Large regular graphs of given valency and second eigenvalue

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Abstract

From Alon and Boppana, and Serre, we know that for any given integer \( k \geq 3 \) and real number \( \lambda < 2\sqrt{k-1} \), there are finitely many \( k \)-regular graphs whose second largest eigenvalue is at most \( \lambda \). In this paper, we investigate the largest number of vertices of such graphs.

1 Introduction

For a \( k \)-regular graph \( G \) on \( n \) vertices, we denote by \( \lambda_1(G) = k > \lambda_2(G) \geq \ldots \geq \lambda_n(G) = \lambda_{\text{min}}(G) \) the eigenvalues of the adjacency matrix of \( G \). For a general reference on the eigen-

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values of graphs, see [8 15].

The second eigenvalue of a regular graph is a parameter of interest in the study of graph connectivity and expanders (see [11 18 20] for example). In this paper, we investigate the maximum order $v(k, \lambda)$ of a connected $k$-regular graph whose second largest eigenvalue is at most some given parameter $\lambda$. As a consequence of work of Alon and Boppana, and of Serre [1 11 23 26 30 31 37], we know that $v(k, \lambda)$ is finite for $\lambda < 2\sqrt{k - 1}$. The recent result of Marcus, Spielman and Srivastava [24] showing the existence of infinite families of Ramanujan graphs of any degree at least 3 implies that $v(k, \lambda)$ is infinite for $\lambda \geq 2\sqrt{k - 1}$.

In this paper, we will investigate $v(k, \lambda)$ for various values of $k$ and $\lambda$. The parameter $v(k, -1)$ can be determined using the fact that a graph with second eigenvalue at most $-1$ is a disjoint union of complete graphs. Indeed, if a graph $G$ is not a disjoint union of complete graphs, then $G$ induces a subgraph isomorphic to $K_{1,2}$, so eigenvalue interlacing (see, for example, [8 Proposition 3.2.1]) implies $\lambda_2(G) \geq \lambda_2(K_{1,2}) = 0$. Thus $v(k, -1) = k + 1$ and the unique graph meeting this bound is $K_{k+1}$. The parameter $v(k, 0)$ can be determined using the fact that a graph with exactly one positive eigenvalue must be a complete multipartite graph (see [6, page 89], for example). Indeed, the largest $k$-regular multipartite graph is clearly the complete bipartite graph $K_{k,k}$, since a $k$-regular $t$-partite graph has $tk/(t - 1)$ vertices. Thus $v(k, 0) = 2k$, and $K_{k,k}$ is the unique graph meeting this bound. The values of $v(k, -1)$ and $v(k, 0)$ also follow from Theorem 2 in Section 2 below.

Results from Bussemaker, Cvetković and Seidel [9] and Cameron, Goethals, Seidel, and Shult [10] give a characterization of the regular graphs with smallest eigenvalue $\lambda_{\min} \geq -2$. Since the second eigenvalue of the complement of a regular graph is $\lambda_2 = -1 - \lambda_{\min}$, the regular graphs with second eigenvalue $\lambda_2 \leq 1$ are also characterized. This characterization can be used to find $v(k, 1)$ (see Section 3).

The values remaining to be investigated are $v(k, \lambda)$ for $1 < \lambda < 2\sqrt{k - 1}$. Reingold, Vadhan and Wigderson [33] used regular graphs with small second eigenvalue as the starting point of their iterative construction of infinite families of expanders using the zig-zag product. The parameter $v(k, \lambda)$ has also been studied by Teranishi and Yasuno [39] from a design theory perspective, Nozaki [32] from a linear programming and association schemes point of view, Koledin and Stanić [21 22 38] from a spectral graph theory perspective, Høholdt and Justesen [19], and Richey, Shutty and Stover [42]. Nozaki called graphs with smallest second eigenvalue for given order and degree extremal expander graphs. Høholdt and Justesen [19] focused on bipartite regular graphs and applied their results to the construction of graph codes while Richey, Shutty, and Stover [42] implemented Serre’s quantitative version of the Alon–Boppanna Theorem [37] to obtain upper bounds for $v(k, \lambda)$ for several values of $k$ and $\lambda$. For certain values of $k$ and $\lambda$, these authors made some conjectures about $v(k, \lambda)$. Guo, Mohar, and Tayfeh-Rezaie [16 27 28] studied a similar problem involving the median
eigenvalues.

In this paper, we determine \( v(k, \lambda) \) explicitly for several values of \((k, \lambda)\), confirming or disproving several conjectures in [12], and we find the graphs (in many cases unique) which meet our bounds. In many cases these graphs are distance-regular. For definitions and notations related to distance-regular graphs, we refer the reader to [8, Chapter 12]. Table 1 contains a summary of the values of \( v(k, \lambda) \) that we found for \( k \leq 22 \). Table 2 contains six infinite families of graphs and seven sporadic graphs meeting the bound \( v(k, \lambda) \) for some values of \( k, \lambda \) due to Theorem 2. Table 3 illustrates that the graphs in Table 2 that meet the bound \( v(k, \lambda) \) also meet the bound \( v(k, \lambda') \) for certain \( \lambda' > \lambda \) due to Proposition 3.

2 Linear programming method

In this section, we give a bound for \( v(k, \lambda) \) by the linear programming method in [32]. Let \( F_i = F_i^{(k)} \) be orthogonal polynomials defined by the three recurrence formula:

\[
F_0^{(k)}(x) = 1, \quad F_1^{(k)}(x) = x, \quad F_2^{(k)}(x) = x^2 - k,
\]

and

\[
F_i^{(k)}(x) = xF_{i-1}^{(k)}(x) - (k - 1)F_{i-2}^{(k)}(x)
\]

for \( i \geq 3 \). The following is called the linear programming bound for regular graphs [32].

**Theorem 1.** Let \( G \) be a connected \( k \)-regular graph with \( v \) vertices. Let \( \lambda_1 = k, \lambda_2, \ldots, \lambda_n \) be the distinct eigenvalues of \( G \). Suppose there exists a polynomial \( f(x) = \sum_{i \geq 0} f_i F_i^{(k)}(x) \) such that \( f(k) > 0, f(\lambda_i) \leq 0 \) for any \( i \geq 2, f_0 > 0, \) and \( f_i \geq 0 \) for any \( i \geq 1 \). Then we have

\[
v \leq \frac{f(k)}{f_0}.
\]

Let \( T(k, t, c) \) be the \( t \times t \) tridiagonal matrix with lower diagonal \((1, 1, \ldots, 1, c)\), upper diagonal \((k, k-1, \ldots, k-1)\), and with constant row sum \( k \), where \( c \) is a positive real number. The following is the main theorem in this section.

