An upper bound on quantum capacity of unital channels

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Abstract

We analyze the quantum capacity of a unital quantum channel, using ideas from the proof of near-optimality of Petz recovery map [Barnum and Knill 2000] and give an upper bound on the quantum capacity in terms of regularized output 2-norm of the channel. We also show that any code attempting to exceed this upper bound must incur large error in decoding, which can be viewed as a weaker version of the strong converse results for quantum capacity. As an application, we find nearly matching upper and lower bounds (up to an additive constant) on the quantum capacity of quantum expander channels. Using these techniques, we further conclude that the ‘mixture of random unitaries’ channels arising in the construction of quantum expanders in [Hastings 2007] show a trend in multiplicativity of output 2-norm similar to that exhibited in [Montanaro 2013] for output $\infty$-norm of random quantum channels.

1 Introduction

One of the most fundamental developments in quantum information theory has been towards an understanding of various capacities of quantum channels. Quantum capacity of a quantum channel is characterized by a well-known quantity called the coherent information [SN96]. Similarly, classical capacity of a quantum channel is characterized by its Holevo information [Hol73]. Unfortunately, a single letter formula for either the quantum capacity or the classical capacity is not known, and a regularization is needed to completely capture these capacities [Smi10].

It was shown by Shor [Sho01] that the problem of regularization of Holevo information is related to various other additivity questions in quantum information and in particular to additivity of the minimum entropy output of a quantum channel. This was combined with an extensive study of multiplicativity of output norms of quantum channels (we discuss output 2-norm and output $\infty$-norm in Section 2; general definition can be found in following references). Violations of multiplicativity of various output norms were shown in a series of results [WH02, HW08, CHL+08], culminating in a proof of violation of additivity of minimum entropy output by Hastings [Has09]. The work [Mon13] studied the output $\infty$-norm of a random quantum channel, where it was shown that most quantum channels still satisfied a weaker version of the multiplicativity of output $\infty$-norm (Theorem 3 in the reference [Mon13]).
In this work, we primarily consider the quantum capacities of unital channels and their output $2$-norms. We provide an upper bound on quantum capacity of such channels in terms of their regularized output $2$-norm. In addition, we prove a result that is reminiscent of the ‘strong converse theorems’, which have received a great deal of attention in recent literature on quantum channel capacity (see for example, [SW13, WWY14, TWW14, WW14, GW15, MW14, CMW16] and references therein).

Results and techniques

We provide an upper bound on the quantum capacity and the zero error classical capacity of a quantum channel (Lemma 3.2 for quantum capacity of a general channel, Corollary 3.3 for quantum capacity of a unital channel and Lemma 3.4 for zero error classical capacity of a general channel). Our bound is inspired from the near-optimality of Petz recovery map due to Barnum and Knill [HB00], which has been well studied in literature, such as for approximate quantum error correction [NM10] and achievability results in quantum channel capacity [BDL16]. Using this bound, we derive an upper bound on quantum capacity of unital channels and also a weak form of strong converse theorem for quantum capacity: for any encoding-decoding operation that attempts to exceed the upper bound on quantum capacity, the success fidelity of decoding the quantum message falls exponentially in number of channel uses (Theorem 4.2).

As an application, we consider the well studied quantum expander channels (various constructions of which have been presented in [AS04, Has07, Har08, GE08, Har09]), and in particular, the mixture of random unitaries as defined in [Has07]. We find an upper and a lower bound on quantum capacities of such random channels, and show that with high probability, the upper and lower bounds differ by a small constant (Lemma 4.4 and Corollary 4.6). Moreover, along the lines of the result shown in [Mon13], we find that the output $2$-norm of such channels is nearly multiplicative (with high probability), with the multiplicativity exponent close to $1$ (Corollary 4.6).

2 Preliminaries

For integer $n \geq 1$, let $[n]$ represent the set \{1, 2, ..., $n$\}. Let $\mathbb{R}$ represent the set of real numbers. We let $\log$ represent logarithm to the base 2 and $\ln$ represent logarithm to the base $e$.

Consider a finite dimensional Hilbert space $\mathcal{H}$ endowed with an inner product $\langle \cdot, \cdot \rangle$. In this paper, we only consider finite dimensional Hilbert spaces. The $\ell_1$ norm of an operator $X$ on $\mathcal{H}$ is $\|X\|_1 \overset{\text{def}}{=} \text{Tr} \sqrt{X^\dagger X}$ and $\ell_2$ norm is $\|X\|_2 \overset{\text{def}}{=} \sqrt{\text{Tr} XX^\dagger}$. A quantum state (or a density matrix or a state) is a positive semi-definite matrix on $\mathcal{H}$ with trace equal to 1. It is called pure if and only if its rank is 1. A sub-normalized state is a positive semi-definite matrix on $\mathcal{H}$ with trace less than or equal to 1. Let $|\psi\rangle$ be a unit vector on $\mathcal{H}$, that is $\langle \psi, \psi \rangle = 1$. With some abuse of notation, we use $\psi$ to represent the state and also the density matrix $|\psi\rangle\langle \psi|$, associated with $|\psi\rangle$. Given a quantum state $\rho$ on $\mathcal{H}$, support of $\rho$, called $\text{supp}(\rho)$ is the subspace of $\mathcal{H}$ spanned by all eigen-vectors of $\rho$ with non-zero eigenvalues.

A quantum register $A$ is associated with some Hilbert space $\mathcal{H}_A$. Define $|A| \overset{\text{def}}{=} \text{dim}(\mathcal{H}_A)$. Let $\mathcal{L}(A)$ represent the set of all linear operators on $\mathcal{H}_A$. We denote by $\mathcal{D}(A)$, the set of quantum states on the Hilbert space $\mathcal{H}_A$. State $\rho$ with subscript $A$ indicates $\rho_A \in \mathcal{D}(A)$. If two registers $A, B$ are associated with the same Hilbert space, we shall represent the relation by $A \equiv B$. Composition of two registers $A$ and $B$, denoted $AB$, is associated with Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. For two quantum
states \( \rho \in \mathcal{D}(A) \) and \( \sigma \in \mathcal{D}(B) \), \( \rho \otimes \sigma \in \mathcal{D}(AB) \) represents the tensor product (Kronecker product) of \( \rho \) and \( \sigma \). The identity operator on \( \mathcal{H}_A \) (and associated register \( A \)) is denoted \( I_A \).

