Exact solutions in massive gravity

Gianmassimo Tasinato\textsuperscript{1}, Kazuya Koyama\textsuperscript{1} and Gustavo Niz\textsuperscript{1,2}

\textsuperscript{1} Institute of Cosmology and Gravitation, University of Portsmouth, Portsmouth, PO1 3FX, UK
\textsuperscript{2} Departamento de Física, Universidad de Guanajuato, DCI, Campus León, CP 37150, León, Guanajuato, Mexico

E-mail: gianmassimo.tasinato@port.ac.uk

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Abstract
Massive gravity is a good theoretical framework to study the modifications of General Relativity. The theory offers a concrete set-up to study models of dark energy, since it admits cosmological self-accelerating solutions in the vacuum, in which the size of the acceleration depends on the graviton mass. Moreover, nonlinear gravitational self-interactions, in the proximity of a matter source, manage to mimic the predictions of linearized General Relativity; hence, agreeing with solar-system precision measurements. In this paper, we review our work in the subject, classifying, on one hand, static solutions, and on the other hand, self-accelerating backgrounds. For the static solutions, we exhibit black hole configurations, together with other solutions that recover General Relativity near a source via the Vainshtein mechanism. For the self-accelerating solutions, we describe a wide class of cosmological backgrounds, including an analysis of their stability.

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(Some figures may appear in colour only in the online journal)

1. Introduction
Can the graviton have a mass? A graviton mass could explain, in a technically natural way, the small size of the present-day acceleration of our Universe. At large scales, gravity is modified with respect to General Relativity (GR), and the theory admits cosmological accelerating solutions in which the size of the acceleration depends on the graviton mass. This way to explain cosmological acceleration is technically natural in the ’t Hooft sense, because in the limit of graviton mass going to zero, one recovers the full diffeomorphism invariance of GR; hence, corrections to the size of dark energy must be proportional to the (tiny) graviton mass itself. Of course, while potentially explaining present-day acceleration, massive gravity does not solve the old cosmological constant problem: that is, it does not explain why the cosmological constant term should not be present in the gravity Lagrangian. We have nothing to say about this issue in this paper.
Not aware of all possible consequences of massive gravity for cosmology, Fierz and Pauli (FP), back in 1939, started the theoretical study of massive gravity from a field theory perspective [1]. They considered a mass term for linear gravitational perturbations, which is uniquely determined by requiring the absence of ghost degrees of freedom. The mass term breaks the gauge (diffeomorphism) invariance of GR, leading to a classical graviton with five degrees of freedom, instead of the two found in GR. There have been intensive studies into what happens beyond the linearized theory of FP. In 1972, Boulware and Deser (BD) found a scalar ghost mode at the nonlinear level, the so-called sixth degree of freedom in the FP theory [2]. This issue has been re-examined using an effective field theory approach, where gauge invariance is restored by introducing Stückelberg fields [3]. In this language, the Stückelberg fields physically play the role of the additional scalar and vector graviton polarizations. They acquire nonlinear interactions which contain more than two time derivatives, signalling the existence of a ghost [3]. In order to construct a consistent theory, nonlinear terms should be added to the FP model, which are tuned to remove the ghost order by order in perturbation theory. Interestingly, this approach sheds light on another famous problem with FP massive gravity; due to contributions of the scalar degree of freedom, solutions in the FP model do not continuously connect to solutions in GR, even in the limit of zero graviton mass. This is known as the van Dam, Veltman and Zakharov (vDVZ) discontinuity [4, 5]. Observations such as light bending in the solar system would exclude the FP theory, no matter how small the graviton mass is. In 1972, Vainshtein [6] proposed a mechanism to avoid this conclusion; in the small mass limit, the scalar degree of freedom becomes strongly coupled and the linearized FP theory is no longer reliable. In this regime, higher order interactions suppress the linear scalar contribution to the gravitational potential, recovering the GR solutions on scales below a certain macroscopic radius around a matter source.

Until recently, it was thought to be impossible to construct a ghost-free theory for massive gravity beyond the linear order [7, 8]. Using an effective field theory approach, one can show that in order to avoid the presence of a ghost, interactions have to be chosen in such a way that the equations of motion for the scalar and vector components of the Stückelberg field contain no more than two time derivatives. Recently, it was shown that there is a finite number of derivative interactions for scalar Lagrangians that give rise to second-order differential equations. These are dubbed Galileon terms because of a symmetry under a constant shift of the scalar field derivative [9]. Therefore, one expects that any consistent nonlinear completion of FP contains these Galileon terms, at least in an appropriate range of scales in which the scalar dynamics can be somehow isolated from the remaining degrees of freedom; this is the so-called decoupling limit [3]. This turns out to be a powerful criterium for building higher order interactions with the desired properties. Indeed, following this route, de Rham, Gabadadze and Tolley constructed a family of ghost-free extensions to the FP theory, which reduce to the Galileon terms in the decoupling limit. We refer to the resulting theory as $\Lambda_3$ massive gravity [10], where $\Lambda_3$ is a cut-off scale above which quantum corrections become important [11], and that will be described in section 4.2.

In this paper, we review our work to build and analyse exact solutions in $\Lambda_3$ massive gravity. As we have briefly explained, nonlinear effects play an essential role to characterize phenomenological consequences of this theory. Then, the analysis of exact solutions of the equations of motion, obtained by imposing appropriate symmetries (spherical symmetry for static spacetimes, or homogeneity and isotropy for cosmological set-ups), manifests, in idealized but representative situations, how the nonlinear dynamics of the graviton degrees of freedom respond to the presence of a source, or, at very large scales, to the graviton mass itself. After all, looking back to the past, we know that the knowledge of exact solutions of nonlinear field equations has been of crucial importance to understand GR. The Schwarzschild
solution leads to the discovery of the concept of black hole and plays an essential role for analysing the dynamics of objects around massive sources in GR; and modern cosmology would be unthinkable without the use of Friedmann–Robertson–Walker (FRW) solutions. Exact solutions in massive gravity might lead to the discovery and understanding of new features and concepts in a theory of gravitation that can lead to important developments for our comprehension of gravitational interactions.

This paper is organized as follows. In section 2, we explicitly construct the $\Lambda_3$ massive gravity theory, while in section 3 the most general ansatz for spherically symmetric solution is introduced. This leads to two branches of static solutions: one exhibiting the Vainshtein mechanism and the other representing a generalization of Schwarzschild–(A)dS black holes. In section 4, we explore cosmological self-accelerating solutions and their stability under perturbations. Finally, in section 5 we conclude, also outlining possible directions for future research.

2. Ghost-free massive gravity

We begin with the covariant FP mass term in four dimensions, given by

$$L_{FP} = m^2 \sqrt{-g} U^{(2)}, \quad U^{(2)} = (H_{\mu\nu} H^{\mu\nu} - H^2),$$

where $m$ is a parameter with units of mass and the tensor $H_{\mu\nu}$ is a covariantization of the metric perturbations, namely

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \equiv H_{\mu\nu} + \Sigma_{\mu\nu}, \quad \Sigma_{\mu\nu} \equiv \partial_\mu \phi^\alpha \partial_\nu \phi^\beta \eta^{\alpha\beta}. \quad (2)$$

The St"uckelberg fields $\phi^\alpha$ are introduced to restore reparametrization invariance, hence transforming as scalar from the point of view of the physical metric [3]. The internal metric $\eta_{\alpha\beta}$ corresponds to a non-dynamical reference metric, usually assumed to be Minkowski spacetime. The dynamics of the St"uckelberg fields $\phi^\alpha$ are at the origin of the two features discussed in the introduction: the BD ghost excitation and the vDVZ discontinuity. With respect to the first issue, as noted by FP, one can remove the ghost excitation, to linear order in perturbations, by choosing the quadratic structure $H_{\mu\nu} H^{\mu\nu} - H^2$. When expressed in the St"uckelberg field language, terms in the action are arranged in a such way so as to constraint one of the four St"uckelberg fields to be non-dynamical. However, when going beyond linear order, this constraint generically disappears, signalling the emergence of an additional ghost mode [3]. Remarkably, reference [12] has shown how to construct a potential, tuned at each order in powers of $H_{\mu\nu}$, to hold the constraint and remove one of the St"uckelberg fields. Even though the potential is expressed in terms of an infinite series of terms for $H_{\mu\nu}$, it can be resumed into the following finite form [10, 13]:

$$U = -m^2 [U_2 + \alpha_3 d_3 + \alpha_4 d_4],$$

where $\alpha_n$ are free dimensionless parameters, $U_n = n! \det_n (K)$ and the tensor $K_{\mu\nu}$ is defined as

$$K_{\mu\nu} \equiv \delta_{\mu\nu} - \left(\sqrt{g^{-1}\Sigma}\right)_\mu^{\nu}. \quad (4)$$

(The square root is formally understood as $\sqrt{K_{\mu\nu}} \sqrt{K_{\nu\mu}} = K_{\mu\nu}$, and one should take all eigenvalues to be positive to recover the FP model (1) at the lowest order.) The relationship of these potentials with a determinant resides on the following property, which holds for squared real matrices and a complex number $z$

$$\det (I + zK) = 1 + \sum_{n=1}^{\infty} z^n \det_n (K),$$

$$\det_3 (K) = \det_3 (K) = \det_4 (K) = 1,$$
where each matrix invariant $\det_n$ can be written in terms of traces as

\[
\det_1(K) = \text{tr} K,
\]
\[
\det_2(K) = (\text{tr} K)^2 - \text{tr}(K^2),
\]
\[
\det_3(K) = (\text{tr} K)^3 - 3(\text{tr} K)(\text{tr} K^2) + 2\text{tr} K^3,
\]
\[
\det_4(K) = (\text{tr} K)^4 - 6(\text{tr} K)(\text{tr} K^2) + 8(\text{tr} K)(\text{tr} K^3) + 3(\text{tr} K^2)^2 - 6\text{tr} K^4.
\]

All terms $\det_n(K)$ with $n > 4$ vanish in four dimensions. If one chooses a sum of determinants of the form $\sum_{n=1}^{4} \det_n(K) - 4$, then one can generate each $\det_n(K)$ term with a separate coefficient $\alpha_n$, provided a solution to $\sum_{n=1}^{4} \alpha_n = 0$ exists, which is guaranteed by the Newton identities. Therefore, the massive gravity theory can be written in full as

\[
L = \frac{M_{Pl}^2}{2} \sqrt{-g} (R - \frac{2}{\Lambda} - m^2 U),
\]

where $U$ is given by (3) and we have introduced an additional bare cosmological constant $\Lambda$. Note that in order to obtain the FP term (1) as the first-order correction to GR, we have ignored the tadpole term $\det_1(K)$.

