Topological Electric Charge

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Abstract

By treating magnetic charge as a gauge symmetry through the introduction of a “magnetic” pseudo four-vector potential, it is shown that it is possible, using the ’t Hooft-Polyakov construction, to obtain a topological electric charge. The mass of this electrically charged particle is found to be on the order of $\frac{1}{137} M_W$ as opposed to the much larger mass (on the order of $137 M_W$) of the magnetic soliton. Some model building possibilities are discussed.

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I. INTRODUCTION

Using a non-Abelian gauge field coupled to a self-interacting scalar field, 't Hooft and Polyakov [1] have shown that it is possible to construct a topological soliton which has a magnetic charge. This magnetic monopole is guaranteed to be stable because of the nontrivial topology of the vacuum expectation value (VEV) of the scalar field. This strange vacuum configuration was called the “hedgehog” solution by Polyakov since the direction in isospin space in which the VEV points is linked to the radial direction of ordinary space. This magnetic soliton has several unique properties. First its magnetic charge is not a Noether charge (i.e. it is not linked with any apparent symmetry), but is a topological charge, which owes its existence to the unusual vacuum of the scalar field. Second the monopole has no singularities in its fields (so long as the self-coupling, $\lambda$, of the scalar field is not zero). Finally, and unfortunately, it is estimated to have a mass on the order of $137M_W$ (where $M_W$ is the mass of the gauge bosons after symmetry breaking). If $M_W$ is of the order of the electroweak gauge bosons ($\sim 100$ GeV) then seeing such magnetically charged objects is out of the question for any current or planned accelerator. The monopoles large mass comes about because of the value of the gauge coupling of the electric $U(1)$ symmetry, which is embedded in the original non-Abelian gauge theory. The gauge coupling of the original non-Abelian gauge symmetry must be chosen to have the value of the electromagnetic coupling $e$. This leads to the monopole having a large magnetic charge of $\frac{4\pi}{e}$, and a large mass, on the order of $\frac{4\pi}{e} M_W \approx 137M_W$.

It might be asked if it is possible to construct such topologically stable solitons which have the far field of an electric charge. Julia and Zee [2] have found field configurations which carry both magnetic and electric charge, which are called dyons. However the electric charge can not exist without an accompanying magnetic charge, and the stability arguments that apply to the purely magnetic solution do not apply to the dyonic solution (although there are plausibility arguments for its stability). Purely electrically charged solitons would be of more interest than magnetically charged solitons since electrically charged particles
are much more common. The reason for thinking that electric solitons are possible is the
dual symmetry \[3\] between electric and magnetic quantities of Maxwell’s equations

\[
\begin{align*}
E & \rightarrow \cos \theta E + \sin \theta B \\
B & \rightarrow - \sin \theta E + \cos \theta B
\end{align*}
\]

and

\[
\begin{align*}
\rho_e & \rightarrow \cos \theta \rho_e + \sin \theta \rho_m \\
J_e & \rightarrow \cos \theta J_e + \sin \theta J_m \\
\rho_m & \rightarrow - \sin \theta \rho_e + \cos \theta \rho_m \\
J_m & \rightarrow - \sin \theta J_e + \cos \theta J_m
\end{align*}
\]

where \(\rho_{e(m)}\) and \(J_{e(m)}\) are the electric (magnetic) charge and current densities. Thus given a
particle with a certain electric and magnetic charge it is possible to use this dual symmetry
to “rotate” the two charges so that the particle ends up with a different electric and magnetic
charge. By properly choosing the angle \(\theta\) a particle can be made to carry only electric charge
or only magnetic charge. The ability to altogether transform away one type of charge
holds only if all particles have the same ratio of electric to magnetic charge, since the dual
transformation of Eq. (2) is global.

Based on the dual symmetry between electric and magnetic quantities it should be possible
to find a topological electric soliton by using a duality transformation on the the magnetic
soliton. However the monopole solution is in terms of the gauge potentials rather than the
\(E\) and \(B\) fields, which are involved in the dual symmetry of Eq. (1). When \(E\) and \(B\) are
written in terms of the four-vector potential, \(A_\mu\), \((E_i = \partial^i A^0 - \partial^0 A^i \text{ and } B_i = -\partial^i A^k + \partial^k A^i)\)
it appears impossible to implement the dual symmetry in terms of the potentials.

