A CARLESON TYPE MEASURE AND A FAMILY OF MÖBIUS INVARIANT FUNCTION SPACES

GUANLONG BAO AND FANGQIN YE

ABSTRACT. For $0 < s < 1$, let $\{z_n\}$ be a sequence in the open unit disk such that $\sum_n (1 - |z_n|^2)^s \delta_{z_n}$ is an $s$-Carleson measure. In this paper, we consider the connections between this $s$-Carleson measure and the theory of Möbius invariant $F(p, p - 2, s)$ spaces by the Volterra type operator, the reciprocal of a Blaschke product, and second order complex differential equations having a prescribed zero sequence.

1. INTRODUCTION

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. A function $\varphi$ is said to be a Möbius map if $\varphi$ is a one-to-one analytic function that maps $\mathbb{D}$ onto itself. Möbius maps form a group with respect to the composition of mappings. This group is said to be the Möbius group and denoted by $\text{Aut}(\mathbb{D})$. For $a \in \mathbb{D}$, the following special Möbius map defined by

$$\sigma_a(z) = \frac{a - z}{1 - \overline{a}z}, \quad z \in \mathbb{D},$$

exchanges points 0 and $a$. It is well known that every $\varphi$ in $\text{Aut}(\mathbb{D})$ can be represented as $\varphi = e^{i\theta} \sigma_a$ for some real number $\theta$ and some $a \in \mathbb{D}$. Clearly, the Möbius group $\text{Aut}(\mathbb{D})$ is homeomorphic to $\mathbb{D} \times \partial \mathbb{D}$.

A classical topic in complex analysis is to investigate the theory of Möbius invariant function spaces. Let $H(D)$ be the space of functions analytic in $D$. A space $X$ contained in $H(D)$ is a Möbius invariant function space if it is equipped with a semi-norm $\rho$ such that $f \circ \varphi \in X$ and $\rho(f \circ \varphi) \lesssim \rho(f)$ for all $f \in X$ and all $\varphi \in \text{Aut}(D)$. In this case, it is known that there is another semi-norm $\rho'$ on $X$ satisfying that $\rho$ and $\rho'$ are equivalent, and $\rho'(f \circ \varphi) = \rho'(f)$ for all $f \in X$ and $\varphi \in \text{Aut}(D)$. See [2] for the theory on Möbius invariant function spaces.

In this paper, we consider Möbius invariant function spaces $F(p, p - 2, s)$. For $0 < p < \infty$ and $0 < s < \infty$, the space $F(p, p - 2, s)$ is the set of functions $f \in H(D)$ for which

$$\|f\|_{F(p, p - 2, s)}^p = \sup_{a \in D} \int_D |f'(z)|^p (1 - |z|^2)^{p-2} (1 - |\sigma_a(z)|^2)^s dm(z) < \infty,$$

where $dm(z) = 1/\pi dx dy$, $z = x + iy$. It is known that if $p + s \leq 1$, then $F(p, p - 2, s)$ is trivial; that is, $F(p, p - 2, s)$ consists only of constant functions.

2010 Mathematics Subject Classification. 30H25; 30J10; 34C10; 47G10.

Key words and phrases. $F(p, p - 2, s)$ space, Carleson type measure, Blaschke product, Volterra type operator, complex differential equation.

The work was supported by NNSF of China (No. 12001352 and No. 12271328) and Guangdong basic and applied basic research foundation (No. 2022A1515012117).
Since
\[ \|f \circ \varphi - f(\varphi(0))\|_{F(p,p-2,s)} = \|f\|_{F(p,p-2,s)} \]
for every \( f \in F(p,p-2,s) \) and \( \varphi \in \text{Aut}(\mathbb{D}) \), \( F(p,p-2,s) \) is Möbius invariant. For \( p = 2 \) and \( s = 1 \), \( F(p,p-2,s) \) is equal to \( BMOA \), the well-known space of analytic functions in the Hardy space \( H^1 \) whose boundary values having bounded mean oscillation on \( \partial \mathbb{D} \) (cf. [4,19]). For \( p = 2 \), \( F(p,p-2,s) \) is the well-studied space \( Q_s \) (cf. [3,36]). For \( s > 1 \), all \( F(p,p-2,s) \) spaces are the same and equal to the Bloch space \( B \). Recall that \( B \) is the space of functions \( f \in H(D) \) such that
\[ \sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| < \infty. \]
\( F(p,p-2,s) \) spaces are special cases of a general family of analytic function spaces \( F(p,q,s) \) that were introduced by R. Zhao [40] and further investigated in [45]. See [44] for a recent survey on \( F(p,q,s) \) spaces. We also refer to [5,7,34] for some recent investigations on \( F(p,p-2,s) \) spaces.

Carleson type measures are key tools for the modern function theory and operator theory. For an arc \( I \) of \( \partial \mathbb{D} \) with arclength \( |I| \), the Carleson box \( S(I) \) is given by
\[ S(I) = \left\{ r \zeta \in \mathbb{D} : 1 - \frac{|I|}{2\pi} < r < 1, \ z \in I \right\}. \]
For \( 0 < s < \infty \), a nonnegative Borel measure \( \mu \) on \( \mathbb{D} \) is said to be an \( s \)-Carleson measure if \( \mu(S(I)) \lesssim |I|^s \) for all \( I \subseteq \partial \mathbb{D} \). When \( s = 1 \), we obtain the classical Carleson measure that was introduced by L. Carleson [9] to solve the problem of interpolation by functions in \( H^\infty \) and the corona problem. Here \( H^\infty \) is the space of bounded analytic functions in \( \mathbb{D} \) and we will write \( \|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)| \) for \( f \in H^\infty \). It is also well known (cf. [36]) that \( \mu \) is an \( s \)-Carleson measure if and only if
\[ \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \left( \frac{1 - |a|^2}{1 - \overline{z}a} \right)^s d\mu(z) < \infty. \] (1.1)

For \( X \subseteq H^\infty \) and a sequence \( \{z_n\} \) in \( \mathbb{D} \), \( \{z_n\} \) is called an interpolating sequence for \( X \) if for every bounded sequence \( \{\zeta_n\} \) of complex numbers, there is \( f \in X \) with \( f(z_n) = \zeta_n \) for each \( n \). By L. Carleson [9], \( \{z_n\} \) is an interpolating sequence for \( H^\infty \) if and only if \( \{z_n\} \) is uniformly separated; that is, there exists a positive constant \( \gamma \) satisfying
\[ \inf_{m} \prod_{n \neq m} \frac{|z_m - z_n|}{1 - \overline{z}_mz_n} \geq \gamma. \] (1.2)
A sequence \( \{z_n\} \) in \( \mathbb{D} \) is said to be separated if \( \inf_{n \neq k} \rho(z_n, z_k) > 0 \), where \( \rho(z_n, z_k) = |\sigma_{z_n}(z_k)| \) is the pseudo-hyperbolic metric between \( z_n \) and \( z_k \). For \( 0 < r < 1 \) and \( a \in \mathbb{D} \), denote by \( \Delta(a,r) = \{z \in \mathbb{D} : |\rho(a,z)| < r \} \) the pseudo-hyperbolic disk of center \( a \) and radius \( r \). It is well known (cf. [18]) that \( \{z_n\} \) in \( \mathbb{D} \) is uniformly separated if and only if \( \{z_n\} \) is separated and \( \sum_n (1 - |z_n|) \delta_{z_n} \) is a Carleson measure, where \( \delta_{z_n} \) is the point mass measure at \( z_n \). For a sequence \( \{z_n\} \) in \( \mathbb{D} \), it is known (cf. [30]) that \( \sum_n (1 - |z_n|) \delta_{z_n} \) is a Carleson measure if and only if \( \{z_n\} \) is finite unions of interpolation sequences for \( H^\infty \). See [14,28,29] for the study of \( \{z_n\} \) in \( \mathbb{D} \) satisfying that \( \sum_n (1 - |z_n|) \delta_{z_n} \) is a Carleson measure.

