The open question of time fractional PDEs: needs to store all data

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Abstract

It is well known that the numerical solution of time fractional PDEs at current stage depends on all preceding time instances. Hence, the storage requirement grows linearly as the number of the time steps increases, which poses a significant challenge especially for high-dimensional problems. To the best of our knowledge, this is still an open question in the time fractional PDEs. In this paper, we propose an efficient and memory free algorithm to answer this open question by using incremental SVD. Numerical experiments show that our new algorithm not only memory free, but also more efficient than standard solvers.

1 A time-fractional PDE

In this paper, we consider the following time-fractional parabolic equation for $u(x, t)$:

$$\partial_t^\alpha u(x, t) - \Delta u(x, t) = f(x, t), \quad (x, t) \in \Omega \times (0, T],$$
$$u(x, t) = 0, \quad (x, t) \in \partial \Omega \times (0, T],$$
$$u(x, 0) = u_0(x), \quad x \in \Omega,$$

where $\Omega$ is a bounded convex polygonal domain in $\mathbb{R}^d$ ($d = 2, 3$) with a boundary $\partial \Omega$ and $u_0$ is a given function defined on the domain $\Omega$ and $T > 0$ is a fixed value. Here $\partial_t^\alpha u$ ($0 < \alpha < 1$) denotes the left-sided Caputo fractional derivative of order $\alpha$ with respect to $t$ and it is defined by

$$\partial_t^\alpha u = \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - s)^{-\alpha} \frac{d}{ds} u(x, s) ds,$$

where $\Gamma(\cdot)$ is Euler’s Gamma function defined by $\Gamma(x) = \int_0^\infty s^{x-1} e^{-s} ds$ for $x > 0$.

Let $T_h$ be a shape regular and quasiuniform triangulation of domain $\Omega$ and define a continuous piecewise linear finite element space $V_h$ by

$$V_h = \{ v_h \in H^1_0(\Omega) : v_h|_K \text{ is a linear function, } \forall K \in T_h \}.$$

Then the semidiscrete Galerkin scheme for problem (1.1) reads: find $u_h(t) \in V_h$ such that

$$(\partial_t^\alpha u_h, \chi) + (\nabla u_h, \nabla \chi) = (f, \chi) \quad \forall \chi \in V_h, \; t > 0.$$
For the time discretization, we divide the interval $[0, T]$ into $N$ equally spaced subintervals with a time step size $\Delta t = T/N$, and $t_n = n\Delta t, n = 0, \ldots, N$. Then the $L_1$ scheme [4,5] approximates the Caputo fractional derivative $\partial_t^\alpha u(x, t_n)$ by

$$\partial_t^\alpha u(x, t_n) \approx \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \frac{\partial u(x, s)}{\partial s} (t_n - s)^{-\alpha} \, ds$$

$$\approx \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{n-1} \frac{u(x, t_{j+1}) - u(x, t_j)}{\Delta t} \int_{t_j}^{t_{j+1}} (t_n - s)^{-\alpha} \, ds$$

$$= \sum_{j=0}^{n-1} \beta_j \frac{u(x, t_{n-j}) - u(x, t_{n-j-1})}{(\Delta t)^\alpha (2-\alpha)} =: \partial_{\Delta t}^\alpha u(t_n),$$

where the weights $\{\beta_j\}$ are given by

$$\beta_j = (j+1)^{1-\alpha} - j^{1-\alpha}, \quad j = 0, 1, \ldots, n-1.$$

Then the fully discrete scheme reads: given $u_0^n = I_h u_0 \in V_h$ with $c_\alpha = \Gamma(2-\alpha)$, find $u_h^n \in V_h$ for $n = 1, 2, \ldots, N$ such that

$$\beta_0 (u_h^n, v_h) + c_\alpha (\Delta t)^\alpha (\nabla u_h^n, \nabla v_h)$$

$$= \beta_{n-1} (u_h^0, v_h) + \sum_{j=1}^{n-1} (\beta_{j-1} - \beta_j) \left( u_h^{n-j}, v_h \right) + c_\alpha (\Delta t)^\alpha (f(t_n), v_h).$$

Here $I_h$ is the standard $L^2$ projection from $L^2(\Omega)$ into $V_h$. The implementation of the scheme is given in [Algorithm 1] and we use $M$ and $A$ to denote the standard mass and stiffness matrices.

