TORUS QUOTIENT OF RICHARDSON VARIETIES IN ORTHOGONAL AND SYMPLECTIC GRASSMANNIANS

ARPITA NAYEK AND S.K. PATTANAYAK

Abstract. For any simple, simply connected algebraic group $G$ of type $B, C$ and $D$ and for any maximal parabolic subgroup $P$ of $G$, we provide a criterion for a Richardson variety in $G/P$ to admit semistable points for the action of a maximal torus $T$ with respect to an ample line bundle on $G/P$.

Keywords: Schubert variety, Richardson variety, Semi-stable point, Line bundle.

2010 Mathematics Subject Classification: 14F15; 20G05; 22E45.

1. Introduction

For the action of a maximal torus $T$ on the Grassmannian $G_{r,n}$, the GIT quotients have been studied by several authors. In [7] Hausmann and Knutson identified the GIT quotient of the Grassmannian $G_{2,n}$ by the natural action of the maximal torus with the moduli space of polygons in $\mathbb{R}^3$ and this GIT quotient can also be realized as the GIT quotient of an $n$-fold product of projective lines by the diagonal action of $PSL(2, \mathbb{C})$. In the symplectic geometry literature these spaces are known as polygon spaces as they parameterize the $n$-sides polygons in $\mathbb{R}^3$ with fixed edge length up to rotation. More generally, $G_{r,n}/T$ can be identified with the GIT quotient of $(\mathbb{P}^{r-1})^n$ by the diagonal action of $PSL(r, \mathbb{C})$ called the Gelfand-MacPherson correspondence. In [13] and [14] Kapranov studied the Chow quotient of the Grassmannians and he showed that the Grothendieck-Knudsen moduli space $\overline{M}_{0,n}$ of stable $n$-pointed curves of genus zero arises as the Chow quotient of the maximal torus action on the Grassmannian $G_{2,n}$.

Let $G$ be a simply connected semi-simple algebraic group over an algebraically closed field $K$. Let $T$ be a maximal torus of $G$ and $B$ be a Borel subgroup of $G$ containing $T$. In [8] and [9], the parabolic subgroups $P$ of $G$ containing $B$ are described for which there exists an ample line bundle $\mathcal{L}$ on $G/P$ such that the semistable points $(G/P)^{ss}_T(\mathcal{L})$ are the same as the stable points $(G/P)^s_T(\mathcal{L})$. In [25] Strickland reproved these results.

In [11], when $G$ is of type $A$, $P$ is a maximal parabolic subgroup of $G$ and $\mathcal{L}$ is the ample generator of the Picard group of $G/P$, it is shown that there exists unique minimal Schubert variety $X(w)$ admitting semistable points with respect to $\mathcal{L}$. For other types of classical groups the minimal Schubert varieties admitting semistable points were described in [12] and [21].
A Richardson variety $X_w$ in $G/P$ is the intersection of the Schubert variety $X_v$ in $G/P$ with the opposite Schubert variety $X^v$ therein. For $G = SL_n$ and $P$ a maximal parabolic in $G$ a criterion for the Richardson varieties in $G/P$ to have nonempty semistable locus with respect to an ample line bundle $\mathcal{L}$ on $G/P$ is given in [10]. In this paper, we give a criterion for a Richardson variety in $G/P$ to have nonempty semistable locus with respect to the action of a maximal torus $T$ on $G/P$, where $G$ is of type $B$, $C$ and $D$ and $P$ is a maximal parabolic subgroup in $G$.

The organisation of the paper is as follows. Section 2 consists of preliminary notions and some terminologies from algebraic groups and Geometric invariant theory. Section 3 gives a necessary condition for a Richardson variety to admit a semistable point. In section 4 we give a sufficient condition for a Richardson variety in type $B$ and $C$ to admit a semistable point and in section 5 a sufficient condition is given for type $D$.

2. Preliminaries and notation

In this section, we set up some notation and preliminaries. We refer to [3], [5], [6] and [24] for preliminaries in Lie algebras and algebraic groups. Let $G$ be a semi-simple algebraic group over an algebraically closed field $K$. We fix a maximal torus $T$ of $G$ and a Borel subgroup $B$ of $G$ containing $T$. Let $U$ be the unipotent radical of $B$. Let $N_G(T)$ (respectively, $W = N_G(T)/T$) be the normalizer of $T$ in $G$ (respectively, the Weyl group of $G$ with respect to $T$). Let $B^-$ be the Borel subgroup of $G$ opposite to $B$ determined by $T$. We denote by $R$ the set of roots with respect to $T$ and we denote by $R^+$ the set of positive roots with respect to $B$. Let $U_\alpha$ denote the one-dimensional $T$-stable subgroup of $G$ corresponding to the root $\alpha$ and we denote $U_\alpha^*$ by the open set $U_\alpha \setminus \{\text{identity}\}$. Let $S = \{\alpha_1, \ldots, \alpha_l\} \subseteq R^+$ denote the set of simple roots and for a subset $I \subseteq S$ we denote by $P_I$ the parabolic subgroup of $G$ generated by $B$ and $\{n_\alpha : \alpha \in I^c\}$, where $n_\alpha$ is a representative of $s_\alpha$ in $N_G(T)$. Let $W^I = \{w \in W : w(\alpha) \in R^+ \text{ for each } \alpha \in I^c\}$ and $W_I$ be the subgroup of $W$ generated by the simple reflections $s_\alpha$, $\alpha \in I^c$. Then every $w \in W$ can be uniquely expressed as $w = w^Iw_I$, with $w^I \in W^I$ and $w_I \in W_I$. Denote by $w_0$ the longest element of $W$ with respect to $S$. Let $X(T)$ (respectively, $Y(T)$) denote the group of all characters of $T$ (respectively, one-parameter subgroups of $T$). Let $E_1 := X(T) \otimes \mathbb{R}$ and $E_2 = Y(T) \otimes \mathbb{R}$. Let $\langle \ldots \rangle : E_1 \times E_2 \to \mathbb{R}$ be the canonical non-degenerate bilinear form. Let $\{\lambda_j : j = 1, 2, \ldots, l\} \subset E_2$ be the basis of $E_2$ dual to $S$. That is, $\langle \alpha_i, \lambda_j \rangle = \delta_{ij}$ for all $1 \leq i, j \leq l$ and let $C := \{\lambda \in E_2|\langle \alpha, \lambda \rangle \geq 0 \forall \alpha \in R^+\}$. Note that for each $\alpha \in R$, there is a homomorphism $SL_2 \xrightarrow{\phi_{\alpha}} G$ (see [2, p.19]). We have $\hat{\alpha} : G_m \to G$ defined by $\hat{\alpha}(t) = \phi_{\alpha}(t \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix})$. We also have $s_\alpha(\chi) = \chi - \langle \chi, \hat{\alpha} \rangle \alpha$ for all $\alpha \in R$ and $\chi \in E_1$.

Set $s_i = s_{\alpha_i}$ for every $i = 1, 2, \ldots, l$. Let $\{\omega_i : i = 1, 2, \ldots, l\} \subset E_1$ be the fundamental weights; i.e. $\langle \omega_i, \hat{\alpha}_j \rangle = \delta_{ij}$ for all $i, j = 1, 2, \ldots, l$.

Let $X_w = BUwB/B$ (respectively, $X^v = B^-vB/B$) denote the Schubert variety corresponding to $w$ (respectively, the opposite Schubert variety corresponding to $v$). Let $X_v := BuB/B \cap B^-vB/B$ denote the Richardson variety corresponding to $v$ and $w$ where $v \leq w$ in the Bruhat order. Such varieties were first considered by Richardson in [22].
who shows that such intersections are reduced and irreducible whereas the cell intersection \( C_w \cap C^n \) have been studied by Deodhar [1]. Richardson varieties have shown up in several contexts: such double coset intersections \( BwB \cap B^{-v}B \) first appear in [15], [16] and their standard monomial theory is studied in [17] and [2]. We refer to [18] for preliminaries in standard monomial theory.

We recall the definition of the Hilbert-Mumford numerical function and the definition of semistable points from [19]. We refer to [20] for notations in geometric invariant theory.

Let \( X \) be a projective variety with an action of a reductive group \( G \). A point \( x \in X \) is said to be semi-stable with respect to a \( G \)-linearized line bundle \( \mathcal{L} \) if there is a positive integer \( m \in \mathbb{N} \), and a \( G \)-invariant section \( s \in H^0(X, \mathcal{L}^m)^G \) with \( s(x) \neq 0 \).

Let \( \lambda \) be a one-parameter subgroup of \( G \). Let \( x \in \mathbb{P}(H^0(X, \mathcal{L})^* \) and \( \hat{x} = \sum_{i=1}^{r} v_i \), where each \( v_i \) is a weight vector of \( \lambda \) of weight \( m_i \). Then the Hilbert-Mumford numerical function is defined by

\[
\mu^\mathcal{L}(x, \lambda) := -\min\{m_i : i = 1, \ldots, r\}
\]

Then the Hilbert-Mumford criterion says that \( x \) is semistable if and only if \( \mu^\mathcal{L}(x, \lambda) \geq 0 \) for all one parameter subgroup \( \lambda \).

We recall the following result from [23] which will be used in section 3.

**Lemma 2.1.** Let \( G \) be a semisimple algebraic group, \( T \) be a maximal torus, \( B \) be a Borel subgroup of \( G \) containing \( T \) and \( \overline{\mathbb{C}} \) be as defined above.

(a) Let \( \mathcal{L} \) be a line bundle defined by the character \( \chi \in X(T) \). Then if \( x \in G/B \) is represented by \( bwB \), \( b \in B \) and \( w \in W \) is represented by an element of \( N \) in the Bruhat decomposition of \( G \) and \( \lambda \) is a one parameter subgroup of \( T \) which lies in \( \overline{\mathbb{C}} \), we have \( \mu^\mathcal{L}(x, \lambda) = -\langle w(\chi), \lambda \rangle \).

(b) Given any set \( S \) of finite number of one parameter subgroup \( \lambda \) of \( T \), there is an ample line bundle \( \mathcal{L} \) on \( G/B \) such that \( \mu^\mathcal{L}(x, \lambda) \neq 0 \) for all \( x \in G/B, \lambda \in S \).

In this paper, we present results for Richardson varieties in the orthogonal and symplectic Grassmannians. For any character \( \chi \) of \( B \), we denote by \( \mathcal{L}_\chi \), the line bundle on \( G/B \) given by the character \( \chi \). We denote by \( (X_w^v)^{ss}_T(\mathcal{L}_\chi) \) the semistable points of \( X_w^v \) for the action of \( T \) with respect to the line bundle \( \mathcal{L}_\chi \). Using the notations from [2] we recall the following theorem which is needed in the proofs of the main theorems.

**Theorem 2.2** ([2], Proposition 6). Let \( \lambda \) be a dominant weight. The restriction to \( X_w^v \) of the \( p_\pi \), where \( v \leq e(\pi) \leq i(\pi) \leq w \) form a basis of \( H^0(X_w^v, \mathcal{L}_\lambda) \).

In the rest of this section we recall Bruhat ordering in the Weyl groups of type \( B \), \( C \) and \( D \) and how it is related to the Bruhat order for the symmetric group \( S_n \).

**Bruhat order for type \( B_n \) or \( C_n \):** We consider \( \alpha_1 \) as special node of Dynkin diagram for type \( B \) or \( C \). So as a set of generators of Weyl group, we take \( S = \{s_1, s_2, \ldots, s_n\} \), where \( s_1 = (1, -1) \) and \( s_i = (i - 1, i) \forall 2 \leq i \leq n \) as in [1].
As in \cite{26} we use a formula for computing the length of $\sigma \in W$ given by

$$l_B(\sigma) = \frac{inv(\sigma) + neg(\sigma)}{2},$$

(2.1)

where \(inv(\sigma) = |\{(i, j) \in [-n, n] \times [-n, n] \setminus \{0\} : i < j, \sigma(i) > \sigma(j)\}|\) and \(neg(\sigma) = |\{i \in [1, n] : \sigma(i) < 0\}|\).

The following result gives a combinatorial characterization of the Bruhat order in $B_n$.

**Lemma 2.3** (\cite{26}, Proposition 2.8). Let $\sigma, \tau \in W$. Then $\sigma \leq \tau$ in the Bruhat order of $B_n$ if and only if $\sigma \leq \tau$ in the Bruhat order of the symmetric group $S_{[-n,n]\setminus\{0\}}$ where $S_{[-n,n]\setminus\{0\}}$ is the permutation group of integers $-n, -(n-1), \ldots, -1, 1, \ldots, n-1, n$.

**Bruhat order for type $D_n$**: As above we consider $\alpha_1$ as special node for Dynkin diagram for type $D$. For a set of generators of Weyl group we have $S = \{s_1, s_2, \ldots, s_n\}$, where $s_1 = (1, -2)(-1, 2)$ and $s_i = (i-1, i)$ $\forall 2 \leq i \leq n$ as in \cite{1}.

