The Green–Tao theorem for primes of the form $x^2 + y^2 + 1$

Yu-Chen Sun$^1$ · Hao Pan$^2$

Received: 23 June 2018 / Accepted: 27 November 2018 / Published online: 15 December 2018
© Springer-Verlag GmbH Austria, part of Springer Nature 2018

Abstract
We prove that the primes of the form $x^2 + y^2 + 1$ contain arbitrarily long non-trivial arithmetic progressions.

Keywords Prime · Arithmetic progression · Pseudorandom measure

Mathematics Subject Classification Primary 11P32; Secondary 11B25, 11B30, 11N36

1 Introduction
Let $\mathcal{P}$ denote the set of all primes. The celebrated Green–Tao theorem [5] asserts that $\mathcal{P}$ contains arbitrarily long non-trivial arithmetic progressions. That is, for any $k \geq 3$, there exists positive integers $a$ and $d$ such that $a, a + d, \ldots, a + (k - 1)d$ are all primes. In fact, they obtained a stronger result. For a subset $A \subseteq \mathcal{P}$, define the relative upper density of $A$ by

$$d_{\mathcal{P}}(A) := \limsup_{X \to \infty} \frac{|A \cap [1, X]|}{|\mathcal{P} \cap [1, X]|}.$$
Green and Tao proved that for any subset \( A \) of primes with \( \overline{d}_{\mathcal{P}}(A) > 0 \), \( A \) contains arbitrarily long non-trivial arithmetic progressions. There are three key ingredients in Green and Tao’s proof: the Szemerédi theorem, a transference principle and a pseudorandom measure for primes.

Nowadays, the Green–Tao theorem has been generalized in different directions \([6,8,9,14,16,18,19]\). For example, a prime \( p \) is called Chen prime provided that \( p + 2 \) has at most two prime factors. The classical Chen theorem says that there exist infinitely many Chen primes. Green and Tao \([5]\) also claimed that using the similar discussions, one can prove that the Chen primes contains arbitrarily long non-trivial arithmetic progressions. And they gave a Fourier proof of the existence of infinitely many non-trivial three-term arithmetic progressions in the Chen primes. Subsequently, the detailed proof of the extension of Green–Tao theorem to the Chen primes was given by Zhou in \([22]\).

On the other hand, let us consider those primes which can be represented as the sum of two squares plus 1. Let \( \mathcal{P}_2 \) denote the set of all such primes, i.e.,

\[
\mathcal{P}_2 := \{ p \text{ prime} : p = x^2 + y^2 + 1, \quad x, y \in \mathbb{N} \}.
\]

Linnik \([10]\) proved that \( \mathcal{P}_2 \) contains infinitely many primes. In \([7]\), Iwaniec proved that for any sufficiently large \( X \),

\[
|\mathcal{P}_2 \cap [1, X]| \geq \frac{CX}{(\log X)^2}
\]

for some constant \( C > 0 \). For more about the primes in \( \mathcal{P}_2 \), the reader may refer to \([11–13,21]\). Recently, Teräväinen \([20]\) proved that there exist infinitely many non-trivial three-term arithmetic progressions in any subset of \( \mathcal{P}_2 \) with a positive relative density, which extends the Roth-type theorem in the primes of Green \([4]\). It is natural to ask whether the Green–Tao theorem also can be extended to the primes in \( \mathcal{P}_2 \). In this paper, we shall give such an extension.

**Theorem 1.1** Suppose that \( A \) is a subset of \( \mathcal{P}_2 \) with the relative density

\[
\overline{d}_{\mathcal{P}_2}(A) := \limsup_{X \to \infty} \frac{|A \cap [1, X]|}{|\mathcal{P}_2 \cap [1, X]|} > 0.
\]

Then \( A \) contains arbitrarily long non-trivial arithmetic progressions.

The key to our proof is to construct a pseudorandom measure for those primes in \( \mathcal{P}_2 \). In the next section, we shall first give the construction of such a pseudorandom measure. Then the proof of Theorem 1.1 can be reduced to a Goldston–Yıldırım-type \([2,3]\) estimation, which will be proved in the third section.

We introduce several notions which will be used later. Suppose that \( S \) is a finite subset and \( f(x) \) is a function over \( S \). Write

\[
E\left( f(x) | x \in S \right) := \frac{1}{|S|} \sum_{x \in S} f(x).
\]
For an assertion $P$, set $1_P = 1$ or $0$ according to whether $P$ holds or not. Also, let $\phi$ denote the Euler totient function and let $\mu$ denote the Möbius function.

## 2 The pseudorandom measure

First, let us introduce the definition of the linear forms condition and the correlation condition. Suppose that $v \in \mathbb{Z}$ and $L_1, \ldots, L_k \in \mathbb{Q}$ are rational numbers whose numerators and denominators are all bounded. We call

$$\psi(x) := L_1 x_1 + \cdots + L_k x_k + v$$

a linear form, where $x = (x_1, \ldots, x_k)$. Suppose that $N$ is a sufficiently large prime. Let $\mathbb{Z}_N := \mathbb{Z}/N\mathbb{Z}$ be the cyclic group of order $N$. Then $\psi$ also can be viewed as a linear form over $\mathbb{Z}_N$. Let $\nu : \mathbb{Z}_N \to \mathbb{R}$ be a non-negative function. Suppose that $h_0, k_0, m_0$ are positive integers. Suppose that $E(\nu(\psi_1(x)) \cdots \nu(\psi_h(x))) | x \in \mathbb{Z}_N^k) = 1 + o_{h_0, k_0, m_0}(1)$ for any $1 \leq h \leq h_0, 1 \leq k \leq k_0$ and the linear forms $\psi_i(x) = L_{i,1} x_1 + \cdots + L_{i,k} x_k + v_i$ with the numerators and denominators of those $L_{i,j}$ all lie in $[-m_0, m_0]$. Then we say that $\nu$ obeys the $(h_0, k_0, m_0)$-linear forms condition. Similarly, suppose that for any $1 \leq k \leq k_0$, there exists a non-negative weight function $\tau_k : \mathbb{Z}_N \to \mathbb{R}$ such that $E(\tau_k(x) | x \in \mathbb{Z}_N) = O_{k,s}(1)$, and

$$E(\nu(x + v_1) \cdots \nu(x + v_k) | x \in \mathbb{Z}_N) \leq \sum_{1 \leq i < j \leq k} \tau_k(v_i - v_j)$$

for any $v_1, \ldots, v_k \in \mathbb{Z}_N$. Then we say that $\nu$ obeys the $k_0$-correlation condition. A function $\nu(x)$ is called a $m$-pseudorandom measure, provided that $\nu$ satisfies both $(2^{m-1}m, 3m - 4, m)$-linear forms condition and $2^{m-1}$-correlation condition.

Green and Tao [5, Theorem 3.5] proved the following relative Szemerédi theorem.

