Regime-switching shot-noise processes
and longevity bond pricing

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Abstract. In this paper, we consider the valuation of longevity bonds under a regime-switching interest rate and a regime-switching force of mortality model. The model assumes that the interest rate is driven by economic and environmental conditions described by a homogenous Markov chain and that the stochastic force of mortality is modeled by the sum of a regime-switching Gompertz–Makeham model and a regime-switching shot-noise process. Using the conditional Laplace transform of the regime-switching shot-noise process, we give a formula for the longevity bond price in terms of a couple of system partial differential equations. The pricing formula is also derived by using the concept of stochastic flows and the idea of change of measure.

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1Introduction

Longevity risk is the risk of more lives surviving than expected or observed death rates being lower than expected. In the past few decades or so, various medical advances and health technologies tend to reduce the number of deaths. As a consequence, life insurers are facing an increasing longevity risk and have experienced losses in their life annuity business. See, for example, China’s average life expectancy increased from 71.5 years in 2000 to 74.83 years in the end of 2010. Insurers, pension funds, and others in the financial markets have always depended on actuarial data that extrapolates from the past, and thus, their reserves are not sufficient to cover their future liabilities. Recently, a proposal asking people to work longer and draw their pensions later

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in China has been raised to cope with the shortfall in pensions. In fact, the underfunding of too many defined-benefit pension systems by policy makers in the past has put employees’ retirement security at risk in many countries. Therefore, how to manage longevity risk has become a hot topic for both academic researchers and practitioners. One of the most discussed methods of managing this risk is securitization, that is, isolating the cash flows linked to longevity risk and repackaging them into cash flows traded in capital markets. The longevity bond is one of the basic mortality-linked securities proposed in the academic literature for hedging longevity risk (see, e.g., [5, 6]). The first modern issue of a longevity bond, with a maturity of 25 years, was announced in November 2004 by the European Investment Bank and BNP Paribas. Unfortunately, such a product was not successfully accepted by the investors. Although the longevity bonds have not been traded in real markets until now, life insurers, annuity writers, and pension fund companies all believe that the longevity bonds are effective instruments to manage longevity risk. So, there is a practical need to study how to price the longevity bonds.

There are three popular approaches for pricing longevity bonds: the risk-neutral valuation approach, the Wang transformation approach, and the Heath–Jarrow–Morton (HJM) methodology. For the risk-neutral valuation approach, we refer to [4, 9, 14, 25]. For the Wang transformation approach, see [10, 13, 24, 32]. For the HJM methodology, we refer the interested readers to [3, 9]. In this paper, we will use the risk-neutral valuation approach to price the longevity bonds. So, modeling the force of mortality under a risk-neutral or risk-adjusted measure is a key factor of pricing the longevity bonds.

The description of observed age patterns of adult mortality with mathematical models is one of the oldest and most important topics in demography. So far, a lot of stochastic mortality models have been proposed to forecast mortality rates over time, such as affine jump-diffusion processes (see, e.g., [4]), CIR model, stochastic Gompertz-type model (see, e.g., [25]), and so on. Recently, Milidonis et al. [26] proposed a Markov regime-switching stochastic mortality model and provided empirical evidence for the switching behavior of the mortality rates by using the 1901–2005 U.S. population mortality index. Their empirical results revealed that the Markov regime-switching stochastic mortality models performed better than some important models, such as Lee–Carter [23] and Lin–Cox [24] models. They also considered the pricing of mortality securities under the Markov regime-switching stochastic mortality model and found that the regime-switching stochastic mortality model could correct some of the pricing biases, such as overestimation of the market price of mortality risk or underpricing mortality-linked securities.

In fact, Markov regime-switching models have been widely used in different branches of modern financial economics; see [8, 19, 27, 31]. In a regime-switching model, the market is assumed to be in different states depending on the state of the economy. Regime shift from one economic state to another may occur due to various financial factors like changes in business conditions, management decisions, and other macroeconomic conditions. Many researchers have empirically verified the advantages of using the Markov regime-switching model. In the bond market, the switching behavior of market interest rate has been well documented in the empirical finance literature. For example, Ang and Bekaert [1, 2] used interest rate data from the United States, Germany, and the United Kingdom to show empirically that the switching behavior of market interest rates was attributed to business cycles. In the stock market, by using monthly returns data from the Standard and Poor’s 500 and the Toronto Stock Exchange 300 indices, Hardy [20] found that the regime-switching lognormal model fitted to the monthly returns data much better than other econometric models, such as the independent lognormal model and the ARCH-type models. In the insurance market, Hardy et al. [21] found that regime-switching models outperformed other important models in modeling long-term equity returns for equity-linked guarantees. Indeed, life insurance products and pension funds are long-term instruments. Therefore, there is a practical need to incorporate the impact of macroeconomic and environmental conditions into modeling the dynamics of force of mortality. Intuitively, the force of mortality should be much lower when the macroeconomic and environmental conditions are both in good states.

Motivated by Milidonis et al. [26] and all of the afore-mentioned researchers, in this paper, we develop a Markov, regime-switching pricing model for valuing the longevity bonds that can incorporate the impact of changes in economic and environmental conditions on the dynamics of stochastic short rate and mortality. Recently, Shen and Siu [30] also proposed a regime-switching model for valuing longevity bonds. In their paper, the dynamics of the force of mortality was described by a Markov, regime-switching, Ornstein–Uhlenbeck (OU) process with jumps. Note that the force of mortality modeled by them may take a negative
value. However, we assume that the force of mortality is governed by the sum of a regime-switching, dynamic Gompertz–Makeham model and a regime-switching shot-noise model with only upward jumps. Consequently, the force of mortality in our model is always positive. Furthermore, we propose a pricing method different from them.

The regime-switching shot-noise process, which is a generalization of the shot-noise process (see, e.g., [11, 12, 15, 22]), can be well used to measure the impact of shock events and the economic and environmental factors. The regime-switching shot-noise process measures the frequency, magnitude, and time period needed to go back to the previous level of force of mortality immediately after shock events occur. As time passes, the regime-switching shot-noise process decreases as people do their best to overcome these difficulties after the arrival of one of the shock events. This decrease continues until another event occurs, which will result in a positive jump associated with the economic and environmental state in the regime-switching shot-noise process. The motivation for adding a regime-switching shot-noise process to the force of mortality is twofold. First, when people suffer some severe contagious diseases, such as SARS coronavirus or natural disasters, the force of mortality may jump upward. As time passes, some treatments can reduce the risk of death. Second, the regime-switching shot-noise process allows us to obtain the Laplace transform of the regime-switching shot-noise process. Based on the Laplace transform, we can derive some formulas for the price of the longevity bond and survival probability. For modeling the short-rate dynamics, following Buffington and Elliott [8], we assume that the interest rate is a deterministic function of economic and environmental conditions. Under the regime-switching pricing model, we provide two different approaches to price the longevity bonds. This paper is organized as follows. Section 2 presents the dynamics of the short interest rate and the force of mortality and derives the pricing formula for the longevity bond based on the conditional Laplace transform of the regime-switching shot-noise process. In Section 3, we provide an alternative approach to evaluate the longevity bond based on the concept of stochastic flows and the change-of-measure technique. Section 4 shows that the regime-switching shot-noise process allows us to obtain the Laplace transform of the regime-switching shot-noise process. Based on the Laplace transform, we can derive some formulas for the price of the longevity bond and survival probability. For modeling the short-rate dynamics, following Buffington and Elliott [8], we assume that the interest rate is a deterministic function of economic and environmental conditions. Under the regime-switching pricing model, we provide two different approaches to price the longevity bonds. This paper is organized as follows. Section 2 presents the dynamics of the short interest rate and the force of mortality and derives the pricing formula for the longevity bond based on the conditional Laplace transform of the regime-switching shot-noise process. In Section 3, we provide an alternative approach to evaluate the longevity bond based on the concept of stochastic flows and the change-of-measure technique. Section 4 shows that the formulas obtained from the two different approaches are exactly equivalent. Section 5 gives some numerical results. Section 6 concludes. The proof of one main result is presented in the Appendix.

