Smooth phase transitions in the early universe

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Abstract

We apply a quadratic teleparallel torsion scalar of the \( f(T) = T + \alpha T^2 \) field equations to the flat FRW model. We assume two perfect fluid components, the matter component has a fixed equation of state (EoS), while the torsion component has a dynamical EoS. We obtain an effective scale factor allowing an early stage of an inflationary vacuum, while it pushes the inflation to end naturally at later stage turning the universe to a matter dominant phase. The model represents an alternative to inflation models. However, it does not require slow roll conditions to end the inflation phase. We perform a standard cosmological study to examine the cosmic evolution. In addition, we study the effective EoS. The study shows consistent results confirming a smooth phase transition from inflation to matter dominant universe. We consider the case when the torsion is made of a scalar field. This treatment enables us to study the quadratic effect of \( f(T) \) on the potential patterns of the scalar field. At the limit \( \alpha \rightarrow 0 \) the potential is dominated by the kinetic term and coincides with the quadratic inflation. Both cosmological and scalar field analysis show consistent results.

1 Introduction

The formulation of the general relativity (GR) theory within the Riemannian geometry is powered by the Levi-Civita connection which gives only the attraction gravity. Nevertheless, other geometries with different qualities may give the repulsive side of gravity [34]. It is known that Levi-Civita connection plays the role of the displacement field in the GR, so we expect different qualities when using Weitzenböck connection of the teleparallel geometry. The use of this geometry has been started by Einstein in order to unify gravity and electromagnetism [6]. Although, this trial did not succeed, the geometrical structure has been developed later [24, 23]. Interestingly, some developments to the telleparallel geometry have been done to install a Finslerian properties to this geometry [36, 33, 41]. Also, it worths to mention the developments of the telleparallel geometry attempting the global approach using arbitrary moving frames instead of the local expressions in the natural basis [40, 39, 41]. Even the first trials of using this geometry to obtain a gauge field theory of gravity have shown a great interest [31, 21, 30, 32, 16]. However, extensions to Einstein’s work allowed a class of theories to appear independently taking the Lagrangian density to be second order in the torsion tensor [25, 14]. Recently, a new class of modified gravity theories which have received an attention are the \( f(T) \) gravity [3, 22]. Recent applications of the \( f(T) \) in cosmology show an interesting results. For example, avoiding the big bang singularity by presenting a bouncing solution [4, 5]. Also, \( f(T) \) cosmology provides an alternative tool to study inflationary models [9, 10, 3, 11, 1, 2, 20, 13, 26, 35, 7, 8].

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Moreover, the problems in \( f(T) \) theories are discussed [27, 19]. In this work we mainly study the quadratic \( f(T) = T + \alpha T^2 \) on the flat FRW models and possible cosmological constraints on the value of the coefficient \( \alpha \). Actually, a similar study to examine the quadratic effect (\( \alpha T^2 \)) has been performed within the solar system range [18]. So the work is organized as follows: In Section 1, we briefly review the teleparallel geometry and the \( f(T) \) gravity. In Section 2, we apply the quadratic \( f(T) \) on the FRW models. In Section 3, we show an extensive study of the obtained solution. In Section 4, we use the results to perform a cosmological study of the model. In Section 5, we consider the case when the torsion potential is made of a scalar field to study the consequences on the cosmic evolution. The work is summarized in Section 6.

1.1 Teleparallel spacetime

This space is described as a pair \((M, h_i)\), where \( M \) is an \( n \)-dimensional smooth manifold and \( h_i \) (\( i = 1, \ldots, n \)) are \( n \) independent vector fields defined globally on \( M \). The vector fields \( h_i \) are called the parallelization vector fields. In the four dimensional manifold the parallelization vector fields are called the tetrad field. Where the covariant derivative of the parallelization vector field should vanish

\[
D_\nu h_i^\mu := \partial_\nu h_i^\mu + \Gamma^\mu_{\lambda \nu} h_i^\lambda = 0, \tag{1.1}
\]

where \( D \) is the covariant derivative operator, \( \partial_\nu = \frac{\partial}{\partial x^\nu} \) and \( \Gamma^\mu_{\lambda \nu} \) define the nonsymmetric affine connection [37].

\[
\Gamma^\lambda_{\mu \nu} := h_i^\lambda \partial_\nu h_i^\mu. \tag{1.2}
\]

