Generalized Kähler Geometry and current algebras in $SU(2) \times U(1)$ $N=2$ superconformal WZW model.

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Abstract

We examine the Generalized Kähler Geometry of quantum $N=2$ superconformal WZW model on $SU(2) \times U(1)$ and relate the right-moving and left-moving Kac-Moody superalgebra currents to the Generalized Kähler Geometry data of the group manifold using Hamiltonian formalism.

1. It is well-known that conformal supersymmetric $\sigma$-models with extended supersymmetry are important in the construction of realistic models of superstring compactification from 10 to 4 dimensions [1]. The $\sigma$-model on the 6-dimensional Calabi-Yau manifold is one of the important examples of the compactification. The Calabi-Yau manifold being a complex Kähler manifold is not accidental but caused by a close relation between the extended supersymmetry and Kähler geometry. Apart from the metric in a more general case the background geometry may include an antisymmetric $B$-field. In that case the corresponding 2-dimensional supersymmetric $\sigma$-model have a second supersymmetry when the target-space has a bi-Hermitian geometry, known also as Gates-Hull-Roček geometry [2]. In this situation the target manifold contains two complex structures with a Hermitian metric with respect to each of the complex structures. Quite recently it has been shown in [3] that these set of geometric objects, metric, antisymmetric $B$-field and two complex structures antisymmetric with respect to the metric have a unified description in the context of Generalized Kähler Geometry (GKG).

The $N=2$ supersymmetric WZW models on the compact groups [4], [5], [6] provide a large class of examples of exactly solvable quantum conformal $\sigma$-models whose targets supports simultaneously GKG geometry causing the extended $(2,2)$-supersymmetry and affine Kac-Moody superalgebra structure ensuring the exact solution [8], [9]. The GKG nature of the $(2,2)$-supersymmetry in these models has been studied in the series of works [10], [11], [12], [13], [14]. The relation between Generalized Complex geometry and Kac-Moody superalgebra symmetry has already been discussed in papers [18], [19]. In this paper I undertake more thorough study of this relation using WZW model on the group $SU(2) \times U(1)$ as an example. In a more general context it would be also important to see if there are GKG targets which allow the $W$-superalgabras conserved currents. Perhaps Kazama-Suzuki coset models [20]-[22] can be related to such targets according to the results presented in [23].

The approach I follow is close to the one used in papers [15], [16], [17] where the Hamiltonian formalism of $N=1$ supersymmetric $\sigma$-model has been applied to show a relation between the extended supersymmetry in phase-space and generalized complex structure of the target space. I intend to show that Hamiltonian approach works as well to relate GK Geometry and Kac-Moody superalgebra structure for the case of $N=2$ superconformal WZW model on a compact group manifold. It is because the right-moving and left-moving Kac-Moody superalgebra currents are
given by the right and left Poisson-Lie group actions coming from GK Geometry of the group manifold. It also allows us to give Poisson-Lie interpretation of Wakimoto formulas.

2. It is convenient to start examination with the supersymmetric WZW model on an arbitrary compact group $G$ using superfield formalism. The super world-sheet of the model is parametrized by the light-cone even coordinates $\sigma^\pm$, and odd coordinates $\theta^\pm$. The super-derivatives are given by

$$D^\pm = \frac{\partial}{\partial \theta^\pm} + \theta^\pm \frac{\partial}{\partial \sigma^\pm}, \quad \partial^\pm = \frac{\partial}{\partial \sigma^\pm} \pm \frac{\partial}{\partial \sigma^\op \pm} \quad (1)$$

The superfield $g(\sigma^\pm, \theta^\pm)$ takes values in the compact group $G$. The action of the model [24] is given by

$$S = -k \int d^2 \sigma d^2 \theta (g^{-1}D_+ g, g^{-1}D_- g) - \int d^2 \sigma d^2 \theta dt (g^{-1}\partial g / \partial t, \{g^{-1}D_- g, g^{-1}D_+ g\}) \quad (2)$$

