Bose-Einstein droplet in free space

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We show that a droplet of a Bose-Einstein condensate can be dynamically stabilized in free space by rapid oscillations of interatomic interactions between attractive and repulsive through, e.g., the Feshbach resonance. Energy dissipation, which is present in realistic situations, is found to play a crucial role to suppress dynamical instabilities inherent in nonlinear nonequilibrium systems.

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I. INTRODUCTION

Matter-wave bright solitons — stable solitary waves whose density is greater than the background one — have recently been realized in a quasi one-dimensional (1D) Bose-Einstein condensate (BEC) [1, 2]. In 1D, the quantum kinetic pressure counterbalances an attractive interatomic interaction, allowing a stable bright soliton to be formed. However, in 2D or higher dimensions, bright solitons are always unstable against collapse or expansion. When a trapping potential vanishes, Section III presents the droplet and shows that the dependence of the dynamics on the oscillation frequency can be eliminated by the scaling property of the GP equation with oscillating interactions. The stability diagrams with respect to the interaction and dissipative parameters are also obtained. Section IV studies the variational analysis, and Sec. V concludes this paper.

II. THE SCALING PROPERTY OF THE GROSS-PITAEVSKII EQUATION

We consider the GP equation in the presence of dissipation given by [10, 11]

\[
(i - \gamma)\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V_{\text{trap}}(r,t)\psi + \frac{4\pi\hbar^2 a(t)}{m} |\psi|^2 \psi + i\gamma |\psi| \psi,
\]

(1)

where \(\gamma\) is a phenomenological dissipation constant which is to be determined experimentally, and the trapping potential \(V_{\text{trap}}(r,t)\) and the s-wave scattering length \(a(t)\) are controlled to vary in time. The last term on the right-hand side of Eq. (1) guarantees the normalization \(\int |\psi|^2 = N\) with \(N\) being the number of atoms, where the chemical potential is given by

\[
\mu = \int d\mathbf{r} \bar{\psi} \left( -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{trap}}(r,t) + \frac{4\pi\hbar^2 a(t)}{m} |\psi|^2 \right) \psi.
\]

For simplicity, we assume an isotropic trapping potential

\[
V_{\text{trap}}(r,t) = f_{\text{trap}}(t) \frac{m\omega^2}{2} r^2,
\]

(3)

where \(f_{\text{trap}}(t)\) is unity at \(t = 0\), then decreases towards zero, and vanishes at some time. The s-wave scattering length is made to oscillate as

\[
a(t) = f_{\text{int}}(t)(a_0 + a_1 \sin \Omega t),
\]

(4)

where \(f_{\text{int}}(t)\) ramps up from 0 to 1. The ramp functions \(f_{\text{trap}}(t)\) and \(f_{\text{int}}(t)\) are introduced to avoid initial nonadiabatic disturbances that cause dynamic instabilities. These functions must be chosen so that neither collapse nor expansion occurs in the course of the ramping.

Normalizing the length, time, energy, and wave function by \(\langle \hbar/m\Omega \rangle^{1/2} \equiv d_0, \Omega^{-1}, \hbar\Omega, \) and \(\sqrt{N}d_0^{-3/2}\), respectively, we obtain the normalized form of the GP equation:
oscillating interaction.

Interaction shown as the dotted curve. This retardation is the droplet lags slightly behind that of the oscillating in-

(b) that the phase of the breathing-mode oscillations of flow persist even after stabilization. We find from Fig. 1

amplitude. This indicates that large oscillations of mass droplet exhibits breathing-mode oscillations with a large

ramps.

