Sum Of Divisors Function Inequality

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Abstract

This short note provides a sharper upper bound of a well known inequality for the sum of divisors function. This is a problem in pure mathematics related to the distribution of prime numbers. Furthermore, the technique is completely elementary.

1 Introduction

Let $n \geq 1$ be an integer, and let $\sum_{d|n} d$ be the sum of divisors function. The earliest work on the extreme values of the sums of divisors functions appears to be the limit supremum

$$\lim_{n \to \infty} \sup \frac{\sigma(n)}{n \log \log n} = e^\gamma,$$

which was established by Gronwall, see [13], [16, p. 350]. Subsequently, conditional on the Riemann hypothesis, Ramanujan proved the sum of divisors function inequality

$$\sigma(n) \leq e^\gamma n \log \log n + O((\log n)^{-1/2})$$

for all integers $n \geq 3$, see [25], [26, Equation 382]. The explicit result, better known as the Ramanujan-Robin inequality, states the same inequality for any integer $n \geq 5041$, see [23]. Various partial proofs are given in the literature, confer [4], [5], [7], [8], [12], [15], [22], [28], and [30].

The extreme values of the sum of divisors function occur on a subset of highly composite numbers. The minima occur on the set of primes, and the maxima occur on the set of colossally abundant integers. Extremely abundant integers, colossally abundant integers, etc, are integers related to the primorial integers $n = 2^{v_2} \cdot 3^{v_3} \cdot \cdots p^{v_p}$, where $p_k$ is the $k$th prime, and $v_k \geq 1$. The precise classifications of the various classes of highly composite integers are given in [25], [3], [17], and [6], et alii. Other related results appear in [5], [8], [30]. This note proposes a proof of the following upper bound.
Theorem 1.1. If $n \geq 3$ is an integer then

$$\frac{\sigma(n)}{n} \leq e^{\gamma} \log \log n + O\left(e^{-c\sqrt{\log \log n}}\right),$$

(3)

where $c > 0$ is an absolute constant.

Currently the best unconditional estimate of this arithmetical function is the following.

Theorem 1.2. Let $n \geq 2521$, then $\sigma(n)/n \leq e^{\gamma} \log \log n + c/(\log \log n)^2$, where $c > 0$ is a small constant.

The same result appears in several papers [23], [22], [2] and very recently in [1], but with different constants. The numerical data for $n \in [5041, 10^{10}]$ was compiled in [6]. Beside the small improvements produced by the small improvements among the zero free regions of the zeta function, see (12), the inequality (3) is the best unconditional result possible.

On the other hand, there are several conditional criteria; some of these are listed below.

Theorem 1.3. ([23]) Let $n \in \mathbb{N}$ be an integer, and let $\sigma(n) = \sum_{d \mid n} d$ be the sum of divisors function. Then

(i) If the Riemann Hypothesis is true, then, for each $n \geq 5041$,

$$\sigma(n) < e^{\gamma} n \log \log n.$$

(ii) If the Riemann Hypothesis is false, then, there exists constants $0 < \beta < 1/2$ and $c > 0$ such that

$$\sigma(n) \geq e^{\gamma} n \log \log n + \frac{cn \log \log n}{(\log n)^\beta}$$

holds for infinitely many $n$.

The parameter $1 - b < \beta < 1/2$ arises from the possibility of a zero $\rho = b + it \in \mathbb{C}$ of the analytic continuation of the zeta function

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \sum_{n \geq 1} \frac{(-1)^{n+1}}{n^s}$$

(4)

on the half plane $b = \Re(\rho) > 1/2$ if the Riemann hypothesis is false. This in turns implies the existence of more or fewer primes than expected in some intervals. For example, under this condition, the number of primes would be

$$\pi(x) = \text{li}(x) + O(x^{b+\varepsilon}) > \text{li}(x) + O(x^{1/2+\varepsilon})$$

(5)

infinitely often, some of this material is discussed in [17]. The effect of the zeros of the zeta function on the distribution of primes is readily revealed by the explicit formulas, consult the literature. All these inequalities involves the Euler constant $\gamma$. Another well known conditional result offers a constant free inequality, and it is written entirely in terms of the integer $n \geq 1$. 
Theorem 1.4. ([17]) Let $H_n = \sum_{m \leq n} 1/m$ be the harmonic series. For each $n \geq 1$, the inequality

$$\sigma(n) < e^{H_n} \log H_n + H_n$$

is equivalent to the Riemann Hypothesis.

The proof of Theorem 1.1 is given in Section 3. The other sections are optional. These sections cover some background materials focusing on the sum of divisors function, and some associated finite sums and products over the prime numbers.

2 Auxiliary Results

For a complex number $s \in \mathbb{C}$, and a real number $x \geq 1$ the rho series is defined by

$$S_s(x) = -s \sum_{\rho} \frac{x^{\rho-s}}{\rho(\rho-s)}, \quad (7)$$

where the index $\rho \in \mathbb{C}$ ranges over the nontrivial zeros of the zeta function, see [26, p. 134].

