ARBOREALIZATION I: STABILITY OF ARBOREAL MODELS

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Abstract. We establish a stability result for the normal forms of arboreal singularities. As our main application of the stability result we give a geometric characterization of arboreal singularities as the closure of the class of smooth germs of Lagrangian submanifolds under the operation of taking iterated transverse Liouville cones. The parametric version of the stability result implies that the space of germs of symplectomorphisms which preserve an arboreal singularity is weakly homotopy equivalent to the space of automorphisms of the corresponding signed rooted tree. Hence the local symplectic topology of an arboreal singularity reduces to combinatorics, even parametrically.

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1. Introduction

1.1. Main results. This is the first in a series of papers by the authors on the arborealization program. The goal of this program is to determine the extent to which a Weinstein manifold can be deformed so that its skeleton becomes a stratified Lagrangian with arboreal singularities, and to establish the uniqueness properties of such an arboreal skeleton. Beyond this, the implications of the existence and uniqueness of an arboreal skeleton for the study of Weinstein manifolds will also be pursued in future work of the authors.

Arboreal singularities were first defined by the third author in the paper [N13], by means of explicit local models. For the purposes of the arborealization program, it is advantageous to have a more geometric characterization of arboreal singularities. This characterization is tailored to the strategy of proof employed by the authors to arborealize the skeleton of a Weinstein manifold in [AGEN20b]. The main result of this paper is to establish such a characterization, which we formulate as follows.

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First, we introduce some auxiliary notions. A closed subset of a symplectic or contact manifold is called *isotropic* if it is stratified by isotropic submanifolds. It is called Lagrangian or Legendrian if it is isotropic and purely of the maximal possible dimension. The germ at the origin of a locally simply-connected isotropic subset $L \subset T^*\mathbb{R}^n$ of the cotangent bundle with its standard Liouville structure $\lambda = pdq$ admits a unique lift to an isotropic germ at the origin $\hat{L} \subset J^1\mathbb{R}^n = T^*\mathbb{R}^n \times \mathbb{R}$ of the 1-jet bundle. Given an isotropic subset $\Lambda \subset S^*\mathbb{R}^n$ of the cosphere bundle, its Liouville cone $C(\Lambda) \subset T^*\mathbb{R}$, i.e. the closure of its saturation by trajectories of the Liouville vector field $Z = p\frac{\partial}{\partial p}$, is an isotropic subset.

**Definition 1.1.** Arboreal Lagrangian (resp. Legendrian) singularities form the smallest class $\text{Arb}_n^{\text{symp}}$ (resp. $\text{Arb}_n^{\text{cont}}$) of germs of closed isotropic subsets in $2n$-dimensional symplectic (resp. $(2n+1)$-dimensional contact) manifolds such that the following properties are satisfied:

(i) (Invariance) $\text{Arb}_n^{\text{symp}}$ is invariant with respect to symplectomorphisms and $\text{Arb}_n^{\text{cont}}$ is invariant with respect to contactomorphisms.

(ii) (Base case) $\text{Arb}_0^{\text{symp}}$ contains $pt = \mathbb{R}^0 \subset T^*\mathbb{R}^0 = pt$.

(iii) (Stabilizations) If $L \subset (X, \omega)$ is in $\text{Arb}_n^{\text{symp}}$, then the product $L \times \mathbb{R} \subset (X \times T^*\mathbb{R}, \omega + dp \wedge dq)$ is in $\text{Arb}_{n+1}^{\text{symp}}$.

(iv) (Legendrian lifts) If $L \subset T^*\mathbb{R}^n$ is in $\text{Arb}_n^{\text{symp}}$, then its Legendrian lift $\hat{L} \subset J^1\mathbb{R}^n$ is in $\text{Arb}_n^{\text{cont}}$.

(v) (Liouville cones) Let $\Lambda_1, \ldots, \Lambda_k \subset S^*\mathbb{R}^n$ be a finite disjoint union of arboreal Legendrian germs from $\text{Arb}_0^{\text{cont}}$ centered at points $z_1, \ldots, z_k \in S^*\mathbb{R}^n$. Let $\pi : S^*\mathbb{R}^n \to \mathbb{R}^n$ be the front projection. Suppose

- $\pi(z_1) = \cdots = \pi(z_k)$.
- For any $i$, and smooth submanifold $Y \subset \Lambda_i$, the restriction $\pi|_Y : Y \to \mathbb{R}^n$ is an embedding (or equivalently, an immersion, since we only consider germs).
- For any distinct $i_1, \ldots, i_t$, and any smooth submanifolds $Y_{i_1} \subset \Lambda_{i_1}, \ldots, Y_{i_t} \subset \Lambda_{i_t}$, the restriction $\pi|_{Y_{i_1} \cup \cdots \cup Y_{i_t}} : Y_{i_1} \cup \cdots \cup Y_{i_t} \to \mathbb{R}^n$ is self-transverse.

Then the union $\mathbb{R}^n \cup C(\Lambda_1) \cup \cdots \cup C(\Lambda_k)$ of the Liouville cones with the zero-section form an arboreal Lagrangian germ from $\text{Arb}_n^{\text{symp}}$.

With the above classes defined, we can also allow boundary by additionally taking the product $L \times \mathbb{R}_{\geq 0} \subset (X \times T^*\mathbb{R}, \omega + dp \wedge dq)$ for any arboreal Lagrangian $L \subset (X, \omega)$, and similarly for arboreal Legendrians.

In Section 3 we prove the Stability Theorem 3.5 for arboreal models. Rather than give the precise formulation here, we state our main application: for fixed dimension $n$ the class of Lagrangian (resp. Legendrian) arboreal singularities as defined above contains only finitely many local models up to ambient symplectomorphism (resp. contactomorphism). More precisely, to each member of the class $\text{Arb}_n^{\text{symp}}$, one can assign a signed rooted tree $T = (T, \rho, \varepsilon)$ with at most $n + 1$ vertices. Here $T$ is a finite acyclic graph, $\rho$ is a distinguished root vertex, and $\varepsilon$ is a sign function on the edges of $T$ not adjacent to $\rho$. This discrete data completely determines the germ:
Theorem 1.2. If two arboreal Lagrangian singularities \( L \subset (X, \omega) \), \( L' \subset (X', \omega') \) have the same dimension and signed rooted tree \( \mathcal{T} \), then there is (the germ of) a symplectomorphism \( (X, \omega) \cong (X', \omega') \) identifying \( L \) and \( L' \).

Similarly, each member of the class \( \text{Arb}_{n}^{\text{cont}} \) is determined by an associated signed rooted tree \( \mathcal{T} = (T, \rho, \varepsilon) \) with at most \( n + 1 \) vertices.

As a representative for each signed rooted tree \( \mathcal{T} \) one may take the local model \( L_{\mathcal{T}} \subset T^{*}\mathbb{R}^{n} \) detailed in Section 2, where \( n = |n(T)| \) is one less than the number of vertices in the tree. The model \( L_{\mathcal{T}} \subset T^{*}\mathbb{R}^{n} \) is given as the positive conormal to an explicit front \( H_{\mathcal{T}} \subset \mathbb{R}^{n} \). In Section 3 we also establish a parametric version of the Stability Theorem, whose main consequence can be formulated as follows:

Theorem 1.3. Fix a signed rooted tree \( \mathcal{T} = (T, \rho, \varepsilon) \), set \( n = |n(T)| \) and consider the arboreal \( \mathcal{T} \)-front \( H_{\mathcal{T}} \subset \mathbb{R}^{n} \). Let \( D(\mathbb{R}^{n}, H_{\mathcal{T}}) \) be the group of germs at 0 of diffeomorphisms of \( \mathbb{R}^{n} \) preserving \( H_{\mathcal{T}} \) as a front, i.e. as a subset along with its coorientation.

Then the fibers of the natural map \( D(\mathbb{R}^{n}, H_{\mathcal{T}}) \to \text{Aut}(\mathcal{T}) \) are weakly contractible.

Hence the local symplectic topology of an arboreal singularity is completely characterized by the combinatorics of the underlying signed rooted tree, even parametrically.

1.2. Arborealization program. Arboreal singularities were introduced by the third author in the paper [N13] as candidates for tractable singularities of skeleton of Weinstein manifolds.

On the one hand, it is possible to calculate the symplectic invariants of arboreal singularities: this was shown in [N13] from the viewpoint of microlocal sheaves. Parallel results also hold in Lagrangian intersection theory, for instance in the language of Fukaya categories one can apply Lefschetz fibration calculations [Se08] to the plumbing characterization of [Sh18].

Moreover, arboreal skeleta arise naturally in symplectic geometry and allow for efficient calculations: for a basic example, the complex pair-of-pants (the complement of \( n + 2 \) general hyperplanes in \( \mathbb{P}^{n} \)) has a natural arboreal Lagrangian retract constructed in [FK14] and confirmed to be a skeleton in [GN20]. As another concrete example, see [C20] for explicit arboreal skeleta (without boundary) associated to the links of plane curve singularities. For a relationship between arboreal singularities and loose Legendrians see [M19].

On the other hand, it was shown in [N15] that singularities of Whitney stratified Lagrangians can always be locally deformed to arboreal Lagrangians in a non-characteristic fashion, i.e. without changing their microlocal invariants. But it remained an open question whether a global theory existed at the level of Weinstein structures. In two dimensions the story is classical: generic ribbon graphs provide arboreal skeleta. In four dimensions, Starkston proved arboreal skeleta always exist for the Weinstein homotopy class of any Weinstein domain [St18]. In the paper [AGEN20b], the authors establish that, in any dimension, a Weinstein manifold \( X \) can be arborealized whenever it admits a polarization, which is a global field of Lagrangian planes in \( TX \). Moreover, for a polarized Weinstein manifold it is shown in [AGEN20b] that the skeleton can be arranged to have only positive arboreal singularities,
which is the distinguished subclass of arboreal singularities corresponding to signed rooted trees with all signs equal to $+1$. Furthermore, the existence of a polarization is a necessary and sufficient condition for existence of a positive arboreal skeleton. This result, as well as forthcoming work of the authors [AGEN21], relies crucially on the Ridgification Theorem proved by the authors in [AGEN19], as well as on the Stability Theorem 3.5 proved in this paper.

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2. Arboreal models

2.1. Quadratic fronts. Before we present the local models for arboreal singularities, we introduce the quadratic fronts out of which the models will be built and discuss some of their basic properties.

2.1.1. Basic constructions. For $i \geq 0$, define functions $h_i : \mathbb{R}^i \to \mathbb{R}$ by the inductive formula

$$h_0 = 0 \quad h_i = h_i(x_1, \ldots, x_i) = x_1 - h_{i-1}(x_2, \ldots, x_i)^2$$

For example, for small $i$, we have

$$h_1(x_1) = x_1 \quad h_2(x_1, x_2) = x_1 - x_2^2 \quad h_3(x_1, x_2, x_3) = x_1 - (x_2 - x_3^2)^2$$

Fix $n \geq 0$. For $i = 0, \ldots, n$, define smooth graphical hypersurfaces

$$n \Gamma_i = \{x_0 = h_i^2\} \subset \mathbb{R}^{n+1}$$

equipped with the graphical coorientation, and consider their union

$$n \Gamma = \bigcup_{i=0}^n n \Gamma_i$$

Note the elementary identities

$$n \Gamma_i = i \Gamma_i \times \mathbb{R}^{n-i} \quad i = 0, \ldots, n$$

$$(n-1) \Gamma_i \cap n \Gamma_0 = n \Gamma_{i-1} \quad i = 1, \ldots, n$$

Let $T^* \mathbb{R}^n$ denote the cotangent bundle with canonical 1-form $pdx = \sum_{i=1}^n p_idx_i$ where $p = (p_1, \ldots, p_n)$ are dual coordinates to $x = (x_1, \ldots, x_n)$. Let $J^1 \mathbb{R}^n = \mathbb{R} \times T^* \mathbb{R}^n$ denote the 1-jet bundle with contact form $dx_0 + pdx = dx_0 + \sum_{i=1}^n p_idx_i$. 
Given a function $f : \mathbb{R}^n \to \mathbb{R}$ with graph $\Gamma_f = \{ x_0 = f(x) \} \subset \mathbb{R} \times \mathbb{R}^n$, we have the conormal Lagrangian of the graph $L_{\Gamma_f} = \{ x_0 = f(x), p_i = -p_0 \partial f / \partial x_i \} \subset T^*\mathbb{R}^{n+1}$, and the conormal Legendrian of the graph $\Lambda_{\Gamma_f} = \{ x_0 = f(x), p_0 = 1, p_i = -\partial f / \partial x_i \} \subset J^1\mathbb{R}^n$.

For $i = 0$, let $^nL_0 = \mathbb{R}^n \subset T^*\mathbb{R}^n$ denote the zero-section. For $i = 1, \ldots, n$, introduce the conormal Lagrangian

$^nL_i = L_{^n-1\Gamma_i-1} \subset T^*\mathbb{R}^n$

of the graph $^n-1\Gamma_i-1 \subset \mathbb{R}^n$, and consider their union

$^nL = \bigcup_{i=0}^n ^nL_i$

Similarly, for $i = 0, \ldots, n$, introduce the conormal Legendrian

$^n\Lambda_i = \Lambda_{^n\Gamma_i} \subset J^1\mathbb{R}^n$

of the graph $^n\Gamma_i \subset \mathbb{R}^{n+1}$, and consider their union

$^n\Lambda = \bigcup_{i=0}^n ^n\Lambda_i$
Note that the Liouville form vanishes on the conical Lagrangian $^{n}L_{i} \subset T^{*}\mathbb{R}^{n}$, hence its lift to $J^{1}\mathbb{R}^{n} = \mathbb{R} \times T^{*}\mathbb{R}^{n}$ with zero primitive is a Legendrian. We have the following compatibility:

**Lemma 2.1.** The contactomorphism

$$S : J^{1}\mathbb{R}^{n} \longrightarrow J^{1}\mathbb{R}^{n}$$

$$S(x_{0}, x, p) = (x_{0} - p_{1}^{2}/4, x_{1} + p_{1}/2, x_{2}, \ldots, x_{n}, p_{1}, \ldots, p_{n})$$

takes the Legendrian $^{n}\Lambda_{i}$ isomorphically to the Legendrian $\{0\} \times ^{n}L_{i}$, and thus the union $^{n}\Lambda$ isomorphically to the union $\{0\} \times ^{n}L$.

**Proof.** Set $h_{i,1} = h_{i-1}(x_{2}, \ldots, x_{i})$ so that $h_{i} = x_{1} - h_{i,1}^{2}$. Observe $^{n}\Lambda_{i} \subset J^{1}\mathbb{R}^{n}$ is given by the equations

$$x_{0} = h_{i}^{2} \quad \text{pdx} = -dh_{i}^{2} = -2h_{i}dh_{i} = -2h_{i}(dx_{1} - 2h_{i,1}dh_{i,1})$$

so in particular $p_{1} = -2h_{i}$ and $\sum_{i=2}^{n} p_{i}dx_{i} = 4h_{i}h_{i,1}dh_{i,1}$.

