PERELMAN’S \( \lambda \)-FUNCTIONAL AND THE SEIBERG-WITTEN EQUATIONS

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Abstract. In this paper we study the supremum of Perelman’s \( \lambda \)-functional \( \lambda_M(g) \) on Riemannian 4-manifold \( M \) by using the Seiberg-Witten equations. We prove among others that, for a compact Kähler-Einstein complex surface \( (M,J,g_0) \) with negative scalar curvature, (i) If \( g_1 \) is a Riemannian metric on \( M \) with \( \lambda_M(g_1) = \lambda_M(g_0) \), then \( \text{Vol}_{g_1}(M) \geq \text{Vol}_{g_0}(M) \). Moreover, the equality holds if and only if \( g_1 \) is also a Kähler-Einstein metric with negative scalar curvature. (ii) If \{\( g_t \), \( t \in [-1,1] \)\}, is a family of Einstein metrics on \( M \) with initial metric \( g_0 \), then \( g_t \) is a Kähler-Einstein metric with negative scalar curvature.

1. Introduction

In his celebrated paper [H] R. Hamilton introduced the Ricci-flow evolution equation

\[
\frac{\partial}{\partial t} g(t) = -2\text{Ric}(g(t))
\]

with initial metric \( g(0) = g \). The Ricci flow is now a fundamental tool to solve the famous Poincaré conjecture and Thurston’s Geometrization conjecture, by the works of G. Perelman [Pe1][Pe2]. A fundamental new discovery of Perelman is to prove the Ricci-flow evolution equation is the gradient flow of a so called \textit{Perelman’s} \( \lambda \)-\textit{functional} of a Riemannian manifold(cf. [Pe1][KL]), which may be described as follows: for a smooth function \( f \in C^\infty(M) \) on a Riemannian \( n \)-manifold with a Riemannian metric \( g \), let

\[
\mathcal{F}(g, f) = \int_M (R_g + |\nabla f|^2) e^{-f} dvol_g,
\]

where \( R_g \) is the scalar curvature of \( g \). The Perelman’s \( \lambda \)-functional is defined by

\[
\lambda_M(g) = \inf_f \{ \mathcal{F}(g,f) \, | \, \int_M e^{-f} dvol_g = 1 \}.
\]

Note that \( \lambda_M(g) \) is the lowest eigenvalue of the operator \( -4\Delta + R_g \). Let

\[
\lambda_M^*(g) = \lambda_M(g) \text{Vol}_g(M)^{\frac{2}{n}}
\]

which is invariant up to rescale the metric. Perelman [Pe1] has established the monotonicity property of \( \lambda_M^*(g_t) \) along the Ricci flow \( g_t \), namely, the function is non-decreasing along the Ricci flow \( g_t \) whenever \( \lambda_M(g_t) \leq 0 \). Therefore, it is interesting to

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study the upper bound of $\lambda_M(g)$. This leads to define a diffeomorphism invariant $\overline{\lambda}_M$ of $M$ due to Perelman (cf. [Pe2] [KL]) by

$$\overline{\lambda}_M = \sup_{g \in \mathcal{M}} \lambda_M(g),$$

where $\mathcal{M}$ is the set of Riemannian metrics on $M$. It is easy to see that $\overline{\lambda}_M = 0$ if $M$ admits a volume collapsing with bounded scalar curvature but does not admit any metric with positive scalar curvature (cf. [KL]). By a deep result of Perelman (cf. [Pe2][KL]), for a 3-manifold $M$ which does not admit a metric of positive scalar curvature, $(-\overline{\lambda}_M)^{\frac{3}{2}}$ is proportional to the minimal volume of the manifold.

The invariant $\overline{\lambda}_M$ may take value $+\infty$, e.g., $M = S^2 \times S^2$. Thus, it seems only interesting when $\overline{\lambda}_M \leq 0$, i.e, when $M$ does not admit any metric of positive scalar curvature.

In this paper we will investigate $\overline{\lambda}_M$ by using the Seiberg-Witten monopole equations for a 4-manifold $M$. We say a Spin$^c$-structure (or equivalently its first Chern class) is a monopole class if the Seiberg-Witten monopole equations has an irreducible solution.

Our first result is as follows:

**Theorem 1.1.** Let $(M, c)$ be a smooth compact closed oriented 4-manifold with a Spin$^c$-structure $c$. If the first Chern class $c_1$ of $c$ is a monopole class of $M$ satisfying that $c_1^2[M] > 0$. Then, for any Riemannian metric $g$,

$$\overline{\lambda}_M(g) \leq -\sqrt{32\pi^2 c_1^2[M]}.$$

Moreover, the equality holds if and only if $g$ is a Kähler-Einstein metric with negative scalar curvature.

By [Ta] the canonical class of a symplectic manifold is a monopole class. Thus, Theorem 1.1 applies to a Kähler minimal surface of general type, since by [BHPV] $K_X^2 > 0$ if $X$ is a minimal surface of general type. We remark that Theorem 1.1 implies that $\overline{\lambda}_M$ is not a topological invariant of the underlying manifold. Indeed, for any pair of positive integers $(m, n)$, so that $\frac{m}{n} \in \left(\frac{1}{3}, 2\right)$, by [BHPV] VII Theorem 8.3 there is a simply connected minimal surface $X$ of general type so that $m = c_2(X), n = c_1^2(X)$. Let $M$ be the blow up of $X$ at one point. Then $c_1^2[M] = n - 1$. By Theorem 1.1 we know that $\overline{\lambda}_M(g) \leq -\sqrt{32\pi^2(n - 1)}$. On the other hand, since $M$ is a simply connected 4-manifold of odd intersection type, by Freedman’s classification it is homeomorphic to the connected sums $k\mathbb{C}P^2 \# l\overline{\mathbb{C}P^2}$ for some positive integers $k, l$. Since the latter admits a metric with positive scalar curvature, $\overline{\lambda}_{k\mathbb{C}P^2 \# l\overline{\mathbb{C}P^2}} > 0$. This shows that $\overline{\lambda}_M$ is not a topological invariant.

A geometric consequence of Theorem 1.1 is the following comparison theorem.

**Corollary 1.2.** Let $(M, J, g_0)$ be a compact Kähler-Einstein complex surface with negative scalar curvature. If $g_1$ is a Riemannian metric on $M$ with $\lambda_M(g_1) = \lambda_M(g_0)$, then

$$\text{Vol}_{g_1}(M) \geq \text{Vol}_{g_0}(M).$$

Moreover, the equality holds if and only if $g_1$ is a Kähler-Einstein metric with negative scalar curvature.
One may wonder whether \((M, g_1)\) and \((M, g_0)\) are isometric in the above theorem when the equality holds. This may not be true. Indeed, there are infinitely many families of Kähler Einstein metrics on a compact complex surface \(M\) in different isometry classes with negative scalar curvature but all the same volume and same \(\lambda_M(\cdot)\).

The following corollary shows a deformation rigidity of Einstein metrics on compact complex Kähler-Einstein surface with negative scalar curvature.

**Corollary 1.3.** Let \((M, J, g_0)\) be a compact Kähler-Einstein complex surface with negative scalar curvature. If \(\{g_t\}, t \in [-1, 1]\), is a family of Einstein metrics on \(M\) with initial metric \(g_0\), then, for any \(t\), \(g_t\) is a Kähler-Einstein metric with negative scalar curvature.

The above Corollary 1.3 should be compared with Corollary D in [G], where the same conclusion was obtained when \(g_0\) has positive scalar curvature. On the other hand, under some additional technical assumptions similar results are obtained in general dimensions in [DWW] and [Ko] along a completely different line.

For a compact symplectic 4-manifold \(N\) with first Chern class \(c_1\), the Riemann-Roch formula implies that \(c_2^1[N] = 2\chi(N) + 3\tau(N)\), where \(\chi(N)\) and \(\tau(N)\) are the Euler characteristic and the signature of \(N\) respectively. If \(N\) admits a Kähler-Einstein metric with negative scalar curvature, we have already known \(\lambda_N = -\sqrt{32\pi^2(2\chi(N) + 3\tau(N))}\). In the next theorem, we will show the exact quantity of \(\lambda_M\) where \(M\) is obtained by blowing-up \(N\) at \(k\) points.

**Theorem 1.4.** Let \((N, \omega)\) be a compact symplectic 4-manifold with \(b_2^+ (N) > 1\). Let \(M = N^k \# k\mathbb{CP}^2\), where \(k \geq 0\). If \(2\chi(N) + 3\tau(N) > 0\), then

\[
\lambda_M \leq -\sqrt{32\pi^2(2\chi(N) + 3\tau(N))}.
\]

Furthermore, the equality holds if \(N\) admits a Kähler-Einstein metric.

It is known that the Seiberg-Witten invariant of connected sums vanishes if both factors have positive \(b_2^+\). In [Ba] [BaF], a refinement of the Seiberg-Witten invariant is defined, which may not vanish for connected sums of few factors. This may be used to improve the above theorem as follows:

**Theorem 1.5.** Let \((N_i, \omega_i), i = 1, \cdots, \ell\), where \(\ell \leq 4\), be compact symplectic 4-manifolds satisfying that \(b_1(N_i) = 0\), \(b_2^+(N_i) \equiv 3\text{mod}4\), and \(\sum_{i=1}^{\ell} b_2^+(N_i) \equiv 4\text{mod}8\). Assume that \(c_2^1[N_i] > 0\) and \(c_2^1[N_i] \geq 0\) for all \(i\). Let \(X\) be a compact oriented 4-manifold with \(b_2^+(X) = 0\), which admits a metric of positive scalar curvature. Let \(M = \#_{i=1}^{\ell} N_i \# X\). Then

\[
\lambda_M \leq -\sqrt{32\pi^2 \sum_{i=1}^{\ell} c_2^1[N_i]}.
\]

Furthermore, the equality holds if \(N_1, \cdots, N_\ell\) admit Kähler-Einstein metrics.

