Generalised Smolin states and their properties

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Four qubit bound entangled Smolin states are generalised in a natural way to even number of qubits. They are shown to maximally violate simple correlation Bell inequalities and, as such, to reduce communication complexity, though they do not admit quantum security. They are also shown to serve for remote quantum information concentration as like in the case of the original four qubits. Application of the information concentration to the process of unlocking of classical correlations and quantum entanglement by quantum bit is pointed out.

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I. INTRODUCTION

Quantum entanglement [1, 2] is a very important resource in quantum information theory (QIT) [3]. It contributes to fundamental quantum information phenomena [4] and represents itself the quality that is not present in classical world. Entanglement of pure states have been shown to be incompatible with any local hidden models since it violates well-known Bell inequalities [5]. It has been also proved to be an optimal resource for quantum information. The case of mixed states is more complicated. Though mixed states in many cases can serve as a QIT resource it is difficult to characterise useful mixed states entanglement in general. In addition the fundamental question initiated in [6] namely which entangled mixed states admit local hidden variable theories remains still open. The very interesting type of entanglement of that serves as an ideal probe for the above analysis is bound entanglement (BE) [7] that can not be distilled to pure entangled form, nevertheless turns out to be useful in some quantum QIT tasks [15, 16, 17, 18, 19, 20]. On the other hand recently, Smolin bound entangled states [21] a few multipartite bound entangled states [22, 23] including especially the case of 6 qubits [24]. Note that this means that BE can serve for reduction of communication complexity in wide class of schemes provided in [25, 26]. The scenario with minimal number of particles \( N = 6 \) required continuous setting Bell inequalities that can not be implemented experimentally. Also no maximal violation of Bell inequalities have been reported for analysed states.

Quite recently, however, Smolin bound entangled states [27] representing for qubit density matrices have been reported [28] to violate Bell inequalities maximally in a very simple setting (similar to CHSH [29] scenario). At the same time the states do not admit multiparty cryptography scenario which means that even maximal violation of Bell inequalities does not imply quantum security if all the parties are in distant labs.

In the present work we generalise Smolin states to any even number of particles calling new states generalised Smolin States (GSS). We show that they maximally violate Bell inequalities as it was in the case of four qubits.

As such they can reduce communication complexity. Still it can be shown, as in four-qubit case that is spite of maximal Bell violation, the states are not useful for quantum security. On the other hand we show that GSS - like the original Smolin states (see [12]) - allow for remote quantum information concentration. Quantum network realising the Smolin states is also designed. Finally we discuss the relation of Bell inequality violation and quantum security. We find a possibility of interesting application of the result of information concentration as an unlocking of large amount of classical information.

II. GENERALIZED SMOLIN STATES

A. Construction

In this section we extend the last developments concerning bound entanglement in context of Bell inequalities [28] to the case of arbitrary even number of particles.

In the very beginning let us define the following class of unitary operations

\[ U_i^{(n)} = \mathbb{I}^\otimes (n-1) \otimes \sigma_i, \quad n = 1, 2, 3, \ldots, \quad i = 0, 1, 2, 3, \quad (1) \]

where \( \sigma_0 = I \) is identity acting on \( \mathbb{C}^2 \) and \( \sigma_i, i = 1, 2, 3 \) are the standard Pauli matrices. Then let us introduce the so-called Bell basis defined on Hilbert space \( \mathbb{C}^2 \otimes \mathbb{C}^2 \) as

\[
\begin{align*}
|\psi^B_0\rangle &= |\psi_-angle = (1/\sqrt{2}) (|01\rangle - |10\rangle), \\
|\psi^B_1\rangle &= |\phi_-angle = (1/\sqrt{2}) (|00\rangle - |11\rangle), \\
|\psi^B_2\rangle &= |\phi_+angle = (1/\sqrt{2}) (|00\rangle + |11\rangle), \\
|\psi^B_3\rangle &= |\psi_+angle = (1/\sqrt{2}) (|01\rangle + |10\rangle). \\
\end{align*}
\]