If we write $(\hat{x}_{0}, \hat{x}, p) = S(x_{0}, x, p)$, for $(x_{0}, x, p) \in ^{n}\Lambda_{i}$, then we have

$$\hat{x}_{0} = x_{0} - p_{1}^{2}/4 = \pm(x_{0} - h_{i}^{2}) = 0 \quad \hat{x}_{1} = x_{1} + p_{1}/2 = x_{1} - h_{i} = x_{1} - (x_{1} - h_{i,1}^{2}) = h_{i,1}^{2}$$

Now it remains to observe $^{n}L_{i} \subset T^{*}\mathbb{R}^{n}$ is given by the equations

$$x_{1} = h_{i,1}^{2} \quad \sum_{i=2}^{n} p_{i}dx_{i} = -p_{1}dh_{i,1}^{2} = -2p_{1}h_{i,1}dh_{i,1}$$

This completes the proof. \qed

### 2.1.2. Distinguished quadrants.

We now specify some distinguished quadrants of the $^{n}\Gamma$ which we will use to define our arboreal models. Which of these quadrants are cut out by our sign conventions will become clearer when the arboreal models are introduced.

For $0 \leq j < i \leq n$, set

$$h_{i,j} := h_{i-j}(x_{j+1}, \ldots, x_{i})$$

so in particular $h_{i,0} = h_{i}(x_{1}, \ldots, x_{i})$ and $h_{i,i-1} = h_{1}(x_{i}) = x_{i}$.

For fixed $0 \leq i \leq n$, consider the collection of functions

$$h_{i,0}, \ldots, h_{i,i-1}$$

Note the triangular nature of the linear terms of the collection: for all $0 \leq j \leq i - 1$, the subcollection

$$h_{i,j} - x_{j+1}, h_{i,j+1}, \ldots, h_{i,i-1}$$

is independent of $x_{j+1}$. Thus the level sets of the collection are mutually transverse.

Fix once and for all a list of signs $\delta = (\delta_{0}, \delta_{1}, \ldots, \delta_{n})$, $\delta_{i} \in \{\pm 1\}$. Define the domain quadrant $^{n}Q_{i}^{\delta} \subset \mathbb{R}^{n}$ to be cut out by the inequalities

$$\delta_{1}h_{i,0} \leq 0, \ldots, \delta_{i}h_{i,i-1} \leq 0$$
By the transversality noted above, $nQ_\delta^i$ is a submanifold with corners diffeomorphic to $\mathbb{R}_{\geq 0}^i \times \mathbb{R}^{n-i}$. Its codimension one boundary faces are given by the vanishing of one of the functions $h_{i,j}$. Note $nQ_\delta^i$ only depends on the truncated list $\delta_1, \ldots, \delta_i$. In particular, it is independent of $\delta_0$ which will enter the constructions next.

Define the cooriented hypersurface $n\Gamma_{i|\delta} \subset \mathbb{R}^{n+1}$ to be the restricted signed graph

$$n\Gamma_{i|\delta} = \{ x_0 = \delta_0 h_i^2 \}_{nQ_\delta^i}$$

with the graphical coorientation.

Thus $n\Gamma_{i|\delta}$ is cut out by the equations

$$x_0 = \delta_0 h_i^2, \quad \delta_1 h_{i,0} \leq 0, \ldots, \delta_i h_{i,i-1} \leq 0$$

Since $n\Gamma_{i|\delta}$ is graphical over $nQ_\delta^i$, it is also a submanifold with corners diffeomorphic to $\mathbb{R}_{\geq 0}^i \times \mathbb{R}^{n-i}$. Likewise, its codimension one boundary faces are given by the vanishing of one of the functions $h_{i,j}$.

Consider as well the union

$$n\Gamma_{i|\delta} = \bigcup_{i=0}^n n\Gamma_{i|\delta}$$

**Remark 2.2.** Note that

$$n\Gamma_{i} = \bigcup_{\delta, \delta_{i-1}} n\Gamma_{i|\delta} \quad \quad n\Gamma = \bigcup_{\delta, \delta_0=1} n\Gamma_{i|\delta}$$

since $x \in n\Gamma$ implies $x \in n\Gamma_{i|\delta}$ where for $1 \leq j \leq i$, we set $\delta_j = -\text{sgn} (h_{i,j}(x))$, when $h_{i,j}(x) \neq 0$, and choose it arbitrarily otherwise.

**Remark 2.3.** Note if we set $\delta' = (\delta_0, \ldots, \delta_{n-1}, -\delta_n)$, then the map $\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$,

$$(x_0, \ldots, x_{n-1}, x_n) \mapsto (x_0, \ldots, x_{n-1}, -x_n),$$

takes $n\Gamma_{\delta}$ isomorphically to $n\Gamma_{\delta'}$ as a cooriented hypersurface. Thus we could always set $\delta_n = 1$ and not miss any new geometry.

Note $n\Gamma_i \cap \{ x_0 < 0 \}$, hence also $n\Gamma_{i|\delta} \cap \{ \delta_0 x_0 < 0 \}$, is empty since $n\Gamma_i$ is the graph of $h_i^2 \geq 0$.

**Lemma 2.4.** Fix $\delta = (\delta_0, \ldots, \delta_n)$, and set $\delta' = (\delta_0, \delta_2, \ldots, \delta_n)$. The homeomorphism

$$s : \delta_0 \mathbb{R}_{\geq 0} \times \mathbb{R}^n \longrightarrow \delta_0 \mathbb{R}_{\geq 0} \times \mathbb{R}^n$$

$$s(x_0, x_1, x_2, \ldots, x_n) = (x_0, \delta_0 \delta_1 (x_1 + \delta_1 \sqrt{\delta_0 x_0}), x_2, \ldots, x_n)$$

gives a cooriented identification

$$s(n\Gamma_{i|\delta} \cap \{ \delta_0 x_0 \geq 0 \}) = \delta_0 \mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1} \Gamma_{i-1|\delta'} \quad 0 < i \leq n$$

**Proof.** Recall $n\Gamma_{i|\delta}$ is defined by

$$x_0 = \delta_0 h_i^2, \quad \delta_1 h_{i,0} \leq 0, \ldots, \delta_i h_{i,i-1} \leq 0$$
in particular
\[ x_0 = \delta_0 h_i^2 \quad \text{and} \quad \delta_1 h_i,0 = \delta_1 h_i \leq 0 \]

Note the functions \( h_{i,1}, \ldots, h_{i,i-1} \) are independent of the coordinates \( x_0, x_1 \).

When \( \delta_0 x_0 \geq 0 \) and \( \delta_1 h_i \leq 0 \), the equation \( x_0 = \delta_0 h_i^2 \) is equivalent to \( \sqrt{\delta_0 x_0} = -\delta_1 h_i \).

Expanding this in terms of the definitions, we can rewrite this in the form
\[ x_1 + \delta_1 \sqrt{\delta_0 x_0} = h_{i-1}(x_2, \ldots, x_i)^2 \]

Thus since \( \delta_0' = \delta_0 \delta_1 \), we see \( s \) takes \( n\Gamma_i|\delta \cap \{ \delta_0 x_0 \geq 0 \} \) into \( \delta_0 \mathbb{R}_{\geq 0} \times \{ x_1 = \delta_0' h_{i-1}^2 \} \).

Moreover, the additional functions \( h_{i,1}, \ldots, h_{i,i-1} \) cutting out \( n^{-1}\Gamma_{i-1}|_{\delta'} \subset \{ x_1 = \delta_0' h_{i-1}^2 \} \) pull back to the same functions \( h_{i,1}, \ldots, h_{i,i-1} \) cutting out \( n\Gamma_i|\delta \).

Finally, the coorientations of \( n\Gamma_i|_{\delta}, n^{-1}\Gamma_{i-1}|_{\delta'} \) are positive on respectively \( \partial x_0, \partial x_1 \). Observe the \( \partial x_1 \)-component of \( s_*\partial x_0 \) is in the direction of \( \partial x_1 \), and hence \( s \) gives a cooriented identification. \( \square \)

2.1.3. Alternative presentation. For compatibility with inductive arguments, it is useful to introduce an alternative sign convention and alternative presentation of the local models.

Fix signs \( \varepsilon = (\varepsilon_0, \ldots, \varepsilon_n) \). Consider the involution \( \sigma_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n \) defined by \( \sigma_\varepsilon(x_1, \ldots, x_n) = (\varepsilon_1 x_1, \ldots, \varepsilon_n x_n) \).

Define the domain quadrant \( n\Gamma_0^\varepsilon \subset \mathbb{R}^n \) cut out by the inequalities
\[ \varepsilon_0 \varepsilon_1 h_{i,0} \circ \sigma_\varepsilon \leq 0, \ldots, \varepsilon_{i-1} \varepsilon_i h_{i,i-1} \circ \sigma_\varepsilon \leq 0 \]

Define the cooriented hypersurface \( n\Gamma_1^\varepsilon \subset \mathbb{R}^{n+1} \) to be the restricted signed graph
\[ n\Gamma_1^\varepsilon = \{ x_0 = \varepsilon_0 h_i^2 \circ \sigma_\varepsilon \}|_{n\Gamma_0^\varepsilon} \]

with the graphical coorientation. Thus \( n\Gamma_1^\varepsilon \) is cut out by the equations
\[ x_0 = \varepsilon_0 h_i^2 \circ \sigma_\varepsilon \quad \text{and} \quad \varepsilon_0 \varepsilon_1 h_{i,0} \circ \sigma_\varepsilon \leq 0, \ldots, \varepsilon_{i-1} \varepsilon_i h_{i,i-1} \circ \sigma_\varepsilon \leq 0 \]

Consider as well the union
\[ n\Gamma^\varepsilon = \bigcup_{i=0}^n n\Gamma_i^\varepsilon \]

Remark 2.5. A simple but important observation: \( n\Gamma_i^\varepsilon \) in fact only depends on \( \varepsilon_0, \ldots, \varepsilon_{i-1} \) and not \( \varepsilon_i \). This is because \( h_{i,i-1} = x_i \) and so \( \varepsilon_{i-1} \varepsilon_i h_{i,i-1} \circ \sigma_\varepsilon = \varepsilon_{i-1} x_i \). In particular, the union \( n\Gamma^\varepsilon \) is independent of \( \varepsilon_n \).

We have the following adaption of Lemma 2.4.

Lemma 2.6. Fix \( \varepsilon = (\varepsilon_0, \ldots, \varepsilon_n) \), and set \( \varepsilon' = (\varepsilon_1, \ldots, \varepsilon_n) \). The homeomorphism
\[ s : \varepsilon_0 \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0} \times \mathbb{R}^n \]
\[ s(x_0, x_1, x_2, \ldots, x_n) = (x_0, x_1 + \varepsilon_0 \sqrt{\varepsilon_0 x_0}, x_2, \ldots, x_n) \]
Proof. Recall \(\Gamma_i^\varepsilon\) is defined by
\[
x_0 = \varepsilon_0 h_i^2 \circ \sigma_\varepsilon \
\]
in particular
\[
x_0 = \varepsilon_0 h_i^2 \circ \sigma_\varepsilon \
\]
Note the functions \(h_{i,1}, \ldots, h_{i,i-1}\) are independent of the coordinates \(x_0, x_1\).

When \(\varepsilon_0 x_0 \geq 0\) and \(\varepsilon_0 \varepsilon_1 h_i \circ \sigma_\varepsilon \leq 0\), the equation \(x_0 = \varepsilon_0 h_i^2 \circ \sigma_\varepsilon\) is equivalent to \(\sqrt{\varepsilon_0 x_0} = -\varepsilon_0 \varepsilon_1 h_i \circ \sigma_\varepsilon\). Expanding this in terms of the definitions, we can rewrite this in the form
\[
x_1 + \varepsilon_0 \sqrt{\varepsilon_0 x_0} = \varepsilon_1 h_i^2 \circ \sigma_{\varepsilon'}
\]
Thus we see \(s\) takes \(\Gamma_i^\varepsilon \cap \{\varepsilon_0 x_0 \geq 0\}\) into \(\varepsilon_0 \mathbb{R} \times \{x_1 = \varepsilon_1 h_i^2 \circ \sigma'_{\varepsilon'}\}\).

Moreover, the additional functions \(h_{i,1}, \ldots, h_{i,i-1}\) cutting out \(n-1\Gamma_i^\varepsilon \subset \{x_1 = \varepsilon_1 h_i^2 \circ \sigma_{\varepsilon'}\}\) pull back to the same functions \(h_{i,1}, \ldots, h_{i,i-1}\) cutting out \(\Gamma_i^\varepsilon\).

Finally, the coorientations of \(n\Gamma_i^\varepsilon, n-1\Gamma_i^\varepsilon\) are positive on respectively \(\partial x_0, \partial x_1\). Observe the \(\partial_{x_1}\)-component of \(s \circ \partial x_0\) is in the direction of \(\partial_{x_1}\), and hence \(s\) gives a cooriented identification.

Here is a useful corollary that “explains” the geometric meaning of the signs \(\varepsilon\).

**Corollary 2.7.** Fix \(\varepsilon = (\varepsilon_0, \ldots, \varepsilon_n)\).

For \(i = 0, \ldots, n-1\), we have \(\varepsilon_i = \pm 1\) if and only if \(n\Gamma_{i+1}\) is on the ±-side of \(n\Gamma_i\) with respect to the graphical \(dx_0\)-coorientation.

Moreover, for \(i = 1, \ldots, n-1\), we have \(\varepsilon_i = \pm 1\) if and only if \(n\Gamma_{i+1} \cap n\Gamma_0\) is on the ±-side of \(n\Gamma_i \cap n\Gamma_0\) with respect to the graphical \(dx_1\)-coorientation.

**Proof.** For \(i = 0\), the first assertion is immediate from the definitions \(n\Gamma_0 = \{x_0 = 0\}\) and \(n\Gamma_1 = \{x_0 = \varepsilon_0 (\varepsilon_1 x_1)^2 = \varepsilon_0 x_1^2, \varepsilon_0 \varepsilon_1 (\varepsilon_1 x_1) = \varepsilon_0 x_1 \leq 0\}\).

For \(i > 0\), both assertions follow by induction from Lemma 2.6.
projecting diffeomorphically to the front $n\Gamma_1^\varepsilon \subset \mathbb{R}^{n+1}$, and consider their union

$$n\Lambda_i^\varepsilon = \bigcup_{i=0}^n n\Lambda_i^\varepsilon$$

We have the following compatibility of the above Lagrangians and Legendrians analogous to Lemma 2.1.

**Lemma 2.8.** Fix signs $\varepsilon = (\varepsilon_0, \ldots, \varepsilon_n)$, and set $\varepsilon' = (\varepsilon_1, \ldots, \varepsilon_n)$. The contactomorphism

$$S_{\varepsilon_0} : J^1\mathbb{R}^n \rightarrow J^1\mathbb{R}^n$$

$$S_{\varepsilon_0}(x_0, x, p) = (x_0 - \varepsilon_0 p_0^2/4, x_1 + \varepsilon_0 p_1/2, x_2, \ldots, x_n, p_1, \ldots, p_n)$$

takes the Legendrian $n\Lambda_i^\varepsilon$ isomorphically to the Legendrian $\{0\} \times n\Gamma_i^\varepsilon$, and thus the union $n\Lambda^\varepsilon$ isomorphically to the union $\{0\} \times n\Gamma^\varepsilon$.