We can write the equations of motion in a more familiar way by using the potential term (3) as the source of peculiar energy–momentum tensor. In this way, the Einstein equations read

\[
G_{\mu\nu} = T^U_{\mu\nu},
\]

where the energy–momentum tensor is defined as

\[
T^U_{\mu\nu} = \frac{m^2}{\sqrt{-g}} \frac{\delta \sqrt{-g} U}{\delta g^{\mu\nu}}.
\]

The theory defined by (6) has Minkowski spacetime as a trivial solution when $\Lambda = 0$; hence, one can rewrite the metric $g_{\mu\nu}$ and the scalars $\phi^\mu$ as deviations from flat space, namely

\[
\begin{align*}
\delta g_{\mu\nu} &= \eta_{\mu\nu} + h_{\mu\nu}, \\
\phi^\mu &= x^\mu - \pi^\mu,
\end{align*}
\]

where $x^\mu$ are the usual cartesian coordinates spanning $\eta_{\mu\nu}$. In what follows, we will use $\phi^\mu$ or $\pi^\mu$, having in mind that (9) relates them. Moreover, the unitary gauge (where $\pi^\mu = 0$) simplifies the potential (3) considerably, and we will start with this choice in what follows.

There have been intensive studies in the issue of the BD ghost in this theory [14]. The general (but not universal [15]) consensus is that there is indeed no BD ghost and the maximum number of propagation modes in this theory is 5. However, this does not preclude a possibility that one of the five modes becomes a ghost around some backgrounds, as it happens with the scalar mode in the FP Lagrangian around the de-Sitter space, for certain values of the graviton’s mass [16].

3. Spherically symmetric solutions

In this section, we review spherically symmetric solutions in the unitary gauge by following references [17, 13]. The most general ansatz with spherical symmetry, before fixing the gauge, is

\[
d\Omega^2 = dt^2 - b(t, r)^2 dr^2 + a(t, r)^2 d\theta^2 + 2d(t, r) dt dr + c(t, r)^2 d\phi^2,
\]

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$, and the Stuckelberg fields have the structure

\[
\begin{align*}
\phi^0 &= f(t, r), \\
\phi^i &= g(t, r) \frac{x^i}{r},
\end{align*}
\]
We start focusing on the unitary gauge ($\pi^\mu = 0$ or, equivalently, $f = t$ and $g = r$) and look for static solutions that do not depend, explicitly, on time. The metric ansatz (10) reduces to

$$\text{d}s^2 = -b(r)^2 \text{d}t^2 + a(r)^2 \text{d}r^2 + 2d(r) \text{d}r + c(r)^2 \text{d}\Omega^2. \tag{12}$$

Furthermore, we choose to write the non-dynamical flat metric in (2) as $\text{d}s^2 = -\text{d}t^2 + \text{d}r^2 + r^2 \text{d}\Omega^2$. It should be noted that this is not a coordinate choice, but a way to simplify the expressions. Indeed, we have chosen the unitary gauge; hence, we are left with a theory which is not manifestly diffeomorphism invariant. Hence, in this context physics does depend on the choice of coordinates: we will further explore this fact in section 4. Any change of coordinate normally breaks the unitary gauge and switches on a non-trivial profile for the St"uckelberg fields. This also implies that the static solutions considered in the unitary gauge do not provide all the spherically symmetric and static solutions in this theory. Other static solutions might exist with non-trivial St"uckelberg fields turned on.

Conscious of these limitations, let us start with this gauge choice: we will relax it in what follows. We plug the previous metric into the Einstein equations (7) and observe that the Einstein tensor $G_{tt}$ satisfies the identity $d(r)G_{tt} + b(r)^2G_{tt} = 0$, which implies the algebraic constraint $0 = d(r)T^{tt}_0 + b(r)^2T_{tt}^0$. This last equation implies

$$d(r)(c_0r - c(r)) = 0, \tag{13}$$

where $c_0$ is a function of $\alpha_3$ and $\alpha_4$ only (see section 3.2). This constraint is solved in two possible ways, defining two branches of solutions:

$$d(r) = 0 \quad \text{or} \quad c(r) = c_0r. \tag{14}$$

In the following sections, we will analyse each of these two branches separately. We will start from the diagonal one $d(r) = 0$ in section 3.1, where the Vainshtein effect takes place and can be analysed in a systematic way. Then, we will proceed in section 3.2 to study the class of solutions with a non-diagonal metric and $c(r) = c_0r$, corresponding to non-asymptotically flat, Schwarzschild–(anti)-de Sitter solutions that can be relevant to explain present-day cosmological acceleration.

### 3.1. Branch I: Vainshtein mechanism at work

The problem of finding exact vacuum solutions in this branch $d(r) = 0$ is an open question, but we can make interesting progresses by considering perturbations (not necessarily small) from flat space. The following ansatz is useful:

$$b(r) = 1 + N(r), \quad a(r) = (1 + F(r))^{-1/2}, \quad c(r) = (1 + H(r))^{-1}. \tag{15}$$

Initially, we want to consider $N, F$ and $H$ small enough to study the theory to linear order. For this, it is convenient to introduce a new radial coordinate $\rho = \frac{r}{1 + H(r)}$, so that the linearized metric is expressed as

$$\text{d}s^2 = -(1 + n) \text{d}t^2 + (1 - f) \text{d}\rho^2 + \rho^2 \text{d}\Omega^2, \tag{16}$$

where $f(\rho) = F(r(\rho)) - 2h(\rho) - 2\rho h'(\rho)$, $n(\rho) = 2N(r(\rho))$, $h(\rho) = H(r(\rho))$ and a prime denotes a derivative with respect to $\rho$. As discussed above, one should be careful with this change of coordinates since, after fixing a gauge, a change of frame in the metric breaks the unitary gauge and switches on the St"uckelberg fields $\pi^a$. This can be seen from expression (9), where we know that the fields $\phi^a$ transform as scalars, which in turn induces a particular transformation of the fields $\pi^a$. Therefore, from now on one can think of $h$ as geometrically corresponding to the only non-zero component of the St"uckelberg field $\pi^a$. At linear order, the equations for the functions $n(\rho)$, $f(\rho)$ and $h(\rho)$ in the new radial variable $\rho$ are

$$0 = (m^2\rho^2 + 2)f + 2\rho(f' + m^2\rho^2h' + 3m^2\rho h), \tag{17}$$
we have to keep all nonlinear terms in \( h \) at the same time ignore higher order terms in \( n \) to the following system of coupled equations for the fields \( f \), \( n \), and \( h \) where nonlinear effects in \( h \) for the metric fields, but keep all nonlinearities in the component \( n \).

To do this, we consider scales below the Compton wavelength \( m \) for this metric component manifests the nonlinear effects responsible for the Vainshtein mechanism. To understand what really happens in this limit, we must carefully analyse the behaviour of the function \( h(r) \) as \( m \to 0 \). The study of the equations of motion for this metric component manifests the nonlinear effects responsible for the Vainshtein mechanism. To do this, we consider scales below the Compton wavelength \( m \rho \ll 1 \), and at the same time ignore higher order terms in \( GM \). Under these approximations, the equations of motion can still be truncated to linear order in \( f \) and \( n \), but since \( h \) is not necessarily small, we have to keep all nonlinear terms in \( h \). In other words, we take the usual weak field limit for the metric fields, but keep all nonlinearities in the component \( h \), since we expect regions where nonlinear effects in \( h \) become important. As shown in [13], the field equations reduce to the following system of coupled equations for the fields \( f \), \( n \), and \( h \):

\[
f = -2\frac{GM}{\rho} - (m\rho)^2 (h - a\beta^3 + \beta^3),
\]

\[
n' = 2\frac{GM}{\rho^2} - m^2 \rho (h - \beta^3),
\]

\[
0 = \frac{3}{2} \beta^2 \rho h^5 (\rho) - (\alpha^2 + 2\beta) \rho \beta^3 (\rho) + 3(\alpha + \beta A(\rho)) h^2 (\rho) - \frac{3}{2} h(\rho) - A(\rho) = 0,
\]

where

\[
\alpha \equiv 1 + 3\alpha_3
\]

\[
\beta \equiv \alpha_3 + 4\alpha_4.
\]

\[
A(\rho) = \left(\frac{\rho}{\rho_v}\right)^3,
\]

\[
\rho_v \equiv (GM/m^2)^{1/3}.
\]

Equation (24) is a quintic algebraic equation in \( h \), except for the special case where \( \beta = 0 \), where it reduces to a cubic equation. Thus, after obtaining a solution for \( h \) from equation (24), one can calculate the gravitational potentials \( f \) and \( n \) using (22) and (23). In the particular case of \( \beta = 0 \), it is possible to describe the solution in a simple way [17]. For large radial \( \rho \) values, one can linearize the equations in \( h \), recovering the solution in equations (20) and (21), to first order in \( m\rho \). On the other hand, the Vainshtein mechanism applies, and below the so-called Vainshtein radius, \( \rho_v = (GM/m^2)^{1/3} \), \( h \) becomes larger than 1 and the nonlinear terms in \( h \) in equation (24) become important, recovering GR close to a matter source. Actually, for \( \rho \ll \rho_v \), the solution for \( h \) is simply given by \( |\hat{h}| = \rho_v/(\alpha \rho) \gg 1 \). The latter solution for \( h \) and equation (24) with \( \beta = 0 \) implies

\[
2\rho n' = \frac{2GM}{\rho} \left(1 + \frac{1}{2\alpha} \left(\frac{\rho}{\rho_v}\right)^2\right), \quad f = -\frac{2GM}{\rho} \left(1 - \frac{1}{2\alpha} \left(\frac{\rho}{\rho_v}\right)^2\right),
\]
Figure 1. We show the radial gradient of the gravitational potentials $f$ and $n$ in equation (16) as a function of the radial distance. The left plot describes these potentials around the Vainshtein radius, while the right plot depicts the region around the Compton wavelength $\rho \sim 1/m$. The quotient $\gamma' \equiv f'/2n'$ measures deviations from the GR solutions, which give $\gamma' = 1$. Region 1 (2) shows how GR solutions are (not) recovered inside (outside) the Vainshtein radius $\rho_V$. Region 3 shows the asymptotic decay of the linear solutions due to a massive graviton (see equations (20)-(21)). Here, $GM = 1$ and $\alpha = 1$.