We assume a four dimensional manifold, the metric tensor \( g_{\mu \nu} \) is defined by

\[
g_{\mu \nu} := \eta_{ij} h_i^\mu h_j^\nu, \tag{1.3}
\]

where \( \eta_{ij} = (+, -, -, -) \) is the metric of Minkowski spacetime. It can be shown that the metricity condition is fulfilled as a consequence of equation (1.1). Interestingly, the connection (1.2) has a vanishing curvature tensor \( R \) identically but a non vanishing Torsion tensor \( T \). While the vanishing of the torsion tensor implies the space to be Minkowskian. We note that, the tetrad field \( h_i^\mu \) determines a unique metric \( g_{\mu \nu} \), while the inverse is incorrect. The torsion \( T \) and the contortion \( K \) tensor fields are

\[
T^\alpha_{\mu \nu} := \Gamma^\alpha_{\nu \mu} - \Gamma^\alpha_{\mu \nu} = h_i^\alpha \left( \partial_\mu h_i^\nu - \partial_\nu h_i^\mu \right),
\]

\[
K^{\mu \nu \lambda} := -\frac{1}{2} \left( T^{\mu \nu \lambda} - T^{\nu \mu \lambda} - T^{\lambda \mu \nu} \right). \tag{1.4}
\]

We introduce the teleparallel torsion scalar which reproduces the teleparallel equivalent to general relativity (TEGR) theory as

\[
T := T^\alpha_{\mu \nu} S_\alpha^{\mu \nu}, \tag{1.5}
\]

where the tensor \( S \) of type \((2, 1)\) is defined as

\[
S_\alpha^{\mu \nu} := \frac{1}{2} \left( K^{\mu \nu \alpha} + \delta^{\mu \nu} T^{\beta \alpha \beta} - \delta^{\nu \mu} T^{\beta \alpha \beta} \right), \tag{1.6}
\]

where \( S \) is skew symmetric in the last two indices.

Dealing with the connection coefficients as displacement fields in the \( f(T) \) theories might lead to new physical insight of these theories. We use (1.4) to reexpress the Weitzenböck connection (1.2) as

\[
\Gamma^\mu_{\nu \rho} = \{\mu \}_{\nu \rho} + K^{\mu}_{\nu \rho}. \tag{1.7}
\]

The first is the Levi-Civita connection of the GR theory, this connection is defined by the metric \( g_{\mu \nu} \) (gravitational potential) and its first derivatives with respect to the coordinates. This connection contributes as the attractive
gravity in the GR. While the second term is made of the contortion (torsion) which consists of the tetrad vector fields \( h_{\mu}^\nu \) and its first derivatives with respect to the coordinates. Similarly, we can think of the second term as a force of gravity. Actually it contribution to the modified geodesics indicates a repulsive gravity [34]. The value added quality of the teleparallel space is its capability to describe two faces of gravity, attractive and repulsive. So torsion gravity can provide a unique source to explain early and late phases of the cosmic accelerating expansion [26].

1.2 \( f(T) \) field equations

Similar to the \( f(R) \) theory one can defines the action of \( f(T) \) theory as

\[
\mathcal{L}(h_{\mu}^\nu, \Phi_A) = \int d^4x \ h \left[ \frac{\mathcal{M}_n}{2} f(T) + \mathcal{L}_{\text{Matter}}(\Phi_A) \right],
\]

where \( \mathcal{M}_n \) is the reduced Planck mass, which is related to the gravitational constant \( G \) by \( \mathcal{M}_n = \sqrt{8\pi G/c^2} \). Assuming the units in which \( G = c = \hbar = 1 \), in the above equation \( h = \sqrt{-\gamma} = \det (h_{\mu}^\nu) \), \( \Phi_A \) are the matter fields. The variation of (1.8) with respect to the field \( h_{\mu}^\nu \) requires the following field equations [3]

\[
S_{\mu}^{\nu\rho} \partial_\rho T f_{TT} + \left[ h^{-1} h_{\mu}^\rho \partial_\rho (h h_{\nu}^\sigma S_{\sigma}^{\mu}) - T_{\alpha\lambda\mu} S_{\alpha}^{\nu} \right] f_{T} - \frac{1}{4} \delta^\nu_{\mu} f = -4\pi T_{\mu}^{\nu},
\]

where \( f = f(T), f_T = \frac{\partial f(T)}{\partial T}, f_{TT} = \frac{\partial^2 f(T)}{\partial T^2} \).

2 \( f(T) \) Cosmological modifications

We apply the \( f(T) \) field equations (1.9) to the FRW universe of a spatially homogeneous and isotropic spacetime, which directly gives rise to the tetrad given by Robertson [28]. This can be written in spherical polar coordinate \((t, r, \theta, \phi)\) as follows:

\[
(h_{\mu}^\nu) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & L_1 \sin \theta \cos \phi & 4a(t) & 1 \\
0 & L_2 \cos \theta \sin \phi & 4r(t) \sin \theta & 0 \\
0 & 0 & -L_2 \sin \theta & \sqrt{r(t)}
\end{pmatrix},
\]

