TORUS ORBIT CLOSURES IN THE FLAG VARIETY

EUNJEONG LEE, MIKIYA MASUDA, AND SEONJEONG PARK

Abstract. The study of torus orbit closures in the (complete) flag variety was initiated by Klyachko and Gelfand–Serganova in the mid-1980s, but it seems that not much has been done since then. In this chapter, we present some of the work by Klyachko and Gelfand–Serganova and our recent work on the topology, geometry, and combinatorics of torus orbit closures in the flag variety.

CONTENTS

1 Introduction ........................................... 2
2 Torus orbit closures (general) ..................................................... 4
  2.1 Torus actions on flag varieties ........................................... 4
  2.2 Moment map .................................................. 5
  2.3 Torus orbit closures and their moment polytopes ......................... 7
  2.4 Coxeter matroids and Gelfand–Serganova theorem ......................... 9
  2.5 Description of the fan of a torus orbit closure ............................ 12
  2.6 Retractions and metric on finite Coxeter groups .......................... 15
3 Generic torus orbit closures in the flag variety .............................. 18
  3.1 Faces of the permutohedron Πn ....................................... 18
  3.2 Eulerian polynomial ........................................... 20
  3.3 Klyachko’s result ........................................... 22
  3.4 Relation to Hessenberg varieties .................................... 23
4 Generic torus orbit closures in Schubert varieties ........................... 23
  4.1 Generic torus orbit closures in Schubert varieties ......................... 24
  4.2 Generalized Eulerian polynomials ...................................... 26
  4.3 Toric Schubert varieties ........................................ 29
  4.4 Schubert varieties of complexity one .................................. 32
5 Generic torus orbit closures in Richardson varieties ....................... 33
  5.1 Richardson varieties and Bruhat interval polytopes ....................... 33
  5.2 Generic torus orbit closures in Richardson varieties ................... 34
  5.3 Toric Bruhat interval polytopes ...................................... 35
  5.4 Conditions on v and w for Qvw to be a cube ............................... 37
  5.5 Toric varieties of Catalan type ..................................... 37
  5.6 Smooth toric Richardson varieties of Catalan type ....................... 41
6 Problems ........................................................................... 43
  6.1 Poincaré polynomial of Yvw .......................................... 43
  6.2 Combinatorics of Qvw ........................................... 43
Appendix A Toric varieties ......................................................... 44
1. Introduction

The study of torus orbit closures in the flag variety was initiated by Klyachko [42] and Gelfand–Serganova [31]. Since then, there are some works on the subject but those are mainly about generic orbits or the normality or smoothness of orbit closures (see [24], [20], [55], [16], [17]). Recently, the authors investigated the topology and geometry of (not necessarily generic) torus orbit closures in connection with combinatorics. In this chapter, we present some of the work by Klyachko and Gelfand–Serganova and our recent works (see [46], [51], [48], [49], [52]) on the topology, geometry, and combinatorics of torus orbit closures in the flag variety. We mainly focus on type $A$ case although some of the arguments work for other Lie types. We give comments for other Lie types when necessary.

Let $\text{Fl}(n)$ be the flag variety defined by

$$\text{Fl}(n) := \{ (V_1, V_2, \ldots, V_n) \subseteq \mathbb{C}^n \mid \dim_{\mathbb{C}} V_i = i \text{ for all } i = 1, \ldots, n \},$$

where each $V_i$ is a linear subspace of $\mathbb{C}^n$. Let $S_n$ be the symmetric group on $[n] = \{1, \ldots, n\}$. An element of $S_n$ determines a permutation flag. The natural action of the general linear group $GL_n(\mathbb{C})$ on $\mathbb{C}^n$ induces an action of $GL_n(\mathbb{C})$ on $\text{Fl}(n)$. Let $T$ be the maximal torus of $GL_n(\mathbb{C})$ consisting of diagonal matrices, which is isomorphic to $(\mathbb{C}^*)^n$ where $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$. Then the set of permutation flags is exactly the $T$-fixed point set $\text{Fl}(n)^T$ in $\text{Fl}(n)$ and we identify $\text{Fl}(n)^T$ with $S_n$. A key tool to study the $T$-action on $\text{Fl}(n)$ is a moment map

$$\mu: \text{Fl}(n) \rightarrow \mathbb{R}^n$$

which satisfies the following properties:

- $\mu(w) = (w^{-1}(1), \ldots, w^{-1}(n))$ for $w \in \text{Fl}(n)^T = S_n$,
- $\mu(\text{Fl}(n))$ is the convex hull of $n!$ points $\mu(\text{Fl}(n)^T)$ in $\mathbb{R}^n$, that is, the permutohedron $\Pi_n$.

With this understanding, we will explain the content of each section.

The content of Section 2. The closure of a $T$-orbit in $\text{Fl}(n)$, denoted by $Y$, is a $T$-variety and has an open dense $T$-orbit, so $Y$ is a toric variety.\footnote{$Y$ is normal in type $A$ but not necessarily normal for other Lie types (see [16]).} Since scalar matrices in $T$ act trivially on $\text{Fl}(n)$, the (complex) dimension of $Y$ is at most $n-1$. By the convexity theorem due to Atiyah [5] (or Guillemin–Sternberg [35]), the image $\mu(Y)$ is the convex hull of the points $\mu(Y^T)$ in $\mathbb{R}^n$. The fan of $Y$ is the normal fan of the polytope $\mu(Y)$, so $\mu(Y)$ determines $Y$ up to isomorphism as a variety. Then two fundamental questions arise:

(Q1) Characterize the polytopes which arise as $\mu(Y)$ for some $T$-orbit closure $Y$.
(Q2) Describe the fan of $Y$ explicitly.

Gelfand–Serganova [31] show that any edge of $\mu(Y)$ must be parallel to an edge of the permutohedron $\Pi_n$. Since the edges of $\Pi_n$ are parallel to roots $\Phi = \{ e_i - e_j \mid i \neq j \in [n] \}$, where $e_1, \ldots, e_n$ denote the standard basis of $\mathbb{R}^n$, a polytope with edges parallel to roots is called a $\Phi$-polytope. Therefore $\mu(Y)$ is a $\Phi$-polytope. Unfortunately, the converse is not true and the complete solution to (Q1) above is unknown.

$\Phi$-polytopes are closely related to Coxeter matroids. A Coxeter matroid we treat here is a subset $\mathcal{M}$ of $S_n$ which satisfies the Maximality (or Minimality) condition (see Definition 2.11). The fixed point set $Y^T$ regarded as a subset of $S_n$ is a Coxeter matroid. One can define a retraction

$$\mathbb{R}_M^n: S_n \rightarrow \mathcal{M} \subset S_n$$

for a Coxeter matroid $\mathcal{M}$ and describe the fan of $Y$ explicitly using the retraction to $Y^T$ (see Corollary 2.21), which answers (Q2) above.
The retraction $R^{\mu}_{MC}$ has two other interpretations, one is geometric and the other is algebraic. It also has the following interesting property. The 1-skeleton of the permutohedron $\Pi_n$ is a graph with $S_n$ as vertices. If $M$ is a Coxeter matroid, then for each $u \in S_n$, a vertex in $M$ closest to $u$ (with respect to the graph metric) is unique and the closest vertex in $M$ is given by $R^{\mu}_{MC}(u)$.

**The content of Section 3.** We discuss the topology of a generic $T$-orbit closure $Y$ in $Fl(n)$, where $Y$ is called generic if

$$Y^T = Fl(n)^T = S_n,$$

in other words, $\mu(Y) = \Pi_n$. Since $\Pi_n$ is simple, $Y$ is smooth and of complex dimension $n-1$. In fact, $Y$ is isomorphic to the permutohedral variety $Perm_n$ which is a toric variety whose fan consists of Weyl chambers. So we may think of the generic $T$-orbit closure $Y$ as $Perm_n$. The cohomology of $Perm_n$ can be explicitly described and its Poincaré polynomial agrees with $A_n(t^2)$ where $A_n(t)$ denotes the $n$th Eulerian polynomial. The fan of $Perm_n$ has the symmetric of $S_n$ which induces an action of $S_n$ on the cohomology of $Perm_n$. Klyachko [42] shows that the image of the restriction map

$$i^*: H^*(Fl(n); \mathbb{Q}) \rightarrow H^*(Perm_n; \mathbb{Q}),$$

where $i: Perm_n \rightarrow Fl(n)$ is the inclusion map, is the ring of $S_n$-invariants $H^*(Perm_n; \mathbb{Q})^{S_n}$. This fact is generalized to the setting of Hessenberg varieties. Here, Hessenberg varieties are subvarieties $X$ where $S_n$ is locally factorial. Moreover, permutations $w$ in other words, $Q_w$ is simple if $Q_w$ is simple at the vertex $\mu(w)$.

In contrast to the conjecture, the Schubert variety $X_w$ is smooth at $w$ and entirely smooth if it is smooth at the identity $e$ (see [10, p.208]) while $Y_w$ is smooth at $e$. Interestingly, the graph $\Gamma_w(w)$ appears in a different context, i.e., it is shown in [66] that $\Gamma_w(w)$ is a forest if and only if $X_w$ is locally factorial. Moreover, permutations $w$ for which $\Gamma_w(w)$ is forest are characterized in terms of pattern avoidance (see [14]).

We introduce a polynomial $A_w(t)$ for $w \in S_n$ purely combinatorially by looking at ascends. The polynomial $A_w(t)$ is the Eulerian polynomial $A_n(t)$ when $w = w_0$. As mentioned in the content of Section 3 above, $A_{w_0}(t^2)$ agrees with the Poincaré polynomial of the permutohedral variety $Perm_n$, which is isomorphic to $Y_{w_0}$. This fact is generalized in such a way that $A_w(t^2)$ agrees with the Poincaré polynomial of $Y_w$ for any $w \in S_n$ (see Theorem 4.17).

We set

$$c(w) := \dim_{\mathbb{C}} X_w - \dim_{\mathbb{C}} Y_w.$$ When $c(w) = 0$, we have $X_w = Y_w$ which means that $X_w$ is a toric variety with the $T$-action. There are several equivalent conditions for $X_w$ to be a toric variety (see Theorem 4.21), e.g., a reduced decomposition of $w$ is a product of distinct adjacent transpositions. We describe the fan of a toric Schubert variety $X_w$ explicitly in terms of $w$ (see Theorem 4.23). This implies that $X_w$...
is a weak Fano Bott manifold, see Appendix A.2 for details of Bott manifolds. We also present a classification result of toric Schubert varieties $X_w$ for Coxeter elements $w$ (see Theorem 4.29).

The case where $c(w) = 1$ is also studied. In this case, $X_w$ is not necessarily smooth. We present several equivalent conditions for $c(w) = 1$ depending on whether $X_w$ is smooth or singular (see Theorems 4.32 and 4.33).

**The content of Section 5.** For a pair $(v, w)$ of permutations with $v \leq w$, the intersection of a Schubert variety $X_w$ and an opposite Schubert variety $w_0 X_w v$ is non-empty and irreducible. The intersection is denoted by $X^{v, w}$ and called a Richardson variety. It is invariant under the $T$-action on $\text{Fl}(n)$. The moment map image of $X^{v, w}$ is the Bruhat interval polytope

$$Q^{v, w}_w := \text{Conv}\{(u^{-1}(1), \ldots, u^{-1}(n)) \in \mathbb{R}^n \mid v \leq u \leq w\},$$

introduced by Kodama–Williams [43]. Note that $Q^{v, w}_w = Q_w$. The vertex $\mu(v)$ of $Q^{v, w}_w$ is not necessarily simple in $Q_w$ while it is simple when $v = e$. A natural generalization of the conjecture mentioned above is that $Q^{v, w}_w$ is simple if the vertices $\mu(v)$ and $\mu(w)$ are both simple in $Q_w$.

We study the case when $X^{v, w}_w$ is a toric variety with the $T$-action. A toric Richardson variety $X^{v, w}_w$ is not necessarily smooth while every toric Schubert variety $X_w$ is smooth. It turns out that $X^{v, w}_w$ is a smooth toric variety if and only if $Q^{v, w}_w$ is combinatorially equivalent to a cube (see Theorem 5.7). Although pairs $(v, w)$ for which $Q^{v, w}_w$ is combinatorially equivalent to a cube are not completely understood, there are many such pairs. Especially, it is shown in [37] that this is the case when

$$w = s_{n-1} s_{n-2} \cdots s_1 v \quad \text{ (or } w = s_1 s_2 \cdots s_{n-1} v) \quad \text{ and } \quad \ell(w) - \ell(v) = n - 1$$

where $s_i$ ($i = 1, \ldots, n - 1$) denotes the adjacent transposition interchanging $i$ and $i + 1$. Toric Richardson varieties $X^{v, w}_w$ for the pairs $(v, w)$ above also arise from polygon triangulations, so we call such toric Richardson varieties of **Catalan type**. They can be classified up to isomorphism and the Wedderburn–Etherington numbers which count unordered binary trees appear in enumerating the isomorphism classes (see Corollary 5.22).

Finally, in Section 6 we pose several problems related to the discussion developed in this chapter. In Appendix A, we briefly review the theory of toric varieties and explain Bott manifolds in detail.

### 2. Torus orbit closures (general)

In this section, we review the torus actions on flag varieties and the closure of a torus orbit. Moreover, we consider properties of the moment map images of torus orbit closures. Indeed, the fixed point set of a torus orbit closure becomes a Coxeter matroid. Finally, we discuss how to describe the fan of a torus orbit closure.

#### 2.1. Torus actions on flag varieties.

Let $G$ be the general linear group $\text{GL}_n(\mathbb{C})$ over $\mathbb{C}$, $B \subset G$ the set of upper triangular matrices, and $T \subset G$ the set of diagonal matrices. Let $B^- \subset B$ be the set of lower triangular matrices. We denote by $S_n$ the symmetric group on $[n] := \{1, \ldots, n\}$. Then $T = B \cap B^- \cong (\mathbb{C}^*)^n$ and $B^- = w_0 B w_0$, where $w_0$ is the longest element $[n, n-1, \ldots, 1] \in S_n$.

The homogeneous space $G/B$ can be identified with the flag variety $\text{Fl}(n)$ defined by

$$\text{Fl}(n) := \{(\{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n) \mid \dim_{\mathbb{C}} V_i = i \text{ for all } i = 1, \ldots, n\},$$

where $\mathbb{C}^n$ is considered as the complex vector space consisting of column vectors. For an element $w \in S_n$, we use the same letter $w$ for the permutation matrix $[e_{w(1)} \cdots e_{w(n)}]$ in $\text{GL}_n(\mathbb{C})$ to simplify notation, where $e_1, \ldots, e_n$ are the standard basis vectors in $\mathbb{C}^n$. The left multiplication by $T$ on $G$ induces the $T$-action on $G/B$.

**Lemma 2.1** ([28, §10.5]). The set of $T$-fixed points in $G/B$ bijectively corresponds to the symmetric group $S_n$ such that each $u \in S_n$ corresponds to $uB \in G/B$. 
For each \( u \in S_n \), there is a \( T \)-invariant local chart \( U_u \) given by
\[
U_u := \left\{ (x_{ij})B \in G/B \ \middle| \ x_{ij} = \begin{cases} 1 & \text{if } j = u^{-1}(i), \\ 0 & \text{if } j > u^{-1}(i) \end{cases} \right\}
\]
(2.1)
\[
= \left\{ (x_{u(i)j})B \in G/B \ \middle| \ x_{u(i)j} = \begin{cases} 1 & \text{if } j = i, \\ 0 & \text{if } j > i \end{cases} \right\},
\]
which is isomorphic to \( \mathbb{C}^{n(n-1)/2} \) since \( x_{u(i)j} \) with \( j < i \) are arbitrary complex numbers. The \( T \)-action on \( G/B \) restricted to \( U_u \) is given by
\[
(t_1, \ldots, t_n) \cdot (x_{ij})B = (t_i t_{u(j)}^{-1} x_{ij})B.
\]
(2.2)
Therefore, \( uB \) is a unique \( T \)-fixed point in \( U_u \), which corresponds to the origin in \( \mathbb{C}^{n(n-1)/2} \) and a 1-dimensional \( T \)-orbit in \( U_u \) corresponds to a 1-dimensional \( T \)-eigenspace in \( \mathbb{C}^{n(n-1)/2} \).

**Example 2.2.** When \( u = 312 \) in one-line notation,
\[
U_{312} = \left\{ \begin{pmatrix} * & 1 & 0 \\ * & * & 1 \\ 1 & 0 & 0 \end{pmatrix} B \ \middle| \ * \in \mathbb{C} \right\} \cong \mathbb{C}^3,
\]
and the \( T \)-action on \( U_{312} \) is given by
\[
(t_1, t_2, t_3) \cdot \begin{pmatrix} * & 1 & 0 \\ * & * & 1 \\ 1 & 0 & 0 \end{pmatrix} B = \begin{pmatrix} t_1 t_3^{-1} & 1 & 0 \\ t_2 t_3^{-1} & t_2 t_1^{-1} & 1 \\ 1 & 0 & 0 \end{pmatrix} B.
\]

### 2.2. Moment map
We describe a moment map \( \mu: G/B \to \mathbb{R}^n \) explicitly using the Plücker coordinates. We define the set
\[
I_{d,n} := \{ \underline{i} = (i_1, \ldots, i_d) \in \mathbb{Z}^d \mid 1 \leq i_1 < \cdots < i_d \leq n \}.
\]
For an element \( x = (x_{ij}) \in G = \text{GL}_n(\mathbb{C}) \), the \( i \)th Plücker coordinate \( p_{\underline{i}}(x) \) of \( x \) is given by the \( d \times d \) minor of \( x \), with row indices \( i_1, \ldots, i_d \) and the column indices \( 1, \ldots, d \) for \( \underline{i} = (i_1, \ldots, i_d) \in I_{d,n} \). The Plücker embedding is defined to be
\[
\psi: \frac{G/B}{\prod_{d=1}^{n-1} \mathbb{C}P^{(2)} -1}, \quad xB \mapsto \prod_{d=1}^{n-1} [p_{\underline{i}}(x)]_{\underline{i} \in I_{d,n}}.
\]
(2.3)
The map \( \psi \) is \( T \)-equivariant with respect to the action of \( T \) on \( \prod_{d=1}^{n-1} \mathbb{C}P^{(2)} -1 \) given by
\[
(t_1, \ldots, t_n) \cdot [p_{\underline{i}}]_{\underline{i} \in I_{d,n}} := [t_{i_1} \cdots t_{i_d} \cdot p_{\underline{i}}]_{\underline{i} \in I_{d,n}}
\]
for \( (t_1, \ldots, t_n) \in T \) and \( \underline{i} = (i_1, \ldots, i_d) \in I_{d,n} \).

Let \( \omega_{FS} \) be the Fubini–Study form on a complex projective space \( \mathbb{C}P^{m-1} \). With respect to the standard action of the compact torus \( (S^1)^m \) on \( \mathbb{C}P^{m-1} \) defined by
\[
(t_1, \ldots, t_m) \cdot [z_1, \ldots, z_m] = [t_1 z_1, \ldots, t_m z_m],
\]
where each \( t_i \) is a complex number with unit length, the moment map of \( (\mathbb{C}P^{m-1}, \omega_{FS}, (S^1)^m) \) is given by
\[
[z_1, \ldots, z_m] \mapsto \frac{-1}{2 \sum_{i=1}^m |z_i|^2} \left( |z_1|^2, \ldots, |z_m|^2 \right) \text{ up to translation.}
\]

See, for example, [6, Example IV.1.2].\(^2\) Because the action of \((S^1)^n\) on the factor \( \mathbb{C}P^{(2)} -1 \) in (2.3) is given through the homomorphism \((S^1)^n \to (S^1)^{(2)}\) sending \((t_1, \ldots, t_n)\) to \((t_1 \cdots t_{i_d})\) in \( I_{d,n} \), the
\[^2\text{Here, we use a different sign convention from that in [6]. That is, our moment map } \mu: (M, \omega, (S^1)^n) \to \text{Lie}((S^1)^n)^* \text{ satisfies the following: For each } X \in \text{Lie}((S^1)^n), \text{ } d\mu_X = x_X \omega, \text{ where } \mu_X(p) = \langle \mu(p), X \rangle \text{ and } X^\# \text{ is the vector field on } M \text{ generated by the one-parameter subgroup } \{ \exp tX \mid t \in \mathbb{R} \} \subset (S^1)^n.\]
moment map of \((\mathbb{C}P(3)^{-1}, 2\omega_{FS}, (S^1)^n)\) is given by

\[
[p_1]_{I_d,n} \mapsto \sum_{k=1}^n \frac{-1}{|p_k|^2} \left( \sum_{1 \leq k \leq I_d,n} |p_k|^2, \ldots, \sum_{n \in I_d,n} |p_n|^2 \right) \quad \text{up to translation.}
\]

By considering a symplectic form \(\omega\) on \(\prod_{d=1}^{n-1} \mathbb{C}P(3)^{-1}\) given by two times of the Fubini–Study form on each complex projective space, a moment map \(\tilde{\mu}: \prod_{d=1}^{n-1} \mathbb{C}P(3)^{-1} \to \mathbb{R}^n\) is given by

\[
(2.4) \quad \prod_{d=1}^{n-1} |p_k|_{I_d,n} \mapsto -\sum_{d=1}^{n-1} \left\{ \frac{1}{\sum_{k \in I_d,n} |p_k|^2} \left( \sum_{1 \leq k \in I_d,n} |p_k|^2, \ldots, \sum_{n \in I_d,n} |p_n|^2 \right) \right\} + c,
\]

where \(c\) is a constant vector in \(\mathbb{R}^n\). We take \(c = (n, \ldots, n)\) in (2.4) and define

\[
(2.5) \quad \mu := \tilde{\mu} \circ \psi,
\]

which is a moment map of \((G/B, \psi^*\omega, T)\).

**Lemma 2.3.** The moment map \(\mu\) sends the fixed point \(uB \in G/B\) to \((u^{-1}(1), \ldots, u^{-1}(n)) \in \mathbb{R}^n\).

**Proof.** For a permutation \(u \in S_n\), the Plücker coordinates \((p_k)_{I_d,n}\) of \(uB\) are given as follows:

\[
(2.6) \quad p_k = \begin{cases} 
1 & \text{if } k = \{u(1), \ldots, u(d)\}^\uparrow, \\
0 & \text{otherwise}, 
\end{cases}
\]

for each \(k \in I_d,n\). Here, for a subset \(S \subset [n]\), we denote by \(S^\uparrow\) the ordered tuple obtained from \(S\) by sorting its elements in ascending order. Therefore, one can see that for a fixed \(d \in [n-1]\),

\[
\sum_{k \in I_d,n} |p_k|^2 = 1 \quad \text{and the entries of the vector}
\]

\[
\left( \sum_{1 \leq k \leq I_d,n} |p_k|^2, \ldots, \sum_{n \in I_d,n} |p_n|^2 \right)
\]

are 1 for coordinates in \(\{u(1), \ldots, u(d)\}\) and 0 otherwise. Hence the summation

\[
-\sum_{d=1}^{n-1} \left\{ \frac{1}{\sum_{k \in I_d,n} |p_k|^2} \left( \sum_{1 \leq k \in I_d,n} |p_k|^2, \ldots, \sum_{n \in I_d,n} |p_n|^2 \right) \right\}
\]

is an integer vector such that the \(u(k)\)-entry is \(- (n-k)\). Therefore, \(\mu(uB)\) is an integer vector whose \(u(k)\)-entry is \(k\) since \(c = (n, \ldots, n)\) in (2.4). This implies that \(\mu(uB) = (u^{-1}(1), \ldots, u^{-1}(n))\) since \(u^{-1}(u(k)) = k\).

**Example 2.4.** When \(G = \text{GL}_3(\mathbb{C})\), the Plücker embedding \(\psi: G/B \to \mathbb{C}P(3)^{-1} \times \mathbb{C}P(3)^{-1}\) maps an element \(x = (x_{ij}) \in \text{GL}_3(\mathbb{C})\) to

\[
([p_1(x), p_2(x), p_3(x)], [p_1(2), p_1(3), p_2(3, x)]) = ([x_{11}, x_{21}, x_{31}], [x_{11}, x_{12} - x_{21}, x_{21}, x_{32} - x_{31}, x_{12}, x_{21}, x_{32} - x_{31}, x_{22}]).
\]

Since the action of \(T\) on \(\text{GL}_3(\mathbb{C})\) is given by

\[
(t_1, t_2, t_3) \cdot \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} = \begin{pmatrix} t_1x_{11} & t_1x_{12} & t_1x_{13} \\ t_2x_{21} & t_2x_{22} & t_2x_{23} \\ t_3x_{31} & t_3x_{32} & t_3x_{33} \end{pmatrix},
\]

one can easily check that the map \(\psi\) is \(T\)-equivariant. The moment map

\[
\tilde{\mu}: \mathbb{C}P(3)^{-1} \times \mathbb{C}P(3)^{-1} \to \mathbb{R}^3
\]
is given by
\[
([p_1, p_2, p_3], [p_{12}, p_{13}, p_{23}]) 
\mapsto -\frac{1}{|p_1|^2 + |p_2|^2 + |p_3|^2} \left( [p_1]^2, [p_2]^2, [p_3]^2 \right)
- \frac{1}{|p_{12}|^2 + |p_{13}|^2 + |p_{23}|^2} \left( |p_{12}|^2 + |p_{13}|^2, |p_{12}|^2 + |p_{23}|^2, |p_{13}|^2 + |p_{23}|^2 \right)
+ (3, 3, 3).
\]

For $312 \in S_3$, we have
\[
\mu(312B) = \tilde{\mu} \circ \psi(312B) = \tilde{\mu}((0, 0, 1), [0, 1, 0])) = -(0, 0, 1) - (1, 0, 1) + (3, 3, 3) = (2, 3, 1).
\]

**Remark 2.5.** The Plücker embedding can be generalized to partial flag varieties and other Lie types (see, for example, [31] and [13]).