For \( 0 < s < 1 \), let \( \{z_n\} \) be a sequence in \( \mathbb{D} \) such that \( \sum_n (1 - |z_n|^2) s \delta_{z_n} \) is an \( s \)-Carleson measure. This kind of Carleson type measures defined by sequences
Ch. Pommerenke \cite{33} first studied

It is clear that

previous results from the literature.

mention the paper \cite{31} studying these operators related to

almost everywhere. A sequence

and the multiplier operator

systematically in \cite{1}. Operators

the Hardy space

1

to another is only the zero function. Ch. Yuan and C. Tong \cite{39}, and R. Qian and F. Ye \cite{44} used these Carleson type measures to describe interpolating sequences for \( H^\infty \cap F(p, p - 2, s) \) with certain ranges of parameters \( p \) and \( s \). These Carleson type measures are also used to investigate solutions of second order complex differential equations having prescribed zeros (cf. \cite{22, 38}). Recently, in \cite{7} the authors applied these Carleson type measures to construct Blaschke products for considering the proper inclusion relation associated with intersections and unions of some \( F(p, p - 2, s) \) spaces.

The aim of this paper is to consider further the connections between the \( F(p, p - 2, s) \) theory and sequence \( \{ z_n \} \) in \( \mathbb{D} \) such that \( \sum_{k=1}^{\infty} (1 - |z_n|^2) \delta_{z_n} \) is a \( s \)-Carleson measure. Our results involve the Volterra type operator from \( H^\infty \) to \( F(p, p - 2, s) \), the reciprocal of a related Blaschke product, and solutions of second order complex differential equations having prescribed zeros \( \{ z_n \} \).

Throughout this paper, we write \( a \lesssim b \) if there exists a positive constant \( C \) such that \( a \leq Cb \). The symbol \( a \approx b \) means \( a \lesssim b \lesssim a \).

2. Volterra type operators from \( H^\infty \) to \( F(p, p - 2, s) \)

In this section, we give a relation between the Volterra type operator from \( H^\infty \) to \( F(p, p - 2, s) \), and \( \{ a_k \} \) in \( \mathbb{D} \) satisfying that \( \sum_{k=1}^{\infty} (1 - |a_k|^2) \delta_{a_k} \) is a \( s \)-Carleson measure. Our proof also yields that the range of the Cesàro operator acting on \( H^\infty \) is contained in every non-trivial \( F(p, p - 2, s) \) space, which strengthens some previous results from the literature.

The Volterra type operator \( J_g \) with symbol \( g \in H(\mathbb{D}) \) defined on \( H(\mathbb{D}) \) by

\[
J_g f(z) = \int_0^z f(w)g'(w)dw, \quad z \in \mathbb{D}, \quad f \in H(\mathbb{D}).
\]

For the companion operator

\[
I_g f(z) = \int_0^z f'(w)g(w)dw, \quad z \in \mathbb{D}, \quad f \in H(\mathbb{D}),
\]

and the multiplier operator

\[
M_g f(z) = g(z)f(z), \quad z \in \mathbb{D}, \quad f \in H(\mathbb{D}),
\]

it is clear that

\[
M_g f(z) = f(0)g(0) + J_g f(z) + I_g f(z).
\]

Ch. Pommerenke \cite{33} first studied \( J_g \) and proved that \( J_g \) is a bounded operator on the Hardy space \( H^2 \) if and only if \( g \in BMOA \). Later, the operator \( J_g \) was studied systematically in \cite{11}. Operators \( J_g \) and \( I_g \) have attracted a lot of interest. Here we mention the paper \cite{51} studying these operators related to \( F(p, q, s) \) spaces.

An inner function \( f \) is an \( H^\infty \) function whose boundary values satisfy \( |f(e^{i\theta})| = 1 \) almost everywhere. A sequence \( \{ a_k \} \) in \( \mathbb{D} \) is called a Blaschke sequence if

\[
\sum_k (1 - |a_k|) < \infty.
\]
It is well known (cf. [42, p. 69] and [42, Lemma 4.30]) that \( B(z) = \prod_{k=1}^{\infty} \frac{|a_k|}{a_k - z} \frac{a_k - z}{1 - a_k \overline{z}}. \)

All Blaschke products are inner functions. If the zero sequence of a Blaschke product \( B \) is uniformly separated, then \( B \) is said to be an interpolating Blaschke product. A Blaschke product is called a Carleson-Newman Blaschke product if it can be expressed as a product of finitely many interpolating Blaschke products. Recall that a Blaschke product associated with a sequence \( \{a_k\} \) in \( \mathbb{D} \) is a Carleson-Newman Blaschke product if and only if \( \sum_k (1 - |a_k|^2) \delta_{a_k} \) is a 1-Carleson measure.

The following result is Theorem 1.4 in [32] characterizing inner functions in some \( F(p, p - 2, s) \) spaces.

**Theorem A.** Let \( 0 < s < 1 \) and \( p > \max\{s, 1 - s\} \). Then an inner function belongs to \( F(p, p - 2, s) \) if and only if it is a Blaschke product associated with a sequence \( \{z_k\}_{k=1}^{\infty} \) in \( \mathbb{D} \) which satisfies that \( \sum_k (1 - |z_k|)^s \delta_{z_k} \) is an \( s \)-Carleson measure.

We give the following conclusion first.

**Theorem 2.1.** Suppose \( g \in H(\mathbb{D}), 0 < p < \infty \) and \( 0 < s < \infty \) satisfying \( p + s > 1 \). Then the following statements hold:

(a) if \( s > 1 \), or \( s = 1 \) and \( p \geq 2 \), then \( I_g \) is a bounded operator from \( H^\infty \) to \( F(p, p - 2, s) \) if and only if \( g \in H^\infty \);

(b) if \( 0 < s < 1 \), or \( s = 1 \) and \( 0 < p < 2 \), then \( I_g \) is a bounded operator from \( H^\infty \) to \( F(p, p - 2, s) \) if and only if \( g = 0 \);

(c) \( J_g \) is a bounded operator from \( H^\infty \) to \( F(p, p - 2, s) \) if and only if \( g \in \bigcup_{p, s} \bigcup_{I \in \mathcal{B}} \mathcal{D}^{2s} \).

(d) if \( s > 1 \), or \( s = 1 \) and \( p \geq 2 \), then \( M_g \) is a bounded operator from \( H^\infty \) to \( F(p, p - 2, s) \) if and only if \( g \in H^\infty \).

(e) if \( 0 < s < 1 \), or \( s = 1 \) and \( 0 < p < 2 \), then \( M_g \) is a bounded operator from \( H^\infty \) to \( F(p, p - 2, s) \) if and only if \( g = 0 \).

**Proof.** (a) Note that \( F(p_1, p_1 - 2, s_1) \subseteq F(p_2, p_2 - 2, s_2) \) for all possible \( 0 < p_1 < p_2 \) and \( 0 < s_1 < s_2 \). It is also known that \( H^\infty \subseteq BMOA \subseteq B, BMOA = F(2, 0, 1), \) and \( F(p, p - 2, s) = B \) for \( s > 1 \). Now consider \( s > 1 \), or \( s = 1 \) and \( p \geq 2 \). Then \( H^\infty \subseteq F(p, p - 2, s) \) for such \( p \) and \( s \). Let \( g \in H^\infty \). For any \( f \in H^\infty \), then \( f \) also belongs to \( F(p, p - 2, s) \).

Therefore, \( I_g f \) is bounded from \( H^\infty \) to \( F(p, p - 2, s) \). On the other hand, suppose \( I_g \) is bounded from \( H^\infty \) to \( F(p, p - 2, s) \). Note that \( \sup_{b \in \mathbb{D}} \|\sigma_b\|_\infty \leq 1 \). Then

\[
\sup_{b \in \mathbb{D}} \|I_g \sigma_b\|^p_{F(p, p - 2, s)} \geq \sup_{b \in \mathbb{D}} \sup_{\Delta(b, r)} \int_{|z| < r} \frac{(1 - |z|^2)^p}{(1 - |z|^2)^p} |g(z)|^p (1 - |z|^2)^{p - 2} (1 - |\sigma_a(z)|^2)^s \, dn(z).
\]

It is well known (cf. [42, p. 69] and [42, Lemma 4.30]) that

\[
1 - |z| \approx 1 - |b| \approx |1 - \overline{b} z|
\]
for all \(z \in \Delta(b, 1/2)\), and \(|1 - \overline{z}a| \approx |1 - \overline{b}a|\) for all \(z \in \Delta(b, 1/2)\) and all \(a \in \mathbb{D}\). Also the area of \(\Delta(b, 1/2)\) is comparable with \((1 - |b|)^2\). Combining these facts and the subharmonicity of \(|g|^p\), we obtain

\[
1 \gtrsim \sup_{a \in \mathbb{D}} |g(b)|^p (1 - |\sigma_a(b)|^2)^s = |g(b)|^p
\]

for all \(b \in \mathbb{D}\). Thus \(g \in H^\infty\).