The technique developed in proving Theorem 1.4 and 1.5 has an easy corollary:
Corollary 1.6. Let \((N_i, g_i), i = 1, \cdots, l_1\), be compact Riemannian 4m-manifolds \((m \geq 2)\) with holonomy \(SU(2m)\) or \(Sp(m)\) or \(Spin(7)\), and \(X_j, j = 1, \cdots, l_2\), be simply connected compact oriented spin 4m-manifolds with vanishing \(\hat{A}\)-genus, \(\hat{A}(X_i) = 0\). If \(M = \#^{l_1}N_i \#^{l_2}X_j\), and \(\hat{A}(M) \neq 0\), then
\[
\lambda_M = 0.
\]

Let \(M\) be a smooth compact oriented 4-manifold. By Perelman [Pe1] a critical point of \(\lambda_M(\cdot)\) is an Einstein metric. Therefore, it is interesting to ask

**Question:** Can one deform a metric \(g\) to an Einstein metric through the Ricci flow, provided \(\lambda_M(g)\) is sufficiently close to the maximum \(\lambda_M\) of the \(\lambda\)-functional?

This may not have a positive answer in general, of course, e.g., for a graph 3-manifold \(M\), by [Pe2][KL] \(\lambda_M = 0\), but \(M\) can not have any Einstein metric except \(M\) is a flat manifold.

To formulate our next result, let us consider the moduli space of metrics
\[
\mathcal{M}_{(\Lambda, D)} = \{g : |K_g| < \Lambda^2, diam_g < D\},
\]
where \(diam_g\) is the diameter, and \(K_g\) is the sectional curvature of \(g\).

**Proposition 1.7.** Let \((M, J)\) be a compact almost complex 4-manifold satisfying that \(\chi(M) \in \left[\frac{3}{2} \tau(M), 3\tau(M)\right]\), and \(\tau(M) > 0\). If the canonical Spin\(^c\)-structure \(\mathfrak{c}\) induced by \(J\) is a monopole class, then there exists a constant \(\varepsilon = \varepsilon(\Lambda, D) > 0\) depending only on \(\Lambda\) and \(D\) such that for any Riemannian metric \(g \in \mathcal{M}_{(\Lambda, D)}\) on \(M\) satisfying that

\[
(1.9) \quad \lambda_M(g) \geq -\sqrt{32\pi^2(2\chi(M) + 3\tau(M))} - \varepsilon
\]

it can be deformed to a complex hyperbolic metric through the Ricci flow.

The rest of the paper is organized as follows: In §2 we recall some facts about Seiberg-Witten equations. In §3 we prove Theorem 1.1, Corollary 1.2 and Corollary 1.3. In §4 we prove Theorem 1.4, Theorem 1.5 and Corollary 1.6. In §5 we prove Proposition 1.7.

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2. Preliminaries

In this section, we recall some facts about Seiberg-Witten equations. More details can be found in [N1] and [Le2].

Let \((M, g)\) be a compact oriented Riemannian 4-manifold with a Spin\(^c\) structure \(\mathfrak{c}\). Let \(b_2^+\) denote the dimension of the space of self-dual harmonic 2-forms in \(M\). Let \(S^+_\mathfrak{c}\) denote the Spin\(^c\)-bundles associated to \(\mathfrak{c}\), and let \(L\) be the determinant line bundle of \(\mathfrak{c}\). There is a well-defined Dirac operator
\[
\mathcal{D}_A : \Gamma(S^+_\mathfrak{c}) \to \Gamma(S^+_\mathfrak{c})
\]
Let \( c : \wedge^* T^* M \longrightarrow \text{End}(S^+_c \oplus S^-_c) \) denote the Clifford multiplication on the Spin\(^c\)-bundles, and, for any \( \phi \in \Gamma(S^\pm) \), let

\[
q(\phi) = \overline{\phi} \otimes \phi - \frac{1}{2} |\phi|^2 \text{id}.
\]

The Seiberg-Witten equations read

\[
\begin{align*}
D_A \phi &= 0 \\
c(F^+_A) &= q(\phi)
\end{align*}
\]  

where the unknowns are a hermitian connection \( A \) on \( L \) and a section \( \phi \in \Gamma(S^+_c) \), and \( F^+_A \) is the self-dual part of the curvature of \( A \).

A resolution of \(2.1\) is called reducible if \( \phi \equiv 0 \); otherwise, it is called irreducible. If \( (\phi, A) \) is a resolution of \(2.1\), then one calculates

\[
|F^+_A|^2 = \frac{1}{2\sqrt{2}} |\phi|^2,
\]

The Bochner formula reads

\[
0 = 2\Delta |\phi|^2 + 4|\nabla^A \phi|^2 + R_g |\phi|^2 + |\phi|^4,
\]

where \( R_g \) is the scalar curvature of \( g \).

The Seiberg-Witten invariant can be defined by counting the irreducible solutions of the Seiberg-Witten equations (cf. [N1] [Le2]).

**Definition 2.1 (K1).** Let \( M \) be a smooth compact oriented 4-manifold. An element \( \alpha \in H^2(M, \mathbb{Z})/\text{torsion} \) is called a monopole class of \( M \) if and only if there exists a Spin\(^c\)-structure \( c \) on \( M \) with the first Chern class \( c_1 \equiv \alpha (\text{mod torsion}) \), so that the Seiberg-Witten equations have a solution for every Riemannian metric \( g \) on \( M \).

Deep results have been found in Seiberg-Witten theory to detect the monopole classes. For example, if \((M, \omega)\) is a compact symplectic 4-manifold with \( b^+_2 > 1 \), the canonical class of \((M, \omega)\) is a monopole class (cf. [Ta], or Theorem 4.2 in [K2]).

A refinement of Seiberg-Witten invariant is defined in [Ba] [BaF], which takes values in a cohomotopy group. The remarkable fact is that this invariant is not killed off by the sort of connected sum operation. If \((N_i, \omega_i), \ i \in \{1, 2, 3, 4\}\), are the same as in Theorem 1.5, then, by Proposition 10 in [IL], \(\#_{i=1}^l \pm c_1(N_i)\) is a monopole class of \(\#_{i=1}^l N_i\) where \(l \leq 4\).

### 3. Proof of Theorems 1.1

Let \((M, g)\) be a Riemannian Spin\(^c\)-manifold of dimension \(n\). To prove Theorem 1.1, we need the following version of Kato’s inequality.

**Lemma 3.1.** Let \( \phi \) be a harmonic Spin\(^c\)-spinor on \((M, g)\), i.e. \( D_A \phi = 0 \), where \( D_A \) is the Dirac operator and \( A \) is a connection on the determinant line bundle. Then

\[
|\nabla|\phi||^2 \leq |\nabla^A \phi|^2
\]

at all points where \( \phi \) is non-zero. Moreover, the equality can only occur if \( \nabla^A \phi \equiv 0 \).
Proof. Fix a point \( p \in M \) at which \( \phi(p) \neq 0 \) so that \( |\phi| \) is differentiable at \( p \). If \( e_1, \ldots, e_n \) is an orthonormal basis of \( T_p M \), then \( |\nabla \phi|^2 = \sum |\frac{\partial}{\partial e_i} \phi|^2 \) and \( |\nabla^A \phi|^2 = \sum |\nabla^A_{e_i} \phi|^2 \). For any \( i \),

\[
|\phi| \frac{\partial}{\partial e_i} |\phi| = \frac{1}{2} \frac{\partial}{\partial e_i} |\phi|^2 = |\text{Re}(\nabla^A_{e_i} \phi, \phi)| \leq |\nabla^A_{e_i} \phi||\phi|,
\]

\[
|\frac{\partial}{\partial e_i} |\phi|| \leq |\nabla^A_{e_i} \phi|, \quad \text{and} \quad |\nabla |\phi|^2 \leq |\nabla^A \phi|^2.
\]

The equality can only occur if there are real numbers \( \alpha_i \) such that \( \nabla^A_{e_i} \phi = \alpha_i \phi \). Since \( \phi \) is a harmonic Spin\(^c\) spinor,

\[
0 = D_A \phi = \sum c(e_i) \nabla^A_{e_i} \phi = c(\sum \alpha_i e_i) \phi = c(w) \phi,
\]

where \( w = \sum \alpha_i e_i \) and \( c \) is the Clifford multiplication. Then

\[
0 = -|w|^2 \phi.
\]

Thus \( w = 0 \), \( \alpha_i = 0 \) and \( \nabla^A \phi = 0 \) at \( p \). Thus we obtain the conclusion. \( \square \)

For any \( \varepsilon > 0 \), let \( |\phi|^2_\varepsilon = |\phi|^2 + \varepsilon^2 \). If \( \phi \) is harmonic, by above lemma,

\[
|\nabla |\phi|^2_\varepsilon \leq |\phi|_\varepsilon^2 |\nabla |\phi|^2 \leq |\nabla^A \phi|^2
\]

at points where \( \phi(p) \neq 0 \). Since \( \{ p \in M | \phi(p) \neq 0 \} \) is dense in \( M \) for harmonic \( \phi \), we conclude that (3.2) holds everywhere in \( M \).

