Revisiting the Hanbury Brown-Twiss set-up for fractional statistics

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The Hanbury Brown-Twiss experiment has proved to be an effective means of probing statistics of particles. Here, in a set-up involving edge-state quasiparticles in a fractional quantum Hall system, we show that a variant of the experiment composed of two sources and two sinks can be used to unearth fractional statistics. We find a clear cut signature of the statistics in the equal-time current-current correlation function for quasiparticle currents emerging from the two sources and collected at the sinks.

The statistics of indistinguishable particles is manifested in the fate of the common wavefunction for two particles located at positions \(r_1\) and \(r_2\) under exchange:

\[
\Psi(r_1, r_2, r_3, \ldots, r_n) = e^{i\pi\nu} \Psi(r_2, r_1, r_3, \ldots, r_n),
\]

where, in general, other particles may be present at positions \(r_i\), \(i \neq 1, 2\). The values \(\nu = 2n\) and \(\nu = 2n + 1\) for integers \(n\) correspond to the familiar instances of bosonic and fermionic statistics, respectively. In two dimensions, where the concept of exchange can be unambiguously defined, \(\nu\) can assume fractional values corresponding to anyonic statistics. A landmark example of this phenomenon occurs for Laughlin states \(l\) in the fractional quantum Hall (FQH) set-up. In this system, the anyonic nature of quasiparticle/quasihole excitations has been demonstrated \(l\) and, in particular, the gaining of the phase factor \(e^{i\pi\nu}\) by quasiparticles \(l\) and quasiholes \(l\) under exchange, where \(\nu\) is the filling fraction. Of late, a variety of novel proposals for testing the statistics of edge-state quasiparticles in Laughlin states have come forth \(l\).

In this Letter, we propose a set-up consisting of two edge state quasiparticle sources and two sinks, and the measurement of current-current correlation for currents emerging at the two sources and collected at the sinks. At equal times, the correlator is found to depend only on average values of currents and a factor \(\cos \pi\nu\) coming from statistics. As a function of time difference for when currents are correlated, it shows oscillations with a period that depends on the fractional charge of the quasiparticle.

Returning to the common wavefunction of Eq. 1, one can extract from it various properties, such as filling in of available states, and \(N\)-point correlation functions. A quantity sensitive to statistics is the two-particle correlation function

\[
g(r_1, r_2) = N(N - 1) \int dr_3 \ldots dr_n |\Psi(r_1, r_2, r_3, \ldots r_n)|^2,
\]

where \(N\) is the total number of particles. In systems composed of a single species of non-interacting particles, for fermions \(g(r_1, r_2)\) necessarily drops to zero at \(r_1 = r_2\) and typically levels off to the uncorrelated value for \(r_1 \neq r_2\) much greater than the mean particle spacing. For uncondensed bosons, wave-function symmetrization allows \(g(r_1, r_2)\) to reach twice its uncorrelated value. Our purpose here is to study processes that enable the probing of anyonic systems for their statistical information. Specifically, two-particle correlations are manifested in events such as the ones shown in Fig 1, where particles need not scatter, but may merely be detected within a correlation region in time and space to feel the effect of statistics \(l\).
Our proposed set-up is as shown in Fig. 2. Four leads at the corners of the Hall bar define four edge states denoted by $\beta = A, B, C, D$. Low-energy excitations of the FQH system correspond to long-wavelength density distortions of the edge. These excitations can be described by the chiral Luttinger liquid model, and each of the edge states is characterized by the Hamiltonian

