Are Pions Perturbative in Effective Field Theory?

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Abstract

It is shown that pions can be included perturbatively into effective field theory only for the external momenta, well below the pion mass. But for such low energies it is not necessary to include pions explicitly.

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I. INTRODUCTION

Weinberg’s original ideas about the chiral perturbation theory approach to processes involving an arbitrary number of nucleons were followed by intensive investigations of various aspects of this approach (see for example and references included therein).

For processes involving more than one nucleon Weinberg suggested to apply the power counting to the potential rather than to the scattering amplitude. For N-nucleon processes the potential is defined as a sum of N-nucleon irreducible time-ordered perturbation theory diagrams. The amplitude is to be found by solving Lippmann-Schwinger equation (or Schrödinger equation).

Recently it was suggested that pions can be included perturbatively into effective field theory calculations for momenta up to the pionic mass and even higher (up to 300 MeV in the centre of mass) [6–10]. In these papers dimensional regularization and the Power Divergent Subtraction (PDS) scheme were used to describe the NN scattering data and electro-magnetic form factors of deuteron. In the PDS scheme the coefficients (coupling constants) of leading and sub-leading order terms in the effective Lagrangian for \(^1S_0\) wave are of the order \(\sim (100 \text{ MeV})^{-2}\) and \(\sim (150 \text{ MeV})^{-4}\) respectively for the normalisation point equal to pion mass. These values of couplings are not encouraging at all. After inclusion of the pion explicitly one would expect that the scale of couplings would be determined by the mass of the lightest integrated particle \(\sim (800 \text{ MeV})\) provided that the normalisation point is taken to be of the order of the pion mass. Impressive numerical fits given in above mentioned papers extend up to the values of momenta where “small” expansion parameters are even larger than 1. All powers are equally important for such values of expansion parameter and it raises a question whether these fits have anything to do with suggested power counting. Of course there could be some magnificent cancellations among higher order terms but power counting does not take into account such accidental cancellations.

In Weinberg’s power counting the one pion exchange potential is of leading order and hence it has to be iterated via the Lippmann-Schwinger equation. Below, to investigate a little further the power counting arguments, pions are included perturbatively using the subtractions at \(p^2 = -\mu^2\). The conclusion is that using the subtractions at \(p^2 = -\mu^2\) pions can be included perturbatively only for momenta well below the mass of the pion. Close investigations demonstrate that the same conclusions are valid for PDS scheme too in contrast to the results of papers [6–10].

II. EXPLICIT CALCULATIONS

Concentrating on the NN scattering problem one could summarise Weinberg’s ideas in the following way:

One should draw all the diagrams for a given process (there will be an infinite number of them). Diagrams with loops will contain divergences. The effective chiral Lagrangian is non-renormalizable in the traditional sense where only a finite number of parameters are involved; rather, it contains all possible terms which are not suppressed by the symmetries of the theory with the ultraviolet divergences being absorbed into an infinite number of parameters.
of the Lagrangian. One removes divergences by subtracting diagrams (or equivalently one could include contributions of counter-terms) and taking normalisation points of the order of external momenta. For these subtracted diagrams the effective cut-off is then of the order of external momenta. Once these diagrams have been renormalised, one can sort them by orders of the small expansion parameter (external momentum, renormalization point, or mass of pion) by applying Weinberg’s power counting rules.

It turns out that the renormalised diagrams contributing up to and including any given order \( n \) in the small parameters consist of a finite number of diagrams which are two-nucleon irreducible, and an infinite number of diagrams which are two-nucleon reducible. Denoting the sum of the contributing irreducible diagrams by \( V^R \) and the sum of the same diagrams before renormalization formally by \( V \), one finds that the series

\[
T^R = V^R + (VG_0 V)^R + (VG_0 VG_0 V)^R + \ldots
\]  

(1)

includes all contributions up to order \( n \) (and some of the contributions of order greater than \( n \)). Here \( G_0 \) is the free two-nucleon Green function and each iteration of \( V \) is renormalised separately (as indicated by a superscript \( R \)). The summation of this series is highly non-trivial, e.g., there is no known way of writing down a closed integral equation for \( T^R \). To make progress one can pursue one of the following:

1. One could try to solve the Lippmann-Schwinger (LS) equation \( T = V + VG_0 T \) analytically with a formal un-subtracted non-regularized potential, or more rigorously, one could regularize the loop integrals in \( V \) and \( VG_0 T \) and then solve the LS equation analytically. In both cases one would need to perform subtractions (renormalization) in the resulting solutions.

