PARAMETRISING THE PROTON STRUCTURE FUNCTION

L. L. Jenkovszky, A. Lengyel, F. Paccanoni

\textit{a} Bogoliubov Institute for Theoretical Physics, 
Academy of Sciences of the Ukrain 
252143 Kiev, Ukrain

\textit{b} Institute of Electron Physics 
Uzhgorod 294016, Universitetska 21, Ukraine

\textit{c} Dipartimento di Fisica, Università di Padova, 
Istituto Nazionale di Fisica Nucleare, Sezione di Padova 
via F. Marzolo 8, I-35131 Padova, Italy

\textbf{Abstract}

We show that simple parametrisations at small $x$ of the proton structure function work so well in limited regions of the $(x,Q^2)$ plane because they are approximately "self-consistent" solutions of the QCD evolution equation. For a class of them, we predict their $Q^2$ dependence and compare the result with experimental data.

PACS numbers: 13.60.Hb, 12.38.Cy

\textsuperscript{†}email address: JENK@GLUK.APC.ORG

\textsuperscript{‡}email address: SASHA@LEN.UZHGOROD.UA

\textsuperscript{*}email address: PACCANONI @PADOVA.INFN.IT
1. Introduction

Since long time non-Regge behaviour of the proton structure function $F_2(x, Q^2)$ in deep inelastic scattering [1] has been discussed on the basis of the scale break predicted from QCD. Recent experimental data [2, 3] provide further evidence for perturbative QCD evolution equations [4, 5] and confirm [2] the scaling behaviour of the singlet proton structure function at small $x$ and large $Q^2$ predicted in ref. [6].

This double asymptotic scaling is derived from the exact solution of the dynamical equations [5] in the region where nonsinglet contributions to $F_2(x, Q^2)$ can be neglected and the lowest moments of the splitting functions dominate.

Empirical expressions for $F_2(x, Q^2)$, where the functional form of the perturbative double scaling is respected, will reproduce well experimental data. An example is the parametrisation of ref. [7], where an interesting similarity in the energy dependence between the average charged multiplicity in $e^+ e^-$ collisions and the proton structure function has been exploited.

Attempts to implement the unitarity condition in deep inelastic scattering [8, 9, 10] lead to a quite different behaviour for the proton structure function. As suggested in [11] double scaling admits local approximations due to the slow variation with $x$ and $Q^2$ of the relevant variables in the asymptotic region. We agree with this interpretation but other reasons can be found for the agreement of these simple parametrisations with HERA data.

If we consider two simple expressions for $F_2(x, Q^2)$:

$$F_2(x, Q^2) = a(Q^2) + b(Q^2) \left( \frac{x_0}{x} \right)^c(Q^2)$$

(1)

and

$$F_2(x, Q^2) = \sum_{i=0}^2 a_i(Q^2) \left( \ln \frac{x_0}{x} \right)^i$$

(2)

and fit them at low $x$, for each $Q^2$ bin, to the experimental data [2, 3], we find that ansatz (2) has a lower total $\chi^2_{d.o.f.}$, while the exponent $c(Q^2)$ in (1) agrees with the
finding of [2]. The good description of data obtained with the parametrisation [2] must find an explanation in terms of the evolution equation [3]. The purpose of this paper is to find a reason for this agreement.

In Section 2 we will discuss new approximate solutions of the standard evolution equations [5] in a form similar to the ones found in [8, 9, 10]. By limiting ourselves to the double logarithmic approximation, an explicit example will be given. In Section 3 we will show that there is agreement between these solutions and an explicit fit to the data. Last Section is devoted to a discussion of the results.

