Velocity correlations in granular materials

Tong Zhou
The James Franck Institute, The University of Chicago, 5640 S. Ellis Avenue, Chicago, IL 60637
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A system of inelastic hard disks in a thin pipe capped by hot walls is studied with the aim of investigating velocity correlations between particles. Two effects lead to such correlations: inelastic collisions help to build localized correlations, while momentum conservation and diffusion produce long ranged correlations. In the quasi-elastic limit, the velocity correlation is weak, but it is still important since it is of the same order as the deviation from uniformity. For system with stronger inelasticity, the pipe contains a clump of particles in highly correlated motion. A theory with empirical parameters is developed. This theory is composed of equations similar to the usual hydrodynamic laws of conservation of particles, energy, and momentum. Numerical results show that the theory describes the dynamics satisfactorily in the quasi-elastic limit, however only qualitatively for stronger inelasticity.

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I. INTRODUCTION

A granular system normally consists of a large number of particles colliding with one another and losing a little energy in each collision. If such a system is shaken to keep it in motion, its dynamics resembles that of fluids, in that the grains move seemingly randomly. Considerable effort has been devoted to the development of a continuum description for hydrodynamic equations \[1–11\].

Two approaches are employed by different authors. One is to set up a Boltzmann equation \[1–4\], and then to calculate hydrodynamic quantities by doing averaging with the distribution function derived from the equation. In this case, the molecular chaos assumption of the Boltzmann equation assumes zero correlations between particles. The other approach is to specify some hydrodynamic quantities, and then write down the conservation equations for them \[12–14\]. Generally, there are three equations: conservation of mass, balance of energy, conservation of momentum. The mass conservation is in the standard hydrodynamic form. The momentum flux balance equation is in the form of Navier-Stokes equation for fluid dynamics. The energy “conservation” equation includes dissipation of energy via collisions.

The failing of Liouville’s theorem for granular systems \[12\] casts doubts on the applicability of conventional approaches to hydrodynamic equations. In stead of writing down such equations based on unjustified assumptions, studying the dynamics with as few assumptions as possible, and trying to develop a theory closely connected to experiments, may be a less ambitious, but more solid approach.

One major consequence of the usual hydrodynamic theories of fluids is the Maxwell-Boltzmann distribution. In the frame in which the average system velocity is zero, this distribution implies no momentum correlations whatsoever among different particles. This result is true for classical particles independent of the strength of the inter-particle potential. In contrast, however, granular materials commonly induce correlated collective behaviors. Think about the surface waves of vibrated sand \[13\], or their convection patterns \[14\]. (for a recent review, see \[15\]). The grains which take part in these collective behaviors do have correlated velocities. We therefore ask: how important are these correlations and how are they built up?

In this paper, we investigate the building up of correlations between velocities of grains. There are two mechanisms upon which correlations can be built up. One is the inelastic collisions between particles—after a collision, the velocity difference between two particles is smaller than that before the collision. This is a local effect, and the correlation is short ranged. The other mechanism is from momentum conservation—the larger the scale of a perturbation producing a localized average velocity, the slower the perturbation decays \[16\]. Fluctuations make the system non-uniform, so that there are localized clusters of particles all moving with about the same velocity. This correlation effect is a result of fluctuations which are neglected in the usual hydrodynamic treatments.

There are hydrodynamic theories ignoring fluctuations which are consistent with numerical results for weak inelasticities, but are quite inaccurate when inelasticities are strong, see, e.g. \[17\]. These theories are attempts to describe velocity fluctuations about some mean flow. They work fine in quasi-elastic regime because correlations are small and negligible. But when inelasticities are strong, collisions can bring groups of neighboring particles to essentially the same velocity and thereby produce a correlated motion which enhances the observable effect of any fluctuations in the system. We shall see this happen in our study.

The boundary conditions and system sizes independence of the essential characteristics of thermodynamic systems is one indication that thermodynamics is a universal description. However, this independence is lost in granular systems. We show that a universal description may still exist for the unconserved modes of the dynamics.
II. THE THIN PIPE MODEL

A. The system

To investigate the validity of a hydrodynamic description, we should study the simplest situation which can show hydrodynamic behavior. In the elastic case, one-dimensional systems have too many conservation laws and do not show a fully ergodic or hydrodynamic behavior [3, 4]. Here, we investigate a two-dimensional system in the form of a long thin pipe (Fig. 1). The grains confined in the pipe are all identical, and the width of the pipe is set so that two grains cannot pass each other. Thus the motion of grains is two dimensional to ensure ergodicity, while at the same time we can order these grains. The passing condition enable this thin pipe model to simplify greatly both numerical and analytical calculations. (A two-dimensional version of this model is studied by Brey and Cubero [4].)

FIG. 1. Snapshots of the thin pipe system. The periodic side walls are indicated by dashed lines. The two end walls are energy sources kept at the same temperature. Notice how most of the particles fall into a cluster, which moves up and back through the system.

The two side walls are periodic—after leaving one side wall a particle comes back through the other. The distance between the side walls is chosen to be 2.5 times the radius of a particle. This choice prevents any passing. Two end walls are energy sources, and are kept at the same temperature.

