MIXED PENTAGON, OCTAGON AND BROADHURST DUALITY EQUATION

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Abstract. This paper is on elimination of defining equations of the cyclotomic analogues, introduced by the first author, of Drinfeld’s scheme of associators. We show that the mixed pentagon equation implies the octagon equation for $N = 2$ and the particular distribution relation. We also explain that Broadhurst duality is compatible with the torsor structure. We develop a formalism of infinitesimal module categories and use it for deriving a proof left implicit in the first named author’s earlier work.

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0. Introduction

The “Grothendieck-Teichmüller theory” was developed by Drinfeld [Dr] with the motivation of quantization of certain Hopf algebras related with the monodromy of the KZ differential system (Kohno-Drinfeld theorem), and in close relation with Grothendieck’s approach to the description of the action of the absolute Galois group of the rational number field $\mathbb{Q}$ on the “Teichmüller tower” ([G]). One of the main results of this theory is a collection of relations between periods of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ called the MZV’s (multiple zeta values) (see (1.10)). These relations are derived from the study of the monodromy of the KZ system, and fall in three classes: two classes of hexagon (1.9) and one class of pentagon relations (1.7). The elimination of the hexagon relations (i.e., the statement that they are consequences of the pentagon relation) was established by the second-named author in [F10a] (see theorem 1.6 below). The proof uses combinatorial arguments based on the cell decomposition of the compactification of the moduli space $\mathcal{M}_{0,5}$.
The Grothendieck-Teichmüller theory was extended in the cyclotomic context by the first-named author \([E]\). In this theory, an integer \(N \geq 1\) is fixed and the analogues of the MZV’s are periods of the motivic fundamental group the algebraic curve \(\mathbb{P}^1 \setminus \{0, \mu_N, \infty\}\) (\(\mu_N\): the group of \(N\)-th roots of unity) called multiple \(L\)-values (see (2.12)). The study of the monodromy of the cyclotomic KZ system yields a collection of relations between these numbers. It is shown that the pro-algebraic variety \(\text{Psdist}(N, k)\) over \(k\) (\(k\): a field of characteristic 0) arising from the ‘cyclotomic KZ’ relations is equipped with a torsor structure over pro-\(k\)-algebraic group \(\text{GRTMD}(N, k)\), which is the extension of \((\mathbb{Z}/N\mathbb{Z})^\times \times \mathbb{G}_m\) by a prounipotent \(k\)-algebraic group, and whose Lie \(k\)-algebra \(\text{grtmd}(N, k)\) is non-negatively graded.

Another family of relations between MZV’s, the ‘double shuffle and regularization relations’ were studied in \([IKZ, R]\); cyclotomic analogues of these relations were discussed in \([R]\). In \([F11, F10b]\), it was proved that these relations are consequences of the ‘KZ’ and ‘cyclotomic KZ’ relations.

For particular values of \(N\), exceptional symmetries of \(\mathbb{P}^1 \setminus \{0, \infty, \mu_N\}\) give rise to additional families of relations. When \(N = 2\), these relations were made explicit by Broadhurst (\([Bh]\)), and for \(N = 4\), by Okuda (\([O]\)); Okuda’s relations allow one to recover Broadhurst’s (see \([O]\) §4). The results of this paper are of four types:

(A) elimination results for ‘cyclotomic KZ’ relations between multiple \(L\)-values
(B) elimination of defining conditions for pro-algebraic groups and Lie algebras
(C) insertion of Broadhurst’s result in the framework of torsors
(D) a theory of infinitesimal module categories, leading to a proof of a result left implicit in \([E]\), and which is also used in the proof of (B).

We now explain these results in more detail.

(A) The ‘cyclotomic KZ’ relations fall in the following classes: the mixed pentagon (2.5), octagon (2.9) and distribution (2.10) classes; there is one class of distribution relations for each \(N'\) dividing \(N\), \(N' \neq N\). Our first result is the implication of the first distribution relation from the mixed pentagon equation:

**Theorem 0.1 (Proposition 2.9).** If a pair of two group-like elements \((g, h)\) \(\in \exp t_{0,1}^N \times \exp t_{0,1}^N\) (for the notations see below) satisfies the mixed pentagon equation (2.5), then it also satisfies the distribution relation (2.10) for \(N' = 1\).

As a consequence, we obtain the equality of two groups

\[\text{GRTMD}_{(1,1)}(N, k) = \text{GRTM}_{(1,1)}(N, k)\]

and of two torsors

\[\text{Psdist}_{(1,1)}(N, k) = \text{Pseudo}_{(1,1)}(N, k)\]

for a prime \(N\) (see Corollary 2.11).

(B) The Lie algebras \(\text{grtmd}(N, k)\) are defined by mixed pentagon (2.4), octagon (2.3), speciality (2.5) and distribution (2.10) equations. It was proved in \([E]\) that the speciality condition implies the octagon one. We prove that for \(N = 2\) the mixed pentagon equation implies the octagon equation:

**Theorem 0.2 (Theorem 3.1).** For \(N = 2\) if a pair of two Lie elements \((\varphi, \psi)\) \(\in t_{0,2} \times t_{0,2}^N\) with \(c_B(0)(\psi) = c_{AB}(0)(\psi) = 0\) (for the notations see below) satisfies the mixed pentagon equation (2.1), then it also satisfies the octagon equation (2.2).

While this result may be viewed as not particularly useful in view of the result of \([E]\), its proof might be of interest as it extends the combinatorial arguments of \([F10a]\) to a Kummer covering \(\mathfrak{M}_{0,5}^2\) of the moduli space \(\mathfrak{M}_{0,5}\).
The pro-algebraic groups $GRTMD(N,k)$ are similarly defined by the mixed pentagon (2.5), octagon (2.6) and speciality equations (2.7). We prove that for $N = 2$ the octagon equation is implied by the other two equations in the setting:

**Theorem 0.3 (Theorem 3.4).** For $N = 2$ if a pair of two group-like element $(g,h) \in \exp t_0^N \times \exp t_0^{B_0}$ with $c_{B_0}(h) = c_{AB_0}(h) = 0$ satisfies the mixed pentagon equation (2.5) and the special action condition (2.7), then it also satisfies the octagon equation (2.6).

(C). In [Bh], Broadhurst introduced a family of ‘duality’ relations among multiple $L$-values for $N = 2$. These relations will be shown to be compatible with the torsor structure of $Psdist_{(1,1)}(2,k)$:

**Theorem 0.4 (Theorem 4.2).** The subset $\text{PseudoB}_{(1,1)}(2,k)$ defined by the Broadhurst duality (4.2) forms a subtorsor of $Psdist_{(1,1)}(2,k)$.

(D). The notion of infinitesimal module categories over braided monoidal categories is introduced in our appendix. It is defined by several axioms including the mixed pentagon axiom. In Proposition A.2 the notion is employed to prove that the set $GRTM_{(1,1)}(N,k)$ forms a group by the multiplication (2.8).

The structure of the paper is the following. §1 and §2 are a review of the Grothendieck-Teichmüller theory in [Dr] and [E]. In §2 elimination result (A) is proved (Proposition 2.9). Results (B) on mixed pentagon and octagon relations are proved in §3 (Theorems 3.1 and 3.4). Result (C) on compatibility of the Broadhurst duality relations with a torsor structure is proved in §4 (Theorem 4.2). Appendix A contains result (D), i.e., the basics of infinitesimal module category and the proof of the fact implicitly used in [E] that $GRTM_{(1,1)}(N,k)$ is a group. Some errors in [E] are corrected in Appendix B.

1. **The Grothendieck-Teichmüller Group**

This section is a short review on Drinfeld’s theory of associators in [Dr].

Let $k$ be a field of characteristic 0. For $n \geq 2$, the Lie algebra $t_n$ of infinitesimal pure braids is the completed $k$-Lie algebra with generators $t_{ij}$ ($i \neq j, 1 \leq i, j \leq n$) and relations

$t_{ij} = t_{ji}, [t_{ij}, t_{ik} + t_{kj}] = 0$ and $[t_{ij}, t_{kl}] = 0$ for all distinct $i, j, k, l$.

We note that $t_2$ is the 1-dimensional abelian Lie algebra generated by $t_{12}$. The element $z_n = \sum_{1 \leq i < j \leq n} t_{ij}$ is central in $t_n$. Put $t_n^0$ to be the Lie subalgebra of $t_n$ with the same generators except $t_{1n}$ and the same relations as $t_n$. Then we have

$t_n = t_n^0 \oplus k \cdot z_n$.

When $n = 3$, $t_3^0$ is the free Lie algebra $\mathfrak{g}_2$ of rank 2 with generators $A := t_{12}$ and $B := t_{23}$.

If $S$ and $T$ are two sets, then a partially defined map $f : S \to T$ means the data of (a) a subset $D_f \subset S$, and (b) a map $f : D_f \to T$. For a partially defined map $f : \{1, \ldots, m\} \to \{1, \ldots, n\}$, the Lie algebra morphism $t_n \to t_m$, $x \mapsto x^f = x_{f^{-1}(1)} \ldots x_{f^{-1}(n)}$ is uniquely defined by

$(t^f)_{ij} = \sum_{i' \in f^{-1}(i), j' \in f^{-1}(j)} t_{i'j'}$. 
Definition 1.1 (Dr). The Grothendieck-Teichmüller Lie algebra $\mathfrak{g}t_1(k)$ is defined to be the set of $\varphi = \varphi(A, B) \in \mathfrak{t}_3$ satisfying the duality and hexagon equations in $\mathfrak{t}_3$

\begin{align*}
(1.1) \quad & \varphi(A, B) + \varphi(B, A) = 0, \quad \varphi(A, B) + \varphi(B, C) + \varphi(C, A) = 0 \\
\text{with } & A + B + C = 0, \text{ the special derivation condition in } \mathfrak{t}_3
\end{align*}

(1.2) \quad [B, \varphi(A, B)] + [C, \varphi(A, C)] = 0,

and the pentagon equation in $\mathfrak{t}_4$

\begin{align*}
(1.3) \quad & \varphi_{1,2,3,4} + \varphi_{12,3,4} = \varphi_{2,3,4} + \varphi_{1,23,4} + \varphi_{1,2,3}.
\end{align*}

It actually forms a Lie algebra with the Lie bracket given by

\begin{align*}
(1.4) \quad & \langle \varphi_1, \varphi_2 \rangle = [\varphi_1, \varphi_2] + D_{\varphi_2}(\varphi_1) - D_{\varphi_1}(\varphi_2),
\end{align*}

where $D_{\varphi}$ is the derivation of $\mathfrak{t}_3$ defined by

\begin{align*}
D_{\varphi}(A) = [\varphi, A] \quad \text{and } D_{\varphi}(B) = 0.
\end{align*}

The Lie algebra structure is realised by the embedding

\begin{align*}
\mathfrak{g}t_1(k) & \hookrightarrow \text{Der}(\mathfrak{t}_3)
\end{align*}

sending $\varphi \mapsto D_{\varphi}$.

