AN ATTEMPT AT AXIOMATIZATION OF EXTENDING MECHANISM OF SOLUTIONS TO THE FLUID DYNAMICAL SYSTEMS

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Abstract. Note that some classic fluid dynamical systems such as the Navier-Stokes equations, Magnetohydrodynamics (MHD), Boussinesq equations, and etc are observably different from each other but obey some energy inequalities of the similar type. In this paper, we attempt to axiomatize the extending mechanism of solutions to these systems, merely starting from several basic axiomatized conditions such as the local existence, joint property of solutions and some energy inequalities. The results established have nothing to do with the concrete forms of the systems and, thus, give the extending mechanisms in a unified way to all systems obeying the axiomatized conditions. The key tools are several new multiplicative interpolation inequalities of Besov type, which have their own interests.

1. Introduction

The fluid dynamical models, just mention some of them, include the Navier-Stokes equations, Magnetohydrodynamics (MHD) and the liquid crystals. Generally speaking, for those models with viscosities, the local well-posedness can be established in somewhat standard way; however, recalling that it is a well-known open question to prove the global existence of smooth solutions to the three-dimensional Navier-Stokes equations, though a lot of attentions have been made, we are still far from the complete mathematical understanding of the the global well-posedness of these systems.

To understand the possible singularity or global regularity, it may be helpful for us to study some blow-up criteria. The aim of establishing the blow-up criteria is to find such conditions as weak as possible that ensure the global regularity of the solutions. There have been a lot of works on establishing the blow-up criteria for the classic fluid dynamical systems, in particular for the systems mentioned above, in the existing literatures. There are two well-known blow-up criteria for the Navier-Stokes equations, the Ladyzhenskaya–Prodi–Serrin type (see, e.g., [14, 15]) and the Beale–Kato–Majda type (see [1]). The Ladyzhenskaya–Prodi–Serrin blow-up criteria imply...
that if the following Serrin condition

\[ u \in L^q(0, T; L^p(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{q} + \frac{3}{p} \leq 1, \quad 3 < p \leq \infty, \]

holds for a positive time \( T \) and a pair of \((p, q)\), then the solution to the Navier-Stokes equations will not blow-up at time \( T \); while the Beale–Kato–Majda blow up criteria tell us that as long as the \( L^1(0, T; BMO(\mathbb{R}^3)) \) norm of the vorticity is finite, then the solution to the Navier-Stokes equations can be extended beyond the time \( T \). Both cases can be proved by the energy method with Gagliardo-Nirenberg interpolation inequality or some embedding inequality of logarithmic type. Note that the endpoint case of the Serrin condition \( u \in L^\infty(0, T; L^3(\mathbb{R}^3)) \) is different, which was proved by Escauriaza, Seregin and Šverák \([6]\) by employing blow-up analysis and the backward uniqueness property of the parabolic operator.

Many generalizations and extensions to other systems of these two kinds of blow-up criteria have been made. For the Navier-Stokes and MHD equations, Ladyzhenskaya–Prodi–Serrin’s and Beale–Kato–Majda’s criteria were considered in Sobolev space or Besov space, for example see \([4, 5, 7–10, 13, 17–19]\) and the references therein.

We note that though the systems from the fluid dynamics may be observably different from each other, they may obey some energy inequalities of the same type. For example, one can easily check that the Navier-Stokes equations, MHD and Boussineq equations satisfy the same type energy inequalities stated in (H4), (H4’) and (H5), below. Therefore, there should be some common features for these systems, such as the well-posedness under the same assumptions on the initial data, and the same type blow-up criteria under the same conditions on the solutions. Keeping this in mind, it will be very interesting to reveal all these common features determined by the same characteristics of the systems from the fluid dynamics, while this paper is employed as the first attempt to exploit a tip of the iceberg of these common features.

Specifically, we list some axiomatized conditions, by which we get a unified proof for blow-up criteria of Ladyzhenskaya–Prodi–Serrin type and Beale–Kato–Majda type in Besov space of the systems satisfying these conditions. Without such restrictions, these systems may have some blow-up solutions. Recently Tao \([16]\) proved that in the averaged Navier-Stokes equations, it is assumed that the nonlinear term satisfies all the classical harmonic analysis estimates and the fundamental cancellation property, so as to construct the exploding solution. Throughout this paper, we use \((\mathcal{P})\) to denote the Cauchy problem of an arbitrary PDE system in \(\mathbb{R}^d\). The problem \((\mathcal{P})\) considered in this paper is supposed to satisfies (H1)–(H3) and at least one of (H4) and (H4’):

(H1) **(Local existence)** For any \( u_0 \in H^1(\mathbb{R}^d) \) (if some other conditions, such as \( \text{div} \ u = 0 \), is included in \((\mathcal{P})\), then \( u_0 \) is required to satisfy the same conditions), there exists a unique solution \( u \) to \((\mathcal{P})\) on \( \mathbb{R}^d \times (0, T_*) \), with
initial data \(u_0\), and for any \(T < T_*\), there hold
\[ u \in C([0, T]; H^1(\mathbb{R}^d)) \cap L^2(0, T; H^2(\mathbb{R}^d)), \quad \partial_t u \in L^2(0, T; L^2(\mathbb{R}^d)), \]
where \(T_*\) is a positive number depending only on the upper bound of the initial norm \(\|u_0\|_{H^1(\mathbb{R}^d)}\).

\[ \text{(H2) (Joint property)} \]
For any two solutions \(u_1\) and \(u_2\) to \((\mathcal{P})\) on \(\mathbb{R}^d \times (0, T_1)\) and on \(\mathbb{R}^d \times (T_1, T_2)\), respectively, with initial data \(u_0\) and \(u_1\), respectively, such that \(u_1 \in C([0, T_1]; H^1(\mathbb{R}^d))\) and \(u_2 \in C([T_1, T_2]; H^1(\mathbb{R}^d))\), then the joint function \(u\) defined as
\[ u := \begin{cases} u_1(x, t), & t \in [0, T_1), \\ u_2(x, t), & t \in [T_1, T_2), \end{cases} \]
is a solution to \((\mathcal{P})\) on \(\mathbb{R}^d \times (0, T_2)\), with initial data \(u_0\).