**Proof.** The proof is the same as that of Lemma 2.1 with the following observations. Consider the additional equations

$$\varepsilon_0 \varepsilon_1 \delta_1 h_{i,0} \circ \sigma_\delta \leq 0, \ldots, \varepsilon_{i-1} \varepsilon_i h_{i,i-1} \circ \sigma_\varepsilon \leq 0$$

First, over $\varepsilon_0 \varepsilon_1 h_{i,0} \circ \sigma_\varepsilon \leq 0$, when $p_1 = -2\varepsilon_0 \varepsilon_1 h_{i,0} \circ \sigma_\varepsilon$, we then have $p_1 = -2\varepsilon_0 \varepsilon_1 h_{i,0} \circ \sigma_\varepsilon \geq 0$, so we obtain the positive conormal direction. Second, the remaining functions $h_{i,1}, \ldots, h_{i,i-1}$ are independent of $x_0, x_1$. Thus $S_{\varepsilon_0}$ indeed takes $n\Lambda_i^\varepsilon$ to $\{0\} \times n\Gamma_i^\varepsilon$.

**Remark 2.9.** By the lemma, we see the Legendrian $n\Lambda_i^\varepsilon \subset J^1\mathbb{R}^n$ is independent of the initial sign $\varepsilon_0$ so only depends on $\varepsilon' = (\varepsilon_1, \ldots, \varepsilon_n)$.

It is also useful to record the following relationship of $n\Gamma^\varepsilon$ with the extended model $n\Gamma$.

**Lemma 2.10.** Fix signs $\varepsilon = (\varepsilon_0, \ldots, \varepsilon_n)$.

Given a contactomorphism $J^1\mathbb{R}^n \rightarrow J^1\mathbb{R}^n$ restricting to a closed embedding $n\Lambda^\varepsilon \subset \varepsilon_0 \cdot n\Lambda$ with $n\Lambda_i^\varepsilon \subset \varepsilon_0 \cdot n\Lambda_i$, for all $i$, consider the front $\Upsilon = \pi(n\Lambda^\varepsilon) \subset \varepsilon_0 \cdot n\Gamma$.

Then either the involution $\sigma_\varepsilon$ or its composition with $x_n \mapsto \pm x_n$ takes $\Upsilon$ to $n\Gamma^\varepsilon$.

**Proof.** Note we have $n\Lambda_0^\varepsilon = \varepsilon_0 \cdot n\Lambda_0 = n\Lambda_0$. Consider the intersection $\Upsilon' = \pi((n\Lambda_\varepsilon \setminus n\Lambda_0) \cap n\Lambda_0)$ as a front inside of $\pi(n\Lambda_0) = n\Gamma_0 = \{x_0 = 0\}$. By induction, either the involution $\sigma_\varepsilon$ or its composition with $x_n \mapsto \pm x_n$ takes $\Upsilon'$ to $n^{-1}\Gamma^{\varepsilon'}$ where $\varepsilon' = (\varepsilon_1, \ldots, \varepsilon_n)$. So we may assume $\Upsilon' = n^{-1}\Gamma^{\varepsilon'}$. Now observe $n\Gamma^\varepsilon$ is the unique way to extend $n^{-1}\Gamma^{\varepsilon'}$ within $\sigma_\varepsilon(\varepsilon_0 \cdot n\Gamma)$ compatible with coorientations.

We also have the following observation about signs.

**Lemma 2.11.** Let $\nu_0$ be the vertical polarization of $T^*\mathbb{R}^n \rightarrow \mathbb{R}^n$.

Then we have $\varepsilon(nL_{1}^{\varepsilon}, \nu_0, nL_{2}^{\varepsilon}) = \varepsilon_0$.

**Proof.** Recall $nL_{1}^{\varepsilon}$ is the positive conormal to the graph $n^{-1}\Gamma_0^\varepsilon = \{x_0 = 0\}$, and $nL_{2}^{\varepsilon}$ is the positive conormal to the graph $n^{-1}\Gamma_1^\varepsilon = \{x_0 = \varepsilon_0 x_1^2\}$. Since $\varepsilon_0 x_1^2$ is an $\varepsilon_0$-definite quadratic form in $x_1$, the assertion follows.
2.2. Arboreal models. We now present the local models for arboreal singularities.

2.2.1. Signed rooted trees.

Definition 2.12. We will use the following terminology throughout:

(i) A tree $T$ is a nonempty, finite, connected acyclic graph.

(ii) A rooted tree $\mathcal{T} = (T, \rho)$ is a pair of a tree $T$ and a distinguished vertex $\rho$ called the root.

(iii) A signed rooted tree $\mathcal{T} = (T, \rho, \varepsilon)$ is a rooted tree $(T, \rho)$ and a decoration $\varepsilon$ of a sign $\pm 1$ on each edge of $T$ not adjacent to the root $\rho$.

![Figure 2.3. A signed rooted tree.](image)

Given a signed rooted tree $\mathcal{T} = (T, \rho, \varepsilon)$, we write $v(T)$ for the set of vertices, $e(T)$ for the set of edges, and $n(T) = v(T) \setminus \rho$ for the set of non-root vertices. We regard $v(T)$ as a poset with unique minimum $\rho$, and in general $\alpha \leq \beta \in v(T)$ when the shortest path connecting $\beta$ and $\rho$ contains $\alpha$. We call a non-root vertex $\beta$ a leaf if exactly one edge of $T$ is adjacent to $\beta$, and write $\ell(T) \subset v(T)$ for the set of leaf vertices.

Remark 2.13. Throughout what follows, for a finite set $S$, we write $\mathbb{R}^S$ for the Euclidean space of $S$-tuples of real numbers. One may always fix a bijection $S \simeq \{1, 2, \ldots, n\}$, for some $n \geq 0$, and hence an isomorphism $\mathbb{R}^S \simeq \mathbb{R}^n$, but it will be convenient to avoid choosing such identifications when awkward. We will most often consider $S = n(T)$ the non-root vertices for some rooted tree $\mathcal{T} = (T, \rho)$. Here if one prefers to fix a bijection $b : n(T) \rightarrow \{1, 2, \ldots, |n(T)|\}$, we recommend choosing $b$ to be order-preserving: if $\alpha \leq \beta$, then one should ensure $b(\alpha) \leq b(\beta)$. This will allow for a clear translation of our constructions.

Definition 2.14. A signed rooted tree $\mathcal{T} = (T, \rho, \varepsilon)$ is called positive if the decoration $\varepsilon$ consists of signs $+1$.

We will associate to any signed rooted tree $\mathcal{T} = (T, \rho, \varepsilon)$, a multi-cooriented hypersurface, conic Lagrangian, and Legendrian $H_T \subset \mathbb{R}^{n(T)}$, $L_T \subset T^*\mathbb{R}^{n(T)}$, $\Lambda_T \subset J^1\mathbb{R}^{n(T)}$. 
where as usual we write \( n(T) = v(T) \setminus \rho \) for the set of non-root vertices.

By definition, the latter two will be determined by the first as follows:

(i) \( L_T \) is the union of the zero-section \( \mathbb{R}^{n(T)} \) and the positive conormal to \( H_T \).

(ii) \( \Lambda_T \) is the Legendrian lift of \( L_T \) with zero primitive.

2.2.2. Type A trees. Let us first consider the distinguished case of \( A_{n+1} \)-trees with extremal root.

**Definition 2.15.** For \( n \geq 0 \), a linear signed \( A_{n+1} \)-rooted tree is a signed rooted tree \( A_{n+1} = (A_{n+1}, \rho, a) \) with vertices \( v(A_{n+1}) = \{0, 1, \ldots, n\} \), edges \( v(A_{n+1}) = \{[i, i+1] | i = 0, \ldots, n-1\} \), and root \( \rho = 0 \).

By definition, the sign \( a \) is a length \( n \) list of signs \((a_{1,2}, \ldots, a_{n-1,n})\). Let us set \( \varepsilon = (\varepsilon_0, \ldots, \varepsilon_{n-1}) = (a_{1,2}, \ldots, a_{n-1,n}, 1) \) to be the length \( n \) list of signs where we pad \( a \) by adding a single 1 at the end.

**Definition 2.16.** The models for \( A_n \)-type arboreal singularities are given as follows:

(i) The arboreal \( A_1 \)-front is the empty set \( H_{A_1} = \emptyset \) inside the point \( R^0 \).

For \( n \geq 1 \), the arboreal \( A_{n+1} \)-front is the cooriented hypersurface

\[
H_{A_{n+1}} = n^{-1} \Gamma_{\varepsilon} \subset \mathbb{R}^n
\]

introduced in Section 2.1.3.

(ii) For \( n \geq 0 \), the arboreal \( A_{n+1} \)-Lagrangian is the union of the zero-section and positive conormal

\[
L_{A_{n+1}} = \mathbb{R}^n \cup T^+_{\mathbb{R}^n} H_{A_{n+1}} \subset T^*\mathbb{R}^n
\]

(iii) For \( n \geq 0 \), the arboreal \( A_{n+1} \)-Legendrian is the lift

\[
\Lambda_{A_{n+1}} = \{0\} \times L_{A_{n+1}} \subset J^1\mathbb{R}^n
\]

\[\text{Figure 2.4. The two } A_3 \text{ fronts with positive and negative sign.}\]
Figure 2.5. Two $A_4$ fronts with different choices of signs. The other two fronts can be obtained from these two by reflections.

Remark 2.17. Following Remark 2.5, the arbitrary choice of the last sign $\varepsilon_{n-1} = 1$ does not affect the arboreal $A_{n+1}$-models.

Recall the linear signed $A_{n+1}$-rooted tree $A_{n+1} = (A_{n+1}, \rho, a)$ has vertices $v(A_{n+1}) = \{0, 1, \ldots, n\}$ with root $\rho = 0$, and so the non-root vertices form the set $n(A_{n+1}) = \{1, \ldots, n\}$. In the above definition, we should more invariantly view the ambient Euclidean space $\mathbb{R}^n$ in the form $\mathbb{R}^{n(A_{n+1})}$ where the ordering of the coordinates matches that of $n(A_{n+1})$.

With this viewpoint, we rename the smooth pieces of the $A_{n+1}$-front, indexing them by non-root vertices

$$H_i = n^{-1} P_{i-1}^e \subset H_{A_{n+1}} \quad i \in n(A_{n+1}) = \{1, \ldots, n\}$$

Likewise, we rename the smooth pieces of the $A_{n+1}$-Lagrangian, indexing them by vertices

$$L_0 = \mathbb{R}^n \subset L_{A_{n+1}}$$

$$L_i = T_{\mathbb{R}^n} H_i \subset L_{A_{n+1}} \quad i \in n(A_{n+1}) = \{1, \ldots, n\}$$

and similarly, we rename the smooth pieces of the $A_{n+1}$-Legendrian, indexing them by vertices

$$\Lambda_i = \{0\} \times L_{A_{n+1},i} \subset \Lambda_{A_{n+1}} \quad i \in v(A_{n+1}) = \{0, 1, \ldots, n\}$$
Lemma 2.18. For $n \geq 1$, and $n \in v(A_{n+1}) = \{0, 1, \ldots, n\}$ the unique leaf vertex, and $\tilde{H}_n \subset H_{A_{n+1}}$ the interior of the corresponding smooth piece, we have

$$H_{A_{n+1}} \setminus \tilde{H}_n = H_{A_n} \times \mathbb{R}$$

inside of $\mathbb{R}^{n(A_{n+1})} = \mathbb{R}^{n(A_n)} \times \mathbb{R}$.

Proof. Recall the other smooth pieces $H_i = n^{-1}P_{i-1}^{ε}$, for $i = 1, \ldots, n - 1$, are independent of the last coordinate $x_n$. $\square$

2.2.3. General trees. Now we consider a general signed rooted tree $\mathcal{T} = (T, ρ, ε)$.

To each leaf $β \in \ell(T)$, we associate the linear signed $A_{n+1}$-rooted tree $A_β = (A_β, ρ, a)$ where $A_β$ is the full subtree of $T$ on the vertices $v(A_β) = \{α \leq β \in v(T)\}$, and $a$ is the restricted sign decoration.

Consider the Euclidean space $\mathbb{R}^{n(T)}$. For each $β \in \ell(T)$, the inclusion $n(A_β) \subset n(T)$ induces a natural projection

$$\pi_β : \mathbb{R}^{n(T)} \longrightarrow \mathbb{R}^{n(A_β)}$$

Definition 2.19. Let $\mathcal{T} = (T, ρ, ε)$ be a signed rooted tree.

(i) The arboreal model $\mathcal{T}$-front is the multi-cooriented hypersurface given by the union

$$H_\mathcal{T} = \bigcup_{β \in \ell(T)} \pi_β^{-1}(H_{A_β}) \subset \mathbb{R}^{n(T)}$$

where $H_{A_β} \subset \mathbb{R}^{n(A_β)}$ is the arboreal $A_β$-front.

(ii) The arboreal model $\mathcal{T}$-Lagrangian is the union of the zero-section and positive conormal

$$L_\mathcal{T} = \mathbb{R}^{n(T)} \cup T^+_{\mathbb{R}^{n(T)}}, H_\mathcal{T} \subset T^*\mathbb{R}^{n(T)}$$

(iii) The arboreal model $\mathcal{T}$-Legendrian is the the lift

$$Λ_\mathcal{T} = \{0\} \times L_\mathcal{T} \subset J^1\mathbb{R}^{n(T)}$$

Arboreal models $H_\mathcal{T}, L_\mathcal{T}$ and $Λ_\mathcal{T}$ corresponding to positive $\mathcal{T}$ are called positive.

Remark 2.20. When $\mathcal{T} = A_{n+1}$, the above definition recovers Definition 2.16 verbatim.

Transporting from the case of $A_{n+1}$, we may naturally index the smooth pieces of the $\mathcal{T}$-front by non-root vertices

$$H_α = \pi_β^{-1}(H_{A_β, α}) \subset H_\mathcal{T} \quad α \in n(T)$$

where $β \in \ell(T)$ is any leaf with $α \leq β$, and $H_{A_β, α} \subset H_{A_β}$ is the corresponding smooth piece. Likewise, we may index the smooth pieces of the $\mathcal{T}$-Lagrangian by vertices

$$L_ρ = \mathbb{R}^{n(T)} \subset L_\mathcal{T}$$
Figure 2.6. Two non $A_n$-type fronts with different choices of signs.

$$L_\alpha = T^+_\mathbb{R}^{n(T)} H_\alpha \subset L_T \quad \alpha \in n(T)$$

and the smooth pieces of the $T$-Legendrian by vertices

$$\Lambda_\alpha = \{0\} \times L_\alpha \subset \Lambda_T \quad \alpha \in v(T)$$

Let us record a basic compatibility of the above Lagrangians and Legendrians.

Fix a signed rooted tree $T = (T, \rho, \varepsilon)$. Let us first consider the situation when there is a single vertex $\rho' \in T$ adjacent to $\rho$. Let $T' = T \setminus \rho$ be the signed rooted tree with root $\rho'$ and restricted signs.

Let $\alpha_1, \ldots, \alpha_k \in T'$ be the vertices adjacent to $\rho'$, and $\varepsilon_1, \ldots, \varepsilon_k$ the signs of $T$ assigned to the respective edges from $\rho'$ to $\alpha_1, \ldots, \alpha_k$.

Let $L^\infty_T \subset S^*\mathbb{R}^{n(T)}$ be the ideal Legendrian boundary of $L_T \subset T^*\mathbb{R}^{n(T)}$. Note that $L^\infty_T$ lies in the open subspace $J^1\mathbb{R}^{n(T')} \simeq \{p_{\rho'} = 1\} \subset S^*\mathbb{R}^{n(T)}$.