Let \( \rho_{AB} \in \mathcal{D}(AB) \). We define

\[
\rho_B \overset{\text{def}}{=} \text{Tr}_A(\rho_{AB}) \overset{\text{def}}{=} \sum_i (\langle i \rangle \otimes I_B) \rho_{AB}(\langle i \rangle \otimes I_B),
\]

where \( \{|i\rangle\}_i \) is an orthonormal basis for the Hilbert space \( \mathcal{H}_A \). The state \( \rho_B \in \mathcal{D}(B) \) is referred to as the marginal state of \( \rho_{AB} \). Unless otherwise stated, a missing register from subscript in a state will represent partial trace over that register. Given a \( \rho_A \in \mathcal{D}(A) \), a purification of \( \rho_A \) is a pure state \( \rho_{AB} \in \mathcal{D}(AB) \) such that \( \text{Tr}_B(\rho_{AB}) = \rho_A \). Purification of a quantum state is not unique.

A quantum map \( \mathcal{E} : A \rightarrow B \) is a completely positive linear map (mapping states in \( \mathcal{D}(A) \) to states in \( \mathcal{D}(B) \)). In this work, we shall also consider maps that do not preserve trace. Trace preserving quantum maps shall be referred to as quantum channels. A unitary operator \( U_A : \mathcal{H}_A \rightarrow \mathcal{H}_A \) is such that \( U_A^\dagger U_A = U_A U_A^\dagger = I_A \). An isometry \( V : \mathcal{H}_A \rightarrow \mathcal{H}_B \) is such that \( V^\dagger V = I_A \) and \( VV^\dagger = I_B \). The set of all unitary operations on register \( A \) is denoted by \( \mathcal{U}(A) \). A quantum channel \( \mathcal{E} : A \rightarrow A \) is said to be unital if it holds that \( \mathcal{E}(I_A) = I_A \).

Given a quantum map \( \mathcal{E} : A \rightarrow B \), maximum output \( \infty \)-norm of \( \mathcal{E} \) is defined as \( \| \mathcal{E} \|_\infty = \max_{\rho \in \mathcal{D}(A)} \{ \| \mathcal{E}(\rho) \| \} \). Here, \( \| . \| \) is the operator norm. We say that \( \mathcal{E} \) obeys \( \infty \)-norm multiplicativity with exponent \( \alpha \) if \( \| \mathcal{E}^{\otimes n} \|_\infty \leq \| \mathcal{E} \|_\infty^\alpha \). Similarly, maximum output \( 2 \)-norm of \( \mathcal{E} \) is defined as \( \| \mathcal{E} \|_2 = \max_{\rho \in \mathcal{D}(A)} \{ \text{Tr}(\mathcal{E}^2(\rho)) \} \). We say that \( \mathcal{E} \) obeys \( 2 \)-norm multiplicativity with exponent \( \alpha \) if \( \| \mathcal{E}^{\otimes n} \|_2 \leq \| \mathcal{E} \|_2^\alpha \).

Following fact says that the optimization in the definition of \( \| \mathcal{E} \|_2 \) is achieved by a pure state.

**Fact 2.1.** For every state \( \rho \), there exists a pure state \(|\sigma\rangle\) such that \( \text{Tr}(\mathcal{E}^2(\rho)) \leq \text{Tr}(\mathcal{E}^2(\sigma)) \).

**Proof.** We consider the eigen-decomposition \( \rho = \sum_i p_i |\sigma_i\rangle \langle \sigma_i| \). Then

\[
\text{Tr}(\mathcal{E}^2(\rho)) = \sum_{i,j} p_i p_j \text{Tr}(\mathcal{E}(\sigma_i)\mathcal{E}(\sigma_j)) \leq \sum_{i,j} p_i p_j \sqrt{\text{Tr}(\mathcal{E}^2(\sigma_i))\text{Tr}(\mathcal{E}^2(\sigma_j))} = \left( \sum_i p_i \sqrt{\text{Tr}(\mathcal{E}^2(\sigma_i))} \right)^2,
\]

where we use the Cauchy-Schwartz inequality \( \text{Tr}(XY) \leq \sqrt{\text{Tr}(X^2)\text{Tr}(Y^2)} \) for hermitian matrices \( X, Y \). Now, using concavity of square-root, we proceed as

\[
\text{Tr}(\mathcal{E}^2(\rho)) \leq \sum_i p_i \text{Tr}(\mathcal{E}^2(\sigma_i)) \leq \max_i \text{Tr}(\mathcal{E}^2(\sigma_i)).
\]

Thus proves the fact. \( \square \)

**Quantum channel capacities.**

Given a quantum channel \( \mathcal{E} : A \rightarrow B \) that serves as noise, we shall be interested in two kinds of capacities: the quantum capacity and the zero error classical capacity. We first describe the quantum capacity. Fix an \( n > 0 \) and consider a \( d_C \) dimensional ‘source’ Hilbert space \( \mathcal{H}_S \) (the \( C \) in the subscript stands for ‘codespace’, as the dimension of the system is equal to the dimension of the codespace used to encode the quantum states in the system). An encoding operation maps the register \( S \) onto registers \( A_1, A_2, \ldots, A_n \) as follows. Alice introduces an ancillary register \( T \), in the state \(|0\rangle |0\rangle_T \) and applies an isometry \( ST \rightarrow A_1 A_2 \ldots A_n T \). Under this isometry, every vector
We define $C_d\psi$ where

\[ C \text{R}_B \text{sends the registers } A \text{.} \]

Lemma 3.1 proof in Appendix A.