Lemma 2.1. Let $x > 1$ be a large number and let $\rho$ denotes the nontrivial zero of the zeta function, then

(i) \[ \sum_{\rho} \frac{x^{\rho-1}}{\rho(1-\rho)} \ll e^{-c\sqrt{\log x}} \]

Unconditionally,

(ii) \[ \sum_{\rho} \frac{x^{\rho-1}}{|\rho|^2} \ll \frac{1}{\sqrt{x}} \]

Conditioned on the RH,

where $c > 0$ is a constant.

Proof. (i) Let $\rho = \sigma + it = 1 - c_0/\log t + it$ be a nontrivial zero of the zeta function, where $t \in [14, T]$, see [9, p. 87], [14], [10], et cetera. For any small number $\varepsilon > 0$, let $T = e^{c_1(\log x)^{\varepsilon}}$. Then

$$x^{-c_0/\log T} \ll e^{-c(\log x)^{1-\varepsilon}}. \quad (8)$$

The choice of $\varepsilon = 1/2$ leads to the standard error term

$$x^{-c_0/\log T} \ll e^{-c\sqrt{\log x}}. \quad (9)$$

where $c_0, c_1, c > 0$ are constants, see the calculation of error term of the prime number theorem in [19, Theorem 6.9, page 181]. Unconditionally, the series over the nontrivial zeros of the zeta function has the value

$$B = \sum_{\rho} \frac{1}{\rho(1-\rho)} \ll 1 \quad (10)$$
and conditioned on the RH, the series over the nontrivial zeros of the zeta function has the value
\[ B = \sum_{\rho} \frac{1}{|\rho|^2} = -\frac{\gamma}{2} - 1 + \frac{1}{2} \log 4\pi \] (11)
see [9, p. 163]. Multiplying (9) and (10) complete the verification. The verification of statement (ii) is similar to the proof for statement (i) but it uses \( \rho = \sigma + it = 1/2 + it \), where \( t \in \mathbb{R} \) and (11).

The sharper estimate
\[ \rho = \sigma + it = 1 - \frac{c_0}{(\log \log t)^{2/3}(\log \log \log t)^{1/3}} + it \] (12)
of the nontrivial zero of the zeta function provides some improvement on these estimates, but these sharper estimates are not required in this note, see [10, Theorem 1] for the precise details.

A well known upper bound of the sum of divisor function has the form shown below.

**Theorem 2.1.** ([26, Equation 382, p. 143]) If \( n \geq 3 \) is an integer then
\[ \frac{\sigma(n)}{n} \leq e^\gamma \left( \log \log n + \frac{a_0}{\sqrt{\log n}} + S_1(\log n) + \frac{a_1}{\sqrt{\log n \log \log n}} \right), \]
where
\[ S_1(x) = \sum_{\rho} x^{\rho - 1}, \]
and \( a_0 < 0 \) and \( a_1 \) are constants.

**Theorem 2.2.** ([26, Equation 382, p. 143]) Assume the RH. If \( n \geq 3 \) is an integer then
\[ \frac{\sigma(n)}{n} \leq e^\gamma \log \log n + \frac{c}{\sqrt{\log n}}, \]
where \( c > 0 \) is a constant.

**Proof.** This follows from Lemma 2.1 and Theorem 2.1.

### 3 The Main Result

The verification of Theorem 1.1 is a straightforward application of the unconditional estimate for the rho series \( S_s(x) \) derived in Section 2.

**Proof of Theorem 1.1.** Let \( n \geq 5041 \) be an integer and let \( x \asymp \log n \). Substituting the unconditional estimate for the rho series
\[ S_1(\log n) \ll e^{-c\sqrt{\log \log n}}, \] (13)
confer Lemma 2.1, into the expression given in Theorem 2.1 yields
\[
\frac{\sigma(n)}{n} \leq e^\gamma \left( \log \log n + \frac{a_0}{\sqrt{\log n}} + S_1(\log n) + \frac{a_1}{\sqrt{\log n \log \log n}} \right) \\
= e^\gamma \log \log n + O \left( e^{-c\sqrt{\log n}} \right)
\]
where \( c > 0 \) is a constant. \( \blacksquare \)

As stated before, there are several upper bounds of \( \sigma(n)/n \) in the literature. All these estimates are of the equivalent form
\[
\frac{\sigma(n)}{n} \leq e^\gamma \log \log n + O \left( \frac{1}{(\log \log n)^2} \right),
\]
see [1, Theorem 1.3], et alii. Since
\[
e^{-c\sqrt{\log \log n}} \ll \frac{1}{(\log \log n)^b}
\]
for any constant \( b \geq 0 \), the result in Theorem 1.1 seems to be the best possible short of the conditional result \( \sigma(n)/n < e^\gamma \log \log n \).

