Generation of $1/f$ noise motivated by a model for musical melodies

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We present a model to generate power spectrum noise with intensity proportional to $1/f$ as a function of frequency $f$. The model arises from a broken-symmetry variable which corresponds to absolute pitch, where fluctuations occur in an attempt to restore that symmetry, influenced by interactions in the creation of musical melodies.

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INTRODUCTION

Correlations of melodies in music follow a power spectrum where the intensity of that spectrum varies in inverse proportion to the frequency $f$, called $1/f$ noise \[1\]. This characteristic spectrum is followed by musical rhythms \[2\], pitch \[3\], and consonance fluctuations in music \[4\]. The power spectrum is related to autocorrelation functions \[2, 6\], and hence the $1/f$ power spectrum signifies a correlation in, for example, the fluctuating pitch on a time scale over which the power spectrum is $1/f^2$. The $1/f$ power spectrum in music produces a balance between the predictable sound intensity of a random walk $1/f^2$ and random white noise $1/f^0$. This balance between the expected and surprise is said to be psychologically pleasing for the audience \[2, 3, 6\].

In this paper we show that this noise spectrum can be recovered if, firstly, we consider pitch as a broken-symmetry variable, which we believe to be a necessary feature of a mathematical theory of musical melodies, and if, secondly, we include interactions between musicians \[10–13\], which we find to be a sufficient feature.

An understanding of music is rooted in the way the brain works \[14\], and is different in kind from most other arts. A useful contrast is to painting where, often, a representation of the outside world is seen, over-colored in red, yellow, and blue due to our eyes’ particular sensitivity to those colors. Of course, a painting is not a random snapshot of what is “outside” in the world, there is an emotional and intellectual resonance “inside” the artist’s mind from which comes the choice of subject or the manner of presentation \[15\]. Therein lies the art, but its root is representational. The same could be said of plays, sculpture, and indeed of most arts: the artifact would be some representation of what is outside — exaggerated or highlighted somehow — which has an emotional and intellectual resonance inside; these arts have an external referent.

Contrast this to music. Imagine watching a film of a jazz quartet, with the sound turned off. Further, imagine that the film shows the score of the piece. As a musician begins playing, and one sees the score, the performance is not that different from watching a mathematician at a blackboard, flawlessly writing complex equations. As the other musicians begin, it would be as if our mathematician is joined by three others, each writing varied but related theorems at other blackboards, all in synchrony. This is strange, but not the strangest part. What is bizarre is the audience, perhaps humming along or tapping their fingers in time as they themselves interact with the musicians \[16\] and even predicting the onset times of upcoming musical events \[17, 18\]. As the equations multiplied on the blackboards, one would notice that the people listening to the four mathematicians would experience intense feelings of joy, followed by feelings of loss, then elation again, and so on, as they followed the tracings of algebra.

As this thought experiment shows, the representational part of music, based on what is “outside” in the world, is minimal. Musical performances are not snapshots of a natural soundscape. Listeners’ intense reaction to, and involvement with music shows the performance is “inside” the audience’s minds. The snapshot taken — however exaggerated or highlighted with emotional and intellectual resonance — is a snapshot of their brains. That is music, and the direct route it has to our brains.

To understand music, then, is to understand some aspects of the neuroscience of the brain \[19–21\]: music provides a concrete way of addressing how, for example, decisions are made in the context of the creation of musical melodies. Furthermore, we note that music is particularly well suited to a mathematical analysis as, unlike other art forms, it is straightforwardly quantifiable in terms of time and pitch \[22, 23\]. As mentioned above, it is well known that musical melodies obey $1/f$ noise. However there is no theory giving rise to this, and demonstrations of artificial music derived from a $1/f$ power spectrum make use of, for example, filtered white noise \[6\] for which no potential neurological significance can be attached. We also note that the phenomena of $1/f$ noise, although well documented, is in seeming contradiction to the notions of time signatures, key signatures, and the different modes and scales in different cultures. We will not address that in this paper, except in the sense of attempting to find the simplest model consistent with musical melodies which gives rise to $1/f$ noise; further,
and crucially, we make use of the fact that a musical melody can be recognized without regard to the key of that melody, demonstrating the imperfect detection of absolute pitch. We will obtain a 1/f spectrum which is naturally motivated by musical melodies: the two essential ingredients of our model are the arbitrariness of absolute pitch, which is a necessary feature of a treatment of music, and interactions, which we find to be a sufficient feature.