2 The model and longevity bond price

Consider a continuous-time model with finite time horizon \( T := [0, T] \), where \( T < \infty \); all random variables in this paper are assumed to be defined on a filtered complete probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathcal{T}}, \mathbb{P}) \). In this paper, we intend to use the risk-neutral valuation approach. So, we suppose that \( \mathbb{P} \) is a risk-neutral probability measure.

Following Shen and Siu [30], we use a continuous-time, finite-state, observable Markov chain to describe the evolution of economic and environmental factors over time. As explained in [30], using two Markov chains, one can model the evolution of the state of an economy over time and to model the state of an environment over time. If the numbers of states of the two chains are equal, then one can combine the two Markov chains to form one Markov chain, which describes the joint states of economic and environmental factors. Let \( X := \{X_t | t \in \mathcal{T}\} \) be a homogenous continuous-time irreducible Markov chain with generator \( \mathcal{Q} = (q_{ij})_{i,j=1,2,...,N} \) generating a filtration \( \mathcal{F}^X = \{\mathcal{F}^X_t | t \in \mathcal{T}\} \). As in [8], the state space of \( X \) can be taken to be, without loss of generality, the set of unit vectors \( \{e_1, e_2, \ldots, e_N\} \), \( e_i = (0, \ldots, 0, 1, 0, \ldots, 0)^* \in \mathbb{R}^N \), where * denotes the transpose of a vector or a matrix. The states of the chain \( X \) represent different combined states of economic and environmental factors. Elliott [16] and Elliott et al. [17] provided the following semimartingale decomposition for \( X \):

\[
X_t = X_0 + \int_0^t Q^* X_s \, ds + M_t, \tag{2.1}
\]

where \( \{M_t | t \in \mathcal{T}\} \) is an \( \mathbb{R}^N \)-valued martingale with respect to the filtration generated by \( \{X_t | t \in \mathcal{T}\} \). Assume that the stochastic interest rate is given by \( r_t = \langle r, X_t\rangle \), where \( r = (r^1, \ldots, r^N)^* \in \mathbb{R}_+^N \) with \( \mathbb{R}_+ := (0, \infty) \). Here \( \langle \cdot, \cdot \rangle \) denotes the Euclidean scalar product in \( \mathbb{R}^N \), that is, for any \( x, y \in \mathbb{R}^N \), \( \langle x, y \rangle = \sum_{i=1}^N x_i y_i \).

Now we model the dynamics of the force of mortality under the risk-neutral measure.
Consider a regime-switching shot-noise process $S := \{S_t \mid t \in \mathcal{T}\}$ on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathcal{T}}, P)$ such that $S$ solves the stochastic differential equation (SDE)

$$dS_t = -\delta S_t \, dt + dJ_t, \quad t \geq 0,$$

where $\delta > 0$ is a constant, and $J_t = \sum_{j=1}^{N_t} Y_j$ is a regime-switching compound Poisson process. Here $N_t$ is a regime-switching Poisson process with intensity $\rho(s) = (\rho, X_s)$, where $\rho = (\rho^1, \ldots, \rho^N) \in \mathbb{R}^N$ with $\rho^i > 0$ for each $i = 1, 2, \ldots, N$. That is to say, if $X_s = e_j$ for all $s$ in a small interval $(t, t + h]$, then $N_{t+h} - N_t$ has a Poisson distribution with parameter $\rho^j$. $Y_n$ is the amount of the $n$th jump that is independent of $N_t$ given the state of the Markov chain at the $n$th claim time. Assume that the jump amount at time $t$ has the density $f_t(\cdot)$ concentrated on $(0, \infty)$, where $f_t(\cdot) = (f(\cdot), X_t)$ with $f(\cdot) = (f^1(\cdot), \ldots, f^N(\cdot)) \in \mathbb{R}^N$. Assume that $J_t$ does not jump at the jump times of $X$. If there is no regime switching, then $S$ is a special case of the shot-noise process (see, e.g., [11, 12, 15]). In this paper, the process $S$ is a regime-switching version of a shot-noise process.

We assume that under the risk-neutral measure $P$, the evolution of the force of mortality for an individual aged $y$ at time $t$, denoted by $\mu_y := \{\mu_y(t) \mid t \in \mathcal{T}\}$, is governed by the following process:

$$\mu_y(t) = \overline{\mu}_t + S_t, \quad t \geq 0,$$

where $\overline{\mu}_t = (\overline{\mu}_t, X_t)$ for a vector $\overline{\mu}_t = (\overline{\mu}_1(t), \ldots, \overline{\mu}_N(t))^* \in \mathbb{R}_+^N$, with $\overline{\mu}_i(t) = h_i + l_i e_i^{y+t} \in \mathbb{R}_+$, $l_i > 0$, $c_i > 1$, $h_i > -l_i$ for each $i = 1, 2, \ldots, N$. Note that, if we remove $S_t$ from $\mu_y(t)$, then the force of mortality is a regime-switching dynamic Gompertz–Makeham model. For the dynamic Gompertz–Makeham model, we refer to [7]. Consequently, (2.3) is a generalized Gompertz–Makeham model. Here, $S_t$ denotes the impact of shocks associated with economic and environmental factors, such as contagious diseases and natural disasters, on the force of mortality. When a shock occurs, this may cause an increase of the force of mortality. With time passing, the regime-switching shot-noise process decreases as people find ways to overcome these difficulties after the arrival of one of the shock events. This decrease continues until another event occurs, which will result in a positive jump in the regime-switching shot-noise process.

**Remark 1.** If the process $\overline{\mu}_t$ in (2.3) is follows the regime-switching Lee–Carter model

$$\overline{\mu}_t = e^{\alpha_y + \beta_y k(t)},$$

where $\alpha_y, \beta_y$ are deterministic constants or functions, and $k(t) = (k_1, \ldots, k_N)$ for a vector $k = (k_1, \ldots, k_N) \in \mathbb{R}^N$, then the following analysis is similar.

**Remark 2.** The process $S_t$ modeled by (2.2) only concentrates on the positive jump component, whereas the results can be easily enriched by relaxing negative jumps or adding a diffusion component. But this generalization will lead to a positive probability that the force of mortality will take a negative value. Therefore, one should adjust the parameters so that the probability of getting a negative value of the force of mortality is small.

