A design method of acoustic metamaterials with buckling vibrators

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Abstract. This paper proposes a periodic acoustic metamaterial with buckling vibrators, and investigates the design method for the band-gap characteristics of the acoustic metamaterial. The unit cell of the acoustic metamaterial is constructed by a continuous bar, curved beams embedded into the bar and the mass supported by the curved beams. The band structure of infinite period and the transmission characteristics of finite period are researched to analyze the effect of the structural and material parameters of the curved beam on the band-gap characteristics of the unit cell. And the effect of pre-strains applied to the curved beam on the band-gap characteristics is also discussed. The research illustrates that the acoustic metamaterial with buckling vibrators has a low frequency band-gap of 100-125Hz; in the frequency range of 0-200Hz, the central frequency of the band-gap increases with the increase of the beam width; the curvature radius of the beam has little influence on the central frequency of the band-gap, and a big bandwidth of the first band-gap can be obtained when the curvature radius is 20 mm; by applying pre-strain to the curved beam in a certain extent, the central frequency of the band-gap can be effectively reduced, and the attenuation more than 100 dB can be guaranteed.

1. Introduction

Vibration and noise has always been an important factor affecting the performances, such as accuracy, safety, reliability and service life, of modern industrial equipments. Vibration and noise can also reduce people's quality of life and endanger their physical and mental health. With the development of science and technology, vibration and noise has been controlled to a certain extent. But it is still difficult to isolate and absorb the low-frequency vibration due to its characteristics of strong penetrating power and long propagation distance.

The research on the low-frequency vibration reduction is of profound significance. In 1993, Kushwaha et al.[1] proposed the concept of a phononic crystal for the first time, and pointed out that the phononic crystal had a complete band-gap characteristic. In 2000, Liu et al.[2] published a paper in Science and first explained the local resonance band-gap mechanism of phononic crystals. The wavelength corresponding to the local resonance band-gap frequency was much larger than the lattice size, which provided a theoretical basis and implementation method for reducing the low-frequency vibration of small-sized (lattice constant) phononic crystals. Hirsekorn[3] further discussed the band-gap mechanism of the local resonant phononic crystal by simplifying the unit cell into a spring mass system and analyzing the start and stop frequencies of the local resonant band-gap. Pai Wang[4] used
the buckling method to adjust the local resonance band-gap. The pre-stress was applied to the acoustic metamaterial to cause buckling, and the band-gap frequency was reduced to a certain extent. Ryan L. Harne[5] designed a hyperdamping metamaterial by applying the pre-stress to reduce the inherent stiffness of the structure and improving the damping ratio of the structure.

Buckling can reduce the natural frequency of the structure while it can also introduce instability factors. So if the buckling was introduced into the unit cell to reduce the lower frequency vibration and noise, the stability of the structure should be considered.

Based on the theory of local resonance, this paper designs a periodic acoustic metamaterial consisting of a continuous bar and buckling vibrators, and investigates the method for adjusting the bandwidth and central frequency of the band-gap of the acoustic metamaterial. First, the vibration reduction principle of elastic wave in one-dimensional metamaterial bar is discussed; second, the unit cell is constructed through embedding curved beams and the mass into the continuous bar; then, the effect of the structural and material parameters of the curved beam in the unit cell on the band-gap is discussed, including the beam width, the elastic modulus and the curvature radius; next, the influence of pre-strains applied to the curved beam on the band-gap is also analyzed; finally, some valuable conclusions are drawn.

2. Theoretical models and analysis methods
By introducing local resonance units, local resonance acoustic metamaterials can achieve negative equivalent mass density at low frequencies and its band-gap frequency can be reduced. The limitation of the Bragg band-gap is broken, and the purpose of controlling the large wavelength propagation through a small size structure is achieved. Based on the concept of conventional mechanical vibration absorbers, the local resonance of the subsystem can be used to generate inertial forces to work against the external load and prevent elastic waves from propagating forward[6].

Here, an acoustic metamaterial is constructed by periodically embedding vibrators into a continuous bar with infinite length, as shown in Figure 1(a). The acoustic metamaterial can be regarded as a periodic structure consisting of the unit cell shown in Figure 1(b). Due to the periodicity, only the unit cell is considered in the modeling and analysis.

