The poset on connected graphs is Sperner

Stephen G.Z. Smith∗         István Tomon†

November 30, 2015

Abstract

Let $G$ be the set of all connected graphs on vertex set $[n]$. Define the partial ordering $<$ on $G$ as follows: for $G, H \in G$ let $G < H$ if $E(G) \subset E(H)$. The poset $(G, <)$ is graded, each level containing the connected graphs with the same number of edges. We prove that $(G, <)$ has the Sperner property, namely that the largest antichain of $(G, <)$ is equal to its largest sized level.

1 Introduction

Let $(P, <)$ be a partially ordered set (poset). We only consider partially ordered sets with finitely many elements. A chain in $P$ is a set $C \subset P$ of pairwise comparable elements. An antichain $A \subset P$ is a set of pairwise incomparable elements. The poset $(P, <)$ is graded if there exists a partition of $P$ into subsets $A_0, ..., A_m$ such that $A_0$ is the set of minimal elements of $P$, and whenever $x \in A_i$ and $y \in A_j$ with $x < y$ and there is no $z \in P$ with $x < z < y$, then we have $j = i + 1$. If such a partition exists, it is unique and the sets $A_0, ..., A_m$ are the levels of $P$.

A graded poset $(P, <)$ is Sperner if the largest antichain in $P$ is the largest sized level.

Let $m$ be a positive integer, $[m] = \{1, ..., m\}$. The Boolean lattice $2^{[m]}$ is the power set of $[m]$ ordered by inclusion, and $[m]^{(k)} = \{A \subset [m] : |A| = k\}$. By the well known theorem of Sperner [11] the poset $(2^{[m]}, \subset)$ is Sperner, the largest antichains being equal to $[m]^{(\lfloor m/2 \rfloor)}$ and $[m]^{(\lceil m/2 \rceil)}$. The question whether certain posets are Sperner is widely studied. For a short list of such results, see [1]. In this paper, we consider the following poset.

Let $n$ be a positive integer, $m = \binom{n}{2}$. If $G$ is a graph, $V(G)$ is the vertex set of $G$, $E(G)$ is the set of its edges, and $e(G) = |E(G)|$.

Let $G$ denote the set of all connected graphs on vertex set $[n]$. Define the partial ordering $<$ on $G$ such that for $G, H \in G$ we have $G < H$ if $E(G) \subset E(H)$. When there is no risk of confusion, we shall simply write $G$ when referring to the poset $(G, <)$. Observe that $G$ is graded, the levels of $G$ being the families $G^{(k)} = \{G \in G : |E(G)| = k\}$ for $k = n - 1, ..., m$. The following question originates from Katona [7].

Question 1. Is $(G, <)$ Sperner?

We prove that the answer is yes. Let $M = \lceil m/2 \rceil$. The main result of this paper is the following theorem.

Theorem 2. If $n$ is sufficiently large, the largest antichain in $G$ is $G^{(M)}$.
Let \( D \) be the set of all graphs on vertex set \([n]\) and extend the ordering \(<\) to \( D \) in the obvious way. Also, for \( k = 0, \ldots, m \), let \( D^{(k)} \) be the set of graphs in \( D \) with \( k \) edges. Observe that \((D, <)\) is isomorphic to \((2^{[m]}, \subset)\), hence \((D, <)\) is Sperner.

Note that \( G \) is a very dense subset of \( D \). As we shall see in Section 2, the size of \( G \) is at least \( 2^m (1 - 2^{-n - o(n)}) \). However, this alone does not guarantee that \( G \) is Sperner, as there exists a set \( S \) of \( O(m) \) elements in \( 2^{[m]} \), such that \( 2^{[m]} \setminus S \) is graded but not Sperner. Still, we should mention the result of Balogh, Mycroft and Treglown [2], who proved that if \( p_m \to \infty \) and \( B_m(p) \) is a random subset of \( 2^{[m]} \), each element chosen with probability \( p \), then \( B_m(p) \) is Sperner with a probability tending to 1. This shows that only a very small proportion of the subfamilies of \( 2^{[m]} \) with size at least \( 2^m (1 - 2^{-n - o(n)}) \) are not Sperner.

A problem similar to Question 1 has been considered in the paper of Jacobson, Kézdy and Seif [6]. Let \( G \) be a connected graph and let \((C(G), <)\) be the poset, whose elements are the connected, vertex-induced subgraphs of \( G \), and \( H < H' \) if \( V(H) \subset V(H') \). In [6], it was proved that this poset need not be Sperner, even if \( G \) is a tree.

This paper is organized as follows. In section 2, we shall prove various bounds on the number of connected graphs with certain properties. These bounds provide us with some of the ingredients needed for the proof of Theorem 2 in Section 3. In the last section, we propose some open problems.

## 2 Connectivity of graphs

In this section, we investigate the following problems. How many edges can a graph \( G \) have, whose removal destroys the connectivity, or 2-connectivity of \( G \)? Also, what is the number of graphs on vertex set \([n]\) for which the answer to the previous question is \( r \).

For the remainder of this paper, log denotes base 2 logarithm. Also, we extend the definition of binomial coefficients, \( \binom{x}{k} \) for any \( k \in \mathbb{N} \), \( x \in \mathbb{R} \), as defined in the Appendix, section 5.

First, let us state the following simple lemma without proof, which we shall use throughout this section.

**Lemma 3.** Let \( a_1, \ldots, a_s \) be positive integers and let \( a_1 + \ldots + a_s = n \). We have

\[
\sum_{i=1}^{s} \binom{a_i}{2} \leq \binom{n - s + 1}{2},
\]

and

\[
\sum_{1 \leq i < j \leq s} a_i a_j \geq (n - s + 1)(s - 1) + \binom{s - 1}{2}.
\]

Also, if \( a_i \leq k \) for \( i \in [s] \), where \( n/2 < k \leq n - s + 1 \), then

\[
\sum_{i=1}^{s} \binom{a_i}{2} \leq \binom{n - k - s + 2}{2} + \binom{k}{2},
\]

and

\[
\sum_{1 \leq i < j \leq s} a_i a_j \geq k(n - k).
\]

The following lemma is a classical result, which is a weak estimate on the number of disconnected graphs. For completeness, we shall provide a short proof. For more on this topic, see [3].
Lemma 4. The number of disconnected graphs on vertex set $[n]$ is less than $2^{(n-1) + o(n)}$.

Proof. A graph $G$ is disconnected if there is a partition of $[n]$ into two nonempty sets $A$ and $B$ such that there are no edges between $A$ and $B$. The number of disconnected graphs, where $|A| = 1$ and $|B| = n - 1$ is at most $n2^{(n-1)}$, as we have $n$ choices for the partition $\{A, B\}$, and $2^{(n-1)}$ number of different choices for the edges in $B$.

