NONIMMERSIONS OF $RP^n$ IMPLIED BY tmf, REVISITED

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Abstract

In a 2002 paper, the authors and Bruner used the new spectrum tmf to obtain some new nonimmersions of real projective spaces. In this note, we complete/correct two oversights in that paper.

The first is to note that in that paper a general nonimmersion result was stated which yielded new nonimmersions for $RP^n$ with $n$ as small as 48, and yet it was stated there that the first new result occurred when $n = 1536$. Here we give a simple proof of those overlooked results.

Secondly, we fill in a gap in the proof of the 2002 paper. There it was claimed that an axial map $f$ must satisfy $f^*(X) = X_1 + X_2$. We realized recently that this is not clear. However, here we show that it is true up multiplication by a unit in the appropriate ring, and so we retrieve all the nonimmersion results claimed in the 2002 paper.

Finally, we completely determine $\text{tmf}^8(RP^\infty \times RP^\infty)$ and $\text{tmf}^*(CP^\infty \times CP^\infty)$ in positive dimensions.

1. Introduction

In [6], the authors and Bruner described a proof of the following theorem, along with some additional nonimmersion results.

**Theorem 1.1 ([6, 1.1]).** Assume that $M$ is divisible by the smallest 2-power greater than or equal to $h$.

- If $\alpha(M) = 4h - 1$, then $P^{8M+8h+2}$ cannot be immersed in $(\mathbb{C}) \mathbb{R}^{16M-8h+10}$.
- If $\alpha(M) = 4h - 2$, then $P^{8M+8h} \subseteq \mathbb{R}^{16M-8h+12}$.

Here and throughout, $\alpha(M)$ denotes the number of 1’s in the binary expansion of $M$, and $P^n$ denotes real projective space.

In [6], the theorem is followed by a comment that this is new provided $\alpha(M) \geq 6$, i.e., $h \geq 2$, and the first new result occurs for $P^{1536}$. In this note, we point out
that 1.1 is valid when \( h = 1 \), and these results are new when \( M \) is even, including new nonimmersions of \( P^n \) for \( n \) as small as 56. A remark in [6, p. 66] that the nonimmersions when \( h = 1 \) were implied by earlier work of the authors was incorrect. Letting \( h = 1 \) in 1.1, we have the following result.

**Corollary 1.2.**

(a) If \( \alpha(M) = 3 \), then \( P^{8M+10} \not\subseteq \mathbb{R}^{16M+2} \).

(b) If \( \alpha(M) = 2 \), then \( P^{8M+8} \not\subseteq \mathbb{R}^{16M+4} \).

Part (a) is new when \( M \) is even. It is two better than the previous best result, proved in [4], and the nonembedding result that it implies is also new, one better than the previous best, proved in [3]. In [7], a table of known nonimmersions, immersions, nonembeddings, and embeddings of \( P^n \) is presented, arranged according to \( n = 2^i + d \) with \( 0 \leq d < 2^i \) and \( d < 64 \). Part (a) enters the table with a new result for \( d = 58 \), applying first to \( P^{122} \).

If \( M \) is even, then 1.2(b) is new, one better than the previous best result, of [13], and the nonembedding result implied is also new. It enters [7] at \( d = 24 \) and 40, with a new result for \( P^n \) with \( n \) as small as 56. The result of 1.2(b) with \( M = 2^i + 1 \) was also proved very recently by Kitchloo and Wilson in [16]. This result for \( P^{2^i+16} \), two better than the previous result of [4] and also new as a nonembedding, enters [7] at \( d = 16 \), and applies for \( n \) as small as 48.

In Section 2, we present a self-contained proof of Corollary 1.2. The primary reason for doing this, which amounts to a reproof of part of [6, 1.1], is that the proof of the general case in [6] requires some extremely elaborate arguments and calculations. Our proof here, which is just for the case \( h = 1 \), is much more comprehensible.

The proof in [6] contained an oversight which we shall correct here. The argument there was that an immersion of \( P^n \) in \( \mathbb{R}^{n+k} \) implies existence of an axial map \( P^n \times P^m \to P^{m+k} \) for an appropriate value of \( m \), and obtains a contradiction for certain \( n, m, \) and \( k \) by consideration of \( \text{tmf}^*(f) \). Here \( \text{tmf} \) is the spectrum of topological modular forms, which was discussed in [6]. A class \( X \in \text{tmf}^8(P^n) \) was described, along with \( X_1 = X \times 1 \) and \( X_2 = 1 \times X \) in \( \text{tmf}^8(P^n \times P^m) \). It was asserted that \( f^*(X) = X_1 + X_2 \), and a contradiction obtained by showing that, for certain values of the parameters, we might have \( X^k = 0 \) but \( (X_1 + X_2)^k \neq 0 \). We recently realized that it is conceivable that \( f^*(X) \) might contain other terms coming from \( \text{tmf}^8(P^n \wedge P^m) \).

In Section 3 (see Thm. 3.5), we perform a complete calculation of \( \text{tmf}^*(P^{\infty} \times P^{\infty}) \) in positive gradings divisible by 8, and in Section 4 we use it to show that effectively \( f^*(X) = u(X_1 + X_2) \), where \( u \) is a unit in \( \text{tmf}^*(P^{\infty} \times P^{\infty}) \), which enables us to retrieve all the nonimmersions of [6].

In Section 5, we compute \( \text{tmf}^*(C P^{\infty} \times C P^{\infty}) \) in positive gradings. The original purpose of doing this was, prior to our obtaining the argument of Section 4, to see whether we might mimic the argument of [2] and [8] to conclude that if \( f \) is an axial map, then \( f^*(X) \) might necessarily equal \( u(X_1 - X_2) \), where \( u \) is a unit in \( \text{tmf}^*(C P^{\infty} \times C P^{\infty}) \). This approach to retrieving the nonimmersions of [6] did not yield the desired result, but the later approach given in Section 4 did. Nevertheless the nice result for \( \text{tmf}^*(C P^{\infty} \times C P^{\infty}) \) obtained in Theorem 5.16 should be of independent interest.
2. Proof of Corollary 1.2

We begin by proving 1.2(a). The following standard reduction goes back at least to [15]. If $P^{8M+10} \subseteq \mathbb{R}^{16M+2}$, then $\text{gcd}((2L+3-8M-11)z_{8M+10}) \leq 8M - 8$; hence this bundle has $(2L+3-16M-3)$ linearly independent sections, and thus there is an axial map

$$P^{8M+10} \times P^{2L+3-16M-4} \xrightarrow{f} P^{2L+3-8M-12}.$$

The bundle here is the stable normal bundle, $L$ is a sufficiently large integer, and $\text{gcd}$ refers to geometric dimension. Let $X$, $X_1$, and $X_2$ be elements of $\text{tmf}^\ast(-)$ described in [6] and also in Section 1. In Section 4, we will show that we may assume that $f^\ast(X) = X_1 + X_2$, as was done in [6], since this is true up to multiplication by a unit. Since $\text{tmf}^{2L+3-8M-8}(P^{2L+3-8M-12}) = 0$, we have

$$0 = f^\ast(0) = f^\ast(X^{2L-M-1})$$

$$= (X_1 + X_2)^{2L-M-1} \in \text{tmf}^{2L+3-8M-8}(P^{8M+10} \times P^{2L+3-16M-4}).$$

Expanding, we obtain $(2L-M-1)X_1^{M+1}X_2^{2L-2M-2} + (2L-M-1)X_1^MX_2^{2L-2M-1}$ as the only terms which are possibly nonzero. Next we note that, with all $u$'s representing odd integers,

$$\left(\frac{2L-M-1}{M+1}\right) = u_1\left(\frac{M+1}{M+1}\right) = 2^{\alpha(M)-\nu(M+1)}u_2 = 2^{3-\nu(M+1)}u_2,$$

where we have used $\alpha(M) = 3$ at the last step. Here and throughout, $\nu(2^e) = e$.

Similarly, $(2L-M-1) = u_3\left(\frac{2M+1}{M+1}\right) = 2^{\alpha(M)}u_4 = 2^{3}u_4$. Thus an immersion implies that in $\text{tmf}^{2L+3-8M-8}(P^{8M+10} \times P^{2L+3-16M-4})$, we have

$$2^{3}\nu(M+1)u_2X_1^{M+1}X_2^{2L-2M-2} + 2^{3}u_4X_1^MX_2^{2L-2M-1} = 0. \quad (1)$$

We recall [6, 2.6], which states that there is an equivalence of spectra

$$P^{k+8}_b \wedge \text{tmf} \simeq \Sigma^8 P^k_b \wedge \text{tmf}.$$

Combining this with duality, we obtain

$$\text{tmf}^{8M+8}(P^{8M+10}) \approx \text{tmf}_{-8M-9}(P_{-8M-11}^{-2}) \approx \text{tmf}_{-1}(P_{-3}^{8M+6}) \approx \text{tmf}_{-1}(P_{-3}) \approx \mathbb{Z}/8,$$

using [12, p.367] for the final isomorphism. Hence $8X_1^{M+1}X_2^{2L-2M-2} = 0$. Here and throughout, $P_n = P^\infty_n = RP^n/RP^{n-1}$. Similarly,

$$\text{tmf}^{2L+3-16M-8}(P_2^{L+3-16M-4}) \approx \text{tmf}_{7}(P_3) \approx \mathbb{Z}/16,$$

and hence $16X_1^MX_2^{2L-2M-1} = 0$. Duality also implies

$$\text{tmf}^{2L+3-8M-8}(P^{8M+10}_n \times P_2^{L+3-16M-4}) \approx \text{tmf}_{14}(P_{-3} \wedge P_3).$$

Calculations such as $E_2(\text{tmf}_s(P_{-3} \wedge P_3))$, the $E_2$-term of the Adams spectral sequence (ASS), were made by Bruner’s minimal-resolution computer programs in our work on [6]. This one is in a small enough range to actually do by hand. The result is given in Diagram 2.1.
Diagram 2.1. $E_2(tmf_3(P_{-3} \wedge P_3)), \ast \leq 15$:

The $\mathbb{Z}/8 \oplus \mathbb{Z}/16$ arising from filtration 0 in grading 14 in 2.1 is not hit by a differential from the class in $(15,0)$ because, as explained in the last paragraph of page 54 of [6], the class in $(15,0)$ corresponds to an easily-constructed nontrivial map. The monomials $X_1^{M+1}X_2^{L-2M-2}$ and $X_1^M X_2^{2L-2M-1}$ are detected in mod-2 cohomology, and so their duals emanate from filtration 0. We saw in the previous paragraph that 8 and 16, respectively, annihilate these monomials, and hence also their duals. Since the chart shows that the subgroup of $tmf_{14}(P_{-3} \wedge P_3)$ generated by classes of filtration 0 is $\mathbb{Z}/8 \oplus \mathbb{Z}/16$, we conclude that 8 and 16, respectively, are the precise orders of the monomials. In particular, the order of $X_1^M X_2^{2L-2M-1}$ is 16, and hence the class in (1) is nonzero since it has a term $8uX_1^M X_2^{2L-2M-1}$, and so (1) contradicts the hypothesized immersion.