\[ \text{(H3) (Basic energy)} \]
For any solution \(u\) to \((\mathcal{P})\) on \(\mathbb{R}^d \times (0, T)\), it satisfies
\[ \sup_{0 \leq t \leq T} \|u\|_{L^2(\mathbb{R}^d)} \leq \mathcal{M}(t), \]
for any \(t \in (0, T)\), where \(\mathcal{M}\) is a continuous nondecreasing function on \([0, \infty)\), determined by the initial data.

\[ \text{(H4) (First energy inequality)} \]
For any solution \(u\) to \((\mathcal{P})\) on \(\mathbb{R}^d \times (0, T)\), it satisfies the energy inequality
\[ \frac{d}{dt} \|\nabla u\|_{L^2(\mathbb{R}^d)}^2 + c_1 \|\Delta u\|_{L^2(\mathbb{R}^d)}^2 \leq c'_1 \int_{\mathbb{R}^d} |u|^2 |\nabla u|^2 dx, \]
for \(t \in (0, T)\), where \(c_1\) and \(c'_1\) are two positive constants.

\[ \text{(H4')} (\text{First energy inequality}') \]
For any solution \(u\) to \((\mathcal{P})\) on \(\mathbb{R}^d \times (0, T)\), it satisfies the energy inequality
\[ \frac{d}{dt} \|\nabla u\|_{L^2(\mathbb{R}^d)}^2 + c_2 \|\Delta u\|_{L^2(\mathbb{R}^d)}^2 \leq c'_2 \|\nabla u\|_{L^2(\mathbb{R}^d)}^2, \]
for \(t \in (0, T)\), where \(c_2\) and \(c'_2\) are two positive constants.

The following hypothesis on the second energy may also be used in some specific case (the case that \(s = 1\) in Theorem 1.2, below):

\[ \text{(H5) (Second energy inequality)} \]
For any solution \(u\) to \((\mathcal{P})\) on \(\mathbb{R}^d \times (0, T)\), it satisfies the energy inequality
\[ \frac{d}{dt} \|\Delta u\|_{L^2(\mathbb{R}^d)}^2 + c_3 \|\nabla \Delta u\|_{L^2(\mathbb{R}^d)}^2 \leq c'_3 \int_{\mathbb{R}^d} (|\nabla u|^2 |\nabla^2 u|^2 + |\nabla u|^4) dx, \]
for \(t \in (0, T)\), where \(c_3\) and \(c'_3\) are two positive constants.

\[ \text{Remark 1.1.} \text{ It can be verified that both the Navier-Stokes system and the MHD system meet the above conditions (H1)-(H4), (H4') and (H5).} \]

The first result of this paper is the following
Theorem 1.1. Given a positive time $T_*$, and let $(\mathcal{P})$ be the Cauchy problem of an arbitrary PDE system, such that it satisfies the hypotheses (H1)–(H4). Let $u$ be a solution to $(\mathcal{P})$ on $\mathbb{R}^d \times (0, T_*)$, satisfying
\[ u \in L^q(0, T_*; B^{-s}_{p,\infty}(\mathbb{R}^d)), \]
for some constants $s, p$ and $q$, such that
\[ \frac{2}{q} + \frac{d}{p} = 1 - s, \quad \text{with } p \in \left( \frac{d}{1 - s}, \infty \right) \text{ and } s \in (0, 1). \]
Then the solution $u$ can be extended uniquely beyond $T_*$. Note that the hypothesis (H4') is stronger than (H4). Therefore, one can expect that if the system $(\mathcal{P})$ satisfies the stronger hypothesis (H4') instead of (H4), then a better result than Theorem 1.1 should hold. In fact, we have the following

Theorem 1.2. Given a positive time $T_*$, and let $(\mathcal{P})$ be the Cauchy problem of an arbitrary PDE system, such that it satisfies the hypotheses (H1)–(H3) and (H4'). Let $u$ be a solution to $(\mathcal{P})$ on $\mathbb{R}^d \times (0, T_*)$, satisfying
\[ u \in L^q(0, T_*; B^s_{p,\infty}(\mathbb{R}^d)), \]
for some constants $s, p$ and $q$, such that
\[ \frac{2}{q} + \frac{d}{p} = 1 + s, \quad \text{with } p \in \left( \frac{d}{1 + s}, \infty \right) \text{ and } s \in (-1, 1]. \]
Then, for the case that $(p, s) \neq (\infty, 1)$, the solution $u$ can be extended uniquely beyond $T_*$. While for the case that $(p, s) = (\infty, 1)$, the solution $u$ can also be extended uniquely beyond $T_*$, if we have further that (H5) holds, and $d = 2, 3$.

Remark 1.2. (1) It should be noticed that if considering the Cauchy problem to the MHD equations in three dimensions, Theorem 2 has already been obtained in [5]; however, comparing with their work, one of our biggest features here is that we need neither the concrete structure of the equations nor the Bony decomposition and commutator estimates.

(2) Since $L^p(\mathbb{R}^d) \subset B^0_{p,\infty}(\mathbb{R}^d)$ for $s = 0$, the classic Ladyzhenskaya–Prodi–Serrin criterion is included in Theorem 1.2. Theorem 1.1–1.2 still hold if replacing the inhomogeneous Besov spaces by the corresponding homogeneous ones. In fact, by checking the proof, it suffices to verify that Lemma 2.2 and Lemma 2.3 continue to hold if replacing the inhomogeneous Besov spaces there by the homogeneous ones. Thanks to this and noticing that $B^s_{p,\infty}(\mathbb{R}^d) \subset B^s_{p,\infty}(\mathbb{R}^d)$ for $s < 0$ and $B^s_{p,\infty}(\mathbb{R}^d) \cap L^p(\mathbb{R}^d) = B^s_{p,\infty}(\mathbb{R}^d)$ for $s > 0$, the condition for $u$ in Theorem 1.2 can be relaxed to
\[ \begin{cases} u \in L^q(0, T_*; B^s_{p,\infty}(\mathbb{R}^d)), & -1 < s \leq 0, \\ u \in L^q(0, T_*; \dot{B}^s_{p,\infty}(\mathbb{R}^d)), & 0 < s \leq 1, \end{cases} \]
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with \( p, q, \) and \( s \) obeying the relations stated in Theorem 1.2. In particular, in the case that \((p, q, s) = (\infty, 1, 1)\), the relaxed condition reduces to \( \nabla u \in L^1(0, T^*; \hat{B}^0_{\infty, \infty} \mathbb{R}^d) \), which is of the well-known Beale–Kato–Majda type criteria.

