Entanglement distillation, the process of distilling from several copies of a noisy entangled state fewer copies of a more entangled state, is a key element of a quantum repeater scheme for entanglement distribution. Quantum scissors, which can effect probabilistic noiseless linear amplification (NLA) on low mean photon number states, has been proposed as a candidate for entanglement distillation from noisy continuous variable (CV) quantum states distributed across a communication channel. Being a non-Gaussian operation though, the quantum scissors is challenging to analyze, especially when more than one is deployed. We present a derivation of the non-Gaussian state heralded by multiple quantum scissors on a pure loss channel. We choose the reverse coherent information (RCI) of the heralded state as our figure of merit. A prerequisite for any entanglement distillation scheme to be useful in a quantum repeater scheme for entanglement distribution over the channel is that the RCI it heralds exceeds the direct transmission repeater-less entanglement distribution capacity. We evaluate a Gaussian lower bound on the RCI for one and two quantum scissors deployed in a pure loss channel and show that there exist parameter regimes where this condition is met. We show that the optimal heralded RCI with two quantum scissors is significantly more than with a single quantum scissors, albeit at the cost of decreased success probability. Our analysis fortifies the possibility of a quantum repeater scheme for continuous variable quantum states based on quantum scissors.

I. INTRODUCTION

Entanglement shared across large distances promises to be a key resource for distributed quantum computation [1–3], distributed quantum sensing [4–6], quantum communication protocols such as quantum key distribution (QKD) [7], teleportation [8], and superdense coding [9]. Entanglement assistance is known to boost the classical capacity of noisy channels [10] and enable higher rate quantum error correcting codes [11]. Optical photons are arguably the best suited carriers of quantum information to create a quantum network of distributed entanglement [12]. The primary challenge, however, is that quantum states of light experience loss and noise during transmission. This limits the rate and range of distribution. The rate over an optical fiber channel, e.g., is known to drop exponentially with distance [13–15].

Quantum repeater schemes have been proposed to mitigate this rate-distance tradeoff [16, 17]. A key element of the original proposal [18] for a quantum repeater is entanglement distillation. It refers to the process of distilling from a large number of imperfect entangled pairs, a fewer pairs of higher fidelity to a perfect maximally entangled state. For continuous variable (CV) states of light, where information is encoded in the quadrature degrees of freedom of the light field, it is known that Gaussian operations alone cannot distill higher quality Gaussian entanglement from noisy Gaussian entangled states [19]. This precludes any untended Gaussian operation acting as a quantum repeater [20]. Ralph [21] outlined a quantum error correction scheme for CV states based on non-Gaussian entanglement distillation using quantum scissors [22, 23] and CV teleportation with dual homodyne detection. The quantum scissors are known to effect noiseless linear amplification on low mean photon number states [22]. Ralph’s scheme was further investigated in [24, 25]. Entanglement distillation using noiseless linear amplification was experimentally demonstrated in [26, 27].

In this paper, we analyze entanglement distillation with quantum scissors across a pure loss channel in a new light, that differs from prior works in the following ways:

- We present a calculation of the heralded non-Gaussian state and the heralding success probability of multiple quantum scissors in a pure loss channel based on characteristic functions and the Husimi Q function. The calculation also applies to the teleportation-based CV error correction scheme [21] with multiple quantum scissors, and is computationally efficient. This is in contrast to the Fock basis calculations presented in [24, 25], which do not scale well with increasing number of quantum scissors.

- While prior works [24, 25] on the topic considered entanglement measures such as the logarithmic negativity [28] and entanglement of formation [29] as the figures of merit, we consider the reverse coherent information (RCI) [30–32]. The RCI is an operationally relevant measure for the tasks of entanglement and secret key distillation across a channel in being an achievable rate. We evaluate a Gaussian lower bound on the RCI for one and two quantum scissors.
• Apart from qualitatively validating the findings presented in [25] for a single quantum scissors, we investigate the performance with two quantum scissors. We show that the addition of a second quantum scissors can boost entanglement distillation significantly, and in some cases can even help “activate” heralded entanglement that exceeds the repeater-less capacity, which could not be achieved with a single quantum scissors. Although, the boost in entanglement distillation comes at the expense of a lower success probability.

The paper is organized as follows. In Section II, we review the basic concept of NLA based on quantum scissors. In Section III, we present our model and methods including the figure of merit in detail. Section IV contains the results. Finally, we summarize our findings in Section V.

II. NOISELESS LINEAR AMPLIFICATION WITH QUANTUM SCISSORS

Noiseless linear amplification (NLA) is a probabilistic amplification scheme [22] that transforms coherent states as

\[ |\alpha\rangle \rightarrow |g\alpha\rangle, \quad (1) \]

where \( g \in \mathbb{R} \) is the gain. NLA can be implemented in a heralded fashion using linear optics, photon injection and detection [22]. The scheme involves splitting the input signal into \( N \) parts of equal intensity and recombining them following a modified quantum scissors operation on each part as shown in Fig. 1 (a).

Quantum scissors, as the name suggests, refers to an operation that truncates a quantum state in Fock space [23]. A variant of the quantum scissors can, in addition to truncation, be used to amplify certain Fock state components of the state relative to others [22]. Consider the scheme shown in Fig. 1 (b) comprising of single photon injection and detection. A single photon (in mode \( c \)) is mixed with vacuum (in mode \( b \)) on a beam splitter of transmissivity \( \kappa = 1/\sqrt{1+g^2} \) (\( g \) being the intended gain of NLA), creating a Bell state in the \( \{|0\rangle_b \otimes |1\rangle_c, |1\rangle_b \otimes |0\rangle_c \} \) subspace. When the signal in mode \( a \) is mixed with mode \( c \) on a 50:50 beam splitter and either one of the two projections \( \{|0\rangle_a \otimes |1\rangle_c, |1\rangle_a \otimes |0\rangle_c \} \) is applied (i.e., when detector \( D_a \) clicks and \( D_c \) doesn’t, or vice versa), the \( \{|0\rangle, |1\rangle \} \) support of the quantum state of the signal is teleported to mode \( b \). Further, the teleported state is such that its \( |1\rangle \) component is amplified relative to the vacuum component depending on the choice of \( \kappa \). In summary, the modified quantum scissors scheme of Fig. 1 on any input signal state \( |\psi\rangle \propto \alpha_0 \langle 0| + \alpha_1 \langle 1| + \cdots \) heralds the truncated and amplified output

\[ \Gamma(g) (\alpha_0 |0\rangle + \alpha_1 |1\rangle + \cdots) = \sqrt{\frac{1}{1+g^2}} (\alpha_0 |0\rangle + g\alpha_1 |1\rangle). \quad (2) \]

When the signal to the modified quantum scissors is sufficiently weak such that its quantum state resides primarily in the \( \{|0\rangle, |1\rangle \} \) subspace, the operation effects NLA; whereas if the state has significant support on higher photon components, then the amplification is not noiseless anymore owing to the excess noise originating from the truncation of the teleported state in Fock space. Thus, in the scheme of Fig. 1 (a), for a given input signal intensity, \( N \) needs to be sufficiently large so that the sub-signals that are inputs to the quantum scissors operations are weak. When all the quantum scissors operations succeed and all-but-one of the outputs of the \( N \)-combs are measured in the vacuum state, the device approaches NLA; i.e., e.g., on a coherent state input, it effects the transformation given in [22].

