Analytic Q-ball solutions and their stability in a piecewise parabolic potential

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Abstract

Explicit solutions for extended objects of a Q-ball type were found analytically in a model describing complex scalar field with piecewise parabolic potential in (3+1)- and (1+1)-dimensional space-times. Such a potential provides a variety of solutions which were thoroughly examined. It was shown that, depending on the values of the parameters of the model and according to the known stability criteria, there exist stable and unstable solutions. The classical stability of solutions in (1+1)-dimensional space-time was examined in the linear approximation and it was shown explicitly that the spectrum of linear perturbations around some solutions contains exponentially growing modes while it is not so for other solutions.

1 Introduction

Q-ball is a nontopological soliton in theories with a global symmetry \cite{1}. A simple example is the model with one complex scalar field \( \phi \) with \( U(1) \)-invariant potential \( V(\phi^*\phi) \) which satisfies several simple conditions derived in \cite{1}. The standard solution for a Q-ball has the form

\[
\phi(t, \vec{x}) = f(r) e^{i\omega t},
\] (1)
where $\omega$ is a constant and $f(r)$ is a monotonically decreasing (in a simplest case) spherically symmetric function, tending to zero at spatial infinity. Different applications in cosmology (see, for example, [2] for review) encourage investigation of specific features of such classical extended objects. However, nonlinearity of classical field equations is a serious obstacle for obtaining analytical solutions even in theories with simple polynomial potentials (see, for example, [3]), the exceptions like the one presented in [4] (the scalar field potential of this model makes it possible to examine analytically even the linear perturbations above the Q-ball solution, see [5]) are very rare. Some results of qualitative and numerical analysis of Q-ball properties can be found in [6].

To obtain an analytic solution the model with parabolic piecewise potential

$$V(\phi^*\phi) = M^2 \left[ \phi^*\phi + 2\epsilon v(v - \sqrt{\phi^*\phi})\theta \left( \frac{\sqrt{\phi^*\phi}}{v} - 1 \right) \right]$$

was considered in [7]. Here $M^2 > 0$, $\theta$ is the Heaviside step function, $v$ is a parameter of the model, $\epsilon$ is a positive dimensionless constant. Indeed, in both regions $f > v$ and $f < v$ the corresponding equation of motion is analytically solvable and the solutions are regular and smooth. Matching conditions determine the point $r = R$ such that $f(R) = v$. Inside the sphere with radius $R$ the value of the field $f$ is larger than $v$ and outside this sphere $f < v$ and exponentially tends to zero. Thus, the charge and the energy are localized inside the ball with the center at the origin $r = 0$.

In a solvable model, one can find the whole spectrum of excitations above a Q-ball solution, examine the modes responsible for possible instabilities and, in addition, demonstrate explicitly the validity of the stability conditions for Q-balls, which can be found in [8, 9, 10]. Indeed, one can examine linear perturbations above a Q-ball solution in analogy with [3], where instability of the solution in a theory with potential of the form $V = \kappa^2 \phi^*\phi - \frac{\mu^2}{2}(\phi^*\phi)^2$ was established numerically. The square root in potential [2] is an obstacle for analytical consideration of linear perturbations above the Q-ball solution. In this paper we propose a model describing complex scalar field with potential of the form

$$V(\phi^*\phi) = M^2 \phi^*\phi \theta \left( 1 - \frac{\phi^*\phi}{v^2} \right) + (m^2 \phi^*\phi + \Lambda)\theta \left( \frac{\phi^*\phi}{v^2} - 1 \right),$$

where $M^2 > 0$, $M^2 > m^2$, $\theta$ is the Heaviside step function with the convention $\theta(0) = \frac{1}{2}$, in theories with three and one spatial dimensions. The constant $\Lambda$ provides continuity of the potential at the point $\phi^*\phi = v^2$, $\Lambda = v^2(M^2 - m^2)$. This potential is presented in Fig. 1. Of course, this piecewise potential should be considered as a limiting case of some smooth and bounded potential.

Below we will examine Q-ball solutions in a model with potential (3) in $(3+1)$- and $(1+1)$-dimensional space-times and show that the model contains solutions possessing dif-
different properties. As will be shown below, some solutions are stable, some of them are not. According to [3], examination of classical stability in the linear approximation will be made for the (1 + 1)-dimensional case and stability conditions, presented in [8, 9, 10], will be illustrated explicitly.

2 Q-ball in four-dimensional space-time

2.1 Analytical Q-ball solution

First let us consider (3+1)-dimensional space-time. The action of the model has the form

\[ S = \int d^4x \left[ \partial_{\mu} \phi^* \partial^{\mu} \phi - M^2 \phi^* \phi \left( 1 - \frac{\phi^* \phi}{v^2} \right) - (m^2 \phi^* \phi + \Lambda) \theta \left( \frac{\phi^* \phi}{v^2} - 1 \right) \right] \] (4)

with \( \Lambda = v^2(M^2 - m^2) \). We consider the standard form of solution (1). The simplest regular spherically symmetric solution to the equation of motion, coming from (4), takes the form

\[ f = B e^{-\sqrt{M^2-\omega^2}r} \]

for \( f^2 < v^2 \),

\[ f = A \sin(\sqrt{\omega^2 - m^2}r) \]

for \( f^2 > v^2 \). (5) (6)

It is clear that if \( m^2 > 0 \), then \( m < |\omega| < M \); if \( m^2 = 0 \), then \( 0 < |\omega| < M \); otherwise \( 0 \leq |\omega| < M \). The continuity of the solution and its first derivative at the point \( r = R \) such that \( f(R) = v \) defines the coefficients \( A \) and \( B \), which read as

\[ B = \frac{vR}{e^{-\sqrt{M^2-\omega^2}R}} \]

\[ A = \frac{vR}{\sin(\sqrt{\omega^2 - m^2}R)} \] (7) (8)
and defines the value of the matching radius \( r = R \):

\[
\tan \left( \sqrt{\omega^2 - m^2} R \right) = -\frac{\sqrt{\omega^2 - m^2}}{\sqrt{M^2 - \omega^2}}.
\] (9)

