Quasinormal modes of black holes absorbing dark energy

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Abstract

We study perturbations of black holes absorbing dark energy. Due to the accretion of dark energy, the black hole mass changes. We observe distinct perturbation behaviors for absorption of different forms of dark energy onto the black holes. This provides the possibility of extracting information whether dark energy lies above or below the cosmological constant boundary $w = -1$. In particular, we find in the late time tail analysis that, differently from the other dark energy models, the accretion of phantom energy exhibits a growing mode in the perturbation tail. The instability behavior found in this work is consistent with the Big Rip scenario, in which all of the bound objects are torn apart with the presence of the phantom dark energy.

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There has been growing observational evidence showing that our universe is accelerated expanding driven by a yet unknown dark energy (DE) \([1, 2, 3]\). The leading interpretation of such dark energy is a cosmological constant with equation of state \(w = -1\). More sophisticated models have been proposed to replace the cosmological constant by either relating the dark energy to a scalar field called quintessence with \(w > -1\), or to an exotic field called phantom with \(w < -1\). But it is doubtful that there is a clear winner in sight to explain the nature of dark energy at the moment. Recently, extensive analysis found that the current data favors dark energy models with equation of state in the vicinity of \(w = -1\) \([4]\), straddling the cosmological constant boundary. This observational value was pinned down through large scale surveys from CMB, large scale structure and SNIa observations.

At present the equation of state is the only antenna to learn the microscopic nature of dark energy. Its observational value is particularly important to determine whether the dark energy is of the phantom type, quintessence type or cosmological constant. Besides the available observational methods, it is of great interest to devise other complementary tools to measure the values of \(\omega\). In this work we use quasinormal modes (QNM) of a black hole to investigate the nature of DE equation of state. As is well known, if the dark energy is modeled as a background cosmic fluid, the flux of dark energy accreted by the black hole will change its mass. Now, it is reasonable to take into account these imprints in perturbations around a black hole and to extract information about \(w\). An attempt in this direction was studied in \([6]\) by exploiting the gravitational wave radiated from a binary of supermassive black holes. It is expected that the binaries observed with LIGO or LISA can distinguish, in the nearly future, more accurately values of \(w\) of dark energy.

To extract information from the gravitational wave observation and pin down the value of \(w\) of dark energy, we need accurate waveforms on the perturbations around black holes. There has been great progress in studying perturbations around black holes recently. In asymptotically flat spacetimes it is possible to get a schematic picture regarding the dynamics of waves outside black holes. After the initial pulse, the perturbation will experience a quasinormal ringing, which describes the damped oscillations under perturbations in the surrounding geometry of a black hole with frequencies and damping times of oscillations entirely fixed by the black hole parameters. The quasinormal modes is believed as a unique fingerprint to directly identify the black hole existence. Detection of these QNM is expected to be realized through gravitational wave observation in the near future \([7]\). At late times, quasinormal oscillations are swamped by the relaxation process. This relaxation is the requirement of the black hole no hair theorem \([8]\). In Anti-de Sitter and de Sitter spacetimes, perturbations around black holes have been shown as a theoretical testing ground to get deeper understandings.
of the AdS/CFT [9, 10, 11, 12, 13], dS/CFT [14] correspondences. More recently, it has been also argued that perturbations in black hole backgrounds is useful to extract information on black hole phase transitions [13, 16].

In the universe filled with dark energy modeled as scalar field, the action has the form

$$ S = \int d^4x \sqrt{-g} \left[ \frac{R}{16\pi G} - \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} \mu^2 \varphi^2 \right]. $$

(1)

where $\mu$ is the mass of the scalar field, and we have taken the metric sign $(-, +, +, +)$. If the dark energy is described by the phantom field, the kinetic term in the action has the positive sign [18]. Varying the action with respect to $\varphi$, we can obtain the wave equation in the curved spacetime

$$ \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \pm \mu^2 \varphi) = 0, $$

(2)

where the $'+'$ sign describes the phantom field while the $'-'$ sign describes the quintessence field. The wave equation of the quintessence field is the same as the massive scalar field discussed in [19].