Denote the filtration by

$$\mathcal{F}_t = \mathcal{F}_t^X \vee \mathcal{F}_t^S,$$

where $\mathcal{F}_t^S = \sigma(S_u: 0 \leq u \leq t)$. Now we establish the definition of a longevity bond. Define the following survivor index:

$$T-t p_{y+t} = \exp \left( - \int_T^t \mu_y(s) \, ds \right).$$  \hfill (2.4)
The survivor index is similar to the survival probability in actuarial mathematics. Obviously, if the force of mortality is a deterministic function, then \( T_{t}^{-1}P_{y+t} \) is the same as the survival probability. But, the survivor index is a random variable in the stochastic mortality environment. A longevity bond is a kind of zero-coupon bonds with the survivor index being its terminal payoff. Then the price at time \( t \) of this longevity bond is

\[
\Pi(t, T) = \mathbb{E}[e^{-\int_t^T (r_s + \mu_x(s)) \, ds} \mid \mathcal{F}_t] = \mathbb{E}[e^{-\int_t^T (r_s + \bar{\mu}_x + S_s) \, ds} \mid \mathcal{F}_t].
\] (2.5)

From (2.2) we can obtain that for \( s \geq t > 0 \),

\[
S_s = e^{-\delta(s-t)} S_t + \int_t^s e^{-\delta(s-u)} \, dJ_u.
\] (2.6)

By using a stochastic Fubini theorem (see [29]), we integrate \( S_u \) from \( t \) to \( T \) and obtain

\[
\int_t^T S_u \, du = \omega(t, T) S_t + \int_t^T \omega(u, T) \, dJ_u,
\] (2.7)

where

\[
\omega(u, T) = \frac{1 - e^{-\delta(T-u)}}{\delta}.
\] (2.8)

Note that \((X, S)\) is a two-dimensional Markov process with respect to \( \mathcal{F} \). Consequently,

\[
\Pi(t, T) = \mathbb{E}[e^{-\int_t^T (r_s + \bar{\mu}_x + S_s) \, ds} \mid X_t, S_t] =: \pi(t, T, X_t, S_t).
\] (2.9)

To derive the expression for \( \pi(t, T, X_t, S_t) \), we first give two useful lemmas.

**Lemma 1.** Let \( S \) be a regime-switching shot-noise process given by (2.2). Then we have

\[
\mathbb{E}[e^{-\int_t^T S_u \, ds} \mid \mathcal{F}_t^S \vee \mathcal{F}_T^X] = e^{-\omega(t, T) S_t + \int_t^T \mathbb{E} [\omega(u, T) \mid \mathcal{F}_u] \, dJ_u}
\] (2.10)

where \( \omega(t, T) \) is given by (2.8), and \( \mathbb{G}_s = (g_1^1, \ldots, g^N_s)^* \in \mathbb{R}^N \) with

\[
g_j^s = \int_0^\infty \rho^j (e^{-\omega(s, T)x} - 1) f_j^1(x) \, dx, \quad j = 1, \ldots, N.
\] (2.11)

**Proof.** By using (2.7) we have

\[
\mathbb{E}[e^{-\int_t^T S_u \, ds} \mid \mathcal{F}_t^S \vee \mathcal{F}_T^X] = e^{-\omega(t, T) S_t + \int_t^T e^{-\omega(u, T) \, dJ_u} \mid \mathcal{F}_t^S \vee \mathcal{F}_T^X]
\]

\[
= e^{-\omega(t, T) S_t} \mathbb{E}[e^{-\int_t^T \omega(u, T) \, dJ_u} \mid \mathcal{F}_T^X].
\] (2.12)

Since the path of the Markov chain \( \{X_s \mid s \in T\} \) is known, denote the jump times in the interval \((t, T]\) of the Markov chain \( X_s \) by \( t = T_0 < T_1 < \cdots < T_k = T \). Then \( J_{T_j} - J_{T_{j-1}}, j = 1, 2, \ldots, k \), are mutually independent, and consequently,

\[
\mathbb{E}[e^{-\int_t^T \omega(u, T) \, dJ_u} \mid \mathcal{F}_T^X] = \prod_{j=1}^k \mathbb{E}[e^{-\int_{T_{j-1}}^{T_j} \omega(u, T) \, dJ_u} \mid X_s = X_{T_{j-1}}, T_{j-1} \leq s < T_j].
\] (2.13)

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It remains to calculate the conditional expectation in (2.13). Conditionally on the number of jumps of \( N_{T_{j}} - N_{T_{j-1}} \), we have

\[
E\left[ e^{-\int_{T_{j-1}}^{T_{j}} \omega(u,T) \, du} \mid X_{s} = X_{T_{j-1}}, \ T_{j-1} \leq s < T_{j} \right] \\
= \sum_{m=0}^{\infty} P(N_{T_{j}} - N_{T_{j-1}} = m) \\
\times E\left[ e^{-\sum_{n=1}^{m} \omega(T_{n},T) Y_{n}} \mid N_{T_{j}} - N_{T_{j-1}} = m, \ X_{s} = X_{T_{j-1}}, \ T_{j-1} \leq s < T_{j} \right],
\]

where \( T_{1}, \ldots, T_{m} \) are the \( m \) jumps of \( N \) falling into \([T_{j-1}, T_{j})\) conditionally on \( N_{T_{j}} - N_{T_{j-1}} = m \). Let \( U_{1}, \ldots, U_{m} \) be independent random variables, uniformly distributed on \([T_{j-1}, T_{j})\). It is well known that, conditionally on \( N_{T_{j}} - N_{T_{j-1}} = m \), \((T_{1}, \ldots, T_{m})\) has the same distribution as \((U_{(1)}, \ldots, U_{(m)})\), where \( U_{(k)} \) denotes the \( k \)-th order statistic. So, we have

\[
E\left[ e^{-\int_{T_{j-1}}^{T_{j}} \omega(u,T) \, du} \mid X_{s} = X_{T_{j-1}}, \ T_{j-1} \leq s < T_{j} \right] \\
= \sum_{m=1}^{\infty} \frac{(\rho(T_{j-1})(T_{j} - T_{j-1}))^{m}}{m!} e^{-\rho(T_{j-1})(T_{j} - T_{j-1})} \\
\times \left( \int_{T_{j-1}}^{T_{j}} \frac{1}{T_{j} - T_{j-1}} \int_{0}^{\infty} e^{-\omega(s,T)x} \langle f_{s}(x), X_{s} \rangle \, dx \, ds \right)^{m} \\
= e^{\int_{T_{j-1}}^{T_{j}} \langle \theta, X_{s} \rangle \, ds}, \quad (2.14)
\]

where the last equality holds because \( \rho(s) = \rho(T_{j-1}) \) for \( s \in [T_{j-1}, T_{j}) \). Then combining (2.12)–(2.14) yields (2.10). The proof is completed. \( \square \)

Now we give another key result. Define the \( \mathbb{R} \)-valued process

\[
V(t, T) = E\left[ e^{-\int_{t}^{T} \theta_{u} \, du} \mid \mathcal{F}_{t}^{X} \right], \quad (2.15)
\]

where \( \theta_{t} = (\theta_{t}, X_{t}) \) with \( \theta_{t} = (\theta_{1}(t), \ldots, \theta_{N}(t))^{*} \in \mathbb{R}^{N} \). Here, for each \( i = 1, \ldots, N \), \( \theta_{i}(t) \) is a continuous deterministic function with values in \((0, \infty)\), so that \( V(t, T) \leq 1 \). Note that \( X \) is a Markov process with respect to \( \mathcal{F}^{X} \). Consequently,

\[
V(t, T) = E\left[ e^{-\int_{t}^{T} \theta_{u} \, du} \mid X_{t} \right] := F(t, T, X_{t}). \quad (2.16)
\]

Assume that \( V(t, T) \) is continuously differentiable with respect to \( t \). For each \( i = 1, 2, \ldots, N \), write \( F_{i}(t, T) = E\left[ e^{-\int_{t}^{T} \theta_{u} \, du} \mid X_{t} = e_{i} \right] \) and

\[
F(t, T) := (F_{1}(t, T), F_{2}(t, T), \ldots, F_{N}(t, T))^{*} \in \mathbb{R}^{N}.
\]

It is clear that \( V(t, T) = F(t, T, X_{t}) = (F(t, T), X_{t}) \). For notational simplicity, we define \( \text{diag}(\eta) \) as the diagonal matrix with the diagonal entries given by a vector \( \eta = (\eta_{1}, \ldots, \eta_{N})^{*} \in \mathbb{R}^{N} \).

