Hofer Growth of $C^1$-generic Hamiltonian flows

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Abstract

We prove that on certain closed symplectic manifolds a $C^1$-generic cyclic subgroup of the universal cover of the group of Hamiltonian diffeomorphisms is undistorted with respect to the Hofer metric.

1 Introduction

1.1 Hofer growth of cyclic subgroups

Let $(M^{2n}, \omega)$ be a closed symplectic manifold. We denote by $\text{Ham}(M)$ the group of Hamiltonian diffeomorphisms and by $\widetilde{\text{Ham}}(M)$ its universal cover. Our notation will be to denote elements in $\text{Ham}(M)$ by Greek letters and elements in $\widetilde{\text{Ham}}(M)$ by English letters. For instance, we will write $\phi \in \widetilde{\text{Ham}}(M)$ or $\{f_t\}_{t \in [0,1]} \in \widetilde{\text{Ham}}(M)$, where $\{f_t\} \subset \text{Ham}(M)$ is a smooth path of Hamiltonian diffeomorphisms with $f_0 = 1$, and $\{f_t\}_{t \in [0,1]}$ stands for the homotopy class with fixed end points. When we write $f$ with no subscript we are referring to the time-1-map $f = f_1$.

The Hofer metric on $\text{Ham}(M)$ is defined by

$$d(g, f) = \inf \left( \int_0^1 \max |H_t| dt \right),$$

where the infimum is taken over all Hamiltonian functions $H$ which generates $fg^{-1}$ as its time-1-map.

We denote the lift of the Hofer metric to $\widetilde{\text{Ham}}(M)$ also by $d$, i.e.

$$d(\psi, \phi) = \inf \left( \int_0^1 \max |H_t| dt \right),$$

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where the infimum is taken over all Hamiltonian functions $H$ which generates a representative of $\phi \psi^{-1}$ as its Hamiltonian flow.

Let $\{f^n\}_{n \in \mathbb{Z}} \subset \text{Ham}(M)$ be a cyclic subgroup. We say that $\{f^n\}_{n \in \mathbb{Z}}$ is undistorted if
\[
\lim_{n \to \infty} \frac{d(\mathbb{I}, f^n)}{n} > 0.
\]
Note that the limit always exists because $d(\mathbb{I}, f^n)$ is a subadditive sequence.

Similarly for a cyclic subgroup $\{\phi^n\}_{n \in \mathbb{Z}} \subset \tilde{\text{Ham}}(M)$, We say that $\{\phi^n\}_{n \in \mathbb{Z}}$ is undistorted if
\[
\lim_{n \to \infty} \frac{d(\mathbb{I}, \phi^n)}{n} > 0.
\]

The distortion of subgroups of $\tilde{\text{Ham}}(M)$ has been studied on various occasions in connection to Hamiltonian dynamics and ergodic theory, see e.g. [10, chapters 8,11].

In the autonomous case, it has been proved that there exists a $C^0$-open and $C^\infty$-dense subset $\mathcal{A}$ of the set of autonomous normalized Hamiltonian functions such that for every $F \in \mathcal{A}$, the cyclic subgroup generated by the Hamiltonian flow of $F$ is undistorted (see [11, chapter 6]).

In this article we give a similar statement for a $C^1$-generic time dependent element in $\tilde{\text{Ham}}(M)$. When we say $C^1$-generic we mean that the set of elements in $\text{Ham}(M)$ that generate undistorted cyclic subgroups has a $C^1$-open and dense subset.

Let us recall the definition of the $C^1$ topology on $\tilde{\text{Ham}}(M)$. It is known (see [8]) that $\text{Ham}(M)$ is locally simply connected. Fix a basis $\{U_i\}$ of simply connected $C^1$-neighborhoods of $\mathbb{I}$ in $\text{Ham}(M)$. Let $\tilde{U}_i$ be the lift of $U_i$ to $\tilde{\text{Ham}}(M)$ that contains $\mathbb{I} \in \tilde{\text{Ham}}(M)$. By definition, the sets $\{\phi \tilde{U}_i\}$ form a basis of $C^1$-neighborhoods of $\phi \in \tilde{\text{Ham}}(M)$.

Before we state the main results of this paper, let us give the following definitions. When we write $H^i(M)$ we mean the $i$-th cohomology group with integer coefficients.

**Definition 1.1.** Let
\[
\alpha \in H^*(M) := \bigoplus_i H^i(M).
\]
When we write $\text{deg}(\alpha)$ we mean the maximal $k$ such that the projection of $\alpha$ to $H^{2k}(M)$ is non-zero.

**Definition 1.2.** Let $M^{2n}$ be a closed symplectic manifold. Let $c(M) \in H^*(M)$ be the full Chern class of $TM$. We say that there exists an even factorization of $c(M)$ if we can write
\[
c(M) = \alpha \beta,
\]
where
\[ \text{deg}(\alpha) + \text{deg}(\beta) \leq n, \]
\[ 0 < \text{deg}(\alpha) < n, \]
and \( \alpha \) has only terms of even degree. In this case we say that \( c(M) = \alpha \beta \) is an even factorization. When we say that \( \alpha \) has only terms of even degree we mean that the projection of \( \alpha \) to \( H^{2k}(M) \) is zero for odd \( k \).

Our motivation for giving this definition is to provide an obstruction to the dynamics of Hamiltonian diffeomorphisms, whose idea has been announced by Bennequin and it appears in [2].

We are now ready to state the main theorem.

**Theorem 1.3.** Let \( M \) be a closed symplectic manifold with \( H^1(M) = 0 \). If the top Chern class \( c_n(M) \neq 0 \) and there is no even factorization of the full Chern class, then the set of elements in \( \widetilde{\text{Ham}}(M) \) that generate undistorted cyclic subgroups has a \( C^1 \)-open and dense subset.

The following theorem is a corollary (for its proofs see Example [3,3]).

**Theorem 1.4.** Let \( M^4 \) be a four dimensional closed symplectic manifold with \( H^1(M) = 0 \), and \( c^2(M) \neq 0 \). Then the set of elements in \( \widetilde{\text{Ham}}(M) \) that generate undistorted cyclic subgroups has a \( C^1 \)-open and dense subset.

