Nonlinear elliptic equations
with high order singularities

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Abstract
We develop a non-variational approach to the theory of elliptic equations with high order singular structures. No boundary data are imposed and singularities occur along an a priori unknown interior region. We prove that positive solutions have a universal modulus of continuity that does not depend on their infimum value. We further obtain sharp, quantitative regularity estimates for non-negative physical solutions. This solves the well known continuity conjecture for weak solutions to elliptic partial differential equations with high order singularities. The results established in this work are new even in for isotropic equations. Of particular interest, it is shown that solutions to symmetric minimal surface type of equations, $|\nabla u|^K \Delta u \approx u^{-1}$, are uniformly Lipschitz continuous along their singular sets, for all $K \geq 0$.

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1 Introduction
In this paper we establish geometric regularity estimates for nonnegative limiting solutions to singular, anisotropic partial differential equations of the form

$u^\alpha \cdot |Du|^\beta \cdot \Delta u = f(X) \cdot \chi_{\{u>0\}}$,

with $0 \leq \beta < +\infty$, $-(1+\beta) < \alpha < +\infty$ and $f$ bounded away from zero and infinity. Solutions are built up through a limiting process out from positive solutions, or alternatively, by solutions of non-singular approximating equations. Compactness for such a family of solutions will be established
by a universal equicontinuity estimate that is independent of their infimum value. Such a device unlocks the study of non-variational solutions to elliptic equations with high order singular structures.

The theory of singular elliptic equations and its innate backgrounds arise in several areas of pure and applied analysis: theory of fluid mechanics, superconductivity, dynamics of thin films, quenching phenomena, microelectromechanical type of systems, rupture problems, geometric measure theory, calculus of variations, differential geometry, free boundary theory, etc. Many of those problems can be modeled in a general framework:

$$\mathcal{L}(X, u, Du, D^2u) = f(X, u), \quad \Omega \subset \mathbb{R}^n,$$

(1.1)

where $\mathcal{L}$ is elliptic with respect to the Hessian dependence; however ellipticity degenerates along the zero set $\mathcal{Z}(u) := \{X \in B_1 \mid u(X) = 0\}$ as well as along the set of critical points $\mathcal{S}(u) := \{X \in B_1 \mid Du(X) = 0\}$. The magnitudes of the degeneracy along $\mathcal{Z}(u)$ and $\mathcal{S}(u)$ are measured by parameters $\alpha$ and $\beta$, as

$$\mathcal{L}(X, u, Du, D^2u) \approx u^\alpha |Du|^\beta F(X, D^2u),$$

(1.2)

for $0 < |u|, |Du| \ll 1$, with $F$ uniformly elliptic in $D^2u$. Physical interpretations of the models usually require sign conditions upon the forcing term $f(X, u)$ and impose sign constraint on existing solution $u$, say, $u > 0$; nonetheless, the key estimates are the ones that do not depend on $\inf_{B_1} u$.

The theory of elliptic equations for which ellipticity degenerates solely at the set of critical points, i.e., $\alpha = 0$ in (1.1), has experienced an impressive progress through this past decade, see [7, 12, 14, 15, 3] among several other works on this subject.

Isotropic singular PDEs, i.e. $\beta = 0$, often arise from critical point theory of non-differentiable functionals. Namely, given a real parameter $t$, one can formally look at energy functionals of the form

$$\mathcal{J}_t(u) := \int (|\nabla u|^2 + \lambda(\zeta)u^t\chi_{\{u>0\}}) \, d\zeta,$$

where $\lambda$ is a bounded function. When $t > 1$, the functional is differentiable and usual methods in the Calculus of Variations can be employed. The case $t = 1$ refers to the well known obstacle problem, see [8, 9] among many other works on this subject. For $0 < t < 1$, the functional $\mathcal{J}_t$ is continuous but non-differentiable, see for instance [2]. Within this range solutions are
continuously differentiable along the interface \( \{ u = 0 \} \). The borderline case \( t = 0 \) bears upon the cavitation problem, [1].

When \( t < 0 \), the problem becomes more involved from the mathematical viewpoint. Formally the Euler-Lagrange equation associated to \( \partial_t \) is

\[
\Delta u = \frac{\lambda}{2} t \cdot u^{t-1}, \text{ in } \{ u > 0 \}
\]  

(1.3)

It is not hard to see that solutions to (1.3) may fail to belong to \( H^1 \) if \( t < -2 \); an initial indication that such a higher order singular equation ought to be treated by non-variational means. One can rewrite (1.3) as

\[
u^{1-t} \cdot \Delta u = \frac{\lambda}{2} t \cdot \chi_{\{u>0\}}
\]

(1.4)

The formal Euler-Lagrange Equation (1.4) can be understood as part of a lofty class of non-variational elliptic equations, with higher order singularity of the type

\[
u^\alpha |Du|^\beta \Delta u = f(X) \cdot \chi_{\{u>0\}},
\]

(1.5)

in the sense of approximating limiting solutions, to be better explained when time comes. For now, it is enlightening to notice that for limiting solutions, the vanishing boundary \( \partial \{ u > 0 \} \) is a priori unknown and no smoothness or geometric information is previously granted upon such a set.

