EINSTEIN–MAXWELL EQUATIONS: SOLUTION-GENERATING METHODS AS “COORDINATE” TRANSFORMATIONS IN THE SOLUTION SPACES

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The solution-generating methods discovered for integrable reductions of the Einstein and Einstein–Maxwell field equations (soliton-generating techniques, Bäcklund transformations, HKX transformations, Hauser–Ernst homogeneous Hilbert problem, and other group-theoretical methods) can be described explicitly as transformations of especially defined “coordinates” in the infinite-dimensional solution spaces of these equations. In general, the role of such “coordinates” for every local solution can be performed by monodromy data of fundamental solutions of the corresponding spectral problems. However, for large classes of fields, these can be the values of Ernst potentials on the boundaries that consist of degenerate orbits of the space–time isometry group such that space–time geometry and the electromagnetic fields behave regularly near these boundaries. In this paper, transformations of such “coordinates” corresponding to different known solution-generating procedures are described by relatively simple algebraic expressions that do not require any particular choice of the initial (background) solution. Explicit forms of these transformations allow us to find the interrelations between the sets of free parameters that arise in different solution-generating procedures and to determine some physical and geometrical properties of each generating solution even before a detail calculations of all its components.

Keywords: gravitational and electromagnetic fields, Einstein–Maxwell equations, integrability, solution-generating methods

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1. Introduction

Commenced more than forty years ago, the development of various approaches to the studies of the internal structure of the Einstein field equations has led different authors to a discovery that in some physically important cases these equations are completely integrable for space–times that satisfy certain space–time symmetry conditions, called $G_2$-symmetries in what follows.

The conditions that a space–time has the $G_2$-symmetry include the existence of a two-dimensional Abelian isometry group with non-null Killing vector fields and supplementary conditions imposed on the structure of the metric and matter field components. More precise descriptions of the necessary symmetry properties of these fields and corresponding references can be found in [1], [2]. In the cases where the $G_2$-symmetry-reduced Einstein field equations are integrable, these equations, similarly to many other known completely integrable equations, admit solution-generating procedures, which allow constructing infinite hierarchies of solutions with an arbitrary (finite) number of free parameters starting from any known solution of these equations.
Group-theoretic approach: the Geroch group and the Kinnersley–Chitre algebra. The development of the group-theoretic approach to the studies of internal symmetries of vacuum Einstein equations and electrovacuum Einstein–Maxwell equations had begun long ago from a beautiful discoveries of symmetry transformations for space–times with at least one Killing vector field. These transformations were found at the end of the 1950s for the vacuum by Ehlers [3], [4] and at the end of the 1960s for the electrovacuum by Harrison [5]. Later, it was found in [6], [7] (also see book [1]) that these transformations represent subgroups in larger groups of symmetries respectively isomorphic to $SU(1, 1)$ for the vacuum and to $SU(2, 1)$ for electrovacuum cases.

In the early 1970s, Geroch published paper [8], where he conjectured that for vacuum space–times with two commuting non-null Killing vector fields for which two certain real constants vanish (these conditions are equivalent to the $G_2$-symmetry conditions mentioned above), there exists an infinite-dimensional group of internal symmetries whose action on the space of solutions is transitive, i.e., every solution can be obtained by applying a symmetry transformation to a chosen solution, e.g., to the Minkowski space–time. Geroch also argued that the corresponding infinite-dimensional algebra of infinitesimal transformations can be build inductively. Later, in [9], Kinnersley extended these ideas to the case of stationary axisymmetric electrovacuum fields.

In subsequent papers [10]–[12], Kinnersley and Chitre presented a systematic study of the infinite-dimensional algebra of infinitesimal symmetries of the Einstein–Maxwell equations for stationary axisymmetric fields. In these papers, infinite hierarchies of complex matrix potentials associated with every particular solution were constructed, and it was shown that these hierarchies of potentials form a representation space of this algebra. Moreover, it was found that in the electrovacuum case, these sets of potentials admit two $(3 \times 3)$ (or $(2 \times 2)$ in the vacuum case) matrix generating functions, one of which happens to satisfy a linear system of equations with an auxiliary complex parameter, and another generating function can be expressed algebraically in terms of the first one. Later, in Julia’s papers [13], [14], the infinite-dimensional symmetry transformations of Geroch and Kinnersley–Chitre were recognized as Kac–Moody symmetries, and in Breitenlohner and Maison’s paper [15], the structure of the corresponding infinite-dimensional Geroch group was described in detail. However, at that time, the problem of exponentiating the Kinnersley–Chitre infinitesimal transformations for obtaining new solutions of the Einstein–Maxwell equations remained unsolved.

Soliton solutions of the Einstein and Einstein–Maxwell field equations. Using a very different approach based on the ideas and methods of the inverse scattering theory, in the pioneer papers [16], [17], Belinski and Zakharov discovered the existence of infinite hierarchies of exact $N$-soliton solutions of the vacuum Einstein equations depending on $2N$ free real parameters. These solitons can be generated on an arbitrarily chosen vacuum background with the $G_2$ symmetry mentioned above. It is also important that explicit expressions were obtained in those papers for all metric components, including the so-called conformal factor, the coefficient in front of the conformally flat part of the metric. This factor was expressed in [16], [17] explicitly in terms of the components of a chosen vacuum background metric and a solution of the corresponding spectral problem. A more compact, determinant form of Belinski and Zakharov’s $N$-soliton solutions was found in our paper [18]. It is also worth mentioning that in [16], [17], a $2 \times 2$ matrix linear singular integral equation with a Cauchy-type kernel was constructed for generating “nonsoliton” vacuum solutions.

Somewhat later, in our papers [19], [20], using the same general ideas and methods of the inverse scattering approach (but for essentially different, complex self-dual form of the Einstein–Maxwell equations

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1 In a later paper [10], Kinnersley and Chitre made an interesting comment: “Our inclusion of electromagnetism throughout this work has been an enormous help rather than a hindrance. It has revealed a striking interrelationship between electromagnetic and gravitational fields that could not possibly have been anticipated.”
found by Kinnersley [9]), the \(N\)-soliton solutions of the Einstein–Maxwell equations depending on \(3N\) free complex or \(6N\) real parameters were constructed starting from an arbitrarily chosen (\(G_2\)-symmetric) electrovacuum background.

Bäcklund transformations. Some other solution-generating methods suggested later for the \(G_2\)-symmetry-reduced vacuum Einstein equations were constructed using the basic ideas of the theory of Bäcklund transformations and other features in the group-theoretic context. Namely, a construction of Bäcklund transformations (for vacuum time-dependent as well as stationary axisymmetric fields) were described by Harrison [21],\(^2\) who used the pseudopotential method of Wahlquist and Estabrook [22]. In [21], the corresponding equations were expressed in terms of a closed ideal of differential 1-forms. Later, Harrison described particular applications of the Bäcklund transformations and generalized his approach to electrovacuum fields [23], [24].

Subsequently, Neugebauer [25] presented his form of the Bäcklund transformations for the vacuum Einstein equations for stationary axisymmetric fields, which were constructed in the spirit of the known theory of Bäcklund transformations for the sine- or sinh-Gordon equation. In a short series of papers [25]–[27], Neugebauer found a compact (determinant) form of the \(N\)-fold Bäcklund transformations expressed in terms of the Ernst potential of some chosen beginning (or “initial,” or “background”) solution and of the solutions of the Riccati equations with the coefficients depending on this choice.

Exponentiating some of the Kinnersley–Chitre infinitesimal symmetries: HKX transformations. Another solution-generating method for vacuum fields was found by Hoenelaers, Kinnersley, and Xanthopoulos. This method, later called HKX transformations and described in [28], [29] (also see book [1]), was derived as a result of the exponentiation of some infinitesimal Kinnersley–Chitre transformations. The corresponding “rank-\(p\)” transformations allow obtaining a new family of solutions of the same type with \(p+1\) arbitrary real parameters from a given initial vacuum stationary axisymmetric solution. The authors also suggested a construction of a superposition of such transformations with different parameters and gave the simplest examples.

Hauser and Ernst’s homogeneous Hilbert problem and the integral equation method for effecting the Kinnersley–Chitre transformations. Hauser and Ernst suggested yet another approach to generate stationary axisymmetric vacuum [30]–[33] and electrovacuum [34], [35] solutions. Within the class of solutions that are regular in some neighborhood of at least one point of the symmetry axis, the problem of “effecting” (i.e., exponentiating) the Kinnersley–Chitre infinite-dimensional algebra of infinitesimal symmetry transformations was reduced to solving a homogeneous Hilbert problem (HHP) on a closed contour on the plane of an auxiliary complex parameter, which was then reduced to solving a matrix linear singular integral equation of the Cauchy type on this contour.\(^3\) Once a solution of this integral equation is found for any chosen initial (“seed”) solution and arbitrarily selected element of the Kinnersley–Chitre algebra, the transformed solution can be calculated explicitly. However, the rational ansatz with too simple an algebraic structure suggested in [34] for solving this integral equation had led to a class of solutions essentially more restricted in the number of free parameters than the class of electrovacuum solitons [19], [20].

Two years later, using the already mentioned pseudopotential method of Wahlquist and Estabrook [22], Kramer and Neugebauer [36] constructed a linear system for another set of pseudopotentials.\(^4\) The integrability condition for that system are also ensured by the Einstein–Maxwell equations for stationary

\(^2\)Harrison already mentioned Belinski and Zakharov’s results in that paper.

\(^3\)Earlier, a construction of \(2 \times 2\) matrix linear singular integral equation with the kernel of a Cauchy type, whose solutions determine the solutions of the appropriate Riemann–Hilbert problem on the spectral plane, was suggested for generating “nonsoliton” vacuum solutions in Belinski and Zakharov’s first paper [16]. However, in [30], the integral equation method of Hauser and Ernst was more elaborated and examples of the construction of exact solutions for the rational choice of arbitrary functions in the kernel were described.

\(^4\)The relations between different matrix potentials suggested for the Einstein–Maxwell equations in different approaches were described by Kramer [37].
axisymmetric electrovacuum fields. Subsequently, the same authors [38] translated the constructions of soliton solutions in the spirit of inverse scattering transform (which was used earlier in [16], [17] for the vacuum and in [19], [20] for electrovacuum fields) to the system in [36]. Nonetheless, in addition to the results in [19], [20], where the expressions for all components of the metric (besides the conformal factor alone) and for the electromagnetic potential of the electrovacuum soliton solutions had been constructed, a useful input from paper [38] consisted in a derivation of compact determinant expressions for the Ernst potentials for stationary axisymmetric electrovacuum solitons.

The construction of Bäcklund transformations for electrovacuum Ernst equations was described in [24]. An interesting feature in that paper is the use of a modified Wahlquist–Estabrook approach, applicable to systems of equations that can be expressed in terms of differential forms generating a closed ideal with constant coefficients (a CC ideal). Many known integrable systems can be represented in such a form. In these cases, it is possible to formulate simple general ansatizes that lead to a construction in some unified form of the associated linear systems (and corresponding “spectral problems”) as well as of the Bäcklund transformations for these systems.

Later, a spectral problem of yet another structure for the same stationary axisymmetric electrovacuum Einstein–Maxwell field equations in the form of a sigma-model was suggested by Eris, Gürses, and Karasu in [40]. In this more geometric framework, the soliton solutions were also calculated in the spirit of the inverse scattering approach, but the spectral problem used there led directly to the calculation not of the metric components but of the Ernst potentials.

On the relations between solution-generating methods. Close interrelations between different approaches to the construction of vacuum solution-generating methods (the inverse scattering method, the theory of Bäcklund transformations, and group-theoretical approach) were described by Cosgrove [41]–[43]. The relations between the associated linear systems (“spectral problems”) used by different authors for generalizations of their approaches to electrovacuum fields were found by Kramer [37].

On the difficulties with explicit applications of solution-generating methods. The general studies of the families of solutions generated with the methods mentioned above are rather difficult because these families of solutions do not admit a representation in the general and explicit form, due to the presence in these solutions, besides a large number of constant parameters, of some functional parameters—the potentials that characterize the chosen initial (background) solution. In each of these methods, these potentials should satisfy some linear systems with the coefficients depending on the choice of the initial solution, but this systems cannot be solved explicitly for any choice of the initial solution. Only in those cases where this linear system can be solved explicitly for a chosen initial (background) solution can one calculate all components of the solutions generated on this background.