(3) Note that the endpoint case \((p, q, s) = (\infty, \infty, -1)\), i.e. the case that \( u \in L^\infty(0, T; B^{-1}_{\infty, \infty}) \), is excluded in both Theorem 1.1 and Theorem 1.2. In fact, the regularity of the Leray-Hopf weak solutions to the three dimensional Navier-Stokes equation was obtained under the condition that \( u \in C((0, T]; B^{-1}_{\infty, \infty}) \), see [3], it is still unknown if it can be relaxed to \( u \in L^\infty(0, T; B^{-1}_{\infty, \infty}) \).

The rest of the paper is organized as follows. In section 2, we introduce the definition of Besov space and several new interpolation inequalities of Besov type, which are our main technical tools. In section 3, we prove Theorem 1.1 and Theorem 1.2 under the abstract assumptions (H1)–(H5).

2. Preliminaries

2.1. Littlewood-Paley decomposition. Let us recall some basic facts on Littlewood-Paley theory (see [2] for more details). Choose two nonnegative radial functions \( \chi, \phi \in \mathcal{S}(\mathbb{R}^n) \) supported respectively in \( \{ \xi \in \mathbb{R}^n, |\xi| \leq \frac{4}{3} \} \) and \( \{ \xi \in \mathbb{R}^n, \frac{3}{4} \leq |\xi| \leq \frac{8}{3} \} \) such that for any \( \xi \in \mathbb{R}^n \),

\[
\chi(\xi) + \sum_{j \geq 0} \phi(2^{-j} \xi) = 1.
\]

The frequency localization operator \( \Delta_j \) and \( S_j \) are defined by

\[
\Delta_j f = \phi(2^{-j} D) f = 2^{nj} \int_{\mathbb{R}^n} h(2^j y) f(x-y) dy, \quad \text{for } j \geq 0,
\]

\[
S_j f = \chi(2^{-j} D) f = \sum_{-1 \leq k \leq j-1} \Delta_k f = 2^{nj} \int_{\mathbb{R}^n} \tilde{h}(2^j y) f(x-y) dy,
\]

\[
\Delta_{-1} f = S_0 f, \quad \Delta_j f = 0 \text{ for } j \leq -2,
\]

where \( h = \mathcal{F}^{-1} \phi, \tilde{h} = \mathcal{F}^{-1} \chi \). With this choice of \( \phi \), it is easy to verify that

\[
\Delta_j \Delta_k f = 0, \quad \text{if } |j-k| \geq 2; \quad \Delta_j (S_{k-1} g \Delta_k f) = 0, \quad \text{if } |j-k| \geq 5. \tag{2.1}
\]

In terms of \( \Delta_j \), the norm of the inhomogeneous Besov space \( B^s_{p,q} \) for \( s \in \mathbb{R} \), and \( p, q \geq 1 \) is defined by

\[
\| f \|_{B^s_{p,q}} \doteq \| \{ 2^j \| \Delta_j f \|_p \}_{j \geq -1} \|_{\ell^q},
\]

and

\[
\| f \|_{B^s_{p,\infty}} \doteq \sup_{j \geq -1} \{ 2^j \| \Delta_j f \|_p \}.
\]

We will constantly use the following Bernstein’s inequality [2].
Lemma 2.1 (Bernstein inequality). Let \( c \in (0, 1) \) and \( R > 0 \). Assume that \( 1 \leq p \leq q \leq \infty \) and \( f \in L^p(\mathbb{R}^n) \). Then

\[
\text{supp} \hat{f} \subset \{ |\xi| \leq R \} \Rightarrow \| \partial^\alpha f \|_q \leq CR^{|\alpha|+n(\frac{1}{p}-\frac{1}{q})} \| f \|_p,
\]

\[
\text{supp} \hat{f} \subset \{ cR \leq |\xi| \leq R \} \Rightarrow \| f \|_p \leq CR^{-|\alpha|} \sup_{|\xi|=|\alpha|} \| \partial^\alpha f \|_p,
\]

where the constant \( C \) is independent of \( f \) and \( R \).

2.2. New interpolation inequalities of Besov type.

Lemma 2.2. Let \( s \in (0, \infty) \). Then, for any nonzero function \( f \in H^2 \cap B^{-s}_{\infty, \infty} \), the following interpolation inequality holds

\[
\| f \|_p \leq C_{n,p,s} \| f \|_{B^{-s}_{\infty, \infty}}^{\frac{1}{2}} \| f \|_{H^1}^{\frac{1}{2}} \| f \|_{H^2}^{\frac{1}{2}},
\]

for any \( p \in (2 + \frac{s}{2}, 2 + \frac{4}{n}) \), where

\[
C_{n,p,s} = C_n \max \left\{ \frac{1}{2^{s(1 - \frac{2}{n})} - \frac{s}{2} - 1}, \frac{1}{1 - 2^{s(1 - \frac{2}{n})} - \frac{s}{2}} \right\},
\]

for a positive constant \( C_n \) depending only on \( n \).

Proof. For any integer \( k \geq 0 \) and \( p \in [2, \infty] \), it follows from the Hölder and Bernstein inequalities that

\[
\| \Delta_k f \|_p \leq \| \Delta_k f \|_{B^{-s}_{\infty, \infty}}^{\frac{1}{2}} \| \Delta_k f \|_{H^1}^{\frac{1}{2}} \| \Delta_k f \|_{H^2}^{\frac{1}{2}} \leq C_n 2^{2k} \| \nabla \Delta_k f \|_{B^{-s}_{\infty, \infty}} \| \Delta_k f \|_{H^1}^{\frac{1}{2}} \| \Delta_k f \|_{H^2}^{\frac{1}{2}},
\]

from which, using the Bernstein inequality again, one arrives at

\[
\| \Delta_k f \|_p \leq C_n 2^{k[\frac{1}{2} - \frac{1}{p} - \frac{s}{2}]} \| \Delta_k f \|_{H^1}^{\frac{1}{2}} \| \Delta_k f \|_{H^2}^{\frac{1}{2}} \| f \|_{B^{-s}_{\infty, \infty}}^{\frac{1}{2}},
\]