where \( L_1 = 4 + k r^2 \) and \( L_2 = 4 - k r^2 \) and \( a(t) \) is the scale factor. The EoS is taken for an isotropic fluid so that the energy-momentum tensor is \( T_{\mu}^{\nu} = \text{diag}(\rho, -p, -p, -p) \). The tetrad (2.1) has the same metric as FRW metric. Substituting from (2.1) into (1.5) we evaluate the torsion scalar as

\[
T = -6H^2(1 + \Omega_k),
\]

where \( H(t) := \frac{\dot{a}(t)}{a(t)} \) is the Hubble parameter and \( \Omega_k := \frac{k}{a^2 H^2} \) is the curvature density parameter. The \( f(T) \) field equations (1.9) read

\[
3H^2 = 8\pi(\rho + \rho_T) - 3\frac{k}{a^2},
\]
\[
3qH^2 = -4\pi [(\rho + \rho_T) + 3(p + p_T)],
\]

where the matter contributes to the total density and pressure as [26]

\[
\rho = \frac{1}{16\pi} (f + 12H^2 f_T),
\]
\[
p = -\frac{1}{16\pi} \left[ (f + 12H^2 f_T) + 4\dot{H}(f_T - 12H^2 f_{TT}) - \frac{4k}{a^2} (f_T + 12H^2 f_{TT}) \right],
\]
and the torsion contributes to the total density and pressure as

$$\rho_T = \frac{1}{8\pi} \left( 3H^2 - f/2 - 6H^2 f_T + \frac{3k}{a^2} \right),$$  \hspace{0.5cm} (2.7)

$$p_T = \frac{-1}{8\pi} \left[ k \frac{1}{a^2} (1 + 2f_T + 24H^2 f_{TT}) + 2\dot{H} + 3H^2 - f/2 - 2(\ddot{H} + 3H^2)f_T + 24\dot{H} H^2 f_{TT} \right].$$  \hspace{0.5cm} (2.8)

### 3 Smooth transitions universe

In order to examine possible higher orders of gravity in the $f(T)$ theory, we take a quadratic polynomial form of the teleparallel torsion scalar $T$ as below

$$f(T) = T + \alpha T^2,$$  \hspace{0.5cm} (3.1)

where $\alpha$ is a constant. It is clear that this form produces TEGR theory when $\alpha = 0$. As the $f(T)$ function in the FRW spacetime is a function of time $f(T \rightarrow t)$, one easily can show that

$$f_T = \dot{f}/\dot{T}, \quad f_{TT} = \left( \ddot{T} \dot{f} - \dot{T} \ddot{f} \right) / \dot{T}^3.$$  \hspace{0.5cm} (3.2)

For the flat space universe ($k = 0$) the equations (2.5)-(2.8) now read

$$\rho = \frac{3}{8\pi} a^2 \left( a^2 - 18\alpha \dot{a}^2 \right),$$  \hspace{0.5cm} (3.3)

$$p = -\frac{1}{8\pi} a^2 \ddot{a}^2 + 2\ddot{a} \dot{a}^3 + \alpha (18\dot{a}^4 - 72\dot{a}^2 \ddot{a}) a^4.$$  \hspace{0.5cm} (3.4)

The matter density and pressure reduces to the GR results when $\alpha = 0$. However, the torsion contributes to the total density and pressure as

$$\rho_T = \frac{27\alpha \dot{a}^4}{4\pi a^4},$$  \hspace{0.5cm} (3.5)

$$p_T = -\frac{9\alpha \ddot{a}^2 (4\ddot{a} \dot{a} - \dot{a}^2)}{4\pi a^4}.$$  \hspace{0.5cm} (3.6)

The torsion contribution vanishes where $\alpha = 0$, so the theory generally produces the GR theory.

#### 3.1 Inflation with no end

Substituting from (3.3) and (3.4) into the continuity equation of the matter

$$\dot{\rho} + 3H(\rho + p) = 0.$$  \hspace{0.5cm} (3.7)

We study the case when the matter is a cosmological constant dark energy (ADE), i.e. the EoS parameter $\omega = -1$. Solving for the scale factor $a(t)$ we get

$$a_{inf}(t) = a_0 e^{H_0(t-t_0)},$$  \hspace{0.5cm} (3.8)

where $a_0 := a(t_0) \text{ and } H_0 := H(t_0)$. Consequently, we can evaluate

$$T = -\frac{1}{6\alpha}, \quad H = \frac{1}{6\sqrt{\alpha}}, \quad q = -1, \quad \omega_T = -1,$$  \hspace{0.5cm} (3.9)