Considering next the effect of the accretion of dark energy onto the black hole, the black hole mass changes at a rate [5]

$$ \frac{dM}{dt} = 4\pi AM^2 \rho_\infty (1 + w), $$

(3)

where $A \simeq 5.6$ is the numerical factor, which define the energy flux of DE onto the black hole, $\rho_\infty$ is the energy density of dark energy far away from the hole. The black hole is no longer static due to the absorption of dark energy, unless the dark energy being cosmological constant. Accreting the quintessence field with $w > -1$, the black hole mass increases. However the black hole mass decreases during the accretion of phantom energy with $w < -1$. It was argued that the black holes are not torn apart, but disappear by the Big Rip due to the accretion of phantom energy [5]. Other discussions on the change rate of the black hole mass due to the absorption of dark energy can be found in [20]. What would be the fate of the perturbation around black holes absorbing dark energy? To answer this question we shall employ the formalism first developed in Ref. [21]. Some other studies on the time-dependent background QNMs can be found in [22].

To describe the time-dependent black hole background, we start with the Vaidya metric [23]

$$ ds^2 = -f dv^2 + 2c dv dr + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), $$

(4)

where $f = 1 - \frac{2M(v)}{r}$ and $M(v)$ is an arbitrary function of time. The coordinate $v$ is usually called “advanced time” and $c = 1$ describes black hole with ingoing radial flow. The horizon of the Vaidya black hole is inferred
from the null hypersurface condition \( g^\mu\nu \frac{\partial f}{\partial x^\mu} \frac{\partial f}{\partial x^\nu} = 0 \) and \( \tilde{f}(r_+, v) = 0 \). In this case, \( r_+(v) \) satisfies
\[
 r_+ - 2M(v) - 2cr_+\dot{r}_+ = 0, \tag{5}
\]
where \( \dot{r}_+ = \frac{dr_+}{dv} \).

Similarly to the static approach, we introduce the tortoise coordinate \( r_* \) as
\[
 r_* = r + \frac{1}{2\kappa} \ln(r - r_+), \tag{6}
\]
where \( \kappa \) is the surface gravity. Since \( r_* = r_*(r, r_+(v)) \), \( dv = dt + dr_* \), one has
\[
 \frac{dM}{dv} = \left( 1 - \frac{\partial r_*}{\partial v} \right) \frac{dM}{dt} \tag{7}
\]
\[
 = \left[ 1 + \frac{\dot{r}_+}{2\kappa(r - r_+)} \right] \frac{dM}{dt}.
\]

Using the Vaidya metric and the tortoise coordinate, the wave equation (2) in the time-dependent black hole background can be written as
\[
 \left[ \frac{r^2}{c^2} \frac{\partial^2}{\partial v \partial r} + \frac{1}{c} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial v} \right) + \frac{1}{c^2} \frac{\partial}{\partial r} \left( r^2 f \frac{\partial}{\partial v} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \pm \mu^2 \right] \psi = 0, \tag{8}
\]
where before \( \mu^2 \) term the ‘+’ sign describes the phantom field while the ‘−’ sign is for the quintessence field. Since the Vaidya metric is spherically symmetric, above equation thus is separable and the radial wave equation can be written as follows
\[
 (1 + \varepsilon_2) \frac{\partial^2 \Psi}{\partial r_*^2} + 2c^2 \frac{\partial^2 \Psi}{\partial v \partial r_*} + \varepsilon_1 \frac{\partial \Psi}{\partial r_*} - V \Psi = 0, \tag{9}
\]
where
\[
 \varepsilon_2 = \frac{f}{A} + 2c \frac{\partial r_*}{\partial v} - 1, \tag{10}
\]
\[
 \varepsilon_1 = f \left( \frac{1}{A} \right)' + f' + 2cA \frac{\partial}{\partial v} \frac{1}{A}, \tag{11}
\]
\[
 V = \frac{Af'}{r} + \lambda \frac{Ac^2}{r^2} \mp c^2 \mu^2 A, \tag{12}
\]
with \( \frac{1}{A} = \frac{\partial r_*}{\partial r} = 1 + \frac{1}{2\kappa(r - r_+)} \). Here in the effective potential \( V \), the ‘−’ sign is for the phantom field and the ‘+’ sign is for the quintessence field. For the convenience of later numerical calculation, we adjust \( \kappa \) in a way that Eq.(9) for each multipole moment becomes the standard wave equation near the horizon \( r_+(v_0) \) [7]. The boundary condition at the horizon reads
\[
 \lim_{r \to r_+ (v_0)} r \to r_+ (v_0) \varepsilon_2 = \lim_{v \to v_0} \varepsilon_1 = \lim_{v \to v_0} V = 0, \tag{13}
\]
which requires \( \kappa = \frac{1}{2r_+} \).