Regularity issues related to such a class of equations are central problems in the modern study of non-linear PDEs. The study of the isotropic case, \( \beta = 0 \), itself has promoted important development of the theory of 2nd order elliptic equations in the last forty years or so. A very large literature dealing with positive solutions with zero boundary data has evolved from the pioneering work [11]. In such eruditions, singularities occur along a prescribed, fixed, smooth boundary, \( \partial \mathcal{O} \). Obtaining appropriate regularity estimates for solutions to singular PDEs with no geometric or smoothness a priori knowledge on the singular set has been a primary problem since then.

While for the variational theory, i.e. the study of solutions coming from a minimization problems, there have been some late advances, see for instance [13, 20], the appropriate non-variational theory has hitherto been unaccessible, despite of great activity on the theory of singular elliptic equations. It is worth pointing out that the non-variational approach to high order singular equations is prosperous even for isotropic problems. For instance
it does not request $H^1$-bounds (which in general should not be expected for physical solutions) nor finiteness on $\int u^{-\alpha}$, as to develop an existence and regularity theory. Assuming the latter, $\int u^{-\alpha}$, imposes severe constrains on the Hausdorff dimension of limiting zero sets, see for instance [18].

Within the theory of singular elliptic equation, it is of particular interest the case when $\alpha = \beta = 1$ in equation (1.5). Such a problem is related to the theory of symmetric minimal surfaces. In [19] it has been shown that sufficiently smooth solutions are (universally) of class $C^{0,\frac{1}{2}}$. The argument is based upon a modified Bernstein technique. The striking feature of such a result is that it does not assume regularity of the boundary of the positive region. Notwithstanding, since then it has been conjectured that solutions to the symmetric minimal surface equation should in fact be $C^{0,1}$-smooth, i.e. Lipschitz continuous, along the singular set.

It is worth commenting that a (positive) function $u$ satisfying Equation (1.5) in the viscosity sense is of class $C^{1,\frac{1}{1+\beta}}$ (see [3] and also [14]). Of course such an estimate blows-up as inf $u \to 0$. The ultimate goal of this current paper is to prove that nonnegative limiting solutions to (1.5) are $C^\gamma$ smooth, along their singular set, for the sharp value $\gamma$ given by the algebraic relation

$$\gamma := \frac{2 + \beta}{1 + \beta + \alpha}. \quad (1.6)$$

Interestingly enough, one verifies that when $\alpha = 1$, the optimal exponent $\gamma$ becomes 1, i.e., solutions are Lipschitz continuous, regardless the value of $\beta \geq 0$. Hence, as a byproduct of the results established in this work, we provide a positive answer to the Lipschitz regularity conjecture for solutions to the symmetric minimal surface equation, in great generality. This also solves a long standing problem concerning Hölder regularity estimates for higher order singular equations, i.e., when $\alpha > 1$. For such family of singular equations, it is also alluring to observe that it follows from (1.6) that fixed $\alpha > 1$, solutions become more regular as the value of $\beta$ increases. In principle this observation is counterintuitive.

We conclude this Introduction by mentioning that the methods designed for the proof of our main Theorems presented in this work are, in their very own nature, of non-linear character. The very same universal continuity property as well as the sharp regularity estimate hold true, with essentially the same proofs, if Laplacian is replaced by a uniform elliptic fully non-linear operator $F(D^2u)$. Of course, in this case, the universal estimates pass to
depend also upon the ellipticity constants of $F$. We have chosen to work on the simpler case, as to highlight the novelties and main ideas herein designed for the proofs of such a results.

2 Approximating singular PDEs

Anisotropic equations as (1.2) are genuinely non-variational, even when diffusion is ruled by the Laplacian. Hence, the notion of solutions cannot be based upon the language of measures or distributions. Rather, the appropriate notion of approximating solutions rests upon the method of building up physical nonnegative solutions as the limit of positive ones. This is done by means of uniform estimates that do not depend upon the infimum of the solutions. We comment that, alternatively, one could consider nonsingular approximating equations, and obtain estimates that do not deteriorate as the smoothing parameter tends to zero. More precisely, given a real number $\phi \in \mathbb{R}$ and a positive parameter $\epsilon > 0$, we define the continuous real cut-off function $\zeta_{\phi, \epsilon}: [0, \infty) \rightarrow [\epsilon^\phi, \infty)$ by

$$
\zeta_{\phi, \epsilon}(s) := \begin{cases} 
s^\phi & \text{if } s \geq \epsilon \\
\epsilon^\phi & \text{if } s < \epsilon.
\end{cases} \quad (2.1)
$$

One could now look at regularity estimates, uniform-in-$\epsilon$, for bounded solutions to

$$
\zeta_{\alpha, \epsilon}(u) \cdot |D u|^\beta \cdot \Delta u = f(X) \quad (2.2)
$$

in $O \subset \mathbb{R}^n$, with $f(X)$ bounded. In either cases, a universal compactness result paves the way for the theory of limiting solutions. This is our first main goal in this current article.