**Lemma 2.1** Suppose that $\delta > 0$ and $m \geq 3$. Let $f(x)$ be a function over $\mathbb{Z}_N$ such that $E(f(x) | x \in \mathbb{Z}_N) \geq \delta$, and $0 \leq f(x) \leq \nu(x)$ for each $x \in \mathbb{Z}_N$, where $\nu$ is a $m$-pseudorandom measure over $\mathbb{Z}_N$. Then as $N \to \infty$,

$$E(f(x) f(x + y) \cdots f(x + (m - 1)y) | x, y \in \mathbb{Z}_N) \geq c(m, \delta) + o_{k, \delta}(1), \quad (2.1)$$

where $c(m, \delta) > 0$ is a constant only depending on $m$ and $\delta$.

In [1], Conlon, Fox and Zhao improved Green–Tao’s transference principle and showed that the requirement concerning the correlation condition is actually unnecessary. Suppose that as $N \to \infty$,
\[
\mathbb{E} \left( \prod_{j=1}^{k} \prod_{\omega \in \{0,1\}^{[1, \ldots, k] \setminus \{j\}}} \nu \left( \sum_{i=1}^{k} (i - j) x_i^{(\omega_i)} \right) \bigg| x_1^{(0)}, x_1^{(1)}, \ldots, x_k^{(0)}, x_k^{(1)} \in \mathbb{Z}_N \right) = 1 + \omega_k(1)
\]
for any choice of \( \omega_j, \omega \in \{0, 1\} \). Then we say \( \nu \) obeys the \( k \)-linear forms condition. Conlon, Fox and Zhao proved that

**Lemma 2.2** Suppose that \( \delta > 0 \) and \( m \geq 3 \). Let \( f(x) \) be a non-negative function over \( \mathbb{Z}_N \) such that \( \mathbb{E} \left( f(x) \big| x \in \mathbb{Z}_N \right) \geq \delta \) and \( f(x) \leq \nu(x) \) for some function \( \nu \) obeys the \( m \)-linear forms condition. Then (2.1) is also valid.

Clearly the \((2m-1, m, 2m)\)-linear forms condition is stronger than the \( m \)-linear forms condition. So for every \( m \geq 3 \), we only need to construct a pseudorandom measure obeying the \((2m-1, m, 2m)\)-linear forms condition for those primes in \( \mathcal{P}_2 \).

For any positive integer \( q \), let

\[
\mathcal{R}_q = \{ p \text{ prime} : p = q^2 x^2 + q^2 y^2 + 1, \text{ where } x, y \in \mathbb{N} \text{ and } (x, y) = 1 \}.
\]

In [7, Theorem 1], Iwaniec proved that

\[
\frac{c_1}{\phi(q^2)} \cdot \frac{X}{(\log X)^{\frac{3}{2}}} \leq |\mathcal{R}_q \cap [1, X]| \leq \frac{c_2}{\phi(q^2)} \cdot \frac{X}{(\log X)^{\frac{3}{2}}} \tag{2.2}
\]

for any sufficiently large \( X \), where \( c_1, c_2 > 0 \) are absolute constants. Of course, by following Iwaniec’s discussions, we may easily obtain that

\[
\frac{C_1}{\phi(q^2)} \cdot \frac{X}{(\log X)^{\frac{3}{2}}} \leq |\mathcal{R}_q \cap [X, 2X]| \leq \frac{C_2}{\phi(q^2)} \cdot \frac{X}{(\log X)^{\frac{3}{2}}} \tag{2.3}
\]

for some absolute constants \( C_1, C_2 > 0 \).

Suppose that \( \mathcal{A} \) is a subset of \( \mathcal{P}_2 \) with a positive relatively upper density. Let

\[
\delta_0 = \frac{d_{\mathcal{P}_2}(\mathcal{A})}{2}.
\]

Assume that \( X \) is sufficiently large and

\[
|\mathcal{A} \cap [X, 2X]| \geq \delta_0 |\mathcal{P}_2 \cap [X, 2X]|.
\]

Let

\[
\eta_0 = \sum_{q=1}^{\infty} \frac{1}{\phi(q^2)}.
\]
Then in view of (2.3),
\[ |P_2 \cap [X, 2X]| = \sum_{q=1}^{\infty} |R_q \cap [X, 2X]| \geq \sum_{q=1}^{\infty} \frac{C_1}{\phi(q^2)} \cdot \frac{X}{(\log X)^{\frac{3}{2}}} = \frac{\eta_0 C_1 X}{(\log X)^{\frac{3}{2}}}. \tag{2.4} \]

Since \( \eta_0 < +\infty \), there exists a constant \( Q_0 > 0 \) such that
\[ \sum_{q > Q_0} \frac{1}{\phi(q^2)} \leq \frac{C_1}{C_2} \cdot \frac{\delta_0 \eta_0}{2}. \tag{2.5} \]

Now by (2.3), (2.4) and (2.5),
\[ \sum_{q > Q_0} |R_q \cap [X, 2X]| \leq \frac{C_2 X}{(\log X)^{\frac{3}{2}}} \sum_{q > Q_0} \frac{1}{\phi(q^2)} \leq \frac{\delta_0 C_1 X}{(\log X)^{\frac{3}{2}}} \sum_{q=1}^{\infty} \frac{1}{\phi(q^2)} \leq \frac{\delta_0 |P_2 \cap [X, 2X]|}{2} \leq \frac{|A \cap [X, 2X]|}{2}. \]

It follows that
\[ \sum_{q \leq Q_0} |R_q \cap A \cap [X, 2X]| \geq \frac{|A \cap [X, 2X]|}{2}. \]

By the pigeonhole principle, there exists \( 1 \leq q_0 \leq Q_0 \) such that
\[ |R_{q_0} \cap A \cap [X, 2X]| \geq \frac{|A \cap [X, 2X]|}{2 Q_0} \geq \frac{\delta_0 |P_2 \cap [X, 2X]|}{2 Q_0} \geq \frac{\delta_0 \eta_0}{2 Q_0} \cdot \frac{C_1 X}{(\log X)^{\frac{3}{2}}}. \tag{2.6} \]

Let \( w := w(X) \) be an increasing function which very slowly tends to \( +\infty \) as \( X \to +\infty \), and let
\[ W = \prod_{p \leq w} p. \]

Let
\[ S_W = \{ 1 \leq b \leq W : (q_0^2 b + 1, W) = 1, (b, W) \text{ has no prime factor of the form } 4k+3 \}. \]

By the Chinese remainder theorem, it is easy to check that
\[ |S_W| = W \prod_{p \equiv 3 \pmod{4}} \left( 1 - \frac{2}{p} \right) \prod_{p \equiv 3 \pmod{4}, p \nmid W, p \nmid q_0} \left( 1 - \frac{1}{p} \right) \prod_{p \equiv 3 \pmod{4}} \left( 1 - \frac{1}{p} \right) \]
\[ = O\left( \frac{q_0}{\phi(q_0)} \cdot \frac{\phi(W)}{W^{\frac{3}{2}}} \right). \tag{2.7} \]
Conversely, if a prime \( p \equiv q_0^2b + 1 \pmod{q_0^2W} \) can be written as \( p = q_0^2(x^2 + y^2) + 1 \) with \( (x, y) = 1 \), then clearly \((b, W)\) has no prime factor of the form \( 4k + 3 \). Hence by (2.6),

\[
\sum_{b \in S_W} \left| \left\{ p \in \mathcal{R}_{q_0} \cap \mathcal{A} \cap [X, 2X] : p \equiv q_0^2b + 1 \pmod{q_0^2W} \right\} \right| \geq \frac{\delta_0\eta_0}{2Q_0} \cdot \frac{C_1X}{(\log X)^\frac{3}{2}}.
\]

By (2.7) and the pigeonhole principle, there exists \( b \in S_W \) such that

\[
\left| \left\{ p \in \mathcal{R}_{q_0} \cap \mathcal{A} \cap [X, 2X] : p \equiv q_0^2b + 1 \pmod{q_0^2W} \right\} \right| \geq \frac{\phi(q_0)}{q_0} \cdot \frac{W^{1/2}}{\phi(W)^{1/2}} \cdot \frac{\delta_0\eta_0}{Q_0} \cdot \frac{C_3X}{(\log X)^\frac{3}{2}} \tag{2.8}
\]

for some constant \( C_3 > 0 \). We emphasize that here \( \delta_0, \eta_0, q_0, Q_0 \) are all positive constants only depending on the subset \( \mathcal{A} \).

