A new tool GSEQ-FRC for two-dimensional field-reversed configuration equilibrium

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Abstract

A new tool (GSEQ-FRC) is developed for solving two-dimensional equilibrium of the field-reversed configuration (FRC) based on fixed boundary and free boundary conditions that includes external coils. Benefiting from the two-parameter modified rigid rotor radial equilibrium model and the numerical approaches presented (Ma et al 2021 Nucl. Fusion 61 036046) in this work, the GSEQ-FRC code is used to study the equilibrium properties of FRC quantitatively and will be used for fast FRC equilibrium reconstruction. In GSEQ-FRC, the FRC equilibrium is determined by two parameters, i.e. the ratio between thermal pressure and magnetic pressure at the separatrix $\beta_s$, and the normalized scrape of layer width $\delta_s$. Examples with fixed and free boundary conditions are given to demonstrate the capability of GSEQ-FRC for equilibrium calculations. This new tool is used to quantitatively study the factors affecting the shape of the FRC separatrix, which reveals how the FRC changes from racetrack-like to ellipse-like.

Keywords: field-reversed configuration, Grad–Shafranov equilibrium, equilibrium design, separatrix shape

(Some figures may appear in colour only in the online journal)

1. Introduction

Field-reversed configurations (FRCs) are considered as a possible approach to achieve fusion energy, either as a magnetic confinement or as a target plasma for magnetized target fusion [1–5]. The plasma equilibrium is an essential element to understand the FRC properties [6], and is a foundation for studying various plasma phenomena, such as magnetohydrodynamic (MHD) instabilities and plasma transport.

Understanding the interior properties of hot, dense FRC plasma is still a challenge. In early FRC experiments, profile information was deduced from measurements of the excluded flux array [7] and single-chord interferometry [8], where the shape of the separatrix can be estimated. In recent FRC experiments, multi-point Thomson scattering [9] and multi-chord interferometry [10] have provided more detailed FRC profile data. However, non-perturbative internal magnetic field measurement is still difficult. Therefore, theoretical/model equilibrium calculations are particularly important in the study of FRCs, where many equilibrium models such as rigid-rotor (RR) [1], two-point equilibrium [11], three-point equilibrium [12], symmetric [13] and modified rigid-rotor (MRR) [6] are proposed.

The Grad–Shafranov (G–S) equation [14, 15] can be utilized to describe the traditional FRC equilibrium [6]. In cylindrical coordinate system ($r, \theta$) with axial symmetry, it can be...
written as:
\[
\Delta^* \psi \equiv r \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \psi}{\partial r} \right) + \frac{\partial^2 \psi}{\partial z^2} = -\mu_0 r^2 P'(\psi),
\]

with
\[
J_0 = r P'(\psi), \quad B_z = \frac{1}{r} \frac{\partial \psi}{\partial z}; \quad B_r = -\frac{1}{r} \frac{\partial \psi}{\partial z},
\]

where \( \psi = \Psi_p / 2\pi = \int_0^1 B_0 r \, dr \) is the normalized poloidal magnetic flux, \( J_0 \) is the toroidal current density of plasma, \( P' = \partial P / \partial \psi \), and \( \mu_0 \) is the vacuum permeability. In this work, since FRCs have almost no toroidal field, \( B_0 = 0 \) is assumed.

Two-dimensional (2D) numerical equilibrium calculation of FRC may appear simpler than the tokamak by omitting \( B_0 \). However, the requirement of including the scrape of layer (SOL) region imposes some extra difficulties. The existence of bifurcation solutions for the FRC equilibrium calculations was a major roadblock encountered by earlier FRC equilibrium research, where a 2D solution was difficult to reach numerically [16, 17]. Later, the problem was overcome by using the total toroidal current \( J_0 \) [18] in the computational region or the area of plasma surrounded by the separatrix \( S \) [19, 20] as global constraints in equilibrium calculations. Based on \( I_b \) and \( S \), it is possible to solve FRC equilibrium with arbitrary elongation ratios. In these equilibrium calculations [12, 19–24], iterative algorithms are used, where the free model parameters usually are not physical parameters. Therefore, post-processing is required during and after the iteration to derive physical parameters that can be compared with experimental data. The indirect algorithms are somewhat inconvenient and inefficient in equilibrium studies. Sometimes more than two parameters are iterated, while there are only two independent parameters needed to specify FRC equilibrium in some models [6, 11]. The coupling of the model parameters could be the reason leading to bifurcated solutions.