2. A theoretical explanation.

Let us consider the singlet evolution equation [5] at leading order, when only the largest singularity in moment space contribute to the splitting functions. Neglecting non-singlet contributions and defining the new variables

$$\xi = \ln \left( \frac{x_0}{x} \right), \text{ and } \zeta = \ln \left( \frac{\ln(Q^2/\Lambda^2)}{\ln(Q_0^2/\Lambda^2)} \right)$$  \hspace{1cm} (3)

a simplified form for the problem of determining the proton structure function can be derived [3] in the double logarithmic approximation

$$F_2^p = \frac{5}{18} Q(\xi, \zeta)$$  \hspace{1cm} (4-a)

$$Q(\xi, \zeta) = Q(\xi, 0) + \frac{f \gamma^2}{9} \int_0^\xi d\zeta' G(\xi, \zeta')$$  \hspace{1cm} (4-b)

$$\frac{\partial^2 G(\xi, \zeta)}{\partial \xi \partial \zeta} + \delta \frac{\partial G(\xi, \zeta)}{\partial \xi} - \gamma^2 G(\xi, \zeta) = 0$$  \hspace{1cm} (4-c)

where $Q(\xi, \zeta)$ and $G(\xi, \zeta)$ are the singlet quark distribution and the gluon distribution multiplied by $x$, respectively. In the equations above, $f$ is the number of flavours and

$$\gamma = \sqrt{\frac{12}{\beta_0}}, \; \delta = \left( 11 + \frac{2f}{27} \right) / \beta_0$$  \hspace{1cm} (5)
with $\beta_0 = 11 - 2f/3$.

As well known [12] the Riemann function for the Goursat problem in eq. (4-c) is

$$U(\xi, \zeta; \xi', \zeta') = e^{\delta(\zeta - \zeta')} I_0[2\gamma \sqrt{(\xi' - \xi)(\zeta' - \zeta)}]$$

(6)

where $I_0(z)$ is the modified Bessel function of order zero. The solution of (4-c) can be written in the form

$$G(P) = (UG)_A + \int_A^{P_1} d\zeta U \left( \delta G + \frac{\partial G}{\partial \zeta} \right) + \int_A^{P_2} d\xi U \frac{\partial G}{\partial \xi}$$

(7)

where $P = (\xi', \zeta')$, $P_1 = (\xi_0, \zeta')$, $P_2 = (\xi', \zeta_0)$ and $A$ is the point $(\xi_0, \zeta_0)$ of the $(\xi', \zeta')$ plane.

The asymptotic form of $I_0(z)$, for large argument, leads from (7) to the double asymptotic scaling [8], for soft boundary conditions along $AP_1$ and $AP_2$. In the asymptotic region, starting scales $x_0$ and $Q_0^2$ are not important and a lower bound on $\sqrt{\xi \zeta}$ has been imposed in [8] in order to get scaling in this variable.

Near the origin of the $(\xi, \zeta)$ plane, eqs. (4-a, 4-b) give a poor approximation of the reality, if $x_0$ is not small enough, because the non singlet contribution becomes sizable. We can try to avoid in part the valence quark region by imposing on the experimental data the cuts: $x \leq 0.05$ and $y > 0.02$ as in ref. [7]. Once $x_0$ has been fixed, $x_0 = 0.05$, with the choice $Q_0^2 = 6.5 GeV^2$ H1 [4] and ZEUS [3] data cover a small strip of the $(\xi, \zeta)$ plane. The maximum value of $y = Q^2/(xs)$ is 0.56 for H1 and 0.9 for ZEUS. $s = 90200 GeV^2$ is the square of the center of mass energy of the lepton-proton collision. With $\Lambda = 200 MeV$ the aforesaid bounds can be translated in the variables $\xi, \zeta$

$$5.1e^\xi + \xi \approx \ln \left( \frac{x_0ys}{\Lambda^2} \right)$$

(8)
and it turns out that, for three (four) flavours, the argument of the Bessel function in (8) reaches a maximum value of 2.8 (3.0) for ZEUS and 2.6 (2.8) for H1.

