Dimensional Reduction of Dirac Operator

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Abstract

We construct an explicit example of dimensional reduction of the free massless Dirac operator with an internal SU(3) symmetry, defined on a 12-dimensional manifold that is the total space of a principal SU(3)-bundle over a four-dimensional (nonflat) pseudo-Riemannian manifold. Upon dimensional reduction the free twelve-dimensional Dirac equation is transformed into a rather nontrivial four-dimensional one: a pair of massive Lorentz spinor SU(3)-octets interacting with an SU(3)-gauge field with a source term depending on the curvature tensor of the gauge field. The SU(3) group is complicated enough to illustrate features of the general case. It should not be confused with the color SU(3) of quantum chromodynamics where the fundamental spinors, the quark fields, are SU(3) triplets rather than octets.

1 Introduction

It is well known that when we look for a solution with some symmetry, we can reduce the number of variables and thus simplify the problem of solving differential equations. The Schwarzschild solution of the nonlinear Hilbert-Einstein equation is a typical example. A point of view, different from this calculational aspect of symmetry, is essential for the so called "Kaluza-Klein approach." It is observed in the pioneer work of Kaluza (1921, English translation in \([1]\)) that there is one-to-one correspondence between the U(1)-invariant metrics on a five-dimensional manifold and the triples \{metric on four-dimensional manifold, linear connection with structure group U(1) (electromagnetic potential), scalar field\}. The scalar curvature of five-dimensional U(1)-invariant metric is equivalent to the Einstein-Maxwell action for the mentioned fields. This action describes the really observed interaction between gravity and electromagnetic field. This demonstrates the general idea: We consider a “simple” field and “simple” equations but in a “multidimensional” universe. Imposing some symmetry conditions, after dimensional reduction we obtain a set of fields with different nature involved in complicated differential equations. Our hope is that the fields and differential equations, obtained in this way, may describe a real process, and that this

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investigation may be a step to the unification of different interactions in nature. The natural generalizations of the Kaluza-Klein ansatz are considered in the literature: the group of symmetry $G$ is arbitrary, the group $G$ acts on a manifold as on a total space of a principal bundle, and the group $G$ acts on a manifold with one type orbits. See, for example [2].

In this paper the starting point is the free Dirac operator with an SU(3) symmetry defined on a twelve-dimensional Minkowski space that is interpreted as an SU(3) principal bundle over four-dimensional Minkowski space. Should we interpret the outcome in physical terms we should relate the structure group with the "flavor SU(3)" of the quark model, identifying the resulting SU(3)-octets Dirac particles with observed baryons. Such an interpretation would again be a nonstandard one however since, unlike the flavor SU(3) of the standard model, our structure group appears as a local gauge group in four-space-time. We prefer, in fact, to view the present paper as a mathematical model illustrating some surprising features of dimensional reduction.

Our purpose is to consider the simplest possible case because then the arising structures after dimensional reduction are imperative. The initial manifold, denoted by $E$ in the text, is the twelve-dimensional total space of a principal SU(3) bundle, which admits a real spinor bundle with standard fibre $\mathbb{R}^{64}$. In the real case, the spinor connection is uniquely defined if it is compatible with the Levi-Civita connection of the metric on $E$. For physical reasons we consider a complex spinor bundle, a complexification of the real-valued one. The spinor connection is also considered as a complexification of the real one. Thus we avoid the necessity to fix a connection with structure group $U(1)$. Further, when we fix the SU(3)-action on spinor fields we choose the trivial lifting. And thus we avoid some additional terms in the reduced Dirac operator. Also the scalar field in the Kaluza-Klein ansatz is taken to be constant - the Killing metric in the Lie algebra of SU(3). In this way, in the reduced Dirac operator there are only structures whose presence is necessary. We also point out the steps in which, imposing the symmetry, the new structures arise (the gauge field with structure group $G = SU(3)$, its curvature tensor, the Clifford algebra for a four-dimensional manifold, the four-dimensional Dirac operator, the spinor octets, the mass term etc.).

We choose the group of symmetry to be SU(3) acting freely on the 12D manifold because of its connection to the standard model and because we wanted the arising after the reduction gauge field to have structure group SU(3). In the same way one can obtain the dimensional reduction of Dirac operator when the symmetry group is an arbitrary connected Lie group acting freely on the multidimensional manifold.

The article is organized as follows.

In Section 2 the necessary constructions from differential geometry and the algebraic origin of the Kaluza-Klein ansatz are presented. We give the coordinate expression of the Levi-Civita connection for the metric in nonholonomic basis. These formulas are applied to the canonical basis of the SU(3)-invariant metric on $E$. This basis determines a horizontal subbundle $T^h(E) \hookrightarrow T(E)$. The subbundle $T^h(E)$ is invariant under the action of SU(3) and defines a linear connection (gauge field with structure group SU(3)). The components of the Levi-Civita connection for the SU(3)-invariant metric are calculated and they contain components of the gauge field and its stress tensor (eq. (16)).

In Section 3 the Dirac operator for the SU(3)-invariant metric (the Kaluza-Klein ansatz) is considered. The crucial moment here is that the sum $T(E) = T^h(E) \oplus T^v(E)$ is orthogonal with respect to the SU(3)-invariant metric. According to the classifying theorem for Clifford algebras, the Clifford algebra $\text{Cl}(T^z_z(E), g(x)) \approx M_{64}(\mathbb{R})$ is realized as a tensor product of the Clifford algebras of $T^{h}_z(E)$ and $T^{v}_z(E)$. So the standard fibre $\mathbb{C}^{64}$ of the spinor bundle on $E$ takes the structure of tensor product $\mathbb{C}^4 \otimes \mathbb{C}^{16}$.

In Section 4 we give the dimensional reduction of the Dirac operator for the SU(3)-invariant metric.
We introduce an action of SU(3) on the spinor bundle, compatible with the action of SU(3) on $T(E)$. This condition of compatibility does not fix uniquely the action of SU(3) on the spinors. So we choose, as we mentioned above, the simplest case in which the lifting of the SU(3) action on $E$ to the total space of the spinor bundle is trivial in the canonical basis.

