Interactive animations as a tool in teaching general relativity to upper secondary school students

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Abstract. One possible and often seen approach to teaching General Relativity are the so-called embedding diagrams. Educators are using them, sometimes perhaps even unknowingly, any time they talk about spacetime around a star being “like a rubber sheet weighed down by a bowling ball”. In this contribution we will briefly remind what these embedding diagrams are and discuss possible problems with their use. Finally, we will introduce interactive computer animations (applets) as a possible approach to overcome some of these problems.

1. Introduction
The topic of presenting General Relativity (GR) to upper secondary school students has been increasingly given attention by physics education researchers in recent years (see for example [1-3]). As a result, new educational tools, such as [4], for teaching GR are being developed, and attempts have been made to introduce GR to even the youngest of students [5]. In this contribution we will discuss a particular visual representation commonly associated with GR, its educational advantages and mainly possible conceptual problems that might arise when using this representation.

When we do an Internet search for “General Relativity” or a similar query and switch over to images, we will most likely get many depictions of curved spacetime in a form of a “cone-like” shape, often accompanied by a gravitating object such as a planet or a star in the middle. This image has become more or less the main visual aid when we wish to address General Relativity, and its popularity is further emphasized also by its frequent use on T-shirts etc. Speaking from a more scientific point of view, embedding diagrams are useful tools that help us get some insight into complex spacetime geometries, which constitute the majority of GR. Therefore, we consider it important to deal with embedding diagrams when presenting GR even in the most basic form.

We will briefly remind deriving the well-known shape from Einstein’s equations and mention a few practical uses of the diagrams, including possible conceptual problems or even misconceptions that may arise. As a possible solution to these problems, we will suggest using interactive animations.

2. Embedding diagrams
The basic idea of embedding diagrams is to visualize the curvature of given spacetime by embedding its selected (mostly two-dimensional) hypersurface into a space (or spacetime) of higher dimension (mostly three-dimensional, because we can easily visualize it). In this contribution, we will consider only the simplest and most known example, spacetime around a nonrotating spherical object (also known as the Schwarzschild spacetime), using the standard Schwarzschild coordinates \((t, r, \theta, \phi)\), where \(t\) represents time, \(r\) is the radial distance from the center, and \(\theta\) and \(\phi\) are spherical angles. Because any
one object in this spacetime will orbit around the central source of gravitation in a single plane, it makes sense to restrict our attention to a two-dimensional motion in the so-called equatorial plane (defined by $\theta = \pi/2$ – we choose this exact value only because it simplifies our equations, due to the spherical symmetry of the problem any given constant value of $\theta$ is equally valid). Furthermore, we will restrict ourselves to a particular time, setting $t = \text{const}$. A more detailed description of the process can be found for example in chapter 2 of [6].

Metric for the equatorial plane of Schwarzschild spacetime for a constant time is

$$ds^2 = \frac{dr^2}{1 - \frac{2GM}{c^2r}} + r^2 d\phi^2,$$

where $G$ is the Newton’s gravitational constant, $c$ is the speed of light and $M$ is the mass of the central source of gravitation. In order to visualize the curvature in the $r$ direction, we embed this surface into the three-dimensional Cartesian space (where $r$ and $\phi$ are identical to polar coordinates and the third, vertical Cartesian coordinate $z$ is used to visualize the actual curvature – see figure 1). As a result, we get an equation for the $z$ coordinate as a function of $r$:

$$z(r) = \sqrt{\frac{8GMr}{c^2} - \frac{16M^2G^2}{c^4}}.$$

Of course, this equation $r$ and $z$ are in meters, which is not very convenient for visualizing large regions of space. For this reason, geometricized units where $c = G = 1$ are often used. Then we get the simpler form $z(r) = \sqrt{8Mr - 16M^2}$. Finally, the shape we see so often when dealing with popularization of GR (such as in figure 2) is made by rotating the $z(r)$ curve around the $z$ axis (thus introducing the $\phi$ coordinate). It should be added that this results in the cone-like shape with a hole in the middle, given by the fact that the original metric includes a coordinate singularity at the Schwarzschild radius. This corresponds to the spacetime around a nonrotating spherical black hole.