Therefore, corrections to the GR solutions are indeed small for $\rho < \rho_V$, as shown in the left plot of figure 1. Note that if we consider a finite size matter source, it was shown that there is no stable solution that interpolates from the Vainshtein region to the asymptotically flat solution, and the Vainshtein region is naturally matched onto a solution that asymptotes to a non-flat cosmological background [18].

For the $\beta \neq 0$ case, analytic solutions of the previous algebraic equations cannot be found in general. In the following, we will follow the approach presented in [19]. It is possible to determine exactly how many local solutions exist in a neighbourhood of infinity at $\rho = +\infty$, which we refer as asymptotic solutions, and moreover how many local solutions exist in a neighbourhood of $\rho = 0^+$, which we call inner solutions. Furthermore, we can find analytically their leading behaviour as a function of $\rho$. Any global solution of (24) should necessarily interpolate between one of the asymptotic solutions and one of the inner solutions. Therefore, our aim is to understand, for each point in the $(\alpha, \beta)$ phase space, whether and how the above solutions match.

In a neighbourhood of $\rho \to +\infty$, there are, depending on the value of $(\alpha, \beta)$, three or five solutions to equation (24). In particular, there is always a decaying solution, which we indicate with $L$. Its asymptotic behaviour is $h(\rho) \sim (\rho_0/\rho)^3$. This solution corresponds to a spacetime which is asymptotically flat. Additionally, there are two or four solutions to equation (24) which tend to a finite, non-zero value as $\rho \to +\infty$. There is always one positive and one negative root to equation (24); we call these solutions $C_+$ and $C_-$, respectively. In addition, for some regions of parameter space there are two additional roots, which lay between $C_+$ and $C_-$, and which we refer as $P_1$ and $P_2$ (further details about this denomination are given in [19]). Their asymptotic behaviour is $h(\rho) = C$, with $C$ being a constant. These solutions correspond to spacetimes which are asymptotically non-flat. Interestingly, the leading term in the gravitational potentials scales as $\rho^2$ for large radii, the same scaling which we find in a de Sitter spacetime. It is worthwhile to point out that, since we are working on scales below the Compton wavelength of the gravitational field, asymptotically non-flat does not really mean...
the real behaviour at infinity. To understand the true asymptotic behaviour of this solution, one should solve the complete, non-truncated equations.

In a neighbourhood of $\rho \to 0^+$, there are either one or three solutions to equation (24). For $\beta > 0$, there are exactly three inner solutions, while for $\beta < 0$, there is only one inner solution. In particular, there is always a diverging solution, which we denote by $D$. Its leading behaviour is $h(\rho) \sim -\sqrt{2/\beta}(\rho_v/\rho)$. This solution exists for both $\beta > 0$ and $\beta < 0$, with opposite signs for each case. Using this solution in equations (22) and (23), one realizes that the $h^3$ term cancels the $GM/\rho$ term, so the gravitational field does not diverge as $\rho \to 0^+$. We call this solution self-shielding, since the gravitational force vanishes as $\rho \to 0$. This solution is in strong disagreement with gravitational observations. For $\beta > 0$, there are two additional solutions to equation (24), which tend to a finite, non-zero value as $\rho \to 0^+$. We indicate these solutions by $F_+$ and $F_-$, and their leading behaviour is $h(\rho) = \pm(3\beta)^{-1/2}$. Note that for $\beta < 0$ there are no solutions to equation (24) which tend to a finite value as $\rho \to 0^+$. Expressions (22) and (23) for the gravitational potentials imply that the metric associated with these solutions ($F_+$ and $F_-$) approximate the linearized Schwarzschild metric as $\rho \to 0^+$.

From the behaviour of the inner solutions, one concludes that only in the $\beta > 0$ part of the phase space solutions may exhibit the Vainshtein mechanism [20], but not necessarily for all values of $\alpha$ [19]. The phase space diagram which displays our results about solution matching is given in figure 2. We discuss separately the $\beta > 0$ and $\beta < 0$ part of the phase space, and refer to the figure for the numbering of the regions. The notation $I \leftrightarrow A$ means that there is matching between the inner solution $I$ and the asymptotic solution $A$.

For $\beta < 0$. In this part of the phase space, there is only one inner solution, $D$, so there can be at most one global solution to (24). There are three distinct regions which differ in the way the matching works (see [19] for details).
• Region 1: D ↔ C+. The boundaries of this region are the line β = 0 for α < 0 and the parabola \( \beta = c_{12}\alpha^2 \) for \( \alpha > 0 \), where \( c_{12} \) is the negative root of the equation 
\[-4 - 8y + 88y^2 - 1076y^3 + 2883y^4 = 0.\]
• Region 2: no matching. The boundaries of this region are the parabolas \( \beta = c_{12}\alpha^2 \) and \( \beta = c_-\alpha^2 \), where \( c_- \) is the only real root of the equation 
\[8 + 48y - 435y^2 + 676y^3 = 0.\]
• Region 3: D ↔ P_2.

\( \beta > 0 \). In this part of the phase space, there are three inner solutions, D, F_+, and F_-, so there can be at most three global solutions to equation (24). There are six distinct regions with different matching properties.

• Region 4: F_- ↔ L, D ↔ C-. The boundaries of this region are the parabola \( \beta = c_{45}\alpha^2 \), where \( c_{45} = 1/12 \) and the line \( \beta = 0 \). In this region, there are Vainshtein solutions which asymptote to a flat spacetime only.
• Region 5: F_+ ↔ C+, F_- ↔ L, D ↔ C-. The boundaries of this region are the parabola \( \beta = c_{45}\alpha^2 \) for \( \alpha > 0 \) and the parabola \( \beta = c_{56}\alpha^2 \) for \( \alpha < 0 \), where \( c_{56} = (5 + \sqrt{13})/24 \). Vainshtein solutions with both, flat and non-flat asymptotics, are present in this region.
• Region 6: D ↔ C-, F_+ ↔ C+. The boundaries of this region are the parabolas \( \beta = c_{56}\alpha^2 \) and \( \beta = c_{67}\alpha^2 \), where \( c_{67} \) is the positive root of the equation 
\[-4 - 8y + 88y^2 - 1076y^3 + 2883y^4 = 0.\]
In this region, there are Vainshtein solutions which asymptote to non-flat spacetimes only.
• Region 7: F_+ ↔ C+. The boundaries of this region are the parabolas \( \beta = c_{67}\alpha^2 \) and \( \beta = c_+\alpha^2 \), where \( c_+ = 1/4 \). In this region, there are Vainshtein solutions which asymptote to non-flat spacetimes only.
• Region 8: F_+ ↔ C+, F_- ↔ P_1, D ↔ P_2. The boundaries of this region are the parabolas \( \beta = c_+\alpha^2 \) and \( \beta = c_{67}\alpha^2 \), where \( c_{67} = (5 - \sqrt{13})/24 \). In this region, there are Vainshtein solutions which asymptote to non-flat spacetimes only.
• Region 9: F_- ↔ P_1, D ↔ P_2. In this region, there are Vainshtein solutions which asymptote to non-flat spacetimes only.

We note that the decaying solution L never connects to the diverging configuration D, so we cannot have a spacetime which is asymptotically flat and exhibits the self-shielding of the gravitational field at the origin. On the other hand, finite non-zero asymptotic solutions (C± or P_{1,2}) can connect to both finite and diverging inner solutions. Therefore, one can have an asymptotically non-flat spacetime which presents self-shielding at the origin, or an asymptotically non-flat spacetime which tends to Schwarzschild spacetime for small radii. More precisely, for \( \beta < 0 \), there are only solutions displaying the self-shielding of the gravitational field, apart from region 2 where there are no global solutions. Therefore, the Vainshtein mechanism never works for \( \beta < 0 \). In contrast, for \( \beta > 0 \), all three kinds of global solutions are present. Solutions with asymptotic flatness and the Vainshtein mechanism are present in regions 4 and 5, while solutions which are asymptotically non-flat and exhibit the Vainshtein mechanism do exist in all (\( \beta > 0 \)) regions but region 4. Finally, solutions which display the self-shielding of the gravitational field are present in all (\( \beta > 0 \)) regions but region 7.

For the sake of clearness, we show one representative plot with the numerical matching solution between the inner and the asymptotic solutions. We consider solutions which recover the Schwarzschild solution near the origin, and which are asymptotically flat (F_- ↔ L in figure 3 (left)) or non-flat (F_+ ↔ C_+ in figure 3 (right)). Finally, it is essential to decide whether these vacuum solutions are indeed consistent with matter sources, as it was done for the \( \beta = 0 \) case in [18].
Figure 3. We show the radial gradient of the gravitational potentials $f$ and $n$ in equation (16), normalized by the GR counterparts $f_{GR} = n_{GR} = -2GM/\rho$ and the St"uckelberg field $h = \pi^\rho/\rho$, as a function of the radial distance. The left plot describes the matching solution $F_{-} \leftrightarrow L$, while the right plot is $F_{+} \leftrightarrow C_{+}$, with $(\alpha, \beta) = (0, 0.1)$. $f_{GR} = n_{GR} = -2GM/\rho$. Solutions clearly present the vDVZ discontinuity with its resolution via the Vainshtein mechanism. Solutions asymptote a flat spacetime, but with a different $\gamma = f/n$ than the Schwarzschild solution in GR (left) and a non-flat spacetime (right).

