ON CRITICAL AND MAXIMAL DIGRAPHS

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A reader may ask a natural question why this memoir has appeared only now, more than twenty five years after publishing a summary of basic results presented here [3].

The fact is that during those years I worked in A.A.Lyapunov’s Laboratory of Theoretical Cybernetics in Academic town in Novosibirsk. The results published in [3] seemed to be very interesting for me and I decided that their detailed proofs should be published in some respectable international journal. But publishing of Soviet mathematicians’s works abroad was connected with certain formal difficulties 25 years ago. Moreover, necessity of changing affiliation after Lyapunov’s death in 1973, leaving for Omsk in 1976, and new topics of my research in the Omsk State University had put the publishing this work far to the periphery. In 90-s I stopped my work in mathematics and my activity focused on absolutely other questions.

Probably this memoir would still remain unpublished but for the many years interest of Professor Ferdinand Gliviak in these results. For a long time he was studying similar problems and together with a group of his disciples and colleagues got a number of interesting results (see, for example, [5] – [7]). Some time ago he asked me again to publish the detailed proofs of results from [3]. I understood that I was indebted to my colleagues mathematicians. And the debts are to be paid back. At last this text is in front of you. But this would not happened, unless my friend and colleague Dr. Victor Il’ev, who works actively in mathematics himself, had found the power and time and had carried out the huge work on preparation of this manuscript for publishing.

I am sincerely grateful for the mentioned above to Professor Ferdinand Gliviak and Dr. Victor Il’ev, and also to Professor Alexander Kolokolov for the opportunity to discuss different scientific and other problems during 25 years.

Author,
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ON CRITICAL AND MAXIMAL DIGRAPHS

G. Š. Fridman ¹

Abstract

This paper is devoted to the study of directed graphs with extremal properties relative to certain metric functionals. We characterize up to isomorphism critical digraphs with infinite values of diameter, quasi-diameter, radius and quasi-radius. Moreover, maximal digraphs with finite values of radius and quasi-diameter are studied.

Introduction

This paper is devoted to the study of directed graphs with extremal properties relative to certain metric functionals.

We introduce a distance function $\rho(x,y)$ on the vertex set of a directed graph without loops $G = (X,U)$ in the following way: $\rho(x,x) = 0$; for $x \neq y$ let $\rho(x,y)$ be equal to the minimum number of arcs in a directed path from $x$ to $y$, if the vertex $y$ is reachable from $x$; otherwise, we set $\rho(x,y) = \infty$. We introduce in addition a function $\rho_m(x,y)$ as follows:

$$\rho_m(x,y) = \min\{\rho(x,y), \rho(y,x)\}.$$  

We define the following quantities of the digraph $G = (X,U)$:

- the diameter $d(G) = \max_{x,y \in X} \rho(x,y)$;
- the quasi-diameter $d_m(G) = \max_{x,y \in X} \rho_m(x,y)$;
- the radius $r(G) = \min_{x \in X} \max_{y \in X} \rho(x,y)$;
- the quasi-radius $r_m(G) = \min_{x \in X} \max_{y \in X} \rho_m(x,y)$.

A digraph $G = (X,U)$ is called $d$-critical, if the addition of an arbitrary missing arc, whose both endpoints belong to $X$, results either in decreasing the number of bicomponents (strong components) or in decreasing the diameter. Analogously we can introduce the notions of $d_m$-critical, $r$-critical, and $r_m$-critical digraphs.

Let $i$ be an invariant of an $n$-vertex digraph $G$. Then $G$ is called maximal by $i$, if it has the maximum number of arcs among all $n$-vertex digraphs with invariant $i$.

In [11] $d$-critical digraphs were studied. Earlier, in [12], the similar problem for undirected graphs was posed and solved. In [13] the least upper bound is found on the number of edges in an $n$-vertex ordinary undirected graph with given finite radius and in [8], [9], upper bounds on the number of arcs in digraphs of the following classes: 1) $n$-vertex digraphs without loops which have $k$ bicomponents and infinite radius, 2) $n$-vertex digraphs without loops which have $k$ bicomponents and given finite radius $r$ and are such that each bicomponent is a complete symmetric digraph.

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Graphs and digraphs, which are extremal with respect to deleting of arcs and vertices, were studied in [5] – [7]. Moreover, in [7] some results on digraphs near to r-critical digraphs were obtained, in [6] the similar problem for undirected graphs was studied.

In [2] all d-critical digraphs of infinite diameter were characterized up to isomorphism. In [3] r-critical digraphs with infinite radius and r_m-critical digraphs with infinite quasi-radius were characterized up to isomorphism. Moreover, in [3] the least upper bound on the number of arcs in an n-vertex digraph with given finite radius r was obtained and all maximal by r digraphs were characterized up to isomorphism. In [4] the similar results on maximal by d_m digraphs were presented.

This paper contains full proofs of all results presented in [2] – [4]. In Sections 1, 2 we shall characterize up to isomorphism critical digraphs with infinite values of d, d_m, r, and r_m. In Section 3 maximal by r and d_m digraphs with finite values of r and d_m are studied.

1. On critical digraphs with infinite values of diameter and quasi-diameter

In this section we shall characterize up to isomorphism critical digraphs with infinite values of diameter and quasi-diameter.

Two vertices x and y in a digraph G = (X, U) are called mutually reachable, if ρ(x, y) < ∞ and ρ(y, x) < ∞. Obviously, the relation of mutual reachability is an equivalence relation. Thus, the vertex set of G is partitioned into equivalence blocks

\[ X = X_1 \cup ... \cup X_k, \quad X_i \cap X_j = \emptyset \text{ for } i \neq j. \]

The subgraph of the digraph G induced by the set X_i is called a strong component (bicomponent). A digraph that has one bicomponent is called biconnected.

It is easy to see that the diameter d(G) of a digraph G is finite if and only if G is biconnected. The quasi-diameter is finite if and only if for any pair of vertices in G at least one of them is reachable from the other.

Let G = (X, U) be a d-critical digraph and d(G) = ∞. Since the operation of the transitive closure does not violate the relation of mutual reachability, it is easy to see that G is a transitive digraph. Hence each bicomponent of G is a complete symmetric digraph; and if an arc goes in G from some vertex of a bicomponent A_i to some vertex of a bicomponent A_j, then G contains all arcs the initial vertices of which belong to A_i and the terminal ones belong to A_j. The same is also valid for d_m-critical digraphs with infinite quasi-diameter.

Let D = (X, U) be an arbitrary digraph and \{A_1, ..., A_k\} be the totality of its bicomponents. We associate with the digraph D a digraph \(\Gamma(D)\) in the following way. The vertices of the digraph \(\Gamma(D)\) are y_1, ..., y_k; an edge (y_i, y_j) belongs to \(\Gamma(D)\) iff there is an arc in D whose initial vertex belongs to A_i and the terminal vertex belongs to A_j. The digraph \(\Gamma(D)\) is called the Hertz graph of the digraph D. Evidently, \(\Gamma(D)\) contains no directed cycles. Furthermore, if D is a transitive digraph, then its Hertz graph is also transitive.

The above considerations show that a d-critical digraph of infinite diameter and a d_m-critical digraph of infinite quasi-diameter can be uniquely up to isomorphism reconstructed from its Hertz graph, whenever the number of vertices in each bicomponent is determined. Hence the problem of describing d-critical digraphs with d = ∞ and d_m-critical digraphs with d_m = ∞ is reduced to describing corresponding Hertz graphs. This is the topic of the following considerations.
Lemma 1 A transitive acyclic digraph is a subgraph of some complete transitive acyclic digraph.

Proof. The set of arcs of a transitive acyclic digraph defines on the set of its vertices a relation of a partial order. Hence the assertion of the lemma is equivalent to the well known fact that any partially ordered finite set can be embedded into a linearly ordered set without violating the partial order. ■

The totality of arcs of a complete acyclic transitive $k$-vertex digraph defines on the set of its vertices a relation of linear order, therefore it is isomorphic to the graph $\Gamma_k = (X_k, U_k)$ where $X_k = \{1, \ldots, k\}$ and $(i, j) \in U_k \iff i < j$. We shall call a digraph that is isomorphic to the digraph $\Gamma_k$ for some $k$ a transitive tournament.

We consider only directed graphs in this paper; henceforth the term “graph” will always mean directed graph.

Theorem 1 The Hertz graph of a $d$-critical digraph of infinite diameter with $k$ bicomponents is isomorphic to the graph $\Gamma_k$.

Proof. Let a graph $\Gamma$ satisfy the hypothesis of the theorem; then it is acyclic transitive digraph and hence, by Lemma 1, is isomorphic to a subgraph $\Gamma'_k$ of the graph $\Gamma_k$. Let an arc $(i, j)$ where $i < j$ be missing in $\Gamma'_k$. Evidently the addition of the arc does not result in the appearance of a directed cycle contrary to $\Gamma$ being $d$-critical with $d = \infty$. Hence $\Gamma'_k = \Gamma_k$. ■

Corollary 1 Let $D$ be a nonbiconnected digraph with the property that the addition of an arbitrary missing arc converts it into a biconnected digraph. Then the Hertz graph of $D$ is isomorphic to the graph $\Gamma_2$.

Proof. By Theorem 1, the Hertz graph of $D$ is isomorphic to the graph $\Gamma_k$ for some $k$. But if $k \geq 3$, then the addition of the arc $(k, k - 1)$ to $\Gamma_k$ does not make it biconnected. Hence the Hertz graph of the digraph $D$ is isomorphic to the graph $\Gamma_2$. ■

We note that the assertion of Corollary 1 contains the Rois theorem [14].

Corollary 2 (See also [9]) The number of arcs in an $n$-vertex digraph of infinite diameter with $k$ bicomponents does not exceed $n(n - k) + \frac{k^2 - k}{2}$.

Proof. Let a digraph $G$ contain the greatest number of arcs among all $n$-vertex digraphs of infinite diameter with $k$ bicomponents $A_1, \ldots, A_k$. Let the bicomponent $A_i$ contain $m_i$ vertices ($i = 1, \ldots, k$).

Evidently, $G$ is a $d$-critical digraph. By Theorem 1, the Hertz graph of $G$ is isomorphic to $\Gamma_k$. Let $2 \leq m_i \leq m_j$. Consider a $d$-critical digraph $G'$ of infinite diameter that is obtained from $G$ by moving a vertex from the bicomponent $A_i$ to the bicomponent $A_j$. In this case the number of arcs in $G'$ is greater than the number of arcs in $G$ by $m_j - (m_i - 1) = m_j - m_i + 1 > 0$, but this contradicts the maximality of the number of arcs in $G$. Hence each bicomponent in $G$, excluding perhaps one, contains one vertex. Then the number of arcs in $G$ equals $(n - 1) + (n - 2) + \ldots + (n - k + 1) + (n - k + 1)(n - k) = n(n - k) + \frac{k^2 - k}{2}$. ■
Since \( \max_{2 \leq k \leq n} \left\{ n(n-k) + \frac{k^2-k}{2} \right\} = (n-1)^2 \), it follows from Corollary 2 that the number of arcs in an \( n \)-vertex nonbiconnected digraph does not exceed \((n-1)^2\).

Let \( D(n, k, d = \infty) \) denote the totality of all nonisomorphic \( n \)-vertex \( d \)-critical digraphs of infinite diameter with \( k \) bicomponents, and \( \beta(n, k) = |D(n, k, d = \infty)| \).

**Corollary 3**

\[ \beta(n, k) = \binom{n-1}{k-1} \]

**Proof.** Let us consider \( k \) boxes numbered 1, ..., \( k \) and \( n \) identical points contained in the boxes. There are no empty boxes. We consider the points as vertices of a graph such that points in a box constitute a complete symmetric digraph, and arcs go from all vertices in the boxes with smaller numbers to all vertices in the boxes with greater numbers. There are no other arcs in this graph. It follows from Theorem 1 that distributing in different ways \( n \) identical points in \( k \) boxes without empty ones and constructing digraphs in the way as before, we can obtain all \( d \)-critical digraphs of the totality \( D(n, k, d = \infty) \). All these digraphs are nonisomorphic. It was shown in [1] that the number of ways of distributing \( n \) identical objects in \( k \) distinct boxes without empty ones equals \( \binom{n-1}{k-1} \). \( \blacksquare \)

**Corollary 4** The number of distinct \( d \)-critical digraphs of infinite diameter with \( k \) bicomponents that can be constructed on given \( n \) numbered vertices equals \( k! S(n, k) \) where \( S(n, k) \) are the Stirling numbers of the second kind.

**Proof.** The proof of Corollary 2 shows that the number of distinct \( d \)-critical digraphs of infinite diameter with \( k \) bicomponents that can be constructed on \( n \) numbered vertices equals the number of ways of distributing \( n \) distinct objects in \( k \) distinct boxes without empty ones. It was shown in [1] that the number equals \( k! S(n, k) \). \( \blacksquare \)

Now we proceed to characterizing \( d_m \)-critical digraphs with infinite quasi-diameter. As we have stated, in order to do that, it suffices to characterize their Hertz graphs. Let \( \Gamma \) be the Hertz graph of a \( d_m \)-critical digraph with \( d_m = \infty \). We noted in the beginning of the section that the quasi-diameter of a digraph is infinite iff there is a pair of vertices, neither of which is reachable from the other. The above notes imply that \( \Gamma \) is acyclic, transitive, and has a pair of vertices, neither of which is reachable from the other. We shall show that there exists exactly one pair of vertices in \( \Gamma \) with this property. Indeed, suppose there exist two such pairs \( \{a, b\} \) and \( \{c, d\} \). Consider the subgraph of the graph \( \Gamma \) induced by the vertices \( a, b, c, d \). It is easy to see that the subgraph is isomorphic to either of the graphs depicted in Figures 1, 2.

