HEAVY TRAFFIC LIMIT FOR THE WORKLOAD PLATEAU PROCESS IN A TANDEM QUEUE WITH IDENTICAL SERVICE TIMES

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Abstract

We consider a two-node tandem queueing network in which the upstream queue is $GI/GI/1$ and each job reuses its upstream service requirement when moving to the downstream queue. Both servers employ the first-in-first-out policy. To investigate the evolution of workload in the second queue, we introduce and study a process $M$, called the plateau process, which encodes most of the information in the workload process. We focus on the case of infinite-variance service times and show that under appropriate scaling, workload in the first queue converges, and although the workload in the second queue does not converge, the plateau process does converge to a limit $M^*$ that is a certain function of two independent Lévy processes. Using excursion theory, we compare a time changed version of $M^*$ to a limit process derived in previous work.

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1 Introduction

The goal of this paper is to establish a stochastic process limit of a tandem network where the first queue is a $GI/GI/1$ queue and customers reuse their service requirement when moving to the second queue. This structure induces a strong dependence between arrivals and services at the second queue, leading to unusual phenomena. The tandem model under consideration was introduced for Poisson arrivals in the PhD thesis of O. Boxma where a rather complete analysis of the invariant distribution was given,
providing a rare example of a non-product form tandem queueing network for which an explicit analysis of the downstream queue is possible.

In addition, this model also shows unusual behavior in heavy traffic. In the finite variance case it is known [12, 13] that the amount of work at the second node is of smaller order than the amount of work at the first node as the system load $\rho$ (which is identical for both queues) tends to 1. For service times with bounded support, it is even shown in [6] that the expected value of the waiting time in the second queue is finite for $\rho = 1$. The intuition behind these results is that the amount of work in the first queue is driven by sums, and in the second queue is driven by maxima, suggesting that both queues should scale identically when the service times have infinite variance.

This behavior has recently been confirmed in our work [10], which is a prequel to the present study. In [10] we investigated the behavior of the workload of the second queue at embedded time points when the first queue empties. It was shown that this embedded Markov chain is sufficiently tractable, and analytic methods were used to investigate the process limit of this embedded Markov chain.

In the present paper we take up the task of analyzing the full workload process at the second node. The challenge is that this process does not converge in any of the usual Skorohod topologies in heavy traffic.

To see this, consider a generic period during which both queues are busy. During an interarrival time for the second queue, its workload will decrease by exactly the duration of this interarrival time. But this time equals the interdeparture time from the first queue, which equals the service time of the job about to transfer. Since this job reuses its service time at the second queue, this also equals the amount of work about to enter the second queue. The effect on the workload process in the second queue is a return to the height attained at the previous arrival time, unless the current job is larger than any previous job in the busy period.

The frequency of return to the same height can be seen in Figure 1, where the workload in the second queue must hit zero before each increase. When compared to the workload in the first queue, it is clear the behavior is very different because the workload in the second queue frequently has consecutive local maxima of the same value. If the new job is larger then all previous jobs, then the workload in the second queue may be zero for a nonzero amount of time, but if a big job is not larger than the current workload in the second queue then, under scaling, the workload process decreases at a rate converging to $\infty$ as $\rho \to 1$, for a period long enough for the workload to return to its previous height when the big job arrives. Thus, the scaled workload process will fail asymptotically to have a left limit at
Figure 1: The workload in both queues with identical service times in each. 1000 Poisson arrivals with parameter 1/3.1 service times are Pareto(1,3/2).
such a point.

We note that this type of behavior has also been mentioned in Whitt’s monograph [16], where new spaces \((E, F)\) to potentially deal with such fluctuations have been suggested. Though an approach using this framework would be interesting, we take a different approach in the present paper which is more tailored to the model we consider. Notice that the silhouette of the workload in the second queue seems to converge under the same scaling as the workload in the first queue. In contrast to the jagged peaks of the workload in the first queue, the silhouette is characterized by rolling hills.

Much of the information about the workload in the second queue is retained if we only keep track of these recurring levels or plateaus. In doing so we eliminate the oscillating behavior that prevents us from working directly with the workload in the second queue.

This is the strategy we follow. We introduce and study a process \(M\), called the plateau process, which encodes most of the information in the workload process. The plateau process is defined to be the workload in the second queue at the time of the most recent arrival. This definition eliminates the difficulty with scaling described above. We show that under the scaling mentioned above the plateau process converges to a limit \(M^*\) that is a certain function of two independent Lévy processes \(U^*\) and \(V^*\).

More explicitly, the \(N\)th job waits in the second queue for a period of time \(F(U, V, 1)(N)\), where \(U\) and \(V\) are the arrival and service processes for the model, and for two functions \(x, y : [0, \infty) \to \mathbb{R}\),

\[
F(x, y, c)(t) = \sup_{0 \leq s \leq t} \left( y(s) - y(s-) + \sup_{0 \leq r \leq s} (x(r) - y([r - c]^+)) \right) \\
- \sup_{0 \leq s \leq t} \left( x(s) - y([s - c]^+) \right).
\]

At time \(t\) the number of jobs that have arrived to the second queue is \(R(t)\), and the above composition of functions is continuous on a relevant set in the Skorohod path space \(\mathbb{D}\). For a sequence of models indexed by \(r\), the plateau process in the \(r\)th model can be written

\[
M^r(t) = F(U^r, V^r, 1)(R^r(t)).
\]

Letting \(\tilde{M}^r(t) = \frac{1}{a_r} M^r(rt)\), we show that

\[
\tilde{M}^r \Rightarrow M^*,
\]

where \(M^*(t) = F(U^* + \gamma \mu e, V^*, 0)(t/\mu)\); see Theorem 2.1 below.
The process appearing in the limit is not Markovian, but a suitable time-change is shown to be. Loosely speaking, our second result establishes that the embedded Markov chain of the limit process coincides with the limit of the embedded Markov chains considered in [10].

The paper is organized as follows. We first carefully define the model and scaling, make mild asymptotic assumptions, and state our main result, Theorem 2.1.

The bulk of the paper, Sections 3 and 4, is devoted to the proof, which is essentially an elaborate application of the continuous mapping theorem. This is a bit delicate because the relevant mapping \( F \) is not continuous everywhere. In particular, we first show in a series of steps that \( M^r(t) \) can indeed be represented as the composition of functions \( F(U^r, V^r, 1)(R^r(t)) \) described above. Then a series of steps shows that \( F \) is continuous on a particular subset of \( D \times D \times \mathbb{R} \) (see Lemma 4.5). Proving that for the limiting primitive processes \( U^* \) and \( V^* \), the triple \((U^* + \gamma \mu e, V^*, 0)\) is almost surely in this set enables a final application of the continuous mapping theorem together with the random time change theorem.

Finally, in Section 6 we develop some ideas from excursion theory to analyze the limit process \( M^* \) in a special case. After performing a time change using a local time derived from \( V^* \), the process becomes Markov. We are then able to apply some excursion theory results to calculate one dimensional distributions and relate this process to the limit derived in [10].

1.1 Notation

The following notation will be used throughout. Let \( \mathbb{N} = \{1, 2, \ldots\} \) and let \( \mathbb{R} \) denote the real numbers. Let \( \mathbb{R}_+ = [0, \infty) \). For \( a, b \in \mathbb{R} \), write \( a \vee b \) for the maximum, and \( a \wedge b \) for the minimum, \( [a]^+ = a \vee 0 \), \( [a]^− = a \wedge 0 \), \( [a] \) for the integer part of \( a \). For \( f : \mathbb{R}_+ \to \mathbb{R} \) let \( f^+(t) = \sup_{0 \leq s \leq t} f(s) \).

Let \( D = D([0, \infty), \mathbb{R}) \) be the space of real valued, right-continuous functions on \( [0, \infty) \) with finite left limits. We endow \( D \) with the skorohod \( J_1 \)-topology which makes \( D \) a Polish space [4]. For \( T \geq 0 \), let \( \rho_T(x, y) = \sup_{s \in [0, T]} |x(s) − y(s)| \). Let \( e \in D \) be the identity function \( e(t) = t \). For \( x \in D \), let \( x(t−) = \lim_{s \uparrow t} x(s) \), and let \( x^−(t) = x(t−) \) for \( t > 0 \) and \( x^−(0) = x(0) \).

Following Ethier and Kurtz [8] let \( \Lambda' \) be the collection of strictly increasing functions mapping \( \mathbb{R}_+ \) onto \( \mathbb{R}_+ \). Let \( \Lambda \subset \Lambda' \) be the set of Lipschitz continuous functions such that \( \lambda \in \Lambda \) implies \( \sup_{s > t \geq 0} \left| \frac{\lambda(s) − \lambda(t)}{s − t} \right| < \infty \).

We will often use [8] Proposition 3.5.3: let \( \{x_n\} \subset D \) and \( x \in D \). Then \( x_n \overset{J_1}{\rightarrow} x \) if and only if for each \( T > 0 \) there exists \( \{\lambda_n\} \subset \Lambda' \)
(possibly depending on $T$) such that $\lim_{n \to \infty} \sup_{0 \leq t \leq T} |\lambda_n(t) - t| = 0$ and $\lim_{n \to \infty} \sup_{0 \leq t \leq T} |x_n(t) - x(\lambda_n(t))| = 0$.

We write $X \sim Y$ if $X$ and $Y$ are equal in distribution. Weak convergence of random elements will be denoted by $\Rightarrow$. We adopt the convention that a sum of the form $\sum_{i=n}^{m}$ with $n > m$, or a sum over an empty set of indices equals zero.

2 Tandem queue model and main result

In this section we give a precise description of the tandem queue, specify our assumptions, and state our main result.