As in \cite{26} we use a formula for computing the length of $\sigma \in W$ given by

$$l_D(\sigma) = \frac{inv(\sigma) - neg(\sigma)}{2},$$

(2.2)

where \(inv(\sigma)\) and \(neg(\sigma)\) are as defined above.

The following result gives a combinatorial characterization of the Bruhat order in $D_n$.

**Lemma 2.4** (\cite{1}, Theorem 8.2.8). Let $\sigma, \tau \in W$. Then $\sigma \leq \tau$ in the Bruhat order of $D_n$ if and only if

(i) $\sigma \leq_B \tau$ (Bruhat order in type $B$) and

(ii) $\forall a, b \in [1, n]$, if $[-a, a] \times [-b, b]$ is an empty rectangle for both $\sigma$ and $\tau$ and $\sigma[-a - 1, b + 1] = \tau[-a - 1, b + 1]$, then \(\sigma[-1, b + 1] \equiv \tau[-1, b + 1] \mod 2\) where $\sigma[i, j] = |\{a \in [-n, n] : a \leq i \text{ and } \sigma(a) \geq j\}|$ for $i, j \in [-n, n]$.

3. A necessary condition for admitting semi-stable points

Let $G$ be a simple simply-connected algebraic group and $P_r$ be a parabolic subgroup of $G$ corresponding to the simple root $\alpha_r$. Let $L_r$ be the line bundle on $G/P_r$ corresponding to the fundamental weight $\omega_r$. In this section, we provide a criterion for Richardson varieties in $G/P_r$ to admit semistable points with respect to $L_r$. This criterion was proved for type $A$ in \cite{10}.

**Proposition 3.1.** Let $G$ be a simple simply connected algebraic group and let $P_r$ be the maximal parabolic corresponding to the simple root $\alpha_r$. Let $L_r$ be the line bundle on $G/P_r$ corresponding to the fundamental weight $\omega_r$. Let $v, w \in W^{P_r}$. If $(X^v_w)^s(L_r) \neq \emptyset$ then $v(n\omega_r) \geq 0$ and $w(n\omega_r) \leq 0$.

**Proof.** Let $\chi = n\omega_r$. Assume that $(X^v_w)^s(L_\chi) \neq \emptyset$. Let $x \in ((BwP_r/P_r) \cap (B^-vP_r/P_r))^s(L_\chi)$. Then by Hilbert-Mumford criterion \cite{19} Theorem 2.1, we have $\mu^\chi(x, \lambda) \geq 0$ for all one parameter subgroups $\lambda$ of $T$. Since $x \in ((BwP_r/P_r) \cap (B^-vP_r/P_r))^s(L_\chi)$, using
Lemma 2.1], we see that \( \mu^L(x, \lambda) = -\langle w(\chi), \lambda \rangle \) for every one parameter subgroup \( \lambda \) in the fundamental chamber associated to \( B \), and \( \mu^L(x, \lambda) = \mu^L(w_0x, w_0\lambda w_0^{-1}) = -\langle w_0v(\chi), w_0(\lambda) \rangle = -\langle v(\chi), \lambda \rangle \) for every one parameter subgroup \( \lambda \) of \( T \) in the Weyl chamber associated to \( B^- \). Since \( x \) is a semistable point, we have \( \mu^L(x, \lambda) \geq 0 \) for every one parameter subgroup \( \lambda \) of \( T \). Hence \( \langle w(\chi), \lambda \rangle \leq 0 \) for all \( \lambda \) in the Weyl chamber associated to \( B^- \). This implies that \( w(\chi) \leq 0 \) and \( v(\chi) \geq 0 \). \( \square \)

For \( G \) is of type \( A \) in [10] it is shown that the above conditions are also sufficient. For type \( B, C \) and \( D \) the example below shows that the conditions \( w(\chi) \leq 0 \) and \( v(\chi) \geq 0 \) are only necessary but not sufficient.

**Example:** Let \( G \) be either of type \( B_4 \) or \( C_4 \) and \( \chi = \omega_3 \). Let \( v = (1, 2, -3, 4) = s_3s_2s_1s_3 \) and \( w = (1, 4, -3, 2) = s_3s_2s_1s_3s_2s_3 \). We have \( v(\omega_3) = \alpha_4 \) and \( w(\omega_3) = -\alpha_3 \). The sections of \( L_\chi \) on \( X_w^v \) are of the form \( p_\chi P_w \) where \( m, n \in \mathbb{N} \). But, \( mw(\chi) + nw(\chi) \neq 0 \) for any \( m, n \in \mathbb{N} \). So these sections are not \( T \)-invariant. So the set \( (X_w^v)^{ss}(L_\chi) \) is empty.

If \( G \) is of type \( D_4 \) and \( \chi = \omega_3 \), we take \( v = (-1, 4, -2, 3) = s_4s_1s_2s_3 \) and \( w = (-1, 2, -4, 3) = s_4s_3s_1s_2s_3 \). Here we have \( v(\omega_3) = \alpha_3 \) and \( w(\omega_3) = -\alpha_4 \). As in the last paragraph, here also we conclude that the set \( (X_w^v)^{ss}(L_\chi) \) is empty.

In order to find a sufficient condition for the Richardson varieties to admit semistable points we first need to classify all \( v, w \in W^P \) satisfying the above conditions. Since \( \chi \) is a dominant weight we have \( w_1(\chi) \leq w_2(\chi) \) for \( w_1 \leq w_2 \). So we just need to describe all maximal \( v \) and minimal \( w \) such that \( v(\chi) \geq 0 \) and \( w(\chi) \leq 0 \). Note that for \( G \) is of type \( A \) since all the fundamental weights are minuscule the maximal \( v \) and minimal \( w \) satisfying the above conditions are unique (see [11]) but for other types this is not the case.

We conclude this section by introducing some notation here:

**Notation:** For \( s, t \in \mathbb{Z} \) such that \( s \leq t \) we set \( \{s, \ldots, t\} \). For \( p \in \mathbb{N} \) we set \( J_p = \{i_1, \ldots, i_p \} \) and \( \bar{J}_p = \{-i_p, -i_{p-1}, \ldots, -i_1 \} \). For a set \( S \subset \mathbb{Z} \), \( S \uparrow \) denotes the integers in the set \( S \) occurring in increasing order and \( S \downarrow \) denotes the integers in the set \( S \) occurring in decreasing order.

4. Type B and C

Now for \( G \) is of type \( B \) and \( C \) and for a fundamental weight \( \omega_r \), we are in a position to describe all the minimal \( w \in W^r \) and maximal \( v \in W^r \) such that \( v(\omega_r) \geq 0 \) and \( w(\omega_r) \leq 0 \).

**Proposition 4.1.** The set of all maximal \( v \in W^r \) such that \( v(\omega_r) \geq 0 \) for type \( B_n \) and \( C_n \) are the following:

(i) Let \( r = 1 \). Then

\[
v = \begin{cases} 
(-(n-1), -(n-3), \ldots, -3, -1, 2, 4, \ldots, n-2, n), & \text{if } n \text{ is even} \\
(-(n-1), -(n-3), \ldots, -4, -2, 1, 3, 5, \ldots, n-2, n), & \text{if } n \text{ is odd}.
\end{cases}
\]
(ii) Let \(2 \leq r \leq n - 1\) and \((n + 1) - r = 2m\). For any \(i = (i_1, i_2, \ldots, i_m) \in J_{m,\lfloor 2, n\rfloor}\), there exists unique maximal \(v_i \in W_{Ir}\) such that \(v_i(\omega_r) = (\sum_{k=1}^{m} \alpha_{i_k})\). We have \(v_i = ([1, n] \setminus \{[\hat{i}], [\hat{i}']\}) \uparrow, -[\hat{i}', \hat{i}]\), where \(\hat{i}' = (i_1 - 1, i_2 - 1, \ldots, i_m - 1) \in J_{m,\lfloor 1, n-1\rfloor}\).

(iii) Let \(2 \leq r \leq n - 1\) and \((n + 1) - r = 2m + 1\). For any \(i = (i_1, i_2, \ldots, i_m) \in J_{m,\lfloor 3, n\rfloor}\), there exists unique maximal \(v_i \in W_{Ir}\) such that \(v_i(\omega_r) = (\alpha_1 + \sum_{k=1}^{m} \alpha_{i_k})\) (for \(B_n\)) and \(v_i(\omega_r) = (\frac{1}{2} \alpha + \sum_{k=1}^{m} \alpha_{i_k})\) (for \(C_n\)). We have \(v_i = ([1, n] \setminus \{[\hat{i}], [\hat{i}']\}) \uparrow, -[\hat{i}', 1, \hat{i}]\), where \(\hat{i}' = (i_1 - 1, i_2 - 1, \ldots, i_m - 1) \in J_{m,\lfloor 2, n-1\rfloor}\).

(iv) Let \(r = n\). Then \(v = (2, 3, 4, \ldots, n - 1, n, 1)\).

We prove this proposition after proving the following lemma.

**Lemma 4.2.** Let \(v, v_i\) are as defined in Proposition 4.1.

(i) Let \(r = 1\). Then \(w > v\) and \(l(w) = l(v) + 1\) iff \(w = s_kv\) where \(k\) takes the following values \(\{k \in \{2, 4, 6, \ldots, n - 2, n\}, \ n\ is\ even\ \} \cup \{k \in \{1, 3, 5, \ldots, n - 2, n\}, \ n\ is\ odd\}\).

(ii) Let \(2 \leq r \leq n - 1\). Then \(w > v_i\) and \(l(w) = l(v_i) + 1\) if and only if \(w\) is either \(s_{\alpha_i}v_i\) or \(s_{\alpha_i + \alpha_{i+1}}v_i\) for some \(i\) such that \(|i_t - i_{t+1}| \geq 3\) or \(s_{\alpha_i + \alpha_{i-1}}v_i\) for some \(i\) such that \(|i_t - i_{t-1}| \geq 3\).

(iii) Let \(r = n\). Then \(w > v\) and \(l(w) = l(v) + 1\) iff \(w = s_1v\).

**Proof.** We will prove this lemma for case (ii) and \((n + 1) - r = 2m\). For other cases the proof is similar.

Let \(w > v_i\) and \(l(w) = l(v_i) + 1\). Then \(w = s_{\beta}v_i\), for some positive root \(\beta\) such that height of \(\beta\) is less than or equal to 2. So, \(\beta\) is either \(\alpha_j\) or \(\alpha_j + \alpha_{j+1}\) for some simple root \(\alpha_j\).

**Case 1:** \(\beta = \alpha_j\).

If \(\beta = \alpha_i\), then \(s_{\beta}v_i(\omega_r) = \sum_{k \neq i} \alpha_{i_k} - \alpha_{i_t}\). Since \(s_{\beta}v_i(\omega_r) < v_i(\omega_r)\) and \(\omega_r\) is a dominant weight we have \(s_{\beta}v_i > v_i\). Since \(\beta\) is a simple root we have \(l(s_{\beta}v_i) = l(v_i) + 1\).

If \(\beta = \alpha_j, j \neq i\), then \(s_{\beta}v_i(\omega_r) \geq v_i(\omega_r)\). So \(s_{\beta}v_i \leq v_i\) in \(W_{Ir}\), a contradiction.

**Case 2:** \(\beta = \alpha_j + \alpha_{j+1}\).

If \(j = i_t\) and \(|i_t + 1 - i_t| \geq 3\) then \(s_{\beta}v_i(\omega_r) = \sum_{k \neq i_t} \alpha_{i_k} - \alpha_{i_t+1}\). So \(s_{\beta}v_i(\omega_r) < v_i(\omega_r)\) and hence \(s_{\beta}v_i > v_i\). Now we will show that \(l(s_{\beta}v_i) = l(v_i) + 1\) for this \(\beta\).

Note that \(s_{\beta}v_i = ([i, \hat{i}]\downarrow, [\hat{i}, \hat{i}']\uparrow, \{1, n\} \setminus \{[\hat{i}], [\hat{i}']\} \uparrow, -[\hat{i}', \hat{i}]\) where \(\hat{i}' = (i_1 - 1, i_2 - 1, \ldots, i_{t-1} - 1, i_t + 1, i_{t+1} - 1, \ldots, i_m - 1)\). In \(v_i\), the position of \(i_t + 1\) is right to the position of \(i_t - 1\) and left to \(i_t\) but in \(s_{\beta}v_i\) the position of \(i_t\) remains unchanged and the positions of \(i_t + 1\) and \(i_t - 1\) are interchanged. Similarly in \(v_i\) the position of \(-i_t - 1\) is right to \(-i_t\) and left to \(-i_t + 1\) but in \(s_{\beta}v_i\) the position of \(-i_t\) remains unchanged and the positions of \(-i_t - 1\) and \(-i_t + 1\) are interchanged. So \(inv(s_{\beta}v_i) = inv(v_i) + 2\). Hence \(l(s_{\beta}v_i) = l(v_i) + 1\).
If \( j = i_t \) and \(|i_{t+1} - i_t| = 2\) then since \( s_\beta v_\omega = v_\omega \), we have \( s_\beta v_\omega = v_\omega \) in \( W^r \), a contradiction.

