Hamilton differential Harnack inequality and $W$-entropy for Witten Laplacian on Riemannian manifolds

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April 24, 2018

Abstract. In this paper, we prove the Hamilton differential Harnack inequality for positive solutions to the heat equation of the Witten Laplacian on complete Riemannian manifolds with the $CD(-K, m)$-condition, where $m \in [n, \infty)$ and $K \geq 0$ are two constants. Moreover, we introduce the $W$-entropy and prove the $W$-entropy formula for the fundamental solution of the Witten Laplacian on complete Riemannian manifolds with the $CD(-K, m)$-condition and on compact manifolds equipped with $(-K, m)$-super Ricci flows.

MSC2010 Classification: primary 53C44, 58J35, 58J65; secondary 60J60, 60H30.

Keywords: Hamilton differential Harnack inequality, $W$-entropy, super Ricci flows.

1 Introduction

Differential Harnack inequality is an important tool in the study of geometric PDEs. Let $M$ be an $n$ dimensional complete Riemannian manifold, $u$ a positive solution to the heat equation

$$\partial_t u = \Delta u.$$  

(1)

In their famous paper [8], Li and Yau proved that if $\text{Ric} \geq -K$, where $K \geq 0$ is a positive constant, then for all $\alpha > 1$,

$$\frac{\lvert \nabla u \rvert^2}{u^2} - \alpha \frac{\partial_t u}{u} \leq \frac{n \alpha^2}{2t} + \frac{n \alpha^2 K}{\sqrt{2}(\alpha - 1)}.$$  

(2)

In particular, if $\text{Ric} \geq 0$, then taking $\alpha \to 1$, the Li-Yau differential Harnack inequality holds

$$\frac{\lvert \nabla u \rvert^2}{u^2} - \frac{\partial_t u}{u} \leq \frac{n}{2t}.$$  

(3)

In [6], Hamilton proved a dimension free differential Harnack inequality on compact Riemannian manifolds with Ricci curvature bounded from below. More precisely, if $M$ is a compact Riemannian manifold with $\text{Ric} \geq -K$,

then, for any positive and bounded solution $u$ to the heat equation [11], it holds

$$\frac{\lvert \nabla u \rvert^2}{u^2} \leq \left( \frac{1}{t} + 2K \right) \log(A/u), \ \forall x \in M, t > 0,$$  

(4)
where $A := \sup\{u(t, x) : x \in M, t \geq 0\}$. Indeed, the same result holds on complete Riemannian manifolds with Ricci curvature bounded from below. Under the same condition $\text{Ric} \geq -K$, Hamilton also proved the following differential Harnack inequality for any positive solution to the heat equation (11)

$$\frac{|\nabla u|^2}{u^2} - e^{2Kt} \frac{\partial_t u}{u} \leq \frac{n}{2t} e^{AKt}. \quad (5)$$

In particular, when $K = 0$, the above inequality reduces to the Li-Yau Harnack inequality (3) on complete Riemannian manifolds with non-negative Ricci curvature. Moreover, Hamilton [6] proved that, on compact Riemannian manifolds with $\text{Ric} \geq -K$, any positive and bounded solution of the heat equation $\partial_t u = \Delta u$ with $0 < u \leq A$ satisfies

$$\frac{\partial_t u}{u} + \frac{|\nabla u|^2}{u^2} \leq \frac{K}{1 - e^{-Kt}} [n + 4 \log(A/u)], \quad \forall t \geq 0. \quad (6)$$

On the other hand, Perelman [22] reformulated the Ricci flow as the gradient flow of the $F$-functional, where $F(g, f) = \int_M (R + |\nabla f|^2) e^{-f} dv$ is defined on the product space of Riemannian metrics and $C^\infty$-functions equipped with the standard $L^2$-Riemannian metric with the constraint that $e^{-f} dv$ does not change, where $R$ is the scalar curvature of $g$. He then introduced the $W$-entropy functional and proved its monotonicity along the conjugate equation coupled with the Ricci flow. The $F$-functional has been used by Perelman to characterize the steady gradient Ricci solitons, and the $W$-entropy has been used to characterize the shrinking gradient Ricci solitons. As an application of the $W$-entropy formula, Perelman [22] proved the non local collapsing theorem for the Ricci flow, which plays an important role for ruling out cigars, one part of the singularity classification for the final resolution of the Poincaré conjecture and geometrization conjecture.

Since Perelman’s preprint [22] was posted on Arxiv in 2002, many people have studied the $W$-like entropy for other geometric flows on Riemannian manifolds. In [20, 21], Ni proved the $W$-entropy formula for the heat equation $\partial_t u = \Delta u$ on compact and complete Riemannian manifolds with non-negative Ricci curvature, where $\Delta$ denotes the usual Laplace-Beltrami operator on Riemannian manifolds. In [18], Li and Xu extended Ni’s $W$-entropy formula to the heat equation $\partial_t u = \Delta u$ on complete Riemannian manifolds with Ricci curvature bounded from below by a negative constant.

From [22, 20, 18, 10, 12, 13], it has been known that there is a close connection between the differential Harnack inequality and the $W$-entropy for the heat equation on Riemannian manifolds. To see this link, let $(M, g)$ be a complete Riemannian manifold with bounded geometry condition, $u$ be a positive solution to the heat equation $\partial_t u = \Delta u$. As in [20, 21], let

$$H_n(u(t)) = -\int_M u \log u dv - \frac{n}{2} (\log(4\pi t) + 1). \quad (7)$$

Then

$$\frac{d}{dt} H_n(u(t)) = \int_M \left[ \frac{|\nabla u|^2}{u^2} - \frac{\partial_t u}{u} - \frac{n}{2t} \right] du.$$

Suppose that $(M, g)$ is a complete Riemannian manifold with non-negative Ricci curvature. Then the Li-Yau Harnack inequality [3] holds. This yields

$$\frac{d}{dt} H_n(u(t)) \leq 0.$$

Let

$$W_n(u(t)) = \frac{d}{dt} (tH_n(u(t))).$$
In [20, 21], Ni proved that
\[
\frac{d}{dt} W_n(u(t)) = -2t \int_M \left[ \nabla^2 \log u + \frac{g}{2t} \right]^2 + \text{Ric}((\nabla \log u, \nabla \log u) \right] u dv.
\]

We now introduce some notations and definitions to develop the main part of this paper. Let \((M, g)\) be a complete Riemannian manifold, \(\phi \in C^2(M)\) and \(d\mu = e^{-\phi} dv\), where \(v\) is the Riemannian volume measure on \((M, g)\). The Witten Laplacian acting on smooth functions is defined by
\[
L = e^{\phi} \text{div}(e^{-\phi} \nabla) = \Delta - \nabla \phi \cdot \nabla.
\]

For any \(u, v \in C^\infty_0(M)\), the integration by parts formula holds
\[
\int_M \langle \nabla u, \nabla v \rangle d\mu = -\int_M Luvd\mu = -\int_M uLvd\mu.
\]
Thus, \(L\) is the infinitesimal generator of the Dirichlet form
\[
\mathcal{E}(u, v) = \int_M \langle \nabla u, \nabla v \rangle d\mu, \quad u, v \in C^\infty_0(M).
\]

In [1], Bakry and Emery proved that for all \(u \in C^\infty_0(M)\),
\[
L|\nabla u|^2 - 2\langle \nabla u, \nabla Lu \rangle = 2|\nabla^2 u|^2 + 2\text{Ric}(L)(\nabla u, \nabla u),
\]
where
\[
\text{Ric}(L) = \text{Ric} + \nabla^2 \phi
\]
is now called the infinite dimensional Bakry-Emery Ricci curvature associated with the Witten Laplacian \(L\). For \(m \in [n, \infty)\), the \(m\)-dimensional Bakry-Emery Ricci curvature associated with the Witten Laplacian \(L\) is defined by
\[
\text{Ric}_{m,n}(L) = \text{Ric} + \nabla^2 \phi - \frac{\nabla \phi \otimes \nabla \phi}{m - n}.
\]
In view of this, we have (see [9])
\[
L|\nabla u|^2 - 2\langle \nabla u, \nabla Lu \rangle \geq \frac{2|Lu|^2}{m} + 2\text{Ric}_{m,n}(L)(\nabla u, \nabla u).
\]
Here we only define \(\text{Ric}_{m,n}(L)\) for \(m = n\) when \(\phi\) is a constant. By definition, we have
\[
\text{Ric}(L) = \text{Ric}_{\infty,n}(L).
\]
Following [1], we say that \((M, g, \mu)\) satisfies the curvature-dimension \(CD(K, m)\)-condition\(^\dagger\) for a constant \(K \in \mathbb{R}\) and \(m \in [n, \infty]\) if and only if
\[
\text{Ric}_{m,n}(L) \geq Kg.
\]