In Section 5 we list the steps in the procedure of dimensional reduction where the new structures presented in the reduced Dirac operator arise.

2 Basic constructions and notations

Let $E$ be a smooth manifold, $g$ a metric on $E$ (with arbitrary signature), and $\nabla$ the corresponding Levi-Civita connection. Let $\{h_t\}$ be a (local) nonholonomic basis of $T(E)$ and $\{h^t\}$ the corresponding dual basis on $T^*(E)$. In this basis we have the following notation:

$\nabla(h_\beta) = \Gamma^\rho_{\alpha\beta}h^\alpha \otimes h_\rho$, 
$\nabla_{h_\alpha}(h_\beta) = \nabla_\alpha(h_\beta) = \Gamma^\rho_{\alpha\beta}h_\rho$, 
$[h_\alpha, h_\beta] = C^\rho_{\alpha\beta}h_\rho$, 
$g(h_\alpha, h_\beta) = g_{\alpha\beta}$,

$g(\nabla_\alpha(h_\beta), h_\gamma) = \Gamma^\rho_{\alpha\beta\gamma}g_{\rho\gamma} = \Gamma_{\alpha\beta\gamma}$, 
$g([h_\alpha, h_\beta], h_\gamma) = C^\rho_{\alpha\beta\gamma}g_{\rho\gamma} = C_{\alpha\beta\gamma}$.

The condition $\nabla(g) = 0$ and the requirement for the vanishing of the torsion reads:

$\nabla_{h_\mu}(h_\nu) - \nabla_{h_\nu}(h_\mu) = [h_\mu, h_\nu] \Rightarrow \Gamma_{\mu\nu\rho} - \Gamma_{\nu\mu\rho} = C_{\mu\nu\rho}$, 
$\nabla(g_{\mu\nu}(h_\rho)) = g(\nabla_\rho(h_\mu), h_\nu) + g(h_\mu, \nabla_\rho(h_\nu)) \Rightarrow \Gamma_{\rho\mu\nu} + \Gamma_{\rho\nu\mu} = h_\rho(g_{\mu\nu})$.

from here it follows that

$2\Gamma_{\alpha\beta\gamma} = C_{\alpha\beta\gamma} + C_{\gamma\alpha\beta} + C_{\gamma\beta\alpha} + h_\alpha(g_{\beta\gamma}) + h_\beta(g_{\alpha\gamma}) - h_\gamma(g_{\alpha\beta})$.

We follow the classical construction of the generalized Kaluza-Klein ansatz. The point structure of the ansatz is the description of the metric on the vector space $L$, which is the middle term in the short exact sequence:

$0 \rightarrow L_0 \stackrel{i}{\rightarrow} L \stackrel{\pi}{\rightarrow} L_1 \rightarrow 0$.

(3)

We realize this in coordinates by choosing a basis $\{e\} = \{f, e\}$ in $L$: $h_\mu = f_\mu$, $\mu = 1, 2, ..., m = \dim(L_1)$, $h_k = e_k$, $k = m + 1, ..., m + n = \dim(L_0)$. The vector space $L_0 = \text{span}(e_1, ..., e_n)$, and the vector space $L_1$ is identified with $\text{span}(f_1, ..., f_m)$ and $i(e_k) = e_k; j(e_k) = 0, j(f_\mu) = f_\mu$. Every splitting of the exact sequence (3) is given by a linear map $S: L_1 \rightarrow L$ with the property $j \circ S = 1$, i.e., is given by defining the vectors:

$\tilde{f}_\mu = S(f_\mu) = f_\mu - A^k_\mu e_k$.

(4)

In these formulas we have summation over repeated indices. Here the matrix $A^k_\mu$ is arbitrary. Every metric $g_k$ on $L$, for which the restriction on $i(L_0)$ is nondegenerate is uniquely determinate by the conditions:

$g_k(\tilde{f}_\mu, \tilde{f}_\nu) = g_{\mu\nu}$,
$g_k(\tilde{f}_\mu, e_k) = 0$,
$g_k(e_k, e_l) = g_{kl}$,

where $g_{0kl}$ and $g_{\mu\nu}$ are metrics on $L_0$ and $L_1$. In this manner we have one-to-one correspondence between the metrics on $L$, nondegenerated on $L_0$, and the triples \{metric on $L_1$, metric on $L_0$, splitting of (3)\}. In the basis $\{f_\mu, e_k\}$ the metric, defined by the equations (5) has components:

$\{g_k\} = \begin{pmatrix}
g_{\mu\nu} + A^i_\mu A^j_\nu g_{0ij} & A^i_\mu g_{0i}
g_{0kl} A^i_v & g_{0kl}
\end{pmatrix}$.

(6)
The above construction is the algebraic origin of the Kaluza-Klein ansatz. In the case of the general Kaluza-Klein ansatz this construction arises in the tangent space of each point of the manifold where the group of symmetry acts. More precisely, let \((E,p,M)\) be a principal bundle with structure group \(G = SU(3)\). We assume for simplicity that the principal bundle is trivial and the manifold \(M\) is isomorphic to \(\mathbb{R}^4\) as a topological manifold. We take a global trivialization \(E = M \times SU(3)\) and the right group action is \(R_\mathbf{g} : (x,z) \mapsto (x, zg)\) where \((x,z) \in M \times SU(3)\) and \(\mathbf{g} \in SU(3)\). We also fix coordinates on the total space \(E\), \((x^\mu, z^k) = (x,z), \mu = 1,2,3,4, k = 5,\ldots, 12, p(x^\alpha, z^k) = (x^\alpha)\). Because of the assumed triviality of \(M\) the coordinates \(x^\alpha\) are global. Let \(T^v(E) \hookrightarrow T(E)\) be the vertical subbundle, \(\mathbf{f}_\mu\) nonholonomic basis of \(T(M)\), \(\mathbf{e}_k\) - the fundamental fields on \(E\) corresponding to a basis \(\mathbf{e}_5, \ldots, \mathbf{e}_{12}\) of the Lie algebra \(su(3)\). The fields \(\mathbf{f}_\mu, \mathbf{e}_k\) form a nonholonomic basis on \(TE = T(M \times SU(3))\), and have the form

\[
\mathbf{f}_\mu = f_{\mu}^\nu(x) \frac{\partial}{\partial x^\nu}, \quad \mathbf{e}_k = e_k^i(z) \frac{\partial}{\partial z^i}.
\]

