Nonlinear Measure about $l^2$-norm with Application in Synchronization Analysis of Complex Networks via the General Intermittent Control

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Abstract—Based on the nonlinear measure about $l^2$-norm, some simple yet generic criteria are derived by using the general intermittent control, which ensure the exponential stability of dynamic systems. The numerical simulations whose theoretical results are applied to robust synchronization of complex networks demonstrate the effectiveness and feasibility of the proposed technique.

Index Terms—nonlinear measure about $l^2$-norm, the general intermittent control, exponential stability, synchronization; complex networks.

I. INTRODUCTION

Since its origins in the work of Fujisaka and Yamada[1-3], Afraimovich, Verichev, and Rabinovich[4], and Pecora and Carroll[5], the study of synchronization of chaotic systems[6-13] is of great practical significance and has received great interest in recent years. In the above literature, the approach applied to stability analysis is basically the Lyapunov’s method. As we all known, the construction of a proper Lyapunov function usually becomes very skillful, and the Lyapunov’s method does not specifically describe the convergence rate near the equilibrium point of the system. Hence, there is little compatibility among all of the stability criteria obtained so far.

The concept named the nonlinear measure about $l^2$-norm[14-19] has been applied to the investigation of the existence, uniqueness or stability analysis of the equilibrium. Intermittent control[20-24] has been used for a variety of purposes in engineering fields such as manufacturing, transportation, air-quality control and communication. A wide variety of synchronization or stabilization using the periodically intermittent control method has been studied (See [21-28]). Compared with continuous control methods[7-15], intermittent control is more efficient when the system output is measured intermittently rather than continuously. All of intermittent control and impulsive control are belong to switch control. But the intermittent control is different from the impulsive control, because impulsive control is activated only at some isolated moments, namely it is of zero duration, while intermittent control has a nonzero control width. Therefore, it is essential and important to investigate the exponential synchronization of networks with mixed delays by periodically intermittent control.

A special case of such a control law is of the form

$$U(t) = \begin{cases} -k(y(t) - x(t)), & (nT \leq t < nT + \delta), \\ 0, & (nT + \delta \leq t < (n+1)T), \end{cases}$$

where $k$ denotes the control strength, $\delta > 0$ denotes the switching width, and $T$ denotes the control period. The general intermittent controller

$$U(t) = \begin{cases} -k(y(t) - x(t)), & (h(n)T \leq t < h(n)T + \delta), \\ 0, & (h(n)T + \delta \leq t < h(n+1)T), \end{cases}$$

or

$$U(t) = \begin{cases} -k(y(t) - x(t)), & (h(n)T + \delta \leq t < h(n+1)T + \delta), \\ 0, & (h(n+1)T + \delta \leq t < h(n+1)T), \end{cases}$$

is designed, where $h(n)$ is a strictly monotone increasing function on $n$, then the sufficient yet generic criteria for synchronization of complex networks with and without delayed item are obtained.

II. PRELIMINARIES

Let $X$ be a Banach space endowed with the $l^2$-norm $\|x\|$, i.e. $\|x\| = \sqrt{x^T x} = \sqrt{\langle x, x \rangle}$, where $\langle \cdot, \cdot \rangle$ is inner product, and $\Omega$ be a open subset of $X$. We consider the following system:

$$\frac{dx}{dt} = F(x(t)) + G(x(t - \tau)),$$

where $F, G$ are nonlinear operators defined on $\Omega$, and $x(t), x(t - \tau) \in \Omega$, and $\tau$ is a time-delayed positive constant, and $F(0) = G(0) = 0$.

Definition[22,24,30] System (1) is called to be exponentially stable on a neighborhood $\Omega$ of the
equilibrium point, if there exist constants $\mu > 0$, $m > 0$, such that
\[
\|x(t)\| \leq m \exp(-\mu t)\|x_0\| \quad (t > 0),
\] (4)
where $x(t)$ is any solution of (1) initiated from $x(t_0) = x_0$.

