Generalizations of Schouten-Nijenhuis B bracket

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The Schouten-Nijenhuis bracket is generalized for the superspace case and for the Poisson brackets of opposite Grassmann parities.

1 Introduction

Recently a prescription for the construction of new Poisson brackets from the bracket with a definite Grassmann parity was proposed [1]. This prescription is based on the use of exterior differentials of diverse Grassmann parities. It was indicated in [1] that this prescription leads to the generalizations of the Schouten-Nijenhuis bracket [2,3,4,5,6,7,8,9] on the both superspace case and the case of the brackets with diverse Grassmann parities. In the present report we give the details of these generalizations1.

2 Poisson brackets related with the exterior differentials

Let us recall the prescription for the construction from a given Poisson bracket of a Grassmann parity 0;1 (mod 2) of another one.

A Poisson bracket, having a Grassmann parity , written in arbitrary non-canonical phase variables $z^a$

$$f_A;B g = A \circ \partial_z \partial_z^{-1} \circ f_B; A; \partial_z^{-1} \circ \partial_z B;$$

where $\circ$ and $\circ$ are right and left derivatives respectively, has the following main properties:

$$g(f_A;B g ) \circ g_a + g_b + (mod 2);$$

$$f_A;B g = ( 1 )^{(g_a + 1)} \circ f_B; A; g ;$$

$$X^{(a b c)} ( 1 )^{(g_a + 1)} \circ f_A; f_B; C g g = 0;$$

which lead to the corresponding relations for the matrix $\cdot^{ab}$

$$g \cdot^{ab} \circ g_a + g_b + (mod 2);$$

$$\cdot^{ab} = ( 1 )^{(g_a + 1)} \circ \cdot^{ba};$$

$$X^{(a b c)} ( 1 )^{(g_a + 1)} \circ \cdot^{ad} \circ \partial_z \partial_z^{-1} \circ \cdot^{bc} = 0;$$

Concerning the generalizations of the Schouten-Nijenhuis bracket see also [10,11].
where \( \theta_z = 0 \theta z^a \) and \( g_z = g(z^a) \), \( g_A = g(A) \) are the corresponding Grassmann parities of phase coordinates \( z^a \) and a quantity \( A \) and a sum with a symbol \((abc)\) under it designates a sum motion over cyclic permutations of \( a, b \) and \( c \). We shall consider the non-degenerated matrix \( !^{ab} \) which has an inverse matrix \( !^{ab}(1)_{bc} \) (a grading factor is chosen for the convenience)

\[
!^{ab}(1)_{bc} = \frac{a}{c}
\]

(there is no sum motion over in the previous relation) with the properties

\[
g(1)_{ab} g_a + g_b \quad (\mod 2);
\]

\[
!^{ab} = \left( 1 \right)^{(g_a + 1)(g_b + 1)}!^{ba};
\]

\[
X_{(abc)} \left( 1 \right)^{(g_a + 1)(g_b + 1)}!^{bc} = 0;
\]

The Hamilton equations for the phase variables \( z^a \), which correspond to a Hamiltonian \( H \)

\[
(g(A) = 1),
\]

\[
\frac{dz^a}{dt} = fz^a; H \quad g = !^{ab} z^a H
\]

(5)

can be represented in the form

\[
\frac{dz^a}{dt} = !^{ab} z^b H \quad !^{ab} (\partial H)_{ab} = (z^a ; d H), \quad (6)
\]

where \( d (0; 1) \) is one of the exterior differentials \( d_z \) or \( d_A \), which have opposite Grassmann parities 0 and 1 respectively and following symmetry properties with respect to the ordinary multiplication

\[
d_z z^a d_z z^b = \left( 1 \right)^{(g_a)_{ab} (d_z z^b); d_z z^a};
\]

\[
d_z z^a d_z z^b = \left( 1 \right)^{(g_a + 1)(g_b + 1)} d_z z^b d_z z^a
\]

(7)

and exterior products

\[
d_z z^a \wedge d_z z^b = \left( 1 \right)^{(g_a)_{ab} (d_z z^b) \wedge d_z z^a};
\]

\[
d_z z^a \wedge d_z z^b = \left( 1 \right)^{(g_a + 1)(g_b + 1)} d_z z^b \wedge d_z z^a
\]

(8)

We use different notations \( \wedge \) and \( \wedge \) for the exterior products of \( d_z z^a \) and \( d_z z^a \) respectively.