On the “coordinates” in the space of solutions. The difficulties just mentioned can be overcome if, instead of the metric and field components, we introduce some “coordinates” on the space of solutions such that, on one hand, these “coordinates” would be related to various physical and geometrical characteristics of the solutions and, on the other hand, different solution-generating procedures could be represented as transformations of these “coordinates.”

In the most general cases, for the \( G_2 \)-symmetry-reduced vacuum Einstein equations and electrovacuum Einstein–Maxwell equations, the role of such “coordinates” in the infinite-dimensional spaces of their local solutions can be played by the monodromy data of the fundamental solutions of the corresponding associated linear systems (“spectral problems”) [44], [39]. However, for large subclasses of field configurations that share the same type of asymptotic behavior near some space–time boundaries, such “coordinates” can be

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5 The structure of this system with its supplementary conditions differs essentially from more simple structure of the spectral problem used for construction of solitons in [19], [20].

6 The expression for conformal factor for electrovacuum solitons was found in [39].
defined in a simpler way. For example, for stationary axisymmetric fields, the role of such “coordinates”
can be played by the values of the Ernst potentials on those parts of the symmetry axis near which the
space–time geometry and electromagnetic fields behave regularly.

In this paper, we consider the classes of fields that, similarly to regular parts of the symmetry axis in
axisymmetric fields, have the boundaries made of degenerate orbits of the isometry group $G_2$ with regular
behavior of the metric and electromagnetic fields near these boundaries (see below for more details). It is
clear that besides the stationary axisymmetric fields near the regular parts of the symmetry axis, these
classes of fields include cylindrical waves and some other types of wave-like or cosmological solutions,
stationary fields with Killing horizons, and some other types of solutions that can, in particular, have
a dynamical nature (like the well-known “C-metrics”). For all these types of fields, the “coordinates”
in the space of solutions can be represented by the functional parameters defined as the values of the Ernst
potentials on the lines in the orbit space that consist of the points (orbits) at which the orbit geometry is
degenerate but the space–time geometry remains regular. These “coordinates” determine the corresponding
local solutions “almost uniquely.”

**Transformations of “coordinates” in the space of solutions.** As we show in what follows,
different solution-generating procedures can be represented explicitly and in a very simple form as trans-
formations of the above-described “coordinates” in the spaces of local solutions. It is very important that
these transformations have a general form that does not require specifying the choice of the initial solution
in advance. The initial solution in the expressions for these transformations is represented by functions
that can be chosen arbitrarily and are similar “coordinates” of the initial solution in the space of solutions.

### 2. Integrable reductions of the Einstein and Einstein–Maxwell
equations

**Metric and electromagnetic potential.** Integrable reductions of the vacuum Einstein equations
and of the electrovacuum Einstein–Maxwell equations arise if the metric and electromagnetic potential
components have the forms

$$
\begin{align*}
\text{Metric components:} & \quad g_{\mu\nu} dx^\mu \, dx^\nu + g_{ab} dx^a \, dx^b, \\
\text{Potential components:} & \quad A_i = \{ A_\mu, A_a \}, \quad A_\mu = 0, \\
& \text{coordinates:} \quad x^i = \{ x^\mu, x^a \}, \\
& \text{indices:} \quad \mu, \nu, \ldots = 1, 2, \\
& \quad a, b, \ldots = 3, 4,
\end{align*}
$$

(1)

where the components of the metric $g_{\mu\nu}$, $g_{ab}$ and the electromagnetic potential $A_a$ are independent of the
coordinates $x^a$ and may depend on coordinates $x^\mu$.

Each of these reductions belongs to one of two types depending on whether the space–time isometry
group $G_2$ admits a time-like Killing vector field (the “elliptic” case) or not (the “hyperbolic” case), i.e.,
whether the signature of the two-dimensional metric $g_{\mu\nu}$ on the space of orbits is respectively Euclidean
or Lorentzian. In the expressions below, the sign symbol $\epsilon$ and its “square root” $j$ remind us about the
difference between these cases:

$$
\epsilon = \begin{cases} 
1, & \text{hyperbolic case}, \\
-1, & \text{elliptic case}, 
\end{cases} \quad j = \begin{cases} 
1, & \epsilon = 1, \\
i, & \epsilon = -1.
\end{cases}
$$

(2)

The metric components $g_{\mu\nu}$ determine the two-dimensional metric on the orbit space of the space–time
isometry group $G_2$. By an appropriate choice of local coordinates $x^i$, the metric $g_{\mu\nu}$ can be presented in

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7For a given local solution, the Ernst potentials are defined up to some gauge freedom, which, however, that does not
change the geometry and physical parameters of the solution.
a conformally flat form, where we use the sign symbols $\epsilon_1$ and $\epsilon_2$ for a unified description of all cases:

$$g_{\mu\nu} = f \eta_{\mu\nu}, \quad \eta_{\mu\nu} = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix}, \quad \epsilon_1 = \pm 1, \quad \epsilon_2 = \pm 1, \quad \epsilon_1 \epsilon_2 = -\epsilon.$$  \hspace{1cm} (3)

Here, by definition, $f > 0$. The relation between $\epsilon_1$, $\epsilon_2$, and $\epsilon$ arises here from the condition of the Lorentz signature $(-+++)$ of the four-dimensional metric.

For the metric components $g_{ab}$ that determine the *metric on the orbits* of the space–time isometry group $\mathcal{G}_2$ and for components of the electromagnetic potential $A_a$, we introduce the parameterizations

$$g_{ab} = \epsilon_0 \begin{pmatrix} H & H \Omega \\ H \Omega & H \Omega^2 + \frac{\epsilon \alpha^2}{H} \end{pmatrix}, \quad A_a = \{A, \tilde{A}\}, \quad \det \|g_{ab}\| \equiv \epsilon \alpha^2,$$  \hspace{1cm} (4)

where, by definition, $H > 0$, $\alpha > 0$ and $\epsilon_0 = \pm 1$. Then the Einstein and Einstein–Maxwell equations for fields (1) imply that the function $\alpha(x^1, x^2)$ defined in (4) is “harmonic,” i.e., it satisfies a linear equation that is the two-dimensional d’Alembert equation in the hyperbolic case and the two-dimensional Laplace equation in the elliptic case. Therefore, for the function $\alpha(x^1, x^2)$, we can define its “harmonically conjugate” function $\beta(x^1, x^2)$ such that

$$\begin{cases} \det \|g_{ab}\| \equiv \epsilon \alpha^2, \\
\eta^{\mu\nu} \partial_\mu \partial_\nu \alpha = 0, \\
\eta^{\mu\nu} \partial_\mu \partial_\nu \beta = 0, \\
\epsilon^{\mu\nu} = \eta_{\mu\gamma} \epsilon^{\gamma\nu}, \quad \epsilon^{\mu\nu} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{cases}$$  \hspace{1cm} (5)

These geometrically defined functions $(\alpha, \beta)$ are used in what follows as local conformal coordinates on the orbit space of $\mathcal{G}_2$. We call them the generalized Weyl coordinates. It is also convenient to use their linear combinations

$$\xi = \beta + j \alpha, \quad \eta = \beta - j \alpha,$$  \hspace{1cm} (6)

which are real null coordinates in the hyperbolic case ($j = 1$) and are complex conjugate to each other in the elliptic case ($j = i$).

**Matrix form of dynamical equations for the vacuum.** For the metric in (1), the dynamical part of the Einstein equations for vacuum gravitational fields can be represented in a $2 \times 2$ matrix form [16], [17]

$$\begin{cases} \eta^{\mu\nu} \partial_\mu (\alpha \partial_\nu g \cdot g^{-1}) = 0, \\
g^T = g, \quad \det g \equiv \epsilon \alpha^2, \end{cases}$$  \hspace{1cm} (7)

where $g = \|g_{ab}\|$, and $T$ means matrix transposition. The electrovacuum Einstein–Maxwell equations for fields (1) can be presented in a similar form but by $3 \times 3$ matrix equations [39]. However, there also exist more convenient forms of the dynamical part of these equations.

**The Ernst equations.** The dynamical part of the electrovacuum Einstein–Maxwell equations for space–times with $\mathcal{G}_2$-symmetry can also be represented in the form of the Ernst equations\(^8\) for the function $\alpha$

\(^8\)These equations were originally derived by Ernst for stationary axisymmetric vacuum fields [45], and were then generalized to the case of stationary axisymmetric electrovacuum fields in [46]. In these equations, the Weyl cylindrical coordinates were used. For these coordinates, $\alpha = \rho$ and $\beta = z$. Similar equations can easily be derived in the hyperbolic case as well, and these are usually called the hyperbolic Ernst equations.
already mentioned above and two scalar complex Ernst potentials \( \mathcal{E} \) and \( \Phi \). In our notation, these equations take the form (the bar on a symbol means complex conjugation)

\[
\begin{align*}
\left( \text{Re } \mathcal{E} + \Phi \right) \eta^{\mu\nu} \left( \partial_\mu \partial_\nu \mathcal{E} + \frac{\partial_\mu \alpha}{\alpha} \partial_\nu \mathcal{E} \right) & - \eta^{\mu\nu} \left( \partial_\mu \mathcal{E} + 2 \Phi \partial_\mu \Phi \right) \partial_\nu \mathcal{E} = 0, \\
\left( \text{Re } \mathcal{E} + \Phi \right) \eta^{\mu\nu} \left( \partial_\mu \partial_\nu \Phi + \frac{\partial_\mu \alpha}{\alpha} \partial_\nu \Phi \right) & - \eta^{\mu\nu} \left( \partial_\mu \mathcal{E} + 2 \Phi \partial_\mu \Phi \right) \partial_\nu \Phi = 0, \\
\eta^{\mu\nu} \partial_\mu \partial_\nu \alpha & = 0.
\end{align*}
\]

For \( \Phi = 0 \), these equations reduce to the vacuum Ernst equation. The relations of \( \mathcal{E} \) and \( \Phi \) to the field components in (1) in the notation (4) take the form

\[
\begin{align*}
\text{Re } \Phi & = A, \\
\partial_\mu (\text{Im } \Phi) & = -\frac{\epsilon_0}{\alpha} \epsilon_\mu^{\nu} (\partial_\nu \tilde{A} - \Omega \partial_\nu A), \\
\text{Re } \mathcal{E} & = -\epsilon_0 H - \Phi \tilde{F}, \\
\text{Im } (\partial_\mu \mathcal{E} + 2 \Phi \partial_\mu \Phi) & = H^2 \epsilon_\mu^{\nu} \epsilon_\nu \Omega.
\end{align*}
\]

These relations allow finding all field components (1) algebraically or in quadratures if the Ernst potentials \( \mathcal{E}, \Phi \) and the function \( \alpha \) are known.

The Kinnersley equations \[9\]. For the fields in (1), using the notation

\[
h_{ab} \equiv g_{ab}, \quad h_a^b = \epsilon^{bc} h_{ac}, \quad h_{ab} = h_a^c \epsilon_{cb}, \quad \epsilon_{ab} = \epsilon^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]

the Einstein–Maxwell equations can be reduced to the self-dual matrix form

\[
\begin{align*}
H_\mu^a & = i \alpha^{-1} \epsilon_\mu^{\nu} h_a^c H_{\nu c}^b, & H_\mu^b & = \partial_\mu H_a^b, \\
\Phi_\mu a & = i \alpha^{-1} \epsilon_\mu^{\nu} h_a^c \Phi_{\nu c}, & \Phi_\mu a & = \partial_\mu \Phi_a, \\
H_\mu^a & = \partial_\mu h_a^b + i \alpha^{-1} \epsilon_\mu^{\nu} h_a^c \epsilon_\nu \partial_\nu h_c^b + 2 \Phi_\mu a \Phi^b.
\end{align*}
\]

The left-hand equations in the first two lines in (11) are the self-duality conditions for \( H_\mu^a \) and \( \Phi_\mu a \), while the existence of their potentials \( H_a^b \) and \( \Phi_a \) follow from the Einstein–Maxwell equations. The equation in the third line in (11) is the definition of \( H_\mu^a \), introduced in \[9\]. Equations (11) were also used in \[19\], \[20\] in the analysis of the integrability of Einstein–Maxwell equations.

