Poisson statistics for beta ensembles on the real line at high temperature

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Abstract

This paper studies beta ensembles on the real line in a high temperature regime, that is, the regime where \( \beta N \to \text{const} \in (0, \infty) \), with \( N \) the system size and \( \beta \) the inverse temperature. In this regime, the convergence to the equilibrium measure is a consequence of a recent result on large deviation principle by Liu and Wu (Stochastic Processes and their Applications (2019)). This paper focuses on the local behavior and shows that the local statistics around any fixed reference energy converges weakly to a homogeneous Poisson point process.

Keywords: beta ensembles ; high temperature ; large deviation principle ; Poisson statistics

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1 Introduction

Let \( V: \mathbb{R} \to \mathbb{R} \) be a measurable function. Let

\[
H_N = H_N(\lambda_1, \ldots, \lambda_N) = \frac{1}{N} \sum_{i=1}^{N} V(\lambda_i) - \frac{1}{N^2} \sum_{i \neq j} \log |\lambda_j - \lambda_i|
\]

\[
= \int V(x) dL_N(x) - \int \int_{x \neq y} \log |x - y| dL_N(x) dL_N(y),
\]

be the energy of the configuration \((\lambda_1, \lambda_2, \ldots, \lambda_N) \in \mathbb{R}^N\) under the external potential \(V\) and the log-interaction, where \(L_N = N^{-1} \sum_{i=1}^{N} \delta_{\lambda_i}\) denotes the empirical distribution with \(\delta_{\lambda}\) the Dirac measure. Beta ensembles are then defined as ensembles of \(N\) particles with the joint density proportional to

\[
e^{-\frac{\beta N^2}{2} H_N} = |\Delta(\lambda)|^\beta e^{-\frac{\beta N}{2} \sum_{i=1}^{N} V(\lambda_i)}.
\]

Here \(\Delta(\lambda) = \prod_{i<j}(\lambda_j - \lambda_i)\) is the Vandermonde determinant. The parameter \(\beta > 0\) is regarded as the inverse temperature of the system.

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Under some mild conditions on $V$, the energy functional

$$E(\mu) = \int \int \left( \frac{1}{2} V(x) + \frac{1}{2} V(y) - \log |x - y| \right) d\mu(x) d\mu(y)$$

$$= \int V(x) d\mu(x) - \int \int \log |x - y| d\mu(x) d\mu(y)$$

which is well-defined on the set $\mathcal{P}(\mathbb{R})$ of probability measures on $\mathbb{R}$ has a unique minimizer $\mu_V$ of compact support, that is,

$$E(\mu_V) = \inf_{\mu \in \mathcal{P}(\mathbb{R})} E(\mu).$$

The minimizer $\mu_V$ is an equilibrium of the system in the sense that for fixed $\beta > 0$, as $N \to \infty$, the empirical distribution $L_N$ converges weakly to $\mu_V$, almost surely. Gaussian fluctuations around the limit were also known [8].

What happens in case the parameter $\beta$ varies a function of $N$? This problem has been considered through several works. For instance, a large deviation principle (LDP) for the sequence of empirical distributions $\{L_N\}$ was established under the condition that $(\beta N)/\log N \to \infty$. As a consequence, the sequence $\{L_N\}$ converges weakly to $\mu_V$, almost surely, as long as $(\beta N)/\log N \to \infty$ [4]. In a particular case of Gaussian beta ensembles ($V(x) = x^2/2$), the convergence to the semi-circle distribution (the equilibrium measure in this case), holds in an optimal condition that $\beta N \to \infty$. It is optimal because when $\beta N$ stays bounded, $L_N = L_{N, \beta}$ converges to Gaussian like probability measures of full-support [2, 3, 7, 10, 12, 13].

This paper deals with the following (scaled) beta ensembles

$$\lambda_1, \lambda_2, \ldots, \lambda_N \propto \frac{1}{Z_N} |\Delta(\lambda)|^\beta e^{-\sum_{i=1}^N V(\lambda_i)}$$

in the regime where $\beta N \to 2c \in (0, \infty)$. Here $Z_N$ is the normalizing constant. A LDP has been recently studied for more general models [9]. It turns out that under the condition

$$\int |x|^k e^{-V(x)} dx < \infty, \quad k = 0, 1, 2, \ldots,$$

the sequence of empirical distributions $\{L_N\}$ satisfies a LDP with the good rate function $I_c(\mu) = E_c(\mu) - \inf_\nu E_c(\nu)$, where the functional $E_c$ is defined for absolutely continuous probability measure $\mu(dx) = \rho(x) dx$,

$$E_c(\mu) = \int \log(\rho(x)) \rho(x) dx + \int V(x) \rho(x) dx - c \int \log |x - y| \rho(x) \rho(y) dxdy.$$

(See Section 2 for a more precise definition.) Such functional has appeared in heuristic saddle point arguments as in [1, 2, 12]. Note that the functional $E_c$ has a unique minimizer, denoted by $\mu_c$ (or $\rho_c$ for the density), because of the strict convexity. Then the LDP implies the almost sure convergence of empirical distributions to the equilibrium measure $\mu_c$.

The aim of this paper is to study the local statistics around a fixed reference energy $E \in \mathbb{R}$,

$$\xi_N(E) = \sum_{i=1}^N \delta_{N(\lambda_i - E)}.$$
We show that the local statistics $\xi_N(E)$ converges weakly to a homogeneous Poisson point process on $\mathbb{R}$. That local behavior in the case of Gaussian beta ensembles was proved in [3] and in [10] by different methods. However, the two approaches relied more or less on both the joint density and the tridiagonal matrix model. This paper refines ideas developed in the two papers to establish the Poisson statistics for beta ensembles with generic potential $V$. Our main result is stated as follows.

**Theorem 1.1.** Assume that the potential $V$ is continuous and that

$$\lim_{x \to \pm \infty} \frac{V(x)}{\log(1 + x^2)} = \infty.$$

Then in the regime where $\beta N \to 2c \in (0, \infty)$, the following hold.

(0) The sequence of empirical distributions $\{L_N\}$ satisfies a LDP with the good rate function $I_c(\mu)$.

