QUANTUM STATES OF THE KAPITZA PENDULUM

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The quantum states of the Kapitza pendulum are investigated within the framework of the effective potential obtained by the method of averaging over fast oscillations. An analytical estimate of the energy spectrum of stabilized states is given using a model potential. For the lowest states of an inverted pendulum, an expression is obtained for the spectrum in terms of energies of a harmonic oscillator corrected using perturbation theory. Tunneling effect corrections to the energies of resonance states in two wells of the effective potential are found. The results of calculations of the structure of vibrational and rotational spectra of the Kapitza pendulum by the semiclassical method and by the Numerov numerical algorithm are compared.

Keywords: Kapitza stabilization, double-well potential, quantum states, energy spectrum, semiclassical description.

INTRODUCTION

In 1951, P. L. Kapitza carried out systematic research of unusual features of the equilibrium of the rigid pendulum when its suspension point vibrated with high frequency [1] thereby making its upper position stable [2, 3]. This phenomenon, explained by the Kapitza analytical method [4] based on the asymptotic separation of fast and slow variables [5], has found wide application in various branches of classical and quantum physics. The dynamics of nonlinear classical mechanical systems in high-frequency fields is described by the effective time-independent Hamiltonian obtained by successive expansion of the solution in inverse powers of the frequency and allows one to investigate the motion in the presence of fast oscillating fields within the framework of the theory developed for autonomous systems [6]. A nonlinear system can also be stabilized by noise, so that the upper equilibrium point of the pendulum becomes stable even when the noise is white and the simple Kapitza pendulum effect is not realized [7]. The stable vertical position of an inverted pendulum is also realized when the suspension point is subjected to the joint effect of high-frequency harmonics and stochastic forces [8]. The stabilization effect of vibration was generalized to complex elastic systems [9–13], and this approach to charged particle motion in electromagnetic waves led to the concept of the Miller force [14, 15] and the optical tweezer principle [16–18].

The phenomena similar to the Kapitza effect arise in non-mechanical systems, for example, when considering the nonlinear Schrödinger equation with a periodically varying dispersion coefficient for pulse stabilization in fiber-optic systems [19] and in structures with a transverse distribution of the refractive index periodically modulated along the longitudinal coordinate leading to dynamic light confinement [20, 21]. The stabilization theory based on averaging method can be generalized to the imaginary oscillating potential. At a high frequency, a bound state is formed in it that can ensure the stability of the optical resonator with variable reflectivity [22]. The unstable mode of resonator solitons is stabilized by the Kapitza effect due to the amplitude and phase modulation [23].

Quantum dynamics in a high-frequency field, like the classical dynamics, is described by the effective time-independent Hamiltonian [24, 25]. This effective Hamiltonian determines the states and the magnitude of the lowest resonance in the atomic trap model [26] experimentally realized as a combination of a static configuration with...
an external alternating electric field [27]. Such an oscillating field captures particles, since at least one bound state always exists in the one-dimensional well of the effective potential [28, 29]. Experimental installations with ultracold atoms based on this principle are convenient tools for studying quantum systems far from the equilibrium. The Floquet substance with the Kapitza stabilization effect for atoms can be produced at maxima of the optical lattice field [30–32].

The dynamic Kapitza stabilization in many-particle systems prevents heating beyond a certain threshold of periodic action [33] and causes a number of other phenomena. In particular, periodic modulation of transverse magnetic field makes it possible to ensure stable trapping of ferromagnetic spin systems around unstable paramagnetic configurations [34]. A generalization of the Kapitza pendulum to a system of many bodies is the sine-Gordon model with a periodic impact dynamically stable under the influence of a finite frequency and amplitude [35]. The collective behavior in the two-mode controllable dynamic Bose–Hubbard model for the Josephson mode of weak non-resonant action with small chaotic component also reproduces the dynamics of the Kapitza pendulum [36].

The classical Kapitza pendulum is experimentally realized at micrometer scale using a colloidal particle suspended in water and trapped by optical tweezers [37]. Quantum dynamics of the Kapitza pendulum can be experimentally realized by transformation of the nanodimensional superconducting rotator into the unusual or inverted quantum pendulum in an adjustable constant field in combination with inertia and excitation [38].

