On quadratic multidimensional type-I BSVIEs, infinite families of BSDEs and their applications

Camilo HERNÁNDEZ*
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Abstract

This paper investigates multidimensional extended type-I BSVIEs and infinite families of BSDEs in the case of quadratic generators. We establish existence and uniqueness results in the case of fully quadratic as well as Lipschitz-quadratic quadratic generators. We also present and discuss a type of flow property satisfied by this family of BSVIEs. As a preliminary step, we establish the well-posedness of a class of infinite families of BSDEs, as introduced in Hernández and Possamaï [20], which are of interest in their own right. Our approach relies on the strategy developed by Tevzadze [49] for quadratic BSDEs and the treatment of Lipschitz extended type-I BSVIEs in [20]. We motivate their analysis of both of these objects by a series of practical applications.

Key words: Backward stochastic Volterra integral equations, representation of partial differential equations, infinite families of BSDEs, Backward stochastic differential equations, risk-sensitive nonzero sum games, time inconsistency.

1 Introduction

This paper studies type-I extended backward stochastic Volterra integral equations, BSVIEs for short, as recently revisited in Hernández and Possamaï [20]. Let $X$ be the solution to a drift-less stochastic differential equation, SDE for short, under a probability measure $\mathbb{P}$, $\mathcal{F}$ be the $\mathbb{P}$-augmentation of the filtration generated by $X$, see Section 2.1 for details. The data of the problem corresponds to a collection of $\mathcal{F}_T$-measurable random variables $(\xi(t))_{t \in [0,T]}$, referred in the literature of BSVIEs as the free term, as well as a generator $g$. A solution to a type-I BSVIEs corresponds to a tuple $(Y, Z, N)$, of appropriately $\mathbb{F}$-adapted and integrable processes, satisfying

$$Y_t = \xi(s) + \int_s^T g_r(s, X, Y_r^s, Z_r^s, Y_r^s, Z_r^s)dr - \int_s^T Z_r^s dX_r - \int_s^T dN_r^s, \quad t \in [0, T], \mathbb{P} - a.s., s \in [0, T]. \quad (1.1)$$

The noticeable feature of (1.1) is the appearance of the ‘diagonal’ processes $(Y_r^s)_{t \in [0,T]}$ and $(Z_r^s)_{t \in [0,T]}$ in the generator. A prerequisite for rigorously introducing these processes is some regularity of the solution. Indeed, the regularity of $s \mapsto (Y^s, Z^s)$ in combination with the pathwise continuity of $Y$ and the introduction of a derivative of $s \mapsto Z^s$, as proposed in [20], are sufficient for the analysis. We also remark that, as we work with a general filtration $\mathcal{F}$, the additional process $N$ corresponds to a martingale process which is $\mathbb{P}$-orthogonal to $X$. In this work, we focus on the, to the best of knowledge, not studied case of multidimensional type-I BSVIEs with quadratic generators. Notably, we extend the analysis in [20] to the multidimensional quadratic case by exploiting the fact that the well-posedness of (1.1) is equivalent to that of an infinite family of backward stochastic differential equations, BSDEs for short.

The study of type-I BSVIEs began with the following set up: on a probability space supporting a Brownian motion $B$, one seeks for a pair $(Y, Z)$ of processes such that

$$Y_t = \xi(t) + \int_t^T g_r(t, Y_r, Z_r^t)dr - \int_t^T Z_r^t dB_r, \mathbb{P} - a.s., t \in [0, T]. \quad (1.2)$$

The first mention of such equations is, to the best of our knowledge, due to Hu and Peng [21] in the context of Hilbert-valued BSDEs, see the discussion following Remark 1.1 therein. Two decades later, Lin [40] considered (1.2) in the case

* Columbia University, IEOR department, USA, camilo.hernandez@columbia.edu. Author supported by the CKGSB fellowship.
\(\xi(t) = \xi, \ t \in [0, T]\). The general study of type-I BSVIEs (1.2) is due to Yong [58, 59]. For completeness, we remark that the concept of type-II BSVIEs, where the term \(Z^r_t\) is also present in the generator, was also been studied in the literature. Type-II BSVIEs are beyond the scope of this paper and we refer the reader the interested reader to [58; 59]. Note that BSDEs correspond to the case in which the data does not depend on the new parameter, i.e.

\[
Y_t = \xi + \int_t^T g_r(Y_r, Z_r)dr - \int_t^T Z_r dB_r, \ t \in [0, T], \ \mathbb{P}-\text{a.s.,}
\]

for which the seminal works of Pardoux and Peng [44] and El Karoui, Peng, and Quenez [12] introduced a systematic treatment and collected a wide range of their properties. Among such properties we recall the so-called flow property, that is to say, for any \(0 \leq r \leq T\),

\[
Y_t(T, \xi) = Y_t(r, Y_r(T, \xi)), \ t \in [0, r], \ \mathbb{P}-\text{a.s.,}
\]

and \(Z_t(T, \xi) = Z_t(r, Y_r(T, \xi))\), \(dt \otimes d\mathbb{P}\)-a.e. on \([0, r] \times \Omega\),

where \((Y(T, \xi), Z(T, \xi))\) denotes the solution to the BSDE with terminal condition \(\xi\) and final time horizon \(T\). We highlight that, without additional assumptions, a solution to a general BSVIE does not satisfy the flow property.

Extended type-I BSVIEs (1.1) provide a rich framework to address new problems in mathematical finance and control. For instance, as initially suggested in Wang and Yong [54], BSVIEs appear in time-inconsistent control problems via either Bellman’s and Pontryagin’s principles, see Yong [60] and Wei, Yong, and Yu [56], respectively. A link was then made rigorous independently by Wang and Yong [52, Section 5] and Hernández and Possamaï [19, Lemma A.2.3]. Although following different approaches, both analyses lead to introduce type-I BSVIEs in which the diagonal of \(Z\) appears in the generator. Likewise, the case of cost functionals given by the \(Y\) component of a type-I BSVIE (1.2), in which \(g\) depends on a control was studied in Hamaguchi [17]. The adjoint equation induced by Pontryagin’s optimal principle solves an type-I BSVIE in which the diagonal of \(Y\) appears in the generator, see also Wang [51].

We remark that, to the best of our knowledge, there are no well-posedness results for multidimensional quadratic type-I BSVIEs (1.2), let alone for extended ones (1.1). In fact, to the best of our knowledge, the study of non-Lipschitz BSVIEs remains limited to Ren [46], Shi and Wang [47], Wang, Sun, and Yong [53], and Wang and Zhang [55]. In [55] and [46], the authors consider solutions to general multidimensional type-I BSVIEs where the generator is increasing and concave in \(y\) and Lipschitz in \(z\). [47] continued the study and settled some flaws in the analysis of [46]. On the other hand, [53] presents the first analysis of scalar BSVIEs whose generator have quadratic growth on \(z\). Indeed, the authors consider a standard one dimensional type-I BSVIE (1.2) in which the generator is Lipschitz in \(y\) and quadratic in \(z\), which we will refer to as the Lipschitz quadratic case, provided the data of the BSVIE is bounded.

We emphasise that the additional assumptions in [53] are due to underlying employed results for scalar quadratic BSDEs.\(^1\) Indeed, extending the ideas in [3; 4] was the strategy behind [53] and it explains the framework of their result, i.e. scalar BSVIEs with Lipschitz quadratic generator and bounded data. Moreover, as [53] states: “The case [of] \(Y\) being higher dimensional will be significantly different in general.” Therefore, in the multidimensional case, new approaches become necessary as tools that are usually used in the analysis of scalar BSDEs, like monotone convergence or Girsanov transform, are no longer available. In fact, Frei and Dos Reis [16], provide a simple example of a multidimensional quadratic BSDE with a bounded terminal condition for which there is no solution. This counterexample shows that a direct generalization of the approaches in [33; 3; 4] would be unsuccessful in the case of extended multidimensional quadratic type-I BSVIEs.

Beyond imposing structural conditions on the generator, the literature on multidimensional quadratic BSDE provides general well-posedness results exploiting the theory of BMO martingales or focusing on Markovian BSDEs.\(^2\) The result in Harter and Richon [18] approximates the solution of a Lipschitz quadratic BSDE, assuming the \emph{a priori} existence of uniform estimates on the BMO norm of the local martingale \(\int_0^T Z^n dB_r\) and exploiting the theory of Malliavin calculus to pass to the limit. Concerning Markovian BSDEs, Xing and Žitković [57] focuses on a class of Markovian systems whose generator satisfy an abstract structural condition. However, neither of these approaches extends properly when considering extended type-I BSDEs. The crux of the problem lies in the fact that (1.1) allows for generators in which the diagonal of both \(Y\) and \(Z\) appear in the generator. Moreover, the approach presented in [20] for Lipschitz extended type-I BSVIEs leverages suitable estimates for both of these processes to establish a fixed point argument.

\(^1\)We recall that the analysis of scalar quadratic BSDEs is much more delicate. The first result, due to Kobylanski [33], recently revisited by Jackson and Žitković [25], holds for bounded and Lipschitz quadratic data, and was extended to the super quadratic case in Lepeltier and San Martín [39] and Delbaen, Hu, and Bao [7]. Briand and Hu [3; 4] showed that imposing sufficiently large exponential moments \(\xi\) is enough.

\(^2\)See Cheridito and Nam [6] for specific choices of generators. For triangular and diagonally quadratic generator see Jackson and Žitković [26], Hu and Tang [23], Hu, Tang, and Wang [24], Jamneshan, Kupper, and Luo [28] and Kupper, Luo, and Tangpi [37], and Luo [41, 42]. For completeness, see also Frei [15] and Kramkov and Pulido [35].
On the other hand, the original method introduced in Tevzadze \cite{49} takes a different view of this problem and presents a fixed-point argument that is able to cover quadratic BSDEs, in both $y$ and $z$, but requires, once again, the data to be bounded and sufficiently small. We stress that, unlike the approaches of \cite{18} or \cite{57}, the approach in \cite{49} works for quadratic BSDEs. In light of our discussion at the end of the previous paragraph, this methodology can be reconcile with the analysis in \cite{20} to obtain a well-posedness result for multidimensional quadratic extended type-I BSVEIs. This constitute the methodological motivation of our approach.

To be able to cover multidimensional quadratic type-I BSVEIs as general as (1.1), following the ideas in \cite{20}, our approach is based on the fact that, for an appropriate choice of data, its well-posedness is equivalent to that of the system of infinite families of BSDEs of the form

$$
\begin{align*}
\mathcal{Y}_t &= \xi(T) + \int_t^T h_r(X, \mathcal{Y}_r, Z_r, \mathcal{Y}_r^r, Z_r^r, \partial \mathcal{Y}_r^r)dr - \int_t^T Z_r^r dX_r - \int_t^T d\mathcal{N}_r, \\
\mathcal{Y}_t^a &= \eta(s) + \int_t^T g_r(s, X, \mathcal{Y}_r^a, Z_r^a, \mathcal{Y}_r, Z_r)dr - \int_t^T Z_r^a dX_r - \int_t^T d\mathcal{N}_r^a, \\
\partial \mathcal{Y}_t^a &= \partial_s \eta(s) + \int_t^T \nabla g_r(s, X, \partial \mathcal{Y}_r^a, \partial Z_r^a, \mathcal{Y}_r, Z_r)dr - \int_t^T \partial Z_r^a dX_r - \int_t^T \partial d\mathcal{M}_r^a.
\end{align*}
$$

for unknown $(\mathcal{Y}, \mathcal{Z}, \mathcal{N}, Y, Z, N, \partial \mathcal{Y}, \partial \mathcal{Z}, \partial \mathcal{N})$ required to have appropriate integrability, see Section 3 for details.

The rest of the paper is organised as follows: To motivate the results of this document, i.e. the study of the well-posedness of infinite families of BSDEs (1.3) and type-I extended BSVEIs (1.1), this introductory section closes with three practical applications of our results in the following. Section 2 introduces the problem’s set up and the appropriate integrability spaces for our analysis. Section 3 presents our well-posedness result for infinite families of multidimensional BSDEs in both the linear quadratic and quadratic whose proofs are deferred to Section 6 and Section 7, respectively. Section 4 establishes the equivalence of the well-posedness of type-I extended BSVEIs with that of a system of the form as those studied in Section 3. We close our study of BSVEIs in Section 5 where we discuss on the nature of the flow property for type-I extended BSVEIs. Lastly, some auxiliary results are presented in the Appendix section.

1.1 Practical motivations

(i) An immediate practical motivation comes from the study of time-inconsistent control problems. In this kind of problems the idea of optimal controls is incompatible with the underlying agent’s preferences and a successful concept of solution is that of consistent plans or equilibria, as initially introduced in Ekeland and Lazrak \cite{9, 10}. This approach is known as that of the sophisticated agent. The analysis in \cite{19}, limited to the Lipschitz case, established the connection with type-I BSVEIs via an extended dynamic principle. For completeness, we precise the dynamics of the controlled process $X$ in a Markovian framework. Let $\sigma_1$ be bounded, and $A := [a_1, a_2] \subseteq \mathbb{R}$ so that

$$
X_t = x_0 + \int_0^t \sigma_1 r - X_r dr + \int_0^t \sigma_2 dW^a_r, \quad t \in [0, T], \quad \mathbb{P}^a\text{-a.s.}
$$

for some $A$-valued process $\alpha$ denoting the agent’s action. $\mathbb{P}^a$ denotes a probability measure governing the distribution of the canonical process $X$ which the agent controls and $W^a$ denotes a $\mathbb{P}^a$–Brownian motion, see Section 2.1. We recall that $\mathbb{P}^a$ is guarantee to exists as the previous Lipschitz SDE has a unique strong solution. Note that $X_T$ is bounded as at $t = T$, we have that $X_T = \alpha_T \in [a_1, a_2]$. Moreover, for real valued $k$, $F$, and $G$ with appropriate, the reward of an agent performing $\alpha$ from time $t$ onwards and current state value $x \in \mathcal{X}$, is given by

$$
J(t, x, \alpha) := \mathbb{E}^\alpha \left[ \int_0^T k_r(t, X_r, \alpha)dr + F(t, X_T) \bigg| \mathcal{F}_t \right] + G(t, \mathbb{E}^\alpha [X_T | \mathcal{F}_t]),
$$

where $\mathbb{E}^\alpha [\cdot | \mathcal{F}_t]$ denote the classic conditional expectation operator under $\mathbb{P}^a$. The noticeable features of this type of rewards are: (a) the dependence of $k$, $F$ and $G$ on $t$ which besides the case of exponential discounting is a source of time-inconsistency; (b) the possible non-linear dependence of $G$ on a conditional expectation of $g(X_T)$, another source of time-inconsistency, which would allow for mean-variance type of criteria.

Following the analysis in \cite{19}, let $\xi(s, x) := F(s, x) + G(s, g(x))$, $g_l(s, x, z, a) := k_l(s, a) + \frac{1}{2} \sigma^2 \sigma^T \sigma z^T \sigma G(t, n)$,

$$
H_t(x, z, u, n, z) := \sup_{\alpha \in A} \left\{ g_l(t, x, z, a) \right\} - u - \partial_s G(t, n) - \frac{1}{2} z^T \sigma u \sigma z \partial^2 n_n G(t, n),
$$

3
and denote by \(a^*(t, x, z)\) the \(A\)-valued measurable mapping attaining the sup in \(H\), assumed to exists. Then, we find that agent’s value function associated to an equilibrium action \(\alpha^*\) correspond to \(\mathcal{V}_t^i\) and \(\alpha^*_t := a^*(t, X_t, Z_t)\), respectively, where
\[
\mathcal{V}_t = \xi(T, X_T) + \int_t^T H_r(X_r, Z_r, \partial Y^*_r, \tilde{N}_r, \tilde{Z}_r)dr - \int_t^T Z^*_r \cdot dX_r,
\]
\[
Y^*_t = \xi(s, X_T) + \int_t^T g_r(s, X_r, Z^*_r, a^*(r, X_r, Z_r))dr - \int_t^T Z^*_r \cdot dX_r,
\]
\[
\tilde{N}_t = g(X_T) + \int_t^T b_r(X_r, a^*(r, X_r, Z_r)) \cdot \sigma^*_r \tilde{Z}_r dr - \int_t^T \tilde{Z}_r \cdot dX_r.
\]

We remark that \(\partial Y\) is defined as in Section 3. Moreover, we highlight that: (a) \(H\) is quadratic in \(z\) and, if, for instance, \(G(s, n) = \phi(s)n^2\), it is quadratic in \(n\) too; (b) the appearance of \(\tilde{N}\) and \(\tilde{Z}\) in the first equation leads to a multidimensional system even if \(Y\) is real values; (c) having access to a well-posedness result for the previous system guarantees the existence and uniqueness of an equilibrium strategy, see [19]. This is particularly important in light of the example, stemming from a mean-variance investment problem, in [38] in which uniqueness of the equilibrium fails. We direct to Section 3 and Theorem 3.6 for details on the following result. We also mention that by construction Assumption B is satisfied.

**Proposition 1.1.** Let Assumption A hold. Suppose \(x \mapsto (\xi(t, x), \partial_x \xi(s, x))\) is monotone and continuous, and Assumption C holds for \(\kappa = 10\). Then, the previous system has a unique solution and there is a unique equilibrium associated to the time-inconsistent control problem faced by the sophisticated time-inconsistent agent.

(ii) Our next motivation comes from the risk-sensitive non-Markovian nonzero-sum game introduced in El Karoui and Hamadène [11] and revisited in [23]. This is a situation in which many individuals are allowed to intervene on a system \(X\), but contrary to the zero-sum case, their preferences are not necessarily antagonistic. In fact, we allow each one to look after her own interest. It was established in [11, Proposition 5.1] that the resolution of this game problem is obtained via multidimensional quadratic BSDE. Said equation is of multidimensional type since there are several players and each one has its own associated reward functional. As first noted in [11], and settled in [18], the existence of a solution for multidimensional BSDEs renders whether the game has an equilibrium point. However, an extension of this model to the time-inconsistent control problem faced by the sophisticated time-inconsistent agent.

\[
\int_t^T \nu \left( g^i(T - t) \xi^i - \int_t^T g^i(r - s) k^i_r (X_{\land r}, \alpha_r, \alpha_r^{-i}) dr \right) dF_t.
\]

In this setting, we obtain that the existence to an equilibrium solution to the nonzero-sum game between time-inconsistent agents is associated to the well-posedness of the \(n\)-dimensional BSIE given by
\[
Y^*_t = \xi + \int_t^T g_r(s, X_{\land r}, Z^*_r, a^*(r, X_{\land r}, Z^*_r))dr - \int_t^T Z^*_r \cdot dX_r.
\]

As in the previous example, our result guarantee the well-posedness of the previous BSIEs. Let us further remark that this class of models can be tailored to cover applications to: financial market equilibrium problems for interacting agents, as in Bielag, Lionet, and Dos Reis [2], Espinosa and Touzi [14, 16, and 15]; price impact models, as in [35] and [34]; and Principal-Agent contracting problems with competitive agents as in Elie and Possamaï [13] to mention just a few.

(iii) One additional motivation for our results builds upon the treatment presented in [53] of scalar continuous-time dynamic risk measures in [12]. This is, it is possible to extending these ideas to the more realistic situation where the risky portfolio is vector-valued? In the static case, Jouini, Meddeb, and Touzi [29] provides a notion of multidimensional coherent risk measure that renders a convenient extension of the real-valued risk measures initially introduced in Artzner, Delbaen, Eber, and Heath [1]. Building upon these ideas, Kulikov [36] presents a model that, for instance, can accommodate the risks of changing currency exchange rates and transaction costs. More generally, exploiting the partial order induced by a given convex cone \(K \subseteq \mathbb{R}^n\), namely, \(y \preceq_K y' \iff y' - y \in K\) for \(y, y' \in \mathbb{R}^n\), the author introduces multidimensional coherent risk measures that extend classic one-dimensional risk measures such as the tail \(V@R\), and weighted \(V@R\).

Let us remark that at the heart of the treatment of continuous-time dynamic risk measures in [53] lies the access to a comparison theorem, which the authors recover for one-dimensional type-I BSIEs (1.2). We emphasise that, as for BSDEs, this is a much harder task in the multidimensional case. Nevertheless, the positive result presented in Hu and Peng [22] for multidimensional BSDEs makes perfectly clear that this follows from the study of the so-called viability property
for (multidimensional) BSDEs as presented, for instance, in Buckdahn, Quincampoix, and Răşcanu [5]. We recall that the approach in [5] is based on the convexity of the distance function induced by $\mathcal{K}$, where, in general, $\mathcal{K}$ can be taken to be any closed convex set. This is certainly the case whenever $\mathcal{K}$ is a convex cone. All things considered, the crux of the problem lies in establishing the appropriate extension of the viability property for multidimensional type-I BSVIEs. One this is available, as shown in [22], one should access a comparison theorem thus answering the aforementioned question.

2 Preliminaries

Notations: we fix a time horizon $T > 0$. Given $(E, |\cdot|)$ a finite-dimensional Euclidean space, a positive integer $d$, and a non-negative integer $q$, $C^q_d(E)$ (resp. $C^q_d(E)$) will denote the space of functions from $E$ to $\mathbb{R}^d$ which are $q$ times continuously differentiable (resp. and bounded with bounded derivatives). When $d = 1$ we write $C_q(E)$ and $C_q,b(E)$. For $\phi \in C_{0,q}([0, T] \times E)$ with $q \geq 2$, if $s \mapsto \phi(s, \alpha)$ is uniformly continuous uniformly in $\alpha$, we denote by $\rho_\phi : [0, T] \longrightarrow \mathbb{R}$ its modulus of continuity. $\partial_\alpha \phi$ and $\partial^2_{\alpha \alpha} \phi$ denote the gradient and Hessian with respect to $\alpha$, respectively. For $(u, v) \in (\mathbb{R}^p)^2$, $u \cdot v$ will denote their usual inner product, and $|u|$ the corresponding norm. $\mathbb{S}^n_+(\mathbb{R})$ denotes the set of $n \times n$ symmetric positive semi-definite matrices, while $\text{Tr}[M]$ denotes the trace of $M \in \mathbb{R}^{n \times n}$, and $|M| := \sqrt{\text{Tr}[M^T M]}$ for $M \in \mathbb{R}^{m \times n}$.