From (1) and (2) one can immediately infer that \( U_i^{(2)} |\psi^B_0\rangle \langle \psi^B_0 | U_i^{(2)} = |\psi^B_i\rangle \langle \psi^B_i | \). For the purposes of further analyzes it is convenient to rewrite the above states using the Hilbert–Schmidt formalism (see [30]). Let us recall that every state \( \rho \) acting on space \( \mathbb{C}^2 \otimes \mathbb{C}^2 \) may be
written in the form

\[ \rho = \frac{1}{4} \left( I \otimes I + r \cdot \sigma \otimes I + I \otimes s \cdot \sigma + \sum_{i,j=1}^{3} t_{ij} \sigma_i \otimes \sigma_j \right), \tag{3} \]

where \( I \) is defined as previous, \( r \) and \( s \) are vectors from \( \mathbb{R}^3 \), \( \sigma \) is vector constructed from Pauli matrices, i.e., \( \sigma = [\sigma_1, \sigma_2, \sigma_3] \). Finally, coefficients \( t_{ij} = \text{Tr}(\rho \sigma_i \otimes \sigma_j) \) form real-valued matrix \( t \).

For Bell states \((2)\), we have the nice geometrical structure \((3)\) that results in particular in:

\[ |\psi_i^B \rangle \langle \psi_i^B | = \frac{1}{4} \left[ I \otimes I + \sum_{i=1}^{3} t^{(i)} \sigma_i \otimes \sigma_i \right], \quad i = 1, \ldots, 4, \]

\[ t^{(0)} = \text{diag}[-1, -1, -1] \]

\[ t^{(1)} = \text{diag}[1, 1, 1] \]

\[ t^{(2)} = \text{diag}[1, -1, -1] \]

\[ t^{(3)} = \text{diag}[1, 1, -1]. \tag{4} \]

Note that for all Bell states vectors \( r \) and \( s \) equal to zero and matrices \( t^{(i)} \) are diagonal. Moreover, all these states maximally violate CHSH-Bell inequality for correlation function \((5)\), i.e., the amount of violation is \( \sqrt{2} \) and it is maximal value achievable by Quantum Mechanics.

Then let us introduce the so-called Smolin state \((6)\) acting on space \((\mathbb{C}^2)^{\otimes 4}:

\[ \rho^S = \frac{1}{4} \sum_{i=0}^{3} |\psi_i^B \rangle \langle \psi_i^B | \otimes 2\]

\[ = \frac{1}{4} \sum_{i=0}^{3} \left( U_i^{(2)} \right)^\dagger |\psi_i^B \rangle \langle \psi_i^B | U_i^{(2)} \) \otimes 2, \tag{5} \]

This state is bound entangled since we cannot distill singlet between any pair of particles. However, the distillation is possible when any two particles are in the same laboratory. As shown in \((23)\), the Smolin state posses the intriguing feature, namely despite being bound entangled it violates maximally the CHSH-type Bell inequality for four particles.

Now, we are in position to present our method. Firstly let us define states by the recursive formulas:

\[ \rho_2 \equiv |\psi_0 \rangle \langle \psi_0 |, \]

\[ \rho_4 = \frac{1}{4} \sum_{i} U_i^{(2)} \rho_2 U_i^{(2)\dagger} \otimes U_i^{(2)} \rho_2 U_i^{(2)\dagger} \equiv \rho^S, \]

\[ \rho_6 = \frac{1}{4} \sum_{i} U_i^{(4)} \rho_4 U_i^{(4)\dagger} \otimes U_i^{(4)} \rho_4 U_i^{(4)\dagger}, \]

\[ \vdots \]

\[ \rho_{2(n+1)} = \frac{1}{4} \sum_{i} U_i^{(2n)} \rho_{2n} U_i^{(2n)\dagger} \otimes U_i^{(2n)} \rho_{2n} U_i^{(2n)\dagger}. \tag{6} \]

This construction starts from one of the Bell states, namely, singlet. Obviously, this state is free entangled state and violates maximally Bell inequalities. This property is crucial for our purposes since, as we will see below, our construction is ‘smuggling’ it to the arbitrary even number of particles. Furthermore, as it is underlined in \((3)\), \( \rho_4 \) is bound entangled and therefore, again because of this specific type of construction, all states from this class with \( n \geq 4 \) are bound entangled. It is interesting that from all these states with \( n \geq 4 \) it is possible to distill only one singlet whenever any subset of \( n-2 \) particles are in the same laboratory.