(b) First, consider \(0 < s < 1\). Suppose \(I_g\) is a bounded operator from \(H^\infty\) to \(F(p, p - 2, s)\). By the proof of (a), we get \(g \in H^\infty\). For every \(e^{i\theta} \in \partial \mathbb{D}\), from Lemma 3.3 in [5], there exists a Blaschke sequence \(\{a_k\}_{k=1}^\infty\) in \(\mathbb{D}\) such that \(e^{i\theta}\) is the unique accumulation point of \(\{a_k\}_{k=1}^\infty\).

\[
\sup_{a \in \mathbb{D}} \sum_{k=1}^\infty (1 - |\sigma_a(a_k)|^2)^{\frac{1+s}{2}} < \infty, \quad (2.2)
\]

and

\[
\sup_{a \in \mathbb{D}} \sum_{k=1}^\infty (1 - |\sigma_a(a_k)|^2)^s = +\infty. \quad (2.3)
\]

Condition (2.2) implies that \(\sum_{k=1}^\infty (1 - |a_k|)\delta_{a_k}\) is a Carleson measure. Hence \(\{a_k\}_{k=1}^\infty\) is finite unions of interpolation sequences for \(H^\infty\). Then there exists a positive integer \(m\) and positive real numbers \(\gamma_i, i = 1, 2, \ldots, m\), such that

\[
\{a_k\} = \bigcup_{1 \leq i \leq m} \{a_{ik}\}, \quad \text{and} \quad \inf_{1 \neq \ell} \prod_{j \neq \ell} \rho(a_{ij}, a_{il}) \geq \gamma_i, \quad i = 1, 2, \ldots, m.
\]

Because of (2.3), among these sequences \(\{a_{ik}\}_{k=1}^\infty, i = 1, 2, \ldots, m\), there exists a sequence \(\{a_{iok}\}_{k=1}^\infty\) such that

\[
\sup_{a \in \mathbb{D}} \sum_{k=1}^\infty (1 - |\sigma_a(a_{iok})|^2)^s = +\infty. \quad (2.4)
\]

Of course, \(e^{i\theta}\) is also the unique accumulation point of \(\{a_{iok}\}_{k=1}^\infty\). Denote by \(B\) the Blaschke product associated with the sequence \(\{a_{iok}\}_{k=1}^\infty\). By a well-known fact of interpolating sequences for \(H^\infty\) (cf. [21], p. 681), we know

\[
\bigcup_{k=1}^\infty \Delta \left( a_{iok}, \frac{\gamma_{i0}}{4} \right) \subseteq \left\{ z \in \mathbb{D} : (1 - |z|)|B'(z)| \geq \frac{\gamma_{i0}(1 - \gamma_{i0})}{8} \right\}.
\]
Clearly, pseudo-hyperbolic disks $\Delta \left( a_{i_0k}, \frac{\gamma(a)}{4} \right)$ are pairwise disjoint. Consequently,

$$\infty > \left\| I_gB \right\|_{F(p,p-2,s)}^p = \sup_{a \in \mathbb{D}} \int |B'(z)|^p |g(z)|^p (1 - |z|^2)^p (1 - |a(z)|^2)^s \, dm(z)$$

$$\geq \sup_{a \in \mathbb{D}} \int \left\{ z \in \mathbb{D}: |z| \geq \frac{\gamma(a)(1 - |a(z)|^2)}{4} \right\} |B'(z)|^p |g(z)|^p \times (1 - |z|^2)^p (1 - |a(z)|^2)^s \, dm(z)$$

$$\geq \sup_{a \in \mathbb{D}} \sum_{k=1}^{\infty} \int \Delta \left( a_{i_0k}, \frac{\gamma(a)}{4} \right) |g(z)|^p \frac{1 - |a(z)|^2}{(1 - |z|^2)^2} \, dm(z)$$

$$\geq \sup_{a \in \mathbb{D}} \sum_{k=1}^{\infty} |g(a_k)|^p (1 - |a_a(a_k)|^2)^s.$$

Combining this with (2.4), we get $|g(e^{i\theta})| = \lim_{k \to \infty} |g(a_{i_0k})| = 0$. Due to the arbitrariness of $e^{i\theta}$ and the maximum modulus principle, we get $g \equiv 0$. Conversely, if $g \equiv 0$, it is clear that $I_g$ is bounded from $H^\infty$ to $F(p, p-2, s)$.

Second, consider the case of $s = 1$ and $0 < p < 2$. Also, $g \equiv 0$ implies clearly the boundedness of $I_g$ from $H^\infty$ to $F(p, p-2, s)$. Next, we follow a method from [11, p. 177]. Let $I_g$ be bounded from $H^\infty$ to $F(p, p-2, s)$. As proved in part (a), $g \in H^\infty$. Suppose $g \not\equiv 0$. Then there exists a set $E$ in $[0, 2\pi]$ with positive Lebesgue measure such that $\lim_{r \to 1} g(re^{i\theta}) \neq 0$ for every $\theta \in E$. For $0 < p < 2$, it is known from [20, Theorem 1] that there exists $f \in H^\infty$ such that

$$\int_0^1 |f'(re^{i\theta})|^p (1 - r)^{p-1} \, dr = +\infty, \quad (2.5)$$

for almost every $\theta \in [0, 2\pi]$; that is, (2.5) holds when $\theta$ belongs to some set $E$ in $[0, 2\pi]$ and the Lebesgue measure of $F$ is $2\pi$. Thus for any $\theta \in E \cap F$, there exists an $r(\theta)$ in $(0, 1)$ satisfying that $\inf_{r(\theta) < r < 1} |g(re^{i\theta})| > 0$ and hence

$$\int_0^1 |f'(re^{i\theta})|^p |g(re^{i\theta})|^p (1 - r)^{p-1} \, dr$$

$$\geq \inf_{r(\theta) < r < 1} |g(re^{i\theta})| \int_0^1 |f'(re^{i\theta})|^p (1 - r)^{p-1} \, dr$$

$$= +\infty.$$

Clearly, the Lebesgue measure of $E \cap F$ is also positive. We get

$$\int_{D} |f'(z)|^p |g(z)|^p (1 - |z|)^{p-1} \, dm(z) = +\infty. \quad (2.6)$$

But the boundedness of $I_g$ from $H^\infty$ to $F(p, p-2, s)$ gives

$$\infty > \left\| I_g f \right\|_{F(p,p-2,1)}^p = \int_{D} |f'(z)|^p |g(z)|^p (1 - |z|^2)^{p-1} \, dm(z),$$

which contradicts (2.6). Hence $g \equiv 0$.

(c) Suppose $J_g$ is bounded from $H^\infty$ to $F(p, p-2, s)$. Set $f(z) \equiv 1$. Then $J_g f \in F(p, p-2, s)$. Since $g(z) = J_g f(z) + g(0)$, we get $g \in F(p, p-2, s)$. 


Conversely, let $g \in F(p, p - 2, s)$. For $h \in H^\infty$, it is clear that $\|J_g h\|_{F(p, p - 2, s)}^p \leq \|h\|_2^p \|g\|_{F(p, p - 2, s)}^p$, which gives the boundedness of $J_g$ from $H^\infty$ to $F(p, p - 2, s)$.

(d) Let $M_g$ be a bounded operator from $H^\infty$ to $F(p, p - 2, s)$. Because constant functions are in $H^\infty$, we get $g \in F(p, p - 2, s)$. By (c), $J_g$ is bounded from $H^\infty$ to $F(p, p - 2, s)$. Because of (2.1), $I_g$ is also bounded from $H^\infty$ to $F(p, p - 2, s)$.

From (a), we get $g \in H^\infty$. Conversely, let $g \in H^\infty$. Since $g > 1$, or $s = 1$ and $p \geq 2$, $g$ also belongs to $F(p, p - 2, s)$. By (a), (c) and (2.1), we get the boundedness of $M_g$ from $H^\infty$ to $F(p, p - 2, s)$.

The proof of (e) is similar to the proof of (d), so we omit it.

The following result gives a connection between the Volterra type operator from $H^\infty$ to $F(p, p - 2, s)$ and the Carleson type measures we considered.

