Abstract

The population protocol model describes a network of anonymous agents that interact asynchronously in pairs chosen at random. Each agent starts in the same initial state $s$. We introduce the dynamic size counting problem: approximately counting the number of agents in the presence of an adversary who at any time can remove any number of agents or add any number of new agents in state $s$. A valid solution requires that after each addition/removal event, resulting in population size $n$, with high probability each agent “quickly” computes the same constant-factor estimate of the value $\log_2 n$ (how quickly is called the convergence time), which remains the output of every agent for as long as possible (the holding time). Since the adversary can remove agents, the holding time is necessarily finite: even after the adversary stops altering the population, it is impossible to stabilize to an output that never again changes.

We first show that a protocol solves the dynamic size counting problem if and only if it solves the loosely-stabilizing counting problem: that of estimating $\log n$ in a fixed-size population, but where the adversary can initialize each agent in an arbitrary state, with the same convergence time and holding time. We then show a protocol solving the loosely-stabilizing counting problem with the following guarantees: if the population size is $n$, $M$ is the largest initial estimate of $\log n$, and $s$ is the maximum integer initially stored in any field of the agents’ memory, we have expected convergence time $O(\log n + \log M)$, expected polynomial holding time, and expected memory usage of $O(\log^2(s) + (\log \log n)^2)$ bits. Interpreted as a dynamic size counting protocol, when changing from population size $n_{prev}$ to $n_{next}$, the convergence time is $O(\log n_{next} + \log \log n_{prev})$.

1 Introduction

A population protocol [6] is a network of $n$ anonymous and identical agents with finite memory called the state. A scheduler repeatedly selects a pair of agents independently and uniformly at random to interact. Each agent sees the entire state of the other agent in the interaction and updates own state in response. Time complexity is measured by parallel time: the number of interactions divided by the population size $n$, capturing the natural time scale in which each agent has $\Theta(1)$ interactions per unit time. The agents collectively do a computation, e.g., population size counting: computing the value $n$. Counting is a fundamental task in distributed computing: knowing an estimate of $n$ often simplifies the design of protocols solving problems such as majority and leader election.

A protocol is defined by a transition function with a pair of states as input and as output (more generally to capture randomized protocols, a relation that can associate multiple outputs to the same input). For example, consider the simple counting protocol with transitions $L_i \rightarrow L_{i+1}, F_{i+j}, L_{i+j}$, with every agent starting in $L_1$. In population size $n$, this protocol converges to a single agent in state $L_n$, with all other agents in state $F_i$ for some $i$. The additional transitions $F_i, F_j \rightarrow F_j, F_j$ for $i < j$ propagate the output $n$ to all agents.
The dynamic size counting problem

In contrast to most work, which assumes the population size $n$ is fixed over time, we model an adversary that can add or remove agents arbitrarily and repeatedly during the computation. All agents start in the same state, including newly added agents. The goal is for each agent to approximately count the population size $n$, which we define to mean that all agents should eventually store the same output $k$ in their states, which with high probability is within a constant multiplicative factor of $\log n$. Once all agents have the same output $k$, they have converged. They maintain $k$ as the output for some time called the holding time (after which they might alter $k$ even if the population size has not changed). In response to a “significant” change in size from $n_{\text{prev}}$ to $n_{\text{next}}$, agents should re-converge to a new output $k'$ of $\log n_{\text{next}}$. (Agents are not “notified” about the change; instead they must continually monitor the population to test whether their current output is accurate.) Note that if $n_{\text{prev}}$ is close to $n_{\text{next}}$ (within a polynomial factor), then $k$ may remain an accurate estimate of $n_{\text{next}}$, so agents may not re-converge in response to a small change.

Ideally the expected convergence time is small, and the expected holding time is large. With a fixed size population, it is common to require the output to stabilize to a value that never again changes after convergence, i.e., infinite holding time. However, this turns out to be impossible with an adversary that can remove agents (Observation 3.4). When changing from size $n_{\text{prev}}$ to $n_{\text{next}}$, our protocol achieves expected convergence time $O(\log n_{\text{next}} + \log \log n_{\text{prev}})$ and expected holding time $\Omega(n_{\text{next}}^c)$, where $c$ can be made arbitrarily large. The number of bits of memory used per agent is $O(\log^2(s) + (\log \log n)^2)$, where $s$ is the maximum integer stored in the agents’ memory after the change.

While it is common to measure population protocol memory complexity by counting the number of states (which is exponentially larger than the number of bits required to represent each state), that measure is a bit awkward here. Our protocol is uniform—the same transition rules for every population size—so has an infinite number of producible states. One could count expected number of states that will be produced, but this is a bit misleading: in time $t$ each agent visits $O(t)$ states on average, so $O(t \cdot n)$ states total. Counting how many bits are required is more accurate metric of the actual memory requirements.

The loosely-stabilizing counting problem

The dynamic size counting problem has an equivalent characterization: rather than removing agents and adding them with a fixed initial state, the loosely-stabilizing adversary sets each agent to an arbitrary initial state in a fixed-size population. A protocol solves the dynamic size counting problem if and only if it solves the loosely-stabilizing counting problem, with the same convergence and holding times (Lemma 3.6). Due to this equivalence, we analyze our protocol assuming a fixed population size and adversarial initial states. In this case our convergence time $O(\log n + \log M)$ is measured as a function of the population size $n$ and the value $M$ that is the maximum estimate value stored in agents’ memory. From the perspective of the dynamic size counting problem, these “adversarial initial states” would correspond to the agent states after correctly estimating the previous population size, just prior to adding or removing agents.

\footnote{Nonuniform protocols require agents to be initialized with an estimate $k$ of $\log n$ in order to accomplish other tasks, such as a “leaderless phase clock”\cite{1}. The bound $k = \Theta(\log n)$ is necessary and sufficient for correctness and speed in most cases\cite{1,2,3,4,5,11,13,16,23,30,34}.}
1.1 Related work

Initialized counting with a fixed size population. In population protocols with fixed size, there is work computing exactly or approximately the population size $n$. For a full review see [19]. Such protocols reach a stable configuration from which the output cannot change. Some of these counting protocols would still solve the counting problem in the presence of an adversary who can only add agents (see Observation 3.3). However, these protocols fail in the presence of an adversary who can also remove agents, since they work only in the initialized setting and rely on reaching a stable configuration (see Observation 3.4).

Self-stabilizing counting with a fixed size population. A population protocol is self-stabilizing if, from any initial configuration, it reaches to a correct stable configuration. Self-stabilizing size counting has been studied [8–10,26], but provably requires adding a “base station” agent that cannot be corrupted by the adversary. In these protocols the base station is the only agent required to learn the population size. Aspnes, Beauquier, Burman, and Sohier [8] showed a time- and space-optimal protocol that solves the exact counting problem in $O(n \log n)$ time, using 1-bit memory for each non-base station agent.

Size regulation in a dynamically sized population. The model described by Goldwasser, Ostrovsky, Scafuro, and Sealfon [24] is close to our setting. They consider the size regulation problem: approximately maintaining a target size (hard-coded into each agent) using $O(\log \log n)$ bits of memory per agent, despite an adversary that (like ours) adds or removes agents. That paper assumes a model variation in which:

- The agents can replicate or self-destruct.
- The computation happens through synchronized rounds of interactions. At each round the scheduler selects a random matching of size $k = O(n)$ agents to interact.
- The adversary’s changes to the population size are limited. The adversary can insert or delete a total of $o(n^{1/4})$ agents within each round.

The latter two model differences above crucially rule out their protocol as useful for our problem. We use the standard asynchronous scheduler, and much of the complexity of our protocol is to handle drastic population size changes (e.g., removing $n - \log n$ agents). Additionally, their protocol heavily relies on flipping coins of bias $\frac{1}{\sqrt{n}}$ that we cannot utilize since the agents don’t start with an estimate of $n$. Moreover, even when the agents compute their estimate, the population size might change.

Loosely-stabilizing leader election. Sudo, Nakamura, Yamauchi, Ooshita, Kakugawa, and Masuzawa [33] introduce loose-stabilization as a relaxation for the self-stabilizing leader election problem in which the agents must know the exact population size to elect a leader. The loosely stabilizing leader election guarantees that starting from any configuration, the population will elect a leader within a short time. After that, the agents hold the leader for a long time but not forever (in contrast with self-stabilization). On the positive side, the agents no longer need to know the exact population size to solve the loosely-stabilizing leader election, but a rough upper bound suffices. Loosely-stabilizing leader election has been studied, providing a time-optimal protocol that solves the leader election problem [32] and a tradeoff between the holding and convergence times [25,35].

Computation with dynamically changing inputs. Alistarh, Töpfer, and Uznanski [5] consider the dynamic variant of the comparison problem. In the comparison problem, a subset of population are in the input states $X$ and $Y$ and the goal is to compute if $X > Y$ or $X < Y$. In the dynamic variant of the comparison problem, they assume an adversary who can change the counts of the input states at any time. The agents should compute the output as long as the counts remain untouched for sufficiently long time. They propose
Dynamic size counting in population protocols

A protocol that solves the comparison problem in \( O(\log n) \) time using \( O(\log n) \) states per agent, assuming \(|X| \geq C_2 \cdot |Y| \geq C_1 \log n \) for some constants \( C_1, C_2 > 1 \).

Berenbrink, Biermeier, Hahn, and Kaaser consider the adaptive majority problem (generalization of the comparison problem [5]). At any time every agent has an opinion from \( \Lambda \) with high probability. For the purpose of representation, we make an exception in our protocol, when we show agents generate nonuniform protocol different transitions are applied for different population sizes. In contrast, in a polynomial protocol, we specify transitions formally with pseudocode that indicate how agents alter each state. We say a configuration \( \Delta \) of a population protocol is a multiset over \( \Lambda \) of size \( n \), giving the states of the \( n \) agents in the population. For a state \( s \in \Lambda \), we write \( c(s) \) to denote the count of agents in state \( s \). A transition is a 4-tuple, written \( \alpha: (r_1, r_2) \rightarrow (p_1, p_2) \), such that \( ((r_1, r_2), (p_1, p_2)) \in \Delta \). If an agent in state \( r_1 \) interacts with an agent in state \( r_2 \), then they can change states to \( p_1 \) and \( p_2 \). This notation omits explicit probabilities; our main protocol’s transitions can be implemented so as to always have either one or two possible outputs for any input pair, with probability 1/2 of each output in the latter case. For every pair of states \( r_1, r_2 \) without an explicitly listed transition \( r_1, r_2 \rightarrow p_1, p_2 \), there is an implicit null transition \( r_1, r_2 \rightarrow r_1, r_2 \) in which the agents interact but do not change state. For our main protocol, we specify transitions formally with pseudocode that indicate how agents alter each independent field in their state. We say a configuration \( \Delta \) is reachable from a configuration \( \Delta \) if applying 0 or more transitions to \( \Delta \) results in \( \Delta \).