Hereafter states \( \rho_{2n} \) for \( n > 2 \) shall call Generalized Smolin States (GSS).

It is worth noticing that all these states, including \( \rho_2 \) are permutationally invariant since we have the following

**Observation 1.** Any state \( \rho_{2n} \) may be written in the form

\[ \rho_{2n} = \frac{1}{2^{2n}} \left[ I \otimes 2n + (-1)^n \sum_{i=1}^{3} \sigma_i^{\otimes 2n} \right], \quad n = 1, 2, 3, \ldots \tag{7} \]

**Proof.** The proof will be established using mathematical induction. For \( n = 1 \) this observation is obvious, since \((4)\) represents Hilbert-Schmidt expression for singlet \((4)\). Therefore for further clarity we investigate the case with \( n = 2 \), i.e., the case of Smolin state. So, our task is to prove that

\[ \rho_4 \equiv \rho^S = \frac{1}{16} \left[ I \otimes 4 + \sum_{i=1}^{3} \sigma_i^{\otimes 4} \right]. \tag{8} \]

To this aim it suffice to substitute Hilbert-Schmidt expansions for all Bell states to \((5)\) and to utilize the facts that

\[ \sum_{i=0}^{3} t^{(i)} = 0, \quad \sum_{i=0}^{3} t^{(i)} \otimes t^{(i)} = 4 \text{diag}[1, \ldots, 1]. \tag{9} \]

Now we assume that for arbitrary natural number \( n \) the thesis \((7)\) if fulfilled. Then we need to prove that

\[ \rho_{2(n+1)} = \frac{1}{2^{(n+1)}} \left[ I \otimes 2(n+1) + (-1)^{n+1} \sum_{i=1}^{3} \sigma_i^{\otimes 2(n+1)} \right]. \tag{10} \]

Firstly, let us recall that by the definition \((6)\) state \( \rho_{2(n+1)} \) may be constructed as follows

\[ \rho_{2(n+1)} = \frac{1}{4} \sum_{i=0}^{3} U_i^{(2n)} \rho_{2n} U_i^{(2n)\dagger} \otimes U_i^{(2n)} \rho_{2n} U_i^{(2n)\dagger}. \tag{11} \]

Secondly, let us note that arbitrary density matrix \( \xi \) describing \( N \) spin one-half particles may be written as

\[ \xi = \frac{1}{2^N} \sum_{i_1, \ldots, i_N = 0}^{3} \lambda_{i_1, \ldots, i_N} \sigma_{i_1} \otimes \cdots \otimes \sigma_{i_N}. \tag{12} \]

Coefficients \( \lambda_{i_1, \ldots, i_N} \) form tensor that we shall denote by \( \Lambda \) and its part responsible for \( i_1, \ldots, i_N = 1, 2, 3 \) by \( T \). Immediate observation is that for states \( U_i^{(2n)} \rho_{2n} U_i^{(2n)} \) all coefficients \( \lambda_{i_1, \ldots, i_N} \) are equal to zero except the cases
where \( i_1 = i_2 = \ldots = i_N \). Moreover, it is clear from (7) and by virtue of the equality \( \sigma_i \sigma_j \sigma_i = 2 \delta_{ij} \sigma_i - \sigma_j \), that states \( U_i^{(2n)} \rho_{2n} U_i^{(2n)} \) have tensors \( T \) of the form
\[
T_{2n}^{(0)} = (-1)^n \text{diag}[1, \ldots, 1, \ldots, 1], \\
T_{2n}^{(1)} = (-1)^n \text{diag}[1, \ldots, -1, \ldots, -1], \\
T_{2n}^{(2)} = (-1)^n \text{diag}[-1, \ldots, 1, \ldots, -1], \\
T_{2n}^{(3)} = (-1)^n \text{diag}[-1, \ldots, -1, \ldots, 1]. \\
\]
(13)

Finally one has
\[
\sum_{i=0}^{3} T_{2n}^{(i)} = 0, \\
\sum_{i=0}^{3} T_{2n}^{(i)} \otimes t^{(i)} = 4(-1)^{n+1} \text{diag}[1, \ldots, 1, \ldots, 1]. \\
\]
(14)

The proof is now completed after substitution of states \( U_i^{(2n)} \rho_{2n} U_i^{(2n)} \) and Bell states (2) to (11) with the aid of (13) and (14).