Since the parameter \( R \) is positive, it should be taken to be

\[
R = \frac{1}{\sqrt{\omega^2 - m^2}} \left( \arctan \left( -\frac{\sqrt{\omega^2 - m^2}}{\sqrt{M^2 - \omega^2}} \right) + \pi \right).
\] (10)

All the features of our solution can be expressed through this parameter \( R \) or, equivalently, through the frequency \( \omega \). Note that the presented solution is the simplest one – the absolute value of the function \( f \) is a monotonically decreasing function (which means that the charge density is a monotonic function of \( r \) as well), which is equal to \( v \) only at the one point \( r = R \). Obviously, there may exist solutions which cross the line \( \sqrt{\phi^* \phi} = v \) several times and have nodes (i.e. such that there exist points \( r_i \neq \infty : f(r_i) = 0 \)). Because of the simplicity of potential (3), such solutions can also be found analytically. It would be interesting to examine their properties in comparison with those of the simplest solution presented above, but this topic lies beyond the scope of the present paper.

One sees that the scalar field potential is unbounded from below if \( m^2 < 0 \). Meanwhile, one can always consider a potential which coincides with the one defined by (3) for \( \phi^* \phi \leq \tilde{v}^2 \) but changes its behavior and becomes growing for \( \phi^* \phi > \tilde{v}^2 \), where \( \tilde{v} > v \). In such a case one should take only those Q-ball solutions for which the maximum absolute value of the scalar field \( |f(0)| \leq \tilde{v} \).

Now let us examine the properties of the solution at hand. It is not difficult to calculate the total charge and the total energy of the Q-ball. They look as follows:

\[
Q = -i \int d^3x \left( \dot{\phi}^* \dot{\phi} - \dot{\phi}^* \phi \right) = 4\pi R^2 \omega v^2 \left( \frac{(M^2 - m^2)}{(\omega^2 - m^2)} \frac{R \sqrt{M^2 - \omega^2} + 1}{\sqrt{M^2 - \omega^2}} \right),
\] (11)

\[
E = \int d^3x \left( \dot{\phi}^* \dot{\phi} + \partial_i \phi^* \partial_i \phi + V(\phi^* \phi) \right) =
= 4\pi R^2 \omega^2 v^2 \left( \frac{(M^2 - m^2)}{(\omega^2 - m^2)} \frac{R \sqrt{M^2 - \omega^2} + 1}{\sqrt{M^2 - \omega^2}} \right) + \frac{4\pi}{3} R^3 v^2 (M^2 - m^2),
\] (12)

where \( R \) is defined by (10). One can see that \( E = \omega Q + 4\pi R^3 v^2 (M^2 - m^2)/3 > \omega Q \). Note that the inequality \( E > \omega Q \) should hold for any Q-ball solution (see simple derivation of this fact in Appendix A). Another interesting observation is that

\[
Q = -v^2 (M^2 - m^2) \frac{d}{d\omega} \left( \frac{4\pi}{3} R^3 \right).
\] (13)
This observation allows one to make another cross-check of our results. Indeed, let us differentiate equation (12) with respect to the charge $Q$:

$$\frac{dE}{dQ} = \omega + \frac{d\omega}{dQ} Q + \frac{4\pi}{3} v^2 (M^2 - m^2) \frac{d(R^3)}{dQ} = \omega + \frac{d\omega}{dQ} Q + \frac{4\pi}{3} v^2 (M^2 - m^2) \frac{d(R^3)}{dQ} \frac{d\omega}{dQ}. \quad (14)$$

Using equation (13) we easily get the well-known relation $[6, 8, 9]$

$$\frac{dE}{dQ} = \omega. \quad (15)$$

which should hold for any Q-ball solution.

2.2 $m^2 > 0$

First, we consider the case $m^2 > 0$. The $E(Q)$ and $E(Q)/Q$ diagrams for this case for $M/m = 3$ (here and below, without loss of generality, we consider $\omega > 0$) are presented in Fig. 2. We see that there are two different branches in this figure, which correspond to solutions which are localized in different ways. Localization of solutions with larger energy is due to the exponential suppression outside the core $[0, R]$ and we will refer to it as a "wide" branch. The size of solutions with smaller energy is just $R$ (see below) and most of their charge is concentrated inside the core $[0, R]$. We will refer to these solutions as a "narrow" branch. We see that there is a solution with nonzero minimal charge and energy. This result

![Figure 2: E(Q) (left plot) and E(Q)/Q (right plot) for the Q-ball in (3+1)-dimensional theory, $m^2 > 0$, $M/m = 3$. The dashed line corresponds to free particles with $E = MQ$.](image)

is very similar to those obtained in $[8, 11]$ where completely different scalar field potentials were utilized. Indeed, one can see that the left plot presented in Fig. 2 looks the same as the corresponding plots in Fig. 2 and Fig. 3 of $[8]$. The right plot in Fig. 2 has the same
form as the one presented in [11]. Of course, these results are also very similar to the results obtained in [7].

The quantitative difference between branches can be expressed through the parameter

$$ g = \sqrt{\frac{M^2 - \omega^2}{\omega^2 - m^2}}. \quad (16) $$

In the case of large $Q$ two different branches, presented in Fig. 2, correspond to $g \gg 1$ and $g \ll 1$. For large $Q$ and $M \sim m$ the properties of the solutions are summarized in Table 1.

| Type of solution for large $Q$ | Wide branch | Narrow branch |
|-------------------------------|-------------|---------------|
| Asymptotics                   | $g \ll 1$   | $g \gg 1$     |
| Soliton radius $L$            | $\sim \frac{1}{g\sqrt{M^2 - m^2}}$ | $\sim R \sim \frac{\pi g}{\sqrt{M^2 - m^2}}$ |
| Soliton energy $E$            | $\sim v^2 L$ | $\sim v^2 M^4 L^5$ |
| Soliton charge $Q$            | $\sim v^2 L/M$ | $\sim v^2 M^3 L^5$ |

Table 1. Properties of the Q-ball solutions for large $Q$; $m^2 > 0$; (3+1)-dimensional space-time.