When discussing random events in a protocol of population size \( n \), we say event \( E \) happens with high probability if \( \Pr[\neg E] = O(n^{-c}) \), where \( c \) is a constant that depends on our choice of parameters in the protocol, where \( c \) can be made arbitrarily large by changing the parameters. For concreteness, we will write a particular polynomial probability such as \( O(n^{-2}) \), but in each case we could tune some parameter (say, increasing the time complexity by a constant factor) to increase the polynomial’s exponent.

To measure time we count the total number of interactions (including null transitions such as \( a, b \rightarrow a, b \) in which the agents interact but do not change state), and divide by the number of agents \( n \).

In a uniform protocol (such as the main one of this paper), the transitions are independent from the population size \( n \) (see [20] for a formal definition). In other words, a single protocol computes the output correctly when applied on any population size. In contrast, in a nonuniform protocol different transitions are applied for different population sizes.

\footnote{For the purpose of representation, we make an exception in our protocol, when we show agents generate a geometric random variable in one line (see Protocol [5]). However, we can assume a geometric random variable is generated through \( O(\log n) \) consecutive interactions with each selecting out of two possible outputs (H or T).}
A protocol stably solves a problem if the agents eventually reach a correct configuration with probability 1, and no subsequent interactions can move the agents to an incorrect configuration; i.e., the configuration is stable. A population protocol is self-stabilizing if from any initial configuration, the agents stably solve the problem.

## 3 Dynamic size counting

In a population of size \( n \), define \( C(n, \delta) \) to be the set of correct configurations \( c \) such that every agent \( u \) in \( c \) obeys \((1 - \delta) \log n < u.\)\text{estimate} < (1 + \delta) \log n\). Let \( t_h \) be any time bound. Moreover, we define \( L(n, \delta, t_h) \subset C(n, \delta) \) the subset of correct configurations such that as the expected time for protocol \( P \) starting from a configuration \( l \in L(n, \delta, t_h) \) to stay in \( C(n, \delta) \) is at least \( t_h(n) \).

▶ **Definition 1.** Let \( n_{\text{prev}} \) and \( n_{\text{next}} \) denote the previous and next population size. A protocol \( P \) solves the dynamic size counting problem if there is a \( \delta > 0 \), called the accuracy, such that if the population size changes from \( n_{\text{prev}} \) to \( n_{\text{next}} \), the protocol reaches a configuration \( l \in L(n, \delta, t_h) \) with high probability. The time needed to do this is called the convergence time. Moreover, \( t_h \), the time that the population stays in \( C(n_{\text{next}}, \delta) \), is called the holding time.

A population protocol is \((t_c(n), t_h(n))\)-loosely stabilizing if starting from any initial configuration, the agents reach a correct configuration in \( t_c(n) \) time and stay in the correct configuration for additional \( t_h(n) \) time \([32,33]\). In contrast to self-stabilizing \([7,17]\), subsequent interactions can move the agents to an incorrect configuration; however, the agents recover quickly from an incorrect configuration.

Given any starting configuration \( s \notin C(n, \delta) \) of size \( n \), we define \( f_c(s, L(n, \delta, t_h)) \) as the expected time to reach a correct configuration in \( L(n, \delta, t_h) \).

▶ **Definition 2.** \([32,33] \) **Definition 2** Let \( t_c(n, M) \) and \( t_h(n) \) be functions of \( n \), the largest integer value \( M \) in the initial configuration \( s \), and the set of correct configuration \( C(n, \delta) \). A protocol \( P \) is a \((t_c(n, M), t_h(n), \delta)\) loosely-stabilizing population size counting protocol if there exists a set \( L(n, \delta, t_h) \) of configurations satisfying:

For every \( n \) and every initial configuration \( s \notin C(n, \delta) \) of size \( n \), \( f_c(s, L(n, \delta, t_h)) \leq t_c(n, M) \)

### 3.1 Basic properties of the dynamic size counting problem

We first observe that the key challenge in dynamic size counting is that the adversary may remove agents. If the adversary can only add agents, the problem is straightforward to solve with optimal convergence and holding times.

▶ **Observation 3.3.** Suppose the adversary in the dynamic size counting problem only adds agents. Then there is a protocol solving dynamic size counting with \( O(\log n) \) convergence time (in expectation and with probability \( \geq 1 - O(1/n) \)) and infinite holding time.

**Proof.** Each agent in the initial state \( s \) generates a geometric random variable. After the last time that the adversary adds agents, resulting in \( n \) total agents, exactly \( n \) geometric random variables will have been generated. Agents propagate the maximum by epidemic using transition \( a, b \rightarrow \max(a, b), \max(a, b) \), taking \( 3 \ln n \) time to reach all agents with probability \( \geq 1 - \frac{1}{n^2} \) \([17, \text{Corollary 2.8}] \). The maximum of \( n \) i.i.d. geometric random variables is in the range \([\log n - \log \ln n, 2 \log n]\) with probability \( \geq 1 - \frac{1}{n} \) \([18, \text{Lemma D.7}] \). ◀
In contrast, if the adversary can remove agents, then even if it is guaranteed to do this exactly once, no protocol can be stabilizing, i.e., have infinite holding time.

**Observation 3.4.** Suppose the adversary in the dynamic size counting problem will remove agents exactly once. Then any protocol solving the problem has finite holding time.

**Proof.** Suppose otherwise. Let the initial population size be \( n \) and the later size be \( n' < n \). The protocol must handle the case where the adversary never removes agents, since in population size \( n \) this is equivalent to an adversary who starts with \( n + 1 \) agents and immediately removes one of them. Thus if the adversary waits sufficiently long before the removal, then all agents stabilize to output \( k = \Theta(\log n) \). In other words, no sequence of transitions can alter the value, including transitions occurring only among any subpopulation of size \( n' \). So after the adversary removes \( n - n' \) agents, the remaining \( n' \) agents are unable to alter the output \( k \), a contradiction if \( n' \) is sufficiently small compared to \( n \).

Lemma 3.6 shows that the dynamic size counting problem is equivalent to the loosely-stabilizing counting problem. Due to this equivalence, our correctness proofs will use the loosely-stabilizing characterization.

To prove Lemma 3.6 we require the following result, proven in [18, Lemma 4.2]. It states that for any finite set \( \Lambda_f \) of states producible in a protocol from a uniform initial state, for any sufficiently large initial population size \( n \), WHP many copies of each state in \( \Lambda_f \) appear.

**Lemma 3.5 ([18]).** Let \( P \) be a population protocol, and let \( \Lambda_f \) be a finite set of states, each producible in \( P \) from sufficiently many agents in state \( s \). Then there are constants \( c, \delta, n_0 > 0 \) such that, for all \( n \geq n_0 \), starting from \( n \) agents in state \( s \), if \( d \) is the configuration reached at time 1, then \( \Pr[|\forall q \in \Lambda_f, d(q) \geq \delta n| \geq 1 - 2^{-cn}] \).

Recall that we define \( M \) as the largest integer value the agents stored in the starting configuration \( s \).

**Lemma 3.6.** A protocol solves the dynamic size counting problem with convergence time \( t_c(n, M) \) and holding time \( t_h(n) \) if and only if it solves the loosely-stabilizing counting problem with convergence time \( t_c(n, M) \) and holding time \( t_h(n) \).

**Proof sketch.** Any states present in an adversarially prepared configuration \( c \) will be produced in large quantities from any sufficiently large initial configuration of all initialized states \( s \) [18, Lemma 4.2]. The dynamic size adversary can then remove agents to result in \( c \), which the protocol must handle, showing it can handle an arbitrary initial configuration.

**Proof.** The easy direction is that if the protocol \( P \) solves the loosely-stabilizing counting problem, then it solves the dynamic size counting problem. Starting from any possible configuration \( s \) of size \( n \), with largest integer \( M \), \( P \) converges in time \( t_c(n, M) \) with holding time \( t_h(n) \). The configuration immediately after the dynamic size adversary adds or removes agents is simply one of the configurations that \( P \) is able to handle.

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3 Lemma 4.2 in [18] is stated slightly differently. Rather than an arbitrary finite set of states \( \Lambda_f \), it considers for some fixed \( m \in \mathbb{N}^+ \) and \( 0 < \rho \leq 1 \), the set of states producible using \( m \) different types of transitions, each having transition probability at least \( \rho \) (necessarily a finite set for fixed \( m \) and \( \rho \)). Setting \( m \) to the maximum number of types of transitions needed to produce any state \( q \in \Lambda_f \), and \( \rho \) to be the minimum transition probability among any of those transitions, we obtain the simpler lemma statement used here. Also, Lemma 4.2 in [18] allowed more general initial configurations, permitting agents in different states, but each having count \( \Omega(n) \) (so-called “dense” configurations).
To see the reverse direction, suppose $P$ solves the dynamic size counting problem; we argue that $P$ also solves the loosely-stabilizing counting problem. Let $c$ be any configuration of $P$. The dynamic size adversary can do the following to reach configuration $c$. Let $A_f = \{ q \mid c(q) > 0 \}$ be the set of states in $c$. Apply Lemma 3.5 with this choice of $A_f$. Choose $n$ sufficiently large that $\delta n \geq \max_{q \in A_f} c(q)$. Then Lemma 3.5 says that starting from $n$ agents in state $s$, with probability at least $1 - 2^{-\epsilon n}$, in the configuration $d$ at time 1, for all $q \in A_f$, $d(q) \geq \delta n \geq c(q)$. Thus $d \geq c$. Now the adversary removes agents from $d$ to result in $c$. Let $n' = ||c||$. The protocol converges from $c$ with largest integer $M'$ in time $t_c(n, M')$, with holding time $t_h(n)$. But since $c$ is an arbitrary configuration of the protocol, this implies that $P$ solves the loosely-stabilizing counting problem.

### 3.2 High-level overview of dynamic size counting protocol

In this section we briefly describe our protocol solving the dynamic size counting, which is defined formally in Section 3.3. By Lemma 3.6 it suffices to design a protocol solving the loosely-stabilizing counting problem for a fixed population size $n$. Our protocol uses the “detection” protocol of [3]. Consider a subset of states designated as a “source”. A detection protocol alerts all agents whether a source state is present in the population.

In Protocol [1] the population maintains several dynamic groups, with the agent’s group stored as a positive integer field $\text{group}$. The $\text{group}$ values are not fixed: each agent changes its $\text{group}$ field on every interaction, with equal probability either incrementing $\text{group}$ or setting it to 1. We show that, no matter the initial group values, after $O(\log n)$ time the group values will be in the range $[1, 8 \log n]$ WHP. The distribution of $\text{group}$ values is very close to that of $n$ i.i.d. geometric random variables, in the sense that each agent’s $\text{group}$ value is independent of every other, with expected $n/2^i$ agents having $\text{group} = i$ if each agent has had at least $i$ interactions.

