The models of delocalized membranes

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Abstract

The generally adopted approach in theory of relativistic strings and membranes, is similar to use of Lagrange coordinates in continious media mechanics. One can use an alternative approach, which is similar to use of Euler coordinates. Under such approach the consideration of thick (delocalized) membranes is natural. Membrane kinematics, which correspond to Euler coordinates is constructed. Variables, similar to Hamiltonian variables, are introduced by means of Legander transformation. The case of free membranes appears to be degenerate. The models of delocalized membranes

1 Introduction

This paper present the Euler coordinate approach to membrane theory using the results of paper [17, 18, 19, 20] as main examples

From kinematical point of view membrane is time-like submanifold \( V \) (world surface) in space-time \( M \), \( \dim V = n < D = \dim M \). Let \( V \) is oriented and \( \partial V = 0 \).

One has two natural methods to describe world surface \( V \)

- introducing coordinates \( \xi = (\xi^0, \ldots, \xi^{n-1}) \) at world surface \( V \), \( V = \{ X \in M \mid x(\xi) \} \);
- introducing set of \( D - n \) scalar fields \( \varphi = (\varphi^1, \ldots, \varphi^{D-n}) \) to describe \( V \) as level surface \( V = \{ X \in M \mid \varphi(X) = \text{const} \} \). This approach describes simultaneously all membranes corresponding to different level surfaces \( \varphi = \text{const} \).

Coordinates \( \xi \) are related to medium (membrane), so one has to consider them as a sort of Lagrange coordinates. Coordinates \( X \) are related to space-time, so one has to consider them as a sort of Euler coordinates.

Under the first approach one can use the following standard action [1, 2, 3, 4]

\[
S_{\text{standard}}[x(\xi)] = -T \int V d^n \xi \sqrt{\det \gamma_{ij}},
\]

where

\[
\gamma_{ij} = g_{MN} \frac{\partial x^M}{\partial \xi^i} \frac{\partial x^N}{\partial \xi^j}.
\]

Unfortunately number is Lagrange coordinates \( \xi \) is \( n \), the number of Euler coordinates \( X \) is \( D \). \( D > n \), so it is not possible to express Euler coordinates in terms of Lagrange ones.

To make number of Lagrange coordinates equal to \( D \) initial action (1) has to be modified. The obvious generalization replaces single membrane \( V \) by set of membranes \( V(\varphi) \), paramerized by variables \( \varphi = (\varphi^1, \ldots, \varphi^{D-n}) \) (simple delocalized membrane)

\[
S[x(\varphi, \xi)] = - \int_{V(\varphi)} d^{D-n} \varphi \cdot f(\varphi) \int V d^n \xi \sqrt{\det \gamma_{ij}}.
\]

Corresponding equations of motion for this action are obviously the same as for standard one, but new analogy of set \( (\xi, \varphi) \) withLargange (related to medium) coordinates is transparent. Like for standard action, we still have \( n \) constrains \( \frac{\delta S(\xi, \varphi)}{\delta x^M} \frac{\partial x^M}{\partial \varphi^k} = 0 \). Let \( \frac{\partial x^M}{\partial \varphi^k} \neq 0 \), then \( (\xi, \varphi) \) are new (Lagrange) coordinates at space-time \( M \), so \( \xi = \xi(X), \varphi = \varphi(X) \).

Now one can transform the action from Lagrange coordinates \( (\xi, \varphi) \) to Euler coordinates \( X = x(\xi, \varphi) \). New action has to involve integration over \( X \) instead of \( (\xi, \varphi) \) and fields \( \xi(X), \varphi(X) \) instead of \( x(\xi, \varphi) \)

\[
S[\varphi(X)] = - \int M d^D X \sqrt{|g|} f(\varphi) \wedge d\varphi \wedge \ldots \wedge d\varphi^{D-n} ||, (3)
\]

where we use norm of \( k \)-form \( ||J|| = (J, J), (A, B) = \sqrt{\frac{1}{n!} A_{N_1 \ldots N_k} B^{N_1 \ldots N_k}} \). Fortunately new action does not involve \( n \) fields \( \xi(X) \) and now we have no constraints.