3. Using the embedding diagram

As mentioned above, any time we talk about spacetime being like a rubber sheet weighed down by a bowling ball, we are (perhaps even unknowingly) basically using the visual representation brought forth by embedding diagrams. There are many advantages to this analogy, the biggest one being that we can

![Figure 1. We use the third dimension to visualize curvature in the radial direction.](image1.png)

![Figure 2. “Cone-like” embedding diagram visualizing the equatorial plane curvature of the Schwarzschild spacetime.](image2.png)
actually build models that enable us to throw marbles and small balls around and watch them move. A video of one fine example in which elastic fabric is used to model space curvature can be found at [7]. Understandably, a weighed-down sheet of fabric only roughly approximates the actual shape that arises from Einstein’s equations and such approximation should be kept in mind.

Another often seen example of visualizing curvature is using regular cones instead of the more complex shape. Cones can be easily made with just a piece of paper and their geometry is quite straightforward, therefore they are (together with spheres) great for some simple explanations about curvature and geodesics. The use of cones in the exposition of general relativity can be found for example in [8] or in a video from the science popularizing Youtube channel called VSauce [9] (the referenced video has had, as of September 2018, nearly 8.4 million views, therefore the importance of its role in popularization of GR should not be overlooked). Mathematically speaking, when replacing the above mentioned embedding diagram by a cone, we are approximating a square root function with a linear one. Surprisingly, even this first approximation enables us to show some aspects of curved space, namely the bending of straight trajectories when spatial curvature is introduced (for more details see the mentioned references).

However, all these uses have the same conceptual problems. First of all, they are based on the embedding diagram that ignores time curvature. General relativity is a theory of spacetime curvature and for most cases, including planetary orbits, curvature in time has a greater effect than space curvature. For this reason, even in the exact computer models, which we will see later, we can never get a stable orbit when dealing only with spatial curvature. One may be, for example, tempted to try to use our embedding diagram to motivate the 2nd Kepler’s law of planetary motion but, as was just said, stable orbits cannot be explained by spatial curvature only. This is the biggest downside of our use of this simplest of all embedding diagrams. There exist of course embedding diagrams including time (such as in [10]) but these are hardly suitable for beginner learners, such as upper-secondary school students, and are not (in the author’s experience) usually part of even undergraduate university courses in GR.

When we try to visualize spatial curvature using solid bend surfaces and marbles, the main obstacle is the actual force of gravity acting on the marbles. We would like to show that gravitation is in fact the curvature of space (spacetime); however, the real motion of the marbles is governed not only by the curved surface under them but also in some non-negligible part by Earth’s gravitation (not to mention the friction forces), pulling the marbles down. Students may not immediately realize this (although some of them might) but we as educators should be aware of that. Ideally, we would need a “marble” that would move “inside” the bend surface, not on it.

Finally, let us remind an important but often not mentioned aspect of our embedding diagram. It is meant to visualize a two-dimensional motion in the equatorial plane, so even though we observe the motion of our particles in three dimensions, whether they are real marbles rolling on a bend surface or part of a computer generated image or animation, the only physically relevant part of the motion is in the $(r, \varphi)$ plane.

4. **Applets using Visual Python**

A possible approach to address the above mentioned conceptual problems is using computer animations. They may not be exactly “hands-on” but what they lack in that aspect, they make up by allowing us to be more precise. As a tool for creating such an animation, we have used Visual Python (also called VPython, for short), a library of the popular programming language Python, that was specifically created for the purpose of enabling us to easily create 3D objects on the screen and manipulate them in any way we like using computer code. A great feature of Visual Python is that the generated 3D scene can be implicitly manipulated with (zoomed, rotated etc.) and we as programmers are spared the tedious work of coding these graphical features ourselves. Visual Python enables us to make not only animations, we can of course add interactivity by adding buttons, sliders etc., making a standalone applet. Furthermore, our work can be then easily shared on a website. More about this open-source project can be found at its website [11].
A basic applet showing us the features of an embedding diagram (see figure 3) can be found at [9], together with a link for sharing the applet and to view its source code. It enables the user to send point-like particles across the embedding diagram and watch their trajectory being bent by the curvature of the surface (as can be seen in figure 3). We shall now discuss the features of the applet in relation with the conceptual problems mentioned in the previous section:

- **Motion of marbles rolling on bend surface affected by actual gravity and friction** – naturally, when we use a simulation, we don’t have to deal with non-ideal conditions of real world experiments. Point-like particles can be made to move strictly in the surface without friction and without any force acting on them, their motion being affected purely by the curvature of the surface. To support the idea that the motion is restricted to the surface, it is beneficial to be able to flip the whole embedding diagram upside down, a feature that is easily achieved with a simulation but hardly with a real aperture.

- **Emphasis on the two-dimensional motion** – as we mentioned above, the only physically relevant part of the motion is in the horizontal plane \((r, \varphi)\) or \((x, y)\), if you will. With a rotatable simulation, we can take advantage of a view from the top (looking in the direction of the \(z\) axis). In this view, we see purely the effect of spatial curvature on the motion. Taking advantage of the simulation further, we can add a second particle which will follow the motion of the first particle but only in the horizontal plane. This way we can point out the difference in the two motions and how they are related. It would be most interesting to see such a setup with the marbles, but one can imagine achieving that could be extremely difficult.

- **Purely spatial curvature** – needless to say, this is the one conceptual obstacle that we cannot get rid of when using this type of an embedding diagram and we need to keep it in mind. However, what we can do is add another particle moving in the horizontal plane whose motion will start with the same initial conditions as the particle on the curved surface but which moves according to the complete set of equations of motion for the equatorial plane of Schwarzschild spacetime. This mainly means adding the time component, which substantially changes the motion. This feature allows the user to, at least qualitatively, compare the “real” motion with the one due to space curvature. A practical note: In order to clearly see the curvature of the embedding diagram and its effect on the motion, we usually visualize a region of space that is very close to the Schwarzschild horizon, which means that the real-motion particle almost immediately falls inside the central object. Therefore this feature serves only as a rough comparison of the two motions.

- **Dynamical changes of the curvature** – lastly, let us mention a nice feature that is again possible only using a computer simulation. By changing the central mass parameter \(M\) of the curvature (see equations above), we can change the curvature dynamically, even during a particles motion. While this hardly corresponds to a real world situation (the central gravitating body would have to loose mass while remaining spherically symmetric and without rotation), it is an interesting feature enabling us to compare trajectories for different curvatures. In other words, we can show the, perhaps intuitive, fact that larger mass curves space around itself more, resulting in stronger curving of the trajectories.
5. Conclusion

We have discussed how the most common embedding diagram of the Schwarzschild spacetime is formed and how it is often used in popular or introductory treatments of General Relativity. However, using these diagrams bears its conceptual difficulties, of which we have mentioned the most common. Some of these difficulties are almost impossible to avoid and need to be kept in mind by both students and teachers. Some can be avoided or stressed out with the use of computer animations or applets. We have introduced such an applet together with one particular tool for creating physical applets and animations, the Visual Python. This programming tool is open-source and so is the embedding diagram applet introduced in this contribution. It can be found at the author’s website [12] and can be used by anyone, or even modified by users with some experience with programming in Python. The discussed features of the applet aim to overcome some of the conceptual difficulties of teaching with embedding diagrams. We invite readers to have a look at the applet and perhaps even use it in their own teaching practise. As we haven’t had the chance to introduce the applet to students yet, any feedback on the use of the applet will be appreciated. If you have any questions or suggestions, you are invited you to write the author of this contribution.

Figure 3. The embedding diagram applet. Visual Python automatically gives us the ability to rotate and zoom the 3D scene. Features of the applet are discussed in the text.
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