For $(\Omega, \mathcal{F})$ a measurable space, $\text{Prob}(\Omega)$ denotes the collection of probability measures on $(\Omega, \mathcal{F})$. For a filtration $\mathcal{F} := (\mathcal{F}_t)_{t \in [0, T]}$ on $(\Omega, \mathcal{F})$, $\mathcal{P}_{\text{pred}}(E, \mathcal{F})$ (resp. $\mathcal{P}_{\text{proj}}(E, \mathcal{F})$, $\mathcal{P}_{\text{opt}}(E, \mathcal{F})$, $\mathcal{P}_{\text{mean}}(E, \mathcal{F})$) denotes the set of $E$-valued, $\mathcal{F}$-predictable processes (resp. $\mathcal{F}$-progressively measurable processes, $\mathcal{F}$-optional processes, $\mathcal{F}$-adapted and measurable). When $\mathcal{F}$ is fixed we remove the dependence, e.g. we write $\mathcal{P}_{\text{opt}}(E)$ for $\mathcal{P}_{\text{opt}}(E, \mathcal{F})$. For $\mathcal{F} \in \text{Prob}(\Omega)$, $\mathcal{F}^\mathbb{F} := (\mathcal{F}_t^\mathbb{F})_{t \in [0, T]}$, denotes the $\mathbb{F}$-augmentation of $\mathcal{F}$. With this, $(\Omega, \mathcal{F}, \mathcal{F}^\mathbb{F}, \mathbb{P})$ can be extended to a complete probability space, see Karatzas and Shreve [30, Chapter II.7]. $\mathcal{F}^\mathbb{F}_t$ denotes the right limit of $\mathcal{F}^\mathbb{F}$, so that $\mathcal{F}^\mathbb{F}_0$ is the minimal filtration that contains $\mathcal{F}$ and satisfies the usual conditions. For $\{s, t\} \subseteq [0, T]$, with $s \leq t$, $\mathcal{T}_s,t(\mathbb{F})$ denotes the collection of $[t, T]$-valued $\mathbb{F}$-stopping times.

2.1 The stochastic basis on the canonical space

We fix two positive integers $n$ and $m$, which represent respectively the dimension of the martingale which will drive our equations, and the dimension of the Brownian motion appearing in the dynamics of the former. We consider the canonical space $\mathcal{X} := \mathcal{C}([0, T], \mathbb{R}^n)$, with canonical process $X$. We let $\mathcal{F}$ be the Borel $\sigma$-algebra on $\mathcal{X}$ (for the topology of uniform convergence), and we denote by $\mathcal{F}^\mathbb{F} := (\mathcal{F}_t^\mathbb{F})_{t \in [0, T]}$ the natural filtration of $X$. We fix a bounded Borel measurable map $\sigma : [0, T] \times \mathcal{X} \longrightarrow \mathbb{R}^{n \times m}$, $\sigma(X) \in \mathcal{P}_{\text{meas}}(\mathbb{R}^{n \times m}, \mathcal{F}^\mathbb{F})$, and an initial condition $x_0 \in \mathbb{R}^n$. We assume there is $\mathcal{P} \in \text{Prob}(\mathcal{X})$ such that $\mathcal{P}[X_0 = x_0] = 1$ and $X$ is martingale, whose quadratic variation, $(X) = ((X)_t)_{t \in [0, T]}$, is absolutely continuous with respect to Lebesgue measure, with density given by $\sigma \sigma^\top$. Enlarging the original probability space, see Stroock and Varadhan [48, Theorem 4.5.2], there is an $\mathbb{R}^m$-valued Brownian motion $B$ with

$$X_t = x_0 + \int_0^t \sigma_r(X_r)dB_r, \ t \in [0, T], \ \mathbb{P}-\text{a.s.}$$

We now let $\mathcal{F} := (\mathcal{F}_t)_{t \in [0, T]}$ be the (right-limit) of the $\mathbb{P}$-augmentation of $\mathcal{F}^\mathbb{F}$. We stress that we will not assume $\mathbb{P}$ is unique. In particular, the predictable martingale representation property for $(\mathcal{F}, \mathbb{P})$-martingales in terms of stochastic integrals with respect to $X$ might not hold.

Remark 2.1. We remark that the previous formulation on the canonical is by no means necessary. Indeed, any probability space supporting a Brownian motion $B$ and a process $X$ satisfying the previous SDE will do, and this can be found whenever that equation has a weak solution.

2.2 Functional spaces and norms

We now introduce the spaces of interest for our analysis. In the following, $(\Omega, \mathcal{F}_T, \mathcal{F}, \mathbb{P})$ denotes the filtered probability space as defined in the introduction. We are given a non-negative real number $c$ and $(E, |\cdot|)$ a finite-dimensional Euclidean space, i.e. $E = \mathbb{R}^d$ for some non-negative integer $d$ and $|\cdot|$ denotes the $L^2$-norm. We also introduce the $\mathcal{L}^\infty$-norm which for an arbitrary $E$-valued random variable $\zeta$ is given by $\|\zeta\|_\infty := \inf\{C \geq 0 : |\zeta| \leq C, \ \mathbb{P}-\text{a.s.}\}$ as well as the spaces

- $\mathcal{L}^{\infty, c}(E)$ of $\zeta \in \mathcal{P}_{\text{meas}}(E, \mathcal{F}_T)$ $\mathbb{P}$-essentially bounded, such that $\|\zeta\|_{\mathcal{L}^{\infty, c}} := \|e^{\frac{2}{c}} T \zeta\|_\infty < \infty$;
- $\mathcal{S}^{\infty, c}(E)$ of $Y \in \mathcal{P}_{\text{opt}}(E)$, with $\mathbb{P}$-a.s. càdlàg paths on $[0, T]$ and $\|Y\|_{\mathcal{S}^{\infty, c}} := \sup_{t \in [0, T]} e^{\frac{2}{c}} |Y_t| < \infty$;
• $S^2,\mathcal{E}(E)$ of $Y \in \mathcal{P}_{\text{opt}}(E)$, with $\mathbb{P}$-a.s. càdlàg paths on $[0, T]$ and $\|Y\|_{S^2,\mathcal{E}} := \mathbb{E}\left[\sup_{t \in [0, T]} e^{\xi t}|Y_t|^2\right] < \infty$;

• $L^{1,\infty,\mathcal{E}}(E)$ of $Y \in \mathcal{P}_{\text{opt}}(E)$ with $\|Y\|_{L^{1,\infty,\mathcal{E}}} := \left\|\int_0^T e^{\xi t}|Y_t|\, dt\right\|_{\infty} < \infty$;

• $H^{2,\mathcal{E}}(E)$ of $Z \in \mathcal{P}_{\text{pred}}(E)$, which are defined $\sigma\sigma_t^T dt$-a.e., with $\|Z\|^2_{H^{2,\mathcal{E}}} := \mathbb{E}\left[\int_0^T e^{\xi r}|\sigma_r Z_r|^2\, dr\right] < \infty$;

• $\mathbb{BMO}^{2,\mathcal{E}}(E)$ of square integrable $E$-valued $(\mathbb{F}, \mathbb{P})$-martingales $M$ with $\mathbb{P}$-a.s. càdlàg paths on $[0, T]$ and

$$\|M\|^2_{\mathbb{BMO}^{2,\mathcal{E}}} := \sup_{\tau \in [0, T]} \mathbb{E}\left[\left(\int_{\tau}^T e^{\xi r}dM_r\right)^2\right]_{\mathcal{F}_T} < \infty$$;

• $\mathbb{BMO}^{2,\mathcal{E}}(E)$ of $Z \in \mathcal{P}_{\text{pred}}(E)$, which are defined $\sigma\sigma_t^T dt$-a.e., with $\|Z\|^2_{\mathbb{BMO}^{2,\mathcal{E}}} := \left\|\int_0^T Z_r dX_r\right\|^2_{\mathbb{BMO}^{2,\mathcal{E}}} < \infty$;

• $M^{2,\mathcal{E}}_\mathbb{BMO}(E)$ of càdlàg martingales $N \in \mathcal{P}_{\text{opt}}(E)$, $\mathbb{P}$-orthogonal to $X$ (that is the product $XN$ is an $(\mathbb{F}, \mathbb{P})$-martingale), with $N_0 = 0$ and $\|N\|^2_{\mathbb{BMO}^{2,\mathcal{E}}} := \|N\|^2_{\mathbb{BMO}^{2,\mathcal{E}}} < \infty$;

• $M^{2,\mathcal{E}}(E)$ of càdlàg martingales $N \in \mathcal{P}_{\text{opt}}(E)$, $\mathbb{P}$-orthogonal to $X$, $N_0 = 0$ and $\|N\|^2_{\mathbb{BMO}^{2,\mathcal{E}}} := \mathbb{E}\left[\int_0^T e^{\xi r}d[N]_r\right] < \infty$;

• $\mathcal{P}^{2,\mathcal{E}}(E, \mathcal{F}_T)$ of two parameter processes $(U^t_s)_{s \in [0, T]} : ([0, T]^2 \times \Omega, \mathcal{B}([0, T]) \otimes \mathcal{F}_T) \rightarrow (E, \mathcal{B}(E))$ measurable.

• $\mathcal{C}^{\infty,\mathcal{E}}(E)$ denotes the space of collections $\eta := (\eta(s))_{s \in [0, T]} \in \mathcal{P}^{2,\mathcal{E}}(E, \mathcal{F})$ such that the mapping $([0, T] \times \Omega, \mathcal{B}([0, T]) \otimes \mathcal{F}) \rightarrow (\mathcal{L}^{\infty,\mathcal{E}}(E, \mathcal{F}_T), \|\cdot\|_{\mathcal{L}^{\infty,\mathcal{E}}})$ : $s \mapsto \eta(s)$ is continuous and $\|\eta\|_{\mathcal{L}^{\infty,\mathcal{E}}} := \sup_{s \in [0, T]} \|\eta(s)\|_{\mathcal{L}^{\infty,\mathcal{E}}} < \infty$.

• Given a Banach space $(\mathcal{F}(E), \|\cdot\|_{\mathcal{F}})$, we define $(\mathcal{C}^{\mathcal{E}}(E), \|\cdot\|_{\mathcal{C}^{\mathcal{E}}})$ the space of $U \in \mathcal{P}^{2,\mathcal{E}}(E, \mathcal{F}_T)$ such that $([0, T], \mathcal{B}([0, T])) \rightarrow (\mathcal{C}^{\mathcal{E}}(E), \|\cdot\|_{\mathcal{C}^{\mathcal{E}}})$ : $s \mapsto U^s$ is continuous and $\|U\|_{\mathcal{C}^{\mathcal{E}}} := \sup_{s \in [0, T]} \|U^s\|_{\mathcal{F}} < \infty$.

For example, $L^{1,\infty,\mathcal{C}^{\mathcal{E}}}(E)$ denotes the space of $U \in \mathcal{P}^{2,\mathcal{E}}(E, \mathcal{F}_T)$ such that $([0, T], \mathcal{B}([0, T])) \rightarrow (L^{1,\infty,\mathcal{E}}(E), \|\cdot\|_{L^{1,\infty,\mathcal{E}}})$ : $s \mapsto U^s$ is continuous and $\|U\|_{L^{1,\infty,\mathcal{E}}} := \sup_{s \in [0, T]} \|U^s\|_{L^{1,\infty,\mathcal{E}}} < \infty$;

• $\mathbb{H}^{2,\mathcal{E}}(E)$ of $(Z^t_s)_{(s,t) \in [0,T]^2} \in \mathcal{P}^{2,\mathcal{E}}(E, \mathcal{F}_T)$ such that $([0, T], \mathcal{B}([0, T])) \rightarrow (\mathbb{H}^{2,\mathcal{E}}(E), \|\cdot\|_{\mathbb{H}^{2,\mathcal{E}}})$ : $s \mapsto Z^s$ is absolutely continuous with respect to the Lebesgue measure, $\mathcal{Z} \in \mathbb{H}^{2,\mathcal{E}}(E)$, where $\mathcal{Z} := (Z^t_s)_{(s,t) \in [0,T]^2}$ is given by

$$Z^t_s := Z^T_s - \int_t^T \partial Z^r_s \, dr,$$

and $\|Z\|^2_{\mathbb{H}^{2,\mathcal{E}}} := \|Z\|^2_{\mathbb{H}^{2,\mathcal{E}}} < \infty$;

• $\mathbb{H}^{2,\mathcal{E}}_{\mathbb{BMO}}(E)$ of $(Z^t_s)_{(s,t) \in [0,T]^2} \in \mathcal{P}^{2,\mathcal{E}}(E, \mathcal{F}_T)$ such that $([0, T], \mathcal{B}([0, T])) \rightarrow (\mathbb{H}^{2,\mathcal{E}}_{\mathbb{BMO}}(E), \|\cdot\|_{\mathbb{H}^{2,\mathcal{E}}_{\mathbb{BMO}}})$ : $s \mapsto Z^s$ is absolutely continuous with respect to the Lebesgue measure, $\mathcal{Z} \in \mathbb{H}^{2,\mathcal{E}}_{\mathbb{BMO}}(E)$, where $\mathcal{Z} := (Z^t_s)_{(s,t) \in [0,T]^2}$ is given by

$$Z^t_s := Z^T_s - \int_t^T \partial Z^r_s \, dr,$$

and $\|Z\|^2_{\mathbb{H}^{2,\mathcal{E}}_{\mathbb{BMO}}} := \|Z\|^2_{\mathbb{H}^{2,\mathcal{E}}_{\mathbb{BMO}}} < \infty$.

Remark 2.2. (i) We remark that the first set of spaces in the previous list, but $M^{2,\mathcal{E}}_{\mathbb{BMO}}(E)$, are the corresponding weighted version of the classic spaces in the literature for BSDEs, which are recovered by taking $c = 0$. Such weighted spaces are known to be more suitable to handle existence results. Moreover, given our assumption of finite time horizon these spaces are known to be isomorphic for any value of $c$.
(ii) The second set of these spaces are weighted versions of suitable extensions of the classical ones, whose norms are tailor–made to the analysis of the systems we will study. Some of these spaces have been previously considered in the literature on BSDEs, see [20], [58] and [52]. Of particular interest are the spaces $\mathbb{H}^{2,c}(E)$ and $\mathbb{H}^{2,c}_{BMO}(E)$. Indeed, the space $\mathbb{H}^{2,c}(E)$ being closed implies $\mathbb{H}^{2,c}_{BMO}(E)$ is a closed subspace of $\mathbb{H}^{2,c}(E)$ and thus a Banach space. Let us recall that the space $\mathbb{H}^{2,c}(E)$ allows us to define a good candidate for $(Z^i_t)_{t \in [0,T]}$ as an element of $\mathbb{H}^{2,c}(E)$. Let $\tilde{\Omega} := [0,T] \times \mathcal{X}$, $\tilde{\omega} := (t,x) \in \tilde{\Omega}$ and

$$3_s(\tilde{\omega}) := Z^T_t(x) - \int_s^T \partial Z^t_r(x) dr, \ dt \otimes d\mathbb{P}-a.e. \ \tilde{\omega} \in \tilde{\Omega}, \ s \in [0,T],$$

so that the Radon–Nikodym property and Fubini’s theorem imply $3_s = Z^*, dt \otimes d\mathbb{P}$-a.e., $s \in [0,T]$. Lastly, as for $\tilde{\omega} \in \tilde{\Omega}$, $s \mapsto 3_s(\tilde{\omega})$ is continuous, we may define

$$Z^t_1 := Z^T_t - \int_t^T \partial Z^t_r dr, \ for \ dt \otimes d\mathbb{P}$-a.e. \ $(t,x) \in [0,T] \times \mathcal{X}$.

(iii) Lastly, we comment on our choice to introduce the spaces $\mathbb{M}^{2,c}_{BMO}(E)$ and $\mathbb{M}^{2,2,c}_{BMO}(E)$. Those familiar with the theory of BSDEs would recognise the integrability in $\mathbb{M}^{2,c}(E)$ as the typical one for orthogonal martingales. However, given the setting of this paper, one might argue whether it would be more natural to require a BMO–type of integrability, as the space $\mathbb{M}^{2,c}_{BMO}(E)$ does. This had been noticed since [49]. Therefore, a natural question is how requiring one specific type of integrability would quantitatively affect our well-posedness results.

### 2.3 Auxiliary inequalities

We list some useful inequalities. Young’s inequality states that for $\varepsilon > 0$, $2ab \leq \varepsilon a^2 + \varepsilon^{-1} b^2$. For any positive integer $n$ and any collection $(a_i)_{1 \leq i \leq n}$ of non–negative numbers it holds that

$$\left( \sum_{i=1}^n a_i \right)^2 \leq n \sum_{i=1}^n a_i^2. \quad (2.1)$$

A particularly useful inequality in our setting is obtained from the so–called energy inequality, see Meyer [43, Chapter VII. Section 6]. For a positive integer $p$ and a potential $X$, i.e. a positive right–continuous super–martingale s.t. $\mathbb{E}[X_t] \to 0$, $t \to \infty$, the $p$-th-energy is defined by

$$e_p(X_t) := \frac{1}{p} \mathbb{E}[(A^c_\infty)^p], \ \text{where} \ A \ \text{is the increasing, right–continuous process appearing in the Doob–Meyer decomposition of } X.$$ The $p$-energy inequality states that

$$e_p(X_t) \leq C^p, \ \text{whenever} \ |X_t| \leq C.$$

In our framework, it leads to obtain the following auxiliary inequalities, whose proof we present in Appendix A.

**Lemma 2.3.** Let $\hat{d}$ be a positive integer.

(i) Let $Z \in \mathbb{H}^{2,\hat{d}}_{BMO}(\mathbb{R}^{\hat{d} \times \hat{d}})$. Then,

$$\mathbb{E} \left[ \left( \int_0^T e^{cu} |\sigma^r_u Z^r_t |^2 dr \right)^p \right] \leq p! \| Z \|^p_{\mathbb{H}^{2,\hat{d}}_{BMO}} \quad (2.2)$$

(ii) Let $Z \in \mathbb{H}^{2,c}(\mathbb{R}^{\hat{d} \times \hat{d}})$, $Z = (Z^i_t)_{t \in [0,T]}$ and $c > 0$, $\varepsilon > 0$. Then, $\mathbb{P}$–a.s.

$$\int_t^T e^{cu} |\sigma^r_u Z^r_u |^2 du \leq \int_t^T e^{cu} |\sigma^r_u Z^r_u |^2 du + \int_t^T \int_t^T \varepsilon e^{cu} |\sigma^r_u Z^r_u |^2 + \varepsilon^{-1} e^{cu} |\sigma^r_u \partial Z^r_u |^2 du dr, \ t \in [0,T].$$

Moreover, for any $t \in [0,T]$

$$\mathbb{E}_t \left[ \left( \int_t^T e^{cu} |\sigma^r_u Z^r_u |^2 du \right)^2 \right] \leq 6 \left( (1 + T^2) \| Z \|^4_{\mathbb{H}^{2,\hat{d}}_{BMO}} + T^2 \| \partial Z \|^4_{\mathbb{H}^{2,\hat{d}}_{BMO}} \right)$$

$$\mathbb{E}_t \left[ \int_t^T e^{cu} |\sigma^r_u Z^r_u |^2 du \right] \leq (1 + T) \| Z \|^2_{\mathbb{H}^{2,\hat{d}}_{BMO}} + T \| \partial Z \|^2_{\mathbb{H}^{2,\hat{d}}_{BMO}}$$
3 The BSDE system

For a fixed integers $d_1$ and $d_2$ we are given jointly measurable mappings $h$, $g$, $\xi$ and $\eta$, such that for any $(y, z, u, v, \omega) \in \mathbb{R}^{d_1} \times \mathbb{R}^{n \times d_1} \times \mathbb{R}^{d_2} \times \mathbb{R}^{n \times d_2} \times \mathbb{R}^{d_2}$

$$
\begin{align*}
  h : [0, T] \times \mathcal{X} \times \mathbb{R}^{d_1} \times \mathbb{R}^{n \times d_1} \times \mathbb{R}^{d_2} \times \mathbb{R}^{n \times d_2} \times \mathbb{R}^{d_2} \rightarrow \mathbb{R}^{d_1}, \\
  g : [0, T]^2 \times \mathcal{X} \times \mathbb{R}^{d_2} \times \mathbb{R}^{n \times d_2} \times \mathbb{R}^{d_1} \times \mathbb{R}^{n \times d_1} \times \mathbb{R}^{d_2} \rightarrow \mathbb{R}^{d_2}, \\
  \xi : [0, T] \times \mathcal{X} \rightarrow \mathbb{R}^{d_1}, \\
  \eta : [0, T] \times \mathcal{X} \rightarrow \mathbb{R}^{d_2}.
\end{align*}
$$

Moreover, throughout this section we assume the following condition on $(\eta, g)$.

**Assumption A.** $(s, u, v, y, z) \mapsto g_t(s, x, u, v, y, z)$ (resp. $s \mapsto \eta(s, x)$) is continuously differentiable, uniformly in $(t, x, y, z)$ (resp. in $x$). Moreover, the mapping $Y_t : [0, T]^2 \times \mathcal{X} \times (\mathbb{R}^{d_2} \times \mathbb{R}^{n \times d_2})^2 \times \mathbb{R}^{d_1} \times \mathbb{R}^{n \times d_1} \rightarrow \mathbb{R}^{d_2}$ defined by

$$
\begin{align*}
  \nabla g_t(s, x, u, v, y, z) := \partial_s g_t(s, x, u, v, y, z) + \partial_u g_t(s, x, u, v, y, z)u + \sum_{i=1}^n \partial_{y_i} g_t(s, x, u, v, y, z)v_i,
\end{align*}
$$

satisfies $\nabla g(s, u, v, y, z) \in \mathcal{P}_{\text{prog}}(\mathbb{R}^{d_2}, \mathbb{F})$, $s \in [0, T]$. Set $(\tilde{h}, \tilde{g}, (s, \cdot), \nabla \tilde{g}, (s)) := (h, (\cdot, 0), g(s, \cdot, 0), \partial_s g(s, \cdot, 0)), \text{ for } 0 := (u, v, y, z)|_{(0, \ldots, 0)}$.

To ease the readability, in the rest of the document we will remove the dependence of the previous spaces on the underlying Euclidean spaces where the processes take value, i.e. we will write $\mathcal{S}^{\infty, c}$ for $\mathcal{S}^{\infty, c}(\mathbb{R}^{d_1})$.

Given $\mathcal{F}_T$–measurable $(\xi, \eta)$ and $h$ and $g$, with $\partial_s \eta$ and $\nabla \tilde{g}$ given by Assumption A, which we will refer as the data, we consider the following infinite family of BSDEs defined for $s \in [0, T]$, $\mathbb{P}$–a.s. for any $t \in [0, T]$ by

$$
\begin{align*}
  Y_t &= \xi(T, X, \xi T) + \int_t^T h_r(X, Y_r, Z_r, U_r, V_r, \partial U_r)dr - \int_t^T \tilde{Z}_r^d dX_r - \int_t^T dN_r, \\
  U_t^s &= \eta(s, X, \xi T) + \int_s^T g_r(s, X, U^s_r, V^s_r, Y_r, Z_r)dr - \int_t^T V^s_r dX_r - \int_t^T dM_r, \\
  \partial U_t^s &= \partial_s \eta(s, X, \xi T) + \int_s^T \nabla g_r(s, X, \partial U^s_r, \partial V^s_r, \partial U^s_r, V^s_r, Y_r, Z_r)dr - \int_t^T \partial V^s_r dX_r - \int_t^T dM_r, \\
  \mathcal{S}^c := \mathcal{S}^{\infty, c} \times \mathbb{H}^{2, c}_{\text{BMO}} \times \mathcal{M}^{2, c} \times \mathcal{S}^{\infty, 2, c} \times \mathbb{H}^{2, 2, c}_{\text{BMO}} \times \mathcal{M}^{2, 2, c} \times \mathcal{S}^{\infty, 2, c} \times \mathbb{H}^{2, 2, c}_{\text{BMO}} \times \mathcal{M}^{2, 2, c}
\end{align*}
$$

and $\|\cdot\|_{\mathcal{H}^c}$ denotes the respective induced norm.