In all the above region the series expansion for $I_0(z)$, truncated to the first three terms, provides an approximation better than the standard asymptotic expansion for all $z$, up to $z \approx 2.8$. However, already at $z = 4$, the relative errors become 20 and 3.6 percent, respectively. This can explain both the limitations of this truncation and the interest for a local parametrisation whose success is doomed to an abrupt failure. Outside the region outlined above, deviations from QCD perturbative evolution are in fact under control in the simple expression for $F_2(x, Q^2)$ we will obtain. The onset of new dynamical effects, like damping by screening corrections [13], can be easily detected and, finally, a reasonable and simple input for evolution is suggested, just at the border of the existing published data.

We return now to eq.(7). Boundary conditions can be given along the lines $\zeta_0 = 0$ and $\xi_0 \approx 2.75$ (3.0) for H1 (ZEUS). This choice satisfies the cut (8) and is equivalent to a shift in the starting point $x_0$ as far as the evolution is concerned. A more refined solution, where part of the boundary is given by eq.(8) with $y = 0.02$, would meet with difficulties in the choice of the initial distributions. Since the series for $I_0(z)$ will be truncated, $I_0(2\sqrt{z}) \approx 1 + z + z^2/4$, the evaluation of $G(\xi, \zeta)$, for given boundary conditions, is immediate. If the starting conditions are

$$G(\xi, 0) = \alpha_1 + \lambda(\xi - \xi_0) \quad (9)$$

and

$$G(\xi_0, \zeta) = \alpha_1 + \alpha_2 \zeta \quad (10)$$

with a n-terms approximation for $I_0(z)$, we end up with a polynomial in $\xi$, of degree $(n+1)$, for $G(\xi, \zeta)$. We argued that a three terms expansion for $I_0(z)$ is
needed to reproduce correctly the $Q^2$ evolution, then $G(\xi, \zeta)$, and hence $F_2$, should be a polynomial of fourth degree in the variable $\xi$ with coefficients depending on $\zeta$.

In order to reduce the number of free parameters we can perform the last integral in eq.(7) obtaining
\[
\int_0^{\zeta'} \frac{e^{-\delta\zeta}}{\sqrt{\zeta}} I_1[2\gamma\sqrt{(\xi' - \xi_0)\zeta}] d\zeta \approx \gamma\sqrt{\xi' - \xi_0} \left( \frac{1}{\delta} (1 - e^{-\delta\zeta'}) + \frac{\gamma^2}{2} \left( \frac{1}{\delta} - e^{-\delta\zeta'} (\zeta' + \frac{1}{\gamma}) \right) \right) (11)
\]
Errors in eq.(11) have been checked numerically and are compatible with the approximation adopted. At high $Q^2$, the result will lose in accuracy but, with this expedient, we can utilize in the fit the $Q^2$ bins where few data appear.

If we rewrite eq.(2) in the form
\[
F_2 = \frac{5}{18} \sum_{i=0}^{2} c_i(\zeta)(\xi - \xi_0)^i
\]
we get the relations ($c_i = c_i(0)$)
\[
c_0(\zeta) = c_0 + \frac{\gamma^2 f}{9} \left( \alpha_1 \zeta + \alpha_2 \frac{\zeta^2}{2} \right)
\]
(13-a)
\[
c_1(\zeta) = c_1 + \frac{\gamma^4 f}{9\delta^2} \left[ \left( -\alpha_1 + \frac{\alpha_2}{\delta} + \frac{\delta \lambda}{2} \right) (1 - e^{-\delta\zeta}) + (\delta \alpha_1 - \alpha_2) \zeta + \frac{\delta \alpha_2 \zeta^2}{2} \right]
\]
(13-b)
\[
c_2(\zeta) = c_2 + \frac{\gamma^6 f}{18\delta^3} \left[ \left( -2\alpha_1 + \frac{3\alpha_2}{\delta} + \frac{\delta^2 \lambda}{\gamma^2} \right) (1 - e^{-\delta\zeta}) + 
+ (\delta \alpha_1 - 2\alpha_2) \zeta + \left( \delta \alpha_1 - \alpha_2 - \frac{\delta^3 \lambda}{\gamma^2} \right) e^{-\delta\zeta} + \frac{\delta \alpha_2 \zeta^2}{2} \right]
\]
(13-c)
Boundary conditions are imposed by giving \( c_0(\zeta) \) (hence \( c_0, \alpha_1 \) and \( \alpha_2 \) are known) and the constants \( c_i \) \((i = 1, 2)\). Finally the constraint