In the MRR model [6], FRC equilibrium can be determined by two physical quantities \( \beta_0 \) and \( \delta_z \), where \( \beta_0 \) is the ratio of the plasma pressure to the magnetic pressure at the separatrix, and \( \delta_z \) is the normalized SOL width. The general solver for equilibrium (GSEQ) code, developed by ENN fusion theory and simulation group, use the MRR model to solve 2D FRC equilibrium, with three steps: specifying the free parameters \( \beta_0 \) and \( \delta_z \); solving other model parameters from 1D equilibrium; and solving 2D equilibrium. Benefiting from this algorithm, the equilibrium calculation almost becomes an algebraic equation solving process with high efficiency. In addition, FRC characteristics are readily obtained when the two parameters \( \beta_0 \) and \( \delta_z \) are determined. This greatly improves the efficiency to arrive at the desired equilibrium. The convenience and efficiency of the GSEQ code can be beneficial in the future for equilibrium fitting of experimental data. One unique feature of the GSEQ-FRC tool is that two key quantities of an FRC equilibrium, the maximum vacuum magnetic field \( B_0 \) (detailed meaning of \( B_0 \) is discussed in appendix A) and the separatrix radius \( R_s \) are scaling factors, which are not involved in the equilibrium solving process, could be specified after the solution. This can significantly improve the computational efficiency on equilibrium design of FRCs.

This paper is organized as follows. Section 2 introduces the algorithms and procedures used in GSEQ-FRC. In section 3, the 2D equilibrium with fixed boundary condition is presented. Examples of free boundary equilibrium are shown in section 4. In section 5, the factors affecting the shape of the FRC separatrix are discussed. Summary and discussion are presented in section 6.

2. The algorithm in GSEQ-FRC

The method for solving the free parameters of equilibrium models, and the initial result of 2D FRC equilibrium using the GSEQ code is presented in [6]. As follows, the 2D equilibrium solver is described in section 2.1. In section 2.2, a step-by-step method for the GSEQ-FRC tool is presented.

2.1. Introduction to the G–S solver

The GSEQ code, a plasma equilibrium solver with external coils, is developed for advanced study of plasma equilibrium. The accuracy of the result from GSEQ code is demonstrated in appendix B. This code supports two distinctive ‘upper’ \( (r \rightarrow r_{\text{wall}}) \) boundary conditions: fixed boundary condition for metal wall as flux-conserver, and free boundary condition for quartz chamber. Most of the algorithms for solving the G–S equation are well known, therefore, only a brief outline will be given. For the flux-conserving metal wall, \( \psi \) of the ‘upper’ boundary condition is calculated using Green’s function [24]:

\[
\psi_b(r, z) = \int \int G(r, z; r', z') J_0(r', z') \, dr' \, dz',
\]

where
\[
G(r, z; r', z') = \frac{1}{2\pi} \sqrt{rr'} \left[ (2 - k^2) K(k^2) - 2E(k^2) \right]
\]

is the free space Green’s function, which gives the poloidal flux at \( (r, z) \) from a unit coil current source at \( (r', z') \). \( K(k^2) \) and \( E(k^2) \) are the elliptic integrals of the first and the second kind respectively, and

\[
k^2 \equiv \frac{4rr'}{(r + r')^2 + (Z - Z')^2}.
\]

For the free boundary condition case, a detailed procedure can be found in [25]. The ‘upper’ boundary varies due to the change of the plasma during the numerical iterations, where the G–S equation is solved inside the plasma-vacuum boundary. The \( \psi_b \) on the edge of a computational domain is calculated as a combination of two parts:

\[
\psi_{b} = \psi_{b}^{\text{ coils}} + \psi_{b}^{p},
\]

where \( \psi_{b}^{\text{ coils}} \) represents the contribution from the currents in the external axial coils, and \( \psi_{b}^{p} \) is from the plasma current using the latest approximation to \( J_0 \) in the calculation procedure. Both terms in equation (7) are computed using the Green’s function for a toroidal current source.
Next, we consider the computational procedures of the plasma subregion. The plasma equilibrium in the GSEQ code is found by solving finite difference approximations to the G–S equation using the successive over relaxation (SOR) method [18, 22]. For the FRC, the MHD equilibrium may be more difficult with \( r \rightarrow 0 \) as the ‘lower’ boundary condition. The numerical procedure is demonstrated in figure 1 and comprises of three steps: in the first step, the information of the external coils including the current, the position and shapes, the computational domain, \( I_0 \) as the global constraints on the plasma equilibrium and \( C = 1 \) is given as the input data, where the guessed \( \psi(r,z) \) is used as the initial \( \psi \) distribution. The second step is the G–S equation solver. The new \( \psi \) is updated using \( P(\psi_{old}) \) and the SOR method \( (\psi = w\psi_{old} + (1-w)\psi_{r}) \), \( w \in (0,1) \) is used as the acceleration parameter) with fixed boundary or free boundary conditions, respectively. \( C = C_{old} \times I_0/I_b \) is used in GSEQ code to implement \( I_0 \) as a global constraint for 2D equilibrium calculation, so that \( I_0 = I_b \) and \( C \sim 1 \) is obtained finally, where \( I_b \) is the total 2D current calculated from the code. The loop is based on the convergence of \( \psi \) and \( \psi_{old} \). In the third step, the updated \( \psi \) is used to satisfy

\[
d = \max |\psi(r,z) - \psi_{old}(r,z)| < \varepsilon, \tag{8}
\]

where \( \varepsilon \) is a specified tolerance, typically \( 10^{-9} \).