**Remark 3.1.** (i) We highlight that system $(S)$ is fully coupled. This means that a solution to the system has to be determined simultaneously. Moreover, the reader might notice our choice to prevent the generator of the first BSDE to depend on the diagonal of $\partial \xi$. This is due to our interest, as in [20], to establish the connection between these systems and type-I BSVIEs (1.1). For this, the presence of the diagonal $\partial U$ plays a key role. It should be clear from our arguments and Lemma 2.3 that these can be easily extended to accommodate this case.

(ii) We remark that for any $c \geq 0$, the space $(\mathcal{H}^c, \|\cdot\|_{\mathcal{H}^c})$ is a Banach space. Indeed, it is clearly a normed space. Moreover, the fact that the spaces $\mathcal{S}^{\infty, c}$ and $\mathcal{M}^{2, c}$ are complete is classical in the literature. The completeness of the spaces $\mathbb{H}^{2, 2, c}_{\text{BMO}}$ and $\mathcal{M}^{2, 2, c}$, which are endowed with BMO–type norms, follows from Dellacherie and Meyer [8, Ch VII, Theorem 88]. Indeed, a dual space is always complete. Finally, the completeness extends clearly to spaces of the form $\mathcal{S}^{\infty, 2, c}, \mathbb{H}^{2, 2, c}_{\text{BMO}}$ and $\mathcal{M}^{2, 2, c}$.

(iii) We also remark that implicit in the definition of $\mathcal{H}^c$ is the fact that $\partial V$ coincides, $dt \otimes d\mathbb{P}$–a.e., with the density with respect to the Lebesgue measure of the application $s \mapsto V^s$ which appears in the definition of the space $\mathbb{H}^{2, c}_{\text{BMO}}$. This is for any $s \in [0, T]$

$$
V^s - V^0 = \int_0^s \partial V^r dr, \text{ in } \mathbb{H}^{2, c}_{\text{BMO}}.
$$

This contrasts with the result in the Lipschitz case obtained in [20] where this was a consequence of the result, see also [20, Remark 4.2]. The reason for this is that the quadratic growth of the generator is incompatible with the contraction specified in the proof of [20, Theorem 3.5].
We now state precisely what we mean by a solution to \((S)\).

**Definition 3.2.** A tuple \(\mathfrak{h} = (Y, Z, N, U, V, M, \partial U, \partial V, \partial M)\) is said to be a solution to \((S)\) with terminal condition \((\xi, \eta)\) and generators \((f, g)\) under \(\mathbb{P}\), if \(\mathfrak{h}\) satisfies \((S)\) \(\mathbb{P}\)-a.s., and, \(\mathfrak{h} \in \mathcal{H}^c\) for some \(c > 0\).

For \(c > 0\) and \(R > 0\), we define \(\mathcal{B}_R \subseteq \mathcal{H}^c\) to be the subset of \(\mathcal{H}^c\) of processes \((Y, Z, N, U, V, M) \in \mathcal{H}^c\) such that
\[
\|(Y, Z, N, U, V, M)\|^2_{\mathcal{H}^c} \leq R^2.
\]

The need to introduce \(\mathcal{B}_R\) is inherent to the quadratic growth nature of \((S)\). Systems of the type of \((S)\) were recently studied in a Lipschitz framework in [19]. By choosing a weight \(c\) large enough, and exploiting the fact that all weighted norms are equivalent, the authors of [19] are able to obtain the well-posedness of \((S)\) in the space \(\mathcal{H}^2 := \mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{M}^2 \times \mathbb{S}^{2, 2} \times \mathbb{H}^{2, 2} \times \mathbb{M}^{2, 2} \times \mathbb{S}^{2, 2} \times \mathbb{H}^{2, 2} \times \mathbb{M}^{2, 2}\). In the setting of this paper, the Lipschitz assumption for the generators is replaced by some local quadratic growth. As a consequence, one cannot recover a contraction by simply choosing a weight large enough. In fact, as noticed in [49] and [32], given our growth assumptions, our candidate for providing a contractive map is no longer Lipschitz continuous, but only locally Lipschitz continuous. The idea, initially proposed in [49], is then to localise the usual procedure to a ball, thus making the application Lipschitz continuous again, and then to choose the radius of such a ball so as to recover a contraction. The crucial contribution of [49] is to show that such controls can be obtained by taking the data of the system small enough in norm.

Our procedure is inspired by this idea and incorporates it into the strategy devised in [20] to address the well-posedness of this kind of systems. We have decided to work on weighted spaces as, in our opinion, it does significantly simplify the arguments in the proof. We also mention that we tried to estimate the greatest ball, i.e. the largest radius \(R\), for which such a localisation procedure leads to a contraction. Details are found in the proof. As we work on weighted spaces, we will find \(c > 0\) large enough so that, given data with sufficiently small norm, \((S)\) has a unique solution in \(\mathcal{B}_R \subseteq \mathcal{H}^c\). In words, throughout the proof, we will accumulate conditions on any candidate value for \(c\) that allows to verify the necessary steps to obtain the result. As such, an appropriate value of \(c\) must satisfy all such conditions. This should be clear from the statement of the result.

### 3.1 The Lipschitz–quadratic case

**Assumption B.** \((i)\) \(\exists \tilde{c} \in \mathbb{R} \) such that \((\xi, \eta, \partial \eta, \tilde{f}, \tilde{g}, \tilde{\nabla} \tilde{g}) \in \mathcal{L}^{\infty, \tilde{c}} \times \mathcal{L}^{\infty, 2, \tilde{c}} \times \mathcal{L}^{1, \infty, \tilde{c}} \times \mathcal{L}^{1, 2, \tilde{c}}\).

\((ii)\) \(\exists (L_y, L_u, L_u) \in \mathbb{R}^3 \) s.t. \(\forall (s, t, x, y, \tilde{g}, u, \tilde{u}, u, \tilde{u}, z, v, v) \in [0, T]^2 \times \mathcal{X} \times (\mathbb{R}^d)^2 \times (\mathbb{R}^d)^4 \times \mathbb{R}^{n \times d_1} \times (\mathbb{R}^{n \times d_2})^2\)
\[
|\tilde{h}_t(x, y, z, u, v, u) - \tilde{h}_t(x, \tilde{y}, \tilde{z}, \tilde{u}, \tilde{v}, \tilde{u})| + |\tilde{g}_t(s, x, u, v, y, z) - \tilde{g}_t(s, \tilde{x}, \tilde{u}, \tilde{v}, \tilde{y}, \tilde{z})|
+ |\nabla \tilde{g}_t(s, x, u, v, u, v, y, z) - \nabla \tilde{g}_t(s, \tilde{x}, \tilde{u}, \tilde{v}, \tilde{y}, \tilde{z})| \leq L_y |y - \tilde{y}| + L_u |u - \tilde{u}| + L_u |u - \tilde{u}|;
\]

\((iii)\) \(\exists (L_z, L_v, L_v) \in \mathbb{R}^3 \), \(\phi \in \mathbb{L}^{2, \tilde{c}}_{BMO}\) s.t. \(\forall (s, t, x, y, u, u, z, \tilde{z}, v, v, \tilde{v}, \tilde{v}) \in [0, T]^2 \times \mathcal{X} \times (\mathbb{R}^d)^2 \times (\mathbb{R}^d)^4 \times (\mathbb{R}^{n \times d_1})^2 \times (\mathbb{R}^{n \times d_2})^4\)
\[
|\tilde{h}_t(x, y, z, u, v, u) - \tilde{h}_t(x, \tilde{y}, \tilde{z}, \tilde{u}, \tilde{v}, \tilde{u})| - (z - \tilde{z})^\top \sigma_r(x) \phi_t + |\tilde{g}_t(s, x, u, v, y, z) - \tilde{g}_t(s, \tilde{x}, \tilde{u}, \tilde{v}, \tilde{y}, \tilde{z})| - (v - \tilde{v})^\top \sigma_r(x) \phi_t
+ |\nabla \tilde{g}_t(s, x, u, v, u, v, \tilde{y}, \tilde{z}) - \nabla \tilde{g}_t(s, \tilde{x}, \tilde{u}, \tilde{v}, \tilde{y}, \tilde{z})| - (v - \tilde{v})^\top \sigma_r(x) \phi_t
\leq L_z |\sigma_r(x) z| + |\sigma_r(x) \tilde{z}| + |\sigma_r(x) (z - \tilde{z})| + L_v |\sigma_r(x) v| + |\sigma_r(x) \tilde{v}| + |\sigma_r(x) (v - \tilde{v})| + L_v |\sigma_r(x) v| + |\sigma_r(x) \tilde{v}| + |\sigma_r(x) (v - \tilde{v})|.
\]

**Remark 3.3.** We now comment on the previous set of assumptions. Assumption B.\((i)\) imposes integrability on the data of the system. We highlight that in our setting we require the integral with respect to the time variable of the generators \((\tilde{f}, \tilde{g}, \tilde{\nabla} \tilde{g})\) to be bounded. This is in contrast to requiring the generators itself to be bounded. On the other hand, Assumption B.\((ii)\) imposes a classic uniformly Lipschitz growth assumption on the \((Y, U, \partial U)\) terms for the system. Finally, Assumption B.\((iii)\) imposes a slight generalisation of a local Lipschitz quadratic growth, this corresponds to the presence of the process \(\phi\), and is similar to the one found in [49]. This property is almost equivalent to saying that the underlying function is quadratic in \(z\). The two properties would be equivalent if the process \(\phi\) was bounded. Here we allow something a bit more general by letting \(\phi\) be unbounded but in \(\mathbb{H}^2_{\text{BMO}}\). As we will see next, since this assumption allows us to apply the Girsanov transformation, we do not need to bound the processes and BMO-type conditions are sufficient. Lastly, we also remark that \(\phi\) is common for the three generators and do not depend on \(s\). This is certainly a limitation in terms of the system \((S)\). Yet, as we are working towards establishing the well-posedness of the BSIE \((4.1)\), we will see in Section 4, namely \((S_f)\), that we will chose \(h\) and \(g\) in such a way that such condition is sensible.

As a preliminary to our analysis, we note that Assumption B.\((iii)\) can be simplified. This is the purpose of the next lemma. Therefore, without lost of generality in the rest of this section we assume \(\phi = 0\).
Lemma 3.4. Let
\[ \hat{h}_t(x, y, z, u, v, u) := h_t(x, y, z, u, v, u) - z^T \sigma_r(x) \phi_t, \quad \hat{g}_t(s, x, u, v, y, z) := g_t(s, x, u, v, y, z) - v^T \sigma_r(x) \phi_t, \]
\[ \nabla \hat{g}_t(s, x, u, v, y, z) := g_t(s, x, u, v, y, z) - v^T \sigma_r(x) \phi_t, \]
Then \((Y, Z, N, U, V, N)\) is a solution to \((S)\) with terminal condition \((\xi, \eta)\) and generator \((f, g)\) under \(P\) if and only if \((Y, Z, N, U, V, N)\) is a solution to \((S)\) with terminal condition \((\xi, \eta)\) and generator \((\hat{f}, \hat{g})\) under \(Q \in \text{Prob}(\Omega)\) given by
\[ \frac{dQ}{dP} = \mathcal{E} \left( \int_0^T \phi_t \cdot dX_t \right). \]

Proof. We first verify that \(Q\) above is well-defined. Indeed, from the fact that \(\phi \in \mathbb{H}^{2,c}_B(\mathbb{R}^m)\) we have that the process defined above is a uniformly integrable martingale and Girsanov’s theorem holds, see [31, Section 1.3]. To verify the assertion of the lemma we note, for instance, that
\[ \mathcal{Y}_t = \xi + \int_t^T \hat{h}_r(X, Y_r, Z_r, U_r, V_r, \partial U_r) dr - \int_t^T \hat{Z}_r (dX_r - \sigma_r \phi_r dr) + \int_t^T dN_r, \quad t \in [0, T], \text{ Q-a.s.} \]

To ease the presentation of our result, for \((\gamma, c, R) \in (0, \infty)^3, \varepsilon_i \in (0, \infty), i \in \{1, \ldots, 11\}, \text{ and } \kappa \in \mathbb{N}\), we define
\[ I_0^\varepsilon := \| \xi \|^2_{L^\infty} + 2 \| \eta \|^2_{L^\infty} + (1 + \varepsilon_1 + \varepsilon_2) \| \partial \eta \|^2_{L^2,2,2} + \varepsilon_3 \| \hat{h} \|^2_{L^2,2,2} + (\varepsilon_4 + \varepsilon_5) \| \hat{g} \|^2_{L^2, \infty, \infty, \infty} + \varepsilon_6 \| \nabla \hat{g} \|^2_{L^2, \infty, \infty, \infty}, \]
\[ c^\varepsilon := \text{max}\{2L_u + (\varepsilon_1 + \varepsilon_2)TL_u + (\varepsilon_3 + \varepsilon_4 + \varepsilon_5)\| \Sigma_u \|^2_{\infty, \infty, \infty}, 2L_u + \varepsilon_6 TL_u + \varepsilon_7 L_u + \varepsilon_8 L_u, 2L_u + \varepsilon_9 L_u, 2L_u + \varepsilon_{10} L_u, 2L_u + \varepsilon_{11} L_u \} \]
\[ L_* := \text{max}\{L_2, L_4, L_5, \}, U(\kappa) := \frac{1}{168\kappa L_*^2}. \]

Remark 3.5. Let us mention that the previous expressions arise in the analysis given our goal of finding the largest ball over which we can guarantee a contraction. In particular, for this reason there are several degrees of freedom, \(\varepsilon_i\)’s, that determine \(c^\varepsilon\). In particular, let us mention that there are many simplifying choices that can be made, at the risk of loosing some flexibility. For instance, letting \(\varepsilon_i \in (0, \infty)^5, i \in \{1, \ldots, 5\}, \varepsilon_1 = \varepsilon_2 = \varepsilon_6 = \tilde{\varepsilon}_1, \varepsilon_3 = \tilde{\varepsilon}_2, \varepsilon_4 = \varepsilon_5 = \tilde{\varepsilon}_3, \varepsilon_7 = \varepsilon_{11} = \tilde{\varepsilon}_4 \text{ and } \varepsilon_8 = \varepsilon_9 = \varepsilon_{10} = \varepsilon_5 \text{ we only need to choose 5 variables and } \]
\[ I_0^\varepsilon := \| \xi \|^2_{L^\infty} + 2 \| \eta \|^2_{L^\infty} + (1 + \varepsilon_1 + \varepsilon_2) \| \partial \eta \|^2_{L^2,2,2} + \varepsilon_3 \| \hat{h} \|^2_{L^2,2,2} + 2\tilde{\varepsilon}_1 \| \nabla \hat{g} \|^2_{L^2, \infty, \infty, \infty} + 3 \varepsilon_1 \| \nabla \hat{g} \|^2_{L^2, \infty, \infty, \infty}, \]
\[ c^\varepsilon := \text{max}\{2L_u + (\varepsilon_1 + \varepsilon_2)TL_u + (\varepsilon_4 + \varepsilon_5)\| \Sigma_u \|^2_{\infty, \infty, \infty}, 2L_u + \varepsilon_6 TL_u + \varepsilon_4 \| \Sigma_u \|^2_{\infty, \infty, \infty}, 2L_u + \varepsilon_9 L_u, 2L_u + \varepsilon_8 L_u, 2L_u + \varepsilon_7 L_u, \}
\[ 2L_u + 2\tilde{\varepsilon}_1 TL_u + \varepsilon_8 L_u + \varepsilon_5 L_u + \varepsilon_6 L_u + \varepsilon_7 L_u \]

Assumption C. Let \((\gamma, c, R) \in (0, \infty)^3, \varepsilon_i \in (0, \infty), i \in \{1, \ldots, 11\}, \kappa \in \mathbb{N}\). We say Assumption C holds for \(\kappa\) if
\[ (\sqrt{\varepsilon_1 + \varepsilon_2 + 3\kappa + \sqrt{3\kappa}})^2 \leq 28 \kappa, \quad I_0^\varepsilon \leq \gamma R^2 / \kappa, \quad R^2 < U(\kappa), \quad c \geq c^\varepsilon. \]

Theorem 3.6. Let Assumption B holds. Suppose Assumption C holds for \(\kappa = 10\). Then, there exists a unique solution to \((S)\) in \(\mathcal{B}_R \subseteq \mathcal{H}^c\) with
\[ R^2 < \frac{1}{168\kappa L_*^2}. \]

Remark 3.7. We would like to comment on both the qualitative and quantitative statements in Theorem 3.6.

(i) As a result of our procedure, and consistent with the results available in the literature, we require the data of \((S)\) to be sufficiently small in order to obtain the well-posedness of \((S)\). We stress this property on the data is not influenced only by the value of the generator \(R\), but also, by the value of \(c\) which determines the norms.

(ii) In addition, we introduced Assumption C which depends on a parameter \(\kappa \in \mathbb{N}\). This parameter is related with the number of processes for which we have to keep track of the integrability of. In particular, we mention that, in the proof we present in Section 6, in addition to the 9 elements that prescribe a solution to the system, we also control the norm of the diagonal process \((V_t^i)_{t \in [0, T]}\) which appears in the definition of \(\mathbb{P}^{\gamma, \varepsilon, c}\). An alternative line of reasoning is available by leaving out the norm of process \((V_t^i)_{t \in [0, T]}\) in the definition of \(\mathbb{P}^{\gamma, \varepsilon, c}\) and exploit the inequality derived in Lemma 2.3.
(iii) We also want to point out that we tried to obtain the largest ball for which the whole procedure goes through. This can be appreciated in Equation (6.8) which introduces an upper bound on $R$ for which our candidate map is well-defined. A word of caution nonetheless, since our bound of $R$ cannot be directly compared to the ones obtained in [49] or [32]. Indeed, we recall that the norms involved depend on our choice of $c$. As such, the radius of the ball, $B_R \subseteq \mathcal{H}^c$, in Theorem 3.6 depends on the choice of $c$. Consequently, if we were to exploit the equivalence between all the spaces $\mathcal{H}^c$ the radius would have to be appropriately adjusted.

**Remark 3.8.** Alternative versions of Theorem 3.6 are available. These are related to the orthogonal martingales.

(i) We first highlight that we are able to show, a posteriori, that the $(\mathcal{N}, M, \partial M)$ part of the solution to system (S) in $\mathcal{H}^c$ actually has a finite $\text{BMO}$--type norm. This is proved in Theorem 3.9. We recall that in this quadratic setting with bounded terminal condition, this might be considered as a more natural type of integrability for the martingale part of the solutions.

(ii) Furthermore, one could wonder about how the statement of Theorem 3.6 changes in the case where $(\mathcal{N}, M, \partial M)$ are required, a priori, to have finite $\text{BMO}$--norm. This would allow, for instance, to keep a control on the $\text{BMO}$--norm inside the resulting ball. We cover this in Theorem 3.10.

(iii) Finally, we consider the case when the representation property for $(\mathcal{F}, \mathcal{P})$--martingales in terms of stochastic integrals with respect to $X$ holds. This is the case, for example, if $\mathbb{P}$ is an extremal point of the convex hull of the set of probability measures satisfying the requirements in Section 2.1, see [27, Theorem 4.29]. In such a scenario, we have that $\mathcal{N} = M = \partial M = 0$. This case is covered in Theorem 3.10.

**Theorem 3.9.** Let Assumption B hold and $\mathcal{H}$ be a solution to (S) in the sense of Definition 3.2. Then

$$\|\mathcal{N}\|_{\text{BMO}}^2 + \|M\|_{\text{BMO}}^2 + \|\partial M\|_{\text{BMO}}^2 < \infty$$

We now address the well–posedness of (S) in the following two scenarios

(i) when $(\mathcal{N}, M, \partial M) \in \mathbb{M}_{\text{BMO}}^{2,c} \times \mathbb{M}_{\text{BMO}}^{2,c} \times \mathbb{M}_{\text{BMO}}^{2,c}$, i.e. $(\mathcal{N}, M, \partial M)$ are required to have finite $\text{BMO}$ norm;

(ii) when the predictable martingale representation property for $(\mathcal{F}, \mathcal{P})$–martingales in term of stochastic integral with respect to $X$ hold, i.e. both $\mathcal{N}$, $M$ and $\partial M$ vanish.

For this, let us introduce the spaces $(\hat{\mathcal{H}}^c, \| \cdot \|_{\hat{\mathcal{H}}^c})$ and $(\mathcal{H}, \| \cdot \|_{\mathcal{H}})$ for $c > 0$, where

$$\hat{\mathcal{H}}^c := \mathbb{S}_{\text{BMO}}^{\infty,c} \times \mathbb{H}_{\text{BMO}}^{2,c} \times \mathbb{M}_{\text{BMO}}^{2,c} \times \mathbb{H}_{\text{BMO}}^{2,c} \times \mathbb{S}_{\text{BMO}}^{\infty,c} \times \mathbb{H}_{\text{BMO}}^{2,c} \times \mathbb{M}_{\text{BMO}}^{2,c},$$

$$\mathcal{H} := \mathbb{S}_{\text{BMO}}^{\infty,c} \times \mathbb{H}_{\text{BMO}}^{2,c} \times \mathbb{M}_{\text{BMO}}^{2,c} \times \mathbb{H}_{\text{BMO}}^{2,c} \times \mathbb{S}_{\text{BMO}}^{\infty,c} \times \mathbb{H}_{\text{BMO}}^{2,c} \times \mathbb{M}_{\text{BMO}}^{2,c},$$

and $\| \cdot \|_{\hat{\mathcal{H}}^c}$ and $\| \cdot \|_{\mathcal{H}}$ denote the associated norms. Moreover, $\mathcal{B}_R \subseteq \mathcal{H}^c$ (resp. $\mathcal{B}_R \subseteq \mathcal{H}$) denotes the ball of radius $R$ for the norm $\| \cdot \|_{\mathcal{H}}$ (resp. $\| \cdot \|_{\hat{\mathcal{H}}^c}$).

**Theorem 3.10.** Let Assumption B hold. If in addition

(i) Assumption C holds for $\kappa = 11$, then, there exists a unique solution to (S) in $\mathcal{B}_R \subseteq (\hat{\mathcal{H}}^c, \| \cdot \|_{\hat{\mathcal{H}}^c})$.

(ii) Assumption C holds for $\kappa = 8$, and the predictable martingale representation property for $(\mathcal{F}, \mathcal{P})$–martingales in term of stochastic integral with respect to $X$ hold. Then, there exists a unique solution to (S) in $\mathcal{B}_R \subseteq (\mathcal{H}, \| \cdot \|_{\mathcal{H}})$.

### 3.2 The quadratic case

Our approach allows us to consider also the case of quadratic generators as follows.