By taking the exterior differential \( d \) from the Hamilton equations (5), we obtain

\[
\frac{d(d z^a)}{dt} = (d !^{ab} z^a H)_{ab} + \left( 1 \right)^{(g_a + 1)} !^{ab} z^a (d H)_{ab} = (d z^a ; d H), \quad (9)
\]

As a result of equations (8) and (9) we have by definition the following binary composition for functions \( F \) and \( H \) of the variables \( z^a \) and their differentials \( d z^a \)

\[
\begin{align*}
\langle F; H \rangle & = F \otimes z^a !^{ab} z^b + \left( 1 \right)^{(g_a + 1)} z^a !^{ab} z^b \\
& + z^a y^z \otimes !^{ab} z^b \otimes y^b \otimes H;
\end{align*}
\]

(10)
By using relations (2)–(4) for the matrix \( t^{ab} \), we can establish the following properties for the binary composition (10):

\[
g(\mathcal{F}; H) = g_{\mathcal{F}} + g_{\mathcal{H}} + \ldots \mod 2;
\]

\[
(\mathcal{F}; H)\, (1) = (1)g_{\mathcal{F}} + (1)g_{\mathcal{H}} + (\mathcal{F}; H) ;
\]

\[
X(\mathcal{F}; H) = (1)g_{\mathcal{F}} + (1)g_{\mathcal{H}} + (\mathcal{G}; (\mathcal{F}; \mathcal{H}) +) = 0;
\]

\[
(\mathcal{G}; \mathcal{F} \mathcal{H})
\]

which means that the composition (10) satisfies all the main properties for the Poisson bracket with the Grassmann parity equal to \( + \). Thus, the application of the exterior differentials of opposite Grassmann parities to the given Poisson bracket results in the brackets of the different Grassmann parities.

By transition to the co-differential variables \( y^a \), related with differentials \( y^a \) by means of the matrix \( t^{ab} \)

\[
y^a = y^b t_{ba};
\]

the Poisson bracket (10) takes a canonical form:

\[
(\mathcal{F}; H) = F \otimes \otimes_y (1)g_{\mathcal{F}} + (1)g_{\mathcal{H}} + \otimes_y; \otimes_y (\mathcal{G}; \mathcal{F} \mathcal{H});
\]

that can be proved with the use of the Jacobi identity (4).

The bracket (10) is given on the functions of the variables \( z^a, y^a \)

\[
F = X \frac{1}{p!} \sum_{a_{1}, \ldots, a_{p}} f_{a_{1}, \ldots, a_{p}} (z); \quad g(f_{a_{1}, \ldots, a_{p}}) = f_{\mathcal{F}} + g_{\mathcal{F}} + \ldots
\]

whereas this bracket, rewritten in the form (12), is given on the functions of variables \( z^a \) and \( y^a \)

\[
F = X \frac{1}{p!} \sum_{a_{1}, \ldots, a_{p}} f^{a_{1}, \ldots, a_{p}} (z); \quad g(f^{a_{1}, \ldots, a_{p}}) = f_{\mathcal{F}} + p + g_{\mathcal{F}} + \ldots
\]

We do not exclude a possibility of the even Grassmann parity \( \mathcal{F} \) for a quantity \( \mathcal{F} \). By taking into account relation (11), we have the following rule for the rising of indices:

\[
f_{a_{1}, \ldots, a_{p}} = \sum_{k=1}^{p} (g_{a_{1}} + g_{a_{k} + k}) (g_{a_{1} + a_{k} + 1}) \ldots (g_{a_{1} + \ldots + a_{k} + 1}) \frac{1}{a_{1}, \ldots, a_{p}} f_{a_{1}, \ldots, a_{p}} ;
\]

Note that the quantities \( f_{a_{1}, \ldots, a_{p}} \) and \( f^{a_{1}, \ldots, a_{p}} \) have in general the different symmetry and parity properties.

