Maximum-Size Envy-free Matchings

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Abstract
We consider the problem of assigning residents to hospitals when hospitals have upper and lower quotas that respectively denote the maximum and minimum number of residents that can be assigned to them in any matching. Apart from this, both residents and hospitals have a preference list which is a strict ordering on a subset of the other side. Stability is a well-known notion of optimality in this setting.

Every Hospital-Residents (HR) instance without lower quotas admits at least one stable matching. When hospitals have lower quotas (HRLQ), there exist instances for which no matching that is simultaneously stable and feasible exists. We investigate envy-freeness which is a relaxation of stability for such instances. Yokoi [24] gave a characterization for HRLQ instances that admit a feasible and envy-free matching. Yokoi’s algorithm gives a minimum size feasible envy-free matching, if there exists one. We investigate the complexity of computing a maximum size envy-free matching in an HRLQ instance (MAXEFM problem) which is equivalent to computing an envy-free matching with minimum number of unmatched residents (MIN-UR-EFM problem).

We show that both the MAXEFM and MIN-UR-EFM problems for an HRLQ instance with arbitrary incomplete preference lists are NP-hard. We show that MAXEFM cannot be approximated within a factor of \(\frac{7}{19}\). On the other hand we show that the MIN-UR-EFM problem cannot be approximated for any \(\alpha > 0\). We present \(\frac{1}{2(\alpha+1)}\) approximation algorithm for MAXEFM when quotas are at most 1 where \(L\) is the length of longest preference list of a resident. We also show that both the problems become tractable with additional restrictions on preference lists and quotas. We also investigate the parameterized complexity of these problems and prove that they are W[1]-hard when deficiency [14] is the parameter. On the positive side, we show that the problems are fixed parameter tractable for several interesting parameters.

2012 ACM Subject Classification Mathematics of computing → Graph theory; Theory of computation → Design and analysis of algorithms

Keywords and phrases Matchings under preferences, Lower quota, NP-hardness, Parameterized complexity

1 Introduction

Gale and Shapley, in their seminal work on college admissions [10], formally introduced the Stable Marriage Problem and its generalization – the Hospital-Residents (HR) Problem. The input to the HR problem is a bipartite graph \(G = (\mathcal{R} \cup \mathcal{H}, E)\) where \(\mathcal{R}\) denotes the
set of residents, $H$ denotes the set of hospitals, and an edge $(r, h) \in E$ denotes that $r$ and $h$ are mutually acceptable. Every vertex in $G$ ranks its neighbors in $G$ in a strict order, referred to as the preference list of the vertex. A hospital $h \in H$ has an associated capacity or upper-quota $q^+(h)$ which denotes the maximum number of residents that can be assigned to $h$. A matching $M \subseteq E$ in $G$ is an assignment of residents to hospitals such that each resident is matched to at most one hospital, and every hospital $h$ is matched to at most $q^+(h)$-many residents. For a matching $M$, let $M(r)$ denote the hospital that $r$ is matched to, and $M(h)$ denote the set of residents matched to $h$ in $M$. We say that a hospital $h$ is under-subscribed in a matching $M$ if $|M(h)| < q^-(h)$. The goal is to match residents to hospitals optimally with respect to the preference lists. Stability is a de-facto notion of optimality in this setting and is defined by the absence of blocking pairs, defined below:

**Definition 1.** A pair $(r, h) \in E \setminus M$ is a blocking pair w.r.t. the matching $M$ if $r$ is either unmatched in $M$ or prefers $h$ over $M(r)$ and $h$ is either under-subscribed in $M$ or prefers $r$ over at least one resident $r' \in M(h)$. A matching $M$ is stable if there is no blocking pair w.r.t. $M$.

It is well-known that every instance of the HR problem admits a stable matching which can be computed in linear time by the Gale and Shapley algorithm [10].

A generalization of the HR problem is to allow hospitals to have demands or lower-quotas (LQ) which denote the minimum number of residents that must be assigned to it in any feasible matching. Thus, in addition to the HR setting, every hospital $h$ has a lower-quota $0 \leq q^-(h) \leq q^+(h)$ associated with it. This is called the HRLQ problem. A hospital $h$ is said to be deficient in a matching $M$ if $|M(h)| < q^-(h)$. The goal is to compute a feasible matching $M$ that is optimal. A matching $M$ in an HRLQ instance is feasible if, for every hospital $h$, $q^-(h) \leq |M(h)| \leq q^+(h)$. In other words, a feasible matching does not leave any hospital deficient. We define deficiency of an HRLQ instance with respect to a stable matching $M$ as follows:

**Definition 2.** Let $G = (R \cup H, E)$ be an HRLQ instance and $M$ be a stable matching in $G$. Then deficiency($M$) = $\sum_{h \in H} \max[0, q^-(h) - |M(h)|]$.

Lower quotas are important from a practical perspective, since it is natural for a hospital to require a minimum number of residents to run the hospital smoothly. The well-known Rural Hospitals Theorem [12] states that every stable matching matches every hospital to the same number of residents. Thus, hospitals which are deficient in one stable matching are deficient in all the stable matchings. Imposing lower quotas offers a way out of this situation since infeasible matchings are no longer acceptable. However, the presence of lower quotas poses new challenges. Unlike the HR setting, there exist simple instances of the HRLQ problem that may not admit any feasible and stable matching, even when a feasible matching exists in the instance. Consider an HRLQ instance with $R = \{r_1, r_2\}$ and $H = \{h_1, h_2\}$ where both hospitals have a unit upper-quota and $h_2$ has a lower-quota of 1. The preferences of the residents and hospitals can be read from Fig. 1. The matching $M_2 = \{(r_1, h_1)\}$ is stable but not feasible since $|M_2(h_2)| < q^-(h_2)$, or $h_2$ is deficient in $M_2$. The matchings $M_1 = \{(r_1, h_2)\}$ and $M_2 = \{(r_1, h_2), (r_2, h_1)\}$ are both feasible but not stable since $(r_1, h_1)$ blocks both of them.

In light of this, relaxations of stability, like popularity and envy-freeness have been proposed in the literature [20] [19] [24]. We consider envy-freeness in this paper, which is defined by the absence of envy-pairs.
Figure 1 An HRLQ instance with no feasible and stable matching.

Figure 2 An HRLQ instance with two envy-free matchings of different sizes.

Definition 3. Given a matching \( M \) in \( G \), a resident \( r \) has a justified envy towards a matched resident \( r' \), where \( M(r') = h \) and \((r,h) \in E\) if either \( r \) is unmatched in \( M \) or \( r \) prefers \( h \) over \( M(r) \) and \( h \) prefers \( r \) over \( r' \). The pair \((r,r')\) is an envy-pair w.r.t. \( M \). A matching \( M \) is envy-free if there is no envy-pair w.r.t. \( M \).

Note that every envy-pair is a blocking pair but the converse is not true and hence envy-freeness is a relaxation of stability. In the example in Fig. 1 the matching \( M_1 \) is envy-free but \( M_2 \) is not, since \((r_1,r_2)\) is an envy-pair w.r.t. \( M_2 \).

Envy-freeness is motivated by fairness from a social perspective. Importance of envy-free matchings has been recognized in the context of constrained matchings \cite{9, 11, 16, 17, 5}, and their structural properties have been investigated in \cite{23}. We remark that, if the example in Fig. 1 is modified such that \( h_1 \) also has a lower-quota of 1, then \( M_2 \) is the unique feasible matching and it is not envy-free. Thus, there exist instances in which no feasible matching is envy-free. They still form an attractive choice since there are HRLQ instances with no stable, feasible matching but that admit envy-free, feasible matchings. In the rest of the paper, envy-free matchings always refer to feasible envy-free matchings.

Recently, Yokoi \cite{24} gave a characterization for HRLQ instances that admit an envy-free matching and a linear-time algorithm to compute an envy-free matching if it exists. The characterization result of Yokoi can be stated as follows: given an HRLQ instance \( G \), construct a modified HR instance \( G' \) where upper-quotas of all the hospitals in \( G' \) are equal to their respective lower-quotas in \( G \). The instance \( G \) admits an envy-free matching if and only if no hospital is under-subscribed in a stable matching in \( G' \).

It is interesting to note that there is a contrast between stability in the HR problem and envy-freeness in the HRLQ problem in terms of size. By the Rural Hospitals Theorem \cite{12}, all the stable matchings in an HR instance have the same size. (This gives a simple characterization of HRLQ instances that admit a stable and feasible matching). On the other hand, even when an HRLQ instance admits an envy-free matching, there may be envy-free matchings of different sizes. Consider the example in Fig. 2 with \( n \) residents \( R = \{r_1, \ldots, r_n\} \) and two hospitals \( H = \{h_1, h_2\} \). The hospital \( h_2 \) has an upper-quota and lower-quota of one each. The instance admits an envy-free matching \( N_1 = \{(r_1, h_2)\} \) of size one and another envy-free matching \( N_n = \{(r_1, h_1), (r_2, h_1), \ldots, (r_{n-1}, h_1), (r_n, h_2)\} \) of size \( n \). Matching \( M \) is a maximal envy-free matching if addition of any edge to \( M \) either violates the property of
matching or the property of envy-freeness. Thus, $N_1$ is a maximal envy-free matching.

We remark that Yokoi’s characterization gives an envy-free matching in an HRLQ instance and it is in fact a minimum size envy-free matching. Furthermore, the output of Yokoi’s algorithm need not be even a maximal envy-free matching. Thus, it is natural to ask the complexity of computing a maximum size envy-free matching in an HRLQ instance when one exists. We denote this as the MAXEFM problem which is the focus of the current paper. Alternatively, one can ask for an envy-free matching that has minimum number of unmatched residents; we denote this as the MIN-UR-EFM problem. For exact solutions, the two problems are equivalent and we show that the problems are NP-hard. Moreover, we show that the MIN-UR-EFM problem does not admit any approximation algorithm. In contrast, our inapproximability result for the MAXEFM problem leaves a hope for some non-trivial approximation algorithms.

1.1 Our results

We show the following new results in this paper.

- **Theorem 4.** The MAXEFM (equivalently MIN-UR-EFM) problem is NP-hard. Moreover, the MIN-UR-EFM problem has no $\alpha$-approximation algorithm for any factor $\alpha > 0$ unless $P = NP$. The results hold even when every resident has a preference list of length at most two.

