Abstract
Motivated by the study of greedy algorithms for graph coloring, we introduce a new graph parameter, which we call weak degeneracy. By definition, every $d$-degenerate graph is also weakly $d$-degenerate. On the other hand, if $G$ is weakly $d$-degenerate, then $\chi(G) \leq d + 1$ (and, moreover, the same bound holds for the list-chromatic and even the DP-chromatic number of $G$). It turns out that several upper bounds in graph coloring theory can be phrased in terms of weak degeneracy. For example, we show that planar graphs are weakly 4-degenerate, which implies Thomassen’s famous theorem that planar graphs are 5-list-colorable. We also prove a version of Brooks’s theorem for weak degeneracy: a connected graph $G$ of maximum degree $d \geq 3$ is weakly $(d - 1)$-degenerate unless $G \cong K_{d+1}$. (By contrast, all $d$-regular graphs have degeneracy $d$.) We actually prove an even stronger result, namely that for every $d \geq 3$, there is $\varepsilon > 0$ such that if $G$ is a graph of weak degeneracy at least $d$, then either $G$ contains a $(d + 1)$-clique or the maximum average degree of $G$ is at least $d + \varepsilon$. Finally, we show that graphs of maximum degree $d$ and either of girth at least 5 or of bounded chromatic number are weakly $(d - \Omega(\sqrt{d}))$-degenerate, which is best possible up to the value of the implied constant.

KEYWORDS
DP-coloring, graph coloring, graph degeneracy, greedy algorithms, maximum degree, weak degeneracy

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1 | INTRODUCTION

All graphs in this paper are finite and simple. Recall that for a graph $G$, $\chi(G)$ denotes its chromatic number, that is, the minimum number of colors necessary to color the vertices of $G$ so that adjacent vertices are colored differently. A well-studied generalization of graph coloring is list coloring, which was introduced independently by Vizing [19] and Erdős et al. [10]. In the setting of list coloring, each vertex $u \in V(G)$ is given a set $L(u)$, called its list of available colors. A proper $L$-coloring is then a function $\varphi$ defined on $V(G)$ such that:

- $\varphi(u) \in L(u)$ for all $u \in V(G)$; and
- $\varphi(u) \neq \varphi(v)$ for all $uv \in E(G)$.

The list-chromatic number of $G$, denoted by $\chi_L(G)$, is the minimum $k$ such that $G$ admits a proper $L$-coloring whenever $|L(u)| \geq k$ for all $u \in V(G)$. Clearly, $\chi_L(G) \geq \chi(G)$ for all graphs $G$.

A further generalization of list coloring is DP-coloring (also known as correspondence coloring), which was recently introduced by Dvořák and Postle [8]. A related notion of local conflict coloring was studied independently from the algorithmic standpoint by Fraigniaud et al. [12]. Just as in list coloring, we assume that every vertex $u \in V(G)$ of a graph $G$ is given a list $L(u)$ of colors to choose from. In contrast to list coloring though, the identifications between the colors in the lists are allowed to vary from edge to edge. That is, each edge $uv \in E(G)$ is assigned a matching $C_{uv}$ (not necessarily perfect and possibly empty) from $L(u)$ to $L(v)$. If $\alpha \beta \in C_{uv}$, we say that $\alpha$ corresponds to $\beta$ (under the correspondence $C$). A proper $(L, C)$-coloring of $G$ is a function $\varphi$ defined on $V(G)$ such that:

- $\varphi(u) \in L(u)$ for all $u \in V(G)$; and
- $\varphi(u)\varphi(v) \notin C_{uv}$ for all $uv \in E(G)$.

List coloring is a special case of this framework, where $\alpha \in L(u)$ corresponds to $\beta \in L(v)$ if and only if $\alpha = \beta$. The DP-chromatic number of $G$, denoted by $\chi_{DP}(G)$, is the minimum $k$ such that $G$ admits a proper $(L, C)$-coloring whenever $|L(u)| \geq k$ for all $u \in V(G)$. Again, it is clear from the definition that $\chi_{DP}(G) \geq \chi_L(G)$.

In this paper, we are interested in greedy algorithms for graph coloring. The basic greedy algorithm considers the vertices of $G$ one at a time. When we get to consider a vertex $u$, we assign to it an arbitrary color, say $\alpha$, from $L(u)$. At this point, to ensure that the coloring is proper, we have to remove the colors corresponding to $\alpha$ from the lists of colors available to the neighbors of $u$. Thus, the list size for every neighbor of $u$ may decrease by 1, while all the other lists remain unchanged. If throughout this process no list size reduces to 0 (i.e., if every uncolored vertex always has at least one available color), then we successfully obtain a proper (DP-)coloring of $G$. This idea is formally captured in the notion of graph degeneracy:

**Definition 1.1** (Degeneracy). Let $G$ be a graph and let $f : V(G) \to \mathbb{N}$ be a function.¹ For a vertex $u \in V(G)$, the operation $\text{DELETE}(G, f, u)$ outputs the graph $G' := G - u$ and the function $f' : V(G') \to \mathbb{Z}$ given by the formula

1In this paper $\mathbb{N} = \{0, 1, 2, \ldots\}$ denotes the set of all nonnegative integers.
\[
f'(v) := \begin{cases} 
  f(v) - 1 & \text{if } uv \in E(G); \\
  f(v) & \text{otherwise}.
\end{cases}
\]

An application of the operation DELETE is legal if the resulting function \(f'\) is nonnegative, that is, if \(f'(v) \geq 0\) for all \(v \in V(G')\). A graph \(G\) is \(f\)-degenerate if it is possible to remove all vertices from \(G\) by a sequence of legal applications of the operation DELETE. Given \(d \in \mathbb{N}\), we say that \(G\) is \(d\)-degenerate if it is degenerate with respect to the constant \(d\) function. The degeneracy of \(G\), denoted by \(d(G)\), is the minimum \(d\) such that \(G\) is \(d\)-degenerate.

It follows from the above discussion that \(\chi_{DP}(G) \leq d(G) + 1\) for every graph \(G\); because of this, the quantity \(d(G) + 1\) is sometimes referred to as the coloring number of \(G\) [9]. It is not hard to see that a graph \(G\) is \(d\)-degenerate if and only if every nonempty subgraph of \(G\) has a vertex of degree at most \(d\) [7, proposition 5.2.2].

The upper bound \(\chi_{DP}(G) \leq d(G) + 1\) is usually not sharp. For instance, if \(G\) is a \(d\)-regular graph, then \(d(G) = d\), which implies that \(\chi_{DP}(G) \leq d + 1\). However, the only connected \(d\)-regular graphs \(G\) with \(\chi_{DP}(G) = d + 1\) are the complete graph \(K_{d+1}\) and—if \(d = 2\)—cycles [4]. (A curious distinction between DP-coloring and list coloring is that \(\chi_{DP}(C_n)\) is 2 if \(n\) is even and 3 if \(n\) is odd, while \(\chi_{DP}(C_n) = 3\) for all \(n \geq 3\) [8, §1.1].) It is therefore interesting to see if we can modify the greedy coloring procedure to “save” some of the colors and get a better bound on \(\chi_{DP}(G)\). Here we investigate a particularly simple (but, as we shall see, already quite powerful) way of doing so. To motivate our main definition, consider a vertex \(u \in V(G)\) and let \(w\) be its neighbor. In general, if we assign a color to \(u\), then \(w\) may lose one of its colors. However, suppose that \(|L(u)| > |L(w)|\), that is, that \(u\) has strictly more available colors than \(w\). In this case, there must be a color in \(L(w)\) that does not correspond to any color in \(L(w)\) and assigning such a color to \(u\) does not affect \(L(w)\) (of course, the other neighbors of \(u\) may still lose a color). In this way, we “save” an extra color for \(w\). This idea naturally leads to the notion that we call weak degeneracy:

**Definition 1.2** (Weak degeneracy). Let \(G\) be a graph and let \(f : V(G) \rightarrow \mathbb{N}\) be a function. For a pair of adjacent vertices \(u, w \in V(G)\), the operation DELSAVE\((G, f, u, w)\) outputs the graph \(G' := G - u\) and the function \(f' : V(G') \rightarrow \mathbb{Z}\) given by the formula

\[
f'(v) := \begin{cases} 
  f(v) - 1 & \text{if } uv \in E(G) \text{ and } v \neq w; \\
  f(v) & \text{otherwise}.
\end{cases}
\]

An application of the operation DELSAVE is legal if \(f(u) > f(w)\) and the resulting function \(f'\) is nonnegative. A graph \(G\) is weakly \(f\)-degenerate if it is possible to remove all vertices from \(G\) by a sequence of legal applications of the operations DELETE and DELSAVE. Given \(d \in \mathbb{N}\), we say that \(G\) is weakly \(d\)-degenerate if it is weakly degenerate with respect to the constant \(d\) function. The weak degeneracy of \(G\), denoted by \(wd(G)\), is the minimum \(d\) such that \(G\) is weakly \(d\)-degenerate.

Again, the above discussion shows that \(\chi_{DP}(G) \leq wd(G) + 1\) for every graph \(G\). Actually, the same bound holds even for the online version of DP-chromatic number called DP-paint number, which was introduced by Kim et al. [15] (see Section 3 for the definition):

**Proposition 1.3.** For every graph \(G\),
\[ \chi(G) \leq \chi_e(G) \leq \chi_{DP}(G) \leq \chi_{DPP}(G) \leq \text{wd}(G) + 1, \]

where \( \chi_{DPP}(G) \) is the DP-paint number of \( G \).

It turns out that the simple way of “saving” colors using the \textsc{DelSave} operation is sufficient for several nontrivial upper bounds. For example, consider the case of planar graphs. It follows from Euler’s formula that planar graphs are 5-degenerate, which gives a simple proof of their 6-colorability (and even 6-DP-colorability). On the other hand, Thomassen [18] proved that every planar graph is 5-list-colorable, and this result was extended to DP-coloring by Dvořák and Postle [8]. The value 5 here is optimal as Voigt [20] constructed planar graphs of list-chromatic number exactly 5. While degeneracy is not sufficient to establish Thomassen’s theorem, we show in Section 4 that weak degeneracy is.

**Theorem 1.4.** Every planar graph is weakly 4-degenerate.

Next, we consider Brooks-type theorems for weak degeneracy. As mentioned earlier, if \( d \geq 3 \), then the only connected graph \( G \) of maximum degree \( d \geq 3 \) with \( \chi_{DP}(G) = d + 1 \) is the complete graph \( K_{d+1} \). We show that there is a corresponding bound on weak degeneracy:

**Theorem 1.5.** If \( G \) is a connected graph of maximum degree \( d \geq 3 \), then either \( G \cong K_{d+1} \) or \( G \) is weakly \((d-1)\)-degenerate.

More generally, suppose that \( G \) is a connected graph and \( |L(u)| \geq \text{deg}_G(u) \) for every vertex \( u \in V(G) \) (i.e., the lower bound on the list size varies depending on the degree of the vertex). In the list-coloring framework, Borodin [5] and, independently, Erdős et al. [10] showed that \( G \) is \( L \)-colorable unless it is a Gallai tree, that is, a connected graph in which every block is either a clique or an odd cycle. In the DP-coloring setting the same result holds, except that the graphs that need to be excluded are the \( GDP \) trees, that is, connected graphs in which every block is either a clique or a cycle (not necessarily odd) [4]. We again establish the corresponding result for weak degeneracy.

