Non-existence of perturbed solutions under a second-order sufficient condition

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We present an optimization problem in infinite dimensions which satisfies the usual second-order sufficient condition but for which perturbed problems fail to possess solutions.

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1 Introduction

For nonlinear optimization problems in finite dimensions, it is well known that uniqueness of the Lagrange multiplier together with the standard second-order sufficient condition (SSC) leads to the stability of the solution under perturbations of the problem and this is a key result for the analysis of SQP methods, see Bonnans, 1994, Proposition 6.3 and Robinson, 1982, Theorem 4.20. For infinite-dimensional problems, such a result is not known and one typically resorts to a strong second-order sufficient condition for the analysis of SQP-type methods, see, e.g., Malanowski, 1992, (2.17) and Alt, 1990, (2.3). In this note, we present an example which satisfies the usual second-order sufficient conditions but for which perturbed problems fail to possess solutions.

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2 Second-order sufficient conditions and perturbations

For convenience, we state the standard SSC for a problem of type

$$\text{Minimize } f(x) \quad \text{s.t. } x \in C,$$

where $X$ is a Banach space, $f: X \to \mathbb{R}$ is twice continuously Fréchet differentiable on $X$ and $C \subset X$ is closed and convex. Due to the simple structure of the constraint, Robinson’s constraint qualification is trivially satisfied by the above problem and we have the following result, see Bonnans, Shapiro, 2000, Theorem 3.63(i), Remark 3.68.

**Theorem 2.1.** Let $\bar{x} \in C$, $\beta, \eta > 0$ be given, such that the conditions

$$\begin{align*}
- f'(\bar{x}) & \in N_C(\bar{x}) & (2.1a) \\
f''(\bar{x})h^2 & \geq \beta \|h\|^2_X & \forall h \in \hat{C}_\eta(\bar{x}) & (2.1b)
\end{align*}$$

hold. Then, a quadratic growth condition is valid at $\bar{x}$, i.e., there exist $\delta, \varepsilon > 0$ such that

$$f(x) \geq f(\bar{x}) + \frac{\delta}{2} \|x - \bar{x}\|^2_X \quad \forall x \in C, \|x - \bar{x}\|_X \leq \varepsilon. \quad (2.2)$$

Here, $T_C(\bar{x})$ and $N_C(\bar{x})$ are the tangent cone and the normal cone in the sense of convex analysis, respectively, and

$$\hat{C}_\eta(\bar{x}) := \{h \in T_C(\bar{x}) \mid f'(\bar{x})h \leq \eta \|h\|_X\}$$

is the so-called approximate critical cone. The role of the Lagrange multiplier is played by $-f'(\bar{x})$ and this element is uniquely determined (for fixed $\bar{x}$).

In finite dimensions, the assertion of Theorem 2.1 also gives stability w.r.t. perturbations. As an example, we give a simple result with linear perturbations.

**Proposition 2.2.** Suppose that $X$ is finite dimensional and that $\bar{x} \in C$, $\delta, \varepsilon > 0$ satisfy (2.2). Then, for any $x^\star \in X^*$ with $\|x^\star\|_{X^*} < \delta \varepsilon / 2$ the perturbed problem

$$\text{Minimize } f(x) + \langle x^\star, x \rangle_X \quad \text{s.t. } x \in C$$

has a local solution $z$ satisfying $\|z - \bar{x}\|_X \leq 2\delta^{-1} \|x^\star\|_{X^*}$.

**Proof.** We consider the localized problem

$$\text{Minimize } f(x) + \langle x^\star, x \rangle_X \quad \text{s.t. } x \in C \cap B_{\varepsilon}(\bar{x}).$$

Since the objective is continuous and since the feasible set is compact, there exists a local
solution \( z \). The optimality of \( z \) together with the growth condition (2.2) implies
\[
\frac{\delta}{2} \| z - \bar{x} \|^2_X \leq f(z) - f(\bar{x}) = (f(z) + \langle x^*, z \rangle_X) - (f(\bar{x}) + \langle x^*, \bar{x} \rangle_X) + \langle x^*, \bar{x} - z \rangle_X
\]
\[
\leq \| x^* \|_{X^*} \| z - \bar{x} \|_X.
\]
Thus, \( \| z - \bar{x} \|_X \leq 2 \delta^{-1} \| x^* \|_{X^*} < \varepsilon \). This shows the desired inequality and since the constraint \( z \in B_\varepsilon(\bar{x}) \) is not binding, \( z \) is a local solution of the perturbed problem.