Figure 1 corresponds to the case, when the sets \( \{a, b\} \) and \( \{c, d\} \) has a vertex in common; Figure 2 corresponds to the case, when the pairs are disjoint. In both cases we add the arc \((c, d)\). It is easy to see that the obtained graphs do not contain directed cycles, and in each graph there exists a pair of vertices neither of which is reachable from the other. Thus, we have shown that in the Hertz graph of a \( d_m \)-critical digraph of infinite quasi-diameter there exists exactly one pair of vertices neither of which is reachable from the other.

We denote by \( \Gamma_{k,i} = (X_k, U_{k,i}) \) a digraph such that

\[ X_k = \{1, ..., k\} \text{ and } (s, j) \in U_{k,i} \iff (s < j) \& [(s, j) \neq (i, i + 1)] \]
that is the graph $\Gamma_{k,i}$ is obtained from the graph $\Gamma_k$ by removal of the arc $(i, i+1)$, $i \in \{1, \ldots, k-1\}$.

**Theorem 2** Let $\Gamma$ be the Hertz graph of a $d_m$-critical digraph of infinite quasi-diameter with $k$ bicomponents, $a$ and $b$ be the vertices of $\Gamma$, neither of which is reachable from the other; the indegree of the vertex $a$ equals $i$, $0 \leq i \leq k-2$. Then $\Gamma$ is isomorphic to the graph $\Gamma_{k,i+1}$.

**Proof.** The above considerations show that the graph $\Gamma$ is acyclic, transitive, and has the property that any two its vertices, excluding the pair $\{a, b\}$, are adjacent. Hence the subgraph $\Gamma'$, that is obtained from $\Gamma$ by removal of the vertex $b$ and arcs incident to it, is a complete acyclic transitive $(k-1)$-vertex digraph. By Lemma 1, $\Gamma'$ is isomorphic to the graph $\Gamma_{k-1}$. Let us renumber the vertices of $\Gamma'$ in the following way: as a number of a vertex $x$ we consider the number $j$ that corresponds to the vertex $x$ under the isomorphism $\Gamma' \leftrightarrow \Gamma_{k-1}$ (evidently, such an isomorphism is unique). It is easy to see that the number of the vertex $a$ is $i+1$. We shall show that $\Gamma$ contains all arcs of the kind $(b, x_j)$ where $j > i + 1$, and all arcs of the kind $(x_t, b)$ where $t < i + 1$. Indeed, if $\Gamma$ contained the arc $(x_l, b)$ where $l > i + 1$, then there would exist a path $\{(x_{i+1}, x_l), (x_l, b)\}$, $x_{i+1} = a$ from $a$ to $b$ contrary to the hypothesis of mutual unreachability of the vertices $a$ and $b$. If $\Gamma$ contained an arc of the kind $(b, x_m)$ where $m < i + 1$, then there would exist a path $\{(b, x_m), (x_m, x_{i+1})\}$ from $b$ to $a$ contrary to the hypothesis again. But the vertex $b$ is adjacent to each vertex in $\Gamma$, except the vertex $a$, hence $\Gamma$ contains all arcs of the kind $(b, x_j)$, $j > i + 1$ and $(x_s, b)$, $s < i + 1$. Now let us add the arc $(a, b)$ to the graph $\Gamma$. One sees that the obtained graph $\Gamma''$ is isomorphic to the graph $\Gamma_k$; under the isomorphism the number $i + 1$ corresponds to the vertex $a$, and $i + 2$ corresponds to $b$. Hence $\Gamma$ is isomorphic to the graph that is obtained from the graph $\Gamma_k$ by the removal of the arc $(i+1, i+2)$. □

From the notes on $d_m$-critical digraphs with $d_m = \infty$ one sees that Theorem 2 gives the description of these digraphs up to isomorphism.

By Corollary 2 of Theorem 1, the greatest number of arcs in a given number of bicomponents equals $n(n-k) + \frac{k^2-k}{2}$. In view of the connection between $d_m$-critical digraphs of infinite quasi-diameter and $d$-critical digraphs of infinite diameter established by Theorems 1 and 2 we obtain

**Corollary 1** If the number of arcs in an $n$-vertex digraph with $k \geq 3$ bicomponents is greater than $n(n-k) + \frac{k^2-k}{2} - 1$, then its quasi-diameter is finite.
It is easy to see that the number of arcs in an \( n \)-vertex digraph of infinite quasi-diameter with two bicomponents does not exceed \((n - 1)(n - 2)\). This result together with Corollary 1 gives

**Corollary 2** If the number of arcs in an \( n \)-vertex digraph is greater than \( n^2 - 3n + 2 \), then its quasi-diameter is finite.

Let \( q(n, k) \) denote the number of nonisomorphic \( d_m \)-critical digraphs having \( k \) bicomponents and infinite quasi-diameter; and \( q^*(n, k) \) denote the number of distinct \( d_m \)-critical digraphs, having \( k \) bicomponents and infinite quasi-diameter, that can be constructed on \( n \) numbered vertices.

**Corollary 3**

\[
q(n, k) = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor & \text{for } k = 2, \\ (k - 1) \sum_{t=2}^{n-k+2} \binom{n - t - 1}{k - 3} \left\lfloor \frac{t}{2} \right\rfloor & \text{for } k > 2. \end{cases}
\]

**Proof.** In the same manner as in the proof of Corollary 3 to Theorem 1 we consider \( k \) boxes numbered 1, \ldots, \( k \) and \( n \) identical points, contained in these boxes without empty ones. We consider the points as vertices of a digraph such that the points in a box constitute a complete symmetric graph, and arcs go from all points in the box numbered \( i \) to all points in the box numbered \( j \) where \( i < j \) and \( j \neq 2 \).

The number of nonisomorphic digraphs obtained in different distributions of \( n \) identical points in given \( k \) boxes for \( k > 2 \) equals

\[
(k - 1) \sum_{t=2}^{n-k+2} \binom{n - t - 1}{k - 3} \left\lfloor \frac{t}{2} \right\rfloor.
\]

Each summand in the sum corresponds to the number of ways of distributing \( t \) points in two identical boxes, and the rest \( n-t \) points in \( k-2 \) distinct boxes. Thus, we have counted the number of nonisomorphic \( d_m \)-critical digraphs, whose Hertz graphs are isomorphic to \( \Gamma_{k,1} \). Evidently, for any \( i \) and \( j \) the number of nonisomorphic \( d_m \)-critical digraphs, whose Hertz graph is isomorphic to \( \Gamma_{k,i} \), equals the number of nonisomorphic digraphs, whose Hertz graph is isomorphic to \( \Gamma_{k,j} \), therefore the number of nonisomorphic \( n \)-vertex \( d_m \)-critical digraphs, having \( k \) bicomponents and infinite quasi-diameter, for \( k > 2 \) equals

\[
(k - 1) \sum_{t=2}^{n-k+2} \binom{n - t - 1}{k - 3} \left\lfloor \frac{t}{2} \right\rfloor.
\]

If \( k = 2 \) then the number of such digraphs is \( \left\lfloor \frac{n}{2} \right\rfloor \).

**Corollary 4**

\[
q^*(n, k) = \begin{cases} 2^{n-1} - 1 & \text{for } k = 2, \\ (k - 1) \sum_{t=2}^{n-k+2} \binom{n}{t} (k - 2)! S(n - t, k - 2)(2^{t-1} - 1) & \text{for } k > 2. \end{cases}
\]

The proof of the corollary is analogous to the proof of Corollary 4 to Theorem 1.
2. On critical digraphs with infinite values of radius and quasi-radius

In this section we shall characterize up to isomorphism critical digraphs with infinite values of radius and quasi-radius.

It is easy to see that \( r \)-critical and \( r_m \)-critical digraphs with infinite values of \( r \) and \( r_m \), like \( d \)-critical and \( d_m \)-critical digraphs with infinite values of \( d \) and \( d_m \), are transitive. Hence the problem of describing these digraphs is reduced to the problem of describing their Hertz graphs.

I. In this subsection we shall characterize \( r \)-critical digraphs of infinite radius.

**Theorem 3** Let \( \Gamma = (X, U) \) be the Hertz graph of a \( r \)-critical digraph \( G \) with \( k \) bicomponents and \( r(G) = \infty \). Then the graph \( \Gamma \) is isomorphic to the graph \( \Gamma_{k,1} \).

**Proof.** Lemma 1 implies that the graph \( \Gamma \) contains a source which is a vertex that has no incoming arcs. Let \( z \) denote the source. We shall show that arcs go from the vertex \( z \) to all but one vertices in \( \Gamma \). Indeed, assume there is a pair of vertices \( u \) and \( v \) that are not reachable from \( z \). Since the graph \( \Gamma \) is antisymmetric, at least one of the arcs \((u, v), (v, u)\) is missing in it. Let for instance, the arc \((u, v)\) be missing. Having added the arc \((z, u)\) we obtain a graph \( \Gamma' \) in which there are no directed cycles and whose radius is infinite. It is impossible, since \( \Gamma \) was a \( r \)-critical digraph. Thus, arcs go from the vertex \( z \) to all vertices in \( \Gamma \), except a vertex \( y \). Obviously, the vertex \( y \) is also a source in \( \Gamma \). Hence arcs go from it to all vertices in \( \Gamma \), except \( z \). Denote by \( X' \) the set \( X \setminus \{z, y\} \). It is easy to see that the addition of an arc, both endpoints of which belong to \( X' \), does not decrease the radius. Since \( \Gamma \) is a \( r \)-critical digraph, the addition of an arc, both endpoints of which belong to \( X' \), must result in the appearance of a directed cycle. Since the subgraph of \( \Gamma \) induced by the set \( X' \) is transitive and antisymmetric, what has been said and Lemma 1 imply that this subgraph is a \((k-2)\)-vertex transitive tournament. This means that \( \Gamma \) is isomorphic to the graph \( \Gamma_{k,1} \).

Theorem 3 is actually contained in the work [8] in a somewhat different form.

We shall cite a few corollaries of Theorem 3.

**Corollary 1** (See also [8]) If an \( n \)-vertex digraph \( G \) with \( k > 1 \) bicomponents contains more than \( \lambda(n, k) \) arcs, then \( r(G) < \infty \) where

\[
\lambda(n, k) = \begin{cases} 
(n - 1)(n - 2), & \text{if } k = 2, \\
n(n - k) + \frac{k^2 - k - 2}{2}, & \text{if } k > 2.
\end{cases}
\]

**Proof.** Let \( G \) be an \( n \)-vertex digraph of infinite radius with \( k \) bicomponents. Let \( G \) contain the greatest number of arcs. Evidently, \( G \) is a \( r \)-critical graph. Hence its Hertz graph is isomorphic to the graph \( \Gamma_{k,1} \). If \( k = 2 \) then the number of arcs in an \( n \)-vertex digraph, whose Hertz graph is isomorphic to \( \Gamma_{2,1} \), is maximal when one of its bicomponents contains \( n - 1 \) vertices. Hence in this case the digraph \( G \) contains \((n - 1)(n - 2)\) arcs. Now let \( k \geq 3 \). We shall show that the number of arcs in a digraph, whose Hertz graph is isomorphic to \( \Gamma_{k,1} \), is maximal in the case when all its bicomponents, excluding one bicomponent not corresponding to the vertices 1 and 2 of \( \Gamma_{k,1} \), are one-vertex. Indeed,
let \(2 < i < j \leq k\) and the number of vertices in the bicomponents, corresponding to the vertices \(i\) and \(j\) of the graph \(\Gamma_{k,1}\), equal \(k_i \geq 2\) and \(k_j \geq 2\), respectively. Let us move all vertices but one from the component corresponding to the vertex \(j\) to the component corresponding to the vertex \(i\). The number of arcs, connecting vertices of these bicomponents to vertices of other bicomponents, evidently does not change. The number of arcs in the subgraph induced by the vertices of these bicomponents was \(k_i(k_i - 1) + k_j(k_j - 1) + k_ikj\), and became \((k_i + k_j - 1)(k_i + k_j - 2) + (k_i + k_j - 1)\); that is it has decreased by \(k_ikj - k_i - k_j + 1 \geq 1\). It is easy to see that if we move all vertices but one from each bicomponent, corresponding to the vertices 1 or 2 of \(\Gamma_{k,1}\), to the bicomponent, corresponding to the vertex \(k\), then the number of arcs in the obtained \(r\)-critical digraph will not decrease. Thus, we have shown that the number of arcs in a \(r\)-critical digraph whose Hertz graph is isomorphic to \(\Gamma_{k,1}\) is maximal, when all its bicomponents, excluding one not corresponding to the vertices 1 and 2 of the graph \(\Gamma_{k,1}\), are one-vertex. The number of arcs in such an \(n\)-vertex digraph equals

\[
2(n - 2) + (n - 3) + ... + (n - k + 1) + (n - k + 1)(n - k) = n(n - k) + \frac{k^2 - k - 2}{2},
\]
as required.

Corollary 2 If a digraph \(G\) has the property that \(r(G) = \infty\) and the addition of an arbitrary missing arc converts it into a digraph with finite radius, then the Hertz graph of \(G\) is isomorphic either to \(\Gamma_{3,1}\) or to \(\Gamma_{2,1}\).

Proof. If the digraph \(G\) has the property stated in Corollary 2, then so does its Hertz graph \(\Gamma\). By Theorem 3, the graph \(\Gamma\) is isomorphic to the graph \(\Gamma_{k,1}\) for some \(k\). If \(k \geq 4\) then the addition of the arc \((4, 3)\) to \(\Gamma_{k,1}\) does not convert it into a digraph with finite radius. Hence \(\Gamma\) is isomorphic either to \(\Gamma_{3,1}\) or to \(\Gamma_{2,1}\). It is easy to see that these graphs possess the property stated in Corollary 2.