Definition 1. A nonnegative function $u \in C(O)$ is said to be a limiting solution to the non-variational singular equation

$$
u_0^\alpha \cdot |D u_0|^\beta \cdot \Delta u_0 = f(X) \chi_{\{u_0 > 0\}} \quad (2.3)
$$

if there exists a sequence of positive functions $u_j$, satisfying

$$
u_j^\alpha \cdot |D u_j|^\beta \cdot \Delta u_j = f(X)
$$

in the viscosity sense, converging locally uniformly to $u_0$ in $O$. 

We comment that fixed $\varepsilon > 0$ and prescribed a continuous boundary datum $g \in C(\partial \Omega)$, Perron’s type method (c.f. [16], [10] and [21]) assures that Equation (2.2) has a continuous viscosity solution $u_\varepsilon$. The key issue – to be addressed here – is whether the family of viscosity solutions generated by Perron’s method is uniformly equicontinuous.

Regarding the nonsingular approximating equation, we should point out that there is nothing special about the cut-off function $\zeta_{\alpha,\varepsilon}$ defined in (2.1). The key analytical feature of the singular approximation is that
\[
t^{-\alpha} \cdot \zeta_{\alpha,\varepsilon}(t) \leq C,
\]
for a constant $C$ that is independent of $\varepsilon$. That is, our estimates hold for equations with degeneracy that are as strong as $u_\alpha$. Likewise, we could also consider $\varepsilon$-family of equations with varying potentials $f_\varepsilon$, as long as $f_\varepsilon$ remains bounded. An important example is the high energy activation problems
\[
Lu_\varepsilon = \beta_\varepsilon(u_\varepsilon),
\]
where $L$ is an elliptic operator and $\beta_\varepsilon(u) := \frac{1}{\varepsilon} \beta_1 \left( \frac{u}{\varepsilon} \right)$, for a smooth function $\beta_1$ supported on $[0,1]$. Notice that if $u_\varepsilon$ solves (2.5), then it also satisfies
\[
\zeta_{1,\varepsilon}(u_\varepsilon) \cdot Lu_\varepsilon = f_\varepsilon(X),
\]
for $f_\varepsilon(X) := \zeta_{1,\varepsilon}(u_\varepsilon) \beta_\varepsilon(u_\varepsilon) \in L^\infty$, with $\|f_\varepsilon\|_\infty \leq \|\beta_1\|_\infty \leq M$, uniform in $\varepsilon$. Thus, equations involving high energy activation fit into the framework set herein. For a detailed study of anisotropic high energy activation problems, we cite [4].

Positive solutions in smooth, bounded domains can be found by solving a boundary value problem with large boundary data. Indeed, within the region $\{u \geq 1\}$ the equation behaves as $|Du|^\beta \Delta u \in L^\infty$. Applying up to the boundary regularity, one sees that $\{u < 1\}$ is empty if the boundary data is large enough. As to obtain a physical solution with non-empty free boundary, one can then solve boundary value problems for a sequence of decreasing boundary data.

### 3 Hypotheses and main results

We start off this section by commenting on the sign assumption assumed on the forcing term $f(X)$. Clearly if $f \leq 0$, positive solutions are superharmonic...
and then the limiting free boundary is empty. The interesting case, even from applied viewpoint, is when we assume the existence of a constant $c_0 > 0$ such that

$$c_0 \leq f(X) \leq c_0^{-1}. \quad (3.1)$$

Such a condition will be enforced hereafter in this paper. In addition, we shall also assume uniform continuity of $f$. Recall that a modulus of continuity is a nondecreasing function $\sigma : (0, \infty) \to (0, \infty)$ satisfying $\sigma(0^+) = 0$. A function is said to be $\sigma$-continuous in $\Omega \subset \mathbb{R}^n$, if

$$|f(X) - f(Y)| \leq \sigma(|X - Y|).$$

Let us look at the appropriate range for the degeneracy exponents $\alpha$ and $\beta$. We are interested in models for which ellipticity degenerates at critical points as well along free boundary points $\partial \{u_0 > 0\}$. Such a consideration gives the range $\beta \geq 0$ and $\alpha \geq 0$. Physical interpretation of the model indicates that when $\alpha < 0$, in fact a stronger diffusion process takes place within the region where $\inf u_0 \downarrow 0$. Hence there must exist a negative lower bound $\alpha_0$, such that if $\alpha \leq \alpha_0$, then any limiting solution vanishing at a point must be identically zero. When $\beta = 0$, a direct application of strong maximum principle yields $\alpha_0 = -1$. For the anisotropic case, $\beta > 0$ one should expect that such lower bound is less than $-1$, simply because the degeneracy along critical points attenuates the strong diffusion process near the free boundary.