The agents store an array of “signal” integers in their $\text{signals}$ field, as a way to track the existing $\text{group}$ values in the population. Each agent in the $i$’th group is responsible to boost the signal associated with $i$. The goal is to have $\text{signals}[i] > 0$ for all agents if and only if some agent has $\text{group} = i$.

The detection protocol of [3], explained below, provides a technique for agents to know which groups are still present. Once a signal for group $k$ fades out, the agents speculate that there is no agent with $\text{group} = k$. Depending on the current value stored as $\text{estimate}$ in agents’ memory and the value $k$, this might cause recalculating the population size. The agents are constantly checking for the changes in the $\text{signals}$. They re-compute $\text{estimate}$ once there is a large gap between $\text{estimate}$ and the first $\text{group}$ $i$ with $\text{signals}[i] = 0$. We call $i$ the first missing value (stored in the field FMV).

The $\text{signals}$ array is updated as follows. An agent with $\text{group} = k$ sets $\text{signals}[k]$ to its maximum possible value $(3k + 1)$; we call this boosting. Other groups $k$ are updated between two agents $u,v$ with $u.\text{signals}[k] = a$ and $v.\text{signals}[k] = b$ via propagation transitions that set both agent’s $\text{signals}[k]$ to $\max(a - 1, b - 1, 0)$. The paper [2] used a nonuniform protocol where each agent already has an estimate of $\log n$. They prove that if the state being detected (in our case, a state with $\text{group} = k$) is absent and the current maximum signal is $c$, then all agents will have signal 0 within $\Theta(c)$ time. However, if the state being detected is present, then the boosting transitions (occurring every $O(1)$ units of parallel time on average

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*The difference is that a geometric random variable $G$ obeys $\Pr[G = j] = 1/2^j$ for all $j \in \mathbb{N}^*$, but after $i$ interactions an agent $u$ can increment $u.\text{group}$ by at most $i$, so $\Pr[u.\text{group} = j] = 0$ if $j \gg i$. 

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in the worst case that its count is only 1) will keep the signal positive in all agents with high probability. For this to hold, it is critical that the maximum value set during boosting is $\Omega(\log n)$; the nonuniform protocol of [3] uses its estimate of $\log n$ for this purpose.

Crucially, our protocol associates smaller maximum signal values to smaller group values (so many are much smaller than $\log n$), to ensure that a signal does not take abnormally long to get to 0 when its associated group value is missing. Otherwise, if we set each signal value to $\Omega(\log n)$ (based on the agent’s current estimate \texttt{estimate} of $\log n$) during boosting, then it would take time proportional to \texttt{estimate} (which could be much larger than the actual value of $\log n$) to detect the absence of a \texttt{group} value. Thus it is critical that we provide a novel analysis of the detection protocol, showing that the signals for smaller group values $k \ll \log n$ remain present with high probability. This requires arguing that the boosting reactions for such smaller values are happening with sufficiently higher frequency, due to the higher count of agents with \texttt{group} = $k$, to compensate for the smaller boosting signal values they use.

### 3.3 Formal description of loosely-stabilizing counting protocol

The DynamicCounting protocol (Protocol [1]) divides agents among $\Theta(\log n)$ groups via the UpdateGroup subprotocol. The agents update their \texttt{group} from $i$ to $i+1$ with probability $1/2$ or reset to group 1 with probability $1/2$. The number of agents at each group and the total number of groups are both random variables that are dynamically changing through time. We show that the total number of groups remains close to $\log n$ at all times with high probability.

The agents start with arbitrary (or even adversarial) \texttt{group} values but we show that WHP the set of \texttt{group} values will converge to $[1, 8\log n]$ within $O(\log n)$ time (Corollary [5.10]). Additionally, each agent stores an array of $O(\log n)$ signal values in their \texttt{signals} field. It is crucial for agents to maintain positive values in the \texttt{signals}[i] if some agent has \texttt{group} = $i$. They use the first \texttt{group} $i$ with \texttt{signals}[i] = 0 (stored in \texttt{FMV}). The agents use \texttt{FMV} as an approximation of $\log n$ and constantly compare it with their \texttt{estimate} value.

Depending on the \texttt{estimate} value stored in agents’ memory, the agents maintain 3 main phases of computation:

**NormalPhase**: An agent stays in the NormalPhase as long as there is a small gap between \texttt{estimate} and \texttt{FMV}: $0.25 \cdot \texttt{estimate} \leq \texttt{FMV} \leq 2.5 \cdot \texttt{estimate}$. Additionally, each agent stores an array of $O(\log n)$ signal values in their \texttt{signals} field. It is crucial for agents to maintain positive values in the \texttt{signals}[i] if some agent has \texttt{group} = $i$. They use the first \texttt{group} $i$ with \texttt{signals}[i] = 0 (stored in \texttt{FMV}). The agents use \texttt{FMV} as an approximation of $\log n$ and constantly compare it with their \texttt{estimate} value.

**WaitingPhase**: An agent switches from NormalPhase to WaitingPhase if it sees a large gap between the \texttt{FMV} and \texttt{estimate}: $\texttt{FMV} \not\in \{0.25 \cdot \texttt{estimate}, \ldots, 2.5 \cdot \texttt{estimate}\}$. The purpose of WaitingPhase is to give enough time to the other agents so that by the end of the WaitingPhase for one agent, with high probability every other agent has also noticed the large gap between the \texttt{FMV} and \texttt{estimate} and entered WaitingPhase.

**UpdatingPhase**: During the UpdatingPhase, every agent uses a new geometric random variable and propagates the maximum by epidemic. We set WaitingPhase long enough so that with high probability when the first agent switches to the UpdatingPhase, the rest of the population are all in WaitingPhase. By the end of UpdatingPhase, every agent switches back to NormalPhase.

Below we explain each subprotocol in more detail.

In every interaction, both sender and receiver update their group according to the rules of the UpdateGroup subprotocol. If we look at the distribution of the \texttt{group} values after $O(\log n)$ time, there are about $n/2$ agents in group 1, $n/4$ agents in group 2 and $n/2^{i}$ agents in group $i$ (see Figure [4]). Note that the number of agents in each group decreases
Protocol 1 DynamicCounting(u, v)

\[
\text{for agent } \in \{u, v\} \text{ do} \\
\quad \text{UpdateGroup(agent)} \\
\text{SignalPropagation(u, v)} \\
\text{for agent } \in \{u, v\} \text{ do} \\
\quad \text{UpdateMV(agent)} \\
\quad \text{SizeChecker(agent)} \\
\quad \text{if agent.phase } \neq \text{ NormalPhase then} \\
\quad \quad \text{TimerRoutine(agent)} \\
\text{PropagateMaxEst(u, v)} \\
\text{for agent } \in \{u, v\} \text{ do} \\
\quad \text{if agent.phase } = \text{ NormalPhase then} \\
\quad \quad \text{agent.estimate } \leftarrow \text{agent.GRV}
\]

exponentially, but we ensure that agents with larger group values use stronger signals to propagate, since there is less support for those groups.

Protocol 2 UpdateGroup(agent u)

\[
\text{u.group } \leftarrow \begin{cases} 
\text{u.group } + 1 & \text{with probability } 1/2 \\
1 & \text{with probability } 1/2
\end{cases}
\]

To notify all agents about the set of all group values that are generated among the population, we use the detection protocol of \[3\] that is also used as a synchronization scheme in \[12\]. The agents store an integer for each group value that is generated by the population. The signals is an array of length $\Theta(\log n)$ such that a positive value in index $i$ represents some agents in the population have generated group $= i$. Note that, as an agent updates its group, it boosts multiple signals based on its group value, e.g., an agent with group $= i$ helped boosting all the indices $1, 2, 3, \ldots, i$ of signals in its last $i$ interactions. We use the SignalPropagation protocol to keep the signal of group $i$ positive as long as some agents have generated group $= i$.

Protocol 3 SignalPropagation(agent u, agent v)

\[
\triangleright \text{Boosting:} \\
\quad \text{u.signals}[\text{u.group}] \leftarrow (3 \cdot \text{u.group}) + 1 \\
\quad \text{v.signals}[\text{v.group}] \leftarrow (3 \cdot \text{v.group}) + 1 \\
\triangleright \text{Propagate signal:} \\
\quad \text{for } i \in \{1, 2, \ldots, \text{Max(|u.signals|, |v.signals|)}\} \text{ do} \\
\quad \quad m \leftarrow \text{Max(u.signals}[i], v.signals[|i|]) \\
\quad \quad u.signals[|i|], v.signals[|i|] \leftarrow \text{Max}(0, m - 1)
\]

Regardless of the initial configuration, the distribution of group values changes immediately (in $O(\log n)$ time) but it might take more time for the signals to get updated. It takes $O(i)$ time for signals[i] to hit zero. The larger the index $i$, signals[i] leaves the population slower. Hence, the agents look at the first missing signal that they observe among the array of all signals.

Once there is a large gap between the first missing group (FMV) and the agents’ estimation
Protocol 4 UpdateMV(agent u)

▷ Finds the first appearance of a zero in u.signals beyond index \([\log(u.estimate)]\)

\[
\begin{align*}
  s &\leftarrow \lceil \log(u.estimate) \rceil \\
  u.FMV &\leftarrow \min\{i \in [s,|u.signals|] \mid u.signals[i] = 0\}
\end{align*}
\]

of \(\log n\) (estimate), each agent individually moves to a waiting phase and waits for other agents to catch the same gap between their estimate and FMV. Note that, we time this phase as a function of FMV and not the estimate since the estimate is not valid anymore and might be much smaller or larger than the true value of \(\log n\).

Protocol 5 SizeChecker(u)

if \(u.\text{phase} = \text{NormalPhase}\) then

\[
\begin{align*}
  \text{if } u.FMV \not\in \{0, 2.5 \cdot u.\text{estimate}, ..., 2.5 \cdot u.\text{estimate}\} \text{ then} \\
  u.\text{phase} &\leftarrow \text{WaitingPhase} \\
  u.\text{timer} &\leftarrow 1
\end{align*}
\]

▷ Waiting for the other agents to detect the size change

Eventually all agents will notice the large discrepancy between FMV and estimate, and move to the WaitingPhase. The WaitingPhase is followed by the UpdatingPhase (explained in the TimerRoutine). In the UpdatingPhase all agents generate one geometric random variable (stored in GRV) and start propagating the maximum value. For the purpose of representation, we assume the agents generate a geometric random variable in one line (line 4 in Protocol 5).

Once the UpdatingPhase is finished all agents will update their estimate to the maximum geometric random variable they have seen and switch to the NormalPhase again.