**Definition 2** \cite{15,16,18,19} Suppose that $\Omega$ is an open subset of $\mathbb{R}^n$, and $G : \Omega \to \mathbb{R}^n$ is an operator. The constant
\[
m_\Omega (G) = \sup_{x,y \in \Omega} \frac{(G(x) - G(y), x - y)}{\|x - y\|^2}
\]
is called the nonlinear measure of $G$ on $\Omega$ with the $l^2$-norm $\| \cdot \|$.

**Lemma 1** \cite{29} Suppose that $\Omega$ is an open subset of $\mathbb{R}^n$, and $F : \Omega \to \mathbb{R}^n$ is a bounded operator. The function
\[
f(r) = \|F+rI\|_{\Omega} - r\|x-y\|, \quad (r \geq 0, x \in \Omega),
\]
is monotone decreasing function on $r$; thus the limit
\[
\lim_{r \to \infty} f(r) = 0,
\]
here, the operator $F+rI$ mapping every point $x \in \Omega$ denotes $F(x)+rx$.

**Lemma 2** \cite{29} If the operator $G$ in the system (3) satisfies
\[
\|G(x) - G(y)\| \leq l\|x - y\|\]
for any $x, y \in \Omega$, where $l$ is a positive constant. The solutions $x(t), y(t)$, initiated from $x(t_0) = x_0, y(t_0) = y_0$ of the system (3)
satisfy $\|x(t) - y(t)\| \leq \|x_0 - y_0\| \exp\{\lambda(t - t_0)\}$, $\forall t \geq 0,$
where $\lambda = m_\Omega (F) + \exp\{-m_\Omega (F)\|r\|\}$.

**Corollary 1** Let $G(x(t - \tau)) = 0$, $\hat{\lambda} = m_\Omega (F)$ be defined as in Definition 2, then the result similar to Lemma 2 is obtained.

### III. SYNCHRONIZATION VIA GENERAL INTERMITTENT CONTROL AND EXAMPLES

Consider a delayed complex dynamical network consisting of $N$ linearly coupled nonidentical nodes described by
\[
\dot{x}_i(t) = f(x_i(t)) + g(x_i(t - \tau)) + \sum_{j=1}^{N} a_{ij} x_j(t) + u_i(t), i = 1, 2, \ldots, N
\]
where $x_i = (x_{i1}, x_{i2}, \ldots, x_{in})^T \in \mathbb{R}^n$ is the state vector of the $i$th node, $f : \mathbb{R}^n \to \mathbb{R}^n$ are nonlinear vector functions, $u_i(t)$ is the control input of the $i$th node, and $A = (a_{ij})_{N \times N}$ is the coupling configuration matrix representing the coupling strength and the topological structure of the complex networks, in which $a_{ii} > 0$ if there is connection from node $i$ to node $j (i \neq j)$, and is zero, otherwise, and the constraint $a_{ij} = - \sum_{l \neq i} a_{ij} = - \sum_{l \neq j} a_{lj}$, $(i,j=1,2,L, N)$, is set.