**2.3. Torus orbit closures and their moment polytopes.** For a point $x \in G/B$, the closure $\overline{T \cdot x}$ of the $T$-orbit $T \cdot x$ is a toric variety in $G/B$. The moment map image $\mu(\overline{T \cdot x})$ is a convex polytope in $\mathbb{R}^n$ with vertices $\mu((\overline{T \cdot x})^T)$ by the convexity theorem (Atiyah [5], Guillemin–Sternberg [35] for symplectic case).

On the other hand, $(\overline{T \cdot x})^T$ can be found as follows. Choose a representative $\tilde{x} \in G_{n}(\mathbb{C})$ of $x$. The non-vanishing of $p_\mathbf{i}(\tilde{x})$ is independent of the choice of the representative $\tilde{x}$ of $x$ and we define
\[
I_d(x) := \{ \mathbf{i} = (i_1, \ldots, i_d) \in I_{d,n} \mid p_\mathbf{i}(\tilde{x}) \neq 0 \}, \quad 1 \leq d \leq n.
\]

**Proposition 2.6 ([31], Proposition 1 in §5.2).** For an element $x \in G_{n}(\mathbb{C})/B$, we have
\[
(\overline{T \cdot x})^T = \{ wB \mid w \in S_n, \{ w(1), \ldots, w(d) \} \uparrow \in I_d(x) \text{ for all } 1 \leq d \leq n \}.
\]

**Proof.** We note that for $\mathbf{i} \in I_{d,n}$, if $p_\mathbf{i}(\tilde{x}) = 0$, then $p_\mathbf{i}(\tilde{y}) = 0$ for any $y \in \overline{T \cdot x}$. Here, we denote by $\tilde{y} \in G_{n}(\mathbb{C})$ a representative of $y$. Therefore we have
\[
\psi((\overline{T \cdot x})^T) \subset \left\{ [p_\mathbf{i}]_{\mathbf{i} \in I_{d,n}} \in \prod_{d=1}^{n-1} \mathbb{C}P(\mathbb{C}) \mid p_\mathbf{i} \neq 0 \text{ if } \mathbf{i} \in I_d(x) \right\}.
\]

This provides
\[
(\overline{T \cdot x})^T \subset \{ wB \mid w \in S_n, \{ w(1), \ldots, w(d) \} \uparrow \in I_d(x) \text{ for all } 1 \leq d \leq n \}
\]
since for $\mathbf{i} \in I_{d,n}$, we have $p_\mathbf{i}(wB) \neq 0$ if and only if $\{ w(1), \ldots, w(d) \} \uparrow = \mathbf{i}$ (cf. (2.6)).

Now we consider the opposite inclusion. Recall that the group of homomorphisms from $T$ to $\mathbb{C}^*$ is naturally isomorphic to $\mathbb{Z}^n$. For each $u = (u_1, \ldots, u_n) \in \mathbb{Z}^n$, we have the one-parameter subgroup $\lambda_u(t) = (t^{u_1}, \ldots, t^{u_n})$ of $T$, where $t \in \mathbb{C}^*$. Suppose that $w \in S_n$ satisfies $\{ w(1), \ldots, w(d) \} \uparrow \in I_d(x)$ for all $1 \leq d \leq n$. Take $u = (u_1, u_2, \ldots, u_n) \in \mathbb{Z}^n$ such that $u_{w(1)} < u_{w(2)} < \cdots < u_{w(n)}$. Then we have
\[
\lim_{t \to 0} \lambda_u(t) \cdot x = wB.
\]

This proves the desired inclusion, so we are done. \[\square\]

**Example 2.7.** Let $x = \begin{pmatrix} \alpha & 1 & 0 \\ \beta & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} B \in G_{3}(\mathbb{C})/B$ with $\alpha, \beta \in \mathbb{C}^*$. Then we have
\[
I_1(x) = \{ (1), (2), (3) \}, \quad I_2(x) = \{ (1, 2), (1, 3) \}, \quad I_3(x) = \{ (1, 2, 3) \}, \quad \text{and}
(\overline{T \cdot x})^T = \{ wB \mid w \in S_3, \{ w(1), \ldots, w(d) \} \uparrow \in I_d(x) \text{ for all } 1 \leq d \leq 3 \} = \{123B, 132B, 213B, 312B\}.
\]

Since
\[
\mu(123B) = (1, 2, 3), \quad \mu(132B) = (1, 3, 2), \quad \mu(213B) = (2, 1, 3), \quad \mu(312B) = (2, 3, 1)
\]
by Lemma 2.3, $\mu(\overline{T \cdot x})$ is the convex hull of the four points in (2.7), see Figure 1. We label each vertex $\mu(uB)$ with $u$ (using a different font) for simplicity.
In general, the moment polytope $\mu(T \cdot x)$ has the following property.

**Proposition 2.8.** If two vertices $\mu(uB)$ and $\mu(vB)$ of $\mu(T \cdot x)$ are joined by an edge of $\mu(T \cdot x)$ for $u, v \in S_n$, then $tu = v$ for some transposition $t$ of $S_n$, in other words, an edge of $\mu(T \cdot x)$ is parallel to $e_i - e_j$ for some $1 \leq i < j \leq n$.

**Proof.** By the convexity theorem, an edge of $\mu(T \cdot x)$ is $\mu(O)$ for some 1-dimensional $T$-orbit $O$ in $\overline{T \cdot x}$. Since $\overline{T \cdot x}$ is invariant under the $T$-action on $G/B$, $O$ is also a 1-dimensional $T$-orbit in $G/B$. Note that $O$ is isomorphic to $\mathbb{C}^*$, $\overline{O}$ is $\mathbb{C}P^1$, and $\overline{O}$ consists of two $T$-fixed points, say $uB$ and $vB$, which map to the two end points of the edge $\mu(O)$ by the moment map $\mu$.

Let $U_u$ be the $T$-invariant chart of $G/B$ centered at $uB$ in (2.1). Then, as noted before, (2.2) implies that the $T$-orbit $O$ is of the form

$$(u + cE_{u(ij)})B \quad \text{for some } i, j \text{ with } j < i,$$

where $c \in \mathbb{C}^*$ and $E_{u(ij)}$ is the $n \times n$ matrix with 1 at the $(u(i), j)$ entry and 0 otherwise. Suppose that $u(i) < u(j)$ (the essentially same argument works when $u(i) > u(j)$). Then we apply the following elementary transformations to $(u + cE_{u(ij)})B$:

1. multiply the $j$th column by $1/c$,
2. subtract the $i$th row from the $j$th row,
3. multiply the $i$th row by $-c$,

and then approach $c$ to $\infty$:

$$
\begin{pmatrix}
c & 1 \\
1/c & 0
\end{pmatrix}B \overset{(1)}{=} \begin{pmatrix}1 & 1 \\ 1/c & 0\end{pmatrix}B \overset{(2)}{=} \begin{pmatrix}1 & 0 \\ 1/c & -1/c\end{pmatrix}B \overset{(3)}{=} \begin{pmatrix}1 & 0 \\ 1/c & 1\end{pmatrix}B \overset{c \to \infty}{\rightarrow} \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}B,
$$

where the $2 \times 2$ matrices above are intersections of the $j$th and $i$th columns ($j < i$) and the $u(i)$th and $u(j)$th rows ($u(i) < u(j)$). The other entries remain unchanged by the elementary transformations above. This shows that when $c$ approaches $\infty$, the point $(u + cE_{u(ij)})B$ approaches the $T$-fixed point $t_{u(i), u(j)}uB$, where $t_{u(i), u(j)}$ is the transposition interchanging $u(i)$ and $u(j)$. Since $vB = t_{u(i), u(j)}uB$, this implies the proposition. 

Since the root system of type $A_{n-1}$ is $\Phi = \{ \pm (e_i - e_j) \mid 1 \leq i < j \leq n \}$, Proposition 2.8 says that an edge vector of the moment polytope $\mu(T \cdot x)$ is parallel to a root in $\Phi$. This fact holds in any Lie type.

**Theorem 2.9** ([13, Theorem on page xii]). Let $G$ be a semisimple algebraic group over $\mathbb{C}$, $B$ a Borel subgroup in $G$ and $T \subset B$ a maximal torus. Let $\Phi$ be the associated root system and $\mu : G/B \to \mathfrak{t}^*$ a moment map, where $\mathfrak{t}^*$ denotes the vector space dual to the Lie algebra $\mathfrak{t}$ of the maximal compact torus of $T$. Then, for any point $x \in G/B$, an edge of the moment polytope $\mu(T \cdot x)$ is parallel to a root in $\Phi$ (such a polytope is called a $\Phi$-polytope, see Definition 2.14).

**Remark 2.10.** Partial flag varieties also have torus actions and one may consider moment maps. We refer the reader to [30] for the combinatorial properties of moment maps of torus orbit closures in Grassmannian.
2.4. Coxeter matroids and Gelfand–Serganova theorem. To study the combinatorial properties of torus orbit closures, Gelfand and Serganova introduced the notion of Coxeter matroids. In this subsection, we recall the definition of Coxeter matroids and the characterization of Coxeter matroids in terms of polytopes by Gelfand–Serganova. It turns out that the $T$-fixed point set of a $T$-orbit closure in the flag variety $G/B$ is a Coxeter matroid.

Let $W$ be a finite Coxeter group, so generators of $W$ are prescribed. The symmetric group $S_n$ on $[n]$ with adjacent transpositions as generators is a typical example of a Coxeter group. For $u \in W$, let $\leq^u$ denote the $u$-shifted order, that is, $v \leq^u w$ means $u^{-1}v \leq u^{-1}w$ in the Bruhat order on $W$. Note that $\leq^u$ is a partial order on $W$ with $u$ as the smallest element.

**Definition 2.11.** A subset $M$ of a finite Coxeter group $W$ is called a **Coxeter matroid** if it satisfies the Maximality Property: for any $u \in W$, there is a unique element $v \in M$ such that $w \leq^u v$ for all $w \in M$.

**Remark 2.12.**
1. Since the multiplication by the longest element of $W$ reverses the Bruhat order on $W$, the Maximality Property is equivalent to the Minimality Property: for any $u \in W$, there exists a unique element $v \in M$ such that $v \leq^u w$ for all $w \in M$.
2. In fact, a Coxeter matroid is defined more generally in [13]. Let $P$ be a parabolic subgroup of a finite Coxeter group $W$. Then the Bruhat order on $W$ induces a partial order on $W/P$ and a subset $M$ of $W/P$ is called a Coxeter matroid if it satisfies the Maximality Property above. A Coxeter matroid in Definition 2.11 is the case where $P$ is the identity subgroup and an ordinary matroid can be regarded as the case where $W = S_n$ and $P$ is a maximal parabolic subgroup (see [13, Theorem 1.3.1]).

**Example 2.13** ([48, Example 2.2]). Let $M = \{213, 132\}$ be a subset of $S_3$. Since $213 \not\leq 132$ and $132 \not\leq 213$, there is no element $v \in M$ such that $w \leq^{123} v$ for all $w \in M$. Hence $M$ is not a Coxeter matroid. However, one can check that $\{231, 321\}$ is a Coxeter matroid of $S_3$.

We recall the characterization of Coxeter matroids in terms of polytopes by Gelfand–Serganova. As is well-known, a finite Coxeter group $W$ can be regarded as a reflection group on a vector space $V$, where the generators of $W$ act on $V$ as reflections (see [39, Section 5.3]). Let $\Phi$ be the set of roots of $W$.

**Definition 2.14.** A convex polytope $\Delta$ in $V$ is called a **$\Phi$-polytope** if every edge of $\Delta$ is parallel to a root in $\Phi$.

Choose a generic point $\nu \in V$ which is not fixed by any reflection in $W$. For a subset $M$ of $W$, we define $\Delta_M$ to be the convex hull of the $M$-orbit $\{w \cdot \nu \mid w \in M\}$ of the point $\nu$ in $V$. When $M = W$, $\Delta_W$ is called the $W$-permutohedron (see [25, Section 2.4] and [38]). Two vertices $v \cdot \nu$ and $w \cdot \nu$ of $\Delta_W$ are joined by an edge of $\Delta_W$ if and only if $v^{-1}w$ is a generator of $W$ (see [25, Lemma 2.13]). Thus any edge of $\Delta_W$ is parallel to a root and vice versa. Therefore, the $W$-permutohedron $\Delta_W$ is a $\Phi$-polytope and we may say that a convex polytope $\Delta$ in $V$ is a $\Phi$-polytope if every edge of $\Delta$ is parallel to an edge of $\Delta_W$.

However, $\Delta_M$ is not necessarily a $\Phi$-polytope unless $M = W$. The following is a part of the Gelfand–Serganova theorem [31] (also see [13, Theorem 6.3.1]).

**Theorem 2.15** (Gelfand–Serganova theorem). A subset $M$ of a finite Coxeter group $W$ is a Coxeter matroid if and only if $\Delta_M$ is a $\Phi$-polytope. Therefore, the $T$-fixed point set of a $T$-orbit closure in the flag variety $G/B$ is a Coxeter matroid by Theorem 2.9.

We provide the proof for the case $W = S_n$. We refer the reader to [13] for more details. Before giving it, we prepare two lemmas. Suppose that

$$ V = \{(a_1, \ldots, a_n) \in \mathbb{R}^n \mid a_1 + \cdots + a_n = 0\} $$

and $S_n$ acts on $V$ as permuting coordinates, i.e., $u \cdot (a_1, \ldots, a_n) = (a_{u^{-1}(1)}, \ldots, a_{u^{-1}(n)})$, in other words,

$$ u \cdot \left( \sum_{i=1}^{n} a_i e_i \right) = \sum_{i=1}^{n} a_{u(i)} e_{u(i)}. $$
We define the ordering $\preceq$ on $V$ by putting $x \preceq y$ for $x, y \in V$ if there exists non-negative constants $c_i \geq 0$ such that
\[
y - x = \sum_{i=1}^{n-1} c_i (e_u(i) - e_u(i+1)),
\]
which is equivalent to saying $u^{-1} \cdot y - u^{-1} \cdot x = \sum_{i=1}^{n-1} c_i (e_i - e_{i+1})$ with the same coefficients $c_i$.

This means
\[
x \preceq y \iff u^{-1} \cdot x \preceq u^{-1} \cdot y.
\]

**Lemma 2.16** ([13, Lemma 6.2.3]). Suppose that $\nu = (\nu_1, \nu_2, \ldots, \nu_n) \in V$ satisfies $\nu_1 < \nu_2 < \cdots < \nu_n$. If $v \preceq w$, then $v \cdot \nu \preceq w \cdot \nu$.

*Proof.* Note that $v \preceq w$ if and only if $u^{-1} v \leq u^{-1} w$, and $v \cdot \nu \preceq w \cdot \nu$ if and only if $u^{-1} v \cdot \nu \preceq u^{-1} w \cdot \nu$ (see (2.8)). Hence it suffices to prove the statement in the case $u = e$. Suppose $v \leq w$. Then we have $v^{-1} \leq w^{-1}$ and hence $\{v^{-1}(1), \ldots, v^{-1}(d)\}\leq \{w^{-1}(1), \ldots, w^{-1}(d)\}$ for $1 \leq d \leq n$ (cf. [10, §3.2]). This implies $\sum_{i=1}^{d} v^{-1}(i) \leq \sum_{i=1}^{d} w^{-1}(i)$ for $1 \leq d \leq n$, so we get
\[
\sum_{i=1}^{d} \nu_{v^{-1}(i)} \leq \sum_{i=1}^{d} \nu_{w^{-1}(i)} \quad \text{for } 1 \leq d \leq n.
\]

On the other hand, we have
\[
w \cdot \nu - v \cdot \nu = (\nu_{w^{-1}(1)}, \ldots, \nu_{w^{-1}(n)}) - (\nu_{v^{-1}(1)}, \ldots, \nu_{v^{-1}(n)})
\]
\[
= \sum_{i=1}^{n} \left( \nu_{w^{-1}(i)} - \nu_{v^{-1}(i)} \right) (e_i - e_{i+1}).
\]

Accordingly, by (2.9), we obtain $v \cdot \nu \preceq w \cdot \nu$ and we are done. \hfill \Box

The converse in Lemma 2.16 does not hold in general. For instance, if $v = 1432$ and $w = 4123$, then $v \not\preceq w$ but since
\[
w \cdot \nu - v \cdot \nu = (v_2, v_3, v_4, v_1) - (v_1, v_4, v_3, v_2)
\]
\[
= (v_2 - v_1)(e_1 - e_2) + (v_2 + v_3 - v_1 - v_4)(e_2 - e_3) + (v_2 - v_1)(e_3 - e_4),
\]
we have $v \cdot \nu \preceq w \cdot \nu$ if $v_2 + v_3 \geq 0$ (note that $v_1 + v_2 + v_3 + v_4 = 0$ because $\nu \in V$).

However, the converse holds in the following special case.

**Lemma 2.17** ([13, Lemma 6.2.5]). Let $\nu$ be as in Lemma 2.16. Suppose that $w \cdot \nu - v \cdot \nu$ is parallel to a root in $\Phi$. Then $v \preceq w$ if $v \cdot \nu \preceq w \cdot \nu$.

*Proof.* As noted in the proof of Lemma 2.16, we may assume $u = e$. If $v \cdot \nu \preceq w \cdot \nu$, then
\[
w \cdot \nu - v \cdot \nu = c(e_i - e_j)
\]
for some $c > 0$ and $i < j$. This implies that $w^{-1} = v^{-1} t_{i,j}$ and $v^{-1}(i) < v^{-1}(j)$ since $c > 0$, where $t_{i,j}$ denotes the transposition interchanging $i$ and $j$. Therefore $v^{-1} \leq w^{-1}$ and hence $v \leq w$. \hfill \Box

*Proof of Theorem 2.15 in type A.* We take $\nu = (\nu_1, \nu_2, \ldots, \nu_n) \in V$ which satisfies $\nu_1 < \nu_2 < \cdots < \nu_n$. We will prove that if $M \subset S_n$ is a Coxeter matroid, then $\Delta_M$ is a $\Phi$-polytope. Assume on the contrary that $\Delta_M$ is not a $\Phi$-polytope. Then there exists an edge $l$ with vertices $v_1 \cdot \nu$ and $v_2 \cdot \nu$ that is not parallel to any root. Consider a linear function $f : V \to \mathbb{R}$ that is constant on $l$ and takes smaller values on the other points of $\Delta_M$.

Since the edge $l$ is not parallel to any root, we may assume that $f$ is not vanishing on any root in $\Phi$. Accordingly, there is a unique simple system of roots $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_{n-1}$ such that $f(\tilde{\alpha}_i) > 0$ for $i = 1, \ldots, n-1$. Moreover, since the group $S_n$ acts transitively on the set of all simple root systems, there exists an element $u \in S_n$ which sends $\{e_1, e_{n+1} | i = 1, \ldots, n-1\}$ to $\{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_{n-1}\}$.

For any $w \in M \setminus \{v_1\}$ we have $f(w \cdot \nu) \leq f(v_1 \cdot \nu)$ and the vector $w \cdot \nu - v_1 \cdot \nu$ has at least one negative coefficient with respect to $\{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_{n-1}\} = \{e_{u(i)} - e_{u(i+1)} | i = 1, \ldots, n-1\}$. Therefore, we have $v_1 \not\preceq w$ and this implies $v_1 \not\preceq w$ by Lemma 2.16. Accordingly, by the Maximality Property, $v_1$ must be the maximal element with respect to the $u$-shifted order. However, by the
similar argument, one can show that \( v_2 \) is also the maximal element with respect to the \( u \)-shifted order. This contradicts the Maximality Property.

Now we prove that if \( \Delta_M \) is a \( \Phi \)-polytope, then \( M \subset S_n \) is a Coxeter matroid. Assume on the contrary that \( M \) is not a Coxeter matroid. Then there are (at least) two maximal elements \( v_1 \) and \( v_2 \) in \( M \) with respect to \( \leq_u \) for some \( u \in S_n \). Let \( x_1 \cdot \nu, \ldots, x_\ell \cdot \nu \) be the vertices of \( \Delta_M \) adjacent to the vertex \( v_1 \cdot \nu \) and \( y_1 \cdot \nu, \ldots, y_m \cdot \nu \) the vertices adjacent to the vertex \( v_2 \cdot \nu \). Set

\[
(2.10) \quad \beta_p = x_p \cdot \nu - v_1 \cdot \nu \quad (p = 1, \ldots, \ell) \quad \text{and} \quad \gamma_q = y_q \cdot \nu - v_2 \cdot \nu \quad (q = 1, \ldots, m).
\]

Since \( \Delta_M \) is a \( \Phi \)-polytope, each \( \beta_p \) or \( \gamma_q \) is parallel to a root in \( \Phi \), and therefore either \( \preceq_u \) \( 0 \) or \( \succeq_u 0 \).

Suppose that \( \beta_p \preceq_u 0 \) for all \( p \). Then \( \Delta_M \) is contained in the cone \( \{ \chi \in V \mid \chi \preceq_u v_1 \cdot \nu \} \). Therefore \( z \cdot \nu \preceq_u v_1 \cdot \nu \) for all \( z \in M \), in particular, \( v_2 \cdot \nu \preceq_u v_1 \cdot \nu \). Similarly, if \( \gamma_q \preceq_u 0 \) for all \( q \), then \( v_1 \cdot \nu \preceq_u v_2 \cdot \nu \). Hence, if \( \beta_p \preceq_u 0 \) and \( \gamma_q \preceq_u 0 \) for all \( p \) and \( q \), then \( v_1 \cdot \nu = v_2 \cdot \nu \) and hence \( v_1 = v_2 \), a contradiction.

Thus, some \( \beta_p \) or \( \gamma_q \) is \( \succeq_u 0 \). Suppose that \( \beta_p \succeq_u 0 \) for some \( p \), i.e.,

\[
x_p \cdot \nu \succeq_u v_1 \cdot \nu.
\]

Then \( v_1 \preceq_u x_p \) by Lemma 2.17. This contradicts the maximality of \( v_1 \) because \( v_1 \neq x_p \) and \( x_p \in M \). The same argument works when \( \gamma_q \succeq_u 0 \) for some \( q \), so we deduce a contradiction in any case. Thus \( M \) is a Coxeter matroid.

**Example 2.18.** Using Theorem 2.15, one can check that subsets \( \{123, 213, 132, 312\} \) and \( \{231, 321\} \) of \( S_3 \) are Coxeter matroids while a subset \( \{213, 132\} \) of \( S_3 \) is not a Coxeter matroid since the edge joining the vertices \( s_1 \cdot \nu \) and \( s_2 \cdot \nu \) is not parallel to any root in \( \Phi = \{ \pm \alpha_1, \pm \alpha_2, \pm (\alpha_1 + \alpha_2) \} \) (see Figure 2 and Example 2.13).

\[
\begin{align*}
(1) \ M &= \{123, 213, 132, 312\} \\
(2) \ M &= \{231, 321\} \\
(3) \ M &= \{213, 132\}
\end{align*}
\]

**Figure 2. Examples of \( \Delta_M \)**

By Theorem 2.15, torus orbit closures provide a bunch of Coxeter matroids. A Coxeter matroid \( M \) of \( W \) is said to be **representable** if \( M \) can be realized as the \( T \)-fixed point set of a \( T \)-orbit closure in the flag variety \( G/B \), that is, there exists a point \( x \in G/B \) such that \( M = (T \cdot x)^T \), where \((G/B)^T\) is identified with \( W \). See [13, §1.7.5, §3.6.2, §3.10.3]. A computer check shows that any Coxeter matroid of \( S_n \) is representable when \( n \leq 4 \). On the other hand, there exists a non-representable Coxeter matroid of \( S_n \) when \( n = 7 \), which we explain in the following.

Let

\[
A = \{\{1, 2, 4\}, \{1, 3, 5\}, \{1, 6, 7\}, \{2, 3, 6\}, \{2, 5, 7\}, \{3, 4, 7\}, \{4, 5, 6\}\}
\]

be a collection of subsets of \( \{1, 2, \ldots, 7\} \). Each subset corresponds to one of the seven lines on the Fano plane in Figure 3. Define a subset \( M \) of \( S_7 \) by

\[
M = \{w \in S_7 \mid \{w(1), w(2), w(3)\} \notin A\}.
\]

Then \( M \) is a Coxeter matroid (see [48]).
Suppose that $\mathcal{M}$ is the $T$-fixed point set of a $T$-orbit closure in $GL_7(\mathbb{C})/B$. Then, by Proposition 2.6, there is an element $x$ in $GL_7(\mathbb{C})/B$ such that
\[ \mathcal{M} = \{ w \in S_7 \mid \{ w(1), \ldots, w(d) \}^t \in I_d(x) \text{ for all } 1 \leq d \leq 7 \}. \]
However, it is known in [65, §16] and easy to check that there is no $7 \times 3$ matrix of rank 3 whose three rows $v_{j_1}, v_{j_2}, v_{j_3}$ are linearly independent if and only if $\{ j_1, j_2, j_3 \} \notin A$. Therefore, $\mathcal{M}$ cannot be obtained as the $T$-orbit closure in $GL_7(\mathbb{C})/B$.