2.1. Vibrator installation
As shown in Figure 1(b), a vibrator is installed at Section 0 of the unit cell. The displacement at Section 0 is continuous, namely, the left displacement \( u^0 \) is equal to the right displacement \( u^o \). However, due to the influence of the vibrator, the load force on the section is not continuous, and the left load \( f^0 \) and the right load \( f^o \) are different, satisfying:

\[
\begin{bmatrix}
  u^0 \\
  f^0 
\end{bmatrix} =
\begin{bmatrix}
  1 & 0 \\
  G(w) & 1 
\end{bmatrix}
\begin{bmatrix}
  u^o \\
  f^o 
\end{bmatrix}
\]

where, \( G(w) \) is the transfer function of the spring-damped vibrator and can be written as:

\[
G(w) = \frac{k\Omega^2(1 + 2\zeta\Omega i)}{1 - \Omega^2 + 2\zeta\Omega i}
\]
where, \( i = \sqrt{-1} \); \( k \) is the spring stiffness; \( \zeta \) is the damping ratio of the vibrator, \( \zeta = \frac{c}{\sqrt{2km}} \); \( c \) is the damping of the vibrator; \( m \) is the mass of the vibrator; \( \Omega \) is the normalized frequency, \( \Omega = \frac{w}{w_0} \); \( w_0 \) is the natural frequency of the vibrator, \( w_0 = \sqrt{km} \); \( w \) is the angular frequency of the wave.

Then, the displacement \( u_L^- \) and the load \( f_L^- \) at Section \( L \) can be calculated by:

\[
\begin{bmatrix}
  u_L^- \\
  f_L^-
\end{bmatrix} = \begin{bmatrix}
  \cos(\beta_0 L) & \frac{\sin(\beta_0 L)}{EA\beta_0} \\
  -EA\beta_0 \sin(\beta_0 L) & \cos(\beta_0 L)
\end{bmatrix}
\begin{bmatrix}
  u_0^+ \\
  f_0^+
\end{bmatrix}
\]

(3)

where, \( E \) is the young's modulus of the continuous bar; \( A \) is the cross-sectional area of the continuous bar; \( \beta_0 \) is the propagation wave number of the continuous bar, \( \beta_0 = \frac{\omega}{c_0} \); \( c_0 \) is the wave velocity, \( c_0 = \sqrt{\frac{E}{\rho}} \); \( \rho \) is the linear mass density of the continuous bar.

Substituting equation (3) into equation (1) yields:

\[
\begin{bmatrix}
  u_L^- \\
  f_L^-
\end{bmatrix} = \begin{bmatrix}
  \cos(\beta_0 L) + \frac{\sin(\beta_0 L)}{EA\beta_0} G(w) & \frac{\sin(\beta_0 L)}{EA\beta_0} \\
  -EA\beta_0 \sin(\beta_0 L) + \cos(\beta_0 L) G(w) & \cos(\beta_0 L)
\end{bmatrix}
\begin{bmatrix}
  u_0^+ \\
  f_0^+
\end{bmatrix}
\]

(4)

Apply periodic boundary conditions:

\[
\begin{bmatrix}
  u_L^- \\
  f_L^-
\end{bmatrix} = e^{-i\beta L} \begin{bmatrix}
  u_0^- \\
  f_0^-
\end{bmatrix}
\]

(5)

The dispersion equation can be obtained:

\[
\lambda^2 \left[ 2\cos(\beta_0 L) + \frac{\sin(\beta_0 L)}{EA\beta_0} G(w) \right] + 1 = 0, \quad \lambda = e^{-i\beta L}
\]

(6)

It could be seen from equation (6), if \( \zeta = 0 \), the embedded spring-damped subsystem in the unit cell is an undamped vibrator. When \( |G(w)| \) is small, \( \lambda \) has a pair of conjugate complex solutions, and \( \beta \) corresponds to a pair of real solutions, showing a propagating wave in the bar. In particular, when \( |G(w)| = 0 \), the solution is \( \beta = \pm \beta_0 \), which is the inherent wave number of the bar. When \( |G(w)| \) is large, \( \lambda \) has a real solution, and \( \beta \) is the imaginary root, corresponding to the vanishing wave in the bar, meaning that the wave with this frequency cannot propagate in the bar.

If \( \zeta \neq 0 \), equation (6) is a complex coefficient equation, the solution \( \lambda \) is a pair of non-conjugate complex numbers, and the corresponding solutions \( \beta \) contain imaginary parts, which represents the absorption of waves.

