ON GRAUERT-RIEMENSCHNEIDER TYPE CRITERIIONS

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Abstract. Let \((X, \omega)\) be a compact Hermitian manifold of complex dimension \(n\). In this article, we first survey recent progress towards Grauert-Riemenschneider type criterions. Secondly, we give a simplified proof of Boucksom’s conjecture given by the author under the assumption that the Hermitian metric \(\omega\) satisfies \(\partial\bar{\partial}\omega^l = 0\) for all \(l\), i.e., if \(T\) is a closed positive current on \(X\) such that \(\int_X T^n_{ac} > 0\), then the class \(\{T\}\) is big and \(X\) is Kähler. Finally, as an easy observation, we point out that Nguyen’s result can be generalized as follows: if \(\partial\bar{\partial}\omega = 0\), and \(T\) is a closed positive current with analytic singularities, such that \(\int_X T^n_{ac} > 0\), then the class \(\{T\}\) is big and \(X\) is Kähler.

1. Introduction

Let \((X, \omega)\) be a compact Hermitian manifold of dimension \(n\), \(\omega\) be a Hermitian metric on \(X\). The Grauert-Riemenschneider conjecture (G-R conjecture for short) \cite{13} reads that: if there is a holomorphic Hermitian line bundle \((L, h) \to X\), such that \(c(L) = \frac{1}{2\pi} \Theta_h \geq 0\) on \(X\) and is strictly positive on a dense open subset of \(X\), then \(X\) is Moishezon. A compact complex manifold is said to be Moishezon, if it is birational to a projective manifold.

This conjecture was first proved by Siu \cite{22} and then by Demailly \cite{8} shortly later.

Siu even proved that if \(c(L) \geq 0\) on \(X\) and strictly positive at least at one point in \(X\), then \(X\) is Moishezon. Siu first proved that, under the curvature condition of G-R conjecture, one has

\[
\dim H^q(X, L^k) = o(k^n), \text{ for } q \geq 1.
\]

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Then, by a Riemann-Roch theorem argument, one can get that the Kodaira dimension of $L$ is of top dimension, i.e., dimension $n$. Thus $X$ is Moishezon.

Demailly \cite{8, 9} established the celebrated holomorphic Morse inequalities. The strong version of holomorphic Morse inequalities is stated as follows:

$$\sum_{0 \leq j \leq q} (-1)^{q-j} \dim \mathbb{C} H^j(X, L^k) \leq (-1)^q \frac{k^n}{n!} \int_{X(\leq q)} (c(L))^n + o(k^n)$$

where $X(\leq q)$ be the set where $c(L)$ is non degenerate and has at most $q$ negative eigenvalue. Take $q = 1$ in above inequalities, one can get that

$$\dim H^0(X, L^k) \geq \dim H^0(X, L^k) - \dim H^1(X, L^k) \geq \frac{k^n}{n!} \int_{X(\leq 1)} (c(L))^n + o(k^n).$$

Under the curvature condition of G-R conjecture, one finds $X(\leq 1)$ is a non-empty open set of $X$, then the integral term $\int_{X(\leq 1)} (c(L))^n > 0$, from which one can easily find the Kodaira dimension of $L$ is $n$, i.e., $X$ is Moishezon. The advantage of Demailly’s holomorphic Morse inequalities is that they give a criterion of a Hermitian holomorphic line bundle to be big by the positivity of an integral of the curvature, which is a little bit weaker than the positivity of the curvature itself.

Around 1990’s, due to the development of several complex variables, singular metrics were introduced to holomorphic line bundles. An Hermitian metric $h$ of a holomorphic line bundle $L$ is said to be singular, if locally we can write $h = e^{-2\varphi}$, with $\varphi \in L^1_{\text{loc}}$. There are many positivities due to the curvature current of the singular metric, e.g., semi positive, nef, pseudo effective and big. In general, one has semi positive $\subset$ nef $\subset$ pseudo effective $\subset$ big.

In \cite{15}, Ji-Shiffman proved that a holomorphic line bundle is big if and only if there is a singular metric of $L$ such that the curvature current is positive definite, i.e., is a Kähler current.

