POSITIVE OR SIGN-CHANGING SOLUTIONS FOR A CRITICAL SEMILINEAR NONLOCAL EQUATION

WEI LONG AND JING YANG

Abstract. We consider the following critical semilinear nonlocal equation involving the fractional Laplacian

$$(-\Delta)^s u = K(|x|)|u|^{2^*_s - 2}u, \quad \text{in} \quad \mathbb{R}^N,$$

where $K(|x|)$ is a positive radial function, $N > 2 + 2s$, $0 < s < 1$, and $2^*_s = \frac{2N}{N - 2s}$. Under some asymptotic assumptions on $K(x)$ at an extreme point, we show that this problem has infinitely many non-radial positive or sign-changing solutions.

Key words : fractional Laplacian; semilinear nonlocal equation; critical; reduction method.

AMS Subject Classifications: 35J20, 35J60

1. Introduction

In this paper, we consider the following critical semilinear nonlocal equation involving the fractional Laplacian

$$(-\Delta)^s u = K(x)|u|^{2^*_s - 2}u, \quad \text{in} \quad \mathbb{R}^N$$  \hspace{1cm} (1.1)

with $N > 2 + 2s$, where $K(x)$ is a positive continuous potential, $0 < s < 1$, $2^*_s = \frac{2N}{N - 2s}$. Here, the fractional Laplacian of a function $f : \mathbb{R}^N \to \mathbb{R}$ is expressed by the formula

$$(-\Delta)^s f(x) = C_{N,s} P.V. \int_{\mathbb{R}^N} \frac{f(x) - f(y)}{|x-y|^{N+2s}} dy = C_{N,s} \lim_{\delta \to 0} \int_{\mathbb{R}^N \setminus B_\delta(x)} \frac{f(x) - f(y)}{|x-y|^{N+2s}} dy, \quad (1.2)$$

where $C_{N,s}$ is some normalization constant.

Recently, a great attention has been focused on the study of problems involving the fractional Laplacian, from a pure mathematical point of view as well as from concrete applications, since this operator naturally arises in several areas such as physics, probability and finance, see for instance [2, 4, 11, 28]. In fact, the fractional Laplacian can be understood as the infinitesimal generator of a stable Lévy process (see [3]). The literature involving the fractional Laplacian is really too wide to attempt any reasonable comprehensive treatment in a single paper. We would just mention some very recent papers which analyze fractional elliptic equations involving the critical Sobolev exponent (cf. [3, 8, 15, 20, 22, 24, 26, 30]).

It is well known, but not completely trivial, that $(-\Delta)^s$ reduces to the standard Laplacian $-\Delta$ as $s \to 1$. Especially, when $s = 1$, equation (1.1) is reduced to the classical semilinear

The authors sincerely thank Professor Shuangjie Peng for helpful discussions and suggestions.
elliptic equation

\[
\begin{cases}
-\Delta u = K(x)|u|^{2s-2}u, & x \in \mathbb{R}^N \\
u \in D^{1,2}(\mathbb{R}^N),
\end{cases}
\] (1.3)

where \(D^{1,2}(\mathbb{R}^N)\) denotes the completion of \(C_0^\infty(\mathbb{R}^N)\) under the norm \(\int_{\mathbb{R}^N} |\nabla u|^2\).

Our main interest in this paper is to investigate the multiplicity of solutions to equation (1.1). Before our study on this problem, we would like draw the reader’s attention to some recent results on the multiplicity of positive solutions to equation (1.3). Amrousi, Azorero and Peral [1], and Cao, Noussair and Yan [7] proved the existence of two or many positive solutions if \(K\) is a perturbation of the constant, i.e.

\[K = K_0 + \varepsilon h(x), 0 < \varepsilon \ll 1.\]

In [19], Li proved that (1.3) has infinitely many positive solutions if \(K(x)\) is periodic, while similar result was obtained in [34] if \(K(x)\) has a sequence of strict local maximum points tending to infinity. In particular, in [31], Wei and Yan gave a very interesting result which says that equation (1.3) with \(K(x)\) being radial has solutions with large number of bumps near infinity and the energy of these solutions can be arbitrarily large. The reader can refer to [9, 32, 33] for the existence of infinitely solutions on semilinear equations involving critical and supercritical exponents.

When \(0 < s < 1\), Chen and Zheng [10] studied the following singularly perturbed problem

\[\varepsilon^{2s}(-\Delta)^s u + V(x)u = |u|^{p-1}u, \quad \text{in } \mathbb{R}^N.\] (1.4)

They showed that when \(N = 1, 2, 3, 1 < p < 2^*_s - 1\) and \(\varepsilon\) is sufficiently small, \(\max\{\frac{1}{2}, \frac{p}{2} - 1\} < s < 1\) and \(V\) satisfies some smoothness and boundedness assumptions, equation (1.4) has a nontrivial solution \(u_\varepsilon\) concentrated to some single point as \(\varepsilon \to 0\). Very recently, in [13], Dávila, del Pino and Wei generalized various existence results known for (1.4) with \(s = 1\) to the case of fractional Laplacian. In [23], the authors gave a result which says that (1.4) with \(V(x)\) being radial has solutions with large number of bumps near infinity and the energy of this solutions can be very large when \(\varepsilon\) is fixed and \(N \geq 2\).

Naturally, one would like know if the critical equation (1.1) has infinitely many non-radial solutions. To the best of our knowledge, it seems that there is no answer for this question. The aim of this paper is to obtain infinitely many non-radial positive (sign-changing) solutions for (1.1) whose functional energy are arbitrarily large, under some assumptions that \(K(x) = K(|x|) > 0\) has a local maximum (minimum) at some point \(r_0 > 0\).

Letting

\[\max\{2, N - 2s - 2 \cdot \frac{(N - 2s)^2}{N + 2s}\} < m < N - 2s,\] (1.5)

we assume that \(0 < K(|x|) \in C(\mathbb{R}^N)\) satisfies the following conditions at \(r_0\)

\((K) : K(r) = K(r_0) - c_0|r - r_0|^m + O(|r - r_0|^{m+\theta}), \quad \text{as } r \in (r_0 - \delta, r_0 + \delta),\)

or

\((K') : K(r) = K(r_0) + c_0|r - r_0|^m + O(|r - r_0|^{m+\theta}), \quad \text{as } r \in (r_0 - \delta, r_0 + \delta),\)
where \( c_0 > 0, \theta > 0, \delta > 0 \) are some constants.

Without loss of generality, in what follows we assume that \( K(r_0) = 1 \).

Our main results in this paper can be stated as follows

**Theorem 1.1.** Suppose that \( N > 2 + 2s, 0 < s < 1, \) and \( K(r) \) satisfies (K). Then problem \( (1.1) \) has infinitely many non-radial positive solutions.

**Theorem 1.2.** Suppose that \( N > 2 + 2s, 0 < s < 1, \) and \( K(r) \) satisfies \( (K') \). Then problem \( (1.1) \) has infinitely many non-radial sign-changing solutions.

For the sake of completeness, we firstly introduce basic theory on fraction Laplacian operator.

The nonlocal operator \((-\Delta)^s\) in \( \mathbb{R}^N \) is defined on the Schwartz class through the Fourier transform,

\[
(-\Delta)^s f(\xi) = (2\pi|\xi|)^{2s} \hat{f}(\xi),
\]
or via the Riesz potential, see for example \([18, 29]\) for the precise formula. As usual, \( \hat{f}(\xi) \) denotes the Fourier Transform of \( f \), \( \hat{f}(\xi) = \int_{\mathbb{R}^N} e^{-2\pi\xi \cdot x} f(x) dx \). Observe that \( s = 1 \) corresponds to the standard local Laplacian. Since the fractional operator is nonlocal, L. Caffarelli and L. Silvestre showed in \([5]\) that any fractional power of the Laplacian can be determined as an operator that maps a Dirichlet boundary condition to a Neumann-type condition via an extension problem. To be more precise, consider the function \( u = u(x, y) : \mathbb{R}^N \times [0, \infty) \to \mathbb{R} \), that solves the boundary value problem

\[
u(x, 0) = f(x) \quad (1.6)
\]

and

\[
\Delta_x u + \frac{1 - 2s}{y} u_y + u_{yy} = 0. \quad (1.7)
\]

Then, up to a multiplicative constant depending only on \( s \),

\[
C(-\Delta)^s f = \lim_{y \to 0^+} -y^{1-2s} u_y = \frac{1}{2s} \lim_{y \to 0^+} \frac{u(x, y) - u(x, 0)}{y^{2s}}.
\]

Thought the rest of this paper, the homogeneous fractional Sobolev space \( D^s(\mathbb{R}^N)(0 < s < 1) \) is given by

\[
D^s(\mathbb{R}^N) = \left\{ u \in L^\frac{2N}{N-2s}(\mathbb{R}^N) : \| u \|_{D^s(\mathbb{R}^N)} := \left( \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 \right)^{\frac{1}{2}} < +\infty \right\}.
\]

Note that \( D^s(\mathbb{R}^N) \) is a Hilbert space equipped with an inner product

\[
\langle u, v \rangle_{D^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\xi|^{2s} \hat{u}(\xi) \hat{v}(\xi).
\]

We also define a fractional Laplace operator on the whole space, \( (-\Delta)^s : D^s(\mathbb{R}^n) \to D^{-s}(\mathbb{R}^N) \) by

\[
\langle (-\Delta)^s u, v \rangle_{D^{-s}(\mathbb{R}^N)} = \langle u, v \rangle_{D^s(\mathbb{R}^N)},
\]
where $D^{−s}(\mathbb{R}^N)$ is the dual of $D^s(\mathbb{R}^N)$. Then, one can easily check that if $u \in D^{2s}(\mathbb{R}^N)$, we have $(-\Delta)^s u \in L^2(\mathbb{R}^N)$ such that

$$(-\Delta)^s u = \mathfrak{F}^{-1}[|\xi|^{2s}\hat{u}(\xi)]$$

where $\mathfrak{F}^{-1}$ denotes the inverse Fourier transform so that we see for $u, v \in D^s(\mathbb{R}^n)$

$$\langle u, v \rangle_{D^s(\mathbb{R}^n)} = \int_{\mathbb{R}^N} (-\Delta)^s u \cdot (-\Delta)^s v$$

and assuming additionally $u \in D^{2s}(\mathbb{R}^N), v \in L^2(\mathbb{R}^N)$, we can integrate by parts:

$$\int_{\mathbb{R}^N} (−\Delta)^s u \cdot (−\Delta)^s v = \int_{\mathbb{R}^N} (−\Delta)^s u \cdot v.$$

In what remains of this paper, we will always mean that the equation

$$(-\Delta)^s u = f \quad \text{in} \quad \mathbb{R}^N,$$

is satisfied if

$$u(x) = (-\Delta)^{−s} f(x) := \frac{1}{\gamma(N, s)} \int_{\mathbb{R}^N} \frac{f(y)}{|x − y|^{N−2s}} dy,$$  \hspace{1cm} (1.8)

as long as $f$ has enough decay for the integral to be well defined. The constant $\gamma(N, s) = \frac{\pi^\frac{N}{2}2^{2s}\Gamma(s)}{\Gamma(\frac{N}{2}−s)}$. There are other definitions of $(-\Delta)^s u$, which are equivalent to (1.8) under suitable assumptions, see [3, 27].

Lieb in 1983 [22] (also see [16, 17]) established that the extremals correspond precisely to functions of the form

$$U_{\varepsilon, \xi}(x) = \alpha\left(\frac{\varepsilon}{\varepsilon^2 + |x−\xi|^2}\right)^{\frac{N−2s}{2}}, \quad \alpha > 0,$$  \hspace{1cm} (1.9)

which for a suitable choice

$$\alpha = \alpha_{N, s} = 2^{\frac{N−2s}{2}} \frac{\Gamma(\frac{N+2s}{2})}{\Gamma(\frac{N}{2})}$$

solve the equation

$$(-\Delta)^s u = u^{\frac{N+2s}{N−2s}}, \quad u > 0, \quad \text{in} \quad \mathbb{R}^N.$$  \hspace{1cm} (1.10)

Very recently, J. DÁvila, M. del Pino and Y. Sire [12] obtained the non-degeneracy of the solutions for (1.10) in arbitrary dimension $N > 2s$.

So, we know that $U_{\varepsilon, \xi}(x)$ is non-degenerate in $D^s(\mathbb{R}^N)$. More precisely, define the functional corresponding to (1.10) as follows

$$f_0(u) = \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^s u|^2 − \frac{1}{2s} \int_{\mathbb{R}^N} |u|^{2s}, \quad u \in D^{2s}(\mathbb{R}^N).$$

Then $f_0$ possesses a finite-dimensional manifold $Z$ of least energy critical points, given by

$$Z = \{U_{\varepsilon, \xi} : \varepsilon > 0, \ \xi \in \mathbb{R}^N\}.$$  \hspace{1cm} (1.11)

Moreover,

$$\text{ker} f''_0(z) = \text{span}_\mathbb{R} \left\{ \frac{\partial U_{\varepsilon, \xi}}{\partial \xi_1}, \ldots, \frac{\partial U_{\varepsilon, \xi}}{\partial \xi_N}, \frac{\partial U_{\varepsilon, \xi}}{\partial \varepsilon} \right\}, \quad \forall \ U_{\xi, \varepsilon} \in Z.$$
The following idea to prove our main results is essentially from [31]. We will use the solution
\[ U_{\varepsilon, \xi}(x) = \alpha_{N,s} \left( \frac{\varepsilon}{\varepsilon^2 + |x - \xi|^2} \right)^{\frac{N-2s}{2}}, \]
of
\[ (-\Delta)^s u = u^{2^*_s - 1}, \quad u > 0, \quad x \in \mathbb{R}^N, \quad u(0) = \max_{\mathbb{R}^N} u(x) \quad (1.11) \]
to build up the approximate solutions for (1.1). From [6, 16, 17, 22], we see that when \( s \in (0, 1) \), the unique ground solution of (1.11) decays like \( \frac{1}{|x|^{N-2s}} \) when \(|x| \to \infty\).

