Strong coupling superconductivity due to massless boson exchange

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We solve the problem of fermionic pairing mediated by a massless boson in the limit of large coupling constant. At weak coupling, the transition temperature is exponentially small and superconductivity is robust against phase fluctuation. In the strong coupling limit, the pair formation occurs at a temperature of the order of the Fermi energy, however, the actual transition temperature is much smaller due to phase and amplitude fluctuations of the pairing gap. Our model calculations describe superconductivity due to color magnetic interactions in quark matter and in systems close to a ferromagnetic quantum critical point with Ising symmetry. Our strong coupling results are, however, more general and can be applied to other systems as well, including the antiferromagnetic exchange in 2D used for description of the cuprates.

I. INTRODUCTION

Strong coupling superconductivity due to the interaction between electrons and lattice vibrations has been successfully studied using the coupled Eliashberg equations for the frequency dependent normal and anomalous self energies of superconductors. The theory finds its justification in the weakness of the corrections to the electron-phonon vertex, caused by the small ratio of the electron and ion masses. This theory inspired numerous efforts to describe superconductivity caused by other bosons, even though its justification turns out to be considerably more subtle in some of those cases. Important progress has been made in the study of pairing due to the exchange of bosons that are collective excitations of the fermions. In this context, the interplay between superconductivity and quantum criticality is particularly interesting, as superconductivity in correlated electron systems often occurs in the proximity of a quantum critical point (QCP). At a QCP, the pairing boson becomes massless, and new and unexpected behavior emerges. A related problem occurs in the theory of quantum chromodynamics at high density where single-gluon exchange becomes dominant. The exchange of gluons is believed to cause color superconductivity. As was pointed out by Son, and later in Refs., the color magnetic interaction in high density QCD is unscreened at low temperatures, i.e., the pairing is mediated by a gapless boson. The pairing problem then becomes formally very similar to superconductivity at a QCP, even though the transition temperatures may be different by a factor as big as 10.

In previous studies of the pairing problem near a QCP, the authors of Refs. assumed that the effective, boson-mediated fermion-fermion interaction $u$ is much smaller than the fermionic bandwidth, $W$ (which is generally of the same order as $E_F$), i.e.,

$$ g \sim \frac{u}{W} \ll 1. \quad (1) $$

The limit $g \ll 1$ is often called weak coupling. This notation is not quite correct, as near a QCP the smallness of $g$ doesn’t imply that the system behaves as a weakly coupled Fermi liquid – the mass renormalization due to the exchange of a gapless boson is still singular in $D \leq 3$ and destroys the Fermi liquid behavior at the QCP (see below). To simplify the notations, we nevertheless refer to $g \ll 1$ as weak coupling and $g \gg 1$ as strong coupling. With this notation the Eliashberg theory for superconductivity due to electron-phonon interaction is in the "weak coupling" limit since $g \propto \frac{v_F}{v_s} \ll 1$. This is due to the smallness of the ratio of the sound and Fermi velocities, while the product $\lambda = \rho_F V_{cp}$ of the electron-phonon interaction $V_{cp}$ and the density of states at the Fermi level $\rho_F$ can be of order unity.

The condition $g \ll 1$ implies that the pairing comes only from fermions in a tiny range of momenta around the Fermi surface, i.e., that the system behavior at energies comparable to $E_F$ is irrelevant for the pairing. This makes the pairing problem universal and allows one to use well-established computational techniques, e.g., the Eliashberg theory. However, for various systems of interest the interaction is not necessarily small. In particular, the same interaction that leads to the pairing is often also responsible for the onset of order at the QCP. Generic density-wave instabilities come from fermions with energies $O(E_F)$ and require $g$ to be of order unity. In the cuprate superconductors, to which the ideas of collective-mode mediated $d$–wave pairing was applied, the Hubbard interaction $U$ is at least comparable to $W$ as is evidenced by e.g. the Heisenberg antiferromagnetism at half-filling. For color superconductivity, the effective coupling $u$ is also not necessarily small compared to $E_F$, and $g$ well may be larger than 1.

These arguments call for an understanding of the pairing problem beyond the “weak coupling” limit. In the present paper we extend previous “weak-coupling” studies of the pairing mediated by a gapless boson to the truly strong coupling limit $g \gg 1$. For definiteness we consider pairing of 3D electrons mediated by a scalar boson which is gapless at $q = 0$. This model describes $p$-wave
superconductivity near a ferromagnetic Ising QCP, and color superconductivity of quarks. However, the results at strong coupling are quite general and can be applied to other systems as well, including the antiferromagnetic exchange in 2D used for description of the cuprates. We discuss applications to other systems in a separate section. There is also a connection between our model and the interaction between conduction electrons, mediated by transverse photons. However, as shown in, the exchange of transverse photons does not lead to superconductivity.

A word of caution. In the context of quark pairing mediated by gluons, the equation for the pairing vertex has only been derived in a gauge invariant manner at weak coupling. At strong coupling, antiparticle pairing, neglected in our model, may come into play. Still, qualitatively, the results obtained assuming only particle pairing likely remain valid at both weak and strong coupling.

The main results of this paper are summarized in the phase diagram, Fig. 1. In the weak coupling limit, we find, in agreement with Son and others, that the transition temperature behaves, to leading exponential order, as

$$ \log \frac{\omega_0}{T_c} = \frac{\pi}{2 \sqrt{g}}, $$

where $\omega_0 \sim E_F/g$. This result is parametrically larger than the usual BCS result, $\log \omega_0/T_c \propto 1/g$, and the difference is due to the gapless nature of the pairing boson (see below). Still, $T_c$ is exponentially small at small $g$.

At $g = O(1)$, $T_c$ becomes of order $E_F$, although the prefactor for $T_c$ is a small number. At even larger $g$, we find two characteristic temperatures. The larger temperature, $T_{pair}$, sets the onset of pairing, and is of order $E_F$ (again, with a small prefactor). The smaller temperature, $T_c$ is of order $\sqrt{\omega_0} E_F \sim E_F/\sqrt{g} \ll E_F$. This temperature is determined by the superfluid stiffness, and sets the scale for phase coherence, i.e., of the actual superconductivity. In between $T_c \sim E_F/\sqrt{g}$ and $T_{pair} \sim E_F$, the system displays pseudogap behavior: pairs of fermions are already formed, but do not move coherently.

The accuracy of the computations is a central issue for the theoretical analysis near a QCP. At small $g$, one can use Eliashberg theory since the relevant bosonic and fermionic frequencies are much smaller than $E_F$. Although the frequency-dependent self-energy is not small due to the near-criticality, vertex corrections and the momentum-dependent self-energy are exponentially small (see below). At $g = O(1)$, typical frequencies become of order $E_F$, vertex corrections become $O(1)$, and the momentum-dependent self-energy becomes of the same order as the frequency-dependent self-energy. In this situation, no reliable theoretical scheme is possible.

A naive expectation would be that at strong coupling, vertex corrections get even stronger. We show, however, that at larger $g$, vertex corrections actually saturate at a value $O(1)$ and do not grow with $g$. At the same time, the momentum dependent term in the self-energy again becomes small compared to its frequency dependence, this time the relative smallness is in $1/g$. Furthermore, the pairing problem at $g \gg 1$ involves fermions with energies below $E_F$ for which the density of states can be approximated by a constant. As a result, the new version of the Eliashberg theory (more accurately, the local theory) becomes qualitatively valid. This new local theory is different from the original Eliashberg theory in that the lattice cannot be neglected, and the existence of a finite bosonic bandwidth now plays a crucial role. Still, like in the Eliashberg theory, we derive closed-form equations for the fermionic self-energy and the pairing vertex. To justify the local approximation at $g > 1$ quantitatively, we extend the model to $N$ fermion flavors and consider the limit of large $N$. In this case, vertex corrections become of order $1/N$ and can be safely neglected.