(7)

The natural exact sequence

\[
0 \rightarrow T^v(E) \rightarrow T(E) \rightarrow p^*(T(M)) \rightarrow 0
\]

(8)

realizes the exact sequence \([3]\) at the tangent space of each point of the manifold \(E\). Here \(p^*(T(M))\) is the pull back of the tangent bundle of \(M\) (see \([3]\)). Each metric \(g_{\mathbf{x}}\) on \(E\) can be written in the form:

\[
\{g_{\mathbf{x}}(x,z)\} = \begin{pmatrix}
g_{\mu\nu}(x,z) + A_{ij}^\mu(x,z) A_{ij}^\nu(x,z) g_{0k\ell}(x,z) & A_{ij}^\mu(x,z) g_{0\ell}(x,z) \\
g_{0k\ell}(x,z) & g_{0k\ell}(x,z)
g_{0\ell}(x,z)
\end{pmatrix},
\]

(9)

where \(g_{\mu\nu}(x,z) = g_{\mu}(x,z)(\mathbf{f}_\mu(x), \mathbf{f}_\nu(x))\), \(g_{0k\ell}(x,z) = g_{\ell}(x,z)(\mathbf{e}_k(x), \mathbf{e}_\ell(x))\). The vector fields \(\mathbf{f}_\mu(x,z) = \mathbf{f}_\mu - A_{k}^{\mu}(x,z)\mathbf{e}_k\) span a horizontal subbundle \(T^h(E) \hookrightarrow T(E)\) orthogonal to \(T^v(E)\) with respect to the metric \([9]\). The ansatz \([9]\) is convenient to describe the metrics on \(E\) invariant under the action of the group \(G = SU(3)\). The invariant metrics \(g\) on \(E\) have the form

\[
\{g_{\mathbf{x}}(x,z)\} = \begin{pmatrix}
g_{\mu\nu}(x) + A_{ij}^\mu(x) A_{ij}^\nu(x) g_{0k\ell}(x) & A_{ij}^\mu(x) g_{0\ell}(x) \\
g_{0k\ell}(x) & g_{0k\ell}(x)
g_{0\ell}(x)
\end{pmatrix},
\]

(10)

This is the Kaluza-Klein ansatz in our case. In this formula \(g_{\mu\nu}(x) = g(\mathbf{f}_\mu, \mathbf{f}_\nu)(x)\) is an arbitrary metric on \(M\). Because \(\mathbf{f}_\mu\) is a nonholonomic basis on \(M\), without loss of generality we can think that \(g_{\mu\nu}\) is in canonical form, i.e., \(\mathbf{f}_\mu\) are tetrad. This will be used in our calculation later. \(g_{0k\ell}(x)\) at each point \(x \in M\) is invariant metric on the Lie algebra \(su(3)\), i.e., \(g_0\) is a field defined on \(M\) taking values in the set of invariant metrics on the Lie algebra \(su(3)\). The vector fields \(\mathbf{f}_\mu(x,z) = \mathbf{f}_\mu(x) - A_{k}^{\mu}(x,z)\mathbf{e}_k\) span orthogonal horizontal subbundle \(T^h(E)\) which is invariant under the action of the structure group of the principal bundle \(G = SU(3)\). So \(A_{k}^{\mu}(x)\) define a linear connection in the principal bundle with a structure group \(G = SU(3)\). A classical result is that there is one-to-one correspondence between the \(G\)-invariant metrics on \(E\) and the triples \{metric on \(M\), linear connection with values in the Lie algebra of \(G\), “scalar field”\}. In the basis \(\{\mathbf{f}_\mu, \mathbf{e}_k\}\) for the metric \([10]\) we have

\[
\begin{align*}
g_{\mathbf{x}}(\mathbf{f}_{\mu}(x), \mathbf{f}_{\nu}(x)) &= g_{\mu\nu}(x), \\
g_{\mathbf{x}}(\mathbf{f}_{\mu}(x), \mathbf{e}_{k}(x)) &= 0, \\
g_{\mathbf{x}}(\mathbf{e}_{k}(z), \mathbf{e}_{\ell}(z)) &= g_{0\ell}(x).
\end{align*}
\]

(11)

The next step is to construct the Dirac operator on \(E\) corresponding to the metric \([10]\). To calculate the Levi-Civita connection \([2]\) of the metric \([10]\) we have to introduce the commutator coefficients for the
basis \{f_\mu, e_k\}:

\[ [f_\mu, f_\nu] (x) = C^\rho_{\mu\nu}(x) f_\rho(x), \]

\[ [f_\mu, e_k] = [f_\mu^\nu(x) \frac{\partial}{\partial x^\nu}, e_k^l(z) \frac{\partial}{\partial x^l}] = 0, \tag{12} \]

\[ [e_k, e_l] (z) = t_{kl}^m e_m(z). \]

Here \(C^\rho_{\mu\nu}(x)\) are determined by the choice of the nonholonomic basis \(f_\mu\) in \(T(M)\), \(t_{kl}^m\) are the structure constants of the Lie algebra \(su(3)\) for the basis \(\mathbf{e}_\gamma, ..., \mathbf{e}_{\gamma_{12}}\). The nonholonomic basis \(\{f_\mu, e_k\}\), because of (1), is convenient for the construction of the Dirac operator. By means of (12) and (1) we calculate

\[ [\hat{f}_\mu, \hat{f}_\nu] (x, z) = C^\rho_{\mu\nu}(x) \hat{f}_\rho(x) - F_{\mu}^k(x) e_k(z), \]

\[ [\hat{f}_\mu, e_k] (x, z) = -A^l_\mu(x) t^m_{lk} e_m(z), \]

\[ [e_k, e_l] (z) = t_{kl}^m e_m. \tag{13} \]