**Lemma 2.21.** The contactomorphism

$$S : J^1\mathbb{R}^{n(T')} \to J^1\mathbb{R}^{n(T')},$$

$$S(x_{\rho'}, x, p) = (x_{\rho'} - \sum_{i=1}^k \varepsilon_i p^2_{\alpha_i}/4, \hat{x}, \hat{p})$$

$$\hat{x}_{\alpha_i} = x_{\alpha_i} + \varepsilon_i p_1/2, \text{ for } i = 1, \ldots, k, \quad \hat{x}_{\beta} = x_{\beta} \text{ else}$$

takes the Legendrian $L^\infty_T$ isomorphically to the Legendrian $\{0\} \times L_{T'}$.

Thus $L^\infty_T$ itself is a model arboreal Legendrian of type $T' = T \setminus \rho$.

**Proof.** For each leaf vertex of $T$, we have a linear signed type $A$ subtree of $T$ given by the vertices running from $\rho$ to the leaf. By Definition 2.19, $L_T$ is the union of the corresponding linear signed type $A$ subcomplexes $L_A$. Each such subcomplex is independent of the coordinate $x_{\beta}$ indexed by vertices $\beta$ not in the subtree, hence lies in the zero locus of the dual
coordinate $p_\beta$. Thus transport of each $L^\infty_A$ under the contactomorphism of the lemma reduces to that of Lemma 2.8.

More generally, suppose $\rho_1, \ldots, \rho_\ell$ are the vertices adjacent to $\rho$. Observe that $T \setminus \rho$ is a disjoint union of signed rooted subtrees $T_j \subset T \setminus \rho$, for $j = 1, \ldots, \ell$, with $\rho_j$ as root and restricted signs. Let $T_j^+ = T_j \cup \rho \subset T$ be the signed rooted subtree with $\rho$ readjoined as root and with restricted signs. Set $c_j = n(T) \setminus n(T_j)$.

Let $L^\infty_T \subset S^*\mathbb{R}^n(T)$ be the ideal Legendrian boundary of $L_T \subset T^*\mathbb{R}^n(T)$. We similarly have $L^\infty_{T_j^+} \subset S^*\mathbb{R}^n(T_j^+)$ the ideal Legendrian boundary of $L_{T_j^+} \subset T^*\mathbb{R}^n(T_j^+)$. Since $\rho_j$ is the unique vertex adjacent to $\rho$ within $T_j^+$, observe that $L_{T_j^+}$ is connected and in fact lies in

$$J^1\mathbb{R}^n(T_j) = \{p_\rho_j = 1\} \subset S^*\mathbb{R}^n(T_j^+).$$

Moreover, observe that $L^\infty_T$ is the disjoint union of the connected components

$$\Lambda_j = L^\infty_{T_j^+} \times \mathbb{R}^{c_j} \subset J^1\mathbb{R}^n(T_j) \times T^*\mathbb{R}^{c_j} = \{p_\rho_j = 1\} \subset S^*\mathbb{R}^n(T).$$

By Lemma 2.21, $L^\infty_{T_j^+} \subset J^1\mathbb{R}^n(T_j)$ is a model arboreal Legendrian of type $T_j$, so $\Lambda_j = L^\infty_{T_j^+} \times \mathbb{R}^{c_j} \subset J^1\mathbb{R}^n(T_j) \times T^*\mathbb{R}^{c_j}$ is a stabilized model arboreal Legendrian of type $T_j$. This proves:

**Lemma 2.22.** Fix a signed rooted tree $T = (T, \rho, \varepsilon)$.

Let $\rho_1, \ldots, \rho_\ell$ be the vertices adjacent to $\rho$. Let $T_j \subset T \setminus \rho$ be the signed rooted subtree with $\rho_j$ as root and restricted signs, and $T_j^+ = T_j \cup \rho \subset T$ the signed rooted subtree with $\rho$ readjoined as root and with restricted signs. Set $c_j = n(T) \setminus n(T_j)$.

Then the ideal Legendrian boundary $L^\infty_T \subset S^*\mathbb{R}^n(T)$ of the model arboreal Lagrangian $L_T \subset T^*\mathbb{R}^n(T)$ of type $T$ is the disjoint union of the Legendrians

$$\Lambda_j = L^\infty_{T_j^+} \times \mathbb{R}^{c_j} \subset S^*\mathbb{R}^n(T),$$

which are stabilized model arboreal Legendrians of type $T_j$.

By Lemma 2.18, we also have the following.

**Corollary 2.23.** For $\beta \in \ell(T)$ a leaf vertex, and $\bar{H}_\beta \subset H_T$ the interior of the corresponding smooth piece, we have

$$H_T \setminus \bar{H}_\beta = H_T \setminus \beta \times \mathbb{R}^\beta$$

inside of $\mathbb{R}^n(T) = \mathbb{R}^n(T \setminus \beta) \times \mathbb{R}^\beta$.

2.2.4. **Extended arboreal models.** It will be useful for us also define extended arboreal models associated with rooted, but not signed trees $\mathcal{F} = (T, \rho)$.

For the unsigned rooted tree $\mathcal{A}_{n+1} = (A_{n+1}, \rho)$ we define

$$H_{\mathcal{A}_{n+1}} := n^{-1} \Gamma \subset \mathbb{R}^n,$$

$$L_{\mathcal{A}_{n+1}} := \mathbb{R}^n \cup T^*_{\mathbb{R}^n} H_{\mathcal{A}_{n+1}} \subset T^*\mathbb{R}^n,$$
\[ \Lambda_{\mathcal{A}_{n+1}} := 0 \times L_{\mathcal{A}_{n+1}} \subset J^1 \mathbb{R}^n. \]

Similarly, for a general rooted tree \( T = (T, \rho) \) we define

\[ H_T = \bigcup_{\beta \in \ell(T)} \pi_{1}^{-1}(H_{\mathcal{A}_\beta}) \subset \mathbb{R}^{n(T)} \]

where \( H_{\mathcal{A}_\beta} \subset \mathbb{R}^{n(\mathcal{A}_\beta)} \) is the arboreal \( \mathcal{A}_\beta \)-front. Furthermore, we define

\[ L_T = \mathbb{R}^{n(T)} \cup T^+_{\mathbb{R}^{n(T)}} H_T \subset T^* \mathbb{R}^{n(T)} \]

and

\[ \Lambda_T = \{0\} \times \Lambda_T \subset J^1 \mathbb{R}^{n(T)} \]

Clearly, for any signed version \( \mathcal{T} \) of the tree \( T \) we have \( H_T \subset H_\mathcal{T}, L_T \subset L_\mathcal{T}, \Lambda_T \subset \Lambda_\mathcal{T} \).

**Lemma 2.24.** Given a closed embedding \( \Lambda_\mathcal{T}^\infty \subset \Lambda^\infty_\mathcal{T} \) with \( \Lambda_{T, \alpha}^\infty \subset \Lambda_{\mathcal{T}, \alpha}^\infty \) for all \( \alpha \), the front \( \pi(\Lambda_\mathcal{T}^\infty) \subset H_T \) is an embedding of \( H_T \).

**Proof.** For each leaf vertex of \( T \), we have a linear signed type \( \mathcal{A} \) subtree of \( T \) given by the vertices running from \( \rho \) to the leaf. By construction, \( \Lambda_\mathcal{T}^\infty \) and \( \Lambda^\infty_\mathcal{T} \) are the union of the corresponding type \( \mathcal{A} \) subcomplexes \( L_\mathcal{A}^\infty \) and \( L^\infty_\mathcal{A} \). Each such subcomplex is independent of the coordinates \( x_{\beta} \) indexed by vertices \( \beta \) not in the subtree. Now Lemma 2.10 confirms \( \pi(L_\mathcal{A}^\infty) \) is the standard embedding of \( H_\mathcal{A} \) after a change of coordinates \( x_{\alpha} \) indexed by vertices \( \alpha \) in the subtree. Moreover, the change of coordinates agrees for \( x_{\alpha} \) indexed by vertices \( \alpha \) in the intersection of such subtrees. By definition, \( H_T \) is the union of the \( H_\mathcal{A} \). \( \square \)

3. The stability theorem

In this section we define arboreal Lagrangian and Legendrian subsets and prove their stability under symplectic reduction and Liouville cone operations.

### 3.1. Arboreal Lagrangians and Legendrians.

**Definition 3.1.** Arboreal Lagrangians and Legendrians are defined as follows:

(a) A closed subset \( L \subset X \) of a \( 2m \)-dimensional symplectic manifold \( (X, \omega) \) is called an

arboreal Lagrangian if the germ of \( (X, L) \) at any point \( \lambda \in L \) is symplectomorphic to the germ of the pair \( (T^* \mathbb{R}^n \times T^* \mathbb{R}^{m-n}, L_T \times \mathbb{R}^{m-n}) \) at the origin, for a signed rooted tree \( T \) with \( n := n(T) \leq m \).

(b) A closed subset \( \Lambda \subset Y \) of a \((2m + 1)\)-dimensional contact manifold \((Y, \xi)\) is called an

arboreal Legendrian if the germ of \( (Y, \Lambda) \) at any point \( \lambda \in \Lambda \) is contactomorphic to the germ of \( (J^1(\mathbb{R}^n \times \mathbb{R}^{m-n}) = J^1 \mathbb{R}^n \times T^* \mathbb{R}^{m-n}, \Lambda_T \times \mathbb{R}^{m-n}) \) at the origin, for a signed rooted tree \( T \) with \( n := n(T) \leq m \).

(c) A closed subset \( H \subset M \) of an \((m + 1)\)-dimensional manifold \( M \) is called an arboreal

front if the germ of \( (M, H) \) at any point \( m \in M \) is diffeomorphic to the germ of \( (\mathbb{R}^{n+1} \times \mathbb{R}^{m-n}, H_T \times \mathbb{R}^{m-n}) \) at the origin, for a signed rooted tree \( T \) with \( n := n(T) \leq m \).
The pair \((T, m)\) is called the arboreal type of the germ of \(L\), \(\Lambda\), or \(H\) at the given point. We say \(L\), \(\Lambda\), or \(H\) is positive if it is locally modeled on positive arboreal models at all points.

**Remark 3.2.** Later we will also allow arboreal Lagrangians to have boundary and even corners, but throughout the present discussion we restrict to the above definition for simplicity.

Given an arboreal Lagrangian we call \(\sup_{\lambda \in L} \{n(T(\lambda))\}\) the maximal order of \(L\), where \(T(\lambda)\) is a the signed rooted tree describing the germ of \(L\) at the point \(\lambda\). Similarly, we define the maximal order of arboreal Legendrians and fronts.

Every arboreal Lagrangian or Legendrian is naturally stratified by isotropic strata indexed by the corresponding tree type. A Lagrangian distribution \(\eta\) in \(X\) is called transverse to an arboreal Lagrangian \(L\) if it is transverse to all top-dimensional strata of \(L\). Similarly a Legendrian distribution \(\eta \subset \xi\) in a contact \((Y, \xi)\) is called transverse to an arboreal Legendrian \(\Lambda\) if it has trivial intersection with tangent planes to all top-dimensional strata of \(\Lambda\).

**Definition 3.3.** A polarization of \(L\) or \(\Lambda\) is a transverse Lagrangian distribution.

**Remark 3.4.** We emphasize the transversality to an arboreal Lagrangian means transversality to its closed smooth pieces, and not just to open strata.

Before we continue we introduce some auxiliary notions. Let \(V\) be a symplectic vector space and \(\ell_1, \ell_2, \ell_3 \subset V\) linear Lagrangian subspaces which are pairwise transverse. We write \(\ell_1 \prec \ell_2 \prec \ell_3\) if \(\ell_3\) corresponds to a positive definite quadratic form with respect to the polarization \((\ell_1, \ell_2)\) of \(V\). Let \(C \subset V\) be a coisotropic subspace. For any linear Lagrangian subspace \(\ell \subset V\) we denote by \([L]^C\) the symplectic reduction of \(\ell\) with respect to \(C\).

Let \(L\) be an arboreal Lagrangian whose germ at a point \(\lambda \in L\) has the type \((T, \rho, \varepsilon), m)\). Let \(L_\rho \subset T_\lambda X\) the tangent plane to the root Lagrangian corresponding to the root \(\rho\). For each vertex \(\alpha\) connected by an edge with \(\rho\) let \(L_\alpha \subset T_\lambda X\) denote the Lagrangian plane tangent to the Lagrangian corresponding to the vertex \(\alpha\). We recall that \(L_\rho\) and \(L_\alpha\) cleanly intersect along a codimension 1 subspace. Consider a coisotropic subspace \(C_\alpha := \text{Span}(L_\rho, L_\alpha) \subset T_\lambda X\). Let \(\eta\) be a Lagrangian distribution in \(X\) transverse to \(L\). Define the sign

\[
\varepsilon(\eta, L, \alpha) = \begin{cases} 
+1, & \text{if } [L_\rho]^{C_\alpha} \prec [L_\alpha]^{C_\alpha} \prec [\eta]^{C_\alpha}; \\
-1, & \text{if } [L_\rho]^{C_\alpha} \prec [\eta]^{C_\alpha} \prec [L_\alpha]^{C_\alpha}.
\end{cases}
\]

Similarly, if \(\Lambda\) is an arboreal Legendrian in a contact manifold \((Y, \xi)\), and \(\eta\) a Legendrian distribution transverse to \(\Lambda\), then for any point \(\lambda \in \Lambda\) of type \(T = (T, \rho, \varepsilon)\) we assign a sign \(\varepsilon(\eta, \Lambda, \alpha)\) for every vertex \(\alpha\) adjacent to the root \(\rho\) as equal to \(\pm 1\) depending on the \(\prec\)-order of the triple \([L_\rho]^{C_\alpha}, [L_\alpha]^{C_\alpha}, [\eta]^{C_\alpha}\) in \([\xi]^{C_\alpha}\).

**3.2. Stability of arboreal Lagrangians and Legendrians.** The following is the main result of this paper.
**Theorem 3.5.** Let $T$ be a signed rooted tree. Let $\rho_1, \ldots, \rho_k$ be vertices adjacent to the root $\rho$ and $T_j$ be subtrees with roots $\rho_j$ (where we removed the decoration of edges $[\rho_j \alpha]$). Let $\phi_j : \mathcal{F}^* \mathbb{R}^m \to J^1 \mathbb{R}^m$, $m \geq n = n(\mathcal{F})$, be germs of Weinstein hypersurface embeddings with disjoint images. Denote $z_j := \phi_j(0)$, $\Lambda^j = \phi_j(L_{T_j} \times \mathbb{R}^{m-n(T_j)})$, $j = 1, \ldots, k$. Suppose that

(i) $\pi(z_j) = 0$;

(ii) the arboreal Legendrian $\Lambda := \bigcup_{j=1}^k \Lambda^j$ projects transversely under the front projection $J^1 \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n$;

(iii) for each edge $[\rho_j \alpha]$ we have $\varepsilon(\nu, \Lambda^j, \alpha) = \varepsilon_{[\rho_j \alpha]}$.

Then $\mathbb{R}^m \cup C(\Lambda)$ is an arboreal Lagrangian of type $(T, m)$ or equivalently, the germ of the front $\pi(\Lambda)$ is diffeomorphic to $H_T \times \mathbb{R}^{m-n(T)}$.

