Two models of partial differential equations with discrete and distributed state-dependent delays

ALEXANDER V. REZOUNENKO
Department of Mechanics and Mathematics, Kharkov University,
4, Svobody Sqr., Kharkov, 61077, Ukraine
E-mail : rezounenko@univer.kharkov.ua

Abstract. This work is the first attempt to treat partial differential equations with discrete (concentrated) state-dependent delay. The main idea is to approximate the discrete delay term by a sequence of distributed delay terms (all with state-dependent delays). We study local existence and long-time asymptotic behavior of solutions and prove that the model with distributed delay has a global attractor while the one with discrete delay possesses the trajectory attractor.

Key words: Partial functional differential equation, state-dependent delay, delay selection, global attractor, trajectory attractor.

Mathematics Subject Classification 2000: 35R10, 35B41, 35K57.

1. Introduction

The theory of delay ordinary differential equations has a rich history and still be one of the actively developing branches of the theory of differential equations. We cite a few monographs which are the classical source of fundamental facts and approaches in this field [Hale (1977), Mishkis (1972), Hale and Lunel (1993), Diekmann et al. (1995), Azbelev et al. (1991)].

Another developed branch of the theory of differential equations is the theory of partial differential equations (PDEs). We refer to [Babin and Vishik (1992), Temam (1988), Chueshov (1999)] where many deep results on the qualitative theory of PDEs are presented.

These fields have very much in common when we are interested in the qualitative theory. It is not surprisingly since both delay equations and PDEs can be treated as abstract dynamical systems in infinite-dimensional spaces. Recently, some efforts have...
been applied to develop the theory of PDEs with delay. Such equations are naturally more difficult since they are infinite-dimensional in both time and space variables. We refer to the monograph [Wu (1996)] and to a few articles which are close to the subject of this work [Travis and Webb (1974), Chueshov (1992), Chueshov and Rezounenko (1995), Boutet de Monvel, Chueshov and Rezounenko (1998), Rezounenko and Wu (2005)].

Recently, the theory of state-dependent ordinary differential equations (equations where delay depends on the state of the system) has attracted attention of many researchers. We refer to [Nussbaum and Mallet (1992), Nussbaum and Mallet (1996), Mallet-Paret et al. (1994), Walther (2002), Walther (2003)] and references therein. The approach in these works essentially based on the Lipschitz continuity in time of solutions of ordinary differential equations. Unfortunately, the last property does not hold for solutions of PDEs, so one has to propose a new approach.

The first attempt to treat PDEs with state-dependent delay has been done in [Rezounenko and Wu (2005)]. There was proposed a model of PDEs with distributed state-dependent (state-selective) delay; the existence and uniqueness of solutions have been proved and the asymptotic behavior of solutions has been studied.

The present article is the first attempt to treat PDEs with discrete (state-dependent) delay. We propose two models of PDEs with discrete and distributed (state-dependent) delays and study their local and long-time asymptotic behavior. The main idea of the present work is to approximate the discrete delay term by a sequence of distributed delay terms (cf. the forms of $F$ and $F_n$ in (1) and (2)). We first develop the techniques for studying PDEs with distributed (state-dependent) delay and than apply it to investigations of PDEs with discrete (state-dependent) delay. We propose a sequence of simple distributed delay terms constructed as integrals over $(-r, 0)$ with step functions as kernels of these integrals. More precisely, using the well-known Lebesgue theorem we approximate the value of $y(s)$ for almost all $s \in (a, b)$ by the sequence $\left\{ \varepsilon_n^{-1} \int_{s-\varepsilon_n}^{s} y(\tau) d\tau \right\}_{n=1}^{\infty}$, where $\varepsilon_n \to 0_+$ as $n \to \infty$. These integrals can be rewritten in the form $\int_{0}^{r} y(s + \theta) \cdot \tilde{\xi}^n(\theta, s) d\theta$, where $\tilde{\xi}^n$ is the step-function $\tilde{\xi}^n(\theta, s) \equiv \varepsilon_n^{-1}$ for $\theta \in [s - \varepsilon_n, s]$ and $\tilde{\xi}^n(\theta, s) \equiv 0$ for $\theta \notin [s - \varepsilon_n, s]$.

For the model with distributed delay we prove (section 3) the existence and uniqueness theorems, construct an evolution semigroup and obtain the existence of global attractor. Since for the model with discrete state-dependent delay the uniqueness of solutions is not assumed, to study the long-time asymptotic dynamics of these solutions we apply (section 4) the theory of trajectory attractors (see [Chepyzhov and Vishik (1997)]) and
references therein).

The obtained results can be applied to the diffusive Nicholson’s blowflies equation (see e.g. So and Yang (1998), So, Wu and Yang (2000)) with state-dependent (both discrete and distributed) delays.

2. Formulation of the models with discrete and distributed delays

Let us start with the following non-local partial differential equation with state-dependent discrete delay

\[
\frac{\partial}{\partial t} u(t, x) + Au(t, x) + du(t, x) = \int_{\Omega} b(u(t - \eta(u(t), u_t), y))f(x - y)dy \equiv (F(u_t))(x), \quad x \in \Omega, \tag{1}
\]

where \( A \) is a densely-defined self-adjoint positive linear operator with domain \( D(A) \subset L^2(\Omega) \) and with compact resolvent, so \( A : D(A) \to L^2(\Omega) \) generates an analytic semigroup, \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^n \), \( f : \Omega - \Omega \to R \) is a bounded function to be specified later, \( b : R \to R \) is a locally Lipschitz bounded map \((|b(w)| \leq C_b \text{ with } C_b \geq 0)\), \( d \) is a positive constant. The function \( \eta(\cdot, \cdot) : L^2(\Omega) \times L^2(\eta(0, 0); L^2(\Omega)) \to R \) represents the state-dependent discrete delay. We denote for short \( H \equiv L^2(\Omega) \times L^2(-r, 0; L^2(\Omega)) \).

Consider the following non-local partial differential equation with state-dependent distributed delay

\[
\frac{\partial}{\partial t} u(t, x) + Au(t, x) + du(t, x) = \int_0^{-r} \{ \int_{\Omega} b(u(t + \theta, y))f(x - y)dy \} \xi^n(\theta, u(t), u_t)d\theta \\
\equiv (F_n(u_t))(x), \quad x \in \Omega, \tag{2}
\]

where the function \( \xi^n(\cdot, \cdot, \cdot) : [-r, 0] \times H \to R \) represents the state-dependent distributed delay.