A compact complex manifold is said to be in the Fujici class $C$, if it is bimeromorphic to a compact Kähler manifold. It was proved by Demailly-Paun in \cite{12} that $X$ is in the Fujiki class $C$ if and only if it carries a Kähler current.

Recently, experts in several complex variables and complex geometry try to relax the assumptions in G-R conjecture, say from the integral class to the transcendental class, and from semi positivities to nefness
and pseudo effectiveness. Along these directions, there are the following conjectures.

**Conjecture 1.1** (Boucksom’s conjecture, cf. [4]). If a compact complex manifold $X$ carries a closed positive $(1, 1)$-current $T$ with $\int_X T^n_{ac} > 0$, then the class $\{T\} \cdot X$ is big, and thus $X$ is in the Fujiki class $\mathcal{C}$.

**Conjecture 1.2** (Demailly-Păun’s conjecture, cf. [12]). Let $X$ be a compact complex manifold of complex dimension $n$ and $\alpha \in H_{\partial\bar{\partial}}^{1,1}(X, \mathbb{R})$ be a nef class, if $\int_X \alpha^n > 0$, then $\alpha$ is big and thus $X$ is in the Fujiki class $\mathcal{C}$.

Recently, there has been progress of the above conjectures.

If the class $\{T\}$ is assumed to be integral (i.e. $T$ is a curvature current associated to a singular metric of a pseudo-effective holomorphic line bundle), the conjecture is proved by Popovici [19] by using singular Holomorphic Morse inequalities of Bonavero [3] and an improved regularization of currents with mass control. But in general case, there is no holomorphic Morse type inequalities for real transcendental $(1, 1)$-class. For this reason, Boucksom-Demailly-Păun-Peternell posed the following

**Conjecture 1.3** (Boucksom-Demailly-Păun-Peternell’s conjecture, cf. [5]). Let $X$ be a compact complex manifold of complex dimension $n$. Let $\alpha$ be a closed real $(1, 1)$-form and let $X(\alpha, \leq 1)$ be the set where $\alpha$ non degenerate and has at most one negative eigenvalue. If $\int_{X(\alpha, \leq 1)} \alpha^n > 0$, then the class $\{\alpha\} \in H_{\partial\bar{\partial}}^{1,1}(X, \mathbb{R})$ is big and

$$\text{vol}(\{\alpha\}) := \sup_{\theta \in [0, \pi]} \int_{X \setminus \text{Sing}(T)} T^n \geq \int_{X(\alpha, \leq 1)} \alpha^n,$$

where $T$ ranges over all Kähler currents $T \in \{\alpha\}$ with analytic singularities.

If $X$ is assumed to be compact Kähler, then

- Conjecture is proved by Demailly-Păun in [12].
- Conjecture is proved by Boucksom in [4], i.e. $\alpha$ is big if and only if $\text{vol}(\alpha) > 0$. It is worth to mention that if $\alpha$ is nef, then $\text{vol}(\alpha) = \int_X \alpha^n$, thus Boucksom’s proof of Conjecture yields a proof of Conjecture.

If $(X, \omega)$ is only assumed to be compact Hermitian, then

- Under the assumption that $\partial\bar{\partial}\omega^l = 0$ for all $l$, Chiose gives a proof of Conjecture in [6] by combining Lamari’s criterion and Tosatti-Weinkove’s solution to complex Monge-Ampère
equation on compact Hermitian manifolds [24]. See also [23] for even more simplified proof in this case. Note that there always exists such a metric on any compact complex surface.

- Under the assumption that $\partial \bar{\partial} \omega = 0$, and the class $\alpha$ is semi-positive, Nguyen proved both Conjecture 1.1 and Conjecture 1.2 in [18] by combining Lamari’s criterion and analysis of degenerate complex Monge-Ampère equations.
- Under the assumption that $\partial \bar{\partial} \omega^l = 0$ for all $l$, the author gives a proof of Conjecture 1.1 by modifying Chiose’s arguments.