III. METHODS

As mentioned earlier, NLA is probabilistic. The success probability of an NLA scheme with \( N \)-quantum scissors is input state dependent and decreases exponentially with \( N \). For an input coherent state \( |\alpha\rangle \), e.g., the success probability of an \( N \)-scissors NLA is given by \( P_s = 1/(1+g^2)^N e^{-(1-g^2)|\alpha|^2} \) [22].
state input that represents continuous-variable entanglement, b) determining the heralding success probability, and c) evaluating a figure of merit for the task.

Figure 2. A pure loss channel of transmissivity $\eta$ with TMSV state input appended by $N-$quantum scissors of gain $g = \sqrt{(1 - \kappa)}/\kappa$ at the output.

Consider the transmission of one share of a TMSV state of mean photon number $\mu$ through a pure loss channel followed by a device that implements NLA using $N-$quantum scissors, as shown in Fig. 2. A pure loss channel of transmissivity $\eta$ from Alice to Bob $N_{A'\rightarrow B}$ is a bosonic channel that can be described by the relation between the quadrature operators of the input and output modes given by $\hat{x}_B = \sqrt{\eta} \hat{x}_A + \sqrt{1 - \eta} \hat{x}_E$, where mode $E$ denotes an environment mode that injects the vacuum state. It is a Gaussian channel and maps Gaussian states at the input to Gaussian states at the output. (See Appendix A for details on Gaussian states, beam splitter unitary, and the pure loss channel.) On the other hand, the quantum scissors operation, and consequently the action of a device comprising of $N-$quantum scissors for any finite $N$, are non-Gaussian. Hence, the quantum state heralded across the channel appended by NLA, when the latter operation is successful, is a non-Gaussian state.

A. Modified Quantum scissors based on heralded single photon injection and ON-OFF photodetection

The calculation of the non-Gaussian state heralded across the channel appended by $N-$quantum scissors quickly becomes cumbersome with increasing (but finite) $N$ (c.f. [24, 25]). In order to make the calculation more tractable, we emulate the quantum scissors by replacing the photon number detectors with ON-OFF photodetection $\{|0\rangle \langle 0|, I - |0\rangle \langle 0|\}$, and the single photon by one share of a weak TMSV state (of mean photon number $\mu_{\text{aux}}$). Fig. 3 shows such an emulation of a single-quantum scissors NLA (denoted as $N = 1-$NLA hereafter) acting on a pure loss channel with TMSV input. The weak TMSV state heralds a single photon state for injection into the quantum scissors when the idler mode is projected onto $I - |0\rangle \langle 0|$. (With out loss of generality, in all the calculations presented in this paper, we choose $\mu_{\text{aux}} = 0.01$.) The success probability of the quantum scissors operation is expected to be enhanced having replaced the $\{|0\rangle, |1\rangle\}$ projection with $\{|0\rangle \langle 0|, I - |0\rangle \langle 0|\}$ detection, whereas the quality of the amplification is expected to be marginally degraded. However, the qualitative behavior of the scheme still remains preserved as will be shown in subsequent analyses.

Figure 3. A pure loss channel of transmissivity $\eta$ with TMSV state input, appended by a $N = 1-$NLA, i.e., a single quantum scissors implemented with a weak EPR source and ON-OFF projections. When the idler mode of the weak TMSV state is projected on $I - |0\rangle \langle 0|$, a single photon is heralded in the signal mode and injected into the quantum scissors.

B. Heralded state and heralding success probability

We now describe the calculation of the heralded non-Gaussian quantum state and heralding success probability for the scheme shown in Fig. 3 where the input to the channel is one share of a TMSV state. The scheme involves a bosonic system of five modes, whose pre-measurement state is Gaussian, meaning the quantum state is completely described by its first two moments. (See Appendix A for the system description, the initial and pre-measurement covariance matrices.)

The quantum scissors operation is successful when, in Fig. 3, either one (but not both) of the modes $C$ and $Y$, along with mode $D$ are measured in the “ON” projection $I - |0\rangle \langle 0|$. The heralded state clearly is non-Gaussian since the $I - |0\rangle \langle 0|$ projection is non-Gaussian. We capture the heralded non-Gaussian state in modes $A$ and $B$ by its Husimi Q function, defined as

$$Q_{\rho}(\alpha, \beta) = \frac{\langle \alpha | \rho | \beta \rangle}{\pi^2}, \alpha, \beta \in \mathbb{C}. \quad (3)$$

It can be determined as an overlap integral between the five mode Gaussian state $\rho$ in modes $ABCDY$ and the projections $|\alpha\rangle \langle \alpha|_A \otimes |\beta\rangle \langle \beta|_B \otimes (I - |0\rangle \langle 0|)^{\otimes 2}|_C \otimes (I - |0\rangle \langle 0|)^{\otimes 2}|_D \otimes (I - |0\rangle \langle 0|)^{\otimes 2}|_Y$, normalized by the probability of the projections $\pi_1 = (I - |0\rangle \langle 0|)^{\otimes 2}|_C \otimes (I - |0\rangle \langle 0|)^{\otimes 2}|_Y$ and $\pi_2 = |0\rangle \langle 0|_C \otimes (I - |0\rangle \langle 0|)^{\otimes 2}_Y$, respectively, which constitute the success probability of the quantum scissors operation. The heralded states corresponding to the two pos-
sible successful projections turn out to be the same up to local phases.

The above overlap integrals are sums of Gaussian integrals that can be evaluated efficiently (See Appendix C). For example, the success probability for the projection $\pi_1$ on modes $CDY$ involves evaluating the overlap integral

$$P_1 = \text{Tr}(\pi_1 \rho_{ABCDY}),$$

where $\chi_{ABCDY}$ and $\chi_0 = \chi_{|0\rangle|0\rangle}$ are the characteristic functions of the heralded non-Gaussian state and the vacuum state, respectively. The success probability associated with the other projection, namely $\pi_2$ also turn out to be the same as (4) due to symmetry between modes $C$ and $Y$, so that the total success probability is $P'_\text{succ} = \text{Tr} ((\pi_1 + \pi_2) \rho) = P_1 + P_2 = 2P_1$. The success probability of a quantum scissors operation with a deterministic single photon injection into mode $C$ can be deduced from the above success probability by renormalizing it with the probability of detecting a single photon in the idler mode $D$. That is,

$$P'_\text{succ} = P'_\text{succ}/\left(\frac{\mu_{\text{aux}}}{\mu_{\text{aux}} + 1}\right)^N,$$

where the scaling factor is the probability of observing the $|1\rangle |1\rangle$ projection on the idler mode of the TMSV state of mean photon number $\mu_{\text{aux}}$.