Ernst equations in terms of a CC ideal of differential forms. Other interesting forms of the vacuum and electrovacuum Ernst equations were obtained by Harrison \[21\], \[24\], who used a modified Wahlquist–Estabrook formalism based on the construction of a closed ideal with constant coefficients (CC ideal) of differential 1-forms. It consists of self-dual \( \eta_1, \eta_2, \eta_5, \eta_7, \eta_8 \) and anti-self-dual \( \eta_3, \eta_4, \eta_6, \eta_9, \eta_{10} \) forms (the vacuum corresponds to \( \eta_7 = \eta_8 = \eta_9 = \eta_{10} = 0 \)):

\[
* \eta_p = \frac{1}{j} \eta_p, \quad p = 1, 2, 5, 7, 8, \quad \text{and} \quad * \eta_p = -\frac{1}{j} \eta_p, \quad p = 3, 4, 6, 9, 10.
\]
Here and hereafter, "d" denotes the Hodge “star operator": \( *d\phi \equiv \varepsilon^{\mu\nu} \partial_\nu \phi d\mu^\nu \). For simplicity in what follows, as in [21], [24], we omit the wedge symbol \( \wedge \) in the external products of forms. If the forms \( \eta_1, \ldots, \eta_{10} \) satisfy the conditions

\[
\begin{align*}
4d\eta_1 &= \eta_1(\eta_4 + \eta_6 - \eta_3) - \eta_3\eta_5 - 4\epsilon_0\eta_1\eta_7, \\
4d\eta_2 &= \eta_2(\eta_3 + \eta_6 - \eta_4) - \eta_4\eta_5 - 4\epsilon_0\eta_2\eta_8, \\
4d\eta_3 &= \eta_3(\eta_2 + \eta_6 - \eta_1) - \eta_1\eta_6 - 4\epsilon_0\eta_3\eta_9, \\
4d\eta_4 &= \eta_4(\eta_1 + \eta_5 - \eta_2) - \eta_2\eta_6 - 4\epsilon_0\eta_4\eta_{10}, \\
2d\eta_5 &= \eta_5\eta_6, & 2d\eta_6 &= -\eta_5\eta_6, \\
8d\eta_7 &= \eta_7(\eta_4 + 2\eta_6 - \eta_3) + 2\eta_3(\eta_1 - \eta_5), \\
8d\eta_8 &= \eta_8(\eta_3 + 2\eta_6 - \eta_4) + 2\eta_4(\eta_2 - \eta_5), \\
8d\eta_9 &= \eta_9(\eta_2 + 2\eta_5 - \eta_1) + 2\eta_1(\eta_3 - \eta_6), \\
8d\eta_{10} &= \eta_{10}(\eta_1 + 2\eta_5 - \eta_2) + 2\eta_2(\eta_4 - \eta_6), \\
\eta_1\eta_2 &= \eta_1\eta_5 = \eta_1\eta_7 = \eta_1\eta_8 = \eta_2\eta_5 = 0, \\
\eta_2\eta_7 &= \eta_2\eta_8 = \eta_5\eta_7 = \eta_5\eta_8 = \eta_7\eta_8 = 0, \\
\eta_3\eta_4 &= \eta_3\eta_6 = \eta_3\eta_9 = \eta_3\eta_{10} = \eta_4\eta_6 = 0, \\
\eta_4\eta_9 &= \eta_4\eta_{10} = \eta_6\eta_9 = \eta_6\eta_{10} = \eta_9\eta_{10} = 0,
\end{align*}
\]

(12)

then there exist (at least locally) two complex functions \( \mathcal{E}(x^\mu), \Phi(x^\mu) \) and two real functions \( \alpha(x^\mu) \) and \( \beta(x^\mu) \) such that these forms can be expressed as

\[
\begin{align*}
\eta_1 &= \frac{2(\mathcal{E}_\xi + 2\Phi_\Phi_\xi)}{\epsilon_0H} d\xi, & \eta_2 &= -\frac{2(\mathcal{E}_\xi + 2\Phi_\Phi_\xi)}{\epsilon_0H} d\xi, & \eta_3 &= \frac{d\xi}{j\alpha}, \\
\eta_3 &= -\frac{2(\mathcal{E}_\eta + 2\Phi_\Phi_\eta)}{\epsilon_0H} d\eta, & \eta_4 &= -\frac{2(\mathcal{E}_\eta + 2\Phi_\Phi_\eta)}{\epsilon_0H} d\eta, & \eta_5 &= -\frac{d\eta}{j\alpha}, \\
\eta_7 &= \frac{2\Phi_\xi}{\sqrt{H}} d\xi, & \eta_8 &= \frac{2\Phi_\xi}{\sqrt{H}} d\xi, & \eta_9 &= \frac{2\Phi_\eta}{\sqrt{H}} d\eta, & \eta_{10} &= \frac{2\Phi_\eta}{\sqrt{H}} d\eta, \\
H &= -\epsilon_0(\mathcal{E} + \bar{\mathcal{E}} + 2\Phi\bar{\Phi}),
\end{align*}
\]

where \( \xi \) and \( \eta \) were defined in terms of \( \alpha \) and \( \beta \) in (6). In addition, it also follows from (12) that the functions \( \mathcal{E} \) and \( \Phi \) satisfy the Ernst equations, which in the coordinates \( (\xi, \eta) \) take the form

\[
\begin{align*}
\left\{ \text{Re} \mathcal{E} + \Phi \bar{\Phi} \right\} \left( 2\mathcal{E}_{\xi\eta} - \frac{\mathcal{E}_\xi - \mathcal{E}_\eta}{\xi - \eta} \right) - (\mathcal{E}_\xi + 2\Phi_\Phi_\xi)\mathcal{E}_\eta - (\mathcal{E}_\eta + 2\Phi_\Phi_\eta)\mathcal{E}_\xi &= 0, \\
\left\{ \text{Re} \mathcal{E} + \Phi \bar{\Phi} \right\} \left( 2\Phi_{\xi\eta} - \frac{\Phi_\xi - \Phi_\eta}{\xi - \eta} \right) - (\mathcal{E}_\xi + 2\Phi_\Phi_\xi)\Phi_\eta - (\mathcal{E}_\eta + 2\Phi_\Phi_\eta)\Phi_\xi &= 0.
\end{align*}
\]

It is worth mentioning in advance that representing the field equations in terms of a closed CC ideal of 1-forms suggests a nice way to construct the associated linear systems and Bäcklund transformations using the ansatzes

\[
d\Psi = \left( \sum_{p=1}^{10} B^p \eta_p \right) \Psi, & \quad \bar{\eta}_p = \sum_{q=1}^{10} A_{p}^{\ d} \eta_q,
\]

where the matrix coefficients \( B^p \) and \( A_{p}^{\ d} \) are functions of the so-called pseudopotentials (see [21], [24] and the corresponding section below).
Calculation of the conformal factor. The Einstein–Maxwell equations for field components (1) also imply the constraint equations that allow calculating the conformal factor $f$ for any solution of dynamical equations (7) or (8) (see, e.g., [16], [39], [2]). In particular, in the generalized Weyl coordinates $x^1 = \alpha$ and $x^2 = \epsilon_1 \beta$, where the factor $\epsilon_1$ arises due to the definition of $\beta$ in (5), we have

$$g_{\mu\nu} \, dx^\mu \, dx^\nu = f(\epsilon_1 \, d\alpha^2 + \epsilon_2 \, d\beta^2)$$

and

$$\frac{\partial_\alpha (fH)}{fH} = \frac{\alpha}{2H^2} (f_\alpha \mathcal{F}_\alpha + \epsilon f_{\beta} \mathcal{F}_{\beta} - \frac{2\epsilon_0 \alpha}{H} (\partial_\alpha \Phi \partial_\alpha \Phi + \epsilon \partial_\beta \Phi \partial_\beta \Phi),$$

$$\frac{\partial_\beta (fH)}{fH} = \frac{\alpha}{2H^2} (f_\alpha \mathcal{F}_\alpha + \epsilon f_{\beta} \mathcal{F}_{\beta} - \frac{2\epsilon_0 \alpha}{H} (\partial_\alpha \Phi \partial_\alpha \Phi + \epsilon \partial_\beta \Phi \partial_\beta \Phi),$$

where

$$\mathcal{F}_\alpha = -\epsilon_0 \partial_\alpha H + i\alpha^{-1} H^2 \partial_\beta \Omega = \partial_\alpha \mathcal{E} + 2\Phi \partial_\alpha \Phi,$$

$$\mathcal{F}_\beta = -\epsilon_0 \partial_\beta H + i\alpha^{-1} H^2 \partial_\alpha \Omega = \partial_\beta \mathcal{E} + 2\Phi \partial_\beta \Phi.$$

and this allows calculating the conformal factor $f$ in quadratures in terms of field components or in terms of the Ernst potentials.

3. Space of solutions that are regular near degenerate orbits of $\mathcal{G}_2$

The orbits of the isometry group $\mathcal{G}_2$ are degenerate if the function $\alpha$ (which characterizes the area element on the orbits) vanishes on these orbits. In the orbit space of the group $\mathcal{G}_2$, the points with $\alpha = 0$ constitute lines. There exists a large class of solutions with regular behavior of space–time geometry and of the electromagnetic fields on the line $\alpha = 0$ or on its finite or semi-infinite intervals. The necessary and sufficient condition for this regularity is the possibility to expand field components (1) near $\alpha = 0$:

$$H = H_0 + H_2 \alpha^2 + H_4 \alpha^4 + \cdots, \quad A = A_0 + A_2 \alpha^2 + A_4 \alpha^4 + \cdots,$$

$$\Omega = \Omega_0 + \Omega_2 \alpha^2 + \Omega_4 \alpha^4 + \cdots, \quad \Phi = \Phi_0 + \Phi_2 \alpha^2 + \Phi_4 \alpha^4 + \cdots,$$

$$f = f_0 + f_2 \alpha^2 + f_4 \alpha^4 + \cdots.$$  

The corresponding expansions of the Ernst potentials are

$$\mathcal{E} = \mathcal{E}_0 + \mathcal{E}_2 \alpha^2 + \mathcal{E}_4 \alpha^4 + \cdots, \quad \Phi = \Phi_0 + \Phi_2 \alpha^2 + \Phi_4 \alpha^4 + \cdots,$$

where the coefficients of these expansions are functions of the generalized Weyl coordinate $\beta$ “harmonically” conjugate to the coordinate $\alpha$.

The substitution of expansions (15) and (16) in (8), (9) and (14) shows that $\Omega_0$ is constant. The values $\mathcal{E}_0(\beta)$ and $\Phi_0(\beta)$ of the Ernst potentials on the boundary $\alpha = 0$ remain arbitrary, while all other coefficients of these expansions are determined uniquely by these boundary values. Thus, the space of solutions regular

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9The constant parameter $\Omega_0$ can be made equal to zero by using appropriate linear transformations with constant coefficients of the Killing vectors $\partial/\partial x^\alpha$. However, if one of the Killing vectors corresponds to axial symmetry with a $2\pi$-periodic angle coordinate $\varphi$, this transformation of Killing vectors is not an admissible global coordinate transformation and it should be regarded as some “cut-and-paste” procedure changing the space–time manifold such that the role of a $2\pi$-periodic angle coordinate is played not by the old coordinate $\varphi$ but by some new angular coordinate $\varphi'$. In addition, several regular intervals separated by the sources may exist on the symmetry axis, and expansions of type (15), (16) may be applicable near each of these intervals. In that case, the constants $\Omega_0$ may be different on different intervals and we cannot make all of them equal to zero simultaneously by any global Killing vector transformation.
on the boundary \( \alpha = 0 \) is infinite dimensional and the functions \( \mathcal{E}_0(\beta) \) and \( \Phi_0(\beta) \) can serve as “coordinates” in this space.\(^{10}\)

In what follows, the action of various known solution-generating procedures is considered in the space of solutions whose Ernst potentials are given by expansions (16) near \( \alpha = 0 \). The “coordinates” of the generated solutions \( \{\mathcal{E}(\beta),\Phi(\beta)\} \) are expressed in terms of arbitrarily chosen “coordinates” of the seed (background) solution \( \{\mathcal{E}_0(\beta),\Phi_0(\beta)\} \) and a set of arbitrary constants.

4. Inverse scattering approach and soliton-generating techniques

As mentioned in Sec. 1, the soliton solutions for the Einstein equations were originally found for vacuum gravitational fields by Belinski and Zakharov [16] for the hyperbolic case \((\epsilon = 1)\), and later in [17] for the elliptic case \((\epsilon = -1); \) we also refer the reader to book [47].