for any integer \( k \geq 0 \). Thus, noticing that \( \| \Delta_k g \|_2 \leq C_n \| g \|_2 \), we obtain

\[
\| \Delta_k f \|_p \leq C_n 2^{k[\frac{1}{2} - \frac{1}{p} - \frac{s}{2}]} \| \nabla f \|_{B^{-s}_{\infty, \infty}}^{\frac{1}{2}} \| f \|_{B^{-s}_{\infty, \infty}}^{\frac{1}{2}}, \quad (2.2)
\]

\[
\| \Delta_k f \|_p \leq C_n 2^{-k[\frac{1}{2} - \frac{1}{p} - \frac{s}{2}]} \| \Delta f \|_{B^{-s}_{\infty, \infty}}^{\frac{1}{2}} \| f \|_{B^{-s}_{\infty, \infty}}^{\frac{1}{2}}, \quad (2.3)
\]

for any integer \( k \geq 0 \) and \( p \in [2, \infty] \). For \( k = -1 \), by the Hölder inequality, and noticing that \( \| \Delta_{-1} f \|_2 \leq C_n \| f \|_2 \), one deduces

\[
\| \Delta_{-1} f \|_p \leq \| \Delta_{-1} f \|_{B^{-s}_{\infty, \infty}}^{\frac{1}{2}} \| \Delta_{-1} f \|_{\infty}^{\frac{1}{2}} \leq C_n 2^{-(\frac{1}{2} - \frac{s}{2})} \| f \|_2 \| f \|_{B^{-s}_{\infty, \infty}}^{\frac{1}{2}} \| f \|_{B^{-s}_{\infty, \infty}}^{\frac{1}{2}}, \quad (2.4)
\]

for any \( p \in [2, \infty] \).
For any \( p \in (2 + \frac{2}{s}, 2 + \frac{4}{s}) \), one can easily verify
\[
s \left( 1 - \frac{2}{p} \right) - \frac{2}{p} > 0 \quad \text{and} \quad \frac{4}{p} - s \left( 1 - \frac{2}{p} \right) > 0. \tag{2.5}
\]

On account of this, for any integer \( k_0 \geq 0 \), it follows from (2.2)–(2.4) that
\[
\|f\|_p = \left\| \sum_{k=1}^{\infty} \Delta_k f \right\|_p \leq \|\Delta_1 f\|_p + \sum_{k=0}^{k_0} \|\Delta_k f\|_p + \sum_{k=k_0+1}^{\infty} \|\Delta_k f\|_p
\]
\[
\leq C_n 2^{-s(1 - \frac{3}{p}) - \frac{2}{p}} \|f\|_p \left\| \sum_{k=1}^{k_0} 2^k \|\Delta f\|_2 \right\|_p \|f\|_p^{1 - \frac{2}{p}} + C_n \sum_{k=0}^{k_0} 2^k \left( 2^{-s(1 - \frac{3}{p}) - \frac{2}{p}} \|f\|_p \right) \|\Delta f\|_2 \|f\|_p^{1 - \frac{2}{p}}
\]
\[
+ C_n \sum_{k=k_0+1}^{\infty} 2^{-k(1 - \frac{3}{p}) - \frac{2}{p}} \|f\|_p \left\| \sum_{k=1}^{k_0} 2^k \|\Delta f\|_2 \right\|_p \|f\|_p^{1 - \frac{2}{p}}
\]
\[
\leq C_n \|f\|_p \left( \sum_{k=1}^{k_0} 2^k \|\Delta f\|_2 \right) \|f\|_p \left( \sum_{k=0}^{k_0} 2^k \right) + C_n \sum_{k=k_0+1}^{\infty} 2^k \left( 2^{-s(1 - \frac{3}{p}) - \frac{2}{p}} \|f\|_p \right) \|\Delta f\|_2 \|f\|_p^{1 - \frac{2}{p}}
\]
\[
\leq C_{n,p,s} \|f\|_p \left( 2^{k_0} \left( 2^{-s(1 - \frac{3}{p}) - \frac{2}{p}} \|f\|_p \right) \|\Delta f\|_2 \right) \|f\|_p^{1 - \frac{2}{p}} + 2^{-s(1 - \frac{3}{p}) - \frac{2}{p}} \|f\|_p \|\Delta f\|_2 \|f\|_p^{1 - \frac{2}{p}}
\]
\[
\tag{2.6}
\end{align}
\]
for any \( p \in (2 + \frac{2}{s}, 2 + \frac{4}{s}) \).

Note that \( \frac{\|\Delta f\|_2 + \|f\|_H^1}{\|f\|_H^1} \geq 1 \), there is a unique integer \( k_0 \geq 0 \), such that
\[
2^{k_0} \leq \frac{\|\Delta f\|_2 + \|f\|_H^1}{\|f\|_H^1} < 2^{k_0+1}.
\]
Choosing such \( k_0 \) in (2.6), and recalling (2.5), we obtain
\[
\|f\|_p \leq C_{n,p,s} \|f\|_p \left( \frac{\|\Delta f\|_2 + \|f\|_H^1}{\|f\|_H^1} \right) \left( 2^{-s(1 - \frac{3}{p}) - \frac{2}{p}} \|f\|_p \right) \|\Delta f\|_2 \|f\|_p^{1 - \frac{2}{p}}
\]
\[
+ C_{n,p,s} \|f\|_p \left( \frac{\|f\|_H^1}{\|f\|_H^1} \right) \left( 2^{-s(1 - \frac{3}{p}) - \frac{2}{p}} \|f\|_p \right) \|\Delta f\|_2 \|f\|_p^{1 - \frac{2}{p}}
\]
\[
\leq C_{n,p,s} \|f\|_p \left( 2^{-s(1 - \frac{3}{p}) - \frac{2}{p}} \|f\|_p \right) \|\Delta f\|_2 \|f\|_p^{1 - \frac{2}{p}}
\]
\[
\leq C_{n,p,s} \|f\|_p \left( 2^{-s(1 - \frac{3}{p}) - \frac{2}{p}} \|f\|_p \right) \|\Delta f\|_2 \|f\|_p^{1 - \frac{2}{p}}
\]
for any \( p \in (2 + \frac{2}{s}, 2 + \frac{4}{s}) \). This completes the proof. \( \square \)

**Lemma 2.3.** Let \( s \in (-\infty, 1) \). Then, for any nonzero function \( f \in B_{s, \infty}^* \) such that \( \nabla f \in H^1 \), we have
\[
\|\nabla f\|_q \leq C_{n,q,s} \|f\|_p \left( 2^{-s(1 - \frac{3}{p}) - \frac{2}{p}} \|f\|_p \right) \|\nabla f\|_2 \|\nabla f\|_H^1 \|\nabla f\|_H^1 \|\nabla f\|_H^1,
\]
for any \( q \in (2, 2 + \frac{2}{1-s}) \), where

\[
C_{n,q,s} = C_n \max \left\{ \frac{1}{2^{1-s} (1 - \frac{2}{q}) - 1}, \frac{1}{1 - 2^{-\frac{2}{q} - (1-s) (1 - \frac{2}{q})}} \right\},
\]

for a positive constant \( C_n \) depending only on \( n \).