**Theorem 2.** Let \( \lambda_2 \) be the second largest eigenvalue of \( T(k, t, c) \). Then we have

\[
v(k, \lambda_2) \leq M(k, t, c) = 1 + \sum_{i=0}^{t-3} k(k-1)^i + \frac{k(k-1)^{t-2}}{c}.
\]

Let \( G \) be a \( k \)-regular connected graph with second largest eigenvalue at most \( \lambda_2 \), valency \( k \), and \( v(k, \lambda_2) \) vertices. Then \( v(k, \lambda_2) = M(k, t, c) \) if and only if \( G \) is distance-regular with quotient matrix \( T(k, t, c) \) with respect to the distance-partition.
Table 1: Summary of our Results for $k \leq 22$

| $(k, \lambda)$ | $v(k, \lambda)$ | $(k, \lambda)$ | $v(k, \lambda)$ | $(k, \lambda)$ | $v(k, \lambda)$ |
|----------------|-----------------|----------------|-----------------|----------------|-----------------|
| $(2, -1)$      | 3               | $(7, 1)$       | 18              | $(14, \sqrt{13})$ | 366             |
| $(2, 0)$       | 4               | $(7, 2)$       | 50              | $(14, \sqrt{26})$ | 4760            |
| $(2, \frac{1}{2} (\sqrt{5} - 1))$ | 5               | $(8, -1)$      | 9               | $(14, \sqrt{39})$ | 804648          |
| $(2, 1)$       | 6               | $(8, 0)$       | 16              | $(15, -1)$       | 16              |
| $(2, \sqrt{2})$ | 8               | $(8, 1)$       | 21              | $(15, 0)$        | 30              |
| $(2, \frac{1}{2} (\sqrt{5} + 1))$ | 10              | $(8, \sqrt{7})$ | 114             | $(15, 1)$        | 32              |
| $(2, \sqrt{3})$ | 12              | $(8, \sqrt{14})$ | 800             | $(16, -1)$       | 17              |
| $(3, -1)$      | 4               | $(8, \sqrt{21})$ | 39216           | $(16, 0)$        | 32              |
| $(3, 0)$       | 6               | $(9, -1)$      | 10              | $(16, 1)$        | 34              |
| $(3, 1)$       | 10              | $(9, 0)$       | 18              | $(16, 2)$        | 77              |
| $(3, \sqrt{2})$ | 14              | $(9, 1)$       | 24              | $(17, -1)$       | 18              |
| $(3, \sqrt{3})$ | 18              | $(9, 2\sqrt{2})$ | 146             | $(17, 0)$        | 34              |
| $(3, 2)$       | 30              | $(9, 4)$       | 1170            | $(17, 1)$        | 36              |
| $(3, \sqrt{6})$ | 126             | $(9, 2\sqrt{6})$ | 74898           | $(18, -1)$       | 19              |
| $(4, -1)$      | 5               | $(10, -1)$     | 11              | $(18, 0)$        | 36              |
| $(4, 0)$       | 8               | $(10, 0)$      | 20              | $(18, 1)$        | 38              |
| $(4, 1)$       | 12              | $(10, 1)$      | 27              | $(18, \sqrt{17})$ | 614            |
| $(4, \sqrt{5} - 1)$ | 10              | $(10, 2)$      | 56              | $(18, \sqrt{34})$ | 10440         |
| $(4, \sqrt{3})$ | 26              | $(10, 3)$      | 182             | $(18, \sqrt{51})$ | 3017196       |
| $(4, 2)$       | 35              | $(10, 3\sqrt{2})$ | 1640            | $(19, -1)$       | 20              |
| $(4, \sqrt{6})$ | 80              | $(10, 3\sqrt{3})$ | 132860          | $(19, 0)$        | 38              |
| $(4, 3)$       | 728             | $(11, -1)$     | 12              | $(19, 1)$        | 40              |
| $(5, -1)$      | 6               | $(11, 0)$      | 22              | $(20, -1)$       | 21              |
| $(5, 0)$       | 10              | $(11, 1)$      | 24              | $(20, 0)$        | 40              |
| $(5, 1)$       | 16              | $(12, -1)$     | 13              | $(20, 1)$        | 42              |
| $(5, 2)$       | 42              | $(12, 0)$      | 24              | $(20, \sqrt{19})$ | 762           |
| $(5, 2\sqrt{2})$ | 170             | $(12, 1)$      | 26              | $(20, \sqrt{38})$ | 14480         |
| $(5, 2\sqrt{3})$ | 2730            | $(12, \sqrt{11})$ | 266             | $(20, \sqrt{57})$ | 5227320       |
| $(6, -1)$      | 7               | $(12, \sqrt{22})$ | 2928            | $(21, -1)$       | 22              |
| $(6, 0)$       | 12              | $(12, \sqrt{33})$ | 354312          | $(21, 0)$        | 42              |
| $(6, 1)$       | 15              | $(13, -1)$     | 14              | $(21, 1)$        | 44              |
| $(6, \sqrt{5})$ | 62              | $(13, 0)$      | 26              | $(22, -1)$       | 23              |
| $(6, \sqrt{10})$ | 312             | $(13, 1)$      | 28              | $(22, 0)$        | 44              |
| $(6, \sqrt{15})$ | 7812            | $(14, -1)$     | 15              | $(22, 1)$        | 46              |
| $(7, -1)$      | 8               | $(14, 0)$      | 28              | $(22, 2)$        | 100             |
| $(7, 0)$       | 14              | $(14, 1)$      | 30              |                |                |
Proof. We first show that the eigenvalues of $T$ coincide with the zeros of $\sum_{i=0}^{t-2} F_i(x) + F_{t-1}(x)/c$ (see [6] Section 4.1 B)). Indeed,

$$[F_0, F_1, \ldots, F_{t-1}/c]T = [xF_0, xF_1, \ldots, xF_{t-2}, (k-1)F_{t-2} + (k-c)F_{t-1}/c],$$

and

$$[F_0, F_1, \ldots, F_{t-1}/c](T - xI) = [0, 0, \ldots, 0, (k-1)F_{t-2} + (-x + k - c)F_{t-1}/c]$$

$$= [0, 0, \ldots, 0, (k-x)(\sum_{i=0}^{t-2} F_i + F_{t-1}/c)]$$

$$= [0, 0, \ldots, 0, (k-x)((c-1)G_{t-2} + G_{t-1})/c]$$

by the three recurrence relation, where $G_t(x) = \sum_{j=0}^{i} F_j(x)$. From this equation, the zeros of $(k-x)((c-1)G_{t-2} + G_{t-1})$ are eigenvalues of $T$. The monic polynomials $G_t$ form a sequence of orthogonal polynomials with respect to some positive weight on the interval $[-2\sqrt{k-1}, 2\sqrt{k-1}]$ [32]. Since the zeros of $G_{t-2}$ and $G_{t-1}$ interlace on $[-2\sqrt{k-1}, 2\sqrt{k-1}]$, the zeros of $(k-x)((c-1)G_{t-2} + G_{t-1})$ are simple. Therefore all eigenvalues of $T$ coincide with the zeros of $(k-x)((c-1)G_{t-2} + G_{t-1})$, and are simple.

Let $\lambda_1 = k > \lambda_2 > \ldots > \lambda_t$ be the eigenvalues of $T$. We show the polynomial

$$f(x) = (x - \lambda_2) \prod_{i=3}^{t}(x - \lambda_i)^2 = \sum_{i=0}^{2t-3} f_i F_i(x)$$

satisfies $f_i > 0$ for $i = 0, 1, \ldots, 2t-3$. Note that it trivially holds that $f(k) > 0$, and $f(\lambda) \leq 0$ for any $\lambda \leq \lambda_2$. The polynomial $f(x)$ can be expressed by

$$f(x) = \frac{(c-1)G_{t-2} + G_{t-1}}{x - \lambda_2} \sum_{i=0}^{t-2} F_i + F_{t-1}/c).$$

By Theorem 3.1 in [12] (or Theorem 4 in [32]), $g(x) = ((c-1)G_{t-2} + G_{t-1})/(x - \lambda_2)$ has positive coefficients in terms of $G_0, G_1, \ldots, G_{t-1}$. This implies that $g(x)$ has positive coefficients in terms of $F_0, F_1, \ldots, F_{t-1}$. Therefore $f_i > 0$ for $i = 0, 1, \ldots, 2t-3$ by Theorem 3 in [32].

The polynomial $g(x)$ can be expressed by $g(x) = \sum_{i=0}^{t-2} g_i F_i(x)$. By Theorem 3 in [32], we have

$$f_0 = \sum_{i=0}^{t-2} g_i F_i(k) = g(k).$$

By the linear programming bound obtained from $f(x)$, we have

$$v(k, \lambda_2) \leq \frac{f(k)}{f_0} = \sum_{i=0}^{t-2} F_i(k) + F_{t-1}(k)/c$$

$$= 1 + \sum_{i=0}^{t-3} k(k-1)^i + \frac{k(k-1)^{t-2}}{c}. $$

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By Remark 2 in [32], the graph attaining the bound has girth at least $2t - 2$, and at most $t$ distinct eigenvalues. Therefore the graph is a distance-regular graph with quotient matrix $T(k, t, c)$ by Theorem 6 in [32] and [13] . Conversely the distance-regular graph with quotient matrix $T(k, t, c)$ clearly attains the bound $M(k, t, c)$.