For a thermodynamic system, the bulk properties should not depend on the details of boundary conditions. However, for some granular properties, boundary conditions can be quite important [2, 14]. We employ two different boundary conditions in the numerical calculations: In both cases, when a particle hits an end wall the direction of its motion is turned around, and the particle is returned to the system. In the fixed speed boundary condition, the returned particles move away from the wall with a unit speed. Alternatively, in the Boltzmann boundary condition the returning speed is picked from a distribution \( P(u) = 2ue^{−u^2} \). All figures describe simulations with the Boltzmann boundary conditions unless otherwise specified.

B. The parameters and variables

We use the simplest model: non-rotating particles. After a collision, the radial relative velocity changes sign, and decreases by a factor of the restitution coefficient \( r \), with \( 0 < r < 1 \). In the collision, the other components of the velocities are unchanged. Thus, \( r = 1 \) is for elastic particles, and \( r = 0 \) for extremely inelastic particles. We also define \( \epsilon = 1 − r \).

The coordinate in the problem is an index, \( i \), which indicates the position of the particle. Suppose there are \( N = 2n \) particles in the thin pipe. They are ordered as

\[-n, -(n-1), \ldots, -1, 0, 1, \ldots, (n-1).\]

By using the particle number, \( i \), as our coordinate, we take advantage of the ‘no-passing’ property of the thin pipe and thereby get a Lagrangian description of the system.

Let us denote the velocity of the \( i \)th particle as \( \vec{u}_i \), the relative velocity between the \( i \)th and the \((i+1)\)th particle as \( \vec{v}_i \equiv \vec{u}_{i+1} − \vec{u}_i \), and the velocity of the center of mass as \( \vec{u} \), the velocity of the \( i \)th particle with respect to the center of mass as \( \vec{u}_i \equiv \vec{u}_i − \vec{u} \). Let us assume the pipe is along the \( x \) direction. Then the \( x \)-component of the velocities are special and we denote them as \( u_i \) and \( v_i \). We use an over-line notation to indicate the root mean square (rms) value of some quantities.

We propose a method to calculate profiles of various quantities throughout the system and the velocity correlations. (A profile is a plot of the value of some averaged quantity as a function of the particle number variable \( i \).) Instead of the strongly correlated velocities \( \vec{u}_i \)'s, we study the relative velocities of neighboring particles, \( \vec{v}_i \). In using \( \vec{v} \) we focus our attention on the relative motion of the particles and away from their collective and correlated motion.

There are four parameters which will describe our system, the particle number, \( N \), the pipe length \( L \), width \( W \), and the inelasticity \( \epsilon \). Of course \( \epsilon \) measures the total amount of inelasticity in one collision. In a system with many particles, the effect of the inelasticity is enhanced by the correlation effects. For this reason, we expect two combinations of \( N \) and \( \epsilon \) to be important. The product \( Ne \) measures the total amount of inelasticity in the system. For a one dimensional system, imagine a particle with a large velocity hitting a group of \( n \) particles, sitting almost at rest. The added momentum will cascade down
the group until at the end of the line the transmitted
momentum will be diminished by a factor \(\exp(-n\epsilon)\). In
addition, a previous calculation [11] showed that dissipa-
tion of energy led to a gradual decay of temperature in
the form of an exponential of \(-c\sqrt{n}\epsilon\) where \(c\) is a con-
stant. Thus we expect a dip in temperature determined
by the combination of parameters, \(\sqrt{\epsilon N}\). Changing the
remaining parameters, \(L\) and \(W\), will only modify some
numerical factors in the theory—but will not change the
qualitative behavior of the system.

The system showed in Figure 1 contains both low den-
sity and intermediate density regimes. There are some
complication in such systems because of different geo-
metrical factors for different density regimes [11]. To
avoid such complication and focus on the dynamics of
the system, we carry out our numerical calculations only
for systems with extremely high density, where the typ-
ical spacing between neighboring particles is about 2% of the radius of a particle; or for systems with extremely
low density, where the spacing at the highest density re-
gion of the system is about 10 times the radius. The
essential characteristics of the dynamics are independent
of density regimes.

C. The steady state

This system can reach a statistical steady state. In this
state, the particles move fast near the hot walls, and the
density is low there. Towards the center of the system,
the density is higher. For quasi-elastic situations, the sys-
tem is relatively uniform; but for stronger inelasticities,
the particles near the center can form a cluster and move
with about the same velocity. The cluster was seen and
understood in previous calculations [11,16]. The relative
motion of particles is reduced by the inelastic collisions
between them. In fact, when \(N\epsilon\) is large, the relative
motion can be very small and then momentum conserva-
tion causes that each particle in the cluster has about
the same velocity which is just the mean velocity of the
cluster.

![Figure 2](image-url) Profiles of a low density system of 100 particles
with \(r = 0.94\), just above the critical value for inelastic col-
lapse \(r_{c}\). \(\bigcirc\) is for \(\langle u_i^2 \rangle\), \(\star\) for \(\langle v_i^2/2 \rangle\), and \(+\) for \(\langle u_i u_{i+1} \rangle\).

