Self-Similar Jordan Arcs Which Do Not Satisfy OSC.

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The problem of finding explicit geometrical criteria for a system $S$ of contraction similarities, implying the open set condition, is discussed since 80-ies and still remains open.

One of such criteria is the finite intersection property: The system $S = \{S_1, ..., S_m\}$ with the attractor $K$, has f.i. property if for any $i \neq j$ the set $S_i(K) \cap S_j(K)$ is finite.

It was proved in 2007 by C.Bandt and H.Rao [2] that if a system $S = \{S_1, ..., S_m\}$ of contraction similarities in $\mathbb{R}^2$ with a connected attractor $K$ has the finite intersection property, then it satisfies OSC. The authors wrote they believe that in $\mathbb{R}^3$ this is not so.

In this paper we prove the following

**Theorem 1.** There is such system $S = \{S_1, ..., S_m\}$ of contraction similarities in $\mathbb{R}^3$, which:

1. does not satisfy OSC,
2. satisfies one-point intersection property and
3. whose attractor is a Jordan arc $\gamma \subset \mathbb{R}^3$.

Notice that the statements (1) and (2) imply that the system $S$ does not satisfy weak separation property WSP.

To show the existence of such self-similar arcs we use the zipper construction [1] and prove the following three statements, which are the foundation of our approach to construction of non-WSP curves:

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First is a general position theorem for fractal curves, which gives the condition specifying how to get rid of intersection by small deformations of pairs of fractal curves within a given family of such pairs \( \{(\varphi(x,t),\psi(x,t))\} \), depending on some parameter \( x \in \mathbb{R}^3 \).

**Theorem 2.** Let \( \varphi(x,t),\psi(x,t) : B^3 \times I \to \mathbb{R}^3 \) be continuous maps which
(1) are \( \alpha \)-Hölder with respect to \( t \) and
(2) satisfy the condition: for any \( t,s \in I \) the function \( f(x,t,s) = \varphi(x,t) - \psi(x,s) \) is bi-Lipschitz with respect to \( x \).

Then Hausdorff dimension of the set \( \Delta = \{ x \in B^3 | \varphi(x,I) \cap \psi(x,I) \neq \emptyset \} \) does not exceed \( 2/\alpha \).

Second is a corollary of Barnsley’s Collage Theorem, that gives the conditions under which a deformation of a fractal is bi-Lipschitz on its certain subpieces.

**Proposition 3.** Let \( S = \{ S_1,\ldots,S_m \}, T = \{ T_1,\ldots,T_m \} \) be systems of contractions in a complete metric space \( X \), and \( q = \max(\text{Lip } S_i,\text{Lip } T_i) \).

Let \( \varphi : I^\infty \to K, \psi : I^\infty \to L \) be index maps for the attractors \( K(S) \) and \( L(T) \). Let \( V \) be such bounded set, that for any \( i = 1,\ldots,m \), \( S_i(V) \subset V \) and \( T_i(V) \subset V \). Put \( \Delta_i(x) = T_i(x) - S_i(x) \) and suppose \( \| \Delta_i(x) \| \leq \delta \) for any \( i \) and any \( x \in V \). Then,

**B1)** If for some multiindex \( j \), \( \delta_1 < \| \Delta_j(x) \| < \delta_2 \) for any \( x \in V \) and \( \delta_1 > \frac{q_j \delta}{1 - q} \), then for \( \sigma \in \tau_j I^\infty \),

\[
\delta_1 - \frac{q_j \delta}{1 - q} \leq \| \psi(\sigma) - \varphi(\sigma) \| \leq \delta_2 + \frac{q_j \delta}{1 - q}
\]

**B2)** If for some multiindices \( i, j \), \( \delta_1 < |\Delta_i(x) - \Delta_j(y)| < \delta_2 \) for any \( x,y \in V \) and \( \delta_1 > \frac{(q_i + q_j)\delta}{1 - q} \), then for \( \tau \in \tau_j I^\infty, \sigma \in \sigma_i I^\infty \),

\[
\delta_1 - \frac{(q_i + q_j)\delta}{1 - q} \leq \| \psi(\sigma) - \varphi(\sigma) - \psi(\tau) + \varphi(\tau) \| \leq \delta_2 + \frac{(q_i + q_j)\delta}{1 - q}
\]

And the third, most delicate, statement allows to make deformations of system \( S \) and the curve \( \gamma \), under which the violation of WSP by the system \( S \) is preserved.
Theorem 4. 1) If two-generator subgroup $G = \langle \xi, \eta, \cdot \rangle$, $\xi = re^{i\alpha}$, $\eta = Re^{i\beta}$ in $\mathbb{C} \setminus \{0\}$ is a dense subgroup of second type, then for any $z_1, z_2 \in \mathbb{C} \setminus \{0\}$ there is such sequence $\{(n_k, m_k)\}$ that

$$
\lim_{k \to \infty} z_1^{e^{i\alpha} n_k} = 1, \quad \lim_{k \to \infty} z_2^{e^{i\beta} m_k} = e^{-i \arg(z_2)}.
$$

2) The set $\{(\xi, \eta)\}$ of all pairs of generators of dense subgroups of the second type, is dense in $\mathbb{C}^2$.

The subgroups mentioned in the theorem were defined in [4] and we give a short summary of the results of this work in the Addendum.

1 Brief outline of the method.

The idea of the example is quite simple and is based on zipper construction introduced by Vladislav Aseev in [1].

Namely, a system $S = \{S_1, ..., S_m\}$ of contractions of a metric space $X$ is called a zipper with vertices $\{z_0, ..., z_m\}$ and signature $\varepsilon = (\varepsilon_1, ..., \varepsilon_m), \varepsilon_i \in \{0, 1\}$, if for any $i = 1, ..., m$, $S_i(z_0) = z_{i-1+\varepsilon_i}$ and $S_i(z_m) = z_{i-\varepsilon_i}$.

We denote the attractor of a zipper $S$ by $\gamma(S)$, or simply by $\gamma$.

Note, that for any zipper $S$ and any linear zipper $T$ on $[0, 1]$ having the same signature $\varepsilon$ there is unique a Hölder continuous $(S, T)$-equivariant map $\varphi_{ST} : [0, 1] \to \gamma$ which is called a linear parametrization of $S$. [1]

We begin with a self-similar zipper $S = \{S_1, ..., S_{2m}\}$ in $\mathbb{R}^3$, whose vertices $z_0, ..., z_{2m}$ and similarities $S_i$ are chosen in such way, that for some closed bicone $V$ with vertices $z_0, z_{2m}$,

(A1) for any $S_i \in S$, $S_i(V) \subset V$;

(A2) for any such $i, j$, that $|j - i| > 1$, $S_i(V) \cap S_j(V) = \emptyset$

(A3) for any $i \neq m + 1$ and $1 \leq i \leq 2m$, $S_{i-1}(V) \cap S_i(V) = \{z_i\}$;

At the same time, inside $S_m(V) \cup S_{m+1}(V)$ we provide that

(A4) There are such subarcs $\gamma_A \supset \gamma_{m-3}$, $\gamma_B \supset \gamma_{m+4}$ and such sequences $\{i_k\}, \{j_k\}$, that the set $S_m(\gamma) \cap S_{m+1}(\gamma) \setminus \{z_m\}$ is a disjoint union of sets

$$S_{m+1}S^k_1(\gamma_A) \cap S_mS^j_k(\gamma_B)$$
The sequence of pairs of maps $S'_k = S_{m+1}^{j_k}, S''_k = S_m S_{2m}^{j_k}$, contains such subsequence $\{S'_{k_n}, S''_{k_n}\}$, that
\[
\lim_{n \to \infty} (S'_{k_n} S_{m-3})^{-1}(S''_{k_n} S_{m+4}) = \text{Id}.
\]

There is such linear zipper $T$ in $[0, 1]$ that the Hölder exponent of the linear parametrization $\varphi_{S'}$ is greater than 3/4.