In particular, for \( M = \mathbb{CP}^2 \), \( C^1 \)-generic elements generate undistorted cyclic subgroups. In fact for \( \mathbb{CP}^n \) we can upgrade the theorem and formulate it with respect to cyclic subgroups of \( \text{Ham}(M) \).

**Theorem 1.5.** The set of Hamiltonian diffeomorphisms in \( \text{Ham}(\mathbb{CP}^n) \) that generate undistorted cyclic subgroups of \( \text{Ham}(\mathbb{CP}^n) \) has a \( C^1 \)-open and dense subset.

1.2 **Idea of the proof**

Consider the set
\[ \chi = \{ \phi \in \widetilde{\text{Ham}}(M) : \sigma(\phi) \neq 0 \}, \]
where \( \sigma \) is an asymptotic spectral invariant (See the definition in Section 1.4). We will show that \( \chi \) is a subset of the set of elements in \( \widetilde{\text{Ham}}(M) \) that generate undistorted cyclic subgroups(See Proposition 2.1), and that in certain manifolds it is a \( C^1 \)-open and dense set in \( \widetilde{\text{Ham}}(M) \).

In Section 2 we prove that \( \chi \) is \( C^1 \)-open.
In our proof that $\chi$ is $C^1$-dense we restrict to the case where the set of elements in $\widehat{\text{Ham}}(M)$ that have an elliptic periodic fixed point is $C^1$-dense. In Section 3 we give a method to check if this is the case by examining the full Chern class of $TM$. We show that if there is no even factorization then the set of elements in $\widehat{\text{Ham}}(M)$ that have an elliptic periodic fixed point is $C^1$-dense. First we follow Bennequin (see [2]) and show that if there is no even factorization then there are no partially hyperbolic symplectomorphisms. Next we use a result by Saghin and Xia ([12]) which states that a $C^1$-generic symplectomorphism which is not partially hyperbolic has an elliptic periodic fixed point (The definitions of elliptic periodic fixed points and partially hyperbolic symplectomorphisms appear in the next subsection).

In Section 4 we deal with the last step of the proof, which is to show that for every $\phi \in \widehat{\text{Ham}}(M)$ with an elliptic periodic fixed point, we can do a small $C^1$-perturbation and get a new element $\tilde{\phi} \in \widehat{\text{Ham}}(M)$ which is $C^1$-close to $\phi$ and such that $\tilde{\phi} \in \chi$. For the main part of the construction of the perturbation we follow Bonnati, Crovisier, Vago and Wilkinson ([1]). This shows that if the set of elements in $\widehat{\text{Ham}}(M)$ that have an elliptic periodic fixed point is $C^1$-dense, then $\chi$ is $C^1$-dense in $\widehat{\text{Ham}}(M)$.

In Section 5 we discuss whether our results could be applied to $\text{Ham}(M)$, that is whether a $C^1$-generic element in $\text{Ham}(M)$ generates an undistorted cyclic subgroup with respect to the Hofer metric.

In Section 6 we give examples for manifolds that satisfy the requirements on the full Chern class. For these manifolds a $C^1$-generic element in $\widehat{\text{Ham}}(M)$ generates an undistorted cyclic subgroup.

1.3 Partially hyperbolic maps and elliptic periodic fixed points

Let $M^{2n}$ be a closed symplectic manifold such that $H^1(M) = 0$, and let $f \in \text{Ham}(M)$. In this section and also throughout the paper we assume that an auxiliary Riemannian metric has been chosen.

A point $p \in M$ is called an elliptic l-periodic fixed point if $f^l(p) = p$ and all of the eigenvalues of $d_p(f^l)$ are simple, non-real and with norm 1.

A continuous splitting of the tangent bundle $TM = A \oplus B$ is called invariant if it is invariant under $df$. For an invariant splitting we say that $A$ dominates $B$ if there exists $m > 0$ such that for each $x \in M$ and two unit vectors $u \in A_x, v \in B_x$, the following inequality holds

$$\|d_x f^m(u)\| \geq 2 \|d_x f^m(v)\|.$$
A diffeomorphism $f$ is called \textit{partially hyperbolic} if the following conditions hold:

1. There is an invariant splitting $TM = E^u \oplus E^c \oplus E^s$ with at least two of them non-trivial.

2. $E^u$ is uniformly expanding, i.e., there exist $\alpha > 1, a > 0$ such that $\| df^k(v) \| \geq a\alpha^k \| v \|$ for all $v \in E^u, k \in \mathbb{N}$.

3. $E^s$ is uniformly contracting, i.e., there exist $\beta > 1, b > 0$ such that $\| df^{-k}(u) \| \geq b\beta^k \| u \|$ for all $u \in E^s, k \in \mathbb{N}$.

4. $E^u$ dominates $E^c$, and $E^c$ dominates $E^s$.

Let $\text{Symp}_1(M)$ be the set of all $C^1$ symplectomorphisms, and

$$ \text{Symp}(M) \subset \text{Symp}_1(M) $$

be the set of all $C^\infty$ symplectomorphisms. A result by Saghin and Xia states that there exists a $C^1$-open and dense $U \subset \text{Symp}_1(M)$ so that for each $f \in U$, $f$ is either partially hyperbolic or it has an elliptic periodic fixed point, see [12]. See also [6] for a similar and independent result by Horita and Tahzibi.

Note that since $U$ is open, $U \cap \text{Symp}(M)$ is $C^1$-dense in $\text{Symp}(M)$. We get that in a $C^1$-dense subset, a $C^\infty$ symplectomorphism which is not partially hyperbolic has an elliptic periodic fixed point. In the case where $H^1(M) = 0$, the result is also true for a $C^1$-dense subset of the set of Hamiltonian diffeomorphisms. This is a simple consequence of the fact that the group of symplectomorphisms is locally path connected and the subgroup of Hamiltonian diffeomorphisms is exactly the connected component of the identity. Since $\pi : \tilde{\text{Ham}}(M) \to \text{Ham}(M)$ is open and continuous, we get the following

\textbf{Theorem 1.6.} The set

$$ \{ \{ f_t \} \in \tilde{\text{Ham}}(M) : \text{ $f_1$ is partially hyperbolic or $f_1$ has an elliptic periodic fixed point} \} $$

is $C^1$-dense in $\tilde{\text{Ham}}(M)$.