All the above positive results are called Grauert-Reimenschneider type criterion. For the convenience of the reader, we state theorems of Nguyen and the author as follows.

**Theorem 1.1** ([18]). Suppose that $X$ is a compact complex manifold equipped with a pluriclosed metric $\omega$, i.e. $\partial \bar{\partial} \omega = 0$, $\{\beta\}$ is a semi-positive class of type $(1,1)$, such that $\int_X \beta^n > 0$, then $\{\beta\}$ contains a Kähler current. Thus this gives a partial solution to the Conjecture 1.2.

**Theorem 1.2** ([25]). Suppose that $X$ is a compact complex manifold equipped with a Hermitian metric $\omega$ satisfying $\partial \bar{\partial} \omega^k = 0$ for all $k$, the volume $\text{vol}(\alpha)$ of a pseudo-effective class $\alpha \in H^{1,1}_{\partial \bar{\partial}}(X, \mathbb{R})$ is defined and proved to be finite. Then it is proved that if $\alpha$ is pseudo-effective and $\text{vol}(\alpha) > 0$, then $\alpha$ is big, thus $X$ is in the Fujiki class $\mathcal{C}$, and finally Kähler. This gives a partial solution to the Conjecture 1.1.

**Remark 1.1.** It is worth to mention that Theorem 1.2 drops the assumption of nefness of the class $\alpha$, which is an improvement of Chiose’s result, since in general the nef cone of a compact Hermitian manifold is a subset of the pseudo effective cone.

In this paper, absorbing new techniques towards the above mentioned conjectures, we first give a simplified proof of Theorem 1.2. Secondly, as an easy observation, we point out that Nguyen’s results can be generalized to the following

**Theorem 1.3.** Let $(X, \omega)$ be a compact complex manifold of complex dimension $n$, $\omega$ be a Hermitian metric with $\partial \bar{\partial} \omega = 0$, and $T$ be a closed positive current with analytic singularities (see Definition 2.2), if $\int_X T_{ac}^n > 0$, then $\{T\}$ is big. Thus $X$ is in the Fujiki class $\mathcal{C}$, and finally Kähler. Where $T_{ac}$ is the absolutely continuous part in the Lebesgue decomposition of $T$ with respect to the Lebesgue measure on $X$.

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2. Preliminaries

2.1. $\partial \bar{\partial}$-cohomology. Let $X$ be an arbitrary compact complex manifold of complex dimension $n$. Since the $\partial \partial$-lemma does not hold in general, it is better to work with $\partial \bar{\partial}$-cohomology which is defined as

$$H^{p,q}_{\partial \bar{\partial}}(X, \mathbb{C}) = \left( C^\infty(X, \Lambda^{p,q} T^*_X) \cap \ker d \right) / \partial \bar{\partial} C^\infty(X, \Lambda^{p-1,q-1} T^*_X).$$

By means of the Frölicher spectral sequence, one can see that $H^{p,q}_{\partial \bar{\partial}}(X, \mathbb{C})$ is finite dimensional and can be computed either with spaces of smooth forms or with currents. In both cases, the quotient topology of $H^{p,q}_{\partial \bar{\partial}}(X, \mathbb{C})$ induced by the Fréchet topology of smooth forms or by the weak topology of currents is Hausdorff, and the quotient map under this Hausdorff topology is continuous and open.

In this paper, we will just need the $(1,1)$-cohomology space $H^{1,1}_{\partial \bar{\partial}}(X, \mathbb{C})$. The real structure on the space of $(1,1)$-smooth forms (or $(1,1)$-currents) induces a real structure on $H^{1,1}_{\partial \bar{\partial}}(X, \mathbb{C})$, and we denote by $H^{1,1}_{\partial \bar{\partial}}(X, \mathbb{R})$ the space of real points. A class $\alpha \in H^{1,1}_{\partial \bar{\partial}}(X, \mathbb{C})$ can be seen as an affine space of closed $(1,1)$-currents. We denote by $\{T\} \in H^{1,1}_{\partial \bar{\partial}}(X, \mathbb{C})$ the class of the current $T$. Since $i \partial \bar{\partial}$ is a real operator (on forms or currents), if $T$ is a real closed $(1,1)$-current, its class $\{T\}$ lies in $H^{1,1}_{\partial \bar{\partial}}(X, \mathbb{R})$ and consists of all the closed currents $T + i \partial \bar{\partial} \varphi$ where $\varphi$ is a real current of degree 0.