It has been shown by various authors \([4], [5], [6]\) that Maxwell’s equations with magnetic
and electric charge can be dealt with by introducing a second, pseudo four-vector potential,
\(C_\mu\), in addition to the usual four-vector potential, \(A_\mu\) (the term pseudo for \(C_\mu\) refers to its
behaviour under parity). This two potential approach has the advantage over the Dirac
string approach \([7]\) or the Wu-Yang fiber bundle approach \([8]\) in that it requires neither a
singular string variable nor a patching of the gauge potential. Using two potentials also
puts magnetic charge on the same footing as electric charge by treating both as $U(1)$ gauge symmetries [9]. The drawback of this approach is that there are two “photons” in the theory rather than the one photon that is observed [10]. This can be overcome in two ways: Either by putting extra conditions on the two gauge fields so that only the number of degrees of freedom necessary for one photon are left [6], or by accepting the other “photon”, but hiding it and the magnetic charge associated with it through the Higgs mechanism [11].

The two potential theory of electric and magnetic charge allows the dual symmetry of Maxwell’s equations to be extended to the level of the gauge fields. Using this with the ’t Hooft- Polyakov construction it is a trivial matter to construct topologically stable electric poles rather than magnetic poles. The major difference between the magnetically charged soliton and an electrically charged soliton is in the enormous difference of their respective masses. We will first review the relevant aspects of the two potential theory.

II. THE DUAL FOUR-VECTOR POTENTIAL

In three-vector notation, Maxwell’s equations with electric and magnetic charge are (in Lorentz-Heavyside units) [3]

\[
\nabla \cdot \mathbf{E} = \rho_e \\
\n\nabla \times \mathbf{B} = \frac{1}{c} \left( \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}_e \right) \\
\n\n\n\nabla \cdot \mathbf{B} = \rho_m \\
\n\n- \nabla \times \mathbf{E} = \frac{1}{c} \left( \frac{\partial \mathbf{B}}{\partial t} + \mathbf{J}_m \right) \\
\]

Introducing two four-vector potentials, $A^\mu = (\phi_e, \mathbf{A})$ and $C^\mu = (\phi_m, \mathbf{C})$, the $\mathbf{E}$ and $\mathbf{B}$ fields can be written as

\[
\mathbf{E} = -\nabla \phi_e - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \nabla \times \mathbf{C} \\
\mathbf{B} = -\nabla \phi_m - \frac{1}{c} \frac{\partial \mathbf{C}}{\partial t} + \nabla \times \mathbf{A} \quad (4)
\]

The usual definitions of $\mathbf{E}$ and $\mathbf{B}$ only involve $\phi_e$ and $\mathbf{A}$. Substituting the above expanded definitions for $\mathbf{E}$ and $\mathbf{B}$ into Maxwell’s equations, Eq. (3), yields (after using some standard vector identities and applying the Lorentz gauge condition to both four-vector potentials)
the wave equation form of Maxwell’s equations for both $A^\mu$ and $C^\mu$. The equation for $A^\mu$ has electric charges and currents ($J_\varepsilon^\mu \equiv (\rho_\varepsilon, J_\varepsilon)$) as sources, while the equation for $C^\mu$ has magnetic charges and currents ($J_m^\mu \equiv (\rho_m, J_m)$) as sources. In the two potential theory all of Maxwell’s equations are dynamical equations.

The two four-vector potentials, $A^\mu$ and $C^\mu$, are similar except for their behaviour under parity transformations. The $\mathbf{E}$-field is an ordinary vector under parity, and the $\mathbf{B}$-field is a pseudovector. The normal definition of the fields in terms of the potentials, implies that $\phi_\varepsilon$ must a scalar and $\mathbf{A}$ must be a vector under parity. In order for the $\mathbf{E}$ and $\mathbf{B}$ fields to retain their parity properties under the expanded definitions of Eq. (4), $\phi_m$ must be a pseudoscalar and $\mathbf{C}$ must be a pseudovector under parity. Therefore $A^\mu$ is a four-vector while $C^\mu$ is a pseudo four-vector.