\textbf{1.4 Asymptotic spectral invariants}

\textbf{Definition 1.7.} Let $(U^{2n}, \omega)$ be an open symplectic manifold. Let $\phi \in \tilde{\text{Ham}}(U)$ be an element such that there is a representative generated by a
compactly supported Hamiltonian function \( \{F_t\}_{t \in [0,1]} \). We define the Calabi homomorphism of \( \phi \) to be

\[
\text{Cal}(\phi) = \int_0^1 \int_U F_t \omega^n \, dt.
\]

It is known that the Calabi homomorphism is well defined and it is indeed a homomorphism from \( \tilde{\text{Ham}}(U) \) to \( \mathbb{R} \) (See [8]).

**Definition 1.8.** A function \( c : \tilde{\text{Ham}}(M) \to \mathbb{R} \) is called a subadditive spectral invariant if

1. (conjugation invariance) \( \forall \phi, \psi \in \tilde{\text{Ham}}(M), c(\phi \psi \phi^{-1}) = c(\psi) \).
2. (subadditivity) \( c(\phi \psi) \leq c(\phi) + c(\psi) \)
3. (stability) \( \int_0^1 \min(F_t - G_t) \, dt \leq c(\phi) - c(\psi) \leq \int_0^1 \max(F_t - G_t) \, dt \),
   where \( \phi \) and \( \psi \) have representatives that are generated by normalized Hamiltonian functions \( F \) and \( G \) respectively.
4. (spectrality) \( c(\phi) \in \text{spec}(\phi) \) for all non-degenerate \( \phi \in \tilde{\text{Ham}}(M) \).

Remember that an element \( \{f_t\} \in \tilde{\text{Ham}}(M) \) is called non-degenerate if the graph of \( f_1 \) in \( M \times M \) is transversal to the diagonal. The action spectrum \( \text{spec}(\{f_t\}) \) is the set of all the actions \( A_F(y, D) \), where \( F \) is a normalized Hamiltonian that generates \( \{f_t\} \) and \( y \) is a fixed point of \( f_1 \).

It is known that for every closed symplectic manifold there exists a subadditive spectral invariant.

For a subadditive spectral invariant \( c \) we can define the asymptotic spectral invariant as

\[
\sigma(\phi) = \lim_{k \to \infty} \frac{c(\phi^k)}{k}.
\]

Every asymptotic spectral invariant is homogeneous and the stability property holds. For an open displaceable set \( U \subset M \), and an element \( \phi \in \tilde{\text{Ham}}(M) \) supported in \( U \) we have

\[
\sigma(\phi) = -V^{-1} \cdot \text{Cal}(\phi),
\]

where \( V = \int_M \omega^n \) and \( \text{Cal}(\phi) \) is the Calabi homomorphism of \( \phi \) if we regard it as an element of \( \tilde{\text{Ham}}(U) \).

Denote

\[
I(\phi, \psi) := |\sigma(\phi \psi) - \sigma(\phi) - \sigma(\psi)|.
\]
It is known that
\[ I \leq \min(q(\phi), q(\psi)), \]
where \( q(\phi) = c(\phi) + c(\phi^{-1}) \). It is also known that for a displaceable set \( U \), one has
\[ \sup q(\phi) < \infty, \]
where the supremum runs over all \( \phi \in \widehat{\text{Ham}}(M) \) supported in \( U \). We denote this value by
\[ w(U) = \sup q(\phi). \]

For the proofs of these facts and for further discussion on spectral invariants see [11].

**Proposition 1.9.** Let \( \phi, \psi \in \widehat{\text{Ham}}(M) \) such that \( \phi \psi = \psi \phi \), and \( \phi \) is supported in a displaceable set \( U \subset M \). Then \( I(\phi, \psi) = 0 \).

**Proof.**
\[ \sigma(\phi \psi) = \frac{\sigma(\phi^k \psi^k)}{k} = \frac{k\sigma(\phi) + k\sigma(\psi) + C(k)}{k}, \]
where \( C(k) \) is a constant depending on \( k \) with \( |C(k)| \leq w(U) \). We get that
\[ \sigma(\phi \psi) = \sigma(\phi) + \sigma(\psi) + \frac{C(k)}{k} \xrightarrow{k \to \infty} \sigma(\phi) + \sigma(\psi). \]

\[ \square \]

2 Proof that \( \chi \) is open

Let \( M \) be a closed symplectic manifold with \( H^1(M) = 0 \).

**Proposition 2.1.** For \( \phi \in \widehat{\text{Ham}}(M) \), if \( \sigma(\phi) \neq 0 \) then \( \{\phi^n\}_{n \in \mathbb{Z}} \) is undistorted.

**Proof.** It is known that (see [11])
\[ d(1, \phi) \geq |\sigma(\phi)|. \]
The completion of the proof is due to the homogeneity of \( \sigma \).

Put
\[ \chi = \{\phi \in \widehat{\text{Ham}}(M) : \sigma(\phi) \neq 0\}. \]

**Theorem 2.2.** The set \( \chi \subset \widehat{\text{Ham}}(M) \) is \( C^1 \)-open.
This is an easy consequence of the following.

**Theorem 2.3.** The function $\sigma: \widetilde{\text{Ham}}(M) \to \mathbb{R}$ is $C^1$-continuous.

**Proof.** From the stability property of $\sigma$ together with the bi-invariance property of the Hofer metric, we get that it is enough to show that for every $\epsilon > 0$ if $\phi$ is $C^1$-close enough to the identity then there exists a Hamiltonian function $H$ that generates a representative such that

$$\max(|H_t|) < \epsilon$$

for each $t$.