### 2.5. Description of the fan of a torus orbit closure.

For a point $x \in G/B$, the closure $Y := T \cdot x$ of its $T$-orbit $T \cdot x$ is a (possibly non-normal) toric variety in $G/B$. Although Proposition 2.6 provides how to find the fixed point set $Y^T$ and the convexity theorem leads us to the moment polytope $\mu(Y)$, it is hard to describe the fan of $Y$ from $\mu(Y)$. In this subsection and the next subsection, we identify $(G/B)^T = W$ for simplicity, so $Y^T \subset W$. We define a retraction $2R_Y$ (called geometric retraction) of the Weyl group $W$ of $G$ onto the $T$-fixed point set $Y^T \subset Y$ by using the Orbit-Cone correspondence in toric variety and describe the fan of $Y$ using the retraction.

We think of the Lie algebra $\mathfrak{t}$ of the maximal compact torus of $T$ as $\text{Hom}(\mathbb{C}^*, T) \otimes \mathbb{R}$ and the lattice $\mathbb{Z}_\mathfrak{t}$ of $\mathfrak{t}$ as $\text{Hom}(\mathbb{C}^*, T)$, where $\text{Hom}(\mathbb{C}^*, T)$ denotes the group of algebraic homomorphisms from $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$ to $T$. The vector space $\mathfrak{t}^*$ dual to $\mathfrak{t}$ can be thought of as $\text{Hom}(T, \mathbb{C}^*) \otimes \mathbb{R}$ and the set $\Phi$ of roots of $G$ is a finite subset of $\text{Hom}(T, \mathbb{C}^*) = \mathbb{Z}_\mathfrak{t}$. The Weyl group $W$ of $G$ acts on $\mathfrak{t}$ as the adjoint action and on its dual space $\mathfrak{t}^*$ as the coadjoint action, i.e.,
\[ (w \cdot f)(x) := f(A_{w^{-1}}x) \quad \text{for } w \in W, \quad f \in \mathfrak{t}^* \text{ and } x \in \mathfrak{t}. \]

The Borel subgroup $B$ determines the set $\Phi^+$ of positive roots and the tangent space $T_u(G/B)$ of $G/B$ at a $T$-fixed point $u \in W$ decomposes as follows:
\begin{equation}
(2.11) \quad T_u(G/B) = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_{-\alpha} \quad \text{as } T\text{-modules for } u \in W,
\end{equation}
where $\mathfrak{g}_{\alpha}$ denotes the root space of $\alpha$, which is the eigenspace of $\mathfrak{g}$ for $\alpha \in \Phi$ (see, for example, [34, §3]).

For each $u \in W$, we define
\[ C(u) := \{ \lambda \in \mathfrak{t} \mid \langle u(\alpha), \lambda \rangle \leq 0 \quad \text{for all simple roots } \alpha \} \]
where $\langle , \rangle$ denotes the natural pairing between $\mathfrak{t}^*$ and $\mathfrak{t}$. The interiors $\text{Int}(C(u))$ of the cones $C(u)$ above form the Weyl chambers. The identity in (2.11) implies that for any $\lambda_u \in \text{Int}(C(u)) \cap \mathbb{Z}_\mathfrak{t}$, we have
\begin{equation}
(2.12) \quad (G/B)^{\lambda_u(\mathbb{C}^*)} = (G/B)^T.
\end{equation}

For each $w \in W = (G/B)^T$, we choose an element $\lambda_w \in \text{Int}(C(w)) \cap \mathbb{Z}_\mathfrak{t}$ and define
\[ S_w := \{ x \in G/B \mid \lim_{t \to 0} \lambda_w(t) \cdot x = w \}, \]
which is independent of the choice of $\lambda_w$. Then $S_w$ is a $T$-invariant affine open subset of $G/B$ and isomorphic to $T_w(G/B)$ as a $T$-variety (see [9]).

![Figure 3. The Fano plane](image-url)
Proposition 2.19 ([48, Proposition 3.1]). Let \( x \) be a point of \( G/B \) and \( Y = \overline{T \cdot x} \). For any \( u \in W \) and \( \lambda_u \in \text{Int}(C(u)) \cap t_\mathbb{Z} \), the limit point \( \lim_{t \to 0} \lambda_u(t) \cdot x \) is an element of \( Y^T \) depending only on \( u \) and \( Y \). Furthermore, if \( u \in Y^T \), then \( \lim_{t \to 0} \lambda_u(t) \cdot x = u \).

Proof. Since \( Y \) is closed, the limit point \( \lim_{t \to 0} \lambda_u(t) \cdot x \) belongs to \( Y \) and clearly remains fixed under the action of \( \lambda_u(C^*) \). Therefore, the limit point is indeed in \( Y^T \) by (2.12). Denote the limit point by \( u \). Since \( \lambda_u(t) \cdot x \in S_w \) and \( S_w \) is \( T \)-invariant, \( x \) belongs to \( S_w \). Moreover, because \( S_w \) is isomorphic to \( T_w(G/B) \) as a \( T \)-variety, it follows from (2.11) that \( \lim_{t \to 0} \lambda_u(t) \cdot x \) is independent of the choice of \( \lambda_u \in \text{Int}(C(u)) \cap t_\mathbb{Z} \).

If \( u \in Y^T \), then \( x \) belongs to \( S_w \) because otherwise \( Y = \overline{T \cdot x} \) does not contain \( u \) (note that \( S_w \) is a \( T \)-invariant open subset of \( G/B \)). Therefore, we obtain \( \lim_{t \to 0} \lambda_u(t) \cdot x = u \). \( \square \)

By Proposition 2.19, the map \( R_Y^u : W \to Y^T \subset W \) defined by
\[
R_Y^u(u) := \lim_{t \to 0} \lambda_u(t) \cdot x
\]
is a retraction of \( W \) onto \( Y^T \), which we call a geometric retraction.

Example 2.20. Take a point \( x = \begin{pmatrix} \alpha & 1 & 0 \\ \beta & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \in \text{GL}_3(\mathbb{C})/B \) in Example 2.7, where \( \alpha, \beta \in \mathbb{C}^* \).

For \( Y = \overline{T \cdot x} \), we have \( Y^T = \{123, 132, 213, 312\} \) as observed in Example 2.7. Choose an element \( \lambda = (\lambda_1, \lambda_2, \lambda_3) \in \text{Int}(C(231)) \cap t_\mathbb{Z} \), that is, \( \lambda_2 < \lambda_3 < \lambda_1 \). Since
\[
(2.14) \quad x = \begin{pmatrix} \alpha \beta^{-1} & 1 & 0 \\ 1 & 0 & 1 \\ \beta^{-1} & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} \alpha \beta^{-1} & 1 & -\alpha \beta^{-1} \\ 1 & 0 & 0 \\ \beta^{-1} & 0 & -\beta^{-1} \end{pmatrix} \quad B = \begin{pmatrix} \alpha \beta^{-1} & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad B,
\]
we have
\[
\lambda(t) \cdot x = \begin{pmatrix} t^{\lambda_1} & 0 & 0 \\ 0 & t^{\lambda_2} & 0 \\ 0 & 0 & t^{\lambda_3} \end{pmatrix} \cdot \begin{pmatrix} \alpha \beta^{-1} & 1 & 0 \\ 1 & 0 & 0 \\ \beta^{-1} & 0 & 1 \end{pmatrix} \quad B
\]
\[
= \begin{pmatrix} t^{\lambda_1} \alpha \beta^{-1} & t^{\lambda_1} & 0 \\ t^{\lambda_2} & 0 & 0 \\ t^{\lambda_3} \beta^{-1} & 0 & t^{\lambda_3} \end{pmatrix} \quad B
\]
\[
= \begin{pmatrix} t^{\lambda_1 - \lambda_2 \alpha \beta^{-1}} & 1 & 0 \\ 1 & 0 & 0 \\ t^{\lambda_3 - \lambda_2 \beta^{-1}} & 0 & 1 \end{pmatrix} \quad B \xrightarrow{t \to 0} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad B \quad (\because \lambda_2 < \lambda_3 < \lambda_1). \]

Therefore, we get \( R_Y^u(231) = 213 \). By a similar computation, we obtain Table 1.

| \( u \) | 123 | 213 | 231 | 312 | 132 |
|---|---|---|---|---|---|
| \( R_Y^u(u) \) for \( Y \) | 123 | 213 | 213 | 312 | 312 | 312 | 132 |

Table 1. \( R_Y^u(u) \) for \( Y = \overline{T \cdot x} \) in Example 2.20.

The following corollary follows from Proposition 2.19 and the Orbit-Cone correspondence of toric varieties (see [19, Proposition 3.2.2 & Theorem 3.4.5]).

Corollary 2.21. (cf. [46, Corollary 3.7]) The maximal cone \( C_Y(y) \) corresponding to \( y \in Y^T \) in the fan of (the normalization of) \( Y \) is given by \( \bigcup_{u \in C_Y} R_Y^u(y) \).

Remark 2.22 ([48, Remark 3.4]). (1) Since the action of \( T \) on \( Y \) is not effective, the ambient space of the fan of \( Y \) is the quotient of \( T \) by the subspace \( \text{Hom}(\mathbb{C}^*, T_Y) \otimes \mathbb{R} \), where \( T_Y \) is the toral subgroup of \( T \) which fixes \( Y \) pointwise. Therefore, to be precise, we need to project the cones \( C_Y(y) \) to this quotient space in the corollary above.
(2) When $G$ is of type $A_n$, $D_4$, or $B_2$, every $T$-orbit closure in $G/B$ is normal ([16, Proposition 4.8]) while when $G = G_2$, non-normal torus orbit closures exist (see [16, Example 6.1] and references therein).

**Example 2.23.** Let $Y$ be the torus orbit closure in Example 2.20. Corollary 2.21 together with Table 1 shows that the fan of $Y$ consists of four maximal cones:

$$C(123), \ C(132), \ C(213) \cup C(231), \ C(312) \cup C(321).$$

See Figure 5. Here, the ambient space of the fan of $Y$ is the quotient space $\mathbb{R}^3/\mathbb{R}(1,1,1)$ and the identification $\mathbb{R}^3/\mathbb{R}(1,1,1) \rightarrow \mathbb{R}^2$ given by $[a_1, a_2, a_3] \mapsto (a_1 - a_2, a_2 - a_3)$ is used in Figures 4 and 5. For instance, the cone $C(213)$ consists of points $[a_1, a_2, a_3]$ satisfying $a_2 \leq a_1 \leq a_3$, so that it corresponds to the set of points $\{(b_1, b_2) \in \mathbb{R}^2 \mid b_1 \geq 0, b_1 + b_2 \leq 0\}$ under the identification.

![Figure 4. Cones $C(v)$ for $v \in S_3$](image1)

![Figure 5. The fan of $Y$](image2)

Now we reformulate the geometric retraction $R^p_\Phi$ using a Bruhat decomposition of $G/B$. It makes the meaning of the geometric retraction more transparent.

For $u \in W$, we set $B_u := uB^-u^{-1}$, where $B^-$ is the opposite Borel subgroup $w_0Bu_0$ and $w_0$ is the longest element of $W$. The Lie algebra of the Borel subgroup $B_u$ is given as follows:

$$\text{Lie}(B_u) = t \oplus \bigoplus_{\alpha \in \Phi^+} g_{-u(\alpha)}.$$

With respect to $B_u$, we obtain the following Bruhat decomposition:

$$G/B = \bigsqcup_{w \in W} B_u \cdot wB/B$$

and we set

\[ (2.15) \quad A^u_w := B_u \cdot wB/B = u \cdot B^-u^{-1}wB/B. \]

Note that $A^u_w$ is the Schubert cell $BwB/B$ and $A^e_w$ is the opposite Schubert cell $B^-wB/B$, where $e$ denotes the identity element of $W$. Similarly to (opposite) Schubert cells, we have

\[ (2.16) \quad A^u_w = \bigsqcup_{w^e \leq u \leq w} A^v_{w^e} \quad \text{and hence} \quad (A^u_w)^T = \{v \in W \mid w \leq u v \leq w^e w_0\}. \]

Indeed, since $B^-wB/B$ is the disjoint union of opposite Schubert cells indexed by elements $z \in W$ satisfying $w \leq z \leq w_0$, it follows from (2.15) that

$$A^u_w = u \cdot B^-u^{-1}wB/B = u \cdot \bigsqcup_{u^{-1}w \leq z \leq w_0} B^-zB/B = \bigsqcup_{u^{-1}w \leq u^{-1}(uz) \leq u^{-1}(uw_0)} u \cdot B^-u^{-1}(uz)B/B = \bigsqcup_{w \leq u v \leq w^e w_0} A^v_{w^e},$$

where $v = u^{-1}(uz)$ for $u^{-1}w \leq u^{-1}(uz) \leq u^{-1}(uw_0)$.
which shows (2.16).

**Proposition 2.24** ([48, Proposition 3.5]). Let \( x \) be a point of \( G/B \) and \( Y = \overline{T \cdot x} \). Then \( x \in A_w^u \) if and only if \( \mathcal{R}_Y(u) = w \).

**Example 2.25.** Take \( u = 231 \in S_3 \). Then the corresponding Borel subgroup \( B_u \) is given by

\[
B_u = u B^{-u} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & * & * \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} * & * & * \\ 0 & * & 0 \\ 0 & * & * \end{pmatrix}.
\]

Here, \( * \in \mathbb{C} \) and \( * \in \mathbb{C}^* \). For \( w = 213 \in S_3 \), the cell \( A_u^w = A_{231}^{213} \) is given by

\[
B_u w B / B = \begin{pmatrix} * & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} B.
\]

Therefore, \( x = \left( \begin{array}{c} \alpha \\ \beta \\ 0 \end{array} \right) \) is a Coxeter matroid by Theorem 2.16. Hence, \( \mathcal{R}_Y(231) = 213 \) for \( Y = \overline{T \cdot x} \) as observed in Example 2.7.

2.6. Retractions and metric on finite Coxeter groups. For a \( T \)-orbit closure \( Y = \overline{T \cdot x} \) in the flag variety \( G/B \), we defined the geometric retraction

\[ \mathcal{R}_Y^q : W \to Y^T \subset W \]

in (2.13) by looking at the limit points of the trajectory of \( x \) by 1-parameter subgroups. Note that \( Y^T \) is a Coxeter matroid by Theorem 2.15.

In general, for a Coxeter matroid \( M \) of a Coxeter group \( W \), there is a unique \( \leq^u \)-minimal element in \( M \) for any element \( u \) of \( W \) (see Remark 2.12). If we denote the minimal element by \( \mathcal{R}^q_M(u) \), then we obtain a map

\[ \mathcal{R}^q_M : W \to M \subset W. \]

One can easily check that \( \mathcal{R}^q_M \) is a retraction on \( W \) and we call \( \mathcal{R}^q_M \) a matroid retraction.\(^3\)

**Theorem 2.26** ([48, Theorem 3.7]). Let \( G \) be a semisimple algebraic group over \( \mathbb{C} \), \( B \) a Borel subgroup of \( G \), and \( T \) a maximal torus of \( G \) contained in \( B \). Then \( \mathcal{R}_Y^q = \mathcal{R}_Y^{q_T} \) for any \( T \)-orbit closure \( Y \) in \( G/B \).

**Proof.** Let \( x \) be a point of \( G/B \) such that \( Y = \overline{T \cdot x} \). For \( u \in W \), let \( \mathcal{R}_Y^q(u) = w \). By Proposition 2.24, \( T \cdot x \subset A_w^u \) and hence \( Y = \overline{T \cdot x} \subset A_{\mathcal{R}_Y^q(u)}^u \). Therefore, we have

\[ Y^T \subset A_{\mathcal{R}_Y^q}^u = \{ v \in W \mid w \leq^u v \leq^u w_0 \}, \]

where the equation above follows from (2.16). Hence, \( w \) is the unique \( \leq^u \)-minimal element in \( Y^T \), proving \( \mathcal{R}_Y^q = \mathcal{R}_Y^{q_T} \).

A finite Coxeter group \( W \) has a metric \( d \) defined by

\[ d(v, w) := \ell(v^{-1} w) = \ell(w^{-1} v) \quad \text{for } v, w \in W \]

where \( \ell( ) \) denotes the length function on \( W \). Note that the metric \( d \) is invariant under the left multiplication of \( W \). For a subset \( M \) of \( W \), we define

\[ d(v, M) := \min \{ d(v, w) \mid w \in M \}. \]

The metric \( d \) can be interpreted in terms of the \( W \)-permutohedron \( \Delta_W \) as follows. As mentioned in Subsection 2.4, the \( W \)-permutohedron \( \Delta_W \) is the convex hull of the \( W \)-orbit of a generic point \( v \) in the vector space \( V \), and two vertices \( v \cdot v \) and \( w \cdot v \) are joined by an edge in \( \Delta_W \) if and only if \( v^{-1} w \) is a generator of \( W \). Therefore, if we identify \( w \cdot v \) with \( w \in W \), then the distance \( d(v, w) \) can be thought of as the minimum length of the paths in \( \Delta_W \) connecting \( v \) and \( w \) through

\(^3\)In [13], a map \( \mu : W \to W \) satisfying the inequality \( \mu(u) \leq^v \mu(v) \) for all \( u, v \in M \) is called a matroid map. For a Coxeter matroid \( M \), the map \( W \to W \) sending \( u \) to the \( \leq^u \)-maximal element of \( M \) is a matroid map. Note that our matroid retraction satisfies the opposite inequality \( \mathcal{R}^q_M(u) \geq^v \mathcal{R}^q_M(v) \) for all \( u, v \in M \).
edges of $\Delta_W$. In other words, the metric $d$ is the graph metric on the graph obtained as the 1-skeleton of $\Delta_W$. For example, if $W = S_4$ and $(v, w) = (1243, 3214)$, then $v^{-1}w = 4213$ and hence $d(v, w) = \ell(4213) = 4$. Figure 6 shows a minimal-length path joining $v$ and $w$ in $\Delta_{S_4}$. In Figure 6, the vertex $v \cdot \nu$ is labeled by $\nu$ for each $v \in S_4$.

**Figure 6.** A minimal-length path between 1243 and 3214 in $\Delta_{S_4}$

For every subset $M$ of $W$ and every $u \in W$, there exists an element $y \in M$ such that $d(u, y) = d(u, M)$ since $M$ is a finite set, but such an element $y$ may not be unique. However, the following proposition says that such an element $y$ is unique if $M$ is a Coxeter matroid.

**Proposition 2.27** ([48, Proposition 2.6]). *If $M$ is a Coxeter matroid of a finite Coxeter group $W$, then $q = R^M(u)$ is the unique element satisfying $d(u, q) = d(u, M)$.*

**Proof.** First we remark that if $v \prec^u w$, then $d(u, v) < d(u, w)$ for $u, v, w \in W$. Indeed, this is shown by the following observation:

\[ v \prec^u w \iff u^{-1}v < u^{-1}w \]
\[ \iff d(e, u^{-1}v) < d(e, u^{-1}w) \]
\[ \iff d(u, v) < d(u, w), \]

where $e$ denotes the identity element of $W$ and the last equivalence follows from the invariance of the metric $d$ under the left multiplication of $W$.

Since $R^M_M(u)$ is the unique $\leq^u$-minimal element in $M$, it follows from the above observation that $d(u, R^M_M(u)) \leq d(u, w)$ for every $w \in M$ and the equality holds only when $w = R^M_M(u)$. Hence $R^M_M(u)$ is the unique element in $W$ closest to $u$. This proves the proposition. \(\square\)

There is an algorithm to find $R^M_M(u)$ when $W$ is a Weyl group of classical Lie type. The Weyl group $W$ of classical Lie type is of the following form:

\[ W = \begin{cases} S_n & \text{if } W \text{ is of type } A_{n-1}, \\ (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n & \text{if } W \text{ is of type } B_n \text{ or } C_n, \\ (\mathbb{Z}/2\mathbb{Z})^{n-1} \rtimes S_n & \text{if } W \text{ is of type } D_n. \end{cases} \]

We denote the set $\{\bar{1}, \ldots, \bar{n}\}$ by $[\bar{n}]$ and regard $\bar{i} = i$. In each type, we will use one-line notation for $u \in W$, i.e.,

\[ u = u(1)u(2)\cdots u(n) \]

where $u(i) \in [n] \cup [\bar{n}]$ and $u(1)u(2)\cdots u(n)$ is a permutation on $[n]$ if we forget the bars. There is no bar in type $A$ and the number of bars in $u(1), \ldots, u(n)$ is even (possibly zero) in type $D$. In types $B$, $C$ and $D$, we have $u(\bar{i}) = u(i)$.

For $u \in W$, we define a linear order $\prec^u$ on the set $[n] \cup [\bar{n}]$ by

\[ u(1) \prec^u \cdots \prec^u u(n) \prec^u u(\bar{1}) \prec^u \cdots \prec^u u(\bar{n}). \]
This induces a $u$-lexicographic order $\preceq^u_{\text{lex}}$ on the set of words of length $n$ in the alphabet $[n] \cup [n]$. Then we obtain a linear order $\prec^u$ on $W$, where $v \prec^u w$ if and only if $v(1) \cdots v(n) \prec^u_{\text{lex}} w(1) \cdots w(n)$. Note that $u$ is the minimal element of $W$ with respect to $\prec^u$.

**Definition 2.28.** Let $W$ be a Weyl group of classical Lie type and $M$ an arbitrary subset of $W$. For each $u \in W$, we define $R_M(u)$ as the $u$-minimal element of $M$ with respect to the order $\prec^u$. Then the map

$$R_M^u : W \to M (\subset W)$$

is a retraction of $W$ onto $M$, which we call an algebraic retraction.

**Example 2.29.**

1. We take a subset $M = \{123, 132, 213, 312\}$ of $S_3$. For $u = 231$, we have a linear order $2 \prec^u 3 \prec^u 1$. Since

$$213 \prec^u_{\text{lex}} 312 \prec^u_{\text{lex}} 123 \prec^u_{\text{lex}} 132,$$

we have $R_M^u(231) = 213$.

2. We take a subset $M = \{1423, 1432, 2413, 3412\}$ of $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_4$. For $u = 2314$, we have a linear order

$$\bar{2} \prec^u 3 \prec^u \bar{1} \prec^u 4 \prec^u \bar{4} \prec^u 1 \prec^u \bar{3} \prec^u 2.$$

Since

$$1432 \prec^u_{\text{lex}} 1423 \prec^u_{\text{lex}} 3412 \prec^u_{\text{lex}} 2413,$$

we have $R_M^u(2314) = 1432$.

The following theorem together with Proposition 2.27 shows that if $M$ is a Coxeter matroid of a Weyl group $W$ of classical Lie type, then $R_M^u(u)$ for $u \in W$ provides the point in $M$ closest to $u$.

**Theorem 2.30 (48, Theorem 4.7).** If $M$ is a Coxeter matroid of a Weyl group $W$ of classical Lie type, then $R_M^u = R_M^m$.

Unless $M$ is a Coxeter matroid, $R_M^u(u)$ does not necessarily provide a point in $M$ closest to $u$ as shown in the following example.

**Example 2.31.** Let $W = S_4$ and $M = \{1423, 2134\}$. Then $M$ is not a Coxeter matroid because the convex hull $\Delta_M$ of $M$, the dotted red line in Figure 7, is not a $\Phi$-polytope since the dotted red line is not parallel to any edge of the $S_4$-permutohedron. On the other hand, one can check that $M$ has the following property: for each $u \in S_4$, there is a unique element in $M$ closest to $u$. However, if we take $u = 1324$ for instance, then $d(u, 2134) = 2$ and $d(u, 1423) = 3$, but $R_M^u(u) = 1423$, see Figure 7. Therefore, $R_M^u(1324)$ is not an element of $M$ closest to 1324.

![Figure 7. $R_M^u(1324)$ is not the element of $M$ closest to 1324.](image)

As mentioned in Proposition 2.27, the following is a necessary condition for $M \subset W$ to be a Coxeter matroid.
For each $u \in W$, there is a unique $q \in M$ such that $d(u, q) = d(u, M)$. However, it is not a sufficient condition as is shown in Example 2.31. On the other hand, Proposition 2.27 and Theorem 2.30 show that if $M$ is a Coxeter matroid of $W$, then the unique element $q$ in ($*$) above must be given by $R^*_M(u)$. We ask whether these two necessary conditions are sufficient:

**Problem 2.32** ([48]). Let $W$ be a Weyl group of classical Lie type. Suppose that a subset $M$ of $W$ satisfies the following two conditions:

1. for each $u \in W$, there is a unique $q \in M$ such that $d(u, q) = d(u, M)$, and
2. $q = R^*_M(u)$.

Then, is $M$ a Coxeter matroid?

For $M \subseteq S_n$, the answer of the above problem is yes when $|M| = 2$ (see [48, Proposition 4.9]) or $n \leq 4$ (by a computer check), but we do not know the answer for an arbitrary subset $M$ of $S_n$ with $n \geq 5$.