The Vainshtein mechanism has been studied intensively in the context of the Dvali–Gabadadze–Porrati braneworld model [26] and Galileon models [21]. In particular, it was shown that the most general second-order scalar tensor theory described by the Hordenski action [22] leads to the same field equations as massive gravity in the decoupling limit [23].

3.2. Branch II: exact solutions

As we learned in the previous section, an essential property of this theory of massive gravity is the strong coupling phenomenon occurring in the proximity of a source. On the other hand, the graviton mass induces nonlinearities in the behaviour of long wavelength gravitons, responsible for the emergence of the second branch of solutions that we are going to study in this section. In an appropriate gauge, these solutions are asymptotically de Sitter or anti-de Sitter, depending on the choice of parameters.

We start with the unitary gauge ($\pi^\mu = 0$) and allow for arbitrary couplings $\alpha_3$ and $\alpha_4$, while from now on we set, for simplicity, the bare cosmological constant to vanish (see [17, 13] for a more complete discussion including a bare cosmological constant). We choose the static ansatz of equation (12) for the metric, and we focus on the second branch of solutions for the constraint equation (13): $c(r) = c_0 r$. Then, the exact solution of field equations is given by [24, 17, 13],

$$
\begin{align*}
    c(r) & = c_0 r, \\
    b(r)^2 & = b_0 + \frac{b_1}{r} + b_2 r^2, \\
    a(r)^2 + b(r)^2 & = Q_0, \\
    \Delta^2(r) + a(r)^2 b(r)^2 & = \Delta_0.
\end{align*}
$$

Moreover, the equations of motion fix the constant parameters $b_0$, $b_2$, $c_0$ and $Q_0$, leaving the values of $b_1$ and $\Delta_0$ free (although their sizes must be contained within certain intervals). Note that in GR, diffeomorphism invariance allows one to choose the function $c(r)$ to be $c(r) = r$,.
so that \( c_0 = 1 \). In this theory of massive gravity, after having fixed the gauge, this choice is no longer possible and the equations of motion determine \( c_0 \). One finds

\[
c_0 = \frac{1 + 6\alpha_3 + 12\alpha_4 \pm \sqrt{1 + 3\alpha_3 + 9\alpha_3^2 - 12\alpha_4}}{3(1 + 3\alpha_3 + 4\alpha_4)} \tag{31}
\]

for non \( \alpha_3 \neq -4\alpha_4 \), and

\[
c_0 = \frac{2}{3} \left( \frac{1 - 12\alpha_4}{1 - 8\alpha_4} \right),
\]

for \( \alpha_3 = -4\alpha_4 \), which in particular includes the case \( \alpha_3 = \alpha_4 = 0 \). After plugging the metric components (30) in the remaining Einstein equations, one can find the values for the other \( \alpha_3 \) and parameters. The corresponding general expressions are quite lengthy, and for this reason we relegate them to the appendix. As a concrete, simple example, in the main text we work out the special case \( \alpha_3 = -4\alpha_4 \), where the parameters are

\[
c_0 = \frac{2}{3} \left( \frac{1 - 12\alpha_4}{1 - 8\alpha_4} \right), \quad b_0 = \frac{\Delta_0}{c_0^2}, \quad b_2 = \frac{m^2\Delta_0}{4(12\alpha_4 - 1)}, \quad Q_0 = \frac{16(1 - 12\alpha_4)^4 + 81(1 - 8\alpha_4)^4\Delta_0}{36(1 + 4\alpha_4(-5 + 24\alpha_4))^2}. \tag{32}
\]

The previous solution is valid for \( \alpha_4 \) in the ranges \( \alpha_4 < 1/12 \) and \( \alpha_4 > 1/8 \). We find that \( b_1 \) and \( \Delta_0 \) are arbitrary; this vacuum solution is then characterized by two integration constants.

The resulting metric coefficients can be rewritten in the following, easier-to-handle form:

\[
a(r)^2 = \frac{9}{4}\Delta_0 \left( \frac{1 - 8\alpha_4}{1 - 12\alpha_4} \right)^2 [p(r) + \gamma + 1], \quad c(r) = 2 \left( \frac{1 - 12\alpha_4}{1 - 8\alpha_4} \right) r,
\]

\[
b(r)^2 = \frac{9}{4}\Delta_0 \left( \frac{1 - 8\alpha_4}{1 - 12\alpha_4} \right)^2 [1 - p(r)], \tag{33}
\]

\[
d(r) = \frac{9\Delta_0}{4} \left( \frac{1 - 8\alpha_4}{1 - 12\alpha_4} \right)^2 \sqrt{p(r)(p(r) + \gamma)}
\]

with \( (\mu = -b_1/b_0) \)

\[
p(r) \equiv \frac{\mu}{r} + \frac{(1 - 12\alpha_4)m^2r^2}{9(1 - 8\alpha_4)^2}, \quad \gamma \equiv \frac{16}{81\Delta_0} \left( \frac{1 - 12\alpha_4}{1 - 8\alpha_4} \right)^4 - 1. \tag{34}
\]

In order to have a consistent solution, we must demand that the argument of the square root appearing in the expression for \( d(r) \), equation (33), is positive. A sufficient condition to ensure this is that \( \mu \geq 0 \), and

\[
0 < \sqrt{\Delta_0} < \frac{9}{4} \left( \frac{1 - 12\alpha_4}{1 - 8\alpha_4} \right)^2. \tag{35}
\]

The metric might be rewritten in a more transparent diagonal form, by means of a coordinate transformation. In the absence of diffeomorphism invariance, any coordinate transformation of time \( t \) forces us to leave the unitary gauge, and to switch on a non-trivial profile for the St"uckelberg field \( \pi^\mu \) of the form \( \pi^\mu = (\pi_0(r), 0, 0, 0) \). One finds that then the metric can be rewritten in a diagonal form as

\[
ds^2 = -b(r)^2 \, dt^2 + \tilde{a}(r)^2 \, dr^2 + c(r)^2 \, d\Omega^2,
\]

while the equations of motion for the fields involved are solved by

\[
\tilde{a}(r)^2 = \frac{4}{9} \left( \frac{1 - 12\alpha_4}{1 - 8\alpha_4} \right)^2 \frac{1}{1 - p(r)}, \quad \pi_0'(r) = -\frac{\sqrt{p(r)(p(r) + \gamma)}}{1 - p(r)}. \tag{37}
\]
Figure 4. Allowed region of parameter space where solutions, which asymptote dS (vertical lines in light or red colour) or AdS (horizontal lines in dark or blue colour), exist. Left (right) plot is for the negative (positive) branch of equation (31). The brane solution is given by the special choice of for \( \alpha_3 = -\frac{1}{3} \) and \( \alpha_4 = \frac{1}{12} \) [13], and the solid lines correspond to the line \( \alpha_3 = -4\alpha_4 \). We have set \( \Lambda_1 = 0 \).

with \( b(r) \), \( c(r) \) and \( p(r) \) being the same as in equation (33). If one then makes a further time-rescaling

\[
t \rightarrow \frac{4(1-12\alpha_4)^2}{9\Delta_0^{1/2}(1-8\alpha_4)^2} t,
\]

then the resulting metric acquires a manifestly de Sitter–Schwarzschild or anti-de Sitter–Schwarzschild form. The choice between these two possibilities depends on whether \( \alpha_4 \) is smaller or larger than \( 1/12 \), as can be seen inspecting the function \( p(r) \) in equation (34). On the other hand, we should point out that this time-rescaling cannot be performed, without further introducing a time-dependent contribution to \( \pi_0 \). As expected, the metric in equation (36) can also be obtained by making the following transformation of the time coordinate \( d\tilde{t} = dr + \pi'_0 dr \) to the original metric (12). This produces a non-zero time component for \( \pi^\mu \) that does not vanish even in the limit \( m \to 0 \).

To summarize so far, we found vacuum solutions in this theory that are asymptotically de Sitter or anti-de Sitter, depending on the choice of the parameters.

Figure 4 shows the allowed parameters \( \alpha_3 \) and \( \alpha_4 \) for the existence of these asymptotically dS or AdS solutions. Further solutions and studies on black holes in this massive gravity theory can be found in [25].

4. Cosmological acceleration

4.1. Self-accelerating solution

One of the interesting features of massive gravity is self-acceleration. The self-accelerating solution was originally found in the DGP braneworld model where the acceleration of the Universe can be realized without introducing the cosmological constant [27]. However, the self-accelerating solution in the DGP model suffers from a ghost instability [28].
The first complete self-accelerating solution in the $\Lambda_3$ massive gravity theory (6) was reported in [13, 17] (the self-accelerating solution in the decoupling limit was first obtained in [29]). This configuration describes an accelerating cosmological universe in the vacuum, in which the rate of acceleration is controlled by the size of the graviton mass. The solution is a coordinate transformation of the exact solution (36)–(37), after having performed the time-rescaling (38) (one can use (A.1)–(A.3) for more general values of $\alpha_3$ and $\alpha_4$). Let us review how to construct it for the simple case of $\alpha_3 = -4\alpha_4$. In the case of the asymptotically de Sitter solution (36)–(37) with $\alpha_4 < 1/12$ and $\mu = 0$, the metric can also be written in a time-dependent form, at the price of switching on the additional components of $\pi^\mu$. After dubbing

$$\tilde{m}^2 \equiv \frac{m^2}{(1 - 12\alpha_4)}$$

we can make the following coordinate transformation $t = F_{\tilde{t}}(\tau, \rho)$ and $r = Fr_{\tilde{r}}(\tau, \rho)$ with

$$F_{\tilde{t}}(\tau, \rho) = \frac{4}{3\Delta_0^{1/2}\tilde{m}} \left( \frac{1 - 12\alpha_4}{1 - 8\alpha_4} \right) \text{arc tanh} \left( \frac{\sinh \left( \frac{\alpha_3}{2} \right)}{\cosh \left( \frac{\alpha_3}{2} \right) - \frac{\tilde{m}^2}{\tilde{m}^2} e^{\alpha_3/2}} \right),$$

$$F_{\tilde{r}}(\tau, \rho) = \frac{3}{2} \left( \frac{1 - 8\alpha_4}{1 - 12\alpha_4} \right) \rho e^{\alpha_3/2}.$$

Then, the metric becomes that of flat slicing of de Sitter

$$ds^2 = -dt^2 + e^{\alpha_3} (d\rho^2 + \rho^2 d\Omega^2),$$

where the Hubble parameter is given by

$$H \equiv \frac{\tilde{m}}{2} \equiv \frac{m}{2(1 - 12\alpha_4)^{1/2}}.$$  

The St"uckelberg fields $\pi^\mu$ are now given by

$$\pi^\mu = (\pi_0 + \tau - F_{\tilde{r}}(\tau, \rho) - F_{\tilde{r}}, 0, 0).$$

Interestingly, the value of the Hubble parameter is ruled by the mass of the graviton: we have a self-accelerating solution, in which the smallness of the acceleration rate can be associated with the smallness of the graviton mass.