Although analytic solutions to the LS equations are rare, such approaches have been successfully applied to the \(^1S_0\) NN problem with (only) contact interaction terms \([2,14]\). For this simple case it is easy to verify the general assumption that solving the LS equation with regularization and after subtracting divergences in the solution is equivalent to the summation of already subtracted diagrams.

Solving the LS equation and after subtracting divergences is equivalent to summing an infinite number of renormalised diagrams. Note that diagrams of all orders (up to infinity) are involved in this summation. The higher order subtracted diagrams include contributions of higher order counter-terms. For example, if one takes the sum of the first two (leading and sub-leading) terms of the potential in the chiral expansion and iterates it using the LS equation, and then renormalises the obtained amplitude, one thereby includes the contributions of an infinite number of counter-terms with all orders of derivatives (up to infinity). As far as power counting must be performed for renormalised diagrams, i.e. after taking into account the contributions of counter-terms, the presence of contributions of an infinite number of counter-terms does not change power counting arguments despite the claims of ref. \([12,10]\) that Weinberg’s power counting is inconsistent. A similar situation arises in the meson sector. As mentioned in Ref. \([12]\), high order terms (high order loops) contribute at low order for calculations (before renormalization) of \( \pi-\pi \) scattering, if one does not make use of dimensional regularization. Does this mean that power counting is not valid in other regularizations and that different regularizations are
not equivalent? Of course not. Power counting is valid after renormalization, i.e. for subtracted diagrams, and hence for regularizations different from dimensional regularization one just has to perform some additional subtractions (over and above what is needed when dimensional regularization is used). One gets the same renormalised diagrams with the same power counting provided that the normalisation conditions are the same.

2. One could utilise cut-off theory \[2,13,15–17\]. The advantage of cut-off theory is that one can solve the LS equation numerically. It is interesting to note that the aforementioned criticism \[11,7,9\] of Weinberg’s power counting has recently been shown not to apply to the cut-off theory \[13\]. The discussion given in Ref. \[13\] and the one above for renormalised theory correspond with each other, supporting the equivalence of renormalised and cut-off theory once again \[15\].

3. And of course one could try to sum renormalised diagrams directly.

To follow the third way one can use the EFT expansion of the quantity \(pcot\delta(p)\) suggested in \[5\]:

\[
pcot\delta(p) = ip + \frac{4\pi}{M} A = ip + \frac{4\pi}{M} \left( \frac{1}{A_0} - \frac{A_2}{A_0^2} + \ldots \right)
\]

where \(A_0\) is non-perturbative amplitude, which is a sum of all diagrams with leading order potential as a vertex. \(A_2\) contains all contributions of leading order potential with one insertion of second order potential etc. \[5\]. One could solve Lippmann-Schwinger equation for the potential with leading and second order contributions and find the inverse amplitude. The difference with \(\frac{1}{A_0} - \frac{A_2}{A_0^2}\) would be of higher order and hence the agreement between this two solutions should be good up to the validity of considered approximation. The large difference would be an indicator that diagrams with many insertions of second order potential are larger than estimated. One can include part of second (or higher) order potential into \(A_0\) non-perturbatively, while including another part as perturbative insertions. As was mentioned in the introduction, power counting arguments do not rely on cancellation between diagrams of the same order, so the higher order contributions coming from that part of second order potential which was included non-perturbatively, should be small.

The contribution of contact interaction terms to the 2-nucleon potential in the partial \(^1S_0\) wave up to (and including) the sub-leading order has the form:

\[
V_C^{(2)}(p, p') = V_0 + V_2 = C_0 + C_2 \left( p^2 + p'^2 \right)
\]

The one pion exchange potential in the \(^1S_0\) channel is:

\[
V_\pi(p, p') = -\frac{g_A^2}{2f^2} \frac{q^2}{q^2 + m_\pi^2} = -\frac{g_A^2}{2f^2} \left( \frac{m_\pi^2}{q^2 + m_\pi^2} - 1 \right)
\]

where \(q = p' - p\), \(g_A = 1.25\) and \(f = 132\) MeV.