initial nonadiabatic disturbances smaller than linear

due to dissipation. From the time evolution of the den-

We use a quadratic function for \( f \) for

t\( \equiv 0.03 \) \([10, 11]\), and the ramp parameters \( \gamma \) \( = 0.01 \) from those of the sta-

\[ T_{\text{trap}} = 16 \] and \( T_{\text{int}} = 10 \). The initial state is chosen to be the noninteracting ground state for a trapping potential with \( \omega = \Omega/30 \). The density and width oscillate at the driving frequency. [The black bands in Fig. 4 (a) represent rapid oscillations of \( \langle r \rangle \) and \( |\psi(0)|^2 \) which are beyond the resolution limit.] We note that the amplitudes and mean values of both \( \langle r \rangle \) and \( |\psi(0)|^2 \) converge to constant values, indicating that a BEC droplet is dynamically stabilized. The initial disturbances last for a relatively long time shown as slow oscillations or ripples of \( \langle r \rangle \) and \( |\psi(0)|^2 \) in Fig. 4 (a), which gradually decay due to dissipation. From the time evolution of the density profile \( |\psi(0)|^2 \) [inset in Fig. 4 (b)], we find that the droplet exhibits breathing-mode oscillations with a large amplitude. This indicates that large oscillations of mass flow persist even after stabilization. We find from Fig. 4 (b) that the phase of the breathing-mode oscillations of the droplet lags slightly behind that of the oscillating interaction shown as the dotted curve. This retardation is considered to be due to dissipation because there is no phase difference in the dissipation-free 2D case \([2]\), where \( |\psi(0)|^2 \) and \( \langle r \rangle \) are in-phase and out-of-phase with the oscillating interaction.

\[
(i-\gamma)\frac{\partial \psi}{\partial t} = -\frac{\nabla^2}{2} \psi + \frac{f_{\text{trap}}(t)}{2} \left( \frac{\omega}{\Omega} \right)^2 \psi^2 + g(t)|\psi|^2 \psi + i\gamma \mu \psi,
\]

where \( g(t) \equiv 4\pi Na(t)/d_0 \equiv f_{\text{int}}(t)(g_0 + g_1 \sin t) \), and the wave function is normalized as \( \int d\bm{r} |\psi|^2 = 1 \). It should be noted that once the trapping potential is switched off \( [f_{\text{trap}}(t) = 0] \), Eq. (4) no longer depends on \( \Omega \) explicitly. This implies that the \( \Omega \) dependence can be eliminated to those at \( \Omega \) by rescaling \( \Omega \) for instance, we have only to increase the strength of interaction beyond the resolution limit.] We note that the ampli-

This implies that the \( \Omega \) dependence can be eliminated to those at \( \Omega \) by rescaling \( \Omega \). For example, if large \( \Omega \) is unavailable

amplitude profile

We numerically solve the GP equation \( \psi \) using the Crank-Nicholson scheme \([12]\), where the following simple forms of the ramp functions are used:

\[
f_{\text{trap}}(t) = \begin{cases} 1 - t/T_{\text{trap}} & (0 \leq t \leq T_{\text{trap}}) \\ 0 & (t > T_{\text{trap}}) \end{cases}
\]

\[
f_{\text{int}}(t) = \begin{cases} 1 - (t/T_{\text{int}} - 1)^2 & (0 \leq t \leq T_{\text{int}}) \\ 1 & (t > T_{\text{int}}) \end{cases}
\]

We use a quadratic function for \( f_{\text{int}}(t) \) because it makes initial nonadiabatic disturbances smaller than linear ramps.