The condition (A5) means that the system $S$ does not satisfy WSP, but (A1) – (A3) do not imply in general, that $\gamma$ is a Jordan arc.

So, the main difficulty is to change the system $S$ slightly in such way that $\gamma$ gets rid of all its self-intersections without violating (A4, A5).

For that reason, instead of a single zipper $S$, in Section 2 we construct a family of self-similar zippers $S_\xi = \{S_1, ..., S_{2m}\}$, depending continuously on a parameter $\xi \in D$, where $D$ is some domain in $\mathbb{R}^3$, so that for any $\xi \in D$, $S_\xi$ satisfies (A1) - (A6) with the same domain $V$, sequences $S'_k, S''_k$ and the same linear zipper $T$ for all $\xi \in D$.

The most delicate problem here is to choose such parameters for the family $\{S_\xi\}$, that the condition (A4) would hold for all $S_\xi$. This is guaranteed by our Theorem 4 on the properties of dense subgroups in $\mathbb{C}^\ast$, proved in [3].

After that we show that the set of the parameters $\xi \in D$, for which $S_\xi$ defines a Jordan arc, is dense in $D$. To obtain this, we apply the general position Theorem 2 the following way:

We take the family $\{\varphi^\xi, \xi \in D\}$ of linear parametrisations $\varphi^\xi(t) : [0, 1] \to \gamma^\xi$ of zippers $S_\xi$ by the zipper $T = \{T_1, ..., T_m\}$. Denoting by $\varphi_k(\xi, t), \psi_k(\xi, t)$ the parametrisations of the subarcs $S'_k(\gamma_A), S''_k(\gamma_B)$, obtained by restriction of $\varphi^\xi(t)$ to the subintervals $T_m T_{i+1}^{j_k}(I_A), T_m \cap T_{2m}^{j_k}(I_B)$ of $I = [0, 1]$, we consider the functions $f_k(\xi, t, s) = \varphi_k(\xi, t) - \psi_k(\xi, s)$. Applying Proposition 3, we show that the function $f_k(\xi, t, s) = \varphi_k(\xi, t) - \psi_k(\xi, s)$, is bi-Lipschitz with respect to $\xi$ for each fixed $t, s$. By our construction, $f_k(\xi, t, s)$ is $\alpha$-Hölder with respect to $t$ and $s$ and $\alpha > 2/3$.

Therefore, applying Theorem 2 to each pair of subarcs from the sequence $S'_k(\gamma_A), S''_k(\gamma_B)$ and using Baire category argument, we get that each ball in the domain $D$ contains such $\xi$, that all the intersections $S'_k(\gamma_A) \cap S''_k(\gamma_B)$ are empty and therefore $\gamma^\xi$ is a Jordan arc. So the set of parameters $\xi$, for which the system $S(\xi)$ has a Jordan attractor $\gamma$ is dense in $D$. 

4
2 Setting the parameters of zippers $S^\xi$.

In this section we define the parameter domain $D$, the zippers $S^\xi$ for any $\xi \in D$ and the bicones $V_i$.

2.1 Polygonal lines defining the zippers $S^\xi$.

First, we define three angles $\beta_1 < \beta_0 < \beta_2$, necessary for our construction. Take $\beta_0 = \arctan(1/2)$; let $\mu = 0.0100512$ and put $\beta_1 = \beta_0 - 2\mu, \beta_2 = \beta_0 + \mu$.

We define the domain $D$ in $\mathbb{R}^3$ by the equation
\[
D = (1/1.02, 1.02) \times (\beta_0 - \mu, \beta_0 - \mu/2) \times (-\mu, \mu)
\]
Denote a point in $D$ by $\xi = (\rho, \theta, \phi)$.

Now we define a family of zippers $S^\xi$ depending on a parameter $\xi \in D$ so for each $\xi \in D$ we define a polygonal line in the plane $XY$ with vertices $z_0, \ldots, z_{2m}$. All its vertices, except $z_{m+1}$, do not depend on $\xi$.

The similarities $S_i$ in each zipper $S^\xi$ are the compositions of XY-plane preserving similarities sending $\{z_0, z_{2m}\}$ to $\{z_{i-1}, z_i\}$ and rotations in some angle $\alpha_i$ around the axis $\{z_{i-1}, z_i\}$.

In this family, only the point $z_{m+1}$ and the maps $S_{m+1}, S_{m+2}$ and $S_{m+4}$ depend on $\xi$, others being the same for any $\xi \in D$.

Mentioning the points, maps or subset of the attractor of each system $S^\xi$ we will write $z^\xi_i$ or $S^\xi_i$ only in the cases when it is needed for our argument, otherwise we will not mention the parameter $\xi$, assuming the dependence by default.

The table below shows the x and y coordinates of the vertices $z_i$, the contraction ratios $q_i = \frac{||z_i - z_{i-1}||}{||z_{2m} - z_0||}$ and rotation angles $\alpha_i$.

| No | 0 | 1 | 2 | .... | m-5 | m-4 | m-3 | m-2 | m-1 | m |
|----|----|----|----|------|-----|-----|-----|-----|-----|---|
| x  | -3 | 6q_1 - 3 | -1 | ... | -1 | -1 | -0.92 | -0.899 | -0.447 | 0 |
| y  | 0.8 | 0.8 | 0.8 | ... | 1.750 | 1.8 | 1.84 | 1.798 | 0.894 | 0 |
| $q_i$ | - | $q_1$ | $\frac{1}{3}$ - $q_1$ | ... | 0.008 | 0.015 | 0.008 | 0.168 | $\frac{1}{6}$ |
| $\alpha_i$ | - | $\alpha_1$ | 0 | ... | 0 | 0 | 0 | 0 | 0 |
Some comments on the values in the table.

1. Since the most significant point is $z_m$, it lies in the origin. Therefore $z_0 = (-3, 0.8, 0)$ and $z_{2m} = (3, 0.8, 0)$.

2. $q_1, \alpha_1, q_{2m}, \alpha_{2m}$: According to the Proposition 23, the pairs of generators of dense subgroups of the second type are dense in $\mathbb{C}^2$. So we take such pair of generators $q_1 e^{i\alpha_1}, q_{2m} e^{i\alpha_{2m}}$ lying in 0.003-neighborhood of the point $1/6$. $q_1$ and $q_{2m}$ define the points $z_1, z_{2m}$ and corresponding contraction ratios.

3. $q_{m+1}, \alpha_{m+1}, q_{m+2}$: The point $z_{m+1}$ has the norm $\rho$ and the angle between $z_m z_{m+1}$ and OY axis is $\theta$, this gives us the values in the column $m + 1$. $q_{m+2}$ is defined as $\| z_{m+2} - z_{m+1}\|$.

4. Symmetry. For any $i \neq 1, m - 1, m + 1, 2m - 1$, the points $z_i$ and $z_{2m-1}$ are symmetric with respect to the y axis.

5. $\beta_0$ triangle. The points $z_{m-4}, z_{m-3}, z_{m+4}, z_{m+3}$ lie on the sides of an isosceles triangle with angles $\beta_0 = \arctan(1/2)$ at its base $z_0 z_{2m}$. 

| No | $m + 1$ | $m + 2$ | $m + 3$ | $m + 4$ | $m + 5$ | $2m - 2$ | $2m - 1$ | $2m$ |
|---|---|---|---|---|---|---|---|---|
| $x$ : | $\rho \sin \theta$ | 0.899 | 0.92 | 1 | 1 | $\ldots$ | 1 | $3 - 6q_{2m}$ | 3 |
| $y$ : | $\rho \cos \theta$ | 1.798 | 1.84 | 1.8 | 1.75 | $\ldots$ | 0.8 | 0.8 | 0.8 |
| $q_i$ : | $\frac{\rho}{6}$ | $q_{m+2}$ | 0.008 | 0.015 | 0.008 | $\ldots$ | $\ldots$ | $\frac{1}{3} - q_{2m}$ | $q_{2m}$ |
| $\alpha_i$ : | $\varphi$ | 0 | 0 | $\alpha_{m+4}$ | 0 | $\ldots$ | $\ldots$ | 0 | $\alpha_{2m}$ |
6. **$\beta_1$ triangle.** At the same time the points $z_{m-5}, z_{m-2}, z_{m+2}, z_{m+5}$ lie on the sides of an isosceles triangle with angles $\beta_1$ at its base $z_0z_{2m}$.