Corollary 3 If an \(n\)-vertex digraph \(G\) contains more than \((n - 1)(n - 2)\) arcs, then \(r(G) < \infty\).

Proof. If the number of bicomponents in the graph \(G\) equals two, then the number of its arcs evidently does not exceed \((n - 1)(n - 2)\), what corresponds to the case, when either of the bicomponents is a complete symmetric \((n - 1)\)-vertex graph, and the other is one-vertex. If the number of bicomponents of \(G\) is greater than two, then by Corollary 1, the number of its arcs does not exceed \(n(n - k) + \frac{k^2 - k - 2}{2}\). The last expression does not exceed \((n - 1)(n - 2)\), if \(k \geq 3\).

Corollary 4 The number of nonisomorphic \(r\)-critical \(n\)-vertex digraphs with \(k\) bicomponents and infinite radius equals

\[
\nu(n, k) = \begin{cases} 
\left\lfloor \frac{n}{2} \right\rfloor & \text{for } k = 2, \\
\sum_{t=2}^{n-k+2} \left\lfloor \frac{t}{2} \right\rfloor \left( \begin{pmatrix} n - t - 1 \\ k - 3 \end{pmatrix} \right) & \text{for } k \geq 3.
\end{cases}
\]
Proof. The Hertz graph of a critical digraph with \( k \) bicomponents and infinite radius is isomorphic to \( \Gamma_{k,1} \). Let the bicomponents, corresponding to the vertices 1 and 2 of \( \Gamma_{k,1} \), contain together \( t \) vertices of a critical digraph and \( k \geq 3 \). Fix the number of vertices in each of the bicomponents, corresponding to the vertices 1 and 2 of \( \Gamma_{k,1} \). Distributing the rest \( n - t \) identical vertices in \( k - 2 \) distinct bicomponents we obtain (see [1]) \( \binom{n-t-1}{k-3} \) nonisomorphic critical digraphs. There are \( \left\lfloor \frac{t}{2} \right\rfloor \) ways of distributing \( t \) vertices in bicomponents, corresponding to the vertices 1 and 2 of \( \Gamma_{k,1} \). Hence the number of nonisomorphic \( r \)-critical digraphs of infinite radius having \( n \) vertices, \( k \) bicomponents, and such that the total number of vertices in bicomponents, corresponding to the vertices 1 and 2 of \( \Gamma_{k,1} \), equals \( t \) is \( \left\lfloor \frac{t}{2} \right\rfloor \binom{n-t-1}{k-3} \). Summing the quantity over \( t \) we obtain what we needed to prove. The proof in the case \( k = 2 \) is obvious. \( \blacksquare \)

Corollary 5 The number of distinct \( r \)-critical digraphs with \( k \) bicomponents and infinite radius that can be constructed on \( n \) numbered vertices equals

\[
\nu^r(n,k) = \begin{cases} 
2^{n-1} - 1 & \text{for } k = 2, \\
\sum_{t=2}^{n-k+2} \binom{n}{t} (2^{t-1} - 1)(k-2)! S(n-t,k-2) & \text{for } k > 2.
\end{cases}
\]

The proof of Corollary 5 is analogous to the proof of Corollary 4, but in this case we should consider the vertices to be distinct.

II. In this subsection we shall take up the study of \( r_m \)-critical digraphs with infinite quasi-radius. As above, we shall characterize the structure of corresponding Hertz graphs.

We call a vertex in a digraph \( G = (X,U) \) a quasi-center, if the relation \( \max_{y \in X} \rho_m(x,y) < \infty \) holds. Evidently, the quasi-radius of a digraph is finite iff the digraph contains a quasi-center.

Lemma 2 Let a \( k \)-vertex graph \( \Gamma \) be the Hertz graph of a \( r_m \)-critical digraph with infinite quasi-radius. In addition, suppose the graph \( \Gamma \) is not weakly connected. Then \( \Gamma \) is isomorphic to the graph \( \Gamma_{k,0} = (X_{k,0},U_{k,0}) \), where

\[
X_{k,0} = \{1,\ldots,k\} \text{ and } (i,j) \in U_{k,0} \iff (i < j) \land (i \neq 1).
\]

Proof. Evidently, the graph \( \Gamma \) can not contain more than two weak components, since the quasi-radius of a graph containing more than one weak component can not be finite. Let \( X_1 \) and \( X_2 \) be the weak components of the graph \( \Gamma \). If we add an arc such that both its endpoints belong to one weak component, then the quasi-radius of the obtained digraph remains infinite. Hence the number of bicomponents must decrease. Then using Lemma 1, we obtain that the subgraphs induced by the sets \( X_1 \) and \( X_2 \) are transitive tournaments. If \( \min\{|X_1|,|X_2|\} = 1 \) then \( \Gamma \) is isomorphic to the graph \( \Gamma_{k,0} \). Let \( |X_1| = l, |X_2| = t \), and \( \min\{l,t\} \geq 2 \). Denote by \( \Gamma^1 \) and \( \Gamma^2 \) the subgraphs of \( \Gamma \) induced by the sets \( X_1 \) and \( X_2 \), respectively. Renumber the vertices in \( \Gamma \) so, that the mapping \( \varphi : x_i \leftrightarrow i, i = 1,\ldots,l \) be an isomorphism of the graphs \( \Gamma^1 \) and \( \Gamma_l \), and the mapping \( \psi : x_{j+t} \leftrightarrow j, j = 1,\ldots,t \) be an isomorphism of the graphs \( \Gamma^2 \) and \( \Gamma_t \). According to the above considerations it is possible. Let us add the arc \((x_1,x_{l+t})\) to the graph \( \Gamma \). Denote the obtained graph by \( \Gamma' \).
It is easy to see that the number of bicomponents does not change. We shall show that the quasi-radius of the obtained graph is infinite. Indeed, the vertex \( x_j \) and each vertex \( x_j, j > l \) are mutually unreachable. Hence no vertex in the subgraph \( \Gamma^2 \) can be a quasi-center in the obtained graph. On the other hand, the vertex \( x_{l+1} \) and each vertex \( x_i, i \leq l \) are mutually unreachable. Hence none of the vertices \( x_i, i \leq l \) can be a quasi-center in the graph \( \Gamma' \). Thus we have proved that \( \Gamma' \) has no quasi-centers, hence its quasi-radius is infinite. This implies that \( \Gamma \) is not \( r_m \)-critical contrary to the hypothesis. ■

**Lemma 3** Let \( \Gamma \) be the Hertz graph of a \( d_m \)-critical digraph with \( r_m = \infty \). Let \( \Gamma \) be weakly connected and each vertex in \( \Gamma \) have either outdegree or indegree equal to zero. Then \( \Gamma \) is isomorphic to the graph \( D_4 = (Y_4, V_4) \) where

\[
Y_4 = \{1, 2, 3, 4\} \text{ and } V_4 = \{(1, 3), (1, 4), (2, 3), (2, 4)\}.
\]

**Proof.** Let \( X_1 \) be the totality of the vertices of the graph \( \Gamma \) whose indegree equals zero; \( X_2 \) be the totality of the vertices of \( \Gamma \) whose outdegree equals zero. Evidently, the sets \( X_1 \) and \( X_2 \) are disjoint. Since \( r(\Gamma) \geq r_m(\Gamma) = \infty \), it follows from Theorem 3 that \( |X_1| \geq 2 \); since otherwise, there exists a vertex in \( X_2 \) whose total degree equals zero contrary to the weak connectivity of \( \Gamma \). Suppose \( |X_2| = 1 \). Since arcs from \( X_1 \) can go only to \( X_2 \); either \( r_m(\Gamma) = 1 \), if an arc goes from each vertex of \( X_1 \) to the only vertex of \( X_2 \), or there exists a vertex in \( X_1 \), whose total degree equals zero, contrary to the assumption of \( \Gamma \) being weakly connected. Hence \( |X_2| \geq 2 \). We shall show that \( |X_1| = |X_2| = 2 \). Let, for instance, \( |X_1| \geq 3 \). Add an arc, both endpoints of which belong to \( X_1 \). Denote the obtained graph by \( \Gamma' \). Evidently, the number of bicomponents in \( \Gamma' \) equals the number of bicomponents in the digraph \( \Gamma \). We shall show that the quasi-radius of \( \Gamma' \) is infinite. Indeed, since \( |X_1| \geq 3 \), for any vertex \( x \in X_1 \) there exists a vertex \( y \in X_1 \) such that \( \rho_m(x, y) = \infty \). Hence \( X_1 \) contains no quasi-centers of \( \Gamma' \). The set \( X_2 \) contains no quasi-center of \( \Gamma' \) either, because any two vertices in \( X_2 \) are mutually unreachable and \( |X_2| \geq 2 \). Hence, if \( |X_1| \geq 3 \), then the graph \( \Gamma \) is not \( r_m \)-critical. Evidently, an analogous argument is valid for the set \( |X_2| \). Thus, we have shown that \( |X_1| = |X_2| = 2 \). It means that \( \Gamma \) is isomorphic to a subgraph of the graph \( D_4 \). However, the graph \( D_4 \) has no directed cycles, and \( r_m(D_4) = \infty \). Hence \( \Gamma \cong D_4 \). On the other hand, it is easy to see that the graph \( D_4 \) is \( r_m \)-critical. ■

**Lemma 4 (basic)** Let \( \Gamma \) be the Hertz graph of a weakly connected \( r_m \)-critical digraph with \( r_m = \infty \). Then the vertex set of the graph \( \Gamma \) can be partitioned into two disjoint subsets such that

a) the subgraphs induced by each of these subsets are \( r_m \)-critical with \( r_m = \infty \);

b) an arc goes from each vertex of the first subset to each vertex of the second subset.

In order to prove Lemma 4 we need a few lemmas.

If each vertex in a graph \( \Gamma \) has either outdegree or indegree equal to zero, then by Lemma 3, \( \Gamma \) is isomorphic to the graph \( D_4 \); and the graph \( D_4 \) satisfies Lemma 4.

So, let \( \Gamma \) contain a vertex \( v \) such that outdegree and indegree of the vertex are greater than zero. Denote by \( B_v \) the set of vertices of \( \Gamma \) reachable from \( v \), by \( A_v \) the set of vertices of \( \Gamma \) from which the vertex \( v \) is reachable, and by \( C_v \) the totality of all other vertices. Denote by \( \Gamma(A_v), \Gamma(B_v), \) and \( \Gamma(C_v) \) the subgraphs of the graph \( \Gamma \) that are induced by the sets \( A_v, B_v, \) and \( C_v \), respectively. Since the graph \( \Gamma \) is transitive, an arc goes from
each vertex of the set \(A_v\) to each vertex of the set \(B_v \cup \{v\}\), and from the vertex \(v\) to each vertex of the set \(B_v\). Obviously, \(C_v \neq \emptyset\) since otherwise, the vertex \(v\) would be a quasi-center in \(\Gamma\).

**Lemma 5** Let \(\Gamma\) be the Hertz graph of a \(r_m\)-critical digraph with infinite quasi-radius; \(A_v, B_v, C_v, \{v\}\) be the sets defined above. Suppose \(r_m(\Gamma(A_v)) = r_m(\Gamma(B_v)) = \infty\). Then
1) An arc goes from each vertex of the set \(A_v\) to each vertex of the set \(C_v\); an arc goes from each vertex of the set \(C_v\) to each vertex of the set \(B_v\).
2) The subgraph \(\Gamma(C_v)\) is a transitive tournament.
3) The subgraphs \(\Gamma(A_v)\) and \(\Gamma(B_v)\) are \(r_m\)-critical.

**Proof.** First we shall show that \(\Gamma(A_v)\) and \(\Gamma(B_v)\) are \(r_m\)-critical. Let us add an arc, both endpoints of which belong to \(B_v\), to the graph \(\Gamma\). Denote the obtained graph by \(\Gamma'\). No vertex of the set \(C_v\) can be a quasi-center in \(\Gamma'\), for each of such vertices and the vertex \(v\) are mutually unreachable, the vertex \(v\) can not be a quasi-center of \(\Gamma'\) for the same reason. To any vertex in the set \(A_v\) there exists a vertex in this set such that these two vertices are mutually unreachable. Hence \(A_v\) contains no quasi-centers of \(\Gamma'\). Since \(\Gamma'\) is \(r_m\)-critical and \(r_m(\Gamma) = \infty\), it follows that either \(B_v\) contains a quasi-center of \(\Gamma'\) that is also a quasi-center in the graph \(\Gamma(B_v)\), or \(\Gamma'\) contains a directed cycle. However, the addition of an arc, both endpoints of which belong to \(B_v\), can result in the appearance of directed cycle only in the subgraph \(\Gamma(B_v)\). This consideration shows that \(\Gamma(B_v)\) is \(r_m\)-critical. An analogous consideration is evidently valid with respect to the subgraph \(\Gamma(A_v)\). Thus, statement 3) of the lemma is proved.