Let \( N \) be a prime lying in \([q_0^{-2}W^{-1}\epsilon_m^{-1}X, q_0^{-2}W^{-1}\epsilon_m^{-1}X + X(\log X)^{-2}]\), where

\[
\epsilon_m = \frac{1}{4^m(m + 4)!}.
\]

According to the prime number theorem with a remainder term, such prime \( N \) always exists whenever \( X \) is sufficiently large. Let

\[
C_0 = \frac{\phi(q_0)}{q_0} \cdot \frac{\delta_0\eta_0}{Q_0} \cdot \frac{C_3}{2},
\]

where \( C_3 \) is the constant in (2.8). As we have shown,

\[
\left| \left\{ n \in [\epsilon_mN, 2\epsilon_mN] : q_0^2(Wn + b) + 1 \in \mathcal{R}_{q_0} \cap \mathcal{A} \right\} \right| \geq \frac{W^{1/2}}{\phi(W)^{1/2}} \cdot \frac{C_0X}{(\log X)^\frac{3}{2}}. \tag{2.9}
\]

Define the Möbius-type function

\[
\mu_3(n) := \begin{cases} 
1, & \text{if } n = 1, \\
(-1)^r, & \text{if } n = p_1 \cdots p_r, \ p_1, \ldots, p_r \text{ are distinct primes with } p_i \equiv 3 \pmod{4}, \\
0, & \text{otherwise}.
\end{cases}
\]

Let

\[
R = N^{\frac{1}{2m + 4}}.
\]
Let \( \chi : \mathbb{R} \rightarrow \mathbb{R} \) be a smooth function such that \( \chi(0) = 1 \) and \( \chi \) is supported on the interval \([-1, 1]\). Define

\[
\Lambda_R(n) := \sum_{d \mid n} \mu(d) \chi \left( \frac{\log d}{\log R} \right)
\]

and

\[
\Lambda_R^*(n) := \sum_{d \mid n} \mu_3(d) \chi \left( \frac{\log d}{\log R} \right).
\]

Evidently, if \( q_0^2(Wn + b) + 1 \in \mathcal{R}_{q_0} \), then

\[
\Lambda_R(q_0^2(Wn + b) + 1) \Lambda_R^*(Wn + b) = 1.
\]

We are ready to define our pseudorandom measure. Let

\[
\alpha_0 = \lim_{s \to 1} \frac{1}{s - 1} \prod_{p \equiv 3 \pmod{4}} \left( 1 - \frac{1}{p^s} \right)^2.
\]

(2.10)

By our knowledge on the Riemann \( \zeta \)-function and Dirichlet \( L \)-function, we have \( \alpha_0 > 0 \) and

\[
\lim_{s \to 1} \frac{1}{s - 1} \prod_{p \equiv 3 \pmod{4}} \left( 1 - \frac{1}{p^s} \right)^2 = \frac{1}{\alpha_0}.
\]

For any \( n \in [\epsilon_m N, 2\epsilon_m N] \), let

\[
v(n) := \frac{(\log R)^2}{\alpha_0^2 \cdot C_\chi} \prod_{p \mid W} \left( 1 - \frac{1}{p} \right) \prod_{p \equiv 3 \pmod{4}} \left( 1 - \frac{1}{p} \right)^2 \cdot \Lambda_R(q_0^2(Wn + b) + 1)^2 \Lambda_R^*(Wn + b)^2,
\]

(2.11)

and let \( v(n) = 1 \) for the other \( n \in \mathbb{Z}_N \), where \( C_\chi > 0 \) is a constant only depending on \( \chi \) which we shall determine later. Let

\[
f(n) = \frac{(\log R)^3}{\alpha_0 \cdot C_\chi} \prod_{p \equiv 3 \pmod{4}} \left( 1 - \frac{1}{p} \right)^2 \prod_{p \mid W} \left( 1 - \frac{1}{p} \right)^2
\]

(2.12)

provided that \( n \in [\epsilon_m N, 2\epsilon_m N] \) and

\[q_0^2(Wn + b) + 1 \in \mathcal{R}_{q_0} \cap \mathcal{A},\]
also, set \( f(n) = 0 \) for the other \( n \in \mathbb{Z}_N \). Clearly \( f(x) \leq \nu(x) \) for each \( x \in \mathbb{Z}_N \) and

\[
\mathbb{E} \left( f(x) \mid x \in \mathbb{Z}_N \right) \geq \frac{\epsilon_m \nu_0 C_0}{2^{2m+8m^2} \cdot \alpha_0^1 C_X} > 0
\]

by (2.9), where

\[
\nu_0 = \min_{x \geq 2} \prod_{p \leq x} \left( 1 - \frac{1}{p} \right)^{-\frac{1}{2}} \prod_{p \equiv 3 \mod 4} \left( 1 - \frac{1}{p} \right)^{\frac{1}{2}}.
\]

hence by lemma 2.2, if \( \nu \) obeys the \( m \)-linear forms condition, then

\[
\mathbb{E} \left( f(x) f(x + y) \cdots f(x + (m - 1)y) \mid x, y \in \mathbb{Z}_N \right) \geq c_m, A
\]

for some constant \( c_m, A > 0 \) only depending on \( m \) and \( A \). since \( \epsilon_m < 1/m, R_{q_0} \cap A \) contains a non-trivial arithmetic progression of length \( m \).

according to green and tao’s discussions in the proof of [5, proposition 9.8], in order to show \( \nu \) obeys the \( m \)-linear forms condition, it suffices to prove the following goldston–yildirim-type estimation.