So far, we used the general static solutions of the previous section to determine time-dependent self-accelerating configurations via suitable coordinate transformations. However, one can also follow another approach, and try to directly find time-dependent, self-accelerating configurations in the theory of $\Lambda_3$ massive gravity without relying on the unitary gauge. The hope, following this second route, is to determine additional cosmological configurations for this theory. Starting from the ansatz we wrote in equation (10), more general cosmological solutions can be obtained by focusing on the ansatz $d(r, t) = 0$ and $c(r, t) = r a(r, t)$ in (10) so that the metric becomes

$$ds^2 = -b(t, r)^2 dr^2 + a(t, r)^2 (dr^2 + r^2 d\Omega^2).$$

Wyman et al [30] showed that self-accelerating configurations are characterized by the following profile for the function $g(r, t)$ characterizing the St"uckelberg field $\phi^i$ (see equation (11)):

$$g(t, r) = c_0^{-1} a(t, r),$$

where $c_0$ is given by (31). The equation of motion for $g$ evaluated on the solution (45) provides a constraint on the function $f$ characterizing the Struckelberg field $\phi^0$:

$$\sqrt{X} X'_1 = (2c_0 P_2 - P'_2) W - P'_0,$$
where the $P_n$ functions

\[
P_0(x) = -12 - 2x(6 - 12(x - 1)(x - 2)\alpha_3 - 24(x - 1)^2\alpha_4,
\]

\[
P_1(x) = 2(3 - 2x) + 6(x - 1)(x - 3)\alpha_3 + 24(x - 1)^2\alpha_4,
\]

\[
P_2(x) = -2 + 12(x - 1)\alpha_3 - 24(x - 1)^2\alpha_4
\]

are evaluated at $x = c_0^{-1}$, $P'_n(x) \equiv dP_n/dx$. Moreover,

\[
X = \left(\frac{\dot{f}}{b} + \mu \frac{\dot{g}}{a}\right)^2 - \left(\frac{\dot{g}}{b} + \mu \frac{f'}{a}\right)^2,
\]

\[
W = \frac{\mu}{ab} (\dot{f}g' - \dot{g}f'),
\]

and $\mu = \text{sgn}(\dot{f}g' - \dot{g}f')$, where primes denote derivatives with respect to $r$ and overdots with respect to $t$. Using these equations, it is possible to show that the Einstein equations are given by

\[
G^\mu_\nu = -3H^2\delta^\mu_\nu,
\]

\[
H^2 = \frac{1 + 3\alpha_3 \pm 2\alpha_5}{3(1 + 3\alpha_3 \pm \alpha_5)^2}m^2,
\]

where

\[
\alpha_5 \equiv 1 + 3\alpha_3 + 9\alpha_3^2 - 12\alpha_4.
\]

Note that there are two branches of solutions. This approach leads to self-accelerating solutions where the Hubble parameter is determined by the mass of the graviton.

There are many possible solutions for the function $f(t, r)$ for given solutions of metric satisfying the Einstein equations (49). For example, in the simple case of $\alpha_3 = \alpha_4 = 0$, the configurations (41) and (43) are given by

\[
b = 1, \quad a = e^{mt/2},
\]

\[
f(t, r) = -\frac{3}{2} e^{mt/2} r + \frac{3}{m} \left(\arctanh \left(\frac{1}{2}e^{mt/2}mr\right) + \arctanh \left(\frac{4 - e^{mt} (4 + m^2 r^2)}{-4 + e^{mt} (-4 + m^2 r^2)}\right)\right).
\]

Note that this self-accelerating solution has a flat FRWL metric; however, one can also write it as an open or closed FRWL spacetime, with the price of changing the St"uckelberg fields accordingly. In all the FRWL frames, the St"uckelberg fields are inhomogeneous. In fact, it was suggested that there was no FRW solution that keeps the FRW symmetry for the fiducial metric $\Sigma_\mu^\nu$ [31]. However, Gumrukcuoglu et al found a special self-accelerating solution which represents an open universe where the fiducial metric $\Sigma_\mu^\nu$ respects the FRW symmetries of the physical metric $g_{\mu\nu}$ [32]. Their solution is given by

\[
b = 1, \quad a = \frac{a_0(t)}{1 - (mr)^2/16}, \quad f(t, r) = \frac{3}{m} a_0(t) \left(\frac{1}{1 - (mr)^2/16}\right),
\]

\[
a_0(t) = \sinh(mt/2).
\]

For this solution, the fiducial metric preserves the FRW symmetry, $\Sigma_\mu^\nu = (9/4)\text{diag}(2a_0/m)^2$, 1, 1, 1. The behaviour of perturbations around this particular self-accelerating solution is very different from the other solutions that break the FRW symmetry for the fiducial metric $\Sigma_\mu^\nu$. At the linear order, scalar and vector perturbations have no kinetic terms; hence, they are strongly coupled [33], which leads to nonlinear instabilities [34, 35]. The absence of the scalar kinetic term originates from the special choice of the solution for $f$ [36] that retains the FRW symmetry for the fiducial metric. This leads to an enhanced symmetry that eliminates the scalar perturbations [37]. In the rest of this paper, we do not consider this class of self-accelerating solution and consider the case where the FRW symmetry is broken.
for the fiducial metric $\Sigma_{\mu\nu}$. However, we emphasize that the physical metric still retains the FRW symmetry in these solutions.

Wyman et al [30] also showed that the ordinary Friedmann equation is obtained even if we add ordinary matter energy density $\rho_m(t)$. The matter only sees the effect of the mass terms as a cosmological constant with no direct coupling to the scalar fields on the exact solution. Cosmological solutions in massive gravity and its extension including a de Sitter fiducial metric and bigravity can be found in [38].

4.2. Decoupling limit solutions and their instability

Once we determined self-accelerating, de Sitter solutions in this model, it is crucial to study their stability: this is the subject of this section. In order to make the analysis manageable, we focus on a convenient limit of Lagrangian (6) which captures most of the dynamics of the helicity-0 and helicity-1 mode, but keeps the linear behaviour of the helicity-2 (tensor) mode [3]. The limit, called the decoupling limit, is defined as

$$m \to 0, \quad M_{pl} \to \infty, \quad \Lambda_3 \equiv m^2 M_{pl} = \text{fixed.}$$ (53)

In order to obtain canonically normalized kinetic terms for the helicity-2 and helicity-1 modes, together with the relevant couplings for the helicity-0 modes, when this limit is taken one needs to canonically normalize the fields in the following way:

$$h_{\mu\nu} \to M_{pl} h_{\mu\nu}, \quad A_\mu \to m M_{pl} A_\mu, \quad \pi \to m^2 M_{pl} \pi,$$ (54)

where we have split the St"uckelberg fields $\pi^\mu$ into a scalar component $\pi$ and a divergenceless vector $A^\mu$ in the usual way, namely

$$\pi^\mu = \eta^{\mu\nu}(\partial_\nu \pi + A_\nu).$$ (55)

In order to take the decoupling limit (53) of the self-accelerating solutions defined by (45), the solution has to be written in a particular frame where the limit is well defined. For example, if one tries to na"ively take the decoupling limit of solution (51), the metric $h_{\mu\nu}$ will diverge as $M_{pl} \to \infty$. A frame, where the limit can be taken is the conformally flat frame, defined as follows:

$$a(t, r) = b(t, r) = \frac{c(t, r)}{r} = \left(1 + H^2 (r^2 - l^2)/4\right)^{-1}.$$ (56)

