An improved method of using equilibrium profile to design radial tires

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Abstract
Tire performances are strongly influenced by cross section profile, and equilibrium profile has been a focus of radial tire researches, however, how to use the profiles to design radial tires has been reported rarely due to reasons of secrecy. This paper describes a practical method of using equilibrium profile to design radial tires, which is based on the restricted dimensions by tire standards. Based on the membrane model and minimum energy principle, equations of equilibrium profiles of radial tires with flat and curved belts are obtained by using variational approach. A method of designing carcass contours by using these equations is developed, in contrast to earlier methods the proposed method makes the width rather than the height of the point with maximum width on the profiles as the input parameter, and the restriction lengths of carcass by the belt are also calculated at the same time. For 175R14 tire, this paper’s results are compared with Akasaka’s, the accuracy of this method is confirmed and the effects of belt radius on profiles are revealed. Four different designs of 295R22.5 tire are analyzed by finite element method to verify the benefits of this method. This method can be used to design both tires with larger and smaller aspect ratios, and it will offer a strong guide for designing radial tires.

Key words: Equilibrium profiles, Radial tires, Design method, Tire design, Tire model

1. Introduction
Radial tires are widely used in modern vehicles due to their good performances, such as excellent durability, outstanding comfortable, powerful traction, flexible handling and lower rolling resistance. There are several factors affecting the performances of tires, for example, materials, structures, and manufacturing processes. Radial tire structures are mainly tire contour, belt and bead. Tire contour is one of the most important factors, because it forms the inner cavity and determines directions of forces caused by inflation pressure (Clark, 1981). In general, sidewall of radial tire is made of a single layer of rubber-cord composite, its bending stiffness is very small, and will have a large deformation when inflated by a higher pressure, however, at last, it will reach to a stable shape, which is called equilibrium profile. So, if the initial contour is equilibrium profile the deformation of tire contour will be minimized, and the stress caused by deformation in rubber materials will become very small, which could improve the durability (Bauer, 2013). That is why so many researchers have carried out a lot of works to study the equilibrium profile (Lee, et al., 2011). Although some optimizations can be taken by using finite element method (Kaliske, et al., 2013; Tanaka and Ohishi, 2010; Yang and Olatunbosun, 2012), a complete global optimization is still difficult due to the complexity of tire structures, so the researches on initial design are necessary.

Generally speaking there are three methods for obtaining the equilibrium profile. One was based on physical...
models, which are mainly membrane (Purdy, 1963; Ghoreishy, 1993), net (Day and Gehman, 1963), shell (Bozdog and Olson, 2005) and composite (Akasaka, 1981) models, force balance principle were used in these models. Another is based on geometric model (Koutny, 1981), minimum energy principle was used and the force balance was not involved, it was much simple with respect to physical models. The third was based on finite element model, and it has been applied to analyze tire successively (Ghoreishy, 2008; Wood et al., 2012). The equations obtained by different methods are various, most of them are very complex, and they are not convenient and intuitive to be used for designing tires.

Currently, there are two main approaches in applying equilibrium profiles to design tires. The first is also the main approach (Rivlin, 1958; Robecchi, 1973) emphasizes that only sidewall could be seen to deform freely and be described by equilibrium profile, so, a demarcation point (point Q in Fig. 1) between crown and sidewall must be defined, then make the curves of crown to be tangent to sidewall at point Q. The other method was used by Frank and Akasaka (Akasaka, 1981), a parameter called allotment ratio of pressure was introduced into the equation, it could control the curve of belt, and the demarcation point between belt and carcass was also needed.

![Fig. 1](image)

The sketch of radial tire structure

However, the demarcation point has great influence on the profiles, it has been given by experience, and researches carried on determining the demarcation point in the stage of initial design are rarely. So, in this paper, a simple method is introduced to use equilibrium profile to design tires, and the demarcation point between carcass and belt or the restriction width of carcass by the belt can be obtained at the same time. The paper is organized as follows. Firstly, the equilibrium profiles of carcass with flat and curved belts are deduced by using geometric models. Then, the method of applying equilibrium profile to tire design is built, and numerical analysis of 175SR14 tire and FEA of four 295R22.5 tire with different curves of carcasses are done. Finally, discussions about the results of numerical examples and FEA simulations are reported.