The Kapitza quantum pendulum is stabilized in the form of quantum states in the vicinity of the local minimum of the effective potential energy. The present work is devoted a theoretical description of the quantum states of this potential and determination of their energy.

1. QUANTUM EQUATION OF MOTION OF THE KAPITZA PENDULUM

Let us consider the Kapitza pendulum, that is, the plane pendulum with length $l$ in a uniform field, the suspension point of which performs high-frequency vertical oscillations $a \cos \omega t$ [5] with amplitude $a \ll l$. The coordinates of the point with mass $m$ are

$$x = l \sin \theta, \quad y = a \cos \omega t + l \cos \theta,$$

(1.1)

and the Lagrange function is

$$L = \frac{J}{2} \dot{\theta}^2 + mla^2 \cos \omega t \cos \theta + mgl \cos \theta,$$

(1.2)

where $J = ml^2$. We consider the force $f = -mla^2 \cos \omega t \sin \theta$ to be rapidly variable with the frequency:

$$\omega >> \sqrt{g/l}.$$

(1.3)

Then the classical equation of motion has the form

$$\ddot{\theta} + l^{-1}(g + a\omega^2 \cos \omega t) \sin \theta = 0,$$

(1.4)

and after averaging over fast oscillations, the effective potential energy takes the form

$$U_{\text{eff}}(\theta) = U \left( 2a \sin^2 \theta - \cos \theta \right),$$

(1.5)
where \( U = mgl \) and \( \alpha = \frac{a^2 \omega^2}{8gl} \). The point \( \sin \theta = 0 \) corresponds to the local minimum of the function \( U_{\text{eff}}(\theta) \). The condition \( \alpha > 1/4 \) provides stability of the upper position \( \theta = \pi \) with \( U_{\text{eff}}(\pi) = U \). The equal global minimum \( -U \) of the potential energy \( U_{\text{eff}}(\theta) \) is reached at \( \theta = 0 \).

The Schrödinger equation for the wave function of the Kapitza quantum pendulum has the form

\[
i \hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2J} \frac{\partial^2 \Psi}{\partial \theta^2} - ml \left( a \omega^2 \cos \omega t \cos \theta + g \cos \theta \right) \Psi.
\]

We use the Cook transform \([39]\) for the wave function

\[
\Psi(\theta, t) = \psi(\theta, t) \exp\left(-i \frac{m l a \omega^2 \cos \theta}{\hbar \omega} \sin \omega t\right).
\]

(1.7)

to exclude fast oscillations. After substitution of expression (1.7) into equation (1.6) and averaging over the period of fast oscillations of the external force, we obtain the Schrödinger equation with the effective potential

\[
i \hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2J} \frac{\partial^2 \psi}{\partial \theta^2} + U_{\text{eff}}(\theta) \psi.
\]

(1.8)

The stationary states \( \phi \) exist in the effective time-independent potential \( U_{\text{eff}}(\theta) \), the equation for which has the form

\[
\frac{d^2 \phi}{d\theta^2} + \frac{2J}{\hbar^2} \left( E - U_{\text{eff}}(\theta) \right) \phi = 0.
\]

(1.9)

This is the so-called Whittaker–Hill equation the general properties of which were well studied in works \([40, 41]\). In the limit \( E >> U \), equation (1.8) describes a plane rigid quantum rotator. Its normalized solutions possess double-periodicity along the angle with the well-known form of the moment quantization

\[
\phi = \frac{1}{\sqrt{2\pi}} \exp(\pm in\theta), \quad n = 0, 1, 2, ...
\]

(1.10)

and the discrete energy spectrum counted from the bottom of the potential well is given by the formula

\[
E_n + U = \frac{\hbar^2 n^2}{2J}.
\]

(1.11)

The dynamics of the plane quantum pendulum in a uniform field without periodic forces was also described in sufficient detail in work \([42]\). Below we consider different modes of motion of the quantum Kapitza pendulum that have not yet been investigated using measurement units in which \( U = 1 \) and \( \hbar = 1 \).
2. VIBRATIONAL STATES

