Differentiations of operator algebras over non-archimedean fields.

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Abstract

Differentiations of operator algebras over non-archimedean spherically complete fields are investigated. Theorems about a differentiation being internal are demonstrated.

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1 Introduction.

Differentiations of operator algebras over the complex field were investigated in [4, 13, 8]. It was shown that derivations of $C^*$-algebras and von Neumann algebras are internal. But the case of operator algebras over non-archimedean fields was not studied yet.

This article continuous previous investigations of operator algebras over non-archimedean fields (see [1] and references therein), where their spectral theory was described. The present paper is devoted to investigations of derivations of operator algebras over infinite spherically complete fields with non-trivial non-archimedean multiplicative norms having values in $\Gamma \cup \{0\}$.

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where $\Gamma$ is a discrete multiplicative group, particularly over locally compact fields. Theorems about a differentiation being internal are demonstrated.

All results of this paper are obtained for the first time.

2 Differentiations of operator algebras

1. Definitions. Suppose that $F$ is an infinite field supplied with a non-archimedean non-trivial multiplicative norm relative to which it is complete as a uniform space. Let $X$ be a Banach space over $F$, denote by $L(X) = L(X, X)$ a Banach space of all continuous $F$-linear operators from $X$ into $X$.

An algebra $\Phi$ contained in $L(X)$ over $F$ such that $A^t \in \Phi$ for each $A \in \Phi$ will be called an algebra with transposition, where a mapping $A \mapsto A^t$ on $\Phi$ is called a transposition, if it is $F$-linear and $(A^t)^t = A$ and $(AB)^t = B^t A^t$ for each $A, B \in \Phi$.

A Banach algebra with transposition is called a $T$-algebra.

A subalgebra of $L(X)$ will be called an operator algebra.

An operator $A \in \Psi$ is called symmetric if $A^t = A$.

An $F$-linear continuous mapping $D : \Psi \to \Psi$ on an algebra $\Psi$ over $F$ is called a derivation of $\Psi$ if $D(AB) = D(A)B + AD(B)$ for each $A, B \in \Psi$.

For subalgebras $\Phi$ and $\Psi$ of $L(X)$ satisfying the condition if $A \in \Phi \cup \Psi$ then $A^t \in L(X)$ let $T\{\Phi, \Psi\}$ denote the minimal $T$ subalgebra in $L(X)$ containing $\Phi$ and $\Psi$.

A norm of an operator $A \in L(X)$ is defined as

$$\|A\| := \sup_{x \neq 0} \|Ax\|/\|x\|.$$  

A homomorphism $\phi : \Psi \to L(X)$ such that it is $F$-linear, $\phi(aA + bB) = a\phi(A) + b\phi(B)$, and multiplicative $\phi(AB) = \phi(A)\phi(B)$ and $\phi(A^t) = [\phi(A)]^t$ for each $A, B \in \Psi$ and $a, b \in F$ is called a representation of a $T$ algebra on $X$. If additionally $\phi$ is bijective, then such representation $\phi$ is called faithful.

2. Remark. Henceforth, operator $T$-algebras are considered. Denote by $c_0(\alpha, F)$ the Banach space of all mappings $x : \alpha \to F$ satisfying the condition that for each $\epsilon > 0$ the set $\{j : j \in \alpha; |x_j| > \epsilon\}$ is finite, where $c_0(\alpha, F)$ is supplied with norm $\|x\| := \sup_{j \in \alpha}|x_j|, \ x_j = x(j), \ \alpha$ is a set. That is either $\Gamma_F := \{|x| : x \in F \setminus \{0\}\}$ is discrete or $\text{card}(\alpha) < \aleph_0$. 

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If $X = c_0(\alpha, F)$ and $A \in L(X)$, then a transposed operator $A^t$ can be defined by the equality $A^t_{j,k} = A_{k,j}$ for each $j, k \in \alpha$, where $Ae_k = \sum_{j \in \alpha} A_{k,j}e_j$ with $A_{k,j} \in F$, $e_j \in c_0(\alpha, F)$ denotes the basic vector $e_j(k) = \delta_{k,j}$ for each $k \in \alpha$, $\delta_{k,j} = 0$ for $j \neq k$, while $\delta_{j,j} = 1$. For $X = c_0(\alpha, F)$ this operation $A \mapsto A^t$ will serve as the transposition if $A$ and $A^t$ are in $L(X)$.

If $F$ is a spherically complete non-archimedean field with discrete multiplicative group $\Gamma_F$ or $X$ is finite dimensional over $F$, then a Banach space $X$ over $F$ is isomorphic with $c_0(\alpha, F)$ (see Theorems 5.13 and 5.16 [12]). Henceforward, it is supposed that either a spherically complete field $F$ is locally compact or it contains a family $\{G_\alpha : \alpha \in \mu\}$ of locally compact subfields $G_\alpha$ such that their union is dense in $F$, that is $\bigcup_\alpha G_\alpha = F$, where $\bar{A}$ denotes the completion of a subset $A$ relative to the uniformity inherited from $F$.

Henceforth, it is supposed that a Banach space $X$ is isomorphic with $c_0(\alpha, F)$.

Let $C_\infty(\Lambda, F)$ denote a Banach algebra of all continuous functions $f : \Lambda \to F$ such that for each $\epsilon > 0$ there exists a compact subset $V$ in $\Lambda$ for which $|f(x)| \leq \epsilon$ for every $x \in \Lambda \setminus V$, where $\Lambda$ is a zero-dimensional locally compact Hausdorff space, while $C(\Lambda, F)$ denotes the algebra of all continuous functions $f : \Lambda \to F$. If a Banach algebra $\Psi$ is isomorphic with $C_\infty(\Lambda, F)$, then it is called a $C$-algebra.

If $F$ is a field with a multiplicative norm and $\Lambda$ is a subset in $F$, a space of all continuous functions $f : \Lambda \to F$ so that for each $\epsilon > 0$ a positive number $0 < r < \infty$ exists for which $|f(x)| < \epsilon$ for each $x \in \Lambda$ with $|x| > r$ is denoted by $C_\infty(\Lambda, F)$. That is $C_\infty(\Lambda, F)$ is a space of continuous functions tending to zero at infinity.

Evidently, each $C$-algebra is a $T$-algebra.

If a Banach space $X$ is over a spherically complete field $F$ and $X^*$ is its topological dual Banach space, i.e. of all continuous $F$-linear functionals $y^* : X \to F$, then each $A \in L(X, Y)$ has an adjoint operator $A^* : Y^* \to X^*$, where $Y$ is a Banach space over $F$, $A^* \in L(Y^*, X^*)$. On the other hand, $X^*$ is the Banach space over the spherically complete field $F$ and hence isomorphic with $c_0(\beta, F)$ for some set $\beta$. But each vector $x \in X = c_0(\alpha, F)$ gives rise to
a continuous \( F \)-linear functional \( x^* z := \sum_{j \in \alpha} x_j z_j \) for each \( z \in X \). Therefore, the natural embedding \( X \hookrightarrow X^* \) exists, that is \( \alpha \subset \beta \). This implies, that the operation \( L(X) \ni A \mapsto A^* \in L(X^*) \) can be considered as an extension of \( A \mapsto A^t \) from \( X \) onto \( X^* \) for each \( A \in L(X,X) = L(X) \) (see also Chapter 3 in [12]).

For a Banach space \( X \) over a spherically complete field \( F \) each closed linear subspace \( Y \) is orthocomplemented in accordance with Theorems 5.13 and 5.16 [12]. Therefore, in such case it is written below for short a projection \( \pi_Y : X \to Y \) instead of an orthoprojection, where \( \pi_Y(X) = Y \) (see also [1]).

3. Lemma. If \( \Phi \) is a \( C \)-algebra over \( F \) and \( D \) is its differentiation, then \( D = 0 \) on it.

Proof. In the space \( C_\infty(\Lambda, F) \) an \( F \)-linear subspace of simple functions

\[
f(x) = \sum_{j=1}^n a_j \chi_{B_j}
\]

is dense, where \( a_j \in F \), \( B_j \) is a clopen subset in \( \Lambda \), \( \chi_B \) denotes the characteristic function of a subset \( B \) in \( \Lambda \), that is \( \chi_B(x) = 1 \) for each \( x \in B \), while \( \chi_B(x) = 0 \) for any \( x \in \Lambda \setminus B \). Then \( D(\chi_B) = D(\chi_B^2) = 2D(\chi_B)\chi_B \), consequently, \( D(\chi_B)(1 - 2\chi_B) = 0 \) and hence \( D(\chi_B) = 0 \). A differentiation \( D \) is \( F \)-linear and continuous, consequently, \( D(f) = 0 \) for each \( f \in C_\infty(\Lambda, F) \).

4. Lemma. If \( A \in L(X) \), where \( X \) is Banach space over a locally compact field \( F \), and an operator \( A \) is such that \( \overline{F(A)} \) is a least closed \( C \)-subalgebra of \( L(X) \) containing \( A \), and a spectrum of \( \overline{F(A)} \) is contained in a closed ball \( B(F, 0, \|A\|) \) containing 0 in \( F \) of radius \( \|A\| \), \( D : \overline{F(A)} \to L(X) \) and \( \pi_{\overline{F(A)}} D : \overline{F(A)} \to \overline{F(A)} \) are differentiations, then \( \pi_{\overline{F(A)}} DB = 0 \) for each \( B \in \overline{F(A)} \), where \( \pi_{\Psi} : L(X) \to \Psi \) denotes an \( F \)-linear projection on a closed subalgebra \( \Psi \) in \( L(X) \).