The classical equations of motion are nothing else but the conservation low equations:

$$D_-(g^{-1}D_+ g) = D_+(D_- g g^{-1}) = 0. \quad (3)$$

3. The author of papers [5],[6] established a correspondence between extended supersymmetric WZNW models and finite-dimensional Manin triples. By the definition [25], a Manin triple $(g, g_+, g_-)$ consists of a Lie algebra $g$, with nondegenerate invariant inner product $\langle, \rangle$ and isotropic Lie subalgebras $g^\pm$ such that $g = g_+ \oplus g_-$ as a vector space. There is a one to one correspondence [5] between a complex Manin triple endowed with antilinear involution which conjugates isotropic subalgebras $\tau : g^\pm \to g^\mp$ and a complex structure on a real Lie algebra of the compact group that ensures connection between Manin triple construction of $N=2$ Virasoro superalgebra currents of [5],[6] and approach [4] based on the complex structure on the Lie algebra. Thus, for an arbitrary compact Lie group $G$ with the complex structure $J$ the complexification $g^C$ of the Lie algebra $g$ of the group has the Manin triple structure $(g^C_+, g^C_-, g^C_-)$ where the isotropic subalgebras are the $\pm i$-eigenspaces of $J$. The Lie group version of this triple is the double Lie group $(G^C, G^+_+, G^-_-)$ [26], [27]. The real Lie group $G$ is extracted from its complexification by hermitian conjugation $\tau$:

$$G = \{g \in G^C | \tau(g) = g^{-1} \} \quad (4)$$

Because of $G^C = G_+ G_-$ each element $g \in G^C$ admits two decompositions

$$g = g_+ g_- \quad (5)$$

where $\tilde{g}_\pm$ are dressing transformed elements of $g_\pm$ [27]. Taking into account (4) and (5) we conclude that the element $g$ from $G^C$ belongs to $G$ iff

$$\tau(g_\pm) = \tilde{g}_\mp^{-1} \quad (6)$$

These equations mean that there is an left and right dressing actions of the complex groups $G_\pm$ on $G$ [26] so the elements of $G$ can be parametrized by the elements from the complex group $G_+$ (or $G_-$).
More generally, one has to consider the set $W$ of classes $G_+\backslash G^C \rightarrow G_-$ and pick up a representative $w$ for each class $[w] \in W$. It gives the stratifications (Bruhat decomposition) of $G^C$ [28]:

$$G^C = \bigcup_{[w] \in W} G_+wG_- = \bigcup_{[w] \in W} G_-wG_+ \quad (7)$$

Thus, in a general case the compact group $G$ can be represented as a set of dressing action orbits endowed with natural complex coordinates [29]. In this note we will not look into this complication.

4. The simplest nonabelian compact group with complex structure is $SU(2) \times U(1)$. Its complexification is $GL(2, \mathbb{C})$. Let us fix the following basis in the Lie algebra $gl(2, \mathbb{C})$

$$e^0 = \frac{1}{2} \begin{pmatrix} 1 + i & 0 \\ 0 & -1 + i \end{pmatrix}, \quad e^1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$e_0 = \frac{1}{2} \begin{pmatrix} 1 - i & 0 \\ 0 & -1 - i \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad (8)$$

The isotropic subalgebras forming the corresponding Manin triple $(gl(2, \mathbb{C}), g_+, g_-)$ are given by

$$g_+ = \mathbb{C}e_0 \oplus \mathbb{C}e_1, \quad g_- = \mathbb{C}e^0 \oplus \mathbb{C}e^1 \quad (9)$$

where the invariant scalar product $\langle , \rangle$ is given by trace so that $\langle e^a, e_b \rangle \equiv Tr(e^a e_b) = \delta^a_b$. The first decomposition from (5) takes the following form

$$g_+ = \begin{pmatrix} \exp(\frac{i}{2}(1-i)x^0) & 0 \\ \exp(\frac{i}{2}(1-i)x^0)x^1 & \exp(-\frac{1}{2}(i+1)x^0) \end{pmatrix},$$

$$g_- = \begin{pmatrix} \exp(\frac{i}{2}(i+1)y^0) & \exp(\frac{i}{2}(i-1)y^0)y^1 \\ 0 & \exp(\frac{i}{2}(i-1)y^0) \end{pmatrix} \quad (10)$$