Figure 2 shows a plot of some profiles in an inelastic
situation with two hot walls. Notice that the profile of
\(\langle u_i^2 \rangle\) has a flat region at the center. This was seen before
[4]. That flattening occurs because the central particles
almost always fall within a cluster, and the cluster moves
with a large average velocity but small relative velocities.
The plot of \(\langle v_i^2 \rangle\) indeed shows that the relative velocity
decreases to a very small value near the center of the sys-
tem. This decay in \(\langle v_i^2 \rangle\) is roughly what we might expect
from a simple hydrodynamic description, in which one
balances energy flux with dissipation [11]. The hydrody-
namics then gives an \(i\)-dependence which is a superposi-
tion of growing and decaying exponentials. That theory
is in some sense a mean field theory which ignores the
correlations between velocities. In much of what we do,
delicate and long-range correlations effects will be very
important for the behavior of \(\vec{u}\)'s but less important for the \(\vec{v}\)'s. In fact we shall see that the rms of \(v_i\) obeys

\[
\frac{\partial^2}{\partial i^2} \bar{v}_i = b^2 \bar{v}_i, \tag{1}
\]

where \(b^2\) is proportional to \(\epsilon\) for small values of the in-
elasticity. The solution to the equation is

\[
\bar{v}_i = \bar{v}_0 \cosh(bi). \tag{2}
\]

Equations (1) and (2) describe a situation in which heat
conduction balances against energy dissipation.

On the other hand, the large degree of correlation be-
tween \(u_i\) and \(u_{i+1}\) is quite unexpected. No such correla-
tion occurs in the usual statistical mechanics. This kind
of correlation effect is not directly contained in any hy-
drodynamic equations. As we shall see, it is a result of
fluctuations not usually included in hydrodynamics.
FIG. 3. Profiles of $\langle u^2_i/2 \rangle$ and $\langle v^2_i/2 \rangle$ for two different boundary conditions. The system has low density 100 particles and $r = 0.94$. Each profile is rescaled by changing the scale of velocity so that $\langle v^2_i/2 \rangle = 1$. There are two lines, which nearly overlap each other, describing $\langle v^2_i/2 \rangle$. The o is for $\langle u^2_i \rangle$ with the Boltzmann boundary condition, and the + is for $\langle u^2_i \rangle$ with fixed speed boundary condition. These two profiles are very different.

Boundary conditions are often important for granular systems. Figure 3 shows the effects of boundary conditions. We see $\langle u^2_i \rangle$ depends sensitively on boundary conditions. In contrast, after a rescaling, $\langle v^2_i \rangle$ is nearly independent of boundary conditions. There is no similar rescaling which can make the profiles for $\langle u^2_i \rangle$ overlap.

D. Correlated motion and random motion

Since
\[
\langle u^2_i \rangle = \langle u^2_i + u^2_{i+1} \rangle - 2\langle u_i u_{i+1} \rangle, \tag{3}
\]
when the correlation $\langle u_i u_{i+1} \rangle$ is weak, we simply have $\overline{u}^2_i = 2\overline{u}_i^2$, assuming a weak $i$-dependence. But when correlation is strong, the relation between $\overline{u}^2_i$ and $\overline{u}_i^2$ is quite different. We shall study that difference in detail. From the mechanism described above, we know near the center, $\overline{u}^2_i$ is roughly constant, independent of $i$, as a consequence of the motion of the cluster. Conversely, $\overline{u}_i^2$ will vary because of energy dissipation. In our considerations, we shall focus upon $\overline{u}_i^2$, which has an average which can be interpreted as a local temperature. We argue that $\overline{u}_i^2$ is a more relevant variable than $\overline{u}_i^2$, since to a large extent, it determines the collision rate, and the effect of a collision. In addition, $\overline{u}^2_i$ behaves as predicted by the simple hydrodynamics theory, it decays exponentially, and forms a hyperbolic cosine curve as a function of $i$. Conversely, $\overline{u}_i^2$ is produced by subtle correlation effects.

We can also write (3) in the form
\[
\frac{1}{2} \langle u^2_i + u^2_{i+1} \rangle = \frac{1}{2} \langle u^2_i \rangle + \langle u_i u_{i+1} \rangle. \tag{4}
\]
The term on the left hand side describes the total motion with respect to the lab frame, the second term on the right hand side describes the correlated motion between particle $i$ and particle $i+1$, and the first term on the right hand side describe the random relative motion between neighboring particles. So put this into words,
\[
\left( \begin{array}{c} \text{total motion} \\ \text{random motion} \end{array} \right) = \left( \begin{array}{c} \text{random motion} \\ \text{correlated motion} \end{array} \right).
\]
The first term on the right can be interpreted as a temperature; the second as a result of the correlated motion of the two particles. In this way, we see that the ratio
\[
R_i = \frac{\langle v^2_i \rangle}{\langle u^2_i + u^2_{i+1} \rangle} = \frac{\text{random motion}}{\text{total motion}}, \tag{5}
\]
indicates the amount of correlation in the motion. When the inelasticity is weak, the velocity correlations are also weak, and this ratio is very close to unity. For strong inelasticity, where correlations are strong, this ratio can be very small.

E. PDF’s of velocities

The probability distribution functions (PDF) for $u_i$ and $v_i$ provide considerable additional insight into the nature of the system. See figures (Fig. 4) and (Fig. 3). In these figures, the variables are normalized to give each PDF the same variance.

