MISCELLANOUS APPLICATIONS OF CERTAIN MINIMAX THEOREMS. I

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ABSTRACT. Here is one of the results of this paper (with the convention $\frac{1}{0} = +\infty$): Let $X$ be a real Hilbert space and let $J : X \to \mathbb{R}$ be a $C^1$ functional, with compact derivative, such that

$$\alpha^* := \max \left\{ 0, \limsup_{\|x\| \to +\infty} \frac{J(x)}{\|x\|^2} \right\} < \beta^* := \sup_{x \in X \setminus \{0\}} \frac{J(x)}{\|x\|^2} < +\infty.$$ 

Then, for every $\lambda \in \left[ \frac{1}{2} \beta^*, \frac{1}{2} \alpha^* \right]$ and for every convex set $C \subseteq X$ dense in $X$, there exists $\tilde{y} \in C$ such that the equation

$$x = \lambda J'(x) + \tilde{y}$$

has at least three solutions, two of which are global minima of the functional $x \to \frac{1}{2} \|x\|^2 - \lambda J(x) - \langle x, \tilde{y} \rangle$.

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1. RESULTS

The present one is the first of a series of notes (with the same title) on new consequences and applications of certain minimax theorems that we have established in the past years ([2]-[10]).

In [8], in particular, we obtained the following result:

**Theorem A.** ([8], Theorem 3.4) Let $X$ be a non-empty set, $Y$ a real inner product space and $I : X \to \mathbb{R}$, $\Phi : X \to Y$ two given functions.

Then, for each $\mu > 0$, at least one of the following assertions holds:

(a) for each filtering cover $\mathcal{N}$ of $X$, there exists $A \in \mathcal{N}$ such that

$$\sup_{y \in Y} \inf_{x \in A} (I(x) + \mu(2\langle \Phi(x), y \rangle - \|y\|^2)) < \inf_{x \in A} \sup_{y \in \Phi(A)} (I(x) + \mu(2\langle \Phi(x), y \rangle - \|y\|^2)).$$

(b) for each global minimum $u$ of $x \to I(x) + \mu \|\Phi(x)\|^2$, one has

$$I(u) \leq I(x) + 2\mu \langle \Phi(x), \Phi(u) \rangle - \|\Phi(u)\|^2)$$

for all $x \in X$.

A cover $\mathcal{N}$ of the set $X$ is said to be filtering if for each $A_1, A_2 \in \mathcal{N}$, there exists $A_3 \in \mathcal{N}$ such that $A_1 \cup A_2 \subseteq A_3$.

In the present short note, we want to highlight two applications of the following consequences of Theorem A:
**Theorem 1.1.** Let $X$ be a non-empty set, $x_0 \in X$, $Y$ a real inner product space, $I : X \to \mathbb{R}$, $\Phi : X \to Y$, with $I(x_0) = 0$, $\Phi(x_0) = 0$, and $\mu > 0$. Assume that

$$\inf_{x \in X} I(x) < 0 \leq \inf_{x \in X} (I(x) + \mu\|\Phi(x)\|^2).$$

Then, for each filtering cover $\mathcal{N}$ of $X$, there exists $A \in \mathcal{N}$ such that

$$\sup_{y \in Y} \inf_{x \in A} (I(x) + \mu(2\langle \Phi(x), y \rangle - \|y\|^2)) < \inf_{x \in A} \sup_{y \in \Phi(A)} (I(x) + \mu(2\langle \Phi(x), y \rangle - \|y\|^2)).$$

**Proof.** The assumptions imply that $x_0$ is a global minimum of $x \to I(x) + \mu\|\Phi(x)\|^2$. But, at the same time, since $\inf_X I < 0$, $x_0$ is not a global minimum of $I$. Hence, (b) of Theorem A does not hold and so (a) holds. △

**Theorem 1.2.** Let $X$ be a non-empty symmetric set in a real vector space, $Y$ a real inner product space, $I : X \to \mathbb{R}$ an even function and $\Phi : X \to Y$ an odd function.

