Bloch’s conjecture revisited

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Abstract

Let \( X \) be a non-singular projective complex surface. We can show that Bloch’s conjecture (i.e., if \( pg = 0 \) then the Albanese kernel vanishes) is equivalent to the following statement:

If \( pg(X) = 0 \) then for any given Zariski open \( U \subset X \) and \( \omega \in H^2(U, \mathbb{C}) \) there is a smaller Zariski open \( V \subset U \) such that

\[
\omega |_V = \omega' + \omega_{\mathbb{Z}}
\]

where \( \omega' \in F^2H^2(V,\mathbb{C}) \) and \( \omega_{\mathbb{Z}} \) is integral.

Let \( X \) denote a complex algebraic surface which is smooth and complete. Let \( A_0(X) \) denote the subgroup of the Chow group of zero cycles given by cycles of degree zero, and let \( J^2(X) \) be the Albanese variety of \( X \), which we regard as the intermediate Jacobian canonically associated with the Hodge structure on \( H^3(X, \mathbb{Z}(2)) \). Let \( \phi : A_0(X) \to J^2(X) \) be induced by the canonical (surjective) mapping to the Albanese variety. The kernel of \( \phi \) is usually called the “Albanese kernel”.

**Bloch’s conjecture**  If \( pg(X) = 0 \) then the “Albanese kernel” vanishes.

The goal of this note is to show the following:

**Theorem 1** Let \( X \) be as above, and assume that \( pg(X) = 0 \). Then

\[
\text{“Albanese kernel”} \cong \lim_{U \subset X} \frac{H^2(U, \mathbb{C})}{F^2H^2(U, \mathbb{C}) + H^2(U, \mathbb{Z}(2))}
\]

where the limit is taken over all non-empty Zariski open subsets of \( X \).

We then clearly have the following:

**Corollary 1** Let \( X \) be as above, and assume that \( pg(X) = 0 \). The “Albanese kernel” vanishes if and only if for any given Zariski open \( U \subset X \) and \( \omega \in H^2(U, \mathbb{C}) \), there is a smaller Zariski open \( V \subset U \) such that

\[
\omega |_V = \omega' + \omega_{\mathbb{Z}}
\]

where \( \omega' \in F^2H^2(V,\mathbb{C}) \) and \( \omega_{\mathbb{Z}} \) is integral (i.e., \( \omega_{\mathbb{Z}} \in \text{im} H^2(V, \mathbb{Z}(2)) \)).
We will make use of the following result only in the particular case when $Y$ is a curve.

**Lemma 1** Let $Y$ be any reduced variety over the complex numbers. Then

\[
\lim_{U \subset Y} \frac{H^1(U, \mathbb{C})}{F^1 H^1(U, \mathbb{C}) + H^1(U, \mathbb{Z}(1))} = 0
\]

where the limit is taken over all non-empty Zariski open subsets of $Y$.

**Proof** Since the statement concerns the generic points of a variety, we may assume that $Y$ is irreducible, non-singular and projective. For any open $U \subset Y$, let $S = Y - U$, and let $\text{Div}^0_S(Y)$ be the group of divisors on $Y$ which are supported on $S$, and are homologous to 0 on $Y$. There is an exact sequence

\[
0 \to H^1(Y, \mathbb{Z}(1)) \to H^1(U, \mathbb{Z}(1)) \xrightarrow{\text{res}} \text{Div}^0_S(Y) \to 0
\]

where \(\text{res} \) denotes the sum of the residue maps associated to components of $S$ which are divisors on $Y$. This underlies an exact sequence of mixed Hodge structures, where $F^1(\text{Div}^0_S(Y) \otimes \mathbb{C}) = \text{Div}^0_S(Y) \otimes \mathbb{C}$ (indeed, for any such divisor $D$, we have that $F^1 H^2_p(Y, \mathbb{C}(1)) \cong F^0 H^0(\tilde{D}, \mathbb{C}(0))$ for a desingularization $\tilde{D}$ of $D$). Now

\[
H^1(Y, \mathbb{C}) \cong H^1(Y, \mathcal{O}_Y),
\]

and since $F^1$ yields an exact functor on the category of mixed Hodge structures (see [D]), we get that

\[
\frac{H^1(Y, \mathbb{C})}{F^1 H^1(Y, \mathbb{C})} \cong \frac{H^1(U, \mathbb{C})}{F^1 H^1(U, \mathbb{C})}.
\]

Hence there is a natural map

\[
\text{Div}^0_S(Y) \to \frac{H^1(U, \mathbb{C})}{F^1 H^1(U, \mathbb{C}) + H^1(Y, \mathbb{Z}(1))} \cong \frac{H^1(Y, \mathbb{C})}{F^1 H^1(Y, \mathbb{C}) + H^1(Y, \mathbb{Z}(1))} \cong \text{Pic}^0(Y).
\]

This map is known to be just the natural map $D \mapsto \mathcal{O}_Y(D)$. Hence

\[
\frac{H^1(U, \mathbb{C})}{F^p H^{2p-1}(U, \mathbb{C}) + H^{2p-1}(U, \mathbb{Z}(p))} \cong \text{Pic}^0(Y)
\]

where the limit is taken over all non-empty Zariski open subsets of $Y$.