The solution of (6) for the group $SU(2) \times U(1)$ is given by

$$y^0 = x^0 + \ln(1 + |x^1|^2), \quad y^1 = \exp(-x^0 + \bar{x}^0)\bar{x}^1$$

$$g_+g_- = (1 + |x^1|^2)^{-\frac{i}{2}}a^{-1-i\frac{1}{2}}a^{-1-i} \begin{pmatrix} 1 & -a^{-2}a^2\bar{x}^1 \\ x^1 & a^{-2}a^2 \end{pmatrix} \in SU(2) \times U(1) \quad (11)$$

where $a = \exp(\frac{i}{2}x^0)$. Hence, the $x^0, x^1$ are the holomorphic coordinates on $SU(2) \times U(1)$ determined by the left dressing action of $G_+$. One can show [29] that these coordinates are holomorphic with respect to the complex structure $J_\rho$ generated by the right translations from the complex structure of the Manin triple $(gl(2, \mathbb{C}), g_+, g_-)$.

We employ these coordinates to rewrite the WZW action (2) on $SU(2) \times U(1)$ in the form of the supersymmetric $\sigma$-model action

$$S = \frac{1}{2} \int d^2\sigma d^2\theta E_{ij} \rho_+^i \rho_-^j \quad (12)$$

Here, the matrix $E_{ij}$ depends on the superfields $X^i$, $\bar{X}^i$ corresponding to the coordinates $x^i$, $\bar{x}^i$. It is related [29] to the Semenov-Tian-Shansky simplectic form $\Omega_{ij}$:

$$E_{ij} = i\Omega_{i(k}(J_\rho)^k_{j)},$$

$$\Omega = \frac{k}{2}(1 + |x^1|^2)^{-1}(x^1 \rho^0 \wedge \rho^1 + \rho^0 \wedge \rho^0 + \rho^1 \wedge \rho^1 - x^1 \rho^0 \wedge \rho^1) \quad (13)$$
The superfields $\rho^i_\pm = (\rho^a_\pm, \bar{\rho}^a_\pm)$ are the world-sheet restrictions of the holomorphic $\rho^a = Tr(e^a dg_+ g^{-1}_+)$ and anti-holomorphic $\bar{\rho}^a = Tr(e^a dg_+ g^{-1}_+)$ 1-forms w.r.t $J_r$. They are given by

$$
\begin{align*}
\rho^0_\pm &= D \pm X^0, \\
\rho^1_\pm &= D \pm X^1 + X^1 D \pm X^0, \\
\bar{\rho}^0_\pm &= D \pm \bar{X}^0, \\
\bar{\rho}^1_\pm &= D \pm \bar{X}^1 + \bar{X}^1 D \pm \bar{X}^0
\end{align*}
$$

(14)

The target space metric $g_{ij}$ and $B$-field $B_{ij}$ can be read off from $E_{ij}$:

$$
\begin{align*}
 g_{ij} &= \frac{1}{2}(E_{ij} + E_{ji}) \\
 B_{ij} &= -\frac{1}{2}(E_{ij} - E_{ji})
\end{align*}
$$

(15)

The relations (13), (15) are nothing else but the Theorem from [12] describing GKG data in terms of gerbes.

5. Now we find the canonically conjugated momentum and super-Hamiltonian following the analysis of [16]. Let us define new set of odd world-sheet coordinates and derivatives

$$
\begin{align*}
\theta^+ &= \frac{1}{\sqrt{2}}(\theta^0 + \theta^1), \\
\theta^- &= \frac{1}{\sqrt{2}}(\theta^1 - \theta^0), \\
D_0 &= \frac{\partial}{\partial \theta^0} + \theta^0 \partial_1 + \theta^1 \partial_0, \\
D_1 &= \frac{\partial}{\partial \theta^1} + \theta^1 \partial_1 + \theta^0 \partial_0
\end{align*}
$$

(16)

so that $D_0^2 = D_1^2 = \partial_1, D_0 D_1 + D_1 D_0 = 2 \partial_0$.