2.2. A step-by-step method for GSEQ-FRC

In this subsection, GSEQ-FRC tool is presented, which combines method for solving the free parameters of the MRR model [6] and the GSEQ code. The flowchart of the GSEQ-FRC tool is illustrated in figure 2.

As described in [6], the MRR-1 model is:

\[
P(\psi) = B^2 e^{\frac{\beta}{2\mu_0}} \cdot \exp \left( -\alpha \frac{\psi}{B_s R_s^2} \right) \cdot \left[ \frac{\psi}{B_s R_s^2} \right]^q + 1 \right]^n, \tag{9}
\]

with \( q = 1, n = 2 \) as default, \( \alpha \) and \( \sigma \) are the free parameters. The free parameters in equation (9) are solved from the following normalized constraint equations:

\[
\Delta_s = -\frac{p}{\psi_{m}} \bigg|_{\psi=\psi_m} = -\frac{\psi_{m}}{\psi_{m}} \bigg|_{\psi=\psi_m} - 1 = 0, \tag{10}
\]

\[
\int_{\psi_m}^{\psi_m} \frac{d\psi}{\sqrt{1 - p(\psi_{m})}} + \frac{1}{4} = 0, \tag{11}
\]

where \( \Delta_s \) is used to characterize the SOL width, and \( \psi_{in} \) is the trapped flux. The dimensionless forms are used. For the radius \( R_s \rightarrow \psi \), the flux \( \psi/(B_s R_s^2) \rightarrow \psi \), and the pressure \( P/P_{in} \rightarrow p \), where \( P_{in} = B^2 e^{\frac{\beta}{2\mu_0}} \). Using equations (10)–(12), the FRC equilibrium is uniquely determined by \( \beta \) and \( \Delta_s \).

In GSEQ-FRC tool, the important quantities \( B_e, R_s \) and \( R_w \) (the chamber radius) are not involved in the equilibrium solving process. This is because the \( I_{1D} = 2\pi \int_0^{R_s} j_0 r dr \) is used as a constraint instead of \( I_0 \) in the 2D equilibrium solving of GSEQ-FRC. The following steps are used: specifying two parameters \( \beta_s \) and \( \Delta_s \) to obtain the normalized 1D equilibrium profiles. With the target values of \( B_e, R_s \) and \( R_w \), the poloidal flux at the chamber wall \( \psi_{in} \) is determined, which is used to calculate the external coil current of the fixed boundary.
as the vacuum flux

In this subsection, the wall is assumed to be an ideal conductor (here we mean that the flux \( \psi \) at the wall is taken as the vacuum flux \( \psi_{\text{colls}} \) from the outside coils), so the equilibrium can be solved with the procedure for fixed boundary case.

### 3. 2D equilibrium with fixed boundary condition

In this section, we apply the GSEQ-FRC to study several typical FRC equilibria.

#### 3.1. 2D equilibrium with fixed boundary condition

A simple FRC machine with three coils is shown in figure 3. The central confinement region has 0.28 m inner diameter and 1.5 m length. The settings of the external coils are presented in table 1. In this subsection, the wall is assumed to be an ideal conductor (here we mean that the flux \( \psi \) at the wall is taken as the vacuum flux \( \psi_{\text{colls}} \) from the outside coils), so the equilibrium can be solved with the procedure for fixed boundary case.

![Figure 3: Schematic of the FRC machine, where gray boxes are the external coils, and solid red line is the wall. The upper half shows the magnetic flux with the FRC included, and the lower half shows the magnetic fluxes of the vacuum field.](image)

![Figure 4: The 2D fixed boundary condition equilibrium with the \( B_e = 1 \) T, \( R_s = 0.075 \) m, \( \beta_e = 0.6 \) and \( \delta_e = 0.0424 \). (a) The contour of magnetic fluxes of 2D equilibrium from GSEQ-FRC. (b), (c) The profiles of \( \psi \) and \( B_z \), respectively, where the red solid line is the profile in vacuum and the green dash line is the profile of final convergent results. (d), (e) The equilibrium pressure and current density profiles of 1D and the 2D in midplane, respectively. The external coil current are all \( 6.45 \times 10^5 \) A for each coil.](image)

The target values of \( R_s \), \( R_a \) and \( R_e \) are not involved in the equilibrium solving when using GSEQ-FRC for the design of the 2D equilibrium. This new tool eliminates the requirement for numerous calculations to match with the target values. In the process of GSEQ-FRC solving 2D equilibrium, the \( I_{1D} \) is used as the global constraint instead of \( I_e \), which is the first time to link 1D equilibrium characteristics with 2D. In the following, \( B_e = 1 \) T and \( R_s = 7.5 \) cm are taken as the target values.