One can express all values in terms of large $Q$. For example, if $m \sim M$ the size of wide Q-ball is $L_w \sim \frac{QM}{v^2}$ and the size of narrow Q-ball is $L_n \sim (\frac{Q}{M^2 m^2})^{1/5}$. This difference is the origin for the titles of the branches – indeed, for large $Q$ and for $M \sim v$ we have $L_w \gg L_n$. Solutions of the wide branch are very similar to the Q-clouds of [11].

It should be mentioned that the usual thin-wall approximation can not be used for solutions presented above. Indeed, for the stable branch even in the limit $\omega \to m$ (in this case the solution most rapidly falls off in the region $r > R$) the solution considerably differs from a constant in the region $[0, R]$.

2.3 $m = 0$

Now we turn to the second case $m = 0$. The scalar field potential in this case contains flat directions, such type of potentials arise in supersymmetric theories. As can be seen from Fig. 3 the properties of the Q-ball solutions in this case look very similar to those discussed in the previous subsection. Meanwhile, there is a considerable difference between the cases $m^2 > 0$ and $m = 0$. In the first case solutions with large charge $Q$ on the lower branch have the following energy-charge dependence: $E \sim Q$. In the second case $m = 0$ one obtains from (11) and (12) for large $Q$ (i.e for $\omega \to 0$) the following energy-charge dependence: $E \sim Q^{3/4}$. The latter relation coincides with the general estimate for potentials of this type, which can be found in [2].
Figure 3: $E(Q)$ (left plot) and $E(Q)/Q$ (right plot) for the Q-ball in (3+1)-dimensional theory, $m = 0$.

2.4 $m^2 < 0$

The third case $m^2 < 0$ appears to be completely different from the previous cases. First, there are two different phases – solutions in the first phase contain three branches in the $E(Q)$ diagram (see Fig. 4), whereas another phase contains only one branch (see Fig. 5).

Figure 4: $E(Q)$ (left plot) and $E(Q)/Q$ (right plot) for the Q-ball in (3+1)-dimensional theory, $m^2 < 0$, $M/|m| = 3$.

The transition between the phases occurs at $\frac{M}{|m|} \approx 0.728$.

We see that in Fig. 4 there exist three branches, two of them intersect each other. There also exists nontrivial (i.e. having nonzero energy) solution with the zero charge $Q$. An analogous case was described in [7] for the appropriate range of the parameters of the scalar field potential. Note that though the scalar field potential in [7] is bounded from below for
any values of the parameters, whereas our potential is unbounded from below for $m^2 < 0$, similar Q-ball solutions having three branches arise in both cases. An interesting observation is that the part of $E(Q)$ dependence containing two branches, i.e. the part starting from $\omega = 0$ and ending at $\omega = \omega_c < M$, where $\omega_c$ corresponds to the solution with minimal (this minimum is local in general) energy, resembles the $E(Q)$ dependence found in Ref. [5] in the model with a completely different potential. The difference between our case and the case discussed in Ref. [5] is that the $E(Q)$ dependence of Ref. [5] has $\omega_c \to \infty$ and the corresponding branch ends at the point $(0; 0)$ on the $Q, E$ plane.

The $E(Q)$ dependence of another phase (presented in Fig. 5) is very similar to the one of the model presented in Ref. [3].

With the help of the explicit solution it can be shown that the thin-wall approximation also does not describe the case $m^2 < 0$.

2.5 Stability of the solutions

Now let us discuss the stability of the Q-ball solutions presented above. The first type of stability is the quantum mechanical stability. We will focus on the case $m^2 > 0$. One sees from Fig. 2 that the lower branch crosses the line $E = MQ$, corresponding to free particles with the rest mass $M$, at some charge, say $Q_x$. Thus, for the region of charges $Q > Q_x$ the energy of the Q-ball is smaller than the energy of free particles and thus such a Q-ball is quantum mechanically stable. In the region $Q_{min} < Q < Q_x$, where $Q = Q_{min} \approx 2.85 \frac{4\pi v^2}{m^2}$ for $M/m = 3$, the Q-ball solution is unstable with respect to radiation of free particles. Q-balls from the upper "wide" branch are always unstable from this point of view.
Now let us consider the Q-ball fission. It is known that Q-balls are stable against fission if \( d^2E/dQ^2 < 0 \) (although this condition is known, we present a simple justification of this fact in Appendix B). This relation holds for the lowest branches in all three cases (for the case \( m^2 < 0 \) this is valid for the phase with \( M/|m| \gtrsim 0.728 \)). Indeed, \( \frac{d^2E}{dQ^2} = \frac{d\omega}{dq} \) (see Eq. (15)), whereas for the lowest branches \( \frac{dQ}{d\omega} < 0 \) because the charge increases while \( \omega \) decreases and thus \( d^2E/dQ^2 < 0 \). This can also be seen from Fig. 2, Fig. 3 and Fig. 4.

The last type of stability, which can be discussed here, is the classical stability, i.e. stability with respect to small perturbations. The classical stability criterion proposed in [8, 9] implies that a Q-ball solution is classically stable if \( \frac{dQ}{d\omega} < 0 \) (for \( \omega > 0 \) and \( Q > 0 \), which is exactly our case), a mathematically rigorous proof of this fact can be found in [10] (see also references therein). This relation, as it was shown above, holds for the lowest branches in all three cases (for the case \( m^2 < 0 \) this is valid for the phase with \( M/|m| \gtrsim 0.728 \)) and thus we can conclude that Q-balls from these branches are classically stable. An interesting observation for the case \( m^2 < 0, M/|m| \gtrsim 0.728 \) is that classically stable Q-ball solutions lie between solutions with locally minimal and locally maximal charges, i.e. there exist stable solution with minimal charge (and energy) and stable solution with maximal charge (and energy).