**Definition 2.1.** Let $(X, \omega)$ be a compact Hermitian manifold. A cohomology class $\alpha \in H^{1,1}_{\partial \bar{\partial}}(X, \mathbb{R})$ is said to be pseudo-effective iff it contains a positive current; $\alpha$ is nef iff, for each $\varepsilon > 0$, $\alpha$ contains a smooth form $\theta_\varepsilon$ with $\theta_\varepsilon \geq -\varepsilon \omega$; $\alpha$ is big iff it contains a Kähler current, i.e. a closed $(1,1)$-current $T$ such that $T \geq \varepsilon \omega$ for $\varepsilon > 0$ small enough. Finally, $\alpha$ is a Kähler class iff it contains a Kähler form.

Since any two Hermitian forms $\omega_1$ and $\omega_2$ are commensurable (i.e. $C^{-1} \omega_2 \leq \omega_1 \leq C \omega_2$ for some $C > 0$), these definitions do not depend on the choice of $\omega$.

2.2. Lebesgue decomposition of a current. In this subsection, we refer to [17]. For a measure $\mu$ on a manifold $M$ we denote by $\mu_{ac}$ and $\mu_{sing}$ the uniquely determined absolute continuous and singular measures (with respect to the Lebesgue measure on $M$) such that

$$\mu = \mu_{ac} + \mu_{sing}$$

which is called the Lebesgue decomposition of $\mu$. If $T$ is a $(1,1)$-current of order 0 on $X$, written locally $T = i \sum T_{ij} dz_i \wedge d\bar{z}_j$, we defines its
absolute continuous and singular components by
\[ T_{ac} = i \sum (T_{ij})_{ac} dz_i \wedge d\bar{z}_j, \]
\[ T_{sing} = i \sum (T_{ij})_{sing} dz_i \wedge d\bar{z}_j. \]
The Lebesgue decomposition of \( T \) is then
\[ T = T_{ac} + T_{sing}. \]
If \( T \geq 0 \), it follows that \( T_{ac} \geq 0 \) and \( T_{sing} \geq 0 \). Moreover, if \( T \geq \alpha \) for a continuous \((1,1)\)-form \( \alpha \), then \( T_{ac} \geq \alpha \), \( T_{sing} \geq 0 \). The Radon-Nikodym theorem insures that \( T_{ac} \) is (the current associated to) a \((1,1)\)-form with \( L^1_{loc} \) coefficients. The form \( T_{ac}(x)^n \) exists for almost all \( x \in X \) and is denoted \( T_{ac}^n \).

Note that \( T_{ac} \) in general is not closed, even when \( T \) is, so that the decomposition doesn’t induce a significant decomposition at the cohomological level. However, when \( T \) is a closed positive \((1,1)\)-current with analytic singularities along a subscheme \( V \), the residual part \( R \) in Siu decomposition (c.f. [21]) of \( T \) is nothing but \( T_{ac} \), and the divisorial part \( \sum_k \nu(T,Y_k)[Y_k] \) is \( T_{sing} \). The following facts are well-known.

**Lemma 2.1 (c.f. [4]).** Let \( f : Y \to X \) be a proper surjective holomorphic map. If \( \alpha \) is a locally integrable form of bidimension \((k,k)\) on \( Y \), then the push-forward current \( f_*\alpha \) is absolutely continuous, hence a locally integrable form of bidimension \((k,k)\). In particular, when \( T \) is a positive current on \( Y \), the push-forward current \( f_*(T_{ac}) \) is absolutely continuous, and we have the formula \( f_*(T_{ac}) = (f_*T)_{ac} \).