(i) The minimizer $\mu_c$ of $\mathcal{E}_c$, or of $I_c$ has continuous density $\rho_c$ satisfying the relation

$$\rho_c(x) = \frac{1}{Z_c} e^{-V(x) + 2c \int \log |x-y| \rho_c(y) dy}, \quad \text{for all } x \in \mathbb{R},$$

where $Z_c$ is a constant.

(ii) The empirical distribution $L_N$ converges weakly to $\mu_c$, almost surely.

(iii) For fixed $E \in \mathbb{R}$, the local statistics $\xi_N(E)$ converges weakly to a homogeneous Poisson point process on $\mathbb{R}$ with density $\rho_c(E) > 0$.

The paper is organized as follows. The next section is devoted to introduce a LDP. Section 3 studies properties of the limiting measure $\mu_c$ and proves some estimates needed for Section 4 in which the Poisson statistics is derived.

## 2 Large deviation principle

### 2.1 Assumption on the potential $V$

Throughout this paper, we assume that the potential $V: \mathbb{R} \to \mathbb{R}$ is measurable, bounded below and

$$\lim_{x \to \pm \infty} \frac{V(x)}{\log(1 + x^2)} = \infty.$$

Under that assumption,

$$\int_{\mathbb{R}} |x|^k e^{-V(x)} dx < \infty, \quad \text{for all } k = 0, 1, 2, \ldots$$

Let $\alpha$ be the probability measure with density $\alpha(x) = Z^{-1} e^{-V(x)}$, where $Z = \int_{\mathbb{R}} e^{-V(x)} dx$ is the normalizing constant. Then all moments of $\alpha$ are finite.
2.2 Energy functionals

Let $\mathcal{P}(\mathbb{R})$ be the set of probability measures on $\mathbb{R}$, endowed with the weak topology. For $\mu, \nu \in \mathcal{P}(\mathbb{R})$, the entropy of $\mu$ relative to $\nu$ (also called the Kullback–Leibler divergence) is defined by

$$H(\mu|\nu) = \begin{cases} \int \frac{d\mu}{d\nu} \log \frac{d\mu}{d\nu} d\nu, & \text{if } \mu \ll \nu, \\ +\infty, & \text{otherwise.} \end{cases}$$  \hspace{1cm} (1)$$

Here $\mu \ll \nu$ means that the measure $\mu$ is absolutely continuous with respect to the measure $\nu$, and $\frac{d\mu}{d\nu}$ denotes the Radon–Nikodym derivative.

(i) $H(\mu|\nu) \geq 0$, (with equality only when $\mu = \nu$);

(ii) $H(\mu|\nu)$ is lower semi-continuous; and

(iii) for $L > 0$, the sublevel set $\{\mu \in \mathcal{P}(\mathbb{R}) : H(\mu|\nu) \leq L\}$ is compact.

In addition, it is also known that

(iv) $H(\mu|\nu)$ is strictly convex on the sublevel set $\{\mu \in \mathcal{P}(\mathbb{R}) : H(\mu|\nu) \leq L\}$, for any $L > 0$.

Let $\{X_i\}_i$ be an i.i.d. (independent identically distributed) sequence of random variables with common distribution $\alpha$. Denote by $\eta_N$ the empirical measure of $\{X_i\}_{i=1}^N$,

$$\eta_N = \frac{1}{N} \sum_{i=1}^N \delta_{X_i}.$$ 

Then Sanov’s theorem states that $\{\eta_N\}_N$ satisfies a LDP on $\mathcal{P}(\mathbb{R})$ with the good rate function $H(\mu|\alpha)$. The functional $H(\mu|\alpha)$ can be expressed as follows

$$H(\mu|\alpha) = \int \rho(x) \log(\rho(x)) dx + \int V(x) \rho(x) dx + \log Z,$$  \hspace{1cm} (2)$$

where $\rho$ is the density of $\mu$, provided that $H(\mu|\alpha) < \infty$.

For $c > 0$, let

$$H_c(\mu) := \begin{cases} H(\mu|\alpha) - c \iint \log |x-y| d\mu(x) d\mu(y), & \text{if } H(\mu|\alpha) < \infty, \\ +\infty, & \text{otherwise.} \end{cases}$$  \hspace{1cm} (3)$$

We claim that the functional $H_c$ is well-defined, that is, $H_c(\mu) \in (-\infty, \infty)$, if $H(\mu|\alpha) < \infty$. This is an easy consequence of the Donsker–Varadhan variational formula. Indeed, the variational formula implies that for a bounded measurable function $\psi: \mathbb{R}^2 \to \mathbb{R}$,

$$\iint \psi(x,y) d\mu(x) d\mu(y) \leq 2H(\mu|\alpha) + \log \iint e^{\psi(x,y)} d\alpha(x) d\alpha(y).$$
It follows that the inequality still holds for any non-negative measurable function by the monotone convergence theorem. Then using that formula for
\[
\psi(x, y) = \begin{cases} \frac{1}{2} |\log |x - y||, & x \neq y, \\ 0, & x = y, \end{cases}
\]
we obtain that
\[
\frac{1}{2} \iint |\log |x - y|| \, d\mu(x) \, d\mu(y) \leq 2H(\mu|\alpha) + \log \iint e^{\frac{1}{2} |\log |x - y||} \, d\alpha(x) \, d\alpha(y) = 2H(\mu|\alpha) + \log \iint \left( |x - y|^\frac{1}{2} \mathbf{1}_{\{|x - y| \geq 1\}} + |x - y|^\frac{1}{2} \mathbf{1}_{\{|x - y| < 1\}} \right) \, d\alpha(x) \, d\alpha(y) < \infty.
\]
Here the integral in the last line is finite because the density \( \alpha(x) \) is bounded and the first moment of \( \alpha \) is finite. Therefore, once the functional \( H_c(\mu) \) is finite, the measure \( \mu \) is absolutely continuous and \( H_c(\mu) \) can be written as a sum of finite integrals
\[
H_c(\mu) = H_c(\rho) = \int \rho(x) \log(\rho(x)) \, dx + \int V(x) \rho(x) \, dx
\]
\[
- c \iint \log |x - y| \rho(x) \rho(y) \, dxdy + \log Z,
\]
with \( \rho \) the density of \( \mu \).