7. **Points near $z_m$.** The points $z_{m-3}, z_{m-2}, z_{m-1}, z_m$ lie on a line, forming the angle $\beta_0$ with the vertical axis and $||z_m - z_{m-1}|| = 1$. The same is true for their symmetrical counterparts $z_{m+3}, z_{m+2}, z_m$. The point $z_{m+1}$ is slightly shifted to the left and upwards.

8. **Omitted entries.** The points $z_3, ..., z_{m-6}$ and $z_{m+6}, ..., z_{2m-3}$ divide the intervals $(z_2, z_{m-5})$ and $(z_{m+5}, z_{2m-2})$ into sufficiently small equal parts, therefore their coordinates and corresponding $q_i$ need not be mentioned.

2.2 **The similarities $S_i$**

Each zipper $S^\xi = \{S_1, ..., S_{2m}\}$ has vertices $(z_0, ..., z_{2m})$ and signature $(0, 0, ..., 0, 1, 0, ..., 0)$, where only $\epsilon_{m+4} = 1$. This means that $S_{m+4}$ reverses the order, i.e. $S_{m+4}(z_0) = z_{m+4}$ and $S_{m+4}(z_{2m}) = z_{m+3}$. See Fig. 6

$S_1$ and $S_{2m}$ are defined as the compositions of contractions with ratios $q_1, q_{2m}$ with fixed points $z_0$ and $z_{2m}$ respectively and rotations in angles $\alpha_1$ and $-\alpha_{2m}$ around the real axis.

The map $S^\xi_{m+1}$ is a composition of a plane similarity sending $z_0$ to $z_m$ and $z_{2m}$ to $z_{m+1}$ and a rotation around the line $z_mz_{m+1}$ in the angle $\phi$, which is the third coordinate of the parameter $\xi$. 

Figure 1:
The map $S_{m+2}$ preserves the plane $XY$, but depends on $\xi$ because its value at $z_0$ is $z_m^\xi_{m+1}$.

Finally, the map $S_{m+4}$ is a composition of a (fixed) similarity map of a plane $XY$ sending $z_0$ to $z_{m+4}$ and $z_{2m}$ to $z_{m+3}$ and a rotation in an angle $\alpha_{m+4}$ around the line $z_{m+3}z_{m+4}$. The angle $\alpha_{m+4}$ depends on the coordinate $\theta$ of $\xi$ and will be defined in the next section.

All the other maps $S_2, ..., S_{2m-1}$ are the similarities, preserving $XY$ plane.

The similarity ratios $q_i$ of $S_i$ are equal to $|z_i - z_{i-1}|/6$. Direct computation shows that when $m \geq 12$ and $q_3 = ... = q_{m-5}$, the similarity dimension of $S$ is less than 1.28 for any $\xi \in D$.

### 2.3 Bicones and the sets $A$ and $B$.

Let $V'$ be a rhombus whose diagonal is $z_0, z_{2m}$ and the angle between diagonal and its sides is $\beta_2$.

The value of $\beta_2$ was chosen as minimal of those ones, for which two small copies of $V'$ with diagonals $z_{m-3}z_{m-4}$ and $z_{m+4}z_{m+3}$ respectively lie inside the large $V'$.

For each other $i$, the copy of $V'$ whose diagonal is $z_{i-1}z_i$ also lies inside
Figure 4: The bicones $V^1 \subset V^0 \subset V$: $\beta_2$ is enlarged to make the inclusion visible.

$V'$.

Getting to the space, we replace $V'$ by a bicone $V$ with the same axis $z_0, z_{2m}$ and angle $\beta_2$ between the axis and a generator.

We denote by $V_i$ the images of the bicone $V$ under similarities sending $z_0, z_{2m}$ to $z_{i-1}, z_i$. They satisfy the relations (A1)–(A3) from Section 1.

These properties are valid for any choice of $\xi \in D$. (See Fig.3)

Along with the bicones $V_i$, we will consider the bicones $V_i^0$ with the same vertices and with the angle $\beta_0$ between axis and generator and the bicones $V_i^1$ with the same vertices and with the angle $\beta_1$ between axis and generator.

**Lemma 5.** For any $\xi \in D$,

(i) for any $i = 1, \ldots, 2m$, $V_i^0 \cap V_i^0 = \{z_i\}$;

(ii) $V_m \cap V_{m+1}^1 = V_m^1 \cap V_{m+1} = \{z_m\}$;

(iii) the dihedral angle with edge $z_m z_{m+1}$ containing common generators of $V_m$ and $V_{m+1}$, is no greater than 0.545

(iv) $V_m^0 \cap V_{m+1}^0 \neq \emptyset$ and the dihedral angle with the edge $z_m z_{m+1}$ and sides, containing common generators of $V_m^0$ and $V_{m+1}^0$, lies between 0.224 and 0.317.

**Proof** Since the angle between the axes is greater than $2\beta - \mu$, we have (i) and (ii).
The upper bound of such angle for \( V_m \cap V_{m+1} \) is \( 2 \arccos \frac{\tan(\beta - \mu/2)}{\tan(\beta + \mu)} = 0.545 \) gives us (iii). Now, the upper and lower bounds for the angle for \( V^0_m \cap V^0_{m+1} \) are \( 2 \arccos \frac{\tan(\beta - \mu/2)}{\tan(\beta)} \) and \( 2 \arccos \frac{\tan(\beta - \mu/4)}{\tan(\beta)} \), which gives (iv).

2.4 The value of \( \alpha_{m+4} \).

Consider the bicones \( V^0_m \) and \( V^0_{m+1} \). They have common vertex \( z_m = 0 \). The bicone \( V^0_m \) stays fixed, while \( V^0_{m+1} \) changes its position depending on its second variable vertex \( z_{m+1}^\xi \), and the angle between the axis of \( V_{m+1} \) and OY is \( \theta \in (\beta - \mu, \beta - \mu/2) \), therefore \( V^0_m \) and \( V^0_{m+1} \) have nonempty intersection.

The angle \( \alpha_{m+4} \) is equal to an angle between the tangent planes to the surfaces of \( V^0_m \) and \( V^0_{m+1} \) at points of their common generator. By spherical cosine theorem, it is equal to

\[
\alpha_{m+4}(\theta) = \arccos(4 - 5 \cos(\beta_0 + \theta)).
\]

The values of the derivative \( \cos(\alpha_{m+4}(\theta))' \) for \( \theta \in [\beta - \mu, \beta - \mu/2] \) lie in the interval \((-20, -14)\). Therefore, for any \( \theta_1, \theta_2 \in (\beta - \mu, \beta - \mu/2) \),

\[
|\alpha_{m+4}(\theta_1) - \alpha_{m+4}(\theta_2)| < 20|\Delta \theta|
\]

2.5 The sets A and B.

There are 2 sets, formed from \( V_i \)-s, which will be needed for our considerations: the set \( A = V_{m-4} \cup V_{m-3} \cup V_{m-2} \) and the set \( B = V_{m+3} \cup V_{m+4} \cup V_{m+5} \).

The sets \( A \) and \( B \) lie outside the bicone \( V^1 \), but intersect \( V^0 \) so that the axes of \( V_{m-3} \) and \( V_{m+4} \) are contained in the boundary of \( V^0 \).