The two potential theory can be cast most simply in four-vector notation. Defining the following two field strength tensors

$$F^{\mu \nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$G^{\mu \nu} = \partial^\mu C^\nu - \partial^\nu C^\mu$$

and their duals

$$F^{\mu \nu} = \frac{1}{2} \varepsilon^{\mu \nu \alpha \beta} F_{\alpha \beta}$$

$$G^{\mu \nu} = \frac{1}{2} \varepsilon^{\mu \nu \alpha \beta} G_{\alpha \beta}$$

(5)

where $\varepsilon^{\mu \nu \alpha \beta}$ is the Levi-Civita tensor, with $\varepsilon^{0123} = +1$ and having total antisymmetry in its indices. The $\mathbf{E}$ and $\mathbf{B}$ fields then can be written as

$$E_i = F^{0i} - G^{0i} = F^{0i} + \frac{1}{2} \varepsilon^{ijk} G_{jk}$$

$$B_i = G^{0i} + F^{0i} = G^{0i} - \frac{1}{2} \varepsilon^{ijk} F_{jk}$$

(7)

Maxwell’s equations in four-vector notation become

$$\partial_\mu F^{\mu \nu} = \partial_\mu \partial^\mu A^\nu = J^\nu_\varepsilon$$

$$\partial_\mu G^{\mu \nu} = \partial_\mu \partial^\mu C^\nu = J^\nu_m$$

(8)
where the Lorentz condition \( \partial_\mu A^\mu = \partial_\mu C^\mu = 0 \) has been imposed on the two potentials in going from the first to the middle expression. Finally the dual symmetry of Eqs. (1), (2) can now be written in terms of the two four-vector potentials and the two four-currents

\[
A^\mu \rightarrow \cos \theta A^\mu + \sin \theta C^\mu \quad C^\mu \rightarrow -\sin \theta A^\mu + \cos \theta C^\mu
\]
\[
J_\mu^e \rightarrow \cos \theta J_\mu^e + \sin \theta J_\mu^m \quad J_\mu^m \rightarrow -\sin \theta J_\mu^e + \cos \theta J_\mu^m
\]

(9)

Eq. (9) extends the dual symmetry of Eq. (1) to the level of the four-vector potentials. This implies that it should be possible to construct a topological electric charge in exactly the same way ’t Hooft and Polyakov constructed a topological magnetic charge. The relevant parts of their solution are reviewed in the next section.

### III. THE ’T HOOFT-POLYAKOV MONOPOLE SOLUTION

’t Hooft and Polyakov [1] independently discovered the possibility of constructing a finite-energy, magnetically charged soliton in a non-Abelian gauge theory coupled to a spontaneous symmetry breaking scalar field. The stability of this field configuration was guaranteed by the nontrivial homotopy of the Higgs field at spatial infinity [12].

In constructing the monopole solution ’t Hooft considered an \( SO(3) \) gauge theory coupled to a triplet scalar field with the following Lagrange density

\[
\mathcal{L} = -\frac{1}{4} H_{\mu \nu}^a H^{a \mu \nu} + \frac{1}{2} D_\mu \Phi^a D^{\mu} \Phi^a + \frac{1}{2} \mu^2 \Phi^a \Phi^a - \frac{1}{4} \lambda (\Phi^a \Phi^a)^2
\]

(10)

where

\[
H_{\mu \nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + g \epsilon^{abc} W_\mu^b W_\nu^c
\]

(11)

and

\[
D_\mu \Phi^a = \partial_\mu \Phi^a + g \epsilon^{abc} W_\mu^b \Phi^c
\]

(12)

\( \epsilon^{abc} \) are the structure constants of \( SO(3) \). The \( SO(3) \) gauge coupling, \( g \), is at this point unspecified. If \( \mu^2 > 0 \) and \( \lambda > 0 \) then the scalar field develops a vacuum expectation value.
of \( v = \frac{\mu}{\sqrt{\lambda}} \). Inserting the following spherically symmetric ansatz into the equations of motion that come from the Lagrangian of Eq. (10)

\[
W_i^a = \epsilon_{aij}x^j[1 - K(r)] \quad W_0^a = 0
\]

\[
\Phi^a = \frac{x^a H(r)}{g r^2}
\]

one arrives at the coupled differential equations for the functions \( K(r) \) and \( H(r) \)

\[
r^2 K'' = K(K^2 + H^2 - 1)
\]

\[
r^2 H'' = 2HK^2 - \mu^2 r^2 H \left( 1 - \frac{\lambda}{g^2 \mu^2 r^2} H^2 \right)
\]