In the case \( n = 1 \), due to relations (6), (8), the term \( s \) in the decomposition of a function \( F (z^a; y^a) \) into degrees \( p \) of the variables \( y^a \)

\[
F = \frac{X}{p!} \sum_{a_{1}, \ldots, a_{p}} f_{a_{1}, \ldots, a_{p}} (z)
\]

can be treated as \( p \)-form \( s \) and the bracket (10) can be considered as a Poisson bracket on \( p \)-form \( s \), so that being taken between a \( p \)-form and a \( q \)-form results in a \((p + q - 1)\)-form. Thus, the bracket (10) is a generalization of the bracket introduced in [13, 14] on the superspace case and on the case of the brackets [12] with arbitrary Grassmann parities \((= 0; 1)\).

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2 There is no summation over \( p \) in relation (11).

3 Concerning a Poisson bracket between \( 1 \)-form \( s \) and its relation with the Lie bracket of vector fields see in the book [12].
3 Generalizations of the Schouten-Nijenhuis bracket

If we take the bracket in the canonical form \( [F, H] \), then we obtain the generalizations of the Schouten-Nijenhuis bracket \( [2,3] \) (see also \( [2,5,6,7,8,9,13] \)) onto the cases of superspace and the brackets of diverse Grassmann parities. Indeed, let us consider the bracket \( [F, H] \) between monomials \( F \) and \( H \) having respectively degrees \( p \) and \( q \).

\[
F = \frac{1}{p!} y_{a_1}^{a_1} z_{a_2}^{a_2} \ldots z_{a_p}^{a_p} (x) \quad g(z_{a_1}^{a_1} z_{a_2}^{a_2} \ldots z_{a_p}^{a_p}) = g_p + p + g_{a_1} + \ldots + g_{a_p}
\]

\[
H = \frac{1}{q!} y_{b_1}^{b_1} z_{b_2}^{b_2} \ldots z_{b_q}^{b_q} (x) \quad g(z_{b_1}^{b_1} z_{b_2}^{b_2} \ldots z_{b_q}^{b_q}) = g_q + q + g_{b_1} + \ldots + g_{b_q}
\]

Then as a result we obtain

\[
(F; H)_{\pm} = \frac{(-1)^{\delta b_1 + \delta b_2 + (q + 1)(q + g_1 + g_2)}}{p! q! 1!}
\]

\[
y_{b_1}^{b_1} y_{b_2}^{b_2} \ldots y_{b_p}^{b_p} z_{b_1}^{b_1} z_{b_2}^{b_2} \ldots z_{b_q}^{b_q} (x)
\]

\[
\sum_{\sigma}(1)^{g_1 + \ldots + g_p + q_1 + \ldots + q_q} (g_1 + \ldots + g_p) + \delta b_1 + \delta b_2 + \ldots + (q + 1) k_1 + (p - 1) k_1
\]

\[
\sum_{\delta}(p - 1) k_1!
\]

\[
y_{b_1}^{b_1} y_{b_2}^{b_2} \ldots y_{b_p}^{b_p} \cdot a_1 f^{a_1 \ldots a_p} \theta_{z_1} h^{b_1 \ldots b_q}
\]

(13)

3.1 Particular cases

Let us consider the formula (13) for the particular values of and .

1. We start from the case which leads to the usual Schouten-Nijenhuis bracket for the skew-symmetric contravariant tensors. In this case, when \( g_1 = 0 \), \( g_2 = 1 \) and the matrix \( \Omega^{ba} (x) = \Omega^{ba} 0 (x) \) corresponds to the usual Poisson bracket for the commuting coordinates \( x^a = x^a \), we have

\[
(F; H)_{1} = \frac{(-1)^{q+1}(q + p)}{p! q! 1!}
\]

\[
y_{b_1}^{b_1} y_{b_2}^{b_2} \ldots y_{b_p}^{b_p} \cdot a_1 f^{a_1 \ldots a_p} \theta_{z_1} h^{b_1 \ldots b_q}
\]

(14)

where \( a_1 \ldots a_p \) are Grassmann co-derivative variables related owing to (11) with the Grassmann differential variables \( a_1 \ldots a_p \).

