Fully nonlinear elliptic and parabolic equations in weighted and mixed-norm Sobolev spaces

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Abstract
We present the first to date weighted and mixed-norm Sobolev estimates for fully nonlinear elliptic and parabolic equations in the whole space under a relaxed convexity condition with almost VMO dependence on space-time variables. The corresponding interior and boundary estimates are also obtained.

Mathematics Subject Classification 35K10 · 35J15 · 60J60

1 Introduction
The goal of this paper is to establish weighted and mixed-norm Sobolev estimates for fully nonlinear second-order elliptic and parabolic equations with almost VMO dependence on space-time variables, under a relaxed convexity condition. The interest in results concerning equations in spaces with mixed Sobolev norms arises, for example, when one wants to have better regularity of traces of solutions of parabolic equations for each time slice while treating linear or nonlinear equations. Weighted estimates are natural especially when one considers the boundary value problem. In this case typically weights are chosen to be powers of the distance to the boundary, so that the derivatives of solutions are allowed to blow up near the boundary. Our work is in part motivated by them.

The usual Sobolev space theory of linear elliptic and parabolic equations with continuous main coefficients has long and rich history reflected in lots of papers and books. In early nineties Chiarenza, Frasca, and Longo, and Bramanti and Cerutti in their classical papers...
[2,7,8] discovered a way which allows main coefficients to be almost in VMO rather than continuous. Their approach was also continued in quite a few papers and books. See, for instance, [23]. As the previous theory, this approach is based on the theory of singular integrals or its versions and explicit integral representation of solutions of model equations. The same approach also works for equations with sufficiently regular coefficients in Sobolev spaces with Muckenhoupt $A_p$-weights, as is shown, for instance, in [6] and the references therein. About ten years ago a different approach was suggested based on the Fefferman–Stein theorem in place of the theory of singular integrals. This approach is more flexible and applies to nonlinear equations as well as to linear ones and does not require any explicit representation of solutions in any model case. For instance, it allowed the authors of [10,11] to generalize the results of the type in [6] to a large extent to a very wide range of linear equations with almost VMO coefficients and, in addition, also derive mixed norms estimates with $A_p$-weights.

Our goal is to prove similar results for fully nonlinear equations in what concerns almost VMO, mixed norms, and $A_p$-weights. As far as the authors are aware, these are the first results in these directions in the theory of fully nonlinear elliptic and parabolic equations. They are obtained by a combination of the ideas, techniques, and results from [11] and [18]. There is a different approach to fully nonlinear equations based on the notion of viscosity solution and it would be extremely interesting to know if any of our results can be obtained by means of this theory. Let $\mathbb{R}^d$ be the $d$-dimensional Euclidean space of points $x = (x_1, \ldots, x_d)$ and $\mathbb{S}$ be the set of $d \times d$ symmetric matrices. For $\delta \in (0, 1)$, by $\mathbb{S}_\delta$ we denote the subset of $\mathbb{S}$ consisting of matrices whose eigenvalues are between $\delta$ and $\delta^{-1}$. We are interested in elliptic operators in the form

$$F[u] := F(D^2 u, x),$$

where $F = F(u'', x), u'' \in \mathbb{S}, x \in \mathbb{R}^d$, is a given function, as well as the corresponding parabolic operators in the form

$$\partial_t u + F[u] := \partial_t u + F(D^2 u, t, x).$$

Here and everywhere below

$$D^2 u = (D_{ij}u), \quad Du = (D_i u), \quad D_i = \frac{\partial}{\partial x_i}, \quad D_{ij} = D_i D_j, \quad \partial_t = \frac{\partial}{\partial t}.$$

Under the assumption that $F$ is Lipschitz continuous with respect to $u''$, $F(0, x) = 0$, $F$ is almost convex in $u''$ and almost VMO in $x$ for large values of $|u''|$, we obtain weighted Sobolev estimates in the whole space with Muckenhoupt $A_p$-weights. See Sect. 3 and Theorem 3.13 for more precise assumptions and the result. By using a powerful extrapolation theorem due to Rubio de Francia [24], we then derive mixed-norm Sobolev estimates in the whole space under some additional conditions. See Theorem 3.18. For operators $F$ which are positive homogeneous of degree one with respect to $u''$, we prove a local mixed-norm estimate. See Theorem 3.27. We also consider fully nonlinear elliptic equations in half spaces, and prove estimates near the boundary with $A_p$-weights on $\mathbb{R}^d_+$ and, as a typical example, weights which are powers of the distance to the boundary. See Sects. 4 and 5. The corresponding estimates for parabolic equations in the whole space, half spaces, balls, and half balls are also established in Sects. 6 and 7. It is worth noting that one can also consider operators $F$ with lower order terms. However, in order not to overburden this paper, we only consider operators which depends only on $D^2 u$ and $x$ (and also $t$ in the parabolic case).