**Theorem 2.2.** Suppose $B$ is a Blaschke product with zeros $\{a_k\}_{k=1}^\infty$. Let $0 < s < 1$ and $p > \max\{s, 1 - s\}$. Then the following conditions are equivalent:

(a) $\sum_{k=1}^\infty (1 - |a_k|^2)^s \delta_{a_k}$ is an $s$-Carleson measure;

(b) $J_B$ is a bounded operator from $H^\infty$ to $F(p, p - 2, s)$.

**Proof.** This result is clearly from Theorem A and Theorem 2.1. □

If $g(z) = -\log(1 - z)$, then $J_g$ is the modified Cesàro operator $\tilde{C}$, namely,

$$\tilde{C}f(z) = \int_0^z \frac{f(w)}{1 - w} dw, \quad z \in \mathbb{D}, \quad f \in H(\mathbb{D}).$$

Note that the Cesàro operator $C$ is defined by

$$Cf(z) = \frac{1}{z} \int_0^z \frac{f(w)}{1 - w} dw, \quad z \in \mathbb{D}, \quad f \in H(\mathbb{D}).$$

N. Danikas and A. Siskakis [12] showed that $\mathcal{C}(H^\infty) \nsubseteq H^\infty$ but $\mathcal{C}(H^\infty) \subseteq BMOA$. M. Essén and J. Xiao [16] obtained that $\mathcal{C}(H^\infty) \subseteq Q_s$ for $0 < s < 1$. Note that $F(p, p - 2, s) \subseteq Q_s$ when $0 < p < 2$ and $\max\{1 - p, 0\} < s \leq 1$. We obtain smaller Möbius invariant function spaces closing to $\mathcal{C}(H^\infty)$ as follows.

**Theorem 2.3.** Let $0 < p < \infty$ and $0 < s < \infty$ with $p + s > 1$. Then $\mathcal{C}(H^\infty) \subseteq F(p, p - 2, s)$.

**Proof.** It is known (cf. [31]) that the function $g(z) = -\log(1 - z)$ belongs to all nontrivial $F(p, p - 2, s)$ spaces. By Theorem 2.1, $J_g$ is a bounded operator from $H^\infty$ to $F(p, p - 2, s)$; that is, $\tilde{C}(H^\infty) \subseteq F(p, p - 2, s)$. Hence $\mathcal{C}(H^\infty) \subseteq F(p, p - 2, s)$. □

One can refer to [6, 8, 17] for more results on the range of the Cesàro operator or Cesàro-like operators acting on $H^\infty$.

3. $s$-CARLESON MEASURE $\sum_{n=1}^\infty (1 - |z_n|^2)^s \delta_{z_n}$ VIA THE RECIPROCAL OF A BLASCHKE PRODUCT IN $F(p, p - 2, s)$

Let $\{z_n\}_{n=1}^\infty$ be a separated Blaschke sequence in $\mathbb{D}$ and let $B$ be the Blaschke product associated with $\{z_n\}_{n=1}^\infty$. Suppose $0 < p < 2$. By Theorem C in [29], $\sum_{n=1}^\infty (1 - |z_n|^2) \delta_{z_n}$ is a Carleson measure if and only if

$$\sup_{\varphi \in \text{Aut}(\mathbb{D})} \int_{\mathbb{D}} \frac{1}{|B(\varphi(z))|^p} dm(z) < \infty.$$ (3.1)
Clearly, (3.1) is equivalent to

\[ \sup_{\varphi \in \text{Aut}(\mathbb{D})} \int_{\mathbb{D}} \left( \frac{1}{|B(\varphi(z))|} - 1 \right)^p \, dm(z) < \infty. \]  

(3.2)

In this section, for \(0 < s < 1\) and a separated Blaschke sequence \(\{z_n\}_{n=1}^\infty\), based on the description of Blaschke products in \(F(p, p - 2, s)\), we give a characterization of \(s\)-Carleson measure \(\sum_{n=1}^\infty (1 - |z_n|^2)^s \delta_{z_n}\) in terms of the reciprocal of the Blaschke product with zeros \(\{z_n\}\). Indeed, this Blaschke product belongs to some \(F(p, p - 2, s)\) spaces.

By (1.1), for \(s > 0\), \(\sum_{n=1}^\infty (1 - |z_n|^2)^s \delta_{z_n}\) is an \(s\)-Carleson measure if and only if

\[ \sup_{\varphi \in \text{Aut}(\mathbb{D})} \sum_{n=1}^\infty (1 - |\varphi(z_n)|^2)^s < \infty. \]  

(3.3)

Let \(\phi \in \text{Aut}(\mathbb{D})\). From (3.3), if \(\sum_{n=1}^\infty (1 - |z_n|^2)^s \delta_{z_n}\) is an \(s\)-Carleson measure, then \(\sum_{n=1}^\infty (1 - |\phi(z_n)|^2)^s \delta_{\phi(z_n)}\) is also an \(s\)-Carleson measure. In this sense, \(s\)-Carleson measure \(\sum_{n=1}^\infty (1 - |z_n|^2)^s \delta_{z_n}\) is Möbius invariant. Hence the equivalent characterization we will give also shows this invariance.

We begin with the following auxiliary result.

**Lemma 3.1.** Let \(0 < p < \infty\), \(0 \leq q < \infty\), and \(0 < s < 1\) such that \(p + s > 1\). Suppose \(B\) is a Blaschke product associated with \(\{z_n\}_{n=1}^\infty\) in \(\mathbb{D}\). If

\[ \sup_{\varphi \in \text{Aut}(\mathbb{D})} \int_{\mathbb{D}} \frac{1 - |B(\varphi(z))|^p}{|B(\varphi(z))|^q} (1 - |\varphi(z)|^2)^{s-2} \, dm(z) < \infty, \]

then \(\sum_{n=1}^\infty (1 - |z_n|^2)^s \delta_{z_n}\) is an \(s\)-Carleson measure.

**Proof.** The Schwarz-Pick Lemma gives

\[ (1 - |z|^2)|B'(z)| \leq 1 - |B(z)|^2 \]

for all \(z \in \mathbb{D}\). Combining this with the change of variables, we deduce

\[ \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |B'(z)|^p (1 - |z|^2)^{p-2} (1 - |\sigma_a(z)|^2)^s \, dm(z) \]
\[ \leq \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \frac{(1 - |B(z)|^p)}{|B(z)|^q} (1 - |\sigma_a(z)|^2)^s \, dm(z) \]
\[ \leq \sup_{\varphi \in \text{Aut}(\mathbb{D})} \int_{\mathbb{D}} \frac{(1 - |B(\varphi(z))|^p)}{|B(\varphi(z))|^q} (1 - |\varphi(z)|^2)^{s-2} \, dm(z) \]
\[ < \infty. \]

Hence \(B \in F(p, p - 2, s)\). Note that \(0 < p < \infty\) and \(0 < s < 1\) with \(p + s > 1\). It follows from Theorem 4.3 in [32] that \(\sum_{n=1}^\infty (1 - |z_n|^2)^s \delta_{z_n}\) is an \(s\)-Carleson measure.

The following useful lemma is from [27].

**Lemma B.** If \(B(z)\) is the Blaschke product associated with a sequence \(\{z_n\}\) satisfying condition (7.2) for a given \(\gamma\), and if \(\epsilon > 0\) is given, there exists a constant \(\gamma_0\) depending only on \(\gamma\) and \(\epsilon\) such that \(|B(z)| \geq \gamma_0\) whenever \(\rho(z, z_n) \geq \epsilon\) for all \(n\).

Lemma B below is also well-known; see [38] p. 100] for a brief proof of it.
Lemma C. Let $B(z)$ be the Blaschke product associated with a sequence $\{z_n\}$ satisfying condition (2) for a given $\gamma$. Then there exists a positive constant $C$ depending only on $\gamma$ such that

$$|B(z)| \geq C \rho(z, z_k)$$

for all $z \in \Delta(z_k, \delta/4)$ and for every $k$.

The following conclusion is Corollary 2.4 in [32].

Lemma D. Let $S$ be an inner function and let $1 \leq p < \infty$, $-2 < q < \infty$, and $0 \leq s, p^* < \infty$ such that $p > q + s + 1 > 0$. Then, for any analytic function $f$ in $\mathbb{D}$ and $a \in \mathbb{D}$, the following quantities are comparable:

(a) $$\int_{\mathbb{D}} |f(z)|^{p^*} (1 - |S(z)|^2)^p (1 - |z|^2)^{q-p}(1 - |\sigma_a(z)|^2)^s \, dm(z);$$

(b) $$\int_{\mathbb{D}} |f(z)|^{p^*} |S'(z)|^p (1 - |z|^2)^q (1 - |\sigma_a(z)|^2)^s \, dm(z).$$

Note that if $0 < s < 1$ and $\sum_{n=1}^{\infty} (1 - |z_n|^2)^s \delta_{z_n}$ is an $s$-Carleson measure, then $\{z_n\}_{n=1}^{\infty}$ is finite unions of uniformly separated sequences. Using Lemma [B] and Lemma [D] we prove the following result.