The number of disconnected graphs where $|A|, |B| \geq 2$ is at most $2^n2^{(n-2)+1}$, as there are at most $2^n$ number of choices for the partition $(A, B)$, and the number of ways to choose the edges inside $A$ and $B$ is at most $2^{(|A|+|B|)} \leq 2^{(n-2)+1}$. Hence, the total number of disconnected graphs is at most $2^{(n-1) + o(n)}$.

If $G$ is a graph and $U \subset V(G)$, $G[U]$ is the subgraph of $G$ induced on the vertex set $U$. If $F \subset E(G)$, then $G - F$ is the graph on vertex set $V(G)$ and edge set $E(G) \setminus F$. If $e \in E(G)$, we simply write $G - e$ instead of $G - \{e\}$. A multi-graph is a graph where we allow multiple edges between a pair of vertices. A cycle is a multi-graph with vertices $v_1, ..., v_s$ for some positive integer $s$, whose edges are $v_i \leftrightarrow v_{i+1}$ for $i = 1, ..., s$, where the indices are meant modulo $s$. We note that two vertices with two different edges connecting them is also a cycle by our definition.

We define the skeleton of a connected graph $G$ as follows. An edge $e \in E(G)$ is a bridge, if $G - e$ is disconnected. Let $B$ be the set of bridges in $G$ and let $A_1, ..., A_t$ be the vertex sets of the components of $G - B$. Then the skeleton of $G$ is $Sk(G) = (B, \{A_1, ..., A_t\})$.

The following simple lemma lists the main properties of the skeleton. This lemma can be found in [4] and also can be easily verified by the reader, so we omit its proof.

Lemma 5. Let $G$ be a connected graph with skeleton $(B, \{A_1, ..., A_t\})$. Then $|B| = t - 1$ and $G[A_i]$ is 2-edge-connected for $i \in [t]$.

A chorded cycle is a multi-graph $C$ on $s$ vertices for some $s \geq 2$ integer which is the disjoint union of a cycle with $s$ vertices and an additional edge. This additional edge is called the chord of
the cycle. Note that there might be more than one way to decompose a chorded cycle into a cycle and a chord. For example, two vertices with 3 edges connecting them is a chorded cycle. However, this should not cause any confusion later on.

If $G$ is a 2-edge-connected graph, let $R(G)$ be the set of edges $f \in E(G)$ such that $G - f$ is not 2-edge-connected. Lemma 6 gives an upper bound on the number of edges whose removal from $G$ destroys its 2-edge-connectedness.

**Lemma 6.** Let $G$ be a 2-edge-connected graph and let $H = G - R(G)$. Denote the number of components of $H$ by $q$. Then $|R(G)| \leq 2q - 2$.

**Proof.**

Let the components of $H$ be $H_1, ..., H_q$. Then any edge in $R(G)$ connects two different components in $H$. Define the multi-graph $K$ on vertex set $[q]$ as follows: if $H_i$ and $H_j$ are connected by $l$ edges in $G$, then $i$ and $j$ are connected by $l$ edges in $K$.

Note that the graph $K$ cannot contain a cycle with a chord. Otherwise, suppose that there is a cycle with vertices $i_1, ..., i_k$ and a chord $i_qi_b$. Let $e \in R(G)$ be an edge connecting $H_{i_a}$ and $H_{i_b}$ in $G$. Then $H_{i_a}$ and $H_{i_b}$ are still connected by at least 2 disjoint paths in $G - e$, hence $e$ cannot be an element of $R(G)$.

Our task is reduced to showing that if a multi-graph on $q$ vertices does not contain a chorded cycle, then it has at most $2q - 2$ edges. We prove this by induction on $q$. For $q = 1$, the statement is trivial. Suppose that the corresponding statement holds for multi-graphs with at most $q - 1$ vertices and let $L$ be a multi-graph on $q$ vertices without a chorded cycle. If $L$ does not contain a cycle, then $|E(L)| \leq q - 1$.

Suppose that $L$ contains a cycle $C$ with vertices $v_1, ..., v_s$, respectively. Define the multi-graph $L'$ by collapsing the cycle $C$ into one vertex in $L$. Formally, let $L'$ be the graph on vertex set $V(L) \setminus V(C) \cup \{v'_C\}$, where $v'_C$ is a new vertex obtained by collapsing $C$. Two vertices $x, y \in V(L) \setminus V(C)$ are joined by $l$ edges in $L'$ if $\overline{xy}$ is an edge $l$ times in $L$, and $x \in V(L) \setminus V(C)$ is joined to $v'_C$ by $l$ edges in $L'$ if there are $l$ edges going from $x$ to $C$ in $L$. Observe that as there is no chorded cycle in $L$, every edge in $L[V(C)]$ is an edge of $C$ as well, so $L'$ has exactly $|E(L)| - s$ edges and $q - s + 1$ vertices.

Also, $L'$ does not contain a chorded cycle. Otherwise, suppose that $D$ is a chorded cycle with vertices $u_1, ..., u_t$, where $\overline{u_iu_{i+1}}$ is an edge for $i = 0, ..., t \mod (t + 1)$, and $\overline{u_au_b}$ is a chord of this cycle. We have to check three cases.

First case, if $v'_C \notin \{u_0, ..., u_t\}$. Then $D$ is also a chorded cycle in $L$, contradiction.

Now suppose that $v'_C = u_d$ for some $d \in [t]$. Then there is an edge $\overline{u_{d-1}u_b}$ in $L'$ connecting $C$ and $u_{d-1}$, and there is an edge $\overline{v_ju_{d+1}}$ in $L$ connecting $C$ and $u_{d+1}$. Second case, if $d \in [t] \setminus \{a, b\}$. Then the sequence of vertices

$$u_1, ..., u_{d-1}, v_i, v_{i+1}, ..., v_j, u_{d+1}, ..., u_t$$

(the indices of the $v$’s are meant modulo $s$) determines a cycle with the chord $\overline{u_au_b}$ in $L$, contradiction.

Third case, if $d \in \{a, b\}$. Without the loss of generality, suppose that $d = a$. Then there exists an edge $\overline{u_bu_k}$ connecting $u_b$ and $C$. Then $k$ is either an element of $\{i, i + 1, ..., j\} \mod s$ or $\{j, j + 1, ..., i\} \mod s$. Suppose that $k \in \{i, i + 1, ..., j\} \mod s$, the other case being similar. Then

$$u_1, ..., u_{a-1}, v_i, v_{i+1}, ..., v_j, u_{a+1}, ..., u_t$$

determines a cycle with the chord $\overline{v_ku_b}$ in $L$, contradiction.
As $L'$ has $q - s + 1$ vertices, where $q - s + 1 < q$, we can apply our induction hypotheses to $L'$.

We get that $|E(L')| \leq 2|V(L')| - 2$, namely that

$$|E(L)| - s \leq 2(q - s + 1) - 2.$$  

Rearranging the terms in the inequality, we have

$$|E(L)| \leq 2q - s \leq 2q - 2.$$

\[\square\]

We remark that if $R(G)$ is non-empty, it has at least 2 elements. This is true because if $e \in R(G)$, then $G - e$ contains a bridge $f$. But then $f \notin R(G)$ as well.