where $q := -\frac{\ddot{a}}{\dot{a}^2}$ is the deceleration parameter. The solution gives an exponential expanding universe with a constant Hubble parameter, i.e. de Sitter universe. So equivalently we have a vacuum dominant universe which
powered the inflation scenario. Recalling (3.1), we find the coefficient \( \alpha \) of the \( T^2 \) higher order gravity contributes to this inflationary phase. The comparison of the results of (3.9) with the de Sitter solution directly relates the coefficient \( \alpha \) to the cosmological constant \( \Lambda \) by \( \alpha = \frac{1}{12\Lambda} \). It is well known that at this early stage the cosmological constant has a value much larger than its present value. The discrepancy between the two values is about 120 orders of magnitude, this is called the cosmological constant problem. According to our analysis here; we expect the value of \( \Lambda \) is very small \( \sim 10^{-73} \) s\(^2\), so that the quadratic contribution in the \( f(T) \) can be ignored. We next examine the second-fluid (torsion) component. Substituting from (3.8) into (3.5) and (3.6) we find that the torsion fluid satisfies the continuity equation as

\[ \dot{\rho}_T + 3H(\rho_T + p_T) \equiv 0, \]  

(3.10)

so we find that the total energy density satisfies the continuity equation. This case shows that both the matter and the torsion fluids having the same behaviour where \( \rho = \rho_T = \frac{1}{T^2c^2} = \frac{1}{10^3} \), which is the density of vacuum. Also, the EoS parameters has been obtained as \( \omega = \omega_T = -1 \). This case gives directly a de Sitter universe, where both matter and torsion act as a vacuum state in an indistinguishable behaviour. Using (3.1) and (3.9) we get \( f(T) = -\frac{3\pi}{Tm} \sim T \). Interestingly, we find that the solution omits the second order contribution of the \( f(T) \) naturally and the \( f(T) \) is a constant, more precisely it acts just like the cosmological constant. This may explain why the two-fluid components have the same behaviour. We saw that the choice of \( \omega = -1 \) of the matter fluid constrains the torsion fluid to acquire the same behaviour producing a pure vacuum universe which represents the key of the inflationary universe. Although the model does not address the cosmological constant issue, it succeeded to predict the exponential cosmic inflation. But it does not provide a mechanism to end the inflation period and resuming the big bang to radiation dominant universe era.

### 3.2 From inflation to matter without slow-roll

Similarly, we take the second case when the EoS parameter of the matter fluid \( \omega \neq -1 \). The asymptotic solution of the continuity equation of matter (3.7) up to \( O(t^4) \) provides the following scale factor

\[
a_{\text{eff}}(t) = e^{\frac{2}{3}(1 + \omega)t} \frac{\epsilon}{c_2} \left( \frac{2c_1(1 + \omega)^2 + (1 + \omega)^2c_1^2 - 8\alpha}{1 + \omega + 2c_1(1 + \omega)^2} \right),
\]  

(3.11)

where \( c_1 \) and \( c_2 \) are constants of integration. The effective scale factor can be decomposed as given in (3.11), where the expression \( t^{\frac{2}{3}(1 + \omega)} \) represents the matter dominant epochs, it takes the values of \( t^{1/2} \) and \( t^{2/3} \) for radiation (\( \omega = 1/3 \)) and dust (\( \omega = 0 \)) states, respectively. While the other expression in (3.11) which is given by an exponential function of time represents an inflationary (vacuum dominant) epoch. So we take \( a_m \) and \( a_v \) in (3.11) to denote the scale factor of matter and vacuum dominant epochs, respectively. In order to describe how the universe evolves according to the scale factor (3.11), we concisely determine the behaviour of the scale factor (3.11) when \( t \to 0 \) and its asymptotic behaviour when \( t \to \infty \). So we get

\[
\lim_{t\to 0} a_m = 0, \quad \lim_{t\to \infty} a_v = 1.
\]

The above limits show that the scale factor is dominated by \( a_v \) where \( a_m \) is negligible at the very early universe time; then the \( a_v \) decays to unity while the effect of \( a_m \) becomes dominant at later phase. So we say that the scale factor starts with an inflationary universe with an exponential scale factor decays smoothly to new phase of matter dominant universe. The plots in Figure 1 show the different evolution patterns of the matter and the vacuum states providing a clear phase transition from an inflation epoch to matter dominant universe. For example, in the inflation to radiation transition plot we see that three different stages. These stages can be summarized as follows:

(i) The first stage at the early time shows that both matter and vacuum having similar behaviour. Although the effective scale factor (3.11) matches the steeper evolution of the vacuum state, so the vacuum dominates radiation at this pre-stage.
Figure 1: Smooth phase transitions in the early universe. The left plot shows a phase transition from inflation to radiation, the dash line shows the behaviour of $\alpha_v$ of the vacuum, the vertical dash line indicates an inflation period then it bends right to end the inflation phase and entering a steady universe phase. The dot line shows the behaviour of $a_m$ when the matter has been chosen as a radiation, i.e. $\omega = 1/3$. The solid line represents the effective scale factor behaviour, its plot initially matches the vacuum phase at the left lower corner, while it matches the radiation phase at latest stage at the right upper corner. The intermediate part of the plot indicates the phase transition from vacuum to radiation. The right plot has the same pattern but with a transition from vacuum to dust universe, i.e. $\omega = 0$. The constants have been chosen as $c_1 = c_2 = 10^{-5}$.