If \( j, j + 1 \neq i_t \) then \( s_\beta v_\omega = v_\omega \). So \( s_\beta v_\omega \leq v_\omega \) in \( W^r \), a contradiction.

**Case 3.** \( \beta = \alpha_{j-1} + \alpha_j \). In this case the proof is similar to the previous case.

The converse part is clear from the definition of \( v_\omega \). \( \square \)

**Proof of proposition 4.1:** We prove case (ii). The proofs of other cases are similar.

We prove that for any \( \omega \in J_{m,[2,n]} \) there exists \( v_\omega \in W^r \) such that \( v_\omega = \sum_{k=1}^m \alpha_{i_k} \).

Note that,

\[
\omega_r = 2m(\alpha_1 + \alpha_2 + \cdots + \alpha_r) + \sum_{i=1}^{2m-1} (2m - i)\alpha_{r+i}, 2 \leq r \leq n - 1.
\]

Now consider the partial order on \( J_{m,[2,n]} \), given by \((i_1, i_2, \ldots, i_m) \leq (j_1, j_2, \ldots, j_m)\) if \( i_k \leq j_k, \forall k \) and \((i_1, i_2, \ldots, i_m) < (j_1, j_2, \ldots, j_m)\) if \( i_k \leq j_k \forall k \) and \( i_k < j_k \) for some \( k \). We will prove the theorem by induction on this order.

For \((j_1, j_2, \ldots, j_m) = (n - (2m - 2), n - (2m - 4), \ldots, n - 2, n)\), we have \( v_\omega = \sum_{k=1}^m \alpha_{i_k} \).

Now consider the partial order on \( J_{m,[2,n]} \), given by \((i_1, i_2, \ldots, i_m) \leq (j_1, j_2, \ldots, j_m)\) if \( i_k \leq j_k, \forall k \) and \((i_1, i_2, \ldots, i_m) < (j_1, j_2, \ldots, j_m)\) if \( i_k \leq j_k \forall k \) and \( i_k < j_k \) for some \( k \). We will prove the theorem by induction on this order.

For \((j_1, j_2, \ldots, j_m) = (n - (2m - 2), n - (2m - 4), \ldots, n - 2, n)\), we have \( v_\omega = \sum_{k=1}^m \alpha_{i_k} \).

Now consider the partial order on \( J_{m,[2,n]} \), given by \((i_1, i_2, \ldots, i_m) \leq (j_1, j_2, \ldots, j_m)\) if \( i_k \leq j_k, \forall k \) and \((i_1, i_2, \ldots, i_m) < (j_1, j_2, \ldots, j_m)\) if \( i_k \leq j_k \forall k \) and \( i_k < j_k \) for some \( k \). We will prove the theorem by induction on this order.

For \((j_1, j_2, \ldots, j_m) = (n - (2m - 2), n - (2m - 4), \ldots, n - 2, n)\), we have \( v_\omega = \sum_{k=1}^m \alpha_{i_k} \).

Now consider the partial order on \( J_{m,[2,n]} \), given by \((i_1, i_2, \ldots, i_m) \leq (j_1, j_2, \ldots, j_m)\) if \( i_k \leq j_k, \forall k \) and \((i_1, i_2, \ldots, i_m) < (j_1, j_2, \ldots, j_m)\) if \( i_k \leq j_k \forall k \) and \( i_k < j_k \) for some \( k \). We will prove the theorem by induction on this order.

For \((j_1, j_2, \ldots, j_m) = (n - (2m - 2), n - (2m - 4), \ldots, n - 2, n)\), we have \( v_\omega = \sum_{k=1}^m \alpha_{i_k} \).

Now consider the partial order on \( J_{m,[2,n]} \), given by \((i_1, i_2, \ldots, i_m) \leq (j_1, j_2, \ldots, j_m)\) if \( i_k \leq j_k, \forall k \) and \((i_1, i_2, \ldots, i_m) < (j_1, j_2, \ldots, j_m)\) if \( i_k \leq j_k \forall k \) and \( i_k < j_k \) for some \( k \). We will prove the theorem by induction on this order.

For \((j_1, j_2, \ldots, j_m) = (n - (2m - 2), n - (2m - 4), \ldots, n - 2, n)\), we have \( v_\omega = \sum_{k=1}^m \alpha_{i_k} \).

Now consider the partial order on \( J_{m,[2,n]} \), given by \((i_1, i_2, \ldots, i_m) \leq (j_1, j_2, \ldots, j_m)\) if \( i_k \leq j_k, \forall k \) and \((i_1, i_2, \ldots, i_m) < (j_1, j_2, \ldots, j_m)\) if \( i_k \leq j_k \forall k \) and \( i_k < j_k \) for some \( k \). We will prove the theorem by induction on this order.

For \((j_1, j_2, \ldots, j_m) = (n - (2m - 2), n - (2m - 4), \ldots, n - 2, n)\), we have \( v_\omega = \sum_{k=1}^m \alpha_{i_k} \).

Now consider the partial order on \( J_{m,[2,n]} \), given by \((i_1, i_2, \ldots, i_m) \leq (j_1, j_2, \ldots, j_m)\) if \( i_k \leq j_k, \forall k \) and \((i_1, i_2, \ldots, i_m) < (j_1, j_2, \ldots, j_m)\) if \( i_k \leq j_k \forall k \) and \( i_k < j_k \) for some \( k \). We will prove the theorem by induction on this order.

For \((j_1, j_2, \ldots, j_m) = (n - (2m - 2), n - (2m - 4), \ldots, n - 2, n)\), we have \( v_\omega = \sum_{k=1}^m \alpha_{i_k} \).

Now consider the partial order on \( J_{m,[2,n]} \), given by \((i_1, i_2, \ldots, i_m) \leq (j_1, j_2, \ldots, j_m)\) if \( i_k \leq j_k, \forall k \) and \((i_1, i_2, \ldots, i_m) < (j_1, j_2, \ldots, j_m)\) if \( i_k \leq j_k \forall k \) and \( i_k < j_k \) for some \( k \). We will prove the theorem by induction on this order.

For \((j_1, j_2, \ldots, j_m) = (n - (2m - 2), n - (2m - 4), \ldots, n - 2, n)\), we have \( v_\omega = \sum_{k=1}^m \alpha_{i_k} \).
Proposition 4.3. The set of all minimal \( w \) in \( W^r \) such that \( w(\omega_r) \leq 0 \) for type \( B_n \) and \( C_n \) are the following:

(i) Let \( r = 1 \). Then

\[
w = \begin{cases} 
    (n, -(n-2), \ldots, -2, 1, 3, \ldots, n-3, n-1), & \text{if } n \text{ is even} \\
    (n, -(n-2), \ldots, -3, -1, 2, 4, \ldots, n-3, n-1), & \text{if } n \text{ is odd.}
\end{cases}
\]

(ii) Let \( 2 \leq r \leq n - 1 \). For \( (n + 1) - r = 2m, i \in J_{m, [2, n]} \) and \( v_i(\omega_r) = \sum_{k=1}^{m} \alpha_{ik} \), we have \( w_\uparrow = s_{i_1} s_{i_2} \ldots s_{i_m} v_\uparrow = ([1, n]\{i, j\} \uparrow, -i, -j) \) where \( j' = (i_1 - 1, i_2 - 1, \ldots, i_m - 1) \). In this case \( w_\uparrow(\omega_r) = -v_i(\omega_r) \).

(iii) Let \( 2 \leq r \leq n - 1 \). For \( (n + 1) - r = 2m + 1, i \in J_{m, [3, n]} \) and \( v_i(\omega_r) = \alpha_1 + \sum_{k=1}^{m} \alpha_{ik} \), we have \( w_\uparrow = s_{i_1} s_{i_2} \ldots s_{i_m} v_\uparrow = ([1, n]\{i, j\} \uparrow, -i, -j) \), where \( j' = (i_1 - 1, i_2 - 1, \ldots, i_m - 1) \) \( J_{m, [2, n-1]} \). In this case also \( w_\uparrow(\omega_r) = -v_i(\omega_r) \).

(iv) For \( r = n \), \( w = (2, 3, \ldots, n - 1, n, 1) \).

Proof. For the proof of minimality of \( w \) refer to [12].

Proposition 4.4. Let \( v, w, v_\downarrow \), and \( w_\downarrow \) be as stated in Proposition 4.1 and Proposition 4.3 respectively.

(i) For \( 2 \leq r \leq n - 1 \), \( X^w_{v_\downarrow} \) is nonempty if and only if \( |i_k - j_k| \leq 1 \) \( \forall 1 \leq k \leq m \).

(ii) For \( r = 1, n \), \( X^w_{v_\downarrow} \) is not empty for any \( v \) and \( w \).

Proof. We will prove this lemma for \( 2 \leq r \leq n - 1 \) and \( (n + 1) - r = 2m \). For other cases the proof is similar.

Let \( X^w_{v_\downarrow} \) be nonempty. So \( v_\downarrow < w_\downarrow \). Since \( l(v_\downarrow) = l(v_\downarrow) \) we have \( l(w_\downarrow) - l(v_\downarrow) = m \). By the repeated application of Lemma 4.2 we have \( w_\downarrow = \prod_{\text{card}(\beta)=m} s_\beta v_\downarrow \) for some \( \beta \) such that \( \beta \)'s is either \( \alpha_i \), or \( \alpha_i + \alpha_{i+1} \) or \( \alpha_i + \alpha_{i-1} \) for some \( i \). So \( w_\downarrow(\omega_r) = -\sum_{j_k:|i_k-j_k| \leq 1} \alpha_{j_k} \).

Hence, \( |i_k - j_k| \leq 1 \) \( \forall 1 \leq k \leq m \).

Conversely, let \( |i_k - j_k| \leq 1 \) for all \( 1 \leq k \leq m \). So, \( w_\downarrow = \prod_{t \in \{t: i_t = j_t\}} s_{\alpha_{i_t}} \prod_{t \in \{t: j_t = i_t - 1\}} s_{\alpha_{i_t} + \alpha_{i_t + 1}} \prod_{t \in \{t: j_t = i_t + 1\}} s_{\alpha_{i_t + 1} + \alpha_{i_t + 1}} v_\downarrow \). The arrows \( \uparrow \) and \( \downarrow \) denote that the reflections in the product are applied in increasing and decreasing order of \( t \) respectively. Since \( i, j \in J_{m, [2, n]} \) and \( |i_k - j_k| \leq 1 \), we see that the sets \( \{ \bigcup_{\{t: j_t = i_t + 1\}} \{i_t - 1, i_t, i_t + 1\} \} \), \( \{ \bigcup_{\{t: j_t = i_t - 1\}} \{i_t - 1, i_t - 1, i_t\} \} \) and \( \{ \bigcup_{\{t: j_t = i_t\}} \{i_t - 1, i_t\} \} \) are mutually disjoint.
In the first step we see that when we multiply \( v_1 \) by \( s_{\alpha_t+\alpha_t+1} \) in increasing order of \( t \in \{ t : j_t = i_t + 1 \} \) then after each multiplication the product is greater than \( v_1 \) and the length increases by 1. Let \( t \) be maximal such that \( j_t = i_t + 1 \). By lemma 4.2, \( s_{\alpha_t+\alpha_t+1}v_1 > v_1 \) and \( l(s_{\alpha_t+\alpha_t+1}v_1) = l(v_1) + 1 \).

Let \( t \) be such that \( i_t-1 = i_t - 2 \) and \( j_{t-1} = i_{t-1} + 1 \). Note that \( s_{\alpha_t+\alpha_t+1}v_1 = (-i, \hat{i}, [-n, -1], \{[\hat{i}], [-\hat{i}] \} \uparrow, [1, n] \{[\hat{i}], \hat{i} \} \uparrow, [-\hat{i}], \hat{i} \} \), where \( \hat{i} = (i_1 - 1, i_2 - 1, \ldots, i_t - 3, i_t + 1, i_t + 1, \ldots, i_{m-1}, 1 \) and \( s_{\alpha_{i_t-2}+\alpha_{i_t-1}}s_{\alpha_{i_t}+\alpha_{i_t+1}}v_1 = (-i, \hat{i}, [-n, -1], \{[\hat{i}], [-\hat{i}] \} \uparrow, [1, n] \{[\hat{i}], \hat{i} \} \uparrow, [-\hat{i}], \hat{i} \} \), where \( \hat{i} = (i_1 - 1, i_2 - 1, \ldots, i_t - 1, i_t + 1, i_t + 1, \ldots, i_{m-1}, 1 \).

