Expressions for two generalized Furdui series

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Abstract

We solve two problems of analysis and special function theory recently posed by Furdui. The series in question are special cases in our solution.

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Statement of results

We let $\Gamma$, $\psi$, and $\psi^{(j)}$ denote the Gamma, digamma, and polygamma functions, respectively \[1\]. We let $\gamma = -\psi(1)$ be the Euler constant. We let $\zeta(z)$ denote the Riemann zeta function, $\zeta(z,a)$ the Hurwitz zeta function, and $\text{Li}_s$ the polylogarithm function \[8\]. The latter functions may be initially defined by the series
\[
\text{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}, \quad |z| \leq 1,
\] (1)
and analytically continued throughout the complex plane. In the case of integral index, as occurs in the following, we also have an expression in terms of the generalized hypergeometric function $\text{}_{p}F_{q}$ \[1\]:
\[
\text{Li}_n(z) = z \cdot _{n+1}F_{n}(1,1,\ldots,1;2,\ldots,2;z).
\] (2)

We have the special case
\[
\text{Li}_1(z) = -\ln(1-z).
\] (3)

We then have

**Proposition 1.** Put for integers $j \geq 0$ and $|z| \geq 1$, $z \neq -1$,
\[
S_j(z) \equiv \sum_{n=1}^{\infty} \frac{(-1)^n}{z^n} \frac{1}{n^j} \left[ \zeta \left(1 + \frac{1}{n}\right) - n - \gamma \right].
\] (4)
Then (a)
\[
S_j(z) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \gamma_k \text{Li}_{j+k} \left(\frac{1}{z}\right),
\] (5)
and (b) (Furdui case \[7\])
\[
S_0(1) = \gamma_1 \ln 2 + \sum_{k=2}^{\infty} \frac{(-1)^k}{k!} \gamma_k (2^{1-k} - 1) \zeta(k),
\] (6)
where \( \{\gamma_k\}_{k=0}^{\infty} \) are the Stieltjes constants for the Riemann zeta function [2, 3, 9].

**Proposition 2.** Put for integers \( j \geq 0 \) and \( |z| \geq 1, z \neq -1 \),

\[
T_j(z) \equiv \sum_{n=1}^{\infty} \frac{(-1)^n}{z^n} \frac{1}{n^j} \left[ n - \Gamma\left( \frac{1}{n} \right) - \gamma \right]. \tag{7}
\]

Let

\[
\Gamma(x) - \frac{1}{x} = \sum_{j=0}^{\infty} \frac{c_j}{(j+1)!} x^j, \quad |x| < 1,
\]

where \( c_0 = -\gamma \) and \( c_1 = \gamma^2 + \zeta(2) \). Then (a)

\[
T_j(z) = -\sum_{k=1}^{\infty} \frac{c_k}{(k+1)!} \text{Li}_{j+k}\left(-\frac{1}{z}\right), \tag{9}
\]

and (b) (Furdui case [7])

\[
T_0(1) = -\sum_{k=2}^{\infty} \frac{c_k}{(k+1)!} (2^{1-k} - 1)\zeta(k) - \frac{c_1}{2} \ln 2. \tag{10}
\]

**Proposition 3.** Let \( \{\gamma_k(a)\}_{k=0}^{\infty} \) be the Stieltjes coefficients for the Hurwitz zeta function [2, 3, 9]. Put for integers \( j \geq 0, \ell \geq 1, |z| \geq 1, z \neq -1, \) and Re \( a > 0 \),

\[
S_{j\ell}(z,a) \equiv \sum_{n=1}^{\infty} \frac{(-1)^n}{z^n} \frac{1}{n^j} \left[ \zeta(\ell) \left( 1 + \frac{1}{n}, a \right) - (-1)^\ell n^{\ell+1} - (-1)^\ell \gamma_\ell(a) \right]. \tag{11}
\]

Then (a)

\[
S_{j\ell}(z,a) = \sum_{k=\ell+1}^{\infty} \frac{(-1)^k}{(k-\ell)!} \gamma_k(a) \text{Li}_{j+k-\ell}\left(-\frac{1}{z}\right), \tag{12}
\]