A complex network is said to achieve asymptotical synchronization if
\[
x_i(t) = x_j(t) = \tilde{x} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n) = \tilde{x}(t) \quad t \to \infty,
\]
where $(t) \in \mathbb{R}^n$ is a solution of a real target node, satisfying
\[
\dot{\tilde{x}}(t) = f(\tilde{x}(t)) + g(\tilde{x}(t - \tau))
\]
For our synchronization scheme, let us define error vector and control input $u_i(t)$ as follows, respectively:
\[
e_i(t) = x_i(t) - x(t), i = 1, 2, L, N.
\]
When $h(n)$ is a strictly monotone increasing function on $n$ with $h(0) = 0, \lim_{n \to +\infty} h(n) = +\infty,$
\[
u_i(t) = \begin{cases} -k_{e_i} \delta_i, & (h(n)T \leq t < h(n)T + \delta_i), \\ 0, & (h(n)T + \delta_i \leq t < h(i + 1)T), \end{cases} \quad (k > 0, i = 1, 2, L, N)
\]
When $h(n)$ is a strictly monotone decreasing function on $n$ with $h(0) = +\infty$, $\lim_{n \to +\infty} h(n) = 0,$
\[
u_i(t) = \begin{cases} -k_{e_i} \delta_i, & (h(n)T \leq t < h(n)T + \delta_i), \\ 0, & (h(n)T + \delta_i \leq t < h(i + 1)T), \end{cases} \quad (k > 0, i = 1, 2, L, N)
\]
In this work, the goal is to design suitable function $h(n)$ and parameters $\delta, T$ and $k$ satisfying the condition (7). The error system follows from the expression
\[
\begin{align*}
\dot{e}_i(t) + f(x_i(t)) - f(\tilde{x}(t)) + g(x_i(t - \tau)) - g(\tilde{x}(t - \tau)) + \sum_{l \neq i} a_{il} e_l(t) + u_i(t), \\
\dot{e}_i(t) + f(x_i(t)) - f(\tilde{x}(t)) + g(x_i(t - \tau)) - g(\tilde{x}(t - \tau)) + \sum_{l \neq i} a_{il} e_l(t) + u_i(t), \\
\vdots
\end{align*}
\]
When $h(n)$ is a strictly monotone increasing function on $n$ with $h(0) = 0$, $\lim_{n \to +\infty} h(n) = +\infty$, we obtain the following result:

**Theorem 1** Suppose that the operator $g$ in the network (6) satisfies condition (5), and $m_\Omega$ is defined as Definition 2, $\hat{\lambda} = m_\Omega (F) + \exp\{-m_\Omega (F)\|r\|\}$, where
\[
m_\Omega (F) = \sup_{x \in \Omega} \frac{\|F(x) - e(t)\|}{\|e(t)\|^2},
\]
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\[ F(t) = ((f(x_i(t)) - 2(s(t)) + \sum_{j=1}^{N} a_{ij}e_j(t))^T, \]

\[ e(t) \in (e_i(t), e_j(t), L, e_{ij}(t))^T, \]

where \( a_{ij} \) and \( L \) are the same as Theorem 3. Then the synchronization of networks (6) is achieved if the parameters \( \delta, T \) and \( k, \eta \) satisfy

\[ \text{inf}(\rho + \lambda) \frac{h^{-1}(t-\delta / T) - \lambda}{t} \geq \eta > 0, \]

(9)

where \( \rho = k - \lambda > 0 \), \( h^{-1}(\rho) \) is the inverse function of the function \( h(\rho) \).

**Proof** From Lemma 2, the conclusion is valid:

\[ \|e(t)\| \leq \|e(h(n)T)\| \exp\{-\lambda(t-h(n)T)\} \]

(10)

for any \( h(n)T \leq t < h(n+1)T + \delta \);

\[ \|e(t)\| \leq \|e(h(n)T + \delta)\| \exp\{-\lambda(t-h(n)T - \delta)\} \]

(11)

for any \( h(n)T + \delta \leq t < h(n+1)T \).

We obtain the result similar to Theorem 1 if the condition (9) or (12), respectively, is satisfied. Then the synchronization of the network (6) is achieved if the parameters \( \delta, T \) and \( k, \eta \) satisfy

\[ (\rho + \lambda) \frac{p_{ij}}{p_i} - \lambda \geq \eta > 0, \]

(13)

**Example 1** Consider a delayed Hopfield neural network\(^{[1,52]}\) with two neurons:

\[ \dot{x}(t) = -Cx(t) + DF(x(t)) + BF(x(t - t)), \]

(14)

where \( x(t) = (x_1(t), x_2(t))^T, f(x(t)) = (\tanh(x_1(t)), \tanh(x_2(t)))^T, \]

\[ C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \]

\[ D = \begin{bmatrix} 2.0 & -0.1 \\ -5.0 & 3.0 \end{bmatrix}, \]

\[ B = \begin{bmatrix} -1.5 & -0.1 \\ -0.2 & -2.5 \end{bmatrix}. \]

It should be noted that the network is actually a chaotic delayed Hopfield neural network.