Part (b) of Corollary 1.2 is proved similarly. If $P^{8M+8}$ immerses in $\mathbb{R}^{16M+4}$, then there is an axial map

$$P^{8M+8} \times P^{2L+3-16M-6} \xrightarrow{f} P^{2L+3-8M-10},$$

and hence, up to odd multiples,

$$2^{2-\nu(M+1)} X_1^{M+1} X_2^{L-2M-2} + 2^2 X_1^M X_2^{2L-2M-1}$$

$$= 0 \in tmf^{2L+3-8M-8}(P^{8M+8} \wedge P^{2L+3-16M-6}),$$

(2)

since $\alpha(M) = 2$. We have $tmf^{8M+8}(P^{8M+8}) \approx tmf_{-1}(P_{-1}) \approx \mathbb{Z}/2$, and

$$tmf^{2L+3-16M-8}(P^{2L+3-16M-6}) \approx tmf_{-1}(P_{-3}) \approx \mathbb{Z}/8.$$

Thus the two monomials in (2) have order at most 2 and 8, respectively. On the other hand, the group in (2) is isomorphic to $tmf_6(P_{-1} \wedge P_{-3})$. A minimal resolution calculation easier than the one in Diagram 2.1 shows that $tmf_6(P_{-1} \wedge P_{-3})$ has $\mathbb{Z}/2 \oplus \mathbb{Z}/8$ emanating from filtration 0 (and another $\mathbb{Z}/2 \oplus \mathbb{Z}/8$ in higher filtration). The monomials of (2) are generated in filtration 0, and since the above upper bound
for their orders equals the order of the subgroup generated by filtration-0 classes, we conclude that the orders of the monomials in (2) are precisely 2 and 8, respectively, and so the term $4X_1^M X_2^{2t-2M-1}$ in (2) is nonzero, contradicting the immersion.

3. tmf-cohomology of $P^\infty \times P^\infty$

In this section, we compute tmf$^*(P^\infty)$ and tmf$^8(P^\infty \times P^\infty)$ in positive gradings. These will be used in the next section in studying the axial class in tmf-cohomology.

There is an element $c_4 \in \pi_8(\text{tmf})$ which reduces to $v_1^4 \in \pi_8(\text{bo})$; it has Adams filtration 4. It acts on tmf$^*(X)$ with degree $-8$. Recall also that $\pi_4(\text{bo}) = \text{bo}_*$ is as depicted in Diagram 5.1. We denote $\text{bo}^* = \text{bo}_*$. We use $P_1$ and $P^\infty$ interchangeably.

Theorem 3.1. There is an element $X \in \text{tmf}^8(P_1)$ of Adams filtration 0, described in [6], such that, in positive dimensions divisible by 8, tmf$^*(P_1)$ is isomorphic as an algebra over $\mathbb{Z}(2)[c_4]$ to $\mathbb{Z}(2)[c_4][X]$. In particular, each tmf$^8(P_1)$ with $i > 0$ is a free abelian group with basis $\{c_4^i X^{i+j} : j \geq 0\}$. There is a class $L \in \text{tmf}^6(P_1)$ such that

- tmf$^6(P_1)$ is a free abelian group with basis $\{L, c_4^i X^j : j \geq 1\}$, and
- $L^2 = 2L$ and $LX = 2X$.

Moreover, in positive dimensions tmf$^*(P_1)$ is isomorphic as a graded abelian group to bo$^*[X]$, and is depicted in Diagram 3.4.

Remark 3.2. No claim is made about the action of elements of tmf$_*$ other than $c_4$ on tmf$^*(P_1)$. A complete description of tmf$^*(P_1)$ as a graded abelian group could probably be obtained using the analysis in the proof which follows, together with the computation of the $E_2$-term of the ASS converging to tmf$_*(P_{-1})$, which was given in [10]. However, this is quite complicated and unnecessary for this paper, and so will be omitted.

Proof. We begin with the structure as a graded abelian group. There are isomorphisms

$$\text{tmf}^*(P_1) \approx \lim_{\to} \text{tmf}^*(P^n_1) \approx \lim_{\to} \text{tmf}_{-s-1}(P_{-n-1}) = \text{tmf}_{-s-1}(P_{-2})_2. \tag{3}$$

Since $H^*(\text{tmf}; \mathbb{Z}_2) \approx A/\mathbb{A}_2$, there is a spectral sequence converging to $\text{tmf}_*(X)$ with $E_2(X) = \text{Ext}_{A_2}(H^*X, \mathbb{Z}_2)$. Here $A_2$ is the subalgebra of the mod 2 Steenrod algebra $A$ generated by $\text{Sq}^1$, $\text{Sq}^2$, and $\text{Sq}^4$. Also $\mathbb{Z}_2 = \mathbb{Z}/2$.

We compute $E_2(P_{-2})$ from the exact sequence

$$E_2^{-1,t}(P_{-2}) \to E_2^{-1,t}(P_{-2}) \to E_2^{0,t}(P_{-2}) \to E_2^{0,t}(P_{-2}) \to E_2^{0,t}(P_{-2}). \tag{4}$$

It was proved in [18] that

$$\text{Ext}_{A_2}(P_{-\infty}, \mathbb{Z}_2) \approx \bigoplus_{i \in \mathbb{Z}} \text{Ext}_{A_1}(\Sigma^8i^{-1}\mathbb{Z}_2, \mathbb{Z}_2).$$

Here we have initiated a notation that $P_n^m := H^*(P^n_m)$. A complete calculation of $\text{Ext}_{A_2}(P_{-\infty}, \mathbb{Z}_2)$ was performed in [10], but all we need here are the first few groups. We can now form a chart for $E_2(P_{-2})$ from (4), as in Diagram 3.3, where $\phi$ indicate
elements of $\text{Ext}_{A_2}(P_{-\infty}, \mathbb{Z}_2)$ suitably positioned, and lines of negative slope correspond to cases of $q_* \neq 0$ in (4).

**Diagram 3.3.** $\text{tmf}^*(P_{-\infty}^{-2}), -17 \leq * \leq 2$:

![Diagram 3.3](image)

Dualizing, we obtain Diagram 3.4 for the desired $\text{tmf}^*(P_{1}^{\infty})$.

**Diagram 3.4.** $\text{tmf}^*(P_{1}^{\infty}), * \geq -2$:

![Diagram 3.4](image)

Naming of the generators $X^i$ is clear since $X$ has filtration 0. The free action of $c_4$ is also clear. The class $L$ is (up to sign) the composite $P_1 \overset{\lambda}{\to} S^0 \to \text{tmf}$, where $\lambda$ is the well-known Kahn-Priddy map. Thus $L$ is the image of a class $\hat{L} \in \pi^0(P_1)$. Lin’s theorem ([17]) says that $\pi^0(P_1) \approx \mathbb{Z}^2$, generated by $\hat{L}$. Since $\pi^0(P_1) \to ko^0(P_1)$ is an isomorphism, and, since $(1 - \xi)^2 = 2(1 - \xi)$ for a generator $(1 - \xi)$ of $ko^0(P_1)$, we obtain $\hat{L}^2 = 2\hat{L}$, and hence also for $L$. We chose the generator to be $(1 - \xi)$ rather than $(\xi - 1)$ to avoid minus signs later in the paper.

To prove the claim about $LX$, first note that, by the structure of $\text{tmf}^8(P_1)$, we must have $LX = p(c_4X)X$ for some polynomial $p$. Multiply both sides by $L$ and apply the result about $L^2$ to get $2LX = p(c_4X)LX$; hence $2p = p^2$, from which we conclude $p = 2$. 

\[\square\]
There is a split short exact sequence of

\[ \text{Theorem 3.5.} \quad \text{In positive dimensions divisible by 8,} \quad \text{tmf}^*(P_1 \times P_1) \text{ is isomorphic as a graded abelian group to a free abelian group on monomials } X_1^iX_2^j \text{ with } i, j > 0 \text{ direct sum with a free } \mathbb{Z}[c_4]-\text{module with basis } \{L_1X_1^i, X_1^iL_2; i \geq 1\}. \text{ The product and } \mathbb{Z}[c_4]-\text{module structure is determined from Theorem 3.1 and} \]

\[ c_4(X_1X_2) = (c_4X_1)X_2 = X_1(c_4X_2) = \sum_{i \geq 0} \gamma_ic_4^1(L_1X_2^{i+1} + X_1^{i+1}L_2), \]

\[ \text{for certain integers } \gamma_i \text{ with } \gamma_0 \text{ divisible by 8.} \]

The proof of this theorem involves a number of subsidiary results. They and it occupy the remainder of this section. We will use duality and exact sequences similar to (4), but to get started, we need Ext.

\[ \text{Proof.} \quad \text{The } \mathbb{Z}_2 \text{ is, of course, the subgroup generated by } x^0, \text{ which is an } A\text{-submodule.} \]

A splitting morphism \( P \otimes P \rightarrow \mathbb{Z}_2 \otimes P \) is defined by \( g(x_1^i \otimes x_2^j) = x_1^0 \otimes x_2^{i+j}. \) This is \( A\)-linear since

\[ g(Sq^k(x_1^i \otimes x_2^j)) = \sum_{\ell} \binom{i}{\ell} \binom{j}{k-\ell} x_1^0 \otimes x_2^{i+j+k} \]

\[ = \binom{i+j}{k} x_1^0 \otimes x_2^{i+j+k} = Sq^k g(x_1^i \otimes x_2^j). \]

The following result is more substantial. We will prove it at the end of this section.

\[ \text{Lemma 3.6 (9).} \quad \text{There is a split short exact sequence of } A\text{-modules} \]

\[ 0 \rightarrow \mathbb{Z}_2 \otimes P \rightarrow P \otimes P \rightarrow (P/\mathbb{Z}_2) \otimes P \rightarrow 0. \]

\[ \text{Proof.} \quad \text{A splitting morphism } P \otimes P \rightarrow \mathbb{Z}_2 \otimes P \text{ is defined by } g(x_1^i \otimes x_2^j) = x_1^0 \otimes x_2^{i+j}. \]

\[ \text{This is } A\text{-linear since} \]

\[ g(Sq^k(x_1^i \otimes x_2^j)) = \sum_{\ell} \binom{i}{\ell} \binom{j}{k-\ell} x_1^0 \otimes x_2^{i+j+k} \]

\[ = \binom{i+j}{k} x_1^0 \otimes x_2^{i+j+k} = Sq^k g(x_1^i \otimes x_2^j). \]

The following result is more substantial. We will prove it at the end of this section.

\[ \text{Proposition 3.7.} \quad \text{There is a short exact sequence of } A_2\text{-modules} \]

\[ 0 \rightarrow C \rightarrow (P/\mathbb{Z}_2) \otimes P \rightarrow B \rightarrow 0, \]

where \( C \) has a filtration with

\[ F_p(C)/F_{p-1}(C) \approx \Sigma^{8p} A_2/\text{Sq}^2, \quad p \in \mathbb{Z}, \]

and \( B \) has a filtration with

\[ F_p(B)/F_{p-1}(B) \approx \bigoplus_{\mathbb{Z} \text{ copies}} \Sigma^{8p-2} A_2/\text{Sq}^1, \quad p \in \mathbb{Z}. \]

The generator of \( F_p(C)/F_{p-1}(C) \) is \( x_1^{1/2}x_2^{8p-1}; \) a basis over \( \mathbb{Z}_2 \) for \( C \) is

\[ \{x_1^{2i+2}x_2^{2j+4}; x_1^{4i}x_2^{4j}; x_1^{8i}x_2^{4j}; i \in \mathbb{Z}\} \]

\[ \cup \{x_1^{2i}x_2^{i-1} + x_1^{4i}x_2^{j-2}, i \neq 0(8)\} \cup \{x_1^{2i}x_2^{8p-1} \in \mathbb{Z}\}. \]

A minimal set of generators as an \( A_2\)-module for the filtration quotients of \( B \) is

\[ \{x_1^{8i-1}x_2^{4j-1}; i, j \in \mathbb{Z}\}. \]
Corollary 3.8. A chart for $\text{Ext}^{s,t}_{A_2}(P \otimes P, \mathbb{Z}_2)$ in $8p - 3 \leq t - s \leq 8p + 4$ is as suggested in Diagram 3.9, for all integers $p$. The big batch of towers in each grading $\equiv 2(4)$ represents an infinite family of towers. The pattern of the other classes is repeated with vertical period 4. Thus, for example, in $8p - 1$ there is an infinite tower emanating from filtration 4 for each $i \geq 0$.

