Robust Stabilization of Nonlinear Systems Using Periodic Event-triggered Control

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Abstract

Periodic event-triggered control (PETC) has the advantages of both sampled-data control and event-triggered control, and is well-suited for implementation in digital platforms. However, existing results on PETC design mainly focus on linear systems, and their extension to nonlinear systems are still sparse. This paper investigates PETC design for general nonlinear systems subject to external disturbances, and provides sufficient conditions to ensure that the closed-loop system implemented by PETC input-to-state stable, using state feedback and observer-based output feedback controllers, respectively. For incrementally quadratic nonlinear systems, sufficient conditions for PETC design are provided in the form of linear matrix inequalities. The sampling period and triggering functions for all the cases considered are provided explicitly. Two examples are given to illustrate the effectiveness of the proposed method.

1 Introduction

Digital control systems are normally executed in a time-triggered fashion, under which the sensors and actuators are accessed periodically. In contrast, event-triggered control (ETC) executes the sensing and actuation only when certain triggering rules are satisfied; this can be seen as adding feedback to the sensing and actuation processes (see a recent survey paper [18] and references therein). The ETC paradigm is designed to avoid unnecessary waste of computation/communication resources by reducing the number of sensing/actuation executions, while still guaranteeing a desirable closed-loop performance [5, 31, 29, 24, 15, 7]; this shows potential in applications for systems with limited energy resources, such as networked control systems and embedded control systems. In reality, the full state information is hard to obtain, so observer-based output feedback ETC design has also been considered; however, output feedback ETC can not be extended straightforwardly from state feedback ETC, especially for nonlinear systems, for which the separation principle does not hold in general [12, 2, 1, 34].

Since the triggering condition of ETC has to be monitored continuously, it is difficult to implement ETC in digital platforms directly. To overcome this problem, periodic event-triggered control (PETC) was proposed [20, 17, 19]. By evaluating the triggering conditions periodically and deciding whether to update the sensing/actuation at each sampling time, PETC inherits both the benefits of ETC and sampled-data control, and can be implemented on a standard digital platform. Furthermore, Zeno phenomenon is avoided since the sampling period is a lower bound for the minimum inter-execution time. Note that although ETC for discrete-time models can be considered as PETC (e.g., see [17, 13]), the inter-sample behavior of the original continuous-time systems are not captured in the discrete-time analysis.

PETC design for continuous-time linear systems was investigated in [20] where three approaches were presented by considering the closed-loop system as an impulsive system, a piecewise linear system, and a perturbed linear system, respectively. PETC design for continuous-time nonlinear systems is more difficult because of an intrinsic difficulty: the discrete-time
dynamics of a nonlinear system can not be exactly known from its continuous-time dynamics in general [28, 14, 6, 32, 1, 36]. State feedback PETC design for nonlinear systems was given in [32] to ensure the globally asymptotically stability of the closed-loop system, with the sampling period bound provided explicitly; state feedback PETC design for nonlinear systems was investigated in [6] by redesigning the event function of an existing continuous ETC system using overapproximation techniques, such that the control performance guarantees for the continuous ETC system can be preserved; output feedback PETC for nonlinear Lipschitz systems was considered using impulsive observers in [14], to guarantee practical stabilization of the closed-loop system. In spite of these interesting results, many PETC design problems for nonlinear systems are largely open and deserve to be further explored; for instance, output feedback PETC design for general nonlinear dynamics subject to external disturbances has not been investigated, and systematic methods to determine the sampling period are still rare.

This paper investigates PETC design for continuous-time nonlinear control systems affected by disturbances using the state feedback and observer-based output feedback controllers, respectively, to ensure input-to-state stability of the closed-loop system implemented by the corresponding event-triggering mechanism (ETM). Specifically, assuming that the continuous-time state feedback controller (respectively, the continuous-time observer and output feedback controller) for the nonlinear system is given, sufficient conditions to determine the sampling period and triggering functions are provided explicitly. Based on that, PETC design for incrementally quadratic nonlinear systems, which is a broad class of nonlinear systems whose nonlinearity is characterized by incremental multiplier matrices and includes Lipschitz nonlinear systems and sector-bounded nonlinear systems as special cases, is considered. A systematic and constructive way to design the sampling period and triggering functions are given as LMI-based sufficient conditions that guarantee input-to-state stability of the closed-loop system. Discussion on applying the results to continuous-time linear control systems is also given. Compared with existing results on PETC design (e.g., [14, 6, 32, 4, 36]), this paper considers more general plant dynamics (i.e., general nonlinear model subject to disturbances) and more general PETC setups (i.e., ETMs exist in both the sensing and actuation channels for the output feedback case), provides verifiable LMI-based sufficient conditions for a broad class of nonlinear systems (i.e., incrementally quadratic systems), and gives explicitly the interval that the sampling period can be chosen from. In addition, although the analysis techniques in this paper also utilize some results from the emulation approach, they do not make the same type of assumptions that was proposed in [8, 20] and used in [2, 3; c.f., Remark 4].

The remainder of the paper is organized as follows: in Section 2, the problem setup and statement are given; in Section 3, state feedback and output feedback PETC design for continuous-time general nonlinear models are presented individually; in Section 3, state feedback and output feedback PETC design for incrementally quadratic nonlinear systems are provided, with corresponding sufficient conditions given in the form of LMIs, respectively, and followed by the discussion on applying the results to continuous-time linear control systems; in Section 5, two simulation examples are used to illustrate the effectiveness of the proposed method.

**Nomenclature.** Denote the set of real numbers, non-negative real numbers and non-negative integers by $\mathbb{R}$, $\mathbb{R}_0$ and $\mathbb{Z}_0$, respectively. Denote the 2-norm by $\|\cdot\|_2$. Given a non-empty and closed set $A$, the point-to-set distance from $x$ to $A$ is denoted by $\|x - A\|_A = \inf_{y \in A} \|y - x\|$. Denote the identity matrix of size $n$ by $I_n$. Denote the zero matrix of size $n_1 \times n_2$ by $0_{n_1 \times n_2}$ and the zero vector of size $n$ by $0_n$; the subscripts will be omitted when clear from context. Given a matrix $M$, $M > 0$, $M \succ 0$, $M < 0$ and $M \preceq 0$ means that $M$ is positive definite positive semi-definite, negative definite, and negative semi-definite, respectively; given two matrices $M_1$ and $M_2$, $M_1 \geq M_2$ means $M_1 - M_2 \geq 0$. Denote the block diagonal matrix by $diag\{M_1, \ldots, M_n\}$ where $M_1, \ldots, M_n$ are square matrices in the diagonal block. For symmetric matrices, $*$ will be used to stand for entries whose values follow from symmetry. A signal $x : \mathbb{R}_0 \to \mathbb{R}^n$ is called left-continuous if $\lim_{s \to t^-} x(s) = x(t)$ for all $t > 0$. “$\forall x$ a.e.” means for every $x \in \mathbb{R}^{n_x}$ except
for a set of zero Lebesgue-measure in $\mathbb{R}^{n_x}$.

A continuous function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{K}$ (denoted as $f \in \mathcal{K}$) if it is strictly increasing and $f(0) = 0$; $f$ belongs to class $\mathcal{K}_\infty$ (denoted as $f \in \mathcal{K}_\infty$) if $f \in \mathcal{K}$ and $f(r) \to \infty$ as $r \to \infty$. A continuous function $f : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{KL}$ (denoted as $f \in \mathcal{KL}$) if for each fixed $s$, function $f(r,s) \in \mathcal{K}_\infty$ with respect to (w.r.t.) $r$ and for each each fixed $r$, function $f(r,s)$ is decreasing w.r.t. $s$ and $f(r,s) \to 0$ as $s \to \infty$. $\nabla f$ is the gradient of a function $f$. Given a system $\dot{x} = f(x,u)$ where $x$ is the state, $u$ is a measurable essentially bounded input, and $f$ is a locally Lipschitz function, it is called input-to-state stable (ISS), if there exist functions $\beta_1 \in \mathcal{KL}$, $\beta_2 \in \mathcal{K}$ such that for every initial state $x_0$, the solution of the system, $x(t,x_0,u)$, satisfies $\|x(t,x_0,u)\| \leq \beta_1(\|x_0\|,t) + \beta_2(\|u\|)$ for all $t \geq 0$, where $\|u\| := \sup_{t \geq 0} \|u(t)\|$.

2 Problem Statement

At first, consider the setups shown in Fig. 1 where the full-state information is available and the state feedback control is used.

![continuous time state feedback control](image1)

(a) Continuous time state feedback control

![state feedback control implemented with ETM](image2)

(b) State feedback control implemented with ETM

Figure 1: Two setups of closed-loop systems with state feedback control.

Fig. 1(a) shows the closed-loop system in continuous time. The plant is a nonlinear system given as

$$\dot{x}(t) = f(x(t), u(t), w(t))$$

where $x \in \mathbb{R}^{n_x}$ is the state, $u \in \mathbb{R}^{n_u}$ is the control input, $w \in \mathbb{R}^{n_w}$ is the disturbance, $f$ is a locally Lipschitz continuous function. The state feedback controller is given as $u(t) = k(x(t))$ where $k$ is a continuous function. Assume that $k(x)$ is known and designed such that the closed-loop system in continuous time is ISS.

Fig. 1(b) shows the closed-loop system implemented with ETM. In the following, denote the sampling period to be $h > 0$, and define the sampling times as $t_k := kh$ for any $k \in \mathbb{Z}_{\geq 0}$. The state of the plant, $x(t)$, is sampled at each sampling time $t_k$. The input to the controller, $\tilde{x}_c(t)$, is updated only when the event-triggering condition for the state is satisfied. Specifically,
$	ilde{x}_c(t)$ is a left-continuous, piecewise constant signal that is defined for $t \in (t_k, t_{k+1}]$ as

$$
\tilde{x}_c(t) = \begin{cases} 
  x(t_k), & \text{if } \Gamma_x(x(t_k), e(t_k)) \geq 0, \\
  \tilde{x}_c(t_k), & \text{if } \Gamma_x(x(t_k), e(t_k)) < 0,
\end{cases}
$$

where

$$
e(t) = x(t) - \tilde{x}_c(t)
$$

and $\Gamma_x(x(t), e(t))$ is the triggering function that will be determined later. The triggering times $t_0^x, t_1^x, t_2^x, \ldots$ are given by $t_0^x = 0$ and

$$
t_{k+1}^x = \min \{ ih \mid ih > t_k^x, \Gamma_x(x(ih), e(ih)) \geq 0 \}.
$$

The control law $u_c(t)$ is $u_c(t) = k(\tilde{x}_c(t))$, and the actuation input to the plant, $u(t)$, is equal to $u_c(t)$, i.e.,

$$
u(t) = k(\tilde{x}_c(t)).
$$

Next, consider the setups shown in Fig. 2 where only the output information is available and output feedback control is used.

![Diagram](image)

(a) Continuous time observer-based output feedback control

(b) Observer-based output feedback control implemented with ETMs in both the sensing and actuation channels

Figure 2: Two setups of closed-loop systems with observer-based output feedback control.