From Theorem 3.5 it follows by induction that the class of arboreal singularities as axiomatically defined in Section 1 coincides with the class of arboreal singularities defined in terms of local models, which is equivalent to the statement of Theorem 1.2. Theorem 3.5 is a corollary of its unsigned version which is the content of the following proposition.

**Proposition 3.6.** Let $\mathcal{F}$ be a rooted tree. Let $\rho_1, \ldots, \rho_k$ be vertices adjacent to the root $\rho$ and let $T_j$ be subtrees with roots $\rho_j$. Let $\phi_j : \mathcal{F}^* \mathbb{R}^m \to J^1 \mathbb{R}^m$, $m \geq n = n(\mathcal{F})$, be germs of Weinstein hypersurface embeddings. Denote $z_j := \phi_j(0)$, $\Lambda^j = \phi_j(L_{T_j} \times \mathbb{R}^{m-n(T_j)})$, $j = 1, \ldots, k$. Suppose that

(i) $\pi(z_j) = 0$;

(ii) the extended arboreal Legendrian $\Lambda := \bigcup_{j=1}^k \Lambda^j$ projects transversely under the front projection $J^1 \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n$;

Then $\mathbb{R}^m \cup C(\Lambda)$ is an extended arboreal Lagrangian of type $(\mathcal{F}, m)$, or equivalently, the germ of the front $\pi(\Lambda)$ is diffeomorphic to $H_{\mathcal{F}} \times \mathbb{R}^{m-n(\mathcal{F})}$. 

![Figure 3.1. The notion of sign for the $A_2$ singularity.](image)
Proof of Theorem 3.5 using Proposition 3.6. Consider the arboreal Legendrian as a closed subcomplex of the extended model. Apply Proposition 3.6 to assume the extended front is in canonical form. Then Lemma 2.24 implies the front of the original arboreal Legendrian is a canonical model.

Proposition 3.6 will be proven below in this section (see Section 3.6) below, but first we discuss some corollaries of Theorem 3.5.

Corollary 3.7. Let \( \Lambda \subset \partial_\infty T^*M \) be an arboreal Legendrian. Suppose that the front projection \( \pi : \Lambda \to M \) is a transverse immersion. Then \( L := C(\Lambda) \cup M \) is an arboreal Lagrangian.

**Figure 3.2.** The zero section union the Liouville cone on a Legendrian whose projection to the zero section is an immersion is arboreal.

**Proof.** The intersection \( H := M \cap \overline{C(\Lambda)} \) is the front of the Legendrian \( \Lambda \). Each point \( a \in H \) has finitely many pre-images \( z_1, \ldots, z_k \in \Lambda \). The germs \( \Lambda^j \) of \( \Lambda \) at \( z_j \) by our assumption are images of arboreal Lagrangian models under Weinstein embeddings of their symplectic neighborhoods. Hence, by Theorem 3.5 the germ of \( L \) at \( z \) is of arboreal type.

It is not a priori clear that even the standard Lagrangian (resp. Legendrian) arboreal models are arboreal Lagrangians (resp. Legendrians). However, the following corollary shows that they are.

**Corollary 3.8.** Consider a model Lagrangian \( L_\mathcal{T} \subset T^*\mathbb{R}^n, n = n(\mathcal{T}) \). Then for any point \( \lambda \in L_\mathcal{T} \) the germ of \( L_\mathcal{T} \) at \( \lambda \) is a \((\mathcal{T}', n)\)-Lagrangian for a signed rooted tree \( \mathcal{T}' \).

**Proof.** We argue by induction in \( n \). The base of the induction is trivial. Assuming the claim for \( n - 1 \) we recall that \( L_\mathcal{T} \) can be presented as \( L_\rho \cup C(\Lambda) \), where \( L_\rho \) is the smooth piece corresponding to the root \( \rho \) of \( \mathcal{T} \) and \( \Lambda \) is a union of model Legendrians of dimension \( n - 1 \) in \( \partial_\infty T^*(\mathbb{R}^n) \). By the induction hypothesis \( \Lambda \) is an arboreal Legendrian, and hence applying Corollary 3.7 we conclude that \( L_\mathcal{T} \) is an arboreal Lagrangian.
Remark 3.9. We will not need it in what follows, so only briefly comment here that it is possible to specify precisely the type \((T', n)\) of the germ of \(LT\) at each point \(\lambda \in LT\). Following [N13] the underlying tree \(T'\) is a canonically defined subquotient of \(T\), in other words, a diagram \(T' \leftarrow S \rightarrow T\), where \(S \rightarrow T\) is a full subtree, and \(S \rightarrow T'\) contracts some edges; conversely, any such subquotient can occur. Furthermore, if we partially order \(T\) with the root \(\rho \in T\) as minimum, then the root \(\rho' \in T'\) is the unique minimum of the natural induced partial order on \(T'\). Finally, to equip \(T'\) with signs, we restrict the signs of \(T\) to the subtree \(S\), then push them forward to \(T'\) using that each edge of \(T'\) is the image of a unique edge of \(S\).

**Corollary 3.10.** Let \(LT \subset T^n\) be a model Lagrangian associated with a signed rooted tree \((T, \rho, \varepsilon)\). Let \(\eta_0, \eta_1\) be two polarizations transverse to \(LT\). Suppose that for any vertex \(\alpha\) of \(T\) adjacent to \(\rho\) we have

\[
\varepsilon(\eta_0, L, \alpha) = \varepsilon(\eta_1, L, \alpha).
\]

Then there is a (germ at the origin of) a symplectomorphism \(\psi : T^*\mathbb{R}^n \to T^*\mathbb{R}^n\) such that \(\psi(L) = L\) and \(d\psi(\eta_0) = \eta_1\) along \(L\).

**Proof.** There exists embeddings \(h_0, h_1 : T^*\mathbb{R}^n \to J^1\mathbb{R}^n\) as Weinstein hypersurfaces, such that \(h_j(\eta_j) = \nu_0, j = 0, 1\), where \(\nu_0\) is the canonical Legendrian foliation of \(J^1\mathbb{R}^n\) by fibers of the front projection to \(\mathbb{R}^n \times \mathbb{R}\). Consider the arboreal Lagrangians \(T_j := C(h_j(L_T)) \cup (\mathbb{R}^n \times \mathbb{R})\), \(j = 0, 1\), and note that their arboreal types are described by the same signed rooted tree \(\mathcal{T}\) obtained from \(\mathcal{T}\) by adding a new root, connecting it by an edge to the old one, and assigning to edges \([\rho\alpha]\) of \(\mathcal{T} \subset \mathcal{T}\) adjacent to the old root \(\rho\) the sign \(\varepsilon(\eta_0, L, \alpha) = \varepsilon(\eta_1, L, \alpha)\). Applying Theorem 3.5 we find the required symplectomorphism \(\psi\).

**Corollary 3.11.** Let \(H \subset M\) be an arboreal front. Then for any submanifold \(\Sigma \subset M\) transverse to (all strata of) \(H\) the intersection \(\Sigma \cap H\) is an arboreal front in \(\Sigma\).

**Proof.** We can assume that \(H\) is an arboreal front germ at a point \(x \in H\), and hence the germ of \((M, H)\) at \(x\) is diffeomorphic to the germ of \((\mathbb{R}^{n(T)}+1 \times \mathbb{R}^k, H_T \times \mathbb{R}^k)\) for some rooted signed arboreal tree \(T\) and \(k = n - n(T)\). Note that the transversality of \(\Sigma\) to \(H\) implies that \(\text{codim} \Sigma \leq k\) and that the projection of \(p : \Sigma \subset \mathbb{R}^{n(T)+1} \times \mathbb{R}^k \to \mathbb{R}^{n(T)+1}\) to the first factor is a submersion, and because we are dealing with germs, it is a trivial fibration. On the other hand, the projection \(p|_{\Sigma \cap H} : \Sigma \cap H \to H_T\) is the restriction of this fibration to \(H_T \subset \mathbb{R}^{N(T)}\).

3.3. **Parametric version.** The following is the parametric version of Theorem 3.5.

**Theorem 3.12.** Let \(T\) be a signed rooted tree. Let \(\rho_1, \ldots, \rho_k\) be vertices adjacent to the root \(\rho\) and \(T_j\) be subtrees with roots \(\rho_j\) (where we removed the decoration of edges \([\rho_j\alpha]\)). Let \(\phi_j^Y : \mathcal{T}^*\mathbb{R}^m \to J^1\mathbb{R}^m, m \geq n = n(\mathcal{T})\), be families of germs of Weinstein hypersurface embeddings with disjoint images, parametrized by a manifold \(Y\). Denote \(z_j^Y := \phi_j^Y(0), A_j^Y = \phi_j^Y(L_{T_j} \times \mathbb{R}^{m-n(T_j)})\), \(j = 1, \ldots, k\). Suppose that
Figure 3.3. Illustration that $\Sigma \cap H$ is an arboreal front in $\Sigma$.

(i) $\pi(z_y^j) = 0$;
(ii) the arboreal Legendrian $\Lambda_y := \bigcup_{j=1}^k \Lambda_y^j$ projects transversely under the front projection $J^1\mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n$;
(iii) for each edge $[\rho_j \alpha]$ we have $\varepsilon(\nu, \Lambda_y^j, \alpha) = \varepsilon[\rho_j \alpha]$.

Then there exists a family of diffeomorphisms $\phi_y$ between $H_T \times \mathbb{R}^{m-n(T)}$ and the front $\pi(\Lambda_y)$. If $K \subset Y$ is a closed subset and the $\phi_y$ are the standard embeddings of the local model for $y \in O_p(K)$, then we may further assume $\phi_y = \text{Id}$ for $y \in O_p(K)$.

The parametric version of Proposition 3.6 is formulated similarly. As a consequence of Theorem 3.12 we get the following result, which was formulated as Theorem 1.3 in Section 1.

**Corollary 3.13.** Fix a signed rooted tree $\mathcal{T} = (T, \rho, \varepsilon)$, set $n = |n(T)|$ and consider the arboreal $\mathcal{T}$-front $H_T \subset \mathbb{R}^n$. Let $D(\mathbb{R}^n, H_T)$ be the group of germs at $0$ of diffeomorphisms of $\mathbb{R}^n$ preserving $H_T$ as a front, i.e. as a subset along with its coorientation.

Then the fibers of the natural map $D(\mathbb{R}^n, H_T) \to \text{Aut}(\mathcal{T})$ are weakly contractible.

**Proof.** We deduce Corollary 3.13 from Theorem 3.12. We will argue for $\mathcal{T} = \mathcal{A}_{n+1}$ when $H_{\mathcal{A}_{n+1}} = \mathbb{n}^{-1}\Gamma$; the case of general $\mathcal{T}$ is similar.

Since $\text{Aut}(\mathcal{A}_{n+1})$ is trivial, we seek to show $D(\mathbb{R}^n, n^{-1}\Gamma)$ is weakly contractible. Note any $\varphi \in D(\mathbb{R}^n, n^{-1}\Gamma)$ preserves $0$, and moreover, preserves the canonical flag in $T_0\mathbb{R}^n$ given by the tangents to the intersections $\bigcap_{i<i_0} n^{-1}\Gamma_i$.

Let $D(\mathbb{R}^n)$ denote the group of germs at $0$ of diffeomorphisms of $\mathbb{R}^n$. Consider a $k$-sphere of maps $f_t \in D(\mathbb{R}^n, n^{-1}\Gamma)$, $t \in S^k$. Since all $f_t$ preserve $0$ and the canonical flag in $T_0\mathbb{R}^n$, there exists a $k+1$-ball of diffeomorphisms $g_t \in D(\mathbb{R}^n)$, $t \in B^{k+1}$, extending $f_t$. Applying Theorem 3.12 to the Weinstein hypersurface embeddings induced by $g_t$, we can find diffeomorphisms $h_t$ such that $h_t$ takes $g_t(n^{-1}\Gamma)$ back to $n^{-1}\Gamma$ and such that $h_t$ is the identity for $t \in S^k$. Then $h_t \circ g_t \in D(\mathbb{R}^n, n^{-1}\Gamma)$, $t \in B^{k+1}$, gives an extension of $f_t$ to the $k+1$-ball. \qed

We also formulate the parametric version of Corollary 3.10.
Corollary 3.14. Let $L_T \subset T^*\mathbb{R}^n$ be a model Lagrangian associated with a signed rooted tree $(T,\rho,\varepsilon)$. Let $\eta_0^y, \eta_1^y$ be two families of polarizations transverse to $L_T$ parametrized by a manifold $Y$. Suppose that for any vertex $\alpha$ of $T$ adjacent to $\rho$ we have

$$\varepsilon(\eta_0^y, L, \alpha) = \varepsilon(\eta_1^y, L, \alpha).$$

Then there is a family of (germ at the origin of) symplectomorphisms $\psi^y : T^*\mathbb{R}^n \to T^*\mathbb{R}^n$ such that $\psi^y(L) = L$ and $d\psi^y(\eta_0^y) = \eta_1^y$ along $L$. Moreover, if $\eta_0^y = \eta_1^y$ for $y \in \mathcal{O}p(K)$ for $K \subset Y$ a closed subset, then we can take $\psi^y = \text{Id}$ for $y \in \mathcal{O}p(K)$.

The proof is just like in the non-parametric case, but applying Theorem 3.12 instead of Theorem 3.5.

3.4. Tangency loci. Before proving Proposition 3.6 and its parametric analogue we need to analyze more closely the geometry of hypersurfaces forming arboreal fronts.

Definition 3.15. Given smooth hypersurfaces $X_1, X_2 \subset \mathbb{R}^{n+1}$, we denote by $T(X_1, X_2) \subset \mathbb{R}^{n+1}$ their tangency locus, i.e. the subset of points $x \in X_1 \cap X_2$ such that $T_x X_1 = T_x X_2$.

Remark 3.16. Given smooth Legendrians $L_1, L_2 \subset J^1\mathbb{R}^n$ whose fronts $X_1 = \pi(L_1), X_2 = \pi(L_2) \subset \mathbb{R}^{n+1}$ are smooth hypersurfaces, note that $T(X_1, X_2) = \pi(L_1 \cap L_2)$.

For $0 \leq j < i \leq n$, recall the notation

$$h_{i,j} := h_{i-j}(x_{j+1}, \ldots, x_i)$$

so in particular $h_{i,0} = h_i(x_1, \ldots, x_i)$ and $h_{i,i-1} = h_1(x_i) = x_i$. Set

$$T_{i,j} = \{h_{i,j} = 0\} \subset \mathbb{R}^{n+1}$$

Note $h_{i,j}$ is independent of $x_0, \ldots, x_j$, and we have

$$T_{i,j} = \mathbb{R}^{j+1} \times \mathbb{R}^{n-j-1} \cap \{h_{i,j} = 0\}$$

Lemma 3.17. For $0 \leq j < i \leq n$, the tangency locus $T^{n\Gamma_i, n\Gamma_j} \subset \mathbb{R}^{n+1}$ is the intersection of either $n\Gamma_i$ or $n\Gamma_j$ with the union

$$\{h_{i,j} = 0\} \cup \bigcup_{k=0}^{j-1} \{h_{i,k} = h_{j,k} = 0\} = T_{i,j} \cup \bigcup_{k=0}^{j-1} (T_{i,k} \cap T_{j,k})$$

Proof. Since $n\Gamma_i, n\Gamma_j$ are the graphs of $h_i^2, h_j^2$, the projection of $T^{n\Gamma_i, n\Gamma_j}$ to the domain $\mathbb{R}^n$ is cut out by

$$h_i^2 = h_j^2 \quad \text{and} \quad dh_i^2 = dh_j^2$$

Note $h_i = h_{i,0} = x_1 - h_{i,1}, h_j = h_{j,0} = x_1 - h_{j,1}$. By examining the $dx_1$-component of $dh_i^2 = dh_j^2$, we see it implies $h_i = h_j$. Thus the projection of $T^{n\Gamma_i, n\Gamma_j}$ is cut out by the single equation $dh_i^2 = dh_j^2$ which in turn implies $h_i = h_j$.