According Weinberg’s power counting criteria this potential is of leading order for the momenta of the order of the mass of pion and it is of the order \(p^2/m_\pi^2\) for the momenta much
FIG. 1. Graphs corresponding to \( A_1 \). Dashed line is the exchange of pion and the four solid line vertex corresponds to the potential \( V_C \).

below the pion mass, provided that the normalisation point is taken of the order of external momenta.

One can include second order contact interaction potential non-perturbatively, pionic potential perturbatively and determine the range of validity of perturbative inclusion of pionic potential. Diagrams including contact interaction and one pion are drawn in FIG. 1. The diagrams in the second row correspond to pure contact interactions. These bubble-chain diagrams are divergent and are to be subtracted. They can be summed up by iterating the contact interaction potential \( V^{(2)}_C \) and subtracting at \( p^2 = -\mu^2 \) \cite{14}.

One gets the following expression for the non-perturbative Feynmann amplitude:

\[ A^{(2)}_0 = - \frac{C_0 + C_2 p^2}{1 + (C_0 + C_2 p^2) \frac{M(\mu + i p)}{4\pi}} \]

The sub-diagram of diagram a), containing the pionic line, is not divergent but has to be subtracted. The point is that the corresponding diagram in relativistic theory is divergent and has to be subtracted. The subtraction of the relativistic diagram at the point \( p^2 = -\mu^2 \) automatically leads to the subtraction of its non-relativistic approximation. A bubble sub-diagram of the diagram b) containing the pionic line has two non-divergent sub-diagrams itself. According to the same argument as above these sub-diagrams, although finite, have to be subtracted before the divergent bubble sub-diagram is subtracted. Note that these subtractions of non-divergent diagrams are necessary not to violate unitarity.

After subtractions one gets the following expression:

\[ A_1 = A_1^{(I)} + A_1^{(II)} + A_1^{(III)} \]

where

\[ A_1^{(I)} = \left( \frac{g_\Lambda^2}{2f^2} \right) \left[ -1 + \frac{m_\pi^2}{4p^2} \ln \left( 1 + \frac{4p^2}{m_\pi^2} \right) \right] \]

\[ A_1^{(II)} = \frac{g_\Lambda^2}{f^2} \left( \frac{m_\pi M A_0}{4\pi} \right) \left\{ -\frac{(\mu + i p)}{m_\pi} + \frac{m_\pi}{2p} \left[ \tan^{-1} \left( \frac{2p}{m_\pi} \right) + \frac{i}{2} \ln \left( 1 + \frac{4p^2}{m_\pi^2} \right) \right] - \frac{m_\pi}{2\mu} \ln \left( 1 + \frac{2\mu}{m_\pi} \right) \right\} \]

\[ A_1^{(III)} = \frac{g_\Lambda^2}{2f^2} \left( \frac{m_\pi M A_0}{4\pi} \right)^2 \times \]

\[ \left\{ -\frac{(\mu + i p)}{m_\pi} \right\} \left\{ \frac{i}{m_\pi} \tan^{-1} \left( \frac{2p}{m_\pi} \right) - \frac{i p}{\mu} \ln \left( 1 + \frac{2\mu}{m_\pi} \right) - \frac{1}{2} \ln \left( 1 + \frac{4p^2}{m_\pi^2} \right) \right\} \]

(7)
FIG. 2. Phase shifts for the subtractions at $p^2 = -\mu^2$. Left figure corresponds to the non-perturbative and right figure to the perturbative inclusion of $C_2 (p^2 + p'^2)$ term. Solid line corresponds to the effective range expansion. Dash-dotted and dotted lines correspond to $\mu = 10$ and $40$ MeV respectively and dashed line corresponds to the floating normalisation point.

