A remark on calibrations and Lie groups

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Dedicated to Blaine Lawson on the occasion of his 80th birthday

Abstract

We use the notion of the principal three-dimensional subgroup of a simple Lie group to identify certain special subspaces of the Lie algebra and address the question of whether these are calibrated for invariant forms on the group.

1 Introduction

The notion of a calibrated differential form $\varphi$, as introduced in [3], has become very important especially in the study of Calabi-Yau, $G_2$ and $Spin(7)$-manifolds, where $\varphi$ is a covariant constant form. On the other hand, the manifolds which have most covariant constant forms, namely compact simple Lie groups $G$, have received less attention, although they are addressed in [12],[8],[9],[11].

Recall that the cohomology of a simple Lie group $G$ of rank $\ell$ is an exterior algebra on $\ell$ generators with harmonic representatives $\varphi_i$ of odd degree $d_i$ which are covariant constant. The Cartan 3-form $\varphi_1$ is the generator of smallest degree and Tasaki [12] showed that this defines a calibration and moreover that a three-dimensional subgroup associated to the highest root is calibrated for this form and is volume-minimizing.

He also showed that the Hodge dual $*\varphi_1$ calibrates the codimension 3 subspace of non-regular elements of $G$.

Amongst the three-dimensional subgroups there is a particularly distinguished one, the principal three-dimensional subgroup, and Kostant showed [6] that under the action of this group the Lie algebra decomposes $\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_\ell$ into irreducible
representations of $SO(3)$ whose dimensions are precisely the degrees $d_i$ of the generators of the cohomology. The author conjectured in [5] that there is an exact fit here – that for each subspace $V_i$ there exists a corresponding generator which restricts nontrivially. To the author’s knowledge this has not yet been confirmed, though there is some information in [1]. In any case, if the restriction is non-zero it opens up the possibility of more complex calibrated submanifolds.

In this paper we observe first that the function defined by $\varphi_i$ on the Grassmannian of oriented subspaces of $\mathfrak{g}$ of dimension $d_i$ has a critical point on $V_i$. If this critical value is nonzero then any submanifold of dimension $d_i$ tangential to a conjugate of $V_i$ will be minimal [11]. If the non-zero value is the maximum then $\varphi_i$ defines a calibration and any such submanifold is volume minimizing.

We then search for non-zero values by using the transitive action of groups on odd-dimensional spheres $S^{2m+1}$, and an argument initiated by X.Liu [8]. This consists of pulling back the volume form on the sphere and averaging over the group to produce an invariant form on $G$ of degree $2m + 1$. We use the well-known list of groups with transitive actions to show that in each case the pull-back of the volume form restricted to a corresponding $V_i$ is non-negative and hence its average is non-zero, providing some evidence for the conjecture. The relevant degrees are $2n - 1$ for $SO(2n)$ and $SU(n)$, $4n - 1$ for $Sp(n)$, $7$ for $Spin(7)$ and $15$ for $Spin(9)$.

Finally we mention the entirely different context [5] in which the conjecture arose, involving the moduli space of stable bundles on a curve $C$.

## 2 Invariant forms

Let $G$ be a compact simple Lie group. The covariant constant forms on $G$ are the bi-invariant forms and these are defined as multilinear alternating forms $\alpha$ on $\mathfrak{g}$ by

$$\alpha(a_1, \ldots, a_{2m+1}) = p(a_1, [a_2, a_3], \ldots, [a_{2m}, a_{2m+1}])$$

where $p$ is an adjoint-invariant polynomial of degree $m + 1$. These polynomials correspond under the Chern-Weil homomorphism to characteristic classes like Chern or Pontryagin classes and we shall often label the invariant forms this way – as classes of degree $2m + 2$ in the cohomology $H^*(B_G)$ of the classifying space. The Killing form is a quadratic polynomial and yields the Cartan 3-form.

The irreducible representations of the three-dimensional group $SU(2)$ are symmetric powers $S^n$ of the standard complex 2-dimensional representation $S$. The space $S^n$ may be thought of as the action on homogeneous polynomials $p(z_1, z_2)$ of degree $n$. 