Meanwhile, it would be interesting to consider linear perturbations explicitly in order to examine the upper branches and their possible instability modes. A complete analytical analysis of linearized theory in the (3+1)-dimensional case seems to be a rather complicated task and it lies beyond the scope of this paper. Nevertheless, we performed analysis of perturbations in the linear approximation in a simpler model in (1+1)-dimensional space-time for the cases \( m^2 > 0 \) and \( m^2 < 0 \). The physical properties of the (1+1)-dimensional model are very similar to those in the (3+1)-dimensional case and we think that results obtained in such a simplified case can be applied to the (3+1)-dimensional case as well.

### 3 Q-ball in two-dimensional space-time

#### 3.1 Analytical solution and its properties

Now the action takes the form

\[
S = \int dtdz \left( \partial_\mu \phi^* \partial^\mu \phi - M^2 \phi^* \phi \theta \left( 1 - \frac{\phi^* \phi}{v^2} \right) - (m^2 \phi^* \phi + \Lambda) \theta \left( \frac{\phi^* \phi}{v^2} - 1 \right) \right)
\]

with \( \Lambda = v^2(M^2 - m^2) \). Again we consider the standard ansatz

\[
\phi(t, z) = f(z)e^{i\omega t}
\]
with a dimensionless even function \( f(z) = f(-z) \). At the points \( z = \pm R \) function \( f(z) \) is equal to \( v \) and there are discontinuities in the ordinary differential equation on \( f(z) \):

\[
- \frac{\partial^2 f}{\partial z^2} - \omega^2 f + M^2 \theta \left( 1 - \frac{f^2}{v^2} \right) f + m^2 \theta \left( \frac{f^2}{v^2} - 1 \right) f = 0.
\] (18)

Inside the interval \((-R, R)\) the function \(|f|\) is larger than \( v \) and outside this interval it falls off exponentially to zero. The corresponding solution to equation of motion (18) takes the form

\[
f = ve^{-\sqrt{M^2 - \omega^2}|z|-R}, \quad \text{for} \quad f^2 < v^2, \quad |z| > R
\] (19)

\[
f = v \frac{\cos(\sqrt{\omega^2 - m^2}z)}{\cos(\sqrt{\omega^2 - m^2}R)}, \quad \text{for} \quad f^2 > v^2, \quad |z| < R
\] (20)

with \( R \) defined as

\[
R = \frac{1}{\sqrt{\omega^2 - m^2}} \arctan \left( \frac{\sqrt{M^2 - \omega^2}}{\sqrt{\omega^2 - m^2}} \right).
\] (21)

The total charge and the total energy look as follows:

\[
Q = 2v^2\omega \left( \frac{(M^2 - m^2) (R\sqrt{M^2 - \omega^2} + 1)}{(\omega^2 - m^2) \sqrt{M^2 - \omega^2}} \right),
\] (22)

\[
E = 2v^2\omega^2 \left( \frac{(M^2 - m^2) (R\sqrt{M^2 - \omega^2} + 1)}{(\omega^2 - m^2) \sqrt{M^2 - \omega^2}} \right) + 2v^2(M^2 - m^2)R,
\] (23)

where \( R \) is defined by (21). Again one sees that \( E = \omega Q + 2v^2(M^2 - m^2)R > \omega Q \). It is straightforward to show that, in analogy with the (3+1)-dimensional case, the following relation holds:

\[
Q = -v^2(M^2 - m^2) \frac{d(2R)}{d\omega}.
\] (24)

Using this relation we easily obtain (15).

First we consider the simpler case \( m^2 > 0 \). The function \( E(Q) \) for \( M/m = 2 \) is presented in Fig. 6. As in the \((3+1)\)-dimensional case, there exist two different branches, their properties again can be expressed through the parameter \( g \) defined by (16) and the effective size \( L \). The differences between solutions are summarized in Table 2.

| Type of solution for large \( Q \) | Wide branch | Narrow branch |
|-----------------------------------|-------------|--------------|
| Asymptotics \( g \ll 1 \) | \( g > 1 \) |
| Soliton size \( L \) \( \sim \frac{1}{g\sqrt{M^2 - m^2}} \) | \( \sim R \sim \frac{g}{2\sqrt{M^2 - m^2}} \) |
| Soliton energy \( E \) \( \sim v^2M^2L \) | \( \sim v^2M^4L^3 \) |
| Soliton charge \( Q \) \( \sim v^2ML \) | \( \sim v^2M^3L^3 \) |

Table 2. Properties of the Q-ball solutions for large \( Q; m^2 > 0; (1+1)\)-dimensional space-time.
Figure 6: $E(Q)$ for the Q-ball in (1+1)-dimensional theory (solid line) and for free particles with $E = MQ$ (dashed line), $m^2 > 0$, $M/m = 2$.

Figure 7: $E(Q)$ for the Q-ball in (1+1)-dimensional theory, $m^2 < 0$, $M/|m| = 2$ (left plot) and $M/|m| = 1$ (right plot).
The case $m^2 < 0$ is also very similar to the one in the $(3+1)$-dimensional case. Again there are two phases, presented in Fig. [7]. The transition between the phases in $(1+1)$-dimensional space-time occurs at $\frac{M}{|m|} \approx 1.496$.