Before presenting the main results proven in this work, let us declare that any constant or entity that depends only on dimension, $\|u\|_\infty$, $\alpha$ and $\beta$, $c_0$ and the modulus of continuity of $f$ will be called universal. We now pass to discuss the central theorems to be delivered in this manuscript. The first key result we prove is a universal compactness theorem for positive solutions to singular equations.

**Theorem 1.** Let $v$ be a bounded positive viscosity solution to

$$v^\alpha |Dv|^{\beta} \Delta v = f(X), \quad (3.2)$$

in $\mathcal{O} \subset \mathbb{R}^n$, with $\beta \geq 0$ and $f$ satisfying (3.1). Given a subdomain $\mathcal{O}' \Subset \mathcal{O}$, there exists a modulus of continuity $\varpi$, depending only on universal parameters and $\mathcal{O}'$, such that $v$ is $\varpi$-continuous in $\mathcal{O}'$. 
The proof of Theorem 1 involves a refinement of the so-called Ishii-Lions method, see for instance [17, 6], and it will be delivered in Section 4. Of particular interest, Theorem 1 allows us to develop the theory of limiting solutions to singular elliptic equations with free boundaries

\[ u_0^\alpha |Du_0|^\beta \Delta u_0 = f(X) \chi_{\{u_0 > 0\}}, \quad \text{in} \quad \mathcal{O} \subset \mathbb{R}^n, \quad (3.3) \]

as forecasted by Definition 1.

Let us now turn our attention to the regularity theory for limiting solutions of the free boundary problem (3.3). It follows from the Theorem proven in [3] that \( u_0 \) is locally of class \( C^{1,1+\beta} \) within its positive set. The major, key issue though is to understand the optimal growth behavior of such a function along the free boundary \( \partial\{u_0 > 0\} \). Such a sharp estimate is given by the next result.

**Theorem 2.** Let \( u_0 \) be a bounded limiting solution to (3.3) with \( \beta \geq 0 \), \( \alpha > -(1+\beta) \) and \( f \) satisfying (3.1). Fixed a subdomain \( \mathcal{O}' \Subset \mathcal{O} \), there exists a constant \( C \geq 1 \), depending only upon universal parameters and \( \text{dist}(\partial \mathcal{O}', \partial \mathcal{O}) \), such that if \( Z \in \partial\{u_0 > 0\} \cap \mathcal{O}' \), then

\[ \sup_{B_r(Z)} u_0 \leq C r^{2+\beta/(1+\beta+\alpha)}, \]

for any \( r < \text{dist}(Z, \partial \mathcal{O}') \).

A direct consequence of Theorem 2 is the sharp control of the value of \( u \) at a point off the free boundary in terms of its distance to the zero set \( \mathcal{Z}(u) := \{u_0 = 0\} \).

**Corollary 3.** Let \( u_0 \) be as in the statement of Theorem 2. Then for any point \( X \in \{u_0 > 0\} \) with \( \text{dist}(X, \mathcal{Z}(u)) < \text{dist}(X, \partial \mathcal{O}) \), there holds

\[ u_0(X) \leq C \cdot \text{dist}(X, \mathcal{Z}(u_0))^{2+\beta/(1+\beta+\alpha)}. \]

We conclude this Section by commenting that nonnegative solutions obtained as limit of Perron’s (minimal) solutions are non-degenerate, i.e.,

\[ \sup_{B_r(Z)} u_0 \sim r^{2+\beta/(1+\beta+\alpha)}, \]
for any free boundary point $Z \in \partial \{u_0 > 0\}$. The proof of this fact follows as in [3, Section 8]. At least for non-degenerate limiting solutions, when $\alpha \geq 0$, a combination of Theorem 2 and [3, Corollary 3.2], gives that $u_0$ is precisely $C^\gamma$ up to the free boundary. Indeed, in such a scenario, for $\text{dist}(Z, \partial \{u_0 > 0\}) \ll 1$, we initially estimate from [3, Corollary 3.2] (see also [14])

$$[u_0]_{C^\gamma(B_{d/4}(Z_0))} \lesssim \frac{1}{d^\gamma} \left( \|u_0\|_{L^\infty(B_{d/2}(Z_0))} + d^{\frac{2+\beta}{1+\beta}} \cdot \|u_0^{-\alpha}\|_{L^\infty(B_{d/2}(Z_0))} \right).$$