**Proof.** For any \( q \in [2, \infty] \) and any integer \( k \geq 0 \), it follows from the H"older and Bernstein inequalities that

\[
\| \nabla \Delta_k f \|_q \leq \| \nabla \Delta_k f \|_2 \| \nabla \Delta_k f \|_\infty^{1 - \frac{2}{q}} \leq C_n 2^{k(1-s) \left( 1 - \frac{2}{q} \right)} \| \nabla \Delta_k f \|_2 \| f \|_B^{1 - \frac{2}{q}},
\]

from which, using the Bernstein inequality again, we have

\[
\| \nabla \Delta_k f \|_q \leq C_n 2^{-k \left( \frac{2}{q} - (1-s) \left( 1 - \frac{2}{q} \right) \right)} \| \Delta \Delta_k f \|_2 \| f \|_B^{1 - \frac{2}{q}}.
\]

Thanks to the above two inequalities, noticing that \( \| \Delta g \|_2 \leq C_n \| g \|_2 \), we get

\[
\| \nabla \Delta_k f \|_q \leq C_n 2^{k \left( 1-s \left( 1 - \frac{2}{q} \right) \right)} \| \nabla f \|_2 \| f \|_B^{1 - \frac{2}{q}}, \tag{2.7}
\]

\[
\| \nabla \Delta_k f \|_q \leq C_n 2^{-k \left( 2 - (1-s) \left( 1 - \frac{2}{q} \right) \right)} \| \Delta f \|_2 \| f \|_B^{1 - \frac{2}{q}}, \tag{2.8}
\]

for any \( q \in [2, \infty] \) and integer \( k \geq 0 \). For \( k = -1 \), by the H"older and Bernstein inequalities, we have

\[
\| \nabla \Delta_{-1} f \|_q \leq \| \nabla \Delta_{-1} f \|_2 \| \nabla \Delta_{-1} f \|_\infty^{1 - \frac{2}{q}} \leq C_n 2^{s \left( 1 - \frac{2}{q} \right)} \| \nabla f \|_2 \| f \|_B^{1 - \frac{2}{q}}
\]

\[
\leq C_n 2^{-\left( 2 - (1-s) \left( 1 - \frac{2}{q} \right) \right)} \| \nabla f \|_2 \| f \|_B^{1 - \frac{2}{q}}, \tag{2.9}
\]

for any \( q \in [2, \infty] \).

For any \( q \in (2, 2 + \frac{2}{1-s}) \), one can easily check that

\[
(1-s) \left( 1 - \frac{2}{q} \right) > 0 \quad \text{and} \quad \frac{2}{q} - (1-s) \left( 1 - \frac{2}{q} \right) > 0.
\]

Thus, it follows from (2.7) – (2.9) that

\[
\| \nabla f \|_q = \left\| \sum_{k=1}^{\infty} \nabla \Delta_k f \right\|_q \leq \sum_{k=1}^{k_0} \| \nabla \Delta_k f \|_q + \sum_{k=k_0+1}^{\infty} \| \nabla \Delta_k f \|_q
\]

\[
\leq C_n \| f \|_B^{1 - \frac{2}{q}} \left( \sum_{k=1}^{k_0} 2^{k(1-s) \left( 1 - \frac{2}{q} \right)} \| \nabla f \|_2^2 + \sum_{k=k_0+1}^{\infty} 2^{-k \left( 2 - (1-s) \left( 1 - \frac{2}{q} \right) \right)} \| \Delta f \|_2^2 \right)
\]

\[
\leq C_{n,p,s} \| f \|_B^{1 - \frac{2}{q}} \left( 2^{k_0(1-s) \left( 1 - \frac{2}{q} \right)} \| \nabla f \|_2^2 + 2^{-\left( k_0+1 \right) \left( 2 - (1-s) \left( 1 - \frac{2}{q} \right) \right)} \| \Delta f \|_2^2 \right), \tag{2.10}
\]
for any $k_0 \geq 0$. Take $k_0 \geq 0$ be the unique integer such that

$$2^{k_0} \leq \frac{\|\Delta f\|_2 + \|\nabla f\|_2}{\|\nabla f\|_2} < 2^{k_0+1},$$

then it follows from (2.10) that

$$\|\nabla f\|_q \leq C_{n,p,s} f \left| B_{1,\infty} \right| \left( \frac{\|\Delta f\|_2 + \|\nabla f\|_2}{\|\nabla f\|_2} \right)^{(1-s)(1-\frac{n}{p})} \|\nabla f\|_2^2 + C_{n,p,s} \left| \left| f \right| \right| \left| B_{1,\infty} \right| (1-s)(1-\frac{n}{q}) \|\Delta f\|_2^q$$

$$\leq C_{n,p,s} f \left| B_{1,\infty} \right| \left| \left| f \right| \right| \left| B_{1,\infty} \right| (1-s)(1-\frac{n}{q}) \|\nabla f\|_2^q,$$

for any $q \in (2, 2 + \frac{2}{1-q})$, proving the conclusion.

We also will use an integral in time version of the logarithmic type inequality stated in the next lemma. Some similar inequalities can be found in Huang–Wang [11] and Hong–Li–Xin [12].

**Lemma 2.4.** Suppose that $n = 2, 3$. Then the following two inequalities hold

$$\int_{t_1}^{t_2} \|\nabla f\|_\infty \, dt \leq C \left( \left( \int_{t_1}^{t_2} \|f\|_{B_{1,\infty}} \, dt \right) \log \left( \int_{t_1}^{t_2} \|\nabla \Delta f\|_2 \, dt + e \right) + 1 \right),$$

for any $f$ such that the quantities in the formulas make sense and are finite, where $C$ is a positive constant independent of $t_1$ and $t_2$.