The effective potential energy of the Kapitza pendulum has the form of one stabilizing shallow well and another deep potential well of finite depth separated by a potential barrier. The tops of the symmetric potential energy maxima $U_{\text{max}}$ are located at the points

$$\theta_{\text{max}} = \pi \mp \arccos(1/4\alpha)$$  \hspace{1cm} (2.1)

and

$$U_{\text{max}} = 2\alpha + \frac{1}{8\alpha}. \hspace{1cm} (2.2)$$

To estimate analytically the energies for the shallow well states, we replace the effective potential in the vicinity of $\theta = 0$ by the model potential

$$U_{\text{well}}(\theta) = -\frac{U_{\text{max}} - 1}{\cosh^2 k_{\text{min}} (\theta - \theta_{\text{min}})} + U_{\text{max}}. $$  \hspace{1cm} (2.3)

The energy spectrum of the bound states in it is described by the formula [28]

$$E_n - U_{\text{max}} = -W \left[ \frac{1 + \frac{U_{\text{max}} - 1}{W}}{1 + (1 + 2n)} \right]^2, \hspace{1cm} (2.4)$$

where $W = \frac{k_{\text{min}}^2}{8J}$. The limiting condition for the existence of only one bound state has the form

$$U_{\text{max}} - 1 < \frac{k_{\text{min}}^2}{J}. \hspace{1cm} (2.5)$$

Near the bottom of the shallow well, it is possible to consider the potential as a square-law function of the deviation angle and to use the approximate expression

$$U_{\text{osc}}(\theta) = \pm 1 + \frac{C(\theta - \theta_{\text{min}})^2}{2}, \quad C = 4\alpha \mp 1, \hspace{1cm} (2.6)$$

where the upper sign corresponds to the shallow well, and the lower sign corresponds to the deep well. Then equation (1.9) takes the form

$$\frac{d^2\phi}{d\xi^2} + 2J \left( E \mp 1 - \frac{C(\theta - \theta_{\text{min}})^2}{2} \right) \phi = 0. \hspace{1cm} (2.7)$$

Let us denote $\xi = \mu(\theta - \theta_{\text{min}}), \quad \mu = (JC)^{1/4}, \quad \lambda = 2(E \mp 1)/\omega_c$, where $\omega_c = ((4\alpha \mp 1)/J)^{1/2}$ is the oscillation frequency of a classical oscillator. In these notations, equation (2.7) is written as

$$\frac{d^2\phi}{d\xi^2} + (\lambda - \xi^2) \phi = 0. \hspace{1cm} (2.8)$$
Its solutions are expressed in terms of the Hermite polynomials of order \( n \) in the form [28]

\[
\phi_n(\theta) = \left(\frac{\mu}{\pi}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\xi) \exp(-\xi^2/2),
\]

(2.9)

and \( E_n = 1 + (n + 1/2)\omega_c \). These energy values systematically overestimate the true values, since the oscillator potential is narrower than the effective potential.

To refine the oscillator energy values, we take advantage of perturbation theory in the basis of functions (2.9). The states of the quantum oscillator are not degenerate; therefore, the nonzero correction to the energy in the first order of perturbations \( H_f(\theta) = U_{\text{eff}}(\theta) - U_{\text{osc}}(\theta) \) is

\[
E_n^{(1)} = \langle n | H_f | n \rangle = \langle n | U_{\text{eff}} | n \rangle - 1 - \frac{1}{2}(n + 1/2)\omega_c.
\]

(2.10)

After calculation of the matrix element \( \langle n | U_{\text{eff}} | n \rangle \), we obtain

\[
E_n^{(1)} = \alpha - 1 - \frac{1}{2}(n + 1/2)\omega_c + \exp(-1/4\mu^2)L_n(1/2\mu^2) - \alpha(\exp(-1/\mu^2)L_n(2/\mu^2),
\]

(2.11)

where \( L_n(x) \) is the Laguerre polynomials. Formula (2.11) can be used to calculate the energy of the lowest levels both in the shallow and deep wells.

