Effective hamiltonians for $1/N$ expansion
in two-dimensional QCD

J. L. F. Barbón\textsuperscript{1} and Krešimir Demeterfi\textsuperscript{2,3}

Joseph Henry Laboratories
Princeton University
Princeton, NJ 08544, U.S.A.

We discuss the general structure of effective hamiltonians for systematic $1/N$ expansion in QCD using the light-cone quantization. These are second-quantized hamiltonians acting on the Fock space of mesons and glueballs defined by the solution of the $N = \infty$ problem. In the two-dimensional case we find only cubic and quartic interaction terms, and give explicit expressions for the vertex functions as integrals of solutions of ’t Hooft equation. As examples of possible applications of our formalism, we study $1/N$ corrections to meson mass and form factors for decays of $Q\bar{q}$ states, recently discussed by Grinstein and Mende in the large-$N$ limit. We find that $1/N$ is a good small expansion parameter.
1. Introduction

It is generally believed that quantum chromodynamics (QCD) is the correct theory of strong interactions. Because of the asymptotic freedom one can use perturbation theory to explain short-distance properties of hadrons. At long distances, however, the theory becomes strongly coupled exhibiting color confinement, and nonperturbative treatment is clearly required.

As originally proposed by ’t Hooft [1], generalizing the gauge group of QCD to SU($N$) provides $1/N$ as the unique small expansion parameter if $N$ is large (in reality, $N = 3$). It is hoped that in the limit $N \to \infty$ the problem simplifies enough to become tractable (while still capturing the essential dynamics), and that a systematic $1/N$ expansion can be developed around such a solution. Large-$N$ expansion has indeed been a very fruitful idea in various branches of physics (see, c.f., ref. [2]). In the physically most interesting case of QCD, however, this approach has been successfully applied only in two space-time dimensions, where ’t Hooft solved the $N = \infty$ problem for mesons by summing an infinite set of planar diagrams [3]. The solution is given in the form of an integral equation for meson spectrum and wave functions. In this approximation one finds an infinite tower of stable mesons with asymptotically linearly growing mass. This model was further studied by Callan, Coote and Gross [4], and by Einhorn [5] who showed that the scattering amplitudes and the electromagnetic form factors are given completely in terms of asymptotic color-singlet bound states, and that no free quarks appear in the physical spectrum. More recently, Grinstein and Mende [6] extended the analysis of form factors to the case of flavor changing currents and used the ’t Hooft model to explicitly illustrate their earlier argument that semileptonic $B$ decays in the four-dimensional theory are dominated by a single pole in the combined limits $N \to \infty, M_b \to \infty$ and $m_\pi \to 0$ [7].

In this paper we discuss effective hamiltonians for systematic $1/N$ expansion in QCD using the light-cone quantization. The advantage of using light-cone quantization comes from the fact that the light-cone vacuum is an exact eigenstate of the full hamiltonian. As a result, light-cone Fock space methods become powerful tools in the study of relativistic bound-state problems [8]. A nice feature of the large-$N$ limit in this formulation is that the corresponding bound-state equations become linear equations for the so-called light-cone wave functions*. The main idea of our approach is to construct a second-quantized

* The Fock space formulation of the $1/N$ expansion of QCD has been advocated some time ago by C. Thorn [9].
hamiltonian acting on the Fock space of mesons and glueballs defined by the solution of the $N = \infty$ problem. As usual, this hamiltonian is defined by requiring the equivalence of its matrix elements between the Fock space states to the matrix elements of the fundamental QCD hamiltonian between asymptotic states. This effectively defines the vertex functions as nested commutators of the light-cone hamiltonian with the operators creating asymptotic states. These commutators can be expressed solely in terms of the solutions of the large-$N$ eigenvalue problem. Because of the structure of the light-cone hamiltonian and the light-cone Fock space, such effective hamiltonian will have only a finite number of interaction terms, but with non-local vertices. Once the vertex functions are known one can develop a systematic $1/N$ expansion by using the standard light-cone perturbation theory. Our method is quite general and can be easily extended to the baryon sector, as well as to higher dimensions, provided the planar solution is known.

We carry out this program explicitly for the meson sector of two-dimensional QCD. Using the hamiltonian light-cone formalism is particularly useful in this case because it eliminates the need to sum gluon diagrams which are summarized in the effective Coulomb interaction by solving the gauge field constraint. We find only cubic and quartic interactions and derive expressions for the vertex functions in terms of ’t Hooft functions. Since solutions of ’t Hooft equation are not known in terms of standard functions, we can only give numerical estimates of the vertex functions. We present several examples of these vertex functions obtained numerically and discuss their properties relevant for the $1/N$ expansion. Using the standard light-cone perturbation theory we study the $1/N$ corrections to the meson spectrum and to the $B$-meson decay form factors, and find that $1/N$ for $N = 3$ is indeed effectively a small parameter.

The outline of our presentation goes as follows. In sect. 2 we discuss the general structure of effective hamiltonians for the $1/N$ expansion in QCD, independent of the dimension of space-time. In sect. 3 we review the light-cone quantization of two-dimensional QCD, and write the full light-cone hamiltonian in terms of color-singlet operators. In sect. 4 we rederive the bound-state equations for mesons and baryons in the large-$N$ limit in order to demonstrate power of the hamiltonian light-cone formalism which does not require summing diagrams. In sect. 5 we present the main result of our paper – construction of the effective hamiltonian for mesons in two-dimensional QCD. As examples of possible applications of our formalism we study $1/N$ corrections to the meson spectrum in sect. 6 and to the form factors in sect. 7. Finally, in sect. 8 we make some comments on the analytic properties, and end with conclusions and a few general remarks in sect. 9. In the appendix we collect explicit expressions for vertex functions.
2. Effective hamiltonians

Given the $N = \infty$ solution of QCD, it would be easy to rewrite the $1/N$ expansion in a formal hamiltonian framework. The planar solution is known to be a free bosonic Fock space of mesons and glueballs. Let us denote their creation operator generically by $A_i$. The asymptotic Hilbert space at $N = \infty$ is then spanned by states of the form

$$A_{i_1} \ldots A_{i_n} |0\rangle,$$

where the indices stand for the whole set of meson/glueball quantum numbers.

The perturbative physics in powers of $1/N$ depends on the matrix elements,

$$\langle 0| A_{j_1} \ldots A_{j_q} H A_{i_p}^\dagger \ldots A_{i_1}^\dagger |0\rangle,$$

where $H$ is the full QCD hamiltonian. An equivalent second quantized hamiltonian acting on the asymptotic Hilbert space (2.1) is easily constructed by matching the matrix elements (2.2). The solution is,

$$H_{\text{eff}} = \sum_{p,q} \frac{1}{p!q!} H_{(p,q)}(j_1, \ldots, j_q; i_1, \ldots, i_p) A_{j_1}^\dagger \ldots A_{j_q}^\dagger A_{i_1} \ldots A_{i_p},$$

with irreducible many-body interactions given by the matrix elements,

$$H_{(p,q)}(j_1, \ldots, j_q; i_1, \ldots, i_p) = \langle 0| A_{j_1} \ldots A_{j_q} [\ldots [H, A_{i_1}^\dagger] \ldots A_{i_p}^\dagger] |0\rangle.$$

These quantities should be calculated from the fundamental quark-gluon theory. In particular, we must know the expressions for $A_i$ in terms of the fundamental variables (wave functions), which in turn is equivalent to diagonalization of the $H_{(1,1)}$ part of the effective hamiltonian (the planar, $N = \infty$, problem). Of course, this is a formidable task in practice. As pointed out by Thorn [9], such Fock space representation of the $1/N$ expansion is very involved in the standard equal-time quantization, although it simplifies considerably if we adopt the light-cone quantization scheme [10] or, in more physical terms, if we look at the system from the infinite momentum frame [11].

Let us briefly review Thorn’s argument. The main simplifying feature of the light-cone hamiltonians is the absence of pure positive/negative frequency terms, with the result that the Fock vacuum becomes an exact eigenstate of the full (interacting) hamiltonian. As
far as $1/N$ power-counting is concerned we can forget about all quantum numbers except color. The SU($N$) structure for the glue sector of QCD is

$$H_{LC} \sim \text{Tr}(a^\dagger a) + \frac{1}{\sqrt{N}}\text{Tr}(a^\dagger a^\dagger a) + \frac{1}{N}\text{Tr}(a^\dagger aa^\dagger a) + \frac{1}{N^2}\text{Tr}(a^\dagger aa^\dagger a) + \text{h.c.} \quad (2.5)$$

where $a^\dagger$ is an $N \times N$ operator-valued hermitian matrix of transverse gluon creators. It is then easy to see that the $N = \infty$ eigenvalue equation closes on the subspace of invariant states of the form

$$G^\dagger|0\rangle = \sum_l f(1, \ldots, l) \frac{1}{N^{l/2}}\text{Tr}(a_1^\dagger \ldots a_l^\dagger)|0\rangle, \quad (2.6)$$

which can be represented graphically as a superposition of closed strings each containing $l$ “partons”. The coefficients $f(1, \ldots, l)$ are usually referred to as the light-cone wave functions. In fact, the complete set of “string” states (2.6) provides a plane-wave basis for Wilson loop operators, and the glueball states are clearly built directly in loop space.

In the equal-time quantization, in addition to terms in eq. (2.5), the hamiltonian contains pure positive frequency pieces, like $\frac{1}{\sqrt{N}}\text{Tr}(a^\dagger a^\dagger a)$. When these terms are included, the $N = \infty$ problem closes on a much more complicated space of coherent states of invariant traces (previous string-like states). A simple 0-dimensional toy model from ref. [9] illustrates this fact. The hamiltonian

$$H = \epsilon_0\text{Tr}(a^\dagger a) + \frac{1}{\sqrt{N}}\left(\text{Tr}(a^\dagger a^\dagger a^\dagger) + \text{h.c.}\right) \quad (2.7)$$

has eigenvalue $E = -N^2\epsilon_0$ for the eigenstate

$$|\psi\rangle = \exp\left\{-\frac{N}{3\epsilon_0}\text{Tr} \left( \frac{a^\dagger}{\sqrt{N}} \right)^3 \right\}|0\rangle.$$

The form of the coherent states of loops for the real problem (2.5) is not known, and the large-$N$ Fock space is too big even for numerical approach. As a result, the only hamiltonian Fock space method that could work in practice is the light-cone one.