2. Nomenclature

- $r$: Radial coordinate of point on the tire profile
- $z(r)$: Abscissa of point on the tire profile
- $L$: Half-length of the profile
- $A$: Highest point of the meridian profile
- $B$: Point of maximum width on the meridian profile
- $C$: Lowest point of tire profile
- $K$: Demarcation point of belt and carcass on the profile
- $r_A$: Radial coordinate of point A
- $r_C$: Radial coordinate of point C
- $r_B$: Radial coordinate of point B
- $r_K$: Radial coordinate of point K
- $z_A$: Abscissa of point A
- $z_C$: Abscissa of point C
- $z_B$: Abscissa of point B
3. Equilibrium profiles of radial tires

Radial tire is an axisymmetric geometry, which can be simplified as a two-dimensional curve rotating about central axis, as shown in Fig. 2. According to the minimum energy principle, if the length of profile is a constant, the volume of the rotating body should be maximum under inflation.

The curve on right side of \( r \) axis can be expressed as \( z = z(r) \), and half-length of the profile is \( L \) (the length of the profile between points A and C), assuming the length does not change before and after inflation. In fact, if the elongation rate of cord under given inflation pressure is known, the hypothesis can be removed, just adding the elongation to \( L \), it becomes \( (1+\varepsilon)L \). The \( L \) can be obtained as

\[
L = \int_{r_c}^{r_t} \sqrt{1+(z')^2} \, dr
\]  

(1)

The volume of the geometry can be given as

\[
V(z) = 4\pi \int_{r_c}^{r_t} rz(r) \, dr
\]

(2)
As the hypothesis mentioned before, the curve should meet the condition as

\[ V(z) \rightarrow \max |L| = \text{constant} \] (3)

The variational approach will be used to solve this problem (variational approach – isoperimetric problem) (Liberzon, 2012). Lagrange function is built as

\[ H = rz + \lambda \sqrt{1 + (z')^2} \] (4)

If the geometry has the largest volume, the Eq. (4) must satisfy the differential equation as

\[ \frac{d}{dr} \left( \frac{\partial H}{\partial z'} \right) - \frac{\partial H}{\partial z} = 0 \] (5)

Substituting Eq. (4) into Eq. (5), we can obtain that

\[ z' = \left( \frac{1}{2} r^2 + c_1 \right) \sqrt{\lambda^2 - \left( \frac{r^2}{2} + c_1 \right)^2} \] (6)

The boundary conditions are \( z'(r_B) = 0 \) and \( z'(r_A) = \infty \), two equations can be established and combined to solve \( \lambda \) and \( c_1 \).

\[
\begin{align*}
\frac{1}{2} r_B^2 + c_1 &= 0 \\
\lambda^2 - \left( \frac{r_B^2}{2} + c_1 \right)^2 &= 0
\end{align*}
\]

\[ \Rightarrow \begin{cases} c_1 = -\frac{r_B^2}{2} \\
\lambda^2 = \left( \frac{r_B^2}{2}, \frac{r_B}{2} \right)^2 \end{cases} \] (7)

So, the curve can be expressed as

\[ z = \int_{r_a}^{r} \frac{r^2 - r_B^2}{\sqrt{(r_B^2 - r^2)^2 - (r^2 - r_B^2)^2}} dr \] (8)

\[ \rho(r) = \frac{(1 + (z')^2)^{3/2}}{z''} = \frac{r - r_B^2}{\lambda^2 + \frac{r_B^2}{2}} \] (9)

Equation (8) has the same form comparing with the equation obtained by Day and Gehman (1963). Equation (9) was also reported by RCOT (Yamagishi et al., 1987) and TCOT (Ogawa et al., 1990). The curve above does not contain the belt, so it can be called tire profile without belt.