(b) for \( j \geq 1 \)

\[
S_{j\ell}(1,a) = \sum_{k=\ell+1}^{\infty} \frac{(-1)^k}{(k-\ell)!} \gamma_k(a) (2^{1+\ell-j-k} - 1)\zeta(j+k-\ell), \tag{13}
\]

and (c)

\[
S_{0\ell}(1,a) = (-1)^\ell \gamma_{\ell+1} \ln 2 + \sum_{k=\ell+2}^{\infty} \frac{(-1)^k}{(k-\ell)!} \gamma_k(a) (2^{1+\ell-k} - 1)\zeta(k-\ell). \tag{14}
\]
Proposition 4. Put for integers \( j \geq 0, \ell \geq 0, \) and \( |z| \geq 1, \ z \neq -1, \)

\[
U_{j\ell}(z) \equiv \sum_{n=1}^{\infty} \frac{(-1)^n}{z^n} \frac{1}{n^j} \left[ \left( \frac{\zeta'(s)}{\zeta} \right)^{(\ell)} \left( 1 + \frac{1}{n} \right) + (-1)^\ell n^{\ell+1} + \ell!\zeta \right].
\]  

(15)

Let

\[
\frac{\zeta'(s)}{\zeta} = -\frac{1}{s-1} - \sum_{j=0}^{\infty} \eta_j (s-1)^j, \quad |s-1| < 3,
\]  

(16)

where \( \eta_0 = -\gamma \) and \( \eta_1 = \gamma^2 + 2\gamma_1 \) \([6, 5]\) (Appendix). Then we have (a)

\[
U_{j\ell}(z) = -\sum_{k=\ell+1}^{\infty} \frac{k!}{(k-\ell)!} \eta_k \text{Li}_{j+k-\ell} \left( \frac{-1}{z} \right),
\]  

(17)

(b) for \( j \geq 1 \)

\[
U_{j\ell}(z) = -\sum_{k=\ell+1}^{\infty} \frac{k!}{(k-\ell)!} \eta_k (2^{1+\ell-j-k} - 1) \zeta(j+k-\ell),
\]  

(18)

and (c)

\[
U_{0\ell}(1) = \eta_{\ell+1} \ln 2 - \sum_{k=\ell+2}^{\infty} \frac{k!}{(k-\ell)!} \eta_k (2^{1+\ell-j-k} - 1) \zeta(j+k-\ell).
\]  

(19)

Proof of Propositions

 Proposition 1. We make use of the well known Laurent expansion \([2, 3, 9]\)

\[
\zeta(s) = \frac{1}{s-1} + \sum_{k=0}^{\infty} \frac{(-1)^k \gamma_k}{k!} (s-1)^k, \quad s \neq 1,
\]  

(20)

where \( \gamma_0 = \gamma. \) Then we have

\[
S_j(z) = \sum_{n=1}^{\infty} \frac{(-1)^n}{z^n} \frac{1}{n^j} \sum_{k=1}^{\infty} \frac{(-1)^k \gamma_k}{k!} \frac{1}{n^k}
\]

\[
= \sum_{k=1}^{\infty} \frac{(-1)^k \gamma_k}{k!} \sum_{n=1}^{\infty} \frac{(-1)^n}{z^n n^{j+k}}
\]

4
\[
\sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \gamma_k \text{Li}_{j+k} \left( -\frac{1}{z} \right),
\]  
\tag{21}
\]
wherein we used the series definition (1). For part (b) we use the alternating zeta function case
\[
\text{Li}_k(-1) = (2^{1-k} - 1)\zeta(k),
\]  
\tag{22}
\]
together with the easily verified limit
\[
\lim_{x \to 1} (2^{1-x} - 1)\zeta(x) = -\ln 2.
\]  
\tag{23}
\]
Alternatively, we could make use of the special case (3) in Eq. (21).

**Remarks.** Numerically, we have \( S_0(1) \simeq -0.0462635927840 \) and \( \gamma_1 \ln 2 \simeq -0.0504720979971 \).

As many series and integral representations for \( \gamma_k \) are known, (e.g. \[2, 3\]) (5) and (6) may be rewritten in a variety of ways.

By the functional equation of the zeta function, the summand of (4) could be written in terms of \( \zeta(-1/n) \).

**Proposition 2.** This Proposition follows similarly, using the defining expansion (8) for the constants \( c_j \). For part (b), we again use the case (22) and the limit (23).

**Remarks.** Numerically, \( T_0(1) \simeq 0.371990830350 \) and \( -c_1 (\ln 2)/2 \simeq -0.685561374577 \).

As a first approximation, one may take \( c_k/(k + 1)! \simeq (-1)^{k+1} \) for all \( k \geq 2 \).

The constants \( c_j \) may be systematically found from polyganmic constants in terms of Bell polynomials. This is because \( \Gamma' = \Gamma\psi \) and we may appeal to Lemma 1 of \[4\].