(14)

where the prime means \( \frac{d}{dr} \). In addition to the pure gauge solution to these equations (i.e. \( K(r) = 0 \) and \( H(r) = \frac{\mu}{\sqrt{\lambda}} r \)) there exists a nontrivial finite energy solution. That such a solution exists can best be seen by calculating the energy of the the field configuration of Eq. (13)

\[
E = \int T^{00}(r) d^3x
\]

\[
= \int \left( \frac{1}{4} H_{ij} H^{aij} - \frac{1}{2} D_i \Phi^a D^i \Phi^a - \frac{1}{2} \mu^2 \Phi^a \Phi^a + \frac{1}{4} \lambda (\Phi^a \Phi^a)^2 \right) d^3x
\]

\[
= \frac{4\pi}{g^2} \int_0^\infty \left( (K')^2 + \frac{(K^2 - 1)^2}{2r^2} + \frac{H^2 K^2}{r^2} + \frac{(r H' - H)^2}{2r^2} - \frac{\mu^2 H^2}{2} + \frac{\lambda H^4}{4g^2 r^2} \right) dr
\]

(15)

Adding the constant term, \( -\frac{1}{4} \lambda v^4 \), to the Lagrangian, allows us to write the scalar field potential as \( \frac{1}{4} \lambda (\Phi^a \Phi^a - v^2)^2 \) which means that the last two terms in Eq. (13) become

\[
+ \frac{\lambda}{4} v^2 \left( \frac{H^2}{g^2 r^2} - v^2 \right)^2
\]

(16)

Thus every term in Eq. (13) is positive-definite. Then, since neither \( K(r) = 0 \) nor \( K(r) = 1 \) is the lowest minimum, the variational principle requires that an intermediate solution must exist. An analytical solution to Eq. (14) has been found [13] in the limit when both \( \mu^2 \) and \( \lambda \) are equal to zero. When \( \mu^2 \) and \( \lambda \) are non-zero the solution must be found numerically, and then Eq. (17) becomes

\[
E = \frac{4\pi}{g^2} M_W f \left( \frac{\lambda}{g^2} \right)
\]

(17)
where \( M_W = \mu g/\sqrt{\lambda} \) is the mass of two of the \( SO(3) \) gauge bosons after symmetry breaking.

The function \( f(\frac{\lambda}{g^2}) \), which must be evaluated numerically, is \( O(1) \).

To embed a \( U(1) \) electromagnetic symmetry into the \( SO(3) \) theory, the following gauge-invariant generalization of the Maxwell field strength tensor is defined

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{1}{g|\Phi|^3} \epsilon_{abc} \Phi^a (\partial_\mu \Phi^b)(\partial_\nu \Phi^c)
\]

\[A_\mu = \frac{1}{|\Phi|} \Phi^a W_{\mu}^a \quad (18)\]

To see how a monopole emerges from this generalized field strength tensor, the asymptotic values of the ansatz of Eq. (13) are inserted into Eq. (18). As \( r \to \infty \) \( K(r) \to 0 \) and \( H(r) \to g \mu \sqrt{\lambda} r \) which means

\[
W_i^a \to \epsilon_{aij} x^j / gr^2 + O(r^{-2})
\]

\[
\Phi^a(r) \to r^a v / r + O(r^{-2}) \quad (19)
\]

The asymptotic configuration of the scalar field is called the “hedgehog” solution, since the Higgs field approaches its vacuum value \( v \) in a peculiar way. Instead of pointing in a fixed direction in isospin space for all points in configuration space (i.e. \( \rho^a(r) = v \delta^a_3 = v[0,0,1] \)) it points in an isospin direction that coincides with the spatial radial direction. This links the internal (isospin) space with the external (configuration) space. Inserting the asymptotic fields of Eq. (19) into the generalized field strength tensor of Eq. (18) yields

\[
B_i = -\frac{1}{2} \epsilon_{ijk} F^{jk} \to \frac{r^i}{gr^3}
\]

\[
E_i = F^{i0} = 0 \quad (20)
\]