Now let us prove statement 2). Add to \(\Gamma\) all missing arcs that go from \(A_v\) to \(C_v\) and from \(C_v\) to \(B_v\). Denote the obtained graph by \(\Gamma''\). Evidently the number of bicomponents of this graph equals the number of bicomponents of the graph \(\Gamma\). The same considerations as in the proof of statement 3) show that quasi-radius of the graph \(\Gamma''\) is infinite. Finally, let us show the graph \(\Gamma(C_v)\) is a transitive tournament. Indeed, the addition of an arc, both endpoints of which belong to \(C_v\), can not result in the appearance of a quasi-center in the obtained graph, since each vertex in \(C_v\) and the vertex \(v\) are mutually unreachable, and \(r_m(\Gamma(A_v)) = r_m(\Gamma(B_v)) = \infty\). Hence it must result in the appearance of a directed cycle. However the subgraph \(\Gamma(C_v)\) is the only subgraph in which a directed cycle can appear. It means that the transitive antisymmetric digraph \(\Gamma(C_v)\) has the property that the addition of an arbitrary arc missing in it results in the appearance of a directed cycle. Thus, it follow from Lemma 1 that this digraph is a transitive tournament. ■

**Lemma 6** Let \(\Gamma\) be the Hertz graph of a \(r_m\)-critical digraph with \(r_m = \infty\). If \(\min\{r_m(\Gamma(A_v)), r_m(\Gamma(B_v))\} < \infty\), then \(|C_v| = 1\).

**Proof.** Let, for instance, \(r_m(\Gamma(A_v)) < \infty\).

We shall show first that the set \(C_v\) contains a vertex that is reachable from no quasi-center of the graph \(\Gamma(A_v)\). Suppose conversely that any vertex of the set \(C_v\) is reachable from some quasi-center of the graph \(\Gamma(A_v)\). Denote by \(Z_v\) the totality of all quasi-centers of the graph \(\Gamma(A_v)\). It is easy to see that the subgraph \(\Gamma(Z_v)\) of the graph \(\Gamma(A_v)\), induced by the set \(Z_v\), is a transitive tournament. Let a vertex \(z_0 \in Z_v\) correspond to the vertex 1 under the isomorphism \(\Gamma(Z_v) \leftrightarrow \Gamma|_{Z_v}\). Then all vertices of the set \(Z_v\) are reachable from \(z_0\) and, by assumption, so are all vertices of the set \(C_v\). But in this case the vertex \(z_0\) is a quasi-center, what is impossible. Thus we have shown that \(C_v\) contains a vertex that
is reachable from no quasi-center of the graph $\Gamma(A_v)$. Now we shall show that the set $C_v$ contains exactly one vertex with the above property.

Suppose there exist two such vertices $y_1$ and $y_2$; suppose the arc $(y_2,y_1)$ is missing in the graph $\Gamma(C_v)$. Let us add an arc to $\Gamma$ that goes from an arbitrary quasi-center of $\Gamma(A_v)$ to the vertex $y_2$. The following consideration shows that we shall obtain a graph $\Gamma$ with infinite quasi-radius.

No vertex in the set $C_v$ can be a quasi-center in $\Gamma^1$, since such a vertex and the vertex $v$ are mutually unreachable, hence the vertex $v$ can not be a quasi-center in $\Gamma^1$ either. None of the quasi-centers of the graph $\Gamma(A_v)$ can be a quasi-center in $\Gamma^1$, since each of them and the vertex $y_1$ are mutually unreachable; no other vertex of the set $A_v$ can be a quasi-center in the graph $\Gamma^1$, since no such vertex is a quasi-center in the graph $\Gamma(A_v)$. Evidently, the set $B_v$ contains no quasi-centers.

Let $y \in C_v$ be that unique vertex that is reachable from no quasi-center of the graph $\Gamma(A_v)$. Suppose, in addition, $C_v \setminus \{y\} \neq \emptyset$. It is easy to see that the vertex $y$ is reachable from no vertex of the set $C_v \setminus \{y\}$.

Consider next two cases.

(i) $r_m(\Gamma(B_v)) = \infty$. Let us add the arc $(y,v)$ to the graph $\Gamma$. We shall show that quasi-radius of the obtained graph $\Gamma^2$ is infinite. The vertex $y$ remains mutually reachable with no quasi-center of the graph $\Gamma(A_v)$, hence neither vertex $y$ nor any vertex in $A_v$ can be a quasi-center in the graph $\Gamma^2$. No vertex of the set $C_v \setminus \{y\}$ is mutually reachable with the vertex $v$, hence no such vertex can be a quasi-center in the graph $\Gamma^2$. No vertex of the set $B_v$ can be a quasi-center in the graph $\Gamma^2$ for the reason that $r_m(\Gamma(B_v)) = \infty$.

(ii) $r_m(\Gamma(B_v)) < \infty$. The above considerations show that if $r_m(\Gamma(A_v)) < \infty$ then the set $C_v$ contains exactly one vertex $y$ that is mutually reachable with no quasi-center of the graph $\Gamma(A_v)$. An analogous consideration shows that if $r_m(\Gamma(B_v)) < \infty$ then the graph $\Gamma(C_v)$ contains exactly one vertex $u$ that is mutually reachable with no quasi-center of the graph $\Gamma(B_v)$. Here evidently no vertex in the set $C_v \setminus \{u\}$ is reachable from the vertex $u$. Suppose first that $y = u$. Then by the above consideration, it is an isolated vertex in the graph $\Gamma(C_v)$. Let us add the arc $(t,v)$ where $t \in C_v \setminus \{y\}$. We shall show that quasi-radius of the obtained graph $\Gamma^3$ is infinite.

No vertex in the set $C_v \cup \{v\}$ is a quasi-center in $\Gamma^3$, since the vertex $y$ is mutually reachable with no vertex in the set $(C_v \cup \{v\}) \setminus \{y\}$. No vertex in the set $A_v \cup B_v$ can be a quasi-center in the graph $\Gamma^3$ for the reason that the vertex $y$ is mutually reachable with no quasi-center in the graphs $\Gamma(A_v)$ and $\Gamma(B_v)$.

Suppose now $y \neq u$. Then by the above considerations, the arc $(u,y)$ is missing in the graph $\Gamma(C_v)$. Let us add the arc $(v,u)$ to the graph $\Gamma$ and verify that quasi-center of the obtained graph $\Gamma^4$ is infinite. Taking into account that the arc $(u,y)$ is missing in the graph $\Gamma(C_v)$ we see that no vertex in the set $A_v$ can be a quasi-center in the graph $\Gamma^4$, since the vertex $y$ is mutually reachable with no quasi-center in the graph $\Gamma(A_v)$. No vertex in the set $B_v \cup \{u\}$ can be a quasi-center in the graph $\Gamma^4$, since the vertex $u$ is mutually reachable with no quasi-center of the graph $\Gamma(B_v)$. And no vertex of the set $(C_v \cup \{v\}) \setminus \{u\}$ can be a quasi-center in $\Gamma^4$ for the reason that the vertex $v$ is mutually reachable with no vertex in the set $C_v \setminus \{u\}$. Thus, in the cases $\{r_m(\Gamma(A_v)) < \infty\} \& \{r_m(\Gamma(B_v)) = \infty\}$ and $\{r_m(\Gamma(A_v)) < \infty\} \& \{r_m(\Gamma(B_v)) < \infty\}$ the assumption $|C_v| \geq 2$ leads to a contradiction.

Let us show that the only unconsidered case $\{r_m(\Gamma(A_v)) = \infty\} \& \{r_m(\Gamma(B_v)) < \infty\}$ can be reduced to case (i). We note that in view of the symmetry of the function $r_m$ the graph $\Gamma$ is $r_m$-critical iff so is the graph $\overline{\Gamma}$ that is obtained by reversing the direction
of the arcs in the graph $\Gamma$. Let $\overline{A}_v$ be the totality of vertices of the graph $\overline{\Gamma}$ from which the vertex $v$ is reachable; $\overline{B}_v$ be the totality of vertices of the graph $\overline{\Gamma}$ reachable from the vertex $v$; $\overline{C}_v$ be the totality of all other vertices of the graph $\overline{\Gamma}$. It is obvious that

$$\overline{A}_v = B_v, \overline{B}_v = A_v, \overline{C}_v = C_v; \ r_m(\overline{A}_v) = r_m(\Gamma(B_v)), \ r_m(\overline{B}_v) = r_m(\Gamma(A_v)).$$

Suppose that $r_m(\Gamma(A_v)) = \infty$ and $r_m(\Gamma(B_v)) < \infty$. Hence $r_m(\Gamma(\overline{B}_v)) = \infty$ and $r_m(\Gamma(\overline{A}_v)) < \infty$.

Then case (i) considered above implies that $|\overline{C}_v| = |C_v| = 1$. ■

**Lemma 7** Let $\min\{r_m(\Gamma(A_v)), r_m(\Gamma(B_v))\} < \infty$, $C_v = \{y\}$. Denote by $A_y$ the set of vertices of the graph $\Gamma$ from which the vertex $y$ is reachable; by $B_y$ the set of vertices reachable from $y$; by $C_y$ the totality of all other vertices. Denote by $\Gamma(A_y), \Gamma(B_y), \Gamma(C_y)$ the subgraphs of the graph $\Gamma$ induced by the sets $A_y, B_y, C_y$, respectively. Then the graph $\Gamma$ satisfies one of the conditions listed below:

1) $A_y \neq \emptyset, B_y \neq \emptyset, r_m(A_y) = r_m(B_y) = \infty$ and the graph $\Gamma$ is of the form indicated in Lemma 5.

2) $A_y = \emptyset, B_y \neq \emptyset, r_m(B_y) = \infty$, the subgraph $\Gamma(B_y)$ is $r_m$-critical, the subgraph $\Gamma(C_y)$ is a transitive tournament, and an arc goes from each vertex of the set $C_y$ to each vertex of the set $B_y$.

3) $A_y \neq \emptyset, B_y = \emptyset, r_m(A_y) = \infty$, the subgraph $\Gamma(A_y)$ is $r_m$-critical, the subgraph $\Gamma(C_y)$ is a transitive tournament, and an arc goes from each vertex of the set $A_y$ to each vertex of the set $C_y$.

**Proof.** First we note that $A_y \subseteq A_v$ and $B_y \subseteq B_v$. Now we set about proving case 2 of the lemma.

Let $A_y = \emptyset, B_y \neq \emptyset$. Then the vertex $y$ is reachable from no vertex of the set $A_v$. In this case the addition of an arc, both endpoints of which belong to the set $A_v$, can not result in the appearance of a quasi-center in the graph $\Gamma$; hence it must decrease the number of bicomponents of the graph $\Gamma(\overline{A}_v)$. Thus, we conclude that the graph $\Gamma(\overline{A}_v)$ is a transitive tournament.

Consider next two cases.

(i) Let $r_m(B_v) = \infty$. Then the addition of an arc that goes from $y$ to some vertex of the set $B_v$ can result neither in the appearance of a quasi-center in the graph $\Gamma$ nor in reducing the number of its bicomponents. Hence $B_y = B_v$. But in this case $C_y = A_v \cup \{v\}$.

The facts that $\Gamma(A_v)$ is a transitive tournament and an arc goes from each vertex of the set $A_v$ to the vertex $v$ imply that $\Gamma(C_y) = \Gamma(A_v \cup \{v\})$ is a transitive tournament, and the graph $\Gamma$ satisfies condition 2 of the lemma.

(ii) Let $r_m(B_v) < \infty$. Denote by $W_v$ the totality of quasi-centers of the graph $\Gamma(B_v)$. The $W_v$ is a transitive tournament. Let $w$ be a vertex in $W_v$ that is reachable from each vertex of the set $W_v$. Denote by $\overline{W}_v$ the totality of vertices of the set $B_v$ from which the vertex $w$ is reachable. Evidently, no vertex in the set $\overline{W}_v$ is adjacent to the vertex $y$, since otherwise, the quasi-radius of the graph $\Gamma$ would be finite. It follows from the transitivity of the graph $\Gamma(B_v)$ and from the definition of the set $\overline{W}_v$ that an arc goes from each vertex of the set $\overline{W}_v$ to each vertex of the set $B_v \setminus \overline{W}_v$ (hence there are no arcs from the set $B_v \setminus \overline{W}_v$ to the set $\overline{W}_v$). It is easy to see that the addition of an arbitrary arc, both endpoints of which belong to $\overline{W}_v$, can not result in the appearance of a quasi-center in the graph $\Gamma$. Hence it must decrease the number of bicomponents of the graph $\Gamma(\overline{W}_v)$. 15
Hence $\Gamma(\overline{W_v})$ is a transitive tournament; taking into account that an arc goes from each vertex of $\overline{W_v}$ to each vertex of $B_v \setminus \overline{W_v}$, we obtain $\overline{W_v} = W_v$. It is easy to see that an arc goes from the vertex $y$ to each vertex of the set $B_v \setminus W_v$. Now let us investigate the graph $\Gamma(B_v \setminus W_v)$. The quasi-radius of the digraph is infinite; since otherwise, taking into account what has been said, we would obtain that quasi-radius of $\Gamma$ is finite. On the other hand, the addition of an arbitrary arc, whose both endpoints belong to $B_v \setminus W_v$, must result in the appearance of a quasi-center in the graph $\Gamma$ or in decreasing the number of bicomponents of the graph $\Gamma$. But there are no arcs that go from the vertices of the set $B_v \setminus W_v$. Hence the addition of an arbitrary arc, both endpoints of which belong to $B_v \setminus W_v$, must result either in decreasing the number of bicomponents of the graph $\Gamma(B_v \setminus W_v)$ or in the appearance of a quasi-center in it. Hence the graph $\Gamma(B_v \setminus W_v)$ is $r_m$-critical with infinite quasi-radius. In order to prove case 2 of the lemma, we just have to note that $C_y = W_v \cup \{v\} \cup A_v$ and the graph $\Gamma(C_y) = \Gamma(A_v \cup \{v\} \cup W_v)$ is a transitive tournament.

Case 3 of the lemma like in the proof of Lemma 6 can be reduced to case 2 with the help of considering the graph $\Gamma$.