The structure of the paper is as follows. In the next section we set up the model and define the large $N$ limit used to perform the strong coupling calculation. In Sec. III we briefly discuss the weak coupling limit and present an alternative derivation of Son’s result for $T_c$. In Sec. IV we solve the pairing problem at strong coupling. In Sec. V we analyze gap fluctuations and demonstrate the existence of two characteristic temperature scales. In Sec VI we justify our computational procedure. In Sec VII we discuss other systems, including the cuprates. The last section presents our conclusions. Several technical details are presented in the Appendix.

II. MODEL AND LARGE $N$ EXPANSION

We consider the pairing problem in which 3D fermions, $\psi_k$, interact via exchanging a massless, bosonic mode

![FIG. 1: Schematic phase diagram for the superconducting transition temperature $T_c$ and the pairing instability temperature $T_{pair}$ as function of the dimensionless coupling constant $g$. While $T_c \approx T_{pair}$ for weak coupling, an intermediate regime $T_c < T < T_{pair}$ with phase (and amplitude) fluctuations occurs in the strong coupling limit.](image)
with a static propagator $D_0(q) = 1/q^2$: 
\begin{equation}
\mathcal{H}_{\text{int}} = -\frac{u}{k_F} \sum_{k,k',q} \psi_{k+q}^\dagger \psi_{k'}^\dagger D_0(q) \psi_{k'} \psi_k.
\end{equation}

Here $u > 0$ is the effective interaction (with the dimension of energy), and $k_F$ is the Fermi momentum. Note the overall sign in (3) is opposite to that in systems with Coulomb interaction. Another energy scale in the problem is the fermionic bandwidth $W$ (roughly, the scale up to which the fermionic dispersion $\varepsilon_k$ can be linearized around Fermi surface, $\varepsilon_k = v_F (k - k_F)$). The ratio of the two characteristic energies defines the dimensionless coupling constant $g$ in Eq. (1).

The interaction (3) leads to pairing, and also gives rise to fermionic and bosonic self-energies, $\Sigma_k(\omega)$ and $\Pi_q(\Omega)$, respectively. The two self-energies are related to fermionic and bosonic propagators via
\begin{equation}
G_k^{-1}(\omega) = i\omega - v_F (k - k_F) - \Sigma_k(\omega)
\end{equation}
\begin{equation}
D_q^{-1}(\Omega) = q^2 + \Pi_q(\Omega).
\end{equation}

In order to perform a controlled calculation at strong coupling, we generalize the model of Eq. (1) to $N$ fermion flavors and rescale $v_F \to v_F N$ and $u \to u N$. In what follows, we assume that the new $v_F$ and $u$ are constants, independent on $N$. The generalized Eliashberg theory is a set of three coupled integral equations for the pairing vertex $\Phi$ and the self-energies $\Sigma$ and $\Pi$. We will primarily be interested in the onset of the pairing and consider the linearized equation for $\Phi$, and normal state expressions for $\Sigma$ and $\Pi$. Then the system of three coupled equations has the form
\begin{equation}
\Phi(\omega) = \frac{u}{k_F} \int_{q,\omega'} D_q(\omega - \omega') \Phi(\omega')
\end{equation}
\begin{equation}
\times G_{kF+q}(\omega') G_{kF+q}(-\omega'),
\end{equation}
\begin{equation}
\Sigma(\omega) = \frac{u}{k_F} \int_{q,\omega'} D_q(\omega - \omega') G_{kF+q}(\omega'),
\end{equation}
\begin{equation}
\Pi_q(\Omega) = \frac{u}{k_F} \int_{k,\omega} G_k(\omega) G_{k+q}(\omega + \Omega).
\end{equation}

We used the notation $\int_{q,\omega} \ldots = \int \frac{d^3q}{(2\pi)^3} T \sum_{\omega_n} \ldots$ with Matsubara frequencies $\Omega_n = 2n\pi T$ and $\omega_n = (2n + 1)\pi T$ for bosons and fermions, respectively.

Eqs. (4) neglect vertex corrections and the momentum dependence of $\Sigma$ and $\Phi$. We will argue below that at large $N$, both approximations hold both at weak and at strong coupling. Physically, these approximations are based on the (verifiable) assumption that bosons are slow modes compared to fermions. This allows one to factorize the momentum integration in Eqs. (5). Namely, for every given $k_F$ along the Fermi surface, the integration over the component $q_\perp$ transverse to Fermi surface in the equations for $\Phi$ and $\Sigma$ involves only fast fermions, while integrating over the remaining two momentum components $q_\parallel$ in the bosonic propagator, one can set $q_\perp = 0$.

This implies that the boson propagator actually only appears in Eqs. (5) through the "local" interaction
\begin{equation}
d(\Omega) = \int_0^{q_0} dq_\parallel D_{q_\parallel} q_\parallel = 0(\Omega),
\end{equation}
where $q_0 \sim k_F$ is the upper cutoff in the integral over $q_\parallel$.

As a result, the equations for $\Phi$ and $\Sigma$ in (5) reduce to
\begin{equation}
\Phi(\omega) = \frac{3g}{2} \int d\omega' \Phi(\omega') d(\omega' - \omega),
\end{equation}
\begin{equation}
\Sigma(\omega) = -\frac{3g}{2} \int d\Omega \text{sign}(\Omega + \omega) d(\Omega),
\end{equation}
where the factor of $\frac{3}{2}$ is for further convenience and the coupling constant $g$ is given as
\begin{equation}
g = \frac{u}{24\pi^2 E_F k_F},
\end{equation}
with $E_F = v_F k_F / 2$. Without further approximation we can explicitly solve for $\Pi(\Omega)$ and find
\begin{equation}
\Pi(q,\Omega) = \frac{|\Omega|}{q},
\end{equation}
where
\begin{equation}
\gamma = 12\pi^2 k_F^3 \frac{g}{E_F}.
\end{equation}

Note that $\gamma$ does not depend on $N$, despite the fact that the Landau damping term contains a flavor index $N$ as an overall factor. The $N$-independence of $\gamma$ is the result of our rescaling: in rescaled variables $\gamma \to N^{-1} u N k_F$ stays finite.

For the "local" interaction (1), we obtain from Eqs. (5) and (6):
\begin{equation}
d(\Omega) = \frac{1}{3} \log \left( 1 + \frac{\omega_0}{|\Omega|} \right),
\end{equation}
where $\omega_0$ is the characteristic frequency of the bosonic degrees of freedom:
\begin{equation}
\omega_0 = \frac{q_0^3}{\gamma} = \frac{E_F}{g}
\end{equation}
and we introduced
\begin{equation}
E_F = \frac{E_F^*}{12} \left( \frac{q_0}{\pi k_F} \right)^3.
\end{equation}

Below we will refer to $E_F$ as to Fermi energy. We should keep in mind however that our $E_F$ depends on the choice of the upper momentum cut off $q_0$ and is only of the same order of magnitude as a actual Fermi energy of the system.

Substituting Eq. (11) into Eq. (5) and integrating over frequency, we obtain
\begin{equation}
i\Sigma(\omega) = \omega g \left( \frac{\omega_0}{|\omega|} \log \frac{\omega_0 + |\omega|}{\omega_0} + \log \frac{\omega_0 + |\omega|}{|\omega|} \right)
\end{equation}
\begin{equation}
= \begin{cases} 
\omega g \log \frac{\omega_0}{|\omega|} & |\omega| \ll \omega_0 \\
\text{sign}(\omega) g \omega_0 \log \frac{|\omega|}{\omega_0} & |\omega| \gg \omega_0 
\end{cases}.
\end{equation}
We see that \( \omega_0 \) sets the scale at which the momentum cutoff in the bosonic propagator begins affecting the fermionic self-energy. At low energies, the cutoff is irrelevant, and the self-energy has the form typical for a marginal Fermi liquid. At \( \omega \gg \omega_0 \), the self-energy almost saturates and only logarithmically depends on frequency. At weak coupling, \( \omega_0 \gg E_F / g > E_F \), and the crossover is meaningless as Eq. (14) only holds up to \( \omega \sim E_F \) (we recall that in obtaining Eq. (14) we approximated the density of states by a constant). The marginal Fermi liquid behavior then extends all the way up to \( E_F \). At strong coupling, \( \omega_0 \ll E_F \), and the crossover in \( \Sigma(\omega) \) occurs well below \( E_F \). In this situation, marginal Fermi liquid behavior only holds at small frequencies \( \omega < E_F / g \), while at \( E_F / g < \omega < E_F \), \( \Sigma(\omega) \) depends logarithmically on frequency (see Fig. III).