Integrating out $\theta^\alpha$-variable the action (12) becomes

$$
S = \int d^2 \sigma d\theta^1 (\partial_0 X^\mu X_\mu^* - \frac{1}{2} g_{\mu\nu} \partial_1 X^\mu D_1 X^\nu \\
- \frac{1}{2} (X_\mu^* - B_{\mu\lambda} D_1 X^\lambda) g^{\mu\nu} D_1 (X^\nu_\tau - B_{\nu\eta} D_1 X^\eta) \\
+ \frac{1}{2} D_1 X^\mu \Gamma^\tau_{\mu\nu}(X^\nu_\lambda - B_{\nu\eta} D_1 X^\eta) g^{\lambda\eta}(X^\tau_\eta - B_{\tau\omega} D_1 X^\omega) \\
- \frac{1}{2} H^{\mu\nu\lambda}(X^\mu_\eta - B_{\mu\eta} D_1 X^\eta)(X^\nu_\tau - B_{\nu\tau} D_1 X^\tau)(X^\lambda_\omega - B_{\lambda\omega} D_1 X^\omega))
$$

(17)

where

$$
X^*_\mu = g_{\mu\nu} D_0 X^\nu + B_{\mu\nu} D_1 X^\nu
$$

(18)

is canonically conjugated momentum to the field $X^\mu$, $\Gamma^\tau_{\mu\nu}$ are the Christoffel symbols of the Levi-Civita connection for the metric $g_{\mu\nu}$:

$$
g_{\mu\eta} \Gamma^\eta_{\nu\lambda} = \frac{1}{2}(g_{\mu\nu,\lambda} + g_{\lambda\mu,\nu} - g_{\nu\lambda,\mu})
$$

(19)

and

$$
H^{\mu\nu\lambda} = \frac{1}{2}(B_{\mu\nu,\lambda} + B_{\lambda\mu,\nu} + B_{\nu\lambda,\mu})
$$

(20)

determines WZW 3-form. From (17) we can also read off the density of the super-Hamiltonian which defines an evolution along the variables $(\sigma_0, \theta^0)$.
Introducing the Semenov-Tian-Shansky Poisson bivector \[2, 6\] is nothing else but the direct sum of Wakimoto realizations \[31\]. The expression (27) gives Poisson-Lie interpretation for the left-moving current algebra in \(\rho\) as the singular part of the following OPE

\[ X^*_\rho(Z_1)X^\nu(Z_2) = \frac{1}{12} \delta^\nu_\rho h + \text{reg.} \]  

(21)

where the new supercoordinate \(Z = (z, \Theta)\), \(z = \exp(\im \sigma_1)\), \(\Theta = (\im z) \frac{\im}{\im \theta} \Theta_1 \Theta_2\).

In what follows we rescale the canonical variables by \(X^*_\rho(Z) \rightarrow \frac{1}{k} X^*_\rho(Z)\), \(X^\nu(Z) \rightarrow X^\nu(Z)\) to get the standard OPE

\[ X^*_\rho(Z_1)X^\nu(Z_2) = \frac{1}{12} \delta^\nu_\rho h + \text{reg.} \]  

(22)

6. The main feature of the superconformal WZW model is that it has two copies of Kac-Moody superalgebra symmetries generated by the left-moving \(D_+ gg^{-1}\) and right-moving \(g^{-1} D_- g\) currents. These two algebras determine the dynamics and \(N = 2\) supersymmetry of the model completely due to the generalized Sugawara construction \([4], [5], [6]\) (see also \([7]\)). Therefore it is important to get these currents in the Hamiltonian approach and then recover the Hamiltonian density as well as the total \(N = (2, 2)\) Virasoro superalgebra.