Inspired by Perelman’s introduction of the modified Ricci flow \(\partial_t g = -2(\text{Ric} + \nabla^2 \phi)\) in [22], we define the \((K, m)\)-(Perelman) Ricci flow and \((K, m)\)-super (Perelman) Ricci flows as follows. We call a manifold \((M, g(t), \phi(t), t \in [0, T])\) equipped with a family of time dependent Riemann metrics \(g(t)\) and \(C^2\)-potentials \(\phi(t)\) a \((K, m)\)-(Perelman) Ricci flow if
\[
\frac{1}{2} \frac{\partial g}{\partial t} + \text{Ric}_{m,n}(L) = Kg, \quad \forall t \in (0, T],
\]
\(^\dagger\)Here the word “\(CD\)” means “curvature-dimension".
and we call \((M, g(t), \phi(t), t \in [0, T])\) a \((K, m)\)-super (Perelman) Ricci flow if
\[ \frac{1}{2} \frac{\partial g}{\partial t} + \text{Ric}_{m,n}(L) \geq K g, \quad \forall t \in (0, T]. \]
See also our previous paper [13] and [14, 15, 16]. When \(\phi\) is a constant and \(m = n\), the \((K, n)\)-(Perelman) Ricci flow is indeed the Hamilton \(K\)-Ricci flow
\[ \frac{1}{2} \frac{\partial g}{\partial t} + \text{Ric} = K g, \quad \forall t \in (0, T], \]
and a \((K, n)\)-super (Perelman) Ricci flow is a Hamilton \(K\)-super Ricci flow
\[ \frac{1}{2} \frac{\partial g}{\partial t} + \text{Ric} \geq K g, \quad \forall t \in (0, T]. \]
While when \(m = \infty\), the \((K, \infty)\)-(Perelman) Ricci flow is indeed the following extension of the modified Ricci flow introduced by Perelman [22] (where \(K = 0\))
\[ \frac{1}{2} \frac{\partial g}{\partial t} + \text{Ric}(L) = K g, \quad \forall t \in (0, T], \]
and a \((K, \infty)\)-super (Perelman) Ricci flow reads as follows
\[ \frac{1}{2} \frac{\partial g}{\partial t} + \text{Ric}(L) \geq K g, \quad \forall t \in (0, T]. \]
We would like to point out that the notion of super Ricci flows has been also independently introduced by K.-T. Sturm on time-dependent metric measure spaces [24]. See also Kopfer-Sturm [7].

In [9], the Li-Yau Harnack inequality [2] has been extended to positive solutions of the heat equation of the Witten Laplacian
\[ \frac{\partial}{\partial t} u = Lu \]
on complete Riemannian manifolds with the \(CD(-K, m)\)-condition, i.e., the \(m\)-dimensional Bakry-Emery Ricci curvature associated with \(L\) satisfies \(\text{Ric}_{m,n}(L) \geq -K g\), where \(m \in [n, \infty)\) and \(K \geq 0\). In particular, on complete Riemannian manifolds with the \(CD(0, m)\)-condition, the classical Li-Yau Harnack inequality [3] has been extended to positive solutions to the heat equation [10] (see [11])
\[ \frac{|\nabla u|^2}{u^2} - \frac{\partial_t u}{u} \leq \frac{m}{2t}. \]
In [12], an improved version of the Hamilton Harnack inequality [4] has been established for positive and bounded solutions to the heat equation [10] on complete Riemannian manifolds with the \(CD(-K, \infty)\)-condition, where \(K \geq 0\) is a constant.

On the other hand, in [10, 12], the \(W\)-entropy formula has been also extended to the heat equation of the Witten Laplacian [9] on complete Riemannian manifolds with non-negative \(m\)-dimensional Bakry-Emery Ricci curvature condition. More precisely, let \((M, g)\) be a complete Riemannian manifold with bounded geometry condition, \(u\) a positive solution to the heat equation [9] of the Witten Laplacian on \((M, g, \mu)\). Let
\[ H_m(u(t)) = - \int_M u \log u d\mu - \frac{m}{2} (\log(4\pi t) + 1). \]
Then
\[ \frac{d}{dt} H_m(u(t)) = \int_M \left[ \frac{|\nabla u|^2}{u^2} - \frac{\partial_t u}{u} - \frac{m}{2t} \right] u d\mu. \]
Suppose that \((M, g)\) is a complete Riemannian manifold with the \(CD(0, m)\)-condition. Then the Li-Yau Harnack inequality \([10]\) yields

\[
\frac{d}{dt} H_m(u(t)) \leq 0.
\]

Let

\[
W_m(u(t)) = \frac{d}{dt} (tH_m(u(t))).
\]

By \([10, 12]\), we have the \(W\)-entropy formula for the heat equation of the Witten Laplacian

\[
\frac{d}{dt} W_m(u(t)) = -2t \int_M \left[ \nabla^2 \log u + \frac{g}{2t} \right]^2 u \mu + 2t \int_M \left( \nabla \log u \cdot \nabla \phi - \frac{m-n}{2t} \right)^2 u \mu.
\]

In particular, \(\frac{d}{dt} W_m(u(t)) \leq 0\) if \(Ric_{m,n}(L) \geq 0\). Moreover, the above definition formulas \([11]\) and \([12]\) indicate also the close connection between the extended Li-Yau Harnack inequality \([10]\) and the \(W\)-entropy on complete Riemannian manifolds with the \(CD(0, m)\)-condition. Moreover, a rigidity theorem for \(W_m\) was also proved in \([10]\) on complete Riemannian manifolds with the \(CD(0, m)\)-condition. See also \([11]\) for the \(W\)-entropy formula for the Fokker-Planck equation on complete Riemannian manifolds with the \(CD(0, m)\)-condition.

In our previous papers \([13, 16]\), we proved the \(W\)-entropy formula for the heat equation of the Witten Laplacian on complete Riemannian manifolds equipped with \((m, n)\)-dimensional Bakry-Emery Ricci curvature condition to the conjugate heat equation of the Witten Laplacian. In \([13]\), \(W\)-entropy formula for the usual Laplacian to \(M \times S^{m-n}\) equipped with a suitable warped product Riemannian metric, and gave a natural geometric interpretation of the \(W\)-entropy formula for the heat equation of the Witten Laplacian. In \([13]\), we have also proved the \(W\)-entropy formula for the heat equation of time dependent Witten Laplacian on Riemannian manifolds equipped with \((K, m)\)-super Ricci flows, where \(m \in [n, \infty)\) and \(K \geq 0\). More precisely, for \(K = 0\), let \((M, g(t), \phi(t), t \in [0, T])\) be a compact manifolds equipped with a family of time dependent metrics \(g(t)\) and \(C^2\)-potentials \(\phi(t), t \in [0, T]\) such that \(dg = e^{-\phi}dv\) is independent of \(t\) (which is equivalent to the conjugate heat equation \(\partial_t \phi = \frac{4}{1} \text{Tr}(\partial_t g)\)), then the \(W\)-entropy defined by \([11]\) and \([12]\) for positive solution to the heat equation \([9]\) of the time dependent Witten Laplacian

\[
L = \Delta_g(t) - \nabla \phi(t) \cdot \nabla g(t)
\]

satisfies

\[
\frac{d}{dt} W_m(u(t)) = -2t \int_M \left[ \nabla^2 \log u + \frac{g}{2t} \right]^2 u \mu + 2t \int_M \left( \nabla \log u \cdot \nabla \phi - \frac{m-n}{2t} \right)^2 u \mu.
\]

In particular, if \((M, g(t), \phi(t), t \in [0, T])\) is a \((0, m)\)-super Ricci flow in the sense that

\[
\frac{1}{2} \frac{\partial g}{\partial t} + Ric_{m,n}(L) \geq 0, \quad \frac{\partial \phi}{\partial t} = \frac{1}{2} \text{Tr} \left( \frac{\partial \phi}{\partial t} \right),
\]

then \(W_m(u(t))\) is decreasing in time \(t\) on \([0, T]\). For general case \(K \geq 0\) and \(m \in [n, \infty]\), see \([13, 16]\).
The purpose of this paper is threefolds. First, we extend the Hamilton differential Harnack inequalities \(^{(9)}\) and \(^{(10)}\) to positive solutions of the heat equation \(^{(9)}\) of the Witten Laplacian on complete weighted Riemannian manifolds with the \(CD(0,m)\)-condition. Second, we use Hamilton’s Harnack inequality to introduce a new \(W\)-entropy formula to positive solutions of the heat equation \(^{(9)}\) of the Witten Laplacian on complete weighted Riemannian manifolds with the \(CD(0,m)\)-condition. Finally, we extend the \(W\)-entropy formula to the heat equation \(^{(9)}\) associated with the time dependent Witten-Laplacian on compact Riemannian manifolds with the \(CD(0,m)\)-condition. As mentioned above, by previous works in \([20,10,12,13]\), there exists an essential link between the \(W\)-entropy and the Li-Yau Harnack inequality \(^{(10)}\) for the heat equation of the Witten Laplacian on complete Riemannian manifolds satisfying the \(CD(0,m)\)-condition. Our result indicates that, when \(m \in [n,\infty)\) and \(K \geq 0\), there still exists an essentially deep connection between the \(W\)-entropy and the Hamilton differential Harnack inequality \(^{(15)}\) for the heat equation of the Witten Laplacian on complete Riemannian manifolds with the \(CD(0,m)\)-condition.