2.1 Definition of the model

We formulate a model equivalent to the one in Boxma [7]. The tandem queueing system consists of two queues Q1 and Q2 in series; both Q1 and Q2 are single-server queues with an unlimited buffer. Jobs enter the tandem system at Q1. After completion of service at Q1 a job immediately enters Q2, and when service at Q2, which is the exact same length as previously experienced in Q1, is completed it leaves the tandem system. Jobs are served individually and at both servers with the first in first out discipline. We assume the system is empty at time zero.

More precisely, at Q1 the exogenous arrival process $E(\cdot)$ is a renewal process. Jump times of this process correspond to times at which jobs enter the system. This renewal process is defined from a sequence of interarrival times $\{u_i\}_{i=1}^{\infty}$, where $u_1$ denotes the time at which the first job to arrive after time zero enters the system and $u_i$, $i \geq 2$, denotes the time between the arrival of the $(i-1)$st and the $i$th jobs to enter the system after time zero. Thus, $U_i = \sum_{j=1}^{i} u_j$ is the time at which the $i$th arrival enters the system, which is interpreted as zero if $i = 0$, and $E(t) = \sup\{i \geq 0 : U_i \leq t\}$ is the number of exogenous arrivals by time $t$. We assume that the sequence $\{u_i\}_{i=1}^{\infty}$ is an independent and identically distributed sequence of nonnegative random variables with $E[u_1] = \mu < \infty$.

At Q1, the service process, $\{V_i, i = 1, 2, \ldots\}$, is such that $V_i$ records the total amount of service required from the server by the first $i$ arrivals. More precisely, $\{v_i\}_{i=1}^{\infty}$ denotes an independent and identically distributed sequence of strictly positive random variables with common distribution function $F$, independent of the collection $\{u_i\}_{i=1}^{\infty}$. We interpret $v_i$ as the amount of processing time that the $i$th arrival requires from both servers.
The $v_i$’s are known as the \textit{service times}. Then, $V_i = \sum_{j=1}^{i} v_j$, which is taken to be zero if $i = 0$. It is assumed that $E[v_1] = \nu < \infty$.

For $t \geq 0$, let

$$ I(t) = \sup_{s \leq t} [V_E(s) - s]^- . $$

We interpret $I(t)$ as the cumulative amount of time that the first server has been idle up to time $t$. For $n \geq 0$, let

$$ I_n = I(U_n). $$

Then $I_n$ is the cumulative amount of time that first server has been idle up to the arrival of the $n$th job in the first queue.

Let $W_i(t)$ denote the (immediate) workload at time $t$ at $Q_i$, $i = 1, 2$, which is the total amount of time that the server must work in order to satisfy the remaining service requirement of each job present in the system at time $t$, ignoring future arrivals. For $t \geq 0$ we define

$$ W_1(t) = V_E(t) - t + I(t). $$

Let $D_n$ be the \textit{transfer time} of the $n$th job. So, the $n$th job exits $Q_1$ and enters $Q_2$ at time $D_n$. Let $d_1 = u_1 + v_1$ and $d_n = D_n - D_{n-1}$ for $n \geq 2$ be the \textit{intertransfer time} between arrivals of the $(n-1)$st and $n$th job to the second queue. For $n \geq 0$ we have

$$ D_n = V_n + I_n. $$

Let $R(t)$ denote the number of transfers to $Q_2$ by time $t$. For $t \geq 0$ we have

$$ R(t) = \sup\{n \geq 0 : D_n \leq t\}. \quad (1) $$

Let $J(t)$ denote the cumulative amount of time that the second server has been idle up to time $t$, and $W_2(t)$ as the workload in $Q_2$ at time $t$. That is, for $t \geq 0$ let

$$ J(t) = \sup_{s \leq t} [V_R(s) - s]^- , $$

$$ W_2(t) = V_R(t) - t + J(t). $$

If $k$ is the index of the first job in a busy period of the first queue then $W_1(t_k) = v_k$. Similarly, $W_2(D_k) = v_k$ if the $k$th job arrives to the second queue at a time when the second queue is empty.

Finally, let $M_n$ denote the workload in the second queue at the time of the arrival of the $n$th job to the second queue, which is just the sojourn time
of the \(n\)th job in the second queue. Let \(M(t)\) be the piecewise constant right continuous function that agrees with the work load in the second queue at each transfer time and whose discontinuities are contained in the transfer times. We call \(M(t)\) the \textit{plateau process}. For integers \(n \geq 0\) and real numbers \(t \geq 0\) we have

\[
M_n = W_2(D_n), \\
M(t) = M_{R(t)}.
\]

Finally, we define for \(t \geq 0\),

\[
U(t) = U_{[t]} \quad \text{and} \quad V(t) = V_{[t]}.
\]

### 2.2 Sequence of models, assumptions, and results

We now specify a sequence of tandem queueing models indexed by \(r \in \mathbb{R}\), where \(r\) increases to \(\infty\) through a sequence in \((0, \infty)\). Each model in the sequence is defined on the same probability space \((\Omega, \mathcal{F}, \mathbb{P})\). The \(r\)th model in the sequence is defined as in the previous section where we add a superscript \(r\) to each symbol. In particular, for \(t \geq 0\) let \(M^r(t)\) denote the plateau process in the \(r\)th system.

Then \(\{v_i^r\}_{i=1}^\infty\) and \(\{u_i^r\}_{i=1}^\infty\) are the service times and interarrival times to the first queue with positive, finite means \(E[v_i^r] = \nu^r\) and \(E[u_i^r] = \mu^r\) for each \(i = 1, 2, \ldots\) independent of each other. Define the following scaled versions of processes in the \(r\)th model for a sequence of positive reals \(a_r \to \infty\) and \(t \geq 0\),

\[
\bar{U}^r(t) = r^{-1}U(rt) \quad \text{and} \quad \bar{V}^r(t) = r^{-1}V(rt) \\
\bar{U}^r(t) = a_r^{-1} (U(rt) - r \mu^r t) \quad \text{and} \quad \bar{V}^r(t) = a_r^{-1} (V(rt) - r \nu^r t) \\
\bar{M}^r(t) = a_r^{-1}M^r(rt).
\]

Asymptotic assumptions. We make the following asymptotic assumptions, as \(r \to \infty\), about our sequence of models. Assume there is a sequence \(\{a_r\}\) such that \(r/a_r \to \infty\), \(\bar{U}^r(1) \Rightarrow U^*\), \(\bar{V}^r(1) \Rightarrow V^*\) in \(\mathbb{R}\). In this case \(U^*\) and \(V^*\) are centered infinitely divisible random variables; see Feller [9] XII.7. Then we have \(U^r \Rightarrow U^*\) and \(V^r \Rightarrow V^*\) in \(\mathbb{D}\), where \(U^*\) and \(V^*\) are Lévy stable motions with \(U^*(1) \sim U^*\) and \(V^*(1) \sim V^*\); see [16] supplement 2.4.1. We further assume \(\lim_{r \to \infty} \mu^r = \lim_{r \to \infty} \nu^r = \mu\) and the traffic intensity parameter for the \(r\)th system \(\rho^r = \mu/\nu^r\) satisfies

\[
\frac{r}{a_r} (1 - \rho^r) \to \gamma \in \mathbb{R}.
\]
Definition 1. Define the mapping \( F : \mathbb{D} \times \mathbb{D} \times \mathbb{R} \to \mathbb{D} \) by

\[
F(x, y, c)(t) = \sup_{0 \leq s \leq t} \left( y(s) - y(s-) + \sup_{0 \leq r \leq s} (x(r) - y([r - c]^+)) \right) \\
- \sup_{0 \leq s \leq t} \left( x(s) - y([s - c]^+) \right)
\]

The following is the main result of the paper.

Theorem 2.1. As \( r \to \infty \),

\( \mathcal{M}^r \Rightarrow \mathcal{M}^* \),

where \( \mathcal{M}^*(t) = F(U^* + \gamma \mu e, V^*, 0)(t/\mu) \).

3 The plateau process as a function of \( U \) and \( V \)

In this section we derive various relationships between the stochastic processes comprising the tandem queueing model. These relationships hold for any of the \( r \) indexed models, so we suppress superscripts referring to a particular model in sequence.

3.1 The idleness process for the first queue

This section is a prerequisite for understanding the arrival process in the second queue. If the cumulative idleness in the first queue is identically zero for all time, then the arrival process to the second queue is just a renewal process formed by the service times. Here we consider the cumulative idleness process in the first queue as a discrete time process. Consider the model defined in section 2.1.

Lemma 3.1. For each \( n \geq 1 \),

\[
I_n = u_1 + \max_{k=1}^{n} \left( \sum_{j=2}^{k} (u_j - v_{j-1}) \right),
\]

for \( n = 1, 2, \ldots \)

Proof. We proceed by induction. First observe that \( \sum_{j=2}^{1} (u_j - v_{j-1}) = 0 \), by convention, so

\[
\max_{k=1}^{n} \left( \sum_{j=2}^{k} (u_j - v_{j-1}) \right) \geq 0
\]
for $n \geq 1$. $I_1 = u_1 + \max_{k=1}^{1} \sum_{j=1}^{k} (u_j - v_{j-1}) = u_1$. For $n = 2$,

$$I_2 = u_1 + [u_2 - v_1]^+ = u_1 + \max_{k=1}^{2} \left( \sum_{j=2}^{k} (u_j - v_{j-1}) \right),$$

since there is no additional idleness if the second job arrives while the first job is in service. This is the base case for the induction.