Diagram 3.9. $\text{Ext}^{s,t}_{A_2}(P \otimes P, \mathbb{Z}_2)$ in $8p - 3 \leq t - s \leq 8p + 4$:

Proof of Corollary 3.8. We first note that $\text{Ext}^{*,*}_{A_2}(P, \mathbb{Z}_2)$ is identical to the left portion of Diagram 3.3 extended periodically in both directions. Also,

$$\text{Ext}^{*,*}_{A_2}(A_2/Sq^1, \mathbb{Z}_2) \approx \text{Ext}^{*,*}_{A_0}(\mathbb{Z}_2, \mathbb{Z}_2)$$

is just an infinite tower, and

$$\text{Ext}^{*,*}_{A_2}(A_2/Sq^2, \mathbb{Z}_2) \approx \text{Ext}^{*,*}_{A_1}(A_1/Sq^2, \mathbb{Z}_2)$$

is given as in Diagram 3.10. We will show at the end of this proof that

$$\text{Ext}^{*,*}_{A_2}(C, \mathbb{Z}_2) \approx \bigoplus_{p \in \mathbb{Z}} \text{Ext}^{*,*}_{A_2}(\Sigma^{8p}A_2/Sq^1, \mathbb{Z}_2)$$

(5)

and similarly

$$\text{Ext}^{*,*}_{A_2}(B, \mathbb{Z}_2) \approx \bigoplus_{p} \bigoplus_{\mathbb{Z}} \text{Ext}^{*,*}_{A_2}(\Sigma^{4p-2}A_2/Sq^1, \mathbb{Z}_2).$$
These would follow by induction on $p$ once you get started, but since $p$ ranges over all integers, that is not automatic.

Thus $\text{Ext}_{A_2}(P \otimes P, \mathbb{Z}_2)$ is formed from

$$\text{Ext}_{A_2}(P, \mathbb{Z}_2) \oplus \bigoplus_{p} \text{Ext}_{A_2}(\Sigma^8 p A_2/Sq^2, \mathbb{Z}_2) \oplus \bigoplus \text{Ext}_{A_2}(\Sigma^{4p-2} A_2/Sq^1, \mathbb{Z}_2),$$

using the sequences in Lemma 3.6 and Proposition 3.7. The Ext sequence of 3.6 must split, and there are no possible boundary morphisms in the Ext sequence of 3.7, yielding the claim of the corollary.

To prove (5), let $(s, t)$ be given, and choose $p_0$ so that $8p_0 < t - 23s + 2$. Since the highest degree element in $A_2$ is in degree 23, $\text{Ext}_{A_2}^{s,t}(F_{p_0}(C), \mathbb{Z}_2) = 0$. Actually a much sharper lower vanishing line can be established, but this is good enough for our purposes. Thus, for this $(s, t)$,

$$\text{Ext}_{A_2}^{s,t}(F_{p_1}(C), \mathbb{Z}_2) \approx \bigoplus_{p \leq p_1} \text{Ext}_{A_2}^{s,t}(\Sigma^8 p A_2/Sq^2)$$

for $p_1 \leq p_0$, as both are 0. Let $p_1$ be minimal such that (6) does not hold. Then comparison of exact sequences implies that

$$\text{Ext}_{A_2}^{s,t-1}(F_{p_1-1}(C), \mathbb{Z}_2) \rightarrow \text{Ext}_{A_2}^{s,t}(F_{p_1}(C)/F_{p_1-1}(C), \mathbb{Z}_2)$$

must be nonzero. But one or the other of these groups is always 0, as both charts $\text{Ext}_{A_2}^{s,t}(F_{p_1-1}(C), \mathbb{Z}_2)$ and $\text{Ext}_{A_2}^{s,t}(F_{p_1}(C)/F_{p_1-1}(C), \mathbb{Z}_2)$ are copies of Diagram 3.10 displaced by four vertical units from one another. Thus (6) is true for all $p_1$, and hence (5) holds. A similar proof works when $C$ is replaced by $B$.

Diagram 3.10. $\text{Ext}_{A_2}(A_2/Sq^2, \mathbb{Z}_2)$:

Now we can prove a result which will, after dualizing, yield Theorem 3.5. The groups $\text{Ext}_{A_1}(\mathbb{Z}_2, \mathbb{Z}_2)$ to which it alludes are depicted in Diagram 5.1. The content of this result is pictured in Diagram 3.14.

**Proposition 3.11.** In dimensions $t - s \equiv 2 \mod 4$ with $t - s \leq -10$, the chart of $\text{Ext}_{A_1}(P^{-\infty} \otimes P^{-\infty}, \mathbb{Z}_2)$ consists of $i$ infinite towers emanating from filtration 0 in dimensions $-8i - 6$ and $-8i - 10$, together with the relevant portion of two copies of $\text{Ext}_{A_1}(\mathbb{Z}_2, \mathbb{Z}_2)$ beginning in filtration 1 in each dimension $-8i - 2$. The generators of

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1Actually this is not quite true; for one family of elements we need to use $h_0$-naturality.
the towers in $-8i - 10$ correspond to cohomology classes $x_1^{-9}x_2^{-8i-1}, \ldots, x_1^{-8i-1}x_2^{-9}$. The generators of the two copies of $\text{Ext}^A_1(Z_2, Z_2)$ in $-8i - 2$ arise from $h_0$ times classes corresponding to $x_1^{-1}x_2^{8i-1}$ and $x_1^{-8i-1}x_2^{-1}$.

**Proof.** Using exact sequences like (4) on each factor, we build

$$\text{Ext}^{*,*}_A(P_{-\infty} \otimes P_{-\infty}, Z_2)$$

from

$$A := \text{Ext}^{*,*}_A(P \otimes P, Z_2), \quad B := \text{Ext}^{*,*}_A(P_{-1} \otimes P, Z_2),$$

$$C := \text{Ext}^{*,*}_A(P \otimes P_{-1}, Z_2), \quad D := \text{Ext}^{*,*}_A(P_{-1} \otimes P_{-1}, Z_2),$$

with possible $d_1$-differential from $A$ and into $D$. In the range of concern, $t - s \leq -9$, the $D$-part will not be present, and the part of Diagram 3.9 in dimension $\not\equiv 2 \mod 4$ will not be involved in $d_1$. Using [18] for $B$ and $C$, the relevant part, namely the portion of $A$ in dimension $\equiv 2 \mod 4$, together with $B$ and $C$, is pictured in Diagram 3.12.

**Diagram 3.12.** Portion of $A + B + C$:

In dimension $8p - 2$, the towers in $A$ arise from all classes $x_1^{-8i-1}x_2^{-8j-1}$ with $i + j = -p$, while in dimension $8p + 2$, they arise from

$$x_1^{8i-1}x_2^{8j+3} \sim x_1^{8i+3}x_2^{-8j-1}.$$

The finite towers in $B$ arise from $x_1^{4i-1}x_2^{8j-1}$ with $i \geq 0$, and those from $C$ from
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\( x_1^{8i-1} x_2^{4j-1} \) with \( j \geq 0 \). The homomorphism

\[ \text{Ext}^0_{A_2}(P \otimes P, \mathbb{Z}_2) \rightarrow \text{Ext}^0_{A_2}(P^\infty \otimes P, \mathbb{Z}_2) \oplus \text{Ext}^0_{A_2}(P \otimes P^\infty, \mathbb{Z}_2), \]

which is equivalent to the \( d_1 \)-differential mentioned above, sends classes to those with the same name. In dimension \( \leq -10 \), this is surjective, with kernel spanned by classes with both components \( < -1 \). In dimension \(-8i - 6\) and \(-8i - 10\), there will be \( i \) such classes. We illustrate by listing the classes in the first few gradings:

- \( -14: x_1^{-9} x_2^{-5} \sim x_1^{-5} x_2^{-9} \)
- \( -18: x_1^{-9} x_2^{-9} \)
- \( -22: x_1^{-17} x_2^{-5} \sim x_1^{-13} x_2^{-9}, x_1^{-9} x_2^{-13} \sim x_1^{-5} x_2^{-17} \)
- \( -26: x_1^{-17} x_2^{-9}, x_1^{-9} x_2^{-17} \).

These kernel classes yield infinite towers emanating from filtration 0.

For each \( p < 0 \), the towers arising from \( x_1^{4j-1} x_2^{8p-1}, j \geq 0 \), in \( A \) combine with those in the \( p \)-summand of

\[ B \cong \bigoplus_{p \in \mathbb{Z}} \text{Ext}_{A_1}(\Sigma^{8p-1} P^\infty_{-1}, \mathbb{Z}_2), \]

as in Diagram 3.13 to yield one of the copies of \( \text{Ext}_{A_1}(\mathbb{Z}_2, \mathbb{Z}_2) \) arising from filtration 1. An identical picture results when the factors are reversed.

\[ \square \]

**Diagram 3.13.** Part of \( \text{Ext}^2_{A_1}(P^{-2}_\infty \otimes P^{-2}_\infty, \mathbb{Z}_2) \):

\[ \text{Putting things together, we obtain that in dimensions less than } -8, \]

\[ \text{Ext}^2_{A_1}(P^{-2}_\infty \otimes P^{-2}_\infty, \mathbb{Z}_2) \]

consists of a chart described in Proposition 3.11 and partially illustrated in Diagram 3.14 together with the classes in Diagram 3.9, which are not part of the infinite sums of towers in dimension \( \equiv 2 \mod 4 \).
Diagram 3.14. Illustration of Proposition 3.11:

\[
\begin{align*}
-26 & \quad -18 & \quad -10 \\
\end{align*}
\]

The only possible differentials in the Adams spectral sequence of \( P_{-2}^{-\infty} \wedge P_{-2}^{-\infty} \wedge \text{tmf} \) involving the classes in dimensions \( 8p - 2 \) with \( p < 0 \) are from the towers in \( 8p - 1 \) in Diagram 3.9, but these differentials are shown to be 0 as in [6, p. 54]. Similarly to (3), we have

\[
\text{tmf}^*(P_1 \wedge P_1) \approx \text{tmf}_{-4-2}(P_{-2}^{-\infty} \wedge P_{-2}^{-\infty}),
\]

and so we obtain a turned-around version of Diagram 3.14, of the same general sort as Diagram 3.4, as a depiction of a relevant portion of \( \text{tmf}^*(P_1 \wedge P_1) \), with the labeled columns in Diagram 3.14 corresponding to cohomology gradings 24, 16, and 8.