### 3. Generic Torus Orbit Closures in the Flag Variety

In this section, we observe that generic torus orbit closures in the flag variety $Fl(n)$ are in fact permutohedral varieties and then discuss their topology. Their Poincaré polynomials turn out to be Eulerian polynomials. We also discuss Klyachko’s theorem which describes the restriction image of the cohomology of $Fl(n)$ to that of the generic torus orbit closure in $Fl(n)$. Finally we discuss a generalization of Klyachko’s theorem to Hessenberg varieties.

The image of the moment map

$$
\mu: Fl(n) = G/B \rightarrow \mathbb{R}^n
$$

defined in (2.5) is the permutohedron

$$
\Pi_n = \text{Conv}\{ (w(1), \ldots, w(n)) \mid w \in S_n \}.
$$

Note that $\Pi_n$ is the $S_n$-permutohedron in Subsection 2.4. The permutohedron $\Pi_n$ lies on the hyperplane

$$
\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 + \cdots + x_n = n(n+1)/2 \}
$$

and one can easily see that $\Pi_n$ is of dimension $n-1$.

**Definition 3.1.** We say that a $T$-orbit $O$ in $Fl(n)$ is *generic* if $\mu(O) = \Pi_n$, in other words if $\overline{O^T} = S_n$.

#### 3.1. Faces of the Permutohedron $\Pi_n$.

We recall some facts on the faces of the permutohedron $\Pi_n$. We refer the reader to [44, Section 5.A] or [68, Theorem 6.1] for more detail.

The permutohedron $\Pi_n$ is contained in the half space of $\mathbb{R}^n$ defined by

$$
\left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \left| \sum_{i=1}^{k} x_i \geq \sum_{i=1}^{k} i \right. \right\} (k = 1, \ldots, n-1)
$$

and the intersection of its boundary with $\Pi_n$, that is,

$$
\left\{ (x_1, \ldots, x_n) \in \Pi_n \left| \sum_{i=1}^{k} x_i = \sum_{i=1}^{k} i \right. \right\} (k = 1, \ldots, n-1)
$$

is a facet of $\Pi_n$ which meets at the vertex $(1, \ldots, n)$. This provides all the facets of $\Pi_n$ which meet at the vertex $(1, \ldots, n)$. Since $\Pi_n$ is invariant under permutations of the coordinates of $\mathbb{R}^n$, it follows that $\Pi_n$ is a simple polytope of dimension $n-1$ and the facets of $\Pi_n$ meeting at the vertex $(w(1), \ldots, w(n))$ of $\Pi_n$ ($w \in S_n$) are written as

$$
\left\{ (x_1, \ldots, x_n) \in \Pi_n \left| \sum_{i=1}^{k} x_{w^{-1}(i)} = \sum_{i=1}^{k} i \right. \right\} (k = 1, \ldots, n-1).
$$
The facet in (3.3) is determined by the subset \(\{w^{-1}(1), \ldots, w^{-1}(k)\}\) of \([n]\) and the observation above shows that there is a bijective correspondence between non-empty proper subsets of \([n]\) and facets of \(\Pi_n\). Indeed, the facet of \(\Pi_n\) associated to a non-empty proper subset \(A\) of \([n]\) is defined by

\[
F(A) := \left\{ (x_1, \ldots, x_n) \in \Pi_n \mid \sum_{a \in A} x_n = \frac{|A|(|A| + 1)}{2} \right\}.
\]

Therefore, there are \(2^n - 2\) facets in \(\Pi_n\) and (3.3) shows that the \(n - 1\) facets of \(\Pi_n\) meeting at the vertex \((w(1), \ldots, w(n))\) are \(F(A)\)'s with \(w(A) = \{1, \ldots, |A|\}\). See Figure 8.

![Figure 8. Facets of the permutohedron \(\Pi_3\)](image)

The following lemma can easily be proved.

**Lemma 3.2.** Let \(A\) and \(B\) be non-empty proper subsets of \([n]\). Then \(F(A) \cap F(B) \neq \emptyset\) if and only if \(A \subseteq B\) or \(B \subseteq A\). Therefore, a codimension \(k\) face of \(\Pi_n\) is \(\bigcap_{i=1}^d F(A_i)\) with \(A_1 \subset \cdots \subset A_k\).

Since \(\Pi_n\) lies on the hyperplane in (3.1) which is perpendicular to the vector \((1, \ldots, 1)\) in \(\mathbb{R}^n\), the normal fan \(\Sigma(\Pi_n)\) of \(\Pi_n\) is defined in the quotient space \(N_{\mathbb{R}} := \mathbb{R}^n / \mathbb{Z}^n\), where the lattice \(N\) of \(N_{\mathbb{R}}\) is the one induced from the standard lattice \(\mathbb{Z}^n\) of \(\mathbb{R}^n\). By (3.4), the primitive (inward) normal vector \(\delta_A\) to the hyperplane in (3.4) defining the facet \(F(A)\) is the \((0,1)\)-vector with 1 in the \(a\)th coordinate for each \(a \in A\) and 0 otherwise. Therefore we have the following.

**Lemma 3.3.** The quotient image of the vector \(\delta_A\) on \(N_{\mathbb{R}} = \mathbb{R}^n / \mathbb{Z}^n\) is the primitive ray vector in the normal fan \(\Sigma(\Pi_n)\) corresponding to the facet \(F(A)\).

The maximal cone in the fan \(\Sigma(\Pi_n)\) corresponding to the vertex \((1, \ldots, n)\) is spanned by the ray vectors corresponding to the \(n - 1\) facets in (3.2) and those ray vectors are the projection image of the following \(n - 1\) vectors on \(N_{\mathbb{R}}\):

\[
(1,0,\ldots,0), \quad (1,1,0,\ldots,0), \quad \ldots, \quad (1,\ldots,1,0).
\]

The other maximal cones in \(\Sigma(\Pi_n)\) are obtained by permuting the above cone. This shows that the collection of maximal cones coincides with the (closures of) Weyl chambers in type \(A\). The maximal cones determine the fan \(\Sigma(\Pi_n)\) and the permutohedral variety \(\text{Perm}_n\) is the toric variety whose fan has the (closure of) Weyl chambers as maximal cones. Therefore, we obtain the following.

**Proposition 3.4.** The maximal cones in the fan of a generic torus orbit closure in the flag variety \(\text{Fl}(n)\) are the (closures of) Weyl chambers. Therefore, the closure of a generic torus orbit in the flag variety \(\text{Fl}(n)\) is isomorphic to the permutohedral variety \(\text{Perm}_n\).

The cohomology ring of a compact smooth toric variety is explicitly described in terms of the associated fan. Applying the general result to our setting together with Lemmas 3.2 and 3.3, we obtain the following theorem.
Theorem 3.5. The cohomology ring of the permutohedral variety $\text{Perm}_n$ has the following presentation:

$$H^*(\text{Perm}_n; \mathbb{Z}) = \mathbb{Z}[\tau_A \mid \emptyset \subset A \subset [n]]/\mathcal{I}$$

where $\deg \tau_A = 2$ and $\mathcal{I}$ is the ideal generated by the following two types of elements:

1. $\tau_A \tau_B$ for $A \not\subset B$ or $B \not\subset A$,
2. $\sum_{p \in A} \tau_A - \sum_{q \in B} \tau_B$ for $p, q \in [n]$.

Example 3.6. Take $n = 3$. Then $H^*(\text{Perm}_3; \mathbb{Z})$ is generated by six elements

$$\tau(1), \tau(2), \tau(3), \tau(1,2), \tau(1,3), \tau(2,3)$$

with relations

1. $\tau(1)\tau(2) = \tau(1)\tau(3) = \tau(2)\tau(3) = \tau(1,2)\tau(1,3) = \tau(2)\tau(1,3) = \tau(3)\tau(1,2) = 0$,
2. $\tau(1) + \tau(1,2) + \tau(1,3) = \tau(2) + \tau(1,2) + \tau(2,3) = \tau(3) + \tau(1,3) + \tau(2,3)$.

It follows that $H^2(\text{Perm}_3)$ is freely generated by four classes, say $\tau(1)$, $\tau(2)$, $\tau(1,2)$, $\tau(2,3)$, and $H^4(\text{Perm}_3)$ is freely generated by one class, say $\tau(1)\tau(1,2)$. Therefore, the Poincaré polynomial of $\text{Perm}_3$ is given by

$$(3.5) \quad \text{Poin}(\text{Perm}_3, t) = 1 + 4t^2 + t^4.$$ 

In fact, since the fan of $\text{Perm}_3$ is isomorphic to the fan of the toric variety obtained by blowing up $\mathbb{CP}^2$ with the standard $T^2$-action at the three fixed points, $\text{Perm}_3$ is isomorphic to $\mathbb{CP}^2#3\mathbb{CP}^2$, where $\mathbb{CP}^2$ denotes $\mathbb{CP}^2$ with the reversed orientation.

3.2. Eulerian polynomial. The ascent $\text{asc}(w)$ of $w \in S_n$ is defined by

$$(3.6) \quad \text{asc}(w) := \# \{ i \in [n-1] \mid w(i) < w(i+1) \}$$

and the Eulerian polynomial $A_n(t)$ is defined by

$$A_n(t) := \sum_{w \in S_n} t^{\text{asc}(w)}.$$ 

Note that $0 \leq \text{asc}(w) \leq n - 1$ for any $w \in S_n$, where the former equality is attained only when $w = [n, n-1, \ldots, 1]$ and the latter equality is attained only when $w$ is the identity. Therefore, the constant term of $A_n(t)$ is 1 and the highest degree term is $t^{n-1}$. Eulerian polynomials $A_n(t)$ for $n \leq 5$ are given as follows:

$$(3.7) \quad A_1(t) = 1, \quad A_2(t) = 1 + t, \quad A_3(t) = 1 + 4t + t^2, \quad A_4(t) = 1 + 11t + 11t^2 + t^3, \quad A_5(t) = 1 + 26t + 66t^2 + 26t^3 + t^4.$$ 

Remark 3.7. (1) The descent $\text{des}(w)$ of $w \in S_n$ is defined by

$$\text{des}(w) := \# \{ i \in [n-1] \mid w(i) > w(i+1) \}$$

and it holds that

$$\sum_{w \in S_n} t^{\text{asc}(w)} = \sum_{w \in S_n} t^{\text{des}(w)}.$$ 

Therefore, we may define the Eulerian polynomial $A_n(t)$ using the descents.

(2) The Eulerian polynomial $A_n(t)$ was originally defined by Euler as the polynomial which appears in the numerator of the following formula:

$$\sum_{k=1}^{\infty} k^n t^k = \frac{tA_n(t)}{(1-t)^{n+1}}.$$ 

As easily checked, the Eulerian polynomials defined this way have the following recurrence relation

$$A_{n+1}(t) = (nt + 1)A_n(t) - t(t-1)\frac{dA_n(t)}{dt}$$

while one can see that the Eulerian polynomials defined in (3.6) satisfy the same recurrence relation as above. Since $A_1(t) = 1$ in both definitions, they define the same family of polynomials.
It follows from (3.5) and (3.7) that $\text{Poin}(\text{Perm}_3, t) = A_3(t^2)$. This is not a coincidence.

**Theorem 3.8 ([42, Theorem 2]).** $\text{Poin}(\text{Perm}_n, t) = A_n(t^2)$ for any $n \geq 1$.

**Remark 3.9.** The polynomials $A_n(t)$ for $n \leq 5$ in (3.7) are palindromic and unimodal. Indeed, this holds for any $A_n(t)$ since the Poincaré duality implies the palindromicity and the hard Lefschetz theorem implies the unimodality because $\text{Perm}_n$ is a smooth projective toric variety.

In order to give a proof of Theorem 3.8, we recall a fact on edges of the permutohedron $\Pi_n$.

**Lemma 3.10.** Two vertices $(u(1), \ldots, u(n))$ and $(v(1), \ldots, v(n))$ of $\Pi_n$ ($u, v \in S_n$) are joined by an edge of $\Pi_n$ if and only if there is an adjacent transposition $s_i = (i, i + 1)$ for $i \in [n - 1]$ such that $v = s_iu$.

**Proof.** We note that

\[(v(1), \ldots, v(n)) - (1, \ldots, n) = \sum_{i=1}^{n-1} \left( \sum_{k=1}^{i} v(k) - \sum_{k=1}^{i} u(k) \right) (e_i - e_{i+1})\]

where $e_1, \ldots, e_n$ denote the standard basis vectors of $\mathbb{R}^n$. The coefficient of $e_i - e_{i+1}$ in (3.8) is non-negative for any $i$ and $v \in S_n$. Moreover, when $v = s_i$, the right hand side at (3.8) is $e_i - e_{i+1}$. This implies the lemma when $u$ is the identity. The general case follows from this special case and the invariance of $\Pi_n$ under permutations of the coordinates of $\mathbb{R}^n$.

**Proof of Theorem 3.8.** Equation (3.8) is generalized to

\[(v(1), \ldots, v(n)) - (u(1), \ldots, u(n)) = \sum_{i=1}^{n-1} \left( \sum_{k=1}^{i} v(k) - \sum_{k=1}^{i} u(k) \right) (e_i - e_{i+1}).\]

By Lemma 3.10, the endpoints of the edges emanating from the vertex $(u(1), \ldots, u(n))$ are $(s_iu(1), \ldots, s_iu(n))$ for $i \in [n - 1]$. Set $p = u^{-1}(i)$ and $q = u^{-1}(i + 1)$. Then, since $u(p) = i + 1$ and the action of $s_i$ from the left interchanges $i$ and $i + 1$, (3.9) reduces to

\[(s_iu(1), \ldots, s_iu(n)) - (u(1), \ldots, u(n)) = e_p - e_q = e_{u^{-1}(i)} - e_{u^{-1}(i+1)}.\]

Take real numbers $a_1 > \cdots > a_n$ and consider a linear function $f: \mathbb{R}^n \to \mathbb{R}$ defined by

\[f(x_1, \ldots, x_n) = a_1x_1 + \cdots + a_n x_n.\]

It follows from (3.10) that

\[f(s_iu(1), \ldots, s_iu(n)) - f(u(1), \ldots, u(n)) = a_{u^{-1}(i)} - a_{u^{-1}(i+1)}\]

\[\begin{cases} > 0 & \text{if } u^{-1}(i) < u^{-1}(i + 1), \\ < 0 & \text{if } u^{-1}(i) > u^{-1}(i + 1). \end{cases}\]

Recall that $\mu(wB) = (w^{-1}(1), \ldots, w^{-1}(n))$ for the moment map $\mu: \text{Fl}(n) \to \mathbb{R}^n$ and we labeled the vertex $(w^{-1}(1), \ldots, w^{-1}(n))$ as $w$. Then, the observation above shows that asc($w$) is the number of edges emanating from the vertex $w$ and increasing with respect to the function $f$. Therefore, discrete Morse theory applied to the function $f$ on the permutohedron $\Pi_n$ produces a decomposition

\[\Pi_n = \bigsqcup_{w \in S_n} P_w, \quad P_w \approx (\mathbb{R}_{\geq 0})^{\text{asc}(w)}\]

where $\approx$ means homeomorphic as a manifold with corners. Moreover, one sees that $\mu^{-1}(P_w) \approx \mathbb{C}^{\text{asc}(w)}$ for $\mu$ restricted to $\text{Perm}_n$ and the decomposition

\[\text{Perm}_n = \mu^{-1}(\Pi_n) = \bigsqcup_{w \in S_n} \mu^{-1}(P_w)\]

gives a cell decomposition of $\text{Perm}_n$. Since the dimension of the cell $\mu^{-1}(P_w)$ is $2\text{asc}(w)$, we obtain

\[\text{Poin}(\text{Perm}_n, t) = \sum_{w \in S_n} t^{2\text{asc}(w)} = A_n(t^2),\]

proving the theorem. \qed
Example 3.11. Take \( n = 3 \). Consider a linear function \( f(x_1, x_2, x_3) = 5x_1 + 4x_2 + x_3 \). The values \( f(\mu(wB)) \) and \( \text{asc}(w) \) are given as follows. Note that the vertex \( \mu(wB) \) of \( \Pi_3 \) is labeled by \( w \) as before.

| \( w \) | 123 | 213 | 132 | 231 | 312 | 321 |
|-------|-----|-----|-----|-----|-----|-----|
| \( \mu(wB) \) | (1, 2, 3) | (2, 1, 3) | (1, 3, 2) | (3, 1, 2) | (2, 3, 1) | (3, 2, 1) |
| \( f(\mu(wB)) \) | 16 | 17 | 19 | 21 | 23 | 24 |
| \( \text{asc}(w) \) | 2 | 1 | 1 | 1 | 1 | 0 |

Accordingly, we have

\[ \text{Poin}(\text{Perm}_3, t) = A_3(t^2) = 1 + 4t^2 + t^4. \]

3.3. Klyachko’s result. Since

\[ \text{Fl}(n) = \{ V_\bullet = (\{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n) \mid \dim \mathbb{C} V_i = i \text{ for all } i = 1, \ldots, n \}, \]

there is a sequence of tautological complex vector bundles on \( \text{Fl}(n) \):

\[ E_1 \subset E_2 \subset \cdots \subset E_n = \text{Fl}(n) \times \mathbb{C}^n \]

where

\[ E_i := \{ (V_\bullet, v) \in \text{Fl}(n) \times \mathbb{C}^n \mid v \in V_i \} \quad (i = 1, \ldots, n). \]

We define

\[ x_i := c_1(E_i/E_{i-1}) \quad \text{for } i = 1, \ldots, n, \]

where \( E_0 = \text{Fl}(n) \), i.e., the 0-dimensional vector bundle over \( \text{Fl}(n) \). Since

\[ E_n \cong \bigoplus_{i=1}^n E_i/E_{i-1} \]

and \( E_n \) is the trivial vector bundle \( \text{Fl}(n) \times \mathbb{C}^n \), we have

\[ 1 = c(E_n) = \prod_{i=1}^n c(E_i/E_{i-1}) = \prod_{i=1}^n (1 + x_i). \]

Therefore, the \( i \)th elementary symmetric polynomial \( e_i(x) \) in \( x_1, \ldots, x_n \) vanishes in \( H^*(\text{Fl}(n); \mathbb{Z}) \) for \( i = 1, \ldots, n \). The following theorem shows that \( H^*(\text{Fl}(n); \mathbb{Z}) \) is generated by \( x_1, \ldots, x_n \) as a ring and any polynomial in \( x_1, \ldots, x_n \) which vanishes in \( H^*(\text{Fl}(n); \mathbb{Z}) \) is generated by \( e_i(x) \)'s.

Theorem 3.12 (Borel [12]). Let \( x_1, \ldots, x_n \) be as above. Then

\[ H^*(\text{Fl}(n); \mathbb{Z}) = \mathbb{Z}[x_1, \ldots, x_n]/(e_1(x), \ldots, e_n(x)) \]

as a ring.

We denote by \( \text{Perm}_n \) a generic torus orbit closure in the flag variety \( \text{Fl}(n) \) because it is isomorphic to the permutohedral variety \( \text{Perm}_n \). The natural action of \( S_n \) on the fan of \( \text{Perm}_n \) induces an action of \( S_n \) on \( \text{Perm}_n \) and hence on its cohomology ring.

Theorem 3.13 (Klyachko [42]). Let \( \iota : \text{Perm}_n \to \text{Fl}(n) \) be the inclusion map. Then the image of the restriction map

\[ \iota^* : H^*(\text{Fl}(n); \mathbb{Q}) \to H^*(\text{Perm}_n; \mathbb{Q}) \]

agrees with the ring of \( S_n \)-invariants \( H^*(\text{Perm}_n; \mathbb{Q})^{S_n} \) and

\[ H^*(\text{Perm}_n; \mathbb{Q})^{S_n} = \mathbb{Q}[\omega_1, \ldots, \omega_{n-1}]/(\omega_i(\omega_{i-1} - 2\omega_i + \omega_{i+1}) \mid i = 1, \ldots, n - 1) \]

where \( \omega_i = x_1 + \cdots + x_i \) and \( \omega_0 = \omega_n = 0 \).

Remark 3.14. (1) The cohomology ring of \( \text{Perm}_n \) is explicitly described in Theorem 3.5. The \( S_n \)-orbit of \( \tau_A \) with \( |A| = i \) is \( \sum_{|B|=i} \tau_B \) which is equal to \( x_i - x_{i+1} \).

(2) If we set \( \alpha_i := x_i - x_{i+1} \), then \( \omega_i - 2\omega_i + \omega_{i+1} = -\alpha_i \). Therefore, we may express the ideal in (3.11) as \( (\omega_i\alpha_i \mid i = 1, \ldots, n - 1) \). Note that \( \omega_i \)'s correspond to the fundamental weights while \( \alpha_i \)'s correspond to the roots in type \( A \).
3.4. Relation to Hessenberg varieties. Theorem 3.13 is generalized to the setting of Hessenberg varieties. Given a square matrix $A$ of size $n$ and a function $h: [n] \to [n]$ (called a Hessenberg function) satisfying
\[ h(1) \leq h(2) \leq \cdots \leq h(n) \quad \text{and} \quad h(j) \geq j \quad \text{for all} \quad j \in [n], \]
the Hessenberg variety $\text{Hess}(A, h)$ is defined as
\[ \text{Hess}(A, h) := \{ V_\bullet \in \text{Fl}(n) \mid A(V_j) \subset V_{h(j)} \text{ for all } j \in [n] \}, \]
where the matrix $A$ is regarded as a linear transformation on $\mathbb{C}^n$. We often express the Hessenberg function $h$ as a vector $(h(1), \ldots, h(n))$ by listing the values of $h$. When $h = (n, \ldots, n)$, it is obvious from the definition that $\text{Hess}(A, h)$ is the flag variety $\text{Fl}(n)$ regardless of $A$.

The Hessenberg variety $\text{Hess}(S, h)$ for a square matrix $S$ of size $n$ with distinct eigenvalues is called regular semisimple. It is known that
\begin{enumerate}
\item $\text{Hess}(S, h)$ is smooth,
\item $\dim \text{Hess}(S, h) = \sum_{j=1}^n (h(j) - j)$,
\item $\text{Hess}(S, h)$ is connected if and only if $h(j) \geq j + 1$ for all $j \in [n - 1]$.
\end{enumerate}
The topology of $\text{Hess}(S, h)$ is independent of the choice of $S$ and we take $S$ to be a diagonal matrix with distinct diagonal entries in the following. Since $S$ commutes with the diagonal torus $T$ of $\text{GL}_n(\mathbb{C})$, the restricted action of $T$ on $\text{Fl}(n)$ leaves $\text{Hess}(S, h)$ invariant. One sees that
\begin{equation}
\text{Hess}(S, h)^T = \text{Fl}(n)^T = S_n.
\end{equation}
When $h = (2, 3, \ldots, n, n)$, it follows from (2) and (3) above that $\dim \text{Hess}(S, h) = n - 1$ and $\text{Hess}(S, h)$ is connected. Although $\dim \text{Hess}(S, h) = n - 1$, the subgroup $D$ of $T$ consisting of scalar matrices acts on $\text{Fl}(n)$ trivially and the induced action of $T/D$ on $\text{Hess}(S, h)$ is effective. Therefore, $\text{Hess}(S, h)$ for $h = (2, 3, \ldots, n, n)$ is a smooth toric variety and hence it is a torus orbit closure in $\text{Fl}(n)$. Moreover, the orbit is generic by (3.12). Therefore, $\text{Hess}(S, h)$ for $h = (2, 3, \ldots, n, n)$ is isomorphic to the permutohedral variety $\text{Perm}_n$.

Using the $T$-action on $\text{Hess}(S, h)$, Tymoczko [63] constructed an action of $S_n$ (called dot action) on $H^*(\text{Hess}(S, h); \mathbb{Z})$ and when $h = (2, 3, \ldots, n, n)$, this action agrees with that on $H^*(\text{Perm}_n; \mathbb{Z})$ induced from the action of $S_n$ on $\text{Perm}_n$. Therefore, the following theorem is a generalization of the former part of Theorem 3.13.

**Theorem 3.15** ([1, Theorem B]). Let $\iota: \text{Hess}(S, h) \to \text{Fl}(n)$ be the inclusion map. Then the image of the restriction map
\[ \iota^*: H^*(\text{Fl}(n); \mathbb{Q}) \to H^*(\text{Hess}(S, h); \mathbb{Q}) \]
agrees with the ring of $S_n$-invariants $H^*(\text{Hess}(S, h); \mathbb{Q})^{S_n}$ for any Hessenberg function $h$.

In fact, it is known that
\begin{equation}
H^*(\text{Hess}(S, h); \mathbb{Q})^{S_n} \cong H^*(\text{Hess}(N, h); \mathbb{Q}),
\end{equation}
where $N$ is a nilpotent matrix with one Jordan block (Hess($N, h$) is called regular nilpotent) and an explicit ring presentation of $H^*(\text{Hess}(N, h); \mathbb{Q})$, which reduces to (3.11) when $h = (2, 3, \ldots, n, n)$, is known. Therefore, Theorem 3.13 is completely generalized to the setting of Hessenberg varieties, see [1, 2, 36] in more detail.

**Remark 3.16.** It is shown in [3] that when $h = (2, 3, \ldots, n)$ (so that $\text{Hess}(S, h) = \text{Perm}_n$), Theorem 3.15 and (3.13) hold with $\mathbb{Z}$-coefficient but (3.11) does not.