4 Representations and Identities

The sum of divisors function \( \sum_{d|n} d \) is ubiquitous in number theory. It appears in the analysis many different problems in pure and applied mathematics. Its product representation in (18) unearths its intrinsic link to the distribution of the prime numbers. The totient function is defined by \( \varphi(n) = \#\{k : \gcd(k, n) = 1\} \).

**Lemma 4.1.** Let \( n \geq 1 \) be an integer, and let the symbol \( p^v \ | \ | n \) denotes the maximum prime power divisor. Then

(i) The Euler totient function has the product formula
\[
\varphi(n) = n \prod_{p|n} \left( 1 - \frac{1}{p} \right).
\]

(ii) The sum of divisors function has the product formula
\[
\sigma(n) = \prod_{p^v|n} \left( 1 + p + p^2 + \cdots + p^v \right).
\]

These representations of the divisor and totient functions are well known and/or easy to establish.

**Lemma 4.2.** Let \( n \geq 1 \) be an integer, and let the symbol \( p^v \ | \ | n \) denotes the maximum prime power divisor. Then, the sum of divisors function has the product formula
\[
\frac{\sigma(n)}{n} = \frac{n}{\varphi(n)} \prod_{p^v|n} \left( 1 - \frac{1}{p^{v+1}} \right).
\]
Proof. Use Lemma 4.1, and the geometric series formula $\sum_{0 \leq k \leq x} r = (1 - r^{x+1})/(1 - r)$ to evaluate the sigma-phi identity

$$\frac{\sigma(n) \varphi(n)}{n} = \prod_{p \mid n} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^v}\right) \prod_{p \mid n} \left(1 - \frac{1}{p}\right) = \prod_{p \mid n} \left(1 - \frac{1}{p^{v+1}}\right). \quad (20)$$

To illustrate the negligible effect of negligible multiplication by a prime power, a quantitative expression is computed in the next result.

**Lemma 4.3.** Let $p \geq 2$ be a prime, and let $t \geq 0$ be an integer. If $p^v \mid n$, then

$$\frac{\sigma(p^t n)}{p^t n} = \frac{\sigma(n)}{n} \cdot \frac{1 - p^{-(t+v+1)}}{1 - p^{-(v+1)}}. \quad (21)$$

**Proof.** By hypothesis $p^v \mid n$. Since $\sigma(n)$ is multiplicative, it is sufficient to observe the effect of extra prime power factor $p^t$ with $t \geq 0$ on the value $\sigma(p^v)/p^v$. To achieve this goal, modify the basic fact $\sigma(p^v) = (p^{v+1} - 1)/(p - 1)$ to isolate the effect of multiplication by $p^t$. That is,

$$\frac{\sigma(p^{t+v})}{p^{t+v}} = \frac{p^{t+v+1} - 1}{p^{t+v}(p - 1)} \cdot \frac{p^{v+1} - 1}{p^v(p - 1)} \cdot \frac{p^v(p - 1)}{p^{v+1} - 1}$$

$$= \frac{\sigma(p^v)}{p^v} \cdot \frac{p^{t+v+1} - 1}{p^{t+v}(p - 1)} \cdot \frac{p^v(p - 1)}{p^{v+1} - 1}$$

$$= \frac{\sigma(p^v)}{p^v} \cdot \frac{p^{t+v+1} - 1}{p^{t+v+1} - p^t} \cdot \frac{1}{p^{v+1} - 1}$$

$$= \frac{\sigma(p^v)}{p^v} \cdot \frac{1 - p^{-(t+v+1)}}{1 - p^{-(v+1)}}. \quad (22)$$

It is immediate that $f(t) = \sigma(p^tn)/p^tn < 2 \log \log n$ is a slowly increasing function of $t \geq 0$, but it is bounded above.

## 5 Extreme Values

The sum of divisors function is an oscillatory function, its values oscillate from its minimum $\sigma(n) = n + 1$ at the prime integers $n$ to its maximum $\sigma(n) < 2n \log \log n$ at the extremely abundant integers $n$. 
5.1 Asymptotic Extrema

The suprema of the sum of divisors function over various subsets of integers are known.

**Theorem 5.1.** ([13]) Let \( n \geq 1 \) be an integer. Then

(i) The limit supremum over the integers is
\[
\lim_{n \to \infty} \sup \frac{\sigma(n)}{n \log \log n} = e^\gamma.
\]

(ii) The limit supremum over the odd integers is
\[
\lim_{\text{odd } n \to \infty} \sup \frac{\sigma(n)}{n \log \log n} = \frac{e^\gamma}{2}.
\]

(iii) The limit supremum over the squarefree integers is
\[
\lim_{\text{squarefree } n \to \infty} \sup \frac{\sigma(n)}{n \log \log n} = \frac{6e^\gamma}{\pi^2}.
\]

Refer to [16, p. 353], and similar references for proofs and other details on the maximal order of this function.

