Duality of Fix-Points for Distributive Lattices

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Abstract

We present a novel algorithm for calculating fix-points. The algorithm calculates fix-points of an endo-function \( f \) on a distributive lattice, by performing reachability computation a graph derived from the dual of \( f \); this is in comparison to traditional algorithms that are based on iterated application of \( f \) until a fix-point is reached.

1 Introduction

In this paper we cast the problem of calculating fix-points in a categorical framework. We first show how fix-points can be expressed as limit constructions within a suitable category (\( \mathcal{C} \) say). We then transport the fix-point calculation problem to the dual category of \( \mathcal{C} \) (\( \mathcal{D} \), say). Within the dual-category \( \mathcal{D} \), the problem gets converted into calculating a co-limit construction. Still working within the dual-category, we show that the co-fix-point object can be calculated using a reachability based iterative algorithm. Finally the resulting dual fix-point object is transported back across the duality to \( \mathcal{C} \) to give us the fix-point object that we initially wanted to calculate.

In this paper, we concentrate on the case where the category \( \mathcal{C} \) above, is the category of finite distributive lattices with homomorphisms, and the dual category \( \mathcal{D} \) is the category of finite partial-orders and monotone functions.

The rest of the paper is structured as follows: Section 2 gives some background on lattice theory and duality-theory for distributive lattices. Section 3 formulates the notion of fix-points within a categorical setting. Section 4 then transports this calculation to the dual-category, and derives an algorithm for computing the dual-fix-point object. We conclude with some comments about future work in Section 6.

2 Background

In this section, we will present some background on lattice theory, leading up to the duality theory of distributive lattices and homomorphisms.
2.1 Lattice Theory

We assume the basic definitions of lattices and develop only the part of the theory required for talking about duality.

**Definition 2.1 (Order Ideal)** Given a poset \(\langle P, \sqsubseteq \rangle\), a subset \(S\) of \(P\) is an order ideal if it is closed under the ordering relation of the poset.

\[ x \in S \land y \sqsubseteq x \Rightarrow y \in S \]

We will denote the set of order-ideals of a poset, \(P\), by \(\mathcal{O}(P)\).

**Definition 2.2 (Join-irreducible elements)** Let \(L\) be a lattice. An element \(x \in L\) is join-irreducible if

1. \(x \neq \bot\) and
2. \(x = a \sqcup b\) implies \(x = a\) or \(x = b\) for all \(a, b \in L\).

Given a lattice \(L\), we denote the set of join-irreducible elements of \(L\) by \(\mathcal{J}(L)\). The set \(\mathcal{J}(L)\) is also a poset and inherits the order relation of \(L\).

2.2 Duality Theory of Distributive Lattices

In this section we present the duality theory of distributive lattices. Distributive lattices exhibit a duality that is a generalization of Stone Duality of Boolean algebras. The results presented in this section are standard results and we refer the reader to [1] for a very fine introduction to this subject.

We begin the section with the definition of distributive lattices and homomorphisms of distributive lattices.

**Definition 2.3 (Distributive Lattice)** Distributive lattices are lattices \(\langle L, \sqsubseteq, \land, \lor, \bot, \top \rangle\) that satisfy the distributive law

\[ a \land (b \lor c) = (a \land b) \lor (a \land c) \]

for any elements \(a, b,\) and \(c\) of the lattice.

**Definition 2.4 (homomorphism)** A function between distributive lattices \(f : L \to K\), is a homomorphism if

\[
\begin{align*}
  f(a \land b) &= f(a) \land f(b) \\
  f(a \lor b) &= f(a) \lor f(b) \\
  f(\bot) &= \bot \\
  f(\top) &= \top
\end{align*}
\]

It can be shown that the set of order-ideals of a finite poset \(P\) ordered by subset inclusion, i.e. \(\langle \mathcal{O}(P), \subseteq, \land, \lor, \emptyset, P \rangle\), forms a distributive lattice [1].
Theorem 2.5 (Priestley’s Theorem (c.f. [1])) Let \( L \) be a finite distributive lattice. Then the map \( \eta : L \to \mathcal{O}(J(L)) \) defined by
\[
\eta(a) = \{ x \in J(L) \mid x \sqsubseteq a \}
\]
is an isomorphism of \( L \) onto \( \mathcal{O}(J(L)) \)

Proposition 2.6 (Priestley duality (c.f. [1])) Let \( P \) and \( Q \) be finite posets and let \( L = \mathcal{O}(P) \) and \( K = \mathcal{O}(Q) \).

Given a homomorphism \( f : L \to K \), there is an associated order-preserving map \( \phi_f : Q \to P \) defined by
\[
\phi_f(y) = \min \{ x \in P \mid y \in f(\downarrow x) \}
\]
for all \( y \in Q \), where \( \downarrow x \) is the principal-ideal of \( x \).