Here

\[ F_{\mu}^m = f_\mu(A_\nu^m) - f_\nu(A_\mu^m) + t_{\mu}^l A^m_\nu + C_{\mu\nu}^\rho A^m_\rho \tag{14} \]

is the curvature tensor of the linear connection, determined by the one-form \(A_\mu = A^m_\mu a_m\). Then the coefficients \(C_{\alpha\beta\gamma}\) in (2) for the basis \(\{f_\mu, e_k\}\) are

\[ C_{\mu\nu\rho} = g([f_\mu, f_\nu], f_\rho) = g_{\alpha\beta} C_{\mu\nu\rho}^\alpha \gamma, \quad C_{\mu\nu\rho} = -g_{0\rho k} F_{\mu\nu}^l = -F_{\mu\nu k}, \]

\[ C_{\mu\nu k} = -A^0_\mu(x) t^m_{\mu\nu k} = -C_{\mu\nu k}, \]

\[ C_{\mu k l} = -A^0_\mu(t_{kml} + t_{mkl}) = -C_{\mu k l}, \tag{15} \]

\[ C_{k l m} = t_{k l m}, \]

where \(t_{mkl} = g_{00} i^l t^{i m k}\). From (2) we obtain the components of Levi-Civita connection in the basis \(\{f_\mu, e_k\}\):

\[ \Gamma^E_{\mu\nu\rho} = \Gamma_{\mu\nu\rho}, \quad \Gamma^E_{\mu k l} = -\frac{1}{2} F_{\mu k l}, \quad \Gamma^E_{\mu k l} = -\frac{1}{2} A^m_\mu(t_{mkl} + t_{mlk}) + \frac{1}{2} F_{\mu}^k(g_{0 k}), \]

\[ \Gamma^E_{k l m} = \frac{1}{2}(t_{k l m} + t_{mkl} + t_{mlk}). \tag{16} \]

In these formulas \(\Gamma^E_{\mu\nu\rho}\) are the components of the Levi-Civita connection in the basis \(\{f_\mu\}\) for the metric \(g\) on \(M\).

3 Dirac operator for the Kaluza-Klein metric

To describe the Dirac operator for the Kaluza-Klein metric (10) we need some preliminary constructions. Let \(L\) be a real vector space and \(g\) - a metric on \(L\) of type \((p, q)\); it is the canonical embedding \(\vartheta : L \rightarrow \text{Cl}(\eta)\)

\[ \vartheta : L \rightarrow \text{Cl}(\eta) \tag{17} \]

of the vector space \(L\) into the corresponding Clifford algebra \(\mathbb{H}\). \(\vartheta(\textbf{x})^2 = g(\textbf{x}, \textbf{x})\mathbf{1}, \textbf{x} \in L\), where \(\mathbf{1}\) is the unit of the algebra \(\text{Cl}(\eta) \equiv \text{Cl}^{p,q}\). If \(a_1, ..., a_n\) is an arbitrary basis of \(L\), \(\vartheta(a_1) = \gamma_1, \gamma_2 = \gamma_1 \gamma_3 = 2\eta_{ij} \mathbf{1}, \eta_{ij} = g(a_i, a_j)\). If \(a_1, ..., a_n\) is oriented orthonormal basis, the volume element \(\omega = \gamma_1...\gamma_n\) is uniquely determined and \(\omega^2 = \pm 1\). We will assume that \(L\) has fixed orientation. The symmetry of the metric on \(L\) gives rise to some structures on the spinor bundle on \(L\). In order to describe them we need some facts for the classification of the Clifford algebras.

The first step in the classification and realizations of the Clifford algebras for arbitrary metric is the following statement (\(\mathbb{H}\), \(\mathbb{K}\)):

If \((L_1 \oplus L_2, \eta_1 \oplus \eta_2)\) is an orthogonal direct sum of metric vector spaces
(L_1, \eta_1) and (L_2, \eta_2) then Cl(\eta_1 \oplus \eta_2) = Cl(\eta_1) \hat{\otimes} Cl(\eta_2), where \hat{\otimes} is the \mathbb{Z}_2-graded tensor product of the naturally \mathbb{Z}_2-graded Clifford algebras. In some exceptional cases the \mathbb{Z}_2-graded tensor product may be replaced with the usual tensor product. In our example is realized one of these exceptional cases.

Let \text{dim}(L) = p + q = 2k be even. We say that Cl(\eta) > 0 or Cl(\eta) < 0 if \omega^2 = +1 or \omega^2 = -1. Let (L_1, \eta_1) and (L_2, \eta_2) be vector spaces with metrics \eta_1 and \eta_2, and (L_1 \oplus L_2, \eta_1 \oplus \eta_2) be an orthogonal direct sum. Then (see [4], [5])

\begin{align}
\text{Cl}(\eta_1) > 0 \iff \text{Cl}(\eta_1 \oplus \eta_2) = \text{Cl}(\eta_1) \otimes \text{Cl}(\eta_2), \\
\text{Cl}(\eta_1) < 0 \iff \text{Cl}(\eta_1 \oplus \eta_2) = \text{Cl}(\eta_1) \otimes \text{Cl}(-\eta_2).
\end{align}

Let \{a_i\}, \{a_j\} be bases in L_1 and L_2. The isomorphisms in (18) are given by

\begin{align}
\gamma_i \otimes 1_2 &\mapsto \gamma_i = \vartheta(a_i, 0), \ i = 1, 2, ..., n_1 = \text{dim}(L_1) \\
\omega_i \otimes \gamma_j &\mapsto \gamma_{n_1+j} = \vartheta(0, a_j), \ j = 1, 2, ..., n_2 = \text{dim}(L_2).
\end{align}

These isomorphisms give the classification of the Clifford algebras.