5.2 Conditional Lower and Upper Bounds

**Theorem 5.2.** ([21]) Let \( n_k = 2 \cdot 3 \cdots p_k \) be the product of the first \( k \geq 1 \) primes.

(i) If the Riemann Hypothesis is true, then, for each \( n_k \geq 5041 \),
\[
\frac{n_k}{\varphi(n_k)} > e^\gamma \log \log n_k
\]
for all \( k \geq 1 \).

(ii) If the Riemann Hypothesis is false, then,
\[
\frac{n_k}{\varphi(n_k)} < e^\gamma \log \log n_k \quad \text{and} \quad \frac{n_k}{\varphi(n_k)} > e^\gamma \log \log n_k
\]
occur for infinitely many \( k \geq 1 \).

**Lemma 5.1.** If the RH is false, then, the ratio \( \sigma(n)/n \) satisfies the upper bound
\[
\frac{\sigma(n)}{n} \leq \log \log n \tag{23}
\]
for infinitely many highly composite integers \( n \geq 1 \).

**Proof.** Apply Theorem 5.2 to the phi-sigma inequality
\[
\frac{\sigma(n)}{n} = \frac{n}{\varphi(n)} \prod_{p \mid n} \left(1 - \frac{1}{p^{e+1}}\right) \tag{24}
\]
\[
\leq \frac{n}{\varphi(n)} \leq \log \log n
\]
for infinitely many highly composite integers \( n \).
5.3 Unconditional Lower and Upper Bounds

Several lower and upper estimates derived by several methods are computed in this subsection.

Upper Bound I. An upper estimate derived from the sigma-phi identity, refer to Lemma 4.2, is computed below.

**Lemma 5.2.** Let \( n \in \mathbb{N} \) and let \( p \mid n \), then the ratio \( \sigma(p^t n)/p^t n \) remains uniformly bounded and independent of the prime power \( p^t \). More precisely

1. For any integer \( t \geq 1 \),
   \[
   \frac{\sigma(p^t n)}{p^t n} < \frac{n}{\varphi(n)}.
   \]
2. For any integer \( r \geq 1 \),
   \[
   \frac{\sigma(n^r)}{n^r} < \frac{n}{\varphi(n)}.
   \]

**Proof.** (i) Take the sigma-phi representation of the sum of divisors function, see Lemma 4.2, and use the fact that \( p \mid n \) to simplify the ratio \( p^t n/\varphi(p^t n) \):

\[
\frac{\sigma(p^t n)}{p^t n} = \frac{p^t n}{\varphi(p^t n)} \prod_{p^v \mid p^t n} \left(1 - \frac{1}{p^{v+1}}\right) < \prod_{p^v \mid n} \left(1 - \frac{1}{p^{v+1}}\right) = \frac{n}{\varphi(n)}.
\]

The product on the right side of third line, which is bounded by 1, absorbs the effect of multiplication by a prime power. (ii) The proof for this case is similar. □

Lower Bound II. An upper estimate derived from the prime divisors of an integer is computed in this subsection. Let \( n \geq 1 \) be a highly composite number, and let \( p \mid n \) be the largest prime divisor of \( n \). For the single prime \( p \approx \log n \), the sum of divisors function has the unconditional upper bound

\[
\sum_{d \mid pn} \frac{1}{d} \geq \left(1 + \frac{1}{2p}\right) \sum_{d \mid n} \frac{1}{d} \geq \left(1 + \frac{1}{4\log n}\right) \sum_{d \mid n} \frac{1}{d} \geq \sum_{d \mid n} \frac{1}{d} \]  

(26)

A more general estimate is given below.
Lemma 5.3. Let \( n \in \mathbb{N} \) be a highly composite integer, and let \( r \geq 2 \). Then

(i) \[
\sum_{d \mid n^r} \frac{1}{d} \geq \left( 1 + \frac{\log \log n + c_0}{4 \log n} + O \left( \frac{1}{(\log n)^2} \right) \right) \sum_{d \mid n} \frac{1}{d},
\]

(ii) \[
\sum_{d \mid n^r} \frac{1}{d} \leq \left( 1 + \frac{4 \log \log n + c_0}{\log n} + O \left( \frac{1}{(\log n)^2} \right) \right) \sum_{d \mid n} \frac{1}{d},
\]

where \( c_0 > 0 \) is a nonnegative constant.

Proof. (i) Expand the sum into several subsums:

\[
\sum_{d \mid n^r} \frac{1}{d} = \sum_{d \mid n} \frac{1}{d} + \sum_{1 \leq t \leq r} \sum_{p^v \mid |n} \frac{1}{p^{v+t}} \sum_{d \mid n} \frac{1}{d} + \cdots
\]

\[
\geq \sum_{d \mid n} \frac{1}{d} + \frac{1}{2} \sum_{1 \leq t \leq r} \sum_{p^v \mid |n} \frac{1}{p^{v+t}} \sum_{d \mid n} \frac{1}{d}
\]

\[
= \left( 1 + \frac{1}{2} \sum_{1 \leq t \leq r} \sum_{p^v \mid |n} \frac{1}{p^{v+t}} \right) \sum_{d \mid n} \frac{1}{d}.
\]

(27)

The triple sum collects some of the divisors \( d \mid n^r \) not included in the basic sum of divisors \( \sum_{d \mid n} 1/d \). By Lemma 8.1, the prime power divisors \( p^v \mid |n \) have the upper bound \( p^v < 2 \log n \), including the largest prime divisor \( p_k < 2 \log n \).