Using eqs. (2.4), (2.5) and (2.6) one easily finds the following structure of the effective light-cone hamiltonian in the glueball sector:

$$H_{\text{eff}} \sim H_{(1,1)}G^\dagger G + \frac{1}{2N}H_{(1,2)}G^\dagger G^\dagger G + \frac{1}{4N^2}H_{(2,2)}G^\dagger G^\dagger G^\dagger G + \frac{1}{6N^2}H_{(1,3)}G^\dagger G^\dagger G^\dagger G + \text{h.c.} \quad (2.8)$$
The planar hamiltonian is defined by the one-particle sector in this expansion, \( H_0 = H_{(1,1)} G^\dagger G \), and its diagonalization constitutes the large-\( N \) solution of the model. In general, \( H_{(1,1)} \) is a complicated integral kernel. For example, for fields in the adjoint representation it is given by an infinite hierarchy of integral operators, accounting for the interaction of partons inside strings (2.6) of arbitrary length [12]. The working hypothesis in this whole approach is that only a small number of partons contributes significantly to the low-lying states. This goes under the name of light-cone Tamm-Dancoff approximation [13] and numerical approaches, like the discretized light-cone method [8], automatically incorporate such Fock space truncations. Encouraging evidence for the validity of this procedure is found in the analysis of simple two-dimensional models with complicated loop space, like two-dimensional QCD with adjoint matter [12].

The actual computation of the kernels (2.4) involves numerical analysis even in the simplest possible case, the ’t Hooft model. However, an interesting general feature of the light-cone effective hamiltonians (2.3) is that they contain only a finite number of vertices, albeit non-local ones. These non-local vertices replace the infinite tower of higher-derivative operators in usual local expansions of effective lagrangians. The relation between our light-cone hamiltonians and the standard covariant effective lagrangians is very much the same as the relation between light-cone string field theory and covariant string field theory: the former has only cubic interactions while the latter has a non-polynomial action.

As a final remark, we point out that by construction (2.4) only depends on the meson/glueball quantum numbers, while the fundamental hamiltonian \( H \) depends on all quark/gluon quantum numbers. Thus, deriving (2.4) involves “integrating out” part of the microscopic degrees of freedom, justifying the use of the word “effective”. However, it is important to note that no low-energy expansion is assumed, at least in principle, as long as the large-\( N \) problem is solved exactly. The nature of the approximation is more similar to the Born-Oppenheimer method, where fast and slow variables are decoupled at the level of wave functions. The effective hamiltonians considered here give the exact answer for perturbative questions about the free asymptotic Fock space, like 1/\( N \) corrections to scattering amplitudes and self-energies, but their description of “nuclear physics” is only approximate. For example, exotics would be treated as meson bound states, rather than four-quark bound states, in exact analogy with the Born-Oppenheimer decoupling in molecular physics.
In the remainder of this paper we study in detail the effective light-cone hamiltonian for two-dimensional QCD. In this case one can integrate out the gluons by solving the light-cone gauge constraint, and the large-$N$ Fock space has a simple structure encoded in the solutions of ‘t Hooft’s bound-state equation. This model illustrates many of the issues involved in the large-$N$ approach and the structure we find is very similar to the one of four-dimensional QCD in the valence approximation (constituent quark model).

3. The two-dimensional model

In this section we review the light-cone quantization of two-dimensional QCD with $N_f$ flavors of quarks in the fundamental representation of SU($N$), and fix the notation to be used throughout the paper. The model is given by the lagrangian,

$$\mathcal{L} = -\frac{1}{4} \text{Tr} F^2 + \sum_{j=1}^{N_f} \bar{\psi}_j (i\not{D} - m_j) \psi_j ,$$

(3.1)

where $iD_\mu = i\partial_\mu - gA^a_\mu T^a$, and the SU($N$) generators, $T^a$, are normalized as

$$\text{Tr} (T^a T^b) = \delta^{ab} ,$$

(3.2a)

$$\sum_a T^a_{\alpha\beta} T^a_{\gamma\delta} = \delta_{\alpha\delta} \delta_{\beta\gamma} - \frac{1}{N} \delta_{\alpha\beta} \delta_{\gamma\delta} .$$

(3.2b)

We shall work in the light-cone coordinates $x^\pm = (x^0 \pm x^1)/\sqrt{2}$, and use the following representation for $\gamma$-matrices: $\gamma^0 = \sigma_2, \gamma^1 = i\sigma_1$. The light-cone hamiltonian, $P^-$, and momentum, $P^+$, are given by

$$P^\mp = \int_{x^+=\text{const.}} T^{+\mp} ,$$

(3.3)

where $T^{\mu\nu}$ is the energy-momentum tensor. In the light-cone quantization one takes $x^+ = \text{const.}$ as initial value surfaces for the light-cone hamiltonian.

In the studies of gauge theories on the light cone, it is most convenient to work in the light-cone gauge, $A_-=0$, which introduces the following constraints:

$$i\gamma^- \partial_- \psi_j = \frac{1}{2} m_j \gamma^- \gamma^+ \psi_j ,$$

$$\quad (i\partial_-)^2 A^a_+ = g \sum_{j=1}^{N_f} \bar{\psi}_j \gamma^+ T^a \psi_j ,$$

(3.4)

$$\sum_{j=1}^{N_f} \int_{x^+ = \text{const.}} \bar{\psi}_j \gamma^+ T^a \psi_j \mid \text{phys.} = 0 .$$
The first constraint projects out a component of the quark field and it is purely kinematical. The solution of the second constraint,

$$A_+^a = g \sum_{j=1}^{N_f} \frac{1}{(i\partial^-)^2} \bar{\psi}_j \gamma^+ T^a \psi_j ,$$  \hspace{2cm} (3.5)

induces the well-known Coulomb potential between the fermion currents, which is responsible for all the static properties of the model. The zero-mode of the gauge field $A_+^a$ does not enter the second constraint. Direct integration over this zero-mode leads to the third constraint which restricts the Fock space states to color singlets. The last two constraints therefore entail the confinement property of the model. After solving constraints (3.4) the light-cone hamiltonian reads,

$$P^- = \int_{x^+ = \text{const.}} \left( \sum_j \frac{m_j^2}{\sqrt{2}} \psi_j^\dagger \frac{1}{i\partial^-} \psi_j + g^2 \sum_{j,k} (\psi_j^\dagger T^a \psi_j) \frac{1}{(i\partial^-)^2} (\psi_k^\dagger T^a \psi_k) \right) ,$$  \hspace{2cm} (3.6)

and the light-cone momentum is given by,

$$P^+ = \frac{i}{\sqrt{2}} \sum_j \int_{x^+ = \text{const.}} \psi_j^\dagger \partial^- \psi_j .$$  \hspace{2cm} (3.7)

In eqs. (3.6) and (3.7) $\psi_j$ represents physical degree of freedom – the projected out fermion field: $\psi_j \rightarrow \frac{1}{2} \gamma^+ \gamma^+ \psi_j$.

In order to exhibit the $1/N$ power counting, it is convenient to rewrite $P^-$ in a Fock space representation. This is straightforward upon introducing the mode expansion:

$$\psi_{j,\alpha} = \frac{1}{2^{1/4}} \int_0^\infty \frac{dk^+}{\sqrt{2\pi}} \left( b_{j,\alpha}(k^+) e^{-ik^+ x^-} + d_{j,\alpha}^\dagger(k^+) e^{ik^+ x^-} \right) ,$$  \hspace{2cm} (3.8)

where $\alpha$ is a color index, and the operators $b$ and $d$ satisfy canonical anticommutation relations:

$$\{b_{i,\alpha}(k^+), b_{j,\beta}^\dagger(p^+)\} = \{d_{i,\alpha}(k^+), d_{j,\beta}^\dagger(p^+)\} = \delta_{\alpha\beta} \delta_{ij} \delta(k^+ - p^+) .$$  \hspace{2cm} (3.9)

(The normalization of $\psi$ is chosen to close the Poincaré algebra.) To simplify notation we introduce a collective index $k \equiv (k^+, i)$ to denote both momentum $k^+$ and the flavor index $i$. In this notation

$$\int dk \equiv \sum_{i=1}^{N_f} \int_0^\infty dk^+ .$$
After normal ordering the fermion bilinears, the light-cone Hamiltonian (3.6) can be written in the creation–annihilation basis in terms of color-singlet operators:

\[ M_{kk'}^\dagger \equiv \sqrt{N} \sum_{\alpha} b^\dagger_{\alpha}(k) d^\dagger_{\alpha}(k') , \]
\[ M_{kk'} \equiv (M_{kk'}^\dagger)^\dagger , \]
\[ B_{kk'} \equiv \sum_{\alpha} b^\dagger_{\alpha}(k) b_{\alpha}(k') , \]
\[ D_{kk'} \equiv \sum_{\alpha} d^\dagger_{\alpha}(k) d_{\alpha}(k') . \]  

These operators have obvious physical meaning. For example, \( M_{kk'}^\dagger \) (\( M_{kk'} \)) creates (annihilates) a meson in which quark of flavor \( i \) carries momentum \( k^+ \), while the antiquark of flavor \( i' \) carries momentum \( k'^+ \). Similarly, \( B_{kk'} \) (\( D_{kk'} \)) counts the number of quarks (antiquarks). It is convenient to write \( P^− \) as a sum of three terms,

\[ P^− = T + V + V_{\text{sing}} . \]  

The kinetic energy, \( T \), is given by

\[ T = \sum_{j=1}^{N_f} \bar{m}_j^2 \frac{\sqrt{N}}{2} \int \frac{dk^+}{k^+} (B_{kk} + D_{kk}) , \]  

with the renormalized mass

\[ \bar{m}_j^2 = m_j^2 - g^2 N \frac{\pi}{4} \left( 1 - \frac{1}{2N^2} \right) . \]  

The potential energy, \( V \), is given by

\[ V = -\frac{g^2 N}{4\pi} \int dk dk' dp dp' \left( K_{MM} M_{kk'}^\dagger M_{pp'} + K_{MB} M_{kk'}^\dagger B_{pp'} ight. \\
\left. + K_{MD} M_{kk'}^\dagger D_{pp'} + K_{BD} B_{kk'} D_{pp'} + K_{BB} B_{kk'} B_{pp'} ight. \\
\left. + K_{DD} D_{kk'} D_{pp'} + \text{h. c.} \right) , \]  