Set \( \nu = k^{\frac{N-2s}{N-2s-m}} \), to be the scaling parameter. Using the transformation \( u(x) \to \nu^{\frac{N-2s}{2}} u(\frac{x}{\nu}) \), we find that (1.1) becomes
\[ (-\Delta)^s u = K(\frac{x}{\nu}) u^{2^*_s - 1}, \quad u > 0, \quad x \in \mathbb{R}^N. \]

Throughout this paper, we always assume that
\[ r \in \left[ r_0^\nu - 1, r_0^\nu + 1 \right], \quad \text{for some small } \bar{\theta} > 0, \]
and
\[ \varepsilon_0 \leq \varepsilon \leq \varepsilon_1, \quad \text{for some constants } \varepsilon_1 > \varepsilon_0 > 0. \]

Set \( x = (x', x'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2} \). Define
\[ \mathcal{H} = \left\{ u : u \in D^{2s}(\mathbb{R}^N), u \text{ is even in } x_j, j = 2, \ldots, N, \right. \]
\[ \left. u(r \cos \theta, r \sin \theta, x'') = u(r \cos(\theta + \frac{2i\pi}{k}), r \sin(\theta + \frac{2i\pi}{k}), x'') \right\}. \]

Write
\[ x^i = \left( r \cos \frac{2(i-1)\pi}{k}, r \sin \frac{2(i-1)\pi}{k}, 0 \right), \quad i = 1, \ldots, k, \]
where 0 is the zero vector in \( \mathbb{R}^{N-2} \). Denote
\[ U_{\varepsilon, r}(x) = \sum_{i=1}^k U_{\varepsilon, x^i}(x). \]

To prove Theorem 1.1 it suffices to verify the following result:

**Theorem 1.3.** Under the assumption of Theorem (1.1), there is an integer \( k_0 > 0 \), such that for any integer \( k \geq k_0 \), (1.1) has a solution \( u_k \) of the form
\[ u_k = U_{\varepsilon, r}(x) + \omega_r \]
where \( \omega_r \in \mathcal{H}, \ r_k \in \left[ r_0^\nu - \frac{1}{\nu^{\bar{\theta}}}, r_0^\nu + \frac{1}{\nu^{\bar{\theta}}} \right] \) for some constants \( \bar{\theta} > 0 \) and \( \varepsilon_0 \leq \varepsilon \leq \varepsilon_1 \).
To consider the sign-changing solutions, for any integer \( k > 0 \), we define
\[
\bar{x}_i = \left( r \cos \left( \frac{(i - 1)\pi}{k} \right), r \sin \left( \frac{(i - 1)\pi}{k} \right), 0 \right), \quad i = 1, \cdots, 2k,
\]
write
\[
\bar{U}_{r,\varepsilon}(x) = \sum_{i=1}^{2k} (-1)^{i-1} U_{\bar{x}_i}(x).
\]

Denote
\[
\mathcal{H}' = \left\{ u : u \in D^{2s}(\mathbb{R}^N), u \text{ is even in } x_j, j = 2, \cdots, N, \right. \\
u(r \cos \theta, r \sin \theta, x''') = (-1)^i u(r \cos(\theta + \frac{i\pi}{k}), r \sin(\theta + \frac{i\pi}{k}), x''') \left. \right\}.
\]

To prove Theorem 1.2, we only need to show that:

**Theorem 1.4.** Under the assumption of Theorem (1.1), there is an integer \( k_0 > 0 \), such that for any integer \( k \geq k_0 \), (1.1) has a solution \( u_k \) of the form
\[
\bar{u}_k = \bar{U}_{r,\varepsilon}(x) + \bar{\omega}_r
\]
where \( \bar{\omega}_r \in \mathcal{H}' \), \( r_k \in \left[ r_0 \nu \frac{1}{\nu'} r_0 \nu + \frac{1}{\nu'} \right] \) for some constants \( \bar{\theta} > 0 \) and \( \varepsilon_0 \leq \varepsilon \leq \varepsilon_1 \).

The rest of the paper is organized as follows. In Sect.2, we will carry out a reduction procedure. We prove our main result in Section 3. Finally, in Appendix, some basic estimates and an energy expansion for the functional corresponding to problem (1.1) will be established.

**2. Finite-dimension Reduction**

In this section, we perform a finite-dimensional reduction. Let
\[
\|u\|_* = \sup_{x \in \mathbb{R}^N} \left( \sum_{i=1}^{k} \frac{1}{(1 + |x - x^i|)^{\frac{N-2s+\tau}{2}}} \right)^{-1} |u(x)|
\]
and
\[
\|f\|_{**} = \sup_{x \in \mathbb{R}^N} \left( \sum_{i=1}^{n} \frac{1}{(1 + |x - x^i|)^{\frac{N+2s+\tau}{2}}} \right)^{-1} |f(x)|,
\]
where \( \tau = \frac{N-2s-m}{N-2s} \). Write
\[
Z_{i,1} = \frac{\partial U_{x^i,\varepsilon}}{\partial r}, \quad Z_{i,2} = \frac{\partial U_{x^i,\varepsilon}}{\partial \varepsilon}.
\]
Consider
\[ \begin{cases} 
(-\Delta)^s \varphi_k - (2^s - 1)K(\frac{\|u\|}{\nu})U^{2^s-2}_{r,\varepsilon} \varphi_k = H_k + \sum_{l=1}^{2} c_l \sum_{i=1}^{k} U^{2^s-2}_{x_i,\varepsilon}Z_{i,l}, \\
\varphi_k \in \mathcal{H}, \\
\langle U^{2^s-2}_{x_i,\varepsilon}Z_{i,l}, \varphi_k \rangle = 0, \quad i = 1, ..., k, l = 1, 2,
\end{cases} \tag{2.3} \]

for some numbers \( c_l \), where \( \langle u, v \rangle = \int_{\mathbb{R}^N} uv \).

**Lemma 2.1.** Suppose that \( \varphi_k \) solves \((2.3)\) for \( H = H_k \). If \( \|H_k\|_{**} \) goes to zero as \( k \) goes to infinity, so does \( \|\varphi_k\|_{*} \).

**Proof.** We will argue by an indirect method. Suppose to the contrary that there exist \( k \to +\infty, H = H_k, \varepsilon_k \in [\varepsilon_0, \varepsilon_1], r_k \in [r_0\nu - \nu^{-\theta}, r_0\nu + \nu^{-\theta}] \) and \( \varphi_k \) solving \((2.3)\) for \( H = H_k, \varepsilon = \varepsilon_k, r = r_k \) with \( \|H_k\|_{**} \to 0 \) and \( \|\varphi_k\|_{*} \geq c > 0 \). We may assume that \( \|\varphi_k\|_{*} = 1 \). For simplicity, we drop the subscript \( k \). By \((1.8)\), we can rewrite \((2.3)\) as

\[ \varphi(x) = \frac{1}{\gamma(N,s)}(2^s - 1) \int_{\mathbb{R}^N} \frac{1}{|y - x|^{N-2s}} K(\frac{|y|}{\nu}) U^{2^s-2}_{r,\varepsilon}(y) \varphi(y) \, dy \]

\[ + \frac{1}{\gamma(N,s)} \int_{\mathbb{R}^N} \frac{1}{|y - x|^{N-2s}} H(y) + \sum_{l=1}^{2} c_l \sum_{i=1}^{k} U^{2^s-2}_{x_i,\varepsilon}(y)Z_{i,l}(y) \, dy, \tag{2.4} \]

Analogously to Lemma \([A.3]\) we have

\[ \left| \int_{\mathbb{R}^N} \frac{1}{|y - x|^{N-2s}} K(\frac{|y|}{\nu}) U^{2^s-2}_{r,\varepsilon}(y) \varphi(y) \, dy \right| \]

\[ \leq C\|\varphi\| \, \int_{\mathbb{R}^N} \frac{1}{|y - x|^{N-2s}} U^{2^s-2}_{r,\varepsilon}(y) \sum_{i=1}^{k} \frac{1}{(1 + |y - x^i|)^{\frac{N-2s}{2} + \tau}} \, dy \tag{2.5} \]

\[ \leq C\|\varphi\| \sum_{i=1}^{k} \frac{1}{(1 + |x - x^i|)^{\frac{N-2s}{2} + \tau + \theta}}. \]

By Lemma \([A.2]\) we get

\[ \left| \int_{\mathbb{R}^N} \frac{1}{|x - y|^{N-2s}} H(y) \, dy \right| \leq C\|H\|_{**} \int_{\mathbb{R}^N} \frac{1}{|x - y|^{N-2s}} \sum_{i=1}^{k} \frac{1}{(1 + |y - x^i|)^{\frac{N+2s}{2} + \tau}} \, dy \]

\[ \leq C\|H\|_{**} \sum_{i=1}^{k} \frac{1}{(1 + |x - x^i|)^{\frac{N-2s}{2} + \tau}} \tag{2.6} \]
and

\[ \left| \int_{\mathbb{R}^N} \frac{1}{|x-y|^{N-2s}} \sum_{i=1}^{k} U_{x_i,\varepsilon}^{2s-2}(y)Z_{i,t}(y) \, dy \right| \]

\[ \leq C \int_{\mathbb{R}^N} \frac{1}{|x-y|^{N-2s}} \sum_{i=1}^{k} U_{x_i,\varepsilon}^{2s-1}(y) \, dy \]

\[ \leq C \int_{\mathbb{R}^N} \frac{1}{|x-y|^{N-2s}} \sum_{i=1}^{k} \frac{1}{(1 + |y-x^i|)^{N+2s}} \, dy \]

\[ \leq C \sum_{i=1}^{k} \frac{1}{(1 + |x-x^i|)^N} \]

\[ \leq C \sum_{i=1}^{k} \frac{1}{(1 + |x-x^i|)^{\frac{N-2s}{2} + \tau}}. \]  

(2.7)

Next, we estimate \( c_{1,l} \), \( l = 1, 2 \). Multiplying (2.3) by \( Z_{1,t} \) \((t = 1, 2)\) and integrating, we see that \( c_l \) satisfies

\[ \sum_{l=1}^{2} c_l \sum_{i=1}^{k} \left\langle U_{x_i,\varepsilon}^{2s-2}Z_{i,t}, Z_{1,t} \right\rangle = \left\langle (-\Delta)^s \varphi - (2s - 1) \right\rangle_{l,\varepsilon} K \left( \frac{|y|}{\nu} U_{x_i,\varepsilon}^{2s-2} \varphi, Z_{1,t} \right) - \left\langle H, Z_{1,t} \right\rangle. \] 

(2.8)

It follows from Lemma A.1 that

\[ | \left\langle H, Z_{1,t} \right\rangle | \leq C \| H \|_{**} \int_{\mathbb{R}^N} \frac{1}{(1 + |y-x^1|)^{N-2s}} \sum_{i=1}^{k} \frac{1}{(1 + |y-x^i|)^{\frac{N+2s}{2} + \tau}} \]

\[ \leq C \| H \|_{**}. \]

Then

\[ \left\langle (-\Delta)^s \varphi - (2s - 1) K \left( \frac{|y|}{\nu} U_{x_i,\varepsilon}^{2s-2} \varphi, Z_{1,t} \right) \right\rangle = \left\langle (-\Delta)^s Z_{1,t} - (2s - 1) K \left( \frac{|y|}{\nu} U_{x_i,\varepsilon}^{2s-2} \varphi \right) \right\rangle \]

\[ = \left\langle (2s - 1) U_{x_i,\varepsilon}^{2s-2} Z_{1,t} - (2s - 1) K \left( \frac{|y|}{\nu} U_{x_i,\varepsilon}^{2s-2} Z_{1,t} \right) \right\rangle \]

\[ = C \| \varphi \|_s \int \left| K \left( \frac{|y|}{\nu} \right) - 1 \right| U_{x_i,\varepsilon}^{2s-2} \frac{1}{(1 + |y-x^1|)^{N-2s}} \sum_{i=1}^{k} \frac{1}{(1 + |y-x^i|)^{\frac{N-2s}{2} + \tau}}. \] 

(2.9)