Lemma 3.2. Suppose $0 < s < 1$ and $\{z_n\}_{n=1}^{\infty}$ is a sequence in $\mathbb{D}$ such that $\sum_{n=1}^{\infty} (1 - |z_n|^2)^s \delta_{z_n}$ is an $s$-Carleson measure. Let $B$ be the Blaschke product associated with $\{z_n\}_{n=1}^{\infty}$. Assume $1 \leq p < \infty$ and $0 \leq q < 2/k$, where $k$ is the number of unions of uniformly separated sequences as which $\{z_n\}_{n=1}^{\infty}$ can be written. Then

$$\sup_{\varphi \in \text{Aut}(\mathbb{D})} \int_{\mathbb{D}} \frac{(1 - |B(\varphi(z))|^p)}{|B(\varphi(z))|^q} (1 - |z|^2)^{s-2} \, dm(z) < \infty.$$  

Proof. Since $\{z_n\}_{n=1}^{\infty}$ is $k$ unions of uniformly separated sequences, there exist positive real numbers $\gamma_i$, $i = 1, 2, \cdots, k$, such that

$$\{z_n\}_{n=1}^{\infty} = \bigcup_{1 \leq i \leq k} \{z_{m_i}\}_{n=1}^{\infty},$$

and

$$\inf \prod_{j \neq \ell} \rho(z_{ij}, z_{il}) \geq \gamma_i, \quad i = 1, 2, \cdots, k.$$  

Denote by $B_i$ the Blaschke product associated with $\{z_{m_i}\}_{n=1}^{\infty}$.

By the change of variables, we get

$$\sup_{\varphi \in \text{Aut}(\mathbb{D})} \int_{\mathbb{D}} \frac{(1 - |B(\varphi(z))|^p)}{|B(\varphi(z))|^q} (1 - |z|^2)^{s-2} \, dm(z) = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \frac{(1 - |B(z)|^p)}{|B(z)|^q} (1 - |\sigma_a(z)|^2)^s \, dm(z). \quad (3.4)$$
Thus, (3.7) and (3.8) yield
\[
\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \frac{(1 - |B(z)|)^p (1 - |\sigma_a(z)|^2)^s}{|B(z)|^q} \frac{1}{(1 - |z|^2)^2} \, dm(z) \\
\leq \sup_{a \in \mathbb{D}} \prod_{i=1}^k \left( \int_{\mathbb{D}} \frac{1}{|B_i(z)|^{q_x_i}} \frac{(1 - |B(z)|)^p (1 - |\sigma_a(z)|^2)^s}{(1 - |z|^2)^2} \, dm(z) \right)^{1/x_i} ,
\] (3.5)
where \(x_1, \ldots, x_k\) are positive numbers such that
\[
\frac{1}{x_1} + \cdots + \frac{1}{x_k} = 1 .
\] (3.6)

Bear in mind that \(0 < s < 1\) and \(1 \leq p < \infty\). It follows from Lemma B and Theorem A that
\[
\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \frac{(1 - |B(z)|)^p (1 - |\sigma_a(z)|^2)^s}{(1 - |z|^2)^2} \, dm(z) \approx \|B\|_{F_{p,p-2,s}}^p < \infty .
\] (3.8)

Thus, (3.7) and (3.8) yield
\[
\sup_{a \in \mathbb{D}} \prod_{i=1}^k \left( \int_{\mathbb{D}} \frac{1}{|B_i(z)|^{q_x_i}} \frac{(1 - |B(z)|)^p (1 - |\sigma_a(z)|^2)^s}{(1 - |z|^2)^2} \, dm(z) \right)^{1/x_i} < \infty .
\] (3.9)

On the other hand, by Lemma C we deduce
\[
\sup_{a \in \mathbb{D}} \prod_{i=1}^k \left( \int_{\mathbb{D}} \frac{1}{|B_i(z)|^{q_x_i}} \frac{(1 - |B(z)|)^p (1 - |\sigma_a(z)|^2)^s}{(1 - |z|^2)^2} \, dm(z) \right)^{1/x_i} \\
\leq \sup_{a \in \mathbb{D}} \prod_{i=1}^k \left( \sum_{j=0}^{\infty} \int_{\Delta(z_{i,j}, T_{\gamma_i})} \frac{(1 - |B(z)|)^p (1 - |\sigma_a(z)|^2)^s}{(1 - |z|^2)^2} \, dm(z) \right)^{1/x_i} \\
\leq \sup_{a \in \mathbb{D}} \prod_{i=1}^k \left( \sum_{j=0}^{\infty} \frac{(1 - |\sigma_a(z_{i,j})|^2)^s}{(1 - |z_{i,j}|^2)^2} \int_{\Delta(z_{i,j}, 1/2)} \frac{(1 - |\sigma_a(z_{i,j})|^2)^s}{(1 - |z_{i,j}|^2)^2} \, dm(z) \right)^{1/x_i} .
\]

If every \(x_i\) satisfies \(qx_i < 2\), then a change of variables yields
\[
\int_{\Delta(z_{i,j}, 1/2)} (\rho(z, z_{i,j}))^{-qx_i} \, dm(z) \approx (1 - |z_{i,j}|^2)^2 \int_0^{1/2} r^{1-qx_i} \, dr \approx (1 - |z_{i,j}|)^2 .
\]
Condition \((3.6)\) and \(qx_i < 2\) for every \(x_i\) yield the assumed condition \(q < 2/k\). In fact, taking \(x_1 = \cdots = x_k = k\) is enough for this proof. Combining these with \(3.3\), we get

\[
\sup_{a \in \mathbb{D}} \prod_{i=1}^{k} \left( \int_{\mathbb{D}\setminus \Omega} \frac{1}{|B_i(z)|^{qx_i}} \frac{(1 - |B(z)|)^p (1 - |\sigma_a(z)|^2)^s}{(1 - |z|^2)^2} \ dm(z) \right)^{1/x_i} \\
\leq \sup_{a \in \mathbb{D}} \prod_{i=1}^{k} \left( \sum_{n=1}^{\infty} (1 - |\sigma_a(z_n)|^2)^s \right)^{1/x_i} < \infty. \tag{3.10}
\]

Joining \((3.4)\), \((3.5)\), \((3.9)\) and \((3.10)\), we get the desired result. \(\square\)

Now we state the main result of this section as follows.

**Theorem 3.3.** Let \(0 < s < 1\) and \(1 \leq p < 2\). Suppose \(\{z_n\}_{n=1}^{\infty}\) is a separated Blaschke sequence in \(\mathbb{D}\) and \(B\) is the Blaschke product associated with \(\{z_n\}_{n=1}^{\infty}\). Then the following conditions are equivalent:

(a) \(\sum_{n=1}^{\infty} (1 - |z_n|^2)^s \delta_{z_n}\) is an \(s\)-Carleson measure;

(b) \(\sup_{\varphi \in \text{Aut}(\mathbb{D})} \int_{\mathbb{D}} \left( \frac{1}{|B(\varphi(z))|} - 1 \right)^p (1 - |z|^2)^{s-2} \ dm(z) < \infty.\)

**Proof.** Note that if \(\{z_n\}_{n=1}^{\infty}\) is a separated Blaschke sequence such that \(\sum_{n=1}^{\infty} (1 - |z_n|^2)^s \delta_{z_n}\) is an \(s\)-Carleson measure for \(0 < s < 1\), then \(\{z_n\}_{n=1}^{\infty}\) is a uniformly separated sequence. Set \(q = p\) in Lemma \(3.1\) and Lemma \(3.2\). The conclusion follows. \(\square\)

Taking \(s = 1\) in (b) of Theorem \(3.3\), we do not get C. Nolder’s condition \((3.2)\). In the proof of Theorem \(3.3\), the characterization of Blaschke products in \(F(p, p-2, s)\) plays an important role, where the condition \(0 < s < 1\) was used.