**Theorem 1.6.** Let \( G \) be a connected graph. The following statements are equivalent:

1. \( G \) is weakly \( f \)-degenerate, where \( f(u) = \text{deg}_G(u) - 1 \) for all \( u \in V(G) \);
2. \( G \) is not a \( GDP \)-tree.

Theorem 1.5 is an immediate corollary of Theorem 1.6. We prove Theorem 1.6 in Section 5.

Recall that the average degree of a nonempty graph \( G \), denoted by \( \text{ad}(G) \), is the average of the degrees of the vertices of \( G \). Equivalently, we have \( \text{ad}(G) = 2E(G)/|V(G)| \). The maximum average degree of \( G \), denoted by \( \text{mad}(G) \), is defined by \( \text{mad}(G) := \max_H \text{ad}(H) \), where the maximum is taken over all nonempty subgraphs \( H \) of \( G \). The maximum average degree of a graph is a natural measure of its local density. There is a close relationship between a graph’s maximum average degree and its degeneracy; namely, we have

\[ 2d(G) \geq \text{mad}(G) \geq d(G). \]
For \( d \)-regular graphs \( G \), \( \text{mad}(G) = d(G) = d \). By contrast, we show that if \( \text{wd}(G) \geq 3 \) and \( G \) contains no \((\text{wd}(G) + 1)\)-clique, then \( \text{mad}(G) \geq \text{wd}(G) + \varepsilon \), where \( \varepsilon > 0 \) only depends on \( \text{wd}(G) \):

**Theorem 1.7.** Let \( G \) be a nonempty graph. If the weak degeneracy of \( G \) is at least \( d \geq 3 \), then either \( G \) contains a \((d + 1)\)-clique or

\[
\text{mad}(G) \geq d + \frac{d - 2}{d^2 + 2d - 2}.
\]

Note that Theorem 1.7 is a strengthening of Theorem 1.5, since \( \text{mad}(G) \) is at most the maximum degree of \( G \). Our proof of Theorem 1.7, which we present in Section 5.2, relies on Theorem 1.6 and follows an approach similar to the one used by Gallai [13] to establish a lower bound on the average degree of critical graphs.

As far as lower bounds on weak degeneracy are concerned, a fairly straightforward double counting argument gives the following:

**Proposition 1.8.** Let \( G \) be a \( d \)-regular graph with \( n \geq 2 \) vertices. Then

\[
\text{wd}(G) \geq d - \sqrt{2n}.
\]

In particular, if \( n = O(d) \), then \( \text{wd}(G) \geq d - O(\sqrt{d}) \). For example, Proposition 1.8 yields the bound \( \text{wd}(K_{d,d}) \geq d - 2\sqrt{d} \) for \( d \geq 2 \). Actually, this can be improved to \( d - \sqrt{2d} - 1 \):

**Proposition 1.9.** If \( G \) is a triangle-free \( d \)-regular graph with \( n \geq 4 \) vertices, then

\[
\text{wd}(G) > d - \sqrt{n} - 1.
\]

In particular, the complete bipartite graph \( K_{d,d} \) with \( d \geq 2 \) satisfies \( \text{wd}(K_{d,d}) > d - \sqrt{2d} - 1 \).

This should be contrasted with the fact that \( \chi(K_{d,d}) = 2 \), \( \chi(P) = (1 + o(1)) \log_2 d \) [10], and \( \chi_{dp}(K_{d,d}) = \Theta(d / \log d) \) [3]. We prove Propositions 1.8 and 1.9 in Section 6.

It seems plausible that every \( d \)-regular graph has weak degeneracy at least \( d - O(\sqrt{d}) \); we leave verifying or refuting this supposition as an open problem:

**Conjecture 1.10.** Every \( d \)-regular graph \( G \) satisfies \( \text{wd}(G) \geq d - O(\sqrt{d}) \).

In view of the above lower bounds, it makes sense to ask, for what classes of graphs \( G \) does the upper bound \( \text{wd}(G) \leq d - \Omega(\sqrt{d}) \) hold, where \( d \) is the maximum degree of \( G \)? Along these lines, we establish the following results:

**Theorem 1.11.** For each integer \( k \geq 1 \), there exist \( c > 0 \) and \( d_0 \in \mathbb{N} \) such that if \( G \) is a graph of maximum degree \( d \geq d_0 \) with \( \chi(G) \leq k \), then \( \text{wd}(G) \leq d - c\sqrt{d} \).

**Theorem 1.12.** There exist \( c > 0 \) and \( d_0 \in \mathbb{N} \) such that if \( G \) is a graph of maximum degree \( d \geq d_0 \) and girth at least 5, then \( \text{wd}(G) \leq d - c\sqrt{d} \).

Theorems 1.11 and 1.12 are proved in Section 7 using probabilistic arguments.
We finish the introduction with another conjecture that implies both Theorems 1.11 and 1.12:

**Conjecture 1.13.** For each integer $k \geq 1$, there exist $c > 0$ and $d_0 \in \mathbb{N}$ such that if $G$ is a graph of maximum degree $d \geq d_0$ and without a $k$-clique, then $\text{wd}(G) \leq d - c\sqrt{d}$.

At present, we do not even know if Conjecture 1.13 holds for $k = 3$, that is, whether $\text{wd}(G) \leq d - \Omega(\sqrt{d})$ for triangle-free graphs of maximum degree $d$.

## 2 PRELIMINARY RESULTS

In this section, we establish several basic results about weak degeneracy that will be used throughout the rest of this article.

**Lemma 2.1** (Weak degeneracy is monotone). Let $G$ be a weakly $f$-degenerate graph. If $g : V(G) \to \mathbb{N}$ is a function such that $g(u) \geq f(u)$ for all $u \in V(G)$, then $G$ is weakly $g$-degenerate.

**Proof.** We wish to show that $G$ is weakly $g$-degenerate by removing its vertices via the same sequence of operations that witnesses that $G$ is weakly $f$-degenerate. The only possible issue is that an application of DELSAVE may become illegal when $f$ is replaced by $g$. Namely, it can happen that DELSAVE$(G, f, u, w)$ is legal, while DELSAVE$(G, g, u, w)$ is not due to the fact that $g(u) \leq g(w)$. However, since $f(u) > f(w)$, this means that $g(w) > f(w)$, so instead of DELSAVE$(G, g, u, w)$ we can simply use DELETE$(G, g, u)$: this replaces $g(w)$ by $g(w) - 1$, which is still at least $f(w)$.

**Lemma 2.2** (Weak degeneracy and DP-coloring). Let $G$ be a weakly $f$-degenerate graph. Suppose that every vertex $u \in V(G)$ is given a list $L(u)$ of available colors and that for each edge $uv \in E(G)$, there is a matching $C_{uv}$ from $L(u)$ to $L(v)$. If $|L(u)| \geq f(u) + 1$ for all $u \in V(G)$, then $G$ admits a proper $(L, C)$-coloring.

**Proof.** This statement was essentially established in the introduction (in the discussion preceding Definition 1.2). We give a more detailed proof here for completeness. Since $G$ is weakly $f$-degenerate, it is possible to remove all vertices from $G$ by a sequence of legal applications of the operations DELETE and DELSAVE. Fix any such sequence $S = (\mathcal{O}_0, \ldots, \mathcal{O}_{n-1})$. Set $(G_0, f_0) := (G, f)$ and for each $0 \leq i \leq n - 1$, let $(G_{i+1}, f_{i+1})$ be the result of applying the operation $\mathcal{O}_i$ to $(G_i, f_i)$. We color the vertices of $G$ one by one, in the order in which they are removed by $S$. Each time a vertex $u$ is assigned a color $\alpha$, we remove the colors corresponding to $\alpha$ from the lists of colors available to the neighbors of $u$, thus ensuring that the resulting coloring is proper. Let $L_i(u)$ be the list of colors available to a vertex $u \in V(G_i)$ at the start of step $i$ (in particular, $L_0(u) := L(u)$). Throughout our coloring procedure, we will maintain the following property:

(P) $|L_i(u)| \geq f_i(u) + 1$ for all $u \in V(G_i)$. 

If we can achieve this, then we will successfully color the entire graph, since no uncolored vertex will ever run out of available colors. Now, property \((P_0)\) holds by assumption. On step \(i\), we assume that \((P_i)\) holds and consider two cases.

**Case 1:** \(\mathcal{O}_i = \text{DELETE}(G_i, f_i, u_i)\).

In this case, we assign to \(u_i\) an arbitrary available color. It is clear that property \((P_{i+1})\) holds regardless of what color is assigned to \(u_i\).

**Case 2:** \(\mathcal{O}_i = \text{DELSAVE}(G_i, f_i, u_i, w_i)\).

If \(|L(w_i)| > f_i(w_i) + 1\), we can, as in Case 1, assign an arbitrary available color to \(u_i\). Now suppose that \(|L(w_i)| = f_i(w_i) + 1\). Then, by \((P_i)\) and since this application of \(\text{DELSAVE}\) is legal, we have \(|L_i(u_i)| > f_i(u_i) + 1 > f_i(w_i) + 1 = |L_i(w_i)|\). This means that \(u_i\) must have an available color \(\alpha \in L_i(u_i)\) that does not correspond to any color in \(L_i(w_i)\). If we assign \(\alpha_i\) to \(u_i\), then the list of available colors for \(w_i\) will not change, and thus \((P_{i+1})\) will not be violated, as desired.

**Lemma 2.3** (Partitioning lemma). Let \(G\) be a weakly \(f\)-degenerate graph. Suppose that functions \(f_1, f_2 : V(G) \rightarrow \mathbb{Z}\) satisfy \(f_1(u) + f_2(u) = f(u) - 1\) for all \(u \in V(G)\). Then there is a partition \(V(G) = V_1 \sqcup V_2\) such that the subgraph \(G[V_i]\) is weakly \(f_i\)-degenerate for each \(i \in \{1, 2\}\).

**Proof.** The proof is by induction on \(|V(G)|\). If \(V(G) = \emptyset\), the statement holds vacuously. Now suppose that \(|V(G)| \geq 1\) and the claim holds for all graphs with \(|V(G)| - 1\) vertices. Since \(G\) is weakly \(f\)-degenerate, there is a legal application of an operation \(\mathcal{O} \in \{\text{DELETE}, \text{DELSAVE}\}\) that produces a pair \((G', f')\) in which the graph \(G'\) is weakly \(f'\)-degenerate. We consider the two cases depending on whether \(\mathcal{O}\) is \(\text{DELETE}\) or \(\text{DELSAVE}\).

**Case 1:** \(\mathcal{O} = \text{DELETE}(G, f, u)\).

Then \(G' = G - u\). Since \(f_1(u) + f_2(u) = f(u) - 1 \geq -1\), we have \(f_1(u) \geq 0\) or \(f_2(u) \geq 0\). For concreteness, say \(f_1(u) \geq 0\). Define a function \(f'_1 : V(G') \rightarrow \mathbb{Z}\) by

\[
    f'(v) := \begin{cases} 
        f_1(v) - 1 & \text{if } uv \in E(G); \\
        f_1(v) & \text{otherwise}.
    \end{cases}
\]

Then \(f'_1 + f_2 = f' - 1\), so, by the inductive hypothesis, there is a partition \(V(G') = V'_1 \sqcup V_2\) such that \(G[V'_1]\) is weakly \(f'_1\)-degenerate and \(G[V_2]\) is weakly \(f_2\)-degenerate. Set \(V_1 := V'_1 \sqcup \{u\}\). We claim that the partition \(V(G) = V_1 \sqcup V_2\) is as desired. Since \(G[V_2]\) is weakly \(f_2\)-degenerate by assumption, we just need to argue that \(G[V_1]\) is weakly \(f_1\)-degenerate. As \(f_1(u) \geq 0\), the function \(f_1\) is nonnegative on \(V_1\). Now we are done since \(\text{DELETE}(G[V_1], f_1, u) = (G[V'_1], f'_1)\) and \(G[V'_1]\) is weakly \(f'_1\)-degenerate.