3. Model construction and band-gap characteristics

Based on the vibration reduction principle of one-dimensional acoustic metamaterial bar, an acoustic metamaterial with buckling vibrators is designed, as shown in Figure 2. In the unit cell, four curved beams are embedded between the frame and the vibrator to provide additional stiffness for the vibrator. When the y-direction load is applied to the acoustic metamaterial, the curved beams will deform and buckle, which can effectively change the natural frequency of the resonance unit to achieve the adjustment of the band-gap.
Figure 2. An acoustic metamaterial with buckling vibrators: (a) an acoustic metamaterial consisting of unit cells; (b) the unit cell.

In the unit cell shown in Figure 2(b), the lattice constant is $a = 50mm$; the size of the mass is $16 \times 10mm$; the thickness of the frame is $b = 5mm$; the width of the beam is $w = 2mm$; the curvature radius of the beam is $r = 20mm$. The material parameters of each component are shown in Table 1.

Table 1. Material parameters.

| Components | Material  | Elastic modulus (Pa) | Poisson's ratio | Density (kg m$^{-3}$) |
|------------|-----------|----------------------|-----------------|-----------------------|
| Beam       | Polyurethane | 50e4                | 0.45            | 1350                  |
| Mass       | Steel     | 2.1e11               | 0.3             | 7850                  |
| Frame      | Aluminum  | 7.0e10               | 0.3             | 2710                  |
| Filler     | Rubber    | 3e4                 | 0.499           | 980                   |

In this paper, two methods are used to analyze band-gap characteristics of acoustic metamaterials: the band structure of infinite period and the transmission characteristics of finite period. According to Bloch's theorem of periodic structures, the eigenmodes of infinite periodic phononic crystals are all Bloch modes, and the corresponding eigenwaves are usually called Bloch waves. Each mode of Bloch waves can be characterized by a Bloch wave vector $k$ (wave number vector). The band structure diagram or dispersion diagram describes the relationship of the wave number vectors $k$ with frequencies [7].

For a finite period acoustic metamaterial shown in Figure 2, an x-direction line displacement load with unit amplitude is input from the left end of the metamaterial structure, and the displacement response is picked up at the other end. The response can be calculated by:

$$TL = 20 \cdot \log\left(\frac{|u_t|}{|u_i|}\right)$$

(7)

where, $TL$ is the amplitude attenuation rate; $u_t$ and $u_i$ are the amplitude of the transmitted and incident waves, respectively.

The band structure diagram and the amplitude-frequency curve are simulated by COMSOL software, and are given in Figure 3. Figure 3(a) is the band structure diagram, where the ordinate is frequency and the abscissa is the $X-T-X$ scan of the wave vector $k$ along the boundary of the irreducible Brillouin zone. The shadow zones in Figure 3(a) are the band-gap ranges of the unit cell, where the band structure curves do not pass and no elastic wave propagates. Figure 3(a) shows three band-gaps, including 87-94.8Hz, 95.1-130Hz and 164-174Hz, evaluated by the method of the band
structure of infinite period. Figure 3(b) is the amplitude-frequency curve and shows that the amplitude attenuation of elastic wave is obvious in the frequency ranges of 90-130Hz and 167-175Hz. And 90-130Hz and 167-175Hz are the band-gaps evaluated by the method of transmission characteristics of finite period.

![Figure 3. Band-gap characteristics: (a) band structure; (b) amplitude-frequency characteristics.](image)

It can be found that there are certain errors in the band-gaps obtained by the two methods of the band structure and the transmission characteristics. The reason is that, the simulation results in Figure 3(a) is achieved by applying periodic boundary conditions to the left and right ends of the unit cell, corresponding to an infinite periodic acoustic metamaterial, however, the band-gaps in Figure 3(b) are the simulation results of a finite periodic acoustic metamaterial consisting of 10 unit cells. The little errors indicate that the band-gaps obtained by the two methods agree well, verifying the results of the two methods in this paper.

4. Effect of beam parameters on band-gap
The band structure of acoustic metamaterials is relatively sensitive to parameter changes. Summarizing the influence of parameters on the band-gap can provide a reference for the structural design and band-gap optimization of acoustic metamaterials. The influence of the parameters of the vibrator and the filling material in the unit cell has been involved in the previous research. For the local resonance acoustic metamaterials, the larger the effective mass density difference between the vibrator and the filling material is, the larger the band-gap width becomes; the smaller the elastic modulus of the filling material is, the lower the band-gap frequency becomes. In this paper, curved beams are embedded in the general local resonance metamaterial, so the effect of the curved beam parameters on its band-gap characteristics will be mainly studied.