Proof. The field \( F \) is locally compact, consequently, it is spherically complete. Therefore, the Banach subspace \( \overline{F(A)} \) in \( L(X) \) is orthocomplemented in the non-archimedean sense and the continuous \( F \)-linear projection \( \pi_{\overline{F(A)}} \) exists (see Chapter 5 [12]). Take a closed ball \( B(F, 0, \|A\|) \) in the field \( F \), where \( B(Y, z, r) := \{ x \in Y : \rho(x, z) \leq r \} \) denotes a closed ball with center \( z \) of radius \( 0 < r \) in a metric space \( Y \) with a metric \( \rho \). Since the field \( F \) is locally compact, this ball \( B(F, 0, \|A\|) \) is compact. So the \( C \)-
algebra \( C(B(\mathbf{F}, 0, \|A\|), \mathbf{F}) \) of all continuous functions \( f : B(\mathbf{F}, 0, \|A\|) \to \mathbf{F} \) exists. For each polynomial \( P_n(x) \) of degree \( n \) on \( B(\mathbf{F}, 0, \|A\|) \) the corresponding operator \( P_n(A) \) is defined, where \( A^0 = I \) is the unit operator, \( A^n x = A(A^{n-1} x) \) for each \( x \in X \). The \( \mathbf{F} \)-linear space of polynomials is dense in \( C(B(\mathbf{F}, 0, \|A\|), \mathbf{F}) \) in accordance with Kaplansky’s theorem 43.3 [13]. By the conditions of this lemma a spectrum of a \( C \)-algebra \( \mathbf{F}(A) \) is contained in a closed ball \( B(\mathbf{F}, 0, \|A\|) \). Therefore, \( f(A) \) is defined for each continuous function \( f : B(\mathbf{F}, 0, \|A\|) \to \mathbf{F} \) and \( \mathbf{F}(A) \) is contained in \( C(B(\mathbf{F}, 0, \|A\|), \mathbf{F}) \) as the closed subalgebra. Certainly, \( \pi_{\mathbf{F}(A)} DA \in \mathbf{F}(A) \) and \( \pi_{\mathbf{F}(A)} DB = \pi_{\mathbf{F}(A)} DA B + A \pi_{\mathbf{F}(A)} DB \) for each \( A, B \in \mathbf{F}(A) \), hence \( \pi_{\mathbf{F}(A)} D \) is the continuous differentiation on \( \mathbf{F}(A) \), since the operators \( \pi_{\mathbf{F}(A)} \) and \( D \) are continuous. A closed subalgebra of a \( C \)-algebra is a \( C \)-algebra by Corollary 6.13 [12]. Therefore, by the preceding lemma the differentiation \( \pi_{\mathbf{F}(A)} \) on \( \mathbf{F}(A) \) is degenerate.

5. **Definition.** Let \( \rho : \Psi \to \mathbf{F} \) be a linear continuous functional on a \( T \)-algebra \( \Psi \) over \( \mathbf{F} \). If \( \rho(A^f) = \rho(A) \) for each \( A \in \Psi \), then \( \rho \) will be called symmetric. If a symmetric continuous functional \( \rho \) is such that \( \rho(I) = 1 \), then \( \rho \) is called a state of \( \Psi \). A state \( \rho \) of a \( T \)-algebra \( \Psi \) is definite on a symmetric operator \( A \), when \( \rho(A^n) = [\rho(A)]^n \) for every natural number \( n \).

A functional \( \rho \) is called multiplicative on a \( T \)-algebra \( \Psi \), if \( \rho(AB) = \rho(A) \rho(B) \) for each \( A, B \in \Psi \).

6. **Lemma.** Let \( A \) be an operator in a Banach algebra \( \Psi \) over a locally compact field \( \mathbf{F} \) and let a continuous linear functional \( \rho : \overline{\mathbf{F}(A)} \to \mathbf{F} \) be multiplicative on \( A^n \) for each \( n \in \mathbb{N} \). Suppose that \( \rho \) has a \( \mathbf{K} \)-linear extension on \( C(B(\mathbf{F}, 0, \|A\|), \mathbf{K}) \), where a field \( \mathbf{K} \) is an extension of a field \( \mathbf{F} \). Then \( \rho(f(A)) = f(\rho(A)) \) for each \( f \in C(B(\mathbf{F}, 0, \|A\|), \mathbf{K}) \).

**Proof.** From \( \rho(A^n) = [\rho(A)]^n \) for every \( n \in \mathbb{N} \) it follows that \( \rho \) is multiplicative on the Banach algebra \( \overline{\mathbf{F}(A)} \) generated by \( A \). But \( \overline{\mathbf{F}(A)} \) is the Banach algebra having an isometric embedding into \( C(B(\mathbf{F}, 0, \|A\|), \mathbf{F}) \). For each polynomial \( P_m(x) \) on \( B(\mathbf{F}, 0, \|A\|) \) with values in \( \mathbf{K} \) due to multiplicativity and \( \mathbf{F} \)-linearity of \( \rho \) one gets \( \rho(P_m(A)) = P_m(\rho(A)) \). From Kaplansky’s theorem and continuity of \( \rho \) it follows that \( \rho(f(A)) = f(\rho(A)) \) for each continuous function \( f : B(\mathbf{F}, 0, \|A\|) \to \mathbf{K} \).
7. Lemma. Let \( A \) be a symmetric operator in a \( T \)-algebra \( \Psi \) over a locally compact field \( F \) and let \( K \) be its extension so that \( \sqrt{x} \in K \) for every \( x \in F \) and for each natural number \( m \geq 2 \), then for a marked natural number \( n \geq 2 \) there exists \( B \in \Psi_K \) such that \( B^n = A \), where \( \Psi_K \) is an extension of an algebra \( \Psi \) over \( K \).

**Proof.** Suppose that a set \( \alpha \) is infinite. A net of projection operators \( \pi_\gamma \) on finite dimensional subspaces \( c_0(\gamma, F) \) in \( c_0(\alpha, F) \) exists, where \( \gamma \) are finite subsets in \( \alpha \) and their family \( \Upsilon \) is partially ordered by inclusion. Then \( B = \{ \beta : \beta = \alpha \setminus \gamma; \gamma \in \Upsilon \} \) is a filter base. For each \( x \in X = c_0(\alpha, F) \) the limit \( \lim_{\mathcal{B}} \pi_\gamma x = x \) exists. Therefore, \( \lim_{\mathcal{B}} \pi_\gamma A \pi_\gamma x = Ax \) for each \( x \in X \).

Each operator \( \pi_\gamma A \pi_\gamma \) is compact from \( X \) into \( X \). In view the decomposition theorem of compact operators (see Lemma 1 and Note 2 of Appendix A in [9]) it has the decomposition

\[
(1) \quad \pi_\gamma A \pi_\gamma = C_{\gamma}^{-1} \Lambda_\gamma C_{\gamma} \quad \text{over} \quad K,
\]

where \( C_\gamma \) is an invertible operator on \( c_0(\gamma, K) \) and \( \Lambda_\gamma \) is a diagonal operator on \( c_0(\alpha, K) \) relative to its standard base, moreover, \( C_\gamma^t = C_\gamma^{-1} \) for a symmetric operator \( \pi_\gamma A \pi_\gamma \). The latter decomposition automatically encompasses the case of finite \( \alpha \) also. Thus \( P_n(\pi_\gamma A \pi_\gamma) \) is correctly defined for each polynomial \( P_n \) on \( F \) with values in \( K \) and \( \lim_{\mathcal{B}} P_n(\pi_\gamma A \pi_\gamma) x = P_n(A) x \) for every vector \( x \in X \).

From the embedding \( \overline{F} \hookrightarrow C(B(F, 0, \|A\|), F) \) (see §6) and the continuity of the function \( F \ni x \mapsto \sqrt{x} \in K \) it follows that \( B = \sqrt{A} \in \overline{K}(A) \). The latter algebra is contained in \( \Psi_K \).

8. Lemma. Let \( D \) be a derivation operator on a \( T \)-algebra \( \Psi \) over a field \( F \), let also \( K \) be an extension of \( F \) complete relative to its uniformity. Then \( D \) has a continuous \( K \)-linear extension on \( \Psi_K \) as a derivation operator.

**Proof.** A field \( K \) contains \( F \) as a subfield, a multiplicative non-archimedean norm on \( F \) has a multiplicative non-archimedean extension on \( K \) (see [12, 14]). The completion \( \hat{K} \) of \( K \) relative to this norm is a field complete relative to the norm. A Banach space \( X \) over a field \( F \) has an \( F \)-linear continuous embedding into \( X_K \), where \( X = c_0(\alpha, F) \) and \( X_K = c_0(\alpha, \hat{K}) \).

Then the closure of the \( \hat{K} \)-linear span of \( \Psi \) in \( L(X_K) \) gives \( \Psi_K \) such that the embedding \( \Psi \hookrightarrow \Psi_K \) is continuous relative to the operator norm. By
the condition of this lemma $K$ is complete relative to its uniformity, hence $K = K$ and $\Psi_K = \Psi_K$.

Put $DbA = bDA$ for each $b \in K$ and $A \in \Psi$. Therefore, $D(bAB) = bD(AB) = b(DA)B + bADB$ for any $b \in K$ and $A, B \in \Psi$. Moreover, $\|D(bAB)\| \leq |b| \max(\|DA\|, \|A(DB)\|)$ and $(DbA + tB) \leq \max(|b| \|DA\|, |t| \|DB\|)$ for each $b, t \in K$ and $A, B \in \Psi$, consequently, $D$ has a continuous $K$-linear extension as a derivation operator on the $K$-linear span of $\Psi$ in $L(X_K)$ and hence on its completion $\Psi_K$.

9. Lemma. Suppose that $D$ is a derivation of a $T$-algebra $\Psi$ over a field $F$ having an extension up to a derivation on $\Psi_K$, where a field $K$ is an extension of $F$. Let $\rho$ be a definite state on a symmetric operator $A$ and $A = B^2$ for some symmetric operator $B \in K(A)$ and $\rho$ has an extension as a state on $\Psi_K$ and either $DB \in K(A)$ or $\rho(B(DB)) = \rho(B)\rho(DB)$ and $\rho((DB)B) = \rho(DB)\rho(B)$. Then $\rho(DA) = 0$.

Proof. The differentiation operator $D$ is $F$-linear, hence $DA = D(A - \rho(A)I)$, since $DI = 0$. Therefore, without loss of generality it is sufficient to consider the case $\rho(A) = 0$, since $\rho(A - \rho(A)I) = 0$. Consider an operator $B = A^{1/2}$, i.e. $A = B^2$. A state $\rho$ is multiplicative on the $T$-algebra $F(A)$ generated by $A$, since $\rho(A^n) = [\rho(A)]^n$ for each natural number $n$. Thus $\rho(B) = 0$ and hence $\rho(DA) = \rho((DB)B) + \rho(B(DB)) = \rho(B)\rho(DB) + \rho(DB)\rho(B) = 0$, since either $DB \in K(A)$ or $\rho(B(DB)) = \rho(B)\rho(DB)$ and $\rho((DB)B) = \rho(DB)\rho(B)$.