### 3.2 Perturbations and classical stability analysis

Now let us consider the linearized theory above the Q-ball solution presented in the previous subsection. We will be interested mainly in exponentially growing modes which indicate the existence of classical instability. We consider small perturbations above the classical solution of the form

$$
\phi(t, z) = e^{i\omega t} f(z) + e^{i\omega t} \eta(t, z).
$$

Formally, the linearized equation of motion takes the form

$$
-i\dot{\eta} - 2i\omega \eta + \omega^2 \eta + \eta'' - (M^2 + (m^2 - M^2)\theta(f^2 - v^2)) \eta - (m^2 - M^2)\delta(f^2 - v^2)f^2(\eta + \eta^*) = 0,
$$

which can be easily brought to the form

$$
-i\dot{\eta} - 2i\omega \eta + \omega^2 \eta + \eta'' - (M^2 + (m^2 - M^2)\theta(f^2 - v^2)) \eta - F\delta(|z| - R)(\eta + \eta^*) = 0 \quad (26)
$$

where

$$
F = F(\omega) = \frac{m^2 - M^2}{2\sqrt{M^2 - \omega^2}}.
$$

(27)

It is worth mentioning that the mixing between $\eta$ and $\eta^*$ occurs for potential (3) only at the points $z = \pm R$ through the term with the $\delta$ function and this fact really simplifies the consideration.

There are two obvious solutions to equation (26). The first one is the translational mode

$$
\eta \sim f'(z) \quad (28)
$$

and the second one corresponds to the existence of the $U(1)$ global symmetry

$$
\eta \sim if(z). \quad (29)
$$

In order to find other possible solutions in the linearized theory we consider the standard ansatz for perturbations (see, for example, [3, 5])

$$
\eta = \psi_1(z)e^{i\gamma t} + \psi_2(z)e^{-i\gamma^* t}. \quad (30)
$$

Substituting this decomposition into (26) one obtains the equations

$$
\begin{align*}
[-\partial_{\bar{z}}^2 + U(z) + F\delta(|z| - R)] \psi_1 + F\delta(|z| - R)\psi_2 &= (\omega + \gamma)^2 \psi_1, \\
[-\partial_{\bar{z}}^2 + U(z) + F\delta(|z| - R)] \psi_2 + F\delta(|z| - R)\psi_1 &= (\omega - \gamma)^2 \psi_2, 
\end{align*}
$$

(31)
where $U(z) = M^2 \theta(|z| - R) + m^2 \theta(R - |z|)$. It should be noted that formally the case $\text{Re}\gamma = 0$ should be considered separately, because in this case there is no separation of the equation (26) into terms proportional to $e^{i(\text{Re}\gamma)t}$ and $e^{-i(\text{Re}\gamma)t}$, which results in two different equations presented above. But it appears that the equation for the spectrum, which will be obtained from (31), is also valid for the case $\text{Re}\gamma = 0$.

First, let us consider normalized solutions to equations (31). They take the form

$$\eta(z) = e^{i\gamma t} e^{i\mu_1 z} a_1 + e^{-i\gamma t} e^{i\mu_2 z} b_1, \quad \text{Im}\mu_1 > 0, \quad \text{Im}\mu_2 > 0 \quad (32)$$

with $(\gamma + \omega)^2 = M^2 + \mu_1^2$ and $(-\gamma^* + \omega)^2 = M^2 + \mu_1^2$ for $|z| > R$ and

$$\eta(z) = e^{i\gamma t} (e^{i\mu_2 z} a_2 + e^{-i\mu_2 z} \tilde{a}_2) + e^{-i\gamma t} (e^{i\mu_2 z} b_2 + e^{-i\mu_2 z} \tilde{b}_2) \quad (33)$$

with $(\gamma + \omega)^2 = m^2 + \mu_2^2$ and $(-\gamma^* + \omega)^2 = m^2 + \mu_2^2$ for $|z| \leq R$. Note that if the conditions $\text{Im}\mu_1 > 0, \text{Im}\mu_2 > 0$ are not fulfilled, the solutions do not fall off at spatial infinity and thus such solutions are not normalizable.

It is convenient to consider modes which are even and odd in $z$ separately from the very beginning. In the first case we have $\eta|_{z=0} = 0$. The matching condition at the point $z = R$, coming from (26), looks as follows:

$$\eta|_{z=R+0} - \eta|_{z=R-0} = F(\eta + \eta^*)|_{z=R} \quad (34)$$

The condition (34) together with the continuity of $\eta$ at the point $z = R$ generate the characteristic equation on $\gamma$. After some calculations (which are straightforward but quite tedious and we do not present the details of calculations here), we can get the following characteristic equation for the spectrum:

$$iF \left( G(\mu_1, \mu_2) Y(-\tilde{\mu}_2^*) + G(-\tilde{\mu}_1^*, -\tilde{\mu}_2^*) Y(\mu_2) \right) - G(\mu_1, \mu_2) G(-\tilde{\mu}_1^*, -\tilde{\mu}_2^*) = 0, \quad (35)$$

where

$$G(\mu_1, \mu_2) = \mu_2 \left( 1 - e^{-i2\mu_2 R} \right) - \mu_1 \left( 1 + e^{-i2\mu_2 R} \right),$$

$$Y(\mu_2) = 1 + e^{-i2\mu_2 R}.$$ 

Note that equation (35) can be used only if the conditions

$$\text{Im}\gamma \neq 0, \quad \text{Im}\gamma = 0 \quad \text{and} \quad \omega - M < \text{Re}\gamma < M - \omega \quad (37)$$

are fulfilled. In this case $\eta$ falls off exponentially with $z \to \pm\infty$ and we have normalized solutions. Otherwise (i.e. if $\text{Im}\gamma = 0$ and $\text{Re}\gamma \geq M - \omega$ or $\text{Re}\gamma \leq -M + \omega$) one should
consider a more general form of perturbations in order to describe modes from the continuous spectrum.

Analogous calculations can be made for the odd modes, for which \( \eta|_{z=0} = 0 \). The matching condition again has the form (34) and we get equation (35), but now with

\[
G(\mu_1, \mu_2) = \mu_1 \left( 1 - e^{-i\mu_2 R} \right) - \mu_2 \left( 1 + e^{-i\mu_2 R} \right),
\]

\[
Y(\mu_2) = -1 + e^{-i\mu_2 R}.
\]

These formulas also can be used only if (36) or (37) are fulfilled.