The heralded state and heralding success probability calculations for NLA with higher number of quantum scissors follow similarly to the calculations for NLA with higher number of quantum scissors deployed) and is not infinite anymore. We determine the optimal RCI of the heralded non-Gaussian channel by optimizing over the mean photon number of the input TMSV state and the NLA gain.

C. Reverse coherent information

Consider a quantum channel $\mathcal{N}_{A'\rightarrow B}$ (a completely positive, trace preserving, linear map from input to output). The reverse coherent information (RCI) of a state $\rho_B = \mathcal{N}_{A'\rightarrow B} (\rho_{A'})$ obtained at the output of the channel when fed with input $\rho_{A'}$ is defined as [30,32]

$$I_R(\mathcal{N},\rho_{A'}) = I(\rho_{A'})_{\rho_{AB}} := H(A)_{\rho_{AB}} - H(AB)_{\rho_{AB}},$$

where $\rho_{AB} = \mathcal{N}_{A'\rightarrow B} (\psi_{AA'})$, $\psi_{AA'}$ being a purification of $\rho_{A'}$, such that $\rho_B = \text{Tr}_A(\rho_{AB})$. The RCI of the channel $\mathcal{N}$ is defined as the RCI maximized over all possible choice of inputs to the channel.

The RCI has an important operational meaning in the context of entanglement and shared secret key distillation across a quantum channel. It is an achievable rate for distilling these resources across a channel from Alice to Bob in the limit of asymptotically many channel uses, when assisted by local operations (LO) and a single feedback (reverse) classical communication (CC) from Bob to Alice [30,32]. The RCI thus also serves as a lower bound for the channel’s capacity for these tasks when performed under unlimited two-way LOCC assistance.

The RCI of a pure loss channel (with no mean energy constraint) is optimized by the Einstein-Podolsky-Rosen (EPR) state, and is known to be $-\log_2(1-\eta)$ ebits per channel use [33]. The RCI rate can be achieved, e.g., when Alice transmits one share of a TMSV state of energy approaching infinity (approaching the EPR state) through the channel and Bob performs heterodyne detection at the output. More recently, the rate $-\log_2(1-\eta)$ was show also to be an upper bound for energy-constrained, two-way unlimited LOCC-assisted entanglement and secret key distillation across a pure loss channel [13] (see also [15] for strong converse theorem, [13,34] for other valid unconstrained upper bounds, and [35] for energy constrained upper bounds). Thus, the RCI captures the highest possible unconstrained rate achievable for entanglement and secret key distillation across a pure loss channel.

With the inclusion of quantum scissors-based NLA in the channel, the effective channel that is heralded upon successful NLA operation is now a non-Gaussian channel. The RCI of this heralded channel is not optimized by the infinite energy TMSV (EPR) state since the dimensionality of the output space is truncated (a function of the number of quantum scissors deployed) and is not infinite anymore. We determine the optimal RCI of the heralded non-Gaussian channel by optimizing over the mean photon number of the input TMSV state and the NLA gain.

D. A lower bound on the heralded reverse coherent information

Evaluating the RCI of the state heralded upon successful operation of quantum scissors-based NLA following the pure loss channel is perceived to be nontrivial. As an interim remedy, we resort to calculating the RCI of the covariance matrix of the heralded state, which by the Gaussian extremality theorem [36] amounts to a lower bound on the RCI of the heralded non-Gaussian state.

The covariance matrix $V(\rho)$ can be determined from the $Q$ function as

$$V_{i,j}(\rho) = 2 \int d\mathbf{r} r_i' r_j Q_\rho(\alpha,\beta) - \delta_{i,j} - 2 \int d\mathbf{r} r_i Q_\rho(\alpha,\beta) \int d\mathbf{r} r_j Q_\rho(\alpha,\beta),$$

where $\mathbf{r} = (x_1, x_2, p_1, p_2)^T \in \mathbb{R}^4$, $\alpha = (x_1 + i p_1)/\sqrt{2}$, $\beta = (x_2 + i p_2)/\sqrt{2}$, and the $Q$ function in real coordinates is $Q_\rho(x_1, x_2, p_1, p_2) = \langle x_1, x_2, p_1, p_2 | \rho | x_1, x_2, p_1, p_2 \rangle / (4\pi^2)$.
Figure 4. Gaussian RCI heralded across a pure loss channel of transmissivity $\eta = 0.01$ appended with $N = 1, 2$–NLA, as a function of the input TMSV state mean photon number and the gain of the NLA. The parameter $\kappa$ is related to the NLA gain by $\kappa = 1/(1 + g^2)$. The floors of the plots correspond to $-\log_2 (1 - \eta)$.

Given the covariance matrix $V(\rho_{AB})$ of a bipartite state $\rho_{AB}$, the RCI of $V(\rho_{AB})$ follows from (6) as

$$I_R(A' \langle B\rangle_{\rho_{AB}} = H(A)_{\rho_{AB}} - H(AB)_{\rho_{AB}} = g(\tilde{\nu}_A) - g(\tilde{\nu}_{AB}),$$

where $g(x) := \left(\frac{x+1}{2}\right) \log_2 \left(\frac{x+1}{2}\right) - \left(\frac{x-1}{2}\right) \log_2 \left(\frac{x-1}{2}\right)$ is the entropy of a thermal state of mean photon number $(x-1)/2$, and $\tilde{\nu}_A$ and $\tilde{\nu}_{AB}$ are the symplectic eigenvalues of the covariance matrices corresponding to mode $A$ and modes $AB$, respectively. We call this the Gaussian RCI of the state $\rho_{AB}$.