8
with the kernels given explicitly by:

\[
K_{MM} = \left[ \frac{\delta_{ij}\delta_{i'j'}}{(k^+ - p^+)^2} + \frac{1}{N} \frac{\delta_{ii'}\delta_{jj'}}{(k^+ + k'^+)^2} \right] \delta(k^+ + k'^+ - p^+ - p'^+),
\]

\[
K_{BB} = K_{DD} = \frac{1}{2N} \left[ \frac{\delta_{ij'}\delta_{jj'}}{(k^+ - p'^+)^2} + \frac{1}{N} \frac{\delta_{ii'}\delta_{jj'}}{(k^+ - k'^+)^2} \right] \delta(k^+ - k'^+ + p^+ - p'^+),
\]

\[
K_{MB} = \frac{2}{\sqrt{N}} \left[ \frac{\delta_{ij'}\delta_{ji'}}{(k^+ + p'^+)^2} + \frac{1}{N} \frac{\delta_{ii'}\delta_{ji'}}{(k^+ + k'^+)^2} \right] \delta(k^+ + k'^+ + p^+ - p'^+),
\]

\[
K_{MD} = -\frac{2}{\sqrt{N}} \left[ \frac{\delta_{ij'}\delta_{ji'}}{(k^+ + p'^+)^2} + \frac{1}{N} \frac{\delta_{ii'}\delta_{ji'}}{(k^+ - k'^+)^2} \right] \delta(k^+ - k'^+ + p^+ - p'^+),
\]

\[
K_{BD} = -\frac{1}{N} \left[ \frac{\delta_{ij}\delta_{i'j'}}{(k^+ + p^+)^2} + \frac{1}{N} \frac{\delta_{ii'}\delta_{jj'}}{(k^+ - k'^+)^2} \right] \delta(k^+ - k'^+ + p^+ - p'^+).
\]

The singular term, \( V_{\text{sing}} \), reads

\[
V_{\text{sing}} = g^2 N \frac{N_f}{4\pi} \frac{N_f}{N} \left( \int \frac{dk_1^+ dk_2^+}{(2\pi)^2} \frac{\delta(k_1^+ - k_2^+)}{(k_1^+ - k_2^+)^2} \right) \int dk (B_{kk} + D_{kk}) + O(N^2).
\]

The constant term \( O(N^2) \), which we did not write down explicitly, is an infinite vacuum energy and appears as a result of our choice of normal ordering. Had we started with the completely normal-ordered hamiltonian, \( :P^- : \), we would obtain a finite result with unshifted mass \( \bar{m}_j = m_j \). In fact, the precise definition of the quark mass (being the mass of a colored object) is a matter of convenience. In the next section we will comment more on this issue. The singular term \((3.16)\) arises because of our convention to write \( BB \) and \( DD \) terms in \((3.14)\) as color singlet operators, and it cancels against similar terms in \( V \) when we compute matrix elements between meson states. This completes the definition of the model which we study in the rest of this paper.

4. The ’t Hooft equation

In this section we summarize the large-\( N \) solution of two-dimensional QCD found by ’t Hooft \cite{tHooft}. We emphasize the power of the light-cone method in deriving bound-state equations, since no diagrams need to be considered. In fact, all gluon diagrams are summed over by means of the effective Coulomb interaction \((3.3)\) (see ref. \cite{ref}).
According to 1/N power counting in eq. (3.13), the dominant term in the interaction hamiltonian corresponds to the $MM$ interaction,

$$V_{MM} = -\frac{g^2 N}{2\pi} \int K_{MM} \bar{M}_{kk'} M_{pp'}.$$

It is now straightforward to recover ’t Hooft equation by evaluating the action of the operator $V_{MM}$ on general one-meson states of the form,

$$|\phi, jj', P^+\rangle = \sqrt{P^+} \int_0^1 dx \phi(x) M^\dagger(x P^+, j; (1 - x) P^+, j') |0\rangle .$$

This is a normalized one-meson state with total light-cone momentum $P^+$, light-cone wave function $\phi(x)$ and flavors $j$ and $j'$ for quark and antiquark, respectively. The variable $x$ has the physical interpretation of the fraction of light-cone momentum carried by the quark.

So, to leading order in $1/N$ the action of $P^-$ on the wave function $\phi(x)$ is given by

$$P^-[\phi] = \frac{1}{2P^+} \left[ \left( \frac{m_j^2}{x} + \frac{m_{j'}^2}{1-x} \right) \phi(x) - \frac{g^2 N}{\pi} \int_0^1 dy \frac{\phi(y) - \phi(x)}{(x-y)^2} \right] + O \left( \frac{1}{N^{1/2}} \right).$$

Here we have absorbed the mass shift $\bar{m}^2 = m^2 - g^2 N/\pi$ into the integral operator by means of the identity,

$$\int_0^1 \frac{dy}{(x-y)^2} = -\frac{1}{x(1-x)} .$$

As it was previously announced, the action of the divergent piece on the hamiltonian $V_{sing}$ on meson states is completely cancelled by contributions from the $BB$ and $DD$ terms. In eq. (4.3) we deal with the infrared divergences of the potential energy by means of the principal value analytic regularization, which is known to give the correct physical results for this theory [13].

There are corrections to the bound-state equation coming from the subleading “singlet” terms in the $MM$, $BB$, $DD$ and $BD$ kernels. Their total effect amounts to a simple renormalization of the coupling constant $g^2 \to \bar{g}^2 = g^2(1 - 1/N^2)$. As a result, the 1/N-corrected bound-state equation, $\mu^2 = 2P^+P^-$, reads:

$$\mu_n^2 \phi_n(x) = \left( \frac{m_j^2}{x} + \frac{m_{j'}^2}{1-x} \right) \phi_n(x) - \frac{g^2 N}{\pi} \int_0^1 dy \frac{\phi_n(y) - \phi_n(x)}{(x-y)^2} .$$

This is the standard ’t Hooft equation with a definition of constituent quark mass compatible with the appearance of a pion in the chiral limit. Namely, as $m \to 0$ then $\phi = 1$
becomes a solution with $\mu = 0$. Other, less “physical”, definitions of the quark mass are also possible. For example, in the totally normal-ordered hamiltonian $m$ is not shifted by the coupling constant and the large-$N$ chiral symmetry point would be at $m^2 = -g^2N/\pi$.

We have seen that $1/N$ corrections to the ’t Hooft equation coming from the one-meson sector are relatively mild and amount simply to a redefinition of the coupling. In fact, these corrections are kinematical and do not occur for a $U(N)$ gauge group, since they ultimately come from the traceless character of the generators (3.2b). The remaining $1/N$ corrections are non-trivial and correspond to meson exchange, as will be discussed in sect. 6.

The power of the light-cone method in deriving bound-state equations is not restricted to the meson sector. As an illustration we quote here the bound-state equation for baryons in the large-$N$ limit [16], which can be derived by acting with $P^-$ on baryon states of the form,

$$|B\rangle = \int \{k\} \phi(k_1, \ldots, k_N) \frac{1}{\sqrt{N!}} \epsilon^{\alpha_1 \cdots \alpha_N} b_{\alpha_1}^\dagger(k_1) \cdots b_{\alpha_N}^\dagger(k_N)|0\rangle.$$

Here we consider a single flavored “proton” for simplicity. Acting on this type of states only the $BB$ term in $V$ contributes to leading order and we obtain,

$$\frac{\mu^2}{2P^+} \phi(k_1, \ldots, k_N) = \sum_{i=1}^{N} \frac{\bar{m}^2}{2k_i} \phi(k_1, \ldots, k_N)$$

$$- \frac{g^2N}{2\pi} \sum_{i<j}^{N} \frac{1}{N} \int dp \frac{\phi(k_1, \ldots, p, \ldots, k_i + k_j - p, \ldots, k_N)}{(p-k_i)^2}.$$

This is a relativistic hamiltonian equation describing pairwise interactions of order $1/N$ among $N$ quarks, as expected on physical grounds. The Hartree approximation proposed by Witten [17] in a non-relativistic context can be implemented in this relativistic model in explicit hamiltonian form. Finally, we note that an effective hamiltonian to compute interactions could be derived along the lines of sect. 2, with selection rules,

$$\frac{1}{N} H_{\text{eff}} \sim H_{(1,1)} B^\dagger B + \frac{1}{4} H_{(2,2)} B^\dagger B B B + \frac{1}{\sqrt{N}} H_{BM}(M^\dagger B^\dagger B + B^\dagger B M),$$

where the last term corresponds to meson emission by baryons.
5. Effective meson hamiltonian

In this section we construct the effective hamiltonian which summarizes the $1/N$ expansion of two-dimensional QCD. The derivation follows the general discussion in sect. 2, with the meson creation operators given by:

$$
\mathcal{M}^\dagger(n, j j', P^+) = \sqrt{P^+} \int_0^1 dx \, \phi_n(x) \, M^\dagger(x P^+, j; (1 - x) P^+, j') ,
$$

where $\phi_n(x)$ is the solution of the bound-state equation (4.4). These operators are normalized to leading order as,

$$
\left[ \mathcal{M}(n, j j', P^+), \mathcal{M}^\dagger(m, l l', Q^+) \right] = \delta_{n m} \delta_{j j'} \delta(P^+ - Q^+) .
$$

The commutator of $\mathcal{M}$ and $\mathcal{M}^\dagger$ contains some subleading extra terms corresponding to the fact that the states $\mathcal{M}^\dagger |0\rangle$ as defined in (5.1) are orthonormal only in the large-$N$ limit. However, within $1/N$ perturbation theory we are interested in matrix elements of the hamiltonian in the $N = \infty$ orthonormal basis. Therefore, when computing the nested commutators in (2.4) we use (5.2), neglecting $1/N$ corrections to the normalization of the asymptotic states. In this way we always work with the planar asymptotic states and ensure the consistency of the method.