If all of \( \theta_{i}(t) \) are constants, then Lemma A.1 in [8] gives

\[
V(t, T) = \langle e^{(Q - \text{diag}(\theta))(T-t)}1, X_{t} \rangle, \quad (2.17)
\]

where \( 1 = (1, \ldots, 1)^{*} \in \mathbb{R}^{N} \). The following lemma generalizes the result of Buffington and Elliott [8].
Lemma 2. Let $V(t, T)$ be defined by (2.16). Then we have

$$V(t, T) = \sum_{i=1}^{N} F_i(t, T) \langle e_i, X_t \rangle,$$

where $F_1, F_2, \ldots, F_N$ satisfy the following system of $N$ coupled partial differential equations (PDEs):

$$-\theta_i(t) F_i(t, T) + \frac{\partial F_i}{\partial t} + \sum_{k=1}^{N} q_{ik} F_k(t, T) = 0, \quad F_i(T, T) = 1, \quad i = 1, \ldots, N. \quad (2.19)$$

Proof. Define

$$Z_t = Y_t V(t, T) = E_\mathcal{F} \left[ e^{-\int_0^T \theta_u \, du} \mid \mathcal{F}_t \right] = Y_t F(t, T, X_t),$$

where $Y_t = e^{-\int_0^t \theta_u \, du}$. Note that $\theta_u > 0$. Consequently, $Z_t$ is a bounded $\mathcal{F}_t$-martingale. Applying integration by parts to $Z_t$ yields

$$dZ_t = -\theta_t Z_t \, dt + \left( \frac{\partial F}{\partial t} + \langle F(t, T), Q X_t \rangle \right) Y_t \, dt + Y_t \langle F(t, T), dM_t \rangle. \quad (2.20)$$

Since $\theta_i(t) > 0$ for each $i = 1, \ldots, N$, the last term of (2.20) is a square-integrable martingale. Hence, we must have

$$-\theta_t F(t, T, x) + \frac{\partial F}{\partial t} + \langle F(t, T), Q^* x \rangle = 0 \quad (2.21)$$

with

$$F(T, T, x) = 1. \quad (2.22)$$

Since $x$ takes one of the values $e_i, i = 1, 2, \ldots, N$, Eqs. (2.21) and (2.22) correspond to Eq. (2.19). So, $V(t, T) = \langle F(t, T), X_t \rangle = \sum_{i=1}^{N} F_i(t, T) \langle e_i, X_t \rangle$. The proof is completed. \qed

The following results present the longevity bond price and survival probability under the risk-neutral measure, respectively.

Theorem 1. The price at time $t$ of the longevity bond is given by

$$\pi(t, T, X_t, S_t) = e^{-\omega(t, T) S_t} \sum_{i=1}^{N} B_i(t, T) \langle e_i, X_t \rangle,$$

where $B_1, B_2, \ldots, B_N$ satisfy the following system of $N$ PDEs:

$$-(r^i + h^i(t) - g^i_j) B_i(t, T) + \frac{\partial B_i}{\partial t} + \sum_{k=1}^{N} q_{ik} B_k(t, T) = 0, \quad B_i(T, T) = 1, \quad i = 1, \ldots, N. \quad (2.24)$$

with $g^i_j, j = 1, \ldots, N$, defined in Lemma 1.
Proof. Using the “tower property” of conditional expectations yields

$$\pi(t, T, X_t, S_t) = E\left[ e^{-\int_t^T (r_s + \bar{\nu}_s + S_s) \, ds} \mid \mathcal{F}_T \right]$$

$$= E\left[ e^{-\int_t^T (r_s + \bar{\nu}_s) \, ds} E\left[ e^{-\int_t^T S_s \, ds} \mid \mathcal{F}_T \right] \mid \mathcal{F}_t \right]$$

$$= e^{-\omega(t,T)S_t} E\left[ e^{-\int_t^T (r_s + \bar{\nu}_s - (g^s, X_s)) \, ds} \mid \mathcal{F}_t \right]$$

$$= e^{-\omega(t,T)S_t} E\left[ e^{-\int_t^T (r + \bar{\nu}_s - (g^s, X_s)) \, ds} \mid \mathcal{F}_t \right],$$

where the third equality follows from Lemma 1. Then by using Lemma 2 we obtain the result. □

Theorem 2. Let $\chi_1(t, T, X_t, S_t) = E\left[ e^{-\int_t^T \mu^s(s) \, ds} \mid \mathcal{F}_t \right]$ be the survival probability under the risk-neutral measure $P$. Then we have

$$\chi_1(t, T, X_t, S_t) = e^{-\omega(t,T)S_t} \sum_{i=1}^N B_i(t, T) \langle e_i, X_t \rangle,$$  \hspace{1cm} (2.25)

where $B_1, B_2, \ldots, B_N$ satisfy the following system of $N$ PDEs:

$$-(\bar{h}_i(t) - g^i_t) B_i(t, T) + \frac{\partial B_i}{\partial t} + \sum_{k=1}^N q_{ki} B_k(t, T) = 0, \quad B_i(T, T) = 1, \quad i = 1, \ldots, N,$$  \hspace{1cm} (2.26)

with $g^i_t, j = 1, \ldots, N$, defined in Lemma 1.

Proof. The proof is similar to the one of Theorem 1, so we omit it. □

3 Another approach to pricing longevity bonds

In this section, we provide an alternative approach to evaluate the longevity bond based on the concept of stochastic flows and the change-of-measure technique. This pricing method is similar to that of Shen and Siu [30]. Since the interest rate is stochastic, the most efficient way to price a contingent is to use the change-of-numéraire technique and to choose an ad hoc zero-coupon bond as a new numéraire. So, we need to know the dynamics of a default free zero-coupon bond with expiry date $T$.

The price of a zero-coupon bond at time $t$ with maturity at time $T$ is

$$P(t, T, X_t, S_t) = E\left[ e^{-\int_t^T r_s \, ds} \mid \mathcal{F}_T \right] = E\left[ e^{-\int_t^T r_s \, ds} \mid X_t, S_t \right].$$  \hspace{1cm} (3.1)

It is clear that $P(t, T, X_t, S_t)$ does not depend on $S_t$, so we write $P(t, T, X_t)$ for $P(t, T, X_t, S_t)$, denote $P_i(t, T) := E\left[ e^{-\int_t^T r_s \, ds} \mid X_t = e_i \right]$ for $i = 1, 2, \ldots, N$, and write

$$P(t, T) = (P_1(t, T), P_2(t, T), \ldots, P_N(t, T))^* \in \mathbb{R}^N.$$  

Theorem 3. The price of a zero-coupon bond can be represented as

$$P(t, T, X_t) = \langle e^{(Q - \text{diag}(r))(T-t)} 1, X_t \rangle = \sum_{i=1}^N P_i(t, T) \langle e_i, X_t \rangle,$$  \hspace{1cm} (3.2)
where \( P_1, P_2, \ldots, P_N \) satisfy the following system of \( N \) coupled PDEs:

\[
\frac{\partial P_i}{\partial t} + \sum_{k=1}^{N} q_{ik} P_k(t, T) - r^i P_i(t, T) = 0, \quad P_i(T, T) = 1, \quad i = 1, \ldots, N. \tag{3.3}
\]