**Assumption D.**

(i) $\exists \bar{c} > 0$ s.t. $(\xi, \eta, \partial \eta, \bar{f}, \bar{g}, \nabla \bar{g}) \in \mathcal{L}^{\infty,\bar{c}} \times \mathcal{L}^{\infty,\bar{c},2} \times \mathcal{L}^{1,\infty,\bar{c}} \times \mathcal{L}^{1,\infty,\bar{c},2} \times \mathcal{L}^{1,\infty,\bar{c},2}$.

(ii) $\exists (L_y, L_u, \bar{L}_u) \in (0, \infty)^3$, s.t. $\forall (s, t, x, y, \bar{y}, u, \bar{u}, \bar{v}, v, v, v) \in [0, T]^2 \times \mathcal{X} \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$,

$$\begin{align*}
|h_t(x, y, z, u, v, v) - h_t(x, \bar{y}, z, \bar{u}, v, \bar{v})| &\leq L_y |y - \bar{y}| |y + |\bar{y}|| + L_u |u - \bar{u}| |u + |\bar{u}|| + L_u |u - \bar{u}|, \\
|g_t(x, \bar{u}, u, y, z) - g_t(x, u, \bar{v}, v, v, z)| &\leq \sqrt{g_t(x, u, v, v, y, z) - \nabla g_t(x, u, v, v, y, z)} \leq L_u |u - \bar{u}| |u + |\bar{u}|| + L_y |y - \bar{y}| |y + |\bar{y}||.
\end{align*}$$
(iii) \( \exists (L_2, L_y, L_v) \in (0, \infty)^3, \phi \in \mathbb{H}^2_{\text{BMO}} \) s.t. \( \forall (s, t, x, y, u, z, \bar{v}, \bar{v}, \bar{\nu}) \in [0,T]^2 \times \mathcal{X} \times \mathbb{R}^d_+ \times (\mathbb{R}^d_+)^2 \times (\mathbb{R}^{n \times d}_+)^2 \times (\mathbb{R}^{n \times d}_+)^4 \)

\[
|\tilde{h}_t(x, y, z, u, v, u) - h_t(x, y, z, u, \tilde{v}, u) - (z - \bar{z})^T \sigma_r(x)\phi_t| + |g_t(s, x, u, y, z) - g_t(s, x, u, \bar{v}, y, \bar{v}) - (v - \bar{v})^T \sigma_r(x)\phi_t| \\
+ |\nabla g_t(s, x, u, v, u, v, y, z) - \nabla g_t(s, x, u, \tilde{v}, u, \tilde{v}, y, \bar{v}) - (v - \tilde{v})^T \sigma_r(x)\phi_t| \\
\leq L_2|\sigma^T_r(x)z| + |\sigma^T_r(x)\tilde{z}| + L_v|\sigma^T_r(x)v| + |\sigma^T_r(x)(v - \tilde{v})| + L_u|\sigma^T_r(x)v| + |\sigma^T_r(x)v||\sigma^T_r(x)(v - \tilde{v})|.
\]

**Remark 3.11.** In the previous set of assumptions we have allowed the generators to have quadratic growth in all of the terms, with the exception of \( h \) on the term \( u \). The reason for this is twofold: (i) for the kind of systems that motivate our analysis the term \( \partial U_i^t \) always appears linearly in the generator, (ii) when making the connection with the type-I BSVIE (1.1) \( \partial U_i^t \) plays the auxiliary role of keeping track of the diagonal processes \( (U_i^t)_{t \in [0,T]} \) and \( (V_i^t)_{t \in [0,T]} \) and for this it suffices that it appears linearly in the generator. This is, the Lipschitz assumption on this variable will suffice for the purposes of our results and the problems that motivated our study.

As before, we need to introduce some auxiliary notation. For \( (\gamma, c, R) \in (0, \infty)^3, \varepsilon_i \in (0, \infty), i \in \{1, \ldots, 6\}, \) and \( \kappa \in \mathbb{N}, \) we let \( I^0_0 \) be as in Section 3.1 and define

\[
L_* := \max\{L_y, L_u, L_u, L_v, L_v, L_v\}, \ c^* := \max\{c_1^{-1} TL_{u, u}^2, c_2^{-1} TL_v, 2L_u\}, \ U(\kappa) := \frac{1}{336\kappa L_2^2 \max\{2, T^2\}}.
\]

**Assumption E.** Let \( (\gamma, c, R) \in (0, \infty)^3, \varepsilon_i \in (0, \infty), i \in \{1, \ldots, 6\}, \) and \( \kappa \in \mathbb{N}. \) We say Assumption E holds for \( \kappa \) if

\[
(\sqrt{\varepsilon_1} + \varepsilon_2 + 3\kappa + \sqrt{3}\kappa)^2 \leq 56\kappa, \ I^0_0 \leq \gamma R^2 / \kappa, \ R^2 < U(\kappa), \ c \geq c^*.
\]

**Theorem 3.12.** Let Assumption D holds. Suppose Assumption E holds for \( \kappa = 10. \) Then, there exists a unique solution to (S) in \( \mathcal{B}_R \subset \mathcal{H}^c \) with

\[
R^2 < \frac{1}{336\kappa L_2^2 \max\{2, T^2\}}
\]

**Remark 3.13.** We remark that the best way to appreciate the previous result is by contrasting it with our well-posedness result in the linear quadratic case. For simplicity, let us assume \( L_y = L_u = L_u = L_v = L_v. \) Regarding the constraint on the weight parameter of the resulting norm, the result is pretty much in the same order of magnitude. Nevertheless, the most noticeable feature is that by allowing the system to have quadratic growth the greatest radius under which our argument is able to guarantee the well-posedness of the solution decreases by a factor \( 2 \max\{2, T^2\}. \) This quantity can be significant in light of its dependence on the time horizon \( T. \)

## 4 Multidimensional type-I BSVIEs of quadratic growth

We now address the well-posedness of multidimensional linear quadratic and quadratic type-I BSVIEs. Let \( d \) be a non-negative integer, and \( f \) and \( \xi \) be jointly measurable functionals such that for any \( (s, y, z, u, v) \in [0, T] \times (\mathbb{R}^d_+ \times \mathbb{R}^{n \times d}_+)^2 \)

\[
f : [0, T]^2 \times \mathcal{X} \times (\mathbb{R}^d_+ \times \mathbb{R}^{n \times d}_+)^2 \longrightarrow \mathbb{R}^d, \ f(s, \cdot, y, z, u, v) \in \mathcal{P}_{\text{prog}}(\mathbb{R}^d, \mathbb{F}),
\]

\[
\xi : [0, T] \times \mathcal{X} \longrightarrow \mathbb{R}^d, \ \xi(s, \cdot) \text{ is } \mathcal{F}_s\text{-measurable.}
\]

The main result in this section follows exploiting the well-posedness of (S). Therefore, we work under the following set of assumptions.

**Assumption F.** \((s, y, z) \mapsto f_t(s, x, y, z, u, v) \) \((\text{resp. } s \mapsto \xi(s, x)) \) is continuously differentiable, uniformly in \((t, x, u, v) \) \((\text{resp. in } x). \) Moreover, the mapping \( \nabla f : [0, T]^2 \times \mathcal{X} \times (\mathbb{R}^d_+ \times \mathbb{R}^{n \times d}_+)^3 \longrightarrow \mathbb{R}^d \) defined by

\[
\nabla f_t(s, x, u, v, y, z, u, v) := \partial_y f_t(s, x, y, z, u, v) + \partial_y f_t(s, x, y, z, u, v)u + \sum_{i=1}^d \partial_{z_i} f_t(s, x, y, z, u, v)\nu_i,
\]

satisfies \( \nabla f(s, \cdot, y, z, u, v, v, u, v) \in \mathcal{P}_{\text{prog}}(\mathbb{R}^d, \mathbb{F}), s \in [0, T]. \) Set \( (\hat{f}, \hat{f}(s), \nabla \hat{f}(s)) := (f(s, \cdot, 0, 0), f(s, \cdot, 0), \partial_t f(s, s, 0)), \) for \( 0 := (u, v, y, z)(0, \ldots, 0). \)

Let \( (\mathcal{H}^c, \|\cdot\|_{\mathcal{H}^c}) \) denote the space of \((Y, Z, N) \in \mathcal{H}^c \) such that \( \| (Y, Z, N) \|_{\mathcal{H}^c} < \infty \) where

\[
\mathcal{H}^c := \mathcal{S}^{\infty, 2_+} \times \mathcal{M}^{2_+} \times \mathcal{M}^{2_+}, \ \|\cdot\|_{\mathcal{H}^c} := \|Y\|_{\mathcal{S}^{\infty, 2_+}}^2 + \|Z\|_{\mathcal{M}^{2_+}}^2 + \|N\|_{\mathcal{M}^{2_+}}^2.
\]

We consider the \( n \)-dimensional type-I BSVIE on \((\mathcal{H}^c, \|\cdot\|_{\mathcal{H}^c}) \)

\[
Y^s_t = \xi(s, X) + \int_t^T f_r(s, X, Y^r_t, Z^r_t, Y^r_t, Z^r_t)dr - \int_t^T Z^r_t dX_r - \int_t^T dN^r_s, \ t \in [0, T], \ \mathbb{P}_s\text{-a.s., } s \in [0, T].
\]

We work under the following notion of solution.
Definition 4.1. We say \((Y, Z, N)\) is a solution to the type-I BSVIE (4.1) if \((Y, Z, N)\) in \(H^*\) verifies (4.1).

We may consider the system, given for any \(s \in [0, T]\) by

\[
\begin{align*}
\mathcal{Y}_t & = \xi(T, X) + \int_t^T \left( f_r(x, X, Y_r, Z_r, Y_r^*, Z_r^*) - \partial Y_r^* \right) dr - \int_t^T Z_r^* dX_r - \int_t^T dN_r, \ t \in [0, T], \ \mathbb{P}\text{-a.s.,} \\
Y_t^s & = \xi(s, X) + \int_s^T f_r(s, X, Y_r^s, Z_r^s, Y_r, Z_r) dr - \int_s^T Z_r^s dX_r - \int_s^T dN_r^s, \ t \in [0, T], \ \mathbb{P}\text{-a.s.,} \\
\partial Y_t^s & = \partial_\xi(s, X) + \int_s^T \nabla f_r(s, X, \partial Y_r^s, \partial Z_r^s, Y_r^s, Z_r^s, Y_r, Z_r) dr - \int_s^T \partial Z_r^s dX_r - \int_s^T d\partial N_r^s, \ t \in [0, T], \ \mathbb{P}\text{-a.s.}
\end{align*}
\]

\((S_f)\)

Remark 4.2. Let us briefly comment that our set-up for the study type-I BSVIE (4.1) is based on the systems introduced in Section 3. As such, the necessity of the set of assumptions in Assumption F is clear.

We are now in position to prove the main result of this paper. The next result shows that under the previous choice of data for \((S_f)\), its solution solves the type-I BSVIE with data \((\xi, f)\) and vice versa in both the linear quadratic and quadratic case. For this we introduce the following set of assumptions.

Assumption G. (i) \(\exists c \in (0, \infty)\) such that \((\xi, \eta, \vartheta, \tilde{f}, \tilde{g}, \nabla \tilde{g}) \in L^{\infty, c, \tilde{c}} \times L^{\infty, 2, \tilde{c}} \times L^{1, \infty, c} \times L^{1, \infty, 2, \tilde{c}}\).

(ii) \(\exists (L_y, L_u, L_v) \in (0, \infty)^3\) s.t. \(\forall (s, t, x, y, \tilde{y}, u, \tilde{u}, u, \tilde{u}, z, v, \tilde{v}, v) \in [0, T]^2 \times \mathcal{X} \times (\mathbb{R}^d)^6 \times \mathbb{R}^{n \times d} \times (\mathbb{R}^{n \times d})^3\)

\[
|f_t(s, x, y, z, u, v) - f_t(s, x, \tilde{y}, z, \tilde{u}, v) - (z - \tilde{z})^T \sigma_t(x)\phi_t| + |\nabla f_t(s, x, u, v, z, u, \tilde{v}) - (v - \tilde{v})^T \sigma_t(x)\phi_t| \\
\leq L_y |y - \tilde{y}| + L_u |u - \tilde{u}| + L_v |u - \tilde{u}|;
\]

(iii) \(\exists (L_z, L_v, L_u) \in (0, \infty)^3\), \(\phi \in \mathbb{H}^{2, \tilde{c}}_{\text{BMO}}\) s.t. \(\forall (s, t, x, y, \tilde{y}, u, \tilde{u}, u, \tilde{u}, z, v, \tilde{v}, v) \in [0, T]^2 \times \mathcal{X} \times (\mathbb{R}^d)^6 \times \mathbb{R}^{n \times d} \times (\mathbb{R}^{n \times d})^3\)

\[
|f_t(s, x, y, z, u, v) - f_t(s, x, \tilde{y}, z, \tilde{u}, v) - (z - \tilde{z})^T \sigma_t(x)\phi_t| + |\nabla f_t(s, x, u, v, z, u, \tilde{v}) - (v - \tilde{v})^T \sigma_t(x)\phi_t| \\
\leq L_y |y - \tilde{y}| + L_u |u - \tilde{u}| + L_v |u - \tilde{u}| + L_u |u - \tilde{u}| + L_u |u - \tilde{u}|;
\]

Assumption H. (i) \(\exists c \in (0, \infty)\) such that \((\xi, \eta, \vartheta, \tilde{f}, \tilde{g}, \nabla \tilde{g}) \in L^{\infty, c, \tilde{c}} \times L^{\infty, 2, \tilde{c}} \times L^{1, \infty, c} \times L^{1, \infty, 2, \tilde{c}}\).

(ii) \(\exists (L_y, L_u, L_v) \in (0, \infty)^3\) s.t. \(\forall (s, t, x, y, \tilde{y}, u, \tilde{u}, u, \tilde{u}, z, v, \tilde{v}, v) \in [0, T]^2 \times \mathcal{X} \times (\mathbb{R}^d)^6 \times \mathbb{R}^{n \times d} \times (\mathbb{R}^{n \times d})^3\)

\[
|f_t(s, x, y, z, u, v) - f_t(s, x, \tilde{y}, z, \tilde{u}, v) - (z - \tilde{z})^T \sigma_t(x)\phi_t| + |\nabla f_t(s, x, u, v, z, u, \tilde{v}) - (v - \tilde{v})^T \sigma_t(x)\phi_t| \\
\leq L_y |y - \tilde{y}| + L_u |u - \tilde{u}| + L_v |u - \tilde{u}| + L_u |u - \tilde{u}|;
\]

(iii) \(\exists (L_z, L_v, L_u) \in (0, \infty)^3\), \(\phi \in \mathbb{H}^{2, \tilde{c}}_{\text{BMO}}\) s.t. \(\forall (s, t, x, y, \tilde{y}, u, \tilde{u}, u, \tilde{u}, z, v, \tilde{v}, v) \in [0, T]^2 \times \mathcal{X} \times (\mathbb{R}^d)^6 \times \mathbb{R}^{n \times d} \times (\mathbb{R}^{n \times d})^3\)

\[
|f_t(s, x, y, z, u, v) - f_t(s, x, \tilde{y}, z, \tilde{u}, v) - (z - \tilde{z})^T \sigma_t(x)\phi_t| + |\nabla f_t(s, x, u, v, z, u, \tilde{v}) - (v - \tilde{v})^T \sigma_t(x)\phi_t| \\
\leq L_y |y - \tilde{y}| + L_u |u - \tilde{u}| + L_v |u - \tilde{u}| + L_u |u - \tilde{u}| + L_v |u - \tilde{u}| + L_v |u - \tilde{u}|.
\]

Theorem 4.3. Let Assumption F hold. Then, the well-posedness of \((S_f)\) is equivalent to that of the type-I BSVIE (4.1) if either:

(i) Assumption G holds and Assumption C holds for \(\kappa = 7\). In such case, there exists a unique solution to the type-I BSVIE (4.1) in \(B_R \subset H^{*c}\) with

\[
R^2 < \frac{1}{108\kappa L_x^2}
\]

(ii) Assumption H holds and Assumption E holds for \(\kappa = 7\). In such case, there exists a unique solution to the type-I BSVIE (4.1) in \(B_R \subset H^{*c}\) with

\[
R^2 < \frac{1}{336\kappa L_x^2 \max\{2, T^2\}}
\]
Proof. Let us first note that the second part of the statements in (i) and (ii) are a direct consequence of Theorem 3.6 and Theorem 3.12, respectively.

Let us first argue why it suffices to have Assumption C and Assumption E hold for $\kappa = 7$ instead of 10. This follows from the specification of data for $(S_f)$. Indeed, following the notation of the proof of Theorem 3.6 in this case we have that $\mathcal{Y} = \mathcal{U}$, $\mathcal{Z} = \mathcal{V}$ and $\mathcal{N} = \mathcal{M}$ so the auxiliary equation introduced in step 1 (ii) is not necessary and, as from (6.6), the argument in the proof holds with 7 instead on 10.

We are only left to argue the equivalence of the solutions. Let us argue (i), the argument for (ii) is analogous. For this we follow [20, Theorem 4.3]. Let $(\mathcal{Y}, \mathcal{Z}, \mathcal{N}, \mathcal{Y}, \mathcal{Z}, \partial \mathcal{Y}, \partial \mathcal{Z}, \partial \mathcal{N}) \in \mathcal{B}_R \subseteq \mathcal{H}\hspace{0.1cm}^\kappa$ be a solution to $(S_f)$. It then follows from [20, Lemma 6.2] that

$$Y_t^i = \xi(T, X) + \int_t^T \left( f_r(r, X, Y_r^i, Z_r^i, \mathcal{Y}_r, \mathcal{Z}_r) - \partial \mathcal{Y}_r^i \right) dr - \int_t^T Z_r^i \, dX_r - \int_t^T d\tilde{N}_r, \ t \in [0, T], \ P\text{-a.s.},$$

where $\tilde{N}_t := N_t^i - \int_0^t \partial N_r^i \, dr, \ t \in [0, T]$, and $\tilde{N} \in \mathbb{M}_{2, c}^{2, c}$. As in Theorem 3.9, we obtain that $\tilde{N} \in \mathbb{M}_{2, BMO}$. This shows that $((Y_t^i)_{t \in [0, T]}, (Z_t^i)_{t \in [0, T]}, \mathcal{Y}_t, \mathcal{Z}_t, (\tilde{N}_t)_{t \in [0, T]}), \ t \in [0, T]$ solves the first BSDE in $(S_f)$. It then follows from the well-posedness of $(S_f)$, which holds by Assumption F, H, E and Theorem 3.6, that $((Y_t^i)_{t \in [0, T]}, (Z_t^i)_{t \in [0, T]}, (\tilde{N}_t)_{t \in [0, T]}), \ t \in [0, T]$ is an $\mathcal{S}^{2, c} \times \mathbb{H}_{BMO}^{2, c} \times \mathbb{M}_{2, c}$ solution to $(S_f)$ and the result follows.

We show the converse result. Let $(\mathcal{Y}, \mathcal{Z}, \mathcal{N}) \in \mathcal{B}_R \subseteq \mathcal{H}\hspace{0.1cm}^\kappa$ be a solution to type-I BSVIE (4.1). It is clear that the processes $\mathcal{Y} := (Y_t^i)_{t \in [0, T]}, \mathcal{Z} := (Z_t^i)_{t \in [0, T]}, \mathcal{N} := (N_t^i)_{t \in [0, T]}$ are well-defined. Then, since Assumption F holds and $(\mathcal{Y}, \mathcal{Z}, \mathcal{Y}, \mathcal{Z}, \mathcal{N}) \in L^{1, \infty, c} \times \mathbb{H}_{BMO}^{2, c} \times \mathbb{S}_{2, 1, c} \times \mathbb{H}_{BMO}^{2, c} \times \mathbb{M}_{2, c}$, we can apply [20, Lemma 6.2] and obtain the existence of $(\partial \mathcal{Y}, \partial \mathcal{Z}, \partial \mathcal{N}) \in \mathbb{S}_{2, 1, c} \times \mathbb{H}_{BMO}^{2, c} \times \mathbb{M}_{2, c}$ such that for $s \in [0, T]$,

$$\partial \mathcal{Y}_s = \partial \xi(s, X) + \int_s^T \nabla f_r(s, X, \partial \mathcal{Y}_r, \partial \mathcal{Z}_r) \, dX_r - \int_s^T \partial \mathcal{Z}_r^\top \, dX_r - \int_s^T \partial \mathcal{N}_r, \ t \in [0, T], \ P\text{-a.s.}$$

Moreover, from the fact that Assumption C holds for $\kappa = 7$ we obtain that $\|\partial \mathcal{Y}\|_{\mathbb{S}_{2, 1, c}} + \|\partial \mathcal{Z}\|_{\mathbb{H}_{BMO}^{2, c}} + \|\partial \mathcal{N}\|_{\mathbb{M}_{2, c}} \leq R^2$.

Let us claim that $\mathcal{Y} := (\mathcal{Y}, \mathcal{Z}, \mathcal{N}, \mathcal{Y}, \mathcal{Z}, \partial \mathcal{Y}, \partial \mathcal{Z}, \partial \mathcal{N})$ is a solution to $(S_f)$, where $N_t := N_t^i - \int_0^t \partial N_r^i \, dr, \ t \in [0, T]$. For this, we first note that in light of [20, Lemma 6.1-6.2] we have that

$$\mathcal{Y}_t = \xi(T, X) + \int_t^T h_r(s, X, \mathcal{Y}_r, \mathcal{Z}_r, \partial \mathcal{Y}_r, \partial \mathcal{Z}_r) \, dX_r - \int_t^T \mathcal{Z}_r^\top \, dX_r - \int_t^T \mathcal{N}_r, \ t \in [0, T], \ P\text{-a.s.}.$$  

(4.2)

Now, $\tilde{N} \in \mathbb{M}_{2, BMO}^{2, c}$ follows as in Theorem 3.9. As in step 1 (iii) in the proof of Theorem 3.6, we obtain that $\mathcal{Y} \in \mathcal{S}^{2, c}$. We are only left to argue $\|h\| \leq R$. This follows readily following step 1 (ii) in the proof of Theorem 3.6 and the fact that $\|\mathcal{Z}\|_{\mathbb{H}_{BMO}^{2, c}} + \|\mathcal{Z}\|_{\mathbb{H}_{BMO}^{2, c}} + \|\partial \mathcal{Z}\|_{\mathbb{H}_{BMO}^{2, c}} \leq R^2$.

5 On the flow property for type-I extended BSVIEs

In this section we present a brief discussion on the so-called flow property in the context of the type-I BSVIEs studied in the previous section. Contrary to the case of BSDEs, said BSVIEs fail to have the so-called flow property, this is

$$Y_t^i \neq Y_s^i + \int_t^s f_r(t, X, Y_r^i, Z_r^i, Y_r^i) \, dX_r - \int_t^s Z_r^i \, dX_r - \int_t^s dN_r.$$  

(5.1)

This is known to be a distinguishing and recurrent feature of Volterra processes. Indeed, Viens and Zhang [50] studied the dynamic backward problems in a framework where the forward state process satisfies a Volterra type SDE. Note that BSVIEs can be interpreted as Volterra-type extensions of the classic dynamic backward problem, e.g. of computing conditional expectations. Thus, the situation described in (5.1) corresponds to the antipodal scenario of [50] in the sense that the forward state process satisfies a classic SDE and the Volterra feature is present in the kind of the process we want to take conditional expectation of. However, beneath the surface of both scenarios is the presence of a certain manifestation of time inconsistency. The main result in [50] is a functional Itô formula. For this, the nature of the forward process, neither Markov processes nor semimartingales, makes necessary to concatenate the observed path up to the current time with a certain smooth observable curve derived from the distribution of the future paths. We stress that this new feature
is due to the underlying time inconsistency. All in all, having access to an Itô formula unravels a type of flow property that holds for this kind of problems.