Let $I_r$ be the set of 2-edge-connected graphs $G$ such that $|R(G)| = r$, and let $I_r^{(k)} = I_r \cap G^{(k)}$. In the next lemma, we give an upper bound on the size of $I_r^{(k)}$.

**Lemma 7.** Let $\epsilon$ be a positive real number. There exists $n_1(\epsilon)$ such that if $n > n_1(\epsilon)$, the following holds. For any positive integers $r$ and $k$ satisfying $2 \leq r \leq n$ and $M \leq k \leq M + n$, we have

$$|I_r^{(k)}| \leq \left(\binom{n-\epsilon r}{2} + \epsilon r n\right).$$

**Proof.**

Let $q = r/2 + 1$. By Lemma 6 if $G$ is a 2-edge-connected graph with $|R(G)| = r$, then $G - R(G)$ has at least $q$ components, and at most $r$ components. We now count the number of graphs $G$ where $G - R(G)$ has exactly $s$ components. Note that $s \leq r$, otherwise the edges of $R(G)$ could not connect all the components of $G - R(G)$.

The number of graphs $G$, for which $|R(G)| = s$, and where the components in $G - R(G)$ have sizes $a_1, \ldots, a_s$ with $e_1, \ldots, e_s$ edges inside them, respectively, is at most

$$\left(\binom{n^2}{r}\right) \left(\binom{n}{a_1, \ldots, a_s}\right) \prod_{i=1}^{s} \left(\binom{a_i}{2}/e_i\right).$$

(1)

Here, $\left(\binom{n^2}{r}\right)$ is an upper bound on the number of ways to pick the edges of $R(G)$, $\left(\binom{n}{a_1, \ldots, a_s}\right)$ is the number of ways to partition $[n]$ into parts of size $a_1, \ldots, a_s$, and $\left(\binom{a_i}{2}/e_i\right)$ is the number of ways to choose the $e_i$ edges in a component of size $a_i$. We shall prove that (1) is at most $\left(\binom{n-s+1}{2}/k\right)2^{3\epsilon r n/5}$. Let us bound the terms in (1).

First, $\left(\binom{n^2}{r}\right) = 2^{2r \log n} < 2^{3\epsilon r n/5}$, if $n$ is sufficiently large given $\epsilon$.

Also, $(a_1, n-a_1) \leq n^2 = 2^n \log s$. Unfortunately, if $r$ is small, we cannot bound this term by $2^{3\epsilon r n}$, where $c$ is some fixed constant. We shall overcome this obstacle later in the proof.

Finally,

$$\prod_{i=1}^{s} \left(\frac{a_i/2}{e_i}\right) \leq \left(\frac{\sum_{i=1}^{s} a_i/2}{k-r}\right) \leq \left(\frac{n-s+1}{2}/k-r\right),$$

where the last inequality holds by Lemma 3. Here,

$$\left(\frac{n-s+1}{2}/k-r\right) < \left(\frac{r + (n-s+1)}{k}\right),$$

5
see (1) in the Appendix. Hence, we have

\[ \prod_{i=1}^{s} \binom{a_i}{2} e_i \leq \binom{n-s+1}{2} + \varepsilon rn/5, \]

provided \( n > 5/\varepsilon \).

First, suppose that \( r \) is such that \( \log r < \varepsilon r/5 \). In this case, we have \( \binom{n}{a_1,\ldots,a_s} \leq 2^{\varepsilon rn/5} \). Hence, (1) is at most \( \left( \binom{n-s+1}{2} + \varepsilon rn/5 \right) 2^{\varepsilon rn/5} \).

Now consider the case when \( \log r > \varepsilon r/5 \). Then \( r < R(\varepsilon) \), where \( R(\varepsilon) \) is a constant only depending on \( \varepsilon \). In this case, we shall bound the product

\[ \left( \frac{n}{a_1,\ldots,a_s} \right) \prod_{i=1}^{s} \binom{a_i}{2} e_i. \] (2)

Without loss of generality, suppose that \( a_1 \geq \ldots \geq a_s \) and observe that \( \binom{n}{a_1,\ldots,a_s} < n^{a_2+\ldots+a_s} \). Hence, if \( a_1 \geq n-4r \), then \( \binom{n}{a_1,\ldots,a_s} < n^{4r} < 2^{\varepsilon rn/5} \), if \( n \) is sufficiently large given \( \varepsilon \). Now suppose that \( a_1 < n-4r \). Applying Lemma 3, we get

\[ \sum_{i=1}^{s} \binom{a_i}{2} \leq \frac{4r-s-2}{2} + \frac{n-4r}{2}, \]

\[ \leq 8r^2 + \frac{n-4r}{2}. \]

Suppose \( n > 20R(\varepsilon) \), then the inequality

\[ 8r^2 + \frac{n-4r}{2} \leq \frac{n-s+1}{2} - 2rn \]

holds as well. Hence,

\[ \prod_{i=1}^{s} \binom{a_i}{2} e_i < \left( \sum_{i=1}^{s} \binom{a_i}{2} \right) \leq \frac{\binom{n-s+1}{2} - 2rn}{k} < \frac{\binom{n-s+1}{2} - 2rn + r}{k}, \]

where the last inequality holds by (1) in the Appendix. Also, using (2) in the Appendix,

\[ \frac{\binom{n-s+1}{2} - 2rn + r}{k} \leq \left( \frac{\binom{n-s+1}{2}}{k} \right) 2^{-rn}. \]

Thus, we can bound (2) from above by \( \left( \frac{\binom{n-s+1}{2}}{k} \right) \), and so (1) is at most \( \left( \frac{\binom{n-s+1}{2} + \varepsilon rn/5}{k} \right) 2^{\varepsilon rn/5} \) in this case as well.

Now let us bound the number of all 2-edge-connected graphs with \( k \) edges, for which \( |R(G)| = r \) and \( G - R(G) \) has \( s \) components. The number of such graphs is at most

\[ \sum_{a_1+\ldots+a_s=n} \sum_{e_1+\ldots+e_s=k-r} \binom{n^2}{r} \binom{n}{a_1,\ldots,a_s} \prod_{i=1}^{s} \binom{a_i}{2} e_i. \] (3)

The first sum has exactly \( \binom{n}{s-1} \) terms, while the second sum has \( \binom{k-r+s}{s-1} \) terms. Hence, (3) is at most

\[ \binom{n}{s-1} \binom{k-r+s}{s-1} \binom{n-s+1}{2} + \varepsilon rn/5 \right) 2^{\varepsilon rn/5}. \]
Here, \( \binom{n}{s-1} \leq 2^{r \log n} \) and \( \binom{k-r+s}{s-1} < 2^{2r \log n} \). Thus, \( \textbf{3} \) is at most
\[
\left( \frac{n-s+1}{2} + \epsilon rn/5 \right) \frac{2^{3\epsilon r n/5}}{k},
\]
provided \( n \) is sufficiently large given \( \epsilon \).
Finally, the number of 2-edge-connected graphs with \( |R(G)| = r \) is at most
\[
\sum_{i=q}^{r} \left( \frac{n-i+1}{2} + \epsilon rn/5 \right) \frac{2^{3\epsilon r n/5}}{k} < \left( \frac{n-q+1}{2} + \epsilon rn/5 \right) \frac{2^{4\epsilon r n/5}}{k}.
\]

Applying \( \textbf{2} \) in the Appendix, we get
\[
|R^{(k)}_r| \leq \left( \frac{n-q+1}{2} + \epsilon rn \right).
\]

\( \square \)

In the proof of Theorem \( \textbf{2} \), we shall also use the following technical lemma.