$$H_0^\beta = \frac{1}{4\pi u} \int (\partial_x \phi_\beta)^2 dx_\beta,$$

where the bosonic fields $\phi$ obey the commutation relations $[\phi_\beta(x), \phi_\gamma(x')] = i\pi \nu \text{sign}(x-x') \delta_{\beta\gamma}$, and their gradients are proportional to density distortions. Here, we have set the edge-state velocity to unity. Gates allow for pinching the edge states close to one another to form the cruciform pattern shown in Fig. 2, thus enabling inter-edge quasiparticle tunneling. For each edge state $\beta$, we assume the tunneling to take place from points $x_j$ where $j = 1, 2, 3, 4$ for $\beta = A, B, C, D$ respectively. Here, we require that the region formed by the tunneling points be comparable to the size of the quasiparticles. Unlike in the bulk, a second-quantized description of edge-state quasiparticles is relatively straightforward to formulate. We describe particles at the tunneling points by the creation operators $\psi_j^\dagger = \kappa_j e^{-i\theta_j(x_j)}$, where the $\kappa$ denote Klein factors. The commutation relations for the bosonic fields of Eq. (3) ensure that these quasiparticles, when exchanged with others residing on the same edge, exhibit the statistics of Eq. (4). The Klein factors ensure that they do so when exchanged with particles from neighboring edge states. We pick the convention

$$\psi_j^\dagger \psi_k^\dagger = e^{-i\nu \mathbf{t}} \psi_k^\dagger \psi_j^\dagger,$$

where $j < k$ for $j = 1, 2, 3$, and $k = 1$ for $j = 4$. Tunneling of these quasiparticles to neighboring edge states can be controlled by means of gate voltages. It is described by the tunneling Hamiltonian

$$H_t^{jk} = u_{jk} \psi_j^\dagger \psi_k + \text{h.c.},$$

where ‘h.c.’ denotes Hermitian conjugation, $\psi$’s are as in Eq. (3), and the $u$’s denote tunable bare tunneling strengths. As a variant of the Hanbury Brown-Twiss (HBT) experiment, we select the points $m = 1, 3$ as quasiparticle sources, and $n = 2, 4$ as sinks by raising the potentials of the edge states $A$ and $C$ with respect to $B$ and $D$ by a voltage $V$. As all tunneling occurs in a fixed geometry, two-particle correlation functions cannot be studied as a function of spatial separation. However, current-current correlations can be measured as a function of temporal separation. These tunneling currents take the form

$$I_{mn}(t) = \frac{ie^*}{\hbar} (u_{mn} \psi_m^\dagger \psi_n e^{i\tilde{V}t} - \text{h.c.}),$$

where $e^* = e\nu$ is the charge of the quasiparticle, $\tilde{V} = eV/\hbar$. Edge states then carry measurable currents

$$I_I = \frac{\nu e^2}{\hbar} V - I_{12} - I_{14}; \quad I_{II} = I_{12} + I_{32}$$

$$I_{III} = \frac{\nu e^2}{\hbar} V - I_{32} - I_{34}; \quad I_{IV} = I_{34} + I_{14}, \quad (7)$$

where in Fig. 2, $I_n$ are currents going into leads ‘$\alpha$’.

The finite-temperature average values of these currents can be calculated using non-equilibrium Keldysh techniques that treat tunneling perturbatively (see, for e.g., Ref. [12]). To summarize the treatment, from Eq. (6), we derive an action for each of the four edge states. Away from the tunneling points $x_j$, the edge states are described by free fields, whose form is explicitly obtained in terms of the fields at the tunneling points using equations of motion. These free fields are integrated out to obtain an effective action described by fields $\phi_j$ at points $x_j$. We then introduce a generating functional in terms of backwards and forwards real-time paths $\phi^\pm_j(t)$, which enables us to obtain expectation values. In equilibrium, the correlation function $C_j = \langle \phi_j^\dagger \phi_j \rangle$ and response function $R_j = \langle \phi_j^\dagger \phi_j \rangle$, where $\phi^\pm_j = \phi_j \mp \frac{i}{\omega} \phi_j$, satisfy the fluctuation-dissipation theorem $C_j(\omega) = \text{coth}(\hbar\omega/2kT) R_j(\omega)$. The effect of tunneling is treated perturbatively.