IV. RESULTS

A. Gaussian Reverse Coherent Information for $N = 1, 2$–NLA-assisted pure loss channel

In Figs. 4 and 5, we plot the RCI of the heralded covariance matrix (referred to as Gaussian RCI of the non-Gaussian state hereafter) and the heralding success probability for a pure loss channel appended with $N = 1$–NLA and $N = 2$–NLA. The quantity is plotted as a function of the mean photon number $\mu$ of the TMSV input and the transmissivity $\kappa = 1/(1 + g^2)$ of the asymmetric beamsplitter in the quantum scissors ($g$ being the NLA gain). We choose a channel of transmissivity $\eta = 0.01$. The choice of a small transmissivity $\eta$, e.g., $\eta = 0.1$, for illustration is so that the amplification due to quantum scissors remains noiseless. The mean photon number of the input TMSV state is optimized to determine the best possible heralded Gaussian RCI. We make the following observations from these Figures. a) The heralded Gaussian RCI exceeds the direct transmission capacity $-\log_2 (1 - \eta)$ [14] (denoted by the floor of both the 3-d plots in Fig. 4) for a certain regime of the parameters $\mu$ and $\kappa$ for both $N = 1, 2$. b) The Gaussian RCI of $N = 2$–NLA is about four times that of $N = 1$–NLA. Additionally, we point out that for a different choice of $\eta$, e.g., $\eta = 0.1$ (Fig. 7), the optimal heralded Gaussian RCI with $N = 1$–NLA never exceeds $\log_2 (1 - \eta)$, whereas with $N = 2$–NLA it exceeds the bound. In other words, the inclusion of a second quantum scissors “activates” a heralded entanglement of higher quality than direct transmission. c) The increase in the Gaussian RCI is accompanied by a steep decrease in the heralding success probability of the NLA, which drops exponentially going from $N = 1$ to $N = 2$–NLA as shown in Fig. 5.
Figure 6. Gaussian RCI heralded across $N = 1, 2$−NLA vs heralding success probability for a pure loss channel of transmissivity $\eta = 0.01$.

Figure 7. Gaussian RCI heralded across a pure loss channel of transmissivity $\eta = 0.1$ appended with $N = 1, 2$−NLA, as a function of the input TMSV state mean photon number $\mu$ and the gain of the NLA. The parameter $\kappa$ is related to the gain of the NLA by $\kappa = 1/(1 + g^2)$. The planes in the plots correspond to $-\log_2 (1 - \eta)$.

B. Entanglement of formation and Reverse coherent information lower bounds for Teleportation through NLA-assisted pure loss channel

In [21], a variant of the scheme in Fig. 2 was studied as quantum error correction for the transmission of quantum continuous variable states over a lossy channel. The scheme is as shown in Fig. 8. Here, direct transmission through the lossy channel is replaced by continuous-variable quantum teleportation over a lossy entangled resource established by sending one share of a finite energy two-mode squeezed vacuum (TMSV) state through the channel followed by NLA. The resource state is referred to as an error correction (EC) box, and is characterized by the mean photon number of the teleportation resource TMSV state $\mu_{\text{res}}$, the number of quantum scissors $N$, and the NLA gain $g$. Using a meticulous Fock basis calculation, Dias and Ralph [25] recently showed that this scheme with $N = 1$−NLA, when NLA is successful, can distill higher entanglement than direct transmission over the lossy channel. They considered the entanglement of formation (EOF) (See Appendix D) of the heralded covariance matrix (Gaussian part of the heralded non-Gaussian state) as the figure of merit, which constitutes a lower bound on the entanglement of formation of the heralded non-Gaussian state by the Gaussian extremality theorem [36].

The calculation of Dias and Ralph, however, is tedious and difficult to be extended to NLA with multiple quan-
Figure 9. Entanglement of formation (EOF) of the covariance matrix heralded across the EC Box of [21] for a TMSV input, as a function of the heralded effective transmission. The bare channel transmissivity is chosen to be \( \eta = 0.01 \), mean photon number of the input TMSV and the teleportation resource TMSV chosen to be equal \( \mu_{\text{res}} = \mu = 0.33 \), \( N = 1, 2 \), and the NLA gain \( g \) is varied. a) The EOF of the covariance matrix of the average state heralded across the error corrected channel. b) The average of EOF of the conditional heralded covariance matrices (conditioned on and averaged over the outcome of the dual homodyne detection). (Black) corresponds to the EOF for transmission across the bare lossy channel.

Figure 10. Success Probability of [21]'s EC box for \( N = 1, 2 \)–NLA as a function of the heralded effective transmission. The bare channel transmissivity is chosen to be \( \eta = 0.01 \), mean photon number of the input TMSV and the teleportation resource TMSV chosen to be equal \( \mu_{\text{res}} = \mu = 0.33 \), \( N = 1, 2 \), and the NLA gain \( g \) is varied.

The method based on characteristic functions described in Section 3.1B offers an efficient alternative means to determine the heralded non-Gaussian state in this case. Using this method along with general ideas from Gaussian conditional dynamics [37, 38] to deal with the teleportation elements such as dual homodyne detection (CV Bell state measurement) and displacement (unitary correction) (See Appendix B), we analyzed the effect of NLA with \( N = 1, 2 \)–quantum scissors in Fig. 8. We calculated the Gaussian lower bound on the entanglement of formation of the heralded state by evaluating yet another measure, the Gaussian entanglement of formation, on the Gaussian part of the heralded state, which equals its Entanglement of formation. (See Appendix D for details.)

In this scheme, the non-Gaussian state heralded upon successful NLA operation for a TMSV input is a function of the dual homodyne detection outcome. (In particular, both its first and second moments are dependent on the outcome.) There are two ways to quantify the performance of the scheme: a) evaluating the figure of merit on the covariance matrix of the average heralded state, or, b) evaluating the average of the figure of merit applied on the conditional heralded covariance matrices, where in both cases the averaging is with respect to the dual homodyne measurement outcomes. We will call these \( q_1 \) and \( q_2 \), respectively. Convexity of the EOF implies that \( q_2 \geq q_1 \). It should be noted that while the displacement correction associated with the teleportation impacts the average heralded state and in turn its EOF, it does not affect the EOF of the conditional heralded states, since the measure is independent of the first moments.

In Fig. 9 we plot \( q_1 \) and \( q_2 \) for a TMSV input of mean photon number \( \mu \) as a function of the effective transmission parameter defined as \( \eta_{\text{effec}} = g^2 \eta \chi^2 \), where \( g \) is related to the gain of the NLA, and \( \chi = \tanh(\sinh^{-1}(\sqrt{\mu_{\text{res}}})) \). The channel transmissivity is chosen to be \( \eta = 0.01 \), and the mean photon number of the TMSV state at the input as well as in the EC box (teleportation resource state) are chosen to be \( \mu_{\text{res}} = \mu = 0.33 \), (which corresponds to \( \chi = 1 \)), and the NLA amplitude gain \( g \) is scanned over. In Fig. 9 (a), the quantity \( q_1 \) is plotted, where it has been optimized over a classical gain tuning parameter that scales the dual homodyne outcome prior to the displacement correction operation. We find that our curve for the \( N = 1 \)–NLA is qualitatively similar, but below what was reported in [25] for the same. This is as expected, since we have considered ON-OFF heralding photodetection instead of perfect photon-number-resolving (PNR) single photon detection in the quantum scissors. In addition, we now have calculated the same figure of merit also for \( N = 2 \)–NLA. We observe that there is a steep
increase in the Gaussian lower on the EOF going from $N = 1$ to $N = 2$. In Fig. 9 (b), the quantity $q_2$ is plotted, and as is expected due to convexity of the EOF, they are higher than the corresponding curves in Fig. 9 (a). In the remainder of this discuss, we will consider the quantity $q_2$ as the sole figure of merit for the scheme.