**Proposition 2.1** Suppose that \( m \) and \( h \) are two positive integers. Let

\[
\psi_i(x) := \sum_{j=1}^{h} L_{ij} x_j + v_i, \quad i = 1, \ldots, m,
\]

be linear forms over \( \mathbb{Z}_N \) such that

(i) the coefficients \( L_{ij} \) are integers and \( |L_{ij}| \leq \sqrt{w}/2 \) for any \( 1 \leq i \leq m \) and \( 1 \leq j \leq h \);

(ii) the \( h \)-tuples \( (L_{ij})_{j=1}^{h} \) are never identically zero, and that no two \( h \)-tuples are rational multiples of each other.

write

\[
\theta_i := q_0^2 (W \psi_i + b) + 1, \quad \theta_{m+i} := W \psi_i + b
\]

for each \( 1 \leq i \leq m \). suppose that \( B \) is a product \( \prod_{i=1}^{h} I_i \subset \mathbb{R}^h \), where each \( I_i \subset \mathbb{R} \) is an interval of length at least \( R^{10m} \). then

\[
\mathbb{E} \left( \Lambda_R(\theta_1(x))^2 \cdots \Lambda_R(\theta_m(x))^2 \Lambda_{R,w}(\theta_{m+1}(x))^2 \cdots \Lambda_{R,w}(\theta_{2m}(x))^2 \mid x \in B \right)
\]

\[
= \left( C_X + o_w(1) \right) \cdot \frac{\alpha_0^1}{(\log R)^{\frac{3}{2}}} \prod_{p \equiv 3 \mod 4} \frac{p}{p-1} \cdot \prod_{p|W} \frac{p^2}{(p-1)^2} \right)^m
\]

(2.13)
where \( C_\chi > 0 \) is a constant only depending on \( \chi \), and \( o_w(1) \) means a term which tends to 0 as \( w \to +\infty \).

3 The Goldston–Yıldırım-type estimation

In this section, the proof of Proposition 2.1 will be given. We shall follow the way of Tao in [17].

**Lemma 3.1** Let \( c_1, c_2, \ldots, c_k \) be some integers. For each prime \( p \), arrange the bounded complex numbers \( c_{p,1}, \ldots, c_{p,k} \) such that

\[
c_{p,j} = c_j + O(p^{-1})
\]

unless \( p \) divides \( W \). Then

\[
\prod_{p \equiv 3 \pmod{4}} \left( 1 - \sum_{j=1}^{k} \frac{c_{p,j}}{p^{s_j}} \right) = G_1 \cdot \left( 1 + O(H) \right) \prod_{j=1}^{k} \left( \alpha_0(s_j - 1) \right)^{\frac{1}{2}c_j},
\]

(3.2)

where \( s_1, \ldots, s_k \) are bounded complex numbers with \( \Re(s_j) \geq 1 \) and

\[
G_1 = \prod_{p \equiv 3 \pmod{4}} \frac{1 - p^{-1}(c_{p,1} + \cdots + c_{p,k})}{(1 - p^{-1})^{c_1 + \cdots + c_k}}, \quad H = \log w \cdot \max_{1 \leq j \leq k} \{|s_j - 1|\}.
\]

Also, we have

\[
\prod_{p \not\equiv 3 \pmod{4}} \left( 1 - \sum_{j=1}^{k} \frac{c_{p,j}}{p^{s_j}} \right) = G_2 \cdot \left( 1 + O(H) \right) \prod_{j=1}^{k} \left( \alpha_0^{-1}(s_j - 1) \right)^{\frac{1}{2}c_j},
\]

(3.3)

where

\[
G_2 = \prod_{p \not\equiv 3 \pmod{4}} \frac{1 - p^{-1}(c_{p,1} + \cdots + c_{p,k})}{(1 - p^{-1})^{c_1 + \cdots + c_k}}.
\]

Furthermore, in (3.2) and (3.3), the implied constants in the symbol \( O \) only depend on \( k \) and bounds of those \( c_{p,j} \) and \( s_j \).

**Proof** According to the definition of \( \alpha_0 \) in (2.10), we have

\[
\prod_{p \equiv 3 \pmod{4}} \prod_{j=1}^{k} \left( 1 - \frac{1}{p^{s_j}} \right)^{c_j} = \left( 1 + o_k(1) \right) \prod_{j=1}^{k} \left( \alpha_0(s_j - 1) \right)^{\frac{1}{2}c_j}
\]
and
\[
\prod_{p \not\equiv 3 \pmod{4}} \prod_{j=1}^{k} \left( 1 - \frac{1}{p^{s_j}} \right)^{c_j} = \left( 1 + o_k(1) \right) \prod_{j=1}^{k} \left( a_0^{-1}(s_j - 1) \right)^{\frac{1}{2}c_j}
\]
as \(s_1, \ldots, s_k \to 1\). So it suffices to show that
\[
\prod_{p \equiv 3 \pmod{4}} \frac{1 - (p^{-s_1}c_{p,1} + \cdots + p^{-s_k}c_{p,k})}{(1 - p^{-s_1})^{c_1} \cdots (1 - p^{-s_k})^{c_k}} = G_1 \cdot (1 + O(H)) \tag{3.4}
\]
and
\[
\prod_{p \not\equiv 3 \pmod{4}} \frac{1 - (p^{-s_1}c_{p,1} + \cdots + p^{-s_k}c_{p,k})}{(1 - p^{-s_1})^{c_1} \cdots (1 - p^{-s_k})^{c_k}} = G_2 \cdot (1 + O(H)). \tag{3.5}
\]
Here we only prove (3.4), since the proof of (3.5) is very similar. Suppose that \(p \nmid W\). By (3.1),
\[
\sum_{j=1}^{k} c_{p,j} p^{s_j} = \sum_{j=1}^{k} c_j p^{s_j} + O\left( \frac{1}{p^2} \right)
\]
since \(\Re(s_1), \ldots, \Re(s_k) \geq 1\). Then we have
\[
\frac{1 - (p^{-s_1}c_{p,1} + \cdots + p^{-s_k}c_{p,k})}{(1 - p^{-s_1})^{c_1} \cdots (1 - p^{-s_k})^{c_k}} = \frac{1 - (p^{-s_1}c_1 + \cdots + p^{-s_k}c_k) + O(p^{-2})}{(1 - p^{-s_1})^{c_1} \cdots (1 - p^{-s_k})^{c_k}}.
\]
Clearly for each \(1 \leq j \leq k\),
\[
\lim_{s_1, \ldots, s_k \to 1} \frac{\partial}{\partial s_j} \left( \frac{1 - (p^{-s_1}c_1 + \cdots + p^{-s_k}c_k)}{(1 - p^{-s_1})^{c_1} \cdots (1 - p^{-s_k})^{c_k}} \right)
= \lim_{s_1, \ldots, s_k \to 1} \frac{\log p \cdot p^{-s_j} c_j \cdot (1 - p^{-s_j}) - (1 - (p^{-s_1}c_1 + \cdots + p^{-s_k}c_k)) \cdot p^{-s_j} c_j}{(1 - p^{-s_1})^{c_1} \cdots (1 - p^{-s_j})^{c_j+1} \cdots (1 - p^{-s_k})^{c_k}}
= O \left( \frac{\log p}{p^2} \right).
\]
It follows that
\[
\frac{1 - (p^{-s_1}c_1 + \cdots + p^{-s_k}c_k)}{(1 - p^{-s_1})^{c_1} \cdots (1 - p^{-s_k})^{c_k}} = \frac{1 - p^{-1}(c_1 + \cdots + c_k)}{(1 - p^{-1})^{c_1 + \cdots + c_k}} + O\left( \frac{\log p}{p^2} \cdot \max_{1 \leq j \leq k} \{|s_j - 1|\} \right).
\]
Similarly, if \( p \) divides \( W \), we also have

\[
1 - \left( \frac{-s_1 c_{p,1} + \cdots + -s_k c_{p,k}}{1 - p^{-s_1} c_1 \cdots (1 - p^{-s_k} c_k)} \right)
\]

\[
= 1 - \left( \frac{-s_1 c_{p,1} + \cdots + -s_k c_{p,k}}{1 - p^{-s_1} c_1 \cdots (1 - p^{-s_k} c_k)} \right) \cdot \prod_{j=1}^{k} (1 - p^{-s_j} c_{p,j} - c_j)
\]

\[
= 1 - \left( \frac{-s_1 c_{p,1} + \cdots + -s_k c_{p,k}}{1 - p^{-1} (c_1 + \cdots + c_k)} \right) + O \left( \sum_{1 \leq j \leq k} |c_{p,j} - c_j| \right).
\]