As such, a natural question in our framework is: can we recover, in a appropriate sense, a flow property for type-I extended BSVIEs? By appropriate sense we mean that the sought version of this property must be compatible with the negative result in (5.1). The answer to this question can be extracted from the proof of Theorem 4.3. Indeed, as \((Y_t, Z_t, \mathcal{N}) = (Y, Z, \mathcal{N}_t) = BSVIEs. Actually , in the context of time-inconsistent control problems such as the ones presented in Section 1.1, [19] leveraged (5.2) to provide a justification of the choice of an equilibrium policies for sophisticated time-inconsistent agents. 

(iii) We also emphasise that this serves as a further motivation for our approach via systems of infinite families of BSDEs such as \((S_t)\). Indeed, it is able to handle the well-posedness of extended type-I BSVIEs, and, as a by product, it leverages (5.2), the underlying flow property, to accommodate BSVIEs where the diagonal of \(Z\) appears in the generator. 

6 Proof of the Linear Quadratic case

Proof of Theorem 3.6. For \(c > 0\), let us introduce the mapping

\[
\mathcal{W} : (B_R, \| \cdot \|_{\mathcal{H}}) \rightarrow (B_R, \| \cdot \|_{\mathcal{H}})
\]

\[
(y, z, u, v, m, d_u, d_v, d_m) \rightarrow (Y, Z, N, U, V, M, dU, dV, dM),
\]

given for any \(s \in [0, T], \mathbb{P}\)-a.s. for any \(t \in [0, T]\) by

\[
Y_t = \xi(T, X, \lambda_T) + \int_t^T h_r(X, Y_r, z_r, U_r^s, v_r^s, \partial U_r^s)dr - \int_t^T Z_r^\top dX_r - \int_t^T d\mathcal{N}_r,
\]

\[
U_t^s = \eta(s, X, \lambda_T) + \int_t^T g_r(s, X, U_r^s, v_r^s, Y_r, z_r)dr - \int_t^T V_r^\top dX_r - \int_t^T dM_r^s,
\]

\[
\partial U_t^s = \partial s \eta(s, X, \lambda_T) + \int_t^T \nabla g_r(s, X, \partial U_r^s, \partial v_r^s, U_r^s, v_r^s, Y_r, z_r)dr - \int_t^T \partial V_r^\top dX_r - \int_t^T dM_r^s.
\]

Step 1: We first argue that \(\mathcal{W}\) is well-defined.

(i) In light of Assumption B, there is \(c > 0\) such that

\[
(\xi, \eta, \partial \eta, \nabla \eta, \nabla \xi) \in \mathcal{L}^{\infty,c} \times (\mathcal{L}^{\infty,2,c})^2 \times \mathcal{L}^{1,\infty,c} \times (\mathcal{L}^{1,\infty,2,c})^2
\]

and

\[
(z, v, \partial v) \in H^{2,c}_{BMO} \times H^{2,2,c}_{BMO} \times H^{2,2,c}_{BMO},
\]

thus, we may use (2.2) to obtain

\[
E \left[ |\xi(T)|^2 + \left| \int_0^T |h_t(0, z_t, 0, v_t^s) dt |^2 \right| \right] + \sup_{s \in [0, T]} E \left[ |\eta(s)|^2 + \left| \int_0^T |g_t(s, 0, v_t^s, 0, z_t) dt |^2 \right| \right]
\]

\[
+ \sup_{s \in [0, T]} E \left[ |\partial \eta(s)|^2 + \left| \int_0^T |\nabla g_t(s, 0, \partial v_t^s, 0, v_t^s, 0, z_t) dt |^2 \right| \right]
\]

\[
\leq E \left[ |\xi(T)|^2 + 3 \int_0^T |h_t|^2 dt + 10L^2 \int_0^T |z_t|^2 dt + 3L^2 \int_0^T |v_t|^2 dt \right]
\]
whose elements we may denote with superscripts, e.g. $H$. In light of Assumption 5 we satisfy the equation $Y$, $N$, $L$, $U$, $\partial U$.

$$
\left\| L \right\|_{\infty} + 3\left\| h \right\|_{L^1,\infty,2.6} + 3\parallel g \parallel^2_{L^1,\infty,2.6} + 4\ ||\nabla g \parallel^2_{L^1,\infty,2.6} < \infty.
$$

Therefore, by [20, Theorem 3.2], $F$ defines a well–posed system of BSDEs with unique solution in the space $\mathcal{S}^2$. We recall the spaces involved in the definition of $\mathcal{S}^2$, and their corresponding norms, were introduced in Section 2.2.

(ii) Arguing as in [20, Lemmata 6.1 and 6.2], we may use Assumption B and $v \in E_B^{2,2,2}$, i.e. $\partial v$ is the density with respect to the Lebesgue measure of $s \rightarrow v_s$, to obtain that $(\mathcal{U}, \mathcal{V}, \mathcal{M}) \in \mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{M}^2$ given by

$$
\mathcal{U}_t := U^t_t, \quad \mathcal{V}_t := V^t_t, \quad \mathcal{M}_t := M^t_t - \int_0^t \partial M^t_r \, dr, \quad t \in [0, T],
$$

satisfy the equation

$$
\mathcal{U}_t = \eta(T, X_{\mathcal{T}}) + \int_t^T (g_t(r, X, \mathcal{U}_r, v^r_r, Y^r_r, \partial Y^r_r) - \partial U^r_r) dr - \int_t^T \mathcal{V}_r \, dX_r - \int_t^T \partial \mathcal{M}_r, \quad t \in [0, T], \mathbb{P}-\text{a.s.}
$$

(iii) We show $(\mathcal{Y}, \mathcal{U}) \in \mathcal{S}^{\infty,\infty} \times \mathcal{S}^{\infty,\infty}$ and $\|U\|_{\mathcal{S}^{\infty,\infty}} + \|\partial U\|_{\mathcal{S}^{\infty,\infty}} < \infty$.

To alleviate the notation we introduce

$$
\begin{align*}
\eta_r &:= h_r(Y_r, z_r, \mathcal{U}_r, v^r_r, \partial U^r_r), \quad g_r := g_r(r, \mathcal{U}_r, v^r_r, Y^r_r, \partial Y^r_r), \\
g_r(s) &:= g_r(s, U^s_r, v^s_r, Y^s_r, \partial Y^s_r), \quad \nabla g_r(s) := \nabla g_r(s, U^s_r, v^s_r, Y^s_r, \partial Y^s_r).
\end{align*}
$$

and,

$$
\mathcal{Y}_r = (\mathcal{Y}, \mathcal{U}, U^s, \partial U^s), \quad \mathcal{Z}_r = (\mathcal{Z}, V^s, \partial V^s), \quad \mathcal{M}_r = (\mathcal{M}, M^s, \partial M^s),
$$

whose elements we may denote with superscripts, e.g. $\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4$ correspond to $\mathcal{Y}, \mathcal{U}, U^s, \partial U^s$. In light of Assumption B, $dr \otimes d\mathbb{P}$–a.e.

$$
\begin{align*}
|h_r| &\leq L_U|\mathcal{Y}_r| + L_2|\nabla Y^r_r|^2 + L_u|\mathcal{U}_r| + Lu|\nabla Y^r_r|^2 + L_u|\partial U^r_r| + |h_r|, \\
g_r &\leq L_u|\mathcal{U}_r| + L_2|\nabla Y^r_r|^2 + L_y|\mathcal{Y}_r| + L_2|\nabla Y^r_r|^2 + \|\partial U^r_r\| + |g_r|, \\
g_r(s) &\leq L_u|U^s_r| + L_2|\nabla U^s_r|^2 + L_y|\mathcal{Y}_r| + L_2|\nabla U^s_r|^2 + |g_r(s)|, \\
|\nabla g_r(s)| &\leq L_u|\partial U^s_r| + L_2|\nabla U^s_r|^2 + L_u|\nabla U^s_r| + L_2|\nabla U^s_r|^2 + L_2|\nabla Y^s_r| + L_2|\nabla U^s_r|^2 + L_2|\nabla U^s_r| + \|\nabla g_r(s)\|.
\end{align*}
$$

Applying Meyer–Itô’s formula to $e^{\mathcal{T}}(|\mathcal{Y}_r| + |\mathcal{U}_r| + |U^r_r| + |\partial U^r_r|)$ we obtain

$$
\begin{align*}
e^{\mathcal{T}}(|\mathcal{Y}_r| + |\mathcal{U}_r| + |U^r_r| + |\partial U^r_r|) &\geq e^{\mathcal{T}}(|\mathcal{Y}_r| + |\mathcal{U}_r| + |U^r_r| + |\partial U^r_r|) + \mathcal{M}_r - \mathcal{M}_T + \mathcal{L}_T^0 \\
&= e^{\mathcal{T}T}(\mathcal{X}) + |\mathcal{Y}_r(T)| + |\mathcal{U}_r(T)| + |U^r_r(T)| + |\partial U^r_r(T)| + | \mathcal{M}_r(T) |
\end{align*}
$$

$$
\begin{align*}
&\quad + \int_t^T e^{\mathcal{V}_r} \left( \operatorname{sgn}(g_r) \cdot h_r + \operatorname{sgn}(g_r) \cdot g_r + \operatorname{sgn}(U^r_r) \cdot g_r(s) + \operatorname{sgn}(\partial U^r_r) \cdot \nabla g_r(s) - \frac{c}{2} \sum_{i=1}^{4} |\mathcal{Y}_r^i| \right) \, dr
\end{align*}
$$

where $\mathcal{L}_T^0 := \mathcal{L}_T^0(\mathcal{Y}, \mathcal{U}, U^s, \partial U^s)$ denotes the non–decreasing and pathwise–continuous local time of the semimartingale $(\mathcal{Y}, \mathcal{U}, U^s, \partial U^s)$ at 0, see [45, Theorem 70], and, we introduced the martingale (recall that (i) and (ii) guarantee $(Z, V, N, M, V, \partial V, M, \partial M)$ $\in (\mathbb{H}^2) \times (\mathbb{M}^2) \times (\mathbb{H}^2)^2 \times (\mathbb{M}^2)^2$)

$$
\begin{align*}
\mathcal{M}_t &:= \sum_{i=1}^{4} \int_0^t e^{\mathcal{T}}\mathcal{Z}_r^i \cdot \operatorname{sgn}(\mathcal{Y}_r^i) \, dX_r + \int_0^t e^{\mathcal{T}}\mathcal{Z}_r \cdot \operatorname{sgn}(\mathcal{Y}_r^i) \, d\mathcal{M}_r^i, \quad t \in [0, T].
\end{align*}
$$

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Again, we take conditional expectations with respect to $\mathcal{F}_t$ in Equation (6.2) and exploit the fact $\widehat{L}_T^0$ is non-decreasing. Moreover, in combination with (6.1) and Lemma B.1, we obtain back in (6.2) that for $C_1 := 4L_y + TL_y + TL_uL_y$, $C_2 := 2L_u + TL_u + TL_uL_u$, and $C_3 := L_u$

$$e^{\mathfrak{F}_t}([\mathcal{U}_t] + |U_t^*| + |\partial U_t^*|) + \mathbb{E}\left[\int_t^T e^{\mathfrak{F}_r}[\mathcal{U}_r](c/2 - C_2)dr\right]$$

\[\text{and } + \sup_{s \in [0,T]} \mathbb{E}_t \left[\int_t^T e^{\mathfrak{F}_r}[U_r^*](c/2 - C_3)dr\right]\]

\[\leq \mathbb{E}_t\left[e^{\mathfrak{F}_T}(|\xi| + |\eta(T)| + |\eta(s)| + |\partial \eta(s)|)\right] + \mathbb{E}_t\left[\int_t^T e^{\mathfrak{F}_r}(|\hat{h}_r| + |\hat{g}_r| + |\nabla \hat{g}_r| + |\nabla \hat{g}_r|)dr\right] + (T + L_uT)\left(\|\partial \eta\|_{L^{\infty}} + \|\nabla \hat{g}\|_{L^{\infty}} + L_u\left(\|\partial v\|_{H^{2,2}}^2 + \|\partial v\|_{H^{2,2}}^2 + \|\partial v\|_{H^{2,2}}^2\right)\right)

\[+ \mathbb{E}_t\left[\int_t^T e^{\mathfrak{F}_r}(2L_y|\sigma_r^\top z_r|^2 + 2L_v|\sigma_r^\top v_r|^2 + 2L_v|\sigma_r^\top v_r|^2 + L_v|\sigma_r^\top \partial v_r|^2)dr\right].\]

where we recall the notation $L_* = \max\{L_z, L_u, L_v\}$. Thus, for

\[c \geq 2\max\{4L_y + TL_y + TL_uL_y, 2L_u, 2L_u + TL_u + TL_uL_u, L_u\}\]

\[= \max\{8L_y + 2TL_y + 2TL_uL_y, 4L_u + 2TL_u + 2TL_uL_u, 2L_u\},

we obtain

\[
\max\{e^{\mathfrak{F}_t}|\mathcal{U}_t|, e^{\mathfrak{F}_t}|U_t^*|, e^{\mathfrak{F}_t}|U_t^*|, e^{\mathfrak{F}_t}|\partial U_t^*|\} \leq e^{\mathfrak{F}_t}\left(|\mathcal{U}_t| + |U_t^*| + |\partial U_t^*|\right)
\]

\[\leq \|\xi\|_{L^{\infty}} + |\hat{h}|_{L^{1,\infty}} + 2(|\eta|_{L^{\infty}} + |\nabla \hat{g}|)_{L^{1,\infty}} + (1 + T + TL_u)\left(\|\partial \eta\|_{L^{\infty}} + \|\nabla \hat{g}\|_{L^{1,\infty}}\right)

\[+ (4 + T + L_uT)L_u\left(\|\partial v\|_{H^{2,2}}^2 + \|\partial v\|_{H^{2,2}}^2 + \|\partial v\|_{H^{2,2}}^2\right)\]

\[(iv) \text{ We show } (Z, \mathcal{V}, \mathcal{N}, \mathcal{M}) \in \left(H^{2,2}_{\text{BMO}}\right)^2 \times (M^{2,2})^2 \text{ and } \|V\|_{L^{2,2}} + \|M\|_{L^{2,2}} < \infty. \text{ Applying Itô's formula to } e^{\mathfrak{F}_t}(|\mathcal{U}_t|^2 + |U_t^*|^2 + |\partial U_t^*|^2) \text{ we obtain}
\]

\[
\sum_{i=1}^4 e^{\mathfrak{F}_t}|\mathfrak{Y}_t^i|^2 + \int_t^T e^{\mathfrak{F}_r}|\sigma_r^\top \mathfrak{Z}_t^i|^2dr + \int_t^T e^{\mathfrak{F}_r-}d\mathfrak{M}_r\big|_{\mathfrak{Z}_t^i} + \mathcal{M}_t - \mathcal{M}_T
\]

\[= e^{\mathfrak{F}_t}\left(|\xi|^2 + |\eta(T)|^2 + \|\mathcal{U}_t^*\|^2 + |\mathcal{M}_t| + \|\partial \mathcal{M}_t\|_{L^{2,2}} + \|\partial \mathcal{M}_t\|_{L^{2,2}} + \|\partial \mathcal{M}_t\|_{L^{2,2}}\right)

\[+ \int_t^T e^{\mathfrak{F}_r}\left(2\mathcal{U}_t^* \cdot h_r + 2\mathcal{U}_t^* \cdot g_r + 2\partial \mathcal{M}_t^* \cdot g_r(s) + 2\partial \mathcal{M}_t^* \cdot \nabla g_r(s) - c \sum_{i=1}^4 |\mathfrak{Y}_r^i|^2\right)dr\]

where for any $s \in [0,T]$ we introduced the martingale

\[
\widetilde{\mathfrak{M}} := \sum_{i=1}^4 \int_0^t e^{\mathfrak{F}_t} \mathfrak{Y}_t^i \cdot dX_r + \int_0^t e^{\mathfrak{F}_r-} \mathfrak{Y}_r^i \cdot d\mathfrak{N}_r, \quad t \in [0,T].
\]

Indeed, Burkholder–Davis–Gundy’s inequality and the fact that $(Y, U, Z, V, U, \partial U, V, \partial V) \in (S^2)^2 \times (S^2)^2 \times (S^2)^2 \times (S^2)^2$, recall (i) and (ii), implies that there exists $C > 0$ such that

\[
\mathbb{E}\left[\sup_{t \in [0,T]} \left|\int_0^t e^{\mathfrak{F}_t} \mathfrak{Y}_t^i \cdot dX_r\right|\right] \leq C \mathbb{E}\left[\int_0^T e^{\mathfrak{F}_t} |\mathfrak{Y}_t^i|^2 (|\mathfrak{Z}_t^i|^2)dr\right] \leq C e^{\mathfrak{F}_T} \mathbb{E}\left[|\mathfrak{Y}_t^i|^2 \mathbb{E}\left[|\mathfrak{Z}_t^i|^2\right]\right], i \in \{1, ..., 4\},
\]

which guarantees each of the processes in $\widetilde{\mathfrak{M}}$ is an uniformly integrable martingale. Thus, taking conditional expectations with respect to $\mathcal{F}_t$ in Equation (6.4), we obtain

\[
\sum_{i=1}^4 e^{\mathfrak{F}_t}|\mathfrak{Y}_t^i|^2 + \mathbb{E}\left[\int_t^T e^{\mathfrak{F}_r}|\mathfrak{Z}_t^i|^2dr + \int_t^T e^{\mathfrak{F}_r-}d\mathfrak{M}_r\big|_{\mathfrak{Z}_t^i}\right].
\]
We now let \( \tau \)

These inequalities in combination with Lemma 2.2, as well as, as we have

These inequalities in combination with Lemma B.1, and Young’s inequality, show that if we define \( C_i := \tilde{C}_{e_i,1} + \varepsilon_8 + \varepsilon_9 + \varepsilon_{10} + (\varepsilon_1 + \varepsilon_2)T L_y^2 \), \( C_2 := \tilde{C}_{e_2,1} + \varepsilon_7 \), \( C_3 := \tilde{C}_{e_5} + \varepsilon_6 + (\varepsilon_1 + \varepsilon_2)T L_y^2 \), \( C_4 := \tilde{C}_{e_{10,1}} \), then for any \( \varepsilon_i > 0 \), \( i \in \{7, \ldots, 15\} \)

We now let \( \tau \in \mathcal{T}_{0,T} \). In light of (6.3), for

(2.2) yields

\[
\sum_{i=1}^{4} e^{\tau t} (|\mathcal{G}|^2) + E_t \left[ \int_0^T e^{\tau t} |\mathcal{G}|^2 \right] + E_t \left[ \int_0^T e^{\tau t} \right]
\]
which in turn leads to
\[
\frac{1}{10} \left( \|Y\|_{S^{\infty,c}}^2 + \|U\|_{S^{\infty,c}}^2 + \|U\|_{S^{\infty,c}}^2 + \|\partial U\|_{S^{\infty,c}}^2 + \|Z\|_{\text{BMO}}^2 \right)
+ \|V\|_{\text{BMO}}^2 + \|\partial V\|_{\text{BMO}}^2 + \|\mathcal{N}\|_{M^{2,c}}^2 + \|M\|_{M^{2,c}}^2 + \|\partial M\|_{M^{2,c}}^2
\leq \|\xi\|_{S^{\infty,c}}^2 + 2\|\eta\|_{S^{\infty,c}}^2 + (1 + \xi_1 + \xi_2)\|\partial \eta\|_{S^{\infty,c}}^2 + \xi_3\|\tilde{h}\|_{S^{\infty,c}}^2 + (\xi_4 + \xi_5)\|\tilde{g}\|_{S^{\infty,c}}^2
+ (\xi_1 + \xi_2 + \varepsilon_6)\|\nabla \tilde{g}\|_{S^{\infty,c}}^2 + 2L^2_v \left( \varepsilon_1 + \varepsilon_2 + \varepsilon_14 + \varepsilon_16 + \varepsilon_18 + \varepsilon_19\right)\|\tilde{g}\|_{\text{BMO}}^2
+ 2L^2_v \left( \varepsilon_1 + \varepsilon_2 + \varepsilon_13 + \varepsilon_15 + \varepsilon_17 + \varepsilon_19\right)\|\partial v\|_{\text{BMO}}^2
+ (\xi_3 - \xi_1 - \xi_1^2 + \varepsilon_1^2)\|Y\|_{S^{\infty,c}}^2 + \left( \varepsilon_4 + \varepsilon_14 + \varepsilon_15\right)\|U\|_{S^{\infty,c}}^2 + (\xi_5 - \xi_16 + \varepsilon_17)\|\partial U\|_{S^{\infty,c}}^2
+ (\xi_1^2 - \xi_1 + 1 - \varepsilon_1 + \varepsilon_1^2)\|\partial \tilde{g}\|_{\text{BMO}}^2
\tag{6.6}
\]

From (6.6) we conclude \((Z, N) \in H^{2,c} \times M^{2,c}, \|V\|_{\text{BMO}}^2 + \|\partial V\|_{\text{BMO}}^2 + \|\partial U\|_{\text{BMO}}^2 + \|\partial M\|_{\text{BMO}}^2 < \infty\).