FIG. 1: QNM behaviors of black holes absorbing dark energy. In plotting the figure we have taken parameters \( l = 2, \rho_\infty = 0.001, A \simeq 5.6, \mu = 0.001 \).

In order to simplify (9), we make a variable transformation \( u = u(r_*, v_*) \) and \( v = v_* \), where the curve \( u(r_*, v_*) = \text{constant} \) is determined by the equation

\[
\frac{dr_*}{dv_*} = \frac{1 + \varepsilon_2}{2c}.
\]
In addition, when \( c = 1 \) and \( \varepsilon_2 \to 0 \), we have back the usual null coordinate \( u \to v + 2r_* \), similar to the static case \([21]\). Using the new variables \( u \) and \( v \), the wave equation changes to

\[
\left( 1 + \varepsilon_2 \right) \left( \frac{\partial u}{\partial r_*} \right)^2 + 2c \frac{\partial u}{\partial v} \frac{\partial^2 \Psi}{\partial u^2} + \left[ \left( 1 + \varepsilon_2 \right) \frac{\partial^2 u}{\partial r_*^2} + 2c \frac{\partial^2 u}{\partial v \partial r_*} + \varepsilon_1 \frac{\partial u}{\partial r_*} \right] \frac{\partial \Psi}{\partial u} + 2c \frac{\partial u}{\partial r_*} \frac{\partial^2 \Psi}{\partial u \partial v} - V \Psi = 0. \tag{15}
\]

In the static limit, \( \varepsilon_2 = \varepsilon_1 = 0 \), and \( u = v + 2r_* \), thus the radial equation returns to its usual form \([7]\)

\[
\frac{\partial^2 \Psi}{\partial u \partial v} + \frac{V}{4} \Psi = 0. \tag{16}
\]

Now for the Vaidya spacetime, \( u(r_*, v_*) \) satisfies

\[
\begin{cases}
(1 + \varepsilon_2) \frac{\partial u}{\partial r_*} + 2c \frac{\partial u}{\partial v} = 0 \\
(1 + \varepsilon_2) \frac{\partial^2 u}{\partial r_*^2} + 2c \frac{\partial^2 u}{\partial v \partial r_*} + \varepsilon_1 \frac{\partial u}{\partial r_*} = 0,
\end{cases} \tag{17}
\]

and Eq. \((15)\) simplifies to

\[
\frac{\partial^2 \Psi}{\partial u \partial v} - \left( 2c \frac{\partial u}{\partial r_*} \right)^{-1} V \Psi = 0. \tag{18}
\]

Numerical solution of \((18)\) proceeds by first integrating \( u(r_*, v_*) \) according to \((14)\), and then using next the finite difference method developed in \([21]\) to solve the wave equation in the Vaidya metric. Denoting \( \Psi \to \Psi_{i,j} = \Psi(u_i, v_j) \), and

\[
\frac{\partial \Psi}{\partial u} \to \frac{\Psi_{i+1,j} - \Psi_{i-1,j}}{\Delta u}, \quad \frac{\partial \Psi}{\partial v} \to \frac{\Psi_{i,j+1} - \Psi_{i,j-1}}{\Delta v},
\]

we can discrete \((18)\) into

\[
\Psi_{i-1,j+1} + \Psi_{i+1,j-1} - \Psi_{i-1,j-1} - \Psi_{i+1,j+1} + \left( 2c \frac{\partial u}{\partial r_*} \right)^{-1} V \Psi_{i,j} = 0. \tag{20}
\]

Taking \( \Psi_N = \Psi_{i+1,j-1} \), \( \Psi_S = \Psi_{i-1,j-1} \), \( \Psi_E = \Psi_{i+1,j-1} \), \( \Psi_W = \Psi_{i-1,j+1} \), and using \( \Psi_{i,j} = \frac{\Psi_E + \Psi_W}{2} \), we have

\[
\Psi_W + \Psi_E - \Psi_S - \Psi_N + \left( 2c \frac{\partial u}{\partial r_*} \right)^{-1} \frac{\Psi_E + \Psi_W}{2} = 0. \tag{21}
\]

FIG. 3: Late time evolution of the perturbation of massive scalar field in the stationary spacetime. We have taken \( w = -1 \), and different values of the field mass \( \mu = 0.001, 0.005 \) and \( 0.01 \) respectively.
FIG. 4: Late time evolution of the perturbation in around the black hole absorbing quintessence and phantom fields. We have taken different values of the field mass as $\mu = 0.001, 0.005$ and $0.01$ respectively.