Then, for each $\mu > 0$, at least one of the following assertions holds:

(a) for each filtering cover $\mathcal{N}$ of $X$, there exists $A \in \mathcal{N}$ such that

$$\sup_{y \in Y} \inf_{x \in A} (I(x) + \mu(2\langle \Phi(x), y \rangle - \|y\|^2)) < \inf_{x \in A} \sup_{y \in \Phi(A)} (I(x) + \mu(2\langle \Phi(x), y \rangle - \|y\|^2));$$

(b) if $u \in X$ is a global minimum of $x \to I(x) + \mu\|\Phi(x)\|^2$, then $\Phi(u) = 0$ and $u$ is a global minimum of $I$.

**Proof.** Assume that (a) does not hold. Let $u \in X$ be any global minimum of $x \to I(x) + \mu\|\Phi(x)\|^2$. Then, by Theorem A, we have

$$I(u) \leq I(x) + 2\mu(\langle \Phi(x), \Phi(u) \rangle - \|\Phi(u)\|^2) \quad (1)$$

for all $x \in X$. So, in particular, we have

$$I(u) \leq I(-u) + 2\mu(\langle \Phi(-u), \Phi(u) \rangle - \|\Phi(u)\|^2) = I(u) - 4\mu\|\Phi(u)\|^2$$

which yields

$$\|\Phi(u)\|^2 = 0.$$  

So, by (1), we get

$$I(u) \leq I(x)$$

for all $x \in X$. △

The application of Theorem 1.1 we wish to present is provided by the following result jointly with Theorem 1.4:

**Theorem 1.3.** Let $X$ be a real inner product space and let $\tau$ be a topology on $X$. Moreover, let $J : X \to \mathbb{R}$ be a functional such that

$$J(0) = 0 < \sup_X J$$

and

$$\beta^* := \sup_{x \in X \setminus \{0\}} \frac{J(x)}{\|x\|^2} < +\infty.$$  

Finally, let $\lambda > \frac{1}{J^*}$ and let $\mathcal{N}$ be a filtering cover of $X$ such that, for each $A \in \mathcal{N}$ and each $y \in X$, the restriction to $A$ of the functional $x \to \|x\|^2 - \lambda J(x) + \langle x, y \rangle$ is $\tau$-lower semicontinuous and $\inf \tau$-compact.

Then, there exists $\bar{A} \in \mathcal{N}$ with the following property: for every convex set $C \subseteq X$ whose closure (in the strong topology) contains $\bar{A}$, there exists $\bar{y} \in C$ such that the restriction to $\bar{A}$ of the functional $x \to \|x\|^2 - \lambda J(x) + \langle x, 2(\beta^* \lambda - 1)\bar{y} \rangle$ has at least two global minima.
Proof. In view of (2), we have
\[ \inf_{x \in X} (\|x\|^2 - \lambda J(x)) < 0, \]
as well as
\[ \inf_{x \in X} (\|x\|^2 - \lambda J(x) + (\beta^* \lambda - 1)\|x\|^2) \geq 0. \]
So, we can apply Theorem 1.1 taking \( Y = X, \)
\[ \mu = \beta^* \lambda - 1, \]
\[ I(x) = \|x\|^2 - \lambda J(x) \]
and
\[ \Phi(x) = x. \]
Therefore, there exists \( \tilde{A} \in \mathcal{N} \) such that
\[ \sup_{y \in Y} \inf_{x \in \tilde{A}} (\|x\|^2 - \lambda J(x) + (\beta^* \lambda - 1)(2\langle x, y \rangle - \|y\|^2)) < \inf_{x \in \lambda \tilde{A}} \sup_{y \in A} (\|x\|^2 - \lambda J(x) + (\beta^* \lambda - 1)(2\langle x, y \rangle - \|y\|^2)). \] (3)

Now, consider the function \( f : X \times X \to \mathbb{R} \) defined by
\[ f(x, y) = \|x\|^2 - \lambda J(x) + (\beta^* \lambda - 1)(2\langle x, y \rangle - \|y\|^2) \]
for all \((x, y) \in X \times X.\) Since \( f(x, \cdot) \) is continuous and \( \tilde{A} \subseteq \overline{C}, \) we have
\[ \sup_{y \in A} f(x, y) = \sup_{v \in \tilde{A}} f(x, y) \leq \sup_{v \in C} f(x, y) = \sup_{y \in C} f(x, y) \]
for all \( x \in X, \) and hence, taking (3) into account, it follows that
\[ \sup_{y \in C} \inf_{x \in \tilde{A}} f(x, y) < \inf_{x, y \in \tilde{A}} \sup_{y \in C} f(x, y) \leq \inf_{x \in \lambda \tilde{A}} \sup_{y \in \tilde{A}} f(x, y). \] (4)