\[
\left. \frac{\partial c_1(\zeta)}{\partial \zeta} \right|_{\zeta=0} = \frac{\gamma^2 f}{9} \lambda
\]

determines \( \lambda \) if the derivative of \( c_1(\zeta) \), for \( \zeta = 0 \), is known from experiment. The application of the above condition is difficult since experimental data have large errors. However gluon density must satisfy other approximate constraints \cite{14, 15, 16} and, since it is universal, its behaviour can be inferred also from vector meson \cite{17} and diffractive jet \cite{18} production.

We notice that the initial conditions (9) and (10) must be considered as an example since (9) should be multiplied by a factor \((1 - x)^\alpha\), that is \((1 - x_0 e^{-\xi})^\alpha\), and (10) is perhaps too naive. As a consequence of the neglect of the term \((1 - x)^\alpha\), predictions for \( F_2(x, Q^2) \) will be above the experimental data for the points with large \( x \). The effect of this approximation will be in fact more important than the neglect of the valence quarks contribution in the selected region.

3. Comparison with a fit to the data.

We argued that eqs.(12) and (13-a-13-c) will reproduce the experimental proton structure function in the region of the \((x, Q^2)\) plane specified above. To substantiate this belief we performed a fit of the experimental data, for each \( Q^2 \) bin separately, to the functional form (12).

To facilitate the comparison with the theoretical predictions, \( \xi_0 \) in (12) has been chosen as \( \xi_0 = \ln(0.05/0.0032) \) for H1 data \cite{2} that will be considered first. The total \( \chi^2_{d.o.f.} \) for all the 16 values of \( Q^2 \), from 6.5 GeV\(^2\) to 500 GeV\(^2\), is 3.9 using the full errors, with a mean value of 0.24 for each \( Q^2 \) bin fitted. Parametrisation
would give instead a total \( \chi^2_{d.o.f.} \) of 4.7. This finding justifies our claim in the Introduction that parametrisation (2) is preferable to (1).

The result from the fit is displayed in fig.1, where the values of the coefficients in eq.(12)

\[
A_i(\zeta) = \frac{5}{18} c_i(\zeta) \quad (i = 0, 1, 2)
\]

have been plotted with closed circles as function of \( Q^2 \). The continuous curves in fig.1 are obtained from eq. (13-a-13-c) with the following boundary conditions

\[
c_0 = 2.24, \quad c_1 = 0.46, \quad c_2 = 0.063
\]

\[
\alpha_1 = 5.17, \quad \alpha_2 = 9.83, \quad \lambda = 2.53
\]

and \( f = 4 \). While the parameters \( \alpha_1 \) and \( \alpha_2 \) have been fixed from a fit to \( F_2(x = 0.0032, Q^2) \), eq.(13-a), a guess for \( \lambda \) has been obtained from the experimental gluon momentum densities [2, 16].

In fig.2 the continuous curves show the gluon momentum density as function of \( x \), obtained with the parameters in (13), at \( Q^2 = 6.5 \text{ GeV}^2 \) and evolved at \( Q^2 = 20 \text{ GeV}^2 \), without any approximation on the last integral in eq.(7). Comparison with the proposal in [2, 16] shows that the slope \( \lambda \) is almost correct, while the value of \( \alpha_1 \) is below what one would expect by a 20 percent. This discrepancy can be ascribed to the difficulty of fitting the proton structure function, at fixed \( x \), as function of \( Q^2 \). Both the simplified input and the large errors in the data contribute to the difference that can be quantified in a leading-order calculation. The dashed curve in fig.2 represents the result for \( G(x, Q^2) \) obtained from eq.(12) with the leading order method of ref. [14]. Next-to-leading order calculation would lower this prediction, but the value for \( \alpha_1 \), suggested by the fit, seems really too
small.