In this frame, all the known self-accelerating solutions lead to the same decoupling limit solution for the St"uckelberg fields, namely

$$\pi^0 = \left(\frac{1 + 3\alpha_3 + \alpha_5}{2 + 3\alpha_3 + \alpha_5} \sqrt{\Delta_0} - 1\right) l - \frac{1}{2} \sqrt{\frac{(1 + 3\alpha_3 + 2\alpha_5)[(1 + 3\alpha_3 + \alpha_5)^4 - (2 + 3\alpha_3 + \alpha_5)^4 \Delta_0]}{3\Delta_0 (1 + 3\alpha_3 + \alpha_5)^2 (2 + 3\alpha_3 + \alpha_5)^2}} m r^2 + \mathcal{O}(m^2),$$

$$\pi^r = -\frac{1}{1 + 3\alpha_3 + \alpha_5} r + \mathcal{O}(m^2),$$ (57)

with $\alpha_5$ given by equation (50). We recall the reader that $Q_0$ is one of the two integration constants of the black hole solution found in the section 3.2, and in the decoupling limit, it sets a background vector charge, as one may appreciate in the following equations. If we split $\pi^\mu$ into a scalar and vector piece as in (55), canonically normalize the fields as in (54) and
take the decoupling limit (53), then one obtains [40, 41]
\[ h_{\mu\nu} = -\frac{1}{2} \Lambda_3 H^2 (r^2 - t^2) \eta_{\mu\nu}, \]
\[ \pi = -\frac{1}{2} \left( 1 - \frac{1}{c_0} \right) \left( 1 + \frac{3 (1 + 3 \alpha_3 + \alpha_5) Q_0^2}{(2 + 3 \alpha_3 + \alpha_5)^2 (1 + 3 \alpha_3 + 2 \alpha_5)} \right) \Lambda_3 r^2 + \frac{c_0 - 1}{2c_0} \Lambda_3 r^2, \]
\[ A_0 = -\frac{Q_0}{2} \Lambda_3 r^2, \] (58)

where \( c_0 \) is given by (31) (or (32) for \( \alpha_3 = -4 \alpha_4 \), and the Hubble parameter \( H \) by
\[ H^2 = \frac{1}{3} (1 + 3 \alpha_3 \pm 2 \alpha_5) m^2, \] (59)

which is the generalization of (42) for arbitrary \( \alpha_3 \) and \( \alpha_4 \). Moreover, there is a relation between \( \Delta_0 \) and \( Q_0 \):
\[ \Delta_0 = \frac{Q_0^2 (2 + 3 \alpha_3 + \alpha_5)^2 (1 + 3 \alpha_3 + \alpha_5)^4 + (2 + 3 \alpha_3 + \alpha_5)^4 (1 + 3 \alpha_3 + 2 \alpha_5)}{(1 + 3 \alpha_3 + \alpha_5)^4}, \] (60)

and in the case of AdS, there is an extra bound given by
\[ Q_0^2 < \frac{(2 + 3 \alpha_3 + \alpha_5)^2 (1 + 3 \alpha_3 + 2 \alpha_5)}{3 (1 + 3 \alpha_3 + \alpha_5)^4}. \] (61)

If one takes the vector charge to zero \( Q_0 = 0 \) (or equivalently if \( \Delta_0 = (1 + 3 \alpha_3 + \alpha_5)^4 / (2 + 3 \alpha_3 + \alpha_5)^4 \)), then these solutions can be written in a simpler covariant way:
\[ h_{\mu\nu} = -\frac{1}{2} \Lambda_3 H^2 (x^\mu x_\mu) \eta_{\mu\nu}, \quad \pi = \frac{c_0 - 1}{2c_0} \Lambda_3 x^\mu x_\mu, \quad A^\mu = 0. \] (62)

Therefore, corrections of order \( m^2 \) in (57), which do not show in the decoupling limit, are the main differences among solutions in the full theory. These solutions in the decoupling theory were also found in [29].

In this decoupling limit, the structure of the Lagrangian becomes much simpler and the various self-accelerating configurations become the same. For these reasons, it is particularly convenient to study the dynamics of perturbations in this limit. If problems or instabilities arise in this limit, then they are unavoidably present also in the full theory outside the decoupling regime. Interestingly, it has been shown in [29] that, in the decoupling limit, the coupling between the scalar mode \( \pi \) and the trace \( T \) of the energy–momentum tensor vanishes around these self-accelerating configurations; hence, the coupling to matter is the same as in GR with no need to implement a Vainshtein mechanism. However, we have shown in [40, 41] that all these backgrounds present instabilities in the vector sector. We present here the main results concerning these instabilities, focusing on the case of \( \alpha_3 = \alpha_4 = 0 \). A generalization to arbitrary values is straightforward and can be found in [40, 41]. We start by considering perturbations of the fields \( h_{\mu\nu}, A^\mu \) and \( \pi \) which only depend on time and radial components. Namely
\[ h_{\mu\nu} = h^0_{\mu\nu} + \hat{h}_{\mu\nu}, \quad A^\mu = A^\mu_0 + \hat{A}^\mu, \quad \pi = \pi_0 + \hat{\pi}, \] (63)

where the background quantities (those with an index 0) are given by the self-accelerating solution (62). The Lagrangian for the tensor and scalar perturbations (without further truncations) reads
\[ \mathcal{L}_{h_{\mu\nu}, \pi} = -\frac{1}{2} \hat{h}_{\mu
u} \hat{e}^{\rho\sigma} \hat{e}^{\rho\sigma} \hat{h}_{\mu\nu} + \hat{h}_{\mu\nu} X_{\mu\nu} - 6H^2 \hat{\pi} \Box \hat{\pi}, \] (64)
where (in units in which $\Lambda_3 = 1$, that we adopt from now on) $H^2 = \frac{1}{2}$ and $\Delta_{\mu\nu}$ is given by

$$\Delta_{\mu\nu} = [\hat{\Pi}_{\mu}^\rho \hat{\Pi}_{\nu} - \hat{\Pi}_{\mu} \hat{\Pi}_{\nu} + \frac{1}{2} \eta_{\mu\nu} (\hat{\Pi}_{\rho}^\sigma \hat{\Pi}_{\sigma}^\rho - \hat{\Pi}_{\rho}^\rho \hat{\Pi}_{\rho}^\rho)] .$$

(65)

We can use the following field redefinition to decouple the helicity-2 from the helicity-0 field:

$$\hat{h}_{\mu\nu} \rightarrow \hat{h}_{\mu\nu} - \partial_\mu \hat{\pi} \partial_\nu \hat{\pi} .$$

(66)

Then, the kinetic terms for tensor and scalar are diagonalized resulting, up to total derivatives, in

$$\mathcal{L}_{h_{\mu\nu}, \pi} = -\frac{1}{2} \hat{h}_{\mu\nu} \mathcal{L}^{\mu\nu\alpha\beta} \hat{h}_{\alpha\beta} - 6 H^2 \hat{\pi} \Box \hat{\pi} - \frac{1}{2} (\Box \hat{\pi})^2 - (\partial_\mu \partial_\nu \hat{\pi} \partial^\mu \partial^\nu \hat{\pi}) .$$

(67)

Let us emphasize that the previous Lagrangian contains terms which are quadratic in $\hat{h}_{\mu\nu}$, but higher orders in the scalar field $\hat{\pi}$. The scalar field terms are the so-called Galileon combinations. In contrast, as mentioned before, the vector piece has an infinite number of interactions [40, 41]. For our purposes, it is enough to stop at fourth order in the fields, resulting in the Lagrangian

$$\mathcal{L}_{A_{\mu}} = \frac{1}{18} \left\{ 3 \hat{\Pi}_{\mu\nu} F_{\rho\nu} - 6 (\hat{\Pi}_{\mu}^\rho F^{\rho\nu} F_{\nu}) \right. \right.
$$

$$- 2 \left[ \hat{\Pi}_{\mu}^\rho \hat{\Pi}_{\nu}^\sigma F_{\rho\sigma} - \hat{\Pi}_{\mu}^\rho F_{\rho\nu} F_{\nu} - \hat{\Pi} (\hat{\Pi}_{\rho}^\mu F^{\rho\nu} F_{\nu}) \right] + \cdots$$

(68)

where $F_{\mu\nu} = \partial_\mu A_{\nu} - \partial_\nu A_{\mu}$. As mentioned above, the kinetic terms for the vectors vanish; however, vectors become dynamical by coupling them with the scalar at third or higher order in fluctuations (this was already pointed out in [29, 40, 42]). Nevertheless, one should worry about higher derivatives in the equations of motion, since the previous Lagrangian contains contributions with two time derivatives in the scalar field $\hat{\pi}$. For systems coupling scalars with vectors, it is possible to find the combination that ensures that the equations of motion do not contain at all terms containing more than two time derivatives. It is a generalization of Galileon combinations which was explored in [43] and dubbed $p$-form Galileons. Up to fourth order in perturbations, the correct combination (without including higher derivatives in $A_{\mu}$) is

$$\mathcal{L}^{\text{p-form}} = a_0 [\text{tr} \Pi F^2 - 2 \text{tr} F^2 F^2] + b_0 [-\text{tr} F^2 [\text{tr} \Pi^2 - (\text{tr} \Pi)^2] - 4 \text{tr} \Pi \text{tr} F^2 + 4 \text{tr} \hat{\Pi}^2 F^2 + 2 \text{tr} \Pi F \Pi F]$$

(69)

where $a_0$ and $b_0$ are arbitrary coefficients, and we have used the tr notation to simplify the index structure. The above third-order action with the aforementioned properties was presented in [43], while the fourth-order one is as far as we know new. Comparing (68) with (69) we note that while the third-order action has the correct structure to avoid higher order time derivatives in the equation of motion, the fourth-order Lagrangian does not seem to satisfy this requirement. However, a suitable field redefinition makes it possible to remove the fourth-order term from the third-order contribution, leaving a healthy Lagrangian without higher derivative equations of motion (in agreement with the ghost-free statement of the theory).