Given an order-preserving map \( \phi : Q \to P \), there is an associated homomorphism \( f_\phi : L \to K \) defined as the direct-image of the relation \( \phi^{-1} \)
\[
f_\phi(a) = \phi^{-1}(a) \text{ for all } a \in L
\]
Equivalently,
\[
\phi(y) \in a \iff y \in f_\phi(a) \text{ for all } a \in L, \ y \in Q
\]

Theorem 2.7 (Duality (c.f. [1])) The mappings defined in Proposition 2.6 define a duality between the category \( \text{Dist}_{\text{fin}} \) of finite distributive lattices and homomorphisms, and the category, \( \text{Ord}_{\text{fin}} \), of finite partial orders and monotone maps.

Proof: The proof of duality between the categories \( \text{Ord}_{\text{fin}} \) and \( \text{Dist}_{\text{fin}} \) requires establishing two functors \( \mathcal{O} : \text{Ord}_{\text{fin}} \to \text{Dist}_{\text{fin}}^{\text{op}} \), and \( J : \text{Dist}_{\text{fin}}^{\text{op}} \to \text{Ord}_{\text{fin}} \), such that \( \mathcal{O} \circ J \cong \text{Id}_{\text{Dist}_{\text{fin}}^{\text{op}}} \) and \( J \circ \mathcal{O} \cong \text{Id}_{\text{Ord}_{\text{fin}}} \). Theorem 2.5 and Theorem 2.7 provide the functors and isomorphisms that allow us to establish the required conditions.

\[ \blacksquare \]

3 Fix-points

In this section, we give an universal characterization of fix-points as the limit of a particular diagram. This characterization of fix-points has been inspired by [3] which characterizes fix-points of an endomorphism as an equalizer.

Lemma 3.1 Given a finite distributive lattice \( L \), and an endomorphism \( f : L \to L \), the fix-points of \( f \) form a distributive lattice.

Proof: The minimum and maximum fix-points of \( f \) are \( \bot_L \) and \( \top_L \) respectively for homomorphisms. Given any two fix-points \( x \) and \( y \), it is trivial to see that \( x \sqcap_L y \) and \( x \sqcup_L y \) are also fix-points
\[
f(x \sqcup_L y) = f(x) \sqcup_L f(y) = x \sqcup_L y
\]
Therefore, we can conclude that the lattice of fix-points of \( f \) are also distributive as any sub-lattice of a distributive lattice is distributive \( \square \).

In categorical language, the category of distributive lattices and structure preserving maps has equalizers \( \square \) of the form:

\[
\begin{array}{ccc}
F & \xrightarrow{\iota} & D \\
\downarrow & & \downarrow \\
F' & \quad & D
\end{array}
\]

where the distributive lattice \( F \) is the lattice of fix-points of the map \( f : D \to D \), and the homomorphism \( \iota \) is an embedding of the lattice of fix-points, \( F \), into \( D \).

4 Fix-point calculation using duality

The duality between distributive lattices and partial orders given in the previous section is a very strong result. All universal properties of distributive lattices and homomorphisms can be equivalently stated and studied as (dual) universal properties of partial-orders and monotone functions. In particular, \( \square \) characterizes the fix-points of a homomorphism as a universal property. By duality we can characterize the fix-points in the category \( \text{Ord}_{\text{fin}} \) of partial-orders and monotone functions as the co-equalizers\(^1\)

\[
\begin{array}{ccc}
C & \xleftarrow{k} & \mathcal{J}(D) \\
\downarrow & & \downarrow \\
\mathcal{J}(D) & \xrightarrow{id} & \mathcal{J}(D)
\end{array}
\]

Further, this co-equalizer object \( C \), is related to the fix-point object \( F \) in \( \text{Dist}_{\text{fin}} \) of \( f \), up to isomorphism by

\[
\mathcal{O}(C) \cong F \tag{1}
\]

Now, given \( f \) and \( g \) of type \( A \to B \) in the category \( \text{Set} \) of sets and functions, their co-equalizer is just a quotient; i.e. the smallest equivalence relation, \( \sim \), on \( B \), such that \( f(x) \sim g(x) \), for every \( x \in A \). We can perform a similar construction for calculating co-equalizers in \( \text{Ord}_{\text{fin}} \), and build the smallest preorder \( \preceq \) that extends the order on \( B \), such that \( f(x) \approx g(x) \) for all \( x \in A \), where \( \approx \) is the equivalence relation \( \preceq \cap \succeq \). The co-equalizer is then given by \( C \cong B/\approx \). It is easy to see that \( C \) is a partial order, with the order relation, \( \subseteq_C \), induced by \( \preceq \):

\[
[x] \approx [y] \iff x \preceq y \tag{2}
\]

\(^1\)The same as the equalizer diagram, but with arrow directions reversed.
Proposition 4.1 (c.f. [2]) The category, $\text{Ord}_{\text{fin}}$, has co-equalizers of $f, g : P \to Q$, given by a partial order whose elements are equivalence classes of elements of $Q$.