Using the basic commutators,

$$
\left[ B_{k k'}, M^\dagger_{p p'} \right] = \delta_{k' p} M^\dagger_{k p'} ,
\left[ D_{k k'}, M^\dagger_{p p'} \right] = \delta_{k' p'} M^\dagger_{p k} ,
$$

the computation of the nested commutators in (2.4) is long but entirely straightforward. We find the following structure:

$$
P_{\text{eff}} = H_{(1,1)} \mathcal{M}^\dagger \mathcal{M} + \frac{1}{2 \sqrt{N}} \left( H_{(1,2)} + \frac{1}{N} H'_{(1,2)} \right) \left( \mathcal{M}^\dagger M^\dagger M + M^\dagger M M \right)
+ \frac{1}{4 N} \left( H_{(2,2)} + \frac{1}{N} H'_{(2,2)} \right) M^\dagger M^\dagger M M ,
$$

where the primes refer to the “singlet” subleading terms in the kernels (3.15). An interesting feature of this hamiltonian is that it has only a finite number of terms (cubic and quartic interaction). This is so because in the light-cone gauge both single-meson states (5.1) and the fundamental QCD hamiltonian (3.12), (3.14), (3.16) are polynomials.
in $M, M^\dagger, B, D$. From the properties of commutators (5.2) and (5.3) it then follows that
the nested commutator will eventually vanish. In the equal-time formalism meson states
are complicated “coherent” states in terms of $M^\dagger$ and it is not a priori obvious that the
effective hamiltonian would terminate.

After solving the ’t Hooft equation, $H_{(1,1)}$ becomes the standard kinetic energy for
free mesons,

$$P_{\text{eff},0}^- = \sum_{n(jj')} \int_0^\infty dP^+ \frac{\mu_n^2(jj')}{2P^+} \mathcal{M}^\dagger(n, jj', P^+) \mathcal{M}(n, jj', P^+) .$$

(5.5)

The three-meson coupling picks contributions only from the $M^\dagger B$ and $M^\dagger D$ terms,

$$\frac{1}{\sqrt{N}} H_{(1,2)} + \frac{1}{N\sqrt{N}} H'_{(1,2)} \sim \langle 0 | \mathcal{M} M [V_{MB} + V_{MD}, M^\dagger] | 0 \rangle ,$$

(5.6)

while the four-meson vertex depends on the $BB$, $DD$ and $BD$ terms,

$$\frac{1}{N} H_{(2,2)} + \frac{1}{N^2} H'_{(2,2)} \sim \langle 0 | \mathcal{M} M [[V_{BB} + V_{DD} + V_{BD}, M^\dagger], M^\dagger] | 0 \rangle .$$

(5.7)

Before writing the answer for the vertices it is convenient to introduce some notational
devices. Let us define the charge conjugation operator, $C$, which interchanges quarks and
antiquarks ($b^\dagger$ and $d^\dagger$ operators). Acting on wave functions and flavor indices we have,

$$C [f(j,j')\phi(x)] = f(j',j)\phi(1-x) .$$

(5.8)

This operator simplifies the computations because it relates the kernels in (3.14) as
$C(K_{MB}) = -K_{MD}$ and $C(K_{BB}) = K_{DD}$. We also define the permutation operators $P_L$
($P_R$) on initial (final) labels, with obvious action on all quantum numbers and variables.

Finally, from all vertices we extract a factor of the form:

$$\frac{1}{N^{\text{power}}} H_{\text{in,out}} = \frac{g^2N}{\pi} \delta(P^+_{\text{in}} - P^+_{\text{out}}) \prod_{\text{legs},l} \sqrt{P^+_{l}} \ [\text{in} \rightarrow \text{out}] .$$

(5.9)

We classify vertices by the different flavor structures. For the leading three-meson
interactions (5.10) we find two independent flavor structures which we write as

$$[L \rightarrow R_1 R_2] = \frac{1}{2\sqrt{N}} (1 + P_R) \delta_{l'r_1} \delta_{l'r_2} \mathcal{T}(L|R_1 R_2; \omega)$$

$$+ \frac{1}{2N\sqrt{N}} (1 + P_R) \delta_{l'r_1'} \delta_{l'r_2'} \mathcal{T}'(L|R_1 R_2; \omega) .$$

(5.10)
For example, the two flavor structures of the leading term in eq. (5.10) explicitly read:

\[ \delta_{\ell r_1} \delta_{\ell' r_2'} \delta_{r_1 r_2} T(L|R_1 R_2; \omega) + \delta_{\ell r_2} \delta_{\ell' r_1'} \delta_{r_2 r_1'} T(L|R_2 R_1; 1 - \omega) . \]

In the above formulae \( \ell, L \) labels refer to initial particles and \( r, R \) to final ones. Also \( \omega \equiv P^+_r/P^+ \) denotes the light-cone momentum fraction carried by particle \( R_1 \). \( T \) is a convolution of wave functions which determines all leading form factors [5], and \( T' \) is the subleading triple-meson vertex. All vertex functions are explicitly listed in Appendix A.

Regarding the four-meson vertices (5.11) we separate \( BB + DD \) and \( BD \) contributions. In the first case there are two flavor structures coming from both the leading and the singlet terms:

\[
\begin{align*}
[L_1 L_2 \to R_1 R_2]_{BD} &= \frac{1}{2N} \left( 1 + \mathcal{P}_L \right) \left[ \delta_{\ell_1 r_1} \delta_{\ell'_1 r'_1} \delta_{\ell_2 r_2} \delta_{\ell'_2 r'_2} F_{BD}(L_1 L_2 ; z| R_1 R_2 ; \omega) \right] \\
&+ \frac{1}{2N^2} \left( 1 + \mathcal{P}_R \right) \left[ \delta_{\ell_1 r_1} \delta_{\ell'_1 r'_1} \delta_{\ell_2 r_2} \delta_{\ell'_2 r'_2} F'_{BD}(L_1 L_2 ; z| R_1 R_2 ; \omega) \right]
\end{align*}
\]

where \( z \equiv P^+_{\ell_1}/P^+_{in} \) is the analog of \( \omega \) for the initial state. Finally, from the \( BD \) terms we find four leading flavor structures and two subleading ones:

\[
\begin{align*}
[L_1 L_2 \to R_1 R_2]_{BD} &= \frac{1}{4N} \left( 1 + \mathcal{P}_L + \mathcal{P}_R + \mathcal{C} \right) \left[ \delta_{\ell_1 r_1} \delta_{\ell'_1 r'_1} \delta_{\ell_2 r_2} \delta_{\ell'_2 r'_2} F_{BD}(L_1 L_2 ; z| R_1 R_2 ; \omega) \right] \\
&+ \frac{1}{2N^2} \left( 1 + \mathcal{P}_R \right) \left[ \delta_{\ell_1 r_1} \delta_{\ell_2 r_2} \delta_{\ell'_2 r'_2} \delta_{\ell'_1 r'_1} F'_ {BD}(L_1 L_2 ; z| R_1 R_2 ; \omega) \right].
\end{align*}
\]

(5.12)

\[ \text{Fig. 1: Graphical representation of vertices (5.10), (5.11) and (5.12).} \]

We represent vertices (5.10), (5.11) and (5.12) graphically as shown in figs. 1a, 1b and 1c, respectively.

With the explicit expressions for the vertex functions, \( T \) and \( F \), listed in Appendix A, we have completed the description of the effective meson hamiltonian of two-dimensional
QCD, which may now be used to compute systematic $1/N$ corrections to meson scattering amplitudes and spectrum.

The solutions of ’t Hooft equation are not known analytically and we can only evaluate our vertex functions numerically. To solve ’t Hooft equation we used the Multhopp’s method to transform it into an infinite system of algebraic equations which is then solved approximately by truncation. This method was first applied to eq. (4.4) in ref. [18], and later used in refs. [6,19]. (For a nice summary of the method which we closely follow, see appendix of ref. [19].) For moderate mass of the light quark results are independent of truncation already with several hundreds of equations. Near the chiral limit ($m_q \to 0$), the convergence becomes quite slow and the discretized light-cone method may be a better option. Some examples of $\mathcal{T}$ functions can be seen in fig. 5 in sect. 8.

Since we have a light-cone hamiltonian we must use light-cone perturbation theory, which is in fact a time-ordered perturbation scheme (see, c.f., ref. [8]). It is very similar to old-fashioned perturbation theory, although it is much simpler due to the absence of vacuum subgraphs (there are no pure positive/negative frequency terms in the light-cone hamiltonian). It is defined through the Born expansion of the complete propagator between light-cone wave functions:

$$\langle \text{out} \left| \frac{1}{E - P_{\text{eff}}^- + i0} \right| \text{in} \rangle = \langle \text{out} \left| \frac{1}{E - P_0^- + i0} \right| \text{in} \rangle + \langle \text{out} \left| \frac{1}{E - P_0^- + i0} P_I^- \frac{1}{E - P_0^- + i0} \right| \text{in} \rangle.$$  (5.13)

Here $E \equiv i\partial/\partial x^+$ is the light-cone energy operator, and we split the effective hamiltonian into free and interaction parts

$$P_{\text{eff}}^- = P_0^- + P_I^-,$$

with $P_0^-$ given by (5.3). We evaluate (5.13) by inserting the spectral decomposition of unity between any two interactions:

$$1 = \sum_{\{n(jj')\}} \int \{dP^+\} \frac{\mathcal{M}_1^\dagger \ldots \mathcal{M}_K^\dagger |0\rangle \langle 0| \mathcal{M}_1 \ldots \mathcal{M}_K}{K!}.$$

This algorithm determines the corresponding graphical rules, which have the same symmetry factors as the Feynman rules once the factor $1/p!q!$ has been extracted from each vertex as in eq. (2.3). One could also recover this perturbative expansion from covariant
perturbation theory in the fundamental quark/gluon variables. After going to the light-cone gauge, the integrals over intermediate $k^-$ components of momenta can be evaluated by residue calculus. For each pole one finds the corresponding time-ordered process of light-cone perturbation theory. Within Feynman’s $i\epsilon$ prescription and 't Hooft’s principal value prescription one can explicitly show the equivalence diagram by diagram [20]. The main advantage of our method is that the gluons are integrated out from the very beginning and there is no need to consider Bethe-Salpeter techniques.

6. 1/$N$ corrections to the spectrum

With the effective hamiltonian (5.4) at hand we may compute the mixing mass matrix for mesons in light-cone perturbation theory by the usual procedure of summing the one-particle irreducible (1PI) geometric series for the propagator:

$$
\left( \frac{1}{p^2 - \mu^2} \right)_{n,n'} = \left( \frac{1}{p^2 - \mu^{(0)}^2 - \Sigma(p^2)} \right)_{n,n'}
$$

and we obtain the mixing matrix:

$$
\mu_{nn'}^2 = \mu^{(0)}_n \delta_{nn'} + \Sigma_{nn'}(p^2),
$$

(6.1)

where $\mu^{(0)}_n^2$ stands for the large-$N$ spectrum of eq. (4.4). The leading 1/$N$ contribution to the self-energy matrix $\Sigma_{nn'}$ comes from bubble diagrams with three-meson vertices (5.10).