2.3 Large deviation principle

Let us consider the following beta ensembles
\[
(\lambda_1, \lambda_2, \ldots, \lambda_N) \propto \frac{1}{Z_N} \Delta(\lambda)^{\beta_N} e^{-\sum_{i=1}^{N} V(\lambda_i)},
\]
in the regime where \( \beta_N N \to 2c \in (0, \infty) \). Then the sequence of empirical distributions \( L_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i} \) satisfies a LDP on \( \mathcal{P}(\mathbb{R}) \) with the good rate function \( I_c \) defined as
\[
I_c(\mu) = H_c(\mu) - \inf_{\nu \in \mathcal{P}(\mathbb{R})} H_c(\nu).
\]
This is a particular case of a general result in [9].

The functional \( H_c(\mu) \) is strictly convex on any sublevel set \( \{ \mu \in \mathcal{P}(\mathbb{R}) : H(\mu|\alpha) \leq L \}, (L > 0) \). This is because the logarithmic energy
\[
- \iint \log |x - y| \rho(x) \rho(y) \, dxdy
\]
is also convex on that set (cf. [11, Lemma 1.8]). Consequently, the minimizer of \( H_c \) is unique. Therefore, the LDP implies the following strong law of large numbers.

**Theorem 2.1.** Let \( \mu_c \) be the unique minimizer of \( H_c \), or of \( I_c \). Then as \( N \to \infty \) with \( \beta N \to 2c \in (0, \infty) \), the empirical distribution \( L_N \) converges weakly to \( \mu_c \), almost surely.
Remark 2.2. By a heuristic saddle point argument, it was shown in [1] that the limiting measure in the regime $\beta N \to 2c \in (0, \infty)$ is a minimizer of the energy functional $H_c$. Then by using functional derivative, an equation to characterize the minimizer $\rho_c$ was derived
\[
\int \frac{V'(x)\rho_c(x)}{x-z} dx + cS^2_c(z) + S'_c(z) = 0, \quad z \in \mathbb{C} \setminus \mathbb{R},
\]
where $S_c(z) = \int \frac{\rho_c(x)dx}{x-z}$ is the Stieltjes transform of $\rho_c$.

3 Equilibrium measures

We have just shown that in the regime where $\beta N \to 2c \in (0, \infty)$, the empirical distribution $L_N$ converges weakly to the limiting measure $\mu_c$ which is the minimizer of the energy functional $H_c$, almost surely. Let $\rho_c$ be the density of $\mu_c$. In this section, we are going to derive an equation characterizing $\rho_c$ in a rigorous way. Let us fix a sequence $\{\beta_N\}$ such that $\beta_N \to 2c$. For simplicity, the subindex $N$ of $\beta_N$ is omitted in formulae.

3.1 Some initial estimates

Let us begin with an estimate for the normalizing constant $Z_N$, also called a partition function
\[
Z_N = \int \cdots \int |\Delta(\lambda)|^\beta e^{-\sum_{i=1}^N V(\lambda_i)} d\lambda_1 \cdots d\lambda_N.
\]
Note that $Z_N$ depends on both $N$ and $\beta$.

Lemma 3.1. For $\kappa > 0$, there is a constant $C_\kappa \in \mathbb{R}$ such that for $\beta N \leq \kappa$,
\[
\frac{\log Z_N}{N} \geq C_\kappa.
\]

Proof. Let us express $Z_N$ in terms of the integral with respect to the probability measure $\alpha$,
\[
Z_N = Z^N \int \cdots \int e^{\frac{\beta}{2} \sum_{i \neq j} \log |\lambda_j - \lambda_i|} d\alpha(\lambda_1) \cdots d\alpha(\lambda_N).
\]
Then a lower bound is just a direct consequence of Jensen’s inequality,
\[
\log Z_N \geq N \log Z + \frac{\beta N(N-1)}{2} \iint \log |x-y| d\alpha(x)d\alpha(y)
\]
\[
\geq N \left( \log Z + \begin{cases} 0, & \text{if } \ell_\alpha = \iint \log |x-y| d\alpha(x)d\alpha(y) \geq 0 \\ \frac{\ell_\alpha}{2}, & \text{otherwise} \end{cases} \right).
\]
The proof is complete. \qed

Lemma 3.2. Assume that $\varphi: \mathbb{R} \to \mathbb{R}$ is a measurable function satisfying
\[
\int_{\mathbb{R}} |x|^k e^{\varphi(x) - V(x)} dx < \infty, \quad k = 0, 1, 2, \ldots.
\]
Let
\[ Z^{(\varphi)}_N = \int \cdots \int |\Delta(\lambda)|^\beta \prod_{i=1}^N e^{(\varphi-V)(\lambda_i)} d\lambda_i. \]

Then for \( \beta N \leq \kappa \),
\[ \log \frac{Z^{(\varphi)}_N}{N} \leq C_{\varphi,\kappa}, \]
where \( C_{\varphi,\kappa} \) is a constant depending on \( \varphi \) and \( \kappa \).

**Proof.** Let \( Z_\varphi \) be the normalizing constant of the probability measure
\[ d\mu(x) = \frac{1}{Z_\varphi} e^{\varphi(x)-V(x)} dx. \]