Proposition 3.2. Let \((M, g)\) be a compact oriented Riemannian 4-manifold, and \( \mathfrak{c} \) be a Spin\(^c\)-structure on \( M \). If there is an irreducible solution \((\phi, A)\) to the Seiberg-Witten equations (2.1) for \( g \) and \( \mathfrak{c} \), then

\[
\overline{\lambda}_M(g) \leq -\sqrt{32\pi^2 |c_1^+|^2[M]},
\]

where \( c_1^+ \) is the self-dual part of the harmonic form representing the first Chern class \( c_1 \) of \( \mathfrak{c} \). When \( |c_1^+| \neq 0 \), equality can only occur if \( g \) is a Kähler metric with constant negative scalar curvature.

Proof. Let \((\phi, A)\) be an irreducible solution to the Seiberg-Witten equations. The Bochner formula implies

\[
0 = \frac{1}{2} \Delta |\phi|^2 + |\nabla^A \phi|^2 + \frac{R_g}{4} |\phi|^2 + \frac{1}{4} |\phi|^4,
\]

Therefore

\[
\int_M (|\nabla^A \phi|^2 + \frac{R_g}{4} |\phi|^2) dvol_g = -\frac{1}{4} \int_M |\phi|^4 dvol_g.
\]

By (3.2),

\[
\int_M (|\nabla |\phi|^2_\varepsilon |^2 + \frac{R_g}{4} |\phi|^2_\varepsilon|^2) dvol_g \leq -\frac{1}{4} \int_M |\phi|^4 dvol_g + \varepsilon^2 \int_M \frac{R_g}{4} dvol_g.
\]

Since \( \lambda_M(g) \) is the lowest eigenvalue of the operator \(-4\Delta + R_g\), we obtain

\[
\lambda_M(g) \int_M |\phi|^2_\varepsilon dvol_g \leq \int_M (4|\nabla |\phi|^2_\varepsilon |^2 + R_g |\phi|^2_\varepsilon|^2) dvol_g.
\]
Note that, for \( \varepsilon \ll 1 \), \( \lambda_M(g) \leq 0 \). By Schwarz inequality,
\[
\lambda_M(g) \text{Vol}_g(M)^{1/2} \left( \int_M |\phi|^4 dvol_g \right)^{1/2} \leq \lambda_M(g) \left( \int_M |\phi|^2 dvol_g \right) \leq - \int_M |\phi|^4 dvol_g + \varepsilon^2 \int_M R_g dvol_g.
\]

Letting \( \varepsilon \to 0 \), we obtain
\[
\overline{\lambda}_M(g) = \lambda_M(g) \text{Vol}_g(M)^{1/2} \leq - \left( \int_M |\phi|^4 dvol_g \right)^{1/2}.
\]

By the second equation in the Seiberg-Witten equations, we get that
\[
\overline{\lambda}_M(g) \leq - \left( \int_M |\phi|^4 dvol_g \right)^{1/2} = - \left( 8 \int_M |F_A^+|^2 dvol_g \right)^{1/2}.
\]

Note that \( c_1^+ \) is the self-dual part of the harmonic form representing the first Chern class \( c_1 \). Clearly \( F_A^+ - 2\pi c_1^+ \) is \( L^2 \)-orthogonal to the harmonic forms space. Thus
\[
\int_M |F_A^+|^2 dvol_g \geq 4\pi^2 \int_M |c_1^+|^2 dvol_g = 4\pi^2 \int_M c_1^+ \wedge c_1^+ = 4\pi^2 [c_1^+]^2
\]
and
\[
\overline{\lambda}_M(g) \leq - \sqrt{32\pi^2 [c_1^+]^2 [M]}
\]
If the equality holds, all of \( \leq \) above are \( \equiv \). By Lemma 3.1 and the Bochner formula, \( \nabla^A \phi \equiv 0 \), \( R_g = - |\phi|^2 = \text{const.} \), and
\[
\nabla F_A^+ \equiv 0.
\]

Note that \( F_A^+ \) is a non-degenerate 2-form since \( \phi \neq 0 \). Thus, \( g \) is a Kähler metric with parallel Kähler form \( \omega = \sqrt{2 \frac{F_A^+}{|F_A^+|}} \). The desired result follows. \( \square \)

**Proof of Theorem 1.1.** From the hypothesis, for any Riemannian metric \( g \), there is a solution \( (\phi, A) \) of the Seiberg-Witten equations. Let \( c_1^+ \) is the self-dual part of the harmonic form representing the first Chern class \( c_1 \) of \( c \). Since
\[
\frac{1}{8} \int_M |\phi|^4 dvol_g = \int_M |F_A^+|^2 dvol_g \geq 4\pi^2 [c_1^+]^2 [M] \geq 4\pi^2 c_1^2 [M] > 0,
\]
the solution \( (\phi, A) \) is irreducible. By Proposition 3.2, we have
\[
\overline{\lambda}_M(g) \leq - \sqrt{32\pi^2 [c_1^+]^2 [M]} \leq - \sqrt{32\pi^2 c_1^2 [M]}
\]
Moreover, if \( [c_1^+]^2 [M] \neq 0 \), equality can only occur if \( g \) is a Kähler metric with constant negative scalar curvature. If \( g \) is a metric such that the equality holds in the above formula, then \( g \) is a critical point of the functional \( \overline{\lambda}_M(\cdot) \). By the claim in §2.3 of [Pe1], \( g \) is a gradient soliton, i.e., we have the following equation
\[
\text{Ric}(g) - cg + \nabla \nabla f = 0,
\]
where \( c \) is a constant, \( f \) satisfies the equation
\[
-4\triangle e^{-\frac{1}{2}f} + R_g e^{-\frac{1}{2}f} = \lambda_M(g) e^{-\frac{1}{2}f}.
\]
Since \( \lambda_M(g) \) is the lowest eigenvalue of the operator \(-4\triangle + R_g \) where \( R_g \) is a constant, we obtain that \( f \) is a constant, and \( g \) is an Einstein metric.
Now assume that \( g \) is a Kähler-Einstein metric with negative scalar curvature. We can assume that the Ricci form \( \rho = -\omega \) where \( \omega \) is the Kähler form associated to \( g \). It is well known that \( \rho \) is self-dual and is the harmonic representative of \( 2\pi c_1 \). We have

\[
(32\pi^2 [c_1^+]^2[M])^{1/2} = (32\pi^2 c_1^2[M])^{1/2} = 4(\frac{1}{2} \int_M \omega^2)^{1/2} = 4\text{Vol}_g(M)^{1/2}.
\]

Since \( \lambda_M(g) \) is the lowest eigenvalue of the operator \(-4\Delta - 4\), \( \lambda_M(g) = -4 \). Thus

\[
\lambda_M(g) = -4\text{Vol}_g(M)^{1/2} = -\sqrt{32\pi^2 c_1^2[M]}.
\]

The desired result follows. \( \square \)

**Proofs of Corollary 1.2 and Corollary 1.3.** From the hypothesis, \((M, J)\) is a complex surface of general type with

\[
c_1^2[M] = 2\chi(M) + 3\tau(M) > 0,
\]

(cf. Corollary 3.5 in [Le2]). By Theorem 4.1 in [Le2], the (mod 2) Seiberg-Witten invariant \( n_c(M) \neq 0 \) where \( c \) is the canonical Spin\(^c\) structure induced by \( J \).

If \( g_1 \) is a Riemannian metric on \( M \), then, by Theorem 1.1,

\[
\lambda_M(g_1)\text{Vol}_{g_1}(M)^{1/2} = \lambda_M(g_1) \leq -\sqrt{32\pi^2 c_1^2[M]} = \lambda_M(g_0)\text{Vol}_{g_0}(M)^{1/2}.
\]

Thus, if \( \lambda_M(g_1) = \lambda_M(g_0) \), we obtain

\[
\text{Vol}_{g_1}(M) \geq \text{Vol}_{g_0}(M)
\]

with equality if and only if \( g_1 \) is a Kähler-Einstein metric with negative scalar curvature. This proves Corollary 1.2.

