Bend-Bounded Path Intersection Graphs: Sausages, Noodles, and Waffles on a Grill*

Steven Chaplick1⋆, Vít Jelínek2, Jan Kratochvíl3, and Tomáš Vyskočil4

1 Department of Mathematics, Wilfrid Laurier University, Waterloo, CA, chaplick@cs.toronto.edu
2 Computer Science Institute, Faculty of Mathematics and Physics, Charles University, Prague, e-mail: jelinek@iuuk.mff.cuni.cz
3 Department of Applied Mathematics, Faculty of Mathematics and Physics, Charles University, Prague, honza@kam.mff.cuni.cz
4 Department of Applied Mathematics, Faculty of Mathematics and Physics, Charles University, Prague, e-mail: whisky@kam.mff.cuni.cz

Abstract. In this paper we study properties of intersection graphs of k-bend paths in the rectangular grid. A k-bend path is a path with at most k 90 degree turns. The class of graphs representable by intersections of k-bend paths is denoted by Bk-VPG. We show here that for every fixed k, Bk-VPG ⊊ Bk+1-VPG and that recognition of graphs from Bk-VPG is NP-complete even when the input graph is given by a Bk+1-VPG representation. We also show that the class Bk-VPG (for k ≥ 1) is in no inclusion relation with the class of intersection graphs of straight line segments in the plane.

1 Introduction

In this paper we continue the study of Vertex-intersection graphs of Paths in Grids (VPG graphs) started by Asinowski et. al [1,2]. A VPG representation of a graph G is a collection of paths of the rectangular grid where the paths represent the vertices of G in such a way that two vertices of G are adjacent if and only if the corresponding paths share at least one vertex.

VPG representations arise naturally when studying circuit layout problems and layout optimization [15] where layouts are modelled as paths (wires) on grids. One approach to minimize the cost or difficulty of production involves minimizing the number of times the wires bend [3,13]. Thus the research has been focused on VPG representations parameterized by the number of times each path is allowed to bend (these representations are also the focus of [12]). In particular, a k-bend path is a path in the grid which contains at most k

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5 The grids to which we refer are always rectangular.
bends where a bend is when two consecutive edges on the path have different horizontal/vertical orientation. In this sense a $B_k$-VPG representation of a graph $G$ is a VPG representation of $G$ where each path is a $k$-bend path. A graph is $B_k$-VPG if it has a $B_1$-VPG representation.

Several relationships between VPG graphs and traditional graph classes (i.e., circle graphs, circular arc graphs, interval graphs, planar graphs, segment (SEG) graphs, and string (STRING) graphs) were observed in [1,2]. For example, the equivalence between string graphs (the intersection graphs of curves in the plane) and VPG graphs is formally proven in [2], but it was known as folklore result [6]. Additionally, the base case of this family of graph classes (namely, $B_0$-VPG) is a special case of segment graphs (the intersection graphs of line segments in the plane). Specifically, $B_0$-VPG is more well known as the 2-DIR. The recognition problem for the VPG = string graph class is known to be NP-Hard by [9] and in NP by [14]. Similarly, it is NP-Complete to recognize 2-DIR = $B_0$-VPG graphs [11]. However, the recognition status of $B_k$-VPG for every $k > 0$ was given as an open problem from [2] (all cases were conjectured to be NP-Complete). We confirm this conjecture by proving a stronger result. Namely, we demonstrate that deciding whether a $B_{k+1}$-VPG graph is a $B_k$-VPG graph is NP-Complete (for any fixed $k > 0$) – see Section 4.

Furthermore, in [12] it is shown that $B_0$-VPG $\subseteq$ $B_1$-VPG $\subsetneq$ VPG and it was conjectured that $B_k$-VPG $\subsetneq$ $B_{k+1}$-VPG for every $k > 0$. We confirm this conjecture constructively – see Section 3.

Finally, we consider the relationship between the $B_k$-VPG graph classes and segment graphs. In particular, we show that SEG and $B_k$-VPG are incomparable through the following pair of results (the latter of which is somewhat surprising): (1) There is a $B_1$-VPG graph which is not a SEG graph; (2) For every $k$, there is a 3-DIR graph which has no $B_k$-VPG representation.