For physical reasons we will consider complex spinor fields and we will need a complexification of the Clifford algebra:

\[ Cl^{p,q} \otimes \mathbb{C} = Cl(L \otimes \mathbb{C}, \eta) = Cl^p, \quad n = p + q. \]

It is known ([4], [5]) that Cl^{2k} \cong M_{2^k}(\mathbb{C}), Cl^{2k+1} \cong M_{2^k}(\mathbb{C}) \oplus M_{2^k}(\mathbb{C}). In our case the Clifford algebra Cl^{12} is even and thus Cl^{12} \cong M_{64}(\mathbb{C}) has only one simple module. We have the following isomorphisms:

\begin{align}
\text{Cl}^{1,3} &\cong M_4(\mathbb{R}), \quad \text{Cl}^{1,3} < 0 \\
\text{Cl}^{0,8} &\cong Cl^{8,0} \cong M_{10}(\mathbb{R}), \quad \text{Cl}^{0,8} > 0, \quad \text{Cl}^{8,0} > 0.
\end{align}

So we have

\[ Cl^{1,11} = Cl^{1,3} \otimes Cl^{8,0} \cong M_4(\mathbb{R}) \otimes M_{10}(\mathbb{R}). \]

Let (E, g) be an oriented even dimensional manifold with metric g, \text{sign}(g) = (p, q). This means that the tangent bundle over E has a cocycle \psi_{\alpha\beta}(x) \in \text{Aut}(\mathbb{R}^{2k}, \eta) = O(p, q). An element \lambda \in \text{Aut}(\mathbb{R}^{2k}, \eta) uniquely determines an element \tilde{\lambda} \in \text{Aut}(Cl^{p,q}). So \tilde{\psi}(x) \in \text{Aut}(Cl^{p,q}) is a cocycle which defines the Clifford bundle Cl(TE) over E. In this bundle the fiber Cl(TE)_x is the Clifford algebra for the vector space (T_x E, g(x)). The standard fiber of the complex Clifford bundle Cl^{C}(TE) is Cl^{2k} = M_{2^k}(\mathbb{C}). A spin^C structure on E is equivalent to a bundle \zeta of simple complex modules over the Clifford bundle Cl^{C}(TE). We will point out some details in the construction of the complexified bundle \zeta because they are important in our later study on the structures arising in \zeta. Let \tilde{L} = \vartheta(L) \hookrightarrow Cl(\eta) be the image of the linear space L in the Clifford algebra. \vartheta : L \longrightarrow \tilde{L} is an isomorphism and we will identify L and \tilde{L} by means of \vartheta. The Clifford group \mathcal{F} is defined by

\[ \mathcal{F}^{p,q} = \{ c \in Cl^*(p, q) \mid c \tilde{L} c^{-1} \subset L \}, \]

where Cl^* is the set of invertible elements. In the complexified case we have similarly

\[ \mathcal{F}^{2k} = \{ c \in Cl^{2k} \mid c \tilde{L} c^{-1} \subset L \}. \]

The linear map \( j(c) : L \longrightarrow \tilde{L} : u \longrightarrow j(c)(u) = c u c^{-1} \) is orthogonal and \( j : \mathcal{F} \longrightarrow O(p, q) \) is surjective and gives the exact sequence

\[ 1 \longrightarrow Cl^* \longrightarrow \mathcal{F}^{2k} \xrightarrow{j} O(p, q) \longrightarrow 1. \]
And in the real case we have

\[ 1 \rightarrow \mathbb{R}^* \rightarrow \mathcal{F}_p^q \rightarrow O(p, q) \rightarrow 1. \]  

(26)

A spin\(^C\) bundle \(\zeta\) is a complex vector bundle on \(E\) and each fiber \(\zeta_{(x,z)}\), \((x,z) \in E = M \otimes G\), is a simple complex module over the algebra \(\text{Cl}(T(x,z)E)\). Let \(\psi_{\alpha\beta}(x) \in \text{Aut}(\mathbb{R}^{2k}, \eta)\) be a cocycle of \(T(E)\) and \(\widetilde{\psi}_{\alpha\beta}(x) \in \text{Aut}(\text{Cl}^{2k})\) be the cocycle of \(\text{Cl}^{2k}(TE)\). Because the standard fiber of \(\text{Cl}(TE)\) is \(\text{Cl}^{2k} \cong M_{2k}(\mathbb{C}) \cong \text{Hom}((\mathbb{C}^{2k}), \mathbb{C}^{2k})\), and \(\mathbb{C}^{2k}\) (the standard fiber of the spinor bundle \(\zeta\)) is the simple module of \(M_{2k}(\mathbb{C})\), the spin\(^C\) bundle has a cocycle \(\varphi_{\alpha\beta}(x) \in \text{Aut}(\mathbb{C}^{2k}) \subset M_{2k}(\mathbb{C}) \cong \text{Cl}^{2k}\) of invertible elements \(\varphi_{\alpha\beta} \in \mathcal{F}^{2k}\) and

\[ j(\varphi_{\alpha\beta}(x)) = \psi_{\alpha\beta}(x). \]  

(27)

In general, not every manifold admits a spin\(^C\) structure, and even if it admits there may exist different spin\(^C\) structures. In our example the base manifold has only one (up to isomorphism) spin\(^C\)-structure.

The Dirac operator

\[ D : C^\infty(\zeta) \rightarrow C^\infty(\zeta) \]  

(28)

is determined by a linear connection on the spinor bundle and the requirement that its symbol \(\sigma(D) : T^*(E) \rightarrow \zeta^* \otimes \zeta\) be the unique irreducible representation of the Clifford algebra \(\text{Cl}^{2k}\) at each fibre (see [11]).