10. Theorem. Suppose that $D$ is a derivation of a $T$-algebra $\Psi$ over a locally compact field $F$ such that $\sqrt{x} \in K$ for each $x \in F$, where a field $K$ is an extension of $F$, and that $\mathcal{Z}$ is a center of $\Psi$. Then $D$ annihilates $\mathcal{Z}$.

Proof. Since $\Psi$ is a Banach algebra, its center $\mathcal{Z}$ is closed in $\Psi$. The field $F$ is spherically complete, consequently, $\mathcal{Z}$ is complemented in $\Psi$. Take $A \in \mathcal{Z}$ and consider $F(A)$ which is complemented in $\mathcal{Z}$ and hence in $\Psi$. Any element $A \in \Psi$ can be presented as $A = A_1 - A_2$, where $A_1^t = A_1$ is symmetric and $A_2^t = -A_2$ is antisymmetric, $A_1 = \frac{A + A^t}{2}$, $A_2 = \frac{A - A^t}{2}$. If $A \in \mathcal{Z}$, then $A_1, A_2 \in \mathcal{Z}$. Then $\sqrt{-1} = i \in K$ and $(iA_2)^t = -A_2$, where $i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ over the field $F$, when $i \notin F$. On the other hand, if $i \in F$, one can consider $\Psi \oplus \Psi$ on $X \oplus X$, where $\begin{pmatrix} 0 & A_2 \\ -A_2 & 0 \end{pmatrix}$ is symmetric, when $A_2$ is antisymmetric.
Therefore, it is sufficient to consider the case of symmetric \( A \in \mathcal{Z} \).

Choose any multiplicative \( F \)-linear continuous functional \( \rho \) on \( \overline{F(A)} \) so that \( \rho(I) = 1 \) and \( \rho(A) \neq 0 \). Consider a projection \( \pi_Y DA \) of \( DA \) onto a Banach subspace \( Y = \Psi \oplus \overline{F(A)} \), i.e. \( \Psi = Y \oplus \overline{F(A)} \). Take any continuous extension of \( \rho \) so that \( \rho(\pi_Y DA) \neq 0 \) and such that \( \rho \) is multiplicative on \( \overline{F(A, \pi_Y DA)} \), where \( \overline{F(A_1, \ldots, A_n)} \) denotes a minimal closed subalgebra of \( \Psi \) containing elements \( A_1, \ldots, A_n \in \Psi \). This is possible, since \( \overline{F(A, \pi_Y DA)} \) is the algebra with two commuting generators \([A, \pi Y DA] = 0\). Moreover, the inclusion \( A \in \mathcal{Z} \) implies \( \overline{F(A)} \subset \mathcal{Z} \) and \( \Psi/\mathcal{Z} = (\Psi/\overline{F(A)})/(\mathcal{Z}/\overline{F(A)}) \) and \( (\pi_Y DA) + \overline{F(A)} = \theta(\pi_Y DA) = DA + \overline{F(A)} \), where \( \theta : \Psi \to \Psi/\overline{F(A)} \) denotes the quotient mapping. Then we also get \( \theta(C) = C + \overline{F(A)} \) and \( D(AB) + \overline{F(A)} = (DA)B + A(DB) + \overline{F(A)} = (DA + \overline{F(A)})(B + \overline{F(A)}) + (A + \overline{F(A)})(DB + \overline{F(A)}) = \theta(D(AB)) = \theta(DA)\theta(B) + \theta(A)\theta(DB) \), consequently, \( \theta \circ D \) is the differentiation on the quotient algebra \( \Psi/\overline{F(A)} \).

If \( V \in \overline{F(\pi_Y DA)} \ominus \overline{F(A)} \) is a non zero element and \( V^n \in \overline{F(A)} \) for some natural number \( n \geq 2 \), then take an algebraically closed field \( K \) containing \( F \) so that \( \sqrt[n]{x} \in K \) for each \( x \in F \). Therefore, \( V \in \overline{K(A)} \) and one can take \( \rho(V) = \sqrt[n]{f(\rho(A))} \) with \( Q = V^n = f(A) \in \overline{F(A)} \), where \( f \) is a continuous function from \( B(F, 0, 1) \) into \( F \) (see Lemmas 6 and 7). Thus it remains to treat the variant when \( V^n \notin \overline{F(A)} \) for each natural number \( n \).

For this it is sufficient to choose a multiplicative extension of \( \rho \) on \( \overline{F(\pi_Y DA)} \ominus \overline{F(A)} \) putting \( \rho(V^n) = [\rho(V)]^n \neq 0 \) for each natural number \( n \) and for some non zero element \( V \in \overline{F(\pi_Y DA)} \ominus \overline{F(A)} \). Indeed, without loss of generality using multiplication on scalars \( A \mapsto bA \) for \( b \in F \setminus \{0\} \) it is possible to restrict on the case \( \max(\|A\|, \|V\|) < 1 \) and choose \( 0 < |\rho(A)| \leq \|A\| \) and \( 0 < |\rho(V)| \leq \|V\| \). The Banach subspace \( \overline{F(A, \pi_Y DA)} \) is closed in \( \Psi \), consequently, by the non-archimedean Hahn-Banach theorem over the spherically complete field \( F \) a functional \( \rho \) has a continuous extension on \( \Psi \) (see [12] and §8.203 in [10]).

The family of such functionals \( \rho \) separates different elements of \( \Psi \) and \( Q \in \Psi \ominus \overline{F(A, \pi_Y DA)} \), hence \( \rho(DA) = 0 \) for each such \( \rho \) if and only if \( DA = 0 \).

Applying Lemmas 6-9 we get the statement of this theorem.
11. Definition. If $\Psi$ is a $T$-algebra on a Banach space $X$ over a field $F$, its strong topology is characterized by a base of neighborhoods $V_{x_1,\ldots,x_n;\epsilon} := \{ A \in \Psi : \|Ax_j\| < \epsilon \ \forall j = 1,\ldots,n \}$ of zero, where $x_1,\ldots,x_n \in X$, $\epsilon > 0$, $n \in \mathbb{N}$. If a field $F$ is spherically complete and $X^*$ is a topological dual space of $X$, a weak topology on $\Psi$ is given by a base of neighborhoods $W_{x_1,\ldots,x_n;y_1,\ldots,y_n;\epsilon} := \{ A \in \Psi : |y_jAx_j| < \epsilon \ \forall j = 1,\ldots,n \}$ of zero, where $x_1,\ldots,x_n \in X$, $y_1,\ldots,y_n \in X^*$, $\epsilon > 0$, $n \in \mathbb{N}$. Denote by $\bar{\Psi}$ the completion of $\Psi$ relative to the weak topology.

12. Lemma. Suppose that $D$ is a derivation of a $T$-algebra $\Psi$ on a Banach space $X$ over a spherically complete field $F$. Then a unique weakly continuous extension $\bar{D}$ of $D$ on $\bar{\Psi}$ exists.

Proof. The mappings $A \mapsto \frac{A^t + A}{2} =: A_1$ and $A \mapsto \frac{A^t - A}{2} =: A_2$ are continuous on a $T$-algebra $\Psi$. An extension $K$ from Lemma 7 of a spherically complete field $F$ can be considered as an $F$-linear space. By Lemma 8 $D$ has a continuous extension on $\Psi_K$ as a derivation operator. As in Lemma 10 it is sufficient to consider a symmetric operator $A$.

Put $S := \{ A \in \Psi : \|A\| \leq 1, \ A^t = A \}$ to be the unit ball of symmetric operators. Then the mapping $S \ni A \mapsto y(D(A^2)x) = y(ADAx + (DA)Ax)$ is strongly continuous at zero, since $|y(ADAx + (DA)Ax)| \leq \|D\| \max(\|Ax\|\|y\|; \|x\|\|Ay\|)$, where $x,y \in X$ and $X$ is embedded into $X^*$. On the other side, the mapping $S \ni A \mapsto A^{1/2}$ is strongly continuous at zero, since $\|A^{1/2}x\| \leq \|Ax\|\|x\|$ due to Formula 7(1), where $x \in X$. Thus $A \mapsto y(DA_1x) - y(DA_2x) = y(DAx)$ is strongly continuous at zero on $S$. This implies that $H := S \cap q^{-1}(B(F,0,1))$ is strongly closed in $S$, where $q(A) := y(DAx)$ for some marked vectors $x,y \in X$, $0 < r < \infty$.

Recall that a subset $U$ of a topological $F$-linear space $Q$ is called absolutely $F$-convex if $B(F,0,1)U + B(F,0,1)U \subset U$.

The norm on $F$ is non-archimedean, i.e. $|a + b| \leq \max(|a|,|b|)$ for each $a,b \in F$. It can be lightly seen, that the set $H$ is absolutely $F$-convex and strongly closed, consequently, $H$ is weakly closed in $S$. Indeed, if a net $T_n \in H$ strongly converges to $T \in H$, then $T_n - T \in H$ for each $n$ and hence the net $(T_n - T)$ strongly converges to zero. Therefore, $y((T_n - T)x)$ converges to zero for each $x \in X$ and $y \in X^*$. 

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By the non-archimedean Hahn-Banach theorem 8.203 [10] the set $H$ is closed relative to a weak topology of finite combinations $b$ of operators relative to the weak operator topology of finite combinations $b$ of operators.

Therefore, the derivation $D$ is weakly continuous on $B(\Phi, 0, 1)$, since the mapping $B(\Phi, 0, 1) \ni A \mapsto y(DAx)$ is continuous for each marked $x, y \in X$ and the derivation operator $D$ is $F$-linear. This means that $D$ is uniformly continuous relative to the weak uniformity on $B(\Phi, 0, 1)$ and implies that $D$ has a continuous extension on $\overline{B(\Phi, 0, 1)}$ and hence on $\overline{\Phi}$ with range in $\overline{\Phi}$ by Theorem 8.3.10 [2], since $\overline{B(\Phi, 0, 1)}$ is the closed absorbing set in $\overline{\Phi}$.