Some inequalities for the sets \( A \) and \( B \).

For any two points \( x \) and \( y \) in the set \( A \) (or both in the set \( B \)), we have

\[
\frac{||x - z_0||}{||y - z_0||} < 1.06 \text{ and } \frac{||x - Pr_1(x)||}{||y - Pr_1(y)||} < 1.095.
\]

At the same time, the dihedral
angle, containing A and B, whose edge is the line $z_0z_{2m}$, is no greater than $2\sqrt{5}\mu$.

Therefore, the set A lies in a domain defined by the inequalities

$$R \leq |x - z_0| \leq 1.06R; \quad -\sqrt{5}\mu \leq \varphi \leq \sqrt{5}\mu; \quad \beta_0 - 2\mu \leq \theta \leq \beta_0 + \mu,$$

where $\varphi$ and $\theta$ are the azimuth and polar angles for the point $x$ in spherical coordinates with the origin $z_0$, real line being the polar axis and the azimuth direction being parallel to $OY$.

Here $R = 2.214$ is the distance from $z_0$ to the nearest point of $A$.

Direct computation shows that

**Lemma 6.** The set $A$ can be covered by a ball $W$ of radius $0.036R$ with the center defined by
$$|x - z_0| = 1.03R; \quad \varphi = 0; \quad \theta = \beta_0 - \mu/2$$

By symmetry, analogous inequalities hold for the points in the set $B$.

3 Checking the properties of the family of zippers $S_\xi$.

Now we can verify the properties $A1$ – $A6$ for any $\xi \in D$ and evaluate the difference $S_\xi^i(x) - S_\eta^i(x)$ for $x \in V$. Fortunately $A1$-$A3$ are obvious so we proceed to $A6$, $A5$, $A4$.

3.1 Defining linear zipper $T$ and checking $A6$.

As it was proved in ([1], Lemma 1.1), for any linear zipper $T = \{T_1, \ldots, T_{2m}\}$ with attractor $K(T) = [0, 1]$ and with the same signature $\xi$ as the zipper $S$, there is unique continuous function $\varphi_{ST} : [0, 1] \to \gamma(S)$ satisfying

$$\varphi_{ST} \circ T_i(t) = S_i \circ \varphi_{ST}(t) \text{ for any } i = 1, \ldots, 2m \text{ and } t \in [0, 1]$$

This map is Hölder continuous. We prove the following theorem which allows to find its Hölder exponent:

**Theorem 7.** Let $S = \{S_1, \ldots, S_m\}$ be a zipper in $\mathbb{R}^n$ and suppose all $S_i$ are similarities.

Let $\varphi : I[0, 1] \to \gamma(S)$ be the linear parametrization of $S$ by a zipper $T = \{T_1, \ldots, T_m\}$. The Hölder exponent $\alpha$ of the map $\varphi$ satisfies

$$\alpha = \min_{i=1,\ldots,m} \frac{\log \text{Lip } S_i}{\log \text{Lip } T_i}$$

**Proof**

Denote $q_i = \text{Lip } S_i$, $p_i = \text{Lip } T_i$, $p_{\min} = \min_{i=1,\ldots,m} p_i$. Let $M$ be the diameter of $\gamma(S)$.

Observe that for any multiindex $i = i_1 \ldots i_k$, $q_i \leq p_{\min}^k$.

Suppose $a, b \in I$, $a < b$. There are two possibilities:
1) For some multiindex $i_1...i_{k+1}$, 

$$T_{i_1...i_{k+1}}(I) \subset [a, b] \subset T_{i_k}(I)$$

In this case, $p_{i_1...i_k} p_{\text{min}} < |a - b| \leq p_{i_1...i_k}$, while $\|f(a) - f(b)\| \leq q_{i_1...i_k} M$. But $q_{i_1...i_k} = p_{i_1...i_k}^{\alpha}$, so

$$\|f(a) - f(b)\| \leq \frac{M}{p_{\text{min}}^{\alpha}} |a - b|^\alpha$$

2) For some pair of multiindices $i_1...i_{k+1}$ and $j_1...j_{l+1}$,

$$T_{i_1...i_{k+1}}(I) \cup T_{j_1...j_{l+1}}(I) \subset [a, b] \subset T_{i_k}(I) \cup T_{j_1...j_l}(I)$$

In this case, $(p_{i_1...i_k} + p_{j_1...j_l}) p_{\text{min}} \leq |a - b| \leq (p_{i_1...i_k} + p_{j_1...j_l}) M$. Suppose $p_{i_1...i_k} \geq p_{j_1...j_l}$, then $p_{i_1...i_k} p_{\text{min}} \leq |a - b| \leq 2p_{i_1...i_k}$

Thus, we have

$$\|f(a) - f(b)\| \leq 2M p_{i_1...i_k} \leq \frac{2M}{p_{\text{min}}^{\alpha}} |b - a|^\alpha$$

The exponent $\alpha$ is exact because for some $k$, $q_k = p_k^{\alpha}$, and therefore $\|f(a) - f(b)\| = L|a - b|^\alpha$ for $a = T^m(0), b = T^n(1), L = \|z_0 - z_m\|$. ■

**Lemma 8.** There exist such linear zipper $\mathcal{T}$ on $[0, 1]$, that for any $\xi \in \mathcal{D}$, the linear parametrisation $\varphi^\xi = \varphi_{\mathcal{S}^\xi \mathcal{T}}$ has Hölder exponent greater than $3/4$.

**Proof** Take some $\xi_0 = (1, 0, \theta_0) \in \mathcal{D}$. Let $q_i = \text{Lip} \mathcal{S}^\xi_i$ and $s$ be the similarity dimension of $\mathcal{S}^{\xi_0}$. Take a zipper $\mathcal{T} = \{T_1, ..., T_{2^n}\}$ on $[0, 1]$ with signature $\varepsilon$ and contraction ratios $p_i = q_i^\varepsilon$. Obviously $\frac{\log q_i}{\log p_i} \equiv 1/s$. There are only two indices $i = m + 1$ and $i = m + 2$, for which $\text{Lip} \mathcal{S}^\xi_i = q_i^\varepsilon$ depends on $\xi$. For both of them

$$|\log q_i / q_i| \leq \log(1.02), \text{ therefore } 1/1.012 < \frac{\log q_i'}{\log q_i} < 1.012.$$  

Therefore $\frac{\log p_i}{\log q_i'} < 1.28 \cdot 1.012 < 4/3$.

It follows that for any $\xi \in \mathcal{D}$, the linear parametrisation $\varphi^\xi$ of $\mathcal{S}^\xi$ by the zipper $\mathcal{T}$ has Hölder exponent greater than $3/4$. ■
3.2 Verification of A5

First, we reformulate the first statement of Theorem 4 in the following way:

**Lemma 9.** Let \( \eta_1, \eta_2 \) be the generators of a dense subgroup of second type, then for any \( z_1, z_2 \in \mathbb{C} \setminus \{0\} \) and any rays \( l_1, l_2 \) issuing from zero, there is such sequence \( \{(n_k, m_k)\} \) that \( \lim_{k \to \infty} \left| \frac{z_1 \eta_1^{n_k}}{z_2 \eta_2^{m_k}} \right| = 1 \), while the angles between \( l_1, l_2 \) and rays passing from 0 to \( z_1 \eta_1^{n_k} \) and \( z_2 \eta_2^{m_k} \) respectively, converge to 0. ■

Second, the statement of the Lemma remains valid if we replace:

1) the plane \( \mathbb{C} \) by a cone \( \mathbb{C} \subset \mathbb{R}^3 \) with the axis \( L \);
2) the products \( z_1 \eta_1^{n_k} \) and \( z_2 \eta_2^{m_k} \) by \( f_1^{n_k}(z_1) \) and \( f_2^{m_k}(z_2) \), where \( f_i \) are compositions of homothety with contraction ratio \( |\eta_i| \) and rotation around the axis \( L \) in the angle \( \arg(\eta_i) \) and
3) Taking for \( l_1, l_2 \) some generators of the cone \( C \).