2.3. Resolution of singularities.

**Definition 2.2 (Currents with analytic singularities).** We say that a function \( \phi \) on \( X \) has analytic singularities along a subscheme \( V(\mathcal{I}) \) (corresponding to a coherent ideal sheaf \( \mathcal{I} \)) if there exists \( c > 0 \) such that \( \phi \) is locally congruent to \( \frac{c}{2} \log(\sum |f_i|^2) \) modulo smooth functions, where \( f_1, \ldots, f_r \) are local generators of \( \mathcal{I} \). Note that a function with analytic singularities is automatically almost plurisubharmonic, and is smooth away from the support of \( V(\mathcal{I}) \).

We say an almost positive \((1,1)\)-current has analytic singularities, if we can find a smooth form \( \theta \) and a function \( \varphi \) on \( X \) with analytic singularities, such that \( T = \theta + dd^c\varphi \). Note that one can always write \( T = \theta + dd^c\varphi \) with \( \theta \) smooth and \( \varphi \) almost plurisubharmonic on a compact complex manifold.

**Definition 2.3 (Pull back of \((1,1)\)-currents).** Let \( f : Y \to X \) be a surjective holomorphic map between compact complex manifolds and \( T \)
be a closed almost positive \((1,1)\)-current on \(X\). Write \(T = \theta + dd^c \varphi\) for some smooth form \(\theta \in \{T\}\), and \(\varphi\) an almost plurisubharmonic function on \(X\). We define its pull back \(f^*T\) by \(f\) to be \(f^*\theta + dd^c f^* \varphi\). Note that this definition is independent of the choices made, and we have \(\{f^*T\} = f^*\{T\}\).

We now use the notations in Definition 2.2. From [14, 1, 2], one can blow-up \(X\) along \(V(\mathcal{I})\) and resolve the singularities, to get a smooth modification \(\pi: \tilde{X} \to X\), where \(\tilde{X}\) is a compact complex manifold, such that \(\pi^{-1}(\mathcal{I}) = O(-D)\) for some simple normal crossing divisor \(D\) upstairs. The pull back \(\pi^*T\) clearly has analytic singularities along \(V(\pi^{-1}(\mathcal{I})) = D\), thus its Siu decomposition writes

\[
\pi^*T = (\pi^*T)_{ac} + (\pi^*T)_{sing} = \theta + c[D],
\]

where \(\theta\) is a smooth \((1,1)\)-form. If \(T \geq \gamma\) for some smooth form \(\gamma\), then \(\pi^*T \geq \gamma\), and thus \(\theta \geq \pi^*\gamma\). We call this operation a resolution of the singularities of \(T\).

2.4. Regularization of currents. The following celebrated regularization theorem is due to Demailly.

**Theorem 2.2** (c.f. [7, 10]). Let \(T\) be a closed almost positive \((1,1)\)-current on a compact Hermitian manifold \((X, \omega)\). Suppose that \(T \geq \gamma\) for some smooth \((1,1)\)-form \(\gamma\) on \(X\). Then there exists a sequence \(T_k\) of currents with analytic singularities in \(\{T\}\) which converges weakly to \(T\), and \(T_{k,ac}(x) \to T_{ac}(x)\) a.e., such that \(T_k \geq \gamma - \varepsilon_k \omega\) for some sequence \(\varepsilon_k > 0\) decreasing to 0, and such that \(\nu(T_k, x)\) increases to \(\nu(T, x)\) uniformly with respect to \(x \in X\).

2.5. Lamari’s criterion.

**Theorem 2.3** (c.f. [16]). Let \(X\) be an \(n\)-dimensional compact complex manifold and let \(\Phi\) be a real \((k,k)\)-form, then there exists a real \((k-1,k-1)\)-current \(\Psi\) such that \(\Phi + dd^c \Psi\) is positive iff for any strictly positive \(\partial \bar{\partial}\)-closed \((n-k, n-k)\)-forms \(\Upsilon\), we have \(\int_X \Phi \wedge \Upsilon \geq 0\).