(ii) The second stage shows that the vacuum evolves much faster than radiation matter providing an inflationary phase; then the vacuum shows a steady expansion ending the inflation period, while the radiation evolves similar to the first stage. By the end of this intermediate stage the two components follow different tracks. The radiation energy decreases as the frequency $\nu \propto a(t)^{−1}$.

$$\frac{a_{\text{end}}}{a_{\text{inf}}} = \frac{\nu_{\text{inf}}}{\nu_{\text{end}}} = \frac{E_{\text{inf}}}{E_{\text{end}}}$$

where $E$ is the energy of the radiation, the subscripts “inf” and “end” denote the beginning and the end of the second stage (inflation phase), respectively. Nevertheless, the vacuum approaches a critical value $\alpha_v \rightarrow 1$ whereas the vacuum has no more repulsive energy to loose during the supercooling process of the inflation. Now the vacuum switch the inflation off as given by the plots of Figure 1.

(iii) In the last stage the radiation evolves much faster than vacuum. The flat plateau of the vacuum in Figure 1 indicates that the vacuum enters a phase transition epoch where the variation of its scale factor vanishes, i.e. $\delta a_v = 0$. The peculiar pattern of the vacuum predicts a constant vacuum density. The vacuum background in its final stage releases its latent energy allowing a new phase to reheat the radiation to produce a matter dominate universe. This is consistent with the late behaviour the effective scale factor that matches perfectly the radiation dominant.

It is well known that the inflation has provided an add-on tool to solve the big bang cosmology problems. In quantum cosmology, the inflationary universe is fairly understood using the scalar field description. But the slow-roll conditions are required to end this inflationary phase. However, this model provides a satisfactory mechanism to enforce the vacuum to end its inflation and enters a phase transition state releasing its latent energy to allow the radiation matter to dominate the universe. So the model provides an alternative mechanism to the turn the universe from an inflationary phase to radiation dominant phase naturally without imposing slow-roll conditions.
The left and right plots show the evolution of the Hubble and the deceleration parameters, respectively. Dot and dash lines denote that the EoS parameter is chosen to describe radiation ($\omega = 1/3$) and dust ($\omega = 0$) matter, respectively. The constants have the same values as in Figure 1.

4 Cosmological study

The standard cosmology gives a set of parameters to study the evolution of the universe through the Hubble and the deceleration parameters, while the other parameters are called density parameters allow studying the composition of the universe. We study some of these parameters correspond to the effective scale factor (3.11) and the assumed two-fluids which are given by the set of equations (3.3)-(3.6). So the Hubble parameter reads

$$H = \frac{2(1 + \omega)^2(t^2 - c_1 t + c_1^2) + 16\alpha}{(1 + \omega)^{3/2}}.$$  (4.1)

One can see that the Hubble parameters counts a large value at the very early time, while it decays at later time, $\lim_{t \to \infty} H = 0$. Interestingly, the above expression shows a time dependent Hubble parameter capable to describe an inflationary and a later matter dominant phases. Also, the deceleration parameter shows a consistent results with the above description. We evaluate the deceleration parameters at both early and late phases as

$$\lim_{t \to 0} q = -1, \quad \lim_{t \to \infty} q = \frac{3}{2}\omega + \frac{1}{2}.$$  (4.2)

This again indicates an accelerating expansion phase as $q < 0$ at the early time, while the universe turns to a decelerating phase at later phase of the matter dominant universe. The asymptotic behaviour is consistent with the known results as $q \to 1/2$ or 1 when the EoS of the matter are chosen as ($\omega = 0$) for dust or ($\omega = 1/3$) for radiation, respectively. Also, equation (4.2) shows that the deceleration $q \to -1$ at all time when the matter is chosen as ADE ($\omega \to -1$). This covers the case which has been studied in Subsection (3.1). A more detailed description can be seen from the plots of Figure 2. Substituting from (4.1) into (2.2), noting that $\Omega_k = 0$ of the flat space, we evaluate the teleparallel torsion scalar as

$$T = -\frac{8}{3}(1 + \omega)^4(t^2 - c_1 t + c_1^2)^2 \frac{(1 + \omega)^{3/2}}{(1 + \omega)^6 t^6}.$$  (4.3)

It is well known that the inflationary models assume a scalar field (inflaton) powering this epoch; then it decays rapidly providing a reheating phase allows the pair production to occur. On the other hand, the torsion tensor field plays the main role in the teleparallel spacetime, i.e. the vanishing of the torsion implies an annihilation of all the...
Figure 3: The plots from left to right show the evolution of the teleparallel torsion scalar field, the torsion fluid EoS and the effective EoS parameters, respectively. Dot and dash lines denote that the EoS parameter is chosen to describe radiation ($\omega = 1/3$) and dust ($\omega = 0$) matter, respectively. The constants have the same values as in Figure 1.