The above NP-hardness and hardness of approximation hold when every hospital has a quota of at most one (however, resident preference lists are no longer of length at most two).

- **Theorem 5.** The MAXEFM problem cannot be approximated within a factor of $\frac{21}{19}$ unless $P = NP$. The result holds when all quotas are at most one.

We complement the above hardness results by the following positive results for restricted HRLQ instances.

- **Theorem 6.** There exists an $\frac{1}{L+1}$ approximation algorithm for MAXEFM when quotas are at most 1 where $L$ is the length of longest preference list of a resident.

In the light of Theorem 1, we consider the case when residents have preference lists of length at most two and hospitals have a quota of at most one. We denote the restriction as 01-HRLQ-2R.

- **Theorem 7.** There is an efficient algorithm for the MAXEFM problem for the 01-HRLQ-2R instances.

Next, we allow residents to have arbitrary length lists and hospitals to have arbitrary quotas, however, every hospital with positive lower-quota has all the residents in its preference list. This restriction is referred to as the CL-restriction in literature [14].

- **Theorem 8.** There is a simple linear-time algorithm for the MAXEFM problem for CL-restricted instances.

**Parametric results:** Besides the above results, we investigate the parameterized complexity of the problem. On the positive side, we show that the MAXEFM problem is fixed parameter tractable (FPT) on several interesting parameters. On the negative side, we show that the problem is W[1]-hard for the natural parameter of deficiency of the HRLQ instance.
1.2 Related work

Various notions of optimality in HRLQ instances have been extensively studied in [8] [13] [24] [20] [19]. A trade-off between envy-freeness and non-wastefulness is considered in [8]. In [14], feasible matchings in the HRLQ instance have been studied for the objective of minimizing the number of blocking pairs or blocking residents. They show hardness for both these objective functions and also give algorithmic results.

Yokoi [24] gives characterization of HRLQ instances that admit envy-free matchings, but the matching is not maximal. Building on this, Krishnapriya et.al. [19] presents an algorithm to compute a maximal envy-free matching containing Yokoi’s matching, and also gives empirical results. Another notion of optimality, namely popularity in the HRLQ problem has been considered in [20]. In a different setting with lower quotas, in which hospitals either fulfill required lower quotas or are closed is studied in [2]. They show that it is NP-complete to decide if a stable matching exists in this setting. Lower quotas are also studied by [15] and [6] in the context of classified stable matchings (CSM).

Organization of the paper: In Section 2 we present our NP-hardness result and inapproximability for MIN-UR-EFM problem. In Section 3 we present the hardness of approximation for the MAXEFM problem. In Section 4 we present two polynomial time algorithms for special cases of MAXEFM problem. In Section 5 we present our results related to parameterized complexity of MAXEFM problem.

2 Inapproximability of MIN-UR-EFM

In order to show the hardness result, we show a reduction from the well-known NP-Complete Independent Set (IND-SET) problem. Let \( (G = (V, E), k) \) be an instance of the Independent Set problem where \(|V| = n\) and \(|E| = m\). The goal is to decide whether \( G \) has an independent set of size \( k \). We will create an instance \( G' = (\mathcal{R} \cup \mathcal{H}, E') \) of the MAXEFM problem as follows.

For every vertex \( v_i \in V \), we have a vertex resident \( r_i \in \mathcal{R} \); for every edge \( e_j \in E \), we have an edge resident \( r'_j \in \mathcal{R} \). Thus \(|\mathcal{R}| = m + n\). The set \( \mathcal{H} \) consists of \( n + 1 \) hospitals, one hospital per vertex \( (h_{v_i} \text{ for } v_i) \) in \( G \) and an additional hospital \( x \). A hospital \( h_{v_i} \) has zero lower-quota and an upper quota equal to \( 1 + |E_i| \) where \( E_i \) denotes the set of edges incident on \( v_i \) in \( G \). We let \( E_i \) denote the set of edge residents corresponding to edges in \( E_i \). The hospital \( x \) has both lower-quota and upper-quota as \( k \).

Preference lists: The preferences of the residents and the hospitals can be found in Fig. 3. A vertex resident \( r_i \) has a length two preference list with \( h_{v_i} \) followed by hospital \( x \). An edge resident \( r'_j \) has a length two preference list with the two hospitals (denoted by \( h_{v_{j1}} \) and \( h_{v_{j2}} \)) corresponding to the end-points \( v_{j1}, v_{j2} \) of the edge \( e_j \) in any order. A hospital \( h_{v_i} \) has the vertex resident \( r_i \) as its top-choice followed by the edge residents in \( E_i \) in any strict order. Finally the hospital \( x \) has in its preference list all the \( n \) vertex residents in any strict order.

Stable Matching in \( G' \): It is straightforward to verify that a stable matching in \( G' \) does not match any resident to \( x \), thus making it infeasible.

Lemma 9 proves the correctness of our reduction.

\( \triangleright \textbf{Lemma 9.} \) \( G \) has an independent set of size \( k \) iff \( G' \) has an envy-free matching of size \( m + n \).

**Proof.** Let \( S \subseteq V \) be an independent set of size \( k \) in \( G \). We will construct an envy-free matching in \( G' \) which matches all residents in \( \mathcal{R} \). For every vertex \( v_i \in S \), match the vertex resident \( r_i \) to the hospital \( x \). Each hospital \( h_{v_i} \) such that \( v_i \in S \) must remain empty to ensure
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We fix an arbitrary ordering on sets $X$, $H$, $E$. A vertex resident $r_j$ has the set $H_j$, followed by set $X$. An edge resident $r'_j$ has two sets of hospitals (denoted by $H_{j1}$ and $H_{j2}$) corresponding to the edge $r'_j = (r_j, e_j)$.

| $R$   | $H$                      |
|-------|--------------------------|
| $r_1 : h_1$, $x$ | $[0, |E_1| + 1]$ $h_1 : r_1$, $E_1$ |
| $r_2 : h_2$, $x$ | $[0, |E_2| + 1]$ $h_2 : r_2$, $E_2$ |
| ...   | ...                      |
| $r_n : h_n$, $x$ | $[0, |E_n| + 1]$ $h_n : r_n$, $E_n$ |
| $r'_1 : h_{11}$, $h_{12}$ | $[k, k]$ $x : r_1$, $r_2$, ..., $r_n$ |
| $r'_2 : h_{21}$, $h_{22}$ |                       |
| ...   | ...                      |
| $r'_m : h_{m1}$, $h_{m2}$ |                       |

Figure 3 Reduced instance $G'$ of MAXEFM from instance $(G, k)$ of IND-SET.

evnonly-freeness. For vertex $v_i \notin S$, match the resident $r_i$ to the hospital $h_i$. Thus, every vertex resident is matched. Since $S$ is an independent set, at least one end-point of every edge is not in $S$. Thus, for an edge $e_j = (v_{j1}, v_{j2})$, the corresponding edge resident $r'_j$ can be matched to at least one of $h_{j1}$ or $h_{j2}$ without causing envy. Thus, every edge resident is matched and we have an $R$-perfect envy-free matching, that is, an envy-free matching of size $m + n$.

For the other direction, let us assume that $G$ does not have an independent set of size $k$. Consider any envy-free matching $M$ in $G'$. Due to the lower-quota of $x$, exactly $k$ vertex residents must be matched to $x$ in $M$. Let $V' = \{v_i \in V \mid M(r_i) = x\}$. Then, $|V'| = k$. Since $V'$ is not an independent set, there exists at least two residents $r_i$ and $r_s$ matched to $x$ in $M$ such that $e_i = (v_{i1}, v_{i2}) \in E$. Due to the preference lists of the hospitals, both $h_t$ and $h_s$ must remain empty to ensure envy-freeness. This implies that the edge resident $r'_j$ must be unmatched in $M$. This implies that $|M| < m + n$. This completes the proof of the lemma.

Thus, MAXEFM is NP-hard. Since, MAXEFM and MIN-UR-EFM match on exact solution, this implies that MIN-UR-EFM is also NP-hard. We observe the following for the MIN-UR-EFM problem. When $G$ has an independent set of size $k$, there are zero residents unmatched in an optimal envy-free matching of $G'$, whereas when $G$ does not admit any independent set of size $k$, every envy-free matching leaves at least one resident unmatched. This immediately implies that any $\alpha$-approximation algorithm for MIN-UR-EFM problem will be able to distinguish between yes and no instances of Independent Set problem. Finally, note that in the reduced instance shown in Fig. 3, every resident has exactly two hospitals in its preference list. This establishes Theorem 4.

Theorem 10. MAXEFM and MIN-UR-EFM are NP-hard when the quotas are at most 1.

Proof. Let $(G = (V, E), k)$ be an instance of the Independent Set problem where $|V| = n$ and $|E| = m$. The goal is to decide whether $G$ has an independent set of size $k$. We will create an instance $G' = (R \cup H, E')$ of the MAXEFM problem as follows. We modify the reduction shown in Fig. 3. We define the vertex residents $r_i$, edge residents $r'_j$ and sets $E_i$, $E_j$ as done earlier. Let $q_i = |E_i| + 1$ for all vertices $v_i \in V$. For every vertex $v_i \in V$, let $H_i = \{h_{i1}^1, h_{i1}^2, \ldots, h_{i1}^{q_i}\}$ be the set of hospitals. Let $X = \{x_1, x_2, \ldots, x_k\}$ be also the set of hospitals. Every hospital in set $H_i$ has zero lower-quota and an upper-quota equal to 1. Every hospital $x_i \in X$ has both lower and upper-quota equal to 1.

Preference lists: The preferences of the residents and the hospitals can be found in Fig. 4. We fix an arbitrary ordering on sets $X$, $H_i$, $E_i$. A vertex resident $r_j$ has the set $H_j$, followed by set $X$. An edge resident $r'_j$ has two sets of hospitals (denoted by $H_{j1}$ and $H_{j2}$) corresponding to the edge $r'_j = (r_j, e_j)$. The preferences of the residents and the hospitals can be found in Fig. 4.
to the end-points $v_{j1}, v_{j2}$ of the edge $e_j$ in any order. Every hospital $h \in H_i$ has the vertex resident $r_i$ as its top-choice followed by the edge residents in $E_i$ in any strict order. Finally the hospitals in $X$ have in their preference lists all the $n$ vertex residents in any strict order.