**Remark 1.** The distance-regular graphs which have $T(k, t, c)$ as a quotient matrix of the distance partition are precisely the distance-regular graphs with intersection array $\{k, k - 1, \ldots, k - 1; 1, \ldots, 1, c\}$.

**Corollary 1.** Let $H$ be a connected $k$-regular graph with at least $M(k, t, c)$ vertices. Let $\lambda_2$ be the second largest eigenvalue of $T(k, t, c)$. Then $\lambda_2 \leq \lambda_2(H)$ holds with equality only if $H$ meets the bound $M(k, t, c)$.

**Proof.** By Theorem 2 if $\lambda_2 > \lambda_2(H)$, then the order of $H$ is at most $M(k, t, c)$. If the order of $H$ is equal to $M(k, t, c)$, then $H$ has at most $t - 1$ distinct eigenvalues by Remark 2 in [32]. However then the order of $H$ is less than $M(k, t - 1, 1)$ by the Moore bound, a contradiction. Therefore if $\lambda_2 > \lambda_2(H)$, then the order of $H$ is less than $M(k, t, c)$. Namely if the order of $H$ is at least $M(k, t, c)$, then $\lambda_2 \leq \lambda_2(H)$. If $\lambda_2 = \lambda_2(H)$ holds, then the order of $H$ is bounded above by $M(k, t, c)$ in Theorem 2 and attains the bound.

We will discuss a possible second eigenvalue $\lambda_2$ of $T(k, t, c)$. Indeed for any $-1 \leq \lambda < 2\sqrt{k - 1}$ there exist $t, c$ such that $\lambda$ is the second eigenvalue of $T(k, t, c)$. Let $\lambda^{(t)}$, $\mu^{(t)}$ be the largest zero of $G_t$, $F_t$, respectively. The zero $\lambda^{(t)}$ can be expressed by $\lambda^{(t)} = 2\sqrt{k - 1} \cos \theta$, where $\pi/(t + 1) < \theta < \pi/t$ [2] Section III.3.

**Proposition 1.** The following hold:

1. $\lambda^{(t)} < \mu^{(t)}$ for any $k$, $t$.
2. $\mu^{(t-1)} < \lambda^{(t)}$ for $k \geq 5$ and any $t$, $k = 4$ and $t \leq 5$, or $k = 3$ and $t \leq 3$.
3. $\mu^{(t-1)} > \lambda^{(t)}$ for $k = 4$ and $t \geq 6$, or $k = 3$ and $t \geq 4$.

**Proof.** Since $F_t(\lambda^{(t)}) = G_t(\lambda^{(t)}) - G_{t-1}(\lambda^{(t)}) = -G_{t-1}(\lambda^{(t)}) < 0$, we have $\lambda^{(t)} < \mu^{(t)}$ for any $k$, $t$. Note that $F_t$ has a unique zero greater than $\lambda^{(t)}$. By the equality $(k - 1)F_{t-1} = (k - 1 - x)G_{t-1} + G_t$, it is satisfied that

$$
(k - 1)F_{t-1}(\lambda^{(t)}) = (k - 1 - \lambda^{(t)})G_{t-1}(\lambda^{(t)}) + G_t(\lambda^{(t)})
= (k - 1 - \lambda^{(t)})G_{t-1}(\lambda^{(t)})
= (k - 1 - 2\sqrt{k - 1} \cos \theta)G_{t-1}(\lambda^{(t)})
\begin{cases}
> (k - 1 - 2\sqrt{k - 1} \cos \frac{\pi}{t+1})G_{t-1}(\lambda^{(t)}) \geq 0 \text{ for } (k, t) \text{ in } (2), \\
< (k - 1 - 2\sqrt{k - 1} \cos \frac{\pi}{t})G_{t-1}(\lambda^{(t)}) \leq 0 \text{ for } (k, t) \text{ in } (3).
\end{cases}
$$

This implies the proposition.
Remark 2. The second largest eigenvalue $\lambda_2(c)$ of $T(k, t, c)$ is the largest zero of $(c−1)G_{t-2} + G_{t-1}$. Since the zeros of $G_{t-2}$ and $G_{t-1}$ interlace, $\lambda_2(c)$ is a monotonically decreasing function in $c$. In particular, $\lim_{c→∞} \lambda_2(c) = \lambda^{(t-2)}$, $\lambda_2(1) = \lambda^{(t-1)}$, and $\lim_{c→0} \lambda_2(c) = \mu^{(t-1)}$.

Note that both $F_i$ and $G_i$ form a sequence of orthogonal polynomials with respect to some positive weight on the interval $[-2\sqrt{k−1}, 2\sqrt{k−1}]$. By Remark 2 possible values $\lambda_2(t, c)$ of $T(k, t, c)$ is between $\lambda_2(2, 1) = −1$ and $\lim_{t→∞} \lambda_2(t, c) = 2\sqrt{k−1}$.

By the following proposition, we may assume $c ≥ 1$ in Theorem 2 to obtain better bounds.

Proposition 2. For any $\lambda$ such that $\lambda^{(t−1)} < \lambda < \mu^{(t−1)}$, there exist $0 < c_1 < 1$, $c_2 > 0$ such that both the second-largest eigenvalues of $T(k, t, c_1)$ and $T(k, t + 1, c_2)$ are $\lambda$. Then we have $M(k, t, c_1) > M(k, t + 1, c_2)$.

Proof. Since $(c_1 − 1)G_{t-2}(\lambda) + G_{t-1}(\lambda) = 0$ holds, we have

$$c_1 = -\frac{G_{t-1}(\lambda) - G_{t-2}(\lambda)}{G_{t-2}(\lambda)} = -\frac{F_{t-1}(\lambda)}{G_{t-2}(\lambda)}.$$ 

Similarly $c_2 = -F_t(\lambda)/G_{t-1}(\lambda)$ holds. Note that $F_{t-1}(\lambda) = -c_1G_{t-2}(\lambda) < 0$ and $F_t(\lambda) = -c_2G_{t-1}(\lambda) < 0$. Therefore it is satisfied that

$$M(k, t, c_1) − M(k, t + 1, c_2) = k(k − 1)^{t−2}(\frac{1}{c_1} − 1 − \frac{1}{c_2}(k − 1))$$

$$= k(k − 1)^{t−2}(−\frac{G_{t-2}(\lambda)}{F_{t-1}(\lambda)} − 1 + (k − 1)\frac{G_{t-1}(\lambda)}{F_t(\lambda)})$$

$$= k(k − 1)^{t−2}(−\frac{G_{t-1}(\lambda)}{F_{t-1}(\lambda)} + (k − 1)\frac{G_{t-1}(\lambda)}{F_t(\lambda)})$$

$$= k(k − 1)^{t−2}\frac{G_{t-1}(\lambda)}{F_{t-1}(\lambda)}(−F_t(\lambda) + (k − 1)F_{t-1}(\lambda))$$

$$= k(k − 1)^{t−2}(k − \lambda)G_{t-1}(\lambda)^2$$

$$> 0.$$ 

Table 2 shows the known examples attaining the bound $M(k, t, c)$. The incidence graphs of $PG(2, q)$, $GQ(q, q)$, and $GH(q, q)$ are known to be unique for $q ≤ 8$, $q ≤ 4$, and $q ≤ 2$, respectively (see, for example, 6 Table 6.5 and the following comments). The incidence graphs of $PG(2, 2)$, $GQ(2, 2)$, and $GH(2, 2)$ are the Heawood graph, the Tutte-Coxeter graph (or Tutte 8-cage), and the Tutte 12-cage, respectively.