Note that by the Mertens theorem,

\[
\sum_{p \mid W} \frac{\log p}{p} = \sum_{p \leq w} \frac{\log p}{p} = O(\log w).
\]

Multiplying all these above estimates together, we may get (3.4).

Now we are ready to prove Proposition 2.1. Let

\[
u_j(n) := \begin{cases} \mu(n), & \text{if } 1 \leq j \leq m, \\ \mu_3(n), & \text{if } m + 1 \leq j \leq 2m. \end{cases}
\]

Clearly the left side of (2.13) coincides with

\[
\mathbb{E} \left( \prod_{j=1}^{2m} \sum_{d_j, e_j \leq R} u_j(d_j) u_j(e_j) \chi \left( \frac{\log d_j}{\log R} \right) \chi \left( \frac{\log e_j}{\log R} \right) |x \in \mathbb{B} \right),
\]

which can be rearranged as

\[
\sum_{d_j, e_j \leq R} \prod_{\mu(d_j) \mu(e_j) \neq 0}^{2m} u_j(d_j) u_j(e_j) \chi \left( \frac{\log d_j}{\log R} \right) \chi \left( \frac{\log e_j}{\log R} \right) \cdot \mathbb{E} \left( \prod_{j=1}^{2m} 1_{d_j, e_j \mid \theta_j(x)} |x \in \mathbb{B} \right).
\]

Suppose that \( p \) is a prime. For each \( \emptyset \neq \mathcal{I} \subseteq \{1, 2, \ldots, 2m\} \), let

\[
\lambda_{\mathcal{I}}(p) := p \cdot \mathbb{E} \left( \prod_{j \in \mathcal{I}} 1_{p \mid \theta_j(x)} |x \in \mathbb{Z}_p^h \right).
\]
In particular, set \( \lambda_\vartheta(p) = 1 \). Assume that \( p \nmid W \). For \( 1 \leq i < j \leq 2m \), since \( \theta_i \) is not a rational multiple of \( \theta_j \), we must have

\[
|\{ x \in \mathbb{Z}_p^h : p \text{ divides both } \theta_i(x) \text{ and } \theta_j(x) \}| = O(p^{h-2}).
\]

So \( \lambda_\vartheta(p) = O(p^{-1}) \) whenever \( |I| \geq 2 \). Of course, if \( |I| = 1 \), then \( \lambda_\vartheta(p) = 1 \). Next, assume that \( p \mid W \). Recall that \( (\varphi_b, W) = 1 \) and \( (b, W) \) has no prime factor of the form \( 4k + 3 \). Then \( \lambda_\vartheta(p) = 0 \) provided that \( p \equiv 3 \) (mod 4), or \( p \equiv 1 \) (mod 4) and \( I \cap \{ 1, \ldots , m \} \neq \emptyset \). Of course, if \( p \equiv 1 \) (mod 4) and \( I \subseteq \{ m + 1, \ldots , 2m \} \), we still have \( \lambda_\vartheta(p) = 1 \) or \( O(p^{-1}) \) according to whether \( |I| = 1 \) or \( |I| \geq 2 \).

Assume that \( d_1, \ldots , d_{2m}, e_1, \ldots , e_{2m} \) are all square-free integers lying in \( [1, R] \). Define

\[
\mathcal{I}_{d_1, \ldots , d_{2m}}(p) := \{ 1 \leq j \leq 2m : p|d_j \}
\]

for each prime \( p \). Let \( D = [d_1, \ldots , d_{2m}, e_1, \ldots , e_{2m}] \) be the least common multiple of those \( d_j, e_j \). Clearly \( D \leq R^{4m} \). Note that \( \mathbf{1}_{d_j, e_j|\theta_j(x)} \) can be viewed as a function over \( \mathbb{Z}_D^h \). So by the Chinese remainder theorem, we have

\[
\mathbb{E}\left( \prod_{j=1}^{2m} \mathbf{1}_{d_j, e_j|\theta_j(x)} | x \in B \right) = \mathbb{E}\left( \prod_{j=1}^{2m} \mathbf{1}_{d_j, e_j|\theta_j(x)} | x \in \mathbb{Z}_D^h \right) + O_{m,h}\left( \frac{D}{\min_{1 \leq i \leq h} |I_i|} \right)
\]

\[
= \prod_{p \text{ prime } |D} \frac{\lambda_{\mathcal{I}_{d_1, \ldots , d_{2m}}(p) \cup \mathcal{I}_{e_1, \ldots , e_{2m}}(p)}{p} + O_{m,h}(R^{-6m}).
\]

If \( p \nmid D \), then clearly \( \mathcal{I}_{d_1, \ldots , d_{2m}}(p) = \mathcal{I}_{e_1, \ldots , e_{2m}}(p) = \emptyset \), i.e., \( \lambda_{\mathcal{I}_{d_1, \ldots , d_{2m}}(p) \cup \mathcal{I}_{e_1, \ldots , e_{2m}}(p)}(p) = 1 \). It follows that

\[
\sum_{d_j, e_j \leq R} \prod_{j=1}^{2m} u_j(d_j)u_j(e_j) \chi\left( \frac{\log d_j}{\log R} \right) \chi\left( \frac{\log e_j}{\log R} \right) \mathbb{E}\left( \prod_{j=1}^{2m} \mathbf{1}_{d_j, e_j|\theta_j(x)} | x \in B \right)
\]

\[
= \sum_{d_j, e_j \leq R} \prod_{j=1}^{2m} u_j(d_j)u_j(e_j) \chi\left( \frac{\log d_j}{\log R} \right) \chi\left( \frac{\log e_j}{\log R} \right)
\]

\[
\prod_{p \text{ prime } |D} \frac{\lambda_{\mathcal{I}_{d_1, \ldots , d_{2m}}(p) \cup \mathcal{I}_{e_1, \ldots , e_{2m}}(p)}{p} + O_{m,h}(R^{-2m})
\]

\[
= \sum_{d_j, e_j \leq R} \prod_{j=1}^{2m} u_j(d_j)u_j(e_j) \chi\left( \frac{\log d_j}{\log R} \right) \chi\left( \frac{\log e_j}{\log R} \right)
\]

\[
\frac{g(d_1, \ldots , d_{2m}, e_1, \ldots , e_{2m})}{[d_1, \ldots , d_{2m}, e_1, \ldots , e_{2m}]} + O_{m,h}(R^{-2m}),
\]