4. Generic torus orbit closures in Schubert varieties

In this section, we consider geometric and topological properties of generic torus orbit closures in Schubert varieties. A generic torus orbit closure $Y_w$ in a Schubert variety $X_w$ is not smooth in general and we discuss how to determine the smoothness of $Y_w$ at a fixed point by considering a certain graph. We study the Poincaré polynomial of $Y_w$ which is a generalization of the Eulerian polynomial. Moreover, we study the fan of a toric Schubert variety and discuss the classification of
toric Schubert varieties. Finally, we summarize the properties of Schubert varieties of complexity one.

4.1. Generic torus orbit closures in Schubert varieties.

**Definition 4.1.** For $w \in S_n$, the Schubert variety $X_w$ is a subvariety of $G/B$ defined by $X_w := BwB/B$.

The action of $T$ on $G/B$ leaves both $X_w$ and $X_w^T$ invariant.

**Definition 4.2.** We say that a $T$-orbit $O$ in $X_w$ is generic if $\mu(O) = \mu(X_w)$, in other words, if $O^T = X_w^T$. We denote by $Y_w$ the closure of a generic torus orbit in the Schubert variety $X_w$.

Since $X_{w_0} = G/B$, the generic torus orbit closure $Y_{w_0}$ in $X_{w_0}$ is the permutohedral variety $\text{Perm}_n$ considered in Section 3 (cf. Definition 3.1), in other words, $Y_{w_0} = \text{Perm}_n$.

We notice that a fixed point $uB$ is contained in the Schubert variety $X_w$ if and only if $u \leq w$ in the Bruhat order; $uB$ is contained in the opposite Schubert variety $X^w$ if and only if $u \geq w$ in the Bruhat order. By Lemma 2.3, we have

$$\mu(X_w) = \mu(Y_w) = Q_w.$$

Motivated by this fact, we provide the following definition.

**Definition 4.3.** For $w \in S_n$, we define a polytope $Q_w$ by

$$Q_w := \text{Conv}\{(u^{-1}(1), \ldots, u^{-1}(n)) \mid u \leq w\} \subset \mathbb{R}^n.$$

We depict the Bruhat interval $[e, 3412] = \{u \in S_4 \mid u \leq 3412\}$ and the polytope $Q_{3412}$ in Figure 9.

![Figure 9. The Bruhat interval $[e, 3412]$ and the polytope $Q_{3412}$](image)

**Remark 4.4.** Note that the polytope $Q_w$ is the $\Phi$-polytope $\Delta_{[e, w^{-1}]}$ in Subsection 2.4, where $W = S_n$ and $\nu = (1, 2, \ldots, n)$.
By definition, $Y_w$ is the toric variety whose fan is the normal fan of the polytope $Q_w$. The polytope $Q_w$ is not simple in general. For example, the polytope $Q_{3412}$ is 3-dimensional, but there are four facets meeting at the vertex 3412 (see Figure 9). At each vertex of $Q_w$, the primitive direction vectors of the edges emanating from the vertex are root vectors, i.e., of the form $e_i - e_j$. Therefore, we obtain the following.

**Proposition 4.5** ([46, §8]). For $w \in S_n$, the toric variety $Y_w$ is smooth if and only if the polytope $Q_w$ is simple.

To each vertex of $Q_w$ we associate a graph which detects the simpleness of the vertex. For $u \leq w$, we set

$$
\tilde{E}_w(u) = \{(u(i), u(j)) \mid i < j, \; t_{u(i), u(j)} u \leq w, \; |\ell(u) - \ell(t_{u(i), u(j)} u)| = 1\},
$$

where $t_{a,b}$ denotes the transposition interchanging $a$ and $b$. Then the digraph $\Gamma_w(u)$ defined by the vertex set $[n]$ and the edge set $\tilde{E}_w(u)$ is acyclic by [62, Theorem 4.19], so it has a unique transitive reduction (see [4]). Here, a *transitive reduction* of a digraph is a digraph with the same vertices and as fewer edges as possible, such that if there is a directed path connecting two vertices, then there is also such a path in the reduction. We denote the transitive reduction of $\Gamma_w(u)$ by $\Gamma_w(u)$ and the edge set of $\Gamma_w(u)$ by $E_w(u)$.

**Example 4.6** ([52, Example 3.2]). Take $w = 3412$ and $u = 2143$. Then we have

$$
\tilde{E}_w(u) = \{(1, 4), (2, 3), (2, 1), (4, 3)\}.
$$

The corresponding digraph $\Gamma_w(u)$ and its transitive reduction $\Gamma_w(u)$ are presented as follows.

$$
\Gamma_w(u) = \begin{array}{c}
1 \rightarrow 2 \rightarrow 3 \rightarrow 4
\end{array} \quad \text{transitive reduction} \quad \begin{array}{c}
1 \rightarrow 2 \rightarrow 3 \rightarrow 4
\end{array} = \Gamma_w(u)
$$

As a result, we have $E_w(u) = \{(1, 4), (2, 1), (4, 3)\}$.

**Example 4.7.** For the longest element $w_0$, the condition $t_{u(i), u(j)} u \leq w_0$ is always satisfied. Therefore, we have

$$
\tilde{E}_{w_0}(u) = \{(u(i), u(j)) \mid i < j, \; |\ell(u) - \ell(t_{u(i), u(j)} u)| = 1\}.
$$

One can easily check that

$$
\{(u(i), u(i + 1)) \mid i = 1, \ldots, n - 1\} \subset \tilde{E}_{w_0}(u).
$$

Indeed, $E_{w_0}(u) = \{(u(i), u(i + 1)) \mid i = 1, \ldots, n - 1\}$.

**Proposition 4.8** ([46, Proposition 7.7]). For $w \in S_n$, two vertices $u$ and $v$ of $Q_w$ are joined by an edge if and only if

$$
v = t_{u(i), u(j)} u(= u t_{i,j}) \quad \text{for } (u(i), u(j)) \in E_w(u).
$$

The proposition above says that the elements in $E_w(u)$ correspond to the edges of $Q_w$ emanating from the vertex $u$. Therefore, one can read from the graph $\Gamma_w(u)$ the number of edges emanating from the vertex $u$ in the polytope $Q_w$.

**Theorem 4.9** ([46, Theorem 1.2]). The generic torus orbit closure $Y_w$ in a Schubert variety $X_w$ is smooth at a fixed point $uB$ in $Y_w$ if and only if the graph $\Gamma_w(u)$ is a forest. Therefore, $Y_w$ is smooth if and only if $\Gamma_w(u)$ is a forest for every $u \leq w$.

The graph $\Gamma_w(u)$ appears in a slightly different viewpoint. Indeed, it is shown in [66] that the Schubert variety $X_w$ is locally factorial if and only if $\Gamma_w(u)$ is a forest. This motivated the authors in [14] to study the graph and they provided a necessary and sufficient condition for $\Gamma_w(u)$ to be a forest in terms of the pattern avoidance.\footnote{We say that a permutation $w \in S_n$ contains the pattern $q = q_1 q_2 \cdots q_k$ if there is a $k$-element set of indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ so that $w(i_r) < w(i_s)$ if and only if $q_r < q_s$ for $1 \leq r < s \leq k$. If a permutation $w$ does not contain $q$, then we say that $w$ avoids the pattern $q$. Similarly, $w$ avoids the pattern 45312 if every occurrence of the pattern 45312 is a subsequence of an occurrence of 45312.} Combining the theorem above with [14, Theorem 1.1], we obtain the following.
Proposition 4.10. The following statements are equivalent.

1. \( \Gamma_w(u) \) is a forest.
2. \( w \) avoids the patterns 4231 and 45312.
3. \( Y_w \) is smooth at \( wB \).
4. \( Q_w \) is simple at \( w \).

About the smoothness of \( Y_w \), we have the following conjecture.

Conjecture 4.11 ([46, Conjecture 7.17]). The graph \( \Gamma_w(u) \) is a forest for any \( u \leq w \) if \( \Gamma_w(w) \) is a forest. (This is equivalent to saying that the generic torus orbit closure \( Y_w \) in the Schubert variety \( X_w \) is smooth if it is smooth at the fixed point \( wB \), in other words, \( Q_w \) is simple if \( Q_w \) is simple at the vertex \( w \).)

4.2. Generalized Eulerian polynomials. In this subsection, we introduce a polynomial \( A_w(t) \) for each \( w \in S_\nu \), which agrees with the Eulerian polynomial \( A_w(t) \) when \( w = w_0 \), and explain that the Poincaré polynomial of \( Y_w \) is given by \( A_w(t^2) \). The tool we use to compute the Poincaré polynomial of \( Y_w \) is discrete Morse theory. We have the moment map

\[
\mu := Y_w \rightarrow Q_w \subset \mathbb{R}^n.
\]

The following is well-known for simple polytopes and is a consequence of discrete Morse theory.

Theorem 4.12 (see [52, Theorem 2.7] for example). Let \( P \subset \mathbb{R}^n \) be a lattice polytope. Suppose that there exists a linear function \( h: \mathbb{R}^n \rightarrow \mathbb{R} \) such that for each vertex \( u \) of \( P \), the direction vectors of ascending edges emanating from \( u \) are linearly independent and form a face \( F(u) \) of \( P \). Then the Poincaré polynomial \( \operatorname{Poin}(X_P, t) \) of the toric variety \( X_P \) whose fan is the normal fan of \( P \) is given by

\[
\operatorname{Poin}(X_P, t) = \sum_{u \in V(P)} t^{2\operatorname{asc}(u)},
\]

where \( V(P) \) is the set of all vertices of \( P \).

Remark 4.13. The vertex \( u \) may not be simple in \( P \) but simple in \( F(u) \). Therefore the number of \( k \)-dimensional faces of \( P \) which are contained in \( F(u) \) and contain \( u \) is \( \binom{\operatorname{asc}(u)}{k} \). This implies that

\[
\sum_{u \in V(P)} (1 + t)^{2\operatorname{asc}(u)} = \sum_{k=0}^{n} f_k(P) t^k =: f_P(t),
\]

where \( n = \dim P \), \( f_k(P) \) denotes the number of \( k \)-dimensional faces of \( P \), and \( f_n(P) = 1 \). The polynomial \( f_P(t) \) is called the \( f \)-polynomial of \( P \) and the polynomial \( h_P(t) := f_P(t - 1) \) is called the \( h \)-polynomial of \( P \). Theorem 4.12 can be restated as

\[
\operatorname{Poin}(X_P, t) = h_P(t^2)
\]

if there exists a linear function \( h \) in Theorem 4.12. When \( P \) is simple, such a linear function \( h \) always exists and (4.1) is well-known in this case. However, it is not true that the formula (4.1) holds for every polytope \( P \). For example, if \( P \) is an octahedron, then its \( f \)-polynomial is

\[
f_P(t) = 6 + 12t + 8t^2 + t^3.
\]

Therefore

\[
h_P(t^2) = 6 + 12(t^2 - 1) + 8(t^2 - 1)^2 + (t^2 - 1)^3 = 1 - t^2 + 5t^4 + t^6.
\]
which cannot be the Poincaré polynomial of \( X_w \) because it has a negative coefficient. In fact, the polynomial \( h_p(t^2) \) is the virtual Poincaré polynomial of \( X_p \). It is known that the virtual Poincaré polynomial agrees with the ordinary Poincaré polynomial for compact smooth toric varieties, see [27, Section 4.5], but Theorem 4.17 below implies that they agree for \( Y_w \) although \( Y_w \) is not necessarily smooth.

**Example 4.14.** Let \( P \) be a pyramid with five vertices:

\[
P = \text{Conv}\{(1,0,0), (0,1,0), (-1,0,0), (0,0,-1), (0,0,1)\}.
\]

Choose a linear function \( h: \mathbb{R}^3 \rightarrow \mathbb{R} \) defined by

\[
h(x_1, x_2, x_3) = -2x_1 - x_2 + 3x_3.
\]

The function \( h \) gives an orientation on edges of \( P \) as displayed in Figure 10.

**Figure 10.** The pyramid with an orientation

Then, for each vertex \( u \) of \( P \), the corresponding face \( F(u) \) and the number of ascending edges are as in Table 2. Therefore, the Poincaré polynomial \( \text{Poin}(X_p, t) \) of \( X_p \) is given by

\[
\text{Poin}(X_p, t) = 1 + t^2 + 2t^4 + t^6.
\]

| \( F(u) \) | \( h(u) \) | asc(u) |
|---|---|---|
| ![Face 1](image1.png) | -2 | 3 |
| ![Face 2](image2.png) | -1 | 2 |
| ![Face 3](image3.png) | 1 | 2 |
| ![Face 4](image4.png) | 2 | 1 |
| ![Face 5](image5.png) | 3 | 0 |

**Table 2.** Faces \( F(u) \) and asc(u) for the pyramid

On the other hand, since \((f_0(P), f_1(P), f_2(P), f_3(P)) = (5, 8, 5, 1)\) for the pyramid \( P \), we have

\[
h_p(t^2) = 5 + 8(t^2 - 1) + 5(t^2 - 1)^2 + (t^2 - 1)^3 = 1 + t^2 + 2t^4 + t^6.
\]

Therefore the formula (4.1) certainly holds in this case.

We shall apply Theorem 4.12 to \( Q_w \) to find the Poincaré polynomial of \( Y_w \). What we have to do is to find a linear function \( h \) on \( \mathbb{R}^n \) which satisfies the conditions in Theorem 4.12.

**Lemma 4.15 ([52, Lemma 3.4]).** Let \( h: \mathbb{R}^n \rightarrow \mathbb{R} \) be a linear function defined by the inner product with a vector \((a_1, \ldots, a_n) \in \mathbb{R}^n \) with \( a_1 > a_2 > \cdots > a_n \). Then, for \((u(i), u(j)) \in E_w(u)\), the edge emanating from the vertex \( u \) to the vertex \( v \) in \( Q_w \), where \( v = t_{u(i), u(j)}u \), is ascending with respect to the function \( h \) if and only if \( u(i) < u(j) \). Here, we consider orientations on edges of \( P \) according to the function \( h \).

Motivated by Lemma 4.15, we define

\[
E_w(u)^+ = \{(u(i), u(j)) \in E_w(u) \mid u(i) < u(j)\},
\]

\[
E_w(u)^- = \{(u(i), u(j)) \in E_w(u) \mid u(i) > u(j)\}.
\]
Then
\[ E_w(u) = E_w(u)^+ \sqcup E_w(u)^{-}. \]
Note that \( E_w(u)^+ \) (resp. \( E_w(u)^- \)) corresponds to the ascending (resp. descending) edges emanating from \( u \) by Lemma 4.15 with respect to the function \( h \).

Now we set
\[ a_w(u) := \lfloor E_w(u)^+ \rfloor \]
and define
\[ A_w(t) := \sum_{u \leq w} t^{a_w(u)}. \]

**Example 4.16.** By Example 4.7, we have
\[ E_{w_0}(u)^+ = \{ (u(i), u(j)) \in E_{w_0}(u) \mid u(i) < u(j) \} \]
\[ = \{ (u(i), u(i+1)) \mid u(i) < u(i+1) \text{ for } i = 1, \ldots, n-1 \}. \]

Accordingly, \( a_{w_0}(u) \) is the number of ascents in \( u \) and
\[ A_{w_0}(t) = A_0(t). \]

On the other hand, \( X_{w_0} = \text{Fl}(n) \) and \( Y_{w_0} \) is the permutohedral variety \( \text{Perm}_n \). Therefore, we have
\[ \text{Poin}(Y_{w_0}, t) = A_{w_0}(t^2) \]
by Theorem 3.8.

One can check that the linear function \( h \) in Lemma 4.15 satisfies the conditions in Theorem 4.12 so that we obtain the following theorem extending Theorem 3.8.

**Theorem 4.17** ([52, Theorem 3.6]). Let \( Y_w \) be the generic torus orbit closure in the Schubert variety \( X_w \) for \( w \in S_n \). Then the Poincaré polynomial of \( Y_w \) is given by
\[ \text{Poin}(Y_w, t) = A_w(t^2). \]

We close this subsection by presenting the Poincaré polynomials of singular toric varieties \( Y_{4231} \) and \( Y_{3412} \) using Theorem 4.17. The moment polytopes of \( Y_{4231} \) and \( Y_{3412} \) are non-simple polytopes \( Q_{4231} \) and \( Q_{3412} \), respectively, see Figure 11. In both polytopes, we take \( h : \mathbb{R}^4 \rightarrow \mathbb{R} \) by the inner product with a vector \( (12, 2, -1, -2) \). For instance, the image of the vertex \( 4132 = (2, 4, 3, 1) \) under \( h \) is given by
\[ h(4132) = ((2, 4, 3, 1), (12, 2, -1, -2)) = 2 \times 12 + 4 \times 2 - 3 - 2 = 27. \]
We number each vertex \( u \) as \( h(\mu(uB)) \). Therefore, the vertex \( 4132 = (2, 4, 3, 1) \) is numbered as 27 (see Figure 11(1)).

![Figure 11. Polytopes \( Q_{4231} \) and \( Q_{3412} \)](image-url)
Using this function $h$, we get the Poincaré polynomials of these singular generic torus orbit closures as follows:

\[ \mathcal{P} \text{oin}(Y_{4231}, t) = 1 + 7t^2 + 11t^4 + t^6, \]
\[ \mathcal{P} \text{oin}(Y_{3412}, t) = 1 + 5t^2 + 7t^4 + t^6. \]

One can easily confirm the formula (4.1) in these two cases. Indeed, the formula (4.1) holds for every $Y_w$, which implies

\[ A_w(t) = h_Q(t) \quad \text{for every } w \in S_n. \]

**Remark 4.18.** For a rational polytope $P$ allowing a retraction sequence, the Poincaré polynomial of the toric variety $X_P$ is the polynomial $h_P(t^2)$. We refer the reader to [7] for a precise definition of retraction sequences. It is shown in [57] that a linear function $h$ in Lemma 4.15 defines a retraction sequence on the polytope $Q_w$. However, it is not known whether all retraction sequences come from linear functions.

There is a geometric interpretation of Theorems 4.12 and 4.17. An algebraic variety $X$ is called *paved by (complex) affine spaces* if $X$ has a filtration

\[ X = X_\ell \supset X_{\ell-1} \supset \cdots \supset X_1 \supset X_0 = \emptyset \]

by closed subvarieties such that $X_i \setminus X_{i-1}$ is isomorphic to a disjoint union of affine spaces. If an algebraic variety is paved by affine spaces, then the cycle map

\[ c_X : A_*(X) \to H_*(X; \mathbb{Z}) \]

is an isomorphism and the Betti numbers are directly computed by counting dimensions of affine spaces. We refer to [29, Example 19.1.11].

**Proposition 4.19.** Let $P \subset \mathbb{R}^n$ be a lattice polytope. Suppose that there exists a linear function $h : \mathbb{R}^n \to \mathbb{R}$ such that for each vertex $u$ of $P$, the direction vectors of ascending edges emanating from $u$ are linearly independent and form a face of $P$. Then the toric variety $X_P$ is paved by affine spaces.

### 4.3. Toric Schubert varieties.

**Definition 4.20.** For $w \in S_n$, we define

\[ c(w) := \dim_{\mathbb{C}} X_w - \dim_{\mathbb{C}} Y_w = \dim_{\mathbb{C}} X_w - \dim_{\mathbb{R}} Q_w = \ell(w) - \dim_{\mathbb{R}} Q_w, \]

and call $c(w)$ the *complexity* of the Schubert variety $X_w$ (or the complexity of $w$).

When $c(w) = 0$, we have $X_w = Y_w$; so the Schubert variety $X_w$ is a toric variety. Let $w = s_{i_1} \cdots s_{i_m}$ be a reduced decomposition of $w$. Then it is known from [23, 41] that the Schubert variety $X_w$ is a toric variety if and only if $i_1, \ldots, i_m$ are distinct. Moreover, all toric Schubert varieties are Bott manifolds, which are smooth projective toric varieties whose moment map images are combinatorially equivalent to cubes. We refer the reader to Appendix A for more details on Bott manifolds.

There are several equivalent conditions to toric Schubert varieties as follows.

**Theorem 4.21.** [23, 41, 60, 61, 50] The following statements are equivalent:

1. $X_w$ is a toric variety (i.e., of complexity zero).
2. $X_w$ is a smooth toric variety.
3. $w$ avoids the patterns 321 and 3412.
4. A reduced decomposition of $w$ consists of distinct letters.
5. $X_w$ is isomorphic to a Bott–Samelson variety.
6. The Bruhat interval $[e, w]$ is isomorphic to the Boolean algebra $\mathfrak{B}_{\ell(w)}$ of rank $\ell(w)$ as posets.
7. $Q_w$ is combinatorially equivalent to the cube of dimension $\ell(w)$. 


The fan of a toric Schubert variety \( X_w \) is the normal fan of the polytope \( Q_w \). Since \( Q_w \) is combinatorially equivalent to a cube as stated in Theorem 4.21, we shall find its normal fan by investigating the primitive direction vectors of the edges emanating from the vertices \( e \) and \( w \) of \( Q_w \). Let \( w = s_{i_1} \cdots s_{i_m} \) be a reduced decomposition of \( w \in S_n \) and we assume that \( i_1, \ldots, i_m \) are distinct. Then \( w^{-1} = s_{i_m} \cdots s_{i_1} \) and the Bruhat interval \([e,w^{-1}]\) has \( m \) many atoms and coatoms:

\[
\text{atoms: } s_{i_k} \quad (k = 1, \ldots, m),
\]

\[
\text{coatoms: } s_{i_m} \cdots \hat{s}_{i_k} \cdots s_{i_1} \quad (k = 1, \ldots, m).
\]

**Lemma 4.22.** Let \( w = s_{i_1} \cdots s_{i_m} \). For \( 1 \leq k \leq m \), we set

\[
a_k = s_{i_1}s_{i_2} \cdots s_{i_{k-1}}(i_k), \quad b_k = s_{i_1}s_{i_2} \cdots s_{i_{k-1}}(i_k + 1)
\]

where we understand \( a_1 = i_1 \) and \( b_1 = i_1 + 1 \). Then we have

\[
s_{i_m} \cdots \hat{s}_{i_k} \cdots s_{i_1} = w^{-1}t_{a_k}b_k.
\]

**Proof.** For any \( v \in S_n \) and transposition \( t_{p,q} \), we have

\[
vt_{p,q}v^{-1} = t_{v(p),v(q)}.
\]

Noting that \( s_{i_k} = t_{i_k,i_k+1} \) and \( s_j^{-1} = s_j \), we apply (4.3) to the right hand side of the equation

\[
s_{i_m} \cdots \hat{s}_{i_k} \cdots s_{i_1} = w^{-1}s_{i_1}s_{i_2} \cdots s_{i_{k-1}}s_{i_k}^{-1} \cdots s_{i_2}s_{i_1}.
\]

Then the lemma immediately follows. \( \Box \)

The Cartan integers \( c_{i,j} \), which are entries of the Cartan matrix, are given by

\[
c_{i,j} = \begin{cases} 
2 & \text{if } i = j, \\
-1 & \text{if } |i - j| = 1, \\
0 & \text{otherwise.}
\end{cases}
\]

**Theorem 4.23.** Let \( w = s_{i_1} \cdots s_{i_m} \) be a reduced decomposition of \( w \in S_n \). Assume that \( i_1, \ldots, i_m \) are distinct. Then the fan of the toric Schubert variety \( X_w \) is isomorphic to the fan in \( \mathbb{R}^m \) such that primitive ray vectors are the 2m column vectors of the following matrix and a subset of the column vectors forms a cone if and only if it does not contain both the \( i \)th column vectors in the left submatrix and the right submatrix for each \( i = 1, \ldots, m \):

\[
\begin{pmatrix}
1 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & -1 & 0 & \cdots & 0 \\
\vdots & \ddots & & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 1 & \ddots & \ddots & \ddots & -1
\end{pmatrix},
\]

where \( a_{j,k} = -c_{i_j,i_k} \) for \( 1 \leq k < j \leq m \). Here, \( c_{i,j} \) are Cartan integers (see (4.4)).

**Proof.** Since the fan of \( X_w \) is the normal fan of \( Q_w \) and \( Q_w \) is combinatorially equivalent to an \( m \)-dimensional cube by Theorem 4.21, it is enough to consider the edge vectors emanating from the vertices \( e \) and \( w \) to find the fan of \( X_w \).

Let \( e_1, \ldots, e_n \) be the standard basis vectors of \( \mathbb{R}^n \) and we set

\[
v_i := e_i - e_{i+1} \quad \text{for } 1 \leq i \leq n - 1.
\]

The dual vector space to the subspace of \( \mathbb{R}^n \) spanned by \( v_1, \ldots, v_{n-1} \) is the quotient space \( \mathbb{R}^n/\mathbb{R}(1, \ldots, 1) \) and

\[
v_i^* := e_1 + \cdots + e_i \quad \text{for } 1 \leq i \leq n - 1
\]

form the basis of \( \mathbb{R}^n/\mathbb{R}(1, \ldots, 1) \) dual to (4.5) through the standard scalar product on \( \mathbb{R}^n \).