At this point, we can highlight a crucial step in this approach. It is clear from (6.6) that the norm of \(\mathcal{F}\) does not have a linear growth in the norm of the input. In the following, we will see that choosing the data of the system small enough and localising \(\mathcal{F}\) will bring us back to the linear growth scenario. For this, we observe that if we define
\[
C_{\varepsilon} := \min \{ 1 - 10(\varepsilon_3 - \varepsilon_12 + \varepsilon_13) , 1 - 10(\varepsilon_4 - \varepsilon_14 + \varepsilon_15) , 1 - 10(\varepsilon_5 - \varepsilon_16 + \varepsilon_17) , 1 - 10(\varepsilon_6 - \varepsilon_18 + \varepsilon_19 + \varepsilon_20) \},
\]
\[
\{ 1 - 10(\varepsilon_3 - \varepsilon_12 + \varepsilon_13) , 1 - 10(\varepsilon_4 - \varepsilon_14 + \varepsilon_15) , 1 - 10(\varepsilon_5 - \varepsilon_16 + \varepsilon_17) , 1 - 10(\varepsilon_6 - \varepsilon_18 + \varepsilon_19 + \varepsilon_20) \} \subseteq (0, 1]^4
\]
and for some \(\gamma \in (0, \infty)\), \(\varepsilon_0 \leq \gamma R^2/10\), we obtain back in (6.6)
\[
\|(Y, Z, N, U, \partial U, \partial V, \partial M)\|_{H^c}^2
\leq C_{\varepsilon}^{-1} \left( 10I_0^2 + 20L^2_v \left( \varepsilon_1 + \varepsilon_2 + \varepsilon_12 + \varepsilon_14 + \varepsilon_16 + \varepsilon_18 \right)\|\tilde{g}\|_{H^{1,c}}^4
+ (\varepsilon_1 + \varepsilon_2 + \varepsilon_12 + \varepsilon_13 + \varepsilon_15 + \varepsilon_17 + \varepsilon_19)\|v\|_{H^{1,c}}^2 \right)
\]
\[
\leq C_{\varepsilon}^{-1} R^2 \left( \gamma + 20L^2_v \left( \varepsilon_1 + \varepsilon_2 + \varepsilon_12 + \varepsilon_14 + \varepsilon_16 + \varepsilon_18 \right)\|\tilde{g}\|_{H^{1,c}}^2
+ (\varepsilon_1 + \varepsilon_2 + \varepsilon_12 + \varepsilon_13 + \varepsilon_15 + \varepsilon_17 + \varepsilon_19)\|v\|_{H^{1,c}}^2 \right)
\]
\[
\leq C_{\varepsilon}^{-1} R^2 \left( \gamma + 20L^2_v R^2 \left( \varepsilon_1 + \varepsilon_2 + \sum_{i=12}^{20} \varepsilon_i \right) \right)
\]

Therefore, to obtain \(\mathcal{F}(B_R) \subseteq B_R\), that is to say that the image under \(\mathcal{F}\) of the ball of radius \(R\) is contained in the ball of radius \(R\), it is necessary to find \(R^2\) such that the term in parentheses above is less or equal than \(C_{\varepsilon}\), i.e.
\[
R^2 \leq \frac{1}{20L^2_v \varepsilon_1 + \varepsilon_2 + \sum_{i=12}^{20} \varepsilon_i}
- C_{\varepsilon} - \gamma
\tag{6.7}
\]
Clearly, there are many choices of \(\varepsilon\)'s so that the above holds. Among such, we wish to choose \(\gamma, \varepsilon_i\), so that the expression to the right in (6.7) is maximal. In light of lemma B.2, we have that \(||(Y, Z, N, U, V, M)\|_{H^c} \leq R\) provided that
\[
R^2 \leq \frac{1}{95 \cdot 3 \cdot 5^2 \cdot 7 \cdot L^2_v}
\tag{6.8}
\]
(v) Lastly, we are left to argue \((U, V, M, \partial U, \partial V, \partial M) \in S^{\infty,c} \times M^{2,c} \times M^{2,c} \times M^{2,c} \times M^{2,c} \times M^{2,c}\). The argument for \((U, V, M)\) is analogous to that of \((\partial U, \partial V, \partial M)\), thus we argue the continuity of the applications \([[0, T], B([0, T]) \mapsto (S^{\infty,c}, \|\cdot\|_{S^{\infty,c}}) \mapsto \text{BMO}^*, \|\cdot\|_{M^{2,c}}^2, \|\cdot\|_{M^{2,c}}^2) \mapsto \varphi^* = U^* \mapsto (R^*, V^*, M^*)\).

Recall \(\rho_g\) denotes the modulus of continuity of \(g\). Let \((s_n) \subseteq [0, T], s_n \xrightarrow{n \to \infty} s_0 \in [0, T]\) and for \(\varphi \in \{U, V, M, u, v, \eta\}\), let \(\Delta \rho^\varphi := \varphi^* - \varphi^0\). Applying Itô's formula to \(e^{it} \|\Delta U^n\|_{L^2_v}^2\), proceeding as in Step 1 (ii) we obtain
\[
\|\Delta U^n\|_{S^{\infty,c}}^2 + \|\Delta V^n\|_{\text{BMO}}^2 + \|\Delta M^n\|_{M^{2,c}}^2 \leq 4 \left( \|\Delta \eta^n\|_{S^{\infty,c}}^2 + 4L^2_v \|\Delta v^n\|_{\text{BMO}}^2 + \rho_g(|s_n - s_0|)^2 \right).
\]
We conclude, \(\mathcal{F}(B_R) \subseteq B_R\) for all \(R\) satisfying (6.8).
Step 2: We now argue that $\mathcal{F}$ is a contraction in $\mathcal{B}_R \subseteq \mathcal{H}$ for the norm $\| \cdot \|_{\mathcal{H}}$.

Let $(Y^n, Z^n, U^n, V^n, M^n, U, V, M, \partial U, \partial V, \partial M) := (y^n, z^n, u^n, v^n, m^n, \partial u^n, \partial v^n, \partial m^n) \in \mathcal{B}_R$, $i = 1, 2$.

For $\varphi \in \{ y, z, n, u, v, m, \partial u, \partial v, \partial m \}$, we denote $\delta \varphi := \varphi - \varphi$ and

\[
\delta h_t := h_t(Y_t^1, z_t^1, u_t^1, v_t^1, U_t^1, V_t^1) - h_t(Y_t^2, z_t^2, U_t^2, V_t^2), \\
\delta g_t := g_t(Y_t^1, U_t^1, e_t^1, Y_t^1, z_t^1) - \partial U_t^2, g_t(Y_t^2, U_t^2, e_t^2, Y_t^2, z_t^2), \\
\delta \xi_t := \delta g_t(Y_t^1, U_t^1, e_t^1, Y_t^1, z_t^1) - \partial U_t^2, g_t(Y_t^2, U_t^2, e_t^2, Y_t^2, z_t^2).
\]

Applying Itô’s formula to $e^{ct} (|\delta Y_t|^2 + |\delta U_t|^2 + |\delta H_t|^2)$, we obtain that for any $t \in [0, T]$

\[
\sum_{i=1}^4 e^{ct} |\delta Y_t|^2 + \int_t^T e^{ct} |\sigma_{t, r}^{(1)} \delta Y_t|^2 dr + \int_t^T e^{ct} - dTr[\delta Y_t] + \delta \eta_t - \delta \eta_T
\]

\[
= \int_t^T e^{ct} \left(2 \delta Y_t \cdot \delta h_t + 2 \delta U_t \cdot \delta g_t + 2 \delta U_t \cdot \delta g_t(s) + 2 \partial \delta U_t \cdot \partial \delta g_t(s) - c \sum_{i=1}^4 |\delta Y_t|^2 \right) dr
\]

where $\delta \eta_t$ denotes the corresponding martingale term. Let $\tau \in \mathcal{T}_{0,T}$ as in Lemma B.1 we obtain for $c > 2L_u$

\[
\mathbb{E}_r \left[ \int_T^e e^{ct} 3T |\delta U_t|^2 dt \right] \leq \sup_{s \in [0, T]} \mathbb{E}_r \left[ \int_T^e \left( e^{ct} |\delta \nabla g_t(s, 0, \partial v_t^0, 0, v_t^0, 0, z_t)| dt \right)^2 \right]
\]

\[
+ TL_u^2 \mathbb{E}_r \left[ \int_T^e e^{ct} |\delta Y_t|^2 dt \right] + TL_u^2 \sup_{s \in [0, T]} \mathbb{E}_r \left[ \int_T^e e^{ct} |\delta U_t|^2 dt \right].
\]

We now take conditional expectation with respect to $\mathcal{F}_r$ in the expression above. In addition, we use Assumption B in combination with (6.9), exactly as in Step 1 (iv). Then we obtain from Young’s inequality that for any $\xi_t \in (0, \infty)$, $i \in \{1, \ldots, 11\}$, and

\[
c \geq \max \left\{ 2L_u + \xi_1 + 3T L_u^2, \xi_8 + \xi_9 + \xi_{10} + \xi_7, 2L_u + \xi_7 + 3T \xi_2 + 1 + \xi_1, 2L_u + \xi_7 L_u^2, 2L_u + \xi_1 L_u^2 + \xi_1 \right\}
\]

it follows that

\[
\sum_{i=1}^4 e^{ct} |\delta Y_t|^2 + \mathbb{E}_r \left[ \int_T^e \left( e^{ct} |\sigma_{t, r}^{(1)} \delta Y_t|^2 + \int_T^e e^{ct} - dTr[\delta Y_t] \right) \right]
\]

\[
\leq \xi_3^{-1} |\delta Y_t|^2 + \xi_4^{-1} |\delta U_t|^2 + \xi_5^{-1} |\delta U_t|^2 + \xi_6^{-1} |\delta U_t|^2 + \xi_7^{-1} |\delta Y_t|^2 + \xi_8^{-1} |\delta Y_t|^2 + \xi_9^{-1} |\delta Y_t|^2 + \xi_10^{-1} |\delta Y_t|^2 + \xi_11^{-1} |\delta Y_t|^2
\]

\[
+ \xi_1 + \xi_2 \sup_{s \in [0, T]} \mathbb{E}_r \left[ \int_T^e e^{ct} |\nabla g_t(s, 0, \partial v_t^0, 0, v_t^0, 0, z_t)| dt \right]^2
\]

\[
+ \xi_3 \sup_{r \in [0, T]} \mathbb{E}_r \left[ \int_T^e e^{ct} |\delta h_t|^2 dt \right] + \xi_4 \sup_{r \in [0, T]} \mathbb{E}_r \left[ \int_T^e e^{ct} |\delta g_t|^2 dt \right]^2
\]

\[
+ \xi_5 \sup_{s \in [0, T]} \mathbb{E}_r \left[ \int_T^e e^{ct} |\delta g_t(s)| dt \right]^2 + \xi_6 \sup_{s \in [0, T]} \mathbb{E}_r \left[ \int_T^e e^{ct} |\delta \nabla g_t(s)| dt \right]^2.
\]
We now estimate the terms on the right side of (6.11). Note that in light of Assumption B.(iii) we have

\[
\max \left\{ \left| \mathbb{E}_\tau \left[ \int_\tau^T e^{\tau t} |\partial \tilde{g}_i(s, \sigma_u, \partial u, \partial v, \partial v^\ast, z)| \, ds \right] \right|^2, \left| \mathbb{E}_\tau \left[ \int_\tau^T e^{\tau t} |\partial \tilde{g}_i(s)| \, ds \right] \right|^2 \right\}
\leq \left| \mathbb{E}_\tau \left[ \int_\tau^T e^{\tau t} \left( L_v \sigma_u^T \partial v + \left( |\sigma_u^T \partial v| + |\sigma_u^T \partial v^*| \right) + L_u \sigma_u^T \partial v^* \left( |\sigma_u^T \partial v^*| + |\sigma_u^T \partial v^*| \right) + \right. \right. \right. \\
\left. \left. \left. + L_z |\sigma_u^T \partial z| \left( |\sigma_u^T \partial z| + |\sigma_u^T \partial z^*| \right) \right) \, ds \right] \right|^2 \right.
\leq 3L^2 \left| \mathbb{E}_\tau \left[ \int_\tau^T e^{\tau t} \left( |\sigma_u^T \partial v| + |\sigma_u^T \partial v^*| \right) \, ds \right] \right|^2
\]
\[
\left( \sqrt{30 + (\xi_1 + \xi_2)} + \sqrt{30} \right)^2 \leq 2^2 \cdot 7 \cdot 10, \Xi \text{ is contractive}, \text{i.e.}
\]

\[
\|\delta Y, \delta Z, \delta N, \delta U, \delta V, \delta M, \delta DU, \delta DV, \delta DM \|_{\mathcal{H}^c}^2 < \|\delta z\|_{\mathcal{H}^c}^2 + \|\delta v\|_{\mathcal{H}^c}^2 + \|\delta \upsilon\|_{\mathcal{H}^c}^2.
\]

**Step 3:** We consolidate our results. To begin with, we collect the constraints of the weight of the norms. In light of (6.5) and (6.10), c satisfies

\[
c \geq \max \{ 2L_y + \epsilon_1^2 TL_1^2 + \epsilon_1 TL_1^2 + \epsilon_2 TL_2^2 + \epsilon_1^2 1L_2^2 \leq \epsilon_8 + \epsilon_8 + \epsilon_8, 2L_u + \epsilon_1 1T_1^2 + \epsilon_2 TL_2^2 + \epsilon_1^2 1L_1^2 + \epsilon_8 + \epsilon_8 + \epsilon_8 + \epsilon_8 + \epsilon_8, \}
\]

\[
\text{where the equality follows from the choice } \epsilon_i = \epsilon_i, i \in \{1, 2, \ldots, 11\}.
\]

All together we find that given \( \gamma \in (0, \infty), \epsilon_i \in (0, \infty), i \in \{1, \ldots, 11\}, \epsilon_i \in (0, \infty), \text{ such that } \epsilon_1 + \epsilon_2 \leq (2\sqrt{70} - \sqrt{30})^2 - 30, \}

\( \Xi \) is a well-defined contraction in \( \mathcal{B}_R \subseteq \mathcal{H}^c \) for the norm \( \|\cdot\|_{\mathcal{H}^c} \) provided: (i) \( \gamma, \epsilon_i, i \in \{1, \ldots, 6\}, \) and the data of the problem satisfy \( I_0^2 \leq \gamma R^2/10; \) (ii) \( c \) satisfies (6.13).

**Proof of Theorem 3.9.** We show for \( \|\delta M\|_{\mathcal{BMO}^{2, \epsilon}}^2 < \infty, \) the argument for \((N, M)\) being completely analogous. Without loss of generality we assume \( c = 0, \) see Remark 2.2. In light of Assumption B, we have that \( dr \otimes d^2\text{-a.e.} \)

\[
|\nabla g_r(s, DU_r^s, DU_r^s, U_r^s, V_r^s, Y_r, Z_r)| \leq L_u|U_r^s| + L_v|\sigma_r^T V_r^s|^2 + L_u|U_r^s| + L_v|\sigma_r^T V_r^s|^2 + L_y|Y_r| + L_z|\sigma_r^T Z_r|^2 + |\nabla g_r(s)|.
\]

Let \( \tau \in T_0.T. \) We now note, recall \( \partial V \in H_2^{2} \)

\[
E_{\tau}\left( \left( \int_{\tau-}^{\tau} \partial V_r^s \partial dX_r \right)^2 \right) = E_{\tau}\left( \left( \int_{\tau-}^{\tau} \partial V_r^s \partial dX_r \right)^2 \right) = E_{\tau}\left( \left( \int_{\tau-}^{\tau} \partial V_r^s \partial dX_r \right)^2 \right).
\]

All together, it follows from (S) and Jensen’s inequality that

\[
E_{\tau}\left( \left( \int_{\tau-}^{\tau} \partial dM_r^s \right)^2 \right) \leq 10E_{\tau}\left[ |\partial \eta(s)|^2 + \left| \nabla g_r(s) \right|^2 + |\partial U_r^s|^2 + T \int_{\tau-}^{\tau} L_u|\partial U_r^s|^2 + L_u|U_r^s|^2 + L_y|Y_r|^2 \right]
\]

\[
+ \left| \int_{\tau-}^{\tau} L_v|\sigma_r^T V_r^s|^2 \right|^2 + \left| \int_{\tau-}^{\tau} L_z|\sigma_r^T Z_r|^2 \right|^2 + \left| \int_{\tau-}^{\tau} \partial V_r^s \partial dX_r \right|^2 \leq 10 \left( |\partial \eta(s)|^2_{2, \infty}^2 + \| \nabla \eta \|^2_{\mathcal{BMO}^{2, \epsilon}} + L_u T^2 \| \partial U \|^2_{\mathcal{BMO}^{2, \epsilon}} + L_u T^2 \| \partial U \|^2_{\mathcal{BMO}^{2, \epsilon}} + L_y T \| \partial V \|^2_{\mathcal{BMO}^{2, \epsilon}} + 2L_x^2 \| \partial \nu \|^2_{\mathcal{BMO}^{2, \epsilon}} + 2L_x^2 \| \partial \nu \|^2_{\mathcal{BMO}^{2, \epsilon}} \right)
\]

**Proof of Theorem 3.10.** Upon close inspection of the proof of Theorem 3.6, we see that the only stages of the argument where both the presence and the norm of \((N, M, \partial M)\) plays a role are in (6.6), (6.7) and (6.12). We address each of them in the following. We consider the case (i). The argument for (ii) follows similarly.

If we were to require \( \mathcal{BMO} \)-norms on \((N, M, \partial M)\) we see that

(i) In (6.6), the presence of the ess\( \sup \) in the \( \mathcal{BMO} \)-norm would require us to consider the estimate right before (6.6) for each of the 9 processes that define the solution to (S). Thus we obtain a factor 11 instead of 10. This yields

\[
C_\epsilon := \min \left\{ 1 - 11(\epsilon_3^{-1} + \epsilon_3^{-1} + \epsilon_3^{-1}), 1 - 11(\epsilon_4^{-1} + \epsilon_4^{-1} + \epsilon_4^{-1}), 1 - 11(\epsilon_5^{-1} + \epsilon_5^{-1} + \epsilon_5^{-1}), 1 - 11(\epsilon_6^{-1} + \epsilon_6^{-1} + \epsilon_6^{-1}) \right\}
\]
\[ I_0^2 \leq \gamma R^2 / 11, \]

and
\[ \|(Y, Z, N, U, V, M, \partial U, \partial V, \partial M)\|_{H^3}^2 \leq C_{\varepsilon}^{-1} R^2 \left( \gamma + 24 L_{\varepsilon}^2 R^2 \left( \varepsilon_1 + \varepsilon_2 + \sum_{i=12}^{20} \varepsilon_i \right) \right). \]

(ii) As a consequence of the previous observation (6.7) would be replace by
\[ R^2 \leq \frac{C_{\varepsilon} - \gamma}{24 L_{\varepsilon} \varepsilon_1 + \varepsilon_2 + \sum_{i=12}^{20} \varepsilon_i} \leq \frac{1}{2^3 \cdot 3 \cdot 7 \cdot 11^2 \cdot L_{\varepsilon}} = \mathcal{U}(11), \]

where the upper bound results from the proper version of the optimisation procedure, i.e. Lemma B.2.

(iii) Likewise, with \( C_{\varepsilon} \) as in the proof, (6.12) is now given by
\[ \|(Y, Z, N, U, V, M, \partial U, \partial V, \partial M)\|_{H^3}^2 \]
\[ \leq 24 C_{\varepsilon}^{-1} L_{\varepsilon}^2 R^2 (3 \varepsilon_1^2 + 3 \varepsilon_2^2 + 2 \varepsilon_4^2 + 2 \varepsilon_5^2 + 3 \varepsilon_6^2) \left( \|\delta z\|_{H^2}^2 + \|\delta v\|_{H^2}^2 + \|\delta \partial v\|_{H^2}^2 \right). \]

where the second inequality follows from the new version of the optimisation procedure, i.e. Lemma B.3.

All things considered, Step 3 will lead to require Assumption C for \( \kappa = 11 \). By assumption the result follows. \( \square \)

7 Proof of the Quadratic case

Proof of Theorem 3.12. For \( c > 0 \), let us introduce the mapping
\[ \Xi : (B_R, \|\cdot\|_{H^3}) \rightarrow (B_R, \|\cdot\|_{H^3}) \]
\[ (y, z, n, u, v, m, \partial u, \partial v, \partial m) \rightarrow (Y, Z, N, U, V, M, \partial U, \partial V, \partial M), \]

with \( \Phi = (Y, Z, N, U, V, M, \partial U, \partial V, \partial M) \) given for any \( s \in [0, T], \mathbb{P}\text{-a.s.} \) for any \( t \in [0, T] \) by
\[ Y_t = \xi(T, X, s, t) + \int_t^T h_r(Y, r, z_r, u^r, v^r, \partial U^r) dr - \int_t^T \hat{Z}^r dX_r - \int_t^T dN_r, \]
\[ U^s_t = \eta(s, X, s, t) + \int_t^T g_r(s, X, u^s_r, v^s_r, y, z_r) dr - \int_t^T \hat{V}^s_r dX_r - \int_t^T dM^s_r, \]
\[ \partial U^s_t = \delta_\eta(s, X, s, t) + \int_t^T \nabla g_r(s, X, u^s_r, v^s_r, y, z_r) dr - \int_t^T \hat{V}^s_r dX_r - \int_t^T dM^s_r. \]

Step 1: We first argue that \( \Xi \) is well-defined.

(i) Let us first remark that for \( u \in S^{\infty, 2, c} \)
\[ E \left[ \int_0^T |u_t|^2 dt \right] \leq T \|u\|_{S^{\infty, 2, c}}^2. \] (7.1)

In light of Assumption D, there is \( c > 0 \) such that \( (\xi, \eta, \partial \eta, \hat{f}, \hat{g}, \nabla \hat{g}) \in \mathcal{C}^{\infty, c} \times \mathcal{L}^{\infty, c} \times \mathcal{L}^{1, \infty, c} \times (\mathcal{L}^{1, \infty, 2, c})^2 \), thus, we may use (2.2) and (7.1) to obtain
\[ E \left[ |\xi(T)|^2 + \int_0^T |h_t(y, t, z_t, u^t, v_t)|^2 |dt| \right]^2 + \sup_{s \in \mathcal{C}^{\infty, c}} E \left[ |\eta(s)|^2 + \int_0^T |g_t(s, u^s_t, v^s_t, y, z_t)|^2 |dt| \right]^2 \]
\[ + \sup_{s \in \mathcal{C}^{\infty, c}} E \left[ |\partial \eta(s)|^2 + \int_0^T |\nabla g_t(s, u^s_t, v^s_t, y, z_t)|^2 |dt| \right]^2 \]
\[ \leq E \left[ |\xi(T)|^2 + 5 \int_0^T |\hat{h}|^2 |dt|^2 + 17 L_{\varepsilon}^2 \int_0^T |y|^2 |dt|^2 + 17 L_{\varepsilon}^2 \int_0^T |z|^2 |dt|^2 + 5 L_{\varepsilon}^2 \int_0^T |u|^2 |dt|^2 + 5 L_{\varepsilon}^2 \int_0^T |v|^2 |dt|^2 \right], \]
\[\sup_{s \in [0, T]} \mathbb{E} \left[ |\eta(s)|^2 + 5 \left( \int_0^T |\tilde{g}_t(s)| dt \right)^2 + 12L_2^2 \left( \int_0^T |u_t^i|^2 dt \right)^2 + 12L_2^2 \left( \int_0^T |v_t^i|^2 dt \right)^2 \right] + \sup_{s \in [0, T]} \mathbb{E} \left[ (\partial\eta(s))^2 + 7 \left( \int_0^T |\nabla\tilde{g}_t(s)| dt \right)^2 + 7L_2^2 \left( \int_0^T |\partial u_t^i|^2 dt \right)^2 + 7L_2^2 \left( \int_0^T |\partial v_t^i|^2 dt \right)^2 \right] \leq \|\xi\|_{S_{\infty}, \infty}^2 + 5\|\tilde{h}\|_{S_{\infty}, \infty}^2 + 5\|\tilde{g}\|_{S_{\infty}, \infty}^2 + 7\|\nabla\tilde{g}\|_{S_{\infty}, \infty}^2 + 17L_2^2 T^2 \|\tilde{G}\|_{S_{\infty}, \infty}^2 + 17L_2^2 T^2 \|\tilde{Y}\|_{S_{\infty}, \infty}^2 + 24L_2^2 \|v\|_{S_{\infty}, \infty}^2 + 7L_2^2 \|\tilde{v}\|_{S_{\infty}, \infty}^2 + 14L_2^2 \|\tilde{v}\|_{\tilde{v}^2_{BMO}} < \infty.\]

Therefore, by [20, Theorem 3.2], there exists a well-posed system of BSDEs with unique solution in the space \(S^2\). We recall the spaces involved in the definition of \(S^2\), and their corresponding norms, were introduced in Section 2.2.