For the inductive step, assume equation (5) holds for $n \geq 2$. There are two cases. In the first case the $(n + 1)$st job arrives before the $n$th service is complete. In this case the first job in the current busy period had index $i \leq n$, arrived at time $t_i$, and the total amount of work that has arrived since $t_i$, $\sum_{k=i}^{n} v_k$ exceeds the amount of time $\sum_{k=i+1}^{n+1} u_k$ since $t_i$. That is,

$$\sum_{k=i+1}^{n+1} u_k - v_{k-1} < 0,$$

for some $i \leq n$. Thus

$$\max_{k=1}^{n+1} \left( \sum_{j=2}^{k} (u_j - v_{j-1}) \right) = \max_{k=1}^{n} \left( \sum_{j=2}^{k} (u_j - v_{j-1}) \right),$$

and the cumulative idle time has not increased

$$I_n = I_{n+1} = u_1 + \max_{k=1}^{n+1} \left( \sum_{j=2}^{k} (u_j - v_{j-1}) \right).$$

In the second case, the $(n + 1)$st job arrives after the $n$th service is complete, so the total idle time just before the arrival of the $n + 1$ job is $u_1 + \sum_{k=2}^{n+1} u_k - v_{k-1}$. In this case, for any job $i \leq n$, the total amount of time $\sum_{k=i+1}^{n+1} u_k$ exceeds the total amount of work $\sum_{k=i}^{n} v_k$ since $t_i$. That is,

$$\sum_{k=i+1}^{n+1} u_k - v_{k-1} \geq 0.$$

Thus,

$$\left( \sum_{j=2}^{k} (u_j - v_{j-1}) \right) \leq \left( \sum_{j=2}^{n+1} (u_j - v_{j-1}) \right)$$
for each \( k = 2, \ldots, n+1 \), and we have
\[
\sum_{j=2}^{n+1} u_j - v_{j-1} = \max_{k=1}^{n+1} \left( \sum_{j=2}^{k} (u_j - v_{j-1}) \right).
\]

Note that the departure process of the first queue is equal to the arrival process \( R(\cdot) \) of the second queue. Since the queueing discipline is FIFO, the number of jobs that have arrived to the second queue by time \( t \) is the greatest number \( N \) such that the total amount of time needed to complete the first \( N \) jobs, \( \sum_{k=1}^{N} v_k \), is less than the amount of time spent working, \( t \) minus the cumulative idle time in the first queue.

### 3.2 Workload in the second queue

In this section we show how to write the plateau process \( M(\cdot) \) as a function of the primitive arrival and service processes. The following formula relates sojourn times in the second queue to service times and idleness in the first queue. It comes from Lindley recursion [1] for a FIFO queue \( W_2(D_{n+1}) = v_{n+1} + [W_2(D_n) - d_{n+1}]^+ \), where no independence needs to be assumed about the intertransfer times \( d_k \) and service times \( v_k \).

**Lemma 3.2.** The sojourn time of the \( n \)th job in the second queue is
\[
M_n = \max_{k=1}^{n} \{ v_k + I_k \} - I_n.
\]

**Proof.** Note that the sojourn time of the \( n \)th job includes its service time. The second queue is initially empty and the service time of the \( n \)th job is the same in both queues. Clearly \( I_1 = u_1 \), since the first queue is empty until the arrival of the first job. So,
\[
M_1 = v_1 = \max_{k=1}^{1} \{ v_k + I_k \} - I_1.
\]

The intertransfer time between the \( n \)th and \((n + 1)\)st job is \( d_{n+1} = v_{n+1} + (I_{n+1} - I_n) \). Proceeding by induction, suppose \( M_n = \max_{k=1}^{n} \{ v_k + I_k \} - I_n \).
Then, Lindley recursion gives

\[
M_{n+1} = v_{n+1} + [M_n - v_{n+1} - (I_{n+1} - I_n)]^+
\]

\[
= v_{n+1} \vee (M_n - (I_{n+1} - I_n))
\]

\[
= v_{n+1} \vee \left( \max_{k=1}^n (v_k + I_k) - I_n - (I_{n+1} - I_n) \right)
\]

\[
= \left[ (v_{n+1} + I_{n+1}) \vee \max_{k=1}^n (v_k + I_k) \right] - I_{n+1}
\]

\[
= \max_{k=1}^{n+1} (v_k + I_k) - I_{n+1}.
\]

Definition 2. Define the translation function \( G : \mathbb{D} \times \mathbb{R} \to \mathbb{D} \) by

\[ G(x, c)(t) = x([t - c]^{+}), \]

and define \( H : \mathbb{D} \times \mathbb{D} \times \mathbb{R}_+ \to \mathbb{D} \) as the composition

\[ H(x, y, c) = (x - G(y, c))^+. \]

More explicitly,

\[ H(x, y, c)(t) = \sup_{0 \leq s \leq t} \left( x(s) - y([s - c]^{+}) \right). \]

We can write \( I_n \) in terms of \( V \) and \( U \) from 3.

Lemma 3.3. For each \( n \geq 1 \),

\[ I_n = H(U, V, 1)(n), \]

Moreover \( H \) is constant on intervals of the form \([n, n+1]\) where \( n \) is an integer, so for each integer \( n \) we have \( H(U, V, n)(\lfloor t \rfloor) = H(U, V, n)(t) \) for all \( t \geq 0 \).

Proof. The processes \( V \) and \( U \) are constant between integers so \( H \) is constant on intervals of the form \([n, n+1]\), where \( n \) is an integer. For an integer \( k \),
Now we can write $R$ in terms of $U$ and $V$.

Corollary 3.4.

$$R(t) = \max \{m \geq 0 : V(m) + H(U,V,1)(m) \leq t\}.$$  

Proof. From Definition 1 we have $R(t) = \max\{N \geq 0 : \sum_{k=1}^{N} v_k + I_N \leq t\}$. We have $\sum_{k=1}^{N} v_k = V(N)$ by Definition 3 and $I_N = H(U,V,1)(N)$ by Lemma 3.3.

We can now write the plateau process in terms of the function $F$ defined in section 2.2. By Definitions 1 and 2,

$$F(x, y, c) = (y - y^- + H(x, y, c))^\uparrow - H(x, y, c),$$

or more explicitly,

$$F(x, y, c)(t) = \sup_{0 \leq s \leq t} (y(s) - y(s-) + H(x, y, c)(s)) - H(x, y, c)(t).$$

Lemma 3.5. For all $t \geq 0$,

$$M_{[t]} = F(U,V,1)(t).$$
Proof. By lemma 3.2,

\[ M_{[t]} = \max_{k=1}^{|t|} (v_k + I_k) - I_{[t]} \]

\[ = \max_{k=1}^{|t|} (V(k) - V(k-) + I_k) - I_{[t]} \]

\[ = \max_{k=1}^{|t|} (V(k) - V(k-) + H(U,V,1)(k)) - H(U,V,1)(\lfloor t \rfloor) \]

by lemma 3.3. For a positive integer \( k \) we have \( H(U,V,1)(t) \) is constant for \( t \) in \([k,k+1)\) and \( V(k) - V(k-) \geq V(t) - V(t-) \) for \( t \) in \([k,k+1)\). Thus,

\[ M_{[t]} = \sup_{0 \leq s \leq t} (V(s) - V(s-) + H(U,V,1)(s)) - H(U,V,1)(t) \]

\[ = F(U,V,1)(t). \]

\[ \square \]

Finally we can express \( M(\cdot) \) as function of \( U \) and \( V \). By Definition (2), \( M(t) \) is the composition \( M(\cdot) \) with the arrival process to the second queue. That is,

\[ M(t) = M_{R(t)} \]

\[ = F(U,V,1)(\max \{ m \geq 0 : V(m) + H(U,V,1)(m) \leq t \}). \]

Notice that the plateau process is greater than or equal to the workload in the second queue at each time, that is \( M(t) \geq W_2(t) \) for each \( t \geq 0 \).

4 Continuity properties of \( G, H, \) and \( F \)

In this section we identify a subset of the domain of \( F \) that almost surely contains the limits of the processes we are interested in and on which \( F \) is continuous. This result is obtained by treating \( F \) as a composition of continuous functions. The strategy of proof is similar to showing addition is continuous on a large subset of \( \mathbb{D} \times \mathbb{D} \) (see e.g. [15]).

Lemma 4.1. For any \( x \in \mathbb{D} \), \( G \) is continuous at \((x,0)\) in the product topology on \( \mathbb{D} \times \mathbb{R} \).

Proof. Let \( c_n \) be a sequence in \( \mathbb{R} \) with \( c_n \to 0 \), and let \( x_n \to x \) in \( \mathbb{D} \). Then for each \( T > 0 \) there exists \( \{ \lambda_n \} \subset \Lambda \) such that \( \sup_{0 \leq t \leq T} |\lambda_n(t) - t| \to 0 \) as \( n \to \infty \) and \( \sup_{0 \leq t \leq T} |x_n(t) - x(\lambda_n(t))| \to 0 \) as \( n \to \infty \).
For each \( n = 1, 2, \ldots \) define
\[
\tilde{\lambda}_n(t) = \begin{cases} 
\lambda_n(t - c_n), & \text{if } t \geq 2|c_n|, \\
\lambda_n \left( \left( 1 - \frac{sgn(c_n)}{2} \right) t \right), & \text{if } t < 2|c_n|,
\end{cases}
\]
where \( sgn(c_n) = -1 \) if \( c_n < 0 \), \( sgn(c_n) = 1 \) if \( c_n > 0 \), and \( sgn(c_n) = 0 \) if \( c_n = 0 \).