Following well known result holds for $Q(E)$ (see [Wil12] for a detailed discussion)

**Fact 2.2** (The Lloyd-Shor-Devetak theorem). [Llo97, Sho02, Dev05] For a quantum channel $E : A \rightarrow B$ introduce a reference register $R$ with dimension of $\mathcal{H}_R$ same as the dimension of $\mathcal{H}_A$. Then

\[ Q(E) \geq \max_{|\Psi_{RA}\rangle \in D(RA)} (S(E(\Psi_A)) - S(I_R \otimes E(\Psi_{RA}))) \]

The zero error classical capacity [MA05] is defined as follows. Given a collection of $M$ messages $\{1, 2, \ldots, M\}$, Alice encodes each message $m$ into a quantum state $\rho_m \in D(A_1A_2 \ldots A_n)$ and sends the registers $A_1, A_2, \ldots, A_n$ sequentially through the channel $E$. Receiving all the registers $B_1, B_2, \ldots, B_n$, Bob applies a decoding operation $R$ that recovers the message $m$ with zero error. We define $C_n(E)$ as the largest possible $\log(M)$ such that there exist quantum states $\{\rho_1, \rho_2, \ldots, \rho_m\}$ and a recovery operation $R$ such that $R(E(\rho_m)) = |m\rangle\langle m|$. Zero error classical capacity of $E$ is now defined as $C(E) = \lim_{n \rightarrow \infty} \frac{1}{n} C_n(E)$.

### 3 Upper bound on capacities using Petz recovery map

Given a noise $E : X \rightarrow Y$ acting on certain register $X$, and any positive semi-definite operator $\Pi$ on register $Y$, we define the following associated map $P_\Pi(\rho) = E!(\Pi^{-1}\rho\Pi^{-1})$. Here, the map $E^! : Y \rightarrow X$ is defined as $\text{Tr}(\sigma E^!(\rho)) = \text{Tr}(E(\sigma)\rho)$ for all $\rho \in D(Y)$ and $\sigma \in D(X)$. The Petz recovery map is a special case when $\Pi$ is chosen to be $E_\Pi$ a quantum channel. The following relation was essentially proved in [HB00] (and elaborated in [NM10]). We reproduce its proof in Appendix A.

**Lemma 3.1** ([HB00, NM10]). For any quantum map $R : Y \rightarrow X$, the noise $E : X \rightarrow Y$, a positive semi-definite operator $\Pi$ on register $Y$ fully supported in the image of $E$ and any state $\psi \in \mathcal{H}_X$, it holds that

\[ F^2(\psi, R(\psi)) \leq \sqrt{\langle \psi | P_\Pi(\psi) \rangle} \langle \psi | R(\Pi^2) | \psi \rangle. \]
Now, as discussed in Section 2, consider the setting of \( n \) registers \( A_1, A_2 \ldots A_n \), such that all \( A_i \equiv A \). Let \( \mathcal{E} : A \rightarrow B \) be a noise, which acts independently on above registers as \( \mathcal{E}^\otimes n : A_1 \otimes A_2 \otimes \ldots A_n \rightarrow B_1 \otimes B_2 \otimes \ldots B_n \). For the operator \( T \), we consider the associated map \( \mathcal{P} \sqrt{d_T} \ket{0} \bra{0} \), which we simply abbreviate as \( \mathcal{P}_T \). Here \( d_T \) is the dimension of \( \mathcal{H}_T \). From the Kraus representation of \( T \) (that is, \( T(\rho) = \sum_i \ket{i} \rho \bra{i} \ket{0} \)), it is easy to observe that \( \mathcal{P}_T(\ket{0} \bra{0}) = T^\dagger(\ket{0} \bra{0}) = \frac{I_T}{d_T} \).

For the channel \( \mathcal{E} \) and operator \( \Pi \) supported on the image of \( \mathcal{E}^\otimes n \), define the following map:

\[
G_{\mathcal{E}, \Pi}(.) = \Pi^{-1/2} \mathcal{E}(.) \Pi^{-1/2}.
\]

Then we have the following lemma.

**Lemma 3.2.** Given a noise \( \mathcal{E} : A \rightarrow B \) such that dimension of \( \mathcal{H}_A \) is \( d \) and a codespace \( C \) (along with register \( T \) and recovery map \( \mathcal{R}_C \)) with average fidelity \( \eta \), we have

\[
d_C \leq \frac{1}{\eta^4} \min_{\Pi} \| G_{\mathcal{E}^\otimes n, \Pi} \|_2 \cdot \text{Tr}(\Pi^2),
\]

where minimization is over all positive semi-definite operators \( \Pi \) that are in the support of image of \( \mathcal{E}^\otimes n \).

**Proof.** Fix an orthonormal basis in \( \mathcal{H}_{A_1, A_2, \ldots A_n, T} : \{ \ket{\phi_1}, \ket{\phi_2} \ldots \ket{\phi_{d_T}} \} \) such that for all \( i \leq d_C \), \( \phi_i \subset C \). Let \( \Pi \) be any operator fully supported in the image of \( \mathcal{E}^\otimes n \). Consider the following map associated to \( \mathcal{E}^\otimes n \):

\[
\mathcal{P}_\Pi(.) \equiv \mathcal{E}^\otimes n(\Pi^{-1}(.) \Pi^{-1}).
\]

We apply Lemma 3.1 to the ‘noise’ \( \mathcal{T} \otimes \mathcal{E}^\otimes n \) and the map \( \mathcal{P}_T \otimes \mathcal{P}_\Pi \):

\[
\sum_i \mathcal{F}_i(C)(\mathcal{T} \otimes \mathcal{E}^\otimes n(\phi_i)) \leq \sum_i \bra{\phi_i} \mathcal{P}_T \otimes \mathcal{P}_\Pi(\mathcal{T} \otimes \mathcal{E}^\otimes n(\phi_i)) \bra{\phi_i} \mathcal{R}_C(d_T \ket{0} \bra{0} \otimes \Pi^2) \ket{\phi_i}.
\]