\( \odot \) Springer
The Green–Tao theorem for primes of the form $x^2 + y^2 + 1$}

where

$$g(d_1, \ldots, d_{2m}, e_1, \ldots, e_{2m}) := \prod_{p \text{ prime}} \lambda_{I_{d_1, \ldots, d_{2m}(p)}, e_1, \ldots, e_{2m}(p)}(p).$$

As we have shown,

$$\lambda_{I}(p) = \begin{cases} 1, & \text{if } I = \emptyset, \\ 0, & \text{if } p \mid W \text{ and } p \equiv 3 \pmod{4}, \\ 0, & \text{if } p \mid W, \ p \equiv 1 \pmod{4} \text{ and } I \cap \{1, \ldots, m\} \neq \emptyset, \\ \lambda_{I} + O(p^{-1}), & \text{otherwise}, \end{cases}$$

where

$$\lambda_{I} := \begin{cases} 1, & \text{if } |I| = 1, \\ 0, & \text{if } |I| \geq 2. \end{cases}$$

Write

$$e^x \chi(x) = \int_{-\infty}^{+\infty} \psi(t)e^{-ixt} dt$$

for some function $\psi$. We know that $\psi$ is rapidly decreasing (cf. [15, Chapter 6, Corollary 2.2]), i.e., obeys the bounds

$$\psi(t) = O_{A}((1 + |t|)^{-A})$$

for any $A > 0$. Then

$$\chi\left(\frac{\log d_j}{\log R}\right) = \int_{-\infty}^{+\infty} d_j^{-\frac{1+it}{\log R}} \psi(t) dt = \int_{-(\log R)^{\frac{1}{2}}}^{(\log R)^{\frac{1}{2}}} d_j^{-\frac{1+it}{\log R}} \psi(t) dt + O_{A,\chi}\left((\log R)^{-A}\right)$$

for any arbitrarily large $A$. Clearly

$$\sum_{\frac{1}{d_1, \ldots, d_{2m}, e_1, \ldots, e_{2m}} \leq R} \prod_{p \text{ prime}} \left(1 + \frac{4m}{p}\right) = O_m((\log R)^{4m}).$$

Also, note that $g(d_1, \ldots, d_{2m}, e_1, \ldots, e_{2m})$ is bounded. Hence for any large $A > 0$,

$$\sum_{\frac{1}{d_1, \ldots, d_{2m}, e_1, \ldots, e_{2m}} \leq R} \prod_{\frac{u_j(d_j)u_j(e_j)}{p \text{ prime}}} \left(\frac{\log d_j}{\log R}\right) \chi\left(\frac{\log d_j}{\log R}\right) \chi\left(\frac{\log e_j}{\log R}\right)$$
\[= \int (\log R)^{\frac{1}{2}} \cdots \int (\log R)^{\frac{1}{2}} \Omega(s_1, \ldots, s_{2m}, t_1, \ldots, t_{2m}) \prod_{j=1}^{2m} \psi(s_j) \psi(t_j) ds_j dt_j + O_{A, \chi}((\log R)^{-A}),\]

where

\[\Omega(s_1, \ldots, s_{2m}, t_1, \ldots, t_{2m}) = \sum_{d_1, \ldots, d_{2m}, e_1, \ldots, e_{2m}} g(d_1, \ldots, d_{2m}, e_1, \ldots, e_{2m}) \prod_{j=1}^{2m} \frac{u_j(d_j)u_j(e_j)}{d_j^{1+\varepsilon_j} e_j^{1+\varepsilon_j}}.\]

Write \(y_j = (1 + is_j)/\log R\) and \(z_j = (1 + it_j)/\log R\) for \(1 \leq j \leq 2m\). Note that

\[I_{d_1, \ldots, d_{2m}}(p) \cup I_{e_1, \ldots, e_{2m}}(p) \subseteq \{1, 2, \ldots, m\}\]

provided \(p \equiv 3 \pmod{4}\). Hence

\[\Omega(s_1, \ldots, s_{2m}, t_1, \ldots, t_{2m}) = \sum_{d_1, \ldots, d_{2m}, e_1, \ldots, e_{2m}} \prod_{p \text{ prime}} \frac{\lambda I_{d_1, \ldots, d_{2m}}(p) \cup I_{e_1, \ldots, e_{2m}}(p)(p)}{p^{1+\varepsilon_j + \varepsilon_j - 1}} \prod_{j=1}^{2m} \frac{u_j(d_j)u_j(e_j)}{d_j^{1+\varepsilon_j} e_j^{1+\varepsilon_j}}.\]

Let

\[\eta_1 = \sum_{\substack{I, J \subseteq \{1, \ldots, 2m\} \\ I \cup J \neq \emptyset}} (-1)^{|I|+|J|} \lambda_{I \cup J}, \quad \eta_2 = \sum_{\substack{I, J \subseteq \{1, \ldots, m\} \\ I \cup J \neq \emptyset}} (-1)^{|I|+|J|} \lambda_{I \cup J},\]

and

\[\kappa_1(p) = \sum_{\substack{I, J \subseteq \{1, \ldots, 2m\} \\ I \cup J \neq \emptyset}} (-1)^{|I|+|J|} \lambda_{I \cup J}(p),\]

\[\kappa_2(p) = \sum_{\substack{I, J \subseteq \{1, \ldots, m\} \\ I \cup J \neq \emptyset}} (-1)^{|I|+|J|} \lambda_{I \cup J}(p).\]
Applying Lemma 3.1, we get

\[
\prod_{p \equiv 3 \pmod{4}} \left( 1 - \sum_{\substack{\mathcal{I}, \mathcal{J} \subseteq \{1, \ldots, 2m\} \\ \mathcal{I} \cup \mathcal{J} \neq \emptyset}} \frac{(-1)^{|\mathcal{I}|+|\mathcal{J}|+1} \lambda_{\mathcal{I} \cup \mathcal{J}}(p)}{p^{1+\sum_{j \in \mathcal{I}} y_j + \sum_{j \in \mathcal{J}} z_j}} \right) \]

\[
= G_1 \cdot \left( 1 + O\left( \frac{\log w}{(\log R)^{\frac{1}{2}}} \right) \right) \prod_{\substack{\mathcal{I}, \mathcal{J} \subseteq \{1, \ldots, 2m\} \\ \mathcal{I} \cup \mathcal{J} \neq \emptyset}} \left( \alpha_0 \cdot \left( \sum_{j \in \mathcal{I}} y_j + \sum_{j \in \mathcal{J}} z_j \right) \right)^{(-1)^{|\mathcal{I}|+|\mathcal{J}|+1} \frac{1}{2} \lambda_{\mathcal{I} \cup \mathcal{J}}},
\]

where

\[
G_1 = \prod_{p \equiv 3 \pmod{4}} \left( 1 + \frac{\kappa_1(p)}{p} \right) \cdot \left( 1 - \frac{1}{p} \right)^{\eta_1}.
\]