In \( s_{\alpha_t+\alpha_t+1}v_1 \), the position of \( i_t - 3 \) is left to the positions of both \( i_t - 1 \) and \( i_t - 2 \) but in \( s_{\alpha_{i_t-2}+\alpha_{i_t-1}}s_{\alpha_{i_t}+\alpha_{i_t+1}}v_1 \), the position of \( i_t - 2 \) remains unchanged and the positions of \( i_t - 1 \) and \( i_t - 3 \) are interchanged. Similarly the positions of \( -i_t + 1 \) and \( -i_t + 2 \) are also interchanged in \( s_{\alpha_{i_t-2}+\alpha_{i_t-1}}s_{\alpha_{i_t}+\alpha_{i_t+1}}v_1 \). So \( inv(s_{\alpha_{i_t-2}+\alpha_{i_t-1}}s_{\alpha_{i_t}+\alpha_{i_t+1}}v_1) = inv(s_{\alpha_{i_t}+\alpha_{i_t+1}}v_1) + 2 \). Hence \( l(s_{\alpha_{i_t-2}+\alpha_{i_t-1}}s_{\alpha_{i_t}+\alpha_{i_t+1}}v_1) = l(s_{\alpha_{i_t}+\alpha_{i_t+1}}v_1) + 1 \). By lemma 2.3 we see that \( s_{\alpha_{i_t}+\alpha_{i_t+1}}v_1 < s_{\alpha_{i_t-2}+\alpha_{i_t-1}}s_{\alpha_{i_t}+\alpha_{i_t+1}}v_1 \).

Repeating this process we can see that \( \prod_{t \in \{ t : j_t = i_t + 1 \} \uparrow} s_{\alpha_t+\alpha_t+1}v_1 > v_1 \) and the length is increased by the number of reflections multiplied.

Since \( \bigcup_{\{ t : j_t = i_t + 1 \}} \{ i_t - 1, i_t, i_t + 1 \} \), \( \bigcup_{\{ t : j_t = i_t - 1 \}} \{ i_t - 2, i_t - 1, i_t \} \) and \( \bigcup_{\{ t : j_t = i_t \}} \{ i_t - 1, i_t \} \) are mutually disjoint, by repeating the above process we see that \( \prod_{t : j_t = i_t} s_{\alpha_t} \prod_{t : j_t = i_t - 1 \downarrow} s_{\alpha_t+\alpha_{t-1}} \)

\( \prod_{t : j_t = i_t \downarrow} s_{\alpha_t+\alpha_{t+1}}v_1 > v_1 \) and the length is increased by the number of reflections multiplied. So \( w_1 > v_1 \) and hence \( X_{w_1}^v \) is nonempty. \( \square \)

Remark: Note that \( \hat{i} \) denotes the positions of the simple roots with nonzero coefficients in \( w_1(\omega_r) \) and similarly, \( \hat{j} \) denotes the positions of the simple roots with nonzero coefficients in \( w_1(\omega_r) \).

**Theorem 4.5.** Let \( v, w, v_1 \) and \( w_1 \) be as stated in Proposition 4.1 and Proposition 4.3 respectively.

(i) For \( 2 \leq r \leq n - 1 \), \( (X_{w_2}^v)^{ss} (L_r) \) is nonempty if and only if \( \hat{i} = \hat{j} \).

(ii) For \( r = 1 \) and \( n \), \( (X_{w_1}^v)^{ss} (L_r) \) is non-empty for any \( v \) and \( w \).

**Proof.** We will prove this theorem for case (i). Proof of case (ii) is similar. Let \( \hat{i} = \hat{j} \). Then we have \( v_1(\omega_r) + w_1(\omega_r) = 0 \). So \( p_{v_1}w_1 \) is a non-zero \( T \)-invariant section of \( L_r \) on \( G/P_r \) which does not vanish identically on \( X_{w_2}^v \). Hence, \( (X_{w_2}^v)^{ss} (L_r) \) is non-empty.

Conversely, if \( \hat{i} \neq \hat{j} \), then there exists \( t \) such that \( j_t \neq i_t \). Since \( X_{w_1}^v \neq \emptyset \), by Proposition 4.4 we have \( j_t = i_t + 1 \) or \( j_t = i_t - 1 \). If \( j_t = i_t + 1 \) then \( w_1(\omega_r) = -\sum_{k \neq t} \alpha_i - \alpha_{i+1} \) and if \( j_t = i_t - 1 \) then \( w_1(\omega_r) = -\sum_{k \neq t} \alpha_i - \alpha_{i-1} \). Let \( u \in W_r^T \) be such that \( v_1 \leq u \leq w_1 \). Then
$u$ is of the form $u = \prod_{\beta} s_{\beta} v_2$, where $\beta$'s are some positive roots. For $j_t = i_t + 1$ at most one $\beta$ can be $\alpha_{i_t} + \alpha_{i_t+1}$ and none of the other $\beta$'s can contain $\alpha_{i_t}$ or $\alpha_{i_t+1}$ as a summand. So in $u(\omega_r)$, the coefficient of $\alpha_{i_t}$ is either zero or one and the coefficient of $\alpha_{i_t+1}$ is either zero or $-1$. Similarly for $j_t = i_t - 1$ at most one $\beta$ can be $\alpha_{i_t} + \alpha_{i_t-1}$ and none of the other $\beta$'s can contain $\alpha_{i_t}$ or $\alpha_{i_t-1}$ as a summand. So in $u(\omega_r)$ the coefficient of $\alpha_{i_t}$ is either zero or one and the coefficient of $\alpha_{i_t-1}$ is either zero or $-1$. For $j_t = i_t + 1$, $u(\omega_r)$ contains either $\alpha_{i_t}$ or $\alpha_{i_t+1}$ as a summand and for $j_t = i_t - 1$, $u(\omega_r)$ contains either $\alpha_{i_t}$ or $\alpha_{i_t-1}$ as a summand. Hence there does not exist a sequence $v_L = u_1 \leq u_2 \leq \ldots \leq u_k = w_L$ such that $\sum_{t=1}^k u_t(\omega_r) = 0$ and so we don’t have a non-zero $T$-invariant section of $L_r$ which is not identically zero on $X_{u_L}^{\nu_L}$. So, we conclude that the set $(X_{u_L}^{\nu_L})^s_{T}(L_r)$ is empty. \hfill \square

We illustrate Proposition 4.4 and Theorem 4.5 with an example.

**Example:** $B_5$, $\omega_4 = (2, 2, 2, 2, 1)$

| $i$ | $v_L$ | $v_L(\omega_2)$ | $w_L(\omega_2)$ | $w_L$ |
|-----|-------|-----------------|-----------------|-------|
| (2) | (3, 4, 5, -1, 2) | (0, 1, 0, 0, 0) | (0, -1, 0, 0, 0) | (3, 4, 5, -2, 1) |
| (3) | (1, 4, 5, -2, 3) | (0, 0, 1, 0, 0) | (0, 0, -1, 0, 0) | (1, 4, 5, -3, 2) |
| (4) | (1, 2, 5, -3, 4) | (0, 0, 0, 1, 0) | (0, 0, 0, -1, 0) | (1, 2, 5, -4, 3) |
| (5) | (1, 2, 3, -4, 5) | (0, 0, 0, 0, 1) | (0, 0, 0, 0, -1) | (1, 2, 3, -5, 4) |

So from the above observation, $X_{w_L}^{v_L(2)}$, $X_{w_L}^{v_L(3)}$, $X_{w_L}^{v_L(4)}$, $X_{w_L}^{v_L(5)}$, $X_{w_L}^{v_L(6)}$, and $X_{w_L}^{v_L(7)}$ are all non-empty. We have $(X_{w_L}^{v_L(2)})^s_T(L_4)$, $(X_{w_L}^{v_L(3)})^s_T(L_4)$, $(X_{w_L}^{v_L(4)})^s_T(L_4)$, $(X_{w_L}^{v_L(5)})^s_T(L_4)$, $(X_{w_L}^{v_L(6)})^s_T(L_4)$ and $(X_{w_L}^{v_L(7)})^s_T(L_4)$ are empty whereas $(X_{w_L}^{v_L(2)})^s_T(L_4)$, $(X_{w_L}^{v_L(3)})^s_T(L_4)$, $(X_{w_L}^{v_L(4)})^s_T(L_4)$ and $(X_{w_L}^{v_L(5)})^s_T(L_4)$ are non-empty.

5. **Type D**

As in types $B$ and $C$ here also for a fundamental weight $\omega_r$, we describe all the maximal $v \in W^{I_r}$ and minimal $w \in W^{I_r}$ such that $v(\omega_r) \geq 0$ and $w(\omega_r) \leq 0$ and then we use the same techniques to describe $v, w \in W^{I_r}$ for which the Richardson variety $X_w^v$ have nonempty semistable locus for the action of a maximal torus $T$ and with respect to the line bundle $L_r$.

**Proposition 5.1.** Let $G$ be of type $D_n$. Let $v \in W^{I_r}$ be maximal such that $v(\omega_r) \geq 0$. Then the description of $v$ is the following:

(i) For $r = 1$, $v(4\omega_1) = \begin{cases} 2\alpha_2 + 2\sum_{i=2}^{n/2} \alpha_{2i}, & n \equiv 0 \pmod{4} \\ 2\alpha_1 + 2\sum_{i=2}^{n/2} \alpha_{2i}, & n \equiv 2 \pmod{4} \\ \alpha_1 + 3\alpha_2 + 2\alpha_3 + 2\sum_{i=2}^{n/2} \alpha_{2i+1}, & n \equiv 1 \pmod{4} \\ 3\alpha_1 + \alpha_2 + 2\alpha_3 + 2\sum_{i=2}^{n/2} \alpha_{2i+1}, & n \equiv 3 \pmod{4}. \end{cases}$
where \( v = \begin{cases} 
(n-1), -(n-3), \ldots, -3, -1, 2, 4, 6, \ldots, n-2, n), & n \equiv 0 \pmod{4} \\
(n-1), -(n-3), \ldots, 3, 1, 2, 4, 6, \ldots, n-2, n), & n \equiv 2 \pmod{4} \\
(n-1), -(n-3), \ldots, -4, -1, 2, 3, 5, \ldots, n-2, n), & n \equiv 1 \pmod{4} \\
(n-1), -(n-3), \ldots, -4, 1, 2, 3, 5, \ldots, n-2, n), & n \equiv 3 \pmod{4}. 
\end{cases} \)

(ii) For \( r = 2 \), \( v(4\omega_2) = \begin{cases} 
2\alpha_1 + 2\sum_{i=2}^{n} \alpha_{2i}, & n \equiv 0 \pmod{4} \\
2\alpha_2 + 2\sum_{i=2}^{n} \alpha_{2i}, & n \equiv 2 \pmod{4} \\
3\alpha_1 + \alpha_2 + 2\alpha_3 + 2\sum_{i=2}^{n-1} \alpha_{2i+1}, & n \equiv 1 \pmod{4} \\
\alpha_1 + 3\alpha_2 + 2\alpha_3 + 2\sum_{i=2}^{n-1} \alpha_{2i+1}, & n \equiv 3 \pmod{4}. 
\end{cases} \)

where \( v = \begin{cases} 
(n-1), -(n-3), \ldots, -3, 1, 2, 4, 6, \ldots, n-2, n), & n \equiv 0 \pmod{4} \\
(n-1), -(n-3), \ldots, -3, -1, 2, 4, 6, \ldots, n-2, n), & n \equiv 2 \pmod{4} \\
(n-1), -(n-3), \ldots, -4, 1, 2, 3, 5, \ldots, n-2, n), & n \equiv 1 \pmod{4} \\
(n-1), -(n-3), \ldots, -4, 1, 2, 3, 5, \ldots, n-2, n), & n \equiv 3 \pmod{4}. 
\end{cases} \)

(iii) Let \( 3 \leq r \leq n-1 \). For \((n+1) - r = 2m\) and for any \( \hat{i} = (i_1, i_2, \ldots, i_m) \in J_{m,[1,n]} \setminus Z \), there exists an unique \( v_\hat{i} \in W^r \) such that \( \sum_{k=1}^{m} \alpha_{i_k} \), where \( Z = \{(1,3,i_1,i_2,\ldots,i_{m-2}) : i_k \in \{5,\ldots,n-1,n\} \text{ and } i_{k+1} - i_k \geq 2, \forall k\} \).

(a) For \( \hat{i} \in J_{m,[3,n]}, v_\hat{i}(\omega_r) = \sum_{k=1}^{m} \alpha_{i_k} \), where \( Z = \{(1,3,i_1,i_2,\ldots,i_{m-2}) : i_k \in \{5,\ldots,n-1,n\} \text{ and } i_{k+1} - i_k \geq 2, \forall k\} \).