The time evolution of the peak density \( |\psi(0)|^2 \) and the monopole moment \( \langle r \rangle = \int d\bm{r} |\psi|^2 \) for \( g_0 = -69 \) and \( g_1 = 155 \). The dissipation constant \( \gamma \) is taken to be 0.03 \([10, 11]\], and the ramp parameters are \( T_{\text{trap}} = 16 \) and \( T_{\text{int}} = 10 \). The initial state is chosen to be the noninteracting ground state for a trapping potential with \( \omega = \Omega/30 \). The density and width oscillate at the driving frequency. [The black bands in Fig. 4 (a) represent rapid oscillations of \( \langle r \rangle \) and \( |\psi(0)|^2 \) which are beyond the resolution limit.] We note that the amplitudes and mean values of both \( \langle r \rangle \) and \( |\psi(0)|^2 \) converge to constant values, indicating that a BEC droplet is dynamically stabilized. The initial disturbances last for a relatively long time shown as slow oscillations or ripples of \( \langle r \rangle \) and \( |\psi(0)|^2 \) in Fig. 4 (a), which gradually decay due to dissipation. From the time evolution of the density profile \( |\psi(0)|^2 \) [inset in Fig. 4 (b)], we find that the droplet exhibits breathing-mode oscillations with a large amplitude. This indicates that large oscillations of mass flow persist even after stabilization. We find from Fig. 4 (b) that the phase of the breathing-mode oscillations of the droplet lags slightly behind that of the oscillating interaction shown as the dotted curve. This retardation is considered to be due to dissipation because there is no phase difference in the dissipation-free 2D case \([2]\), where \( |\psi(0)|^2 \) and \( \langle r \rangle \) are in-phase and out-of-phase with the oscillating interaction.

![Figure 1](image1.png)

**FIG. 1:** (a) Time evolution of the peak density \( |\psi(0)|^2 \) (left axis) and monopole moment \( \langle r \rangle = \int d\bm{r} |\psi|^2 \) (right axis) for the oscillating interaction \( g(t) = -69 + 155 \sin t \) with \( \gamma = 0.3 \). The initial state is the noninteracting ground state in a trapping potential \( r^2/1800 \). Then the interaction is gradually turned on, and the trapping potential is simultaneously turned off as shown in the inset. (b) A magnification of (a). The inset shows the density profile \( |\psi(r)|^2 \) from \( t = 8000 \) to \( t = 8020 \). The dotted line is a sine function for comparison of the phase.

We gradually changed \( g_0 \) and \( g_1 \) from those of the stable droplet state in Fig. 4 and found distinct types of instabilities as shown in Fig. 5. When \( g_0 \) is decreased, a droplet expands because of a decrease in an attractive interaction as shown in Fig. 5 (a). The expansion occurs also by an increase in \( g_1 \), since the effective repulsive interaction due to the oscillation is proportional to \( g_1^2 \) [see Eq. (11)]. When \( g_0 \) is increased and \( g_1 \) is decreased, the system becomes dynamically unstable against the slow oscillation and eventually expands away as shown in Fig. 5 (b). The expansion occurs, despite the fact that an increase in \( g_0 \) enhances the attraction and a decrease in \( g_1 \) suppresses the repulsion. The slow oscillation is seen in the profile of the maxima of \( |\psi(0)|^2 \) (the
squares and the density profiles in (b) and (c) are magnifications of in Fig. 1, and change the parameters to various values to destabilization of the droplet. show that higher radial modes are excited, which leads parametric resonance. The density profiles in the inset frequency grows, this instability is considered to arise from for large ripple in the upper edge), while the minima is almost constant. Figure 2 (c) shows an instability that arises for large \(|g_0|\) and \(g_1\). Since the modulation at half frequency grows, this instability is considered to arise from parametric resonance. The density profiles in the inset show that higher radial modes are excited, which leads to destabilization of the droplet.

We prepare a stable droplet in the same manner as in Fig. 1 and change the parameters to various values to obtain the stability diagram shown in Fig. 3. The parameters \(g_0\) and \(g_1\) are changed linearly during \(0 \leq t \leq 600\). We define the lifetime of the droplet as the duration between \(t = 600\) and the time at which the droplet begins to expand indefinitely. The regions referred to as “instability I, II, and III” in Fig. 3 correspond to the instabilities demonstrated in Fig. 2 (a), (b), and (c). We find from Fig. 3 (a) that the stable region is surrounded by the three distinct types of instabilities. The boundary between the “stable” and “instability I” regions are sharp while the “instability II” gradually sets in as \(|g_0|\) increases and \(g_1\) decreases. The collapse occurs for large \(|g_0|\) and small \(g_1\), but the corresponding parameter regime lies outside Fig. 3.