The left and right boundaries in the calculation of fixed-boundary condition equilibrium are set to \( d\psi/dr = 0 \), i.e. periodic boundary conditions, and the lower boundary is \( \psi(r = 0) = 0 \). The 2D equilibrium of FRC using fixed boundary condition is demonstrated in figure 4. The contour of the 2D equilibrium is shown in figure 4(a). The axial distribution of \( \psi \) and magnetic field \( B_z \) at the wall are shown in figures 4(b) and (c), with external current of \( 6.45 \times 10^5 \) A for each coil. The magnetic field at the ‘upper’ boundary of the midplane in the presence of plasma in the chamber is \( B_{\text{ch}} = 0.997 \) T obtained from figure 4(c). The 1D equilibrium is the input data in the process of solving 2D equilibrium using GSEQ-FRC tool. Comparison of pressure and current density profiles from solve_MRR module (1D) and the midplane profiles (2D) of 2D equilibrium are shown in figures 4(d) and (e) respectively as a confidence check.

| Table 1. The external coils parameters. |
|----------------------------------------|
| Coi No | Z (m) | R (m) | Width (m) | Thickness (m) |
|--------|------|------|----------|--------------|
| 1      | −0.7 | 0.15 | 0.1      | 0.006        |
| 2      | 0.0  | 0.15 | 1.2      | 0.006        |
| 3      | 0.7  | 0.15 | 0.1      | 0.006        |
3.2. Equilibrium with complex fixed boundary condition

In this subsection, the complex equilibrium of asymmetric settings of the external coil currents and the FRCs merging case is calculated, based on the geometry configurations of the external coils and chamber of the FIX device [26], which is shown in figure 5. The equilibrium of the two merging FRCs is demonstrated in figure 5(a). It should be noticed that the currents of the external coils are asymmetric, which can be seen from $\psi$ and $B_z$ profile at the walls in figures 5(b) and (c). The complex equilibrium is solved using $I_{1D}$ from MRR_module as a constraint with the total current at the O-point of left-side FRC. The pressure and current density profiles of 1D equilibrium are consistent with the poloidal profiles of 2D equilibrium at the O-point, which is presented in figures 5(d) and (e), respectively.

In the 2D equilibrium with fixed boundary conditions, the magnetic flux within the boundary is fixed. Being a good conductor, the presence of the FRC plasma in the vacuum chamber will expel the magnetic flux from the space it occupies. Therefore, $B_z$ will increase between the chamber wall and the plasma boundary as shown in figures 4(c) and 5(c). The $B_z$ enhancement is due to the eddy current in the chamber wall and the plasma diamagnetic current, where null is about 10 to 20 cm beyond the FRC ends. In figure 5(c), plasma is elongated to near $z = \pm 2$ m, where the chamber wall radius is decreasing, so that the $B_z$ enhancement factor is larger in these regions.

4. 2D equilibrium with free boundary condition

In this section, the 2D equilibrium with free boundary condition is calculated using GSEQ-FRC. The left and right boundaries are calculated as the upper boundary in the free-boundary condition equilibrium, i.e. $\psi_b = \psi_b^{\text{coll}} + \psi_b^p$, and the lower boundary is $\psi(r = 0) = 0$. As shown in figure 6, the 2D free boundary condition equilibrium is presented with the same settings as those in the fixed boundary condition case, except the setting of boundary conditions. The final axial profiles of $\psi$ and $B_z$ at the upper boundary of the chamber are demonstrated in figures 6(b) and (c), respectively. Both compare the cases with and without the plasma. $B_z$ at the ‘upper’ boundary increases somewhat with the presence of plasma shown in figure 6(c), which is because the presence of plasma compresses the magnetic field lines outside the separatrix, causing the increase of $B_z$. It is found in figure 6(c) that the upper boundary magnetic field $B_u = 0.88147$ T at the midplane in the presence of plasma, not the preset $B_u = 1$ T, which is explained in appendix A. Also, the pressure and current density profiles of 1D equilibrium match well with the midplane profiles of 2D equilibrium, illustrated in figures 6(d) and (e). The external coil current required to form the target FRC shown in figure 6 is $8.179 \times 10^7$ A for each coil at free boundary equilibrium condition.

The $B_z$ radial profiles near the left boundary ($z = -0.72$ m) from fixed and free boundary are shown in figure 7. It is found that $B_z$ in the free boundary case is larger, i.e. the magnetic pressure is larger, for which the plasma is more compressed axially, so that the elongation ratio of FRC from free boundary condition is smaller than that from fixed boundary, which is found by comparing figure 6(a) with figure 4(a).

We have found [6] that four independent parameters $(\beta_s, \delta_s, B_u, R_F)$ can capture most of the FRC 1D equilibrium profile features. Whereas, five independent parameters $(\beta_s, \delta_s, B_u, R_F, l_e)$ can probably roughly describe a 2D FRC equilibrium. In the present work, we can control the first four parameters $(\beta_s, \delta_s, B_u, R_F)$ well. However, how to control the FRC length $l_e$ is still an open question, especially in the free boundary condition case.