(3.4)

Applying Theorem 2 and non-degeneracy respectively, we estimate

$$\|u_0\|_{L^\infty(B_{d/2}(Z_0))} \lesssim d^\gamma$$

and

$$\|u_0^{-\alpha}\|_{L^\infty(B_{d/2}(Z_0))} \lesssim d^{\frac{\alpha}{1+\beta}}.$$ (3.5)

Finally, plugging (3.5) into (3.4), and taking into account the precise value of $\gamma$, (1.6), we verify that the $C^\gamma$ norm of non-degenerate limiting solutions is under control up the the free boundary.

For improved estimates that hold exclusively along the free boundaries, see [23, 24].

4 Universal Compactness

This Section is devoted to the proof of Theorem 1, where it is established the key universal compactness property for solutions, independent of the infimum of $u$.

We restrict the proof to the case $\alpha > 0$, as the complementar range follows from [14, 3]. Fixed $X_0 \in \mathcal{O}$, let us denote by $d := \text{dist}(X_0, \partial \mathcal{O})$; if $\text{dist}(X_0, \partial \mathcal{O}) = +\infty$, then we fix $d$ to be any arbitrary large (but finite) number. With no loss of generality, we will assume $X_0 = 0$. We will show that for any $\delta > 0$ given, there exist $\rho_\delta \ll 1$ and $L_\delta \gg 1$ such that

$$\sup_{\Omega \times \bar{\Omega}} \{v(\rho_\delta X) - v(\rho_\delta Y) - L_\delta \omega(|X - Y|) - \kappa \cdot (|X|^2 + |Y|^2)\} \leq \delta,$$ (4.1)

where $\omega(\zeta) = \begin{cases} \zeta - \frac{1}{10^4} \zeta^{3/2}, & \text{for } \zeta \leq d, \\ \frac{9}{10} d, & \text{for } \zeta \geq d. \end{cases}$. The parameter $\kappa > 0$ is chosen such that

$$\frac{8\|v\|_{L^\infty}}{d^2} \leq \kappa.$$ (4.2)

Notice that in the case $d = +\infty$, then $\kappa > 0$ can be chosen arbitrarily small.
In order to verify (4.1), let us suppose, for the sake of contradiction, that
\[
\sup_{\Omega \times \Omega} \left\{ v(\varrho_\delta X) - v(\varrho_\delta Y) - L_\delta \omega(|X - Y|) - \kappa \cdot (|X|^2 + |Y|^2) \right\} > \delta. \tag{4.3}
\]
We shall reach an inconsistency by selecting \(L_\delta\) big enough and \(\varrho_\delta\) tiny. Let \((\bar{X}, \bar{Y})\) denote a pair of points where such a maximum is attained. It follows from the thesis that
\[
\kappa \cdot (|\bar{X}|^2 + |\bar{Y}|^2) < 2\|v\|_{\infty}, \tag{4.4}
\]
thus, \(\bar{X}\) and \(\bar{Y}\) are interior points and \(|\bar{X} - \bar{Y}| < d\). Clearly \(\bar{X} \neq \bar{Y}\), otherwise we obtain a direct contradiction on (4.3). Define in the sequel the vectors
\[
\xi_X := L_\delta \omega'(|\bar{X} - \bar{Y}|) \eta + 2\kappa \bar{X}, \tag{4.5}
\]
\[
\xi_Y := L_\delta \omega'(|\bar{X} - \bar{Y}|) \eta - 2\kappa \bar{Y}, \tag{4.6}
\]
where \(\eta := \frac{\bar{X} - \bar{Y}}{|\bar{X} - \bar{Y}|}\). Notice that
\[
\omega'(|\bar{X} - \bar{Y}|) \geq \omega'(r) = \frac{17}{20}. \tag{4.7}
\]
Also,
\[
2\kappa \cdot \max \{||X||, ||Y||\} < \frac{8\|v\|_{\infty}}{d}, \tag{4.8}
\]
thus
\[
\xi_X \approx \sigma_0 L_\delta + O\left(\frac{\|v\|_{\infty}}{d}\right), \tag{4.9}
\]
\[
\xi_Y \approx \sigma_0 L_\delta + O\left(\frac{\|v\|_{\infty}}{d}\right), \tag{4.10}
\]
for a nonzero vector \(\sigma_0\). From Jensen-Ishii’s approximation Lemma, see [10, Theorem 3.2], for \(i > 0\) small enough, it is possible to find matrices \(M_X\) and \(M_Y\) with
\[
(\xi_X, M_X) \in J^- (v, \bar{X}), \tag{4.11}
\]
\[
(\xi_Y, M_Y) \in J^+ (v, \bar{Y}), \tag{4.12}
\]
where \(J^-\) and \(J^+\) denote the subjet and superjet respectively, verifying the following matrix inequality
\[
\begin{pmatrix}
M_X & 0 \\
0 & -M_Y
\end{pmatrix} \leq \begin{pmatrix}
Z & -Z \\
-Z & Z
\end{pmatrix} + (2\kappa + i) \cdot \text{Id}_{2n \times 2n}, \tag{4.13}
\]
where
\[ Z = L_\delta \omega''(|\bar{X} - \bar{Y}|) \frac{(\bar{X} - \bar{Y}) \otimes (\bar{X} - \bar{Y})}{|\bar{X} - \bar{Y}|^2} + \frac{\omega'(|\bar{X} - \bar{Y}|)}{|\bar{X} - \bar{Y}|} \left\{ \text{Id}_{n \times n} - \frac{(\bar{X} - \bar{Y}) \otimes (\bar{X} - \bar{Y})}{|\bar{X} - \bar{Y}|^2} \right\}. \] (4.14)