To prove Corollary 1.3, let \( \{g_t\}, t \in [0, 1] \), be a family of Einstein metrics starting at \( g_0 \) on \( M \), i.e. \( \text{Ric}(g_t) = \frac{R}{4} g_t \). Let \( f_t \in C^\infty(M) \) such that, for any \( t \), \( e^{-f_t} \) is the eigenfunction of the lowest eigenvalue of the operator \(-4\Delta + R_{g_t} \) normalized by \( \int_M e^{-f_t}dvol_{g_t} = 1 \). Note that \( \lambda_M(g_t) = R_{g_t} \), and \( f_t \) is a constant function for any \( t \in [0, 1] \). For a \( t_0 \in [0, 1] \), if \( v_{ij} = \frac{d}{dt}g_{t_0, ij}|_{t=t_0} \), \( h = \frac{d}{dt}f_t|_{t=t_0} \), then we have \( \frac{d}{dt}dvol_{g_t}|_{t=t_0} = \frac{1}{2}vdvol_{g_{t_0}} \) where \( v = g_{t_0}^{ij}v_{ij} \), and

\[
\int_M e^{-f_{t_0}}(\frac{1}{2}v - h)dvol_{g_{t_0}} = 0.
\]

By the first formula in Section 1 of [Pe1],

\[
\frac{d}{dt}\lambda_M(g_t)|_{t=t_0} = \int_M e^{-f_{t_0}}[-v_{ij}\text{Ric}_{g_{t_0, ij}} + (\frac{1}{2}v - h)R_{g_{t_0}}]dvol_{g_{t_0}}
\]

\[
= -\int_M e^{-f_{t_0}}v_{ij}\text{Ric}_{g_{t_0, ij}}dvol_{g_{t_0}}
\]

\[
= -\int_M e^{-f_{t_0}}\frac{R_{g_{t_0}}}{4}dvol_{g_{t_0}}.
\]
\[ \frac{d}{dt} \lambda_M(g_t) \bigg|_{t=t_0} = \frac{d}{dt} \lambda_M(g_t) \text{Vol}_{g_{t_0}}(M) \sqrt{2} + \frac{1}{2} \lambda_M(g_t) \text{Vol}_{g_{t_0}}(M) \frac{d}{dt} \text{Vol}_{g_t}(M) \bigg|_{t=t_0} \\
= -\text{Vol}_{g_{t_0}}(M) \sqrt{2} \frac{R_{g_{t_0}}}{4} \int_M v d\text{Vol}_{g_{t_0}} \\
+ \frac{1}{4} \lambda_M(g_t) \text{Vol}_{g_{t_0}}(M) \sqrt{2} \int_M v d\text{Vol}_{g_{t_0}} \\
= 0. \]

Hence
\[ \frac{d}{dt} \lambda_M(g_t) \equiv 0. \]
and so
\[ \lambda_M(g_t) \equiv \lambda_M(g_0) = -\sqrt{32\pi^2 c_1^2[M]}, \]
for any \( t \). Therefore, by Theorem 1.1 \( g_t \) is a Kähler-Einstein metric with negative scalar curvature. Corollary 1.3 follows. \( \square \)

4. Proofs of Theorem 1.4 and 1.5

To prove Theorem 1.4 and Theorem 1.5 we need the following proposition.

**Proposition 4.1.** Let \( N \) and \( X \) be two smooth compact oriented \( n \)-manifolds, \( n \geq 3 \), and \( M \) be the connected sum of \( N \) and \( X \), i.e. \( M = N \# X \).

(i) If \( X \) admits a metric with positive scalar curvature, then
\[ \overline{\lambda}_N \leq \overline{\lambda}_M. \]
(ii) If \( \overline{\lambda}_N \leq 0, \overline{\lambda}_X \leq 0, \overline{\lambda}_M \leq 0, \) and \( n = 4m \), then
\[ -(\overline{\lambda}_X^{2m} + \overline{\lambda}_N^{2m})^{\frac{1}{2m}} \leq \overline{\lambda}_M. \]

We remark that the inequality above is often a strict inequality, e.g. if \( N \) is a simply connected Spin-manifold of dimension \( 4m \geq 5 \) with \( \hat{A} \)-genus nonzero and \( X = \mathbb{C}P^{2m} \), clearly \( \overline{\lambda}_N \leq 0 \), however, by [GL][SY] it is well-known that \( N \# \mathbb{C}P^{2m} \) admits a metric with positive scalar curvature, therefore \( \overline{\lambda}_{N \# \mathbb{C}P^{2m}} > 0 \).

**Lemma 4.2.** Let \( (X, h) \) be an oriented compact Riemannian \( n \)-manifold with positive scalar curvature, \( N \) be an oriented smooth compact \( n \)-manifold, \( n \geq 3 \), and \( M = N \# X \). Then, for any metric \( g \) on \( N \) and \( 0 < \varepsilon \ll 1 \), there exists a metric \( g_\varepsilon \) on \( M \) such that
\[ \lambda_N(g) - \varepsilon \leq \lambda_M(g_\varepsilon), \quad \text{and} \quad |\text{Vol}_g(N) - \text{Vol}_{g_\varepsilon}(M)| \leq \varepsilon. \]

**Remark.** The fact that \( M \) admits a metric \( g_\varepsilon \) such that \( \lambda_M(g_\varepsilon) \) is close to \( \lambda_N(g) \) is an easy consequence of Theorem 3.1 in [BD2]. Here we must construct \( g_\varepsilon \) carefully such that \( \text{Vol}_{g_\varepsilon}(M) \) is close to \( \text{Vol}_g(N) \).
Proof. For a $p \in N$, denote $U(r) = \{x|\text{dist}_g(x,p) < r\}$. By Lemma 3.7 in [BD2], there exists a $0 < \tau < 1$ such that, for any $0 < r < \frac{\tau}{2}$ and any smooth function $u$ on $A((r,2r)\pi)$, the following holds
\[
(4.3) \quad \|u\|^2_{L^2(A((r,2r)\pi))} \leq 10r^{\frac{\delta}{2}}\|u\|^2_{L^2(A((r,2r)\pi))},
\]
if $\int_{\partial U(\rho)} u\partial_r u\,dA \geq 0$ holds for all $\rho \in [r, (2r)\pi]$. Here $A((r,2r)\pi) = \{x|r \leq \text{dist}_g(x,p) \leq 2r\}$, and $\nu$ is the unite normal vector field of $\partial U(\rho)$ pointing away from $p$. Let $\Lambda$ be a positive constant bigger than the lowest eigenvalue of the operator $-\Delta + R_g$ on $(N\setminus U(\tau), g)$ with Dirichlet boundary conditions. Let $R_0$ be a lower bound of the scalar curvature $R_g$ of $(N, g)$, and $R_1$ be a number such that
\[
(4.4) \quad R_1 > \min\{0, \Lambda\}, \quad \Lambda\frac{\Lambda - R_0}{R_1 - \Lambda} \leq \frac{\varepsilon}{2}.
\]
By the arguments in the proof of Theorem 3.1 in [BD2] or Proposition 2.1 of [BD1], there is a metric $g'$ on $N$ arbitrarily close to $g$ in the $C^1$-topology such that $R_{g'} \geq R_0$ and $R_{g'} \geq 2R_1$ on a neighborhood $U_0$ of $p$. Since both $\lambda_N(g)$ and $\text{Vol}_g(N)$ depend continuously on $g$ in the $C^1$-topology (See Lemma 3.4 in [BD2]), we may without loss of generality assume that $R_g \geq R_0$ and $R_g \geq 2R_1$ on a neighborhood $U_0$ of $p$.

Now we choose $r > 0$ and $\zeta > 0$ so small that
\[
(i) \quad \frac{R_1 - R_0}{R_1 - \Lambda}((\Lambda + 1 - R_0)\zeta + \zeta^2) \leq \frac{\varepsilon}{2},
\]
\[
(ii) \quad 2\sqrt{10}r^{\frac{1}{2}} < \zeta,
\]
\[
(iii) \quad U((2r)\pi) \subset U_0,
\]
\[
(iv) \quad (2r)\pi \pi \leq \tau,
\]
\[
(v) \quad \text{Vol}_g(U(r)) < \frac{\varepsilon}{2}.
\]
Let $\eta$ be a smooth cut-off function such that
\[
(i) \quad 0 \leq \eta \leq 1 \text{ on } N,
\]
\[
(ii) \quad \eta \equiv 0 \text{ on } U(r),
\]
\[
(iii) \quad \eta \equiv 1 \text{ on } N\setminus U(2r),
\]
\[
(iv) \quad |d\eta| \leq \frac{2}{\tau} \text{ on } N.
\]

**Lemma 4.3.** For any $0 < \theta_0 \ll 1$, there is a metric $\tilde{g}_{\theta_0}$ on $A((r,\frac{r}{2})\pi) = U(r)\setminus U(\frac{r}{2})$ satisfying that $R_{\tilde{g}_{\theta_0}} \geq R_1$, $\tilde{g}_{\theta_0}$ agrees with $g$ near the boundary $\partial U(r)$, and $\tilde{g}_{\theta_0}$ agrees with $dt^2 + \delta^2 g_{0,1}$ near the boundary $\partial U(\frac{r}{2}) \simeq S^{n-1}(1)$, where $\delta = \delta(\theta_0)$ is a function of $\theta_0$ such that $\delta \ll \theta_0$, and $g_{0,1}$ is the standard metric of sectional curvature 1 on $S^{n-1}(1)$. Furthermore,
\[
|\text{Vol}_{\tilde{g}_{\theta_0}}(A(r,\frac{r}{2})) - \text{Vol}_g(U(r))| \longrightarrow 0,
\]
if $\theta_0 \longrightarrow 0$. 


Proof. We will use Gromov-Lawson’s construction here (See Theorem A of [GL], and Theorem 3.1 of [RS]). The key idea of the proof of Theorem A in [GL] is to choose a suitable curve $\gamma$ in the $t-\varrho$ plane, and to consider

$$T_\gamma = \{(t,x) \in \mathbb{R} \times U(r) | (t, \text{dist}_g(x,p) = \varrho) \in \gamma\},$$

with the induced metric, where $\mathbb{R}$ is given the Euclidean metric and $\mathbb{R} \times U(r)$ is given the natural product metric $dt^2 + g$. The scalar curvature is given by

$$R_{\gamma} = R_g + ((n-1)(n-2)\frac{1}{\varrho^2} + O(1)) \sin^2 \theta - (n-1)(\frac{1}{\varrho} + O(\varrho))k \sin \theta$$

$$\geq 2R_1 + ((n-1)(n-2)\frac{1}{\varrho^2} - C) \sin^2 \theta - (n-1)(\frac{1}{\varrho} + C' \varrho)k \sin \theta,$$

where $C, C'$ are constants depending only on the curvature of $g$, $k$ is the curvature of $\gamma$, and $\theta$ is the angle between $\gamma$ and the $\varrho$-axis (See the formula (1) in [GL]). There are several steps to construct $\gamma$.