The bounds in Table 2 solve conjectures of Richey, Shutty, and Stover 42. Richey, Shutty, and Stover show that $v(3, 2) ≤ 105$, but they note that the largest 3-regular graph with $\lambda_2 ≤ 2$ they are aware of is the Tutte-Coxeter graph on 30 vertices. They conjectured that $v(3, 2) = 30$. They show that $v(4, 2) ≤ 77$ and conjecture that the largest 4-regular graph with $\lambda_2 ≤ 2$ is the so-called rolling cube graph on 24 vertices (that is, the bipartite
Table 2: Known graph meeting the bound $M(k, t, c)$

| $(k, \lambda)$ | $v(k, \lambda)$ | Graph meeting bound | Unique? | Ref. |
|----------------|------------------|---------------------|---------|------|
| $(2, 2 \cos(2\pi/n))$ | $n$ | $n$-cycle $C_n$ | yes | |
| $(k, -1)$ | $k + 1$ | Complete graph $K_{k+1}$ | yes | |
| $(k, 0)$ | $2k$ | Complete bipartite graph $K_{k,k}$ | yes | |
| $(q + 1, \sqrt{q})$ | $2(q^2 + q + 1)$ | incidence graph of $PG(2, q)$ | ? | [6, 36] |
| $(q + 1, \sqrt{2q})$ | $2(q + 1)(q^2 + 1)$ | incidence graph of $GQ(q, q)$ | ? | [3, 6] |
| $(q + 1, \sqrt{3q})$ | $2(q + 1)(q^4 + q^2 + 1)$ | incidence graph of $GH(q, q)$ | ? | [3, 6] |
| $(3, 1)$ | 10 | Petersen graph | yes | [18] |
| $(4, 2)$ | 35 | Odd graph $O_4$ | yes | [29] |
| $(7, 2)$ | 50 | Hoffman–Singleton graph | yes | [18] |
| $(5, 1)$ | 16 | Clebsch graph | yes | [15, 35] |
| $(10, 2)$ | 56 | Gewirtz graph | yes | [7, 14] |
| $(16, 2)$ | 77 | $M_{22}$ graph | yes | [5, 17] |
| $(22, 2)$ | 100 | Higman–Sims graph | yes | [5, 17] |

$PG(2, q)$: projective plane, $GQ(q, q)$: generalized quadrangle, $GH(q, q)$: generalized hexagon, $q$: prime power

Since the order of a graph is an integer, $v(k, \lambda)$ can be bounded above by $\lceil M(k, t, c) \rceil$. The graphs meeting the bound $M(k, t, c)$ can be maximal under the assumption of a larger second eigenvalue.

**Proposition 3.** Let $\lambda_1$, $\lambda_2$ be the second largest eigenvalues of $T(k, t + 1, c_1)$, $T(k, t, c_2)$, respectively. Suppose there exists a graph which attains the bound $M(k, t, c)$. Then the following hold:

1. If $c = 1$ holds, then $v(k, \lambda_1) = v(k, \lambda)$ for $c_1 > k(k-1)^{t-1}$. Moreover if $M(k, t, c)$ is even, and $k$ is odd, then $v(k, \lambda_1) = v(k, \lambda)$ for $c_1 > k(k-1)^{t-1}/2$.

2. If $c > 1$ holds, $v(k, \lambda_2) = v(k, \lambda)$ for $c_2 > c - c^2/(k(k-1)^{t-2} + c)$. Moreover if $M(k, t, c)$ is even, and $k$ is odd, then $v(k, \lambda_2) = v(k, \lambda)$ for $c_2 > c - 2c^2/(k(k-1)^{t-2} + 2c)$. 

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Proof. We show only (1) because (2) can be proved similarly. For $c_1 > k(k-1)^{t-1}$, we have

$$M(k, t, c) = v(k, \lambda) \leq v(k, \lambda_1) \leq \lceil M(k, t, c_1) \rceil = M(k, t, c).$$

Therefore $v(k, \lambda) = v(k, \lambda_1)$. If $k$ is odd, $v(k, \lambda_1)$ must be even. For $c_1 > k(k-1)^{t-1}/2$, we have

$$M(k, t, c) = v(k, \lambda) \leq v(k, \lambda_1) \leq \lceil M(k, t, c_1) \rceil = M(k, t, c) + 1.$$

Thus if $M(k, t, c)$ is even, then $v(k, \lambda) = v(k, \lambda_1)$.

The larger second eigenvalues in Proposition 3 are calculated in Table 3. The graphs in Table 3 meet $v(k, \lambda)$ for any $\lambda_2 \leq \lambda < \lambda'$, where $\lambda'$ is the largest zero of $f(x)$ in the table.

By Theorem 2, we can obtain an alternative proof of the theorem due to Alon and Boppana, and Serre.

**Corollary 2** (Alon–Boppana, Serre). For given $k$, $\lambda < 2\sqrt{k-1}$, there exist finitely many $k$-regular graphs whose second largest eigenvalue is at most $\lambda$.

**Proof.** The second largest eigenvalue $\lambda_2(t)$ of $T(k, t, 1)$ is equal to the largest zero of $G_{t-2}$. The zero is expressed by $\lambda_2(t) = 2\sqrt{k-1} \cos \theta$, where $\theta$ is less than $\pi/(t-2)$ [2, Section III.3]. This implies that there exists a sufficiently large $t'$ such that $\lambda_2(t') > \lambda$. Therefore we have

$$v(k, \lambda) \leq v(k, \lambda_2(t')) \leq 1 + \sum_{i=0}^{t'-2} k(k-1)^i.$$ 

\[\square\]

**3 Second largest eigenvalue 1**

In this section, we classify the graphs meeting $v(k, 1)$. The complement of a regular graph with second eigenvalue at most 1 has smallest eigenvalue at least $-2$. The graph structure is obtained from a subset of a root system, and it is characterized as a line graph except for sporadic examples [6, Theorem 3.12.2]. The following theorem is immediate by Theorem 3.12.2 in [6].

**Theorem 3.** Let $G$ be a connected regular graph with $v$ vertices, valency $k$, and second largest eigenvalue at most 1. Then one of the following holds:

1. $G$ is the complement of the line graph of a regular or a bipartite semiregular connected graph.

2. $v = 2(k-1) \leq 28$, and $G$ is a subgraph of the complement of $E_7(1)$, switching-equivalent to the line graph of a graph $\Delta$ on eight vertices, where all valencies of $\Delta$ have the same parity (graphs nos. 1–163 in Table 9.1 in [9]).
(3) \( v = 3(k-1) \leq 27, \) and \( G \) is a subgraph of the complement of the Schl"{a}fli graph (graphs nos. 164–184 in Table 9.1 in [9]).