It is interesting to note that, according to (36) and (37), the unstable mode (if exists) is normalized. As already mentioned, the non-normalized modes from the continuous spectrum exist if \( \text{Im}\gamma = 0 \) and if \( M - \omega \leq \text{Re}\gamma \) or \( \text{Re}\gamma \leq \omega - M \).

Now let us turn to an examination of the discrete spectrum of our model. First, it is necessary to note that there exist obvious solutions to equation (35) in the case of the odd modes. Indeed, if \( \text{Im}\gamma = 0 \), these solutions to (35) are simply \( \mu_2 = 0 \) or \( \tilde{\mu}_2 = 0 \), which leads to \( \text{Re}\gamma = -\omega - m \), \( \text{Re}\gamma = -\omega + m \), \( \text{Re}\gamma = \omega + m \), and \( \text{Re}\gamma = \omega - m \). But it can be shown that for these values of \( \gamma \) the only solution to the initial system of linearized equations of motion is \( \eta \equiv 0 \). Thus, these roots are unphysical. Note that depending on the value of \( M/m \) these roots corresponding to unphysical modes may formally lie in the continuous spectrum.

Equation (35) was examined numerically for \( M/m = 2 \). We have failed to find solutions to (35) for \( \text{Re}\gamma \neq 0 \) and \( \text{Im}\gamma \neq 0 \). For the case \( \text{Im}\gamma = 0 \) we have found that solutions to equation (35) may exist (or may not exist) depending on the value of \( \omega \). The case \( \text{Re}\gamma = 0 \) and \( \text{Im}\gamma \neq 0 \) was examined separately for different values of \( M/m \). We have failed to find any exponentially growing odd mode. For \( \omega > \omega_c \), where \( \omega_c = \omega_c (M/m) \) is the frequency of the Q-ball solution corresponding to the minimal charge and energy (for \( M/m = 2 \) this frequency is \( \omega_c \approx 1.803 \, m \)), only one exponentially growing even mode was found (it is evident that modes with nonzero imaginary part of \( \gamma \) correspond to classical instability). For \( \omega < \omega_c \) we have failed to find any exponentially growing even mode. Thus, the numerical analysis shows that the upper ("wide") branch in Fig. 6 is always classically unstable, whereas the lower ("narrow") branch, for which \( \frac{dQ}{d\omega} < 0 \), is always classically stable, which coincides with the classical stability criterion of [8, 9]. An example of a nontrivial numerical solution for \( \text{Im}\gamma \) for even excitations together with the function \( \frac{dQ}{d\omega} \) is presented in Fig. 8. The results of our analysis also show that Q-ball solutions with minimal energy are always classically stable.

The numerical search for exponentially growing modes was also made for the case \( m^2 < 0 \). We restricted ourselves only to examining the even modes with \( \text{Re}\gamma = 0 \). The results of the numerical search for \( \text{Im}\gamma \) for even excitations together with the function \( \frac{dQ}{d\omega} \) are presented in
Figure 8: The value of $|\text{Im}\gamma|$ as a function of $\omega$ for the case $m^2 > 0$, $M/m = 2$. Solid line – numerical calculations; dotted line – the function $\xi|F(\omega)|/m$. The dashed line corresponds to the function $\frac{dQ}{d\omega}$. 
Figure 9: The value of $|\text{Im}\gamma|$ as a function of $\omega$ for the case $m^2 < 0$, $M/|m| = 2$. Solid line – numerical calculations; dotted line – the function $\xi|F(\omega)|/|m|$. The dashed line corresponds to the function $dQ/d\omega$.

Figure 10: The value of $|\text{Im}\gamma|$ as a function of $\omega$ for the case $m^2 < 0$, $M/|m| = 1$. Solid line – numerical calculations; dotted line – the function $\xi|F(\omega)|/|m|$. The dashed line corresponds to the function $dQ/d\omega$. 
Fig. 9 and Fig. 10. We see that in the regions where \( \frac{dQ}{d\omega} < 0 \) exponentially growing modes are absent, whereas such modes exist if \( \frac{dQ}{d\omega} > 0 \). The latter means that Q-ball solutions in the phase presented on the right plot in Fig. 7 are always classically unstable, because \( \frac{d^2 E}{dQ^2} = \frac{d\omega}{dQ} > 0 \) in this case. This result coincides with the results of numerical analysis in the model of [3] in (3 + 1)-dimensional space-time with an analogous \( E(Q) \) dependence.

In the limit of large charges this instability can be obtained analytically in all three cases \( (m^2 > 0, m = 0, m^2 < 0) \). Let us show it explicitly. Suppose that \( \gamma \) is purely imaginary and \( |\gamma| \gg M \). In this case equation (26) can be rewritten as

\[
-|\gamma|^2 \eta + \eta'' - F\delta(|z| - R) (\eta + \eta^*) \approx 0,
\]

which suggests that the function \( \eta \) can be chosen to be real. The even solution to equation (38) takes the form

\[
\eta(z) = \alpha e^{\gamma|t|} \cosh(|\gamma|z), \quad |z| < R, \tag{39}
\]

\[
\eta(z) = \beta e^{\gamma|t|} e^{-|\gamma z|}, \quad |z| > R. \tag{40}
\]

The continuity of \( \eta \) at the points \( z = \pm R \) and discontinuity of its first derivative at these points (see equation (34)) results in

\[
|\gamma| (\text{th}(|\gamma|R) + 1) = -2F. \tag{41}
\]

Let us parametrize \( |\gamma| \) as

\[
|\gamma| = -\xi F \tag{42}
\]

For the solutions at large \( Q \), which are supposed to be unstable, we have \( \omega \sim M \) and

\[
R \approx \frac{\sqrt{M^2 - \omega^2}}{M^2 - m^2}. \tag{43}
\]

Substituting (42) and (43) into (41) we arrive at

\[
\xi (\text{th}(\xi/2) + 1) = 2, \tag{44}
\]

which results in \( \xi \approx 1.2785 \). The function \( \xi|F(\omega)|/|m| \) together with the numerical solution for \( |\text{Im}\gamma| \) is presented in Fig. 8, Fig. 9 and Fig. 10.