Protocol 6 TimerRoutine(u)

\[
\begin{align*}
  u.\text{timer} &\leftarrow u.\text{timer} + 1 \\
  \text{if } u.\text{timer} > 12 \cdot u.FMV \text{ then} \\
  \text{if } u.\text{phase} = \text{WaitingPhase} \text{ then} \\
  u.GRV &\leftarrow \text{a new geometric random variable} \\
  u.\text{phase} &\leftarrow \text{UpdatingPhase} \\
  u.\text{timer} &\leftarrow 1 \\
  \text{if } u.\text{phase} = \text{UpdatingPhase} \text{ then} \\
  u.\text{estimate} &\leftarrow u.GRV \\
  u.\text{phase} &\leftarrow \text{NormalPhase} \\
  u.\text{timer} &\leftarrow 0
\end{align*}
\]

Recall that the agents remain in the NormalPhase as long as their FMV and estimate are fairly close. They continue changing their group values and send group signals as described earlier.

Intuitively, for each group value, about \(n/2^i\) agents will hold \(\text{group} = i\), and try to boost \(\text{signals}[i]\) by setting it to the max = \(\Theta(i)\). As the value of \(i\) grows, the number of agents with \(\text{group} = i\) decreases, but their signals get stronger since the agents enhance a group signal \(i\) proportional to \(i\). In a normal run of the protocol, the agents expect to have positive values in \(\text{signals}[i]\) for group values between \([\log \ln n, \log n]\).

Alternatively, the agents could generate a geometric random variable through \(O(\log n)\) consecutive interactions, each selecting a random coin flip (H or T). In this alternative version, we should make the WaitingPhase longer.
PropagateMaxEst(agent u, agent v)
if u.phase = v.phase & u.phase ≠ WaitingPhase then
u.GRV, v.GRV ← max(u.GRV, v.GRV)

4 Simulation results

In this section, we present our simulation results for Protocol 1. We present two separate simulation results, one starting with a uniformly random initial configuration (Figure 1) in which each agent starts with a random number (bounded by 60) in each of their group and `estimate` fields. Additionally, we set a random integer of $Θ(i)$ in each index of their `signals`. In Figure 1 we depict the group and `signals` fields of the agents in a population size $n = 10^6$. We can observe the changes in the group and `signals` within multiple snapshots from the population.

In our second simulation, we initialized the population with default values (starting in the initialized setting), however, after the population converges to an `estimate` of $O(\log n)$ we remove agents uniformly at random to simulate an adversarial initialized population. The result of this simulation is shown in Figure 2.

5 Analysis of DynamicCounting

5.1 Useful time bounds

Let $n$ and $A$ be the population size and the set of agents in the population respectively. Let $I(t, u)$ represent the number of interactions involving agent $u$ by the time $t$. Also, let $G_{u,t}$ represent the group value for agent $u$ at time $t$ calculated via the rules defined in Protocol 2.

Lemma 5.1. For any $d ≥ 3$ and $2d/3 ≥ c ≥ 1$ during $dn \ln n$ interaction, all agents have at least $2(d - \sqrt{dc}) \ln n$ and at most $2(d + \sqrt{3dc}/2) \ln n$ interactions with arbitrary large probability $1 - n^{1-c}$.

Proof. Let us consider a fixed agent $u$. Recall that we defined $I(t, u)$ to be the number of interactions involving agent $u$ in $t$ time. Note that $I(t, u)$ has a binomial distribution with parameters $B(dn \ln n, 2/n)$ with $\mu = 2d \ln n$ for $t = d \ln n$. We can get a tight bound on the expected value using a straightforward Chernoff bound:

For the lower bound:

\[ \Pr[I(t, u) ≤ (1 - \delta)\mu] ≤ e^{-\delta^2 \mu/2} \]  
(1)

\[ \Pr[I(t, u) ≤ (1 - \delta)2d \ln n] ≤ e^{-\delta^2 (2d \ln n)/2} \]  
(2)

\[ \Pr[I(t, u) ≤ (1 - \delta)2d \ln n] ≤ n^{-\delta^2 d} \]  
(3)

\[ \Pr[I(t, u) ≤ 2(d - \sqrt{dc}) \ln n] ≤ n^{-c} \]  
setting $\delta = \sqrt{c/d} ≤ 1$  
(4)

With a union bound we can show that $\Pr[I(t, u) ≤ 2(d - \sqrt{dc}) \ln n] ≤ n^{1-c}$ for all agents. and for the upper bound:

\[ \Pr[I(t, u) ≥ (1 + \delta)\mu] ≤ e^{-\delta^2 \mu/(2+\delta)} \]  
(5)

\[ \Pr[I(t, u) ≥ (1 + \delta)2d \ln n] ≤ e^{-\delta^2 (2d \ln n)/3} \]  
for $0 ≤ \delta ≤ 1$  
(6)

\[ \Pr[I(t, u) ≥ (1 + \delta)2d \ln n] ≤ n^{-2\delta^2 d/3} \]  
(7)

\[ \Pr[I(t, u) ≥ 2(d + \sqrt{3dc}/2) \ln n] ≤ n^{-c} \]  
setting $\delta = \sqrt{3c/2d} ≤ 1$  
(8)
The x-axis shows all the indices in the or UpdatingPhase with probability 1/3. Plots of the signals for a randomly initialized population. The x-axis shows all the indices in the signals of the agents (bounded by 40 in the simulation). On the y-axis, and every index i, we take the minimum pairwise value of u signals[i] for all u ∈ A.

Figure 1 Simulation results for population size = 10^6. Initializing each agent with a random 1 ≤ group < 30, and for every 1 ≤ i ≤ 60, 0 ≤ signals[i] ≤ 3 · i + 1 in NormalPhase, WaitingPhase, or UpdatingPhase with probability 1/3. Plots of the signals for a randomly initialized population. The x-axis shows all the indices in the signals of the agents (bounded by 40 in the simulation). On the y-axis, and every index i, we take the minimum pairwise value of u signals[i] for all u ∈ A.
(a) The distribution of estimate during the course of a computation. First, the agents agree on \(\text{estimate} = 20\) and keep their estimate throughout the computation. This simulation shows how the agents update their \(\text{estimate}\) once we removed \(O(n)\) agents from the original population.

(b) The distribution of phase values. This simulation shows how the agents update their \(\text{estimate}\) by going through WaitingPhase and UpdatingPhase consecutively.

**Figure 2** Simulation results for population size \(n = 400000 \approx 2^{18}\). Initializing each agent with group = 1, and an empty signals in NormalPhase. First, the agents calculate the \(\text{estimate}\) for \(n = 2^{18}\), then at time 1350 we remove all but 500 agents randomly which results in updating the agents’ \(\text{estimate}\) from 20 to 10.
A union bound shows that \( \Pr[I(t, u) \geq 2(d + \sqrt{3dc/2}) \ln n] \leq n^{1-c} \) for all agents.

**Corollary 5.2.** For any \( d \geq 3 \), during \( dn \ln n \) interaction, all agents have at least \( 0.2d \ln n \) and at most \( 4d \ln n \) interactions with probability \( 1 - n^{1-2d/3} \).

**Proof.** Setting \( c = 2d/3 \) in Lemma 5.1 results in the above bounds.

### 5.2 Bound on the group values

Recall that the agents calculate a dynamic group value by following the rules of Protocol 2. As described in this protocol, the agents move through different group values according to the Markov chain shown in Figure 3.

![Figure 3](image)

**Figure 3** The infinite chain of group values.

In this part, we analyze the distribution of group values. Note that at the very beginning of the protocol, the group values are rather chaotic since the agents might start holding any arbitrary group values that are much larger than \( \log n \). However, after all agents reset back to \( \text{group} = 1 \), we can show for each \( \text{group} = k \), \( \Pr[\text{group} = k] \approx \frac{1}{2^k} \).

In the rest of this section we assume the initialized setting for simplicity. Later on we show how we can generalize our results to any arbitrary initial configuration. Recall that \( G_{u,t} \) stands for the group value of agent \( u \) at time \( t \) and \( I(t, u) \) shows the number of interactions involving this agent by the time \( t \). Note that with this definition, \( G_{u,t} \) is equal to \( k \) (for \( k < I(t, u) \)) if and only if agent \( u \) generates the sequence of \([HTTT...T]\) (H followed by \( k-1 \) Ts) during its last \( k \) interactions. Thus, we have:

\[
\forall k \in \mathbb{N}, \quad 1 \leq k < I(t, u) : \Pr[G_{u,t} = k] = \frac{1}{2^k} \tag{9}
\]

With this definition \( G_{u,t} \) is undefined for any agents that has not generated \( H \) yet. In other words, the values \( G_{u,t} \) are “close to geometric” in the sense that they are independent and have probability equal to a geometric random variable on all values \( k < I(t, u) \).

**Observation 5.3.** For agents \( u_1, u_2, \ldots, u_n \), and the values \( k_i < I(t, u_i) \), for \( 1 \leq i \leq n \):

\[
\Pr[G_{u_1,t} = k_1, G_{u_2,t} = k_2, \ldots, G_{u_n,t} = k_n] = \prod_{i=1}^{n} \Pr[G_{u_i,t} = k_i]
\]

\[6\] The truncated chain mapping all states \( k+1, k+2, \ldots \) to \( k+1 \) is also known as the “winning streak.”
Next we bound the maximum group value that has been generated by any agent. Let $M_t = \max_{u \in A} G_{u,t}$ be the maximum value of $G_{u,t}$ across the population at time $t$.

\textbf{Lemma 5.4.} Let $c \geq 2$ and let $t$ be a time such that all agents have at least $c \log n$ interactions. In a population of size $n$, $\frac{1}{c} \log n \leq M_t$ with probability at least $1 - \exp\left(-n^{1-1/c}\right)$ and $M_t < c \log n$ with probability at least $1 - n^{1-c}$.

\textbf{Proof.} Recall that $G_{u,t}$ is the group value of agent $u$, $I(t,u)$ is the number of interactions that this agent had by the time $t$, and $M_t$ is the maximum value of $G_{u,t}$ for all $u \in A$ at time $t$. Since all agents have had at least $c \log n$ interactions, for all values of $k < c \log n$,\
\[
\Pr \left[ G_{u,t} = k \right] = \frac{1}{2^k}.
\]
To have $M_t \geq c \log n$, at least one agent must have generated a group value greater than or equal to $c \log n$:
\[
\Pr \left[ M_t \geq c \log n \right] = \Pr \left[ \exists u \in A \right] G_{u,t} \geq c \log n
\]
\[
\leq n \cdot \left( \frac{1}{2} \right)^{c \log n}
\]
by the union bound
\[
= n^{1-c}
\]
For the other direction, to have $M_t > \frac{1}{d} \log n$, at least one agent must have generated group $> \frac{1}{d} \log n$. By Observation 5.3
\[
\Pr \left[ M_t < \frac{1}{d} \log n \right] = \Pr \left[ \forall u \in A \right] G_{u,t} < \frac{1}{d} \log n
\]
\[
= \left( 1 - \frac{1}{2^{1/d \log n}} \right)^n
\]
\[
= \left( 1 - \frac{1}{n^{1/d}} \right)^n
\]
\[
= \left( 1 - \frac{n^{1-1/d}}{n} \right)^n
\]
\[
\leq \exp\left(-n^{1-1/d}\right).
\]
\textbf{Lemma 5.5.} Let $\delta > 0$, $0 < \epsilon < 1$ and let $t$ be a time such that all agents have at least $(1 + \delta) \log n$ interactions. Define $\text{FMV}_t = \min \{ k \in \mathbb{N} \mid (\forall u \in A) u.\text{group} \neq k \}$ at time $t$. Then, $\text{FMV}_t > (1 - \epsilon) \log n$ with probability at least $(1 - \epsilon) \log(\log(n)) \cdot \exp\left(-n^\epsilon\right)$ and $\text{FMV}_t \leq (1 + \delta) \log n$ with probability at least $1 - \left( \frac{1}{n^{\epsilon/2}} \right)^{(2+\delta) \log n}$.