(ii) For \(U_t := U_t^1, \tilde{V}_t := V_t^1, M_t := M_t^1 - \int_0^t \partial M_t^i \, dr, t \in [0, T]\), \((U, V, M) \in \mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{M}^2\) and satisfy the equation

\[\mathcal{U}_t = \eta(T, X, \lambda_T) + \int_t^T (g_r(r, X, u_r^i, v_r^i, y_r, z_r) - \partial U_r^i) \, dr - \int_t^T \tilde{V}_r \, dX_r - \int_t^T dM_r, t \in [0, T], \mathbb{P}\text{-a.s.}\]

(iii) \((Y, U) \in \mathbb{S}_{\infty, \infty} \times \mathbb{S}_{\infty, \infty}\) and \(\|U\|_{S_{\infty, \infty}} + \|U\|_{S_{\infty, \infty}} < \infty.\)

In light of Assumption D, \(dr \otimes d\mathbb{P}\text{-a.e.}\)

\[|h_t| \leq L_2|y_t|^2 + L_2|\sigma_r^T z_r|^2 + L_2|u_t|^2 + L_2|\sigma_r^T v_t|^2 + L_2|\partial U|^2 + |\tilde{h}_t|,\]

\[|g_t| \leq L_2|u_t|^2 + L_2|\sigma_r^T v_t|^2 + L_2|y_t|^2 + L_2|\sigma_r^T z_r|^2 + |\partial U|^2 + |\tilde{g}_t|,\]

\[|g_t(s)| \leq L_2|u_t|^2 + L_2|\sigma_r^T v_t|^2 + L_2|y_t|^2 + L_2|\sigma_r^T z_r|^2 + |\tilde{g}_t(s)|,\]

\[|\nabla g_t(s)| \leq L_2|u_t|^2 + L_2|\sigma_r^T v_t|^2 + L_2|y_t|^2 + L_2|\sigma_r^T z_t|^2 + |\tilde{g}_t(s)|.\]

Again, we apply Meyer–Itô’s formula to \(e^{\tilde{\xi}^T} (|\mathcal{Y}_t| + |\mathcal{U}_t| + |\partial U_t^i|)\) and take conditional expectations with respect to \(\mathcal{F}_t\) in Equation (6.2). Moreover, in combination with (7.2) and Lemma B.1, we obtain back in (6.2) that

\[e^{\tilde{\xi}^T} (|\mathcal{Y}_t| + |\mathcal{U}_t| + |\partial U_t^i|) + \mathbb{E} \left[ \int_t^T \frac{c}{2} e^{\tilde{\xi}^T} |\mathcal{Y}_r| \, dr \right] + \mathbb{E} \left[ \int_t^T \frac{c}{2} e^{\tilde{\xi}^T} |\partial U_r^i| \, dr \right] \leq \mathbb{E} \left[ e^{\tilde{\xi}^T} \left( |\xi| + \|\eta(T)| + \|\eta(s)\| + |\partial \eta(s)| \right) \right] + \mathbb{E} \left[ \int_t^T e^{\tilde{\xi}^T} \left( |\tilde{h}_t| + |\tilde{g}_t| + |\tilde{y}_t| + |\nabla \tilde{g}_t(s)| \right) \, dr \right] + \mathbb{E} \left[ \int_t^T e^{\tilde{\xi}^T} \left( L_2|y_t|^2 + L_2|\sigma_r^T y_t|^2 + 2L_2|\sigma_r^T u_t|^2 \right) \, dr \right] + \mathbb{E} \left[ \int_t^T e^{\tilde{\xi}^T} \left( L_2|\sigma_r^T u_t|^2 + 4L_2|\sigma_r^T z_t|^2 + 2L_2|\sigma_r^T v_t|^2 \right) \, dr \right].\]

where we recall the notation \(L_* = \max\{L_y, L_u, L_{\sigma_r}, L_{\sigma_r^T z_t}, L_{\sigma_r^T v_t} \}\). Thus, for any \(c > 0\) we obtain

\[\max \{e^{\tilde{\xi}^T} |\mathcal{Y}_t|, e^{\tilde{\xi}^T} |\mathcal{U}_t|, e^{\tilde{\xi}^T} |\partial U_t^i| \} \leq \|\xi\|_{L_{\infty, \infty}} + \|\tilde{h}\|_{L_{\infty, \infty}} + \|\tilde{g}\|_{L_{\infty, \infty}} + (1 + T + L_2 T) \left( \|\partial \eta\|_{L_{\infty, \infty}} + \|\nabla \tilde{g}\|_{L_{\infty, \infty}} \right) + (4 + T + L_2 T) L_* \left( \|y\|_{S_{\infty}, \infty} + \|u\|_{S_{\infty}, \infty} + \|v\|_{S_{\infty}, \infty} + \|\partial u\|_{S_{\infty}, \infty} + \|\partial v\|_{S_{\infty}, \infty} \right) + \|M\|_{S_{\infty}, \infty} + \|\partial M\|_{S_{\infty}, \infty} < \infty.\]

(iv) We show \((Z, V, N, M) \in (\mathbb{H}^2_{BMO})^2 \times (\mathbb{M}^2_{BMO})^2\) and \(\|V\|_{S_{\infty}, \infty}^2 + \|M\|_{S_{\infty}, \infty}^2 + \|\partial V\|_{S_{\infty}, \infty}^2 + \|\partial M\|_{S_{\infty}, \infty}^2 < \infty.\)
\[ \sum_{i=1}^{4} e^{cT} |\mathcal{Y}_t|^2 + \mathbb{E}_t \left[ \int_t^T e^{cT} |\sigma_r^T 3_r^i|^2 \, dr + \int_t^T e^{cT} \, d\mathcal{T}[\mathcal{Y}_t] \right] \]

\[ = \mathbb{E}_t \left[ \int_t^T e^{cT} \left| \nabla \tanh(\mathcal{Y}_t) \right|^2 \, dr + \frac{e}{e^2} \int_t^T e^{cT} \mathcal{U}_r^2 \, dr \right] \]

Equation (2.2) yields

\[ c \geq \max \{ \varepsilon^{-1} 7T L_u^2, \varepsilon^{-1} 7T, 2L_u \}, \quad (7.3) \]
which in turn leads to

\[
\frac{1}{10} \left( \| \nabla \|_{BMO}^{2} + \| U \|_{BMO}^{2} + \| U \|_{\mathcal{M}}^{2} + \| \partial U \|_{BMO}^{2} + \| \partial M \|_{\mathcal{M}}^{2} \right) \\
+ \| V \|_{BMO}^{2} + \| \partial V \|_{BMO}^{2} + \| \nabla V \|_{\mathcal{M}}^{2} + \| M \|_{\mathcal{M}}^{2} + \| \partial M \|_{\mathcal{M}}^{2} \\
\leq \| \varepsilon \|_{BMO}^{2} + 2\| \nabla \|_{BMO}^{2} + (1 + \varepsilon_1 + \varepsilon_2) \| \partial \|_{BMO}^{2} + \varepsilon_3 \| \partial \|_{\mathcal{M}}^{2} \\
+ (\varepsilon_4 + \varepsilon_5) \| \partial \|_{\mathcal{M}}^{2} + (\varepsilon_1 + \varepsilon_2 + \varepsilon_6) \| \nabla \|_{BMO}^{2} \\
+ L^2 T^2 (\varepsilon_1 + \varepsilon_2 + \varepsilon_7 + \varepsilon_8 + \varepsilon_9 + \varepsilon_10) \| y \|_{BMO}^{2} + L^2 T^2 (\varepsilon_1 + \varepsilon_2 + \varepsilon_9) \| u \|_{\mathcal{M}}^{2} \\
+ L^2 T^2 (\varepsilon_1 + \varepsilon_2 + \varepsilon_25) \| \partial u \|_{BMO}^{2} + 2L^2 (\varepsilon_1 + \varepsilon_2 + \varepsilon_8 + \varepsilon_12 + \varepsilon_16 + \varepsilon_20) \| z \|_{BMO}^{2} \\
+ 2L^2 (\varepsilon_1 + \varepsilon_2 + \varepsilon_10 + \varepsilon_14 + \varepsilon_18 + \varepsilon_22) \| y \|_{BMO}^{2} + 2L^2 (\varepsilon_1 + \varepsilon_2 + \varepsilon_24) \| \partial v \|_{BMO}^{2} \\
+ (\varepsilon_3^{-1} + \varepsilon_7^{-1} + \varepsilon_8^{-1} + \varepsilon_9^{-1} + \varepsilon_10^{-1}) \| y \|_{BMO}^{2} + (\varepsilon_4^{-1} + \varepsilon_11^{-1} + \varepsilon_12^{-1} + \varepsilon_13^{-1} + \varepsilon_14^{-1}) \| u \|_{\mathcal{M}}^{2} \\
+ (\varepsilon_5^{-1} + \varepsilon_15^{-1} + \varepsilon_16^{-1} + \varepsilon_17^{-1} + \varepsilon_18^{-1}) \| u \|_{\mathcal{M}}^{2} + (\varepsilon_6^{-1} + \varepsilon_19^{-1} + \varepsilon_20^{-1} + \varepsilon_21^{-1} + \varepsilon_22^{-1} + \varepsilon_23^{-1} + \varepsilon_24^{-1}) \| \partial U \|_{BMO}^{2} \\
\tag{7.4}
\] 

From (7.4) we conclude \((Z, N) \in \mathbb{H}^{2,c} \times \mathbb{M}^{2,c}, \| V \|_{BMO}^{2} + \| \partial V \|_{BMO}^{2} + \| M \|_{\mathcal{M}}^{2} + \| \partial M \|_{\mathcal{M}}^{2} < \infty. \) 

Defining \(C_{\varepsilon} \) analogously and if for some \(\gamma \in (0, \infty)\)

\[
I_0 = \gamma R^2 / 10, \\
\tag{7.5}
\]

we obtain back in (7.4)

\[
\| (Y, Z, N, U, V, M, \partial U, \partial V, \partial M) \|_{BMO}^{2} < \varepsilon \]

\[
\leq C_{\varepsilon} \left( \frac{1}{10 I_0^2} + 10 L^2 \max \{ 2, T^2 \} \left( (\varepsilon_1 + \varepsilon_2 + \varepsilon_7 + \varepsilon_8 + \varepsilon_9 + \varepsilon_11 + \varepsilon_12 + \varepsilon_13 + \varepsilon_14 + \varepsilon_15 + \varepsilon_16 + \varepsilon_17 + \varepsilon_18 + \varepsilon_19) \right) \right) \| y \|_{BMO}^{2} \\
+ (\varepsilon_1 + \varepsilon_2 + \varepsilon_25) \| \partial u \|_{BMO}^{2} + (\varepsilon_1 + \varepsilon_2 + \varepsilon_8 + \varepsilon_12 + \varepsilon_16 + \varepsilon_20) \| z \|_{BMO}^{2} \\
+ (\varepsilon_1 + \varepsilon_2 + \varepsilon_10 + \varepsilon_14 + \varepsilon_18 + \varepsilon_22) \| y \|_{BMO}^{2} + (\varepsilon_1 + \varepsilon_2 + \varepsilon_24) \| \partial v \|_{BMO}^{2} \\
\leq C_{\varepsilon}^{-1} \frac{R^2}{\gamma} \left( \gamma + 10 L^2 \max \{ 2, T^2 \} \right) \frac{C_{\varepsilon} - \gamma}{\varepsilon_1 + \varepsilon_2 + 24 \sum_{i=7}^{24} \varepsilon_i} \\
\right)
\]

Therefore, to obtain \(\mathcal{B}(B_R) \subseteq B_R, \) that is to say that the image under \(\mathcal{B}\) of the ball of radius \(R\) is contained in the ball of radius \(R, \) it is necessary to find \(R^2\) such that the term in parentheses above is less or equal than \(C_{\varepsilon}, \) i.e.

\[
R^2 \leq \frac{1}{10 L^2 \max \{ 2, T^2 \} \varepsilon_1 + \varepsilon_2 + 24 \sum_{i=7}^{24} \varepsilon_i} C_{\varepsilon} - \gamma \\
\tag{7.6}
\]

which after optimising the choice of \(\varepsilon\)'s renders

\[
R^2 \leq \frac{1}{2^6 \cdot 3 \cdot 5^2 \cdot 7 \cdot L^2 \max \{ 2, T^2 \}} \\
\tag{7.6}
\]

\((v)\) The continuity of the applications \((0, T], B(0, T]) \rightarrow (S^{\infty,c}, \| \cdot \|_{S^{\infty,c}}) (\text{resp. } (\mathbb{H}^{2,c}, \| \cdot \|_{\mathbb{H}^{2,c}}), (\mathbb{M}^{2,c}, \| \cdot \|_{\mathbb{M}^{2,c}}); s \mapsto v^s \) for \(v = U^s, \partial U^s \) (resp. \(v^s, \partial v^s, M^s, \partial M^s \)) follows analogously as in the proof Theorem 3.6.

We conclude, \(\mathcal{B}(B_R) \subseteq B_R\) for all \(R\) satisfying (7.6).

**Step 2:** We now argue that \(\mathcal{B}\) is a contraction in \(B_R \subseteq \mathcal{H}\) for the norm \(\| \cdot \|_{\mathcal{H}^c}.\) Let

\[
\delta h_i := h_i(y_i, z_i, u_i, v_i, \partial U_i^1, \partial U_i^2), \\
\delta g_i := g_i(y_i, z_i, u_i, v_i, \partial U_i^1, \partial U_i^2), \\
\delta h_i := h_i(y_i, z_i, u_i, v_i, \partial U_i^1, \partial U_i^2), \\
\delta g_i := g_i(y_i, z_i, u_i, v_i, \partial U_i^1, \partial U_i^2),
\]

for \(i = 1, 2, \ldots, n.\)
\[ \delta \tilde{g}_t(s) := g_t(s, u_1^{1,s}, v_1^{1,s}, y_1^{1}, z_1^{1}) - g_t(s, u_1^{2,s}, v_1^{2,s}, y_1^{2}, z_1^{2}), \]

\[ \delta \nabla \delta \tilde{g}_t(s) := \nabla g_t(s, \partial u_1^{1,s}, \partial v_1^{1,s}, u_1^{1,s}, v_1^{1,s}, y_1^{1}, z_1^{1}) - g_t(s, \partial u_1^{2,s}, \partial v_1^{2,s}, u_1^{2,s}, v_1^{2,s}, y_1^{2}, z_1^{2}). \]

Applying Itô’s formula we obtain that for any \( t \in [0, T] \)

\[ \sum_{i=1}^{4} e^{ct} \kappa_i^2 \left( \int_{t}^{T} e^{ct} |\sigma_r \delta \beta_i^r|^2 \, dr + \int_{t}^{T} e^{ct} - d\tau |\delta \nu_i^r | + \delta \tilde{\nu}_t - \delta \tilde{\nu}_0 \right) \]

\[ = \int_{t}^{T} e^{ct} \left( 2 \delta \nu_r \cdot \delta h_r + 2 \delta U_r \cdot \delta g_r + 2 \delta U_s^i \cdot \delta \tilde{g}_r(s) + 2 \delta \nu_r \cdot \delta \nabla \delta \tilde{g}_r(s) \right) \, \, dr \]

\[ \leq \int_{t}^{T} e^{ct} \left( 2 |\delta \nu_r| (L_u |\delta \nu_r^2| + |\delta h_r|) + 2 |\delta U_r| (|\delta \nu_r^2| + |\delta \tilde{g}_r(s)|) + 2 |\delta U_s^i| |\delta \nabla \delta \tilde{g}_r(s)| - e \sum_{i=1}^{4} |\delta \beta_i^r|^2 \right) \, \, dr \]

where \( \delta \tilde{\nu}_0 \) denotes the corresponding martingale term. Let \( \tau \in T_{0,T} \), as in Lemma B.1 we obtain for \( c > 2L_u \)

\[ \mathbb{E}_\tau \left[ \int_{\tau}^{T} \frac{e^{ct}}{3T} |\delta \nu_r^2| \, dr \right] \leq \sup_{s \in [0,T]} \mathbb{E}_\tau \left[ \int_{\tau}^{T} e^{ct} |\delta \nabla \delta \tilde{g}_r(s)| \, dr \right]^2 \]

We now take conditional expectation with respect to \( \mathcal{F}_\tau \) in the expression above and use Assumption B. in combination with (7.7). We then obtain from Young’s inequality that for any \( \epsilon \in (0, \infty), i \in \{1, 2, \} \), and

\[ c \geq \max \{ \epsilon^{-1} 3T L_u^2, 3T \epsilon^{-1}, 2L_u \}, \]

it follows that

\[ \sum_{i=1}^{4} e^{ct} |\delta \beta_i^r|^2 + \mathbb{E}_\tau \left[ \int_{t}^{T} e^{ct} |\sigma_r \delta \beta_i^r|^2 \, dr + \int_{t}^{T} e^{ct} - d\tau |\delta \nu_i^r | \right] \]

\[ \leq \bar{\epsilon}^{-1} |\delta \beta_i^r|^2 \mathbb{E}_\tau \left[ \int_{t}^{T} e^{ct} |\delta \nabla \delta \tilde{g}_r(s)| \, dr \right]^2 + \bar{\epsilon} \sup_{\tau \in T_{0,T}} \mathbb{E}_\tau \left[ \int_{\tau}^{T} e^{ct} |\delta h_r| \, dr \right]^2 \]

\[ \bar{\epsilon} \sup_{\tau \in T_{0,T}} \mathbb{E}_\tau \left[ \int_{\tau}^{T} e^{ct} |\delta \tilde{g}_r| \, dr \right]^2 \]

We now estimate the terms on the right side of (7.9). Note that in light of Assumption B.(iii) we have

\[ \mathbb{E}_\tau \left[ \int_{\tau}^{T} e^{ct} |\delta \nabla \delta \tilde{g}_r(s)| \, dr \right]^2 \]

\[ \leq \mathbb{E}_\tau \left[ \int_{\tau}^{T} e^{ct} \left( |\partial u_1^{1,s}| + |\partial u_2^{1,s}| + L_v |\sigma_r | \delta v_1^r + |\sigma_r | \delta v_2^r \right) \right] \]

\[ + L_u |\delta u_1^r| \left( |u_1^r| + |u_2^r| \right) + L_v |\sigma_r | \delta v_1^r \left( |v_1^r| + |v_2^r| \right) \]

\[ L_v |\delta g_r| \left( |y_1^r| + |y_2^r| \right) + L_z |\sigma_r | \delta z_r \left( |z_1^r| + |z_2^r| \right) \]

\[ \leq 6L_u^2 \mathbb{E}_\tau \left[ \int_{\tau}^{T} e^{ct} |\delta \nu_r^2| \, dr \right] \mathbb{E}_\tau \left[ \int_{\tau}^{T} e^{ct} \left( |\partial u_1^{1,s}| + |\partial u_2^{1,s}| \right)^2 \, dr \right] \]

\[ + 6L_u^2 \mathbb{E}_\tau \left[ \int_{\tau}^{T} e^{ct} |\sigma_r | \delta v_1^r | \delta v_2^r | \, dr \right] \mathbb{E}_\tau \left[ \int_{\tau}^{T} e^{ct} \left( |\sigma_r | \delta v_1^r | + |\sigma_r | \delta v_2^r | \right)^2 \, dr \right] \]

\[ + 6L_u^2 \mathbb{E}_\tau \left[ \int_{\tau}^{T} e^{ct} |\delta u_1^r | \delta u_2^r | \, dr \right] \mathbb{E}_\tau \left[ \int_{\tau}^{T} e^{ct} \left( |u_1^r | + |u_2^r | \right)^2 \, dr \right] \]

\[ + 6L_u^2 \mathbb{E}_\tau \left[ \int_{\tau}^{T} e^{ct} |\sigma_r | \delta v_1^r | \delta v_2^r | \, dr \right] \mathbb{E}_\tau \left[ \int_{\tau}^{T} e^{ct} \left( |\sigma_r | \delta v_1^r | + |\sigma_r | \delta v_2^r | \right)^2 \, dr \right] \]
Thus, letting choosing $(\varepsilon, \gamma)$ satisfy
\[
(\varepsilon, \gamma) \in \left\{ \varepsilon_1, \varepsilon_2 \right\} \quad \text{satisfy} \quad (2.1)
\]
and Cauchy–Schwartz’s inequality. Similarly
\[
\max \left\{ \left| \mathbb{E}_\tau \left[ \int_t^T e^{c_T} |\delta h| \, d\tau \right] \right|^2, \left| \mathbb{E}_\tau \left[ \int_t^T e^{c_T} |\delta g| \, d\tau \right] \right|^2 \right\} \leq 4L^2T^2 \max \left\{ \mathbb{E}_\tau \left[ \int_T^T e^{c_T} |\delta h| \, d\tau \right] \right|^2 \]
Over all, we obtain back in (7.9) that
\[
\sum_{i=1}^4 e^{c_i} |\delta h_i|^2 + \mathbb{E}_\tau \left[ \int_t^T e^{c_T} |\delta h| \, d\tau \right] \leq \varepsilon_\delta^{-1} |\delta Y| \mathbb{T}_{\mathbb{E}_\tau}^2 + \varepsilon_\delta^{-1} |\delta U| \mathbb{T}_{\mathbb{E}_\tau}^2 + \varepsilon_\delta^{-1} |\delta U| \mathbb{T}_{\mathbb{E}_\tau}^2 + \varepsilon_\delta^{-1} |\delta U| \mathbb{T}_{\mathbb{E}_\tau}^2
\]
\[
+ 6(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) \mathbb{E}_\tau \left[ \int_T^T e^{c_T} |\delta h| \, d\tau \right] \leq 20C_{\mathbb{E}_\tau}^{-1} L^2T^2 \max \left\{ (2, T^2) \right\} \]
Minimising for $\varepsilon_1$ and $\varepsilon_2$ fixed, we find that letting
\[
R^2 < \frac{1}{2^{0.3 \cdot 5^2 \cdot 7 \cdot L^2 \cdot \max \left\{ (2, T^2) \right\} }}, \quad c \geq \max \left\{ \varepsilon_1^{-1} 7T L^2 \varepsilon_2^{-1} 7T, \varepsilon_1^{-1} 3T L^2 \varepsilon_2^{-1} 2L \right\}
\]
we have that
\[
|\delta \mathbb{F} |^2 \mathbb{T}_{\mathbb{E}_\tau} \leq \frac{20}{2^{0.3 \cdot 7 \cdot L^2 \cdot \max \left\{ (2, T^2) \right\} }}, \quad c \geq \max \left\{ \varepsilon_1^{-1} 7T L^2 \varepsilon_2^{-1} 7T, \varepsilon_1^{-1} 3T L^2 \varepsilon_2^{-1} 2L \right\}
\]
Thus, letting choosing $(\sqrt{30} + \varepsilon_1 + \varepsilon_2) + \sqrt{30} \leq 2^{3 \cdot 7 \cdot 10}$, $\mathbf{T}$ is a contraction.

