Nonlinear magnetically charged black holes in 4D Einstein-Gauss-Bonnet gravity

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In this letter we present an exact spherically symmetric and magnetically charged black hole solution with exponential model of nonlinear electrodynamics [S. Kruglov, Annals Phys. 378, 59-70 (2017)] in the context of 4D Einstein-Gauss-Bonnet (EGB) gravity. We show that our −ve branch, in the limit of GB coupling coefficient $\alpha \to 0$ and the nonlinear parameter $\beta \to 0$, reduces to the magnetically charged black hole of Einstein-Maxwell gravity in GR. In addition we study the embedding diagram of the black hole geometry and the thermodynamic properties such as the Hawking temperature and the heat capacity of our black hole solution.

I. INTRODUCTION

It is well known that the EGB theory is topological in 4D as the GB Lagrangian is a total derivative and, therefore, it does not contribute to the gravitational dynamics in 4D. In a recent work Glavan & Lin [3] proposed an idea based on the rescaling the Gauss-Bonnet coupling constant $\alpha$ as $\alpha/(D - 4)$, and taking the limit $D \to 4$ at the level of the field equation to obtain in this way a non-trivial contribution in 4D. This novel 4D EGB gravity has interesting properties such as bypasses the conclusions of Lovelock’s theorem and avoids Ostrogradsky instability. Furthermore the static and spherically symmetric vacuum black holes found in [3] have interesting properties, for example the gravitational force is repulsive at short distance and thus an infalling particle never reaches $r = 0$ point. In other words, the theory is free from singularity problem. This is in contrast to Einstein’s general relativity, where an infalling particle will eventually hit the singularity and effective theory breaks down, this is also the case in HD black holes [1].

After the work of Glavan and Lin [3] the 4D EGB theory received compelling attention. The charged AdS black hole was obtained in Ref. [5], black holes in the four-dimensional Einstein-Lovelock gravity, [6], clouds of string in the novel 4D EGB black holes [7], a Vaidya metric Ref. [8], generating black holes solution was also addressed in Ref. [14], Hayward and Bardeen black holes in 4D EGB theory [9, 10], rotating black holes using Newman-Janis algorithm [11, 12], rotating black hole as particle accelerator [13], thermodynamical properties of AdS black hole were studied in Ref. [15], QNMS, stability and shadows [16], gravitational lensing by black holes [17], strong gravitational lensing in homogeneous plasma [18], stability of the Einstein Static Universe in 4D EGB [19], QNMs and Strong Cosmic Censorship [20], wormholes in 4D EGB [21], thin shell wormholes [22], relativistic stars [23], 4D EGB as heat engine [24], the innermost stable circular orbit and shadow [25], greybody factor and power spectra of the Hawking radiation in the novel 4D EGB de-Sitter gravity [26], superradiance and stability of the novel 4D charged EGB black hole [27], weak cosmic censorship conjecture for the novel 4D charged EGB black hole with test scalar field and particle [28], extended thermodynamics and microstructures in AdS space [29], spinning test particle in 4D EGB [30], perturbative and nonperturbative QNMs of 4D EGB [31], regularized Lovelock gravity [32], thin accretion disk around 4D EGB [33] and many other studies.

In the same time, objections on the 4D EGB theory were reported in the work of Gurses et al. [34] and Ref. [35]. Importantly, in a very recent works, it was argued that a well-defined $D \to 4$ limit of EGB gravity can be obtained and a regularized field equations has been reported in Refs. [36, 37]. More specifically in Ref. [37] authors employed the Mann-Ross method [38] and found that the limit $D \to 4$ is a special case of the scalar-tensor theory of the Horndeski type obtained by a dimensional reduction method. In addition it is pointed out that the spherically symmetric spacetimes in 4D the black hole solution should remain valid in these regularised theories, however by going beyond the spherically symmetric cases the solutions are not valid.