The classical conserved left-moving currents are given by

\[ L^a \equiv Tr(e^a(D_0 + D_1)gg^{-1}), \quad L_a \equiv Tr(e_a(D_0 + D_1)gg^{-1}), \ a = 0, 1 \]  

(23)

In what follows we will use the common notation \(L_i\) for \((L^a, L_a)\). Using (11) one finds that

\[ L_i = \Omega_{ij}(g^j k^i(\rho^*_k - B_{kn}\rho^n) + \rho^i) = \Omega_{ij} g^{j k}(\rho^*_k + E_{k n}\rho^n) \]  

(24)

here \(\rho^i = (Tr(e^a D_1 g_+ g_+), Tr(e_a D_1 g_+ g_+))\) are the super-circle restrictions of the corresponding 1-forms and \(\rho^*_k\) are the conjugated to \(\rho^i\) superfields:

\[
\begin{align*}
\rho_0^* &= X^0_0 - X^1 X^*_1, \quad \rho_1^* = X^*_1 \\
\rho_0^* &= X^0_0 - \bar{X}^1 X^*_1, \quad \rho_1^* = \bar{X}^*_1
\end{align*}
\]

(25)

Introducing the Semenov-Tian-Shansky Poisson bivector [26]

\[ P \equiv \Omega^{-1} = \frac{2}{k} (X^1 \rho_0^* \rho_1 + \rho_0^* \rho_0 + \rho_1^* \rho_1 - \bar{X}^1 \rho_0^* \rho_1) \]  

(26)

and lifting up the indeces in (24) one gets

\[ L^i = -\frac{k}{2} (P^i(\rho^i) + \im J_\rho^i) \]  

(27)

Notice that formula close to (27) has been obtained in the context of topological \(\sigma\)-models [30]. The expression (27) gives Poisson-Lie interpretation for the left-moving current algebra in \(N = 2\) superconformal WZW model. Rewriting (27) by the fields \(X^i(Z), X^*_i(Z)\) we find that it is nothing else but the direct sum of Wakimoto realizations [31]

\[
\begin{align*}
L^1 &= X^1_1 - \bar{X}^1 X^*_0 + (\bar{X}^1)^2 X^*_1 - \frac{k}{2} D\bar{X}^1 - \frac{k}{2} \bar{X}^1 D\bar{X}^0, \\
L^0 &= -X^1 X^*_1 + \bar{X}^*_0 + \bar{X}^1 \bar{X}^*_1 - \frac{k}{2} D\bar{X}^0 \\
L_0 &= -X^1 X^*_1 - \bar{X}^*_0 + \bar{X}^1 \bar{X}^*_1 + \frac{k}{2} DX^0 \\
L_1 &= -(X^1)^2 X^*_1 + X^1 X^*_0 - \bar{X}^*_1 + \frac{k}{2} DX^1 + \frac{k}{2} X^1 DX^0
\end{align*}
\]  

(28)
The quantum versions of (27), (28) differ from the classical ones by the normal ordering and reproduce the correct OPE’s due to (22):

\[ L^0(Z_1)L_0(Z_2) = Z_{12}^{-\frac{1}{2}}k + \text{reg.}, \]
\[ L^1(Z_1)L_1(Z_2) = Z_{12}^{-\frac{1}{2}}k + Z_{12}^{-\frac{1}{2}}(L^0 + L_0)(Z_2) + \text{reg.}, \]
\[ L^0(Z_1)L_1(Z_2) = -Z_{12}^{-\frac{3}{2}}L_1(Z_2) + \text{reg.}, \]
\[ L_0(Z_1)L_1(Z_2) = Z_{12}^{-\frac{3}{2}}L^1(Z_2) + \text{reg.}, \]
\[ L_0(Z_1)L_1(Z_2) = -Z_{12}^{-\frac{1}{2}}L_1(Z_2) + \text{reg.}. \]

(29)

Recovering the Plank constant for the currents (28) one sees that the level \( k \) is scaled as \( \frac{1}{\hbar} \).