3. SEMICLASSICAL DESCRIPTION

The number and density of levels increase with \( J \), and the behavior of the system tends to the classical motion. Now we consider the bound states in the double-well potential in the semiclassical approximation [43]. The stable classical motion of the inverted pendulum with the angular momentum

\[
p(E,\theta) = \sqrt{2J(E - U_{\text{eff}}(\theta))}
\]

(3.1)
in potential (1.5) is allowed between the classical turning points \( \theta_1 \) and \( \theta_2 \) in the shallow well in which \( p(E,\theta_1,2) = 0 \). Between the boundaries of the potential barrier \( \theta_2 \) and \( \theta_3 \), the motion is classically forbidden, and penetration through the barrier has the character of the quantum tunnel effect. The solution of the equation \( E - U_{\text{eff}}(\theta) = 0 \) determines the position of the barrier turning points

\[
\cos \theta_{2,3} = -\frac{1}{4\alpha} \pm \sqrt{\left|1 - \frac{E}{2\alpha} + \frac{1}{16\alpha^2}\right|}.
\]

(3.2)

For \( E = U_{\text{max}} \), the turning points merge into one, the potential barrier disappears, and the stabilization is absent.

To determine the energy \( E_n \) in the potential well, the semiclassical quantization rule can be written in the form

\[
S_{12} = (n + 1/2)\pi,
\]

(3.3)
where the phase integral is $S_{12} = S(E, \theta_1, \theta_2) = \int_{\theta_1}^{\theta_2} p(E, \theta) d\theta$ and $n = 0, 1, 2, \ldots$. For finite motion, the maximum value of the phase integral $S_{12}$ is reached when the classical turning points are located at symmetrical maxima $U_{\text{max}}$. Quantum effects are most pronounced when $\sqrt{2J(U_{\text{max}} - U_{\text{min}})} \sim 1$. With an increase in this parameter, the energy spectrum gradually turns into a continuous one.

In two adjacent wells of the Kapitza pendulum, states with close energies can exist. In this case, the resonant tunneling of the pendulum from one well to another ensures the formation of two bi-localized states. To find their energies, we use the transformation matrix of solutions for the period [44]. The periodic Floquet solution of the Hill equation satisfies to the condition

$$\lambda \phi(\theta + 2\pi) = \phi(\theta). \quad (3.4)$$

For the stationary states on a circle, the multiplicator $\lambda = 1$. In the semiclassical approximation, the transformation matrix of the wave function for the examined potential obtained in work [45] is well known. To find the energies of the stationary states for $\theta_1 < \theta < \theta_2$, we choose two linearly independent solutions [46]

$$u = p^{-1/2} \exp(iS(E, \theta_1, \theta)), \quad v = u^*. \quad (3.5)$$

The transformation of these functions over the period takes the form

$$u(\theta + 2\pi) = Du(\theta) + Rv(\theta), \quad (3.6)$$

$$v(\theta + 2\pi) = D^*v(\theta) + R^*u(\theta).$$

Condition (3.4) leads to the equation

$$\begin{vmatrix} D - \lambda & R \\ R^* & D^* - \lambda \end{vmatrix} = 0. \quad (3.7)$$

Taking this into account, the spectral equation assumes the form

$$\lambda^2 - 2 \Re D\lambda + 1 = 0 \quad (3.8)$$

because $DD^* - RR^* = 1$ due to the Wronskian properties, and

$$\Re D = \frac{1}{2} \beta \cos S_{12} \cos S_{34} - \sin S_{12} \sin S_{34}, \quad (3.9)$$

where $\beta = 4\sigma^2 + \frac{1}{4\sigma^2}$ and $\sigma(E) = \exp\left(\frac{1}{\hbar} \int_{\theta_1}^{\theta_2} |p(E, \theta)| d\theta\right)$. With allowance for $\lambda = 1$, the condition for the determination of energies assumes the form

$$\Re D = 1. \quad (3.10)$$
If the energies of states in these wells differ by more than $\beta^{-1}$, these states are practically independent. For closely spaced levels, their mutual influence can be considered as a perturbation taking into account their action as a small deviation $\delta_{ij}$ of the phase from the quantization condition

$$S_{ij} = (n + 1/2)\pi + \delta_{ij}.$$  \hspace{1cm} (3.11)

Here $n = n_1$ for the shallow well and $n = n_2$ for the deep well. Using representation (3.9) and taking into account equation (3.10), we obtain

$$\frac{1}{2} \beta \delta_{12} \delta_{34} = 1 + (-1)^{n_1+n_2}.$$  \hspace{1cm} (3.12)