**Algorithm 1** (Finite element and $L_1$ scheme for solving time fractional heat equation)

**Input:** $c_\alpha, \alpha, \Delta t, M \in \mathbb{R}^{m \times m}, A \in \mathbb{R}^{m \times m}, u_0$

1. Set $u_0 = M u_0; \overline{A} = M + c_\alpha (\Delta t)^\alpha A; \beta = \text{zeros}(m, 1); \overline{U} = \text{zeros}(m, n)$
2. for $i = 1$ to $m$
3. \hspace{1em} $\beta(i) = i^{1-\alpha} - (i-1)^{1-\alpha}; \beta(1) = 1$;
4. Get the load vector at $t_i$, denoted by $b_i$;
5. $b_i = \beta(i) \overline{u}_0 + c_\alpha (\Delta t)^\alpha b_i + \sum_{j=1}^{i-1} (\beta(j) - \beta(j+1)) \overline{U}(, i-j)$;
6. Solve $\overline{A} u_i = b_i$;
7. $\overline{U}(, i) = M u_i$;
8. end for

**Output:** $u_m$

The computational cost of [Algorithm 1] is obvious: To compute the numerical solution at $t_n$, the solutions at all preceding time instances are required, which poses a significant challenge especially for high-dimensional problems and multi-query applications.

## 2 Incremental SVD

In 2002, Brand [1] proposed an incremental algorithm to find the SVD of a low rank matrix. However, the algorithm needs to compute thousands or millions of orthogonal matrices and their
multiplications together. Those multiplications may corrode the orthogonality and many reorthogonalizations are needed in practice.

Very recently, Zhang [7] improved the incremental SVD and could avoid computing those large amount orthogonal matrices. Hence, the expensive reorthogonalizations are not needed. Furthermore, it is proved that the modification does not change the accuracy of the algorithm.

Let us start by introducing some notation needed throughout this work. For convenience, we adopt the Matlab notation herein. Given a matrix $U$, we use $U(:, 1 : r)$ and $U_r$ to denote the first $r$ columns of $U$, and $U(1 : r, 1 : r)$ be the leading principal minor of $U$ of order $r$. Next, we follow [7] and briefly discuss the incremental SVD algorithm.

### 2.1 Initialization

Assume that the first column of $U$ is nonzero, i.e., $u_1 \neq 0$, we initialize the core SVD of $u_1$ by setting

$$
\Sigma = (u_1^\top u_1)^{1/2}, \quad Q = u_1 \Sigma^{-1}, \quad R = 1.
$$

The algorithm is shown in Algorithm 2.

**Algorithm 2 (Initialize ISVD)**

**Input:** $u_1 \in \mathbb{R}^m$

1. $\Sigma = (u_1^\top u_1)^{1/2}; \quad Q = u_1 \Sigma^{-1}; \quad R = 1$;

**Output:** $Q, \Sigma, R$.

Suppose we already have the rank-$k$ truncated SVD of the first $\ell$ columns of $U$:

$$
U_\ell = Q \Sigma R^\top, \quad \text{with} \quad Q^\top Q = I, \quad R^\top R = I, \quad \Sigma = \text{diag} (\sigma_1, \ldots, \sigma_k),
$$

where $\Sigma \in \mathbb{R}^{k \times k}$ is a diagonal matrix with the $k$ (ordered) singular values of $U_\ell$ on the diagonal, $Q \in \mathbb{R}^{m \times k}$ is the matrix containing the corresponding $k$ left singular vectors of $U_\ell$, and $R \in \mathbb{R}^{\ell \times k}$ is the matrix of the corresponding $k$ right singular vectors of $U_\ell$.