The first inner sum for \( t = 1 \) has the lower estimate

\[
\sum_{p^v \mid |n} \frac{1}{p^{v+1}} \geq \sum_{p^v \mid |n} \frac{1}{2 \log n \ p}
\]

\[
= \frac{1}{2 \log n} \sum_{p \leq \log n} \frac{1}{p}
\]

\[
\geq \frac{\log \log n + \gamma + O(1/\log n)}{2 \log n}
\]

\[
\geq \frac{\log \log n + \gamma}{2 \log n} + O \left( \frac{1}{(\log n)^2} \right).
\]

(28)
The other inner double sum for the range $2 \leq t \leq r$ has the lower estimate
\[
\sum_{p^{i} \mid n, 2 \leq t \leq r} \frac{1}{p^{t+i}} \geq \sum_{p^{i} \mid n, 2 \leq t \leq r} \frac{1}{2 \log n \ p^{t}}
= \frac{1}{2 \log n} \sum_{p \leq \log n, 2 \leq t \leq r} \frac{1}{p^{t}}
\geq \frac{1}{2 \log n} \sum_{p \leq \log n} \frac{1}{p^{2}} \left( \frac{1-p^{-r-1}}{1-p^{-1}} \right)
\geq \frac{1}{2 \log n} \sum_{p \leq \log n} \frac{1}{p^{2}}
= O\left( \frac{1}{(\log n)^{2}} \right),
\]
where $\pi(\log n) \geq \log n/2 \log \log n$. Summing everything yields
\[
\left( 1 + \frac{1}{2} \sum_{1 \leq t \leq r} \sum_{p^{i} \mid n} \frac{1}{p^{t+i}} \right) \sum_{d \mid n} \frac{1}{d} = \left( 1 + \frac{\log \log n + c_{0}}{4 \log n} + O\left( \frac{1}{(\log n)^{2}} \right) \right) \sum_{d \mid n} \frac{1}{d},
\]
where $c_{0} = \gamma > 0$ is a constant. The proof of (ii) is similar. □

### 6 Prime Harmonic Sums

**Lemma 6.1.** ([20, Lemma 2.6.]) For a positive integer $q \geq 1200$ and $x > 50q^{2}$,
\[
\sum_{\substack{50q^{2} < p \leq x \mod q \leq 200}} \frac{1}{p} < \frac{1}{\varphi(q)} \left( \log \log x - \log \log 50q^{2} - \frac{1.5}{\log x} + \frac{2.5}{(\log x)^{2}} + \frac{1.5}{\log 50q^{2}} \right).
\]

**Lemma 6.2.** If $x > 200$ is a real number, then
\[
\sum_{2 \leq p \leq x} \frac{1}{p} < \log \log x + 0.564754 - \frac{1.5}{\log x} + \frac{2.5}{(\log x)^{2}}.
\]

**Proof.** Set $q = 2$ in Lemma 6.1 to obtain
\[
\sum_{200 < p \leq x} \frac{1}{p} < \log \log x - \log \log 200 - \frac{1.5}{\log x} + \frac{2.5}{(\log x)^{2}} + \frac{1.5}{\log 200}.
\]
Completing the last finite sum yields
\[
\sum_{2 \leq p \leq x} \frac{1}{p} < \sum_{2 \leq p \leq 200} \frac{1}{p} + \log \log x - \log \log 200 - \frac{1.5}{\log x} + \frac{2.5}{(\log x)^{2}} + \frac{1.5}{\log 200}
= \log \log x + 0.564754 - \frac{1.5}{\log x} + \frac{2.5}{(\log x)^{2}}
\]
(32)
since
\[
\sum_{2 \leq p \leq 200} \frac{1}{p} - \log \log 200 + \frac{1.5}{\log 200} = 0.564754 \ldots.
\]
(33)
7 Prime Harmonic Products

The prime harmonic products are sine qua non in number theory. It has a natural link to the totient function $\varphi(n)$, the sieve of Eratosthenes, and other related concepts. Accurate estimates of this product and related products are essential in a variety of calculations in number theory.

**Lemma 7.1.** ([24, Theorems 7, 8]) Let $x \geq 2$ be a large number. Then

(i) \[ \left| \frac{1}{e^\gamma \log x} \prod_{p \leq x} \left( 1 - \frac{1}{p} \right)^{-1} - 1 \right| < \frac{1}{2(\log x)^2}. \]

(ii) \[ \left| (e^\gamma \log x) \prod_{p \leq x} \left( 1 - \frac{1}{p} \right) - 1 \right| < \frac{1}{2(\log x)^2}. \]

These are improved versions of the original works by Mertens, see [24], and [29] for more details. The Euler constant is defined by $\gamma = \lim_{x \to \infty} \left( \sum_{n \leq x} \frac{1}{n} - \log x \right) = 0.577215665\ldots$, see [11, p. 28], [18, Theorem 2.2.1] for several definitions of this number and similar references.