The classical conserved right-moving currents are given by

\[ R^a \equiv Tr(e^a g^{-1}(D_0 + D_1)g), \quad R_a \equiv Tr(e_a g^{-1}(D_0 + D_1)g), a = 0, 1 \]

Introducing the common notation \( R_i \) for the currents \( (R^a, R_a) \) one can obtain

\[ R^i = -\frac{k}{2}(P(p^i) - iJ_i p^i) \]

(31)

where \( p^i = (Tr(e^a D_1 \tilde{g}_+ \tilde{g}_+), Tr(e_a D_1 \tilde{g}_+ \tilde{g}_+)) \) are the super-circule restrictions of holomorphic and anti-holomorphic 1-forms w.r.t. the complex structure \( J_i \) which is generated by the left translations on the group \( SU(2) \times U(1) \). The group element \( \tilde{g}_+ \) is determined by the second decomposition from (5) and parametrized by the dressing transformed coordinates \( \tilde{x}^0, \tilde{x}^1 \) which are related to the \( x^0, x^1 \) by

\[ \tilde{x}^0 = -x^0 - \ln(1 + |x^1|^2), \quad \tilde{x}^1 = -\exp(x^0 - \tilde{x}^0)x^1 \]

(32)

We also have

\[ \tilde{p}^0 = D_1 \tilde{X}^0, \quad \tilde{p}^1 = D_1 \tilde{X}^1 + \tilde{X}^1 D_1 \tilde{X}^0, \quad \tilde{p}^a = \tilde{p}^a \]

(33)

where \( \tilde{X}^i \) are the superfields corresponding to the coordinates \( \tilde{x}^i \). The expression (31) implies that the canconically conjugated superfields \( \tilde{X}_i^* \) (as well as the fields \( \tilde{p}_i^* \) ) have been introduced so that their commutation relations coincide with (22) after the quantization. Then, one can see that quantum versions of the currents (31) written in terms of these new canonical variables \( \tilde{X}^i(Z), \tilde{X}_i^*(Z) \) nearly match with the Wakimoto formulas (28). The only difference is that the level of the right-moving algebra is equal \(-k\) and this is the only case when the right-moving currents commute to the left ones [32].

To summarize, the expressions (27), (31) determine the left-moving and right-moving current superalgebras in terms of GKG data of the group manifold. It is natural to expect that they are valid for an arbitrary \( N = 2 \) supersymmetric WZW model.

7. Now we reproduce the Sugawara construction for the left-moving and right-moving \( N = 2 \) Virasoro superalgebra currents from GKG data: \( \sigma \)-model metric \( g_{ij} \), B-field \( B_{ij} \) and complex structures \( J_i, J_r \).

Let us define two generalized complex structures [16], [14] acting in the direct sum of tangent and cotangent bundles on the group:

\[
J_1 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \begin{pmatrix} J_l + J_r & -\omega^{-1} - \omega^{-1} \\ \omega_l - \omega_r & -J_l^T - J_r^T \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix}
\]
\[
J_2 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \begin{pmatrix} J_l - J_r & -\omega^{-1} + \omega^{-1} \\ \omega_l + \omega_r & -J_l^T + J_r^T \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix}
\]

(34)
where \( \omega_{t,r} = g J_{t,r}, \omega_{t,r}^{-1} = -J_{t,r} g^{-1} \). According to [16], [14] the left-moving \( K_l(Z) \) and right-moving \( K_r(Z) \) \( U(1) \) supercurrents of the \( N = 2 \) Virasoro superalgebra are given by

\[
K_l = i < (DX, X^*)_l, (J_1 + J_2)(DX, X^*)_r >= \quad \omega_l^{-1} ij(X^*_l + (g - B)_{ik} DX^k)(X^*_r + (g - B)_{jk} DX^k)
\]

\[
K_r = i < (DX, X^*)_l, (J_1 - J_2)(DX, X^*)_r >= -i \omega_r^{-1} ij(X^*_l - (g + B)_{ik} DX^k)(X^*_r - (g + B)_{jk} DX^k)
\]

Using (13), (26) as well as the fact that \( \rho \)

Using (13), (26) as well as the fact that \( \rho, \bar{\rho} \), \( a, 0, 1 \) are holomorphic 1-forms w.r.t. \( J_r, J_l \) respectively one can deduce

\[
K_l = \frac{i}{2k} \omega_l(L, L), \quad K_r = -\frac{i}{2k} \omega_r(R, R)
\]

where we introduced Lie algebra valued currents \( L = L_a e^a + L^a e_a \) and \( R = R_a e^a + R^a e_a \).