4. \(s\)-Carleson Measure \(\sum_{n=1}^{\infty} (1 - |z_n|^2)^s \delta_{z_n}\) and \(F(p, p-2, s)\) via \(f'' + Af = 0\)

In this section, for \(0 < s < 1\), \(p > \max\{s, 1 - s\}\), \(q > \max\{s, 1 - s\}\) and a separated sequence \(\{z_n\}_{n=1}^{\infty}\) in \(\mathbb{D}\) satisfying that \(\sum_{n=1}^{\infty} (1 - |z_n|^2)^s \delta_{z_n}\) is an \(s\)-Carleson measure, we show that there exists a function \(A\) analytic in \(\mathbb{D}\) such that \(|A(z)|^q (1 - |z|^2)^{2q-2+s} \ dm(z)\) is an \(s\)-Carleson measure and the equation \(f'' + Af = 0\) admits a nontrivial solution \(f \in F(p, p-2, s) \cap H^\infty\) whose zero-sequence is \(\{z_n\}_{n=1}^{\infty}\). From the perspective of larger ranges of parameters \(p\) and \(q\) given here, our result improves some previous conclusions.

For \(A \in H(\mathbb{D})\), it is well known that all solutions of the second order complex differential equation

\[
f'' + Af = 0 \tag{4.1}
\]

belong to \(H(\mathbb{D})\). For a sequence \(\{z_n\}\) of distinct points in \(\mathbb{D}\), if there exists \(A \in H(\mathbb{D})\) such that \(4.1\) has a solution \(f\) with zeros precisely at the points \(z_n\), then \(\{z_n\}\) is said to be a prescribed zero sequence (cf. \([24, 26]\)). See \([25]\) for a historical review in this area. In fact, by \(4.1\), \(A = -f''/f\), which gives that any \(z_n\) must be a simple zero of \(f\); otherwise \(A\) is not analytic at \(z_n\). We refer to \([23]\) for some recent results on \(4.1\) associated with Carleson measures.
For any positive integer $n$, recall that the Bloch space $B$ is also equal to the set of functions $f \in H(\mathbb{D})$ such that

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)^n |f^{(n)}(z)| < \infty.$$  

For $0 \leq \alpha < \infty$, the growth space $H^\alpha_\mathbb{D}$ is the set of functions $g \in H(\mathbb{D})$ satisfying

$$\|g\|_{H^\alpha_\mathbb{D}} = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |g(z)| < \infty.$$  

The following result is Lemma 1 in [22].

**Lemma E.** Let $g \in H^\alpha_\mathbb{D}$ for $0 \leq \alpha < \infty$, and let $0 < \gamma < 1$. If $g(z_0) = 0$ for some $z_0 \in \mathbb{D}$, then there exists a positive constant $C = C(\alpha, \gamma)$ such that

$$|g(z)| \leq \frac{C \|g\|_{H^\alpha_\mathbb{D}} \rho(z, z_0)}{(1 - |z_0|^2)^\alpha}, \quad z \in \Delta(z_0, \gamma).$$

Proposition 4.1 below is of interest only if $H^\infty \not\subseteq X$. Its proof is based on a theoretical abstraction from a well-known method in the filed of second order complex differential equations with prescribed zero sequences (cf. [25, p. 47] or [22, pp. 301-302]).

**Proposition 4.1.** Let $X$ be a vector space of analytic functions in $\mathbb{D}$ satisfying conditions (a), (b) and (c) below.

(a) Suppose $B$ is a Blaschke product associated with a sequence $\{a_k\}_{k=1}^\infty$ in $\mathbb{D}$ satisfying $\inf_m \prod_{n \neq m} \rho(a_n, a_m) \geq \gamma$ for some $\gamma \in (0, 1)$ and $I$ is a Blaschke product associated with a sequence $\{b_k\}_{k=1}^\infty$ in $\mathbb{D}$ such that $\rho(a_k, b_k) \leq s$, $k = 1, 2, \ldots$, for some constant $s \in (0, \gamma/2)$. If $B \in X$, then $I$ also belongs to $X$.

(b) If both $B$ and $I$ are interpolating Blaschke products in $X$ and $c$ is a complex constant, then the function $\exp(cB)$ belongs to $X$.

(c) If both $f$ and $g$ are in $H^\infty \cap X$, then $fg \in H^\infty \cap X$.

If $J$ is an interpolating Blaschke product associated with a sequence $\{z_k\}_{k=1}^\infty$ in $\mathbb{D}$ and $J \in X$, then there exists a function $A$ analytic in $\mathbb{D}$ such that $\sup_{z \in \mathbb{D}} (1 - |z|^2)^2 |A(z)| < \infty$ and the equation $f'' + Af = 0$ admits a nontrivial solution $f \in H^\infty \cap X$ whose zero-sequence is $\{z_k\}_{k=1}^\infty$.

**Proof.** Since $J$ is an interpolating Blaschke product associated with $\{z_k\}_{k=1}^\infty$, there is a positive constant $\gamma_1$ such that

$$\inf_n \prod_{m \neq n} \frac{z_m - z_n}{1 - \overline{z_m}z_n} = \inf_n (1 - |z_n|^2) |J'(z_n)| \geq \gamma_1.$$  

Note that any bounded analytic function is a Bloch function. Then

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)^2 |J''(z)| < \infty.$$  

Hence,

$$\sup_n \frac{|J''(z_n)|}{|J'(z_n)|^2} < \infty.$$  

Set

$$\zeta_n = \frac{J''(z_n)}{2|J'(z_n)|^2}, \quad n = 1, 2, \ldots$$
Then \( \{ \zeta_n \} \) is a bounded sequence. By Earl’s constructive proof of Carleson’s interpolating theorem for \( H^\infty \) (see [15]), there exists a complex constant \( c \) depending only on \( \gamma_1 \) and a Blaschke product

\[
J_1(z) = \prod_{k=1}^{\infty} \frac{|\eta_k|}{\eta_k} \frac{\eta_k - z}{1 - \eta_k \bar{z}}
\]

such that

\[
c(\sup_k |\zeta_k|) J_1(z_n) = \zeta_n, \quad n = 1, 2, \ldots.
\]

Moreover, the zeros \( \{ \eta_n \} \) of the Blaschke product \( J_1(z) \) can be chosen to satisfy

\[
\rho(z_n, \eta_n) \leq \frac{\gamma_1}{3}, \quad n = 1, 2, \ldots.
\]

By \( J \in X \) and condition (a), we see that \( J_1 \in X \).

If \( n \neq k \), then

\[
\rho(\eta_n, \eta_k) \geq \rho(z_n, \eta_k) - \rho(\eta_n, z_n) \geq \rho(z_n, \eta_k) - \rho(z_k, \eta_k) - \rho(\eta_n, z_n) \geq \gamma_1 - \frac{2}{3} \gamma_1.
\]

Thus \( \{ \eta_n \} \) is separated. Since \( \{ z_k \}_{k=1}^{\infty} \) is uniformly separated, \( \sum_n (1 - |z_n|^2)^2 \delta z_n \) is a 1-Carleson measure, namely,

\[
\sup_{a \in \mathbb{D}} \sum_n \frac{(1 - |z_n|^2)(1 - |a|^2)}{|1 - \bar{a}z_n|^2} < \infty.
\]

Note that \( \rho(z_n, \eta_n) \leq \gamma_1/3 \) for all \( n \). It is known (cf. [42, p. 69] and [42, Lemma 4.30]) that

\[
1 - |z_n|^2 \approx 1 - |\eta_n|^2, \quad |1 - \bar{\eta} z_n| \approx |1 - \bar{\eta} \eta_n|,
\]

for all \( n \) and \( a \in \mathbb{D} \). Consequently,

\[
\sup_{a \in \mathbb{D}} \sum_n \frac{(1 - |\eta_n|^2)(1 - |a|^2)}{|1 - \bar{a} \eta_n|^2} < \infty.
\]

Then \( \{ \eta_n \}_{n=1}^{\infty} \) is also uniformly separated and hence \( J_1 \) is also an interpolating Blaschke product.

Set \( f = Je^h \), where \( h = c(\sup_k |\zeta_k|) J_1 J \). Then \( f \) satisfies the equation \( f'' + Af = 0 \), where \( A \in H(\mathbb{D}) \) and

\[
A = -\frac{f''}{f} = -\frac{J''}{J} + 2\frac{J'h'}{J} - (h')^2 - h''.
\]

Condition (b) yields that \( e^h \in X \). Clearly, \( e^h \in H^\infty \). We get \( f \in H^\infty \cap X \) from condition (c). It is also clear that the zero set of \( f \) is \( \{ z_k \}_{k=1}^{\infty} \).