Applying inequality (4.13) to vectors of the form \((\xi, \xi)\), we conclude
\[ \text{Spect}(M_Y - M_X) \in (-4\kappa - i, +\infty). \] (4.15)

However, if we apply to the special vector \((\eta, -\eta)\), we conclude
\[ \text{Spect}(M_Y - M_X) \cap \left( \frac{c}{\sqrt{d}} L_\delta - 4\kappa - i, +\infty \right) \neq \emptyset, \] (4.16)

for a universal number \(c > 0\), depending only upon our choice for \(\omega\). Combining (4.15) and (4.16), we end up with
\[ \left( \frac{c}{\sqrt{d}} L_\delta - n(4\kappa + i) \right) + \text{Trace}(M_X) < \text{Trace}(M_Y). \] (4.17)

Notice that if we choose
\[ L_\delta \gtrsim \|v\|_\infty d^{-3/2} + i\sqrt{d}, \] (4.18)

then the term \(\frac{c}{\sqrt{d}} L_\delta - n(4\kappa + i)\) becomes positive. Also, since from our thesis
\[ v(\delta \bar{X}) - v(\delta \bar{Y}) > \delta, \] (4.19)

we have the upper control
\[ v^{-1}(\delta \bar{X}) < \frac{1}{\delta}. \] (4.20)

It also follows directly from (4.19) the lower control
\[ v^{-1}(\delta \bar{Y}) > \frac{\delta}{\|v\|^2_\infty} + v^{-1}(\delta \bar{X}). \] (4.21)

In the sequel, we use Equation satisfied by \(v(\delta \bar{X})\), namely,
\[ u^\alpha |D u|^\beta \Delta u = \delta^{2+\beta} f(\delta \bar{X}) =: g(X) \] (4.22)
together with (4.11) and (4.12) to write up the following pointwise inequalities

\[
\text{Trace}(M_X) \geq g(\bar{X}) \cdot |\xi_X|^{-\beta} \cdot v^{-\alpha}(\theta_\delta \bar{X}),
\]

(4.23)

\[
\text{Trace}(M_Y) \leq g(\bar{Y}) \cdot |\xi_Y|^{-\beta} \cdot v^{-\alpha}(\theta_\delta \bar{Y}),
\]

(4.24)

Combining (4.23), (4.24) and (4.17), taking into account (4.20), we obtain

\[
g(\bar{Y}) \cdot |\xi_Y|^{-\beta} \cdot v^{-\alpha}(\theta_\delta \bar{Y}) \leq g(\bar{X}) \cdot |\xi_X|^{-\beta} \cdot v^{-\alpha}(\theta_\delta \bar{X})
\]

(4.25)

In view of (4.9) and (4.10), given \(\iota \ll 1\), to be chosen \textit{a posteriori} depending only on \(\|v\|_\infty\) and \(\delta\), for \(L_\delta \gg 1\), and \(\theta_\delta \ll 1\) depending on the modulus of continuity of \(f\) and \(\iota\) – but choices independent of the infimum of \(v\) – there holds

\[
f(\theta_\delta \bar{X}) \cdot |\xi_X|^{-\beta} \leq f(\theta_\delta \bar{Y}) \cdot |\xi_Y|^{-\beta} \leq (1 + \iota)^{1/\alpha}.
\]

(4.26)

Thus, from (4.25), taking into account (4.20) we find

\[
v^{-1}(\theta_\delta \bar{Y}) \leq (1 + \iota)v^{-1}(\theta_\delta \bar{X}).
\]

(4.27)