In this paper we aim to find an exact black hole solution in the context of nonlinear electrodynamics supported with magnetic charge with a lagrangian density proposed in Ref. [39]. This model was subsequently used in Ref. [40], while in Ref. [41] a different lagrangian has been used to obtain regular magnetic black holes. The paper is organized as follows: In Sec. 2, we solve the field equations in the novel 4D EGB gravity to ob-
tain magnetically charged black hole solution. In Sec. 3, we explore embedding diagram. In Sec. 4, we study the Hawking temperature and the heat capacity. Finally we comment on our results in Sec. 5.

II. MAGNETICALLY CHARGED BLACK HOLES IN 4D EGB GRAVITY

Let us begin by writing the action in the EGB gravity in D-dimensions to derive the equations of motion. The gravitational action is given by

\[ I_A = \frac{1}{16\pi} \int d^D x \sqrt{-g} \left[ R + \frac{\alpha}{D-4} \mathcal{L}_{\text{GB}} \right] + \mathcal{I}_{\text{NED}} \]  

(1)

in which \( g \) is the determinant of the metric \( g_{\mu\nu} \) while \( \alpha \) is the GB coupling coefficient and has dimensions of \([\text{length}]^2\). The Lagrangian density of exponential electrodynamics on the other hand reads [39]

\[ \mathcal{L}_{\text{NED}} = -F \exp(-\beta F), \]  

(2)

where

\[ F = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \]  

(3)

is the Maxwell invariant with a pure magnetic field given by the 2-form

\[ F = q \sin \theta d\theta \wedge d\phi. \]  

(4)

In particular the parameter \( \beta \) possesses the dimension of the \([\text{length}]^4\) and the upper bound was reported \( \beta \leq 1 \times 10^{-25} \text{T}^2 \) from PVLAS experiment. The term \( \mathcal{L}_{\text{GB}} \) is the Lagrangian defined and is given by

\[ \mathcal{L}_{\text{GB}} = R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - 4R^{\mu\nu} R_{\mu\nu} + R. \]  

(5)

The variation of (1) with respect to metric \( g_{\mu\nu} \) gives the field equations [8]

\[ G_{\mu\nu} + \frac{\alpha}{D-4} H_{\mu\nu} = 8\pi T^{\text{NED}}_{\mu\nu}, \]  

(6)

where the energy momentum tensor in our case reads [39]

\[ T^{\text{NED}}_{\mu\nu} = \frac{1}{4\pi} \exp(-\beta F) \left[ (1 - \beta F) F^{\mu\lambda} F_{\lambda}^{\nu} - g^{\mu\nu} F \right], \]  

(7)

along with the following expression

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}, \]  

\[ H_{\mu\nu} = 2 \left( R R_{\mu\nu} - 2 R_{\mu\nu} R^\rho \right) - 2 R_{\mu\nu\rho\sigma} R^{\rho\sigma} - R_{\mu\nu\rho\sigma} R^{\rho\sigma}, \]  

(8)

In the these equations \( R \) is the Ricci scalar, \( R_{\mu\nu} \) the Ricci tensor, \( R_{\mu\nu}\) is the so-called Lanczos tensor and finally \( R_{\mu\nu\rho\sigma} \) the Riemann tensor. As we already pointed out the GB term is total derivative and does not contribute to the field equations in 4D. But if we re-scaled the coupling constant \( \alpha/(D-4) \), and considering maximally symmetric spacetimes with curvature scale \( K \) [8], we obtain

\[ \frac{g_{\mu\nu}}{\sqrt{-g}} \frac{\delta \mathcal{L}_{\text{GB}}}{\delta g_{\mu\nu}} = \frac{\alpha(D-2)(D-3)}{2(D-1)} K^2 \delta_{\mu
u}, \]  

(9)

hence one can see that the variation of the GB action does vanish in \( D = 4 \) due to the re-scaled coupling constant [3]. The general static and spherically symmetric metric in \( D \)-dimensions reads

\[ ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2_{D-2}. \]  