Our proofs of weighted estimates are based on mean oscillation estimates proved earlier in [17,18], the Hardy–Littlewood maximal function theorem, and a local version of the
Fefferman–Stein sharp functions theorem with $A_p$-weights, which is one of our main results and is stated in Corollary 2.10 below. Such local version of the Fefferman–Stein sharp functions theorem allows us to derive estimates without relying on a partition of unity argument, which is not applicable to general fully nonlinear operators. The key ingredients in the proof of mean oscillation estimates in [17,18] are the Evans–Krylov theorem and a $W^2_2$ estimate for equations with measurable coefficients, which is originally due to Lin [22]. For mixed-norm estimates, we follow the argument in [11] by using a generalized extrapolation theorem, Theorem 7.11, in the spirit of Rubio de Francia [24].

The interior (usual) $W^2_2$ estimates for fully nonlinear elliptic equations were derived in [5], basically, under the convexity assumption on $F$ with respect to $u''$ and almost continuity assumption with respect to $x$. In [25] global estimates were obtained under the same kind of assumption. See also recent [3] for an extension to weighted Sobolev spaces. These results were obtained by using the theory of viscosity solutions. The same theory applied in [9] to parabolic case yields similar results under similar assumptions as in the elliptic case.

For elliptic Bellman’s equations with VMO dependence on the independent variables the interior $W^2_2$ estimates were first obtained in [21]. Later, boundary and similar estimates for parabolic equations, as well as a solvability result, were obtained in [12]. The relaxed convexity and VMO conditions (Assumptions 3.1 and 5.1) in the current paper are adopted from [17], in which the existence of $W^2_2$ solutions for fully nonlinear elliptic equations in domains was proved. See [17, Remarks 2.4, 2.5, and Example 2.2] for some interesting discussions and examples regarding these conditions, and also [18] for a result for parabolic equations. This paper is a continuation of this line of research in the weighted and mixed-norm settings. For other relevant results in the literature, we refer the reader to [4,12,17,18,21,26] and a recent book [19] by the second named author. We note that in [4] although the results are stated for viscosity solutions, the authors actually considered $W^2_2$ strong solutions, not viscosity solutions.

The remaining part of the paper is organized as follows. In the next section, we recall some definitions and facts from Chapter 3 of [16] and prove a local version of the Fefferman–Stein sharp function theorem. We consider elliptic equations in the whole space and in balls in Sect. 3, and in half spaces and in half balls in Sects. 4 and 5. In Sects. 6 and 7, we prove analogous results for parabolic equations. In the “Appendix”, we state and prove a generalized extrapolation theorem, Theorem 7.11.

In the proofs of various results in this paper, we use the symbol $N$ sometimes with indices to denote finite nonnegative constants which may change from one occurrence to another and we do not always specify on which data these constants depend. In these cases the reader should remember that, if in the statement of a result there are constants called $N$ which are claimed to depend only on certain parameters, then in the proof of the result the constants $N$ also depend only on the same parameters unless specifically stated otherwise. Of course, if we write $N = N(\ldots)$, this means that $N$ depends only on what is inside the parentheses.

2 Partitions and sharp functions

For reader’s convenience, we first recall some definitions and facts from Chapter 3 of [16]. Let $(\Omega, \mathcal{F}, \mu)$ be a complete measure space with a $\sigma$-finite measure $\mu$, such that

$$\mu(\Omega) = \infty.$$
Let $\mathcal{F}_0$ be the subset of $\mathcal{F}$ consisting of all sets $A$ such that $\mu(A) < \infty$. By $\mathbb{L}$ we denote a fixed dense subset of $L_1(\Omega) = L_1(\Omega, \mathcal{F}, \mu)$. For any $A \in \mathcal{F}$ we set 

$$|A| = \mu(A).$$

For $A \in \mathcal{F}_0$ and functions $f$ summable on $A$ we use the notation 

$$f_A = \int_A f \mu(dx) := \frac{1}{|A|} \int_A f(x) \mu(dx) \quad (0 := 0)$$

for the average value of $f$ over $A$. We write $f \in L^1_{1,\text{loc}}(\Omega)$ if $f|_A \in L^1(\Omega)$ for any $A \in \mathcal{F}_0$.