To second order in tunneling, the technique described above gives the following form for the average currents:

$$\langle I_{mn}(t) \rangle = \nu e \left(\frac{2u_{mn}}{\hbar}\right)^2 \int dt_a \times \sin \tilde{V}(t_a - t) \sin F(t_a - t) e^{-f(t_a - t)}; \quad (8)$$

where $F(t) \equiv \nu \int d\omega \omega^{-1} \sin \omega t$ and $f(t) \equiv 2\nu \int d\omega \omega^{-2} \text{coth}(\hbar\omega/2kT) \sin(\omega t/2)^2$ originate from response and correlation functions respectively. Evaluating the above gives for the differential conductance contributions,

$$\frac{dI}{dV} = \nu e V (4e V / kT), \quad G(x \rightarrow \infty) \rightarrow \text{const.},$$

which increases with decreasing voltage. Thus, at low temperatures, the conductance can be controlled through magnetic flux.
temperatures, which is desirable for keeping thermal noise minimal, the voltage \( V \) must be held large compared to the bare tunneling strength for the perturbative treatment to remain valid.

We now analyze current correlations of particles emerging from the two sources detected at the two sinks. In principle, a variety of current-current correlators contain information on statistics. As was originally shown by Hanbury Brown and Twiss, even particles from a single source can be distributed into two detectors to exhibit statistical correlations \([15]\). Thus, focusing on one set of source-drain edge states and measuring correlations between the transmitted and reflected currents \([16]\) along the source edge state, as was done for the integer quantum Hall system \([11]\), can give statistical information. Alternatively, the current-current correlations between currents from one source collected at two different drains can be calculated \([17]\), as has been done explicitly for the FQH set-up \([8]\). Here, we find that a clear-cut signature of anyonic statistics comes from events shown in Fig. 4. To extract information on statistics from these events, we propose the measurement of the following time-translation invariant current-current correlator:

\[
\mathcal{C}(t - t') = \langle \Delta I_{12}(t) \Delta I_{34}(t') + \Delta I_{14}(t) \Delta I_{32}(t') \rangle,
\]

where \( \Delta I \equiv I - \langle I \rangle \).

The correlator \( \mathcal{C} \) can be obtained from three sets of measurements. The first would measure correlations \( \langle \Delta I_{12}(t) \Delta I_{14}(t') \rangle \) for currents \( I_{12} \) and \( I_{14} \) measured \( \Delta t = t - t' \) apart. The other two sets of measurements would be performed in the absence of sources ‘1’ and ‘3’, respectively, realized by controlling the appropriate tunneling strengths \( u_{mn} \) by means of gate voltages. It is important to note that these other sets of measurements do not require a change in sample, but can be achieved merely by applying the required gate voltages in a single sample. In each of these instances, currents into leads \( I_{1} \) and \( I_{2} \) would have the form \( I_{12} = I_{m2} \) and \( I_{14} = I_{m4} \) with \( m = 3 \) and \( m = 1 \), respectively. Then, one could measure cross-correlations \( \langle C \rangle \), for current from one source held at the same potential \( V \) as in the first case, into two drains, where \( \tilde{C}_m(\Delta t) = \langle \Delta I_{m2}(t) \Delta I_{m4}(t') \rangle \), with \( m = 3, 1 \) respectively, and \( \Delta t = t - t' \). These correlations themselves carry statistical information, but are complicated by the fact that the source edge states are endowed with their own dynamics. Nevertheless, as seen in Ref. \([8]\), one can procure valuable information from them similar to that contained in our sought-after correlator \( \mathcal{C}(t - t') \) of Eq. 3. This correlator \( \mathcal{C} \) can now be obtained by subtracting the contributions of the latter two measurements from the first:

\[
\mathcal{C}(\Delta t) = \langle \Delta I_{12}(t) \Delta I_{14}(t') \rangle - \tilde{C}_1(\Delta t) - \tilde{C}_3(\Delta t).
\]

In fact, Ref. \([15]\) proposes completely analogous sets of measurements in a similar four point tunneling set-up in the integer quantum Hall system, and there too, a correlator analogous to \( \mathcal{C} \) provides key information on statistics.