It remains to consider case 1. Let $A_y \neq \emptyset$, $B_y \neq \emptyset$. Let $Z_v$ be the totality of quasi-centers of the graph $\Gamma(A_v)$, and $W_v$ be the totality of quasi-centers of the graph $\Gamma(B_v)$ (either of these sets can be empty, in this case the corresponding graph $\Gamma(A_v)$ or $\Gamma(B_v)$ is $r_m$-critical with $r_m = \infty$). Denote by $\overline{Z_v}$ the totality of vertices of the set $A_v$ that are reachable from at least one vertex of the set $Z_v$; denote by $\overline{W_v}$ the totality of vertices of the set $B_v$ that are reachable from at least one vertex of the set $W_v$. It is easy to show in the same manner as above that $\overline{Z_v} = Z_v$ and $\overline{W_v} = W_v$. Thus, we obtain that an arc goes from each vertex of the set $A_v \setminus Z_v$ to each vertex of the set $Z_v$, and an arc goes from each vertex of the set $W_v$ to each vertex of the set $B_v \setminus W_v$. Now taking into account that the sets $A_v \setminus Z_v$ and $B_v \setminus W_v$ contain no quasi-centers of the graphs $\Gamma(A_v)$ and $\Gamma(B_v)$ respectively; one obtains easily that $A_y = A_v \setminus Z_v$ and $B_y = B_v \setminus W_v$. Here $\Gamma(A_v \setminus Z_v)$ and $\Gamma(B_v \setminus W_v)$ are $r_m$-critical digraphs with $r_m = \infty$, and $C_y = Z_v \cup \{v\} \cup W_v$. But $\Gamma(Z_v)$ and $\Gamma(W_v)$ are transitive tournaments, an arc goes from each vertex of the set $Z_v$ to the vertex $v$, and from the vertex $v$ to each vertex of the set $W_v$. Hence $\Gamma(Z_v \cup \{v\} \cup W_v)$ is a transitive tournament. Thus, we obtain that $\Gamma(A_y)$ and $\Gamma(B_y)$ are $r_m$-critical with $r_m = \infty$, $\Gamma(C_y)$ is a transitive tournament; and an arc goes from each vertex of the set $A_y$ to each vertex of the set $C_y$, and from each vertex of the set $C_y$ to each vertex of the set $B_y$.

**Proof of Lemma 4.** It follows from Lemmas 3, 5, 7 that the Hertz graph of a weakly connected $r_m$-critical digraph with infinite quasi-radius is either isomorphic to the graph $D_4$ or is of the form represented in Figures 3, 4, 5.

Here the graphs $\Gamma(A_y)$ and $\Gamma(B_y)$ are $r_m$-critical with $r_m = \infty$, the graph $\Gamma(C_y)$ is a transitive tournament; the arrows mean that arcs go from each vertex of the set, from which an arrow issues, to each vertex of the set, which the arrow enters. The decomposition of the vertex set of the graph $D_4$ into the subsets $\{1, 2\}$ and $\{3, 4\}$ evidently satisfies the assertion of Lemma 4. Suppose the graph $\Gamma$ is of the form represented in Figure 3. Consider the decomposition of the vertex set of the graph into the subsets $A_y$ and $C_y \cup \{y\} \cup B_y$. The graph $\Gamma(A_y)$ is $r_m$-critical with $r_m = \infty$ and an arc goes from each vertex of the set $A_y$ to each vertex of the set $C_y \cup \{y\} \cup B_y$. We shall show that the graph $\Gamma(C_y \cup \{y\} \cup B_y)$ is $r_m$-critical with $r_m = \infty$. 


It is obvious that \( r_m(\Gamma(C_y \cup \{y\} \cup B_y)) = \infty \). The addition of an arbitrary arc, both endpoints of which belong to \( B_y \), results either in decreasing the number of bicomponents or in the appearance of a quasi-center. The addition of an arc going from \( B_y \) to \( C_y \cup \{y\} \) decreases the number of bicomponents. The addition of an arc connecting the vertex \( y \) to some vertex of the set \( C_y \) results in a vertex of \( C_y \) adjacent to \( y \) becoming a quasi-center in the graph \( \Gamma(C_y \cup \{y\} \cup B_y) \). Hence the graph \( \Gamma(C_y \cup \{y\} \cup B_y) \) is \( r_m \)-critical with \( r_m = \infty \).

Let us now consider the case corresponding to Figure 4. In this case the desired decomposition is given by the sets \( C_y \cup \{y\} \) and \( B_y \); since the graph \( \Gamma(C_y \cup \{y\}) \) is isomorphic to the graph \( \Gamma_k \) for some \( k \), and hence it is \( r_m \)-critical with \( r_m = \infty \), and \( B_y \) has the property by the hypothesis. The case corresponding to Figure 5 can be reduced to the case corresponding to Figure 4 by reversing the direction of the arcs.

Now we can proceed with proving the principal result of the subsection.

**Theorem 4** Let \( \Gamma \) be the Hertz graph of a \( r_m \)-critical digraph with infinite quasi-radius. Then the vertex set of the graph \( \Gamma \) can be partitioned into \( s \) disjoint subsets \( X_1, \ldots, X_s \) for some \( s \) such that

a) \( 1 \leq s \leq \left\lfloor \frac{k}{2} \right\rfloor \) where \( k \) is the number of vertices in the graph \( \Gamma \);

b) \( |X_i| = k_i \geq 2 \) and the graph \( \Gamma(X_i) \) is isomorphic to the graph \( \Gamma_{k_i,0} \) for \( i = 1, \ldots, s \);

c) an arc goes from each vertex of the set \( X_i \) to each vertex of the set \( X_j \) for \( 1 \leq i < j \leq s \);

d) there are no arc, besides those named, in the graph \( \Gamma \).

**Proof.** If the graph \( \Gamma \) is not weakly connected, then by Lemma 2, it is isomorphic to the graph \( \Gamma_{k,0} \). Hence it satisfies the assertion of the theorem for \( s = 1 \). Therefore in the sequel we may confine ourselves to considering only weakly connected graphs.

First we shall show that the vertex set of the graph \( \Gamma \) can be partitioned into two subsets \( X_1 \) and \( M_1 \) such that the graph \( \Gamma(X_1) \) is isomorphic to the graph \( \Gamma_{t,0} \) for some \( t \), the graph \( \Gamma(M_1) \) is \( r_m \)-critical with \( r_m = \infty \), and an arc goes from each vertex of the set \( X_1 \) to each vertex of the set \( M_1 \). By Lemma 4, the vertex set of the graph \( \Gamma \) can be partitioned into two subsets \( Y_1 \) and \( N_1 \) such that the graphs \( \Gamma(Y_1) \) and \( \Gamma(N_1) \) are
$r_m$-critical with $r_m = \infty$ and an arc goes from each vertex of the set $Y_1$ to each vertex of the set $N_1$. If the graph $\Gamma(Y_1)$ is not weakly connected, then the assertion is proved; otherwise, applying Lemma 4 to the graph $\Gamma(Y_1)$ we obtain a decomposition of the set $Y_1$ into subsets $Y_2$ and $N_2$ with corresponding properties. It is easy to see that the graph $\Gamma(N_2 \cup N_1)$ is $r_m$-critical with $r_m = \infty$. We continue this procedure until at some step $m$ we obtain a graph $\Gamma(Y_m)$ that is not weakly connected and hence is isomorphic to the graph $\Gamma|_{Y_m,0}$. It is easy to show, by induction, that the graph $\Gamma(N_1 \cup \ldots \cup N_m)$ is $r_m$-critical with $r_m = \infty$. Setting $X_1 = Y_m$ and $M_1 = N_1 \cup \ldots \cup N_m$ completes the proof of the assertion.

Now we continue proving the theorem. If the graph $\Gamma(M_1)$ is not weakly connected, then the graph $\Gamma(M_1)$ is isomorphic to the graph $\Gamma|_{|M_1|,0}$, and the theorem is proved. Otherwise, applying the preceding assertion to the graph $\Gamma(M_1)$ we obtain a decomposition of the vertex set of the graph $\Gamma$ into three subsets $X_1, X_2, M_2$; here the graph $\Gamma(X_2)$ is isomorphic to the graph $\Gamma|_{|X_2|,0}$, the graph $\Gamma(M_2)$ is $r_m$-critical with $r_m = \infty$, and an arc goes from each vertex of the set $X_1$ to each vertex of the set $X_2 \cup M_2$ and from each vertex of the set $X_2$ to each vertex of the set $M_2$. Continue this process by induction until at some step $s$ we obtain that the graph $\Gamma(M_s)$ is not weakly connected. Then setting $X_s = M_s$ we obtain that the totality of sets $\{X_1, \ldots, X_s\}$ is the desired decomposition of the vertex set of the graph $\Gamma$.

Note that if we introduce a relation $T$ on the vertex set of the graph $\Gamma$ in the following way:

$$xTy \iff \text{both vertices } x \text{ and } y \text{ belong to one set in the decomposition } \{X_1, \ldots, X_s\};$$

then it follows from Theorem 4 that the factor graph of the graph $\Gamma$ relative to the relation $T$ is isomorphic to the graph $\Gamma_s$.

Now we cite a few corollaries of Theorem 4.

**Corollary 1** Let $G$ be an $n$-vertex digraph with $k \geq 2$ bicomponents. If the number of arcs in $G$ is greater than $n(n-k-1) + \left\lfloor \frac{k^2}{2} \right\rfloor$, then $r_m(G) < \infty$.

**Proof.** Denote by $\Gamma_{k,s;k_1,\ldots,k_s}$ a $k$-vertex digraph, whose vertex set is partitioned into $s$ nonintersecting classes $Y_1, \ldots, Y_s$, $|Y_\alpha| = k_\alpha \geq 2$, $\alpha = 1, \ldots, s$; the subgraph induced by the set $Y_\alpha$ is isomorphic to $\Gamma_{k_\alpha,0}$, and, in addition, if $\alpha_1 < \alpha_2$ then an arc goes from each vertex of the class $Y_{\alpha_1}$ to each vertex of the class $Y_{\alpha_2}$; there are no other arcs in the digraph $\Gamma_{k,s;k_1,\ldots,k_s}$.

Obviously, an $n$-vertex digraph $D$ of infinite quasi-radius having the greatest number of arcs is $r_m$-critical. By Theorem 4, the Hertz graph of the graph $D$ is isomorphic to the graph $\Gamma_{k,s;k_1,\ldots,k_s}$ for some $s, k_1, \ldots, k_s$. Let $\{Y_1, \ldots, Y_s\}$ be the decomposition of the vertex set of the graph $\Gamma_{k,s;k_1,\ldots,k_s}$ described above. The graph induced by the set $Y_\alpha$ is isomorphic to $\Gamma_{k_\alpha,0}$. Let $D_\alpha$ be the subgraph of the graph $D$ induced by bicomponents corresponding to vertices of the set $Y_\alpha$. It is easy to see that all bicomponents of the graph $D_\alpha$, excluding perhaps one not corresponding to the vertex 1 of the graph $\Gamma_{k_\alpha,0}$, are one-vertex. If it is not so, then we replace the subgraph $D_\alpha$ of $D$ by a graph $D'_\alpha$ possessing the above property. Then evidently, the number of arcs connecting vertices of the subgraph $D_\alpha$ to other vertices of the graph $D$ equals the number of arcs connecting vertices of the subgraph $D'_\alpha$ to other vertices of the graph $D$; and the number of arcs in the graph
\[\text{D}a'\] is greater than the number of arcs in the graph \(D_\alpha\). Suppose that bicomponents corresponding to the vertices \(k_\alpha\) of the graphs \(\Gamma_{k_\alpha,0}\) are the only bicomponents that can have more than one vertex. Let \(|Y_i| \geq 4\). Consider then a new decomposition \(\{Y_1, \ldots, Y_{l-1}, Y'_1, Y''_1, Y_{l+1}, \ldots, Y_s\}\) of the vertex set of the graph \(\Gamma_{k,s;k_1,\ldots,k_s}\). Here \(Y'_1\) consists of the two vertices of the set \(Y_i\) corresponding to the vertices 2, 3 of the graph \(\Gamma_{k_1,0}\), and \(Y''_1\) consists of the rest vertices of the set \(Y_i\). The difference between the graphs \(\Gamma_{k,s;k_1,\ldots,k_s}\) and \(\Gamma_{k,s+1;k_1,\ldots,k_{l-1},2,k_l-2,k_{l+1},\ldots,k_s}\) is that the arc \((2,3)\) is missing in the subgraph induced by the set \(Y_i\) in the graph \(\Gamma_{k,s;k_1,\ldots,k_s}\); but the arcs \((2,1)\) and \((3,1)\) that are missing in the graph \(\Gamma_{k,s;k_1,\ldots,k_s}\) are present in \(\Gamma_{k,s+1;k_1,\ldots,k_{l-1},2,k_l-2,k_{l+1},\ldots,k_s}\). Taking into account that bicomponents of the graph \(D\) corresponding to the vertices 1, 2, 3 of the graph \(\Gamma_{k_1,0}\) are one-vertex, we see that the number of arcs in the graph \(D\) increases by one after the above operation. Thus, we have shown that \(|Y_\alpha| \leq 3\) \((\alpha = 1, \ldots, s)\) in the Hertz graph of the graph \(D\). Suppose now that the Hertz graph \(\Gamma\) of the graph \(D\) contains two subsets \(Y_i\) and \(Y_j\) such that \(|Y_i| = |Y_j| = 3\). Let us consider the sets \(\{2^i, 2^j\}, \{1^i, 3^j\}, \{1^j, 3^i\}\) instead of the sets \(Y_i\) and \(Y_j\) where \(\alpha^\beta\) is the vertex of the graph \(\Gamma\) corresponding to the vertex \(\alpha\) of the graph \(\Gamma_{k_\alpha,0}\). Here if \(i < j\) then the arcs \((2^i, 2^j), (1^i, 2^j), (3^i, 2^j)\) vanish from the graph \(\Gamma\), and the arcs \((2^i, 1^i), (2^j, 1^i), (2^i, 3^i), (2^j, 1^j)\) appear in it. Note that the bicomponents of the graph \(D\) corresponding to the vertices 1, 2, 1, 2 are one-vertex. Hence the described operation results in increasing the number of arcs of the graph \(D\) by one.