Quantization of these expressions may lead to quantum corrections, but the result is known due to [5], [6], (see also [7]):

\[
K_l(Z) = \frac{1}{k}(L^a L_a + f_a DL^a - f^a DL_a), \quad K_r(Z) = -\frac{1}{k}(R^a R_a + f_a DR^a - f^a DR_a)
\]

where the normal ordering is implied, \( f_a = f_{ab}^b, f^a = f^{ab}_b \) and \( f_{ab}^c, f^{ab}_c \) are the structure constants of the isotropic subalgebras \( g_{\pm} \). The geometric meaning of the quantum corrections was found in [18], [19].

One can check the commutation relations for the modes of the currents (37) and find remaining generators of \( N = 2 \) Virasoro superalgebra using the realization [8]. We will begin looking into this realization for the currents \( L^a(Z), L_a(Z) \):

\[
L^a = \sqrt{k} \psi^a + \Theta(j^a + f_{ac} \psi^c \psi_b - \frac{1}{2} f^{bc}_a \psi^b \psi^c),
\]

\[
L_a = \sqrt{k} \psi_a + \Theta(j_a + f_{ab} \psi_b \psi_b - \frac{1}{2} f^{bc}_a \psi_b \psi^c)
\]

where \( \psi^a(z), \psi_a(z) \) are the free fermionic currents and \( j^a(z), j_a(z) \) are the bosonic currents determined by the OPE:

\[
\psi^a(z_1)\psi_b(z_2) = z_1^{-1} \delta^a_b + ...
\]

\[
j^a(z)j^b(w) = -(z-w)^{-2}\frac{1}{2}(e^a, e^b) + (z-w)^{-1} f_{bc}^e j^c(w) + \text{reg},
\]

\[
j_a(z)j_c(w) = -(z-w)^{-2}\frac{1}{2}(e_a, e_b) + (z-w)^{-1} f_{bc}^a j_c(w) + \text{reg},
\]

\[
j^a(z)j_b(w) = (z-w)^{-2}\frac{1}{2}(2k \delta^a_b - (e^a, e_b)) + (z-w)^{-1} (f_{bc}^a j^c - f_{bc}^a j_c)(w) + \text{reg},
\]

where \( (,)_l \) is a Killing form. Taking into account (39) the currents (38) satisfy (29) thus it is easy to deduce

\[
K_l = I(z) + \Theta \sqrt{\frac{k}{2}} (G^+ - G^-)(z),
\]

\[
I(z) = \psi^a \psi_a + \frac{1}{k} f_a (j^a + f_{ac} \psi^c \psi_b) - \frac{1}{k} f^{ac}_a (j_a + f_{bc} \psi^b \psi^c),
\]

\[
G^+ = \sqrt{\frac{2}{k}} (j^a \psi_a - \frac{1}{2} f^{ad}_c \psi^a \psi^d \psi^c),
\]

\[
G^- = \sqrt{\frac{2}{k}} (j_a \psi^a - \frac{1}{2} f^{ad}_c \psi^a \psi^d \psi^c)
\]
We see that even part \( I(z) \) of \( K_l \) neither more nor less than the \( U(1) \)-current of \( N = 2 \) Virasoro superalgebra while the odd contribution is given by spin-\( \frac{3}{2} \) currents \( G^\pm \) of the superalgebra. It corresponds with the expressions found in [5], [6], [7].

The right-moving currents of \( N = 2 \) Virasoro superalgebra can be obtained analogously. Thus we recovered all currents of the \( N = (2,2) \) Virasoro superalgebra and hence, the density of the quantum super-Hamiltonian from the GK Geometry of \( SU(2) \times U(1) \) group manifold. We also showed that left-moving and right-moving Kac-Moody superalgebra symmetries in quantum \( N = 2 \) superconformal WZW model are determined by left and right Poisson-Lie group actions on the group manifold and lead to the Wakimoto-type formulas. Taking into account that our analysis has local nature it would be important to consider global construction of the Kac-Moody superalgebra currents in terms of GKG data. Perhaps the most appropriate way to do that is to use gerbe description of GK Geometry found in [12].

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