We also observe that since (by construction) $\preceq$ extends $\sqsubseteq$,

\[ x \sqsubseteq_B y \Rightarrow [x]_\approx \sqsubseteq_C [y]_\approx \quad (3) \]

Applying the above construction for our special case of co-equalizer of $\text{id}$ and $\phi_f = f$, we observe that the construction boils down to computing the connected subsets of the map of $\phi_f$ – i.e. the co-equalizer is the set of equivalence classes of elements $P$, obtained by identifying $x, \phi_f(x)$, and $\phi_f^{-1}(x)$ for all $x \in P$.

We can now state the relation between the co-equalizer in the category $\text{Ord}_{\text{fin}}$ and the equalizer in the dual category $\mathcal{O}(\text{Ord}_{\text{fin}})$.

Proposition 4.2 Let $P$ be a finite ordered set and let $L = \mathcal{O}(P)$. Given a homomorphism $f : L \to L$

- let $\text{Fix}(f)$ represent the sub-lattice of $L$, consisting of the fix-points of $f$
- Let $\phi_f : P \to P$ be as defined in Proposition 2.6
- Let $C = \text{Eq}(\phi_f)$, be the co-equalizer of $\phi_f$, and $\text{id}_P$, calculated as equivalence classes of $P$
- Let $E = \bigcup \circ \mathcal{O}(C)$, i.e. the distributive lattice obtained by calculating the set-union for each order-ideal (consisting of sets of equivalence classes) of $C$

Now, given any element $X$ of $L$

\[ f(X) = X \Leftrightarrow X \in E \]

Proof:

$(\Leftarrow)$ An order-ideal of $C$, is a set of subsets (equivalence classes) of the partial order $P$ whose elements are the join-irreducible elements of $L$. The equivalence classes are not themselves down-closed by the ordering on $P$. But it is fairly easy to see, from the construction of $C$, that every order-ideal of $C$, under $\cup$, gives rise to an order-ideal of $P$.

To show that $f(X) = X$, we observe that $X$ is obtained as a union of equivalence classes of $P$ in $C$. Each of these equivalence classes $X_i$ are themselves closed under $\phi_f$ and $\phi_f^{-1}$ (by the co-limit construction). Therefore since $f = f_{\phi_f} = \phi_f^{-1}$ (Proposition 2.6), each equivalence class satisfies the property $f(X_i) = X_i$ and the result follows.

$(\Rightarrow)$ Given an order ideal, $X$, such that $f(X) = X$. By Proposition 2.6 this implies that $\phi_f^{-1}(X) = x$; in other words

\[ \forall x \in X \bullet \exists y \in X \bullet y = \phi_f(x) \]
Therefore, since $\phi f$ is a function, $X$ is closed under $\phi f$.

Therefore, in general, $X$ is an union of equivalence classes of $P$, that are closed under $\phi f$ and $\phi f^{-1}$; i.e. $X$ is an union of elements of $C$ (by the co-equalizer construction of Proposition 4.1). Now, Equation 3 tells us that this set of elements of $C$ is an order-ideal in $C$. Therefore $X$ is an element of $E$.

\section{Algorithm for fix-point calculation}

Putting all the pieces presented so far, we obtain an algorithm for calculating fix-points using dual representation of lattices and homomorphisms.

\begin{algorithm}
\caption{}
\begin{algorithmic}[1]
\Require $f : L \to L$
\State calculate $P = \mathcal{J}(L)$ and $\phi f : P \to P$ (Proposition 2.6)
\State construct the (undirected) graph $G$ of $\phi f$ over $P$
\State calculate the connected components of $G$, and its ordering based on $P$
\State select any order-ideal $M$ of $G$
\State return $\bigcup L_\downarrow (\bigcup M)$
\end{algorithmic}
\end{algorithm}

 Basically, the algorithm calculates the join-irreducible elements, $\mathcal{J}(L)$ of a distributive lattice, $L$; and the dual, $\phi f$, of a lattice-homomorphism, $f$. It then computes the connected components, $C$ say, of the un-directed graph of $\phi f$ on the elements of $\mathcal{J}(L)$. The set $C$ is partially-ordered and inherits its ordering from the partial order of $\mathcal{J}(L)$. Finally, the fix-points of $f$ are computed from the ideals of $C$, as the least-upper-bound of the sets in a given ideal.

\section{Conclusion and Future Work}

The work presented in this paper is only a starting point for investigations into alternate algorithms for fix-point calculation. Future work includes generalizing the framework presented here to hemimorphisms – this particular generalization will find immediate application in the area of data-flow analysis of programs. Another topic to be further investigated is the connection between the framework presented here and model-checking algorithms; and relating the search algorithm of model-checkers over a Kripke-structure to the equivalence-class calculation performed by the algorithm presented in this paper.
References

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[2] J. Goubault-Larrecq. Why is cpo co-complete. Technical report, Laboratoire Spéciﬁcation et Vérification, Ecole Normale Supérieure de Cachan, 2002.

[3] C. B. Jay. Tail recursion through universal invariants. *Theoretical Computer Science*, 115(1):151–189, 1993.