This extension is $F$-linear as well, since
\[
\lim_n D(bT_n + H_n) = b \lim_n DT_n + \lim_n DH_n \quad \text{for each } b \in F \text{ and } T_n, H_n \in \Phi
\]
with $\lim_n T_n = T \in \Phi$ and $\lim_n H_n = H \in \Phi$. Moreover, $y(D(T_nH_n)x) = y((DT_n)H_nx) + y(T_n(DH_n)x)$ for each $x \in X$ and $y \in X^*$, consequently,
\[
\lim_n \lim_k y((DT_n)H_kx) + y(T_n(DH_k)x) - y(D(T_nH_k)x) = y(D(TH)x) - y(D(TH)x) = 0,
\]
where $D$ is the bounded operator on $\Phi$ and weakly continuous on $\overline{\Phi}$, hence $D$ is the derivation on $\overline{\Phi}$ as well.

13. Definitions. A $T$-algebra of bounded operators on a Banach space $X$ over a spherically complete field $F$ closed relative to the weak operator topology and containing the unit operator will be called a $W^T$-algebra. For an operator $A \in L(X)$ and a $W^T$-algebra $\Psi$ let $\overline{\Psi}(A)$ denote the closure relative to the weak operator topology of finite combinations $b_1 B_1 + \ldots + b_n B_n$ of operators $B_j = V_j A V_j^T$, where $V_j$ is an isometry operator on $X$ for every $j$, i.e. $\|V_j x\| = \|x\|$ for each $x \in X, b_1, \ldots, b_n \in B(F, 0, 1)$.

If $\Upsilon$ is a family of operators in $L(X)$, then $\Upsilon' := \{C : C \in L(X); [C, T] = 0 \ \forall T \in \Upsilon\}$ denotes the commutant of $\Upsilon$, where $[C, T] = CT - TC$ is the commutator of two operators.

A center $Z(\Psi)$ of an algebra $\Psi$ is a set of all its elements commuting with each element in $\Psi$. An element $A \in Z(\Psi)$ in the center is called central.

14. Lemma. Let $A$ be a linear continuous operator $A : X \to X$ on a Banach space over a spherically complete field $F$ and let $G$ be a locally compact field contained in $F$. Suppose that $f \in C_\infty(F, F)$ is a continuous
function tending to zero at infinity the restriction of which \( f|_G \) belongs to \( C_\infty(G, G) \). Then a linear continuous bounded operator \( f(A) \in L(X) \) exists.

**Proof.** The field \( \mathbb{Q}_p \) of \( p \)-adic numbers is locally compact. Let \( G \) be a locally compact field so that \( \mathbb{Q}_p \subset G \subset \mathbb{F} \), i.e. either a locally compact field \( G \) containing \( \mathbb{Q}_p \) or the \( p \)-adic field itself. Then each \( \mathbb{F} \)-linear operator is also \( G \)-linear.

Let \( X_G \) denote the Banach space over \( G \) obtained from the Banach space \( X \) over \( \mathbb{F} \) considering \( \mathbb{F} \) as the Banach space over \( G \), i.e. by the restriction of the field of scalars. Take \( P \) a projection \( P \in P_G \) on a finite-dimensional over \( G \) subspace in \( X_G \) with \( P_G \) denoting the family of all projections having finite dimensional ranges partially ordered by inclusion of their ranges in \( X_G \).

Then each operator \( P AP \) can be reduced to the diagonal form

(1) \( PAP = CTE \)

over \( G \) by a lower and upper triangular operators \( C \) and \( E \) respectively invertible on \( PX \) with diagonal operator \( T \) such that \((C - I)\) and \((E - I)\) are nilpotent operators on \( PX_G \) (see Lemma 1 of Appendix A in [9]).

In accordance with E. Zermelo’s theorem on each set \( \Lambda \) a relation exists, which well orders \( \Lambda \) (see [2]). Suppose that \( P_\beta \) is a family of projections on a Banach space over a spherically complete field \( \mathbb{F} \), where \( \beta \in \Lambda \) and a set \( \Lambda \) is well ordered and \( P_\alpha \leq P_\beta \) for each \( \alpha \leq \beta \). Denote by \( \land_{\alpha \in \Lambda} P_\alpha \) an projection from \( X \) onto the subspace \( \bigcap_{\alpha \in \Lambda} P_\alpha X \), while defining \( \lor_{\alpha \in \Lambda} P_\alpha := I - \land_{\alpha \in \Lambda} (I - P_\alpha) \), where \( I \) is the unit operator on \( X \), \( Ix = x \) for each \( x \in X \). Then the family \( Q_\alpha := P_\alpha - \lor_{\beta < \alpha} P_\beta \) consists of mutually orthogonal projections on \( X \) such that its sum is \( \lor_{\beta \in \Lambda} Q_\beta = \lor_{\beta \in \Lambda} P_\beta =: P \).

Indeed, \( Q_\beta \perp P_\alpha \) are orthogonal for each \( \alpha < \beta \) and \( Q_\beta \perp Q_\alpha \), since \( Q_\alpha \subseteq P_\alpha \), i.e. \( Q_\alpha X \subseteq P_\alpha X \). Therefore, \( \lor_{\alpha \in \Lambda} Q_\alpha \subseteq P \). If \( \alpha_1 \) is the least element of \( \Lambda \), then \( P_{\alpha_1} = Q_{\alpha_1} \). Suppose that

\[ P_\beta = \lor_{\alpha \leq \beta, \alpha \in \Lambda} Q_\alpha \]

for each \( \beta < \gamma \in \Lambda \). From the definition of \( Q_\gamma \) it follows, that

\[ Q_\gamma = P_\gamma - \lor_{\alpha < \gamma, \alpha \in \Lambda} Q_\alpha \], consequently,

\[ P_\gamma = I - \land_{\alpha \leq \gamma, \alpha \in \Lambda} (I - Q_\alpha) = \lor_{\alpha \leq \gamma, \alpha \in \Lambda} Q_\alpha \].

Thus by transfinite induction the latter equality is fulfilled for each \( \gamma \in \Lambda \), hence \( P \subseteq \lor_{\alpha \in \Lambda} Q_\alpha \), together with the opposite inclusion this implies \( P =
\( \forall \alpha \in \Lambda Q_\alpha. \)

The field \( \mathbf{F} \) is spherically complete and considered as a Banach space over \( \mathbf{G} \) is isomorphic with \( c_0(\beta, \mathbf{G}) \) for some set \( \beta \) by Theorems 5.13 and 5.16 [12]. Therefore, \( \lim_{p_{\mathbf{G}}} \mathbf{P} \mathbf{A} \mathbf{P} \mathbf{x} = \mathbf{A} \mathbf{x} \) for each \( \mathbf{x} \in \mathbf{X} \). To an operator \( \mathbf{Y} \in L(\mathbf{X}) \) an operator \( \mathbf{Y}_\mathbf{G} \in L(\mathbf{X}_\mathbf{G}) \) corresponds such that to each matrix element \( e_j^* \mathbf{Y} e_k \) over \( \mathbf{F} \) an operator block on \( c_0(\beta, \mathbf{G}) \) is posed.

Then \( \mathbf{C} - \mathbf{I} \) and \( \mathbf{E} - \mathbf{I} \) are nilpotent operators such that \( (\mathbf{C} - \mathbf{I})^l = 0 \) and \( (\mathbf{E} - \mathbf{I})^l = 0 \) for each \( l \geq m \), where \( m \) is an order of a square \( m \times m \) matrix with entries in \( \mathbf{G} \), i.e. \( m = \text{dim}_G \mathbf{P} \mathbf{X}_G \) is a dimension of \( \mathbf{P} \mathbf{X}_G \) over the field \( \mathbf{G} \), operators \( \mathbf{C} \) and \( \mathbf{E} \) are as in Formula 14(1). Therefore,

\[
(2) \quad C^k = \sum_{0 \leq h \leq \min(m,k)} \binom{k}{h} (C - I)^h, \\
\]

where \( (C - I)^0 = \mathbf{I} \) is the unit operator, as usually \( \binom{k}{h} = k!/(h!(k - h)!) \) denotes the binomial coefficient. Since \( \binom{k}{h} \) are integers, it follows that \( \binom{k}{h} \in \mathbf{G} \leq 1 \) and hence \( \|S(C)\| \leq \sup_{0 \leq h \leq \min(m,n)} |s_h| \|C - I\|^h < \infty \) for each polynomial

\[
(3) \quad S(x) = \sum_{k=0}^{n} s_k x^k \\
\]

on \( \mathbf{G} \) with coefficients \( s_k \in \mathbf{G}, \ s_n \neq 0 \), of degree \( n = \deg S \). Moreover, \( S(T) = \text{diag}(S(t_1), ..., S(t_m)) \) for the diagonal operator \( T = \text{diag}(t_1, ..., t_n) \) in the corresponding non-archimedean orthonormal basis in the subspace \( \mathbf{P} \mathbf{X}_\mathbf{G} \) over the field \( \mathbf{G} \), where \( t_1, ..., t_m \in \mathbf{G} \). On the other hand, applying Theorems 5.4, 5.11 and 5.16 [1] we get:

\[
(4) \quad \|S(P\mathbf{A}P)\| \leq \sup_{t \in \mathbf{G}, ||t|| \leq ||P\mathbf{A}P||} |S(t)|, \\
\]

since \( \|P\mathbf{A}P\| = \sup_{1 \leq v, t \leq m} |q_v^* P\mathbf{A}P q_t|, \ \|P\| = 1 \) for each non-degenerate projection operator, where \( q_j \) is a non-archimedean orthonormal basis in \( \mathbf{P} \mathbf{X}_\mathbf{G} \), \( q_j^* \in \mathbf{P} \mathbf{X}_\mathbf{G}^* \) denotes a \( \mathbf{G} \) linear functional corresponding to \( q_j \).