**Stable Matching in $G'$:** It is straightforward to verify that a stable matching in $G'$ does not match any resident to any hospital in set $X$, thus making it infeasible.

| $R$ | $H$ |
|-----|-----|
| $r_1 : H_1 X$ | $[0, 1] h_1^1 : r_1 E_1$ |
| $r_2 : H_2 X$ | $[0, 1] h_1^q : r_1 E_1$ |
| $\vdots$ | $\vdots$ |
| $r_n : H_n X$ | $[0, 1] h_n^1 : r_n E_n$ |
| $r'_{1,i} : H_{11} H_{12}$ | $[1,1] x_1 : r_1 r_2 \ldots r_n$ |
| $r'_{2,i} : H_{21} H_{22}$ | $[1,1] x_2 : r_1 r_2 \ldots r_n$ |
| $\vdots$ | $\vdots$ |
| $r'_{m,i} : H_{m1} H_{m2}$ | $[1,1] x_m : r_1 r_2 \ldots r_n$ |

![Figure 4](image-url) Reduced instance $G'$ of MAXEFM from instance $\langle G, k \rangle$ of IND-SET.

Note that every vertex $v_i$ has $q_i$ many hospitals - each with an upper quota of 1, so the set of hospitals in $H_i$ together have enough quota to get matched with the corresponding vertex resident $r_i$ and all the edge residents corresponding to the edges incident on $v_i$. Lemma 11 proves the correctness of the reduction. Hence, MAXEFM and MIN-UR-EFM are NP-hard even if all the quotas are at most 1.

**Lemma 11.** $G$ has an independent set of size $k$ iff $G'$ has an envy-free matching of size $m + n$.

**Proof.** Let $S \subseteq V$ be an independent set of size $k$ in graph $G$. We construct an envy-free matching in $G'$ which matches all the residents in $R$. Let $T$ be the set of residents corresponding to the vertices $v_i \in S$ i.e. $T = \{r_i \mid v_i \in S\}$. Match $T$ with $X$ using Gale and Shapley stable matching algorithm. Let $T'$ be the set of vertex residents $r_i$ such that $v_i \notin S$ i.e. $T' = \{r_1, r_2, \ldots, r_n\} \setminus T$. Let $H'$ be the set of hospitals appearing in sets $H_i$ such that $v_i \notin S$ i.e. $H' = \bigcup_{i:v_i \in S} H_i$. Match $T' \cup \{r'_{1}, \ldots, r'_{m}\}$ with $H \setminus H' \setminus X$ using Gale and Shapley stable matching algorithm.

We now prove that the matching is envy-free. No pair of residents in $T$ together form an envy-pair because we computed a stable matching between $T$ and $X$. No pair of residents in $T' \cup \{r'_{1}, \ldots, r'_{m}\}$ together form an envy-pair because we computed a stable matching between this set and $H \setminus H' \setminus X$. Since, all hospitals in $H'$ are forced to remain empty, no resident in set $T$ can envy a resident in set $\{r'_{1}, \ldots, r'_{m}\}$. A resident in $T'$ is matched to a higher preferred hospital than any hospital in $X$, hence such resident cannot envy any resident in $T$. Thus, the matching is envy-free.

We now prove that the matching size is $m + n$. Every vertex resident $r_i$ is matched either with some hospital in $X$ or some hospital in $H_i$. Since, $S$ is an independent set, at least one end point of every edge is not in $S$. So for every edge $e_t = (v_{j1}, v_{j2})$, there is at least one
hospital in sets $H_{t1}, H_{t2}$ that is not in $H'$. Thus, every edge resident is also matched. Thus, we have an envy-free matching of size $m + n$.

For the other direction, let us assume that $G$ does not have an independent set of size $k$. Consider an arbitrary envy-free matching $M$ in $G'$. Due to the unit lower-quota of every $x_i \in X$, exactly $k$ vertex residents must be matched to hospitals in $X$. Let $S \subseteq V$ be the set of vertices $v_i$ such that the corresponding vertex resident $r_i$ is matched to some hospital in $X$ in $M$, i.e. $S = \{v_i \mid M(r_i) \in X\}$. So, $|S| = k$. Since, $S$ is not an independent set, there exists at least two vertex residents $r_s$ and $r_t$ matched to some hospital in $X$ such that the edge $e_j = (v_s, v_t) \in E$. Due to the preference lists of the hospitals, all the hospitals in both $H_s$ and $H_t$ sets must remain empty in $M$ to ensure envy-freeness. This implies that the edge resident $r_j^*$ must be unmatched. This implies that $|M| < m + n$. This completes the proof of the lemma. ▲

3 Hardness of approximation for MAXEFM

In this section, we prove Theorem 5 from Section 1 i.e. our hardness of approximation result for the MAXEFM problem. We show a reduction from the Minimum Vertex Cover (MVC) problem to the MAXEFM problem in which all hospitals have a lower quota of either zero or one. Our reduction is inspired by the reduction showing inapproximability of the maximum size weakly stable matching problem in the presence of ties and incomplete lists (MAX SMTI) by Halldórsson et al. [13]. They show that MAX SMTI cannot be approximated to a factor of $\frac{21}{19} - \epsilon$ for any $\epsilon > 0$ unless P=NP. They also show a lower bound of 1.25 on the approximation ratio of MAX SMTI, under the conjecture [15] that MVC is hard to approximate within a factor of $2 - \epsilon$, $\epsilon > 0$. We show the same inapproximability ratios for the MAXEFM problem by a suitable reduction from MVC to MAXEFM.

**Reduction**: Given a graph $G = (V, E)$, which is an instance of the MVC problem, we construct an instance $G'$ of the MAXEFM problem. Thus $G'$ is an HRLQ instance. Let $VC(G)$ denote a minimum vertex cover of $G$ and $OPT(G')$ denote a maximum size envy-free matching in $G'$. Also, let $n = |V|$.

Corresponding to each vertex $v_i$ in $G$, $G'$ contains a gadget with three residents $A_i, B_i, C_i$, and four hospitals $a_i, b_i, c_i, x_i$. All hospitals have an upper-quota of one, $a_i, b_i, x_i$ have a lower-quota of zero and $c_i$ has a lower-quota of 1. Assume that the vertex $v_i$ has $d$ neighbors in $G$, namely $v_{i1}, v_{i2}, \ldots, v_{id}$. The preference lists of the three residents and four hospitals, are as in Fig. 4. We impose an arbitrary but fixed ordering of the neighbors of $v_i$ in $G$ which is used as a strict ordering of neighbors in the preference lists of resident $C_i$ and hospital $a_i$ in the HRLQ instance $G'$. Note that instance $G'$ has $N = 3|V|$ residents and $\frac{4N}{3}$ hospitals. This completes the description of the HRLQ instance $G'$ for the MAXEFM problem.

▶ Lemma 12. The instance $G'$ does not admit any stable and feasible matching.

**Proof.** The matching $M_s = \{(A_i, a_i), (B_i, b_i), (C_i, x_i) \mid i = 1, \ldots, n\}$ is stable in $G'$ since
every resident gets the first choice. Since $M_x$ leaves $c_i$ deficient for each $i$, it is not feasible. By the Rural Hospitals Theorem \[\text{Lemma 13.}\] we conclude that $G'$ does not admit any stable and feasible matching.

\[\implies\text{Lemma 13.}\] Let $G'$ be the instance of the MAXEFM problem constructed as above from an instance $G = (V, E)$ of the MVC problem. If $VC(G)$ denotes a minimum vertex cover of $G$ and $OPT(G')$ denotes a maximum size envy-free matching in $G'$, then $|OPT(G')| = 3|V| - |VC(G)|$.

**Proof.** We first prove that $|OPT(G')| \geq 3|V| - |VC(G)|$. Given a minimum vertex cover $VC(G)$ of $G$, we construct an envy-free matching $M$ for $G'$ as follows: $M = \{(B_i, c_i), (C_i, x_i) | v_i \in VC(G)) \cup \{(A_i, a_i), (B_i, b_i), (C_i, c_i) | v_i \notin VC(G)) \}$. Thus, for a vertex $v_i$ in the vertex cover, $M$ leaves the resident $A_i$ unmatched, thereby matching only two residents in the gadget corresponding to $v_i$. For a vertex $v_i$ that is not in the vertex cover, $M$ matches all the three residents in the gadget corresponding to $v_i$. Hence $|OPT(G')| \geq |M| = 2|VC(G)| + 3(|V| - |VC(G)|) = 3|V| - |VC(G)|$.

\[\implies\text{Claim 14.}\] $M$ is envy-free in $G'$.

**Proof.** It is straightforward to verify that there is no envy-pair consisting of two residents associated with the same vertex $v_i \in G$. Now, without loss of generality, assume that $C_i$ envies a resident matched to hospital $a_j$. By construction of our preference lists, $(v_i, v_j)$ is an edge in $G$. Thus, at least one of $v_i$ or $v_j$ must belong to $VC(G)$. If $v_i \in VC(G)$, then by the construction of $M$, $C_i$ is matched to its top choice hospital $x_i$ in $M$ and hence $C_i$ cannot participate in an envy-pair. Also, $a_i$ is left unmatched, hence $C_j$ can not form an envy-pair with $M(a_i)$.

Now we prove that $OPT(G') \leq 3|V| - |VC(G)|$. Let $M = OPT(G')$ be a maximum size envy-free matching in $G'$. Consider a vertex $v_i \in V$ and the corresponding residents and hospitals in $G'$. Note that $c_i$ must be matched in $M$ for $i = 1, \ldots, n$. Hence following two cases arise. Refer Fig. 7 for the patterns mentioned below.

**Case 1:** $M(c_i) = C_i$. Then either $M(B_i) = b_i, M(A_i) = a_i$ which is pattern 1 or $M(B_i) = a_i$ and $A_i$ is unmatched (pattern 2), or $M(B_i) = b_i$ and $M(a_i) = C_j$ for some $(v_i, v_j) \in E$ (pattern 3).