In our example we choose \(E\) to be a total space of a principal SU(3) bundle over \(M\) and \(M\) to be isomorphic to \(\mathbb{R}^4\) as a topological manifold. We also choose the metric on \(M\) to be the Killing metric on the Lie algebra \(su(3)\). And in the real case we have \(\text{Cl}(TE) = T^h(E) \oplus T^v(E)\) is orthogonal. And from (17)

\[ \text{Cl}(TE) = \text{Cl}(T^h E) \otimes \text{Cl}(T^v E), \]  

(29)

i.e., the standard fiber of \(\text{Cl}(TE)\) is \(\text{Cl}^{1,3} \otimes \text{Cl}^{0,8}\) in the real (Majorana) case and \(\mathbb{C}^4 \otimes \mathbb{C}^8\) in the complexified case. So the standard fiber of the spinor bundle \(\zeta\) is isomorphic to \(\mathbb{R}^4 \otimes \mathbb{R}^{16}\) in the Majorana case and \(\mathbb{C}^4 \otimes \mathbb{C}^8\) in the complexified case. In the nonholonomic basis [7] of \(TE\) according to (19) we have

\[ f_{\mu}(x) \mapsto \gamma_{\mu} \otimes 1 \in \text{Cl}(T^h_{(x,z)}E) \oplus T^v_{(x,z)}E) = \text{Cl}(T^h_{(x,z)}E, g) \otimes \text{Cl}(T^h_{(x,z)}E, -g) \]

\[ e_k(x) \mapsto \omega \otimes \gamma_k \in \text{Cl}(T^h_{(x,z)}E) \oplus T^v_{(x,z)}E) = \text{Cl}(T^h_{(x,z)}E, g) \otimes \text{Cl}(T^h_{(x,z)}E, -g). \]  

(30)

In the real case the Levi-Civita connection determines a unique connection on the Majorana spinor bundle. Let \(\nabla\) be the Levi-Civita connection for the metric \(g_\eta\) on \(E\) and \(h_\alpha\) be an orthonormal with respect of \(g_\eta\) nonholonomic basis of \(TE\). If

\[ \nabla_{\eta_{\alpha}}(h_\beta) \equiv \nabla_{\alpha}(h_\beta) = \gamma_{\alpha \beta}^E h_\rho, \]  

(31)

then in the real case there is a unique corresponding connection on \(\zeta\),

\[ \nabla_{h_\alpha} = h_\alpha + S_\alpha, \]

\[ S_\alpha = -\frac{1}{4} \Gamma_{\rho \sigma}^E \gamma^\rho \gamma^\sigma = -\frac{1}{4} \Gamma_{\rho \sigma} \gamma^\rho \gamma^\sigma, \]

\[ \Gamma_{\alpha \beta \gamma} = g^E_{\sigma \rho} \Gamma_{\alpha \beta}^E, \quad g^E_{\alpha \beta} = g^E(h_\alpha, h_\beta) = \text{const}. \]  

(32)

If in the complexified case we want to have a soldering between the parallel transport in \(TE\) and \(\zeta\) there remains freedom for the choice of the \(U(1)\)-connection. In our example we fix the complexification of
Then the Dirac operator reads:

$$D = \Gamma^\mu E_\mu = g^{E\mu}E_{\mu} = \gamma^E\mu(h_\mu + S_\mu), \quad \Gamma^\mu E_\mu = \gamma^E\mu(h_\mu + S_\mu). \quad (33)$$

$$\text{For the Kaluza-Klein metric we specify } g_{\mu \nu} = g(f_\mu, f_\nu) = \text{const} = \text{diag}(-1, 1, 1, 1), \quad \text{on } M. \text{ For simplicity we take the "scalar fields" } g_{0kl} \text{ to be constant: the Killing form in the Lie algebra } su(3). \text{ In the global basis } \{f_\mu, e_k\} \text{ the metric } (10) \text{ has the form:}$$

$$\{g^E\} = \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
\end{pmatrix}. \quad (34)$$

The coefficients of the Levi-Civita connection of the metric (34) are defined in the general case in (16). Because of the form of the metric in (34) the Clifford bundle \(C(TE)\) has global generators:

$$\gamma^E_\mu = \gamma^E \otimes 1, \quad \mu = 1, 2, 3, 4, \quad \gamma^E_k = \omega \otimes \gamma_k, \quad k = 5, ..., 12, \quad (35)$$

where \(\gamma_\mu\) are generators of \(C^{1,3}\), \(\gamma_k\) are generators of \(C^{0,8}\). After these specifications the Dirac operator (32), (33) for the Kaluza-Klein metric (10) reads:

$$D = \Gamma^E_\mu \nabla f_\mu + \gamma^E_k \nabla f_k = (\gamma^E \otimes 1)(f_\mu + S_\mu) + (\omega \otimes \gamma^E)(e_k + S_k)$$

$$= (\gamma^E \otimes 1)(f_\mu - A^E_k e_k) + (\omega \otimes \gamma^E)e_k$$

$$+ (\gamma^E \otimes 1)(\Gamma^E_{\mu \nu \rho} \gamma^\nu \gamma^\rho \otimes 1 + \Gamma^E_{\mu \kappa \lambda} \gamma^\kappa \otimes 1)(\omega \otimes \gamma^E)$$

$$+ \Gamma^E_{\mu \kappa \lambda}(\omega \otimes \gamma^E)(\gamma^\nu \otimes 1) + \Gamma^E_{\mu \kappa \lambda}(\omega \otimes \gamma^E)(\omega \otimes \gamma^E)$$

$$+ \Gamma^E_{\kappa \lambda}(\omega \otimes \gamma^E)(\gamma^\mu \otimes 1) + \Gamma^E_{\kappa \lambda}(\omega \otimes \gamma^E)(\omega \otimes \gamma^E)) \quad (36)$$

Here \(\gamma^E = g^{E\sigma} \gamma_\sigma\) and \(-g_{0k} \gamma_k\). Using (16) we obtain

$$D = (\gamma^E \otimes 1)(f_\mu - A^E_k e_k) + (\omega \otimes \gamma^E)e_k$$

$$- \frac{1}{4} (\gamma^E \otimes 1)(\Gamma^E_{\mu \nu \rho} \gamma^\nu \gamma^\rho \otimes 1 + F_{\mu \nu \rho} \omega \gamma^\nu \otimes \gamma^\rho$$

$$- \frac{1}{2} A^E_{\mu}(t_{mkl} - t_{mk}) + \gamma^E \otimes 1) - \frac{1}{4} (\omega \otimes \gamma^E)(\frac{1}{2} F_{\mu \nu \rho} \gamma^\nu \gamma^\rho \otimes 1$$

$$- A^E_\mu (t_{mk} + t_{ml}) - \omega \gamma^E \otimes \gamma^E - \frac{1}{2} (t_{klt} + t_{kl} + t_{km}) 1 \otimes \gamma^E) \quad (37)$$