Definition 1.2 (Dr). The Grothendieck-Teichmüller group $\text{GRT}_1(k)$ is defined to be the set of series $g \in \exp \mathfrak{t}_3$ satisfying the duality and hexagon equations in $\exp \mathfrak{t}_3$

\begin{align*}
(1.5) \quad & g(A, B)g(B, A) = 1, \quad g(C, A)g(B, C)g(A, B) = 1 \\
\text{with } & A + B + C = 0, \text{ the special action condition in } \exp \mathfrak{t}_3
\end{align*}

\begin{align*}
(1.6) \quad & A + g(A, B)^{-1}Bg(A, B) + g(A, C)^{-1}Cg(A, C) = 0,
\end{align*}

and the pentagon equation in $\exp \mathfrak{t}_4$

\begin{align*}
(1.7) \quad & g_{1,2,3,4}g_{12,3,4} = g_{2,3,4}g_{1,23,4}g_{1,2,3}.
\end{align*}

It forms a group by the multiplication

\begin{align*}
(1.8) \quad & g_1 \circ g_2 = g_2(A, B) \cdot g_1(A, g_2^{-1}Bg_2).
\end{align*}

The group structure is realised by the embedding (but anti-homomorphism)

\begin{align*}
\text{GRT}_1(k) \hookrightarrow \text{Aut}(\mathfrak{t}_3)
\end{align*}

sending $g$ to the automorphism $A_g$ defined by

\begin{align*}
A \mapsto A \quad \text{and } B \mapsto g^{-1}Bg.
\end{align*}

We note that its associated Lie algebra is $\mathfrak{g}t_1(k)$.

Definition 1.3 (Dr). The associator set $M_1(k)$ is defined to be the set of series $g \in \exp \mathfrak{t}_3$ satisfying the pentagon equation (1.7) and the following variant of hexagon equations

\begin{align*}
(1.9) \quad & g(A, B)g(B, A) = 1, \quad \exp\{\frac{A}{2}\}g(C, A)\exp\{\frac{C}{2}\}g(B, C)\exp\{\frac{B}{2}\}g(A, B) = 1 \\
\text{with } & A + B + C = 0.
\end{align*}

It forms a right $\text{GRT}_1(k)$-torsor by (1.8) with $g_1 \in M_1(k)$ and $g_2 \in \text{GRT}_1(k)$.
Remark 1.4. It is shown that the special derivation condition \([\text{1.2}]\) for \(\varphi \in \mathfrak{t}_3^{0}\) (resp.\([\text{1.3}]\) for \(g \in \exp \mathfrak{t}_3^{0}\)) follows from duality and hexagon equations \([\text{1.1}]\) and the pentagon equation \([\text{1.3}]\) in \([\text{Dr}]\) proposition 5.7. (resp.\([\text{1.5}]\) and \([\text{1.7}]\) in \([\text{Dr}]\) proposition 5.9.)

Remark 1.5. A typical example of elements in \(M_1(\mathbb{C})\) is

\[
\varphi_{KZ}(A, B) = \Phi_{KZ}\left(\frac{A}{2\pi\sqrt{-1}}, \frac{B}{2\pi\sqrt{-1}}\right)
\]

constructed in \([\text{Dr}]\), where \(\Phi_{KZ}(A, B)\) is the Drinfeld associator. This series has the following expression:

\[
\Phi_{KZ}(A, B) = 1 + \sum (-1)^m \zeta(k_1, \ldots, k_m) A^{k_m-1} B \cdots A^{k_1-1} + \text{(regularized terms)}
\]

where \(\zeta(k_1, \ldots, k_m)\) are multiple zeta values defined by the following series

\[
(1.10) \quad \zeta(k_1, \ldots, k_m) = \sum_{0 < n_1 < \cdots < n_m} \frac{1}{n_1^{k_1} \cdots n_m^{k_m}}
\]

for \(m, k_1, \ldots, k_m \in \mathbb{N}(= \mathbb{Z}_{>0})\) with \(k_m \neq 1\) and for the regularised terms see \([F03a]\).

For a monic monomial \(W\) in \(U\mathfrak{t}_3^{0} = k\langle(A, B)\rangle\), \(c_W(g)\) for \(g \in U\mathfrak{t}_3^{0}\) means the coefficient of \(W\) in \(g\). On pentagon and hexagon equations we have the following.

Theorem 1.6 (\([\text{F10a}]\).) (1). Let \(\varphi\) be an element of \(\mathfrak{t}_3^{0}\) with \(c_B(\varphi) = c_{AB}(\varphi) = 0\). If \(\varphi\) satisfies the pentagon equation \([\text{1.3}]\), then it also satisfies duality and hexagon equations \([\text{1.1}]\).

(2). Let \(g\) be an element of \(\exp \mathfrak{t}_3^{0}\) with \(c_B(g) = c_{AB}(g) = 0\). If \(g\) satisfies the pentagon equation \([\text{1.3}]\), then it also satisfies duality and hexagon equations \([\text{1.1}]\).

(3). Let \(g\) be an element of \(\exp \mathfrak{t}_3^{0}\) with \(c_B(g) = 0\) and \(c_{AB}(g) \in k^x\). If \(g\) satisfies the pentagon equation \([\text{1.3}]\), then the duality and hexagon equations \([\text{1.1}]\) hold for \(g(\frac{A}{\mu}, \frac{B}{\mu})\) with \(\mu = \pm \sqrt{24c_{AB}(g)} \in k\).

Remark 1.7. In \([\text{F11}]\) it is shown that the pentagon equation \([\text{1.7}]\) implies the double shuffle relation and the regularization relation, which are one of the fundamental relations among multiple zeta values.

2. The cyclotomic Grothendieck-Teichmüller group

This section is a review of the first named author’s theory on the cyclotomic analogues of associators in \([E]\).

Here we recall the notations\(^1\) in \([E]\): For \(n \geq 2\) and \(N \geq 1\), the Lie algebra \(\mathfrak{t}_{n,N}\) is the completed \(k\)-Lie algebra with generators

\(t^{ij}\) (\(2 \leq i \leq n\)), and \(t(a)^{ij}\) (\(i \neq j, 2 \leq i, j \leq n, a \in \mathbb{Z}/N\mathbb{Z}\))

and relations

\[
t(a)^{ij} = t(-a)^{ji},
\]

\[
[t(a)^{ij}, t(a+b)^{jk} + t(b)^{jk}] = 0,
\]

\[
[t^{ii} + t^{jj} + \sum_{c \in \mathbb{Z}/N\mathbb{Z}} t(c)^{ij}, t(a)^{ij}] = 0,
\]

\(^1\) Several of them are changed for our convenience.
\[
[t^{1i}, t^{lj}] + \sum_{c \in \mathbb{Z}/N \mathbb{Z}} t(c)^{ij} = 0,
\]

\[t^{1i}, t(a)^{jk} = 0 \quad \text{and} \quad t(a)^{ij}, t(b)^{k}\]

for all \(a, b \in \mathbb{Z}/N \mathbb{Z}\) and all distinct \(i, j, k, l\) \((2 \leq i, j, k, l \leq n)\).

We note that \(t_{n,1} \) is equal to \(t_n\) for \(n \geq 2\). We have a natural injection \(t_{n-1,N} \hookrightarrow t_{n,N}\). The Lie subalgebra \(f_{n,N}\) of \(t_{n,N}\) generated by \(t^{1n}\) and \(t(a)^{in}\) \((2 \leq i \leq n-1, a \in \mathbb{Z}/N \mathbb{Z})\) is free of rank \((n - 2)N + 1\) and forms an ideal of \(t_{n,N}\). Actually it shows that \(t_{n,N}\) is a semi-direct product of \(f_{n,N}\) and \(t_{n-1,N}\). The element \(z_{n,N} = \sum_{1 \leq i < j \leq n} t^{ij}\) with \(t^{ij} = \sum_{a \in \mathbb{Z}/N \mathbb{Z}} t(a)^{ij}\) \((2 \leq i < j \leq n)\) is central in \(t_{n,N}\). Put \(t_{n,N}^0\) to be the Lie subalgebra of \(t_{n,N}\) with the same generators except \(t^{1n}\) and the same relations as \(t_{n,N}\). Then we have

\[t_{n,N} = t_{n,N}^0 \oplus k \cdot z_{n,N}\]

Especially when \(n = 3\), \(t_{3,N}^0\) is free Lie algebra \(\mathfrak{F}_{N+1}\) of rank \(N + 1\) with generators \(A := t^{12}\) and \(B(a) = t(a)^{23}\) \((a \in \mathbb{Z}/N \mathbb{Z})\).