We consider equations (1) or (2) with the following initial conditions

\[
u(0+) = u^0 \in L^2(\Omega), \quad u|_{(-r,0)} = \varphi \in L^2(0, T; L^2(\Omega)). \tag{3}
\]

3. Distributed delay problem

In this section we study the existence and properties of solutions for distributed delay problem (2), (3).

**Definition 1.** A function \( u \) is a weak solution of problem (2) subject to the initial conditions (3) on an interval \([0, T]\) if \( u \in L^\infty(0, T; L^2(\Omega)) \cap L^2(-r, 0; L^2(\Omega)) \cap L^2(0, T; D(A_1^2)), \)
Theorem 1. Assume that $u(\theta) = \varphi(\theta)$ for $\theta \in (-r, 0)$ and

$$-\int_0^T \langle u, \dot{v} \rangle dt + \int_0^T \langle A^{\frac{1}{2}}u, A^{\frac{1}{2}}v \rangle dt + \int_0^T \langle du - F_n(u_t), v \rangle dt = -\langle u^0, v(0) \rangle$$

(4)

for any function $v \in L^2(0, T; D(A^{\frac{1}{2}}))$ with $\dot{v} \in L^2(0, T; D(A^{-\frac{1}{2}}))$ and $v(T) = 0$.

Proof of Theorem 1. Let us denote by $A_{e^k}$ a Galerkin approximate solution of order $m$ for the problem (2), subject to the initial conditions (3) has a weak solution $u$.

(i) $b : R \rightarrow R$ is locally Lipschitz and bounded i.e., there exists a constant $C_b$ so that $|b(w)| \leq C_b$ for all $w \in R$;

(ii) $f : \Omega \rightarrow R$ is bounded;

(iii) $\xi^n : [-r, 0] \times L^2(\Omega) \times L^2(-r, 0; L^2(\Omega)) \rightarrow R$ satisfies the following conditions:

a) for any $M > 0$ there exists $L_{\xi,M,n}$ so that for all $(v^i, \psi^i) \in H$ satisfying $||v^i||^2 + \int_0^r ||\psi^i(s)||^2 ds \leq M^2$, $i = 1, 2$ one has

$$\int_{-r}^0 |\xi^n(\theta, v^1, \psi^1) - \xi^n(\theta, v^2, \psi^2)| d\theta \leq L_{\xi,M,n} \left( ||v^1 - v^2||^2 + \int_{-r}^0 ||\psi^1(s) - \psi^2(s)||^2 ds \right)^{1/2},$$

(5)

b) there exists $C_{\xi,1} > 0$ so that

$$||\xi^n(\cdot, v, \psi)||_{L^1(-r,0)} \leq C_{\xi,1} \text{ for all } (v, \psi) \in H.$$  

(6)

Then for any $(u^0, \varphi) \in H \equiv L^2(\Omega) \times L^2(-r, 0; L^2(\Omega))$ the problem (2) subject to the initial conditions (3) has a weak solution $u(t)$ on every given interval $[0, T]$ which satisfies

$$u(t) \in C([0, T]; L^2(\Omega)).$$

(7)

Proof of Theorem 1. Let us denote by $\{e_k\}_{k=1}^\infty$ an orthonormal basis of $L^2(\Omega)$ such that $Ae_k = \lambda_k e_k$, $0 < \lambda_1 < \ldots < \lambda_k \rightarrow +\infty$. We say that function $u^m(t, x) = \sum_{k=1}^m g_{k,m}(t)e_k(x)$ is a Galerkin approximate solution of order $m$ for the problem (2), (3) if

$$\begin{cases}
\langle u^m + Au^m + du^m - F_n(u_t^m), e_k \rangle = 0, \\
\langle u^m(0+), e_k \rangle = \langle u^0, e_k \rangle, \quad \langle u^m(\theta), e_k \rangle = \langle \varphi(\theta), e_k \rangle, \quad \forall \theta \in (-r, 0)
\end{cases}$$

(8)

$k = 1, \ldots, m$. Here $g_{k,m} \in C^1(0, T; R) \cap L^2(-r, T; R)$ with $g_{k,m}(t)$ being absolutely continuous.
Equations \([8]\) for fixed \(m\) and \(n\) can be rewritten as the following system for the \(m\)-dimensional vector-function \(v(t) = v^m(t) = (g_{1,m}(t), \ldots, g_{m,m}(t))^T\):

\[
\dot{v}(t) = \hat{f}(v(t)) + \int_0^t p(v(t + \theta))\tilde{\xi}^n(\theta, v(t), v_t)d\theta,
\]

where function \(\tilde{\xi}^n\) satisfies properties similar to \([5]\), \([6]\) if one uses \(|\cdot|_{R^m}\) instead of \(|\cdot|_{L^2(\Omega)}\).

We notice that \(\|u^m(t, \cdot)\|_{L^2(\Omega)}^2 = \sum_{k=1}^m g_{k,m}(t) = |v(t)|_{R^m}^2\).

Under the above assumptions, the functions \(\hat{f}\) and \(p\) are locally Lipschitz, \(|p(s)| \leq c_2\) for \(s \in R\), Therefore, for any initial data \(\varphi \in L^2(-r, 0; R^m)\), \(a \in R^m\) Theorem 6 and Remark 9 from Rezounenko (2004) give that there exists \(\alpha > 0\) and a unique solution of \([9]\) \(v \in L^2(-r, \alpha; R^m)\) such that \(v_0 = \varphi\) and \(v(0) = a\), and \(v|_{[0, \alpha]} \in C([0, \alpha]; R^m)\) (for more details see Rezounenko (2004)).

