INFINITE-DIMENSIONAL MANIFOLDS
AS RINGED SPACES

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Abstract. We analyze the possibility of defining infinite-dimensional manifolds as ringed spaces. More precisely, we consider three definitions of manifolds modeled on locally convex spaces: in terms of charts and atlases, in terms of ringed spaces, and in terms of factored spaces, as introduced by Douady in his thesis. It is shown that for large classes of locally convex model spaces (containing Fréchet spaces and duals of Fréchet-Schwartz spaces), the three definitions are actually equivalent. The equivalence of the definition via charts with the definition via ringed spaces is based on the fact that for the classes of model spaces under consideration, smoothness of maps turns out to be equivalent to their scalarwise smoothness (that is, the smoothness of their composition with smooth real-valued functions).

1. Introduction

Finite-dimensional manifolds, whether smooth, real- or complex-analytic, are commonly defined via charts and atlases. The other standard way of defining them relies on a dual point of view, focussing on the functions rather than on the points themselves, and this is achieved via a sheaf-theoretical approach. More precisely, a smooth n-dimensional manifold \(M\) is then defined as a locally ringed space \((M_0, \mathcal{O}_M)\) that is locally isomorphic to the locally ringed space \((\mathbb{R}^n, \mathcal{C}^\infty_{\mathbb{R}^n})\). The sheaf-theoretical approach is hardly avoidable when one wants to deal with singular generalizations of manifolds (varieties or schemes for instance), or with “non-reduced situations”, such as supermanifolds, where the rings of “functions” have nilpotents. In this last example, a section of the structural sheaf is not determined by its values on the points of the underlying topological space, which makes the sheaf-theoretical approach particularly relevant in defining supermanifolds.

In finite dimensions, the two definitions of manifolds (via atlases and as certain locally ringed spaces) are well-known to be equivalent. In infinite dimensions, the situation is quite different. Infinite-dimensional manifolds, whether locally modeled on Banach spaces, Fréchet spaces or general locally convex spaces, have almost always been defined in terms of charts and atlases. One reason for that is the belief, following the thesis of Douady...

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[Dou], that the sheaf of scalar-valued functions does not give sufficient information to define the morphisms (contrary to the finite-dimensional case, where defining the smooth functions valued in $\mathbb{R}$ suffices to determine the morphisms valued in $\mathbb{R}^k$ for every natural number $k$). In [Maz], Mazet defines a category of infinite-dimensional analytic spaces, precisely in terms of ringed spaces. However, his category leads to pathologies (such as the sum of two analytic maps not necessarily being analytic). Douady avoids these pathologies by introducing a third approach for capturing the notion of space, which he uses to define his category of Banach analytic spaces. Namely, given a category $C$, Douady defines a $C$-functor $X$ to be a pair $(X_0, \mathcal{O}_X^C)$ where $X_0$ is a topological space, and $\mathcal{O}_X^C$ is a covariant functor from $C$ to the category of sheaves of sets on $X_0$. In this way, for every object $F$ in $C$ (thought of as a possible target), one associates a sheaf of sets $\mathcal{O}_X^C(F)$ (thought of as the sheaf of $F$-valued morphisms on $X_0$). Compared to a ringed space, a functor $X$ encodes already in its “structural functor” the definition of the morphisms valued in any target space (from a certain category).

The functor space approach obviously adds a supplementary “technical layer”, which can be felt already when defining the local models for Banach analytic spaces (as functor spaces). Thus, unless the recourse to functor spaces is absolutely necessary, it is preferable to deal with the more traditional setting of ringed spaces, i.e. to associate only a single structure sheaf instead of a sheaf-valued functor to each space. We are thus lead to the question whether the insufficiency of the sheaf of scalar-valued functions pointed out by Douady (and the related pathologies) appears also in the non-singular setting of infinite-dimensional smooth manifolds.

In this paper, we address this question by observing that the obstruction to define infinite-dimensional manifolds as ringed spaces boils down to the failure of a scalarwise smooth map between open sets of locally convex spaces to be smooth. More precisely, given two locally convex spaces $E$ and $F$, and given an open subset $U$ of $E$, we adopt a standard notion of smoothness for maps $\Phi : U \to F$ (cf. Definition 3.1), following Bastiani, Hamilton, Milnor, Neeb, ... Then, a map $\Phi$ is said to be scalarwise smooth if the function $f \circ \Phi : U \to \mathbb{R}$ is smooth for every smooth function $f : F \to \mathbb{R}$. The chain rule implies clearly that smooth maps are scalarwise smooth. The converse is easily seen to be true in finite dimensions (just take the linear forms $e^*_i$ dual to a basis $\{e_i : 1 \leq i \leq \dim F\}$). In infinite dimensions, the converse is non-trivial. In the first main result of this paper (Theorem 3.19), we prove that it is true whenever $F$ is Mackey-complete and the Mackey-closure topology on $E$ coincides with the given topology (the necessary definitions will be recalled throughout the paper, but let us already note that the assumption on $E$ corresponds to a large class of locally convex spaces which contains Fréchet spaces and duals of Fréchet-Schwartz spaces, and that every complete locally convex space is Mackey-complete). In the second main result of the paper (Theorem 4.12 and Corollary 4.13), we show that for infinite-dimensional manifolds modeled on locally convex spaces $E$
satisfying the assumptions of Theorem 3.19, the definition via charts and atlases is equivalent to the definition via ringed spaces. Finally, we complete our comparison of the various definitions of infinite-dimensional manifolds by showing that the definition via charts and atlases is equivalent to the one based on functored spaces.

Our paper is organized as follows. In Section 2, we present a proof of the special case of Theorem 3.19, when the domain is $\mathbb{R}$. More precisely, we show that for a Mackey-complete locally convex space $E$, a curve $c : \mathbb{R} \to E$ is smooth if and only if it is scalarwise smooth. While this result and its idea of proof are not new (compare [KM]), they are crucial to our proof of Theorem 3.19; we recall them in a concise but self-contained way for the convenience of the reader, which gives us also the opportunity to introduce our notations for the rest of the paper. In Section 3, we recall the definition of smooth maps that we will be using, and prepare and prove Theorem 3.19 using as an intermediate step the calculus of convenient smoothness studied by Kriegl and Michor. In Section 4, we present in detail the three definitions of infinite-dimensional manifolds under investigation here, and prove the comparison results (Theorem 4.12 and Corollary 4.13) mentioned above.

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2. Smoothness of curves

The goal of this section is to recall the proof of the following result: if $c$ is a map from $\mathbb{R}$ to a complete real (Hausdorff) locally convex space $E$, and $E'$ is the continuous dual of $E$, then smoothness of $\ell \circ c$ for every $\ell \in E'$ implies the smoothness of $c$ (Theorem 2.11). While this is obvious if $E$ is finite-dimensional (it is enough to take the projections $e_i^* : E \to \mathbb{R}$ where $\{e_i : 1 \leq i \leq \dim E\}$ is an arbitrary basis of $E$), proving that it remains true in the general case requires more work. The strategy (essentially taken from [KM]), is the following.

In any locally convex space, there is a natural notion of bounded set. The collection of these bounded sets (“the von Neumann bornology”) is not very sensitive to the locally convex topology: a classical theorem of Mackey in functional analysis shows that if one varies the topology while keeping the same dual space, the bounded sets remain the same. As a consequence, one can view the bounded sets from the perspective of the weak topology instead of the given topology. In the weak topology, it is natural and immediate that a subset of $E$ whose image by every linear functional is bounded must be itself bounded.

On the other hand, for curves, being $C^\infty$ is ultimately a bornological concept: the $C^\infty$ curves remain the same if one changes the locally convex topology, while keeping the same underlying bornology. This follows from the fact that a $C^1$ curve is locally Lipschitz (by the mean value theorem), and the Lipschitz condition (which is essentially bornological), implies continuity.
Translating smoothness in terms of Lipschitz conditions (involving bounded sets), it becomes possible to use the dual characterization of boundedness given by Mackey’s theorem, to obtain a dual characterization of smoothness.

In what follows, we recall, for the convenience of the reader and for later reference, the details of the above arguments, starting with Mackey’s theorem, the cornerstone in proving Theorem 2.11 as well as other results in this paper. For a proof of the former theorem, see, e.g., Theorem 36.2 in [Tre] or Theorem 8.3.4 in [Jar].

**Theorem 2.1.** (Mackey’s theorem) Let $E$ be a locally convex space, and $B$ a subset of $E$. If $\ell(B)$ is bounded for every $\ell \in E'$, then $B$ is bounded.