(10)

with the unite sphere in \( D \) dimensions

\[ d\Omega^2_{D-2} = d\theta^2 + \sum_{i=2}^{D-2} \prod_{j=1}^{i-1} \sin^2 \theta_j d\theta_i^2. \]

Using the energy-momentum for the energy density it follows [39]

\[ \rho = \frac{q^2}{2\pi^4} \exp\left(-\frac{\beta q^2}{2\pi^4}\right), \]  

(11)

where for pure magnetic field in the spherically symmetric spacetime is given as

\[ F = \frac{q^2}{2\pi^4}. \]  

(12)

The \((t-t)\) component of the Einstein field equations yields

\[ -q^2 \exp\left(-\frac{\beta q^2}{2\pi^4}\right) - r(-2f(r) + r^2 + 2\alpha) f'(r) \]  

(13)

\[ -(f(r) - 1)(r^2 + \alpha f(r) - \alpha) = 0 \]

Solving this equation we find the two branches

\[ f(r) = 1 + \frac{r^2}{2\alpha} \left( 1 \pm \sqrt{1 + \frac{8M\alpha}{r^3} - \frac{23/4}{r^4} \exp\left(-\frac{\beta q^2}{2\pi^4}\right) \Xi q^2 \alpha} \right) \]  

(14)

where

\[ \Xi = -2^{7/8} \text{WhittakerM}\left(\frac{1}{8}, \frac{5}{8}, \frac{q^2}{2\pi^4}\right) \exp\left(\frac{\beta q^2}{4\pi^4}\right) + \frac{5^{3/4}}{4} \left(\frac{q^2}{2\pi^4}\right)^{1/8}. \]  

(15)
FIG. 1. Upper left panel: Plot for \( f(r) \) for chosen \( \alpha = 0.5, \beta = 0.1, M = 1 \) and \( q = 0.2 \). Upper right panel: Plot for \( f(r) \) for chosen \( \alpha = 0.5, \beta = 0.1, M = 1 \) and \( q = 0.6 \). Button left panel: Plot for \( f(r) \) for chosen \( \alpha = 0.5, \beta = 0.1, M = 1 \) and \( q = 0.62 \). Button right panel: Plot for \( f(r) \) for chosen \( \alpha = 0.5, \beta = 0.1, M = 1 \) and \( q = 0.7 \). Depending on the parameter values the spacetime can have two horizons known as the Cauchy and event horizons, then an extremal black hole with degenerate horizons and finally no horizons at all.

FIG. 2. The BH spacetime embedded in a three-dimensional Euclidean space. Left panel: We choose \( M = 1, \alpha = 0.2, \beta = 0.1 \) and \( q = 0.2 \). Right panel: We choose \( M = 1, \alpha = 0.2, \beta = 0.1 \) and \( q = 0.6 \).

Taking the limit \( \beta \to 0 \) we obtain

\[
\lim_{\beta \to 0} f(r) = 1 + \frac{r^2}{2\alpha} \left( 1 \pm \sqrt{1 + \frac{8M\alpha}{r^3} - \frac{4q^2\alpha}{r^4}} \right)
\]

which is the charged solution in 4D EGB with a vanishing cosmological constant reported in Ref. [5]. The \( \pm \) sign in Eq. (14) refers to two different branches of solution. Boulware and Deser [1] have demonstrated that EGB black holes with +ve branch sign are unstable and the graviton degree of freedom is a ghost, while the branch with −ve sign is stable and is free of ghosts. In our case, in the limit \( \alpha \to 0 \), the +ve positive branch leads to
FIG. 3. Left panel: Plot for $T_H$ as a function of $r = r_+$ for chosen $\alpha = 0.1, \beta = 0.1, M = 1$ and $q = 0.2$. Right panel: Plot for $T_H$ as a function of $r = r_h$ for chosen $\alpha = 0.5, \beta = 0.1, M = 1$ and $q = 0.5$.