Next, the profiles with belts will be studied. There are two types of belts: curved and flat, as shown in Fig. 3, in order to simplify calculation the curved belts are also assumed to be arcs (Akasaka, 1981), in fact, they can be any curves. Radial tire with curved belt is abbreviated as RTCB, and radial tire with flat belt is abbreviated as RTFB. The
belt is assumed to be rigid, the part of the profile contacting with belt must coincide with belt. In other words, the belt and profile must be tangent at the demarcation point K.

![Diagram of radial tires with different belts: (a) is RTCB and (b) is RTFB](image)

Fig. 3 Radial tires with different belts: (a) is RTCB and (b) is RTFB

The volume of radial tires with belt can be divided into two parts \( V_1 \) and \( V_2 \), such as Fig. 3, when the length and radius of belt are given, the volume under belt can be regarded as a constant \( V_2 \), and the total volume is

\[
V = 2(V_1 + V_2) \tag{10}
\]

According to above, maximizing \( V \) is equal to maximize \( V_1 \), so, the profile between point K and point C must be in accordance with Eq. (8).

For RTFB, the boundary conditions are \( z'(r_B)=0 \) and \( z'(r_K)=\infty \), \( r_K=r_A \), so the profile of RTFB can also be expressed by Eq. (8).

For RTCB, the boundary conditions are \( z'(r_B)=0 \) and \( z'(r_K)=k \). Substituting the boundaries into Eq. (6),

So, the equilibrium profile of RTCB can be expressed as

\[
z = z_K + \int_{r_K}^{r_B} \frac{r^2 - r_B^2}{\sqrt{k^2 + 1 - k^2 (r^2 - r_B^2)^2 - (r^2 - r_K^2)^2}} \, dr \tag{11}
\]

\[
\rho(r) = \frac{\sqrt{k^2 + 1} \, r_K^2 - r_B^2}{2r} \tag{12}
\]

It is worth noting that the radius of curvature of points (the point should not be on the belt) on the profile for RTCB is different from that of RTFB. The expression is different from RCOT (Yamagishi et al., 1987) and TCOT (Ogawa et al., 1990), there is an additional coefficient comparing with Eq. (9), which indicates that the \( k \) also effects \( \rho(r) \).

4. How to use the equilibrium profiles to design radial tires

When designing a new tire, there are several constrains of dimensions according to European Tire and Rim Technical Organization, The Tire and Rim Association of America or the standards of other countries. For example, section width, aspect ratio and overall diameter, and rim dimensions, the sectional profile of tire must meet these constraints. So, some parameters values of the inner contour can be determined by the limits, for example, \( r_A \) can be obtained by subtracting the overall radius with tread thickness, \( z_B \) can be obtained by subtracting the section width with sidewall thickness, as shown in Fig. 4. The position of point C can be determined according to the rim and the section
width empirically (Bauer, 2013). The radius of belt \( (R) \) is also an important parameter, it affects the traction performance and the belt edge durability (Clark, 1981), it is also determined empirically.

![The sectional dimension of a tire](image)

Now, the parameters values of \( r_A \), \( z_B \), \( r_C \), \( z_C \) and \( R \) are given, Eq. (11) can be used to calculate the equilibrium profile. Composite Simpson’s rule (Gautschi, 2012) for numerical integration is used to calculate the Eq. (11).

![The flowchart of the method](image)

There are four restrictions of the profile: (1) the profile must pass through point C; (2) the section width of the profile must be \( z_B \); (3) the radius of the highest point on the profile must be \( r_A \); (4) and the profile must be tangent to belts at the demarcation point.