The crossover in the self-energy at strong coupling parallels the crossover in the “local” bosonic propagator \( d(\omega) \) in Eq. (11)

\[
d(\Omega) = \begin{cases} \frac{1}{2} \log \frac{\Omega}{\omega_0} & |\Omega| \ll \omega_0 \ , \\ \frac{1}{2} \log \frac{\omega_0}{|\Omega|} & |\Omega| \gg \omega_0 \ . \end{cases}
\]

Like for the self-energy, this crossover is meaningful only at strong coupling, when \( \omega_0 < E_F \).

Substituting the self-energy and \( d(\Omega) \) into the equation for the pairing vertex, we obtain

\[
\Phi(\omega) = \frac{g}{2} \int d\omega' \frac{\Phi(\omega')}{|\omega' + i\Sigma(\omega')|} \log \left( 1 + \frac{\omega_0}{|\omega - \omega'|} \right).
\]

Strictly speaking, we have to evaluate this equation at finite \( T \), because the linearized equation for \( \Phi \) is only valid at the onset temperature for the pairing. By reasons that we outline below, we label this temperature as \( T_{\text{pair}} \) rather than \( T_c \). As we will only be interested in the order of magnitude estimate for \( T_{\text{pair}} \), we adopt a simplified approach, and instead of performing the discrete Matsubara sum, use Eq. (14) at finite \( T \), but introduce a lower frequency cutoff at \( \omega \sim T \). In the weak coupling limit, this procedure was shown earlier to yield the same \( T_{\text{pair}} \) (modulo a numerical prefactor), as one would obtain by performing an explicit summation over discrete Matsubara frequencies. In Appendix B we show that the same holds for large \( g \). With this simplification we have to solve

\[
\Phi(\omega) = g \int_{T_{\text{pair}}}^{\infty} d\omega' \frac{\Phi(\omega')}{|\omega' + i\Sigma(\omega')|} K(\omega, \omega') ,
\]

with bosonic kernel

\[
K(\omega, \omega') = \frac{1}{2} \log \left[ \left( 1 + \frac{\omega_0}{|\omega - \omega'|} \right) \left( 1 + \frac{\omega_0}{|\omega + \omega'|} \right) \right].
\]

In what follows we solve this equation, first in the weak coupling limit \( g \ll 1 \), where we reproduce the results of Ref. 32, and then in the strong coupling limit \( g \gg 1 \).

![FIG. 2: Characteristic energy scales for the weak (\( g \ll 1 \)), intermediate (\( g \sim 1 \)) and strong (\( g \gg 1 \)) coupling limit. While at weak coupling, \( \omega_0 \approx E_F / g \) is large compared to \( E_F \) and thus irrelevant, it emerges as a new low energy scale in the strong coupling limit.](image)

### III. PAIRING PROBLEM AT WEAK COUPLING

At weak coupling, one obviously expects \( T_{\text{pair}} \) to be much smaller than \( \omega_0 \) (see Fig. III). This in turn implies that only frequencies \( \omega \ll \omega_0 \) are relevant. For these frequencies, the self-energy \( \Sigma(\omega) \) and the kernel \( K(\omega, \omega') \) in (15) can be simplified to

\[
\Sigma(\omega) = -i\omega \log \frac{\omega_0}{|\omega|}
\]

\[
K(\omega, \omega') = \log \frac{\omega_0}{\sqrt{\omega^2 - \omega'^2}}.
\]

Eq. (17) then becomes

\[
\Phi(\omega) = g \int_{T_{\text{pair}}}^{\omega_0} d\omega' \frac{\Phi(\omega')}{\omega'(1 + g\omega' \log \frac{\omega_0}{\omega'})}.
\]

This equation yields \( T_{\text{pair}} \) for the pairing in a marginal Fermi liquid. Eq. (20) was solved numerically in Ref. 32. We show that an analytic solution is also possible. Our computational procedure is similar to the one used by Son. In addition to the approach of Ref. 32 we also analyze the pairing susceptibility.

If the two logarithmic terms in the r.h.s. of Eq. (20) were absent, the equation for the pairing vertex would be the same as in BCS theory\(^{30}\), and \( T_{\text{pair}} \) would scale as \( \omega_0 e^{-1/g} \). However, as Son demonstrated, the presence of the logarithm in the pairing kernel substantially enhances \( T_{\text{pair}} \) at weak coupling and changes its functional form to \( T_{\text{pair}} \propto \omega_0 e^{-\pi/(2\sqrt{g})} \). The easiest way to see this is to introduce logarithmic variables: \( x = \log \left( \frac{\omega}{\omega_0} \right) \), \( x' = \log \left( \frac{\omega'}{\omega_0} \right) \), \( x_F = \log \left( \frac{\omega_F}{\omega_0} \right) \), and re-write Eq. (20) with logarithmic accuracy as

\[
\Phi(x) = g \int_{x_0}^{x_F} dx' \bar{\Phi}(x') \left( \frac{x'}{1 + gx'} \right) + g x \int_{x_0}^{x_F} dx' \bar{\Phi}(x') \left( \frac{x'}{1 + gx'} \right).
\]

Differentiating both sides of Eq. (21) over \( x \), we find

\[
\frac{d\Phi(x)}{dx} = g \int_{x_0}^{x_F} \Phi(x') \left( \frac{1}{1 + gx} \right).
\]
Differentiating one more time, we find that the integral equation for the anomalous vertex reduces to a second-order differential equation:
\[ \frac{d^2 \Phi(x)}{dx^2} = -g \frac{\Phi(x)}{1 + gx}. \] (23)

The \( gx \) term in the r.h.s. of Eq. (23) is due to the fermionic self energy. We assume and verify afterwards that \( gx \ll 1 \) for all relevant \( x \), and drop this term from (23). The solution of Eq. (23) is then elementary:
\[ \Phi(x) = A \cos(\sqrt{gx}x) + B \sin(\sqrt{gx}x). \] (24)

The two boundary conditions
\[ \Phi(x = 0) = 0, \] \[ \frac{d\Phi(x)}{dx} \bigg|_{x=x_T} = 0 \] (25)
follow from (21) and (22), respectively. They yield \( A = 0 \), and
\[ \cos(\sqrt{gx_T}) = 0. \] (26)

The onset temperature \( T_{\text{pair}} \) corresponds to the smallest \( x_T \) that satisfies (20), i.e., \( gx_T \ll 1 \). This justifies dropping the \( gx \) term (i.e., fermionic self-energy) from (23). The first order differential equation reduces to a second-order differential equation for the anomalous vertex reduces to a second-order differential equation. In distinction to the weak coupling case, however, the term \( gx \), coming from the self-energy, is now the dominant term in the denominator in the r.h.s. of (23). Leaving only this term,

\[ \chi_{pp}(\omega, T) = \frac{\cos(\sqrt{g}\text{log}(\omega/T_{\text{pair}}))}{\cos(\sqrt{g}\text{log}(\omega/T_{\text{pair}}))}. \] (29)

Note that \( T \) and \( \omega_0 \) are lower and upper limits of the integration over \( \omega \) in (20), hence the pairing susceptibility is only defined in the interval \( T < \omega < \omega_0 \). We see from (20) that at the upper boundary, \( \omega = \omega_0 \), \( \chi_{pp} = 1 \) at any \( T \). This is a clear distinction to the BCS limit. As long as \( T > T_{\text{pair}} \), the pairing susceptibility remains positive everywhere in the interval \( T < \omega < \omega_0 \) despite the fact that the solution of the differential equation (23) for \( \Phi(\omega) \) is formally an oscillating function of frequency. At \( T_{\text{pair}} \), \( \log(T_{\text{pair}}) = \frac{\omega_0}{g} \), and \( \chi_{pp} \) diverges for all \( \omega \), except \( \omega = \omega_0 \). Below \( T_{\text{pair}} \), \( \chi_{pp} \) is negative at low frequencies, implying that the system is unstable towards pairing. We show the behavior of \( \chi_{pp}(\omega, T) \) as function of temperature and frequency in Fig.11.