Let us consider, as an example, a two-flavor model with heavy ($Q$) and light ($q$) quarks. We can then form two-dimensional analogs of pions ($q\bar{q}$), $B$ mesons ($Q\bar{q}$), $\bar{B}$ mesons ($q\bar{Q}$) and $\eta_b$ mesons ($Q\bar{Q}$).

Fig. 2: Diagrams contributing to $B_n-B_{n'}$ mixing.
For example, to leading order there is no $B-\pi$ or $B-\eta$ mixing. The leading contributions to $B_n-B_{n'}$ mixing are given by the diagrams in fig. 2. For the physical values of the masses, diagrams in fig. 2b are strongly suppressed compared to those in fig. 2a. We quote here the result for $\pi, B$ exchange – diagram 2a:

$$
\Sigma_{nn'}(p^2)_{B,\pi \text{ channel}} = -\frac{1}{N} \left( \frac{g^2 N}{\pi} \right)^2 \frac{1}{p^2} \sum_{k,m} \int_0^1 dz \frac{T(B_n|B_k \pi_m; z)T(B_{n'}|B_k \pi_m; z)}{(z-z_+(z-z_-)} ,
$$

(6.2)

where

$$z_{\pm} = \frac{1}{2p^2} \left( p^2 + \mu_+ \mu_- \pm \sqrt{(p^2 - \mu_+^2)(p^2 - \mu_-^2)} \right) ,$$

and $\mu_{\pm} \equiv \mu_{k}^{(0)} \pm \mu_{m}^{(0)}$, with the standard $i\epsilon$ prescription implicit in (6.2). This formula clearly exhibits the threshold structure. In particular, in a low-energy approximation we may simplify expression (6.2) by setting the vertex functions equal to the on-shell values $T_{nkm}$, and we get the standard form of the bubble diagram:

$$
\Sigma_{nn'}(p^2) \simeq -\frac{1}{N} \left( \frac{g^2 N}{\pi} \right)^2 \sum_{k,m} T_{nkm} T_{n'km} \log \left[ \frac{\sqrt{\mu_+^2 - p^2} + \sqrt{\mu_-^2 - p^2}}{\sqrt{\mu_+^2 - p^2} - \sqrt{\mu_-^2 - p^2}} \right]^2 ,
$$

where we recognize the usual threshold function.

We have calculated numerically several elements of the $\Sigma$-matrix as given by formula (5.2) for three different values of the light quark mass: $m_q^2 = 0.3136, 0.1, 0.01$, and for the heavy quark mass $m_Q^2 = 2000$. In the units $g^2 N/\pi = 1$, the large-$N$ value of the $B$ meson ground state mass is found to be $\mu_B^{(0)} = 2121, 2101, 2089$, respectively. The on-shell values of $\Sigma_{00}$ for $N = 3$ are found to be: $-52, -318, -1560$. (Note the negative sign, typical of second order perturbation theory about the ground state.) We have checked that the contributions with larger values of the excitation numbers, $k$ and $m$, in eq. (5.2) quickly become very small. We note, however, that the largest contributions to $\Sigma_{00}$ come from diagrams involving odd $B$-resonances in the intermediate state, and similar results are obtained for $\Sigma_{11}$ and $\Sigma_{22}$. The vertex functions with odd $B$-resonances in the final state become very large as we approach the chiral limit for the light quark mass. In addition, the bubble diagram for the diagonal elements of the self-energy matrix is anomalously large, because the $T$ functions enter squared, and no destructive interference between the vertices occurs. This symmetry enhancement is not present for the off-diagonal elements of the self-energy matrix. For example, for the mixing between the first $B$-resonance and
the fundamental state we find $\Sigma_{01} = 7, 116, 760$ for the different light quark masses. We see that the mixings are much smaller than the diagonal level shifts, but still, they seem to blow up as we approach the chiral limit $m_q \to 0$. These uncontrollable large corrections for light quark mass do not mean that the $1/N$ expansion is sick, but rather that the chiral limit does not survive $1/N$ corrections. In fact, the renormalization procedure that fixes the quark masses must be repeated at any order in the perturbative expansion, and the large values of the self-energy matrix translate into large renormalizations of the quark masses, but still small overall corrections to the physical spectrum.

To see how this works, let us denote by $\sigma_n(p^2)$ the eigenvalues of the self-energy matrix. These quantities depend on $p^2$ and also on the dimensionless parameters of the t'Hooft equation:

$$x_q = \frac{\pi m_q^2}{g^2N}, \quad x_Q = \frac{\pi m_Q^2}{g^2N}.$$  \hspace{1cm} (6.3)

To any order in $1/N$ the quark masses are determined by fitting the mass of the fundamental states $\pi_0$ and $B_0$ to the “experimental” pion and $B$ meson masses:

$$\mu_0^{(0)2}(x_q, x_Q) + \sigma_0(x_q, x_Q, \mu_0^2) = \mu_0^2.$$  \hspace{1cm} (6.4)

Taking $\mu_0$ to be the mass of pion or $B$ meson we get two equations that fix $x_q$ and $x_Q$. The rest of the spectrum $\mu_n^2$ is then obtained from the following implicit equation:

$$\mu_n^{(0)2}(x_q, x_Q) + \sigma_n(x_q, x_Q, \mu_n^2) = \mu_n^2.$$  \hspace{1cm} (6.5)

We can illustrate the renormalization of the quark masses in a simple example. For a qualitative discussion, let us neglect the heavy quark mass running and fix $x_Q = 2000$. We will consider the contribution to the light quark renormalization coming from the $B$-resonances only, and furthermore neglect the off-diagonal mixings. We define the mass of the fundamental $B$ meson to be:

$$2101 = \mu_0^{(0)2}(x_q = 0.1).$$

This equation means that the mass of the light quark at $N = \infty$ is taken to be $x_q = 0.1$. The $1/N$-corrected value of $x_q$ is obtained by solving the equation (6.4):

$$2101 = \mu_0^{(0)2}(x_q, p^2 = 2101).$$

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A simple estimate follows from linear interpolation between two points: at \( x_q = 0.1, \sigma_0(p^2 = 2101) \sim \Sigma_{00}(p^2 = 2101) \sim -320 \), and at \( x_q = 0.3136, \sigma_0(p^2 = 2101) \sim -52 \). This results in an estimate of the renormalized quark mass of \( x_q \simeq 0.34 \). We obtain a larger mass, for which the self-energy corrections are again small. For this value of the quark masses, the typical shifts found from equation (6.5) for \( N = 3 \) are,

\[
\frac{\Delta \mu_n^2}{\mu_n^2} \sim 2.5\%.
\]

So, taking into account that the self-energy diagram is anomalously large, as explained before, we see that the typical \( 1/N \) corrections are small, and the apparent singularity for very light quarks simply means that the interacting theory restores the chiral symmetry, if broken in the free \( N = \infty \) theory. In general, the fact that the self-energy corrections are large and negative as \( m_q \to 0 \), implies that the chiral symmetry renormalization point \( m_q = 0 \) is unstable under \( 1/N \) corrections, and the whole picture agrees with Coleman’s theorem, which forbids spontaneous symmetry breaking in two dimensions.

As a final comment, we point out that \( \sigma_n \) also induces a wave function renormalization,

\[
Z_n = \frac{1}{1 - \frac{\partial \sigma_n}{\partial p^2}(p^2 = \mu_n^2)}, \tag{6.6}
\]

which in turn amounts to extra \( 1/N \) corrections to the vertices, to be added to the trivial \( 1/N^2 \) correction coming from the renormalization of \( g \to \bar{g} \):

\[
\mathcal{T}(L|R_1 R_2) \to \sqrt{Z_L Z_{R_1} Z_{R_2}} \mathcal{T}(L|R_1 R_2), \tag{6.7}
\]

and similarly for the \( \mathcal{F} \) functions.

### 7. \( 1/N \) corrections to form factors

The planar limit of the electromagnetic form factors of \((1 + 1)\)-dimensional QCD was derived in ref. [5]. Recently, in ref. [7], the analysis was extended to the case of flavor changing currents, in order to simulate semileptonic heavy meson decays in two dimensions. According to ref. [7] the form factors for semileptonic decays of \( B \) mesons are single-pole dominated in the combined limits: \( N \to \infty, M_b \to \infty \) and \( m_\pi \to 0 \) in the four-dimensional theory. The argument goes schematically as follows: one first writes the form factor as a
superposition of pole terms, according to the spectral decomposition at \( N = \infty \) (narrow width approximation),
\[
F(q^2) \sim \sum_n \frac{C_n}{q^2 - \mu_n^2},
\]
and then uses the heavy quark limit to prove the vanishing of all \( C \)'s but one, by means of chiral Ward identities.

This argument was tested in ref. [6] in planar two-dimensional QCD. It turns out that the heavy quark mass limit is not necessary in two dimensions due to the constrained kinematics, and the single pole dominance is easily obtained in the chiral limit for the pions. The nature of the \( 1/N \) corrections to this behavior can be easily studied using the formalism we have developed here. As an example, we will again consider a two-flavor model as in sect. 6. Since two-dimensional vector and axial currents are dual of each other, we only need to consider the coupling of mesons to the flavor-changing vector current:
\[
J_{Qq}^\mu = \bar{\psi}_Q \gamma^\mu \psi_q.
\]

In order to derive effective meson–current vertices to be used in the \( 1/N \) perturbation theory, we will couple the quark current to a background vector field in a minimal fashion:
\[
P_{Qq}^- = e \int_{x^+ = \text{const.}} \left( J_{Qq}^\mu W_\mu + \text{h.c.} \right).
\]

After solving for the physical fermion fields we have,
\[
P_{Qq}^- = e\sqrt{2} \int_{x^+ = \text{const.}} \left( : \psi_Q^\dagger \psi_q : W_+ + \frac{m_q m_Q}{2} : \left( \frac{1}{i \partial_-} \psi_Q \right)^\dagger \left( \frac{1}{i \partial_-} \psi_q \right) : W_- + \text{h.c.} \right). \quad (7.2)
\]