The above considerations imply that if an \(n\)-vertex graph \(D\) of infinite quasi-radius with \(k\) bicomponents contains the greatest number of arcs, then its Hertz graph is isomorphic to the graph \(\Gamma_{k,s;k_1,\ldots,k_s}\) where \(s = \left\lfloor \frac{k}{2} \right\rfloor\). It is easy to see that the number of arcs in the \(n\)-vertex digraph \(D\) whose Hertz graph is isomorphic to \(\Gamma_{k,s;k_1,\ldots,k_s}\), \(s = \left\lfloor \frac{k}{2} \right\rfloor\) is maximal, if all bicomponents of the graph \(D\), excluding perhaps one not corresponding to the vertex 1 of the subgraph of \(\Gamma_{k,s;k_1,\ldots,k_s}\) induced by the set \(Y_i\), \(|Y_i| = 3\), are one-vertex. In this case a simple calculation shows that the graph \(D\) contains \(n(n-k-1) + \left\lfloor \frac{k^2}{2} \right\rfloor\) arcs.

**Corollary 2** If a digraph \(G\) with \(k\) bicomponents and infinite quasi-radius has the property that the addition of an arbitrary missing arc converts it into a digraph with finite quasi-radius, then

1. \(k = 2l\);
2. the Hertz graph of \(G\) is isomorphic to \(\Gamma_{k,k/2,2,\ldots,2}\).

**Proof.** Suppose that the Hertz graph of the digraph \(G\) is isomorphic to \(\Gamma_{k,s;k_1,\ldots,k_s}\) and \(\exists i (k_i \geq 3)\). As we have stated, the subgraph induced by the set \(Y_i\) is isomorphic to \(\Gamma_{k_i,0}\). It is easy to see that if we add the arc \((3^i, 2^i)\) to the Hertz graph of the graph \(G\), then the quasi-radius of the obtained graph is infinite. Hence \(\forall i (k_i = 2)\) and \(k = 2l\).

Let \(\mathcal{P}_n(n,k)\) denote the number of nonisomorphic \(n\)-vertex \(r_m\)-critical digraphs with \(k\) bicomponents and infinite quasi-radius.

**Corollary 3**

\[
\mathcal{P}(n,k) = \sum_{l=1}^{k/2} \sum_{s=3(l-k)_+}^l \sum_{t=2s}^{n-k+2s} \prod_{p_1+\ldots+p_s=t} \left( \frac{l}{s} \right) \left( \frac{k-2l-1}{l-s-1} \right) \left( \frac{n-t-1}{k-2s-1} \right) \left\{ \frac{s}{2} \right\},
\]

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where \( m_+ = \max\{m, 0\} \), \( \binom{0}{k} \equiv 0, k \neq 0, 1, k = 0, \binom{-1}{k} \equiv 1 \), and \( \prod_{i=1}^{0} \left\lfloor \frac{p_i}{2} \right\rfloor \equiv 1 \).

Proof. First we note that if the Hertz graphs of digraphs \( G_1 \) and \( G_2 \) are nonisomorphic, then the digraph \( G_1 \) and \( G_2 \) are not isomorphic either. Let us count the number of nonisomorphic \( n \)-vertex \( r_m \)-critical digraphs with \( k \) bicomponents and \( r_m = \infty \) such that their Hertz graphs have \( l \) blocks \( Y_1, \ldots, Y_l \), exactly \( s \) of which are two-vertex. This number equals the number of ways of distributing \( k \) identical particles in \( l \) distinct boxes such that each box contains at least two particles and exactly \( s \) boxes contain two particles each. The latter number equals (see [1])

\[
\binom{l}{s} \binom{k - 2l - 1}{l - s - 1}.
\]

We note that distinct distributions (by number) of vertices in nontwo-vertex blocks correspond to nonisomorphic graphs.

Let us consider some two-vertex block \( Y_i \) that contains \( p_i \) vertices. Suppose the position of vertices in all other blocks is fixed. Then varying the number of vertices only in the vertices of the block \( Y_i \) we can obtain \( \left\lfloor \frac{p_i}{2} \right\rfloor \) nonisomorphic graphs. Let the number of vertices of the digraph contained in the two-vertex blocks equal \( p_1 + \ldots + p_s = t \). The number of ways of distributing the rest \( n - t \) vertices in the vertices of nontwo-vertex blocks equals \( \binom{n - t - 1}{k - 2s - 1} \); and the number of ways of distributing \( t \) vertices in the vertices of \( s \) two-vertex blocks (such that the obtained graphs are nonisomorphic) equals

\[
\sum_{p_1+\ldots+p_s=t} \left\{ \prod_{i=1}^{s} \left\lfloor \frac{p_i}{2} \right\rfloor \right\}.
\]

Finally we obtain that the number of nonisomorphic \( n \)-vertex \( r_m \)-critical digraphs with \( k \) bicomponents and \( r_m = \infty \), the Hertz graphs of which have \( l \) blocks exactly \( s \) of which are two-vertex, equal

\[
\sum_{p_1+\ldots+p_s=t} \binom{l}{s} \binom{k - 2l - 1}{l - s - 1} \binom{n - t - 1}{k - 2s - 1} \left\{ \prod_{i=1}^{s} \left\lfloor \frac{p_i}{2} \right\rfloor \right\}.
\]

Summing the quantity over \( t, s, l \) we obtain what we needed to prove.

Let \( \xi(n, k) \) denote the number of distinct \( r_m \)-critical digraphs with \( k \) bicomponents and infinite quasi-radius that can be constructed on \( n \) numbered vertices.

Corollary 4

\[
\xi(n, k) = \sum_{l=1}^{[k/2]} \sum_{s=(3l-k)_+}^{n-k+2s} \sum_{t=2s}^{n-k+2s} \sum_{p_1+\ldots+p_s=t} \binom{l}{s} \binom{k - 2l - 1}{l - s - 1} \binom{n - t}{k - 2s} \times (k - 2s)! S(n - t, k - 2s) \frac{t!}{p_1! \ldots p_s!} \left\{ \prod_{i=1}^{s} \left( 2^{p_i-1} - 1 \right) \right\},
\]

where \( S(u, v) \) are the Stirling numbers of the second kind.

Proof. The proof of the corollary is analogous to the proof of Corollary 3; we just note that \( (k - 2s)! S(n - t, k - 2s) \) is the number of ways of distributing \( n - t \) numbered vertices in \( k - 2s \) bicomponents contained in nontwo-vertex blocks;
\[
\sum_{p_1 + \ldots + p_s = t} \frac{t!}{p_1! \ldots p_s!} \left\{ \prod_{i=1}^{s} (2^{p_i} - 1) \right\} \text{ is the number of ways of distributing } t \text{ numbered vertices in } 2s \text{ bicomponents contained in two-vertex blocks, and there are } \binom{n}{n-t} \text{ ways to choose } t \text{ vertices contained in bicomponents of two-vertex blocks.} \]

3. On maximal digraphs of finite radius and quasi-diameter

In the preceding sections we have characterized up to isomorphism critical digraphs
with infinite values of \(d, d_m, r, r_m\). \(d\)-critical digraphs of finite diameter were characterized
up to isomorphism by L.S. Mel’nikov [11]. The solution to the corresponding problem
for critical digraphs with finite values of \(d_m, r, r_m\) has not been found yet. However, we
succeeded in obtaining the least upper bounds on the number of arcs in \(n\)-vertex digraphs
with given finite values of radius and quasi-diameter, and characterizing the digraphs for
which the bounds are achieved. We should remark in this connection that the least upper
bound on the number of arcs in an \(n\)-vertex digraph of given finite radius was found by
S.M. Ismailov [10]. However, there are significant ambiguities in the proof of the result in
[10] and it seems to us that the method used by author can’t lead to the proof. The least
upper bound on the number of edges in an ordinary undirected \(n\)-vertex graph of given
radius was obtained by V.G. Vizing [13].

I. Let \(G\) be an \(n\)-vertex directed graph without loops and \(r(G) = k < \infty\). We remind
that the digraph \(G\) is said to be maximal if it has the maximum number of arcs among
all \(n\)-vertex digraphs of radius \(k\).

In this subsection we shall obtain the least upper bound on the number of arcs in an
\(n\)-vertex digraph of radius \(k\), and characterize all maximal digraphs.

Let a digraph \(G = (X, U)\) have radius \(k\). A vertex \(x_1\) of the graph \(G\) is called a center,
if \(\max_{y \in X} \rho(x_1, y) = r(G) = k\). There exists a vertex \(x_{k+1}\) in \(G\) such that \(\rho(x_1, x_{k+1}) = k\).

Let \(\{x_1, \ldots, x_{k+1}\}\) be a naturally ordered totality of vertices of some shortest directed path
going from \(x_1\) to \(x_{k+1}\). Let vertices \(x_1, \ldots, x_t\) belong to one bicomponent, and vertices \(x_{t+1}, \ldots, x_{k+1}\) do not belong to this bicomponent. Denote the set \(\{x_1, \ldots, x_{k+1}\}\) by \(M\).

Partition the set \(X \setminus M\) into two subsets \(S\) and \(Y\) in the following way: the set \(S\) consists of the vertices that belong to the bicomponent of \(G\) containing the vertex \(x_1\), and the set \(Y\) consists of all other vertices of the set \(X \setminus M\). Let the set \(S\) contain \(s\) vertices.

Denote by \(g(n, k)\) the maximum number of arcs in a directed \(n\)-vertex graph of radius \(k\). It is obvious that \(g(n, 1) = n(n-1)\), and \(g(n, 2) = n(n-2)\); and a digraph of radius two is maximal iff the outdegree of each vertex of the graph equals \(n-2\). Thus, it remains to consider the case \(k \geq 3\).

Let us define a function \(F(n, k, s, t)\) in the following way:
\[
F(n, k, s, t) = n(n-k) + \frac{k^2 - k - 2}{2} + \left[ -n(s + t + 2) + s^2 + ts + t^2 + 3k + 2 + (n-k-s-1) \max\{t, 3\} + s \max\{k, t+2\} \right].
\]
Lemma 8 If $1 \leq t \leq k$ then the number of arcs in an $n$-vertex digraph of radius $k$ does not exceed $\max_{1 \leq t \leq k} \max_{0 \leq s \leq n-k-1} F(n, k, s, t)$.

Proof. Let us consider Figure 6.

(i) Let us bound the number of arcs connecting the sets $M$ and $Y$. Since all vertices of the set $Y$ do not belong to the bicomponent containing the vertex $x_1$, no arc goes from a vertex of the set $Y$ to a vertex of the set $\{x_1, ..., x_t\}$. Let $y \in Y$ and the vertex $y$ be adjacent to some vertex $x_l \in M$. Then, a fortiori, no arc goes from the vertex $y$ to the vertices of $M$ whose numbers are greater than $l + 2$. Taking this into account, one obtains easily that the number of arcs connecting the vertex $y$ to vertices of the set $M$ does not exceed $(k + 1 - t) + \max\{t, 3\}$; hence the number of arcs connecting the sets $M$ and $Y$ does not exceed $(n - s - k - 1)(k + 1 - t + \max\{t, 3\})$. In addition, the number of arcs in the subgraph induced by the set $Y$ does not exceed $(n - s - k - 1)(n - s - k - 2)$.

(ii) Let us bound the number of arcs incident to the set $S$. Let $z$ be an arbitrary vertex in the set $S$.

a) Suppose that no arc goes from vertices of the set $M$ to the vertex $z$. Here if $k + 1$ arcs go from the vertex $z$ to the set $M$, then there is a vertex in the set $(S \cup Y) \setminus \{z\}$ that is not a terminal vertex of an arc issuing from $z$; since otherwise, the vertex $z$ would be a center in $G$, and the radius of $G$ would equal 1, contrary to the hypothesis. Hence in this case the number of arcs going from the vertex $z$ to other vertices of $G$ does not exceed $k + (s - 1) + (n - k - s - 1) = n - 2$.

b) Suppose there are arcs in $G$ that go from the set $M$ to the vertex $z$. We note that the vertices $x_1, ..., x_l$ are the only vertices with this property. Moreover, if an arc goes from a vertex $x_l$ to the vertex $z$, then no arc can go from the vertex $z$ to the vertex $x_i$ for $i > l + 2$; since otherwise, $\rho(x_l, x_{k+1})$ would be less than $k$. So it is easy to obtain that the number of arcs connecting the vertex $z$ to the vertices of $M$ does not exceed $t + 3$. If this number equals $t + 3$ then arcs go from the vertex $z$ to the vertices $x_1, x_2, x_3, \max_{1 \leq i \leq k+1} \rho(z, x_i) < k$; and in order that radius of the graph $G$ be greater or equal to $k$ it is necessary that at least one arc of the kind $(z, u)$ where $u \in (S \cup Y) \setminus \{z\}$ is missing in $G$. Then the total number of arcs connecting the vertex $z$ to the set $M$ and going from $z$ to the set $S \cup Y$ does not exceed $(t + 2) + (s - 1) + (n - s - k - 1) = n + t - k$; and the number of arcs incident to the set $S$ does not exceed $s(n + t - k)$ in this case. Combining cases a) and b) we see that the number of arcs incident to the set $S$ does not exceed $s \max\{n - 2, n + t - k\}$.