teleparallel geometric structure producing a Minkowskian geometry. Most of the quantum applications require this Minkowskian background, while one cannot accept this assumption easily at the early time when the matter is condensed in a tiny space producing a highly curved background. We see that the teleparallel geometry, powerfully, gives a description of gravity better and more complete (attractive and repulsive) than the Riemannian’s. In addition, the vanishing of the curvature tensor provides a good chance to perform some quantum applications. Interestingly, we show that the teleparallel torsion scalar field of this theory shares the inflaton its rapid decaying character, since

$$\lim_{t \to 0} T = -\infty, \quad \lim_{t \to \infty} T = 0.$$ 

A more detailed description can be seen in Figure 2. The plot shows a rapid decay of the torsion scalar which contributes directly to explain the huge difference between the large cosmological constant at the early universe and its small present value which is known as the cosmological constant problem, for more details see [8]. In the next Section, we analog description using scalar field treatment. This leads to study the torsion fluid as a physical vacuum phase, so we evaluate the EoS parameter of the torsion fluid ($\omega_T := p_T/\rho_T$) at the early and late stages of the universe. Using (3.5) and (3.6) we obtain

$$\lim_{t \to 0} \omega_T = -1, \quad \lim_{t \to \infty} \omega_T = 2\omega + 1. \quad (4.4)$$

The above evaluation indicates that the torsion fluid initially acts as $\Lambda$DE ($\omega_T = -1$) in all cases, crosses the quintessence limit to an ordinary matter with $\omega > 0$ at later stages. We study when the matter is dust and radiation so the asymptotic behaviour of the torsion EoS approaches the values of $\omega_T \to 1$ or $\omega_T \to 5/3$, respectively. Moreover, the equation (4.4) shows that the torsion does not evolve, $\omega \to -1$ as $t \to \infty$, when the second fluid is also $\Lambda$DE. So the torsion may show different evolution according to the type of the associated matter, see Figure 2, which strongly recommends a certain interaction between the two fluid components of the universe. Finally, we study the effective EoS $\omega_{eff} := \frac{p_{eff} + \rho_{eff}}{\rho_{eff}}$ at both early and late time. Using the equations (3.3)-(3.6) we obtain

$$\lim_{t \to 0} \omega_{eff} = -1, \quad \lim_{t \to \infty} \omega_{eff} = \omega \Rightarrow \omega_{eff} : \omega_T \to \omega.$$ 

The above limits show that the effective EoS initially starts as $\Lambda$DE evolves to match perfectly the matter EoS at later stages. So the universe turns ends its inflationary (torsion dominant) phase, smoothly turns itself to matter dominant phase with no need to slow-roll conditions. This is consistent with the results of the previous Section.
5 Torsion potential

We showed that the contortion contribute as a repulsive force of gravity in the modified geodesic equation. This repulsive gravity has the upper hand to power the early or the late cosmic accelerating expansion. On the hand, the scalar field analysis is the most powerful tool to describe the cosmic inflation at the early universe. So we consider the approach that has been purposed by \[38\], by introducing sixteen fields \( t^i_\mu \) that are called “torsion potential”. These fields form a quadruplet basis vectors, so we write the following linear transformation:

\[
  h_i = t^\mu_i \partial_\mu, \quad h^i = t^i_\mu dx^\mu, \]

the torsion potential \( t^i_\mu \) and its inverse satisfy

\[
  t = \det(t^i_\mu) \neq 0, \quad t^\mu_i t^j_\nu = \delta^\mu_j \delta^i_\nu. \]

Then the torsion can be written as \[38\]

\[
  T^\alpha_{\mu\nu} = t^\alpha_i (\partial_\nu t^i_\mu - \partial_\mu t^i_\nu). \tag{5.1}
\]

Generally, the torsion potential \( t^i_\mu \) can be reformed by a physical scalar, vector or tensor fields. In the previous sections we show that the torsion gravity has derived the cosmic evolution from inflation to matter dominant universe. However, the cosmic inflation is powered by a spin-0 matter, we assume the case when the torsion potential is constructed by a scalar field \( \varphi(x) \) with a Minkowiskian background in order to relate to reformulate the the \( f(T) \) results using a scalar field, then we take

\[
  t^\mu_i = \delta^\mu_i \varphi, \quad t^i_\mu = \delta^i_\mu \varphi^{-1},
\]

where \( \varphi \) is a non-vanishing scalar field. Then the torsion is expressed as

\[
  T^\alpha_{\mu\nu} = \delta^\alpha_i \varphi_{,\mu} - \delta^\alpha_i \varphi_{,\nu}, \quad K^\alpha_{\mu\nu} = \eta^{\alpha\beta} \left( \eta_{\mu\beta \nu} - \eta_{\mu\nu \beta} \right). \tag{5.2}
\]