IV. PAIRING PROBLEM AT STRONG COUPLING

We next analyze the strong coupling limit \( g \gg 1 \). In distinction to the weak coupling regime, we now have two characteristic energy scales in the problem, \( \omega_0 = E_F/g \ll E_F \), and \( E_F \), which is the ultimate upper cutoff in the theory (see Fig.II). The issue then is which of the two scales determines the onset of the pairing.

Suppose momentarily that only frequencies \( \omega \leq \omega_0 \) contribute to the pairing. At \( \omega < \omega_0 \), the pairing kernel and the self-energy can still be approximated by Eq. (19), and the equation for the pairing vertex can be reduced to the differential equation (23). In distinction to the weak coupling case, however, the term \( gx \) coming from the self-energy, is now the dominant term in the denominator in the r.h.s. of (23). Leaving only this term,
we arrive at
\[ \frac{d^2 \Phi (x)}{dx^2} = - \frac{\Phi (x)}{x}. \] (30)

Note that \( g \) drops from this equation because of cancellation between \( g \) factors in the effective interaction and the self-energy.

The solution of Eq. (29) with \( \Phi (x = 0) = 0 \) is \( \Phi (x) \propto \sqrt{\pi} J_1 (2 \sqrt{x}) \), where \( J_1 \) is a Bessel function. Substituting this solution back into (24) and assuming that the upper limit in the frequency integral in (21) is still \( x_F \) (i.e., that only \( \omega \leq \omega_0 \) are relevant for the pairing), we obtain \( x_T = 3.670(5) \). This leads to \( T_{\text{pair}} \approx 0.025 \omega_0 \), i.e., to a pairing instability at a temperature which is a fraction of \( \omega_0 \).

This result is similar to McMillan’s \( T_{\text{pair}} \sim \omega_D e^{-(1+g)/g} \sim \omega_D \) for strongly coupled phonon superconductors \[ (\omega_D \text{ is Debye frequency}). \] However, like for phonons, there is actually no reason to restrict the frequency integral to \( \omega < \omega_0 \sim E_F/g \), since for strong coupling there also exists a wide frequency range \( \omega_0 < \omega < E_F \) where, on the one hand, the pairing kernel and the self-energy are different from (19), and, on the other hand, typical frequencies are still below \( E_F \), i.e., a low-energy description is at least qualitatively valid. The existence of this extra range raises the possibility that the onset of pairing may occur at a temperature of order \( E_F \), not of order \( \omega_0 \sim E_F/g \). Note that for the electron-phonon case, the scale which sets the ultimate upper cutoff for the pairing (the analog of \( E_F \) in our case) is \( \omega_D \sqrt{T/(1.49.50.51)} \).

To verify whether \( T_{\text{pair}} \) scales as \( E_F \), not as \( \omega_0 \), we analyze the equation for \( \Phi (\omega) \) assuming that all characteristic frequencies are larger than \( \omega_0 \). At these frequencies, the pairing kernel and the self-energy are given by
\[ \Sigma (\omega) = -i \text{sign} (\omega) g \omega_0 \log \left| \frac{\omega}{\omega_0} \right| \]
\[ K (\omega, \omega') = \frac{\omega_0}{2} \left( \frac{1}{|\omega - \omega'|} + \frac{1}{|\omega + \omega'|} \right). \] (31)

Now the pairing kernel scales as \( 1/\omega \), while the self-energy is nearly a constant, and only logarithmically depends on frequency.

Substituting the pairing kernel and the self-energy into (17) and using the fact that for all \( \omega < E_F \) the self-energy \( \Sigma (\omega) \) exceeds the bare \( \omega \), we obtain
\[ \Phi (\omega) = \int_T^{E_F} \frac{d\omega' \Phi (\omega')}{2 \log \frac{\omega_0}{\omega_0}} \left( \frac{1}{|\omega - \omega'|} + \frac{1}{|\omega + \omega'|} \right). \] (32)

The logarithmic divergence in the r.h.s. of (32) at \( \omega = \omega' \) can easily be regularized as the \( 1/|\omega - \omega'| \) form of the kernel is only valid at \( |\omega - \omega'| > \omega_0 \).

We see that the dimensionless ratio \( T/E_F \) is the only parameter in Eq. (32), except for the \( \log \frac{\omega_0}{\omega_0} \) term in the denominator in (32). Hence, if this equation has a solution at some finite value of this parameter, the pairing instability should occur at \( T \sim E_F \).

The analysis of Eq. (32) requires special care because of the interplay between the \( 1/\omega \) dependence of the pairing kernel and logarithmic behavior of the self-energy. The discussion is somewhat technical, and we moved it into Appendix A. We find there that the solution of (32) at frequencies between \( T \) and \( E_F \) is
\[ \Phi (\omega) = A \frac{E_F}{\sqrt{\omega}} \cos \left( \beta \log \frac{\omega}{E_F} + \phi \right), \] (33)
where \( \beta = 0.148(2) \) is determined from the solution of a transcendental equation, and \( A, \phi \) are real constants.

Like at weak coupling, the two limits of the integration over \( \omega' \) in Eq. (32) for \( \Phi \) imply two boundary conditions for \( \Phi (\omega) \) from (33). One of them determines the phase \( \phi \), while the other determines the pairing instability temperature (the overall factor \( A \) in (33) cannot be determined from the linearized gap equation). For a simple estimate of \( T_{\text{pair}} \), we use the same boundary conditions as in the weak coupling limit, i.e., (i) assume that frequencies larger than \( E_F \) are irrelevant for the pairing and set \( \Phi (\omega = E_F) = 0 \), and (ii) assume that \( \frac{d^2 \Phi (\omega)}{d\omega^2} |_{\omega = T_{\text{pair}}} = 0 \). We then obtain \( \phi = \frac{\pi}{2} \) and
\[ T_{\text{pair}} = E_F e^{-\frac{\beta}{4}} \approx 0.0676 E_F, \] (34)
where \( 1/\beta = (1/\beta) \arccos \left( -\frac{1}{\sqrt{1+4\beta^2}} \right) \approx 0.37 \). In Appendix B we demonstrate that the same result, modulo a numerical prefactor, is obtained by solving explicitly the linearized gap equation for discrete Matsubara frequencies.

We see therefore that at strong coupling, the pairing instability temperature \( T_{\text{pair}} \) is indeed of the order of the Fermi energy \( E_F \), although numerically it is still much smaller than \( E_F \). This temperature is larger by a factor \( g \) than the McMillan-type estimate, \( T_{\text{pair}} \sim \omega_0 \), which ignores the pairing interaction at energies larger than the characteristic bosonic frequency. We emphasize that in order to obtain Eq. (34), it was crucial that we included into consideration the normal state self-energy renormalization. Had we ignored it, an oscillating solution for \( \Phi (\omega) \) at \( \omega > \omega_0 \) would not have been possible, i.e., no pairing instability would occur at \( T > \omega_0 \) (see Ref. 33). Alternatively speaking, \( T_{\text{pair}} \sim E_F \) is the result of the interplay between a non-Fermi liquid behavior of the fermions caused by the logarithmic self energy \( \Sigma (\omega) \propto g \omega_0 \log \frac{\omega_0}{|\omega'|} \) and a retarded pairing interaction governed by a “local” boson susceptibility \( d(\omega) \propto \frac{1}{\omega} \).