**Lemma 7.2.** ([27, Corollary 3]) Let $x \geq 14$ be a large number. If the Riemann hypothesis holds, then

(i) \[ \left| \frac{1}{e^\gamma \log x} \prod_{p \leq x} \left( 1 - \frac{1}{p} \right)^{-1} - 1 \right| < \frac{\log x + 5}{8\pi \sqrt{x}}. \]

(ii) \[ \left| (e^\gamma \log x) \prod_{p \leq x} \left( 1 - \frac{1}{p} \right) - 1 \right| < \frac{3 \log x + 5}{8\pi \sqrt{x}}. \]

**Theorem 7.1.** (Mertens) Let $n \geq 1$ be an integer. Then

(i) The limit supremum over the prime is

\[ \lim_{x \to \infty} \sup \frac{1}{\log x} \prod_{p \leq x} \left( 1 - \frac{1}{p} \right)^{-1} = e^\gamma. \]

(ii) The limit supremum over the prime is

\[ \lim_{x \to \infty} \sup \frac{1}{\log x} \prod_{p \leq x} \left( 1 + \frac{1}{p} \right)^{-1} = \frac{6e^\gamma}{\pi^2}. \]

Some references and other information on these limits are available in see [11, p. 31].
8 Highly Composite Numbers

Let \( p_k \) be the \( k \)th prime in increasing order, and let \( v_p = \max\{m : p^m \mid n\} \) is the \( p \)-adic valuation. Extremely abundant integers, and colossally abundant integers are related to the primorial integers \( n = 2^{v_2} \cdot 3^{v_3} \cdots p^{v_p} \), but the exponents have certain multiplicative structure \( 1 \leq v_p \leq \cdots \leq v_3 \leq v_2 \).

**Definition 8.1.** Let \( d(n) = \sum_{d \mid n} 1 \). An integer \( n \in \mathbb{N} \) is called highly composite if and only if \( d(m)1 < d(n) \) for all integers \( m < n \).

**Definition 8.2.** Let \( \sigma(n) = \sum_{d \mid n} d \). An integer \( n \in \mathbb{N} \) is called colossally abundant if and only if

\[
\frac{\sigma(m)}{m^{1+\varepsilon}} < \frac{\sigma(n)}{n^{1+\varepsilon}}
\]

for all integers \( m < n \), and some small number \( \varepsilon > 0 \).

**Definition 8.3.** An integer \( n \geq 10080 \) is called extremely abundant if and only if

\[
\frac{\sigma(m)}{m \log \log m} < \frac{\sigma(n)}{n \log \log n}
\]

for all integers \( m < n \).

These numbers are studied in [25], [3], [17], [6], [22], et alii.

Fixed a highly composite integer \( n \geq 1 \), and let \( n_r = n^r \) with a parameter \( r \geq 1 \).

The sequence of functions

\[
\frac{\sigma(n_r)}{n_r} < \frac{\sigma(n_{r+1})}{n_{r+1}} < \frac{\sigma(n_{r+1})}{n_{r+1}} < \cdots
\]

is strictly monotonically increasing, but bounded above by \( e^\gamma \log \log n \), see equation (36).

**Lemma 8.1.** ([3, Theorem 2]) Let \( n \geq 1 \) be a large highly composite integer, then

(i) Unconditionally, the largest prime divisor \( p \mid n \) has the asymptotic

\[
p = (\log n) \left( 1 + O\left( \frac{1}{(\log \log n)^2} \right) \right).
\]

(ii) Modulo the Riemann hypothesis, the largest prime divisor \( p \mid n \) has the asymptotic

\[
p = (\log n) \left( 1 + O\left( \frac{\log \log n}{(\log n)^{1/2}} \right) \right).
\]

**Lemma 8.2.** Let \( n \geq 1 \) be a large highly composite integer, then

\[
\frac{\sigma(n)}{n} < e^\gamma \log(x) \left( 1 + \frac{0.5}{(\log x)^2} \right),
\]

where \( x \ll \log n \).
Proof. Let \( v = v_p \geq 0 \) be the valuation of \( n \) at \( p \). Suppose that each prime divisor \( p \mid n \) satisfies \( p \leq x \).

\[
\frac{\sigma(n)}{n} = \prod_{p^v \mid n} \left( \frac{1 - p^{-v-1}}{1 - p^{-1}} \right)
\]

\[
= \prod_{p^v \mid n} \left( 1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^v} \right)
\]

\[
< \prod_{p \leq x} \left( 1 - \frac{1}{p} \right)^{-1} .
\]

By Lemma 7.1, it follows that

\[
\frac{\sigma(n)}{n} < e^{\gamma} \log(x) \left( 1 + \frac{5}{(\log x)^2} \right) ,
\]

where \( x \ll \log n \). \( \square \)

Lemma 8.3. Let \( n \geq 1 \) be a large highly composite integer, then

\[
\log \log n > \log(x) \left( 1 + O \left( \frac{1}{(\log x)^3} \right) \right) ,
\]

where \( x \ll \log n \).