**Definition 2.2.** Let $E$ be a locally convex space, and $c : \mathbb{R} \to E$ a curve.

1. If $J$ is an open subset of $\mathbb{R}$, we say that $c$ is **Lipschitz** on $J$ if the set $\left\{ \frac{c(t_2) - c(t_1)}{t_2 - t_1} ; t_1, t_2 \in J \text{ and } t_1 \neq t_2 \right\}$ is bounded in $E$.

2. We say that $c$ is **locally Lipschitz** if every point in $\mathbb{R}$ has a neighborhood on which $c$ is Lipschitz.

**Definition 2.3.** Let $E$ be a locally convex space, and $c : \mathbb{R} \to E$ a curve. For $k \in \mathbb{N}$, we say that $c$ is of class $\text{Lip}^k$ if all the derivatives of $c$ up to order $k$ exist, and $c^{(k)} : \mathbb{R} \to E$ is locally Lipschitz.

If $A$ is any subset of $E$, we will denote by $\langle A \rangle$ the absolute convex hull of the closure of $A$. We will need the following version of the mean value theorem, for curves in a locally convex space.

**Theorem 2.4.** (Mean Value Theorem) Let $E$ be a locally convex space, and $c : [a, b] \to E$ a curve which is continuous on $[a, b]$, and differentiable on $]a, b[$. Then

$$\frac{c(b) - c(a)}{b - a} \in \langle \{ c'(t) ; a < t < b \} \rangle$$

**Proof:** cf. [KM], I.1.4.

**Corollary 2.5.** Let $E$ be a locally convex space, and $c : \mathbb{R} \to E$ a curve which is differentiable on an open interval $J \subset \mathbb{R}$. If $c'$ is bounded on $J$, then $c$ is Lipschitz on $J$.

**Proof:** Let $t_1, t_2 \in J$ with $t_1 \neq t_2$. By the mean value theorem,

$$\frac{c(t_2) - c(t_1)}{t_2 - t_1} \in \langle \{ c'(t) ; t \in J \} \rangle$$

So $\left\{ \frac{c(t_2) - c(t_1)}{t_2 - t_1} ; t_1, t_2 \in J \text{ and } t_1 \neq t_2 \right\} \subset \langle \{ c'(t) ; t \in J \} \rangle$, and this last set is bounded since the absolute convex hull of any bounded set is bounded. □
Proposition 2.6. Let $E$ be a locally convex space, and $c: \mathbb{R} \to E$ a curve.

1. If $c$ is of class $C^k$, then $c$ is of class $\text{Lip}^{k-1}$.
2. If $c$ is of class $\text{Lip}^{k-1}$, then $c$ is of class $C^{k-1}$.

Proof: 1. If $c$ is $C^k$, then $c^{(k-1)}$ is $C^1$. This implies (via the preceding corollary) that $c^{(k-1)}$ is locally Lipschitz. Thus, $c$ is $\text{Lip}^{k-1}$.

2. If $c$ is $\text{Lip}^{k-1}$, then $c^{(k-1)}$ is locally Lipschitz. This implies that $c^{(k-1)}$ is continuous. Thus, $c$ is $C^{k-1}$. □

Corollary 2.7. Let $E$ be a locally convex space, and $c: \mathbb{R} \to E$ a curve. Then $c$ is of class $C^\infty$ if and only if $c$ is of class $\text{Lip}^{k}$ for all $k \in \mathbb{N}$.

Proof: If $c$ is smooth, then $c$ is $C^k$ for every $k$. The first part of Proposition 2.6 implies then that $c$ is $\text{Lip}^{k-1}$ for every $k$. Conversely, if $c$ is $\text{Lip}^{k}$ for every $k$, then the second part of Proposition 2.6 implies that $c$ is $C^k$ for every $k$, and so $c$ is smooth. □

Definition 2.8. Let $E$ be a locally convex space. A curve $c: \mathbb{R} \to E$ is said to be weakly smooth if $\ell \circ c$ is smooth for every $\ell \in E'$.

Definition 2.9. Let $E$ be a locally convex space.

1. A sequence $(x_n)$ in $E$ is said to be Mackey-convergent to a point $x \in E$ if there exists an absolutely convex bounded set $B \subset E$, and a sequence $(\mu_n)$ of real numbers converging to 0, such that $x_n - x \in \mu_n B$ for all $n$.

2. A sequence $(x_n)$ in $E$ is said to be Mackey-Cauchy if there exists an absolutely convex bounded set $B \subset E$, and a double sequence $(\mu_{n,m})$ of real numbers converging to 0, such that $x_n - x_m \in \mu_{n,m} B$ for all $n,m$.

We also have the same notions for nets (just replace “sequence” by “net” everywhere in the preceding definition).

3. $E$ is called Mackey-complete if every Mackey-Cauchy net in $E$ converges.

Remark 2.10. Every complete locally convex space is sequentially complete, and in turn sequential completeness implies Mackey-completeness. For metrizable locally convex spaces the three notions of completeness coincide. Given a locally convex space $E$, the completion $\hat{E}$ yields a complete (and thus Mackey-complete) locally convex space together with a continuous linear embedding $j: E \hookrightarrow \hat{E}$ having dense image. (Compare, e.g. Theorem 5.2 in [Tre].) A continuous curve $c: \mathbb{R} \to E$ yields then a continuous curve $\hat{c} = j \circ c: \mathbb{R} \to \hat{E}$, and $c$ is weakly smooth if and only if $\hat{c}$ is weakly smooth.

Theorem 2.11. Let $E$ be a Mackey-complete locally convex space and $c: \mathbb{R} \to E$ a curve. Then $c$ is smooth if and only if it is weakly smooth.
Using again Mackey’s theorem, we deduce that $B$ open subset notably $[\text{Ham}]$, $[\text{Mil}]$, $[\text{Ne1}]$, $[\text{Ne2}]$. The goal of this section is to generalize applicable theory of infinite-dimensional manifolds and Lie groups, compare $c(\{J\}_{t})$ for each $(\ell \in E')$ true for $\ell \circ c$ is smooth. We first prove that $c$ is differentiable. Let $t_{0} \in \mathbb{R}$, and $J$ a compact interval about $t_{0}$. Set $q(t) := \frac{c(t) - c(t_{0})}{t - t_{0}}$ for all $t \in J - \{t_{0}\}$. We need to show that $q$ has a limit as $t \to t_{0}$. Let $B := \left\{ \frac{q(t) - q(t')}{t - t'} ; t, t' \in J - \{t_{0}\} \text{ and } t \neq t' \right\}$. Then $\ell(B) = \left\{ \frac{(\ell \circ q)(t) - (\ell \circ q)(t')}{t - t'} ; t, t' \in J - \{t_{0}\} \text{ and } t \neq t' \right\}$.

Now for each $t \in J - \{t_{0}\}$, we have $$(\ell \circ q)(t) = (\ell \circ c)(t) - (\ell \circ c)(t_{0}) \cdot \frac{t - t_{0}}{t - t_{0}}.$$ Since $\ell \circ c$ is smooth, the same must be true for $\ell \circ q$. In particular, the mean value theorem implies the existence of $\tau \in [0, 1]$ such that $$(\ell \circ q)(t) - (\ell \circ q)(t') = (t - t') (\ell \circ q)'(t') + \tau(t - t').$$ Now $t' + s(t - t') \in J$ for every $s \in [0, 1]$ (by convexity of $J$). Since $\ell \circ q'$ is continuous and $J$ is compact, we deduce that there is $M > 0$ such that $|(\ell \circ q)'(t') + \tau(t - t')| \leq M$ for all $s \in [0, 1]$. But then, $|(\ell \circ q)(t) - (\ell \circ q)(t')| \leq M|t - t'|$. Thus, $\ell(B)$ is bounded. Since $\ell$ was arbitrary, we deduce by Mackey’s theorem that $B$ is bounded as well, i.e. $q$ is Lipschitz on $J - \{t_{0}\}$. Otherwise stated, the net $(q_{t})_{t \in J - \{t_{0}\}}$ with $q_{t} := q(t)$ is Mackey-Cauchy. Mackey-completeness of $E$ implies that $q$ has a continuous extension to $J$, which allows us to define $c'(t_{0})$ as $q(t_{0})$. Thus, $c$ is differentiable, and for every $\ell \in E'$, we have $(\ell \circ c)' = \ell \circ c'$ (by chain rule and linearity of $\ell$).