Defining a Fock space for the vector bosons,
\[
W^+ = (W^-)^\dagger = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{dk^+}{\sqrt{k^+}} \left( a(k^+) e^{-ik^+ x^-} + a^\dagger(k^+) e^{ik^+ x^-} \right),
\]
\[
[a(k^+), a^\dagger(k'^+)] = \delta(k^+ - k'^+),
\]
we derive a meson-vector boson effective Hamiltonian along the lines of sect. 2 with the result:
\[
P_{Qq,\text{eff}}^- = \sqrt{N} \left( H_{(M \rightarrow W)} a^\dagger M + H_{(W \rightarrow M)} M^\dagger a \right)
+ H_{(M \rightarrow M + W)} a^\dagger M^\dagger M + H_{(M + W \rightarrow M)} M^\dagger M a. \quad (7.4)
\]
An interesting feature of this hamiltonian is the absence of direct $W \rightarrow M + M$ or $M + M \rightarrow W$ transitions. This fact will be of some significance in the sequel. The effective vertices in (7.4) may be computed from the general formula (2.4), with the result:

- $B - W^\pm$ mixing:
  \[ H^+ \left[ B_n(P^+) \rightarrow W^+(q^+) \right] = \frac{e}{2\pi} \delta(P^+ - q^+) \frac{1}{\sqrt{P^+}} \int_0^1 dx \phi_{B_n}(x) , \]
  \[ H^- \left[ B_n(P^+) \rightarrow W^-(q^+) \right] = -\frac{e}{2\pi} \delta(P^+ - q^+) \frac{m_q m_Q}{2q^+ \sqrt{P^+}} \int_0^1 dx \frac{\phi_{B_n}(x)}{x(1 - x)} , \]

- $W^\pm$ emission by $Q$ ($z \equiv q^+/P_B^+$):
  \[ H^+ \left[ B_n(P_B^+) \rightarrow \pi_m(P_\pi^+) + W^+(q^+) \right] = \frac{e}{2\pi} \delta(P_B^+ - q^+ - P_\pi^+) \sqrt{\frac{P_\pi^+}{P_B^+}} \frac{1}{q^+} \]
  \[ \times \int_0^1 dx \phi_{B_n}(z + (1 - z)x)\phi_{\pi_m}(x) , \]
  \[ H^- \left[ B_n(P_B^+) \rightarrow \pi_m(P_\pi^+) + W^-(q^+) \right] = -\frac{e}{2\pi} \delta(P_B^+ - q^+ - P_\pi^+) \sqrt{\frac{P_\pi^+}{P_B^+}} \frac{1}{q^+} \]
  \[ \times \frac{m_q m_Q}{2P_\pi^+ P_B^+} \int_0^1 dx \frac{\phi_{B_n}[z + (1 - z)x]\phi_{\pi_m}(x)}{(1 - x)[1 - z - (1 - z)x]} . \]

Analogous expressions hold for $W^\pm$ emission by $\bar{q}$, with the following modifications of (7.4): there is an overall minus sign, $\pi_m$ is substituted by $\eta_m$ in the final state, and $z + (1 - z)x \rightarrow (1 - z)x$ inside the integrals.

Given the complete effective hamiltonian, including (7.4), we are in position to compute form factors. Let us consider the time-like form factor defined as the light-cone 1PI amplitude for $W^\pm$ decay into on-shell $B$ and $\pi$ mesons:

\[ F^\pm(q^2) = C \left< B(\omega q^+); \pi((1 - \omega)q^+) \right| \frac{1}{q^- - P_0^-} \left| q^- - P_0^- \right> W^\pm(q^+) \right>_{1PI} . \]

In this formula the factors $(q^- - P_0^-)$ amputate external propagators, and the constant

\[ C = \frac{4\pi}{e} q^+ \sqrt{P_B^+ P_\pi^+} \]

defines $F^\pm$ compatible with the conventions in ref. [3]. Also, $\omega \equiv P_B^+/q^+$ is the light-cone momentum fraction carried by the $B$ meson. It becomes a function of $q^2$ once conservation of light-cone energy is enforced for on-shell $B$ and $\pi$ mesons,

\[ q^2 = \frac{\mu_B^2}{\omega} + \frac{\mu_\pi^2}{1 - \omega} . \]
Fig. 3: Form factor diagram for the decay $W^\pm \rightarrow B + \pi$.

The $1/N$ expansion of the form factor is generated by expanding the propagator in (7.7). The first term (free propagation) vanishes because of the kinematics, and the second term (one intermediate interaction) also vanishes because there are no couplings for direct decay of $W^\pm$ into two mesons in our effective hamiltonian (7.4). So, the $O(1)$ contribution involves at least two intermediate interactions, and we can separate it from the $1/N$ corrections as (see fig. 3):

$$F^\pm(q^2) = \sum_{M_1, M_2} \langle B\pi | (q^- - P^-_0) \frac{1}{q^- - P^-_0 + i\epsilon} (q^- - P^-_0) \rangle_{1PI}$$

$$\times C \left( \frac{1}{q^- - P^-_0 + i\epsilon} \frac{H(2 \leftrightarrow 1)}{2\sqrt{N}} \frac{1}{q^- - P^-_0 + i\epsilon} \sqrt{N} H_{B_n \rightarrow W^\pm} \right) W^\pm \rangle_{1PI},$$

where $H(2 \leftrightarrow 1)$ stands for $H_{(2,1)} + H'_{(2,1)}/N$. The intermediate state of two mesons is either a $B_\ell$, $\pi_m$ state or a $B_\ell$, $\eta_m$ state, and the corresponding propagator and measure contribute a factor:

$$q^+ \int_0^1 dz \frac{2q^+}{q^2 - \mu_\ell^2/z - \mu_m^2/(1-z)},$$

where $\mu_\ell$ is the mass of $B_\ell$ and $\mu_m$ that of $\pi_m$ or $\eta_m$. Also, $z \equiv P_{B_\ell}^+/q^+$ is the light-cone momentum fraction of the $B_\ell$ resonance, and the extra factor $q^+$ is the Jacobian from the intermediate light-cone momenta to the momentum fraction $z$. The remaining propagator in formula (7.9) carries a $B_n$ resonance of mass $\mu_n$, created out of the vacuum by the current, and it contributes a factor $2q^+/(q^2 - \mu_n^2)$.

Now, putting everything together, in terms of the meson decay constant, $f_n \equiv \int dx \phi_n(x)$, we can write:

$$F^+(q^2) = 2q^+ \frac{g^2 N}{\pi} \sum_n \frac{f_n T_n(q^2)}{q^2 - \mu_n^2},$$

$$F^-(q^2) = -\frac{1}{q^+} \frac{g^2 N}{\pi} \sum_n \frac{(-)^n f_n \mu_n^2 T_n(q^2)}{q^2 - \mu_n^2}.$$
In the second relation we have made use of the so-called parity relation,

\[ m_q m_\ell \int_0^1 dx \frac{\phi_n(x)}{x(1-x)} = (-)^n \mu_n^2 f_n. \]

which was proved in ref. [3]. The function \( T_n(q^2) \) contains all 1/N corrections and is of the form:

\[ T_n(q^2) = T(B_n|B \pi;\omega) + \frac{1}{N} T'(B_n|B \pi;\omega) + \Delta T_n(q^2). \]  

(7.11)

Here we separated the leading term from the first correction, given by the singlet three-meson coupling, and \( \Delta T_n \) which includes meson exchange:

\[
\Delta T_n(q^2) = \sum_{\ell,m} \int_0^1 dz \left[ T(B_n|B_\ell \pi_m; z) + \frac{1}{N} T'(B_n|B_\ell \pi_m; z) \right] \\
\times \sqrt{\frac{\omega(1 - \omega)}{z(1-z)}} \frac{2(q^+)^2}{q^2 - \frac{\mu_\ell^2}{z} - \frac{\mu_m^2}{1-z}} \left[ G(B_\ell \pi_m; z|B \pi;\omega) + G(\pi_m B_\ell; 1 - z|B \pi;\omega) \right] \\
+ \sum_{\ell,m} \int_0^1 dz \left[ T(B_n|\eta_m B_\ell; 1 - z) + \frac{1}{N} T'(B_n|\eta_m B_\ell; z) \right] \\
\times \sqrt{\frac{\omega(1 - \omega)}{z(1-z)}} \frac{2(q^+)^2}{q^2 - \frac{\mu_\ell^2}{z} - \frac{\mu_m^2}{1-z}} \left[ G(B_\ell \eta_m; z|B \pi;\omega) + G(\eta_m B_\ell; 1 - z|B \pi;\omega) \right].
\]

(7.12)

\( G(2\text{meson}|B \pi;\omega) \) represents the amputated Green’s function for the \((2 \text{meson} \rightarrow B \pi)\) transition in formula (7.9). To leading order \((1/N)\), there are nine diagrams contributing to the various four-point Green’s functions \( G \). They are listed in fig. 4 with the corresponding combinatorial factors. We have either irreducible four-meson vertices for direct processes like \((B_\ell, \pi_m) \rightarrow (B, \pi)\), or two successive three-meson interactions coming with two different time orderings, like \((B_\ell, \pi_m) \rightarrow (B_\ell, \pi_k, \pi) \rightarrow (B, \pi)\) and \((B_\ell, \pi_m) \rightarrow (B, \pi_k, \pi_m) \rightarrow (B, \pi)\). For example, the total \( O(1/N) \) contribution to \( \Delta T_n \) due to \( B, \eta \) exchange is given by the diagrams 4c, 4h and 4i. The remaining six diagrams correspond to \( B, \pi \) exchange. Because of the propagator suppression diagrams 4f and 4g are smaller than diagrams 4a, 4b, 4d and
Fig. 4: Diagrams contributing to various four-point Greens's functions.
\[
\Delta T_n(q^2)_{B,\pi \text{ channel}} = -\frac{1}{N} \frac{g^2 N}{\pi} \sum_{\ell,m} \int_0^1 dz \frac{T(B_n|B_{\ell} \pi_m; z)}{q^2(z-z_+)(z-z_-)}
\]

\[
\times \left\{ F_{B_2+D^2}(B_{\ell} \pi_m; z|B \pi; \omega) + \frac{1}{2} F_{BD}(B_{\ell} \pi_m; z|B \pi; \omega) \right. \\
- 2 \frac{g^2 N}{\pi} \sum_{k+m=\text{odd}} \left[ \theta(\omega - z) \omega z \frac{T(B|B_{\ell} \pi_m; \frac{z}{\omega})T(\pi|\pi_k \pi_m; \frac{\omega-z}{1-z})}{\mu_B^2(z-\omega z_+)(z-\omega z'_+)} \right. \\
+ \theta(z - \omega)(1 - \omega)(1 - z) \frac{T(B_{\ell}|B \pi_m; \frac{\omega}{z})T(\pi|\pi_k \pi_m; \frac{\omega-z}{1-z})}{\mu_\pi^2(z - (1 - \omega)z''_+)(z - (1 - \omega)z''_-)} \left. \right]\right\} + O\left(\frac{1}{N^2}\right),
\] (7.13)

where \(z_\pm\) variables are defined in terms of the function

\[ Z_{ab}(p^2) \equiv \frac{1}{2p^2} \left( p^2 + \mu_+ \mu_- \pm \sqrt{(p^2 - \mu_+^2)(p^2 - \mu_-^2)} \right), \quad \mu_\pm \equiv \mu_a \pm \mu_b, \] (7.14)

by the equations:

\[ z_\pm = Z_{\ell m}(q^2), \quad z'_\pm = Z_{mk}(\mu_B^2), \quad z''_\pm = Z_{\ell k}(\mu_\pi^2). \]

We see how light-cone perturbation theory makes explicit the spectral decomposition (unitarity). Looking at the cuts induced by the functions (7.14) we identify the thresholds for \(B_n \rightarrow B_{\ell}, \pi_m\) in all diagrams [from the common factor in (7.13)]. The thresholds for \(\pi_m, B_{\ell} \rightarrow B\) and \(\pi_k, \pi_m \rightarrow \pi\) can also be identified in the last two diagrams, as functions of \(\mu_B^2\) and \(\mu_\pi^2\), respectively. As usual, the pseudo-thresholds at \(\mu_-\) should fall on unphysical sheets.