The paper is organized as follows. In Section 2 we introduce the Noodle-Forcing Lemma, which is the key to restricting the topological structure of VPG representations. In Section 3 we introduce the “sausage” structure which is the crucial gadget that we use for the hardness reduction and which by itself shows that $B_k$-VPG is strict subset of $B_{k+1}$-VPG. We also demonstrate the incomparability of $B_k$-VPG and SEG in Section 3. The NP-hardness reduction is presented in Section 4. We end the paper with some remarks and open problems.

2 Noodle-Forcing Lemma

In this section, we present the key lemma of this paper (see Lemma 1). Essentially, we prove that, for “proper” representations $R$ of a graph $G$, there is a graph $G'$ where $G$ is an induced subgraph of $G'$ and $R$ is “sub-representation”

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6 Note: a $k$-DIR graph is an intersection graph of straight line segments in the plane with at most $k$ distinct directions (slopes).

7 This was inspired by the order forcing lemma of [12].

8 This gadget is named due to its VPG representation resembling sausage links.
of every representation of $G'$ (i.e., all representations of $G'$ require the part corresponding to $G$ to have the “topological structure” of $R$). We begin this section with several definitions.

Let $G = (V, E)$ be a graph. A representation of $G$ is a collection $R = \{R(v), v \in V\}$ of piecewise linear curves in the plane, such that $R(u) \cap R(v)$ is nonempty iff $uv$ is an edge of $G$.

An intersection point of a representation $R$ is a point in the plane that belongs to (at least) two distinct curves of $R$. Let $\text{In}(R)$ denote the set of intersection points of $R$.

A representation is proper if

1. each $R(v)$ is a simple curve, i.e., it does not intersect itself,
2. $R$ has only finitely many intersection points (in particular no two curves may overlap) and finitely many bends, and
3. each intersection point $p$ belongs to exactly two curves of $R$, and the two curves cross in $p$ (in particular, the curves may not touch, and an endpoint of a curve may not belong to another curve).

Let $R$ be a proper representation of $G = (V, E)$, let $R'$ be another (not necessarily proper) representation of $G$, and let $\phi$ be a mapping from $\text{In}(R)$ to $\text{In}(R')$. We say that $\phi$ is order-preserving if it is injective and has the property that for every $v \in V$, if $p_1, p_2, \ldots, p_k$ are all the distinct intersection points on $R(v)$, then $\phi(p_1), \ldots, \phi(p_k)$ all belong to $R'(v)$ and they appear on $R'(v)$ in the same relative order as the points $p_1, \ldots, p_k$ on $R(v)$. (If $R'(v)$ visits the point $\phi(p_i)$ more than once, we may select one visit of each $\phi(p_i)$, such that the selected visits occur in the correct order $\phi(p_1), \ldots, \phi(p_k)$.)

For a set $P$ of points in the plane, the $\varepsilon$-neighborhood of $P$, denoted by $N_\varepsilon(P)$, is the set of points that have distance less than $\varepsilon$ from $P$.

**Lemma 1 (Noodle-Forcing Lemma).** Let $G = (V, E)$ be a graph with a proper representation $R = \{R(v), v \in V\}$. Then there exists a graph $G' = (V', E')$ containing $G$ as an induced subgraph, which has a proper representation $R' = \{R'(v), v \in V'\}$ such that $R(v) = R'(v)$ for every $v \in V$, and $R'(w)$ is a horizontal or vertical segment for $w \in V' \setminus V$. Moreover, for any $\varepsilon > 0$, any (not necessarily proper) representation of $G'$ can be transformed by a homeomorphism of the plane and by circular inversion into a representation $R' = \{R'(v), v \in V'\}$ with these properties:

1. for every vertex $v \in V$, the curve $R'(v)$ is contained in the $\varepsilon$-neighborhood of $R(v)$, and $R(v)$ is contained in the $\varepsilon$-neighborhood of $R'(v)$.
2. there is an order-preserving mapping $\phi: \text{In}(R) \to \text{In}(R')$, with the additional property that for every $p \in \text{In}(R)$, the point $\phi(p)$ coincides with the point $p$.

**Proof.** Suppose we are given a proper representation $R$ of a graph $G$. We say that a point in the plane is a special point of $R$, if it is an endpoint of a curve in $R$, a bend of a curve in $R$, or an intersection point of $R$.