(4) \( v = 4(k-1) \leq 16, \) and \( G \) is a subgraph of the complement of the Clebsch graph (graphs

| Graph               | \( t \) | \( c \) | \( f(x) \) | \( \lambda' \) |
|---------------------|--------|-------|-----------|-----------|
| \( K_{k+1} \) (k: even) | 2      | 1     | \( x^2 - (k - k^2)x + k^2 - 2k \) | \( \lambda' \) |
| \( K_{k,k} \) (k: odd)          | 2      | 1     | \( 2x^2 - (k - k^2)x + k^2 - 3k \) | \( \lambda' \) |
| \( K_{k,k} \) (k: even)          | 3      | \( k \) | \( x^2 - (1 - k)x - 1 \) | \( \lambda' \) |
| \( PG(2,q) \) (q + 1: even)     | 4      | \( q+1 \) | \( (q^2 + 1)x^3 + (q^3 + q^2)x^2 \) | \( \lambda' \) |
| \( PG(2,q) \) (q + 1: odd)      | 4      | \( q+1 \) | \( + (-q^3 - 2q - 1)x - q^4 - q^3 \) | \( \lambda' \) |
| \( GQ(q,q) \) (q + 1: even)     | 5      | \( q+1 \) | \( (-q^2 + q - 1)x^4 - q^3x^3 \) | \( \lambda' \) |
| \( GQ(q,q) \) (q + 1: odd)      | 5      | \( q+1 \) | \( + (2q^3 - 2q^2 + 2q + 1)x^2 \) | \( \lambda' \) |
| \( GH(q,q) \) (q + 1: even)     | 7      | \( q+1 \) | \( (-q^4 + q^3 - q^2 + q - 1)x^6 \) | \( \lambda' \) |
| \( GH(q,q) \) (q + 1: odd)      | 7      | \( q+1 \) | \( + (4q^5 - 4q^4 + 4q^3 - 4q^2 + 4q + 1)x^4 \) | \( \lambda' \) |
|                      |        |       | \( + (-3q^6 + 3q^5 - 3q^4 + 3q^3 - 3q^2 - 3q)x^2 \) | \( \lambda' \) |
|                      |        |       | \( -q^7x^5 + 4q^6x^3 - 3q^7x + q^2 \) | \( \lambda' \) |
| \( \text{Petersen} \) | 3      | 1     | \( x^3 + 12x^2 + 7x - 24 \) | 1.11207 |
| \( \text{Odd graph } O_4 \)      | 4      | 2     | \( 19x^3 + 36x^2 - 97x - 108 \) | 2.02156 |
| \( \text{Hoffman–Singleton} \)     | 3      | 1     | \( x^3 + 126x^2 + 113x - 756 \) | 2.02845 |
| \( \text{Clebsch} \)               | 3      | 2     | \( 3x^2 + 5x - 10 \) | 1.1736  |
| \( \text{Gewirtz} \)               | 3      | 2     | \( 23x^2 + 45x - 185 \) | 2.02182 |
| \( M_{22} \)                      | 3      | 4     | \( 61x^2 + 240x - 736 \) | 2.02472 |
| \( \text{Higman–Sims} \)           | 3      | 6     | \( 13x^2 + 77x - 209 \) | 2.0232  |

\( \lambda' \) is the largest zero of \( f(x) \)
The following theorem shows the classification of graphs meeting $v(k, 1)$.

**Theorem 4.** Let $G$ be a connected $k$-regular graph with second largest eigenvalue at most 1, with $v(k, 1)$ vertices. Then the following hold:

1. $v(2, 1) = 6$, and $G$ is the hexagon.
2. $v(3, 1) = 10$, and $G$ is the Petersen graph.
3. $v(4, 1) = 12$, and $G$ is the complement of the graph no. 186 in Table 9.1.
4. $v(5, 1) = 16$, and $G$ is the Clebsch graph.
5. $v(6, 1) = 15$, and $G$ is the complement of the line graph of the complete graph with 6 vertices, or the complement of one of the graphs nos. 171–176 in Table 9.1.
6. $v(7, 1) = 18$, and $G$ is the complement of one of the graphs nos. 177–180 in Table 9.1.
7. $v(8, 1) = 21$, and $G$ is the complement of one of the graphs nos. 181, 182 in Table 9.1.
8. $v(9, 1) = 24$, and $G$ is the complement of the graph no. 183 in Table 9.1.
9. $v(10, 1) = 27$, and $G$ is the complement of the Schl"afli graph.
10. $v(k, 1) = 2k + 2$ for $k \geq 11$, and $G$ is the complement of the line graph of $K_{2,k+1}$.

**Proof.**

1. A connected 2-regular graph is an $n$-cycle, whose eigenvalues are $2 \cos(2\pi j/n)$ ($j = 0, 1, \ldots, n-1$). This implies (1).

2. By Theorem 2 for $T(k, 3, (k−1)/2)$, we have $v(k, 1) \leq 3k + 1$. The two graphs are unique graphs attaining this bound [18, 32, Theorem 10.6.4].

3. The complement of the line graph of $K_{2,k+1}$ is of degree $k$ and has $2k + 2$ vertices for any $k$. We will prove that there exists no graph with at least $2k + 2$ vertices except for these graphs for $k \geq 11$. In the case of Theorem 3 (3) (4), we have no graph for $k \geq 11$. In the case of Theorem 3 (2), trivially $v = 2(k−1) < 2k + 2$. We consider the case of Theorem 3 (1). Let $G$ be the complement of the line graph of a $t$-regular graph with $u$ vertices. Then $G$ is of degree $k = (u/2−2)t + 1$, and has $v = ut/2$ vertices. Therefore $v = ut/2 = u(k−1)/(u−4) \leq 2(k−1) < 2k + 2$ because $u \geq 8$ for $k \geq 11$. Let $G$ be the complement of the line graph of a bipartite semiregular connected graph $(V_1, V_2, E)$. Let $|V_i| = u_i$ and the degree of $x \in V_i$ be $t_i$, where we suppose $t_1 \geq t_2$. Then $G$ is of degree.
$k = (u_1 - 1)t_1 - t_2 + 1 \geq (u_1 - 2)t_1 + 1$, and has $v = u_1t_1$ vertices. If $u_1 = 1$ holds, then $G$ has no edge. For $u_1 > 3$, it is satisfied that
\[ v \leq (1 + \frac{2}{u_1 - 2})(k - 1) \leq 2(k - 1) < 2k + 2 \] (1)
for any $k$. For $u_1 = 3$, we have $t_2 \leq u_1 = 3$ and
\[ v = 3t_1 = \frac{3}{2}(k + t_2 - 1) \leq \frac{3}{2}(k + 2) < 2k + 2 \] (2)
for $k > 2$. For $u_1 = 2$, similarly $t_2 \leq u_1 = 2$ and
\[ v = 2t_1 = 2(k + t_2 - 1) \leq 2k + 2 \] (3)
for any $k$, with equality only if $t_1 = k + 1$, $t_2 = 2$, $u_1 = 2$ and $u_2 = k + 1$. Thus (10) holds.

(3), (5)–(9): Every candidate of maximal graphs comes from Theorem 3 (3) or (4) except for the case of the complete graph in (5). We prove that there does not exist a larger graph which comes from Theorem 3 (1). By inequalities (11–13), the complement of the line graph of a bipartite semiregular graph is not maximal for $k > 2$. We consider the case of the complements of the line graphs of $t$-regular graphs with $u$ vertices. Since $v = k - 1 + 2t$ is at least 12, 15, 18, 21, 24, 27, we have $u - 1 \geq t \geq 5, 5, 6, 7, 8, 9$ for $k = 4, 6, 7, 8, 9, 10$, respectively. Therefore $k = (u/2 - 2)t + 1 \geq (t - 2)(t - 1)/2 \geq 6, 6, 10, 15, 21, 28$ for $k = 4, 6, 7, 8, 9, 10$, respectively. The only parameter $(v, k, u, t) = (15, 6, 6, 5)$ satisfies the conditions and it corresponds to the case of the complete graph in (5).

4 Other Values of $v(k, \lambda)$

When no graph meets the bound given by Theorem 2, other techniques may be necessary to find $v(k, \lambda)$. However, the bound is still useful in reducing the size of graphs which must be checked. In this section we describe several tools which we will use, and then find $v(k, \lambda)$ in a few more cases.

Let $n(k, g)$ denote the minimum possible number of vertices of a $k$-regular graph with girth $g$. A $(k, g)$-cage is a graph which attains this minimum. The following lower bound on $n(k, g)$ due to Tutte [11] will be useful.