On the other hand, although our scalar–vector Lagrangian (68) does not lead to a propagation of a sixth ghost mode, it does generally lead to a ghost-like instability around self-accelerating configurations, in which the ghost is one of the available vector modes. (Another issue with this theory is acasuality, associated with the propagation of superluminal signals, as recently pointed out in [39].) It has been shown in [40] that, when turning on a non-trivial profile for the background vector field, the corresponding Lagrangian for perturbations around the resulting configuration acquires kinetic terms for the vector with the wrong sign. Here, following [41], we instead directly point out the instability by analysing the Hamiltonian associated with the Lagrangian obtained by combining the third-order Lagrangian contained in (68) with the scalar kinetic term:

$$\mathcal{L}^{\text{third}} = -3 H^2 \Box \hat{\pi} - \frac{1}{2} \left[ \Box \hat{\pi} F_{\mu\nu} F^{\mu\nu} - 2 \partial_\mu \hat{\pi} F^{\mu\rho} F_{\rho} \right].$$

(70)
where we have removed the hats over the field to simplify the notation. We choose, for simplicity, the gauge $A_0 = 0$, $\partial_i A_i = 0$, and by doing a standard $(3 + 1)$-decomposition, the previous Lagrangian reads

$$ \mathcal{L} = -3H^2\dot{\pi}^2 + \frac{1}{12}[2\pi \dot{A}_i \Delta A_i + \Delta \pi \dot{\pi} - \pi_{ij} \dot{A}_i \dot{A}_j] + \cdots, $$

(71)

where the dots represent the terms without time derivatives, which we do not include since they do not play a role in the present discussion. The conjugate momenta to $\pi$ and $A_i$ are

$$ \Pi_\pi = -6H^2\left(\pi - \frac{1}{9H^2} \dot{A}_i \Delta A_i\right), \quad \Pi_{A_i} = \frac{2}{3}[\dot{\pi} \Delta A_i + \Delta \pi \dot{A}_i - \pi_{ij} \dot{A}_j]. $$

(72)

In order to analyse the associated Hamiltonian, it is convenient to introduce the matrix $\kappa_{ij} \equiv \Delta \pi \delta_{ij} - \pi_{ij}$. If $\kappa_{ij} = 0$, then we can easily invert the relations that define the conjugate momenta, and obtain the following Hamiltonian:

$$ \mathcal{H} = -\frac{\Pi_\pi^2}{12H^2} + \frac{1}{12H^2} \left( \Pi_\pi + \frac{9H^2 A_i \Pi_{A_i}}{A_i \Delta A_i} \right)^2 + \cdots, $$

(73)

$$ = \frac{3}{2} \frac{\Pi_\pi A_i \Pi_{A_i}}{A_i \Delta A_i} + \frac{27H^2}{4} \left( \frac{A_i \Pi_{A_i}}{A_i \Delta A_i} \right)^2 + \cdots, $$

(74)

where the dots represent terms without momentum variables. The previous Hamiltonian is linear in $\Pi_\pi$; hence, it is unbounded from below. Note that this argument holds even in the limit in which $H^2$ vanishes. In conclusion, perturbations of the background self-accelerating solution, along the direction of scalar fluctuations such that $\kappa_{ij} = 0$, admit unstable directions along which the system falls towards regions where the energy is unbounded from below. Similar conclusions hold for more generic $\kappa_{ij}$. Let us, for example, consider a $\kappa_{ij}$ that is non-vanishing and invertible. Then, after straightforward manipulations, one can show that the Hamiltonian can be written as

$$ \frac{4}{3} \mathcal{H} = -\frac{1}{9H^2 + \Delta A_i \kappa_{ij}^{-1} \Delta A_j} \left( \Pi_\pi - \Delta A_i \kappa_{ij}^{-1} \Pi_{A_i} \right)^2 + \Pi_{A_i} \kappa_{ij}^{-1} \Pi_{A_j} + \cdots, $$

(75)

where, again, the dots represent terms without momentum variables. It is not difficult to see that there are many unstable directions associated with this Hamiltonian. For example, make a choice for the vector $\Delta A_i$ so that the scalar combination $\mathcal{C} \equiv \Delta A_i \kappa_{ij}^{-1} \Delta A_j$ is non-vanishing and has a given sign. For definiteness, the magnitude of $\Delta A_i$ is chosen such that the denominator of the first term has the same sign of $\mathcal{C}$. Accordingly, choose the vector $\Pi_{A_i}$ such that $\Pi_{A_i} \kappa_{ij}^{-1} \Pi_{A_j}$ has the same sign of $\mathcal{C}$ (for example, choose it in the same direction of the $\Delta A_i$). Then, by choosing a suitable magnitude for $\Pi_\pi$, it is possible to make one of the two terms in the previous Hamiltonian arbitrarily negative—hence the Hamiltonian is unbounded from below. Other cases, such as the case in which $\kappa_{ij}^{-1}$ is non-vanishing but not invertible, can be treated in a similar way. Furthermore, while here we focussed on the case $\alpha_3 = \alpha_4 = 0$ it is straightforward to extend this analysis to the more general case, obtaining the same conclusion (see [41] for details).

To summarize, one generically expects instabilities around the self-accelerating solutions discussed so far: there are many directions in the moduli space of fluctuations along which the energy is unbounded from below, and towards which the system can be driven into dangerous regions. On the other hand, to close this section with a positive perspective, it might very well be that suitable deformations of known solutions (or even completely new configurations) exist that, renouncing to the symmetries imposed on the ans"atze considered so far, do not present the problems discussed above. Very recently, a proposal in this direction has been pushed forward in [35], which considers the possibility of breaking the isotropy of three-dimensional spatial slices to find stable configurations. Still much work is needed to clarify this subject and analyse phenomenological consequences of these solutions.
5. Future directions

Massive gravity is a good theoretical laboratory to study the modifications of GR with interesting phenomenological consequences. Nonlinear self-interactions of massive gravity in proximity of a source manage to mimic the predictions of linearized GR, hence agreeing with solar-system precision measurements. Moreover, massive gravity offers a concrete set-up for studying models of dark energy in modified gravity scenarios. Indeed, at large distances gravity is modified with respect to GR, and the theory admits cosmological accelerating solutions in the vacuum in which the size of acceleration depends on the graviton mass. Dark energy models built in this way have the opportunity to be technically natural in the ’t Hooft sense: in the limit of graviton mass going to zero one gains a symmetry, by recovering the full diffeomorphism invariance of GR. Consequently, assuming that the bare cosmological constant is zero, any corrections to the size of dark energy must be proportional to the (tiny) graviton mass itself.

Hence, nonlinear effects play a crucial role for characterizing phenomenological consequences of massive gravity. Motivated by this fact, the analysis of exact solutions of the equations of motion, obtained by imposing appropriate symmetries (spherical symmetry for static spacetimes, or homogeneity and isotropy for cosmological set-ups), manifests, in idealized but representative situations, how the nonlinear dynamics of the graviton degrees of freedom respond to the presence of a source, or, at very large scales, to the graviton mass itself. This has been the argument of this paper, in which we reviewed our works on these topics.

Much interesting work is left for the future: our results can be extended in various directions that will improve our understanding of massive gravity and, in general, of consistent infrared modifications of GR. From one side, it would be interesting to find new stationary configurations renouncing spherical symmetry and to test analytically the effectiveness of Vainshtein mechanism when spherical symmetry is broken. As a concrete example, it would be interesting to find the analogues of the Kerr geometry in this scenario, in which frame dragging effects can be quantitatively analysed. Also, it would be interesting to determine cosmological configurations that break the isotropy or homogeneity of the cosmological solutions analysed until now. Indeed, working in a suitable decoupling limit, we have shown that the cosmological self-accelerating configurations studied so far are characterized by instabilities in the vector sector. Given the recent results of [35], we speculate that these instabilities can be possibly avoided by renouncing some of the symmetries that characterize the solutions (for example isotropy of the three spatial directions). It would be interesting to determine stable self-accelerating backgrounds following this route and study their consequences for what respect the dynamics of cosmological fluctuations. We hope to be able to develop all these questions in our future work.

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Appendix. General exact solution

From the general Lagrangian (6), and using the non-diagonal ansatz (12) together with Einstein equations (7), one can show that there are two branches of solutions for non-vanishing $\alpha_3$ and
\(\alpha_4\) as was done in section 3. Here, we only consider the branch with a non-diagonal metric, where analytic solutions can be found. Since the combination \(\sqrt{1 + 3\alpha_3 + 9\alpha_3^2 - 12\alpha_4}\) is always present in the solution of this branch (see (31)), it is convenient to map the \((\alpha_3, \alpha_4)\) parameters into \((\alpha_3, \alpha_5)\), where \(\alpha_5^2 = 1 + 3\alpha_3 + 9\alpha_3^2 - 12\alpha_4\). In this new set of parameters, the combination, \(d(r)G_{rt} + b(r)^2G_{tt} = 0\), fixes \(c(r)\) as a function of \(r\) in the following way:

\[
c(r) = c_0r = \frac{(1 + 3\alpha_3 + \alpha_5)}{(2 + 3\alpha_3 + \alpha_5)}r.
\] (A.1)

The rest of Einstein equations give

\[
b(r)^2 = \frac{\Delta_0}{c_0^2}(1 - p), \quad a(r)^2 = \frac{\Delta_0}{c_0^2}(p + \gamma + 1), \quad d(r) = \sqrt{\Delta_0 - a(r)^2b(r)^2}, \] (A.2)

where

\[
p = \mu + \left(\frac{1 + 3\alpha_3 + 2\alpha_5}{2(2 + 3\alpha_3 + \alpha_5)}\right)m^2r^2, \quad \gamma + 1 = \frac{(1 + 3\alpha_3 + \alpha_5)}{\Delta_0(2 + 3\alpha_3 + \alpha_5)^2}.
\] (A.3)

Just like in the \(\alpha_3 = \alpha_4 = 0\) \((\alpha_5 = 1)\) case, there are two integration constants, \(\mu\) and \(\Delta_0\), but in order to have a positive argument for the square root in \(d(r)\), \(\Delta_0\) has to run from \(\Delta_0 = 0\) to \(\Delta_0 = c_0^2\). If we focus on the massless case \(\mu = 0\) only, then the solution describes the static patch of the de Sitter or anti-de Sitter spacetime.