Let us recall some facts about symplectomorphisms which are $C^1$-close to the identity. Let $\Delta \subset (M \times M, -\omega \oplus \omega)$ be the diagonal. There is a symplectomorphism $\Psi$ from a small neighborhood of the diagonal $N(\Delta) \subset M \times M$ to a small neighborhood of the zero section $N(M_0) \subset T^*M$ with the symplectic form $d\lambda_{\text{can}}$ defined on $T^*M$. For a $C^1$-small Hamiltonian diffeomorphism $f$, the image $\Psi(\text{graph}(f))$ is a graph of an exact 1-form $dF$ (see [3]).

For a smooth path of exact 1-forms $dG_t$ such that for each $t$, $dG_t$ is close enough to the zero section and $G_0 = 0$, there exist a Hamiltonian isotopy $\{g_t\}$ such that $\text{graph}(g_t) = \Psi^{-1}(\text{graph}(dG_t))$. In addition, every loop of exact 1-forms $dG_t$ is homotopic to the zero section by the homotopy $\{d(s \cdot G_t)\}_{s \in [0,1]}$.

This proves the following

**Proposition 2.4.** Let $f_t$ and $g_t$ be two paths of Hamiltonian diffeomorphisms with $f_0 = g_0 = 1$ and $f_1 = g_1$ that are $C^1$-close enough to the identity. Then they are homotopic with fixed end points.

Let $\{g_t\}$ be a representative of $\phi \in \widetilde{\text{Ham}}(M)$, which is $C^1$-close to 1. We get that $\Psi(\text{graph}(g_t)) = \text{graph}(dG_t)$ for some $G_t : M \to \mathbb{R}$. Denote $F = G_1$. There is a Hamiltonian isotopy $\{f_t\}$, such that $\Psi(\text{graph}(f_t)) = \text{graph}(d(t \cdot F))$. From Proposition 2.3, we get that $\{f_t\}$ is a representative of $\phi$.

We would like to show that $\|\frac{d}{dt}\phi\|$ is arbitrarily small (when we choose $\{f_t\}$ to be $C^1$-close enough to 1). Assuming this, we would get that there
exists a Hamiltonian $H$ that generates $\{f_t\}$ which is a representative of $\phi$, such that $\|\text{grad}H_t\|$ is arbitrarily small. There is a constant $K$ such that for each Hamiltonian function $H$,

$$|H_t| < K \cdot \sup \|\text{grad}H_t\|.$$ 

It follows that if we choose $\{f_t\}$ to be $C^1$-close enough to $\mathbf{1}$, we would get that

$$\max(H_t) < \epsilon$$

for each $t$. This completes the proof under the assumption that we can make $\|\partial f_t/\partial t\|$ be arbitrarily small.

Let us prove this assumption. From the fact that $\Psi(\text{graph}(f_t)) = \text{graph}(d(t \cdot F))$, there exists a path of diffeomorphisms $h_t : M \rightarrow M$ such that

$$\Psi \circ \text{gr}f_t = d(t \cdot F) \circ h_t,$$

where $\text{gr}f_t : M \rightarrow M \times M$ is defined by

$$\text{gr}f_t(x) = (x, f_t(x)).$$

Denote $\pi_1, \pi_2 : M \times M \rightarrow M$ to be the projections to the first and second copies of $M$ respectively. We get that

$$\mathbf{1} = \pi_1 \circ \Psi^{-1} \circ d(tF) \circ h_t,$$

$$f_t = \pi_2 \circ \Psi^{-1} \circ d(tF) \circ h_t.$$

By differentiating both equations by $t$, one can check that we can express $\|\partial f_t/\partial t\|$ as a sum of arguments that become arbitrarily small when we make $\{f_t\}$ be $C^1$-closer to $\mathbf{1}$, and make $F$ smaller (Note that for $v \in T\text{graph}(f_t)$, $\|\pi_{1\ast}v\| - \|\pi_{2\ast}v\| \|v\|$ is arbitrarily small).

We get that $\|\partial f_t/\partial t\|$ is arbitrarily small and this completes the proof.

\[\square\]

### 3 Obstruction to the existence of a partially hyperbolic symplectomorphism

The next theorem provides an obstruction to the existence of a partially hyperbolic symplectomorphism and it will enable us to give examples for manifolds that do not admit partially hyperbolic symplectomorphisms. From Theorem 1.6, we get that in these manifolds the set of elements that their time-1-maps have elliptic periodic fixed points is $C^1$-dense in $\widehat{\text{Ham}}(M)$.

The idea of the obstruction has been announced by Bennequin (oral communication) and it is presented in [2].
Theorem 3.1. Let $M^{2n}$ be a closed symplectic manifold with a non-vanishing top Chern class, and $f \in \text{Symp}(M)$ a partially hyperbolic Hamiltonian diffeomorphism. Then there exists an even factorization of the full Chern class of $M$ $c(M) = \alpha \beta$.

Theorem 3.2. Let $M^{2n}$ be a closed symplectic manifold and suppose that there is an isotropic subbundle $L$, i.e. $L \subset L^\omega$, and $\text{rank}(L) = i$. Then there exists a factorization of the full Chern class $c(M) = \alpha \beta$ where $\alpha$ has only terms of even degree, $\deg(\alpha) \leq i$ and $\deg(\beta) \leq n - i$.

Proof that Theorem 3.2 implies Theorem 3.1. There exists a constant $Q > 0$ such that for all $v_1, v_2 \in TM$,

$$\omega(v_1, v_2) \leq Q \|v_1\| \|v_2\|.$$  

Let $x \in M$ and $u_1, u_2 \in E^s_x$.

$$|\omega(u_1, u_2)| = |\omega(d_x f^k(u_1), d_x f^k(u_2))| \leq Q \|d_x f^k(u_1)\| \|d_x f^k(u_2)\| \leq b \beta - 2k Q \|u_1\| \|u_2\| \xrightarrow{k \to \infty} 0.$$  

We get that $\omega(u_1, u_2) = 0$, so $E^s \subset (E^s)^\omega$. We get that $E^s$ is an isotropic subbundle, and so there exists a factorization of the full Chern class $c(M) = \alpha \beta$ where $\alpha$ has only terms of even degree. Let us prove that $\deg(\alpha)$ could not be zero or $n$, and this will show that $c(M) = \alpha \beta$ is an even factorization.