The method is running in Matlab codes and making \( R_s=1e-6 \). The flowchart of using the equilibrium profile for designing a tire is shown in Fig. 5. \( z_C1 \) and \( z_B1 \) are the calculated values of \( z_C \) and \( z_B \) in the calculation process, adjusting \( z_K \) and \( r_B \) to make the absolute differences of \( z_C1 \) and \( z_C \), \( z_B1 \) and \( z_B \) within \( \epsilon_1 \) and \( \epsilon_2 \). When the values of \( z_K \) and \( r_B \) are known, the value of \( k \) can be calculated, and the profile can be obtained by integrating Eq. (11).

5. Numerical examples and FEA analysis

175SR14 tire is chosen to be a numerical example, parameter values of the tire are shown in Table 1, which are extracted from Akasaka’s article (Akasaka, 1981).
Table 1 The parameter values of 175SR14 tire.

| Parameter | Value [mm] |
|-----------|------------|
| $r_A$     | 305.30     |
| $r_C$     | 178.23     |
| $z_C$     | 56.00      |
| $z_B$     | 87.3       |
| $R$       | 240.00     |

In this calculation, $\varepsilon_1$ and $\varepsilon_2$ (seen in Fig. 5) are set to be equal to 0.2, the smaller the value, the more accurate the results, however, the longer the calculation time. The numerical calculations are carried on a computer with 8G RAM and 8-core processors, run time for the calculation is about 6 minutes. The profile obtained by using this paper’s method will be compared with the Akasaka’s measurement and calculations.

Four different 295R22.5 tires are designed to verify the benefits of this method, as shown in Fig. 6. The first is equilibrium profile obtained by this paper’s method, Design 1 is partial equilibrium profile with a larger restriction length, Design 2 is partial equilibrium profile with a smaller restriction length and Design 3 is non-equilibrium profile. Design 1 and Design 2 are both modified from the equilibrium profile.

The parameter values of 295R22.5 tire are shown in Table 2.

Table 2 Parameter values of 295R22.5 tire.

| Parameter | Value [mm] |
|-----------|------------|
| $r_A$     | 305.30     |
| $r_C$     | 178.23     |
| $z_C$     | 56.00      |
| $z_B$     | 87.3       |
| $R$       | 240.00     |

The finite element analyses are taken by ABAQUS software, and REBAR elements are used to represent the cords in carcass, belts, and chafer. The four designs have the same mesh except small differences at shoulders, and they have the same materials. The inflation pressure is 0.9 MPa, parts and meshes of the tires are shown in Fig. 7.
The strain energy density function $W$ of rubber materials can be expressed as the Rivlin form:

$$W = \sum_{i,j=0}^{2} C_{ij} (I_1 - 3)^i (I_2 - 3)^j$$  \hspace{1cm} (13)$$

Where

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$  \hspace{1cm} (14)$$

$$I_2 = \lambda_1^2 \lambda_2^2 + \lambda_2^2 \lambda_3^2 + \lambda_3^2 \lambda_1^2$$  \hspace{1cm} (15)$$

Because of the is incompressible approximately

$$\lambda_1 \lambda_2 \lambda_3 = 1$$  \hspace{1cm} (16)$$

The cubic equation of the Rivlin model is

$$W = C_{10} (I_1 - 3) + C_{01} (I_2 - 3) + C_{11} (I_1 - 3)(I_2 - 3) + C_{20}(I_1 - 3)^2 + C_{12}(I_1 - 3)(I_2 - 3)^2 + C_{21}(I_1 - 3)(I_2 - 3)^2 + C_{22}(I_1 - 3)^2 (I_2 - 3)$$  \hspace{1cm} (17)$$

And Yeoh model are simplified from Rivlin’s as

$$W = C_{10} (I_1 - 3) + C_{20}(I_1 - 3)^2 + C_{30}(I_1 - 3)^3$$  \hspace{1cm} (18)$$

The strain energy density function $W$ of Mooney-Rivlin model is

$$W = C_{10} (I_1 - 3) + C_{01} (I_2 - 3)$$  \hspace{1cm} (19)$$

The parameter values of rubber materials are shown in Table 3.