**Case 2:** $M(c_i) = B_i$. Then $(A_i, a_i) \notin M$, otherwise $B_i$ has a justified envy towards $A_i$. Also, $(C_i, b_i) \notin M$ otherwise $B_i$ has a justified envy towards $C_i$. Hence $M(C_i) = x_i$ (pattern 5) or $M(C_i) = a_j$ for some $(v_i, v_j) \in E$ (pattern 4).
Maximum-Size Envy-free Matchings

Figure 7 Five patterns caused by $v_i$

Vertex cover $C$ of $G$ corresponding to $M$: Using $M$, we now construct the set $C$ of vertices in $G$ which constitute a vertex cover of $G$. If $v_i$ is matched as pattern 1 then $v_i \notin C$, else $v_i \in C$. From the following claim, it follows that $C$ is a vertex cover of $G$.

$\triangleright$ Claim 15. If $(v_i, v_j) \in E$, then the gadgets corresponding to both of them can not be matched in pattern 1 in any envy-free matching $M$.

Proof. Let, if possible, there exist an edge $(v_i, v_j) \in E$ such that the gadgets corresponding to both $v_i$ and $v_j$ are matched in pattern 1 in $M$. Thus $M(C_i) = c_i$ and $M(a_j) = A_j$. But then $C_i$ has justified envy towards $A_j$ (via hospital $a_j$), contradicting the envy-freeness of $M$. $\triangleright$

Size of $C$: Each gadget could be matched in any of the patterns. Patterns 3 and pattern 4 occur in pairs for a pair of vertices $v_i, v_j$, that is, $M(a_i) = C_j$ or vice-versa. It can be verified that there is no envy-pair among the six residents corresponding to the vertices $v_i, v_j$ matched as pattern 3 and pattern 4 respectively. We say that pattern 3 contributes 2.5 edges to $M$ and pattern 4 contributes 1.5 edges. Hence together they contribute to an average matching size of 2. Only pattern 1 contributes 3 edges to $M$. Now it is straightforward to see that $|OPT(G')| = 2|C| + 3(|V| - |C|) = 3|V| - |C|$. Thus $|VC(G)| \leq |C| = 3|V| - |OPT(G')|$. This completes the proof of the lemma. $\triangleright$

Now we prove the hardness of approximation for the MAXEFM problem. These results (Lemma 17 and Theorem 18 and their proofs) are analogous to Theorem 3.2 and Corollary 3.4 from [13].

$\triangleright$ Proposition 16. [3] For any $\epsilon > 0$ and $p < \frac{3-\sqrt{5}}{2}$, the following statement holds: If there exists a polynomial time algorithm that, given a graph $G = (V, E)$, distinguishes between the following two cases, then $P = NP$.

1. $|VC(G)| \leq (1 - p + \epsilon)|V|$
2. $|VC(G)| > (1 - \max\{p^2, 4p^3 - 3p^4\} - \epsilon)|V|$

Proposition 16, Lemma 13 and $N = 3|V|$ together imply the following lemma.

$\triangleright$ Lemma 17. For any $\epsilon > 0$ and $p < \frac{3-\sqrt{5}}{2}$, the following statement holds: If there exists a polynomial time algorithm that, given a MAXEFM instance $G'$ consisting of $N$ residents and $\frac{4N}{3}$ hospitals, distinguishes between the following two cases, then $P = NP$. 

(a) pattern 1  (b) pattern 2  (c) pattern 3  (d) pattern 4  (e) pattern 5
Thus, effectively we have constructed a vertex cover with a fixed constant, with an approximation factor of at most $\frac{\delta}{19}$. 

**Proof.** As in [13], we substitute the MAXEFM one. In Section 4.3 we consider all residents have preference lists of length two and the quotas of all hospitals are at most $3$.

In this section we present our efficient algorithms for two special cases of the MAXEFM problem. We first present an algorithm for computing a maximal envy-free matching. In Section 4.2, we show that the same algorithm computes a maximum envy-free matching if all residents have preference lists of length two and the quotas of all hospitals are at most one. In Section 4.3, we consider CL-restricted instances and present an efficient algorithm for the MAXEFM problem.

1. $|OPT(G')| \geq \frac{2r+\epsilon}{3}N$
2. $|OPT(G')| < \frac{19+\epsilon}{27}N$

We assume without loss of generality that an approximation algorithm for the MAXEFM problem computes a maximal envy-free matching.

**Theorem 18.** It is NP-hard to approximate the MAXEFM problem within a factor of $\frac{21}{19} - \delta$, for any constant $\delta > 0$, even when the quotas of all hospitals are either $0$ or $1$.

**Proof.** As in [13], we substitute $p = \frac{1}{r}$ in Lemma 17 to obtain the simplified cases as follows.

1. $|OPT(G')| \geq \frac{21}{27}N$
2. $|OPT(G')| < \frac{19+\epsilon}{27}N$

Now, suppose we have a polynomial-time approximation algorithm for the MAXEFM problem with an approximation factor of at most $\frac{21}{19} - \delta$, $\delta > 0$. Consider the above two cases with a fixed constant, $\epsilon < \frac{3611}{3615}$. For an instance of case (1), this algorithm outputs a matching of size $\geq \frac{21}{27}N\frac{1}{19-\delta}$, and for an instance of case (2), it outputs a matching of size $< \frac{19+\epsilon}{27}N$. By our setting of $\epsilon$, we can easily verify that $\frac{21}{27}N\frac{1}{19-\delta} > \frac{19+\epsilon}{27}N$. Hence using this approximation algorithm, we can distinguish between instances of the two cases, implying that $P = NP$. This completes the proof of the theorem.

**Remark 19.** A long standing conjecture [13] states that MVC is hard to approximate within a factor of $2 - \epsilon$, $\epsilon > 0$. We obtain a lower bound of $1.25$ on the approximation ratio of MAXEFM, modulo this conjecture.

**Proof.** Suppose we have an $r$-approximation algorithm for the MAXEFM problem. Let $G = (V, E)$ be such that $|VC(G)| \geq \frac{|V|}{r}$. Approximability of MVC for general graphs is equivalent to the approximability of MVC for graphs with this property [21].

Using reduction provided in Lemma 17 obtain a MAXEFM instance $G'$. We showed that $|OPT(G')| = 3|V| - |VC(G)|$. Suppose we are given a maximal envy-free matching, $M$ for $G'$, obtained using this algorithm, we can construct a vertex cover $C$ for $G$ with $|C| \leq 3|V| - |M|$. Since $M$ is an $r$-approximation to $OPT(G')$, we have $|M| \geq \frac{|OPT(G')|}{r}$. Combining these constraints we get,

$$|C| \leq 3|V| - |M| \leq \left(6 - \frac{5}{r}\right)|VC(G)|$$

On substituting $r = 1.25 - \delta$ in the above equation ($0 < \delta \leq 0.25$), we see that $1 \leq 6 - \frac{5}{r} < 2$. Thus, effectively we have constructed a vertex cover $C$, which is a $k$-approximation to $VC(G)$, where $1 \leq k < 2$. This contradicts the conjecture that MVC is hard to approximate within a factor of $2 - \epsilon$, $\epsilon > 0$. This completes the proof.

### 4 Efficient algorithms for special cases

In this section we present our efficient algorithms for two special cases of the MAXEFM problem. We first present an algorithm for computing a maximal envy-free matching. In Section 4.2, we show that the same algorithm computes a maximum envy-free matching if all residents have preference lists of length two and the quotas of all hospitals are at most one. In Section 4.3, we consider CL-restricted instances and present an efficient algorithm for the MAXEFM problem.
4.1 Maximal Envy-free matching

In this section, we present an algorithm that computes maximal envy-free matching in an HRLQ instance. Let $G$ be an HRLQ instance which admits an envy-free matching – this can be verified efficiently using Yokoi’s algorithm. Recall that Yokoi’s algorithm outputs a minimum cardinality envy-free matching. Given our hardness results in the previous sections we relax the requirement to output a maximal envy-free matching. We remark that the straightforward approach of starting with an empty matching and adding edges as long as the matching is envy-free does not work. This is because we need to satisfy the lower-quotas of the hospitals in addition to maintaining envy-freeness. In [19], the authors consider the problem of computing maximal envy-free matchings. However, they start with Yokoi’s output $M_0$ and compute a maximal envy-free matching containing $M_0$. To see that this can be quite restrictive in terms of size, consider the example in Fig. 2. The matching $N_1 = \{(r_1, h_2)\}$ is the maximal matching containing Yokoi’s output. In contrast the matching $N_2$ of size $n$ is also maximal (in fact maximum) but does not contain Yokoi’s output. Our algorithm in this section outputs the matching $N_2$ for the example in Fig. 2. Though a maximal matching could be arbitrarily small compared to a maximum matching, we show below that it is a $\frac{1}{L+1}$-approximation, if $L$ is the length of longest preference list of a resident and all quotas are at most 1.

\begin{lemma}
Let $L$ be the length of longest preference list of any resident and suppose all the quotas are at most 1. Then, a maximal envy-free matching is $\frac{1}{L+1}$-approximation of MAXEFM.
\end{lemma}

\textbf{Proof.} Let $M$ be a maximal envy-free matching and $OPT$ be an optimal solution for MAXEFM. Let $R_{OPT}$ and $R_M$ denote the set of residents matched in $OPT$ and $M$ respectively. We will upper bound $|R_{OPT}|$ in terms of $|R_M|$. Let $X_1$ be the set of matched residents in both $M$ and $OPT$. Due to maximality of $M$, we have one of the two conditions true for a resident that is matched in $OPT$ but unmatched in $M$. For such resident $r$,

- Either there exists an unmatched hospital $h$ in the neighborhood of $r$ but adding $(r, h)$ to $M$ causes envy to some matched resident $r'$. We denote the set of such residents $r$ as $X_2$.
- Or all the hospitals in the neighborhood of $r$ are fully subscribed. We denote the set of such residents $r$ as $X_3$.