In view of Kaplansky’s theorem a family of polynomials is dense in \( C(B(\mathbf{G}, 0, r), \mathbf{G}) \) for each \( 0 < r < \infty \) for the locally compact field, since the ball \( B(\mathbf{G}, 0, r) \) is compact. For every \( f \in C_\infty(\mathbf{G}, \mathbf{G}) \) and each \( r = p^j \in \Gamma_\mathbf{G} := \{|x| : x \in \mathbf{G} \setminus \{0\}\} \) a sequence \( \{S_n(x) : k\} \) of polynomials exists uniformly converging to \( f \) on \( B(\mathbf{G}, 0, p^j) \), where \( n_j(k) < n_j(k + 1) \) for each \( k \in \mathbf{N}, \ n = \deg S_n. \)
By induction construct them such that \( \{n_{j+1}(k) : k \in \mathbb{N}\} \subset \{n_j(k) : k \in \mathbb{N}\} \) for each natural number \( j \in \mathbb{N} \). Choosing the diagonal subsequence \( \{n_j(j) : j \in \mathbb{N}\} \) one gets a sequence of polynomials \( S_{n_j(j)} \) point wise converging to \( f \) on \( G \) and uniformly converging to \( f \) on each bounded ball \( B(G, 0, r) \), since \( \lim_{|t| \to \infty} f(t) = 0 \). Since \( \|A\| < \infty \), the function

\[
(5) \quad f(A)x = \lim_{j \to \infty} \lim_{P \in \mathbb{P}_G} S_{n_j(j)}(C)S_{n_j(j)}(T)S_{n_j(j)}(E)x
\]

exists for each \( x \in X \), where \( C, T \) and \( E \) correspond to \( PAP \), \( P \in \mathbb{P}_G \).

Evidently it is linear by \( x \in X \), since \( \lim_{P \in \mathbb{P}_G} S_{n_j(j)}(C)S_{n_j(j)}(T)S_{n_j(j)}(E) \) is a linear operator on \( X \) over \( F \) for each \( j \). Since \( G \subset F \), Formulas (1 – 5) imply that

\[
(6) \quad \|f(A)\| \leq \sup_{t \in F, \|t\| \leq \|A\|} |f(t)| < \infty.
\]

15. **Theorem.** Suppose that \( \Phi \) is an algebra with transposition of bounded linear operators on a Banach space \( X \) over a spherically complete field \( F \), then each \( A \in B(\bar{\Phi}, 0, 1) \) belongs to the strong operator closure \( B(\bar{\Phi}, 0, 1) \) of the unit ball \( B(\bar{\Phi}, 0, 1) \) of \( \Phi \). If \( Q \) is a symmetric operator in \( B(\bar{\Phi}, 0, 1) \), then \( Q \) is in the strong-operator closure of the set of symmetric operators in \( B(\bar{\Phi}, 0, 1) \).

**Proof.** As in section 12 for an absolutely convex subset \( E \) of \( L(X) \) the weak- and strong-operator closures coincide, since \( X \) is a Banach space over a spherically complete field \( F \). Indeed, for each proper norm closed linear subspace \( Y \) of \( X \) and a point \( x \in X \setminus Y \) a continuous linear functional \( f : X \to F \) exists such that \( f(x) = 1 \) and \( f(Y) = 0 \) due to the Hahn-Banach theorem over \( F \) (see §8.203(f) [10]). For each point \( x \) outside the norm closure \( cl_n U \) of a subset \( U \) in \( X \) there exists a closed ball \( B(X, x, r) := \{z \in X : \|z-x\| \leq r\} \) containing \( x \) of radius \( 0 < r < \infty \) such that the intersection \( (cl_n U) \cap B(X, x, r) = \emptyset \) is void with \( r \in \Gamma_F \), where \( \Gamma_F := \{|b| : b \in F \setminus \{0\}\} \) is a multiplicative group contained in \( R \). The multiplicative norm on \( F \) is non-trivial, consequently, zero is the limit point of \( \Gamma_F \) in \( R \). Therefore, a radius \( r > 0 \) can be chosen so that \( \inf_{y \in cl_n U} \|x - y\| > r \).

If \( V \) is an absolutely convex norm closed subset of \( X \) and \( x \in X \setminus V \), there exists a hyperplane \( y + Y \) in \( X \) which does not contain \( x \) and does not intersect \( V \), where \( y = \lambda x \) for some \( \lambda \in F \), \( 0 < |\lambda| \leq 1 \), \( X = Y \oplus F \). 13
The topological dual space $X'$ of all continuous linear functionals $f : X \to \mathbb{F}$ separates points in $X$, consequently, there exists a family $\{f_\beta\} \subset X'$ of continuous linear functionals and closed subsets $K_\beta$ in the field $\mathbb{F}$ such that $V = \bigcap_\beta f_\beta^{-1}(K_\beta)$.

Evidently, if $A$ is in the strong operator closure of $E$, then it is in the weak operator closure of $E$. Let now $A$ be in the weak-operator closure of $E$. Consider vectors $x_1, \ldots, x_n \in X$ and the $n$-fold direct sum $X^\oplus n = X \oplus \ldots \oplus X$. An operator $G$ on $X$ induces and operator $\tilde{G} = G \oplus \ldots \oplus G$ on $X^\oplus n$. Therefore, $\{\tilde{G} : G \in E\} =: \tilde{E}$ is an absolutely convex subset of $X^\oplus n$, hence $\tilde{E} \tilde{x}$ is an absolutely convex subset of $X^\oplus n$, where $\tilde{x} = (x_1, \ldots, x_n)$. If $\tilde{A}$ is in the weak-operator closure of $\tilde{E}$, $\tilde{A} \tilde{x}$ is in the weak closure of $\tilde{E} \tilde{x}$, hence in the norm closure of $\tilde{E} \tilde{x}$ in $X^\oplus n$ due to the fact demonstrated above.

That is for each $\epsilon > 0$ there exist $T \in E$ such that $\|Tx_j - Ax_j\| < \epsilon$ for each $j = 1, \ldots, n$. Thus the weak-operator closure and the strong-operator closure of $E$ coincide.

In view of Lemma 14, an operator $f(A)$ is defined for each symmetric bounded operator $A \in L(X)$ and hence $f(A)$ for each $A$ in $\bar{\Psi}$, since $\lim_{|t| \to \infty} f(t) = 0$. Moreover, for each bounded symmetric operator $A$ a symmetric operator $A_G$ on $X_G$ corresponds, since $x^t = x$ for each $x \in \mathbb{F}$.

Let $Q$ be a symmetric operator in $\bar{\Psi}$, let also $K_b$ be a net of operators in $\Psi$ weak-operator converging to $Q$. Then $(K_b + K_b')/2$ is a net of symmetric operators in $\Psi$ converging to $Q$ relative to the weak-operator topology. But the set of symmetric operators in $\Psi$ is absolutely convex and from the fact demonstrated above $Q$ is in its strong-operator closure.

Consider a symmetric operator $Q \in B(\bar{\Psi}, 0, 1)$ and a net of symmetric operators $M_b \in \Psi$ strong-operator converging to $Q$. Let $p$ be a prime number so that $\mathbb{F}$ is an extension of the $p$-adic field $\mathbb{Q}_p$, hence up to an equivalence of multiplicative norms on $\mathbb{F}$ we have $|p| := |p|_{\mathbb{F}} = 1/p$ (see [14, 15]). Take a continuous function $f : \mathbb{F} \to \mathbb{F}$ so that $f(t) = t$ on $B(\mathbb{F}, 0, 1)$, while $f(t) = p^{2k-1}t$ on $B(\mathbb{F}, 0, p^k) \setminus B(\mathbb{F}, 0, p^{k-1})$ for each natural number $k \in \mathbb{N} := \{1, 2, 3, \ldots\}$, since the ball $B(\mathbb{F}, 0, r)$ is clopen (simultaneously closed and open) in $\mathbb{F}$, where $r > 0$. The function $f$ has the natural extension on the field $\mathbb{K}$ containing $\mathbb{F}$ so that $\sqrt[k]{x} \in \mathbb{K}$ for each $x \in \mathbb{F}$, putting $f(t) = t$
on $B(K,0,1)$, while $f(t) = p^{2k-1}t$ on $B(K,0,p^k) \setminus B(K,0,p^{k-1})$ for every $k \in \mathbb{N}$. Since $sp(Q) \subset B(K,0,1)$ (see [12, 1]), it follows that $f(Q) = Q$. Moreover, the function $f$ is strong-operator continuous on the set of symmetric operators in $\Psi$. The inequality $|f(t)| \leq 1$ for each $t$ implies that $\|f(M_b)\| \leq 1$ for each $b$. If $x, y \in K$ and $|x - y| < |x|$, then $|y| = |x|$ due to the non-archimedean inequality $|x + y| \leq \max(|x|, |y|)$ for each $x, y \in K$. Therefore, $f(M_b)$ is strong-operator converging to $f(Q)$, since $\lim_b S_{n_j(b)}(M_b) = S_{n_j(b)}(Q)$ for each $j$ and $P \in P_G$. Thus $Q$ is in the strong-operator closure of the set of symmetric operators from $B(cl_0,\Psi,0,1)$ and hence the strong operator limit of symmetric elements in $B(\Psi,0,1)$.

Generally if $A \in B(\Psi,0,1)$, then form an operator $A' := \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$ on $X \oplus X$ which is symmetric. Then $A' \in B(\Psi_2,0,1)$, where $\Psi_2$ denotes the family of all operators on $X \oplus X$ presented as $2 \times 2$ matrices with entries in $\Psi$. From the proof above it follows that $A'$ is in the strong-operator closure of $\Psi_2$. Particularly each entry of $A'$ is in the strong-operator closure of $B(\Psi,0,1)$, since each entry in $B(\Psi_2,0,1)$ is in $B(\Psi,0,1)$.

16. Definition. A derivation $D$ of a subalgebra $\Upsilon$ in $L(X)$ is called spatial, if an operator $B \in L(X)$ exists such that $D = ad B|_{\Upsilon}$.

17. Theorem. Let $\Psi$ be a $T$-algebra on a Banach space over a spherically complete field $F$, let also $D$ be a derivation of $\Psi$. Then for each commutative $W^*$-subalgebra $\Phi$ in a commutant $\Psi'$ a bounded $F$-linear operator $B = B_\Phi \in L(X)$ exists such that $B$ commutes with $\Phi$ and $D = ad B|_{\Psi'}$.