We shall upper bound each term \( \bra{\phi_i} \mathcal{P}_T \otimes \mathcal{P}_\Pi(\mathcal{T} \otimes \mathcal{E}^\otimes n(\phi_i)) \ket{\phi_i} \) as follows.

\[
\begin{align*}
\bra{\phi_i} \mathcal{P}_T \otimes \mathcal{P}_\Pi(\mathcal{T} \otimes \mathcal{E}^\otimes n(\phi_i)) \ket{\phi_i} \\
= \bra{\phi_i} \mathcal{P}_T(\ket{0} \bra{0} \otimes \mathcal{P}_\Pi(\mathcal{E}^\otimes n(\text{Tr}_T \phi_i))) \ket{\phi_i} \\
(\text{as } \mathcal{T} \text{ traces out register } T \text{ and replaces it with the state } \ket{0} \bra{0}) \\
= \bra{\phi_i} \mathcal{T}_T^{d_T} \otimes \mathcal{P}_\Pi(\mathcal{E}^\otimes n(\text{Tr}_T \phi_i))) \ket{\phi_i} \\
(\text{as } \mathcal{P}_T \text{ replaces the state } \ket{0} \bra{0} \text{ with the maximally mixed state on register } T) \\
\leq \frac{1}{d_T} \max_{\phi_i} \text{Tr}((\text{Tr}_T \phi_i) \mathcal{P}_\Pi(\mathcal{E}^\otimes n(\text{Tr}_T \phi_i))) \\
= \frac{1}{d_T} \max_{\phi_i} \text{Tr}(\mathcal{E}^\otimes n(\text{Tr}_T \phi_i) \Pi^{-1} \mathcal{E}^\otimes n(\text{Tr}_T \phi_i) \Pi^{-1}) \\
(\text{follows by incorporating the definition of the map } \mathcal{P}_\Pi) \\
= \frac{1}{d_T} \max_{\phi_i} \text{Tr}(\Pi^{-1/2} \mathcal{E}^\otimes n(\text{Tr}_T (\phi_i)) \Pi^{-1/2} \Pi^{-1/2} \mathcal{E}^\otimes n(\text{Tr}_T (\phi_i)) \Pi^{-1/2}) \\
(\text{writing } \Pi^{-1} = \Pi^{-1/2} \Pi^{-1/2} \text{ and then using cyclicity of trace}) \\
= \frac{1}{d_T} \max_{\phi_i} \text{Tr}((G_{\mathcal{E}^\otimes n, \Pi}(\text{Tr}_T (\phi_i)))^2) \leq \frac{1}{d_T} \| G_{\mathcal{E}^\otimes n, \Pi} \|_2.
\end{align*}
\]
Applying it in Equation 1, we obtain

\[ \sum_i F^4(\phi_i, R_C(T \otimes E^\otimes n(\phi_i))) \leq \frac{\|G_{E^\otimes n, \Pi}\|_2}{d_T} \sum_i \langle \phi_i | R_C(d_T |0\rangle \langle 0| \otimes \Pi^2) | \phi_i \rangle \]

(as each term \( \langle \phi_i | R_C(d_T |0\rangle \langle 0| \otimes \Pi^2) | \phi_i \rangle \) is positive)

\[ = \|G_{E^\otimes n, \Pi}\|_2 \cdot \text{Tr}(R_C(|0\rangle \langle 0| \otimes \Pi^2) \sum_i \phi_i) \]

\[ = \|G_{E^\otimes n, \Pi}\|_2 \cdot \text{Tr}(R_C(|0\rangle \langle 0| \otimes \Pi^2)) \]

\[ = \|G_{E^\otimes n, \Pi}\|_2 \cdot \text{Tr}(\Pi^2) \]

(\( \sum \phi_i = I^\otimes n \otimes I_T \), since \( \phi_i \) form an orthonormal basis)

\[ = \|G_{E^\otimes n, \Pi}\|_2 \cdot \text{Tr}(\Pi^2) \]

(as the map \( R_C \) is a trace preserving quantum map)

On the other hand,

\[ \sum_i F^4(\phi_i, R_C(T \otimes E^\otimes n(\phi_i))) \geq d_C \cdot \frac{\sum_{i \leq d_C} F^4(\phi_i, R_C(T \otimes E^\otimes n(\phi_i)))}{d_C}. \]

Thus, we obtain

\[ \|G_{E^\otimes n, \Pi}\|_2 \cdot \text{Tr}(\Pi^2) \geq d_C \cdot \frac{\sum_{i \leq d_C} F^4(\phi_i, R_C(T \otimes E^\otimes n(\phi_i)))}{d_C}. \]

Now, averaging over all possible basis in codespace \( C \), we find that

\[ \|G_{E^\otimes n, \Pi}\|_2 \cdot \text{Tr}(\Pi^2) \geq d_C \cdot \int_{\phi \in C} F^4(\phi_i, R_C(T \otimes E^\otimes n(\phi))) d\phi \geq d_C \cdot \left( \int_{\phi \in C} F(\phi_i, R_C(T \otimes E^\otimes n(\phi))) d\phi \right)^4, \]

where last inequality follows by convexity of the function \( x \to x^4 \). This proves the lemma, by incorporating the definition of average fidelity \( \eta \) and optimizing over all possible positive semi-definite operators \( \Pi \) supported in the image of \( E^\otimes n \).

We have the following corollary of above lemma, which gives an upper bound on the quantum capacity and also says that exceeding this upper bound leads to decrease in average fidelity exponentially in \( n \). Since we shall use this corollary in later sections for unital channels, we have restricted its statement to such channels.