For a partially defined map \(f : \{1, \ldots, m\} \rightarrow \{1, \ldots, n\}\) such that \(f(1) = 1\), the Lie algebra morphism \(t_{n,N} \rightarrow t_{m,N}: x \mapsto x^f = x^f^{-1}(1) \ldots f^{-1}(n)\) is uniquely defined by

\[(t(a)^{ij})^f = \sum_{i' \in f^{-1}(i), j' \in f^{-1}(j)} t(a)^{i'j'} \quad (i \neq j, 2 \leq i, j \leq n)\]

and

\[(t^{ij})^f = \sum_{j' \in f^{-1}(j)} t^{ij'} + \frac{1}{2} \sum_{j'' \in f^{-1}(j)} \sum_{i' \in f^{-1}(i), j' \in \mathbb{Z}/N \mathbb{Z}} t(c)^{i'j''} + \sum_{i' \neq 1 \in f^{-1}(1), j' \in f^{-1}(j)} \sum_{c \in \mathbb{Z}/N \mathbb{Z}} t(c)^{i'j'}\]

\((2 \leq j \leq n)\). Again for a partially defined map \(g : \{2, \ldots, m\} \rightarrow \{1, \ldots, n\}\), the Lie algebra morphism \(t_n \rightarrow t_{m,N}: x \mapsto x^g = x^g^{-1}(1) \ldots g^{-1}(n)\) is uniquely defined by

\[(t^{ij})^g = \sum_{i' \in g^{-1}(i), j' \in g^{-1}(j)} t(0)^{i'j'} \quad (i \neq j, 1 \leq i, j \leq n)\]

**Definition 2.1 (\cite{E}).** For \(N \geq 1\), the Lie algebra \(\mathfrak{g} \mathfrak{r} \mathfrak{t} \mathfrak{m}_{[1,1]}(N, k)\) is defined to be the set of pairs \((\varphi, \psi) \in t_{3,N}^0 \times t_{3,N}^0\) satisfying \(\varphi \in \mathfrak{g} \mathfrak{r} \mathfrak{t}_1(k)\), the **mixed pentagon equation** in \(t_{3,N}^0\)

\[\psi^{1,2,3,4} + \psi^{12,3,4} = \varphi^{2,3,4} + \psi^{1,23,4} + \psi^{1,2,3}\]

the **octagon equation** in \(t_{3,N}^0\)

\[\psi(A, B(0), B(1), \ldots, B(i), \ldots, B(N - 1))
- \psi(A, B(1), B(2), \ldots, B(i + 1), \ldots, B(0))
+ \psi(C, B(1), B(0), \ldots, B(N + 1 - i), \ldots, B(2))
- \psi(C, B(0), B(N - 1), \ldots, B(N - i), \ldots, B(1)) = 0\]
with $A + \sum_{a \in \mathbb{Z}/N\mathbb{Z}} B(a) + C = 0$, 

the special derivation condition in $t^0_{3,N}$

$$
(2.3) \quad \sum_{a \in \mathbb{Z}/N\mathbb{Z}} \left[ \psi(A, B(a), B(a+1), \ldots, B(a+i), \ldots, B(a-1)), B(a) \right] \\
+ \left[ \psi(A, B(0), B(1), \ldots, B(i), \ldots, B(N-1)) \\
- \psi(C, B(0), B(N-1), \ldots, B(N-i), \ldots, B(1)), C \right] = 0
$$

and $c_B(0)(\psi) = 0$.  

Here for any $k$-algebra homomorphism $\iota : U^{\mathfrak{g}+1}_N \to S$ the image $\iota(\varphi) \in S$ is denoted by $\varphi(\iota(A), \iota(B(0)), \ldots, \iota(B(N-1)))$. The Lie algebra structure is given by

$$
(2.4) \quad \langle (\varphi_1, \psi_1), (\varphi_2, \psi_2) \rangle = \langle \psi_1, \psi_2 \rangle
$$

with

$$
\langle \varphi_1, \varphi_2 \rangle = [\varphi_1, \varphi_2] + D_{\varphi_2}(\varphi_1) - D_{\varphi_1}(\varphi_2) \text{ and } \langle \psi_1, \psi_2 \rangle = [\psi_1, \psi_2] + \bar{D}_{\psi_2}(\psi_1) - \bar{D}_{\psi_1}(\psi_2).
$$

Here $D_\psi$ means the derivation of $t^0_{3,N}$ defined by

$$
\bar{D}_\psi(A) = [\psi, A], \quad \bar{D}_\psi(B(a)) = [\psi - \psi(A, B(a), B(a+1), \ldots, B(a-1)), B(a)]
$$

for $a \in \mathbb{Z}/N\mathbb{Z}$ and

$$
\bar{D}_\psi(C) = [\psi(C, B(0), B(N-1), \ldots, B(1)), C].
$$

The Lie algebra structure is realised by the embedding

$$
\text{grtm}_{(1,1)}(N, k) \hookrightarrow \text{Der}(t^0_3) \times \text{Der}(t^0_{3,N})
$$

sending $(\varphi, \psi) \mapsto (D_\varphi, \bar{D}_\psi)$.

**Remark 2.2.** It is shown in [E] that the special derivation condition (2.3) for $\psi$ implies the octagon equation (2.2).

**Definition 2.3** ([E]). For $N \geq 1$, the group $\text{GRTM}_{(1,1)}(N, k)$ is defined to be the set of pairs $(g, h) \in \exp t^0_3 \times \exp t^0_{3,N}$ satisfying $g \in \text{GRT}_1(k)$, $c_B(0)(h) = 0$, the mixed pentagon equation in $\exp t^0_{4,N}$

$$
(2.5) \quad h^{1,2,3,4}h^{1,2,3,4} = g^{2,3,4}h^{1,2,3,4},
$$

the octagon equation in $\exp t^0_{3,N}$

$$
(2.6) \quad h(A, B(1), B(2), \ldots, B(0))^{-1}h(C, B(1), B(0), \ldots, B(2)) \cdot h(C, B(0), B(N-1), \ldots, B(1))^{-1}h(A, B(0), B(1), \ldots, B(N-1)) = 1
$$

with $A + \sum_{a \in \mathbb{Z}/N\mathbb{Z}} B(a) + C = 0$ and

the special action condition in $\exp t^0_{3,N}$

$$
(2.7) \quad A + \sum_{a \in \mathbb{Z}/N\mathbb{Z}} Ad(\tau_a h^{-1})(B(a)) + Ad(h^{-1} \cdot h(C, B(0), B(N-1), \ldots, B(1)))(C) = 0
$$

---

\footnote{For our convenience, we slightly change the original definition by adding the small condition $c_B(0)(\psi) = 0$. The relation to the original Lie algebra is the direct sum decomposition of Lie algebras $g_{\text{original}} = \mathfrak{g} \oplus k \cdot B(0)$, where $\mathfrak{g} = \text{grtm}_{(1,1)}(N, k)$.}
where $\tau_a (a \in \mathbb{Z}/N\mathbb{Z})$ is the automorphism defined by $A \mapsto A$ and $B(c) \mapsto B(c+a)$ for all $c \in \mathbb{Z}/N\mathbb{Z}$.

It forms a group by the multiplication

$$
(g_1, h_1) \cdot (g_2, h_2) = \left( g_2(A, B) \cdot g_1(A, Ad(g_2^{-1})(B)), h_2(A, B(0), B(1), \ldots, B(N-1)) \right).
$$

$$
h_1(A, Ad(h_2^{-1})(B)(0), Ad(\tau h_2^{-1})(B)(1), \ldots, Ad(\tau_{N-1} h_2^{-1})(B)(N-1)) \right).
$$

The group structure is realised by the embedding (but opposite homomorphism)

$$
\text{GRTM}_{(1,1)}(N, k) \hookrightarrow \text{Aut}_{\mathfrak{t}}^0 \times \text{Aut}_{\mathfrak{t}}^0_{\mathfrak{g} \mathfrak{t} \mathfrak{m}},
$$

sending $(g, h)$ to the automorphism $(A_g, A_h)$ where $A_h$ is defined by $A \mapsto A$ and $B(a) \mapsto Ad(\tau h^{-1})(B)(a))$ for $a \in \mathbb{Z}/N\mathbb{Z}$. We note that its associated Lie algebra is $\mathfrak{gtm}_{(1,1)}(k)$.

**Definition 2.4 (E).** The torsor $\text{Pseudo}_{(1,1)}(N, k)$ is defined to be the set of pairs $(g, h) \in \exp \mathfrak{t} \times \exp \mathfrak{g} \mathfrak{t} \mathfrak{m}$ satisfying $g \in M_1(k)$, $c_{B(0)}(h) = 0$, the mixed pentagon equation (2.5) and the following variant of octagon equation in $\exp \mathfrak{t}_{3, N}$

$$
h(A, B(1), B(2), \ldots, B(0))^{-1} \exp \left\{ \frac{B(1)}{2} \right\} h(C, B(1), B(0), \ldots, B(2)) \exp \left\{ \frac{C}{2} \right\}.
$$

$$
h(C, B(0), B(N-1), \ldots, B(1))^{-1} \exp \left\{ \frac{B(0)}{2} \right\} \cdot \exp \left\{ \frac{B(1)}{2} \right\} \cdot \exp \left\{ \frac{A}{2} \right\} = 1.
$$

It forms a right $\text{GRTM}_{(1,1)}(N, k)$-torsor by (2.8) with $(g_1, h_1) \in \text{Pseudo}_{(1,1)}(N, k)$ and $(g_2, h_2) \in \text{GRTM}_{(1,1)}(N, k)$.

**Remark 2.5.** In contrast with remark 1.4, it is not known if (2.3) and (2.7) follow from the rest of the equations (cf. [E] remark 7.8).

**Remark 2.6.** In [F10b] it is shown that the mixed pentagon equation (2.5) implies the double shuffle relation and the regularization relation among multiple $L$-values.

Let $N, N' \geq 1$ with $N' \mid N$. Put $d = N/N'$. The morphism $\pi_{N,N'} : t_{n,N} \to t_{n,N'}$ is defined by

$$
t^{ij} \mapsto dt^{ij} \quad \text{and} \quad t^{ij}(a) \mapsto t^{ij}(\bar{a}) \quad (i \neq j, 2 \leq i, j \leq n, a \in \mathbb{Z}/N\mathbb{Z}),
$$

where $\bar{a} \in \mathbb{Z}/N'\mathbb{Z}$ means the image of $a$ under the map $\mathbb{Z}/N\mathbb{Z} \to \mathbb{Z}/N'\mathbb{Z}$.