Second, we prove that $c'$ is locally Lipschitz. Let $t_{0} \in \mathbb{R}$, and $J$ a compact interval about $t_{0}$. Then for every $\ell \in E'$, the function $(\ell \circ c)'$ is Lipschitz on $J$ (since $(\ell \circ c)'$ is continuous, is bounded on $J$). Let $B_{1} = \left\{ \frac{c'(t_{2}) - c'(t_{1})}{t_{2} - t_{1}} ; t_{1}, t_{2} \in J \text{ and } t_{1} \neq t_{2} \right\}$. Then, since $\ell \circ c' = (\ell \circ c)'$, we have $\ell(B_{1}) = \left\{ \frac{(\ell \circ c)'(t_{2}) - (\ell \circ c)'(t_{1})}{t_{2} - t_{1}} ; t_{1}, t_{2} \in J \text{ and } t_{1} \neq t_{2} \right\}$. Since $(\ell \circ c)'$ is Lipschitz on $J$, we have that $\ell(B_{1})$ is bounded. But $\ell$ is arbitrary. Using again Mackey’s theorem, we deduce that $B_{1}$ is bounded, hence $c'$ is Lipschitz on $J$, and so $c'$ is locally Lipschitz, that is, $c$ is $\text{Lip}^{1}$.

Replacing $c$ by $c'$ in the above chain of arguments, we arrive at the conclusion that $c'$ is $\text{Lip}^{1}$ as well, i.e. $c$ is $\text{Lip}^{2}$. By induction, we conclude that $c$ is $\text{Lip}^{k}$ for all $k$, i.e. $c$ is smooth. \hfill $\square$

3. Smoothness of maps

In this section, we start by recalling the notion of a smooth map from an open subset $U$ of a locally convex space $E$ into a locally convex space $F$. Among the various existing notions of (differentiable and) smooth maps, we choose what is called $C_{\infty}^{\infty}$ in the book of Keller (Ker), going back at least to Bastiani (compare Definitions II.3.1 and II.2.2 in Bas). This definition of smoothness turns out to be appropriate for the construction of an applicable theory of infinite-dimensional manifolds and Lie groups, compare notably Ham, Mil, Ne1, Ne2. The goal of this section is to generalize
Theorem 2.11 to maps having infinite-dimensional source. Namely, we want to show that if $E$ is for example a Fréchet space or the continuous dual of a Fréchet-Schwartz space (cf. below for details), a map $\Phi : E \supset U \rightarrow F$ is smooth if and only if it is scalarwise smooth. By scalarwise smooth, we mean that $f \circ \Phi \in \mathcal{C}^\infty(U, \mathbb{R})$ for every $f \in \mathcal{C}^\infty(F, \mathbb{R})$.

As a matter of fact, this result is very easy to prove if we replace the notion of smoothness of Bastiani et al. by another one: the “convenient smoothness” considered by Frélicher, Kriegl and Michor (compare notably [KM]). A map $\Phi : U \rightarrow F$ is said to be conveniently smooth if it sends smooth curves to smooth curves. Since the notion of convenient smoothness relies on curves, Theorem 2.11 immediately yields the equivalence between convenient smoothness and scalarwise convenient smoothness of maps.

For applications, it is thus highly desirable to establish that convenient smoothness coincides with smoothness in the sense of Bastiani-Hamilton-Milnor-Neeb, for relevant classes of locally convex spaces (“generalized Bochner theorem”). That this is indeed the case was already observed in [Ne2] in case where the source $E$ is a Fréchet space. Below we will first prove a more general result of this type, before giving our crucial characterization of smoothness in terms of weak resp. scalarwise smoothness.

Let $E$ and $F$ be locally convex spaces, and $U$ an open subset of $E$. For a continuous map $\Phi : U \rightarrow F$, the Gâteaux derivative of $\Phi$ at a point $x \in U$ in the direction of a vector $v \in E$ is defined by

$$d\Phi|_x(v) = \lim_{t \to 0} \frac{\Phi(x+tv) - \Phi(x)}{t}$$

provided the limit exists.

**Definition 3.1.** The map $\Phi$ is said to be of class $\mathcal{C}^1$ if $d\Phi : U \times E \rightarrow F$, $(x,v) \mapsto (d\Phi)(x,v) := d\Phi|_x(v)$ exists and is continuous. We define inductively a map $\Phi$ to be of class $\mathcal{C}^{k+1}$ if it is of class $\mathcal{C}^1$ and $d\Phi$ is of class $\mathcal{C}^k$. Furthermore, a map is called being of class $\mathcal{C}^\infty$ or smooth if it is $\mathcal{C}^k$ for all $k \in \mathbb{N}$.

**Remark 3.2.** If $E$ and $F$ are Banach spaces, being $\mathcal{C}^1$ in the above sense is weaker than the usual notion of $\mathcal{C}^1$ in the sense of Fréchet differentiability, which requires the map $x \mapsto d\Phi|_x$ to be continuous as map from $U$ to $\mathcal{L}(E, F)$, equipped with the operator norm topology. However, $\mathcal{C}^2$ in the above sense implies $\mathcal{C}^1$ in the usual Fréchet differentiability sense (cf. Proposition 2.7.1 in [Kel], or Theorem I.7 in [Ne1]), so that in Banach spaces, the two definitions lead to the same smooth maps.

Next we turn to the notion of convenient smoothness. Before we recall its definition and main properties, let us already note that in general, it is possible to find conveniently smooth maps which are not even continuous!
This hints to the fact that there should be a different topology which is more adapted to convenient smoothness, and for which conveniently smooth maps are automatically continuous. Since the definition of convenient smoothness relies on smooth curves, it is natural to use smooth curves to define this topology.

**Definition 3.3.** Let $E$ be a locally convex space. The Mackey-closure topology (also called $c^\infty$-topology) is the finest topology on $E$ making all the smooth curves $c : \mathbb{R} \rightarrow E$ continuous.

**Remark 3.4.** The $c^\infty$-topology on $E$ is clearly finer than the given locally convex topology. Note that if $E$ is a Fréchet space or the dual of a Fréchet-Schwartz space, then the two topologies coincide ([KM], Theorem I.4.11). For Fréchet spaces, we give a short and elementary derivation of this fact in this appendix.

Note also that there are locally convex spaces such that the Mackey-closure topology is not even a vector space topology and thus a fortiori does not equal the initially given topology. Examples of this phenomenon are strict inductive limits of strictly increasing sequences of infinite-dimensional Fréchet spaces, as e.g. $\mathcal{D}(M)$, the space of compactly supported smooth functions on a noncompact finite-dimensional smooth manifold. (Compare [KM], Proposition I.4.26.)

In Section 2, we have seen that smoothness of curves is ultimately a bornological concept, in the sense that the requirement of continuity of the derivatives can be replaced by that of being locally Lipschitz, a notion that relies on the bounded sets. Consequently, it should not be very surprising that the $c^\infty$-topology can be characterized, as the next two propositions will show, in terms of Mackey-convergent sequences (which, by the following lemma, can be thought of as the sequential counterpart of locally Lipschitz maps).

**Lemma 3.5.** Let $E$ be a locally convex space, and $c : \mathbb{R} \rightarrow E$ a locally Lipschitz curve. For any convergent sequence of real numbers $(t_n)$ in $E$ is Mackey-convergent.

**Proof:** Let $a = \lim t_n$. Since $c$ is locally Lipschitz, there exists a neighborhood $J$ of $a$ such that the set $\left\{ \frac{c(t'') - c(t')}{{t''} - t'} ; t', t'' \in J \text{ and } t' \neq t'' \right\}$ is bounded. Let $N \in \mathbb{N}$ be such that $n \geq N$ implies $t_n \in J$. Then $\left\{ \frac{c(t_n) - c(a)}{t_n - a} ; n \geq N \right\}$ is bounded, which proves that $c(t_n)$ Mackey-converges to $c(a)$. \hfill \Box

Recall that a sequence $(x_n)_{n \in \mathbb{N}^*}$ of points of a locally convex space $E$ converges fast to a point $x \in E$ if for every $\alpha \in \mathbb{N}$, the set $\{n^\alpha(x_n - x) ; n \in \mathbb{N}^*\}$ is bounded in $E$. Therefore, a sequence $(x_n)_{n \in \mathbb{N}^*}$ converges fast to $x$ if and only if, for each $\alpha \in \mathbb{N}$, there exists a bounded subset $B_\alpha$ of $E$ such that...
Lemma 3.6. (Special curve lemma) Let $E$ be a locally convex space, $(x_n)_{n \in \mathbb{N}^*}$ a sequence in $E$, and $x$ a point of $E$. If $(x_n)_{n \in \mathbb{N}^*}$ converges fast to $x$, then there exists a smooth curve $c : \mathbb{R} \to E$ such that $c(1/n) = x_n$ for every $n \in \mathbb{N}^*$ and $c(0) = x$.