**Vacuum solitons of Belinski and Zakharov.** The construction of solitons suggested in [16], [17] was based on a representation of field equations (7) as the compatibility conditions of the following linear system with a complex (“spectral”) parameter \( \lambda \):  
\[
(\lambda \delta^{\nu}_{\mu} - \epsilon \alpha \epsilon^{\nu}_{\mu}) \partial_{\nu} \Psi = V_{\mu} \Psi, \quad V_{\mu} = -\epsilon \alpha \epsilon^{\nu}_{\mu} \partial_{\nu} g \cdot g^{-1}. \tag{17}
\]
Here, \( \beta_{\mu} = \partial_{\mu} \beta \) and the functions \( \alpha(x^\mu) \) and \( \beta(x^\mu) \) are defined in (5). To construct soliton solutions of system (17), the following ansatz for \( \Psi(x^\mu,\lambda) \) was used in [16], [17]:  
\[
\Psi = \mathcal{X} \cdot \Psi^0, \quad \mathcal{X} = \text{I} + \sum_{k=1}^{N} \frac{R_k}{\lambda - \mu_k}, \quad \mathcal{X}^{-1} = \text{I} + \sum_{\ell=1}^{N} \frac{S_\ell}{\lambda - \nu_\ell}, \tag{18}
\]
where \( \Psi^0(x^\mu,\lambda) \) is a fundamental solution of system (17), corresponding to some known vacuum solution chosen as the background for solitons. (Here and hereafter, the symbol \( \Psi^0 \) on a letter denotes all functions characterizing this choice of the background solution.)

The functions \( \mu_k \) and \( \nu_\ell \) as well as the matrices \( R_k \) and \( S_\ell \) in (18) are independent of \( \lambda \); they are unknown functions of \( x^\mu \). Substituting (18) in Eqs. (17) and the solving the relations derived in this way, one can calculate all metric components and the Ernst potential for the \( N \)-soliton solution on a chosen vacuum background.

**Determinant form of the vacuum \( N \)-soliton solutions.** All metric components and the Ernst potential for the Belinski–Zakharov vacuum \( N \)-soliton solutions can be expressed in a more compact (determinant) form that was found by the author in [18]. Here, we present only the general determinant expression for the Ernst potential of the \( N \)-soliton solution used below:

\[
\text{Re} \mathcal{E} = \left( \prod_{k=1}^{N} \lambda_k \right) \left( \frac{\Delta_{\text{Re}}}{} \right) \text{Re} \mathcal{E}^0, \quad \text{Im} \mathcal{E} = \text{Im} \mathcal{E}^0 - \frac{\Delta_{\text{Im}}}{\Delta}. \tag{19}
\]
In these expressions, \( N \) is the number of solitons, which is equal to the number of poles of the “dressing” matrix function \( \mathcal{X} \) and which should be even for the Belinski–Zakharov solitons,\(^{11}\) and \( \Delta, \Delta_{\text{Re}}, \) and \( \Delta_{\text{Im}} \)
denote the determinants of \( N \times N \) matrices \((i,j,\ldots=1,\ldots,N)\),
\[
\Delta_{ij} = \frac{\lambda_i \lambda_j}{\lambda_i \lambda_j - \epsilon} (\mathbf{m}_i \cdot \mathbf{p}_j), \quad \Delta_{[Re]}_{ij} = \Delta_{ij} - (\text{Re} \mathcal{E})^{-1}(\mathbf{p}_i \cdot \mathbf{e}_1)(\mathbf{p}_j \cdot \mathbf{e}_1), \quad \Delta_{[Im]}_{ij} = \Delta_{ij} + \mu_j(\mathbf{p}_i \cdot \mathbf{e}_1)(\mathbf{m}_j \cdot \mathbf{e}_2),
\]
where \((\mathbf{m}_i \cdot \mathbf{p}_j)\) is the scalar products of vectors \(\mathbf{m}_i\) and \(\mathbf{p}_j\), while the vectors \(\mathbf{e}_1\) and \(\mathbf{e}_2\) have the components \(\mathbf{e}_1 = \{1,0\}\) and \(\mathbf{e}_2 = \{0,1\}\). The functions \(\lambda_k = \mu_k/\alpha\) and \(\mu_k(\alpha, \beta)\) are solutions of the algebraic equations
\[
\mu_k + 2\beta + \frac{\epsilon \alpha^2}{\mu_k} = 2w_k,
\]
and the components of the vectors \(\mathbf{m}_k\) and \(\mathbf{p}_k\) are
\[
\mathbf{p}_k = \mathring{\mathbf{g}} \cdot \mathbf{m}_k^T, \quad \mathbf{m}_k = \{1, c_k\} \cdot \mathring{\mathbf{M}}_k.
\]
In these expressions, \(w_k\) and \(c_k\) are constants that can be chosen arbitrarily if the sets \(w_k\) and \(c_k\), as well as the corresponding functions \(\mu_k\) consist of pairs of real and/or mutually complex conjugate functions.

The \(2 \times 2\) matrix \(\mathring{\mathbf{g}}\) consists of components \(\mathring{g}_{ab}\) of the chosen background metric, and each matrix \(\mathring{\mathbf{M}}_k\) is a fundamental solution of the linear system
\[
(\mu_k \delta^\nu_\mu - \epsilon \alpha \varepsilon^\nu_\mu) \partial_\nu \mathring{\mathbf{M}}_k + \mathring{\mathbf{M}}_k \mathring{\mathbf{V}}_\mu = 0, \quad \mathring{\mathbf{V}}_\mu = -\epsilon \alpha \varepsilon^\nu_\mu \partial_\nu \mathring{\mathbf{g}} \cdot \mathring{\mathbf{g}}^{-1}.
\]

The most difficult part in these calculations is finding the fundamental solution \(\mathring{\mathbf{M}}_k\) of Eqs. (22). This system cannot be solved explicitly for any chosen background solution \(\mathring{\mathbf{g}}\), and this does not allow expressing the \(N\)-soliton solution explicitly in terms of the components of the background metric and a set of arbitrary constant parameters. Even in the explicitly solvable cases, the possible complexity of the derived expression can make the analysis of the asymptotic behavior of the solutions useful.

**Asymptotics of the vacuum \(N\)-soliton solution near \(\alpha = 0\).** Rather than specifying the initial (background) solution, we assume only its asymptotic form (15), (16) near \(\alpha = 0\). In addition, we can restrict our consideration by the minimal case \(N = 2\) because, as shown in [17], the solution-generating procedure can be constructed iteratively, i.e., the \(N\)-soliton solution can be constructed by generating two solitons on the background with \(N - 2\) solitons.

For any (real or complex) \(w_k\), the functions \(\mu_k(\alpha, \beta)\) can have one of two possible asymptotic forms as \(\alpha \to 0\), which we distinguish by the superscripts “+” or “−”:
\[
\mu_k^+ = \frac{\epsilon \alpha^2}{2(w_k^+ - \beta)} \left[ 1 + \frac{\epsilon \alpha^2}{4(w_k^+ - \beta)^2} + \cdots \right],
\]
\[
\mu_k^- = 2(w_k^- - \beta) \left[ 1 - \frac{\epsilon \alpha^2}{4(w_k^- - \beta)^2} + \cdots \right].
\]

The calculations using general expressions (19)–(22) should therefore be divided into subcases, and even for \(N = 2\) we have to consider three cases separately, depending on whether \(\mu_1\) and \(\mu_2\) have the same type of asymptotics, \(\mu_1^+\), \(\mu_2^-\) or \(\mu_1^+\), \(\mu_2^+\), or a “mixed” case \(\mu_1^-, \mu_2^-\) occurs. In the zeroth-order terms with respect to \(\alpha\), we then obtain the corresponding boundary values of the Ernst potential \(\mathcal{E}_{++}(\beta), \mathcal{E}_{--}(\beta),\) and \(\mathcal{E}_{+-}(\beta)\) for the vacuum two-soliton solution on the vacuum background described by an arbitrary boundary value

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of its Ernst potential $\hat{\mathcal{E}}(\beta)$:

$$
\mathcal{E}_{++}(\beta) = \frac{i[c_2^+(w_1^+ - \beta) - c_2^+(w_2^+ - \beta)]\hat{\mathcal{E}}(\beta) + w_1^+ - w_2^+}{c_1^+ c_2^+(w_1^+ - w_2^+)\hat{\mathcal{E}}(\beta) - ic_1^+(w_1^+ - \beta) + ic_2^+(w_2^+ - \beta)},
$$

$$
\mathcal{E}_{--}(\beta) = \frac{-i[c_1^- (w_1^- - \beta) - c_2^- (w_2^- - \beta)]\hat{\mathcal{E}}(\beta) + c_1^- c_2^- (w_1^- - w_2^-)}{(w_1^- - w_2^-)\hat{\mathcal{E}}(\beta) - ic_1^- (w_2^- - \beta) + ic_2^- (w_1^- - \beta)},
$$

$$
\mathcal{E}_{+-}(\beta) = \frac{[w_+ - \beta + c_+ c_- (w_- - \beta)]\hat{\mathcal{E}}(\beta) + ic_- (w_+ - w_-)}{w_- - \beta + c_+ c_- (w_+ - \beta) - ic_+ (w_+ - w_-)\hat{\mathcal{E}}(\beta)}.
$$

We must note here that for $N = 2$ in the mixed case $(+-)$, the constants $w_+$ and $w_-$, as well as the constants $c_+$ and $c_-$ must be real, while in the $(++)$ and $(- -)$ cases, the pairs of constants $(w_1^+, w_2^+)$ and $(c_1^+, c_2^+)$, as well as $(w_1^-, w_2^-)$ and $(c_1^-, c_2^-)$ can be chosen real or complex conjugate to each other. However, it is easy to see that the solutions with boundary values of the Ernst potential $\mathcal{E}_{++}(\beta)$ and $\mathcal{E}_{--}(\beta)$ are not different, and these can be transformed into each other by the parameter substitution

$$
w_1^- = w_1^+, \quad w_2^- = w_2^+, \quad c_1^- = \frac{1}{c_1^+}, \quad c_2^- = -\frac{1}{c_2^+}.
$$

The two-soliton solutions of the mixed type with the boundary value of the Ernst potential $\mathcal{E}_{+-}(\beta)$ turn out to be a part of the family $\mathcal{E}_{++}$ that corresponds to real parameters $w_1^+, w_2^+$ and $c_1^+, c_2^+$. To see this, it suffices to substitute

$$
w_1^+ = w_+, \quad w_2^- = w_-, \quad c_1^+ = c_+, \quad c_2^- = -\frac{1}{c_+}.
$$

Thus, the most general vacuum two-soliton solution generated on an arbitrary vacuum background with the metric of form (1) is determined by the Ernst potential with the boundary value at $\alpha = 0$ of the form

$$
\mathcal{E}(\beta) = \frac{i[c_2(w_1 - \beta) - c_1(w_2 - \beta)]\hat{\mathcal{E}}(\beta) + w_1 - w_2}{c_1 c_2(w_1 - w_2)\hat{\mathcal{E}}(\beta) - ic_1 (w_1 - \beta) + ic_2 (w_2 - \beta)},
$$

(23)

where $\hat{\mathcal{E}}(\beta)$ is the value of the Ernst potential of the background metric at $\alpha = 0$; $(w_1, w_2)$ and $(c_1, c_2)$ are the pairs of arbitrarily chosen constant parameters, which must be real or complex conjugate to each other.

We also note that the cases where the parameters in the pairs $(w_1, w_2)$ and $(c_1, c_2)$ are chosen real or complex conjugate to each other are essentially different. In particular, in the stationary axisymmetric case with real $w_k$ and $c_k$, the two-soliton solution describes the interaction of a rotating black hole (the Kerr–NUT source with a horizon) with the background field, and the choice of these parameters complex conjugate in pairs leads to the solution that describes the interaction of a naked singularity without a horizon (an “overextreme” Kerr–NUT source) with the same background.

Subsequent applications of Belinski–Zakharov vacuum soliton-generating transformations lead to solutions with any even number of solitons on a chosen background. The corresponding boundary values of these soliton solutions can easily be derived by successive transformations of type (23) for the boundary data for the Ernst potential on $\alpha = 0$.

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12Here and below, “NUT” means one of the parameters that characterizes the source in the Kerr–NUT solution (besides the mass $m$ and angular momentum $a$ parameters) and which was named after Newman, Tamburino, and Unti; see book [1] for the details.
Electrovacuum Einstein–Maxwell solitons. Later attempts to directly generalize the soliton-generating transformations from vacuum to electrovacuum fields were unsuccessful. Another method for generating electrovacuum solitons on an arbitrarily chosen electrovacuum background was suggested in our papers [19], [20] (see [39] for more details). This method is based on a complex form of the Einstein–Maxwell equations expressed in the form of duality equations. These equations were presented as integrability conditions of a linear system with a constant (“spectral”) parameter $E$.