An important remark concerning the validity of the linear approximation is in order. We examine classical perturbations, and if the scalar field potential is smooth, we can consider linear approximation without any restrictions (we can always choose small enough amplitude of perturbations which does not destroy linear approximation). Our case is rather nonstandard because of the Heaviside step function in the scalar field potential. The \( \theta \) function can be easily regularized as

\[
\theta \left( \frac{\dot{\phi}^* \dot{\phi}}{v^2} - 1 \right) \rightarrow \frac{1}{2} \left( 1 + \text{th} \left[ \alpha \left( \frac{\dot{\phi}^* \dot{\phi}}{v^2} - 1 \right) \right] \right), \tag{45}
\]
where $\alpha$ is a large dimensionless parameter. In the linear approximation it can be expanded as

$$
\frac{1}{2} \left( 1 + \text{th} \left[ \alpha \left( \frac{\phi^* \phi}{v^2} - 1 \right) \right] \right) \approx \frac{1}{2} \left( 1 + \text{th} \left[ \alpha \left( \frac{f^2}{v^2} - 1 \right) \right] \right) + \frac{\alpha f (\eta + \eta^*)}{2v^2 \cosh^2 \left( \alpha \left( \frac{f^2}{v^2} - 1 \right) \right)}. \quad (46)
$$

Let us take perturbations at the points $|z| = R$. It is clear that the relation $\frac{\alpha (\eta + \eta^*)}{v} \ll 1$ should hold in order not to break down the linear approximation. In the limit $\alpha \to \infty$ (which is exactly the case for which the linearized equation (26) was obtained), this relation obviously leads to the constraint $\eta + \eta^*|_{|z|=R} = 0$, otherwise formally we get the breakdown of the linear approximation. Thus, let us consider the linearized theory with the constraint

$$
\eta + \eta^*|_{|z|=R} = 0. \quad (47)
$$

Substituting (30) into (47) and taking into account the fact that the relation (47) should hold at any moment of time, we get

$$
\psi_2(\pm R) = -\psi_1(\pm R). \quad (48)
$$

Now we take the first equation of (31). Taking into account (48), it can be rewritten as

$$
\hat{L}\psi_1 = (\omega + \gamma)^2 \psi_1 \quad (49)
$$

and

$$
\hat{L}\psi_1^* = (\omega + \gamma^*)^2 \psi_1^*, \quad (50)
$$

where the operator $\hat{L}$ is defined as $\hat{L} = -\partial_z^2 + U(z)$. Multiplying (49) by $\psi_1^*$, integrating over the coordinate $z$ and using (50), we get

$$
(\omega + \gamma)^2 = (\omega + \gamma^*)^2. \quad (51)
$$

The latter equation has two solutions. The first one leads to $\text{Im}\gamma = 0$, which means that exponentially growing modes, indicating instability, are absent in this case. The second solution is $\gamma = -\omega + i\gamma_i$, where $\gamma_i = \text{Im}\gamma$. Using (49), we easily obtain

$$
\langle \psi_1|\hat{L}|\psi_1 \rangle = -\gamma_i^2 \langle \psi_1|\psi_1 \rangle. \quad (52)
$$

But this relation can not be fulfilled, because $\langle \psi_1|\hat{L}|\psi_1 \rangle$ is nonnegative for any $\psi_1$. Indeed, the eigenfunction of the operator $\hat{L}$, corresponding to the eigenvalue $\omega^2 > 0$, is $f(z)$. This eigenfunction has no nodes, which means that it is the lowest mode and thus all other eigenvalues are also larger than zero, leading to non-negativity of $\langle \psi_1|\hat{L}|\psi_1 \rangle$. 

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Analogous calculations, leading to the same result, can be performed for the function $\psi_2$. Thus, we have shown that formally there are no modes indicating instability if constraint (17) is imposed. Meanwhile, this constraint is a consequence of the existence of a generalized function in our potential. A physically reasonable potential should be smooth, leading to nonzero, though sometimes very small, amplitude of perturbations. It this case in the linearized equations of motion (26) delta-function should be replaced by some smooth function containing some parameter of regularization $\alpha$ (like the one in (15)). For very large, but finite $\alpha$, the linearized equation of motion, as well as corresponding solutions for perturbations and eigenvalues $\gamma$, may look very similar to those in the $\alpha \to \infty$ limiting case. Of course, most probably such an equation can not be solved analytically. But the difference between the exact solution and the solution in the $\alpha \to \infty$ limiting case is controlled by the parameter $\sim 1/\alpha$, and for very large $\alpha$ the corrections to the case $\alpha \to \infty$ are supposed to be very small. This reasoning justifies the use of the "relaxed" linearized theory described by equation (26), without constraint (17). This "relaxed" theory allows one to see what could happen with the stability in a more realistic case of a smooth scalar field potential.

4 Conclusion

In the present paper we discussed Q-ball solutions in a model describing complex scalar field with a piecewise parabolic potential (3). It was shown that due to the simplicity of the potential Q-ball solutions can be found analytically, which really simplifies the analysis of their properties – the spectra of solutions were obtained analytically in (3+1)- and (1+1)-dimensional space-times for different values of the parameters of the model. For $\omega \to M$ the Q-ball solutions are very close to the condensate line $E = MQ$ and this fact can be interesting for examining the Q-ball production. It should be noted that the Q-ball solutions presented in this paper can not be described by the standard thin-wall approximation.

The stability of the obtained solutions was also examined. In the simpler theory in (1+1)-dimensional space-time it was shown explicitly that solutions with $dQ/d\omega > 0$ are always unstable, whereas solutions $dQ/d\omega < 0$ do not contain exponentially growing modes, at least of the simplest form $\sim e^{\gamma t}$.