Since we assume that the matrix $U$ is low rank, then it is reasonable to anticipate that most vectors of $\{u_{\ell+1}, u_{\ell+2}, \ldots, u_n\}$ are linear dependent or almost linear dependent with the vectors in $Q \in \mathbb{R}^{m \times k}$. Without loss of generality, we assume that the next $s$ vectors, $u_{\ell+1}, \ldots, u_{\ell+s}$, their residuals are less than the tolerance when project them onto the space spanned by the columns of $Q$. While the residual of $u_{\ell+s+1}$ is greater than the tolerance, i.e.,

$$
\|u_i - QQ^\top u_i\| < \text{tol}, \quad i = \ell + 1, \ldots, \ell + s,
$$

$$
\|u_i - QQ^\top u_i\| \geq \text{tol}, \quad i = \ell + s + 1.
$$

### 2.2 Update the SVD of $U_{\ell+s}$

By (2.1) and the above assumption, we have

$$
U_{\ell+s} = [U_\ell \mid u_{\ell+1} \mid \ldots \mid u_{\ell+s}]
$$

$$
= [Q \Sigma R^\top \mid u_{\ell+1} \mid \ldots \mid u_{\ell+s}]
$$

$$
\approx [Q \Sigma R^\top \mid QQ^\top u_{\ell+1} \mid \ldots \mid QQ^\top u_{\ell+s}]
$$

$$
= Q \left[ \begin{array}{c|c|c|c}
\Sigma & Q^\top u_{\ell+1} & \cdots & Q^\top u_{\ell+s}
\end{array} \right] \left[ \begin{array}{c|c|c}
R & 0 \\
0 & I_s
\end{array} \right]^\top
$$
We use the Matlab built-in function \texttt{svd}(Y, ‘econ’) to find the SVD of \( Y \) since the matrix \( Y \) is always short-and-fat. We let \( Y = Q_Y \Sigma_Y R_Y^\top \) and split \( R_Y \) into \( \begin{bmatrix} R_Y^{(1)} \\ R_Y^{(2)} \end{bmatrix} \), then we update the SVD of \( U_{\ell + s} \) by
\[
Q \leftarrow QQ_Y, \quad \Sigma \leftarrow \Sigma_Y, \quad R \leftarrow \begin{bmatrix} R_R^{(1)} \\ R_R^{(2)} \end{bmatrix} \in \mathbb{R}^{k \times (k + s)}.
\]

2.3 Update the SVD of \( U_{\ell + s + 1} \)

The SVD of \( U_{\ell + s + 1} \) can be constructed by

1. letting \( e = u_{\ell + s + 1} - QQ^\top u_{\ell + s + 1} \) and let \( p = \|e\| \) and \( \bar{e} = e/p \),
\[
U_{\ell + s + 1} = [U_{\ell + s} \mid u_{\ell + 1}] = \begin{bmatrix} Q \Sigma R^\top \mid p\bar{e} + QQ^\top u_{\ell + s + 1} \end{bmatrix}
\]
\[
= \begin{bmatrix} Q \mid \bar{e} \end{bmatrix} \begin{bmatrix} \Sigma & Q^\top u_{\ell + s + 1} \\ 0 & p \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}^\top,
\]

2. finding the full SVD of \( \bar{Y} = \begin{bmatrix} \Sigma & Q^\top u_{\ell + s + 1} \\ 0 & p \end{bmatrix} \), and let \( \bar{Q} \Sigma \bar{R}^\top \) be the SVD of \( \bar{Y} \), then

3. updating the SVD of \( [U_{\ell + s} \mid u_{\ell + s + 1}] \) by
\[
[U_{\ell + s} \mid u_{\ell + s + 1}] = ([Q \mid \bar{e}] \bar{Q}) \Sigma \left( \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix} \bar{R} \right)^\top. \quad (2.2)
\]

For many data sets, they may have a large number of nonzero singular values but most of them are very small. Without truncating those small singular values, the incremental SVD algorithm maybe keep computing them and hence the computational cost is huge. Hence, we need to set another truncation if the last few singular values are less than a tolerance.