The classes \( X_i^1X_j^2 \) described in Theorem 3.5 are detected by the S-duals of the classes from which the filtration-0 towers in dimensions \( 8p - 2 \) in Diagram 3.14 arise, and so they can be chosen to be the corresponding elements of \( \text{tmf}^8(P_1 \wedge P_1) \). Similarly, the classes \( L_1X_i^1 \) and \( X_i^1L_2 \) have Adams filtration 1, and so one would anticipate that they represent the duals of the generators of the two towers in dimension \( 8p - 2 \) with \( p < 0 \) in Diagram 3.14. This seems a bit harder to prove using the Adams spectral sequence; however, the Atiyah-Hirzebruch spectral sequence shows this quite clearly. The class \( X_1^1 \) is detected by \( H^{8i}(P_1; \pi_0(\text{tmf})) \), while \( L \) is detected by \( H^1(P_1; \pi_1(\text{tmf})) \). Under the pairing, their product is detected in \( H^{8i+1}(P_1; \pi_1(\text{tmf})) \), clearly of Adams filtration 1.

The last part of Theorem 3.5 deals with the action of \( c_4 \) on the monomials \( X_1^1X_2^2 \). Since \( \text{tmf} \) is a commutative ring spectrum, \( \text{tmf}^*(P_1 \wedge P_1) \) is a graded commutative algebra over \( \text{tmf}^* \). The action \( c_4(X_1X_2) \) must be of the form

\[
\sum_{i \geq 0} \gamma_i c_4^i(L_1X_2^1 + X_1^1L_2)
\]

as these are the only elements in \( \text{tmf}^8(P_1 \wedge P_1) \), and the class must be invariant under reversing factors. The divisibility of \( \gamma_0 \) by 8 follows since \( c_4 \) has Adams filtration 4.

Having just completed the proof of Theorem 3.5, we conclude this section with the postponed proof of Proposition 3.7.
Proof of Proposition 3.7. Let $C$ denote the $A_2$-submodule of $(\mathcal{P}/\mathbb{Z}_2) \otimes \mathcal{P}$ generated by all $x_1^{8p-1}x_2^e$, $p \in \mathbb{Z}$. Note that $\text{Sq}^2(x_1^{8p-1}x_2^e) = \text{Sq}^4 \text{Sq}^6(x_1^{8p-9}x_2^e)$. Thus a basis of $A_2/\text{Sq}^2$ acting on all $x_1^{8}x_2^e$ spans $C$. The 24 elements in a basis of $A/\text{Sq}^2$ acting on $x_1^{8}x_2^e$ yield

\[
x_1^{x_1^8}, \quad x_1^{x_1^8} + x_1^{x_2^8}, \quad x_1^{x_2^8} + x_1^{x_1^8}, \quad x_1^{x_1^8} + x_1^{x_2^8}, \quad x_1^{x_2^8} + x_1^{x_1^8},
\]

These classes with second components shifted by all multiples of 8 exactly comprise for $C$ described in the proposition.

The procedure to establish the structure of $B = (\mathcal{P}/\mathbb{Z}_2) \otimes \mathcal{P})/C$ is similar but more elaborate. For the 32 elements $\theta$ in a basis of $A_2/\text{Sq}^4$, we list $\theta(x_1^{8}x_2^e)$ and $\theta(x_1^{8}x_2^e)$. Then we show that these, with each component allowed to vary by multiples of 8, together with $C$, fill out all of $(\mathcal{P}/\mathbb{Z}_2) \otimes \mathcal{P}$.

It is convenient to let $Q$ denote the quotient of $(\mathcal{P}/\mathbb{Z}_2) \otimes \mathcal{P}$ by $C$ and all elements $\theta(x_1^{8}x_2^{j-1})$ and $\theta(x_1^{8}x_2^{j+3})$. We will show $Q = 0$. This will complete the proof of Proposition 3.7, implying in particular that $\text{Sq}^1(x_1^{8}x_2^{j-1})$ and $\text{Sq}^1(x_1^{8}x_2^{j+3})$ are decomposable over $A_2$.

A separate calculation is performed for each mod 8 value of the degree. Here we use repeatedly that the $A_2$-action on $x^i$ depends only on $i$ mod 8. We illustrate with the case in which degree $\equiv 0$ mod 8. The other seven congruences are handled similarly, although some are a bit more complicated.

A basis of $A_2/\text{Sq}^4$ in degree $\equiv 0$ mod 8 acting on $x_1^{8}x_2^e$ yields the following elements:

\[
x_1^{x_1^8} + x_1^{x_2^8}, \quad x_1^{x_2^8} + x_1^{x_1^8}, \quad x_1^{x_1^8} + x_1^{x_2^8}, \quad x_1^{x_2^8} + x_1^{x_1^8}, \quad x_1^{x_1^8} + x_1^{x_2^8}, \quad x_1^{x_2^8} + x_1^{x_1^8},
\]

We will use these relations to show that all classes (in degree $\equiv 0$ mod 8) are 0 in $Q$. First, $R_8$ implies that all classes $X_8$ are congruent to one another. Since $X_0$ is 0 in the
quotient due to \( P/\mathbb{Z}_2 \), we conclude that all classes \( X_{8i} \) are zero in \( \mathbb{Q} \). Next, \( R_4 \) implies that all \( X_{8i+4} \) are congruent to one another. Since \( X_4 + X_8 \in C \), and we have just shown that \( X_8 \equiv 0 \) in \( \mathbb{Q} \), we deduce that all \( X_{8i+4} \) are zero in \( \mathbb{Q} \). Now we use \( R_2 + R_7 \) to see that all \( X_{8i+2} + X_{8i+4} \) are congruent to one another, then that \( X_2 + X_4 \in C \) to deduce all \( X_{8i+2} + X_{8i+4} \equiv 0 \), and finally the result of the previous sentence to conclude all \( X_{8i+2} \equiv 0 \). Then \( R_2 \) implies all \( X_{8i+6} \equiv 0 \). Now \( R_1 + R_3 + R_5 \), together with relations previously obtained, implies all \( X_{8i+1} \) are congruent to one another, and since \( X_1 \in C \), we conclude all \( X_{8i+1} \equiv 0 \). Finally, \( R_1 \) implies \( X_{8i-1} \equiv 0 \), \( R_6 \) implies \( X_{8i+5} \equiv 0 \), and then \( R_3 \) implies \( X_{8i+3} \equiv 0 \).

\[ \square \]

4. Careful treatment of the axial class

In this section, we fill the gap in the proof in [6] of its Theorem 1.1 by careful consideration of the possible “other terms” in the axial class discussed in the introduction. We show that, at least as far as the monomials \( c X_1^i X_2^j \) in its powers are concerned, the axial class equals \( u(X_1 + X_2) \), where \( u \) is a unit in \( \text{tmf}^0(P^\infty \times P^\infty) \). Thus the \( \ell \)-th power of the axial class is nonzero in \( \text{tmf}^{8k}(P^n \times P^m) \) if and only if \( (X_1 + X_2)^\ell \) is nonzero there, and the latter is the condition which yielded the nonimmersions of \([6, 1.1]\). Thus we have a complete proof of \([6, 1.1]\).

If \( P^n \times P^m \xrightarrow{f} P^{m+k} \) is an axial map, then there is a commutative diagram

\[
\begin{array}{ccc}
P^n \times P^m & \xrightarrow{f} & P^{m+k} \\
\downarrow & & \downarrow \\
P^\infty \times P^\infty & \xrightarrow{g} & P^\infty \end{array}
\]

where \( g \) is the standard multiplication of \( P^\infty \), since \( P^\infty = K(\mathbb{Z}_2, 1) \). Since \( X \in \text{tmf}^{8k}(P^{m+k}) \) has been chosen to extend over \( P^\infty \), we obtain that \( f^* (X) \) is the restriction of \( g^* (X) \). By Theorem 3.5 and the symmetry of \( g \), we must have

\[
g^* (X) = X_1 + X_2 + \sum_{i \geq 0} \kappa_i c_i (L_1 X_1^{i+1} + X_2^{i+1} L_2), \tag{7}
\]

for some integers \( \kappa_i \). This is what we call the “axial class.” Then \( g^* (X^\ell) \) equals the \( \ell \)-th power of (7). Using the formulas for \( L_1^2, L_1 X_1, \) and \( c_4(X_1 X_2) \) in Theorems 3.1 and 3.5 and the binomial theorem, this \( \ell \)-th power can be written in terms of the basis described in 3.5. If some \( \kappa_i \)'s are nonzero, then the coefficients of \( X_1^i X_2^{\ell-i} \) in \( g^* (X^\ell) \) will not equal \( \ell \), as was claimed in [6]. We will study this possible deviation carefully.

One simplification is to treat \( L_1 \) and \( L_2 \) as being just 2. Note that \( L_i \) acts like 2 when multiplying by \( X_i \), and if, for example, \( L_1 \) is present without \( X_1 \), then the terms \( c_i L_1 X_2^j \) cannot cancel our \( X_1^i X_2^\ell \)-classes because both are separate parts of the basis. You have to carry the terms along, because they might get multiplied by an \( X_1 \), and then it is as if \( L_1 = 2 \). We will incorporate this important simplification throughout the remainder of this section.
For example, one easily checks that, using $L_2^2 = 2L_1$ and $L_1X_1 = 2X_1$, we obtain

$$(X_1 + X_2 + L_1X_2)^4 = (X_1 + 3X_2)^4 - 80X_2^4 + 40L_1X_2^4.$$  

The exponent of 2 in each monomial of $(X_1 + 3X_2)^4 - 80X_2^4$ is the same as that in $(X_1 + X_2)^4$, and $L_1X_2^4$ is a separate basis element.

With this simplification, the axial class in (7) becomes

$$X_1 + X_2 + 2\sum_{i>0} \kappa_i c^4_i (X_1^{i+1} + X_2^{i+1})$$  

for some integers $\kappa_i$. There was another term $2\kappa_0(X_1 + X_2)$, but it can be incorporated into the leading $(X_1 + X_2)$. The odd multiple that it can create is not important.

From Theorem 3.5, we have

$$c^4(X_1X_2) = 16(X_1 + X_2) + 2\sum_{k>0} \gamma_k c^4_k (X_1^{k+1} + X_2^{k+1}),$$  

for some integers $\gamma_k$. The 16 comes from $\gamma_0 = 8$ and $L_i = 2$. Actually we do not really know that $\gamma_0 = 8$, even just up to multiplication by a unit, but it is divisible by 8 and the possibility of equality must be allowed for. This gives

$$c^4(X_1^{i+1}X_2^{j+1}) = 16(X_1^{i+1}X_2^j + X_1^iX_2^{j+1}) + 2\sum_{k>0} \gamma_k c^4_k (X_1^{i+k+1}X_2^j + X_1^iX_2^{j+k+1}).$$  

(10)

Here we use that in a graded $\text{tmf}^*$-algebra $\text{tmf}^*(X)$ with even-degree elements, $c(xy) = cx \cdot y$, for $c \in \text{tmf}^*$ and $x, y \in \text{tmf}^*(X)$.