5.3 \textbf{Bounds on the first missing group}

In this part, we analyze the bounds for the first group value that has no support, i.e., the value $\min \{ k \in \mathbb{N}^+ \mid (\forall u \in A) u.\text{group} \neq k \}$. Considering $n$ i.i.d. geometric random variables, the first missing value to be the smallest integer not appearing among the random variables.

The first missing value has been studied in the literature \cite{27,28,31} as the “the first empty urn” (see also “probabilistic counting” \cite{22}) but for simplicity we use a loose bound for our analysis.

\textbf{Lemma 5.5.} Let $\delta > 0$, $0 < \epsilon < 1$ and let $t$ be a time such that all agents have at least $(1 + \delta) \log n$ interactions. Define $\text{FMV}_t = \min \{ k \in \mathbb{N} \mid (\forall u \in A) u.\text{group} \neq k \}$ at time $t$. Then, $\text{FMV}_t > (1 - \epsilon) \log n$ with probability at least $(1 - \epsilon) \log(\log(n)) \cdot \exp\left(-n^\epsilon\right)$ and $\text{FMV}_t \leq (1 + \delta) \log n$ with probability at least $1 - \left( \frac{1}{n^{\epsilon/2}} \right)^{(2+\delta) \log n}$.
Proof. Recall that we use $I(u,t)$ to represent the number of interactions agent $u$ had by time $t$. Also, we use $G_{u,t}$ to show $u$'s group at time $t$.

For any group value $k$ that $k < I(u,t)$, we have $\Pr [G_{u,t} = k] = \frac{1}{2^k}$ (Equation (9)). We say group $i$ is missing if $i \neq G_{u,t}$ for all $u \in A$. Since the $G_{u,t}$'s are independent (across different agents $u$ for a fixed $t$), For any $1 \leq i \leq c \log n$, the probability that $i$ is missing is $(1 - \frac{1}{2^k})^n \approx \exp(-\frac{n^2}{i})$. We take a union bound on all $1 \leq i \leq \epsilon \log n$:

$$\Pr \left[ \text{FMV}_t \leq (1 - \epsilon) \log n \right] \leq \sum_{i=1}^{(1-\epsilon) \log n} \left(1 - \frac{1}{2^i}\right)^n \leq \sum_{i=1}^{(1-\epsilon) \log n} \exp \left(-\frac{n}{2^i}\right) \leq (1 - \epsilon) \log(n) \cdot \exp \left(-\frac{n}{2(1-\epsilon) \log n}\right) \text{ since } \left(1 + \frac{x}{n}\right)^n \leq \exp(x)$$

$$= (1 - \epsilon) \log(n) \cdot \exp(-n')$$

For the upper bound,

$$\Pr \left[ \text{FMV}_t > U \right] \leq \prod_{i=1}^{U} \Pr \left[ (\exists u \in A) G_{u,t} = i \right]$$

$$\leq \prod_{i=1}^{U} \left(n \cdot \left(\frac{1}{2^i}\right)\right) \text{ By the union bound}$$

$$= n^U \cdot \prod_{i=1}^{U} \frac{1}{2^i}$$

$$= n^U \cdot \frac{1}{2^{U(U+1)/2}}$$

$$= \left(\frac{n}{2^{U(U+1)/2}}\right)^U$$

Substituting $U = (2 + \delta) \log n$,

$$\Pr \left[ \text{FMV}_t > (2 + \delta) \log n \right] \leq \left(\frac{n}{2^{(1+(2+\delta) \log n)/2}}\right)^{(2+\delta) \log n}$$

$$= \left(\frac{n}{\sqrt{2} \cdot 2^{(1+\delta/2) \log n}}\right)^{(2+\delta) \log n}$$

$$= \left(\frac{n}{\sqrt{2} \cdot n^{1+\delta/2}}\right)^{(2+\delta) \log n}$$

$$= \left(\frac{1}{\sqrt{2} \cdot n^{\delta/2}}\right)^{(2+\delta) \log n}$$

$$< \left(\frac{1}{n^{\delta/2}}\right)^{(2+\delta) \log n}$$
Corollary 5.6. Let \( c \geq 2 \) and let \( t \) be a time such that all agents have at least \( c \log n \) interactions. Define \( \text{FMV}_t = \min \{ k \in \mathbb{N} \mid (\forall u \in \mathcal{A}) \, u.\text{group} \neq k \} \) at time \( t \). Then, \( \text{FMV}_t > 0.9 \log n \) with probability at least \( 1 - 0.9 \log n \cdot \exp \left(-n^{0.1}\right) \) and \( \text{FMV}_t \leq 3 \log n \) with probability at least \( 1 - n^{-3 \log n^2} \).

Proof. Set \( \epsilon = 0.1 \) and \( \delta = 1 \) in Lemma 5.5.

### 5.4 Distribution of the group values

![Figure 4](image-url)  
**Figure 4** Showing the distribution of group values after 300 parallel-time in a population of size \( n = 10^9 \). The x-axis indicates the different group values while the y-axis indicates the number of agents in each group. Note that, we are using log-scale for the y-axis. In this snapshot of the population, \( \text{FMV} = 29 \). Even though, the maximum group value is 35 and is much larger than \( \text{FMV} \).

So far we proved bounds on the existing group values. However, in general, we need to show that at a given time \( t = \Omega(\log n) \), there are about \( \frac{n}{k^2} \) agents having \( \text{group} = k \) WHP. The following lemma gives us a lower and upper bound for number of agents in each group:

Lemma 5.7. Let \( c \geq 2, \, 0 < \epsilon < 1, \, 0 \leq \delta \leq 1, \) and let \( t \) be a time such that all agents have at least \( c \log n \) interactions. Let \( 1 \leq k \leq (1 - \epsilon) \log n \), then, the number of agents who hold \( \text{group} = k \), is at least \( L_k = (1 - \delta) \frac{n}{2^k} \) with probability at least \( 1 - \exp \left(-\frac{\delta^2 n^2}{2}\right) \) and at most \( U_k = (1 + \delta) \frac{n}{2^k} \) with probability at least \( 1 - \exp \left(-\frac{\delta^2 n^2}{3}\right) \).

Proof sketch. The fraction of agents with \( \text{group} = k \) is equal to the fraction of heads in of a binomial distribution \( B(n, 2^{-k}) \) with \( \mu = \frac{n}{2^k} \), so the Chernoff bound applies.

Proof. Again, we consider group values that are less than \( I(t, u) \) for all \( u \in \mathcal{A} \). Recall that at time \( t \), agent \( u \) has \( \text{group} = k \) with probability \( 1/2^k \) for \( k < I(t, u) \). In Observation 5.3 we showed the event of having agents holding different group values are independent, thus, the fraction of agents holding value \( \text{group} = k \) is equal to the fraction of heads in of a binomial distribution \( B(n, 2^{-k}) \) with \( \mu = \frac{n}{2^k} \). Thus, we can use Chernoff bound for the tail bounds. For the upper bound and all \( 0 \leq \delta \leq 1 \) we have:
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\[
\Pr \left[ B(n, 2^{-k}) \geq (1 + \delta) \frac{n}{2^k} \right] \leq \exp \left( -\frac{\delta^2 \mu}{3} \right) \\
\leq \exp \left( -\frac{\delta^2 \cdot n}{3 \cdot 2^k} \right) \\
\leq \exp \left( -\frac{\delta^2 \cdot n}{3 \cdot 2^{(1 - \epsilon) \log n}} \right) \text{ since } k \leq (1 - \epsilon) \log n \\
\leq \exp \left( -\frac{\delta^2 \cdot n^c}{3} \right)
\]

Similarly for the lower bound and all \(0 \leq \delta \leq 1\) we can write:

\[
\Pr \left[ B(n, 2^{-k}) \leq (1 - \delta) \frac{n}{2^k} \right] \leq \exp \left( -\frac{\delta^2 \mu}{2} \right) \\
\leq \exp \left( -\frac{\delta^2 \cdot n}{2^{k+1}} \right) \\
\leq \exp \left( -\frac{\delta^2 \cdot n}{2^{(1 - \epsilon) \log n + 1}} \right) \text{ since } k \leq (1 - \epsilon) \log n \\
\leq \exp \left( -\frac{\delta^2 \cdot n^c}{2} \right).
\]

Setting \(\epsilon = 0.1\) and \(\delta = 1/4\) gives the following corollary.

**Corollary 5.8.** Let \(c \geq 2\), and let \(t\) be a time such that all agents have at least \(c \log n\) interactions. Let \(1 \leq k \leq 0.9 \log n\), then, the number of agent who hold \(\text{group} = k\), is at least \(L_k = \frac{3}{2^{k+1}} n\) and at most \(U_k = \frac{5}{2^{k+2}} n\) with probability at least \(1 - \exp \left( -\frac{n^{0.1}}{32} \right)\) and \(1 - \exp \left( -\frac{n^{0.1}}{32} \right)\) respectively.

Let us summarize what we know about the distribution of the \(\text{group}\) values and the number of agents holding each \(\text{group}\) value at time \(t\) in the following theorem.

**Theorem 5.9.** Fix a time \(t \geq d \ln n\) for \(d > 30\), let \(M^*_t\) and \(\text{FMV}_t\) be the maximum \(\text{group}\) value and the \(\text{FMV}\) at this time respectively. Then,

- \(0.9 \log n \leq M^*_t < 0.1d \log n\) with probability at least \(1 - 2 \cdot n^{1-d/10} - 2 \cdot n^{1-2d/3}\).
- \(0.9 \log n \leq \text{FMV}_t < 3 \log n\) with probability at least \(1 - 4 \cdot n^{1-2d/3}\).
- The number of agents who hold \(\text{group} = k\) for \(1 \leq k \leq 0.9 \log n\), is in \([\frac{3}{2^{k+1}} n, \frac{5}{2^{k+2}} n]\) with probability at least \(1 - 4 \cdot n^{1-2d/3}\).