3 Counterexample

We present a counterexample which shows that Proposition 2.2 fails in infinite dimensions. We define the operator \( S : L^2(0, 1) \to L^2(0, 1) \) via \((Su)(x) = \int_0^x u(\xi) \, d\xi\). We further define
\[
X := \mathbb{R} \times L^2(0, 1), \quad C := \{(t, u) \in X \mid |u| \leq t \ \text{a.e. on} \ (0, 1)\},
\]
\[
f_h(t, u) := t^2 + \| Su \|^2_{L^2(0, 1)} - \frac{1}{2} \| u \|^2_{L^2(0, 1)} - ht \quad \forall (t, u) \in X,
\]
where \( h \geq 0 \) is a linear perturbation parameter and \( h = 0 \) corresponds to the unperturbed problem. It is clear that \( X \) is a Banach space, \( C \subset X \) is closed and convex and \( f_h \) is twice Fréchet differentiable with
\[
\langle f''_h(\bar{t}, \bar{u}), (t, u) \rangle_X = 2\bar{t}t + 2\langle Su, Su \rangle_{L^2(0, 1)} - \langle \bar{u}, u \rangle_{L^2(0, 1)} - ht
\]
and
\[
f''_h(\bar{t}, \bar{u})(t, u)^2 = 2t^2 + 2\| Su \|^2_{L^2(0, 1)} - \| u \|^2_{L^2(0, 1)}.
\]
We start by checking that the point \((\tilde{t}, \tilde{u}) = (0, 0)\) satisfies the SSC from Theorem 2.1 for the unperturbed objective \( f_0 \). It is clear that \((0, 0) \in C \) and \( f_0''((0, 0)) = 0 \in \mathcal{N}_C(0, 0) \). Since \( C \) is a closed, convex cone, we have \( T_C(0, 0) = C \), thus, the approximate critical cone is given by \( \hat{C}_C(0, 0) = C \). It remains to verify (2.1b). Let \((t, u) \in C \) be arbitrary. Then, \( |u| \leq t \) implies \( \|(Su)(x)\| \leq tx \). Thus,
\[
f_0''(0, 0)(t, u)^2 = 2t^2 + 2\| Su \|^2_{L^2(0, 1)} - \| u \|^2_{L^2(0, 1)}
\]
\[
\geq 2t^2 - 2 \int_0^1 (tx)^2 \, dx - \int_0^1 t^2 \, dx = (2 - 2/3 - 1)t^2 = \frac{1}{3} t^2
\]
\[
\geq \frac{1}{6} \left( t^2 + \| u \|^2_{L^2(0, 1)} \right) \geq \frac{1}{6} \| (t, u) \|^2_X
\]
and this verifies (2.1b) with \( \beta = 1/6 \). Thus, \((\tilde{t}, \tilde{u}) = (0, 0)\) is a local minimizer of \( f_0 \) on \( C \) and one can show that it is even a global minimizer on \( C \).

It remains to compute the minimizers of \( f_h \) on \( C \). Thus, let \( h \geq 0 \) be given and let \((\tilde{t}, \tilde{u}) \in C \) be a local minimizer of \( f_h \) on \( C \).
First, we consider the case \( \tilde{t} > 0 \). Then, \( \tilde{u} \) is a minimizer of
\[
\text{Minimize } \|S\tilde{u}\|_{L^2(0,1)}^2 - \frac{1}{2} \|u\|_{L^2(0,1)}^2 \quad \text{s.t. } -\tilde{t} \leq u \leq \tilde{t} \text{ a.e. on } (0,1).
\]
Now, Pontryagin’s maximum principle implies that \( \tilde{u}(x) \) solves
\[
\text{Minimize } 2(S^*S\tilde{u})(x)u - \frac{1}{2}u^2 \quad \text{s.t. } u \in [-\tilde{t}, \tilde{t}]
\]
for almost every \( x \in (0,1) \). Here, \( S^*S\tilde{u} \) is given by
\[
(S^*S\tilde{u})(x) = \int_0^1 (S\tilde{u})(\xi) d\xi.
\]
In particular, we get the implications
\[
(S^*S\tilde{u})(x) > 0 \quad \Rightarrow \quad \tilde{u}(x) = -\tilde{t}
\]
\[
(S^*S\tilde{u})(x) < 0 \quad \Rightarrow \quad \tilde{u}(x) = \tilde{t}.
\]
We argue that \( S^*S\tilde{u} = 0 \). Let \( x \in [0,1] \) be a maximizer of the continuous function \( S^*S\tilde{u} \). Towards a contradiction, suppose that \( (S^*S\tilde{u})(x) > 0 \). Then, we get \( \tilde{u} = -\tilde{t} \) in a neighborhood of \( x \) and this implies
\[
(S^*S\tilde{u})'(\xi) = -(S\tilde{u})(\xi) \quad \text{and} \quad (S^*S\tilde{u})''(\xi) = -\tilde{u}(\xi) = \tilde{t} > 0.
\]
for all \( \xi \) in a neighborhood of \( x \). Together with \( (S\tilde{u})(0) = 0 \), this contradicts the assumption that \( x \) is a maximizer of \( S^*S\tilde{u} \). Hence, \( S^*S\tilde{u} \leq 0 \). Similarly, we can show \( S^*S\tilde{u} \geq 0 \). Thus, \( S^*S\tilde{u} = 0 \) and this gives \( \tilde{u} = 0 \) which is, obviously, not a minimizer. Hence, every local minimizer \( (\tilde{t}, \tilde{u}) \) satisfies \( \tilde{t} = 0 \) and, consequently, \( \tilde{u} = 0 \). This shows that \( \tilde{t} = 0 \) is a local minimizer of
\[
\text{Minimize } t^2 - ht \quad \text{s.t. } t \geq 0.
\]
This implies \( h = 0 \).

To summarize, \( (0,0) \) is the only (local) minimizer of \( f_0 \) on \( C \) and satisfies the second-order sufficient conditions. However, the linearly perturbed functionals \( f_h, h > 0 \), do not admit local minimizers on \( C \).

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