Fig. 2 (a) shows the closed-loop system in continuous time where the plant is given in (1) and the output is given as

$$
y(t) = g(x(t))
$$

with $y \in \mathbb{R}^{n_y}$ and $g$ a continuous function. The observer is given as

$$
\dot{\hat{x}} = \varphi(\hat{x}, u, y)
$$
where $\dot{x} \in \mathbb{R}^{n_x}$, $\varphi$ is a continuously differentiable function, and the observer-based controller is given as $u(t) = k(\hat{x}(t))$ where $k$ is a continuous function. Assume that $\varphi(\cdot)$ and $k(\cdot)$ are designed for (1) and (5) such that $\dot{x}$ asymptotically converges to $x$ when $w = 0$, and the system (1) implementing the observer-based controller $u(t)$ is ISS.

Fig. 2 (b) shows the closed-loop system implemented with ETMs in both the sensing and actuation channels. The output of the plant, $y(t)$, is sampled at each sampling time $t_k$. The input to the observer, $y_e(t)$, is updated only when the event-triggering condition for the output is satisfied. Specifically, $y_e(t)$ is a left-continuous, piecewise constant signal that is defined for $t \in (t_k, t_{k+1}]$ as

$$y_e(t) = \begin{cases} y(t_k), & \text{if } \Gamma_y(y(t_k), y_e(t_k)) \geq 0, \\ y_e(t_k), & \text{if } \Gamma_y(y(t_k), y_e(t_k)) < 0, \end{cases} \quad (7)$$

where

$$\Gamma_y(y(t), y_e(t)) = y_e(t) - y(t) \quad (8)$$

and $\Gamma_y(y(t), y_e(t))$ is the triggering function of the output that will be determined later. The triggering times $t^y_0, t^y_1, t^y_2, \ldots$ are given by $t^y_0 = 0$ and

$$t^y_{k+1} = \min_{i \in \mathbb{Z}_{\geq 0}} \{ih \mid ih > t^y_k, \Gamma_y(y(ih), y_e(ih)) \geq 0\}. \quad (9)$$

Under ETMs, the observer [6] becomes

$$\dot{x} = \varphi(x, u, y_e) \quad (10)$$

and the observer-based controller generates a continuous control law $u_e(t) = k(\hat{x}(t))$ which is sampled at each sampling time $t_k$. The input to the plant, $u(t)$, is updated only when the event-triggering condition for the input is satisfied. Specifically, define a left-continuous, piecewise constant signal $\hat{x}_e(t)$ for $t \in (t_k, t_{k+1}]$ as

$$\hat{x}_e(t) = \begin{cases} \dot{x}(t_k), & \text{if } \Gamma_u(\dot{x}(t_k), x_e(t_k)) \geq 0, \\ \hat{x}_e(t_k), & \text{if } \Gamma_u(\dot{x}(t_k), x_e(t_k)) < 0, \end{cases} \quad (11)$$

where

$$x_e(t) = \hat{x}_e(t) - \dot{x}(t) \quad (12)$$

and $\Gamma_u(\dot{x}(t), x_e(t))$ is the triggering function of the input that will be determined later. The triggering times $t^u_0, t^u_1, t^u_2, \ldots$ are given by $t^u_0 = 0$ and

$$t^u_{k+1} = \min_{i \in \mathbb{Z}_{\geq 0}} \{ih \mid ih > t^u_k, \Gamma_u(\dot{x}(ih), x_e(ih)) \geq 0\}. \quad (13)$$

The input to the plant, $u(t)$, is given as

$$u(t) = k(\hat{x}_e(t)). \quad (14)$$

Systems that are implemented with ETMs are impulsive systems, which evolve continuously based on ODEs most of the time and exhibit impulses at some instances [16, 37]. Clearly, for systems implemented with ETMs, the impulses happen when the triggering conditions are met.

Inspired by [21] and [23], the input-to-state stability of impulsive systems w.r.t. a given set is defined below.
Definition 1. Consider the following impulsive system

\[
\begin{aligned}
\dot{x}(t) &= f(x(t), u(t)), t \in (T_i, T_{i+1}], \\
x^+(t) &= g(x(t), u(t)), t = T_i,
\end{aligned}
\]

(13)

where \(i \in \mathbb{Z}_{\geq 0}, \{T_0, T_1, T_2, \ldots \} \) is a sequence of impulsive times with \(T_0 < T_1 < \ldots \), the state \(x(t) \in \mathbb{R}^n\) is absolutely continuous between impulses, \(u(t) \in \mathbb{R}^m\) is a locally bounded Lebesgue-measurable input, and \(x^+(t) := \lim_{s \to t^+} x(s)\). Given a time sequence \(\{T_i\}\), the impulsive system \((13)\) is ISS w.r.t. a given non-empty and closed set \(A\) if there exist functions \(\beta \in \mathcal{KL}\) and \(\gamma \in \mathcal{K}_\infty\), such that for every initial condition \(x(T_0)\) and every admissible input \(u\), the solution to \((13)\) exists globally and satisfies

\[
\|x(t)\|_A \leq \beta(\|x(T_0)\|_A, t - T_0) + \gamma(u\|T_0, T\))
\]

(14)

where \(\|\cdot\|_I\) denotes the supremum norm on an interval \(I\). The impulsive system \((13)\) is uniformly ISS w.r.t. a given class \(\mathcal{S}\) of admissible sequences of impulse times if there exist functions \(\beta \in \mathcal{KL}\) and \(\gamma \in \mathcal{K}_\infty\) that are independent of the choice of the time sequence, such that \(14\) holds for every time sequence in \(\mathcal{S}\).

In the following, the closed-loop system implemented with ETMs is called uniformly ISS, or just ISS for short, w.r.t. a given (non-empty and closed) set \(A\), if it is uniformly ISS over all impulsive times generated by the periodic event-triggering mechanisms. It should be noted that the impulsive times generated by the periodic event-triggering mechanisms have no accumulation point (i.e. Zeno phenomenon is avoided) since the inter-execution times are lower bounded by the sampling period.

The PETC design problems that will be investigated in this paper are the following:
1. Given the setup in Fig. 1 (b) where the full-state information is available, design the sampling period \(h\) and the triggering function \(\Gamma_x(x, e)\) such that the closed-loop system is ISS;
2. Given the setup in Fig. 2 (b) where the output information is available, design the sampling period \(h\) and the triggering functions \(\Gamma_y(y, ye), \Gamma_u(\hat{x}, xe)\) such that the closed-loop system is ISS.

In Section 3, problem 1 and problem 2 will be studied for the plant being a general nonlinear system. In Section 4, problem 1 and problem 2 will be studied for the plant being an incrementally quadratic nonlinear system, for which constructive ways based on convex programs to determine the sampling period and the triggering functions will be provided.

3 Robust Stabilization of Nonlinear Systems Using PETC

In this section, PETC design for the nonlinear system \((1)\) with output \((5)\) will be investigated.

One key technique that will be used is from the emulation approach, which has been widely used to analyze the stability property of a system under sampling \([22, 25]\). Computation of the maximum allowable sampling period in the emulation approach was investigated in \([2, 8, 20]\), which will be also used later to determine the sampling period \(h\) in PETC design.

The following lemma from \([8]\) will be used in analysis later. This lemma gives the explicit time for the solution of a special ODE to decrease monotonically from \(\lambda^{-1}\) to \(\lambda\) where \(0 < \lambda < 1\).

Lemma 1. \([8]\) Let \(\phi: [0, \hat{T}] \rightarrow \mathbb{R}\) be the solution of the following ODE:

\[
\dot{\phi} = -2\mu \phi - \gamma(\phi^2 + 1)
\]

(15)
with \( \phi(0) = \lambda^{-1}, 0 < \lambda < 1, \mu > 0, \gamma > 0, \) and
\[
\tilde{T}(\mu, \gamma, \lambda) = \begin{cases} 
\frac{1}{\mu} \arctan \left( \frac{r(1-\lambda)}{2 + \lambda (\frac{\mu}{1-\lambda}) + \lambda} \right), & \gamma > \mu, \\
\frac{1}{\mu} \frac{1 - \lambda}{1 + \lambda}, & \gamma = \mu, \\
\frac{1}{\mu} \arctanh \left( \frac{r(1-\lambda)}{2 + \lambda (\frac{\mu}{1-\lambda}) + \lambda} \right), & \gamma < \mu,
\end{cases}
\]
(16)

\[
r = \sqrt{\left( \frac{2}{\mu} \right)^2 - 1}.
\]
(17)

Then, \( \phi(\tau) \in [\lambda, \lambda^{-1}] \) for all \( \tau \in [0, \tilde{T}] \), and \( \phi(\tilde{T}) = \lambda \).

With \( r \) given in (17), define \( T(\mu, \gamma) \) as
\[
T(\mu, \gamma) = \begin{cases} 
\frac{1}{\mu} \arctan(r), & \gamma > \mu, \\
\frac{1}{\mu}, & \gamma = \mu, \\
\frac{1}{\mu} \arctanh(r), & \gamma < \mu.
\end{cases}
\]
(18)

**Remark 1.** Clearly, the functions \( T(\mu, \gamma) \) and \( \tilde{T}(\mu, \gamma, \lambda) \) are both positive, and \( T(\mu, \gamma) = \tilde{T}(\mu, \gamma, 0) \). Furthermore, for fixed \( \mu, \gamma \), \( \tilde{T}(\mu, \gamma, \cdot) \) is a strictly decreasing function, and \( \tilde{T}(\mu, \gamma, \lambda) \to 0 \) as \( \lambda \to 1 \).

### 3.1 State Feedback PETC Design For Nonlinear Systems

In this subsection, PETC design is considered for the setup in Fig. 1 (b) where the plant is given by (1) and the state feedback controller is given by (4). The dynamics of the closed-loop system can be expressed as an impulsive model as follows:

\[
\begin{align*}
\dot{x}_s &= F_s(x, e, w) := \begin{pmatrix} \tilde{f}_s(x, e, w) \\ 1 \end{pmatrix}, \quad t \in (t_k, t_{k+1}], \\
 x_{s\uparrow} &= G_s(x, e) := \begin{pmatrix} x \\ g_s(x, e) \\ 0 \end{pmatrix}, \quad t = t_k,
\end{align*}
\]
(19, 20)

where \( \tau \in \mathbb{R}_{\geq 0} \) is a clock variable, \( e \) is defined in (3), and
\[
\begin{align*}
\begin{pmatrix} x(t) \\ e(t) \\ \tau(t) \end{pmatrix}, & \quad i, \\
\begin{pmatrix} x(t^+) \\ e(t^+) \\ \tau(t^+) \end{pmatrix}, & \quad i^+.
\end{align*}
\]
(21)

where \( f_s(x, e, w) = f(x, k(x - e), w), \)
\[
g_s(x, e) = \begin{cases} 
0, & \text{if } \Gamma_x(x, e) \geq 0, \\
e, & \text{if } \Gamma_x(x, e) < 0.
\end{cases}
\]

Recall that \( t_k \) above is the sampling times defined as \( t_k := kh \) for any \( k \in \mathbb{Z}_{\geq 0} \); the subscript \( s \) in notations above stands for *state*.