The Einstein–Maxwell equations expressed in the form of duality equations. These equations were presented as integrability conditions of a linear system with a constant (“spectral”) parameter $w$ for a $3 \times 3$ matrix function $\Psi(x^\mu, w)$, which was supplied with the condition for the existence of a $3 \times 3$ matrix integral $K(w)$ of Hermitian structure,

$$2i[(w - \beta)\delta_\mu^\nu - \epsilon\alpha\varepsilon_\mu^\nu]\partial_\nu \Psi = U_\mu \Psi, \quad \Psi^\dagger W \Psi = K(w), \quad (24)$$

where “$^\dagger$” denotes Hermitian conjugation. In the vacuum limit, this system does not coincide with the Belinski–Zakharov linear system [16], [17].

The complex $3 \times 3$ matrices $U_\mu$, $\mu = 1, 2$, are independent of the complex parameter $w$ and their components can be expressed in terms of the metric and electromagnetic potential components and their first derivatives. The components of the Hermitian $3 \times 3$ matrix $W$ can be expressed algebraically in terms of the metric components $g_{ab}$ and components of the complex electromagnetic potential $\Phi_a$. In contrast to [19], [20], and [39], the explicit expressions for the matrices $U_\mu$ and $W$ are given below in the case of the “maximally positive” metric signature ($- + +$) used here (for more notation, see (10) and (11)),

$$U_\mu = \begin{pmatrix} -\Phi^\mu a & \Phi^\mu a \\ -2\Phi^\mu b & 2\Phi^\mu b \end{pmatrix}, \quad W = 4i(w - \beta)\Omega + \begin{pmatrix} 4\Phi^a b + 4\Phi^a \Phi^b & -2\Phi^a \\ -2\Phi^b & 1 \end{pmatrix},$$

where $\mu = 1, 2$ and $a, b, \ldots = 3, 4$. The matrix $\Omega$ and the reduced form of the integral $K(w)$ are expressed as

$$\Omega = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad K(w) = \begin{pmatrix} 0 & 4i & 0 \\ -4i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (25)$$

The soliton solutions of the Einstein–Maxwell equations arise as the solutions of spectral problem (24) with an ansatz similar to that in (18),

$$\Psi = \chi \cdot \Psi, \quad \chi = I + \sum_{k=1}^N \frac{R_k}{w - w_k}, \quad \chi^{-1} = I + \sum_{\ell=1}^N \frac{S_\ell}{w - w_\ell}, \quad (26)$$

where $w_k$ are arbitrary complex constants, the $3 \times 3$ matrix functions $R_k(x^\mu)$ and $S_\ell(x^\mu)$ are the unknowns, which are independent of $w$, and $\Psi(x^\mu, w)$ is a fundamental solution of system (24), corresponding to an arbitrarily chosen ($G_2$-symmetric) electrovacuum background for solitons.

In accordance with [19], [20], the expressions for the Ernst potentials for the $N$-soliton solution of the electrovacuum Einstein–Maxwell equations can be presented in the form

$$\mathcal{E} = \mathcal{E} - 2i \sum_{k, \ell=1}^N \Delta^{-1}_{k\ell}(e_1 \cdot p_\ell)(m_k \cdot e_2),$$

$$\Phi = \Phi + 2i \sum_{k, \ell=1}^N \Delta^{-1}_{k\ell}(e_1 \cdot p_\ell)(m_k \cdot e_3), \quad (27)$$

where $\mathcal{E}$ and $\Phi$ are the potentials of the chosen background solution; $(e_1 \cdot p_\ell)$, $(m_k \cdot e_2)$, and $(m_k \cdot e_3)$ are respectively the first component of each of the vectors $p_\ell$ and the second and third components of each
of the vectors \( \mathbf{m}_k \) (the indices \( k, \ell = 1, 2, \ldots, N \) enumerate not the components but the three-dimensional vectors \( \mathbf{p}_\ell \) and \( \mathbf{m}_k \)). The components of the matrix \( \| \Delta_{k\ell} \| \) are

\[
\Delta_{k\ell} = \frac{(\mathbf{m}_\ell \cdot \mathbf{p}_k)}{w_{\ell} - \bar{w}_k}, \quad w_k \neq \bar{w}_k, \quad k, \ell = 1, 2, \ldots, N,
\]

where \( \{ w_k \} \) is a set of \( N \) arbitrary complex constants, which should not be chosen real; \( (\mathbf{m}_\ell \cdot \mathbf{p}_k) \) is a scalar product of vectors \( \mathbf{m}_\ell \) and \( \mathbf{p}_k \). The components of the vectors \( \mathbf{m}_\ell \) and \( \mathbf{p}_k \) can be expressed in terms of the fundamental solution \( \hat{\Psi}(x^\mu, w) \) of system (24) and the value of its matrix integral \( \hat{\mathbf{K}}(w) \) corresponding to the chosen background solution as

\[
\mathbf{m}_\ell = \mathbf{k}_\ell \cdot \hat{\Psi}^{-1}(x^\mu, w = w_\ell), \quad \mathbf{k}_\ell = \{ 1, \ell, d_\ell \}, \quad \mathbf{p}_k = \hat{\Psi}(x^\mu, w = \bar{w}_k) \cdot \mathbf{l}_k, \quad \mathbf{l}_k = \left( \begin{array}{c} -\bar{c}_k \\ 1 \\ 4i\bar{d}_k \end{array} \right),
\]

where \( \{ c_\ell \} \) and \( \{ d_\ell \} \) are two sets of \( N \) arbitrary complex constants each, and the last expression for the components of the vectors \( \mathbf{l}_k \) corresponds to the choice of \( \hat{\mathbf{K}}(w) \) in form (25). The \( N \)-soliton electrovacuum solutions described above arise as the result of soliton-generating transformations of an arbitrarily chosen electrovacuum solution with \( G_2 \)-symmetry, which plays the role of background for solitons. These solutions depend on the \( 3N \) arbitrary complex constants \( c_k, d_k, \) and \( w_k \).

**Asymptotics of the Einstein–Maxwell \( N \)-soliton solutions near \( \alpha = 0 \).** System (24), as well as system (22), does not admit an explicit solution in general form without a particular choice of the background solution. But for any solution whose gravitational and electromagnetic fields exhibit regular behavior as \( \alpha \to 0 \), there exists a fundamental solution of system (24), whose boundary value at \( \alpha = 0 \) can be presented in the form

\[
\hat{\Psi}(\beta, w) = \begin{pmatrix}
\frac{1}{w - \beta} & \frac{\hat{\mathcal{E}}(\beta)}{2i(w - \beta)} & \frac{\hat{\Phi}(\beta)}{2i(w - \beta)} \\
0 & \frac{1}{\hat{\mathcal{E}}(\beta)} & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

where \( \hat{\mathcal{E}}(\beta) \) and \( \hat{\Phi}(\beta) \) are the boundary values at \( \alpha = 0 \) of the Ernst potentials of the background solution arbitrarily chosen within the class of solutions that are regular near \( \alpha = 0 \). In the case \( N = 1 \), using the electrovacuum soliton-generating transformation, in the leading term as \( \alpha \to 0 \), we obtain a relation between the boundary values of the Ernst potentials for the background and the generated one-soliton solution on this background,

\[
\begin{align*}
\mathcal{E} &= \frac{4c(w_1 - \bar{w}_1)(\bar{c} - 2\bar{d}\hat{\Phi}) + 2i\hat{\mathcal{E}}[\bar{c}(\bar{w}_1 - \beta) - c(w_1 - \beta) - 4id\bar{d}(\bar{w}_1 - \beta)]}{(w_1 - \bar{w}_1)(\hat{\mathcal{E}} - 4id\hat{\Phi}) + 2i[\bar{c}(w_1 - \beta) - c(\bar{w}_1 - \beta) - 4id(\bar{w}_1 - \beta)]}, \\
\Phi &= \frac{2id(w_1 - \bar{w}_1)(\hat{\mathcal{E}} - 2i\hat{\mathcal{E}}) - 2i\hat{\mathcal{E}}[(c - \bar{c})(w_1 - \beta) + 4id(w_1 - \beta)]}{(w_1 - \bar{w}_1)(\hat{\mathcal{E}} - 4id\hat{\Phi}) + 2i[\bar{c}(w_1 - \beta) - c(\bar{w}_1 - \beta) - 4id(\bar{w}_1 - \beta)]},
\end{align*}
\]

where, in accordance with (28), \( c = c_1 \) and \( d = d_1 \). It is interesting to note that transformation (29) of boundary values of the Ernst potentials has an important property that if the pole becomes real and the other parameters satisfy the condition \( c - \bar{c} + 4id \neq 0 \), then the soliton disappears, i.e.,

\[
\mathcal{E} \to \hat{\mathcal{E}} \quad \text{and} \quad \Phi \to \hat{\Phi} \quad \text{for} \quad w_1 \to \bar{w}_1.
\]
5. Bäcklund transformations

Other solution-generating methods for vacuum and electrovacuum fields with components (1) were based on the theory of Bäcklund transformations. The first of these were found by Harrison in [21], where both the hyperbolic and elliptic cases of the vacuum Ernst equation were considered. The procedure of constructing Bäcklund transformations suggested in [21] was not presented in its final form, and some examples were considered later in [23]. The Bäcklund transformations for Einstein–Maxwell equations were constructed by the same author a few years later, in [24].

Another form of Bäcklund transformations for vacuum fields was found by Neugebauer and was described in a series of papers [25]–[27], where the considerations were restricted to stationary axisymmetric fields. These transformations also lead from any chosen initial solution to new families of vacuum solutions.

Harrison’s Bäcklund transformations for vacuum fields. In [21], Harrison used the Wahlquist–Estabrook pseudopotential method [22]. Later, he described these transformations using the modified Wahlquist–Estabrook method [24], which was based on the field equations in the form of a closed CC ideal of 1-forms. This form of field equations was described above in Eqs. (12) and (13), where in the vacuum case, we should set Φ = 0 and therefore η7 = η8 = η9 = η10 = 0. In the vacuum case, the 1-forms generating the CC ideal are

\[ \eta_1 = \frac{\mathcal{E}'}{\text{Re} \mathcal{E}} d\xi, \quad \eta_2 = \frac{\mathcal{E}'}{\text{Re} \mathcal{E}} d\xi, \quad \eta_3 = \frac{\mathcal{E}'}{\text{Re} \mathcal{E}} d\eta, \quad \eta_4 = \frac{\mathcal{E}'}{\text{Re} \mathcal{E}} d\eta, \]

and the equation for the pseudopotential q found in [21] takes the form

\[ 4dq = (1 + q\zeta \left( q\eta_1 - \frac{1}{\zeta}\eta_2 \right) + (q + \zeta) \left( -\eta_2 + \frac{q}{\zeta}\eta_4 \right) + (1 - q^2) \left( \zeta\eta_5 + \frac{1}{\zeta}\eta_6 \right), \]

where \( \zeta = \sqrt{(w-\eta)/(w-\xi)} \) with w an arbitrary real constant. For Harrison’s Bäcklund transformations [21], the transformed 1-forms are

\[ \tilde{\eta}_1 = -\frac{q(1 + q\zeta)}{q + \zeta} \eta_1 + (1 + q\zeta)\eta_3, \quad \tilde{\eta}_2 = -\frac{q + \zeta}{q(1 + q\zeta)} \eta_2 + \frac{q + \zeta}{q} \eta_5, \]

\[ \tilde{\eta}_3 = -\frac{(1 + q\zeta)}{q(q + \zeta)} \eta_3 + \frac{(1 + q\zeta)}{q\zeta} \eta_6, \quad \tilde{\eta}_4 = -\frac{q(q + \zeta)}{1 + q\zeta} \eta_4 + \frac{(q + \zeta)}{\zeta} \eta_6. \]