First, let $\gamma_0$ be the bent line segment given by $\{(t, \varrho) | \varrho = -\coth \theta_0 t + \frac{r}{4}\}$ on $\mathbb{R} \times [\frac{r}{4}, 0]$, and a smooth curve with angle between $\gamma_0$ and the $\varrho$-axis less than $\theta_0$ on $\mathbb{R} \times [r, \frac{r}{4}]$. From the proof of Theorem A in [GL] or the proof of Theorem 3.1 in [RS], we can choose $0 < \theta_0 \ll 1$ such that $R_{\gamma_0} \geq R_1$.

Following the arguments in P359 of [RS], we choose a $\varrho_0$ with $0 < \varrho_0 < \min(\sqrt{\frac{1}{4C}}, \sqrt{\frac{1}{2C'}})$. Then, for $0 < \varrho \leq \varrho_0$, we have

$$R_{\gamma} \geq 2R_1 + (n-1)\frac{3}{4\varrho^2} \sin^2 \theta - (n-1)\frac{3}{2\varrho}k \sin \theta.$$ 

Let $\gamma$ be $\gamma_0$ on $\mathbb{R} \times [\varrho_0, \varrho]$, and be a curve satisfying $k = \frac{\sin \theta}{2\varrho}$ on $\mathbb{R} \times [\varrho_0, 0]$. By the arguments in P359 of [RS], $\gamma$ is given by the graph of function $\varrho = f(t)$ with...
Thus the metric \( (4.10) \):

\[
\gamma \quad \text{after} \\
\delta < (4.7)
\]

Thus we have

\[
(4.6) \quad \delta = \frac{t_0^2 \theta_0}{(\theta_0 - \frac{r}{2})^2 + t_0^2}, \quad \text{and} \quad B = t_0 + \sqrt{4\delta(\theta_0 - \delta)}.
\]

By taking \( \theta_0 \ll r \) and \( \phi_0 \ll r \), we obtain

\[
(4.7) \quad \delta < 4\theta_0^2 \theta_0, \quad \text{and} \quad B < r\theta_0 + 4\theta_0 \theta_0.
\]

After \( \gamma \) reach \((B, \delta)\), let \( \gamma \) be \([B, 2B] \times \{\delta\}\). Now we have constructed a metric on \( T_\gamma \), denoted by \( g_\gamma \), satisfying that \( R_\gamma \geq R_1 \), \( g_\gamma \) agrees with \( g \) near \( \partial U(r) \), \( g_\gamma \) agrees with the product metric induced by \( \mathbb{R} \times U(r) \) near the other boundary of \( T_\gamma \), \([2B] \times \partial U(\delta)\). Furthermore, if we let \( \theta_0 \longrightarrow 0 \), then, by (4.7),

\[
(4.8) \quad |\text{Vol}_{g_\gamma}(T_\gamma) - \text{Vol}_g(U(r))| \longrightarrow 0.
\]

Note that \( \partial U(\delta) \cong S^{n-1}(\delta) = \{y \in \mathbb{R}^n||y|| = \delta\} \). If \( g_{0,1} \) is the standard metric of sectional curvature \( 1 \) on \( S^{n-1}(1) \), then \( \frac{1}{\delta^2} g_{|\partial U(\delta)} \) converges to \( g_{0,1} \) in the \( C^2 \)-topology by Lemma 1 in [GL], i.e. there is a 2-tensor \( \alpha(\delta) \) on \( S^{n-1}(1) \) with \( \frac{1}{\delta^2} g_{|\partial U(\delta)} - g_{0,1} = \alpha(\delta) \) and \( ||\alpha(\delta)||_{C^2} \longrightarrow 0 \) when \( \delta \longrightarrow 0 \). Let \( \sigma(t) \) be a smooth function such that \( \sigma(t) \equiv 1 \) on \([0, \frac{1}{4}]\), \( \sigma(t) \equiv 0 \) on \([\frac{1}{4}, 1]\), and \( |\frac{d}{dt}\sigma(t)| \leq 4 \) on \([0, 1]\). Define a metric on \([0, \frac{1}{4}] \times S^{n-1}(1) \) by \( g_\delta = dt^2 + g_{0,1} + \sigma(\delta t)\alpha(\delta) \). When \( \delta \ll 1 \), \( R_{g_\delta} > \frac{1}{4} \). Define \( g_\delta = \delta^2 g_\delta \) on \([2B, 1] \times \partial U(\delta) \) which satisfies that \( g_\delta = dt^2 + g_{|\partial U(\delta)} \) near \( \{2B\} \times \partial U(\delta) \), \( g_\delta = dt^2 + \delta^2 g_{0,1} \) near \( \{1\} \times \partial U(\delta) \), \( R_{g_\delta} > \frac{1}{4\delta^2} \), and \( \text{Vol}_{g_\delta}([2B, 1] \times \partial U(\delta)) = O(\delta^{-n-1}) = O(\theta_0^{2n-2}) \). Let \( \tilde{T}_\gamma \) be the manifold obtained by gluing \( T_\gamma \) and \([2B, 1] \times \partial U(\delta) \) at \( \{2B\} \times \partial U(\delta) \), i.e.

\[
(4.9) \quad \tilde{T}_\gamma = T_\gamma \cup [2B, 1] \times \partial U(\delta),
\]

and \( \tilde{g}_\gamma \) be a metric on \( \tilde{T}_\gamma \) such that \( \tilde{g}_\gamma = g_\gamma \) on \( T_\gamma \), and \( \tilde{g}_\gamma = g_\delta \) on \([2B, 1] \times \partial U(\delta)\). Thus the metric \( \tilde{g}_\gamma \) satisfies that \( R_{\tilde{g}_\gamma} \geq R_1 \) and

\[
(4.10) \quad |\text{Vol}_{\tilde{g}_\gamma}(\tilde{T}_\gamma) - \text{Vol}_g(U(r))| \longrightarrow 0,
\]

when \( \theta_0 \longrightarrow 0 \). Since \( A(r, \frac{1}{2}) \simeq \tilde{T}_\gamma \), we obtain the conclusion by letting \( \tilde{g}_{\theta_0} = \tilde{g}_\gamma \).

Let’s continue to prove Lemma 4.2. Let \( \tilde{U} \) be the connected sum of \( U(r) \) and \( X \). Now let’s consider \((X, h)\). By the proof of Theorem A in [GL], we have a compact manifold \( \tilde{X} \) with boundary \( \partial \tilde{X} = S^{n-1}(\varsigma) \), which is obtained by deleting a small disc from \( X \), and a metric \( \tilde{h} \) on \( \tilde{X} \) such that the scalar curvature \( R_{\tilde{h}} \) is positive, and \( \tilde{h} = dt^2 + g_{0,\varsigma} \) near the boundary \( \partial \tilde{X} \), where \( g_{0,\varsigma} \) is the standard metric of sectional curvature \( \frac{1}{\varsigma} \) on \( S^{n-1}(\varsigma) \). By letting \( \theta_0 \ll \min\{\varsigma, \min R_h\} \), we obtain that \( \delta \ll \min\{\varsigma, \min R_h\} \), and the metric \( (\frac{\varsigma}{2})^2 \tilde{h} \) satisfies that the scalar curvature of \( (\frac{\varsigma}{2})^2 \tilde{h} \) is bigger than \( R_1 \), \( (\frac{\varsigma}{2})^2 \tilde{h} = dt^2 + g_{0,\delta} \) near the boundary \( \partial \tilde{X} \), and

\[
\text{Vol}_{(\frac{\varsigma}{2})^2 \tilde{h}}(\tilde{X}) \longrightarrow 0,
\]

if \( \delta \longrightarrow 0 \).
Note that \( \tilde{U} \) is obtained by gluing \( A(r, \frac{r}{2}) \) and \( \tilde{X} \) at \( \partial U(\frac{r}{2}) \cong \partial \tilde{X} \), i.e. \( \tilde{U} = A(r, \frac{r}{2}) \cup \tilde{X} \). For any \( \theta_0 \ll 1 \), define a metric \( \widetilde{g}_{\theta_0} \) on \( \tilde{U} \) such that \( \widetilde{g}_{\theta_0} = \tilde{g}_{\theta_0} \) on \( A(r, \frac{r}{2}) \), where \( \tilde{g}_{\theta_0} \) is the metric obtained in Lemma 4.3, and \( \tilde{g}_{\theta_0} = (\frac{\lambda}{\varepsilon})^2 \tilde{h} \) on \( \tilde{X} \), which satisfies that \( R_{\tilde{g}_{\theta_0}} \geq R_1 \) and

\[
|\text{Vol}_{\tilde{g}_{\theta_0}}(\tilde{U}) - \text{Vol}_{g}(U(r))| < |\text{Vol}_{\tilde{g}_{\theta_0}}(A(r, \frac{r}{2})) - \text{Vol}_{g}(U(r))| + \text{Vol}_{(\frac{\lambda}{\varepsilon})^2 \tilde{h}}(\tilde{X}) \to 0,
\]

when \( \theta_0 \to 0 \).