Lemma 1. ([7] Lemma 5.1) Assume that \( \Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_k) \) with \( \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_k \), and \( \bar{\Sigma} = \text{diag}(\mu_1, \mu_2, \ldots, \mu_k) \) with \( \mu_1 \geq \mu_2 \geq \ldots \geq \mu_k \). Then we have
\[
\mu_{k + 1} \leq p, \quad (2.3a)
\]
\[
\mu_{k + 1} \leq \sigma_k \leq \mu_k \leq \sigma_{k - 1} \leq \ldots \leq \sigma_1 \leq \mu_1. \quad (2.3b)
\]

Inequality (2.3a) implies that the last singular value of \( \bar{Y} \) can possibly be very small, no matter how large of \( p \). That is to say the tolerance we set for \( p \) can not avoid the algorithm keeping compute very small singular values. Hence, another truncation is needed if the data has a large number of very small singular values. Inequality (2.3b) guarantees us that only the last singular value of \( \bar{Y} \) can possibly less than the tolerance. Hence, we only need to check the last one.

(i) If \( \Sigma_Y(k + 1, k + 1) \geq \text{tol} \sum_{j=1}^{k+1} \Sigma_Y(j, j) \), then
\[
Q \leftarrow [Q \mid \bar{e}] Q_Y, \quad \Sigma \leftarrow \Sigma_Y, \quad R \leftarrow \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix} R_Y. \quad (2.4)
\]

(ii) If \( \Sigma_Y(k + 1, k + 1) < \text{tol} \sum_{j=1}^{k+1} \Sigma_Y(j, j) \), then
\[
Q \leftarrow [Q \mid \bar{e}] Q_Y(:,1:k), \quad \Sigma \leftarrow \Sigma_Y(1:k,1:k), \quad R \leftarrow \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix} R_Y(:,1:k). \quad (2.5)
\]
2.4 Reorthogonalization

Theoretically, the matrix $[Q \mid \tilde{e}] Q_Y$ is orthogonal. However, in practice, small numerical errors cause a loss of orthogonality. Without a reorthogonalization, the decomposition of the incremental SVD algorithm is not orthogonal. Therefore, the singular values are not true and truncation is not reliable.

For the reorthogonalization to $[Q \mid \tilde{e}] Q_Y$, Oxberry et al. [6] performed QR and Fareed et al. [2] used modified Gram–Schmidt if the inner product between $\tilde{e}$ and the first column of $Q$ is larger than some tolerance. The computational cost can be high if the rank of the matrix is not extremely low. In [7], Zhang suggests reorthogonalize it recursively: only apply the Gram-Schmidt to the new adding vector $\tilde{e}$ since the matrix $Q$ has already been reorthogonalized in previous steps.

Algorithm 3 (Update ISVD)

Input: $Q \in \mathbb{R}^{m \times k}$, $\Sigma \in \mathbb{R}^{k \times k}$, $R \in \mathbb{R}^{l \times k}$, $u_{\ell+1} \in \mathbb{R}^{m}$, $\text{tol}$, $V$, $Q_0$, $q$