Actually, this form of the torsion tensor has been used successfully to make the torsion satisfy the gauge invariance and the minimal coupling principles. This helped to get a dynamical torsion gauge invariant and minimally coupled electromagnetism \[29, 17, 12, 15\]. Using the above equations and (1.6), the teleparallel torsion scalar (1.5) can be written in terms of the scalar field \( \varphi \) as

\[
  T = -9 \varphi_{,\mu} \varphi^\mu. \tag{5.3}
\]

where \( \varphi^\mu = \eta^{\mu\alpha} \varphi_{,\alpha} \). The above treatment shows that the torsion acquires dynamical properties and it propagates through space. Using equations (4.3) and (5.3); then solving for \( \varphi \). We obtain

\[
  \varphi = \varphi_0 \pm \frac{2 \sqrt{6}}{9(1 + \omega)} \left[ \ln(t) + \frac{4\alpha/(1 + \omega)^2 - c_1(t - c_1/2)}{t^2} \right], \tag{5.4}
\]

where \( \varphi_0 \) is a constant of integration. We first study the leading term of (5.4) by taking

\[
  \varphi = \varphi_0 \pm \frac{2 \sqrt{6}}{9(1 + \omega)} \ln(t), \tag{5.5}
\]

alternatively, we write

\[
  t = e^{\pm \frac{3\sqrt{6}}{4}(1+\omega)(\varphi-\varphi_0)}, \tag{5.6}
\]

We introduce a new scalar field \( \psi := \pm \frac{3\sqrt{6}}{4}(1+\omega)(\varphi-\varphi_0) \) to simplify expressions. Equation (5.6) enables us to investigate the transformation between the teleparallel torsion scalar \( T \) and the scalar \( \psi \) fields. Substituting from (5.6) into (4.3) so

\[
  T(\psi) = -\frac{8e^{\pm 6\psi}}{3(1 + \omega)^6} \left[ 8\alpha + (1 + \omega)^2 \left( e^{\mp 2\psi} - c_1 e^{\mp \psi} + c_1^2 \right) \right]^2,
\]
Figure 4: The potential pattern corresponds to the leading term of the scalar field (5.5). The right plot shows that the false vacuum is separated by a potential barrier at $\psi \sim 3.11$. The right plot shows that the potential slowly rolls to its effective minimum of the true vacuum at $\psi \sim 4.67$ allowing a tunneling event from high energy false vacuum. The constants have been chosen as $\omega = 1/3$, $c_1 = 10^{-5}$ whereas the dash and dot lines correspond to value of $\alpha = 0$, $10^{-4}$, respectively.

As shown in the previous Section that the teleparallel torsion scalar has a decaying performance while the Weitzenböck connection shows a capability to explain the repulsive gravity. On the other hand, it well known that the cosmic inflation is powered by assuming a spinless particles at this early universe stage. Combining this to the above mentioned analysis of the scalar field $\phi$ leads us to reformulate the Friedmann equations of the torsion contribution as an inflationary background in terms of the scalar field $\psi$. So we consider the Lagrangian density of a homogeneous scalar field $\psi$

$$L = \frac{1}{2} \partial_{\mu} \psi \partial^{\mu} \psi - V(\psi). \quad (5.7)$$

Where the kinetic energy is proportional to the teleparallel torsion scalar (5.3), while the potential energy $V(\psi)$ is mainly constructed from the $f(T)$ gravity so that the two components of the Lagrangian density are made of the torsion gravity. The variation of the Lagrangian density with respect to the Minkowisky metric directly reads the scalar field pressure as

$$p_{\psi} = \frac{1}{2} \dot{\psi}^2 - V(\psi). \quad (5.8)$$

The temporal gradient of the scalar field (5.4) is evaluated as

$$\dot{\psi} = \pm \frac{2\sqrt{6}}{9(1 + \omega)t} = \pm \frac{2\sqrt{6}e^{\pm \psi}}{9(1 + \omega)}. \quad (5.9)$$