**Remark 2.1.** Theorem 2.3 also holds for closed positive \((1,1)\)-currents.

2.6. Complex Monge-Ampère equations. The following theorem on complex Monge-Ampère equations on compact Hermitian manifolds is due to Tosatti-Weinkove [24].

**Theorem 2.4** ([24]). Let \((X, \omega)\) be a compact Hermitian manifold. For any smooth real-valued function \(F\) on \(X\), there exist a unique real
number $C > 0$ and a unique smooth real-valued function $\phi$ on $X$ solving
\[(\omega + i\partial\bar{\partial}\phi)^n = Ce^F\omega^n,\]
with $\omega + i\partial\bar{\partial}\phi > 0$ and $\sup_X \phi = 0$. Furthermore, if $\partial\bar{\partial}\omega^k = 0$ for $1 \leq k \leq n - 1$, then we have
\[C = \frac{\int_X \omega^n}{\int_X e^F\omega^n}.
\]

3. A simplified proof of Theorem 1.2

From the definition of $\text{vol}(\alpha)$, one can find a positive closed current $S \in \alpha$ such that
\[\int_X S_{ac}^n > \sqrt[2]{\text{vol}(\alpha)} > 0.\]
Then by Demailly’s regularization theorem (Theorem 2.2), combined with Fatou’s lemma, we can find a sequence $T_k$ of closed currents with analytic singularities in $\alpha$ such that
- $T_k \geq -\varepsilon_k \omega$, where $\varepsilon_k \searrow 0$ as $k \to \infty$;
- $\int_X T_k^n_{ac} \geq c > 0$.

For each $k$, we choose a smooth proper modification $\mu_k : X_k \to X$ such that
- $\mu_k^* T_k = \theta_k + D_k$, where $\theta_k \geq -\varepsilon_k \mu_k^* \omega$ is a smooth closed form and $D_k$ is a real effective divisor.
- $\int_X T_k^n_{k,ac} = \int_{X_k} \theta_k^n \geq c > 0$.

Lemma 3.1 (c.f. [11]). Suppose that $(X, \omega)$ is a compact Hermitian manifold with a Hermitian metric $\omega$ satisfying $\partial\bar{\partial}\omega^k = 0$ for all $k$. Let $\pi : \tilde{X} \to X$ be a smooth modification (a tower of blow-ups). Then there exists a Hermitian metric $\Omega$ on $\tilde{X}$ such that
- $\partial\bar{\partial}\Omega^k = 0$ for all $k$ on $\tilde{X}$;
- for any give $\varepsilon > 0$, one can make the inequality $\|\Omega - \mu^* \omega\| < \varepsilon$
holds.

Proof. Suppose that $\tilde{X}$ is obtained as a tower of blow-ups
\[\tilde{X} = X_N \to X_{N-1} \to \cdots \to X_1 \to X_0 = X,\]
where $X_{j+1}$ is the blow-up of $X_j$ along a smooth center $Y_j \subset X_j$. Denote by $E_{j+1} \subset X_{j+1}$ the exceptional divisor, and let $\pi_j : X_{j+1} \to X_j$ be the blow-up map. The line bundle $\mathcal{O}(-E_{j+1})|_{E_{j+1}}$ is equal to $\mathcal{O}_{P(N_j)}(1)$ where $N_j$ is the normal bundle to $Y_j$ in $X_j$. Pick an arbitrary smooth Hermitian metric on $N_j$, use this metric to get an induced Fubini-Study metric on $\mathcal{O}_{P(N_j)}(1)$, and finally extend this metric as a smooth Hermitian metric on the line bundle $\mathcal{O}(-E_{j+1})$. Such a metric has positive curvature along tangent vectors of $X_{j+1}$ which are tangent
to the fibers of $E_{j+1} = P(N_j) \to Y_j$. Assume further that $\omega_j$ is a Gauduchon metric satisfying assumption (*) on $X_j$. Then

