Applications of the duality method to generalizations of the Jordan canonical form

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We show how Ptak’s duality method leads to short proofs of two extensions of the Jordan canonical form, viz. the normal form for a matrix over an arbitrary (not necessarily algebraically closed) field under similarity and the canonical form for a pair of matrices under contragredient equivalence.

The duality method is summarized in the following.

Lemma. Let $V$ be a finite-dimensional space over a field $F$, let $A : V \rightarrow V$ be a linear map, and $S \subset V$ be an $A$-invariant subspace of $V$. If $T \subset V^*$ is an $A^*$-invariant subspace of the dual $V^*$ of $V$ such that

\begin{align*}
s \in S, & \quad \langle s, t \rangle = 0 \quad \forall t \in T \implies s = 0, \\
t \in T, & \quad \langle s, t \rangle = 0 \quad \forall s \in S \implies t = 0,
\end{align*}

then $V = S + \text{ann}(T)$ is an $A$-invariant direct sum decomposition of $V$, with $\text{ann}(T) = \{ v \in V : \langle v, t \rangle = 0 \quad \forall t \in T \}$ the annihilator of $T$.

We give a proof for the sake of completeness.

Proof. The condition (1) implies that the sum $S + \text{ann}(T)$ is direct. If $\dim T \geq \dim S$ and \{t_j\}_{j=1}^{\dim T} (\{s_j\}_{j=1}^{\dim S}) is a basis of $T$ ($S$), then the matrix $G = (\langle s_i, t_j \rangle : i = 1, \ldots, \dim S, \ j = 1, \ldots, \dim T)$ has fewer rows than columns, hence the equation $Gx = 0$ has a nontrivial solution, so (2) fails. In other words, (2) implies that $\dim T \leq \dim S$, hence $\dim \text{ann}(T) \geq \dim V - \dim S$. Thus, $V = S + \text{ann}(T)$. Since $T$ is $A^*$-invariant, $\text{ann}(T)$ is $A$-invariant, which completes the proof.

\[\Box\]

1 The analogue of the Jordan form for an arbitrary field

Theorem 1. Let $V$ be a finite-dimensional linear space over a field $F$ and let $A : V \rightarrow V$ be a linear map. There exists a basis of $V$ such that the representation of $A$ with respect to that basis has the form

\[
\text{diag}(A_1, \ldots, A_p),
\]

where

\[\text{diag}(A_1, \ldots, A_p),\]
This form is unique up to reordering of the blocks $A_1, \ldots, A_p$.

**Proof.** Since the space of all linear maps on $V$ is finite-dimensional, there exists $k \in \mathbb{N}$ such that $A^k \in \text{span}\{I, A, \ldots, A^{k-1}\}$, so $f(A) = \{0\}$, hence some monic polynomial in $F[x]$ annihilates $A$.

Let $f \in F[x]$ be the monic polynomial of minimal degree such that $f(A) = \{0\}$ and let $f = (f_1)^{k_1} \cdots (f_r)^{k_r}$ be its decomposition into powers of distinct (monic) primes $f_i$, $i = 1, \ldots, r$. Let $g_i = \prod_{j=1,j \neq i}^{r} (f_j)^{k_j}$. Since $F[x]$ is a Euclidean domain and $\gcd(g_1, \ldots, g_r) = 1$, it follows that $g_1 h_1 + \cdots + g_r h_r = 1$ for some $h_1, \ldots, h_r \in F[x]$, hence $v = h_1(A)g_1(A)v + \cdots + h_r(A)g_r(A)v$ for any $v \in V$. But $h_i(A)g_i(A)V \subseteq V_i = \ker(f_i(A))^{k_i}$, so $V = V_1 + \cdots + V_r$.

Suppose $v \in V_i \cap \sum_{j \neq i} V_j$. As the polynomials $(f_i)^{k_i}$ and $\prod_{j \neq i} (f_j)^{k_j}$ are relatively prime, there exist $s_{i,1}, s_{i,2} \in F[x]$ such that $s_{i,1}(f_i)^{k_i} + s_{i,2} \prod_{j \neq i} (f_j)^{k_j} = 1$, hence $v = s_{i,1}(A)(f_i(A))^{k_i}v + s_{i,2}(A) \prod_{j \neq i} (f_j(A))^{k_j}v = 0$, since $V_i = \ker(f_i(A))^{k_i}$ and $\sum_{j \neq i} V_j \subseteq \ker \prod_{j \neq i} (f_j(A))^{k_j}$. So, $V = V_1 + \cdots + V_r$ is a (n $A$-invariant) direct sum decomposition of $V$. The arguments given so far are standard.