**Corollary 3.3.** Suppose the channel \( E \) is unital. Then we have that

\[ Q(E) \leq \log(d \cdot \lim_{n \to \infty} \|E^\otimes n\|_2^{1/n}). \]

Furthermore, let \( C \) be any codespace of dimension \( d_C = d^n \|E^\otimes n\|_2 (1 + \beta)^n \), for some \( \beta > 0 \). Then the average fidelity \( \eta \) satisfies the following relation, irrespective of the recovery map:

\[ \eta^4 \leq \frac{1}{(1 + \beta)^n}. \]
Proof. In Lemma 3.2 we choose \( \Pi = I \otimes n \). This gives \( G_{E \otimes n, \Pi} = E \otimes n \) and we find that 
\[
\frac{1}{n} \log(d_C) \leq \log(d \cdot \lim_{n \to \infty} \| E \otimes n \|_2^{1/n} / \eta^4).
\]
Now we take the limit \( n \to \infty \) and then take \( \eta \to 1 \). Second part of the corollary proceeds by direct substitution in Lemma 3.2, with the choice of \( \Pi = I \otimes n \).

For the zero error classical capacity of \( E \), similar result is shown to hold.

Lemma 3.4. It holds that
\[
C(E) \leq \lim_{n \to \infty} \frac{1}{n} \log(\min \| G_{E \otimes n, \Pi} \|_2 \cdot \text{Tr}(\Pi^2)).
\]

Proof. Given the constraint \( R_{C/E}(\rho) = |m\rangle \langle m| \), we find that \( \rho_m \rho_{m'} = 0 \) if \( m \neq m' \). Now, for the mapping \( m \to \rho_m \), we consider a purifying register \( T \) such that \( \psi_m \in D(A_1A_2...A_nT) \) is a purification of \( \rho_m \). Clearly, \( \psi_1, \psi_2, ... \psi_M \) form a basis in a \( M \)-dimensional subspace of \( \mathcal{H}_{A_1A_2...A_nT} \). Thus, we can repeat the analysis in Lemma 3.2 with \( \eta = 1 \), from which this lemma follows.

4 Regularized 2-norm for unital channels and capacity of expanders

In this section, we shall restrict ourselves to unital channels acting on a \( d \)-dimensional Hilbert space. Let the Kraus decomposition of \( E : A \to A \) be \( E(.) = \sum_i E_i(.) E_i^\dagger \). Since \( E \) is unital, \( I \) is a fixed point of \( E \) with eigenvalue 1. Second largest singular value of \( E \) is defined as \( \lambda_2(E) \overset{\text{def}}{=} \max_{\rho : \text{Tr}(\rho) = 1} \sqrt{\text{Tr}(E(\rho) E(\rho)^\dagger)} \). Then we have the following lemma, proved in Appendix B.

Lemma 4.1. Let \( E \) be a unital channel with second largest singular value \( \lambda_2 \overset{\text{def}}{=} \lambda_2(E) < 1 \). For all \( n \geq 1 \), it holds that
\[
\| E \otimes n \|_2 \leq (\frac{1}{d} + \lambda_2^2)^n.
\]
In particular, \( \lim_{n \to \infty} \| E \otimes n \|_2^{1/n} \leq (\frac{1}{d} + \lambda_2^2) \).

Combining with Corollary 3.3 we obtain our main theorem in a straightforward manner.

Theorem 4.2. Let \( E \) be a unital channel. Then we have that
\[
Q(E) \leq \log(1 + d \cdot \lambda_2^2).
\]
Furthermore, let \( C \) be any codespace of dimension \( d_C = (1 + d\lambda_2^2)^n (1 + \beta)^n \), for some \( \beta > 0 \). Then the average fidelity \( \eta \) satisfies the following relation, irrespective of the recovery map:
\[
\eta^4 \leq \frac{1}{(1 + \beta)^n}.
\]

4.1 Expander channels

Definition 4.3. A unital quantum channel \( E : A \to A \) with \( k \) Kraus operators \( \{ E_i \}_{i=1}^k \) and acting on a \( d \)-dimensional Hilbert space \( \mathcal{H}_A \) is said to be a \( (C, k, d) \)-expander if it holds that \( \lambda_2^2(E) = \frac{C}{k} \).

Under this definition, we obtain the following Lemma.
Lemma 4.4. Given a channel $\mathcal{E} : A \to A$ that is a $(C,k,d)$-expander. Then following properties hold for $\mathcal{E}$.

- The quantum capacity $Q(\mathcal{E})$ is upper bounded by $\log(d) - \log(k) + \log(C + \frac{k}{d})$ and lower bounded by $\log(d) - \log(k) - \log(4 + 4\varepsilon,k,d)$.
- If $\frac{\log(dC)}{n} = \log(d) - \log(k) + \log((4 + \varepsilon)(1 + \beta))$, then average fidelity $\eta$ satisfies $\eta^4 < (1 + \beta)^{-n}$.
- For all $n$, it holds that $\|\mathcal{E}^\otimes n\|_2^{1/n} \leq \frac{1}{d} + \frac{C}{k}$ and $\|\mathcal{E}\|_2 \geq \frac{1}{k}$.

Proof. We prove each item separately.

- The upper bound on $Q(\mathcal{E})$ follows from Theorem 4.2 and the assumption in Definition 4.3 that $\lambda^2_3(\mathcal{E}) = \frac{C}{d}$. For the lower bound, we recall from the Lloyd-Shor-Devetak theorem (Fact 2.2) that $Q(\mathcal{E}) \geq \max_{\Psi_{RA}}(S(\mathcal{E}(\Psi_{A})) - S(I_R \otimes \mathcal{E}(\Psi_{RA})))$. Now let $\Psi_A \overset{def}{=} \frac{1}{d}$. Then $S(\mathcal{E}(\Psi_{A})) = S(\Psi_{A}) = \log(d)$. On the other hand, $S(I_R \otimes \mathcal{E}(\Psi_{RA})) \leq \log(k)$ as $\Psi_{RA}$ is a pure state and $\mathcal{E}$ is composed of $k$ Kraus operators (which means that $\mathcal{E}(\Psi_{RA})$ is a convex combination of $k$ pure states). Hence $Q(\mathcal{E}) \geq \log(d) - \log(k)$.