In the PDF plots for $u_i$, we see a fundamentally Gaussian behavior inside the cluster. Outside the cluster, the part of the curve shown is Gaussian but there is a strong high velocity tail.

FIG. 4. Time-averages probability for $u_i$ in a low density system with $N = 100, r = 0.94$. Five such curves are shown, which are for $i = 0, -10, -20, -35, -45$. The first three in this list are close to Gaussian and they lie almost on top of one another. The other two, are quite different.

In contrast, the PDF plots for $v_i$ show a structure which is essentially the same inside and outside the cluster. Thus, all over the system, the $v'$s behave in the same way, but this behavior is quite non-trivial.
ory and Equation (2), a hyperbolic cosine function of these suggest that plus its weak dependence on the boundary conditions—some simple hydrodynamics equations.

We will use the constancy of the PDF of \( v_i \) (in the whole system) and \( u_i \) (in the interior of the system) to develop our theoretical model.

### III. MOTION IN THE CENTER OF MASS FRAME

Figure 3 suggests that we can decompose the dynamics of the system into two parts: I) the motion of grains in the center of mass frame and II) the motion of the center of mass itself. Part I is independent of boundary conditions and all the effects of boundary conditions are attributed to Part II. Part I is described in terms of the variables \( v_i \) which may be considered to be weakly correlated with one another. Part II involves variables \( u_i \), and strong correlations among the variables. In this section, we focus our attention upon the effects of conservation laws upon the system, and particularly on motion of Part I.

#### A. Theoretical calculation

Since the number of degrees of freedom of Part I is equal to the number of \( v_i \)'s, this part of motion can be described in terms of \( v_i \)'s. So the problem can be solved in two steps: the \( \text{rms} \) of \( v_i \)'s and the correlations between \( v_i \)'s. Our interests in the variable \( v_i \)'s are also based on the numerical results showed in the previous section that the profile of \( \pi_i \) is, in accordance to hydrodynamics theory and Equation (2), a hyperbolic cosine function of \( i \), plus its weak dependence on the boundary conditions—these suggest that \( v_i \) can form the basis of a solution to some simple hydrodynamics equations.

1. **Profile of \( \pi_i \)**

Collisions

For the steady state, mass conservation is reduced to a trivial statement that \( \langle u'_i \rangle = 0 \) and \( \langle v'_i \rangle = 0 \). The momentum and energy transfer between particles are results of collisions between them. So to investigate momentum and energy conservation, we study the effects of a single collision first.

Let us consider a collision between the \( i \)th and the \((i+1)\)th particle, during which \( u'_i \), \( u'_{i+1} \), and \( v'_i \) change to \( u''_i \), \( u''_{i+1} \), and \( v''_i \) respectively. According to the inelastic collision rule,

\[
\begin{align*}
    u''_i &= u'_i + \frac{1+r}{2} \hat{n} v_{i,n}, \\
    u''_{i+1} &= u'_{i+1} - \frac{1+r}{2} \hat{n} v_{i,n}, \\
    v''_i &= v'_i -(1+r) \hat{n} v_{i,n},
\end{align*}
\]

where \( \hat{n} \) denotes a unit vector, pointing in the direction of the line of centers at the point of collision while \( v_{i,n} \) is the component of \( v_i \) in that direction.

**Pressure**

The collision described above results in a change in the momentum of particle \( i + 1 \),

\[
\hat{n} \cdot [u'_i - u'_{i+1}] = -\frac{1+r}{2} \hat{n} v_{i,n}.
\]

In a long time interval \( t \), the momentum change of particle \( i + 1 \) from collisions between particle \( i \) and particle \( i + 1 \)

\[
P_i W t = -\frac{1+r}{2} \sum^{(i)} (\hat{n} \cdot \hat{x}) v_{i,n},
\]

where \( \sum^{(i)} (\cdots) \) is summation over all the collisions between the \( i \)th particle and the \((i+1)\)th particle. The \( x \) component in Equation (4) is the direction along the pipe.

In writing Equation (4) we have identified the rate of momentum transfer from particle \( i \) to particle \( i + 1 \) as an average pressure, \( P_i \) times the pipe-width, \( W \), while \( t \) is the time for the summation. We shall be dealing a lot with sums over collisions as in equation (4). To understand them, we should realize that \( \sum^{(i)} (\cdots)/t \) can be written as the rate of collisions between \( i \) and \( i + 1 \), \( c_i \), times an average over collisions \( (\cdots) \) of this type. Notice that the average over collisions is very different from the time-average \( \langle \cdots \rangle \). For example, \( \langle v_{i} \rangle \) must be zero in any steady state situation. However, since \( v_i \) must be negative for a collision to occur, then \( \langle v_{i} \rangle \) must be negative.