So as \( r \to \infty \) the fields rapidly approach those of a Coulomb magnetic field and zero electric field. Taking the non-Abelian gauge coupling to be equal to the usual “electric” \( U(1) \) coupling \((g = e)\) the magnetic charge implied by the far fields of Eq. (20) is \( 4\pi/e \).
IV. ELECTRICALLY CHARGED SOLITON

The definition of the \( U(1) \) field strength tensor and the identification of \( W^a_\mu \Phi^a/|\Phi| \) with \( A_\mu \), in Eq. (18), are arbitrary. The term, \( W^a_\mu \Phi^a/|\Phi| \), could just as easily have been used to define the “magnetic” gauge potential, \( C_\mu \). The only change this would require is that either the \( SO(3) \) gauge fields, \( W^a_\mu \), or the scalar fields, \( \Phi^a \), would have to be pseudo quantities, since \( C_\mu \) is a pseudo four-vector. Using the dual symmetry of Eq. (9), the two four-vector potentials can be transformed into one another by taking \( \theta = \frac{\pi}{2} \). This gives \( A_\mu \rightarrow C_\mu \), and from Eq. (8) it also changes the “electric” field strength tensor into the “magnetic” field strength tensor \( (F_{\mu\nu} \rightarrow G_{\mu\nu}) \). In this way Eq. (18) becomes

\[
G_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu - \frac{1}{g|\Phi|^3} \epsilon_{abc} \Phi^b (\partial_\mu \Phi^c) (\partial_\nu \Phi^e) \\
C_\mu = \frac{1}{|\Phi|} \Phi^a W^a_\mu \\
\]  

(21)

As with the expression for the “electric” field strength tensor, \( G_{\mu\nu} \) is gauge-invariant. Inserting the asymptotic values of the non-Abelian gauge fields and of the scalar fields (which now define the magnetic gauge potential, \( C_\mu \)) from Eq. (19) into Eq. (21) one finds that the far fields of this soliton are

\[
B_i = G^{i0} = 0 \\
E_i = \frac{1}{2} \epsilon^{ijk} G_{jk} \rightarrow - \frac{r^i}{gr^3} \\
\]  

(22)

where the expanded definitions of the \( E \) and \( B \) fields from Eq. (7) have been used. Requiring that the charge of the \( E \) field in Eq. (22) have the magnitude of the charge of an electron (or proton) leads to the requirement that \( g = 4\pi/e \) (where \( e \) is the magnitude of the electron’s charge). This means that the original \( SO(3) \) coupling, \( g \), must be large. In the case of the magnetic soliton the \( SO(3) \) gauge coupling was taken to be \( g = e \), since there one wanted to embed the electric \( U(1) \) symmetry (with gauge boson, \( A_\mu \), and coupling, \( e \)) into the non-Abelian theory. In the present case, the coupling of the magnetic \( U(1) \) symmetry, which is embedded in the \( SO(3) \) theory, is fixed by the condition that the electric charge of the soliton be that of observed particles.
The main difference between the electric soliton and the magnetic soliton is in their masses. By equating the energy in the fields with the mass of the soliton one finds, from Eq. (17), that the magnetically charged soliton (which has $g = e$) has a mass of

$$M_m = E = \frac{4\pi}{e^2} M_W f\left(\frac{\lambda}{e^2}\right) \approx 137 M_W$$  \hspace{1cm} (23)

since numerically $f(\frac{\lambda}{e^2})$ is $O(1)$. In contrast the electric soliton (which has $g = 4\pi/e$) has a mass of

$$M_e = E = \frac{e^2}{4\pi} M_W f\left(\lambda e^2\right) \approx \frac{1}{137} M_W$$  \hspace{1cm} (24)