Proof : cf. [KM], I.2.8. \hfill \Box

Lemma 3.7. Let $E$ be a locally convex space. Every sequence in $E$ that Mackey-converges to some point $x \in E$ has a subsequence that converges fast to $x$.

Proof : Let $(x_n)$ be a sequence in $E$ that Mackey-converges to some point $x \in E$. By definition, there exists an absolutely convex bounded set $B \subset E$, and a sequence $(\mu_n)$ of real numbers converging to 0, such that $x_n - x \in \mu_n B$ for all $n$. W.l.o.g. the $\mu_n$ can be chosen to be positive. Since $(\mu_n)$ converges to 0, there exists a positive integer $n_1$ such that $\mu_{n_1} < \frac{1}{2}$. Then, there exists a positive integer $n_2 > n_1$ such that $\mu_{n_2} < \frac{1}{2^2}$. By induction, one obtains in this way a strictly increasing sequence of positive integers $n_1 < n_2 < \cdots < n_k < \cdots$ such that $\mu_{n_k} < \frac{1}{2^k}$ for every $k \geq 1$. Since $B$ is balanced, this implies that $x_{n_k} - x \in \frac{1}{2^k} B$ for every $k \geq 1$. For each $\alpha \in \mathbb{N}$, choose $\lambda_\alpha > 0$ such that $\frac{1}{2^k} \leq \frac{\lambda_\alpha}{\lambda_{n_k}}$ for every $k \geq 1$, and set $B_\alpha := \lambda_\alpha B$. Using again the fact that $B$ is balanced, we deduce that $x_{n_k} - x \in \frac{1}{2^k} B_\alpha$. Since $B_\alpha$ is bounded, this shows that the subsequence $(x_{n_k})$ converges fast to $x$. \hfill \Box

Proposition 3.8. Let $E$ be a locally convex space. A subset $U$ of $E$ is $c^\infty$-open if and only if for every point $x \in U$ and every sequence $(x_n)$ in $E$ that Mackey-converges to $x$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $x_n \in U$.

Proof : Suppose $U$ is $c^\infty$-open, let $x \in U$ and $(x_n)$ a sequence in $E$ that Mackey-converges to $x$. Assume that the set $\{n \in \mathbb{N}^* \mid x_n \notin U\}$ is infinite. Then there exists $(x_{n_k}) = (x_{n_k})_{k \geq 1}$, a subsequence of $(x_n)$, such that none of its points $x_{n_k}$ is in $U$. A fortiori, $(x_{n_k})$ Mackey-converges to $x$ as well. It follows from the above lemma that $(x_{n_k})$ has a subsequence $(x_{n_{k_j}})$ that converges fast to $x$. By Lemma 3.6 there exists a smooth curve $c : \mathbb{R} \to E$ such that $c(1/j) = x_{n_{k_j}}$ for every $j \in \mathbb{N}^*$ and $c(0) = x$. Since $U$ is $c^\infty$-open, $c^{-1}(U)$ is open in $\mathbb{R}$. Moreover, $0 \in c^{-1}(U)$. Therefore, there exists $N \in \mathbb{N}$ such that $j \geq N$ implies $\frac{1}{j} \in c^{-1}(U)$. Consequently, $j \geq N$ implies $x_{n_{k_j}} \in U$. This contradicts the fact that none of the $x_{n_k}$ belongs to $U$. Conversely, suppose that for every point $x \in U$ and every sequence $(x_n)$ in $E$ that Mackey-converges to $x$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $x_n \in U$. Let $c$ be a smooth curve, $t \in c^{-1}(U)$, and $(t_n)$ a sequence of real numbers converging to $t$. Being smooth, $c$ is locally Lipschitz, and so by Lemma 3.6 the sequence $(c(t_n))$ Mackey-converges to $c(t) \in U$. By our assumption, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $(c(t_n)) \in U$. Thus,
\[ n \geq N \] implies \( t_n \in c^{-1}(U) \), which shows that \( c^{-1}(U) \) is open in \( \mathbb{R} \). \( \square \)

**Proposition 3.9.** Let \( E \) be a locally convex space. A subset \( A \) of \( E \) is \( c^\infty \)-closed if and only if for every Mackey-convergent sequence \((x_n)\) of points of \( A \), the limit of \((x_n)\) belongs to \( A \).

**Proof:** Suppose \( A \) is \( c^\infty \)-closed, and let \( x \) be the limit of a Mackey-convergent sequence \((x_n)\) of points of \( A \). Let \( U = E - A \), so that \( U \) is \( c^\infty \)-open. If \( x \notin A \), then \( x \in U \), and so there exists \( N \in \mathbb{N} \) such that \( n \geq N \) implies \( x_n \in U \). This clearly contradicts the fact that \( x_n \in A \) for all \( n \). Thus, \( x \in A \). Conversely, suppose \( A \) has the stated property. If \( A \) is not \( c^\infty \)-closed, then \( U = E - A \) is not \( c^\infty \)-open. This implies the existence of a point \( x \) in \( U \) and a sequence \((x_n)\) in \( E \) that Mackey-converges to \( x \), such that the set \( \{ n \in \mathbb{N} \mid x_n \in A \} \) is infinite. This is tantamount to the existence of a subsequence \((x_{n_k})\) of \((x_n)\) whose terms belong to \( A \). Since \((x_{n_k})\) Mackey-converges to \( x \) the same holds true for \((x_{n_k})\). Since \( x_{n_k} \in A \) for each \( k \), the property assumed on \( A \) implies that \( x \in A \), contradiction. \( \square \)

We are ready to recall the precise definition and some of the properties of conveniently smooth maps. (cf. notably [KM]).

**Definition 3.10.** Let \( E \) and \( F \) be locally convex spaces, \( U \) a \( c^\infty \)-open subset of \( E \), and \( \Phi : U \rightarrow F \) a map. We say that \( \Phi \) is **conveniently smooth** if for every smooth curve \( c : \mathbb{R} \rightarrow U \), the curve \( \Phi \circ c : \mathbb{R} \rightarrow F \) is smooth.

**Remark 3.11.** It is easy to see that the composition of two conveniently smooth maps is conveniently smooth.

**Proposition 3.12.** Let \( E \) and \( F \) be locally convex spaces, and \( U \) an open subset of \( E \). Any conveniently smooth map \( \Phi : U \rightarrow F \) is continuous when \( E \) is equipped with the \( c^\infty \)-topology.

**Proof:** Let \( V \) be an open subset of \( F \). To show that \( \Phi^{-1}(V) \) is \( c^\infty \)-open in \( U \), we need to show that \( c^{-1}(\Phi^{-1}(V)) \) is open in \( \mathbb{R} \) for every \( c \in C^\infty(\mathbb{R}, U) \). But \( c^{-1}(\Phi^{-1}(V)) = (\Phi \circ c)^{-1}(V) \), which is clearly open in \( \mathbb{R} \) since the curve \( \Phi \circ c \) is smooth, and therefore continuous. \( \square \)

We denote by \( C^\infty_{\text{conv}}(U, F) \) the set of conveniently smooth maps from \( U \) to \( F \).

The equivalence between smoothness and convenient smoothness, when it holds, is not trivial to establish, even for functions from \( \mathbb{R}^d \) to \( \mathbb{R} \). In this case, it was first proved by Boman in 1967 [Bom].

**Theorem 3.13.** (Boman’s theorem)

\[
C^\infty_{\text{conv}}(\mathbb{R}^2, \mathbb{R}) = C^\infty(\mathbb{R}^2, \mathbb{R}).
\]
**Proof**: cf. [Bom] or [KM], I.3.4.

Now we concentrate on the structure of $C^\infty_{\text{conv}}(U, F)$.

**Proposition 3.14.** Let $E$ and $F$ be locally convex spaces, $U$ a $c^\infty$-open subset of $E$. Then $C^\infty_{\text{conv}}(U, F)$ is a locally convex space for the coarsest topology making the maps $c^\alpha : C^\infty_{\text{conv}}(U, F) \to C^\infty(\mathbb{R}, F)$, $\Phi \mapsto c^\alpha \Phi = \Phi \circ c$ for all $c \in C^\infty(\mathbb{R}, U)$ continuous.