Proof. Assume that \( n = \prod_{p^v \mid n} 2^{v_2} \cdot 3^{v_3} \cdots p^{v_p} \). Clearly, for any highly composite integer \( v_2 \geq 2 \). Hence,

\[
\log n = \log \prod_{p^v \mid n} 2^{v_2} \cdot 3^{v_3} \cdots p^{v_p}
\]

\[
= \sum_{p^v \mid n} \log p^v_p
\]

\[
> (v_2 - 1) \log 2 + \sum_{p \mid n} \log p
\]

\[
\geq \log 2 + \sum_{p \leq x} \log p ,
\]

where \( x \ll \log n \). By the prime number theorem

\[
\sum_{p \leq x} \log p = x + O \left( \frac{x}{(\log x)^2} \right) \geq x - \left( \frac{c_0 x}{(\log x)^2} \right) ,
\]

where \( c_0 > 0 \) is a constant. Thus,
\[ \log \log n > \log \left( \log 2 + x + O(x(\log x)^{-2}) \right) \]
\[ \geq \log \left( \log 2 + x - (c_0 x)(\log x)^{-2} \right) \]
\[ = \log \left( x \left( 1 + \frac{\log 2 - (c_0 x)(\log x)^{-2}}{x} \right) \right) \]
\[ = \log(x) \left( 1 + \frac{\log \left( 1 + \frac{\log 2 - (c_0 x)(\log x)^{-2}}{x} \right)}{\log x} \right) \]
\[ = \log(x) \left( 1 - \left( \frac{c_1}{(\log x)^3} \right) \right), \]

where \( c_1 > 0 \) is a constant.

9 Other Result

Another technique for proving the sum of divisors inequality for the sequence of integers \( n_r = n^r \), with \( r \geq 2 \) is provided in this section. The verification involves the assumption that the inequality is false for all colossally abundant integers to derive a reductio ad absurdum.

Theorem 9.1. Let \( n \geq 5041 \) be a sufficiently large integer, and let \( r \geq 2 \). Then

\[ \sigma(n^r) < e^\gamma n \log \log n. \tag{44} \]

Proof. Let \( n \geq 5041 \) be a large colossally abundant integer, and let \( r \geq 2 \). Now, suppose that

\[ e^\gamma n^r \log \log n^r \leq \sigma(n^r) \tag{45} \]

for all \( r \geq 2 \).

I. Applying Lemma 7.1 and Lemma 8.1 in succession reduce the right side to

\[ \frac{\sigma(n^r)}{n^r} < \prod_{p \mid n^r} \left( 1 - \frac{1}{p} \right)^{-1} \]
\[ = \prod_{p \mid n} \left( 1 - \frac{1}{p} \right)^{-1} \]
\[ \leq \prod_{p \leq x} \left( 1 - \frac{1}{p} \right)^{-1} \]
\[ < e^\gamma \log x \left( 1 + \frac{1}{2(\log x)^2} \right) \]
\[ = e^\gamma \log \log n \left( 1 + \frac{1}{2(\log \log n)^2} \right) + O \left( \frac{1}{(\log \log n)^2} \right), \]
where \( x = (\log n)(1 + O(1/(\log \log n)^2)) \), this follows from Lemma 8.1, and routine calculations.

II. By the hypothesis in equation (45), and elementary calculations, the left side reduces to

\[
\frac{\sigma(n^r)}{n^r} \geq e^\gamma \log \log n^r = e^\gamma \log \log n + e^\gamma \log r.
\] (47)

III. Combining equations (46) and (47) yield

\[
e^\gamma \log \log n \left(1 + \frac{1}{2(\log \log n)^2}\right) + O \left(\frac{1}{(\log \log n)^2}\right) > \frac{\sigma(n^r)}{n^r} \geq e^\gamma \log \log n + e^\gamma \log r.
\] (48)

Re expressing (??) in the equivalent form

\[
1 + \frac{1}{2(\log \log n)^2} + O \left(\frac{1}{(\log \log n)^2}\right) > 1 + \frac{\log r}{\log \log n}
\] (49)

shows that the left side converges to 1 at a faster rate than the right side by a factor of \(1/\log \log n\). Clearly, there is a contradiction for all \( r \geq 2 \). Ergo, \( \sigma(n^r) < e^\gamma n \log \log n \) for all sufficiently large integers \( n \geq 1 \), and a fixed parameter \( r \geq 2 \).

\[\blacksquare\]

References

[1] Axler, C.; Nicolas, J. Large values of \( n/\varphi(n) \) and \( \sigma(n)/n \). Acta Arithmetica, First Online Preprint, September, 2023. 2, 5