**Lemma 1.** Define $n_l(k, g)$ by
\[ n_l(k, g) = \begin{cases} \frac{k(k-1)(g-1)^2 - 2}{k-2} & \text{if } g \text{ is odd,} \\ \frac{2(k-1)^g - 2}{k-2} & \text{if } g \text{ is even.} \end{cases} \]
Then $n(k, g) \geq n_l(k, g)$.
The following lemma is easily verified.

**Lemma 2.** Each of the graphs in Figure 1 has spectral radius greater than 2.

For a graph $G$, a vertex $v \in V(G)$, and a subset $U \subset V(G)$, define the distance $\text{dist}(v, U) = \min_{u \in U} \text{dist}(u, v)$. For an induced subgraph $H$ of $G$, let $\Gamma_i(H)$ and $\Gamma_{\geq i}(H)$ be the sets of vertices in $G$ at distance exactly $i$ and at least $i$ from $V(H)$ in $G$, respectively. Let $\rho(G)$ and $d(G)$ denote the spectral radius and average degree of $G$, respectively. Note that $d(G) \leq \rho(G)$.

**Lemma 3.** Suppose $G$ is a connected, $k$-regular graph with second largest eigenvalue $\lambda_2(G) \leq \lambda < k$, and $H$ is an induced subgraph of $G$ with $d(H) \geq \lambda$. Then for the subgraph $K$ induced by $\Gamma_{\geq 2}(H)$ we have $d(K) \leq \lambda$, with equality only if $d(H) = \lambda_2(G) = \lambda$.

**Proof.** Consider the quotient matrix $Q$ of the partition $\{V(H), \Gamma_1(H), \Gamma_{\geq 2}(H)\}$ of $V(G)$. We have

$$Q = \begin{pmatrix} \alpha & k - \alpha & 0 \\ \gamma & k - (\gamma + \epsilon) & \epsilon \\ 0 & k - \beta & \beta \end{pmatrix},$$

where $\alpha = d(H)$, $\beta = d(K)$, and $\gamma$ and $\epsilon$ are the average numbers of neighbors in $H$ and $K$, respectively, of the vertices in $\Gamma_1(H)$. The eigenvalues of $Q$ interlace those of $G$, so we must have $\lambda_2(Q) \leq \lambda_2(G) \leq \lambda$. It is straightforward to verify that $\lambda_1(Q) = k$ and

$$\lambda_2(Q) = \frac{1}{2} \left( \alpha + \beta - (\gamma + \epsilon) + \sqrt{\Delta} \right),$$

where $\Delta = (\alpha + \beta - (\gamma + \epsilon))^2 - 4(\alpha \beta - \beta \gamma - \alpha \epsilon)$. By hypothesis we have $\alpha \geq \lambda$. If also $\beta \geq \lambda$, then we find that $\alpha = \beta = \lambda_2(Q) = \lambda$, as we will prove below.

Indeed, if both $\alpha > \lambda$ and $\beta > \lambda$, then by Cauchy interlacing $\lambda_2(G) \geq \lambda_2(H + K) > \lambda$, where $H + K$ is the disjoint union of $H$ and $K$, a contradiction. Suppose $\alpha \geq \lambda$ and $\beta \geq \lambda$. If $\alpha = \beta = \lambda$, then (4) becomes $\lambda_2(Q) = \lambda$. Otherwise we must have $\alpha > \beta = \lambda$ or $\beta > \alpha = \lambda$. If $\sqrt{\Delta} \geq \gamma + \epsilon$, then clearly $\lambda_2(Q) > \lambda$, a contradiction. If $\sqrt{\Delta} < \gamma + \epsilon$, then $\Delta < (\gamma + \epsilon)^2$, which implies $(\alpha - \beta)(\alpha - \beta + 2(\epsilon - \gamma)) < 0$. Thus we have either $\alpha > \beta$ and $\epsilon < \gamma - \frac{1}{2}(\alpha - \beta)$, or $\beta > \alpha$ and $\gamma < \epsilon - \frac{1}{2}(\beta - \alpha)$. Suppose the former is true. Then $\beta = \lambda$ and we can write $\alpha = \beta + s = \lambda + s$ and $\epsilon = \gamma - \frac{s}{2} - t$ for some $s, t > 0$. Then (4) becomes

$$\lambda_2(Q) = \frac{1}{4} \left( 4\lambda - 4\gamma + 3s + 2t + \sqrt{\Delta^2} \right),$$

Figure 1: Graphs with spectral radius greater than 2.
where \( \Delta' = 16\gamma^2 + (s - 2t)^2 - 8\gamma(s + 2t) \). If \( \sqrt{\Delta'} > 4\gamma - 3s - 2t \), then clearly \( \lambda_2(Q) > \lambda \), a contradiction. If \( \sqrt{\Delta'} \leq 4\gamma - 3s - 2t \), then \( \Delta' \leq (4\gamma - 3s - 2t)^2 \), which implies \( \gamma \leq \frac{t}{2} + t \).

However, this implies \( \epsilon = \gamma - \frac{t}{2} - t \leq 0 \), a contradiction. If \( \beta > \alpha \) and \( \gamma < \epsilon - \frac{1}{2}(\beta - \alpha) \), the same argument holds (simply swap the roles of \( \alpha \) and \( \beta \) and of \( \gamma \) and \( \epsilon \) in the above argument). Thus we cannot have \( \alpha \geq \lambda \) and \( \beta \geq \lambda \) unless \( \alpha = \beta = \lambda \), so we must have \( \beta < \lambda \) or \( \alpha = \beta = \lambda_2(Q) = \lambda \).

**Lemma 4.** Suppose \( G \) is a connected, \( k \)-regular graph with second largest eigenvalue \( \lambda_2(G) \leq \lambda < k \). If \( G \) contains an induced subgraph \( H \) on \( s \) vertices with \( t \) edges and either \( d(H) \geq \lambda \) or \( \rho(H) > \lambda \), then

\[
|V(G)| \leq s + \frac{2k - \lambda - 1}{k - \lambda}(ks - 2t). \tag{5}
\]

**Proof.** Since \( G \) is \( k \)-regular, there are \( ks - 2t \) edges from \( H \) to \( \Gamma_1(H) \), which implies \( |\Gamma_1(H)| \leq ks - 2t \). We will show that \( |\Gamma_{\geq 2}(H)| \leq \frac{k - 1}{k - \lambda}|\Gamma_1(H)| \), which completes the proof that (5) holds.

First, note that each vertex in \( \Gamma_1(H) \) has a neighbor in \( H \), so each such vertex has at most \( k - 1 \) neighbors in \( \Gamma_{\geq 2}(H) \). Then there are at most \( (k - 1)|\Gamma_1(H)| \) edges from \( \Gamma_1(H) \) to \( \Gamma_{\geq 2}(H) \). If \( d(H) \geq \lambda \) then by Lemma 3 we have \( d(K) \leq \lambda \), where \( K \) is the subgraph induced by \( \Gamma_{\geq 2}(H) \). If not, then \( \rho(H) > \lambda \), so \( \rho(K) \leq \lambda \) (and so also \( d(K) \leq \lambda \)) by eigenvalue interlacing. Since \( G \) is \( k \)-regular, this implies that the average number of neighbors in \( \Gamma_1(H) \) of the vertices in \( \Gamma_{\geq 2}(H) \) is at least \( k - \lambda \), so there are at least \( (k - \lambda)|\Gamma_{\geq 2}(H)| \) edges from \( \Gamma_{\geq 2}(H) \) to \( \Gamma_1(H) \). This completes the proof. \qed

**Proposition 4.** If \( G \) is a connected, 3-regular graph with \( \lambda_2(G) > 1 \), then \( \lambda_2(G) \geq \sqrt{2} \), with equality if and only if \( G \) is the Heawood graph.