For the case of free boundary condition, the method used in the present version of GSEQ-FRC is similar to the one that solves the free boundary equilibrium of traditional tokamak [25]. One difference between FRC and traditional tokamak is that plasma current flows outside the FRC separatrix so that axial boundaries in the left and right will truncate edge plasma current. Therefore, the impact of selecting different axial positions on the calculated equilibrium is discussed in appendix C. We also notice some previous studies of 2D FRC equilibria [17, 22, 23, 27], which are relevant to free boundary condition equilibria. However, they are different from our treatment or the standard treatment in tokamak [25] of the free boundary equilibrium. Therefore, the 2D free boundary condition equilibrium of FRC requires further investigation.
5. Factors affecting the shape of FRC separatrix

In early 2D numerical equilibrium studies, the racetrack-like separatrix is often obtained. However, both racetrack-like and ellipse-like separatrices were shown in early experiments. Spencer et al [28] investigated the factors affecting the shape of the separatrix based on MHD equilibrium, and concluded that steeper $P(\psi)$ can cause more ellipse-like separatrix. However, Suzuki et al [20,29] also studied this problem but obtained the opposite conclusion, i.e. the shape of the separatrix becomes racetrack-like as $P(\psi)$ becomes steeper at the separatrix. If we check how [20, 28, 29] obtained their conclusions, we find those conclusions are only qualitative. For example, the method that caused the steep $P(\psi)$ was not mentioned clearly in [28]. Whereas, the parameter $\gamma$ increases that changes others parameters to achieve the ellipse-like shape in [20, 29] (cf the model parameters of subplots (a) in figures 5 and 6 from [20]).

The factors affecting the shape of the separatrix are investigated quantitatively in this subsection to resolve the above. We use the following equation to describe the shape of the separatrix, which is defined in [30] with

$$\frac{r^2}{a^2} + \frac{|z|^m}{b^m} = 1,$$  \hspace{1cm} (13)

where $a$ is the radius of the separatrix at the midplane, $b$ is the half-length of the separatrix, and $m$ is the shape index. As parameter $m$ is larger, the shape of the separatrix becomes more racetrack-like. $m = 2$ is the elliptical shape in equation (13).

The fixed boundary condition equilibrium are shown in figure 8 with varying $\delta$, but fixed $\beta_s$. As $\delta$ decreases, the shape of the separatrix changes from ellipse-like to racetrack-like, and the parameter $m$ that best fits the shape of the separatrix with equation (13) increases as shown in figure 8(b). In this subsection, we use $-dp/dr$ to represent the pressure gradient. The pressure gradient near the separatrix is larger with the shape of separatrix becoming racetrack-like, which is demonstrated in figure 8(c).

Figure 9 illustrates the changes of fixed boundary condition equilibrium for different $\beta_s$ with fixed $\delta$. When $\beta_s$ increases, the shape of the separatrix changes from ellipse-like to racetrack-like, and the parameter $m$ also increases, which are presented in figures 9(a) and (b), respectively. Once again, as
shown in figure 9(c), the pressure gradient becomes larger as the shape of separatrix becomes racetrack-like.

From figures 8 and 9, we find that decreasing $\delta_s$ or increasing $\beta_s$ make the pressure gradient near the separatrix larger, and the shape of the separatrix becomes more racetrack-like.

Next, we do an analysis of the results in [20, 28, 29], where the derivative of the pressure with respect to $\psi$ is

$$
\frac{dP(\psi)}{d\psi} = \begin{cases} 
-c(1 + \epsilon \psi) & (\psi \leq 0) \\
-c e^{-\gamma \psi} & (\psi > 0)
\end{cases}
$$

where $c$, $\epsilon$ and $\gamma$ are constants. $\delta_s$ from equation (A.1) is expressed as

$$
\delta_s = \frac{1}{\gamma B_s R_s^2 \sqrt{1 - \beta_s}}
$$

From equation (15), it is found that an increase in $\gamma$ causes a decrease in $\delta_s$, thus a steeper pressure near the separatrix, and the shape of the separatrix changes from ellipse-like to racetrack-like. Equation (15) reasonably explains the conclusions in [20, 29]. The change of $\delta_s$ can also cause a change of $\beta_s$, which is presented in equation (15). However, when $\delta_s$ is fixed, the pressure gradient increases with $\beta_s$, and the shape of the separatrix becomes racetrack-like, which would be similar to figure 9. Thus, the results in [20, 29] are consistent with ours. However, our results are more clear and conclusive by controlling the pressure gradient parameter explicitly and quantitatively by using $\delta_s$ and $\beta_s$. A simple physics picture is that racetrack-like equilibrium has a larger axial gradient than the ellipse-like equilibrium, and therefore corresponds to a higher radial gradient.

6. Summary and conclusion

The GSEQ-FRC tool (https://github.com/hsxie/gseq), a 2D FRC G–S equilibrium simulation tool, has been developed and is applied to FRC equilibrium design. Unlike the conventional method, in GSEQ-FRC, the equilibria are solved with two parameters using the physical properties of FRC. Several examples with fixed boundary and free boundary conditions have been demonstrated. The properties of the FRC equilibrium with fixed and free boundaries are investigated systematically. Furthermore, the factors affecting the shape of the FRC separatrix are discussed quantitatively, and it is found that steeper pressure at the separatrix causes the shape of the separatrix to change from ellipse-like to racetrack-like. The GSEQ-FRC tool has advantages in equilibrium calculations, which can also be extended to FRC equilibrium reconstruction.