**BROKEN-SYMMETRY VARIABLES**

In this section, we will review the properties of broken-symmetry variables [24–29]. When, for example, a free energy has a continuously broken symmetry, this gives rise to a broken-symmetry variable $h$, whose fluctuations act to restore that symmetry. The modes of a broken-symmetry variable have no energy gap to begin propagating, and hence the correlations of $h$ are power-laws, without scale. In particular, in Fourier space, as a function of wavenumber $q$, it can be shown that a broken-symmetry variable, often called a Goldstone variable, satisfies\(\langle |h(q)|^2 \rangle \propto 1/q^2\), for small $q$, where $h$ is the Fourier transform of $h$, defined below, and the brackets denote an average.

We will now give a description of the broken-symmetry variable associated with surfaces of coexisting phases, and then adapt it to musical melodies. For surfaces, the broken symmetry is translational invariance in the direction normal to the interface, the broken-symmetry variable is the local height of the interface, and the dynamical modes are called capillary waves or ripplons. Consider a surface in the $d$-dimensional $\vec{x}$ plane of a $(d + 1)$-dimensional system $\vec{r} = (\vec{x}, y)$. The position of the surface, assuming small fluctuations and therefore no droplets or overhangs, is $y = h(\vec{x})$, where $h$ is the local height of the surface. The probability of being in a state $h(\vec{x})$ is proportional to $e^{-F(h)}$, where $F$ is the free energy in units of the temperature. If the free energy does not depend on the absolute position of the interface, which corresponds to translational invariance of space in the direction normal to the interface, it is invariant under $h \to h + \text{const}$. Hence it can only depend on $h$ through gradients. To lowest order in gradients, we have

$$F \propto \int d\vec{x}(\nabla h)^2. \quad (1)$$

Note that the free energy is minimized by minimizing variations in $h$, any constant $h$ minimizes $F$. This form, being quadratic, can be diagonalized and solved as a product of independent Gaussians by using Fourier transforms:

$$\hat{h}(\vec{q}) = \int d\vec{x} e^{-i\vec{q}\cdot\vec{x}} h(\vec{x}) \quad (2)$$

where we can write

$$h(\vec{x}) = \int \frac{d\vec{q}}{(2\pi)^d} e^{i\vec{q}\cdot\vec{x}} \hat{h}(\vec{q}), \quad (3)$$

since the Dirac delta function ensures completeness,

$$\delta(\vec{x} - \vec{x}') = \int \frac{d\vec{q}}{(2\pi)^d} e^{i\vec{q} \cdot (\vec{x} - \vec{x}')}. \quad (4)$$

The solution is

$$\langle h(\vec{q}) h(\vec{q}') \rangle = (2\pi)^d \delta(q + q') \hat{G}(q), \quad (5)$$

where

$$\hat{G}(q) \propto \frac{1}{q^2}. \quad (6)$$

This is the standard result for a broken-symmetry Goldstone variable [29]. As the integrals are Gaussian, all moments can be obtained from the second moment. In real space, correlation functions satisfy

$$\langle h(\vec{x}) h(0) \rangle = \int \frac{d\vec{q}}{(2\pi)^d} e^{i\vec{q} \cdot \vec{x}} \hat{G}(q). \quad (7)$$

Translational and rotational invariance of space in the plane parallel to the interface gives rise to the delta function $\delta(\vec{q} + \vec{q}')$, and implies that $\langle h(\vec{x}) h(0) \rangle = \langle h(\vec{x} + \vec{x}_0) h(\vec{x}_0) \rangle$, where $\vec{x}_0$ corresponds to the arbitrary origin, and can be ignored.

This general result can be obtained in other ways. In particular, and which is necessary for our purposes, the noise source does not need to be thermal: the derivation of the Kardar-Parisi-Zhang equation [27–29] provides an example. If the rate in change of time $\tau$ of a broken-symmetry variable has the invariance $h \to h + \text{const}$, and we endeavour to find the rate of change of $h$ via a gradient expansion in $h$, and to lowest order in time derivatives, we obtain

$$\frac{\partial h}{\partial \tau} \propto \nabla^2 h + \mu, \quad (8)$$

where $\mu$ is a noise. By the central-limit theorem, the only nontrivial moment of the noise satisfies:

$$\langle \mu(\vec{x}, \tau) \mu(\vec{x}', \tau') \rangle \propto \delta(\vec{x} - \vec{x}') \delta(\tau - \tau'). \quad (9)$$

Note that, for our purposes, we will not need to consider the higher-order term in the Kardar-Parisi-Zhang equation which breaks the symmetry $h \to -h$. In steady state, one again recovers the same probability and free energy, and the same results, including $\hat{G} \propto 1/q^2$.