The action of this form was initially considered by Hosotani et.al. [5, 6]. Backer and Fairlie [11, 12] suggested an alternative interpretation for the same action as action for \( (D-n) \)-dimensional membrane. The same action (3) in the case \( f(\varphi) = \delta^{D-n}(\varphi) \) was used.
by Morris and Gee [8, 9, 10] in the context of string theory quantisation. The action (3) was also used to handle with Hamilton-Jacobi equations for strings and p-branes [7, 12].

2 Kinematics

This section presents more formal view on membrane kinematics in Euler (space-time) coordinates.

Kinematics in Euler coordinates involves two spaces and one map from one space, to other one

- space-time $\mathbf{M}$, $\text{dim} \mathbf{M} = D$, $X^M$ are coordinates on $\mathbf{M}$, $g_{MN}$ is metric;
- auxiliary space $\mathbf{F}$, $\text{dim} \mathbf{F} = D - n < D$, $\phi^\alpha$ are coordinates on $\mathbf{F}$, $\Omega = f(\phi) d\phi^1 \wedge \ldots \wedge d\phi^{D-n}$ is volume form;
- map $\phi : \mathbf{M} \rightarrow \mathbf{F}$, $\phi$ is “potential”.

Using these basic objects we can introduce the following notions

- $\varphi^{-1}(\phi) = \mathbf{V}(\phi), \quad \text{dim} \mathbf{V}(\phi) = n < D,$
  $\mathbf{V}(\phi)$ is membrane number $\phi$;
- $\varphi^* : \Omega \rightarrow J = f(\phi(X)) d\phi^1(X) \wedge \ldots \wedge d\phi^{D-n}(X)$,
  $J$ is “field intensity”;
- $\ast J$ is membrane current.

$$dJ = 0 \iff \delta(\ast J) = 0$$

is “kinematic conservation law” (conservation of membrane current);

- $\int_\mathbf{U} J$ is membrane flux through $\mathbf{U}$, $\text{dim} \mathbf{U} = D - n$;

$$g^{\alpha\beta} = g^{MN} \partial_M \varphi^\alpha \partial_N \varphi^\beta$$

is inverse metric generated by $\varphi$ at space $\mathbf{F}' = \varphi(\mathbf{M})$;
- $G_{\alpha\beta}$ is matrix inverse to $g^{\alpha\beta}$ ($G_{\alpha\beta} G^{\beta\gamma} = \delta^\gamma_\alpha$);

$$\|J\| = f(\varphi) \sqrt{\det(g^{\alpha\beta})}$$

is density of membranes in attendant coordinates;
- $n = \frac{1}{\|u\|}$ is unit normal form ($\|n\|^2 = 1$);

- $u = *n$ is unit tangent form (“velocity form”, $\|u\|^2 = -1$);

- $\bar{P}_{MN} = \frac{g^{MN}}{n_{MK_2 \ldots K_{D-n} n_{K_2 \ldots K_{D-n}}} (D-n-1)!}$ is projector to normal directions of membrane;

- $P_{MN} = g_{MN} \bar{P}_{MN} = -u_{MK_2 \ldots K_n} u_{NK_2 \ldots K_n} (n-1)!$ (7)

is projector to tangent directions of membrane (projector to membrane world surface).

In some problem it is useful to use attendant (Lagrange) coordinates $X = (\xi, \phi)$. Metric $g_{MN}$ and inverse metric $g^{MN}$ are splitted into four blocks:

$$g_{MN} = \begin{pmatrix} g_{mn} & g_{m\nu} \\ g_{\mu n} & g_{\mu\nu} \end{pmatrix}, \quad g^{MN} = \begin{pmatrix} g^{mn} & g^{m\nu} \\ g^{\mu n} & g^{\mu\nu} \end{pmatrix},$$

is matrix at membrane world surface $V(\phi)$.

$g^{\mu\nu}$ is inverse metric at auxiliary space $\mathbf{F}'$.