Since for any two sequences \( f_k, g_k \) of orthogonal transformations in \( \mathbb{R}^n \) (and therefore for similarities having origin as a fixed point), the convergence of the sequence \( f_k^{-1} g_k \) to \( \text{Id} \) is equivalent to the convergence of \( g_k f_k^{-1} \) to \( \text{Id} \), we have the following

**Lemma 10.** Let \( PQ \) be a segment in \( \mathbb{R}^3 \) and \( n \) be some its normal vector. Let \( D_k, D_k' \) be two sequences of points in \( \mathbb{R}^3 \setminus \{0\} \) and \( n_k, n_k' \) sequences of unit normal vectors to the segments \( OD_k, OD_k' \). Let \( f_k \) (resp. \( g_k \)) be the similarities which map \( P \) to \( O \), \( PQ \) to \( OD_k \) (resp. \( OD_k' \)) and whose orthogonal parts send \( n \) to \( n_k \) (resp. \( n_k' \)). Then

\[
f_k^{-1} g_k \to \text{Id} \quad \text{iff} \quad \frac{|OD_k|}{|OD_k'|} \to 1, \quad (n_k, n_k') \to 1, \quad \angle D_k OD_k' \to 0. \]

**Lemma 11.** For any \( \xi \in D \), there is such sequence \( \Sigma_\xi = \{(i_k, j_k)\} \) that the similarities \( S'_k = S_m S_{2m}^i \) and \( S''_k = S_{m+1} S_{1k}^j \) satisfy

\[
\lim_{k \to \infty} (S'_k S_{m+4})^{-1} (S''_k S_{m-3}) = \text{Id}. \]

**Proof** Observe that \( S_{m+4}^{-1}(z_{2m}) = S_{m-3}^{-1}(z_0) = (-153, 0.8, 0) \). Denote this point by \( P \). Let \( Q = (3, 0, 8, 0) \). Each map \( S_m S_{2m}^k S_{m+4} \) sends \( P \) to the point \( z_m = (0, 0, 0) \) and the point \( Q \) to some point on the surface of the cone \( S_m(V^0) \). Each map \( S_{m+1} S_{1k}^i S_{m-3} \) sends \( P \) to the point \( z_m = (0, 0, 0) \)
and the point $Q$ to some point on the surface of the cone $S_{m+1}(V^0)$. The maps $f_1 = S_m S_{2m} S_m^{-1}$ and $f_2 = S_{m+1} S_1 S_{m+1}^{-1}$ are the similarities preserving the cones $S_m(V^0)$ and $S_{m+1}(V^0)$ respectively, defined by two generators of a dense group of second kind.

Let $l$ be a common generator of these intersecting cones.

Denoting $w_1 = S_m \circ S_{m+4}(Q)$ and $w_2 = S_{m+1} \circ S_{m-3}(Q)$, rewrite

$$S_m S_{2m} S_m^{-1}(Q) = f_1^k(S_m \circ S_{m+4}(Q)) = f_1^k(w_1)$$

and

$$S_{m+1} S_1 S_{m+1}^{-1}(Q) = f_2^k(S_{m+1} \circ S_{m-3}(Q)) = f_2^k(w_2)$$

Since $w_1, w_2$ lie on the surfaces of the cones $S_m(V^0)$ and $S_{m+1}(V^0)$, we may apply Lemma 9 to get subsequences $i_k, j_k$ for which $\lim_{k \to \infty} ||f_1^k(w_1)|| = 1$, while the angles between $l$ and rays passing from 0 to $f_1^k(w_1)$ and $f_2^k(w_2)$ respectively, converge to 0.

At the same time, since $S_{m+4}$ contains a rotation in the angle $-\alpha_{m+4}$, which is the angle between normals to these cones at points of $l$, the angle between the images of the normal $n$ to the segment $PQ$ under the maps $S_k' S_{m+4} = S_m S_{2m} S_{m+4}$ and $S_k'' S_{m-3} = S_{m+1} S_1 S_{m-3}$ converges to 0. Therefore, by Lemma 10,

$$\lim_{k \to \infty} (S_k' S_{m+4})^{-1}(S_k'' S_{m-3}) = \text{Id}. \quad \blacksquare$$

### 3.3 Dividing $S_m(\gamma) \cap S_{m+1}(\gamma)$ to a sequence of disjoint pieces: checking (A4).

**Lemma 12.** There is such sequence $\Sigma = \{(i_k, j_k)\}$ in $\mathbb{N} \times \mathbb{N}$ that:

1. For any $\xi \in D$,
   $$S_m(\gamma) \cap S_{m+1}(\gamma) = \{z_m\} \cup \left( \bigcup_{i,j=1}^{\infty} (S_{m+1} S_1^{i_k}(\gamma_A) \cap S_m S_2^{j_k}(\gamma_B)) \right)$$

2. For any $k$ there is such $\xi \in D$ that
   $$S_{m+1} S_1^{i_k}(A) \cap S_m S_2^{j_k}(B) \neq \emptyset$$

3. If a pair $(i, j) \notin \Sigma$, then for any $\xi \in D$,
   $$S_{m+1} S_1^i(A) \cap S_m S_2^j(B) = \emptyset$$

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The sequences \(\{(i_k)\}\) and \(\{(j_k)\}\) are strictly increasing and both projections \(Pr_1 : (i, j) \rightarrow i, Pr_2 : (i, j) \rightarrow j\) are injective on \(\Sigma\).

For any \(\xi \in D\), \(\Sigma_\xi \subset \Sigma\).

**Proof**

For any system \(S_\xi, \xi \in D\), its invariant set \(\gamma_\xi\) is a Jordan arc if \(S_m(\gamma) \cap S_{m+1}(\gamma) = \{z_m\}\).

Consider the set \(S_m(\gamma) \cap S_{m+1}(\gamma)\). It is contained in \(S_m(V) \cap S_{m+1}(V)\).

By Lemma 5, \(S_m(V^1) \cap S_{m+1}(V^1) = S_m(V) \cap S_{m+1}(V^1) = \{z_m\}\). Therefore, \(S_m(V) \cap S_{m+1}(V) = S_m(V \setminus \hat{V}^1) \cap S_{m+1}(V \setminus \hat{V}^1)\).

The intersection of \(\gamma \setminus \{z_0, z_{2m}\}\) and \(V \setminus \hat{V}^1\) lies in the set
\[
\left( \bigcup_{n=0}^{\infty} S^m_{1}(A) \right) \cup \left( \bigcup_{n=0}^{\infty} S^m_{2m}(B) \right)
\]

therefore
\[
S_m(\gamma) \cap S_{m+1}(\gamma) \subset \{z_m\} \cup \left( \bigcup_{i,j=1}^{\infty} (S_m S_{1}^i(A) \cap S_{m} S_{2m}^j(B)) \right)
\]

Suppose \(x \in A\), \(y \in B\) and \(S_{m+1} S_{1}^i(x) = S_m S_{2m}^j(y)\). Denote \(a = \log(\|x - z_0\|), b = \log(\|y - z_{2m}\|). Then \log(q_{m+1}) + i \log(q_1) + a = \log(q_m) + j \log(q_{2m}) + b, or
\[
|i \log(q_1) - j \log(q_{2m})| \leq |b - a| + \log(q_{m+1}/q_m)
\]

According to the inequality (1), \(|b - a| < \log 1.06 < 0.06\). At the same time, \(|\log q_{m+1}/q_m| < 0.04\). Therefore \(i\) and \(j\) are the solutions of the inequality
\[
|i \log(q_1) - j \log(q_{2m})| < 0.1
\]

(**)

Both \(q_1\) and \(q_{2m}\) lie between 1/5 and 1/7, so absolute values of their logarithms are greater than 1.