Finally, let us define a partial Fourier transform using $\vec{q} = (q_x, \vec{k})$, such that $q_x$ is one-dimensional, and $\vec{k}$ is $(d - 1)$-dimensional. Namely, let

$$P(x, \vec{k}) = \int \frac{dq_x}{2\pi} e^{iq_x x} \hat{G}(q_x, \vec{k}). \quad (10)$$
Using $\hat{G} = 1/q^2 = 1/(q_x^2 + k^2)$ gives straightforwardly,

$$P(x, \vec{k}) = e^{-|k|x}/|k|, \quad (11)$$

so that

$$P(0, \vec{k}) = 1/|k|. \quad (12)$$

Note that the $x = 0$ axis is equivalent to any constant $x$ axis from translational invariance in the plane parallel to the interface.

**MODEL OF MELODY**

A necessary feature of a theory of music is the imperfect detection of absolute pitch. Unlike vision, where the wavelengths corresponding to red, blue and yellow are privileged for most people, the absolute frequencies of notes are arbitrary for most. For example, a musical melody is recognized without regard to the key in which it is performed. Having a specific pitch, then is the same as having a specific value for a broken-symmetry variable. Hence we identify absolute pitch, in essence the key of a melody, as a broken-symmetry variable, and so the highness $h$ of a pitch in a melody will behave with the correlations of a broken-symmetry variable.

This is a necessary component to understanding musical melodies. We can imagine a solitary musician sitting by the piano, playing notes at random, close to a previous pitch. Following the nergy function above, the decision on what note to follow a given note is given by the constraint enforced by gradients of $h$ above, which favours nearby notes; the concept of nearby is used in the most simple sense here, and more refined descriptions exist. If the highness of the pitch is also called $h$, the same notation as for the example involving surfaces above, and the space in which the notes are played is one-dimensional time $t$, we have

$$\langle h(t) h(0) \rangle = \int df (2\pi)^{-1} e^{-f t} \hat{G}(f), \quad (13)$$

where we have introduced the frequency $f$ replacing the Fourier wavenumber $q$ above, and

$$\hat{G}(f) \propto 1/f^2. \quad (14)$$

This is the frequency spectrum of a random walk, not $1/f$ noise. Hence, recovering $1/f$ noise will require more than the imperfect detection of pitch: we will introduce another element, interactions, which will act to temper these variations in frequency.

As such, let us insist that the solitary musician listens to other musicians, all of whom are also attempting to play melodies in the same manner as our first musician. Further, we will assume the musicians temper each others’ choices of notes such that notes chosen are constrained not only to be close to a previous note, but to be close to those played by the other musicians. Now the dimension of space is two-dimensional and the frequency spectrum changes: one dimension is time $t$ and the second dimension is the direction of interaction with other musicians at a given time called $x$, where we can conveniently measure the strength of the interactions in units of $t$ and $x$. This model provides a minimal interaction that one musician can have with others. From above, we immediately obtain the correlation function

$$P(x, f) = e^{-f x}/f. \quad (15)$$

Hence, considering the notes on the $x = 0$ axis corresponding to a chosen musician, we have

$$P(0, f) = 1/f. \quad (16)$$

This is our main result, recovering $1/f$ noise. It arises from one necessary condition, imperfect absolute pitch detection, and one sufficient and we believe plausible condition, interactions, which has been identified and noted in the literature. There are two further implications of our analysis which are testable. Firstly, the exponentially-damped correlations of Eq. (15) correspond to correlations across nonzero $x$ and frequency. Secondly, for $t = 0$, which corresponds to any fixed time, we find that there are power-law correlations in the Fourier transform of $x$, $q_x$, following $1/|q_x|$.

FIG. 1: The power $P$ spectrum of the solid-on-solid model showing $1/f$ noise as a function of $\log(P)$ versus $\log(f)$.

As an example of how readily a $1/f$ spectrum can be obtained, we show the results of a Monte Carlo simulation of the equilibrated solid-on-solid model, which is known to be a good model of capillary wave fluctuations. Each lattice site of a two-dimensional square grid can have any integer height $h_i$, where $i = 1, 2, \ldots N$
runs over all sites of the square lattice of \(N\) sites. The energy of interaction is proportional to \(\sum_i |h_i - h_{nn}|\), where \(nn\) denotes the nearest neighbours of \(i\). In Figure 1 we show the power spectrum of the equilibrated solid-on-solid model with \(N = 1024 \times 1024\) at noise intensity \(T = 4\), in units of the interaction constant. This is the averaged power spectrum of one “row” of the model, corresponding to the time axis of the evolving melody.

In conclusion, we have presented a simple model which recovers \(1/f\) noise for musical melodies from one necessary condition, imperfect absolute pitch detection, and one plausible sufficient condition, interactions. A more refined theory of musical melodies would be of value, but this study provides a first step, and we believe the features we have identified will play roles in any such refined theories.

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