(b) For \( \hat{i} \in J_{m-1,[4,n]}, v_{\hat{i}}(\omega_r) = \alpha_1 + \sum_{k=1}^{m-1} \alpha_{i_k} \), with

\[
v_{\hat{i}} = \begin{cases} 
([3, n] \backslash \{\hat{i},\hat{\omega}'\}) \uparrow, -\hat{i}', 1, 2, \hat{i}, & m \text{ is odd} \\
(-t, [3, n] \backslash \{t, \hat{i}, \hat{\omega}'\}) \uparrow, -\hat{i}', 1, 2, \hat{i}, & m \text{ is even},
\end{cases}
\]

where \( t = \min\{[3, n] \backslash \{\hat{i}, \hat{\omega}'\}\} \).

(c) For \( \hat{i} \in J_{m-1,[4,n]}, v_{\hat{i}}(\omega_r) = \alpha_2 + \sum_{k=1}^{m-1} \alpha_{i_k} \), with

\[
v_{\hat{i}} = \begin{cases} 
([t, [3, n] \backslash \{t, \hat{i}, \hat{\omega}'\}) \uparrow, -\hat{i}', 1, 2, \hat{i}, & m \text{ is odd} \\
([3, n] \backslash \{\hat{i}, \hat{\omega}'\}) \uparrow, -\hat{i}', 1, 2, \hat{i}, & m \text{ is even},
\end{cases}
\]

where \( t = \min\{[3, n] \backslash \{\hat{i}, \hat{\omega}'\}\} \).

(iv) Let \( 3 \leq r \leq n-1 \). For \((n+1) - r = 2m+1\) and for any \( \hat{i} = (i_1, i_2, \ldots, i_{m+1}) \in J_{m,[4,n]}, \) there exists unique \( v_\hat{i} \in W^r \) such that \( v_\hat{i}(\omega_r) = \frac{1}{2} \alpha_1 + \frac{1}{2} \alpha_2 + \sum_{k=1}^{m} \alpha_{i_k} \). Also, for any \( \hat{i} = (i_1, i_2, \ldots, i_{m-1}) \in J_{m-1,[5,n]}, \) there exists unique \( v_{\hat{i}} \in W^r \) such that \( v_{\hat{i}}(\omega_r) = \frac{3}{2} \alpha_1 + \frac{1}{2} \alpha_2 + \alpha_3 + \sum_{k=1}^{m-1} \alpha_{i_k} \). We have:

\[
(a) v_\hat{i} = \begin{cases} 
(-1, [3, n] \backslash \{\hat{i}, \hat{\omega}'\}) \uparrow, -\hat{i}', 2, \hat{i}, & m \text{ is odd} \\
(1, [3, n] \backslash \{\hat{i}, \hat{\omega}'\}) \uparrow, -\hat{i}', 2, \hat{i}, & m \text{ is even}.
\end{cases}
\]
(b) \(v_{2,1} = \begin{cases} (4, [5, n]) & \text{if } m \text{ is odd} \\ (-4, [5, n]) & \text{if } m \text{ is even} \end{cases}\)

(c) \(v_{2,2} = \begin{cases} (4, [5, n]) & \text{if } m \text{ is odd} \\ (-4, [5, n]) & \text{if } m \text{ is even} \end{cases}\)

(v) For \(r = n, v = (1, 3, 4, 5, \ldots, n - 1, n, 2)\) with \(v(\omega_n) = \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2\).

Proof. The proof of the proposition is similar to the proofs for type B and C which uses the following crucial lemma. \(\square\)

Lemma 5.2. Let \(v, v_{1,1}, v_{1,2}, v_{2,1}, v_{2,2}\) are as defined in Proposition 5.1.

(i) Let \(r = 1\). Then \(w > v\) and \(l(w) = l(v) + 1\) if \(w = s_kv\) where \(k\) takes the following values:

\[
\begin{align*}
&k \in \{2, 4, 6, \ldots, n - 2, n\}, \quad n \equiv 0(\mod 4) \\
&k \in \{1, 4, 6, \ldots, n - 2, n\}, \quad n \equiv 2(\mod 4) \\
&k \in \{2, 5, 7, \ldots, n - 2, n\}, \quad n \equiv 1(\mod 4) \\
&k \in \{1, 5, 7, \ldots, n - 2, n\}, \quad n \equiv 3(\mod 4).
\end{align*}
\]

(ii) Let \(r = 2\). Then \(w > v\) and \(l(w) = l(v) + 1\) if \(w = s_kv\) where \(k\) takes the following values:

\[
\begin{align*}
&k \in \{2, 4, 6, \ldots, n - 2, n\}, \quad n \equiv 0(\mod 4) \\
&k \in \{1, 4, 6, \ldots, n - 2, n\}, \quad n \equiv 2(\mod 4) \\
&k \in \{2, 5, 7, \ldots, n - 2, n\}, \quad n \equiv 1(\mod 4) \\
&k \in \{1, 5, 7, \ldots, n - 2, n\}, \quad n \equiv 3(\mod 4).
\end{align*}
\]

(iii) Let \(3 \leq r \leq n - 1\) and \((n + 1) - r = 2m + 1\). Then,

(a) \(w > v_{1,1}\) and \(l(w) = l(v_{1,1}) + 1\) if \(w = s_{a_{it}}v_{1,1}\) or \(s_{a_{it} + a_{it+1}}v_{1,1}\) for \(i_t: |i_t - i_{t+1}| \geq 3\) or \(s_{a_{it} + a_{it-1}}v_{1,1}\) for \(i_t: |i_t - i_{t-1}| \geq 3\) or \(s_{a_{it} + a_{it+1} + a_{it-1}}v_{1,1}\) for \(i_t: |i_t - i_{t+1}| \geq 3\) with \(i_t \geq 5\) or \(s_{a_{it} + a_{it+1}}v_{1,1}\) for \(i_t: |i_t - i_{t+1}| \geq 3\) with \(i_t \geq 5\) or \(s_{a_{it} + a_{it+1} + a_{it-1}}v_{1,1}\) for \(i_t: |i_t - i_{t+1}| \geq 3\) with \(i_t \geq 5\) or \(s_{a_{it} + a_{it+1} + a_{it-1} + a_{it-2}}v_{1,1}\) for \(i_t: |i_t - i_{t+1}| \geq 3\) with \(i_t \geq 5\) or \(s_{a_{it} + a_{it+1} + a_{it-1} + a_{it-2}}v_{1,1}\) for \(i_t: |i_t - i_{t+1}| \geq 3\) with \(i_t \geq 5\).

(b) \(w > v_{1,2}\) and \(l(w) = l(v_{1,2}) + 1\) if \(w = s_{a_{it}}v_{1,2}\) for \(k \in \{1, i_t\}\) or \(s_{a_{it} + a_{it+1}}v_{1,2}\) for \(i_t: |i_t - i_{t+1}| \geq 3\) or \(s_{a_{it} + a_{it-1}}v_{1,2}\) for \(i_t: |i_t - i_{t-1}| \geq 3\) with \(i_t \geq 5\) or \(s_{a_{it} + a_{it+1} + a_{it-1}}v_{1,2}\) for \(i_t: |i_t - i_{t+1}| \geq 3\) with \(i_t \geq 5\) or \(s_{a_{it} + a_{it+1} + a_{it-1} + a_{it-2}}v_{1,2}\) for \(i_t: |i_t - i_{t+1}| \geq 3\) with \(i_t \geq 5\) or \(s_{a_{it} + a_{it+1} + a_{it-1} + a_{it-2}}v_{1,2}\) for \(i_t: |i_t - i_{t+1}| \geq 3\) with \(i_t \geq 5\).

(c) \(w > v_{2,1}\) and \(l(w) = l(v_{2,1}) + 1\) if \(w = s_{a_{it}}v_{2,1}\) for \(k \in \{1, i_t\}\) or \(s_{a_{it} + a_{it+1}}v_{2,1}\) for \(i_t: |i_t - i_{t+1}| \geq 3\) or \(s_{a_{it} + a_{it-1}}v_{2,1}\) for \(i_t: |i_t - i_{t-1}| \geq 3\) with \(i_t \geq 6\) or \(s_{a_{it} + a_{it+1} + a_{it-1}}v_{2,1}\) for \(i_t: |i_t - i_{t+1}| \geq 3\) with \(i_t \geq 6\) or \(s_{a_{it} + a_{it+1} + a_{it-1} + a_{it-2}}v_{2,1}\) for \(i_t: |i_t - i_{t+1}| \geq 3\) with \(i_t \geq 6\).

(d) \(w > v_{2,2}\) and \(l(w) = l(v_{2,2}) + 1\) if \(w = s_{a_{it}}v_{2,2}\) for \(k \in \{2, i_t\}\) or \(s_{a_{it} + a_{it+1}}v_{2,2}\) for \(i_t: |i_t - i_{t+1}| \geq 3\) or \(s_{a_{it} + a_{it-1}}v_{2,2}\) for \(i_t: |i_t - i_{t-1}| \geq 3\) with \(i_t \geq 6\) or \(s_{a_{it} + a_{it+1} + a_{it-1}}v_{2,2}\) for \(i_t: |i_t - i_{t+1}| \geq 3\) with \(i_t \geq 6\) or \(s_{a_{it} + a_{it+1} + a_{it-1} + a_{it-2}}v_{2,2}\) for \(i_t: |i_t - i_{t+1}| \geq 3\) with \(i_t \geq 6\).

(v) Let \(r = n\). Then \(w > v\) and \(l(w) = l(v) + 1\) if \(w = s_1v\) or \(s_2v\).
Proof. We prove this lemma only for case (iv) and part (b). Proofs of the other cases are similar to this case.

Let \( w \in W^{I_r} \) such that \( w > v_{l,1} \) and \( l(w) = l(v_{l,1}) + 1 \). Then \( w = s_\beta v_{l,1} \), with \( ht(\beta) \leq 2 \) or \( \beta = \alpha_1 + \alpha_3 + \alpha_4 \) when \( i_1 \geq 6 \). Note that \( v_{l,1}(\omega_r) = \frac{3}{2} \alpha_1 + \frac{1}{2} \alpha_2 + \alpha_3 + \sum_{k=1}^{m-1} \alpha_{i_k} \).

Case 1. \( \beta = \alpha_k \), a simple root.

If \( k = i_t \), then \( s_{\alpha_t} v_{l,1}(\omega_r) = \frac{3}{2} \alpha_1 + \frac{1}{2} \alpha_2 + \alpha_3 + \sum_{k \neq t} \alpha_{i_k} - \alpha_{i_t} \). Since \( s_\beta v_{l,1}(\omega_r) < v_{l,1}(\omega_r) \) and \( \omega_r \) is a dominant weight we have \( s_\beta v_{l,1} > v_{l,1} \). Since \( \beta \) is a simple root so \( l(s_\beta v_{l,1}) = l(v_{l,1}) + 1 \).

If \( \beta = \alpha_1 \), then \( s_1 v_{l,1}(\omega_r) = -\frac{1}{2} \alpha_1 + \frac{1}{2} \alpha_2 + \alpha_3 + \sum_{k \neq t} \alpha_{i_k} \). So \( s_1 v_{l,1}(\omega_r) < v_{l,1}(\omega_r) \). Hence \( s_1 v_{l,1} > v_{l,1} \) with \( l(s_3 v_{l,1}) = l(v_{l,1}) + 1 \).

If \( \beta = \alpha_2 \) or \( \alpha_3 \), then \( s_\beta v_{l,1} = v_{l,1} \) in \( W^{I_r} \), a contradiction.

If \( k \notin \{1, 2, 3, i_t\} \), then \( s_\beta v_{l,1}(\omega_r) > v_{l,1}(\omega_r) \). Hence \( s_\beta v_{l,1} < v_{l,1} \), a contradiction.

Case 2. \( \beta = \alpha_k + \alpha_{k+1} \), a positive root of height 2.

If \( k = i_t \) with \( |i_t - i_{t+1}| \geq 3 \), then \( s_\beta v_{l,1}(\omega_r) = \frac{3}{2} \alpha_1 + \frac{1}{2} \alpha_2 + \alpha_3 + \sum_{k \neq t} \alpha_{i_k} - \alpha_{i_{t+1}} < v_{l,1}(\omega_r) \). So \( s_\beta v_{l,1} > v_{l,1} \). Now we prove that \( l(s_\beta v_{l,1}) = l(v_{l,1}) + 1 \).

We will prove this case for \( m \) is odd. Note that \( s_\beta v_{l,1} = (-\hat{i}, -3, -2, -1, \hat{\beta}, [-n, -5] \setminus \{[-l], [-\hat{\beta}]\} \uparrow, -4, 4, [5, n] \setminus \{(l), \hat{\beta}\} \uparrow, -\hat{\beta}, 1, 2, 3, \hat{i}) \) where \( \hat{\beta} = (i_1 - 1, i_2 - 1, \ldots, i_{t-1} - 1, i_t + 1, i_{t+1} - 1 \ldots, i_m - 1) \). In \( v_{l,1} \), the position of \( i_t + 1 \) is right to the position of \( i_t - 1 \) and left to the position of \( \hat{i}_t \) but in \( s_\beta v_{l,1} \), the position of \( i_t \) remains unchanged and the positions of \( i_t + 1 \) and \( i_t - 1 \) are interchanged. Similarly in \( v_{l,1} \) the position of \( -i_t - 1 \) is right to the position of \( -i_t \) and left to the position of \( -i_t + 1 \) but in \( s_\beta v_{l,1} \) the position of \( -i_t \) remains unchanged and the positions of \( -i_t - 1 \) and \( -i_t + 1 \) are interchanged. So \( inv(s_\beta v_{l,1}) = inv(v_{l,1}) + 2 \) and hence \( l(s_\beta v_{l,1}) = l(v_{l,1}) + 1 \).