Before we describe the graph $G'$, we first construct an auxiliary graph $H$ which is a subdivision of a 3-connected plane graph whose drawing overlays the representation $R$ and has the following properties.
P1 The edges of $H$ are drawn as vertical and horizontal segments, and every internal face of $H$ is a rectangle (possibly containing more than four vertices of $H$ on its boundary). The outer face of $H$ is not intersected by any curve of $R$.

P2 No curve of $R$ passes through a vertex of $H$, and no edge of $H$ passes through a special point of $R$.

P3 Every face of $H$ contains at most one special point of $R$, and no two faces containing a special point are adjacent.

P4 Every edge of $H$ is intersected at most once by the curves of $R$.

P5 Every face of $H$ is intersected by at most two curves of $R$, and if a face $f$ is intersected by two curves of $R$, then the two curves intersect inside $f$.

P6 Every curve of $R$ intersects the boundary of a face of $H$ at most twice.

We construct the plane graph $H$ in several steps. In the first step, we produce a square grid $H_0$ such that the whole representation $R$ is contained in the interior of $H_0$, i.e., no part of $R$ intersects the outer face of $H_0$. We may further assume that the grid $H_0$ is fine enough so that it has the properties P1–P3 above. See Fig. 1.

Fig. 1. An embedding of a graph $H_0$ over a representation $R$.

If the graph $H_0$ does not satisfy all the properties P1–P6, we will further subdivide some of the faces of $H_0$. Suppose first that $H_0$ has an edge $e$ that is intersected more than once by the curves of $R$ (see Fig. 2). We then add a new edge $e'$ into the drawing of $H_0$, which is parallel to $e$ and embedded very close to $e$, thus splitting a face adjacent to $e$ into two new faces. We then split the face incident to $e$ and $e'$ by new edges perpendicular to $e$ and $e'$, in such a way that each intersection point of $e$ or $e'$ with a curve of $R$ belongs to a different edge in the new graph. Performing this operation for every edge of $H_0$, we obtain a new graph $H_1$, which satisfies the properties P1–P4.

If the graph $H_1$ fails to satisfy P5 or P6, it means that $H_1$ has a face $f$ whose intersection with $R$ contains two disjoint curves $p$ and $q$, each of which is a subcurve of a curve from $R$ (see Fig. 3). In this case, we draw in $f$ a piecewise linear curve $c$ with horizontal and vertical segments which cuts $f$ into two pieces.

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such that one contains $p$ and the other $q$. We then extend the segments of $c$ to form a grid-like subdivision of the face $f$ into subfaces, each of which is only intersected by at most one of $p$ and $q$. If necessary, we may further subdivide the newly created faces to make sure they do not violate P4. In this way, we obtain the graph $H$ satisfying all the properties P1–P6.

We now transform $H$ into an arrangement $S$ of horizontal and vertical segments. See Figure 4. For every vertex $v \in V(H)$, the set $S$ contains two segments $S_1(v)$ and $S_2(v)$ which intersect close to the point $v$. We call these two segments the vertex-segments of $v$. We assume that the vertex segments do not intersect any of the curves of $R$, and they do not overlap with any edge of $H$. For every
edge \( e \in E(H) \), we further put into \( S \) a segment \( S(e) \), called the \textit{edge-segment of} \( e \), which partially overlaps with \( e \), intersects exactly those curves of \( R \) that \( e \) intersects, and does not intersect any vertex-segment or any other edge-segment of \( S \). Finally, for any vertex \( v \) that is incident to an edge \( e \) of \( H \), we put into \( S \) a segment \( S(v,e) \) that intersects both \( S(e) \) and the vertex segment of \( v \) that is parallel to \( S(e) \), and does not intersect any other segment of \( S \) or curve of \( R \). We call \( S(v,e) \) the \textit{connector of} \( v \) and \( e \).

![Diagram](image)

\textbf{Fig. 4.} Transforming the graph \( H \) into a segment arrangement \( S \).

Define a representation \( R' = R \cup S \), and let \( G' \) be the graph determined by this representation. Clearly, \( G \) is an induced subgraph of \( G' \), and the representation \( R' \) has the properties stated in the lemma. It remains to show that every representation of \( G' \) can be transformed into a representation \( R'' \) that has the two properties stated in the lemma. We will first show that every representation of \( G' \) can be transformed into a representation that satisfies the first property, and then we will show that the second property follows from the first one.