**Case 2:** \(\mathcal{O} = \text{DELSAVE}(G, f, u, w)\).
Again we have \( G' = G - u \). It will be convenient to assume that \( f_1(w), f_2(w) \geq -1 \). If this is not the case and, say, \( f_1(w) < -1 \), then we replace \( f_1 \) and \( f_2 \) by the functions \( f_1^*, f_2^* : V(G) \to \mathbb{Z} \) given by \( f_1^*(w) := -1, f_2^*(w) := f(w) \), and \( f_i^*(v) := f_i(v) \) for all \( i \in \{1, 2\} \) and \( v \neq w \). We can do this because every weakly \( f_i^* \)-degenerate subgraph of \( G \) is also weakly \( f_i \)-degenerate. For \( i = 2 \) this follows from Lemma 2.1 since \( f_2 \geq f_2^* \) by definition. On the other hand, if a subgraph \( H \) of \( G \) is weakly \( f_1^* \)-degenerate, then \( w \not\in V(H) \) since \( f_1^*(w) < 0. \) As \( f_1^* \) and \( f_1 \) agree on all vertices except \( w \), \( H \) must be weakly \( f_1 \)-degenerate as well. Since this application of \( \mathrm{DELSAVE} \) is legal, we have \( f(u) > f(w) \), which implies that \( f_1(u) > f_1(w) \) or \( f_2(u) > f_2(w) \). For concreteness, say \( f_1(u) > f_1(w) \). Define \( f'_1 : V(G') \to \mathbb{Z} \) by

\[
f'_1(v) := \begin{cases} f_1(v) - 1 & \text{if } uv \in E(G) \text{ and } v \neq w; \\ f_1(v) & \text{otherwise.} \end{cases}
\]

Then \( f_1' + f_2 = f' - 1 \), so, by the inductive hypothesis, there is a partition \( V(G') = V_1' \sqcup V_2 \) such that \( G[V_1'] \) is weakly \( f_1' \)-degenerate and \( G[V_2] \) is weakly \( f_2 \)-degenerate. Set \( V_i := V_i' \cup \{u\} \). We claim that the partition \( V(G) = V_1 \cup V_2 \) is as desired. We just need to argue that \( G[V_i] \) is weakly \( f_i \)-degenerate. Since \( f_1(u) > f_1(w) \geq -1 \), we have \( f_1(u) \geq 0 \). Hence, \( f_1 \) is nonnegative on \( V_i \). It remains to observe that by a legal application of one of the operations \( \mathrm{DELETE}, \mathrm{DELSAVE} \) it is possible to reduce the pair \((G[V_1], f_1) \) to \((G[V_1'], f_1') \). Indeed, if \( w \not\in V_1 \), then \((G[V_1'], f_1') = \mathrm{DELET}(G[V1], f_1, u) \), while if \( w \in V_1 \), then \((G[V_1'], f_1') = \mathrm{DELSAVE}(G[V_1], f_1, u, w). \)

\[ \square \]

3 | ONLINE DP-COLORING AND WEAK DEGENERACY

As mentioned in the introduction, DP-paint number is an online version of DP-chromatic number introduced by Kim et al. in [15]. It is defined by means of a certain game on a graph \( G \).

**Definition 3.1** DP-painting game. Let \( G \) be a graph and let \( g : V(G) \to \mathbb{N} \) be a function. The **DP-painting game on \((G, g)\)** is played between two players—Lister and Painter—as follows. The game proceeds in rounds, starting with Round 0. At the start of Round \( i \), we have a graph \( G_i \), where we initially set \( G_0 := G \). Lister then picks a list \( L_i(u) \) of colors for each vertex \( u \in V(G_i) \) and assigns to every edge \( uv \in E(G_i) \) a matching \( C_{i,uv} \) from \( L_i(u) \) to \( L_i(v) \) (the matching \( C_{i,uv} \) need not be perfect and, in particular, may be empty). In response, Painter picks a function \( \varphi_i \) defined on some subset \( U_i \subseteq V(G_i) \) with the following properties:

- \( \varphi_i(u) \in L_i(u) \) for all \( u \in U_i \) (in particular, \( L_i(u) \neq \emptyset \) for all \( u \in U_i \)); and
- \( \varphi_i(u) \varphi_i(v) \not\in C_{i,uv} \) for all \( u, v \in U_i \) that are adjacent in \( G_i \).

Then we set \( G_{i+1} := G_i - U_i \) and proceed to Round \( i + 1 \). Lister wins the game if for some \( i \in \mathbb{N} \), there is a vertex \( u \in V(G_i) \) with \( \sum_{j<i} |L_j(u)| \geq g(u) \); otherwise, Painter wins.
A graph $G$ is $g$-DP-paintable if Painter has a winning strategy in the DP-painting game on $(G, g)$. Given $k \in \mathbb{N}$, we say that $G$ is $k$-DP-paintable if it is DP-paintable with respect to the constant $k$ function. The DP-paint number $\chi_{DPP}(G)$ of $G$ is the least $k$ such that $G$ is $k$-DP-paintable.

Take $k \in \mathbb{N}$ and consider the DP-painting game on $(G, k)$, where $k$ is the constant $k$ function. On Round 0, Lister may decide to give each vertex $u \in V(G)$ a list $L_0(u)$ of colors of size $L_0(u) = k$. Then Painter must immediately assign a color to every vertex. Therefore, Painter can win only if $\chi_{DP}(G) \leq k$, which shows that $\chi_{DP}(G) \leq \chi_{DPP}(G)$ for all $G$. On the other hand, if Lister always plays so that $|L_i(u)| \leq 1$ for all $i$ and $u \in V(G_i)$, then Painter can win if and only if $\chi_p(G) \leq k$, where $\chi_p(G)$ is the classical paint number of $G$, that is, the online analog of list-chromatic number (see [15, §2] for details). Thus, $\chi_p(G) \leq \chi_{DPP}(G)$ as well, so $\chi_{DPP}(G)$ provides a common upper bound on $\chi_{DP}(G)$ and $\chi_p(G)$. It is shown in [15] that either inequality $\chi_{DP}(G) \leq \chi_{DPP}(G)$ and $\chi_p(G) \leq \chi_{DPP}(G)$ can be strict; however, it is unknown if both of them can be strict at the same time. It is also not known if the difference $\chi_{DPP}(G) - \chi_{DP}(G)$ can be arbitrarily large.

The goal of this section is to prove Proposition 1.3, which says that the DP-paint number is bounded above by weak degeneracy plus 1. We prove it in the following stronger form:

**Proposition 1.3.** If $G$ is a weakly $f$-degenerate graph, then $G$ is $(f + 1)$-DP-paintable.

**Proof.** The strategy for Painter is to pick functions $\varphi_i$ so as to maintain the following property:

$$(P_i) \text{ } G_i \text{ is weakly } f_i\text{-degenerate, where } f_i(u) := f(u) - \sum_{j<i} |L_j(u)| \text{ for all } u \in V(G_i).$$

If this can be achieved, then Painter will never lose, since for all $u \in V(G_i)$, we will have $f_i(u) \geq 0$, or, equivalently, $\sum_{j<i} |L_j(u)| < f(u) + 1$. Since $f_0 = f$, property $(P_0)$ holds by assumption, so it remains to argue that if $(P_i)$ holds at the start of Round $i$, then Painter will be able to pick $\varphi_i$ so that $(P_{i+1})$ holds.

Suppose Lister assigned a list $L_i(u)$ of colors and a matching $C_{i,uv}$ to each vertex $u \in V(G_i)$ and edge $uv \in E(G_i)$ respectively. For all $u \in V(G_i)$, let

$$f_{i,1}(u) := |L_i(u)| - 1 \text{ and } f_{i,2}(u) := f_i(u) - |L_i(u)|.$$ 

Since $f_{i,1}(u) + f_{i,2}(u) = f_i(u) - 1$ for all $u \in V(G_i)$, Lemma 2.3 yields a partition $V(G_i) = U_i \sqcup W_i$ such that $G[U_i]$ is weakly $f_{i,1}$-degenerate, while $G[W_i]$ is weakly $f_{i,2}$-degenerate. By Lemma 2.2, $G[U_i]$ admits a proper $(L_i, C_i)$-coloring $\varphi_i$. Painter plays this coloring $\varphi_i$. Then $G_{i+1} = G[W_i]$, so, to establish $(P_{i+1})$, we need to show that $G[W_i]$ is weakly $f_{i+1}$-degenerate. To this end, note that for each $u \in W_i$,

$$f_{i+1}(u) = f(u) - \sum_{j \leq i} |L_j(u)| = f_i(u) - |L_i(u)| = f_{i,2}(u),$$

and $G[W_i]$ is indeed weakly $f_{i,2}$-degenerate by construction. \qed
In this section we prove the analog of Thomassen’s theorem [18] on 5-list-colorability of planar graphs in the context of weak degeneracy:

Theorem 1.4. Every planar graph is weakly 4-degenerate.

As in the proof of Thomassen’s theorem, we use induction to establish a technical lemma, which then easily yields Theorem 1.4. First, we need a definition. Let $G$ be a graph and let $f : V(G) \rightarrow \mathbb{N}$ be a function. Given a subset $U \subseteq V(G)$, we say that $G$ is $U$-safely weakly $f$-degenerate if, starting with $(G, f)$, it is possible to remove all vertices from $G$ by a sequence of legal applications of the operations delete and delsave, so that every vertex in $U$ is removed using the delete operation. In particular, $G$ is $(V(G))$-safely weakly $f$-degenerate if and only if $G$ is $f$-degenerate.

Lemma 4.1. Let $G$ be a planar graph on at least three vertices where every nonouter face is triangular and the outer face is a cycle $C$ of length $k$. Let the vertices of $C$ in the natural order be $v_1, \ldots, v_k$. Define $f : V(G) \setminus \{v_1, v_2\} \rightarrow \mathbb{Z}$ by $f(u) := \begin{cases} 2 - |N_G(u) \cap \{v_1, v_2\}| & \text{if } u \in V(C); \\ 4 - |N_G(u) \cap \{v_1, v_2\}| & \text{otherwise}. \end{cases}$

Then $G - v_1 - v_2$ is $(V(C) \setminus \{v_1, v_2\})$-safely weakly $f$-degenerate.

Proof. We proceed by induction on $|V(G)|$. If $|V(G)| = 3$, then $G - v_1 - v_2$ comprises a single vertex, which is 0-degenerate, as desired. Now suppose that $|V(G)| \geq 4$ and that the induction hypothesis holds for smaller graphs. We consider two cases.

Case 1: $C$ has a chord $v_a v_b$.