**Proof.** By Corollary 5.2 at time \(t \geq d \ln n\) all agents have at least \(0.2d \ln n > 0.1d \log n\) interactions with probability \(1 - n^{1-2d/3}\). Conditioning on all agents having \(I_U = 0.1d \log n\) interactions, we can use Lemma 5.4 and Lemma 5.7 to prove that the maximum \(\text{group}\) value is at most \(U = 0.1d \log n\) using the law of total probability:

Let \(\mathcal{E}\) denote the event \((\forall a \in A) I(t, a) > I_U\), all agents have at least \(I_U\) interactions. Let \(\mathcal{D}\) denote an arbitrary event over the population. By the law of total probability we have:

\[
\Pr[\mathcal{D}] = \Pr[\mathcal{D} | \mathcal{E}] \cdot \Pr[\mathcal{E}] + \Pr[\mathcal{D} | \bar{\mathcal{E}}] \cdot \Pr[\bar{\mathcal{E}}] \\
\geq \Pr[\mathcal{D} | \mathcal{E}] \left(1 - n^{1-2d/3}\right) + 0 \cdot n^{1-2d/3}
\]

by Corollary 5.2.
Substitute $\mathcal{D}$ with event $M_t^* < U$. Thus, for the maximum group value at time $t$ we have:

$$\Pr[M_t^* < 0.1 \log n] \geq \left(1 - n^{1-0.1d}\right) \cdot \left(1 - n^{-\frac{2d}{3}}\right)$$

by Lemma 5.3

$$\geq 1 - n^{1-0.1d} - n^{1-\frac{2d}{3} + \frac{2d}{3}}$$

$$\geq 1 - 2 \cdot n^{1-0.1d}$$

since $n^{1-0.1d} > n^{1-\frac{2d}{3}}$

Similarly, for the other direction we get:

$$\Pr[M_t^* > 0.9 \log n] \geq \left(1 - \exp\left(-n^{0.1}\right)\right) \cdot \left(1 - n^{-\frac{2d}{3}}\right)$$

by Lemma 5.4

$$\geq 1 - \exp\left(-n^{0.1}\right) - n^{1-\frac{2d}{3}} + O(1/n)$$

for $d > 3$

$$\geq 1 - 2 \cdot n^{1-\frac{2d}{3}}$$

since $\exp\left(-n^{0.1}\right) < n^{1-\frac{2d}{3}}$

We follow a similar calculation to prove upper and lower bounds on the $FMV$ at time $t$:

$$\Pr[FMV_t > 0.9 \log n] \geq \left(1 - 0.9 \log n \cdot \exp\left(-n^{0.1}\right)\right) \cdot \left(1 - n^{-\frac{2d}{3}}\right)$$

by Corollary 5.6

$$\geq 1 - 0.9 \log n \cdot \exp\left(-n^{0.1}\right) - n^{1-\frac{2d}{3}} + O(1/n)$$

for $d > 4$

$$\geq 1 - 2 \cdot n^{1-\frac{2d}{3}}$$

and for the other direction:

$$\Pr[FMV_t < 3 \log n] \geq \left(1 - n^{-\frac{3 \log n}{2d}}\right) \cdot \left(1 - n^{1-\frac{2d}{3}}\right)$$

by Corollary 5.6

$$\geq 1 - n^{-\frac{3 \log n}{2d}} - n^{1-\frac{2d}{3}} + O(1/n)$$

for $d > 3$

$$\geq 1 - 2 \cdot n^{1-\frac{2d}{3}}$$

Finally, we prove the number of agents holding group $= k$ is between $[L_k^*, U_k^*]$. Let $I(G_k, t)$ denote the number of agents having group $= k$ at time $t$, then:

$$\Pr\left[I(G_k, t) < \frac{5 \cdot n}{2^{k+2}}\right] \geq \left(1 - \exp\left(n^{0.1}/48\right)\right) \cdot \left(1 - n^{-\frac{2d}{3}}\right)$$

by Corollary 5.8

$$\geq 1 - 2 \cdot n^{1-\frac{2d}{3}}$$

and

$$\Pr\left[I(G_k, t) > \frac{3 \cdot n}{2^{k+2}}\right] \geq \left(1 - \exp\left(n^{0.1}/32\right)\right) \cdot \left(1 - n^{-\frac{2d}{3}}\right)$$

by Corollary 5.8

$$\geq 1 - 2 \cdot n^{1-2d/3}.$$

Let us summarize Theorem 5.9 in the corollary.

**Corollary 5.10.** Let $t \geq 30 \ln n$. Then, for large values of $n$, with probability at least $1 - 4/n^{10}$, we have:

- The $FMV$ will remain in $[0.9 \log n, 3 \log n]$.
- The number of agents who hold group $= k$ for $1 \leq k \leq 0.9 \log n$, will remain in $\left[\frac{3n}{2^{k+2}}, \frac{5n}{2^{k+2}}\right]$.

**Proof.** Let $d = 30$ in Theorem 5.9.
5.5 Group detection

In the previous section we show that the set of present group values among the population will quickly (in $O(\log n)$ time) enter a small interval of values $([1, 8 \cdot \log n])$ consistent with the population size. In this section, we will prove the following:

- The agents agree about the presence of group values that are in $[\log \ln n, 0.9 \log n]$ after $O(\log n)$ time WHP.
- For a non-existing group value $q$, each agent will have signals[$q$] = 0 in $O(q + \log n)$ time WHP.

We designed Protocol 3 such that each agent in the $i$'th group boosts the associated signal value by setting signals[$i$] = $B_i$ (recall $B_i = \Theta(i)$). We will show by having at least $L_i$ agents boosting signals[$i$], the whole population learns about the existence of the $i$'th group in $O(\log n)$ time with high probability. Intuitively, although signals[$i$] starts lower than signals[$j$] for $i < j$, so potentially dies out more quickly, it is also boosted more often since more agents have group value $i$. Concretely, with $L_i$ agents responsible to boost signal $i$, and for all indices $\log n < i < 0.9 \log n$ in the signals of the agents, $\Pr[\text{signals}[i] = 0] < \exp\left(-\frac{2^n}{n^{\log \log n}}\right)$.

Intuitively, the next lemma shows that if the group values are distributed as in Lemma 5.7 then the whole population will learn about all the present group values above $\log \ln n$ within $O(\log n)$ time. Note that $\Pr[u.\text{signals}[i] = j]$ is the probability that the agent $u$ has value $j$ in the $i$'th index of its signals. The following lemma is a restatement from [5, Section 5.1].

**Lemma 5.11.** In the execution of Protocol 3 suppose that for each group value $\log \ln n < i < 0.9 \log n$, at least $A_i$ agents hold group $i$. For every agent $u \in A$ let $u.\text{signals}[i] = r_i$ when $u.\text{group} = i$. Assuming each agent has at least $r_i$ interactions, then for a fixed agent $u$ and index $i$, $\Pr[u.\text{signals}[i] = 0] \leq \left(1 - \frac{A_i}{n}\right)^{2^{r_i-1}}$.

**Proof.** We are analyzing each signals[$i$] field independently since the agents update each index with $a, b \to \max(a-1, b-1, 0)$. Note that for the analysis of this protocol, we don’t need to know the exact counts of the agents who have value $j$ for $0 < j \leq r_i$ for each signal $i$; instead, we show that the agents’ values remains above 0 for all indices $\log \ln n < i < 0.9 \log n$ using a backward induction.

Fix agents $u_1$, $u_2$, and $i$, let $s_1 = u_1.\text{signals}[i]$ and $s_2 = u_2.\text{signals}[i]$ be the values associated to index $i$ of their signals before their interaction. Let $s'_i = u_1.\text{signals}[i]$ and $s'_2 = u_2.\text{signals}[i]$ represent the same field after their interaction. Note that $s'_i \geq j$ if and only if max($s_1, s_2) > j$. To show the base case of our induction, consider the interaction between agent $u_1$ and agent $u_2$, assuming both had at least 1 interaction:

$$\Pr[s'_i < r_i - 1] = \Pr[s_1 < r_i] \cdot \Pr[s_2 < r_i] \leq \left(1 - \frac{A_i}{n}\right)^2$$

Assuming all agents had at least $r_i - j$ interactions we can calculate $\Pr[s'_i < j]$ recursively:

$$\Pr[s'_1 < j] = \Pr[s_1 < j + 1] \cdot \Pr[s_2 < j + 1] \leq \left(1 - \frac{A_i}{n}\right)^{2^{r_i-j}}.$$
Specifically, for $j = 1$, conditioning on all agents having at least $r_i$ interactions, we have:

$$
\Pr [s'_1 < 1] = \Pr [s_1 < 2] \cdot \Pr [s_2 < 2] 
\leq \left(1 - \frac{A_i}{n}\right)^{2^{n_i - 1}}.
$$

To use the previous lemma, we need to make sure that the agents wait for sufficiently long time such that each agent has at least $r_i$ interactions. The next corollary uses Lemma 5.11 to derive bounds for the entire protocol using bounds from Lemma 5.7 for the distribution of the group values. Also, Lemma 5.11 takes a union bound over all agents and group values $i$, and uses the concrete value $r_i = B_i = 3 \cdot i + 1$ used in our protocol.