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or more conveniently the polynomial $p(z) = p(z_1/z_2, 1)$ and is therefore of dimension $n + 1$. Since $-1 \in SU(2)$ acts trivially if $n$ is even, these are the irreducibles for $SO(3)$ and are real. When $n$ is odd they are quaternionic representations of $SU(2)$.

The Clebsch-Gordon formula tells us how to decompose a tensor product: if $m \geq n$ then

$$S^m \otimes S^n = S^{m+n} \oplus S^{m+n-2} \oplus \cdots \oplus S^{m-n}.$$  

The decomposition involves contraction with the skew form on $S$ and it follows then that $S^n \otimes S^n = S^{2n} \oplus S^{2n-2} \oplus \cdots$ and the skew part $\Lambda^2S^n = S^{2n-2} \oplus S^{2n-6} \oplus \cdots$.

The generators of the cohomology $H^*(G)$ have degrees $d_i = 2\lambda_i + 1$ where $\lambda_i$ are the exponents of the Lie algebra. For completeness we list them:

- $A_\ell : 1, 2, 3, \ldots, \ell,$
- $B_\ell : 1, 3, 5, \ldots, 2\ell - 1,$
- $C_\ell : 1, 3, 5, \ldots, 2\ell - 1.$
- $D_\ell (\ell \text{ odd}) : 1, 3, 5, \ldots, 2\ell - 3,$
- $F_4 : 1, 5, 7, 11,
- $G_2 : 1, 5.$
- $E_6 : 1, 4, 5, 7, 8, 11,$
- $E_7 : 1, 5, 7, 9, 11, 13, 17,$
- $E_8 : 1, 7, 11, 13, 17, 19, 23, 29.$

In this list for each group the exponents are distinct, but for $D_\ell$ where $\ell$ is even the exponent $\ell - 1$ occurs twice. In terms of $SO(4n)$ characteristic classes the two invariants can be taken to be the Euler class and a Pontryagin class of the same degree. The generators are not unique, just as we can take a basis of invariant polynomials for $SU(n)$ as $\text{tr} a^k (k = 2, \ldots, n)$ or the coefficients of $\det(\lambda - a)$.

Kostant’s theorem [6] tells us that under the action of the principal three-dimensional subgroup, which is unique up to conjugation, $g = V_1 \oplus V_2 \oplus \cdots \oplus V_\ell$ where $V_i \cong S^{2\lambda_i}$. Clearly $\lambda_1 = 1$ gives the Lie algebra of the subgroup.

As an example, the irreducible representation $S^n$ defines a homomorphism $SU(2) \to SU(n+1)$ whose image is the principal three-dimensional subgroup and the Lie algebra $\mathfrak{su}(n+1)$ is isomorphic to the trace zero elements in $\text{Hom}(S^n, S^n) \cong S^n \otimes S^n$. The Clebsch-Gordon formula gives $S^2 \oplus \cdots \oplus S^{2n}$ as the decomposition $V_1 \oplus V_2 \oplus \cdots \oplus V_\ell$.

### 3 Critical points

Given an invariant form $\varphi_i$ of degree $d_i$ we can evaluate it on an oriented $d_i$-dimensional subspace of $g$ to obtain a function $f_i$ on the oriented Grassmannian $\widetilde{Gr}(d_i, g)$ of such subspaces.

**Theorem 1** The function $f_i$ has a critical point at $[V_i]$. 

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Proof: Using the metric on the Grassmannian, the gradient of $f_i$ at $[V_i]$ is a tangent vector which, by virtue of the adjoint invariance of $\varphi_i$, is invariant under the action of $SU(2)$ which stabilizes $[V_i]$. The tangent space of the Grassmannian at $[V_i]$ is isomorphic to $\text{Hom}(V_i, g/V_i)$, but as we have seen, except for the case $D_\ell$ where $\ell$ is even, the exponents are distinct and so the irreducible $V_i$ does not occur in the decomposition of $g/V_i$. By $SU(2)$-invariance, the homomorphism is zero and so the gradient is zero. It therefore remains to consider the case of $SO(4n)$.