The morphism $\delta_{N,N'} : t_{n,N} \to t_{n,N'}$ is defined by

$$
t^{ij} \mapsto t^{ij} \quad \text{and} \quad t^{ij}(a) \mapsto \begin{cases} t^{ij}(a/d) & \text{if} \ \left( \begin{array}{c} d \mid a \\ i \neq j \end{array} \right), \\ t^{ij}(a) & \text{if} \ \left( \begin{array}{c} d \nmid a \\ i = j, 2 \leq i, j \leq n, a \in \mathbb{Z}/N\mathbb{Z} \end{array} \right). \end{cases}
$$

(i \neq j, 2 \leq i, j \leq n, a \in \mathbb{Z}/N\mathbb{Z}).$ For $\psi \in \mathfrak{t}_{3, N}^0$, put $\rho_{N,N'}(\psi) = c_{B(0)}(\pi_{N,N'}(\psi)) - c_{B(0)}(\psi)$.

The morphism $\pi_{N,N'}$ (resp. $\delta_{N,N'}$) : $t_{n,N} \to t_{n,N'}$ induces the morphisms

$\text{grtm}_{(1,1)}(N, k) \Rightarrow \text{grtm}_{(1,1)}(N', k)$, $\text{GRTM}_{(1,1)}(N, k) \Rightarrow \text{GRTM}_{(1,1)}(N', k)$ and $\text{Pseudo}_{(1,1)}(N, k) \Rightarrow \text{Pseudo}_{(1,1)}(N', k)$ which we denote by the same symbol $\pi_{N,N'}$ (resp. $\delta_{N,N'}$). We also remark that

$\text{grtm}_{(1,1)}(1, k) = \text{grt}_1(k)$, $\text{GRTM}_{(1,1)}(1, k) = \text{GRT}_1(k)$ and $\text{Pseudo}_{(1,1)}(1, k) = M_1(k)$. 
Definition 2.7. (1) For $N \geq 1$, $\text{grtm}_{(1,1)}(N,k)$ is the Lie subalgebra of $\text{grtm}_{(1,1)}(N,k)$ defined by imposing the distribution relation in $t_{3,N}^0$ for all $N'|N$

$$\pi_{NN'}(N) - \delta_{NN'}(\psi) = \rho_{NN'}(\psi)B(0).$$

(2). For $N \geq 1$, $\text{GRTMD}_{(1,1)}(N,k)$ is the subgroup of $\text{GRTM}_{(1,1)}(N,k)$ defined by imposing the distribution relation in $exp t_{3,N}^0$ for all $N'|N$

$$\pi_{NN'}(h) = e^{\rho_{NN'}(h)}B(0)\delta_{NN'}(h).$$

(3). For $N \geq 1$, the $\text{GRTMD}_{(1,1)}(N,k)$-torsor $P_{\text{dist}}(1,1)(N,k)$ is the subtorsor of $\text{Psdist}_{(1,1)}(N,k)$ defined by imposing the distribution relation (2.11) in $\text{exp t}_{3,N}^0$ for all $N'|N$.

Remark 2.8. A typical example of an element of $\text{Psdist}_{(1,1)}(N,C)$ is

$$\varphi_N^{12}(A,B(0),\cdots,B(N-1)) = \Phi_KZ \left( \begin{array}{cccc} A & B(0) & \cdots & B(N-1) \\ 2\pi & \sqrt{-1} & \cdots & 2\pi & \sqrt{-1} \end{array} \right)$$

where $\Phi_KZ(A,B(0),\cdots,B(N-1))$ is the cyclotomic Drinfeld associator constructed in [E]. It has the following expression:

$$\Phi_KZ = 1 + \sum (-1)^m L(k_1,\cdots,k_m;\zeta_1,\cdots,\zeta_m)A^{k_m-1}B(a_{m})\cdots A^{k_1-1}B(a_1)$$

+ (regularized terms)

where $\zeta_1 = \zeta_N^{a_1}, \cdots, \zeta_{m-1} = \zeta_N^{-a_{m-1}}, \zeta_m = \zeta_N^{-a_m}$ with $\zeta_N = \exp\left(\frac{2\pi i}{N}\right)$ and $L(k_1,\cdots,k_m;\zeta_1,\cdots,\zeta_m)$ are multiple $L$-values defined by the following series

$$L(k_1,\cdots,k_m;\zeta_1,\cdots,\zeta_m) := \sum_{0 < n_1 < \cdots < n_m} \zeta_{n_1}^{k_1}\cdots\zeta_{n_m}^{k_m}$$

for $m, k_1,\cdots,k_m \in \mathbb{N}(= \mathbb{Z}_{>0})$ and $\zeta_1,\cdots,\zeta_m \in \mu_N$ with $(k_m,\zeta_m) \neq (1,1)$.

The following says that the distribution relation for $N' = 1$ follows from the mixed pentagon equation. Note that the distribution relation for $N' = N$ is automatically satisfied.

Proposition 2.9. (1). Suppose that $(\varphi, \psi) \in t_{3,1}^0 \times t_{3,1}^0$ satisfies the mixed pentagon equation (2.1) in $t_{3,1}^0$. Then it also satisfies the distribution relation (2.10) for $N' = 1$ in $t_{3,1}^0 = t_{3,1}^0$.

(2). Suppose that $(g, h) \in \exp t_{3,1}^0 \times \exp t_{3,1}^0$ satisfies the mixed pentagon equation (2.4) in $\exp t_{3,1}^0$. Then it also satisfies the distribution relation (2.11) for $N' = 1$ in $\exp t_{3,1}^0 = \exp t_{3,1}^0$.

Proof. (1). By taking the image of (2.1) by the composition of $\pi_{N1}$ with the projection $t_{4,1}^0 \rightarrow t_{3,1}^0$ eliminating the first strand, we get

$$\pi_{N1}(\psi) = \varphi + Ne_A(\psi)A + c_B(N_{N1}(\psi))B.$$ 

Next by taking the image of (2.1) by the composition of $\delta_{N1}$ with the projection, we get

$$\delta_{N1}(\psi) = \varphi + c_A(\psi)A + c_B(\psi)B.$$ 

By the lemma below these two equations give (2.10) for $N' = 1$.

(2). Similarly we obtain $\pi_{N1}(h) = e^{c_B(\pi_{N1}(h))B}g$ and $\delta_{N1}(h) = e^{c_B(\pi_{N1}(h))B}g$ from (2.4), which implies the claim. □
\textbf{Lemma 2.10.} Suppose that \((\varphi, \psi) \in t_3^{1} \times t_3^{1,2}\) (resp. \( \exp t_3^{1} \times \exp t_3^{1,2}\)) satisfies the mixed pentagon equation \((2.1)\) in \(t_3^{1,2}\) (resp. \((2.5)\) in \(\exp t_3^{1,2}\)). Then \(c_A(\psi) = 0\).

\textit{Proof.} It can be proved directly by inspecting the terms of degree 1. \qed

As a corollary, we have

\textbf{Corollary 2.11.} For a prime \(p\), we have

\(\text{grtm}_d(1,1)(p, k) = \text{grtm}_d(1,1)(p, k)\),

\(\text{GRTMD}(1,1)(p, k) = \text{GRTM}(1,1)(p, k)\),

\(\text{Psdist}(1,1)(p, k) = \text{Pseudo}(1,1)(p, k)\).

\textbf{Remark 2.12.} In \([\text{DeG}]\) Deligne and Goncharov construct the motivic fundamental group \(\pi_1^M(\mathbb{P}^1 \setminus \{0, 1, \mu_N\}, 1_0)\) (\(\mu_N\): the group of \(N\)-th roots of unity) with the tangential base point \(1_0\) at 0, which determines a pro-object of the \(\mathbb{Q}\)-linear category \(\text{MT}(\mathbb{Z}[\mu_N, \frac{1}{N}])\) of mixed Tate motives of \(\mathbb{Z}[\mu_N, \frac{1}{N}]\). This causes the morphism

\(\varphi_N : \text{LieGal}^M(\mathbb{Z}[\mu_N, \frac{1}{N}]) \to \text{Dert}_{1,2}^{0}\),

where \(\text{LieGal}^M(\mathbb{Z}[\mu_N, \frac{1}{N}])\) is the motivic Lie algebra of the category. It is a graded free Lie algebra with \(\text{rk} \mathbb{K}_{2n-1}(\mathbb{Z}[\mu_N, \frac{1}{N}])\) generators in each degree \(n > 0\). The map \(\varphi_N\) is shown to be injective for \(N = 1\) in \([\text{DW}]\) and for \(N = 2, 3, 4\) and 8 in \([\text{De}]\). For \(N = 6\), a certain modification of the map \(\varphi_N\) is shown to be injective in \([\text{De}]\). Partial injectivity results for \(N = 2p\) (\(p\): a prime) were obtained in \([\text{DW}]\). Because all the defining equations of \(\text{grtm}_d(1,1)(N, k)\) are geometric, it can be shown that \(\text{Im} \varphi_N\) is embedded in \(\text{grtm}_d(1,1)(N, k) \subset \text{Dert}_{3,2}^{0}\). It is one of the fundamental questions to ask if they are equal or not.

3. Mixed pentagon and octagon equations

In this section, we focus on the case \(N = 2\) and prove that the mixed pentagon equation implies the octagon equation.