**Proof.** Theorem 3 is a direct consequence of Lemma 2. \( \square \)

Define the forward measure \( P^T \) equivalent to \( P \) on \( \mathcal{F}_T \) by

\[
\frac{dP^T}{dP} \bigg|_{\mathcal{F}_T} = A_T, \quad A_T = \frac{e^{-\int_0^T r_s \, ds}}{P(0, T, X_0)}, \tag{3.4}
\]

and

\[
A_t = E[A_T \mid \mathcal{F}_t] = e^{-\int_0^t r_s \, ds} \frac{P(t, T, X_t)}{P(0, T, X_0)}. \tag{3.5}
\]

Consequently, changing the risk-neutral measure by the measure \( P^T \) yields

\[
\pi(t, T, X_t, S_t) = E\left[ e^{-\int_t^T (r_s + \bar{\eta}_s + S_s) \, ds} \mid \mathcal{F}_t \right] = P(t, T, X_t) E^T\left[ e^{-\int_t^T (\bar{\eta}_s + S_s) \, ds} \mid \mathcal{F}_t \right] \tag{3.6}
\]

where \( E^T[\cdot] \) is the expectation under \( P^T \), and \( \chi_2(t, T, X_t, S_t) \) represents the survival probability under the forward measure \( P^T \). To obtain \( \chi_2(t, T, X_t, S_t) \), we must derive the model dynamics under the forward measure \( P^T \).

**Theorem 4.** Under the forward measure \( P^T \), the generator of the chain \( X \) is \( Q^T(t) := \{q^T_{ij}(t)\}_{i,j=1,2,\ldots,N} : \)

\[
q^T_{ij}(t) = \begin{cases} q_{ij} P_j(t, T) / P_i(t, T), & i \neq j, \\ -\sum_{k \neq i} q_{ik}^T(t), & i = j. \end{cases} \tag{3.7}
\]

where \( P_i(t, T), i = 1, \ldots, N, \) are given in Theorem 3, and the semimartingale dynamics for the chain \( X \) is given by

\[
X_t = X_0 + \int_0^t \left( Q^T(s) X_s + M^T_t \right) ds,
\]

where \( \{M^T_t \mid t \in T\} \) is an \( \mathbb{R}^N \)-valued \((\mathcal{F}_t, P^T)\)-martingale. Furthermore, the probability law of \( J_t \) remains unchanged under \( P^T \).

**Proof.** From (3.5) we have

\[
\frac{dA_t}{A_t} = -r_t - dt + \frac{dP(t, T, X_t)}{P(t, T, X_t)}. \tag{3.8}
\]

In order to derive \( dP(t, T, X_t) \), define the discounted bond price process as

\[
\overline{P}(t, T, X_t) = e^{-\int_0^t r_s \, ds} P(t, T, X_t) = E\left[ e^{-\int_0^T r_s \, ds} \mid \mathcal{F}_t \right].
\]
Proposition 5.1 in [28] or Corollary A.1 in [30] we conclude the result.

Applying Itô's differentiation rule to the discounted bond price process \( P(t, T, X_t) \), we obtain

\[
d\bar{P}(t, T, X_t) = \langle \bar{P}(t, T), dM_t \rangle + \frac{\partial \bar{P}}{\partial t} dt + \langle \bar{P}(t, T), Q^* X_t \rangle dt.
\]

Since \( \bar{P}(t, T, X_t) \) is an \((\mathfrak{F}, \mathcal{P})\)-martingale, we must have

\[
\frac{\partial \bar{P}}{\partial t} + \langle \bar{P}(t, T), Q^* X_t \rangle = 0.
\]

Consequently,

\[
d\bar{P}(t, T, X_t) = \langle \bar{P}(t, T), dM_t \rangle.
\]  \tag{3.8}

From the relationship between \( \bar{P}(t, T, X_t) \) and \( P(t, T, X_t) \) we obtain

\[
dP(t, T, X_t) = r_t P(t, T, X_t) dt + \langle P(t, T), dM_t \rangle.
\]

So,

\[
\frac{dA_t}{A_t} = \frac{\langle P(t, T), dM_t \rangle}{P(t, T, X_t)} = \frac{\langle P(t, T), dX_t \rangle}{P(t, T, X_t)} - \langle P(t, T), Q^* X_t \rangle dt
\]

\[
= \frac{dP(t, T, X_t) - (\partial P(t, T, X_t)/\partial t) dt + \langle P(t, T), Q^* X_t \rangle dt}{P(t, T, X_t)}
\]

where the last equality holds because \( dP(t, T, X_t) = (\partial P(t, T, X_t)/\partial t) dt + \langle P(t, T), dX_t \rangle \). Then from Proposition 5.1 in [28] or Corollary A.1 in [30] we conclude the result. \( \Box \)

Define \( H(t, T, X_t, S_t) = E^T[e^{-\int_t^T \bar{\pi}_s ds} \mid X_t, S_t] \). It is obvious that \( H(t, T, X_t, S_t) \) does not depend on \( S_t \), so we write \( H(t, T, X_t) \) for \( H(t, T, X_t, S_t) \). Write

\[
H_i(t, T) := E^T[e^{-\int_t^T \bar{\pi}_s ds} \mid X_t = e_i], \quad i = 1, 2, \ldots, N,
\]

and

\[
H(t, T) = (H_1(t, T), H_2(t, T), \ldots, H_N(t, T)) \in \mathbb{R}^N.
\]

To derive \( \chi_2(t, T, X_t) \), we define another new probability \( \mathcal{P}^1 \) equivalent to \( \mathcal{P}^T \) through the Radon–Nikodym derivative

\[
\left. \frac{d\mathcal{P}^1}{d\mathcal{P}^T} \right|_{\mathfrak{F}_T} = \mathcal{A}_T, \quad \mathcal{A}_T = e^{-\int_0^T \bar{\pi}_s ds} / H(0, T, X_0).
\]  \tag{3.9}

and

\[
A_t = E[\mathcal{A}_T \mid \mathfrak{F}_t] = e^{-\int_0^t \bar{\pi}_s ds} \frac{H(t, T, X_t)}{H(0, T, X_0)}. \tag{3.10}
\]

From Lemma 2 we obtain an explicit formula for \( H(t, T, X_t) \), which is presented in the following theorem.
Theorem 5. Let $\bar{\omega}_t = (\bar{h}_1(t), \ldots, \bar{h}_N(t))^* \in \mathbb{R}^N$ with $\bar{h}_i(t) = h_i + l_i \omega^{y+t}$. Then

$$ H(t, T, X_t) = \sum_{i=1}^{N} H_i(t, T) \langle e_i, X_t \rangle, $$

where $H_1, H_2, \ldots, H_N$ satisfy the following system of $N$ coupled PDEs:

$$ \frac{\partial H_i}{\partial t} + \sum_{k=1}^{N} q_{ik}^T(t) H_k(t, T) - \bar{h}_i(t) H_i(t, T) = 0, \quad H_i(T, T) = 1, \quad i = 1, \ldots, N. $$

Proof. The result is a direct consequence of Lemma 2.