The function, $f(\lambda e^2)$, is $O(1)$ since the energy of the fields (Eq. (13)) is invariant under the duality transformation that turns the the magnetic soliton into an electric soliton. One might worry that the argument of $f$ is different in the two cases. It can be shown that $f(0) = 1$  \cite{13}, and increases monotonically with the argument. Thus for a given $\lambda$ the argument of the electric case is always closer to zero and the value of the function $f$ is closer to 1. From Eqs. (23), (24) it is seen that the mass of the electric soliton is over $10^4$ times smaller than that of the magnetic soliton. If the mass of the gauge boson, $M_W$, is taken to be of the order of the electroweak gauge bosons (i.e. $O(100)$ GeV) then such an electrically charged, spin zero particle should have already been detected. This would seem to imply that if such electric solitons exist, the non-Abelian gauge group, into which they are embedded, must undergo symmetry breaking in such a way that the mass of the gauge bosons are at least several orders of magnitude greater then the masses of the electroweak gauge bosons. Even if $M_W$ were in the range of 50 TeV it might be possible to see such an electric soliton at some reasonably extrapolated future accelerator. An alternative possibility would be to use the spin from isospin mechanism  \cite{14} and form bound states out of particles with various combinations of topological electric charge and gauge magnetic charge. These bound states would carry a spin of 1/2, obey Fermi-Dirac statistics  \cite{15}, and be in the mass range of the baryons. In this paper, however, we simply want to show the theoretical possibility of obtaining a topological electric charge from a non-Abelian gauge theory, since the $SO(3)$
group which is used here is apparently not a theory picked by nature. We will leave for
a future work the task of building a more realistic model through the use of a larger non-
Abelian symmetry.

In standard electrodynamics only the $\mathbf{B}$ field can be written as a curl. Including magnetic
charge in electrodynamics as a gauge charge, by introducing a second four-vector potential,
then requires that part of the $\mathbf{E}$ field be given by the curl of this second potential. The crucial
element in constructing a finite-energy, stable field configuration, with either a Coulomb
electric or magnetic far field, is being able to write that field as the curl of some vector
potential. A general argument can be given [16] that shows this. In order for the energy of
the soliton, Eq. (15), to be finite the covariant derivative of the scalar field must satisfy the
following boundary condition as $r \to \infty$

$$D_\mu \Phi^a = \partial_\mu \Phi^a + g e^{abc} W^b_\mu \Phi^c \to O(r^{-2}) \tag{25}$$

In order for there to be a Coulomb far field (either electric or magnetic) the gauge fields,
$W^a_\mu$, must go like $r^{-1}$ as $r \to \infty$. In addition the magnitude of the scalar field must approach
a constant (its VEV) as $r \to \infty$. Then even though each of the two separate terms need not
approach zero like $r^{-2}$, some cancellation can occur between the terms such that $D_\mu \Phi^a \to 0$
like $r^{-2}$. This is what happens with the ansatz of Eq. (13). For time-independent fields the
time component of the first term of Eq. (25) is zero, so no cancellation can occur between
the two terms. Therefore $W^a_0$ must go to zero faster than $r^{-1}$ and does not give rise to a
Coulomb far field. When the $U(1)$ gauge field is identified with $W^a_\mu \Phi^a/|\Phi|$, as in Eqs. (18)
or (21), this implies that the time component of the $U(1)$ gauge field also will not yield a
Coulomb far field. In standard electrodynamics, where the $\mathbf{E}$ field is defined only by $F^{i0}$,
a Coulomb field is only possible if $A_0 \neq 0$ (in fact if $A_0 = 0$ and only static solutions are
considered then $\mathbf{E} = 0$). In the two potential theory, however, the $\mathbf{E}$ field also has a part
that is the curl of a vector potential (i.e. $E_i = 1/2 \epsilon^{ijk} G_{jk}$). A Coulomb far field is then
possible if the spatial components of the non-Abelian gauge field (and therefore the spatial
components of the embedded $U(1)$ gauge field) go to zero like $r^{-1}$, as is the case for the ansatz of Eq. (13).

Looking in detail at where the Coulomb far fields come from, it appears as if they are due entirely to the scalar fields. Inserting the asymptotic field conditions of Eq. (19), into the generalized field strength tensors of Eqs. (18) or (21), it is found that the Coulomb fields come only from the last term of the generalized field strength tensors, which involve only the scalar fields. This makes it appear that whether the soliton has a magnetic charge or an electric charge is completely independent of the type of $U(1)$ gauge field (either $A_\mu$ or $C_\mu$) that is embedded into the non-Abelian gauge theory. This is not the case. It has been shown [12] that by performing a gauge transformation to the unitary or Abelian gauge, it is possible to rotate the scalar field into the more common asymptotic vacuum configuration

$$\Phi^a(r) \rightarrow v^3 = v(0, 0, 1)$$

Using this asymptotic scalar field in the generalized field strength tensors gives