Denote $\text{rank}(E^s) = i > 0$. Note that on the one hand $\deg(c(M)) = n$ because $c_n(M) \neq 0$. On the other hand,

$$\deg(c(M)) \leq \deg(\alpha) + \deg(\beta) \leq i + (n - i) = n.$$  

We get that all the inequalities are actually equalities and $\deg(\alpha) = i > 0$.

If $\deg(\alpha) = n$, we get that $\text{rank}(E^s) = n$. From the fact that $f$ is symplectic, we get that $\text{rank}(E^u) = n$ (see [12]), and $\text{rank}(E^c) = 0$. On the other hand $f$ is isotropic to $1$, so $\text{rank}(E^c) > 0$ (see [13]) and this is a contradiction. We get that $0 < \deg(\alpha) < n$. This completes the proof.

Proof of Theorem 3.2. Let $J$ be a compatible almost complex structure. The subbundle $L \oplus JL$ is symplectic and $L, JL$ are its Lagrangian subbundles. The subbundle $L \oplus JL$ is also isomorphic to the complexification of $L$, so we get that $c(L \oplus JL)$ has only terms of even degree(see [9] chapter 15). We can write

$$TM = (L \oplus JL) \oplus (TM/(L \oplus JL)),$$

where the subbundle $(TM/(L \oplus JL))$ is also symplectic. Put $\alpha = c(L \oplus JL)$ and $\beta = c(TM/(L \oplus JL))$. This completes the proof.
4 \textit{C}^1\text{-Generic elements generate undistorted cyclic subgroups}

Let $M$ be a closed symplectic manifold with $H^1(M) = 0$, and let $\sigma$ be an asymptotic spectral invariant.

The following theorem shows that if the manifold satisfies that a $C^1$-generic Hamiltonian diffeomorphism has an elliptic periodic fixed point, then the set $\chi$ is $C^1$-dense in $\text{Ham}(M)$.

**Theorem 4.1.** Let $M$ be a closed symplectic manifold with $H^1(M) = 0$, and $\phi \in \text{Ham}(M)$ such that its time-1-map has an elliptic periodic fixed point. Then for every $C^1$-open neighborhood $\mathcal{U} \subset \text{Ham}(M)$ of $\phi$, there exists $\psi \in \mathcal{U}$ such that $\sigma(\psi) \neq 0$.

**Proof of Theorem 4.1.** Theorem 2.2 and Theorem 4.1 shows that if a $C^1$-generic element satisfies that its time-1-map has an elliptic periodic fixed point then the set $\chi$ is $C^1$-open and dense in $\text{Ham}(M)$. From proposition 2.1 we get that

$$\chi \subset \{ \phi \in \text{Ham}(M) : \{ \phi^n \}_{n \in \mathbb{Z}} \text{ is undistorted} \}.$$  

Finally, from Theorem 3.1 we get that if $c_n(M) \neq 0$ and there is no even factorization then there are no partially hyperbolic symplectomorphisms and from Theorem 1.6 we get that a $C^1$-generic Hamiltonian diffeomorphism has an elliptic periodic fixed point.

The rest of this section is dedicated to the proof of Theorem 4.1, in which we follow the construction that appears in [1].

The idea of the proof is to first construct an element $[[g_t]] \in \text{Ham}(M)$ which is $C^1$-close to $\phi$ and such that there exists a small open set $U \subset M$ and an integer $k \in \mathbb{N}$ such that $g^k|_U = 1$ and $g^j(U) \cap U = \emptyset$ for all $j < k$.

The second step will be to perturb $g$ inside $U$ in order to change the value of the asymptotic spectral invariant.

We start with the following lemma.

**Lemma 4.2.** Let $[[f_t]] \in \text{Ham}(M)$, and denote $f = f_1$. Let $p \in M$ be an elliptic $l$-periodic fixed point of $f$. Then for any $C^1$-open neighborhood $\mathcal{U}$ of $[[f_t]]$ and an open neighborhood $V \subset M$ of $p$, there exists an element $[[g_t]] \in \text{Ham}(M)$ and $\delta_1 > \delta_2 > 0$ such that $B_{\delta_1}(p)$ is inside a Darboux chart around $p$ and
1. $\{g_t\} \in \mathcal{U}$.

2. $B_{\delta_1}(p) \subset V$.

3. $g$ agrees with $f$ on the orbit of $p$, i.e. $g^i(p) = f^i(p), \forall i \in \{1, \ldots, l\}$.

4. $g$ agrees with $f$ outside the orbit of $B_{\delta_1}(p)$, i.e. $g|_{M \cup \bigcup_{i=1}^{l} f_i(B_{\delta_1}(p))} = f|_{M \cup \bigcup_{i=1}^{l} f_i(B_{\delta_1}(p))}$.

5. $g|_{B_{\delta_2}(p)} = T$, where $T$ is linear with simple, non real eigenvalues of the form $e^{\alpha_j 2\pi \sqrt{-1}}$ with $\alpha_j$ rational. In this case we say that $T$ has eigenvalues with rational angles.

**Sketch of proof.** The idea of the proof is to perturb the generating function of $f$. We divide the proof into three parts. The first step is to construct a symplectomorphism that would be arbitrarily $C^1$-close to $f$ and such that all the properties needed from the time-1-map in the lemma would hold, except that maybe its eigenvalues would not have rational angles. The second step is to do another perturbation to get a symplectomorphism $g$ that would satisfy all the conditions from the time-1-map in the lemma. The last step will be to define a Hamiltonian isotopy from $f$ to $g$, and define $\{g_t\}$ to be the juxtaposition between $\{f_t\}$ and this Hamiltonian isotopy.