**Proof.** We first prove the first inequality. By the definition of $B_{1,\infty}$, and the Bernstein inequality, it is obvious that

$$\|\nabla \Delta_k f\|_\infty \leq C^k \|\Delta_k f\|_\infty \leq C \|f\|_{B_{1,\infty}},$$

(2.11)

for any integer $k \geq -1$. Using the Bernstein inequality again, and noticing that $\|\Delta_k g\|_2 \leq C_n \|g\|_2$, we deduce

$$\|\nabla \Delta_k f\|_\infty \leq C \left( \frac{2}{1-}\frac{2}{k} \right)^k \|\nabla \Delta_k f\|_2 \leq C \left( \frac{2}{1-}\frac{2}{k} \right)^k \|\nabla f\|_2,$$

(2.12)

for any integer $k \geq 0$. With the aid of the above two inequalities, we have

$$\|\nabla f\|_\infty = \left| \sum_{k=-1}^{\infty} \nabla \Delta_k f \right| _\infty \leq \sum_{k=-1}^{k_0} \|\nabla \Delta_k f\|_\infty + \sum_{k=k_0+1}^{\infty} \|\nabla \Delta_k f\|_\infty$$

$$\leq C(k_0 + 1) \|f\|_{B_{1,\infty}} + C \left( \sum_{k=k_0+1}^{\infty} 2^{-k(\frac{2}{1-}\frac{2}{k})} \|\nabla f\|_2 \right)$$

$$\leq C \left[ (k_0 + 1) \|f\|_{B_{1,\infty}} + 2^{-(k_0+1)(\frac{2}{1-}\frac{2}{k})} \|\nabla f\|_2 \right].$$
for any integer $k_0 \geq 0$. Integrating the above inequality with respect to $t$ over the interval $(t_1, t_2)$, and choosing $k_0 \geq 0$ be the unique integer such that

$$k_0 \leq \left[ \left( 2 - \frac{n}{2} \right) \log 2 \right]^{-1} \log \left( \int_{t_1}^{t_2} \| \nabla \Delta f \|_2 + 1 \right) < k_0 + 1,$$

then we obtain

$$\int_{t_1}^{t_2} \| \nabla f \|_\infty dt \leq C(k_0 + 1) \int_{t_1}^{t_2} \| f \|_{B_{p,\infty}^s} dt + C 2^{-(k_0+1)(2-\frac{n}{2})} \int_{t_1}^{t_2} \| \nabla \Delta f \|_2 dt$$

$$\leq C \left( \int_{t_1}^{t_2} \| f \|_{B_{p,\infty}^s} dt \right) \left[ \log \left( \int_{t_1}^{t_2} \| \nabla \Delta f \|_2 + 1 \right) + 1 \right]$$

$$+ Ce^{-[(2-\frac{n}{2}) \log 2(k_0+1)]} \int_{t_1}^{t_2} \| \nabla \Delta f \|_2 dt$$

$$\leq C \left( \int_{t_1}^{t_2} \| f \|_{B_{p,\infty}^s} dt \right) \log \left( \int_{t_1}^{t_2} \| \nabla \Delta f \|_2 + e \right) + 1, \quad \text{proving the first inequality.}$$

3. Proofs of Theorem 1.1 and Theorem 1.2

In this section, we give the proof of Theorem 1.1 and Theorem 1.2.

**Proof of Theorem 1.1.** Note that $B_{p,\infty}^{-s} \hookrightarrow B_{p,\infty}^{-s-\frac{n}{p}}$, it follows that

$$L^q(0, T; B_{p,\infty}^{-s}) \hookrightarrow L^q(0, T; B_{p,\infty}^{-s_1}), \quad s_1 := s + \frac{n}{p}.$$ 

By assumption

$$\frac{2}{q} + \frac{n}{p} = 1 - s, \quad \text{with } p \in \left( \frac{n}{1 - s}, \infty \right) \text{ and } s \in (0, 1),$$

it has

$$s_1 := s + \frac{n}{p} = 1 - \frac{2}{q} \in [s, 1) \subseteq (0, 1).$$

Thus, by the aid of the assumption in Theorem 1.1 we always has $u \in L^q(0, T; B_{\infty,\infty}^{-s_1})$, for some $s_1 \in (0, 1)$. As a result, to prove Theorem 1.1, it suffice to prove the case that $p = \infty$. Because of this, without loss of generality, we suppose that $u \in L^q(0, T; B_{\infty,\infty}^{-s})$ for some $s \in (0, 1)$ in the following proof.

We first note that for any $s \in (0, 1)$, one can choose the numbers $p_s \in (2 + \frac{2}{s}, 2 + \frac{4}{s})$ and $q_s \in (2, 2 + \frac{2}{s+1})$, such that

$$\frac{2}{p_s} + \frac{2}{q_s} = 1.$$ 

In fact, when $p$ and $q$ run in the intervals $(2 + \frac{2}{s}, 2 + \frac{4}{s})$ and $(2, 2 + \frac{2}{s+1})$, respectively, then the quantity $\frac{2}{p} + \frac{2}{q}$ runs in the interval $(\frac{1+2s}{2+s}, 1 + \frac{s}{s+1})$, and each point in this
interval can be arrived at by $\frac{2}{p} + \frac{2}{q}$. The validity of the above equality then follows from the observation that $\frac{1+\frac{2s}{2+s}}{2} < 1 < 1 + \frac{s}{s+1}$ for any $s \in (0, 1)$. By Lemma 2.2 and Lemma 2.3, we have the following estimates
\[
\|u\|_{p_s} \leq C\|u\|_{B_{s,\infty}^{2,s}} \|u\|_{H^1}^{1-\frac{2}{p_s}} \|u\|_{H^2}^{s(1-\frac{2}{p_s})-\frac{2}{p_s}},
\]
\[
\|\nabla u\|_{q_s} \leq C\|u\|_{B_{s,\infty}^{2,s}} \|u\|_{H^1}^{1-\frac{2}{q_s}} \|u\|_{H^2}^{s(1-\frac{2}{q_s})-(1+s)(1-\frac{2}{q_s})},
\]
and thus
\[
\|u\|_{p_s} \|\nabla u\|_{q_s} \leq C\|u\|_{B_{s,\infty}^{2,s}} \|u\|_{H^1}^{2-\frac{2}{p_s} - \frac{2}{q_s}} \|u\|_{H^2}^{(1-s)(1-\frac{2}{p_s}) + \frac{2}{q_s} - (1+s)(1-\frac{2}{q_s})}
\times \|u\|_{H^2}^{s(1-\frac{2}{p_s})-\frac{2}{p_s} + (1+s)(1-\frac{2}{q_s})} = C\|u\|_{B_{s,\infty}^{2,s}} \|u\|_{H^1}^{1-s} \|u\|_{H^2}^{s},
\]
where $C$ is a positive constant.