Based on the formula for $H(t, T, X_t)$, we can obtain the dynamics of $X_t$ and $S_t$ under the measure $P^1$:

Theorem 6. Under the forward measure $P^1$, the generator of the chain $X$ is $Q^1(t) := [q_{ij}^1(t)]_{i,j=1,2,\ldots,N}$:

$$ q_{ij}^1(t) = \begin{cases} q_{ij}^T(t) H_j(t, T) / H_i(t, T), & i \neq j, \\ - \sum_{k \neq i} q_{ik}^1(t), & i = j, \end{cases} $$

where $H_i(t, T), i = 1, \ldots, N$, are given in Theorem 5, and the semimartingale dynamics for the chain $X$ is given by

$$ X_t = X_0 + \int_0^t (Q^1(s))^T X_s \, ds + M_t^1, $$

where $\{M^1(t) \mid t \in \mathcal{T}\}$ is an $\mathbb{R}^N$-valued $(\mathcal{F}, P^1)$-martingale. Furthermore, the probability law of $J_t$ remains unchanged under $P^1$.

Proof. The proof is similar to that of Theorem 4, so we omit it.

Then, the survival probability under the forward measure $P^T, \chi_2(t, T, X_t, S_t)$, can be expressed as

$$ \chi_2(t, T, X_t, S_t) = H(t, T, X_t) E^1 [e^{-\int_t^T S_s \, ds} \mid \mathcal{F}_t] := H(t, T, X_t) \chi_3(t, T, X_t, S_t), $$

where $E^1[\cdot]$ is the expectation under $P^1$. So, if the formula for $\chi_3(t, T, X_t, S_t)$ can be derived, then the pricing formula for the longevity bond and the survival probability under the forward measure $P^T$ can be fully solved. Instead of using Lemma 1 and Lemma 2, we follow the same augments as in [30] to derive $\chi_3(t, T, X_t, S_t)$.

Theorem 7. Let $\chi_3(t, T, X_t, S_t) = E^1 [e^{-\int_t^T S_s \, ds} \mid \mathcal{F}_t]$. Then we have

$$ \chi_3(t, T, X_t, S_t) = e^{\omega(t, T) S_t} \sum_{i=1}^{N} \bar{A}_i(t, T) \langle e_i, X_t \rangle, $$

where $\omega(t, T) = (1 - e^{-\delta(T-t)})/\delta$, and $\bar{A}_1, \bar{A}_2, \ldots, \bar{A}_N$ satisfy the following system of $N$ coupled PDEs:

$$ \frac{\partial \bar{A}_i}{\partial t} + \sum_{k=1,k \neq i}^{N} q_{ik}^1(t) \bar{A}_k(t, T) + (q_{ii}^1(t) + g_i^1) \bar{A}_i(t, T) = 0, \quad i = 1, \ldots, N. $$

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with
\[
\tilde{A}_i(T, T) = 1, \quad i = 1, \ldots, N.
\]

Proof. The proof is presented in the Appendix. \qed

The next result gives the pricing formula for the longevity bond and the survival probability under the forward measure \( P^T \).

**Theorem 8.** Let \( \chi_2(t, T, X_t, S_t) = \mathbb{E}^T[e^{-\int_t^T \mu(y) \, ds} \mid X_t, S_t] \) represent the survival probability under the measure \( P^T \). Then we have
\[
\chi_2(t, T, X_t, S_t) = H(t, T, X_t)\chi_3(t, T, X_t, S_t) = e^{-\omega(t,T)S_t} \sum_{i=1}^N H_i(t, T)\tilde{A}_i(t, T) \langle e_i, X_t \rangle,
\]
and the price at time \( t \) of the longevity bond is given by
\[
\pi(t, T, X_t, S_t) = e^{-\omega(t,T)S_t} \sum_{i=1}^N P_i(t, T)H_i(t, T)\tilde{A}_i(t, T) \langle e_i, X_t \rangle,
\]
where \( H_i(t, T) \) and \( \tilde{A}_i(t, T) \) are given in Theorems 5 and 7, respectively.

Proof. Theorem 8 is a direct consequence of Theorem 7. \qed

### 4 Comparison between two pricing formulas

In the last two sections, we use different approaches to derive the pricing formulas for the longevity bond given by (2.23) and (3.19), respectively. It is not obvious that (2.23) and (3.19) are equivalent, so we now show that the two formulas are equivalent.

In order to prove that (2.23) and (3.19) are equivalent, we shall prove that \( B_i \) and \( P_iH_i\tilde{A}_i, \ i = 1, \ldots, N \), satisfy the same system of \( N \) PDEs:
\[
- \left(r^i + \bar{h}_i(t) - g_i^t\right)C_i(t, T) + \frac{\partial C_i}{\partial t} + \sum_{k=1}^N q_{ik} C_k(t, T) = 0
\]
with
\[
C_i(T, T) = 1, \quad i = 1, \ldots, N,
\]
and the solution to (4.1)–(4.2) is unique. Write
\[
C(t, T) = (C_1(t, T), \ldots, C_N(t, T))^* \in \mathbb{R}^N
\]
and define \( \Delta(t) = Q - \text{diag}(r + \bar{h}(t) - \bar{g}(t)) \). Then (4.1)–(4.2) can be expressed as
\[
\frac{\partial C(t, T)}{\partial t} + \Delta(t)C(t, T) = 0, \quad C(T, T) = 1.
\]
Since \( \Delta(t) \) is continuous, there exists a unique solution of (4.3) over the finite time interval \( T \). It remains to show that \( B_i \) and \( P_iH_i\tilde{A}_i, \ i = 1, \ldots, N \), both satisfy (4.1)–(4.2). From Theorem 1 we can easily see that \( B_1, \ldots, B_N \) satisfy (4.1)–(4.2).
From Theorems 3, 5, and 7 we have, for $i = 1, 2, \ldots, N$,

\[
\frac{\partial P_i}{\partial t} + \sum_{k=1, k\neq i}^N q_{ik} P_k(t, T) + (q_{ii} - r^i) P_i(t, T) = 0,
\]

\[
\frac{\partial H_i}{\partial t} + \sum_{k=1, k\neq i}^N q_{ik}^T(t) H_k(t, T) + (q_{ii}^T(t) - \overline{h}_i(t)) H_i(t, T) = 0,
\]

\[
\frac{\partial \tilde{A}_i}{\partial t} + \sum_{k=1, k\neq i}^N q_{ik}^1(t) \tilde{A}_k(t, T) + (q_{ii}^1(t) + g_i^1) \tilde{A}_i(t, T) = 0,
\]

with

\[
P_i(T, T) = 1, \quad H(T, T, e_i) = 1, \quad \tilde{A}_i(T, T) = 1.
\]

Differentiating $P_i H_i \tilde{A}_i$ with respect to $t$ gives

\[
\frac{\partial (P_i H_i \tilde{A}_i)}{\partial t} = H_i(t, T) \tilde{A}_i(t, T) \frac{\partial P_i}{\partial t} + P_i(t, T) \tilde{A}_i(t, T) \frac{\partial H_i}{\partial t} + P_i(t, T) H_i(t, T) \frac{\partial \tilde{A}_i}{\partial t}.
\]

Then substituting (4.4) into (4.6) yields

\[
\frac{\partial (P_i H_i \tilde{A}_i)}{\partial t} = -H_i \tilde{A}_i \left( \sum_{k=1, k\neq i}^N q_{ik} P_k(t, T) + (q_{ii} - r^i) P_i(t, T) \right) - P_i(t, T) \tilde{A}_i(t, T)
\]

\[
\times \left( \sum_{k=1, k\neq i}^N q_{ik}^T(t) H_k(t, T) + (q_{ii}^T(t) - \overline{h}_i(t)) H_i(t, T) \right) - P_i(t, T) H_i(t, T)
\]

\[
\times \left( \sum_{k=1, k\neq i}^N q_{ik}^1(t) \tilde{A}_k(t, T) + (q_{ii}^1(t) + g_i^1) \tilde{A}_i(t, T) \right).
\]

Recall that $q_{ij}^T(t) = (P_j(t, T)/P_i(t, T)) q_{ij}$ and $q_{ij}^1(t) = (H_j(t, T)/H_i(t, T)) q_{ij}^T(t) = (H_j(t, T)/H_i(t, T)) \times (P_j(t, T)/P_i(t, T)) q_{ij}$ for $i \neq j$. Consequently, (4.7) changes into

\[
\frac{\partial (P_i H_i \tilde{A}_i)}{\partial t} = -P_i(t, T) H_i(t, T) \tilde{A}_i(t, T) (q_{ii} - r^i - \overline{h}_i(t) + g_i^1)
\]

\[
- \sum_{k=1, k\neq i}^N q_{ik} P_k(t, T) H_k(t, T) \tilde{A}_k(t, T)
\]

with

\[
P_i(T, T) H_i(T, T) \tilde{A}_i(T, T) = 1.
\]

Comparing (4.8)–(4.9) with (4.1)–(4.2), we conclude (2.23) and (3.19) are equivalent.