B. Violation of Bell inequality

Quite recently Brukner et al. showed that aside from being one of the most important tools in detection of quantum non-locality, Bell inequalities constitute criterion of usefulness of the quantum states in reducing communication complexity. The prove is constructive since for every Bell inequality and for broad class of quantum protocols they propose a multi-party communication complexity problem. Quantum protocols for this problem are more efficient when one uses quantum state violating that inequality. Therefore the next step of our analyzes is to show explicitly violation of one chosen inequality by GSS.

To this aim we consider standard scenario in which \( j \)-th party \( (j = 1, 2, \ldots, 2n) \) can choose between two dichotomic observables \( \hat{O}_{k_j}^{(j)}, \ k_j = 1, 2 \). Then all parties measure simultaneously one of arbitrarily chosen observable. After many runs of experiment, trying to prove that there does not exist any LHV model for a given state, they must show violation of arbitrary Bell inequality. For our purposes it suffice to consider CHSH-type Bell inequality of the form:
\[
|E(1, \ldots, 1, 1) + E(1, \ldots, 1, 2)_{2n-1} \\
+ E(2, \ldots, 2, 1) - E(2, \ldots, 2, 2)_{2n-1}| \leq 2. \\
\]
(15)

which may be derived from the more general set of Bell inequalities or using the same technique as for two-particle CHSH Bell inequality. Function \( E \) appearing in (15) is so-called correlation classically defined as an average of the measurement outputs taken over many runs of experiment:
\[
E(k_1, \ldots, k_N) = \left\langle \prod_{j=1}^{N} \hat{O}_{k_j}^{(j)} \right\rangle_{\text{avg}}. \\
\]
(16)

In quantum regime the definition is
\[
E_{QM}(k_1, k_2, \ldots, k_{2n}) = \text{Tr}[\rho \hat{O}_{k_1}^{(1)} \otimes \hat{O}_{k_2}^{(2)} \otimes \ldots \otimes \hat{O}_{k_{2n}}^{(2n)}], \\
\]
(17)

where in case of spin-\( \frac{1}{2} \) particles dichotomic observables are of the form
\[
\hat{O}_{k_j}^{(j)} = \hat{n}_{k_j}^{(j)} \cdot \sigma, \quad k_j = 1, 2, \quad j = 1, 2, \ldots, 2n, \\
\]
(18)

where \( \hat{n}_{k_j}^{(j)} \) denote vectors from \( \mathbb{R}^3 \), obeying \( |\hat{n}_{k_j}^{(j)}| = 1 \). Let us choose the vectors
\[
\hat{n}_1^{(j)} = \hat{x}, \quad \hat{n}_2^{(j)} = \hat{y}, \quad j = 1, 2, \ldots, 2n - 1, \\
\hat{n}_1^{(2n)} = \frac{\hat{x} + \hat{y}}{\sqrt{2}}, \quad \hat{n}_2^{(2n)} = \frac{\hat{x} - \hat{y}}{\sqrt{2}}, \\
\]
(19)

where \( \hat{x} \) and \( \hat{y} \) stand for unity vectors directed along, respectively, \( OX \) and \( OY \) axes. The above choice gives the value
\[
E_{QM}(1, \ldots, 1, 1)(\rho_{2n}) + E_{QM}(1, \ldots, 1, 2)(\rho_{2n}) \\
+ E_{QM}(2, \ldots, 2, 1)(\rho_{2n}) - E_{QM}(2, \ldots, 2, 2)(\rho_{2n}) = (-1)^n 2 \sqrt{2}, \\
\]
(20)

that obviously violate Bell inequality. Moreover, this violation is maximal which can be easily shown by Tsirelson bound since for this purpose in each term (20) we can combine all 2n-1 local operators into one dichotomic operator.