The correlation can be evaluated in the perturbative Keldysh approach outlined above. To lowest non-vanishing order, i.e., fourth order in tunneling, it takes the form

\[
\mathcal{C}(\Delta t) = \langle I_{12}(t) \rangle \langle I_{34}(t') \rangle + \langle I_{14}(t) \rangle \langle I_{32}(t') \rangle + C_\circ(\Delta t).
\]

(11)

The function \( C_\circ(\Delta t) \) is the piece in the perturbation that connects all points 1–4 of Fig. 2 and thus contains information on the statistics. Explicitly, it is given by

\[
C_\circ(\Delta t) = \cos \pi \nu (e^*)^2 \prod_{m,n} 2^{\nu m n} \frac{\gamma}{h} \int dt_a dt_b \times \left[ \cos \left( \frac{1}{2} \int_0^{\infty} V t' - t_a - t_b \cos \bar{F} e^{-\bar{f}} \right) \right],
\]

where \( m = 1, 3 \) and \( n = 2, 4 \). Here, \( \bar{F} \equiv 1/2 \sum_{a=1,3} (F(t - t_a) + F(t' - t_a)) \) and similarly \( \bar{f} \), involve the functions \( F \) and \( f \), which appear in Eq. 3. When the time difference \( \Delta t \) is small, i.e. \( \hbar/\Delta t \gg k T, e^* V \), one expects the correlations to be maximal \([10]\). In fact, in this limit and for uniform scattering \( u_{mn} = u \) (which we assume from here on), upon evaluating Eq. 12, the current correlation defined in Eq. 9 reduces to the simple and suggestive form

\[
\mathcal{C}(\Delta t \rightarrow 0) = 2 \left[ 1 + \cos \pi \nu \right] \langle I_{12} \rangle \langle I_{34} \rangle,
\]

(13)

where the behavior of the average currents \( \langle I \rangle \) is given in Eq. 8. This is consistent with the fermionic limit \( \mathcal{C}(0) = 0, \nu = 1 \), which reflects the fact that two electrons cannot be in the same place simultaneously, and the bosonic limit of maximal “bunching” for \( \nu = 0 \).

The function \( \mathcal{C}(\Delta t) \) for finite \( \Delta t \) carries telling information on two-particle correlations for edge-state quasiparticles. At \( T = 0 \), the integral of Eq. 12 can be evaluated using contour integration to give

\[
\frac{C_\circ(\Delta t)}{2 \cos \pi \nu} = \left[ \frac{4 \pi^2 e^*}{\hbar^2 \left( \frac{\left( V^2 - \epsilon_0^{*2} - \epsilon_0^* V / e \right) \pi \cos \bar{V} \Delta t / 2}{V \Gamma(2 \nu) \Gamma(1 - 2 \nu)} \right)^2} \times \left[ \text{Re} \left( e^{-i \bar{V} \Delta t / 2} \int_0^\infty e^{-\nu \bar{V} - \nu(t + i \bar{V} \Delta t - \nu)} \right) \right]^2, \]

(14)

where \( \epsilon_0 \), the excitation gap for the bulk Hall fluid, acts as a high-energy cut off. The resulting behavior of the current-current correlations is shown in Fig. 3.
identifies $e^* = \nu e$ as the quantum of quasiparticle charge that couples to the applied voltage. Note that for Laughlin quasiparticles, statistics and charge are directly related to one another, in that the phase factor acquired under exchange may be interpreted as the Aharonov-Bohm term picked up by the charged quasiparticle. However, as the connection between charge and statistics is more complicated for non-Laughlin states, in these cases, $C(\Delta t)$ becomes important in bearing information on both aspects, distinct from one another. While the dependence on charge and statistics ought to hold regardless of the effective theory used to describe the FQH system, the third feature reflects the chiral Luttinger liquid description of the edge state; at large separation time, $C$ decays to the uncorrelated value in the power-law form $C_0(\Delta t) \sim |V\Delta t|^{-2\nu}$, where the power-law behavior is characteristic of Luttinger liquids. [Note also that the static correlation function of Eq. \ref{eq:correlation_function} decays as $g(r_1, r_2) \sim |r_1 - r_2|^{-2\nu}$ within a single edge state, in contrast to the $|r_1 - r_2|^{-2}$ decay appropriate to electrons in a 1-dimensional Fermi liquid.] At finite temperatures, as seen from Eq. \ref{eq:correlation_temperature} and Eq. \ref{eq:correlation_temperature2}, we expect the maximal correlation function $C(\Delta t = 0)$ to cross over to its uncorrelated value at temperatures $kT \approx e^*V$.