**Lemma 8.** Let \( n > 150 \). Let \( G \) be a non-2-edge-connected graph on vertex set \( [n] \) such that \( e(G) \geq M \) and \( Sk(G) = (B, \{A_1, ..., A_t\}) \). Suppose that \( t \geq 3 \) or \( \min\{|A_1|, |A_2|\} > 1 \). Then
\[
\sum_{1 \leq i < j \leq t} |A_i||A_j| - 2(t-1) - \sum_{i=1}^{t} |R(G[A_i])| \geq n. \tag{4}
\]

**Proof.**
By Lemma \( \textbf{4} \) we have \( |R(G[A_i])| < 2|A_i| \). Hence, \( \sum_{i=1}^{t} |R(G[A_i])| < 2n \).
First, suppose that \( \max\{|A_1|, ..., |A_t|\} \leq n - 6 \). By Lemma \( \textbf{3} \) we have
\[
\sum_{1 \leq i < j \leq t} |A_i||A_j| \geq 6(n-6) \geq 5n.
\]
Hence, using the trivial bound \( t-1 < n \), we have that \( \textbf{4} \) holds.
Now suppose that \( |A_1| \geq n - 5 \). In this case, we have \( t \leq 6 \). Let \( H = G[A_1] \). Every edge of \( G \) not contained in \( H \) is either in \( B \) or it is an edge of \( G[[n] \setminus A_1] \). Hence, the number of edges not contained in \( H \) is at most 20, so \( e(H) \geq M - 20 \).
Let \( H_1, ..., H_q \) be the vertex sets of the components of \( H - R(H) \). Then the number of edges of \( H \) is at most
\[
2q - 2 + \sum_{i=1}^{q} \left( \frac{|H_i|}{2} \right) < 2n + \left( \frac{n-q+1}{2} \right),
\]
where the inequality holds by Lemma \( \textbf{3} \) Comparing the lower and upper bounds on \( e(H) \) we get the inequality
\[
M - 20 < 2n + \left( \frac{n-q+1}{2} \right).
\]
If \( q > n/3 \), the right hand side of the inequality is at most \( 2n^2/9 + 3n \), while the left hand side is larger than \( n^2/4 - n \). This is contradiction, noting that \( 2n^2/9 + 3n < n^2/4 - n \) for \( n > 150 \). Hence, we have \( q < n/3 \), implying \( |R(H)| < 2n/3 \). This gives
\[
\sum_{i=1}^{t} |R(G[A_i])| \leq |R(H)| + 2(|A_2| + ... + |A_t|) < 2n/3 + 10.
\]
If \( |A_1| \leq n - 2 \), then \( \sum_{1 \leq i < j \leq t} |A_i||A_j| \geq 2(n - 2) \) by Lemma 3, so (4) holds.

We deduce that the only case when (4) can fail is when the largest of \( |A_1|, \ldots, |A_t| \) is \( n - 1 \), so \( t = 2 \) and \( \min\{|A_1|, |A_2|\} = 1 \).

\[ \square \]

3 Matchings between levels

In this section, we prove Theorem 2.

Let \( n - 1 \leq k, l \leq m \). We say that there is a complete matching from \( G^{(k)} \) to \( G^{(l)} \), if there is an injection \( f: G^{(k)} \to G^{(l)} \) such that \( G \) and \( f(G) \) are comparable for all \( G \in G^{(k)} \). The next Lemma states that to prove Theorem 2, it is enough to find a complete matching from the smaller sized level to the larger sized level for any two consecutive levels. Due to its simplicity, we shall only sketch the proof of this lemma.

**Lemma 9.** Suppose that there is a complete matching from \( G^{(k)} \) to \( G^{(k+1)} \) for \( k = n - 1, \ldots, M - 1 \), and there is a complete matching from \( G^{(l+1)} \) to \( G^{(l)} \) for \( l = M, \ldots, m - 1 \). The largest antichain in \( G \) is \( G^{(M)} \).

**Proof.**

Using the complete matchings, one can build a chain partition of \( G \) into \( |G^{(M)}| \) number of chains. But the size of the maximal antichain in \( G \) is at most the number of chains in any chain partition of \( G \).

\[ \square \]

First, we show that if we are below the middle level \( G^{(M)} \), or at least \( n \) above the middle level, then it is easy to prove the existence of a complete matching.

Let \( X \subset G^{(k)} \) for some \( n - 1 \leq k \leq m \). The shadow of \( X \) is

\[ \triangle(X) = \{ G \in G^{(k-1)} : \exists H \in X, G < H \}, \]

and the upper shadow of \( X \) is

\[ \nabla(X) = \{ G \in G^{(k+1)} : \exists H \in X, H < G \}. \]

In our proofs, we shall apply Hall’s theorem [5], which states the following. Let \( G = (A, B, E) \) a bipartite graph. There is a complete matching in \( G \) from \( A \) to \( B \) if and only if \( |X| \leq |\Gamma(X)| \) for all \( X \subset A \), where \( \Gamma(X) \) denotes the neighbourhood of \( X \).

**Lemma 10.** There is a complete matching from \( G^{(k)} \) to \( G^{(k+1)} \) for \( k = n - 1, \ldots, M - 1 \).

**Proof.**

Let \( X \subset G^{(k)} \). By Hall’s theorem, it is enough to show that \( |X| \leq |\nabla(X)| \). Let \( B \) be the bipartite graph with color classes \( X \) and \( \nabla(X) \), and the edges of \( B \) being the comparable pairs. If \( G \in X \), the degree of \( G \) is \( m - k \). Also, if \( H \in \nabla(X) \), the degree of \( H \) is at most \( k + 1 \).

Let \( e \) be the number of edges of \( B \). Then counting \( e \) from \( X \) and then from \( \nabla(X) \), we have

\[ |X|(m - k) = e, \]

and

\[ e \leq |\nabla(X)|(k + 1). \]

Hence,
\[ |X| \leq |\nabla(X)|(k+1)/(m-k) \leq |\nabla(X)|. \]

Using similar ideas, now we show that if we are above the middle level by at least \( n \), then there is a matching from \( G^{(k+1)} \) to \( G^{(k)} \).

