Research Article

On the Stability of Quadratic Functional Equations

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Received 5 October 2007; Revised 27 November 2007; Accepted 4 January 2008

Let $X, Y$ be vector spaces and $k$ a fixed positive integer. It is shown that a mapping $f(kx + y) + f(kx - y) = 2k^2f(x) + 2f(y)$ for all $x, y \in X$ if and only if the mapping $f : X \to Y$ satisfies $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ for all $x, y \in X$. Furthermore, the Hyers-Ulam-Rassias stability of the above functional equation in Banach spaces is proven.

1. Introduction

The stability problem of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms. Hyers [2] gave a first affirmative answer to the question of Ulam for Banach spaces. Hyers’ theorem was generalized by Aoki [3] for additive mapping and by Th. M. Rassias [4] for linear mappings by considering an unbounded Cauchy difference. The paper of Th. M. Rassias [4] has provided a lot of influence in the development of what we now call Hyers-Ulam-Rassias stability of functional equations. Th. M. Rassias [5] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. Gajda [6], following the same approach as in [4], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [6] as well as by Rassias and Šemrl [7] that one cannot prove a Th.M. Rassias’ type theorem when $p = 1$. J. M. Rassias [8], following the spirit of the innovative approach of Th. M. Rassias [4] for the unbounded Cauchy difference, proved a similar stability theorem in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$.

The functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y)$$

(1.1)

is called a quadratic functional equation. In particular, every solution of the quadratic functional
equation is said to be a \textit{quadratic function}. A Hyers-Ulam-Rassias stability problem for the quadratic functional equation was proved by Skof [9] for mappings \( f : X \to Y \), where \( X \) is a normed space and \( Y \) is a Banach space. Cholewa [10] noticed that the theorem of Skof is still true if the relevant domain \( X \) is replaced by an Abelian group. In [11], Czerwik proved the Hyers-Ulam-Rassias stability of the quadratic functional equation. Several functional equations have been investigated in [12–17].

Throughout this paper, assume that \( k \) is a fixed positive integer.

In this paper, we solve the functional equation

\[
    f(kx + y) + f(kx - y) = 2k^2 f(x) + 2f(y) \tag{1.2}
\]

and prove the Hyers-Ulam-Rassias stability of the functional equation (1.2) in Banach spaces.

2. Hyers-Ulam-Rassias stability of the quadratic functional equation

\textbf{Proposition 2.1.} Let \( X \) and \( Y \) be vector spaces. A mapping \( f : X \to Y \) satisfies

\[
    f(kx + y) + f(kx - y) = 2k^2 f(x) + 2f(y) \tag{2.1}
\]

for all \( x, y \in X \) if and only if the mapping \( f : X \to Y \) satisfies

\[
    f(x + y) + f(x - y) = 2f(x) + 2f(y) \tag{2.2}
\]

for all \( x, y \in X \).

\textbf{Proof.} Assume that \( f : X \to Y \) satisfies (2.1).

Letting \( x = y = 0 \) in (2.1), we get \( f(0) = 0 \).

Letting \( y = 0 \) in (2.1), we get \( f(kx) = k^2 f(x) \) for all \( x \in X \).

Letting \( x = 0 \) in (2.1), we get \( f(-y) = f(y) \) for all \( y \in X \).

It follows from (2.1) that

\[
    f(kx + y) + f(kx - y) = 2k^2 f(x) + 2f(y) = 2f(kx) + 2f(y) \tag{2.3}
\]

for all \( x, y \in X \). So the mapping \( f : X \to Y \) satisfies

\[
    f(x + y) + f(x - y) = 2f(x) + 2f(y) \tag{2.4}
\]

for all \( x, y \in X \).

Assume that \( f : X \to Y \) satisfies \( f(x + y) + f(x - y) = 2f(x) + 2f(y) \) for all \( x, y \in X \).

We prove (2.1) for \( k = j \) by induction on \( j \).

For the case \( j = 1 \), (2.1) holds by the assumption.