**Remark:** Let $Y$ be a non-singular projective complex variety. By the same argument in the proof of the Lemma, one can see that

\[
\lim_{U \subset Y} \frac{H^{2p-1}(U, \mathbb{C})}{F^p H^{2p-1}(U, \mathbb{C}) + H^{2p-1}(U, \mathbb{Z}(p))} = 0
\]
Proof of the Theorem Let $\mathcal{H}^2(Z(2))$ and $\mathcal{H}^2/F^2$ be the Zariski sheaves on $X$ associated to $U \mapsto H^2(U, Z(2))$ and $U \mapsto H^2(U, C)/F^2$ respectively. Let $\mathcal{H}^3(Z(2)_D)$ denote the Deligne–Beilinson cohomology Zariski sheaf on $X$ (cf. [G] and [E]). We then have the following diagram of sheaves

\[
\begin{array}{cccccc}
0 & \to & H^2(Z(2)) & \to & H^2(C(X), Z(2)) & \to & \bigoplus_{x \in X^1} i_x H^1(C(x), Z(1)) & \to & \cdots \\
& \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & \to & H^2/F^2 & \to & H^2(C(X))/F^2 & \to & \bigoplus_{x \in X^1} i_x H^1(C(x))/F^1 & \to & 0 \\
& \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & \to & H^3(Z(2)_D) & \to & H^3(C(X), Z(2)_D) & \to & \bigoplus_{x \in X^1} i_x H^2(C(x), Z(1)_D) & \to & 0 \\
& 0 & 0 & 0 & 0 & 0 & 0 & 0 & 
\end{array}
\]

obtained by comparing the arithmetic resolutions of the left hand column (see [BV] for that of $H^2/F^2$) via the standard long exact sequences in Deligne-Beilinson cohomology (and taking into account that singular cohomology of an affine vanishes in degrees greater than its algebraic dimension). Now, because of the Lemma, we have that

\[
H^2(C(x), Z(1)_D) \defeq \lim_{U \subset X} H^2(U, Z(1)_D) \cong \lim_{U \subset X} \frac{H^1(U, C)}{F^1 H^1(U, C) + H^1(U, Z(1))} = 0.
\]

Thus the right-bottom corner in the diagram above vanishes and the sheaf $\mathcal{H}^3(Z(2)_D)$ is then identified by the constant sheaf associated with the group

\[
H^3(C(X), Z(2)_D) \defeq \lim_{U \subset X} H^3(U, Z(2)_D) \cong \lim_{U \subset X} \frac{H^2(U, C)}{F^2 H^2(U, C) + H^2(U, Z(2))}
\]

where the limit is taken over all non-empty Zariski open subsets of $X$. Finally, if $p_g(X) = 0$, it is known (see [BV] and [E]) that

\[
H^0(X, \mathcal{H}^3(Z(2)_D)) \cong \ker(A_0(X) \xrightarrow{\partial} J^2(X)),
\]

yielding the claimed identification.

Remark: Note that for a surface $X$ with $p_g \neq 0$ we have that

\[
\lim_{U \subset X} \frac{H^2(U, C)}{F^2 H^2(U, C) + H^2(U, Z(2))} \neq 0,
\]

since this group maps onto the “Albanese kernel”, which is non-zero by [M].

For a surface $X$ with $p_g = 0$, the canonical “residue map”

\[
H^2(C(X))/F^2 \to \prod_{x \in X^1} H^1(C(x))/F^1
\]
is injective, with quotient group isomorphic to $H^3(X, \mathbb{C})/F^2$ (see [BV]); thus the integral lifting of the “residue” of a given $\omega \in H^2(\mathbb{C}(X))/F^2$ always exists, but, in order to lift $\omega$ itself to a class in $H^2(\mathbb{C}(X), \mathbb{Z}(2))$, one should know that the lifting of the “residue” of $\omega$ yields zero in the group of zero-cycles as well as in $H^3(X, \mathbb{Z}(2))$.

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