**Definition 2.1** Let $\mathbb{Z} = \{n : n = 0, \pm 1, \pm 2, \ldots\}$ and let $(\mathbb{C}_n, n \in \mathbb{Z})$ be a sequence of partitions of $\Omega$ each consisting of countably many disjoint sets $C \in \mathbb{C}_n$ and such that $\mathbb{C}_n \subset \mathcal{F}_0$ for each $n$. For each $x \in \Omega$ and $n \in \mathbb{Z}$ there exists (a unique) $C \in \mathbb{C}_n$ such that $x \in C$. We denote this $C$ by $C_n(x)$.

We call the sequence $(\mathbb{C}_n, n \in \mathbb{Z})$ a filtration of partitions if the following conditions are satisfied.

(i) The elements of partitions are “large” for big negative $n$’s and “small” for big positive $n$’s:

$$\inf_{C \in \mathbb{C}_n} |C| \to \infty \quad \text{as} \quad n \to -\infty, \quad \lim_{n \to +\infty} f_{C_n(x)} = f(x) \quad \text{(a.e.)} \quad \forall f \in \mathbb{L}.$$

(ii) The partitions are nested: for each $n$ and $C \in \mathbb{C}_n$ there is a (unique) $C' \in \mathbb{C}_{n-1}$ such that $C \subset C'$.

(iii) The following regularity property holds: for any $n, C$, and $C'$ as in (ii) we have

$$|C'| \leq N_0 |C|,$$

where $N_0$ is a constant independent of $n, C, C'$.

We set

$$\mathbb{C}_\infty = \bigcup_n \mathbb{C}_n.$$

**Definition 2.2** Let $\mathbb{C}_n, n \in \mathbb{Z}$, be a filtration of partitions of $\Omega$.

(i) Let $\tau = \tau(x)$ be a function on $\Omega$ with values in $\{\infty, 0, \pm 1, \pm 2, \ldots\}$. We call $\tau$ a stopping time (relative to the filtration) if, for each $n = 0, \pm 1, \pm 2, \ldots$, the set

$$\{x : \tau(x) = n\}$$

is either empty or else is the union of some elements of $\mathbb{C}_n$.

(ii) For a function $f \in L^1_{1,\text{loc}}(\Omega)$ and $n \in \mathbb{Z}$, we denote

$$f_{|n}(x) = \int_{C_n(x)} f(y) \mu(dy).$$

If we are also given a stopping time $\tau$, we let

$$f_{|\tau}(x) = f_{|\tau(x)}(x)$$

for those $x$ for which $\tau(x) < \infty$ and $f_{|\tau}(x) = f(x)$ otherwise.
The simplest example of a stopping time is given by $\tau(x) \equiv 0$.

We are going to use the following simple properties of the objects introduced above.

**Lemma 2.3** Let $C_n, n \in \mathbb{Z}$, be a filtration of partitions of $\Omega$.

(i) Let $f \in L_1,_{\text{loc}}(\Omega), f \geq 0$, and let $\tau$ be a stopping time. Then

$$
\int_{\Omega} f|_{\tau}(x) I_{\tau<\infty} \mu(dx) = \int_{\Omega} f(x) I_{\tau<\infty} \mu(dx),
$$

$$
\int_{\Omega} f|_{\tau}(x) \mu(dx) = \int_{\Omega} f(x) \mu(dx).
$$

(ii) Let $g \in L_1(\Omega), g \geq 0$, and let $\lambda > 0$ be a constant. Then $\tau(x) := \inf\{n : g|_n(x) > \lambda\}$ (inf $\emptyset \equiv \infty$) \hspace{1cm} (2.1)

is a stopping time. Furthermore, we have

$$
0 \leq g|_{\tau}(x) I_{\tau<\infty} \leq N_0 \lambda, \quad \{|x : \tau(x) < \infty|\} \leq \lambda^{-1} \int_{\Omega} g(x) I_{\tau<\infty} \mu(dx).
$$

Define the maximal function of $f$ by

$$
\mathcal{M} f(x) = \sup_{n<\infty} |f|_n(x),
$$

so that $\mathcal{M} f = \mathcal{M}|f|$.

Notice that Lemma 2.3 implies the following.

**Corollary 2.4** (Maximal inequality) For $\lambda > 0$ and nonnegative $g \in L_1(\Omega)$, the maximal inequality holds:

$$
|\{x : \mathcal{M} g(x) > \lambda\}| \leq \lambda^{-1} \int_{\Omega} g(x) I_{\mathcal{M} g > \lambda} \mu(dx).
$$

Indeed, for $\tau$ as in (2.1), we have

$$
\{x : \mathcal{M} g(x) > \lambda\} = \{x : \tau(x) < \infty\}.
$$

**Corollary 2.5** Let $p \in (1, \infty), g \in L_1(\Omega), g \geq 0$. Then

$$
\|\mathcal{M} g\|_{L^p(\Omega)} \leq q \|g\|_{L^p(\Omega)},
$$

where $q = p/(p - 1)$.