Then $C + v_a v_b$ is the union of two cycles $C_1, C_2$ with $E(C_1) \cap E(C_2) = \{v_a v_b\}$. Without loss of generality, suppose $v_1 v_2 \in E(C_1)$ (and so $v_1 v_2 \not\in E(C_2)$). Let $G_1, G_2$ be the respective induced subgraphs of $G$ on the vertices of each $C_i$ along with the vertices on the interiors of each cycle. Let $f_1 := f|_{V(G_1) \setminus \{v_1, v_2\}}$ and define $f_2 : V(G_2) \setminus \{v_1, v_2\} \rightarrow \mathbb{Z}$ by $f_2(u) := \begin{cases} 2 - |N_G(u) \cap \{v_a, v_b\}| & \text{if } u \in V(C_2); \\ 4 - |N_G(u) \cap \{v_a, v_b\}| & \text{otherwise}. \end{cases}$

By the induction hypothesis, starting with $(G_1 - v_1 - v_2, f_1)$, we can remove all vertices from $V(G_1) \setminus \{v_1, v_2\}$ via legal applications of the operations delete and delsave, where each vertex in $V(C_1) \setminus \{v_1, v_2\}$ is removed using delete. Applying the same sequence of operations but starting with $(G - v_1 - v_2, f)$ yields the pair $(G_2 - v_a - v_b, f_2)$. By the inductive hypothesis again, we can now remove every remaining vertex via a sequence of legal applications of delete and delsave, with every vertex in $V(C_2) \setminus \{v_a, v_b\}$ removed using delete, as desired.

Case 2: $C$ has no chord.
Since every nonouter face of $G$ is a triangle, the neighbors of $v_k$ form a path $u_1 \ldots u_\ell$, where $u_1 = v_1$ and $u_\ell = v_{k-1}$. The assumption that $C$ has no chord implies that $u_2, \ldots, u_{\ell}$ belong to the interior of $C$. Then the cycle $C' := u_1 \ldots u_\ell v_{k-2} \ldots v_1$ bounds the outer face of $G' := G - v_k$. Applying the induction hypothesis to $G'$ shows that $G'' := G - v_1 - v_2 - v_k$ is $(V(C')\setminus \{v_1, v_2\})$-safely weakly $f'$-degenerate, where $f' : V(G'') \to \mathbb{Z}$ is defined by:

$$f'(u) = \begin{cases} 2 - |N_{G'}(u) \cap \{v_1, v_2\}| & \text{if } u \in V(C'); \\ 4 - |N_{G'}(u) \cap \{v_1, v_2\}| & \text{otherwise}. \end{cases}$$

In other words, starting with $(G'', f')$, we can remove every vertex by a sequence of legal applications of DELETE and DELSAVE, where each vertex in $(V(C')\setminus \{v_1, v_2\})$ is removed using DELETE. Since $f' \leq f$, we may apply the same sequence of operations starting with $(G - v_1 - v_2, f')$ instead (see Lemma 2.1). Moreover, we can accrue some extra savings for the vertex $v_k$, as follows. Consider any $u_i$ with $2 \leq i \leq \ell - 1$. By assumption, $u_i$ is removed from $G''$ using the DELETE operation, but since $u_i \notin V(C)$, we are now allowed to remove $u_i$ using DELSAVE. Notice that $f'(u_i) = f(u_i) - 2$, because $u_i$ is in $V(C')$ but not in $V(C)$. When $u_i$ was removed from $G''$, the value of the function at $u_i$ was at least 0, which means that at the same stage of the process on $G - v_1 - v_2$, the value of the function at $u_i$ is at least 2. On the other hand, since $v_k \in V(C)$ and $v_1 \in N_G(v_k)$, we have $f(v_k) \leq 1 < 2$. This means that instead of using the operation DELETE, we can legally remove $u_i$ using DELSAVE($\ldots, u_\ell, v_k$). Upon performing this modified sequence of operations, we only have $v_k$ left to remove, so we just need to check that the value of the function at $v_k$ is at least 0. To this end, note that $f'(v_k)$ is 1 if $v_{k-1} \neq v_2$ and 0 otherwise. Since the only neighbor of $v_k$ that may be removed without saving $v_k$ is $v_{k-1}$, and that can only happen when $v_{k-1} \neq v_2$, it follows that the value at $v_k$ cannot drop below 0, as desired. \hfill $\square$

We now complete the proof of the theorem.

\textbf{Proof of Theorem 1.4.} Since adding vertices or edges cannot decrease the weak degeneracy of a graph, it suffices to prove the theorem for maximal planar graphs $G$ on at least three vertices. Then $G$ is a planar triangulation. Let $v_1, v_2$ be adjacent vertices on its outer face. Removing $v_1$ and $v_2$ using DELETE and then applying Lemma 4.1, we see that $G$ is weakly 4-degenerate, as desired. \hfill $\square$

\section{Brooks-Type Results}

\subsection{Weakly (deg$-1$)-degenerate graphs}

We say that a graph $G$ is weakly (deg$-1$)-degenerate if it is weakly degenerate with respect to the function $u \mapsto \deg_G(u) - 1$. Recall that a GDP tree is a connected graph in which every block is either a clique or a cycle. The main result of this section is the following characterization of connected weakly (deg$-1$)-degenerate graphs:

\textbf{Theorem 1.6.} Let $G$ be a connected graph. The following statements are equivalent:

(1) $G$ is weakly (deg$-1$)-degenerate;
To begin with, we need the following standard fact:

**Lemma 5.1.** Let $G$ be a connected graph and let $f : V(G) \to \mathbb{N}$ be a function. Suppose that:

(a) $f(u) \geq \deg_G(u) - 1$ for all $u \in V(G)$; and
(b) $f(x) \geq \deg_G(x)$ for some $x \in V(G)$.

Then $G$ is $f$-degenerate.

**Proof.** Fix a vertex $x$ witnessing (b) and list the vertices of $G$ as $u_1, u_2, \ldots, u_n$ in order of decreasing distance to $x$, resolving ties arbitrarily. Then $u_n = x$ and, for each $1 \leq i \leq n - 1$, the vertex $u_i$ has at least one neighbor among $u_{i+1}, \ldots, u_n$. We can now remove all vertices from $G$ by applying the operation DELETE to them in this order. □

The next lemma contains the central part of our argument:

**Lemma 5.2.** Let $G$ be a connected graph that is not weakly $(\deg - 1)$-degenerate. Then every connected induced subgraph of $G$ without cut vertices is regular.

**Proof.** Take a subset $A \subseteq V(G)$ such that the subgraph $G[A]$ has no cut vertices and suppose, toward a contradiction, that $G[A]$ is not regular. Define $f : V(G) \to \mathbb{N}$ by $f(u) = \deg_G(u) - 1$ for all $u \in V(G)$. Our goal is to show that $G$ is weakly $f$-degenerate. Note that every connected component of $G - A$ contains at least one vertex $v$ that has a neighbor in $A$ and hence satisfies $f(v) \geq \deg_{G-A}(v)$. Therefore, by Lemma 5.1, we can remove all vertices from $G - A$ using only the operation DELETE. After this, the graph $G$ will be replaced by $G' = G[A]$ and the function $f$ by $f' : A \to \mathbb{N} : u \mapsto \deg_{G[A]}(u) - 1$. Since the graph $G[A]$ is connected and not regular, we can pick two adjacent vertices $x, y \in A$ with $\deg_{G[A]}(x) < \deg_{G[A]}(y)$ and hence $f'(x) < f'(y)$. Now we let

\[
(G'', f'') := \text{DELSAVE}(G[A], f', y, x).
\]

Since $f'(y) > f'(x)$, this is a legal application of DELSAVE. As the graph $G[A]$ has no cut vertices, the graph $G'' = G[A] - y$ is connected. It remains to observe that $G''$ is $f''$-degenerate by Lemma 5.1, where condition (b) is witnessed by the vertex $x$. □

It remains to characterize the graphs satisfying the conclusion of Lemma 5.2:

**Lemma 5.3.** Let $G$ be a connected graph such that every connected induced subgraph of $G$ without cut vertices is regular. Then $G$ is a GDP-tree.

**Proof.** Suppose, toward a contradiction, that $G$ is a counterexample with the fewest vertices. Note that $|V(G)| \geq 4$, since all connected graphs on at most 3 vertices are GDP-trees. By the minimality of $|V(G)|$, every proper connected induced subgraph of $G$ must be a GDP-tree.
We claim that $G$ is 2-connected. Otherwise, every block in $G$ would be a proper connected induced subgraph of $G$, hence a GDP-tree. The only GDP-trees without cut vertices are cliques and cycles, so this implies that every block in $G$ is a clique or a cycle, that is, $G$ is a GDP-tree.

Since $G$ is 2-connected, it must be regular. Let $d$ be the common degree of the vertices of $G$. Then $d \geq 2$ by 2-connectedness. Furthermore, if $d$ were equal to 2, then $G$ would be a cycle and hence a GDP-tree. Therefore, $d \geq 3$.

Pick an arbitrary vertex $x \in V(G)$ and consider the graph $G' := G - x$. Then $G'$ is connected, so it is a GDP-tree. Since $G$ is regular and not a clique, not every vertex in $V(G')$ is adjacent to $x$. This implies that $G'$ is not regular, so it must have a cut vertex and at least two blocks.

Let $B$ be an arbitrary leaf block in $G'$ and let $c \in V(B)$ be the cut vertex of $G'$ in $B$. The graph $B$ is regular, so let $k$ be the common degree of every vertex of $B$. The degree of a vertex $u \in V(B) \setminus \{c\}$ in $G$ is either $k + 1$ or $k$, depending on whether $u$ is adjacent to $x$ or not. Since $G$ is 2-connected, $x$ must be adjacent to at least one vertex in $V(B) \setminus \{c\}$, which, since $G$ is $d$-regular, implies that $k + 1 = d$ and $x$ is in fact adjacent to every vertex in $V(B) \setminus \{c\}$. Hence, $x$ has at least $|V(B)| - 1 \geq k = d - 1$ neighbors in $B$.

Finally, as there are at least two distinct leaf blocks in $G'$, we conclude that $x$ has at least $2(d - 1)$ neighbors. Therefore, $d \geq 2(d - 2)$, that is, $d \leq 2$, which is a contradiction. □

Theorem 1.6 now follows easily:

Proof of Theorem 1.6. The implication (2) $\Rightarrow$ (1) is a combination of Lemmas 5.2 and 5.3. The implication (1) $\Rightarrow$ (2) follows since GDP-trees are not DP-degree-colorable [4, theorem 9]. That is, if $G$ is a GDP-tree, then it is possible to give each vertex $u \in V(G)$ a list $L(u)$ of available colors of size $|L(u)| \geq \deg_G(u)$ and assign to each edge $uv \in E(G)$ a matching $C_{uv}$ from $L(u)$ to $L(v)$ so that $G$ is does not admit a proper $(L, C)$-coloring. By Lemma 2.2, this implies that $G$ is not weakly $(\deg - 1)$-degenerate. □

## 5.2 | Weak degeneracy and maximum average degree

Here we establish a lower bound on the maximum average degree of a graph in terms of its weak degeneracy.

Theorem 1.7. Let $G$ be a nonempty graph. If the weak degeneracy of $G$ is at least $d \geq 3$, then either $G$ contains a $(d + 1)$-clique or

$$\mathrm{mad}(G) \geq d + \frac{d - 2}{d^2 + 2d - 2}.$$  

We derive Theorem 1.7 from Theorem 1.6. Our argument is closely analogous to the proof of the lower bound on the average degree of DP-critical graphs due to Kostochka, Pron, and the first named author [4, corollary 10], which in turn is based on earlier work of Gallai [13].