It is easy to get from the boundedness of \(b\) and \([6]\) that

\[
|\langle F_n(u_t), v \rangle_{L^2(\Omega)}| \leq M_f|\Omega|^{3/2}C_bC_{\xi,1} \cdot \|v\|.
\]

Now, we try to get an \(a\)-priori estimate for the Galerkin approximate solutions for the problem \([2], [3]\). We multiply \([8]\) by \(g_{k,m}\) and sum over \(k = 1, \ldots, m\). Hence for \(u(t) = u^m(t)\) and \(t \in (0, \alpha] \equiv (0, \alpha(m)]\), the local existence interval for \(u^m(t)\), we get

\[
\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \|A^{1/2}u(t)\|^2 + d\|u(t)\|^2 \leq |\langle F_n(u_t), u(t) \rangle|.
\]

Using \([10]\), we obtain

\[
\frac{d}{dt}\|u(t)\|^2 + 2\|A^{1/2}u(t)\|^2 \leq \tilde{k}_1\|u(t)\|^2 + \tilde{k}_3.
\]

Since \(\frac{d}{dt}\|u(t)\|^2 + 2\|A^{1/2}u(t)\|^2 = \frac{d}{dt}\left(\|u(t)\|^2 + 2\int_0^t \|A^{1/2}u(\tau)\|^2d\tau + \tilde{k}_3\right)\), we denote by \(\chi(t) \equiv \|u(t)\|^2 + 2\int_0^t \|A^{1/2}u(\tau)\|^2d\tau + \tilde{k}_3\) and rewrite the last estimate as follows \(\frac{d}{dt}\chi(t) \leq \tilde{k}_1 \cdot \chi(t)\). Multiplying it by \(e^{-\tilde{k}_1t}\), one gets \(\frac{d}{dt}\left(e^{-\tilde{k}_1t}\chi(t)\right) \leq 0\). Integrating from 0 to \(t\) and then multiplying by \(e^{\tilde{k}_1t}\), we obtain \(\chi(t) \leq \left(\|u(0)\|^2 + \tilde{k}_3\right)e^{\tilde{k}_1t}\). So, we have the \(a\) -priori estimate

\[
\|u(t)\|^2 + 2\int_0^t \|A^{1/2}u(\tau)\|^2d\tau \leq \left(\|u(0)\|^2 + \tilde{k}_3\right)e^{\tilde{k}_1t} - \tilde{k}_3.
\]

Estimate \([13]\) gives that, for \(u^0 \in L^2(\Omega)\) the family of approximate solutions \(\{u^m(t)\}_{m=1}^\infty\) is uniformly (with respect to \(m \in N\)) bounded in the space \(L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; D(A^{1/2}))\), where \(D(A^{1/2})\) is the domain of the operator \(A^{1/2}\) and \([0, T]\) is the local
existence interval. From (13) we also get the continuation of \( u^m(t) \) on any interval, so (13) holds for all \( t > 0 \).

Using the definition of Galerkin approximate solutions (8) and their property (13), we can integrate over \([0, T]\) to obtain
\[
\int_0^T \| A^{-1/2} \dot{u}^m(\tau) \|^2 d\tau \leq C_T \text{ for any } T.
\]
These properties of the family \( \{u^m(t)\}_{m=1}^{\infty} \) give that \( \{(u^m(t); \dot{u}^m(t))\}_{m=1}^{\infty} \) is a bounded sequence in the space
\[
X_T \equiv L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; D(A^{1/2})) \times L^2(0, T; D(A^{-1/2})).
\]
Then there exist a function \((u(t); \dot{u}(t))\) and a subsequence \(\{u^{m_k}\} \subset \{u^m\}\) such that
\[
(u^{m_k}; \dot{u}^{m_k}) \text{-weakly converges to } (u; \dot{u}) \text{ in the space } X_T.
\]
By a standard argument (using the strong convergence \( u^{m_k} \to u \) in the space \( L^2(0, T; L^2(\Omega)) \) which follows from (15) and the Doubinskii’s theorem, one can show (see e.g. Lions (1969), Chueshov (1999) and Rezounenko (1997)) that any *-weak limit is a solution of \( (2) \) subject to the initial conditions \( (3) \). To prove the continuity of weak solutions we use the well-known

**Proposition 1** (see e.g., Proposition 1.2 in [Showalter (1997)]). Let the Banach space \( V \) be dense and continuously embedded in the Hilbert space \( X \); identify \( X = X^* \) so that \( V \hookrightarrow X \hookrightarrow V^* \). Then the Banach space \( W_p(0, T) \equiv \{u \in L^p(0, T; V) : \dot{u} \in L^q(0, T; V^*)\} \) (here \( p^{-1} + q^{-1} = 1 \)) is contained in \( C([0, T]; X) \).

In our case \( X = L^2(\Omega), V = D(A^{1/2}), V^* = D(A^{-1/2}), p = q = 1/2 \) (see (14),(15)). Hence Proposition 1 gives (7). The proof of Theorem 1 is complete. ■

Now we describe a sufficient condition for the uniqueness of weak solutions.

**Theorem 2.** Assume that functions \( b \) and \( f \) are as in Theorem 1 (satisfy properties (i),(ii)), function \( \xi^n \) satisfies property (iii)-a) and
\[
\xi^n(\cdot, v, \psi) \in L^\infty(\Omega) \text{ for all } (v, \psi) \in H.
\]
Then solution of (2), (3) given by Theorem 1 is unique.

**Proof of Theorem 2.** Let \( u^1 \) and \( u^2 \) be two solutions of (2), (3). Below we denote for short \( w(t) = w^{n,m}(t) = u^{1,n,m}(t) - u^{2,n,m}(t) \) - the difference of corresponding Galerkin approximate solutions. Hence
\[
\frac{d}{dt} ||w(t)||^2 + 2 ||A^{1/2}w(t)||^2 + 2d ||w(t)||^2 = \langle F_n(u^1_t) - F_n(u^2_t), w(t) \rangle.
\]
Let us consider the difference $\langle F_n(u^1) - F_n(u^2), w(t) \rangle$ in details (see (1), (2)).

$$\langle F_n(u^1) - F_n(u^2), w(t) \rangle \equiv \int_{\Omega} \left[ \int_{-r}^{0} \left\{ \int_{\Omega} b(u (t + \theta, y)) f(x - y)dy \right\} \xi^n(\theta, u^1(t), u^1_\theta) d\theta - \int_{-r}^{0} \left\{ \int_{\Omega} b(u^2(t + \theta, y)) f(x - y)dy \right\} \xi^n(\theta, u^2(t), u^2_\theta) d\theta \right] \cdot w(t, x) dx$$

$$= \int_{\Omega} \left[ \int_{-r}^{0} \left\{ \int_{\Omega} b(u(t + \theta, y)) f(x - y)dy \right\} \xi^n(\theta, u^1(t), u^1_\theta) d\theta - \int_{-r}^{0} \left\{ \int_{\Omega} b(u^2(t + \theta, y)) f(x - y)dy \right\} \xi^n(\theta, u^1(t), u^1_\theta) d\theta \right] \cdot w(t, x) dx,$n

$$+ \int_{\Omega} \left[ \int_{-r}^{0} \left\{ \int_{\Omega} b(u^2(t + \theta, y)) f(x - y)dy \right\} \xi^n(\theta, u^2(t), u^2_\theta) d\theta - \int_{-r}^{0} \left\{ \int_{\Omega} b(u^2(t + \theta, y)) f(x - y)dy \right\} \xi^n(\theta, u^2(t), u^2_\theta) d\theta \right] \cdot w(t, x) dx.$$n