In Fig. 10, the heralding success probability of the scheme is plotted as a function of the effective transmission for the same set of parameter values as chosen in Fig. 9. We observe that our curve for the $N = 1$ case is qualitatively similar, but slightly above the one reported in [25]. This is again consistent with our choice of ON-OFF photodetection in place of single photon detection in the quantum scissors.

In Fig. 11, the average of the heralded conditional EOF and the heralding success probability of NLA are plotted as a function of the NLA intensity gain $g^2$. The EOF of direct transmission through the lossy channel forms the benchmark. This is exceeded by teleportation of the input through EC box with $N = 1, 2$–NLA. The figure also shows the performance of ideal NLA (corresponding to $N \rightarrow \infty$) for comparison.

Though, the EOF is a valid entanglement measure, it is not operationally the most relevant measure when it comes to characterizing entanglement distribution rates across a channel. For this reason, we analyze the average RCI of the heralded conditional covariance matrices across the EC box. One key mathematical difference between the measures is that while the EOF is non-negative by definition, the RCI can take on negative values for separable states. We find that averaging over the dual homodyne outcomes is severely detrimental to the RCI and leaves the average RCI negative for nearly all choice of parameters. Post-selecting on the dual homodyne outcome over a small range of values around 0 (whose probability of occurrence is maximal among all possible outcomes of the measurement), we plot the average of the RCI of the heralded conditional covariance matrices in Fig. 12 for EC box with $N = 1, 2$–NLA. These curves are identical to the ones in Fig. 4, but are scaled with respect to higher values of mean photon number $\mu$. This is consistent with the fact that the teleportation of two TMSV heralds a new TMSV of a different mean photon number, and hence, the teleportation through the EC box converges to transmission of a different TMSV through the NLA assisted lossy channel.
V. CONCLUSIONS

We studied continuous-variable entanglement distillation across a pure loss channel with quantum scissors-based noiseless linear amplification (NLA). We presented a calculation based on phase space characteristic functions and the Husimi-Q function to determine the non-Gaussian state heralded by the quantum scissors operation. The complexity of the method scales efficiently when a few and finite number of quantum scissors are placed in the channel. Having determined the non-Gaussian state, we quantified the amount of entanglement that can be distilled from the state (optimized over the input and the NLA gain) by evaluating a Gaussian lower bound on its reverse coherent information (RCI) \[30\] \[32\].

We also applied the calculation to the proposal of \[21\] that replaces transmission through a lossy channel with teleportation over a NLA-error corrected lossy entangled resource state as shown in Fig. \[8\]. Previous studies on this scheme had considered entanglement measures such as the logarithmic negativity \[21\] or the entanglement of formation (EOF) \[25\]. We validated some of these findings with our method and further studied a pure loss channel with two quantum scissors with EOF as the figure of merit. Additionally, we calculated a Gaussian lower bound on the RCI \[30\] \[32\] for the scheme with one and two quantum scissors.

To highlight our main conclusions, we found that a) increasing the number of scissors amounts to higher quality of distilled entanglement as witnessed by an increase in the Gaussian RCI (a lower bound on the RCI) of the heralded state. The increase in heralded entanglement though comes at the expense of decreased success probability. b) In some cases, a second quantum scissors in the channel can help herald an RCI that exceeds the repeater-less bound on RCI, while a single quantum scissors could not help exceed the bound. c) In the NLA-CV error correction scheme of Fig. \[8\] the Gaussian lower bound on the heralded RCI by the scheme, on average (over the teleportation dual homodyne detection outcomes), does not exceed the repeater-less bound. Yet, when post-selected over a narrow window of the teleportation dual homodyne measurement outcomes around zero, the heralded RCI can exceed the bound. In this limit of a small window of outcomes, the scheme converges to the scheme in Fig. \[2\] a pure loss channel appended by quantum scissors-based NLA with an entangled state input, with only higher optimal input mean photon numbers.

Note added- During completion of this work, we became aware of the work of \[39\], which investigates the use of quantum scissors-based NLA in the context of CV quantum key distribution over a thermal loss channel using a method similar to ours, and shows improved distance of transmission. They lower bound the secret key generation rate in terms of the difference of a mutual information and a Holevo information, whereas we have lower bounded the entanglement and secret key distillation rates over a pure loss channel using the reverse coherent information.

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Appendix A: Mathematical Description of the System

Gaussian States of a Bosonic Continuous-Variable (CV) system. A system of $M$ bosonic modes can be described by the creation and annihilation operators $\hat{a}_i^\dagger, \hat{a}_i$, such that $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{i,j}$, $[\hat{a}_i, \hat{a}_j] = [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0 \forall i,j \in \{1, \ldots, M\}$, and the corresponding quadrature operators $\hat{x}_i = (\hat{a}_i + \hat{a}_i^\dagger) / \sqrt{2}$, $\hat{p}_i = (\hat{a}_i - \hat{a}_i^\dagger) / (i\sqrt{2})$, such that $[\hat{x}_i, \hat{p}_j] = i\delta_{i,j}$.

For a quantum state $\hat{\rho}$ defined on the $M$-mode Hilbert space $\mathcal{H}^{\otimes M}$, a characteristic function can be defined as the following operator Fourier transform (c.f., [40])

$$\chi (\xi) = \text{Tr} \left( \hat{\rho} \hat{W} (\xi) \right), \quad (A1)$$

where $\hat{W} (\xi)$ is the Weyl operator

$$\hat{W} (\xi) = \exp \left( -i \xi^T \hat{r} \right), \quad (A2)$$

and $\hat{r} = (\hat{x}_1, \ldots, \hat{x}_M, \hat{p}_1, \ldots, \hat{p}_M)^T$, $\xi = (\xi_1, \ldots, \xi_2M)^T$, $\xi_i \in \mathbb{R} \ \forall i \in \{1, \ldots, M\}$. The characteristic function of a quantum Gaussian state by definition is Gaussian, i.e., it can be written as

$$\chi (\xi) = \exp \left( -\frac{1}{4} \xi^T V \xi - i s^T \xi \right), \quad (A3)$$

where $V$ is the $2M \times 2M$ real symmetric covariance matrix defined as $V_{i,j} = \langle \{ \hat{x}_i, \hat{x}_j \} \rangle_\rho - 2 \langle \hat{x}_i \rangle_\rho \langle \hat{x}_j \rangle_\rho$ and $s = \langle \hat{r} \rangle_\rho$ is the $2M$-dimensional mean displacement vector.