Under these notations, the following theorem is given for the state feedback PETC design.

**Theorem 1.** Consider the setup shown in Fig. 1 (b), in which the plant and the controller are given by (1) and (4), respectively. Suppose that there exist positive numbers \( \mu, \gamma, \alpha, d, \) and
a differentiable, positive definite, radially unbounded function \( V_1(x) : \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0} \) such that \( \forall x, e.c., \forall w, \)

\[
\nabla V(x_s) F_s(x, e, w) \leq -\alpha V(x_s) + d\|w\|^2
\]

where \( V(x_s) = V_1(x) + V_2(e, \tau), \) \( V_2(e, \tau) = \phi(\tau)e^\top e, \) \( \phi \) is the solution of ODE \[15\]. Choose positive numbers \( \alpha_0, s, h, \lambda \) satisfying \( \alpha_0 < \alpha, \lambda < 1 \) and

\[
\frac{\log(1 + s)}{\alpha_0} < h < T(\mu, \gamma),
\]

\[
h = \tilde{T}(\mu, \gamma, \lambda),
\]

\[
(1 + s)\lambda^2 < 1,
\]

where \( \tilde{T}(\mu, \gamma, \lambda) \) and \( T(\mu, \gamma) \) are defined in \[16\] and \[18\], respectively. Let the initial condition of \( \phi \) be \( \phi(0) = \lambda^{-1} \). If the triggering function is chosen as

\[
\Gamma(x, e) = (\lambda^{-1} - (1 + s)\lambda)\|e\|^2 - sV_1(x),
\]

then the closed-loop system \[19\]–\[20\] is ISS w.r.t. the set \( \{(x, e, \tau)| (x, e) = (0, 0)\} \).

Proof. By Lemma \[1\] \( \phi(\tau) \in [\lambda, \lambda^{-1}] \) for any \( \tau \in [0, h] \), and \( \phi(h) = \lambda \). Because \( V_1 \) and \( V_2 \) are both positive definite, the function \( V \) is positive definite w.r.t. \( x \) and \( e \) (i.e., \( V(x_s) \geq 0 \) for any \( x, e \in \mathbb{R}^{n_x} \), and \( V(x_s) = 0 \) when \( x = e = 0, V(x_s) \neq 0 \) otherwise). Furthermore, \( V(x_s) \) is differentiable and radially unbounded for any \( x, e \in \mathbb{R}^{n_x} \).

During the continuous dynamics when \( t \in (t_k, t_{k+1}] \), inequality \[22\] implies

\[
V(x_s(t)) \geq \frac{d}{\alpha - \alpha_0} \|w(t)\|^2 \Rightarrow \dot{V}(x_s(t)) \leq -\alpha_0 V(x_s(t)), \forall t \in (t_k, t_{k+1}] \text{ a.e.}
\]

where \( \dot{V}(x_s) \) is the derivative of \( V \) along the trajectory of \[19\].

At the impulse time when \( t = t_k \), there are two cases. Note that \( (1 + s)\lambda^2 < 1 \) implies \( \lambda^{-1} - (1 + s)\lambda > 0 \).

(i) If \( \Gamma(x, e) < 0 \), the triggering condition is not met. Since \( \Gamma(x, e) < 0 \) implies \( \lambda^{-1}\|e\|^2 < (1 + s)\lambda\|e\|^2 + sV_1(x) \), the following holds:

\[
V(x_s^+) = V_1(x) + \lambda^{-1}\|e\|^2 < (1 + s)V(x_s).
\]

(ii) If \( \Gamma(x, e) \geq 0 \), the triggering condition is met. Then from \[20\] and since \( e(t_k^+) = 0 \), it holds that

\[
V(x_s^+) = V_1(x) \leq V(x_s).
\]

In summary, at the impulse time when \( t = t_k \),

\[
V(x_s^+) \leq (1 + s)V(x_s) = e^{\log(1+s)}V(x_s).
\]

Then a bound for \( V(x_s(t)) \) can be shown using \[27\] and \[28\] as follows. Clearly, there exists a sequence of times \( t_0 := t_0 \leq t_1 < \hat{t}_1 < \hat{t}_2 < \hat{t}_2 \ldots \) such that

\[
V(x_s(t)) \geq \frac{d}{\alpha - \alpha_0} \|w(t)\|^2_{[t_0, t]}, \forall t \in (\hat{t}_i, \hat{t}_{i+1}], i = 0, 1, 2, ...
\]

\[
V(x_s(t)) \leq \frac{d}{\alpha - \alpha_0} \|w(t)\|^2_{[t_0, t]}, \forall t \in (\hat{t}_i, \hat{t}_i], i = 1, 2, ...
\]
Now consider the case when the first interval \((t_0, \hat{t}_1]\) is non-empty, i.e., \(\hat{t}_1 > t_0\). If \(\hat{t}_1 < \infty\), then between any two consecutive impulses \(t_{k-1}, t_k \in (t_0, \hat{t}_1]\), from (27) and (29), it follows that \(V(x_s(t)) \leq -\alpha_0 V(x_s(t)), \forall t \in (t_{k-1}, t_k)\) a.e., which implies that
\[
V(x_s(t_k)) \leq e^{-\alpha_0 h} V(x_s(t_{k-1})).
\]

From (28), it follows that
\[
V(x_s(t_{k+})) \leq e^{\log(1+s)} V(x_s(t_k)).
\]

Therefore, for any \(t \in (t_0, \hat{t}_1]\), it holds that
\[
V(x_s(t)) \leq e^{\log(1+s)} \frac{d}{\alpha - \alpha_0} \|w\|_2^2_{[t_0, \hat{t}_1]}.
\]

If \(\hat{t}_1 = \infty\), then it is easy to see that (31) holds for any \(t \in (t_0, \infty)\). Note that \(\frac{\log(1+s) - \alpha_0 h}{h} < 0\) by the choice of \(h\) in (23).

Next, consider the case when \(t \geq \hat{t}_1\). For any subinterval \((\hat{t}_i, \hat{t}_i], i \geq 1\), where \(\hat{t}_i < \infty\), inequality (30) holds. If \(\hat{t}_i\) is not an impulse time, then (31) holds for \(t = \hat{t}_i\). If \(\hat{t}_i\) is an impulse time, then (28) implies that
\[
V(x_s(\hat{t}_i^+)) \leq e^{\log(1+s)} \frac{d}{\alpha - \alpha_0} \|w\|_2^2_{[t_0, \hat{t}_i]}.
\]

In either case, inequality (32) holds. For any subinterval \((\hat{t}_i, \hat{t}_i], i \geq 1\), where \(\hat{t}_i = \infty\), it is easy to see that inequality (32) also holds. In summary, (32) holds for any subinterval \((\hat{t}_i, \hat{t}_i], i \geq 1\).

For any subinterval \((\hat{t}_i, \hat{t}_i], i \geq 1\), using the same argument that derives (31), the following inequality holds for any \(t \in (\hat{t}_i, \hat{t}_i] \):
\[
V(x_s(t)) \leq e^{\log(1+s)} \frac{d}{\alpha - \alpha_0} \|w\|_2^2_{[\hat{t}_i, \hat{t}_i]}.
\]

Therefore, combining (31), (32) and (33), the following bound can be shown for \(V(x_s)\):
\[
V(x_s(t)) \leq \max\{e^{\log(1+s)} \frac{d}{\alpha - \alpha_0} \|w\|_2^2_{[t_0, t]}, e^{\log(1+s)} \frac{d}{\alpha - \alpha_0} \|w\|_2^2_{[t_0, t]}\}, \forall t \geq t_0.
\]

Since \(\frac{\log(1+s) - \alpha_0 h}{h} < 0\), the function \(e^{\frac{\log(1+s) - \alpha_0 h}{h} (t-t_0)}\) is strictly decreasing for \(t \geq t_0\). Since \(V\) is positive definite and radially unbounded for any \(x, e \in \mathbb{R}^{n_x}\), by the standard argument for ISS (e.g., see [20, 23, 21]), it can be concluded that (14) holds. This completes the proof. \(\square\)

Theorem 1 provides a systematic way to determine the sampling period \(h\) and triggering function \(\Gamma_x(x, e)\) for the state feedback PETC design. The key is to find a function \(V(x_s)\), which is in fact an exponential ISS-Lyapunov function of the impulsive system (19)-(20), such that (22) holds. The given structure of \(V(x_s)\) makes it possible to construct \(V(x_s)\) in a systematic way. If, for example, the dynamics (1) is polynomial, the sum-of-squares optimization can be used to find \(V(x_s)\) (see Example 1 in Section 5); the case when (1) is an incrementally quadratic nonlinear system will be discussed in Section 4.

Remark 2. A set of Lyapunov-based sufficient conditions for the input-to-state stability of impulsive systems were given in [22, 27]. Particularly, when the continuous dynamics are exponentially ISS but the impulses are destabilizing, it was shown in [21] that the impulsive system is uniformly ISS if some average dwell-time condition is satisfied, which was relaxed to be a generalized average dwell-time condition in [27]. These important results rely on the existence of (exponential) ISS-Lyapunov functions. Part of the proof of Theorem 1 in [22], and the sampling period \(h\) in PETC design is a lower bound for the dwell-time.
Remark 3. In Theorem 1, there always exist $\alpha_0, s, h, \lambda$ that satisfy (23), (24) and (25). Specifically, since $\log(1 + s) \to 0$ as $s \to 0^+$, there always exist $\alpha_0, s, h$ satisfying (23). Because $T(\mu, \gamma)$ and $\hat{T}(\mu, \gamma, 0)$ have the properties stated in Remark 1, there always exists $\lambda$ satisfying (24). If (25) does not hold with such $s$ and $\lambda$, then it is always possible to find a smaller $s$ such that (25) holds, while still guaranteeing that (23) holds. Therefore, the values $\alpha_0, s, h, \lambda$ in Theorem 1 always exist.

Different choices of $s, h, \lambda$ will result in different triggering frequencies. For instance, if $h$ is chosen, which implies $\lambda$ is fixed, then a smaller $s$ will render the triggering condition (26) easier to be met, which will tend to increase the triggering frequency, while a larger $s$ will tend to decrease the triggering frequency; if $s$ is fixed, then a smaller $h$ will result in a larger $\lambda$, which will tend to decrease the triggering frequency, while a larger $h$ will tend to increase the triggering frequency. The effect of choosing different parameters will be demonstrated by Example 1 in the simulation section.

3.2 Output Feedback PETC Design For Nonlinear Systems

In this subsection, PETC design is considered for the setup in Fig. 2 (b) where the plant is given by (1), the output is given by (5), the observer is given by (9) and the observer-based output feedback controller is given by (12).