The Dirac operator (38) for the Kaluza-Klein metric (10) acts on spinor fields which have 64 components. Due to (39) the standard fiber of the complex spinor bundle is \(\mathbb{C}^4 \otimes \mathbb{C}^{16}\).
4 Dimensional reduction of the Dirac operator

We need to specify the action of the symmetry group $G = SU(3)$ on the spinor bundle. The action $R_g(x, z) = (x, zg)$ on the base $E = M \times SU(3)$ of the spinor bundle $\zeta$ must be lifted to a bundle morphism action on the spinor bundle $\zeta$. This lifting must be in agreement with the action of $SU(3)$ on $T(E)$. More precisely, let $R_g : (x, z) \to (x, zg)$ be the action of $g \in G = SU(3)$ on $E$. For the Kaluza-Klein metric (10) the tangent lifting $R_g^T : T(x, z)(E) \to T(x, zg)(E)$ is an isometry. In our trivialization, in the basis \{\mathbf{f}_\mu, \mathbf{e}_k\} the tangent lifting $R_g^T : \mathbb{R}^{12} \to \mathbb{R}^{12}$ is the identity. Let $F : \mathbb{C}^4 \otimes \mathbb{C}^{16} \to \mathbb{C}^4 \otimes \mathbb{C}^{16}$ be the lifting of the action $L_g$ to the complexified spinor bundle $\zeta^C$. $F_g$ must satisfy (in the same trivialization)

$$j^C(F_g) = R_g^T = 1$$

with $j^C$ given from (25). Our purpose is to construct explicitly the simplest example of the Dirac operator acting on spinors over four-dimensional manifold. So we fix $F_g = 1$ and then the action of $G = SU(3)$ on spinor fields, i.e., the sections on $\zeta$, is

$$R_g(\psi)^{\mu a}(x, z) = \psi^{\mu a}(x, zg) .$$

(40)

The Dirac operator (38) for the Kaluza-Klein metric (10) is $SU(3)$-invariant, when the action of $SU(3)$ on spinor fields is specified as in (40). For the invariant spinor fields, from (40) we have

$$R_g(\psi)^{\mu a} = \psi^{\mu a} \Rightarrow \psi^{\mu a}(x, zg) = \psi^{\mu a}(x, e) \equiv \psi^{\mu a}(x) .$$

(41)

The set of all invariant spinor fields $C^\infty(\zeta |_M)$ is identified, due to (41), with $C^\infty(\zeta |_{M \times \{e\}}) \equiv C^\infty(\zeta |_{M}).$ The dimensional reduction of the Dirac operator (38) is a restriction of (38) on the set of $SU(3)$-invariant spinor fields and we obtain the reduced Dirac operator $D_r$:

$$D_r : C^\infty(\zeta |_{M}) \to C^\infty(\zeta |_{M}).$$

(42)

To calculate $D_r$ we have to put in (38) a $SU(3)$ invariant spinor field. For invariant spinor fields $\psi^{\mu a}(x)$ we have from (41), $\mathbf{e}_k(\psi^{\mu a}) = 0$. For the reduced Dirac operator $D_r$ we obtain

$$D_r = (\gamma^\mu \otimes 1)(\mathbf{f}_\mu - \frac{1}{4} \Gamma_{\mu \nu \rho}(x) \gamma^\nu \gamma^\rho) - \frac{1}{8} F_{\mu \nu k}(x) \gamma^\mu \gamma^\nu \otimes \gamma^k + \frac{1}{4} A^{m}_\mu(x) t_{mlk} \gamma^\mu \otimes \gamma^k \gamma^l + \frac{1}{4} t_{mlk} \omega \otimes \gamma^k \gamma^l \gamma^m,$$

(43)

where the coefficients $t_{mlk}$ are totally antisymmetric in all indices. The reduced Dirac operator (43) acts on the sections $\psi \in C^\infty(\zeta |_{M}).$ The standard fiber is $\mathbb{C}^4 \otimes \mathbb{C}^{16}$. The bundle $\zeta |_{M}$ is canonically isomorphic to $\zeta^M \otimes \zeta^{SU(3)}$:

$$\zeta |_{M} \approx \zeta^M \otimes \zeta^{SU(3)} ,$$

(44)

where $\zeta^M$ is the (complex) spinor bundle on $M$ and $\zeta^{SU(3)}$ is a vector bundle on $M$ with standard fiber $\mathbb{C}^{16}$ considered as a simple module of the Clifford algebra $\mathfrak{Cl}^{0,8}$ corresponding to the Lie algebra $su(3)$ with the Killing metric $g$. \{\mathbf{f}_\mu\} is a nonholonomic global basis of $T(M)$ and tetrada for the metric $g$. $\Gamma_{\mu \nu \rho}(x)$ are the components of the Levi-Civita connection of $g$ in the basis \{\mathbf{f}_\mu\} and $-\frac{1}{4} \Gamma_{\mu \nu \rho}(x) \gamma^\nu \gamma^\rho$ are the components of the spinor connection in $\zeta^M$, so

$$_{M}D = (\gamma^\mu \otimes 1)(\mathbf{f}_\mu - \frac{1}{4} \Gamma_{\mu \nu \rho}(x) \gamma^\nu \gamma^\rho),$$

(45)

is the Dirac operator for the metric $g$ acting on spinor fields with isotopic indices. $A^{m}_\mu(x)$ is a gauge field with values in the Lie algebra $su(3)$ and $F_{\mu \nu k}(x)$ is its curvature tensor. We can write (43) in the form

$$D_r = (\gamma^\mu \otimes 1)(\mathbf{f}_\mu - \frac{1}{4} \Gamma_{\mu \nu \rho}(x) \gamma^\nu \gamma^\rho + \frac{1}{4} A^{m}_\mu t_{mlk} 1 \otimes \gamma^k \gamma^l) - \frac{1}{8} F_{\mu \nu k} \omega \gamma^\mu \gamma^\nu \otimes \gamma^k + \omega \otimes \epsilon ,$$

(46)
The group of symmetry $T^v$ vertical subbundle $g$ for arbitrary metric structures of new type arise. In this example, $E$ fields. In the spirit of this idea we comment here on the steps in the reduction procedure, where the one-type field in a multidimensional case, but having some symmetry, and considered only on the invariant involved in complicated differential equations may be considered as “simple” differential equations for $\epsilon = \frac{1}{2} t_{kln} \gamma^k \gamma^l \gamma^n$.