For the case \( j = 2 \), since

\[
    f(2x + y) + f(2x - y) = f(x + y + x) + f(x - y + x) \\
    = 2f(x + y) + 2f(x) - f(y) + 2f(x - y) + 2f(x) - f(-y) \\
    = 2f(x + y) + 2f(x - y) + 4f(x) - 2f(y) \\
    = 4f(x) + 4f(y) + 4f(x) - 2f(y) \\
    = 8f(x) + 2f(y) \tag{2.5}
\]

for all \( x, y \in X \), then (2.1) holds.
Theorem 2.2. For all Banach space with norm \( \| \cdot \| \), for all \( x, y \in X \), (2.1) holds for \( j = n - 2 \) and \( j = n - 1 \) \((2 < n \leq k)\). By the assumption,

\[
\begin{align*}
 f(nx + y) + f(nx - y) &= f((n-1)x + y + x) + f((n-1)x - y + x) \\
 &= 2f((n-1)x + y) + 2f(x) - f((n-2)x + y) \\
 &\quad + 2f((n-1)x - y) + 2f(x) - f((n-2)x - y) \\
 &= 4(n-1)^2 f(x) + 4f(y) + 4f(x) - 2(n-2)^2 f(x) - 2f(y) \\
 &= 2n^2 f(x) + 2f(y)
\end{align*}
\]

for all \( x, y \in X \), (2.1) holds for \( j = n \). Hence the mapping \( f : X \to Y \) satisfies (2.1) for \( j = k \). \( \square \)

From now on, assume that \( X \) is a normed vector space with norm \( \| \cdot \| \) and that \( Y \) is a Banach space with norm \( \| \cdot \| \). For a given mapping \( f : X \to Y \), we define

\[
Df(x, y) := f(kx + y) + f(kx - y) - 2k^2 f(x) - 2f(y)
\]

for all \( x, y \in X \).

Now we prove the Hyers-Ulam-Rassias stability of the quadratic functional equation \( Df(x, y) = 0 \).

**Theorem 2.2.** Let \( f : X \to Y \) be a mapping with \( f(0) = 0 \) for which there exists a function \( \varphi : X^2 \to [0, \infty) \) such that

\[
\bar{\varphi}(x, y) := \sum_{j=0}^{\infty} \frac{1}{k^{2j}} \varphi(k^j x, k^j y) < \infty,
\]

\[
\|Df(x, y)\| \leq \varphi(x, y)
\]

for all \( x, y \in X \). Then there exists a unique quadratic mapping \( Q : X \to Y \) such that

\[
\|f(x) - Q(x)\| \leq \frac{1}{2k^2} \bar{\varphi}(x, 0)
\]

for all \( x \in X \).

**Proof.** Letting \( y = 0 \) in (2.9), we get

\[
\|2f(kx) - 2k^2 f(x)\| \leq \varphi(x, 0)
\]

for all \( x \in X \). So

\[
\left\| f(x) - \frac{1}{k^2} f(kx) \right\| \leq \frac{1}{2k^2} \varphi(x, 0)
\]

for all \( x \in X \). Hence

\[
\left\| \frac{1}{k^{2j}} f(k^j x) - \frac{1}{k^{2m}} f(k^m x) \right\| \leq \sum_{j=1}^{m-1} \frac{1}{2k^{2j+2}} \varphi(k^j x, 0)
\]

(2.13)
Moreover, letting

\[ \text{Theorem 2.4.} \]

\[ \text{Corollary 2.3.} \]

for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x \in X \). It follows from (2.13) that the sequence \( \{(1/k^{2n})f(k^n x)\} \) is Cauchy for all \( x \in X \). Since \( Y \) is complete, the sequence \( \{(1/k^{2n})f(k^n x)\} \) converges. So one can define the mapping \( Q : X \to Y \) by

\[
Q(x) := \lim_{n \to \infty} \frac{1}{k^{2n}} f(k^n x)
\]

(2.14)

for all \( x \in X \).

By (2.8),

\[
\|DQ(x,y)\| = \lim_{n \to \infty} \frac{1}{k^{2n}} \|Df(k^n x, k^n y)\| \leq \lim_{n \to \infty} \frac{1}{k^{2n}} \varphi(k^n x, k^n y) = 0
\]

(2.15)

for all \( x, y \in X \). So \( DQ(x,y) = 0 \). By Proposition 2.1, the mapping \( Q : X \to Y \) is quadratic. Moreover, letting \( l = 0 \) and passing the limit \( m \to \infty \) in (2.13), we get (2.10).

Now, let \( T : X \to X \) be another quadratic mapping satisfying (2.1) and (2.10). Then we have

\[
\|Q(x) - T(x)\| = \frac{1}{k^{2n}} \|Q(k^n x) - T(k^n x)\|
\]

\[
\leq \frac{1}{k^{2n}} \left( \|Q(k^n x) - f(k^n x)\| + \|T(k^n x) - f(k^n x)\| \right)
\]

(2.16)

\[
\leq \frac{1}{k^{2n+2}} \bar{\varphi}(k^n x, 0),
\]

which tends to zero as \( n \to \infty \) for all \( x \in X \). So we can conclude that \( Q(x) = T(x) \) for all \( x \in X \). This proves the uniqueness of \( Q \). So there exists a unique quadratic mapping \( Q : X \to Y \) satisfying (2.10).

\[ \square \]

**Corollary 2.3.** Let \( p < 2 \) and \( \theta \) be positive real numbers, and let \( f : X \to Y \) be a mapping such that

\[ \|Df(x,y)\| \leq \theta(\|x\|^p + \|y\|^p) \]

(2.17)

for all \( x, y \in X \). Then there exists a unique quadratic mapping \( Q : X \to Y \) such that

\[ \|f(x) - Q(x)\| \leq \frac{\theta}{8 - 2^{p+1}} \|x\|^p \]

(2.18)

for all \( x \in X \).