If $\gamma_{mn} \neq 0$, then in attendant coordinates

$$g_{\text{attendant}} = \frac{\det \gamma_{mn}}{\det g^{\mu\nu}}.$$

Equation (8) allows to derive action (3) from (2).

Identity is (8) is trivial consequence of the following statement:

Let $G$ is $N \times N$-matrix, $\det G \neq 0$.

$$G = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad G^{-1} = \begin{pmatrix} K & L \\ M & N \end{pmatrix},$$

where $A, B, C, D$ are matrices $n \times n$, $n \times (N - n)$, $(N - n) \times n$ and $(N - n) \times (N - n)$ respectively. $K$, $L$, $M$, $N$ are matrices of the same dimensions as $A$, $B$, $C$, $D$. If $\det A \neq 0$, then

$$\det G = \frac{\det A}{\det N}.$$ (8)

3 Dynamics

In this section we continue the analogy above between membrane theory in Euler coordinates and classical mechanics for dynamics.

Let us consider the action

$$S[\varphi(X)] = \int_\mathbf{M} d^D X \sqrt{|g|} L(\varphi, J).$$ (9)

One can introduce “momentum” $K$ by following formula

$$K = \frac{\partial L}{\partial \dot{J}}.$$ (10)
Variation of action gives
\[ \frac{1}{\sqrt{|g|}} \frac{\delta S}{\delta \varphi^\alpha} = \frac{\partial L}{\partial \varphi^\alpha} - (J_{(\alpha)}, \delta K), \] (11)
where
\[ J_{(\alpha)} = (-1)^{D+n+\alpha} f(\varphi) \, d\varphi^1 \wedge \ldots \wedge \hat{d}\varphi^\alpha \ldots \wedge d\varphi^{D-n}. \] (12)
The term under the hat has to be skipped.

By means of Legendre transformation one can define “Hamiltonian”
\[ H_{\text{mem.}}(\varphi, K) = (K, J(\varphi, K)) - L(\varphi, J(\varphi, K)); \] (13)
\[ \frac{\partial H_{\text{mem.}}}{\partial \varphi^\alpha} = J, \] (14)
\[ \frac{\partial H_{\text{mem.}}}{\partial K} = -(J_{(\alpha)}, \delta K). \] (15)
System (15) of \( D - n \) equations contains not only \( \varphi^\alpha \) and \( J \), but also \( \partial M \varphi^\alpha \).

For some application it would be useful to modify system (15) in following way.
\[ \frac{\partial H_{\text{mem.}}}{\partial \varphi^\alpha} \frac{\partial M}{\partial \varphi^\alpha} \delta \varphi^\alpha = -(J_{(\alpha)}, \delta K) \partial M \varphi^\alpha = -(J, \delta K)_M. \]

Here we introduce new notation \( (J, \delta K)_M = (\frac{1}{D-n-1} \int_{\mathbb{R}^{D+n-1}} J_{K_1 \ldots K_{D-n-1}, M}(\delta K)^{K_1 \ldots K_{D-n-1}}. \) If \( ||J|| \neq 0 \), then matrix \( \partial M \varphi^\alpha \) has rank \( D - n \), i.e. new system is equivalent to (15). Taking into account
\[ \nabla_M H_{\text{mem.}} = \frac{\partial H_{\text{mem.}}}{\partial J} \frac{\partial M}{\partial J} + \left( \frac{\partial H_{\text{mem.}}}{\partial J}, \nabla_M J \right) \]
we write
\[ \nabla_M H_{\text{mem.}} = (K, \nabla_M J) - (J, \delta K)_M. \] (16)
System (16) is equivalent to (15).

For simple delocalized membrane \( L(\varphi, J) = -||J||, \)
\( K = -n \), where \( n = \frac{J}{||J||} \), so \( J \) is not expressible in terms of \( K \) (information on \( ||J|| \) does not included to \( n \)), and “Hamiltonian” does not exist. I.e. simple delocalized membrane is degenerate case.