Since \( J \) and \( h \) are in \( H^\infty \) which is a subset of \( \mathcal{B} \), \( (1 - |z|^2)^2 |h'(z)|^2 \) and \( (1 - |z|^2)^2 |h''(z)| \) are uniformly bounded on \( \mathbb{D} \). Write

\[
\Omega = \mathbb{D} \setminus \left( \bigcup_{n=1}^{\infty} \Delta(z_n, \frac{\gamma_1}{4}) \right).
\]

By Lemma [13]

\[
\sup_{z \in \Omega} (1 - |z|^2)^2 \left| \frac{J''(z) + 2J'(z)h'(z)}{J(z)} \right| < \infty.
\]
Remark 1. It is easy to see that (a) and (b) in Proposition 4.1 can be replaced by (c) and (d) below simultaneously.

(c) If an interpolating Blaschke product \( B \) associated with a sequence \( \{a_k\}_{k=1}^{\infty} \) belongs to \( X \), then the Blaschke product \( I \) associated with a sequence \( \{b_k\}_{k=1}^{\infty} \) also belongs to \( X \), where \( \rho(a_k, b_k) < s, k = 1, 2, \cdots \), for some constant \( s \in (0, 1) \).

(d) Suppose \( B \) is an interpolating Blaschke product in \( X \), \( I \) is a Carleson-Newman Blaschke product in \( X \) and \( c \) is a complex constant. Then the function \( \exp(cBI) \) belongs to \( X \).

Remark 2. For \( X \subseteq H(\mathbb{D}) \), denote by \( M(X) \) the set of multipliers on \( X \); that is,

\[
M(X) = \{ f \in X : \ f g \in X \text{ for all } g \in X \}.
\]

From [13, Lemma 11], if \( X \) is a Banach space of analytic functions on which point evaluations are bounded, then \( M(X) \subseteq H^{\infty} \). Replacing \( X \) in Proposition 4.1 by \( M(X) \), we get a corresponding result on \( M(X) \) immediately.

Remark 3. For a special space \( X \) satisfying assumptions in Proposition 4.1 it is interesting to consider further the pointwise growth condition or integrated growth condition of the function \( A \) in Proposition 4.1.

In 2019 J. Gröhn [22] gave several interesting results for solutions of (4.1) having prescribed zeros in \( \mathbb{D} \). In particular, he obtained the following result.

Theorem F. Let \( 0 < s \leq 1 \). If \( \Lambda \subseteq \mathbb{D} \) is a separated sequence such that \( \sum_{z_n \in \Lambda} (1 - |z_n|)^s \delta_{z_n} \) is an \( s \)-Carleson measure, then there exists a function \( A \) analytic in \( \mathbb{D} \) such that \( |A(z)|^p (1 - |z|^2)^{2p-2s} \sigma_{\lambda}(z) \) is an \( s \)-Carleson measure and the equation \( f'' + Af = 0 \) admits a nontrivial solution \( f \in Q_s \cap H^{\infty} \) whose zero-sequence is \( \Lambda \).

For \( 0 < p_1 < p_2 < \infty \) and \( 0 < s \leq 1 \) with \( p_1 + s > 1 \), it is known that \( F(p_1, p_1 - 2, s) \subseteq F(p_2, p_2 - 2, s) \). For \( s > 1 \), all nontrivial \( F(p, p - 2, s) \) spaces are equal to the Bloch space. Suppose \( 0 < p < \infty \) and \( 0 < s < \infty \) with \( 2p + s > 1 \). For \( A \in H(\mathbb{D}) \), \( |A(z)|^p (1 - |z|^2)^{2p-2s} \sigma_{\lambda}(z) \) is an \( s \)-Carleson measure if and only if

\[
\sup_{w \in \mathbb{D}} \int_{\mathbb{D}} |A(z)|^p (1 - |z|^2)^{2p-2}(1 - |\sigma_{\lambda}(z)|^2)^s \sigma_{\lambda}(z) < \infty.
\]

Denote by \( N_{p,s} \) the space of analytic functions \( A \) satisfying the formula above. Clearly, if \( 0 < p < \infty \) and \( 0 < s < \infty \) with \( 2p + s > 1 \), then \( N_{p,s} \) contains only constant functions. For \( 0 < p < \infty \) and \( s > 1 \), \( N_{p,s} \) is equal to the Bloch type space \( B^3 \) consisting of functions \( f \in H(\mathbb{D}) \) with

\[
\sup_{z \in \mathbb{D}} (1 - |z|^2)^3 |f'(z)| < \infty.
\]
From [33] Remark 3, if $0 < p_1 < p_2 < \infty$ and $0 < s \leq 1$ with $2p_1 + s > 1$, then $N_{p_1,s} \subset N_{p_2,s}$. Thus the following theorem from [33] is a proper generalization of the case of $0 < s < 1$ in Theorem [F].

**Theorem G.** Let $0 < s < 1 < p < \infty$ and $1 < q < \infty$. If $A \subset \mathbb{D}$ is a separated sequence such that $\sum_{n \in A}(1 - |z_n|)^s \delta_{z_n}$ is an $s$-Carleson measure, then there exists a function $A$ analytic in $\mathbb{D}$ such that $|A(z)|^q(1 - |z|^2)^{2q - 2 + s} dm(z)$ is an $s$-Carleson measure and the equation $f'' + Af = 0$ admits a nontrivial solution $f \in F(p, p - 2, s) \cap H^\infty$ whose zero-sequence is $\Lambda$.

Theorems [4.2] below is a sequel to Theorem [F] and Theorem [G]. Form the explanations after Theorem [F] we see that Theorem 4.2 below strengthens Theorem [G] and the case of $0 < s < 1$ in Theorem [F].

**Theorem 4.2.** Suppose $0 < s < 1$, $p > \max\{s, 1 - s\}$ and $q > \max\{s, 1 - s\}$. If $A \subset \mathbb{D}$ is a separated sequence such that $\sum_{n \in A}(1 - |z_n|)^s \delta_{z_n}$ is an $s$-Carleson measure, then there exists a function $A$ in $H(\mathbb{D})$ such that $|A(z)|^q(1 - |z|^2)^{2q - 2 + s} dm(z)$ is an $s$-Carleson measure and the equation $f'' + Af = 0$ admits a nontrivial solution $f \in F(p, p - 2, s) \cap H^\infty$ whose zero-sequence is $\Lambda$.

**Proof.** Note that $0 < s < 1$ and $p > \max\{s, 1 - s\}$. We first show that (a), (b) and (c) in Proposition 4.1 hold by taking $X = F(p, p - 2, s)$. Using Theorem [A] and checking the proof of Proposition 4.1 we get that (a) in Proposition 4.1 holds when $X = F(p, p - 2, s)$. Clearly, if $X = F(p, p - 2, s)$, then (c) in Proposition 4.1 also holds. Suppose $B$ and $I$ are interpolating Blaschke products in $F(p, p - 2, s)$ and $c$ is a complex constant. Then

$$
\sup_{z \in \mathbb{D}} \left| e^{CB(I)(z)} \right|^p \left(1 - |z|^2\right)^{p - 2} \left(1 - |\sigma_c(z)|^2\right)^s dA(z)
\leq \|I\|_{F(p, p - 2, s)}^p \sup_{z \in \mathbb{D}} \left| e^{CB(I)(z)} \right|^p + \|B\|_{F(p, p - 2, s)}^p \sup_{z \in \mathbb{D}} \left| e^{CB(I)(z)} \right|^p < \infty,
$$

which gives $e^{CBI} \in F(p, p - 2, s)$. Thus (b) in Proposition 4.1 also holds when $X = F(p, p - 2, s)$.