- Second item again follows from Theorem 4.2

- For the third item, we observe that $\|\mathcal{E}\|_2 \geq \frac{1}{k}$ for any channel. This follows because $\|\mathcal{E}\|_2 = \max_\psi \text{Tr}(\mathcal{E}(\psi)^2)$. Now, $\mathcal{E}(\psi)$ is a convex combination of $k$ pure states and hence $\text{Tr}(\mathcal{E}(\psi)^2) > \frac{1}{k}$. This proves the item when combined with Lemma 4.1.

A well known example of expander construction is due to Hastings [Has07], who showed the following theorem.

Theorem 4.5 ([Has07]). Pick $k/2$ unitary operators $\{U_1, U_2, \ldots, U_{k/2}\}$ (each acting on $d$ dimensional Hilbert space) from the Haar measure and construct the quantum channel $\mathcal{E}(\rho) \overset{def}{=} \frac{1}{k} \sum_i (U_i \rho U_i^\dagger + U_i^\dagger \rho U_i)$. Then for every $\varepsilon > 0$, with probability at least $1 - e^{-\varepsilon \cdot d^{2/15}}$, $\mathcal{E}$ is a $(4 + 4\varepsilon,k,d)$-expander.

Combining this with Lemma 4.4 we obtain the following straightforward corollary. The third item below is similar in spirit to the result in [Mon13].

Corollary 4.6. Consider a random channel $\mathcal{E}$ as constructed in Theorem 4.5. Then for every $\varepsilon > 0$, setting $d > \frac{4}{\varepsilon}$, the following holds with probability at least $1 - e^{-\varepsilon \cdot d^{2/15}}$.

- The quantum capacity $Q(\mathcal{E})$ is upper bounded by $\log(d) - \log(k) + \log(4 + 5\varepsilon) + \log(4 + 4\varepsilon,k,d)$ and lower bounded by $\log(d) - \log(k) - \log(4 + 4\varepsilon,k,d)$.
- If $\frac{\log(dC)}{n} = \log(d) - \log(k) + \log((4 + 5\varepsilon)(1 + \beta))$, then average fidelity $\eta$ decays as $\eta^4 < (1 + \beta)^{-n}$.
- $\|\mathcal{E}^\otimes n\|_2^{1/n} \leq \|\mathcal{E}\|_2^{n(1 + \frac{1}{\log(4\varepsilon)})}$. 


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References

[AS04] Andris Ambainis and Adam Smith. Small pseudo-random families of matrices: Derandomizing approximate quantum encryption. *Proceedings of RANDOM 2004*, 2004.

[BDL16] Salman Beigi, Nilanjana Datta, and Felix Leditzky. Decoding quantum information via the petz recovery map. *Journal of Mathematical Physics*, 57(8), 2016.

[CHL+08] Toby Cubitt, Aram W. Harrow, Debbie Leung, Ashley Montanaro, and Andreas Winter. Counterexamples to additivity of minimum output p-ryeni entropy for p close to 0. *Communications in Mathematical Physics*, 284(1):281–290, 2008.

[CMW16] Tom Cooney, Milán Mosonyi, and Mark M. Wilde. Strong converse exponents for a quantum channel discrimination problem and quantum-feedback-assisted communication. *Communications in Mathematical Physics*, 344(3):797–829, 2016.

[Dev05] I. Devetak. The private classical capacity and quantum capacity of a quantum channel. *IEEE Transactions on Information Theory*, 51(1):44–55, Jan 2005.

[GE08] David Gross and Jens Eisert. Quantum margulis expanders. *Quantum Information and Computation*, 8:722, 2008.

[GW15] Manish K. Gupta and Mark M. Wilde. Multiplicativity of completely bounded p-norms implies a strong converse for entanglement-assisted capacity. *Communications in Mathematical Physics*, 334(2):867–887, 2015.

[Har08] Aram Harrow. Quantum expanders from any classical cayley graph expander. *Quantum Information and Computation*, 8:715–721, 2008.

[Har09] M.B. Hastings; Aram Harrow. Classical and quantum tensor product expanders. *Quantum Information and Computation*, 9:336, 2009.

[Has07] M.B. Hastings. Random unitaries give quantum expanders. *Phys. Rev. A*, 76:032315, 2007.

[Has09] M. B. Hastings. Superadditivity of communication capacity using entangled inputs. *Nature Physics*, 5:255–257, 2009.

[HB00] E. Knill H. Barnum. Reversing quantum dynamics with near-optimal quantum and classical fidelity. arXiv:quant-ph/0004088, 2000.
[Hol73] A. S. Holevo. Statistical problems in quantum physics. In Proceedings of the Second Japan–USSR Symposium on Probability Theory, volume 330, pages 104–119. Springer-Verlag, Berlin, 1973.

[HW08] Patrick Hayden and Andreas Winter. Counterexamples to the maximal p-norm multiplicativity conjecture for all $p > 1$. Communications in Mathematical Physics, 284(1):263–280, 2008.

[Llo97] Seth Lloyd. Capacity of the noisy quantum channel. Phys. Rev. A, 55:1613–1622, Mar 1997.

[MA05] Rex A. C. Medeiros and Francisco M. De Assis. Quantum zero-error capacity. International Journal of Quantum Information, 3(1):135, 2005.

[Mon13] Ashley Montanaro. Weak multiplicativity for random quantum channels. Communications in Mathematical Physics, 319(2):535–555, 2013.

[MW14] C. Morgan and A. Winter. Pretty strong converse for the quantum capacity of degradable channels. IEEE Transactions on Information Theory, 60(1):317–333, Jan 2014.

[NM10] Hui Khoon Ng and Prabha Mandayam. Simple approach to approximate quantum error correction based on the transpose channel. Phys. Rev. A, 81, 2010.