1: Set $d = Q^\top u_{\ell+1}$; $e = u_{\ell+1} - Qd$; $p = (e^\top e)^{1/2}$;
2: if $p < \text{tol}$ then
3: $q = q + 1$;
4: Set $V\{q\} = d$;
5: else
6: if $q > 0$ then
7: Set $Y = [\Sigma \mid \text{cell2mat}(V)]$;
8: $[Q_Y, \Sigma_Y, R_Y] = \text{svd}(Y, \text{\textquoteleft econ\textquoteright})$;
9: Set $Q_0 = Q_0 Q_Y$, $\Sigma = \Sigma_Y$, $R_1 = R_Y(1 : k, :)$, $R_2 = R_Y(k + 1 : \text{end}, :)$, $R = \begin{bmatrix} RR_1 \\ R_2 \end{bmatrix}$;
10: Set $d = Q_Y^\top d$
11: end if
12: Set $e = e/p$;
13: if $|e^\top W Q(:, 1)| > \text{tol}$ then
14: $e = e - Q(Q^\top e)$; $p_1 = (e^\top e)^{1/2}$; $e = e/p_1$;
15: end if
16: Set $Y = \begin{bmatrix} \Sigma & d \\ 0 & p \end{bmatrix}$;
17: $[Q_Y, \Sigma_Y, R_Y] = \text{svd}(Y)$;
18: Set $Q_0 = \begin{bmatrix} Q_0 & 0 \\ 0 & 1 \end{bmatrix} Q_Y$;
19: if $\Sigma_Y(k + 1, k + 1) > \text{tol} \sum_{j=1}^{k+1} \Sigma_Y(j, j)$ then
20: $Q = [Q \mid e]Q_0$, $\Sigma = \Sigma_Y$, $R = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix} R_Y$, $Q_0 = I_{k+1}$;
21: else
22: Set $Q = [Q \mid e]Q_0(:, 1 : k)$, $\Sigma = \Sigma_Y(1 : k, 1 : k)$, $R = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix} R_Y(:, 1 : k)$, $Q_0 = I_k$
23: end if
24: $V = []$; $q = 1$;
25: end if

Output: $Q$, $\Sigma$, $R$, $V$, $Q_0$, $q$

Remark 1. Some care should be taken to assemble the matrix $Y$ in line 7 of Algorithm 3. In
Matlab, it is extremely expensive if we update the matrix $V$ by

$$V \leftarrow [V \mid d].$$

We save the vector $d$ in a cell array and then use \texttt{cell2mat} to convert it into a matrix.

2.5 Final check and full implementation

We note that the output of Algorithm 3, $V$, may be not empty. This implies that the output of Algorithm 3 is not the SVD of $U$. Hence we have to update the SVD for the vectors in $V$. We give the implementation of this step in Algorithm 4.

Algorithm 4 (ISVD final check)

\begin{itemize}
  \item \textbf{Input:} $Q \in \mathbb{R}^{m \times k}$, $\Sigma \in \mathbb{R}^{k \times k}$, $R \in \mathbb{R}^{\ell \times k}$, $V$, $q$
  \item if $q > 0$ then
    \item Set $Y = \left[ \Sigma \mid \text{cell2mat}(V) \right]$;
    \item $[Q_Y, \Sigma_Y, R_Y] = \text{svd}(Y, \text{'econ'});$ \\
    \item Set $Q = QQ_Y$, $\Sigma = \Sigma_Y$, $R_1 = R_Y(1 : k,:)$, $R_2 = R_Y(k + 1 : \text{end},:)$, $R = \left[ \begin{array}{c} R R_1 \\ R_2 \end{array} \right]$;
  \item end if
\end{itemize}

\textbf{Output:} $Q$, $\Sigma$, $R$.

Next, we complete the full implementation in Algorithm 5.

Algorithm 5 (Full ISVD)

\begin{itemize}
  \item \textbf{Input:} tol
  \item Get $u_1$;
  \item $[Q, \Sigma, R] = \text{InitializeISVD}(u_1)$; % Algorithm 2
  \item Set $V = []$; $Q_0 = 1$; $q = 0$;
  \item for $\ell = 2, \ldots, n$ do
    \item Get $u_\ell$
    \item $[q, V, Q_0, Q, \Sigma, R] = \text{UpdateISVD}(q, V, Q_0, Q, \Sigma, R, u_\ell, \text{tol})$; % Algorithm 3
  \item end for
  \item $[Q, \Sigma, R] = \text{ISVDfinalcheck}(q, V, Q_0, Q, \Sigma, R)$; % Algorithm 4
\end{itemize}

\textbf{Output:} $Q$, $\Sigma$, $R$.