If the quantum numbers $n_1$ and $n_2$ have different parities, the signs of the interaction of the states in the two subbarrier regions are opposite, and energy splitting is absent. For the states with the same parities,

$$\frac{dS_{12}}{dE} \frac{dS_{34}}{dE} (\Delta E)^2 = \frac{4}{\beta}.$$  \hspace{1cm} (3.13)

Given that $\frac{dS_{ij}}{dE} = \frac{\pi}{\omega_{ij}}$, the energy splitting of the states is expressed in the form

$$\Delta E = \pm \frac{2}{\pi} \sqrt{\frac{\omega_{12} \omega_{34}}{\beta}},$$  \hspace{1cm} (3.14)

where $\omega_{12}$ is the classical oscillation frequency in the shallow well, and $\omega_{34}$ is the classical oscillation frequency in the deep well. For $\sigma \gg 1$, we obtain

$$\Delta E = \pm \frac{\sqrt{\omega_{12} \omega_{34}}}{\pi} \sigma^{-1},$$  \hspace{1cm} (3.15)

which is twice the value of the tunneling splitting of energy levels found for an asymmetric double well in the semiclassical approximation \[47\] and by the method of two-level approximation \[48\] due to double subbarrier overlap of the wave functions.

For the overbarrier rotational motion of the Kapitza pendulum with energy $E > U_{\text{max}}$, the condition for the applicability of the procedure of state averaging over fast oscillations of the external field takes the form

$$\omega \gg \omega_r,$$  \hspace{1cm} (3.16)

where $\omega_r$ is the classical rotation frequency of the pendulum. The semiclassical Bohr–Sommerfeld quantization condition for rotational motion in the effective potential is written as

$$S(E, 0, 2\pi) = 2\pi n.$$  \hspace{1cm} (3.17)

It allows one to find the energies of states taking into account the action of the potential and the periodicity of motion.
4. RESULTS OF NUMERICAL CALCULATIONS AND DISCUSSIONS

As an example illustrating the general theoretical description, we consider the Kapitza pendulum with parameters $\alpha = 1/2$ and $J = 270$. This choice of the parameter values ensures the presence of several quantum levels in the shallow well. The effective potential energy and model potential (2.3) with $\kappa_{\text{min}} = 1.8$ together with six discrete energy levels are shown in Fig. 1.

Table 1 presents results of calculations of the state energies for the model effective potentials in the semiclassical approximation by the shooting method [49] using the numerical Numerov algorithm for integration of the Schrödinger equation [50, 51] and in the harmonic oscillator approximation corrected for the effective potential. The underbarrier asymptotic for the starting values of the numerical Numerov procedure was taken in the semiclassical approximation, thereby limiting the accuracy of calculations by this method. The wave function of the ground state of the oscillator was a Gaussian function. Considering its dispersion as a parameter, the variational method was used to determine the energy of the lowest state in the shallow well which coincided with the results of calculations by the shooting method and the first order perturbation theory for the harmonic oscillator.

Our comparison of the results of calculations of the spectrum by different methods shows that the model potential is suitable for the general estimation of the energy level positions in the shallow well; it ensures the stability of the inverted pendulum, but indicates the additional state which is not reproduced by semiclassical or direct numerical

| Model potential | Semiclassical approximation | Shooting method | Harmonic oscillator | Harmonic oscillator with correction |
|-----------------|-----------------------------|----------------|---------------------|----------------------------------|
| 1.0358          | 1.0300                      | 1.0296         | 1.0304              | 1.0296                           |
| 1.1015          | 1.0875                      | 1.0869         | 1.1520              | 1.0874                           |
| 1.1552          | 1.1409                      | 1.1403         | 1.0912              | 1.1421                           |
| 1.1969          | 1.1896                      | 1.1892         | 1.2128              | 1.1939                           |
| 1.2266          | 1.2317                      | 1.2304         | 1.2736              | 1.2429                           |
| 1.2443          |                             |                |                     |                                  |

Fig. 1. Dependences of the effective potential (the solid curve), model potential (points), and bound state energy levels (dashed straight lines) on the angle $\theta$. 
calculations. This difference with the semiclassical calculations for the given model potential is explained by the well-known limitations of the Wetzel–Kramer–Brillouin (WKB) method accuracy. However, additional research taking into account the features of the wave function behavior near the top of the potential barrier is required for more accurate conclusions related directly to such states in the effective potential of the Kapitza pendulum.