**Proof**: It is not difficult to check that this topology on $C^\infty_{\text{conv}}(U, F)$ is defined by the family of seminorms $P = \{ p_{K,\alpha,q,c} : K$ compact subset of $\mathbb{R}, \alpha \in \mathbb{N}, q$ continuous seminorm on $F, c \in C^\infty(\mathbb{R}, U) \}$, where we set $p_{K,\alpha,q,c}(\Phi) = \sup_{t \in K} q((\Phi \circ c)(\alpha)(t))$ for $\Phi \in C^\infty_{\text{conv}}(U, F)$. Moreover, if $\Phi \in C^\infty_{\text{conv}}(U, F) - \{0\}$, let $x \in U$ be such that $\Phi(x) \neq 0$, and $c := k_x : \mathbb{R} \to U$ the constant curve at $x$. There exists a continuous seminorm $q$ on $F$ such that $q(\Phi(x)) \neq 0$. Take $\alpha = 0$ and $K = \{0\}$. Then $p_{K,\alpha,q,c}(\Phi) = \sup_{t \in [0]} q((\Phi \circ c)(t)) = q((\Phi \circ c)(0)) = q(\Phi(x)) \neq 0$. Thus, $P$ is separating. □

One of the major benefits of working with conveniently smooth maps is the fact that they give rise to a Cartesian closed category, as the next theorem will show.

**Theorem 3.15.** Let $E_1$, $E_2$ and $F$ be locally convex spaces, $U_1$ and $U_2$ $c^\infty$-open subsets of $E_1$ and $E_2$ respectively. Then, as sets, $C^\infty_{\text{conv}}(U_1 \times U_2, F) \cong C^\infty_{\text{conv}}(U_1, C^\infty_{\text{conv}}(U_2, F))$.

**Proof**: cf. Theorem I.3.12 in [KM]. □

A first consequence of Cartesian closedness is a the following generalization of Boman’s theorem.

**Corollary 3.16.** Let $E$ be a locally convex space. Then, $C^\infty_{\text{conv}}(\mathbb{R}^n, E) = C^\infty(\mathbb{R}^n, E)$

**Proof**: Let $\Phi \in C^\infty_{\text{conv}}(\mathbb{R}^n, E)$. By Theorem 3.15, we have $C^\infty_{\text{conv}}(\mathbb{R}^n, E) \cong C^\infty_{\text{conv}}(\mathbb{R}^{n-1}, \mathbb{R}, E))$. In particular, for every $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \in \mathbb{R}$, the partial map $\Phi(x_1, \ldots, \bullet, \ldots, x_n), y \mapsto \Phi(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n)$ lies in $C^\infty(\mathbb{R}, E)$. Then $\frac{\partial}{\partial x_i}(x_1, \ldots, x_i, \ldots, x_n) =$

$$\lim_{t \to 0} \frac{\Phi(x_1, \ldots, x_i + t, \ldots, x_n) - \Phi(x_1, \ldots, x_i, \ldots, x_n)}{t} = \frac{\Phi(x_1, \ldots, \bullet, \ldots, x_n)(x_i + t) - \Phi(x_1, \ldots, \bullet, \ldots, x_n)(x_i)}{t} = \Phi(x_1, \ldots, \bullet, \ldots, x_n)'(x_i).$$

Thus, all first-order partial derivatives of $\Phi$ exist. Inductively, one obtains the existence of all higher-order partial derivatives of $\Phi$. This proves that $\Phi \in C^\infty(\mathbb{R}^n, E)$. Conversely, if $\Phi \in C^\infty(\mathbb{R}^n, E)$, then $\Phi \in C^\infty_{\text{conv}}(\mathbb{R}^n, E)$ by
the chain rule.

Another consequence of Cartesian closedness is the convenient smoothness of the differential.

**Proposition 3.17.** Let $E$ and $F$ be locally convex spaces, $U$ a $c^\infty$-open subset of $E$, and $\Phi \in C^\infty_{\text{conv}}(U, F)$. For every $x \in U$ and $v \in E$, set $d\Phi|_x(v) := \lim_{t \to 0} \frac{\Phi(x + tv) - \Phi(x)}{t}$. Then $d\Phi \in C^\infty_{\text{conv}}(U \times E, F)$.

**Proof:** We claim that the map $\delta : C^\infty_{\text{conv}}(U, F) \times U \times E \to F$ defined by $\delta(\Phi, x, v) := d\Phi|_x(v) = \lim_{s \to 0} \frac{\Phi(x + sv) - \Phi(x)}{s}$ is conveniently smooth. Indeed, let $c = (\tilde{\Phi}, \tilde{x}, \tilde{v}) : \mathbb{R} \to C^\infty_{\text{conv}}(U, F) \times U \times E$ be a smooth curve, and set $h(t, s) := \tilde{\Phi}(t)(\tilde{x}(t) + s\tilde{v}(t))$. Then $(\delta \circ c)(t) = \delta(c(t)) = \delta(\tilde{\Phi}(t), \tilde{x}(t), \tilde{v}(t)) = lim_{s \to 0} \frac{\tilde{\Phi}(t)(\tilde{x}(t) + s\tilde{v}(t)) - \tilde{\Phi}(t)(\tilde{x}(t))}{s} = lim_{s \to 0} \frac{h(t, s) - h(t, 0)}{s} = \frac{\partial h}{\partial s}(t, 0)$. Since $h$ is clearly conveniently smooth (and smooth by Corollary 3.16), we deduce that the curve $\delta \circ c$ is smooth. Thus, $\delta \in C^\infty_{\text{conv}}(C^\infty_{\text{conv}}(U, F) \times U \times E, F)$, and therefore $d := \delta \in C^\infty_{\text{conv}}(C^\infty_{\text{conv}}(U, F), C^\infty_{\text{conv}}(U \times E, F))$. In particular, $d\Phi \in C^\infty_{\text{conv}}(U \times E, F)$.

An immediate consequence of the above proposition, already observed in [Ne2] for Fréchet spaces, is a further generalization of Boman’s theorem, stating that for a reasonable class of locally convex spaces, convenient smoothness coincides with smoothness (in the sense of the above definition).

**Proposition 3.18.** Let $E$ and $F$ be locally convex spaces, and $U$ an open subset of $E$. Assume that the given topology on $E$ coincides with the Mackey-closure topology. Then $C^\infty_{\text{conv}}(U, F) = C^\infty(U, F)$.

**Proof:** By the chain rule, smoothness implies convenient smoothness. For the nontrivial direction, suppose $\Phi : U \to F$ is conveniently smooth. By the preceding proposition, $d\Phi : U \times E \to F$ is conveniently smooth as well. Proposition 3.12 implies then that $d\Phi$ is continuous for the $c^\infty$-topology, therefore continuous by our assumption on $E$. Thus, $\Phi$ is $C^1$, and by induction, one obtains that $\Phi$ is smooth.

We are ready to state and prove the main result of this section.

**Theorem 3.19.** Let $E$ and $F$ be locally convex space spaces, and $U$ an open subset of $E$. Assume that the given topology on $E$ coincides with the Mackey-closure topology and that $F$ is Mackey-complete. Let $\Phi : U \to F$ be a continuous map. Then the following are equivalent:

(i) $\Phi$ is smooth.
(ii) For every open subset $V$ of $F$ containing $\Phi(U)$ and every $f \in C^\infty(V, \mathbb{R})$, we have $f \circ \Phi \in C^\infty(U, \mathbb{R})$.

(iii) For every $f \in C^\infty(F, \mathbb{R})$, we have $f \circ \Phi \in C^\infty(U, \mathbb{R})$ (i.e., $\Phi$ is scalarwise smooth).

(iv) For every $\ell \in F'$, we have $\ell \circ \Phi \in C^\infty(U, \mathbb{R})$ (i.e., $\Phi$ is weakly smooth).

**Proof:** $(i) \implies (ii)$ is evident by chain rule, and $(ii) \implies (iii)$ follows upon taking $V = F$. $(iii) \implies (iv)$ follows from the fact that every continuous linear map is smooth. It remains to show $(iv) \implies (i)$. Suppose that for every $\ell \in F'$, we have $\ell \circ \Phi \in C^\infty(U, \mathbb{R})$. For every smooth curve $c : \mathbb{R} \to U$, the function $\ell \circ (\Phi \circ c)$ is smooth since $\ell \circ (\Phi \circ c) = (\ell \circ \Phi) \circ c$. By Theorem 2.11, the curve $\Phi \circ c : \mathbb{R} \to F$ is smooth. This means that the map $\Phi : U \to F$ is conveniently smooth. By the above proposition, we conclude that $\Phi$ must be smooth. \hfill \qed

4. INFINITE-DIMENSIONAL MANIFOLDS

**Definition 4.1.** A class of l.c. model spaces or l.c. models is a subclass $E$ of the class of real (Hausdorff) locally convex spaces such that $E$ contains the numerical spaces $\mathbb{R}^n$ for all $n \in \mathbb{N}$.