Now go back to Equation (4). For the steady state, the momentum flux must be a constant, so this summation over a long time interval must be independent of
averages of \( \vec{v} \) pend weakly on energy flux. However, the above expressions involve the energy dissipation with the difference of the average of relative velocity squared is the temperature, in fact, the momentum flux is proportional to \( \frac{\partial T}{\partial y} \), which are correlated and do not belong to the motion in the center of mass frame. To find a consistent description, we wish to simplify our energy-flow equation by reducing it to an equation for \( \langle v_i v_{i+1} \rangle \). However, correlations between \( v_i \) and \( v_{i+1} \) appears in \( \langle \rangle \). We must eliminate these terms. For an elastic uniform system, this correlation takes a simple form,

\[
\langle v_i v_{i+1} \rangle = \langle (u_{i+1} - u_i)(u_{i+2} - u_{i+1}) \rangle
= -\langle u_{i+1}^2 \rangle
= -\langle v_i^2 \rangle
\]

In the elastic case, it is equally true for the usual time-weighted average or for the collision weighted average, as

\[
\frac{(i+1)}{2} \sum (v_i v_{i+1,n}) + \frac{(i-1)}{2} (v_i v_{i-1,n}) = -\frac{1}{2}\langle v_i^2 \rangle v_{i+1}
\]

where \( v_i^c \equiv \sqrt{\sum (v_i^2 c_i t/n_t^i)} \), and \( n_t^i \) is the total number of collisions between particle \( i \) and particle \( i+1 \), \( n_t^i = c_it \).

As defined here, \( v_i^c \) is an collision average of \( v_i \) just before collisions.

Equation (13) has scalars on the left and right hand side. There are corrections to this relation for inelastic particles and when there is a spatial variation in the averages. The corrections must be scalars and of order \( \rho \). One correction is of order of the order of \( \epsilon T^2 \). In the other correction, \( \mu^2 \) is applied to \( \rho^2 \). However, in virtue of the result in Equation (1) these two terms are really the same. Consequently, we need only one of these two corrections. We write the resulting structure in an even parity form as

\[
\vec{v}_i^2 - \vec{v}_i^2 = -(1 - r^2) \rho^2, \\
\vec{v}_{i+1}^2 - \vec{v}_{i+1}^2 = (1 + r) \rho^2 v_{i+1, n} + \frac{(1 + r)^2}{4} \rho^2, \\
\vec{v}_{i-1}^2 - \vec{v}_{i-1}^2 = (1 + r) \rho^2 v_{i-1, n} + \frac{(1 + r)^2}{4} \rho^2.
\]

For the steady state, the total change in \( v_i^2 \) should vanish,

\[
\sum (v_i^2 - v_i^2) + \sum (v_i^2 - v_i^2) + \sum (v_i^2 - v_i^2) = 0,
\]

or equivalently,

\[
(1 - r)^i \sum v_i^2 + (i+1) \sum (v_i^2 - v_i^2) + (i-1) \sum (v_i^2 - v_i^2) = 0.
\]

The first term is from energy dissipation, while the other terms take the form of energy transfer.

**Profile of \( \rho \)**

We wish to simplify our energy-flow equation by reducing it to an equation for \( \langle v_i v_{i+1} \rangle \). However, correlations between \( v_i \) and \( v_{i+1} \) appears in \( \langle \rangle \). We must eliminate these terms. For an elastic uniform system, this correlation takes a simple form,

\[
\langle v_i v_{i+1} \rangle = \langle (u_{i+1} - u_i)(u_{i+2} - u_{i+1}) \rangle
= -\langle u_{i+1}^2 \rangle
= -\langle v_i^2 \rangle
\]

In the elastic case, it is equally true for the usual time-weighted average or for the collision weighted average, as

\[
\frac{(i+1)}{2} \sum (v_i v_{i+1,n}) + \frac{(i-1)}{2} (v_i v_{i-1,n}) = -\frac{1}{2}\langle v_i^2 \rangle v_{i+1}
\]

where \( v_i^c \equiv \sqrt{\sum (v_i^2 c_i t/n_t^i)} \), and \( n_t^i \) is the total number of collisions between particle \( i \) and particle \( i+1 \), \( n_t^i = c_it \).

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Equation (13) has scalars on the left and right hand side. There are corrections to this relation for inelastic particles and when there is a spatial variation in the averages. The corrections must be scalars and of order \( \rho \). One correction is of order of the order of \( \epsilon T^2 \). In the other correction, \( \mu^2 \) is applied to \( \rho^2 \). However, in virtue of the result in Equation (1) these two terms are really the same. Consequently, we need only one of these two corrections. We write the resulting structure in an even parity form as
\[
\sum_{i=1}^{(i+1)} (v_{i,n}v_{i+1,n}) + \sum_{i=1}^{(i-1)} (v_{i,n}v_{i-1,n}) = -\frac{1}{2} (v_{i+1}^c v_{i+1}^c + v_{i-1}^c v_{i-1}^c),
\]

(14)

Now Equation (14) can be used to transform equation (11) into the form

\[
-\epsilon v_i^2 + \frac{1+r}{4} (v_{i+1}^c v_{i+1}^c + v_{i-1}^c v_{i-1}^c) - \frac{1}{2} a_1 \epsilon (v_{i+1}^c v_{i+1}^c + v_{i-1}^c v_{i-1}^c) = 0,
\]

or, since \(v_i^c\) is a constant independent of \(i\), (see Equation (14)), we find a heat flow equation

\[
(6 - 4a_1) \frac{\epsilon}{1 + r} v_i = \frac{d^2 v_i}{dt^2}.
\]