Using the local Lipschitz property of $b$, (13) and (5), one easily checks that there are positive constants $C_3, C_4$ such that

$$|\langle F_n(u^1) - F_n(u^2), w(t) \rangle| \leq C_3 \|w(t)\|^2 + C_4 \int_{-r}^{0} \|w(t + \theta)\|^2 d\theta$$

$$\leq C_3 \|w(t)\|^2 + C_4 \int_{-r}^{t} \|w(s)\|^2 ds = C_3 \|w(t)\|^2 + C_4 \left( \int_{-r}^{0} \|w(\theta)\|^2 d\theta + \int_{0}^{t} \|w(s)\|^2 ds \right).$$

The last estimate, (17) and $\|A^{1/2}v\|^2 \geq \lambda_1 \|v\|^2$ give

$$\frac{d}{dt} \|w(t)\|^2 + 2(\lambda_1 + d) \|w(t)\|^2 \leq C_3 \|w(t)\|^2 + C_4 \left( \int_{-r}^{0} \|w(\theta)\|^2 d\theta + \int_{0}^{t} \|w(s)\|^2 ds \right).$$

We rewrite this as

$$\frac{d}{dt} \left[ \|w(t)\|^2 + 2(\lambda_1 + d) \int_{0}^{t} \|w(s)\|^2 ds \right] \leq C_3 \|w(t)\|^2 + C_4 \left( \int_{-r}^{0} \|w(\theta)\|^2 d\theta + \int_{0}^{t} \|w(s)\|^2 ds \right).$$

Hence there exists $C_5 > 0$, such that for $Z(t) \equiv \|w(t)\|^2 + 2(\lambda_1 + d) \int_{0}^{t} \|w(s)\|^2 ds$, we have

$$\frac{d}{dt} Z(t) \leq C_5 Z(t) + C_4 \int_{-r}^{0} \|w(\theta)\|^2 d\theta.$$

Gronwall lemma implies

$$Z(t) \leq \left( \|w(0)\|^2 + C_4 C_5^{-1} \int_{-r}^{0} \|w(\theta)\|^2 d\theta \right) \cdot e^{C_5 t}. \quad (18)$$

The last estimate allows one to apply the well-known

**Proposition 2.** [Yosida (1965), Theorem 9] Let $X$ be a Banach space. Then any *-weak convergent sequence $\{w_k\}_{n=1}^\infty \in X^*$ *-weak converges to an element $w_\infty \in X^*$ and $\|w_\infty\|_X \leq \liminf_{n \to \infty} \|w_n\|_X$. 

7
Hence, for the difference \( u^1(t) - u^2(t) \) of two solutions we have

\[
\|u^1(t) - u^2(t)\|^2 + 2(\lambda_1 + d) \int_0^t \|u^1(s) - u^2(s)\|^2 ds \\
\leq \left( \|u^1(0) - u^2(0)\|^2 + C_4C_5^{-1} \int_r^0 \|\varphi^1(\theta) - \varphi^2(\theta)\|^2 d\theta \right) \cdot e^{C_5 t}.
\]

(19)

We notice that by (1) the difference \( \|u^1(t) - u^2(t)\| \) makes sense for all \( t \in [0, T], \forall T > 0 \). The last estimate gives the uniqueness of solutions and completes the proof of Theorem 2.

Theorems 1 and 2 allow us to define the evolution semigroup \( S_t : H \to H \), with \( H \equiv L^2(\Omega) \times L^2(-r, 0; L^2(\Omega)) \), by the formula \( S_t(u^0; \varphi) \equiv (u(t); u(t + \theta)), \theta \in (-r, 0) \), where \( u(t) \) is the weak solution of (2), (3). The continuity of the semigroup with respect to time follows from (7), and with respect to initial conditions from (19).

For the study of long-time asymptotic properties of the above evolution semigroup we recall (see e.g. Babin and Vishik (1969), Temam (1988))

**Definition 2.** A global attractor of the semigroup \( S_t \) is a closed bounded set \( U \) in \( H \),

strictly invariant \((S_tU = U \text{ for any } t \geq 0)\), such that for any bounded set \( B \subset H \) we have

\[
\lim_{t \to +\infty} \sup \{ \text{dist}_H(S_t y, U), y \in B \} = 0.
\]

As in [Babin and Vishik (1992), Chueshov (1992), Chueshov (1999), Rezounenko (1997)] (see also [Rezounenko and Wu (2005)]) we prove

**Theorem 3.** Under the assumptions of Theorems 1 and 2, the dynamical system \((S_t; H)\) has a compact global attractor \( U \) which is a bounded set in the space \( H_1 \equiv D(A^\alpha) \times W \),

where \( W = \{ \varphi : \varphi \in L^\infty(-r, 0; D(A^\alpha)), \dot{\varphi} \in L^\infty(-r, 0; D(A^{\alpha-1})) \}, \alpha \leq \frac{1}{2} \).

4. Discrete delay problem

Consider the function \( \eta : H \to R \) (as before \( H \equiv L^2(\Omega) \times L^2(-r, 0; L^2(\Omega)) \)) which represents the state-dependent discrete delay in equation (1). Let us fix a positive sequence \( \{\varepsilon_n\}_{n=1}^\infty \subset R_+ \) such that \( \varepsilon_n \to 0_+ \) and define the sequence of functions \( \xi^n : [-r, 0] \times H \to R \) as follows

\[
\xi^n(\theta, a, \varphi) = \begin{cases} 
1/\varepsilon_n, & \theta \in [-\eta(a, \varphi) - \varepsilon_n, -\eta(a, \varphi)]; \\
0, & \theta \notin [-\eta(a, \varphi) - \varepsilon_n, -\eta(a, \varphi)]; \\
\end{cases} \quad \text{with} \quad \varepsilon_n > 0.
\]

(20)