To satisfy $dh_i^2 = dh_j^2$, so in particular $h_i = h_j$, there are two possibilities: (i) $h_i = h_j = 0$; or (ii) $h_i = h_j \neq 0$. In case (i), we find the subset $\{h_{i,0} = h_{j,0} = 0\}$ appearing in the union of the
assertion of the lemma. In case (ii), we observe \( dh_i^2 = dh_j^2 \) is then equivalent to \( dh_{i,1}^2 = dh_{j,1}^2 \) which in turn implies \( h_{i,1} = h_{j,1} \).

Now we repeat the argument. To satisfy \( dh_{i,1}^2 = dh_{j,1}^2 \), so in particular \( h_{i,1} = h_{j,1} \), there are two possibilities: (i) \( h_{i,1} = h_{j,1} = 0 \); or (ii) \( h_{i,1} = h_{j,1} \neq 0 \). In case (i), we find the subset \( \{ h_{i,1} = h_{j,1} = 0 \} \) appearing in the union of the assertion of the lemma. In case (ii), we observe \( dh_{i,1}^2 = dh_{j,1}^2 \) is then equivalent to \( dh_{i,2}^2 = dh_{j,2}^2 \) which in turn implies \( h_{i,2} = h_{j,2} \).

Iterating this argument, we obtain the subset \( \bigcup_{k=0}^{j-1} \{ h_{i,k} = h_{j,k} = 0 \} \), and arrive at the final equation \( dh_{i,j}^2 = 0 \). By examining the \( dx_{j+1} \)-term, we see \( dh_{i,j}^2 = 0 \) holds if and only if \( h_{i,j} = 0 \), which gives the remaining subset of the assertion of the lemma. \( \square \)

Remark 3.18. The only evident redundancy in the description of the lemma is \( T_{i,j-1} \cap T_{j,j-1} \subset T_{i,j} \) since \( h_{i,j-1} = x_j - h_{i,j}^2 \), \( h_{j,j-1} = x_j \), so their vanishing implies the vanishing of \( h_{i,j} \).

We will be particularly interested in the locus \( T_{i,j} \subset T(\text{^n}_i, \text{^n}_j) \) and formalize its structure in the following definition.

**Definition 3.19.** Given smooth hypersurfaces \( X_1, X_2 \subset \mathbb{R}^{n+1} \), we denote by \( \tau^o(X_1, X_2) \subset T(X_1, X_2) \) the subset of points \( x \in X_1 \cap X_2 \) where in some local coordinates we have \( X_1 = \{ x_0 = 0 \} \), \( X_2 = \{ x_0 = x_1^2 \} \). We write \( \tau(X_1, X_2) \subset T(X_1, X_2) \) for the closure of \( \tau^o(X_1, X_2) \), and refer to it as the primary tangency of \( X_1, X_2 \).

Remark 3.20. Given smooth Legendrians \( L_1, L_2 \subset J^1 \mathbb{R}^{n} \) whose fronts \( X_1 = \pi(L_1), X_2 = \pi(L_2) \subset \mathbb{R}^{n+1} \) are smooth hypersurfaces, note that \( \tau^o(X_1, X_2) \) is the front projection of where \( L_1, L_2 \) intersect cleanly in codimension one.

We have the following consequence of Lemma 3.17.

**Corollary 3.21.** For \( 0 \leq j < i \leq n \), the primary tangency \( \tau(\text{^n}_i, \text{^n}_j) \subset \mathbb{R}^{n+1} \) is the intersection of either \( \text{^n}_i \) or \( \text{^n}_j \) with \( T_{i,j} \).

Before continuing, let us record the following for future use.

**Lemma 3.22.** Fix \( 0 \leq k < j \leq n - 1 \).

We have

\[
\tau(\Gamma_n, \text{^n}_k) \cap \tau(\text{^n}_j, \text{^n}_k) = \tau(\text{^n}_n, \text{^n}_j) \cap \tau(\text{^n}_j, \text{^n}_k)
\]

where the primary tangency of \( \tau(\Gamma_n, \text{^n}_k) \), \( \tau(\text{^n}_j, \text{^n}_k) \) of the left hand side is calculated in \( \text{^n}_k \approx \mathbb{R}^n \).

**Proof.** By the preceding corollary, the left hand side is the intersection \( \text{^n}_k \cap \tau(T_{n,k}, T_{j,k}) \).

Note \( \text{^n}_k \cap T_{j,k} = \tau(\text{^n}_j, \text{^n}_k) = \text{^n}_j \cap T_{j,k} \). Hence

\[
\text{^n}_k \cap \tau(T_{n,k}, T_{j,k}) = \tau(\text{^n}_j, \text{^n}_k) \cap \tau(T_{n,k}, T_{j,k})
\]

since \( y \in \text{^n}_k \cap \tau(T_{n,k}, T_{j,k}) \iff y \in \text{^n}_k \cap T_{j,k}, y \in \tau(T_{n,k}, T_{j,k}) \iff y \in \text{^n}_j \cap T_{j,k}, y \in \tau(T_{n,k}, T_{j,k}) \iff y \in \text{^n}_j \cap T_{j,k} \).
Figure 3.4. Verification of the conclusion of Lemma 3.22 for \( n = 2 \), in this case both the right and left hand sides of the equality \( \tau(\tau(2\Gamma_2, 2\Gamma_0), \tau(2\Gamma_1, 2\Gamma_0)) = \tau(2\Gamma_2, 2\Gamma_1) \cap \tau(2\Gamma_1, 2\Gamma_0) \) consist of the origin.

Next, recall
\[
T_{n,k} = \mathbb{R}^{k+1} \times \frac{n-k-1}{n-1} \mathbb{R}_{n-k-1} \\
T_{j,k} = \mathbb{R}^{k+1} \times \frac{n-k-1}{n-1} \mathbb{R}_{j-k-1}
\]

Hence by the preceding corollary, we have
\[
\tau(T_{n,k}, T_{j,k}) = T_{j,k} \cap \{ h_{n,j} = 0 \}
\]

Thus the left hand side is given by \( n\Gamma_j \cap T_{j,k} \cap T_{n,j} \).

On the other hand, by the preceding corollary, the right hand side is also given by \( n\Gamma_j \cap T_{n,j} \cap T_{j,k} \). \( \square \)

3.4.1. More on distinguished quadrants.

**Corollary 3.23.** For \( 0 \leq j < i \leq n \), we have
\[
{n\Gamma}_i \cap {n\Gamma}_j = T({n\Gamma}_i^c, {n\Gamma}_j^c) = \tau({n\Gamma}_i^c, {n\Gamma}_j^c)
\]

and they coincide with the closed boundary face of \( {n\Gamma}_i^c \) cut out by \( h_{i,j} = 0 \).

**Proof.** For \( j = 0 \), we have \( {n\Gamma}_0^c = {n\Gamma}_0 = \{ x_0 = 0 \} \). From the definitions, we have
\[
{n\Gamma}_i \cap {n\Gamma}_0 = T({n\Gamma}_i^c, {n\Gamma}_0) = \tau({n\Gamma}_i^c, {n\Gamma}_0)
\]

which is cut out of \( nP_i^c \) by \( h_{i,0} = h_i = 0 \).

For \( j > 0 \), the assertions follow from Lemma 2.4 by induction on \( n \). \( \square \)

**Remark 3.24.** Note for any \( 0 \leq j < i \leq n \), we have
\[
\tau({n\Gamma}_i, {n\Gamma}_j) = \bigcup_{\epsilon} \tau({n\Gamma}_i^\epsilon, {n\Gamma}_j^\epsilon)
\]
To see this, consider \( x \in \tau(n_1, n_j) \), so that \( h_{i,j}(x) = 0 \) by Corollary 3.21. Choose \( \varepsilon \) so that \( x \in n_{i,j} \). Then by Corollary 3.23, we have \( x \in \tau(n_{i,j}, n_{i,j}) \).

For \( i = 0 \), let \( n_{i,0} = \mathbb{R}^n \subset T^*\mathbb{R}^n \) denote the zero-section. For \( i = 1, \ldots, n \), consider the conormal bundles

\[
\gamma_i = T_{n-1}n_i \subset T^*\mathbb{R}^n
\]

and their union

\[
\gamma = \bigcup_{i=0}^n \gamma_i
\]

Similarly, for \( i = 0, \ldots, n \), consider the smooth Legendrian

\[
\Lambda_i \subset J^1\mathbb{R}^n
\]

that maps diffeomorphically to \( n_{i,j} \subset \mathbb{R}^{n+1} \) under the front projection \( \pi : J^1\mathbb{R}^n \rightarrow \mathbb{R}^{n+1} \), and their union

\[
\Lambda = \bigcup_{i=0}^n \Lambda_i
\]

Note the contactomorphism of Lemma 2.1 takes \( \Lambda_i \subset J^1\mathbb{R}^n \) isomorphically to \( \{0\} \times n_i \subset \{0\} \times T^*\mathbb{R}^n \), and thus \( \Lambda \subset J^1\mathbb{R}^n \) isomorphically to \( \{0\} \times \gamma \subset \{0\} \times T^*\mathbb{R}^n \).

We have the following topological consequence of Lemma 2.4.

**Corollary 3.25.** As a union of smooth manifolds with corners, \( n_{i,j} \subset \mathbb{R}^{n+1} \) is given by the gluing

\[
n_{i,j} = \left((n_{i,j} \times \mathbb{R} \setminus \{0\}) \bigcup_{n_{i,j} \times \{0\}} \mathbb{R} \right) \times \{0\}
\]

where \( \varepsilon' = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n) \). The front projection takes \( n_i \subset J^1\mathbb{R}^n \) homeomorphically to \( n_{i,j} \subset \mathbb{R}^{n+1} \).

Before continuing, let us record the following for future use.

**Corollary 3.26.** For \( 0 < j < i \leq n \), the closure of the codimension one clean intersection of \( n_{i,j} \) is precisely \( n_{i,j} \).

**Proof.** The closure of the codimension one clean intersection of \( n_{i,j} \) is conic and projects to the primary tangency of \( n_{i-1,j-1} \). By Corollary 3.21, the primary tangency of \( n_{i-1,j-1} \) is cut out by \( h_{i,j-1} = 0 \). By Corollary 3.23, this is precisely the tangency \( T(n_{i-1,j-1}) \) and hence lifts precisely to the conic intersection \( n_{i,j} \).

**3.5. The case of \( \mathcal{A}_{n+1} \)-tree.** The following Theorem 3.27 will play a key role in proving Proposition 3.6.

**Theorem 3.27.** Let \( \varphi : T^*\mathbb{R}^n \rightarrow J^1\mathbb{R}^n \) be an embedding as a Weinstein hypersurface. Assume that the image of \( n \) under \( \varphi \) is transverse to the fibers of the projection \( J^1\mathbb{R}^n \rightarrow \mathbb{R}^n \). Let \( \mathcal{Y} = \pi(\varphi(n)) \subset \mathbb{R} \times \mathbb{R}^n \) be (the germ of) the front at the central point.

Then there exists a diffeomorphism \( \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R} \times \mathbb{R}^n \) taking \( \mathcal{Y} \) to the germ at the origin of \( n \subset \mathbb{R} \times \mathbb{R}^n \).
The proof of Theorem 3.27 will proceed by induction on the dimension \( n \). At each stage, we will prove the fully parametric version:

**Theorem 3.28.** Let \( \varphi^y : T^*\mathbb{R}^n \to J^1\mathbb{R}^n \) be a family of Weinstein hypersurface embeddings parametrized by a manifold \( Y \). Assume that the image of \( ^nL \) under \( \varphi^y \) is transverse to the fibers of the projection \( J^1\mathbb{R}^n \to \mathbb{R}^n \). Let \( \Upsilon^y = \pi(\varphi^y(\cdot L)) \subset \mathbb{R} \times \mathbb{R}^n \) be (the germs of) the fronts at the central points.

Then there exists a family of diffeomorphisms \( \psi^y : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n \) taking \( \Upsilon^y \) to the germ at the origin of \( ^n\Gamma \subset \mathbb{R} \times \mathbb{R}^n \). If \( \varphi^y = \text{Id} \) for \( y \in \mathcal{O}_p(K) \), where \( K \subset Y \) is a closed subset, then we may assume \( \psi^y = \text{Id} \) for \( y \in \mathcal{O}_p(K) \).

As usual the case of general pairs \((Y,K)\) follows from the case \( Y = D^k \) and \( K = S^{k-1} \).

**Base case** \( n = 0 \). The \( k \)-parametric version states: the germ of any graphical hypersurface \( \Upsilon \subset \mathbb{R} \times \mathbb{R}^k \) is diffeomorphic to the germ of the zero-graph \( ^0\Gamma \times \mathbb{R}^k = \{0\} \times \mathbb{R}^k \). This can be achieved by an isotopy generated by a time-dependent vector field of the form \( h_t \partial_{x_0} \). This vector field is zero at infinity if \( \Upsilon \) is standard at infinity.

**Case** \( n = 1 \). The next case of the induction \( n = 1 \) is elementary but slightly different from the others, so it is more convenient to treat separately.

With the setup of the theorem, consider the front \( \Upsilon = \pi(\cdot \Lambda) \subset \mathbb{R}^2 \), and assume without loss of generality that the origin is the central point. By induction, we may assume, the front takes the form \( \Upsilon = \Gamma_0 \cup \Upsilon_1 \subset \mathbb{R}^2 \) where \( \Gamma_0 = \{x_0 = 0\} \). Near the origin, the intersection \( \Gamma_0 \cap \Upsilon_1 \) and tangency locus \( T(\Gamma_0, \Upsilon_1) \) coincide and consist of the origin alone. Moreover, by construction, the origin is a simple tangency, and so \( \Upsilon_1 = \{x_0 = \alpha x_1^2\} \) with \( \alpha(0) \neq 0 \). Now it is elementary to find a time-dependent vector field of the form \( h_t x_1 \partial_{x_1} \), hence vanishing on \( \Gamma_0 \), generating an isotopy taking \( \Upsilon_1 \) to either \( \Gamma_1 = \{x_0 = x_1^2\} \) or \( -\Gamma_1 = \{x_0 = -x_1^2\} \). In the former case, we are done; in the latter case, we may apply the diffeomorphism \((x_0,x_1) \mapsto (-x_0,x_1)\) to arrive at the configuration \( \Gamma_0 \cup \Gamma_1 \). Finally, it is evident the prior constructions can be performed parametrically, with the vector field zero at infinity if \( \Upsilon \) is standard at infinity.