If \( k = i_t \) with \( |i_t - i_{t+1}| = 2 \), then \( s_\beta v_{l,1} = v_{l,1} \) in \( W^{I_r} \) since \( s_\beta v_{l,1}(\omega_r) = v_{l,1}(\omega_r) \), a contradiction.

If \( \beta = \alpha_k + \alpha_{k+1} \) and \( k \notin \{2, 3, i_t\} \) then \( s_\beta v_{l,1}(\omega_r) \geq v_{l,1}(\omega_r) \) we have \( s_\beta v_{l,1} \leq v_{l,1} \) in \( W^{I_r} \), a contradiction.

If \( k = 3 \) then \( \beta = \alpha_3 + \alpha_4 \). So \( s_\beta v_{l,1}(\omega_r) > v_{l,1}(\omega_r) \) and hence \( s_\beta v_{l,1} < v_{l,1} \), a contradiction.

If \( k = 2 \) then \( \beta = \alpha_2 + \alpha_3 \), then \( s_\beta v_{l,1}(\omega_r) = v_{l,1}(\omega_r) \) and hence \( s_\beta v_{l,1} = v_{l,1} \) in \( W^{I_r} \), a contradiction.

If \( j, j + 1 \neq i_t \) then \( s_\beta v_{l,1}(\omega_r) \geq v_{l,1}(\omega_r) \) and \( s_\beta v_{l,1} \leq v_{l,1} \) in \( W^{I_r} \), a contradiction.

Case 3. If \( \beta = \alpha_{k-1} + \alpha_k \), a positive root of height 2 the proof is similar to above case.

Case 4. If \( \beta = \alpha_1 + \alpha_3 \), then \( s_\beta v_{l,1} = s_3 s_1 v_{l,1} \). From the reduced expression of \( v_{l,1} \) we see that \( l(s_3 v_{l,1}) = l(v_{l,1}) + 2 \), a contradiction.
Case 5. If $\beta = \alpha_1 + \alpha_3 + \alpha_4$ with $i_1 \geq 6$, then $s_{\beta v_{1,1}}(\omega_r) = \frac{1}{2} \alpha_1 + \frac{1}{2} \alpha_2 - \alpha_4 + \sum \alpha_{i_k} < v_{1,1}(\omega_r)$. So $s_{\beta v_{1,1}} > v_{1,1}$. We show that $l(s_{\beta v_{1,1}}) = l(v_{1,2}) + 1$.

We will prove this case for $m$ is odd. Note that $s_{\beta v_{1,1}} = (-1, -3, -2, 4, \{\lceil i \rceil, \lceil i' \rceil \}) \uparrow , \lceil i \rceil , -4, 2, 3, \lceil i' \rceil )$. So $inv(s_{\beta v_{1,1}}) = inv(v_{1,2}) + 4$ and $neg(s_{\beta v_{1,1}}) = neg(v_{1,2}) + 2$. Hence $l(s_{\beta v_{1,1}}) = l(v_{1,2}) + 1$.

Proof of the converse is clear from the definition of maximal $v$ such that $v(\omega_r) \geq 0$. $\square$

Proposition 5.3. Let $G$ be of type $D_n$ and let $v, v_{1,1}, v_{1,2}, v_{1,3}, v_{1,4}$ be as defined in Proposition 5.1. Let $w \in W^r$ be minimal such that $w(\omega_r) \leq 0$. Then the description of $w$ is the following:

(i) For $r = 1$, $w = \begin{cases} (-n, -(n - 2), \ldots, -4, -2, 1, 3, 5, \ldots, n - 3, n - 1), & n \equiv 0(\text{mod } 4) \\ (-n, -(n - 2), \ldots, -4, -2, -1, 3, 5, \ldots, n - 3, n - 1), & n \equiv 2(\text{mod } 4) \\ (-n, -(n - 2), \ldots, -5, -3, -2, 1, 4, \ldots, n - 3, n - 1), & n \equiv 1(\text{mod } 4) \\ (-n, -(n - 2), \ldots, -5, -3, -2, 1, 4, \ldots, n - 3, n - 1), & n \equiv 3(\text{mod } 4). \end{cases}$

(ii) For $r = 2$, $w = \begin{cases} (n, -(n - 2), \ldots, -4, -2, -1, 3, 5, \ldots, n - 3, n - 1), & n \equiv 0(\text{mod } 4) \\ (n, -(n - 2), \ldots, -4, -2, -1, 3, 5, \ldots, n - 3, n - 1), & n \equiv 2(\text{mod } 4) \\ (n, -(n - 2), \ldots, -3, -2, 1, 4, 6, \ldots, n - 3, n - 1), & n \equiv 1(\text{mod } 4) \\ (n, -(n - 2), \ldots, -3, -2, 1, 4, 6, \ldots, n - 3, n - 1), & n \equiv 3(\text{mod } 4). \end{cases}$

(iii) Let $3 \leq r \leq n - 1$ and $(n + 1) - r = 2m$.

If $\lceil i \rceil \in J_m[3,n]$ then $w_{\lceil i \rceil} = s_t s_r \ldots s_1 v_\lceil i \rceil$ with $w_{\lceil i \rceil}(\omega_r) = -v_\lceil i \rceil(\omega_r)$ and

\[
w_{\lceil i \rceil} = \begin{cases} (1, [2,n]\{[\lceil i \rceil, \lceil i' \rceil] \} \uparrow , -i, r), & m \text{ is odd} \\ (1, [2,n]\{[\lceil i \rceil, \lceil i' \rceil] \} \uparrow , -i, r), & m \text{ is even}. \end{cases}
\]

If $\lceil i \rceil \in J_{m-1}[4,n]$ then $w_{\lceil i \rceil} = s_t s_r \ldots s_1 v_{1,\lceil i \rceil}$ with $w_{1,\lceil i \rceil}(\omega_r) = -v_{1,\lceil i \rceil}(\omega_r)$ and

\[
w_{1,\lceil i \rceil} = \begin{cases} (1, [3,n]\{[\lceil i \rceil, \lceil i' \rceil] \} \uparrow , -i, -2, -1, r), & m \text{ is odd where } t = min[3,n]\{\lceil i \rceil, \lceil i' \rceil\}. \\ (1, [3,n]\{[\lceil i \rceil, \lceil i' \rceil] \} \uparrow , -i, -2, -1, r), & m \text{ is even}. \end{cases}
\]

If $\lceil i \rceil \in J_{m-1}[4,n]$ then $w_{2,\lceil i \rceil} = s_t s_r \ldots s_1 v_{2,\lceil i \rceil}$ with $w_{2,\lceil i \rceil}(\omega_r) = -v_{2,\lceil i \rceil}(\omega_r)$ and

\[
w_{2,\lceil i \rceil} = \begin{cases} (1, [3,n]\{[\lceil i \rceil, \lceil i' \rceil] \} \uparrow , -i, -2, -1, r), & m \text{ is odd where } t = min[3,n]\{\lceil i \rceil, \lceil i' \rceil\}. \\ (1, [3,n]\{[\lceil i \rceil, \lceil i' \rceil] \} \uparrow , -i, -2, -1, r), & m \text{ is even}. \end{cases}
\]

(iv) Let $3 \leq r \leq n - 1$ and $(n + 1) - r = 2m + 1$.

For $\lceil i \rceil \in J_m[4,n]$ and $v_{1,\lceil i \rceil}(\omega_r) = \frac{1}{2} \alpha_1 + \frac{1}{2} \alpha_2 + \sum_{k=1}^{m} \alpha_{i_k}$, we have $w_{\lceil i \rceil} = s_t s_r \ldots s_1 v_{\lceil i \rceil}$.

In this case $w_{\lceil i \rceil}(\omega_r) = -v_{\lceil i \rceil}(\omega_r)$ and

\[
w_{\lceil i \rceil} = \begin{cases} (1, [3,n]\{[\lceil i \rceil, \lceil i' \rceil] \} \uparrow , -i, -2, r), & m \text{ is odd} \\ (1, [3,n]\{[\lceil i \rceil, \lceil i' \rceil] \} \uparrow , -i, -2, r), & m \text{ is even}. \end{cases}
\]

For $\lceil i \rceil \in J_{m-1}[5,n]$ and $v_{1,\lceil i \rceil}(\omega_r) = \frac{3}{2} \alpha_1 + \frac{1}{2} \alpha_2 + \alpha_3 + \sum_{k=1}^{m} \alpha_{i_k}$, we have $w_{2,\lceil i \rceil} = s_t s_r \ldots s_1 v_{2,\lceil i \rceil}$.

\[
w_{2,\lceil i \rceil} = \begin{cases} (4, [5,n]\{[\lceil i \rceil, \lceil i' \rceil] \} \uparrow , -i, -3, -2, r), & m \text{ is odd} \\ (4, [5,n]\{[\lceil i \rceil, \lceil i' \rceil] \} \uparrow , -i, -3, -2, r), & m \text{ is even}. \end{cases}
\]
For $i \in J_{m-1, [5,n]}$ and $v_{i,2}(\omega_r) = \frac{1}{2} \alpha_1 + \frac{3}{2} \alpha_2 + \alpha_3 + \sum_{k=1}^{m-1} \alpha_{ik}$, we have $w_{i,1} = s_1 s_3 s_2 s_i s_{i+1} \ldots s_{i-1} v_{i,2}$.

In these cases $w_{i,1}(\omega_r) = -v_{i,1}(\omega_r)$ and $w_{i,2}(\omega_r) = -v_{i,2}(\omega_r)$ and

$$w_{i,1} = \begin{cases} (-4, [5, n] \setminus \{i, j\} \uparrow \rightarrow, -i, -3, -2, -1, j), & m \text{ is odd} \\ (4, [5, n] \setminus \{i, j\} \uparrow \rightarrow, -i, -3, -2, -1, j), & m \text{ is even.} \end{cases}$$

(v) For $r = n$, we have $w = s_1 s_2 v$ and in this case $w(\omega_r) = -v(\omega_r)$.

Proof. For the proof of minimality of $w$ refer to [12].

Proposition 5.4. Let $v, w, v_{i,1}, w_{i,1}, v_{i,2}, w_{i,2}, v_{i,1}, w_{i,1}, v_{i,2}, w_{i,2}$ be defined in Proposition 5.1 and Proposition 5.3. Let $w \in W^{P_r}$ be minimal and $v \in W^{P_r}$ be maximal such that $w(\omega_r) \leq 0$ and $v(\omega_r) \geq 0$. Then $X^v_w \neq \emptyset$ iff the pair $(v, w)$ is one of the following:

(i) Let $3 \leq r \leq n - 1$. For $(n+1) - r = 2m$:

$$(v, w) = \begin{cases} (v_{i,1}, w_{i,2}), & s.t. |i_k - j_k| \leq 1 \forall 1 \leq k \leq m \\ (v_{i,1}, w_{i,2}), (v_{i,2}, w_{i,2}), & s.t. |i_{k+1} - j_k| \leq 1 \forall 1 \leq k \leq m - 1 \text{ with } i_1 = 3 \\ (v_{i,1}, w_{i,2}), (v_{i,2}, w_{i,2}), & s.t. |i_k - j_{k+1}| \leq 1 \forall 1 \leq k \leq m - 1 \text{ with } j_1 = 3. \end{cases}$$

For $(n+1) - r = 2m + 1$:

$$(v, w) = \begin{cases} (v_{i,1}, w_{i,2}), & s.t. |i_k - j_k| \leq 1 \forall 1 \leq k \leq m \\ (v_{i,1}, w_{i,2}), (v_{i,2}, w_{i,2}), & s.t. |i_{k+1} - j_k| \leq 1 \forall 1 \leq k \leq m - 1 \text{ with } i_1 = 4 \\ (v_{i,1}, w_{i,2}), (v_{i,2}, w_{i,2}), & s.t. |i_k - j_{k+1}| \leq 1 \forall 1 \leq k \leq m - 1 \text{ with } j_1 = 4. \end{cases}$$

(ii) For $r = 1, 2$ or $n$, $X^v_w \neq \emptyset$ for any $v$ and $w$.