[2] Axler, Christian. A new upper bound for the sum of divisors function. Bull. Aust. Math. Soc. 96 (2017), no. 3, 374-379. 2

[3] L. Alaoglu and P. Erdos, On highly composite and similar numbers. Trans. Amer. Math. Soc. 56 (1944), 448-469. 1, 12

[4] Akbary A., Friggstad Z. and Juricevic R., Explicit upper bounds for \( \prod_{x \leq x} p/(p - 1) \). Contrib. Discrete Math. 2(2) (2007), 153–160. 1

[5] William D. Banks, Derrick N. Hart, Pieter Moree, C. Wesley Nevans, The Nicolas and Robin inequalities with sums of two squares. Monatsh. Math. 157, no. 4,(2009) 303-322. http://arxiv.org/abs/0710.2424. 1

[6] Briggs, K. Abundant numbers and the Riemann hypothesis. Experiment. Math. 15 (2006), no. 2, 251-256. 1, 2, 12

[7] Broughan K. and Trudgian T., Robins inequality for 11-free integers. Integers 15 (2015), Article ID A12, 5 pages. 1
[8] Y.-J. Choie, N. Lichiardopol, P. Moree, P. Sole, *On Robin’s criterion for the Riemann Hypothesis*. J. Theor. Nombres Bordeaux 19 (2007), no. 2, 357-372. http://arxiv.org/abs/math/0604314.

[9] Davenport, Harold. *Multiplicative number theory*. Springer-Verlag Berlin, Heidelberg, New York, 1980. 3, 4

[10] Ford, K. *Zero-free regions for the Riemann zeta function*. http://arxiv.org/abs/1910.08205. 3, 4

[11] Finch, S.R. *Mathematical constants*. Encyclopedia of Mathematics and its Applications 94, (Cambridge University Press, Cambridge, 2003). 11

[12] A. Grytczuk, *Upper bound for sum of divisors function and Riemann Hypothesis*. Tsukuba J. Math, Vol. 31 No.1,(2007) 67-75. 1

[13] Gronwall, T. H. *Some asymptotic expressions in the theory of numbers*. Trans. Amer. Math. Soc. 14 (1913), no. 1, 113-122. 1, 7

[14] Ivic, Aleksandar. *The Riemann zeta-function. Theory and applications*. Wiley, New York; Dover Publications, Inc., Mineola, NY, 2003. 3

[15] Alexander Hertlein, *Robin’s inequality for new families of integers*. http://arxiv.org/abs/1612.05186. 1

[16] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*. 5th ed., Oxford University Press, Oxford, 1979. 1, 7

[17] Jeffrey C. Lagarias, *An Elementary Problem Equivalent to the Riemann Hypothesis*. http://arxiv.org/abs/math/0008177. 1, 2, 3, 12

[18] Jeffrey C. Lagarias, *Euler’s constant: Euler’s work and modern developments*. http://arxiv.org/abs/1303.1856. 11

[19] Montgomery, Hugh L.; Vaughan, Robert C. *Multiplicative number theory. I. Classical theory*. Cambridge University Press, Cambridge, 2007. 3

[20] Kinlaw, P.; Kobayashi, M.; Pomerance, C. *On the equation \( \varphi(n) = \varphi(n + 1) \).* Acta Arithmetica 196 (2020), 69-92. 10

[21] J. L. Nicolas, *Petites valeurs de la fonction \( d(n) \) Euler*. J. Number Theory 17 (1983) 375-388. 7

[22] S. Nazardonyavi, *Superabundant numbers, their subsequences and the Riemann*. http://arxiv.org/abs/1211.2147. 1, 2, 12

[23] Guy Robin, *Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann*. J. Math. Pures Appl. (9) 63 (1984), 187-213. 1, 2

[24] J.B. Rosser and L. Schoenfeld, *Approximate formulas for some functions of prime numbers*. Illinois J. Math. 6 (1962) 64-94. 11
[25] Ramanujan, S. *Highly composite numbers*. Proc. London Math. Soc. (2) 14 (1915), 347-409. 1, 12

[26] Ramanujan, S. *Highly composite numbers*. Ramanujan J. 1 (1997), 119-153. 1, 3, 4

[27] Schoenfeld, Lowell. *Sharper bounds for the Chebyshev functions \( \theta(x) \) and \( \psi(x) \).* II. Math. Comp. 30 (1976), no. 134, 337-360. 11

[28] Patrick Sole, Michel Planat, *Robin inequality for 7-free integers.* [http://arxiv.org/abs/1012.0671](http://arxiv.org/abs/1012.0671). 1

[29] Mark B. Villarino, *Mertens’ Proof of Mertens’ Theorem.* [http://arxiv.org/abs/math/0504289](http://arxiv.org/abs/math/0504289). 11

[30] Marek Wojtowicz, *Robins inequality and the Riemann hypothesis.* Proc. Japan Acad. Ser. A Math. Sci. Volume 83, Number 4 (2007), 47-49. 1