Similarly,

\[
\prod_{p \not\equiv 3 \pmod{4}} \left( 1 - \sum_{\substack{\mathcal{I}, \mathcal{J} \subseteq \{1, \ldots, 2m\} \\ \mathcal{I} \cup \mathcal{J} \neq \emptyset}} \frac{(-1)^{|\mathcal{I}|+|\mathcal{J}|+1} \lambda_{\mathcal{I} \cup \mathcal{J}}(p)}{p^{1+\sum_{j \in \mathcal{I}} y_j + \sum_{j \in \mathcal{J}} z_j}} \right) \]

\[
= G_2 \cdot \left( 1 + O\left( \frac{\log w}{(\log R)^{\frac{1}{2}}} \right) \right) \prod_{\substack{\mathcal{I}, \mathcal{J} \subseteq \{1, \ldots, 2m\} \\ \mathcal{I} \cup \mathcal{J} \neq \emptyset}} \left( \alpha_0^{-1} \cdot \left( \sum_{j \in \mathcal{I}} y_j + \sum_{j \in \mathcal{J}} z_j \right) \right)^{(-1)^{|\mathcal{I}|+|\mathcal{J}|+1} \frac{1}{2} \lambda_{\mathcal{I} \cup \mathcal{J}}},
\]

where

\[
G_2 = \prod_{p \not\equiv 3 \pmod{4}} \left( 1 + \frac{\kappa_2(p)}{p} \right) \cdot \left( 1 - \frac{1}{p} \right)^{\eta_2}.
\]

Suppose that \( p \equiv 3 \pmod{4} \). Clearly \( \kappa_1(p) = 0 \) if \( p \mid W \). Assume that \( p \nmid W \). By (3.1),

\[
\kappa_1(p) = \sum_{\mathcal{I}, \mathcal{J} \subseteq \{1, \ldots, 2m\} \atop |\mathcal{I} \cup \mathcal{J}|=1} (-1)^{|\mathcal{I}|+|\mathcal{J}|} + O_m(p^{-1}) = -2m + O_m(p^{-1}).
\]

Similarly, we also have \( \eta_1 = -2m \). So

\[
G_1 = \prod_{p \equiv 3 \pmod{4} \atop p \mid W} \left( 1 - \frac{1}{p} \right)^{-2m} \cdot \prod_{p \equiv 3 \pmod{4} \atop p \nmid W} \left( 1 - \frac{2m + O(p^{-1})}{p} \right) \left( 1 - \frac{1}{p} \right)^{-2m}
\]
\[ = (1 + a_w(1)) \prod_{p \equiv 3 \pmod{4}} \left( 1 - \frac{1}{p} \right)^{-2m}. \]

Suppose that \( p \not\equiv 3 \pmod{4} \). Clearly we still have \( \kappa_2(p) = 0 \) for those \( p \mid W \). If \( p \nmid W \), then
\[
\kappa_2(p) = \sum_{I, J \subseteq \{1, \ldots, m\}} (-1)^{|I|+|J|} + O_m(p^{-1}) = -m + O_m(p^{-1}).
\]

Also, \( \eta_2 = -m \). Thus
\[
G_2 = (1 + a_w(1)) \prod_{p \equiv 3 \pmod{4}} \left( 1 - \frac{1}{p} \right)^{-m}.
\]

On the other hand, clearly
\[
\prod_{I, J \subseteq \{1, \ldots, 2m\}} \left( \alpha_0 \cdot \left( \sum_{j \in I} y_j + \sum_{j \in J} z_j \right) \right)^{(-1)^{|I|+|J|+1} \frac{1}{2} \lambda_{I \cup J}}
\]
\[
= \left( \frac{\alpha_0}{\log R} \right)^{-\frac{1}{2} \eta_1} \prod_{I, J \subseteq \{1, \ldots, 2m\}, I \cup J \neq \emptyset} \left( \log R \sum_{j \in I} y_j + \log R \sum_{j \in J} z_j \right)^{(-1)^{|I|+|J|+1} \frac{1}{2} \lambda_{I \cup J}}
\]
\[
= \frac{\alpha_0^m}{(\log R)^m} \cdot \Psi_1(s_1, \ldots, s_{2m}, t_1, \ldots, t_{2m}),
\]

where
\[
\Psi_1(s_1, \ldots, s_{2m}, t_1, \ldots, t_{2m})
\]
\[
= \prod_{I, J \subseteq \{1, \ldots, 2m\}, I \cup J \neq \emptyset} \left( \sum_{j \in I} (1 + is_j) + \sum_{j \in J} (1 + it_j) \right)^{(-1)^{|I|+|J|+1} \frac{1}{2} \lambda_{I \cup J}}. \quad (3.6)
\]

Similarly, since \( \eta_2 = -m \), we also have
\[
\prod_{I, J \subseteq \{1, \ldots, m\}} \left( \alpha_0^{-1} \cdot \left( \sum_{j \in I} y_j + \sum_{j \in J} z_j \right) \right)^{(-1)^{|I|+|J|+1} \frac{1}{2} \lambda_{I \cup J}}
\]
\[
= \frac{1}{\alpha_0^m (\log R)^m} \cdot \Psi_2(s_1, \ldots, s_m, t_1, \ldots, t_m),
\]

\( \pentagon \) Springer
where

\[
\Psi_2(s_1, \ldots, s_m, t_1, \ldots, t_m) = \prod_{\mathcal{I}, \mathcal{J} \subset \{1, \ldots, m\}} \left( \sum_{j \in \mathcal{I}} (1 + is_j) + \sum_{j \in \mathcal{J}} (1 + it_j) \right)^{(-1)^{|\mathcal{I}|+|\mathcal{J}|+1} \frac{1}{2} \lambda_{\mathcal{I} \cup \mathcal{J}}}. \tag{3.7}
\]

Thus we get

\[
\begin{aligned}
\Omega(s_1, \ldots, s_{2m}, t_1, \ldots, t_{2m}) &= (1 + o_w(1)) \prod_{p=3 \pmod{4}} \left(1 - \frac{1}{p}\right)^{-2m} \prod_{p \equiv 3 \pmod{4}} \left(1 - \frac{1}{p}\right)^{-m} \\
&\quad \cdot \frac{1}{(\log R)^{2m}} \cdot \Psi_1(s_1, \ldots, s_{2m}, t_1, \ldots, t_{2m}) \Psi_2(s_1, \ldots, s_m, t_1, \ldots, t_m).
\end{aligned}
\]

Let

\[
C_{m, \chi} = \int_{-(\log R)^{\frac{1}{2}}}^{(\log R)^{\frac{1}{2}}} \cdots \int_{-(\log R)^{\frac{1}{2}}}^{(\log R)^{\frac{1}{2}}} \Upsilon(s_1, \ldots, s_{2m}, t_1, \ldots, t_{2m}) \prod_{j=1}^{2m} \psi(s_j) \psi(t_j) ds_j dt_j,
\]

where

\[
\Upsilon(s_1, \ldots, s_{2m}, t_1, \ldots, t_{2m}) = \Psi_1(s_1, \ldots, s_{2m}, t_1, \ldots, t_{2m}) \Psi_1(s_2, \ldots, s_m, t_1, \ldots, t_m).
\]