W hen Grassmann parities of the quantities \( f \) and \( h \) are equal to zero \( g_1 = g_2 = 0 \), we obtain from (13)

\[
(F; H)_{1} \overset{\text{def}}{=} \frac{(-1)^{p+1}(p + 1)}{a_p q!} a_1 f^{a_1 \ldots a_p} \cdot h^{b_1 \ldots b_q}
\]

where \( f^{a_1 \ldots a_p} \cdot h^{b_1 \ldots b_q} \) are components of the usual Schouten-Nijenhuis bracket (see, for example, [5]), for the contravariant antisymmetric tensors. This bracket has the following symmetry property

\[
[F, H] = (-1)^{pq} [H, F]
\]

\(^{4}\text{Here and below we use the same notation } [F, H] \text{ for the different brackets. We hope that this will not lead to the confusion.}\)
and satisfies the Jacobi identity
\[ X \left( (1)^{p_0} [F;H]_E \right) = 0; \]

where \( s \) is a degree of a monomial \( E \).

2. In the case \( s = 0 \) and \( \overset{1}{!}_{0}^{ab}(x) = \overset{1}{!}_{0}^{ba}(x) \) we obtain the bracket for symmetric contravariant tensors (see, for example, [7])

\[
[F;H]_0 = \frac{1}{p!(q-1)!} Y_{b_1}^{b_2} \cdots Y_{a_p}^{a_1} \delta_{a_1}^{a_p} \delta_{a_1}^{a_p} f^{a_1 \cdots a_p} (x) h^{b_1 \cdots b_q} \]

where commuting co-diagonals \( y_0^a \) connected with commuting di diagonals \( x^a \) in accordance with [11]

\[ y_0^a = y_0^b \overset{1}{!}_{0}^{ba} \]

and the bracket \( [F;H]^{y_0^a \cdots y_0^q} \) has the following symmetry property

\[ [F;H] = (1)^{q-1} [H;F] \]

and satisfies the Jacobi identity

\[ X \left( (1)^{q-1} [E;F;H] \right) = 0; \]

3. By taking the Martin bracket \( [15] \) \( \overset{1}{!}_{0}^{ab}(x) = \overset{1}{!}_{0}^{ba}(x) \) with Grassmann coordinates \( z^a = a \) \( (g_a = 1) \) as an initial bracket [1], we have in the case \( s = 0 \) for antisymmetric contravariant tensors on the Grassmann algebra

\[
[F;H]_0 = \left( \frac{1}{p!(q-1)!} \right) \delta_{a_1}^{a_p} \delta_{a_1}^{a_p} f^{a_1 \cdots a_p} (x) h^{b_1 \cdots b_q} \]

where the Grassmann co-diagonals \( \overset{1}{!}_{0}^{a} \) related with the Grassmann di diagonals \( a \) as

\[ \overset{1}{!}_{0}^{a} = \overset{1}{!}_{0}^{ba} \]

The bracket \( [F;H] \) has the following symmetry property

\[ [F;H] = (1)^{q-1} [H;F] \]

and satisfies the Jacobi identity

\[ X \left( (1)^{q-1} [E;F;H] \right) = 0; \]