Let us begin with a simpler case. Let $f : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a symplectomorphism with $f(0) = 0$. Consider the symplectic matrix $df_0 : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$. We wish to construct a symplectomorphism $g$ such that for a small $\delta > 0$,

$$g|_{B_{\delta}(0)} = df_0,$$

$$g|_{\mathbb{R}^{2n} \setminus B_{\delta}(0)} = f.$$

Recall that in a small neighborhood of 0, there exists a generating function for $f$. Denote $f(p_1, q_1) = (p_2, q_2)$. Let $S : \mathbb{R}^{2n}(q, q') \to \mathbb{R}$ be the generating function of $f$, i.e.

$$\frac{\partial S}{\partial q}(q_1, q_2) = -p_1,$$

$$\frac{\partial S_i}{\partial q'}(q_1, q_2) = p_2.$$

Since $f(0) = 0$ we can write

$$S = \langle q, M_1 q \rangle + \langle q, M_2 q' \rangle + \langle q', M_3 q' \rangle + k(q, q').$$
where $M_1, M_2, M_3$ are matrices and $k(q, q')$ has only terms of order greater than two. Define a smoothened step function $a_δ : \mathbb{R} \rightarrow \mathbb{R}$

$$a_δ(x) = \begin{cases} 0 & x < δ \\ 1 & x > 2δ \end{cases}.$$ 

Write 

$$\tilde{S} = \langle q, M₁q \rangle + \langle q, M₂q' \rangle + \langle q', M₃q' \rangle + a_δ(||(q, q')||)k(q, q').$$

We can choose $δ$ to be small enough so that there exists a symplectomorphism $g$ such that $\tilde{S}$ is its generating function in a neighborhood that contains $B_{2δ}(0)$. Note that 

$$g|_{B_{δ}(0)} = df₀, \quad g|_{\mathbb{R}^{2n}\setminus B_{2δ}(0)} = f.$$ 

In order for $g$ to be $C^1$-close to $f$, we need $\tilde{S}$ to be $C^2$-close to $S$. One can check that the norm of the difference between the second derivative of $\tilde{S}$ and the second derivative of $S$ is $O(δ)$. So we can choose $δ$ to be small enough so that $g$ would be arbitrarily $C^1$-close to $f$. Note that this construction fails if we would try to make $g$ be $C^k$-close to $f$, for $k > 1$.

Let us return to the proof of the lemma. For each $0 ≤ i < l$ choose a Darboux chart $U_i$ around $f^i(p)$ such that $U_i \cap U_j = \emptyset$, and $f^i(p)$ is identified with $0 \in \mathbb{R}^{2n}$. Take a small enough ball $B₀ \subset U₀$ such that $f^i(B₀) \subset U_i$, and $f^i(B₀) \subset U₀$. Since $f^i(B₀)$ and $f^{i+1}(B₀)$ are subsets of Darboux charts, we can treat $f|_{f^i(B₀)} : f^i(B₀) \rightarrow f^{i+1}(B₀)$ as a symplectomorphism between subsets of $\mathbb{R}^{2n}$. From the construction above, we get a symplectomorphism $\tilde{g}$ which is a linear map in a small ball inside $f^i(B₀)$ for each $i$, $\tilde{g} = f$ outside a larger ball inside $f^i(B₀)$ for each $i$, and $\tilde{g}(0) = 0$ for each $i$, that is $\tilde{g}^i(p) = f^i(p)$ for each $i$. Note also that in a small ball $B$ inside $B₀$, $\tilde{g}^i : B \rightarrow \tilde{g}^i(B)$ is the multiplication of all the matrices $d_{f^i(p)}(f|_{f^i(B₀)})$, so it is also linear. Denote this linear map by $\tilde{T}$. We get that the symplectomorphism $\tilde{g}$ satisfies almost all the properties needed from the time-1-map in the conditions of the lemma. The only property that does not hold, is that the eigenvalues of $\tilde{T}$ are not necessarily with rational angles.

Our next task is to find a $C^∞$ perturbation $g$, so that $g^l$ restricted to a small enough ball inside $B₀$ would be a matrix $T$ whose eigenvalues are with rational angles. Choose such a symplectic matrix $T$ which is close to $\tilde{T}$. Denote $H₁$ to be the Hamiltonian function defined on $f^{l⁻¹}(B₀)$ that generates $d_{f^{l⁻¹}(p)}(f|_{f^{l⁻¹}(B₀)})$ as its time-1-map. Find a small symplectic matrix $Q$ such that $TQ = T$. Find a Hamiltonian function $H₂$ so that $H₁ + H₂$ would generate $d_{f^{l⁻¹}(p)}(f|_{f^{l⁻¹}(B₀)})Q$ as its time-1-map. Choose a cutoff function $a$.
supported in a small ball inside $f^{l-1}(B_0)$, and define the Hamiltonian function of the perturbed symplectomorphism in $f^{l-1}(B_0)$ to be $H_1 + a \cdot H_2$. Denote this new symplectomorphism by $g$. Note that outside a small ball inside $f^{l-1}(B_0)$, $g = \bar{g}$. Since $H_2$ can be chosen to be arbitrarily small, we get that $g$ would be arbitrarily $C^\infty$-close to $\bar{g}$. Note that

$$
\bar{T} = \prod_{i=0}^{l-1} d_{f_i(p)}(f_i|_{f_i(B_0)}),
$$

so we get that in a small ball inside $B_0$,

$$
g^j = \left( \prod_{i=0}^{l-2} d_{f_i(p)}(f_i|_{f_i(B_0)}) \right) d_{f_i(p)}(f_i|_{f_i(B_0)})Q = \bar{T}Q = T.
$$

Hence we can construct a symplectomorphism $g$ such that it is arbitrarily $C^1$-close to $f$, and it satisfies all the conditions in the lemma.

Since $g$ is $C^1$-close to $f$, we can construct a path of symplectomorphisms from $f$ to $g$, such that all of the symplectomorphisms in the path are $C^1$-close to $f$ (see [8, Theorem 10.1 and its proof]). From the fact that $H^1(M) = 0$, we get that this path is a Hamiltonian isotopy. Define $\{g_t\}$ to be the juxtaposition of $\{f_t\}$ and this Hamiltonian isotopy.