We need the following result, essentially established by Gallai in [13] (Gallai’s paper is in German; see [4, appendix] for a proof in English):
Lemma 5.4 [4, lemma 20]. Let $T$ be a GDP-tree of maximum degree at most $d \geq 3$ and without a $(d + 1)$-clique. Then $\text{ad}(T) \leq d - 1 + 2/d$.

We say that $G$ is a **minimal** graph of weak degeneracy $d$ if $\text{wd}(G) = d$ and $\text{wd}(H) < d$ for every proper subgraph $H$ of $G$.

**Lemma 5.5.** Let $G$ be a minimal graph of weak degeneracy $d \geq 3$.

(a) The minimum degree of $G$ is at least $d$.

(b) Let $U := \{u \in V(G) : \deg_G(u) = d\}$. Then every component of $G[U]$ is a GDP-tree.

**Proof.** (a) Suppose that there is a vertex $u \in V(G)$ with $\deg_G(u) \leq d - 1$. We will show that $G$ is weakly $(d - 1)$-degenerate. Let $f$ be the constant $d - 1$ function on $V(G)$. By the minimality of $G$, we may remove every vertex from $(G - u, f)$ via a sequence of legal applications of the operations $\text{DELETE}$ and $\text{DELSAVE}$. Since $\deg_G(u) \leq d - 1$, we may use the same sequence of operations to remove every vertex except $u$ from $(G, f)$ (at which point the function $f$ will be replaced by the map sending $u$ to $d - 1 - \deg_G(u)$) and then remove $u$ using the operation $\text{DELETE}$.

(b) Let $C$ be a connected component of $G[U]$ and let $f$ be the constant $d - 1$ function on $V(G)$. By the minimality of $G$, we may remove every vertex from $(G - V(C), f)$ via a sequence of legal applications of the operations $\text{DELETE}$ and $\text{DELSAVE}$. If we perform the same sequence of operations on $(G, f)$, then the graph $G$ will be replaced by $C$, while the function $f$ will be replaced by the map sending each $u \in V(C)$ to $d - 1 - \deg_{G-V[C]}(u) = \deg_C(u) - 1$. Since $G$ is not weakly $(d - 1)$-degenerate, this implies that $C$ is not weakly $(\deg - 1)$-degenerate. Hence, by Theorem 1.6, $C$ is a GDP-tree. □

**Proof of Theorem 1.7.** Fix $d \geq 3$. It suffices to argue that every minimal graph $G$ of weak degeneracy $d$ and without a $(d + 1)$-clique satisfies

$$\text{ad}(G) \geq d + \frac{d - 2}{d^2 + 2d - 2}.$$ 

To this end, we use discharging. Let the initial charge of each vertex $u \in V(G)$ be $\text{ch}(u) := \deg_G(u)$. The only discharging rule is: Every vertex $u \in V(G)$ with $\deg_G(u) > d$ sends to each neighbor the charge $\frac{d - 2}{d^2 + 2d - 2}$. Let the new charge of each vertex $u$ be $\text{ch}^*(u)$. Note that

$$\text{ad}(G) \geq \left| V(G) \right| = \sum_{u \in V(G)} \text{ch}(u) = \sum_{u \in V(G)} \text{ch}^*(u).$$

For any vertex $u$ with $\deg_G(u) > d$, we have

$$\text{ch}^*(u) \geq \deg_G(u) - c\deg_G(u) \geq \left(1 - \frac{d}{d^2 + 2d - 2}\right)(d + 1) = d + \frac{d - 2}{d^2 + 2d - 2}.$$
Let \( C \) be any connected component of \( G[U] \), where \( U \) is the set of all vertices of degree \( d \) in \( G \). By Lemma 5.5(b), \( C \) is a GDP-tree. Hence, by Lemma 5.4, 
\[
\text{ad}(C) \leq d - 1 + 2/d.
\]
Therefore,
\[
\sum_{u \in V(C)} \text{ch}^*(u) = d|V(C)| + c \sum_{u \in V(C)} (d - \deg_C(u)) \geq \left( d + \frac{d}{d^2 + 2d - 2} \right)|V(C)| = \left( d + \frac{d - 2}{d^2 + 2d - 2} \right)|V(C)|.
\]

The above bounds imply that
\[
\sum_{u \in V(G)} \text{ch}^*(u) \geq \left( d + \frac{d - 2}{d^2 + 2d - 2} \right)|V(G)|,
\]
which yields the desired result. \( \square \)

6 | LOWER BOUNDS FOR REGULAR GRAPHS

In this section, we establish lower bounds on weak degeneracy for regular graphs.

**Proposition 1.8.** Let \( G \) be a \( d \)-regular graph with \( n \geq 2 \) vertices. Then \( \text{wd}(G) \geq d - \sqrt{2n} \).

**Proof.** Let \( k := \text{wd}(G) \). Set \( G_0 := G \) and let \( f_0 \) be the constant \( k \) function on \( V(G) \). By definition, starting with \( (G_0, f_0) \), it is possible to remove all vertices from \( G \) via a sequence of legal applications of the operations DELETE and DELSAVE. Fix any such sequence \( S = (O_i, \ldots, O_{n-1}) \). For each \( 0 \leq i \leq n - 1 \), let \( (G_{i+1}, f_{i+1}) \) be the result of applying \( O_i \) to \( (G_i, f_i) \). Then we can write
\[
O_i = \text{DELETE}(G_i, f_i, u_i) \quad \text{or} \quad O_i = \text{DELSAVE}(G_i, f_i, u_i, w_i).
\]
For each \( 0 \leq i \leq n - 1 \), define
\[
d_i := |\{ j < i : u_ju_i \in E(G) \}| \quad \text{and} \quad \sigma_i := |\{ j < i : u_ju_i \in E(G) \}|.
\]
(So \( \sigma_i \) is the number of vertices that “save” \( u_i \).) Then \( 0 \leq f_i(u_i) = k - d_i + \sigma_i \) and thus
\[
k \geq d_i - \sigma_i. \tag{1}
\]
Adding (1) up over the interval \( n - t \leq i \leq n - 1 \) for some integer \( 1 \leq t \leq n \) yields
\[
kt \geq \left( n-1 \sum_{i=n-t}^{n-1} d_i \right) - \left( n-1 \sum_{i=n-t}^{n-1} \sigma_i \right). \tag{2}
\]
Each index \( j \) contributes to \( \sigma_i \) for at most one \( i \), so \( \sum_{i=0}^{n-1} \sigma_i \leq n \). Also, since \( G \) is \( d \)-regular,

\[
\sum_{i=n-t}^{n-1} d_i = dt - |E(G[u_{n-t}, \ldots, u_{n-1}])| \geq dt - \left( \frac{t}{2} \right).
\]

Therefore, (2) implies that

\[ k \geq d - \frac{t - 1}{2} - \frac{n}{t}. \]

Finally, taking \( t := \lceil \sqrt{2n} \rceil \) gives

\[ k \geq d - \frac{\lceil \sqrt{2n} \rceil - 1}{2} - \frac{n}{\lceil \sqrt{2n} \rceil} \geq d - \frac{\sqrt{2n}}{2} - \frac{n}{\sqrt{2n}} = d - \sqrt{2n}, \]

as desired. \( \square \)

**Proposition 1.9.** If \( G \) is a triangle-free \( d \)-regular graph with \( n \geq 4 \) vertices, then \( \text{wd}(G) > d - \sqrt{n} - 1 \).

**Proof.** The argument is virtually the same as in the proof of Proposition 1.8, except that we use Mantel’s theorem to replace the bound (3) by \( \sum_{i=n-t}^{n-1} d_i \geq dt - t^2/4 \). This yields

\[ \text{wd}(G) \geq d - \frac{t - 1}{4} - \frac{n}{t}. \]

Now we set \( t := \lceil 2\sqrt{n} \rceil \) and conclude that

\[ \text{wd}(G) \geq d - \frac{\lceil 2\sqrt{n} \rceil}{4} - \frac{n}{\lceil 2\sqrt{n} \rceil} \geq d - 2\sqrt{\frac{n + 1}{4}} - \frac{n}{2\sqrt{n}} \geq d - \sqrt{n} - \frac{1}{4}, \]

as desired. \( \square \)

### 7 | GOING BELOW THE MAXIMUM DEGREE

#### 7.1 | Preliminaries

In this section, we review some necessary background facts. First, we will need the Lovász Local Lemma, in the following form:

**Theorem 7.1** (Lovász Local Lemma [1, lemma 5.1.1]). Let \( \mathcal{X} \) be a finite family of random events such that each \( X \in \mathcal{X} \) has probability at most \( p \) and is mutually independent from all but \( \Delta \) other events in \( \mathcal{X} \). If \( ep(\Delta + 1) \leq 1 \), then the probability that no event in \( \mathcal{X} \) happens is positive.

We shall also use the Chernoff bound for binomial random variables:
**Theorem 7.2** (Chernoff bound [17, p. 43]). If $X \sim \text{Bin}(n, p)$ is a binomial random variable, then for all $0 \leq t \leq np$,

$$\mathbb{P}[|X - np| > t] < 2 \exp\left(-\frac{t^2}{3np}\right).$$

Next, we need a quantitative version of the Central Limit Theorem due to Berry and Esseen.

**Theorem 7.3** (Berry–Esseen [11, §XVI.5]). There is a universal constant $A > 0$ with the following property. Let $X_1, ..., X_n$ be independent identically distributed random variables such that $\mathbb{E}[X_i] = 0$, $\mathbb{E}[X_i^2] = \sigma^2 > 0$, and $\mathbb{E}[|X_i|^3] = \rho < \infty$. Then for all $t \in \mathbb{R}$,

$$\left| \mathbb{P}\left[ \frac{\sum_{i=1}^n X_i}{\sigma \sqrt{n}} \leq t \right] - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t e^{-x^2/2} dx \right| \leq \frac{A \rho}{\sigma^3 \sqrt{n}}.$$

In particular, if $X \sim \text{Bin}(n, p)$ is a binomial random variable, then for any $\beta > 0$,

$$\mathbb{P}[X \leq np - \beta \sqrt{n}] = \int_{-\infty}^{\beta/\sqrt{p(1-p)}} e^{-x^2/2} dx + O\left(\frac{1 - 2p(1-p)}{\sqrt{p(1-p)n}}\right). \quad (4)$$

This means that for large $n$, $\mathbb{P}[X \leq np - \beta \sqrt{n}]$ is separated from 0. By applying this result to the random variable $n - X \sim \text{Bin}(n, 1 - p)$, we see that $\mathbb{P}[X \geq np + \beta \sqrt{n}]$ is separated from 0 as well.

The following is a standard consequence of Hall's theorem:

**Lemma 7.4.** Let $G$ be a graph and let $A, B \subseteq V(G)$ be disjoint sets. Suppose that each vertex in $A$ has at most $d_1$ neighbors in $B$, while each vertex in $B$ has at least $d_2$ neighbors in $A$. Let $t \in \mathbb{N}$ satisfy $td_1 \leq d_2$. Then there exists a partial function $s : A \longrightarrow B$ such that:

- for all $u \in A$, if $s(u)$ is defined, then $s(u)$ is a neighbor of $u$;
- the preimage of every vertex $w \in B$ under $s$ has cardinality exactly $t$.