Note that \( M \) is obtained by gluing \( N \setminus U(r) \) and \( \tilde{U} \) at \( \partial U(r) \), i.e.

\[
M = (N \setminus U(r)) \cup \tilde{U}.
\]

Define metrics \( g_{\theta_0} \) on \( M \) by \( g_{\theta_0} = g \) on \( N \setminus U(r) \) and \( g_{\theta_0} = \tilde{g}_{\theta_0} \) on \( \tilde{U} \), which satisfy

\[
|\text{Vol}_{g}(N) - \text{Vol}_{g_{\theta_0}}(M)| \leq |\text{Vol}_{g}(U(r))| + |\text{Vol}_{g}(U(r)) - \text{Vol}_{\tilde{g}_{\theta_0}}(\tilde{U})| \to 0,
\]

when \( \theta_0 \to 0 \). Thus, for any \( 0 < \varepsilon \ll 1 \), there is a \( \theta_0 \) such that

\[
(4.11) \quad |\text{Vol}_{g}(N) - \text{Vol}_{g_{\theta_0}}(M)| < \varepsilon.
\]

By defining \( g_{\varepsilon} = g_{\theta_0} \) on \( M \), we obtain the volumes inequality.

**Lemma 4.4.**

\[
\lambda_N(g) - \varepsilon \leq \lambda_M(g_{\varepsilon}).
\]

**Proof.** The following arguments is the same as the proof of Theorem 3.1 in [BD2]. But for reader’s convenience, we present the proof here. Let \( u \) be the eigenfunction of \( \lambda_M(g_{\varepsilon}) \) on \( (M, g_{\varepsilon}) \). The function \( v = \eta u \) can be regarded as a function on \( (N, g) \). Thus

\begin{equation}
(4.12) \quad \lambda_N(g) \leq \frac{\int_N (4|dv|^2 + R_g v^2) dv \text{vol}_g}{\int_N v^2 dv \text{vol}_g}.
\end{equation}

Since \( \Lambda \) is larger than the lowest eigenvalue of the operator \(-4\Delta + R_g \) on \( (N \setminus U(\bar{r}), g) \) with Dirichlet boundary conditions, we have \( \lambda_M(g_{\varepsilon}) \leq \Lambda \) by Lemma 92.5 in [KL]. Thus

\[
R_1 \int_{\tilde{U} \cup A(r, 2r)} u^2 dv \text{vol}_{g_{\varepsilon}} + R_0 \int_{M \setminus U \cup A(r, 2r)} u^2 dv \text{vol}_{g_{\varepsilon}} \leq \int_M (4|du|^2 + R_g u^2) dv \text{vol}_{g_{\varepsilon}} \leq \Lambda \int_M u^2 dv \text{vol}_{g_{\varepsilon}}.
\]

Hence

\begin{equation}
(4.13) \quad \int_{\tilde{U} \cup A(r, 2r)} u^2 dv \text{vol}_{g_{\varepsilon}} \leq \frac{\Lambda - R_0}{R_1 - R_0} \int_M u^2 dv \text{vol}_{g_{\varepsilon}}.
\end{equation}

We have

\begin{equation}
(4.14)
\end{equation}
\[ \|v\|_{L^2(N)}^2 = \|\eta u\|_{L^2(N)}^2 \geq \|u\|_{L^2(N \setminus U(2r))}^2 \]
\[ \geq (1 - \frac{\Lambda - R_0}{R_1 - R_0})\|u\|_{L^2(M)}^2 \]
\[ \geq \frac{R_1 - \Lambda}{R_1 - R_0}\|u\|_{L^2(M)}^2. \]

For a \( \rho \in [r, (2r)^{\frac{1}{11}}] \), set \( \tilde{U}_\rho = \tilde{U} \cup A(r, \rho) \). Since \( R_{g_\varepsilon} \geq R_1 \) on \( \tilde{U}_\rho \), we have
\[ \lambda_M(g_\varepsilon)\|u\|_{L^2(\tilde{U}_\rho)}^2 = 4 \int_{\tilde{U}_\rho} \langle \Delta u, u \rangle dvol_{g_\varepsilon} + \int_{\tilde{U}_\rho} R_{g_\varepsilon}u^2 dvol_{g_\varepsilon} \]
\[ = 4 \int_{\tilde{U}_\rho} |du|^2 dvol_{g_\varepsilon} - 4 \int_{\partial \tilde{U}_\rho} u \partial_\nu u dA + \int_{\tilde{U}_\rho} R_{g_\varepsilon}u^2 dvol_{g_\varepsilon} \]
\[ \geq -4 \int_{\partial \tilde{U}_\rho} u \partial_\nu u dA + R_1 \|u\|_{L^2(\tilde{U}_\rho)}^2. \]

Hence
\[ 4 \int_{\partial \tilde{U}_\rho} u \partial_\nu u dA \geq (R_1 - \lambda_M(g_\varepsilon))\|u\|_{L^2(\tilde{U}_\rho)}^2 \geq (R_1 - \Lambda)\|u\|_{L^2(\tilde{U}_\rho)}^2 \geq 0. \]

By Lemma 3.7 in [BD2],
\[ \|u\|_{L^2(A(r, 2r))} \leq 10r^{\frac{2}{7}}\|u\|_{L^2(A(r, (2r)^{\frac{1}{11}}))} \leq 10r^{\frac{2}{7}}\|u\|_{L^2(M)}. \]

Thus
\[ (4.15) \]
\[ \frac{2}{\rho}\|u\|_{L^2(A(r, 2r))} \leq \zeta\|u\|_{L^2(M)}. \]

We have
\[ \|dv\|_{L^2(N)}^2 = \|d(\eta u)\|_{L^2(N)}^2 \]
\[ \leq (\|\eta du\|_{L^2(N)}^2 + \|d\eta u\|_{L^2(N)}^2)^2 \]
\[ \leq (\|du\|_{L^2(M)}^2 + \zeta\|u\|_{L^2(M)}^2)^2 \]
\[ \leq (1 + \zeta)\|du\|_{L^2(M)}^2 + \zeta(1 + \zeta)\|u\|_{L^2(M)}^2 \]
\[ \leq (1 + \zeta)\|du\|_{L^2(M)}^2 \]
\[ \leq \Lambda + 1 - R_0 \zeta + \zeta^2\|u\|_{L^2(M)}^2. \]

Since \( R_{g_\varepsilon} > R_1 > 0 \) on \( \tilde{U} \cup U(2r) \),
\[ (4.16) \]
\[ (R_{g_\varepsilon}v, v)_{L^2(N)} \leq (R_{g_\varepsilon}\eta u, \eta u)_{L^2(M)} \leq (R_{g_\varepsilon}u, u)_{L^2(M)}. \]

We obtain
\[ 4\|dv\|_{L^2(N)}^2 + (R_{g_\varepsilon}v, v)_{L^2(N)} \]
\[ \leq 4(1 + \zeta)\|du\|_{L^2(M)}^2 + \zeta(1 + \zeta)\|u\|_{L^2(M)}^2 \]
\[ \leq (1 + \zeta)\Lambda(g_\varepsilon)\|u\|_{L^2(M)}^2 - \zeta R_0\|u\|_{L^2(M)}^2 + \zeta(1 + \zeta)\|u\|_{L^2(M)}^2 \]
\[ \leq [\Lambda(g_\varepsilon) + (\Lambda + 1 - R_0)\zeta + \zeta^2]\|u\|_{L^2(M)}^2. \]
Hence we have
\[
\lambda_N(g) \leq \frac{\int_N (4|dv|^2 + R_g v^2) dvol_g}{\int_N v^2 dvol_g}
\leq \frac{R_1 - R_0}{R_1 - \lambda} \left[ \lambda_M(g_\varepsilon) + (\Lambda + 1 - R_0) \zeta + \zeta^2 \right]
= \lambda_M(g_\varepsilon) + \frac{\Lambda - R_0}{R_1 - \lambda} \lambda_M(g_\varepsilon) + \frac{R_1 - R_0}{R_1 - \lambda} [(\Lambda + 1 - R_0) \zeta + \zeta^2]
\leq \lambda_M(g_\varepsilon) + \varepsilon,
\]
by (4.4), (4.5) and (4.14). Thus both Lemma 4.4 and Lemma 4.2 are proved. □

**Lemma 4.5.** Let \( N_1 \) and \( N_2 \) be two compact oriented \( 4m \)-manifolds with \( \overline{\lambda}_{N_1} \leq 0, \overline{\lambda}_{N_2} \leq 0 \). Let \( M = N_1 \# N_2 \). Assume that \( \overline{\lambda}_M \leq 0 \). For any metrics \( g_1 \) and \( g_2 \) on \( N_1 \) and \( N_2 \) respectively with \( \lambda_{N_1}(g_1) = \lambda_{N_2}(g_2) = -1 \), and \( 0 < \varepsilon \ll 1 \), there is a metric \( g_\varepsilon \) on \( M \) such that
\[
(1 + \varepsilon)^2 (\overline{\lambda}_{N_1}(g_1)^2 + \overline{\lambda}_{N_2}(g_2)^2 + \varepsilon) \geq \overline{\lambda}_M(g_\varepsilon)^2.
\]