For example, such functions can be constructed as composition

\[
\xi^n(\theta, a, \varphi) = \tilde{\xi}^n(\theta, -\eta(a, \varphi)),
\]

(21)
where \( \tilde{\xi}_n(\theta, s) : [-r, 0] \times R \to R \) are the step-functions (see figure below)

\[
\tilde{\xi}_n(\theta, s) \equiv \begin{cases} 
1/\varepsilon_n, & \theta \in [s - \varepsilon_n, s]; \\
0, & \theta \notin [s - \varepsilon_n, s],
\end{cases} \quad \text{with } \varepsilon_n > 0.
\]

\[\begin{array}{c}
\tilde{\xi}_n(\theta, s) = \frac{1}{\varepsilon_n}, \quad \tilde{\xi}_n^{k}(\theta, s) = \frac{1}{\varepsilon_k} \\
\tilde{\xi}_k(\theta, s) = \frac{1}{\varepsilon_k}, \quad \tilde{\xi}_n^k(\theta, s) = \frac{1}{\varepsilon_n}
\end{array}\]

Figure 1: Graph of functions \( \tilde{\xi}_n(\theta, s) \) and \( \tilde{\xi}_k(\theta, s) \) with \( 0 < \varepsilon_k < \varepsilon_n \).

**Remark 1.** It is easy to see that functions \( \xi^n \), defined in (20), satisfy all the assumptions of Theorem 2, hence weak solution of (2), (3) is unique for all \( n \).

**Remark 2.** We notice that functions \( \xi^n \), defined in (20) do not satisfy neither assumption (iii)-a) nor assumption (iii)-b) of Theorem 1 from [Rezounenko and Wu (2005)], but do satisfy the ones of Theorem 1 given in Section 3.

Our idea is to use the sequence of equations (2) with the right-hand sides \( F_n \) (with functions \( \xi^n \) defined in (20)) to treat equation (1). This idea is based on the following well-known fact.

Let \( X \) be a Banach space, \( y \in L^1(0, T; X) \) and define the primitive \( Y(t) \equiv \int_0^t y(s)ds, 0 \leq t \leq T < +\infty \).

**Proposition 3. (Lebesgue theorem)** [Yosida (1965), Showalter (1997)] At a.e., \( t \in (0, T) \), \( Y \) is (strongly) differentiable with \( Y'(t) \equiv \lim_{\varepsilon \to 0} \varepsilon^{-1}(Y(t + \varepsilon) - Y(t)) = y(t) \).

That gives for any \( (a, \varphi) \in H \)

\[
y(t - \eta(a, \varphi)) = \lim_{n \to \infty} \int_{-r}^0 y(t + \theta) : \tilde{\xi}_n(\theta, -\eta(a, \varphi))d\theta = \lim_{n \to \infty} \int_{-r}^0 y(t + \theta) : \xi^n(\theta, a, \varphi)d\theta. \tag{22}
\]

**Remark 3.** Let us explain how to get (22). If we choose in Lebesgue theorem \( \varepsilon = -\varepsilon_n < 0 \) and \( t = s \), one obtains

\[
y(s) = \lim_{\varepsilon \to 0_-} \varepsilon^{-1} \int_s^{s+\varepsilon} y(\theta)d\theta = (\varepsilon)^{-1} \int_{s+\varepsilon}^s y(\theta)d\theta = \lim_{\varepsilon \to 0_-} (\varepsilon)^{-1} \int_\varepsilon^0 y(s + \theta)d\theta
\]
We use the last equality for \( h = \eta(a, \varphi) \) and \( s = t - h = t - \eta(a, \varphi) \) to get the first equality in (22). The second equality in (22) follows from (21).

In Definitions 1 and 3 we are interested in \( \eta \). The last property reflects the main idea to approximate the discrete delay term (with delay \( \eta(a, \varphi) \)) by a sequence of distributed delay terms (cf. the forms of \( F \) and \( F_n \) in (1) and (2)). In Definitions 1 and 3 we are interested in \( X = L^2(\Omega) \).

**Definition 3.** A function \( u \) is a weak limiting solution of problem (1) subject to the initial conditions (3) on an interval \([0, T]\) if \( u \in L^\infty(0, T; X) \cap L^2(\Omega) \cap L^2(0, T; D(A^{1/2})) \) and \( \hat{u} \) weakly converges to \( (u, \hat{u}) \) in the space \( X_T \), when \( \min\{n, m\} \to \infty \) (see (14), (15)). Here \( u^{n,m} \) is the Galerkin approximate solution (see (3)) of order \( m \) for the problem (2), (3) (the right-hand side of (2) is \( F_n \) with functions \( \xi^n \) defined in (20)).

**Remark 4.** It is important to note that if we formulate in the similar manner the definition for a fixed \( n \) and condition \( m \to \infty \) instead of \( \min\{n, m\} \to \infty \) (see (23)), then we get the definition of a weak solution of (2), (3) which is equivalent to Definition 1. It follows from (14) and the uniqueness of weak solutions (see Remark 1).

Now we prove

**Theorem 4.** Assume that functions \( b \) and \( f \) are as in Theorem 1 (satisfy properties (i),(ii)) and function \( \eta : H \to [0, r] \) is locally Lipschitz i.e. for any \( M > 0 \) there exists \( L_{n,M} \) so that for all \( (v^i, \psi^i) \in H \) satisfying \( ||v^i||^2 + \int_0^r ||\psi^i(s)||^2 ds \leq M^2, i = 1, 2 \) one has

\[
|\eta(v^1, \psi^1) - \eta(v^2, \psi^2)| \leq L_{n,M} \left( ||v^1 - v^2||^2 + \int_{-r}^0 ||\psi^1(s) - \psi^2(s)||^2 ds \right)^{1/2}, \tag{24}
\]

Then for any \((u^0, \varphi) \in H\) the problem (1) subject to the initial conditions (3) has a weak limiting solution on every given interval \([0, T]\) and satisfies \( u(t) \in C([0, T]; L^2(\Omega)). \)
Proof of Theorem 4. It is easy to check that property (24) implies that all functions $\xi^n$, defined in (20), satisfy properties a), b) of Theorem 1 with $L_{\xi,M,n} = 2\varepsilon^{-1}_n L_{0,M}$ and $C_{\xi,1} = 1$ for all $n$.