With the aid of the above estimate, by hypothesis (H4), it follows from the Hölder and Young inequalities that
\[
\frac{d}{dt}\|\nabla u\|_2^2 + c_1\|\Delta u\|_2^2 \leq \frac{c_1}{2}\|u\|_2^2 \|\nabla u\|_2^2 \leq C\|u\|_{B_{s,\infty}^{2,s}} \|u\|_{H^1}^{2(1-s)} \|u\|_{H^2}^{2s}
\leq C\|u\|_{B_{s,\infty}^{2,s}} \|\nabla u\|_{H^2}^2 + \|u\|_{H^1}^2
\leq C\|\Delta u\|_2^2 + C(\|u\|_{B_{s,\infty}^{2,s}} \|u\|_{H^1}) \|\nabla u\|_{H^1}^2
\leq \frac{c_1}{2}\|\Delta u\|_2^2 + C(\|u\|_{B_{s,\infty}^{2,s}}^2 + 1)\|u\|_{H^1}^2,
\]
and thus
\[
\frac{d}{dt}\|\nabla u\|_2^2 + c_1\|\Delta u\|_2^2 \leq C(\|u\|_{B_{s,\infty}^{2,s}}^2 + 1)(\|\nabla u\|_2^2 + \|u\|_2^2),
\]
for any $t \in (0, T)$. Applying the Gronwall inequality to the above inequality and recalling the hypothesis (H3), we obtain
\[
\sup_{0 \leq \tau \leq t} \|\nabla u\|_2^2 + \frac{c_1}{2}\|\Delta u\|_2^2 \leq e^{C \int_0^t (1+\|u\|_{B_{s,\infty}^{2,s}}^2) d\tau} \left(\|\nabla u_0\|_2^2 + \mathcal{M}(t) \int_0^t (1+\|u\|_{B_{s,\infty}^{2,s}}^2) d\tau\right)
\leq e^{C \int_0^T (1+\|u\|_{B_{s,\infty}^{2,s}}^2) d\tau} \left(\|\nabla u_0\|_2^2 + \mathcal{M}(T) \int_0^T (1+\|u\|_{B_{s,\infty}^{2,s}}^2) d\tau\right),
\]
for any $t \in (0, T)$. On account of this, and recalling the hypothesis (H3), we obtain the a priori estimate
\[
\sup_{0 \leq t < T} \|u\|_{H^1}^2 + \int_0^T \|u\|_{H^2}^2 dt \leq G_1 < \infty,
\]
for any $t \in (0, T)$, where $G_1$ is a positive constant.

Choose a positive time $T^{**} \in (0, T_*)$. By the hypothesis (H1) and recalling the above a priori estimate, starting from the time $T^{**}$, there is a unique solution $u^{**}$ to $(\mathcal{P})$ on $\mathbb{R}^n \times (T^{**}, T^{**} + T)$, with initial data $u(T^{**})$, such that $u^{**} \in C([T^{**}, T^{**} + T]; H^1)$, where $T = T(G_1)$. Define
\[
\tilde{u} = \begin{cases} 
  u(x, t), & t \in [0, T^{**}), \\
  u^{**}(x, t), & t \in [T^{**}, T^{**} + T(G_1)].
\end{cases}
\]
then, by hypothesis (H2), $\tilde{u}$ is a solution to $(\mathcal{P})$ on $\mathbb{R}^n \times (0, T^{**} + T(G_1))$. Note that $T(G_1)$ is independent of $T^{**}$, therefore, one can choose $T^{**}$ close enough to $T_*$, such that $T^{**} + T > T_*$, and as a result $\tilde{u}$ is an extension of $u$ to a time beyond $T_*$. This completes the proof. □

**Proof of Theorem 1.2** Similar to the situation encountered in the proof of Theorem 1.1, it suffices to consider the case $p = \infty$. Besides, without loss of generality, we suppose that
\[
\sup_{0 \leq t \leq T} \|\nabla u\|_2^2 \geq 1,
\]
otherwise, we then already have the a priori estimate $\sup_{0 \leq t \leq T} \|\nabla u\|_2^2 \leq 1$, on account of which, by the same argument as that in the last paragraph of the proof of Theorem 1.1, one can easily prove the conclusion.

We first consider the case that $s \in (-1, 1)$. By Lemma 2.3 it follows from the Young inequality that
\[
\|\nabla u\|_3^3 \leq C \|u\|_{B^{s, \infty}} \|\nabla u\|_2^{1+s} \|\nabla u\|_2^{1-s} \\
\leq C \|u\|_{B^{s, \infty}} (\|\nabla u\|_2^2 + \|\nabla u\|_2^{1+s} \|\Delta u\|_2^{1-s}) \\
\leq \frac{c_2}{2} \|\Delta u\|_2^2 + C (\|u\|_{B^{s, \infty}} + \|u\|_{B^{s, \infty}}^{1+s}) \|\nabla u\|_2^2 \\
\leq \frac{c_2}{2} \|\Delta u\|_2^2 + C (1 + \|u\|_{B^{s, \infty}}^{1+s}) \|\nabla u\|_2^2,
\]
and thus it follows from the hypothesis (H4') that
\[
\frac{d}{dt} \|\nabla u\|_2^2 + \frac{c_2}{2} \|\Delta u\|_2^2 \leq C (1 + \|u\|_{B^{s, \infty}}^{1+s}) \|\nabla u\|_2^2,
\]
for any $t \in (0, T)$. Applying the Gronwall inequality to the above inequality yields
\[
\sup_{0 \leq \tau \leq t} \|\nabla u\|_2^2 + \frac{c_2}{2} \int_0^t \|\Delta u\|_2^2 d\tau
\]
≤ e^{\int_0^t (1 + \|u\|_{B^1_{\infty, \infty}}^2) dt} \|\nabla u_0\|_2^2 \leq e^{\int_0^t (1 + \|u\|_{B^1_{\infty, \infty}}^2) dt} \|\nabla u_0\|_2^2,$