**Proof.** Let \( N = N_1 \cup N_2 \), and \( p_i \in N_i, i = 1, 2 \). Notations, \( r, \bar{r}, U(r), R_0, R_1, \Lambda, \) and \( \zeta \), are the same as in the proof of Lemma 4.2. Here the only difference is that we use the set \( \{p_1, p_2\} \) instead of \( \{p_1\} \) in step of \( N \). Denote \( U_i(r) = \{x \in N_i| \text{dist}_{g_i}(x, p_i) \leq r\} \). By Lemma 4.3, for any \( 0 < \theta_0 \ll 1 \), for each \( i \), there is a metric \( \bar{g}_{i, \theta_0} \) on \( A_i(r, \frac{r}{2}) = U_i(r) \cup \bigcup_{j \neq i} U_j(\frac{r}{2}) \) satisfying that \( R_{\bar{g}_{i, \theta_0}} \geq R_i, \bar{g}_{i, \theta_0} \) agrees with \( g_i \) near the boundary \( \partial U_i(r) \), and \( \bar{g}_{i, \theta_0} \) agrees with \( dt^2 + \delta(\theta_0)^2 g_{0,1} \) near the boundary \( \partial U_i(r) \approx S^{n-1}(1) \), where \( g_{0,1} \) is the standard metric of sectional curvature 1 on \( S^{n-1}(1) \). From (4.6), we can choose \( \delta = \delta(\theta_0) \) as a function of \( \theta_0 \) in-dependent of \( i \) such that \( \delta \ll \theta_0 \). Furthermore,
\[
|\text{Vol}_{\bar{g}_{i, \theta_0}}(A_i(r, \frac{r}{2})) - \text{Vol}_{g_i}(U_i(r))| \to 0,
\]
if \( \theta_0 \to 0 \). Note that \( M \) is obtained by gluing \( N_1 \setminus U_1(\frac{r}{2}) \) and \( N_2 \setminus U_2(\frac{r}{2}) \) at \( U_1(\frac{r}{2}) \), i.e. \( M = N_1 \setminus U_1(\frac{r}{2}) \cup N_2 \setminus U_2(\frac{r}{2}) \). Define a metric \( g_{\theta_0} \) on \( M \) by \( g_{\theta_0} = g_i \) on \( N_i \setminus U_i(r) \), and \( g_{\theta_0} = \bar{g}_{i, \theta_0} \) on \( A_i(r, \frac{r}{2}) \), which satisfies
\[
|\text{Vol}_{g_{\theta_0}}(M) - \sum \text{Vol}_{g_i}(N_i)| \leq \sum |\text{Vol}_{\bar{g}_{i, \theta_0}}(A_i(r, \frac{r}{2})) - \text{Vol}_{g_i}(U_i(r))| \to 0,
\]
when \( \theta_0 \to 0 \). For any \( \varepsilon > 0 \), by letting \( \theta_0 \ll 1 \) and \( g_{\varepsilon} = g_{\theta_0} \), we find a metric \( g_{\varepsilon} \) on \( M \) with
\[
|\text{Vol}_{g_{\varepsilon}}(M) - \sum \text{Vol}_{g_i}(N_i)| \leq \varepsilon.
\]
By the same arguments as in the proof of Lemma 4.4, we have
\[
-1 - \varepsilon \leq \lambda_M(g_{\varepsilon}).
\]
Thus
\[
(1 + \varepsilon)^2 (\overline{\lambda}_{N_1}(g_1)^2 + \overline{\lambda}_{N_2}(g_2)^2 + \varepsilon) \geq \overline{\lambda}_M(g_{\varepsilon})^2.
\]
□
Proof of Proposition 4.1. First, we assume that there is a metric $h$ on $X$ with positive scalar curvature. By Lemma 4.2, for any metric $g$ on $N$ and $0 < \varepsilon \ll 1$, there exists a metric $g_\varepsilon$ on $M$ such that

$$\lambda_N(g) - \varepsilon \leq \lambda_M(g_\varepsilon), \quad \text{and} \quad |\text{Vol}_g(N) - \text{Vol}_{g_\varepsilon}(M)| \leq \varepsilon.$$  

Thus we have

$$(\lambda_N(g) - \varepsilon)(\text{Vol}_g(N) + \varepsilon)\frac{\tau}{2} \leq \lambda_M(g_\varepsilon)\text{Vol}_{g_\varepsilon}(M)\frac{\tau}{2} \leq \overline{\lambda}_M,$$

or

$$(\lambda_N(g) - \varepsilon)(\text{Vol}_g(N) - \varepsilon)\frac{\tau}{2} \leq \lambda_M(g_\varepsilon)\text{Vol}_{g_\varepsilon}(M)\frac{\tau}{2} \leq \overline{\lambda}_M.$$  

By letting $\varepsilon \to 0$,

$$\lambda_N(g)\text{Vol}_g(N)\frac{\tau}{2} \leq \overline{\lambda}_M.$$  

Thus

$$\overline{\lambda}_N = \sup_{g \in \mathcal{M}} \lambda_N(g) \leq \overline{\lambda}_M,$$

where $\mathcal{M}$ is the set of Riemannian metrics on $N$. Hence, we obtain (4.1).

Now we assume that $\overline{\lambda}_N \leq 0$, $\overline{\lambda}_X \leq 0$, $\overline{\lambda}_M \leq 0$, and $n = 4m$. We can choose any two metrics $g_1$ and $g_2$ on $N$ and $X$ respectively with $\overline{\lambda}_N(g_1) < 0$ and $\overline{\lambda}_X(g_2) < 0$. After re-scaling them, we can assume $\lambda_N(g_1) = \lambda_X(g_2) = -1$. By Lemma 4.5, for any $\varepsilon > 0$, there exists a metric $g_\varepsilon$ on $M$ such that

$$-(1 + \varepsilon)(\overline{\lambda}_N(g_1))^{2m} + (\overline{\lambda}_X(g_2))^{2m} + \varepsilon\frac{\tau}{2m} \leq \lambda_M(g_\varepsilon) \leq \overline{\lambda}_M.$$  

By letting $\varepsilon \to 0$, we obtain

$$-(\overline{\lambda}_N(g_1))^{2m} + (\overline{\lambda}_X(g_2))^{2m} \frac{\tau}{2m} \leq \overline{\lambda}_M.$$  

Thus

$$-(\overline{\lambda}_N^{2m} + \overline{\lambda}_X^{2m}) \frac{\tau}{2m} = -((\sup_{g_1 \in \mathcal{M}_1} \overline{\lambda}_N(g_1))^{2m} + (\sup_{g_2 \in \mathcal{M}_2} \overline{\lambda}_X(g_2))^{2m}) \frac{\tau}{2m} \leq \overline{\lambda}_M.$$  

□

Proof of Theorem 1.4. By Theorem 1 in [Ta], the Spin$^c$ structure induced by a compatible almost complex structure on $(N, \omega)$ has Seiberg-Witten invariant equal to $\pm 1$. By Lemma 1 in Section 3 of [Le4], for any metric $g$ on $M$, we can choose a Spin$^c$ structure on $M$ with non-vanishing Seiberg-Witten invariant, and

$$[c_1^+]^2[M] \geq 2\chi(N) + 3\tau(N) > 0.$$  

Thus, by Proposition 3.2, we obtain

$$\overline{\lambda}_M(g) \leq -\sqrt{32\pi^2[c_1^+]^2[M]} \leq -\sqrt{32\pi^2(2\chi(N) + 3\tau(N))}.$$  

Hence

$$\overline{\lambda}_M \leq -\sqrt{32\pi^2(2\chi(N) + 3\tau(N))}.$$  

If $N$ admits a Kähler-Einstein metric, then, by Proposition 4.1,

$$-\sqrt{32\pi^2(2\chi(N) + 3\tau(N))} = \overline{\lambda}_N \leq \overline{\lambda}_M \leq -\sqrt{32\pi^2(2\chi(N) + 3\tau(N))}.$$  

Hence we obtain the conclusion. □
Lemma 4.6. Let $M$ be a spin manifold with non-vanishing $\hat{A}$-genus, i.e. $\hat{A}(M) \neq 0$. Then

$$\overline{\lambda}_M \leq 0.$$  

Proof. Since $\hat{A}(M) \neq 0$, for any metric $g'$ on $M$, there is a non-vanishing harmonic spinor $\phi \in \Gamma(S)$ with $\int_M |\phi|^2 dvol_{g'} = 1$, where $S$ is the spin bundle. The Bochner formula implies that

$$0 = D^2 \phi = \nabla^* \nabla \phi + \frac{R_{g'}}{4} \phi,$$

where $D : \Gamma(S) \rightarrow \Gamma(S)$ is the Dirac operator. By (3.2),

$$\lambda_M(g') \leq \int_M (4|\nabla |\phi||_\varepsilon|^2 + R_{g'} |\phi|^2_\varepsilon) dvol_{g'}$$

$$\leq \int_M (4|\nabla |\phi|^2 + R_{g'} |\phi|^2) dvol_{g'} + \varepsilon^2 \int_M R_{g'} dvol_{g'} = \varepsilon^2 \int_M R_{g'} dvol_{g'}.$$