The principal three-dimensional subgroup in $SO(4n)$ acts reducibly on $\mathbb{R}^{4n}$. It is the representation $1 \oplus S^{4n-2}$ and so $g \cong \Lambda^2(1 \oplus S^{4n-2}) = S^{4n-2} \oplus \Lambda^2(S^{4n-2})$. Denote by $V$ the first subspace here. Using the Clebsch-Gordan decomposition we have $\Lambda^2(S^{4n-2}) = S^{8n-6} \oplus S^{8n-10} \oplus \cdots \oplus S^2$ which contains a copy of $S^{4n-2}$ which we call $V'$. If $e_0, e_1, \ldots$ is an orthonormal basis of $1 \oplus S^{4n-2}$ with $e_0$ spanning the trivial component then $(e_0, e_1, \ldots) \mapsto (-e_0, e_1, \ldots)$ is an orientation-reversing involution $\sigma$ commuting with $SO(3)$ and acting as $-1$ on $V$ and $+1$ on $V'$. The invariant polynomial on $\mathfrak{so}(4n)$ defined by the Pfaffian $\sqrt{\det a}$ changes sign under change of orientation so it defines an invariant form $\varphi$ such that $\sigma^* \varphi = -\varphi$, hence $\varphi$ evaluated on $V'$ is zero since $\sigma = 1$ there. We therefore associate $V$ to $\varphi$ and $V'$ to $\varphi'$, defined by the Pontryagin class, and consider the corresponding functions $f, f'$. Pontryagin classes are of course orientation-independent. The function $f'$ is $\sigma$-invariant and so its gradient at $[V']$ is an invariant element of $\text{Hom}(V', V)$, but the action here is $-1$, so the gradient vanishes and this is a critical point. The case of $f$ is similar, taking into account the fact that $\sigma$ changes orientation on $V$. 

\[ \square \]

4 Groups acting on spheres

4.1 The invariant forms

We focus now on a family of covariant constant forms which arise geometrically. If a simple group $G$ acts transitively on an odd-dimensional sphere then we have the projection $p : G \to S^{2m+1} = G/H$ and averaging over $G$ the pull-back $p^* \omega$ of the volume form on $S^{2m+1}$ gives an invariant $(2m+1)$-form. Since $p^* \omega$ is $H$-invariant this is equivalent to averaging over the sphere as in [8]. We know in advance that this form is non-zero for, by [7] (see also [10]), the stabilizer $H$ is not homologous to zero and so the cohomology class $[p^* \omega] \neq 0$.

The groups acting transitively on spheres are well-known, especially from their appearance as special holonomy groups. For a simple group $G$ and an odd-dimensional
sphere we have:

\[ \text{SO}(2n), \quad \text{SU}(n), \quad \text{Sp}(n), \quad \text{Spin}(7) \subset \text{SO}(8), \quad \text{Spin}(9) \subset \text{SO}(16). \]

A universal multiple of the invariant form which the averaging produces can be labelled by a characteristic class which restricts to zero in the cohomology \( H^*(B_H) \) of the classifying space of the stabilizer \( H \) of the action. The group \( H \) stabilizes a vector in an even-dimensional space so this is the Euler class for \( \text{SO}(2n) \), the Chern class \( c_n \) for \( \text{SU}(n) \), the Chern class \( c_{2n} \) for \( \text{Sp}(n) \subset \text{SU}(2n) \). The last two examples in the list are stabilizers of a vector in the spin representation and expressing the Euler class for the spin representation in terms of the basic weights gives multiples of \( p_1^2 - 4p_2 \) for \( \text{Spin}(7) \) and \( p_1^4 - 8p_1^2p_2 + 16p_2^2 - 64p_4 \) in the case of \( \text{Spin}(9) \) (see also [2]).

We want to prove that the invariant form is non-zero on the component \( S^{2m} \subset g \), the tangent space at the identity. As in [3], the translate of \( p^*\omega \) from a general point \( g \) with \( p(g) = v \in S^{2m+1} \subset \mathbb{R}^{2m+2} \) to the identity gives a form on the Lie algebra which, evaluated on \( (a_1, \ldots, a_{2m+1}), a_i \in g \), is \( \det(v, a_1v, a_2v, \ldots, a_{2m+1}v) \). If \( (a_1, \ldots, a_{2m+1}) \) forms a basis for \( S^{2m} \) and this is nonnegative and not identically zero for all \( v \) in the sphere, then the average will be positive and the invariant form will be nonzero. We proceed to consider the different cases.