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Appendix A. The accuracy of the quasi-1D equilibrium

From equation (B.4) in [6], the general 1D force balance equation from $\mathbf{J} \times \mathbf{B} = \nabla P$ is

$$
P(r) + \frac{B_e^2(r)}{2 \mu_0} = \frac{B_m^2}{2 \mu_0} + \int \frac{B_z}{\mu_0} \left( \frac{\partial B_z}{\partial z} \right) dr.
$$

The second term on the right side is the magnetic field curvature effect at the midplane. In equation (A.1), we define $B_e \equiv \sqrt{2 \mu_0 P_m}$, where $P_m$ is the maximal pressure, i.e. the pressure at the O-point. The quasi-1D equilibrium $P + \frac{B_e^2}{2 \mu_0} = \frac{B_m^2}{2 \mu_0} = P_m$ ignoring the curvature effect can describe elongated FRCs well, where $B_e$ can be defined as $B_z$ at the wall as $P \to 0$. The quasi-1D equilibrium approximation is commonly used in both theoretical and experimental studies.

Figure A1 shows the radial profiles of the free boundary equilibrium at the midplane in figure 6. From the red dashed line and blue solid line in figure A1, it is found that the quasi-1D equilibrium matches well inside the separatrix, but the deviation increases outside the separatrix. The left and right sides of equation (A.1) are well balanced, demonstrated by the red dashed line and the black dashed line.

Although it is found from figure A1 that the magnetic field curvature effect is significant outside the separatrix in the 2D free boundary condition equilibrium, the 2D result of GSEQ for $P(r)$ and $J_\theta(r)$ still match well with the 1D result shown in figures 6(d) and (e). This implies that the solution approach used in GSEQ-FRC based on MRR model is robust, even when the midplane magnetic field curvature effect cannot be ignored.
The pressure profile with influences, and the midplane in figure 6. The GSEQ code is benchmarked with grass_ft [31]. The Appendix B. The benchmark of the GSEQ code

The GSEQ code is benchmarked with grass_ft [31]. The grass_ft is an FRC G-S equation solver with fixed boundary condition. In grass_ft code, the ‘upper’ boundary $\psi(r_w, z)$ is set with $r_w = \text{const}$:

$$\psi = \begin{cases} 
\psi_w, & (0 \leq z \leq z_c) \\
\frac{r_w}{2} + \frac{1}{\psi_w} \ln \left( \frac{p_w}{p_b} \right) \psi & (z > z_c)
\end{cases}$$

(B.1)

Here, $z_{\text{mir}}$ is the axial length from the midplane to the mirror end, $z_c$ is the axial position at which the mirror field critically influences, and $r_w$ is a control parameter for the mirror ratio. The pressure profile with $\psi$ is

$$p(\psi) = \begin{cases} 
\frac{p_s}{\psi_w} \ln \left( \frac{p_w}{p_s} \right) \psi & (\psi \geq 0) \\
\frac{1}{2} \frac{p_s}{\psi_w^2} \ln \left( \frac{p_w}{p_s} \right)^2 \psi^2 & (\psi < 0)
\end{cases}$$

(B.2)

where $p_s$, $\psi_w$, $p_w$ are coefficients. In equation (B.2), the normalized parameter of $p(\psi)$ is $\psi_w^2/2 \mu_0 r_w^2$. Note that in grass_ft code, $\psi > 0$ is inside the separatrix as shown in equation (B.2), which is opposite to the default definition in GSEQ code. Thus, changes are made in the GSEQ code to benchmark.

The results calculated by GSEQ and grass_ft code are shown in figure B1, where the free parameters in equation (B.2) are set with $p_s = 5.5$, $\psi_w = 1$ and $p_w = 1 \times 10^{-5}$. Figure B1(a) is the specific fixed boundary set by equation (B.1). Figures B1(b) and (c) show the contour of $\psi(r, z)$ from GSEQ and grass_ft equilibrium codes with same parameters.

Equilibrium with the axial boundary $z_{w} = \pm 0.75$ m is calculated in figures 4 and 6. A natural divertor (jet) structure exists at the axial end of the FRC. We would like to know how the setting of different axial boundaries influences the calculated equilibrium. The equilibrium for setting different axial boundaries is calculated in figure C1, where figure C1(a) illustrates the fixed boundary condition case and (b) shows the free boundary condition case. It can be concluded that the selection of the axial boundary position does not affect the separatrix in the calculation of the GSEQ-FRC tool.

Figure C2 illustrates the radial profiles of $B_z$ at different axial positions ($z = 0.7, 0.45, 0.25, 0$ m) from figure C1. The results indicate that the selection of different axial boundary positions only affects the $B_z$ profile near the axial boundary ($z = 0.7$ m for $z_{w} = 0.72$ m and $z = 0.45$ m for $z_{w} = 0.5$ m) in the fixed boundary condition case, and has little influence on the proximity to the FRC and free boundary condition case. Figures C1 and C2 suggest that the GSEQ-FRC tool is

Figure A1. The radial profiles of the free boundary equilibrium in the midplane in figure 6.