Define the estimation error of the observer as

\[
\hat{e}(t) = x(t) - \hat{x}(t)
\]

and

\[
\xi(t) = \begin{pmatrix} x(t) \\ \hat{e}(t) \end{pmatrix}.
\]

Define the sampling induced error as

\[
\eta(t) = \begin{pmatrix} y_e(t) \\ x_e(t) \end{pmatrix}.
\]

where $y_e, x_e$ are defined in (8) and (11), respectively.

Then dynamics of the closed-loop system can be expressed as an impulsive model as follows:

\[
\dot{x}_o(t) = F_o(\xi, \eta, w) := \begin{pmatrix} \tilde{f}_1^1(\xi, \eta, w) \\ \tilde{f}_2^2(\xi, \eta, w) \\ 1 \end{pmatrix}, \quad t \in (t_k, t_{k+1}],
\]

\[
x_o^+(t) = G_o(\xi, \eta) := \begin{pmatrix} \xi \\ \eta \\ \tau \end{pmatrix} = \begin{pmatrix} \xi(h^+) \\ \eta(h^+) \\ \tau(h^+) \end{pmatrix}, \quad t = t_k,
\]

where $\tau$ is a clock variable, and

\[
\tilde{f}_1^1(\xi, \eta, w) = \frac{f(x, k(\hat{x}_e), w)}{f(x, k(\hat{x}_e), w) - \varphi(\hat{x}, k(\hat{x}_e), w)},
\]

\[
\tilde{f}_2^2(\xi, \eta, w) = \frac{\nabla g(x) \cdot f(x, k(\hat{x}_e), w)}{-f(x, k(\hat{x}_e), w) + \varphi(\hat{x}, k(\hat{x}_e), w)},
\]

\[
g_o(\xi, \eta) = \begin{pmatrix} g_1^o(\xi, \eta) \\ g_2^o(\xi, \eta) \end{pmatrix}.
\]
\[
\begin{align*}
g_\omega^1(\xi, \eta) &= \begin{cases} 
0, & \text{if } \Gamma_y(y, y_e) \geq 0, \\
\eta_e, & \text{if } \Gamma_y(y, y_e) < 0,
\end{cases} \\
g_\omega^2(\xi, \eta) &= \begin{cases} 
0, & \text{if } \Gamma_u(\dot{x}, x_e) \geq 0, \\
x_e, & \text{if } \Gamma_u(\dot{x}, x_e) < 0.
\end{cases}
\end{align*}
\]

The subscript \( o \) in notations above stands for \textit{output}.

Under these notations, the following theorem is given for observer-based output feedback PETC design.

\textbf{Theorem 2.} Consider the setup shown in Fig. 2 (b) where the plant, output, observer and controller are given by Eqs. \( 1, 3, 10 \) and \( 12 \), respectively. Suppose that there exist positive numbers \( \mu, \mu_2, \gamma_1, \gamma_2, c_1, c_2, \alpha, d, \) and differentiable, positive definite, radially unbounded functions \( V_1(\xi) : \mathbb{R}^{2n_x} \to \mathbb{R}_{\geq 0}, V_3(y) : \mathbb{R}^{n_y} \to \mathbb{R}_{\geq 0} \) and \( V_4(\dot{x}) : \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0} \) such that \( \forall x_o \text{ a.e., } \forall \omega, \)

\[
\nabla V(x_o)F_o(\xi, \eta, \omega) \leq -\alpha V(x_o) + d\|w\|^2, \\
c_1V_3(g(x)) + c_2V_4(\dot{x}) \leq V_1(\xi),
\]

where

\[
\begin{align*}
V(x_o) &= V_1(\xi) + V_2(\eta, \tau), \\
V_2(\eta, \tau) &= c_1\phi_1 y_e^\top y_e + c_2\phi_2 x_e^\top x_e,
\end{align*}
\]

and \( \phi_i(i = 1, 2) \) are the solutions of the following ODEs:

\[
\dot{\phi}_i = -2\mu_i\phi_i - \gamma_i(\phi_i^2 + 1).
\]

Choose positive numbers \( \alpha_0, s, h, \lambda_1, \lambda_2 \) satisfying \( \alpha_0 < \alpha, \lambda_1 < 1, \lambda_2 < 1, \) and

\[
\begin{align*}
\frac{\log(1 + s)}{\alpha_0} < h < \min\{\mathcal{T}(\mu_1, \gamma_1), \mathcal{T}(\mu_2, \gamma_2)\}, \\
h = \mathcal{T}(\mu_1, \gamma_1, \lambda_1), \quad h = \mathcal{T}(\mu_2, \gamma_2, \lambda_2), \\
(1 + s)\lambda_1^2 < 1, \quad (1 + s)\lambda_2^2 < 1,
\end{align*}
\]

where \( \mathcal{T}(\mu, \gamma, \lambda) \) are defined in (16) and (18), respectively. Let the initial condition of \( \phi_i \) be \( \phi_i(0) = \lambda_i^{-1} \) for \( i = 1, 2. \) If the triggering functions are chosen as

\[
\begin{align*}
\Gamma_y(y, y_e) &= (\lambda_1^{-1} - (1 + s)\lambda_1)\|y_e\|^2 - sV_3(y), \\
\Gamma_u(\dot{x}, x_e) &= (\lambda_2^{-1} - (1 + s)\lambda_2)\|x_e\|^2 - sV_4(\dot{x}),
\end{align*}
\]

then the closed-loop system \( 37-38 \) is ISS w.r.t. the set \( \{(x, \dot{x}, \tau)| (x, \dot{x}, \tau) = (0, 0)\} \).

\textbf{Proof.} By Lemma 1 \( \phi_i(\tau) \in [\lambda_i, \lambda_i^{-1}] \) for any \( \tau \in [0, h] \), and \( \phi_i(h) = \lambda_i, i = 1, 2. \) Because \( V_1, V_2 \) are both positive definite, the function \( V \) is positive definite w.r.t. \( \xi \) and \( \eta \). (i.e., \( V(x_o) \geq 0 \) for any \( \xi \in \mathbb{R}^{2n_x}, \eta \in \mathbb{R}^{n_y+n_y}, \) and \( V(x_o) = 0 \) when \( \xi = \eta = 0, V(x_o) \neq 0 \) otherwise). Furthermore, \( V(x_o) \) is differentiable and radially unbounded for any \( \xi, \eta \).

During the continuous dynamics when \( t \in (t_k, t_{k+1}] \), the inequality (40) holds. Hence,

\[
V(x_o(t)) \geq \frac{d}{\alpha - \alpha_0}\|w(t)\|^2 \Rightarrow \dot{V}(x_o(t)) \leq -\alpha_0 V(x_o(t)), \forall t \in (t_k, t_{k+1}] \text{ a.e.}
\]

where \( \dot{V}(x_o) \) is the derivative of \( V \) along the trajectory of (37).

At the impulse time when \( t = t_k \), there are four cases regarding satisfaction of the input and output triggering conditions. Note that \((1 + s)\lambda_i^2 < 1 \) implies \( \lambda_i^{-1} - (1 + s)\lambda_i > 0, \) for \( i = 1, 2. \)
(i) If $\Gamma_y(y, y_c) < 0$ and $\Gamma_u(\hat{x}, x_c) < 0$, the output and input triggering conditions are not met. Since $\Gamma_y(y, y_c) < 0$, $\lambda_1^{-1}\|y_c\|^2 < (1 + s)\lambda_1\|y_c\|^2 + sV_3(y)$; since $\Gamma_u(\hat{x}, x_c) < 0$, $\lambda_2^{-1}\|x_c\|^2 < (1 + s)\lambda_2\|x_c\|^2 + sV_4(\hat{x})$. Therefore,

$$V(x_o^+)= V_1(\xi) + c_1\lambda_1^{-1}\|y_c\|^2 + c_2\lambda_2^{-1}\|x_c\|^2 < V_1(\xi) + c_1(1 + s)\lambda_1\|y_c\|^2 + c_1 sV_3(y)$$

$$+ c_2(1 + s)\lambda_2\|x_c\|^2 + c_2 sV_4(\hat{x}) = V_1(\xi) + s(c_1 V_3(y) + c_2 V_4(\hat{x})) + (1 + s)(c_1\lambda_1\|y_c\|^2 + c_2\lambda_2\|x_c\|^2) \leq (1 + s)V(x_o).$$

(ii) If $\Gamma_y(y, y_c) < 0$ and $\Gamma_u(\hat{x}, x_c) \geq 0$, then

$$V(x_o^+) = V_1(\xi) + c_1\lambda_1^{-1}\|y_c\|^2 < V_1(\xi) + c_1(1 + s)\lambda_1\|y_c\|^2 + c_1 sV_3(y) \leq (1 + s)V(x_o).$$

(iii) If $\Gamma_y(y, y_c) \geq 0$ and $\Gamma_u(\hat{x}, x_c) < 0$, then

$$V(x_o^+) = V_1(\xi) + c_2\lambda_2^{-1}\|x_c\|^2 < V_1(\xi) + c_2(1 + s)\lambda_2\|x_c\|^2 + c_2 sV_4(\hat{x}) \leq (1 + s)V(x_o).$$

(iv) If $\Gamma_y(y, y_c) \geq 0$ and $\Gamma_u(\hat{x}, x_c) \geq 0$, then

$$V(x_o^+) = V_1(\xi) \leq V(x_o).$$

In summary, at the impulse time when $t = t_k$,

$$V(x_o) \leq (1 + s)V(x_o) = e^{log(1+s)}V(x_o). \quad (49)$$

From (48) and (49), the same argument as in the proof of Theorem 1 can be used to show that

$$V(x_o(t)) \leq \max\{e^{log(1+s)\cdot\alpha h\cdot\sqrt{\gamma}(t-t_0)}V(x_o(t_0)), e^{log(1+s)}\cdot\frac{d}{\alpha - \alpha_0}\|w\|^2\}, \forall t \geq t_0.$$  

Since $\frac{log(1+s)\cdot\alpha h}{h} < 0$ by the choice of $h$, the function $e^{\frac{log(1+s)\cdot\alpha h\cdot\sqrt{\gamma}(t-t_0)}{h}}$ is strictly decreasing for $t \geq t_0$. Since $V$ is positive definite and radially unbounded for any $\xi, \eta$, by the standard argument for ISS, it can be concluded that the closed-loop system (37)-(38) is ISS w.r.t. the set $\{((\xi, \eta, \tau)|(\xi, \eta) = (0, 0))\}$, and therefore, it is ISS w.r.t. the set $\{(x, \hat{e}, \tau)|(x, \hat{e}) = (0, 0)\}$. This completes the proof. $\square$

Theorem 2 provides a systematic way to determine the sampling period $h$ and triggering functions $\Gamma_y(y, y_c), \Gamma_u(\hat{x}, x_c)$ for the observer-based output feedback PETC design. Similar to the discussion in Subsection 3.1, there always exist $\alpha_0, s, h, \lambda_1, \lambda_2$ that satisfy conditions (13)–(15) in Theorem 2 and different choices of $s, h, \lambda_1, \lambda_2$ will result in different triggering frequencies.