### 4.2 The case \( \text{SO}(2n) \)

As noted above, the principal 3-dimensional subgroup in this case arises from a reducible representation \( 1 \oplus S^{2n-2} \) and the subspace \( V_i \subset \mathfrak{so}(2n) \) of dimension \( 2n - 1 \) is spanned by \( a_i = e_0 \otimes e_i - e_i \otimes e_0 \) for \( 1 \leq i \leq 2n - 1 \). Then \( a_i(v) = v_ie_0 - v_0e_i \) and, since \( \|v\|^2 = 1 \),

\[
v \wedge a_1v \wedge \cdots \wedge a_{2n-1}v = v_0^{2n-2}e_0 \wedge e_1 \wedge \cdots \wedge e_{2n-1}.
\]

This is non-negative hence the average is non-zero.

This formula is Example 3.7 in [3], where Lemma 3.5 in that paper shows that in \( \mathfrak{so}(2n) \) for general \( a_i \)

\[
\det(v, a_1v, a_2v, \ldots, a_{2n-1}v) = \|v\|^2Q_{2n-2}(v) \tag{1}
\]

where \( Q_{2n-2}(v) \) is homogeneous in \( v \) of degree \( 2n - 2 \). In our situation where \( a_1, \ldots, a_{2n-1} \) span one of the spaces \( V_i \), this will be an invariant of the \( \text{SU}(2) \) action on \( \mathbb{R}^{2n} \) and the focus of our attention in the other cases.
4.3 The case \( SU(n) \)

Here the principal three-dimensional subgroup is the action of \( SU(2) \) in its irreducible representation \( \mathbf{S}^{n-1} \), and so its image in \( SU(n) \) is a copy of \( SU(2) \) for \( n \) even and \( SO(3) \) for \( n \) odd. The \( 2n - 1 \)-dimensional subspace \( V_i \) is \( \mathbf{S}^{2n-2} \) and so we have an inclusion

\[
\mathbf{S}^{2n-2} \subset \text{Hom}(\mathbf{S}^{n-1}, \mathbf{S}^{n-1}) \cong \mathbf{S}^{n-1} \otimes \mathbf{S}^{n-1}
\]

and we can recognize this from the Clebsch-Gordon formula.

In terms of polynomials \( p(z) \) it is the adjoint of the multiplication map, but a more convenient description is to identify \( \mathbf{S}^m \) with \( H^0(P^1, \mathcal{O}(m)) \), holomorphic sections of the line bundle of degree \( m \) on the projective line. Since each \( \mathbf{S}^m \) has either a nondegenerate skew or symmetric form we also have an invariant identification \( \mathbf{S}^m \cong H^1(P^1, \mathcal{O}(-m-2)) \) by Serre duality. Then we have a natural tensor product map

\[
H^1(P^1, \mathcal{O}(-2n)) \otimes H^0(P^1, \mathcal{O}(n-1)) \rightarrow H^1(P^1, \mathcal{O}(-n-1)) \cong H^0(P^1, \mathcal{O}(n-1))
\]

which realizes the map \( \mathbf{S}^{2n-2} \otimes \mathbf{S}^{n-1} \rightarrow \mathbf{S}^{n-1} \). This is the action of \( V_i \subset \mathfrak{su}(n) \) on \( \mathbb{C}^n \).

Consider first the case where \( n = 2m + 1 \) is odd, then \( \mathbf{S}^{n-1} = \mathbf{S}^{2m} \) is even and has a real structure and so we can write a complex vector \( v = v_1 + iv_2 \) where \( v_1, v_2 \) are real. Of course \( SU(n) \) does not preserve the real structure, only the three-dimensional subgroup does. Now \( \mathbf{S}^{2m} \subset \mathbf{S}^{2m} \otimes \mathbf{S}^{2m} \) is symmetric and real and elements of \( V_i \subset \mathfrak{su}(2m + 1) \) are of the form \( iA \) for a real symmetric matrix \( A \).