Concluding, we have just shown that any of states violates Bell inequality maximally. However, for \( n = 1 \) it is obvious since for this value of \( n \) we have one of the Bell states, for \( n > 1 \) this violation is surprising in the light of the fact that all these states are bound entangled.

C. Noisy states

Trying to generalize the above considerations, we investigate some of the properties of the GSS in presence of noise. In other words below we characterize states
\[
\rho_{2n}(p) = (1 - p)I_{2n} + p \rho_{2n}, \quad 0 \leq p \leq 1 \\
\]
(21)

where \( I \) as previous is identity acting on one-qubit space and bound entangled states \( \rho_{2n} \) are defined by (10). Below
we show that this family of states has similar separability properties and violate Bell inequality in the same regime with respect to \( p \) as two-qubit Werner states \( \Phi \).

In the first step let us observe that by virtue of (7) we may rewrite (21) as follows

\[
\varrho_{2n}(p) = \frac{1}{2^{2n}} \left[ I^@2n + (-1)^n p \sum_{i=1}^{3} \sigma_i^{\otimes 2n} \right] \tag{22}
\]

To investigate separability properties of \( \varrho_{2n}(p) \) let us introduce projectors

\[
P_{k}^{(\pm)} = \frac{1}{2} (I \pm \sigma_k) \tag{23}
\]

as corresponding to eigenvectors of \( \sigma_k \) with eigenvalues \( \pm 1 \). Then let us consider two-qubit mixed separable states introduced in (39):

\[
\varrho_{k}^{(\pm)} = \frac{1}{2} \left[ P_{k}^{(\pm)} \otimes P_{k}^{(\pm)} + P_{k}^{(-)} \otimes P_{k}^{(+)} \right] \tag{24}
\]

Please notice that these states may be easily generalized to arbitrary amount of particles. To this aim let us introduce the following notations:

\[
\eta^{(\pm)}_{k,1} \equiv P_{k}^{(\pm)},
\eta^{(\pm)}_{k,2} = \frac{1}{2} \left[ \eta_{k,1}^{(+)} \otimes P_{k}^{(\pm)} + \eta_{k,1}^{(-)} \otimes P_{k}^{(\mp)} \right] \equiv \varrho_{k}^{(\pm)},
\eta^{(\pm)}_{k,3} = \frac{1}{2} \left[ \eta_{k,2}^{(+)} \otimes P_{k}^{(\pm)} + \eta_{k,2}^{(-)} \otimes P_{k}^{(\mp)} \right],
\eta^{(\pm)}_{k,n-1} = \frac{1}{2} \left[ \eta_{k,n-1}^{(+)} \otimes P_{k}^{(\pm)} + \eta_{k,n-1}^{(-)} \otimes P_{k}^{(\mp)} \right],
\eta^{(\pm)}_{k,n} = \frac{1}{2} \left[ \eta_{k,n-1}^{(+)} \otimes P_{k}^{(\pm)} + \eta_{k,n-1}^{(-)} \otimes P_{k}^{(\mp)} \right]. \tag{25}
\]

From that construction it is obvious that all states \( \eta^{(\pm)}_{k,n} \) are fully separable. Moreover, taking into account expression (28) we may constitute the following

**Observation 3.** All states \( \eta^{(\pm)}_{k,n} \) have the form

\[
\eta^{(\pm)}_{k,n} = \frac{1}{2^{n}} \left( I^@n \pm \sigma_k^@n \right). \tag{26}
\]

**Proof.** Since the above observation is rather obvious, we decided to present below proof for \( n = 2 \). Generalization to arbitrary \( n \) is straightforward. From the definition (28) we infer

\[
\eta^{(\pm)}_{k,2} = \varrho^{(\pm)}_{k} = \frac{1}{2} \left[ P_{k}^{(\pm)} \otimes P_{k}^{(\pm)} + P_{k}^{(-)} \otimes P_{k}^{(\mp)} \right], \tag{27}
\]

and then application of (28) to the above yields

\[
\eta^{(\pm)}_{k,2} = \frac{1}{8} \left[ (I + \sigma_k) \otimes (I \pm \sigma_k) + (I - \sigma_k) \otimes (I \mp \sigma_k) \right] = \frac{1}{2^2} \left[ I^@2 \pm \sigma_k^@2 \right]. \tag{28}
\]