FIG. 4. Upper left panel: Plot for $C_+$ for chosen $\alpha = 0.1, \beta = 0.1, M = 1$ and $q = 0.2$. Upper right panel: Plot for $C_+$ for chosen $\alpha = 0.1, \beta = 0.1, M = 1$ and $q = 0.4$. Button left panel: Plot for $C_+$ for chosen $\alpha = 0.1, \beta = 0.1, M = 1$ and $q = 0.6$. Button right panel: Plot for $C_+$ for chosen $\alpha = 0.1, \beta = 0.1, M = 1$ and $q = 0.8$.

\[ f(r) = \frac{r^2}{\alpha} + \frac{2M}{r} - \frac{q^2 \exp \left( -\frac{\beta q^2}{2r^2} \right) \left[ 4\text{WhittakerM} \left( \frac{1}{8}, \frac{5}{8}, \frac{\beta r^2}{2} \right) \exp \left( \frac{\beta q^2}{4r^2} \right) \right] 2^{1/8} + 5 \left( \frac{q^2 \beta}{r^4} \right)^{1/8}}{5 \left( \frac{q^2 \beta}{r^4} \right)^{1/8} r^2} + \ldots \ldots \]  

which is a wormhole solution in a de-Sitter/anti-de Sitter spacetimes depending on the sign of $\alpha$. On the other hand, in the limit $\alpha \to 0$, the $-ve$ goes over

\[ f(r) = 1 - \frac{2M}{r} + \frac{q^2 \exp \left( -\frac{\beta q^2}{2r^2} \right) \left[ 4\text{WhittakerM} \left( \frac{1}{8}, \frac{5}{8}, \frac{\beta r^2}{2} \right) \exp \left( \frac{\beta q^2}{4r^2} \right) \right] 2^{1/8} + 5 \left( \frac{q^2 \beta}{r^4} \right)^{1/8}}{5 \left( \frac{q^2 \beta}{r^4} \right)^{1/8} r^2} + \ldots \ldots \]
FIG. 5. Upper left panel: Plot for $C_+$ for chosen $\alpha = 0.1$, $\beta = 0.1$, $M = 1$ and $q = 0.6$. Upper right panel: Plot for $C_+$ for chosen $\alpha = 0.4$, $\beta = 0.1$, $M = 1$ and $q = 0.6$. Button left panel: Plot for $C_+$ for chosen $\alpha = 0.6$, $\beta = 0.1$, $M = 1$ and $q = 0.6$. Button right panel: Plot for $C_+$ for chosen $\alpha = 0.8$, $\beta = 0.1$, $M = 1$ and $q = 0.6$.

From the last two equation we can perform the limit $\beta \to 0$ to find

$$f(r) = \frac{r^2}{\alpha} + \frac{2M}{r} - \frac{q^2}{r^2} + \ldots,$$  \hspace{1cm} (18)

for the $+$ve branch sign and

$$f(r) = 1 - \frac{2M}{r} + \frac{q^2}{r^2} + \ldots,$$  \hspace{1cm} (19)

the $-$ve branch sign, respectively. The last result is nothing but the charged black hole solution of GR. In that sense, if $q$ is replaced by the electric charge our solution (14) is a generalization of the recent work presented in Ref. [5] when the cosmological constant vanishes. We notice that in Refs. [36, 37] a well well defined $D \to 4$ limit of EGB gravity was presented. Importantly in the case of spherically symmetric spacetimes in $4D$ our black hole solution should remain valid in these regularised theories, however by going beyond the spherically symmetric cases the solutions are not valid.