(iii) Let us now bound the number of arcs in the subgraph induced by the set $M$. Since $t \leq k$, hence no arc goes from a vertex whose number is greater than $t$ to a vertex whose number is less or equal to $t$. Hence the number of arcs in this subgraph does not
exceed

\[ k + (t-1)+(t-2)+\ldots+1+(k-t)+(k-t-1)+\ldots+1 = k + \frac{t(t-1)}{2} + \frac{(k-t+1)(k-t)}{2}. \]

It is easy to see that in these three cases we have considered all arcs in the graph \( G \). Hence if \( t < k \) and \( s \) is fixed \((0 \leq s \leq n-k-1)\), then the number of arcs in \( G \) does not exceed

\[
(n-s-k-1)(k+1-t+\max\{t,3\}) + (n-s-k-1)(n-s-k-2) + \\
+k + t^2 - t - kt + \frac{k^2 + k}{2} + s \max\{n-2, n + t - k\} = \\
= n(n-k) + \frac{k^2-k-2}{2} + [(n-s-k-1)\max\{t,3\} + s \max\{n-2, n + t - k\} + \\
+3k + 2 + s^2 + st + sk + 2s - 2sn + t^2 - tn - 2n] = F(n, k, s, t). \]

**Lemma 9** For \( 1 \leq t \leq k-1 \)

\[ F(n, k, s, t) \leq n(n-k) + \frac{k^2-k-2}{2}, \]

and an equality holds iff \( t = 1 \) and \( s = 0 \).

**Proof.** a) Suppose \( t < 3, k \geq t+2 \). Then

\[
F(n, k, s, t) = n(n-k) + \frac{k^2-k-2}{2} + 3n - 3s - 3k - 3 + sn - 2s + \\
+3k + 2 + s^2 + sk + st + 2s - 2sn + t^2 - tn - 2n = \\
= n(n-k) + \frac{k^2-k-2}{2} - s(n-s-k-t+3) - (t-1)(n-t-1) \leq \\
\leq n(n-k) + \frac{k^2-k-2}{2}.
\]

Here an equality in the last inequality holds iff \( t = 1 \) and \( s = 0 \).

b) Suppose \( t \geq 3, k \geq t+2 \). Then

\[
F(n, k, s, t) = ns - st - kt - t + sn - 2s + 3k + 2 + s^2 + sk + \\
+st + 2s - 2sn + t^2 - tn - 2n + n(n-k) + \frac{k^2-k-2}{2} = \\
= n(n-k) + \frac{k^2-k-2}{2} - s(n-k-s) - 2(n-k-1) - (t-1)(k-t) < \\
< n(n-k) + \frac{k^2-k-2}{2}.
\]

c) Suppose \( t < 3, k < t+2 \). Then \( t = 2, k = 3 \) and hence

\[
F(n, 3, s, 2) = n(n-3) + \frac{3^2 - 3 - 2}{2} - n + 3 - s(n-s-3) < \\
< n(n-3) + \frac{3^2 - 3 - 2}{2} = n^2 - 3n + 2.
\]
d) Suppose \( t \geq 3, k < t + 2 \). Since \( t \leq k - 1 \) in order to prove the lemma it suffices to consider the case \( t = k - 1 \). Then

\[
F(n, k, s, k - 1) = n(n - k) + \frac{k^2 - k - 2}{2} + \left[-n(s + k + 1) + s^2 + (k - 1)s + (k - 1)^2 + 3k + 2 + (n - k - s - 1)(k - 1) + s(k + 1)\right] = \\
= n(n - k) + \frac{k^2 - k - 2}{2} - s(n - s - k - 1) - 2n + k - 4 < \\
< n(n - k) + \frac{k^2 - k - 2}{2}. \quad \blacksquare
\]

**Lemma 10** Let \( G \) be an \( n \)-vertex digraph with radius \( k < \infty \), \( A_1 \) be the bicomponent that contains a center. Then the outdegree of any vertex in \( A_1 \) does not exceed \( n - k \).

**Proof.** Let \( v \) be an arbitrary vertex in the bicomponent \( A_1 \), and \( B_v \) be the totality of those vertices in \( G \) to which arcs go from \( v \). \( |B_v| = m \) is the outdegree of the vertex \( v \). We note that any vertex in the graph \( G \) is reachable from the vertex \( v \), and the shortest path from \( v \) to an arbitrary vertex in \( G \) contains at most one vertex of the set \( B_v \). Hence the path contains at most \( n - m + 1 \) vertices, and its length does not exceed \( n - m \). It means \( r(G) \leq n - m \); since \( r(G) = k \) we have \( m \leq n - k \). \( \blacksquare \)

**Remark.** It follows from Lemma 10 that the number of arcs in an \( n \)-vertex biconnected digraph of radius \( k \) does not exceed \( n(n - k) \).

**Theorem 5** The following equalities hold:

\[
g(n, 1) = n(n - 1), \quad g(n, 2) = n(n - 2), \\
g(n, k) = n(n - k) + \frac{k^2 - k - 2}{2} \quad \text{for } k \geq 3.
\]

**Proof.** We only need to prove the third relation.

Denote the quantity \( n(n - k) + \frac{k^2 - k - 2}{2} \) by \( \varphi(n, k) \). First we shall show that \( g(n, k) \leq \varphi(n, k) \). It follows from Lemmas 8, 9 that if \( t < k \) in an \( n \)-vertex digraph of radius \( k \), then this inequality holds. Partition the vertex set of the graph \( G \) into two subsets \( A_1 \) and \( A_2 \), where \( A_1 \) is the bicomponent containing center, and \( A_2 \) consists of all other vertices. If \( t \geq k \) then \( |A_1| = p \geq k \). The number of arcs in the subgraph induced by the set \( A_2 \) does not exceed \( (n - p)(n - p - 1) \). Now we note that if we sum outdegrees of all vertices in the set \( A_1 \), then we obtain the number of all other arcs in the graph \( G \). But by Lemma 10, the outdegree of any vertex in the set \( A_1 \) does not exceed \( n - k \). Hence the number of arcs in the graph \( G \) does not exceed \( p(n - k) + (n - p)(n - p - 1) = n(n - k) - (p - k + 1)(n - p) \). Taking into account that in our case \( n \geq p \geq k \) we obtain that the latter expression does not exceed \( n(n - k) \). The inequality \( g(n, k) \leq \varphi(n, k) \) is proved.

In order to prove the reverse inequality we consider the following \( n \)-vertex digraph \( D_0 \): The vertex set of the digraph is partitioned into \( k + 1 \) disjoint subsets \( X_1, \ldots, X_{k+1} \) where \( |X_1| = |X_2| = \ldots = |X_{k+1}| = 1 \) and \( |X_2| = n - k \). An arc goes from each vertex in \( X_i \) to each vertex in \( X_{i+1}, i = 1, \ldots, k \); if \( 1 < i < j \leq k + 1 \) then an arc goes from each vertex in \( X_j \) to each vertex in \( X_i \). The subgraph induced by the set \( X_2 \) is complete symmetric. There are no other arcs in the digraph \( D_0 \). Evidently, the radius of the graph equals \( k \).
A simple calculation shows that the number of arcs in this graph equals \( \varphi(n, k) \). This proves the inequality \( g(n, k) \geq \varphi(n, k) \). \( \blacksquare \)

Now we take up characterizing maximal digraphs of given radius. It follows from the proof of Theorem 5 that in a maximal digraph \( S = \emptyset, t = 1 \), the subgraph induced by the set \( Y \) is complete symmetric, and the subgraph induced by the set \( M \) has the following properties: an arc goes from the vertex \( x_i \) to the vertex \( x_{i+1} \); if \( 1 < i < j \leq k+1 \) then an arc goes from the vertex \( x_j \) to the vertex \( x_i \); there are no other arcs in the subgraph. Let \( x_i \) be the first vertex from which arcs go to the set \( Y \), and \( Y_i \) be the totality of vertices in \( Y \) to which arcs go from \( x_i \). Thus, if \( i = 1 \) then each vertex of the set \( Y_1 \) is connected by a pair of antiparallel arcs to the vertices \( x_2, x_3, x_4 \), and an arc goes from each vertex of the set \( \{x_4, \ldots, x_{k+1}\} \) to each vertex of the set \( Y_1 \). Each vertex in the set \( Y \setminus Y_1 \) is connected by a pair of antiparallel arcs to each of the vertices \( x_2, x_3, x_4 \), and an arc goes from each vertex in the set \( \{x_5, \ldots, x_{k+1}\} \) to each vertex in the set \( Y \setminus Y_1 \). Partition the vertex set of this graph into \( k + 1 \) subsets \( X_1, \ldots, X_{k+1} \) as follows:

\[
X_1 = \{x_1\}, \quad X_2 = \{x_2\} \cup Y_1, \quad X_3 = \{x_3\} \cup (Y \setminus Y_1), \quad X_4 = \{x_4\}, \ldots, \quad X_{k+1} = \{x_{k+1}\}.
\]

We see that the subgraphs induced by each of these subsets are complete symmetric; an arc goes from each vertex in the set \( X_i \) to each vertex in the set \( X_{i+1} \); if \( 1 < i < j \leq k+1 \) then an arc goes from each vertex in the set \( X_j \) to each vertex in the set \( X_i \); there are no other arcs in this graph.

Let \( x_i \) be the first vertex in the set \( M \) from which arcs go to the set \( Y \), and \( i > 1 \). Then each vertex in the set \( Y_i \) has the following properties: an arc goes from each vertex in the set \( Y_i \) to each vertex in the set \( \{x_2, \ldots, x_{i-1}\} \); each vertex in the set \( Y_i \) is connected by a pair of antiparallel arcs to the vertices \( x_i, x_{i+1}, x_{i+2} \); and an arc goes from each vertex in the set \( \{x_{i+3}, \ldots, x_{k+1}\} \) to each vertex in the set \( Y_i \). Each vertex in the set \( Y \setminus Y_i \) has the following properties: an arc goes from each vertex in the set \( Y \setminus Y_i \) to each vertex in the set \( \{x_2, \ldots, x_i\} \), each vertex in the set \( Y \setminus Y_i \) is connected by a pair of antiparallel arcs to each vertex in the set \( \{x_{i+1}, x_{i+2}, x_{i+3}\} \); an arc goes from each vertex in the set \( \{x_{i+4}, \ldots, x_{k+1}\} \) to each vertex in the set \( Y \setminus Y_i \). Let us partition the vertex set of this graph into nonempty disjoint subsets \( X_1, \ldots, X_{k+1} \) as follows:

\[
X_1 = \{x_1\}, \ldots, X_i = \{x_i\}, \quad X_{i+1} = \{x_{i+1}\} \cup Y_i, \\
X_{i+2} = \{x_{i+2}\} \cup (Y \setminus Y_i), \quad X_{i+3} = \{x_{i+3}\}, \ldots, X_{k+1} = \{x_{k+1}\}.
\]

The decomposition possess the same properties as those of the decomposition in the preceding case. Thus we have proved

**Theorem 6** All \( n \)-vertex maximal digraphs \( D \) of finite radius \( k \) are exhausted by the following:

a) if \( k = 1 \), then \( D \) is a complete symmetric digraph;

b) if \( k = 2 \), then \( D \) is a digraph such that the outgoing semidegree of each vertex equal \( n - 2 \);

c) if \( k \geq 3 \), then the vertex set of \( D \) can be partitioned into \( k + 1 \) nonempty disjoint subsets \( X_1, \ldots, X_{k+1} \) such that \( |X_1| = |X_{k+1}| = 1 \), all other subsets, excluding perhaps two with consecutive indices, are singletons; if \( 1 < i < j \leq k \), then an arc goes from each vertex of the set \( X_j \) to each vertex of the set \( X_i \); an arc goes from each vertex of the set \( X_i \) to each vertex of the set \( X_{i+1} \), \( i = 1, \ldots, k \); the subgraphs induced by each of the subsets \( X_i \) are complete symmetric; there are no other arcs in this graph besides those listed above.
Corollary 1 For \( k \geq 3 \) the number of nonisomorphic \( n \)-vertex maximal digraph of radius \( k \) equals \( (n - k - 1)(k - 2) + 1 \).

Proof. It follows from Theorem 6 that the number of such graphs equals the number of ways of distributing \( n - 2 \) identical particles in \( k - 1 \) nonnumbered boxes without empty ones such that each box, excluding perhaps two neighboring ones, contains one particle. For \( n = k + 1 \) the number equals 1; for \( n = k + 1 \) it equals \( k - 1 \). For \( n \geq k + 3 \) the number of ways such that there is a box containing \( n - k - 1 \) particles equals \( k - 1 \); and the number of ways such that exactly two neighboring boxes contain more than one particle equals \( (n - k - 2)(k - 2) \). Hence the total number equals \( (n - k - 2)(k - 2) + k - 1 = (n - k - 1)(n - 2) + 1 \). The last formula covers all the three cases. ■

Corollary 2 Let \( \chi(n,k) \) denote the number of maximal \( n \)-vertex graphs of radius \( k \) that can be constructed on given \( n \) numbered vertices. Then

\[
\chi(n, k) = \begin{cases} 
1 & \text{for } k = 1, \\
(n - 1)^n & \text{for } k = 2, \\
(k - 1)! \binom{n}{k} + (k - 2)(k - 1)! \binom{n}{k - 1} (2^{n-k+1} - 2n + 2k - 4) & \text{for } k \geq 3.
\end{cases}
\]

The proof of the corollary is analogous to the proof of Corollary 1.