Thus, $|R_{OPT}| = |X_1| + |X_2| + |X_3|$. Since $X_1 = R_{OPT} \cap R_M \subseteq R_M$, so $|X_1| \leq |R_M|$. For every resident $r \in X_2$, we will charge a matched resident $r'$ in $M$ where $r'$ is defined above. Note that, each such $r'$ can envy a resident $r \in X_2$ via some hospital $h$ such that $h$ prefers $r'$ over $r$ and $r'$ prefers $h$ over $M(r')$. Thus, $|X_2| \leq |R_M| \ast (L - 1)$ because for $r'$ there are at most $L - 1$ higher preferred hospitals than $M(r')$ and each one of them could be matched to at most one resident in $OPT$ and thus contributes to $X_2$. Note that every resident in $X_3$ has all the neighboring hospitals full in $M$. The union of neighboring hospitals of every resident in $X_3$ is a subset of matched hospitals in $M$. Since each of these hospitals is matched to exactly one resident in $M$, $|X_3| \leq |R_M|$. Thus, $|R_{OPT}| \leq |R_M| \ast (L + 1)$, i.e. $M$ is $\frac{1}{L+1}$-approximation. This completes the proof and established Theorem 6. Also note that if $X_2$ is empty, then $M$ is 2-approximation since $|R_{OPT}| \leq 2 \ast |R_M|$. \hfill \qed

Now we present our algorithm that computes a maximal envy-free matching which need not strictly contain Yokoi’s output. At a high-level our algorithm works as follows: It starts with any feasible envy-free matching (possibly Yokoi’s output) and computes an envy-free augmenting path with respect to the current matching. An augmenting path $P$ with respect to an envy-free matching $M$ is envy-free is $M \oplus P$ is envy-free. In order to do so, it deletes
a set of edges from the graph. To define these deletions, we need the following definition from [19]. Let \( M \) be any envy-free matching in \( G \). If resident \( r \) is unmatched in \( M \) we let \( M(r) = \perp \) and \( \perp \) is considered as the least preferred choice by any resident.

**Definition 21.** [19] Let \( h \) be any hospital in \( G \), a threshold resident \( r' \) for \( h \), if one exists, is the most preferred resident in the preference list of \( h \) such that \( r' \) prefers \( h \) over \( M(r') \). If no such resident exists, we assume a unique dummy resident \( d_h \) at the end of \( h \)’s preference list to be the threshold resident for hospital \( h \).

In every iteration of our algorithm, we compute the threshold resident for every hospital. Observe that unless the threshold resident \( r' \) is matched to \( h \) or to a hospital higher preferred than \( h \), no resident lower preferred to \( r' \) by hospital \( h \) can be matched to \( h \). Thus, in our algorithm, we delete the set of edges \( E_1 \) (see Step 5) which correspond to lower preferred residents than the threshold resident for a hospital. Next, we consider for a hospital \( h \) its (matched) residents more preferred than its threshold resident \( r' \). For each such resident \( r \) we delete the edge \((r, h)\) if \( r \) is matched in \( M \) and \( M(r) \neq h \) and \( r \) prefers \( M(r) \) over \( h \). We denote such deletions as the set of edges \( E_2 \). Note that both these deletions ensure that after augmentation, a resident never gets demoted (matched to a lower preferred hospital). It is easy to see that the algorithm has at most \( n \) iterations each taking \( O(m + n) \) time, thus the overall running time is \( O(mn) \).

**Algorithm 1** Maximal envy-free matching

1: Input : \( G = (R \cup H, E) \)
2: Let \( M \) be any feasible envy-free matching in \( G \).
3: do
4: Compute the threshold resident \( r' \) for all \( h \in H \) w.r.t. \( M \)
5: Let \( G' = (R \cup H, E') \) be an induced sub-graph of \( G \), where \( E' = E \setminus (E_1 \cup E_2) \),
\( E_1 = \{(r, h)\} | (r, h) \in E \setminus M, h \) prefers threshold resident \( r' \) over \( r \)\),
\( E_2 = \{(r, h)\} | (r, h) \in E \setminus M, h \) prefers \( r \) over threshold resident \( r' \) and \( r \) prefers \( M(r) \) over \( h \)\}
6: if there exists an augmenting path \( P \) w.r.t. \( M \) in \( G' \) then
7: \( M_P = M \oplus P \)
8: \( M = M_P \)
9: else
10: Exit the loop.
11: while true
12: Return \( M \)

**Lemma 22.** Matching \( M \) produced by Algorithm 1 is maximal envy-free matching.

**Proof.** We first argue that \( M \) is envy-free. Assume not. Consider the first iteration (say \( i \)-th iteration) during the execution of the algorithm in which an envy pair is introduced. Call the matching at the end of iteration \( i \) as \( M_i \). Let that envy pair be \((r, r')\) w.r.t. \( M_i \) where \( M_i(r') = h \), and \( h \) prefers \( r \) over \( r' \) and \( r \) prefers \( h \) over \( M_i(r) \). Thus \( r \) envies \( r' \) w.r.t. \( M_i \). We consider two cases depending on whether or not the edge \((r', h)\) belongs to \( M_{i-1} \).

**Case 1:** Assume \((r', h) \in M_{i-1} \). In this case, \( r \) cannot be unmatched or matched to a lower preferred hospital than \( h \) in \( M_{i-1} \). Else \( r \) envies \( r' \) w.r.t. \( M_{i-1} \) a contradiction that \( i \) is the first iteration when an envy pair was introduced. Thus \( M_{i-1}(r) = h' \) where \( r \) prefers \( h' \) over \( h \). Now since the algorithm never demotes any resident, it implies that \( r \) prefers \( M_i(r) \) over
2 01-HRLQ-2R restricted instances

In Section 2 we proved that the MAXEFM is NP-hard when every resident has two hospitals in its preference list. However, in our hard instance hospitals have arbitrary quotas. Now consider the case where every resident has at most two length list and every hospital has at most one upper-quota, that is 01-HRLQ-2R instances. We show in Lemma 23 that in such instances Algorithm 1 computes a maximum size envy-free matching. For that, we need Lemma 24.

Lemma 23. If $M$ is an envy-free matching in a 01-HRLQ-2R instance and $|M^*| > |M|$ is another envy-free matching, then $M$ admits an augmenting path $P$ such that $M \oplus P$ is envy-free.

Proof. Consider the symmetric difference $M \oplus M^*$. There must exist an augmenting path $P = (r_1, h_1, r_2, h_2, \ldots, r_n, h_n)$ w.r.t. $M$ where $r_1$ is unmatched and $h_n$ under-subscribed in $M$. Further, for each $i = 1, \ldots, n$, $M^*(r_i) = h_i$. We note that $r_1$ prefers $M^*$ over $M$ (being matched versus being unmatched) and since $M^*$ is envy-free, it must be the case that $h_1$ prefers $r_2$ over $r_1$ (else $r_1$ envies $r_2$ w.r.t. $M^*$.) Since $M$ is also envy-free, we conclude that every resident $r_i$ in $P$ prefers $M^*(r_i)$ over $M(r_i)$; and every hospital $h_i$ prefers $r_{i+1}$ over $r_i$.

If $M \oplus P$ is envy-free, we are done. Therefore assume that $M^* = M \oplus P$ is not envy-free. Let $r$ have justified envy towards $r'$ w.r.t. $M'$. We first note that if both $r$ and $r'$ belong to $P$, then $M^*$ is not envy-free. Similarly if both $r$ and $r'$ do not belong to $P$, then $M$ is not envy free. Thus exactly one of $r$ or $r'$ belong to $P$.

We now claim that no resident $r_i$, for $i = 1, \ldots, n$ belonging to the path $P$ can have justified envy to any resident outside the path. Note that every resident in $P$ gets promoted in $M'$ as compared to $M$. Thus if some $r_i = r$ envys $r'$ w.r.t. $M'$, the same envy pair exists w.r.t. to $M$, a contradiction. Thus it must be the case that a resident $r$ not belonging to $P$ envies a resident $r' = r_i$ belonging to $P$. We argue that $r$ must be unmatched in $M$. First note that since $r$ envies $r_i$, there exists an edge $(r, h_i)$ in the graph. Note that the edge $(r, h_i)$ neither belongs to $M$ nor to $M^*$, because every $h_i$ except $h_n$ is matched in both and $M$ and $M^*$ along residents in $P$. Consider $M^*(r) = h$, then there exists an edge $(r, h)$. Since $h_i \neq h$ and the length of preference list of $r$ is at most two, we conclude that $r$ must be unmatched in $M$. Thus any resident $r$ that envies a resident $r'$ along $P$ is unmatched in $M$.

We now show that if $M \oplus P$ is not envy-free, we can construct another path $P'$ starting at such an unmatched resident such that $M \oplus P'$ is envy-free. Recall the augmenting path $P = (r_1, h_1, r_2, h_2, \ldots, r_n, h_n)$ w.r.t. $M$. Let $h_i$ be the hospital closest to $h_n$ along this path such that there exists an unmatched resident $r$ such that $(r, h_i)$ is in the graph and $r$ envies $r_i$ w.r.t. $M'$. If there are multiple such residents, pick the one that is most preferred by $h_i$. 

$h'$, a contradiction to the fact that $r$ envies $r'$ w.r.t. $M_i$. Case 2: Assume $(r', h) \notin M_{i-1}$. If $r$ was unmatched in $M_{i-1}$ or matched to a lower-preferred hospital then either $r$ or a higher preferred resident than $r$ is a threshold for $h$. Thus the edge $(r', h) \in E_1$ and hence gets deleted. Else $r$ is matched to a higher preferred hospital than $h$ in $M_{i-1}$. In this case by the same argument as above, $r$ does not envy $r'$. Thus in both the cases, we prove that $M$ is an envy-free matching.

To complete the proof we argue that $M$ is maximal. Assume not and let $M \cup e$ be an envy-free matching. Clearly $e$ does not belong to the last iteration of our algorithm, else we have an augmenting path. Thus $e \in E_1 \cup E_2$. Note that all edges in $E_2$ are of the form $(r, h)$ where $r$ is matched in $M$. Thus $e \notin E_2$. Finally for every edge $e \in E_1$, we know that $M \cup e$ is not envy-free. Thus $M$ is a maximal envy-free matching in $G$. ▶
Now consider the path $P' = \langle r, h_i, r_{i+1}, \ldots, r_n, h_n \rangle$. Note that by the choice of the hospital $h_i$, $M \oplus P'$ is envy-free. This gives us the desired path $P'$.

**Lemma 24.** Algorithm 2 produces maximum size envy-free matching if every resident’s preference list has length at most 2 and all the quotas are at most 1.