**Proof of Theorem 4.1.** By Lemma 4.2 we can take $\{g_t\} \in \hat{\text{Ham}}(M)$ such that it satisfies all the conditions in the lemma. If $\sigma(\{g_t\}) \neq 0$ then we are done, so suppose $\sigma(\{g_t\}) = 0$. Because of the fact that all of the eigenvalues of $T$ have rational angles, there is an integer $q$ (the smallest common multiple of the denominators) such that

$$
g^{q}|_{B_{\delta^2}(p)} = 1.
$$

Let $k \in \mathbb{N}$ be the smallest number such that $g^k|_{B_{\delta^2}(p)} = 1$. There exists $x \in B_{\delta^2}(p)$ such that $g^j(x) \neq x$ for all $0 < j < k$. From continuity there is a ball $B \subset B_{\delta^2}(p)$ around $x$, such that

$$
g^j(B) \cap B = \emptyset,
$$

for all $0 < j < k$. We can choose $B$ such that the open set $\cup_{j=1}^k g^j(B)$ would be displaceable.

Let $H$ be a positive time independent Hamiltonian function supported in $B$. For $\epsilon > 0$, let $\{h_t^\epsilon\}$ be the Hamiltonian isotopy generated by $\epsilon \cdot H$. Put

$$
h_t^\epsilon(x) = \begin{cases} 
  g^j \circ h_t^\epsilon \circ g^{-j}(x) & x \in g^j(B), j = 0, \ldots, k - 1 \\
  x & \text{otherwise}
\end{cases}
$$

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Note that since $\bigcup_{j=1}^{k} g^j(B)$ is displaceable,

$$\sigma([\{h_t^j\}]) = k \text{Cal}(h^\kappa) > 0.$$  

The important part of this construction is that we get that the time-1-maps commute, i.e. $h_t^1 \circ g_1 = g_1 \circ h_t^1$.

**Claim:** For a small enough $\epsilon$,

$$[\{g_t h_t^1\}] = [\{h_t^1 g_t\}].$$

**Proof:** For a small enough $\epsilon$, the path $\{g_t h_t^1 g_t^{-1} (h_t^1)^{-1}\}$ is arbitrarily $C^1$-close to $1$. From this and from Proposition 2.4, we get that

$$[\{g_t h_t^1 g_t^{-1} (h_t^1)^{-1}\}] = 1.$$  

This completes the proof of the claim.

From proposition 1.9 we get that

$$\sigma([\{g_t h_t^1\}]) = \sigma([\{g_t\}]) + \sigma([\{h_t^1\}]) = \sigma([\{h_t^1\}]) > 0.$$  

We can choose $\epsilon$ to be small enough so that $[\{g_t \circ h_t^1\}] \in \mathcal{U}$. This completes the proof.

\[ \square \]

5 \hspace{1cm} \widetilde{\text{Ham}} \hspace{1cm} \text{vs.} \hspace{1cm} \text{Ham}

Throughout this paper, we discussed the notion of the distortion of cyclic subgroups of $\widetilde{\text{Ham}}(M)$. One could ask if the same construction works if we consider undistorted cyclic subgroups of $\text{Ham}(M)$ equipped with Hofer’s metric (also denoted $d$).

Let $\pi : \widetilde{\text{Ham}}(M) \to \text{Ham}(M)$ be the projection. Since $\pi$ is continuous and open, we get that if a set $S \subset \widetilde{\text{Ham}}(M)$ is open or dense in $\widetilde{\text{Ham}}(M)$, then $\pi(S) \subset \text{Ham}(M)$ will be open or dense respectively. In the case where $\sigma$ descends to $\text{Ham}(M)$ we get that $\pi(\chi) \subset \text{Ham}(M)$ is a $C^1$-open and dense subset of the set of Hamiltonian diffeomorphisms that generate undistorted cyclic subgroups. From this we get that in the case where $\sigma$ descends our results extend to $\text{Ham}(M)$.

**Theorem 5.1.** Let $M$ be a closed symplectic manifold such that

1. $H^1(M) = 0$.  

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2. The top Chern class does not vanish, \( c_n(M) \neq 0 \).

3. The full Chern class does not have an even factorization.

4. There exists an asymptotic spectral invariant that descends to \( Ham(M) \).

Then the set of elements in \( Ham(M) \) that generates undistorted cyclic subgroups has a \( C^1 \)-open and dense subset.

In [7] McDuff gives conditions under which the asymptotic spectral invariants descend to \( Ham(M) \). In particular, we get that in \( \mathbb{CP}^n \) the asymptotic spectral invariants descend to \( Ham(\mathbb{CP}^n) \). In Example 6.3 we show that there is no even factorization of \( c(T\mathbb{CP}^n) \) and this proves Theorem 1.5.

6 Examples

In this section we give examples of manifolds that satisfy the requirements of Theorem 1.3. For these manifolds a \( C^1 \)-generic element of \( \widetilde{Ham}(M) \) generates an undistorted cyclic subgroup.

Example 6.1. Let \( M = S^2 \) be the 2-sphere. Note that \( c_1(M) \neq 0 \) and \( H^1(M) = 0 \), and obviously there is no even factorization of the full Chern class.

Example 6.2 (Proof of Theorem 1.4). Let \( M^4 \) be a closed symplectic 4-dimensional manifold such that \( H^1(M) = 0 \) and \( M \) has a non-vanishing top Chern class, \( c_2(M) \neq 0 \). Suppose that there is an even factorization \( c(M) = \alpha \beta \). This means that \( 0 < \text{deg}(\alpha) < 2 \) and \( \alpha \) has only terms of even degree and this is a contradiction. This proves Theorem 1.4.