**Inductive step.** The inductive step takes the following form. Suppose the fully parametric assertion has been established for dimension \( n - 1 \). Starting from \( \cdot \Lambda \subset T^*\mathbb{R}^n \), remove the last smooth piece to obtain \( \cdot \Lambda' = \cdot \Lambda \setminus \cdot \Lambda_n \), and consider the corresponding front \( \Upsilon' = \pi(\cdot \Lambda') \). Note that \( \cdot \Lambda' = (\cdot \Lambda - 1) \times \mathbb{R} \subset T^*(\mathbb{R}^{n-1} \times \mathbb{R}) \), and so by an inductive application of the \( 1 \)-parametric version of the theorem, we may assume

\[
\Upsilon' = \cdot \Lambda' \times \mathbb{R}
\]

Set \( \Upsilon_n = \pi(\cdot \Lambda_n) \). We will find a diffeomorphism \( \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \) that preserves \( \Upsilon' \) (as a subset, not pointwise), and takes \( \Upsilon_n \) to \( ^n\Gamma_n \). Moreover, it will be evident the diffeomorphism can be constructed in parametric form, including the relative parametric form. This will complete the inductive step and prove the theorem.
3.5.1. Two propositions. The proof of the inductive step is based on the following 2 propositions.

**Proposition 3.29.** Fix $n \geq 2$.

With the setup of Theorem 3.27, suppose $\Upsilon = \bigcup_{i=0}^{n-1} \Gamma_i \cup \Upsilon_n$ where $\Upsilon_n = \pi(\mathcal{N}_n)$. Suppose in addition $\Upsilon_n$ has primary tangency loci satisfying

$$
\tau(\Upsilon_n, {n\Gamma}_i) \supset \tau(\mathcal{N}_n, {n\Gamma}_i) \quad i = 0, \ldots, n-1
$$

Then $\Upsilon_n = \{x_0 = \alpha h_n^2\}$ where

$$
\alpha = 1 + \beta \prod_{j=1}^{n-1} h_n^2_{n,j} = 1 + \beta h_n^2_{n,1} \cdots h_n^2_{n,n-1}
$$

Moreover, the same holds in parametric form.

**Proof.** We have $\Upsilon_n = \{x_0 = g\}$ for some $g$. Since $\tau(\Upsilon_n, {n\Gamma}_0) \supset \tau(\mathcal{N}_n, {n\Gamma}_0) = \{h_n = 0\}$, we must have $g$ is divisible by $h_n^2$, hence $g = \alpha h_n^2$, for some $\alpha$. Next, for any $j \neq 0, n$, by Lemma 3.17, $\tau(\mathcal{N}_n, {n\Gamma}_j)$ is cut out by $h_{n,j} = 0$. Since $\tau(\Upsilon_n, {n\Gamma}_j) \supset \tau(\mathcal{N}_n, {n\Gamma}_j)$, and $h_{n,j} \neq 0$ along a dense subset of $\{h_{n,j} = 0\}$, taking the ratio $g/h_n^2$ shows we must have $\alpha = 1 + h_{n,j}^2 \beta_j$, for some $\beta_j$. Repeating this argument, and using the transversality of the level-sets of the collection $h_{n,j}$, we arrive at the assertion. \qed

**Proposition 3.30.** Fix $n \geq 2$.

With the setup of Theorem 3.27, suppose $\Upsilon = \bigcup_{i=0}^{n-1} \Gamma_i \cup \Upsilon_n$ where $\Upsilon_n = \pi(\mathcal{N}_n)$. Suppose in addition $\Upsilon_n = \{x_0 = \alpha h_n^2\}$ where

$$
\alpha = 1 + \beta \prod_{j=1}^{n-1} h_n^2_{n,j} = 1 + \beta h_n^2_{n,1} \cdots h_n^2_{n,n-1}
$$

Consider the family $\Upsilon_n = \{x_0 = (1 - t + t\alpha) h_n^2\}$ so that $\Upsilon_n^0 = {n\Gamma}_n, \Upsilon_n^1 = \Upsilon_n$.

Then there exist functions $h_t : \mathbb{R}^{n+1} \to \mathbb{R}$ such that the vector fields

$$
h_t v_{n-1} = h_t \sum_{i=0}^{n-1} x_i \frac{1}{2^i} \partial x_i = h_t x_0 \partial x_0 + \frac{1}{2} h_t x_1 \partial x_1 + \cdots + \frac{1}{2^{n-1}} h_t x_{n-2} \partial x_{n-2}
$$

generate an isotopy $\varphi_t : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that $\varphi_t(\Upsilon_n^0) = \Upsilon_n^t$.

In addition, the functions $h_t$, hence vector fields $h_t v_{n-1}$, are divisible by the product $\prod_{j=1}^{n-1} h_{n,j}$.

Moreover, all of the above holds in parametric form.

The following lemmas are needed for the proof of Proposition 3.30.

**Lemma 3.31.** For all $0 \leq i \leq n$, the vector field

$$
v_i = \sum_{j=0}^{n} x_j \frac{1}{2^j} \partial x_j = x_0 \partial x_0 + \frac{1}{2} x_1 \partial x_1 + \cdots + \frac{1}{2^i} x_i \partial x_i
$$

preserves each $n\Gamma_j \subset \mathbb{R}^{n+1}$, for $j = 0, \ldots, i$. 
Proof. Since \( n \Gamma_j \subset \mathbb{R}^{n+1} \) is independent of \( x_{j+1}, \ldots, x_n \), it suffices to prove the case \( i = j = n \). Recall \( n \Gamma_n \) is the zero-locus of \( f = x_0 - h^2_n \). We will show \( v(h_n) = \frac{1}{2}h_n \) and so \( v(f) = f \).

Recall \( h_n = h_{n,0} = x_1 - h^2_n \), and in general \( h_{n,j} = x_{j+1} - h^2_{n,j+1} \) with \( h_{n,-1} = x_n \). Thus \( v_n(h_{n,-1}) = \frac{1}{2}h_{n,-1} \), and by induction, \( v(h_{n,j}) = \frac{1}{2}h_{n,j} \), so in particular \( v(h_{n,0}) = v(h_n) = \frac{1}{2}h_n. \)

\( \square \)

Remark 3.32. In the context of the inductive step outlined above, we will use Lemma 3.31 in particular the vector field

\[
\sum_{i=0}^{n-1} x_i \frac{\partial}{\partial x_i} = x_0 \frac{\partial}{\partial x_0} + \frac{1}{2} x_1 \frac{\partial}{\partial x_1} + \cdots + \frac{1}{2^{n-1}} x_{n-1} \frac{\partial}{\partial x_{n-1}}
\]

to move \( \Upsilon_n \) to \( n \Gamma_n \). The lemma confirms we will preserve \( \Upsilon' = \bigcup_{i=0}^{n-1} n \Gamma_i \).

Lemma 3.33. For any \( 0 \leq j < i \leq n \), and \( 1 \leq k \leq i \), we have

\[
\frac{\partial h^2_i}{\partial x_k} = -(-2)^k \prod_{j=0}^{k-1} h_{i,j} = -(-2)^k h_{i,0} h_{i,1} \cdots h_{i,k-1}
\]

Proof. Recall \( h_i = h_{i,0} \) and the inductive formulas \( h_{i,j} = x_{j+1} - h^2_{i,j+1} \) with \( h_{i,-1} = x_i \). Thus we have

\[
\frac{\partial h^2_{i,j}}{\partial x_{j+1}} = 2h_{i,j} \quad \frac{\partial h^2_{i,j}}{\partial x_k} = -2h_{i,j} \frac{\partial h^2_{i,j+1}}{\partial x_k} \quad k > j + 1
\]

and the assertion follows. \( \square \)

Proof of Proposition 3.30. Suppose \( \Upsilon = \bigcup_{i=0}^{n-1} n \Gamma_i \cup \Upsilon_n \) where \( \Upsilon_n \) is the graph of

\[
h_\beta = (1 + \beta \prod_{j=1}^{n-1} h^2_{n,j}) h^2_n = (1 + \beta h^2_{n,1} \cdots h^2_{n,n-1}) h^2_n
\]

Our aim is to find a normalizing isotopy, generated by a time-dependent vector field \( v_t \), taking the graph \( \Upsilon_n = \{ x_0 = h_\beta \} \) to the standard graph \( n \Gamma_n = \{ x_0 = h^2_n \} \), i.e. to the graph where \( \beta = 0 \), while preserving \( \bigcup_{i=0}^{n-1} n \Gamma_i \). Thus for any infinitesimal deformation in the class of functions \( h_\beta \), we seek a vector field \( v \) realizing the deformation and preserving the functions \( h_0, \ldots, h_{n-1} \), i.e. we seek to solve the system

\[
\begin{align*}
\dot{h}_i &= 0, \quad i = 0, \ldots, n-1 \\
\dot{h}_\beta &= \gamma \prod_{j=0}^{n-1} h^2_{n,j} = \gamma h^2_{n,0} \cdots h^2_{n,n-1}
\end{align*}
\]

where \( \dot{h}_\beta \) denotes the derivative of \( h_\beta \) with respect to \( v \), and \( \gamma \) is any given smooth function.

Let \( \Lambda_\beta \subset T^* \mathbb{R}^{n+1} \) denote the conormal to the graph of \( h_\beta \). Any vector field \( v = \sum_{j=0}^{n} v_j \partial/\partial x_j \) on \( \mathbb{R}^{n+1} \) extends to a Hamiltonian vector field \( v_H \) on \( T^* \mathbb{R}^{n+1} \) with Hamiltonian \( H = \sum_{j=0}^{n} p_j v_j \). We will find \( v \) deforming the graph of \( h_\beta \) by finding \( H \) so that \( v_H \) deforms the conormal to the graph \( \Lambda_\beta \).
In general, for a function \( f : \mathbb{R}^n \to \mathbb{R} \), with graph \( \Gamma_f = \{ x_0 = f \} \subset \mathbb{R}^{n+1} \), denote the conormal to the graph by \( T^*_\Gamma_f \subset T^*\mathbb{R}^{n+1} \). With respect to the contact form \( p_1 dx_1 + \ldots + p_n dx_n - x_0 dp_0 \), the conormal \( T^*_\Gamma_f \) is given by the generating function \( F(x_1, \ldots, x_n) = -p_0 f(x_1, \ldots, x_n) \), i.e. it is cut out by the equations

\[
p_i = -p_0 \frac{\partial f}{\partial x_i}, \quad i = 1, \ldots, n
\]

\[
x_0 = f(x_1, \ldots, x_n)
\]

Hence given a Hamiltonian \( H = \sum_{j=0}^n p_j v_j \), its restriction to the conormal \( T^*_\Gamma_f \) is given by

\[
H|_{T^*_\Gamma_f} = p_0 v_0|_{x_0=f} - p_0 \sum_{j=1}^n \frac{\partial f}{\partial x_j} v_j|_{x_0=f} = v_0|_{x_0=f} - \dot{f}
\]

Let us apply the above to \( h_\beta \) and \( h_i \), for \( i = 0, \ldots, n-1 \). It allows us to transform system (2) into the system

\[
v_0(x_1, \ldots, x_n, h_i) - \sum_{j=1}^n \frac{\partial h_i}{\partial x_j} v_j = 0, \quad i = 0, \ldots, n-1
\]

(3)

\[
v_0(x_1, \ldots, x_n, h_\beta) - \sum_{j=1}^n \frac{\partial h_\beta}{\partial x_j} v_j = \gamma \prod_{j=0}^{n-1} h_{n,j}^2
\]

Note we can reformulate Lemma 3.31 from this viewpoint: when \( \beta = \gamma = 0 \), given any function \( h = h(x_1, \ldots, x_n) \), the functions

(4) \[
v_0 = x_0 h, v_1 = \frac{x_1}{2} h, v_2 = \frac{x_2}{4} h, \ldots, v_n = \frac{x_n}{2^n} h
\]

satisfy system (3).

Now let us choose \( v_0, v_1, \ldots, v_{n-1} \) as in (4) but set \( v_n = 0 \). This will satisfy the first \( n \) equations of system (3), independently of \( \beta, \gamma \). From hereon, we will restrict to this class of vector fields and focus on the last equation of system (3).

Let us first set \( \beta = 0 \), so that \( h_\beta = h_n^2 \), and solve system (3) in this case. Using Lemma 3.33, we can then rewrite the left-hand side of the last equation of system (3) in the form

\[
v_0(x_1, \ldots, x_n, h_n^2) - \sum_{j=1}^{n-1} \frac{\partial h_n^2}{\partial x_j} v_j = h \left( h_n^2 - \sum_{j=1}^{n-1} \frac{\partial h_n^2}{\partial x_j} 2^j \right) = h \left( h_n^2 + \sum_{j=1}^{n-1} (-1)^j x_j \prod_{k=0}^{j-1} h_{n,k} \right)
\]
Using $h_n = h_{n,0}, h_{n,k} - x_{k+1} = -h_{n,k-1}^2$, we can inductively simplify the term in parentheses

\[
h_n^2 + \sum_{j=1}^{n-1} (-1)^j x_j \prod_{k=0}^{j-1} h_{n,k} = h_n(h_n - x_1 + \sum_{j=2}^{n-1} (-1)^j x_j \prod_{k=1}^{j-1} h_{n,k})
\]

\[
= h_n(-h_{n,1}^2 + \sum_{j=2}^{n-1} (-1)^j x_j \prod_{k=1}^{j-1} h_{n,k})
\]

\[
= h_n h_{n,1}(-h_{n,1} + x_2 + \sum_{j=3}^{n-1} (-1)^j x_j \prod_{k=2}^{j-1} h_{n,k})
\]

\[\cdots\]

\[
= (-1)^{n-1} h_{n,1} h_{n,2} \cdots h_{n,n-1} = (-1)^{n-1} \prod_{j=0}^{n-1} h_{n,j}
\]

Thus for $\beta = 0$, the last equation of system (3) reduces to

\[
(-1)^{n-1} h \prod_{j=0}^{n-1} h_{n,j} = \gamma \prod_{j=0}^{n-1} h_{n,j}^2
\]

and hence can be solved by

\[
h = (-1)^{n-1} \gamma \prod_{j=0}^{n-1} h_{n,j}
\]

Now for general $\beta$, we will similarly calculate the left-hand side of the last equation of system (3). To simplify the formulas, set

\[
F = \prod_{j=0}^{n-1} h_{n,j} \quad \theta = \beta F^2
\]

Thus we have $h_{\beta} = (1 + \theta)h_n^2$, and our prior calculation showed when $\beta = 0$, the last equation of system (3) took the form

\[
(-1)^{n-1} h F = \gamma F^2
\]

so was solved by $h = (-1)^{n-1} \gamma F$.

For general $\beta$, after factoring out the function $h$ to be solved for, the left-hand side of the last equation of system (3) takes the form

\[
(-1)^{n-1}(1 + \theta) F - h_n^2 \sum_{j=1}^{n-1} \frac{1}{2j} \frac{\partial \theta}{\partial x_j} x_j
\]

Thus the equation itself takes the form

\[
((-1)^{n-1}(1 + \theta) F - h_n^2 \sum_{j=1}^{n-1} \frac{1}{2j} \frac{\partial \theta}{\partial x_j} x_j) h = \gamma F^2
\]
Since $\theta = \beta F^2$, we have

$$\frac{\partial \theta}{\partial x_j} = F^2 \frac{\partial \beta}{\partial q_j} + \beta \frac{\partial F^2}{\partial q_j} = F^2 \frac{\partial \beta}{\partial q_j} + 2F\beta \frac{\partial F}{\partial q_j}$$

and hence $\frac{\partial \theta}{\partial x_j}$ is divisible by $F$. Thus we can divide equation (5) by $F$, and after renaming $\gamma$, write equation (5) in the form

$$(1 + O(x))h = \gamma F$$

where $O(x)$ vanishes at the origin. We conclude we can solve the equation by $h = (1 + O(x))^{-1}\gamma F$.