**Proof.** Let $M$ be the output of Algorithm 1 on a restricted instance $G$. Lemma 22 shows that $M$ is envy-free. For the sake of contradiction, assume that $M$ is not maximum size envy-free. Let $M^*$ be an envy-free matching in $G$ with $|M^*| > |M|$. Consider the symmetric difference $M \oplus M^*$. There must exist an augmenting path $P = \langle r_1, h_1, r_2, h_2, \ldots, r_n, h_n \rangle$ w.r.t. $M$ where $r_1$ and $h_n$ are unmatched in $M$. Moreover $M \oplus P$ is envy-free by Lemma 23. Further, for each $i = 1, \ldots, n$, $M^*(r_i) = h_i$. We note that $r_1$ prefers $M^*$ over $M$ (being matched versus being unmatched) and since $M^*$ is envy-free, it must be the case that $h_1$ prefers $r_2$ over $r_1$ (else $r_1$ envies $r_2$ w.r.t. $M^*$). Since $M$ is also envy-free, we conclude that every resident $r_i$ in $P$ prefers $M^*(r_i)$ over $M(r_i)$; and every hospital $h_i$ prefers $r_{i+1}$ over $r_i$. If all the $M^*$ edges in this path belong to $E'$ in the final iteration of Algorithm 1, then we arrive at a contradiction. Hence there exists some edge $(r_i, h_i) \in M^* \cap P$ such that $(r_i, h_i) \notin E'$; that is $(r_i, h_i)$ was deleted either as an $E_1$ or $E_2$ edge in Step 5 of Algorithm 1.

Suppose $(r_i, h_i) \in E_1$, then there exists a threshold resident for $h_i$, say $r'$ which $h_i$ prefers over $r_i$. Note that, by definition of threshold resident, $r'$ is matched in $M$ to some hospital $h'$ that it prefers lower over $h_i$. Since preference lists of residents are at most length two, $r'$ is not adjacent to any other hospital. We now contradict that $M^*$ is envy-free by observing that $q^+(h_i) = 1$ and $M^*(r_i) = h_i$. Thus for $M^*$ to be envy-free, $r'$ must be matched in $M^*$ to some hospital that is higher preferred than $h_i$. However, no such hospital exists, which implies that $r'$ must remain unmatched in $M^*$ and thus envies $r_i$. This implies that $M^*$ is not envy-free; a contradiction. Now, let $(r_i, h_i) \in E_2$. It implies that $r_i$ was higher preferred by $h_i$ than its threshold resident in the last iteration and $M(r_i)$ was higher preferred by $r_i$ than $h_i = M^*(r_i)$. This is a contradiction to the fact that each $r_i$ in $P$ prefers its $M^*(r_i)$ over $M(r_i)$. This completes the proof and establishes Theorem 2.

**Remark 25.** Both the restrictions in 01-HRLQ-2R are necessary for Lemma 23 to hold. See examples in Fig. 8a and Fig. 8b for which Algorithm 1 fails to compute maximum envy-free matching. In Fig. 8a, suppose we start with Yokoi’s matching $\langle (r_1, h_3) \rangle$ in line 2 of Algorithm 1. Then, there are two augmenting paths $P_1 = \langle h_1, r_1, h_4, r_2 \rangle$ and $P_2 = \langle h_2, r_2 \rangle$. If we chose $P_1$ then we get $M = \{ (r_1, h_1), (r_2, h_2) \}$ and the algorithm terminates but the maximum size envy-free matching in this instance is $M^* = \{ (r_1, h_4), (r_2, h_2), (r_3, h_3) \}$. Note that $M \oplus M^*$ contains the augmenting path $\langle r_3, h_3 \rangle$ which is not envy-free unless we first alternate along $\langle h_1, r_1, h_4, r_2, h_2 \rangle$. In Fig. 8b suppose we start with Yokoi’s matching $\{ (r_1, h_2) \}$. Then, we have two augmenting paths $P_1 = \langle r_3, h_2, r_1, h_1 \rangle$ and $P_2 = \langle r_3, h_3 \rangle$. If $P_2$ is chosen, we get $M = \{ (r_1, h_2), (r_3, h_3) \}$ and the algorithm terminates. The maximum envy-free matching in this instance is $M^* = \{ (r_1, h_1), (r_2, h_1), (r_3, h_2) \}$. Again note that $M \oplus M^*$ contains an augmenting path $\langle r_2, h_1 \rangle$ which is not envy-free unless we first alternate along $\langle h_3, r_3, h_2, r_1, h_1 \rangle$.

**Remark 26.** We stated earlier that Algorithm 1 also computes maximal envy-free matching. But, we note that for 01-HRLQ-2R instance, maximal matching computed by Algorithm 1 is maximum matching as well. However, Algorithm in 19 does not compute maximum size matching. Consider the example in Fig. 9 for which Algorithm in 19 computes $M = \{ (r_1, h_2) \}$ but our Algorithm 1 computes maximum matching $M^* = \{ (r_1, h_1), (r_2, h_2) \}$. 


Maximum-Size Envy-free Matchings

(a) HRLQ instance with 3 hospitals in preference list of a resident
(b) HRLQ instance with quota greater than 1

Figure 8 Inapplicability of Lemma 23 for unrestricted HRLQ instances

Figure 9 HRLQ instance for which matching computed by [10] is not maximum but Algorithm 1 computes maximum matching

4.3 CL-restricted instances

In this section, we consider the MAXEFM problem on CL-restricted HRLQ instances, that is, instances in which every hospital with positive lower-quota has all the residents in its preference list. As mentioned earlier, this restriction was considered earlier by Hamada et al. [14] in the context of computing almost stable matchings for HRLQ instances.

We first note that every HRLQ instance with CL-restriction admits a feasible envy-free matching (recall the characterization by Yokoi [24]). We now present a simple modification to the standard Gale and Shapley algorithm [10] which outputs a maximum size envy-free matching which is feasible. Our algorithm (pseudo-code in Algorithm 2) begins with all residents unmatched and an empty matching $M$. Throughout the algorithm, we maintain two parameters:

- $d$: denotes the deficiency of the matching $M$, that is, the sum of deficiencies of all hospitals with positive lower-quota.
- $k$: the number of unmatched residents w.r.t. $M$.

In every iteration, an unmatched resident who has not yet exhausted its preference list, proposes to the most preferred hospital $h$. If $h$ is deficient w.r.t. $M$, $h$ accepts the proposal from $r$. In case $h$ is not deficient we consider two cases. Firstly, assume $h$ is under-subscribed w.r.t. $M$. In this case $h$ accepts the proposal from $r$ only if there are enough unmatched residents to satisfy the deficiency of the other hospitals, that is, $k > d$. Next assume that $h$ is fully-subscribed. In this case, $h$ rejects the least preferred resident in $M(h) \cup r$. The algorithm terminates when either all residents are matched or every unmatched resident has exhausted its preference list.

We observe the following about the algorithm. Assuming that $G$ admits a feasible matching, we start with $k \geq d$ and this inequality is maintained throughout the algorithm. To see this, note that either $k$ and $d$ are reduced together in line 6 or only $k$ is reduced in line 17 if $k > d$ prior to this. If at some point during the course of the algorithm, $k = d$, then the equality continues to hold after that point. Furthermore, if no resident is rejected due to $k = d$ in line 11, Algorithm 2 then our algorithm degenerates to the Gale and Shapley algorithm [10] and hence our algorithm outputs a stable matching. We use Lemma 27 to
Algorithm 2 MAXEFM in C\(L\)-restricted HRLQ instances.

Input: HRLQ instance with C\(L\)-restriction
Output: Maximum size feasible envy-free matching
1: let \( M = \emptyset \); \( \delta = \sum_{h,q \in \mathcal{H}, q(h) \geq 0} q^{-}(h) \); \( k = |\mathcal{R}| \)
2: while there is an unmatched resident \( r \) which has at least one hospital not yet proposed
3: \( r \) proposes to the most preferred hospital \( h \)
4: if \(|M(h)| < q^{-}(h)\) then
5: \[ M = M \cup \{(r,h)\} \]
6: reduce \( \delta \) and \( k \) each by 1
7: else
8: if \(|M(h)| = q^{+}(h)\) then
9: let \( r' \) be the least preferred resident in \( M(h) \cup r \)
10: \[ M(h) = M(h) \cup r \backslash r' \]
11: else if \(|M(h)| < q^{+}(h)\) and \( k = d \) then
12: let \( r' \) be the least preferred resident in \( M(h) \cup r \)
13: \[ M(h) = M(h) \cup r \backslash r' \]
14: else
15: // we have \(|M(h)| < q^{+}(h)\) and \( k > d \)
16: \[ M = M \cup \{(r,h)\} \]
17: reduce \( k \) by 1
18: Return \( M \)

prove the correctness of our algorithm.

\textbf{Lemma 27.} Assuming \( G \) admits a feasible matching, \( M \) computed by \textbf{Algorithm 2} is feasible and maximum size envy-free.

\textbf{Proof.} We first prove that the output is feasible. Assume not. Then at termination, \( \delta > 0 \), that is, there is at least one hospital \( h \) that is deficient w.r.t. \( M \). However, since our algorithm started with \( k \geq d \), and this invariant is maintained throughout, it implies that \( k \geq 1 \). Thus there is some resident \( r \) unmatched w.r.t. \( M \). Note that \( r \) could not have been rejected by every hospital since \( h \) appears in the preference list of \( r \) and \( h \) is deficient at termination. This contradicts the termination of our algorithm and proves the feasibility of our matching.

Next, we prove that \( M \) is envy-free. Suppose for the sake of contradiction, \( M \) contains an envy pair \((r', r)\) such that \((h, r) \in M\) where \( h \) prefers \( r' \) over \( r \) and \( r' \) prefers \( h \) over \( M(r') \). This implies that \( r' \) must have proposed to \( h \) and \( h \) rejected it. If \( h \) rejected it because \(|M(h)| = q^{+}(h)\), \( h \) is matched with better preferred residents than \( r' \), a contradiction to the fact that \( h \) prefers \( r' \) over \( r \). If \( h \) rejected \( r' \) because \( k = d \), then there are two cases. Either \( r \) was matched to \( h \) when \( r' \) proposed to \( h \). In this case, in line 11 our algorithm rejected the least preferred resident in \( M(h) \). This contradicts that \( h \) prefers \( r' \) over \( r \). Similarly if \( r \) proposed to \( h \) later, since \( k = d \), the algorithm rejected the least preferred resident again contradicting the presence of any envy-pair.