**Proposition 3.** We have from \[2, 3, 9\]
\[
\zeta(s) = \frac{1}{s-1} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \gamma_k(a)(s-1)^k, \quad s \neq 1,
\]  
\tag{24}
\]
where $\gamma_0(a) = -\psi(a)$, for $\ell \geq 1$

$$\zeta^{(\ell)}(s, a) = \frac{(-1)^\ell \ell!}{(s - 1)^{\ell + 1}} + \sum_{k=\ell}^{\infty} \frac{(-1)^k}{k!} \gamma_k(a) k(k-1) \cdots (k-\ell+1)(s-1)^{k-\ell}, \quad s \neq 1, \quad (25)$$

Therefore, we have

$$\zeta^{(\ell)} \left( 1 + \frac{1}{n}, a \right) - (-1)^\ell n^{\ell + 1} - (-1)^\ell \gamma_\ell(a) = \sum_{k=\ell+1}^{\infty} \frac{(-1)^k}{(k-\ell)!} \frac{\gamma_k(a)}{n^{k-\ell}}, \quad (26)$$

giving

$$S_{j\ell}(z, a) = \frac{\sum_{k=\ell+1}^{\infty} \frac{(-1)^k}{(k-\ell)!} \gamma_k(a) \sum_{n=1}^{\infty} \frac{(-1)^n}{z^n} \frac{1}{n^{j+k-\ell}}}{\sum_{k=\ell+1}^{\infty} \frac{(-1)^k}{(k-\ell)!} \gamma_k(a) \text{Li}_{j+k-\ell} \left( -\frac{1}{z} \right)} . \quad (27)$$

This proves part (a). For part (b) we use relation (22). For part (c) in turn we use the limit (23).

**Proposition 4.** We have from (16)

$$\left( \frac{\zeta'}{\zeta} \right)^{(\ell)}(s) = -\frac{(-1)^\ell \ell!}{(s-1)^{\ell+1}} - \sum_{j=\ell}^{\infty} \eta_j j(j-1) \cdots (j-\ell+1)(s-1)^{j-\ell}, \quad |s-1| < 3, \quad (28)$$

giving

$$\left( \frac{\zeta'}{\zeta} \right)^{(\ell)} \left( 1 + \frac{1}{n} \right) + (-1)^\ell \ell! n^{\ell + 1} + \ell! \eta_\ell = -\sum_{j=\ell+1}^{\infty} \frac{j!}{(j-\ell)!} \frac{\eta_j}{n^{j-\ell}}, \quad (29)$$

Then we find

$$U_{j\ell}(z) = -\sum_{k=\ell+1}^{\infty} \frac{k!}{(k-\ell)!} \eta_k \sum_{n=1}^{\infty} \frac{(-1)^n}{z^n} \frac{1}{n^{k+j-\ell}}$$

$$= -\sum_{k=\ell+1}^{\infty} \frac{k!}{(k-\ell)!} \eta_k \text{Li}_{j+k-\ell} \left( -\frac{1}{z} \right) . \quad (30)$$

For part (b) we may use (22) and for part (c) (23).
Remarks. A known recursion relation [5] (Appendix) systematically gives the $\eta_j$ constants in terms of the Stieltjes constants.

Numerically we have $\eta_1 \ln 2 \simeq -0.129997$ and $U_{00}(1) \simeq 0.0975567$.

Similarly we may generalize Proposition 2 to sums containing derivatives of the $\Gamma$ function,

\[
T_{j\ell}(z) \equiv -\sum_{n=1}^{\infty} \frac{(-1)^n}{z^n} \frac{1}{n^\ell} \left[ (-1)^\ell \ell! n^{\ell+1} + \Gamma^{(\ell)} \left( \frac{1}{n} \right) - \frac{c_\ell}{\ell + 1} \right]
\]

\[
= -\sum_{k=\ell+1}^{\infty} \frac{c_k}{(k + 1) (k - \ell)!} \frac{1}{(1/n) - \ell} \right] \frac{1}{(k + 1) (k - \ell)!} \left( \frac{1}{z} \right).
\]

Moreover, we may extend our method to sums with other analytic function summands, including for instance $\zeta^2 + \zeta' - 2\gamma\zeta$ and $\zeta^2 - (\zeta'/\zeta)' - 2\gamma\zeta$. We could also similarly perform sums over derivatives of the Lerch zeta function $\Phi$. 

\[7\]
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