Example 6.3 (Proof of Theorem 1.5). Let \( M = \mathbb{CP}^n \). The full Chern class is

\[
c(M) = (1 + a)^{n+1} - a^{n+1},
\]

where \( a \) is a generator of \( H^2(M) \). The top Chern class is

\[
c_n(M) = n + 1 \neq 0.
\]

We can find the roots of the polynomial and write

\[
c(M) = C \prod_{i=1}^{n} (a - a_i),
\]
where $C$ is a constant and

$$a_i = \frac{1}{z_{n+1}^i - 1},$$

where $z_{n+1}$ is a primitive $n+1$-th root of unity. Suppose that there is an even factorization $c(M) = \alpha \beta$. Note that we assume that $\deg(\alpha) + \deg(\beta) \leq n$ so when calculating the multiplication $\alpha \beta$ we get that the term $a^{n+1}$ would not appear, so in our calculation we can ignore the relation $a^{n+1} = 0$, and consider $c(M), \alpha, \beta$ as elements in the polynomials ring in the variable $a$.

Because we assume that $\alpha$ is not trivial, we get that there exists a root $x$ of the polynomial $c(M)$ such that $x$ is a root of $\alpha$. Because $\alpha$ has only terms of even degree, we get that $-x$ is also a root of $\alpha$ and hence a root of $c(M)$. From that we get that there are $0 < i_1, i_2 < n + 1$ such that $a_{i_1} = -a_{i_2}$.

$$\frac{1}{z_{n+1}^{i_1} - 1} = \frac{-1}{z_{n+1}^{i_2} - 1}$$

$$z_{n+1}^{i_1} + z_{n+1}^{i_2} = 2.$$

Note that $|z_{n+1}^{i_1}| = 1$ and $|z_{n+1}^{i_2}| = 1$ but their sum is 2 so we get that both are equal to 1 and this is a contradiction. From this we get that $\mathbb{CP}^n$ satisfies the requirements of Theorem 1.3. This together with Theorem 5.1, proves Theorem 1.5.

**Example 6.4.** Let $M$ be the 1-point blow-up of $\mathbb{CP}^3$. We will show that $M$ satisfy the conditions of Theorem 1.3. The cohomology ring of $M$ is generated by 2 generators, $a \in H^2(M)$-the pull back of a generator of $H^*(\mathbb{CP}^3)$, and $b \in H^2(M)$-the Poincaré dual of the exceptional divisor, with the relations

$$ab = 0, b^3 = a^3.$$

The full Chern class of $M$ is

$$c(M) = 1 + 4a + 6a^2 + 6a^3 - 2b.$$  

To see this, denote $\tilde{a}$ to be the corresponding generator of $H^*(\mathbb{CP}^3)$, and write

$$c(\mathbb{CP}^3) = 1 + 4\tilde{a} + 6\tilde{a}^2 + 4\tilde{a}^3.$$  

See [9] for the calculation of the full Chern class of $\mathbb{CP}^n$. One can use this to compute the first and second Chern classes of $M$ by a formula that appears in [1] pp. 608-609]. To calculate the top Chern class, one needs to know the alternating sum of the Betti numbers. In our situation the odd
cohomology groups vanishes, so we only need to count the dimensions of the cohomology groups. By performing the blow-up we added an additional two dimensions (generated by \( b \) and \( b^2 \)). This gives the formula for the full Chern class. See also [3] for a general formula for calculating Chern classes of blow-ups.

Suppose that \( c(M) = \alpha \beta \) is an even factorization. For a general \( \alpha \in H^*(M) \) with even degrees and a general \( \beta \in H^*(M) \), one can write

\[
\alpha = 1 + n_1a^2 + n_2b^2.
\]

\[
\beta = 1 + m_1a + m_2b.
\]

Calculate

\[
\alpha \beta = (1 + n_1a^2)(1 + m_1a) + n_2b^2 + m_2b + n_2m_2b^3 = c(M).
\]

We get that \( m_2 = -2, n_2 = 0 \). Hence

\[
(1 + n_1a^2)(1 + m_1a) = 1 + 4a + 6a^2 + 6a^3.
\]

Denote \( q(a) = 1 + 4a + 6a^2 + 6a^3 \). We get from the factorization of \( q \) above, that there exist two roots of \( q \), \( a_i, a_j \) so that \( a_i = -a_j \). The roots of the polynomial \( q(a) = 1 + 4a + 6a^2 + 6a^3 \) are

\[
a_1 \approx -0.38839, a_2 \approx -0.30581 - 0.57932\sqrt{-1}, a_3 \approx -0.30581 + 0.57932\sqrt{-1}.
\]

We get that \( q \) does not have roots such that \( a_i = -a_j \), so this is a contradiction to the existence of the factorization.

Hence \( M \) satisfies the conditions of Theorem 1.3

**Example 6.5.** Let \( M = \mathbb{CP}^2 \times \mathbb{CP}^2 \). The cohomology of \( M \) is generated by two generators \( a, b \) with the relations

\[
a^3 = b^3 = 0.
\]

The full Chern class is

\[
c(M) = (1 + 3a + 3a^2)(1 + 3b + 3b^2).
\]

Write a general even factorization

\[
c(M) = \alpha \beta.
\]

Since \( 0 < \text{deg}(\alpha) < 4 \) and it is even, we get that \( \text{deg}(\alpha) = 2 \). From the equation \( c(M) = \alpha \beta \) we can deduce that \( \text{deg}(\alpha) + \text{deg}(\beta) \geq 4 \). From the
definition of an even factorization $\deg(\alpha) + \deg(\beta) \leq 4$ and hence $\deg(\beta) = 2$. One can write

$$
\alpha = 1 + c_1 a^2 + c_2 b^2 + c_3 ab,
\beta = 1 + d_1 a + d_2 a^2 + d_3 b + d_4 b^2 + d_5 ab.
$$

Look at the equality $c(M) = \alpha \beta$. Each coefficient in $c(M)$ gives us an equation for the variables

$$c_1, c_2, c_3, d_1, d_2, d_3, d_4, d_5.$$

Hence, we have 8 equations and 8 variables. One can solve this equations and get two sets of solutions where each of them has non-integer values. This is a contradiction since the cohomology groups are with integer coefficients. This implies that there are no classes $\alpha, \beta \in H^*(M)$ so that $\alpha$ has only terms of even degree, and $c(M) = \alpha \beta$. Hence $M$ satisfies the conditions of Theorem 1.3.

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