We have \( \{\tilde{\lambda}_n\} \subset \Lambda \) because each \( \tilde{\lambda}_n \) is the composition of two functions in \( \Lambda \). Now,
\[
\sup_{0 \leq t \leq T} |\tilde{\lambda}_n(t) - t| = \left( \sup_{0 \leq t < 2|c_n|} \left| \lambda_n \left( \left( 1 - \frac{sgn(c_n)}{2} \right) t \right) - t \right| \right) \vee \left( \sup_{2|c_n| \leq t \leq T} \left| \lambda_n(t - c_n) - t \right| \right)
\leq \left( \sup_{0 \leq t < 2|c_n|} \left| \lambda_n \left( \left( 1 - \frac{sgn(c_n)}{2} \right) t \right) - \left( 1 - \frac{sgn(c_n)}{2} \right) t \right| \right)
+ \left( \sup_{0 \leq t < 2|c_n|} \left| \left( 1 - \frac{sgn(c_n)}{2} \right) t - t \right| \right) \vee \left( \sup_{2|c_n| \leq t \leq T} \left| \lambda_n(t - c_n) - (t - c_n) \right| + |c_n| \right).
\]

When \( 0 \leq t < 2|c_n| \) we have \( 0 \leq \left( 1 - \frac{sgn(c_n)}{2} \right) t \leq 3|c_n| \), so
\[
\sup_{0 \leq t \leq T} |\tilde{\lambda}_n(t) - t| \leq \left( \sup_{0 \leq t < 3|c_n|} |\lambda_n(t) - t| + 3|c_n| \right) \vee \left( \sup_{2|c_n| - c_n \leq t \leq T - c_n} |\lambda_n(t) - t| + |c_n| \right)
\leq \sup_{0 \leq t \leq T} |\lambda_n(t) - t| + 3|c_n|,
\]
so \( \sup_{0 \leq t \leq T} |\tilde{\lambda}_n(t) - t| \to 0 \) as \( n \to \infty \).

Now, it suffices to show \( \sup_{0 \leq t \leq T} |G(x_n, c_n)(t) - G(x, 0)(\tilde{\lambda}_n(t))| \to 0 \) by [8], Proposition 3.5.3. We have

\[
15
\]
So it suffices to show $\sup_{2|c_n| \leq t \leq T} \left| G(x_n, c_n)(t) - G(x, 0)(\lambda_n(t)) \right|$

$$= \sup_{2|c_n| \leq t \leq T} \left| x_n([t - c_n]^+) - x(\lambda_n(t)) \right|$$

$$= \sup_{2|c_n| \leq t \leq T} \left| x_n(t - c_n) - x(\lambda_n(t - c_n)) \right|$$

$$= \sup_{2|c_n| - c_n \leq t \leq T - c_n} \left| x_n(t) - x(\lambda_n(t)) \right| \to 0 \quad (6)$$

So it suffices to show $\sup_{0 \leq t < 2|c_n|} \left| G(x_n, c_n)(t) - G(x, 0)(\lambda_n(t)) \right| \to 0$.

Fix $\epsilon > 0$ and let $\eta > 0$ such that $\sup_{0 \leq t \leq \eta} |x(0) - x(t)| < \epsilon$ by right continuity of $x$ at zero. Now, for $n$ so large that $|c_n| < \min(T/3, \eta/6)$, $\sup_{0 \leq t \leq \eta} |\lambda_n(t) - t| < \epsilon \wedge \eta/2$, and $\sup_{0 \leq t \leq T} |x_n(t) - x(\lambda_n(t))| < \epsilon$ consider the $c_n < 0$, $c_n > 0$, and $c_n = 0$ cases.

If $c_n < 0$,

$$\sup_{0 \leq t < 2|c_n|} \left| G(x_n, c_n)(t) - G(x, 0)(\lambda_n(t)) \right|$$

$$= \sup_{0 \leq t < 2|c_n|} \left| x_n([t - c_n]^+) - x(\lambda_n(t)) \right|$$

$$= \sup_{0 \leq t < -2c_n} \left| x_n(t - c_n) - x(\lambda_n(t - c_n)) \right|$$

$$\leq \sup_{0 \leq t < -2c_n} \left| x_n(t - c_n) - x(\lambda_n(t - c_n)) \right| + |x(\lambda_n(t - c_n)) - x(\lambda_n(3t/2))|$$

$$\leq \sup_{0 \leq t \leq T} |x_n(t) - x(\lambda_n(t))| + \sup_{0 \leq t < -2c_n} \left| x(\lambda_n(t - c_n)) - x(\lambda_n(3t/2)) \right|$$

$$\leq \sup_{0 \leq t \leq T} |x_n(t) - x(\lambda_n(t))| + \sup_{0 \leq t < -2c_n} |x(\lambda_n(t - c_n))| + \sup_{0 \leq t < -2c_n} |x(\lambda_n(3t/2))|$$.

We have $(t - c_n) \vee (3t/2) \leq -3c_n$ for $0 \leq t < -2c_n$, and so

$$\lambda_n(t - c_n) \vee \lambda_n(3t/2) \leq \lambda_n(-3c_n) \leq -3c_n + \eta/2 \leq \eta.$$

Thus,

$$\sup_{0 \leq t < 2|c_n|} \left| G(x_n, c_n)(t) - G(x, 0)(\lambda_n(t)) \right|$$

$$\leq \epsilon + \sup_{0 \leq t < -2c_n} |x(\lambda_n(t - c_n))| + \sup_{0 \leq t < -2c_n} |x(\lambda_n(3t/2))|$$

$$\leq \epsilon + \sup_{0 \leq t \leq \eta} |x(t)| + \sup_{0 \leq t \leq \eta} |x(t)| \leq 3\epsilon$$
If \( c_n > 0 \),

\[
\sup_{0 \leq t < 2c_n} \left| G(x_n, c_n)(t) - G(x, 0)(\lambda_n(t)) \right|
\]

\[
= \sup_{0 \leq t < 2c_n} \left| x_n([t - c_n]^+) - x(\lambda_n(t)) \right|
\]

\[
= \sup_{0 \leq t < 2c_n} \left| x_n([t - c_n]^+) - x(\lambda_n(t/2)) \right|
\]

\[
\leq \sup_{0 \leq t < c_n} |x_n(0) - x(\lambda_n(t/2))| \vee \sup_{c_n \leq t < 2c_n} |x_n(t - c_n) - x(\lambda_n(t/2))|.
\]

(7)

For the first term,

\[
\sup_{0 \leq t \leq c_n} |x_n(0) - x(\lambda_n(t/2))| \leq \sup_{0 \leq t < c_n} |x_n(0) - x(0)| + |x(0) - x(\lambda_n(t/2))|
\]

\[
= |x_n(0) - x(\lambda_n(0))| + \sup_{0 \leq t < c_n} |x(0) - x(\lambda_n(t/2))|
\]

\[
\leq \sup_{0 \leq t \leq T} |x_n(t) - x(\lambda_n(t))| + \sup_{0 \leq t \leq \eta} |x(0) - x(t)| \leq \epsilon,
\]

since \( \lambda_n(t/2) \leq \lambda_n(c_n/2) \leq c_n/2 + \eta/2 \leq \eta \) for \( 0 \leq t \leq c_n \). For the second term,

\[
\sup_{c_n \leq t < 2c_n} |x_n(t - c_n) - x(\lambda_n(t/2))| = \sup_{0 \leq t < c_n} \left| x_n(t) - x\left(\lambda_n\left(\frac{t + c_n}{2}\right)\right)\right|
\]

\[
\leq \sup_{0 \leq t < c_n} |x_n(t) - x(\lambda_n(t))| + \left|x(\lambda_n(t)) - x\left(\lambda_n\left(\frac{t + c_n}{2}\right)\right)\right|
\]

\[
\leq \epsilon + \sup_{0 \leq t < c_n} \left|x(\lambda_n(t)) - x(0) + x(0) - x\left(\lambda_n\left(\frac{t + c_n}{2}\right)\right)\right|
\]

\[
\leq \epsilon + \sup_{0 \leq t < c_n} |x(\lambda_n(t)) - x(0)| + \sup_{0 \leq t < c_n} \left|x(0) - x\left(\lambda_n\left(\frac{t + c_n}{2}\right)\right)\right|
\]

\[
\leq \epsilon + 2 \sup_{0 \leq t \leq \eta} |x(0) - x(t)| \leq 3\epsilon,
\]

since \( \lambda_n(t) \vee \lambda_n\left(\frac{t + c_n}{2}\right) \leq \lambda_n(c_n) \leq c_n + \eta/2 \leq \eta \) for \( 0 \leq t \leq c_n \).

If \( c_n = 0 \) then \( \lambda_n = \lambda_n^{0} \) so \( G(x_n, c_n)(t) - G(x, 0)\left(\tilde{\lambda}_n(t)\right) = x_n(t) - x(\lambda_n(t)) \),

which converges to zero uniformly by assumption.

So in all three cases we have

\[
\sup_{0 \leq t < 2c_n} \left| G(x_n, c_n)(t) - G(x, 0)(\tilde{\lambda}_n(t)) \right| \leq 3\epsilon.
\]
Together with (6) and since $\epsilon$ was arbitrary, we have
\[
\sup_{0 \leq t \leq T} \left| G(x_n, c_n)(t) - G(x, 0)(\tilde{\lambda}_n(t)) \right| \to 0
\]
as $n \to \infty$.

So we have $G(x_n, c_n) \to G(x, 0)$ on $D$. \hfill \blacksquare

For $x \in D$, let $\text{Disc}(x)$ denote the set of discontinuities of $x$.

**Lemma 4.2.** $H$ is continuous at $(x, y, 0)$ for all $x, y \in D$ such that
\[
\text{Disc}(x) \cap \text{Disc}(y) = \emptyset.
\]

**Proof.** Let $c_n \in \mathbb{R}$ with $c_n \to 0$ and let $x_n$ and $y_n$ be in $D$ such that $x_n \to x$ and $y_n \to y$ and fix a time $T > 0$. Let $z_n = y_n - x_n$ and $z = y - x$.