At the vertex \( e \), the primitive vectors of outgoing edges are \( v_{i_1}, \ldots, v_{i_m} \) because the atoms of the Bruhat interval \([e,w^{-1}]\) are \( s_{i_j} \)'s by (4.2). Hence the primitive facet normal vectors at \( e \) are \( v_{i_1}^*, \ldots, v_{i_m}^* \). On the other hand, in order to find the facet normal vectors at the vertex \( w \), we consider the following two elements by (4.2) and Lemma 4.22:

\[
w^{-1}t_{a_k,b_k} = w^{-1}(1) \cdot w^{-1}(2) \cdots w^{-1}(b_k) \cdots w^{-1}(a_k) \cdots w^{-1}(n),
\]

\[
w^{-1} = w^{-1}(1) \cdot w^{-1}(2) \cdots w^{-1}(a_k) \cdots w^{-1}(b_k) \cdots w^{-1}(n).
\]
Since \( w^{-1} > w^{-1}t_{ak,bk} \), we have \( w^{-1}(ak) > w^{-1}(bk) \). Moreover, we get \( ak < bk \) because \( ak \leq ik \) and \( bk \geq ik + 1 \) by the definition of \( ak \) and \( bk \) in Lemma 4.22. Hence the primitive vector of the outgoing edge at \( w \) to the vertex corresponding to \( w^{-1}t_{ak,bk} \) is \( -e_{ak} + e_{bk} \), which is the same as

\[
-s_{i1}s_{i2}\cdots s_{ik-1}(e_{ik} - e_{ik+1}) = -s_{i1}s_{i2}\cdots s_{ik-1}(v_{ik}).
\]

Here, we regard \( s_i \) as the reflection in \( R^n \) which interchanges the \( i \)th and the \((i+1)\)st coordinates, namely \( s_i(v) = v - (v,e_i - e_{i+1})(e_i - e_{i+1}) \) for \( v \in R^n \), where \( (, ) \) denotes the standard scalar product on \( R^n \). Now we set

\[
w_k := -s_{i1}s_{i2}\cdots s_{ik-1}(v_{ik})
\]

for \( 1 \leq k \leq m \). Then, since \( c_{i,j} = (e_i - e_{i+1}, e_j - e_{j+1}) \), we have

\[
w_k = -s_{i1}s_{i2}\cdots s_{ik-1}(v_{ik})
\]

\[
= -s_{i1}s_{i2}\cdots s_{ik-2}(v_{ik} - c_{ik,ik-1}v_{ik-1})
\]

\[
= -s_{i1}s_{i2}\cdots s_{ik-3}(v_{ik} - c_{ik,ik-2}v_{ik-2}) - c_{ik,ik-1}w_{k-1}
\]

\[
\vdots
\]

\[
= -v_{ik} - c_{ik,ik}w_1 - c_{ik,ik-1}w_2 - \cdots - c_{ik,ik-k}w_{k-2} - c_{ik,ik-1}w_{k-1}.
\]

The facet normal vectors at \( w \) are the dual basis \( w^*_1, \ldots, w^*_m \) to \( w_1, \ldots, w_m \), and the above computation implies that

\[
w^*_k = -v^*_ik - c_{ik+1,ik}v^*_iik+1 - \cdots - c_{im,ik}v^*_im.
\]

This proves the theorem.

We call the right half of the matrix in Theorem 4.23 the reduced characteristic matrix of \( X_w \).

Example 4.24. For \( w = s_1s_2s_3s_4 \), the reduced characteristic matrix of \( X_w \) is

\[
\begin{bmatrix}
-1 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 1 & -1
\end{bmatrix}
\]

In general, the reduced characteristic matrix of \( X_w \) for \( w = s_1s_2\cdots s_{n-1} \) (or \( w = s_{n-1}s_{n-2}\cdots s_1 \)) has 1’s just below the diagonal and 0 at the other off diagonal entries. Therefore, such \( X_w \) is isomorphic to the bounded flag manifold \( BF_{n-1} \). Here, a bounded flag in \( C^n \) is a complete flag

\[
(V_1 \subset V_2 \subset \cdots \subset V_n = C^n)
\]

such that each \( V_k \) (\( 2 \leq k \leq n-1 \)) contains the coordinate subspace \( C^{k-1} = (e_1, \ldots, e_{k-1}) \) spanned by the first \( k-1 \) standard basis vectors, and the bounded flag manifold \( BF_{n-1} \) consists of all bounded flags in \( C^n \), see [15, Section 7.7] for more details on the bounded flag manifold.

A reduced characteristic matrix of a toric Schubert variety \( X_w \) depends on the choice of a reduced decomposition of \( w \) as is seen in Example 4.25, but it determines \( X_w \).

Example 4.25. For \( w = s_1s_3s_2s_4 \), the reduced characteristic matrix of \( X_w \) is

\[
\begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
1 & 1 & -1 & 0 \\
0 & 1 & 0 & -1
\end{bmatrix}
\]

There are four more reduced decompositions: \( w = s_3s_1s_2s_4 = s_1s_3s_4s_2 = s_2s_1s_4s_2 = s_3s_4s_1s_2 \). For each case we have the following reduced characteristic matrices:

\[
\begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
1 & 1 & -1 & 0 \\
1 & 0 & 0 & -1
\end{bmatrix},
\begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 1 & -1 & 0 \\
1 & 1 & 0 & -1
\end{bmatrix},
\begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
1 & 0 & 1 & -1
\end{bmatrix},
\begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
1 & 1 & 0 & -1
\end{bmatrix},
\begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
1 & 1 & 1 & -1
\end{bmatrix}
\]

Applying Proposition A.7 in Appendix A to the above five matrices, one can easily see that all of them determine the same toric variety up to isomorphism.
A reduced decomposition \( w = s_{i_1} \cdots s_{i_m} \) defines a sequence \( i = (i_1, \ldots, i_m) \) and vice versa. So we also call the sequence a reduced decomposition of \( w \). We introduce a digraph \( G_i \) associated with a sequence \( i \), which does not depend on the choice of a reduced decomposition of \( w \), i.e., depends only on \( w \).

**Definition 4.26.** For \( i = (i_1, \ldots, i_m) \), the vertex set and the edge set of a digraph \( G_i \) are defined by

- \( V(G_i) = [m] \);
- \( (k, j) \in E(G_i) \) if and only if \( |i_k - i_j| = 1 \) for \( 1 \leq k < j \leq m \).

**Example 4.27.** For \( i = (1, 2, 3, 4) \) and \( i' = (1, 3, 2, 4) \) corresponding to Examples 4.24 and 4.25, we have the following digraphs:

\[
G_i = 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \quad \text{and} \quad G_{i'} = 1 \rightarrow 2 \rightarrow 3 \rightarrow 4
\]

For \( i = (1, 2, 4, 5) \), the digraph \( G_i \) is not connected as follows:

\[
G_i = 1 \rightarrow 2 \rightarrow 3 \rightarrow 4
\]

The digraphs in Example 4.27 are unions of directed path graphs so that each vertex has at most two outgoing edges. This always holds when \( i \) has distinct components and it implies the following.

**Theorem 4.28.** Let \( i = (i_1, \ldots, i_m) \) be a reduced decomposition of \( w \in S_n \). Suppose that \( i_1, \ldots, i_m \) are distinct. Then the toric Schubert variety \( X_w \) is weak Fano. Moreover, it is Fano if and only if each vertex of the graph \( G_i \) has at most one outgoing edge.

**Proof.** We apply the criterion of Fano or weak Fano by Batyrev to our case, see Appendix A.2 for the criterion. It follows from Theorem 4.23 that a primitive collection of the fan of \( X_w \) consists of the \( k \)th column in the left matrix and the \( k \)th column in the right matrix in Theorem 4.23 for each \( k = 1, \ldots, m \). Then the criterion by Batyrev says that \( X_w \) is Fano if and only if 1 appears in the \( k \)th column at most once for each \( k \) in the reduced characteristic matrix of \( X_w \). Moreover, \( X_w \) is weak Fano if and only if 1 appears in the \( k \)th column at most twice for each \( k \) in the reduced characteristic matrix of \( X_w \). This implies the theorem. \( \square \)

We assume \( n \geq 3 \) in the following. An element of \( S_n \) is called a **Coxeter element** if it can be written as a product of all adjacent transpositions \( s_1, \ldots, s_{n-1} \). Let \( \text{Cox}_n \) denote the set of all Coxeter elements in \( S_n \).

**Theorem 4.29 ([47]).** Let \( w, w' \in \text{Cox}_n \) and let \( i, i' \) be reduced decompositions of \( w, w' \), respectively. The following statements are equivalent:

1. \( X_w \) and \( X_{w'} \) are isomorphic as toric varieties.
2. \( H^*(X_w; \mathbb{Z}) \cong H^*(X_{w'}; \mathbb{Z}) \) as graded rings.
3. \( G_i \cong G_{i'} \) as digraphs.
4. \( w' = w \) or \( w' = w_0ww_0 \).

**Remark 4.30.** The digraph \( G_i \) in the theorem above is a directed path graph with \( n - 1 \) vertices. This can be thought of as a directed Dynkin diagram of type \( A_{n-1} \). It turns out that directed Dynkin diagrams appear in the classification of toric Schubert varieties for other Lie types (see [47]).

4.4. **Schubert varieties of complexity one.** In this subsection, we consider Schubert varieties of complexity one. All toric Schubert varieties are smooth while Schubert varieties of complexity one are not necessarily smooth. We provide similar statements for Schubert varieties of complexity one to Theorem 4.21.

We recall generalized Bott–Samelson varieties from [40]. Let \( G = \text{GL}_n(\mathbb{C}) \) as before. For a permutation \( w \in S_n \), the subvariety \( P_w \) of \( G \) corresponding to \( w \) is defined by

\[
P_w = BwB \subseteq G.
\]
Definition 4.31 (cf. [40, §13.4], [26, Definition 2.1], and [33]). Let \((w_1, \ldots, w_r)\) be a sequence of elements in \(S_n\). The \textit{generalized Bott–Samelson variety} \(Z_{(w_1, \ldots, w_r)}\) is defined by the orbit space

\[ Z_{(w_1, \ldots, w_r)} := (P_{w_1} \times \cdots \times P_{w_r})/\Theta, \]

where the right action \(\Theta\) of \(B^r := B \times \cdots \times B\) on \(\prod_{k=1}^{r} P_{w_k}\) is defined by

\[ \Theta((p_1, \ldots, p_r), (b_1, \ldots, b_r)) = (p_1b_1, b_1^{-1}p_2b_2, \ldots, b_{r-1}^{-1}p_rb_r) \]

for \((p_1, \ldots, p_r) \in \prod_{k=1}^{r} P_{w_k}\) and \((b_1, \ldots, b_r) \in B^r\). When \(\ell(w_1) = \cdots = \ell(w_r) = 1\), \(Z_{(w_1, \ldots, w_r)}\) is called a \textit{Bott–Samelson variety}.

Theorem 4.32 ([21, 32, 45, 49, 61]). For a permutation \(w\) in \(S_n\), the following statements are equivalent:

1. \(X_w\) is smooth and of complexity one.
2. \(w\) contains the pattern 321 exactly once and avoids the pattern 3412.
3. There exists a reduced decomposition of \(w\) containing \(s_is_{i+1}s_i\) as a factor and no other repetitions.
4. \(X_w\) is isomorphic to a generalized Bott–Samelson variety \(Z_{(w_1, \ldots, w_r)}\) such that \(r = \ell(w) - 2\), \(w_k = s_js_{j+1}s_j\) for some \(1 \leq k \leq r\), \(w_i = s_i\), for \(i \neq k\), and \(j_1, \ldots, j_{k-1}, j_{k+1}, \ldots, j_r, j, j+1\) are pairwise distinct.
5. The Bruhat interval \([e, w]\) is isomorphic to \(S_3 \times \mathcal{B}_{\ell(w)-3}\) as posets.
6. The polytope \(Q_w\) is combinatorially equivalent to the product of the hexagon and the cube of dimension \(\ell(w) - 3\).

Theorem 4.33 ([21, 32, 45, 49, 61]). For a permutation \(w\) in \(S_n\), the following statements are equivalent:

1. \(X_w\) is singular and of complexity one.
2. \(w\) contains the pattern 3412 exactly once and avoids the pattern 321.
3. There exists a reduced decomposition of \(w\) containing \(s_{i+1}s_is_{i+2}s_{i+1}\) as a factor and no other repetitions.
4. \(X_w\) is isomorphic to a generalized Bott–Samelson variety \(Z_{(w_1, \ldots, w_r)}\) such that \(r = \ell(w) - 3\), \(w_k = s_{j+1}s_js_{j+2}s_{j+1}\) for some \(1 \leq k \leq r\), \(w_i = s_j\), for \(i \neq k\), and \(j_1, \ldots, j_{k-1}, j_{k+1}, \ldots, j_r, j, j+1, j+2\) are pairwise distinct.
5. The Bruhat interval \([e, w]\) is isomorphic to \([e, 3412] \times \mathcal{B}_{\ell(w)-4}\) as posets.
6. The polytope \(Q_w\) is combinatorially equivalent to the product of \(Q_{3412}\) and the cube of dimension \(\ell(w) - 4\).

5. \textbf{Generic torus orbit closures in Richardson varieties}

In this section, we consider generic torus orbit closures in Richardson varieties. The topology and geometry of generic torus orbit closures in Richardson varieties are related to the combinatorics of Bruhat interval polytopes. Studying the combinatorial properties of Bruhat interval polytopes, we show that every smooth toric Richardson variety is a Bott manifold. We also give a sufficient condition for a pair \((v, w) \in S_n \times S_n\) to give rise to a smooth toric Richardson variety \(X^v_w\). Motivated by this sufficient condition, we define a toric variety of Catalan type and then show that the number of isomorphism classes of smooth toric Richardson varieties of Catalan type is the Wedderburn–Etherington number.

5.1 \textbf{Richardson varieties and Bruhat interval polytopes}. For \(v, w \in S_n\) with \(v \leq w\) in the Bruhat order, the Richardson variety \(X^v_w\) is defined to be the intersection of the Schubert variety \(X_w\) and the opposite Schubert variety \(X^v = w_0X_{w_0v}\). It is known that every Richardson variety \(X^v_w\) is a \(T\)-invariant irreducible subvariety of \(\text{Fl}(n)\) and

\[ \dim_{\mathbb{C}} X^v_w = \ell(w) - \ell(v). \]
Moreover, a $T$-fixed point $uB$ is contained in $X^v_w$ if and only if $v \leq u \leq w$ in the Bruhat order. Accordingly, the set of $T$-fixed points of $X^v_w$ is identified with the Bruhat interval

$$[v, w] = \{ u \in S_n \mid v \leq u \leq w \}.$$ 

Recall that the moment map $\mu$ in (2.4) sends $uB$ to $u = (u^{-1}(1), \ldots, u^{-1}(n))$. Hence we have

$$\mu(X^v_w) = \text{Conv}\{(u^{-1}(1), \ldots, u^{-1}(n)) \mid u \in [v, w]\}.$$ 

Motivated by this fact, we provide the following definition.

**Definition 5.1** ([43]). For elements $v$ and $w$ in $S_n$ with $v \leq w$, we define a polytope $Q^v_w$ by

$$Q^v_w := \text{Conv}\{(u^{-1}(1), \ldots, u^{-1}(n)) \mid u \in [v, w]\}.$$ 

This polytope is called a *Bruhat interval polytope*. Note that $Q^v_w$ is the polytope $Q_w$ introduced in Section 4.

By the definition of $Q^v_w$, we have

$$\mu: X^v_w \rightarrow Q^v_w \subset \mathbb{R}^n.$$ 

**Remark 5.2.** Bruhat interval polytopes were introduced by Kodama–Williams [43] and their combinatorial properties are studied in [62] and [50]. For $v, w \in S_n$ with $v \leq w$, the notation

$$Q_{v, w} := \text{Conv}\{(u(1), \ldots, u(n)) \mid u \in [v, w]\}$$ 

is used in [43, 62, 50]. Therefore we have $Q^v_w = Q_{v^{-1}, w^{-1}}$.

Bruhat interval polytopes $Q^v_w$ and $Q^{v^{-1}}_{w^{-1}}$ are not necessarily combinatorially equivalent even though the intervals $[v, w]$ and $[v^{-1}, w^{-1}]$ are isomorphic as posets. For example, one can check that the polytopes $Q_{35412}^{12345}$ and $Q_{45132}^{13245}$ have different numbers of edges using a computer program like SageMath. However, they have the same dimension.

**Proposition 5.3** ([50, Proposition 3.4]). Bruhat interval polytopes $Q^v_w$ and $Q^{v^{-1}}_{w^{-1}}$ have the same dimension.

5.2. *Generic torus orbit closures in Richardson varieties.* A torus orbit $T \cdot x$ for $x \in X^v_w$ is said to be *generic* if $(T \cdot x)^T = (X^v_w)^T$. Every Richardson variety admits a generic torus orbit. Its proof is given in [46, Proposition 3.8] when $v = e$ and a similar argument works for any Richardson variety $X^v_w$. We will denote by $Y^v_w$ the closure of a generic $T$-orbit in $X^v_w$. Then $Y^v_w$ is the projective toric variety defined by the polytope $Q^v_w$ since $\mu(Y^v_w) = \mu(X^v_w) = Q^v_w$.

Note that the family of Bruhat interval polytopes forms a subfamily of the $\Phi$-polytopes. Hence every edge of a Bruhat interval polytope is parallel to a vector of the form $e_i - e_j$. This implies that $Y^v_w$ is smooth at a fixed point $uB$ if and only if $Q^v_w$ is simple at the vertex $u$ (see [46, Section 8]).

While $Y_w = Y^e_w$ is smooth at $eB$, not every toric variety $Y^v_w$ is smooth at $vB$. For instance, the polytope $Q_{3412}^{13245}$ is not simple at 1324, see Figure 12. A natural generalization of Conjecture 4.11 is the following.

**Conjecture 5.4.** The generic torus orbit closure $Y^v_w$ in the Richardson variety $X^v_w$ is smooth if it is smooth at the fixed points $vB$ and $wB$, in other words, $Q^v_w$ is simple if it is simple at the vertices $v$ and $w$. 

5.3. Toric Bruhat interval polytopes. Since $Q_w^v$ is the moment map image of a toric variety $Y_w^v$, we get

$$\dim Q_w^v = \dim \mathcal{C} Y_w^v \leq \dim \mathcal{C} X_w^v = \ell(w) - \ell(v).$$

Motivated by this observation, we call a Bruhat interval polytope $Q_w^v$ toric if $\dim Q_w^v = \ell(w) - \ell(v)$. Hence a Richardson variety $X_w^v$ is a toric variety with respect to the $T$-action, that is, $X_w^v = Y_w^v$, if and only if the Bruhat interval polytope $Q_w^v$ is toric.

Tsukerman and William [62] showed that every face of the Bruhat interval polytope $Q_w^v$ is realizable as a subinterval of $[v^{-1}, w^{-1}]$, but the converse is not true. For instance, the subinterval $[123, 312]$ of $[123, 321] = [123^{-1}, 321^{-1}]$ does not form a face of the polytope $Q_{123}^{321}$. See Figure 1. However, the converse holds when $Q_w^v$ is toric. More strongly, we have the following.

**Theorem 5.5** ([50, Theorem 5.1]). For a Bruhat interval polytope $Q_w^v$, the following statements are equivalent:

(1) $Q_w^v$ is toric (i.e., $\dim Q_w^v = \ell(w) - \ell(v)$).
(2) $Q_y^x$ is a face of $Q_w^v$ for any $[x^{-1}, y^{-1}] \subseteq [v^{-1}, w^{-1}]$.

Hence, if $Q_w^v$ is toric, then its combinatorial type is determined by the poset structure of $[v^{-1}, w^{-1}]$, so $Q_w^v$ and $Q_{w^{-1}}^v$ are combinatorially equivalent. Here, the assumption of “toric” cannot be removed. Indeed, the interval $[1324, 4231]$ is a Boolean algebra of rank 4, but the corresponding Bruhat interval polytope is of dimension 3. See Figure 13. On the other hand, for every positive integer $m$ there is an example of an interval $[v, w]$ of rank $m$ such that $[v, w]$ is a Boolean algebra and $Q_w^v$ is toric. For instance, when $w = s_m s_{m-1} \cdots s_2 s_1$ and $v = e$, the Bruhat interval polytope $Q_w^v$ is combinatorially equivalent to a cube of dimension $m$ (simply, $m$-cube).
In the following, for simplicity, when a polytope $Q$ is combinatorially equivalent to a cube (or a $d$-cube), we say that $Q$ is a cube (or a $d$-cube). We also say that an interval $[v, w]$ is Boolean if it is a Boolean algebra.

Using the following two facts

1. if $Q$ is a simple polytope of dimension $\geq 2$ and every 2-face of $Q$ is a 2-cube, then $Q$ is a cube ([69, Problems and Exercises 0.1 in p.23] and [67, Appendix]), and
2. every 2-interval is a diamond ([11, Lemma 2.7.3]),

we can prove:

**Proposition 5.6** ([50, Proposition 5.6]). Suppose that $Q_{1324}^{4231}$ is toric. Then $Q_{1324}^{4231}$ is a cube if and only if $Q_{1324}^{4231}$ is simple. (This is equivalent to saying that a toric Richardson variety is a Bott manifold if and only if it is smooth.)

The following gives a characterization of when $Q_{1324}^{4231}$ is a cube.

**Theorem 5.7** ([50, Theorem 5.7]). A Bruhat interval polytope $Q_{1324}^{4231}$ is a cube if and only if it is toric and $[v, w]$ is Boolean. (This is equivalent to saying that a Richardson variety $X_{1324}^{4231}$ is a Bott manifold if and only if it is toric and $[v, w]$ is Boolean.)

In the above theorem, we cannot drop either toric or Boolean.

**Example 5.8.**

1. When $v = 1324$ and $w = 4231$, the interval $[v, w]$ is Boolean of length 4 but $\dim Q_{w}^{v} = 3$, so $Q_{w}^{v}$ is not toric. Since the vertices $v$ and $w$ have degree 4, $Q_{w}^{v}$ is not a cube.

2. When $v = 1324$ and $w = 3412$, we have $\ell(w) - \ell(v) = 4 - 1 = 3$ and $\dim Q_{w}^{v} = 3$. Hence $Q_{w}^{v}$ is toric. However, $[v, w]$ is a 4-crown (see [11, p.52]) and not Boolean. The vertices $v$ and $w$ have degree 4 and the others are simple vertices, so $Q_{w}^{v}$ is not a cube. See Figure 12.

It is shown in [58, Theorem 3.5.2] that the largest rank of Boolean Bruhat intervals in $S_{n+1}$ is at least $n + \left\lfloor \frac{n-1}{2} \right\rfloor$ by finding a sufficient condition on $v$ and $w$ for $[v, w]$ to be Boolean. This implies that there are infinitely many Boolean Bruhat intervals which are not toric like the Boolean interval $[1324, 4231]$. In fact, for any non-negative integer $k$, there is a Boolean interval $[v, w]$ such that $\dim Q_{w}^{v} = \ell(w) - \ell(v) - k$. See [50, Proposition 6.4] for details.
5.4. **Conditions on \( v \) and \( w \) for \( Q^v_w \) to be a cube.** Now we find a sufficient condition for a Bruhat interval polytope \( Q^v_w \) to be a cube.

It is shown in [37, \$5 and \$6] that \( Q^v_w \) is toric (in fact, a cube) if \( v^{-1} = [a_1, \ldots, a_{n-1}, n] \) and \( w^{-1} = [n, a_1, \ldots, a_{n-1}] \) or \( w^{-1} = [1, b_2, \ldots, b_n] \) and \( w^{-1} = [b_2, \ldots, b_n, 1] \). In these cases,

\[
\begin{align*}
  w &= s_1 s_2 \cdots s_{n-1} v & \ell(w) - \ell(v) &= n - 1, \\
  w &= s_{n-1} s_{n-2} \cdots s_1 v & \ell(w) - \ell(v) &= n - 1.
\end{align*}
\]

These examples and Theorem 4.21(3) motivate us to study the following case:

\[ (5.1) \quad w = s_{j_1} s_{j_2} \cdots s_{j_m} v \text{ or } v s_{j_1} s_{j_2} \cdots s_{j_m} \text{ where } \ell(w) - \ell(v) = m \text{ and } j_1, \ldots, j_m \text{ are distinct.} \]

**Proposition 5.9 ([50, Proposition 7.1]).** Suppose that \( w = s_{j_1} s_{j_2} \cdots s_{j_m} v \) or \( v s_{j_1} s_{j_2} \cdots s_{j_m} \) with \( \ell(w) - \ell(v) = m \). Then \( j_1, \ldots, j_m \) are distinct if and only if \( Q^v_w \) is toric.

Note that not every Bruhat interval polytope in the above proposition is a cube. For instance, if \( v = 1324 \) and \( w = 3412 \), then \( v = s_2 \) and \( w = s_2 s_3 s_1 s_2 \), so \( w = s_2 s_3 s_1 v \). However, the polytope \( Q^v_w \) is not a cube as in Figure 12. On the other hand, for \( v = 1243 \) and \( w = 3412 \), since \( v = s_3 \) and \( w = s_2 s_3 s_1 s_2 = s_2 s_1 s_3 s_2 \), neither \( w v^{-1} \) nor \( v^{-1} w \) is a product of distinct adjacent transpositions. However, the Bruhat interval polytope \( Q^v_w \) is a 3-cube. These two examples show that it seems difficult to characterize \( v \) and \( w \) for which \( Q^v_w \) is a cube.