for any $t \in (0, T)$. With the aid of this a priori estimate, recalling the hypothesis (H3), we have the a priori estimate

$$\sup_{0 \leq \tau \leq t} \|u\|_{H^1}^2 + \int_0^t \|u\|_{H^2}^2 d\tau \leq G_2 < \infty,$$

for any $t \in (0, T)$, where $G_2$ is a positive constant. Thanks to this a priori estimate, the same argument as that for the proof of Theorem 1.1 yields the conclusion for the case $s \in (-1, 1)$.

Now we consider the case that $(p, s) = (\infty, 1)$. Let $\varepsilon$ be a sufficiently small positive number which will be determined later. Since $u \in L^1(0, T_*, B^1_{\infty, \infty}(\mathbb{R}^d))$, by the absolutely continuity of the integrals, there is a positive constant $\delta > 0$, such that

$$\int_{T_* - \delta}^{T_*} \|u\|_{B^1_{\infty, \infty}} dt \leq \varepsilon,$$

from which, by Lemma 2.4, we have

$$\int_{T_* - \delta}^{t} \|\nabla u\|_{\infty} ds \leq C \left[ \varepsilon \log \left( \int_{T_* - \delta}^{t} \|\nabla u\|_2 ds + e \right) + 1 \right], \quad (3.2)$$

for any $t \in [T_* - \delta, T_*)$. We set

$$f_1(t) = \|\nabla u\|_2^2(t) + c_2 \int_{T_* - \delta}^{t} \|\nabla u\|_2^2 ds,$$

$$f_2(t) = \|\Delta u\|_2^2(t) + c_3 \int_{T_* - \delta}^{t} \|\nabla \Delta u\|_2^2 ds,$$

for any $t \in [T_* - \delta, T_*]$.

By the Ladyzhenskaya and Sobolev embedding inequalities, we deduce

$$\int_{T_* - \delta}^{t} \|\nabla u\|_d^2 ds \leq C \int_{T_* - \delta}^{t} \|\nabla u\|_2^2 \|\Delta u\|_2^2 ds \leq C \sup_{T_* - \delta \leq s \leq t} f_1^2(s), \quad t \in [T_* - \delta, T_*], \quad (3.3)$$

for $d = 2$, and

$$\int_{T_* - \delta}^{t} \|\nabla u\|_3^4 ds \leq C \int_{T_* - \delta}^{t} \|\nabla u\|_2 \|\Delta u\|_2^2 ds \leq C \int_{T_* - \delta}^{t} f_1^2(s) f_2^2(s) \|\Delta u\|_2^2 ds \leq C f_1(t) \sup_{T_* - \delta \leq s \leq t} f_1^2(s) f_2^2(s), \quad t \in [T_* - \delta, T_*], \quad (3.4)$$

for $d = 3$. It follows from (H4') and (H5) that

$$f_1(t) \leq C \|\nabla u\|_{\infty} f_1(t),$$
\[ f'_2(t) \leq C \| \nabla u \|_\infty f_2(t) + C \| \nabla u \|_4^4, \]

from which, by the Gronwall inequality, recalling (3.2)–(3.4), and choosing \( \varepsilon \) small enough, we have

\[
\begin{align*}
  f_1(t) &\leq C \left( \int_{T_*-\delta}^{t} \| \nabla u \|_2 ds + 1 \right)^{C\varepsilon} f_1(T_* - \delta) \\
  &\leq C(f_2(t) + 1)^{C\varepsilon} \leq C(f_2(t) + 1)^{\frac{1}{6}}, \tag{3.5}
\end{align*}
\]

and

\[
\begin{align*}
  f_2(t) &\leq C \left( \int_{T_*-\delta}^{t} \| \nabla u \|_2 ds + 1 \right)^{C\varepsilon} \left( f_2(T_* - \delta) + \int_{T_*-\delta}^{t} \| \nabla u \|_4^4 ds \right) \\
  &\leq \begin{cases} 
  C(f_2(t) + 1)^{\frac{1}{2}} \left( 1 + \sup_{T_*-\delta \leq s \leq t} f_1^2(s) \right), & d = 2, \\
  C(f_2(t) + 1)^{\frac{1}{2}} \left( 1 + f_1(t) \sup_{T_*-\delta \leq s \leq t} f_1^2(s) f_2^2(s) \right), & d = 3,
\end{cases} \tag{3.6}
\end{align*}
\]

for any \( t \in [T_* - \delta, T_*) \).

Define

\[
\begin{align*}
  F_1(t) &= \sup_{T_* - \delta \leq s \leq t} f_1(s) + 1, \\
  F_2(t) &= \sup_{T_* - \delta \leq s \leq t} f_2(s) + 1,
\end{align*}
\]

for any \( t \in [T_* - \delta, T_*) \). Then, it follows from (3.5) that \( F_1(t) \leq CF_1^\frac{1}{2}(t) \), and thus it follows from (3.6) that \( F_2(t) \leq CF_2^\frac{1}{2}(t) \), for \( d = 2 \); similarly, \( F_2(t) \leq CF_2^\frac{1}{4}(t) F_2^\frac{1}{2}(t) F_2^\frac{1}{4}(t) = CF_2^\frac{3}{4}(t) \), for \( d = 3 \). Hence \( F_1(t), F_2(t) \leq C < \infty \), for any \( t \in [T_* - \delta, T_*) \). On account of this uniform in time estimates, and recalling the hypothesis (H3), one can easily obtain the a priori estimate

\[
\sup_{\delta \leq \tau \leq t} \| u \|_{H^2}^2 + \int_{\delta}^{t} \| u \|_{H^3}^2 d\tau \leq G_3 < \infty,
\]

for any \( t \in (T_* - \delta, T_*) \), where \( G_3 \) is a positive constant. On account of this estimate, the same argument as before yields the conclusion.

\[\square\]

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