Since $\text{Disc}(x) \cap \text{Disc}(y) = \emptyset$, \textbf{[15]} Theorem 4.1 tells us that there exists $\{\lambda_n\} \subset \Lambda'$ such that $\rho_T(\lambda_n, e) \to 0$ and $\rho_T(z_n, z \circ \lambda_n) \to 0$. Since $G$ is continuous at $(z, 0)$ by lemma \textbf{[4.1]} and $(z_n, c_n) \to (z, 0)$ we have $\{\lambda_n\} \subset \Lambda'$ such that $\rho_T(\lambda_n, e) \to 0$ and $\rho_T(G(z_n, c_n), z \circ \lambda_n) \to 0$. In fact, we may construct $\lambda_n$ as in the proof of \textbf{[4.1]} since $x \to x^\uparrow$ is continuous on $D$ and $(x)^\uparrow \circ \tilde{\lambda} = (x \circ \tilde{\lambda})^\uparrow$, we have $\rho_T(H(x_n, y_n, c_n), H(x, y, 0) \circ \lambda_n) \to 0$. Since $T$ was arbitrary we have $H$ is continuous $(x, y, 0)$. \hfill \blacksquare

**Lemma 4.3.** For all $x, y \in D$,
\[
\text{Disc}(H(x, y, 0)) \subset \{ t : y(t) - y(t^{-}) > 0 \} \cup \{ t : x(t) - x(t^{-}) < 0 \}.
\]

In particular, if $\{ t : x(t) - x(t^{-}) < 0 \} = \emptyset$, then
\[
\text{Disc}(H(x, y, 0)) \subset \text{Disc}(y).
\]

**Proof.** $\text{Disc}(H(x, y, 0)) = \{ t : H(x, y, 0)(t) - H(x, y, 0)(t^{-}) \neq 0 \} = \{ t : H(x, y, 0)(t) - H(x, y, 0)(t^{-}) > 0 \}$ since $H(x, y, 0)$ is nondecreasing. Thus,
\[
\text{Disc}(H(x, y, 0)) \subset \{ t : (y - x)(t) - (y - x)(t^{-}) > 0 \}
\]
\[
\subset \{ t : y(t) - y(t^{-}) > 0 \} \cup \{ t : x(t) - x(t^{-}) < 0 \}.
\]

\hfill \blacksquare

**Lemma 4.4.** Let $\lambda_n$ and $\gamma_n$ be strictly increasing homeomorphisms from $[0, T]$ onto $[0, T]$ and $x_n, x \in D$ such that for some finite collection $\{t_j\}_{j=0}^N \subset [0, T]$ with
(i) \( 0 = t_0 < t_1 < \cdots < t_N = T \) we have \( \lambda_n^{-1}(t_j) = \gamma_n^{-1}(t_j) \) for each \( j = 0, 1, 2, \ldots, N \),

(ii) \( \rho_T(x_n, x \circ \lambda_n) < \epsilon \), and

(iii) \( w(x, [t_{j-1}, t_j)) = \sup \{ |x(t) - x(s)| : t, s \in [t_{j-1}, t_j) \} < \epsilon \) for each \( j = 1, 2, \ldots, N \),

then

\[
\rho_T(x_n, x \circ \gamma_n) < 3 \epsilon.
\]

Proof. Let \( r_j = \gamma_n^{-1}(t_j) = \lambda_n^{-1}(t_j) \) for \( j = 0, 1, \ldots, N \), so that \( \bigcup_{j=1}^N [r_{j-1}, r_j) = \)
\[ \bigcup_{j=1}^{N} [t_{j-1}, t_j) = [0, T) \]. Then

\[
\rho_T(x_n, x \circ \gamma_n) = \sup_{0 \leq t \leq T} |x_n(t) - x(\gamma_n(t))| \\
= \max_{k=1}^{N} \sup_{r_{j-1} \leq t < r_j} |x_n(t) - x(\gamma_n(t))| \lor |x_n(T) - x(T)| \\
= \max_{k=1}^{N} \sup_{t_{j-1} \leq t < t_j} |x_n(\gamma_n^{-1}(t)) - x(t)| \lor |x_n(T) - x(T)| \\
= \max_{k=1}^{N} \sup_{t_{j-1} \leq t < t_j} |x_n(\gamma_n^{-1}(t)) - x(t_{j-1}) + x(t_{j-1}) - x(t)| \\
\lor |x_n(T) - x(T)| \\
\leq \max_{k=1}^{N} \left( \sup_{t_{j-1} \leq t < t_j} |x_n(\gamma_n^{-1}(t)) - x(t_{j-1})| + w(x, [t_{j-1}, t_j]) \lor |x_n(T) - x(T)| \right) \\
\lor |x_n(T) - x(T)| \\
\leq \max_{k=1}^{N} \left( \sup_{r_{j-1} \leq t < r_j} |x_n(t) - x(\lambda_n(r_{j-1}))| \lor |x_n(T) - x(T)| \right) \\
\lor |x_n(T) - x(T)| \\
\leq \max_{k=1}^{N} \left( \sup_{r_{j-1} \leq t < r_j} |x_n(t) - x(\lambda_n(t))| \\
+ |x(\lambda_n(t)) - x(\lambda_n(r_{j-1}))| + \epsilon \lor |x_n(T) - x(T)| \right) \\
\lor |x_n(T) - x(T)| \\
\leq \max_{k=1}^{N} \left( \sup_{r_{j-1} \leq t < r_j} |x_n(t) - x(\lambda_n(t))| + w(x, [t_{j-1}, t_j]) + \epsilon \lor |x_n(T) - x(T)| \right) \\
\lor |x_n(T) - x(T)| \\
\leq \max_{k=1}^{N} \left( \sup_{r_{j-1} \leq t < r_j} |x_n(t) - x(\lambda_n(t))| + 2\epsilon \lor |x_n(T) - x(T)| \right) \\
\leq \rho_T(x_n, x \circ \lambda_n) + 2\epsilon \\
\leq 3\epsilon.
\]

Finally, we prove that \( F \) is continuous on a relevant set.

**Lemma 4.5.** \( F \) is continuous at \((x, y, 0)\) in the product topology on \( \mathbb{D} \times \mathbb{D} \times \mathbb{R} \),
for all \( x \) and \( y \in \mathbb{D} \) with \( \text{Disc}(x) \cap \text{Disc}(y) = \emptyset \) and

\[
\{ t : y(t) - y(t-) < 0 \} = \emptyset.
\]

**Proof.** Let \( T > 0 \), let \( \rho_T \) be the uniform metric on function from \([0, T] \) to \( \mathbb{R} \), and fix \( \epsilon > 0 \). Apply Lemma 1 on page 110 of [4] to construct finite subsets \( A_1 = \{ t_j' \} \) and \( A_2 = \{ s_j \} \) of \([0, T] \) such that \( 0 = t_0' < \cdots < t_k' = T \), \( 0 = s_0 < \cdots < s_m = T \), \( w(y; [t_j'_{j-1}, t_j']) = \sup \{ |y(s) - y(t)| : s, t \in [t_j'_{j-1}, t_j'] \} < \epsilon \) and \( w(H(x, y, 0); [s_j-1, s_j]) < \epsilon \) for all \( j \). Since \( \text{Disc}(y) \cap \text{Disc}([H(x, y, 0)] \subset \text{Disc}(x) \cap \text{Disc}(y) = \emptyset \), the two sets \( A_1 \) and \( A_2 \) can be chosen so that \( A_1 \cap A_2 = [0, T] \). Note that \( w(y; [t_j-1, t_j]) < \epsilon \) and \( w(H(x, y, 0); [t_j-1, t_j]) < \epsilon \) for \( \{ t_j \} = A_1 \cup A_2 \). Let \( 2\delta \) be the distance between the closest two points in \( A_1 \cup A_2 \). Choose \( n_0 \) and homeomorphisms \( \lambda_n \) and \( \mu_n \) in \( 
abla \) so that

(i) \( \rho_T(y_n, y \circ \lambda_n) < (\delta \wedge \epsilon) \),

(ii) \( \rho_T(\lambda_n, \epsilon) < (\delta \wedge \epsilon) \),

(iii) \( \rho_T(H(x_n, y_n, c_n), H(x, y, 0) \circ \mu_n) < (\delta \wedge \epsilon) \), and

(iv) \( \rho_T(\mu_n, \epsilon) < (\delta \wedge \epsilon) \)

for \( n \geq n_0 \). Thus for \( n \geq n_0 \)

\[
\lambda_n^{-1}(A_1) \cap \mu_n^{-1}(A_2) = \{ 0, T \}
\]

and \( \{ r_j \} = \lambda_n^{-1}(A_1) \cup \mu_n^{-1}(A_2) \) has corresponding points in the same order as \( \{ t_j \} = A_1 \cup A_2 \). Let \( \gamma_n \) be homeomorphisms of \([0, T] \) defined by

\[
\gamma_n(r_j) = t_j
\]

for corresponding points \( r_j \in \lambda_n^{-1}(A_1) \cup \mu_n^{-1}(A_2) \) and \( t_j \in A_1 \cup A_2 \) and by linear interpolation elsewhere.