Since \( T_{\text{pair}} \sim E_F \), it is inevitable that the magnitude of \( T_{\text{pair}} \) is affected by the system behavior at high energies, i.e., at lattice scales in the condensed matter context. We assumed above that the fermionic density of states is a constant. This is indeed only approximately valid at \( \omega \sim E_F \). To determine \( T_{\text{pair}} \) beyond an order of magnitude estimate, one then needs to solve the full microscopic problem. Still, lattice effects only modify the prefactor in \( T_{\text{pair}} \); the relation \( T_{\text{pair}} \sim E_F \) is generic and survives lattice corrections.
V. THE ROLE OF GAP FLUCTUATIONS

A. Phase fluctuations

In the weak coupling limit it is known that the transition temperature, determined from the linearized gap equation, coincides with the temperature where global phase coherence sets in. This can easily be seen by evaluating the phase stiffness \( \rho_s \) defined as

\[
E_{\text{phase}} = \rho_s \int d^3x (\nabla \varphi)^2. \tag{35}
\]

At weak coupling, \( \rho_s \approx E_F k_F \). Eq. (35) can then be considered as the continuum limit of an XY-spin model on a three dimensional lattice with lattice constant \( \approx k_F^{-1} \) and exchange interaction \( \approx E_F \). Fluctuation effects in this model become relevant well below the onset of the pair-breaking, and by conventional reasoning [14] [15], the phase coherence is established as soon as Cooper pairs are formed, i.e., \( T_{\text{pair}} = T_c \).

Consider next the strong coupling limit, where \( T_{\text{pair}} \sim E_F \). In what follows we argue that at strong coupling, \( \rho_s / k_F \approx E_F / \sqrt{g} \ll T_{\text{pair}} \). In this situation, phase fluctuations become relevant below the onset of the pairing, and by conventional reasoning [52, 53, 54], phase coherence sets in at

\[
T_c \approx E_F / \sqrt{g} \ll T_{\text{pair}} \approx E_F. \tag{36}
\]

This new energy scale is the characteristic energy of a boson in the gapped state below \( T_{\text{pair}} \). In between \( T_{\text{pair}} \) and \( T_c \), the system displays a pseudogap behavior: the density of states develops a maximum at a finite frequency (the tunneling gap), and the spectral weight is transformed from frequencies below the gap to frequencies above the gap. However, the superconducting order parameter only develops at \( T_c \).

We now show how we arrived at \( \rho_s / k_F \approx E_F / \sqrt{g} \). The superfluid stiffness at \( T = 0 \) is obtained by evaluating the sum of fermionic bubbles made of normal and anomalous Green's functions, and is given by

\[
\rho_s = \rho_s^0 \int_{0}^{\infty} d\omega \frac{\Phi^2(\omega)}{[\omega Z(\omega)]^2 + \Phi^2(\omega)}^{3/2}, \tag{37}
\]

where \( \rho_s^0 \sim E_F k_F \) is the stiffness of a BCS superconductor. \( \Phi(\omega) \) is the pairing vertex at \( T = 0 \) and we introduced

\[
Z(\omega) = 1 - \frac{\Sigma(\omega)}{\omega}. \tag{38}
\]

Using the relation between \( \Phi(\omega) \) and the gap function \( \Delta(\omega) = \frac{\Phi(\omega)}{Z(\omega)} \), one can write Eq. (37) as

\[
\rho_s = \rho_s^0 \int_{0}^{\infty} d\omega \frac{\Delta^2(\omega)}{[\omega^2 + \Delta^2(\omega)]^{3/2}}. \tag{39}
\]

For a BCS superconductor, \( Z = 1 \), and \( \Delta \) does not depend on frequency. The frequency integration in (39) then yields \( \rho_s = \rho_s^0 \), independent on \( \Delta \). This essentially implies that at \( T = 0 \), the superfluid density equals the full density.

To obtain \( \rho_s \) at strong coupling, we need to know \( \Delta(\omega, T = 0) \) and \( Z(\omega, T = 0) \). The gap \( \Delta(\omega, T = 0) = \Delta(\omega) \) is obtained by solving the nonlinear gap equation

\[
\Delta(\omega) = \frac{3g}{2} \int_{-\infty}^{\infty} \frac{\Delta(\omega') - \Delta(\omega) \omega'}{\sqrt{\omega'^2 + \Delta(\omega')^2}} d\omega'. \tag{40}
\]

where \( d_{sc}(\Omega) \) is the "local" boson propagator in a superconductor. In the normal state, \( d(\Omega) \) is given by Eqs. (36) [11]. In the presence of \( \Delta \), the bosonic spectrum itself changes due to feedback from the gap opening, and the Landau damping transforms into \( \Pi(\Omega) \sim \gamma \frac{\Omega^2}{\Omega^2 + \Delta^2} \) (Ref. 24). This leads to

\[
d_{sc}(\Omega) = \frac{1}{3} \log \left( 1 + \frac{\omega_{0,sc}^2}{\Omega^2} \right), \tag{41}
\]

where

\[
\omega_{0,sc} \sim \frac{\sqrt{\omega_0 E_F}}{\sqrt{g}} \sim \frac{E_F}{\sqrt{g}}. \tag{42}
\]

is the characteristic energy of the bosons in a state where fermions are gapped – it has the same physical meaning as \( \omega_0 \) above \( T_{\text{pair}} \). For frequencies \( E_F > |\Omega| > \omega_{0,sc} \) we have

\[
d_{sc}(\Omega) \approx \frac{1}{3} \left( \frac{\omega_{0,sc}}{\Omega} \right)^2. \tag{43}
\]

Despite the \( 1/\Omega^2 \) dependence of \( d_{sc}(\Omega) \), the integral over \( \omega' \) in Eq. (40) remains convergent since the numerator vanishes at \( \omega = \omega' \). We can then safely use Eq. (40) for \( d_{sc}(\Omega) \) and drop the restriction that this form is only valid above \( \omega_{0,sc} \). The gap equation then contains only \( E_F \) as the energy scale. Accordingly, \( \Delta(\omega) \) can only be of order \( E_F \), if the gap equation has a solution. We verified that the solution of (40) does indeed exist and yields \( \Delta(\omega) = E_F f(\omega/E_F) \).

The expression for \( Z(\omega) \) follows from the formula for the self-energy

\[
Z(\omega) = 1 + \frac{3g}{2\omega} \int_{-\infty}^{\infty} \frac{d(\omega - \omega') \omega' d\omega'}{\sqrt{\omega'^2 + \Delta(\omega')^2}}. \tag{44}
\]

Here the restriction that Eq. (43) is only valid at frequencies above \( \omega_{0,sc} \) becomes crucial, otherwise the integral over \( \omega' \) in Eq. (44) would diverge. Beyond this, the evaluation of the integral is straightforward, and we obtain

\[
Z(\omega < E_F) \approx g^{1/2}, \quad Z(\omega \gg E_F) \approx 1. \tag{45}
\]

Substituting this \( Z \) into (43), we find

\[
\rho_s(T = 0) \approx \rho_s^0 / \sqrt{g} \approx \omega_{0,sc} k_F. \tag{46}
\]
We see that \( \rho_s(T = 0)/k_F \) is much smaller than \( T_{\text{pair}} \). The exchange constant of the XY-model is therefore \( \omega_{0,sc} \ll E_F \). This leads to our estimate of \( T_c \) in Eq. (46). This estimate is further supported by the fact that a finite \( T \), we found that the leading temperature dependence of the stiffness varies as a function of \( T/\omega_{0,sc} \), i.e., thermal corrections to the stiffness indeed become relevant at \( T \simeq \omega_{0,sc} \).