The vacuum state is a Gaussian state with a covariance matrix equal to the identity operator $I$. The two-mode squeezed vacuum (TMSV) state of mean photon number $\mu = \sinh^2 (r)$ ($r$ being the squeezing parameter) is a Gaussian state with

$$V^{\text{TMSV}} (\mu) = \begin{pmatrix} V^+ (\mu) & 0 \\ 0 & V^- (\mu) \end{pmatrix}, \quad V^\pm (\mu) = \begin{pmatrix} 2\mu + 1 & \pm 2\sqrt{\mu (\mu + 1)} \\ \pm 2\sqrt{\mu (\mu + 1)} & 2\mu + 1 \end{pmatrix}, \quad (A4)$$

and $s = \vec{0}$. The vacuum state is a special case of the TMSV with $r = \mu = 0$.

Gaussian Unitaries. Unitary operators of the form $\hat{U}_{s,S} = \exp \left( i \hat{H} \right)$, where $\hat{H}$ is a Hamiltonian that is at most quadratic in $\hat{r}$ are called Gaussian unitaries. They map quantum Gaussian states into quantum Gaussian states. An arbitrary Gaussian unitary operator can be decomposed as

$$\hat{U}_{s,S} = \hat{D}_{-s} \hat{U}_S, \quad (A5)$$

where $s \in \mathbb{R}^{2M}$, $\hat{D}_{-s} = \otimes_{j=1}^M \hat{D}_{-(s_j, s_{M+j})}$ is the displacement operator such that

$$\hat{D}_{-(s_j, s_{M+j})} = \exp \left( i (s_{M+j} \hat{r}_j - s_j \hat{r}_{M+j}) \right), \quad (A6)$$

and $\hat{U}_S$ is a canonical Gaussian unitary operator generated by a purely quadratic Hamiltonian.

A canonical Gaussian unitary operator $\hat{U}_{s,S}$ transforms the quadrature operators as

$$\hat{r} \rightarrow \hat{U}_{s,S} \hat{r} \hat{U}_{s,S}^\dagger = S \hat{r} + s, \quad (A7)$$

where $S$ is a $2M \times 2M$ symplectic matrix and $s \in \mathbb{R}^{2M}$. Consequently, it transforms the first two statistical moments of an arbitrary quantum state as

$$s \rightarrow Ss, \quad V \rightarrow SVS^T, \quad (A8)$$

where $s \in \mathbb{R}^{2M}$ is the mean vector and $V$ the $2M \times 2M$ covariance matrix.

The two-mode beam splitter transformation is a canonical Gaussian unitary transformation given by

$$\hat{U}_{BS} = \exp \left( i \theta (\hat{x}_1 \hat{p}_2 - \hat{p}_1 \hat{x}_2) \right), \quad (A9)$$
where \( t = \cos^2 \theta \in [0, 1] \) is the transmissivity of the beamsplitter. The corresponding symplectic matrix is given by

\[
S^{(t)} = \begin{pmatrix}
\sqrt{t} & \sqrt{1-t} & 0 & 0 \\
-\sqrt{1-t} & \sqrt{t} & 0 & 0 \\
0 & 0 & \sqrt{t} & \sqrt{1-t} \\
0 & 0 & -\sqrt{1-t} & \sqrt{t}
\end{pmatrix}.
\]

(A10)

**Pure loss channel.** The pure loss channel of transmissivity \( \eta \) can be modeled as a beam splitter unitary transformation of the same transmissivity between the lossy mode and an environment mode that is in the vacuum state. The action of the pure loss channel on the lossy mode is obtained by tracing out the environment mode, and can be expressed as

\[
\mathcal{N}^{(\eta)}: V \rightarrow X^T V X + Y,
\]

(A11)

where \( X = \sqrt{\eta}I \) and \( Y = (1 - \eta) I \).

**Initial and pre-measurement state in Fig. 3.** The pure loss channel is a Gaussian channel and the beam splitter transformation is a Gaussian unitary operation. Hence, in the scheme depicted in Fig. 3, the quantum state across the five modes initially, and prior to measurements in modes \( A, B, C, Y, D \), are both Gaussian state with zero displacement and covariance matrices given by

\[
V_{\text{initial}} = V_{\text{TMSV}}^{AA'} (\mu) \otimes V_{\text{TMSV}}^{CB'} (\mu_{\text{aux}}) \otimes I_{B'},
\]

(A12)

\[
V_{\text{pre-meas}} = S^{(1/2)}_{Y',C'} S^{(\eta)}_{B',C} N_{A'\rightarrow Y'}^{\eta} (V_{\text{initial}}) \left( S^{(1/2)}_{B',C'} \right)^T \left( S^{(\eta)}_{Y',C'} \right)^T,
\]

(A13)

respectively.

**Appendix B: Gaussian Measurements, Conditional Dynamics and CV Teleportation**

A Gaussian measurement is a projection onto a quantum Gaussian state, and thus is completely characterized by a mean vector and a covariance matrix.

**Homodyne and Heterodyne Detection.** Homodyne detection on a single-mode, say of the \( x \)-quadrature, is the projection on to the Gaussian state with mean vector and covariance matrix

\[
r_{\text{hom}} = (x_{\text{hom}}, 0)^T,
\]

(B1)

\[
V_{\text{hom}} = \lim_{r \rightarrow \infty} \begin{pmatrix}
\exp(-2r) & 0 \\
0 & \exp(+2r)
\end{pmatrix},
\]

(B2)

respectively, where \( x_{\text{hom}} \) is measurement outcome and \( r \in \mathbb{R} \) is the squeezing parameter. Heterodyne detection, likewise, is the projection on to a coherent state with mean vector and covariance matrix, respectively being,

\[
r_{\text{het}} = (x_{\text{het}}, y_{\text{het}})^T, \quad V_{\text{het}} = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix},
\]

(B3)

where \( x_{\text{het}} + iy_{\text{het}} \in \mathbb{C} \) is the measurement outcome.

