Selective phase rotation quantum gate entangler

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Abstract

We construct a quantum gate entangler based on selective phase rotation transform. In particular, we established a relation between quantum integral transform and quantum gates entangler in terms of universal controlled gates for multi-qubit states. Our construction is also related to the geometrical structure of multipartite quantum systems.

1 Introduction

In quantum computing we are interested in manipulated multipartite quantum states by quantum gates. One very important types of such gates are called quantum gate entangler which creates entangle states from product states. Recently, topological quantum gate entangler for two-qubit state has been constructed in [1]. These topological operators are called braiding operators that can entangle quantum states and are also unitary solution of Yang-Baxter equation. Thus, topological unitary transformations are very suitable for application in the field of quantum computing. Recently, we have also have constructed such quantum gate entangler for multi-qubit states and general multipartite states [2, 3]. In this paper we will construct quantum gate entangler based on unitary selective phase rotation transform for multi-qubit states. In section 2 we will give a short introduction to the quantum integral transformation and selective phase rotation transform. In section 3 we will recall a theorem on the construct quantum gate entangler for multi-qubit state and then use this theorem to construct a quantum gate entangler based on selective rotation transformation. We also in detail discuss the construction of such a entangler for three-qubit states.

2 Selective phase rotation transform

In this section we will define quantum fourier transform and selective phase rotation transform [4]. Let \( m \in \mathbb{N} \) and \( S_m = \{0, 1, \ldots, 2^m\} \) be a set of integers. Then, for any function \( f : S_m \rightarrow \mathbb{C} \), a discrete integral transform \( \tilde{f} : S_m \rightarrow \mathbb{C} \) with kernel \( K : S_m \times S_m \rightarrow \mathbb{C} \) is defined by

\[
\tilde{f}(y) = \sum_{x=0}^{2^m-1} K(x, y) f(x).
\]  (2.0.1)
In the matrix form the kernel \( K \), the function \( f \), and \( \tilde{f} \) are given by \( K = (K_{\mu\nu})_{0 \leq \mu, \nu \leq 2^m - 1} = (f(0), f(1), \ldots, f(2^m - 1))^T \), and \( \tilde{f} = Kf \), respectively. Moreover, if the kernel \( K \) is unitary, then the inverse transform \( \tilde{f} \rightarrow f \) of a discrete integral transform exists and is given by \( f(x) = \sum_{x=0}^{2^m-1} K^\dagger(x, y)f(y) \). For example, let \( U \) be a unitary matrix acting on the \( m \)-qubit space \( \mathcal{H}_Q \) and \( \{|x\rangle = |x_{m-1}, x_{m-2}, \ldots, x_0\rangle \}_{x_i \in \{0,1\}} \), be a binary basis of \( \mathcal{H} \), where \( x = x_{m-1}2^{m-1} + x_{m-2}2^{m-2} + \cdots + x_02^0 \). Then, \( U|x\rangle = \sum_{y=0}^{2^m-1} U(x, y)|y\rangle \), where \( U(x, y) = \langle x|U|y\rangle \) is the \( (x, y) \)-component of the matrix \( U \). Moreover, the unitary matrix \( U \) implementing a discrete integral transform as follows

\[
U \left( \sum_{x=0}^{2^m-1} f(x)|x\rangle \right) = \sum_{y=0}^{2^m-1} \tilde{f}(y)|y\rangle
\]

is called the quantum integral transform.

Now, let \( K_m(x, y) = e^{i\varphi x} \delta_{xy} \), for all \( x, y \in S_m \) and \( \varphi_x \in \mathbb{R} \). Then the discrete integral transform

\[
\tilde{f}(y) = \sum_{x=0}^{2^m-1} K(x, y)f(x) = \sum_{x=0}^{2^m-1} e^{i\varphi x} \delta_{xy} f(x) = e^{i\varphi} f(y)
\]

is called the selective phase rotation transform. Note that \( K_m \) is a unitary transformation. For example, in matrix form \( K_1 \) and \( K_2 \) are given by

\[
K_1 = \begin{pmatrix} e^{i\varphi_0} & 0 \\ 0 & e^{i\varphi_1} \end{pmatrix}, \quad K_2 = \begin{pmatrix} e^{i\varphi_0} & 0 & 0 & 0 \\ 0 & e^{i\varphi_1} & 0 & 0 \\ 0 & 0 & e^{i\varphi_2} & 0 \\ 0 & 0 & 0 & e^{i\varphi_3} \end{pmatrix} = L_0 L_1,
\]

where

\[
L_0 = A_0 \otimes U_0 + A_1 \otimes I_{2 \times 2} = \begin{pmatrix} U_0 & 0_{2 \times 2} \\ 0_{2 \times 2} & I_{2 \times 2} \end{pmatrix}, \quad L_1 = A_0 \otimes I_{2 \times 2} + A_1 \otimes U_1 = \begin{pmatrix} I_{2 \times 2} & 0_{2 \times 2} \\ 0_{2 \times 2} & U_1 \end{pmatrix},
\]

and

\[
U_0 = \begin{pmatrix} e^{i\varphi_0} & 0 \\ 0 & e^{i\varphi_1} \end{pmatrix}, \quad U_1 = \begin{pmatrix} e^{i\varphi_2} & 0 \\ 0 & e^{i\varphi_3} \end{pmatrix}, \quad A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Note also that \( L_0 \) and \( L_1 \) gates could be implemented with a set of universal gates, e.g., \( L_0 \) is a controlled-\( U_0 \) gate where the control bit is negated and \( L_1 \) is a controlled-\( U_1 \) gate.