| Components       | $C_{10}$ [MPa] | $C_{01}$ [MPa] | $C_{20}$ [MPa] | $C_{30}$ [MPa] | Density [10$^3$ kg/m$^3$] |
|------------------|----------------|----------------|----------------|----------------|--------------------------|
| Tread            | 0.4032847      | 0.30562482     |                |                | 1.08                     |
| Sidewall         | 0.1864789      | 0.33359788     |                |                | 1.09                     |
| Inside liner     | 0.2033023      | 0.24005268     |                |                | 1.18                     |
| Chafer           | 1.0730716      | 0.10774168     |                |                | 1.15                     |
| Apex             | 3.4205411      | -0.564561      | 0.2895460      |                | 1.16                     |
| Belt             | 1.5439468      | -0.0660395     | 0.1272045      |                | 1.19                     |
| Carcass          | 1.7591524      | -0.358691      |                | 0.1272045      | 1.19                     |
| Bead             | 1.54286        | -0.2819        |                |                | 1.14                     |
The reinforcement materials are modelled by linearly elastic. The parameter values of reinforcement materials are shown in Table 4.

### Table 4 Parameter values of reinforcement

| Components  | Young's modulus $[10^4 \text{MPa}]$ | Poisson's ratio | Density $[10^3 \text{kg/m}^3]$ |
|-------------|--------------------------------------|-----------------|-----------------|
| Chafer      | 8.624                                | 0.3             | 1.12            |
| First Belt  | 9.355                                | 0.3             | 1.09            |
| Second Belt | 9.355                                | 0.3             | 1.09            |
| Third Belt  | 5.388                                | 0.3             | 1.09            |
| Fourth Belt | 5.388                                | 0.3             | 1.09            |
| Carcass     | 9.355                                | 0.3             | 1.09            |
| Bead ring   | 21                                   | 0.3             | 6.49            |

6. Result and discussion

Figure 8 is profiles of 175SR14 tire with different belt radius, and they are obtained by using the method above. All these profiles satisfy the four restrictions above, and $z_B$ is 87.3 mm.

![Fig. 8 Profiles of 175SR14 tire with different belt radius](image)

It can be found that the effects of belt radius on shoulder are largely, however there are little effects on bead and sidewall. The demarcation points between belt and carcass, the cord lengths (CL), $r_B$ and $k$ change with changes of belt radius, in order to study intuitively, the dates are extracted and shown in Fig. 9 and Fig. 10.

![Fig. 9 Cord lengths (CL) and restriction lengths (RL) of carcass by the belt of profiles in Fig. 8](image)
The belt radius will also affect the values of $k$ and $r_B$, as shown in Fig. 11. The $r_B$ has a small increment about 0.8 mm when the belt radius increases 50 mm. The $k$ is linearly related to the belt radius, it can be expressed as

$$k = -0.03633R + 2.3708$$

(20)

From Fig. 9 and Fig. 10, we can find that $r_B$ can be estimated when point C, point A and $z_B$ are given, because $r_B$ only changes 2 mm when $R$ changes from 150 mm to infinity. For tires with large aspect ratio, the $r_B$ can be estimated by ellipse, however, for the tires with small aspect ratio, the ellipse cannot give an effective estimation.

It is worth noting that, a small change in $r_B$ will cause a larger change in the profiles, as shown in Fig. 11. Figure 11 is tire profiles with different $r_B$. The effects of $r_B$ on bead and lower sidewall are larger than shoulder, when $r_B$ changes 1 mm $z_C$ will change more than 3 mm, and the $z_B$ will also deviate. So, it should give the $z_B$ rather than $r_B$ at the stage of initial design.