This implies that for any \(i \in \mathbb{N}\) there is at most one \(j \in \mathbb{N}\) for which the inequality (**) holds and vice versa. The solutions of this inequality are the same for all values of \(q_{m+1}\) satisfying \(0.98 < \frac{q_{m+1}}{q_{m}} < 1.02\).

Therefore there is a subsequence \(\{(i_k, j_k)\}\) of \(\Sigma\) which runs through all non-negative solutions of the inequality (**) for any value of \(q_{m+1}\); both
sequences $i_k$ and $j_k$ being strictly increasing. Obviously, it will contain $\Sigma_\xi$ for any $\xi \in D$. Discarding those entries $\{(i_k, j_k)\}$, for which $S_{m+1}^{i_k}(A) \cap S_m^{j_k}(B) = \emptyset$ for any $\xi \in D$, we get the desired sequence.

Thus, for any value of $q_{m+1}$, the set $S_m(\gamma) \cap S_{m+1}(\gamma) \setminus \{z_m\}$ is the disjoint union $\bigcup_{i=0}^{\infty} (S_{m+1}^{i_k}(\gamma_A) \cap S_m^{j_k}(\gamma_B))$.

4 Proof of the main Theorem

To prove Theorem 1 we make the following steps:

First we make the estimates for the difference $\|S^\xi_1(x) - S^n_1(x)\|$ on the set $V$ for any given $\xi, \eta \in D$ for $i = 1, \ldots, 2m$. They are quite simple for $i \neq m+1$ (Lemma 13), so most of the work is done for $i = m+1$. For each $k \in \mathbb{N}$ we express these estimates in terms of the displacement $\delta^*_{\xi_k} = x^\xi_k - x^n_k$ of the point $x_k = S'i_k(z_{m-4})$. We prove in Lemma 17 that the dependence of $x^\xi_k$ on $\xi$ is bi-Lipschitz. In Lemma 18 we estimate $\max_{i=1,\ldots,2m,x\in V} \|S^\xi_i(x) - S^n_i(x)\|$ in terms of $\delta^*_{\xi_k}$. In Lemma 19 we get upper and lower bounds for $\|S^\xi_{m+1}(x) - S^n_{m+1}(x)\|$ on $S^n_{1}(A)$ in terms of $\delta^*_{\xi_k}$. In the Subsection 4.3 we prove Theorem 2 and Proposition 3 in Lemma 20 we prove that the function $f(\xi, s, t)$ used in the Theorem 2 is bi-Lipschitz with respect to $\xi$ and $\alpha$-H"older with respect to $s, t$ which gives us the proof of the main Theorem.

Some notation. Let $\xi = (\rho_1, \varphi_1, \theta_1), \quad \eta = (\rho_2, \varphi_2, \theta_2)$ and $\Delta \xi = (\Delta \rho, \Delta \varphi, \Delta \theta) = \xi - \eta$. Denote by $x^\xi_k$ the point $S'_{\xi_k}(z_{m-4}) = S'^{\xi}_{m+1}S'^{\xi}_1(z_{m-4})$ and let $\delta^*_{\xi_k} = \|x^\xi_k - x^n_k\|$.

Fixing the value of $\xi$, we consider spherical coordinate system whose polar axis is $z_{m}z^\xi_{m+1}$ and whose azimuth direction is a perpendicular to $z_{m}z^\xi_{m+1}$ lying in the right half-plane of the plane XY. We’ll denote these coordinates by $\varrho, \phi, \theta$ and call them $\xi$-coordinates.

It is more convenient to represent the difference $\|S^\xi_1(x) - S^n_1(x)\|, x \in A$ as $\|S^{\xi}_{m+1}(S^{\xi}_{m+1})^{-1}(x) - x\|$, where $x \in S^{\xi}_{m+1}S^{\xi}_{1}(A)$. Denote $F_{\xi, \eta} = S^{\xi}_{m+1}(S^{\xi}_{m+1})^{-1}$.

Each of the maps $F_{\xi, \eta}$ may be viewed as a composition of a rotation $R_\varphi$ in an angle $\Delta \varphi$ with respect to the line $z_{m}z^\xi_{m+1}$, a rotation $R_\theta$ in an angle $\Delta \theta$ with respect to $Z$ axis and homothety $H_\rho$ with ratio $\rho_2/\rho_1$. 

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4.1 Transition maps from $S^\xi$ to $S^\eta$ and their estimates.

First we consider all $i \neq m + 1$:

**Lemma 13.** For any $x \in V$, $\xi, \eta \in D$,

(a) If $i \neq m + 1, m + 2$ or $m + 4$, then $S^\xi_i(x) - S^\eta_i(x) \equiv 0$;

(b) $\|S^\xi_{m+4}(x) - S^\eta_{m+4}(x)\| < 0.46|\Delta \theta|$;

(c) $\|S^\xi_{m+2}(x) - S^\eta_{m+2}(x)\| \leq \|z_{m+1}^\xi - z_{m+1}^\eta\|

**Proof.** For $i = m + 4$,

$$\|S^\xi_{m+4}(x) - S^\eta_{m+4}(x)\| \leq q_{m+4}|\alpha_{m+4}^\xi - \alpha_{m+4}^\eta|r_x,$$

where $r_x$ is a distance from the point $x$ to the OX axis. Therefore,

$$\|S^\xi_{m+4}(x) - S^\eta_{m+4}(x)\| < 20q_{m+4}|\Delta \theta|r_x < 0.298|\Delta \theta|r_x$$

Thus, $r_x \leq 1.538$ implies $\max_{x \in V}\|S^\xi_{m+4}(x) - S^\eta_{m+4}(x)\| < 0.46|\Delta \theta|$.

The similarity $S^\xi_{m+2}$ depends only on a position of the point $z_{m+1}$, therefore

$$\|S^\xi_{m+2}(x) - S^\eta_{m+2}(x)\| \leq \frac{\|x - z_{m+2}\|}{\|z_0 - z_{m+2}\|}\|z_{m+1}^\xi - z_{m+1}^\eta\|.$$\hfill ■

The estimates for $i = m + 1$ are more complicated.

The following two lemmas will help us to estimate maximal and minimal displacement for different points in $V_{m+1}$:

**Lemma 14.** Let $S$ be a similarity in $\mathbb{R}^3$ which is a composition of a homothety with a fixed point $c$ and a rotation around line $l \ni c$ and let $P$ be a plane containing line $l$. Let $B(a, r)$ be a closed ball disjoint from $P$. Then

$$\max_{x, y \in B} \frac{\|S(x) - x\|}{\|S(y) - y\|} \leq \frac{d(a, P) + r}{d(a, P) - r}.$$\hfill ■

**Lemma 15.** Suppose $0 < \alpha < \pi/2$, and $e_1, e_2, e_3$ are such unit vectors that the angles between each $e_i, e_j$ lie between $\pi/2 - \alpha$ and $\pi/2 + \alpha$. Let $\lambda_1 < r_i < \lambda_2$. Then, for any $x = (x_1, x_2, x_3) \in \mathbb{R}$,

$$\lambda_1(\sqrt{1 - 2\sin \alpha})\|x\| \leq \sum_{i=1}^{3} x_i r_i e_i \leq \lambda_2(\sqrt{1 + 2\sin \alpha})\|x\|.$$\hfill ■
Using Lemma 12 and Lemma 5, we find possible $\xi$-coordinates of the point $x^\xi_k$:

**Lemma 16.** If $S'_k(A) \cap S''_k(B) \neq \emptyset$, then the point $x^\xi_k$ has $\xi$-coordinates $(r_k, \phi, \beta_0)$, where $r_k = \sqrt{5 \text{Lip}(S'_k)}$, $-\alpha_c < \phi < \alpha_c$, and

$$\alpha_c = \arccos \frac{\tan(\beta - \mu/2)}{\tan(\beta + \mu)} + \sqrt{5\mu} = 0.295.$$

Using last three Lemmas we show that the map, which assigns to each $\xi \in D$ the point $x^\xi_{k+1}$, is bi-Lipschitz:

**Lemma 17.** For any $k$, the map $g_k(\xi) = S^\xi_{m+1}S^\mu_{k}(z_{m-4})$ is bi-Lipschitz with respect to $\xi$ on $D$.