Now show how to split the subspaces $V_i$. Let $\tilde{V}$ stand for $V_1$, $\tilde{A}$ for $A|_{V_1}$, $\tilde{f}$ for $f_1$, $k$ for $k_1$, $d$ for $\deg f_1$. Since $f$ is the minimal polynomial annihilating $A$, $\tilde{f}^k$ is the minimal polynomial annihilating $\tilde{A}$, so there exists $v \in \tilde{V}$ such that $w = (\tilde{f}(\tilde{A}))^{k-1}\tilde{A}^{d-1}v \neq 0$.

We claim that $w \notin \text{span}\{(\tilde{f}(\tilde{A}))^{k-1}\tilde{A}^jv : j = 0, \ldots, d-2\}$. Indeed, if $w$ were in that span, it would imply $h(\tilde{A})(\tilde{f}(\tilde{A}))^{k-1}v = 0$ for some polynomial $h$ of degree $d-1$. But any polynomial of degree $d-1$ is coprime to $f$, so there would exist a combination of $h$ and $f$ (with coefficients from $F[x]$) equal to 1, which would yield $(\tilde{f}(\tilde{A}))^{k-1}v = 0$, contradicting $w \neq 0$. Hence the claim follows.

So, there exists $v' \in \tilde{V}^*$ such that

$$
\langle (\tilde{f}(\tilde{A}))^{k-1}\tilde{A}^jv, v' \rangle =
\begin{cases}
0 & \text{if } j = 0, \ldots, d-2 \\
\neq 0 & \text{if } j = d-1.
\end{cases}
$$

Let

$$
W_1 = \text{span}\{(\tilde{f}(\tilde{A}))^{i_1-1}\tilde{A}^{i_2-1}v : i_1 = 1, \ldots, k, \ i_2 = 1, \ldots, d\},
$$
\[ W'_i = \text{span}\{ (f(\tilde{A}^*))^{i_1-1} (\tilde{A}^*)^{i_2-1} v' : i_1 = 1, \ldots, k, \ i_2 = 1, \ldots, d \}. \]

Notice that
\[ g_{(i_1,i_2),(j_1,j_2)} = \langle (f(\tilde{A}))^{i_1-1} \tilde{A}^{d-i_2} v, (f(\tilde{A}^*))^{k-j_1} (\tilde{A}^*)^{j_2-1} v' \rangle \neq 0 \]
only if \((i_1,i_2) \preceq (j_1,j_2)\) (in lexicographic order). So, the \(kd \times kd\)-matrix \(g_{(i_1,i_2),(j_1,j_2)} : i_1,j_1 = 1, \ldots, k, i_2,j_2 = 1, \ldots, d\) is upper triangular with nonzero diagonal elements, hence, by the Lemma, \(\tilde{V} = + \text{ann}(W'_i)\) is an \(\tilde{A}\)-invariant direct sum decomposition of \(\tilde{V}\). The matrix representation of \(\tilde{A}|_{W_1}\) with respect to the basis \((f(\tilde{A}))^{i_1-1} \tilde{A}^{d-i_2} v, (f(\tilde{A}^*))^{k-j_1} (\tilde{A}^*)^{j_2-1} v' \rangle \neq 0\)

Splitting the spaces \(\text{ann}(W'_1)\), \(V_2, \ldots, V_r\) in the same way as above, we obtain a direct sum \(V = W_1 + \cdots + W_p\) of \(\tilde{A}\)-invariant indecomposable subspaces and a basis in each so that the matrix representation of \(A\) with respect to the concatenation of the bases of \(W_i\)'s has the form \([3]\).

Since the minimal polynomial \(f\) of \(A\) is unique, the (monic) prime factors \(f_i\) and the powers \(k_i\) with which they occur in \(f\) are determined uniquely. Let
\[ n^i_j = \dim \ker(f_i(A))^j = \sum_{W_i \subseteq \ker(f_i(A))^{k_i}} \min(\dim W_i, j \deg f_i), \quad i = 1, \ldots, r, \ j = 1, \ldots, k_i. \]

Then \(\Delta n^i_j = n^i_{j+1} - n^i_j\) is the number of blocks for \(f_i\) of order greater than \(j \cdot \deg f_i\), so the number of blocks of order \(j \cdot \deg f_i\) equals \(-\Delta^2 n^i_{j-1} / \deg f_i = (\Delta n^i_{j-1} - \Delta n^i_j) / \deg f_i\).

Since the numbers \(n^i_j\) are uniquely determined by the map \(A\), this completes the proof of the uniqueness of \([3]\).

**Remarks.** 1. The arguments in the two preceding paragraphs are variations of those due to de Boor \([1]\). 2. If \(F\) is algebraically closed, the polynomials \(f_i\) are of degree 1, so \([3]\) becomes the Jordan normal form of \(A\). 3. In the proof above, all the factors of the minimal polynomial are treated in the same way in contrast to the proof in \([7]\) where the canonical splitting is first given for the nilpotent part of \(A\) and then follows for all other parts by shifting \(A\) by an eigenvalue \(\lambda\) (for that completion of the proof in \([7]\), see \([1]\)). 4. Theorem 1 is classical and can be found, e.g., in \([5, \text{pp.} 92-97]\). In the sequel, we refer to a matrix in the form \([3]\) as being in the Jordan normal form for the field \(F\), and as the Jordan normal form of the operator \(A\).