Consider an arbitrary representation \( R'' \) of \( G' \). The curves in \( R'' \) that correspond to the vertex-segments, edge-segments and connectors will be referred to as vertex-curves, edge-curves and connector curves. Let \( S''(e) \) be an edge-curve that represents an edge \( e \in E(H) \) in \( R'' \). Let \( c_1 \) and \( c_2 \) be its two adjacent connector curves. Let \( \gamma(e) \) be a minimal subcurve of \( S''(e) \) with the property that its two endpoints belong respectively to \( c_1 \) and \( c_2 \). In particular, \( \gamma(e) \) has no other intersections with the connector curves apart from its endpoints. Call \( \gamma(e) \) the \textit{main part} of \( S''(e) \). Next, for each connector curve, consider its minimal subcurve whose one endpoint belongs to the main part of its adjacent edge-curve, and the other endpoint belongs to its adjacent vertex-curve. Call this subcurve the \textit{main part} of the connector curve.

If, in the representation \( R'' \), we contract each intersecting pair of vertex-curves into a single point, and then replace each connector curve and edge-curve by their main parts, we obtain a drawing of the graph \( H \), where a vertex is represented by the contracted pair of vertex-curves, and an edge corresponds to the main part of its edge-curve together with the adjacent connectors. Since \( H \) is a subdivision of a 3-connected graph, each of its drawings can be transformed into any other drawing by a circular inversion and a homeomorphism. In particular, we may transform \( R'' \) in such a way that the main part of any
edge-curve is contained inside the corresponding edge-segment \( S(v) \), the main part of any connector curve is contained inside the corresponding connector, and every vertex-curve representing a vertex \( v \in V(H) \) is embedded in a small neighborhood of \( S_1(v) \cup S_2(v) \). Suppose therefore that \( R' \) has been transformed in this way.

Let \( f \) be an internal face of \( H \), whose boundary is formed by \( k \) edges \( e_1, \ldots, e_k \) and \( k \) vertices \( v_1, \ldots, v_k \). Consider the union of the main parts \( \gamma(e_1), \ldots, \gamma(e_k) \), together with the main parts of their adjacent connectors and together with the vertex curves representing the vertices \( v_1, \ldots, v_k \). In the union of these curves, there is a unique bounded region that is adjacent to all of the curves \( \gamma(e_1), \ldots, \gamma(e_k) \). We call this region the \textit{pseudo-face} corresponding to \( f \), denoted by \( \bar{f} \). Notice that an edge-curve \( R''(e) \) may only intersect the two pseudo-faces that correspond to the two faces adjacent to \( e \) in \( H \).

Let \( v \) be a vertex of \( G \). Let \( f_0, f_1, \ldots, f_k \) be the faces of \( H \) intersected by the curve \( R(v) \) in the order from one endpoint to the other. Let \( e_1, \ldots, e_k \) be the edges of \( H \) crossed by \( R(v) \), where \( e_i \) is adjacent to \( f_{i-1} \) and \( f_i \). Notice that we have \( k \geq 2 \) due to the property P3.

Since the curve \( R''(v) \) must intersect the edge-curve \( S''(e_i) \) for any \( i = 1, \ldots, k \), it must intersect at least one of the two pseudo-faces \( \bar{f}_{i-1} \) and \( \bar{f}_i \). Also, since \( R''(v) \) does not intersect any edge-curve other than the \( k \) curves \( S''(e_i) \) for \( i = 1, \ldots, k \), it cannot enter into any pseudo-face other than \( \bar{f}_i \) for \( i = 0, \ldots, k \).

Note, however, that \( R''(v) \) does not necessarily intersect the main part of \( S''(e_1) \) and that it does not necessarily penetrate into \( f_0 \), and similarly for \( S''(e_k) \) and \( f_k \) (see Fig. [1]). Overall, we see that \( R''(v) \) enters a pseudo-face \( \bar{f} \) if and only if \( R(v) \) enters the face \( f \), except possibly for the two faces \( f_0 \) and \( f_k \). We also see that \( R''(v) \) crosses the main part of an edge-curve \( S''(e) \) if and only if \( R(v) \) crosses \( e \), except possibly for the two edges \( e_1 \) and \( e_k \). From this it is easy to see that for any \( \epsilon > 0 \), we may deform \( R''(v) \) so that it will belong to \( \mathcal{N}_\epsilon(R(v)) \), without changing the boundary of any pseudo-face. Moreover, by possibly creating a ‘bulge’ in \( \gamma(e_0) \) and \( \gamma(e_k) \) as shown in Fig. [2] we may ensure that every point of \( R(v) \) has a point of \( R''(v) \) in its \( \epsilon \)-neighborhood. This shows that \( R'' \) is homeomorphic to a representation \( R' \) having the first property stated in the lemma.