Proof. Let $X^v_w$ be non-empty. So, $v \leq w$. We prove this part for case (i), $(n+1) - r$ is odd and $(v, w) = (v_{i,1}, w_{i,2})$. For other cases the proofs are similar. Let $(n+1) - r = 2m + 1$. We have $v_{i,1} < w_{i,1}$. Now assume that $|i_{k+1} - j_k| \leq 1, \forall 1 \leq k \leq m - 1$. We need to show that $i_1 = 4$. If not, then $i_1 > 4$. This implies that the coefficient of $\alpha_4$ in $v_{i,1}(\omega_r)$ is zero. Since $l(w_{i,1}) - l(v_{i,2}) = m + 2$ and $w_{i,1} > v_{i,2}$ there exists $m + 2$ positive roots $\beta$ such that $w_{i,1} = \prod_{\text{card(\beta)=m+2}} s_\beta v_{i,1}$. On the other hand we have $v_{i,2}(\omega_r) = \frac{1}{2} \alpha_1 + \frac{1}{2} \alpha_2 + 0.\alpha_3 + 0.\alpha_4 + \sum_{k=1}^{m} \alpha_{ik}$ and $w_{i,2}(\omega_r) = -\frac{3}{2} \alpha_1 - \frac{1}{2} \alpha_2 - \alpha_3 - \sum_{k=1}^{m-1} \alpha_{jk}$. So, $w_{i,2} = \prod_{\text{te\{ij=it+1\}}} s_{\alpha_{it+1}} \prod_{\text{te\{ij=it+1\}}} s_{\alpha_{it+1} + \alpha_{it+1} - 1} s_{\alpha_{it+1} + \alpha_{it+1} + 1} s_{s_1 s_3 s_1 s_3 s_2 s_5 s_5 \ldots s_{i-1} s_{i+1}}(v_{i,1} \ldots v_{i,1})$. Note that the expression in the square bracket contains exactly $m - 1$ reflections corresponding to $m - 1$ distinct positive roots and it is independent of the word in between this expression and $v_{i,1}$. So we have, $w_{i,1} = \prod_{\text{card(\beta)>m+2}} s_\beta v_{i,1}$, a contradiction.

So, $i_1 = 4$. 


Let $i_1 = 4$. We will show that $v_2 < w_{j,1}$ implies that $|i_{k+1} - j_k| \leq 1$, $\forall 1 \leq k \leq m - 1$. Since $l(w_{j,1}) - l(v_2) = m + 2$, by Lemma 5.2 we have $w_{j,1} = (\prod_{\text{card}(\beta) = m-1} s_{s_1}s_3s_4s_3s_2v_1)$ for positive roots $\beta$ such that $\beta$ is either $\alpha_i$, or $\alpha_i + \alpha_{i+1}$ or $\alpha_i + \alpha_{i-1}$ for some $i$, $2 \leq t \leq m$. So $w_{j,1}(\omega_r) = -\frac{3}{2}\alpha_1 - \frac{1}{2}\alpha_2 - \alpha_3 - \sum_{j_k : |i_{k+1} - j_k| \leq 1} \alpha_{j_k}$. Hence, $|i_{k+1} - j_k| \leq 1$, $\forall 1 \leq k \leq m - 1$.

Conversely, let $v = v_2$, $w = w_{j,1}$ with $|i_{k+1} - j_k| \leq 1$, $\forall 1 \leq k \leq m - 1$ and $i_1 = 4$. Then we need to show that $v_2 < w_{j,1}$. Note that in this case $w_{j,1} = \prod_{\{t,j=t_i+1\} \uparrow} s_{\alpha_{t_{i+1}}} \prod_{\{t,j=t_i+1\} \downarrow} s_{\alpha_{t_{i+1}} + \alpha_{t_{i+1} + 1}} (s_{\alpha_1})(s_{\alpha_2}) v_2$. We claim that in this product multiplication of each reflection to $v_2$ amounts to increase the length by one and the product is greater than $v_2$. Since $j \in J_{m,[4,n]}$, $j \in J_{m-1,[5,n]}$ and $|i_{k+1} - j_k| \leq 1$, $\forall 1 \leq k \leq m - 1$ with $i_1 = 4$ we observe that the sets $\bigcup_{\{t,j=t_i+1\} \downarrow} \{i_{t+1} - 1, i_{t+1}, i_{t+1} + 1\}$ and $\bigcup_{\{t,j=t_i+1\} \uparrow} \{i_{t+1} - 1, i_{t+1}\}$ are mutually disjoint.

It is easy to see that $s_{\alpha_1}s_{\alpha_3 + \alpha_4} s_{\alpha_2} v_2 > v_2$ and $l(s_{\alpha_1}s_{\alpha_3 + \alpha_4} s_{\alpha_2} v_2) = l(v_2) + 3$.

Now we claim that $s_{\alpha_1}s_{\alpha_2} s_{\alpha_2} v_2 < \prod_{\{t,j=t_i+1\} \uparrow} s_{\alpha_{t_{i+1}} + \alpha_{t_{i+1} + 1}} s_{\alpha_1}s_{\alpha_3 + \alpha_4} s_{\alpha_2} v_2$ and the length is increased by the number of reflections multiplied. Let $t$ be maximal such that $j_t = i_{t+1} + 1$. Since the sets $\{1,2,3,4\}$ and $\bigcup_{\{t,j=t_i+1\} \downarrow} \{i_{t+1} - 1, i_{t+1}, i_{t+1} + 1\}$ are mutually disjoint, from Lemma 5.2 we see that $s_{\alpha_1}s_{\alpha_3 + \alpha_4} s_{\alpha_2} v_2 < s_{\alpha_{t_{i+1} + \alpha_{t_{i+1} + 1}}} s_{\alpha_1}s_{\alpha_3 + \alpha_4} s_{\alpha_2} v_2$.

Let $t$ be such that $i_t = i_{t+1} - 2$ and $j_{t-1} = i_t + 1$. We have $s_{\alpha_1}s_{\alpha_{t+1}} s_{\alpha_{t+1} + \alpha_{t+1} + 1}s_{\alpha_1}s_{\alpha_3 + \alpha_4} s_{\alpha_2} v_2(\omega_r) = -\frac{3}{2}\alpha_1 - \frac{1}{2}\alpha_2 - \alpha_3 - \sum_{k=2,\neq (t,t+1)} \alpha_{i_k} - \alpha_{i_{t+1} + 1} - \alpha_{i_{t+1}}$. So $s_{\alpha_1}s_{\alpha_{t+1}} s_{\alpha_{t+1} + \alpha_{t+1} + 1}s_{\alpha_1}s_{\alpha_3 + \alpha_4} s_{\alpha_2} v_2 < s_{\alpha_{i_{t+1} + \alpha_{i_{t+1} + 1}}} s_{\alpha_1}s_{\alpha_3 + \alpha_4} s_{\alpha_2} v_2(\omega_r)$. Hence $s_{\alpha_1}s_{\alpha_{t+1}} s_{\alpha_{t+1} + \alpha_{t+1} + 1}s_{\alpha_1}s_{\alpha_3 + \alpha_4} s_{\alpha_2} v_2 > s_{\alpha_{i_{t+1} + \alpha_{i_{t+1} + 1}}} s_{\alpha_1}s_{\alpha_3 + \alpha_4} s_{\alpha_2} v_2$. From the one line notations of these two elements we see that the length is increasing by 1.

Repeating this process we conclude that $v_2 < \prod_{\{t,j=t_i+1\} \uparrow} s_{\alpha_{t_{i+1}}} \prod_{\{t,j=t_i+1\} \downarrow} s_{\alpha_{t_{i+1} + \alpha_{t_{i+1} + 1}}} s_{\alpha_1}s_{\alpha_3 + \alpha_4} s_{\alpha_2} v_2$. □

**Theorem 5.5.** Let $G$ be of type $D_n$ and let $P_r$ be the maximal parabolic subgroup corresponding to the simple root $\alpha_r$. Let $L_r$ be the line bundle corresponding to the fundamental weight $\omega_r$. Then $(X^v)^*_T(L_r)$ is non-empty if and only if the pair $(v,w)$ is one of the following:
(i) For $3 \leq r \leq n - 1$,

$$(n + 1) - r = 2m: \quad (v, w) \text{ s.t. } \begin{cases} v \leq v_{11} \text{ and } w \geq w_{11} \\ v \leq v_{12} \text{ and } w \geq w_{12} \\ v \leq v_{21} \text{ and } w \geq w_{21}. \end{cases}$$

$$(n + 1) - r = 2m + 1: \quad (v, w) \text{ s.t. } \begin{cases} v \leq v_{11} \text{ and } w \geq w_{11} \\ v \leq v_{12} \text{ and } w \geq w_{12} \\ v \leq v_{21} \text{ and } w \geq w_{21}. \end{cases}$$

(ii) For $r = 1, 2$ and $n$, $(X^v_w)^{ss}(L_r)$ is non-empty for any $v$ and $w$.

where $v, w, v_{11}, v_{12}, v_{21}, v_{22}, v_{11}', v_{12}', v_{21}', v_{22}'$ are as in Proposition 5.3.

Proof. Now since $X^v_w \subseteq X^v_w$ implies $(X^v_w)^{ss}(L_r) \subseteq (X^v_w)^{ss}(L_r)$, we can assume that $v$ is maximal and $w$ is minimal having the property that $v(\omega_r) \geq 0$ and $w(\omega_r) \leq 0$. We prove the theorem for case (i) and $(n + 1) - r$ is odd. For other cases the proof is similar.

Let $(n + 1) - r = 2m + 1$. For each pair $(v, w)$, we construct a non-zero $T$-invariant section of $L_r$ on $G/P_r$ which is not identically zero on $X^v_w$.

For $(v, w) = (v_{12}, v_{21})$ we have $v_{12}(\omega_r) + w_{21}(\omega_r) = 0$. So $p_{v_{12}}p_{w_{21}}$ is a non-zero $T$-invariant section of $L_r$ on $G/P_r$ which is not identically zero on $X^v_w$.

For $(v, w) = (v_{12}, v_{21})$, we consider the sequence $v_{12} \leq s_1v_{12} \leq s_1s_3s_1v_{12} \leq s_2s_1s_3s_1\alpha_{i_1}s_\alpha_{i_2} \ldots s_{\alpha_{i_{m-1}}}s_3s_1v_{12} = w_{21}$. We have $v_{12}(\omega_r) + s_1v_{12}(\omega_r) + s_1s_3s_1\alpha_{i_1}s_\alpha_{i_2} \ldots s_{\alpha_{i_{m-1}}}s_3s_1v_{12}(\omega_r) + w_{21}(\omega_r) = 0$ and so $p_{v_{12}}p_{w_{21}}p_{s_1v_{12}}p_{s_1s_3s_1\alpha_{i_1}s_\alpha_{i_2} \ldots s_{\alpha_{i_{m-1}}}s_3s_1v_{12}}$ is a non-zero $T$-invariant section of $L_r$ on $G/P_r$ which is not identically zero on $X^v_w$.

For $(v, w) = (v_{12}, v_{21})$, we consider the sequence $v_{12} \leq s_2v_{12} \leq s_1s_\alpha_{i_1}s_\alpha_{i_2} \ldots s_{\alpha_{i_{m-1}}}s_3s_2v_{12} = w_{12}$. We have $v_{12}(\omega_r) + s_2v_{12}(\omega_r) + s_1s_\alpha_{i_1}s_\alpha_{i_2} \ldots s_{\alpha_{i_{m-1}}}s_3s_2v_{12}(\omega_r) + w_{12}(\omega_r) = 0$ and so $p_{v_{12}}p_{w_{12}}p_{s_1s_\alpha_{i_1}s_\alpha_{i_2} \ldots s_{\alpha_{i_{m-1}}}s_3s_2v_{12}}$ is a non-zero $T$-invariant section of $L_r$ on $G/P_r$ which does not vanish identically zero on $X^v_w$.

So, in all these cases we conclude that $(X^v_w)^{ss}(L_r) \neq \emptyset$.

Conversely, let $(X^v_w)^{ss}(L_r)$ be non-empty.