**Remark 3.** In general, we cannot give a closed-form formula for the longevity bond price, so we need to use some numerical algorithms to solve the PDEs. Comparing (2.23) with (3.19), the pricing formula (2.23) is much easier to implement. Note that, Shen and Siu [30] also use the change-of-measure technique to derive the longevity bond price represented as the product of two exponential affine forms whose coefficients are fundamental matrix solutions of linear matrix-valued ODEs. Their formula can be simplified by using the method presented in Section 2 in order to accelerate the implementation.
5 Numerical results

In this section, we shall make some numerical analysis to illustrate our theoretical results. Since we focus on investigating the impact of the model parameters on the longevity bond price, we just make some numerical analysis without doing the calibration in this paper. In our future research agenda, we plan to use the population data to empirically test our model.

For ease of illustration, we consider $N = 2$, that is, $X$ only switches between two states, where state $e_1$ and state $e_2$ represent “good” and “bad” economic and environmental conditions, respectively. The parameters are fixed unless otherwise noted: $h = (0.006, 0.007)^*$, $l = (0.00000004, 0.00000045)^*$, $x = 60$, $T = 10$, $S_0 = 0$, $q_{12} = q_{21} = 0.5$, $r = (0.05, 0.02)^*$, $\delta = 1$, $\rho = (0.5, 1)^*$, $f^i(x) = \alpha_i e^{-\alpha_i x}$, $x > 0$, $\alpha = (\alpha_1, \alpha_2)^* = (5000, 2000)^*$. To investigate the regime-switching effect, we compare the regime-switching intensities model with the one that has no regime switching. So, for each $\eta_t = \langle \eta, X_t \rangle$ with $\eta = (\eta_1, \eta_2)^*$, in the model without regime switching, we choose a constant $\eta$ such that $e^{-\eta T} = E[e^{\int_0^T \eta_t dt} \mid X_0 = e_1]$.

Figures 1–3 present how the values of the model parameters impact the survival probability under the risk-neutral measure. From them we can conclude that the survival probabilities are much larger when we start at $X_0 = e_1$. We can also see that the survival probability in the no-regime-switching model is much larger than that in the regime-switching model. Figures 1–2 show that the survival probabilities increase in $\delta$ and $\alpha$ because higher $\delta$ or $\alpha$ decreases the value of the regime-switching shot-noise process. From Fig. 3 we can see...
that $\tau P_{0}$ is an increasing function of transition rate when we start from $X_{0} = e_{2}$ because a higher transition rate leads to an increasing probability of switching to the state $e_{2}$. On the contrary, if we start from $X_{0} = e_{1}$, then $\tau P_{0}$ decreases in transition rate because a higher transition rate leads to an increasing probability of switching to the state $e_{1}$.

Figures 4–8 show that the impact of the values of the model parameters on the longevity bond price. From them we can see that the price is much larger when we start at $X_{0} = e_{2}$ because the price not only depends on the force of mortality but also on the interest rate. Although the survival probabilities should be smaller when $X_{0} = e_{2}$, note that $r^{2}$ is also much smaller than $r^{1}$. We can also see that the price in the no-regime-switching model is larger than that in the regime-switching model.

From Figs. 4–5 we can see that the price is an increasing function of $q_{12}$ and a decreasing function of $q_{21}$ because higher $q_{12}$ leads to an increasing probability of switching to the state $e_{2}$, whereas higher $q_{21}$ leads to an increasing probability of switching to the state $e_{1}$. Figures 6–8 show the impact of the regime-switching shot-noise process on the longevity price. From them we can find that the price increases in $\delta$ and $\alpha$ and decreases in the jump intensity $\rho$ because higher $\delta$ or $\alpha$ decreases the value of the regime-switching shot-noise process, whereas a higher jump intensity increases the value of $S$.

Therefore, numerical results reveal that if we do not incorporate the changes of market regimes and environmental conditions into the pricing model, we will overestimate or underestimate the survival probability and the longevity bond price.
6 Conclusions

Since the longevity bonds are long-term instruments, this paper develops a Markov regime-switching pricing model for evaluating the longevity bonds considering the changes of market regimes or environments. In our model, the interest rate and the force of mortality both depend on the economic and environmental conditions described by a homogenous Markov chain. Under the assumptions that the interest rate is a deterministic function of Markov chain and the force of mortality is governed by the sum of a regime-switching dynamic Gompertz–Makeham model and a regime-switching shot-noise model, we use two different approaches to derive the semianalytical solution for the longevity bond price.

Since it is not difficult to derive the conditional Laplace transform of the regime-switching shot-noise process, the first approach we use to derive the pricing formula is based on the Laplace transform. However, the most efficient way to price derivatives in a stochastic interest rate environment is to use the change-of-numéraire technique. Therefore, we also adopt the change-of-measure technique to derive the pricing formula. In addition, motivated by Shen and Siu [30], who explore the concept of stochastic flows to derive Markovian regime-switching exponential affine forms for the bond prices under a Markovian regime-switching jump-augmented Vasicek model, we use the concept of stochastic flows to derive the Laplace transform of the regime-switching shot-noise process under the second approach. Although the formulas for the longevity bond price obtained from the two approaches have different forms of expression, we can prove that the two formulas are equivalent. Comparing the two approaches, the second approach, which is based on the change-of-measure technique, does not impose any restriction on the dynamics of the force of mortality. But we need to know the dynamics of the force of mortality under the new measures, and sometimes it is not easy to specify them. The first approach is valid only for some special cases that depend on the choice of the dynamics of the force of mortality. Under the model we propose in this paper, the first approach is much easier to implement.

In our future research agenda, we plan to empirically test our model. Furthermore, we will model the dependent forces of mortalities of multiple lives by using regime-switching shot-noise models, so that we can price insurance products involving multiple lives.

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Appendix

Proof of Theorem 7. Let $S_{t,v}(s)$ be a version of the process $S_{t,v}, v \geq t$, with initial condition $S_{t,t}(s) = s$. Then from (2.6) we have

$$S_{t,v}(s) = s - \delta \int_{t}^{v} S_{t,u-}(s) \, du + \int_{t}^{v} dJ_{u}. \quad (A.1)$$

Using similar arguments as in [18], we get that there exists a flow of diffeomorphisms $s \rightarrow S_{t,v}(s)$, $P^{1}$-a.s., associated with the SDE for the Markov regime-switching jump-augmented model in (A.1). Write

$$D_{t,v} = \frac{\partial S_{t,v}(s)}{\partial s}$$

for the derivative of the map $s \rightarrow S_{t,v}(s)$. Differentiating (A.1) with respect to $s$ gives

$$D_{t,v} = 1 - \delta \int_{t}^{v} D_{t,u} \, du \quad (A.2)$$

with initial condition $D_{t,t} = 1$. So, we obtain

$$D_{t,v} = e^{-\delta(v-t)}.$$
Define
\[ \omega(t, T) = \int_t^T D_{t,u} \, du = \frac{1 - e^{-\delta(T-t)}}{\delta}, \]
which is a real-valued deterministic process.