Now we consider any fixed sequence $\{(n_i; m_i)\}_{i=1}^{\infty}$ such that $\text{min}\{n_i; m_i\} \to \infty$ as $i \to \infty$. Consider the family of Galerkin approximate solutions $\{u^{n_i; m_i}\}_{i=1}^{\infty}$ (see (8)) all constructed for the same initial data (3). The a-priori estimate (13) with constants $\tilde{k}_1, \tilde{k}_3$ independent of $n$ and $m$ gives that $\{(u^{n_i; m_i}; \dot{u}^{n_i; m_i})\}_{i=1}^{\infty}$ is a bounded sequence in the space $X_T$ (see (14)). Hence there exists a *-weak convergent subsequence, which converges (by Definition 3) to a weak limiting solution of (1), (3). The continuity of a weak limiting solution follows from Proposition 1. The proof of Theorem 4 is complete.

To study the long-time asymptotic dynamics of solutions to (1), (3) we apply the theory of trajectory attractors (see Chepyzhov and Vishik (1997) and references therein).

Consider the following Banach space

$$\mathcal{F}^b_+ \equiv \left\{ w(\cdot) \mid w(\cdot) \in L^b_2(R^+; D(A^{1/2})) \cap L_\infty(R^+; L^2(\Omega)), \quad \dot{w}(\cdot) \in L^b_2(R^+; D(A^{-1/2})) \right\}$$

with the norm

$$||w||_{\mathcal{F}^b_+} = ||w||_{L^b_2(R^+; D(A^{1/2}))} + ||w||_{L_\infty(R^+; L^2(\Omega))} + ||\dot{w}||_{L^b_2(R^+; D(A^{-1/2}))},$$

where

$$||w||_{L^b_2(R^+; D(A^\alpha))} \equiv \sup_{h \geq 0} \int_h^{h+1} ||A^\alpha w(s)||^2 ds, \quad \text{for } \alpha = 1/2 \text{ and } \alpha = -1/2.$$

Now we consider a wider space

$$\mathcal{F}^\text{loc}_+ \equiv \left\{ w(\cdot) \mid w(\cdot) \in L^\text{loc}_2(R^+; D(A^{1/2})) \cap L^\text{loc}_\infty(R^+; L^2(\Omega)), \quad \dot{w}(\cdot) \in L^\text{loc}_2(R^+; D(A^{-1/2})) \right\}$$

equipped with the local *-weak topology and denote it by $\Theta^\text{loc}_+$. More precisely, a sequence $\{w^k\} \subset \mathcal{F}^\text{loc}_+$ converges in $\Theta^\text{loc}_+$ to $w \in \mathcal{F}^\text{loc}_+$ when $k \to \infty$ if for any $T > 0$

$$(w^k; \dot{w}^k) \quad \text{*-weakly converges to } \quad (w; \dot{w}) \quad \text{in the space } \quad X_T \quad (25)$$

(see (14)).

Remark 5. It is easy to see that $\mathcal{F}^b_+ \subset \Theta^\text{loc}_+$ and any ball $B_R = \{ w(\cdot) \in \mathcal{F}^b_+ \mid ||w||_{\mathcal{F}^b_+} \leq R \}$ is compact in $\Theta^\text{loc}_+$.
Definition 4. The translation semigroup \(\{T(h), h \geq 0\}\) acting on the space \(L^2_0(R_+; D(A^{1/2})) \cap L^\infty_0(R_+; L^2_0(\Omega))\) is defined as the set of translations along the time axis i.e.,

\[ T(h)w(\cdot) \equiv w(\cdot + h), \ h \geq 0. \]

It is evidently that the family \(\{T(h), h \geq 0\}\) is indeed a semigroup i.e., \(T(h_1 + h_2) = T(h_1)T(h_2)\) for any \(h_1, h_2 \geq 0\) and \(T(0) = Id\) – identical operator. It is also easy to see that the semigroup \(\{T(h), h \geq 0\}\) is continuous in the topology \(\Theta^\text{loc}_+\).

Definition 5. Trajectory space \(K^+\) for equation (1) is the space of functions \(u \in F^b_+\) such that for any \(T > 0\) the restriction \(u|_{[0,T]}\) is a weak limiting solution of (1), (3).

Remark 6. It is easy to see that trajectory space \(K^+\) is invariant under the translation semigroup \(\{T(h), h \geq 0\}\) i.e., \(T(h)K^+ \subset K^+, \forall h \geq 0\).

Remark 7. By Definition 3, it is not too hard to prove that trajectory space \(K^+\) is closed in the topology \(\Theta^\text{loc}_+\).

Definition 6. A set \(P \subset F^b_+\) is said to be an attracting set for the semigroup \(\{T(h), h \geq 0\}\) on \(K^+\) in the topology \(\Theta^\text{loc}_+\) if for any bounded in \(F^b_+\) set \(B \subset K^+\) one has \(T(h)B \to P\) in the topology \(\Theta^\text{loc}_+\) when \(h \to +\infty\).

Definition 7. A set \(U \subset K^+\) is said to be a trajectory attractor of semigroup \(\{T(h), h \geq 0\}\) if

1) \(U\) is bounded in \(F^b_+\) and compact in \(\Theta^\text{loc}_+\);

2) \(U\) is strictly invariant under the semigroup \(\{T(h), h \geq 0\}\) i.e., \(T(h)U = U\) for all \(h \geq 0\);

3) \(U\) is an attracting set for the semigroup \(\{T(h), h \geq 0\}\) on \(K^+\) in the topology \(\Theta^\text{loc}_+\).

Theorem 5. Under the assumptions of Theorem 4 the semigroup \(\{T(h), h \geq 0\}\) on \(K^+\) possesses the trajectory attractor \(U\).