**Proof.** We have already seen (Table 2) that \( v(3, \sqrt{2}) = 14 \) and the Heawood graph (the incidence graph of \( PG(2,2) \)) is the unique graph meeting this bound. Thus we need only show that no 3-regular graph has second eigenvalue between 1 and \( \sqrt{2} \). Suppose \( G \) is a 3-regular graph with \( 1 < \lambda_2(G) < \sqrt{2} \). We will show that this yields a contradiction. We have immediately that \( |V(G)| < 14 \). Since \( G \) is 3-regular, this implies \( |V(G)| \leq 12 \).

We first note that the average degree of any cycle is \( 2 > \sqrt{2} > \lambda_2(G) \). If \( G \) has girth 3, then Lemma 4 implies \( |V(G)| \leq \frac{6}{\sqrt{2}}(\sqrt{2} + 10) \approx 9.78 \). Since \( G \) is 3-regular, this implies \( |V(G)| \leq 8 \). Lemma 1 implies that a graph with girth more than 5 has at least 14 vertices, so \( G \) has girth at most 5.

We partition the vertices of \( G \) by \( P_1 = \{V(H), \Gamma_1(H), \Gamma_{\geq 2}(H)\} \), where \( H \) is a subgraph of \( G \) isomorphic to \( C_m \), where \( m \) is the girth of \( G \). This partition has quotient matrix \( Q \) given by

\[
Q = \begin{pmatrix}
2 & 1 & 0 \\
\gamma & 3 - (\alpha + \gamma) & \alpha \\
0 & \beta & 3 - \beta
\end{pmatrix},
\]
where $\gamma |\Gamma_1(H)| = m$ (by counting edges from $H$ to $\Gamma_1(H)$) and $\alpha |\Gamma_1(H)| = \beta |\Gamma_{\geq 2}(H)|$ (by counting edges from $\Gamma_1(H)$ to $\Gamma_{\geq 2}(H)$).

We first suppose $G$ has girth 3. Then $4 \leq |V(G)| \leq 8$. If $|V(G)| = 4$, then $G \cong K_4$, and we have $\lambda_2(G) = -1$. If $|V(G)| = 6$, it is straightforward to show that $G \cong C_3 \square K_2$, where $\square$ denotes the graph Cartesian product, and we have $\lambda_2(G) = 1$. Either case is a contradiction.

If $|V(G)| = 8$ then $\Gamma_1(H)$ has 2 or 3 vertices. If $|\Gamma_1(H)| = 2$, then we have $|\Gamma_{\geq 2}(H)| = 3$, $\gamma = 3/2$, and depending on whether there is an edge in $\Gamma_1(H)$ or not we have $\alpha = 1/2$ or $3/2$, $\beta = 1/3$ or 1, and $\lambda_2(Q) = \frac{1}{3}(\sqrt{13} + 4) \approx 2.54$ or 2, respectively. Either case is a contradiction. If $|\Gamma_1(H)| = 3$, then $|\Gamma_{\geq 2}(H)| = 2$, $\gamma = 1$, and depending on whether there is an edge in $\Gamma_{\geq 2}(H)$ or not we have $\beta = 2$ or 3, $\alpha = 4/3$ or 2, and $\lambda_2(Q) = 5/3$ or $\frac{1}{2}(\sqrt{17} - 1) \approx 1.56$, respectively. Either case is a contradiction. Thus $G$ cannot have girth 3.

Suppose $G$ has girth 4. Then we have $6 \leq |V(G)| \leq 12$. If $|V(G)| = 6$, then $G \cong K_{3,3}$ and we have $\lambda_2(G) = 0$. If $|V(G)| = 8$, then it is straightforward to verify that $G$ must either be the 3-cube $Q_3$ or the graph in Figure 2. In either case we have $\lambda_2(G) = 1$, a contradiction.

![Figure 2: A 3-regular graph on 8 vertices with girth 4.](image)

If $|V(G)| = 10$, then $\Gamma_1(H)$ has 2, 3, or 4 vertices. If $|\Gamma_1(H)| = 2$, then $|\Gamma_{\geq 2}(H)| = 4$, $\gamma = 2$, $\alpha = 1$, $\beta = 1/2$, and $\lambda_2(Q) = \frac{41}{3}(\sqrt{41} + 3) \approx 2.35$, a contradiction. If $|\Gamma_1(H)| = 3$, then $|\Gamma_{\geq 2}(H)| = 3$, $\gamma = 4/3$, and $\alpha = \beta$. Then $\alpha \leq 5/3$ (since $3 - (\alpha + \gamma) \geq 0$) implies $\beta \leq 5/3$, which implies $\Gamma_{\geq 2}(H)$ has at least 2 edges. Since $G$ has girth 4, $\Gamma_{\geq 2}(H)$ cannot have 3 edges, so $\Gamma_{\geq 2}(H)$ has exactly 2 edges, $\alpha = \beta = 5/3$, and $\lambda_2(Q) = \frac{1}{2}((\sqrt{241} + 7) \approx 1.88$, a contradiction. If $|\Gamma_1(H)| = 4$, then $|\Gamma_{\geq 2}(H)| = 2$, $\gamma = 2$, and depending on whether there is an edge in $\Gamma_{\geq 2}(H)$ or not we have $\beta = 2$ or 3, $\alpha = 1$ or 3/2, and $\lambda_2(Q) = \frac{1}{2}((\sqrt{5} + 1) \approx 1.62$ or 3/2, respectively. Either case is a contradiction. If $|V(G)| = 12$, then $\Gamma_1(H)$ must be a coclique on 4 vertices (otherwise there are at most 6 edges from $\Gamma_1(H)$ to $\Gamma_{\geq 2}(H)$, so Lemma 3 implies $|\Gamma_{\geq 2}(H)| < 6/(3 - \sqrt{2}) \approx 3.78$, which implies $|V(G)| < 11.78$, a contradiction). Then we have $|\Gamma_1(H)| = |\Gamma_{\geq 2}(H)| = 4$, $\gamma = 1$, $\alpha = \beta = 2$, and $\lambda_2(Q) = \sqrt{3}$. This is a contradiction, so $G$ cannot have girth 4.

Suppose $G$ has girth 5. Then $10 \leq |V(G)| \leq 12$. The Petersen graph with 10 vertices and $\lambda_2 = 1$ is the unique $(3,5)$-cage (see [18]), so $G$ must have 12 vertices. Note we must have $|\Gamma_1(H)| = 5$ and $\gamma = 1$, since vertices in $H$ cannot have common neighbors outside of $H$. Since $|V(G)| = 12$, we have $|\Gamma_{\geq 2}(H)| = 2$, and depending on whether there is an edge in $\Gamma_{\geq 2}(H)$ or not we have $\beta = 2$ or 3, $\alpha = 4/5$ or 6/5, and $\lambda_2(Q) = \frac{1}{5}(2\sqrt{6} + 3) \approx 1.58$ or $\frac{1}{10}(\sqrt{241} - 1) \approx 1.45$, respectively. Either case is a contradiction.

Thus $G$ cannot exist as described, which completes the proof. \qed
Proposition 5. If $G$ is a connected, 4-regular graph with $\lambda_2(G) > 1$, then $\lambda_2(G) \geq \sqrt{5} - 1$, with equality if and only if $G$ is either the graph in Figure 3 or the circulant graph $C_{10}(1, 4)$.

Figure 3: The 4-regular graph $G$ on 8 vertices with $\lambda_2(G) = \sqrt{5} - 1$.