**Proof.** Since $\xi, \eta \in D$, $|\Delta \rho| < 0.02\rho$, $|\Delta \varphi| < 2\mu$, $|\Delta \theta| < \mu/2$, the distances from $x_k$ to the closest fixed point of the maps $H_\rho, R_\varphi$ and $R_\theta$ respectively are $r_k$, $r_k \sin \beta_0 = \frac{r_k}{\sqrt{5}}$ and $r_k \sqrt{\cos^2 \beta_0 + \sin^2 \beta_0 \cos^2 \varphi}$. Each of these values lies between $r_k/\sqrt{5}$ and $r_k$.

The displacements vectors corresponding to each of these maps, have norms $\frac{r_k \Delta \rho}{\rho}, \frac{2r_k \Delta \varphi}{\sqrt{5}}$ and $2r_k \sqrt{\frac{5 - 4 \sin^2 \varphi}{5}} \cdot \sin^2 \frac{\Delta \theta}{2}$, whose values, divided by $r_k \Delta \rho$, $r_k \Delta \varphi$ and $r_k \Delta \theta$ respectively lie in the interval $\left(\frac{0.98}{\sqrt{5}}, 1\right)$.

The angles between each two of these vectors belong to $\left(\frac{\pi}{2} - \alpha_c, \frac{\pi}{2} + \alpha_c\right)$. Applying Lemma 15 and taking into account, that $\sin \alpha_c < 0.295$, we have

$$\frac{0.98r_k}{\sqrt{5}} \sqrt{0.41} \|\Delta \xi\| < \left\|x^\rho_k - x^\xi_k\right\| < r_k \sqrt{1.59} \|\Delta \xi\|.$$

From the last inequality in the proof we obtain

$$1.26 \frac{\delta^*_k}{r_k} < \sqrt{\Delta \rho^2 + \Delta \varphi^2 + \Delta \theta^2} < 3.56 \frac{\delta^*_k}{r_k}.$$

The maximal displacement in the subset $V_{m+1}$ is reached at the point $z_{m+1}$ and is less or equal to $\rho \sqrt{\Delta \rho^2 + \Delta \theta^2} < 3.56 \frac{\delta^*_k}{r_k} \rho < 3.64 \frac{\delta^*_k}{r_k}$.

Since $\Delta \theta < 3.56 \frac{\delta^*_k}{r_k}$, by Lemma 13 for any $x \in V$, $\left\|S^\xi_{m+4}(x) - S^\eta_{m+4}(x)\right\| < 1.64 \frac{\delta^*_k}{r_k}$, so we have
Lemma 18. For any $x \in V$ and any $i = 1, \ldots, 2m$,

$$\|S_1^\eta (x) - S_1^\xi (x)\| < \frac{3.64 \delta_k^*}{r_k}$$

Denote this upper bound for the displacement, $\frac{3.64 \delta_k^*}{r_k}$ by $\delta_k$.

4.2 Estimates for $F_{\xi \eta}$

Now we take $(i_k, j_k) \in \Sigma$, $\xi, \eta \in D$ and estimate the distances between the points of $S_{m+1}^\xi S_1^{i_k} (\gamma_{A}^\xi)$ and $S_{m+1}^\eta S_1^{i_k} (\gamma_{A}^\eta)$ having the same addresses.

To apply the Proposition 3 we first prove the following

Lemma 19. For any $x \in S_1^{i_k} (A)$, $\delta_k^*/1.19 < \|S_{m+1}^\xi (x) - S_{m+1}^\eta (x)\| < 1.19 \delta_k^*$

Proof

The map $F_{\xi \eta}$ is a composition of a homothety with ratio $\rho_2/\rho_1$ and a rotation $R_{\xi \eta}$. The map $F_{\xi \eta}$ sends the point with spherical coordinates $(\rho_1, 0, 0)$ to the point $(\rho_2, 0, \Delta \theta)$, so $R_{\xi \eta}$ sends $(1, 0, 0)$ to $(1, 0, \Delta \theta)$. Therefore the axis $l$ of the rotation $R_{\xi \eta}$ lies in the middle bisector plane for these points and the unit normal vector to this plane has the coordinates $\left(1, 0, \frac{\pi + \Delta \theta}{2}\right)$.

Therefore the set $S_{m+1}^\xi S_1^{i_k} (A)$ can be covered by a ball $W_k$ whose radius is $0.036 R_k$ (where $R_k = q_{m+1}^\xi q_1^{i_k} R$) and whose center has a coordinate $(1.03 R_k, \phi, \beta_0 - \mu/2)$. It follows from Lemma 16 that $-\alpha_c \leq \phi \leq \alpha_c$.

The maximal angle between the coordinate polar axis $l_\xi$ and the plane $P_l$, containing the axis $l$ of $F_{\xi \eta}$ is $\mu/4$, therefore minimal possible distance between the center of $W_k$ and the plane $P_l$ is greater than $0.43 R_k$.

It follows from the Lemma 14 that $\max_{x,y \in W_k} \frac{\|F_{\xi \eta} (x) - x\|}{\|F_{\xi \eta} (y) - y\|} \leq 1.19$.

Taking $x = x_k$ and $\|F_{\xi \eta} (x_k) - x_k\| = \delta_k^*$, for any other point $y \in S_{m+1}^\xi S_1^{i_k} (A)$, we have

$$\delta_k^*/1.19 < \|F_{\xi \eta} (y) - y\| < 1.19 \delta_k^*$$

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4.3 Proof of Theorems 2 and Proposition 3: proof of Theorem 1

Proof of the Theorem 2

Since functions $\varphi(x,t)$ and $\psi(x,t)$ are $\alpha$-Hölder with respect to $t$, then $f(x,s,t)$ is also $\alpha$-Hölder with respect to $(s,t)$.

Since $f(x,s,t)$ is biLipschitz with respect to $x$, it is clear that if $f(x_1,s,t) = f(x_2,s,t)$, then $x_1 = x_2$, so the equation $f(x,s,t) = 0$ specifies an implicit function $x = g(s,t)$ defined on some closed subset $P \subset [0,1] \times [0,1]$.

Take $x_1 = g(s_1,t_1)$ and $x_2 = g(s_2,t_2)$ in $g(P)$. Since $f$ is bi-Lipschitz w.r.t. $x$, $\|f(x_2,s_1,t_1)\| \geq \|x_1 - x_2\|/L$. Since $f$ is $\alpha$-Hölder w.r.t. $(s,t)$, $\|f(x_2,s_1,t_1)\| \leq M\|(s_1 - s_2,t_1 - t_2)\|^\alpha$.

Therefore $|x_1 - x_2| \leq LM\|(s_1 - s_2,t_1 - t_2)\|^\alpha$.

Thus, the function $g(s,t)$ is $\alpha$-Hölder on the set $P$.

From $\dim_H \leq 2$ we obtain $\dim_H(g(P)) \leq 2/\alpha$. ■

Proof of the Proposition 3. For $\sigma = i_1i_2i_3\ldots$ denote $\sigma_k = i_{k+1}i_{k+2}\ldots$ For $j = j_1\ldots j_k$, $\hat{\sigma}_j$ is an operator sending $i_1i_2i_3\ldots$ to $j_1\ldots j_k i_1i_2i_3\ldots$.