### B. A relation to Eliashberg theory

We emphasized above that our strong coupling theory is a local theory, but not an Eliashberg theory. Indeed, in our case, the interaction is larger than the Fermi energy, and the presence of the momentum cutoff in the bosonic propagator is crucial. This distinctness becomes particularly important if we compare our result for \( \rho_s \) with the conventional Eliashberg theory. There \( E_F \) is the largest scale in the problem, even if the “local” interaction \( d(\omega) \) scales as \( 1/\omega \) or even faster (as, e.g., \( 1/\omega^2 \) for phonon superconductors). Once \( E_F \) is the largest energy scale, \( \rho_s/k_F \) is always larger than \( T_{\text{pair}} \), and phase fluctuations are weak. Indeed, according to Eq. (47), \( \rho_s \) scales as

\[
\rho_s \sim \rho_s^0 \frac{\Delta}{\Sigma(\omega) \sim \Delta},
\]

where, as before, \( \rho_s^0 \sim E_F k_F \) is the stiffness of the weak coupling limit. The ratio \( \Delta/\Sigma \) can be quite small if the pairing occurs in the quantum-critical regime and involves near-massless bosons. In particular, for phonon superconductors, when the Debye frequency \( \omega_D \) is much smaller than electron-phonon interaction \( u \), \( \Delta \sim T_{\text{pair}} \sim u \) (see Ref. [48]), and \( \Sigma(\omega \sim u) \sim u^2/\omega_D >> \Delta \). Then \( \rho_s \sim \rho_s^0 (\omega_D/u) \ll \rho_s^0 \). Still, the condition that \( E_F \) is the largest energy scale implies that \( \Sigma(\omega) < E_F \), i.e., \( u^2/\omega_D < E_F \). Then, even though \( \rho_s \) is reduced from its weak coupling value, it still holds that

\[
\rho_s/k_F \sim T_{\text{pair}} \frac{E_F \omega_D}{u^2} > T_{\text{pair}}.
\]

This implies that the exchange coupling in the corresponding XY model is still larger than the onset temperature for the pairing. As a result, within Eliashberg theory one can expect at most modest changes in the transition temperature due to phase fluctuations. In our case, we remind, at strong coupling \( E_F \) is no longer the largest energy scale in the problem, and the \( 1/\omega \) form of \( d(\omega) \) in the strong coupling limit emerges once one imposes a cutoff in the integration over bosonic momenta.

### C. Longitudinal gap fluctuations

In previous subsections we discussed the role of phase fluctuations. They are sufficient to destroy superconducting order between \( T_c \) and \( T_{\text{pair}} \). There also exist, however, longitudinal fluctuations of the pairing gap, and it is instructive to consider how strong they are.

Longitudinal gap fluctuations generally reflect how shallow the profile of the free energy with respect to deviations of \( \Delta(\omega) \) from its equilibrium value is. A shallow profile implies that the superconducting order is weak as different \( \Delta(\omega) \) have almost the same condensation energy. A situation with a shallow profile emerges when, in real frequencies, the attractive part of \( \text{Red}_{sc}(\omega) \) is weak, and the pairing predominantly comes from \( \text{Im}d_{sc}(\omega) \). The imaginary part of a “local” interaction describes purely retarded interaction between fermions. This interaction then does not contribute to the superconducting order parameter, which is an equal time correlator. Accordingly, the slope of the free energy is determined only by a weak \( \text{Red}_{sc}(\omega) \).

A simple estimate of the energy scale at which longitudinal gap fluctuations become relevant can be obtained by analyzing the form of \( \text{Red}_{sc}(\omega) \). Converting Eq. (11) to real frequencies yields

\[
d_{sc}(\Omega) = \frac{1}{3} \log \left( 1 - \frac{\omega_{0,sc}^2}{\Omega^2} \right) + i \frac{\pi \text{sign} \Omega}{3} \left( \Omega^2 - \omega_{0,sc}^2 \right).
\]

We see that \( \text{Red}_{sc}(\omega) \) remains attractive up to a frequency \( \omega_{0,sc}/\sqrt{2} \), and is repulsive at larger frequencies. This means that frequencies above \( \omega_{0,sc}/\sqrt{2} \) do not contribute to the superconducting order parameter, although they do contribute to the pairing itself via \( \text{Im}d_{sc}(\omega) \). This in turn implies that longitudinal gap fluctuations become strong at \( T \geq \omega_{0,sc} \sim E_F/\sqrt{2} \). Comparing this result with Eq. (46), we see that in our strong coupling limit, phase and amplitude fluctuations of the gap are equally important, as the corrections to the superconducting order parameter from both fluctuations become \( O(1) \) at \( T \sim T_c \sim E_F/\sqrt{7} \). One can equally argue that \( T_c \ll T_{\text{pair}} \) is the result of strong phase fluctuations, or the result of soft longitudinal gap fluctuations brought about by the absence of a repulsive component of \( \text{Red}_{sc}(\omega) \) at \( \Omega > \omega_{0,sc} \).

### VI. Migdal parameter

As we discussed in the Introduction, the coupled equations [3] for the pairing vertex and fermionic and bosonic self-energies are valid if vertex corrections and the momentum dependent part of the self energy can be neglected. In case of electron-phonon interaction, this approximation was justified by Migdal. Below we evaluate the leading corrections to our local theory, both in the Eliashberg limit, and at strong coupling. For definiteness, we focus on vertex corrections \( \delta \Gamma \) of the total vertex

\[
\Gamma = \sqrt{\frac{u}{k_F}} \left( 1 + \delta \Gamma \right).
\]
generally, the correction to the interaction vertex between fermions and gapless bosons, depend on the interplay between the bosonic momentum and frequency. In particular, Ward identities imply that vertex corrections in the limit of vanishing bosonic momentum are of the same order as the fermionic self-energy, and not necessary small. However, for the pairing problem, we need to analyze vertex corrections for typical bosonic energies, \( \Omega_{\text{typ}} \), and for typical bosonic momenta, \( q_{\text{typ}} \), that contribute to the pairing.

The leading vertex correction in the normal state is presented in Fig. 4 and is given by:

\[
\delta \Gamma_q (\omega, \Omega) = \frac{Nu}{k_F} \int_{k'-\omega} \frac{D_{k-k'} (\omega - \omega') G_{k'} (\omega')} {G_{k' + q} (\omega' + \Omega)}.
\]

(50)

Here, \( \omega \) is the external fermionic frequency. In principle, \( \delta \Gamma \) depends on two momenta – the bosonic momentum \( q \) and the external fermionic momentum \( k \). However, the dependence on \( k \) is weak and thus irrelevant, and we will neglect it. Performing the momentum integration in (50), we obtain

\[
\delta \Gamma_q (\omega, \Omega) \simeq \frac{3g}{\Omega} \int_0^{\Omega} d(\omega + \omega') \frac{d\omega'} {\sqrt{\Omega^2 + (Nv_F q)^2}}.
\]

(51)

The factor \( N \) in the denominator is a consequence of the rescaling that we performed in Sec. II. In the limit \( q \to 0 \)

\[
\delta \Gamma_{q \to 0} (\omega, \Omega) = \frac{3g}{\Omega} \int_0^{\Omega + \omega} d(\omega + \omega') \frac{d\omega'} {\sqrt{(\Omega + \omega)^2 + (Nv_F q)^2}}
\]

(52)

\[
\simeq \frac{\Sigma (\Omega + \omega) - \Sigma (\omega)} {-i\Omega}.
\]

This is the Ward identity relating the homogeneous vertex with the self energy. We see from (52) that static vertex corrections do not depend on \( N \) and are not small at moderate and strong coupling. The situation, however, changes when we evaluate \( \delta \Gamma_q (\omega, \Omega) \) at \( q_{\text{typ}} \) and \( \Omega_{\text{typ}} \) relevant to the pairing problem. In what follows we evaluate \( \delta \Gamma \) for weak, intermediate and strong coupling, and specify in each case the relevant bosonic momentum and frequency.