In writing the last structure we have noticed that the different kinds of collision averages all have the same \(i\) dependence. Now we can phrase our result in a continuum form

\[
(6 - 4a_1) \frac{\epsilon}{1 + r} v_i = \frac{d^2 \overline{v}_i}{dt^2}.
\]

In this way, we obtain

\[
\overline{v}_i = \overline{v}_0 \cosh(b_i),
\]

(15)

where

\[
b_i^2 = (3 - 2a_1) \epsilon.
\]

(16)

2. Correlations between velocities

Correlations between \(v_i\)'s are short ranged. Let us only consider the nearest neighbor correlation. When there is no dissipation, the only non-vanishing correlation of the \(v_i\)'s is the nearest neighbor average of Equation (12). For \(r < 1\), there is a small correction to that relation. Just as before, (see Equation (14)), we write an equation for the average of a nearest neighbor product in the same form as in the elastic case, but with a correction proportional to \(\epsilon\):

\[
\langle v_i v_{i+1} \rangle = -\frac{1}{2} a_2 \epsilon \overline{v}_i v_{i+1},
\]

(17)

where the averages are time average.

This assumption, with the profile of \(\overline{v}_i\) determined above, completes a description of the motion of grains in the center of mass frame, i.e. Part I of the dynamics described before. As an example, let us calculate the correlations between \(u_i^r\)'s, the velocities of particles in the center of mass frame. To illustrate the similarity between this part of the dynamics and conventional thermodynamics, i.e., the independence of boundary conditions and system sizes, we consider the center of mass frame of the \(2m\) particles at the center of the system. Keep in mind that rather than fixed, \(m\) can be treated as a variable in the following calculation.

Express \(u_i^r\) in terms of \(v_i\)'s,

\[
u_i^r = -\frac{1}{2m} \left[ \sum_{j=i}^{m-1} (m-j)v_j - \sum_{j=-(m-1)}^{i-1} (m+j)v_j \right].
\]

So

\[
\frac{2u_i^ru_{i+1}^r}{u_i^2 + u_{i+1}^2} = \frac{A - m^2 v_i^2}{A + m^2 v_i^2},
\]

where

\[
A = \left( \sum_{j=i+1}^{m-1} (m-j)v_j - \sum_{j=-(m-1)}^{i-1} (m+j)v_j - iv_i \right)^2.
\]

Let us calculate the correlation between \(u_0^r\) and \(u_1^r\). Only keeping the correlations between nearest neighbor, we have

\[
\langle (m-j)v_j - \sum_{j=-(m-1)}^{i-1} (m+j)v_j \rangle^2
\]

\[
= 2 \left( \sum_{j=1}^{m-1} (m-j)^2 v_j^2 + \sum_{j=1}^{m-2} (m-j)(m-j-1)v_jv_{j+1} \right)
\]

\[
= (m-1)^2 \overline{v}_0^2 + \overline{v}_{m-1}^2
\]

\[
+ \sum_{j=1}^{m-2} [(m-j)\overline{v}_j - (m-j-1)\overline{v}_{j+1}]^2
\]

\[
+ 2a_2 \epsilon \sum_{j=1}^{m-2} (m-j)(m-j-1)\overline{v}_j\overline{v}_{j+1}
\]

\[
= A_1 \overline{v}_0^2.
\]

So

\[
\frac{\langle 2u_0^ru_1^r \rangle}{(u_0^r)^2 + (u_1^r)^2} = \frac{A_1 - m^2}{A_1 + m^2} (18)
\]

Obviously, when \(m = 1\), the above ratio is \(-1\), because \(u_0^r = -u_1^r\) at all time. For elastic particles, the ratio can be calculated analytically to be \(-2m-1\). The inelasticity changes this dependence. Let us call \(2m\) the ‘cluster size’, since it corresponds to the usual practice of defining a ‘cluster’ then separate the motion of particles into mean flow and fluctuations.

From expression (18), we see that when the cluster is large enough, \(A_1\) can be big comparing to \(m^2\), then the correlations between velocity fluctuations can be big.
B. Numerical results

We carry out numerical simulations to investigate the statistical steady state of the system. Here we compare the numerical results with the above theory describing the motion in the center of mass frame.

1. Quasi-elastic situations

First let us look at the quasi-elastic situations, i.e. very small $\epsilon$. Before testing the profile of $v_i$'s, we examine the crucial assumption, (14).

![Graph showing numerical results for $1.5 - a_1$ for a high density system with 100 particles averaged over $2 \times 10^9$ collisions. $a_1$ is defined in Equation (14). The ○ is for $r = 0.995$; the + is for $r = 0.95$. The line is for $1.5 - a_1 = 0.029$ from the fit in Figure 8.]

Now let us look more sharply at the data. To find $a_1$, we do a very accurate determination of the ratio of averages from the left and the right hand sides of equation (14). This equation is then solved at each $i$-value to find a local value of $a_1$. The result is shown in Figure 6. The theory is right if $a_1$ is independent of $i$ and wrong if it has an important $i$-dependence. The figure seems to show that there is an excellent fit for the smaller value of $\epsilon$, and a bad fit for the larger.