## 2 The canonical form under contragredient equivalence

Two pairs of matrices, \((A, B)\) and \((C, D)\), are called contragrediently equivalent if \(A, C \in F^{m \times n}\), \(B, D \in F^{n \times m}\), and \(A = SCT^{-1}\), \(B = TDS^{-1}\) for some invertible \(S \in F^{m \times m}\), \(T \in F^{n \times n}\).

The problem of classification of pairs of matrices under contragredient equivalence can be restated as follows. Given an \(n\)-dimensional linear space \(V\) and an \(m\)-dimensional linear space \(W\) and linear maps \(A : V \rightarrow W\), \(B : W \rightarrow V\), choose bases of \(V\) and \(W\) so that the pair \((A, B)\) has a simple representation with respect to these bases.
Theorem 2. Let $V$, $W$ be finite-dimensional linear spaces over a field $F$ and let $A : V \to W$, $B : W \to V$ be linear maps. There exist bases of $V$ and $W$ such that, with respect to those bases, the pair $(A, B)$ has the representation

$$(\text{diag}(I, A_1, \ldots, A_p, 0), \text{diag}(J_{AB}, B_1, \ldots, B_p, 0))$$

where $J_{AB}$ is the nonsingular part of the Jordan form of $AB$, $A_i, B_i \in F^{m_i \times m_i}$, $|m_i - n_i| \leq 1$, and

$$\{(I_{m_i - 1} - 0), \begin{pmatrix} 0 & \vdots \\ \vdots & 0 \\ & I_{m_i - 1} \end{pmatrix}, \begin{pmatrix} 0 & \vdots \\ \vdots & 0 \\ & I_{m_i - 1} \end{pmatrix}, (I_{m_i - 1} - 0), (I_{m_i} - 0), (J_{m_i}, J_{m_i})\}$$

where $J_k$ denotes the $k \times k$-matrix with ones on the first subdiagonal and zeros elsewhere. The representation (4) is unique up to the order of the pairs of blocks $(A_i, B_i), i = 1, \ldots, p$. Two pairs $(A, B)$ and $(C, D)$ are contragrediently equivalent if and only if $AB$ is similar to $CD$ and

$$\begin{array}{ll}
\text{rank } A = \text{rank } C, & \text{rank } BA = \text{rank } DC, \ldots, \text{rank}(BA)^t = \text{rank}(DC)^t, \\
\text{rank } B = \text{rank } D, & \text{rank } AB = \text{rank } CD, \ldots, \text{rank}(AB)^t = \text{rank}(CD)^t, \\
t = \min\{m, n\}.
\end{array}$$

Proof. Step 1. By Theorem 1 of [1] (whose proof holds over an arbitrary field), there exist $V_1$ ($W_1$) and $V_2$ ($W_2$) such that $BA$ ($AB$) is invertible on $V_1$ ($W_1$) and nilpotent on $V_2$ ($W_2$) and $V = V_1 + V_2$ ($W = W_1 + W_2$). Moreover, $V_1 = \text{range}(BA)^r$, $V_2 = \ker(AB)^r$, $W_1 = (AB)^r$, $W_2 = \ker(AB)^r$ for some $r \in \mathbb{N}$. If $x \in V_1$, then $x = (BA)^r y$ for some $y \in V$, hence $(AB)^r Ax = Ay = Ax$, that is, $Ax \in W_i$. Analogously, $By \in V_i$ whenever $y \in W_1$. So, $V = V_1 + V_2, W = W_1 + W_2, A$ maps $V_i$ to $W_i, B$ maps $W_i$ to $V_i$ for $i = 1, 2$.

If $x \in V_2$, then $(AB)^r Ax = 0$, so $Ax \in W_2$. If $x \in V_1$ and $Ax = 0$, then $BAx = 0$, therefore, $x = 0$, since $BA$ is invertible on $V_1$. So, $A$ induces a one-one map from $V_1$ to $W_1$. Likewise, $B$ induces a one-one map from $W_1$ to $V_1$. So, $V_1$ and $W_1$ have the same dimension and the induced maps are also onto.

This step of the proof not only uses Theorem 1 of [1], but also parallels it.

Now one can choose bases of $V_1$ and $W_1$ so that $A|_{V_1}$ is the identity matrix and $B|_{W_1}$ is in Jordan normal form (which is the nonsingular part of the Jordan normal form of $AB$).