There is an iterative nature to the action of $c_4$ in (10), but the leading coefficient 16 enables us to keep track of 2-exponents of leading terms in the iteration. (As observed above, the leading coefficient might be an even multiple of 16, which would make the terms even more highly 2-divisible. We assume the worst, that it equals 16.) We obtain the following key result about the action of $c_4$ on monomials in $X_1$ and $X_2$.

**Theorem 4.1.** There are 2-adic integers $A_i$ such that

$$c_4 = \sum_{i>0} 2^{4+i} A_i \left( \frac{1}{X_1} \left( \frac{X_2}{X_1} \right)^i + \frac{1}{X_2} \left( \frac{X_1}{X_2} \right)^i \right).$$

**Remark 4.2.** This formula will be evaluated on (i.e. multiplied by) monomials $X_1^iX_2^j$. One might worry that the negative powers of $X_1$ or $X_2$ in Theorem 4.1 will cause nonsensical negative powers in $c_4X_1^iX_2^j$. This will, in fact, not occur because the monomials on which we act always have total degree greater than the dimension of either factor. Thus if, after multiplication by $c_4$, a term with negative exponent of $X_i$ appears, then the accompanying $X_{3-i}$-term will be 0 for dimensional reasons.
Proof of Theorem 4.1. The defining equation (9) may be written, with \( \theta = c_4 \sqrt{X_1 X_2} \) and \( z = \sqrt{X_1/X_2} \), as
\[
\theta = 16(z + z^{-1}) + \sum_{i>0} 2\gamma_i \theta^i (z^{i+1} + z^{-(i+1)}).
\]
(11)

Let \( p_i = z^i + z^{-i} \). We will show that
\[
\theta = \sum_{i \geq 0} 2^{4+i} A_i p_{2i+1}
\]
for certain 2-adic integers \( A_i \), which interprets back to the claim of 4.1.

Note that \( p_i p_j = p_{i+j} + p_{|i-j|} \), and hence
\[
p_1 e_1 \cdots p_k e_k = p_{\Sigma e_i} + \mathcal{L},
\]
where \( \mathcal{L} \) is a sum of integer multiples of \( p_j \) with \( j < \sum i e_i \) and \( j \equiv \sum i e_i \mod 2 \).

We will ignore for awhile the coefficients \( \gamma_i \) which occur in (11). This is allowable if we agree that when collecting terms, we only make crude estimates about their 2-divisibility. We have
\[
\theta = 16p_1 + 2\theta p_2 + 2\theta^2 p_3 + 2\theta^3 p_4 + \cdots
\]
\[
= 16p_1 + 2p_2(16p_1 + 2p_2(16p_1 + \cdots)) + 2p_3(16p_1 + \cdots)^2 + \cdots
\]
\[
+ 2p_3(16p_1 + 2p_2(16p_1 + \cdots) + \cdots)^2 + \cdots.
\]

Note that the only terms that actually get evaluated must end with a 16\( p_1 \) factor.

Now let \( T_1 = 16p_1 \) and, for \( i \geq 2 \), let \( T_i = 2\theta^{i-1} p_i \). Each term in the expansion of \( \theta \) involves a sequence of choices. First choose \( T_i \) for some \( i \geq 1 \), and then if \( i > 1 \) choose \((i-1)\) factors \( T_j \), one from each factor of \( \theta^{i-1} \). For each of these \( T_j \) with \( j > 1 \), choose \( j - 1 \) additional factors, and continue this procedure. This builds a tree, and we do not get an explicit product term until every branch ends with \( T_1 \). Each selected factor \( T_j \) with \( j > 1 \) contributes a factor \( 2p_j \). There will also be binomial coefficients and the omitted \( \gamma_i \)'s occurring as additional factors.

For example, Diagram 4.3 illustrates the choices leading to one term in the expansion of \( \theta \). This yields the term \( 2p_2 \cdot 2p_4 \cdot 16p_1 \cdot 2p_2 \cdot 16p_1 \cdot 2p_1 \cdot 2p_2 \cdot 16p_1 \), which equals \( 2^{21}(p_{17} + \mathcal{L}) \), where \( \mathcal{L} \) is a sum of \( p_i \) with \( i < 17 \) and \( i \) odd. By induction, one sees in general that the sum of the subscripts emanating from any node, including the subscript of the node itself, is odd.

Diagram 4.3. A possible choice of terms:

```
    T_1
   /   \
 T_2 - T_4  T_3 - T_2 \\
     /      \
    T_1   T_1
```

The important terms are those in which \( T_2 \) is chosen \( k \) times \( (k \geq 0) \) and then \( T_1 \) is chosen. These give \( (2p_2)^k p_1 \) with no binomial coefficient. This term is...
2^{k+4}(p_{2k+1} + \mathcal{L}). Note that a term $2^{k+4}p_{2k+1}$ with $i < k$ obtained from $\mathcal{L}$ will be more 2-divisible than the $2^{i+4}p_{2i+1}$ term that was previously obtained. Thus it may be incorporated into the coefficient of that term.

All other terms will be more highly 2-divisible than these. For example, the first would arise from choosing $T_3$ then two copies of $T_1$. This would give $2p_3 \cdot 2^4p_1 \cdot 2^4p_1 = 2^9p_5 + \mathcal{L}$, and the $2^9p_5$ can be combined with the $2^6p_5$ obtained from choosing $T_2$ then $T_2$ then $T_1$. Incorporating $\gamma_i$'s may make terms even more divisible, but the claim of (12) is only that $p_{2i+1}$ occurs with coefficient divisible by $2^{4+i}$.

Now we incorporate Theorem 4.1 into (8) to obtain the following key result, which we prove at the end of the section.

**Theorem 4.4.** The monomials $c_iX_1^iX_2^{n-i}$ in the $n$th power of the axial class in $\text{tmf}^{8n}(P^n \times P^n)$ are equal to those in the $n$th power of

$$(X_1 + X_2) \left( u + \sum_{i \geq 1} 2^{4+i} \alpha_i \left( \frac{X_1}{X_2} \right)^i \right),$$

(13)

where $u$ is an odd 2-adic integer and $\alpha_i$ are 2-adic integers.

The factor which accompanies $(X_1 + X_2)$ in (13) is a unit in $\text{tmf}^*(P^n \times P^n)$; we referred to it earlier as $u$. Indeed, its inverse is a series of the same form, obtained by solving a sequence of equations. This justifies the claim in the first paragraph of this section regarding retrieval of the nonimmersions of $[6, 1.1]$.

We must also observe that restriction to $\text{tmf}^{8\ell}(P^n \times P^n)$ of the non-$X_1^1X_2^{\ell-i}$ parts of the basis of $\text{tmf}^{8\ell}(P^n \times P^n)$ cannot cancel the $X_1^1X_2^{\ell-i}$ terms essential for the non-immersion. This is proved by noting that these elements such as $L_1X_2^{1}$ and $c_iL_1X_2^{j+i}$ will restrict to a class of the same name in $\text{tmf}^{8\ell}(P^n \times P^n)$, and will be 0 there for dimensional reasons, since $8\ell > n$.

**Proof of Theorem 4.4.** Let $g^*(X)$ denote the axial class as in (7). From (8) and Theorem 4.1, the difference $g^*(X) - (X_1 + X_2)$ equals

$$2 \sum_{i \geq 1} \kappa_i (X_1^{i+1} + X_2^{i+1})2^{ti} \left( \sum_{j \geq 0} 2^j A_j \left( \frac{1}{X_1} \left( X_2 \right)^j + \frac{1}{X_2} \left( X_1 \right)^j \right) \right)^i.$$  

We let $z = \sqrt{X_1/X_2}$ and $p_j = z^j + z^{-j}$ as in the proof of 4.1.

The summand with $i = 2t$ becomes

$$2\kappa_i (X_1 + X_2) \sum_{s} X_1^{2t-s} X_2^s 2^{ti} \left( \sum_{j \geq 0} 2^j A_j p_{2j} \right)^i = 2\kappa_i (X_1 + X_2)(p_{2t} + \mathcal{L})2^{4t} \sum_k c_k 2^k (p_{2k+1} + \mathcal{L}).$$

Here $k$ is a sum of $j$-values taken from the various factors in the $i$th power. Also, in $p_j + \mathcal{L}$, $\mathcal{L}$ denotes a combination of $p_i$'s with $t < j$. Noting $(p_{2t} + \mathcal{L})(p_{2k+1} + \mathcal{L}) =$
\[ p_{2k+2i} + L, \text{ this becomes} \]
\[
2(X_1 + X_2)2^{4i} \sum c_k 2^k (p_{2k+2i} + L). \tag{14}
\]

The argument when \( i = 2t + 1 \) is similar but slightly more complicated because \( (X_1^{i+1} + X_2^{i+1}) \) is not divisible by \( (X_1 + X_2) \). We obtain
\[
2\kappa_i \frac{X_1 + X_2}{(\sqrt{X_1X_2})^{2t+1}} 2^{4i} \sum c_k 2^k (p_{2k+1} + L)^i.
\]

For one of the factors of the \( i \)th power, say the first, we treat \( p_{2j+1} \) as \( \frac{X_1 + X_2}{\sqrt{X_1X_2}} (p_{2j+1} + L) \).

The expression then becomes
\[
2(X_1 + X_2)p_{i+1} 2^{4i} \sum c_k 2^k (p_{2k+i+1} + L),
\]
where \( k \) is obtained as in the previous case. We again obtain (14).

Thus when \( g^*(X) - (X_1 + X_2) \) is written as \( (X_1 + X_2) \sum \beta_j p_{2j} \), the coefficient \( \beta_j \) satisfies \( \nu(\beta_j) \geq (j-1) + 4 + 1 \). Here the \( (j-1) + 4 \) comes from the case \( i = 1 \), \( k = j-1 \) in (14), and the extra +1 is the factor 2 which has been present all along. This yields the claim of (13). \( \square \)

5. \textbf{tmf-cohomology of} \( CP^\infty \times CP^\infty \)

In [2, 4], and [8], it was noted, first by Astey, that the axial class using \( BP \) (or \( BP\langle 2 \rangle \)) was \( u(X_2 - X_1) \), where \( u \) is a unit in \( BP^*(RP^\infty \wedge RP^\infty) \). In this section, we review that argument and consider the possibility that it might be true when \( BP \) is replaced by \( tmf \), which would render the considerations of the previous section unnecessary. To do this, we calculate \( tmf^*(CP^\infty) \) and \( tmf^*(CP^\infty \times CP^\infty) \) in positive dimensions. (See Theorems 5.13 and 5.16.) Although our conclusion will be that Astey’s \( BP \)-argument cannot be adapted to \( tmf \), nevertheless these calculations may be of independent interest.