### A. Vertex corrections at weak coupling

We first consider the limit of weak coupling, \( g \ll 1 \). The typical bosonic energy \( \Omega_{\text{typ}} \) is of order \( T_c \simeq (E_F / g) e^{-\pi/(4\sqrt{g})} \). On the other hand, typical momenta \( q_{\text{typ}} \) are obtained from the condition that the momentum and frequency dependent term in the bosonic propagator \( D(q, \Omega) \) are of the same order, i.e. \( q_{\text{typ}} \simeq (\gamma \Omega_{\text{typ}})^{1/3} \). Together with Eq. (10) for \( \gamma \) this yields

\[
v_F q_{\text{typ}} \simeq E_F e^{-\pi/(6\sqrt{g})}.
\]

(53)

Comparing \( \Omega \) and \( v_F q_{\text{typ}} \), we see that

\[
v_F q_{\text{typ}} \simeq \Omega_{\text{typ}} \left[ g e^{\pi/(3\sqrt{g})} \right] \gg \Omega_{\text{typ}}.
\]

(54)

We then obtain from Eq. (51)

\[
\delta \Gamma \simeq \frac{g}{Nv_F q_{\text{typ}}} \int_0^{\Omega_{\text{typ}}} \log \frac{\omega}{\omega'} d\omega' = \frac{\pi}{2N} e^{-\frac{\pi}{\sqrt{g}}}.
\]

(55)

We see that vertex correction is exponentially small for small \( g \). The extension to large \( N \) is in fact not needed as vertex corrections are already negligible.

### B. Vertex corrections at intermediate coupling

At intermediate \( g = O(1) \), \( \omega_0 \) and \( E_F \) become of the same order, i.e., there is only one characteristic energy scale in the problem. Vertex corrections are \( O(1) \) for \( N = 1 \), but are still small in \( 1/N \) if we extend the theory to large \( N \).

### C. Vertex corrections at strong coupling

A naive expectation would be that vertex corrections gradually increase with \( g \) and eventually overcome the overall smallness in \( 1/N \). This would invalidate our local theory at sufficiently large \( g \). It turns out, however, that vertex corrections freeze at \( O(1/N) \) and do not grow with \( g \). The saturation originates from the form of the bosonic propagator \( D(q, \Omega) \), which is determined by the self energy \( \simeq \gamma \frac{\Omega}{q} \). As \( \gamma \) by itself scales with the boson-fermion coupling, \( D(q, \omega) \) scales inversely with \( g \) and cancels out the overall factor \( g \) in Eq. (51).

To see this explicitly we note that at strong coupling,\( q_{\text{typ}} \) is of order \( k_F \), hence \( v_F q_{\text{typ}} \) is of order \( E_F \). Typical frequencies \( \Omega_{\text{typ}} \) are also \( O(E_F) \). The external fermionic frequency \( \omega \) in Eq. (51) is also of order of the Fermi
energy. Evaluating the vertex correction diagram using \( d(\Omega) \approx \frac{\omega_0}{\Omega} \), we obtain for these \( g_{\text{typ}} \) and \( \Omega_{\text{typ}} \)

\[
\delta \Gamma \approx g \int_{E_F}^{E_F + \omega_0} \frac{\omega^2 + \omega \Omega}{E_F \sqrt{1 + \Omega^2}} \approx \frac{1}{N} \quad (56)
\]

We see that vertex corrections indeed do not depend on \( g \) and remain small (at large \( N \)) for arbitrary strong coupling.

At large \( N \), vertex corrections remain small for all values of \( g \), i.e., our local theory is valid both in the weak and the strong coupling limit.

\section{OTHER SYSTEMS}

As we discussed in the introduction, the problem discussed in this paper describes \( p \)-wave superconductivity in condensed matter systems close to a ferromagnetic quantum critical point with Ising symmetry, and superconductivity due to color magnetic interactions in dense quark matter\textsuperscript{32,37,38}. As pointed out above, for color superconductivity, antiparticle pairing that has been neglected in Eqs. \textsuperscript{4}, \textsuperscript{5}, may come into play at strong coupling\textsuperscript{58}. We nevertheless believe that the main results of this paper are still relevant to this case.

However, the results obtained in the strong coupling limit are much more general. The key aspect of our theory is that the boson propagator \( D(q_\parallel, q_\perp, \Omega) \) becomes completely local above a certain frequency. We considered a particular case when boson dynamics is set by Landau damping, and

\[
D(\Omega) \propto \frac{\omega_0}{|\Omega|} \quad (57)
\]

where \( \omega_0 \) is a characteristic upper cut off scale of the boson system. However, our results will be equally valid for any \( D(\Omega) \) in the form \( D(\Omega) \propto 1/|\Omega|^\gamma \) with \( \gamma \geq 1 \).

One of the possible applications of our strong coupling result is pairing by antiferromagnetic fluctuations, which has been discussed in great detail in the context of cuprate superconductors\textsuperscript{34}. That analysis was, however, performed within the low-energy spin-fermion model, which is only valid at \( u < E_F \), i.e., in the weak coupling limit, as we defined it in this paper. In the cuprates, as is well known, the Hubbard interaction is comparable to \( E_F \), i.e., \( g \geq 1 \). Our finding that a large pseudogap regime with phase incoherent pairs is inevitable at strong coupling is quite intriguing in view of the numerous observations of pseudogap physics in this class of materials. To make a connection to other studies of pseudogap within Hubbard model, we note that our effective interaction \( u \approx U^2/t \) and \( E_F \approx t \), where \( U \) and \( t \) are the local Coulomb repulsion and tight binding hopping element, respectively. Then \( g = \left( \frac{4}{N} \right)^2 \) and \( \omega_0 \sim E_F/g \sim \frac{4}{N} J \), and \( \omega_{0,s} \sim E_F/\sqrt{g} \sim J \), where \( J = \frac{4t^2}{U} \) is the antiferromagnetic exchange interaction between spins. Accordingly, it follows from our analysis that the pairing sets in at \( T_{\text{pair}} \sim t \), and the pairing gap is \( \Delta \sim t \), while coherent superconductivity occurs at \( T_c \propto J \). We note in this regard that numerical investigations of variational wave functions designed to cover the strong coupling limit\textsuperscript{62,63,64,65} do indeed yield a zero temperature gap \( \Delta \sim t \) and a considerably reduced superfluid stiffness \( \rho_s \textsuperscript{58,59,60} \). Also, our result \( T_c \sim J \) is consistent with the observation that \( T_c \) in underdoped cuprates scales with the neutron peak frequency. The latter turns out to be of order \( J \), if, e.g., one extends the exciton scenario for the resonance peak to the strong coupling limit\textsuperscript{62,63,64,65}.

There exists some similarity between our results and those obtained for the crossover from BCS-type behavior at weak coupling and Bose Einstein condensation (BEC) of pairs at strong coupling\textsuperscript{62,63,64,65}. In particular, for large \( g \), when \( \Delta \) is or order \( E_F \), our pair coherence length \( \xi = \frac{\omega_F}{\Delta} \) becomes of the order of the typical distance between fermions \( \sim k_F^{-1} \). This is similar to the findings for the BCS-BEC crossover\textsuperscript{62,63,64,65}. An important distinction between the two theories is that in BCS-BEC crossover, the pairing interaction is static, while in our case it is dynamic and strongly retarded. The transition at large \( g \) in our case should therefore not be considered as condensation of almost free bosons.

\section{CONCLUSION}

In this paper we considered pairing of 3D fermions, due to an exchange of massless bosons. The model we considered describes \( p \)-wave superconductivity in itinerant fermionic systems close to a ferromagnetic quantum critical point with Ising symmetry, and superconductivity due to color magnetic interactions in quark matter.

At weak coupling, we find that an exchange of a massless boson enhances \( T_c \) compared to the BCS expectation \( T_c^{\text{BCS}} \propto e^{-\frac{1}{a}} \), and the actual transition temperature is \( T_c \approx \omega_0 e^{-\frac{1}{\sqrt{a}}} \), where \( \omega_0 = E_F/g \) is a characteristic energy scale of the bosons. This result agrees with previous calculations\textsuperscript{32,37,38}.

At strong coupling, we find that pairing emerges at a temperature \( T_{\text{pair}} \approx 0.06E_F \), which is only numerically smaller than the Fermi energy. In addition, we find that the phase stiffness behaves as \( \rho_s \approx T_{\text{pair}}/\sqrt{g} \), i.e., the typical energy scale where phase fluctuation become important is parametrically smaller than \( T_{\text{pair}} \). The small phase stiffness implies that coherent superconductivity only emerges at \( T_c \approx T_{\text{pair}}/\sqrt{g} \ll T_{\text{pair}} \). In between \( T_c \) and \( T_{\text{pair}} \), the system displays pseudogap behavior with preformed pairs. The width of the pseudogap regime widens as \( g \) grows.