Step 2. The spaces $V_2$ and $W_2$ are further split as follows. Let $l$ be the length of the longest nonzero product of the form $\cdots \cdot ABA$ or $\cdots \cdot BAB$. Call such a product $C$ and suppose it ends in $A$. Pick $x \in V_2$ so that $Cx \neq 0$ and form the sequence $x, Ax, BAx, \ldots, Cx$, whose elements are alternately in $V_2$ and $W_2$. Let $V_3$ ($W_3$) be the span of the elements of the sequence belonging to $V_2$ ($W_2$).

If $l$ is even, then $\dim V_3 = \dim W_3 + 1 = 1 + l/2$. Pick $x' \in V_2^*$ so that $\langle Cx, x' \rangle \neq 0$. Form the sequence $x', A^* x', \ldots, A^* B^* x', \ldots, C^* x'$. Let $V_4$ ($W_4$) be the annihilator in $V_2$ (in $W_2$) of the elements of the sequence that lie in $V_2^*$ ($W_2^*$). The $(1 + l/2) \times (1 + l/2)$-matrix $(\langle (BA)^j x, (A^* B^*)^{1/2-j} x' \rangle : i, j = 1, \ldots, 1 + l/2)$ is upper triangular with nonzero diagonal entries, hence, by the Lemma, $V_2 = V_3 + V_4$. This argument is exactly the same as the corresponding argument in [1].
Analogously, \( W_2 = W_3 + W_4 \). Moreover, \( A \) maps \( V_i \) to \( W_i \), \( B \) maps \( W_i \) to \( V_i \), \( i = 3, 4 \), and the pair \( (A|_{V_3}, B|_{W_3}) \) has the form

\[
\begin{pmatrix}
I_{l/2} & 0 \\
0 & I_{l/2}
\end{pmatrix}.
\]

If \( l \) is odd, then \( \dim V_3 = \dim W_3 = (1 + l)/2 \), and the above construction gives \( V_2 = V_3 + V_4 \), \( W_2 = W_3 + W_4 \) with \( A \) mapping \( V_i \) to \( W_i \), \( B \) mapping \( W_i \) to \( V_i \), \( i = 3, 4 \), the pair \( (A|_{V_3}, B|_{W_3}) \) having the form \((I_{(1+l)/2}, J_{(1+l)/2})\).

If \( C \) ends in \( B \), then \((A|_{V_3}, B|_{W_3})\) has the form \((\begin{pmatrix} 0 & I_{l/2} \\ I_{l/2} & 0 \end{pmatrix}, (J_{1+l/2}, I_{1+l/2}))\) or \((J_{1+l/2}, I_{1+l/2})\). This step of the proof parallels, with necessary modifications, Theorem 2 of [7]. The problem is now reduced to splitting \( V_4 \) and \( W_4 \) in the same way. The splitting process ends at the \( j \)-th stage if \( A|_{V_2j} = 0 \) and \( B|_{W_2j} = 0 \).

Thus one obtains the canonical form [1]. It is completely determined by the nonsingular part of the Jordan form of \( AB \) and the ranks \( \text{rank}(A) \), \( \text{rank}(BA) \), \( \text{rank}(ABA) \), \ldots, \( \text{rank}(B) \), \( \text{rank}(AB) \), \( \text{rank}(BAB) \), \ldots. Since the rank of any such product equals the size of \( J_{AB} \) if the length of the product exceeds \( 2 \min\{m, n\} \), the infinite sequences above can be terminated at \((BA)^{\min\{m, n\}}, (AB)^{\min\{m, n\}}\). It follows that 1) the representation [1] is unique up to the order of the pairs of blocks and that 2) two pairs \((A, B)\) and \((C, D)\) are contragrediently equivalent if and only if \( AB \) is similar to \( CD \) and \((5)\) holds.

**Remarks.** 1. Ptak’s duality method was rediscovered by I. Kaplansky [6], who also described how to derive the canonical form [1]. The same form was first published by N. T. Dobrovol’skaya and V. A. Ponomarev [2]. J. Gelonch and P. Rubió i Diaz [3, Theorem 2] proved that the pair \((A, B)\) can be represented as

\[
(\text{diag}(A_1, \ldots, A_q), \text{diag}(B_1, \ldots, B_q))
\]

where \( A_i \) and \( B_i^* \) are of the same size and

\[
(\dim \ker A_i, \dim \ker B_i) \in \{(0, 1), (1, 0)\} \quad \text{unless} \quad A_i = 0, B_i = 0.
\]

R. Horn and D. Merino derived the canonical form [1] in [4, Theorem 5]. All the derivations (in [2, 3, 4], and [1]) were for the field \( \mathbb{C} \). 2. Observe that the canonical form of the pair \((I, A)\) under contragredient equivalence is \((I, J_A)\), where \( J_A \) is the Jordan normal form of \( A \). This and many other applications of the canonical form [1] are discussed in [4].

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