The interpretation of (46) is that the reduced free massless SU(3)-invariant Dirac operator on the manifold $\mathbb{C}^6$ is isomorphic to $\mathbb{R}^{16}$. $\mathbb{C}^{16}$ is the unique simple module of the Clifford algebra $\text{Cl}(\mathfrak{su}(3), -g_0) = \mathbb{C}^{0,8} = M_{16}(\mathbb{R})$. The algebra $\mathfrak{su}(3)$, as a real vector space, is isomorphic to $\mathbb{R}^8$ and $g_0$ is the negative defined Killing metric. In the chosen basis $\{\hat{e}_k\}$ of $\mathfrak{su}(3)$, $g_0 = \text{diag}(-1,\ldots,-1)$. The group of symmetry of the Killing metric $g_0$ on $\mathfrak{su}(3)$ (considered as a vector space, isomorphic to $\mathbb{R}^8$) is $O(8)$. According to the standard procedure ([4], [5]), the Lie algebra $\mathfrak{o}(8)$ has a complex spinor representation in $\mathbb{C}^{16}$, which is a direct sum of two irreducible representations and $\mathbb{C}^{16} = \mathbb{C}^8 \oplus \mathbb{C}^8$. These representations are realized on the eigenspaces of the operator $\omega_0 = \prod_{k=4}^{12} \gamma^k$. Let $\rho : O(8) \to \text{End}(\mathbb{C}^{16})$ be the spinor representation, $s \in \mathfrak{o}(8)$ and $s(\hat{e}_i) = s_i^j \hat{e}_j$. Then

$$\rho(s) = -\frac{1}{4} s_{ij} \gamma^i \gamma^j. \quad (47)$$

But the Lie algebra $\mathfrak{su}(3)$ has a natural adjoint representation: $b \in \mathfrak{su}(3), ad(b) \in \text{End}(\mathfrak{su}(3) \approx \mathbb{R}^6)$,

$$ad(\hat{e}_k)(\hat{e}_i) = [\hat{e}_k, \hat{e}_i] = t^j_{ki} \hat{e}_j. \quad (48)$$

The adjoint representation $ad$ takes values in the Lie algebra $\mathfrak{so}(8)$, i.e. $\{t_k\} \in \mathfrak{so}(8)$. So we can take the composition of the two natural representations:

$$\rho \circ ad : \mathfrak{su}(3) \to \text{End}(\mathbb{C}^{16})$$

$$(\rho \circ ad)(\hat{e}_k) = -\frac{1}{4} t_{ijkl} \gamma^i \gamma^j. \quad (49)$$

This is a representation of $\mathfrak{su}(3)$ in $\mathbb{C}^{16}$ which is a direct sum of two eight-dimensional representations of $\mathfrak{su}(3)$. So the bundle $\zeta^{\mathfrak{su}(3)}$ on $M$ is a Whitney sum of two eight-dimensional bundles. Due to this fact, the SU(3) invariant spinors, i.e., the sections $C^\infty(\zeta \mid_M) = C^\infty(C^M \otimes \zeta^{\mathfrak{su}(3)})$ have the natural interpretation as two $\mathfrak{su}(3)$ spinor octets.

The main result of this paper is that the free massless SU(3)-invariant Dirac operator on the manifold $E$ (the total space of principal SU(3)-bundle on four-dimensional manifold $M$) after dimensional reduction is equivalent to the Dirac operator on four-dimensional manifold $M$, acting on two SU(3) spinor octets, in the presence of gravitational field, external SU(3) gauge field with a source depending on the curvature tensor of the SU(3) gauge field and mass term as it is in (10).

5 Comments

One of the ideas of the “Kaluza-Klein approach” is that a collection of fields with different nature involved in complicated differential equations may be considered as “simple” differential equations for one-type field in a multidimensional case, but having some symmetry, and considered only on the invariant fields. In the spirit of this idea we comment here on the steps in the reduction procedure, where the structures of new type arise. In this example $E$ is 12-dimensional manifold with trivial tangent bundle. For arbitrary metric $g_{\xi}$ on $E$ there is just one spinor structure and the spinor fields have 64-components. The group of symmetry $G = \text{SU}(3)$ acts on $E$ as on a total space of a principal bundle. This separate the vertical subbundle $T^v(E) \hookrightarrow T(E)$. The metric under consideration $g_{\xi}$ determines a horizontal subbundle
$T^h(E) \hookrightarrow T(E)$ as an orthogonal complement of $T^v(E)$. The metric $g_x$ is $G$-invariant, so the horizontal subbundle $T^h(E)$ is $G$-invariant and is a linear connection with structure group $G = SU(3)$. Further, the linear connection in the spinor bundle, coming from the Levi-Civita connection for $g_x$ and needed for the Dirac operator, is expressed in terms of this SU(3)-connection. This leads to the appearance of the SU(3) gauge field and its stress tensor. These are classical results for the Kaluza-Klein ansatz. The orthogonal splitting $T(E) = T^h(E) \oplus T^v(E)$ according to the classifying theorem for Clifford algebras leads to the representation (29) and to appearance of spinors on four-dimensional base manifold after the reduction. The metric in the vertical subspace is the Killing metric. There is a natural adjoint representation of Lie algebra $su(3)$ on itself, orthogonal to the Killing metric and a natural representation of the orthogonal group of the Killing metric on corresponding 16-dimensional spinors. Due to the representations (29), this leads to the appearance of two SU(3) spinor octets after the reduction. Finally, the $G$-invariance of the spinor fields in the simple case that we consider leads to vanishing of the vertical derivatives and we obtain the reduced operator acting on the fields defined on the four-dimensional base manifold.

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