We will now show that if \( \epsilon \) is small enough, then every representation \( R' \) that satisfies the first property in the statement of the lemma must admit an order-preserving mapping \( \phi : \text{In}(R) \to \text{In}(R') \).

Let \( N(u) \) denote the set \( \mathcal{N}_\epsilon(R(u)) \). We call \( N(u) \) the \textit{noodle} of \( u \). Suppose that two curves \( R(u) \) and \( R(v) \) have \( k \) mutual crossing points \( p_1, \ldots, p_k \). If \( \epsilon \) is small enough, the intersection \( N(u) \cap N(v) \) has \( k \) connected components \( P_1, \ldots, P_k \), each of them being an open parallelogram, and each \( P_i \) containing a unique intersection point \( p_i \). We call \( P_i \) the \textit{zone} of \( p_i \). By making \( \epsilon \) small, we may assume that the zones of the points in \( \text{In}(R) \) are disjoint, that every noodle has a boundary that is a simple closed curve, and that the boundary of a noodle does not intersect any of the zones.
Consider now the curve $R^*(u)$ for some vertex $u \in V$. We may assume that the endpoints of $R^*(u)$ coincide with the endpoints of $R(u)$ (otherwise we may shorten $R^*(u)$ and deform it in a neighborhood of the endpoints). Choose an orientation of $R(u)$ (i.e., choose an initial endpoint), and suppose that $R^*(u)$ has the same orientation. Suppose that the curve $R(u)$ has $m$ crossing points $q_1, \ldots, q_m$, encountered in this order. Let $Q_1, \ldots, Q_m$ be their zones. Fix a zone $Q_i$ and consider the intersection $R^*(u) \cap Q_i$. This intersection is a union of subcurves of $R^*(u)$. Choose such a subcurve whose endpoints lie on the opposite sides of $Q_i$; if there are more such subcurves, choose the first such subcurve visited when $R^*(u)$ is traced in the direction of its orientation. Call the chosen subcurve the representative of $R^*(u)$ in $Q_i$, denoted by $r_i(u)$. Note that when $R^*(u)$ is traced from beginning to end, the representatives are visited in the order $r_1(u), r_2(u), \ldots, r_m(u)$.

We now define the order-preserving mapping $\phi$. See Fig. 5. Let $p \in \text{In}(R)$ be an intersection of two curves $R(u)$ and $R(v)$, and let $P$ be the zone of $p$. Let $r(u)$ and $r(v)$ be the representatives of $R(u)$ and $R(v)$ inside $P$. Define $\phi(p)$ to be an arbitrary intersection of $r(u)$ and $r(v)$. We may apply a homeomorphism inside $P$ to make sure that $\phi(p)$ coincides with $p$. By the choice of representatives, $\phi$ is order-preserving. This proves the lemma.

**3 Relations between classes**

With the Noodle-Forcing Lemma, we can prove our separation results.
Theorem 1. For any $k \geq 1$, there is a graph $G'$ that has a proper representation using $k$-bend axis-parallel curves, but has no representation using $(k - 1)$-bend axis-parallel curves.

Proof. Consider the graph $K_2$ consisting of a single edge $uv$, with a representation $R$ in which both $u$ and $v$ are represented by weakly increasing $k$-bend staircase curves that have $k + 1$ common intersections $p_1, \ldots, p_{k+1}$, in left-to-right order, see Fig. 7. We refer to this representation as a sausage due to it resembling sausage links.