4. By taking the Martin bracket again, in the case \( s = 1 \)
\[ \overset{1}{!}_{0}^{a} x^a = x_0^a \overset{1}{!}_{0}^{ba} \]
we obtain for the symmetric tensors on Grassmann algebra

\[
\begin{align*}
(\mathcal{F};H)_{1} &= \frac{1}{p!(q - 1)!}X_{b_{q-1}b_{q-2}\ldots b_{1}} a_{1} \mathcal{F}_{a_{1}^{0}a_{2}^{0}} \mathcal{H}_{b_{1}^{0}b_{2}^{0}b_{3}^{0}b_{4}^{0}} \\
&= \frac{1}{(p - 1)!q!}X_{b_{q-1}b_{q-2}\ldots b_{1}} a_{2} \mathcal{E}_{a_{1}^{0}a_{2}^{0}a_{3}^{0}} \mathcal{H}_{b_{1}^{0}b_{2}^{0}b_{3}^{0}b_{4}^{0}} \\
\end{align*}
\]

\[
\text{def} \quad X_{a_{p+q}} a_{2} \mathcal{F};H \mathcal{J}_{a_{1}^{0}a_{2}^{0}a_{3}^{0}} : \]

The bracket \( [\mathcal{F};H] \) has the following symmetry property

\[
[\mathcal{F};H] = (1)(q_{r} + p_{1} + q_{r} + 1)(1) [\mathcal{H};\mathcal{F}] \]

and satisfies the Jacobi identity

\[
\begin{align*}
(1)(q_{r} + s_{1} + q_{r} + 1)(1)[E;[\mathcal{F};H]] &= 0; \\
\end{align*}
\]

5. In general, if we take the even bracket in superspace with coordinates \( z^{a} = (x; \lambda) \), where \( \lambda \) and \( x \) are respectively commuting and anticommuting Grassmann variables, then in the case \( = 1 \) we have

\[
\begin{align*}
(\mathcal{F};H)_{1} &= \frac{1}{p!(q - 1)!}X_{b_{q-1}b_{q-2}\ldots b_{1}} a_{1} \mathcal{F}_{a_{1}^{0}a_{2}^{0}a_{3}^{0}} \mathcal{H}_{b_{1}^{0}b_{2}^{0}b_{3}^{0}b_{4}^{0}} \\
&= \frac{1}{(p - 1)!q!}X_{b_{q-1}b_{q-2}\ldots b_{1}} a_{2} \mathcal{E}_{a_{1}^{0}a_{2}^{0}a_{3}^{0}} \mathcal{H}_{b_{1}^{0}b_{2}^{0}b_{3}^{0}b_{4}^{0}} \\
\text{def} \quad y_{a_{p+q}}^{1} a_{2} \mathcal{F};H \mathcal{J}_{a_{1}^{0}a_{2}^{0}a_{3}^{0}} ;
\end{align*}
\]

where

\[
\text{def} \quad y_{z}^{a} = y_{b}^{a} : \]

The bracket \( [\mathcal{F};H] \) has the following symmetry property

\[
[\mathcal{F};H] = (1)(q_{r} + p_{1} + q_{r} + 1)(1) [\mathcal{H};\mathcal{F}] \]

and satisfies the Jacobi identity

\[
\begin{align*}
(1)(q_{r} + s_{1} + q_{r} + 1)(1)[E;[\mathcal{F};H]] &= 0; \\
\end{align*}
\]

6. In the case of the even bracket in superspace as initial one with \( = 0 \) we obtain

\[
\begin{align*}
(\mathcal{F};H)_{0} &= \frac{1}{p!(q - 1)!}X_{b_{q-1}b_{q-2}\ldots b_{1}} a_{1} \mathcal{F}_{a_{1}^{0}a_{2}^{0}a_{3}^{0}} \mathcal{H}_{b_{1}^{0}b_{2}^{0}b_{3}^{0}b_{4}^{0}} \\
&= \frac{1}{(p - 1)!q!}X_{b_{q-1}b_{q-2}\ldots b_{1}} a_{2} \mathcal{E}_{a_{1}^{0}a_{2}^{0}a_{3}^{0}} \mathcal{H}_{b_{1}^{0}b_{2}^{0}b_{3}^{0}b_{4}^{0}} \\
\text{def} \quad y_{a_{p+q}}^{0} a_{2} \mathcal{F};H \mathcal{J}_{a_{1}^{0}a_{2}^{0}a_{3}^{0}} ;
\end{align*}
\]

where

\[
\text{def} \quad y_{z}^{a} = y_{b}^{a} : \]
Generalizations of Schouten-Nijenhuis Brackets