The same fit has been repeated for the ZEUS data [3]. The total $\chi^2_{d.o.f.}$ increases, with a mean value of 0.8 for each $Q^2$ bin fitted. $A_0(\zeta)$ and $A_1(\zeta)$ appear as in fig.1, but the fit for $A_2(\zeta)$ deteriorates somewhat. While the spread of the fitted points increases, the curves defined in eqs.(13-a-13-d) represent always well the general behaviour. The different regions of the $(\xi, \zeta)$ plane covered by the two sets of data account only partly for the difference in the two fits. However, other parametrisations presented us with the same problem.

Going back to the H1 data, examples of the proton structure function $F_2(x, Q^2)$, obtained with this method, are given in fig.3 with continuous curves, at different $Q^2$ values, as function of the variable $\sqrt{s_{eq}}$ introduced in ref. [7],

$$\sqrt{s_{eq}} = \frac{2Q_1}{\sqrt{x}}.$$ 

The choice of this variable, where $Q_1 = 270 \text{ MeV}$, allows for a comparison of the energy dependence of the mean charged multiplicity with the $x$ behaviour of $F_2$.

Quadratic fits in $\ln s$ to the multiplicity data for $p-p$ scattering are rather old [19, 20] and their derivation in the framework of a Regge model has been suggested [21]. Till 1992 the phenomenological fit of ref. [20] has been shown in the Review of Particle Properties [22] and, from then, an analogous version adapted to $e^+ - e^-$ annihilation became available [23].

In fig.3 we have superimposed to the data also the predictions for the mean charged multiplicity of ref. [20] (dashed line) and of ref. [23] (dotted line) normalized to the $F_2$ data at each $Q^2$ value. The idea of similarity between the two observables, advanced in ref. [7], seems confirmed in this more qualitative approach.
Remarkably, this similarity could be process independent at high energy. Notice that the empirical logarithmic solution has a better $\chi^2_{d.o.f.}$ than the leading-log approximation proposed in [24] and provides a fit to the mean charged multiplicity comparable to the modified leading-log approximation [25] in the energy range under consideration.

4. Conclusions

Motivations for the success of simple parametrisations for the proton structure function have been found in the structure of the QCD evolution equations. In this paper we have shown that a region of the $(x, Q^2)$ plane exists where both asymptotic and truncated series expansions give the answer with a comparable accuracy. In the double logarithmic approximation, we are considering, this region encloses almost all HERA data while, at smaller $x$, the asymptotic solution would certainly be the only appropriate.

Except for an exact next-to-leading order computer calculation, the logarithmic fit appears to be at the level of other simple parametrisations, at least in the explored domain of the relevant variables. This phenomenological finding has a correspondence in the theoretical picture of deep inelastic scattering, where $F_2(x, Q^2)$ is obtained from the discontinuity of the Pomeron exchanged between the proton and a quark loop coupled to the photon. Representing the Pomeron as a gluonic ladder, the Born term, corresponding to the exchange of two gluons, would produce roughly an $x$ independent asymptotic structure function. Every rung added to the ladder brings about a term increasing like $\ln(1/x)$ when $x$ tends to zero [26]. Hence, a two-rung gluonic ladder is suggested from our logarithmic fit.

Two important points must however be noticed. First, at the border of the
$(\xi, \zeta)$ region studied, this approach shows his limitations and the fit deteriorates somewhat. The second point regards the possibility to extend the range of validity of this method by keeping more terms in the expansion for the Bessel function. The answer is negative. Mathematical tables $[27]$ show that, in the double logarithmic approximation, double scaling becomes asymptotically ineluctable.
References

[1] A. De Rujula et al., Phys Rev. D10, 1649 (1974)
    Yu. L. Dokshitzer, Sov. Phys JETP 46, 641 (1977).