Recall that
\[ \chi_3(t, T, x, s) = E^1 \left[ e^{-\int_t^T S_{t,u}(s) \, du} \mid X_t = x, S_{t,t}(s) = s \right], \]
where \( S_{t,u}(s) \) is given in (A.1). Differentiating both sides of this equality yields
\[
\frac{\partial \chi_3(t, T, x, s)}{\partial s} = -\int_t^T D_{t,u} \, du \, E^1 \left[ e^{-\int_t^T S_{t,u}(s) \, du} \mid X_t = x, S_{t,t}(s) = s \right] = -\omega(t, T)\chi_3(t, T, x, s). \tag{A.3}
\]
Integrating (A.3) with respect to \( s \) then gives
\[
\chi_3(t, T, x, s) = e^{A(t,T,x) - \omega(t,T)s} := \tilde{A}(t, T, x)e^{-\omega(t,T)s}, \tag{A.4}
\]
where \( \tilde{A}(t, T, x) = e^{A(t,T,x)} \). In the sequel, we write \( A_i(t, T) := A(t, T, e_i) \), \( \tilde{A}_i(t, T) := \tilde{A}(t, T, e_i) \) for \( i = 1, 2, \ldots, N \), and
\[
A(t, T) = (A_1(t, T), A_2(t, T), \ldots, A_N(t, T))^* \in \mathbb{R}^N, \\
\tilde{A}(t, T) = (\tilde{A}_1(t, T), \tilde{A}_2(t, T), \ldots, \tilde{A}_N(t, T))^* \in \mathbb{R}^N.
\]
Define
\[
\overline{\chi}_3(t, T, x, s) = e^{-\int_{0}^{t} S_{0,u}(S_0) \, du} \chi_3(t, T, x, s).
\]
Write \( \chi_3^i(t, T, s) = \chi_3(t, T, e_i, s), \overline{\chi}_3^i(t, T, s) = \overline{\chi}_3(t, T, e_i, s) \) for each \( i = 1, 2, \ldots, N \) and write
\[
\chi_3(t, T, s) = (\chi_3^1(t, T, s), \chi_3^2(t, T, s), \ldots, \chi_3^N(t, T, s))^* \in \mathbb{R}^N, \\
\overline{\chi}_3(t, T, s) = (\overline{\chi}_3^1(t, T, s), \overline{\chi}_3^2(t, T, s), \ldots, \overline{\chi}_3^N(t, T, s))^* \in \mathbb{R}^N.
\]
Applying Itô’s differentiation rule to \( \overline{\chi}_3(t, T, X_t, S_t) \) yields
\[
d\overline{\chi}_3(t, T, X_t, S_t) = \frac{\partial \overline{\chi}_3}{\partial t} \, dt + \delta \frac{\partial \overline{\chi}_3}{\partial X} \chi_3(t, T, X_t, S_t) \, dt + \langle \overline{\chi}_3(t, T, S_t), dM_t^I \rangle \\
+ \langle \overline{\chi}_3(t, T, S_t), (Q^I(t))^* X_t \rangle \, dt + \langle \overline{\chi}_3(t, T, X_t, S_t) - \overline{\chi}_3(t, T, X_t, S_t^-) \rangle \, dJ_t.
\]
Note that \( \overline{\chi}_3(t, T, X_t, S_t) \) is a bounded \((\mathbb{S}, \mathbb{P}^1)\)-martingale. So, we have
\[
\int_0^\infty \langle \overline{\chi}_3(t, T, x, s + y) - \overline{\chi}_3(t, T, x, s) \rangle \rho(t)f_t(y) \, dy \left( \chi_3(t, T, x, s) \right) \\
+ \frac{\partial \overline{\chi}_3}{\partial t} + \langle \chi_3(t, T, s), (Q^I(t))^* X_t \rangle - \delta \frac{\partial \chi_3}{\partial s} \chi_3(t, T, x, s) = 0.
\]
From the relationship between \( \chi_3(t, T, x, s) \) and \( \chi_3(t, T, x, s) \) we immediately obtain the Markov regime-switching PDE
\[
\int_0^\infty \left( \chi_3(t, T, x, y) - \chi_3(t, T, x, s) \right) \rho(t) f_{t}(y) \, dy \\
+ \frac{\partial \chi_3}{\partial t} - s \chi_3(t, T, x, s) + \left\langle \chi_3(t, T, s), (Q^1(t) \times X_t) \right\rangle - \delta \frac{\partial \chi_3}{\partial s} s = 0 \tag{A.5}
\]
with terminal condition
\[
\chi_3(T, T, x, s) = 1. \tag{A.6}
\]
Recall that the bond price has the following Markov regime-switching exponential-affine form:
\[
\chi_3(t, T, x, s) = e^{A(t,T,x) - \omega(t,T)s}.
\]
So,
\[
\frac{\partial \chi_3}{\partial t} = \chi_3(t, T, x, s) \left( \frac{\partial A}{\partial t} - \frac{\partial \omega}{\partial t} \right), \quad \frac{\partial \chi_3}{\partial s} = -\omega(t, T) \chi_3(t, T, x, s). \tag{A.7}
\]
Then, substituting (A.7) into (A.5), we have the following Markov regime-switching PDE for \( A(t, T, x) \):
\[
\frac{\partial A}{\partial t} + e^{-A(t,T,x)} \left\langle \tilde{A}(t, T), (Q^1(t) \times x) \right\rangle + \int_0^\infty \left( e^{-\omega(t,T)y} - 1 \right) \rho(t) f_{t}(y) \, dy = 0 \tag{A.8}
\]
with
\[
A(T, T, x) = 0. \tag{A.9}
\]
Substituting \( \tilde{A}(t, T, x) = e^{A(t,T,x)} \) into (A.8) and (A.9) yields
\[
\frac{\partial \tilde{A}}{\partial t} + \left\langle \tilde{A}(t, T), (Q^1(t) \times x) \right\rangle + \tilde{A}(t, T, x) \int_0^\infty \left( e^{-\omega(t,T)y} - 1 \right) \rho(t) f_{t}(y) \, dy = 0 \tag{A.10}
\]
with
\[
\tilde{A}(T, T, x) = 1. \tag{A.11}
\]
Since \( x \) takes one of the values \( e_i \) for \( i = 1, 2, \ldots, N \), Eqs. (A.10)–(A.11) correspond to
\[
\frac{\partial \tilde{A}_i}{\partial t} + \sum_{k=1, k \neq i}^N q_{ik}^1(t) \tilde{A}_k(t, T) + (q_{ii}^1(t) + g_{ii}^1) \tilde{A}_i(t, T) = 0, \quad \tilde{A}_i(T, T) = 1.
\]
Then from (A.4) we obtain
\[
\chi_3(t, T, X_t, S_t) = e^{-\omega(t,T)S_t} \left\langle \tilde{A}(t, T), X_t \right\rangle = e^{-\omega(t,T)S_t} \sum_{i=1}^N (e_i, X_t) \tilde{A}_i(t, T).
\]
The proof is completed. \( \square \)
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