In conclusion, we have seen that the principle of extracting information on statistics by means of two-point measurements can be applied equally well to fractional particles realized in current laboratory conditions as to the fermions and bosons found in Nature. Characteristic correlations in the detection of two particles at zero separation in space and time, their decay in space or time, and their oscillations over conjugate sets of variables, be they energy and time or position and momentum, hitherto observed for fermions and bosons, are seen to be manifest in processes involving anyons. Given the current cutting-edge experimental developments in quantum Hall physics, measurements on edge-state quasiparticles such as the ones proposed here and in other work, ought to be within experimental reach, and thus may provide signatures of fractional statistics for the first time.

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\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{correlation_function}
\caption{Normalized correlation function of Eq.\ref{eq:correlation_function}, $C(\Delta t)/(2(I_{12}/I_{13}))$, at zero temperature, as a function of separation time $\Delta t$ for filling fraction $\nu = 1/3$. Here we have chosen $V = 60$ in dimensionless units.}
\end{figure}

[1] R. B. Laughlin, Phys. Rev Lett. 50, 1395 (1983); Rev. Mod. Phys. 71, 863 (1999).
[2] B. I. Halperin, Phys. Rev. Lett. 52, 1583 (1984).
[3] J. K. Jain, G. S. Jeon and K. L. Graham, Phys. Rev. Lett. 91, 036801 (2003).
[4] D. Arovas, J. R. Schrieffer and F. Wilczek, Phys. Rev. Lett. 53, 722 (1984).
[5] X. G. Wen, Adv. Phys. 44, 405 (1995)
[6] E. Fradkin, in Quantum Physics at Mesoscopic Scales, edited by C. Glattli, M. Sanquer and J. T. T. Van (EDP Sciences, 2000).
[7] C. L. Kane, cond-mat/0210621 C. Chamon, D. E. Freed, S. A. Kivelson, S. L. Sondhi and X. G. Wen, Phys. Rev B 55, 2331-2343 (1997).
[8] I. Safi, P. Devillard and T. Martin, Phys. Rev. Lett 86, 4628 (2001).
[9] R. Hanbury Brown and R. Q. Twiss, Phil. Mag., Ser. 7 45, 663 (1954); Nature 177, 27 (1956); Nature 178, 1046 (1956); E. Purcell, Nature 178, 1449 (1956).
[10] G. Baym, Acta Physica Polonica B 29, 1 (1998).
[11] H. Kiesel, A. Renz and F. Hasselbach, Nature 418, 392 (2002); M. Henny et al., Science 284, 296 (1999); W. D. Oliver, J. Kim, R. C. Liu and Y. Yamamoto, Science 284, 299 (1999); R. C. Liu, B. Odom, Y. Yamamoto and S. Tarucha, Nature 391, 263 (1998).
[12] C. L. Kane and M. P. A. Fisher, Phys. Rev. Lett. 72, 724 (1994).
[13] L. Saminadayar, D. C. Glattli, Y. Jin and B. Etienne, Phys. Rev. Lett. 79, 2526 (1997); R. de Picciotto, M. Reznikov, M. Heiblum, V. Umansky, G. Bunin, and D. Mahalu, Nature 389, 162 (1997).
[14] X.G. Wen, Phys. Rev. B 43, 11025 (1991); Phys. Rev. Lett. 64, 2206 (1990); Phys. Rev. B 44, 5708 (1991).
[15] M. Buttiker, Phys. Rev. B 46, 12485 (1992); Y. M. Blanter and M. Buttiker, Phys. Reps. 336, 1 (2000), and references therein.
[16] J. K. Jain, Phys. Rev. Lett. 63, 199 (1989); J. K. Jain, Physics Today 53, 39 (2000).
[17] For instance, for states described by the composite fermion approach, the Aharonov-Bohm phase would have to involve an effective magnetic field. Alternatively, in an edge-state description of quasiparticles in these states [see, for e.g., in Lopez and Fradkin, Phys. Rev. B 59, 15323 (1999)], one would require charge and topological sectors, both of which would contribute to the phase factor.