If \( |y| - \nu r_0 \geq \sqrt{\nu} \), then

\[ ||y| - x^1| \geq |y| - \nu r_0| - |x^1| - \nu r_0| \geq \sqrt{\nu} - \frac{1}{\nu^{\theta}} \geq \frac{1}{2} \sqrt{\nu}. \]
Computing as Lemma A.3, we see
\[
\int_{|y|<\nu r_0} \left| K\left(\frac{|y|}{\nu}\right) - 1 \right| U_{r_0}^{2^* - 2} \frac{1}{(1 + |y - x^1|)^{N-2s}} \sum_{i=1}^{k} \frac{1}{(1 + |y - x^i|)^{\frac{N-2s}{2} + \tau}} dy \\
\leq \frac{C}{\nu^\sigma} \int_{\mathbb{R}^N} U_{r,\varepsilon}^{2^* - 2} \frac{1}{(1 + |y - x^1|)^{N-2s}} \sum_{i=1}^{k} \frac{1}{(1 + |y - x^i|)^{\frac{N-2s}{2} + \tau}} dy \\
\leq \frac{C}{\nu^\sigma},
\]
where \(\sigma\) is a small constant. On the other hand, as Lemma A.3, we have
\[
\int_{|y|<\nu r_0} \left| K\left(\frac{|y|}{\nu}\right) - 1 \right| U_{r_0}^{2^* - 2} \frac{1}{(1 + |y - x^1|)^{N-2s}} \sum_{i=1}^{k} \frac{1}{(1 + |y - x^i|)^{\frac{N-2s}{2} + \tau}} dy \\
\leq \left(\frac{C}{\nu^\sigma}\right)^m \int_{\mathbb{R}^N} U_{r,\varepsilon}^{2^* - 2} \frac{1}{(1 + |y - x^1|)^{N-2s}} \sum_{i=1}^{k} \frac{1}{(1 + |y - x^i|)^{\frac{N-2s}{2} + \tau}} dy \\
\leq \frac{C}{\nu^\sigma},
\]
So, together with (2.9), we obtain
\[
\langle (-\Delta)^s \varphi - (2^*_s - 1) K\left(\frac{|y|}{\nu}\right) U_{r,\varepsilon}^{2^* - 2} \varphi, Z_{1,t} \rangle = o(1) \|\varphi\|_*.
\]
But
\[
\sum_{i=1}^{k} \langle U_{x^i,\varepsilon}^{2^* - 2} Z_{1,t}, Z_{1,t} \rangle \\
\leq \sum_{i=1}^{k} C \int_{\mathbb{R}^N} \left( \frac{1}{(1 + |x - x^i|)^{N-2s}} \right)^{2^* - 1} \frac{1}{(1 + |x - x^i|)^{\frac{N-2s}{2} + \tau}} dx \\
\leq C.
\]
Hence it follows from (2.8) that
\[
c_t = O\left( \frac{1}{\nu^\sigma} \|\varphi\|_* + \|H\|_{**} \right) = o(1).
\]
So,
\[
\|\varphi\|_* \leq \left( \|H\|_{**} + \sum_{i=1}^{k} \frac{1}{(1+|y-x^i|)^{\frac{N-2s}{2} + \tau + \theta}} \right) \sum_{i=1}^{k} \frac{1}{(1+|y-x^i|)^{\frac{N-2s}{2} + \tau}}.
\]
Since \(\|\varphi\|_* = 1\), we obtain from (2.12) that there is \(R > 0\) such that
\[
\|\varphi(x)\|_{L^\infty(B_R(x^1))} \geq a > 0,
\]
where
for some $i$. But $\tilde{\varphi}(x) = \varphi(x - x^i)$ converges uniformly in any compact set to a solution $u$ of
\[
(-\Delta)^s u - (2^*_s - 1)U_{x^i,\varepsilon}^{2^*_s - 2} u = 0, \quad \text{in } \mathbb{R}^N, \tag{2.14}
\]
for some $\varepsilon \in [\varepsilon_0, \varepsilon_1]$ and $u$ is perpendicular to the kernel of (2.14). So $u = 0$. This is a contradiction to (2.12).

**Proof.** Following from [14], let us consider the space
\[
H = \left\{ \varphi \in D^{2s}(\mathbb{R}^N) \mid \left\langle U_{x^i,\varepsilon}^{2^*_s - 2} Z_{i,l}, \varphi \right\rangle = 0, \quad i = 1, \ldots, k, l = 1, 2, \right\}
\]
endowed with usual inner product $[\varphi, \psi] = \int_{\mathbb{R}^N} (-\Delta)^{s/2} \varphi (-\Delta)^{s/2} \psi$. Problem (2.3) expressed in weak form is equivalent to finding a $\varphi \in H$ such that
\[
[\varphi, \psi] = \left\langle (2^*_s - 1)K \left( \frac{|x|}{\nu} \right) U_{x^i,\varepsilon}^{2^*_s - 2} \varphi + H, \psi \right\rangle, \quad \forall \psi \in H.
\]
With the aid of Riesz’s representation theorem, this equation gets rewritten in $H$ in the operational form
\[
\varphi = T_k(\varphi) + \bar{H} \tag{2.17}
\]
with certain $\bar{H} \in H$ which depends linearly in $H$ and where $T_k$ is a compact operator in $H$. Fredholm’s alternative guarantees unique solvability of this problem for any $H$ provided that the homogeneous equation
\[
\varphi = T_k(\varphi)
\]
has only the zero solution in $H$. Let us observe that this last equation is equivalent to
\[
\begin{cases}
(-\Delta)^s \varphi - (2^*_s - 1)K \left( \frac{|x|}{\nu} \right) U_{x^i,\varepsilon}^{2^*_s - 2} \varphi = \sum_{l=1}^{2} c_l \sum_{i=1}^{k} U_{x^i,\varepsilon}^{2^*_s - 2} Z_{i,l}, & x \in \mathbb{R}^N, \\
\varphi \in \mathcal{H}, \\
\left\langle U_{x^i,\varepsilon}^{2^*_s - 2} Z_{i,l}, \varphi \right\rangle = 0, & i = 1, \ldots, k, l = 1, 2,
\end{cases} \tag{2.18}
\]
for certain constants $c_l$. Assume it has a nontrivial solution $\varphi = \varphi_k$, which with no loss of generality may be taken so that $\|\varphi_k\|_* = 1$. But this makes the Lemma 2.1 so that necessarily $\|\varphi_k\|_* \to 0$. This is certainly a contradiction that proves that this equation only has the trivial solution in $H$. We conclude then that for each $H$, problem (2.3) admits a unique solution. We check that
\[
\|\varphi\|_* \leq \|H\|_{**}.
\]
We assume again the opposite. In doing so, we find a sequence $H_k$ with $\|H\|_{**} = o(1)$ and solution $\varphi_k \in H$ of problem (2.3) with $\|\varphi\|_* = 1$. Again this makes the Lemma 2.1 and a
contradiction has been found. This proves estimates (2.15). Estimate (2.16) follows from this and relation (2.11). This concludes this proof of the proposition.

Now, we consider

\[
\begin{align*}
(-\Delta)^s \varphi - (2_s^* - 1) K \left( \frac{|x|}{\nu} \right) U_{r,\varepsilon}^{2_s^*-2} \varphi &= N(\varphi) + l_k + \sum_{i=1}^{2} c_i \sum_{l=1}^{k} U_{x_i,\varepsilon}^{2_s^*-2} Z_{i,l}, \quad x \in \mathbb{R}^N, \\
\varphi &\in \mathcal{H}, \\
\left\langle U_{x_i,\varepsilon}^{2_s^*-2} Z_{i,l}, \varphi \right\rangle = 0, \quad i = 1, \ldots, k, l = 1, 2,
\end{align*}
\]

where

\[
N(\varphi) = K \left( \frac{|x|}{\nu} \right) \left( (U_{r,\varepsilon} + \varphi)^{2_s^*-1} - U_{r,\varepsilon}^{2_s^*-1} - (2_s^* - 1) U_{r,\varepsilon}^{2_s^*-2} \varphi \right)
\]

and

\[
l_k = K \left( \frac{|x|}{\nu} \right) U_{r,\varepsilon}^{2_s^*-1} - \sum_{i=1}^{k} U_{x_i,\varepsilon}^{2_s^*-1}.
\]

Next, we estimate \( N(\varphi) \) and \( l_k \).

**Lemma 2.3.** We obtain

\[
\| N(\varphi) \| \leq C \| \varphi \| \min(2_s^*-1,2),
\]

**Proof.** Firstly, we deal with the case \( 2_s^* \leq 3 \). Since

\[
\sum_{i=1}^{k} \frac{1}{(1 + |x - x^i|)^{\frac{N-2s-\alpha}{N-2s}}} \leq C \sum_{i=2}^{k} \frac{C}{|x^1 - x^i|^\frac{N-2s-\alpha}{N-2s}} \leq C,
\]

we find by Hölder inequality,

\[
|N(\varphi)| \leq C \| \varphi \|_s^{2_s^*-1} \left( \sum_{i=1}^{k} \frac{1}{(1 + |x - x^i|)^{\frac{N-2s-\alpha}{2} + \tau}} \right)^{2_s^*-1} \\
\leq C \| \varphi \|_s^{2_s^*-1} \sum_{i=1}^{k} \frac{1}{(1 + |x - x^i|)^{\frac{N+2s-\alpha}{2} + \tau}} \left( \sum_{i=1}^{k} \frac{1}{(1 + |x - x^i|)^{\tau}} \right)^{\frac{4s}{N-2s}} \\
\leq C \| \varphi \|_s^{2_s^*-1} \sum_{i=1}^{k} \frac{1}{(1 + |y - x^i|)^{\frac{N+2s-\alpha}{2} + \tau}}.
\]

Using the same argument, for the case \( 2_s^* > 3 \), we also can obtain that

\[
|N(\varphi)| \leq C \| \varphi \|_s^2 \left( \sum_{i=1}^{k} \frac{1}{(1 + |x - x^i|)^{N-2s}} \right)^{2_s^*-3} \left( \sum_{j=1}^{k} \frac{1}{(1 + |x - x^j|)^{\frac{N-2s-\alpha}{2} + \tau}} \right)^2 \\
+ \| \varphi \|_s^{2_s^*-1} \left( \sum_{i=1}^{k} \frac{1}{(1 + |x - x^i|)^{\frac{N-2s-\alpha}{2} + \tau}} \right)^{2_s^*-1}.
\]
\[
\begin{align*}
\leq \ & C(\|\phi\|_2^{2s-1} + \|\phi\|_2^2) \left( \sum_{i=1}^{k} \frac{1}{(1 + |x - x^i|)^{N-2s + \tau}} \right)^{2s-1} \\
\leq \ & C\|\phi\|_2^2 \sum_{i=1}^{k} \frac{1}{(1 + |x - x^i|)^{N-2s + \tau}}.
\end{align*}
\]

Lemma 2.4. Assume that \(\|x^1 - \nu r_0\| \leq \frac{1}{\sqrt{\nu}}\), where \(\bar{\theta} > 0\) is a fixed small constant. Then there is a small \(\sigma > 0\), such that

\[
\|l_k\|_{\ast\ast} \leq C \left( \frac{1}{\nu} \right)^{\frac{m}{2} + \sigma}.
\]

Proof. Define

\[
\Omega_i = \left\{ x : x = (x', x'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2}, \left\langle x', x'' \right\rangle = \cos \frac{\pi}{k} \right\}.
\]

We have

\[
l_k = K \left( \frac{|x|}{\nu} \right) \left( U_{r, \varepsilon}^{2s-1} - \sum_{i=1}^{k} U_{x^i, \varepsilon}^{2s-1} \right) + \sum_{i=1}^{k} U_{x^i, \varepsilon}^{2s-1} \left( K \left( \frac{|x|}{\nu} \right) - 1 \right) := J_1 + J_2.
\]

By symmetry, we can assume that \(x \in \Omega_1\). Then

\[
|x - x^i| \geq |x - x^1|, \ \forall \ x \in \Omega_1.
\]

In order to estimate \(J_1\), we have

\[
|J_1| \leq C \left( \sum_{i=1}^{k} \frac{1}{(1 + |x - x^i|)^{N-2s}} \right)^{2s-1} - \sum_{i=1}^{k} \frac{1}{(1 + |x - x^i|)^{N-2s}} \left( \sum_{i=1}^{k} \frac{1}{(1 + |x - x^i|)^{N-2s}} \right)^{2s-1} \leq C \frac{1}{(1 + |x - x^i|)^{4s}} \sum_{i=2}^{k} \frac{1}{(1 + |x - x^i|)^{N-2s}} + C \left( \sum_{i=1}^{k} \frac{1}{(1 + |x - x^i|)^{N-2s}} \right)^{2s-1}.
\] (2.20)
By Lemma A.1 taking any $1 < \alpha \leq \frac{N+2s}{2}$, we obtain that for any $x \in \Omega_1$,

$$
\frac{1}{(1 + |x - x^i|)^{4s}} \sum_{i=2}^{k} \frac{1}{(1 + |x - x^i|)^{N-2s}} \leq \frac{1}{(1 + |x - x^i|)^{N+2s-\alpha}} \sum_{i=2}^{k} \frac{1}{(1 + |x - x^i|)^{N+2s-\alpha}} \leq C \sum_{i=2}^{k} \left[ \frac{1}{(1 + |x - x^i|)^{N+2s-\alpha}} + \frac{1}{|x^i - x^1|^\alpha} \right] \frac{1}{|x^i - x^1|^\alpha}
$$

$$
\leq C \frac{1}{(1 + |x - x^i|)^{N+2s-\alpha}} \sum_{i=2}^{k} \frac{1}{|x^i - x^1|^\alpha}
$$

$$
\leq C \frac{1}{(1 + |x - x^1|)^{N+2s-\alpha}} \left( \frac{k}{\nu} \right)^\alpha.
$$

We can choose $\alpha > \frac{N-2s}{2}$ with $N + 2s - \alpha \geq \frac{N+2s}{2} + \tau$ since $m \geq N - 2s - 2s(\frac{N-2s}{N+2s}) > N - 2s - 2s(N - 2s)$. Hence

$$
\frac{1}{(1 + |x - x^1|)^{4s}} \sum_{i=2}^{k} \frac{1}{(1 + |x - x^i|)^{N-2s}} \leq C \frac{1}{(1 + |x - x^1|)^{\frac{N+2s}{2} + \tau}} \left( \frac{1}{\nu} \right)^{\frac{m}{2} + \epsilon}.
$$

On the other hand, for $x \in \Omega_1$, by Lemma A.1 again, we get

$$
\frac{1}{(1 + |x - x^i|)^{N-2s}} \leq \frac{1}{(1 + |x - x^i|)^{\frac{N-2s}{2}}} \frac{1}{(1 + |x - x^i|)^{\frac{N-2s}{2}}}
$$

$$
\leq \frac{C}{|x^i - x^1|^{\frac{N-2s}{2}} \frac{N-2s}{N+2s} \tau} \left( \frac{1}{(1 + |x - x^1|)^{\frac{N-2s}{2}} + \frac{1}{(1 + |x - x^1|)^{\frac{N-2s}{2} + \frac{N-2s}{N+2s} \tau}} + \frac{1}{(1 + |x - x^1|)^{\frac{N-2s}{2} + \frac{N-2s}{N+2s} \tau}} \right)
$$

$$
\leq \frac{C}{|x^i - x^1|^{\frac{N-2s}{2}} \frac{N-2s}{N+2s} \tau} \frac{1}{(1 + |x - x^1|)^{\frac{N-2s}{2} + \frac{N-2s}{N+2s} \tau}}.
$$