**Proof.** The proof follows from Theorem 2.2 by taking

\[ \varphi(x,y) := \theta(\|x\|^p + \|y\|^p) \]

(2.19)

for all \( x, y \in A \).

\[ \square \]

**Theorem 2.4.** Let \( f : X \to Y \) be a mapping with \( f(0) = 0 \) for which there exists a function \( \varphi : X^2 \to [0, \infty) \) satisfying (2.9) such that

\[
\bar{\varphi}(x,y) := \sum_{j=0}^{\infty} k^{2j} \varphi \left( \frac{x}{k^j}, \frac{y}{k^j} \right) < \infty
\]

(2.20)

for all \( x, y \in X \). Then there exists a unique quadratic mapping \( Q : X \to Y \) such that

\[ \|f(x) - Q(x)\| \leq \frac{1}{2} \bar{\varphi} \left( \frac{x}{k}, 0 \right) \]

(2.21)

for all \( x \in X \).
Proof. It follows from (2.11) that

$$\left\| f(x) - k^2 f \left( \frac{x}{k} \right) \right\| \leq \frac{1}{2} \varphi \left( \frac{x}{k}, 0 \right)$$

(2.22)

for all $x \in X$. Hence

$$\left\| k^{2l} f \left( \frac{x}{k^{2l}} \right) - k^{2m} f \left( \frac{x}{k^{2m}} \right) \right\| \leq \sum_{j=l}^{m-1} \frac{k^{2j}}{2} \varphi \left( \frac{x}{k^{j+1}}, 0 \right)$$

(2.23)

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (2.23) that the sequence $\{k^{2n} f(x/k^n)\}$ is Cauchy for all $x \in X$. Since $Y$ is complete, the sequence $\{k^{2n} f(x/k^n)\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n \to \infty} k^{2n} f \left( \frac{x}{k^n} \right)$$

(2.24)

for all $x \in X$.

By (2.20),

$$\left\| DQ(x, y) \right\| = \lim_{n \to \infty} k^{2n} \left\| Df \left( \frac{x}{k^n}, \frac{y}{k^n} \right) \right\| \leq \lim_{n \to \infty} k^{2n} \varphi \left( \frac{x}{k^n}, \frac{y}{k^n} \right) = 0$$

(2.25)

for all $x, y \in X$. So $DQ(x, y) = 0$. By Proposition 2.1, the mapping $Q : X \to Y$ is quadratic. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.23), we get (2.21).

The rest of the proof is similar to the proof of Theorem 2.2. ~\(\square\)

**Corollary 2.5.** Let $p > 2$ and $\theta$ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying (2.17). Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\left\| f(x) - Q(x) \right\| \leq \frac{\theta}{2^{p+1} - 8} \left\| x \right\|^p$$

(2.26)

for all $x \in X$.

*Proof.* The proof follows from Theorem 2.4 by taking

$$\varphi(x, y) := \theta \left( \left\| x \right\|^p + \left\| y \right\|^p \right)$$

(2.27)

for all $x, y \in A$. ~\(\square\)

From now on, assume that $k = 2$.

**Theorem 2.6.** Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : X^2 \to [0, \infty)$ satisfying (2.9) such that

$$\tilde{\varphi}(x, y) := \sum_{j=0}^{\infty} \frac{1}{3^j} \varphi(3^j x, 3^j y) < \infty$$

(2.28)

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\left\| f(x) - Q(x) \right\| \leq \frac{1}{9} \varphi(x, x)$$

(2.29)

for all $x \in X$. 
Proof. Letting $y = x$ in (2.9), we get
\[\|f(3x) - 9f(x)\| \leq \varphi(x, x)\]  
for all $x \in X$. So
\[\|f(x) - \frac{1}{9}f(3x)\| \leq \frac{1}{9}\varphi(x, x)\]  
for all $x \in X$. Hence
\[\left\| \frac{1}{9^m}f(3^n x) - \frac{1}{9^n}f(3^n x) \right\| \leq \frac{1}{9^n} \sum_{j=1}^{m-1} \frac{1}{9^{j+1}} \varphi(3^j x, 3^j x)\]  
for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (2.32) that the sequence \{(1/9^n)f(3^n x)\} is Cauchy for all $x \in X$. Since $Y$ is complete, the sequence \{(1/9^n)f(3^n x)\} converges. So one can define the mapping $Q : X \to Y$ by
\[Q(x) := \lim_{n \to \infty} \frac{1}{9^n}f(3^n x)\]  
for all $x \in X$.