For simplicity we write all expressions in terms of the scalar field $\psi$. So using (3.11), (3.6) and (5.6), the torsion pressure (3.6) can be reexpressed in terms of the scalar field $\psi$ as

$$p_{\psi} = -\frac{32\alpha}{3\pi} \sum_{n=4}^{12} \beta_n e^{\pm n\psi}, \quad (5.10)$$

where $\beta_n$ are known constant coefficients. Substituting from (5.9) and (5.10) into (5.8), we evaluate the potential of the scalar field $\psi$ as

$$V(\psi) = \frac{4e^{\pm 2\psi}}{27(1 + \omega)^2} + \frac{32\alpha}{3\pi} \sum_{n=4}^{12} \beta_n e^{\pm n\psi}. \quad (5.11)$$
Assuming the associated matter is radiation and choosing the constants $c_1 = 10^{-5}$ and $\alpha = 10^{-4}$, the potential has in general ten critical points with two real values of $\psi$ at $\approx 3.11$ and $\approx 4.67$. This can be seen clearly in the plots of Figure 4. Whereas the false vacuum is separated by a potential barrier. The false vacuum decay is followed by slow roll inflation allowing a tunneling event from the high energy false vacuum. The potential plots of Figure 5 show that the potential of the scalar field has a flat plateau near the false vacuum ($\varphi = 0$) allowing the inflation to occur, while it slowly rolls to its effective minimum of the true vacuum at $\varphi > 0$. It worths to mention that the potential can be rewritten in the form

$$V(\Omega) \propto \Omega^{\pm 2} + \alpha \sum_{n=4}^{12} \Omega^{\pm n},$$

where $\Omega := e^{\pm \psi}$. At the limit $\alpha \to 0$ the kinetic term is dominant, and therefore the potential coincides with quadratic inflation model. This also explains the rushing of the torsion EoS to $\omega_T > 1$, equivalently the effective EoS $\omega_{eff} : -1 \to \omega$, in Figure 3. Similar calculations have been performed including the full expression of the scalar field (5.4). The potential patterns are plotted verses the scalar field $\varphi$ in Figure 5 including non vanishing values of $\alpha$. The plots show a de Sitter inflation pattern when $\alpha = 0$, while the non vanishing $\alpha$ shows how the values of the effective minimum potential vary correspond to different values of $\alpha$. It is interesting to mention that the non vanishing value of $\alpha$ allows the field to oscillate about its minimum potential, so the value of $\alpha$ can be constrained by the energy scale of the reheating process.

6 Summary

We have applied a quadratic $f(T)$ field equations to the flat FRW universe with a perfect fluid. The choice of $f(T) = T + \alpha T^2$ has enabled us to modify the Friedmann equations, whereas the density and pressure of two fluids of matter and torsion reproduce the GR as $\alpha = 0$. We have assumed the matter is governed by the continuity equation.

We have studied two cases of the matter choices:

(i) The matter is assumed to be $\Lambda DE$ with EoS $\omega = -1$. The solution provided an exponential scale factor $a_{inf}(t) = a_0 e^{H_0(t-t_0)}$, which fixes the torsion EoS to a value of $\omega_T = -1$. The standard cosmological study has given Hubble and deceleration parameters as $H = \text{const}$. and $q = -1$. So this model is consistent with $\Lambda$ de Sitter model. Also, we found that the choice of $\omega = -1$ implying that $f(T) \sim T \sim \Lambda$. Consequently,
the coefficient $\alpha$ has been estimated as $\alpha = \frac{1}{12} \sim 10^{-73} \text{ s}^2$. So the model omitted the quadratic term $T^2$ anyways. We found that the model can predict an endless inflationary scenario.

(ii) The matter is assumed to have an EoS $\omega \neq -1$. In particular, we have studied the radiation matter when $\omega = 1/3$. The solution provided a decomposable effective scale factor, $a_{\text{eff}}(t) \propto a_v a_m$, where $a_v$ having an exponential form of time allows an inflation period limited to a specific time interval, whereas $a_m = t^{3(1+\omega)/2}$ matches perfectly the matter dominant universe epoch. So the effective scale factor allows inflation to end without slow roll conditions providing a smooth transition from inflation to matter dominant universe. The extensive scale factor analysis indicates that the vacuum ends the inflation phase then enters an endless phase transition releasing its latent heat. By the end of this stage the effective scale factor now matches perfectly the matter scale factor announcing a matter dominant universe to show up.

We finally applied a special treatment by constructing the torsion tensor from a scalar field $\varphi$ to study the scalar field contribution of the torsion density and pressure. The scalar field analysis shows a contribution of a kinetic energy rushing the effective EoS from $\Lambda$DE to matter dominant universe, i.e. $\omega_{\text{eff}} : \omega_T = -1 \rightarrow \omega$. So the results are consistent with the effective scale factor analysis. Moreover, the decaying behaviour of the teleparallel torsion scalar has been linked to the requirements of the scalar field inflation’s models. We finally addressed that the vanishing of $\alpha$ gives a potential pattern of the old inflation models. The potential plots of the non vanishing values of the coefficient $\alpha$ show a false vacuum decay followed by a slow roll inflation with a tunnel event dragging the vacuum to its effective minimum potential of the true vacuum. The energy scale of the reheating process can constrain the coefficient $\alpha$ to a specific value to quantify the quadratic $f(T)$ gravity contribution to the early universe models.

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