\section{Quantum gate entangler for multi-qubit states}

In this section we will review the construction of a unitary operator that entangle a multi-qubit state \([5]\). Then, we will construct quantum gate entangler for multi-qubit states based on selective phase rotation transform. We will also in detail discuss this operator for three-qubit states. For a multi-qubit state \(|\Psi\rangle =
\[ \sum_{x_m-1 \ldots x_0=0}^1 \alpha_{x_m-1 x_m-2 \ldots x_0} |x \rangle \] a geometrical unitary transformation \( R_{2^m \times 2^m} \) that create multipartite entangled state is defined by \( R_{2^m \times 2^m} = R^d_{2^m \times 2^m} + R^{ad}_{2^m \times 2^m} \), where

\[
R^d_{2^m \times 2^m} = \text{diag}(\alpha_{00 \cdots 0}, 0, \ldots, 0, \alpha_{11 \cdots 0}, 0, \ldots, 0, \alpha_{11 \cdots 1}) \quad (3.0.7)
\]
is a diagonal matrix and

\[
R^{ad}_{2^m \times 2^m} = \text{antidiag}(0, \alpha_{11 \cdots 0}, \ldots, \alpha_{10 \cdots 0}, \alpha_{01 \cdots 1}, \ldots, \alpha_{00 \cdots 1}, 0) \quad (3.0.8)
\]
is an anti-diagonal matrix. Then, we have the following theorem for a multi-qubit state.

**Theorem 3.0.1** If elements of \( R_{2^m \times 2^m} \) satisfy

\[
\alpha_{x_m-1 x_m-2 \ldots x_0} = 0 \quad (3.0.9)
\]

\[
\neq \alpha_{x_m-1 x_m-2 \ldots x_j+1 y_j x_j-1 \ldots x_0} \quad (3.0.10)
\]

then the state \( R_{2^m \times 2^m} (|\psi \rangle \otimes |\psi \rangle \otimes \cdots \otimes |\psi \rangle) \), with \( |\psi \rangle = |0 \rangle + |1 \rangle \) is entangled.

The proof of this theorem is based on the Segre variety

\[
\bigcap_{\forall j} V(\alpha_{x_m-1 x_m-2 \ldots x_j+1 y_j x_j-1 \ldots x_0} \alpha_{y_m-1 y_m-2 \ldots y_j y_j-1 \ldots y_0}) \quad (3.0.11)
\]

which is the image of the Segre embedding and is an intersection of families of quadric hypersurfaces in \( \mathbb{P}_{c}^{2m-1} \) that represents the completely decomposable tensor. Note that we need to imposed some constraints on the parameters \( \alpha_{x_m-1 x_m-2 \ldots x_0} \) to ensure that our \( R_{2^m \times 2^m} \) is unitary. Note also that for a multi-qubit state

\[
R_{2^m \times 2^m} = R_{2^m \times 2^m} P_{2^m \times 2^m} = \text{diag}(\alpha_{00 \cdots 0}, 0, \ldots, 0, \alpha_{11 \cdots 1}) \quad (3.0.12)
\]

is a \( 2^m \times 2^m \) phase gate and \( P_{2^m \times 2^m} \) is \( 2^m \times 2^m \) swap gate.

Now, we will construct a quantum gate entangler based on the kernel of selective phase transform kernel for a multi-qubit state. Let the unitary transformation \( K_m(x, y) \) be defined as in section 2. Moreover, let

\[
K_{2^m \times 2^m} = \frac{1}{\sqrt{2^m}} \begin{pmatrix} e^{i\varphi_{00 \cdots 0}} & 0 & \cdots & 0 \\ 0 & e^{i\varphi_{00 \cdots 1}} & 0 & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & e^{i\varphi_{11 \cdots 1}} \end{pmatrix} \quad (3.0.13)
\]

Then, we also have the following theorem.