$$\Omega_{j+1} = \pi_j^* \omega_j - \varepsilon_{j+1} u_{j+1}$$

where $\pi_j^* \omega_j$ is semi-positive on $X_{j+1}$, positive definite on $X_{j+1} \setminus E_{j+1}$, and also positive definite on tangent vectors of $T_{X_{j+1}} \setminus E_{j+1}$ which are not tangent to the fibers of $E_{j+1} \to Y_j$. It is then easily to see that $\Omega_{j+1} > 0$ by taking $\varepsilon_{j+1} \ll 1$. Thus our final candidate $\Omega$ on $\tilde{X}$ has the form $\Omega = \pi^* \omega - \sum \varepsilon_j \tilde{u}_j$, where $\pi_j = (\pi_{N-1} \circ \cdots \circ \pi_1) u_j$. Since every $u_j$ is a curvature term of a line bundle, the term $\sum \varepsilon_j \tilde{u}_j$ is $d$-closed, thus $\partial \bar{\partial} \Omega^k = 0$. Furthermore, by choosing $\varepsilon_j$ sufficiently small, one can make $\|\Omega - \mu^* \omega\| < \varepsilon$ for any given $\varepsilon > 0$. □

Select on each $X_k$ a Gauduchon metric $\tilde{\omega}_k$ which satisfies $\partial \bar{\partial} \tilde{\omega}_k = 0$ for all $l$, and $\|\tilde{\omega}_k - \mu_k^* \omega\| < \varepsilon$ for any given $\varepsilon > 0$.

In the sequel, we will show that for $k$ large, the class $\theta_k$ is big on $X_k$. We argue by contradiction. By applying Lamari’s criterion (Theorem 2.5), and suppose to the contrary, for any $k$, and any $m \in \mathbb{N}$, there is a Gauduchon metric $\omega_{k,m}$ on $X_k$ such that

$$\int_{X_k} \theta_k \wedge \omega_{k,m}^{n-1} \leq \frac{1}{m} \int_{X_k} \tilde{\omega}_k \wedge \omega_{k,m}^{n-1}.$$ 

Without loss of generality, we assume that

$$\int_{X_k} \tilde{\omega}_k \wedge \omega_{k,m}^{n-1} = 1$$

and therefore

$$\int_{X_k} \theta_k \wedge \omega_{k,m}^{n-1} \leq \frac{1}{m}.$$ 

From Theorem 2.4, we solve the following equation on $X_k$,

$$(\theta_k + \varepsilon_k \mu_k^* \omega + \frac{1}{m} \tilde{\omega}_k + dd^c \varphi_{k,m})^n = C_{k,m} \omega_{k,m}^{n-1} \wedge \tilde{\omega}_k.$$ 

Set $\alpha_{k,m} = \theta_k + \varepsilon_k \mu_k^* \omega + \frac{1}{m} \tilde{\omega}_k + dd^c \varphi_{m}$, then $\alpha_{k,m} > 0$. We need the following

**Lemma 3.2** (c.f. [18, 20]). For every $k, m \in \mathbb{N}^*$, we have that

$$\left( \int_{X_k} \alpha_{k,m} \wedge \omega_{k,m}^{n-1} \right) \left( \int_{X_k} \alpha_{k,m}^{n-1} \wedge \tilde{\omega}_k \right) \geq \frac{1}{n} \left( \int_{X_k} \sqrt{\frac{\alpha_{k,m} \omega_{k,m}^{n-1} \wedge \omega_{k,m}^{n-1}}{\tilde{\omega}_k \tilde{\omega}_k}} \right)^2 = C_{k,m} \frac{n}{n}.$$
Firstly, we have that

\[ C_{k,m} = \int_{X_k} \left( \theta_k + \varepsilon_k \mu_k^* \omega + \frac{1}{m} \tilde{\omega}_k + dd^c \varphi_{k,m} \right)^n \]

\[ = \int_{X_k} \left( \theta_k + \varepsilon_k \mu_k^* \omega + \frac{1}{m} \tilde{\omega}_k \right)^n \geq \int_{X_k} \left( \theta_k + \varepsilon_k \mu_k^* \omega \right)^n \]

\[ = \int_X (T_{k,ac} + \varepsilon_k \omega)^n = \int_X T^n_{k,ac} + O(\varepsilon_k) \]

\[ \geq C > 0 \]

for \( k \) sufficiently large, where \( C \) is a uniform constant independent of \( k \) and \( m \).