For convenience, we write $\Lambda = \{z_n\}_{n=1}^\infty$. Denote by $J$ the Blaschke product associated with the sequence $\{z_n\}_{n=1}^\infty$. Since $\{z_n\}$ is separated and $\sum_{n \in A}(1 - |z_n|)^s \delta_{z_n}$ is an $s$-Carleson measure, $J$ is an interpolating Blaschke product and it follows from Theorem [A] that $J \in F(p, p - 2, s)$. By Proposition 4.1 there exists a function $A$ analytic in $\mathbb{D}$ such that the equation $f'' + Af = 0$ admits a nontrivial solution $f \in H^\infty \cap F(p, p - 2, s)$ whose zero-sequence is $\{z_n\}_{n=1}^\infty$. As shown in the proof of Proposition 4.1,

$$
A = -\frac{J'' + 2J'h'}{J} - (h')^2 - h'', \quad h = CJ_1J,
$$

where $C$ is a complex constant and $J_1$ is a Blaschke product whose zero-sequence $\{\eta_n\}_{n=1}^\infty$ satisfies

$$
\rho(z_n, \eta_n) \leq \frac{\gamma}{3}, \quad n = 1, 2, \cdots
$$

for some $\gamma \in (0, 1)$. For $q > \max\{s, 1 - s\}$, repeating the arguments of the proof of Theorem 1.1 in [33] (i.e. Theorem [G] stated in this paper), we see that $|A(z)|^q(1 - |z|^2)^{2q - 2 + s} dm(z)$ is an $s$-Carleson measure. We finish the proof. \(\square\)
Let $0 < s < 1, q > \max\{s, 1 - s\}$ and $A \in H(D)$. Suppose $|A(z)|^q(1 - |z|^2)^{2q}dm(z)$ is an $s$-Carleson measure. By (1.1) and the subharmonicity of $|A|^q$, we deduce
\[
\sup_{w \in D} \int_{\Delta(w, 1/2)} \left( \frac{1 - |w|^2}{1 - |w|^2 \overline{z}|^2} \right)^s |A(z)|^q(1 - |z|^2)^{2q}dm(z)
\]
which implies the condition of $A$ appeared in Proposition 4.1. In other words, the condition of $A$ in Theorem 4.2 is stronger than that in Proposition 4.1.

It is known (cf. [32, 40]) that $H^\infty \subseteq F(p, p - 2, 1)$ when $2 \leq p < \infty$, but for $0 < p < 2$, $H^\infty \nsubseteq F(p, p - 2, 1)$ and $F(p, p - 2, 1) \nsubseteq H^\infty$. It is natural to consider the case of $s = 1$ in Theorem 4.2 via replacing $Q_1 \cap H^\infty$ (i.e. $H^\infty$) by $F(p, p - 2, 1) \cap H^\infty$ for $0 < p < 2$. For this purpose, one should characterize interpolating Blaschke products $B$ in $F(p, p - 2, 1)$ for $0 < p < 2$ via the distribution of zeros of $B$. It is still open to find this characterization (cf. [22, p.755]).

**Data Availability.**

All data generated or analyzed during this study are included in this article and in its bibliography.

**Conflict of Interest.**

The authors declared that they have no conflict of interest.

**REFERENCES**

[1] A. Aleman and A. G. Siskakis, An integral operator on $H^p$, *Complex Variables Theory Appl.*, 28 (1995), 149-158.

[2] J. Arazy, S. Fisher, and J. Peetre, Möbius invariant function spaces, *J. Reine Angew. Math.*, 363 (1985), 110-145.

[3] R. Aulaskari, J. Xiao, and R. Zhao, On subspaces and subsets of $BMOA$ and $UBC$, *Analysis*, 15 (1995), 101-121.

[4] A. Baernstein II, Analytic functions of bounded mean oscillation, *Aspects of Contemporary Complex Analysis, Academic Press*, 1980, 3-36.

[5] G. Bao and J. Pau, Boundary multipliers of a family of Möbius invariant function spaces, *Ann. Acad. Sci. Fenn. Math.*, 41 (2016), 199-220.

[6] G. Bao, F. Sun, and H. Wulan, Carleson measures and the range of a Cesàro-like operator acting on $H^\infty$, *Anal. Math. Phys.*, 12 (2022), Paper No. 142.

[7] G. Bao, H. Wulan, and F. Ye, Intersections and unions of a general family of function spaces, *Proc. Amer. Math. Soc.*, 149 (2021), 3307-3315.

[8] G. Bao, H. Wulan, and F. Ye, The range of the Cesàro operator acting on $H^\infty$, *Canad. Math. Bull.*, 63 (2020), 633-642.

[9] L. Carleson, An interpolation problem for bounded analytic functions, *Amer. J. Math.*, 80 (1958), 921-930.

[10] L. Carleson, Interpolation by bounded analytic functions and the Corona problem, *Ann. of Math.*, 76 (1962), 547-559.

[11] C. Chatzifountas, D. Girela, and J. Peláz, Multipliers of Dirichlet suspces of the Bloch space, *J. Operator theory*, 72 (2014), 159-191.

[12] N. Danikas and A. Siskakis, The Cesàro operator on bounded analytic functions, *Analysis*, 13 (1993), 295-299.

[13] P. Duren, B. Romberg, and A. Shields, Linear functionals on $H^p$ spaces with $0 < p < 1$, *J. Reine Angew. Math.*, 238 (1969), 32-60.

[14] P. Duren and A. Schuster, Finite unions of interpolation sequences, *Proc. Amer. Math. Soc.*, 130 (2002), 2609-2615.

[15] J. Earl, On the interpolation of bounded sequences by bounded functions, *J. London Math. Soc.*, 2 (1970), 544-548.
[16] M. Essén and J. Xiao, Some results on $Q_p$ spaces, $0 < p < 1$, J. Reine Angew. Math., 485 (1997), 173-195.
[17] P. Galanopoulos, D. Girela, and N. Merchán, Cesàro-like operators acting on spaces of analytic functions, Anal. Math. Phys., 12 (2022), Paper No. 51.
[18] J. Garnett, Bounded Analytic Functions, Springer, New York, 2007.
[19] D. Girela, Analytic functions of bounded mean oscillation. In: Complex Function Spaces, Mekrijärvi 1999 Editor: R. Aulaskari. Univ. Joensuu Dept. Math. Rep. Ser., 4, Univ. Joensuu, Joensuu, (2001) pp. 61-170.
[20] D. Girela, Growth of the derivative of bounded analytic functions, Complex Variables Theory Appl, 20 (1992), 221-227.
[21] J. Gröhn, Solutions of complex differential equation having pre-given zeros in the unit disc, Constr. Approx., 49 (2019), 295-306.
[22] J. Gröhn, Converse growth estimates for ODEs with slowly growing solutions, Math. Z., 298 (2021), 419-450.
[23] J. Heittokangas, Solutions of $f'' + A(z)f = 0$ in the unit disc having Blaschke sequences as the zeros, Comput. Methods Funct. Theory, 5 (2005), 49-63.
[24] J. Heittokangas, A survey on Blaschke-oscillatory differential equations, with updates, Blaschke products and their applications, 43-98, Fields Inst. Commun., 65, Springer, New York, 2013.
[25] J. Heittokangas and I. Laine, Solutions of $f'' + A(z)f = 0$ with prescribed sequences of zeros, Acta Math. Univ. Comenian. (N.S.), 74 (2005), 287-307.
[26] A. Kerr-Lawson, Some lemmas on interpolating Blaschke products and a correction, Canadian J. Math., 21 (1969), 531-534.
[27] A. Nicolau, Finite products of interpolating Blaschke products, J. London Math. Soc., 50 (1994), 520-531.
[28] C. Nolder, An $L^p$ definition of interpolating Blaschke products, Proc. Amer. Math. Soc., 128 (2000), 1799-1806.
[29] G. McDonald and C. Sundberg, Toeplitz operators on the disc, Indiana Univ. J. Math., 28 (1979), 595-611.
[30] J. Pau and R. Zhao, Carleson measures, Riemann-Stieltjes and multiplication operators on a general family of function spaces, Integral Equations Operator Theory, 78 (2014), 483-514.
[31] F. Pérez-González and J. Rättyä, Inner functions in the Möbius invariant Besov-type spaces, Proc. Edinb. Math. Soc., 52 (2009), 751-770.
[32] Ch. Pommerenke, Schlichte Funktionen und analytische Funktionen von beschränkter mittlerer Oscillation, Comment. Math. Helv., 52 (1997), 591-602.
[33] R. Qian and F. Ye, Interpolating sequences for some subsets of analytic Besov type spaces, J. Math. Anal. Appl., 507 (2022), Paper No. 125838, 14 pp.
[34] Ch. Yuan and C. Tong, On analytic campanato and $F(p, q, s)$ spaces, Complex Anal. Oper. Theory, 12 (2018), 1845-1875.
[35] K. Zhu, Operator Theory in Function Spaces, American Mathematical Society, Providence, RI, 2007.
Fangqin Ye, Shantou University, Shantou 515063, Guangdong, China
Email address: fqye@stu.edu.cn