Now, in view of (4), taking into account that \( f_{\tilde{A} \times C} \) is \( \tau \)-lower semicontinuous and \( \tau \)-compact in \( \tilde{A}, \) and continuous and concave in \( C, \) we can apply Theorem 3.2 of [8] to \( f_{\tilde{A} \times C}. \) Consequently, there exists \( \tilde{y} \in C \) such that \( f_{\tilde{A}}(\cdot, \tilde{y}) \) has at least two global minima, and the proof is complete. \( \triangle \)

From Theorem 1.3, in turn, we obtain the following result (with the convention \( \frac{1}{0} = +\infty):\)

**Theorem 1.4.** Let \( X \) be a real Hilbert space and let \( J : X \to \mathbb{R} \) be a \( C^1 \) functional, with compact derivative, such that
\[ \alpha^* := \max \left\{ 0, \lim_{\|x\| \to +\infty} \frac{J(x)}{\|x\|^2} \right\} < \beta^* := \sup_{x \in X \setminus \{0\}} \frac{J(x)}{\|x\|^2} < +\infty. \]

Then, for every \( \lambda \in \left[ \frac{1}{2\beta^*}, \frac{1}{2\alpha^*} \right] \) and for every convex set \( C \subseteq X \) dense in \( X, \) there exists \( \tilde{y} \in C \) such that the equation
\[ x = \lambda J'(x) + \tilde{y} \]
has at least three solutions, two of which are global minima of the functional \( x \to \frac{1}{2}\|x\|^2 - \lambda J(x) - \langle x, \tilde{y} \rangle. \)

**Proof.** Fix \( \lambda \in \left[ \frac{1}{2\beta^*}, \frac{1}{2\alpha^*} \right] \) and a convex set \( C \subseteq X \) dense in \( X. \) For each \( y \in X, \) we have
\[ \lim_{\|x\| \to +\infty} \left( 1 - 2\lambda \frac{J(x)}{\|x\|^2} - \frac{\langle x, y \rangle}{\|x\|^2} \right) = 1 - 2\lambda \lim_{\|x\| \to +\infty} \frac{J(x)}{\|x\|^2} > 0. \]
So, from the identity
\[ \|x\|^2 - 2\lambda J(x) - \langle x, y \rangle = \|x\|^2 \left( 1 - 2 \lambda \frac{J(x)}{\|x\|^2} - \frac{\langle x, y \rangle}{\|x\|^2} \right) \]
it follows that
\[ \lim_{\|x\| \to +\infty} (\|x\|^2 - 2\lambda J(x) - \langle x, y \rangle) = +\infty . \] (5)

Since \( J' \) is compact, \( J \) is sequentially weakly continuous ([12], Corollary 41.9). Then, in view of (5) and of the Eberlein-Smulian theorem, for each \( y \in X \), the functional \( x \to \|x\|^2 - 2\lambda J(x) + \langle x, y \rangle \) is inf-weakly compact in \( X \). So, we can apply Theorem 1.3 taking the weak topology as \( \tau \) and \( N = \{ X \} \). Consequently, since the set \( \frac{1}{1-2\beta^*\lambda}C \) is convex and dense in \( X \), there exists \( \hat{y} \in \frac{1}{1-2\beta^*\lambda}C \) such that the functional \( x \to \|x\|^2 - 2\lambda J(x) + \langle x, 2(2\beta^*\lambda - 1)\hat{y} \rangle \) has at least two global minima in \( X \) which are two of its critical points.

Since the same functional satisfies the Palais-Smale condition ([12], Example 38.25), it has a third critical point in view of Corollary 1 of [1]. Clearly, the conclusion follows taking \( \tilde{y} = (1 - 2\beta^*\lambda)\hat{y} \).