\[\frac{1}{u^2} \left| \nabla u \right|^2 - e^{2Kt} \frac{\partial_t u}{u} \leq \frac{m}{2t} e^{4Kt}. \tag{15}\]

In particular, if \(\text{Ric}_{m,n}(L) \geq 0\), then the Li-Yau differential Harnack inequality holds

\[\frac{1}{u^2} \left| \nabla u \right|^2 - \frac{\partial_t u}{u} \leq \frac{m}{2t}. \]

Integrating the differential Harnack inequality along the geodesic on the space time, Theorem 2.1 implies the following Harnack inequality.

**Theorem 2.2** Under the same condition and notation as in Theorem 2.1, for all \(x, y \in M\), \(0 < \tau < T\), we have

\[ \frac{u(x, \tau)}{u(y, T)} \leq \left( \frac{T}{\tau} \right)^{m/2} \exp \left\{ \frac{1}{4} e^{2K\tau} \left[ 1 + 2K(T - \tau) \right] \frac{d^2(x, y)}{T - \tau} + \frac{m}{2} \left[ e^{2K\tau} - e^{2K\tau} \right] \right\}. \]

The following result extends Hamilton’s estimate \(^{(9)}\) to the heat equation of the Witten Laplacian on complete Riemannian manifolds with the \(CD(-K,m)\)-condition.

**Theorem 2.3** Let \((M, g)\) be a complete Riemannian manifold with bounded Riemannian curvature tensor, \(\phi \in C^2(M)\) such that \(\nabla \phi\) and \(\nabla^2 \phi\) are uniformly bounded on \(M\). Suppose that there exist some constants \(m \in [n,\infty)\) and \(K \geq 0\) such that \(\text{Ric}_{m,n}(L) \geq -K\). Then for any bounded and positive solution \(u\) to the heat equation \(^{(9)}\) with \(A = \sup\{u(x,t), (x,t) \in M \times [0,T]\} < \infty\), it holds

\[ \frac{\partial_t u}{u} + \frac{\left| \nabla u \right|^2}{u^2} \leq \frac{K}{1 - e^{-Kt}} \left[ m + 4\log(A/u) \right], \quad \forall t \in [0,T]. \tag{16}\]
In particular, for $t \in [0, T]$, we have
\[
\frac{\partial_t u}{u} \leq \left( K + \frac{1}{t} \right) \left[ m + 4 \log(A/u) \right].
\] (17)

We would like to mention that, as was pointed out in the report of an anonymous referee, the above estimates are central tools in the study of the classical heat equation as well as Ricci flow and Ricci solitons, so in principle there are similar applications waiting to follow in this area for the heat equation of the Witten Laplacian as well as $(K, m)$-Ricci flow and $(K, m)$-Ricci solitons.

As an application of Theorem 2.1 and Theorem 2.3, we can derive the following bound for the time derivative of the logarithm of the heat kernel of the Witten Laplacian on complete Riemannian manifolds with the $CD(-K, m)$-condition, we have
\[
-\frac{m}{2t} e^{2Kt} \leq \partial_t \log p_t(x, y) \leq \left( K + \frac{1}{t} \right) \left[ m + 4 \log \sup_{x \in M} \frac{p_t(x, y)}{\inf_{x \in M} p_t(x, y)} \right].
\]

Using the upper bound and lower bound estimates of the heat kernel $p_t(x, y)$ on complete Riemannian manifolds with the $CD(-K, m)$-condition obtained in [9, 10, 12], we can derive the following estimate, which seems new in the literature.

**Theorem 2.4** Under the same condition and notation as in Theorem 2.3, for all fixed $T > 0$ and $t \in (0, T]$, we have
\[
-\frac{m}{2t} e^{2Kt} \leq \partial_t \log p_t(x, y) \leq C_{m,n,K,T} \left( 1 + \frac{1}{\sqrt{t}} + \frac{d(x, y)}{t} \right)^2,
\]
where $p_t(x, y)$ denotes the heat kernel of the Witten Laplacian $L$ with respect to $\mu$ on $(M, g)$.

**Remark 2.5** In [12], it has been proved that under the condition $(M, g)$ is a complete Riemannian manifold with bounded geometry condition (i.e., the Riemannian curvature tensor as well its $k$-th covariant derivatives are uniformly bounded up to the 3-rd order), $\phi \in C^4(M)$ such that $\nabla^k \phi$ are uniformly bounded on $M$ for $1 \leq k \leq 4$, then
\[
|\partial_t \log p_t(x, y)| \leq C_{m,n,K,T} \left( 1 + \frac{1}{\sqrt{t}} + \frac{d(x, y)}{t} \right)^2.
\]

While Theorem 2.4 need only to assume the Riemannian curvature tensor $\text{Riem}$ is uniformly bounded, $\text{Ric}_{m,n}(L) \geq -K g$ and $\phi \in C^2(M)$ such that $\nabla \phi$ and $\nabla^2 \phi$ are uniformly bounded on $M$.

The following result indicates the close connection between the Hamilton differential Harnack inequality [5] and the $W$-entropy for the heat equation of the Witten Laplacian on complete Riemannian manifolds with the $CD(-K, m)$-condition. When $K = 0$, it reduces to the $W$-entropy formula [13] for the heat equation of the Witten Laplacian on complete Riemannian manifolds with the $CD(0, m)$-condition.

**Theorem 2.6** Let $(M, g)$ be a complete Riemannian manifold with the bounded geometry condition and $\phi \in C^4(M)$ such that $\nabla^k \phi$ are uniformly bounded on $M$ for $1 \leq k \leq 4$. Let $u$ be the heat kernel of the Witten Laplacian $L = \Delta - \nabla \phi \cdot \nabla$. Let
\[
H_{m,K}(u, t) = -\int_M u \log u d\mu - \Phi_{m,K}(t),
\]

The following result indicates the close connection between the Hamilton differential Harnack inequality [5] and the $W$-entropy for the heat equation of the Witten Laplacian on complete Riemannian manifolds with the $CD(-K, m)$-condition. When $K = 0$, it reduces to the $W$-entropy formula [13] for the heat equation of the Witten Laplacian on complete Riemannian manifolds with the $CD(0, m)$-condition.
where $\Phi_{m,K} \in C((0, \infty), \mathbb{R})$ satisfies

$$\Phi'_{m,K}(t) = \frac{m}{2t} e^{4Kt}, \quad \forall t > 0.$$  

Define the W-entropy by the Boltzmann formula

$$W_{m,K}(u,t) = \frac{d}{dt}(tH_{m,K}(u,t)).$$

Then

$$\frac{d}{dt}W_{m,K}(u,t) = -2t \int_M \left| \nabla^2 \log u + \left( \frac{K}{2} + \frac{1}{2t} \right) g \right|^2 u\mu - 2t \int_M (\text{Ric}_{m,n}(L) + Kg) (\nabla \log u, \nabla \log u) u\mu - \frac{2t}{m-n} \int_M \left| \nabla \phi \cdot \nabla \log u - \frac{(m-n)(1+Kt)}{2t} \right|^2 u\mu - \frac{m}{2t} \left[ e^{4Kt}(1+4Kt) - (1 + Kt)^2 \right].$$

In particular, if $\text{Ric}_{m,n}(L) \geq -Kg$, then, for all $t > 0$, we have

$$\frac{d}{dt}W_{m,K}(u,t) \leq -m \frac{e^{4Kt}(1+4Kt) - (1 + Kt)^2}{2t}.$$  

Moreover, the equality holds at some time $t = t_0 > 0$ if and only if $(M, g, \phi)$ is a fixed point of the $(-K, m)$-Ricci flow, called $(-K, m)$-Ricci soliton or $(-K, m)$-quasi-Einstein manifold,

$$\text{Ric}_{m,n}(L) = -Kg,$$

the potential function $f = -\log u$ satisfies the shrinking soliton equation with respect to $\text{Ric}_{m,n}(L)$, i.e.,

$$\text{Ric}_{m,n}(L) + 2\nabla^2 f = \frac{g}{t},$$

and moreover

$$\nabla \phi \cdot \nabla f = -\frac{(m-n)(1+Kt)}{2t}.$$  

The following result extends Theorem 2.6 to the heat equation of the time dependent Witten Laplacian on compact manifolds equipped with $(-K, m)$-super Ricci flows. When $K = 0$, it is the W-entropy formula (14) on the $(0, m)$-super Ricci flows, which was proved in our previous paper [13].