Secondly, we have the following observation

\[ \int_{X_k} \alpha_{k,m}^{n-1} \wedge \tilde{\omega}_k = \int_{X_k} \left( \theta_k + \varepsilon_k \mu_k^* \omega + \frac{1}{m} \tilde{\omega}_k + dd^c \varphi_{k,m} \right)^{n-1} \wedge \tilde{\omega}_k \]

\[ = \int_{X_k} \left( \theta_k + \varepsilon_k \mu_k^* \omega + \frac{1}{m} \tilde{\omega}_k \right)^{n-1} \wedge \tilde{\omega}_k \]

\[ \leq \int_X (T_{k,ac} + \varepsilon_k \omega + \frac{2}{m} \omega)^{n-1} \wedge (2\omega) \leq M(>0) \]

for \( k \) sufficiently large, where \( M \) is a uniform constant independent of \( k \) and \( m \).

Thirdly, we see that for \( k \) large

\[ \int_{X_k} \alpha_{k,m} \wedge \omega_{k,m}^{n-1} = \int_{X_k} \left( \theta_k + \varepsilon_k \mu_k^* \omega + \frac{1}{m} \tilde{\omega}_k + dd^c \varphi_{k,m} \right) \wedge \omega_{k,m}^{n-1} \]

\[ = \int_{X_k} \left( \theta_k + \varepsilon_k \mu_k^* \omega + \frac{1}{m} \tilde{\omega}_k \right) \wedge \omega_{k,m}^{n-1} \]

\[ \leq \int_{X_k} \theta_k \wedge \omega_{k,m}^{n-1} + \frac{2}{m} \int_{X_k} \tilde{\omega}_k \wedge \omega_{k,m}^{n-1} \leq \frac{3}{m}. \]

Finally, from Lemma 3.2, (1), (2) and (3), we obtain that for \( k \) large

\[ \frac{3M}{m} \geq \frac{C'}{n}, \]

which is absurd for \( m >> 0 \).

Now we arrive at the fact that, for \( k \) large, \( \{\theta_k\} \) is a big class on \( X_k \). Take a Kähler current \( \Theta \) in \( \{\theta_k\} \), one can get a Kähler current \( (\mu_k)_*(\Theta + D_k) \) on \( X \), which belongs to the class \( \alpha \). Thus \( X \) is in the Fujiki class \( C \), which will force \( X \) to be Kähler, since it supports a \( \partial \bar{\partial} \)-closed Hermitian metric. Thus we complete the proof of Theorem 1.2.
4. Proof of Theorem 1.3

Now let \( \pi : \tilde{X} \to X \) be the resolution of singularities such that \( \pi^{-1}\mathcal{I} \) is just \( \mathcal{O}(-D) \) for some simple normal crossing divisor \( D \) upstairs. Thus we have

\[
\pi^*T = (\pi^*T)_{ac} + (\pi^*T)_{sing} = \theta + c[D].
\]

Now we are on the way to prove Theorem 1.3. Since \( \int_X T_{ac}^n > 0 \), we have that

\[
\int_{\tilde{X}} \theta^n = \int_{\tilde{X}} (\pi^*T)^n_{ac} = \int_X T_{ac}^n > 0,
\]

where second equality follows from Lemma 2.1.

From Lemma 3.1, there is a Hermitian metric \( \Omega \) on \( \tilde{X} \) such that \( \partial \bar{\partial} \Omega = 0 \).

Since \( \theta \) is smooth and semi-positive, \( \int_{\tilde{X}} \theta^n > 0 \), from Nguyen’s Theorem 1.1 one can conclude that the class \( [\theta] \) is big. Thus the class \( [\pi_*(\pi^*T)] = [T] = [\pi_*(\theta + c[D])] \) is big. So \( X \) is in the Fujiki class \( \mathcal{C} \), and finally Kähler for the same reason as above, which completes the proof of Theorem 1.3.

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