All sections below deals with membranes of this sort, which are described by the following equations of motion
\[ (n, \delta n)_M = 0. \] (17)

4 Localization

To reproduce the case of unit tension single free membrane with world surface \( a(x) = 0 \) one has to set
\[ J = \delta(a) \, da^1 \wedge \ldots \wedge da^{D-n}. \] (18)

There are two obvious possibilities to reproduce this form of \( J \). \( \delta \)-function can originate from form of action (from function \( f(\phi) \)) or from form of field \( \varphi \)

1. set \( f(\phi) = \delta(\phi), \varphi = a [8, 9] \), or
2. set \( f(\phi) = 1, \varphi^\alpha = \theta(\alpha^\alpha) \) [17]

Field (18) is singular, but in the case of simple delocalized membrane (3) one could consider equations of motion \( (n, \delta n)_M = 0 \) for field of this form, because to calculate \( (n, \delta n)_M \) in some point of membrane one need only values of \( n \) along the same membrane.

5 Energy

For simple delocalized membrane (3) energy-momentum tensor has the form
\[ T_{MN} = -\rho P_{MN}, \]
where \( \rho = ||J|| \) is density of membrane matter (6), and \( P_{MN} \) is projector to membrane world surface (7).

Equation of motion for free membrane is equivalent to conservation of energy-momentum tensor
\[ \nabla_M T^{MN} = 0. \]

Hamiltonian density (it is not “Hamiltonian” from section 3) in Minkowski space-time with signature \((-;++;+)\) is
\[ H = \sqrt{\det(\partial_m \varphi^\alpha \partial_m \varphi^\beta) + \partial_m \varphi^\alpha \partial_m \varphi^\beta p_\alpha p_\beta}, \] (19)
where index \( m = 1, \ldots, D - 1 \) numerate space coordinates, \( p_\alpha \) are canonical momenta.

6 Generalized Majumdar-Papapetru solution

In this section we build explicitly simplest delocalized \( p \)-brane solution, the generalized Majumdar-Papapetru solution. It describes static charged dust cloud with charge density equal (in “natural” units) to mass density. In classical Majumdar-Papapetru solution, instead if dust, one has set of charged extremal black holes with charge equal to mass.

Delocalized \( p \)-brane solutions for arbitrary \( p \) in arbitrary dimensions were presented in papers [17, 18]. In paper [19] intersecting delocalized \( p \)-brane solutions were built (intersecting \( p \)-brane solutions with no delocalised sources are reviewed in papers [13, 14, 15])

Let us consider Einstein-Maxwell action with dust source
\[ S = \int d^4x \sqrt{-g} \left( \frac{R}{2} - \frac{\|F\|^2}{2} - ||J|| + \frac{1}{\sqrt{2}} (*J, A) \right), \]
where \( A \) is electromagnetic potential (1-form), \( F = dA \) is electromagnetic field, \( J = d\varphi^1 \wedge d\varphi^2 \wedge d\varphi^3 \) is 0-brane (dust) field. \( \frac{1}{\sqrt{2}} *J \) is 4-dimensional current density.
By variation over fields $g_{MN}$, $A_M$ and $\varphi^\alpha$ one finds the following equations of motion
\[
R_M^N - \frac{R}{2} \delta_M^N = F_{MK} F^{NK} - \frac{\|F\|^2}{2} \delta_M^N - \|J\|^2 \delta_M^N, \quad (20)
\]
\[
\left( \frac{\delta J}{\|J\|} + \frac{F}{\sqrt{2}} \right) \wedge d\varphi^\alpha \wedge d\varphi^\beta = 0, \quad \delta F = - \frac{\ast J}{\sqrt{2}} \delta F - \frac{\ast J}{\sqrt{2}}
\]
The fields defined by the following equations solve the equations of motion
\[
s^2 = -H^{-2} dX^0 dX^0 + H^2 \delta_{\alpha \beta} dX^\alpha dX^\beta,
\]
\[
A = \frac{\sqrt{2}}{H} dX^0, \quad J = -2\Delta H dX^1 \wedge dX^2 \wedge dX^3.
\]
Here $H$ is smooth positive function of space coordinates $X^\alpha$, $\alpha, \beta = 1, 2, 3$, $\Delta = \partial^2_1 + \partial^2_2 + \partial^2_3$. Calculating $\|J\|$, one has to choose a branch of square root, which gives dust density $\|J\| = -2H^{-3} \Delta H$.