**Dual Homodyne Detection.** Dual homodyne detection is the continuous-variable analog of a Bell state measurement between two modes \( A \) and \( B \). It is a projection of the two modes onto the squeezing TMSV state, which is realized by mixing the two modes on a 50:50 beamsplitter, following by orthogonal homodyne detections on the two modes (\( \hat{x} \) measurement on one mode and \( \hat{p} \) measurement on the other). The mean vector and covariance matrix of the measurement after the beam splitter transformation of the two modes is given by

\[
r_{\text{Dual-hom}} = (\gamma_x, 0, 0, \gamma_y)^T,
\]

(B4)

\[
V_{\text{Dual-hom}} = \lim_{r \rightarrow \infty} \begin{pmatrix}
\exp(-2r) & 0 & \exp(-2r) & 0 \\
0 & \exp(+2r) & 0 & \exp(-2r)
\end{pmatrix} \oplus \begin{pmatrix}
\exp(-2r) & 0 & \exp(+2r) & 0 \\
0 & \exp(-2r) & 0 & \exp(+2r)
\end{pmatrix},
\]

(B5)

where \( \gamma_x + i\gamma_y \in \mathbb{C} \) is the measurement outcome.
Gaussian Conditional Dynamics and overlap integrals. Consider a continuous-variable system of $n$ modes. Let $AB$ be a bipartition of the modes such that subsystem $B$ consists of $m$ modes and subsystem $A$ consists of the remaining $n - m$ modes. Let

$$s = \begin{pmatrix} s_A \\ s_B \end{pmatrix}, \quad V = \begin{pmatrix} V_A & V_{AB} \\ V_{AB}^T & V_B \end{pmatrix}$$

(B6)

be the mean vector and covariance matrix of a quantum Gaussian state $\hat{\rho}$ over the systems $A$ and $B$. The quantum state obtained in mode $A$ by tracing out subsystem $B$, namely $\hat{\rho}_A = \text{Tr}_B (\hat{\rho}_{AB})$ is also a quantum Gaussian state with mean vector and covariance matrix given by

$$s = s_A, \quad V = V_A,$$

(B7)

respectively. On the other hand, when the subsystem $B$ is measured by a Gaussian projective operator $\hat{\rho}^G_B$ of mean vector $r_m \in \mathbb{R}^{2m}$ and covariance matrix $V_m$, then the quantum state $\hat{\rho}_A$ conditioned on the measurement outcome $r_m \in \mathbb{R}^{2m}$ is a quantum Gaussian state too, but its mean vector and covariance matrix are given by

$$s = s_A + V_{AB} \frac{1}{V_B + V_m} (r_m - s_B),$$

$$V = V_A - V_{AB} \frac{1}{V_B + V_m} V_{AB}^T,$$

(B8)

where the probability density function of the outcome $r_m$ is given by the Gaussian overlap integral $p(r_m) = \text{Tr} (\hat{\rho}^G_B \hat{\rho}_{AB})$, which evaluates to

$$p(r_m) = \frac{\exp \left( - (r_m - s_B)^T \frac{1}{V_B + V_m} (r_m - s_B) \right)}{\pi^m \sqrt{\det (V_B + V_m)}}.$$  

(B9)

CV Teleportation of a TMSV across the EC Box of \cite{21}. Consider the scheme in Fig. 8. Since the dual homodyne detection, the lossy channel, and the beamsplitters in the quantum scissors are all Gaussian operations, the joint quantum state across the modes prior to the measurements in the quantum scissors is Gaussian. The mean and covariance matrix of this Gaussian can be written down using (A4), (A10), (A11) and (B8).

Based on the observed dual homodyne outcome $\gamma$, after the NLA operation, a displacement correction unitary is applied on the modes $A$ and $B$, where these modes are displaced back by $g_A (-\gamma_x, -\gamma_y)$ and $g_B (-\gamma_x, +\gamma_y)$. Here $g_A, g_B$ are classical gain parameters, which can be optimized over.

Appendix C: Non-Gaussian Measurement based on ON-OFF Photodetection & Gaussian Overlap Integrals

ON-OFF photodetection is a measurement scheme described by the positive operator valued measure (POVM) elements

$$\Pi_0 = |0\rangle \langle 0|, \quad \Pi_1 = I - \Pi_0,$$

(C1)

where the projective measurement $\Pi_0$ is Gaussian, but $\Pi_1$ is not. In the modified quantum scissors operation considered in this work, both in Figs. 8 and 9 the heralding measurements of NLA are based on ON-OFF photodetection.

When the subsystem $B$ consisting of $m$ out of $n$ modes of a CV system $AB$ in a quantum Gaussian state $\hat{\rho}_{AB}$ is measured with OFF photodetection ($\Pi_0$ projection) on all the $m$ modes, the conditional (Gaussian) quantum state on subsystem $A$ and the probability of obtaining the OFF outcome across the $m$ modes follow from (B8) and (B9), respectively, with $V_m = I^\otimes m$ and $r_m = 0$. The latter is the overlap integral $\text{Tr} (\Pi_0^\otimes m \hat{\rho}_{AB})$, and simplifies to

$$p_0 = \text{Tr} \left( (\Pi_0^\otimes m)_B \hat{\rho}_{AB} \right) = \frac{2^m \exp \left( - s_B^T V_B + I^\otimes m s_B \right)}{\sqrt{\det (V_B + I^\otimes m)}},$$

(C2)

where $s_B$ and $V_B$ are the mean and covariance matrix of the modes in $B$. 

Likewise, the probability of observing $\Pi_1$ is all the $m$ modes is given by

$$p_1 = \text{Tr} \left( (\Pi_1^{\otimes m})_B \hat{\rho}_{AB} \right)$$

$$= \text{Tr} \left( (\Pi_1^{\otimes m})_B \hat{\rho}_B \right)$$

$$= \text{Tr} \left( (I - \Pi_0)^{\otimes m} \hat{\rho}_B \right)$$

$$= \sum_{\tau \in P(K)} (-1)^{1|\tau|} \frac{2^{1|\tau|} \exp \left( -\frac{s_T^T \mathbf{I} + I_{1|\tau|} - s_r}{\sqrt{\det (V + I_{1|\tau|})}} \right)}{\sqrt{\det (V + I_{1|\tau|})}},$$

where $K$ is the set of all $m$ modes contained in system $B$, $P(K)$ the powerset of $K$, i.e., the set of all subsets of $K$ (inclusive of the null element), $s_r$ and $V_\tau$ are the mean vector and covariance matrix of the reduced quantum state on the modes in element $\tau \in P(K)$ and $I_{1|\tau|}$ is the identity matrix of dimension $|\tau|$. Though, in this case the post measurement state on subsystem $A$ is non-Gaussian, and hence cannot be captured using [13] anymore. Nevertheless, the Husimi $Q$ function of the non-Gaussian state on the modes in subsystem $A$ can be written down, e.g., when $A$ consists of two modes $\hat{a}, \hat{b}$, as

$$Q(\alpha, \beta) = \frac{\text{Tr} \left( (|\alpha\rangle \langle \alpha| \otimes |\beta\rangle \langle \beta|)_A \otimes (\Pi_1^{\otimes m})_B \hat{\rho}_{AB} \right)}{\pi^2 p_1}$$

$$\Rightarrow Q(\alpha_x, \beta_x, \alpha_y, \beta_y) = \frac{\sum_{\tau \in P(K)} (-1)^{|\tau|} \frac{2^{1|\tau|+2} \exp \left( -\frac{(s_{r\cup A} - r_{r\cup A})^T v_{r\cup A}^{-1} (s_{r\cup A} - r_{r\cup A})}{\sqrt{\det (V_{r\cup A} + I_{1|\tau|+2})}} \right)}{4\pi^2 p_1}}{\sqrt{\det (V_{r\cup A} + I_{1|\tau|+2})}},$$

where $\alpha = (\alpha_x + i\alpha_y)/\sqrt{2}$ (and likewise $\beta$, and $r_{r\cup A}$ is the zero vector except for the entries corresponding to the modes in $A$, which take the values $(\alpha_x, \beta_x, \alpha_y, \beta_y)$).