3 Incremental SVD for the time fractional PDEs

In this section, we shall apply the incremental SVD for the time fractional heat equation (1.1). Suppose we have already computed $u_1$, \ldots, $u_\ell$, and have the rank-$k$ truncated SVD of $U_\ell = [u_1 \mid u_2 \mid \cdots u_\ell]$ by using the incremental SVD algorithm:

$$U_\ell = Q \Sigma R^T, \quad \text{with} \quad Q^T Q = I, \quad R^T R = I, \quad \Sigma = \text{diag}(\sigma_1, \ldots, \sigma_k). \quad (3.1)$$

We do not store the data \{u_\ell\}_{\ell=1}^\ell, instead, we keep the matrices $Q \in \mathbb{R}^{n \times k}$, $\Sigma \in \mathbb{R}^{k \times k}$ and $R \in \mathbb{R}^{\ell \times k}$. Since $k \ll \ell$, then we remedy the memory issue of the standard method.
To compute \( u_{\ell+1} \), we need to extract the data from the matrices \( Q, \Sigma \) and \( R \), i.e.,

\[
u_j = Q\Sigma R_j^T, \quad j = 1, 2, \ldots, \ell,
\]

and then compute

\[
Mu_j = MQ\Sigma R_j^T, \quad j = 1, 2, \ldots, \ell.
\]

The matrix \( MQ \) stays unchanged unless the residual of the new coming data \( u_{\ell+1} \) is greater than the tolerance, when we project it onto the space spanned by the columns of \( Q \).

**Algorithm 6** (Solve a fractional PDE with ISVD)

**Input:** \( c_\alpha, \alpha, \Delta t, m, M \in \mathbb{R}^{m \times m}, A \in \mathbb{R}^{m \times m}, u_0, u_1, \text{tol} \)

1: Set \( V = [] \); \( Q_0 = 1 \); \( q = 0 \); \( \tilde{u}_0 = Mu_0 \); \( \tilde{A} = M + c_\alpha(\Delta t)^\alpha A \); \( \beta = \text{zeros}(m, 1) \);

2: \([Q, \Sigma, R] = \text{InitializeISVD}(u_1)\);  \hfill \% Algorithm 2

3: for \( i = 2 \) to \( m \) do

4: \( \beta(i) = i^{1-\alpha} - (i-1)^{1-\alpha} \);

5: Get the load vector at \( t_i \), denoted by \( b_i \);

6: if \( q = 0 \) then

7: \( MQ = MQ \); \( C = \Sigma R^T \)

8: \( b_i = \beta(i)\tilde{u}_0 + c_\alpha(\Delta t)^\alpha b_i + MQ\left(\sum_{j=1}^{i-1}(\beta(j) - \beta(j + 1))C(:, i - j)\right)\);

9: Solve \( \tilde{A}u_i = b_i \);

10: \([q, V, Q_0, \Sigma, R] = \text{UpdateISVD}(q, V, Q_0, Q, \Sigma, R, u_i, \text{tol})\); \hfill \% Algorithm 3

11: if \( q > 0 \) then

12: \( C = \Sigma R^T \)

13: end if

14: else

15: \( D = [C \quad \text{cell2mat}(V)] \);

16: \( b_i = \beta(i)\tilde{u}_0 + c_\alpha(\Delta t)^\alpha b_i + MQ\left(\sum_{j=1}^{i-1}D(:, i - j)\beta(j)\right)\);

17: Solve \( \tilde{A}u_i = b_i \);

18: \([q, V, Q_0, \Sigma, R] = \text{UpdateISVD}(q, V, Q_0, Q, \Sigma, R, u_i, \text{tol})\); \hfill \% Algorithm 3

19: end if

20: end for

21: \([Q, \Sigma, R] = \text{UpdateISVDcheck}(q, V, Q_0, Q, \Sigma, R)\); \hfill \% Algorithm 4

**Input:** \( Q, \Sigma, R \)

4  **Numerical experiments**

In this section, we test the efficiency of our Algorithm 6.