\textbf{Theorem 3.1.} Let \((\varphi, \psi) \in t_3^{1} \times t_3^{1,2}\) be a pair satisfying \(c_{B(0)}(\psi) = c_{AB(0)}(\psi) = 0\) and the mixed pentagon equation \((2.1)\) in \(t_3^{1,2}\), i.e.

\[\psi(t^{12}, t^{23} + t^{24}, t^{23} + t^{24}) + \psi(t^{13} + t^{23} + t^{23} + t^{34}, t^{24}, t^{34}) = \varphi(t^{23}, t^{34}) + \psi(t^{12}, t^{13} + t^{23} + t^{24} + t^{24}, t^{24} + t^{34} + t^{34}) + \psi(t^{12}, t^{23}, t^{23})\]

where \(t^{ij} = t^{ij}(0)\) and \(t^{ij} = t^{ij}(1)\). Then \(\psi\) satisfies the octagon equation \((2.2)\).

\textit{Proof.} By taking the image of \((2.1)\) by \(\delta_{21}\) and eliminating the first strand we get \(\delta_{21}(\psi) = \varphi + c_A(\psi)A\), which by lemma \(2.10\) implies \(\delta_{21}(\psi) = \varphi\). Then applying again \(\delta_{21}\) to the mixed pentagon equation \((2.1)\), we get the equation \((2.3)\) for \(\varphi\). Then by our assumption \(c_B(\varphi) = c_{AB}(\varphi) = 0\) and Theorem \(1.6\) (1), we have \((1.1)\) for \(\varphi\).

For \((\varphi, \psi) \in t_3^{1} \times t_3^{1,2}\), put

\[\Pi = \varphi^{2,3,4} + \psi^{1,23,4} + \psi^{1,2,3} - \psi^{1,2,3,4} - \psi^{12,3,4}\]
in $\mathfrak{t}^0_{1,2}$. Let $S_3$ be the group of permutations of $\{1, 2, 3, 4\}$ which fix $\{1\}$. Then
\[
\sum_{\sigma \in S_3} \epsilon(\sigma)\Pi^{1,\sigma(2),\sigma(3),\sigma(4)} = (\psi^{1,2,3} - \psi^{1,3,2}) + (\psi^{14,3,2} - \psi^{14,2,3}) + (\psi^{13,2,4} - \psi^{13,4,2})
+ (\psi^{12,4,3} - \psi^{12,3,4}) + (\psi^{1,3,4} - \psi^{1,4,3}) + (\psi^{1,4,2} - \psi^{1,2,4}) + \sum_{\sigma \in S_3} \epsilon(\sigma)\varphi^{\sigma(2),\sigma(3),\sigma(4)}.
\]

There is a unique automorphism $s$ of the Lie algebra $\mathfrak{t}^0_{4,2}$ such that
\[
s(t^{12}) = t^{13}, s(t^{13}) = t^{12}, s(t^{14}) = t^{14}, s(t^{23}_+) = t^{23}_+, s(t^{24}_+) = t^{34}_+ \text{ and } s(t^{24}_-) = t^{24}_-.
\]
Then $s^4 = id$ and
\[
s(\psi^{12,3,4}) = \psi^{13,2,4}, s(\psi^{12,4,3}) = \psi^{13,4,2}, s(\psi^{1,4,3}) = \psi^{1,4,2}, s(\psi^{1,3,4}) = \psi^{1,2,4}.
\]
It follows that
\[
\sum_{\sigma \in S_3} \epsilon(\sigma)\Pi^{1,\sigma(2),\sigma(3),\sigma(4)} = (\psi^{1,2,3} - \psi^{1,3,2}) + (\psi^{14,3,2} - \psi^{14,2,3})
+ (s - id)(\psi^{12,3,4} - \psi^{12,4,3} + \psi^{1,4,3} - \psi^{1,3,4}) + \sum_{\sigma \in S_3} \epsilon(\sigma)\varphi^{\sigma(2),\sigma(3),\sigma(4)}.
\]
Hence
\[
(id + s + s^2 + s^3) \sum_{\sigma \in S_3} \epsilon(\sigma)\Pi^{1,\sigma(2),\sigma(3),\sigma(4)} = (id + s + s^2 + s^3)(\psi^{1,2,3} - \psi^{1,3,2})
+ (s - id)(\psi^{12,3,4} - \psi^{12,4,3} + \psi^{1,4,3} - \psi^{1,3,4}) + \sum_{\sigma \in S_3} \epsilon(\sigma)\varphi^{\sigma(2),\sigma(3),\sigma(4)}.
\]
By (2.2), $\Pi = 0$. By (1.1), the last term is 0. So we have
\[
(id + s + s^2 + s^3)(\psi^{1,2,3} - \psi^{1,3,2} + \psi^{14,3,2} - \psi^{14,2,3}) = 0.
\]
Since $s^2(X) = X$ for $X = \psi^{1,2,3}, \psi^{1,3,2}, \psi^{14,3,2}$ and $\psi^{14,2,3}$,
\[
(id + s)(\psi^{1,2,3} - \psi^{1,3,2} + \psi^{14,3,2} - \psi^{14,2,3}) = 0.
\]
Let $s'$ be the automorphism of $\mathfrak{t}^0_{3,2}$ uniquely defined by $s$ sending $s(t^{12}) = t^{13}, s(t^{13}) = t^{12}$ and $s(t^{23}) = t^{23}$.

Then the above equation can be read as
\[
\Omega^{1,2,3} = \Omega^{14,2,3} \text{ in } \mathfrak{t}^0_{4,2}
\]
where
\[
\Omega := \psi^{1,2,3} - \psi^{1,3,2} + s'(\psi)^{1,2,3} - s'(\psi)^{1,3,2} \in \mathfrak{t}^0_{3,2}.
\]
By the lemma below, $\Omega$ is described as $\Omega = r(t^{23}_+, t^{23}_-)$ for $r \in S_2$. So
\[
\psi(t^{12}, t^{23}_+, t^{23}_-) = \psi(t^{13}, t^{23}_+, t^{23}_+) + \psi(t^{13}, t^{23}_-, t^{23}_-) = \psi(t^{12}, t^{23}_-, t^{23}_+) = r(t^{23}_+, t^{23}_-).
\]
By the identifications $\mathfrak{t}^0_{3,2}/(t^{12}) \simeq \mathfrak{g}_2 \simeq \mathfrak{t}^0_{1,2}/(t^{13})$, we have
\[
\psi(0, t^{23}_+, t^{23}_-) - \psi(-t^{23}_+, t^{23}_+, t^{23}_+) + \psi(-t^{23}_-, t^{23}_-, t^{23}_+) - \psi(0, t^{23}_-, t^{23}_+) = r(t^{23}_+, t^{23}_-),
\]
\[
\psi(-t^{23}_+, t^{23}_+, t^{23}_-) - \psi(0, t^{23}_+, t^{23}_+) + \psi(0, t^{23}_-, t^{23}_-) - \psi(-t^{23}_-, t^{23}_-, t^{23}_+) = r(t^{23}_-, t^{23}_+).
\]
These equalities give $r = 0$, which means $\Omega = 0$. It yields the validity of the octagon equation (2.2) for $\psi$. \qed
Lemma 3.2. If $X \in \mathfrak{t}_{4,2}^0$ satisfies $X^{1,2,3} = X^{14,2,3}$ in $\mathfrak{t}_{4,2}^0$, then $X$ belongs to the free Lie subalgebra $\mathfrak{F}_3$ of rank 2 with generators $t_{23}^2$ and $t_{23}^3$.

Proof. Consider the linear map $F : \mathfrak{t}_{4,2}^0 \to \mathfrak{t}_{4,2}^0$ sending $h \mapsto h_{12,3} - h_{14,2,3}$. Its image is contained in the Lie subalgebra of $\mathfrak{t}_{4,2}^0$ generated by $t_{12}^2$, $t_{23}^2$, $t_{24}^2$. According to [E], this Lie algebra is freely generated by these 5 elements. On the other hand, $t_{0,2}^3$ can be identified with the free Lie algebra $\mathfrak{F}_3$ generated by $t_{12}^2$, $t_{23}^2$. It follows that ker $F$ is equal to the kernel of the map $F : \mathfrak{F}_3 \to \mathfrak{F}_0$ sending $h \mapsto h(t_{12}^2, t_{23}^2, t_{23}^2) - h(t_{12}^2 + t_{24}^2 + t_{24}^2, t_{23}^2, t_{23}^3)$.

If $X \in \ker \tilde{F}$, then $X(0, t_{23}^2, t_{23}^3) = X(t_{24}^2, t_{24}^3, t_{23}^3)$, which implies, as the Lie subalgebra of $\mathfrak{F}_3$ generated by $t_{24}^2$, $t_{23}^2$ and $t_{23}^3$ is isomorphic to $\mathfrak{F}_3$, that $X$ belongs to the Lie subalgebra $\mathfrak{F}_3 \subset \mathfrak{F}_3$ freely generated by $t_{23}^2$ and $t_{23}^3$. \hfill \Box

The following is a geometric interpretation of our arguments above.