[Sho02] Peter Shor. The quantum channel capacity and coherent information. Lecture Notes, MSRI Workshop on Quantum Computation., 2002.

[Sho04] Peter W. Shor. Equivalence of additivity questions in quantum information theory. Communications in Mathematical Physics, 246(3):473–473, 2004.

[Smi10] Graeme Smith. Quantum channel capacities. https://arxiv.org/abs/1007.2855, 2010.

[SN96] Benjamin Schumacher and M. A. Nielsen. Quantum data processing and error correction. Phys. Rev. A, 54:2629–2635, Oct 1996.

[SW13] Naresh Sharma and Naqueeb Ahmad Warsi. Fundamental bound on the reliability of quantum information transmission. Phys. Rev. Lett., 110:080501, Feb 2013.

[TWW14] Marco Tomamichel, Mark M. Wilde, and Andreas Winter. Strong converse rates for quantum communication. https://arxiv.org/abs/1406.2946, 2014.

[WH02] R.F. Werner and A.S. Holevo. Counterexample to an additivity conjecture for output purity of quantum channels. https://arxiv.org/abs/quant-ph/0203003, 2002.

[Wil12] Mark M. Wilde. Quantum Information Theory:. Cambridge University Press, Cambridge, 12 2012.

[WW14] M. M. Wilde and A. Winter. Strong converse for the classical capacity of the pure-loss bosonic channel. Problems of Information Transmission, 50(2):117–132, 2014.

[WWY14] Mark M. Wilde, Andreas Winter, and Dong Yang. Strong converse for the classical capacity of entanglement-breaking and hadamard channels via a sandwiched rényi relative entropy. Communications in Mathematical Physics, 331(2):593–622, 2014.
A  Proof of Lemma 3.1

Proof. Let \{R_i\}, \{E_i\} be respective Kraus operators for \mathcal{R} and \mathcal{E}. That is, \mathcal{R}(\rho) = \sum_k R_k \rho R_k^\dagger and similarly for \mathcal{E}. Then we have \(F^2(\psi, \mathcal{R}(\psi)) = \sum_{i,j} |\langle \psi | R_j E_i | \psi \rangle|^2\). Consider the matrix \(X_{ij} \overset{\text{def}}{=} \langle \psi | R_j E_i | \psi \rangle\). By singular-value decomposition, there exist unitaries \(U, V\) with respective entries \(\{u_{k,i}\}_{k,i}, \{v_{l,j}\}_{l,j}\) such that \(Y_{k,l} \overset{\text{def}}{=} \sum_{i,j} X_{ij} u_{k,i} v_{l,j}\) is a diagonal matrix and \(\sum_k |Y_k|^2 = \sum_{i,j} |X_{ij}|^2\).

Let \(E_k = \sum_i u_{k,i} E_i\) and \(R'_l = \sum_j v_{l,j} R_j\) be new Kraus operators for \(\mathcal{R}\) and \(\mathcal{E}\) respectively. Then we have that

\[
F^2(\psi, \mathcal{R}(\psi)) = \sum_k |Y_k|^2 = \sum_k |\langle \psi | R_k E_k' | \psi \rangle|^2
\]

\[
= \sum_k |\text{Tr}(\psi R_k' E_k' \psi)|^2 = \sum_k |\text{Tr}(\psi R_k' \Pi^{1/2} \Pi^{-1/2} E_k' \psi)|^2
\]

(as \(\Pi\) is fully supported in the image of \(\mathcal{E}\))

\[
\leq \sum_k \text{Tr}(\psi \Pi_k^' \Pi_k \psi) \text{Tr}(\Pi_k \Pi_k E_k' E_k)^2
\]

(\text{using Cauchy-Schwartz inequality})

\[
\leq \sqrt{\sum_k |\langle \psi | R_k' \Pi_k \psi \rangle|^2} \sqrt{\sum_k |\langle \psi | E_k' \Pi_k E_k \psi \rangle|^2}
\]

\[
\leq \sqrt{\langle \psi | \mathcal{R}(\Pi^2) | \psi \rangle} \sqrt{\sum_k |\langle \psi | E_k' \Pi_k E_k \psi \rangle|^2}
\]

\[
\leq \sqrt{\langle \psi | \mathcal{R}(\Pi^2) | \psi \rangle} \sqrt{\sum_k |\langle \psi | E_k' \Pi_k E_k \psi \rangle|^2} = \sqrt{\langle \psi | \mathcal{R}(\Pi^2) | \psi \rangle} \sqrt{\langle \psi | \mathcal{R}(\mathcal{E}(\psi)) | \psi \rangle}
\]

We explain the second last inequality, which says that \(\sum_k |\langle \psi | R_k' \Pi_k \psi \rangle|^2 \leq \sum_k \langle \psi | R_k' \Pi_k \psi \rangle\).

Consider

\[
\sum_k |\langle \psi | R_k' \Pi_k \psi \rangle|^2 \leq \sum_{k,l} |\langle \psi | R_k' \Pi_k \psi \rangle|^2 = \sum_{k,l} \langle \psi | R_k' \Pi_k \psi \rangle \langle \psi | R_l' \psi \rangle \langle \Pi_k \psi | \Pi_l \psi \rangle.
\]

Let \(|\phi_k\rangle \overset{\text{def}}{=} \Pi_k \psi\). Observe that \(\sum_l R_l' \langle \psi | R_l' \psi \rangle \leq \sum_l R_l' = \mathbb{I}\), since \(\mathbb{I} - |\psi\rangle \langle \psi|\) is a positive semidefinite operator and hence \(R_l' (\mathbb{I} - |\psi\rangle \langle \psi|) R_l'\) is a positive semidefinite operator. This implies

\[
\sum_k |\langle \psi | R_k' \Pi_k \psi \rangle|^2 \leq \sum_k \langle \phi_k | \sum_l R_l' \langle \psi | R_l' \psi \rangle | \phi_k \rangle \leq \sum_k \langle \phi_k | \phi_k \rangle = \sum_k \langle \psi | R_k' \Pi_k \psi \rangle.
\]