**Lemma 11.** There is a complete matching from \( G^{(k+1)} \) to \( G^{(k)} \) for \( k = M + n, \ldots, m \).

**Proof.**
Let \( X \subseteq G^{(k+1)} \). By Hall’s theorem, it is enough to show that \( |X| \leq |\Delta(X)| \). Let \( B \) be the bipartite graph with color classes \( X \) and \( \Delta(X) \), and the edges of \( B \) being the comparable pairs. If \( G \in \Delta(X) \), then the degree of \( G \) in \( B \) is at most \( m - k \).

Now let \( G \in X \). If \( e \in E(G) \) such that \( G - e \) is not an element of \( G \), then \( e \) is a bridge of \( G \). However, by Lemma \ref{lem:bridges} the number of bridges of \( G \) is at most \( n - 1 \). Hence, the degree of \( G \) is at least \( k + 2 - n \). Counting the number of edges of \( B \) two ways, we get
\[ |X|(k + 2 - n) \leq |E(B)|, \]
and
\[ |\Delta(X)|(m - k) \geq |E(B)|. \]
Hence,
\[ \frac{|\Delta(X)|}{|X|} \geq \frac{k + 2 - n}{m - k} \geq 1. \]

Proving that there is a matching from \( G^{(k)} \) to \( G^{(k-1)} \) for the values of \( k \) that are slightly larger than \( M \) is more difficult. We shall deal with this problem in the rest of this section.

Let \( M + 1 \leq k < M + n \). Our strategy for proving the existence of a complete matching from \( G^{(k)} \) to \( G^{(k-1)} \) is briefly as follows. By Hall’s theorem, we will show that, for every \( X \subseteq G^{(k)} \), we have \( |\Delta(X)| \geq |X| \). We shall write \( X \) as \( Y \cup Z \), where \( Y \) is the set of 2-edge-connected graphs in \( X \) and \( Z \) is the rest of \( X \). First, we show that if \( Y \) and \( Z \) do not have roughly the same size, then the size of the shadow of the larger one of them is already larger than \( |X| \). After that, we show that \( |\Delta(Y)| \) is at least \( |Y|(1 + c_1) \) with some positive constant \( c_1 \) depending on the size of \( |Y| \). Also, \( |\Delta(Z)| \) is at least \( |Z|(1 + c_2) \), where \( c_2 \) is some positive constant depending on the size of \( Z \). Moreover, we show that \( |\Delta(Y) \cap B| \) is at least \( (1 - c_3)|Y| \), where \( c_3 \) is a positive constant depending on the size of \( Y \). As we shall see, after these preparations we only need to show that \( c_3 \leq c_2 c_1 \) to establish \( |X| \leq |\Delta(X)| \).

We remind the reader that \( D \) is the family of all graphs on vertex set \([n]\). For \( X \subseteq G^{(k)} \), let \( \partial(X) = \{H \in D^{(k-1)} : \exists \ G \in X, H < G\} \). As \((D, \prec)\) is isomorphic to \((2^{[n]}, \subset)\), the Kruskal-Katona theorem \cite{KP78} tells us which subfamily of \( D^{(k)} \) of given size minimizes the lower shadow. However, we apply Lovász’s lemma \cite{Lovasz} to get a lower bound on the size of \( \partial(X) \), which is a weaker, but computationally more convenient form of the Kruskal-Katona theorem.
Lemma 12. (Lovász [10]) Let $X \subset G^{(k)}$ of size $|X| = \binom{x}{k}$. Then

$$|\partial(X)| \geq \binom{x}{k-1}.$$ 

In particular,

$$\frac{|\partial(X)|}{|X|} \geq \frac{k}{x - k + 1}.$$ 

Let $\mathfrak{B}$ be the set of 2-edge-connected graphs in $G$ and let $\mathfrak{B}^{(k)} = G^{(k)} \cap \mathfrak{B}$. If $X \subset \mathfrak{B}^{(k)}$, then $\Delta(X) = \partial(X)$. Hence, we can use Lemma 12 to get a lower bound for the size of $\Delta(X)$.

In the next Lemma we show that if the size of $X \in G^{(k)}$ is sufficiently large, then we have $|\Delta(X)| \geq |X|$.

Lemma 13. Let $\epsilon > 0$. There exists $n_2(\epsilon)$ such that if $n > n_2(\epsilon)$ the following holds. Let $M + 1 \leq k < M + n$ and let $|X| = \binom{x}{k}$, where $x > \binom{n-1}{2} + \epsilon n$. We have $|\Delta(X)| > |X|$.

Proof.

By Lemma 12

$$|\partial(X)| \geq \binom{x}{k-1}.$$ 

Let $D$ be the set of disconnected graphs with $k - 1$ edges. By Lemma 4

$$|D| \leq 2^{\binom{n-1}{2} + o(n)}.$$ 

Also,

$$|\Delta(X)| = |\partial(X) \setminus D| \geq |\partial(X)| - |D| \geq \binom{x}{k-1} - 2^{\binom{n-1}{2} + o(n)}.$$ 

Thus, we get

$$|\Delta(X)| - |X| \geq \binom{x}{k-1} - \binom{x}{k} - 2^{\binom{n-1}{2} + o(n)} =$$

$$= \left( \frac{x}{k-1} \right) \frac{2k - x - 1}{k} - 2^{\binom{n-1}{2} + o(n)} > \left( \frac{n-1}{2} \right) \binom{\epsilon n}{k-1} \frac{1}{n^2} - 2^{\binom{n-1}{2} + o(n)}.$$ 

By (2) in the Appendix, we have $\left( \frac{n-1}{2} \right) \binom{\epsilon n}{k-1} \geq \left( \frac{n-1}{k-1} \right)^2$. Also, $\left( \frac{n-1}{k-1} \right) = 2^{\binom{n-1}{2} + o(n)}$ holds by Stirling’s formulae. Hence, we have

$$|\Delta(X)| - |X| \geq 2^{\binom{n-1}{2} + o(n) + \epsilon n + o(n)} - 2^{\binom{n-1}{2} + o(n)}.$$ 

Thus, if $n$ is sufficiently large given $\epsilon$, $|\Delta(X)| > |X|$.