**Example 1.** Let \( \Omega = (0, 1) \times (0, 1) \), we consider the equation (1.1) with

\[
\alpha = 0.1, \quad T = 1, \quad u_0 = xy(1-x)(1-y), \quad f = 100\sin(2\pi t(x+y))x(1-x)y(1-y).
\]

We use the linear finite element for the spatial discretization with different time steps \( \Delta t \), here \( h \) is the mesh size (max diameter of the triangles in the mesh). For the incremental SVD, we take \( \text{tol} = 10^{-12} \) in Algorithm 6. For solving linear systems, we apply the Matlab built-in solver
backslash ($\setminus$). We report the wall time (seconds)\(^1\) of Algorithm 1 and Algorithm 6 in Table 1. We also compute the \(L^2\)-norm error between the solutions of Algorithm 1 and Algorithm 6 at the final time \(T = 1\), we see that the error is close to the machine error. Furthermore, Algorithm 6 is more efficient than Algorithm 1. In Table 2 we see that the standard solver has an issue to save the data. However, our new Algorithm 6 avoids storing the huge data.

| \(h\)  | \(1/2^1\) | \(1/2^2\) | \(1/2^3\) | \(1/2^4\) | \(1/2^5\) | \(1/2^6\) | \(1/2^7\) |
|-------|----------|----------|----------|----------|----------|----------|----------|
| Wall time of Algorithm 1 | 0.3 | 0.6 | 2 | 10 | 36 | 162 | 680 |
| Wall time of Algorithm 6 | 0.4 | 0.8 | 2 | 8 | 36 | 146 | 670 |
| Error | 8E-15 | 2E-14 | 3E-14 | 3E-14 | 3E-14 | 3E-14 | 3E-14 |

Table 1: Example 1 with \(\Delta t = 10^{-3}\): The wall time and the \(L^2\)-norm error between the solutions of Algorithm 1 and Algorithm 6

| \(h\)  | \(1/2^3\) | \(1/2^4\) | \(1/2^5\) | \(1/2^6\) | \(1/2^7\) | \(1/2^8\) | \(1/2^9\) |
|-------|----------|----------|----------|----------|----------|----------|----------|
| Wall time of Algorithm 1 | 11 | 15 | 26 | 83 | 330 | 1405 | 6424 |
| Wall time of Algorithm 6 | 1E-12 | 2E-14 | 3E-14 | 3E-14 | 3E-14 | 3E-14 | 3E-14 |
| Error | — | 25835 | — | — | — | — | — |

Table 2: Example 1 with \(\Delta t = 10^{-4}\): The wall time and the \(L^2\)-norm error between the solutions of Algorithm 1 and Algorithm 6 — denotes we have the memory issue.

**Example 2.** In this example, we use the incremental SVD algorithm to study the time fractional diffusion-wave equation:

\[
\partial_t^\alpha u(x, t) - \Delta u(x, t) = f(x, t), \quad (x, t) \in \Omega \times (0, T],
\]

\[
u(x, t) = 0, \quad (x, t) \in \partial \Omega \times (0, T],
\]

\[
u(x, 0) = u_0(x), \quad x \in \Omega,
\]

\[
u_t(x, 0) = v_0(x), \quad x \in \Omega,
\]

where \(\Omega \subset \mathbb{R}^2\) is a bounded convex polygonal region with boundary \(\partial \Omega\), \(x = (x, y)\), \(u_0(x)\), \(v_0(x)\) and \(f(x, t)\) are given functions, and \(\partial_t^\alpha u\) \((1 < \alpha < 2)\) denotes the left-sided Caputo fractional derivative of order \(\alpha\) with respect to \(t\) and it is defined by

\[
\partial_t^\alpha u(x, t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{\partial^2 u(x, s)}{\partial s^2} \frac{ds}{(t-s)^{\alpha-1}}, \quad 1 < \alpha < 2.
\]