**References**

[1] Fierz M and Pauli W 1939 Proc. R. Soc. Lond. A 173 211–32
[2] Boulware D G and Deser S 1972 Phys. Rev. D 6 3368–82
[3] Arkani-Hamed N, Georgi H and Schwartz M D 2003 Ann. Phys. 305 96–118 (arXiv:hep-th/0210184)
[4] van Dam H and Veltman M J G 1970 Nucl. Phys. B 22 397–411
[5] Zakharov V I 1970 JETP Lett. 12 312
[6] Vainshtein A I 1972 Phys. Lett. B 39 393–4
[7] Creminelli P, Nicolis A, Papucci M and Trincherini E 2005 J. High Energy Phys. JHEP09(2005)003 (arXiv:hep-th/0505147)
[8] Deffayet C and Rombouts J-W 2005 Phys. Rev. D 72 044003 (arXiv:gr-qc/0505134)
[9] Nicolis A, Rattazzi R and Trincherini E 2009 Phys. Rev. D 79 064036 (arXiv:0811.2197 [hep-th])
[10] de Rham C, Gabadadze G and Tolley A J 2011 Phys. Rev. Lett. 106 231101 (arXiv:1101.1232 [hep-th])
[11] Hinterbichler K 2012 Rev. Mod. Phys. 84 671 (arXiv:1105.3735 [hep-th])
[12] de Rham C and Gabadadze G 2010 Phys. Rev. D 82 044020 (arXiv:1007.0443 [hep-th])
[13] Koyama K, Niz G and Tasinato G 2011 Phys. Rev. D 84 064033 (arXiv:1104.2143 [hep-th])
[14] Hassan S F and Rosen R A 2012 J. High Energy Phys. JHEP04(2012)123 (arXiv:1111.2070 [hep-th])
[15] Hassan S F and Rosen R A 2012 J. High Energy Phys. JHEP02(2012)126

Kluson J 2012 arXiv:1204.2957 [hep-th]
Kluson J 2012 arXiv:1202.5899 [hep-th]
Mirkabuyi M 2011 arXiv:1112.1435 [hep-th]
Hassan S F, Rosen R A and Schmidt-May A 2012 J. High Energy Phys. JHEP02(2012)026
Kluson J 2012 J. High Energy Phys. JHEP01(2012)013
de Rham C, Gabadadze G and Tolley A J 2011 J. High Energy Phys. JHEP11(2011)093 (arXiv:1108.4521 [hep-th])
Hassan S F and Rosen R A 2012 J. High Energy Phys. JHEP04(2012)123 (arXiv:1111.2070 [hep-th])
Kluson J 2012 arXiv:1209.3612 [hep-th]
Nomura K and Soda J 2012 arXiv:1207.3637 [hep-th]
Deffayet C, Moursad J and Zahariaide G 2012 arXiv:1207.6338 [hep-th]
Deffayet C, Moursad J and Zahariaide G 2012 arXiv:1208.4493 [gr-qc]
[16] Chamseddine A H and Mukhanov V 2013 J. High Energy Phys. JHEP03(2013)092 (arXiv:1302.4367 [hep-th])
[17] Higuchi A 1987 Nucl. Phys. B 282 397
[18] Koyama K, Niz G and Tasinato G 2011 Phys. Rev. Lett. 107 131101 (arXiv:1103.4708 [hep-th])
[19] Bereziani L, Chkareuli G and Gabadadze G 2013 arXiv:1302.0549 [hep-th]
[20] Shiba F, Niz G, Koyama K and Tasinato G 2012 Phys. Rev. D 86 024033 (arXiv:1204.1193 [hep-th])
[20] Chkareuli G and Pirtskhalava D 2012 Phys. Lett. B 713 99 (arXiv:1105.1783 [hep-th])
[21] Wyman M 2011 Phys. Rev. Lett. 106 201102 (arXiv:1101.1295[astro-ph.CO])
Kaloper N, Padilla A and Tanahashi N 2011 J. High Energy Phys. JHEP10(2011)148 (arXiv:1106.4827 [hep-th])
Hiramatsu T, Hu W, Koyama K and Schmidt F 2012 arXiv:1209.3364 [hep-th]
Belikov A V and Hu W 2012 arXiv:1212.0831 [gr-qc]
de Rham C, Tolley A J and Wesley D H 2013 Phys. Rev. D 87 044025 (arXiv:1208.0580 [gr-qc])
De Felice A, Kase R and Tsujikawa S 2012 Phys. Rev. D 85 044059 (arXiv:1111.5090 [gr-qc])
Belikov A V and Hu W 2012 arXiv:1212.0831 [gr-qc]
Li B, Zhao G-B and Koyama K 2013 arXiv:1303.0008[astro-ph.CO]
[22] Horndeski G W 1974 Int. J. Theor. Phys. 10 363–84
[23] Narikawa T, Kobayashi T, Yamauchi D and Saito R 2013 arXiv:1302.2311[astro-ph.CO]
[24] Salam A and Strathdee J A 1977 Phys. Rev. D 16 2668
[25] Gurses M 1979 Phys. Rev. D 20 1019
Nieuwenhuizen T M 2011 Phys. Rev. D 84 024038 (arXiv:1103.5912 [hep-th])
[26] Dvali G R, Gabadadze G and Porrati M 2000 Phys. Lett. B 485 208 (arXiv:hep-th/0005016)
[27] Deffayet C 2001 Phys. Lett. B 502 199 (arXiv:hep-th/0010186)
[28] Koyama K 2005 Phys. Rev. D 72 123511 (arXiv:hep-th/0503191)
Gorbunov D, Koyama K and Sibiryakov S 2006 Phys. Rev. D 73 044016 (arXiv:hep-th/0512097)
Charmousis C, Gregory R, Kaloper N and Padilla A 2006 J. High Energy Phys. JHEP10(2006)066 (arXiv:hep-th/0604086)
Nicolis A and Rattazzi R 2004 J. High Energy Phys. JHEP06(2004)059 (arXiv:hep-th/0404159)
Luty M A, Porrati M and Rattazzi R 2003 J. High Energy Phys. JHEP09(2003)029 (arXiv:hep-th/0303116)
[29] de Rham C, Gabadadze G, Heisenberg L and Pirtskhalava D 2011 Phys. Rev. D 83 103516 (arXiv:1010.1780 [hep-th])
[30] Gratia P, Hu W and Wyman M 2012 Phys. Rev. D 86 061504 (arXiv:1205.4241 [hep-th])
[31] D’Amico G, de Rham C, Dubovsky S, Gabadadze G, Pirtskhalava D and Tolley A J 2011 Phys. Rev. D 84 124046 (arXiv:1108.5231 [hep-th])
[32] Gumrukcuoglu A E, Lin C and Mukohyama S 2011 J. Cosmol. Astropart. Phys. JCAP11(2011)030 (arXiv:1109.3845 [hep-th])
[33] Gumrukcuoglu A E, Lin C and Mukohyama S 2012 J. Cosmol. Astropart. Phys. JCAP03(2012)006 (arXiv:1111.4107 [hep-th])
[34] De Felice A, Gumrukcuoglu A E, Mukohyama S, Gumrukcuoglu A E and Mukohyama S 2012 Phys. Rev. Lett. 109 171101 (arXiv:1206.2080 [hep-th])
[35] De Felice A, Gumrukcuoglu A E, Lin C and Mukohyama S 2013 arXiv:1303.4154 [hep-th]
[36] Wyman M, Hu W and Gratia P 2012 arXiv:1211.4576 [hep-th]
[37] Khosravi N, Koyama K, Niz G and Tasinato G in preparation
[38] Gumrukcuoglu A E, Lin C and Mukohyama S 2012 arXiv:1206.2723 [hep-th]
Volkov M S 2012 arXiv:1205.5713 [hep-th]
Kobayashi T, Sinno M, Yamaguchi M and Yoshida D 2012 arXiv:1205.4938 [hep-th]
Comelli D, Crisostomi M, Nesti F and Pilo L 2012 arXiv:1204.1027 [hep-th]
Khosravi N, Sepangi H R and Shahidi S 2012 arXiv:1202.2767 [gr-qc]
Comelli D, Crisostomi M, Nesti F and Pilo L 2012 J. High Energy Phys. JHEP03(2012)067 (arXiv:1111.1983 [hep-th])
Comelli D, Crisostomi M, Nesti F and Pilo L 2012 J. High Energy Phys. JHEP06(2012)020 (erratum) von Strauss M, Schmidt-May A, Enander J, Mortsell E and Hassän S F 2012 J. Cosmol. Astropart. Phys. JCAP03(2012)042 (arXiv:1111.1655 [gr-qc])

Chamseddine A H and Volkov M S 2011 Phys. Lett. B 704 652 (arXiv:1107.5504 [hep-th])
Fasiello M and Tolley A J 2012 arXiv:1206.3852 [hep-th]
Volkov M S 2012 J. High Energy Phys. JHEP01(2012)035 (arXiv:1110.6153 [hep-th])
Vakili B and Khosravi N 2012 Phys. Rev. D 85 083529 (arXiv:1204.1456 [gr-qc])
de Rham C and Heisenberg L 2011 Phys. Rev. D 84 043503 (arXiv:1106.3312 [hep-th])
D’Amico G, Gabadadze G, Hui L and Pirtskhalava D 2012 arXiv:1206.4253 [hep-th]
Langlois D, Naruko A and Naruko A 2012 Class. Quantum Grav. 29 202001 (arXiv:1206.6810 [hep-th])
Saridakis E N 2012 arXiv:1207.1800 [gr-qc]
Gong Y 2012 arXiv:1207.2726 [gr-qc]
Volkov M S 2012 arXiv:1207.3723 [hep-th]
Cai Y-F, Gao C and Saridakis E N 2012 arXiv:1207.3786 [astro-ph.CO]
Motohashi H and Suyama T 2012 arXiv:1208.3019 [hep-th]
Gumrukcuoglu A E, Kuroyanagi S, Lin C, Mukohyama S and Tanahashi N 2012 arXiv:1208.5975 [hep-th]
Akrami Y, Kotivisto T S and Sandstad M 2012 arXiv:1209.0457 [astro-ph.CO]
[39] Deser S and Waldron A 2013 Phys. Rev. Lett. 110 111101 (arXiv:1212.5835 [hep-th])
[40] Koyama K, Niz G and Tasinato G 2011 J. High Energy Phys. JHEP12(2011)065 (arXiv:1110.2618 [hep-th])
[41] Tasinato G, Koyama K and Niz G 2012 arXiv:1210.3627 [hep-th]
[42] D’Amico G 2012 arXiv:1206.3617 [hep-th]
[43] Deffayet C, Deser S and Esposito-Farese G 2010 Phys. Rev. D 82 061501 (arXiv:1007.5278 [gr-qc])