Now we are in position to finish considerations respecting separability properties of (21). Since the Werner state \( \varrho^{W}(p) \) and the Smolin state \( \varrho^{S}(p) \) are separable for \( p = 1/3 \) we may conjecture that all GSS for \( n > 2 \) are also separable for such value of \( p \). Indeed, we have

**Observation 4.** For \( p = \frac{1}{3} \) states \( \varrho_{2n}(\frac{1}{3}) \) are separable and are of the form

\[
\varrho_{2n}(\frac{1}{3}) = \frac{1}{3^2} \sum_{k=1}^{3} \left\{ \eta_{k,n}^{(+)} \otimes \eta_{k,n}^{(-)} + \eta_{k,n}^{(-)} \otimes \eta_{k,n}^{(+)} \right\} \tag{29}
\]

**Proof.** The proof is rather technical, so we restrict our considerations to the case of odd number of particles. After application of (26) to (29) we obtain

\[
\varrho_{2n}(\frac{1}{3}) = \frac{1}{2^{2n}} \left[ I^@2n + \frac{1}{3} \sum_{k=1}^{3} \sigma_k^@2n \right], \quad n = 1, 3, 5, \ldots \tag{30}
\]

The same procedure for the even number of particles gives expression with minus before the sum. Thus, rewriting these two relations in generalized form

\[
\varrho_{2n}(\frac{1}{3}) = \frac{1}{2^{2n}} \left[ I^@2n + (-1)^n \frac{1}{3} \sum_{k=1}^{3} \sigma_k^@2n \right], \tag{31}
\]

completes the proof. Remark that using LOCC we may always add some noise and therefore noisy GSS become separable for all \( p \in [0, 1/3] \). Subsequently, using observables defined by (19), we can see that violation of (17) by (21) is for \( p \in (1/ \sqrt{2}, 1] \).

**III. APPLICATIONS**

**A. Communication Complexity**

It is quite remarkable that (28) despite being bound entangled Smolin states can reduce communication complexity. This fact shows that, however, bound entangled states are not distillable, they allow to solve some tasks with the same efficiency like free entangled states. Here we show that the Smolin state is not an isolated case, where bound entangled states are equal to free entangled states in context of reducing communication complexity.

As proven by (27) the necessary and sufficient condition for being useful in reducing communication complexity is violation of one arbitrary Bell inequalities. In the light of the former we can see that all GSS are useful with the same efficiency as free entangled states, since we have already proven that this violation is maximal.
B. Remote information concentration

Before we prove the utility of GSS in remote information concentration we focus on telecloning scheme proposed by Murao et al. [43]. This scheme, involving quantum teleportation and cloning allow a sender to teleport an unknown qubit state to spatially separated receivers. Of course, in virtue of no-cloning theorem received qubits are no longer perfect clones of teleported one. On the other hand it is shown that fidelities achievable in such a scheme are sufficient. Suppose that Alice wishes to teleport one qubit $|\phi\rangle_X$ to her spatially separated friends $B_1, \ldots, B_M$. After all (for more details see [45]) all of them share so-called optimal cloning state

$$|\Psi_e\rangle = a|\phi_0\rangle_{AC} + b|\phi_1\rangle_{AC},$$

where

$$|\phi_0\rangle_{AC} = \sum_{j=0}^{M-1} \alpha_j |A_j\rangle_A \otimes |0, M-j\rangle, \{1, j\}\rangle_C$$

$$|\phi_1\rangle_{AC} = \sum_{j=0}^{M-1} \alpha_j |A_{M-j}\rangle_A \otimes |0, j\rangle, \{1, M-j\}\rangle_C$$

$$\alpha_j = \sqrt{\frac{2(M-j)}{M(M+1)}},$$

and

$$|A_j\rangle_A = |0, M-j\rangle, \{1, j\}\rangle_A.$$  \hspace{1cm} (34)