**Corollary 5.12.** For all $i > 0$ and for every agent $u \in A$, assuming $B_i = 3 \cdot i + 1$ let $u.\text{signals}[i] = B_i$ if $u.\text{group} = i$. Suppose that for each group value $\log n < i < 0.9 \log n$, at least $L_i$ agents hold $\text{group} = i$. Let $\beta \geq 8$; then after $\beta \log n$ time, we have:

$$
\Pr \left[ (\exists u \in A)(\exists i \in \{\log n, \ldots, 0.9 \log n\}) \ u.\text{signals}[i] = 0 \right] \leq 2 \cdot n^{1-0.9\beta}
$$

**Proof.** Let us assume at least $L_i$ agents have $\text{group} = i$ (recall definition of $L_i$ in Lemma 5.7). Suppose all agents having $B_i$ interactions, by Lemma 5.11 for a fixed agent $u$ and for all values of $\log n < i < 1/2 \log n$, we have $\Pr [u.\text{signals}[i] = 0] \leq (1 - \frac{2}{n})^{2^{n_i - 1}}$. By Lemma 5.7 we know $L_i \geq \frac{3/4 \cdot n}{2^i}$ with probability at least $1 - 2 \cdot \exp \left(\frac{n^{0.1}}{n^{0.3}}\right)$, thus we can bound $\Pr [u.\text{signals}[i] = 0]$ as follows:

$$
\begin{align*}
\Pr [u.\text{signals}[i] = 0] &\leq \left(1 - \frac{3/4 \cdot 2^i}{2^i}\right)^{2^{n_i - 1}} \\
&\leq \left(1 - \frac{3/4}{2}\right)^{2^{n_i - 1}} \\
&\leq \exp \left(\frac{-3/4 \cdot 2^i}{2}\right) \\
&\leq \exp \left(\frac{-2^{\log n - n - 1}}{2}\right) \quad \text{since } i > \log n \\
&\leq \exp \left(\frac{-2^{\log n - n - 1}}{2}\right) \\
&= \exp \left(\frac{-2^{\log n - n - 1}}{2}\right) \\
&= n^{-\frac{\log n}{2}} \\
&= n^{-\frac{\log n}{2}}.
\end{align*}
$$

By the union bound over all agents $u \in A$ and all values of $i$, we have:

$$
\Pr \left[ (\exists u \in A)(\exists i \in \{\log n, \ldots, 0.9 \log n\}) \ u.\text{signals}[i] = 0 \right] \leq n^{2-\frac{\log n}{2}}.
$$

Now, we set $\beta$ such that each agent has at least $B_i = 3 \cdot i + 1$ interactions for all values of $i$ in $[\log n, 0.9\log n]$. Let $\beta \geq 8$, by Corollary 5.2 after $\beta \log n$ time (equivalent to $\frac{\beta}{n^{0.2}} \log n$) all agents have at least $0.3 \beta \log n$ interactions with probability at least $1 - n^{1-0.9\beta}$.

By the union bound, $\Pr [u.\text{signals}[i] = 0] \leq n^{1-0.9\beta} + n^{2-\frac{\log n}{2}} \leq 2 \cdot n^{1-0.9\beta}$ for sufficiently large $n$. 

In the following lemma, we show that when there is no agent holding $\text{group} = i$, then $\text{signals}[i]$ will become zero in all agents “quickly” with arbitrary large probability. To be
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precise, with no agent boosting signal \( i \), \( \Pr [ u.\text{signals}[i] = 0 ] \geq 1 - n^{-\alpha} \) within \( \Theta(B_i + \alpha \ln n) \) time WHP in which \( B_i \) is the maximum value for signal \( i \). The lemma is a restatement from [17] Lemma 3.3] and [3 Lemma 1].

Lemma 5.13. For every agent \( u \in \mathcal{A} \) let \( u.\text{signals}[i] = B_i \) when \( u.\text{group} = i \). Assume that no agent sets its group to \( i \) from this point on. Then for all \( \alpha \geq 1 \), all agents will have \( \text{signals}[i] = 0 \) after \( 3n \ln(n^{\alpha} \cdot 3^{B_i}) \) interactions with probability at least \( 1 - n^{-\alpha} \).

Proof. Set \( t = 3n \ln(n^{\alpha} \cdot 3^{B_i}) \) and \( R_{\text{max}} = B_i \) in the proof of [17 Lemma 3.3]. ◀

5.6 Dynamic size counting protocol analysis

Let \text{estimate} denote the estimate of \( \log n \) in agents’ memory. Let \( n \) be the true population size. In the previous section we show that the set of present \text{group} values among the population will quickly (in \( O(\log n) \) time) enter a small interval of consecutive values \([1, 3 \cdot \log n]\) consistent with the population size. In this section, we will show that the \text{group} values will remain in that interval (with high probability for polynomial time Theorem 5.18). Moreover, the next two lemmas, show how the agents update their \text{estimate} in case it far from \( \log n \).

Assuming the agents’ \text{estimate} is much smaller than \( \log n \), the next lemma shows that all the agents will notice the large gap between \text{estimate} and FMV. Hence, they will re-calculate their population size estimate.

Lemma 5.14. Let \( M = \max_{u \in \mathcal{A}} u.\text{estimate} \). Assuming \( M \leq 0.22 \log n \), then the whole population will enter WaitingPhase in \( O(\log n) \) time with probability at least \( 1 - O(n^{-2}) \).

Proof. 1. The whole population learns (i.e., \( u.\text{estimate} = M \) for all \( u \in \mathcal{A} \)) about \( M \) by epidemic in less than \( 3 \ln n \) time with probability at least \( 1 - n^{-2} \) [17 Corollary 2.8].

2. By Corollary 5.12 for all \text{group} values in \([\log \ln n, 0.9 \log n]\), the agents will have a non-zero value in their \text{signals} after \( 10 \log n \) time with probability at least \( 1 - 2 \cdot n^{-8} \) (setting \( \beta = 10 \)). Thus, all agents have their FMV > 0.9 log n by this time.

3. Since \( M \leq 0.22 \log n \) by hypothesis, and \( \log n < \text{FMV}/0.9 \) with probability \( \geq 1 - n^{-\frac{3 \log n}{2}} \) by Corollary 5.6, we have \( M \leq 0.22 \log n < 0.22 \cdot \text{FMV}/0.9 < 0.25 \cdot \text{FMV} \).

The lemma holds by the union bound over all three events. ◀

For the other direction, assume the population size estimate in agents’ memory is much larger than \( \log n \). We prove in the the following lemma that all the agents will notice the large gap between \text{estimate} and FMV. Hence, they will re-calculate their population size estimation.

Note that in the Corollary 5.12 we proved for all \text{group} values \( i \) for \( \log \ln n \geq i \), the \text{signals}[i] will have a positive value in \( O(\log n) \) time. However, we could not prove the same bound for values that are less than \( \log \ln n \). So, inevitably the agents ignore their \text{signals} for values that are less than \( \log \ln n \). Since the agents have no access to the value of \( \log n \), they have to use \text{estimate} as an approximation of \( \log n \). Thus, they ignore indices that are less than \( \log M \) in \text{signals}. Making the FMV a function of \( \max(\log M, \log n) \). For example, let us the true population size is \( n \) but \( M > 2^n \), then, the agents should ignore appearance of a zero in their \text{signals} for all indices \( i \) that are \( \leq \log(M) = n \). The correct FMV happens at index \( j = \Theta(\log n) \) but the agents stay in the NormalPhase as long as \text{signals}[i] for \( i \geq n \) are positive. Since for each \text{signals}[i] it takes \( O(i) \) time to hit zero, it takes \( O(n) \) time for the agents to switch to WaitingPhase.

Note that with our current detection scheme, for indices \( i \) that are less than \( \log \ln n \), the event of \text{signals}[i] = 0 happens frequently.
Lemma 5.15. Let \( M = \max_{u \in A} u.\text{estimate} \). Assuming \( M \geq 7.5 \cdot \log n \), then the whole population will enter WaitingPhase in \( O(\log n + \log M) \) time with probability at least \( 1 - O(n^{-2}) \).

Proof. 1. The whole population learn about \( M \) by epidemic in less than \( 3 \log n \) time with probability at least \( 1 - n^{-2} \) [17 Corollary 2.8].

2. By Corollary 5.12 for all group values in \([\log \ln n, 0.9 \log n]\), the agents will have a non-zero value in their signals after \( 10 \log n \) time with probability at least \( 1 - 2 \cdot n^{-8} \) (setting \( \beta = 10 \)). Moreover, any signal associated to a non-existing group value \( q \), will be gone in \( O(\alpha \ln n + q) \) time with probability \( 1 - n^{-\alpha} \) (Lemma 5.13).

3. Additionally, by Corollary 5.6 the first missing group value FMV is less than \( 3 \log n \) with probability at least \( 1 - n^{-3} \). However, since Corollary 5.12 only works for group values greater than \( \log \ln n \), an agent \( u \) ignores any \( u.\text{signals}[i] = 0 \) for \( i < \log(u.\text{estimate}) \leq \log M \).

4. Since \( M \geq 7.5 \log n \) by hypothesis, and \( \log n > \text{FMV}/3 \) with probability \( \geq 1 - n^{-\frac{\text{FMV}}{12}} \) by Corollary 5.6 we have \( M \geq 7.5 \log n > 7.5 \cdot \text{FMV}/3 > 2.5 \cdot \text{FMV} \).

In the next theorem, we will show once there is large gap between the maximum estimate among the population and the true value of \( \log n \), the agents update their estimate in \( O(\log n + \log M) \) time.

Theorem 5.16. Let \( M = \max_{u \in A} u.\text{estimate} \). Assuming \( \text{estimate} \geq 7.5 \log n \) or \( \text{estimate} \leq 0.2 \log n \), then every agent replaces its estimate with a new value that is in \([\log n - \log \ln n, 2 \log n]\) with probability \( 1 - O(1/n) \) in \( O(\log n + \log M) \) time. 

Proof sketch. By Lemmas 5.14 and 5.15 once an agent notices the large gap between estimate and FMV they switch to WaitingPhase. We set WaitingPhase long enough so when the first agent moves to UpdatingPhase, there is no agent left in the NormalPhase. Thus, they all re-generate a new geometric random variable and store the maximum as their estimate.

Proof. Recall that by Lemmas 5.14 and 5.15 once an agent notices the large gap between estimate and FMV they switch to WaitingPhase (i.e., if \( \text{estimate} > 2.5 \text{FMV} \) or \( \text{estimate} < \text{FMV}/4 \)).

By Corollary 5.12 \( 10 \log n \) time is long enough so that all agents have \( u.\text{signals}[i] > 0 \) for \( \log \ln n < i < 0.9 \log n \) with probability \( 1 - n^{-8} \) (setting \( \beta = 10 \)).

By Lemma 5.1 during \( 10 \log n \leq 15 \ln n \) time each agent has at most \( 60 \ln n \leq 44 \log n \) interactions with probability \( \geq 1 - n^{-3} \) (setting \( d = 15 \) and \( c = 10 \)).

Thus, each agent should count up to \( 28 \log n \) during WaitingPhase to ensure that to ensure that with high probability, all other agents have also entered WaitingPhase before switching to UpdatingPhase. By Protocol 6 once the WaitingPhase is finished, an agent switches to UpdatingPhase.

We set \( t_w = 60 \cdot \text{FMV} = O(\log n + \log M) \) be the number of interactions each agent spends in phase WaitingPhase in Protocol 6. So, after \( 10 \log n \) time, every agent has \( \text{FMV} > 0.9 \log n \) (see Protocol 6). Eventually, all agents switch from WaitingPhase to UpdatingPhase to re-generate a new geometric random variable for estimate. By the end of UpdatingPhase all agents have generated a new geometric random variable and propagate the maximum by epidemic in at most \( 3 \ln n \) time with probability at least \( 1 - \frac{1}{n} \) [17 Corollary 2.8]. By [18 Lemma D.7], the maximum of \( n \) i.i.d. geometric random variables is in \([\log n - \log \ln n, 2 \log n]\) with probability at least \( 1 - \frac{1}{n} \).
Finally, in the following lemma, we show that the holding time of our protocol is polynomial.