Now we show that if $X$ is a set of 2-edge-connected graphs in $G^{(k)}$, then the number of 2-edge-connected graphs in the shadow of $X$ cannot be much less than $|X|$.
Lemma 14. Let $\epsilon > 0$. There exists $n_3(\epsilon)$ such that if $n > n_3(\epsilon)$, the following holds. Let $M < k < M + n$ and let $X \subset \mathcal{B}^{(k)}$. Let $|X| = \binom{x}{k}$ and let $r$ be a positive integer with $r < n$. If $x > \left(\frac{n-(r+1)/2}{2}\right) + \epsilon n$, then

\[
\frac{\left|\Delta(X) \cap \mathcal{B}^{(k-1)}\right|}{|X|} > 1 - \frac{4r}{n^2}.
\]

Proof.
Let $B$ be the bipartite graph with color classes $X$ and

\[ U = \Delta(X) \cap \mathcal{B}^{(k-1)}, \]

the edges being the comparable pairs. Every element of $U$ has degree at most $m - k + 1$ in $B$. Also, the degree of a graph $G$ in $X$ is exactly $k - |R(G)|$ in $B$. Let $a$ be the number of graphs in $X$ with degree at most $k - r - 1$ and let $a'$ be the number of graphs in $\mathcal{B}^{(k)}$ with $|R(G)| \geq r + 1$. Then $a < a'$ and by Lemma 7 we have 

\[ a' > \left(\binom{n-(r+1)/2}{k} + \epsilon n/2\right) \]

provided $n > n_1(\epsilon/2)$. Also, we have the following bounds on the number of edges of $B$:

\[ (k - r)(|X| - a') \leq e(B) \leq (m - k + 1)|U|. \]

Hence,

\[ \frac{|U|}{|X| - a'} \geq \frac{k - r}{m - k + 1}. \]

Here, $k \geq m/2 + 1$, so

\[ \frac{|U|}{|X| - a'} \geq \frac{m/2 - r + 1}{m/2} \geq 1 - \frac{4r - 4}{n(n - 1)}. \]

If $|X| > 8n^3a'$, we get $\frac{|U|}{|X|} \geq 1 - \frac{4r}{n^2}$, using that $r \leq n - 1$. But note that if $n$ is sufficiently large given $\epsilon$, then $8n^3 < 2^{\epsilon n/2}$, which means that

\[ 8n^3a' < \left(\binom{n-r/2}{k} + \epsilon n/2\right)2^{\epsilon n/2} < \left(\binom{n-r/2}{k} + \epsilon n/2\right) < \binom{x}{k}. \]

In the next lemma, we show that if $X \subset \mathcal{G}^{(k)}$ is a set of non-2-edge-connected graphs, then the size of the shadow of $X$ is slightly larger than $|X|$.

Lemma 15. Let $\epsilon$ be a positive real number such that $\epsilon < 1/2$. There exists $n_4(\epsilon)$ such that if $n > n_4(\epsilon)$, the following holds. Let $k$ be a positive integer with $M < k < M + n$ and let $X \subset \mathcal{G}^{(k)} \setminus \mathcal{B}^{(k)}$. Let $|X| = \binom{x}{k}$ and let $r$ be a positive integer such that $r < n$ and $x > \left(\frac{n-(r+1)/2}{2}\right) + \epsilon n$. Then

\[
\frac{|\Delta(X)|}{|X|} > 1 + \frac{4 - 4r/n}{n}.
\]

Proof.
Observe $x \geq k \geq (n^2 - n)/4$, so we can always find $r \leq n - 1$ satisfying $x > \left(\frac{n-(r+1)/2}{2}\right) + \epsilon n$. Hence, it is enough to prove this lemma in case $r \leq n - 1$.

Define the bipartite graph $B$ between $X$ and $U = \Delta(X)$ as follows. Let $G \in X$ and $H \in \Delta(X)$ be connected by an edge if $H < G$ and $Sk(G) = Sk(H)$. If $T = (C, \{A_1, \ldots, A_t\})$ is the skeleton of some graph, let $X(T)$ be the set of graphs in $X$ with skeleton $T$, and define $U(T)$ similarly. Let
$B(T)$ be the bipartite subgraph of $B$ induced on $X(T) \cup U(T)$, and let us estimate $|U(T)|/|X(T)|$. If $H \in U(T)$ and $e \in [n]^2 \setminus E(H)$ is an edge connecting $A_i$ and $A_j$ with $i \neq j$, then the skeleton of $H' = H \cup \{e\}$ differs from $T$. Hence, the degree of $H$ in this bipartite graph is at most

$$u_T = m - k + 1 - \sum_{1 \leq i < j \leq t} |A_i||A_j| + t - 1.$$ 

Now let $G \in X(T)$ and $e \in E(G)$. We have $Sk(G - e) = T$ if and only if $e \in G[a_i] \setminus R(G[A_i])$ for some $i \in [t]$. Hence, the degree of $G$ is

$$x_T = k - \sum_{i=1}^{t} |R(G[A_i])| - (t - 1).$$

Bounding the edges of $B(T)$ two different ways, we get

$$|X(T)|x(T) \leq e(T) \leq |U(T)|u(T).$$

Thus,

$$\frac{|U(T)|}{|X(T)|} \geq \frac{x(T)}{u(T)}.$$ 

Suppose that $t \geq 3$ or $\min\{|A_1|, |A_2|\} > 1$. Write $\frac{x(T)}{u(T)} = 1 + \frac{x(T) - u(T)}{u(T)}$. Here,

$$x(T) - u(T) \geq \sum_{1 \leq i < j \leq t} |A_i||A_j| - 2(t - 1) - \sum_{i=1}^{t} |R(G[A_i])| \geq n,$$

where the last inequality holds by Lemma 8. Also, we have $u(T) < m/2$, so

$$1 + \frac{x(T) - u(T)}{u(T)} \geq 1 + 2\frac{n}{m} > 1 + \frac{4}{n}.$$ 

Look at the remaining case, when $t = 2$ and $\min\{|A_1|, |A_2|\} = 1$. Without the loss of generality, let $|A_1| = 1$. We have $u(T) \leq M - (n - 2)$, while $x(T) \geq M - |R(G[A_2])|$. Let $a$ be the number of graphs $G$ in $X(T)$ with $|R(G[A_2])| \geq r + 3$. By Lemma 7, we have

$$a < \left(\binom{n-1}{2} + \frac{\epsilon rn}{2}\right),$$

if $n > n_1(\epsilon/2)$.

Counting the number of edges of $B(T)$ two ways, we get the following bounds:

$$|X(T)| - a)(M - r + 2) \leq e(B(T)) \leq (M - (n - 2))U(T)|.$$

Hence,

$$\frac{|U(T)|}{|X(T)| - a} \geq \frac{M - r + 2}{M - (n - 2)} = 1 + \frac{(n - r)}{M - (n - 2)} > 1 + \frac{4(n - 4r)}{n(n - 1)}.$$ 

If $|X(T)| > 2n^2a$, this implies $\frac{|U(T)|}{|X(T)|} \geq 1 + \frac{4n - 4r}{n^2 - n}$.