This completes the proof of Proposition 3.30. \qed

3.5.2. Proof of Theorem 3.27. In this section, we use Propositions 3.29 and Proposition 3.30 to complete the inductive step outlined in 3.5, and thus, complete the proof of Theorem 3.27. Let us assume $n \geq 2$.

Then $\Upsilon = \Upsilon' \cup \Upsilon_n$ where $\Upsilon' = \bigcup_{i=0}^{n-1} \Gamma_i$, $\Upsilon_n = \pi(\Lambda_n)$. We will implement the following strategy. Suppose for some $0 < k < n - 1$, we have moved $\Upsilon_n$, while preserving $\Upsilon'$, so that we have the relation of primary tangencies

$$\tau(\Upsilon_n, \Gamma_j) \supset \tau(\Upsilon_n, \Gamma_j)$$

Then using Proposition 3.29 and Proposition 3.30, or alternatively, the cases $n = 0, 1$ when respectively $k = n - 1, n - 2$, we will move $\Upsilon_n$, while preserving $\Upsilon'$, so that we have the relation of primary tangencies

$$\tau(\Upsilon_n, \Gamma_j) \supset \tau(\Upsilon_n, \Gamma_j)$$

Proceeding in this way, we will arrive at $k = 0$, where all primary tangencies have been normalized. Then a final application of Proposition 3.29 and Proposition 3.30 will complete the proof.

To pursue this argument, we need the following control over primary tangencies.

**Lemma 3.34.** Fix $0 \leq k < j \leq n - 1$.

We have

$$\tau(\pi(\Upsilon_n, \Gamma_k), \tau(\Upsilon_n, \Gamma_j)) \supset \tau(\Upsilon_n, \Gamma_j) \cap \tau(\Upsilon_n, \Gamma_k)$$

Moreover, when $k = n - 2$, the tangency of $\tau(\Upsilon_n, \Gamma_{n-2})$ and $\tau(\Upsilon_n, \Gamma_{n-2})$ is nondegenerate.

**Proof.** We will assume $k > 0$ and leave the case $k = 0$ as an exercise.

Fix a point

$$y \in \tau(\Upsilon_n, \Gamma_j) \cap \tau(\Upsilon_n, \Gamma_k)$$

In particular $y \in \Upsilon_n$ and so $y = \pi(\tilde{y})$ for some $\tilde{y} \in \Lambda_n$. Recall $\Lambda_n = \bigcup_\varepsilon \Lambda_n^\varepsilon$ and so $\tilde{y} \in \Lambda_n^\varepsilon$, for some $\varepsilon$. 
Figure 3.5. The strategy of the proof: inductively normalize tangencies.

Note $y \in \tau(\Gamma_n, \Gamma_j)$ implies $\tilde{y}$ is in the closure of the clean codimension one intersection of $\Lambda_n \cap \Lambda_j$.

By Corollary 3.26, this locus intersects $\Lambda_n \cap \Lambda_j$ precisely along $\Lambda_n \cap \Lambda_j$ and so $\tilde{y} \in \Lambda_n \cap \Lambda_j$.

Similarly, note $y \in \tau(\Gamma_n, \Gamma_j)$ implies $\tilde{y}$ is in the closure of the clean codimension one intersection of $\Lambda_n \cap \Lambda_j$. By Corollary 3.26, this locus intersects $\Lambda_n \cap \Lambda_j$ precisely along $\Lambda_n \cap \Lambda_j$ and so $\tilde{y} \in \Lambda_n \cap \Lambda_j$.

Thus altogether $\tilde{y} \in \Lambda_n \cap \Lambda_j = (\Lambda_n \cap \Lambda_j) \cap (\Lambda_n \cap \Lambda_j)$.

By Corollary 3.26, the intersections $\Lambda_n \cap \Lambda_j$ and $\Lambda_n \cap \Lambda_j$ are closures of clean codimension one intersections, hence their projections lie in the primary tangencies $\tau(\Gamma_n, \Gamma_k)$ and $\tau(\Gamma_j, \Gamma_k)$. Moreover, $\Lambda_n \cap \Lambda_j$ and $\Lambda_n \cap \Lambda_j$ intersect along their primary tangency. Since $\pi$ restricted to $\Lambda_k$ has no critical points, the projection of this primary tangency is again a primary tangency. Hence $y \in \tau(\gamma(\Gamma_n, \Gamma_k), \Gamma_j)$, proving the asserted containment.

We leave the nondegeneracy of the case $k = n - 2$ to the reader. $\square$

Now we are ready to inductively normalize the primary tangencies.

**Lemma 3.35.** Fix $0 \leq k < n - 1$.

Suppose

$$\tau(\Gamma_n, \Gamma_j) = \tau(\Gamma_n, \Gamma_j) \quad j > k$$

Then there exists a diffeomorphism $\psi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ preserving $\gamma = \bigcup_{i=0}^{n-1} \Gamma_i$ such that

$$\tau(\psi(\Gamma_n), \Gamma_j) = \tau(\Gamma_n, \Gamma_j) \quad j \geq k$$
Moreover, when \( k \neq n - 2 \), the diffeomorphism is an isotopy.

**Proof.** We will assume \( k < n - 3 \). We leave the elementary cases \( k = n - 2, n - 3 \) to the reader. They can be deduced from the parametric versions of the cases \( n = 0, 1 \) presented in \( 3.5, 3.5 \) respectively.

Throughout what follows, we use the projection \( \mathbb{R}^{n+1} \to \mathbb{R}^n \) to identify \( n \Gamma_k = \mathbb{R}^n \).

On the one hand, we have

\[
\tau(n \Gamma_j, n \Gamma_k) = \mathbb{R}^k \times n-k-1 \Gamma_{j-k-1} \quad k < j < n
\]

On the other hand, by Lemma 3.34 and assumption, we have

\[
\tau(\tau(\Upsilon_n, n \Gamma_k), \tau(n \Gamma_j, n \Gamma_k)) = \tau(\tau(n \Gamma_n, n \Gamma_j) \cap n \Gamma_k, n \Gamma_k) \quad k < j < n
\]

Hence within \( n \Gamma_k = \mathbb{R}^n \), the loci \( \tau(\Upsilon_n, n \Gamma_k) \) and \( \tau(n \Gamma_n, n \Gamma_k) \) have the same tangencies with \( \tau(n \Gamma_j, n \Gamma_k) = \mathbb{R}^k \times n-k-1 \Gamma_{j-k-1} \quad k < j < n \)

Thus Proposition 3.29 and Proposition 3.30 provide a time-dependent vector field of the form

\[
v_t = h_t \sum_{i=0}^{n-1} \frac{1}{2^i} x_i \partial x_i
\]

generating an isotopy \( \varphi : \mathbb{R}^{n-k} \to \mathbb{R}^{n-k} \) satisfying

\[
\varphi(\tau(\Upsilon_n, n \Gamma_k)) = \tau(n \Gamma_n, n \Gamma_k)
\]

In addition, the function \( h_t \), hence vector field \( v_t \), is divisible by the product \( \prod_{j=k+1}^{n-1} h_{n,j} \), and thus \( \varphi \) preserves its zero-locus.

Let us complete \( v_t \) to the vector field

\[
V_t = h_t \sum_{i=0}^{n-1} \frac{1}{2^i} x_i \partial x_i
\]

and consider the isotopy \( \psi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \) generated by \( V_t \).

Then \( \psi \) satisfies

\[
\psi(\tau(\Upsilon_n, n \Gamma_k)) = \tau(n \Gamma_n, n \Gamma_k)
\]

It also preserves \( n \Gamma_i \), for \( 0 \leq i \leq n - 1 \), as well as \( \tau(\Upsilon_n, n \Gamma_j) = \tau(n \Gamma_n, n \Gamma_j) \), for \( j > k \). In addition, it preserves

\[
\tau(n \Gamma_j, n \Gamma_k) = \mathbb{R}^k \times n-k-1 \Gamma_{j-k-1} \quad k < j < n
\]

since this is the zero-locus of \( h_{n,j} \). \( \square \)

Finally, let us use the lemma to complete the inductive step of the proof of Theorem 3.27 as outlined above. Suppose for some \( 0 < k \leq n - 1 \), we have moved \( \Upsilon_n \), while preserving \( \Upsilon' \),
so that we have the sought-after primary tangencies
\[ \tau(\Upsilon_n, n^j \Gamma_j) = \tau(n^j \Gamma_n, n^j \Gamma_j) \quad j > k \]
Then using Lemma 3.35, we can move \( \Upsilon_n \), while preserving \( \Upsilon' \), so that we have the sought-after primary tangencies
\[ \tau(\Upsilon_n, n^j \Gamma_j) = \tau(n^j \Gamma_n, n^j \Gamma_j) \quad j \geq k \]
Proceeding in this way, we arrive at \( k = 0 \), where all primary tangencies have been normalized. Now a final application of Proposition 3.29 and Proposition 3.30 move \( \Upsilon_n \) to \( n^j \Gamma_n \), while preserving \( \Upsilon' \), and thus complete the proof of Theorem 3.27.

3.6. Conclusion of the proof. We are now ready to prove Proposition 3.6. As a consequence we establish Theorem 3.5, and by the analogous parametric argument this also establishes the parametric version Theorem 3.12.

Proof of Proposition 3.6. Take any point \( \lambda \) in the front \( H := \pi(\Lambda) \) and let \( \pi^{-1}(\lambda) = \{\lambda_1, \ldots, \lambda_k\} \). Let \( \Lambda_1, \ldots, \Lambda_k \) be germs of \( \Lambda \) at these points of arboreal types \( (\mathcal{F}, n) \), \( n(\mathcal{F}_j) = n_j \). We need to show that the germ of the front \( H \) at \( \lambda \) is diffeomorphic to the germ of a model front \( H_{\mathcal{F}} \), where \( \mathcal{F} \) is a signed rooted tree obtained from \( \bigsqcup T_j \) by adding the root \( \rho \) and adjoining it to the roots \( \rho_j \) of the trees \( T_j \) by edges \( [\rho \rho_j] \). The signs of all edges of the trees \( T_j \) are preserved, while previously unsigned edges \( \rho_j \alpha \) get a sign \( \varepsilon(\nu, L, \alpha) \), see (1).

We proceed by induction on the number of vertices in the signed rooted tree \( \mathcal{F} = (T, \rho, \varepsilon) \).

The base case of a \((\mathcal{A}_1, m)\)-front \( H \subset \mathbb{R}^m \) is the same geometry as appearing in 3.5: any graphical hypersurface \( H \subset \mathbb{R} \times \mathbb{R}^{m-1} \) is isotopic to the germ of the zero-graph \( \{0\} \times \mathbb{R}^{m-1} \).

For the inductive step, fix a rooted tree \( \mathcal{F} = (T, \rho, \varepsilon) \), and as usual set \( n = |n(\mathcal{F})| \).

Consider a \((\mathcal{F}, m)\)-front \( H \subset \mathbb{R}^m \), with by necessity \( m \geq n \).

Fix a leaf vertex \( \beta \in \ell(\mathcal{F}) \), which always exists as long as \( \mathcal{F} \neq \mathcal{A}_1 \). Consider the smaller signed rooted tree \( \mathcal{F}' = \mathcal{F} \setminus \beta \), and the corresponding \((\mathcal{F}', m)\)-front \( H' = H \setminus \hat{H}[\beta] \subset \mathbb{R}^m \), where \( \hat{H}[\beta] \subset H \) is the interior of the smooth piece indexed by \( \beta \). By induction, we may assume
\[ H' = H_{\mathcal{F}'} \times \mathbb{R}^{m-n+1} \subset \mathbb{R}^m \]
Thus it remains to normalize the smooth piece \( H[\beta] \).

Let \( \mathcal{A}_\beta = (A_\beta, \rho, \varepsilon_\beta) \) be the linear signed rooted subtree of \( \mathcal{F} = (T, \rho, \varepsilon) \) with vertices \( v(A_\beta) = \{\alpha \in v(T) \mid \alpha \leq \beta\} \). Set \( d = v(\mathcal{F}) \setminus v(\mathcal{A}_\beta) = n(\mathcal{F}) \setminus n(\mathcal{A}_\beta) \) to be the complementary vertices.

Consider the \((\mathcal{A}_\beta, m)\)-front \( K \subset H \) given by the union \( K = \bigcup_{\alpha \in n(\mathcal{A}_\beta)} K[\alpha] \) of the smooth pieces of \( H \subset \mathbb{R}^m \) indexed by \( \alpha \in n(\mathcal{A}_\beta) \). Note for \( \mathcal{A}_\beta' = \mathcal{A}_\beta \cap \mathcal{F}' \), and \( K' = K \cap H' \), we already have
\[ K' = H_{\mathcal{A}_\beta'} \times \mathbb{R}^{m-n+1+d} \subset \mathbb{R}^m \]
and seek to normalize the smooth piece \( K[\beta] = H[\beta] \).
Now we can apply Theorem 3.27 to normalize $K[\beta]$ viewed as the final smooth piece of $K$. More specifically, we can apply Theorem 3.27 to normalize $K[\beta]$ while preserving $K'$ and viewing the complementary directions $\mathbb{R}^{m-n+1+d}$ as parameters, see Figure 3.6. This ensures we preserve $H'$ and hence do not disturb its already arranged normalization.

This concludes the proof of Proposition 3.6. □

![Figure 3.6. Treating the complementary directions as parameters.](image)

**References**

[AGEN19] D. Alvarez-Gavela, Y. Eliashberg, and D. Nadler, *Arborealization II: Geomorphology of Lagrangian ridges*, arXiv:1912.03439 4

[AGEN20b] D. Alvarez-Gavela, Y. Eliashberg, and D. Nadler, *Arborealization III: Positive arborealization of polarized Weinstein manifolds*, arXiv:1912.03439 1, 3

[AGEN21] D. Alvarez-Gavela, Y. Eliashberg, and D. Nadler, *Arborealization IV: Reidemeister moves for positive arboreal skeleta*, in preparation. 4

[C20] R. Casals, *Lagrangian skeleta and plane curve singularities*, arXiv:2009.06737 3

[GN20] B. Gammage and D. Nadler, *Mirror symmetry for honeycombs*, Trans. Amer. Math. Soc. 373 (2020), 71–107. 3

[FK14] M. Futaki, K. Ueda, *Tropical amoeba and torus-equivariant homological mirror symmetry for the projective space*, Communications in Mathematical Physics, 332(1):53–87, Nov 2014. 3

[M19] E. Murphy, *Arboreal singularities and loose Legendrians I*, arXiv:1902.05056 3

[N13] D. Nadler, *Arboreal Singularities*, arXiv:1309.4122. 1, 3, 21

[N15] D. Nadler, *Non-characteristic expansion of Legendrian singularities*, arXiv:1507.01513. 3

[N16] D. Nadler, *Wrapped microlocal sheaves on pairs of pants*, arXiv:1604.00114.
[Se08] P. Seidel *Fukaya categories and Picard-Lefschetz theory*, Zurich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich, 2008.

[Sh18] V. Shende, *Arboreal singularities from Lefschetz fibrations*, arXiv:1809.10359.

[St18] L. Starkston, *Arboreal Singularities in Weinstein Skeleta*, Selecta Mathematica, 24, 4105–4140 (2018)

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