Let \( \Phi(x,y) = \log |x-y| \vee 0 = \max\{\log |x-y|, 0\} \). Then it is clear that
\[ Z^{(\varphi)}_N = (Z_\varphi)^N \mathbb{E} \left[ \exp \left( \frac{\beta}{2} \sum_{i \neq j} \log |X_i - X_j| \right) \right] \leq (Z_\varphi)^N \mathbb{E} \left[ \exp \left( \frac{\beta}{2} \sum_{i \neq j} \Phi(X_i, X_j) \right) \right], \]
where \( \{X_i\}_{i=1}^N \) is an i.i.d. sequence of random variables with common distribution \( \mu \). Let \( \{Y_i\}_{i=1}^N \) be an independent copy of \( \{X_i\}_{i=1}^N \). Then it follows from Proposition 3.3 (decoupling inequality) and Lemma 3.4 in [9] that
\[ \mathbb{E} \left[ \exp \left( \frac{\beta}{2} \sum_{i \neq j} \Phi(X_i, X_j) \right) \right] \leq \mathbb{E} \left[ \exp \left( 4 \beta \sum_{i \neq j} \Phi(X_i, Y_j) \right) \right] \leq \exp \left( (N-1) \log \mathbb{E} \left[ e^{4\beta N \Phi(X_1, Y_1)} \right] \right) \leq \exp \left( N \log \mathbb{E} \left[ e^{4\kappa \Phi(X_1, Y_1)} \right] \right) \leq \exp \left( N \log \mathbb{E} \left[ 1 + |X_1 - Y_1|^{4\kappa} \right] \right). \]

An upper bound is then obtained by taking the logarithm
\[ \log \frac{Z^{(\varphi)}_N}{N} \leq \log Z_\varphi + \log \left( 1 + \mathbb{E}[|X_1 - Y_1|^{4\kappa}] \right) =: C_{\varphi,\kappa}, \]
which completes the proof. \( \square \)

**Lemma 3.3.** Assume that \( \varphi: \mathbb{R} \to \mathbb{R} \) is a measurable function satisfying
\[ \int_\mathbb{R} |x|^ke^{\varphi(x)-V(x)} dx < \infty, \quad k = 0, 1, 2, \ldots. \]

Then for \( \beta N \leq \kappa \),
\[ \mathbb{E} \left[ e^{\frac{1}{N} \sum_{i=1}^N \varphi(\lambda_i)} \right] \leq \left( \mathbb{E} \left[ e^{\frac{1}{N} \sum_{i=1}^N \varphi(\lambda_i)} \right] \right)^{1/N} \leq M_{\varphi,\kappa}, \]
where \( M_{\varphi,\kappa} \) is a constant depending on \( \varphi \) and \( \kappa \).

**Proof.** By noting that
\[ \mathbb{E} \left[ e^{\sum_{i=1}^N \varphi(\lambda_i)} \right] = \frac{Z^{(\varphi)}_N}{Z_N}, \]
this lemma is a direct consequence of the above two lemmas. \( \square \)
Lemma 3.4. For $\kappa > 0$, there is a constant $M = M(\kappa)$ such that for $\beta N \leq \kappa$,
\[
\mathbb{E} \left[ \prod_{i=1}^{N} |x - \lambda_i|^\beta \right] \leq M (1 + x^2)^{\frac{\beta}{2}}. \tag{6}
\]

Proof. It follows from the inequality
\[
|x - \lambda|^2 \leq (1 + x^2) (1 + \lambda^2),
\]
that
\[
\beta \sum_{i=1}^{N} \log |x - \lambda_i| \leq \frac{\beta N}{2} \log(1 + x^2) + \frac{1}{N} \sum_{i=1}^{N} \frac{\beta N}{2} \log(1 + \lambda_i^2)
\]
\[
\leq \frac{\kappa}{2} \log(1 + x^2) + \frac{1}{N} \sum_{i=1}^{N} \frac{\kappa}{2} \log(1 + \lambda_i^2).
\]

Therefore,
\[
\mathbb{E} \left[ \prod_{i=1}^{N} |x - \lambda_i|^\beta \right] \leq \mathbb{E} \left[ e^{\frac{\beta}{N} \sum_{i=1}^{N} \frac{\beta}{2} \log(1 + \lambda_i^2)} \right] \times (1 + x^2)^{\frac{\beta}{2}}.
\]

The expectation on the right hand side is bounded by a constant $M = M(\kappa)$ by using Lemma 3.3 for the function $\varphi(\lambda) = \frac{\kappa}{2} \log(1 + \lambda^2)$. The proof is complete. \qed

Lemma 3.5. Let $\kappa > 0$ and $L > 0$. Then there is a number $\theta = \theta(\kappa, L)$ such that for $\beta k \lor \beta N \leq \kappa$, and any $x_1, x_2, \ldots, x_k \in [-L, L]$,
\[
\mathbb{E} \left[ \prod_{j=1}^{k} \prod_{i=1}^{N} |x_j - \lambda_i|^\beta \right] \leq \theta^k.
\]

Proof. The proof is similar to that of Lemma 3.4. We begin with the following inequality
\[
\beta \sum_{j=1}^{k} \sum_{i=1}^{N} \log |x_j - \lambda_i| \leq \frac{\beta}{2} \sum_{j=1}^{k} \sum_{i=1}^{N} \left( \log(1 + x_j^2) + \log(1 + \lambda_i^2) \right)
\]
\[
\leq \frac{k \beta N}{2} \log(1 + L^2) + \frac{k \beta}{2} \sum_{i=1}^{N} \log(1 + \lambda_i^2)
\]
\[
\leq \frac{k \kappa}{2} \log(1 + L^2) + \left\{ \begin{array}{ll}
\frac{k}{N} \sum_{i=1}^{N} \frac{\kappa}{2} \log(1 + \lambda_i^2), & \text{if } k \leq N, \\
\sum_{i=1}^{N} \frac{\kappa}{2} \log(1 + \lambda_i^2), & \text{if } k > N.
\end{array} \right.
\]

It follows that
\[
\mathbb{E} \left[ \prod_{j=1}^{k} \prod_{i=1}^{N} |x_j - \lambda_i|^\beta \right] \leq (1 + L^2)^{\frac{k \kappa}{2}} \times \left( \mathbb{E} \left[ e^{\frac{k}{N} \sum_{i=1}^{N} \frac{\kappa}{2} \log(1 + \lambda_i^2)} \right] \right)^k, \quad \text{if } k \leq N,
\]
\[
\leq (1 + L^2)^{\frac{k \kappa}{2}} \times \left( \mathbb{E} \left[ e^{\sum_{i=1}^{N} \frac{\kappa}{2} \log(1 + \lambda_i^2)} \right] \right)^k, \quad \text{if } k > N,
\]
\[
\leq \theta^k,
\]
for some constant $\theta = \theta(\kappa, L) > 0$. Here in case $k \leq N$, Hölder’s inequality has been used. The proof is complete. \qed
Corollary 3.6. For $x \in \mathbb{R}$, let

$$X_N(x) = \prod_{i=1}^{N} |x - \lambda_i|^\beta.$$ 

Then for any compact set $K \subset \mathbb{R}$,

$$\sup_{\beta N \leq \kappa, x \in K} E[(X_N(x))^2] < \infty.$$ 

In particular, for any bounded sequence $\{x_N\} \subset \mathbb{R}$, in the regime where $\beta N \to 2c \in (0, \infty)$, the sequence $\{X_N(x_N)\}$ is uniformly integrable.