**Step 3:** We consolidate our results. In light of (7.3) and (7.8), taking $\varepsilon_i = \varepsilon_i, i \in \{1, 2\}, c$ must satisfy
\[
c \geq \max \left\{ \varepsilon_1^{-1} 7T L^2 \varepsilon_2^{-1} 7T, \varepsilon_1^{-1} 3T L^2 \varepsilon_2^{-1} 2L \right\} \quad \text{max} \left\{ \varepsilon_1^{-1} 7T L^2 \varepsilon_2^{-1} 7T, \varepsilon_1^{-1} 3T L^2 \varepsilon_2^{-1} 2L \right\}
\]
All together we find that given $\gamma \in (0, \infty), \varepsilon_i \in (0, \infty), i \in \{1, 2\}, c \in (0, \infty)$, such that $\varepsilon_1 + \varepsilon_2 \leq (4\sqrt{35} - \sqrt{30})^2 - 30$, $\mathbf{T}$ is a well–defined contraction in $B_R \subseteq \mathbb{E}_\tau$ for the norm $\| \cdot \|_{\mathbb{E}_\tau}$ provided: (i) $\gamma, \varepsilon_i, i \in \{1, 2\}$, and the data of the problem satisfy (7.5); (ii) $c$ satisfies (7.11).

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A Proofs of Section 2

Proof of Lemma 2.3. First note that for $Z \in \mathbb{H}^2_{BMO}(\mathbb{R}^{n \times d})$, $Z \cdot X$ is a continuous local martingale, thus we have that

$$\|Z \cdot X\|_{BMO^2} = \sup_{\tau \in \mathcal{T}_{0,T}} \left\| E \left[ \langle e^{\frac{\tau}{2}} Z \cdot X \rangle_T - \langle e^{\frac{\tau}{2}} Z \cdot X \rangle_\tau \mid \mathcal{F}_\tau \right] \right\|_\infty < \infty.$$  

Therefore, letting $X_t := E \left[ \langle e^{\frac{\tau}{2}} Z \cdot X \rangle_T - \langle e^{\frac{\tau}{2}} Z \cdot X \rangle_\tau \mid \mathcal{F}_\tau \right]$, we have: (i) $|X_t| \leq \|Z \cdot X\|_{BMO^2} = \|Z\|^2_{\mathbb{H}^2_{BMO}}$; (ii) $A = \langle e^{\frac{\tau}{2}} Z \cdot X \rangle_T$.

Indeed, note $X_t = E \left[ \langle e^{\frac{\tau}{2}} Z \cdot X \rangle_T \mid \mathcal{F}_t \right] - \langle e^{\frac{\tau}{2}} Z \cdot X \rangle_\tau$. The result then follows immediately from the energy inequality, i.e.

$$E \left[ \left( \int_0^T e^{\sigma_r^T Z_r}^2 dr \right)^p \right] = E \left[ (A)^p_\infty \right] \leq p! \|Z\|^2p_{\mathbb{H}^2_{BMO}}.$$
To obtain the second part of the statement, recall that by definition of $H^{2,2,c}(\mathbb{R}^{n+\hat{d}})$, $s \mapsto \partial Z^s$ is the density of $s \mapsto Z^s$ with respect to the Lebesgue measure and $Z$ is given as in Remark 2.2. By definition of $Z$, Fubini's theorem and Young's inequality we have that for $\varepsilon > 0$

$$\int_t^T e^{cu}|\sigma_u^T Z_u|^2 - e^{cu}|\sigma_u^T Z_u'|^2 du = \int_t^T \int_r^T 2e^{cu} \text{Tr}[Z_r^T \sigma_u \sigma_u^T Z_r'] dudr \leq \int_t^T \int_r^T \varepsilon e^{cu}|\sigma_u^T Z_r|^2 + e^{-\varepsilon} e^{cu}|\sigma_u^T \partial Z_r'^2| dudr.$$

This proves the first first statement. For the second claim, we may use (2.1) and (2.2) to obtain

$$\mathbb{E}_t \left[ \left( \int_t^T e^{cu}|\sigma_u^T Z_u|^2 du \right)^2 \right] \leq 3 \mathbb{E}_t \left[ \left( \int_t^T e^{cu}|\sigma_u^T Z_u'|^2 du \right)^2 \right] + T \int_t^T \mathbb{E}_t \left[ \left( \int_t^T e^{cu}|\sigma_u^T Z_u'|^2 du \right)^2 \right] dr + T \int_t^T \mathbb{E}_t \left[ \left( \int_t^T e^{cu}|\sigma_u^T \partial Z_r'^2| dudr \right)^2 \right] dr \leq 6(1 + T^2)\|Z\|_{B^2,\infty}^4 + T^2\|\partial Z\|_{B^2,\infty}^4.$$

The inequality for the $H^2$ norm is argued similarly taking expectations.

\[\square\]

**B Proofs of Section 6**

We next lemma helps derive appropriate auxiliary estimates of the terms $U_t^s$ and $\partial U_t^s$ as in Section 6.

**Lemma B.1.** Let $\partial U$ satisfy the equation

$$\partial U^s = \partial_s \eta(s, X_{\lambda,T}) + \int_s^T \nabla g_r(s, X, \partial U_r^s, \partial v_r^s, v_r^s, \partial Y_r, z_r) dr - \int_s^T \partial V^s_r \partial X_r - \int_s^T d\partial M_r^s,$$

and $c \geq \max\{2L_u, 2L_a\}$, the following estimates hold for $t \in [0, T]$

$$\mathbb{E}_t \left[ \left( \int_t^T e^{ct}|\partial U_r^s|^2 dr \right)^2 \right] \leq \|\partial_s \eta\|_{L^\infty,\infty,2,c}^2 + \|\nabla g\|_{L^2,\infty,2,c}^2 + TL^2 u \mathbb{E}_t \left[ \int_t^T e^{ct}|Y_r|^2 dr \right] + T L^2 u \sup_{s \in [0,T]} \mathbb{E}_t \left[ \int_t^T e^{ct}|U_r^s|^2 dr \right]$$

$$+ 2L^2 \|\partial v\|_{B^2,\infty}^2 + \|z\|_{B^2,\infty}^2$$

$$\mathbb{E}_t \left[ \int_t^T e^{ct}|\partial U_r^s| dr \right] \leq \|\partial_s \eta\|_{L^\infty,\infty,2,c} + \|\nabla g\|_{L^1,\infty,2,c} + L_u \mathbb{E}_t \left[ \int_t^T e^{ct}|Y_r|^2 dr \right] + \sup_{s \in [0,T]} \mathbb{E}_t \left[ \int_t^T e^{ct}|U_r^s|^2 dr \right]$$

$$+ L_\ast \left( \|\partial v\|_{B^2,\infty}^2 + \|v\|_{B^2,\infty}^2 + \|z\|_{B^2,\infty}^2 \right).$$

**Proof.** By Meyer–Itô’s formula for $e^{ct}|\partial U_r^s|^2$, see Protter [45, Theorem 70]

$$e^{ct}|\partial U_r^s|^2 + L^2 u - \int_t^T e^{ct} \text{sgn}(\partial U_r^s) \cdot \partial V^s_r \partial X_r - \int_t^T e^{ct} \text{sgn}(\partial U_{r-}^s) \cdot d\partial M_r^s$$

$$= e^{ct} \partial_s \eta(s) + \int_t^T e^{ct} \left( \text{sgn}(\partial U_r^s) \cdot \nabla g_r(s, \partial U_r^s, \partial v_r^s, v_r^s, \partial Y_r, z_r) - \frac{c}{2} \partial U_r^s \right) dr, \quad t \in [0, T],$$

where $L^0 := L^0(\partial U^s)$ denotes the non-decreasing and pathwise-continuous local time of the semi-martingale $\partial U^s$ at 0, see [45, Chapter IV, pp. 216]. We also notice that for any $s \in [0, T]$ the last two terms on the left-hand side are martingales, recall that $\partial V^s \in H^2$ by [20, Theorem 3.5].

In light of Assumption B, letting $\nabla g_r(s) := \nabla g_r(s, \partial U_r^s, \partial v_r^s, v_r^s, Y_r, z_r)$, we have that $dt \otimes d\mathbb{P}$–a.e.

$$|\nabla g_r(s)| \leq L_u|\partial U_r^s| + L_v|\sigma_r^y \partial U_r^s|^2 + L_u|U_r^s| + L_v|\sigma_r^y v_r^s|^2 + L_y|Y_r| + L_z|\sigma_r^y z_r|^2 + |\nabla \bar{g}_r(s)|,$$

(B.2)

We now take conditional expectation with respect to $\mathcal{F}_t$ in Equation (B.1). We may use (B.2) and the fact $\bar{L}^0$ is non-decreasing to derive that for $c > 2L_u$ and $t \in [0, T]$

$$e^{ct}|\partial U_r^s| \leq \mathbb{E}_t \left[ e^{ct} \left( \partial_s \eta(s) + \int_t^T e^{ct} \left( |\nabla \bar{g}_r(s)| + L_v|\sigma_r^y \partial U_r^s|^2 + L_u|U_r^s| + L_v|\sigma_r^y v_r^s|^2 + L_y|Y_r| + L_z|\sigma_r^y z_r|^2 \right) dr \right].$$

(B.3)
Squaring in (B.3), we may use (2.1) and Jensen’s inequality to derive that for \( t \in [0, T] \)
\[
\frac{\epsilon^T}{T} \langle \partial \eta(t) \rangle^2 \leq \mathbb{E}_t \left[ \int_t^T e^{\epsilon T} |\nabla \hat{g}_r(t)|^2 \, dt \right]^2 + TL^2_u \int_t^T e^{\epsilon T} |U_r|^2 \, dt + T \mathbb{E}_t \left[ \int_t^T e^{\epsilon T} |Y_r|^2 \, dt \right]
+ L^2 \left( \int_t^T e^{\epsilon T} |\sigma_r^T \partial v_r|^2 \, dt \right)^2 + L^2 \left( \int_t^T e^{\epsilon T} |\sigma_r^T z_r|^2 \, dt \right)^2
\]

By integrating the previous expression and taking conditional expectation with respect to \( \mathcal{F}_t \), it follows from the tower property that for any \( t \in [0, T] \)
\[
\frac{1}{T} \mathbb{E}_t \left[ \int_t^T e^{\epsilon T} |\partial U_r|^2 \, dt \right] \leq \mathbb{E}_t \left[ \int_t^T e^{\epsilon T} |\partial \eta(r)|^2 \, dr \right] + \mathbb{E}_t \left[ \int_t^T \left( \int_r^T e^{\epsilon T} |\nabla \hat{g}_u(r)|^2 \, du \right) \, dr \right]
+ TL^2_u \mathbb{E}_t \left[ \int_t^T \int_r^T e^{\epsilon T} |U_r|^2 \, du \, dr \right] + T \mathbb{E}_t \left[ \int_t^T e^{\epsilon T} |Y_r|^2 \, dt \right]
+ L^2 \int_t^T \mathbb{E}_t \left[ \left( \int_r^T e^{\epsilon T} |\sigma_u^T \partial v_u|^2 \, du \right) \, dr \right] + L^2 \int_t^T \mathbb{E}_t \left[ \left( \int_r^T e^{\epsilon T} |\sigma_u^T z_u|^2 \, du \right) \, dr \right]
+ T \sup_{r \in [0, T]} \left\{ \|e^{\epsilon T} |\eta(r)|^2\|_{\infty} + \left\| \int_r^T e^{\epsilon T} |\nabla \hat{g}_u(r)|^2 \, du \right\|_{\infty} \right\} + T^2 L^2_u \mathbb{E}_t \left[ \int_t^T e^{\epsilon T} |Y_r|^2 \, dt \right]
+ T^2 L^2_u \sup_{r \in [0, T]} \left\{ \mathbb{E}_t \left[ \int_r^T e^{\epsilon T} |U_r|^2 \, du \right] \right\} + T L^2_u \sup_{r \in [0, T]} \left\{ \mathbb{E}_t \left[ \left( \int_r^T e^{\epsilon T} |\sigma_u^T \partial v_u|^2 \, du \right) \, dr \right] \right\}
+ T L^2_u \sup_{r \in [0, T]} \left\{ \mathbb{E}_t \left[ \left( \int_r^T e^{\epsilon T} |\sigma_u^T z_u|^2 \, du \right) \, dr \right] \right\},
\]
and by (2.2) we obtain for \( c > 2L_u \), and any \( t \in [0, T] \)
\[
\mathbb{E}_t \left[ \int_t^T e^{\epsilon T} |\partial U_r|^2 \, dt \right] \leq \|\partial \eta\|_{H^{1,\infty,2,2}} + \|\nabla \hat{g}\|_{H^{1,2,2}} + T L^2_u \mathbb{E}_t \left[ \int_t^T e^{\epsilon T} |Y_r|^2 \, dt \right] + T^2 L^2_u \sup_{r \in [0, T]} \mathbb{E}_t \left[ \int_t^T e^{\epsilon T} |U_r|^2 \, du \right]
+ 2L^2_u \left\{ \|\partial v\|_{H^{1,2,2}} + \|\hat{g}\|_{H^{1,2}} + \|v\|_{H^{1,2}} \right\}.
\]
Evaluating at \( s = t \) in (B.3) and integrating with respect to \( t \) we derive the second estimate. \( \square 

**Lemma B.2** (Optimal upper bound for \( R \). (OPT1) = \( 1/(2^4) \), where
\[
\min_{\{\alpha(\epsilon_3, \epsilon_{12}, \epsilon_{13}), \alpha(\epsilon_4, \epsilon_{14}, \epsilon_{15}), \alpha(\epsilon_5, \epsilon_{16}, \epsilon_{17}), \alpha(\epsilon_6, \epsilon_{18}, \epsilon_{19}, \epsilon_{20})\} - \gamma}
\]
\[
\text{s.t. } \alpha(\epsilon_8, \epsilon_{12}, \epsilon_{13}) = 1 - 10(\epsilon_8^1 + \epsilon_{12}^-), \quad \alpha(\epsilon_9, \epsilon_{14}, \epsilon_{15}) = 1 - 10(\epsilon_9^1 + \epsilon_{14}^-), \quad \alpha(\epsilon_{10}, \epsilon_{16}, \epsilon_{17}) = 1 - 10(\epsilon_{10}^- + \epsilon_{16}^- + \epsilon_{17}^-), \quad \alpha(\epsilon_{11}, \epsilon_{18}, \epsilon_{19}, \epsilon_{20}) = 1 - 10(\epsilon_{11}^- + \epsilon_{18}^- + \epsilon_{19}^- + \epsilon_{20}^-), \quad \gamma \in (0, \infty),
\]
\[
\epsilon_{12} = \epsilon_{13} = 2\alpha_1, \quad \epsilon_{14} = \epsilon_{15} = 2\alpha_2, \quad \epsilon_{16} = \epsilon_{17} = 2\alpha_3, \quad \epsilon_{18} = \epsilon_{19} = \epsilon_{20} = 3\alpha_4, \quad \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \in (0, \infty)^4.
\]

**Proof.** We begin by noticing that as a function of \((\gamma, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6)\) the objective is bounded by the value when \((\gamma, \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6) \rightarrow (0, 0, 0, \infty, \infty, \infty, \infty)\). Thus, we will maximise
\[
\min_{\{1 - 10(\epsilon_{12}^1 + \epsilon_{13}^-), \quad 1 - 10(\epsilon_{14}^1 + \epsilon_{16}^- + \epsilon_{17}^-), \quad 1 - 10(\epsilon_{18}^1 + \epsilon_{19}^- + \epsilon_{20}^-)\}}
\]
\[
\sum_{i=12}^{20} \epsilon_i
\]
From this we observe that the optimal value is positive. Indeed, there is a feasible solution with positive value, and the min in the objective function does not involve common \( \epsilon_i \) terms, so the minima is attained at one of the terms. Since the value function is symmetric in each of the variables inside each term of the mean we can assume with out lost of generality
\[
\epsilon_{12} = \epsilon_{13} = 2\alpha_1, \quad \epsilon_{14} = \epsilon_{15} = 2\alpha_2, \quad \epsilon_{16} = \epsilon_{17} = 2\alpha_3, \quad \epsilon_{18} = \epsilon_{19} = \epsilon_{20} = 3\alpha_4, \quad \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \in (0, \infty)^4
\]

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So we can write the objective function as \( \min \{ 1 - 10\alpha_1^{-1}, 1 - 10\alpha_2^{-1}, 1 - 10\alpha_3^{-1}, 1 - 10\alpha_4^{-1} \} / (4\alpha_1 + 4\alpha_2 + 4\alpha_3 + 9\alpha_4) \). Now, without lost of generality the min is attained by the first quantity. This is, the optimisation problem becomes
\[
\sup \frac{1 - 10\alpha_1^{-1}}{4\alpha_1 + 4\alpha_2 + 4\alpha_3 + 9\alpha_4} \quad \text{s.t.} \quad \alpha_1 \leq \min \{ \alpha_2, \alpha_3, \alpha_4 \}, 1 - 10\alpha_i^{-1} \in (0, 1], \alpha_i \in (0, \infty), i \in \{1, 2, 3, 4\}.
\]
Now, as the objective function is decreasing in \( \alpha_2, \alpha_3, \alpha_4 \), and \( \alpha_1 \leq \min \{ \alpha_2, \alpha_3, \alpha_4 \} \), we see \( \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 \). Thus
\[
\sup \frac{1 - 10\alpha_1^{-1}}{21\alpha_1} \quad \text{s.t.} \quad 1 - 10\alpha_1^{-1} \in (0, 1], \alpha_1 \in (0, \infty).
\]
Let \( f(\alpha_1) := \frac{\alpha_1 - 10}{21\alpha_1} \). By first order analysis
\[
\partial_{\alpha_1} f(\alpha_1) = \frac{-\alpha_1(\alpha_1 - 20)}{21\alpha_1^4} = 0, \text{ yields, } \alpha_1 \in \{0, 20\}
\]
By inspecting the sign of the derivative, one sees that \( \alpha_1 = 0 \) corresponds to a minima and \( \alpha_1 = 20 \) is the maximum and it is feasible. Thus we obtain that
\[
f(\alpha_1^*) = \frac{1}{23 \cdot 3 \cdot 5 \cdot 7}.
\]
We conclude the maxima when \( (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7, \varepsilon_8, \varepsilon_9, \varepsilon_{10}, \varepsilon_{11}) = (40, 40, 40, 40, 40, 40, 40, 60, 60, 60) \). Evaluating the value function in these values and letting \( (\gamma, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7, \varepsilon_8, \varepsilon_{10}, \varepsilon_{11}) \to (0, 0, 0, \infty, \infty, \infty, \infty) \), we obtain this bound. This is, \( f \) does not attain its maximum value, but in the feasible region it can get as close as possible.

**Lemma B.3 (Minimal bound for Contraction).** (OPT2) = \( 3(\sqrt{30} + (\bar{\varepsilon}_1 + \bar{\varepsilon}_2) + \sqrt{30})^2 \), where
\[
(\text{OPT2}) := \inf \left\{ \left(3\bar{\varepsilon}_1 + 3\bar{\varepsilon}_2 + 2\bar{\varepsilon}_3 + 2\bar{\varepsilon}_4 + 2\bar{\varepsilon}_5 + 3\bar{\varepsilon}_6\right) \min \left\{ \frac{\bar{\varepsilon}_3}{\bar{\varepsilon}_3 - 10}, \frac{\bar{\varepsilon}_4}{\bar{\varepsilon}_4 - 10}, \frac{\bar{\varepsilon}_5}{\bar{\varepsilon}_5 - 10}, \frac{\bar{\varepsilon}_6}{\bar{\varepsilon}_6 - 10} \right\} \right\}
\text{s.t.} \quad 1 - 10\bar{\varepsilon}_i^{-1} \in (0, 1], \bar{\varepsilon}_i \in (0, \infty), i \in \{3, 4, 5, 6\}.
\]

**Proof.** Without lost of generality let us assume the min is attained by the first quantity, i.e. the optimisation problem becomes
\[
\inf \left\{ \left(3\bar{\varepsilon}_1 + 3\bar{\varepsilon}_2 + 2\bar{\varepsilon}_3 + 2\bar{\varepsilon}_4 + 2\bar{\varepsilon}_5 + 3\bar{\varepsilon}_6\right) \frac{\bar{\varepsilon}_3}{\bar{\varepsilon}_3 - 10}, \right\} \text{s.t.} \quad \bar{\varepsilon}_8 \leq \min \{\bar{\varepsilon}_4, \bar{\varepsilon}_5, \bar{\varepsilon}_6\}, 1 - 10\bar{\varepsilon}_3^{-1} \in (0, 1], \bar{\varepsilon}_i \in (0, \infty) \forall i.
\]
As the value function is increasing in \( (\bar{\varepsilon}_4, \bar{\varepsilon}_5, \bar{\varepsilon}_6) \), \( \bar{\varepsilon}_3 \leq \min \{\bar{\varepsilon}_4, \bar{\varepsilon}_5, \bar{\varepsilon}_6\} \) implies we must have \( \bar{\varepsilon}_3 = \bar{\varepsilon}_4 = \bar{\varepsilon}_5 = \bar{\varepsilon}_6 \) a thus we minimise
\[
f(\bar{\varepsilon}) := 3\bar{\varepsilon}^2 + \bar{\varepsilon}(\bar{\varepsilon}_1 + \bar{\varepsilon}_2).
\]
First order analysis renders
\[
\partial_{\bar{\varepsilon}} f(\bar{\varepsilon}) = \frac{9\bar{\varepsilon} - 180\bar{\varepsilon} - 30(\bar{\varepsilon}_1 + \bar{\varepsilon}_2)}{\bar{\varepsilon} - 10} = 0, \text{ yields, } \bar{\varepsilon} \pm = 10 \pm \frac{1}{6} \sqrt{60^2 + 120(\bar{\varepsilon}_1 + \bar{\varepsilon}_2)}.
\]
The minimum occurs at \( \bar{\varepsilon}^* = 10 + \frac{1}{6} \sqrt{60^2 + 120(\bar{\varepsilon}_1 + \bar{\varepsilon}_2)} \), and \( f(\bar{\varepsilon}^*) = 3(\sqrt{30} + (\bar{\varepsilon}_1 + \bar{\varepsilon}_2) + \sqrt{30})^2. \) We conclude the minima occurs when \( (\bar{\varepsilon}_3, \bar{\varepsilon}_4, \bar{\varepsilon}_5, \bar{\varepsilon}_6) = (20, 20, 20, 20) \). Evaluating the value function in these values and letting \( (\bar{\varepsilon}_1, \bar{\varepsilon}_2) \to (0, 0) \), we obtain this bound. This is, \( f \) does not attain its minimum value, but in the feasible region it can get as close as possible.