We now grill the sausage (i.e., we apply the Noodle-Forcing Lemma to $K_2$ and $R$) to obtain a graph $G'$ with a $k$-bend representation $R'$. We claim that $G'$ has no $(k-1)$-bend representation. Assume for contradiction that there is a $(k-1)$-bend
Lemma 1 then implies that there is an order-preserving mapping \( \phi : \text{In}(R) \to \text{In}(R'') \). Let \( s_i(u) \) be the subcurve of \( R''(u) \) between the points \( \phi(p_i) \) and \( \phi(p_i+1) \), and similarly for \( s_i(v) \) and \( R''(v) \). Consider, for each \( i = 1, \ldots, k \), the union \( c_i = s_i(u) \cup s_i(v) \). We know from Lemma 1 that \( s_i(u) \) and \( s_i(v) \) cannot completely overlap, and therefore the closed curve \( c_i \) must surround at least one nonempty bounded region of the plane. Therefore \( c_i \) contains at least two bends different from \( \phi(p_i) \) and \( \phi(p_i+1) \). We conclude that \( R''(u) \) and \( R''(v) \) together have at least \( 2k \) bends, a contradiction. \( \square \)

A straightforward consequence is the following.

**Corollary 1.** For every \( k \), \( B_k\text{-VPG} \subset B_{k+1}\text{-VPG} \).

Because two straight-line segments in the plane cross at most once, the Noodle-Forcing Lemma also implies the following.

**Corollary 2.** For every \( k \geq 1 \), \( B_k\text{-VPG} \not\subset \text{SEG} \).

This raises a natural question: Is there some \( k \) such that every SEG graph is contained in \( B_k\text{-VPG} \)? The following theorem answers it negatively.

**Theorem 2.** For every \( k \), there is a graph which belongs to 3-DIR but not to \( B_k\text{-VPG} \).

**Proof.** We fix an arbitrary \( k \). Consider, for an integer \( n \), a representation \( R \equiv R(n) \) formed by \( 3n \) segments, where \( n \) of them are horizontal, \( n \) are vertical and \( n \) have a slope of 45 degrees. Suppose that every two segments of \( R \) with different slopes intersect, and their intersections form the regular pattern depicted in Figure 8 (with a little bit of creative fantasy this pattern resembles a waffle, especially when viewed under a linear transformation).

Note that the representation \( R \) forms \( \Omega(n^2) \) empty internal triangular faces bounded by segments of \( R \), and the boundaries of these faces intersect in at most a single point. Suppose that \( n \) is large enough, so that there are more than \( 3kn \) such triangular faces. Let \( G \) be the graph represented by \( R \).

The representation \( R \) is proper, so we can apply the Noodle-Forcing Lemma to \( R \) and \( G \), obtaining a graph \( G' \) together with its 3-DIR representation \( R'' \). We claim that \( G' \) has no \( B_k\text{-VPG} \) representation.

Suppose for contradiction that there is a \( B_k\text{-VPG} \) representation \( R'' \) of \( G' \). We will show that the \( 3n \) curves of \( R'' \) that represent the vertices of \( G \) contain together more than \( 3kn \) bends.

From the Noodle-Forcing Lemma, we deduce that there exists an order-preserving mapping \( \phi : \text{In}(R_n) \to \text{In}(R'_n) \). Let \( T \) be a triangular face of the representation \( R \). The boundary of \( T \) consists of three intersection points \( p, q, r \in \text{In}(R) \) and three subcurves \( a, b, c \). The three intersection points \( \phi(p) \), \( \phi(q) \) and \( \phi(r) \) determine the corresponding subcurves \( a'', b'' \) and \( c'' \) in \( R'' \).

The Noodle-Forcing Lemma implies that there is a homeomorphism \( h \) which sends \( a'', b'' \) and \( c'' \) into small neighborhoods of \( a \), \( b \) and \( c \), respectively. This shows that each of the three curves \( a'', b'' \) and \( c'' \) contains a point that does not
belong to any of the other two curves. This in turn shows that at least one of the three curves is not a segment, i.e., it has a bend in its interior.

Since the triangular faces of $R$ have non-overlapping boundaries, and since $\phi$ is order-preserving, we see that for each triangular face of $R$ there is at least one bend in $R''$ belonging to a curve representing a vertex of $G$. Since $G$ has $3n$ vertices and $R$ determines more than $3kn$ triangular faces, we conclude that at least one curve of $R''$ has more than $k$ bends, a contradiction. \hfill $\square$

4 Hardness Results

In this section we strengthen the separation result of Corollary 1 by showing that not only are the classes $B_k$-VPG and $B_{k+1}$-VPG different, but providing a $B_{k+1}$-VPG representation does not help in deciding $B_k$-VPG membership. This also settles the conjecture on NP-hardness of recognition of these classes stated in [2], in a considerably stronger form than it was asked.