By Barnsley Collage Theorem, $|\psi(\sigma) - \varphi(\sigma)| \leq \frac{\delta}{1 - q}$.

Let $i = i_1\ldots i_k$, $j = j_1\ldots j_l$.

Write $\psi(\sigma) - \varphi(\sigma) = (T_i\psi(\sigma_k) - S_i\psi(\sigma_k)) + (S_i\psi(\sigma_k) - S_i\varphi(\sigma_k))$. We have $\delta_1 \leq \|T_i\psi(\sigma_k) - S_i\psi(\sigma_k)\| \leq \delta_2$ and $\|S_i\psi(\sigma_k) - S_i\varphi(\sigma_k)\| \leq \frac{q_1\delta}{1 - q}$, which implies B1.

The same way $\psi(\sigma) - \varphi(\sigma) - \psi(\tau) + \varphi(\tau) = [(T_i\psi(\sigma_k) - S_i\psi(\sigma_k)) - (T_j\psi(\tau_k) - S_j\psi(\tau_k))] + [(S_i\psi(\sigma_k) - S_i\varphi(\sigma_k)) - (S_j\psi(\tau_k) - S_j\varphi(\tau_k))]$.

The norm of first brackets lies between $\delta_1$ and $\delta_2$, the norm of second brackets is no greater than $\frac{(q_1 + q_2)\delta}{1 - q}$, which gives us B2. ■

Let $\varphi^\xi : [0,1] \rightarrow \gamma^\xi$ be the linear parametrization of the zipper $\delta^\xi$ defined in Lemma 8. Let $I_{Ak} = T_{m+1}T_1(\gamma_A)$ and $I_{Bk} = T_mT_2(\gamma_B)$ be the subintervals of $I = [0,1]$, for which $\varphi^\xi(I_{Ak}) = S_mS_1(\gamma_A)$ and $\varphi^\xi(I_{Bk}) = S_mS_2(\gamma_B)$. Denote $\varphi^\xi|_{I_{Ak}}$ by $\varphi(\xi,t)$ and $\varphi^\xi|_{I_{Bk}}$ by $\psi(\xi,t)$. Take $s \in I_{Ak}$, $t \in I_{Bk}$.

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Lemma 20. The function \( f(\xi, s, t) = \varphi(\xi, s) - \psi(\xi, t) \) is bi-Lipschitz with respect to \( \xi \) for any \( s \in I_Ak, t \in I_Bk \).

**Proof**

Observe that if \( S^{m+1}_{1^+} \cap S^{j_k}_{2m+1} \neq \emptyset \), then \( \frac{q_{m+1}q_{2m}}{q_{m+1}q_{1^k}} < 1.06 \).

We apply the Proposition to \( S^{m+1}_{1^+} \) and \( S^{j_k}_{2m+1} \).

Notice that \( q_{m+4} \) is greater than \( q_{m+3} \) and \( q_{m+5} \) and the same is true for their symmetric counterparts, so we use \( q_{m+4} \) (which is equal to \( q_{m-4} \)) for both \( S^{m+1}_{1^+} \) and \( S^{j_k}_{2m+1} \).

By Proposition 3, for any \( s \in I_Ak, t \in I_Bk \),

\[
\frac{\delta^*_k}{1.19} - \frac{q_{m+1}q_{1^k}}{1-q}(q_{m+1}q_{1^k} + q_{m}q_{2m}) \leq |\varphi(\xi, s) - \varphi(\eta, s) - \psi(\xi, t) + \psi(\eta, t)| \leq 1.19\delta^*_k + \frac{q_{m+1}q_{1^k}}{1-q}(q_{m+1}q_{1^k} + q_{m}q_{2m})
\]

Evaluating \( \frac{q_{m+4}}{1-q}(q_{m+1}q_{1^k} + q_{m}q_{2m}) \delta^*_k < 0.02 \cdot 2.06q_{m+1} \cdot 3.64\delta^* < 0.03\delta^*_k \), we get

\[
0.8\delta^*_k < |\varphi(\xi, s) - \varphi(\eta, s) - \psi(\xi, t) + \psi(\eta, t)| < 1.22\delta^*_k
\]

Since for the point \( x_k \), the function \( \varphi(\xi, \eta) = F_{\xi\eta}(x_k) - x_k \) is bi-Lipschitz with respect to \( \xi \) and \( \eta \), and since \( \|F_{\xi\eta}(x_k) - x\| = \delta^*_k \), the last inequality shows that the function \( f(\xi, s, t) = \varphi(\xi, s) - \psi(\xi, t) \) is bi-Lipschitz with respect to \( \xi \).

\[ \blacksquare \]

5 Addendum: Dense groups of second type and their generators.

We will remind several facts about dense 2-generator subgroups in \( \mathbb{C}^* \). See [3] for details.

As it follows from Kronecker’s Theorem

**Proposition 21.** Let \( u, v \in \mathbb{C}, Im\frac{u}{v} \neq 0, \alpha u + \beta v = 1, \alpha, \beta \in \mathbb{R}, \xi = e^{2\pi i u}, \eta = e^{2\pi iv} \).

A group \( G = \langle \xi, \eta \rangle \) is dense in \( \mathbb{C} \) iff for any integers \( k, l, m, \)

\[ k\alpha + l\beta + m = 0 \implies k = l = m = 0. \]
[3]
Then \( G = \langle \xi, \eta, \cdot \rangle \) is called a *dense 2-generator multiplicative group* in \( \mathbb{C} \).

Notice that if \( G = \langle \xi, \eta, \cdot \rangle \) is dense in \( \mathbb{C} \), the formula \( \psi(\xi^n \eta^m) = \left( \frac{\xi}{|\xi|} \right)^m \) defines a homomorphism \( \psi \) of a group \( G = \langle \xi, \eta, \cdot \rangle \) to the unit circle \( S^1 \subset \mathbb{C} \). Put

\[
H_G = \bigcap_{\varepsilon > 0} \psi(B(1, \varepsilon) \cap G)
\]

In other words, \( H_G \) is the set of limit points of all those sequences \( \{e^{i n_k \arg(\xi)}\} \), for which \( \{n_k\} \) are the first coordinates of such sequence \( \{(n_k, m_k)\} \), that \( \lim_{k \to \infty} \xi^{n_k} \eta^{m_k} = 1 \).

The set \( H_G \) is a closed topological subgroup of the unit circle \( S^1 \), so it is either finite cyclic or infinite.

**Definition 22.** A dense 2-generator subgroup \( G \) is called the group of first type, if \( H_G \) is finite, and the group of second type, if \( H_G = S^1 \).

So \( G \) is of second type iff for some \( \alpha \notin \mathbb{Q} \), \( e^{2\pi i \alpha} \in H_G \).

Therefore, if \( \xi, \eta \) are the generators of a group of second type, then for any rational \( p, q \), the numbers \( \xi^p, \eta^q \) also generate a group of the second type. This implies

**Proposition 23.** The set of pairs \( \xi, \eta \) of generators of the groups of second type is dense in \( \mathbb{C}^2 \). \qed

The groups of second type have a significant geometric property:

**Theorem 24.** If the group \( G = \langle \xi, \eta, \cdot \rangle \), \( \xi = re^{i\alpha}, \eta = Re^{i\beta} \) is of second type, then for any \( z_1, z_2 \in \mathbb{C} \setminus \{0\} \) there is such sequence \( \{(n_k, m_k)\} \) that

\[
\lim_{k \to \infty} \frac{z_1 \xi^{n_k}}{z_2 \eta^{m_k}} = 1, \quad \lim_{k \to \infty} e^{im_k \alpha} = e^{-i \arg(z_1)}, \quad \lim_{k \to \infty} e^{in_k \beta} = e^{-i \arg(z_2)}.
\]

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