△

We now give a specific application of Theorem 1.4.

Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with smooth boundary. On the Sobolev space \( H^1_0(\Omega) \), we consider the scalar product
\[ \langle u, v \rangle = \int_{\Omega} \nabla u(x) \nabla v(x) dx \]
with the induced norm
\[ \|u\| = \left( \int_{\Omega} |\nabla u(x)|^2 dx \right)^{\frac{1}{2}} . \]

We denote by \( H^{-1}(\Omega) \) the dual of \( H^1_0(\Omega) \).

If \( n \geq 2 \), we denote by \( \mathcal{A} \) the class of all Carathéodory functions \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) such that
\[ \sup_{(x,\xi) \in \Omega \times \mathbb{R}} \frac{|f(x,\xi)|}{1 + |\xi|^q} < +\infty , \]
where \( 0 < q < \frac{n+2}{n-2} \) if \( n > 2 \) and \( 0 < q < +\infty \) if \( n = 2 \). While, when \( n = 1 \), we denote by \( \mathcal{A} \) the class of all Carathéodory functions \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) such that, for each \( r > 0 \), the function \( x \to \sup_{|\xi| \leq r} |f(x,\xi)| \) belongs to \( L^1(\Omega) \).

Given \( f \in \mathcal{A} \) and \( \varphi \in H^{-1}(\Omega) \), consider the following Dirichlet problem
\[ \begin{cases} -\Delta u = f(x, u) + \varphi & \text{in } \Omega \\ u = 0 & \text{on } \partial \Omega . \end{cases} \] (\( P_{f,\varphi} \))

Let us recall that a weak solution of \( (P_{f,\varphi}) \) is any \( u \in H^1_0(\Omega) \) such that
\[ \int_{\Omega} \nabla u(x) \nabla v(x) dx - \int_{\Omega} f(x, u(x)) v(x) dx - \varphi(v) = 0 \]
for all \( v \in H^1_0(\Omega) \).

Let \( \Psi, J_f : H^1_0(\Omega) \to \mathbb{R} \) be the functionals defined by
\[ \Psi(u) = \frac{1}{2} \|u\|^2 \]
\[ J_f(u) = \int_{\Omega} F(x, u(x)) dx , \]
where
\[ F(x, \xi) = \int_0^\xi f(x, t) dt . \]
Notice that \( J_f \) is a \( C^1 \) functional whose derivative is given by

\[
J'_f(u)(v) = \int_\Omega f(x, u(x))v(x)dx
\]

for all \( u, v H^1_0(\Omega) \). Consequently, the weak solutions of problem \( (Pf, \varphi) \) are exactly the critical points in \( H^1_0(\Omega) \) of the functional \( \Psi - J_f - \varphi \). Moreover, \( J'_f \) is compact.

Furthermore, if \( f \in A \) and the set

\[
\left\{ x \in \Omega : \sup_{\xi \in \mathbb{R}} F(x, \xi) > 0 \right\}
\]

has a positive measure, we have \( \sup_{H^1_0(\Omega)} J_f > 0 \) (see the proof of Theorem 2 of [9]).

We now state the following

**Theorem 1.5.** Let \( f \in A \) be such that the set

\[
\left\{ x \in \Omega : \sup_{\xi \in \mathbb{R}} F(x, \xi) > 0 \right\}
\]

has a positive measure and

\[
\limsup_{|\xi| \to +\infty} \sup_{x \in \Omega} \frac{F(x, \xi)}{\xi^2} \leq 0 . \tag{6}
\]

Then, for every \( \lambda > 0 \) large enough and for every convex set \( C \subset H^{-1}(\Omega) \) dense in \( H^{-1}(\Omega) \), there exists \( \varphi \in C \) such that the problem

\[
\begin{align*}
-\Delta u &= \lambda f(x, u) + \varphi \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega
\end{align*}
\]

has at least three weak solutions, two of which are global minima in \( H^1_0(\Omega) \) of the functional \( \Psi - \lambda J_f - \varphi \).

**Proof.** Condition (6) clearly implies that

\[
\limsup_{\|u\| \to +\infty} \frac{J_f(u)}{\|u\|^2} \leq 0 .
\]

Then, in view of the above-recalled preliminaries, we can apply Theorem 1.4 taking \( X = H^1_0(\Omega) \), \( J = J_f \), and the conclusion directly follows.