Recall that \(\lambda_I = 0\) whenever \(|I| \geq 2\). By (3.6) and (3.7), it is easy to check that

\[
\Psi_1(s_1, \ldots, s_{2m}, t_1, \ldots, t_{2m}) = \prod_{j=1}^{2m} (2 + is_j + it_j)^{-\frac{1}{2}} \prod_{j=1}^{2m} (1 + is_j)^{\frac{1}{2}} \prod_{j=1}^{2m} (1 + it_j)^{\frac{1}{2}},
\]

and

\[
\Psi_2(s_1, \ldots, s_m, t_1, \ldots, t_m) = \prod_{j=1}^{m} (2 + is_j + it_j)^{-\frac{1}{2}} \prod_{j=1}^{m} (1 + is_j)^{\frac{1}{2}} \prod_{j=1}^{m} (1 + it_j)^{\frac{1}{2}}.
\]

It follows that
\[ |\Upsilon(s_1, \ldots, s_{2m}, t_1, \ldots, t_{2m})| \leq 2m \prod_{j=1}^{2m} (1 + |s_j|) \prod_{j=1}^{2m} (1 + |t_j|). \]

Recall that \( \psi(t) \ll_{A} (1 + |t|)^{-A} \) for arbitrarily large \( A > 0 \). So by choosing a sufficiently constant \( A > 0 \), we get

\[
\int_{-(\log R)^{1/2}}^{(\log R)^{1/2}} \cdots \int_{-(\log R)^{1/2}}^{(\log R)^{1/2}} \Upsilon(s_1, \ldots, s_{2m}, t_1, \ldots, t_{2m}) \prod_{j=1}^{2m} \psi(s_j) \psi(t_j) \, ds_j \, dt_j = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \Upsilon(s_1, \ldots, s_{2m}, t_1, \ldots, t_{2m}) \prod_{j=1}^{2m} \psi(s_j) \psi(t_j) \, ds_j \, dt_j + O_A((\log R)^{-A}).
\]

And by (3.8) and (3.9), we have

\[
\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \Upsilon(s_1, \ldots, s_{2m}, t_1, \ldots, t_{2m}) \prod_{j=1}^{2m} \psi(s_j) \psi(t_j) \, ds_j \, dt_j = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{j=m+1}^{2m} \left( \frac{1 + is_j}{2 + is_j + it_j} \right)^{1/2} \left( \frac{1 + it_j}{2 + is_j + it_j} \right)^{1/2} \prod_{j=1}^{m} \psi(s_j) \psi(t_j) \, ds_j \, dt_j = C^m_{\chi},
\]

where

\[
C_{\chi} = \int_{\mathbb{R}^4} \frac{(1 + is_1)(1 + it_1) \cdot (1 + is_2)(1 + it_2)}{(2 + is_1 + it_1) \cdot (2 + is_2 + it_2)} \psi(s_1) \psi(t_1) \psi(s_2) \psi(t_2) \, ds_1 \, dt_1 \, ds_2 \, dt_2.
\]

That is,

\[
C_{m,\chi} = C_{\chi}^m + O_A((\log R)^{-A})
\]

for any \( A > 0 \). Finally, recalling that \( 0 \leq f(x) \leq \nu(x) \) and \( E(f(x)|x \in \mathbb{Z}_N) > 0 \), clearly we must have \( C_{\chi} > 0 \). Then Proposition 2.1 is concluded. \( \square \)

**Acknowledgements** We are grateful to the anonymous referee for his/her very helpful comments. We also thank Professor Henryk Iwaniec for his helpful explanation on Theorem 1 of [7]. The work is supported by National Natural Science Foundation of China (Grant No. 11671197). The second author is the corresponding author.
The Green–Tao theorem for primes of the form $x^2 + y^2 + 1$

References

1. Conlon, D., Fox, J., Zhao, Y.: A relative Szemerdi theorem. Geom. Funct. Anal. 25, 733–762 (2015)
2. Goldston, D.A., Yildirim, C.Y.: Higher correlations of divisor sums related to primes, I: triple correlations. Integers 3, 66pp (2003)
3. Goldston, D.A., Pintz, J., Yildirim, C.Y.: Primes in tuples. I. Ann. Math. 170(2), 819–862 (2009)
4. Green, B.: Roth’s theorem in the primes. Ann. Math. 161(2), 1609–1636 (2005)
5. Green, B., Tao, T.: The primes contain arbitrarily long arithmetic progressions. Ann. Math. 167(2), 481–547 (2008)
6. Green, B., Tao, T.: Linear equations in primes. Ann. Math. 171(2), 1753–1850 (2010)
7. Iwaniec, H.: Primes of the type $\phi(x, y) + A$ where $\phi$ is a quadratic form. Acta Arith. 21, 203–234 (1972)
8. Lê, T.-H.: Green-Tao theorem in function fields. Acta Arith. 147, 129–152 (2011)
9. Lê, T.-H., Wolf, J.: Polynomial configurations in the primes, Int. Math. Res. Not. IMRN 2014, 6448–6473
10. Linnik, J.V.: An asymptotic formula in an additive problem of Hardy–Littlewood. Izv. Akad. Nauk SSSR Ser. Mat. bf 24, 629–706 (1960)
11. Matomäki, K.: Prime numbers of the form $p = m^2 + n^2 + 1$ in short intervals. Acta Arith. 128, 193–200 (2007)
12. Matomäki, K.: The binary Goldbach problem with one prime of the form $p = k^2 + l^2 + 1$. J. Number Theory 128, 1195–1210 (2008)
13. Motohashi, Y.: On the distribution of prime numbers which are of the form $x^2 + y^2 + 1$. Acta Arith. 16 (1969/1970), 351–363
14. Pintz, J.: Polignac Numbers, Conjectures of Erdős on Gaps Between Primes, Arithmetic Progressions in Primes, and the Bounded Gap Conjecture, From Arithmetic to Zeta-functions, pp. 367–384. Springer, Berlin (2016)
15. Stein, E.M., Shakarchi, R.: Fourier Analysis. An introduction. Princeton Lectures in Analysis 1. Princeton University Press, Princeton, NJ (2003)
16. Tao, T.: The Gaussian primes contain arbitrarily shaped constellations. J. Anal. Math. 99, 109–176 (2006)
17. Tao, T.: A remark on Goldston–Yildirim correlation estimates, preprint, available on: http://www.math.ucla.edu/~tao/preprints/Expository/gy-corr.dvi
18. Tao, T., Ziegler, T.: The primes contain arbitrarily long polynomial progressions. Acta Math. 201, 213–305 (2008)
19. Tao, T., Ziegler, T.: A multi-dimensional Szemerédi theorem for the primes via a correspondence principle. Israel J. Math. 207, 203–228 (2015)
20. Teräväinen, J., The Goldbach problem for primes that are sums of two squares plus one, Mathematika, 64(1), 20–70. https://doi.org/10.1112/S0025579317000341 (to appear)
21. Wu, J.: Primes of the form $p = 1 + m^2 + n^2$ in short intervals. Proc. Am. Math. Soc. 126, 1–8 (1998)
22. Zhou, B.-B.: The Chen primes contain arbitrarily long arithmetic progressions. Acta Arith. 138, 301–315 (2009)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.