Proof of Theorem 5. Since any weak limiting solution of (11), (3) is a *-weak limit of Galerkin approximate solutions to (2), (3) we deduce some estimates for these approximate solutions. Using (10) and (11), we have for \(u(t) = u^{m,n}(t)\) :

\[
\frac{d}{dt}\|u(t)\|^2 + 2\|A^{1/2}u(t)\|^2 + 2d\|u(t)\|^2 \leq 2M_f|\Omega|^{3/2}C_bC_{\xi,1}\|u(t)\| \leq d\|u(t)\|^2 + \frac{M_f^2|\Omega|^3C_b^2C_{\xi,1}^2}{d}.
\]
This and $\|A^{1/2}u(t)\|^2 \geq \lambda_1 \|u(t)\|^2$ give

$$
\frac{d}{dt}\|u(t)\|^2 + 2(d + \lambda_1)\|u(t)\|^2 \leq \frac{M_f^2|\Omega|^3C_b^2C_{\xi,1}}{d}.
$$

Multiplying by $e^{(d+2\lambda_1)t}$ and integrating from 0 to $t$, one obtains

$$
\|u(t)\|^2 \leq \|u(0)\|^2 \cdot e^{-2(d+\lambda_1)t} + \frac{M_f^2|\Omega|^3C_b^2C_{\xi,1}}{d(d+2\lambda_1)}, \quad t \geq 0. \tag{26}
$$

Now we multiply (8) by $\dot{g}_{k,m}(t)$ and take the sum over $k = 1, \ldots, m$, and then we multiply (8) by $g_{k,m}(t)$ and take the sum again over $k = 1, \ldots, m$. The sum of the obtained equations is (for $u = u^m$)

$$
\langle F_n(u_t), \dot{u}(t) + u(t) \rangle = \frac{1}{2} \frac{d}{dt} \left\{ \|A^{1/2}u(t)\|^2 + (d + 1)\|u(t)\|^2 \right\} + \|\dot{u}(t)\|^2 + \|A^{1/2}u(t)\|^2 + d\|u(t)\|^2.
$$

Using (10), we obtain positive constants $\gamma_1, d_1$ (independent of $m$ and $n$) such that

$$
\frac{d}{dt}\Psi(t) + \gamma_1\Psi(t) \leq d_1, \quad \text{where} \quad \Psi(t) \equiv \|A^{1/2}u(t)\|^2 + (d + 1)\|u(t)\|^2. \tag{27}
$$

Multiplying it by $e^{\gamma_1 t}$ and integrating from $\tau > 0$ to $\tau + h$ ($h > 0$), we get

$$
\Psi(\tau + h) e^{\gamma_1(\tau + h)} \leq \Psi(\tau) e^{\gamma_1(\tau)} + d_1 \gamma_1^{-1} e^{\gamma_1(\tau + h)}. \quad \text{It gives} \quad \Psi(\tau + h) \leq \Psi(\tau) e^{-\gamma_1 h} + d_1 \gamma_1^{-1}. \quad \text{Integrating from} \quad \tau = 0 \quad \text{to} \quad \tau = 1, \quad \text{one gets}
$$

$$
\int_0^1 \Psi(\tau + h) d\tau = \int_0^{h+1} \Psi(s) ds \leq e^{-\gamma_1 h} \cdot \int_0^1 \Psi(s) ds + d_1 \gamma_1^{-1}.
$$

Using the last inequality and definition of $\Psi$ (see (27)), we obtain

$$
\int_0^{h+1} (\|A^{1/2}u(s)\|^2 + (d + 1)\|u(s)\|^2) ds \leq e^{-\gamma_1 h} \cdot \int_0^1 (\|A^{1/2}u(s)\|^2 + (d + 1)\|u(s)\|^2) ds + d_1 \gamma_1^{-1}.
$$

This and estimate (13) give that

$$
\int_0^{h+1} \|A^{1/2}u(s)\|^2 ds \leq e^{-\gamma_1 h} \cdot \left( d + \frac{3}{2} \right) \left( \|u(0)\|^2 + \tilde{k}_3 \right) e^{\tilde{k}_1 - \tilde{k}_3} + d_1 \gamma_1^{-1}. \tag{28}
$$

It is important to note that all the constants $\gamma_1, d_1, \tilde{k}_1, \tilde{k}_3$ are independent of $m$ and $n$.

In the same way, properties (26), (28) allow one to get from (8) the following estimate

$$
\int_0^{h+1} \|A^{-1/2}\dot{u}(s)\|^2 ds \leq e^{-\gamma_2 h} \cdot d_2 \left( \|u(0)\|^2 + 1 \right) + d_3 \tag{29}
$$

with positive constants $\gamma_2, d_2, d_3$ independent of $m$ and $n$. 
Estimates (26), (28) and (29) imply that any approximate solution \( u^m = u^{n,m} \) belongs to \( \mathcal{F}_b^+ \). Moreover, there exists \( R_1 > 0 \), such that the ball \( B_{R_1} = \{ w(\cdot) \in \mathcal{F}_b^+ \mid ||w||_{\mathcal{F}_b^+} \leq R_1 \} \) is an absorbing set for all the approximate solutions \( u^m = u^{n,m} \) of the problem (2), (3).

The constant \( R_1 > 0 \) is independent of \( m \) and \( n \) which gives that the ball \( B_{R_1} = \{ w(\cdot) \in \mathcal{F}_b^+ \mid ||w||_{\mathcal{F}_b^+} \leq R_1 \} \) is also absorbing for any \(^*\)-weak limit in the space \( X_T \) (see (14), (15)) of a subsequence \( \{ u^{n_k,m_k} \} \subset \{ u^{n,m} \} \). Particularly, this ball is absorbing for any weak solution of (2), (3) and for any weak limiting solution of (1), (3). Hence it is an attracting (in the topology \( \Theta_{loc}^{+} \)) set and by Remark 5 it is compact in \( \Theta_{loc}^{+} \). These properties together with Remarks 6,7 allow us to apply Theorem 3.1 from Chepyzhov and Vishik (1997) which completes the proof of Theorem 5.

As an application we can consider the diffusive Nicholson’s blowflies equation (see e.g. So and Yang (1998), So, Wu and Yang (2000)) with state-dependent (both discrete and distributed) delays. More precisely, we consider equations (1) and (2) where \(-A\) is the Laplace operator with the Dirichlet boundary conditions, \( \Omega \subset \mathbb{R}^n_0 \) is a bounded domain with a smooth boundary, the function \( f \) can be a constant as in So and Yang (1998), So, Wu and Yang (2000) which leads to the local in space coordinate term or, for example, \( f(s) = \frac{1}{\sqrt{4\pi \alpha}} e^{-s^2/4\alpha} \), as in So, Wu, Zou (2001) which corresponds to the non-local term, the nonlinear function \( b \) is given by \( b(w) = p \cdot we^{-w} \). Function \( b \) is bounded, so the conditions of Theorems 1-4 are satisfied. As a result, we conclude that for any functions \( \xi^n \) satisfying conditions of Theorems 1 and 2 the dynamical system \( (S_t, H) \) has a global attractor (Theorem 3). Assuming the discrete delay \( \eta \) is locally Lipschitz we get (Theorem 4) the existence of weak limiting solutions of (1), (3) and the existence of trajectory attractor (Theorem 5) for the corresponding translation semigroup.