Figure B1. Comparison of the results of GSEQ and grass_ft code. (a) The ‘upper’ boundary with $r_v = -4$, $\psi_w = -1$, $z_c = 0.25$ and $z_{\text{mir}} = 0.5$, (b) the contour of $\psi(r, z)$ from GSEQ code, (c) the contour of $\psi(r, z)$ from grass_ft code, and (d) the difference of $\psi$ between GSEQ and grass_ft code. Other values are set to $R_w = 0.17$ m and $Z_w = 1.5$ m.

Appendix C. The influence of selecting the axial boundary on the equilibrium

The midplane influence, the axial position at which the mirror field critically influences, and $r_w$ is a control parameter for the mirror ratio. The pressure profile with $\psi$ is

$$\psi = \begin{cases} 
\psi_w, & (0 \leq z \leq z_c) \\
\frac{r_w}{2} + \frac{1}{\psi_w} \ln \left( \frac{p_w}{p_b} \right) \psi & (z > z_c)
\end{cases}$$

(B.1)

Here, $z_{\text{mir}}$ is the axial length from the midplane to the mirror end, $z_c$ is the axial position at which the mirror field critically influences, and $r_w$ is a control parameter for the mirror ratio. The pressure profile with $\psi$ is

$$p(\psi) = \begin{cases} 
\frac{p_s}{\psi_w} \ln \left( \frac{p_w}{p_s} \right) \psi & (\psi \geq 0) \\
\frac{1}{2} \frac{p_s}{\psi_w^2} \ln \left( \frac{p_w}{p_s} \right)^2 \psi^2 & (\psi < 0)
\end{cases}$$

(B.2)

where $p_s$, $\psi_w$, $p_w$ are coefficients. In equation (B.2), the normalized parameter of $p(\psi)$ is $\psi_w^2/2 \mu_0 r_w^2$. Note that in grass_ft code, $\psi > 0$ is inside the separatrix as shown in equation (B.2), which is opposite to the default definition in GSEQ code. Thus, changes are made in the GSEQ code to benchmark.

The results calculated by GSEQ and grass_ft code are shown in figure B1, where the free parameters in equation (B.2) are set with $p_s = 5.5$, $\psi_w = 1$ and $p_w = 1 \times 10^{-5}$. Figure B1(a) is the specific fixed boundary set by equation (B.1). Figures B1(b) and (c) show the contour of $\psi(r, z)$ from GSEQ and grass_ft equilibrium codes with same parameters.

Equilibrium with the axial boundary $z_{w} = \pm 0.75$ m is calculated in figures 4 and 6. A natural divertor (jet) structure exists at the axial end of the FRC. We would like to know how the setting of different axial boundaries influences the calculated equilibrium. The equilibrium for setting different axial boundaries is calculated in figure C1, where figure C1(a) illustrates the fixed boundary condition case and (b) shows the free boundary condition case. It can be concluded that the selection of the axial boundary position does not affect the separatrix in the calculation of the GSEQ-FRC tool.

Figure C2 illustrates the radial profiles of $B_z$ at different axial positions ($z = 0.7, 0.45, 0.25, 0$ m) from figure C1. The results indicate that the selection of different axial boundary positions only affects the $B_z$ profile near the axial boundary ($z = 0.7$ m for $z_{w} = 0.72$ m and $z = 0.45$ m for $z_{w} = 0.5$ m) in the fixed boundary condition case, and has little influence on the proximity to the FRC and free boundary condition case. Figures C1 and C2 suggest that the GSEQ-FRC tool is
Figure C1. The shape of the separatrix with fixed boundary condition and free boundary condition equilibrium for different axial boundary positions, the blue solid line is $z_w = \pm 0.85$ m, the dark red dashed line is $z_w = \pm 0.75$ m, the orange dashed line is $z_w = \pm 0.50$ m for fixed-boundary and $z_w = \pm 0.35$ m for free-boundary equilibrium, $\psi_{in}$ and $\psi_{out}$ are the $\psi$ inside and outside the separatrix, respectively. (a) Fixed boundary equilibrium, (b) free boundary equilibrium.

Figure C2. Radial magnetic field profiles at different axial positions for fixed and free boundary condition equilibrium. (a) The axial position $z = 0.7$ m, (b) the axial position $z = 0.45$ m, (c) the axial position $z = 0.25$ m, and (d) the axial position $z = 0$ m.

Robust, and that the jet has little impact on the equilibrium from GSEQ-FRC tool.

Detailed examination shows that the plasma current density outside the separatrix is one order of magnitude smaller, and only exists in a small region near the machine axis. Therefore, the truncation of this jet current has a negligible effect on the FRC equilibrium calculations.