Typical examples are the class of numerical spaces $\mathbb{R}^n$ for all $n \in \mathbb{N}$, the class of finite-dimensional vector spaces, the class of Banach spaces, the class of Fréchet spaces, and the class of all locally convex spaces.

**Definition 4.2.** Let $E$ be a class of l.c. model spaces, and $M_0$ a Hausdorff topological space.

(1) A smooth $E$-atlas on $M_0$ is a family of pairs $A = \{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$ such that $\{U_\alpha : \alpha \in A\}$ is an open cover of $M_0$ and for every $\alpha \in A$, there exists a space $E_\alpha$ from the class $E$ and a homeomorphism $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha) \subset E_\alpha$ such that the following compatibility condition is satisfied: for every $\alpha, \beta \in A$ such that $U_{\alpha \beta} = U_\alpha \cap U_\beta \neq \emptyset$, the transition map $\varphi_{\alpha \beta} = \varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ is smooth (in the sense of Definition 2.7).

(2) Given a smooth $E$-atlas $A$ on $M_0$, a (compatible) chart on $M_0$ is a pair $(U, \varphi)$ where $U$ is an open subset of $M_0$ and $\varphi$ is a homeomorphism from $U$ onto an open subset $\varphi(U)$ of some space from the class $E$ such that the transition map between $(U, \varphi)$ and every $(U_\alpha, \varphi_\alpha) \in A$ is smooth.

(3) A smooth $E$-atlas $A$ is said to be maximal if any chart $(U, \varphi)$ compatible with $A$ belongs already to $A$. A maximal atlas is also called a smooth structure, the class $E$ being understood.

**Definition 4.3.** Let $E$ be a class of l.c. model spaces. A smooth $E$-manifold is a pair $M = (M_0, A)$ where $M_0$ is a Hausdorff topological space and $A$ is a maximal smooth $E$-atlas on $M_0$. 
Remark 4.4.

(1) Especially in finite dimensions, $M_0$ is often required to be second countable. However, as we are mainly interested in the generalization to infinite dimensions, we do not insist on this condition here. Let us remark that we cannot insist on first countability neither, since a non-metrizable locally convex space is not first countable (see e.g. [MV], Proposition 25.1).

(2) Given a smooth $E$-atlas $A$, there is always a unique maximal smooth $E$-atlas containing $A$, obtained by adjoining to $A$ all the charts that are compatible with $A$.

(3) If $E$ and $E'$ are classes of l.c. models such that $E \subset E'$, then a smooth $E$-manifold is obviously a smooth $E'$-manifold. Denoting the class of all locally convex spaces by LCS, every smooth $E$-manifold is then a smooth LCS-manifold.

Definition 4.5. Given classes $E$ and $F$ of l.c. models, let $M = (M_0, A)$, resp. $N = (N_0, B)$ be a smooth $E$- resp. $F$-manifold, and $\Phi : M_0 \to N_0$ a continuous map. We say that $\Phi$ is smooth if the following condition is fulfilled: for every $(U, \varphi) \in A$ and every $(V, \psi) \in B$ such that $\Phi(U) \subset V$, the following map (between open sets in locally convex spaces) is smooth:

$$\psi \circ \Phi_U \circ \varphi^{-1} : \varphi(U) \to \psi(V).$$

Remark 4.6.

(1) If $A' \subset A$ resp. $B' \subset B$ is a smooth atlas (not necessarily maximal) of $M$ resp. $N$, it is enough to check the above condition for charts in $A'$ and $B'$.

(2) We notably obtain the notion of a smooth function $f$ on $M$, namely a function $f : M_0 \to \mathbb{R}$ such that for every chart $(U, \varphi) \in A'$ (atlas contained in $A$), $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}$ is a smooth function on the open set $\varphi(U)$ (contained in some space from the class $E$).

(3) Since an open subset $U \subset M_0$ inherits obviously a smooth structure upon restricting the smooth structure to $U$, we have a notion of smooth map/function defined on $U$.

Definition 4.7. Given a class of l.c. model spaces $E$, let $M = (M_0, A)$ be a smooth $E$-manifold. The sheaf $C^\infty_M$ of smooth functions on $M$ is the subsheaf of the sheaf $C^0_M$ of continuous (real-valued) functions on the topological space $M_0$, defined as the contravariant functor

$$C^\infty_M : \text{Open}(M_0) \to \text{RAlg}_{\text{com}}$$

assigning to every open subset $U$ of $M_0$ the commutative $\mathbb{R}$-algebra $C^\infty_M(U)$ of smooth functions on $U$. Here, $\text{Open}(M_0)$ is the category of open subsets of $M_0$ with inclusions as morphisms, and $\text{RAlg}_{\text{com}}$ is the category of unital commutative associative $\mathbb{R}$-algebras with unital $\mathbb{R}$-algebra homomorphisms as morphisms.
Remark 4.8. Obviously, \((M_0, C^\infty_M)\) is a ringed space, with the stalks \(C^\infty_M)_p\) being local unital \(\mathbb{R}\)-algebras with maximal ideals \(m_p = \{ f_p \in (C^\infty_M)_p \mid f(p) = 0 \}\). In short, \((M_0, C^\infty_M)\) is a locally ringed space. Furthermore, \((M_0, C^\infty_M)\) is, as a locally ringed space, locally isomorphic to models \((D_0, (C^\infty_E)_p)\), where \(E\) is a space from the class \(\mathcal{E}\), and \(D_0\) is an open subset of \(E\). The atlas \\{(D_0,j_{D_0})\\}, where \(j_{D_0} : D_0 \hookrightarrow E\) is the canonical inclusion, is obviously smooth and thus contained in a maximal atlas \(\mathcal{A}_{D_0}\). Calling \(D\) the smooth manifold given by \((D_0, \mathcal{A}_{D_0})\), we obviously have \((C^\infty_E)_p|_{D_0} = C^\infty_D\).

Recall that if \((X_0, \mathcal{O}_X)\) and \((Y_0, \mathcal{O}_Y)\) are locally ringed spaces, a morphism of locally ringed spaces between them is a pair \(\Phi = (\Phi_0, \Phi^\sharp)\), where \(\Phi_0 : X_0 \rightarrow Y_0\) is a continuous map and \(\Phi^\sharp : \Phi_0^{-1} \mathcal{O}_Y \rightarrow \mathcal{O}_X\) is a morphism of sheaves of unital \(\mathbb{R}\)-algebras such that for every \(x \in X_0\), the induced unital \(\mathbb{R}\)-algebra homomorphism \(\Phi^\sharp_x : (\mathcal{O}_Y)_{\Phi_0(x)} \rightarrow (\mathcal{O}_X)_x\) is local, i.e. sends maximal ideal to maximal ideal. Let us also recall that \(\Phi^\sharp\) can be equivalently viewed upon as a sheaf morphism \(\mathcal{O}_Y \rightarrow (\Phi_0)_* \mathcal{O}_X\). In the sequel, we apply both formulations without further comment.

A locally ringed space \((X_0, \mathcal{O}_X)\) is here said to be reduced if \(\mathcal{O}_X\) is a subsheaf of the sheaf of continuous functions \(C^0_{X_0}\). If \(\Phi = (\Phi_0, \Phi^\sharp) : (X_0, \mathcal{O}_X) \rightarrow (Y_0, \mathcal{O}_Y)\) is a morphism between reduced locally ringed spaces, then \(\Phi^\sharp\) is given by \(\Phi_0^*\), the pullback by \(\Phi_0\) (and so the morphism \(\Phi\) is completely determined by the underlying continuous map \(\Phi_0\)). Consequently, we often write \(\Phi\) instead of \(\Phi_0\) when the spaces are reduced. Of course, we never do so in the non-reduced case, since \(\Phi^\sharp\) is then part of the data.

Definition 4.9. Let \(\mathcal{E}\) be a class of l.c. models. A structure sheaf-smooth \(\mathcal{E}\)-manifold is a reduced locally ringed space which, as a locally ringed space, is locally isomorphic to models \((D_0, C^\infty_E)\), where \(D_0\) is an open subset of a space from the class \(\mathcal{E}\).

Remark 4.10. By Remark 4.8, every smooth \(\mathcal{E}\)-manifold is, in a natural way, a structure sheaf-smooth \(\mathcal{E}\)-manifold. The converse is nontrivial. Using the preceding section, we can nevertheless show the following results.

Notation 4.11. The class of l.c. models made of all the Mackey-complete locally convex spaces will be denoted by \(\mathcal{M}\).