Proof. Evidently $\Upsilon'$ from Definition 13 is weakly closed in $L(X)$, particularly, $\Psi'$ is weakly closed. Let $\Xi$ be a maximal commutative subalgebra of $\Psi'$, hence it is weakly closed in the commutant $\Psi'$. Consider a lattice $\mathcal{P}$ of projection operators in $\Xi$ which corresponds to $\Psi$ (see Theorems 5.4, 5.11 and 5.16 in [11]).

The central carrier of an operator $A \in \Psi$ is defined to be $(I - P)$, where $P = \bigcup_{\beta} P_{\beta}$ and $P_{\beta}$ is from the set of all central projections in $\Psi$ such that $P_{\beta}A = 0$, i.e. every $P_{\beta}$ is in the center $Z(\Psi)$ of $\Psi$. Denote by $C_A$ the central carrier of $A$, then $C_A A = A$, since $A$ is continuous and $Ax$ is orthogonal to the range of $P_{\beta}$ for each $\beta$, but $Range(P_{\beta}) \subset Range(P)$.

Suppose that $B_{j,k} \in \Psi$ and $Q_{j,k} \in \Psi'$ are operators, then
(i) \( \sum_k B_{j,k} Q_{k,l} = 0 \) if and only if central operators \( A_{j,k} \in \Psi \) exist satisfying the properties:

(ii) \( \sum_k B_{j,k} A_{k,l} = 0 \) and \( \sum_k A_{j,k} Q_{k,l} = Q_{j,l} \) for each \( j, l = 1, \ldots, n \). Particularly, \( BQ = 0 \) for \( B \in \Psi \) and \( Q \in \Psi' \) if and only if \( C_B C_Q = 0 \).

Indeed, from the properties

\[
\sum_k B_{j,k} A_{k,l} = 0 \quad \text{and} \quad \sum_k A_{j,k} Q_{k,l} = Q_{j,l}
\]

of central operators \( A_{j,k} \in \Psi \) it follows that

\[
\sum_k B_{j,k} Q_{k,j} = \sum_k B_{j,k} \sum_t A_{k,t} B_{t,j} = \sum_t \sum_k B_{j,k} A_{k,t} B_{t,j} = 0.
\]

On the other side, if \( \sum_k B_{j,k} Q_{k,l} = 0 \), then one can consider the ring \( \text{Mat}_n(\Psi') \) of all \( n \times n \) matrices with entries in \( \Psi' \) and the union of all projections \( T_n = (A_{j,k}) \) in \( \text{Mat}_n(\Psi') \) which are annihilated under the left multiplication \( BT_n = 0 \) by \( B \), where \( A_{j,k} \in \Psi' \) for each \( j, k \). Consider a diagonal matrix \( E_n \) with entries being projections in \( \Psi' \). Then \( BE_n T_n = 0 \), consequently, \( T_n E_n T_n = E_n T_n \) and hence \( T_n E_n = (T_n E_n T_n)^t = E_n T_n \). Thus \( A_{j,k} \in Z(\Psi') \).

Then the equality \( BQ = 0 \) implies \( T_n Q = Q \), that is, \( \sum_k A_{j,k} Q_{k,l} = Q_{j,l} \) for each \( j, l = 1, \ldots, n \). Particularly, if \( C_B C_Q = 0 \), then \( BQ = BC_B C_Q Q = 0 \).

When \( BQ = 0 \), a central projection \( P \) in \( \Psi \) exists such that \( PB = 0 \) and \( PQ = Q \), consequently, \( PC_B = 0 \) and \( \text{Range}(C_B) \subset \text{Range}(P) \), hence \( C_B C_Q = 0 \).

Recall that vectors \( y_1, \ldots, y_n, \ldots \) are called mutually orthogonal in the non-archimedean sense, if \( \| t_1 y_1 + \ldots + t_k y_k \| = \max_{j=1}^k \| t_j y_j \| \) for each \( t_1, \ldots, t_k \in F \) and \( k \in \mathbf{N} \). Two subspaces \( U \) and \( W \) in a normed space \( Y \) are called orthogonal and denoted \( U \perp W \) if each vector \( x \in U \) is orthogonal to every vector \( y \in W \), \( x \perp y \).

A closed \( F \)-linear subspace \( U \) in a normed space \( Y \) is complemented, if a closed \( F \)-linear subspace \( V \) in \( Y \) exists so that \( U \cap V = \{0\} \) and \( U + V = Y \). It is orthocomplemented if it is complemented and in addition orthogonal \( U \perp V \) to its complement \( V \).

We say, that \( E_1, \ldots, E_j \) are (mutually) complemented, if \( E_l E_k = 0 \) for each \( 1 \leq l \neq k \leq j \).

A projection operator \( E : Y \to Y \) is called an orthoprojection if \( E(Y) \perp E^{-1}(0) \).
By Theorem 3.9 [12] a closed linear subspace $U$ of a Banach space $X$ is complemented if and only if a projection $P : X \to U$ exists. Theorem 3.10 [12] asserts, that a closed linear subspace $U$ of a Banach space $X$ over a non-archimedean field is orthocomplemented if and only if an orthoprojection $E$ of $X$ on $U$ exists. In view of Theorems 5.13 and 5.16 [12] each closed linear subspace of a Banach space over a spherically complete field is orthocomplemented. On the other hand, each closed linear subspace of a Banach space over a spherically complete field has an orthogonal basis which can be extended to an orthogonal basis of the entire Banach space. Therefore, without loss of generality we consider the family $P$ of all orthoprojections $E : X \to X$ (for short of projections).

Then we define a new operator $D_1$ by the formula:

$$(iii) \quad D_1(A_1 E_1 + \ldots + A_n E_n) = \tilde{D}(A_1) E_1 + \ldots + \tilde{D}(A_n) E_n,$$

where $E_1, \ldots, E_n \in P$, $A_1, \ldots, A_n \in \Psi$, $n \in \mathbb{N}$, $\tilde{D}$ is an extension of $D$ from $\Psi$ onto $\tilde{\Psi}$ in accordance with Lemma 12. If $A_1 E_1 + \ldots + A_n E_n = 0$, then from the proof above it follows that central operators $C_{j,k} \in Z(\tilde{\Psi})$ exist so that $\sum_{k=1}^n C_{j,k} E_k = E_j$ and $\sum_{j=1}^n A_j C_{j,k} = 0$. In view of Theorem 10 $\sum_{j=1}^n \tilde{D}(A_j) C_{j,k} = 0$, consequently, $\sum_j \tilde{D}(A_j) E_j = 0$ by $(i, ii)$. This means that $D_1$ is single-valued. Denote by $\Phi$ an algebra over $\mathbf{F}$ of all elements of the form $A_1 E_1 + \ldots + A_n E_n$ with $A_j$ and $E_j$ as above. It is indeed an algebra, since $A_j E_j A_k E_k = A_j A_k E_j E_k$ for each $j, k$.

The definition of $D_1$ implies that this operator is $\mathbf{F}$-linear and bounded on $\Phi$ due to Formula $(iii)$. Next we verify, that $D_1$ is a derivation of $\Phi$.

If projections $E_1, \ldots, E_j$ are complemented, take $F_{j+1} = E_{j+1} - E_{j+1} (E_1 + \ldots + E_j)$ and so on by induction. From $E_l(X) \perp E_l^{-1}(0)$ for each $l = 1, \ldots, j+1$ and $F_{j+1} = (I - E_1 - \ldots - E_j) E_{j+1}$ it follows, that $(I - E_1 - \ldots - E_j)(X) \perp (I - E_1 - \ldots - E_j)^{-1}(0)$ and $(I - E_1 - \ldots - E_j)(E_{j+1} X) \perp E_{j+1}^{-1}(I - E_1 - \ldots - E_j)^{-1}(0)$, consequently, $F_{j+1}$ is also the projection. Then $A_l E_l + A_{j+1} E_{j+1} = A_l (E_l - E_{j+1} E_l) + (A_l + A_{j+1}) E_{j+1} E_l + A_{j+1} (E_{j+1} - E_{j+1} E_j)$ for each $l \leq j$ by induction, consequently, this induces the decomposition $A_1 E_1 + \ldots + A_n E_n = B_1 F_1 + \ldots + B_n F_n$ with complemented projections $F_1, \ldots, F_n \in P$ and $B_1, \ldots, B_n \in \tilde{\Psi}$.

When $E_1, \ldots, E_n$ are complemented projections and $x = \sum_{j=1}^n E_j x$ is a
vector in \(X\) of unit norm \(\|x\| = 1\), then

\[
\| (A_1 E_1 + \ldots + A_n E_n) x \| = \max_{j=1}^n \| A_j E_j x \|,
\]

since \(A_j E_j x = E_j A_j x\) are mutually orthogonal in the non-archimedean sense vectors. Moreover,

\[
\| A_j E_j x \| \leq \| A_j E_j \| \| E_j x \| \leq \max_{\ell=1}^n \| A_\ell E_\ell \|,
\]

since \(\max_j \| E_j x \| = \| x \| = 1\), hence

\[
\| A_1 E_1 + \ldots + A_n E_n \| \leq \max_{j=1}^n \| A_j E_j \|.
\]

At the same time

\[
\max_j \| A_j E_j \| \leq \| A_1 E_1 + \ldots + A_n E_n \|
\]
due to the non-archimedean orthogonality of \(E_j\). This implies

\[
\| D_1(A_1 E_1 + \ldots + A_n E_n) \| = \max_{j=1}^n \| (\bar{D} A_j) E_j \|.\]

Considering orthogonal central projections, one gets as a central carrier \(Q_j\) of \(E_j\) in \(\Psi\) as a projection.