and

$$A^{-1} = -\frac{1}{C_0 + C_2 p^2} - \frac{M}{4\pi} (\mu + ip) - \frac{g_A^2}{f^2} \left[ \frac{m_\pi^2}{8p^2} \ln \left(1 + \frac{4p^2}{m_\pi^2}\right) \left(\frac{1}{C_0} + \frac{M\mu}{4\pi}\right)^2 - \frac{1}{2C_0^2} \right]$$

$$+ \frac{g_A^2}{f^2} \left[ \frac{m_\pi^2 M^2}{128\pi^2} \ln \left(1 + \frac{4p^2}{m_\pi^2}\right) + \frac{m_\pi^2 M}{4\pi} \left(\frac{1}{C_0} + \frac{M\mu}{4\pi}\right) \left(\frac{1}{2p} \tan^{-1}\left(\frac{2p}{m_\pi}\right) - \frac{1}{2\mu} \tan^{-1}\left(\frac{2\mu}{m_\pi}\right)\right) \right]$$

(8)

While the whole amplitude $A$ does not depend on $\mu$, the approximate expression (8) does and by good choice of the value of this parameter one should make contributions of higher order terms small. Matching to the effective range expansion one can determine $C_0$ and $C_2$ for particular values of normalisation point $\mu$. Using these values for $C_0$ and $C_2$, one can find the phase shifts from (8). The phase shifts for $\mu = 10$ MeV and $\mu = 40$ MeV are plotted in FIG.2. One could choose the most natural “floating” normalisation condition $\mu \sim p$. The phase shifts for $\mu = p$ are also plotted in FIG.2. It is seen that for $\mu = 40$ MeV and for the floating normalisation condition the agreement between the effective range expansion and EFT is quite satisfactory for momenta up to 60 MeV while the deviation is significant for higher momenta. The values of couplings for $\mu = 40$ MeV $C_0 \sim (57 \text{ MeV})^{-2}$, $C_2 \sim (80 \text{ MeV})^{-4}$ are almost satisfactory. For $\mu \sim 140$ MeV one expects couplings to be much smaller, $\sim (800 \text{ MeV})^n$ ($n$ is determined by the dimension of coupling constant). Note that for such $\mu$ pions cannot be included perturbatively. The fit to the effective range expansion is also unsatisfactory for a normalisation point exceeding 60 MeV.

One can include second order contact interaction potential perturbatively and calculate phase shifts. The results are plotted in FIG.2. It is seen that the difference between phase shifts for perturbatively and non-perturbatively included second order contact interaction potential is significant for momenta above $\sim 50$ MeV indicating that mentioned potential should be included non-perturbatively.
The above results are not surprising. One could hardly expect perturbative inclusion of pions to be satisfactory unless $p^2/m_\pi^2$ is sufficiently small. But these conclusions do not agree with the results of ref. [6–10]. Should one conclude that PDS is a much better scheme than the subtraction at $p^2 = -\mu^2$?

In [6] second order contact interaction and pionic potentials were included perturbatively and also to make the amplitude $\mu$-independent (up to the desired order) the contact interaction vertex proportional to $D_2 m_\pi^2$ was introduced and included perturbatively.

The inclusion of some terms perturbatively makes sense only if their higher order contributions are small. Note once again that power counting does not rely on cancellations between different contributions, so the higher order contributions of low order terms should be small themselves. It is quite easy to include small contact terms (second order contact interaction term and the term proportional to $Dm_\pi^2$) non-perturbatively while pionic potentials is taken perturbatively. Doing so one is lead to the diagrams depicted in FIG.1. It is straightforward to apply PDS scheme to these diagrams.

Fitting parameters to the effective range expansion (normalisation point is taken equal to $m_\pi$) one gets the phase shifts for $^1S_0$ which are plotted in FIG.3 together with the phase shifts got from the perturbative inclusion of second order contact interaction term ($Dm_\pi^2$ included non-perturbatively) and phase shifts from effective range expansion. It is seen that difference between phase shifts for perturbatively and non-perturbatively included second order contact term is significant for momenta $\sim 100$MeV, demonstrating that for such momenta the mentioned term should be included non-perturbatively. Moreover the phase shifts got using PDS scheme deviate from effective range expansion results already at $\sim 30$MeV.

It was mentioned in [6] that the fit to the effective range parameters reproduces the data very well up to centre of mass momentum $p \sim 150$ MeV. However this observation is not quite reliable. In [6] the phase shifts were determined using formula:

$$\delta = \frac{1}{2i} \ln \left(1 + i\frac{MP}{2\pi}A\right)$$  \hfill (9)
FIG. 4. Phase shifts in PDS scheme for “best fit” parameters. Solid line corresponds to the effective range expansion. Dotted line corresponds to the “best fit” by KSW, long-dashed and dashed lines correspond to perturbative and non-perturbative inclusion of second order small contact terms (\( Dm_\pi^2 \) and \( C_2 (p^2 + p'^2) \) terms) respectively.