Let $(v, w) = (v_{12}, w_{21})$. If $i \neq j$, then there exists $t$ such that $j_t \neq i_t$. Since $X^v_w \neq \emptyset$, by proposition 5.4 we have either $j_t = i_t + 1$ or $j_t = i_t - 1$. If $j_t = i_t + 1$ then $w_{j_t}(\omega_r) = -\frac{1}{2}\alpha_1 - \frac{3}{2}\alpha_2 - \alpha_3 - \sum_{k=1, \neq t}^{m-1} \alpha_{i_k} - \alpha_{i_t + 1}$ and if $j_t = i_t - 1$ then $w_{j_t}(\omega_r) = -\frac{1}{2}\alpha_1 - \frac{3}{2}\alpha_2 - \alpha_3 - \sum_{k=1, \neq t}^{m-1} \alpha_{i_k} - \alpha_{i_t - 1}$. Let $u \in W^J_r$ be such that $v_{12} \leq u \leq w_{21}$. Then $u$ is of the form $u = (\prod_{\beta} s_\beta)v_{12}$, where $\beta$'s are some positive roots. For $j_t = i_t + 1$ at most one $\beta$ can be $\alpha_{i_t} + \alpha_{i_t + 1}$ and none of the other $\beta$'s contain $\alpha_{i_t}$ or $\alpha_{i_t + 1}$ as a summand. So in $u(\omega_r)$ the coefficient of $\alpha_{i_t}$ is either zero or one and the coefficient of $\alpha_{i_t + 1}$ is either zero or $-1$. Similarly for $j_t = i_t - 1$ at most one $\beta$ can be $\alpha_{i_t} + \alpha_{i_t - 1}$ and none of the other $\beta$'s contain $\alpha_{i_t}$ or $\alpha_{i_t - 1}$ as a summand. So in $u(\omega_r)$ the coefficient of $\alpha_{i_t}$ is either zero or one and the coefficient of $\alpha_{i_t - 1}$ is either zero or $-1$. For $j_t = i_t + 1$, $u(\omega_r)$ contains either $\alpha_{i_t}$ or $\alpha_{i_t + 1}$ as a summand and for $j_t = i_t - 1$, $u(\omega_r)$ contains either $\alpha_{i_t}$ or $\alpha_{i_t - 1}$.
as a summand. So, there does not exist any sequence $v_{i,1} = u_1 \leq u_2 \leq \ldots \leq u_k = w_{j,2}$ such that $\sum_{t=1}^{k} u_t(\omega_r) = 0$. So we don’t have a nonzero $T$-invariant section which is not identically zero on $X_w^v$.

Let $(v, w) = (v_{i,2}, w_{j,1})$ where $i = (i_1, i_2, \ldots, i_m)$ and $j = (j_1, j_2, \ldots, j_m)$. Then $v_{i,2}(\omega_r) = \frac{1}{2} \alpha_1 + \frac{1}{2} \alpha_2 + \alpha_4 + \sum_{k=2}^{m} \alpha_i$ and $w_{j,1}(\omega_r) = -\frac{3}{2} \alpha_1 - \frac{1}{2} \alpha_2 - \alpha_3 - \sum_{k=2}^{m} \alpha_i$. Then any $u \in W^r$ such that $v_{i,2} \leq u \leq w_{j,1}$ is of the form $u = (\prod \beta_i) v_{\beta,2}$ where $\beta$’s are some positive roots. At most one $\beta$ can be $\alpha_3 + \alpha_4$ and none of the other $\beta$’s can contain $\alpha_3$ or $\alpha_4$ as a summand. So, the coefficient of $\alpha_4$ in $u(\omega_r)$ is either zero or one and the coefficient of $\alpha_3$ in $u(\omega_r)$ is either zero or $-1$. So for any such $u$, $u(\omega_r)$ contains either $\alpha_3$ or $\alpha_4$ as a summand. So, in this case also there is no non zero $T$-invariant section which is not identically zero on $X_w^v$.

Let $(v, w) = (v_{i,2}, w_{j,1})$ with $i_1 = 4$. If $(i_2, i_3, \ldots, i_m) \neq j$, then there exists $t$ such that $j_t \neq i_{t+1}$. Since $X_{w_{j,1}}^{v_{i,2}} \neq \emptyset$, by proposition 5.4 we have $j_t = i_{t+1} + 1$ or $j_t = i_{t+1} - 1$. If $j_t = i_{t+1} + 1$ then $w_{j,1}(\omega_r) = -\frac{3}{2} \alpha_1 - \frac{1}{2} \alpha_2 - \alpha_3 - \sum_{k=1, k \neq t}^{m-1} \alpha_i - \alpha_{i_{t+1}+1}$ and if $j_t = i_{t+1} - 1$ then $w_{j,1}(\omega_r) = -\frac{3}{2} \alpha_1 - \frac{1}{2} \alpha_2 - \alpha_3 - \sum_{k=1, k \neq t}^{m-1} \alpha_i - \alpha_{i_{t+1}-1}$. Then any $u \in W^r$ such that $v_{i,2} \leq u \leq w_{j,1}$ is of the form $u = (\prod \beta_i) v_{\beta,2}$, where $\beta$’s are some positive roots. For $j_t = i_{t+1} + 1$ at most one $\beta$ can be $\alpha_{i_{t+1}+1} + \alpha_{i_{t+1}+1}$ and none of the other $\beta$’s contain $\alpha_{i_{t+1}}$ or $\alpha_{i_{t+1}+1}$ as a summand. So, in $u(\omega_r)$ the coefficient of $\alpha_{i_{t+1}}$ is either zero or one and the coefficient of $\alpha_{i_{t+1}+1}$ is either zero or $-1$. Similarly for $j_t = i_{t+1} - 1$ at most one $\beta$ can be $\alpha_{i_{t+1}} + \alpha_{i_{t+1}-1}$ and none of the other $\beta$’s contain $\alpha_{i_{t+1}}$ or $\alpha_{i_{t+1}-1}$ as a summand. So, in $u(\omega_r)$ the coefficient of $\alpha_{i_{t+1}}$ is either zero or one and the coefficient of $\alpha_{i_{t+1}-1}$ is either zero or $-1$. For $j_t = i_{t+1} + 1$, $u(\omega_r)$ contains either $\alpha_{i_{t+1}}$ or $\alpha_{i_{t+1}+1}$ as a summand and for $j_t = i_{t+1} - 1$, $u(\omega_r)$ contains either $\alpha_{i_{t+1}}$ or $\alpha_{i_{t+1}-1}$ as a summand. So, like in previous cases here also we don’t have a non zero $T$-invariant section which is not identically zero on $X_w^v$.

For the pair $(v, w) = (v_{i,2}, w_{j,1})$ with $i \neq j$ the proof is similar as in the cases in type $B$ and $C$. 

We illustrate Proposition 5.4 and Theorem 5.5 with an example.

**Example:** $D_5$, $\omega_3 = (\frac{3}{2}, \frac{3}{2}, 3, 2, 1)$

| $v$  | $v(\omega_3)$ | $w(\omega_3)$ | $w$ |
|------|---------------|----------------|------|
| $v_4$ = $(-1, 5, -3, 2, 4)$ | $(\frac{1}{2}, \frac{1}{2}, 0, 1, 0)$ | $(-\frac{1}{2}, -\frac{1}{2}, 0, -1, 0)$ | $(1, 5, -4, -2, 3) = w(4)$ |
| $v_5$ = $(-1, 3, -4, 2, 5)$ | $(\frac{1}{2}, \frac{1}{2}, 0, 0, 1)$ | $(-\frac{1}{2}, -\frac{1}{2}, 0, 0, -1)$ | $(1, 3, -5, -2, 4) = w(5)$ |
| $v_1$ = $(4, 5, 1, 2, 3)$ | $(\frac{3}{2}, \frac{3}{2}, 1, 0, 0)$ | $(-\frac{1}{2}, -\frac{3}{2}, -1, 0, 0)$ | $(4, 5, -3, -2, 1) = w_2$ |
| $v_2$ = $(-4, 5, -1, 2, 3)$ | $(\frac{3}{2}, \frac{3}{2}, 1, 0, 0)$ | $(-\frac{3}{2}, -\frac{1}{2}, 1, 0, 0)$ | $(-4, 5, -3, -2, -1) = w_1$ |
So from the above observation, $X_{v_1}^{(4)}$, $X_{w_1}^{(4)}$, $X_{v_1}^{(5)}$, $X_{w_1}^{(5)}$, $X_{v_1}^{(4)}$, $X_{w_1}^{(4)}$, $X_{v_2}^{(5)}$, $X_{w_2}^{(5)}$, $X_{v_1}^{(4)}$, $X_{w_1}^{(4)}$, $X_{v_2}^{(4)}$, $X_{w_2}^{(4)}$, $X_{v_1}^{(5)}$ and $X_{v_2}^{(5)}$ are all non-empty. We have $(X_{v_1}^{(5)})^s_{T}(L_3)$, $(X_{w_1}^{(4)})^s_{T}(L_3)$, $(X_{v_2}^{(4)})^s_{T}(L_3)$, and $(X_{v_1}^{(5)})^s_{T}(L_3)$ are empty whereas $(X_{v_1}^{(4)})^s_{T}(L_3)$, $(X_{w_2}^{(4)})^s_{T}(L_3)$, and $(X_{v_2}^{(4)})^s_{T}(L_3)$ are non-empty.

References

[1] A. B. Jörner, F. Brenti, (2005) Combinatorics of Coxeter Groups, Graduate Texts in Mathematics, Springer-Verlag, New York.
[2] M. Brion, V. Lakshmibai, A geometric approach to standard monomial theory, Represent. Theory 7 (2003), 651-680 (electronic).
[3] R. W. Carter (1985), Finite groups of Lie type, conjugacy classes and complex characters, John Wiley and Sons.
[4] V. Deodhar, On some geometric aspects of Bruhat orderings. I. A finer decomposition of Bruhat cells, Invent. Math. 79 (1985), no. 3, 499-511.
[5] J. E. Humphreys (1972). Introduction to Lie algebras and representation theory, Springer, Berlin Heidelberg.
[6] J. E. Humphreys (1975), Linear Algebraic Groups. Springer, Berlin.
[7] Hausmann, C and Knutson, A. Polygon spaces and Grassmannians. L’Enseignement Mathematique 43 (1997), no. 1-2, 173-198.
[8] S. S. Kannan, Torus quotients of homogeneous spaces, Proc. Indian Acad. Sci. (Math. Sci.) 108(1) (1998) 1–12.
[9] S. S. Kannan, Torus quotients of homogeneous spaces-II, Proc. Indian Acad. Sci. (Math. Sci.) 109(1) (1999) 23-39.
[10] S. S. Kannan, S. K. Pattanayak, K. Paramasamy, S. Upadhyay, Torus quotients of Richardson varieties, Comm.in algebra 119(4) (2009) 469-485.
[11] S. S. Kannan, Pranab Sardar, Torus quotients of homogeneous spaces of the general linear group and the standard representation of certain symmetric groups, Proc. Indian Acad. Sci. (Math. Sci.) 119(1) (2009) 81–100.
[12] S. S. Kannan, S. K. Pattanayak, Torus quotients of homogeneous spaces-minimal dimensional Schubert varieties admitting semi-stable points, Proc. Indian Acad. Sci. (Math. Sci.) 119(4) (2009) 469-485.
[13] Kapranov, M. M. Chow quotients of Grassmannians-I, I. M. Gelfand Seminar, Adv. Soviet Math. 16, Part 2, Amer. Math. Soc., Providence, 1993, 29-110.
[14] Kapranov, M. M. Veronese curves and Grothendieck-Knudsen moduli space $\overline{M}_{0,n}$, J. Algebraic Geom. 2 (1993), 239-26.
[15] D. Kazhdan, G. Lusztig, Representations of Coxeter groups and Hecke algebras, Inv. Math., 53 (1979), 165-184.
[16] D. Kazhdan, G. Lusztig, Schubert varieties and Poincare duality, Proc. Symp. Pure. Math., A.M.S., 36 (1980), 185-203.
[17] V. Lakshmibai, P. Littelmann, Equivanant K-theory and Richardson varieties, Journal of Algebra 260 (2003) 230-260.
[18] V. Lakshmibai, K. N. Raghavan, (2008). Standard Monomial Theory- Invariant theoretic approach, Encyclopaedia of Mathematical Sciences, 137. Invariant Theory and Algebraic Transformation Groups, 8, Springer-Verlag, Berlin.
[19] D. Mumford, J. Fogarty, F. Kirwan, (1994) Geometric Invariant theory (Third Edition), (Berlin Heidelberg, New York: Springer-Verlag).
[20] P. E. Newstead, (1978). Introduction to Moduli Problems and Orbit Spaces, TIFR Lecture Notes.
[21] S. K. Pattanayak, Minimal Schubert Varieties admitting semistable points for exceptional cases, Comm. Algebra, Vol. 42, no. 9 (2014), 3811-3822.
[22] R. W. Richardson, Intersections of double cosets in algebraic groups, Indag. Math., (N.S.), 3 (1992), 69-77.

[23] C. S. Seshadri, Quotient spaces modulo reductive algebraic groups, Ann. Math. 95 (1972), 511–556.

[24] T. A. Springer, (2009) Linear algebraic groups, Modern Birkhäuser Classics, Boston, MA: Birkhäuser Boston Inc.

[25] E. Strickland, Quotients of flag varieties by a maximal torus, Math. Z. 234(1), 1-7 (2000).

[26] F. Incitti, (2006) Bruhat order on the involutions of classical Weyl groups. Adv. in Appl. Math. 37, no. 1, 68111.

Arpita Nayek, Department of Mathematics and Statistics, Indian Institute of Technology, Kanpur, Kanpur-208016, India, Email: anayek@iitk.ac.in,

S.K. Pattanayak, Department of Mathematics and Statistics, Indian Institute of Technology, Kanpur, Kanpur-208016, India, Email: santosha@iitk.ac.in,