As in equation \( \Box \) we are concerned with the expression \( v \wedge a_1 v \wedge \cdots \wedge a_{2n-1} v \) considering \( \mathbb{C}^n \) as a real vector space where the \( a_j \) lie in \( V_i \). This vanishes when some linear combination of the \( a_i \) has \( v \) as a real eigenvector. But the \( a_i \) are skew adjoint so it can only be the zero eigenvalue. Now each \( a \in V_i \) is of the form \( iA \) for \( A \) real, and so \( iA(v_1 + iv_2) = -Av_2 + iAv_1 \) and if this vanishes then \( Av_1 = 0 = Av_2 \).

Represent \( A \) as an element \( [A] \) of \( H^1(P^1, \mathcal{O}(-2n)) \) and \( v_1 \) as a section \( s \) of \( \mathcal{O}(n-1) \) then \( Av_1 = 0 \) has an interpretation in algebraic geometry: consider the exact sequence of sheaves

\[
0 \rightarrow \mathcal{O}(-2n) \xrightarrow{s} \mathcal{O}(-n-1) \rightarrow \mathcal{O}_D(-n-1) \rightarrow 0
\]

where \( D \) is the divisor of zeros of \( s \). Then the long exact cohomology sequence gives

\[
0 \rightarrow H^0(D, \mathcal{O}_D(-n-1)) \xrightarrow{\delta} H^1(P^1, \mathcal{O}(-2n)) \xrightarrow{\delta} H^1(P^1, \mathcal{O}(-n-1)) \rightarrow 0
\]

so that \( [A]s = 0 \) if and only if \( [A] = \delta t \) for a section \( t \) of \( \mathcal{O}(-n-1) \) on the zero-dimensional cycle \( D \).
Let $s_1$ and $s_2$ be two sections representing $v_1, v_2$ which have a common zero $x$ then the cycles $D_1, D_2$ intersect and taking $t$ as a section of $\mathcal{O}(-n - 1)$ on $x$ defines $\delta(t) = [\alpha]$ which annihilates both $s_1$ and $s_2$. Hence $[\alpha]$ represents a linear combination of $a_j$ such that $v \wedge a_1 v \wedge \cdots \wedge a_{2n-1} v$ vanishes when $v = v_1 + iv_2$ and $v_1, v_2$ are represented by $s_1, s_2$ which have a common zero. These are polynomials $p_1(z), p_2(z)$ of degree $n - 1$ and the condition for a common zero is the vanishing of the resultant

$$R(p_1, p_2) = a_0^{n-1} b_0^{n-1} \prod_{i,j} (\lambda_i - \mu_j) = a_0^{n-1} \prod_{i} p_2(\lambda_i)$$

where $\lambda_i, \mu_j$ are the roots of $p_1(z) = a_0 z^{n-1} + \cdots + a_{n-1}, p_2(z) = b_0 z^{n-1} + \cdots + b_{n-1}$. This is a polynomial in $v = v_1 + iv_2$ homogeneous of degree $2n - 2$. Its vanishing implies $Q_{2n-2}$ from equation (1) vanishes, but these two invariant polynomials have the same degree and the resultant is irreducible hence they are multiples of each other.

The real structure on $\mathbf{S}^{n-1}$ is inherited from the quaternionic structure of $\mathbf{S}$ so a real polynomial of degree $2m$ satisfies $p(-1/z) = z^{-2m} p(z)$ and there is a free involution $\lambda \mapsto -1/\bar{\lambda}$ on the roots of $p$. Let $\lambda_1, \ldots, \lambda_m, -1/\bar{\lambda}_1, \ldots, -1/\bar{\lambda}_m$ be the roots of $p_1$, then

$$R(p_1, p_2) = a_0^{2m} \prod_{i=1}^{m} p_2(\lambda_i) p_2(-1/\bar{\lambda}_i) = (a_0 \prod_{i=1}^{m} \bar{\lambda}_i^{-1})^{2m} \prod_{i=1}^{m} |p_2(\lambda_i)|^2.$$ 

Reality implies $a_{2m} = \bar{a}_0$ so that the product of the roots is $\bar{a}_0/a_0$ and $a_0 \prod_{i=1}^{m} \bar{\lambda}_i^{-1}$ is real. Hence the resultant is non-negative and averaging gives a non-zero evaluation of the form.