We begin by reviewing Astey’s argument. Whereas in previous sections we have used \( P \) to denote real projective spaces, in this section we use \( RP \), to distinguish them from complex projective spaces, which are denoted by \( CP \). There is a commutative diagram
\[
\begin{array}{cccccc}
RP^\infty & \xrightarrow{d_R} & RP^\infty \times RP^\infty & \xrightarrow{m_R} & RP^\infty \\
h \downarrow & & h \times h \downarrow & & h \downarrow \\
CP^\infty & \xrightarrow{d_C} & CP^\infty \times CP^\infty & \xrightarrow{m_C} & CP^\infty \\
1 \times (-1) \downarrow & & 1 \downarrow & & \\
CP^\infty \times CP^\infty & \xrightarrow{m_C} & CP^\infty.
\end{array}
\]

The generator \( X_R \in BP^2(RP^\infty) \) satisfies \( X_R = h^*(X) \). We also have that
\[
m_C \circ (1 \times (-1)) \circ d_C
\]
is null-homotopic. The key fact, which will fail for tmf, is

$$BP^*(CP^\infty \times CP^\infty) \approx BP^*[X_1, X_2].$$

The axial class is $m^*_R(X_R)$. It equals $(h \times h)^*(1 \times (-1))^*m^*_C(X)$, but

$$(1 \times (-1))^*m^*_C(X) \in \ker(d^*).$$

By the above “key fact,” $d^*$ is the projection $BP^*[X_1, X_2] \to BP^*[X]$ in which each $X_i \mapsto X$. The kernel of this projection is the ideal $(X_2 - X_1)$. To see this, just note that in grading $2n$ a kernel element must be $\sum c_i X_1^i X_2^{n-i}$ with $\sum c_i = 0$, and hence

$$\sum c_i(X_1^i X_2^{n-i} - X_1^n) = \sum c_i X_1^i (X_2 - X_1) \sum X_1^j X_2^{n-i-1-j}.$$  

Thus $(1 \times (-1))^*m^*_C(X) = (X_2 - X_1)u$ for some $u \in BP^*(CP^\infty \times CP^\infty)$. This $u$ is a unit by consideration of its reduction to $H^*(-; \mathbb{Z})$, as in [2]. Since $h^*(u)$ will then be a unit in $BP^*(RP^\infty \times RP^\infty)$ and $h^*(X_1) = X_{R_1}$, we obtain the claim about the axial class being a unit times $X_{R_2} - X_{R_1}$.

In order to see if there is any chance of adapting this to tmf, we compute $tmf^*(CP^\infty)$ and $tmf^*(CP^\infty \times CP^\infty)$ in positive gradings. We begin with the relevant Ext calculations.

Let $bo = Ext^*_{A_1}(\mathbb{Z}_2, \mathbb{Z}_2)$. Recall that a chart for this is given as in Diagram 5.1, extended with period $(t - s, s) = (8, 4)$.

Diagram 5.1. $Ext^*_{A_1}(\mathbb{Z}_2, \mathbb{Z}_2)$:

![Diagram 5.1](image)

Let $M_{10}$ denote the $A_2$-module $(1, Sq^4, Sq^2 Sq^4, Sq^4 Sq^2 Sq^4)$.

**Lemma 5.2.** There is an additive isomorphism

$$Ext^*_{A_1}(M_{10}, \mathbb{Z}_2) \cong bo[v_2],$$

where $v_2 \in Ext^{1,7}(-)$.

Thus the chart for $Ext^*_{A_2}(M_{10}, \mathbb{Z}_2)$ consists of a copy of $bo$ shifted by $(t - s, s) = (6i, i)$ units for each $i \geq 0$. 

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Proof. There is a short exact sequence of $A_2$-modules

$$0 \to \Sigma^7 M_{10} \to A_2//A_1 \to M_{10} \to 0.$$  

This yields a spectral sequence which builds $\text{Ext}^{*,*}_{A_2}(M_{10}, \mathbb{Z}_2)$ from

$$\bigoplus_{i \geq 0} \text{Ext}^{*-i,*-7i}_{A_2}(A_2//A_1, \mathbb{Z}_2).$$

Since $\text{Ext}^{*,*}_{A_2}(A_2//A_1, \mathbb{Z}_2) \approx \text{bo}$, one easily checks that there are no possible differentials in this spectral sequence.

Let $C^m_n = H^*(CP^m_n; \mathbb{Z}_2)$.

Theorem 5.3. There is an additive isomorphism

$$\text{Ext}^{*,*}_{A_2}(C^\infty_{-\infty}, \mathbb{Z}_2) \approx \bigoplus_{p \in \mathbb{Z}} \Sigma^{8p-2}\text{bo}[v_2].$$

Of course $\Sigma$ applied to a module or an Ext group just means to increase the $t$-grading by 1.

Proof. There is a filtration of $C^\infty_{-\infty}$ with $F_p/F_{p-1} \approx \Sigma^{8p-2}M_{10}$ for $p \in \mathbb{Z}$. We have $\text{Sq}^2 \iota_{8p-2} = \text{Sq}^4 \text{Sq}^2 \text{Sq}^4 \iota_{8p-10}$. The same argument used in the last paragraph of the proof of Corollary 3.8 works to initiate an inductive proof of the Ext-isomorphism claimed in the theorem.

Corollary 5.4. In gradings $(t-s)$ less than $-1$,

$$\text{Ext}^{*,*}_{A_2}(C^{-2}_{-\infty}, \mathbb{Z}_2) \approx \bigoplus_{p < 0} \Sigma^{8p-2}\text{bo}[v_2].$$

Proof. There is an exact sequence

$$\to \text{Ext}^{s-1,t}_{A_2}(C^{-1}_{-\infty}, \mathbb{Z}_2) \to \text{Ext}^{s,t}_{A_2}(C^{-2}_{-\infty}, \mathbb{Z}_2) \to \text{Ext}^{s,t}_{A_2}(C^\infty_{-\infty}, \mathbb{Z}_2) \overset{q_*}{\to} \text{Ext}^{s,t}_{A_1}(C^\infty_{-1}, \mathbb{Z}_2).$$

The result is immediate from this and Theorem 5.3, since $q_*$ sends the initial tower in $F_0/F_{-1}$ isomorphically to the initial tower in $\text{Ext}^{s,t}_{A_1}(C^\infty_{-1}, \mathbb{Z}_2)$.

The $A$-modules $C^\infty_1$ and $\Sigma^2 C^{-2}_\infty$ are dual. Thus, by [9, Prop. 4],

$$\text{Ext}^{s,t}_{A_2}(Z_2, C^\infty_1) \approx \text{Ext}^{s,t}_{A_2}(\Sigma^2 C^{-2}_\infty, Z_2).$$

There is a ring structure on $\text{Ext}^{*,*}_{A_2}(Z_2, C^\infty_1)$. We deduce the following result, which is pictured in Diagram 5.10.

Corollary 5.5. In $(t-s)$ gradings $\leq 0$, there is a ring isomorphism

$$\text{Ext}^{*,*}_{A_2}(Z_2, C^\infty_1) \approx \text{bo}[v_2][X],$$

where $X \in \text{Ext}^{0,-8}$.  

Proof. We apply the duality isomorphism to 5.4. The multiplicative structure is obtained from the observation that the powers of the class in $\text{Ext}^{0,-8}$ equal the class in $\text{Ext}^{0,-8i}$ for each $i > 0$. 

\begin{flushright}$\square$\end{flushright}
The Ext groups computed here are $E_2$ of the ASS converging to $\text{tmf}^{-s}(CP^\infty)$. We will consider the differentials in this spectral sequence after performing the Ext calculation relevant for $\text{tmf}^{-s}(CP^\infty \times CP^\infty)$.

Now we consider $C_{-\infty}^- \otimes C_{-\infty}^-$, and let $x_1$ and $x_2$ denote elements of $H^2(CP^\infty; Z_2)$. Let $E_2$ denote the exterior subalgebra generated by the Milnor primitives of grading 1, 3, and 7. Note that $A_2 // E_2$ has a basis with elements of grading 0, 2, 4, 6, 8, 10, and 12. Finally we note that for any $j \equiv -2 \mod 8$ with $j \leq -10$, there is a nontrivial $A_2$-morphism $C_{-\infty}^- \xrightarrow{\rho} \Sigma^j Z_2$.

**Lemma 5.6.** Let

$$K = \ker(C_{-\infty}^- \otimes C_{-\infty}^- \xrightarrow{\rho} C_{-\infty}^- \otimes \Sigma^{-10} Z_2).$$

Let $S$ denote the set of all classes $x_i^{8j+2} x_j^{8j-2}$ with $i \leq -1$ and $j \leq -2$, together with the classes $x_1^{8j+2} x_2^{8j-2}$ with $i \leq -1$ and $j \leq -1$. Then $K$ is the direct sum of a free $A_2 // E_2$-module on $S$ with a single relation $Sq^4 Sq^2 Sq^4 (x_1^{10} x_2^{-6}) = 0$.

**Proof.** Since the generators of $E_2$ have odd grading, $A_2 // E_2$ acts on any element of these evenly-graded modules. The action of $A_2 // E_2$ on $x_1^{-2} x_2^{-2}$ yields the additional elements $x_1^{-2} x_2^0 + x_1^0 x_2^{-2}, x_1^{-2} x_2^2 + x_1^0 x_2, x_1^0 x_2^2, x_1^0 x_2^4 + x_1^4 x_2, x_1^4 x_2^2, x_1^4 x_2^4$. These actions of $A_2 // E_2$ on $x_1^{-2} x_2^j$ yields the exponents $x_1^0 x_2^2 + x_1^2 x_2^4, x_1^2 x_2^2 + x_1^4 x_2^4, x_1^4 x_2^2$, and $x_1^4 x_2^4$. Each exponent can be decreased by any multiple of 8.

One can easily check that in each grading all classes in $C_{-\infty}^- \otimes C_{-\infty}^-$ are obtained exactly once from the described elements in $K$ together with $C_{-\infty}^- \otimes \Sigma^{-10} Z_2$. There are four cases, for the four even mod 8 values. We illustrate with the case of grading 4 mod 8. We will just consider the specific value $-28$, but it will be clear that it generalizes to all gradings $\equiv 4 \mod 8$. Letting $X_i$ denote $x_1^{28-i}$, we have:

1. From generators in $-28$, we obtain just $X_{-10}$ in $K$. The class $X_{-18}$ is in $C_{-\infty}^- \otimes \Sigma^{-10} Z_2$.
2. From generators in $-32$, we obtain $X_{-8} + X_{-6}$, $X_{-16} + X_{-14}$, and $X_{-24} + X_{-22}$.
3. From generators in $-36$, we obtain $X_{-8} + X_{-4}$ and $X_{-16} + X_{-12}$.
4. From generators in $-40$, we obtain $X_{-4}, X_{-12} + X_{-8}, X_{-20} + X_{-16},$ and $X_{-24}$.

Note in (4) that $X_0$ and $X_{-28}$ do not appear because each component must be $\leq -4$ and the components sum to $-28$.