Let $T_0$ be the set of pairs $T = (C, \{A_1, A_2\})$ satisfying the following conditions: $T$ is the skeleton of some graph in $G$, $|A_1| = 1$, and $|X(T)| \leq 2na$. Let $X_0 = \bigcup_{T \in T_0} X(T)$. Note that $|T_0| < n^2$ as
we have at most \( n \) choices for \( A_1 \) and at most \( n - 1 \) choices for the one edge in \( C \). Hence, we have \(|X_0| \leq 2n^3a\). This gives the following bound on the size of \( \triangle(X) \).

\[
|\triangle(X)| \geq \left( 1 + \frac{4n - 4r}{n^2 - 1} \right) \left( |X| - |X_0| \right) \geq \left( 1 + \frac{4n - 4r}{n^2 - 1} \right) |X| - 4n^5 \left( (\frac{n - (r+1)/2}{2}) + \frac{\epsilon rn}{2} \right).
\]

Hence, if \(|X| \geq 4n^9 \left( (\frac{n - (r+1)/2}{2}) + \frac{\epsilon rn}{2} \right)\), then

\[
\frac{|\triangle(X)|}{|X|} \geq 1 + \frac{4n - 4r}{n^2}.
\]

But if \( n \) is sufficiently large given \( \epsilon \), we have

\[
4n^9 \left( (\frac{n - (r+1)/2}{2}) + \frac{\epsilon rn}{2} \right) < \left( (\frac{n - (r+1)/2}{2}) + \frac{\epsilon rn}{2} \right) \leq |X|.
\]

\[
\square
\]

In the next lemma, we show that if the number of 2-edge-connected graphs in \( X \) is not in the same range as the number of non-2-edge-connected graphs in \( X \), then \(|X| < |\triangle(X)|\).

**Lemma 16.** There exists \( n_5 \) such that if \( n > n_5 \), the following holds. Let \( M + 1 \leq k < M + n \), \( X \subset \mathcal{G}(k) \) and \( Y = X \cap \mathcal{B} \), \( Z = X - Y \). Suppose that \(|Z| > n|Y|\) or \(|Y| > n|Z|\). Then \(|\triangle(X)| > |X|\).

**Proof.**

If \(|X| \geq (\frac{m - n/2}{k})\), we are done by Lemma 13. So we can suppose that \(|X| < (\frac{m - n/2}{k})\).

Firstly, consider the case when \(|Y| > n|Z|\). Let \( Y = \binom{y}{k} \), then \( y < m - n/2 \). As \( \partial(Y) = \triangle(Y) \), we can apply Lemma 12 to get

\[
\frac{|\triangle(Y)|}{|Y|} > \frac{k}{m - n/2 - k} \geq 1 + \frac{2}{n}.
\]

Hence, \(|\triangle(X)| \geq |\triangle(Y)| > |Y| + 2|Y|/n > |Y| + |Z|\).

Now consider the case when \(|Z| > n|Y|\). Let \( |Z| = \binom{z}{k} \), then \( z \geq k\) \( = n^2/4 + O(n)\).

Set \( \epsilon = 1/40 \) and \( r = \left\lceil \frac{2n}{3} \right\rceil \). We chose \( r \) and \( \epsilon \) such that \( z > (\frac{n - (r+1)/2}{2}) + \epsilon \) \( r \) holds. Hence, by Lemma 15 we have

\[
\frac{|\triangle(Z)|}{|Z|} \geq 1 + \frac{4 - 4r/n}{n} \geq 1 + \frac{4}{3n},
\]

for \( n \) sufficiently large. Estimating the size of the shadow of \( X \) with \(|\triangle(Z)|\), we get

\[
|\triangle(X)| \geq |\triangle(Z)| \geq |Z| + \frac{4|Z|}{3n} \geq |Z| + |Y| = |X|.
\]

\[
\square
\]

We also need the following technical lemma, which tells us what conditions need to be satisfied for the sizes of the shadows of \( Y, Z \) to have \(|X| < |\triangle(X)|\).
Lemma 17. Let $a, b, A_1, A_2, B, c_1, c_2, c_3$ be positive reals such that $A_1 = a(1 - c_1)$, $A_2 = a(1 + c_2)$ and $B = (1 + c_3)$. If $c_1 \leq c_2c_3$, then

$$a + b \leq A_1 + \max\{B, A_2 - A_1\}.$$ 

Proof. We need to show that $ac_1 + b < \max\{B, A_2 - A_1\}$. Observe that we can suppose that $B = A_2 - A_1$. If $B < A_2 - A_1$, we can substitute $b$ with $b' > b$ and $B$ with $B' = (1 + c_3)b'$ satisfying $B' = A_2 - A_1$. Then the left hand side of the inequality increases, while the right hand side does not change. We can proceed similarly if $A_2 - A_1 < B$.

If $B = A_2 - A_1$, then $b = \frac{c_1 + c_2}{1 + c_3}a$. Hence, our inequality becomes

$$ac_1 + \frac{c_1 + c_2}{1 + c_3}a \leq (c_1 + c_2)a.$$ 

Simplifying this inequality, we get that it is equivalent with $c_1 \leq c_2c_3$. 

Now we are ready to show the existence of a complete matching between the levels close to the middle level.

Theorem 18. There exists $n_0$ such that if $n \geq n_0$, the following holds. If $M + 1 \leq k < M + n$, then there exists a complete matching from $G^{(k)}$ to $G^{(k-)}$.

Proof. By Hall’s theorem it is enough to prove that for any $X \subset G^{(k)}$ we have $|X| \leq |\triangle(X)|$. Fix $\epsilon = 1/18$. Let $|X| = \binom{n}{k}^2$. By Lemma 13, if $x > \binom{n-1}{2} + \epsilon$, we are done if $n \geq n_2$. Now suppose that $x \leq \binom{n-1}{2} + \epsilon$. Let $Y = X \cap \mathcal{B}$ and $Z = X - Y$. Let $|Y| = \binom{y}{k}$, $|Z| = \binom{z}{k}$ and suppose that $n > n_5$. By Lemma 16 if $|Y| > n|Z|$ or $|Z| > n|Y|$, we are done. Hence, we can suppose that $x - en < y, z \leq x$, if $n$ is sufficiently large.

Let $U = \triangle(Y) \cap \mathcal{B}$ and $|Y| = \binom{y}{k}$. Let $r$ be a positive integer satisfying $\binom{n-(r+1)/2}{2} + \epsilon(r+1)n \leq x < \binom{n-r/2}{2} + \epsilon rn$. One can easily check that as $k \leq x < \binom{n-1}{2} + \epsilon$ and $\epsilon < 1/4$, such an $r$ always exists, it is unique and $r < n$. Also, $y, z > \binom{n-(r+1)/2}{2} + \epsilon rn$.

By Lemma 14 if $n > n_3(\epsilon)$, we have

$$\frac{|U|}{|Y|} > 1 - \frac{4r}{n^2}.$$ 

Also, by Lemma 12

$$\frac{|\triangle(Y)|}{|Y|} > \frac{k}{y-k+1} > \frac{k}{(n-r/2) + 2\epsilon rn - k} > \frac{m/2}{(n-r/2) + 2\epsilon rn - m/2} = \frac{1}{1 - r(2n-1)/n(n-1) + r^2/2n(n-1) + 8\epsilon r/(n-1)} > \frac{1}{1 - 2r/n + r^2/2n^2 + 9\epsilon r/n},$$

where the last inequality holds if $n$ is sufficiently large.