Let us find a sufficient condition on \( v \) and \( w \) for \( Q^v_w \) to be a cube. For that, we prepare some notations. For \( p, q \in [n-1] \), we set

\[
s(p, q) = \begin{cases} 
  s_p s_{p+1} \cdots s_q & \text{if } p \leq q, \\
  s_p s_{p-1} \cdots s_q & \text{if } p \geq q.
\end{cases}
\]

For each \( s(p, q) \), we also set

\[
\bar{p} = \min\{p, q\}, \quad q = \max\{p, q\}.
\]

We note that if \( j_1, \ldots, j_m \in [n-1] \) are distinct, then we have a minimal expression

\[ (5.2) \quad s_{j_1} s_{j_2} \cdots s_{j_m} = s(p_1, q_1) s(p_2, q_2) \cdots s(p_r, q_r) \]

where the intervals \([\bar{p}_1, \bar{q}_1], \ldots, [\bar{p}_r, \bar{q}_r] \) are disjoint and \( r \) is the minimum among such expressions.

**Example 5.10.** Here are examples of minimal expressions.

\[
\begin{align*}
  (1) \quad & s_1 s_2 \cdots s_{n-1} = s(1, n-1), \quad s_{n-1} s_{n-2} \cdots s_1 = s(n-1, 1), \\
  (2) \quad & s_1 s_3 s_8 s_2 s_4 s_7 s_6 = s_3 s_4 s_1 s_2 s_8 s_7 s_5 = s(3, 4) s(1, 2) s(8, 6), \\
  (3) \quad & s_2 s_3 s_8 s_7 s_1 s_2 = s_2 s_1 s_8 s_7 s_6 = s(2, 1) s(4, 4) s(8, 6).
\end{align*}
\]

We say that the product \( s_{j_1} s_{j_2} \cdots s_{j_m} \) in (5.2) is proper if no two intervals among \([\bar{p}_1, \bar{q}_1], \ldots, [\bar{p}_r, \bar{q}_r] \) are adjacent, in other words, the cycles defined by \( s(p_1, q_1), \ldots, s(p_r, q_r) \) are disjoint. In Example 5.10, (1) and (3) are proper, but (2) is not because the intervals \([3, 4] \) and \([1, 2] \) are adjacent.

The following provides a sufficient condition for \( v \) and \( w \) such that the Richardson variety \( X^v_w \) is smooth and toric.

**Proposition 5.11 ([50, Proposition 7.3]).** Suppose that \( s_{j_1} s_{j_2} \cdots s_{j_m} \) is a proper minimal expression. If \( w = s_{j_1} s_{j_2} \cdots s_{j_m} v \) or \( v s_{j_1} s_{j_2} \cdots s_{j_m} \) with \( \ell(w) - \ell(v) = m \), then the Bruhat interval polytope \( Q^v_w \) is a cube.

In fact, the converse of Proposition 5.11 is also true. That is, a Bruhat interval polytope \( Q^v_w \) is a cube for any \( v \) and \( w \) in (5.1) if and only if the product \( s_{j_1} s_{j_2} \cdots s_{j_m} \) is proper. See [50, Corollary 7.7].

5.5. **Toric varieties of Catalan type.** Many smooth toric Richardson varieties arise from polygon triangulations. This subsection is a preparation for that. Namely we explain how to associate a compact smooth toric variety with a polygon triangulation.

Let \( P_{n+2} \) denote a convex polygon in the plane with \( n + 2 \) vertices (or convex \((n + 2)\)-gon for simplicity). We label the vertices from 0 to \( n + 1 \) in counterclockwise order. A triangulation
of $P_{n+2}$ is a decomposition of $P_{n+2}$ into a set of $n$ triangles by adding $n-1$ diagonals of $P_{n+2}$ which do not intersect in their interiors. Then the Catalan number

$$C_n = \frac{1}{n+1} \binom{2n}{n}$$

is the number of triangulations of $P_{n+2}$. Figure 14 shows the triangulations of $P_5$.

![Figure 14. Triangulations of $P_5$](image)

There are many interpretations of Catalan numbers such as binary trees, Dyck paths, and binary operations, see [59]. In the following, we review the correspondence between polygon triangulations and binary trees and then define a toric variety of Catalan type.

A binary tree is a rooted plane tree with at most two children at each vertex. The set of triangulations $\mathcal{T}$ of a convex polygon $P_{n+2}$ with $n+2$ vertices and the set of binary trees $B_\mathcal{T}$ with $n$ vertices have the following bijective connection. Assume that $\mathcal{T}$ is a triangulation of $P_{n+2}$. We put a vertex of $B_\mathcal{T}$ in the interior of each triangle of $\mathcal{T}$. The root vertex of $B_\mathcal{T}$ corresponds to the vertex in the triangle having the side $\{0, n+1\}$. If two triangles of $\mathcal{T}$ are adjacent on a side, then the corresponding two vertices of $B_\mathcal{T}$ are connected by an edge in $B_\mathcal{T}$. The binary trees associated with the triangulation of $P_5$ in Figure 14 are shown in Figure 15. The root vertex is the vertex with the additional circle.

![Figure 15. Binary trees with three vertices](image)

If a binary tree has zero or two children at each vertex, then it is called full. Similarly to $B_\mathcal{T}$, a full binary tree $C_\mathcal{T}$, which contains $B_\mathcal{T}$ as a subgraph, is associated with a triangulation $\mathcal{T}$ of $P_{n+2}$. Indeed, in addition to the vertices of $B_\mathcal{T}$, we place a vertex outside the side $\{i, i+1\}$ of $P_{n+2}$ for each $i = 0, \ldots, n$, and connect it to the vertex in the interior of the triangle having the side $\{i, i+1\}$ by an edge. This produces the full binary tree $C_\mathcal{T}$. The root vertex of $C_\mathcal{T}$ is the same as that of $B_\mathcal{T}$. The full binary trees associated with the triangulation of $P_5$ in Figure 14 are shown in Figure 16, where the colored vertices represent the vertices placed outside the polygon.

![Figure 16. Full binary trees with three vertices](image)

Now we introduce the left tree and the right tree of a triangulation $\mathcal{T}$ of $P_{n+2}$ using the corresponding full binary tree $C_\mathcal{T}$. The left tree $\mathcal{T}^L$ (resp., right tree $\mathcal{T}^R$) consists of the sides in $\mathcal{T}$ intersecting with an edge of $C_\mathcal{T}$ connecting a vertex and its left child (resp., right child). See Figure 17.
Figure 17. The left tree (colored in blue) and the right tree (colored in red and dashed) of a triangulation of $P_{10}$. Here, we color the vertices $V(C_T) \setminus V(B_T)$ purple.

From the definition of the left and the right trees, we obtain the following lemma.

**Lemma 5.12** ([51, Lemma 4.5]). Let $T$ be a triangulation of $P_{n+2}$. For each $1 \leq k \leq n$, there is only one edge $\{k_L, k\}$ with $k_L < k$ in the left tree $T_L$ and similarly there is only one edge $\{k, k_R\}$ with $k < k_R$ in the right tree $T_R$. Moreover, $k_L, k, k_R$ are the vertices of a triangle in $T$.

For each $k = 1, \ldots, n$, we define
(1) $p_k = e_{k+1} - e_{k+1}$,
(2) $q_k = -e_k + e_{k+1}$,
where $\{e_1, \ldots, e_{n+1}\}$ is the standard basis of $\mathbb{Z}^{n+1}$. The vectors $p_1, \ldots, p_n$ (similarly, $q_1, \ldots, q_n$) form a basis of the $n$-dimensional lattice
$$M = \{(x_1, \ldots, x_{n+1}) \in \mathbb{Z}^{n+1} \mid x_1 + \cdots + x_{n+1} = 0\}.$$ Through the dot product on $\mathbb{Z}^{n+1}$, the dual lattice $N$ of $M$ can be identified with the quotient lattice of $\mathbb{Z}^{n+1}$ by the sublattice generated by $(1, \ldots, 1)$, i.e.,
$$N = \mathbb{Z}^{n+1}/\mathbb{Z}(1, \ldots, 1).$$

Let $\varpi_i$ ($i = 1, \ldots, n$) be the quotient image of $e_1 + \cdots + e_i$ in $N$. Then $\{\varpi_1, \ldots, \varpi_n\}$ is a basis of $N$. For convenience, we set $\varpi_0 = \varpi_{n+1} = 0$.

To each side $\{a, b\}$ in the triangulation $T$ with $a < b$, we assign the vector $\varpi_a - \varpi_b$ and denote it by $v_a$ when $\{a, b\} \in T_R$ and $w_b$ when $\{a, b\} \in T_L$, in other words,
$$v_k = \varpi_k - \varpi_{k+1},$$
$$w_k = \varpi_k - \varpi_k,$$ for $k = 1, \ldots, n$ by Lemma 5.12. Note that the zero vector $0$ is assigned to the distinguished side $\{0, n+1\}$ because $\varpi_0 = \varpi_{n+1} = 0$. Then for the paring $\langle , \rangle$ between $N$ and $M$ induced from the dot product on $\mathbb{Z}^{n+1}$, we have
$$\langle v_i, p_j \rangle = \langle w_i, q_j \rangle = \delta_{ij},$$
where $\delta_{ij}$ denotes the Kronecker delta. See Figure 18 for the assignment of the vectors $w_k$’s and $v_k$’s to the left and right trees of the triangulation in Figure 17. One can check that $\langle v_i, p_j \rangle = \langle w_i, q_j \rangle = \delta_{ij}$ using Table 3.
The collection of cones spanned by \( u_i \) \((i \in I)\), where \( u_i \) is either \( v_i \) or \( w_i \) and \( I \) runs over all subsets of \([n]\), forms a complete non-singular fan \( \Sigma_\tau \) in \( N \otimes \mathbb{R} \).

The above lemma says that the underlying simplicial complex of the fan \( \Sigma_\tau \) is the boundary complex of an \( n \)-dimensional cross-polytope. It is known from \([54, \text{Corollary 3.5}]\) that such a fan is indeed the normal fan of an \( n \)-cube and the smooth compact toric variety \( X(\Sigma_\tau) \) associated with the fan \( \Sigma_\tau \) is a Bott manifold (cf. Proposition A.9).

**Definition 5.14.** Since \( \Sigma_\tau \) is associated with a polygon triangulation \( \mathcal{T} \), we say that the fan \( \Sigma_\tau \) and the corresponding (smooth compact) toric variety \( X(\Sigma_\tau) \) are of Catalan type.

Since the vertices \( k_L, k, k_R \) form a triangle for each \( k \in [n] \), there is a unique \( k_0 \in [n] \) such that \( v_{k_0} + w_{k_0} = 0 \), and for \( k \in [n] \setminus \{k_0\} \) we have

\[
\begin{align*}
v_k + w_k &= \begin{cases} 
v_{k_L} & \text{if } \{k_L, k_R\} \in E(\mathcal{T}^R), \\
w_{k_R} & \text{if } \{k_L, k_R\} \in E(\mathcal{T}^L). \end{cases}
\end{align*}
\]

Note that \( k_0 \) is the remaining vertex of the triangle containing the distinguished side \( \{0, n+1\} \).

See Figure 18 and \([51, \text{Lemma 5.1}]\) for more details. Hence the set of primitive collections of \( \Sigma_\tau \) is

\[\{\{v_k, w_k\} \mid k \in [n]\}.\]

Then Batyrev’s criterion (\([8, \text{Proposition 2.3.6}]\), see also Appendix A.2) implies the following:

**Lemma 5.15** (\([51, \text{Lemma 6.4}]\)). The toric variety \( X(\Sigma_\tau) \) is Fano.

The toric varieties of Catalan type are classified as follows.

**Theorem 5.16** (\([51, \text{Theorem 6.5}]\)). Let \( \mathcal{T} \) and \( \mathcal{T}' \) be triangulations of \( \mathbb{P}^{n+2} \). Then the fans \( \Sigma_\tau \) and \( \Sigma_{\tau'} \) are isomorphic (equivalently, the toric varieties \( X(\Sigma_\tau) \) and \( X(\Sigma_{\tau'}) \) are isomorphic) if and only if the binary trees \( \mathcal{B}_\tau \) and \( \mathcal{B}_{\tau'} \) are isomorphic as unordered rooted trees.

The above theorem implies that the number of isomorphism classes of \( n \)-dimensional toric varieties of Catalan type is the same as that of unordered binary trees with \( n \) vertices. It is known that the latter is the Wedderburn–Etherington number \( b_{n+1} \).

**Corollary 5.17** (\([51, \text{Corollary 6.6}]\)). The number of isomorphism classes of \( n \)-dimensional toric varieties of Catalan type is the Wedderburn–Etherington number \( b_{n+1} \).

Here, the Wedderburn–Etherington number \( b_n \) \((n \geq 1)\) is the number of ways to parenthesize a string of \( n \) letters subject to a commutative (but nonassociative) binary operation and it appears in counting several different objects (see Sequence A001190 in OEIS \([56, [59, A56 in p.133]\)). The
generating function \( B(x) = \sum_{n \geq 1} b_n x^n \) of the Wedderburn–Etherington numbers satisfies the functional equation

\[
B(x) = x + \frac{1}{2} B(x)^2 + \frac{1}{2} B(x^2),
\]

which was the motivation of Wedderburn in his work [64] and was considered by Etherington [22]. This functional equation is equivalent to the recurrence relation

\[
b_{2m-1} = \sum_{i=1}^{m-1} b_i b_{2m-i-1} \quad (m \geq 2), \quad b_{2m} = b_m (b_m + 1) / 2 + \sum_{i=1}^{m-1} b_i b_{2m-i}
\]

with \( b_1 = 1 \). Using this recurrence relation, one can calculate the Wedderburn–Etherington numbers, see Table 4.

| \( n \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---------|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|
| \( b_n \) | 1 | 1 | 2 | 3 | 6 | 11 | 23 | 46 | 98 | 207 | 451 | 983 | 2179 | 4850 |

Table 4. Wedderburn–Etherington numbers \( b_n \) for small values of \( n \)

5.6. Smooth toric Richardson varieties of Catalan type. We say that a smooth toric Richardson variety \( X^r_w \) is of Catalan type if it is of Catalan type as a toric variety, in other words, if the normal fan of \( Q_w^r \) is of Catalan type. In this subsection, we show that every toric variety of Catalan type appears as a smooth toric Richardson variety. However, the converse is not true, i.e., there are smooth toric Richardson varieties which are not of Catalan type. For instance, \( X_{1142} \) is a (smooth) toric Schubert variety but not Fano, so it is not of Catalan type by Lemma 5.15.

For a permutation \( u \in S_n \), we define permutations \( u_h \) and \( u_t \) in \( S_{n+1} \) by

\[
u_h(i) = \begin{cases} 
1 & \text{if } i = 1, \\
u(i-1) + 1 & \text{if } 2 \leq i \leq n + 1,
\end{cases} \quad u_t(i) = \begin{cases} 
u(i) & \text{if } 1 \leq i \leq n, \\
n + 1 & \text{if } i = n + 1.
\end{cases}
\]

For example, if \( u = 2314 \), then \( u_h = 13425 \) and \( u_t = 23145 \). As one may see, the permutation \( u_h \) is obtained from \( u \) by putting the additional number 1 at the head (with the original numbers increased by 1) while \( u_t \) is obtained from \( u \) by putting the additional number \( n + 1 \) at the tail. In the notation, \( h \) stands for head and \( t \) stands for tail.

We set

\[
s(n, 1) = s(n-1, 1) \cdots s_1, \quad s(1, n) = s_1 s_2 \cdots s_n.
\]

One notes that if a pair \((v, w)\) of elements in \( S_n \) satisfies

\[
w = s(1, n) v \quad \text{(resp. } w = s(1, n) v) \quad \text{and } \ell(w) = \ell(v) + n,
\]

then \( v(1) = 1 \) (resp. \( v(n+1) = n + 1 \)), so that

\[
(v^{-1}, w^{-1}) = (u_h, w_h s(1, n)) \quad \text{(resp. } (v^{-1}, w^{-1}) = (u_t, u_t s(1, n)) \text{) for some } u \in S_n.
\]

By Proposition 5.11, the Bruhat interval polytope \( Q_w^r \) for the pair \((v, w)\) in (5.3) is an \( n \)-cube and our concern is the pairs in (5.3). We first consider the former case \((v^{-1}, w^{-1}) = (u_h, u_h s(1, n))\).

We recall a surjection from \( S_n \) to the set of binary trees with \( n \) vertices (cf. [53, Appendix A]). To a permutation \( u \in S_n \), we first associate a binary tree \( \psi(u) \) with vertex labels by finding the smallest number in the one-line notation of \( u \) inductively. We start with the one-line notation \( u(1) u(2) \cdots u(n) \) of \( u \). The smallest integer, say \( u(p) \), in the sequence (which is 1 here) becomes the root of the binary tree \( \psi(u) \) with \( n \) vertices. Then the subsequence \( u(1) \cdots u(p-1) \) will provide the left subtree of the root vertex, and the subsequence \( u(p+1) \cdots u(n) \) will provide the right subtree of the root vertex. More precisely, the smallest integer in the subsequence \( u(1) \cdots u(p-1) \) presents the root of a binary tree with \( p-1 \) vertices, and it is the left child of the root vertex of \( \psi(u) \). On the other hand, the smallest integer in the subsequence \( u(p+1) \cdots u(n) \) presents the root of a binary tree with \( n - p \) vertices, and it is the right child of the root vertex of
ψ(u). Continuing this process, we obtain the binary tree ψ(u). Finally, erasing the vertex labels, we obtain the desired binary tree ψ(u) with n vertices.

For example, if $u = 31687524$, the root of the binary tree $\tilde{\psi}(u)$ is $\psi(2)$, and its left and right subtrees have the roots $\psi(1)$ and $\psi(7)$, respectively. Continuing this process, we first get the binary tree $\tilde{\psi}(u)$, and then by erasing the labels we obtain the binary tree $\psi(u)$, see Figure 19. Note that $v = 21687534$ gives the same binary tree as $u = 31687524$, i.e., $\psi(v) = \psi(u)$.

![Figure 19. The binary trees $\tilde{\psi}(u)$ and $\psi(u)$ for $u = 31687524$](image)

Through the canonical bijection between the set of binary trees with n vertices and that of triangulations of $P_{n+2}$, the assignment

$\psi: S_n \to \{\text{binary trees with n vertices}\} = \{\text{triangulations of } P_{n+2}\}$

is surjective (cf. [53, Appendix A]).

**Proposition 5.18 ([51, Proposition 7.2]).** Let $u \in S_n$ and let $T = \psi(u)$ be the corresponding triangulation of $P_{n+2}$. We denote by $T^L$ and $T^R$ the left and right trees of $T$ as before. Then the edges of $T^L$ correspond to the atoms of the Bruhat interval $[u_h, u_h s(1, n)]$ while the edges of $T^R$ correspond to the coatoms of the Bruhat interval $[u_h, u_h s(1, n)]$. More precisely,

$\{(i, j) \mid u_h < u_h t_{i,j} \leq u_h s(1, n)\} = \{(i, j) \mid \{i - 1, j - 1\} \in E(T^L)\},$

$\{(i, j) \mid u_h \leq u_h s(1, n) t_{i,j} < u_h s(1, n)\} = \{(i, j) \mid \{i, j\} \in E(T^R)\}.$

Here, $x < y$ means that $x < y$ and there is no $z$ such that $x < z < y$.

**Example 5.19.** For $u = 31687524$, the triangulation $T = \psi(u)$ is as shown in Figure 17. Since $v^{-1} = u_h = 142798635$ and $w^{-1} = u_h s(1, 8) = 427986351$, there are eight atoms of the interval $[v^{-1}, w^{-1}]$ given by $v^{-1} t_{i,j}$, where $(i, j)$ is one of the following pairs:

$(1, 2), (1, 3), (3, 4), (4, 5), (4, 6), (3, 7), (3, 8), (8, 9).$

These pairs provide the edges of $T^L$ by subtracting 1 from every component. On the other hand, there are eight coatoms given by $w^{-1} t_{i,j}$, where $(i, j)$ is one of the following pairs:

$(1, 2), (2, 9), (3, 6), (4, 5), (5, 6), (6, 7), (7, 9), (8, 9).$

These pairs are the edges of $T^R$.

**Theorem 5.20 ([51, Theorem 7.4]).** For $u \in S_n$, the normal fan of the Bruhat interval polytope $Q^+_{P_{n+2}}$ for $(v^{-1}, w^{-1}) = (u_h, u_h s(1, n))$ is the fan $\Sigma_{\psi(u)}$ associated with the triangulation $\psi(u)$ of $P_{n+2}$.

Hence, any n-dimensional fan of Catalan type is realized as the normal fan of $Q^+_{P_{n+2}}$ with $(v^{-1}, w^{-1}) = (u_h, u_h s(1, n))$ for some $u \in S_n$. As for the latter case $(v^{-1}, w^{-1}) = (u_t, u_t s(n, 1))$, we have the following.

**Theorem 5.21 ([51, Theorem 7.5]).** For $u \in S_n$, the normal fan of the Bruhat interval polytope $Q^+_{P_{n+2}}$ for $(v^{-1}, w^{-1}) = (u_t, u_t s(n, 1))$ is isomorphic to the fan $\Sigma_{\psi(u_t u_t w_0)}$ associated with the triangulation $\psi(u_t u_t w_0)$ of $P_{n+2}$, where $w_0$ denotes the longest element of $S_n$. 
The following is a direct consequence of Corollary 5.17.

**Corollary 5.22** ([51, Corollary 8.2]). The number of isomorphism classes of \( n \)-dimensional smooth toric Richardson varieties of Catalan type is the Wedderburn–Etherington number \( b_{n+1} \).

### 6. Problems

The study of torus orbit closures in the flag variety is related to the geometry of Schubert varieties and the combinatorics of Bruhat interval polytopes. In this section, we pose some possible avenues for further exploration related to the discussions in this chapter.

#### 6.1. Poincaré polynomial of \( Y_w^v \).

It is known that a Schubert variety \( X_w \) is smooth (in type A) if its Poincaré polynomial is palindromic (see [10, Theorem 6.0.4 and p.208]). Analogously, we ask

**Problem 6.1.** Is the generic torus orbit closure \( Y_w \) in \( X_w \) smooth if its Poincaré polynomial is palindromic?

Recall that \( Y_w^v \) is the generic torus orbit closure in the Richardson variety \( X_w^v \). The Poincaré polynomial of \( Y_w^v \) is computable by Theorem 4.17 but such a formula is not known for \( Y_w^v \) in general unless \( Y_w^v \) is smooth.

**Problem 6.2.**

1. Does the virtual Poincaré polynomial of \( Y_w^v \) agree with the Poincaré polynomial \( \text{Poin}(Y_w^v, t) \) of \( Y_w^v \)? (See Remark 4.13 for virtual Poincaré polynomials.)
2. Is there a polynomial \( A_w^v(t) \) defined similarly to \( A_w(t) \) such that \( \text{Poin}(Y_w^v, t) = A_w^v(t^2) \)?
3. Similarly to Problem 6.1, is \( Y_w^v \) smooth if \( \text{Poin}(Y_w^v, t) \) is palindromic?

#### 6.2. Combinatorics of \( Q_w^v \).

We pose five problems on the combinatorics of Bruhat interval polytopes.

There are many pairs \((v, w)\) such that the Bruhat interval polytope \( Q_w^v \) is a cube. However, those pairs are not completely understood.

**Problem 6.3.** Find all pairs \((v, w)\) such that \( Q_w^v \) is combinatorially equivalent to a cube, equivalently \( X_w^v \) is a smooth toric variety.

The following is a restatement of Conjecture 5.4.

**Problem 6.4.** Is \( Q_w^v \) simple when the two vertices \( v \) and \( w \) are simple in \( Q_w^v \)?

Since \( Q_w^v \) and \( Q_{w-1}^v \) have the same dimension, \( Q_w^v \) is toric if and only if \( Q_{w-1}^v \) is toric. Moreover, when \( Q_w^v \) is toric, its combinatorial type is determined by the poset structure of \([v, w]\). Therefore, when \( Q_w^v \) is toric, \( Q_{w-1}^v \) is simple if \( Q_w^v \) is simple. We ask whether we can drop the condition toric in this statement.

**Problem 6.5.** Is \( Q_{w-1}^v \) simple if \( Q_w^v \) is simple?

In the proof of Theorems 4.32 and 4.33, we use the fact that if \( s_r \) does not appear in a reduced decomposition of \( w \) and \( v \leq w \), then \( Q_{s_r}^v \) and \( Q_{s_r}^w \) are combinatorially equivalent to \( Q_v^w \times Q_{s_r} \) (see [49, Proposition 5.7]). We wonder whether this fact can be generalized as follows.

**Problem 6.6.** Consider \( u, v, w \in S_n \) satisfying \( v \leq w \) and \([e, u] \cap [e, w] = \{e\}\). Are \( Q_{wu}^v \) and \( Q_{uw}^v \) combinatorially equivalent to \( Q_v^w \times Q_u \)?

We set

\[
c(v, w) := \ell(w) - \ell(v) - \dim Q_w^v.
\]

It is the complexity of the \( T \)-action on the Richardson variety \( X_w^v \). Note that \( c(e, w) = c(w) \), see Definition 4.20 for \( c(w) \).