Let

$$Z_{N-1}^{(1)} = \int \cdots \int |\Delta(\lambda)|^\beta e^{-\sum_{i=1}^{N-1} V(\lambda_i)} d\lambda_1 \cdots d\lambda_{N-1}, \quad (\beta = \beta_N).$$

Note that $Z_{N-1}^{(1)}$ is the normalizing constant with parameters $(\beta_N, N - 1)$ which is different from $Z_{N-1}$. Then the following relation holds

$$Z_N = Z_{N-1}^{(1)} \int e^{-V(x)} E_{\beta, N-1} \left[ \prod_{i=1}^{N-1} |x - \lambda_i|^\beta \right] dx$$

$$= Z_{N-1}^{(1)} \int e^{-V(x)} E_{\beta, N-1} \left[ e^{\beta \sum_{i=1}^{N-1} \log |x - \lambda_i|} \right] dx.$$ 

Here $E_{\beta, N-1}[^{\cdot}]$ denotes the expectation with respect to the ensemble with parameters $(\beta, N - 1)$. We also write $\mathbb{P}_{\beta, N-1}(\cdot)$ for the probability measure associated with that ensemble.

Let $\rho_N(x)$ be the first point function of the beta ensemble (5)

$$\rho_N(x) = \frac{1}{Z_N} e^{-V(x)} \int \cdots \int \left( \prod_{i=1}^{N-1} |x - \lambda_i|^\beta \right) |\Delta(\lambda)|^\beta e^{-\sum_{i=1}^{N-1} V(\lambda_i)} d\lambda_1 \cdots d\lambda_{N-1}$$

$$= \frac{Z_{N-1}^{(1)}}{Z_N} e^{-V(x)} E_{\beta, N-1} \left[ \prod_{i=1}^{N-1} |x - \lambda_i|^\beta \right].$$

Then for any integrable function $f$,

$$E[(L_N, f)] = \int f(x) \rho_N(x) dx.$$ 

Here $(\mu, f)$ denotes the integral $\int f(x) d\mu(x)$ for a probability measure $\mu$ and a measurable function $f$. Note that Lemma 3.4 and the assumption on $V$ imply that $\rho_N(x) dx$ is bounded (in $x$), and all moments of $\rho_N(x)$ are finite. And thus, $\int \log |x - y| \rho_N(y) dy < \infty$, for any $x \in \mathbb{R}$.

The following estimate is analogous to Lemma 4.4 in [8].

Lemma 3.7. For $\kappa > 0$, there is a constant $C = C(\kappa) > 0$ such that for $\beta N \leq \kappa$,

$$\frac{Z_{N-1}^{(1)}}{Z_N} \leq C.$$
Proof. Let \( \hat{\rho}_{N-1} \) be the first point function of the ensemble (5) with parameters \((\beta, N - 1)\). Then Jensen’s inequality implies that
\[
\mathbb{E}_{\beta, N-1} \left[ e^{\beta \sum_{i=1}^{N-1} \log |x - \lambda_i|} \right] \geq e^{\beta(N-1) \int \log |x-y| \hat{\rho}_{N-1}(y) dy}.
\]
Using Jensen’s inequality again, we deduce that
\[
\frac{Z_N}{Z_{(1)}_{N-1}} = \int e^{-V(x)} \mathbb{E}_{\beta, N-1} \left[ e^{\beta \sum_{i=1}^{N-1} \log |x - \lambda_i|} \right] dx
\geq Z \int e^{\beta(N-1) \int \log |x-y| \hat{\rho}_{N-1}(y) dy} d\alpha(x)
\geq Z e^{\beta(N-1) \int \int \log |x-y| \hat{\rho}_{N-1}(y) dy d\alpha(x)}.
\]
Let \( \|\alpha\|_{\infty} = Z^{-1} \sup_x e^{-V(x)} < \infty \). Then it is clear that
\[
\int \int \log |x-y| \hat{\rho}_{N-1}(y) dy d\alpha(x) = \int \left( \int \log |x-y| d\alpha(x) \right) \hat{\rho}_{N-1}(y) dy
\geq \int \left( \int_{|x-y| \leq 1} \log |x-y| d\alpha(x) \right) \hat{\rho}_{N-1}(y) dy
\geq \int \left( \int_{|x-y| \leq 1} \log |x-y| \|\alpha\|_{\infty} dx \right) \hat{\rho}_{N-1}(y) dy
= -2\|\alpha\|_{\infty}.
\]
Since \( \beta(N - 1) < \beta N \leq \kappa \), we conclude that
\[
\frac{Z_N}{Z_{(1)}_{N-1}} \geq Ze^{-2\|\alpha\|_{\infty}}.
\]
The proof is complete. \( \square \)

Lemma 3.8. For any \( \kappa > 0 \) and \( k > 0 \),
\[
\sup_{\beta N \leq \kappa} \mathbb{E}[(L_N, |x|^k)] < \infty. \tag{7}
\]
Proof. Note that \( \beta(N - 1) < \beta N \leq \kappa \). Thus, Lemma 3.4 is applicable and gives us the bound
\[
\mathbb{E}_{\beta, N-1} \left[ \prod_{i=1}^{N-1} |x - \lambda_i|^{\beta} \right] \leq M(1 + x^2)^{\frac{k}{2}},
\]
for some constant \( M = M(\kappa) \). Together with Lemma 3.7, it then follows that
\[
\rho_N(x) \leq MC(1 + x^2)^{\frac{k}{2}} e^{-V(x)}.
\]
Recall that all moments of \( \alpha \) are finite. Therefore, for \( \beta N \leq \kappa \),
\[
\mathbb{E}[(L_N, |x|^k)] = \int \int |x|^k \rho_N(x) dx \leq MC \int |x|^k (1 + x^2)^{\frac{k}{2}} e^{-V(x)} dx < \infty,
\]
which completes the proof. \( \square \)

From the proof of the above lemma, we extract the following, which may be called Wegner’s bound for general beta ensembles.