**Lemma 5.17.** Consider the population after $45 \ln n$ time. Let $0.75 \log n \leq \text{estimate} \leq 2.25 \log n$ for all agents. Then, an agent will remain in the NormalPhase with probability at least $1 - 3 \cdot n^{-17}$.

**Proof.** An agent will change its phase from NormalPhase to WaitingPhase if $\text{estimate} < \text{FMV}/4 < 0.75 \log n$ or $\text{estimate} > 2.5 \text{FMV} > 2.25 \log n$ which will not happen as long as $\text{FMV} \in [0.9 \log n, 3 \log n]$ with $0.75 \log n \leq \text{estimate} \leq 2.25 \log n$. By Corollary 5.10 $\text{FMV} \in [0.9 \log n, 3 \log n]$ with probability at least $1 - n^{-\frac{2.5 \log n}{\log n}}$.

However, there are two scenarios in which $u \text{. signals}[i] = 0$ for some agent $u$ and $\log n < i < 0.9 \log n$:

1) There are less than $\frac{3}{16} n$ agents setting their group equal to $i$; which happens with probability less than $4 \cdot n^{-20}$ by Corollary 5.10.

2) The robust signaling protocol has failed to propagate signal $i$ which happens with arbitrary small probability $3 \cdot n^{-17}$ after $20 \log n(14 \ln n)$ time (setting $\beta = 20$ in Corollary 5.12).

By the union bound over these two cases, the agents will remain in the NormalPhase with probability at least $1 - 3 \cdot n^{-17}$.

**Theorem 5.18.** Consider the population after $45 \ln n$ time. Let $0.75 \log n \leq \text{estimate} \leq 2.25 \log n$ for all agents. Then, the agents will remain in the NormalPhase with probability at least $1 - O(n^{-1})$ for $O(n^{15})$ time.

**Proof.** By Lemma 5.17, the agents stay in NormalPhase with probability at least $1 - 3 \cdot n^{-17}$. By the union bound over all $n^{16}$ interactions in $n^{15}$ time, an agent might leave the NormalPhase with probability at least $1 - O(1)/n$ during this time.

In the next lemma we calculate the space complexity of our protocol. Note that, the adversary can initialize agents with large integer values to increase the memory usage arbitrarily.

**Lemma 5.19.** Assuming in the initial configuration of Protocol 1 for every agent $u \in A$, we have $\max(u \text{. estimate, u.GRV, u.group, u.signals.size}) < s$. then Protocol 1 uses $O(\log^2 s + \log n \log \log n)$ bits with probability at least $1 - O(1)/n$.

**Proof.** The set of values that an agent stores are given through the different fields that are use by Protocol 1:

- $\text{estimate} \in [0.2 \log n, 7.5 \log n] = O(\log n)$ bits by Theorem 5.16
- $\text{GRV} \in [\log n - \log \ln n, 2 \log n] = O(\log n)$ bits by Lemma 5.4
- $\text{group} = O(\log \log n)$ bits by Lemma 5.16
- $\text{timer} = O(\log \log n)$ bits by Theorem 5.16
- $\text{signals} = O(\log n \cdot \log \log n)$ bits by Lemma 5.4 and Corollary 5.12

**Theorem 5.20.** Let $M = \max u \text{. estimate}$ for all $u \in A$. There is a uniform leaderless loosely-stabilizing population protocol that WHP:
1. If $M > 7.5 \log n$ or $M < 0.2 \log n$ reaches to a configuration with all agents set their \textit{estimate} with a value in $[\log n - \log \ln n, 2 \log n]$ in $O(\log n + \log M)$ parallel time.

2. If $0.75 \log n < M < 2.25 \log n$, then the agents hold a stable \textit{estimate} during the following $O(n^{15})$ parallel time.

3. Assuming for every agent $u \in A$, $\max(u.\text{estimate}, u.\text{GRV}, u.\text{group}, u.\text{signals}.\text{size}()) < s$ in the initial configuration, then the protocol uses $O(\log^2(s) + \log n \log \log n)$ bits per agent.

### 5.7 Space optimization

In this section we explain how to reduce the space complexity of the protocol from $O(\log^2(s) + \log n \log \log n)$ to $O(\log^2(s) + (\log \log n)^2)$ bits per agent.

In Protocol 1 the agents keep track of all the present \textit{group} values using an array of size $O(\log n)$ (stored in \textit{signals}) by mapping every \textit{group} $i$ to \textit{signals}[i]. We can reduce the space complexity of the protocol by reducing the \textit{signals}’s size. Let the agents map a \textit{group} $= i$ to \textit{signals}[[log $i$]]. So, instead of monitoring all $O(\log n)$ group values, they keep $O(\log \log n)$ indices in their \textit{signals}. Thus, reducing the space complexity to $O(\log^2(s) + (\log \log n)^2)$ bits per agent.

Recall that in Protocol 1 there are $\approx \frac{n}{s}$ agents with \textit{group} $= i$ for $i \leq 0.9 \log n$ that help keeping \textit{signals}[i] positive. However, with this technique, there will be $\approx \sum_{i=2}^{2^{i+1}} \frac{n}{s}$ agents that are helping \textit{signals}[i] to stay positive. So, every lemmas in Section 5.5 about Protocol 3 hold.

Finally, we update Protocol 5 so that the agents compare their \textit{estimate} with $2^{\text{LFMV}}$ in which LFMV is the smallest index $i > \log \log M$ such that \textit{signals}[i] = 0.

On the negative side of this optimization, we get a protocol that is less sensitive about the gap between agents’ \textit{estimate} and $\log n$.

\textbf{Theorem 5.21.} Let $M = \max u.\text{estimate}$ for all $u \in A$. There is a uniform leaderless loosely-stabilizing population protocol that \textit{WHP}:

1. If $M > 15 \log n$ or $M < 0.1 \log n$ reaches to a configuration with all agents set their \textit{estimate} with a value in $[\log n - \log \ln n, 2 \log n]$ in $O(\log n + \log M)$ parallel time.

2. If $0.75 \log n < M < 2.17 \log n$, then the agents hold a stable \textit{estimate} during the following $O(n^{15})$ parallel time.

3. Assuming for every agent $u \in A$, $\max(u.\text{estimate}, u.\text{GRV}, u.\text{group}, u.\text{signals}.\text{size}()) < s$ in the initial configuration, then the protocol uses $O(\log^2(s) + (\log \log n)^2)$ bits per agent.

\textbf{Proof.} Let LFMV = $\lfloor \log(\text{FMV}) \rfloor$, by Corollary 5.6 $0.9 \log n < \text{FMV} \leq 3 \log n$ \textit{WHP}. Equivalently, we can derive bounds for LFMV:

$$\log(0.9 \log n) < \log(\text{FMV}) \leq \log(3 \log n)$$

$$\log \log n - 0.2 \leq \log(\text{FMV}) \leq \log \log n + \log 3$$

$$\log \log n - 1.2 \leq \lfloor \log(\text{FMV}) \rfloor \leq \log \log n + \log 3$$

As the agents compare their \textit{estimate} with $2^{\text{LFMV}}$, we need to calculate the bounds for $2^{\text{LFMV}}$.

$$\log \log n - 1.2 \leq \lfloor \log(\text{FMV}) \rfloor \leq \log \log n + \log 3$$

$$\frac{\log n}{2.3} \leq 2^{\lfloor \log(\text{FMV}) \rfloor} = 2^{\text{LFMV}} \leq 3 \log n$$
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which updates the bounds in Lemmas 5.14 and 5.15 and Theorem 5.16 with the following:

Having $M < 0.1 \log n \leq 0.1 \cdot 2.3 \cdot 2^{\text{LPFW}} \leq 0.23 \cdot 2^{\text{LPFW}}$. On the other side, having $M > 15 \log n > 15 \cdot 2^{\text{LPFW}}/3 \geq 5 \cdot 2^{\text{LPFW}}$.

Finally, our optimization updates the bounds in and Theorem 5.18 as follows: an agent will change its phase from NormalPhase to WaitingPhase if estimate $< 2^{\text{LPFW}}/4 < 0.75 \log n$ or estimate $> 5 \cdot 2^{\text{LPFW}} > 2.17 \log n$ which will not happen as long as $2^{\text{LPFW}} \in [\log n/(2.3), 3 \log n]$ with $0.75 \log n \leq \text{estimate} \leq 2.17 \log n$.

For the space complexity, note that bounds on signals in Lemma 5.19 changes to $O(\log \log n \cdot \log \log n)$ since there are about $O(\log \log n)$ indices with each index $i$ having a value that is $\Theta(i)$.

6 Conclusion and open problems

In this paper, we introduced the dynamic size counting problem. Assuming an adversary who can add or remove agents, the agents must update their estimate according to the changes in the population size. There are a number of open questions related to this problem.

Reducing convergence time. Our protocol’s convergence time depends on both the previous ($n_{\text{prev}}$) and next ($n_{\text{next}}$) population sizes, though exponentially less on the former: $O(\log n_{\text{next}} + \log \log n_{\text{prev}})$. Is there a protocol with optimal convergence time $O(\log n_{\text{next}})$?

Increasing holding time. Observation 3.3 states only that the holding time must be finite, but it is likely that much longer holding times than $\Omega(n^c)$ for constant $c$ are achievable. For the loosely-stabilizing leader election problem, there is a provable tradeoff in the sense that the holding time is at most exponential in the convergence time $2^{23}$. Does a similar tradeoff hold for the dynamic size counting problem?

Reducing space. Our main protocol uses $O(s + (\log n)^{\log n})$ states (equivalent to $O(\log^2(s) + \log n \log \log n)$) bits. In Section 5.7 we showed how we can reduce the state complexity of our protocol to $O(n^c)$ (equivalent to $O(\log^2(s) + (\log \log n)^2))$ bits by mapping more than one group to each index of the signals which reduces the size of the signals from $O(\log n)$ to $O(\log \log n)$. Another interesting trick is to replace our $O(\log n)$ detection scheme to $O(1)$ detection protocol of [21] which puts a constant threshold on the values stored in each index. So, it may be possible to reduce the space complexity even more to $O(2^{\log(\log n)})$ (with all $O(\log n)$ indices present) or $O(c^{\log(\log n)})$ = polylog(n) (using our optimization technique to have $O(\log \log n)$ indices in the signals).

However, the current protocol of [21] has a one-sided error that makes it hard to compose with our protocol. With probability $\epsilon > 0$, the agents might say signal $i$ has disappeared even though there exists agents with $\text{group} = i$ in the population.

In the presence of a uniform self-stabilizing synchronization scheme, one could think of consecutive rounds of independent size computation. The agents update their output if the new computed population size drastically differs from the previously computed population size. Note that, the self-stabilizing clock must be independent of the population size since we are allowing the adversary to change the value of $\log n$ by adding or removing agents. To our knowledge, there is no such synchronization scheme available to population protocols.

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