Kets $|A_j\rangle_A$ represent $M$ normalized and orthogonal states of ancilla involving $M-1$ qubits. The subscript $C$ refers to $M$ qubits holding the clones and finally ket $|\{0, M-j\}, \{1, j\}\rangle$ stands for normalized and symmetric state of $M$ qubits. Let us notice that

$$\sigma_3 \otimes \cdots \otimes \sigma_3 |\phi_1\rangle_{AC} = (-1)^j |\phi_1\rangle_{AC},$$

$$\sigma_2 \otimes \cdots \otimes \sigma_2 |\phi_1\rangle_{AC} = (-1)^{M+1} |\phi_{@1}\rangle_{AC},$$

$$\sigma_1 \otimes \cdots \sigma_1 |\phi_1\rangle_{AC} = |\phi_{@1}\rangle_{AC}, \quad l = 0, 1, \ldots, 2M-1$$

where $\otimes \equiv \oplus \bmod 2$.

Murao and Vedral proved [45] that even if clones are not perfect replicas of teleported qubit, it is still possible to recover information included in optimal cloning state to Charlie using only LOCC. To show it explicitly they used unlockable bound entangled Smolin state $\rho_S$. Now we show that all GSS are useful to perform such a task.

At the very beginning let us assume that the optimal cloning state is distributed among Alice and her friends $B_1, \ldots, B_M$ in such a way that the former posses $M-1$ ancilla qubits (generally these qubits may be also spatially separated) and the latter $M$ qubits of clones. Subsequently, they wish to recreate the original qubit to Charlie using as a quantum channel GSS distributed previously among all actors. To complete this goal Alice and $B_1, \ldots, B_M$ perform a Bell measurement between their qubits, one from optimal cloning state, and one from GSS. After that they arrive at the state $\varrho_{k_1 \ldots k_N}$ given by

$$\varrho_{k_1 \ldots k_N} = \frac{1}{p_{k_1 \ldots k_N}} \times \mbox{Tr}_{A_1 \ldots A_{M-1}, B_1 \ldots B_M} \left[ \bigotimes_{i=1}^{N} P_{k_i} \otimes U_{k_i k_2 \ldots k_N} \right]$$

$$\left| \Psi_e \right\rangle \left\langle \Psi_e \right| \otimes \rho_{2M}^{M} \left[ \bigotimes_{i=1}^{N} P_{k_i} \otimes U_{k_1 k_2 \ldots k_N} \right],$$

where

$$\varrho_{k_1 \ldots k_N} = \frac{1}{p_{k_1 \ldots k_N}} \times \mbox{Tr}_{C} \left[ \varrho_{k_1 \ldots k_N} \right],$$

$$\times \sum_{m_1 \ldots m_N} \lambda_{m_1 \ldots m_N}$$

$$\times \left[ \prod_{i=1}^{N} \mbox{Tr}[P_{k_i}(\sigma_{m_i} \otimes I)]I + (-1)^M$$

$$\times \sum_{r=1}^{3} \prod_{i=1}^{N} \mbox{Tr}[P_{k_i}(\sigma_{m_i} \otimes \sigma_r)]U_{k_1 k_2 \ldots k_N} \sigma_r U_{k_1 k_2 \ldots k_N}^\dagger \right]$$

$$\left(37\right)$$

Since

$$\mbox{Tr}[P_{k_i}(\sigma_{m_i} \otimes I)] = \delta_{m_0}$$

$$\mbox{Tr}[P_{k_i}(\sigma_{m_i} \otimes \sigma_r)] = \sum_{r=1}^{3} t_{i}^{(k_i)} \delta_{m_0} \delta_{ir} \quad \left(38\right)$$

we may rewrite $\left(37\right)$ as

$$\varrho_{k_1 \ldots k_N} = \frac{1}{p_{k_1 \ldots k_N}} \times \left[ \lambda_{0 \ldots 0} I \right.$$

$$+ \left(\lambda_{1 \ldots 1}, \lambda_{2 \ldots 2}, \lambda_{3 \ldots 3}\right) \right] \times \left[ \lambda_{1 \ldots 1}, \lambda_{2 \ldots 2}, \lambda_{3 \ldots 3}\right]$$.  \hspace{1cm} (39)