Finally, by Lemma 15 if $n > n_4(\epsilon)$, we have

$$\frac{|\triangle(Z)|}{|Z|} > 1 + \frac{4 - 4r/n}{n}.$$
Figure 2: The comparability graph between $X$ and its shadow

Now we are ready to estimate $|\triangle(X)|$. We have

$$\triangle(X) = U \cup ((\triangle(Y) \setminus U) \cup \triangle(Z)),$$

where $\cup$ denotes disjoint union. Hence,

$$|\triangle(X)| \geq |U| + \max\{|\triangle(Y)| - |U|, |Z|\}.$$

Also, $|X| = |Y| + |Z|$. Let $c_1 = 4r/n^2$, $c_2 = 2r/n - r^2/2n^2 - 9\epsilon r/n$ and $c_3 = \frac{4 - 4r/n}{n}$. We have $|U| > (1 - c_1)|Y|$, $|\triangle(Y)| > (1 + c_2)|Y|$ and $|\triangle(Z)| > (1 + c_3)|Z|$. Hence, by Lemma 17 our task is reduced to proving that $c_1 \leq c_2 c_3$. Namely,

$$\frac{4r}{n^2} \leq \frac{2r/n - r^2/2n^2 - 9\epsilon r/n}{1 - 2r/n + r^2/2n^2 + 9\epsilon r/n} \frac{4 - 4r/n}{n}.$$

Simplifying this inequality, we get

$$1 - 2r/n + r^2/2n^2 + 9\epsilon r/n \leq (2 - r/2n - 9\epsilon)(1 - r/n).$$

For simplicity, let $\alpha = r/n$. The previous inequality writes as

$$1 - 2\alpha + \alpha^2/2 + 9\epsilon \alpha \leq (2 - \alpha/2 - 9\epsilon)(1 - \alpha),$$

which reduces to

$$\alpha + 18\epsilon \leq 2.$$

As $\alpha < 1$ and $\epsilon = 1/18$, this inequality holds. Hence, if $n$ is sufficiently large, we have $|X| \leq |\triangle(X)|$.

Proof of Theorem 2. Let $n > n_6$, where $n_6$ is the constant given in Lemma 18. By Lemma 9 it is enough to prove that for $k = 1, ..., M - 1$ there is a complete matching from $G^{(k)}$ to $G^{(k+1)}$, and for $k = M + 1, ..., m$, there is a complete matching from $G^{(k)}$ to $G^{(k-1)}$. But we proved exactly this statement in Lemma 10, Lemma 11 and Theorem 18. □
4 Open problems

In this section, we propose several open problems.

The first problem we propose is inspired by the question investigated in [6], which we mentioned in the Introduction. Let $G$ be a connected graph and let $C'(G)$ be the family of subgraphs of $G$ that are connected on the vertex set $V(G)$. Define the partial ordering $< \circ E$ on $C'(G)$ as usual: $H < H'$ if $E(H) \subset E(H')$.

**Question 19.** Let $G$ be a connected graph. Is $(C'(G), <)$ Sperner?

We believe that there should be graphs $G$ for which $(C'(G), <)$ is not Sperner. Unfortunately, even for small graphs, it is difficult to check this property.

We also propose another variation of Question 1. Let $P$ be a graph property (a family of graphs closed under isomorphism) and let $P_n$ denote the family of graphs in $P$ with vertex set $[n]$. Also, for $k = 0, \ldots, \binom{n}{2}$ let $P_n^{(k)}$ be the set of graphs in $P_n$ with $k$ edges. Define the partial ordering $<$ on $P_n$ as usual. The poset $(P_n, <)$ might not be graded, however it still makes sense to ask the following question. For which graph properties $P$ is it true that the largest antichain in $(P_n, <)$ is $P_n^{(k)}$ for some $k$. To ask a more specific question, we propose the following problem.

**Question 20.** Let $P$ be the family of Hamiltonian graphs. Is $(P_n, <)$ Sperner?

Finally, we suggest the following variation of Question 1. Suppose we do not distinguish graphs that are isomorphic. More precisely, define the equivalence relation $\sim$ on $G$ such that $G \sim H$ if $G$ and $H$ are isomorphic, and let $G_0$ be the set of equivalence classes of $G$. Define $<$ on $G_0$ such that for $\bar{G}, \bar{H} \in G_0$ we have $\bar{G} < \bar{H}$ if there exists $G \in \bar{G}$ and $H \in \bar{H}$ satisfying $G < H$ in $(\bar{G}, <)$.

**Question 21.** Is $(G_0, <)$ Sperner?

We conjecture that the answer should be yes to this question.

5 Appendix

Let $k$ be a positive integer and $x$ be a positive real number. We define

\[
\binom{x}{k} = \begin{cases} 
\frac{x(x-1)\ldots(x-k+1)}{k!} & \text{if } k \geq x \\
0 & \text{if } k < x.
\end{cases}
\]

The following properties of $\binom{x}{k}$ can be easily proved.

1. If $x \geq k$, we have $\binom{x}{k-1}/\binom{x}{k} = \frac{k}{x-k+1}$.

2. Let $\delta > 0$ and suppose that $k \leq x \leq 2k - \delta$. Then $\binom{x+\delta}{k} > 2^\delta \binom{x}{k}$.

3. Let $x \geq k$. We have $\binom{x+1}{k} < x\binom{x}{k}$, and $\binom{x}{k} < x\binom{x}{k+1}$.

4. $\binom{x}{k} \leq \binom{x+1}{k}$ and $\binom{x}{k+1} \leq \binom{x+1}{k+1}$.
References

[1] I. Anderson, *Combinatorics of Finite Sets*, Oxford University Press (1987).

[2] J. Balogh, R. Mycroft, A. Treglown, *A random version of Sperner’s theorem*, Journal of Combinatorial Theory, Series A, 128 (2014), 104-110.

[3] B. Bollobás, *Random Graphs*, Cambridge University Press (2001).

[4] B. Bollobás, *Modern Graph Theory*, Springer (1998).

[5] P. Hall, *On Representatives of Subsets*, J. London Math. Soc., 10 (1) (1935): 26-30.

[6] M. S. Jacobson, A. E. Kézdy, S. Seif, *The poset on connected induced subgraphs of a graph need not be Sperner*, Order, 12 (3) (1995): 315-318.

[7] Gy. O. H. Katona, *Personal communication*.

[8] Gy. O. H. Katona, *A theorem of finite sets*, Theory of Graphs, Akadémia Kiadó, Budapest (1968): 187-207.

[9] J. B. Kruskal, *The number of simplicies in a complex*, Mathematical Optimization Techniques, Univ. of California Press (1963): 251-278.

[10] L. Lovász, *Combinatorial Problems and Exercises*, North-Holland, Amsterdam (1993).

[11] E. Sperner, "Ein Satz über Untermengen einer endlichen Menge", Mathematische Zeitschrift (in German), 27 (1) (1928): 544-548.