**Theorem 2.7** Let $(M, g(t), \phi(t), t \in [0, T])$ be a compact manifold with a family of Riemannian metrics and $C^\infty$-potentials $(g(t), \phi(t), t \in [0, T])$. Suppose that

$$\frac{\partial \phi}{\partial t} = \frac{1}{2} \text{Tr} \left( \frac{\partial g}{\partial t} \right).$$

Let $u$ be a positive solution to the heat equation (9) of the time dependent Witten Laplacian

$$L = \Delta_{g(t)} - \nabla_{g(t)} \phi(t) \cdot \nabla_{g(t)}.$$
Let $H_{m,K}(u,t)$ and $W_{m,K}(u,t)$ be as in Theorem 2.6. Then

$$\frac{d}{dt}W_{m,K}(u,t) = -2t \int_M \left| \nabla^2 \log u + \left( \frac{K}{2} + \frac{1}{2t} \right) g \right|^2 ud\mu$$
$$-2t \int_M \left\{ \frac{1}{2} \frac{\partial g}{\partial t} + \text{Ric}_{m,n}(L) + Kg \right\} (\nabla \log u, \nabla \log u) ud\mu$$
$$- \frac{2t}{m-n} \int_M \left\{ \nabla \phi \cdot \nabla \log u - \frac{(m-n)(1+Kt)}{2t} \right\}^2 ud\mu$$
$$- \frac{m}{2t} \left[ e^{4Kt}(1+4Kt) - (1+Kt)^2 \right].$$

In particular, if $(M, g(t), \phi(t), t \in [0, T])$ is a compact manifolds equipped with a $(-K, m)$-super Ricci flow in the sense that

$$\frac{1}{2} \frac{\partial g}{\partial t} + \text{Ric}_{m,n}(L) \geq -K, \quad \frac{\partial \phi}{\partial t} = \frac{1}{2} \text{Tr} \left( \frac{\partial g}{\partial t} \right),$$

then for all $t \in (0, T]$, we have

$$\frac{d}{dt}W_{m,K}(u,t) \leq -\frac{m}{2t} \left[ e^{4Kt}(1+4Kt) - (1+Kt)^2 \right].$$

Moreover, the equality holds on $(0, T]$ if and only if $(M, g(t), \phi(t), t \in [0, T])$ is a $(-K, m)$-Ricci flow in the sense that

$$\frac{\partial g}{\partial t} = -2(\text{Ric}_{m,n}(L) + Kg),$$
$$\frac{\partial \phi}{\partial t} = -R - \Delta \phi - \frac{\left| \nabla \phi \right|^2}{m-n} - nK,$

the potential function $f = -\log u$ satisfies the Hessian equation

$$\nabla^2 f = \left( \frac{K}{2} + \frac{1}{2t} \right) g,$$

and moreover

$$\nabla \phi \cdot \nabla f = -\frac{(m-n)(1+Kt)}{2t}.$$

We can also extend the Hamilton Harnack inequalities to positive solutions to the heat equation $\partial_t u = Lu$ associated with the time dependent Witten Laplacian $L = \Delta - \nabla \phi \cdot \nabla$ on compact or complete Riemannian manifolds equipped with a variant of $(-K, m)$-super Ricci flow. To save the length of the paper, we will do it in a forthcoming paper. See [14].

The rest of this paper is organized as follows. In Section 3, we prove Theorem 2.4, Theorem 2.2 and Theorem 2.3. In Section 4, we prove Theorem 2.6 and Theorem 2.7. In Section 5, we compare the $W$-entropy in Theorem 2.6 and Theorem 2.7 with the $W$-entropy defined in our previous paper [13].

This paper is an improved version of a part of our previous preprint [14]. Due to the limit of the length of the paper, we split [14] into several papers. See also [15, 16, 17].

3 Hamilton Harnack inequalities for Witten Laplacian

3.1 Proof of Theorem 2.1

By the generalized Bochner-Weitzenböck formula, we have

$$(L - \partial_t) \frac{\left| \nabla u \right|^2}{u} = \frac{2}{u} \left| \nabla^2 u - \frac{\nabla u \otimes \nabla u}{u} \right|^2 + \frac{2}{u} \text{Ric}(L)(\nabla u, \nabla u).$$
Taking trace in the first quantity on the right hand side, we can derive

\[(L - \partial_t) \frac{\nabla u^2}{u} \geq \frac{2}{nu} \left| \Delta u - \frac{\nabla u^2}{u} \right|^2 + \frac{2}{u} Ric(L)(\nabla u, \nabla u).\]

Applying the inequality

\[(a + b)^2 \geq \alpha^2 + \frac{b^2}{\alpha}, \quad \forall \alpha > 0,
\]
to \(a = \partial_t u - \frac{\nabla u^2}{u}\), \(b = \nabla \phi \cdot \nabla u\), and \(\alpha = \frac{m-u}{m}\), we have

\[(L - \partial_t) \frac{\nabla u^2}{u} \geq \frac{2}{nu} \left| \partial_t u - \frac{\nabla u^2}{u} \right|^2 + \frac{2}{u} Ric_{m,n}(L)(\nabla u, \nabla u).\]

Hence, under the condition \(Ric_{m,n}(L) \geq -K\), it holds

\[(L - \partial_t) \frac{\nabla u^2}{u} \geq \frac{2}{nu} \left| \partial_t u - \frac{\nabla u^2}{u} \right|^2 - \frac{2K|\nabla u|^2}{u}.\]

Let

\[h = \frac{\partial u}{\partial t} - e^{-2Kt} \frac{|\nabla u|^2}{u} + \frac{e^{2Kt}m}{2t}u.\]

Then \(\lim_{t \to 0^+} h(t) = +\infty\), and

\[(\partial_t - L)h \geq \frac{2}{nu} e^{-2Kt} \left| \partial_t u - \frac{\nabla u^2}{u} \right|^2 - \frac{e^{2Kt}m}{2t^2}u.\]  \hspace{1cm} (18)

We now prove that \(h \geq 0\) on \(M \times \mathbb{R}^+\). In compact case, suppose that \(h\) attends its minimum at some \((x_0, t_0)\) and \(h(x_0, t_0) < 0\). Then, at \((x_0, t_0)\), it holds

\[\frac{\partial h}{\partial t} \leq 0, \quad \Delta h \geq 0, \quad \nabla h = 0.\]

Thus at \((x_0, t_0)\), \((\partial_t - L)h \leq 0\). On the other hand, as \(h(x_0, t_0) < 0\), we have

\[0 \leq e^{2Kt} \frac{m}{2t}u < e^{-2Kt} \frac{|\nabla u|^2}{u} - \frac{\partial u}{\partial t} \leq \frac{|\nabla u|^2}{u} - \frac{\partial u}{\partial t},\]

and hence by (18) we have

\[(\partial_t - L)h > 0.\]

This finishes the proof of Theorem 2.1 in compact case.