As we shall see below it suffice for Charlie to perform an operation

$$U_{k_1 \ldots k_N} = \sigma_{M \otimes 1} \sigma_1 \ldots \sigma_k$$

in order to obtain an original qubit. Since

$$\sum_{k=0}^{3} \sigma_k (t^{(k)} \tilde{\lambda}) \cdot \tilde{\sigma} \sigma_k = -4 \tilde{\lambda} \cdot \tilde{\sigma} \quad \left(40\right)$$

we finally obtain

$$\sum_{k_1 \ldots k_N} \sum_{k_1 \ldots k_N} \varrho_{k_1 \ldots k_N} \varrho_{k_1 \ldots k_N}$$

$$= \frac{1}{2} \frac{1}{\lambda_{0 \ldots 0} I + \left(\lambda_{1 \ldots 1}, \lambda_{2 \ldots 2}, \lambda_{3 \ldots 3}\right) \cdot \tilde{\sigma} \sigma_{M \otimes 1}} \quad \left(41\right)$$
Let us consider some of the properties of generalised Smolin states. It is interesting to understand why maximal Bell violation does not imply quantum security in this case. The naive approach would say that there is some correlations that (i) are strictly nonlocal (since all the measurements in Bell measurement are performed locally) (ii) are not accessible to Eve (since Bell inequalities are violated). One could argue that any violation singles out one particle versus the remaining ones. However still the remaining parties perform measurements locally which means that one the correlations are stronger than just as if they were interpreted in terms of entanglement of single particle versus all other parties taken together. On the other hand we have obvious argument against security since the states are biseparable (separable against any (2)-(2n-2) particles cut) which means that no security can be distilled even if some parties can communicate quantumly. Most probably the reason is that the present states, despite violating Bell inequalities do not have any set of axes that provide perfect correlations between all the parties. Thus, in a sense, the quantum correlations even if nonlocal are completely useless for establishing the correlated data.

Such a set of axes with corresponding maximally correlated probabilities is possessed by GHZ state Hilbert-Schmidt representation of which has nonvanishing coefficients not only at $\sigma^\otimes_{2n}$ operators (as GSS have) but also at all permutations of operators $\sigma^\otimes_{2k} \otimes I^\otimes_{2(n-k)}$. This allows very easily to design Ekert scheme \cite{5} with any of observers choosing randomly one of the following four axes: $\hat{x}$, $\hat{y}$, $(\hat{x} + \hat{y})/\sqrt{2}$, $(\hat{x} - \hat{y})/\sqrt{2}$. This is not the case in GSS case where only few terms survive in Hilbert-Schmidt representation.

In presence of recent important results on security in post quantum theories \cite{10}, following the above discussion it is reasonable to conjecture that any physical system, even in post-quantum theory, that maximally violate Bell inequalities leads to cryptographic security if only has one pair of axes with maximal correlations. It would be also interesting to consider the cases when the presence of maximal correlations is accompanied non-maximal Bell inequalities violation.

Let us pass to the another interesting issue - remote quantum information concentration. One can very easily to see that it can have application if we apply notion of locking of classical correlations \cite{47}. Namely suppose that some huge amount of classical correlations (secure or not) between 2n observers is locked by single qubit that is further deliberately encoded into many qubits send to different them. It happens then that GSS allows them to unlock this classical information in a simple way using remote information concentration and further „unlocking” measurement of the qubit. It is interesting to note that in this case quantumness of remotely concentrated information is important because of quantum entanglement of this qubit with another quantum system that contains the classical locked information. The above application would be quite powerful if one could rigorously show that it is impossible to unlock the quantum information by simulation of the unlocking quantum measurement on the distributed qubit. The reasoning applies immediately to entanglement locking effect \cite{48} since in that case one also have to localise qubit in one of distant labs.

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