When $m > 2$, $\tau = \frac{N-2s-m}{N-2s}$, we can obtain $\frac{N-2s}{2} - \frac{N-2s}{N+2s} \tau > 1$. Thus

$$
\sum_{i=2}^{k} \frac{1}{(1 + |x - x^i|)^{N-2s}} \leq C \left( \frac{k}{\nu} \right)^{\frac{N-2s}{2} - \frac{N-2s}{N+2s} \tau} \frac{1}{(1 + |x - x^1|)^{\frac{N-2s}{2} + \frac{N-2s}{N+2s} \tau}},
$$

which gives

$$
\left( \sum_{i=2}^{k} \frac{1}{(1 + |x - x^i|)^{N-2s}} \right)^{2s-1} \leq C \left( \frac{k}{\nu} \right)^{\frac{N-2s-\tau}{2} - \frac{N-2s}{N+2s} \tau} \frac{1}{(1 + |x - x^1|)^{\frac{N-2s}{2} + \frac{N-2s}{N+2s} \tau}}.
$$

So,

$$
\|J_1\|_{\ast \ast} \leq C \left( \frac{1}{\nu} \right)^{\frac{m}{2} + \epsilon}.
$$
Next, we estimate $J_2$. For any $x \in \Omega_1$ and $i = 2, \cdots, k$, applying Lemma [A.1] we have
\[
U_{x_i, \varepsilon}^{2s-1}(x) \leq C \frac{1}{(1 + |x - x_i|)^{N+2s}} \frac{1}{(1 + |x - x|)^{N+2s}} \leq C \left( \frac{1}{(1 + |x - x_i|)^{\frac{N+2s}{2}}} + \frac{1}{(1 + |x - x_i|)^{\frac{N+2s}{2} + \tau}} \right) \frac{1}{|x_i - x^1|^{\frac{N+2s}{2} - \tau}} \leq C \frac{1}{(1 + |x - x_i|)^{\frac{N+2s}{2} + \tau}|x^1 - x|^{\frac{N+2s}{2} - \tau}},
\]
which implies that
\[
\left| \sum_{i=2}^{k} \left( K \left( \frac{|x|}{\nu} \right) - 1 \right) U_{x_i, \varepsilon}^{2s-1} \right| \leq C \frac{1}{(1 + |x - x_i|)^{\frac{N+2s}{2} + \tau}} \sum_{i=1}^{k} \frac{1}{|x_i - x^1|^{\frac{N+2s}{2} - \tau}} \leq C \frac{1}{(1 + |x - x^1|)^{\frac{N+2s}{2} + \tau}} \left( \frac{k}{\nu} \right)^{\frac{N+2s}{2} - \tau}.
\]
For $x \in \Omega_1$ and $|x| - \nu_0 \geq \delta \nu$, where $\delta > 0$ is a fixed constant, then
\[
|x| - |x^1| \geq |x| - \nu_0 - |x^1| - \nu_0 \geq \frac{\delta \nu}{2}.
\]
As a result,
\[
\left| \left( K \left( \frac{|x|}{\nu} \right) - 1 \right) U_{x_i, \varepsilon}^{2s-1} \right| \leq C \frac{1}{(1 + |x - x_i|)^{N+2s}} \leq C \frac{1}{(1 + |x - x^1|)^{\frac{N+2s}{2} + \tau}} \left( \frac{1}{\nu} \right)^{\frac{N+2s}{2} - \tau}.
\]
If $x \in \Omega_1$ and $|x| - \nu_0 \leq \delta \nu$, then
\[
\left| K \left( \frac{|x|}{\nu} \right) - 1 \right| \leq C \frac{|x|}{\nu} - \nu_0 |m| \leq C \frac{\nu^m}{\nu^m} \left( |x| - |x^1| |^m + |x^1| - \nu_0 |^m \right) \leq C \frac{\nu^m}{\nu^m} |x| - |x^1| |^m + C \frac{\nu^m}{\nu^m} \delta \nu,
\]
and
\[
|x| - |x^1| \leq |x| - \nu_0 + \nu_0 - |x^1| \leq 2\delta \nu.
\]
As a result,
\[
\frac{|x| - |x^1| |^m}{\nu^m} \leq C \frac{1}{(1 + |x - x_i|)^{N+2s}} \leq C \frac{\nu^m}{\nu^m} \left( |x| - |x^1| |^m + \nu_0 \right) \leq C \frac{\nu^m}{\nu^m} \left( |x| - |x^1| |^m + \nu_0 \right) \leq C \frac{\nu^m}{\nu^m} \left( |x| - |x^1| |^m + \nu_0 \right).
\]
\[ \leq \frac{C}{\nu^{\frac{n}{2} + \epsilon}} \left( \frac{1}{1 + \|x - x^1\|^\frac{N+2s}{2}} \right)^{\frac{N+2s}{2} - \tau - \frac{m^2}{2} - \epsilon} \]

since \( \frac{N+2s}{2} - \tau - \frac{m^2}{2} - \epsilon \geq 0 \). Thus, we obtain

\[ \left| (K\left(\frac{|x|}{\nu}\right) - 1) \right| \leq \frac{C}{\nu^{\frac{n}{2} + \epsilon}} \left( \frac{1}{1 + \|x - x^1\|^\frac{N+2s}{2}} \right) \] \(|x| - \nu r_0| \leq \delta \nu.\]

Hence, \( \|J_2\| \leq C \left( \frac{1}{\nu} \right)^{\frac{m^2}{2} + \epsilon} \).

Therefore, \( \|l_k\| \leq C \left( \frac{1}{\nu} \right)^{\frac{m^2}{2} + \epsilon} \).

From Lemmas 2.3 and 2.4, we have

**Proposition 2.5.** There is an integer \( k_0 > 0 \), such that for each \( k \geq k_0 \), \( \varepsilon_0 \leq \varepsilon \leq \varepsilon_1 \), \( |r - \nu r_0| \leq \frac{1}{\nu^\alpha} \), where \( \bar{\theta} > 0 \) is a fixed constant, (2.19) has a unique solution \( \varphi = \varphi(r, \varepsilon) \) satisfying

\[ \|\varphi\| \leq C \left( \frac{1}{\nu} \right)^{\frac{m^2}{2} + \epsilon}, \quad \|c_l\| \leq C \left( \frac{1}{\nu} \right)^{\frac{m^2}{2} + \epsilon} \]

where \( \epsilon > 0 \) is a small constant.

**Proof.** First we recall that \( \nu = k^{N-2s-\frac{m}{2}} \). Set

\[ \mathcal{N} = \left\{ w : w \in C^\alpha(\mathbb{R}^N) \cap \mathcal{H}, \|u\| \leq \frac{1}{\nu^\alpha}, \int_{\mathbb{R}^N} U_{x^i,\varepsilon}^{2^*_s - 2} Z_k w = 0 \right\} \]

where \( 0 < \alpha < s \) and \( i = 1, 2, \ldots, k, l = 1, 2 \). Thus from Proposition 2.2, (2.19) is equivalent to

\[ \varphi = A(\varphi) =: L_k(N(\varphi)) + L_k(l_k) \]

\( L_k \) is defined in Proposition 2.2. We obtain

\[ \|A(\varphi)\| \leq C \|N(\varphi)\| + C \|l_k\| \leq C \|\varphi\|^{\min\{2^*_s - 1, 2\}} + C \left( \frac{1}{\nu} \right)^{\frac{m^2}{2} + \epsilon} \]

\[ \leq C \left( \frac{1}{\nu} \right)^{\frac{m^2}{2} + \epsilon} \]

\[ \leq \frac{1}{\nu^\alpha}. \]

Hence, \( A \) maps \( \mathcal{N} \) to \( \mathcal{N} \). On the other hand,

\[ |N'(t)| \leq C |t|^{\min\{2^*_s - 2, 1\}}. \]
We get
\[ \|A(\varphi_1) - A(\varphi_2)\|_* \leq \|L_k(N(\varphi_1)) - L_k(N(\varphi_2))\|_* \leq C\|N(\varphi_1) - N(\varphi_2)\|_{**} \]
\[ \leq C|N'(\varphi_1 + \theta \varphi_2)|\varphi_1 - \varphi_2| \]
\[ \leq C(\|\varphi_1\|_{\text{min}(2^*_s - 2, 1)} + \|\varphi_2\|_{\text{min}(2^*_s - 2, 1)})\|\varphi_1 - \varphi_2\|_* \]
\[ \leq C(\|\varphi_1\|_{\text{min}(2^*_s - 2, 1)} + \|\varphi_2\|_{\text{min}(2^*_s - 2, 1)})\|\varphi_1 - \varphi_2\|_* \left( \sum_{i=1}^{k} \frac{1}{(1 + |x - x_i|)^{\frac{N-2s}{2} + \gamma}} \right)^{\text{min}(2^*_s - 1, 2)} \]
\[ \leq \frac{1}{2}\|\varphi_1 - \varphi_2\|_* . \]

Thus \( A \) is a contraction map. Therefore, \( A \) is a contraction map from \( N \) to \( N \). Now applying the contraction mapping theorem, we can find a unique \( \varphi = \varphi(r, \varepsilon) \in N \) such that
\[ \varphi = A(\varphi). \]

Moreover, by Proposition 2.2 we have
\[ \|\varphi\|_* \leq C \left( \frac{1}{\nu} \right)^{\frac{m}{2} + \varepsilon}. \]

Moreover, we get the estimate of \( c_l \) from (2.11). We also can see (2.16).

\[ \square \]

3. PROOF OF THE MAIN RESULT

Let \( F(r, \varepsilon) = I(U_{r, \varepsilon} + \varphi) \), where \( r = |x^1| \), \( \varphi \) is the function obtained in Proposition 2.5, and
\[ I(u) = \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 - \frac{1}{2s} \int_{\mathbb{R}^N} K \left( \frac{|x|}{\nu} \right) |u|^{2s}. \]

Proposition 3.1. We have
\[ F(r, \varepsilon) = I(U_{r, \varepsilon}) + O \left( \frac{k}{\nu^{m+\varepsilon}} \right) \]
\[ = k \left( A + \frac{B_0}{\varepsilon^m \nu^m} + \frac{B_1}{\varepsilon^{m-2} \nu^m} \left( \nu r_0 - |x^1| \right)^2 \right. \]
\[ - \sum_{i=2}^{k} \frac{B_2}{\varepsilon^{N-2s} |x^1 - x^i|^{N-2s}} + O \left( \frac{1}{\nu^{m+\varepsilon}} + \frac{1}{\nu^{2s}} \right) \]
\[ = k \left( A + \frac{B_0}{\varepsilon^m \nu^m} + \frac{B_1}{\varepsilon^{m-2} \nu^m} \left( \nu r_0 - r \right)^2 \right. \]
\[ - \sum_{i=2}^{k} \frac{B_2}{\varepsilon^{N-2s} r^{N-2s}} + O \left( \frac{1}{\nu^{m+\varepsilon}} + \frac{1}{\nu^m} \left| \nu r_0 - |x^1| \right|^3 + \frac{k}{\nu^{N-2s}} \right) . \]

where \( \varepsilon > 0 \) is a fixed constant, \( B_i > 0, i = 0, 1, 2, 3 \) are some constants.
Proof. Since
\[ \langle I'(U_{r,\varepsilon} + \varphi), \varphi \rangle = 0, \ \forall \varphi \in \mathcal{N}, \]
there is \( t \in (0, 1) \) such that
\[
F(r, \varepsilon) = I(U_{r,\varepsilon} + \varphi)
= I(U_{r,\varepsilon}) - \frac{1}{2} D^2 I(U_{r,\varepsilon} + t\varphi)(\varphi, \varphi)
= I(U_{r,\varepsilon}) - \frac{1}{2} \int_{\mathbb{R}^N} \left( |\nabla \varphi|^2 - (2^*_s - 1) K\left(\frac{|x|}{\nu}\right) (U_{r,\varepsilon} + t\varphi)^{2^*_s - 2} \right) \varphi^2
dfrac{1}{2} \int_{\mathbb{R}^N} (N(\varphi) + l_k) \varphi
+ \frac{2^*_s - 1}{2} \int_{\mathbb{R}^N} K\left(\frac{|x|}{\nu}\right) ((U_{r,\varepsilon} + t\varphi)^{2^*_s - 2} - U_{r,\varepsilon}^{2^*_s - 2}) \varphi^2
= I(U_{r,\varepsilon}) + O\left(\int_{\mathbb{R}^N} \left( |\varphi|^{2^*_s} + |N(\varphi)||\varphi| + |l_k||\varphi| \right) \right).
\]

However,
\[
\int_{\mathbb{R}^N} (|N(\varphi)||\varphi| + |l_k||\varphi|)
\leq C\left(\|N(\varphi)\|_{**} + \|l_k\|_{**}\right) \|\varphi\|_{*} \int_{\mathbb{R}^N} \sum_{i=1}^{k} \frac{1}{(1 + |x - x^i|)^{\frac{N+2s}{2} + \tau}} \sum_{j=1}^{k} \frac{1}{(1 + |x - x^j|)^{\frac{N-2s}{2} + \tau}}.
\]