In Fig. 3 (b), \(\gamma\) is linearly decreased from 0.03 to 0.01 as \(|g_0|\) and \(g_1\) are (also linearly) changed. The instability regions enlarge as compared with those of Fig 3 (a), and the stability region disappears. From this result, we conclude that the stability region exists only for \(\gamma \neq 0\). However, we cannot exclude the possibility that other
stable states exist, since we do not investigate the entire functional space.

In numerical calculations of the time evolution, we must pay special attention to the boundary effect. Since the trapping potential is absent, the atoms that escape from the droplet spread out, reflect at the boundary, and return to the droplet region, producing spurious boundary effects. For example, when the Dirichlet boundary condition is imposed at $r = 116$, the lifetime of the droplet (whose definition is the same as in Fig. 3) for $g_0 = -69$ and $g_1 = 134$ becomes $\approx 4100$, while the correct value is $\approx 3600$. Therefore, the spatial cutoff in the numerical analysis must be much larger than the size of the droplet.

The numerical simulations that we have presented so far have been carried out under the assumption of the spherical symmetry of the system. However, this geometry cannot generate multipole dynamical instabilities which destroy a BEC droplet. In order to check the absence of such multipole instabilities, we have performed full 3D calculations by discretizing the space of $100 \times 100 \times 100$ size in our dimensionless unit into a $256 \times 256 \times 256$ mesh. A limited spatial size due to our computational power causes spurious boundary effects and a rough mesh produces numerical errors. Nevertheless, as shown in Fig. 1 snapshots of time evolution of a BEC droplet appear isotropic and do not show any multipole instability. We have thus confirmed that a BEC droplet can be stabilized without multipole instabilities.

IV. VARIATIONAL ANALYSIS

The Gaussian variational wave function well describes the dynamical stabilization of BECs in 2D free space. We examine a variational method in 3D using the Gaussian trial function as

$$\psi_{\text{var}}(r, t) = \frac{1}{\pi^{3/4} R^{3/2}} \exp \left( -\frac{r^2}{2R^2} + i \frac{\dot{R} r^2}{2R} \right),$$

where $R(t)$ is the variational parameter that characterizes the size of the condensate. Substituting Eq. (8) into the action

$$S = \int dt d\mathbf{r} \psi^* \left( -i \frac{\partial}{\partial t} - \frac{\nabla^2}{2} + \frac{g}{2} |\psi|^2 \right) \psi,$$

we obtain the equation of motion for $R$ as

$$\ddot{R} = -\frac{d}{dR} \left( \frac{1}{2R^2} + \frac{G}{6R} \right),$$

where $G(t) \equiv (g_0 + g_1 \sin t)/(2^{1/2} \pi^{3/2}) \equiv G_0 + G_1 \sin t$.

We separate $R$ into the slowly varying part $R_0$ and the rapidly oscillating part $\rho$ as $R = R_0 + \rho$. According to Ref. [1], an effective potential for $R_0$ is given by $f_1^2/(4\Omega^2)$, where $f_1$ is the amplitude of the oscillating “force” for $R_0$ and $\Omega$ is its frequency. From Eq. (10), $f_1$ corresponds to $G_1/(2R_0^2)$. Since we are using a system of units in which $\Omega = 1$, the effective potential becomes $G_1^2/(16R_0^4)$, and then the equation of motion for $R_0$ reads

$$\ddot{R}_0 = -\frac{d}{dR_0} \left( \frac{1}{2R_0^2} + \frac{G_0}{6R_0^2} + \frac{G_1^2}{16R_0^4} \right) \equiv -\frac{dU_{\text{eff}}}{dR_0}.$$ (11)

This equation agrees with that in Ref. [4] when $\Omega$ is much larger than the characteristic frequencies of the system. Thus, the oscillating interaction blocks the collapse by the effective potential proportional to $R_0^{-8}$.