### 7 Black hole with radial strings

Black holes with radial string were recently considered in papers [16, 20] (more detailed bibliography is available in these papers), where some very general models were presented. In this section we present the simplest black hole solution with arbitrary continuous distribution of strings. More general case of charged black hole and multidimensional case were considered in paper [20].

Let us consider the following action
\[
S = \int_M d^4X \sqrt{-g} \left( \frac{R}{2} - \|J\|^2 \right), \quad (21)
\]
where 2-form $J = d\varphi^3 \wedge d\varphi^2$ is string field. Norm $\|J\|$ represents density of string matter.

By variation of fields $g_{MN}$ and $\varphi^\alpha$, $\alpha = 1, 2$ one can find equations of motion
\[
R_M^N - \frac{1}{2} \delta_M^N R = -\|J\|^2 \delta_M^N, \quad \frac{\delta J}{\|J\|} \wedge d\varphi^\alpha = 0. \quad (22)
\]

Let space-time $M$ is warped product with warp factor $r^2$ of 2D black hole $B$:
\[
s^2_B = -\left( 1 - \frac{2M}{r} \right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}},
\]
and an arbitrary 2D Riemannian manifold $S$ with metric $ds^2_S = e^{2f(u,v)} (du^2 + dv^2)$, i.e. $ds^2 = ds^2_B + r^2 ds^2_S$. Every metric of this form is a solution of equations of motion (22)
\[
ds^2 = -\left( 1 - \frac{2M}{r} \right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 e^{2f(u,v)} (du^2 + dv^2), \quad (23)
\]
\[
J = (K - 1) \Omega_S,
\]
where $\Omega_S = e^{2f(u,v)} du \wedge dv$ is 2-form of volume at $S$, and
\[
K = -e^{-2f(u,v)} \Delta f(u, v) \quad (24)
\]
is a half of Riemannian curvature of $S$ ($\Delta = \partial^2_1 + \partial^2_2$). In definition of norm $\|J\|$, one have to choose branch of square root, which corresponds to $\|J\| = \frac{K - 1}{4\pi}$, so the case $K < 1$ corresponds to negative string tension. This solution represents black hole with horizon at $r = 2M$. Every surface $u = const, \; v = const$ is string world surface. The density of string distribution is $\|J\|$. Number (total tension) of string intersecting some 2-dimensional surface $U$ is determined by equation (5).

For compact $S$ the total string tension is (5)
\[
\mu_S = \int_S J = 2\pi \chi(S) - A(S), \quad (25)
\]
where $\chi(S)$ is Euler characteristic of $S$, and $A(S) > 0$ is area of $S$. If one require string tension to be positive, then $\mu_S > 0$, and $\chi(S) > 0$ is necessary condition on topology of $S$.

### 8 Conclusion

The paper presents a wide class of models of delocalized membranes using Euler coordinate approach. In Euler coordinates one has only bulk equations and no worldsurface ones. Euler coordinate approach involves no constraints, it allows to build Hamiltonian formalism in a straightforward manner, which could be useful for quantization.

Simple delocalized membrane is a degenerate case, which has the most direct relation to commonly used Lagrange coordinate approach.

Membrane delocalization is useful as method of removing of singular sources from field equations. To recover singular sources one could consider appropriate limit. Nevertheless this procedure is rather nontrivial. E.g. classical Majumdar-Papapetru solution has horizons, meanwhile generalized Majumdar-Papapetru solution has no horizons. Another difficulty is related with definition of delocalized membrane equations of motion in points with zero or infinite membrane density. It is interesting that the difficulties mentioned have not only mathematical, but also physical interpretation. The existing of physical viewpoint enhance our intuition in handling with delocalized membrane models.

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