Since \( \tau = \frac{N - 2s - m}{N - 2s} \sum_{i=2}^{k} \frac{1}{|x^i - x^1|} \leq C \), and Lemma B.1. we obtain
\[
\sum_{i=1}^{k} \frac{1}{(1 + |x - x^i|)^{\frac{N+2s}{2} + \tau}} \sum_{j=1}^{k} \frac{1}{(1 + |x - x^j|)^{\frac{N-2s}{2} + \tau}}
= \sum_{i=1}^{k} \frac{1}{(1 + |x - x^i|)^{N+2s + \tau}} + \sum_{j=1}^{k} \sum_{j \neq i} \frac{1}{(1 + |x - x^i|)^{\frac{N+2s}{2} + \tau}} \frac{1}{(1 + |x - x^j|)^{\frac{N-2s}{2} + \tau}}
\leq \sum_{i=1}^{k} \frac{1}{(1 + |x - x^i|)^{N+2s + \tau}} + C \sum_{i=1}^{k} \frac{1}{(1 + |x - x^i|)^{N+\tau}} \sum_{j=2}^{k} \frac{1}{|x^j - x^1|^\tau}
\leq C \sum_{i=1}^{k} \frac{1}{(1 + |x - x^i|)^{N+\tau}}.
\]

Therefore, we see
\[
\int_{\mathbb{R}^N} (|N(\varphi)||\varphi| + |l_k||\varphi|) \leq C\left(\|N(\varphi)\|_{**} + \|l_k\|_{**}\right) \|\varphi\|_{*} \int_{\mathbb{R}^N} \sum_{i=1}^{k} \frac{1}{(1 + |x - x^i|)^{N+\tau}}
\]
\[ \leq C k (\| N(\phi) \|_{**} + \| l_k \|_{**}) \| \phi \|_* \leq C k \left( \frac{1}{\nu^{m+\epsilon}} \right). \]

On the other hand, by Hölder inequality, we obtain
\[
\int |\phi|^2_* \leq C \| \phi \|_{**}^2 \int \left( \sum_{i=1}^{k} \frac{1}{(1 + |x - x^i|)^{N-2s}} \right)^{2^*_*} \\
\leq C \| \phi \|_{**}^2 \int \left( \sum_{i=1}^{k} \frac{1}{(1 + |x - x^i|)^{N+s}} \right) \left( \sum_{i=1}^{k} \frac{1}{(1 + |x - x^i|)^{s}} \right)^{2^*_*-1} \\
\leq C' \| \phi \|_{**}^2 \int \sum_{i=1}^{k} \frac{1}{(1 + |x - x^i|)^{N+s}} \\
\leq C k \| \phi \|_{**}^2 \\
\leq C k \left( \frac{1}{\nu^{m+\epsilon}} \right).
\]

Therefore applying Proposition A.4 we have
\[
F(r, \epsilon) = I(U_{r, \epsilon}) + O \left( \frac{k}{\nu^{m+\epsilon}} \right) \\
= k \left( A + \frac{B_0}{\nu^{m+\epsilon}} + \frac{B_1}{\nu^{m-2s+1} |x^1|^N} \left( \nu r_0 - |x^1| \right)^2 \\
- \sum_{i=2}^{k} \frac{B_2}{\nu^{N-2s} |x^i - x^1|^N} + O \left( \frac{1}{\nu^{m+\epsilon}} + \frac{1}{\nu^{1+\epsilon}} |x^1|^3 \right) \right).
\]

There is a constant \( B_3 > 0 \), such that
\[
\sum_{i=2}^{k} \frac{1}{|x^i - x^1|^N} = \frac{B_3 k^{N-2s}}{|x^1|^N} + O \left( \frac{k}{|x^1|^N} \right).
\]

So
\[
F(r, \epsilon) = k \left( A + \frac{B_0}{\nu^{m+\epsilon}} + \frac{B_1}{\nu^{m-2s+1} |x^1|^N} \left( \nu r_0 - r \right)^2 \\
- \frac{B_3 k^{N-2s}}{\nu^{N-2s+1} |x^1|^N} + O \left( \frac{1}{\nu^{m+\epsilon}} + \frac{1}{\nu^m} |\nu r_0 - |x^1|^3 + \frac{k}{\nu^{N-2s}} \right) \right).
\]

\[ \Box \]

**Proposition 3.2.**

\[
\frac{\partial F(r, \epsilon)}{\partial \epsilon} = k \left( - \frac{B_0 m}{\nu^{m+\epsilon}} + \sum_{i=2}^{k} \frac{B_2 (N-2s)}{\nu^{N-2s+1} |x^i - x^1|^N} \\
+ O \left( \frac{1}{\nu^{m+\epsilon}} + \frac{1}{\nu^m} |\nu r_0 - |x^1|^3 \right) \right) \]
\[ k \left( - \frac{B_0 m}{\epsilon^{m+1} \nu^m} + \frac{B_3 (N - 2s) k^{N-2s}}{\epsilon^{N-2s} + \nu^{N-2s}} \right) + O \left( \frac{1}{\nu^{m+\epsilon}} + \frac{1}{\nu^m |\nu r_0 - |x^1||^2} + \frac{k}{\nu^{N-2s}} \right), \]

where \( \epsilon > 0 \) is a fixed constant.

**Proof.** We have

\[
\frac{\partial F(r, \epsilon)}{\partial \epsilon} = \left\langle I'(U_{r, \epsilon} + \varphi), \frac{\partial U_{r, \epsilon}}{\partial \epsilon} + \frac{\partial \varphi}{\partial \epsilon} \right\rangle
\]

\[
= \left\langle I'(U_{r, \epsilon} + \varphi), \frac{\partial U_{r, \epsilon}}{\partial \epsilon} \right\rangle + \sum_{i=1}^{k} \sum_{l=1}^{k} c_i \left\langle U_{x^i, \epsilon} Z_{i,l}, \frac{\partial \varphi}{\partial \epsilon} \right\rangle
\]

\[
= \frac{\partial I(U_{r, \epsilon})}{\partial \epsilon} - \int_{\mathbb{R}^N} K \left( \frac{|x|}{\nu} \right) [(U_{r, \epsilon} + \varphi) \cdot \rho_{2s-1} - U_{r, \epsilon}^{2s-1}] \frac{\partial U_{r, \epsilon}}{\partial \epsilon}
\]

\[
+ \sum_{l=1}^{2} \sum_{i=1}^{k} c_i \left\langle U_{x^i, \epsilon} Z_{i,l}, \varphi \right\rangle.
\]

Note that

\[ \left\langle U_{x^i, \epsilon}^{2s-2} Z_{i,l}, \frac{\partial \varphi}{\partial \epsilon} \right\rangle = -\left\langle \frac{\partial (U_{x^i, \epsilon}^{2s-2} Z_{i,l})}{\partial \epsilon}, \varphi \right\rangle. \]

Thus,

\[
\left| \sum_{i=1}^{k} c_i \left\langle U_{x^i, \epsilon}^{2s-2} Z_{i,l}, \frac{\partial \varphi}{\partial \epsilon} \right\rangle \right|
\]

\[
\leq C |c_i| \|\varphi\| \int_{\mathbb{R}^N} \sum_{i=1}^{k} \frac{1}{(1 + |x - x^j|)^{N+2s}} \sum_{l=1}^{k} \frac{1}{(1 + |x - x^j|)^{N+2s}}
\]

\[
= C |c_i| \|\varphi\| \sum_{i=1}^{k} \left( \int_{\mathbb{R}^N} \frac{1}{(1 + |x - x^j|)^{N+2s}} + \int_{\mathbb{R}^N} \frac{1}{(1 + |x - x^j|)^{N-2s}} \sum_{j \neq i} \frac{1}{(1 + |x - x^j|)^{N+2s}} \right)
\]

\[
\leq C |c_i| \|\varphi\| \sum_{i=1}^{k} \int_{\mathbb{R}^N} \left( \frac{1}{(1 + |x - x^j|)^{N+2s}} + \frac{1}{(1 + |x - x^j|)^{N+2s}} \sum_{i=2}^{k} \frac{1}{|x^i - x^j|^{N+2s}} \right)
\]

\[
\leq C \frac{k}{\nu^{m+\epsilon}}.
\]

On the other hand, from \( \varphi \in \mathcal{N} \) we have

\[
\int_{\mathbb{R}^N} K \left( \frac{|x|}{\nu} \right) [(U_{r, \epsilon} + \varphi) \cdot \rho_{2s-1} - U_{r, \epsilon}^{2s-1}] \frac{\partial U_{r, \epsilon}}{\partial \epsilon}
\]

\[
= \int_{\mathbb{R}^N} (2s - 1) K \left( \frac{|x|}{\nu} \right) U_{r, \epsilon}^{2s-1} \frac{\partial U_{r, \epsilon}}{\partial \epsilon} \varphi + O \left( \int_{\mathbb{R}^N} |\varphi|^2 \right)
\]
Thus, So

Define

where \( \bar{\nu} \)

This completes our proof.

Let \( \varepsilon_0 \) be the solution of

\[
-\frac{B_0 m}{\varepsilon^{m+1}} + \frac{B_3 (N-2s)}{\varepsilon^{N-2s+1} r_0^{N-2s}} = 0.
\]

Then

\[
\varepsilon_0 = \left( \frac{B_3 (N-2s)}{B_0 \bar{\theta} r_0^{N-2s}} \right)^{\frac{1}{N-2s-m}}.
\]

Define

\[
\Omega = \left\{ (r, \varepsilon) : r \in \left[ \nu r_0 - \frac{1}{\nu^\bar{\theta}}, \nu r_0 + \frac{1}{\nu^\bar{\theta}} \right], \varepsilon \in \left[ \varepsilon_0 - \frac{1}{\nu^\bar{\theta}}, \varepsilon_0 + \frac{1}{\nu^\bar{\theta}} \right] \right\},
\]

where \( \bar{\theta} > 0 \) is a small constant.

For any \( (r, \varepsilon) \in \Omega \), we have

\[
r = r_0 + O\left( \frac{1}{\nu^{1+\bar{\theta}}} \right).
\]

Thus,

\[
r^{N-2s} = \nu^{N-2s} \left( r_0^{N-2s} + O\left( \frac{1}{\nu^{1+\bar{\theta}}} \right) \right).
\]

So

\[
F(r, \varepsilon) = k \left( A + \left( \frac{B_0}{\varepsilon^m} - \frac{B_3}{\varepsilon^{N-2s} r_0^{N-2s}} \right) \frac{1}{\nu^m} + \frac{B_3}{\varepsilon^{m-2} \nu^m} (\nu r_0 - r)^2
+ O\left( \frac{1}{\nu^{m+\bar{\theta}}} + \frac{1}{\nu^m} |\nu r_0 - r|^3 + \frac{k}{\nu^{N-2s}} \right) \right), \quad (r, \varepsilon) \in \Omega,
\]

(3.1)
and
\[
\frac{\partial F(r, \varepsilon)}{\partial \varepsilon} = k \left( \left( - \frac{B_0 m}{\varepsilon^{m+1}} + \frac{B_3 (N - 2s)}{\varepsilon^{N-2s+1} r_0^{N-2s}} \right) \frac{1}{\nu^m} 
+ O \left( \frac{1}{\nu^m + \varepsilon} |\nu r_0 - r|^2 + \frac{k}{\nu^{N-2s}} \right) \right) (r, \varepsilon) \in \Omega.
\]

(3.2)

Now, we define
\[
\tilde{F}(r, \varepsilon) = -F(r, \varepsilon), \quad (r, \varepsilon) \in \Omega.
\]

Let
\[
\alpha_1 = k \left( -A - \left( \frac{B_0}{\varepsilon_0^m} - \frac{B_3}{\varepsilon_0^{N-2s} r_0^{N-2s}} \right) \frac{1}{\nu^m} - \frac{1}{\nu^m + \theta} \right), \quad \alpha_2 = k(-A + \eta),
\]

where \(\eta > 0\) is a small constant. For \(c \in \mathbb{R}\), define
\[
\tilde{F}^c = \left\{ (r, \varepsilon) \in \Omega, \tilde{F}(r, \varepsilon) \leq c \right\}.
\]

Consider
\[
\begin{align*}
\frac{dr}{dt} &= -D_r \tilde{F}, \quad t > 0; \\
\frac{d\varepsilon}{dt} &= -D_\varepsilon \tilde{F}, \quad t > 0; \\
(r, \varepsilon) &\in \tilde{F}^{\alpha_2}.
\end{align*}
\]

Then we have

**Lemma 3.3.** The flow \((r(t), \varepsilon(t))\) does not leave \(\Omega\) before it reaches \(\tilde{F}^{\alpha_1}\).

**Proof.** If \(\varepsilon = \varepsilon_0 + \frac{1}{\nu^m}\), observing that \(|r - \nu r_0| \leq \frac{1}{\nu^m}\), it follows from (3.2) that
\[
\frac{\partial \tilde{F}(r, \varepsilon)}{\partial \varepsilon} = k \left( c' \frac{1}{\nu^m + \theta} + O \left( \frac{1}{\nu^m + \theta} \right) \right) > 0.
\]

Hence the flow does not leave \(\Omega\).

Similarly, if \(\varepsilon = \varepsilon_0 - \frac{1}{\nu^m}\), it follows from (3.2) that
\[
\frac{\partial \tilde{F}(r, \varepsilon)}{\partial \varepsilon} = k \left( -c' \frac{1}{\nu^m + \theta} + O \left( \frac{1}{\nu^m + \theta} \right) \right) < 0.
\]

Therefore the flow does not leave \(\Omega\).