The last result is an application of Theorem 1.2.

Let \((T, F, \mu)\) be a measure space, \( E \) a real Banach space and \( p \geq 1 \).

As usual, \( L^p(T, E) \) denotes the space of all (equivalence classes of) strongly \( \mu \)-measurable functions \( u : T \to E \) such that \( \int_T \| u(t) \|^p d\mu < +\infty \), equipped with the norm

\[
\| u \|_{L^p(T, E)} = \left( \int_T \| u(t) \|^p d\mu \right)^{\frac{1}{p}} .
\]

A set \( D \subseteq L^p(T, E) \) is said to be decomposable if, for every \( u, v \in D \) and every \( A \in F \), the function

\[
t \to \chi_A(t)u(t) + (1 - \chi_A(t))v(t)
\]

belongs to \( D \), where \( \chi_A \) denotes the characteristic function of \( A \).
Theorem 1.6. Let $(T, F, \mu)$ be a non-atomic measure space, with $0 < \mu(T) < +\infty$, and $E$ a real Banach space. Let $f : E \to \mathbb{R}$ be a lower semicontinuous and even function which has no global minima. Let $g : T \times E \to \mathbb{R}$ be a Carathéodory function such that $g(x, \cdot)$ is odd for all $t \in T$. Moreover, assume that
\[
\max \left\{ \sup_{x \in E} \frac{|f(x)|}{1 + \|x\|^p}, \sup_{(t, x) \in T \times E} \frac{|g(t, x)|}{1 + \|x\|^p} \right\} < +\infty
\]
for some $p \geq 1$. For each $u \in L^p(T, E)$, set
\[
J(u) = \int_T f(u(t))d\mu + \left( \int_T g(t, u(t))d\mu \right)^2.
\]
Then, the restriction of the functional $J$ to any symmetric and decomposable subset of $L^p(T, E)$ containing the constant functions has no global minima.

Proof. Let $X \subseteq L^p(T, E)$ be a symmetric and decomposable set containing the constant functions. Arguing by contradiction, assume that $\hat{u} \in X$ is a global minimum of $J|_X$. For each $h \in L^p(T)$, set
\[
X_h = \{ u \in X : \|u(t)\| \leq h(t) \text{ a.e. in } T \}.
\]
Clearly, $X_h$ is decomposable and the family $\{X_h\}_{h \in L^p(T)}$ is a filtering covering of $X$. Let $I, \Phi : L^p(T, E) \to \mathbb{R}$ be the functionals defined by
\[
I(u) = \int_T f(u(t))d\mu
\]
and
\[
\Phi(u) = \int_T g(t, u(t))d\mu
\]
for all $u \in L^p(T, E)$. Of course, $I$ is lower semicontinuous and even, while $\Phi$ is continuous and odd. Fix $h \in L^p(T)$. Notice that $X_h$ is connected ([11]). Therefore, $\Phi(X_h)$ is an interval, and so
\[
\Lambda_h := \overline{\Phi(X_h)}
\]
is a compact interval. Now, we can apply Theorem 1.D of [6]. Thanks to it, we then have
\[
\sup_{\lambda \in \Lambda_h} \inf_{u \in X_h} (I(u) + 2\lambda \Phi(u) - \lambda^2) = \inf_{u \in X_h} \sup_{\lambda \in \Lambda_h} (I(u) + 2\lambda \Phi(u) - \lambda^2).
\]
Hence, (with $\mu = 1$) assertion $(a_1)$ of Theorem 1.2 does not hold, and so assertion $(b_1)$ must hold. Therefore, $\hat{u}$ is a global minimum of $I|_X$ and $\Phi(\hat{u}) = 0$. Since $X$ contains the constants, we have
\[
I(\hat{u}) \leq \mu(T) \inf_E f.
\]
This clearly implies that
\[
\int_T (f(\hat{u}(t)) - \inf_E f)d\mu = 0,
\]
and so $\hat{u}(t)$ would be a global minimum of $f$ for $\mu$-almost $t \in T$, against the assumptions. \triangle

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