \mathbb{R}, (z_{0,\varphi})_\varphi \) a representative of \( z_0 \), and the sequences in \( \mathbb{R} : (c_n)_{n \in \mathbb{N}}, \) and \((p_n)_{n \in \mathbb{N}}, \) such that \( \lim_{n \to \infty} (p_n + k_n) = \infty \) with
\[
|f^{(n)}_{\varphi_\nu}\left(z_{0,\varphi_\nu}\right)| \leq c_n \varepsilon^{p_n}, \ \forall \varepsilon \in I_{\eta_n}, n \in \mathbb{N}.
\]
Thus, if \( 0 < r < \|\dot{\alpha}_r\| = e^{-|k|} \), then
\[
\lim_{n \to \infty} \left\| \frac{\kappa(f)(z_0)^n}{n!}(z - z_0)^n \right\| \leq \lim_{n \to \infty} e^{-(p_n + k_n)} = 0, \ \forall z \in B_r(z_0).
\]
Hence,
\[
\sum_{n \geq 0} \frac{\kappa(f)(z_0)^n}{n!}(z - z_0)^n
\]
converges uniformly in \( B_r(z_0) \). Let \( \nu \in \mathbb{N} \) and \( s \in \mathbb{R}^*_+ \) such that \( z_{0,\varphi_\nu} \in K_\nu \subset \tilde{K}_{\nu + 1} \) and define
\[
\tilde{D}_s(z_{0,\varphi_\nu}) := \{ \lambda \in \mathbb{C} : |\lambda - z_{0,\varphi_\nu}| \leq s \} \subset \tilde{K}_{\nu + 1}, \ \forall \varepsilon \in I.
\]
By the fact all the elements of \( B_1 \) are associated to zero we have that
\[
f_{\nu\varphi_\nu}(x) = \sum_{n \geq 0} \frac{f_{\nu\varphi_\nu}(z_{0,\varphi_\nu})}{n!}(x - z_{0,\varphi_\nu})^n, \ \forall x \in \tilde{D}(z_{0,\varphi_\nu})
\]
and \( \varepsilon \) sufficiently small. Consequently, we conclude that
\[
\kappa(f)(z) = \sum_{n \geq 0} \frac{\kappa(f)(z_0)^n}{n!}(z - z_0)^n, \ \forall z \in B_r(z_0).
\]
Moreover, as
\[
\frac{\partial}{\partial \bar{z}} (\kappa(f)) = \kappa \left( \frac{\partial}{\partial \bar{z}} f \right) = 0,
\]
we have that \( \frac{\partial}{\partial \bar{z}} (\kappa(f)) = 0 \).

Note that in the proof of the above theorem we actually obtain a lower bound for the radius of convergence in each point.

We finish this section with the following result that is an easy consequence of the last theorem.

**Corollary 35.** Let \( f \in \mathcal{H}\mathcal{G}(\Omega) \). Then \( \kappa(f) \) is analytic if and only if \( f \) is sub-linear.

### 6.1 Some applications

In [8] the authors used ([7], Proposition 2.5) to show the non existence of a solution for a certain first-order partial differential equation. In this article the following question is still raised: The result of no solution existence for such an equation can be generalized to any linear operator with constant coefficients? In [6] generalizes and gives an answer to this question, making use of the Open mapping Theorem in a simple, but interesting way into \( \mathbb{K} \). In this paper we extend this result to \( \mathbb{K} \) using the results of simplified generalized holomorphic functions of [6], that have been extended here for the case of Colombeau’s full algebras.

**Theorem 36.** Let \( \Omega \) be a connected open subset of \( \mathbb{C}^n \), \( f \in \mathcal{H}(\Omega) \) non-constant, \( L = \sum_{1 \leq k \leq m} a_k \frac{\partial}{\partial x_k} \) a linear differential operator with constant coefficients \( a_1, a_2, \ldots, a_m \) belonging to \( \mathbb{K} \) and \( \mathfrak{J} \) the ideal generated by \{\( a_1, a_2, \ldots, a_m \)\}. If there exists \( u \in \mathcal{G}(\Omega) \) such that \( L(u) = f \), then \( \mathfrak{J} = \mathbb{K} \).

**Proof.** Suppose \( \mathfrak{J} \) is an ideal of \( \mathbb{K} \). If there exists \( u \in \mathcal{G}(\Omega) \) such that \( L(u) = f \), then \( \text{Im}(\kappa(f)) \subset \mathfrak{J} \). By ([4], Proposition 2.6 item i)), we have that \( \mathfrak{J} \) is a rare subset of \( \mathbb{K} \). So, \( \text{Im}(\kappa(f)) \) would not be open which contradicts the Theorem 28.

**Proposition 37.** Let \( A \in \mathcal{S}_f \) and \( L \) the differential operator defined by \( L = (\chi_A D^2 + \chi_{A^c} \text{id}) \). Then the solutions of \( L(u) = 0 \) are all of the form \( \chi_A f \) with \( D^2 f \neq 0 \) and \( \chi_A \lambda^2 + \chi_{A^c} = 0 \), \( \forall \lambda \in \mathbb{K} \).

**Proof.** If \( D^2 f = 0 \), then we have that \( L(\chi_A f) = 0 \). Now, let \( f \) be such that \( L(f) = 0 \). Since \( \chi_A D^2 f + \chi_{A^c} f = 0 \), then we have that \( \chi_A (\chi_A D^2 f + \chi_{A^c} f) = 0 \). Thus, \( \chi_A D^2 f = 0 \) and \( \chi_{A^c} f = 0 \). By the fact that \( 1 = \chi_A + \chi_{A^c} \) we have that \( f = \chi_A f \) and \( D^2 f = 0 \). Note that if \( \chi_A \lambda^2 + \chi_{A^c} = 0 \), then \( \chi_{A^c} (\chi_A \lambda^2 + \chi_{A^c}) = 0 \) which implies that \( \chi_{A^c} = 0 \), this contradicts the fact that \( \chi_{A^c} \neq 0 \). So, \( \chi_A \lambda^2 + \chi_{A^c} \neq 0 \), \( \forall \lambda \in \mathbb{K} \).

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