Note that for each \( r_j \in \lambda_n^{-1}(A_1) \cup \mu_n^{-1}(A_2) \) either

\[
\lambda_n(r_j) = t_j \quad \text{or} \quad \mu_n(r_j) = t_j.
\]

Since \( t \mapsto |\gamma_n(t) - t| \) is continuous the maximum is attained at some critical point (exposed point) \( r_j \), so \( \rho_T(\gamma_n, \epsilon) < \rho_T(\lambda_n, \epsilon) \vee \rho_T(\mu_n, \epsilon) < \epsilon \). Now,

\[
\rho_T(F(x_n, y_n, c_n), F(x, y, 0) \circ \gamma_n)
\leq \rho_T(\left( y_n - y_n + H(x_n, y_n, c_n) \right), H(x, y, 0)) \circ \gamma_n) + \rho_T(\{ H(x_n, y_n, c_n), (H(x, y, 0)) \circ \gamma_n) .
\]
For the first term we have
\[
\rho_T \left( (y_n - y_\gamma_n + H(x_n, y_n, c_n))^\dagger, (y - y_\gamma + H(x, y, 0))^\dagger \right) \circ \gamma_n
\]
\[
\leq \rho_T \left( y_n - y_\gamma_n + H(x_n, y_n, c_n), (y - y_\gamma + H(x, y, 0)) \circ \gamma_n \right),
\]
and
\[
\rho_T \left( y_n - y_\gamma_n + H(x_n, y_n, c_n), (y - y_\gamma + H(x, y, 0)) \circ \gamma_n \right)
\]
\[
\leq \rho_T \left( y_n, y \circ \gamma_n \right) + \rho_T \left( y_n, y - y_\gamma \circ \gamma_n \right) + \rho_T \left( H(x_n, y_n, c_n), H(x, y, 0) \circ \gamma_n \right).
\]

Since \( \gamma_n \) is strictly increasing,
\[
\rho_T \left( y_n - y_\gamma \circ \gamma_n \right) = \sup_{0 \leq t \leq T} \left| \lim_{s \to t} y_n(s) - \lim_{r \to \gamma_n(t)} y(r) \right|
\]
\[
= \sup_{0 \leq t \leq T} \left| \lim_{s \to t} y_n(s) - \lim_{r \to t} y(\gamma_n(r)) \right|
\]
and so
\[
\rho_T \left( y_n, y - y_\gamma \circ \gamma_n \right) \leq \sup_{0 \leq t \leq T} \left| y_n(t) - y(\gamma_n(t)) \right|
\]
since the left limit of \( y_n \) and \( y \circ \gamma_n \) exist at each \( t \). Therefore,
\[
\rho_T \left( y_n, y - y_\gamma \circ \gamma_n \right) \leq \rho_T \left( y_n, y \circ \gamma_n \right)
\]
Combining (4,8,9,10) we have,
\[
\rho_T \left( F(x_n, y_n, c_n), F(x, y, 0) \circ \gamma_n \right)
\]
\[
\leq \rho_T \left( (y_n - y_\gamma_n + H(x_n, y_n, c_n))^\dagger, (y - y_\gamma + H(x, y, 0))^\dagger \right) \circ \gamma_n
\]
\[
+ \rho_T \left( H(x_n, y_n, c_n), H(x, y, 0) \circ \gamma_n \right)
\]
\[
\leq 2\rho_T \left( y_n, y \circ \gamma_n \right) + 2\rho_T \left( H(x_n, y_n, c_n), H(x, y, 0) \circ \gamma_n \right)
\]
\[
\leq 12\epsilon,
\]
by lemma 4.3

5 Scaling limit of the plateau process

In this section we prove several results concerning the sequence of models, and then combine these to prove Theorem 2.1. We begin by showing that the function \( H \) scales nicely when no centering is required.
Lemma 5.1. For positive constants $a_n$ and $n$,
\[ a_n^{-1}H(x, y, c)(nt) = H(x^n, y^n, c/n)(t), \]
for all $t \geq 0$, where $x^n(t) = a_n^{-1}x(nt)$ and $y^n(t) = a_n^{-1}y(nt)$.

Proof. By definition,
\[ a_n^{-1}H(x, y, c)(nt) = a_n^{-1} \sup_{0 \leq s \leq nt} (x(s) - y([s - c]_+)) \]
\[ = \sup_{0 \leq s \leq t} (a_n^{-1}x(ns) - a_n^{-1}y([ns - c]_+)) \]
\[ = \sup_{0 \leq s \leq t} (a_n^{-1}x(ns) - a_n^{-1}y(n[s - c/n]_+)) \]
\[ = \sup_{0 \leq s \leq t} (x^n(s) - y^n([s - c/n]_+)) \]
\[ = H(x^n, y^n, c/n)(t) \]
\[ \square \]

Lemma 5.2. The set $\mathcal{K} = \{ x \in \mathbb{D} : x(t) - x(t-) \geq 0 \text{ for each } t \in (0, \infty) \}$ is closed in $\mathbb{D}$.

Proof. Let $\{ x_n \}$ be a sequence in $\mathcal{K}$ such that $x_n \to x$. Fix $t_0 \in (0, \infty)$ with $x(t_0) - x(t_0-) \neq 0$. There exists $t_n \to t_0$ with $x_n(t_n) - x_n(t_n-) \to x(t_0) - x(t_0-)$ by [11] proposition VI.2.1. We have $x_n(t_n) - x_n(t_n-) \geq 0$ for each $n$ since $x_n \in \mathcal{K}$, so $x(t_0) - x(t_0-) \geq 0$ and we must have $x \in \mathcal{K}$.

The next Lemma establishes a joint convergence involving the primitive input processes. Recall that $\bar{U}^r \Rightarrow U^*$ and $\bar{V}^r \Rightarrow V^*$ in $\mathbb{D}$.

Lemma 5.3. For any sequence of real numbers $c_r \to c$,
\[ (\bar{U}^r + c_r e, \bar{V}^r, 1/r) \Rightarrow (U^* + ce, V^*, 0), \]
in the product topology on $\mathbb{D} \times \mathbb{D} \times \mathbb{R}$. Moreover,
\[ \text{Disc}(U^* + ce) \cap \text{Disc}(V^*) = \emptyset \text{ a.s.} \]
and $\{ t : V^*(t) - V^*(t-) < 0 \} = \emptyset \text{ a.s.}$

Proof. Since $ce$ is continuous, $\bar{U}^r \Rightarrow U^*$, and $c_r e \Rightarrow ce$ we have $\bar{U}^r + c_r e \Rightarrow U^* + ce$ by [15]. We have joint convergence $(\bar{U}^r + c_r e, \bar{V}^r) \Rightarrow (U^* + ce, V^*)$ since $\bar{V}^r$ is independent of $\bar{U}^r$ and therefore $\bar{U}^r + c_r e$ is independent of $\bar{V}^r$ because $c_r$ is constant in $\omega$. [16] Theorem 11.4.4, moreover $\bar{U}^r$ is independent
of $V^*$. Since $1/r$ is constant in $\omega$ we have $1/r \to 0$ in probability so [4] Theorem 4.4 gives joint convergence
\[
(V^r + c, e, U^r, 1/n) \Rightarrow (U^* + ce, V^*, 0).
\]

$V^*$ is a stable Lévy motion by 2.4.1 of the online supplement to [16]. So $V^*$ has no fixed discontinuities: $\mathbb{P}\{U^*(t) = U^*(t^-)\} = 1$ for all $t \in (0, \infty)$. By [15] Lemma 4.3, gives $\mathbb{P}\{\text{Disc}(U^*) \cap \text{Disc}(V^*) = \emptyset\} = 1$ and since $ce$ is continuous we have
\[
\mathbb{P}\{\text{Disc}(U^* + ce) \cap \text{Disc}(V^*) = \emptyset\} = 1.
\]

Finally, $\mathbb{P}\{\bar{V}^r \in \mathcal{K}\} = 1$, $\bar{V}^r \Rightarrow \mu e$ in $\mathbb{D}$. Similarly, $\bar{V}^r \Rightarrow \mu e$ in $\mathbb{D}$. Now compute
\[
\bar{R}^r(t) = \frac{1}{r} \sup \{m \geq 0 : V^r(m) + H(U^r, V^r, 1)(m) \leq t\}
\]
\[
= \sup \{x/r \geq 0 : V^r(x) + H(U^r, V^r, 1)(x) \leq rt\}
\]
\[
= \sup \left\{ \frac{y}{r} \geq 0 : \frac{V^r(ry)}{r} + \frac{1}{r} H(U^r, V^r, 1)(ry) \leq t \right\}
\]
\[
= \sup \left\{ y \geq 0 : \frac{V^r(y)}{r} + \frac{1}{r} H(U^r, V^r, 1)(y) \leq t \right\},
\]

Finally, $\bar{V}^r \Rightarrow \mu e$ in $\mathbb{D}$. Similarly, $\bar{V}^r \Rightarrow \mu e$ in $\mathbb{D}$. Now compute
\[
\bar{R}^r(t) = \frac{1}{r} \sup \{m \geq 0 : V^r(m) + H(U^r, V^r, 1)(m) \leq rt\}
\]= \sup \{x/r \geq 0 : V^r(x) + H(U^r, V^r, 1)(x) \leq rt\}
\]= \sup \left\{ \frac{y}{r} \geq 0 : \frac{V^r(ry)}{r} + \frac{1}{r} H(U^r, V^r, 1)(ry) \leq t \right\}
\]= \sup \left\{ y \geq 0 : \frac{V^r(y)}{r} + \frac{1}{r} H(U^r, V^r, 1)(y) \leq t \right\}.
by lemma 5.1. We have \((\bar{U}^r, \bar{V}^r, 1/r) \Rightarrow (\mu e, \mu e, 0)\) in \(\mathbb{D}\) since the processes are independent. The function \(H\) is continuous at \((\mu u e, \mu v e, 0)\), and addition is continuous at continuous elements of \(\mathbb{D}\), so
\[
\bar{V}^r + H(\bar{U}^r, \bar{V}^r, 1/r) \Rightarrow \mu e
\]
in \(\mathbb{D}\). The result follows because \(\mu e\) is in the set of continuity for the function \(x \mapsto \sup\{y \geq 0 : x(y) \leq t\}\) by Corollary 13.6.4 in [16]. ■

We now prove the main result.