Proof. It is straightforward to verify that the second eigenvalue of $T(4, 3, (4 - (\sqrt{5} - 1)^2)/\sqrt{5}) = \sqrt{5} - 1$ and $M(4, 3, (4 - (\sqrt{5} - 1)^2)/\sqrt{5}) = 5 + 12\sqrt{5}/(4 - (\sqrt{5} - 1)^2) \approx 15.85$, so by Theorem 2 we have $v(4, \sqrt{5} - 1) \leq 15$. We checked by computer all 4-regular graphs on at most 15 vertices and found that, in each case where $\lambda_2(G) > 1$, we have $\lambda_2(G) \geq \sqrt{5} - 1$, with equality if and only if $G$ is either the graph in Figure 3 or the circulant graph $C_{10}(1, 4)$. \[\square\]

Note that this implies $v(4, \sqrt{5} - 1) = 10$. It would be interesting to find a proof of Proposition 5 which does not require a computer search. For the proof above the computer must check 906,331 graphs.

Richey, Shutty, and Stover conjectured that $v(3, 1.9) = 18$. We confirm this conjecture, and show that there are exactly two graphs meeting this bound.

Proposition 6. If $G$ is a connected, 3-regular graph with second largest eigenvalue $\lambda_2(G) \leq 1.9$, then $|V(G)| \leq 18$, with equality if and only if $G$ is the Pappus graph (see Figure 4(a)) or the graph in Figure 4(b).

Figure 4: The 3-regular graphs on 18 vertices with $\lambda_2 < 1.9$. 

(a) The Pappus graph with second eigenvalue $\sqrt{3}$.

(b) A graph with $\lambda_2 = \gamma \approx 1.8662$, the largest root of $f(x) = x^3 + 2x^2 - 4x - 6$. 

Proof. It is straightforward to verify that the second eigenvalue of \( T(3, 4, 2641/3510) = 19/10 = 1.9 \) and \( M(3, 4, 2641/3510) = 68530/2641 \approx 25.95 \), so by Theorem 2 we have \( v(3, 1.9) \leq 25 \). Since \( G \) is 3-regular, this implies \( v(3, 1.9) \leq 24 \). We note again that any cycle has spectral radius 2. Then, by Lemma 4 if \( G \) has girth 3, 4, 5, or 6, then \( G \) has at most 11.45, 15.27, 19.09, or 22.91 vertices, respectively. Since \( G \) is 3-regular, this implies \( G \) has at most 10, 14, 18, or 22 vertices, respectively. A 3-regular graph of girth 8 has at least 30 vertices by Lemma 1 (or note that the Tutte-Coxeter graph is the unique (3,8)-cage, see [40] and [41]), so we have shown that a 3-regular graph \( G \) with \( \lambda_2(G) \leq 1.9 \) and more than 18 vertices must have girth 6 or 7.

If \( G \) has girth 7, we note that the McGee graph on 24 vertices is the unique (3,7)-cage ([25] and [41]), so \( G \) must be the McGee graph. Since the McGee graph has second eigenvalue 2, we have proved that \( G \) does not have girth 7.

Now, if \( G \) has more than 18 vertices then \( G \) must have girth 6 and at most 22 vertices. Among 3-regular graphs, we checked by computer the 32 graphs with girth 6 on 20 vertices and the 385 graphs with girth 6 on 22 vertices and found that each has second eigenvalue more than 1.9. Thus \( G \) has at most 18 vertices. If \( G \) has 18 vertices, then \( G \) must have girth 5 or 6. Among 3-regular graphs, we checked by computer the 450 graphs with girth 5 on 18 vertices and found that each has second eigenvalue more than 1.9. We checked the 5 graphs with girth 6 on 18 vertices and found that all but two of them have second eigenvalue more than 1.9. The exceptions were the Pappus graph with second eigenvalue \( \sqrt{3} \) and the graph in Figure 4(b) with second eigenvalue \( \gamma \), where \( \gamma \approx 1.8662 \) is the largest root of \( f(x) = x^3 + 2x^2 - 4x - 6 \).

Note that this implies \( v(3, \sqrt{3}) = 18 \) and \( v(3, \gamma \approx 1.8662) = 18 \) (and, of course, \( v(3, 1.9) = 18 \)). It would be nice to find a proof of Proposition 6 without a computer search.

5 Final Remarks

We conclude the paper with some questions and problems for future research.

Problem 1. Determine \( v(k, \sqrt{k}) \) for \( k \geq 3 \).

We have \( T(k, 4, k - \sqrt{k}) = \sqrt{k} \) and \( M(k, 4, k - \sqrt{k}) = 2k^2 + k^{3/2} - k - \sqrt{k} + 1 \), which yields 
\[
v(k, \sqrt{k}) \leq 2k^2 + k^{3/2} - k - \sqrt{k} + 1.
\]
The Odd graph \( O_4 \) meets this bound (see Table 2). We do not know what other graphs, if any, meet this bound. Odd graphs, in general, do not have \( T(k, t, c) \) as a quotient matrix.

Problem 2. Determine \( v(k, \sqrt{2}) \) for \( k \geq 3 \).
Recall that for $k = 3$ we have $v(3, \sqrt{2}) = 14$ and the Heawood is the unique graph meeting this bound. For $k > 3$ we note that Lemma 4 with $H = K_3$ implies that a graph $G$ with $\lambda_2(G) \leq \sqrt{2}$ and girth 3 satisfies $|V(G)| \leq 3(k-1) \left(1 + \frac{k-2}{k-\sqrt{2}}\right)$, and Lemma 4 with $H = K_{1,3}$ implies that such a graph with girth more than 3 satisfies $|V(G)| \leq 4 + 2(2k-3) \left(1 + \frac{k-1}{k-\sqrt{2}}\right)$ (note that in both cases we have $\rho(H) > \lambda_2(G)$). Combining this with Lemma 1 allows one to restrict the search to graphs with certain girth. For $k \geq 7$, $n_l(k, g)$ is larger than these bounds unless the girth is at most 4, and for $k = 4, 5, or 6$, $n_l(k, g)$ is larger than these bounds unless the girth is at most 5. Thus the graphs sought in Problem 2 must have girth at most 5 for $k = 4, 5, 6$ and girth at most 4 for $k \geq 7$.

Problem 3. Among regular graphs, what is the smallest second eigenvalue larger than 1?

Yu [43] found a 3-regular graph $G$ on 16 vertices (see Figure 5) with smallest eigenvalue $\lambda_{\min} = \gamma \approx -2.0391$, where $\gamma$ is the smallest root of $f(x) = x^6 - 3x^5 - 7x^4 + 21x^3 + 13x^2 - 35x - 4$, and moreover proved that there is no connected, 3-regular graph with smallest eigenvalue in the interval $(\gamma, -2)$ (that is, among all connected, 3-regular graphs $G$ has the largest least eigenvalue less than $-2$). Since the second eigenvalue of the complement of a regular graph is $\lambda_2 = -1 - \lambda_{\min}$, the complement $\overline{G}$ of $G$, a 12-regular graph on 16 vertices, has second eigenvalue $\lambda_2(\overline{G}) = -1 - \gamma \approx 1.0391$. We do not know if $\overline{G}$ has smallest second eigenvalue larger than 1 among regular graphs, but it is not unique. Indeed, the complement of the disjoint union $G + kK_4$ of $G$ and $k$ copies of $K_4$ is a connected, $(12 + 4k)$-regular graph on $16 + 4k$ vertices with second eigenvalue $\lambda_2(G + kK_4) = -1 - \gamma$, so we have found an infinite family of regular graphs with second eigenvalue $-1 - \gamma$.

Problem 4. For any integer $k \geq 2$, let $\lambda(k) := (-1 + \sqrt{4k-3})/2$. Then we find that $v(k, \lambda(k)) \leq k^2 + 1$ with equality if and only if the associated graph is a Moore graph of diameter 2. Moore graphs of diameter 2 only exists for $k = 2, 3, 7$, and possibly 57. If $k$ is not 2, 3, 7, 57, then $v(k, \lambda(k)) \leq k^2$. Determine the exact value of $v(k, \lambda(k))$ in these cases.

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