Theorem 3. For every $k \geq 0$, deciding membership in $B_k$-VPG is NP-complete even if the input graph is given with a $B_{k+1}$-VPG representation.

Proof. It is not difficult to see that recognition of $B_k$-VPG is in NP and therefore we will be concerned in showing NP-hardness only. We use the NP-hardness reduction developed in [11] for showing that recognizing grid intersection graphs is NP-complete. Grid intersection graphs are intersection graphs of vertical and horizontal segments in the plane with additional restriction that no two segments of the same direction share a common point. Thus these graphs are formally close but not equal to $B_0$-VPG graphs (where paths of the same direction are allowed to overlap). However, bipartite $B_0$-VPG graphs are exactly grid intersection...
graphs. This follows from a result of Bellantoni et al. [4] who proved that bipartite intersection graphs of axes parallel rectangles are exactly grid intersection graphs.

![Graph](image_url)

**Fig. 9.** The clause gadget reprinted from [11]

The reduction in [11] constructs, given a Boolean formula $\Phi$, a graph $G_\Phi$ which is a grid intersection graph if and only if $\Phi$ is satisfiable. In arguing about this, a representation by vertical and horizontal segments is described for a general layout of $G_\Phi$ for which it is also shown how to represent its parts corresponding to the clauses of the formula, referred to as clause gadgets, if at least one literal is true. The clause gadget is reprinted with a generous approval of the author in Fig. 9, while Fig. 10 shows the grid intersection representations of satisfied clauses, and Fig. 11 shows the problem when all literals are false. In Fig. 12 we show that in the case of all false literals, the clause gadget can be represented by grid paths with at most 1 bend each. It follows that $G_\Phi \in B_1$-VPG and a 1-bend representation can be constructed in polynomial time. Thus, recognition of $B_0$-VPG is NP-complete even if the input graph is given with a $B_1$-VPG representation.
Fig. 11. The problem preventing the representation of an unsatisfied clause reprinted from [11].

Fig. 12. The representation of an unsatisfied clause gadget via curves with one bend.
We use a similar approach for arbitrary $k > 0$ with a help of the Noodle-Forcing Lemma. We grill the same representation $R$ of $K_2$ as in the proof of Theorem 1. We call the resulting graph $P(u)$ where $u$ is one of the vertices of the $K_2$, the one whose curve in $R$ is ending in a boundary cell denoted by $\alpha$ in the schematic Fig. 7. We call this graph the pin since it follows from Lemma 1 that it has a $B_k$-VPG representation such that the bounding paths of the cell $\alpha$ wrap around the grill and the last segment of $R(u)$ extends arbitrarily far (see the schematic Fig. 13). We will refer to this extending segment as the tip of the pin. It is crucial to observe that in any $B_k$-VPG representation $R'$ of $P(u)$ all bends of $R'(u)$ are consumed between the crossing points with the curve representing the other vertex of $K_2$ and hence the part of $R'(u)$ that lies in the $\alpha$ cell of $R'$ is necessarily straight.

Next we combine two pins together to form a clothespin. The construction is illustrated in the schematic Fig. 14. We start with a $K_4$ whose edges are subdivided by one vertex each. Every STRING representation of this graph contains 4 basic regions which correspond to the faces of a drawing of the $K_4$ (this is true for every 3-connected planar graph and it is seen by contracting the curves corresponding to the degree 2 vertices, the argument going back to Sinden [15]). We add two vertices $x_1, x_2$ that are connected by paths of length 2.
to the boundary vertices of two triangles, say $\beta_1$ and $\beta_2$. The curves representing $x_1$ and $x_2$ must lie entirely inside the corresponding regions. Then we add two pins, say $P(u_1)$ and $P(u_2)$, connect the vertices of the boundary of $\alpha_i$ to $x_i$ by paths of length 2 and make $u_i$ adjacent to a vertex on the boundary of $\beta_i$ (for $i = 1, 2$). Finally, we add a third pin $P(u_3)$ and make $u_3$ adjacent to $u_1$ and $u_2$. We denote the resulting graph by $CP(u)$.

It is easy to check that the clothespin has a $B_k$-VPG representation $\tilde{R}$ such that the tips of $\tilde{R}(u_1)$ and $\tilde{R}(u_2)$ are parallel and extend arbitrarily far from the rest of the representation, as indicated in Fig. 14.