Lemma 3.9. In the regime \( \beta N \to 2c \), there is a constant \( \Lambda > 0 \) such that
\[
\rho_N(x) \leq \Lambda, \quad \text{for any } x \in \mathbb{R}, \text{ and any } N.
\]
3.2 Continuous functions of polynomial growth

We consider here a weaker type of convergence, convergence in probability. We will show that the sequence of moments of the empirical distribution $L_N$ also converges. Note that the weak convergence of probability measures does not imply the convergence of moments.

**Lemma 3.10.** For any $k \in \mathbb{N}$, as $N \to \infty$ with $\beta N \to 2c \in (0, \infty)$,
\[
\langle L_N, x^k \rangle \to \langle \mu_c, x^k \rangle \quad \text{in } L^1, \text{ and in probability.}
\]

**Proof.** In the regime where $\beta N \to 2c \in (0, \infty)$, recall from Lemma 3.8 that for $k \in \mathbb{N}$,
\[
M_{2k} = \sup_N \mathbb{E}[\langle L_N, x^{2k} \rangle] < \infty.
\]
For $L > 0$, let
\[
f_L(x) = (x^k \wedge L) \vee (-L) = \begin{cases} 
   x^k, & \text{if } -L \leq x^k \leq L, \\
   -L, & \text{if } x^k \leq -L, \\
   L, & \text{if } x^k \geq L.
\end{cases}
\]
Since $f_L(x)$ is a bounded continuous function, as $N \to \infty$,
\[
\langle L_N, f_L \rangle \to \langle \mu_c, f_L \rangle \quad \text{in } L^1.
\]
It then follows that $\langle \mu_c, x^k \rangle = \lim_{L \to \infty} \langle \mu_c, f_L \rangle \leq M_k$, for any even $k$. Consequently, all moments of $\mu_c$ are finite.

It is clear that
\[
|\langle L_N, x^k \rangle - \langle L_N, f_L \rangle| \leq \langle L_N, |x|^k 1_{\{|x|^k > L\}} \rangle \leq \frac{1}{L} \langle L_N, x^{2k} \rangle.
\]
Then, by taking the expectations of both sides, we obtain that
\[
0 \leq \mathbb{E}[|\langle L_N, x^k \rangle - \langle L_N, f_L \rangle|] \leq \frac{\mathbb{E}[\langle L_N, x^{2k} \rangle]}{L} \leq \frac{M_{2k}}{L}. \quad (8)
\]
Next, the triangular inequality yields
\[
\mathbb{E}[|\langle L_N, x^k \rangle - \langle \mu_c, x^k \rangle|] \leq \mathbb{E}[|\langle L_N, x^k \rangle - \langle L_N, f_L \rangle|] + \mathbb{E}[|\langle L_N, f_L \rangle - \langle \mu_c, f_L \rangle|] + |\langle \mu_c, x^k - f_L \rangle|.
\]
The first term and the third term are bounded by $M_{2k}/L$ by (8). The second term converges to zero as $N \to \infty$. These imply that $\mathbb{E}[|\langle L_N, x^k \rangle - \langle \mu_c, x^k \rangle|] \to 0$ as $N \to \infty$. The proof is complete. \(\Box\)

A consequence of the convergence of moments is the convergence for continuous test functions of polynomial growth. For the proof of the derivation, see Lemma 2.2 in [6], for example.

**Corollary 3.11.** Let $f$ be a continuous function of polynomial growth. Then in the regime where $\beta N \to 2c$,
\[
\langle L_N, f \rangle \to \langle \mu_c, f \rangle \quad \text{in probability.}
\]
3.3 log functions

Lemma 3.12. Let \( x \in \mathbb{R} \) be given. Then as \( \beta N \to 2c \),

\[
\frac{1}{N} \sum_{i=1}^{N} \log |x - \lambda_i| \to \int \log |x - y|\rho_c(y)dy \quad \text{in probability.}
\]

Recall that \( \rho_c \) is the density of the limiting measure \( \mu_c \).

Proof. We first remark that since \( \rho_N(x) \leq \Lambda \), and since the sequence of probability measures \( \{\rho_N(x)dx\} \) converges weakly to \( \rho_c(x)dx \), it follows that the density \( \rho_c \) is also bounded by \( \Lambda \) (almost everywhere). In addition, recall that all moments of \( \rho_c \) are finite. Thus, for any \( x \in \mathbb{R} \),

\[
\int \log |x - y|\rho_c(y)dy < \infty.
\]

For \( L > 0 \), the truncation \( f_L(y) = \log |x - y| \vee (-L) \) is a continuous function of polynomial growth, and hence, as \( N \to \infty \),

\[
\frac{1}{N} \sum_{i=1}^{N} \log |x - \lambda_i| \vee (-L) \to \int (\log |x - y| \vee (-L))\rho_c(y)dy \quad \text{in probability.}
\]

Next, we use the inequality

\[
|\log |x - y| - f_L(y)| \leq -\log |x - y|1_{|y - x| \leq e^{-L}}
\]

to deduce that

\[
\mathbb{E}[\langle L_N, \log |x - \cdot| \rangle - \langle L_N, f_L \rangle] \leq \mathbb{E}[\langle L_N, -\log |x - \cdot|1_{|\cdot - x| \leq e^{-L}} \rangle]
\]

\[
= \int_{[x-e^{-L},x+e^{-L}]} (-\log |x - y|)\rho_N(y)dy
\]

\[
\leq M \int_{[x-e^{-L},x+e^{-L}]} (-\log |x - y|)dy
\]

\[
= 2M(1 + L)e^{-L}
\]

\[
\to 0 \quad \text{as} \quad L \to \infty.
\]