**Proof.** Let $H$ be the maximal bipartite subgraph of $G$ with parts $A$ and $B$, and let $H^*$ be obtained from $H$ by replacing every vertex $w \in B$ by $t$ copies, denoted $w_1, ..., w_t$. By construction, $H^*$ is a bipartite graph with parts $A$ and $B^* := \{w_j : w \in B, 1 \leq j \leq t\}$. For all $u \in A$, $\deg_{H^*}(u) \leq td_1 \leq d_2$. On the other hand, every vertex $w_j \in B^*$ satisfies $\deg_{H^*}(w_j) \geq d_2$. These inequalities, together with Hall's theorem [7, theorem 2.1.2], imply that $H^*$ has a matching $M$ that saturates $B^*$. We can now define the desired function $s : A \longrightarrow B$ by mapping each $u \in A$ that is covered by $M$ to the unique $w \in B$ such that $uw_j \in M$ for some $j$. \qed

It will be convenient for us to work with $d$-regular graphs rather than graphs of maximum degree $d$. To this end, we shall employ the following facts:
Lemma 7.5 (Chartrand–Wall [6]). If \( G \) is a graph of maximum degree \( d \) and chromatic number at most \( k \), then \( G \) can be embedded into a \( d \)-regular graph \( G^* \) of chromatic number at most \( k \).

Lemma 7.6 ([17, exercise 12.4]). If \( G \) is a graph of maximum degree \( d \) and girth at least \( g \), then \( G \) can be embedded into a \( d \)-regular graph \( G^* \) of girth at least \( g \).

Proof. This fact is well-known, but we include a proof for completeness. We use a simplified version of the construction from [2, proposition 4.1]. Set

\[
N := \sum_{u \in V(G)} (d - \deg_G(u))
\]

and let \( \Gamma \) be an \( N \)-regular graph of girth at least \( g \), which exists by [14, 16]. We may assume that \( V(\Gamma) = \{1, \ldots, q\} \), where \( q := |V(\Gamma)| \). Take \( q \) vertex-disjoint copies of \( G \), say \( G_1, \ldots, G_q \) and define \( S_i := \{u \in V(G_i) : \deg_{G_i}(u) < d\} \) for every \( 1 \leq i \leq q \). The graph \( G^* \) is obtained from the disjoint union of \( G_1, \ldots, G_q \) by performing the following sequence of operations once for each edge \( ij \in E(\Gamma) \), one edge at a time:

1. Pick arbitrary vertices \( u \in S_i \) and \( v \in S_j \).
2. Add the edge \( uv \) to \( E(G^*) \).
3. If \( \deg_{G^*}(u) = d \), remove \( u \) from \( S_i \).
4. If \( \deg_{G^*}(v) = d \), remove \( v \) from \( S_j \).

It is clear that the resulting graph \( G^* \) is as desired. \( \square \)

7.2 Removal schemes

In the next definition we introduce the technical notion of a removal scheme on graph \( G \). Roughly speaking, a removal scheme records the order in which we attempt to remove the vertices from \( G \). Additionally, it indicates whether each vertex is removed using a DELETE or a DELSAVE operation, and in the latter case, what other vertex we “save” an extra color for.

Definition 7.7 (Removal schemes). Fix a graph \( G \). A removal scheme on \( G \) is a pair \((<, \text{save})\), where \(< \) is a linear ordering on \( V(G) \) and \( \text{save} : V(G) \to V(G) \) is a partial function such that for every vertex \( u \in V(G) \), if \( \text{save}(u) \) is defined, then it is a neighbor of \( u \) and \( u < \text{save}(u) \). For convenience, we write \( \text{save}(u) = \text{blank} \) if \( \text{save}(u) \) is undefined. Given a removal scheme \((<, \text{save})\), we call \(< \) the removal order and say that a vertex \( u \) saves the vertex \( \text{save}(u) \). A removal scheme \((<, \text{save})\) is legal if for all \( u, w \in V(G) \) such that \( w = \text{save}(u) \), we have

\[
|\{v \in N_G(u) : v < u \text{ and } \text{save}(v) \neq u\}| < |\{v \in N_G(w) : v < u \text{ and } \text{save}(v) \neq w\}|.
\]

The gap of a vertex \( u \) with respect to a removal scheme \((<, \text{save})\) is the quantity
We also let $\text{gap}(\prec, \text{save}) := \min\{\text{gap}(u; \prec, \text{save}) : u \in V(G)\}.$

**Lemma 7.8.** Let $G$ be a graph of maximum degree at most $d$ and let $(\prec, \text{save})$ be a legal removal scheme on $G.$ Then $G$ is weakly $(d - \text{gap}(\prec, \text{save}))$-degenerate.

**Proof.** For brevity, let $g := \text{gap}(\prec, \text{save}).$ Let $u_0, ..., u_{n-1}$ be the vertices of $G$ listed in the order given by $\prec.$ Define a sequence $(G_i, f_i, 0 \leq i \leq n - 1$ by setting $(G_0, f_0) := (G, d - g)$ and

$$(G_{i+1}, f_{i+1}) := \begin{cases} \text{DELETE}(G_i, f_i, u_i) & \text{if } \text{save}(u_i) = \text{blank}; \\ \text{DELSAVE}(G_i, f_i, u_i, \text{save}(u_i)) & \text{otherwise}. \end{cases}$$

We claim that this construction yields a sequence of legal applications of DELETE and DELSAVE that removes every vertex from $G.$ Indeed, consider any vertex $u_i.$ Note that

$$f_i(u_i) = d - g - |\{v \in N_G(u_i) : v > u_i\} + |\{v \in N_G(u_i) : \text{save}(v) = u_i\}| \
\geq \text{gap}(u_i; \prec \text{ save}) - g \geq 0.$$  

This shows that the functions $f_i$ are nonnegative. Now suppose that $\text{save}(u_i) = w \in V(G).$ Then, by definition, $w$ is a neighbor of $u_i$ that appears after $u_i$ in the ordering $\prec,$ and thus the operation $\text{DELSAVE}(G_i, f_i, u_i, w)$ may be applied. Furthermore, $f_i(u_i) > f_i(w)$ by (5), so this application of is DELSAVE legal, as desired. $\square$

### 7.3 Regular sets

Let $G$ be a graph. Given a vertex $u \in V(G)$ and a set $A \subseteq V(G),$ we let $N_A(u) := N_G(u) \cap A$ denote the set of all neighbors of $u$ in $A$ and write $\deg_A(u) := |N_A(u)|.$ Several times in our arguments, we will need to perform the following operation: given a set $A$ and a number $p \in [0, 1],$ we will need to pick a subset $A' \subseteq A$ such that every vertex $u \in V(G)$ has roughly $p\deg_A(u)$ neighbors in $A'.$ Formally, we introduce the following definition:

**Definition 7.9 (Regular sets).** Fix a graph $G$ of maximum degree $d$ and a subset $A \subseteq V(G).$ Given $p, \varepsilon \in (0, 1],$ a $(p, \varepsilon)$-regular subset of $A$ is a set $A' \subseteq A$ such that every vertex $u \in V(G)$ satisfies one of the following conditions:

- either $\deg_A(u) < 9 \log d/(\varepsilon^2 p)$ (i.e., $u$ has very few neighbors in $A'),$
- or $(1 - \varepsilon)p\deg_A(u) \leq \deg_A(u) \leq (1 + \varepsilon)p\deg_A(u)$ (i.e., $\deg_A(u) \approx p\deg_A(u)).$

Using the Lovász Local Lemma, it is not hard to prove that $(p, \varepsilon)$-regular subsets exist.

**Lemma 7.10.** Let $G$ be a graph of maximum degree $d.$ Fix $p, \varepsilon \in (0, 1].$ Then every set $A \subseteq V(G)$ has a $(p, \varepsilon)$-regular subset.
Proof. We may assume \( d > 8 \), as otherwise \( \deg_A(u) \leq d < 9 \log d \) for all \( u \in V(G) \), so any subset \( A' \subseteq A \) is \((p, \varepsilon)\)-regular. Form a random set \( A' \subseteq A \) by picking each vertex independently with probability \( p \). We shall use the Lovász Local Lemma (Theorem 7.1) to argue that \( A' \) is \((p, \varepsilon)\)-regular with positive probability. Let \( U \subseteq V(G) \) be the set of all vertices \( u \in V(G) \) with \( \deg_A(u) \geq 9 \log d / (\varepsilon^2 p) \). For each \( u \in U \), let \( X_u \) be the random event that

\[
\deg_A(u) \not\in [(1 - \varepsilon)p\deg_A(u), (1 + \varepsilon)p\deg_A(u)].
\]

We need to argue that with positive probability, none of the events \( X_u \) happen. By the Chernoff bound (Theorem 7.2), for each \( u \in U \) we have

\[
P[X_u] < 2\exp\left(-\frac{\varepsilon^2 p\deg_A(u)}{3}\right) \leq 2\exp(-3\log d) = 3d^{-3}.
\]

Each event \( X_u \) is mutually independent from all the events \( X_v \) corresponding to the vertices \( v \) that do not share a neighbor with \( u \). Since there are at most \( d(d - 1) \) vertices that share a neighbor with \( u \) (not including \( u \) itself), the Lovász Local Lemma shows that with positive probability none of the events \( X_u, u \in U \) happen provided that

\[
e \cdot 3d^{-3} \cdot (d(d - 1) + 1) < 1.
\]

This inequality holds for all \( d > 8 \), and the proof is complete.

\[\square\]

7.4 | Graphs of bounded chromatic number

**Theorem 1.11.** For each integer \( k \geq 1 \), there exist \( c > 0 \) and \( d_0 \in \mathbb{N} \) such that if \( G \) is a graph of maximum degree \( d \geq d_0 \) with \( \chi(G) \leq k \), then \( \text{wd}(G) \leq d - c\sqrt{d} \).

Let \( G \) be a graph of maximum degree \( d \) and chromatic number at most \( k \), where we assume that \( d \) is sufficiently large in terms of \( k \). Upon replacing \( G \) with a supergraph if necessary, we may assume that \( G \) is \( d \)-regular (Lemma 7.5). Let \( c \) be a sufficiently small positive quantity depending on \( k \) (but not on \( d \)). We will construct a legal removal scheme \( (<, \text{save}) \) on \( G \) such that \( \text{gap}(<, \text{save}) \geq c\sqrt{d} \). By Lemma 7.8, this will yield the desired result.

We start by applying Lemma 7.10 to obtain a \((2/\sqrt{d}, 1/2)\)-regular subset \( B \) of \( V(G) \). Since \( G \) is \( d \)-regular and \( d \geq 18\sqrt{d} \log d \), every vertex \( u \in V(G) \) satisfies

\[
\sqrt{d} \leq \deg_B(u) \leq 3\sqrt{d}.
\]

Set \( A := V(G) \setminus B \). We will find a legal removal scheme \( (<, \text{save}) \) on \( G \) such that:

(a) In the ordering \( < \), every vertex in \( A \) comes before every vertex in \( B \).
(b) Every vertex in \( B \) is saved at least \( c\sqrt{d} \) times.

Notice that if \( (<, \text{save}) \) satisfies conditions (a) and (b), then \( \text{gap}(<, \text{save}) \geq c\sqrt{d} \), which is the property we want. Indeed, take any vertex \( u \in V(G) \). If \( u \in A \), then, by (a) and (6),

\[
\deg_B(u) \leq 3\sqrt{d}.
\]
On the other hand, if \( u \in B \), then, by (b),

\[
gap(u; <, \text{save}) \geq \max \{ |v \in N_G(u) : v > u| \} \geq \deg_B(u) \geq \sqrt{d}.
\]

Assuming \( c < 1 \), we have \( \gap(u; <, \text{save}) \geq c \sqrt{d} \) in both cases, as desired.