Recall from Theorem 4.32 that a polytope \( Q_w \) is simple and \( c(w) = 1 \) if and only if \( Q_w \) is combinatorially equivalent to the product of the hexagon and the cube of dimension \( \ell(w) - 3 \). Moreover, a polytope \( Q_w \) is simple and \( c(v, w) = 0 \) if and only if \( Q_w \) is combinatorially equivalent to a cube (see Theorem 4.21 and Theorem 5.7). Therefore it is natural to ask the following:

**Problem 6.7.** Is a polytope \( Q_w \) simple and \( c(v, w) = 1 \) if and only if \( Q_w \) is combinatorially equivalent to the product of the hexagon and the cube of dimension \( \ell(w) - \ell(v) - 3 \)?
A. Toric varieties. We recall the background of the theory of toric varieties from [19]. We first recall the definition of toric varieties.

**Definition A.1 ([19, Definition 3.1.1]).** A toric variety of complex dimension $n$ is a normal algebraic variety containing an algebraic torus $T := (\mathbb{C}^*)^n$ as a Zariski open dense subset such that the action of the torus on itself extends to the whole variety.

**Definition A.2 ([19, Definition 3.1.2]).** Let $N$ be a lattice. A fan $\Sigma$ in $N_R := \mathbb{N} \otimes \mathbb{Z} \mathbb{R}$ is a finite collection of cones $\sigma \subseteq N_R$ such that:

1. Every $\sigma \in \Sigma$ is a strongly convex rational polyhedral cone, i.e., for each $\sigma$ there exists a finite set $S \subset \mathbb{N}$ such that $\sigma = \text{Conv}(S) := \left\{ \sum_{u \in S} c_u u \bigg| c_u \geq 0 \right\} \subset \mathbb{N}_R$ and $\sigma \cap (-\sigma) = \{0\}$.
2. For all $\sigma \in \Sigma$, each face of $\sigma$ is also in $\Sigma$.
3. For all $\sigma_1, \sigma_2 \in \Sigma$, the intersection $\sigma_1 \cap \sigma_2$ is a face of each.

Let $\Sigma$ be a fan in $N_R$. Let $M$ be a dual lattice of $N$ and we set $M_R := M \otimes \mathbb{Z} \mathbb{R}$. We denote by $\langle \cdot, \cdot \rangle$ the pairing between $M_R := M \otimes \mathbb{Z} \mathbb{R}$ and $N_R$. Each cone $\sigma \in \Sigma$ gives an affine toric variety $U_\sigma = \text{Spec}(\mathbb{C}[\sigma^\vee \cap M])$, where

$$\sigma^\vee := \{ m \in M_R \mid \langle m, u \rangle \geq 0 \text{ for all } u \in \sigma \}.$$  

By gluing these affine toric varieties, we get a variety $X_\Sigma$. It turns out that this variety is a toric variety and there is a correspondence between normal separated toric varieties and fans.

**Theorem A.3 ([19, Theorem 3.1.5 and Corollary 3.1.8]).** For a fan $\Sigma$, the variety $X_\Sigma$ is a normal separated toric variety. Conversely, for a normal separated toric variety $X$, there exists a fan $\Sigma$ such that $X$ is isomorphic to $X_\Sigma$.

A convex polytope is the convex hull of a finite set of points in the Euclidean space $\mathbb{R}^n$. It is well known that every convex polytope is a bounded intersection of finitely many half-spaces. Two polytopes are combinatorially equivalent if their face posets are isomorphic. A lattice polytope is a convex polytope whose vertices are in the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$.

For a full dimensional lattice polytope $P \subset \mathbb{R}^n$, one can associate a fan, called the normal fan of $P$. Consider the presentation of $P$ given by the intersection of half-spaces:

$$P = \{ m \in \mathbb{R}^n \mid \langle m, u_F \rangle \geq -a_F \text{ for every facet } F \}.$$  

For a face $Q$ of $P$, we set

$$\sigma_Q := \text{Cone}(u_F \mid F \text{ contains } Q).$$

Thus the cone $\sigma_F$ is the ray generated by $u_F$ for a facet $F$ and $\sigma_P = \{0\}$. It is known that $\Sigma_P := \{ \sigma_Q \mid Q \text{ is a face of } P \}$ becomes a fan (see [19, Theorem 2.3.2]) and we call it the normal fan of $P$. Moreover, we have the following correspondence.

**Theorem A.4 ([19, Theorem 6.2.1]).**

$$\begin{align*}
\{ P \subset \mathbb{R}^n \mid P \text{ is a full-dimensional lattice polytope} \} \quad &\downarrow_{1-1} \\
\{ (X_P, D) \mid X_P = X_{\Sigma_P} \text{ is a projective toric variety, } D \text{ is a } (\mathbb{C}^*)^n \text{-invariant ample divisor on } X_P \}. 
\end{align*}$$
Furthermore, the torus invariant subvarieties in \( X_P \) correspond to faces of the polytope \( P \). For example, the vertices of \( P \) correspond to the \( T \)-fixed points of \( X_P \).

For a vertex \( v \) of a polytope \( P \), the degree \( d(v) \) of \( v \) is the number of edges meeting at \( v \). For an \( n \)-dimensional polytope \( P \), a vertex \( v \) of \( P \) is said to be simple if \( d(v) = n \). When all the vertices of \( P \) are simple, we call \( P \) a simple polytope.

A vertex \( v \) of a lattice polytope \( P \) is said to be smooth if it is simple and the primitive direction vectors of the edges emanating from \( v \) form a basis for \( \mathbb{Z}^n \). We call a vertex of \( P \) singular if it is not smooth. A lattice polytope \( P \) is said to be smooth if all the vertices of \( P \) are smooth. We call a lattice polytope \( P \) is singular if some vertex of \( P \) is singular. See Figure 20.

![Figure 20. Examples of singular or smooth lattice polytopes.](image)

As one may expect, there are geometric interpretations of these terminologies.

**Proposition A.5** ([19, Theorem 2.4.3]). Let \( P \) be a lattice polytope and let \( X_P \) be the corresponding toric variety. A vertex \( v \) of \( P \) is smooth if and only if \( X_P \) is smooth at the corresponding fixed point. Moreover, \( X_P \) is smooth if and only if the polytope \( P \) is smooth.

There is a combinatorial way to determine whether a smooth compact toric variety is Fano. For a fan \( \Sigma \), a subset \( R \) of the primitive ray vectors is called a primitive collection of \( \Sigma \) if

\[
\text{Cone}(R) \notin \Sigma \quad \text{but} \quad \text{Cone}(R \setminus \{u\}) \in \Sigma \quad \text{for every} \quad u \in R.
\]

Note that primitive collections of \( \Sigma_P \) correspond to the minimal non-faces of \( P \). For a primitive collection \( R = \{u_1', \ldots, u_r'\} \), we get \( u_1' + \cdots + u_r' = 0 \) or there exists a unique cone \( \sigma \) such that \( u_1' + \cdots + u_r' \) is in the interior of \( \sigma \). That is,

\[
\begin{equation}
A.1 \quad u_1' + \cdots + u_r' = \begin{cases} 
0, \\
(a_1u_1 + \cdots + a_ru_r),
\end{cases}
\end{equation}
\]

where \( u_1, \ldots, u_r \) are the primitive generators of \( \sigma \) and \( a_1, \ldots, a_r \) are positive integers. We call (A.1) a primitive relation, and the degree \( \deg R \) of a primitive collection \( R \) is defined to be \( \ell - (a_1 + \cdots + a_r) \). Batyrev [8] gave a criterion for a projective toric variety to be Fano or weak Fano.

**Proposition A.6** ([8, Proposition 2.3.6]). A smooth compact toric variety \( X_\Sigma \) is Fano (respectively, weak Fano) if and only if \( \deg(R) > 0 \) (respectively, \( \deg(R) \geq 0 \)) for every primitive collection \( R \) of \( \Sigma \).

We can also distinguish two smooth Fano toric varieties using the primitive relations.

**Proposition A.7** ([8, Proposition 2.1.8 and Theorem 2.2.4]). Two smooth Fano toric varieties \( X_\Sigma \) and \( X_{\Sigma'} \) are isomorphic as toric varieties if and only if there is a bijection between the sets of rays of \( \Sigma \) and \( \Sigma' \) inducing a bijection between maximal cones and preserving the primitive relations.

### A.2. Fano Bott manifolds

One of the interesting families of smooth toric varieties is the family of Bott manifolds. In this subsection, we first recall the definition of a Bott manifold, and then we characterize Fano Bott manifolds and classify them.

**Definition A.8** ([33]). A Bott tower is an iterated \( \mathbb{C}P^1 \)-bundle starting with a point:

\[
\begin{array}{cccccccc}
B_n & \longrightarrow & B_{n-1} & \longrightarrow & \cdots & \longrightarrow & B_1 & \longrightarrow & B_0, \\
\bigg\downarrow \quad P(\mathbb{C} \oplus \xi_n) & \bigg\downarrow & \bigg\downarrow & \bigg\downarrow & \bigg\downarrow & \bigg\downarrow & \bigg\downarrow & \bigg\downarrow & \{\text{a point}\}
\end{array}
\]
where each $B_i$ is the complex projectivization $P(\mathbb{C} \oplus \xi_i)$ of the Whitney sum of a holomorphic line bundle $\xi_i$ and the trivial line bundle $\mathbb{C}$ over $B_{i-1}$. The total space $B_n$ is called a Bott manifold.

Each Bott manifold is a smooth projective toric variety associated with a smooth lattice polytope combinatorially equivalent to a cube, and the converse also holds.

**Proposition A.9** ([54, Corollary 3.5]). If a smooth lattice polytope $P$ is combinatorially equivalent to a cube, then the toric variety $X_P$ is a Bott manifold. Indeed, the family of Bott manifolds is

$$\{X_P \mid P \text{ is a smooth lattice polytope that is combinatorially equivalent to a cube}\}.$$  

Let $B_n$ be a Bott manifold. Then there is a smooth lattice polytope $P$ combinatorially equivalent to a cube of dimension $n$ such that $B_n = X_P$. The polytope $P$ has $2n$ facets and there are $n$ pairs of facets not intersecting with each other. Let $\{v_1, \ldots, v_n, w_1, \ldots, w_n\}$ be the ray generators of the normal fan $\Sigma_P$ of $P$ such that $v_i$ and $w_i$ do not form a cone in $\Sigma_P$ for each $i \in [n]$. That is, each primitive collection of $\Sigma_P$ corresponds to the set $\{v_i, w_i\}$. By Proposition A.6, we obtain

$$(A.2) \quad B_n \text{ is Fano if and only if } v_i + w_i \text{ is either } v_j \text{ or } w_j \text{ for some } j \text{ unless } v_i + w_i = 0.$$  

Using $A.2$, to a Fano Bott manifold $B_n$, we associate a signed rooted forest with the vertex set $[n]$ as follows:

- vertex $i$ is a root if $v_i + w_i = 0$,
- we draw an edge with $+$ sign between $i$ and $j$ if $v_i + w_i = v_j$, and
- we draw an edge with $-$ sign between $i$ and $j$ if $v_i + w_i = w_j$.

We can also construct a Fano Bott manifold from a signed rooted forest up to isomorphism.

Let $F$ be a signed rooted forest with vertex set $[n]$. For each $i$, by changing the signs of all edges connecting $i$ and its children simultaneously, we get a new signed rooted forest $r_i(F)$. Then the Bott manifold $B_{r_i(F)}$ corresponding to $r_i(F)$ is isomorphic to the Bott manifold $B_F$ corresponding to $F$ by Proposition A.7. Let $\mathcal{SF}_n$ be the set of all signed rooted forests on the vertex set $[n]$. Denote by $\sim$ the equivalence relation on $\mathcal{SF}_n$ generated by $r_i$’s for all $i$’s.

**Theorem A.10** ([18, Theorem 3.2]). The isomorphism classes in Fano Bott manifolds of complex dimension $n$ bijectively correspond to $\mathcal{SF}_n/\sim$.

We provide all signed rooted forests having three vertices in Figure 21. Among these ten signed rooted forests, there are four equivalence classes in $\mathcal{SF}_3/\sim$ as follows:

$$(1) \sim (2) \sim (3) \sim (4), \quad (5) \sim (7), \quad (6), \quad (8) \sim (9), \quad (10)$$

In the above, the first and the third equivalence classes arise as toric Richardson varieties of Catalan type, and the last two arise as products of Catalan type. However, the second class does not arise as (products of) Catalan type.

![Figure 21. Signed rooted forests with 3 vertices](image)

Similarly, there are thirteen equivalence classes in $\mathcal{SF}_4/\sim$ as shown in Figure 22, where $+$ signs on edges are omitted. In the figure, (7), (9), (11) arise as toric Richardson varieties of Catalan type, and (1), (2), (3), (4), (6) arise as products of Catalan type. However, the remaining ones do not arise as (products of) Catalan type.
Figure 22. Representatives of $\mathcal{SF}_4/\sim$.

REFERENCES

[1] Hiraku Abe, Megumi Harada, Tatsuya Horiguchi, and Mikiya Masuda. The cohomology rings of regular nilpotent Hessenberg varieties in Lie type A. *Int. Math. Res. Not. IMRN*, (17):5316–5388, 2019.

[2] Hiraku Abe and Tatsuya Horiguchi. A survey of recent developments on Hessenberg varieties. In *Schubert calculus and its applications in combinatorics and representation theory*, volume 332 of *Springer Proc. Math. Stat.* pages 251–279. Springer, Singapore, [2020] ©2020.

[3] Hiraku Abe and Haozhi Zeng. The integral cohomology rings of Peterson varieties in type a. arXiv:2203.02629, 2022.

[4] Alfred V. Aho, Michael R. Garey, and Jeffrey David Ullman. The transitive reduction of a directed graph. *SIAM J. Comput.*, 1(2):131–137, 1972.

[5] Michael Francis Atiyah. Convexity and commuting Hamiltonians. *Bull. London Math. Soc.*, 14(1):1–15, 1982.

[6] Michèle Audin. *Torus actions on symplectic manifolds*, volume 93 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, revised edition, 2004.

[7] Anthony Bahri, Soumen Sarkar, and Jongbaek Song. On the integral cohomology ring of toric orbifolds and singular toric varieties. *Algebr. Geom. Topol.*, 17(6):3779–3810, 2017.

[8] Victor V. Batyrev. On the classification of toric Fano 4-folds. volume 94, pages 1021–1050. 1999. Algebraic geometry, 9.

[9] Andrzei S. Białynicki-Birula. Some theorems on actions of algebraic groups. *Ann. of Math. (2)*, 98:480–497, 1973.

[10] Sara Billey and V. Lakshmibai. *Singular loci of Schubert varieties*, volume 182 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 2000.

[11] Anders Björner and Francesco Brenti. *Combinatorics of Coxeter groups*, volume 231 of *Graduate Texts in Mathematics*. Springer, New York, 2005.

[12] Armand Borel. Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts. *Ann. of Math. (2)*, 57:115–207, 1953.

[13] Alexandre V. Borovik, I. M. Gelfand, and Neil White. *Coxeter matroids*, volume 216 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 2003.

[14] Mireille Bousquet-Mélu and Steve Butler. Forest-like permutations. *Ann. Comb.*, 11(3-4):335–354, 2007.

[15] Victor M. Buchstaber and Taras E. Panov. *Toric topology*, volume 204 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2015.

[16] James B. Carrell and Alexandre Kurth. Normality of torus orbit closures in $G/P$. *J. Algebra*, 233(1):122–134, 2000.

[17] James B. Carrell and Jochen Kuttler. Smooth points of $T$-stable varieties in $G/B$ and the Peterson map. *Invent. Math.*, 151(2):353–379, 2003.

[18] Yunhyung Cho, Eunjeong Lee, Mikiya Masuda, and Seonjeong Park. On the enumeration of Fano Bott manifolds. arXiv:2106.12788v1, to appear in Fields Institute Communications “Toric Topology and Polyhedral Products”.
[19] David A. Cox, John B. Little, and Henry K. Schenck. *Toric varieties*, volume 124 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2011.

[20] Romuald Dabrowski. On normality of the closure of a generic torus orbit in $G/P$. *Pacific J. Math.*, 172(2):321–330, 1996.

[21] Daniel Daly. Reduced decompositions with one repetition and permutation pattern avoidance. *Graphs Combin.*, 29(2):173–185, 2013.

[22] Ivor Malcolm Haddon Etherington. Non-associate powers and a functional equation. *The Mathematical Gazette*, 21(242):36–39, 1937.

[23] C. Kenneth Fan. Schubert varieties and short braidedness. *Transform. Groups*, 3(1):51–56, 1998.

[24] Hermann Flaschka and Luc Haine. Torus orbits in $G/P$. *Pacific J. Math.*, 149(2):251–292, 1991.

[25] Sergey Fomin and Nathan Reading. Root systems and generalized associahedra. In *Geometric combinatorics*, volume 13 of IAS/Park City Math. Ser., pages 63–131. Amer. Math. Soc., Providence, RI, 2007.

[26] Naoki Fujita, Eunjeong Lee, and Dong Youp Suh. Algebraic and geometric properties of flag Bott–Samelson varieties and applications to representations. *Pacific J. Math.*, 309(1):145–194, 2020.

[27] William Fulton. *Introduction to toric varieties*, volume 131 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry.

[28] William Fulton. *Young tableaux*, volume 35 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1997. With applications to representation theory and geometry.

[29] William Fulton. *Intersection theory*, volume 2 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, second edition, 1998.

[30] Izrail’ Moiseevich Gel’fand, Mark Goresky, Robert D. MacPherson, and Vera V. Serganova. Combinatorial geometries, convex polyhedra, and Schubert cells. *Adv. in Math.*, 63(3):301–316, 1987.

[31] Izrail’ Moiseevich Gel’fand and Vera V. Serganova. Combinatorial geometries and the strata of a torus on homogeneous compact manifolds. *Uspekhi Mat. Nauk*, 42(2(254)):107–134, 287, 1987.

[32] Richard M. Green and Jozsef Losonczy. Freely braided elements of Coxeter groups. *Ann. Comb.*, 6(3-4):337–348, 2002.

[33] Michael Grossberg and Yael Karshon. Bott towers, complete integrability, and the extended character of representations. *Duke Math. J.*, 76(1):23–58, 1994.

[34] Victor W. Guillemin, Tara Suzanne Holm, and C˘ at˘ alin Zara. A GKM description of the equivariant cohomology ring of a homogeneous space. *J. Algebraic Combin.*, 23(1):21–41, 2006.

[35] Victor W. Guillemin and Shlomo Sternberg. Convexity properties of the moment mapping. *Invent. Math.*, 67(3):491–513, 1982.

[36] Megumi Harada and Tatsuya Horiguchi. The cohomology rings of regular nilpotent Hessenberg varieties. Chapter 9 of *Handbook of Combinatorial Algebraic Geometry: Subvarieties of the flag variety*. 2019.

[37] Megumi Harada, Tatsuya Horiguchi, Mikiya Masuda, and Seonjeong Park. The volume polynomial of regular semisimple Hessenberg varieties and the Gelfand–Tsetlin polytope. *Proc. Steklov Inst. Math.*, 305(Algebraicheskaya Topologiya Kombinatorika i Matematicheskaya Fizika):344–373, 2019.

[38] Christophe Hohlweg, Carsten E. M. C. Lange, and Hugh Thomas. Permutahedra and generalized associahedra. *Adv. Math.*, 226(1):608–640, 2011.

[39] James E. Humphreys. *Reflection groups and Coxeter groups*, volume 29 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1990.

[40] Jens Carsten Jantzen. *Representations of algebraic groups*, volume 107 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, second edition, 2003.

[41] Paramasamy Karuppachamy. On Schubert varieties. *Comm. Algebra*, 41(4):1365–1368, 2013.

[42] Aleksandr Anatol’evich Klyachko. Orbits of a maximal torus on a flag space. *Functional Analysis and Its Applications*, 19(1):65–66, 1985.

[43] Yuji Kodama and Lauren Williams. The full Kostant–Toda hierarchy on the positive flag variety. *Comm. Math. Phys.*, 335(1):247–283, 2015.

[44] Shintaro Kuroki, Eunjeong Lee, Jongbaek Song, and Dong Youp Suh. Flag Bott manifolds and the toric closure of a generic orbit associated to a generalized Bott manifold. *Pacific J. Math.*, 308(2):347–392, 2020.

[45] Venkatramani Lakshmibai and H. Sandhya. Criterion for smoothness of Schubert varieties in $SL(n)/B$. *Proc. Indian Acad. Sci. Math. Sci.*, 100(1):45–52, 1990.

[46] Eunjeong Lee and Mikiya Masuda. Generic torus orbit closures in Schubert varieties. *J. Combin. Theory Ser. A.*, 170:105143, 44pp, 2020.

[47] Eunjeong Lee, Mikiya Masuda, and Seonjeong Park. Toric Schubert varieties and directed Dynkin diagrams. *in preparation*.

[48] Eunjeong Lee, Mikiya Masuda, and Seonjeong Park. Torus orbit closures in flag varieties and retractions on Weyl groups. arXiv:1908.08310v3, to appear in International Journal of Mathematics.

[49] Eunjeong Lee, Mikiya Masuda, and Seonjeong Park. On Schubert varieties of complexity one. *Pacific J. Math.*, 315(2):419–447, 2021.

[50] Eunjeong Lee, Mikiya Masuda, and Seonjeong Park. Toric Bruhat interval polytopes. *J. Combin. Theory Ser. A.*, 179:105387, 41pp, 2021.
[51] Eunjeong Lee, Mikiya Masuda, and Seonjeong Park. Toric Richardson varieties of Catalan type and Wedderburn–Etherington numbers. arXiv:2105.12274v1, 2021.

[52] Eunjeong Lee, Mikiya Masuda, Seonjeong Park, and Jongbaek Song. Poincaré polynomials of generic torus orbit closures in Schubert varieties V. A. Rokhlin-Memorial. In Topology, geometry, and dynamics, volume 772 of Contemp. Math., pages 189–208. Amer. Math. Soc., [Providence, RI, [2021] ©2021.

[53] Jean-Louis Loday, Alessandra Frabetti, Frédéric Chapoton, and François Guichot. Dialgebras and related operads, volume 1763 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2001.

[54] Mikiya Masuda and Taras E. Panov. Semi-free circle actions, Bott towers, and quasitoric manifolds. Mat. Sb., 190(8):95–122, 2008.

[55] Jacqueline Morand. Closures of torus orbits in adjoint representations of semisimple groups. C. R. Acad. Sci. Paris Sér. I Math., 328(3):197–202, 1999.

[56] N. J. A. Sloane, editor. The On-Line Encyclopedia of Integer Sequences (OEIS). published electronically at https://oeis.org (accessed April 13, 2022).

[57] Seonjeong Park and Jongbaek Song. Conic decomposition of a toric variety and its application to cohomology. arXiv:2106.04429v1, 2021.

[58] Nathan Paul Reading. On the structure of Bruhat order. ProQuest LLC, Ann Arbor, MI, 2002. Thesis (Ph.D.)–University of Minnesota.

[59] Richard P. Stanley. Catalan numbers. Cambridge University Press, New York, 2015.

[60] Bridget Eileen Tenner. Pattern avoidance and the Bruhat order. J. Combin. Theory Ser. A, 114(5):888–905, 2007.

[61] Bridget Eileen Tenner. Repetition in reduced decompositions. Adv. in Appl. Math., 49(1):1–14, 2012.

[62] Emmanuel Tsukerman and Lauren K. Williams. Bruhat interval polytopes. Adv. Math., 285:766–810, 2015.

[63] Julianna S. Tymoczko. Permutation actions on equivariant cohomology of flag varieties. In Toric topology, volume 460 of Contemp. Math., pages 365–384. Amer. Math. Soc., Providence, RI, 2008.

[64] Joseph Henry Maclagan Wedderburn. The functional equation \( g(x^2) = 2ax + [g(x)]^2 \). Ann. of Math. (2), 24(2):121–140, 1922.

[65] Hassler Whitney. On the abstract properties of linear dependence. Amer. J. Math., 57(3):509–533, 1935.

[66] Alexander Woo and Alexander Yong. When is a Schubert variety Gorenstein? Adv. Math., 207(1):205–220, 2006.

[67] Li Yu and Mikiya Masuda. On Descriptions of Products of Simplices. Chin. Ann. Math. Ser. B, 42(5):777–790, 2021.

[68] Andrei Zelevinsky. Nested complexes and their polyhedral realizations. Pure Appl. Math. Q., 2(3, Special Issue: In honor of Robert D. MacPherson. Part 1):655–671, 2006.

[69] Günter M. Ziegler. Lectures on polytopes, volume 152 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1995.

(E. Lee) DEPARTMENT OF MATHEMATICS, CHUNGBUK NATIONAL UNIVERSITY, CHEONGJU 28644, REPUBLIC OF KOREA

Email address: eunjeong.lee@chungbuk.ac.kr

(M. Masuda) OSACA CITY UNIVERSITY ADVANCED MATHEMATICS INSTITUTE (OCAMI) & DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, OSACA CITY UNIVERSITY, SUMIYOSHI-KU, SUGIMOTO, 558-8585, OSACA, JAPAN

Email address: mikiyamsd@gmail.com

(S. Park) DEPARTMENT OF MATHEMATICS EDUCATION, JEONJU UNIVERSITY, JEONJU 55069, REPUBLIC OF KOREA

Email address: seonjeongpark@jj.ac.kr