The bracket $[F; H]$ has the following symmetry property

$$[F; H] = (1)^9 g_{9h} [H; F]$$

and satisfies the Jacobi identity

$$X (1)^9 g_{9h} [E; [F; H]] = 0.$$  

(8.6)

7. Taking as an initial bracket the odd Poisson bracket in superspace with coordinates $z^a$, for the case $= 0$ we have

$$[F; H]_1 = \frac{(1)(g_{h_1} + g_{q_1} + g_{l_1})(g_{r_1} + g_{l_1})}{p!(q 1)!} y_{bq_1}^{l_1} b_{wq_1}^{l_1} a_1 f_{a_1}^{a_2 h_1} @ z_1) h_{b_1}^{b_1} h_1^{l_1} (p 1) h!$$

$$y_{bq_1}^{l_1} b_{wq_1}^{l_1} a_1 f_{a_1}^{a_2 h_1} @ z_1) h_{b_1}^{b_1} h_1^{l_1} \text{ def } m y_{a_2}^{q_2} b_{wq_1}^{l_1} a_1 f_{a_1}^{a_2 h_1} @ z_1) h_{b_1}^{b_1} h_1^{l_1}$$

where

$$aq_2 = y_{b_1}^{l_1} + b a_1$$

The bracket $[F; H]$ has the following symmetry property

$$[F; H] = (1)^9 (g_{h_1} + 1)(g_{h_1} + 1) [H; F]$$

and satisfies the Jacobi identity

$$X (1)^9 g_{9h} [E; [F; H]] = 0.$$  

(8.6)

8. At last for the odd Poisson bracket in superspace, taking as an initial one, we obtain in the case $= 1$

$$[F; H]_1 = \frac{(1)(g_{h_1} + g_{h_1} + g_{l_1})(g_{r_1} + p)}{p!(q 1)!} y_{bq_1}^{l_1} b_{wq_1}^{l_1} a_1 f_{a_1}^{a_2 h_1} @ z_1) h_{b_1}^{b_1} h_1^{l_1} (p 1) h!$$

$$y_{bq_1}^{l_1} b_{wq_1}^{l_1} a_1 f_{a_1}^{a_2 h_1} @ z_1) h_{b_1}^{b_1} h_1^{l_1} \text{ def } m y_{a_2}^{q_2} b_{wq_1}^{l_1} a_1 f_{a_1}^{a_2 h_1} @ z_1) h_{b_1}^{b_1} h_1^{l_1}$$

where

$$aq_2 = y_{b_1}^{l_1} + b a_1$$

The bracket $[F; H]$ has the following symmetry property

$$[F; H] = (1)^9 g_{h_1} [H; F]$$

and satisfies the Jacobi identity

$$X (1)^9 g_{9h} [E; [F; H]] = 0.$$  

(8.6)

Thus, we see that the formula contains as particular cases quite a number of the Schouten-Nijenhuis type brackets.
4 Conclusion

We give the prescription for the construction from a given Poisson bracket of the de Rham Grassmann parity another bracket. For this construction we use the exterior di differentials with di even Grassmann parities. We proved that the resulting Poisson bracket essentially depends on the parity of the exterior di differential in spite of these di differentials give the same exterior calculus \[\text{[1]}\]. The prescription leads to the set of di even generalizations for the Schouten-Nijenhuis bracket. Thus, we see that the Schouten-Nijenhuis bracket and its possible generalizations are particular cases of the usual Poisson brackets of di even Grassmann parities \[\text{[12]}\]. We hope that these generalizations will find their own application for the deformation quantization (see, for example, \[\text{[8,16]}\]) as well as the usual Schouten-Nijenhuis bracket.

Acknowledgements

We are sincerely grateful to J. Stashe for the interest to the work and stimulating remarks. One of the authors (V A S.) sincerely thanks L. Bonora for the fruitful discussions and warm hospitality at the SISSA/ISAS (Trieste) where this work has been completed.

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