$$G_{\mu\nu} = \partial_\mu W^3_\nu - \partial_\nu W^3_\mu$$

$$C_\mu = W^3_\mu$$

or

$$F_{\mu\nu} = \partial_\mu W^3_\nu - \partial_\nu W^3_\mu$$

$$A_\mu = W^3_\mu$$

where the $U(1)$ gauge bosons are now associated exclusively with the third isospin component $SO(3)$ gauge boson. Since the expression for the generalized field strength tensors is gauge-invariant, one still gets a Coulomb far field. This comes about, even though the field strength tensors of Eqs. (27), (28) are of the form that usually preclude the existence of a Coulomb field coming from the spatial part of the tensor, because the gauge transformation that takes the “hedgehog” gauge to the Abelian gauge is singular along the positive z-axis. In this way a connection between the ‘t Hooft-Polyakov monopole and Dirac’s monopole can be seen.
V. CONCLUSIONS

In this paper it has been shown that it is possible to construct a finite-energy, topologically stable soliton with an electric charge. This is accomplished through the application of the ’t Hooft-Polyakov monopole solution to the two potential theory of electric and magnetic charge, where the two types of charges are treated as gauge charges through the introduction of two four-vector potentials. In the case of the magnetically charged soliton one starts with some non-Abelian gauge theory which is coupled to a scalar field that breaks the gauge symmetry. By embedding the electric $U(1)$ symmetry in the non-Abelian theory, through the introduction of a generalized field strength tensor, and taking the scalar field to go to the “hedgehog” solution as $r \to \infty$, it is found that a stable, finite-energy, magnetically charged soliton emerges. The gauge coupling, $g$, of the non-Abelian group is required to satisfy, $g = e$, in order that the embedded $U(1)$ symmetry may be identified with the usual electric $U(1)$ gauge field. This leads to both the magnetic charge of the monopole ($= 4\pi/e$) and the mass of the monopole ($\sim 4\pi/e^2$) being extremely large. The magnetic charge carried by the soliton is not a Noether charge but a topological charge, which is due to the nontrivial topology of the scalar field, rather than from a symmetry of the Lagrangian. Thus by starting with an electric gauge charge and using the ’t Hooft-Polyakov ansatz one ends up with a topological magnetic charge.

Based on the dual symmetry between electric and magnetic quantities it is reasonable to believe that this construction can be reversed – i.e. start with a magnetic gauge charge and using the ’t Hooft-Polyakov ansatz, end up with a topological electric charge. This is in fact trivially possible if magnetic charge is treated, like electric charge, as a gauge symmetry by introducing the pseudo four-vector potential, $C_\mu$. Using the dual transformation in terms of the potentials to “rotate” the electric potential, $A_\mu$, into the magnetic potential, $C_\mu$, it is found that the magnetic soliton gets transformed into an electric soliton. Requiring that the electric charge of this soliton be equal in magnitude to the charge of other electrically charged particles (e.g. electrons, protons) we found that the original non-Abelian gauge
coupling must satisfy $g = 4\pi/e$. This made the mass of the electrically charged soliton several orders of magnitude lighter than its magnetic counterpart ($M_e \sim \frac{1}{137} M_W$ compared to $M_m \sim 137 M_W$). Taking $M_W$ to be of the order of the electroweak gauge boson masses, such an electrically charged, spin zero particle should have been observed. This might be taken to imply that such electric solitons are only of theoretical interest. However by using the spin from isospin mechanism \[14\] it may be possible to construct bound states of topological electric charge and gauge magnetic charge, which behave like spin 1/2 fermions \[15\], and have masses roughly in the range of the baryon masses. Here, however, our goal was simply to show that it is possible to get a topological electric charge from a non-Abelian gauge theory, since the $SO(3)$ group is currently not thought to play a fundamental role in particle physics.

Julia and Zee \[2\] have expanded the ansatz of Eq. (13) by setting the time component of the $SO(3)$ gauge field equal to $x^a J(r)/gr^2$. In this way they found that the field equations yielded a solution having both magnetic and electric charge (i.e. a dyon). However the Julia and Zee solution does not allow for the existence of an electric charge without the presence of a magnetic charge. In addition the topological stability arguments \[12\] that apply to the pure magnetic or electric solution do not apply to the dyonic solution, although plausibility arguments for its stability are given.

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