175SR14 tire profiles derived by this paper are in the comparison of Akasaka’s, as shown in Fig. 12. The result shows that this paper’s calculation is in better agreement with Akasaka’s measurement than his calculations (Akasaka, 1981). The Akaska’s calculation 1 deviates inward from the measured profile at shoulder and bead, however, the Akasaka’s calculation 2 deviates outward at shoulder and inward at bead. The RL of this paper’s calculation is close to Akasaka’s calculation 1, and it is smaller than Akasaka’s calculation 2 and measurement. The $z_B$ of Akasaka’s
calculation 2 is bigger than others.

Fig. 12 The profiles of 175SR14 tire derived by this paper and Akasaka’s (Akasaka, 1981)

These results may be caused by the following reasons:

1) The allotment ratio of pressure was used in Aksaka’s calculation 1, however, it was difficult to obtain its accurate value by calculation or measurement, and if it is a constant is unknown yet. The deviations of Aksaka’s calculation 1 from measurement were caused by the incorrect value of allotment ratio of pressure.

2) The traditional method was used in Aksaka’s calculation 2, at first, a point Q was chosen (Fig. 1) empirically, however, it was not the demarcation point between belt and carcass, then, a value of \( r_B \) was given artificially. The deviations of Aksaka’s calculation 2 from measurement were caused by the improper point Q and value of \( r_B \). (Fig. 11).

3) The differences between the values of \( r_B \) in Aksaka’s calculations and measurement were very small (Fig. 12), the reason is that \( r_B \) might be used as an input parameter in Aksaka’s calculation, and its value was given according to the measurement. However, there would be measurement error in \( r_B \), and the \( r_B \) has great influence on tire profile (Fig. 11).

4) Due to the allotment ratio reflects the effects of belt on carcass, the demarcation point in Aksaka’s calculation 1 is point K (Fig. 3), and however, in Aksaka’s calculation 2 is point Q (Fig. 1). The demarcation point in Aksaka’s calculation is close to the demarcation point, which verifies the calculation of RL is correct by this paper’s method.

Figure 13 shows the displacements of 295R22.5 tire carcasses under inflation.

Fig. 13 The displacements of 295R22.5 tire carcasses of four designs
The displacement of equilibrium profile is smaller and more uniform than the others, which verifies the benefits of this paper’s method. Owing to the expansion of belt under inflation pressure, all these designs have large deformations at tread. The bending of belt lead to decrease of belt radius and increase of RL (Fig. 9), Design 1 has a larger RL, which is the reason of smaller displacement at shoulder of Design 1, however, the change of RL will lead to a change of $k$, which will affect the profile in bead. Design 3 is non-equilibrium profile, and the displacement of carcass is very large.

Through the analysis and calculations above, the benefits and rationality of this paper’s method are proved. At the same time, some aspects in tire design should be draw attention to:

1. The stiffness of belt is very important, the deformation of tread should be as small as possible;
2. Let $r_c$, $z_c$ and $z_b$ be the boundary conditions is more rational than $r_B$.

In this paper only the inflation condition was studied, the load case was not considered, because the load performances are not only related to the inner profile but also related to the tread profile, however, the inflation state with little to the tread profile. The further study will be taken on how to combine the tread profile with inner equilibrium profile to obtain good load performances.

7. Conclusion

In this paper, the inextensible membrane model and variational approach are used to obtain the equilibrium profile of radial tires, a more rational method is proposed for initial design, the new profile is in good agreement with Akasaka’s measurement, and the finite element analysis show that the equilibrium profile has a small displacement when inflated. The main conclusions are:

1. The equilibrium profile of RTFB is different from RTCB, it is related to the position and slope of the demarcation point beteen carcass and belt;
2. The demarcation point between carcass and belt should be calculated rather given empirically;
3. The $r_B$ has great influence on tire profile, so $z_c$ should be selected as the initial input parameter rather than $r_B$;
4. The effects of belt radius on $r_B$ is very small, however, on shoulder profile is bigger;
5. This paper’s method is more rational than traditional method for tire design.

This paper’s method can be applied to both tires with big and small aspect ratio, and it will offer a guide for designing radial tires.

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