Now suppose that \(|r - \nu r_0| = \frac{1}{\nu^m}\). Since \(|\varepsilon - \varepsilon_0| \leq \frac{1}{\nu^m}\), we see
\[
\frac{B_0}{\varepsilon^m} - \frac{B_3}{\varepsilon^{N-2s} r_0^{N-2s}} = \frac{B_0}{\varepsilon_0^m} - \frac{B_3}{\varepsilon_0^{N-2s} r_0^{N-2s}} + O(|\varepsilon - \varepsilon_0|^2)
= \frac{B_1}{\varepsilon_0^m} - \frac{B_3}{\varepsilon_0^{N-2s} r_0^{N-2s}} + O \left( \frac{1}{\nu^{2\theta}} \right).
\]
So it follows from (3.1) that
\[
\tilde{F}(r, \Lambda) = k\left(-A - \left(\frac{B_1}{\varepsilon_0^m} - \frac{B_4}{\varepsilon_0^{N-2s}}\right) \frac{1}{\nu^m} - \frac{B_2}{\varepsilon_0^{m-2}}(\nu r_0 - r)^2 + O\left(\frac{1}{\nu^{m+3\theta}}\right)\right) \\
\leq k\left(-A - \left(\frac{B_1}{\varepsilon_0^m} - \frac{B_4}{\varepsilon_0^{N-2s}}\right) \frac{1}{\nu^m} - \frac{B_2}{\varepsilon_0^{m-2}} + O\left(\frac{1}{\nu^{m+3\theta}}\right)\right) \\
< \alpha_1.
\]

(3.3)

\[\square\]

**Proof of Theorem 1.3** We will prove that \(\tilde{F}\), and therefore \(F\) has a critical point in \(\Omega\). Define
\[
\Gamma = \left\{ g : g(r, \varepsilon) = (g_1(r, \varepsilon), g_2(r, \varepsilon)) \in \Omega, (r, \varepsilon) \in \Omega, \right. \\
g(r, \varepsilon) = (r, \varepsilon), \text{if } |r - \nu r_0| = \frac{1}{\nu^p}\}
\]

Denote
\[
c = \inf_{g \in \Gamma} \max_{(r, \varepsilon) \in \Omega} \tilde{F}(g(r, \varepsilon)).
\]

We claim that \(c\) is a critical value of \(\tilde{F}\). In order to prove this, we have to prove
\[
(i) \quad \alpha_1 < c < \alpha_2; \\
(ii) \quad \sup_{|r - \nu r_0| = \frac{1}{\nu^p}} \tilde{F}(g(r, \varepsilon)) < \alpha_1, \quad \forall g \in \Gamma.
\]

In order to prove (ii), let \(g \in \Gamma\). Then for any \(r\) with \(|r - \nu r_0| = \frac{1}{\nu^p}\), we have \(g(r, \varepsilon) = (r, \varepsilon)\).

Hence, from (3.3), we obtain
\[
\tilde{F}(g(r, \varepsilon)) = \tilde{F}(r, \varepsilon) < \alpha_1.
\]

Now we prove (i). It is obvious that \(c < \alpha_2\). For any \(\gamma = (\gamma_1, \gamma_2) \in \Gamma\), then \(g_1(r, \varepsilon) = r\), if \(|r - \nu r_0| = \frac{1}{\nu^p}\). Define
\[
\tilde{g}_1(r) = g_1(r, \varepsilon_0).
\]

Then \(\tilde{g}_1(r) = r\) if \(|r - \nu r_0| = \frac{1}{\nu^p}\). Hence there is an \(\tilde{r} \in \left(\nu r_0 - \frac{1}{\nu^p}, \nu r_0 + \frac{1}{\nu^p}\right)\) such that
\[
\tilde{g}_1(\tilde{r}) = \nu r_0.
\]

Let \(\tilde{\varepsilon} = g_2(\tilde{r}, \varepsilon_0)\). Then it follows from (3.1) that
\[
\max_{(r, \varepsilon) \in \Omega} \tilde{F}(g(r, \varepsilon)) \geq \tilde{F}\left(g\left(\tilde{r}, \varepsilon_0\right)\right) = \tilde{F}(\nu r_0, \tilde{\varepsilon}) \\
= k\left(-A - \left(\frac{B_0}{\varepsilon_0^m} - \frac{B_3}{\varepsilon_0^{N-2s}}\right) \frac{1}{\nu^m} + O\left(\frac{1}{\nu^{m+\theta}} + \frac{k}{\nu^{N-2s}}\right)\right) \\
\geq k\left(-A - \left(\frac{B_0}{\varepsilon_0^m} - \frac{B_3}{\varepsilon_0^{N-2s}}\right) \frac{1}{\nu^m} + O\left(\frac{1}{\nu^{m+3\theta}}\right)\right) > \alpha_1.
\]
\[\square\]
Now we come to give the sketch of proof for Theorem 1.4. Recall that
\[ \bar{U}_{r, \varepsilon}(x) = \sum_{i=1}^{2k} (-1)^{i-1} U_{x^i}(x). \]
We will seek for a solution for equation (1.1) of the form $\bar{U}_{r, \varepsilon}(x) + \varphi$ with $\varphi = \varphi(r, \varepsilon)$ solved (2.19). To this end, we should also perform the same procedure as the proof of Theorem 1.3.

Proceeding as we proved Propositions 3.1 and 3.2, we conclude that
\[ F(r, \varepsilon) := I(\bar{U}_r(x) + \varphi) = k \left( A - \frac{B_0^m}{\varepsilon^m \nu^m} - \frac{B_1^m}{\varepsilon^{m-2} \nu^m} \left( \nu r_0 - r \right)^2 \right. \]
\[ + \frac{B_3^m}{\varepsilon^{N-2s} \nu^{N-2s}} + O \left( \frac{1}{\nu^m + \varepsilon} + \frac{1}{\nu^m} \left| \nu r_0 - \bar{x}^1 \right|^3 + \frac{k}{\nu^{N-2s}} \right) \]
and
\[ \frac{\partial F(r, \varepsilon)}{\partial \varepsilon} = k \left( \frac{B_0^m}{\varepsilon^m + 1} \nu^m - \frac{B_3^m}{\varepsilon^{N-2s} \nu^{N-2s}} + O \left( \frac{1}{\nu^m + \varepsilon} + \frac{1}{\nu^m} \left| \nu r_0 - \bar{x}^1 \right|^2 + \frac{k}{\nu^{N-2s}} \right) \right). \]

Let $\varepsilon_0$ and $\Omega$ be given as above. Define
\[ \alpha_1 = k \left( A - \frac{B_0^m}{\varepsilon_0^m} - \frac{B_3^m}{\varepsilon_0^{N-2s} r_0^{N-2s}} \right) \frac{1}{\nu^m} + \frac{1}{\nu^{m+\theta}} \right), \quad \alpha_2 = k(A + \eta), \]
where $\eta > 0$ is a small constant so that $\alpha_1 < \alpha_2$.

Then proceeding as done in the proof of Theorem 1.3, we can prove Theorem 1.4.

APPENDIX A. ENERGY EXPANSION

In this section, we will give some basic estimates and the energy expansion for the approximate solutions. Recall
\[ x^i = \left( r \cos \frac{2(i - 1) \pi}{k}, r \sin \frac{2(i - 1) \pi}{k}, 0 \right), \quad i = 1, \ldots, k, \quad 0 \in \mathbb{R}^{N-2}, \]
\[ \Omega_i = \left\{ x = (x', x^m) \in \mathbb{R}^2 \times \mathbb{R}^{N-2} : \frac{x'}{\left| x' \right|}, \frac{x^i}{\left| x^i \right|} \geq \cos \frac{\pi}{k} \right\}, \quad i = 1, 2, \ldots, k, \]
\[ U_{r, \varepsilon}(x) = C_{N,s} \varepsilon \sum_{i=1}^{k} \frac{\varepsilon^{N-2s}}{(1 + \varepsilon^2 \left| x - x^i \right|)^{N-2s}}, \]
and
\[ I(u) = \frac{1}{2} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} - \frac{1}{2s} \int_{\mathbb{R}^N} K\left( \frac{|x|}{\nu} \right) |u|^{2s} = \frac{1}{2} \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{s}{2}} u \right|^2 - \frac{1}{2s} \int_{\mathbb{R}^N} K\left( \frac{|x|}{\nu} \right) |u|^{2s}. \]

Similar Lemma B.1 and B.2 in [31], we have,
Lemma A.1. For any constant $0 < \sigma \leq \min\{\alpha, \beta\}$, there is a constant $C > 0$, such that
\[
\frac{1}{(1 + |y - x^i|)^{\alpha}} \frac{1}{(1 + |y - x^j|)^{\beta}} \leq \frac{C}{|x^i - x^j|^{\alpha + \beta - \sigma}} \left(1 + |y - x^i|^{\alpha + \beta - \sigma}ight).
\]

Lemma A.2. For any constant $0 < \kappa < N - 2s$, there is a constant $C > 0$, such that
\[
\int_{\mathbb{R}^N} \frac{1}{|x|^N} \frac{1}{(1 + |y - x|)^{2s + \kappa}} dx \leq \frac{C}{(1 + |y|)^\kappa}.
\]

Lemma A.3. There is a small $\alpha > 0$, such that
\[
\int_{\mathbb{R}^N} \frac{1}{|x - y|^{N-2s}} U_{r,\varepsilon}^{N-2s} (y) \sum_{i=1}^k \frac{1}{(1 + |y - x^i|)^{\frac{N-2s}{2} + \alpha}} dy \leq C \sum_{i=1}^k \frac{1}{(1 + |x - x^i|)^{\frac{N-2s}{2} + \alpha + \alpha}}.
\]

Proof. Since
\[
|x^i - x^1| = 2|x^1| \sin \frac{(i-1)\pi}{k}, \quad i = 2, \ldots, k,
\]
we have
\[
\sum_{i=2}^k \frac{1}{|x^i - x^1|^\eta} = \frac{1}{(2|x^1|)^\eta} \sum_{i=2}^k \frac{1}{\sin \left(\frac{(i-1)\pi}{k}\right)^\eta} = \begin{cases}
\frac{2}{(2|x^1|)^\eta} \sum_{i=2}^\left\lfloor \frac{k}{2} \right\rfloor \frac{1}{\sin \left(\frac{(i-1)\pi}{k}\right)^\eta} + \frac{1}{(2|x^1|)^\eta}, & \text{if } k \text{ is even}, \\
\frac{2}{(2|x^1|)^\eta} \sum_{i=2}^\left\lfloor \frac{k}{2} \right\rfloor \frac{1}{\sin \left(\frac{(i-1)\pi}{k}\right)^\eta}, & \text{if } k \text{ is odd}.
\end{cases}
\]

But
\[
0 < C \leq \frac{\sin \left(\frac{(i-1)\pi}{k}\right)}{\frac{1}{2} \frac{k}{\sin \left(\frac{(i-1)\pi}{k}\right)}} \leq C', \quad i = 2, \ldots, \left\lfloor \frac{k}{2} \right\rfloor.
\]

So, for $\nu = k^{\frac{N-2s}{2s-m}}$ and any $\eta \geq \frac{N-2s-m}{N-2s}$,
\[
\sum_{i=2}^k \frac{1}{|x^i - x^1|^{\nu\eta}} \leq \frac{C k^{\eta}}{\nu^{\eta}} \sum_{i=2}^k \frac{1}{i^{\eta}} = \begin{cases}
\frac{C k^{\eta} \ln k}{\nu^{\eta}} \leq C, & \eta \geq 1, \\
\frac{C k^{\eta}}{\nu^{\eta}}, & \eta < 1.
\end{cases}
\]

For any $x \in \Omega_1$, we have for $i \neq 1, |x - x^i| \geq |x - x^1|$. By using Lemma A.1, we obtain
\[
\sum_{i=2}^k \frac{1}{(1 + |x - x^i|)^{N-2s}} \leq \sum_{i=2}^k \frac{1}{(1 + |x - x^1|)^{N-2s-\eta}} \frac{1}{(1 + |x - x^i|)^{\eta}} \leq \frac{C}{(1 + |x - x^1|)^{N-2s-\eta}} \sum_{i=2}^k \frac{1}{|x^1 - x^i|^{\eta}} \leq \frac{C}{(1 + |x - x^1|)^{N-2s-\eta}}.
\]
Thus,

\[ U_{r,\varepsilon}^{\frac{4s}{N-2s}} \leq \frac{C}{(1 + |x - x^1|)^{4s - \frac{4s\eta}{N-2s}}}. \]

It follows from Lemmas A.1 and A.2 that

\[ \int_{\Omega_1} \frac{1}{|y - x|^{N-2s}} U_{r,\varepsilon}^{\frac{4s}{N-2s}} \sum_{i=1}^{k} \frac{1}{(1 + |x - x^i|)^{\frac{N-2s}{2} + \sigma}} \]

\[ \leq C \int_{\Omega_1} \frac{1}{|y - x|^{N-2s}} \left( \frac{1}{(1 + |x - x^1|)^{\frac{N+6s}{2} + \sigma - \frac{4s\eta}{N-2s}}} + \frac{1}{(1 + |x - x^1|)^{\frac{N+6s}{2} + \sigma} - \frac{(N+2s)\eta}{N-2s}} \sum_{i=2}^{k} \frac{1}{|x^i - x^1|^{\eta}} \right) \]

\[ \leq C \left( \frac{1}{(1 + |y - x^1|)^{\frac{N+2s}{2} + \sigma - \frac{(N+2s)\eta}{N-2s}}} \right) \]

\[ \leq C \left( \frac{1}{(1 + |y - x^1|)^{\frac{N-2s}{2} + \sigma + \alpha}} \right), \]

since \( 2s - \frac{(N+2s)\eta}{N-2s} > 0 \) when \( m > N - 2s(1 + \frac{(N-2s)^2}{N+2s}) \).