The desired result follows easily from the triangular inequality. The proof is complete. \( \square \)

3.4 Partition functions

Similar to the statement of Lemma 3.12, under \( \mathbb{P}_{\beta,N-1} \), it also holds that for fixed \( x \in \mathbb{R} \), as \( \beta N \to 2c \),

\[
\frac{1}{N-1} \sum_{i=1}^{N-1} \log |x - \lambda_i| \to \int \log |x - y|\rho_c(y)dy \quad \text{in probability.}
\]

From which, we get the following results.
Lemma 3.13. For fixed $x \in \mathbb{R}$, as $\beta N \to 2c \in (0, \infty)$,
\[
\prod_{i=1}^{N-1} |x - \lambda_i|^\beta = e^{\beta \sum_{i=1}^{N-1} \log |x - \lambda_i|} \to e^{2c \int \log |x - y| \rho_c(y) dy}
\]
in probability under $\mathbb{P}_{\beta,N-1}$ by the continuous mapping theorem, and then
\[
\mathbb{E}_{\beta,N-1} \left[ \prod_{i=1}^{N-1} |x - \lambda_i|^\beta \right] \to e^{2c \int \log |x - y| \rho_c(y) dy}, \quad (9)
\]
by the uniform integrability.

Lemma 3.14. As $N \to \infty$,
\[
\frac{Z_N}{Z_N^{(1)}} = \int e^{-V(x)} \mathbb{E}_{\beta,N-1} \left[ \prod_{i=1}^{N-1} |x - \lambda_i|^\beta \right] dx \to \int e^{-V(x) + 2c \int \log |x - y| \rho_c(y) dy} dx.
\]

Proof. The desired result follows from (6) and (9) by using Lebesgue’s dominated convergence theorem.

3.5 The first joint functions

Recall that the first point function $\rho_N(x)$ can be expressed as
\[
\rho_N(x) = \frac{Z_N^{(1)}}{Z_N} e^{-V(x)} \mathbb{E}_{\beta,N-1} \left[ \prod_{i=1}^{N-1} |x - \lambda_i|^\beta \right].
\]

Then it follows from Lemma 3.13 and Lemma 3.14 that as $N \to \infty$,
\[
\rho_N(x) \to \frac{e^{-V(x) + 2c \int \log |x - y| \rho_c(y) dy}}{\int e^{-V(t) + 2c \int \log |t - y| \rho_c(y) dy} dt} = \frac{1}{Z_c} e^{-V(x) + 2c \int \log |x - y| \rho_c(y) dy} =: \tilde{\rho}_c(x).
\]

Recall that $\rho_N(x) \leq \Lambda$ for any $x \in \mathbb{R}$ and any $N$. Thus, for any $-\infty < a < b < \infty$,
\[
\int_a^b \rho_N(x) dx \to \int_a^b \tilde{\rho}_c(x) dx,
\]
by the bounded convergence theorem. On the other hand, since the sequence of measures $\{\rho_N(x) dx\}$ converges weakly to $\rho_c(x) dx$, it follows that
\[
\int_a^b \rho_N(x) dx \to \int_a^b \rho_c(x) dx.
\]

Therefore, $\rho_c(x) = \tilde{\rho}_c(x)$, for almost every $x \in \mathbb{R}$. Modify the density $\rho_c(x)$ by taking $\rho_c(x) = \tilde{\rho}_c(x)$ for all $x \in \mathbb{R}$, we get the relation
\[
\rho_c(x) = \frac{1}{Z_c} e^{-V(x) + 2c \int \log |x - y| \rho_c(y) dy}, \quad \text{for all } x \in \mathbb{R}.
\]

This implies that $\rho_c(x) > 0$ for all $x \in \mathbb{R}$, meaning that the limiting measure $\mu_c$ has full support. Combining all the discussions in this section, we arrive at the following result.
Theorem 3.15. The limiting measure \( \mu_c \) in the regime where \( \beta N \to 2c \in (0, \infty) \) has bounded density \( \rho_c \) which can be chosen to satisfy the following equation
\[
\rho_c(x) = \frac{1}{Z_c} e^{-V(x) + 2c \int \log |x-y| \rho_c(y) dy}, \quad \text{for all } x \in \mathbb{R}.
\]
Moreover, for any continuous function \( f \) of polynomial growth, as \( N \to \infty \) with \( \beta N \to 2c \),
\[
\langle L_N, f \rangle = \frac{1}{N} \sum_{i=1}^{N} f(\lambda_i) \to \langle \rho_c, f \rangle \quad \text{in probability.}
\]

4 Poisson statistics

To study the local statistics, we assume further that the potential \( V \) is continuous. In this case, the density \( \rho_c(x) \) chosen in Theorem 3.15 is continuous.

For \( E \in \mathbb{R} \), let \( \xi_N(E) \) be the local statistics around \( E \),
\[
\xi_N(E) = \sum_{i=1}^{N} \delta_{N(\lambda_i - E)}.
\]

Then the \( k \)th correlation function \( R_N^{(k)} \) of \( \xi_N(E) \) is given by
\[
R_N^{(k)}(x_1, x_2, \ldots, x_k) = \frac{N!}{N^k(N-k)!} \rho_N^{(k)}(E + \frac{x_1}{N}, E + \frac{x_2}{N}, \ldots, E + \frac{x_k}{N}),
\]
where \( \rho_N^{(k)} \) is the \( k \)-point function of the beta ensembles (5)
\[
\rho_N^{(k)}(x_1, \ldots, x_k) = \frac{1}{Z_N} |\Delta(x)|^\beta e^{-\sum_{j=1}^{k} V(x_j)}
\]
\[
\times \int \cdots \int \left( \prod_{1 \leq j < k, 1 \leq i \leq N-k} |x_j - \lambda_i|^\beta \right) |\Delta(\lambda)|^\beta e^{-\sum_{i=1}^{N-k} V(\lambda_i)} d\lambda_1 \cdots d\lambda_{N-k}
\]
\[
= \frac{Z_N^{(k)}}{Z_N} |\Delta(x)|^\beta e^{-\sum_{j=1}^{N-k} V(x_j)} E_{\beta, N-k} \left[ \prod_{j=1}^{k} \prod_{i=1}^{N-k} |x_j - \lambda_i|^\beta \right],
\]
with \( Z_N^{(k)} \) being the normalizing constant of the ensemble with parameters \( \beta = \beta_N \) and \( N-k \).