When $n = 2m$ is even, $\mathbf{S}^{2m-1}$ has a complex symplectic structure and a quaternionic structure: an antilinear involution $J$ with $J^2 = -1$. Then $\mathbf{S}^{4m-2} \subset \mathbf{S}^{2m-1} \otimes \mathbf{S}^{2m-1}$ is symmetric which places it in the Lie algebra of complex symplectic transformations. But it is also real and so commutes with $J$. In this case if a linear combination of the $a_i$ annihilates $v$ it annihilates $Jv$ so we again have a 2-dimensional kernel and the criterion is the vanishing of the resultant of two polynomials — $p$ and its transform $p^*$ by $J$ where $p^*(z) = z^{2m-1} p(-1/\bar{z})$. Then the resultant $R(p, p^*)$ is

$$(a_0 \bar{a}_{2m-1})^{2m-1} \prod_{i,j} (\lambda_i + \bar{\lambda}_j^{-1}) = (a_0 \bar{a}_{2m-1})^{2m-1} \prod_{i} (|\lambda_i|^2 + 1) \prod_{i<j} |\lambda_i \bar{\lambda}_j + 1|^2 \left( \prod_{j} \bar{\lambda}_j^{-1} \right)^{2m-1}$$

and since $\prod_j \bar{\lambda}_j = -\bar{a}_{2m-1}/a_0$ this expression is non-positive. Again the average is non-zero.
4.4 The case $Sp(n)$

The group $Sp(n) \subset SU(2n)$ is the subgroup which commutes with a quaternionic structure $J$ and we have just observed that the appropriate $V_i$ does just that, so that it lies in the Lie algebra $\mathfrak{sp}(n)$. The result follows from the previous section.

4.5 The case $Spin(7)$

Here the principal three-dimensional subgroup of $Spin(7)$ projects to the principal one in $SO(7)$. This is the irreducible representation $S_6$ and from the characters we deduce that the 8-dimensional spin representation is $1 \oplus S_6$. This means that the subgroup fixes a spinor and so lies in the stabilizer $G_2$.

The Lie algebra of $G_2$ decomposes as $S^2 \oplus S^{10}$ and so $\mathfrak{so}(7) = S^2 \oplus S^6 \oplus S^{10}$ with respect to the same 3-dimensional group. It follows that $S^6$ is the orthogonal complement of $g_2$. Translated around $Spin(7)$ this is the horizontal subspace for the fibration $p : Spin(7) \to S^7$. This is a Riemannian submersion so $p^* \omega$ is always non-zero on this subspace.

4.6 The case $Spin(9)$

The defining 9-dimensional representation is here $S^8$ and, from the characters again, the 16-dimensional spin representation is $S^{10} \oplus S^4$. In the Lie algebra $\mathfrak{so}(9) \cong \Lambda^2 S^8$ the 15-dimensional component is $S^{14}$ and we are concerned with its action on $S^{10} \oplus S^4$. Since $\Lambda^2(S^{10} \oplus S^4) \cong \Lambda^2(S^{10}) \oplus (S^{10} \otimes S^4) \oplus \Lambda^2 S^4$ there are copies of $S^{14}$ in the first two summands and the action is a linear combination of the two.

We consider again when a linear combination of $a_1, \ldots, a_{15} \in V_i$ has a non-trivial kernel. Suppose $(p, q) \in S^{10} \oplus S^4$ are polynomials in the kernel of $a \in S^{14}$ then we may write this as $(Ap + Bq, -B^T p) = 0$ where $a = (A, B) \in \Lambda^2(S^{10}) \oplus (S^{10} \otimes S^4)$. Now $B^T : S^4 \to S^{10}$ is given by the map

$$H^1(P^1, O(-16)) \otimes H^0(P^1, O(4)) \to H^1(P^1, O(-12))$$

as in Section 4.3 and $B$ by the map

$$H^1(P^1, O(-16)) \otimes H^0(P^1, O(10)) \to H^1(P^1, O(-6))$$

for a class $[\beta] \in H^1(P^1, O(-16)) \cong S^{14}$. If $p, q$ have a common zero then there exists $[\beta]$ with $Bp = 0, B^T q = 0$ represented by a class supported at a single point in $P^1$, 8
the common zero. If we take this point to be $z = 0$ then $[\beta]$ can be identified with the polynomial $z^{14} \in S^{14}$.