\(^{13}\)It seems useful to clarify here that the number of solitons means the number of simple poles in the dressing matrix \( X \) on the spectral plane \( \lambda \) in the case of Belinski–Zakharov solitons (18) and on the spectral plane \( w \) in the case of electrovacuum solitons (26). Because of the obvious difference between these two techniques and of the structures of the “spectral” planes \( \lambda \) and \( w \), the vacuum part of solutions with \( N \) electrovacuum solitons should be compared with solutions with \( 2N \) Belinski–Zakharov vacuum solitons. This comparison shows that in contrast to Belinski–Zakharov vacuum solitons, the number \( N \) of solitons (i.e., poles) in (26) can be not only even, but can also be odd. The vacuum restriction of electrovacuum \( N \)-soliton solution (26) coincides with the vacuum Belinski–Zakharov \( 2N \)-soliton solution with complex conjugate pairs of poles, while the electrovacuum generalization of the Belinski–Zakharov vacuum solitons with pairs of real poles does not arise in this way. However, the electrovacuum solutions of soliton type with real poles can arise (at least for special choices of the background solutions) as a result of analytic continuations of electrovacuum soliton solutions with complex poles in the space of their constant parameters.
Asymptotics of Harrison’s vacuum Bäcklund transformations near $\alpha = 0$. For vacuum fields that are regular near degenerate orbits with $\alpha = 0$ and which admit expansions (15), (16), the above expressions can be used to obtain the expansions for $\zeta$ and for the pseudopotential $q$,

$$
\zeta = 1 + \frac{j\alpha}{w - \beta} + \frac{\epsilon \alpha^2}{2(w - \beta)^2} + \cdots, \quad q = -1 + q_1 \alpha + q_2 \alpha^2 + \cdots,
$$

with the coefficients $q_1 = j(\mathcal{E}_0 - \tilde{\mathcal{E}}_0 + 2ik_0)/(\mathcal{E}_0 + \tilde{\mathcal{E}}_0)(w_0 - \beta)$ and $q_2 = -q_1^2/2$. The function $\mathcal{E}_0(\beta)$ is the value of the Ernst potential on the boundary $\alpha = 0$ for the solution chosen for the application of the Bäcklund transformation. The expressions given just above lead to expressions for the boundary value of the Ernst potential of the transformed solution,

$$
\tilde{\mathcal{E}}_0 + \tilde{\mathcal{E}}_0 = k_1(\mathcal{E}_0 + \tilde{\mathcal{E}}_0)(w_0 - \beta)/(\tilde{\mathcal{E}}_0 - i\beta_0), \quad \tilde{\mathcal{E}}_0 = k_1 \frac{w_0 - \beta}{\tilde{\mathcal{E}}_0 + i\beta_0} + ik_2,
$$

where $k_0, k_1, k_2$, and $w_0$ are arbitrary real constants. To compare these transformations with the Belinski–Zakharov soliton-generating transformations on the boundary $\alpha = 0$ given by (23), we consider a pair of successive transformations (30). Below, the constants $w_0, k_0, k_1$, and $k_2$ correspond to the first transformation and $\tilde{w}_0, \tilde{k}_0, \tilde{k}_1$, and $\tilde{k}_2$, to the second one:

$$
\tilde{\mathcal{E}}_0 = \frac{\tilde{k}_1(w_0 - \beta)}{(\tilde{\mathcal{E}}_0 + i\tilde{k}_0)} + i\tilde{k}_2 = \frac{k_1(\tilde{w}_0 - \beta)}{k_1(w_0 - \beta)/(\mathcal{E}_0 + i\beta_0) + ik_2 + i\beta_0} + i\tilde{k}_2.
$$

This transformation of the boundary values for the Ernst potential coincides with that for the Belinski–Zakharov two-soliton transformations described by (23) if we choose the following relations between the constants:

$$
w_0 = w_1, \quad \tilde{w}_0 = w_2, \quad k_1 = k_1 = -\frac{(c_1 - c_2)(\tilde{k}_0 + k_2)}{c_1c_2(w_1 - w_2)}, \quad \tilde{k}_2 = -k_0 = \frac{1}{c_1}.
$$

Thus, applying a pair of successive Harrison’s Bäcklund transformations with different values of the integration constants is equivalent to generating a pair of Belinski–Zakharov solitons. We recall here that this equivalence of Bäcklund transformations and the soliton-generating technique was affirmed earlier by Cosgrove [41] using more sophisticated and less explicit considerations.

Neugebauer’s Bäcklund transformations for vacuum fields. Another form of Bäcklund transformations for vacuum fields was presented by Neugebauer, who restricted all his consideration to the stationary axisymmetric fields. These transformations also lead from any chosen seed solution to some new family of solutions. In our notation, we should put $\epsilon = -1, \alpha = \rho$, and $\beta = z$ for stationary axisymmetric fields. The expression for the Ernst potential $\mathcal{E}(\rho, z)$ for solutions derived after a series of successive Bäcklund transformations of the seed solution $\tilde{\mathcal{E}}(\rho, z)$ was presented in [25]–[27] in the form (where we made slight changes in notation)

$$
\begin{vmatrix}
\mathcal{E} & 1 & 1 & \cdots & 1 \\
\mathcal{E} + \tilde{\mathcal{E}} & a_1 \gamma_1 & a_2 \gamma_2 & \cdots & a_N \gamma_N \\
\mathcal{E} - \tilde{\mathcal{E}} & \gamma_1^2 & \gamma_2^2 & \cdots & \gamma_N^2 \\
\mathcal{E} + \tilde{\mathcal{E}} & a_1 \gamma_1^3 & a_2 \gamma_2^3 & \cdots & a_N \gamma_N^3 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\mathcal{E} - \tilde{\mathcal{E}} & \gamma_1^N & \lambda_2^N & \cdots & \gamma_N^N \\
\end{vmatrix}
= 0,
$$

(31)
where $N$ must be even; the functions $\gamma_1, \gamma_2, \ldots, \gamma_N$, and $a_1, a_2, \ldots, a_N$ are the values at some chosen points $w_1, w_2, \ldots, w_N$ of two auxiliary functions of $\rho$ and $z$ and of a constant complex parameter $w$, such that

$$\gamma \equiv \gamma(w, \zeta, \bar{\zeta}) = \left(\frac{w - i\bar{\zeta}}{w + i\zeta}\right)^{1/2}$$

and $a \equiv a(w, \zeta, \bar{\zeta})$, where

$$\begin{cases} 
\zeta = \rho + iz, \\
\bar{\zeta} = \rho - iz. 
\end{cases}$$

The constants $w_1, w_2, \ldots, w_N$ can be chosen arbitrarily as long as some of them are real and the others comprise pairs of complex conjugate parameters. For real $w_k$, the quantities $\gamma_k \equiv \gamma(w_k, \zeta, \bar{\zeta})$ must satisfy the condition $\gamma_k\bar{\gamma}_k = 1$, but for complex conjugate $w_l$ and $w_k$ the corresponding values of $\gamma$ must be chosen such that $\gamma_l\bar{\gamma}_k = 1$. All $a_k = a(w_k, \zeta, \bar{\zeta})$ are solutions of the same system of Riccati equations, but correspond to different integration constants,

$$\begin{align*}
\frac{\partial a_k}{\partial \zeta} &= (\bar{\mathcal{E}} + \mathcal{E})^{-1} \left[ (a_k - \gamma_k) \frac{\partial \mathcal{E}}{\partial \zeta} + a_k (a_k \gamma_k - 1) \frac{\partial \bar{\mathcal{E}}}{\partial \zeta} \right], \\
\frac{\partial a_k}{\partial \bar{\zeta}} &= (\bar{\mathcal{E}} + \mathcal{E})^{-1} \left[ (a_k - \frac{1}{\gamma_k}) \frac{\partial \mathcal{E}}{\partial \bar{\zeta}} + a_k \left( \frac{a_k}{\gamma_k} - 1 \right) \frac{\partial \bar{\mathcal{E}}}{\partial \bar{\zeta}} \right],
\end{align*}$$

where the integration constants must be chosen such that for a real $w_k$, the solution $a_k$ satisfies the condition $a_k\bar{a}_k = 1$, and for complex conjugate $w_l$ and $w_k$, the corresponding solutions satisfy the condition $a_l\bar{a}_k = 1$.

Equations (32), as well as Eqs. (22), do not necessarily admit an explicit solution for any chosen seed solution $\mathcal{E}(\zeta, \bar{\zeta})$. Therefore, in order to not subject the choice of the seed solution to further restrictions, we have to closely follow the soliton-generating strategy discussed above, using an asymptotic representation of these Bäcklund transformations and restricting them to the simplest case $N = 2$. For this, we represent solution (32) near $\rho = 0$ as a series $a = a_0(z) + \rho^2 a_1(z) + \cdots$ and, using expansions (16) restricted to the vacuum case, obtain an equation for the leading term $a_0$,

$$\frac{\partial}{\partial z} a_0(z) = \frac{a_0(z) - 1}{\bar{\mathcal{E}}(z) + \mathcal{E}(z)} \left[ \frac{\partial \mathcal{E}}{\partial z} + a_0(z) \frac{\partial \bar{\mathcal{E}}}{\partial z} \right], \quad \mathcal{E}_0(z) = a_0^{-1}(z),$$

where $\mathcal{E}(z)$ is the value of the Ernst potential of the seed solution on the boundary $\rho = 0$. This equation admits an explicit solution

$$a_0(z) = \frac{i\ell + \mathcal{E}}{i\ell - \bar{\mathcal{E}}},$$

where $\ell$ is an arbitrary real integration constant. Successively setting $\ell$ equal to two real values $\ell_1$ and $\ell_2$ and taking the expansion $\lambda_k = 1 - i\rho/(z - w_k) + \cdots$ with real $w_k$ as $\rho \to 0$, we can use (31) to obtain a relation between the boundary values of the Ernst potentials corresponding to the Bäcklund transformations with $N = 2$:

$$\mathcal{E}_{\text{BT}}(z) = \frac{i[\ell_2(z - w_2) - \ell_1(z - w_1)]\mathcal{E}(z) + \ell_1\ell_2(w_1 - w_2)}{(w_1 - w_2)\mathcal{E}(z) + i[\ell_2(z - w_1) - \ell_1(z - w_2)]}. \quad (33)$$

It is easy to see that this transformation of the boundary values of vacuum Ernst potentials coincides with two-soliton transformation (23), with real $w_1$ and $w_2$, if we set $c_1 = 1/\ell_1$ and $c_2 = 1/\ell_2$ there. It can also be shown that applying successive Bäcklund transformations that lead to (31) is equivalent to generating a vacuum $N$-soliton solution on the same seed (background) solution. Thus, within the class of stationary axisymmetric vacuum gravitational fields with regular behavior near some part of the axis $\rho = 0$, for any choice of the seed (background) solution, the Bäcklund transformations described in [25] lead to the same transformation of the space of solutions as does the soliton-generation procedure found previously in [16], [17], [21] and restricted here to stationary axisymmetric fields for comparison.
6. Exponentiating the Kinnersley–Chitre algebra of symmetries

As mentioned in Sec. 1, in [9]–[12] Kinnersley and Chitre constructed a representation of an infinite-dimensional algebra of infinitesimal symmetries of the Einstein–Maxwell equations for stationary axisymmetric fields using infinite hierarchies of fields and potentials associated with every solution. They also found that for vacuum fields these hierarchies admit generating functions determined by solutions of a linear system with a free complex parameter. Some special kinds of the Kinnersley–Chitre infinitesimal symmetry transformations were exponentiated in [28], [29].

Infinite hierarchies of potentials and their generating functions. The hierarchies of fields \( \{ \mathcal{H}_{ab} \} \) and \( \{ \Phi_a \} \) (\( a, b = 3, 4; m, n = 1, 2, \ldots \)), associated with any given solution satisfy the same field equations (11)

\[
\nabla \mathcal{H}_{ab} = i\rho^{-1} h_a \nabla \mathcal{H}_{cb}, \quad \nabla \Phi_a = i\rho^{-1} h_a \Psi \Phi_a,
\]
and the fields of these hierarchies are defined recursively by the relations

\[
\begin{align*}
\Phi_a &= i(M_a + 2\Phi_a \nabla \mathcal{H}_{ab}), \\
\mathcal{H}_{ab} &= \mathcal{H}_{ab},
\end{align*}
\]

where the hierarchies of potentials \( K, L, M, N \) are defined by the equations

\[
\begin{align*}
\nabla K &= \Phi^c \nabla \Phi^c, \\
\nabla L &= \Phi^c \nabla \mathcal{H}_{cb}, \\
\nabla M &= \Phi^{ca} \nabla \Phi^c, \\
\nabla N &= \mathcal{H}_{ca} \nabla \mathcal{H}_{cb},
\end{align*}
\]

where * denotes complex conjugation, \( \nabla \) is the gradient operator such that \( \nabla = \{ \partial_\rho, \partial_\ell \} \) in Weyl coordinates, and \( \partial_\rho = \{ \partial_\rho, \partial_\ell \} \) is the dual operator. We also have \( \mathcal{H}_{ab} = -\mathcal{H}_{ba} \), with the \( 2 \times 2 \) matrix potential \( \mathcal{H}_{ab} \) and the complex electromagnetic potential \( \Phi_a \) defined in (11). The action of the generators of infinitesimal symmetries on these fields and potentials was described in [11].