In complete non-compact case, let \(f = \log u\), and let

\[F = te^{-2Kt}(e^{-2Kt}|\nabla f|^2 - f_t) = te^{-4Kt}|\nabla f|^2 - te^{-2Kt}f_t.\]

Obviously, \(F(0, x) \equiv 0\). We shall prove that

\[F \leq \frac{m}{2}.\]

By direct calculation

\[LF = te^{-4Kt}L|\nabla f|^2 - te^{-2Kt}Lf_t,\]

\[\partial_t F = (1 - 4Kt)e^{-4Kt}|\nabla f|^2 + (2Kt - 1)e^{-2Kt}f_t + te^{-4Kt}\partial_t|\nabla f|^2 - te^{-2Kt}f_{tt},\]
we have
\[ (L - \partial_t)F = te^{-4Kt}(L - \partial_t)\vert \nabla f \vert^2 - te^{-2Kt}(L - \partial_t)f_t + (4Kt - 1)e^{-4Kt}\vert \nabla f \vert^2 - (2Kt - 1)e^{-2Kt}f_t. \]

By the generalized Bochner formula, it holds
\[ (L - \partial_t)\vert \nabla f \vert^2 = 2\vert \nabla^2 f \vert^2 + 2Ric(L)(\nabla f, \nabla f) - 4\nabla^2 f(\nabla f, \nabla f). \]

Note that
\[
L f_t = L \left( \frac{Lu}{u} \right) = \frac{L^2 u}{u} - 2\langle \nabla Lu, \frac{\nabla u}{u^2} \rangle + Lu \left( -\frac{Lu}{u^2} + 2\frac{\vert \nabla u \vert^2}{u^3} \right),
\]
\[
\partial_t f_t = \partial_t \left( \frac{Lu}{u} \right) = \frac{L^2 u}{u} - \left| \frac{Lu}{u} \right|^2,
\]
which yields
\[ (L - \partial_t)f_t = \frac{2Lu\vert \nabla u \vert^2}{u^3} - 2\langle \nabla Lu, \frac{\nabla u}{u^2} \rangle - 4\nabla^2 f(\nabla f, \nabla f) - 2\langle \nabla Lf, \nabla f \rangle. \]

Hence
\[ (L - \partial_t)F = 2te^{-4Kt}[\vert \nabla^2 f \vert^2 + 2(e^{2Kt} - 1)\nabla^2 f(\nabla f, \nabla f)] + 2te^{-4Kt}Ric(L)(\nabla f, \nabla f) + 2te^{-2Kt}(\nabla Lf, \nabla f) + (4Kt - 1)e^{-4Kt}\vert \nabla f \vert^2 - (2Kt - 1)e^{-2Kt}(Lf + \vert \nabla f \vert^2). \]

Now
\[
F = te^{-4Kt}(1 - e^{2Kt})\vert \nabla f \vert^2 - te^{-2Kt}Lf,
\]
\[
\langle \nabla F, \nabla f \rangle = 2te^{-4Kt}(1 - e^{2Kt})\nabla^2 f(\nabla f, \nabla f) - te^{-2Kt}(\nabla Lf, \nabla f).
\]

Therefore
\[ (L - \partial_t)F = 2te^{-4Kt}\vert \nabla^2 f \vert^2 - 2\langle \nabla F, \nabla f \rangle + 2te^{-4Kt} \left( Ric(L)(\nabla f, \nabla f) + K\vert \nabla f \vert^2 \right) + \frac{(2Kt - 1)}{t}F. \]

Note that
\[ \vert \nabla^2 f \vert^2 \geq \frac{1}{m}\vert \Delta f \vert^2 \geq \frac{1}{m}\vert Lf \vert^2 - \frac{1}{m - n}\nabla \phi \otimes \nabla \phi(\nabla f, \nabla f). \]

Thus
\[ (L - \partial_t)F \geq 2te^{-4Kt}\frac{\vert Lf \vert^2}{m} - 2\langle \nabla F, \nabla f \rangle + 2te^{-4Kt} \left( Ric_m,n(L)(\nabla f, \nabla f) + K\vert \nabla f \vert^2 \right) + \frac{(2Kt - 1)}{t}F \]
\[ \geq 2te^{-4Kt} \left[ \frac{[te^{-2Kt}(e^{-2Kt} - 1)\vert \nabla f \vert^2 - F_1]^2}{t^2e^{-4Kt}} \right] - 2\langle \nabla F, \nabla f \rangle + \frac{(2Kt - 1)}{t}F \]
\[ \geq 2 \frac{[te^{-2Kt}(e^{-2Kt} - 1)\vert \nabla f \vert^2 - F_1]^2}{mt} - 2\langle \nabla F, \nabla f \rangle + \frac{(2Kt - 1)}{t}F. \]

Similarly to [3], let \( \eta \) be a \( C^2 \)-function on \([0, \infty)\) such that \( \eta = 1 \) on \([0, 1]\) and \( \eta = 0 \) on \([2, \infty)\), with \(-C_1\eta^{1/2}(r) \leq \eta'(r) \leq 0\), and \( \eta''(r) \geq C_2\), where \( C_1 > 0 \) and \( C_2 > 0 \) are
two constants. Let \( \rho(x) = d(o, x) \) and define \( \psi(x) = \eta(\rho(x)/R) \). Since \( \rho \) is Lipschitz on the complement of the cut locus of \( o \), \( \psi \) is a Lipschitz function with support in \( B(o, 2R) \times [0, \infty) \). As explained in Li and Yau \[8\], an argument of Calabi allows us to apply the maximum principle to \( \psi F \). Let \( (x_0, t_0) \in M \times [0, T] \) be a point where \( \psi F \) achieves the maximum. Then, at \( (x_0, t_0) \),

\[
\partial_t(\psi F) \geq 0, \quad \Delta(\psi F) \leq 0, \quad \nabla(\psi F) = 0.
\]

This yields

\[
(L - \partial_t)(\psi F) = \Delta(\psi F) - \nabla \phi \cdot \nabla(\psi F) - \partial_t(\psi F) \leq 0.
\]

Similarly to \[9\], we have

\[
(L - \partial_t)(\psi F) = \psi(L - \partial_t)F + (L\psi)F + 2\nabla \psi \cdot \nabla F
\geq \psi(L - \partial_t)F - A(R)F + 2\nabla \psi \cdot \nabla F
\geq \psi(L - \partial_t)F - A(R)F + 2(\nabla \psi, \nabla(\psi F))\psi^{-1} - 2F|\nabla \psi|^2\psi^{-1}.
\]

where we use

\[
L\psi \geq -A(R) := -\frac{C_1}{R}(m - 1)\sqrt{K} \coth(\sqrt{K}R) - \frac{C_2}{R^2},
\]

and for some constant \( C_3 > 0 \)

\[
\frac{|\nabla \psi|^2}{\psi} \leq \frac{C_3}{R^2}.
\]

Let \( C(n, K, R) = \frac{C_1}{R}(m - 1)\sqrt{K} \coth(\sqrt{K}R) + \frac{C_2 + C_3}{R^2} \). At the point \( (x_0, t_0) \), we have

\[
0 \geq \psi(L - \partial_t)F - (A(R) + 2|\nabla \psi|^2\psi^{-1})F
\geq \psi \left[ \frac{2[te^{-2Kt}(e^{-2Kt} - 1)]|\nabla F|^2 - F^2}{mt} - 2(\nabla F, \nabla f) + \frac{(2Kt - 1)}{t} \right] - C(n, K, R)F
\geq \psi \left[ \frac{2}{mt} F^2 + \frac{4e^{-2Kt}(1 - e^{-2Kt})}{m} |\nabla F|^2 - 2F(\nabla \psi, \nabla f) + \left[ (2K - \frac{1}{t})\psi - C(n, K, R) \right] F \right]
\geq \psi \left[ \frac{2}{mt} F^2 + \frac{4e^{-2Kt}(1 - e^{-2Kt})}{m} |\nabla F|^2 - 2\frac{C_2}{R} F\psi^{1/2} |\nabla F| + \left[ (2K - \frac{1}{t})\psi - C(n, K, R) \right] F \right].
\]

Multiplying by \( t \) on both sides, and using the Cauchy-Schwartz inequality, we get

\[
0 = \psi \left[ \frac{2}{m} F^2 + tF \left[ \frac{4e^{-2Kt}(1 - e^{-2Kt})}{m} |\nabla F|^2 - 2\frac{C_2}{R} \psi^{1/2} |\nabla F| \right] + \left[ (2Kt - 1)\psi - C(n, K, R)t \right] F \right]
\geq \psi \left[ \frac{2}{m} F^2 + \left[ (2Kt - 1)\psi - C(n, K, R)t \right] t - \frac{C_{2m}}{4e^{-2Kt}(1 - e^{-2Kt})R^2} F \right].
\]

Notice that the above calculation is done at the point \( (x_0, t_0) \). Since \( \psi F \) reaches its maximum at this point, we can assume that \( \psi F(x_0, t_0) > 0 \). Thus

\[
0 \geq \frac{2}{m}(\psi F)^2 - \left[ 1 + C(n, K, R)t + \frac{C_{2m}}{4e^{-2Kt_0}(1 - e^{-2Kt_0})R^2} t \right] (\psi F),
\]

which yields that, for any \( (x, t) \in B_R \times [0, T] \),

\[
F(x, t) \leq (\psi F)(x_0, t_0) \leq \frac{m}{2} \left[ 1 + C(n, K, R)t_0 + \frac{C_{2m}}{4e^{-2Kt_0}(1 - e^{-2Kt_0})R^2} t_0 \right]
\leq \frac{m}{2} \left[ 1 + C(n, K, R)T + \max_{t \in [0, T]} \frac{C_{2m}}{4e^{-2Kt}(1 - e^{-2Kt})R^2} \right].
\]
Let $R \to \infty$, we obtain
\[ F \leq \frac{m}{2}. \]

The proof of Theorem 2.1 is completed.