This completes the proof. \(\square\)

B  Proof of Lemma 4.1

Proof. We consider the mapping

\[
|i\rangle \langle j| \rightarrow |i\rangle |j\rangle.
\]
Under this mapping, a matrix \( A = \sum_{i,j} a_{ij} |i \rangle \langle j | \) goes to a ‘vector’ |\( A \rangle \rangle = a_{ij} |i \rangle \langle j | \) and a rank-1 state |\( \phi \rangle \langle \phi | \) goes to |\( \phi \rangle \langle \phi | \phi^* \rangle \rangle. The inner product becomes \( \langle B|A \rangle = \sum_{i,j} b_{ij}^* a_{ij} = \text{Tr}(B^\dagger A) \) which is the usual Hilbert-Schmidt inner product. The channel \( \mathcal{E} \) gets mapped to the matrix \( E = \sum_i E_i \otimes E_i^* \).

The fact that \( I \) is a fixed point of \( \mathcal{E} \) implies that for |\( I \rangle = \sum |i \rangle |i \rangle \), we have \( \mathcal{E}(I) = I \).

Second largest singular value of \( E \) (which we call \( \lambda_2 \)) is the second largest eigenvalue of \( \sqrt{E^\dagger E} \). Let \( P_0 \equiv \frac{1}{d} |I\rangle \langle I| \) be projector onto the vector |\( I\rangle \langle I| \) (it is easy to check that \( P_0 \cdot P_0 = \frac{1}{d} |I\rangle \langle I| I\rangle \langle I| = \frac{1}{d} |I\rangle \langle I| \) and \( P_1 \equiv I - P_0 \) be projector onto subspace orthogonal to |\( I\rangle \langle I| \). Then we have the following relations

\[
P_1 E_i^\dagger E P_1 < \lambda_2^2 P_1, P_0 E_1^\dagger E P_0 = P_0, P_1 E_1^\dagger E P_0 = 0. \tag{2}
\]

Consider the quantity \( \| E^\otimes n \|_2 \) and recall that the optimisation in its definition is achieved by a pure state (Fact 2.1). Let the optimal pure state be |\( \phi \rangle \rangle. We note that the state \( \mathcal{E}^\otimes n (|\phi \rangle \langle \phi|) \) gets mapped to the vector \( E^\otimes n |\phi \rangle \langle \phi| \phi^* \rangle \).

Thus, we have \( \| E^\otimes n \|_2 = \max_\phi \langle \phi | (E_1^\dagger E)^\otimes n |\phi \rangle \langle \phi| \phi^* \rangle \). For a string \( s \in \{0,1\}^n \), define \( P_s \equiv P_{s_0} \otimes P_{s_1} \otimes \ldots P_{s_n} \). This implies

\[
\langle \phi | (E_1^\dagger E)^\otimes n |\phi \rangle \langle \phi| \phi^* \rangle = \sum_{s,s' \in \{0,1\}^n} \langle \phi | (E_1^\dagger E)^\otimes n P_s \langle \phi| P_{s'} |\phi^* \rangle \quad (\text{Resolution of Identity})
\]

\[
= \sum_{s \in \{0,1\}^n} \langle \phi | (E_1^\dagger E)^\otimes n P_s |\phi^* \rangle \quad \text{as } P_1 E_1^\dagger E P_0 = 0\]

\[
\leq \sum_{s \in \{0,1\}^n} \lambda_2^{2|s|} \langle \phi | (E_1^\dagger E)^\otimes n P_s |\phi^* \rangle \quad \text{(Equation 2)}
\]

\[
\leq \sum_{s \in \{0,1\}^n} \lambda_2^{2|s|} \frac{1}{d^{n-|s|}} \langle \phi | (E_1^\dagger E)^\otimes n P_s |\phi^* \rangle \quad (P_1 < I, P_0 = \frac{1}{d} |I\rangle \langle I|)
\]

\[
= \sum_{s \in \{0,1\}^n} \lambda_2^{2|s|} \frac{1}{d^{n-|s|}} \text{Tr}(\otimes_{i:s_i=0} |I\rangle \langle I| \cdot |\phi \rangle \langle \phi|^* \cdot |\phi^* \rangle \langle \phi^*|)
\]

Now fix an \( s \) and let \( J_s \) be set of qudits on which \( s_i = 0 \). Let \( \bar{J}_s \) be rest of the qudits. Let \( |I\rangle_{J_s} \equiv \otimes_{i \in J_s} |I\rangle_i = \sum_{t \in \{1,2,\ldots,d\}} \otimes_{i \in J_s} |t\rangle_{J_s} |t\rangle_{J_s} \) be the maximally entangled unnormalized state on qudits in \( J_s \). Let \( \rho \equiv \text{Tr}_{\bar{J}_s} |\phi \rangle \langle \phi| \). Then

\[
\text{Tr}(\otimes_{i:s_i=0} |I\rangle \langle I| \cdot |\phi \rangle \langle \phi|^* \cdot |\phi^* \rangle \langle \phi^*|) = |I\rangle_{J_s} \rho \otimes \rho^* |I\rangle_{J_s} = \sum_{t,t'} \langle t| \rho \rangle \langle t'| \rho^* |t'\rangle
\]

\[
= \sum_{t,t'} \langle t| \rho \rangle \langle t'| \rho^* |t\rangle = \text{Tr}(\rho^2) < 1.
\]

The last inequality follows since \( \rho \) is a quantum state.

This gives

\[
\| \mathcal{E}^\otimes n \|_2 \leq \sum_{s \in \{0,1\}^n} \lambda_2^{2|s|} \frac{1}{d^{n-|s|}} = \sum_{|s|=0}^{n} \binom{n}{|s|} \lambda_2^{2|s|} \frac{1}{d^{n-|s|}} = (\frac{1}{d} + \lambda_2^2)^n.
\]

\[\square\]