The convergence of correlation functions follows from the following result, which is a generalization of Lemma 3.12.

Lemma 4.1. Let \( E \in \mathbb{R} \) and \( x \in \mathbb{R} \) be given. Then as \( N \to \infty \) with \( \beta N \to 2c \),
\[
\frac{1}{N} \sum_{i=1}^{N} \log \left| E + \frac{x}{N} - \lambda_i \right| \to \int \log |y| \rho_c(y) dy \quad \text{in probability.}
\]

Proof. With Lemma 3.12 in mind, it suffices to show that for \( 1/2 < \delta < 1 \),
\[
S_N := \mathbb{E} \left[ \left( \frac{1}{N} \sum_{i=1}^{N} \log |E - \lambda_i| - \log \left| E + \frac{x}{N} - \lambda_i \right|^\delta \right)^\delta \right] \to 0 \quad \text{as } N \to \infty.
\]
It follows from the following inequality

\[ | \log |u| - \log |v|| \leq |u - v| \left( \frac{1}{|u|} + \frac{1}{|v|} \right), \]

that

\[
\left| \frac{1}{N} \sum_{i=1}^{N} \log |E - \lambda_i| - \log |E + \frac{x}{N} - \lambda_i| \right| \leq \frac{1}{N^\delta} \sum_{i=1}^{N} \left| \log |E - \lambda_i| - \log |E + \frac{x}{N} - \lambda_i| \right|^\delta
\]

\[ \leq \frac{x^\delta}{N^{2\delta}} \sum_{i=1}^{N} \left( \frac{1}{|E - \lambda_i|} + \frac{1}{|E + \frac{x}{N} - \lambda_i|} \right). \]

Thus,

\[ S_N \leq \frac{x^\delta}{N^{2\delta-1}} \left( \int \frac{1}{|E - y|^{\delta}} \rho_N(y) dy + \int \frac{1}{|E + \frac{x}{N} - y|^{\delta}} \rho_N(y) dy \right). \]

We can bound the first integral as follows

\[
\int \frac{1}{|E - y|^{\delta}} \rho_N(y) dy = \int_{|E-y|\leq 1} \frac{1}{|E-y|^{\delta}} \rho_N(y) dy + \int_{|E-y|> 1} \frac{1}{|E-y|^{\delta}} \rho_N(y) dy
\]

\[ \leq \Lambda \int_{|E-y|\leq 1} \frac{1}{|E-y|^{\delta}} dy + \int_{|E-y|> 1} \rho_N(y) dy
\]

\[ \leq \frac{2\Lambda}{1 - \delta} + 1, \]

which is bounded as \( N \to \infty \). The same estimate holds for the second integral. Therefore \( S_N \to 0 \) as \( N \to \infty \). The proof is complete. \( \square \)

Analogous to Lemma 4.1, it holds that under \( P_{\beta,N-k} \), as \( N \to \infty \) with \( \beta N \to 2c \),

\[ \frac{1}{N} \sum_{i=1}^{N-k} \log \left| E + \frac{x}{N} - \lambda_i \right| \to \int \log |E - y| \rho_c(y) dy \text{ in probability,} \]

and hence, for fixed \( E \), and fixed \( x_1, \ldots, x_k \)

\[ \frac{1}{N} \sum_{j=1}^{k} \sum_{i=1}^{N-k} \log \left| E + \frac{x_j}{N} - \lambda_i \right| \to k \int \log |E - y| \rho_c(y) dy \text{ in probability.} \]

Note that Lemma 3.5 implies the uniform integrability. Then we can deduce that

\[ \rho_N^{(k)} \left( E + \frac{x_1}{N}, E + \frac{x_2}{N}, \ldots, E + \frac{x_k}{N} \right) \to \rho_c(E)^k \text{ as } N \to \infty \text{ with } \beta N \to 2c. \]

Consequently, as \( N \to \infty \) with \( \beta N \to 2c \),

\[ R_N^{(k)}(x_1, x_2, \ldots, x_k) \to \rho_c(E)^k. \]
It also follows from Lemma 3.5 that in the regime where $\beta N \to 2c$, for any compact set $K \subset \mathbb{R}$, there is a constant $\theta_K$ such that
\[ R_N^{(k)}(x_1, x_2, \ldots, x_k) \leq (\theta_K)^k, \quad \text{for } k \leq N, \text{ for any } x_i \in K. \quad (11) \]

Finally, the two equations (10) and (11) together are a sufficient condition for the convergence of the local statistics $\xi_N(E)$ to a homogeneous Poisson point process on $\mathbb{R}$ with density $\rho_c(E)$ (see the Appendix in [3]). Thus, we have finished the proof of the following main result.

**Theorem 4.2.** Assume that the potential $V$ is continuous and that
\[ \lim_{x \to \pm\infty} \frac{V(x)}{\log(1 + x^2)} = \infty. \]
Then for any fixed $E \in \mathbb{R}$, the local statistics $\xi_N(E)$ converges weakly to a homogeneous Poisson point process on $\mathbb{R}$ with density $\rho_c(E) > 0$.

We conclude this paper by the following remark.

**Remark 4.3.** (i) The arguments in this paper can be generalized to show the joint convergence of $(\xi_N(E), \xi_N(E'))$ to independent homogeneous Poisson point processes with densities $\rho_c(E)$ and $\rho_c(E')$, respectively. Here $E \neq E'$ are fixed reference energies.

(ii) We can also prove the following result in the “mesoscopic regime”: for fixed $E_0 \in \mathbb{R}$, and fixed $0 < \gamma < 1$, the point processes $(\xi_N(E_0 - N^{-\gamma}), \xi_N(E_0 + N^{-\gamma}))$ converge weakly to independent Poisson point processes with the same density $\rho_c(E_0)$.

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