**Proof of Theorem 2.1.** By Lemma 3.5
\[
M(t) = F(U^r, V^r, 1)(R(t)).
\]
Under fluid scaling \(\bar{R}^r \Rightarrow e/\mu\) by 5.4. We first consider the scaling limit for \(F\), before composing with \(R\).
\[
a_r^{-1} F(U^r, V^r, 1)(rt) = a_r^{-1} \sup_{0 \leq s \leq rt} (V^r(s) - V^r(s-) + H(U^r, V^r, 1)(s))
= a_r^{-1} H(U^r, V^r, 1)(rt)
= \sup_{0 \leq s \leq t} (a_r^{-1} V^r(s) - a_r^{-1} V^r(s-) + a_r^{-1} H(U^r, V^r, 1)(s))
= a_r^{-1} H(U^r, V^r, 1)(rt).
\]
t \(\mapsto \nu^r t\) is continuous so \(\nu^r(r) - \nu^r(r-) = 0\) and
\[
a_r^{-1} F(U^r, V^r, 1)(rt) = \sup_{0 \leq s \leq t} (V^r(s) - V^r(s-) + a_r^{-1} H(U^r, V^r, 1)(s))
= a_r^{-1} H(U^r, V^r, 1)(rt).
\]
Now, we address the idleness part of (11) that occurs twice.
\[ a_r^{-1}H(U^r, V^r, 1)(rt) \]
\[ = a_r^{-1} \sup_{0 \leq s \leq rt} \left( U^r(s) - V^r([s - 1]^+) \right) \]
\[ = \sup_{0 \leq s \leq t} \left( a_r^{-1}U^r(rs) - a_r^{-1}V^r(r[s - 1/r]^+) \right) \]
\[ = \sup_{0 \leq s \leq t} \left( a_r^{-1}(U^r(rs) - r\mu^r s) + a_r^{-1} r\mu^r s \right. \]
\[ \left. - a_r^{-1}(V^r(r[s - 1/r]^+) - rv^r[s - 1/r]^+) - a_r^{-1}rv^r[s - 1/r]^+ \right) \]
\[ = \sup_{0 \leq s \leq t} \left( \tilde{U}^r(s) + a_r^{-1}r\mu^r s - V^r([s - 1/r]^+) - a_r^{-1}rv^r[s - 1/r]^+ \right) \]
\[ = \sup_{0 \leq s \leq t} \left( \tilde{U}^r(s) + a_r^{-1}r(\mu^r - v^r)s + a_r^{-1}rv^r(s - [s - 1/r]^+) \right. \]
\[ \left. - \tilde{V}^r([s - 1/r]^+) \right). \]

Since
\[ a_r^{-1}rv^r(s - [s - 1/r]^+) = a_r^{-1}rv^r(1/r \wedge s) = a_r^{-1}v^r(1 \wedge rs), \]
we have
\[ a_r^{-1}H(U^r, V^r, 1)(rt) \]
\[ = H(\tilde{U}^r + a_r^{-1}r(\mu^r - v^r)e + a_r^{-1}v^r(1 \wedge re), \tilde{V}^r, 1/r)(t). \]

Putting this expression back into (11),
\[ a_r^{-1}F(U^r, V^r, 1)(rt) = \sup_{0 \leq s \leq t} \left[ \tilde{V}^r(s) - \tilde{V}^r(s-) \right. \]
\[ + H(\tilde{U}^r + a_r^{-1}r(\mu^r - v^r)e + a_r^{-1}v^r(1 \wedge re), \tilde{V}^r, 1/r)(s) \]
\[ \left. - H(\tilde{U}^r + a_r^{-1}r(\mu^r - v^r)e + a_r^{-1}v^r(1 \wedge re), \tilde{V}^r, 1/r)(t) \right) \]
\[ = F(\tilde{U}^r + a_r^{-1}r(\mu^r - v^r)e + a_r^{-1}v^r(1 \wedge re), \tilde{V}^r, 1/r)(t). \]

By Lemma 5.3 we have \((U^* + \gamma e, V^*, 0)\) satisfies the continuity criterion of Lemma 4.5. By the continuous mapping theorem
\[ F(\tilde{U}^r + a_r^{-1}r(\mu^r - v^r)e + a_r^{-1}v^r(1 \wedge re), \tilde{V}^r, 1/r) \Rightarrow F(U^* + \gamma e, V^*, 0). \]
Finally, the scaled plateau process is a composition of $F$ with $R$, 

$$a_r^{-1} F(U^r, V^r, 1)(R(rt)) = a_r^{-1} F(U^r, V^r, 1)(rR^r(t)).$$

Composition is continuous on $(\mathbb{D} \times C_0)$ by [15] Theorem 3.1, where $C_0 \subset \mathbb{D}$ denotes the strictly increasing, continuous functions. So the continuous mapping theorem yields 

$$a_r^{-1} M^r(r \cdot) = M^r \Rightarrow M^* = F(U^* + \gamma \mu e, V^*, 0)(\cdot/\mu).$$

\[\blacksquare\]

### 6 Analysis of the limit process

In this section we derive some properties of the stochastic process $M^*$ that appears as the scaling limit of the plateau process. This allows us in particular to relate the process to the results obtained in [10]; we show that an appropriately time-changed version of $M^*$ is the limit of the correspondingly time-changed processes studied in [10]. Our analysis indicates that while the limit process $M^*$ is not Markov, it has some regenerative properties in a special case.

We focus on the special case where the interarrival time distribution has finite variance, leading to a trivial limit for the arrival process $U^*(t) \equiv 0$. In this case we are using the function

$$F(\gamma \mu e, y, 0)(t/\mu) = \sup_{0 \leq s \leq t/\mu} [y(s) - y(s-) + \sup_{0 \leq r \leq s} [\gamma \mu r - y(r)] - \sup_{0 \leq s \leq t/\mu} [\gamma \mu s - y(s)],$$

where $y$ is replaced by the $\alpha$-stable process $V^*$. If we define $X(t) = V^*(t) - \gamma \mu t$, this expression reduces to

$$F(\gamma \mu e, V^*, 0)(t/\mu) = \sup_{0 \leq s \leq t/\mu} [X(s) - X(s-) - \inf_{0 \leq r \leq s} X(r)] + \inf_{0 \leq s \leq t/\mu} X(s).$$

Write $\overline{X}(t) = \inf_{0 \leq s \leq t} X(s)$ so that this expression reduces to

$$\sup_{0 \leq s \leq t/\mu} [X(s) - X(s-) + \overline{X}(t/\mu) - \overline{X}(s)]. \quad (12)$$

Observe now that $L(t) = -\overline{X}(t)$ is the local time at zero for the reflected process $Y(t) = X(t) - \overline{X}(t)$ associated with $X$. For $v \geq 0$ define

$$Z(v) = \sup_{0 \leq s \leq L^{-1}(v)} [X(s) - X(s-) - (v - L(s))],$$

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where $L^{-1}$ denotes the right-continuous inverse. Then from (12) we see that for times $t$ such that $L(t/\mu) = v$, $M^*(t) = Z(v)$. Put another way, we have for all $t \geq 0$ that $M^*(\mu L^{-1}(L(t/\mu))) = Z(L(t/\mu))$. That is, the process $Z(v) = M^*(\mu L^{-1}(v))$ is a certain time-changed (and embedded) version of the process $M^*$, evaluated at times (scaled by $\mu$) when the local time of $Y$ has attained the level $v$. We now examine the one-dimensional distributions of the process $Z$.

For each $v \geq 0$ we will derive the distribution function $F_v(y), y \geq 0,$ of $Z(v)$ using some calculations from excursion theory (note that $Z(v)$ is a nonnegative random variable). As in Chapter 4 of Bertoin [3], define $N = \{(v, \varepsilon(v)), v \geq 0\}$ as the Poisson point process of excursions away from 0 for the reflected process $Y$. That is, $(v, \varepsilon(v))$ takes values in $[0, \infty) \times \mathcal{E}$, where $\mathcal{E}$ is the space of excursions from zero, and $\varepsilon(v)$ corresponds to the excursion of $Y$ beginning when its local time has attained level $v$. Let $\ell$ denote Lebesgue measure and denote by $n$ the excursion measure of $Y$, which is the sigma-finite measure on $[0, \infty) \times \mathcal{E}$ of the Poisson random measure $N$.

Defining $\Delta(v) = \Delta(\varepsilon(v))$ to be the largest jump made during the excursion $\varepsilon(v)$ (which we set to be 0 if there is no excursion at $v$), we see that

$$Z(v) = \sup_{0 \leq u \leq v} [\Delta(u) - (v - u)]. \quad (13)$$

Since $N' = \left(\sum_{v} \delta_{(v, \Delta(v))}, v \geq 0\right)$ is a Poisson point process on $[0, \infty) \times [0, \infty)$, the process $Z$ is Markov. Note that for any $w \in [0, v]$,

$$Z(v) = \max\{\sup_{0 \leq u \leq w} [\Delta(u) - (w - u)] - (v - w), \sup_{w \leq u \leq v} [\Delta(u) - (v - u)]\}$$

$$= \max\{Z(w) - (v - w), \sup_{w \leq u \leq v} [\Delta(u) - (v - u)]\}$$

$$\sim \max\{Z(w) - (v - w), \sup_{u \in [0, v-w]} [\Delta(u) - u]\}.$$

In particular, taking $w = 0$, we obtain

$$Z(v) \sim \sup_{0 \leq u \leq v} [\Delta(u) - u]. \quad (14)$$

Define $A = A_{v,y} = \{(u, \varepsilon) \in [0, \infty) \times \mathcal{E} : u \in [0, v], \Delta(\varepsilon) > y + u\}$. Then using standard results (e.g. Section 0.5 of Bertoin [3]), we see that for $y > 0$,

$$P(Z(v) > y) = P(N(A) \geq 1). \quad (15)$$
The random variable $N(A)$ is Poisson with mean

$$\lambda(v, y) = (\ell \times n)(A) = \int_0^v n(\Delta(\varepsilon) > y + u)du = \int_y^{y+v} n(\Delta(\varepsilon) > q)dq. \quad (16)$$

So the distribution function of $Z(v)$ is $F_v(y) = \exp(-\lambda(v, y))$, $y > 0$, which is explicit as long as we can derive an expression for $n(\Delta(\varepsilon) > q)$ for each $q > 0$.