When the system (1) has no disturbance (i.e., $w = 0$), Theorem 1 or Theorem 2 guarantees that the closed-loop system with the corresponding ETM is exponentially stable.

Remark 4. The emulation approach has also been used in the event-triggered control design in papers such as [23]. Those papers formulated the system with triggering mechanisms as hybrid systems, and used the techniques of non-smooth analysis and hybrid systems approach to
accomplish the Lyapunov analysis. A key difference between the methods of [2, 32] and results above is that the type of assumptions (for ISS-Lyapunov functions) proposed in [8, 26] are not used in this paper (refer to Assumption 1 in [32], Assumption 1 and 2 in [2]); instead, the ISS-Lyapunov function is assumed to have a special structure and satisfies the ISS condition directly. Though different analysis tools are utilized, in the simulation section, the same example in [2] will be used to illustrate the theoretical results above.

4 PETC Design for Incrementally Quadratic Nonlinear Systems

In this section, PETC design will be investigated for incrementally quadratic nonlinear control systems, which include Lipschitz nonlinear systems and sector bounded nonlinear systems as special cases. As in Section 3, the setups in Fig. 1 for the state feedback control and Fig. 2 for the output feedback control will be considered separately. Sufficient conditions in the form of LMIs will be given for both cases.

Consider the following incrementally quadratic nonlinear control system:

\[
\begin{align*}
\dot{x} &= Ax + Bu + Ep(q) + E_w w, \\
q &= C_q x,
\end{align*}
\]

where \(x \in \mathbb{R}^{n_x}\) is the state, \(u \in \mathbb{R}^{n_u}\) is the control input, \(p : \mathbb{R}^{n_q} \to \mathbb{R}^{n_p}\) is a function representing the known nonlinearity, \(w \in \mathbb{R}^{n_w}\) is the unknown external disturbance, and \(A \in \mathbb{R}^{n_x \times n_x}, B \in \mathbb{R}^{n_x \times n_u}, C_q \in \mathbb{R}^{n_q \times n_x}, E \in \mathbb{R}^{n_x \times n_p}, E_w \in \mathbb{R}^{n_x \times n_w}\) are constant matrices with proper sizes. The characterization of the nonlinearity \(p\) is based on the incremental multiplier matrix defined below.

**Definition 2.** [3, 10] Given a function \(p : \mathbb{R}^{n_q} \to \mathbb{R}^{n_p}\), a symmetric matrix \(M \in \mathbb{R}^{(n_q + n_p) \times (n_q + n_p)}\) is called an incremental multiplier matrix for \(p\) if it satisfies the following incremental quadratic constraint:

\[
\begin{pmatrix} \delta q \\ \delta p \end{pmatrix}^\top M \begin{pmatrix} \delta q \\ \delta p \end{pmatrix} \geq 0, \quad \forall q_1, q_2 \in \mathbb{R}^{n_q}
\]

where \(\delta q = q_2 - q_1\), \(\delta p = p(q_2) - p(q_1)\).

The output of the system (50) is given by

\[
y = Cx
\]

where \(y \in \mathbb{R}^{n_y}, C \in \mathbb{R}^{n_y \times n_x}\).

Given a nonlinearity \(p\), its incremental multiplier matrix that satisfies (51) is not unique. Particularly, if \(M\) is an incremental multiplier matrix for \(p\), then \(\lambda M\) is also a incremental multiplier matrix for \(p\) for any \(\lambda > 0\). Assume that \(p\) satisfies \(p(0_{n_q}) = 0_{n_p}\) in the following: therefore, it holds that

\[
\begin{pmatrix} q \\ p \end{pmatrix}^\top M \begin{pmatrix} q \\ p \end{pmatrix} \geq 0, \quad \forall q \in \mathbb{R}^{n_q}
\]

**Remark 5.** Many nonlinearities can be characterized using the incremental multiplier matrices, such as the globally Lipschitz nonlinearity, the sector bounded nonlinearity, and the polytopic Jacobian nonlinearity. For instance, the sector bounded nonlinearity

\[
(p(t, x) - K_1q(x))^\top S(p(t, x) - K_2q(x)) \leq 0
\]
where $S$ is a symmetric matrix and $K_1, K_2$ are given matrices can be expressed in the form of (51) with

$$M = \begin{pmatrix} -K_1^T SK_2 - K_2^T SK_1 & * \\ S(K_1 + K_2) & -2S \end{pmatrix}.$$ 

Further details can be found in [3, 10, 34, 9, 35].

4.1 State Feedback PETC Design For Incrementally Quadratic Nonlinear Systems

In this subsection, PETC design will be investigated for the setup shown in Fig. 1 (b) where the plant is given as (50)-(51) and the state feedback control is used.

At first, consider the setup shown in Fig. 1 (a) where the plant is given as (50)-(51), and the controller has the following form

$$u = K_1 x + K_2 p(C_q x)$$

(54)

where $K_1 \in \mathbb{R}^{n_u \times n_x}$, $K_2 \in \mathbb{R}^{n_u \times n_p}$. The dynamics of this continuous-time closed-loop system are expressed as

$$\dot{x} = (A + BK_1)x + (E + BK_2)p + E_w w.$$ 

(55)

The following lemma provides sufficient conditions on $K_1, K_2$ such that (55) is ISS.

**Lemma 2.** Consider the plant given by (50) with $p$ satisfying (51) and the controller given by (54). Given $\alpha > 0$, suppose that there exist matrices $P_1 \in \mathbb{R}^{n_x \times n_x}, P_1 = P_1^T > 0$, $P_2 \in \mathbb{R}^{n_u \times n_x}, K_2 \in \mathbb{R}^{n_u \times n_p}$ and positive numbers $d, \sigma$ such that

$$\begin{pmatrix} \Phi & E + BK_2 & E_w \\ * & 0 & 0 \\ * & * & -dI_{n_w} \end{pmatrix} + \sigma S^T M S \preceq 0$$ 

(56)

where $M$ is given in (2), and

$$\Phi = AP_1 + P_1 A^T + BP_2 + P_2^T B^T + \alpha P_1,$$

$$S = \begin{pmatrix} C_q & 0_{n_q \times n_p} & 0_{n_q \times n_w} \\ 0_{n_p \times n_x} & I_{n_p} & 0_{n_p \times n_w} \end{pmatrix}.$$ 

If $K_1$ is chosen as $K_1 = P_2 P_1^{-1}$, then (55) is ISS.

**Proof.** Define $z = (x^T, p^T, w^T)^T$, and $V = x^T P x$ where $P = P_1^{-1}$. Multiply the left-hand side and the right-hand side of (56) by $z^T \text{diag}(P_1^{-1}, I_{n_p}, I_{n_w})$ and its transpose, respectively. Noting that $P_2 = K_1 P_1$, $Sz = \begin{pmatrix} q \\ p \end{pmatrix}$ and (55) holds, it follows that $\dot{V} \leq -\alpha V + d\|w\|^2$. This completes the proof. 

In the following, matrices $K_1, K_2$ are assumed to be known and chosen such that (55) is ISS.

Now consider the closed-loop system implemented with ETM shown in Fig. 1 (b), where the plant is given by (50) with $p$ satisfying (51) for a given $M$, the triggering function is $\Gamma_x(x, e)$, and the controller is

$$u(t) = K_1 \hat{x}_c(t) + K_2 p(C_q \hat{x}_c(t))$$

(57)
Theorem 3. Consider the setup in Fig. 1(b) where the plant is given by (50) with \( p \) satisfying (51) for a given \( M \) and the controller is given by (57). Given \( \alpha > 0 \), suppose that there exist positive numbers \( \mu, \gamma, d \), non-negative numbers \( \sigma_1, \sigma_2 \), matrix \( P \in \mathbb{R}^{n_x \times n_x} \) where \( P = P^\top > 0 \), such that (59) holds where \( \Psi, S_1, S_2 \) are given as

\[
\begin{pmatrix}
\Psi & -PBK_1 & P(E+BK_2) & PBK_2 & (A+BK_1)^\top & PE_w \\
* & -\gamma I & 0 & 0 & -(BK_1)^\top + (\frac{\alpha}{2} - \mu)I & 0 \\
* & * & 0 & 0 & (E+BK_2)^\top & 0 \\
* & * & * & 0 & -\gamma I & E_w \\
* & * & * & * & -dI & 0
\end{pmatrix} + \sigma_1 S_1^\top MS_1 + \sigma_2 S_2^\top MS_2 \preceq 0
\]

(59)

Choose positive numbers \( \alpha_0, s, h, \lambda \) satisfying \( \alpha_0 < \alpha, \lambda < 1 \) and

\[
\begin{cases}
\frac{\log(1+s)}{\alpha_0} < h < \mathcal{T}(\mu, \gamma), \\
h = \tilde{\mathcal{T}}(\mu, \gamma, \lambda), \\
(1+s)\lambda^2 < 1,
\end{cases}
\]

(61)

where \( \tilde{\mathcal{T}}(\mu, \gamma, \lambda) \) and \( \mathcal{T}(\mu, \gamma) \) are defined in (10) and (18), respectively. If the triggering function is chosen as

\[
\Gamma_{x}(x, e) = (\lambda^{-1} - (1+s)\lambda)\|e\|^2 - sx^\top Px,
\]

then the closed-loop system (19)-(20) is ISS w.r.t. the set \( \{(x, e, \tau) | (x, e) = (0, 0)\} \).

Proof. Define \( V(x_s) = V_1(x) + V_2(e, \tau) \) where \( V_1(x) = x^\top Px \), \( V_2(e, \tau) = \phi(\tau)e^\top e \), \( x_s \) is defined in (21), and \( \phi \) is the solution of ODE (15) with the initial condition \( \phi(0) = \lambda^{-1} \). It is easy to see that if (22) holds during the flow (i.e., when \( t \in (t_k, t_{k+1}] \)), then all the conditions of Theorem 1 hold with \( \Gamma_{x} \) defined in (26) and the conclusion follows immediately.
Define $\varrho = \phi e$ and $\zeta = (x^T, e^T, p^T, \delta p^T, \varrho^T, w^T)^T$. Clearly, $V_2(e, \tau) = e^T \varrho$. During the flow,

$$\langle \nabla V(x_\tau), F_s(x, e, w) \rangle = \frac{\partial V_1}{\partial x} f_s(x, e, w) + \frac{\partial V_2}{\partial e} f_s(x, e, w) + e^T \frac{dQ}{d\tau} e$$

$$= 2(x^T P + \varrho^T)((A + BK_1)x - BK_1e + (E + BK_2)p + BK_2\delta \tilde{p} + E_w w) + e^T (-2\mu \phi - \gamma(\phi^2 + 1)) e. \quad (62)$$

Note that $\zeta$ is generated by the right-hand side of (59) by $\zeta^T$ and $\zeta$, respectively, it follows that $2(x^T P + \varrho^T)((A + BK_1)x - BK_1e + (E + BK_2)p + BK_2\delta \tilde{p} + E_w w) + e^T (-2\mu \phi - \gamma(\phi^2 + 1)) e + \alpha x^T P x + \alpha e^T \varrho - d||w||^2 + \sigma_1 \zeta^T S_1^T MS_1 \zeta + \sigma_2 \zeta^T S_2^T MS_2 \zeta \leq 0$. Therefore, it is easy to obtain that the right-hand side of (62) is less than or equal to $-\alpha x^T P x + e^T \varrho + d||w||^2$, which is equal to $-\alpha V(x_\tau) + d||w||^2$. In summary, (22) holds and this completes the proof.