On the other hand, in any $B_k$-VPG representation $R'$ of $CP(u)$, if a curve crosses $R'(u_1)$ and $R'(u_2)$ and no other path of $R'(CP(u))$, then it must cross the tips of $R'(u_1)$ and $R'(u_2)$. This follows from the fact that for $i = 1, 2$, $R'(x_i)$ must lie in $\alpha_i$ (to be able to reach all its bounding curves), and hence, by circle inversion, all bends of $R'(u_i)$ are trapped inside $\beta_i$. If a curve crosses both $R'(u_1)$ and $R'(u_2)$, it must cross them outside $\beta_1 \cup \beta_2$, and hence it only may cross their tips.

Now we are ready to describe the construction of $G'_\Phi$. We take $G_\Phi$ as constructed in [11] replace every vertex $u$ by a clothespin $CP(u)$, and whenever $uv \in E(G_\Phi)$, we add edges $u_i v_j, i, j = 1, 2$. Now we claim that $G'_\Phi \in B_k$-VPG if and only if $\Phi$ is satisfiable, while $G'_\Phi \in B_{k+1}$-VPG is always true.

On one hand, if $G'_\Phi \in B_k$-VPG and $R'$ is a $B_k$-VPG representation of $G'_\Phi$, then the tips of $R'(u_1), u \in V(G_\Phi)$ form a 2-DIR representation of $G_\Phi$ ($R'(u_1)$ and $R'(v_1)$ may only intersect in their tips) and $\Phi$ is satisfiable.

On the other hand, if $\Phi$ is satisfiable, we represent $G_\Phi$ as a grid intersection graph and replace every segment of the representation by a clothespin with slim parallel tips and the body of the pin tiny enough so that does not intersect anything else in the representation. Similarly, if $\Phi$ is not satisfiable, we modify a 1-bend representation of $G_\Phi$ by replacing the paths of the representation by clothespins with 1-bend on the tips, thus obtaining a $B_{k+1}$-VPG representation of $G_\Phi$. The representation consists of a large part inherited from the representation of $G_\Phi$ and tiny parts representing the heads of the pins, but these can be made all of the same constant size and thus providing only a constant ratio refinement of the representation of $G_\Phi$. The representation is thus still of linear size and can be constructed in polynomial time.

5 Concluding Remarks

In this paper we have affirmatively settled two main conjectures of Asinowski et al [2] regarding VPG graphs. We have also demonstrated the relationship between $B_k$-VPG graphs and segment graphs.

The first conjecture that we settled claimed that $B_k$-VPG is a strict subset of $B_{k+1}$-VPG for all $k$. We have proven this constructively. Previously only the following separation was known: $B_0$-VPG $\subsetneq B_1$-VPG $\subsetneq$ VPG.

The second conjecture claimed that the $B_k$-VPG recognition problem is NP-Complete for all $k$. We have actually proven a stronger result; namely, that the
The $B_k$-VPG recognition problem is NP-Complete for all $k$ even when the input graph is a $B_{k+1}$-VPG graph. Previously only the NP-Completeness of $B_0$-VPG (from 2-DIR [11]) and VPG (from STRING [9,14]) were known.

Finally due to the close relationship between VPG graphs and segment graphs (i.e., since $B_0$-VPG = 2-DIR, and SEG $\subseteq$ STRING = VPG) we have considered the relationship between these classes. In particular, we have shown that:

- There is no $k$ such that 3-DIR is contained in $B_k$-VPG (i.e., SEG is not contained in $B_k$-VPG for any $k$).
- $B_1$-VPG is not contained in SEG.

Thus, to obtain polynomial time recognition algorithms, one would need to restrict attention to specific cases with (potentially) useful structure. In this respect, in [8], certain subclasses of $B_0$-VPG graphs have been characterized and shown to admit polynomial time recognition; namely split, chordal claw-free, and chordal bull-free $B_0$-VPG graphs are discussed in [8]. Additionally, in [5], $B_0$-VPG chordal and 2-row $B_0$-VPG have been shown to have polynomial time recognition algorithms. In particular, we conjecture that applying similar restrictions to the $B_k$-VPG graph class will also yield polynomial time recognition algorithms. It is interesting to note that since our separating examples are not chordal it is also open whether $B_k$-VPG chordal $\subset$ $B_{k+1}$-VPG chordal.

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