From equation (16) we see that the important combination determining the properties of the profile of $\bar{v}$ is $3 - 2a_1$. But $a_1$ is very close to 1.5, as shown in Figure 6. Then the $a_1$ effect changes the prefactor in equation (16) from 3 to $3 - 2a_1$, i.e. by a factor of 50. The velocity correlations renormalize $\epsilon$, and reduce the energy dissipation.

Also $a_1$ is essentially a local correlation effect originated from the inelastic collisions. For an elastic system with comparable inhomogeneity, there is also a correction to the factor $-\frac{1}{2}$ in Equation (14), but the correction is usually an order of magnitude smaller than the effects we are seeing here.

2. Stronger inelasticity regime

We look at smaller $r$'s. To avoid inelastic collapses, we limit our $r$ to be greater than $r_c$. For a system of 100 particles with extremely high density, $r_c \approx 0.95$.

When $r$ gets smaller, there is a cluster of particles moving around the center of the pipe, all with about the same velocity. The system is in a state far away from...
equilibrium. Also, it is very nonuniform—the particles around the center are highly correlated while those near the boundaries move independently; the energy flux is strong near the end walls, but rather weak inside the system. As a consequence, the PDF’s of quantities change significantly from particles near the center to those near the boundaries, e.g. the PDF’s of $u_i$’s, though there is no big change in the PDF’s of $v_i$’s.

FIG. 9. Fit to a hyperbolic cosine curve of the profile of $v_i$ for a high density system of 100 particles and $r = 0.95$. This is a higher-{$\epsilon$} analog of Figure 7. The straight line corresponds to a hyperbolic cosine profile-curve, and its slope is 0.054, a value extrapolated from the expression for quasi-elastic cases (Fig. 8). However, the straight-line fit is not very good, especially near the boundary.

Figure 9 once again plots a quantity which should be linear in $i$ if the theory, equations (13), is right. Now, for this larger values of {$\epsilon$}, there are substantial variations in slope. It appears that the theory does not apply for the fifteen particles nearest to each of the boundaries and that it might have small troubles elsewhere. This discrepancy is also shown when we plot the slope, calculated from doing numerical derivatives on Figure 9 to give $b$ as a function of $i$. This plot is given as Figure 10.

FIG. 10. The position dependence of the prefactor $b$ in a high density system with 100 particles and $r = 0.95$. The line is $b = 0.054$ extrapolated from the quasi-elastic cases.

The discrepancy between the theory and numerical results for strong inelasticity is not surprising. Though taking into account the correlations between fluctuations, the theory is still based on concepts of conventional fluids—no internal structures are considered. However, when inelasticity is strong, the dynamics is affected by intrinsic structures of the collection of the particles, and the whole system may belong a different phase [12]. A satisfactory theory must incorporate this feature.

Now let us look at the velocity correlations. Only the nearest neighbor correlation (Equation (17)) is considered. The theory leads to the expression (18) of the correlation between $u_i^0$ and $u_i^1$, which is independent of system sizes or boundary conditions. To test this expression, we calculate numerically this correlation with respect to different cluster sizes, i.e. different $m$, with (15) and with the profile of $\tau_i$ calculated numerically. The comparison between theory and numerical result is showed in Figure 11. We see the correlation increases with increasing cluster size. The comparison is the best for $a_2 = 0.6$. When the cluster size is big enough, most part of the total motion belongs to the correlated motion. We want to point out that this curve is independent of boundary conditions. Also for systems with different sizes, we get sections of different length from this same curve, as shown in the figure.

FIG. 11. The cluster size dependence of the ratio (18). The system is in a low density regime, with $r = 0.94$. * is from time average results of a simulation with 100 particles and ⋄ is from a simulation with 60 particles, and the curve is from (15) with $a_2 = 0.6$.

We want to point out that the major point of Figure 11 is to demonstrate that part of the dynamics, the motion in the center of mass frame, is independent of boundary conditions and system sizes. The agreement between theory and numerical results can not be viewed as a strong support for the details of the theory because the profile of $\tau_i$ is from numerical calculations, rather than (15), also
the value \( a^2 = 0.6 \) is a fitting parameter. The theory captures some qualitative features of the dynamics, but is still incomplete.

IV. MOTION OF THE CENTER OF MASS

Because the total momentum of the system can be only changed by the collisions between the outermost particles and the walls, and the motion of the outermost particles is close to that of a elastic system, the motion of the center of mass should also be close to that of a elastic system. For a elastic system,

\[
\langle u^2 \rangle = \langle (\sum_i u_i/N)^2 \rangle = u^*^2/N
\]  

(19)

where \( u^* \) is the rms speed of the outermost particle. From Figure 12 we see this estimate is about right, though the numerical factor must be calculated from detailed distributions. The result also seems sensitive to \( \epsilon \). This is because the PDF for the velocity of the outermost particle is more skewed for higher value of \( \epsilon \), and so the ratio between \( u^* \) and the momentum transfered into the system from the wall depends on \( \epsilon \).

Notice that the motion of the center of mass depends strongly on the boundary condition.

\[
\langle u^2 \rangle = \langle (\sum_i u_i/N)^2 \rangle = u^*^2/N
\]  

FIG. 12. Test of (19) for two boundary conditions for high density systems with \( N = 100 \). The ratios are all around 1, as we expect from our order of magnitude argument. The \( * \) is for the Boltzmann boundary condition, and the \( \circ \) is for the fixed speed condition. The motion of the center of mass depends strongly on the boundary conditions.