The same approach can be used to the construct the $Q$ function that is heralded when some of the modes in $B$ are projected onto $\Pi_0$, while some others are projected onto $\Pi_1$, which is how we construct the $Q$ function heralded by the $N$-quantum scissors NLA operations.

### Appendix D: Entanglement of Formation

**Definition 1.** The entanglement of formation (EOF) of a bipartite state $\rho_{AB}$ is defined as [20]

$$E_F(\rho_{AB}) := \inf \left\{ \sum_k \lambda_k E(\Psi_k) \right\} \left| \rho_{AB} = \sum_k \lambda_k |\Psi_k\rangle \langle \Psi_k| \right\},$$

where $|\Psi_k\rangle$ are entangled pure states and $E(\Psi_k)$ is the entanglement entropy of $|\Psi_k\rangle$.

It is the minimum amount of pure entanglement required to construct the state $\rho_{AB}$. The EOF is non-increasing under local operations and classical communication (LOCC).

**Definition 2.** The Gaussian entanglement of formation (GEOF) of a bipartite state $\rho_{AB}$ of mean vector $d$ and $4n \times 4n$ dimensional covariance matrix $V$ (2n total modes) is defined as [11]

$$E_G(\rho_{AB}(V, d)) := \inf_{\lambda} \left\{ \int \lambda (dV_p, d\xi) E\left( \Psi_{AB}^G(V_p, \xi) \right) \right\} \left| \rho_{AB} = \int \lambda (dV_p, d\xi) \Psi_{AB}^G(V_p, \xi) \right\},$$

where $\Psi_{AB}^G$ are entangled Gaussian pure states and $\lambda$ is a measure in probability space. For a $n|n$ -mode bipartite state (total $2n$ modes), the GEOF is given by

$$E_G(\rho_{AB}(V, d)) = \sum_{k=1}^n H(r_k),$$

$$H(r) = \cosh^2(r) \log_2 \left( \cosh^2(r) \right) - \sinh^2(r) \log_2 \left( \sinh^2(r) \right).$$

This is so because every $n|n$ -mode bipartite pure Gaussian state is a tensor product of $n$ two mode squeezed states with squeezing parameters $r_k, k \in \{1, \ldots, n\}$ up to a local GLOCC unitary operation, and the entanglement of a TMS state with squeezing $r$ is $H(r)$ as above.

It is the minimum amount of pure Gaussian entanglement required to construct the state $\rho_{AB}$. The GEOF is non-increasing under Gaussian local operations and classical communication (GLOCC).
Figure 13. Relation between entanglement of formation and Gaussian entanglement of formation. $\tau$ represents a bipartite non-Gaussian state, while $\tau^G$ denotes a Gaussian state whose covariance matrix is the same as that of $\tau$. The numberings correspond to the numberings of the text in Appendix D.

**Corollary 3.** The Gaussian entanglement of formation is at least as large as the entanglement of formation

$$E_G(\rho_{AB}) \geq E_F(\rho_{AB}). \tag{D5}$$

*Proof.* This follows from Definitions 2 and 1. The former is an infimum over a restricted set of possible decompositions of the state than the latter, and hence is equal or larger than the latter. \hfill \Box

**Lemma 4.** (Gaussian extremality of EOF and GEOF) Among the set of all quantum states with covariance matrix $V$, and arbitrary mean and other moments, the (Gaussian) entanglement of formation is minimized by the Gaussian states whose covariance matrix equals $V$, i.e.,

$$E_{F/G}(\rho_{AB}(V)) \geq E_{F/G}(\rho^G_{AB}(V)) \tag{D6}$$

*Proposition 5.* For any two-mode Gaussian state $\rho^G_{AB}$, the Gaussian entanglement of formation equals its entanglement of formation, i.e.,

$$E_G(\rho^G_{AB}) = E_F(\rho^G_{AB}) \tag{D7}$$

*Proof.* See [42]. \hfill \Box

**Lemma 6.** The Gaussian entanglement of formation of a bipartite Gaussian state of mean $d$ and covariance matrix $V$

$$E_G(\rho^G_{AB}(V,d)) := \inf_{V_p} \{ E(\rho^G_{AB}(V_p,0)) \mid V_p \leq V \}, \tag{D8}$$

where $\rho^G_{AB}(V_p,0)$ are pure entangled states, and $E$ is the entanglement entropy.

*Proof.* See [41, 42]. We use the results in [42] to evaluate the (Gaussian) entanglement of formation of the heralded covariance matrix in the NLA-assisted communication schemes. \hfill \Box

**Remark 7.** Evidently, from Lemma 6, the Gaussian entanglement of formation is independent of displacements and equals the entanglement entropy of a TMSV state, wherein the infimum picks the TMSV state with the smallest possible squeezing.

**Corollary 8.** When the covariance matrix of a non-Gaussian quantum state $\rho_{AB}$ is $V(\gamma)$, where $\gamma$ is some complex parameter distributed according to $P(\gamma)$, we have that

$$\int d\gamma P(\gamma) E_{F/G}(\rho_{AB}(V(\gamma))) \geq \int d\gamma P(\gamma) E_{F/G}(\rho^G_{AB}(V(\gamma))). \tag{D9}$$

*Proof.* This follows from Lemma 4 and the fact that $P(\gamma) \geq 0$ and $E_{F/G} \geq 0$ for any state. \hfill \Box

**Remark 9.** We use the lower bound in Corollary 8 (with $E_G$) as our figure of merit for the scheme depicted in Fig. 8.

**Remark 10.** A deterministic displacement operation affects only the mean of a generic quantum state, and doesn’t change its covariance matrix or higher moments.

**Corollary 11.** Given a generic conditional state $\rho_{AB}(\gamma)$ of mean $d(\gamma)$ and covariance matrix $V(\gamma)$, conditioned on a parameter $\gamma$ (e.g., $\gamma$ could be the outcome of a dual homodyne detection), the action of conditional displacements $D(g\gamma)$ on the state doesn’t change its GEOF or the average GEOF of Corollary 8.

**Remark 12.** Thus, teleportation displacement correction does not affect the ergodic average GEOF lower bound we calculate for Ralph’s scheme.