By (2.28),
\[\|DQ(x, y)\| = \lim_{n \to \infty} \frac{1}{9^n}\|Df(3^n x, 3^n y)\| \leq \lim_{n \to \infty} \frac{1}{9^n}\varphi(3^n x, 3^n y) = 0\]  
for all $x, y \in X$. So $DQ(x, y) = 0$. By Proposition 2.1, the mapping $Q : X \to Y$ is quadratic. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.32), we get (2.29).

The rest of the proof is similar to the proof of Theorem 2.2. \qed

**Corollary 2.7.** Let $p < 1$ and $\theta$ be positive real numbers, and let $f : X \to Y$ be a mapping such that
\[\|Df(x, y)\| \leq \theta \cdot ||x||^p \cdot ||y||^p\]  
for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that
\[\|f(x) - Q(x)\| \leq \frac{\theta}{9 - 9^p} ||x||^{2p}\]  
for all $x \in X$.

Proof. The proof follows from Theorem 2.6 by taking
\[\varphi(x, y) := \theta \cdot ||x||^p \cdot ||y||^p\]  
for all $x, y \in A$. \qed
Theorem 2.8. Let \( f : X \to Y \) be a mapping with \( f(0) = 0 \) for which there exists a function \( \varphi : X^2 \to [0, \infty) \) satisfying (2.9) such that
\[
\varphi(x, y) := \sum_{j=0}^{\infty} 9^j \varphi \left( \frac{x}{3^j}, \frac{y}{3^j} \right) < \infty
\] (2.38)
for all \( x, y \in X \). Then there exists a unique quadratic mapping \( Q : X \to Y \) such that
\[
\|f(x) - Q(x)\| \leq \varphi \left( \frac{x}{3}, \frac{x}{3} \right)
\] (2.39)
for all \( x \in X \).

Proof. It follows from (2.30) that
\[
\left\|f(x) - 9f \left( \frac{x}{3} \right) \right\| \leq \varphi \left( \frac{x}{3}, \frac{x}{3} \right)
\] (2.40)
for all \( x \in X \). Hence
\[
\left\|9^j f \left( \frac{x}{3^j} \right) - 9^m f \left( \frac{x}{3^m} \right) \right\| \leq \sum_{j=m}^{\infty} 9^j \varphi \left( \frac{x}{3^j}, \frac{x}{3^j} \right)
\] (2.41)
for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x \in X \). It follows from (2.41) that the sequence \( \{9^n f(x/3^n)\} \) is Cauchy for all \( x \in X \). Since \( Y \) is complete, the sequence \( \{9^n f(x/3^n)\} \) converges. So one can define the mapping \( Q : X \to Y \) by
\[
Q(x) := \lim_{n \to \infty} 9^n f \left( \frac{x}{3^n} \right)
\] (2.42)
for all \( x \in X \).

By (2.38),
\[
\|DQ(x, y)\| = \lim_{n \to \infty} \frac{1}{9^n} \|Df(3^n x, 3^n y)\| \leq \lim_{n \to \infty} \frac{1}{9^n} \varphi(3^n x, 3^n y) = 0
\] (2.43)
for all \( x, y \in X \). So \( DQ(x, y) = 0 \). By Proposition 2.1, the mapping \( Q : X \to Y \) is quadratic. Moreover, letting \( l = 0 \) and passing the limit \( m \to \infty \) in (2.41), we get (2.39).

The rest of the proof is similar to the proof of Theorem 2.2. \( \square \)

Corollary 2.9. Let \( p > 1 \) and \( \theta \) be positive real numbers, and let \( f : X \to Y \) be a mapping satisfying (2.35). Then there exists a unique quadratic mapping \( Q : X \to Y \) such that
\[
\|f(x) - Q(x)\| \leq \frac{\theta}{9^p - 9} \|x\|^{2p}
\] (2.44)
for all \( x \in X \).

Proof. The proof follows from Theorem 2.8 by taking
\[
\varphi(x, y) := \theta \cdot \|x\|^p \cdot \|y\|^p
\] (2.45)
for all \( x, y \in A \). \( \square \)
Acknowledgments

Jung Rye Lee was supported by Daejin University grants in 2007. The authors would like to thank the referees for a number of valuable suggestions regarding a previous version of this paper.

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