Suppose the motion of particles in the center of mass frame is independent of the motion of the center of mass itself, i.e. \( u \) is uncorrelated to \( v_i \)'s, then

\[
\langle u^2 \rangle = \langle u_2^2 \rangle + \langle u^2 \rangle
\]

Simulations show that the profile of \( v_i \) is nearly independent of boundary conditions, so is the motion of the system in the center of mass frame. However, \( \langle u^2 \rangle \) depends sensitively on the boundary conditions, and so does the motion of the particles in the lab frame, i.e. the profile of \( u^2_i \) (Fig. 13).

Due to the motion of the center of mass, the correlations between \( u_i \)'s are enhanced, comparing to those between \( u^*_i \)'s.

\[
\frac{\langle 2u_i u_{i+1} \rangle}{\langle u_i^2 + u_{i+1}^2 \rangle} = \frac{2\langle u_i^2 u_{i+1}^2 \rangle + 2\langle u^2 \rangle}{\langle u_i^2 + u_{i+1}^2 \rangle + 2\langle u^2 \rangle}
\]

or

\[
\frac{v_i^2}{\langle u_i^2 + u_{i+1}^2 \rangle} = \frac{v_i^2}{\langle u_i^2 + u_{i+1}^2 \rangle + 2\langle u^2 \rangle} = R_i.
\]

The ratio, \( R_i \), between random motion and total motion was defined by us in Equation (19).

Behavior of the ratio \( R_0 \)

When \( n\epsilon \) is small, we can expand expression (20) using Equations (13), (18) and (19). Keeping terms linear in \( \epsilon \), we have,

\[
- \log(R_0) = \frac{\epsilon}{2n} \{ [ (n - \alpha)^2 + (n - 1) ] (3 - 2a_1) \\
+ 2(n - 1)(n - 2)a_2/3 \},
\]

(21)

where \( 0 < \alpha < 1 \). From Equation (21) we see that when \( n\epsilon \) is small, \( - \log(R_0) \) is proportional to \( n\epsilon \).

Numerical results of \( - \log(R_0) \) are showed in Figure 13. We do see that \( - \log(R_0) \) is proportional to \( \epsilon \) for very small \( \epsilon \). However, when \( \epsilon \) is big, where we expect strong nonlinear effects, it is proportional to \( \epsilon^2 \).
As we argued in Section II, there are two important combinations of $N$ and $\epsilon$. The product $N\sqrt{\epsilon}$ describes how temperature decays towards the center of the system, which agrees excellently with the numerical results when $\epsilon$ is very small. However, Equation (21) shows that in this limit, only the product $N\epsilon$ appears in the final expression for $R_0$. This seems to suggest that $R_0$, i.e., the degree of the coherence of the particles’ motion, is determined by the product $N\epsilon$ (Fig. 14 and Fig. 15). These two figures exhibit rather interesting features of the dynamics [20], though we do not have a satisfactory understanding of them.

V. CONCLUSION

In this paper, we investigated the steady state of a forced granular system in a thin pipe. Correlations between velocities of granular particles are shown to be important for a correct understanding of such systems. For systems in quasi-elastic regime, correlation is small, but not negligible because the deviation from equilibrium is also small. For systems with stronger inelasticity, correlation is crucial for a correct theory. Our theory describes the dynamics satisfactorily in the quasi-elastic limit. For stronger inelasticities, numerical results show quite interesting behaviors of the system, however, our theoretical understanding is only qualitative at this stage.

Characteristically for granular systems, fluctuations are important at all scales, enhanced by the combined effects of momentum conservation and non-uniformity. Also, the separation between fluctuations and mean flow is quite nontrivial. Because if the mean flow is an average of a collection of particles, the correlations between the fluctuations of velocities can be big if the collection is big.

An important issue is the existence of a universal description, which is not common for nonequilibrium systems. The separation of the dynamics into motion in the center of mass frame and the motion of the center of mass itself is quite suggestive.

The motion of the center of mass can not be universal. Momentum conservation decides that the velocity of the center of mass can be changed only by the interaction between particles and external effects. So it depends sensitively on the details of boundary conditions, as shown in the paper, and can not be universal.

This is true for both elastic systems and inelastic ones. However, in elastic systems, every mode has the same strength due to equal-partition law of the energy. The motion of the center of mass is just one mode out of $Nd$ modes, and its effect is negligible for a macroscopic system. In a dissipative system, on the other hand, being the only conserved mode, it can dominate over all other modes. Consequently, a universal description does not exist for the dynamics as a whole.

Still, if we look at the other $N-1$ modes which are perpendicular to this non-universal mode, we may discover some universal features. The independence of the motion in the center of mass frame on the boundary conditions and system sizes is a hint that this part of the dynamics may be universal. Further study is being carried out.

The thin pipe model used here simplifies greatly both the numerical and analytical calculations. The low density version of it may not have higher dimensional analogies, where the sequence of particles is necessarily broken. However, the high density version can be modified for a higher dimension situation, where the sequence can be kept.

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