**Theorem 3.0.2** If elements of \( K_{2^m \times 2^m} \) satisfy

\[
eq e^{i\varphi_{x_m-1 x_m-2 \ldots x_0}} \quad (3.0.14)
\]

\[
eq e^{i\varphi_{y_m-1 y_m-2 \ldots y_0}} \quad (3.0.15)
\]

\[
eq e^{i\varphi_{x_m-1 x_m-2 \ldots x_j+1 y_j x_j-1 \ldots x_0}} \quad (3.0.16)
\]

\[
eq e^{i\varphi_{y_m-1 y_m-2 \ldots y_j y_j-1 \ldots y_0}} \quad (3.0.17)
\]

then the state \( K_{2^m \times 2^m} (|\psi \rangle \otimes |\psi \rangle \otimes \cdots \otimes |\psi \rangle) \), with \( |\psi \rangle = |0 \rangle + |1 \rangle \) is entangled.
The proof of this theorem also follows from the decomposable tensor in terms of the Segre variety. As an example, we will construct such quantum gate entangler for a three-qubit state. In this case, $K_m(x, y)$ is given by

$$K_{8 \times 8} = \frac{1}{\sqrt{8}} \begin{pmatrix} e^{i\varphi_{000}} & 0 & \cdots & 0 \\ 0 & e^{i\varphi_{001}} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & e^{i\varphi_{111}} \end{pmatrix} = I_0^3 I_1^3 I_2^3 I_3^3,$$  \hspace{1cm} (3.0.14)

where $L_0^3, L_1^3, L_2^3,$ and $L_3^3$ are constructed by universal controlled gates as follows

$$L_0^3 = \begin{pmatrix} L_0^2 & I_{4 \times 4} \\ I_{4 \times 4} & I_{4 \times 4} \end{pmatrix}, \quad U_0 = \begin{pmatrix} e^{i\varphi_{000}} & 0 \\ 0 & e^{i\varphi_{001}} \end{pmatrix},$$ \hspace{1cm} (3.0.15)

$$L_1^3 = \begin{pmatrix} L_1^2 & I_{4 \times 4} \\ I_{4 \times 4} & I_{4 \times 4} \end{pmatrix}, \quad U_1 = \begin{pmatrix} e^{i\varphi_{010}} & 0 \\ 0 & e^{i\varphi_{011}} \end{pmatrix},$$ \hspace{1cm} (3.0.16)

$$L_2^3 = \begin{pmatrix} L_0^2 & I_{4 \times 4} \\ I_{4 \times 4} & I_{4 \times 4} \end{pmatrix}, \quad U_2 = \begin{pmatrix} e^{i\varphi_{100}} & 0 \\ 0 & e^{i\varphi_{101}} \end{pmatrix},$$ \hspace{1cm} (3.0.17)

and

$$L_3^3 = \begin{pmatrix} L_1^2 & I_{4 \times 4} \\ I_{4 \times 4} & L_1^2 \end{pmatrix}, \quad U_3 = \begin{pmatrix} e^{i\varphi_{110}} & 0 \\ 0 & e^{i\varphi_{111}} \end{pmatrix},$$ \hspace{1cm} (3.0.18)

where $0_{2 \times 2}$ and $0_{4 \times 4}$ are 2-by-2 and 4-by-4 zero matrices and $I_{2 \times 2}$ and $I_{4 \times 4}$ are 2-by-2 and 4-by-4 identity matrices. Following this construction of quantum gate entangler for three-qubit states based on universal controlled gates we can also write

$$K_{2^m \times 2^m} = L_0^m L_1^m \cdots L_{2^{m-1}-1}^m,$$ \hspace{1cm} (3.0.19)

where $L_i^m$, for $1 \leq i \leq 2^{m-1} - 1$ can be express in terms of universal controlled gates, e.g.,

$$L_0^m = \begin{pmatrix} U_0 & 0_{2 \times 2} \\ 0_{2 \times 2} & I_{2^{m-1} \times 2^{m-1}} \end{pmatrix}, \quad U_0 = \begin{pmatrix} e^{i\varphi_{000}} & 0 \\ 0 & e^{i\varphi_{001}} \end{pmatrix},$$ \hspace{1cm} (3.0.20)

where $0_{2^{m-1} \times 2^{m-1}}$ and $I_{2^{m-1} \times 2^{m-1}}$ are $2^{m-1}$-by-$2^{m-1}$ zero and identity matrices, respectively. Thus we have following corollary.

**Corollary 3.0.3** If elements of universal controlled gates $L_0^m L_1^m \cdots L_{2^{m-1}-1}^m$ satisfy

$$e^{i\varphi_{x_{m-1}} e^{i\varphi_{y_{m-2}}} \cdots e^{i\varphi_{y_{1}} \cdot e^{i\varphi_{y_{0}}}}},$$

$$\neq e^{i\varphi_{x_{m-1}} e^{i\varphi_{y_{m-2}}} \cdots e^{i\varphi_{y_{m-2}}} \cdot e^{i\varphi_{y_{j-1}} \cdot e^{i\varphi_{y_{j-2}}} \cdots e^{i\varphi_{y_{0}}}}}$$

then the state $L_0^m L_1^m \cdots L_{2^{m-1}-1}^m (|\psi\rangle \otimes |\psi\rangle \times \cdots \otimes |\psi\rangle)$ is entangled.
In this paper we have investigated the construction of quantum gate entangler based on selective phase rotation transform. We have shown that this quantum gate entangler can be constructed by universal controlled gates for multi-qubit states. We also have established a relation between geometrical structures of multi-qubit states and such a quantum gate entangler. These results give new way of constructing topological and geometrical quantum gates entangler for multipartite quantum systems.

References

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