By letting $\varepsilon \rightarrow 0$, we obtain that

$$\lambda_M(g') \leq 0,$$

and

$$\overline{\lambda}_M \leq 0.$$  

Proof of Theorem 1.5. By Theorem 1 in [Ta], for any $i$, the Spin$^c$ structure induced by a compatible almost complex structure on $(N_i, \omega_i)$ has Seiberg-Witten invariant equal to $\pm 1$. By Corollary 11 in [IL], for any metric $g$ on $M$, there is a monopole class $\alpha$ of $M$ satisfying that

$$[\alpha^+][M] \geq \sum_{i=1}^\ell c_1^2[N_i].$$

Thus, by Proposition 3.2, we obtain

$$\overline{\lambda}_M(g) \leq -\sqrt{32\pi^2[\alpha^+][M]} \leq -\sqrt{32\pi^2 \sum_{i=1}^\ell c_1^2[N_i]}.$$ 

Hence

$$\overline{\lambda}_M \leq -\sqrt{32\pi^2 \sum_{i=1}^\ell c_1^2[N_i]}.$$ 

Now, we assume that, for each $i$, $N_i$ admits a Kähler-Einstein metric $g_i$. If the scalar curvature of $g_i$ is negative, we have already known that $\overline{\lambda}_{N_i} = \overline{\lambda}_{N_i}(g_i) = -\sqrt{32\pi^2 c_1^2(N_i)}$ by Theorem 1.1. If the scalar curvature of $g_i$ is zero, then $N_i$ is a $K3$-surface from the hypothesis (cf. [BHPV]). By Lemma 4.6, we obtain $0 = \overline{\lambda}_{N_i}(g_i) \leq \overline{\lambda}_{N_i} \leq 0$. Thus

$$\overline{\lambda}_{N_i} = 0 = -\sqrt{32\pi^2 c_1^2(N_i)}.$$ 

By Proposition 4.1,

$$-\sqrt{32\pi^2 \sum_{i=1}^\ell c_1^2[N_i]} = -\sqrt{\sum_{i=1}^\ell \overline{\lambda}_{N_i}^2} \leq \overline{\lambda}_M \leq -\sqrt{32\pi^2 \sum_{i=1}^\ell c_1^2[N_i]}.$$ 

This proves the desired result.  

□
Proof of Corollary 1.6. Note that \( N_i \) are spin manifolds (See [J]), and thus is \( M \). By Lemma 4.6, \( \lambda_M \leq 0 \) as \( \hat{A}(M) \neq 0 \). Since \( X_1 \cdots X_l \) are simply connected compact oriented spin n-manifolds with \( \hat{A}(X_j) = 0 \), \( n \geq 8 \), and \( n = 0 \mod 4 \), for any \( X_j \), there is a metric \( h_j \) on \( X_j \) with positive scalar curvature from Theorem A in [St]. Note that \( (N_i, g_i) \) are Ricci-flat Einstein manifolds with \( \hat{A}(N_i) \neq 0 \) (See [J]). By Lemma 4.6,

\[
0 = \lambda_{N_i}(g_i) \leq \lambda_M \leq 0.
\]

By Proposition 4.1,

\[
0 \leq \lambda_{\sum_{i=1}^{l} N_i} \leq \lambda_M \leq 0.
\]

We obtain the conclusion. \( \square \)

5. Proof of Proposition 1.7

Proof of Proposition 1.7. If it is not true, there exists a sequence of metrics \( \{g_k\} \subset \mathcal{M}_{(\Lambda,D)} \) such that

\[
-\frac{\sqrt{32\pi^2(2\chi(M) + 3\tau(M))}}{2} - \frac{1}{k} \leq \lambda_M(g_k)
\]

but \( g_k \) can never be deformed to a complex hyperbolic metric through the Ricci flow for every \( k \).

Since \( \chi(M) > 0 \), there is a positive constant \( v \) independent of \( k \) such that \( \text{Vol}_{g_k}(M) \geq v \) by the Gauss-Bonnet-Chern theorem. By the Cheeger-Gromov theorem (cf [A]), \( \{g_k\} \) has a \( C^{1,\alpha} \)-convergence subsequence, denoted by \( \{g_k\} \) also. Therefore, there are diffeomorphisms \( F_k \) of \( M \) such that a subsequence of \( \{F_k^* g_k\} \) converges, in the \( C^{1,\alpha} \)-topology on \( M \), to a \( C^{1,\alpha} \)-metric \( g_\infty \). In fact, \( \{F_k^* g_k\} \) converges in the \( L^2,p \)-topology, for any \( p \geq 1 \), and \( g_\infty \) is a \( L^2,p \)-metric (See [A] for details). Thus \( \lambda_M(g_\infty) \) is well defined satisfying that

\[
-\frac{\sqrt{32\pi^2(2\chi(M) + 3\tau(M))}}{2} \leq \lambda_M(g_\infty).
\]

This together with Theorem 1.1 implies that

\[
\lambda_M(g_\infty) = -\frac{\sqrt{32\pi^2(2\chi(M) + 3\tau(M))}}{2},
\]

and \( g_\infty \) is a Kähler-Einstein metric with negative scalar curvature. By Theorem 5 in [Le1] \( \chi(M) \geq 3\tau(M) \). This together with the assumption \( \chi(M) \in \left[ \frac{3}{2} \tau(M), 3\tau(M) \right] \) implies that \( \chi(M) = 3\tau(M) \). By Theorem 5 in [Le1] once again we know that \( g_\infty \) is a complex hyperbolic metric.

To prove the metric can be deformed to a complex hyperbolic metric through the Ricci flow, we need to smooth the \( C^{1,\alpha} \)-convergence to a \( C^2 \)-convergence by Ricci flow. By the main theorem in [BOR] (See Theorem 5.1 in [Fu] for this version), given any \( 1 \gg \epsilon > 0 \) and \( j \in \mathbb{N} \), there exists a constant \( C(j, \epsilon) \) and a smoothing operator \( S_\epsilon : \mathcal{M}_{(\Lambda,D)} \rightarrow \mathcal{M}_{(2\Lambda,2D)} \) such that

(i) \( \|S_\epsilon(g) - g\|_{C^0} < \epsilon \),
(ii) \( \|\nabla S_\epsilon(g) - \nabla g\|_{C^0} < \epsilon \),
(iii) \( \|\nabla^j \text{Rm}(S_\epsilon(g))\|_{C^0} < C(j, \epsilon)\|\text{Rm}(g)\|_{C^0} \),
where \( \text{Rm}(g) \) is the curvature operator of \( g \). The proof of this result is by considering the Ricci-flow evolution equation with initial metric \( g \in \mathcal{M}(\Lambda,D) \)

\[
\frac{\partial}{\partial t} g(t) = -2\text{Ric}(g(t)) \\
g(0) = g,
\]

and letting \( S_\epsilon(g) = g(\epsilon) \). By using the operator \( S_\epsilon \) to metrics \( g_k \), we obtain a sequence of metrics \( \{ S_\epsilon(g_k) \} \subset \mathcal{M}(2\Lambda,2D) \). Let \( \tilde{g}_k = S_\epsilon(g_k) \). By the claim in §2.3 of [Pe1], \( \lambda_M(g) \) is non-decreasing along the Ricci flow if \( \lambda_M(g) \leq 0 \). Thus

\[
-\sqrt{32\pi^2(2\chi(M) + 3\tau(M))} - \frac{1}{k} \leq \lambda_M(g_k) \leq \lambda_M(\tilde{g}_k) .
\]

By the Cheeger-Gromov Theorem again, there are diffeomorphisms \( \tilde{F}_k \) of \( M \) such that a subsequence of \( \{ \tilde{F}_k g_k \} \), saying \( \{ \tilde{F}_k g_k \} \) again, which converges in the \( C^{1,\alpha} \)-topology in \( M \) to a \( C^{1,\alpha} \)-metric \( \tilde{g}_\infty \), a Kähler-Einstein metric with negative scalar curvature by Theorem 1.1. Since \( \| \nabla\text{Rm}(g_k) \|_{C^0} < C(1,\epsilon)\Lambda \), by the Arzela-Ascoli Theorem we get a sub-sequence of \( \{ \text{Rm}(\tilde{F}_k^{-1} g_k) \} \) which \( C^0 \)-converges to \( \text{Rm}(\tilde{g}_\infty) \). Therefore, \( \{ \tilde{F}_k^{-1} g_k \} \) \( C^2 \)-converges to \( \tilde{g}_\infty \). As above by [Le1] \( \tilde{g}_\infty \) is a complex hyperbolic metric. Note that the sectional curvature \( K(\tilde{g}_\infty) \) of a complex hyperbolic metric is negative, i.e. there are constants \( \mu_1, \mu_2 \) such that \( -\mu_1^2 \leq K(\tilde{g}_\infty) \leq -\mu_2^2 \). Thus, for \( k \gg 1 \), we have \( -2\mu_1^2 \leq K(\tilde{g}_k) \leq -\frac{1}{2}\mu_2^2 \). Moreover, the Einstein tensors satisfy that

\[
T_{\tilde{g}_k} = \text{Ric}(\tilde{g}_k) - \frac{R_{\tilde{g}_k}}{4} \tilde{g}_k \longrightarrow 0
\]

in the \( C^0 \)-sense when \( k \longrightarrow \infty \). By the corollary of Theorem 1.1 in [Ye], for a \( k \gg 1 \), \( \tilde{g}_k \) can be deformed to an Einstein metric, which is complex hyperbolic metric by [Le1] again.

Note that we first deform \( g_k \) to \( \tilde{g}_k \) through the Ricci flow, then deform \( \tilde{g}_k \) to a complex hyperbolic metric through the Ricci flow again. A contradiction. The desired result follows. \( \square \)

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