Two \(T\)-algebras \(\Psi\) and \(\Theta\) are called \(T\)-isomorphic, if an \(F\)-linear multiplicative bijective surjective mapping \(\theta : \Psi \to \Theta\) exists continuous together with its inverse mapping,

\[
\theta(\alpha B) = \alpha \theta(B), \quad \theta(AB) = \theta(A) \theta(B) \quad \text{and} \quad \theta(A^t) = [\theta(A)]^t \quad \text{for each} \quad \alpha \in F,
\]

\(A, B \in \Psi\). Since \(\theta\) and \(\theta^{-1}\) are continuous and multiplicative, then \(\| \theta(A) \| = \| A \|\) is an isometry, since

\[
\| \theta(\alpha A^n) \| = |\alpha|^n \| \theta(A) \|^n \| \leq |\alpha|^n \| \theta(A) \|^n
\]

for each \(A \in \Psi\) and \(\alpha \in F\). The \(T\) algebras \(\bar{\Psi} E_j\) and \(\bar{\Psi} Q_j\) are \(T\)-isomorphic, since \(E_j, I - E_j, Q_j, I - Q_j\) and \(E_j^t, (I - E_j)^t = I - E_j^t, Q_j, I - Q_j^t \in \Psi\), where \(\Psi\) is topologically complete. Then

\[
\| (\bar{D} A_j) E_j \| = \| (\bar{D} A_j) Q_j \| = \| D(A_j Q_j) \| \leq \| D \| \| A_j Q_j \| = \| D \| \| A_j E_j \|.
\]

From Theorem 10 it is known that \(\bar{D}\) annihilates the center \(Z(\bar{\Psi})\) of \(\bar{\Psi}\). Therefore,

\[
\| D_1(A_1 E_1 + \ldots + A_n E_n) \| \leq \| \bar{D} \| \max_j \| A_j E_j \| = \| \bar{D} \| \| A_1 E_1 + \ldots + A_n E_n \|,
\]

consequently, \(D_1\) is bounded. Thus \(D_1\) has a bounded extension being a derivation \(D_1 : \Upsilon \to \bar{\Upsilon}\). In view of Lemma 12 it has a continuous extension \(\bar{D}_1\) defined on \(\bar{\Upsilon}\).

On the other hand, \(\bar{\Upsilon}\) is a \(T\)-algebra containing \(\Psi\) and \(\Xi\), since it contains \(\mathcal{P}\), the projection lattice of \(\Xi\), hence \(\Upsilon' \subset \Psi'\) and \(\Upsilon'\) commutes with \(\Xi\). But \(\Xi\) is a maximal commutative subalgebra in \(\Psi'\), we get \(\Upsilon' = \Xi\).

Recall that a vector \(x \in X\) is topologically cyclic relative to the action of \(\Psi\) for a closed linear subspace \(Y\) over \(F\) if \(\Psi x = \{Ax : A \in \Psi\}\) is
everywhere dense in $Y$. A subspace $Y$ is called invariant relative to $\tilde{\Psi}$, if $AY \subset Y$ for each $A \in \tilde{\Psi}$. A closed linear subspace $Y$ in $X$ over $\mathbf{F}$ is called topologically irreducible relative to $\tilde{\Psi}$, if $Y$ is invariant relative to $\tilde{\Psi}$ and each non zero vector $x \in Y \setminus \{0\}$ is topologically cyclic relative to $\tilde{\Psi}$. If $Y$ is a topologically irreducible subspace, it has and orthocomplement $X \ominus Y$. So $X \ominus Y$ has another topologically invariant subspaces and the process can be done by transfinite induction (see [2]). Therefore, the sum of all topologically irreducible subspaces in $X$ relative to $\tilde{\Psi}$ is everywhere dense in $X$.

For any topologically irreducible subspace $Y$ relative to $\tilde{\Psi}$ consider the restriction $\tilde{\Psi}|_Y = \{A|_Y : A \in \tilde{\Psi}\}$. Since $\tilde{D}_1 A \in \tilde{\Psi}$ for each $A \in \tilde{\Psi}$, the subspace $Y$ is invariant relative to $\tilde{D}_1 \tilde{\Psi}$ also. The algebra $\Psi$ and the Banach space $X$ are over the spherically complete field $\mathbf{F}$. Take an (ortho)projection $P$ from $X$ onto a finite dimensional over $\mathbf{F}$ subspace $P X$ of a topologically irreducible subspace $Y$. This induces the finite dimensional over $\mathbf{F}$ subalgebra $P \Psi P = \{P A P : A \in \Psi\}$. Then the differentiations $P D P : P \Psi P \rightarrow P \Psi P$ and $P D_1 P : P \Psi P \rightarrow P \Psi P$ act on it.

Let $J_P$ be the center of $P \Psi P$. Then the differentiation operator $P D P$ annihilates $J_P$ due to Theorem 10 and hence $P D_1 P$ annihilates $J_P \cap \Upsilon$, consequently, $P D P$ and $P D_1 P$ are defined on the quotient algebras $(P \Psi P)/J_P$ and $(P \Upsilon P)/J_P$ correspondingly. Introduce on $(P \Psi P)/J_P$ the Lie algebra $\Psi_P$ structure by $[A, B] = AB - BA$ for each $A, B \in (P \Psi P)/J_P$. Traditionally $ad B$ denotes $ad B(A) = [B, A]$ for each $A \in L(X)$. The latter Lie algebra $\Psi_P$ is non degenerate, i.e. has a non degenerate Killing form $tr(adA adB)$, where $(adA)(E) = [A, E]$ for each $A, E \in \Psi_P$. Then $P D P$ is the differentiation of the Lie algebra $\Psi_P$ so that $P D P[A, B] = [P D P A, B] + [A, P D P B]$ and analogously $P D_1 P$ is the differentiation of $\Upsilon_P$. In view of Theorem 1.5.8 [3] the Lie algebra $\Psi_P$ is complete, i.e. its center is zero and each its differentiation is internal, $der(\Psi_P) = ad(\Psi_P)$, also $\Upsilon_P$ is complete. Thus $P D P$ and $P D_1 P$ are internal derivations of $P \Psi P$ and $P \Upsilon P$ respectively.

Particularly, if $\mathbf{F}$ is a locally compact field take $G_\alpha = \mathbf{F}$. Generally we consider a family $\{G_\alpha : \alpha \in \mu\}$ of locally compact subfields such that $\bigcup_{\alpha \in \mu} G_\alpha = \mathbf{F}$. Since $\mathbf{F}$ is spherically complete and $G_\alpha$ is locally compact, then $G_\alpha$ is spherically complete. This family of subfields is naturally directed.
by inclusion which induces a direction on \( \mu \) such that \( \alpha \leq \beta \) if and only if \( G_\alpha \subset G_\beta \). Consider \( \Psi \) over \( G_\alpha \) and denote it by \( \Psi_\alpha \). In view of Alaoglu-Bourbaki’s theorem (see §9.202 [10]) each bounded closed ball \( B((\Psi_\alpha)’, z, r) \) of radius \( 0 < r < \infty \) and containing \( z \) in \( (\Psi_\alpha)’ \) is weak-operator closed, since \( G_\alpha \) is a locally compact field.

From the proof above it follows that \( der(\Psi_\alpha, P) = ad(\Psi_\alpha, P) \) for each \( \alpha \in \mu \) and \( P \) as above on \( X_\alpha \), where \( X_\alpha \) is the Banach space \( X \) considered over \( G_\alpha \). The set \( P_\alpha \) of projections \( P \) on \( X_\alpha \) is also directed by \( P \leq Q \) if and only if \( P(X_\alpha) \subset Q(X_\alpha) \). There are natural connecting continuous \( G_\alpha \)-linear mappings \( \pi_\beta : X_\beta \to X_\alpha \) for each \( \alpha \leq \beta \in \mu \). Put \( B_\alpha \) to be the projective limit \( B_\alpha = \lim \leftarrow P_\alpha B_\alpha,P \) which exists in \( (\Psi_\alpha)’ \). Then we put \( B = \lim \leftarrow \mu B_\alpha \). These projective limits exist relative to the weak-operator topology due to Proposition 2.5.6 and Corollary 2.5.7 [2]. This operator \( B \) is \( F \) linear, since it is \( G_\alpha \) linear on \( X_\alpha \) for each \( \alpha \) and \( \bigcup_{\alpha \in \mu} G_\alpha = F \).

Considering all possible topologically invariant subspaces and all (ortho)projections \( P \) with finite dimensional over \( F \) ranges one gets due to Theorem 15, that \( \bar{D}_1 \) is the internal derivation of \( \bar{\Psi} \), since the family of all finite dimensional over \( F \) subalgebras \( P\Psi P \) is everywhere dense in \( \bar{\Psi} \) relative to the weak-operator topology. Then \( D = adB \) on \( \Psi \) for some \( B \in \Xi’ \), since

\[
BT - TB = \bar{D}_1(T) = D(I)T = 0 \text{ for each } T \in \mathcal{P}.
\]

18. Definition. A derivation \( D \) of an algebra \( \Psi \) is called inner, if \( D = adB|_{\Psi} \) for some element \( B \in \Psi \) of this algebra.

19. Lemma. Each derivation \( adB \) of a \( T \) algebra \( \Psi \) induces a derivation of \( \Psi’ \). A derivation \( adB \) of \( \bar{\Psi} \) is inner if and only if it induces an inner derivation of \( \Psi’ \).

Proof. For every \( A \in \Psi \) and \( T \in \Psi’ \) one gets

\[
(BT - TB)A - A(BT - TB) = BTA - TBA - ABT + ATB = (BA - AB)T - T(BA - AB) = 0,
\]

since \( [B, A] \in \Psi \). In the case when \( adB \) induces an inner derivation of \( \bar{\Psi} \) so that \( adB = adE \) on \( \bar{\Psi} \) with \( E \in \bar{\Psi} \) this implies that \( (B - E) \) commutes with \( \bar{\Psi} \). Therefore, \( (B - E) \in \Psi’ \). The inclusion \( E \in \bar{\Psi} \) implies that \( ad(B - E) = adB \) on \( \Psi’ \). That is \( adB \) induces an inner derivation of \( \Psi’ \).

20. Definitions. Suppose that \( X \) is a Banach space over a field \( F \) and \( P \) is a projection on \( X \), \( P : X \to X \), and \( \Psi \) is a \( W^1 \) subalgebra
in $L(X)$, $P \in \Psi$. A projection $P$ is called cyclic in $\Psi$ (or under $\Psi'$), if $PX = cl_X \text{span}_F \Psi'x$ for some vector $x \in X$, where $\text{span}_F U := \{ y \in X : y = b_1x_1 + \ldots + b_nx_n; b_1, \ldots, b_n \in F, x_1, \ldots, x_n \in X \}$, $cl_X U$ denotes the closure of a subset $U$ in $X$ relative to the norm topology. Such vector $x$ is called a generating vector under $\Psi'$.