\[ \square \]

3.2 Proof of Theorem 2.2

The proof is as the same as the one of Corollary 2.2 in [6]. For the completeness we reproduce it as follows. Let \( l(x, t) = \log u(x, t) \). Then the Hamilton Harnack inequality is equivalent to
\[ \frac{\partial l}{\partial t} - e^{-2Kt} |\nabla l|^2 + \frac{e^{2Kt} m^2}{2t} \geq 0. \]  
(19)

Let \( \gamma : [0, T] \to M \) be a geodesic with reparametrization by arc length \( s : [\tau, T] \to [0, T] \) so that \( \gamma(s(\tau)) = x \) and \( \gamma(s(T)) = y \). Let \( S(t) = \frac{d\gamma(s(t))}{dt} = \dot{\gamma}(s(t)) \dot{s}(t) \). Then \( |\dot{\gamma}(s(t))| = 1 \). Integrating along \( \gamma(s(t)) \) from \( t = \tau \) to \( t = T \), we have
\[ l(y, T) - l(x, \tau) = \int_{\tau}^{T} \left[ \frac{\partial l}{\partial t} + \nabla l \cdot S \right] dt. \]

By the Cauchy-Schwartz inequality
\[ e^{-2Kt} |\nabla l|^2 + \frac{1}{4} e^{2Kt} |S|^2 \geq \nabla l \cdot S \]
From this and (19) we obtain
\[ l(y, T) - l(x, \tau) \geq - \frac{1}{4} \int_{\tau}^{T} e^{2Kt} |S|^2 dt - \int_{\tau}^{T} \frac{m}{2t} e^{2Kt} dt. \]
Note that \( d(x, y) = \int_{\tau}^{T} |S| dt = \int_{\tau}^{T} ds(t) \). Choosing \( s(t) = a[e^{-2Kt} - e^{-2K\tau}] \), with
\[ a = \frac{d(x, y)}{e^{-2K\tau} - e^{-2K\tau}}, \]
we have
\[ \int_{\tau}^{T} e^{2Kt} |S|^2 dt = \int_{\tau}^{T} e^{2Kt} s^2(t) dt = \frac{2Kd^2(x, y)}{e^{-2K\tau} - e^{-2K\tau}}. \]

Therefore
\[ l(y, T) - l(x, \tau) \geq - \frac{1}{4} \int_{\tau}^{T} e^{2Kt} s^2(t) dt - \int_{\tau}^{T} \frac{m}{2t} e^{2Kt} dt \]
\[ = - \frac{Kd^2(x, y)}{2(e^{-2K\tau} - e^{-2K\tau})} - \int_{\tau}^{T} \frac{m}{2t} e^{2Kt} dt. \]

Note that \( \int_{\tau}^{T} \frac{e^{2Kt}}{t} dt \leq \log \left( \frac{T}{\tau} \right) + e^{2Kt} - e^{2K\tau} \). Thus
\[ \log u(y, T) - \log u(x, \tau) \geq - \frac{Kd^2(x, y)}{2(e^{-2K\tau} - e^{-2K\tau})} - \frac{m}{2} \left[ \log \left( \frac{T}{\tau} \right) + e^{2Kt} - e^{2K\tau} \right]. \]
Using \( \frac{1}{1-e^{-x}} \leq \frac{1+x}{x} \), we can derive the desired estimate.
3.3 Proof of Theorem 2.3

Let \( \psi(t) = \frac{1 - e^{-Kt}}{K} \), and \( h = \psi \left[ Lu + \frac{\nabla u^2}{u} \right] - u[m + 4 \log(A/u)] \). Then

\[ \psi' + K \psi = 1. \]

By (18), under the assumption \( \text{Ric}_{m,n}(L) \geq -K \) we have

\[ (\partial_t - L)\frac{\nabla u^2}{u} \leq \frac{2}{mu} \left| Lu - \frac{\nabla u^2}{u} \right|^2 + 2K\frac{\nabla u^2}{u}, \]

which yields

\[ (\partial_t - L)h \leq \frac{2\psi}{mu} \left| Lu - \frac{\nabla u^2}{u} \right|^2 + \psi \left[ Lu - \frac{\nabla u^2}{u} \right] - 2\frac{\nabla u^2}{u}. \]

By analogue of Hamilton [6], we can verify that

\[ \frac{\partial h}{\partial t} \leq Lh \quad \text{whenever} \quad h \geq 0. \]

Indeed, we can verify this by examining three cases:

(i) If \( Lu \leq \frac{\nabla u^2}{u} \), then \( (\partial_t - L)h \leq 0 \) since \( \psi' \geq 0 \).

(ii) If \( 3\frac{\nabla u^2}{u} \leq Lu \leq 3\frac{\nabla u^2}{u} \), then \( (\partial_t - L)h \leq 0 \) since \( \psi' \leq 1 \).

(iii) If \( 3\frac{\nabla u^2}{u} \leq Lu \), then whenever \( h \geq 0 \), we have

\[ 2 \left[ Lu - \frac{\nabla u^2}{u} \right] \geq Lu + \frac{\nabla u^2}{u} = \frac{h}{\psi} + \frac{mu + 4u \log(A/u)}{\psi} \geq \frac{mu}{\psi}, \]

which yields, since \( \psi' \leq 1 \), we have

\[ (\partial_t - L)h \leq (\psi' - 1) \left[ Lu - \frac{\nabla u^2}{u} \right] - 2\frac{\nabla u^2}{u} \leq 0. \]

Note that \( h \leq 0 \) at \( t = 0 \). By the weak maximum principle on complete Riemannian manifolds, see e.g. Theorem 12.10 in [5], we conclude that \( h \leq 0 \) for all \( t \in [0, T] \). Thus

\[ \frac{Lu}{u} + \frac{\nabla u^2}{u^2} \leq \frac{K}{1 - e^{-Kt}} \left[ m + 4 \log(A/u) \right]. \]

This completes the proof of Theorem 2.3. \( \square \)

3.4 Proof of Theorem 2.4

The lower bound estimate of (18) follows from [15]. It remains to prove the upper bound estimate. Recall the following

**Proposition 3.1** ([12]) Suppose that there exist some constants \( m \geq n, m \in \mathbb{N} \) and \( K \geq 0 \) such that \( \text{Ric}_{m,n}(L) \geq -K \). Then, for any small \( \varepsilon > 0 \), there exist some constants \( C_i = C_i(m, n, K, \varepsilon) > 0 \), \( i = 1, 2 \), such that for all \( x, y \in M \) and \( t > 0 \),

\[ p_t(x, y) \leq \frac{C_1}{\mu(B_y(\sqrt{t}))} \exp \left( -\frac{d^2(x, y)}{4(1 + \varepsilon)t} + \alpha \varepsilon Kt \right) \times \left( \frac{d(x, y) + \sqrt{t}}{\sqrt{t}} \right)^{m/2} \exp \left( \sqrt{(m - 1)Kd(x, y)} \right). \]

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where \( \alpha \) is a constant depending only on \( m \), and

\[
p_t(x, y) \geq C_2 e^{-\left(1+\varepsilon\right)\lambda_{K,m} t} \mu^{-1} \left(B_y(\sqrt{t})\right) \exp \left( -\frac{\sqrt{K}d(x, y)}{4(1-\varepsilon)t} \right) \left[ \frac{\sqrt{K}d(x, y)}{\sinh \sqrt{K}d(x, y)} \right]^{m-1},
\]

where

\[
\lambda_{K,m} = \frac{(m - 1)^2 K}{8}.
\]

Fix \( T > 0 \), and let \( u(t, x) \) be a positive and bounded solution to the heat equation \( \partial_t u = Lu, \ t \in (0, t_1) \). Let

\[
A := \sup \{ u(t, x) : 0 \leq t \leq t_1, x \in M \}.
\]

By Hamilton’s Harnack inequality (17), we have

\[
t \partial_t \log p_t(x, y) \leq (1 + Kt) \left[ m + 4 \log \left( A/u(t, x) \right) \right], \ \forall (t, x) \in [0, t_1] \times M.
\]

Let \( s \in (0, T], \ y \in M, \ t_1 = s/2 \) and \( u(t, x) = p_{s/2+t}(x, y) \). By (20) and using the upper bound and lower bound estimates of the heat kernel \( p_t(x, y) \) in Proposition 3.1, we have

\[
\frac{t}{2} \partial_t \log p_{s/2+t}(x, y) \leq C_{K,m,T} \left( 1 + Kt/2 \right) \left[ 1 + \frac{d(x, y)}{\sqrt{t}} + \log \left( \frac{C_1 \mu(B(y, \sqrt{s/2+t}))}{C_2 \mu(B(y, \sqrt{s/2}))} \right) \exp \left( \frac{C_3 d^2(x, y)}{s/2 + t} + C_4 d(x, y) \right) \right] .
\]