One easily checks that the 11 classes listed above, including $X_{-18}$, form a basis for the space spanned by $X_{-4}, \ldots, X_{-24}$, in an orderly fashion that clearly generalizes to any grading $\equiv 4 \mod 8$. A similar argument works in the other three congruences. There are some minor variations in the top few dimensions. \hfill $\square$

Now we dualize. There is a pairing

$$\text{Ext}_{A_2}(Z_2, C_1^\infty) \otimes \text{Ext}_{A_2}(Z_2, C_1^\infty) \to \text{Ext}_{A_2}(Z_2, C_1^\infty \otimes C_1^\infty).$$

Let $X_i$ denote the class in grading $-8$ coming from the $i$th factor. Then we obtain
Theorem 5.7. The algebra $\text{Ext}_{A_2}^{0,*}(\mathbb{Z}_2, C_1^\infty \otimes C_1^\infty)$ in gradings $\leq -8$ is isomorphic to $\mathbb{Z}_2[X_1, X_2](X_1 X_2, y_{-12})$ with $y_{-12}^2 = X_1^2 X_2 + X_1 X_2^2$. The monomials of the form $X_1^i X_2^j y_{-12}$ are acted on freely by $\mathbb{Z}_2[v_0, v_1, v_2]$. Let $S_n$ denote the $\mathbb{Z}_2$-vector space with basis the monomials $X_1^i X_2^j$, and define a homomorphism $\epsilon : S_n \rightarrow \mathbb{Z}_2$ by sending each monomial to 1. Then $\mathbb{Z}_2[v_0, v_1, v_2]$ acts freely on $\ker(\epsilon)$, while $\text{bo}[v_2]$ acts freely on $S_n/\ker(\epsilon)$. Thus in dimensions $t - s \leq -8$, $\text{Ext}_{A_2}^{s,*}(\mathbb{Z}_2, C_1^\infty \otimes C_1^\infty)$ has, for each $i > 0$, $i$ copies of $\Sigma^{-si-4} \mathbb{Z}_2[v_0, v_1, v_2]$ and $i$ copies of $\Sigma^{-si-16} \mathbb{Z}_2[v_0, v_1, v_2]$, and also one copy of $\Sigma^{-8i-8} \text{bo}[v_2]$.

Here by $\mathbb{Z}_2[X_1, X_2](X_1 X_2, y_{-12})$ we mean a free $\mathbb{Z}_2[X_1, X_2]$-module with basis $\{X_1 X_2, y_{-12}\}$.

Proof. The structure as graded abelian group is straightforward from Lemma 5.6, Corollary 5.5, and the duality isomorphism

$$\text{Ext}_{A_2}^{s,*}(\mathbb{Z}_2, C_1^\infty \otimes C_1^\infty) \approx \text{Ext}_{A_2}^{s-*}(C_{-\infty}^\infty \otimes C_{-\infty}^\infty, \mathbb{Z}_2).$$

We use that $\text{Ext}_{A_2}(A_2/E_2, \mathbb{Z}_2) \approx \mathbb{Z}_2[v_0, v_1, v_2]$. The reason that we only assert the structure in dimension $\leq -8$ is due to the $\Sigma^{-10}$ in the cokernel part of Lemma 5.6, and that Theorem 5.5 was only valid in dimension $\leq 0$. In the range under consideration, the relation on the top class in Lemma 5.6 does not affect Ext.

The ring structure in filtration 0 comes from $\text{Hom}_{A_2}(\mathbb{Z}_2, C_1^\infty \otimes C_1^\infty)$ being isomorphic to elements of $C_1^\infty \otimes C_1^\infty$ annihilated by $\text{Sq}^2$ and $\text{Sq}^4$, which has as basis all elements $x_1^{4i} \otimes x_2^{4j}$ and $(x_1^{4i} \otimes x_2^{4j})(x_1^{4i} \otimes x_2^{4j} + x_1^{4i} \otimes x_2^{4j})$.

Now we show that $\text{Ext}_{A_2}^{1, -8n+2}(\mathbb{Z}_2, C_1^\infty \otimes C_1^\infty) = \mathbb{Z}_2$, and $h_1$ times each monomial in $\text{Ext}_{A_2}^{1, -8n}(\mathbb{Z}_2, C_1^\infty \otimes C_1^\infty)$ equals the nonzero element here. An element in $\text{Ext}_{A_2}^{1, -8n+2}(\mathbb{Z}_2, C_1^\infty \otimes C_1^\infty) = \mathbb{Z}_2$ is an equivalence class of morphisms

$$\Sigma^2 A_2 \oplus \Sigma^4 A_2 \xrightarrow{h} C_1^\infty \otimes C_1^\infty,$$

which increase grading by $8n - 2$, and yield a trivial composite when preceded by

$$\Sigma^4 A_2 \oplus \Sigma^8 A_2 \xrightarrow{\begin{pmatrix} \text{Sq}^2 & \text{Sq}^6 \\ 0 & \text{Sq}^4 \end{pmatrix}} \Sigma^2 A_2 \oplus \Sigma^4 A_2.$$

Morphisms $h$ which can be factored as

$$\Sigma^2 A_2 \oplus \Sigma^4 A_2 \xrightarrow{\text{Sq}^4, \text{Sq}^4} A_2 \xrightarrow{k} C_1^\infty \otimes C_1^\infty$$

are equivalent to 0 in Ext.

We illustrate with the case $n = 3$. There are $A_2$-morphisms increasing grading by 22 sending either $\Sigma^2 A_2$ or $\Sigma^4 A_2$ to any one of the following classes:

$$x_1^{12} x_2^{12}, x_1^{10} x_2^{10}, x_1^8 x_2^8, x_2^8 x_1^8, x_1^6 x_2^6, x_2^6 x_1^6, x_1^4 x_2^4, x_2^4 x_1^4, x_1^2 x_2^2, x_2^2 x_1^2, x_1 x_2, x_2 x_1.$$ (15)
succession:
\[ x_1^3 x_2^4, x_1^2 x_2^3, x_1^1 x_2^2, x_1^0 x_2^1, x_1^{-1} x_2^0, x_1^{-2} x_2^{-1}, x_1^{-3} x_2^{-2}, x_1^{-4} x_2^{-3}. \]

For example, \((\text{Sq}^2, \text{Sq}^4)(x_1^3 x_2^4) = (x_1^5 x_2^6, x_1^4 x_2^5, x_1^3 x_2^4, x_1^2 x_2^3, x_1^1 x_2^2, x_1^0 x_2^1). \) Thus all classes in (16) are equivalent to one another.

Usual Yoneda product considerations show that \(h_1\) times any monomial \(x_1^n x_2^{n-1}\) equals this nonzero element of \(\text{Ext}^{1,8n+2}_{A_2}(\mathbb{Z}_2, \mathcal{C}_1^\infty \otimes \mathcal{C}_1^\infty)\). Indeed, if
\[
0 \leftarrow \mathbb{Z}_2 \leftarrow C_0 \leftarrow C_1 \leftarrow \]
is the beginning of a minimal \(A_2\)-resolution, with \(C_1 = \Sigma^1 A_2 \oplus \Sigma^2 A_2 \oplus \Sigma^4 A_2\), then \(h_1 x_1^n x_2^{n-1}\) is represented by the composite \(C_1 \rightarrow C_0 \rightarrow \mathcal{C}_1^\infty \otimes \mathcal{C}_1^\infty\) sending \(v_2 \mapsto \iota \mapsto x_1^n x_2^{n-1}\), and this is equivalent to the element described in the previous paragraph.

Here is a schematic way of picturing Theorem 5.7. We first list the generators in grading greater than \(-32\). Then for each of the two types of generators, we list the structure arising from them in the first ten dimensions. The \(\text{bo}\{v_2\}\)-structure in the left half of Diagram 5.9 arises from one tower in dimensions \(-24\) and \(-16\), while the \(\mathbb{Z}_2\{v_0, v_1, v_2\}\)-structure in the right half of Diagram 5.9 arises from the other towers in Diagram 5.8.

**Diagram 5.8. Generators of \(\text{Ext}_{A_2}(\mathbb{Z}_2, \mathcal{C}_1^\infty \otimes \mathcal{C}_1^\infty)\):**

**Diagram 5.9. Structure on two types of generators:**
Now we consider the differentials in the ASS converging to $\text{tmf}^*(CP^\infty)$ and then for $\text{tmf}^*(CP^\infty \wedge CP^\infty)$. The gradings are negated when considered as tmf-cohomology groups. Corollary 5.5 gives the $E_2$-term converging to $[\Sigma^* CP_1^\infty, \text{tmf}] \approx \text{tmf}^{-*}(CP_1^\infty)$. We will maintain the homotopy gradings until just before the end. In Diagram 5.10, we depict a portion of the $E_2$-term of this ASS in gradings $-16$ to $1$. There are also classes in higher filtration arising from powers of $v_1^4$ and $v_2$ acting on generators in lower grading. The elements indicated by $\bullet$'s are involved in differentials, as explained later.

**Diagram 5.10.** A portion of $E_2$ for $[\Sigma^* CP^\infty, \text{tmf}]$:

We will prove the following key result about differentials in this ASS.

**Theorem 5.11.** The nonzero differentials in the ASS converging to $[\Sigma^* CP^\infty, \text{tmf}]$, $* < 1$, are given by

\[ d_2(h_1^i v_1^{4i} v_2^j X^{-2k+1}) = h_1^{i+1} v_1^{4i} v_2^{j+1} X^{-2k} \]

for $\epsilon = 0, 1$, $i, j \geq 0$, $k \geq 1$.

Here $h_1$, $v_1^4$, and $v_2$ have the usual $\text{Ext}^{s,t}$ gradings $(s,t) = (1,2), (4,12), (1,7)$, respectively.

Diagram 5.10 pictures the situation for $k = 1$ and small values of $i$ and $j$. The elements indicated by $\bullet$'s are involved in the differentials. The resulting picture is nicer if the filtrations of all classes built on $X^{-2k+1}$ are increased by $1$. There is a nontrivial extension (multiplication by $2$) in dimension $-6$ due to the preceding differential. This is equivalent to the way that $bu_*$ is formed from $bo_*$ and $\Sigma^2 bo_*$. We obtain Diagram 5.12 from Diagram 5.10 after the differentials, extensions, and filtration shift are taken into account.
Diagram 5.12. Diagram 5.10 after differentials and filtration shift.

The regular sequence of towers in the chart beginning in filtration 1 in dimension $-10$ is interpreted as $v_i^j v_2$, $i \geq 0$.

After negating dimensions to switch to cohomology indexing, we obtain the following result, which is immediate from Theorem 5.11 after the extensions such as just seen are taken into account.

Theorem 5.13. In positive gradings, there is an isomorphism of graded abelian groups

$$\text{tmf}^*(CP_1^\infty) \approx \mathbb{Z}_{(2)}[Z_{16}](b_0^* \oplus v_2 \mathbb{Z}_{(2)}[v_1, v_2]).$$

Here $Z_{16} \in \text{tmf}^{16}(CP_1^\infty)$, and $|v_1| = -2$ and $|v_2| = -6$.

Recall that $b_0^* = b_0_{\infty}$ with $b_0_{*}$ as suggested in Diagram 5.1. Much of the ring structure of $\text{tmf}^*(CP_1^\infty)$ is described in 5.13, since $b_0_{*}$ and $v_2 \mathbb{Z}_{(2)}[v_1, v_2]$ are rings, and it is quite clear how to multiply an element in $b_0_{*}$ by one in $v_2 \mathbb{Z}_{(2)}[v_1, v_2]$. Because of the filtration shift that led to the identification of some of the classes in $v_2 \mathbb{Z}_{(2)}[v_1, v_2]$, we hesitate to make any complete claims about the ring structure.