An orthoprojection $P$ in $\Psi$ over a spherically complete field $F$ is called countably decomposable relative to $\Psi$, if every orthogonal family of non zero suborthoprojections of $P$ in $\Psi$ is countable. When the unit operator $I$ is countably decomposable relative to $\Psi$, one says that the $W^t$ algebra $\Psi$ is countably decomposable.

21. Lemma. Let $P$ be a central (ortho)projection in a $W^t$ algebra $\Psi$ over a spherically complete field $F$. This projection $P$ is the central carrier of a cyclic projection in $\Psi$ if and only if $P$ is countably decomposable relative to the center $Z(\Psi)$ of $\Psi$. Moreover, a cyclic projection in $\Psi$ is countably decomposable; two projections $P$ and $Q$ with the same generating vector in $\Psi$ and $\Psi'$ have the same central carrier.

Proof. Consider a central projection $T$ in $\Psi$ with generating vector $x \in X$ and $P = C_T$. Consider the case when there are orthogonal families $P_\alpha$ and $T_\beta$ of (ortho)projections in $Z(\Psi)$ and $\Psi$ respectively contained in $P$ and $T$ correspondingly. The field $F$ is spherically complete and the Banach space $X$ is isomorphic with $c_0(\omega, F)$ for some set $\omega$. Each closed linear subspace in $X$ has an orthonormal basis which can be completed to an orthonormal basis in $X$. If $y \in X$, then there are convergent series $y = \sum P_\alpha y$ and $y = \sum T_\beta y$, where $P_\alpha y \perp P_\beta y$ and $T_\alpha y \perp T_\beta y$ are orthogonal in the non-archimedean sense for each $\alpha \neq \beta$. The convergence of these series is equivalent to that for each $\epsilon > 0$ sets $\{ \alpha : \| P_\alpha y \| > \epsilon \}$ and $\{ \beta : \| T_\beta y \| > \epsilon \}$ are finite. Thus these series may have only countable sets of non zero additives.

When $T_\beta y = 0$ the equalities $\{0\} = cl_X \text{span}_F \Psi' T_\beta y = cl_X \text{span}_F T_\beta \Psi' y$ and $T_\beta T y = T_\beta y$ are valid, if $P_\alpha y = 0$ analogously $P_\alpha y = 0$. That is $P_\alpha P y = P_\alpha y = 0$ due to the equivalence of conditions (i) and (ii) in Section 17. Thus the families $\{ P_\alpha \}$ and $\{ T_\beta \}$ have at most countable subsets of non zero elements, consequently, $P$ and $T$ are countably decomposable.

On the other hand, if $P$ is countably decomposable and $\{ P_\alpha \}$ is a count-
able set of projections cyclic under \((Z(\Psi))'\) with generating vectors \(x_n\) of unit norm and with sum \(\forall n P_n = P\). The field \(F\) is of zero characteristic and contains the \(p\)-adic field \(Q_p\) for some prime number \(p\). Take the vector \(x = \sum_n p^n x_n\), where \(n \in \mathbb{N}\). This sum or series converges, since \(\|p^n x_n\| = \|x_n\| p^{-n}\) up to an equivalence of norms on \(F\). Therefore the equality is valid \(cl_X \span_F (Z(\Psi))'x = PX\), since \(cl_X \span_F (Z(\Psi))'x\) contains \(cl_X \span_F (Z(\Psi))'P_n x = cl_X \span_F (Z(\Psi))'x_n = P_n X\) for each \(n\). Putting \(T\) to be an projection from \(X\) onto \(cl_X \span_F (\Psi)'x\) one gets \(T \subseteq P\), since \(\Psi' \subseteq (Z(\Psi))'\), that is \(PT = T\). Suppose that \(Q \in Z(\Psi)\) and \(QT = T\). This implies that \(Q x = x\) and \(cl_X \span_F (Z(\Psi))'x = cl_X \span_F (Z(\Psi))'Q x = cl_X \span_F Q (Z(\Psi))'x\), consequently, \(P = Q P\). This means that \(P = C_T\) with \(T\) cyclic in \(\Psi\). Therefore, the projection \(P\) is the central carrier of \(cl_X \span_F \Psi x\).

22. **Theorem.** If \(\Psi\) is a \(W'\) algebra on a Banach space over a spherically complete field \(F\) and \(D\) is a derivation of \(\Psi\), then \(D\) is inner.

**Proof.** In view of Theorem 17 a derivation \(D\) has the form \(D = adB|_{\Psi}\) for some bounded linear operator \(B \in L(X)\). Then \(- (BA' - A'B')^t = B' A - AB' \in \Psi\) for each \(A \in \Psi\), consequently, the mapping \(adB^t : \Psi \to \Psi\) is also the differentiation of \(\Psi\). Therefore, \(ad(B + B')\) and \(ad(B - B')\) are derivations of \(\Psi\). If each of these derivations \(ad(B + B')\) and \(ad(B - B')\) is inner, then \(adB\) is inner as well. Mention that the operator \(ad(\lambda I + B)\) is the derivation together with \(adB\) for each \(\lambda \in F\). In accordance with Theorem 10 \(B \Psi'\) is the center of the \(W'\) algebra \(\Psi\), where \(\Phi = Z(\Psi)\).

If \(\{P_\beta : \beta \in \Lambda\}\) is a family of projections on \(X\) so that its sum \(I = \sum_{\beta \in \Lambda} P_\beta\) is the unit operator and \(adB|_{\Psi P_\beta} = adE_\beta|_{\Psi P_\beta}\) for every \(\beta\) and \(sup_{\beta \in \Lambda} \|E_\beta\| < \infty\), where \(\Lambda\) is a suitable set, \(E_\beta \in \Psi P_\beta\), then \(adB|_{\Psi} = adE|_{\Psi}\) for \(E = \sum_{\beta \in \Lambda} E_\beta\).

Take \(Q_\alpha\) a cyclic projection under \((Z(\Psi))'\) for each \(\alpha\). It is sufficient to prove this assertion for countably decomposable center \(Z(\Psi)\) due to Lemma 21. For this one takes a cyclic projection \(T\) in \(\Psi'\) with central carrier \(I\) considering the faithful representation \(\Psi T\) of \(\Psi\) on \(T(X)\). The commutant is \(T \Psi' T\) and so it is sufficient to consider that \(\Psi'\) is countably decomposable.

Let \(G\) be a locally compact field contained in \(F\) and consider the spheric-
cally complete field $F$ as the Banach space over $G$ isomorphic with $c_0(\omega, F)$ for some set $\omega$ (see §21). Then the Banach space $X$ over $F$ has the structure of the Banach space $X_G$ over $G$ as well. To each operator bounded linear operator $A \in L(X)$ a bounded operator $A_G \in L(X_G)$ corresponds. Due to Alaoglu-Bourbaki’s theorem (see §9.202 [10]) a closed bounded ball $B(X_G, x, r) := \{ y \in X_G : \|y - z\| \leq r \}$ in $X_G$ is weakly compact and a bounded closed ball $B(L(X_G), A, r) := \{ C \in L(X_G) : \|C - A\| \leq r \}$ in $L(X_G)$ is compact relative to the weak operator topology, where $0 < r < \infty$. Therefore, $B(L(X_G), A, r) \cap \Psi_G$ is also compact relative to the weak operator topology, where $\Psi_G$ is the $W^t$ algebra $\Psi$ considered over the field $G$, i.e. by narrowing the field from $F$ to $G$ so that $\Psi_G \subset L(X_G)$.

A system of algebras $\{ \Psi_P : P \in \mathcal{P} \}$ and a family of locally compact subfields $\{ G_\alpha : \alpha \in \mu \}$ from §17 gives rise to the projective limit decomposition of each operator $A \in \Psi$ or $E \in \Psi'$ and for the differentiation operator $D$ as well, since $\Psi = \overline{\Psi}$ by the conditions of this theorem. Finally, from Proposition 2.5.6 and Corollary 2.5.7 [2] the assertion follows.

References

[1] B. Diarra, S.V. Ludkovsky. ”Spectral integration and spectral theory for non-archimedean Banach spaces” // Intern. J. of Mathem. and Mathem. Sciences 31: 7 (2002), 421-442.

[2] R. Engelking. ”General topology” (Mir: Moscow, 1986).

[3] M. Goto, F.D. Grosshans. ”Semisimple Lie algebras” (Marcel Dekker, Inc.: New York, 1978).

[4] R.V. Kadison. ”Derivations of operator algebras”// Annals of Mathem. 83: 1 (1966), 280-293.

[5] R.V. Kadison. ”Unitary invaraints for representations of operator algebras”// Annals of Mathem. 66: 2 (1957), 304-379.

[6] R.V. Kadison, J.R. Ringrose. ”Fundamentals of the theory of operator algebras” (Academic Press: New York, 1983).
[7] J. Kakol. "Remarks on spherical completeness of non-archimedean valued fields" // Indag. Mathem. 5: 3 (1994), 321-323.

[8] V. Losert. "The derivation problem for group algebras" // Annals of Mathem. 168: 1 (2008), 221-246.

[9] S.V. Ludkovsky. "Quasi-invariant and pseudo-differentiable measures with values in non-archimedean fields on a non-archimedean Banach space" // J. Mathem. Sci. 128: 6 (2005), 3428-3460.

[10] L. Narici, E. Beckenstein. "Topological vector spaces" (Marcel Dekker, Inc.: New York, 1985).

[11] M. van der Put. "Difference equations over p-adic fields" // Math. Ann. 198 (1972), 189-203.

[12] A.C.M. van Rooij. "Non-Archimedean functional analysis" (Marcel Dekker, Inc.: New York, 1978).

[13] S. Sakai. "Derivations of \( W^* \)-algebras" // Annals of Mathem. 83: 1 (1966), 273-279.

[14] W.H. Schikhof. "Ultrametric calculus" (Cambridge Univ. Press: Cambridge, 1984).

[15] A. Weil. "Basic number theory" (Springer: Berlin, 1973).