Thus, we obtain

\[ \int_{\mathbb{R}^N} \frac{1}{|y - x|^{N-2s}} U_{r,\varepsilon}^{\frac{4s}{N-2s}} \sum_{i=1}^{k} \frac{1}{(1 + |x - x^i|)^{\frac{N-2s}{2} + \sigma}} \]

\[ = \sum_{j=1}^{k} \int_{\Omega_j} \frac{1}{|y - x|^{N-2s}} U_{r,\varepsilon}^{\frac{4s}{N-2s}} \sum_{i=1}^{k} \frac{1}{(1 + |x - x^i|)^{\frac{N-2s}{2} + \sigma}} \]

\[ \leq C \sum_{i=1}^{k} \left( \frac{1}{(1 + |y - x^i|)^{\frac{N-2s}{2} + \sigma + \alpha}} \right). \]

\[ \square \]

Proposition A.4. We have

\[ I(U_{r,\varepsilon}) = \sum \left( A + \frac{B_0}{\varepsilon^{m \mu m}} + \frac{B_1}{\varepsilon^{m-2 \mu m}} (\nu r_0 - r)^2 - \sum_{i=2}^{k} \frac{B_2}{\varepsilon^{N-2s} |x^1 - x^i|^N |x^1 - x^i|^{2+\sigma}} \right) \]

\[ + O\left( \frac{1}{\varepsilon^{m+\sigma}} + \frac{1}{\varepsilon^{m \mu m}} |\nu r_0 - r|^2 + \frac{1}{\varepsilon^{m+\sigma}} \right) \]

where \( B_0 \in [C_1, C_2], \ 0 < C_1 < C_2, \ A = \frac{2}{N} \int_{\mathbb{R}^N} U_{0,1}^{2s}, \ B_i, C_i, i = 1, 2, \) are some constants, and \( r = |x^1| \).
Proof. Using the symmetry, we have

\[
\langle U_r, U_r \rangle_s = \sum_{i=1}^{k} \sum_{j=1}^{k} \int_{\mathbb{R}^N} U_{x^j,\varepsilon}^{2s-1} U_{x^i,\varepsilon}
\]

\[
= k \left( \int_{\mathbb{R}^N} U_{0,1}^{2s} + \sum_{i=2}^{k} \int_{\mathbb{R}^N} U_{x^i,\varepsilon}^{2s-1} U_{x^i,\varepsilon} \right) \tag{A.1}
\]

\[
= k \int_{\mathbb{R}^N} U_{0,1}^{2s} + k \sum_{i=2}^{k} \int_{\mathbb{R}^N} U_{x^i,\varepsilon}^{2s-1} U_{x^i,\varepsilon}.
\]

It follows from Lemma [A.1] that

\[
\sum_{i=2}^{k} \int_{\mathbb{R}^N} U_{x^i,\varepsilon}^{2s-1} U_{x^i,\varepsilon} = C \int_{\mathbb{R}^N} \frac{1}{\left(1 + \varepsilon|x - x^1|\right)^{N-2s}} \sum_{i=2}^{k} \frac{1}{\left(1 + \varepsilon|x - x^1|\right)^{N-2s}}
\]

\[
\leq C \sum_{i=2}^{k} \frac{1}{\varepsilon^{N-2s}|x^1 - x^i|N-2s} \int_{\mathbb{R}^N} \frac{1}{\left(1 + \varepsilon|x - x^1|\right)^{N-2s}(2s-1)} + O\left(\sum_{i=2}^{k} \frac{1}{\varepsilon^{N-2s+\sigma}|x^1 - x^i|N-2s+\sigma}\right)
\]

\[
= \sum_{i=2}^{k} \frac{C_1}{\varepsilon^{N-2s}|x^1 - x^i|N-2s} + O\left(\sum_{i=2}^{k} \frac{1}{|x^1 - x^i|N-2s+\sigma}\right).
\]

However,

\[
\sum_{i=2}^{k} \int_{\mathbb{R}^N} U_{x^i,\varepsilon}^{2s-1} U_{x^i,\varepsilon} = \int_{\mathbb{R}^N} \frac{1}{\left(1 + \varepsilon|x - x^1|\right)^{N-2s}} \sum_{i=2}^{k} \frac{1}{\left(1 + \varepsilon|x - x^1|\right)^{N-2s}}
\]

\[
\geq \sum_{i=2}^{k} \int_{B_{|x^1 - x^i|}(x^1)} \frac{1}{\left(1 + \varepsilon|x - x^1|\right)^{N-2s}} \sum_{i=2}^{k} \frac{1}{\left(1 + \varepsilon|x - x^1|\right)^{N-2s}}
\]

\[
+ \sum_{i=2}^{k} \int_{B_{|x^1 - x^i|}(x^1)} \frac{1}{\left(1 + \varepsilon|x - x^1|\right)^{N-2s}} \sum_{i=2}^{k} \frac{1}{\left(1 + \varepsilon|x - x^1|\right)^{N-2s}}
\]

\[
\geq \sum_{i=2}^{k} \frac{C_2}{\varepsilon^{N-2s}|x^1 - x^i|N-2s} + O\left(\sum_{i=2}^{k} \frac{1}{|x^1 - x^i|N-2s+\sigma}\right).
\]

Hence, there exists $B_0$ in $[C_2, C_1]$, where $C_1$ and $C_2$ are independent of $k$, such that

\[
\sum_{i=2}^{k} \int_{\mathbb{R}^N} U_{x^i,\varepsilon}^{2s-1} U_{x^i,\varepsilon} = \sum_{i=2}^{k} \frac{B_0}{\varepsilon^{N-2s}|x^1 - x^i|N-2s} + O\left(\sum_{i=2}^{k} \frac{1}{|x^1 - x^i|N-2s+\sigma}\right). \tag{A.2}
\]
Now, by symmetry, we see

\[
\int_{\mathbb{R}^N} K\left(\frac{|x|}{\nu}\right) U_r^{2s} = k \int_{\Omega_1} K\left(\frac{|x|}{\nu}\right) U_{x^i,\varepsilon}^{2s} + k2_s^* \int_{\Omega_1} K\left(\frac{|x|}{\nu}\right) \sum_{i=2}^k U_{x^i,\varepsilon}^{2s-1} U_{x^i,\varepsilon}
\]

\[
+ k \begin{cases} 
O\left( \int_{\Omega_1} U_{x^i,\varepsilon}^{2s} \left( \sum_{i=2}^k U_{x^i,\varepsilon} \right) \right), & \text{if } 2 < 2_s^* < 3, \\
O\left( \int_{\Omega_1} U_{x^i,\varepsilon}^{2s-2} \left( \sum_{i=2}^k U_{x^i,\varepsilon} \right)^2 \right), & \text{if } 2_s^* \geq 3.
\end{cases}
\]

For \( x \in \Omega_1, |x - x^i| \geq \frac{1}{2} |x^i - x^1| \) and \(|x - x^i| \geq |x - x^1|\), we obtain

\[
\sum_{i=2}^k U_{x^i,\varepsilon} \leq C \sum_{i=2}^k \frac{1}{(1 + |x - x^1|)^{N-2s-\kappa}} \frac{1}{(1 + |x - x^1|)^\kappa}
\]

\[
\leq C \sum_{i=2}^k \frac{1}{|x^1 - x^i|^{N-2s-\kappa}} \frac{1}{(1 + |x - x^1|)^\kappa},
\]

where \( s < \kappa < \min\{\frac{4}{3}s, \frac{N-2s}{2}\} \). Hence, we get

\[
\int_{\Omega_1} U_{x^i,\varepsilon}^{2s} \left( \sum_{i=2}^k U_{x^i,\varepsilon} \right)^{\frac{2s}{2}} \leq C \int_{\Omega_1} \frac{1}{(1 + |x - x^1|)^{\frac{(N-2s)s}{2}}} \left( \sum_{i=2}^k \frac{1}{|x^1 - x|^{N-2s-\kappa}} \right)^{\frac{2s}{2}} \frac{1}{(1 + |x - x^1|)^{\frac{2s}{2}+\kappa}}
\]

\[
= C \left( \sum_{i=2}^k \frac{1}{|x^1 - x^i|^{N-2s-\kappa}} \right)^{\frac{2s}{2}} \int_{\Omega_1} \frac{1}{(1 + |x - x^1|)^{\frac{(N-2s)s}{2}+\frac{2s}{2}+\kappa}}
\]

\[
\leq C \left( \sum_{i=2}^k \frac{1}{|x^1 - x^i|^{N-2s-\kappa}} \right)^{\frac{2s}{2}} \leq C \left( \frac{k}{\nu} \right)^{N-2s+\sigma}
\]

and

\[
\int_{\Omega_1} U_{x^i,\varepsilon}^{2s-2} \left( \sum_{i=2}^k U_{x^i,\varepsilon} \right)^2 \leq C \int_{\Omega_1} \frac{1}{(1 + |x - x^1|)^{(N-2s)(2s-2)}} \left( \sum_{i=2}^k \frac{1}{|x^1 - x^i|^{N-2s-\kappa}} \right)^2 \frac{1}{(1 + |x - x^1|)^{2\kappa}}
\]

\[
= C \left( \sum_{i=2}^k \frac{1}{|x^1 - x^i|^{N-2s-\kappa}} \right)^2 \int_{\Omega_1} \frac{1}{(1 + |x - x^1|)^{(N-2s)(2s-2)+2\kappa}}
\]
On the other hand,
\[
\int_{\Omega_1} K\left(\frac{|x|}{\nu}\right) U_{x_1,e}^{2s-1} \sum_{i=2}^k U_{x_i,e} = \int_{\Omega_1} U_{x_1,e}^{2s-1} \sum_{i=2}^k U_{x_i,e} + \int_{\Omega_1} (K\left(\frac{|x|}{\nu}\right) - 1) U_{x_1,e}^{2s-1} \sum_{i=2}^k U_{x_i,e}.
\]
But, from Lemmas A.1 and A.2,
\[
\int_{\Omega_1} U_{x_1,e}^{2s-1} \sum_{i=2}^k U_{x_i,e} = \int_{\mathbb{R}^N} U_{x_1,e}^{2s-1} \sum_{i=2}^k U_{x_i,e} - \int_{\mathbb{R}^N \setminus \Omega_1} U_{x_1,e}^{2s-1} \sum_{i=2}^k U_{x_i,e}
\]
\[
= \int_{\mathbb{R}^N} U_{x_1,e}^{2s-1} \sum_{i=2}^k U_{x_i,e} + O\left(\left(\frac{k}{\nu}\right)^{\sigma} \sum_{i=2}^k \frac{1}{|x_1 - x_i|^{N-2s}} \int_{\mathbb{R}^N \setminus \Omega_1} \left(U_{x_1,e}^{2s-1-\sigma} + U_{x_i,e}^{2s-1-\sigma}\right)\right)
\]
\[
= \sum_{i=2}^k e^{N-2s}|x_1 - x_i|^N - \sum_{i=2}^k \frac{B_0'}{e^{N-2s}|x_1 - x_i|^N - 2s} + O\left(\left(\frac{k}{\nu}\right)^{N-2s+\sigma}\right),
\]
where \(0 < \sigma < \frac{2s}{N-2s}\). Moreover, similarly,
\[
\int_{\Omega_1} \left| K\left(\frac{|x|}{\nu}\right) - 1 \right| U_{x_1,e}^{2s-1} \sum_{i=2}^k U_{x_i,e}
\]
\[
= \int_{\mathbb{R}^N} \left| K\left(\frac{|x|}{\nu}\right) - 1 \right| U_{x_1,e}^{2s-1} \sum_{i=2}^k U_{x_i,e} - \int_{\mathbb{R}^N \setminus \Omega_1} \left| K\left(\frac{|x|}{\nu}\right) - 1 \right| U_{x_1,e}^{2s-1} \sum_{i=2}^k U_{x_i,e}
\]
\[
\leq \int_{B_{\frac{1}{2}}(x_1)} \left| K\left(\frac{|x|}{\nu}\right) - 1 \right| U_{x_1,e}^{2s-1} \sum_{i=2}^k U_{x_i,e} + C \int_{\mathbb{R}^N \setminus B_{\frac{1}{2}}(x_1)} U_{x_1,e}^{2s-1} \sum_{i=2}^k U_{x_i,e} + O\left(\left(\frac{k}{\nu}\right)^{N-2s+\sigma}\right)
\]
\[
\leq \frac{C}{\nu^m} \int_{\mathbb{R}^N} U_{x_1,e}^{2s-1} \sum_{i=2}^k U_{x_i,e} + O\left(\left(\frac{k}{\nu}\right)^{\sigma} \sum_{i=2}^k \frac{1}{|x_1 - x_i|^{N-2s}} \int_{\mathbb{R}^N \setminus B_{\frac{1}{2}}(x_1)} \left(U_{x_1,e}^{2s-1-\sigma} + U_{x_i,e}^{2s-1-\sigma}\right)\right)
\]
\[+ O\left(\left(\frac{k}{\nu}\right)^{N-2s+\sigma}\right) + \frac{1}{\nu^{m+\sigma}}\right).
\]
Hence,
\[
\int_{\Omega_1} K\left(\frac{|x|}{\nu}\right) U_{x_1,e}^{2s-1} \sum_{i=2}^k U_{x_i,e} = \frac{\sum_{j=2}^k B_0'}{e^{N-2s}|x_1 - x_i|^{N-2s}} + O\left(\left(\frac{k}{\nu}\right)^{N-2s+\sigma} + \frac{1}{\nu^{m+\sigma}}\right).
\]
Finally,
\[\int_{\Omega_1} K\left(\frac{|x|}{\nu}\right)U^2_{x^1,\varepsilon} = \int_{\Omega_1} U^2_{x^1,\varepsilon} - \frac{C}{\nu^m} \int_{\Omega_1} |x| - \nu r_0|^m U^2_{x^1,\varepsilon} + O\left(\nu^{-m-\theta} \int_{\Omega_1} |x| - \nu r_0|^{m+\theta} U^2_{x^1,\varepsilon}\right)\]
\[= \int_{\mathbb{R}^N} U^2_{0,1} - \frac{C}{\nu^m} \int_{\mathbb{R}^N} |x - x^1| - \nu r_0|^m U^2_{0,\varepsilon} + O\left(\frac{1}{\nu^{m+\theta}}\right).\]

But
\[\frac{C}{\nu^m} \int_{\mathbb{R}^N \setminus B_{\frac{x^1}{2}}(0)} |x - x^1| - \nu r_0|^m U^2_{0,\varepsilon} \leq C \int_{\mathbb{R}^N \setminus B_{\frac{x^1}{2}}(0)} \left(\frac{|x|^m}{\nu^m} + 1\right) U^2_{0,\varepsilon} \leq \frac{C}{\nu^N}.\]