Though the potential (3) is very simple, it provides a variety of Q-ball solutions of different types. We hope that the existence of exact and simple analytical Q-ball solutions with known properties (such as their stability and $E(Q)$ dependence) allows one to consider the model, presented in this paper, as a useful tool for examining different phenomenological scenarios involving Q-balls.
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5 Appendix A

Let us prove that inequality $E > \omega Q$ holds for any Q-ball solution. Consider (D+1)-dimensional space-time. According to (1) we can use the time-independent effective action

$$S_{\text{eff}} = \int d^Dx \left( \omega^2 f^2 - \partial_i f \partial_i f - V(f) \right),$$

where $i = 1, ..., D$, instead of the initial one. Suppose that there exists a solution $f(\vec{x})$ to the corresponding equation of motion. Let us apply the scale transformation $f(\vec{x}) \to f_\lambda(\vec{x}) = f(\lambda \vec{x})$ to this solution and substitute $f_\lambda(\vec{x})$ into the effective action instead of $f(\vec{x})$ (this technique was used in [12] to show the absence of time-independent soliton solutions in some models with a nonlinear scalar field). Now it takes the form

$$S_{\text{eff}}^\lambda = \int d^Dx \left( \omega^2 f^2(\lambda \vec{x}) - \lambda^2 \frac{\partial}{\partial(\lambda x^i)} f(\lambda \vec{x}) \frac{\partial}{\partial(\lambda x^i)} f(\lambda \vec{x}) - V(f(\lambda \vec{x})) \right),$$

where we have passed to the new variables $\lambda \vec{x} \to \vec{x}$ in the second integral. According to the principle of least action we have

$$\left. \frac{S_{\text{eff}}^\lambda}{d\lambda} \right|_{\lambda=1} = 0$$

and thus

$$\int d^Dx V(f) = \omega^2 \int d^Dx f^2 - \frac{D-2}{D} \int d^Dx \partial_i f \partial_i f.$$

Substituting the latter equation into the definition of energy and taking into account the definition of charge $Q = 2\omega \int d^Dx f^2$, we get

$$E = \int d^Dx \left( \omega^2 f^2 + \partial_i f \partial_i f + V(f) \right) = \omega Q + \frac{2}{D} \int d^Dx \partial_i f \partial_i f > \omega Q.$$
Let us show that if the condition $d^2E/dQ^2 < 0$ is fulfilled, then the Q-ball is stable against fission, i.e. against a decay into Q-balls with smaller charges. Suppose $E(Q)$, where $Q$ is supposed to be nonnegative, is a positive monotonically increasing function in the region of charges we are interested in. In this case condition $d^2E/dQ^2 < 0$ means that

\[
\left.\frac{E(Q)}{dQ}\right|_{Q=Q_1} > \left.\frac{E(Q)}{dQ}\right|_{Q=Q_1+\tilde{Q}}
\]

for $\tilde{Q} > 0$. Let us integrate this inequality over the coordinate $\tilde{Q}$ in the region $[Q_1, Q_2]$. We get

\[
E(Q_2) - E(Q_1) > E(Q_2 + \tilde{Q}) - E(Q_1 + \tilde{Q}),
\]

which can be rewritten as

\[
E(Q_2) + E(Q_1 + \tilde{Q}) > E(Q_2 + \tilde{Q}) + E(Q_1),
\]

If we take $Q_1 = 0$ and if $E(0) = 0$, then (60) leads to

\[
E(Q_2) + E(\tilde{Q}) > E(\tilde{Q} + Q_2),
\]

which means that Q-ball fission is energetically forbidden. But in many models including the one discussed in the present paper there exists a minimal charge $Q_{\min} \neq 0$ such that $E(Q_{\min}) = E_{\min} \neq 0$. In such a case, one can try to redefine the function $E(Q)$ in the region $[0, Q_{\min}]$ in order to get an auxiliary function $E_{aux}(Q)$: $E_{aux}(0) = 0$, $E_{aux}(Q)$ is a smooth monotonically increasing function for $Q > 0$, $d^2E_{aux}(Q)/dQ^2 < 0$ and $E_{aux}(Q) = E(Q)$ for $Q \geq Q_{\min}$. If it is possible to construct the function $E_{aux}(Q)$, then inequality (61) is valid for $\tilde{Q}, Q_2 \geq 0$ and, consequently, for $\tilde{Q}, Q_2 \geq Q_{\min}$ (of course, any Q-ball with the charge $Q < 2Q_{\min}$ is always stable against fission regardless of the sign of $d^2E/dQ^2$).

To show that one can always construct such an auxiliary function for a Q-ball let us consider the function $E(Q) = \tilde{\omega}Q$, where the constant $\tilde{\omega}$ is defined by $\tilde{\omega} = E_{\min}/Q_{\min}$. Recall that $dE/dQ|_{Q=Q_{\min}} = \omega_{\min}$. It is evident that if $dE/dQ|_{Q=Q_{\min}} = \tilde{\omega} > \omega_{\min} = dE/dQ|_{Q=Q_{\min}}$ then one can always construct a function $E_{aux}(Q)$, otherwise it is impossible, see examples in Fig. [III]

Using equation (57) we get

\[
E_{\min} = \omega_{\min}Q_{\min} + \frac{2}{D} \int d^Dx \partial_i f \partial_i f = \tilde{\omega}Q_{\min}
\]

and thus $\tilde{\omega} > \omega_{\min}$. The latter means that we can always construct a function $E_{aux}(Q)$ and inequality (61) fulfills if $d^2E/dQ^2 < 0$ is valid.
Figure 11: Solid lines correspond to functions $E(Q)$ and $\dot{E}(Q)$, dashed line corresponds to continuation of the function $E(Q)$ (i.e., to the function $E_{\text{aux}}(Q)$).

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