Expanding both sides to a given order with \( \delta = (\delta_{(0)} + \delta_{(1)} + ...) \) the quantities of the same order were equated to each other thus determining the phase shifts up to the desired order. Such an expansion makes sense only if higher order terms (including high degrees of low order terms which occur due to the expansion of ln function) are small. One can check for which values of external momenta is this condition satisfied just comparing phase shifts obtained by expanding in powers of small parameter with ones obtained by solving \( \delta \) analytically. Large difference between these two results is indication that higher order terms are not negligible by any means. One could think that higher degrees of low order amplitude (obtained while expanding ln-function) which turned out to be large will be cancelled by higher order terms of the amplitude, but this immediately means that these higher order terms of the amplitude are large. If such a cancellation takes place, it is just accidental and has nothing to do with power counting.

Substituting numerical values with best fit from [6] for coupling constants \( C_0 (m_\pi) = -3.34 \text{ fm}^2 \), \( D_2 (m_\pi) = -0.42 \text{ fm}^4 \), \( C_2 (m_\pi) = 3.24 \text{ fm}^4 \) one gets the phase shifts for perturbatively included small contact terms using exact formula (4) and also the above mentioned expansion of the exact formula in powers of small parameter. The results are plotted in FIG.4 together with the phase shifts defined from exact formula for non-perturbatively included small contact terms. It should be clear that the “best fit” obtained in above mentioned papers is not reliable at all for momenta \( \sim 100 \text{ MeV} \) (and higher) and hardly best for lower momenta.

From FIG.3 and FIG.4 one sees that small contact terms become quite large already at the momenta \( \sim 60 \text{ MeV} \). After inclusion of pions explicitly one expects these terms to become non-perturbative for considerably higher momenta. The only explanation of this failure of effective field theory approach is that pions can be included perturbatively only for very low external momenta. FIG.3 suggests that it is still questionable whether the inclusion of pions perturbatively within PDS scheme is consistent for any values of momenta (good fit for extremally low
momenta could be just due to the fit to low order parameters).

The very reasonable explanation of inconsistency of perturbative inclusion of the pion within PDS scheme could be given by noting that PDS scheme like $\overline{MS}$ puts the scale of the loop integrals equal either to the mass of the pion or the normalisation point. So the expansion parameter when pions are included perturbatively and the normalisation point is taken equal to the pionic mass is $\sim m_\pi/\Lambda_{NN}$ ($\Lambda_{NN} \sim 300$ MeV) even for very low external momenta, as was mentioned in ref. [7]. In this work it was pointed that the Yukawa piece of the one pion exchange interaction supports a bound state (and consequently becomes non-perturbative) only when $m_\pi/\Lambda_{NN} \sim 1.7$. But the presence of $\delta$-function part of one pion exchange interaction, which is known to describe attraction [4], makes entire potential completely different from Yukawa piece, so the above argument is hardly relevant.

The related problems encountered while including pions perturbatively within PDS scheme were addressed in recent papers by J.V.Steele and R.J.Furnstahl [18] and T.D. Cohen and J.M.Hansen [19].

III. CONCLUSIONS

For external momenta well below the mass of the pion one could include pions perturbatively in effective field theory using subtractions at $p^2 = -\mu^2$. For such low energies effective theory fits $^1S_0$ wave $NN$ scattering data quite well but the explicit inclusion of pions is not necessary. For higher energies (external momenta) it is necessary to include pionic potential non-perturbatively (iterating Lippmann-Schwinger or Schrödinger equation) in accordance with Weinberg’s power counting.

The close examination demonstrates that the perturbative inclusion of the pion into $NN$ problem within dimensional regularization combined with Power Divergent Subtraction scheme is also consistent only for very low energies.

One should conclude that the cutoff effective field theory based on Weinberg’s power counting (see for ex. [3,13,15–17]) still remains the only systematic way of incorporation of Weinberg’s ideas for not very low energies.

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