In the sequel we select

\[
\iota \leq \frac{\delta^2}{10\|v\|_\infty^2},
\]

(4.28)

which gives a contradiction on (4.21). We have shown that, for any pair of interior points, \(X, Y \in \mathcal{O} \subseteq \mathcal{O}\), and any positive number \(\delta > 0\), there holds

\[
|v(X) - v(Y)| \leq g_\delta^{-1} \max\{L_\delta, \tau, \frac{\|v\|_\infty^4}{\delta^4}\} \cdot |X - Y| + 2\kappa \theta_\delta^{-2} \cdot |X - Y|^2 + \delta,
\]

(4.29)

where \(\tau\) accounts the choices made at (4.18) and (4.26). In particular, all choices are independent of infimum of \(v\). It is easy to check that (4.29) gives a universal modulus of continuity for \(v\). The proof of Theorem follows. \(\square\)

We conclude this Section by commenting on the universal modulus of continuity \(\varpi\) found in Theorem. As inspection on (4.29) reveals the existence of a strictly decreasing function \(\Omega: (0, 1) \rightarrow (0, \infty)\), such that, for any \(0 < \delta < 1\), there holds

\[
|v(X) - v(Y)| \leq \Omega(\delta) \cdot |X - Y| + \delta,
\]

(4.30)
provided \( |X - Y| \leq 1 \). Indeed, \( \Omega(\delta) \approx \varrho_\delta^{-1} L_\delta \), with \( \varrho_\delta \) depending only on universal parameters and on the modulus of continuity of \( f \) and \( L_\delta \) depending only on universal numbers. Let us define the (increasing) function \( \Xi(\delta) := \delta^{-1} \Omega(\delta) \) and in the sequel we set
\[
\varpi(t) := \Xi^{-1}\left(\frac{1}{t}\right),
\]
which is a modulus of continuity. From the very definition of \( \varpi(t) \), it follows that
\[
\frac{1}{\varpi(t)} \Omega(\varpi(t)) = \Xi(\varpi(t)) = \frac{1}{t}.
\]
Now, given two points \( X, Y \) with \( 0 < |X - Y| \leq 1 \), select in (4.30) \( \delta = \varpi(|X - Y|) \), taking into account (4.32), and estimate:
\[
|v(X) - v(Y)| \leq \Omega(\varpi(|X - Y|)) \cdot |X - Y| + \varpi(|X - Y|)
\]
Hence \( 2 \varpi \) is a modulus of continuity for \( v \), which is universal.

Let us further mention that when \( \Omega(\delta) \approx \delta^{-M} \) – which is the case when \( f \) is Hölder continuous – then Theorem \( \bullet \) provides a universal \( C^{0, \frac{1}{M+1}} \)-Hölder continuity estimate for \( v \).

5 Regularity along the free boundary

In this Section we aim to prove that limiting solutions to (3.3) are locally of class \( C^\gamma \) along the free boundary \( \{u_0 > 0\} \). The strategy will be based on a new flatness improvement device which allows us to control the growth of \( u_0 \) in proper geometric dyadic balls. Initially we need a Lemma.

**Lemma 4.** Let \( u : B_1 \to \mathbb{R} \) be a positive solution to
\[
u^\alpha |Du|^{\beta} \Delta u = f(X), \quad \text{in } B_1,
\]
satisfying \( |u| \leq 1 \). Given a positive number \( \theta > 0 \), there exists \( \eta = \eta(\theta) > 0 \) depending only on \( \theta \) and the universal parameters, such that if
\[
\inf_{B_{1/2}} u \leq \eta \quad \& \quad \|f\|_{L^\infty(B_1)} \leq \eta^{1+\alpha+\beta},
\]
then
\[
\sup_{B_{1/4}} u \leq \theta.
\]
Proof. Let us suppose, for the sake of contradiction, that the thesis of the Lemma fails to hold. That is, there exist a sequence of positive functions \( u_j : B_1 \to \mathbb{R} \), satisfying,

\[
|u_j| \leq 1; \quad \iota_j := \inf_{B_{1/2}} u_j = o(1), \quad \|f_j(X)\|_{L^\infty(B_1)} \leq \iota_j^{1+\alpha+\beta},
\]

where \( f_j(X) =: |\nabla u_j|_\beta u_j^\alpha \Delta u_j \) in the viscosity sense; and \( \theta_0 > 0 \), such that

\[
\sup_{B_{1/4}} u_j \geq \theta_0.
\]

In the sequel, we define the normalized function

\[
v_j(X) := \frac{u_j(X)}{\iota_j}.
\]

It is clear that \( \inf v_j = 1 \). Direct computations yield

\[
|Dv_j|_\beta \Delta v_j = \frac{v_j^{-\alpha}}{\iota_j^{1+\alpha+\beta}} \cdot f_j(X) =: g_j(X),
\]

which is a bounded function in \( B_{1/2} \). Now, applying Harnack inequality, see for instance [15], we deduce

\[
C \geq \sup_{B_{1/4}} v_j(X) \geq \frac{\sup_{B_{1/4}} u_j(X)}{\iota_j},
\]

which gives a contradiction if \( \iota_j \ll 1 \).