A legal removal scheme \((<, \text{save})\) satisfying (a) and (b) is constructed as follows. For \( 1 \leq i \leq k \), we recursively define the following numerical parameters:

\[
N_1 := 1 \quad \text{and} \quad N_i := 20k \sum_{j=1}^{i-1} N_j \text{ for } i \geq 2.
\]

Set \( p_i := N_i/(6kN_k) \). Note that \( p_1 < p_2 < \cdots < p_k = 1/(6k) \). We shall assume \( c \) is so small that

\[
32kc < p_1. \tag{7}
\]

Since \( \chi(G) \leq k \), we can partition \( A \) into \( k \) independent sets \( A_1, \ldots, A_k \). Let \( C_i \) be a \((p_i, 1/2)\)-regular subset of \( A_i \) and let \( D_i \) be a \((p_i, 1/2)\)-regular subset of \( A_i \setminus C_i \). The ordering \(<\) is defined by listing the elements of \( V(G) \) in the following order:

\[
C_1, D_1, C_2, D_2, \ldots, C_k, D_k, A \setminus \bigcup_{i=1}^{k} (C_i \cup D_i), B.
\]

(The order of the elements in each set in this list is arbitrary.) Since the elements of \( B \) appear last in this ordering, condition (a) is fulfilled.

Now we need to define the function \( \text{save} \) so that condition (b) holds. We start by recording the following observation:

**Claim 7.11.** Every vertex \( u \in V(G) \) satisfies

\[
\deg_{C_i}(u) \leq \frac{3p_i d}{2} \quad \text{and} \quad \deg_{D_i}(u) \leq \frac{3p_i d}{2}.
\]

**Proof.** Immediate from the definitions of \( C_i \) and \( D_i \) and since the maximum degree of \( G \) is \( d \). \( \square \)

By (6), each vertex \( u \in V(G) \) has at least \( d - 3\sqrt{d} \geq d/2 \) neighbors in \( A \). Therefore, we may partition \( B \) into \( k \) sets \( B_1, \ldots, B_k \) so that each vertex in \( B_i \) has at least \( d/(2k) \) neighbors in \( A_i \). This implies that every vertex in \( B_i \) has many neighbors in \( C_i \) and \( D_i \).

**Claim 7.12.** Every vertex \( w \in B_i \) satisfies

\[
\deg_{C_i}(w) \geq \frac{p_i d}{4k} \quad \text{and} \quad \deg_{D_i}(w) \geq \frac{p_i d}{8k}.
\]
Proof. The first inequality holds since $C_i$ is a $(p_i, 1/2)$-regular subset of $A_i$ and $\deg_{A_i}(w) \geq d/(2k)$. The second inequality follows similarly since, by Claim 7.11,

$$\deg_{A_i \setminus C_i}(w) \geq \frac{d}{2k} - \frac{3p_i}{2}d \geq \frac{d}{4k}. \quad \square$$

Note that, by (6), each vertex in $D_i$ has at most $3\sqrt{d}$ neighbors in $B_i$. On the other hand, by Claim 7.12, every vertex in $B_i$ has at least $p_i d/(8k) \geq p_i d/(8k)$ neighbors in $D_i$. Since, by (7),

$$\lceil c\sqrt{d} \rceil \cdot 3\sqrt{d} < 4cd < \frac{p_i d}{8k},$$

we can apply Lemma 7.4 to find a partial function $s_i : D_i \rightarrow B_i$ such that:

- for all $u \in D_i$, if $s_i(u)$ is defined, then $s_i(u)$ is a neighbor of $u$;
- the preimage of every vertex $w \in B_i$ under $s_i$ has cardinality $\lceil c\sqrt{d} \rceil$.

Now we can define $\text{save} : V(G) \rightarrow V(G)$ by

$$\text{save}(u) := \begin{cases} s_i(u) & \text{if } u \in D_i \text{ and } s_i(u) \text{ is defined;} \\ \text{blank} & \text{otherwise.} \end{cases}$$

By the choice of $s_i$, $(<, \text{save})$ is a removal scheme that satisfies (b). It remains to verify that this removal scheme is legal. To this end, take any $u, w \in V(G)$ such that $\text{save}(u) = w$. By construction, this means that $u \in D_i$ and $w \in B_i$ for some $i$. The vertices that precede $u$ in the ordering $<$ are the ones in $C_i, D_i, ..., C_{i-1}, D_{i-1}, C_i$, plus possibly some vertices in $D_i$. Since the set $A_i$ is independent, $u$ has no neighbors in $C_i \cup D_i$, and hence, by Claim 7.11,

$$\lvert \{v \in N_G(u) : v < u\} \rvert = \sum_{j=1}^{i-1} (\deg_{C_j}(u) + \deg_{D_j}(u)) \leq \sum_{j=1}^{i-1} 3p_jd = \frac{3p_i}{20k}d < \frac{p_i}{4k}d.$$

On the other hand, since no vertex in $C_i$ saves $w$, Claim 7.12 yields

$$\lvert \{v \in N_G(w) : v < u \text{ and } \text{save}(v) \neq w\} \rvert \geq \deg_{C_i}(w) \geq \frac{p_i}{4k}d.$$

Therefore, inequality (5) holds, and the proof of Theorem 1.11 is complete.

7.5 | Graphs of girth at least 5

Theorem 1.12. There exist $c > 0$ and $d_0 \in \mathbb{N}$ such that if $G$ is a graph of maximum degree $d \geq d_0$ and girth at least 5, then $w(G) \leq d - c\sqrt{d}$.

Let $G$ be a graph of maximum degree $d$ and girth at least 5, where $d$ is sufficiently large. Upon replacing $G$ with a supergraph if necessary, we may assume that $G$ is $d$-regular (Lemma 7.6). Let $c$ be a sufficiently small positive constant. As in the proof of Theorem 1.11, we
will construct a legal removal scheme \((<, \text{save})\) on \(G\) such that \(\text{gap}(<, \text{save}) \geq c\sqrt{d}\). By Lemma 7.8, this will yield the desired result.

By Lemma 7.10, there is a \((2/\sqrt{d}, 1/2)\)-regular subset \(B\) of \(V(G)\). Then for every vertex \(u \in V(G)\),

\[
\sqrt{d} \leq \deg_B(u) \leq 3\sqrt{d}.
\]  

(8)

Set \(A := V(G) \setminus B\). Every vertex in \(A\) has at most \(3\sqrt{d}\) neighbors in \(B\), while every vertex in \(B\) has at least \(d - 3\sqrt{d} \geq d/2\) neighbors in \(A\). Since \([\sqrt{d}/8] \cdot 3\sqrt{d} < d/2\), Lemma 7.4 gives a partial function \(s : A \to B\) such that:

- for all \(u \in A\), if \(s(u)\) is defined, then \(s(u)\) is a neighbor of \(u\);
- the preimage of each \(w \in B\) under \(s\) has cardinality \([\sqrt{d}/8]\).

For each \(w \in B\), let \(S_w\) denote the preimage of \(w\) under \(s\); for \(u \in A\), set \(S_u := \emptyset\).

Now we assemble a removal scheme \((<, \text{save})\) using a randomized procedure. Pick a linear ordering \(<\) of \(A\) uniformly at random. The ordering \(<\) will start with the vertices of \(A\) listed according to \(<\), followed by the vertices of \(B\) in some order (to be specified shortly). Intuitively, we imagine that every vertex \(u \in A\) attempts to save the vertex \(s(u)\). This attempt only succeeds if condition (5) is satisfied. Formally, we say that \(u \in A\) with \(s(u) \neq \emptyset\) is successful if

\[
|\{v \in N_A(u) : v < u\}| < |\{v \in N_A(u) \setminus S_v : v < u\}|.
\]

If \(u \in A\) is successful, then we write \(\text{save}(u) := s(u)\); for all other vertices \(u\) we set \(\text{save}(u) := \text{blank}\).

Say that a vertex \(w \in B\) is happy if its preimage under the function \(\text{save}\) has cardinality at least \(c\sqrt{d}\). Let \(H \subseteq B\) be the set of all happy vertices. The ordering \(<\) consists of \(A\) listed according to \(<\), followed by \(B \setminus H\) in an arbitrary order, and then by \(H\) in an arbitrary order. By construction, \((<, \text{save})\) is a legal removal scheme, and we claim that \(\text{gap}(<, \text{save}) \geq c\sqrt{d}\) with positive probability. The key fact we need to establish is the following:

Claim 7.13. With positive probability, every vertex of \(G\) has at least \(c\sqrt{d}\) neighbors in \(H\).

Let us see why Claim 7.13 implies the desired result. Suppose that every vertex of \(G\) has at least \(c\sqrt{d}\) neighbors in \(H\). Take any \(u \in V(G)\). If \(u \not\in H\), then

\[
\text{gap}(u; <, \text{save}) \geq |\{v \in N_G(u) : v > u\}| \geq \deg_H(u) \geq c\sqrt{d}.
\]

On the other hand, if \(u \in H\), then, by the definition of \(H\),

\[
\text{gap}(u; <, \text{save}) \geq |\{v \in N_G(u) : \text{save}(v) = u\}| \geq c\sqrt{d}.
\]

In either case, \(\text{gap}(u; <, \text{save}) \geq c\sqrt{d}\), as desired.

In the remainder of this section, we prove Claim 7.13. It will be convenient to assume that the random ordering \(<\) is sampled according to the following procedure: each vertex \(u \in A\) picks a real number \(\tilde{s}(u) \in [0, 1]\) uniformly at random, and then we set \(u_1 < u_2\) if and only if \(\tilde{s}(u_1) < \tilde{s}(u_2)\) (note that \(\tilde{s}(u_1) \neq \tilde{s}(u_2)\) with probability 1). For each \(u \in V(G)\), let \(X_u\) be the
random event that $\deg_H(u) < c\sqrt{d}$. It is clear that $X_u$ only depends on the values of the function $\vartheta$ on the vertices at distance at most 3 from $u$. Therefore, $X_u$ is mutually independent from the events $X_v$ corresponding to the vertices $v$ at distance more than 6 from $u$. Hence, by the Lovász Local Lemma, to prove that with positive probability none of the events $X_u$ happen it suffices to show that

$$\mathbb{P}[X_u] = o(d^{-6}).$$

(9)

The proof of (9) is somewhat technical, so before getting into its details, let us briefly explain the intuition behind our approach. Assuming $c$ is small enough, it is possible to show that for each $w \in B$, $\mathbb{P}[w \text{ is happy}] = \Omega(1)$. Since every vertex $u \in V(G)$ has at least $\sqrt{d}$ neighbors in $B$, we have $\mathbb{E} [\deg_H(u)] = \Omega(\sqrt{d})$. Ideally, we would now argue that the random variable $\deg_H(u)$ is close to its expected value with very high probability. One way to achieve this would be to show that the random events “$w$ is happy” for $w \in N_B(u)$ are close to being mutually independent and then apply the Chernoff bound or some other similar result. This strategy indeed works in the case when $G$ has girth at least 7. This is because for each $w \in N_B(u)$, the event “$w$ is happy” is determined by the values of $\vartheta$ in the radius-2 ball around $w$, and the girth-7 assumption implies that the radius-2 balls around the vertices in $N_B(u)$ do not overlap too much.