The effective potential $U_{\text{eff}}$ has a local minimum for appropriate values of $G_0$ and $G_1$. For instance, when $G_0 = -70/(2^{1/2} \pi^{3/2})$ and $G_1 = 140/(2^{1/2} \pi^{3/2})$, $U_{\text{eff}}$ has a local minimum at $R_0 \approx 2.4$. In fact, by numerically solving the equation of motion (11) for $R$, we find that a stable solution exists. However, Fig. 3(b) implies that the system is unstable against the dynamical instability for $\gamma = 0$. This indicates that the simple Gaussian function is insufficient to describe the instability of the system.

Figure 4 illustrates the difference between the density profiles of the stable droplet ($t \gtrsim 8000$ in Fig. 4) and the Gaussian functions fitted to them. We adopt the least squares fitting of $r^2 |\psi(r)|^2$ to $r^2/(\pi^{3/2} \sigma^3) \exp(-r^2/\sigma^2)$ with fitting parameter $\sigma$, which is an appropriate fitting in 3D. Figure 4 shows that the numerically exact wave functions significantly deviate from the Gaussian functions even in the stable droplet. We note that the Gaussian function in Fig. 5(a) is almost the same as that in Fig. 5(b). This indicates that the outer region, which has large weight in the fitting due to the factor $r^2$, is almost stationary, and the mass flow occurs mainly around the center of the droplet. Thus, more appropriate functions are needed to correctly describe the phenomena.

V. CONCLUSIONS

We have studied a Bose-Einstein condensate with an oscillating interaction with dissipation, and shown that a BEC droplet is stabilized in 3D free space with realistic
FIG. 5: The density profiles $|\psi(r)|^2$ (solid curves) of the stable droplet for $g_0 = -69$ and $g_1 = 155$ ($t \approx 8000$ in Fig. 4) when the peak density becomes (a) maximal and (b) minimal. The dashed curves are the Gaussian functions fitted to the density profiles.

dissipation, $\gamma = 0.03$. The frequency of the oscillating interaction must be much faster than the characteristic frequency of the system, but it can be a moderate frequency according to the scaling property discussed in Sec. II. For instance, the situation in Fig. 1 can be realized in a condensate of $10^4 \text{ } ^{85}\text{Rb}$ atoms by the s-wave scattering length $a(t) = -0.6 + 1.35 \sin \Omega t$ [nm] with $\Omega = 100 \times 2\pi$ Hz, where a trapping potential with frequency $3.3$ Hz is removed in 25 ms, and the final size of the droplet becomes a few micrometers. Such an oscillation of interaction can be easily realized in experiments using the Feshbach resonance [12].

Under gravity, a condensate falls after the trapping potential is turned off. In the above example, the gravitational sag in the initial trapping potential is $\approx 2$ cm, and the condensate falls about 1 mm until the trapping potential vanishes. The effect of gravity can be canceled out using the magnetic levitation [10], which enables us to observe the longtime behavior of the droplet.

A “gaseous BEC droplet” found in this paper is different from the usual condensate in that it coheres by itself without the help of the trapping potential. This self-trapped matter wave might exhibit interesting dynamics, such as in collective mode, collapsing dynamics, and vortex nucleation, which will be discussed elsewhere.

Note added. While preparing the present paper, a similar paper [17] appeared which also reports stabilization of a BEC in 3D free space by an oscillating interaction. The strength of interaction studied in Ref. [17] is much larger than that of ours. In our parameter regime, a droplet can be stabilized adiabatically by simple ramp functions with realistic dissipation. On the other hand, in Ref [17] a droplet is stabilized without dissipation by following a complicated ramp scheme to reach a final state.

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