In order to estimate the size of \(1/N\) corrections to the form factors, we evaluated numerically some contributions to \(\Delta T_n(q^2)\) for \(B, \pi\) channel given by expression (7.13). We performed calculations for \(n = 2\) (which corresponds to the first state above threshold), and at the on-shell value of \(q^2 = \mu_B^2\). The bare quark masses we used are: \(m_q^2 = 0.3136\) and \(m_Q^2 = 2000\). The leading order contribution to the form factor in this case is \(T(B_2, B\pi; \omega_{\text{on-shell}}) \approx 37\). We estimated a number of terms in the sum in eq. (7.13) and found that the largest contributions from the \(T \times F\) terms are \(\approx 0.5 - 0.1\), while those of \(T \times T \times T\) are \(\approx 5 \times 10^{-3}\). Also, these values decrease for higher resonances in the intermediate states. We also found that not all terms we calculated have the same sign.
which makes bounding the total contribution a little difficult. However, it is safe to say that
the correction is not larger than $1 - 2\%$. We also mention that the $1/N$ correction coming
from the subleading three-meson coupling $\frac{1}{N} T'(B_2, B\pi; \omega_{\text{on-shell}}) \approx 0.8$ (for $N = 3$).
Finally, we expect these corrections to become much larger as we approach the chiral limit
(as was the case with corrections to the spectrum discussed in sect. 6). When computing
form factors, however, one should use the renormalized values of the quark masses, and
our previous analysis suggests that even if we start with a light quark mass close to the
chiral limit, $1/N$ corrections will renormalize it to $\approx 0.3 - 0.4$. This explains our choice of
parameters mentioned above. Although it might be interesting to explore these questions
in more detail, as well as to estimate $\Delta T_n(q^2)$ for different values of $q^2$ (as is shown for
the leading contribution in Fig. 5), it is beyond our computational resources at this time
to perform such calculations.

The $1/N$ expansion we have derived for $F^\pm(q^2)$ is all we need to get both vector and
axial current form factors. It is convenient to define the combinations $f_\pm(q^2)$ through:

$$q_\mu F^\mu(q^2) = (\mu_B^2 - \mu_\pi^2) f_+(q^2) + q^2 f_-(q^2) ,$$

(7.15)

$$\epsilon_{\mu\nu} q^\mu F^\nu(q^2) = 2 \epsilon_{\mu\nu} p_\mu^{\perp} p_\nu^{\perp} f_+(q^2) .$$

In solving for $f_\pm$ from (7.15) and (7.10) one encounters some non-trivial steps. Namely,
non-pole terms vanish because of the identity*

$$\sum_{n=0}^\infty f_n T_n(q^2) = 0 .$$

This relation can be proved using the explicit form of $T(B_n|2 \text{ mesons})$ and $T'(B_n|2 \text{ mesons})$
listed in Appendix A, and the identity:

$$\sum_n f_n \phi_n(x) = 1 ,$$

(7.16)

which for $x \in [0, 1]$ follows from the definition of $f_n$ and the completeness of the wave
functions $[\mathbb{I}]$.

Defining the generalized coupling $g_n$ as

$$g_n(q^2) = \begin{cases} \frac{2\mu_n^2}{\mu_B^2 - q^2\omega} \frac{g^2 N}{\pi} T_n(q^2) , & \text{for even } n , \\ \frac{g^2 N}{\pi} T_n(q^2) , & \text{for odd } n . \end{cases}$$

(7.17)

* The same identity was found to leading order in ref. $[\mathbb{I}]$. 

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the final result for $f_{\pm}$ reads:

$$f_{+}(q^2) = \sum_{n \text{ even}} \frac{f_ng_n}{q^2 - \mu^2_n},$$

$$f_{-}(q^2) = \frac{1}{q^2} \sum_{n \text{ odd}} \frac{f_ng_n}{q^2 - \mu^2_n} - \frac{\mu_B^2 - \mu^2_\pi}{q^2} \sum_{n \text{ even}} \frac{f_ng_n}{q^2 - \mu^2_n}.$$  \hspace{1cm} (7.18)

So, we see that the $1/N$ expansion we derived for $T_n(q^2)$ contains the corrections to $f_{\pm}(q^2)$. In particular, keeping the leading term in (7.11) and recalling the expression for the three-point vertex $T$ (Appendix A) we make contact with the result of Grinstein and Mende [6].

The couplings $g_n$ for even and odd $n$ (odd and even parity, respectively) are related to the couplings appearing in a low energy meson effective lagrangian. In the notation of ref. [3] we have the tree level structures:

$$\mathcal{L}_{\text{int}} = \sum_{abc} \frac{1}{\sqrt{N}} \hat{g}_{abc} \phi_a \phi_b \phi_c,$$

for even parity couplings, and

$$\mathcal{L}_{\text{int}} = \sum_{abc} \frac{1}{\sqrt{N}} \hat{g}_{abc} \epsilon^{\mu\nu} \partial_\mu \phi_a \partial_\nu \phi_b \phi_c,$$

for the odd parity couplings. Here $\phi_a$ represent meson interpolating fields and $\hat{g}_{nB\pi} = g_n(q^2 = \mu^2_n)/\mu^2_n$. According to the analysis in ref. [3], all the on-shell effective couplings vanish in the chiral limit except the first one, $n = 0$, which approaches $\hat{g}_{BB\pi} \approx 1$. It is interesting to note though, that for non-zero pion mass the even and odd effective couplings have quite different sizes. Because no extra kinematical factor appears in the definition (7.17) for even parity couplings, $\hat{g}_n$ for odd $n$ is typically ten times bigger than $\hat{g}_n$ for even $n$.

The mechanism for the chiral supression is very interesting in itself. For resonances above threshold, $\mu^2_n > (\mu_B + \mu_\pi)^2$, the on-shell coupling can be directly obtained from the value of the vertex function, $T$ at the point $\omega = \omega(q^2 = \mu^2_n)$, as solved from eq. (7.8). In fig. 5 we localize this point on the $T$ graph for two different light quark masses. It is amusing to see that the source for the supression is not the shrinking of the $T$ function itself, but the purely kinematical displacement of on-shell value of $\omega$ so that the value of the on-shell coupling “rolls down” to the left of the vertex peak as we approach the chiral limit. The vertex function itself actually grows in absolute value as we set $m_q \to 0$. This means
Fig. 5: Vertex function $T(B_2|B_0, \pi_0; \omega)$ for the heavy quark mass $m_Q^2 = 2000$ and two different values of the light quark mass: (a) $m_q^2 = 0.3136$, (b) $m_q^2 = 0.01$. Dots mark the values of the on-shell couplings.

that non-local effects, missed by the local expansion of an effective lagrangian, may become very important for certain values of the parameters. When computing $1/N$ corrections in the full theory, it is the whole $T$ function what enters as integrand in loop diagrams, and we see that an estimate of the corrections based on the size of the effective low-energy coupling might not be correct, as compared to the complete non-local computation.
8. A comment on analyticity

When using the formulas of the preceding section for \( f_\pm(q^2) \) inside dispersion relations one should shift to the pole: \( q^2 \to \mu_n^2 \). Actually, the question of analyticity is a subtle one in this formalism. Because we work in a non-covariant gauge, in a singular frame (the light-cone quantization surface), and with non-local effective interactions (the Coulomb force), the analyticity of the off-shell Green’s functions is not manifest. In fact, as shown in refs. [4,5], analyticity can be directly verified in particular examples, although it follows from rather non-trivial identities satisfied by the bound-state wave functions.

From the point of view of our effective hamiltonian (7.4) the problem is very clear. Since we have vertices for \( W \) emission by mesons, but no possible \( W \) decay into two mesons, we see that crossing symmetry is not manifest at tree level. To be more explicit, consider, for example, the space-like form factor \( F^+(q^2) \), defined as the 1PI amplitude for \( B \) decay into \( W^+ \) and \( \pi \):

\[
F^+(q^2)_{s-l} = C \left< W^+(q^+); \pi(P^+ - q^+)|\frac{1}{q^+ - P_{\text{eff}} + i0}(q^- - P_0^-)|B(P^+)\right>_{1\text{PI}}.
\]

(8.1)

Now we have an extra \( O(1) \) contribution to (8.1) coming from the direct vertex \( B \to W^+ + \pi \) with the final result:

\[
F^+(q^2)_{s-l} = 2q^+ \frac{g^2N}{\pi} \sum_n f_n T(B|B_n \pi; z) \frac{1}{q^2 - \mu_n^2} + 2P^+ (1 - z) \int_0^1 dx \phi_B(z + (1 - z)x) \phi_\pi(x),
\]

(8.2)

where \( z \equiv q^+/P^+ \). So, in addition to the superposition of resonances of the same form as in eq. (7.10), we get a quark model-type contribution due to the bare emission of \( W^+ \) by the heavy quark. (Formula (8.2) agrees with the Bethe-Salpeter analysis of ref. [5].)