III. EMBEDDING DIAGRAM

In this section, we shall explore the geometry of our black hole solution by embedding it into a higher-dimensional Euclidean space. To simplify the problem let us consider the equatorial plane $\theta = \pi/2$ at a fixed moment $t = $ Constant, in that case we have

$$ds^2 = \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2 d\phi^2,$$  \hspace{1cm} (20)

where

$$b(r) = r(1 - f(r)).$$  \hspace{1cm} (21)

Let us embed this black hole metric into three-dimensional Euclidean space in the cylindrical coordinates,

$$ds^2 = dz^2 + dr^2 + r^2 d\phi^2$$  \hspace{1cm} (22)

From Eqs. (20) and (22), we find that

$$\frac{dz}{dr} = \pm \sqrt{\frac{r}{r - b(r)}} - 1,$$  \hspace{1cm} (23)
where \( b(r) \) is given by Eq.(21). Note that the integration of the last expression cannot be accomplished analytically. Invoking numerical techniques allows us to illustrate the embedding diagrams given in Fig. 2.

IV. BLACK HOLE THERMODYNAMICS

In this section we shall discuss the thermodynamical properties of our magnetically charged black hole solution. Toward this goal we first compute the gravitational mass of a black hole by solving \( f(r_+) = 0 \), yielding

\[
M(r_+) = \frac{2 \exp \left( -\frac{5q^2}{2r^2} \right) \left[ 2^{1/8} q^2 \text{WhittakerM} \left( \frac{1}{8}, \frac{5}{8}, \frac{5q^2}{2r^2} \right) \right]}{5(\frac{q^2}{4r^2})^{1/8} r^2} \bigg|_{r_+}
\]

In particular if we now take the limit \( \beta \to 0 \), we obtain

\[
\lim_{\beta \to 0} M(r_+) = \frac{q^2 + r^2 + \alpha}{2r} \bigg|_{r_+}
\]

a well known result. The Hawking temperature associated to our black hole solution can be found by using the relation

\[
T_H = \frac{f'(r)}{4\pi} \bigg|_{r_+}
\]

Due to the complicated and long expression for \( T_H \), in Fig. 2 we show the plots of Hawking temperature as a function of \( r = r_+ \). The thermodynamical stability of the black hole can be found by using the heat capacity. The stability of the black hole is related to sign of the heat capacity \( C_+ \). The stability of the black hole is related to sign of the heat capacity. In particular when \( C_+ > 0 \) the black hole is stable while in the case \( C_+ < 0 \) the black hole is unstable. The heat capacity of the black hole is given [7]

\[
C_+ = \frac{\partial M_+}{\partial T_+} = \frac{\partial M_+}{\partial r_+} \frac{\partial r_+}{\partial T_+}
\]

Again due to the long and complicated expression in Figs. 4 and 5 we plot the heat capacity for different values of parameters. In Fig. 4 we keep \( \alpha \) constant and we increase the magnetic charge \( \beta \). In Fig. 5 on the other hand we keep constant the magnetic charge \( q \) and increase \( \alpha \), respectively. It is observed from both plots that \( C_+ \) exhibits discontinuous at some critical radius \( r = r_c \). In particular there is a flip of sign in the heat capacity around \( r_c \). The black hole is thermodynamically stable for \( r_+ < r_c \) whereas it is thermodynamically unstable for \( r_+ > r_c \). In other words there is a phase transition at \( r_c \) from the stable to unstable phases.

V. CONCLUSION

In this work we have found an exact solution of magnetically charged black holes with exponential model of nonlinear electrodynamics in the context of 4D EGB gravity. We have shown that our \(-\)ve branch results, in the limit \( \alpha \to 0 \) and \( \beta \to 0 \), reduced exactly to the well known magnetically charged black hole of GR. We have analyzed the black hole geometry by embedding into three-dimensional Euclidean space. We have also explored the thermodynamic properties such as the Hawking temperature and thermal stability of regular black holes. It found that that there is a phase transition at \( r_c \), in which the black hole is thermodynamically stable for \( r_+ < r_c \) and thermodynamically unstable for \( r_+ > r_c \). We notice that in Ref. [36] a well well defined \( D \to 4 \) limit of EGB gravity and the spherically symmetric 4D black hole solution should remain valid in these regularised theories, but not beyond spherical symmetry.

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