Because the beams increase the effective stiffness of the vibrator, the width of the beams has a great effect on the band-gap of the unit cell. By changing the width of the beams from 1mm to 5mm, the band-gaps are calculated and the simulation results are given in Figure 4.

It can be seen from Figure 4, as the width of the beam increases, the band-gap frequency moves high as a whole. The main reason is that the beam width increases and the stiffness also increases, resulting in the increases in the equivalent stiffness of the unit cell and overall eigenfrequency. In order to study the absorption of low-frequency elastic waves, the first band-gap is mainly concerned. With the increase of beam width, the frequency of the first band-gap not only increases but also tends
to narrow. This is because the increase of beam width has a great influence on the stiffness of the vibrator. The first band-gap starts from the second eigenfrequency and ends at the third eigenfrequency. The increase of the vibrator stiffness causes the bending of the third eigenfrequency, leading to the narrowing of the first band-gap. When the beam width increases to 3mm, the fifth-band gap decreases obviously. Two eigenfrequency curves around the 165Hz become flat, and a new band-gap is generated. The reason is that the local resonance of the beam occurs in this frequency range.

Figure 4. Effect of beam width on bandwidth.

The material of the beam is polyurethane with viscoelasticity and high elastic modulus and the elastic modulus of polyurethane beam also has a great effect on the band-gap of the unit cell. By changing the elastic modulus of the beam from 0.5MPa to 8Mpa, the band-gaps are calculated and the simulation results are given in Figure 5.

The elastic modulus of the beam describes its ability to resist deformation under external forces, which can be transformed into lateral and rotational stiffness of the vibrator in the unit cell. Because of the change in stiffness, as the elastic modulus of the beam increases, the frequency of the band-gap generally rises. When the elastic modulus increases to a certain extent, the vibrator will become very rigid. The vibrator, beam and frame are integrated so that the local resonance phenomenon of the vibrator disappears. At this time, the eigenfrequency curves become flat, and the width of the band-gap increases. It can be seen from the figure that the width of the first band-gap at 0.5MPa is equivalent to that at 8Mpa, but the central frequency is 30Hz lower, so the beam with the elastic modulus of 0.5MPa is more suitable for the vibration reduction of low-frequency elastic wave.

Figure 5. Effect of the elastic modulus of the beam on bandwidth.

In addition, the influence of the curvature radius of the beams on the band-gap is also considered. The minimum value of the curvature radius of the beam is set as 7.5mm, equal to a half of the distance
from the vibrator to the frame. The maximal value of the curvature radius is infinite, which transforms the curved beam into a straight beam. By changing the curvature radius in this range, the band-gaps are calculated and the simulation results are given in Figure 6.

Because the curvature of the beam has little effect on the natural frequency of the unit cell, it has little effect on the central frequency of the band-gap, but has a great effect on its width. Considering the first band-gap, when the curvature radius is less than 10 mm, the bending of the beam is large, and it can hardly provide stiffness for the vibrator. Although the frequency of the first band-gap is low, but its width is very little, only about 1 Hz. When the curvature radius of the beam is bigger than 50 mm and the beam almost becomes straight, the first band-gap gradually narrows. Therefore, for the low-frequency band-gap, the beam with the curvature radius of 20 mm has better band-gap characteristics.

Figure 6. Effect of the curvature radius of the beam on bandwidth.

5. Effect of pre-strain on band-gap
Although the band-gaps can be adjusted by means of changing structural and material parameters of the unit cell, the common problems are that band-gap frequency is limited to decrease and the band-gap is very narrow. In order to solve the problems, we will try to broaden bandwidth and decrease the central frequency of the band-gap by applying pre-strain to the beam in the unit cell. In the simulation, the y-direction pre-strain is directly applied to the beam, and the pre-strains are set as $\varepsilon = 0.01$, $\varepsilon = 0.02$ and $\varepsilon = 0.03$, respectively. The band structure curves are given in Figure 7, including with and without pre-strain cases.

Figure 7. Band structure curves of unit cell under pre-strain.
Figure (7) shows that, under the effect of pre-strain, the band structure curves decrease significantly and the central frequency of the first band-gap decreases from 115Hz in Figure 7(a) to about 52Hz in Figure 7(d), which can't be achieved by only changing the structural and material parameters of the unit cell. Changing the pre-strain in the beam has no effect on the mass and elastic modulus of the unit cell, so the decrease of the frequency is only related to the stiffness of the vibrator. In order to further study the reason of band structure curves decline, the first four order mode shapes of unit cell with and without pre-strain are selected for comparison. The eigenfunction corresponding to the eigenfrequency in the band structure represents the propagation mode of elastic wave, so each band-gap represents a unit cell vibration mode [8]. The mode shapes are shown in Figure 8 and Figure 9, where the direction and length of each arrow respectively denote the displacement direction and amplitude.