Finally, we show that \( M \) is a maximum size envy-free matching. We have \( k \geq d \) at the start of the algorithm. If during the algorithm, \( k = d \) at some point, then at the end of the algorithm we have \( k = d = 0 \), implying that, we have an \( \mathcal{R} \)-perfect matching and hence the maximum size matching. Otherwise, \( k > d \) at the end of the algorithm and then we output a stable matching which is maximum size envy-free by Corollary 29 of Lemma 28 below. ▶
We need the following lemma from which it follows that a stable matching is a maximum size envy-free matching in an HR instance.

Lemma 28. If a resident is unmatched in stable matching, then in every envy-free matching that resident is unmatched.

Proof. We will prove the contra-positive. Let $M_e$ be an envy-free matching. Since, the set of residents matched in stable matching is invariant of the matching (by Rural Hospital Theorem), let’s pick an arbitrary stable matching $M_s$. Assume that resident $r_1$ is matched to hospital $h_1$ in $M_e$ and unmatched in $M_s$. Then, hospital $h_1$ must be full in $M_s$ and all the residents in $M_s(h_1)$ are more preferred by $h_1$ than $r_1$, otherwise $(r_1, h_1)$ blocks $M_s$. In $M_e$ at least one of the residents from $M_s(h_1)$ is not matched to $h_1$. Let that resident be $r_2$. Then, $r_2$ must be matched with a higher preferred hospital $h_2$ than $h_1$ in $M_e$ otherwise $r_2$ envies $r_1$. By similar argument as earlier, hospital $h_2$ must be full in $M_e$ and all the residents in $M_s(h_2)$ are better preferred by $h_2$ than $r_2$.

We prove that such process can terminate neither at a hospital nor at a resident - thus a resident like $r_1$ cannot exist. It is clear that once the process hits a resident $r_i$, it must find a higher preferred hospital $h_i$ than $h_{i-1}$ otherwise $r_i$ envies $r_{i-1}$. While at a hospital, the process always hits a new resident because a resident is matched to at most one hospital in a matching but while at a resident, it may hit some hospital more than once. We will prove that in the latter case also, eventually it must find a distinct resident.

Assume that for some resident $r_i$, we have $M_s(r_i) = h_k \in \{h_1, h_2, \ldots, h_{i-2}\}$. We saw that $h_k$ is full in $M_s$. Also, $h_k$ is matched to $r_i$ and $r_k$ in $M_e$ and matched to $r_{k+1}$ in $M_s$. If $r_{k+1}$ is the only resident matched to hospital $h_k$ in $M_s$, then $(r_k, h_k)$ and $(r_i, h_k)$ block $M_s$. Thus, there must exist another resident $r'$ distinct from $r_1$ to $r_i$ such that $r'$ is higher preferred than $r_i$ and $r_k$ by hospital $h_k$ and $r' \in M_s(h_k)$. Thus, we proved that even at the repeated hospital $h_k$, the process finds a distinct resident.

Corollary 29. A stable matching is maximum size envy-free matching in an HR instance.

Proof. By Lemma 28 it is clear that the set of unmatched residents in a stable matching is a subset of unmatched residents in any envy-free matching. Hence, size of every envy-free matching is at most the size of stable matching.

Running time: Algorithm 2 is an adaptation of Gale and Shapley algorithm [10] and runs in linear time in the size of the instance. This establishes Theorem 8.

Parameterized complexity

In this section, we investigate parameterized complexity of the MAXEFM and MIN-UR-EFM problems. We refer the reader to the comprehensive literature on parametric algorithms and complexity [4, 22, 7] for standard notation used in this section. Since the difficulty of MAXEFM lies in the instances where stable matchings are not feasible, we choose parameters related to those hospitals which have a positive lower quota, referred to as the LQ hospitals in this section. In particular, the deficiency of a given HR-LQ instance (see Definition 2 from Section 1) is a natural parameter. Unfortunately, the problem turns out to be $W[1]$-hard for this parameter.

Theorem 30. The MAXEFM and MIN-UR-EFM are $W[1]$-hard when deficiency is the parameter. The hardness holds even if all quotas are 0 or 1.
Proof. Consider the parameterized version of IND-SET problem i.e. a graph $G$ and solution size $k$ as the parameter. Let $(G', k')$ be parameterized reduced instance of MAXEFM, where $k'$ is the deficiency of $G'$ and let $k'$ be the parameter. From the NP-hardness reduction given in section 2 we already saw that the stable matching in $G'$ has deficiency $k$. Then, with $k' = k$, the same reduction is a valid FPT reduction implying that MAXEFM and MIN-UR-EFM are W[1]-hard as follows. If we had an FPT for MAXEFM (MIN-UR-EFM) with deficiency as parameter, we could use it to find maximum size envy-free matching and check if it’s size is $m + n$ (equivalently if the number of unmatched residents is 0). Then, we could answer whether the IND-SET instance $G$ has a $k$-clique. This contradicts since IND-SET is W[1]-hard.

5.1 A polynomial size kernel

First, we give a kernelization result for HRLQ instances with all quotas either 0 or 1. Given an HRLQ instance $G$ where all quotas are 0 or 1 and a integer $k$, the goal is to decide whether $G$ admits an envy-free matching of size at least $k$. We have shown in Section 2 that MAXEFM is NP-hard for this case as well. We consider the following three parameters.

- $\ell$: The size of a maximum matching in a given HRLQ instance.
- $p$: The highest rank of any LQ hospital in any resident’s preference list.
- $t$: Maximum number of non-LQ hospitals shared by the preference lists of any pair of residents.

Given the graph $G$ and $k$ we construct a graph $G'$ such that $G$ admits an envy-free matching of size $k$ if $G'$ admits an envy-free matching of size $k$.

Construction of the graph $G'$: We start by computing a stable matching $M_s$ in $G(V, E)$. If $|M_s| < k$, we have a “No” instance by Lemma 28. If $|M_s| \geq k$ and $M_s$ is feasible, we have a “Yes” instance. Otherwise, $|M_s| \geq k$ but it is infeasible. We know that $|M_s| \leq \ell$. We construct the graph $G'$ as follows.

Let $X$ be the vertex cover computed by picking matched vertices in $M_s$. Then, $|X| \leq 2\ell$. Since, $M_s$ is maximal, $I = V \setminus X$ is an independent set. We now use the marking scheme below to mark edges of $G$ which will belong to the graph $G'$.

Marking scheme: Our marking scheme is inspired by the marking scheme for a kernelization result in 2. Every edge with both end points in $X$ is marked. If $h \in X$ is a hospital, we mark all edges with other end-point in the independent set $I$ if the number of such edges are at most $\ell + 1$. Otherwise we mark the edges corresponding to the highest preferred $(\ell + 1)$ residents of $h$. If $r \in X$ is a resident then we do following: Let $p_r$ denote the highest rank of any LQ hospital in the preference list of $r$. Every edge between $r$ and an hospital at rank 1 to $p_r$ is marked. There can be at most $p$ edges marked in this step. We now construct a set of hospitals $C_r$ corresponding to $r$. The set $C_r$ consists of non-LQ hospitals which are common to the preference list of $r$ and some matched resident in $M_s$. That is,

$$C_r = \{h \in \mathcal{H} \mid q^-(h) = 0 \text{ and } \exists r' \in X, r' \neq r \text{ and } h \text{ is in preference list of both } r \text{ and } r'\}.$$

Mark all edges of the form $(r, h)$ where $h \in C_r$, if not already marked. Now amongst the unmarked edges incident on $r$ (if any exists) mark the edge to the highest preferred hospital $h$. We are now ready to state the reduction rules using the above marking scheme.

Reduction rules: We apply the following reduction rules as long as they are applicable.

1. If $v \in G$ is isolated, delete it.
2. If $(r, h)$ edge is unmarked, delete it.

Thus, we obtain an instance $G' = (V', E')$ where $V' = X \cup I$ and $E' = E(X, X) \cup E(X, I)$, where $E(A, B)$ is the set of edges with one end point in $A$ and other in $B$. 
Lemma 31 below bounds the size of the kernel \( G' \).

\[ \text{Lemma 31.} \quad \text{The graph } G' \text{ has } \text{poly}(\ell, p, t)\text{-size.} \]

**Proof.** We know that \( |X| \leq 2\ell \). Thus, \( E(X, X) = O(\ell^2) \). Let \( X_H \subset X \), be the set of hospitals in \( X \) and \( X_R \subset X = X \setminus X_H \) be the set of residents in \( X \). Then, \( |X_H| = |X_R| \leq \ell \).

For a hospital \( h \in X_H \), we have at most \( \ell + 1 \) marked edges having its other end-point in the independent set \( I \). For a resident \( r \in X_R \), we retained edges with at most \( p + t(\ell - 1) + 1 \) hospitals in independent set \( I \). Hence, \( |E(X, I)| \leq |X_R| \cdot (p + \ell t - t + 1) + |X_H| \cdot (\ell + 1) = O(\ell(p + \ell t - t + 1) + \ell^2) \).

Since \( I \) is independent set, \( |I| = |E(X, I)| \). Thus, the size of \( G' \) is \( O(\ell^2 + \ell(p + \ell t - t + 1)) \).

Safeness of first reduction rule is trivial. Lemma 32 and Lemma 33 prove that the second reduction rule is safe. So, \( G' \) is a kernel.

\[ \text{Lemma 32.} \quad \text{If } G' \text{ has a feasible envy-free matching } M' \text{ such that } |M'| \geq k \text{ then } M' \text{ is feasible and envy-free in } G. \]

**Proof.** Since, \( M' \subseteq E' \subseteq E \), so feasibility in \( G \) follows. Suppose for the contradiction that \( M' \) is not envy-free in \( G \). Then there exists a deleted edge \( (x, y) \) such that it causes envy.

Suppose \( x \) is a hospital. Since, \( (x, y) \) is deleted, there are \( \ell + 1 \) marked neighbors of \( x \), all more preferred than \( y \). Since size of maximum matching is at most \( \ell \), there exists a marked neighbor of \( x \), say \( y' \) who is unmatched in \( M' \). Since, \( x \) prefers \( y' \) over \( y \) it implies, \( x \) prefers \( y' \) over \( M'(x) \) implying that in \( G' \), \( y' \) envies \( M'(x) \) – a contradiction since \( M' \) is envy-free.