Now we report our numerical results on the perturbations around the black hole with accretion of different types of dark energy. Fig.1 illustrates the QNMs behavior. The solid line represents the case of accretion of phantom type dark energy with $w = -1.1$, the dashed line represents the dark energy being the cosmological constant, and the dotted line shows the result of absorbing quintessence field with $w = -0.9$. The details of quasinormal frequencies $\omega_R, \omega_I$ as functions of $v$ are shown in Fig.2. When the black hole absorbs the cosmological constant, its mass does not change so that both the real and imaginary parts of quasinormal frequencies remain constant, in agreement with the static black hole cases. The black hole mass increases due to the accretion of quintessence field. Differently from the stationary black hole case, we observed that both the real part and the absolute value of the imaginary part of quasinormal frequencies decrease with the time evolution. However due to the accretion of phantom field, which causes the black hole mass to decrease, both the real part and the absolute value of the imaginary part of quasinormal frequencies increase with time. Comparing the values of the imaginary parts of QNM, we found that $|\omega_I|$ is smaller for black holes absorbing quintessence than phantom field. This explains what we observed in Fig.1, in which the effect of perturbation can last longer in the black hole background due to the accretion of quintessence type dark energy.

We have also investigated the late time tail behavior of the perturbation in the black hole due to the accretion of dark energy. If the dark energy is cosmological constant, the result is shown in Fig.3. Since
the black hole mass do not change in this case, our result gives the objective picture of the late time tail behavior for the massive scalar field perturbation in the stationary black hole background. Differently from massless scalar field perturbation, we observed oscillations in the tail of massive scalar field perturbation. The frequency of the oscillation increases with the value of mass of scalar field $\mu$, which confirms the argument proposed in [19].

The late time tails of perturbations around black holes absorbing quintessence field and phantom field are shown in Fig.4. The accretion of quintessence field has the similar oscillatory decay behavior in the late time tail as that observed for black hole absorbing dark energy with $w = -1$. Interestingly, the accretion of phantom onto the black hole presents us completely different behavior in the late time tail. Instead of decay, we saw the growing modes in the tail. The same phenomenon has also been observed for static black holes surrounded with phantom type DE [24]. Here we found that the growing appears earlier and faster when $\mu$ is bigger. The growing modes will make the black hole unstable. Thus from the tail of the perturbation we saw that it is not peaceful for the black hole to disappear by the Big Rip due to the accretion of phantom energy as argued in [5]. In the later moment when the black hole has accumulated enough phantom energy, it will “explode”. To find physical reason for this “explosion”, let us examine the behavior of the effective potential in the wave equation. Since the black hole mass will change due to the accretion of phantom or quintessence fields, the effective potential is time-dependent. Choosing $v = 100$, the behaviors of the potential are shown in Fig.5. For the accretion of phantom, the effective potential outside the black hole approaches a negative value, which is different from the case with accretion of the quintessence. Actually this can be seen from the expression of the potential [12], whose behavior does not change with the time evolution. The existence of the negative part in the effective potential can form true bound state which leads to the growing modes [25].

In summary we have investigated perturbations around black holes absorbing dark energy. It is well known that the mass of a black hole changes due to the accretion of dark energy of quintessence and phantom
types\[5\]. The study of perturbation in this black hole background is more complicated than in the usual stationary situation. Using the formalism developed in \[21\], we have observed different QNM behaviors for the accretion of different types of dark energy onto the black hole. The QNM results discussed here are sufficient to illustrate the possibility to distinguish whether dark energy lies above or below the cosmological constant boundary $w = -1$ in the future observation by exploiting the perturbations around black holes. However since the change rate of the mass due to the accretion of dark energy is extremely slow, to extract very accurate information of $w$ from gravitational waves emitted from black holes while avoiding background noise still remains a challenge.

The late time tail behavior of the perturbation around black hole absorbing dark energy presents us interesting results. Instead of the anticipated oscillatory decay in the background of black hole absorbing cosmological constant and quintessence, the accretion of phantom energy onto the black hole exhibits a growing mode in the perturbation. This growing tail tells us that due to the accretion of phantom energy the black hole does not disappear peacefully as argued in \[5\]. Instead, the black hole will explode after getting enough phantom energy. The result is consistent with the Big Rip scenario, i.e, all of the bound objects are to be torn apart in the presence of the phantom dark energy.

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