We now proceed into the proof of Theorem 2. Hereafter, let us label:

\[
\eta_* := \eta(4^{-\gamma}); \quad (5.2)
\]

i.e., \( \eta_* \) is the positive number given by Lemma 4 when one takes \( \theta = 4^{-\gamma} \). Recall

\[
\gamma = \frac{2 + \beta}{1 + \alpha + \beta},
\]

gives the aimed optimal regularity.

Our initial observation is that if \( u \) is a positive solution to Equation (5.1), then the scaled function \( v(X) := u(\varrho X) \), with

\[
\varrho := \eta_*^{1/\gamma} \left( \frac{2 + \beta}{\alpha + \beta} \right)^{1/(1+\alpha+\beta)} \sqrt{\|f\|_{L^\infty}},
\]
solves the same equation in $B_1$ with the right hand side bounded by $\eta_*^{1+\alpha+\beta}$. Hence, from now on, we can assume, modulo a fixed zoom-in, that the RHS of Equation (5.1) is bounded by $\eta_*^{1+\alpha+\beta}$.

Now, let $u_0$ be a nonnegative limiting solution and assume $0 \in \partial\{u_0 > 0\}$. We want to show that there exists a universal constant $C > 0$ such that

$$\sup_{B_r} u_0(X) \leq Cr^\gamma.$$  

For that, let $u_j$ be a sequence of positive solutions converging locally uniformly to $u_0$. Since $u_j(0) = o(1)$, there exists $j_0 \in \mathbb{N}$, such that if $j \geq j_0$,

$$\inf_{B_{1/2}} u_j \leq u_j(0) \leq \eta_*.$$  

It then follows from Lemma 4 that

$$\sup_{B_{1/4}} u_j(X) \leq 4^{-\gamma}, \quad \forall j \geq j_1. \quad (5.3)$$  

In the sequel, we define $v_j^1 : B_1 \to \mathbb{R}$, by

$$v_j^1(X) := 4^\gamma u_j(\frac{1}{4}X).$$  

It follows from (5.3) that $v_j^1$ is a normalized function satisfying

$$\left|(v_j^1)^\alpha|Dv_j^1|^\beta \Delta v_j^1\right| \leq \eta_*^{1+\alpha+\beta},$$  

in the viscosity sense. We can now choose $j_1 > j_0$, such that

$$\inf_{B_{1/2}} v_j^1 \leq 4^\gamma u_j(0) \leq \eta_*.$$  

Applying Lemma 4 to $v_j^1$ and rescaling it back to $u_j$, gives

$$\sup_{B_{1/16}} u_j(X) \leq 16^{-\gamma}, \quad \forall j \geq j_1. \quad (5.4)$$  

Continuing the reasoning, we define $v_j^2 : B_1 \to \mathbb{R}$, by

$$v_j^2(X) := 4^\gamma v_j^1(\frac{1}{4}X) = 16^\gamma u_j(\frac{1}{16}X).$$
Again, through scaling arguments, one verifies that 
\[
\left| (v_j^2)^\alpha |Dv_j^2|^{\beta} \Delta v_j^2 \right| \leq \eta^1_{*}^{1+\alpha+\beta},
\]
in the viscosity sense. For another natural number, \( j_2 > j_1 \), there holds 
\[
\inf_{B_{1/2}} v_j^2 \leq 16^\gamma \cdot u_j(0) \leq \eta_{*},
\]
for all \( j \geq j_2 \). Applying Lemma 4 to \( v_j^2 \) and rescaling the estimate back to \( u_j \) gives
\[
\sup_{B_{1/64}} u_j(X) \leq 64^{-\gamma}, \quad \forall j \geq j_2. \quad (5.5)
\]
Proceeding inductively, we prove that for any natural number \( k \geq 1 \), there exists \( j_k \in \mathbb{N} \) such that
\[
\sup_{B_{4^{-k}}} u_j(X) \leq 4^{-k\gamma}, \quad \forall j \geq j_k. \quad (5.6)
\]
Finally, given \( 0 < r < 1/4 \), let \( k \) be such that \( 4^{-(k+1)} < r \leq 4^{-k} \). We can estimate
\[
\sup_{B_r} u_0 \leq \sup_{B_{4^{-k}}} u_0 \leq 4^{-k\gamma} \leq 4^\gamma \cdot r^\gamma,
\]
and the Theorem is proven. \( \square \)

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