It can be seen from the vibration mode diagram that, the first-order eigenmode corresponds to the up and down vibration of the vibrator, which does not couple with the elastic wave in the x-direction, so the first band curve passes through the whole Brillouin region without band-gaps. The band curve corresponding to the second-order eigenmode is the starting position of the first band-gap, which is the vibration of the vibrator in the x-direction. The strong coupling effect with the elastic traveling wave generates the first band-gap. Comparing the second-order eigenmode diagrams with or without pre-strain, it can be found that under the effect of pre-strain, the stiffness of the vibrator is reduced, so the corresponding eigenfrequency is reduced from 96.33Hz to 40.93Hz. However, the coupling effect is weakened after pre-strain is applied, so the attenuation capacity here is reduced, which can be seen in the following frequency response curve in Figure 10. Modes 3 and 4 show that the torsional resonance mode is weakly coupled with the elastic wave propagation in the x-direction, so their influence on the band-gap characteristics is limited. In Figure 7, the dotted lines are shown as pass-bands in the band structure diagram, but they are shown as two small attenuation valleys in the frequency response curve in Figure 10, and there is still a large attenuation in this frequency range.
In order to further study the effect of pre-strain on the band-gap and vibration reduction, the lateral stiffness of the vibrator and the frequency response of the unit cell applied different pre-strains are calculated, as shown in Figure 11 and Figure 12. It can be found that the increase of pre-strain can greatly change the lateral stiffness of the vibrator, resulting in the reduction of the natural frequency of the unit cell. In Figure 10 (a), when there is no strain, there are two obvious band-gaps of 102-124Hz and 203.8-206.5Hz. The width of the first band-gap is 22 Hz, and the attenuation of elastic wave reaches 200 dB, so the vibration reduction performance is good. The effect of pre-strain on the band-gap is given in Figure 11. It can be seen that the central frequency of the first band-gap decreases with the increasing strain, but the bandwidth with more than 100 dB amplitude attenuation is shrinking. When the pre-strain is increased to 0.03, the first band-gap central frequency is reduced to about 50Hz, but its bandwidth is only 4Hz and the amplitude attenuation is less than 100dB, which can't achieve the expected vibration isolation effect. When the pre-strain is greater than 0.033, the first band-gap of the unit cell disappears, resulting in no band-gap below 200 Hz, and its amplitude attenuation is less than 80 dB. It can be concluded that the band-gap can be completely suppressed by applying a certain amount of pre-strain to the unit cell.

In Figure 13, the displacement contour is plotted in x-direction at 80Hz. When no pre-strain is applied, 80Hz is in the band-pass frequency range, and there is no vibration isolation effect. From the displacement contour, it can be seen that the propagation of elastic wave is unimpeded. When the strain is 0.02, the central frequency of the first band-gap is reduced, and the attenuation at 80Hz can reach 100dB. In the displacement contour, the elastic wave passed through only two unit cells and then
is suppressed. It is further verified that the central frequency of band-gap can be effectively reduced by applying pre-strain, and the low-frequency vibration and noise reduction can be realized.

Figure 13. Displacement contour with $\varepsilon = 0$ (top) and $\varepsilon = 0.02$ (bottom) at 80 Hz.

6. Conclusions
This paper constructs an acoustic metamaterial through periodically embedding buckling vibrators into a continuous bar, and investigates the effect of the structural and material parameters as well as the pre-strains applied to the vibrator on the band-gap of the metamaterial. The conclusions are as follows:

(1) Through analyzing the band structure and the frequency response, it is concluded that the acoustic metamaterial with buckling vibrators has a low frequency band-gap of 100-125Hz;

(2) In the frequency range of 0-200Hz, the central frequency of the band-gap increases with the increase of the beam width in the unit cell, and the bandwidth of the first band-gap decreases slightly;

(3) As the elastic modulus of the beam increases, the bandwidth of the first band-gap decreases first and then increases;

(4) The curvature radius of the beam has little influence on the central frequency of the band-gap, and a big bandwidth of the first band-gap can be obtained when the curvature radius is 20 mm;

(5) By applying pre-strain to the beam in a certain extent, the central frequency of the band-gap can be effectively reduced, and the attenuation more than 100 dB can be guaranteed.

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