Theorem 4.12. Let \(\mathcal{E}\) be the class of locally convex spaces whose topology coincides with the Mackey-closure topology. If \(M = (M_0, \mathcal{A})\) is a smooth \(\mathcal{E}\)-manifold, \(N = (N_0, \mathcal{B})\) a smooth \(\mathcal{M}\)-manifold, and \(\Phi : M_0 \rightarrow N_0\) a continuous map, then \(\Phi\) is smooth if and only if \(\Phi : (M_0, C^\infty_M) \rightarrow (N_0, C^\infty_N)\) is a morphism of locally ringed spaces.

Proof: By the chain rule, smooth maps are morphisms of locally ringed spaces. Assume now that \(\Phi\) is a morphism of locally ringed spaces. Let
manifold in the structure-sheaf sense (with respect to the natural structure sheaves). Thus, $\tilde{\Phi}$ satisfies condition (ii) of Theorem 3.19, which in turn implies that $\tilde{\Phi}$ is smooth. It follows that $\Phi$ itself is smooth. \hfill $\Box$

The above theorem has the following very important consequence.

**Corollary 4.13.** Let $\mathcal{E}$ be again the class of locally convex spaces whose topology coincides with the Mackey-closure topology. For every smooth $\mathcal{E}$-manifold in the structure-sheaf sense $(M_0, \mathcal{O}_M)$, there is a canonical maximal atlas $\mathcal{A}$ on $M_0$ such that $M = (M_0, \mathcal{A})$ is a smooth $\mathcal{E}$-manifold, and $\mathcal{C}_M^\infty = \mathcal{O}_M$.

**Proof:** Let $\mathcal{A}' = \{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$ be a family of pairs such that $\{U_\alpha : \alpha \in A\}$ is a covering of $M_0$ and for all $\alpha \in A$, $\varphi_\alpha : (U_\alpha, (\mathcal{O}(M)|_{U_\alpha}) \longrightarrow (\mathcal{E}_\alpha, (\mathcal{C}_\alpha)|_{\varphi_\alpha(U_\alpha)})$ (with $E_\alpha \in \mathcal{E}$ and $\varphi_\alpha(U_\alpha) \subset E_\alpha$) is an isomorphism of locally ringed spaces. Furthermore, the continuous transition map $\varphi_{\alpha\beta} : \varphi_\beta(U_{\alpha\beta}) \longrightarrow \varphi_\alpha(U_{\alpha\beta})$, the latter map is an isomorphism of the locally ringed spaces $(\varphi_\beta(U_{\alpha\beta}), (\mathcal{E}_\alpha)|_{\varphi_\beta(U_{\alpha\beta})})$ and $(\varphi_\alpha(U_{\alpha\beta}), (\mathcal{E}_\alpha)|_{\varphi_\alpha(U_{\alpha\beta})})$. By the preceding theorem, the map $\varphi_{\alpha\beta}$ is then already smooth. It follows that $\mathcal{A}'$ is a smooth atlas on $M_0$. We observe that the unique maximal atlas $\mathcal{A}$ containing $\mathcal{A}'$ is canonically associated to the given locally ringed space. Furthermore, the corresponding sheaf of smooth functions $\mathcal{C}_M^\infty$ is equal to the structure sheaf $\mathcal{O}_M$. $\Box$

**Remark 4.14.** The preceding theorem and its corollary show that for important classes of l.c. model spaces, as e.g. the class of Fréchet spaces, we can encode smoothness completely in sheaf-theoretic language. This approach simplifies the verification and application of smoothness in infinite dimensions, and allows generalizations to the non-reduced case, as e.g. for “Fréchet supermanifolds”. In general, i.e., for arbitrary classes of l.c. model spaces, resp. for our (regular) local models replaced by more general local models, as e.g., complex-analytic sets in open subsets of complex locally convex spaces, this encoding might not be possible anymore. In order to circumvent this problem, Douady introduced in [Dou] the notions of the next definition.

**Remark 4.15.** Let $\mathcal{C}$ be a category, and denote, for a topological space $Z_0$, the category of sheaves (of sets) on $Z_0$ by $\mathbf{Sh}_{Z_0}$. If $\Phi_0 : X_0 \longrightarrow Y_0$ is a continuous map of topological spaces and $\mathcal{G} : \mathcal{C} \longrightarrow \mathbf{Sh}_{Y_0}$ is a covariant functor, then $\Phi_0^{-1}\mathcal{G} : \mathcal{C} \longrightarrow \mathbf{Sh}_{X_0}$ defined by $(\Phi_0^{-1}\mathcal{G})(A) = \Phi_0^{-1}(\mathcal{G}(A))$ is a covariant functor, called the inverse image of $\mathcal{G}$ by $\Phi_0$.

**Definition 4.16.** Let $\mathcal{C}$ be a category.
(1) A \( \mathcal{C} \)-functored space is a pair \((X_0, \mathcal{O}_X^C)\) where \(X_0\) is a topological space, and \(\mathcal{O}_X^C : \mathcal{C} \rightarrow \text{Sh}_{X_0}\) is a covariant functor from \(\mathcal{C}\) to the category of sheaves on \(X_0\). We say that \(\mathcal{O}_X^C\) is the structure functor of \((X_0, \mathcal{O}_X^C)\).

(2) If \((X_0, \mathcal{O}_X^C)\) and \((Y_0, \mathcal{O}_Y^C)\) are \(\mathcal{C}\)-functored spaces, a morphism of \(\mathcal{C}\)-functored spaces between them is a pair \(\Phi = (\Phi_0, \Phi^c)\) where \(\Phi_0 : X_0 \rightarrow Y_0\) is a continuous map and \(\Phi^c : (\Phi_0)^{-1}\mathcal{O}_Y^C \rightarrow \mathcal{O}_X^C\) is a natural transformation between the two functors \(\Phi_0^{-1}\mathcal{O}_Y^C : \mathcal{C} \rightarrow \text{Sh}_{X_0}\) and \(\mathcal{O}_X^C : \mathcal{C} \rightarrow \text{Sh}_{X_0}\).

Remark 4.19. If \((X_0, \mathcal{O}_X^C)\) is a reduced \(\mathcal{F}\)-functored space, then \((X_0, \mathcal{O}_X)\) is a reduced locally ringed space.

Definition 4.20. Given a class \(\mathcal{F}\) of l.c. models, we continue to use the symbol \(\mathcal{F}\), by a slight abuse of notation, to denote the category whose objects are open subsets of spaces from the class \(\mathcal{F}\), and whose morphisms are the smooth maps between them (in the sense of Definition 3.1). Such a category \(\mathcal{F}\) will be referred to as a “category of l.c. models”.

Definition 4.18. Given a category of l.c. models \(\mathcal{F}\), an \(\mathcal{F}\)-functored space \((X_0, \mathcal{O}_X^F)\) is said to be reduced if for every \(V \in \text{Ob}(\mathcal{F})\) and every open set \(U \subset X_0\), we have \(\mathcal{O}_X^F(V)(U) \subset \mathcal{C}^0(U, V)\), and \(\mathcal{O}_X := \mathcal{O}_X^F(\mathbb{R})\) is a sheaf having local unital \(\mathbb{R}\)-algebras as stalks.

Remark 4.21. Obviously, for any smooth \(\mathcal{E}\)-manifold \(M\), \((M_0, (\mathcal{C}^\infty)_M^F)\) is a reduced \(\mathcal{F}\)-functored space. Furthermore, \((M_0, (\mathcal{C}^\infty)_M)\) is, as a \(\mathcal{F}\)-functored space, locally isomorphic to model \(\mathcal{F}\)-functored spaces \((D_0, (\mathcal{C}^\infty)_D^F)\), where \(D_0\) is an open subset of a space from the class \(\mathcal{E}\).

Definition 4.22. Let \(\mathcal{E}\) be a class of l.c. models. A structure functor-smooth \(\mathcal{E}\)-manifold is a reduced LCS-functored space \((M_0, \mathcal{O}_M^{LCS})\) which, as a LCS-functored space, is locally isomorphic to model functors spaces \((D_0, (\mathcal{C}^\infty)_D^{LCS})\), where \(D_0\) is an open subset of a space from the class \(\mathcal{E}\).

Remark 4.23. By Remark 4.21 for \(\mathcal{F} = \text{LCS}\), every smooth \(\mathcal{E}\)-manifold is, in a natural way, a structure functor-smooth \(\mathcal{E}\)-manifold.