On the other hand, if \(x \in B_{\frac{x^1}{2}}(0), x = (x_1, x^*)\), then \(|x^1| - x_1 \geq \frac{|x^1|}{2} > 0\). We know
\[|x - x^1| = |x^1| - x_1 + O\left(\frac{|x^*|^2}{|x^1| - x^1}\right) = |x^1| - x_1 + O\left(\frac{|x^*|^2}{|x^1|^2}\right).\]

So,
\[|x - x^1| - \nu r_0|^m = |x^1| - x_1 + O\left(\frac{|x^*|^2}{|x^1|}\right) - \nu r_0|^m\]
\[= |x_1|^m + m|x_1|^{m-2}x_1\left(\nu r_0 - |x^1| + O\left(\frac{|x^2|^2}{|x_1|^2}\right)\right) + \frac{1}{2} m(m - 1)|x_1|^{m-2}\left(\nu r_0 - |x^1| + O\left(\frac{|x^2|^2}{|x_1|^2}\right)^2\right)\]
\[+ O\left(\left(\nu r_0 - |x^1| + O\left(\frac{|x^2|^2}{|x_1|^2}\right)^{2+\sigma}\right)\right),\]

and using
\[\int_{B_{\frac{x^1}{2}}(0)} |x_1|^{m-2}x_1 U^2_{0,\varepsilon} = 0,\]

we get
\[\int_{B_{\frac{x^1}{2}}(0)} \left(|x - x^1| - \nu r_0\right|^m U^2_{0,\varepsilon} = \int_{\mathbb{R}^N} |x^1|^m U^2_{0,\varepsilon} + \frac{1}{2} m(m - 1) \int_{\mathbb{R}^N} |x_1|^{m-2} U^2_{0,\varepsilon} (\nu r_0 - |x^1|)^2\]
\[+ O\left(\nu r_0 - |x^1|^{2+\sigma}\right).\]
Therefore,
\[
\int_{\mathbb{R}^N} K\left(\frac{|x|}{\nu}\right) U^2_{r,\varepsilon} \, \nu = k\left(\int_{\mathbb{R}^N} U_{0,1}^2 - \frac{c_0}{\varepsilon m \nu^m} \int_{\mathbb{R}^N} |x_1|^{m-2} U^2_{0,1} \right.
- \frac{c_0}{\varepsilon m \nu^m} \frac{1}{2} \int_{\mathbb{R}^N} |x_1|^{m-2} U^2_{0,1} (\nu r_0 - |x|^1)^2
+ 2\varepsilon^{-2s} \sum_{i=2}^{k} B_i \varepsilon^{-N-2s} |x_1 - x|^1 N-2s + O\left(\frac{1}{\nu^{m+\sigma}}\right). \tag{A.3}
\]

Now, inserting (A.1)–(A.3) into \(I(U_{r,\varepsilon})\), we complete the proof. \(\square\)

Similar to Proposition A.4 and Proposition A.2 in [31], we have

**Proposition A.5.**

\[
\frac{\partial I(U_{r,\varepsilon})}{\partial \varepsilon} = k\left(\frac{-1}{\varepsilon^{m+1}} + \sum_{i=2}^{k} \frac{B_i (N - 2s)}{\varepsilon^{N-2s+1}} \right) + O\left(\frac{1}{\nu^{m+\sigma}} + \frac{1}{\nu^m} |\nu r_0 - |x|^1|^2\right)
\]

where \(B_0 \in [C_1, C_2], 0 < C_1 < C_2, A = \frac{\nu}{N} \int_{\mathbb{R}^N} U_{0,1}^2, B_i, C_i, i = 1, 2, \) are some constants, and \(r = |x|^1\).

Applying the same argument as in the proof of Proposition A.4 in [23], we also can prove the following result.

**Proposition A.6.** We have

\[
I(\bar{U}_{r,\varepsilon}) = k\left(A - \frac{B_0}{\varepsilon m \nu^m} - \frac{B_1}{\varepsilon m \nu^m} (\nu r_0 - r)^2 \right) + \sum_{i=2}^{2k} \frac{B_i' k^{N-2s}}{\varepsilon^{N-2s}} \frac{1}{\varepsilon^{N-2s}} \frac{1}{\nu^{m+\sigma}} + \frac{k}{\nu^{N-2s}} \right) \right)
\]

where \(B_0' \in [C_1', C_2'], 0 < C_1' < C_2', A = \frac{\nu}{N} \int_{\mathbb{R}^N} U_{0,1}^2, B_i', C_i', i = 1, 2, \) are some constants, and \(r = |\bar{x}|^1\).

**Proof.** Similar to Proposition A.4, we have

\[
I(\bar{U}_{r,\varepsilon}) = k\left(A - \frac{B_0}{\varepsilon m \nu^m} - \frac{B_1}{\varepsilon m \nu^m} (\nu r_0 - r)^2 \right) - \sum_{i=2}^{2k} \frac{(-1)^{j-1} B_2}{\varepsilon^{N-2s}} \frac{1}{\varepsilon^{N-2s}} \frac{1}{\nu^{m+\sigma}} \right) \right)
\]

where \(B_0 \in [C_1, C_2], 0 < C_1 < C_2, A = \frac{\nu}{N} \int_{\mathbb{R}^N} U_{0,1}^2, B_i, C_i, i = 1, 2, \) are some constants, and \(r = |\bar{x}|^1\). Therefore, it is sufficient to prove that

\[
\sum_{j=2}^{2k} \frac{(-1)^j}{\varepsilon^{N-2s}} = \frac{B_0' k^{N-2s}}{\nu^{N-2s}} + O\left(\frac{k}{\nu^{N-2s}}\right),
\]
for some $\tilde{B}_0' > 0$, which can be easily verified by using the following facts

$$
\sum_{i=2}^{2k} \frac{1}{|x^i - x^1|^{N-2s}} = \begin{cases} 
2 \sum_{i=2}^{k} \frac{(-1)^i}{|x^i - x^1|^{N-2s}} + \frac{1}{(2|x^1|)^{N-2s}}, & \text{if } k \text{ is odd} \\
2 \sum_{i=2}^{k} \frac{(-1)^i}{|x^i - x^1|^{N-2s}} - \frac{1}{(2|x^1|)^{N-2s}}, & \text{if } k \text{ is even}
\end{cases}
$$

and

$$
0 < c' \leq \frac{\sin \left( \frac{(i-1)\pi}{2k} \right)}{\sin \left( \frac{i\pi}{2k} \right)} \leq C''', \quad i = 2, \cdots, k.
$$

Similar to Proposition A.5 and A.6, we have

Proposition A.7.

$$
\frac{\partial I(U_{r, \varepsilon})}{\partial \varepsilon} = k \left( \frac{mB_0'}{\varepsilon^{m+1}N} - \frac{B_0'(N - 2s)}{\varepsilon^{N-2s} + 1} + O \left( \frac{1}{\varepsilon^{m+\sigma}} + \frac{1}{\nu r(1 + |x^1|^2 + \frac{k}{\nu r^{N-2s}})} \right)\right)
$$

where $B_0' \in [C_1', C_2']$, $0 < C_1' < C_2'$, $A = \frac{2}{N} \int_{\mathbb{R}^N} U_{0,1}^{2s} \delta_{1}, \quad B_i', \quad C_i', \quad i = 1, 2,$ are some constants, and $r = |x^1|$.

REFERENCES

[1] A. Ambrosetti, J. Garcia Azorero, I. Peral, Perturbation of $-\Delta u = \nu \frac{\delta_{r-2s}}{\nu}$, the scalar curvature problem in $\mathbb{R}^N$ and related topics, J. Funct. Anal. 165 (1999) 117–149.

[2] D. Applebaum, Lévy processes and stochastic calculus, Second edition, Cambridge Studies in Advanced Mathematics, 116. Cambridge University Press, Cambridge, 2009.

[3] B. Barrios, E. Colorado, A. de Pablo, U. Sánchez, On some critical problems for the fractional Laplacian operator, J. Differential Equations 252 (2012) 6133–6162.

[4] J. Bertoin, Lévy processes, Cambridge Tracts in Mathematics, 121. Cambridge University Press, Cambridge, 1996.

[5] L. Caffarelli, L. Silvestre, An extension problem related to the fractional Laplacian, Comm. Partial Differential Equations 32 (2007) 1245–1260.

[6] E.A. Carlen, M. Loss, Extremals of functionals with competing symmetries, J. Funct. Anal. 88 (1990) 437–456.

[7] D. Cao, E. Noussair, S. Yan, On the scalar curvature equation $-\Delta u = (1 + \varepsilon K)u^{\frac{N+2}{N-2}}$ in $\mathbb{R}^N$, Calc. Var. Partial Differential Equations 15 (2002) 403–419.

[8] W. Chen, C. Li, B. Ou, Classification of solutions for an integral equation, Comm. Pure Appl. Math. 59 (2006) 330–343.

[9] W. Chen, J. Wei, S. Yan, Infinitely many solutions for the Schrödinger equations in $\mathbb{R}^N$ with critical growth, J. Differential Equations 252 (2012) 2425–2447.

[10] G. Chen, Y. Zhang, Concentration phenomenon for fractional nonlinear Schrödinger equations, arXiv: 1305.4426.

[11] E. Colorado, I. Peral, Semilinear elliptic problems with mixed Dirichlet-Neumann boundary conditions. J. Funct. Anal. 199 (2003) 468–507.
[12] J. Dávila, M. Del Pino, Y. Sire, Non degeneracy of the bubble in the critical case for non local equation, Proc. Amer. Math. Soc. 141 (2) (2013) 3865–3870.
[13] J. Dávila, M. Del Pino, J. Wei, Concentrating standing waves for fractional nonlinear Schrödinger equation, J. Differential Equations, 256 (2014) 858–892.
[14] M. del Pino, P. Felmer, M. Musso, Two bubble solutions in the super-critical Bahri-Coron’s problem, Calc. Var. Partial Differential Equations 16 (2003) 113–145.
[15] S. Dipierro, G. Palatucci, E. Valdinoci, Existence and symmetry results for a Schrödinger type problem involving the fractional Laplacian, Matematiche, 68 (2013) 201–216.
[16] R.L. Frank, E.H. Lieb, Inversion positivity and the sharp Hardy-Littlewood-Sobolev inequality, Calc. Var. Partial Differential Equations 39 (2010) 85–99.
[17] R.L. Frank, E.H. Lieb, A new, rearrangement-free proof of the sharp Hardy-Littlewood- Sobolev inequality. Spectral Theory, Function Spaces and Inequalities, 55C67, Oper. Theory Adv. Appl., 219, Birkhäuser/Springer Basel AG, Basel, 2012.
[18] D. Gilbarg, N.S. Trudinger, Elliptic partial differential equations of second order. Reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001.
[19] Y.Y. Li, On \(-\Delta u = K(x)u^5\) in \(R^3\), Comm. Pure Appl. Math. 46 (1993) 303–340.
[20] Y.Y. Li, Remark on some conformally invariant integral equations: the method of moving spheres, J. Eur. Math. Soc. (JEMS) 6 (2004) 153–180.
[21] Y.Y. Li, M. Zhu, Uniqueness theorems through the method of moving spheres, Duke Math. J. 80 (1995) 383–417.
[22] E.H. Lieb, Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities, Ann. of Math, (2) 118 (1983) 349–374.
[23] W. Long, S. Peng, J. Yang, Infinitely many positive solutions and sign-changing solutions for nonlinear fractional scalar field equations, Discret. Contin. Dynam. Syst. 36 (2016) 917–939.
[24] R. Servadei, E. Valdinoci, Variational methods for non-local operators of elliptic type, Discrete Contin. Dyn. Syst. 33 (2013) 2105–2137.
[25] R. Servadei, E. Valdinoci, A Brezis-Nirenberg result for non-local critical equations in low dimension, Commun. Pure Appl. Anal. 12 (2013) 2445–2464.
[26] X. Shang, J. Zhang, Ground states for fractional Schrödinger equations with critical growth, Nonlinearity 27 (2014) 187–207.
[27] E.M. Stein, Singular integrals and differentiability properties of functions, Princeton Mathematical Series, No. 30 Princeton University Press, Princeton, N.J. 1970 xiv+290 pp. (Reviewer: R. E. Edwards) 46.38 (26.00).
[28] M. Struwe, Variational methods. Applications to nonlinear partial differential equations and Hamiltonian systems, Springer-Verlag, Berlin, 2000.
[29] M. Picone, Sui valori eccezionali di un parametro da cui dipende una equazione differenziale lineare ordinaria del secondo ordine, Ann. Scuola. Norm. Pisa., 11 (1910) 1–144.
[30] J. Tan, The Brezis-Nirenberg type problem involving the square root of the Laplacian, Calc. Var. Partial Differential Equations 42 (2012) 21–41.
[31] J. Wei, S. Yan, Infinite many positive solutions for the prescribed scalar curvature problem on \(S^N\), J. Funct. Anal. 258 (2010) 3048–3081.
[32] J. Wei, S. Yan, Infinity many positive solutions for an elliptic problem with critical or supercritical growth, J. Math. Pures Appl. 96 (2011) 307–333.
[33] J. Wei, S. Yan, Infinitely many nonradial solutions for the Hénon equation with critical growth, Rev. Mat. Iberoam. 29 (2013) 997–1020.
[34] S. Yan, Concentration of solutions for the scalar curvature equation on \(R^N\), J. Differential Equations 163 (2000) 239–264.
School of Mathematics and Statistics, Central China Normal University, Wuhan, 430079, P. R. China
College of Mathematics and Information Science, Jiangxi Normal University, Nanchang, Jiangxi 330022, P. R. China

E-mail address: hopelw@126.com

School of Mathematics and Statistics, Central China Normal University, Wuhan, 430079, P. R. China

E-mail address: yyangecho@163.com