We reach the value 1 < 9.15, \( m_2(F) \leq 0.7993 \), here \( F(x(t)) = -Cx(t) + DF(x(t)) + BF(x(t - t)) \), the function \( h(n) = n^2 / (n+1) \), \( h(n) = 0.3 / n \), which are the strictly monotone increasing or decreasing restriction on \( n \), respectively, then they can be clearly seen that the synchronization of network (6) is realized in Fig.1. Fig.2 (A=0), where \( A = \text{diag}(0.2P, 0.2P) \), and Fig.3 . Fig.4 (A=0), respectively.
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Figure 1. Synchronization error $x_i - x_{i1}$, $x_2 - x_{20}, (i=2,3,L,40)$ when $h(n) = n^2 / (n+1), A \neq 0$.

Figure 2. Synchronization error $x_i - x_{i1}$, $x_2 - x_{20}, (i=2,3,L,40)$ when $h(n) = 0.3 / n, A \neq 0$.

Figure 3. Synchronization error $x_i - x_{i1}$, $x_2 - x_{20}, (i=2,3,L,40)$ when $h(n) = n^2 / (n+1), A = 0$.

Figure 4. Synchronization error $x_i - x_{i1}$, $x_2 - x_{20}, (i=2,3,L,40)$ when $h(n) = 0.3 / n, A = 0$.

Example 2 Consider hyper-chaotic Chen system[33]:

\[
\begin{align*}
\dot{x} &= 35(y-x) + w, \\
\dot{y} &= 7x - xz + 12y, \\
\dot{z} &= xy - 3z, \\
w &= yz + 0.5w.
\end{align*}
\]

We reach the value $m_{44}(F) \leq 5.0304$, here $F(t) = (35(y-x) + w, 7x - xz + 12y, xy - 3z, yz + 0.5w)^T$.

We choose the function $h(n) = \ln(n+1)$, $h(n) = 0.3 / n$, which are strictly monotone increasing or decreasing function on $n$, respectively, then they can be clearly seen that the synchronization of network (6) is realized in Fig.5, Fig.6 (A\neq0), where

\[
A = \begin{pmatrix}
0.2(P + Q) & 0.2(P - Q) \\
0.2(P - Q) & 0.2Q
\end{pmatrix},
\]

and Fig.7, Fig.8 (A=0), respectively.

Figure 5. Synchronization error $x_i - x_{i1}, (i=2,3,L,40), j=1,2,3,4)$ when $h(n) = \ln(n+1), A \neq 0$.

Figure 6. Synchronization error $x_i - x_{i1}, (i=2,3,L,40), j=1,2,3,4)$ when $h(n) = 0.3 / n, A \neq 0$.

Figure 7. Synchronization error $x_i - x_{i1}, (i=2,3,L,40), j=1,2,3,4)$ when $h(n) = \ln(n+1), A = 0$.

Figure 8. Synchronization error $x_i - x_{i1}, (i=2,3,L,40), j=1,2,3,4)$ when $h(n) = 0.3 / n, A=0$.
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IV. CONCLUSION

Approaches for synchronization of complex networks via general intermittent which use the nonlinear operator named the measure about L^2-norm have been presented in this paper. Strong properties of global and exponential synchronization have been achieved in a finite number of steps. The techniques have been successfully applied to Chaotic delayed Hopfield neural networks and hyper-chaotic Chen system. Numerical simulations have verified the effectiveness of the method.

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