4.2 Output Feedback PETC Design For Incrementally Quadratic Nonlinear Systems

In this subsection, PETC design will be investigated for the setup shown in Fig. 2(b) where the plant is given as (50)-(51) with the output given in (52).

At first, consider the setup in Fig. 2(a) where the plant is given as (50)-(51), the output is given as (52), the observer has the following form

$$\begin{align*}
\dot{x} &= A \hat{x} + Bu + E p(q + L_1(\hat{y} - y)) + L_2(\hat{y} - y), \\
\dot{\hat{y}} &= C \hat{x}, \\
\dot{\hat{q}} &= C_\varrho \hat{x},
\end{align*} \quad (63)$$

with $L_1 \in \mathbb{R}^{n_x \times n_y}, L_2 \in \mathbb{R}^{n_x \times n_y}$, and the controller $u$ has the following form

$$u(t) = K_1 \hat{x}(t) + K_2 p(C_\varrho \hat{x}(t)) \quad (64)$$

with $K_1 \in \mathbb{R}^{n_u \times n_x}, K_2 \in \mathbb{R}^{n_u \times n_y}$. The dynamics of this continuous-time closed-loop system can be expressed as

$$\dot{\xi} = A_1 \xi + H_1 p + H_2 \delta p + H_3 \Delta p + H_4 w \quad (65)$$

where $\xi$ is defined in (55), and

$$\begin{align*}
\Delta p &= p(\hat{q}) - p(q), \\
\delta p &= p(q + \delta q) - p(q), \\
\delta q &= -(C_\varrho + L_1 C) \hat{e}, \\
A_1 &= \begin{pmatrix} A + BK_1 & -BK_1 \\ 0 & A + L_2 C \end{pmatrix}, \\
H_1 &= \begin{pmatrix} E + BK_2 \\ 0 \end{pmatrix}, H_2 = \begin{pmatrix} 0 \\ -E \end{pmatrix}, \\
H_3 &= \begin{pmatrix} BK_2 \\ 0 \end{pmatrix}, H_4 = \begin{pmatrix} E_w \\ E_w \end{pmatrix}.
\end{align*} \quad (66)$$
LMI-based sufficient conditions to design $L_1, L_2, K_1, K_2$ are given in Theorem 1 and Theorem 2 in [34] for the case where the set of incremental multiplier matrices for $p$ have a block diagonal parameterization and a block anti-triangular parameterization, respectively. In the following, matrices $L_1, L_2, K_1, K_2$ are assumed to be known and designed such that (75) is ISS.

Now consider the closed-loop system implemented with ETMs shown in Fig. 2 (b). The observer in Fig. 2 (b) becomes

\[
\begin{align*}
\dot{x} &= A\hat{x} + Bu + E(p(\hat{q} + L_1(\hat{y} - y_c))) + L_2(\hat{y} - y_c), \\
\dot{y} &= C\hat{x}, \\
\hat{q} &= C_q\hat{x},
\end{align*}
\]

where $y_c$ is defined in (7). The observer-based controller is now given as

\[
u(t) = K_1\hat{x}_c(t) + K_2p(C_q\hat{x}_c(t)).
\]

Recall that $\xi = \begin{pmatrix} x \\ e \end{pmatrix}$ and $\eta = \begin{pmatrix} y_c \\ x_c \end{pmatrix}$ as defined in (35) and (36), respectively. Then the closed-loop system in Fig. 2 (b), where the plant is given by (50)-(52), the observer is given by (68), and the triggering functions are $\Gamma_y, \Gamma_u$, can be expressed in the form of impulsive system (37)-(38) with

\[
\hat{f}_0^1(\xi, \eta, w) = A_1\xi + A_2\eta + H_1p + H_2\delta p + H_3\delta \hat{p} + H_4w,
\]

\[
\hat{f}_0^2(\xi, \eta, w) = A_3\xi + A_4\eta + H_5p + H_6\delta \hat{p} + H_7\delta \hat{p} + H_8w,
\]

matrices $A_1, H_1, H_2, H_3, H_4$ given in (67), and

\[
\begin{cases}
\delta \hat{p} = p(\xi + \delta \hat{y}) - p(q), \\
\delta \hat{q} = C_q(x_c - \hat{e}), \\
\delta \hat{y} = -(C_q + L_1C)e - L_1y_c,
\end{cases}
\]

\[
\begin{align*}
A_2 &= \begin{pmatrix} 0 & BK_1 \\ L_2 & 0 \end{pmatrix}, & A_4 &= \begin{pmatrix} 0 & -CBK_1 \\ L_2 & -BK_1 \end{pmatrix}, \\
A_3 &= -C(A + BK_1) \begin{pmatrix} CBK_1 \\ 0 \end{pmatrix}, & \text{such that } (73) &\text{ holds where } R_1, R_2, R_3, S_1, S_2, S_3, \text{ and } \sigma, \sigma_1, \sigma_2, \sigma_3, \text{ and } \text{matrix } P \in \mathbb{R}^{2n_x \times 2n_y}, \text{ and } P = P^T > 0, \\
H_5 &= -C(E + BK_2) \begin{pmatrix} 0 \\ E \end{pmatrix}, & H_6 &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\
H_7 &= -CBK_2 \begin{pmatrix} 0 \\ -BK_2 \end{pmatrix}, & H_8 &= \begin{pmatrix} -CEw \\ 0 \end{pmatrix}.
\end{align*}
\]

The following theorem is given for observer-based output feedback PETC design for incrementally quadratic nonlinear systems.

**Theorem 4.** Consider the setup in Fig. 2 (b) where the plant is given by (50) with $p$ satisfying (54) for a given $M$, the output is given by (52), the observer is given by (68), and the controller is given by (69). Given $\alpha > 0$, suppose that there exist positive numbers $\mu_1, \mu_2, a_1, a_2, b_1, b_2, d, \sigma_1, \sigma_2, \sigma_3$, and matrix $P \in \mathbb{R}^{2n_x \times 2n_y}, P = P^T > 0$, such that (73) holds where $R_1, R_2, R_3, S_1, S_2, S_3$ are given in (74). Suppose that there exist matrices $P_1 \in \mathbb{R}^{n_x \times n_y}, P_1 = P_1^T > 0, P_2 \in \mathbb{R}^{n_x \times n_x}, P_2 = P_2^T > 0, such that

\[
\begin{pmatrix} c_1C^TP_1C & 0 \\ 0 & c_2P_2 \end{pmatrix} \preceq \begin{pmatrix} I_{n_x} & I_{n_x} \\ I_{n_x} & -I_{n_x} \end{pmatrix} P \begin{pmatrix} I_{n_x} & 0 \\ I_{n_x} & -I_{n_x} \end{pmatrix}
\]

(75)
where \( c_1 = \sqrt{a_1/b_1}, c_2 = \sqrt{a_2/b_2} \). Choose positive numbers \( \alpha_0, s, h, \lambda_1, \lambda_2 \) satisfying \( \alpha_0 < \alpha \), \( \lambda_1 < 1, \lambda_2 < 1 \), and

\[
\begin{aligned}
\frac{\log(1+s)}{\alpha_0} < h < \min\{T(\mu_1, \gamma_1), T(\mu_2, \gamma_2)\}, \\
h = \tilde{T}(\mu_1, \gamma_1, \lambda_1), \quad h = \tilde{T}(\mu_2, \gamma_2, \lambda_2), \\
(1+s)\lambda_1^2 < 1, \quad (1+s)\lambda_2^2 < 1,
\end{aligned}
\]

(76)

where \( \gamma_1 = \sqrt{a_1b_1}, \gamma_2 = \sqrt{a_2b_2} \), and \( \tilde{T}(\mu, \gamma, \lambda), \tilde{T}(\mu, \gamma) \) are defined in (16) and (18), respectively. If the triggering functions are chosen as

\[
\begin{aligned}
\Gamma_y(y, y) &= (\lambda_1^{-1} - (1+s)\lambda_1)\|y_e\|^2 - sy^TP_1y, \\
\Gamma_u(\hat{x}, x_e) &= (\lambda_2^{-1} - (1+s)\lambda_2)\|x_e\|^2 - s\hat{x}^TP_2\hat{x},
\end{aligned}
\]

then the closed-loop system shown in Fig. 2(b) is ISS w.r.t. the set \( \{(x, \hat{e}, \tau)| (x, \hat{e}) = (0, 0)\} \).

Proof. Define \( V(x_{\hat{a}}) = V_1(\xi) + V_2(\eta, \tau) \) where \( V_1(x) = \xi^TP_1\xi \), \( V_2(\eta, \tau) = c_1\phi_1y_e^ty_e + c_2\phi_2x_e^tx_e \), \( x_{\hat{a}} \) is defined in (39), and \( \phi_i \) is the solution of ODE \( \dot{\phi}_i = -2\mu_i\phi_i - \gamma_i(\phi_i^2 + 1) \) with the initial condition \( \phi_i(0) = \lambda_i^{-1} \), for \( i = 1, 2 \). Define \( V_3(y) = y^T P_1 y \) and \( V_4(\hat{x}) = \hat{x}^T P_2 \hat{x} \). It is easy to see that if (40) and (41) hold during the flow (i.e., when \( t \in (t_k, t_{k+1}) \)), then all the conditions of Theorem 2 hold with \( \Gamma_u, \Gamma_y \) given in (40)-(47), and the conclusion follows immediately.