Generating functions for hierarchies of fields and potentials in the vacuum. The next very interesting step was made when two \( 2 \times 2 \) matrix generating functions

\[
\begin{align*}
F_{ab}(t) &= \sum_{m=0}^{\infty} t^m \mathcal{H}_{ab}, \\
F_{ab}(0) &= i\epsilon_a b,
\end{align*}
\]

\[
\begin{align*}
G_{ab}(s, t) &= \sum_{m, n=0}^{\infty} s^m t^n \mathcal{H}_{ab}, \\
G_{ab}(0, t) &= -iF_{ab}(t)
\end{align*}
\]

were introduced in [11], [12] for the calculation of the hierarchies of fields \( \mathcal{H}_{ab} \) and potentials \( \mathcal{H}_{ab} \) for the vacuum. Here, \( t \) and \( s \) are auxiliary complex parameters and the coordinate dependence of the generating functions and coefficients is omitted. A beautiful discovery in [11], [12] was that \( F_{ab}(t) \) must satisfy a matrix linear equation and the function \( G_{ab} \) allows a very simple expression in terms of \( F_{ab} \),

\[
\begin{align*}
\nabla F_{ab} &= it S^{-2}(1 - 2t^2) \nabla \mathcal{H}_{ab} - 2it \mathcal{H}_{ab} \nabla \mathcal{H}_{ab}, \\
F_{ab}(0) &= i\epsilon_a b,
\end{align*}
\]

\[
\begin{align*}
G_{ab}(s, t) &= (s - t)^{-1} [s \epsilon_a b + tS(s) F_{ca}(s) F_{cb}(t)], \\
G_{ab}(0, t) &= -iF_{ab}(t),
\end{align*}
\]

where \( S^2(t) = (1 - 2t^2)^2 + (2\rho)^2 \).

\[\text{[14]}\] We note that our notation introduced in the foregoing differs in some points from Kinnersley and Chitre’s notation. In particular, we use the metric signature \((-+, +, +)\) instead of \((+, - , - , -)\) used in their papers. Our enumeration of coordinates and notation for indices in (1)–(4) are also different from those in [10]–[12]. As a result, for the metrics on orbits, we use \( h_{ab} \) and hence \( f_{AB} \to -h_{ab} \). In this section, however, in contrast to the rest of the paper, we let * denote complex conjugation, \( \nabla \) denote the gradient operator (instead of \( \partial_\rho \)) and \( \nabla \) denote the dual operator (instead of our usual \( \epsilon_{\mu} \partial_\nu \)). In addition, for stationary axisymmetric fields (to which Kinnersley and Chitre restricted themselves), we should set \( \epsilon = \epsilon_0 = -1 \) and \( \epsilon_1 = \epsilon_2 = 1 \) in notation.
**HKX vacuum-to-vacuum transformations of rank** \( p \geq 0 \). Using the generating functions \( F_{ab} \) and \( G_{ab} \) for the hierarchies of fields and potentials, Hoenselaers, Kinnersley, and Xanthopoulo[28], [29] were able to exponentiate some special kinds of the Kinnersley–Chitre “vacuum-to-vacuum” infinitesimal symmetry transformations and obtain a series of finite symmetry transformations of different “ranks” \( p \geq 0 \). Namely, for the simplest case of transformations of rank \( p = 0 \), they obtained the Ernst potential in the form

\[
E = \mathcal{E} + \frac{i\gamma G(0,u)}{1 - \gamma G(u,u)} \left[ \partial_t G(u,t) \right]_{t=0},
\]

(36)

where \( \gamma \) and \( u \) are arbitrary real constants and \( G(s,t) \) is the upper left element of the matrix \( G_{ab}(s,t) \) defined in (35).

For transformations of rank \( p \geq 1 \), the expressions are more involved,

\[
E = \mathcal{E} + i\gamma^{(p)} \sum_{k,l=0}^{p} G_{0,p-k}(0,u) M_{-k,l}^{-1}(u) \left[ \partial_t G_{l,0}(u,t) \right]_{t=0},
\]

(37)

where the elements of the matrices \( M_{ik}(u) \) and \( G_{ij}(s,t) \) are defined as

\[
M_{ik}(u) = \delta_{ik} - \gamma^{(p)} G_{i,p-k}(u,u), \quad G_{ij}(s,t) = \frac{s}{i!} \frac{t}{j!} \left( \frac{\partial}{\partial s} \right)^i \left( \frac{\partial}{\partial t} \right)^j G(s,t),
\]

(38)

and \( \gamma^{(p)} \) and \( u \) are arbitrary real constants.

In the most general case, the authors of [29] suggested to consider a combined transformations whose infinitesimal versions can include finite sums of transformations of different ranks with different values of the parameter \( u \). However, below we consider only transformations (36) and (37), although our analysis can also be applied to more general cases.

**HKX transformations of the axis values of the Ernst potentials.** In this subsection, in line with our preceding discussion of solution-generating methods, we consider the transformations of the axis data for the Ernst potentials corresponding to HKX transformations.

**Generating functions for hierarchies of potentials on the symmetry axis.** Using the asymptotic behavior of metric components and Ernst potentials (15), (16) near the regular parts of the symmetry axis, where we should set \( \alpha = \rho \) and \( \beta = z \) for stationary axisymmetric vacuum fields, and assuming that all electromagnetic components vanish, we can solve Eqs. (35) asymptotically; on the axis \( \rho = 0 \), we then obtain

\[
F_{ab}(t) = \begin{pmatrix}
  t E(z) & i \\
 1 - 2tz & 1 - 2tz \\
-1 & 0
\end{pmatrix}, \quad G_{ab}(s,t) = -i F_{ab}(t),
\]

(39)

i.e., the matrix \( G_{ab} \) on the axis \( \rho = 0 \) is independent of \( s \). This is a significant simplification for calculations of the HKX transformations of the axis data.

**Rank-0 HKX transformations in terms of Ernst potentials on the axis.** From the general form of rank-0 transformations [29] shown in (36)\(^{15} \) and expressions (39) for the generating functions on the symmetry axis, we obtain the generated vacuum solution on the axis in the form

\[
\mathcal{E}(z) = \frac{(w_0 - z) E(z)}{w_0 - z + i(\gamma/2) E(z)}
\]

(40)

\(^{15}\)We recall that \( G \) in (36) is the upper left entry of the matrix \( G_{ab} \).
for any seed vacuum solution characterized on the axis by the Ernst potential \( \vec{E}(z) \). In (40), we put \( u = 1/2w_0 \). The parameter \( w_0 \) can be eliminated after a shift of the coordinate \( z \) along the axis: \( z \to z + w_0 \). Thus, this transformation depends only on one essential real parameter \( \gamma \). From the physical standpoint, this transformation leads to the generation of a solution corresponding to a superposition of the background field of the seed solution and the field of an extreme Kerr source restricted to a strange subcase with zero mass and equal angular momentum and NUT parameters, \( a = b = -\gamma/4 \). This is a very restricted subcase of vacuum two-soliton transformations (23).

**Combined HKX-transformations in terms of Ernst potentials on the axis.** Following the set of examples considered in [29] (applied there to the flat space–time only), we consider a combined transformation made of two rank-0 transformations applied to an arbitrary seed solution with the Ernst potential \( \vec{E}(z) \) on the axis. In this case, we have the transformed solution

\[
\mathcal{E}(z) = \frac{(w_1 - z)(w_2 - z)\vec{E}(z)}{(w_1 - z)(w_2 - z) + (i/2)[\gamma_1(w_2 - z) + \gamma_2(w_1 - z)]\vec{E}(z)},
\]

where \( w_1 = 1/2u_1 \), \( w_2 = 1/2u_2 \), \( \gamma_1 \), and \( \gamma_2 \) are four arbitrary real constants. One of the constants \( w_1 \) and \( w_2 \) or their combination can be eliminated by a shift of the coordinate \( z \) along the axis. This transformation generates the fields of two extreme objects of type (36) interacting with each other and with the seed metric characterized by the Ernst potential \( \vec{E}(z) \) on the axis. This is also a very restricted subcase of vacuum 4-soliton transformations.

**Rank-\( p \) (\( p \geq 1 \)) HKX transformations of the Ernst potential on the axis.** Rank-\( p \) HKX transformations of the Ernst potentials are defined in (37). Calculating these transformations on the symmetry axis, we have to take into account that the only nonvanishing term in the sum (37) corresponds to the indices \( k = l = 0 \). To explain this, we note that in each product in the sum in (37), the last factor is nonvanishing on the axis only for \( l = 0 \) because \( G(s, t) \) is independent of \( s \) on the axis due to (39). Also, it is easy to see that for the same reason, the matrix \( M_{ik} \) is upper triangular on the axis. Therefore, in this matrix and in its inverse, the first column has only one component that is nonvanishing on the axis. This is \( M_{i0} \). The same is true for the matrix \( M_{i0}^{-1} \), whose upper left element on the axis is \( M_{i0}^{-1,0} = 1/M_{00} \).

Therefore, we obtain the following form of (37) on the axis:

\[
\mathcal{E}(z) = \frac{(w_0 - z)^{p+1}\vec{E}(z)}{(w_0 - z)^{p+1} + i(\gamma^{(p)}/2)w_0z^{p-1}\vec{E}(z)}.
\]

The rank-\( p \) transformation depends on two essential real parameters \( \gamma^{(p)} \) and \( w_0 \). The parameter \( w_0 = 1/2u \) cannot be eliminated by a shift of the origin of \( z \) along the axis. Nonetheless, the absence of a sufficiently large number of free parameters in the transformation does not allow these solutions to be given a rich physical interpretation as in the soliton-generating cases. We also note that if the transformed solution is chosen asymptotically flat, the expansion of the transformed on-axis Ernst potential (42) as \( z \to \infty \) allows determining changes in the multipole moments of the transformed solution compared with the one before the transformation. In particular, it is easy to see that transformation (42) does not change the parameter of mass, but not the NUT parameter. It is clear that for generating physically more interesting solutions by this method, one should consider the combined HKX transformations of different ranks and with different values of the parameter \( w = 1/2u \). However, the construction of such transformations is much more complicated and relevant examples were not considered in [30], [34].

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7. Hauser and Ernst “effectivization” of infinitesimal Kinnersley–Chitre transformations

A powerful approach to exponentiation (“effectivization”) of the Kinnersley–Chitre algebra of infinitesimal symmetries of stationary axisymmetric Einstein–Maxwell equations was developed by Hauser and Ernst in [30]–[35].

**Basic assumptions.** In the Hauser–Ernst approach, all stationary axisymmetric solutions under consideration are assumed to be regular in some neighborhood of at least one point on the symmetry axis.

**Hauser–Ernst approach to the exponentiation of vacuum symmetries.** Within this class of vacuum fields that are regular on the axis for the Kinnersley–Chitre matrix potentials \( F(t) \) depending on Weyl coordinates \( \rho \) and \( z \) and on a free complex parameter \( t \) (with the dependence on the Weyl coordinates to be omitted hereafter), a \( 2 \times 2 \) matrix homogeneous Hilbert problem (HHP) was proposed in [30] and a \( 2 \times 2 \) matrix system of linear singular integral equation was derived for solving this problem.

### 3 \times 3 matrix \( F \)-potential for electrovacuum fields.

In [34], this approach in [30] was generalized to a similar class of electrovacuum fields. Instead of the Kinnersley–Chitre vacuum matrix potential, these authors constructed a \( 3 \times 3 \) matrix potential \( F(t) \) that satisfies a generalized linear system with a complex parameter \( t \). It was also argued that for the fields that are regular on some part of the axis, the gauge can be chosen such that \( F(t) \) is holomorphic for all \( t \) except two branching points \( 2t = 1/(z + i\rho) \) and \( 2t = 1/(z - i\rho) \) and \( t = \infty \), but \( F(t) \cdot \text{diag}\{1,1,1\} \) is holomorphic at \( t = \infty \). In addition, such \( F \) potentials can be chosen such that

\[
F(0) = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

**Hauser–Ernst homogeneous Hilbert problem.** The HHP was formulated in [34], [32] on a closed contour \( L \) in the \( t \) plane, which is symmetric with respect to the real axis and encompasses \( t = 0 \) such that the branch points \( 2t = 1/(z + i\rho) \) and \( 2t = 1/(z - i\rho) \) are outside this contour. Then the HHP takes the form

\[
X_-(t) = X_+(t)G(t), \quad t \in L, \quad X_+(0) = I,
\]

where \( X_+(t) \) is holomorphic on \( L_+ \) and \( X_-(t) \)—on \( L_- \) (here, \( L_+ \) denotes the region on and inside \( L \), and \( L_- \) denotes the region on and outside \( L \) including \( t = \infty \)). The matrix \( G(t) \) connects \( X_+(t) \) and \( X_-(t) \) on the contour \( L \).