Theorem 4.24. Let \(\mathcal{E}\) be a class of l.c. models. If \(M = (M_0, \mathcal{A})\) and \(N = (N_0, \mathcal{B})\) are smooth \(\mathcal{E}\)-manifolds, and \(\Phi : M_0 \rightarrow N_0\) a continuous
map, then $\Phi$ is smooth if and only if $\Phi : (M_0, (C^\infty)_M^{\text{LCS}}) \rightarrow (N_0, (C^\infty)_N^{\text{LCS}})$ is a morphism of $\text{LCS}$-functored spaces.

**Proof:** Note that a morphism $(\Phi_0, \Phi^*)$ between reduced functored spaces is completely determined by the underlying continuous map $\Phi_0$, the information contained in the natural transformation $\Phi^*$ being that of all possible pullbacks by $\Phi_0$. Consequently, as in the case of reduced locally ringed spaces, we write $\Phi$ instead of $(\Phi_0, \Phi^*)$. Suppose now that $\Phi : M \rightarrow N$ is smooth. Then, for any open subset $V$ of a locally convex space $F$ and for any open subset $W$ of $N_0$, the pullback $\Phi^* : C^\infty(W,V) \rightarrow C^\infty(\Phi^{-1}(W),V)$ is well-defined by the chain rule. We claim that $\Phi^* : (C^\infty)_M^{\text{LCS}} \rightarrow (C^\infty)_N^{\text{LCS}}$, which assigns to every $V$ the sheaf map $\Phi^*(V) : (C^\infty)_N^{\text{LCS}}(V) \rightarrow \Phi_*((C^\infty)_M^{\text{LCS}}(V))$ given by the pullbacks $\Phi^*$, is a natural transformation. Indeed, we have to show that given $V'$ (resp. $V''$) open subset of a locally convex space $F'$ (resp. $F''$), and given a smooth map $\chi : V' \rightarrow V''$, we have

$$(C^\infty)_M^{\text{LCS}}(\chi)(\Phi^{-1}(W)) \circ \Phi^*(V')(W) = \Phi^*(V'')(W) \circ (C^\infty)_N^{\text{LCS}}(\chi)(W)$$

for every open set $W$ in $N_0$. But this is true since both LHS and RHS, when evaluated at an element $\Psi \in (C^\infty)_N^{\text{LCS}}(V')(W) = C^\infty(W,V')$, are equal to $\chi \circ \Psi \circ \Phi$. Thus, $\Phi$ is a morphism of $\text{LCS}$-functored spaces. Now we prove the converse of the theorem. Assume that $\Phi$ induces, via the pullbacks $\Phi^*$, a morphism of $\text{LCS}$-functored spaces. This means that for a germ of a smooth map defined on $N_0$ having values in an open subset of an arbitrary locally convex space, the pullback is the germ of a smooth map on $M_0$. Since this property as well as the condition of smoothness are local, we replace w.l.o.g. $M_0$ and $N_0$ by open subsets $U$ resp. $U'$ in locally convex spaces $E, E'$ in $E$. Taking now $\Psi = \text{Id}_{U''}$, the pullback $\Phi^*(\Psi) = \Psi \circ \Phi$ equals $\Phi$ and thus $\Phi$ is smooth. 

The above theorem has the following important consequence.

**Corollary 4.25.** Let $E$ be a class of l.c. models. For every structure functor-smooth $E$-manifold $(M_0, \mathcal{O}_M^{\text{LCS}})$, there is a canonical maximal atlas $\mathcal{A}$ on $M_0$ such that $M = (M_0, \mathcal{A})$ is a smooth $E$-manifold, and $\mathcal{O}_M^{\text{LCS}} = (C^\infty)_M^{\text{LCS}}$.

**Proof:** Mutatis mutandis the proof of Corollary 4.13 shows this corollary as well.

One could also take $E$ to be the class of all complex locally convex spaces. The notion of smooth $E$-manifold is then replaced by that of complex-analytic locally convex manifold. To such a manifold $M = (M_0, \mathcal{A})$, one can then associate a complex-analytic locally convex manifold in the structure-sheaf sense $(M_0, \mathcal{C}_M^\omega)$, where $\mathcal{C}_M^\omega$ is the sheaf of complex-analytic functions on $M$. One has then the following result.

**Theorem 4.26.** Let $M = (M_0, \mathcal{A})$ and $N = (N_0, \mathcal{B})$ be complex-analytic locally convex manifolds, and $\Phi : M_0 \rightarrow N_0$ a continuous map. Then $\Phi$ is complex-analytic if and only if $\Phi : (M_0, \mathcal{C}_M^\omega) \rightarrow (N_0, \mathcal{C}_N^\omega)$ is a morphism of $\text{LCS}$-functored spaces.
locally ringed spaces.

**Proof:** By the chain rule, analytic maps are morphisms of locally ringed spaces. Assume now that $\Phi$ is a morphism of locally ringed spaces. Since the condition of analyticity is local, we can proceed as in the proof of Theorem 4.12 which amounts to replace $M_0$ and $N_0$ by open subsets $D_E$ and $D_F$ of complex locally convex spaces $E$ and $F$ respectively. This gives a map $\tilde{\Phi} : D_E \to D_F$ which is scalarwise analytic. In particular, viewed as a map from $D_E$ to $F$, $\tilde{\Phi}$ is weakly analytic. By [Maz] (Part II, Proposition 1.6), we deduce that $\tilde{\Phi}$ is analytic. It follows that $\Phi$ itself is analytic. \(\square\)

**Remark 4.27.** If, instead of complex-analytic locally convex manifolds, one considers more generally complex-analytic subsets of locally convex spaces (and analytic spaces modelled on such analytic sets), then pathologies appear. There are examples of reduced analytic sets (in the ringed space-sense) with continuous maps into some complex Banach space which are weakly analytic but not analytic (see e.g. [Maz], pp. 73-80). This phenomenon is avoided by defining analytic sets and spaces as functored spaces in [Dou].

5. Appendix

For the reader’s convenience, we give a simple proof of the fact that the Mackey-closure topology on a Fréchet space coincides with the given topology.

**Proposition 5.1.** Let $E$ be a Fréchet space. Then the given topology on $E$ coincides with the Mackey-closure topology.

**Proof:** We know that the $c^\infty$-topology is finer than the given topology. We need to show that conversely, any $c^\infty$-open subset of $E$ is open. Let $U$ be an absolutely convex $c^\infty$-neighborhood of 0. First we show that being $c^\infty$-open, $U$ must absorb every bounded set.

Suppose that there exists a bounded set $B$ which is not absorbed by $U$. Then for every $\lambda > 0$, we have that $B$ is not contained in $\lambda U$. In particular, if we consider a sequence $(\mu_n)$ of positive real numbers converging to 0, then for every $n$, the set $B$ is not contained in $\frac{1}{\mu_n} U$. So for every $n$, we can choose $b_n \in B$ such that $b_n \notin \frac{1}{\mu_n} U$. Now $\mu_n b_n \in \mu_n B$ for every $n$, therefore the sequence $(\mu_n b_n)$ Mackey-converges to 0. Since $U$ is $c^\infty$-open, this implies the existence of $N_1 \in \mathbb{N}$ such that $n \geq N_1 \implies \mu_n b_n \in U$. On the other hand, since $\mu_n \to 0$, there exists $N_2 \in \mathbb{N}$ such that $n \geq N_2 \implies \mu_n < 1$. For $n \geq \max\{N_1, N_2\}$, we have $\mu_n b_n \in \mu_n U \subset U$, and so $b_n \in \frac{1}{\mu_n} U$, contradiction. Thus, $U$ absorbs every bounded set.

Now assume $U$ is not a neighborhood of 0. Since $E$ is Fréchet, it is metrizable and therefore first-countable. Let $(V_n)$ be a decreasing countable basis of neighborhoods of 0 in $E$. We may assume each $V_n$ to be balanced, in which
case \( \frac{1}{n} V_n \) is also a basis of neighborhoods of 0. Since \( U \) is not a neighborhood of 0, for every \( n \), the basic neighborhood \( \frac{1}{n} V_n \) is not contained in \( U \). For each \( n \), choose \( x_n \in \frac{1}{n} V_n \) such that \( x_n \notin U \). It is easy to see that \( nx_n \to 0 \), and so the set \( \{ nx_n : n \in \mathbb{N} \} \) is bounded. By what we have seen above, \( U \) will absorb this set, and so there exists \( \lambda > 0 \) such that \( nx_n \in \lambda U \) for all \( n \). For \( n \) sufficiently large, we will then have \( x_n \in \frac{1}{n} U \subset U \), contradiction. \( \square \)

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