Define \( \varphi = \begin{pmatrix} \varphi_y \\ \varphi_x \end{pmatrix} := \begin{pmatrix} c_1\phi_1y_e \\ c_2\phi_2x_e \end{pmatrix} \) and

\[
\zeta = (\xi^T, \eta^T, p^T, \delta p^T, \dot{\varphi}^T, \varphi^T, w^T)^T.
\]

Clearly, \( \varphi = Q\eta \), which implies that \( V_2(\eta, \tau) = \eta^T\varphi \).

\[
\begin{pmatrix} PA_1 + A_1^TP + \alpha P & PA_2 & PH_1 & PH_2 & PH_3 & A_3^T & PH_4 \\
* & R_1 & 0 & 0 & 0 & A_3^T + R_3^T + \frac{\alpha}{2}I & 0 \\
* & * & 0 & 0 & 0 & H_5^T & 0 \\
* & * & * & 0 & 0 & H_6^T & 0 \\
* & * & * & * & 0 & R_2 & H_7 \\
* & * & * & * & * & * & -dI \\
PQ_1^T M S_1 + \sigma_2 S_2^T M S_2 + \sigma_3 S_3^T M S_3 \leq 0
\end{pmatrix}
\]

(73)

\[
\begin{aligned}
R_1 &= \begin{pmatrix} -a_1I_{n_y} & 0 \\ 0 & -a_2I_{n_x} \end{pmatrix}, \quad R_2 = \begin{pmatrix} -b_1I_{n_y} & 0 \\ 0 & -b_2I_{n_x} \end{pmatrix}, \quad R_3 = \begin{pmatrix} -\mu_1I_{n_y} & 0 \\ 0 & -\mu_2I_{n_x} \end{pmatrix}, \\
S_1 &= \begin{pmatrix} C_q, 0_{n_y \times (3n_x + 2n_y + 3n_p + n_w)} \\ 0_{n_y \times (3n_x + n_y)}, I_{n_p}, 0_{n_y \times (n_x + n_y + 2n_p + n_w)} \end{pmatrix}, \\
S_2 &= \begin{pmatrix} 0_{n_y \times n_x}, -(C_q + L_1C_1), -L_1, 0_{n_y \times (2n_x + n_y + 3n_p + n_w)} \\ 0_{n_y \times (3n_x + n_y + n_p)}, I_{n_p}, 0_{n_y \times (n_x + n_y + n_p + n_w)} \end{pmatrix}, \\
S_3 &= \begin{pmatrix} 0_{n_y \times n_x}, -C_q, 0_{n_y \times n_y}, C_q, 0_{n_y \times (n_x + n_y + 3n_p + n_w)} \\ 0_{n_y \times (3n_x + n_y + 2n_p + n_w)}, I_{n_p}, 0_{n_y \times (n_x + n_y + n_w)} \end{pmatrix}.
\end{aligned}
\]

(74)
During the flow (37),
\[
\langle \nabla V(x_o), F_o(\xi, \eta, w) \rangle = \frac{\partial V_1}{\partial \xi} \dot{f}_o^1(\xi, \eta, w) + \frac{\partial V_2}{\partial \eta} \dot{f}_o^2(\xi, \eta, w) + \eta^\top \frac{\partial Q}{\partial \tau} \eta \\
= 2\xi^\top P(A_1 \xi + A_2 \eta + H_1 p + H_2 \delta \dot{p} + H_3 \delta \dot{p} + H_4 w) \\
+ 2\eta^\top Q(A_3 \xi + A_4 \eta + H_5 p + H_6 \delta \dot{p} + H_7 \delta \dot{p} + H_8 w) \\
+ c_1 y_e (2\mu_1 \phi_1 - \gamma_1 (\phi_2 + 1)) y_e \\
+ c_2 x_e (-2\mu_2 \phi_2 - \gamma_2 (\phi_2 + 1)) x_e \\
= 2\xi^\top P(A_1 \xi + A_2 \eta + H_1 p + H_2 \delta \dot{p} + H_3 \delta \dot{p} + H_4 w) \\
+ 2\eta^\top Q(A_3 \xi + A_4 \eta + H_5 p + H_6 \delta \dot{p} + H_7 \delta \dot{p} + H_8 w) \\
+ \eta^\top R_1 \eta + \rho^\top R_2 \rho + 2\eta^\top R_3 \rho.
\]

(77)

Note that \((q, p) = S_1 \xi, (\delta q, \delta p) = S_2 \eta, (\delta q, \delta p) = S_3 \xi\). Since \(p\) satisfies (31) and (53), it holds that
\[
\frac{\partial q}{\partial p} M \frac{\partial p}{\partial q} \geq 0, \quad \frac{\partial q}{\partial p} M \frac{\partial p}{\partial q} \geq 0,
\]
which implies that \(\sigma_1 \xi^\top S_1^\top M S_1 \xi \geq 0, \quad \sigma_2 \eta^\top S_2^\top M S_2 \eta \geq 0, \quad \sigma_3 \xi^\top S_3^\top M S_3 \xi \geq 0\). Multiplying the left-hand side and the right-hand side of (73) by \(\xi^\top\) and \(\xi\), respectively, it follows that
\[
2\xi^\top P(A_1 \xi + A_2 \eta + H_1 p + H_2 \delta \dot{p} + H_3 \delta \dot{p} + H_4 w) + 2\eta^\top Q(A_3 \xi + A_4 \eta + H_5 p + H_6 \delta \dot{p} + H_7 \delta \dot{p} + H_8 w) + \eta^\top R_1 \eta + \rho^\top R_2 \rho + 2\eta^\top R_3 \rho \leq 0.
\]
Therefore, it is easy to obtain that the right-hand side of (77) is less than or equal to \(-\alpha (\xi^\top P_x + \eta^\top \theta) + d ||w||^2\), which is equivalent to \(-\alpha V(x_o) + d ||w||^2\). Therefore, (40) holds during the flow.

Since \(\xi = \begin{pmatrix} x-n_x \ 0 \ x \end{pmatrix}, \frac{\partial}{\partial x}\) and its transpose to the left-hand side and the right-hand side of (75), respectively, it follows that \(c_1 \xi^\top C^\top P_1 C x + c_2 \xi^\top P_2 \xi \leq \xi^\top P x\), which is equivalent to \(c_1 y^\top P_1 y + c_2 \xi^\top P_2 \xi \leq \xi^\top P x\). Therefore, (75) implies that (41) holds with \(V_3 = y^\top P_1 y, V_4 = \xi^\top P_2 \xi\). This completes the proof.

Conditions of Theorem 4 can be simplified by letting \(\gamma_i = a_i = b_i = c_i = 1\) for \(i = 1, 2\); in this case, the function \(V_2(\eta, \tau)\) in the proof becomes \(V_2(\eta, \tau) = \phi_1 y_e^\top y_e + \phi_2 x_e^\top x_e\).

As discussed in Section 3 there always exist \(\alpha_0, \sigma, \lambda, \lambda\) that satisfy the conditions (61) in Theorem 3 and there always exist \(\alpha_0, \sigma, h, s, \lambda_1, \lambda_2\) that satisfy the conditions (70) in Theorem 4.

4.3 Special Case: Continuous-time Linear Systems

PETC design for continuous-time linear control systems was investigated in [20]. In fact, results in preceding subsections can be applied to linear control systems directly. Specifically, when \(E = 0\), dynamics of (50) becomes
\[
\dot{x} = Ax + Bu + E_w w
\]
where \(A \in \mathbb{R}^{n_x \times n_x}, B \in \mathbb{R}^{n_x \times n_u}, E_w \in \mathbb{R}^{n_x \times n_w}\). Two corollaries about PETC design for system (78) will be shown in this subsection.

For the setup in Fig. 1(b), suppose that the state feedback controller implemented with ETM is given as
\[
u(t) = K \ddot{z}_c(t)
\]
where \(K \in \mathbb{R}^{n_w \times n_x}\) and \(\ddot{z}_c\) is defined in (2). The following corollary, which follows from Theorem 3 provides sufficient conditions based on LMIs for the state feedback PETC design for linear control systems.
Corollary 1. Consider the setup in Fig. 7 (b), where the plant is given by (78) and the controller is given by (79). Given \( \alpha > 0 \), suppose that there exist positive numbers \( \mu, \gamma, d \), a matrix \( P = P^T > 0 \) such that
\[
\begin{pmatrix}
\Psi & -PBK & (A + BK)^T & PE_w \\
* & -\gamma I & -(BK)^T + (\frac{a}{2} - \mu)I & 0 \\
* & * & -\gamma I & 0 \\
* & * & * & -dI \\
\end{pmatrix} \preceq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
\]
where \( \Psi = P(A + BK) + (A + BK)^TP + \alpha P \). Choose positive numbers \( \alpha_0, s, h, \lambda \) satisfying \( \alpha_0 < \alpha, \lambda < 1, \frac{\log(1+s)}{\alpha_0} < h < T(\mu, \gamma), h = \tilde{T}(\mu, \gamma, \lambda), \) and \( (1+s)\lambda^2 < 1 \). If the triggering function is chosen as \( \Gamma_\epsilon(x, e) = (\lambda^{-1} - (1+s)\lambda)\|e\|^2 - sx^TPx \), then the closed-loop system shown in Fig. 4 (b) is ISS w.r.t. the set \( \{(x, e, \tau)|(x, e) = (0, 0)\} \).

Similarly, for the setup in Fig. 2 (b), suppose that the output of (78) is \( y = Cx \) with \( C \in \mathbb{R}^{n_y \times n_x} \), the observer is given as
\[
\dot{x} = A\hat{x} + Bu + L(C\hat{x} - y_c)
\]
where \( L \in \mathbb{R}^{n_x \times n_y} \), \( y_c \) is defined in (7), and the controller is given as
\[
u(t) = K\hat{x}_c(t)
\]
where \( \hat{x}_c \) is defined in (10). The following corollary, which follows from Theorem 4, provides sufficient conditions based on LMIs for observer-based output feedback PETC design for linear control systems.

Corollary 2. Consider the setup in Fig. 3 (b), where the plant is given by (78) with the output \( y = Cx \), the observer is given by (80), and the controller is given by (81). Given \( \alpha > 0 \), suppose that there exist positive numbers \( \mu_1, \mu_2, a_1, a_2, b_1, b_2, d \), non-negative numbers \( \sigma_1, \sigma_2, \sigma_3 \), a matrix \( P = P^T > 0 \) such that
\[
\begin{pmatrix}
\Psi & PA_2 & A_3^T & PH_3 \\
* & R_1 & A_3^T + R_3^T + \frac{d}{2}I & 0 \\
* & * & R_2 & H_5 \\
* & * & * & -dI \\
\end{pmatrix} \preceq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
\]
where \( \Psi = PA_1 + A_1^TP + \alpha P, A_i (i = 1, 2, 3, 4) \) and \( H_4, H_5 \) are those given in (67) and (72), \( R_i (i = 1, 2, 3) \) are those given in (74). Suppose that there exist symmetric and positive definite matrices \( P_1, P_2 \) such that (75) holds where \( c_1 = \sqrt{a_1/b_1}, c_2 = \sqrt{a_2/b_2} \). Choose positive numbers \( \alpha_0, s, h, \lambda_1, \lambda_2 \) satisfying \( \alpha_0 < \alpha, \lambda_1 < 1, \lambda_2 < 1, \) and \( \frac{\log(1+s)}{\alpha_0} < h < \min\{T(\mu_1, \gamma_1), T(\mu_2, \gamma_2)\} \), \( h = \tilde{T}(\mu, \gamma, \lambda) \), \( (1+s)\lambda_1^2 < 1 \) for \( i = 1, 2 \), where \( \gamma_1 = \sqrt{a_1/b_1}, \gamma_2 = \sqrt{a_2/b_2} \). If the triggering functions are chosen as \( \Gamma_y(y, y_c) = (\lambda_1^{-1} - (1+s)\lambda_1)\|y_c\|^2 - sy^TPy, \Gamma_y(\dot{x}, \dot{x}) = (\lambda_2^{-1} - (1+s)\lambda_2)\|\dot{x}\|^2 - s\dot{x}^TP\dot{x} \), then the closed-loop system shown in Fig. 4 (b) is ISS w.r.t. the set \( \{(x, \dot{e}, \tau)|(x, \dot{e}) = (0, 0)\} \).