A complete computation of $\text{tmf}^*(CP_1^\infty)$ was made in [5]; see especially Theorem 7.1 and Diagram 7.1. At first glance, the two descriptions appear quite different, but they seem to be compatible.

Proof of Theorem 5.13. We first prove that there is a nontrivial class in $[\Sigma^{-16}CP_1^\infty, \text{tmf}]$ detected in filtration 0. This is obtained by using the virtual bundle $8(H - 1) - (H^3 - H)$, where $H$ denotes the complex Hopf bundle. Considered as a real bundle $\theta$, this bundle satisfies $w_2(\theta)$ and $p_1(\theta) = 0$. Here we use from [19] that $p_1$ generates the infinite cyclic summand in $H^4(BSO; \mathbb{Z})$ and satisfies $\rho^*(p_1) = c_1^2 - 2c_2$ under $BU \xrightarrow{\rho} BSO$, and $\rho^*(p_1) = 2e_1$ under $BSpin \xrightarrow{\rho} BSO$, where $H^4(BSpin; \mathbb{Z})$ is
an infinite cyclic group generated by $e_1$. The total Chern class of $9H - H^3$ is 
\[ (1 + x)^9 (1 + 3x)^{-1} = 1 + 6x + 18x^2 + \cdots, \]
and hence 
\[ r^*(p_1(\theta)) = (c_1(9H - H^3))^2 - 2c_2(9H - H^3) = (6x)^2 - 2 \cdot 18x^2 = 0. \]
Thus $e_1(\theta) = 0$, hence $CP^\infty \xrightarrow{\theta} BSpin \to K(\mathbb{Z}, 4)$ is trivial, and so $\theta$ lifts to a map $CP^\infty \to BO[8]$. Hence its Thom spectrum induces a degree-1 map $T(\theta) \to MO[8]$. Since $\psi^3(H) = H^3 - H$, by [20] $\theta$ is $J(2)$-equivalent to $S(H - 1)$, and hence its Thom spectrum is $T(S(H - 1)) = \Sigma^{-16}CP_8^\infty$. Using the Ando-Hopkins-Rezk orientation ([1]) $MO[8] \to tmf$, we obtain our desired class as the composite 
\[ \Sigma^{-16}CP_1^\infty \xrightarrow{col} \Sigma^{-16}CP_8^\infty \xrightarrow{T(\theta)} MO[8] \to tmf. \] (17)

We will deduce our differentials from the $d_3$-differential $E_3^{4,21} \to E_3^{7,23}$ in the ASS converging to $\pi_8(tmf)$. This can be seen in [14, p. 37] or [11, Theorem 2.2]; see Remark 5.14 for additional explanation. It is not difficult to show that, with $M_{10}$ as in Lemma 5.2, the morphism 
\[ Ext_{A_2}^{s,t}(Z_2, Z_2) \to Ext_{A_2}^{s,t}(M_{10}, Z_2), \]
induced by the nontrivial $A_2$-map $M_{10} \to Z_2$, sends the $Z_2$ in $Ext_{A_2}^{7,23}(Z_2, Z_2)$ which is not part of the infinite tower to $h_1^2v_4^3v_2$.

We prefer to think about the ASS for $tmf_*(\Sigma^2CP_\infty^\infty)$, which, as we have noted, is isomorphic to that of $[\Sigma^*CP_1^\infty, tmf]$. The $E_2$-term was described in 5.4. Let $S^{-16} \to \Sigma^2CP_\infty^\infty \wedge tmf$ correspond to the map in (17). Since $E_2(CP_\infty^\infty \wedge tmf)$ in negative dimensions is built from copies of $Ext_{A_2}(M_{10}, Z_2)$, we deduce from the previous paragraph that $h_1^2v_4^3v_2g_{-16}$ in the ASS for $tmf_*(\Sigma^2CP_\infty^\infty)$ must be hit by a $d_2$- or $d_3$-differential, since it is the image of a class hit by a $d_3$. The only possibility is that it be $d_2$ from $h_1v_4^3g_{-8}$, as indicated by the dotted line in Diagram 5.10.

Naturality of differentials with respect to $h_1$ and $v_4^3$ implies the differentials of 5.11 for $\epsilon = 0, 1$, all $i, j = 0$, and $k = 1$. Using the diagonal map of $CP_1^\infty$ and the multiplication of $tmf$, powers of (17) give similar nontrivial elements in $[\Sigma^{-16k}CP_1^\infty, tmf]$ for all $k \geq 1$, and by the argument just presented, we establish the differentials of Theorem 5.11 for all $k$ (with $j = 0$ still).

The only possible differentials on $v_2g_{-16}$ would be some $d_r$ with $r > 2$ hitting an element which is acted on nontrivially by $h_1$. However $h_1v_2g_{-16}$ has become 0 in $E_3$ since it was hit by a $d_2$-differential. Thus a nonzero differential on $v_2g_{-16}$ would contradict naturality of differentials with respect to $h_1$-action. Hence there is a map $S^{-10} \to \Sigma^2CP_\infty^\infty \wedge tmf$ hitting $v_2g_{-16}$, and the argument of the previous paragraph implies that $d_2(h_1v_4^3v_2g_{-8}) = h_1v_4^3v_2g_{-16}$ and then other related differentials. This now establishes the differentials of 5.11 when $j = 1$, and sets in motion an inductive argument to establish these differentials for all $j \geq 1$.

No further differentials in the spectral sequence are possible, by dimensional and $h_1$-naturality considerations.

Remark 5.14. The proof of the key $d_3$-differential in the ASS of $tmf$ from the 17-stem to the 16-stem, which was cited above, has not had a thorough proof in the literature.
Giambalvo’s original argument was incorrect and his correction merely refers to “a homotopy argument.” The current authors cited Giambalvo’s result in [11] without additional argument. We provide some more detail here regarding this differential.

The relevant portion of the ASS of tmf appears in Diagram 5.15. In [11] and [14], this was pictured as the ASS of $MO[8]$, but through dimension 18,

$$\text{Ext}^*_A(H^*(MO[8]), \mathbb{Z}_2) \approx \text{Ext}^*_A(\mathbb{Z}_2 \oplus \Sigma^{16}\mathbb{Z}_2, \mathbb{Z}_2).$$

One way of obtaining the differentials from 15 to 14, as in [14], is to note that the $[8]$-cobordism group of 14-dimensional manifolds is $\mathbb{Z}_2$, and so the top two elements must be killed by differentials. It is not difficult to compute in Ext the Massey product formula $B = \langle A, h^0, h^1 \rangle$, where $A$ and $B$ are as in Diagram 5.15. This can be seen as $v^1_4$ times a similar formula between classes in dimensions 6 and 8. Since $A$ is 0 in homotopy, the associated Toda bracket formula says that $B$ must be divisible by $\eta$. But only 0 can be divisible by $\eta$ in dimension 16 here. Thus $B$ must be killed by a differential, and the depicted way is the only way this can happen.

**Diagram 5.15. Portion of ASS of tmf:**

The differentials in the ASS converging to $\text{tmf}_*(CP^\infty_1 \wedge CP^\infty_1)$ are implied by the same considerations that worked for $CP^\infty_1$. The $\mathbb{Z}_2[v_0, v_1, v_2]$-parts in Theorem 5.7 cannot support differentials by dimensionality and $h_1$-naturality. For the $bo$-like part, we prefer thinking about it as $[\Sigma^{s+4}CP^\infty_1 \wedge CP^\infty_1, \text{tmf}] \approx \text{tmf}^{s-4}(CP^\infty_1 \wedge CP^\infty_1)$, where the product structure is more apparent.

Let $Z_n$ denote the nonzero element of $\text{Ext}^A_*(\mathbb{Z}_2, C^\infty_1 \otimes C^\infty_1)/\ker(h_1)$. By Theorem 5.7, $Z_n$ can be represented by $X^i_1 X^{n-i}_1$ for any $1 \leq i < n$. If $n$ is even and $n \geq 4$, choosing $i$ even, $Z_n$ is an infinite cycle because it is an external product of infinite cycles. Hence by the proof of Theorem 5.11,

$$d_2(h^1_1 v^4_1 v^2_2 Z_{2k-1}) = h^{1+1}_1 v^4_1 v^2_2 Z_{2k}$$

for $\epsilon = 0, 1$, $i, j \geq 0$, and $k \geq 2$. 
Finally, $X_1X_2$ is an infinite cycle since there is nothing that it can hit. Also, $h_1v_2X_1X_2$ and $h_1^2v_2X_1X_2$ are not hit by differentials since

$$\text{Ext}^{0,-8}_\mathbb{A}_2(\mathbb{Z}_2, \mathbb{C}_1^\infty \otimes \mathbb{C}_1^\infty) = 0$$

by Theorem 5.7. We obtain the following.

**Theorem 5.16.** In grading $\geq 10$, there is an isomorphism of graded abelian groups

$$\text{tmf}^*(CP_1^\infty \vee CP_2^\infty) \cong y\mathbb{Z}_2[v_1, v_2, X_1, X_2] \oplus \bigoplus_{n \geq 3} I_n \cdot \mathbb{Z}_2[v_1, v_2] \oplus \mathbb{Z}_2[Z](bo^* \oplus v_2\mathbb{Z}_2[v_1, v_2]),$$

where $|y| = 12$, $|X_1| = 8$, $|Z| = 16$, $|v_1| = -2$, and $|v_2| = -6$. Here $I_n = \ker(F_n \xrightarrow{\epsilon} \mathbb{Z})$, where $F_n$ is a free abelian group with basis $\{X|X_2^{n-i}: 1 \leq i < n\}$, and $\epsilon(X_2^{n-i}) = 1$.

Thus $I_n$ consists of all polynomials of grading $n$ with sum of coefficients equal to 0. We could have extended the description in 5.16 down to grading 8, but the description would have been slightly more complicated, since it would include $h_1v_2Z$ and $h_1^2v_2Z$.

The motivation for this section was to see if perhaps $\ker(\text{tmf}^*(CP_1^\infty \vee CP_2^\infty))$ might be something nice like the $I(X_1 - X_2)$, which was the case for $BP^*(-)$. In Theorem 5.16, we described $\text{tmf}^*(CP_1^\infty \vee CP_2^\infty)$. To obtain $\text{tmf}^*(CP_1^\infty \times CP_2^\infty)$, we add on two copies of $\text{tmf}^*(CP_1^\infty)$, which was described in Theorem 5.13. Denote by $Z_1$ and $Z_2$ the generators in $\text{tmf}^16(CP_1^\infty \times CP_2^\infty)$. Monomials $Z_1^nZ_2^{n-i}$ should equal $Z^n$ of 5.16 plus perhaps elements of $I_{2n}$ of 5.16. The class $y$ of 5.16 plus perhaps a sum of elements of higher filtration is in $\ker(d^*)$ and not in the ideal generated by $(Z_1 - Z_2)$. Thus, as expected, $\ker(d^*)$ does not have the nice form that it did for $BP^*(-)$, and so we cannot use this argument to show that the axial class in $\text{tmf}^*(RP_1^\infty \times RP_2^\infty)$ is $u(X_1 - X_2)$. However, we showed something like this by a completely different method in Theorem 4.4. We feel that the results obtained in Theorems 5.13 and 5.16 should be of independent interest.

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