Acknowledgements. The work was supported in part by INTAS. The author wishes to thank Professor Hans-Otto Walther for bringing state-dependent delay differential equations to his attention during the visit to the Institute of Mathematics at University of Giessen in 2002.

References

[Azbelev et al. (1991)] N.V. Azbelev, V.P. Maksimov and L.F. Rakhmatullina, Introduction to the theory of functional differential equations, Moscow, Nauka, 1991.
[Babin and Vishik (1992)] A. V. Babin, and M. I. Vishik, Attractors of Evolutionary Equations, Amsterdam, North-Holland, 1992.

[Boutet de Monvel, Chueshov and Rezounenko (1998)] L. Boutet de Monvel, I. D. Chueshov and A. V. Rezounenko, Inertial manifolds for retarded semilinear parabolic equations, *Nonlinear Analysis*, 34 (1998), 907-925.

[Chepyzhov and Vishik (1997)] V.V. Chepyzhov and M.I. Vishik, Evolution equations and their trajectory attractors, *J. Math. Pures Appl.*, 76 (1997), 913-964.

[Chueshov (1999)] I. D. Chueshov, Introduction to the Theory of Infinite-Dimensional Dissipative Systems, Acta, Kharkov (1999), (in Russian). English transl. Acta, Kharkov (2002) (see [http://www.emis.de/monographs/Chueshov](http://www.emis.de/monographs/Chueshov)).

[Chueshov (1992)] I. D. Chueshov, On a certain system of equations with delay, occurring in aeroelasticity, *J. Soviet Math.* 58, 1992, p.385-390.

[Chueshov and Rezounenko (1995)] I. D. Chueshov, A. V. Rezounenko, Global attractors for a class of retarded quasilinear partial differential equations, *C.R.Acad.Sci.Paris, Ser.I* 321 (1995), 607-612, (detailed version: *Math.Physics, Analysis, Geometry, Vol.2, N.3* (1995), 363-383).

[Diekmann et al. (1995)] O. Diekmann, S. van Gils, S. Verduyn Lunel, H-O. Walther, Delay Equations: Functional, Complex, and Nonlinear Analysis, Springer-Verlag, New York, 1995.

[Hale (1977)] J. K. Hale, Theory of Functional Differential Equations, Springer, Berlin-Heidelberg- New York, 1977.

[Hale and Lunel (1993)] J. K. Hale and S. M. Verduyn Lunel, Theory of Functional Differential Equations, Springer-Verlag, New York, 1993.

[Krisztin, Walther and Wu (1999)] T. Krisztin, H.-O. Walther and J. Wu, Shape, Smoothness and Invariant Stratification of an Attracting Set for Delayed Monotone Positive Feedback, *Fields Institute Monographs, 11, AMS, Providence, RI*, 1999.

[Lions (1969)] J. L. Lions, Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires, Dunod, Paris, 1969.
[Mishkis (1972)] A.D. Mishkis, Linear differential equations with retarded argument. 2nd edition, Nauka, Moscow, 1972.

[Nussbaum and Mallet (1992)] J. Mallet-Paret and R. D. Nussbaum, Boundary layer phenomena for differential-delay equations with state-dependent time lags I, Archive for Rational Mechanics and Analysis 120 (1992), 99-146.

[Nussbaum and Mallet (1996)] J. Mallet-Paret and R. D. Nussbaum, Boundary layer phenomena for differential-delay equations with state-dependent time lags II, J. Reine Angew. Math., 477 (1996), 129-197.

[Mallet-Paret et al. (1994)] J. Mallet-Paret, R. D. Nussbaum, P. Paraskevopoulos, Periodic solutions for functional-differential equations with multiple state-dependent time lags, Topol. Methods Nonlinear Anal. 3 (1994), no. 1, 101–162.

[Rezounenko (1997)] A. V. Rezounenko, On singular limit dynamics for a class of retarded nonlinear partial differential equations, Matematicheskaya fizika, analiz, geometriya. -1997. N.4 (1/2). -C.193-211.

[Rezounenko (2004)] A.V. Rezounenko, A short introduction to the theory of ordinary delay differential equations. Lecture Notes. Kharkov University Press, Kharkov, 2004.

[Rezounenko and Wu (2005)] A.V. Rezounenko and J. Wu, A non-local PDE model for population dynamics with state-selective delay: local theory and global attractors, Journal of Computational and Applied Mathematics. -2005 (in press).

[So, Wu and Yang (2000)] J. W. -H. So, J. Wu and Y. Yang, Numerical steady state and Hopf bifurcation analysis on the diffusive Nicholson’s blowflies equation. Appl. Math. Comput. 111 (2000), no. 1, 33–51.

[So, Wu and Zou (2001)] J. W. -H. So, J. Wu and X.Zou, A reaction diffusion model for a single species with age structure. I. Travelling wavefronts on unbounded domains, Proc. Royal .Soc. Lond. A (2001) 457, 1841-1853.

[So and Yang (1998)] J. W.- H. So and Y. Yang, Dirichlet problem for the diffusive Nicholson’s blowflies equation, J. Differential Equations 150 (1998), no. 2, 317–348.
[Temam (1988)] R. Temam, Infinite Dimensional Dynamical Systems in Mechanics and Physics, Springer, Berlin-Heidelberg-New York, 1988.

[Travis and Webb (1974)] C. C. Travis and G. F. Webb, Existence and stability for partial functional differential equations, Transactions of AMS 200, (1974), 395-418.

[Walther (2002)] H. -O. Walther, Stable periodic motion of a system with state dependent delay, Differential and Integral Equations 15 (2002), 923-944.

[Walther (2003)] H.-O. Walther, The solution manifold and $C^1$-smoothness for differential equations with state-dependent delay, J. Differential Equations 195 (2003), no. 1, 46–65.

[Wu (1996)] J. Wu, Theory and Applications of Partial Functional Differential Equations, Springer-Verlag, New York, 1996.

[Yosida (1965)] K. Yosida, Functional analysis, Springer-Verlag, New York, 1965.

[Showalter (1997)] R.E. Showalter, Monotone operators in Banach space and nonlinear partial differential equations, AMS, Mathematical Surveys and Monographs, vol. 49, 1997.

March 22, 2005