We further argued that several aspects of the strong coupling solution, particularly the existence of two temperatures \( T_{\text{pair}} \) and \( T_c \), are valid for a much larger class of problems in which boson propagator becomes completely local above a certain frequency. The existence of a cut off energy scale, which corresponds to the finite lattice...
constant in the condensed matter context or to the large density in case of color superconductivity, is crucial for the relevance of phase fluctuations. For the cuprates, our results imply that $T_{\text{pair}}$ and the pairing gap $\Delta$ scale with the hopping integral $t$, while $T_c$ scales with the exchange interaction $J$.

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APPENDIX A: PAIRING WITH $|\omega| \sim |\omega'|$ KERNEL

In this appendix we obtain the solution of the Eliashberg equation, Eq. (32), in the strong coupling regime. The analysis of a pairing problem with a kernel $1/|\omega - \omega'|$ is nontrivial and it is useful to solve a more general problem first\textsuperscript{60}. We consider:

$$h_\gamma (\omega) = \frac{1 - \gamma}{2} \int_0^\infty \frac{d\omega' h_\gamma (\omega') k_\gamma (\omega, \omega')}{(\omega')^{1-\gamma}} \tag{A1}$$

with

$$k_\gamma (\omega, \omega') = \frac{1}{|\omega - \omega'|^\gamma} + \frac{1}{|\omega + \omega'|^\gamma} \tag{A2}$$

and argue below that $\Phi (\omega) \sim \lim_{\gamma \to -1} h_\gamma (\omega)$.

The integral equation (A1) is scale invariant suggesting a power-law solution

$$h_\gamma (\omega) = A \omega^{-b}. \tag{A3}$$

Inserting this ansatz into Eq. (A1) we obtain

$$1 = \frac{1 - \gamma}{2} \int_0^\infty dt \frac{1}{t^{1+b-\gamma}} \left( \frac{1}{|t - 1|^\gamma} + \frac{1}{|t + 1|^\gamma} \right), \tag{A4}$$

where $t = \omega' / \omega$. This determines the exponent $b(\gamma)$. The integral over $t$ can be performed explicitly. In the limit where $\gamma$ is close to 1, Eq. (A4) reduces to

$$1 = 1 + (1 - \gamma) y(b), \tag{A5}$$

where $y(b) = \gamma E + \psi(b) - \frac{\pi}{2} \tan \left( \frac{\pi b}{2} \right)$, and $\psi(b)$ is the di-gamma function. While $b$ is undetermined for $\gamma = 1$ it must hold that $y(b) = 0$ for any $\gamma \neq 1$. For real $b$ the condition $y(b) = 0$ cannot be fulfilled. However, for a complex $b = \alpha + i \beta$, we find that the imaginary part of $y(b)$ vanishes if $\alpha = \frac{1}{2}$, i.e. if $y\left( \frac{1}{2} + i \beta \right)$ is purely real. Using this fact and substituting $b = 1/2 + i \beta$ into (A5), we obtain that $\beta$ is determined from

$$\text{Re}\left( \gamma E + \psi\left( \frac{1}{2} + i \beta \right) \right) = \frac{\pi}{2} \tan\left( \frac{\pi}{4} + i \frac{\pi \beta}{2} \right). \tag{A6}$$

This equation is easily solved graphically and yields $\beta = \pm 0.7923(2)$. Therefore

$$h_\gamma (\omega) = A \omega^{-\frac{1}{2} - i \beta} + A^* \omega^{-\frac{1}{2} + i \beta}. \tag{A7}$$

The overall constant $A$ is chosen such that $h_\gamma (\omega)$ is real.

In order to show that Eq. (A7) is indeed the solution of Eq. (32) we discuss more carefully why $\lim_{\gamma \to -1} h_\gamma (\omega)$ gives the desired $\Phi (\omega)$. Eq. (A2) can be re-expressed as

$$\Phi (\omega) = \lim_{\gamma \to -1} \frac{1 - \gamma}{2} \int_T^{E_F} \frac{d\omega' \Phi (\omega') k_\gamma (\omega, \omega')}{\omega'^{1-\gamma} - \omega_0^{1-\gamma}}. \tag{A8}$$

This equation coincides with (A1) if we neglect $\omega_1^{1-\gamma}$ in the denominator. We now recall that at strong coupling, $T_{\text{pair}} \gg \omega_0$. Then for all $\omega' > T$ in (A8) holds that $\omega' \gg \omega_0$, and we can safely neglect $\omega_0^{1-\gamma}$ for any $\gamma \neq 1$.

APPENDIX B: LINEARIZED GAP EQUATION FOR DISCRETE MATSUBARA FREQUENCIES

In the computation of $T_{\text{pair}}$ in the main text we imposed the lower cutoff in the zero-temperature equation for the pairing vertex. In the weak coupling limit, this procedure was shown earlier\textsuperscript{38} to yield the same $T_{\text{pair}}$ (modulo a numerical prefactor), as one would obtain by performing an explicit summation over discrete Matsubara frequencies. In this appendix we demonstrate that the same is true in the strong coupling limit.

The most straightforward way to analyze the linearized pairing problem is by considering the gap function

$$\Delta_n = \frac{\Phi (\omega_n) \Sigma (\omega_n)}{Z (\omega_n)}, \tag{B1}$$

where $Z (\omega_n) = 1 - \Sigma (\omega_n)$, and determine the temperature at which $\Delta_n \neq 0$ for the first time. The advantage of analyzing $\Delta$ instead of $\Phi$ is that the corresponding linearized gap equation does not explicitly contain the fermionic self-energy and can more easily be solved numerically. The equation for $\Delta_n$ is straightforwardly obtained from the equations for $\Phi (\omega_n)$ and $\Sigma (\omega_n)$:

$$\Delta_n = 3 \pi g T \sum_m \frac{\Delta_m - \Delta_n}{\omega_m - \omega_n} \times \text{sign} (\omega_m) d (\omega_m - \omega_n). \tag{B2}$$
In the limit where $T_{\text{pair}} > \omega_0$, relevant $|\omega_n| > \omega_0$, we can approximate $d(\Omega_n)$ by $d(\Omega_n) \approx \frac{\omega_n}{\pi T_{\text{pair}}}$, then

$$\Delta_n = \frac{E_F}{4\pi T_{\text{pair}}} \sum_{m \neq n} \left( \frac{\Delta_m}{m + 1/2} - \frac{\Delta_n}{n + 1/2} \right) \times \frac{\text{sign}(m + 1/2)}{|m - n|}. \tag{B3}$$

The summation over discrete Matsubara frequencies is convergent for large $n$. Thus, no regularization or cut off is needed to solve for $\Delta_m$. The only dimensionless parameter in Eq. (B3) is $E_F/T_{\text{pair}}$, and the non-trivial solution of Eq. (B3) does indeed exist at $T_{\text{pair}} \approx 0.064E_F$. This is even quantitatively close to the estimate $T_{\text{pair}} \approx 0.067E_F$ obtained in the main text.

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In the exciton scenario, the resonance peak frequency is $\Omega_{res} \sim (\Delta \omega_{sf})^{1/2}$ where $\Delta$ is a gap, and $\omega_{sf}$ is the spin-fluctuation frequency. At weak coupling, $\Delta \sim \bar{g}$, and $\omega_{sf} \sim v_F^2 \xi^{-2}/\bar{g}$, where $v_F \sim t$ is Fermi velocity, $\bar{g} \sim u^2/t$, and $\xi$ is a magnetic correlation length. Accordingly, $\Omega_{res} \sim v_F \xi^{-1}$.

At strong coupling, $\Delta \sim t$, $\omega_{sf} \sim t^3/u^2$, and $\Omega_{res} \sim J \xi^{-1}$.

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