**F-potentials from the solution of a homogeneous Hilbert problem.** Various \( F \) potentials are related to solutions of HHP (43) as

\[
\begin{align*}
F(t) &= X_+(t) \cdot \bar{F}(t), & t \in L_+, \\
F(t) \cdot u(t) &= X_-(t) \cdot \bar{F}(t), & t \in L_-,
\end{align*}
\]

where \( \bar{F}(t) \) is the \( F \) potential of the seed solution. The matrix \( u(t) \) is independent of space–time coordinates \( \rho \) and \( z \) and satisfies the conditions

\[
\begin{align*}
u^1(t) \cdot \mathcal{G} \cdot u(t) &= \mathcal{G}, \\
\det u(t) &= 1,
\end{align*}
\]

where

\[
\mathcal{G} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & t/2i \end{pmatrix}.
\]

In addition, \( u(t) \) must be holomorphic in \( L_- \), and the products with its components \( tu^2_1, tu^3_1, t^{-1}u^1_2, t^{-1}u^1_3 \) must be holomorphic at \( t = \infty \).
Linear singular integral equation solving the homogeneous Hilbert problem. In [32], [34], a $3 \times 3$ matrix linear singular integral equation solving Eq. (43) was derived as the holomorphicity condition for $X_-(t)$ in $\mathcal{L}_-$:

$$
\int_L \frac{F(s) \cdot u(s) \cdot F^{-1}(s)}{s(s-t)} ds = 0. \tag{45}
$$

Hauser and Ernst thus constructed a solution-generating method in which for every choice of the seed solution potential $\tilde{F}(t)$ and every chosen $u(t)$ with the analytic properties described above, the $F$-potential can be determined from integral equation (45). In this method, the selection of the $3 \times 3$ matrix $u(t)$ corresponds to some particular element of the Kinnersley–Chitre algebra of infinitesimal symmetry transformations.

Hauser–Ernst solution-generating method as transformations of axis data. In Hauser–Ernst gauges, the $F$ potential on the axis was chosen as

$$
F(t) = \begin{pmatrix}
0 & i & 0 \\
-1/2t & t \mathcal{E}(z) & t \mathcal{F}(z) \\
0 & 1/2t & 1
\end{pmatrix}. \tag{46}
$$

As shown in [32], Eq. (45) considered on the axis implies the condition

$$
C(t) \cdot u(t) \cdot D(t) = 0,
$$

$$
C(t) = \{-i, t \mathcal{E}(1/2t), t \mathcal{F}(1/2t)\},
$$

$$
D(t) = \begin{pmatrix}
-t^{\mathcal{E}}(1/2t) & -it^{\mathcal{F}}(1/2t) \\
-i & 0 \\
0 & 1
\end{pmatrix}, \tag{47}
$$

where $^{\mathcal{E}}(z)$ and $^{\mathcal{F}}(z)$ are the Ernst potentials of the chosen seed solution, and $\mathcal{E}(z)$ and $\Phi(z)$ are the Ernst potential for the generated solution, all considered on the axis $\rho = 0$ and with the substitution $z \to 1/2t$.

To explicitly obtain the transformation of the axis data for the Ernst potentials corresponding to the Hauser–Ernst solution-generating method, we note that condition (47) can be solved explicitly, whence

$$
\mathcal{E}(z) = -2iz \frac{G(z)}{D(z)},
$$

$$
\Phi(z) = -2iz \frac{F(z)}{D(z)},
$$

$$
\{D(z), G(z), F(z)\} = \{2iz, -^{\mathcal{E}}(z), -^{\mathcal{F}}(z)\} \cdot u^{-1}\left(\frac{1}{2z}\right), \tag{48}
$$

where we made the substitution $t \to 1/2z$ in $u(t)$. However, it is clear that we cannot choose the components of $u(t)$ arbitrarily in these expressions because these components must satisfy certain restrictions (see Eq. (44) and two lines below it). To solve these conditions, we use the Gauss decomposition for $u(t)$:

$$
u(t) = \begin{pmatrix} 1 & 0 & 0 \\ p_1 & 1 & 0 \\ p_2 & p_3 & 1 \end{pmatrix} \cdot \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & 1/d_1d_2 \end{pmatrix} \cdot \begin{pmatrix} 1 & q_1 & q_2 \\ 0 & 1 & q_3 \\ 0 & 0 & 1 \end{pmatrix}. \tag{49}
$$

All eight complex parameters are here analytic functions of $t$ in $\mathcal{L}_-$. This expression already takes into account that $\det u(t) = 1$. The condition imposed on the behavior of the components of $u(t)$ as $t \to \infty$, mentioned after (44), implies that $t p_1(t), t p_2(t), t^{-1} q_1(t),$ and $t^{-1} q_2(t)$ must be analytic as $t \to \infty$.

After the substitution of (49) in the first condition in (44), the corresponding equations can be solved explicitly, but four complex functions of $t$ (or of $1/2z$) remain arbitrary. Therefore, too many arbitrary
functions remain in expressions (48), because to transform the axis data of the seed Ernst potentials \( \tilde{E}(z) \) and \( \tilde{\Phi}(z) \) into any arbitrarily chosen axis values of transformed Ernst potentials \( E(z) \) and \( \Phi(z) \), we need only two analytic complex functions (or four such functions that take real values on the real axis). We can therefore impose restrictions on the functional parameters in (49) to exclude pure gauge transformations and to simplify expressions (48). In apparently the simplest case, we obtain a particular solution of (44) in the form

\[ p_1 = p_2 = q_3 = 0, \quad p_3 = -4izd_1q_2^\dagger, \quad d_1 = d_1^1, \quad d_2 = d_1^{-1}, \quad q_1 - q_1^\dagger = -4izq_2q_2^\dagger, \]

where \( \dagger \) denotes complex conjugation of a function evaluated at a complex-conjugate point, e.g., \( d_1^1(t) = \overline{d_1(t)} \). In this case, using simplifying the notation, we obtain the axis form of the Hauser–Ernst solution-generating transformations

\[
\begin{align*}
\mathcal{E}(z) &= k^2(z)[\tilde{E}(z) + i\delta(z) - 2c^1(z)\tilde{\Phi}(z) - c(z)c^1(z)], \\
\tilde{\Phi}(z) &= k(z)[\tilde{\Phi}(z) + c(z)],
\end{align*}
\]

(50)

where

\[ k(z) \equiv d_1\left(\frac{1}{2z}\right), \quad \delta(z) \equiv z\left[q_1\left(\frac{1}{2z}\right) + q_1^\dagger\left(\frac{1}{2z}\right)\right], \quad c(z) \equiv 2izq_2\left(\frac{1}{2z}\right). \]

The functions \( k(z) \) and \( \delta(z) \) must take the real values for real arguments. In the case where \( \tilde{\Phi}(z) = 0 \) and \( c(z) = 0 \), Eq. (50) describes a vacuum-to-vacuum transformation. It is determined (besides the seed \( \tilde{E}(z) \)) by two functions \( k(z) \) and \( \delta(z) \).

In general, the analytic functions given by the components of the matrix \( u(t) \) determined by the functions \( k(z) \), \( \delta(z) \), and \( c(z) \) (together with the seed \( F \)-potential matrix) must be used in the kernel of the Hauser–Ernst integral equation (45) whose solution determines the corresponding generated solution outside the symmetry axis. Thus, at least in principle, the Hauser–Ernst integral equation method allows generating any solutions from the class of stationary axisymmetric electrovacuum solutions with regular behavior at least on some part of the symmetry axis and corresponding to any element of the Kinnersley–Chitre algebra of internal symmetries of stationary axisymmetric Einstein–Maxwell equations. But solving this integral equation for some nontrivial input data is not a simple task. Only very simple examples with rational input data can be found in the literature, and these examples are certainly covered by the vacuum soliton-generating techniques described in the foregoing and by the electrovacuum soliton-generating methods (in the latter case, using the possibility of the analytic continuation of solutions in their parameter spaces).

8. Summary and conclusions

As is known today, the \( G_2 \)-symmetry-reduced vacuum Einstein equations or the electrovacuum Einstein–Maxwell equations are integrable and admit various solution-generating procedures, which allow constructing large families of exact solutions starting from an arbitrarily chosen “seed” (“background”) solution. Each of such solution-generating procedures can be regarded as transformations of the corresponding solution spaces described in terms of transformations of the “coordinates” characterizing every local solution. In the entire solution spaces of the integrable reductions of the Einstein field equations, the monodromy data for the fundamental solution of the associated “spectral problems” can be used as the “coordinates” in the infinite-dimensional solution spaces [44], [39], [48].

In this paper, we considered a simpler construction of such “coordinates” that exist in the (infinite-dimensional) subspaces of electrovacuum solutions for which the gravitational and electromagnetic fields...
behave regularly near degenerate orbits of the space–time isometry group $G_2$. In the corresponding two-dimensional orbit space, the degenerate orbits constitute lines that can be regarded as boundaries of this orbit space. In the infinite-dimensional spaces of solutions of these types, the values of the Ernst potentials on such boundaries in the orbit spaces can serve as “coordinates” of each solution. The solution-generating transformations considered above were described in terms of transformations of these “coordinates” determined by simple expressions in which a particular choice of the initial (background) solution is not assumed. The initial solution in these expressions is represented also by its boundary values of the Ernst potentials, which can be chosen as arbitrary functions of the parameter along the boundary. It is clear that many physical parameters of the generated solutions can be calculated directly from these boundary values of the Ernst potentials and the detailed knowledge of the components of the solution on the entire orbit space is then not necessary. In addition, the explicit form of each of these solution-generating procedures in terms of these “coordinates” allows comparing different suggested solution-generating procedures, finding the relations between numerous constant parameters introduced by different methods, and determining various physical and geometrical properties of the generated solutions (e.g., cylindrical wave profiles on the symmetry axis, multipole moments of asymptotically flat fields, and the appearance of horizons in stationary axisymmetric fields) even before a detailed calculation of all components of the generated solution.

We note that we have not discussed the known integral equation methods that are not solution-generating methods but serve for a direct construction of solutions. These include a scalar linear integral equation method that was suggested by Sibgatullin [49] for constructing stationary axisymmetric electrovacuum fields. Actually, Sibgatullin started from the solution-generating method proposed earlier by Hauser and Ernst [30]–[35] for effecting the Kinnersley–Chitre transformations and considerably reduced their matrix linear singular integral equation to a scalar one, using the choice of Minkowski space–time as the initial solution.

Another method for a direct construction of solutions of integrable reductions of the Einstein field equations that we did not discuss here, but which is not limited to vacuum and electrovacuum fields, is the monodromy transform approach [44], [39]. This approach is also based on a reformulation of integrable reductions of the Einstein field equations in terms of linear singular integral equations, but in contrast to the Hauser and Ernst approach, the construction of these integral equations does not assume any restrictions on the entire space of solutions. One of general applications of this approach, which can be found in [50], is the construction in a unified (determinant) form of a huge class of electrovacuum solutions with an arbitrary rational structure of the Ernst potentials on degenerate orbits of the space–times isometry group $G_2$. This class includes hierarchies of solitons and nonsoliton solutions, stationary axisymmetric solutions (which are not necessarily asymptotically flat), and various types of waves and inhomogeneous cosmological solutions that admit the $G_2$ symmetry. Using this method, more singular types of solutions for interacting waves and inhomogeneous cosmologies were found in [51].

In addition, there are some other types of methods (also based on the integrability of the equations under consideration) that allow a direct construction of multiparameter families of solutions, but do not admit an arbitrary choice of some initial (background) solution. Among these, we mention the algebro-geometric ansatz [52], methods for constructing solutions of boundary value problems [53], [54], and solutions of the characteristic initial value problems [55]. A detailed consideration and further comparison of all these methods and the results of their applications are beyond the scope of this paper.

**Conflicts of interest.** The author declares no conflicts of interest.

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16Some examples of such subspaces of solutions are cylindrical waves, stationary axisymmetric fields created by compact sources and considered near some intervals on the axis between or outside the sources, some cosmological-like solutions, plane waves near the “focusing singularities,” solutions with Killing horizons, and some others.
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