5 Simulation Examples

In this section, two examples will be presented to illustrate the theoretical results in preceding sections: Example 1 illustrates the state feedback PETC design for a polynomial nonlinear system using Theorem 1 and Example 2 illustrates the state feedback and the output feedback PETC design for an incrementally quadratic nonlinear system, using Theorem 3 and Theorem 4 respectively.
Example 1. Consider the following example in [6][20]. The continuous time plant is given as
\[
\dot{x} = x^2 - x^3 + u + 0.1w
\] (82)
and the continuous time state feedback controller is given as \(u(t) = -2x(t)\). When the closed-loop system is implemented with ETM as shown in Fig. 7 (b), the controller becomes \(u(t) = -2\bar{x}_c(t)\) as in (4) where \(\bar{x}_c(t)\) is defined in (2). Then the closed-loop system can be expressed as impulsive model (19)-(20) with \(f(x, e, w) = x^2 - x^3 - 2x + 2e + w\) where \(e = x - \bar{x}_c\) as defined in (3). Since the dynamics of (82) are polynomial, the SOSTOOLS toolbox (see [27]) is used to find out that (22) holds with \(V_1(x) = 1.0192x^2 - 0.1298x^3 + 0.4784x^4\), \(\mu = 0.4941\), \(\gamma = 4.4302\), \(\alpha = 1.2\), \(d = 0.1\). Since \(\mathcal{T}(\mu, \gamma) = 0.3314\), pick \(s = 0.1\), \(\alpha_0 = 1.1\), which implies that \(\frac{\log(1+s)}{\alpha_0} = 0.0866\), and the sampling period \(h = 0.1\), such that \(\frac{\log(1+s)}{\alpha_0} < h < \mathcal{T}(\mu, \gamma)\). Then there exists \(\lambda = 0.6\) such that \(h = \mathcal{\tilde{T}}(\mu, \gamma, \lambda)\), and one can verify that \((1 + s)\lambda^2 < 1\). By Theorem 7, the triggering condition is chosen as
\[
\Gamma_x(e, x) = 1.0067e^2 - 0.1V_1(x).
\]

The simulation results for two sets of initial states and disturbance bounds are shown in Fig 3 where trajectories of the state \(x\) and the input \(u\) are depicted. The red lines (resp. blue lines) indicate the simulation with the initial state \(x(0) = 0.3\) (resp. \(x(0) = -0.4\)) where the disturbance \(w\) is generated randomly and satisfies \(\|w\|_{\infty} \leq 0.8\) (resp. \(\|w\|_{\infty} \leq 0.2\)). In the top subfigure, it can be observed that the state \(x\) is eventually bounded in the presence of disturbances, and a larger bound of \(w\) results in a larger ultimate bound of \(x\); in the bottom subfigure, the input \(u\) is piecewise-constant and it changes its value at each \(t_k\) such that \(\Gamma_x(e(t_k), x(t_k)) \geq 0\).

Since any \(h\) that satisfies \(\frac{\log(1+s)}{\alpha_0} < h < \mathcal{T}(\mu, \gamma)\) can be chosen to be the sampling period, different values of \(h\) and the corresponding triggering functions can be computed, with the same \(V_1(x)\) and \(\mu, \gamma, \alpha, d, s, \alpha_0\) shown above. Let \(w\) satisfy \(\|w\|_{\infty} \leq 0.3\), \(x(0)\) chosen randomly from the set \([-0.5, 0.5]\), and run 100 simulations for \(h = 0.1, 0.15, 0.2, 0.25\) seconds. Table 3 summarizes the average triggering frequency \(f_{avg}\), which is the frequency that \(\Gamma_x(x, e) \geq 0\) during the simulations averaged by 100. It can be observed that \(f_{avg}\) increases as \(h\) increases, which is consistent with the discussion in Remark 3.

| \(h\) | 0.1s | 0.15s | 0.2s | 0.25s |
|-------|-------|-------|-------|-------|
| \(f_{avg}\) | 66.6% | 83.1% | 89.1% | 92.9% |

Table 1: Values of the average triggering frequency \(f_{avg}\) (i.e., the average frequency that \(\Gamma_x(x, e) \geq 0\)) based on 100 simulations.

Example 2. Consider the following dynamical model of the single-link robot arm that was given in [22]:
\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -\sin(x_1) + u + w,
\end{align*}
\] (83)
where \(x = (x_1, x_2)^T\) is the state representing the angle and the rotational velocity, \(u\) is the input representing the torque, and \(w\) is the unknown disturbance. The system can be written in the form of [50] with \(A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\), \(B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\), \(E = \begin{pmatrix} 0 \\ -1 \end{pmatrix}\), \(E_w = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\), \(q = x_1\), \(C_q = (1, 0)\), \(p(q) = \sin(q)\). The nonlinearity \(p\) is globally Lipschitz and satisfies the incremental quadratic constraint [51] with \(M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\).
Figure 3: Trajectories of the state $x$ and the input $u$ in the simulation of Example 1. The sampling period is chosen as $h = 0.1$ seconds. For $x(0) = 0.3$ and $\|w\|_\infty \leq 0.8$, the trajectories are depicted in red, and for $x(0) = -0.4$ and $\|w\|_\infty \leq 0.2$, the trajectories are depicted in blue.

First, consider the setup in Fig. 1 and assume that the continuous-time controller $u$ has the form as given in (54). By Lemma 2, $K_1 = (-11.2257, -5.5774)$ and $K_2 = 1$ can be chosen such that the closed-loop system in Fig. 2 (a) is ISS. By letting $\alpha = 1.2$, the LMI (59) in Theorem 3 can be solved that yields $\mu = 5, \gamma = 20, d = 0.6$, and $P = \begin{pmatrix} 6.5131 & 0.6581 \\ 0.6581 & 0.7294 \end{pmatrix}$.

Choose $h = 0.04, s = 0.04, \lambda = 0.31, \alpha_0 = 1$ such that (61) holds. By Theorem 3, the triggering function $\Gamma_x$ is chosen as

$$\Gamma_x(x, e) = 2.9e^2 - 0.04x^\top Px.$$ 

Choose the initial state as $x_1(0) = 0.5, x_2(0) = -0.5$, and the disturbance to be randomly generated and satisfies $\|w\|_\infty \leq 0.1$. The simulation results are shown in Fig. 4, where trajectories of $x$ and $u$ are shown, respectively.

Consider now the setup in Fig. 2 where the output information $y = x_1$ is available with $C = (1, 0)$. Assume that the continuous-time observer has the form given in (63) and the continuous-time controller has the form given in (64). By the results of (74), $K_1 = (-7.3936, -3.9937)$,
Figure 4: Trajectories of the state $x$ and the input $u$ in the state feedback PETC simulation of Example 2. The sampling period is chosen as $h = 0.04$ seconds, the initial state is chosen as $x_1(0) = 0.5, x_2(0) = -0.5$ and the disturbance satisfies $\|w\|_\infty \leq 0.1$.

$K_2 = 1$, $L_1 = -1$, $L_2 = \begin{pmatrix} -5.1294 \\ -18.0352 \end{pmatrix}$ can be chosen such that the closed-loop system in Fig. 2 (a) is ISS. By letting $\alpha = 1.1$, the LMI (73) in Theorem 4 is solved, which yields the values of $a_1, a_2, b_1, b_2, \mu_1, \mu_2, d, \gamma_1, \gamma_2, c_1, c_2$, from which $T(\mu_1, \gamma_1) = 0.0751, T(\mu_2, \gamma_2) = 0.0639$. Then, solve the LMI (75) to obtain the matrices $P_1 = 0.1462$ and $P_2 = \begin{pmatrix} 0.6307 & 0.1195 \\ 0.1195 & 0.1434 \end{pmatrix}$. Choose the sampling period $h = 0.02, s = 0.02, \lambda_1 = 0.627, \lambda_2 = 0.575, \alpha_0 = 1$ such that (76) hold. By Theorem 4, the triggering functions $\Gamma_y, \Gamma_u$ are chosen as

$$
\Gamma_y(y, y_e) = 0.9554\|y_e\|^2 - 0.02y^\top P_1 y,
\Gamma_u(\hat{x}, \hat{x}_e) = 1.1526\|x_e\|^2 - 0.02\hat{x}^\top P_2 \hat{x}.
$$

Choose the initial state as $x_1(0) = -0.2, x_2(0) = 0.6, \dot{x}_1(0) = -0.3, \dot{x}_2(0) = 0.7$, and let the disturbance be randomly generated and satisfies $\|w\|_\infty \leq 0.05$. The simulation results are shown in Fig. 3 where the trajectories of $x, \dot{x}$ and $u$ are plotted, respectively. It can be seen that $x_1, x_2, \dot{x}$ all eventually go to a neighborhood of the origin.

To show how the sampling period affect the triggering frequencies, different values of $h$ can be chosen, as in Example 7. With other parameters chosen the same as above, 100 simulations...
Figure 5: Trajectories of the state $x$ and the input $u$ in the output feedback PETC simulation of Example 2. The sampling period is chosen as $h = 0.02$ seconds, the initial state is chosen as $x_1(0) = -0.2, x_2(0) = 0.6, \dot{x}_1(0) = -0.3, \dot{x}_2(0) = 0.7$ and the disturbance satisfies $\|w\|_\infty \leq 0.05$.

are done with the initial state $x(0), \dot{x}(0)$ chosen randomly from the set $[-0.5, 0.5]$ and $\|w\|_\infty \leq 0.05$ for $h = 0.005, 0.1, 0.15, 0.2, 0.025$ seconds, respectively. Table 2 summarizes the average triggering frequencies $f_{avg}^y, f_{avg}^u$, which are the average frequencies such that $\Gamma(y, y_e) \geq 0$ and

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\[ \Gamma_{\nu}(\dot{x}, x_e) \geq 0 \text{ during the simulations, respectively. It can be observed that } f^y_{\text{avg}} \text{ and } f^u_{\text{avg}} \text{ both increase when } h \text{ increases, which are consistent with the discussion in Remark 3.} \]

| \( h \) | 0.005s | 0.01s | 0.015s | 0.02s | 0.025s |
|-----|-------|------|-------|------|-------|
| \( f^y_{\text{avg}} \) | 22.3% | 40.0% | 51.2% | 59.9% | 66.8% |
| \( f^u_{\text{avg}} \) | 16.1% | 26.1% | 32.1% | 38.0% | 41.7% |

Table 2: Values of the average triggering frequencies \( f^y_{\text{avg}}, f^u_{\text{avg}} \) based on 100 simulations.

6 Conclusion

This paper investigated periodic event-triggered control design for nonlinear systems subject to disturbances. Sufficient conditions that ensure the closed-loop system input-to-state stable were proposed for state feedback and observer-based output feedback controllers, respectively. LMI-based sufficient conditions for PETC design for incrementally quadratic nonlinear systems were also proposed. For all the cases considered, the sampling period and triggering functions were provided explicitly.

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