Figure 2 shows the dependences of the reduced phase \( n' = (S - 1/2)/\pi \), expressed in terms of the magnitude of the classical action \( S \) between the turning points, on the energy for the shallow and deep wells. The semiclassical energies obtained in this way at \( n' = n \) correspond to 5 levels in the shallow well and 28 states in the deep well.

Five semiclassical states (from 23 to 28) in the deep well at energies from the bottom to the top of the shallow well potential are candidates for resonant state interaction. The uppermost states with \( n_1 = 4 \) and \( n_2 = 27 \) are the closest to the resonance; however, they have different parities, their interaction is zero, and consequently, no energy splitting is observed for them due to resonant tunneling. The wave functions of these states possessing different symmetries and calculated by the Numerov algorithm are shown in Fig. 3.

The energies of the upper states in the deep well and the energies of the rotational states calculated using semiclassical quantization and direct numerical solution by the shooting method are presented in Table 2. As expected, the accuracy of the semiclassical approximation increases with energy.
TABLE 2. Energy of Levels in the Deep Well and Rotational States

| Semiclassical approximation for the deep well | Shooting method for the deep well | Semiclassical approximation for the deep well | Shooting method for rotational states |
|---------------------------------------------|---------------------------------|---------------------------------------------|-----------------------------------|
| 1.0160                                      | 1.0154                          | 1.2619                                      | 1.2629                           |
| 1.0743                                      | 1.0736                          | 1.3034                                      | 1.2827                           |
| 1.1287                                      | 1.1280                          | 1.3537                                      | 1.3004                           |
| 1.1786                                      | 1.1782                          | 1.4106                                      | 1.3239                           |
| 1.2225                                      | 1.2215                          | 1.4735                                      | 1.3503                           |
|                                             | 1.5418                          |                                             | 1.3777                           |
|                                             | 1.6152                          |                                             | 1.465                            |
|                                             | 5.182 (n = 50)                  |                                             | 5.0699                           |
|                                             | 7.218 (n = 60)                  |                                             | 7.1890                           |

Fig. 4. Reduced phase $n' = S / 2\pi$ as a function of the energy for the rotational motion of the Kapitza pendulum. Here the solid curve is for the semiclassical approximation, and the dashed curve is for the free rotation.

Figure 4 shows the result of semiclassical calculation of the reduced phase $n' = S / 2\pi$ as a function of the energy and its comparison with dependence (1.11) for the free rotation. As the energy increases, the two curves converge demonstrating the asymptotic behavior of the spectra.

CONCLUSIONS

The action of a high-frequency force on the inverted quantum linear oscillator leads to stabilization of the center of the wave packet retaining its dispersive spreading [52]. In the Kapitza quantum pendulum, the spreading of states is suppressed due to the nonlinearity, and their destruction, without taking into account the stochastic thermal effect, occurs due to the resonant tunnel passage of the barrier separating the local minimum of the effective potential energy in the upper position of the pendulum from the global minimum in its lower position.
The comparison of the results obtained by different methods shows that the energy of the ground state under condition of stabilization can be calculated confidently within the framework of the oscillator models with the first-order correction of perturbation theory. For the remaining states, it is better to use the semiclassical approximation or numerical calculations by the Numerov method. The semiclassical approximation reproduces well the states not too close to the top of the potential, and its accuracy increases for a large number of levels. The significant discrepancy between results of numerical calculations and simple analytical model estimates arises near the top of the potential barrier where the standard WKB method is violated. A detailed calculation of states near the top of the potential barrier requires separate consideration and suggests a more accurate account of the behavior of the wave function in this energy range. This can be made, for example, by using the inverted oscillator potential near the top of the effective potential [47, 53]. This is especially important for the shallow wells with one weakly bound state when the usual semiclassical approximation is inapplicable.

In conclusion, we note that the problem of motion of the system localized on a circle considered by us is close to the problem of particle motion in the infinitely long periodic Kapitza potential. However, for the infinite potential, the spectrum of solutions found by the Floquet method has, in the presence of resonant states, the structure of the allowed energy bands the centers of which in the range of vibrational states coincide with the energies calculated by us. In this case, the overbarrier motion is characterized by a continuous energy spectrum.

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