Suppose \( x \) is a resident. Given that \( (x, y) \) was deleted, \( y \) is non-LQ hospital. Since \( x \) participates in an envy pair, there are at least two residents \( x \) and \( M'(y) \) which have a common hospital \( y \) in their preference list. Thus by our marking scheme, \( (x, y) \) is not deleted – a contradiction.

\[ \text{Lemma 33.} \quad \text{If } G \text{ has a feasible envy-free matching } M \text{ such that } |M| \geq k \text{ then there exists a feasible envy-free matching } M' \text{ in } G' \text{ such that } |M'| \geq k. \]

**Proof.** If all the edges in \( M \) are present in \( G' \) then \( M' = M \) and we are done. Suppose not, then there exists an edge \( (x, y) \in M \setminus E' \). Let \( x \) be a hospital, then since \( (x, y) \) was deleted \( y \in I \). Note that \( y \) is unmatched in \( M_x \) in \( G \). By Lemma 28 \( y \) cannot be matched in any envy-free matching, which contradicts that \( (x, y) \in M \). Thus \( x \) must be a resident. Since \( (x, y) \) is deleted, there then exists a hospital \( h \) present only in preference list of \( x \) such that \( (x, y) \in E' \). By the marking scheme, \( x \) prefers \( h \) over \( y \). Thus, \( M' = M \setminus (x, y) \cup (x, h) \), which is envy-free since there is no other resident in the preference list of \( h \) other than \( x \).

### 5.2 A maximum matching containing a given envy-free matching

The \textsc{MAXEFM} problem has a polynomial-time algorithm when either there are no LQ hospitals or when all the LQ hospitals have complete preference lists. This fact suggests two parameters – number of LQ hospitals in a given instance \( (q) \), and maximum length of the preference list of any LQ hospital \( (t) \). Our parameterized algorithm for the parameters \( q \) and \( t \) and other parameterized algorithms, described in Section 5.3, make crucial use of an algorithm to extend an envy-free matching \( M \) to a maximum size envy free matching \( M^* \), such that \( M \subseteq M^* \). This algorithm was presented in [19] where it was proved that it produces a maximal envy-free matching containing the given envy-free matching \( M \). We present the
algorithm of [19] for completeness and prove that it outputs a maximum size envy-free matching containing $M$.

We recall their algorithm below as Algorithm 3. However, unlike Algorithm 2 in [19], where they start with Yoki’s output [21], we start with any feasible envy-free matching $M$. Since $M$ need not be a minimum size envy-free matching, in line 4 we set $q^+(h)$ in $G'$ as $q^+(h) = |M(h)|$.

**Algorithm 3** Maximum size envy-free matching containing $M$ [19]

1. Input : $G = (R \cup H, E), M$
2. Let $R'$ be the set of residents unmatched in $M$.
3. Let $H'$ be the set of hospitals such that $|M(h)| < q^+(h)$ in $G$.
4. Let $G' = (R' \cup H', E')$ be an induced sub-graph of $G$, where $E' = \{(r, h) \mid r \in R', h \in H', h \text{ prefers } r \text{ over its threshold resident } r_h\}$. Set $q^+(h)$ in $G'$ as $q^+(h)$ in $G$. Set $q^+(h)$ in $G'$ as $q^+(h) - |M(h)|$ in $G$. Set $q^+(h)$ in $G'$ as $q^+(h) - |M(h)|$ in $G$.
5. Each $h$ has the same relative ordering on its neighbors in $G'$ as in $G$.
6. Compute a stable matching $M_s$ in $G'$.
7. Return $M^* = M \cup M_s$.

Corollary [29] proves that a stable matching is a maximum size envy-free matching in an HR instance. This is used to prove that the output of the above algorithm is a maximum size envy-free matching containing $M$.

**Lemma 34.** The matching $M^*$ output by Algorithm 3 is a maximum size envy-free matching with the property $M \subseteq M^*$.

**Proof.** Assume for the sake of contradiction that $M'$ is an envy-free matching in $G$ such that $M' = M \cup M_s$ and $|M'| > |M^*|$. We first claim that $M_s \subseteq E'$. If not, then there exists $(r, h) \in M_s$ where $(r, h) \notin E'$. However, note that $(r, h)$ does not belong to $E'$ implies that there is a threshold resident $r_h$ such that $r_h$ prefers $h$ over $M(r_h)$. Thus, $r_h$ has justified envy towards $r$ w.r.t. $M'$. This contradicts the assumption that $M'$ is envy-free.

Recall that $G'$ is an HR instance and $M_s$ is a stable matching in $G'$. To complete the proof it suffices to note that a stable matching in $G'$ is a maximum size envy-free matching in $G'$ by Corollary [29] of Lemma [28].

5.3 FPT algorithms for MAXEFM

In this section, we give FPT algorithms for the MAXEFM problem on different parameters. Our first set of parameters is number of LQ hospitals $q$ and the maximum length of the preference list of any LQ hospital $\ell$. The algorithm is simple: it tries all possible assignments $M_e$ of residents to LQ hospitals. If some assignment is not envy-free we discard it. Otherwise we use Algorithm 3 to output a maximum size envy-free matching containing $M_e$. Since our algorithm tries out all possible assignments to LQ hospitals, and the extension of $M_e$ is a maximum cardinality envy-free matching containing $M_e$ (by Lemma 34) it is clear that the algorithm outputs a maximum size envy-free matching.

**Theorem 35.** The MAXEFM problem is FPT when the parameters are the number of LQ hospitals $q$ and length of the longest preference list of any LQ hospital $\ell$.

**Proof.** For an LQ hospital $h$, there are at most $2^\ell$ possible ways of assigning residents to $h$. Since the number of LQ hospitals is $q$, our algorithm considers $2^\ell q$ many different matchings.
Testing whether a matching $M_e$ is envy-free and to extend it to a maximum size envy-free matching containing $M_e$ using Algorithm 3 needs linear time. Thus we have an $O^*(2^{\ell}q)$ time algorithm for the MAXEFM problem. Here $O^*$ hides polynomial terms in $n$ and $m$. ◀

Besides the above, we give the following FPT results. Let $R_d$ be the set of residents that are acceptable to at least one deficient hospital. Let $s = |R_d|$. We denote the deficiency of the given HRLQ instance by $d$. We prove that the MAXEFM problem is FPT if parameters are $s$ and $d$.

▶ Theorem 36. The MAXEFM problem is FPT when the number of deficient hospitals ($d$) and the total number of residents acceptable to deficient hospitals ($s$) are parameters.

Proof. We use bounded branching algorithm as presented in Algorithm 4. Matching in line 2 is computed using EF-HR-LQ algorithm in [24].

Algorithm 4 FPT algorithm for MAXEFM parameterized in $s,d$

Input: HRLQ instance containing feasible envy-free matching
Output: Maximum size feasible envy-free matching
1: Let $H' = \{h \in H \mid h$ is deficient $\}$
2: $M^* =$ Yokoi’s matching.
3: while $H'$ is not empty do
4: Pick $h \in H'$
5: if Preference list of $h'$ is empty then
6: Mark the branch “invalid”.
7: For every resident $r$ in $h$’s preference list, create a branch and match $(r, h)$.
8: In every branch, prune the instance to remove future envy i.e. $E = E \setminus \{(r', h') \mid h' \text{ is more preferred by } r \text{ than } h \text{ and } r' \text{ is less preferred by } h' \text{ than } r\}$
9: for every valid leaf $l$ do
10: Let $A_l$ be the assignment to deficient hospitals.
11: Let $G_l$ be the pruned instance.
12: Find stable matching $M$ in $G_l$ where only unmatched residents and under-subscribed hospitals participate.
13: Let $M_l = M \cup A_l$.
14: if $|M_l| > |M^*|$ then
15: $M^* = M_l$
16: return $M^*$

Correctness: At every step, the instance is pruned to remove future envy, so it is easy to see that the matching output is envy-free. For the sake of contradiction, assume that there exists another larger envy-free matching $M^*$ than the matching $M$ output by Algorithm 4. There must exist at least one hospital $h \in H'$ which has at least one different partner in $M$ and $M^*$. But, since we are considering all possible assignments to deficient hospitals, we must have considered the assignment in $M^*$ as well. So, we could not have missed out a larger envy-free matching.

Running time: There are at most $s$ possible branches and depth is $d$. So, there are at most $s^d$ possible assignments. Removing future envy and computing stable matching takes $O(m)$. So, overall running time is $O(s^d \cdot m)$.
Hence, MAXEFM is FPT if parameters are number of deficient hospitals \((d)\) and the total number of unique residents acceptable to deficient hospitals \((s)\).

**Theorem 37.** The MAXEFM problem parameterized on the total number of residents acceptable to LQ hospitals is FPT.

**Proof.** Consider \(R' = \{r \in R \mid \exists h \in H_{LQ} \text{ such that } (r, h) \in E\}\). Thus, \(R'\) is the set of residents acceptable to at least one LQ hospital. Algorithm 5 is FPT for the parameter \(|R'|\). Yokoi’s matching in line 1 is computed using Yokoi’s EF-HR-LQ algorithm [23]. Matching in line 5 is computed using Algorithm 3.

**Algorithm 5** FPT algorithm for MAXEFM parameterized in \(|R'|\)

| Input: HRLQ instance containing feasible envy-free matching |
| Output: Maximum size feasible envy-free matching |
| 1: \(M^* = \) Yokoi’s matching |
| 2: for assignment \(A\) between \(R'\) and \(H_{LQ}\) do |
| 3: if \(A\) is not feasible or not envy-free then |
| 4: discard \(A\) |
| 5: else |
| 6: Compute maximum size envy-free matching \(M\) containing \(A\). |
| 7: if \(|M| > |M^*|\) then |
| 8: \(M^* = M\) |
| 9: return \(M^*\) |

**Correctness:** We consider all possible assignments to residents in \(R'\) using branching. We discard an assignment that is infeasible or not envy-free. Thus, we consider all possible envy-free and feasible assignment and extend them using Algorithm 3. By Lemma 34, \(M\) is maximum size envy-free matching that contains \(A\) and we pick the largest among them.

**Running time:** There are \(|R'|!\) possible assignments to check. Finding if an assignment is feasible and envy-free takes \(O(m)\). Computing maximum size envy-free matching containing a given assignment takes \(O(|M| m)\). So, overall running time is \(O(|R'|! m)\).

Hence, MAXEFM is FPT if parameter is the number of residents acceptable to LQ hospitals.

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