It turns out that, with a more clever argument, we can reduce the girth requirement from 7 to 5. The idea is to define a certain property of vertices $w \in B$, which we call being powerful (or, more accurately, $\varepsilon$-powerful for some $\varepsilon > 0$), so that the following statements hold:

(a) the event “$w$ is powerful” is determined by the values of $\vartheta$ on the neighbors of $w$;
(b) the probability that $w$ is powerful is at least $\Omega(1)$ (Claim 7.15);
(c) if $w$ is powerful, then $w$ is happy with very high probability (Claim 7.16).

Thanks to (b), the expected number of powerful neighbors for each vertex $u \in V(G)$ is $\Omega(\sqrt{d})$. Using (a) and the girth-5 assumption, we can show that in fact $u$ has $\Omega(\sqrt{d})$ powerful neighbors with very high probability. Finally, according to (c), once $u$ has $\Omega(\sqrt{d})$ powerful neighbors, it is extremely likely that it has $\Omega(\sqrt{d})$ happy neighbors as well.

Let us now begin the formal proof. We start by associating to each vertex of $G$ a (random) vector with entries in $[0,1]$ by setting, for every $u \in V(G)$,

$$x_u := (\vartheta(v) : v \in N_A(u) \setminus S_u).$$

(Recall that $S_u = \emptyset$ for $u \in A$.) Now we introduce the following definitions:

**Definition 7.14** (Powerful vectors and vertices). Given a vector $x = (x_1, ..., x_k) \in [0,1]^k$ and a real number $\alpha \in [0,1]$, let the $\alpha$-power of $x$ be the quantity

$$\pi(x, \alpha) := |\{i : x_i < \alpha\}|.$$

For $\varepsilon > 0$, we say that a vector $x \in [0,1]^k$ is $\varepsilon$-powerful if the following statement holds: If we pick a real number $\alpha \in [0,1]$ and a vector $y \in [0,1]^d$ uniformly at random, then
\[ \mathbb{P} [\pi (y, \alpha) < \pi (x, \alpha)] \geq \varepsilon. \tag{10} \]

A vertex \( w \in B \) is \( \varepsilon \)-powerful if the vector \( x_w \) is \( \varepsilon \)-powerful.

We remark that if \( x \in [0, 1]^k \) is \( \varepsilon \)-powerful, then (10) also holds for \( y \) drawn uniformly at random from \([0, 1]^{\ell}\) for any \( \ell \leq d \). Similarly, if an \( \varepsilon \)-powerful vector \( x \) is obtained from another vector \( x' \) by removing some of the coordinates, then \( x' \) is \( \varepsilon \)-powerful as well, since \( \pi (x', \alpha) \geq \pi (x, \alpha) \) for all \( \alpha \).

Using this notation, we can say that a vertex \( u \in A \) with \( s(u) = w \) is successful if and only if

\[ \pi (x_u, \hat{s}(u)) < \pi (x_w, \hat{s}(u)). \]

**Claim 7.15.** There exists a constant \( \varepsilon > 0 \) such that if \( k \geq d - 5 \sqrt{d} \), then the probability that a uniformly random vector \( x \in [0, 1]^k \) is \( \varepsilon \)-powerful is at least \( \varepsilon \).

**Proof.** For \( \varepsilon > 0 \), let \( p_\varepsilon \) denote the probability that a uniformly random vector \( x \in [0, 1]^k \) is \( \varepsilon \)-powerful. If we sample \( x \in [0, 1]^k \), \( \alpha \in [0, 1] \), and \( y \in [0, 1]^d \) uniformly at random, then

\[ \mathbb{P} [\pi (y, \alpha) < \pi (x, \alpha)] = \mathbb{P} [x \text{ is } \varepsilon \text{-powerful}] \mathbb{P} [\pi (y, \alpha) < \pi (x, \alpha) \mid x \text{ is } \varepsilon \text{-powerful}] \]
\[ + \mathbb{P} [x \text{ is not } \varepsilon \text{-powerful}] \mathbb{P} [\pi (y, \alpha) < \pi (x, \alpha) \mid x \text{ is not } \varepsilon \text{-powerful}] \]
\[ \leq p_\varepsilon + (1 - p_\varepsilon) \varepsilon \leq p_\varepsilon + \varepsilon. \tag{11} \]

We now prove a lower bound on the left-hand side of (11). We sample \( \alpha \in [0, 1] \) first. Note that with probability 1/3, we get \( 1/3 \leq \alpha \leq 2/3 \). Now \( \pi (x, \alpha) \) and \( \pi (y, \alpha) \) are independent random variables sampled from the binomial distributions Bin\( (k, \alpha) \) and Bin\( (d, \alpha) \), respectively. It follows from the Berry–Esseen theorem (specifically from equation (4)) that there exists a constant \( \gamma > 0 \) such that, assuming \( d \) is large enough and \( 1/3 \leq \alpha \leq 2/3 \), we have

\[ \mathbb{P} [\pi (x, \alpha) > \alpha k] \geq \gamma \quad \text{and} \quad \mathbb{P} [\pi (y, \alpha) < \alpha (d - 5 \sqrt{d})] \geq \gamma. \]

Since \( \alpha (d - 5 \sqrt{d}) \leq \alpha k \), we conclude that

\[ \mathbb{P} [\pi (y, \alpha) < \pi (x, \alpha)] \geq \frac{\gamma^2}{3}. \]

By (11), setting \( \varepsilon := \gamma^2/6 \) finishes the proof. \( \square \)

In the remainder of the proof we fix a constant \( \varepsilon \) satisfying the conclusion of Claim 7.15. We shall assume that the ratio \( c/\varepsilon \) is sufficiently small, say \( c < \varepsilon/10 \). To simplify the presentation, we will use the asymptotic notation \( O(\cdot) \) to hide positive constant factors (which may be computed as functions of \( \varepsilon \) and \( c \)).

**Claim 7.16.** For every vertex \( w \in B \), we have
\[ \mathbb{P} [ w \text{ is happy } | w \text{ is } \varepsilon\text{-powerful}] \geq 1 - \exp(-O(\sqrt{d})). \]

**Proof.** Let us fix the values \( \vartheta(v) \) for \( v \in N_A(w) \setminus S_w \) so that the vector \( x_w \) is \( \varepsilon \)-powerful. Now consider any \( u \in S_w \). The value \( \vartheta(u) \) is chosen uniformly at random from \([0, 1]\). Moreover, since \( G \) is triangle-free, \( u \) and \( w \) have no common neighbors, which means that the values \( \vartheta(v) \) for \( v \in N_A(u) \) have not yet been determined. In other words, \( x_u \) is a uniformly random vector from \([0, 1]^{\deg_A(u)}\). Since \( x_w \) is \( \varepsilon \)-powerful and \( \deg_A(u) \leq d \),

\[ \mathbb{P} [u \text{ is successful}] = \mathbb{P} [\pi(x_u, \vartheta(u)) < \pi(x_w, \vartheta(u))] \geq \varepsilon. \]

Since \( G \) has girth at least 5, the vertices in \( S_w \) have no common neighbors except \( w \), and thus the random events “\( \pi(x_u, \vartheta(u)) < \pi(x_w, \vartheta(u)) \)” for \( u \in S_w \) are mutually independent. Therefore, the random variable \( \xi \) equal to the cardinality of the preimage of \( w \) under the function \( \text{save} \) is bounded below by a binomial random variable with distribution \( \text{Bin}(|S_w|, \varepsilon) \). Hence, we may apply the Chernoff bound (Theorem 7.2) and the inequality \( \frac{\xi}{d} \geq \frac{\sqrt{d}}{8} \) to conclude that

\[ \mathbb{P} [w \text{ is not happy}] = \mathbb{P} [\xi < c\sqrt{d}] < 2 \exp\left(-\left(\frac{\varepsilon}{8} - c\right)^2 \frac{8\sqrt{d}}{3\varepsilon}\right) \leq \exp(-O(\sqrt{d})). \]

\[ \square \]

For a vertex \( u \in V(G) \), define

\[ P_\varepsilon(u) := \{ w \in N_B(u) : w \text{ is } \varepsilon\text{-powerful} \}. \]

**Claim 7.17.** For every vertex \( u \in V(G) \), we have

\[ \mathbb{P} [|P_\varepsilon(u)| \geq c\sqrt{d}] \geq 1 - \exp(-O(\sqrt{d})). \]

**Proof.** A slight technical issue here arises from the fact that the vectors \( x_w \) for \( w \in N_B(u) \) may not be probabilistically independent of each other, since each of them may include \( \vartheta(u) \) as one of the coordinates. To remedy this, we define for every \( w \in N_B(u) \) a vector \( x_w' \) as follows:

\[ x_w' := (\vartheta(v) : v \in N_A(w) \setminus (S_w \cup \{u\})). \]

That is, \( x_w' \) is obtained from \( x_w \) by deleting the coordinate corresponding to \( u \). Let

\[ P'_\varepsilon(u) := \{ w \in N_B(u) : x_w' \text{ is } \varepsilon\text{-powerful} \}. \]

Then \( P'_\varepsilon(u) \subseteq P_\varepsilon(u) \), so it suffices to argue that

\[ \mathbb{P} [|P'_\varepsilon(u)| \geq c\sqrt{d}] \geq 1 - \exp(-O(\sqrt{d})). \]
For \( w \in N_B(u) \), let \( k(w) := |N_A(w) \setminus (S_w \cup \{u\})| \). Then, by (8) and since \(|S_w| = \lceil \sqrt{d}/8 \rceil\), we have

\[
k(w) \geq d - 3\sqrt{d} - \lceil \sqrt{d}/8 \rceil - 1 \geq d - 5\sqrt{d}.
\]

By the choice of \( \varepsilon \) and since \( x'_w \) is drawn uniformly at random from \([0, 1]^{k(w)}\), we conclude that

\[
P[x'_w \text{ is } \varepsilon \text{-powerful}] \geq \varepsilon.
\]

As \( G \) has girth at least 5, the vertices in \( N_B(u) \) have no common neighbors except \( u \), so we can apply the Chernoff bound and the inequality \( \deg_\mathcal{B}(u) \geq \sqrt{d} \) to obtain the desired bound

\[
P[|P'_\varepsilon(u)| < c\sqrt{d}] < 2\exp\left(-\frac{(\varepsilon - c)^2 \sqrt{d}}{3\varepsilon}\right) \leq \exp(-O(\sqrt{d})�).
\]

Finally, we can bound the probability of each event \( X_u \):

**Claim 7.18.** Let \( u \in V(G) \). Recall that \( X_u \) is the event that \( \deg_H(u) < c\sqrt{d} \). Then

\[
P[X_u] \leq \exp(-O(\sqrt{d})�).
\]

**Proof.** By Claim 7.16, for each \( w \in N_B(u) \), we have

\[
P[w \in P'_\varepsilon(u) \setminus H] = P[w \text{ is } \varepsilon \text{-powerful but not happy}] \leq \exp(-O(\sqrt{d})�).
\]

Therefore, by the union bound,

\[
P[P'_\varepsilon(u) \setminus H \neq \emptyset] \leq \deg_\mathcal{B}(u) \cdot \exp(-O(\sqrt{d})) \leq \exp(-O(\sqrt{d})�).
\]

Hence, by Claim 7.17,

\[
P[X_u] \leq P[|P'_\varepsilon(u)| < c\sqrt{d}] + P[P'_\varepsilon(u) \setminus H \neq \emptyset] \leq \exp(-O(\sqrt{d})�).
\]

The upper bound on \( P[X_u] \) given by Claim 7.18 implies the asymptotic bound (9). As discussed earlier, this yields Claim 7.13 and completes the proof of Theorem 1.12.

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