Consider now $A : S^{10} \to S^{10}$ defined by $z^{14}$. This consists of contracting in $S^{14} \otimes S^{10}$ seven pairs of terms and symmetrizing. If $p$ vanishes at 0, contraction with $z^{14}$ vanishes also. We deduce that the vanishing of the resultant $R(p, q)$ is a condition for the existence of $a \in V_i$ which annihilates $(p, q)$. This is a polynomial in the coefficients of degree $4 + 10 = 14$. But $Q_{2m-2}(v) = Q_{14}(v)$ in (1) is of degree 14 and so $Q_{14}$ is a multiple of the resultant of two real polynomials $u, v$ of even degrees 4, 10. As in Section 4.3 this is non-negative.

5 Conclusion

We have shown that in certain degrees and certain groups there exists an invariant form which is nonvanishing on $V_i$. This is true for $V_i$ for any $G$, where of course the Cartan three-form restricts non-trivially to any three-dimensional subgroup, not just the principal one. When $G$ has rank $\ell = 2$ we have $g = V_1 \oplus V_2$, an orthogonal decomposition, and the Hodge star of the Cartan 3-form calibrates $V_2$ so all cases are covered. Another example is the group $SU(4)$ which acts transitively on $S^7$ and also on $S^5$ under the homomorphism $SU(4) \to SO(6)$, identifying $SU(4)$ with $Spin(6)$, so we have forms in all degrees 3, 5, 7 in this case, but for higher rank the arguments in this article only relate to a restrictive number of forms.

6 Polyvector fields

We conclude with a brief discussion of the origin in [5] of the conjecture that for each subspace $V_i$ there is an invariant form $\varphi_i$ on $g$ which restricts nontrivially. The context is a Riemann surface $C$ of genus $g > 1$ and the moduli space $M$ of stable holomorphic principal $G\mathfrak{c}$-bundles $P$ on $C$ for a complex simple Lie group $G\mathfrak{c}$. The cotangent space at a point of $M$ is isomorphic to $H^0(C, \text{ad}(P) \otimes K)$ where $K$ is the canonical bundle and evaluating an invariant polynomial $p$ of degree $k$ defines a holomorphic section of $K^k$ on $C$. Taking the dual of $H^0(C, K^k)$ this yields a map $H^1(C, K^{1-k}) \to H^0(M, S^kT)$ which is well-known to be injective and to generate holomorphic sections of the symmetric powers $S^kT$ of the tangent bundle which commute using the Schouten-Nijenhuis bracket [4], or equivalently define Poisson-commuting functions on the cotangent bundle $T^*M$.

If we now use an invariant alternating form $\varphi$ of degree $d$ then evaluation yields a
section of $K^d$ and dually we have a map $H^1(C, K_{1-d}^d) \to H^0(M, \Lambda^dT)$ into the space of polyvector fields on $M$ and these also Schouten-commute [5]. However, whereas using the spectral curve one can see that in the symmetric case the map is injective, for the skew-symmetric case this is not apparent. Instead consider the $G^c$-bundle associated to a rank 2 stable bundle $V$ by the principal homomorphism $SL(2, C) \to G^c$ then we can restrict a form $\varphi_i$ to the subspace $H^0(C, S^{2\lambda_i} V \otimes K) \subset H^0(C, \text{ad}(P) \otimes K)$. By Riemann-Roch this has dimension $(2\lambda_i + 1)(g - 1)$ so if the conjecture held then choosing $n = 2\lambda_i + 1$ holomorphic sections $s_j$ with $s_1 \wedge s_2 \wedge \cdots \wedge s_n$ not identically zero, we could deduce that $\varphi_i$ gives a nonzero section of $K^{d_i}$. There may of course be simpler ways of achieving this.

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