In particular, taking \( t = s/2 \) and changing \( s \) by \( t \) we get

\[
\frac{t}{2} \partial_t \log p_t(x, y) \leq C_{K,m,T} \left( 1 + Kt/2 \right) \left[ 1 + \frac{d(x, y)}{\sqrt{t}} + \log \left( \frac{C_1 \mu(B(y, \sqrt{t}))}{C_2 \mu(B(y, \sqrt{t/2}))} \exp \left( \frac{C_3 d^2(x, y)}{t} + C_4 d(x, y) \right) \right] .
\]

By the generalized Bishop-Gromov volume comparison theorem for weighted volume measure, see [23, 19, 9, 25], as \( \text{Ric}_{m,n}(L) \geq -K \), for all \( R > r > 0 \) and \( y \in M \), we have

\[
\frac{\mu(B(y, R))}{\mu(B(y, r))} \leq \left( \frac{R}{r} \right)^m \exp \left( \sqrt{(m-1)KR} \right).
\]

It follows that

\[
t \partial_t \log p_t(x, y) \leq C_{K,m,T} \left[ 1 + \frac{d^2(x, y)}{t} + \frac{d(x, y)}{\sqrt{t}} + d(x, y) \right],
\]

which yields

\[
\partial_x \log p_t(x, y) \leq C_{K,m,T} \left[ 1 + \frac{d(x, y)}{t} + \frac{1}{\sqrt{t}} \right]^2.
\]

This completes the proof of Theorem 2.4. \( \square \)
4 \ W\text{-entropy for Witten Laplacian with } CD(-K, m)\text{-condition}

Recall that, Perelman [22] introduced the W-entropy and proved its monotonicity along the conjugate heat equation associated to the Ricci flow. In [20, 21], Ni proved the monotonicity of the W-entropy for the heat equation of the usual Laplace-Beltrami operator on complete Riemannian manifolds with non-negative Ricci curvature. In [10, 12], the second author of this paper proved the W-entropy formula and its monotonicity and rigidity theorems for the heat equation of the Witten Laplacian on complete Riemannian manifolds satisfying the CD(0, m)-condition and gave a probabilistic interpretation of the W-entropy for the Ricci flow. In [13], we gave a new proof of the W-entropy formula obtained in [10] for the Witten Laplacian by using Ni’s W-entropy formula to the Laplace-Beltrami operator on \( M \times S^{m-n} \) equipped with a suitable warped product Riemannian metric, and further proved the monotonicity of the W-entropy for the heat equation of the time dependent Witten Laplacian on compact Riemannian manifolds equipped with the super Ricci flow with respect to the \( m \)-dimensional Bakry-Emery Ricci curvature. As we have already seen in Section 1, there is a close connection between the Perelman W-entropy for the heat equation of the Witten Laplacian and the Li-Yau Harnack inequality (10) on complete Riemannian manifolds satisfying the CD(0, m)-condition. In this section, we will introduce the Perelman W-entropy and prove its monotonicity for the heat equation of the Witten Laplacian on complete Riemannian manifolds with the CD(−K, m)-condition.

Recall the following entropy dissipation formulas for the heat equation of the Witten Laplacian on complete Riemannian manifolds with bounded geometry condition. In the case of compact Riemannian manifolds, it is a well-known result due to Bakry and Emery [1].

**Theorem 4.1** ([10, 12, 13]) Let \((M, g)\) be a complete Riemannian manifold with bounded geometry condition, and \( \phi \in C^4(M) \) such that \( \nabla^k \phi \) are uniformly bounded on \( M \) for \( 1 \leq k \leq 4 \). Let \( u \) be the fundamental solution to the heat equation \( \partial_t u = Lu \). Let

\[
H(u(t)) = - \int_M u \log u \, d\mu.
\]

The first order entropy dissipation formula (21) holds if \( \phi \in C^2(M) \) such that \( \text{Ric} \geq -K \).

\[
\frac{d}{dt} H(u(t)) = \int_M | \nabla \log u |^2 u \, d\mu,
\]

(21)

\[
\frac{d^2}{dt^2} H(u(t)) = -2 \int_M \Gamma_2(\nabla \log u, \nabla \log u) u \, d\mu,
\]

(22)

where

\[
\Gamma_2(\nabla \log u, \nabla \log u) = | \nabla^2 \log u |^2 + \text{Ric}(L)(\nabla \log u, \nabla \log u).
\]

Let \((M, g, \phi)\) be as in Theorem 4.1. Inspired by [22, 20, 10, 12, 13], we define

\[
H_{m,K}(u, t) = - \int_M u \log u \, d\mu - \Phi_{m,K}(t),
\]

where \( \Phi_{m,K} \in C((0, \infty), \mathbb{R}) \) satisfies

\[
\Phi'_{m,K}(t) = \frac{m}{2t} e^{4Kt}, \quad \forall t > 0.
\]

The first order entropy dissipation formula (21) holds if \( \phi \in C^2(M) \) such that \( \text{Ric}_{m,n}(L) \geq -K \).
Proposition 4.2 Let \((M,g)\) be a complete Riemannian manifold with bounded geometry condition, \(\phi \in C^4(M)\) be such that \(\nabla^k \phi\) are uniformly bounded on \(M\) for \(1 \leq k \leq 2\). Then, under the condition \(\text{Ric}_{m,n}(L) \geq -K\), we have

\[
\frac{d}{dt} H_{K,m}(u,t) \leq 0.
\]

Proof. By the entropy dissipation formulas in Theorem 4.1 and using the fact \(\int_M \partial_t u d\mu = \int_M L u d\mu = 0\), we have

\[
\frac{d}{dt} H_{m,K}(u,t) = \int_M \left[ \frac{|\nabla u|^2}{u^2} - \frac{m}{2t} e^{4Kt} \right] u d\mu
\]

By the Hamilton Harnack inequality in Theorem 2.1, we have

\[
\frac{d}{dt} H_{m,K}(u,t) \leq 0.
\]

□

Proposition 4.3 Under the same condition as in Theorem 2.6, we have

\[
\frac{d^2}{dt^2} H_{m,K}(u,t) = -2 \int_M \left[ |\nabla^2 \log u|^2 + \text{Ric}(L)(\nabla \log u, \nabla \log u) \right] u d\mu - \left( \frac{2mK}{t} - \frac{m}{2t^2} \right) e^{4Kt}.
\]

Proof. Indeed, by the second order dissipation formula of the Boltzmann entropy in Theorem 4.1 we have

\[
\frac{d}{dt} \int_M \frac{|\nabla u|^2}{u} d\mu = -2 \int_M \left[ |\nabla^2 \log u|^2 + \text{Ric}(L)(\nabla \log u, \nabla \log u) \right] u d\mu.
\]

Combining this with (23), Proposition 4.3 follows. □

Based on the Hamilton differential Harnack inequality (15) in Theorem 2.1 we now introduce the \(W\)-entropy for the heat equation (9) of the Witten Laplacian on complete Riemannian manifolds with the \(CD(-K,m)\)-condition as follows

\[
W_{m,K}(u,t) = \frac{d}{dt}(t H_{m,K}(u,t)).
\]

By the entropy dissipation formula in Theorem 4.1, we have

\[
W_{m,K}(u,t) = \int_M \left[ t(|\nabla \log u|^2 - \Phi_{m,K}'(t)) - \log u - \Phi_{m,K}(t) \right] u d\mu
\]

We are now in a position to prove the main result of this section, i.e., Theorem 2.6

Proof of Theorem 2.6 By (23) and Proposition 4.3 we have

\[
\frac{d}{dt} W_{m,K}(u,t) = -2t \left[ \int_M |\nabla^2 \log u|^2 + \text{Ric}(L)(\nabla \log u, \nabla \log u) u d\mu + \left( \frac{mK}{t} - \frac{m}{4t^2} \right) e^{4Kt} \right]
\]

We are now in a position to prove the main result of this section, i.e., Theorem 2.6
Note that
\[ \left| \nabla^2 \log u + \left( \frac{e^{2Kt}}{2t} + a(t) \right) g \right|^2 = \left| \nabla^2 \log u \right|^2 + 2 \left( \frac{e^{2Kt}}{2t} + a(t) \right) \Delta \log u + n \left( \frac{e^{2Kt}}{2t} + a(t) \right)^2. \]