[2] S. Aid et al, H1 Coll., Nucl. Phys. B47, 3 (1996).

[3] M. Derrick et al., ZEUS Coll., Z. Phys. C72, 399 (1996).

[4] V.N. Gribov and L.N. Lipatov, Sov. J. Nucl. Phys. 15, 438 and 675 (1972).

[5] G. Altarelli and G. Parisi, Nucl. Phys. B126, 298 (1977).

[6] R.D. Ball and S. Forte, Phys. Lett. B335, 77 (1994); Phys. Lett B 336, 77  
    (1994).

[7] A. De Roeck and E.A. De Wolf, Phys. Lett. B388, 188 (1996).

[8] M. Bertini et al., preprint LYCEN 9366, 1993

[9] L.L. Jenkovszky, A.V. Kotikov and F. Paccanoni, Phys. Lett. B314, 421 (1993);  
    F. Paccanoni, Proceedings of the workshop Hadrons-94, Uzhgorod, ed. G. Bugrij  
    et al.; E.S. Martynov, ibidem.

[10] W. Buchmüller and D. Haidt, hep-ph/9605428.

[11] S.Forte and R.D. Ball, hep-ph/9610268.

[12] F.G. Tricomi, Equazioni a derivate parziali, (ed. Cremonese, Roma) 1957.

[13] L.V. Gribov, E.M. Levin and M.G. Ryskin, Phys. Rep. 100, 1 (1983).

[14] K. Prytz, Phys. Lett B311, 286 (1993), Phys. Lett. B332, 393 (1994).
[15] R.K. Ellis, Z. Kunszt and E.M. Levin, Nucl. Phys. B420, 517 (1994).

[16] M. Derrick et al., ZEUS Coll., Phys. Lett. B345, 576 (1995).

[17] M.G. Ryskin, Z. Phys. C57, 89 (1993).

[18] N.N. Nikolaev and B.G. Zakharov, Phys. Lett. B332, 177 (1994).

[19] E. Albini et al., Nuovo Cimento 32A, 101 (1976).

[20] W. Thomè et al., Nucl. Phys. B129, 365 (1977).

[21] N.A. Kobylnsky, E.S. Martynov and V.P. Shelest, Z. Phys. C28, 143 (1985).

[22] K. Hikasa et al., PDG, Phys. Rev. D45, No 11 (1992).

[23] P.D. Acton et al., OPAL Coll., Z. Phys. C53, 539 (1992).

[24] W. Furmanski, R. Petronzio and S. Pokorski, Nucl. Phys. B155, 253 (1979);
A. Bassetto et al., Nucl. Phys. B207, 189 (1982).

[25] V.A. Khoze and W. Ochs, Int. J. Mod. Phys. A12, 2949 (1997) and references quoted there.

[26] B.M. McCoy and T.T. Wu, Phys. Rev. D12, 3257 (1975).

[27] M. Abramowitz and I.A. Stegun eds., Handbook of Mathematical Functions (Dover, New York, 1965).
Figure Captions

fig. 1: Coefficients of the logarithmic fit, plotted as function of $Q^2$ for the data in [2], the continuous line shows the prediction of the model (see text). The point at 500 GeV$^2$ for $A_1$ has been shifted to the right for the sake of clarity.

fig. 2: The gluon momentum density in the model, as function of $x$, at $Q^2 = 6.5$ GeV$^2$ and evolved at $Q^2 = 20$ GeV$^2$ (continuous line). Dashed line shows the result of a leading order calculation [14].

fig. 3: Proton structure function as function of $\sqrt{s_{eq}}$ (see text) at different $Q^2$ values, data are from [2] and continuous lines from our model. Predictions obtained from the mean charged multiplicity, normalized to $F_2(x,Q^2)$, are from [20] (dashed line) and from [23] (dotted line).