II. In this subsection we shall obtain the least upper bound on the number of arcs in an \( n \)-vertex digraph of quasi-diameter \( k < \infty \), and characterize maximal digraphs of finite quasi-diameter.

We note that the only maximal digraph of quasi-diameter 1 is a complete symmetric graph; a digraph is maximal of quasi-diameter 2 iff it is isomorphic to a complete symmetric graph with a pair of arcs of the kind \((x, y), (y, x)\) removed. Thus it only remains to consider the case \( k \geq 3 \).

Let \( G = (X, U) \) be an \( n \)-vertex digraph and \( d_m(G) = k \). Hence there exists a pair of vertices \( x_1, x_{k+1} \) in \( G \) such that \( \rho_m(x_1, x_{k+1}) = \min\{\rho(x_1, x_{k+1}), \rho(x_{k+1}, x_1)\} = k \). Let for instance, \( \rho(x_1, x_{k+1}) = k \), and \( \{x_1, x_2, ..., x_k, x_{k+1}\} \) be the naturally ordered totality of vertices of some shortest directed path from \( x_1 \) to \( x_{k+1} \). Denote the set \( \{x_1, ..., x_{k+1}\} \) by \( M \), and the set \( X \setminus M \) by \( B \). Evidently, the quasi-diameter of the subgraph induced by the set \( M \) equals \( k \). Let us bound the number of arcs in the subgraph induced by the set \( M \).

Lemma 11 A \((k + 1)\)-vertex digraph \( D \) of quasi-diameter \( k \) contains at most \( \frac{k^2 + k}{2} \) arcs.

Proof. (i) Suppose that the graph \( D \) is not biconnected; let vertices \( x_1, ..., x_p \) be all vertices of one of its bicomponents. It is easy to see that the number of arcs in the graph \( D \) does not exceed

\[
k + (p - 1) + (p - 2) + ... + 1 + (k - p) + (k - p - 1) + ... + 1 = \\
k + \frac{p^2 - p}{2} + \frac{(k - p + 1)(k - p)}{2} \leq \frac{k^2 + k}{2},
\]

Here the equality holds iff \( p = 1 \) or \( p = k \).
(ii) Suppose that the graph $D$ is biconnected. Then it is easy to see that both outdegree and indegree of the vertices $x_1$ and $x_{k+1}$ do not exceed 1. Hence the number of arcs in the digraph $D$ does not exceed

$$(k - 2) + [(k - 2) + (k - 3) + ... + 1] + 4 = \frac{k^2 + k}{2} - (k - 3) \leq \frac{k^2 + k}{2},$$

and the equality holds iff $k = 3$. ■

Now let us bound the number of arcs that connect the sets $M$ and $B$. It is easy to see that each vertex in the set $B$ can be connected by a pair of antiparallel arcs to at most three vertices in the set $M$. Hence the number of arcs connecting an arbitrary vertex $z \in B$ to the set $M$ does not exceed $k + 3$; the number of arcs connecting the sets $M$ and $B$ does not exceed $(k + 3)(n - k - 1)$; and the number of arcs in the subgraph induced by the set $B$ does not exceed $(n - k - 1)(n - k - 2)$. Thus, the number of arcs in an $n$-vertex graph of quasi-diameter $k$ does not exceed

$$\frac{k^2 + k}{2} + (n - k - 1)(n + 1) = n(n - k) + \frac{k^2 - k - 2}{2}$$

Denote by $f(n,k)$ the maximum number of arcs in an $n$-vertex digraph of quasi-diameter $k$.

**Theorem 7**

$$f(n,1) = n(n - 1), \quad f(n,2) = n(n - 1) - 2,$$

$$f(n,k) = n(n - k) + \frac{k^2 - k - 2}{2} \text{ for } k \geq 3.$$

**Proof.** We only need to prove the last relation, moreover, the inequality $f(n,k) \leq n(n - k) + \frac{k^2 - k - 2}{2}$ has already been proved. In order to prove the reverse inequality we note that for $k \geq 3$ the quasi-diameter of any maximal $n$-vertex digraph of radius $k$ equals $k$. But by Theorem 5, the number of arcs in a maximal $n$-vertex digraph of radius $k$ equals $n(n - k) + \frac{k^2 - k - 2}{2}$. ■

Now we proceed to characterizing maximal digraphs of given quasi-diameter. Suppose $k \geq 4$. Then it follows from Lemma 11 that the subgraph induced by the set $M$ is not biconnected, and $p = 1$ or $p = k$. If $p = 1$ then it is easy to see that maximal digraphs of quasi-diameter $k$ coincide with maximal digraphs of radius $k$ described in Theorem 6. Suppose now $p = k$. We note that in view of the symmetry of the function $\rho_m(x,y)$ a graph $D$ is maximal of quasi-diameter $k$ iff so is the graph $\overline{D}$ that is obtained from the graph $D$ by reversing the direction of its arcs. But if we reverse the direction of arcs in the graph $D$ in which $p = k$, then we obtain the graph $\overline{D}$ in which $p = 1$. Thus, we have shown that for $k \geq 4$ a maximal digraph of quasi-diameter $k$ is either a maximal digraph of radius $k$ or is obtained from a maximal digraph of radius $k$ by reversing the direction of its arcs.
Let us now consider the case $k = 3$. We note that if a maximal digraph of quasi-diameter 3 is not biconnected, then all the above considerations are applicable to it, and hence it is either a maximal digraph of radius 3 or is obtained from a maximal digraph of radius 3 by reversing the direction of its arcs.

Let a maximal digraph $G = (X, U)$ of quasi-diameter 3 be biconnected. Then it is easy to verify that there exist vertices $z, y, u, v$ in the graph such that $\rho_m(z, v) = \rho(z, v)$ and the subgraph induced by the set $\{z, y, u, v\}$ is biconnected. Then the subgraph is isomorphic to either of the graphs in Figures 7, 8.

![Fig. 7](image7.png)

![Fig. 8](image8.png)

Note that the subgraph in $G$ induced by the set $X \setminus \{z, y, u, v\}$ is complete symmetric, and each vertex in the set $X \setminus \{z, y, u, v\}$ is connected to the set $\{z, y, u, v\}$ by exactly six arcs. Therefore, the vertex set of the graph $G$ can be partitioned into subsets $\{z\}, X_1, X_2, X_3, X_4, \{v\}$ such that the subgraph induced by the set $X_1 \cup X_2 \cup X_3 \cup X_4$ is complete symmetric, each vertex in the set $X_1$ is connected by a pair of antiparallel arcs to the vertex $z$, each vertex in the set $X_2$ is connected by a pair of antiparallel arcs to the vertex $v$, arcs go from each vertex in the set $X_3$ to the vertices $z, v$, arcs go from the vertices $z, v$ to each vertex in the set $X_4$; there are no other arcs in the graph $G$. The decomposition is depicted in Figure 9.

![Fig. 9](image9.png)

We note that some subsets in the decomposition can be empty, but the relation $|X_1||X_2| + |X_3||X_4| > 0$ must hold. Thus we have proved

**Theorem 8** For $k \geq 4$ maximal digraphs of quasi-diameter $k$ are either maximal digraphs of radius $k$ or are obtained from them by reversing the direction of all arcs.

For $k = 3$ three cases arise.
1) Maximal digraph $D$ of quasi-diameter 3 is a maximal digraph of radius 3;
2) Maximal digraph $D$ of quasi-diameter 3 is obtained from a maximal digraph of radius 3 by reversing the direction of its arcs.

3) The vertex set of digraph $D$ can be partitioned into disjoint subsets $\{z\}, X_1, X_2, X_3, X_4, \{v\}$ such that:

   a) $|X_1||X_2| + |X_3||X_4| > 0$;

   b) the subgraph induced by the set $X_1 \cup X_2 \cup X_3 \cup X_4$ is complete symmetric;

   c) each vertex in the set $X_1$ is connected by a pair of antiparallel arcs to the vertex $z$, each vertex in the set $X_2$ is connected by a pair of antiparallel arcs to the vertex $v$;

   d) arcs go from each vertex in the set $X_3$ to each of the vertices $z, v$ and from each of the vertices $z, v$ to each vertex in the set $X_1$; there are no other arcs in the graph $D$.

Maximal digraph of quasi-diameter 2 is obtained from a complete symmetric digraph by the removal of a pair of antiparallel arcs; and for $d_m = 1$ a complete symmetric digraph is the only maximal digraph.

Thus, we have obtained a description of all maximal digraphs of finite quasi-diameter.

Let $\nu(n, k)$ denote the number of nonisomorphic maximal $n$-vertex digraphs of quasi-diameter $k$, $\mu(n, k)$ denote the number of distinct digraphs of quasi-diameter $k$ that can be constructed on $n$ numbered vertices.

**Corollary 1**

$$
\nu(n, k) = \begin{cases} 
1 & \text{for } 1 \leq k \leq 2, \\
\frac{(n - 3)(n + 4)}{2} + \sum_{t=1}^{n-4} \left\lfloor \frac{t}{2} \right\rfloor (n - t - 1) + \left\lfloor \frac{n - 3}{2} \right\rfloor & \text{for } k = 3, \\
2(n - k - 1)(k - 2) + 2 & \text{for } k \geq 4.
\end{cases}
$$

**Proof.** Theorem 6 and Corollary 1 of Theorem 6 imply that the assertion is true for $1 \leq k \leq 2$ and $k \geq 4$. It is easy to see that for $k = 3$ the number of nonisomorphic nonbiconnected $d_m$-critical digraphs equals $2n - 6$. Hence it remains to count the number of nonisomorphic biconnected $d_m$-critical graphs with $d_m = 3$. Let $G = (X, U)$ be a critical biconnected graph of quasi-diameter 3 and $|X_1||X_2||X_3||X_4| > 0$, $|X_1| + |X_2| = t$.

Considering Figure 9 we see that if we interchange the sets $X_1$ and $X_2$, then we obtain a graph isomorphic to the original. Hence the number of nonisomorphic biconnected critical graphs of quasi-diameter 3, in which $|X_1||X_2||X_3||X_4| > 0$ and $|X_1| + |X_2| = t$, equals $\left\lfloor \frac{t}{2} \right\rfloor (n - t - 3)$, and the total number of such critical digraphs in which $|X_1||X_2||X_3||X_4| > 0$ equals $\frac{n-4}{2} \left\lfloor \frac{t}{2} \right\rfloor (n - t - 3)$. It is easy to see that the number of nonisomorphic biconnected critical graphs with $d_m = 3$ in which $|X_1||X_2| = 0$ equals $\sum_{t=2}^{n-2} (t - 1)$, and the number of such nonisomorphic graphs in which $|X_3||X_4| = 0$ equals $2 \sum_{t=2}^{n-3} \left\lfloor \frac{t}{2} \right\rfloor + \left\lfloor \frac{n - 2}{2} \right\rfloor$. Therefore,

$$
\nu(n, 3) = 2n - 6 + \sum_{t=2}^{n-4} \left\lfloor \frac{t}{2} \right\rfloor (n - t - 3) + 2 \sum_{t=2}^{n-3} \left\lfloor \frac{t}{2} \right\rfloor + \left\lfloor \frac{n - 2}{2} \right\rfloor + \sum_{t=2}^{n-2} (t - 1) = \frac{(n - 3)(n + 4)}{2} + \sum_{t=2}^{n-4} \left\lfloor \frac{t}{2} \right\rfloor (n - t - 1) + \left\lfloor \frac{n - 3}{2} \right\rfloor \right].
$$
Corollary 2

\[
\mu(n, k) = \begin{cases} 
1 & \text{for } k = 1, \\
\frac{n(n-1)}{2} & \text{for } k = 2, \\
n(n-1)(2^{2n-5} - 2) & \text{for } k = 3, \\
2 \left\{ (k-1)k! \binom{n}{k} + (k-2)(k-1)! \binom{n}{k-1} (2^{n-k+1} - 2n + 2k - 4) \right\} & \text{for } k \geq 4.
\end{cases}
\]

Proof. The assertion of the corollary for \(1 \leq k \leq 2\) and \(k \geq 4\) is an immediate consequence of Theorem 8 and Corollary 2 of Theorem 6. It remains to prove the assertion for \(k = 3\).

By an argument analogous to that used in the proof of Corollary 1 we can show that on given \(n\) numbered vertices one can construct

\[n(n-1)(2^{n-1} - 4)\] nonbiconnected critical graphs of quasi-diameter 3;

\[2n(n-1)(2^{n-3} - 1)\] biconnected critical graphs with \(d_m = 3\) in which \((|X_1| + |X_2|)(|X_3| + |X_4|) = 0;\)

\[2 \sum_{t=2}^{n-3} n(n-1) \binom{n-2}{t} (2^t - 2)\] biconnected critical graphs of quasi-diameter 3 in which exactly one of the sets \(X_1, X_2, X_3, X_4\) is empty;

\[n-4 \sum_{t=2}^{n-4} n(n-1) \binom{n-2}{t} (2^{t-1} - 1)(2^{n-t-2} - 2)\] critical biconnected graphs in which \(|X_1||X_2||X_3||X_4| > 0.\)

Here we used the same notation as in Figure 9. Thus we obtain that

\[
\mu(n, k) = n(n-1)(2^{n-1} - 4) + 2n(n-1)(2^{n-3} - 1) + 2 \sum_{t=2}^{n-3} n(n-1) \binom{n-2}{t} (2^t - 2) + \sum_{t=2}^{n-4} n(n-1) \binom{n-2}{t} (2^{t-1} - 1)(2^{n-t-2} - 2).
\]

One sees easily that the last expression equals \(n(n-1)(2^{2n-5} - 2).\)

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