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**References**

[1] Armstrong W.T. et al 1981 Field-reversed experiments (FRX) on compact toroids Phys. Fluids 24 2068

[2] Finn J.M. et al 1982 Field-reversed configurations with a component of energetic particles Nucl. Fusion 22 1433

[3] Tuszewski M. 1988 Field reversed configurations Nucl. Fusion 28 2033

[4] Steinhauer L.C. 2011 Review of field-reversed configurations Phys. Plasmas 18 070501

[5] Guo H.Y. et al 2015 Achieving a long-lived high-beta plasma state by energetic beam injection Nat. Commun. 6 6897

[6] Ma H.J. et al 2021 Two-parameter modified rigid rotor radial equilibrium model for field-reversed configurations Nucl. Fusion 61 036046

[7] Tuszewski M. 1981 Excluded flux analysis of a field-reversed plasma Phys. Fluids 24 2126

[8] Okada S., Kiso Y., Goto S. and Ishimura T. 1989 Reduction of the density profile of a field-reversed configuration plasma from detailed interferometric measurements J. Appl. Phys. 65 4625

[9] Deng B.H., Kinley J.S. and Schroeder J. 2012 Electron density and temperature profile diagnostics for C-2 field reversed configuration plasmas Rev. Sci. Instrum. 83 10E339

[10] Deng B.H., Beall M., Schroeder J., Settles G., Feng P., Kinley J.S., Gota H. and Thompson M.C. 2016 High sensitivity far infrared laser diagnostics for the C-2U advanced beam-driven field-reversed configuration plasmas Rev. Sci. Instrum. 87 11E125

[11] Steinhauer L.C. and Intrator T.P. 2009 Equilibrium paradigm for field-reversed configurations and application to experiments Phys. Plasmas 16 072501

[12] Steinhauer L. et al 2014 Two-dimensional interpreter for field-reversed configurations Phys. Plasmas 21 082516

[13] Lee K.Y. 2020 Generalized radial profile of field-reversed configurations based on symmetrical properties Nucl. Fusion 60 046010

[14] Grad H. and Rubin H. 1958 Hydromagnetic equilibria and force-free fields J. Nucl. Energy 7 284

[15] Shafranov V.D. et al 1963 Equilibrium of a toroidal plasma in a magnetic field J. Nucl. Energy, Part C Plasma Phys. 5 251

[16] Marder B. and Weitzner H. 1970 A bifurcation problem in E-layer equilibria Plasma Phys. 12 435

[17] Spencer R.L. et al 1982 Free boundary field-reversed configuration (FRC) equilibria in a conducting cylinder Phys. Fluids 25 1365

[18] Hewett D.W. et al 1985 Two-dimensional equilibria of field-reversed configurations in a perfectly conducting cylindrical shell Phys. Fluids 26 1299

[19] Suzuki K. 1991 Effect of the mirror field on the averaged $\beta$ value in field reversed configuration J. Phys. Soc. Japan 60 3186
[20] Suzuki Y., Okada S. and Goto S. 2000 Two-dimensional numerical equilibria of field-reversed configuration in the strong mirror field *Phys. Plasmas* **7** 4062

[21] Steinhauer L.C., Roche T. and Steinhauer J.D. 2020 Anatomy of a field-reversed configuration *Phys. Plasmas* **27** 112508

[22] Kako M., Ishimura T. and Amano T. 1983 Equilibria of field-reversed configuration with subsidiary coils *J. Phys. Soc. Japan* **52** 3056

[23] Kanki T., Suzuki Y., Okada S. and Goto S. 1999 Numerical simulation of magnetic compression on a field-reversed configuration plasma *Phys. Plasmas* **6** 4672

[24] Gerhardt S.P., Belova E., Inomoto M., Yamada M., Ji H., Ren Y. and Kuritsyn A. 2006 Equilibrium and stability studies of oblate field-reversed configurations in the magnetic reconnection experiment *Phys. Plasmas* **13** 112508

[25] Johnson J.L. *et al* 1979 Numerical determination of axisymmetric toroidal magnetohydrodynamic equilibria *J. Comput. Phys.* **32** 2

[26] Yambe K., Inomoto M. and Okada S. 2013 Influence of bias magnetic field configuration on equilibrium of field-reversed configuration plasma sustained by rotating magnetic field *Fusion Sci. Technol.* **63** 147

[27] Fuentes N.O. and Gavarini H.O. 1995 ECMC, a portable two-dimensional code for plasma equilibrium computation on coaxial-multiple-coil systems *Comput. Phys. Commun.* **90** 169–88

[28] Spencer R.L. and Tuszewski M. 1985 Experimental and computational equilibria of field-reversed configurations *Phys. Fluids* **28** 1810

[29] Suzuki Y. *et al* 1999 Analysis of averaged B value in two dimensional equilibrium of a field-reversed configuration with end mirror fields *J. Plasma Fusion Res. Ser.* **2** 218

[30] Ohkuma Y., Hiroi M., Ikeyama T. and Nogi Y. 2010 Separatrix shape of field-reversed configuration *Phys. Plasmas* **17** 042502

[31] Takahashi T., Inoue K., Iwasawa N., Ishizuka T. and Kondoh Y. 2004 Losses of neutral beam injected fast ions due to adiabaticity breaking processes in a field-reversed configuration *Phys. Plasmas* **11** 3131