Remark 3.3. Put $\mathfrak{M}_{0,4}^2 := \{ z \in \mathbb{A}^1 | z \neq 0, \pm 1 \}$ and $\mathfrak{M}_{0,5}^2 := \{ (x, y) \in \mathbb{A}^2 | xy \neq \pm 1, x, y \neq 0, \pm 1 \}$. These are the the Kummer coverings of the moduli spaces $\mathfrak{M}_{0,4}^2 := \{ z \in \mathbb{A}^1 | z \neq 0, 1 \}$ with $\mathfrak{M}_{0,4}^2 \to \mathfrak{M}_{0,4}^0 : z \mapsto z^2$ and $\mathfrak{M}_{0,5}^2 := \{ (x, y) \in \mathbb{A}^2 | xy \neq 1, x, y \neq 0, 1 \}$ with $\mathfrak{M}_{0,5}^2 \to \mathfrak{M}_{0,5}^0 : (x, y) \mapsto (x^2, y^2)$ respectively. The Lie algebras $t_{0,2}^0$, $t_{0,3}^0$, $t_{0,4}^0$ and $t_{0,2}^4$ are associated with the fundamental groups of $\mathfrak{M}_{0,4}$, $\mathfrak{M}_{0,4}^2$, $\mathfrak{M}_{0,5}$ and $\mathfrak{M}_{0,5}^2$ respectively. The picture of $\mathfrak{M}_{0,5}^5$ in Figure 1 is obtained by blowing-ups of $\mathbb{A}^2$ at $(x, y) = (0, 0), (\pm 1, \pm 1)$ and $(\infty, \infty)$. Our $H$ above corresponds to the pentagon near origin surrounded by $\varphi^{2,3,4}$, $\psi^{1,2,3,4}$, $\psi^{1,2,3}$ and $\psi^{12,3,4}$. Our $\sum_{\sigma \in \mathfrak{S}_3} \epsilon(\sigma) \Pi^{1, \sigma(2), \sigma(3), \sigma(4)}$ above corresponds to the six pentagons in the first quadrant and $\sum_{\sigma \in \mathfrak{S}_3} \epsilon(\sigma) \varphi^{(2), \sigma(3), \sigma(4)}$ corresponds to the hexagon there. Our $(id + s + s^2 + s^3) \sum_{\sigma \in \mathfrak{S}_3} \epsilon(\sigma) \Pi^{1, \sigma(2), \sigma(3), \sigma(4)}$ stands for all the $(24)$-pentagons in the picture and $(id + s + s^2 + s^3)(\psi^{1,2,3} - \psi^{1,3,2} + \psi^{14,3,2} - \psi^{14,2,3})$ means the two octagons near $(0, 0)$ and $(\infty, \infty)$.

Next we show an analogue of Theorem 3.1 for group-like series.

Theorem 3.4. Let $(g, h) \in \exp t_{0,4}^0 \times \exp t_{0,2}^0$ be a pair satisfying $c_{B(0)}(h) = c_{AB(0)}(h) = 0$, the mixed pentagon equation (2.16) and the special action condition (2.24). Then $h$ satisfies the octagon equation (2.6).

Proof. By taking the image of $\delta_{g_1}$ by $\delta_{g_2}$ and eliminating the first strand, we get $\delta_{g_1}(h) = g$ because of Lemma 2.10 and then the pentagon equation (1.7) for $g$. By $c_{B(0)}(h) = 0$ and (1.7), the linear terms of $g$ are all zero. Hence by $c_{AB(0)}(h) = 0$, its quadratic terms are all zero. By Theorem 1.6(2), $g \in \text{GRT}_1(k)$. So it suffices to prove $(g, h) \in \text{GRT}_{1}(2, k)$. The proof can be done by induction on degree. Suppose that we have (2.6) for $(g, h)$ modulo degree $n$, which we denote as $(g, h) \pmod{\text{deg } n} \in \text{GRT}_{1}(2, k)(n)$.

Then there is a pair $(g_1, h_1) \in \text{GRT}_{1}(2, k)$ with $(g, h) \equiv (g_1, h_1) \pmod{\text{deg } n}$ by Lemma A.3. Let $(g_0, h_0)$ be the pair defined by $(g, h) = (g_0, h_0) \circ (g_1, h_1)$. 
Then the pair \((g_0, h_0)\) lies in \(\exp \mathfrak{t}_0 ^g \times \exp \mathfrak{t}_{1,2}^g\) and satisfies \(c_B(0)(h_0) = c_{AB}(0)(h_0) = 0\), \(\mathfrak{t}_0^g\) and \(\mathfrak{t}_{1,2}^g\) by \((g_1, h_1) \in \text{GRTM}_{(1,1)}(2, k)\). By \((g, h) \equiv (g_1, h_1) \pmod{\deg n}\), we have \((g_0, h_0) \equiv (1, 1) \pmod{\deg n}\). Denote the degree \(n\)-part of the pair \((g_0, h_0)\) by \((\varphi, \psi)\). The pair \((\varphi, \psi)\) lies in \(\mathfrak{t}_0 ^g \times \mathfrak{t}_{1,2}^g\) and satisfies \((\varphi, \psi) \in \mathfrak{t}_0^g \times \mathfrak{t}_{1,2}^g\) by \text{(2.3)} and \text{(2.7)} for \((g, h)\) and \((g_0, h_0)\), which is obtained by comparing the lowest differing terms of the equations. Then by Theorem \text{(3.1)} we have

\[(\varphi, \psi) \in \text{grtm}_{(1,1)}(2, k).\]

Let \((g_0', h_0')\) be the element in \(\text{GRTM}_{(1,1)}(2, k)\) which corresponds to \((\varphi, \psi) \in \text{grtm}_{(1,1)}(2, k)\) by the exponential map. Since \((g_0, h_0) \equiv (g_0', h_0') \pmod{\deg n + 1}\),

\[(g_0, h_0) \pmod{\deg n + 1} \in \text{GRTM}_{(1,1)}(2, k)^{(n+1)}\]

Therefore \((g, h) \pmod{\deg n + 1} \in \text{GRTM}_{(1,1)}(2, k)^{(n+1)}\). □
Remark 3.5. We note that in Theorem 3.1 we do not assume the special condition (2.3), on the other hand, in Theorem 3.4 we assume the special condition (2.7). The analogue of Theorem 1.6 (3) might be the implication of (2.9) from (2.1) but we do not know whether this implication holds.

4. Broadhurst duality

We will show that the Broadhurst duality relation is compatible with the torsor structure of Pseudo$\mathcal{B} (1,1) (2, k)$.

Let $\tau$ be the involution of $t_{1,2}^0$ defined by $\tau : A \leftrightarrow B(0)$ and $B(1) \leftrightarrow C$.

Definition 4.1. (1). The set $\text{grtm}_{(1,1)} (2, k)$ is defined as the set of all $\psi \in \text{grtm}_{(1,1)} (2, k)$ such that the Broadhurst duality relation

$$(4.1) \quad \tau (\psi) + \psi + \alpha_\psi (A + B(0)) = 0$$

holds for some $\alpha_\psi \in k$.

(2). The set $\text{GRTMB}_{(1,1)} (2, k)$ is defined as the set of all $(g, h) \in \text{GRTM}_{(1,1)} (2, k)$ such that the Broadhurst duality relation

$$(4.2) \quad \tau (h) e^{\alpha_h B(0)} h e^{\alpha_h A} = 1$$

holds for some $\alpha_h \in k$.

(3). The set $\text{PseudoB}_{(1,1)} (2, k)$ is defined as the set of all $(g, h) \in \text{Pseudo}_{(1,1)} (2, k)$ such that the Broadhurst duality relation (4.2) holds for some $\alpha_h \in k$.

Actually $\alpha_\psi$ and $\alpha_h$ are equal to the coefficients of $B(1)$ in $\psi$ and $h$ respectively.

Theorem 4.2. (1). The set $\text{grtm}_{(1,1)} (2, k)$ forms a Lie algebra by the Lie bracket (1.8).

(2). The set $\text{GRTMB}_{(1,1)} (2, k)$ forms an algebraic group by the multiplication (1.8) and its associated Lie algebra is $\text{grtm}_{(1,1)} (2, k)$.

(3). The set $\text{PseudoB}_{(1,1)} (2, k)$ forms a right $\text{GRTMB}_{(1,1)} (2, k)$-torsor by (1.8).

Proof. (1). Put $\text{OutDer}(t_{1,2}^0) = (\text{Der}/\text{Int})(t_{1,2}^0)$. This quotient forms a Lie algebra with the involution induced by $\tau$. Its invariant part $\text{OutDer}^+ (t_{1,2}^0)$ again forms a Lie algebra. The embedding sending $(\varphi, \psi) \rightarrow (D_\varphi, D_\psi)$ induces the embedding $\text{grtm}_{(1,1)} (2, k) \hookrightarrow \text{Der}(t_{1,2}^0) \times \text{OutDer}(t_{1,2}^0)$. It can be checked that (4.1) is the condition for $(\varphi, \psi)$ to belong to the intersection of two Lie algebras $\text{grtm}_{(1,1)} (2, k)$ and $\text{Der}(t_{1,2}^0) \times \text{OutDer}^+ (t_{1,2}^0)$.

(2). It can be proved similarly. Put $\text{Out}(t_{1,2}^0) = (\text{Aut}/\text{Inn})(t_{1,2}^0)$, the outer automorphism group of $t_{1,2}^0$, the group of automorphisms modulo inner automorphisms. This quotient forms a group with the involution induced from $\tau$. Its invariant part $\text{Out}^+(t_{1,2}^0)$ again forms a group. The embedding sending $(g, h) \rightarrow (A_g, A_h)$ induces the embedding $\text{GRTM}_{(1,1)} (2, k) \hookrightarrow \text{Aut}(t_{1,2}^0) \times \text{Out}(t_{1,2}^0)$. It can be checked that (1.2) is the condition for $(g, h)$ to belong to the intersection of two group $\text{GRTM}_{(1,1)} (2, k)$ and $\text{Aut}(t_{1,2}^0) \times \text{Out}^+(t_{1,2}^0)$. Since $\text{grtm}_{(1,1)} (2, k)$ and $\text{GRTMB}_{(1,1)} (2, k)$ are the associated Lie algbras with these two groups, $\text{grtm}_{(1,1)} (2, k)$ is associated with $\text{GRTMB}_{(1,1)} (2, k)$.