To this end, fix $q > 0$. The idea is to compare the set of excursions with a jump bigger than $q$ to the set of excursions of a modified process, whose lifetimes are longer than the exponential waiting time until the first $q$-jump of the original process. The modified process $\tilde{Y}$ is obtained from $Y$ by thinning all jumps of size greater than $q$, yielding a Lévy process for which the Lévy measure is now restricted to $[0, q]$, so that we may apply a formula of Baurdoux [2] for excursion lifetimes.

In more detail, write $X = \tilde{X} + J_q$, where $J_q$ is a pure jump process independent of $\tilde{X}$ with all jumps greater than $q$, and $\tilde{X}$ almost surely has all jumps bounded by $q$. Define the modified process $\tilde{Y}(t) = \tilde{X}(t) - \tilde{X}(t)$ and let $\tilde{n}$ denote the excursion measure on $\mathcal{E}$ of the process $\tilde{Y}$.

The Laplace exponent of the Lévy process $X$ is $\Psi(s) = s + s^\alpha$, and the corresponding Lévy measure $\nu(dx) = c_\alpha x^{-\alpha-1}dx$, for a strictly positive constant $c_\alpha$ (an expression is given in Exercise 1.4 of [12]). So the Lévy measures of $\tilde{X}$ and $J_q$ are $\nu$ restricted to $[0, q]$ and $(q, \infty)$ respectively. The Lévy exponent of $\tilde{X}(t)$ can be written as

$$\tilde{\Psi}_q(s) = s + s^\alpha + c_\alpha \int_q^{\infty} (1 - e^{-sx})x^{-\alpha-1}dx. \quad (17)$$

Define

$$e_q = \inf\{t \geq 0 : Y(t) - Y(t-) > q\}$$

as the waiting time until the first jump of $Y$ of size greater than $q$. Then $e_q = \inf\{t \geq 0 : J_q(t) > J_q(t-), t \geq 0\}$, and since $J_q$ is independent of $\tilde{X}$, the random variable $e_q$ is exponential with rate $\beta_q = \nu(q, \infty) = c_\alpha q^{-\alpha}/\alpha$ and is independent of $\tilde{X}$.

**Lemma 6.1.** For each $q > 0$,

$$n(\Delta(\varepsilon) > q) = \int_{\mathcal{E}} (1 - e^{-\beta_q|\varepsilon|})d\tilde{n}(\varepsilon), \quad (18)$$

where $|\varepsilon|$ denotes the lifetime of an excursion $\varepsilon \in \mathcal{E}$.
Proof. We show that both expressions are equal to $1/E[L(e_q)]$. Beginning with the left side, multiply and divide by $E[L(e_q)]$ to obtain

$$n(\Delta(\varepsilon) > q) = \int_{\varepsilon} 1_{\{\Delta(\varepsilon) > q\}} d\tilde{n}(\varepsilon)$$

$$= \frac{1}{E[L(e_q)]} E \left[ \int_0^\infty \int_{\varepsilon} 1_{\{\Delta(\varepsilon) > q\}} 1_{\{s \leq e_q\}} d\tilde{n}(\varepsilon) dL(s) \right].$$

We show the second expectation on the right equals one. Since the function $G(s, \omega, \varepsilon) = 1_{\{\Delta(\varepsilon) > q\}} 1_{\{s \leq e_q(\omega)\}}$ on $[0, \infty) \times \Omega \times \mathcal{E}$ is measurable and almost surely left-continuous in $s$, the compensation formula in excursion theory (see Corollary 11 in Section IV.4 of [3]) yields

$$E \left[ \int_0^\infty \int_{\varepsilon} 1_{\{\Delta(\varepsilon) > q\}} 1_{\{s \leq e_q\}} d\tilde{n}(\varepsilon) dL(s) \right] = E \left[ \sum_g 1_{\{\Delta(\varepsilon_g) > q\}} 1_{\{g \leq e_q\}} \right],$$

where for each sample path, the sum is over the left endpoints of all excursion intervals $(g, d)$ and $\varepsilon_g$ is the excursion of $Y$ beginning at time $g$. But since $e_q$ falls during the first excursion with a jump greater than $q$, the sum equals one almost surely.

Turning to the right side of (18), we again multiply and divide, noting that $E[L(e_q)] = E[\tilde{L}(e_q)]$, where $\tilde{L}(t) = -\tilde{X}(t)$ is the local time for $\tilde{Y}$, because the sample paths of $Y$ and $\tilde{Y}$ are identical up to time $e_q$. This gives

$$\int_{\varepsilon} (1 - e^{-\beta_q|\varepsilon|}) d\tilde{n}(\varepsilon) = \frac{1}{E[L(e_q)]} E \left[ \int_0^\infty \int_{\varepsilon} (1 - e^{-\beta_q|\varepsilon|}) 1_{\{s \leq e_q\}} d\tilde{n}(\varepsilon) d\tilde{L}(s) \right],$$

and we must show the second expectation on the right equals one. Using the compensation formula,

$$E \left[ \int_0^\infty \int_{\varepsilon} (1 - e^{-\beta_q|\varepsilon|}) 1_{\{s \leq e_q\}} d\tilde{n}(\varepsilon) d\tilde{L}(s) \right] = E \left[ \sum_g (1 - e^{-\beta_q|\varepsilon_g|}) 1_{\{g \leq e_q\}} \right],$$

where this time the sum is over all excursion intervals $(g, d)$ of $\tilde{Y}$ and $\varepsilon_g$ are the corresponding excursions. Since $e_q$ is independent of $\tilde{Y}$, the expectation on the right can be computed as an iterated integral over $D \times [0, \infty)$ with respect to the product law $P_{\tilde{Y}} \times P_e$ of the random pair $(\tilde{Y}, e_q)$. This yields

$$E_{\tilde{Y}} E_e \left[ \sum_g (1 - e^{-\beta_q|\varepsilon_g|}) 1_{\{g \leq e_q\}} \right] = E_{\tilde{Y}} \left[ \sum_g (1 - e^{-\beta_q|\varepsilon_g|}) P_e(g \leq e_q) \right]$$

$$= E_{\tilde{Y}} \left[ \sum_g P_e(|\varepsilon_g| > e_q) P_e(e_q > g) \right].$$
Note that for each excursion interval \((g, d)\), the lifetime \(|\varepsilon_g| = d - g\). So by the memoryless property of the exponential and since the excursion intervals are disjoint, the right side above is equal to

\[
E \tilde{Y} \left[ \sum_g P_e(e_q < d \mid e_q > g)P_e(e_q > g) \right] = E \tilde{Y} \left[ \sum_g P_e(e_q \in (g, d)) \right] = E \tilde{Y} \left[ P_e(e_q \in [0, \infty) \setminus \mathcal{F}) \right],
\]

where \(\mathcal{F}\) denotes the closure of the zero set of \(\tilde{Y}\). Since \(X = V^* - \gamma \mu t\) is not a monotone or pure jump process, this set has Lebesgue measure zero.

Since the Lévy measure of \(\tilde{X}\) has bounded support, we can apply Equation (3.3) of [2] to the right side of (18), which in the notation of [2] would be written “\(\tilde{n}(|\varepsilon| > e_q)\).” Let \(P_x\) denote the law of \(\tilde{X} + x\) and \(\tau^x_q = \inf\{t \geq 0 : \tilde{X}(t) + x = 0\}\) be the hitting time of zero. Then (18) combined with [2] Equation (3.3) in our setting (in particular \(h(x)\) there is simply \(x\) here) yields

\[
n(\Delta(\varepsilon) > q) = \lim_{x \downarrow 0} P_x(\tau^x_q > e_q) = \lim_{x \downarrow 0} \frac{1 - E_x[e^{-\beta_q \tau^x_q}]}{x}.
\]

Observe that

\[
E_x[e^{-\beta_q \tau^x_q}] = e^{-x \phi_q(\beta_q)},
\]

with \(\phi_q\) the right inverse of \(\tilde{\Psi}_q\).

Thus, we obtain

\[
n(\Delta(\varepsilon) > q) = \phi_q(\beta_q) = \phi_q(c_\alpha q^{-\alpha}/\alpha) =: h(q).
\]

(19)

Rewrite the last expression using (17) to get

\[
c_\alpha q^{-\alpha}/\alpha = \tilde{\Psi}_q(h(q)) = h(q) + h(q)^\alpha + c_\alpha \int_q^{\infty} (1 - e^{-h(q)x}) x^{-\alpha-1} dx.
\]

This can be simplified to

\[
h(q) + h(q)^\alpha = c_\alpha \int_q^{\infty} e^{-h(q)x} x^{-\alpha-1} dx.
\]

Defining \(\kappa(q) = h(q)q\), performing a change of variables \(t = x/q\), and letting \(T_\alpha\) be a Pareto distributed random variable with index \(\alpha\) we obtain

\[
q^{\alpha-1} \kappa(q) + \kappa(q)^\alpha = \frac{c_\alpha}{\alpha} E[e^{-\kappa(q)T_\alpha}].
\]

(20)
This equation can be transformed into Equation (7) in [10] for \( \kappa(y) \) (using \( \lambda = 1 \) and \( \gamma = -\Gamma(1 - \nu) \) there), and so we see by Lemma 3.9 of [10] that (20) has a unique solution \( \kappa(q) \), which by Lemma 3.11 in [10] is a continuous, bounded, regularly varying function of \( q \) with index \( 1 - \alpha \). Combining (16) with (19) we have for all \( v \geq 0 \),

\[
F_v(y) = \exp \left( - \int_{y}^{y+v} \frac{\kappa(q)}{q} dq \right), \quad (y \geq 0).
\]

Comparing with Theorem 2.2 in [10], we see that the above one dimensional distributions of our time changed limit process \( Z(v) = M^*(\mu L^{-1}(v)) \) are precisely the limiting laws of the one dimensional distributions of the process studied in [10], in which the analogous time change was performed on the original (prelimit) plateau process before scaling and taking the limit.

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