Let \(0 = t_0 < t_1 < \cdots < t_N = T\) be a given partition of the time interval. Then we have the time step \(\Delta t = T/N\), \(t_n = n\Delta t\) \((n = 0, 1, \ldots, N)\), and define \(\partial_t^\alpha u^{n+1}\) by

\[
\partial_t^\alpha u(x, t_{n+1}) = \frac{1}{\Gamma(2-\alpha)} \int_0^{t_{n+1}} \frac{\partial^2 u(x, \xi)}{\partial s^2} \frac{(t_{n+1} - s)^{1-\alpha} ds}{(t_n + s)^{\alpha-1}}.
\]

\[
= \frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^{n} \int_{t_k}^{t_{k+1}} \frac{\partial^2 u(x, s)}{\partial s^2} \frac{(t_{n+1} - s)^{1-\alpha} ds}{(t_n + s)^{\alpha-1}}
\]

\[
\approx \frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^{n} \frac{u_{k+1} - 2u_k + u_{k-1}}{\Delta t^2} \int_{t_k}^{t_{k+1}} (t_{n+1} - s)^{1-\alpha} ds,
\]

\(^1\)All the code for all examples in the paper has been created by the authors using Matlab R2020b and has been run on a laptop with MacBook Pro, 2.3 Ghz-8-Core Intel Core i9 with 64GB 2667 Mhz DDR4. We use the Matlab built-in function tic-toc to denote the real simulation time.
where $u_k = u(x, t_k), k = 0, 1, 2, \ldots, N$. In the above equation, the integral is easily obtained as
\[
\int_{t_k}^{t_{k+1}} (t_{n+1} - s)^{1-\alpha} ds = \frac{1}{(2-\alpha)} \Delta t^{2-\alpha} \left[(n-k+1)^{2-\alpha} - (n-k)^{2-\alpha}\right].
\]
Let $\beta_k = (k+1)^{2-\alpha} - k^{2-\alpha}$ and define $\tilde{\partial}_t u_h^{n+1}$ by
\[
\tilde{\partial}_t u_h^{n+1} = \frac{\Delta t^{-\alpha}}{\Gamma(3-\alpha)} \sum_{k=0}^n \beta_k [u_{n-k+1}^{n+1} - 2u_{n-k}^{n+1} + u_{n-k-1}^{n+1}],
\]
where $u_0^0 = I_h u_0, u_0^{-1} = u_0^0 - \Delta t I_h v_0$. Then the fully discrete scheme reads: given $u_h^{-1}, u_h^0 \in V_h$, find $u_h^{n+1} \in V_h$ for $n = 0, 1, 2, \ldots$ such that
\[
(\tilde{\partial}_t u_h^{n+1}, v_h) + (\nabla u_h^{n+1}, \nabla v_h) = (f^{n+1}, v_h), \quad \forall v_h \in V_h.
\]

Next, we consider the above wave equation with
\[
\Omega = (0, 1)^2, \quad \alpha = 1.5, \quad u_0 = xy(1-x)(1-y), \quad v_0 = 0, \quad f = 1.
\]
The mesh and the finite element space are the same with Example 1. We report the wall time (seconds) and errors at the final time $T = 1$ in Table 3. Again, our incremental SVD not only memory free, but also more efficient than the standard solver.

| $h$ | $1/2^3$ | $1/2^4$ | $1/2^5$ | $1/2^6$ | $1/2^7$ | $1/2^8$ | $1/2^9$ | $1/2^{10}$ |
|-----|---------|---------|---------|---------|---------|---------|---------|-----------|
| Wall time of Algorithm 1 | 6 | 19 | 77 | 330 | 1002 | 3183 | 13175 | — |
| Wall time of Algorithm 6 | 24 | 63 | 111 | 230 | 862 | 2645 | 12748 | 60234 |
| Error | 1E-09 | 4E-10 | 4E-10 | 3E-10 | 3E-10 | 3E-10 | 6E-10 | — |

Table 3: Example 2 with $\Delta t = 10^{-4}$: The wall time and the $L^2$-norm error between the solutions of Algorithm 1 and Algorithm 6 — denotes we have the memory issue.

5 Conclusion

In this paper, we propose an efficient and memory free algorithm for solving the time-fractional PDEs by using the incremental SVD algorithm. Although we only test the fractional heat equation, we believe this algorithm could be easily to extend to other time-fractional PDEs such as the Cole-Cole Maxwell’s equations [4]. Furthermore, we will investigate time-dependent PDEs with memory effects, since they share the same challenge that a whole past history of the solution is needed to be stored and used.

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