(3). By direct calculation, it can be shown that

$$\tau (h_3) e^{(\alpha_{h_1} + \alpha_{h_2}) B(0)} h_3 e^{(\alpha_{h_1} + \alpha_{h_2}) A} = 1$$
for \((g_3,h_3) = (g_1,h_1) \circ (g_2,h_2)\) with \((g_1,h_1) \in \text{PseudoB}_{(1,1)}(2,k)\) and \((g_2,h_2) \in \text{GRTM}_{(1,1)}(2,k)\), which shows that \(\text{PseudoB}_{(1,1)}(2,k)\) is a \(\text{GRTM}_{(1,1)}(2,k)\)-space. To prove that it forms a torsor, it suffices to show that the action is transitive. Assume that \((g_1,h_1)\) and \((g_3,h_3)\) belong to \(\text{PseudoB}_{(1,1)}(2,k)\) and they are equal \(\mod \deg n - 1\). Then the degree \(n\)-part \(\psi\) of their difference satisfies \(\tau(\psi) + \psi = 0\). So it gives an element \((\varphi,\psi) \in \text{grtmb}_{(1,1)}(2,k)\). Put

\[
(g_2^{(n)}, h_2^{(n)}) := \text{Exp}(\varphi, \psi) \in \text{GRTM}_{(1,1)}(2,k).
\]

Let \((g_2,h_2) \in \text{GRTM}_{(1,1)}(2,k)\) be the element uniquely determined by \((g_3,h_3) = (g_1,h_1) \circ (g_2,h_2)\). Then \((g_2,h_2) \equiv (g_2^{(n)}, h_2^{(n)}) \mod \deg n\). By approximation methods replacing \((g_1,h_1)\) by \((g_1,h_1) \circ (g_2^{(n)}, h_2^{(n)})\), we can show \((g_2,h_2)\) belongs to \(\text{GRTM}_{(1,1)}(2,k)\).

We note that by Corollary 2.11

\[
\text{grtmb}_{(1,1)}(2,k) = \text{grtmb}_{(1,1)}(2,k),
\]

\[
\text{GRTMDB}_{(1,1)}(2,k) = \text{GRTM}_{(1,1)}(2,k),
\]

\[
\text{PdistB}_{(1,1)}(2,k) = \text{PseudoB}_{(1,1)}(2,k).
\]

**Remark 4.3.** (i). The equation (1.1) holds for \(h = \Phi_{KZ}^N\) with \(\alpha = \log 2\) and \(N = 2\) (cf. [O] §4). Here \(\Phi_{KZ}^N\) means an \(N\)-cyclotomic analogue of the Drinfeld associator, whose all coefficients are multiple \(L\)-values (see also [E]). It is explained in [Q] that the equation yields the Broadhurst duality relation [Bh] (127) of multiple \(L\)-values with signature \(\{\pm\}\).

(ii). In [LNS], a subgroup \(\Gamma\) of the pro-finite Grothendieck-Teichmüller group \(\tilde{GT}\) is introduced, with the properties of both containing the absolute Galois group \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\) of the rational number field \(\mathbb{Q}\) and of acting on the pro-finite completion of all the mapping class groups. One of its main defining conditions is Equation (IV) from [LNS], an equivalent form of which was found in [F03a] Equation (3). The latter equation is a pro-finite analogue of (1.2).

Since (1.1) is geometric, \(\text{grtmb}_{(1,1)}(2,k)\) contains the free Lie algebra \(\text{LieGal}^{1H}(\mathbb{Z}[\frac{1}{\varphi}])\) with one free generator in each degree 1, 3, 5, 7, ... It is fundamental to ask if they are equal or not. Namely

**Question 4.4.** Is the Lie algebra \(\text{grtmb}_{(1,1)}(2,k)\) free with one generator in each degree 1, 3, 5, 7, ...?

**APPENDIX A. INFINITESIMAL MODULE CATEGORIES**

In this appendix basics of infinitesimal module categories are given. We prove the fact implicitly employed in [E] that \(\text{GRTM}_{(1,1)}(N,k)\) forms a group by using its action on infinitesimal module categories.

**A.1. Infinitesimal braided monoidal categories.** An infinitesimal braided monoidal category (IBMC for short) is a set

\[
\mathcal{C} = (\mathcal{C}, \otimes, I, a, c, l, r, U, t)
\]

consisting of a category \(\mathcal{C}\), a bi-functor \(\otimes : \mathcal{C}^2 \to \mathcal{C}\), \(I \in \text{Ob}\mathcal{C}\), functorial assignments \(a_{XYZ} \in \text{Isom}_\mathcal{C}(X \otimes (Y \otimes Z), (X \otimes Y) \otimes Z)\) and \(c_{XY} \in \text{Isom}_\mathcal{C}(X \otimes Y, Y \otimes X)\),
$l_X \in \text{Isom}_C(I \otimes X, X)$, $r_X \in \text{Isom}_C(X \otimes I, X)$, a normal subgroup $U_X$ of $\text{Aut}_C X$ and $t_{XY} \in \text{Lie}_C(X \otimes Y)$ for all $X, Y, Z \in \text{Ob} C$ which satisfies the following:

(i). It forms a braided monoidal category $\mathbf{M}$ (quasi-tensor category [Dr]): the triangle, the pentagon and the hexagon axioms hold for $a, c, l, r$ and $I$ (cf. [Dr] (1.7)-(1.9b)).

(ii). The group $U_X$ is a pro-unipotent $k$-algebraic group and $fU_X f^{-1} = U_Y$ for any $f \in \text{Isom}_C(X, Y)$ holds and any $X, Y \in \text{Ob} C$.

(iii). The map $t_{XY}$ is functorial on $X$ and $Y$ and satisfies

$t_{XY} \in G$ for any $f \otimes G$ where $t$ and $g$ yield a morphism $\text{Mor}(X, X) \in G$ for all $a, b \in C$ with relations

$m, N \in \{\text{parenthesizations of the word } \bullet \cdot \cdot \cdot \}$

and for $X, X' \in \text{Ob} C$:

$\text{Mor}_{\text{univ}}(X, X') = \begin{cases} G_{a, N} & \text{if their lengths } |X| \text{ and } |X'| \text{ are equal to } n, \\ \emptyset & \text{if their lengths are different.} \end{cases}$

Let $\otimes : (\text{Ob} C) \times \text{Ob} C \to \text{Ob} C$ be the map induced by the concatenation and $G_{m, N} \times G_{m, N} \to G_{m + n, N}$ be the homomorphism induced by the juxtaposition. They yield a morphism $\text{Mor}(X, X') \times \text{Mor}(Y, Y') \to \text{Mor}(X \otimes Y, X' \otimes Y')$. Put $a_{XYZ} := 1 \in G_{|X| + |Y| + |Z|, N}$ and

$c_{XY} := \sigma_{|X|, |Y|} \in \xi_{|X| + |Y|} \subset G_{|X| + |Y|, N} \subset G_{|X| + |Y|, N}$

where $\sigma_{|X|, |Y|}$ means the permutation interchanging $X$ and $Y$. Set $I = \emptyset$ (the empty word) $\in \text{Ob} C$, $l_X$ and $r_X$ to be the identity maps. Finally we put $U_X = U_{m, N} \in \text{Aut}_{\text{univ}}(X)$ and

$t_{XY} := \sum_{1 \leq i \leq m, 1 \leq j \leq n, a \in C_N} t(a)^{i, m + j} \in U_{m + n, N}$

for $|X| = m$ and $|Y| = n$. Then $C_{\text{univ}}$ forms a $C_N$-IBMC, which is universal in the following sense: if $C$ is a $C_N$-IBMC with a distinguished object $X$, then there exists a unique functor $C_{\text{univ}} \to C$ of $C_N$-IBMC's which sends $\bullet$ to $X$.
A.2. Infinitesimal module categories over $C_N$-braided monoidal categories.

Let $\mathbb{C} = (\mathcal{C}, \otimes, I, a, c, l, r, U, t, \sigma)$ be a $C_N$-IBMC. An infinitesimal (right) module category (IMC for short) over $\mathbb{C}$ is a set

$$\mathcal{M} = (\mathcal{M}, \otimes, b, r, V, t)$$

consisting of a category $\mathcal{M}$, a bi-functor $\otimes : \mathcal{M} \otimes \mathcal{C} \to \mathcal{M}$, functorial assignments $b_{MXY} \in \text{Isom}_{\mathcal{M}}(M \otimes (X \otimes Y), (M \otimes X) \otimes Y)$ and $r_M \in \text{Isom}_{\mathcal{M}}(M \otimes I, M)$, a normal subgroup $V_M$ of $\text{Aut}_{\mathcal{M}}(M)$ and $t_{MX} \in \text{Lie}_{V_M}X$ for all $M \in \text{Ob}\mathcal{M}$ and $X, Y \in \text{Ob}\mathcal{C}$ which satisfies the following:

(I). It forms a right module category over $(\mathcal{C}, \otimes, I, a, c, l, r)$: the mixed pentagon axiom

$$(b_{MXY} \otimes id_Z)b_{M \otimes X \otimes Y, Z} = b_{M \otimes X, Y, Z}b_{M, X \otimes Y}(id_M \otimes a_{XYZ})$$

and the triangle axioms

$$r_{M \otimes X}b_{MIX} = id_M \otimes r_X \quad \text{and} \quad (r_M \otimes id_X)b_{MIX} = id_M \otimes l_X$$

hold for all $M \in \text{Ob}\mathcal{M}$ and $X, Y, Z \in \text{Ob}\mathcal{C}$.

(II). The octagon axiom

$$id_{M \otimes X} \otimes \sigma_Y = b_{M, M \otimes X}((id_M \otimes c_{XY})b_{MYX}^{-1}(id_M \otimes c_{XY})b_{MYX}^{-1})$$

holds for all $M \in \text{Ob}\mathcal{M}$ and $X, Y \in \text{Ob}\mathcal{C}$.

(III). The group $V_M$ is a pro-unipotent $k$-algebraic group and $fV_Mf^{-1} = V_{M'}$ holds for any $f \in \text{Isom}_{\mathcal{M}}(M, M')$ and any $M, M' \in \text{Ob}\mathcal{M}$.