As a consequence, eq. (8.2) seems to favor a two-component model for form factors (with bare, quark-model terms). In fact, there is no contradiction between (8.2) and (7.10), due to the subtle analytic properties of the three-vertex functions. We see that the time-like kinematics,

\[
q^2 = \frac{\mu_B^2}{\omega} + \frac{\mu_\pi^2}{1 - \omega},
\]

and the space-like kinematics,

\[
\mu_B^2 = \frac{q^2}{z} + \frac{\mu_\pi^2}{1 - z},
\]
are related by crossing: $B_n \leftrightarrow B$; $q^2 \leftrightarrow \mu_B^2$; $z \leftrightarrow \omega$. If we apply the crossing transformation to the three-vertex $T(B_n|B\pi;\omega)$ appearing in (7.10), and analytically continue wave functions through 't Hooft equation for $x \not\in [0,1]$:

$$\phi_n(x) = \left(\frac{\mu_n^2}{g^2N/\pi} - \frac{xQ-1}{x} - \frac{xq-1}{1-x}\right)^{-1} \int_0^1 dy \frac{\phi_n(y)}{(y-x)^2},$$

and further use the identity (7.16) we easily recover result (8.2).

Thus, in this formalism, the two-component model picture and the pure pole-dominance picture are complementary. Direct computation of the decay in the real kinematical region $q^2 < (\mu_B - \mu_\pi)^2$ leads to eq. (8.2) and it looks like a two-component model. However, one cannot relate directly the three-meson coupling function $T(B|B_n\pi;z)$ appearing in (8.2) to the three-meson coupling in a low-energy lagrangian because $q^2 = \mu_n^2$ cannot be reached with $z \in [0,1]$. When the appropriate analytic extension to $q^2 > (\mu_B + \mu_\pi)^2$ is considered, the quark model-type term is already included in the definition of $T(B_n|B\pi;\omega)$, which is directly related to the low-energy coupling when put on-shell, $q^2 = \mu_n^2$. As it was pointed out by Jaffe and Mende [19], inside dispersion relations one must use a sum rule to compute the residue at the poles outside the time-like kinematical region, $\mu_n^2 < (\mu_B + \mu_\pi)^2$, because the interval $(\mu_B - \mu_\pi)^2 < q^2 < (\mu_B + \mu_\pi)^2$ is not covered by either of the definitions (7.7) or (8.1).

9. Conclusions and outlook

In this paper we have discussed some aspects of the interplay between the large-$N$ expansion and light-cone techniques in QCD. Because of the special properties of light-cone Fock spaces the large-$N$ bound-state problem reduces to linear equations for the asymptotic bound states, without any need for complicated combinatorial arguments involving Feynman diagrams. Once the large-$N$ solution is known, the definition of effective many-body hamiltonians renders the $1/N$ expansion in terms of the asymptotic degrees of freedom completely systematic, and provides a diagrammatic technique for the meson-glueball theory.

Although the general features of the method are largely independent of the space-time dimension, quantitative results were reported only for the case of two-dimensional QCD with fermions in the fundamental representation. This is because the effective hamiltonians
must be computed in terms of the large-$N$ solution, and the ’t Hooft model is the only QCD-like theory for which a complete and simple parametrization of the large-$N$ physics is known.

The main result of our analysis of $1/N$ corrections in QCD$_2$ is that $1/N = 1/3$ is effectively a small parameter, at least for moderately heavy quarks. The effective meson hamiltonian is weakly coupled with $1/N$ corrections to spectrum and form factors of order a few percent at most. However, the chiral limit looks rather singular as far as $1/N$ corrections are concerned. In particular, it seems unlikely that the chiral suppression of higher resonances in $B$-meson form factors, as discussed in ref. [6], could survive $1/N$ corrections in the strict chiral limit. This is in fact very natural because Coleman’s theorem forbids spontaneous symmetry breaking in the interacting model. It is well known that the $N = \infty$ theory of mesons is non-interacting, and it simulates a scenario of chiral symmetry breaking, but we do not expect the two limits to commute. Indeed, from our analysis of spectrum corrections in sect. 6 we find that $1/N$ corrections renormalize the light quark masses in such a way that the chiral symmetry point becomes unstable. As a result, QCD$_2$ does not seem to be a good model for chiral dynamics beyond the planar limit.

A general feature of the light-cone effective hamiltonians discussed here is that no low-energy expansion is required in principle, not even in practice for the two-dimensional case. In particular, we have pointed out some non-local effects for light quark masses that could be underestimated by local effective lagrangians. We also explained how some standard analyticity properties are obtained in a very indirect way in this formalism, in terms of non-trivial identities satisfied by the bound-state wave functions.

An interesting point not developed in this paper is the formal analogy between our effective hamiltonians and those of light-cone string field theory. This analogy is based on both the structure of the states in the Fock space representation, and also the perturbation technique. In ref. [21] it was shown that a Nambu-Goto open string with massive ends, quantized in the light-cone gauge, leads to ’t Hooft bound-state equation as mass-shell condition. Thus, we can conjecture that our effective meson hamiltonian provides the string field theory construction for this string in the light-cone gauge. However, ’t Hooft wave functions enter our construction only as an input. As a consequence, no further insights are provided towards a string interpretation of QCD$_2$ in terms of the Gross-Taylor mapping rules [22], which can be considered as the {	extit{ab initio}} definition of the pure Yang-Mills string.
Finally, regarding the four-dimensional generalization of these techniques, we may say that any practical application of effective light-cone hamiltonians to the computation of hadron interactions requires a good quantitative understanding of the large-$N$ bound-state problem. Exact bound-state integral equations can be derived within the light-cone method, and the main simplification attained by the large-$N$ limit is not a smaller hamiltonian, but a much simpler Fock space structure. This fact, combined with the structure of the corrections outlined in this paper, makes clear that the $1/N$ expansion is a very interesting tool in the light-cone approach to QCD.

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Appendix A. Vertex functions

We collect here the explicit expressions for the vertex functions as convolutions of wave functions. It is appropriate to define some related useful quantities like the meson decay constant,

$$f_n \equiv \int_0^1 dx \ \phi_n(x), \quad (A.1)$$

the quark-antiquark-meson vertex,

$$\Phi_n(x) \equiv \int_0^1 dy \ \frac{\phi_n(y)}{(x-y)^2}, \quad (A.2)$$

and the quark-antiquark-meson-meson vertex,

$$\Psi_{n,m}(x; z, \omega) \equiv \int_0^1 dy \ \frac{\phi_n(y)\phi_m(z y)}{(x-y)^2}. \quad (A.3)$$

We also recall the action of the symmetry operators: the charge conjugation operator,

$$\mathcal{C} : \phi(x) \rightarrow \phi(1-x),$$
the permutation operator on the right,

\[ \mathcal{P}_R \equiv (R_1 \leftrightarrow R_2; \omega \leftrightarrow 1 - \omega) , \]

and similarly for \( \mathcal{P}_L \) acting on \( L \)-labels and \( z \).

We can now write the triple-meson vertex functions:

\[
T(L| R_1 R_2; \omega) = (1 - CP_R) \frac{1}{1 - \omega} \int_0^\omega dx \phi_L(x) \phi_{R_1} \left( \frac{x}{\omega} \right) \Phi_{R_2} \left( \frac{x - \omega}{1 - \omega} \right) \\
= \frac{1}{1 - \omega} \int_0^\omega dx \phi_L(x) \phi_{R_1} \left( \frac{x}{\omega} \right) \Phi_{R_2} \left( \frac{x - \omega}{1 - \omega} \right) \\
- \frac{1}{\omega} \int_0^1 dx \phi_L(x) \phi_{R_2} \left( \frac{x - \omega}{1 - \omega} \right) \Phi_{R_1} \left( \frac{x}{\omega} \right) ,
\]  
(A.4a)

and

\[
T'(L| R_1 R_2; \omega) = (1 - C) \frac{f_{R_2}}{1 - \omega} \int_0^\omega dx \phi_L(x) \phi_{R_1} \left( \frac{x}{\omega} \right) .
\]  
(A.4b)

The four-meson vertices coming from the \( BB + DD \) terms read:

\[
\mathcal{F}_{B^2+D^2}(L_1 L_2; z| R_1 R_2; \omega) = (1 + CP_R) \left[ -\theta(\omega - z) \frac{1 - \omega}{z} \int_0^1 dx \phi_{L_2} \left( \frac{1 - \omega}{1 - z} x \right) \phi_{R_2}(x) \\
\times \Psi_{L_1 R_1} \left( 1 - (1 - x) \frac{1 - \omega}{z}; z, \omega \right) + (L \leftrightarrow R, \omega \leftrightarrow z) \right] ,
\]  
(A.5a)

and

\[
\mathcal{F}'_{B^2+D^2}(L_1 L_2; z| R_1 R_2; \omega) = (1 + C) \left[ -\theta(\omega - z) \frac{z(1 - \omega)}{(z - \omega)^2} \int_0^1 dx \phi_{L_1}(x) \phi_{R_1} \left( \frac{z}{\omega} x \right) \\
\times \int_0^1 dy \phi_{L_2} \left( \frac{1 - \omega}{1 - z} y \right) \phi_{R_2}(y) + (L \leftrightarrow R, \omega \leftrightarrow z) \right] .
\]  
(A.5b)

Finally, the four-point vertices from the \( BD \) terms are given by:

\[
\mathcal{F}_{BD}(L_1 L_2; z| R_1 R_2; \omega) = \left[ 2\theta(\omega - z) \frac{1 - \omega}{z} \int_0^1 dx \phi_{L_2} \left( 1 - \frac{1 - \omega}{1 - z} x \right) \phi_{R_2}(1 - x) \\
\times \Psi_{L_1 R_1} \left( \frac{\omega}{z} + (1 - x) \frac{1 - \omega}{1 - z}; z, \omega \right) + (L \leftrightarrow R, \omega \leftrightarrow z) \right] ,
\]  
(A.6a)
and

\[
F'_{BD}(L_1L_2;z|R_1R_2;\omega) = (1 + C) \left[ \theta(\omega - z) \frac{z(1 - \omega)}{(z - \omega)^2} \int_0^1 dx \phi_{L_1}(x)\phi_{R_1}\left(\frac{z}{x}\right) \times \int_0^1 dy \phi_{L_2}\left(1 - \frac{1 - \omega}{1 - z}y\right)\phi_{R_2}(1 - y) + (L \leftrightarrow R, \omega \leftrightarrow z) \right].
\]  

(A.6b)

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