By direct calculation, we have
\[
\frac{d}{dt} W_{m,K}(u,t) = -2t \int_M \left( \nabla^2 \log u + \left( \frac{e^{2Kt}}{2t} + a(t) \right) g \right)^2 u d\mu \\
-2t \int_M \left( Ric_{m,n}(L) + \left( 2a(t) - \frac{1 - e^{2Kt}}{t} \right) g \right) (\nabla \log u, \nabla \log u) u d\mu \\
+2nt \left( \frac{e^{2Kt}}{2t} + a(t) \right)^2 - \frac{me^{4Kt}}{2t} - 2mKe^{4Kt} \\
+2(e^{2Kt} + 2a(t)) \int_M \nabla \phi \cdot \nabla \log u d\mu - 2t \int_M \frac{\nabla \phi \cdot \nabla \log u}{m-n} u d\mu.
\]

Let \( a(t) \) be chosen such that \( 2a(t) - \frac{1 - e^{2Kt}}{t} = K \). Then
\[
\frac{d}{dt} W_{m,K}(u,t) = -2t \int_M \left[ \nabla^2 \log u + \left( \frac{K}{2} + \frac{1}{2t} \right) g \right]^2 u d\mu \\
+2nt \left( \frac{1}{2t} + \frac{K}{2} \right)^2 - \frac{me^{4Kt}}{2t} - 2mKe^{4Kt} \\
+2(1 + Kt) \int_M \nabla \phi \cdot \nabla \log u d\mu - 2t \int_M \frac{\nabla \phi \cdot \nabla \log u}{m-n} u d\mu.
\]

Combining this with
\[
\frac{1}{m-n} \int_M \left| \nabla \phi \cdot \nabla \log u - \frac{(m-n)(1+Kt)}{2t} \right|^2 u d\mu = \frac{(m-n)(1+Kt)^2}{4t^2} - \frac{1+Kt}{t} \int_M \nabla \phi \cdot \nabla \log u d\mu + \int_M \frac{\nabla \phi \cdot \nabla \log u}{m-n} u d\mu,
\]
and noting that
\[
2nt \left( \frac{1}{2t} + \frac{K}{2} \right)^2 - \frac{me^{4Kt}}{2t} - 2mKe^{4Kt} + \frac{(m-n)(1+Kt)^2}{2t} = \frac{m}{2t} \left((1+Kt)^2 - e^{4Kt}(1+4Kt) \right),
\]
we can derive the desired \( W \)-entropy formula. The rest of the proof is obvious. \( \square \)

In particular, taking \( m = n \), \( \phi \equiv 0 \) and \( g \) is a fixed Riemannian metric, we have the following \( W \)-entropy formula for the heat equation of the Laplace-Beltrami operator on Riemannian manifolds, which extends Ni’s result in [20] for \( K = 0 \).

**Theorem 4.4** Let \( (M,g) \) be a complete Riemannian manifold with bounded geometry condition. Let \( u \) be the fundamental solution to the heat equation \( \partial_t u = \Delta u \). Then
\[
\frac{d}{dt} W_{n,K}(u,t) = -2t \int_M \left[ \nabla^2 \log u + \left( \frac{K}{2} + \frac{1}{2t} \right) g \right]^2 u d\mu \\
-\frac{n}{2t} \left[ e^{4Kt}(1+4Kt) - (1+Kt)^2 \right].
\]
In particular, if $\text{Ric} \geq -K$, then, for all $t \geq 0$, we have

$$
\frac{d}{dt} W_{n,K}(u, t) \leq -\frac{n}{2t} \left[e^{4Kt}(1 + 4Kt) - (1 + Kt)^2\right].
$$

Moreover, the equality holds at some time $t = t_0 > 0$ if and only if $M$ is an Einstein manifold, i.e., $\text{Ric} = -Kg$, and the potential function $f = -\log u$ satisfies the shrinking soliton equation, i.e.,

$$
\text{Ric} + 2\nabla^2 f = g_t.
$$

By analogue of the $W$-entropy for the heat equation of the Witten Laplacian on complete Riemannian manifolds with the $\text{CD}(-K, m)$-condition, we can prove the $W$-entropy formula for the heat equation of the time dependent Witten Laplacian on compact manifolds equipped with a $(-K, m)$-super Ricci flow. To do so, let us recall the entropy dissipation formula on compact manifolds with time dependent metrics and potentials.

**Theorem 4.5** ([13]) Let $(M, g(t), t \in [0, T])$ be a family of compact Riemannian manifolds with potential functions $\phi(t) \in C^\infty(M)$, $t \in [0, T]$. Suppose that $g(t)$ and $\phi(t)$ satisfy the conjugate equation

$$
\frac{\partial \phi}{\partial t} = \frac{1}{2} \text{Tr} \left( \frac{\partial g}{\partial t} \right).
$$

Let

$$
L = \Delta_{g(t)} - \nabla_{g(t)} \phi(t) \cdot \nabla_{g(t)}
$$

be the time dependent Witten Laplacian on $(M, g(t), \phi(t))$. Let $u$ be a positive solution of the heat equation

$$
\partial_t u = Lu
$$

with initial data $u(0)$ satisfying $\int_M u(0) d\mu(0) = 1$. Let

$$
H(u, t) = -\int_M u \log u d\mu
$$

be the Boltzmann-Shannon entropy for the heat equation $\partial_t u = Lu$. Then

$$
\frac{\partial}{\partial t} H(u, t) = \int_M |\nabla \log u|_{g(t)}^2 u d\mu,
$$

$$
\frac{\partial^2}{\partial t^2} H(u, t) = -2 \int_M \left[|\nabla^2 \log u|^2 + \left(\frac{1}{2} \frac{\partial g}{\partial t} + \text{Ric}(L)\right)(\nabla \log u, \nabla \log u)\right] u d\mu.
$$

**Proof of Theorem 2.7**. Based on the entropy dissipation formulas in Theorem [13], the proof of Theorem 2.7 is similar to the one of Theorem 2.6. See [13] for the case $K = 0$. □

5 **Comparison with another $W$-entropy functional**

To end this paper, let us mention that in our previous paper [13] we introduced another $W$-entropy functional for the heat equation associated with the Witten Laplacian on complete Riemannian manifolds satisfying the $\text{CD}(-K, m)$-condition as follows

$$
\widetilde{W}_{m,K}(u) = \frac{d}{dt}(t\widetilde{H}_{m,K}(u))
$$

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where
\[ H_{m,K}(u) = -\int_M u \log u \, d\mu - \left[ \frac{m}{2t}(1 + \log(4\pi t)) + \frac{mKt}{2}(1 + \frac{1}{6}Kt) \right], \]
and we proved that
\[ \frac{d}{dt} W_{m,K}(u) = -2t \int_M \left\| \nabla^2 \log u + \left( \frac{K}{2} + \frac{1}{2t} \right) g \right\|^2 + (Ric_{m,n}(L) + Kg)(\nabla \log u, \nabla \log u) \right] u \, d\mu - \frac{2t}{m-n} \int_M \left\| \nabla \phi \cdot \nabla \log u - \frac{(m-n)(1 + Kt)}{2t} \right\|^2 \, u \, d\mu. \]

It is interesting to compare the $W$-entropy defined in [13] with the $W$-entropy defined in this paper, and to compare the $W$-entropy formula proved in [13] with the $W$-entropy formula obtained in Theorem 2.6. Indeed, letting
\[ \Psi_{m,K}(t) = \Phi_{m,K}(t) - \left[ \frac{m}{2t}(1 + \log(4\pi t)) + \frac{mKt}{2}(1 + \frac{1}{6}Kt) \right], \]
we have
\[ \tilde{W}_{m,K}(u) - W_{m,K}(u) = \frac{d}{dt}(t \Psi_{m,K}(t)). \]
Moreover, by direct calculation we have
\[ \frac{d}{dt}(\tilde{W}_{m,K}(u) - W_{m,K}(u)) = \frac{d^2}{dt^2}(t \Psi_{m,K}(t)) = \frac{m}{2t} \left[ e^{4Kt}(1 + 4Kt) - (1 + Kt)^2 \right]. \]

This explains clearly the difference between the $W$-entropy defined in [13] and the $W$-entropy defined in this paper, and the difference between the $W$-entropy formula proved in [13] with the $W$-entropy formula obtained in Theorem 2.6.

Similarly, we can reformulate Theorem 2.7 in terms of $\tilde{W}_{m,K}$. See [13].

Acknowledgement. Part of this work was done when the second author visited the Institut des Hautes Etudes Scientifiques and the Max-Planck Institute for Mathematics Bonn. The authors would like to thank Professors D. Bakry, J.-M. Bismut, M. Ledoux, N. Mok, K.-T. Sturm, A. Thalmaier, F.-Y. Wang and Dr. Yuzhao Wang for their interests and helpful discussions during the preparation of this paper. We are very grateful to anonymous referee for his careful reading and for his very nice comments which lead us to improve the writing of this paper.

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