(IV). The map $t_{MX}$ is functorial on $\mathcal{M}$ and $X$ and satisfies

$$t_{M \otimes X, Y} = b_{MXY}(id_M \otimes c_{XY})b_{MYX}^{-1}(id_M \otimes t_{XY})b_{MYX}^{-1} + \sum_{a \in C_N} (id_{M \otimes X} \otimes \sigma_Y^a) \cdot b_{MXY}(id_M \otimes t_{XY})b_{MYX}^{-1} \cdot (id_{M \otimes X} \otimes \sigma_Y^{-a})$$

and

$$t_{M \otimes X, Y} + t_{MX} \otimes id_Y = b_{MXY}t_{M, X \otimes Y}b_{MYX}^{-1}.$$

We can formulate the notion of functors between two IMC’s over $C_N$-IBMC’s. We note that such category forms a braided module category in the sense of $[E]$. A natural morphism $u_{n, N} \to t_{n+1, N}$ is obtained by shifting indices by $1$. By the morphism we extend the $G_{n, N}$-action on $u_{n, N}$ into on $t_{n+1, N}$ via

$$(c_1, \cdots, c_n) \cdot t_1^{i+1} = t_1^{i+1} \quad \text{and} \quad \sigma(t_1^{i+1}) = t_1^{\sigma(i)+1}$$

for $c_1, \cdots, c_n \in C_N$, $\sigma \in S_n$ and $1 \leq i \leq n$. Put $\hat{U}_{n+1, N} := \text{exp}t_{n+1, N}$ and $\hat{G}_{n+1, N} := \hat{U}_{n+1, N} \rtimes G_{n, N}$. We now construct an IMC $\mathcal{M}_\text{univ}$ over the $C_N$-IBMC $\mathbb{C}_\text{univ}$. Set

$$\text{Ob}\mathcal{M}_\text{univ} = \text{Ob}\mathcal{C}_\text{univ}$$

and for $M, M' \in \text{Ob}\mathcal{M}_\text{univ}$,

$$\text{Mor}_{\mathcal{M}_\text{univ}}(M, M') = \begin{cases} \hat{G}_{n+1, N} & \text{if their lengths } |M| \text{ and } |M'| \text{ are equal to } n, \\ \emptyset & \text{if their lengths are different.} \end{cases}$$

Let $\otimes : \text{Ob}\mathcal{M}_\text{univ} \times \text{Ob}\mathcal{C}_\text{univ} \to \text{Ob}\mathcal{C}_\text{univ}$ be the map induced by the concatenation and $\hat{G}_{n+1, N} \times \hat{G}_{n, N} \to \hat{G}_{n+1, N}$ be the homomorphism induced by the juxtaposition. They yield a morphism $\text{Mor}(M, M') \times \text{Mor}(X, X') \to \text{Mor}(M \otimes X, M' \otimes X')$. 
Put $b_{MXY} := 1 \in \mathcal{G}_{M\lvert X\rvert+Y\rvert+1,N}$ and $r_M := id_M$. Finally we put $V_M = \hat{\mathcal{U}}_{m+1,N} \subset \hat{\mathcal{G}}_{m+1,N} = \text{Aut}_M(M)$ and
\[
t_{Mx} := \sum_{1 \leq j \leq n} \sum_{0 \leq i \leq m+j-1} t^{i+1,j+m+1} \in \mathcal{G}_{m+n+1,N}
\]
for $|M| = m$ and $|X| = n$. Then it can be shown that $M_{\text{univ}}$ forms an IMC over the $C_N$-IBMC $C_{\text{univ}}$, which is universal in the following sense: if $M$ is an IMC over a $C_N$-IBMC $\mathcal{C}$ with distinguished objects $M \in \text{Ob} M$ and $X \in \text{Ob} C$, then there exists unique functors $M_{\text{univ}} \to M$ and $C_{\text{univ}} \to C$ of IMC’s over $C_N$-IBMC’s which send $\bullet$ to $M$ and $X$ respectively.

A.3. Automorphisms. Let $\mathcal{C} = (\mathcal{C}, \otimes, I, a, c, l, r, U, t)$ be an IMC. Let $g \in \exp t_{\mathcal{C}}^0$. Set
\[
\tilde{a}_{\mathcal{C}XY} := a_{\mathcal{C}XYZ} g(a_{\mathcal{C}XY}^{-1}(t_{\mathcal{C}XY} \otimes id_Z) a_{\mathcal{C}XY}^{-1}, \id_{\mathcal{C}X} \otimes t_{\mathcal{C}YZ})^{-1}.
\]
Then the set $\hat{\mathcal{C}} = (\mathcal{C}, \otimes, I, \tilde{a}, c, l, r, U, t)$ is an IMC if and only if $g \in \text{GRT}_{\mathcal{C}}(\mathbf{k})$, i.e. it satisfies (1.5)-(1.7) (c.f. [Dr]). This yields that $\text{GRT}_{\mathcal{C}}(\mathbf{k})$ forms a group by (1.8).

Let $g \in \text{GRT}_{\mathcal{C}}(\mathbf{k})$ and $h \in \exp t_{\mathcal{C},N}^0$. Let $\mathcal{C} = (\mathcal{C}, \otimes, I, a, c, l, r, U, t, \sigma)$ be a $C_N$-IBMC and $\hat{\mathcal{M}} = (\mathcal{M}, \otimes, \hat{b}, r, V, t)$ be an IMC over it. Define $\hat{\mathcal{C}}$ as above. Put $\hat{\mathcal{M}} = (\mathcal{M}, \otimes, \hat{b}, r, V, t)$ with
\[
\tilde{b}_{MXY} = b_{MXY} \cdot h(b_{MXY}^{-1}(t_{MXY} \otimes id_Y) b_{MXY}, \id_M \otimes t_{MXY}, b_{MXY}^{-1}(id_M \otimes \sigma_Y) b_{MXY}(id_M \otimes t_{MXY}), \ldots, b_{MXY}^{-1}(id_M \otimes \sigma_Y^{-1}) b_{MXY}(id_M \otimes t_{MXY}))^{-1}.
\]

**Lemma A.1.** The new set $\hat{\mathcal{M}}$ is an IMC over $\hat{\mathcal{C}}$ if and only if $(g, h) \in \text{GRT}_{\mathcal{C},(1,1)}(N, \mathbf{k})$.

**Proof.** The equation (2.5), (2.6) and (2.7) guarantee respectively the mixed pentagon axiom in (I), the octagon axiom in (II) and the first equality in (IV). As for the second equality in (IV), it is automatic because $t_{MXY} \otimes id_Y$ and $(id_M \otimes \sigma_Y) b_{MXY}(id_M \otimes t_{MXY}) b_{MXY}^{-1}$ commute with $t_{MXY} + t_{MXY} \otimes id_Y$.

Conversely by taking $(\mathcal{C}, M) = (\mathcal{C}_{\text{univ}}, M_{\text{univ}})$, one sees that the presentations (I)-(IV) imply the relations (2.5)-(2.7).

As a corollary we get

**Proposition A.2.** The set $\text{GRT}_{\mathcal{C},(1,1)}(N, \mathbf{k})$ forms a group by the multiplication (2.8).

For $n \geq 1$ define $\text{GRT}_{\mathcal{C},(1,1)}(N, \mathbf{k})^{(n)}$ to be the set of $(g, h) \mod \text{deg } n \in \exp t_{\mathcal{C}}^0 \times \exp t_{\mathcal{C},N}^0 \mod \text{deg } n$ which satisfies all the defining equations of $\text{GRT}_{\mathcal{C},(1,1)}(N, \mathbf{k})$ modulo deg $n$. By considering all IMC over $C_N$-IBMC $\mathcal{C}$ such that $\Gamma^{n+1}U_X = \Gamma^n V_M = \{1\}$ ($\Gamma^n$: the $n$-th term of lower central series) holds for any $X \in \text{Ob} C$ and $M \in \text{Ob} M$, we see that $\text{GRT}_{\mathcal{C},(1,1)}(N, \mathbf{k})^{(n)}$ forms an algebraic group. The following was required to prove Theorem 3.3.

**Lemma A.3.** The natural morphism $\text{GRT}_{\mathcal{C},(1,1)}(N, \mathbf{k}) \to \text{GRT}_{\mathcal{C},(1,1)}(N, \mathbf{k})^{(n)}$ is surjective.
Proof: This is a morphism of pro-unipotent algebraic group, which induces a Lie algebra morphism
\[ \grtm_{(1,1)}(N,k) = \prod_{k=1}^{\infty} \grtm_{(1,1)}(N,k)^{(k)} \rightarrow \grtm_{(1,1)}^{(n)}(N,k) = \oplus_{k=1}^{n} \grtm_{(1,1)}(N,k)^{(k)}. \]
Here \( \grtm_{(1,1)}(N,k)^{(k)} \) means the degree \( k \)-component of \( \grtm_{(1,1)}(N,k) \). Since the Lie algebra morphism is surjective, so is the pro-algebraic group morphism. □

**APPENDIX B. ERRATUM OF [E]**

We use this opportunity to correct some errors in [E].

1. The first two formulæ in [E] page 400 should be replaced by
\[ \Phi_{KZ}^{0,1,23} \Phi_{KZ}^{01,2,3} = \Phi_{KZ}^{1,2,3} \Phi_{KZ}^{0,1,2} \]
and
\[ \Psi_{KZ}^{0,1,23} \Psi_{KZ}^{01,2,3} = \Phi_{KZ}^{1,2,3} \Psi_{KZ}^{0,1,2}. \]

2. For a ring \( R \) and \( N \geq 1 \) the definition of the ring \( R(N) \) in [E]§6.2 should be read as follows. \( R(N) = (\mathbb{Z}/N\mathbb{Z}) \times R \), with the following operations. The sum is given by
\[ (a, r) + (a', r') = (a + a', r + r' + \sigma(a, a')) \]
and the product is given by
\[ (a, r)(a', r') = (aa', \bar{a}r' + \bar{a}'r + Nrr' + \pi(a, a')) , \]
where \( a \mapsto \bar{a} \) is the map \( (\mathbb{Z}/N\mathbb{Z}) \rightarrow \{0, 1, \ldots, N-1\} \) inverse to the ‘reduction modulo \( N \)’ map, and \( \sigma, \pi : (\mathbb{Z}/N\mathbb{Z})^2 \rightarrow \mathbb{Z} \) are defined by
\[ \bar{a} + \bar{a}' = \bar{a} + \bar{a}' + N\sigma(a, a'), \bar{a}a' = \bar{aa}' + N\pi(a, a'). \]

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