The following extends Corollary 2.5 to $g \in L_p(\Omega)$.

**Theorem 2.6** For any $p \in (1, \infty)$ and $g \in L_p(\Omega)$,

$$
\|\mathcal{M} g\|_{L^p(\Omega)} \leq q \|g\|_{L^p(\Omega)}.
$$

Let $w = w(x)$ be a nonnegative function on $\Omega$, such that $\chi(C) < \infty$ for any $C \in C_\infty$, where $\chi(A) := \int_A w \mu(dx)$.

For $\beta \in (0, 1]$, we say that $w$ is of $\beta$-type if

$$
\frac{\chi(A)}{\chi(C)} \leq N_{w, \beta} \frac{|A|^\beta}{|C|^\beta}
$$
for any measurable $A \subset C$ and $C \in \mathbb{C}_\infty$, where $N_{w,\beta}$ is a (finite) constant independent of $C$ and $A$.

**Remark 2.7** In some of our applications $\Omega$ will be a linear metric space with filtration of either dyadic standard or parabolic cubes and $w$ will be an $A_p$-weight with respect to the corresponding metric. It is well known that in such situations if $w \in A_p$ and $[w]_p \leq K_0$, where $K_0$ is a constant, then $w$ is of $\beta$-type for an appropriate $\beta$ and $N_{w,\beta}$ both depending only on $K_0$ and the metric.

The following is a combination of Theorem 2.5 of [11] and Lemma 5.1 of [21].

**Lemma 2.8** Let $\gamma \in (0, 1]$, $v \in L_{1,\text{loc}}(\Omega)$, and let $v_n \to 0$ as $n \to -\infty$ on $\Omega$. Assume that $|u| \leq v$ and for any $C \in \mathbb{C}_\infty$ there exists a measurable function $u^C$ given on $C$ such that $|u| \leq u^C \leq v$ on $C$ and, for any $x \in C$

$$\left( \int_C \int_C |u^C(z) - u^C(y)|^\gamma \mu(dz)\mu(dy) \right)^{1/\gamma} \leq g(x). \quad (2.2)$$

Let $w$ be of $\beta$-type. Then for any $\lambda > 0$ we have

$$\chi \{ x : |u(x)| \geq \lambda \} \leq N_{w,\beta} v^{-\beta} \lambda^{-\gamma \beta} \int_\Omega g^{\gamma \beta}(x) I_{Mv(x) > \alpha \lambda} \chi(dx),$$

where $\alpha = (2N_0)^{-1}$ and $\nu = 1 - 2^{-\gamma}$.

**Proof** Obviously we may assume that $u \geq 0$. Fix a $\lambda > 0$ and define

$$\tau(x) = \inf \{ n \in \mathbb{Z} : v_n(x) > \alpha \lambda \}.$$

We know that $\tau$ is a stopping time and if $\tau(x) < \infty$, then

$$v_n(x) \leq \lambda/2, \quad \forall n \leq \tau(x).$$

We also know that $v_n \to v \geq u$ (a.e.) as $n \to \infty$ (the Lebesgue differentiation theorem). It follows that (a.e.)

$$\{ x : u(x) \geq \lambda \} = \{ x : u(x) \geq \lambda, \tau(x) < \infty \} = \{ x : u(x) \geq \lambda, v_{\tau(x)} \leq \lambda/2 \} = \bigcup_{n \in \mathbb{Z}} \bigcup_{C \in \mathcal{F}_n} A_n(C),$$

where

$$A_n(C) := \{ x \in C : u(x) \geq \lambda, v_n(x) \leq \lambda/2 \},$$

and $\mathcal{F}_n$ is the family of disjoint elements of $\mathbb{C}_n$ such that

$$\{ x : \tau(x) = n \} = \bigcup_{C \in \mathcal{F}_n} C.$$

Next, for each $n \in \mathbb{Z}$ and $C \in \mathbb{C}_n$ on the set $A_n(C)$, if it is not empty, we have $v_n = v_C$ and on $A_n(C)$

$$u^{\gamma} - (v_C)^{\gamma} \geq \lambda^{\gamma}(1 - 2^{-\gamma}) = v^{\gamma} \lambda^{\gamma}.$$
We use this and the inequality \(|a - b| \gamma \geq |a|^\gamma - |b|^\gamma\) and conclude that for \(x \in A_n(C)\),

\[
\int_C |u^C(x) - u^C(y)|^\gamma \mu(dy) \geq (u^C(x))^\gamma - \int_C (u^C(y))^\gamma \mu(dy)
\geq u^\gamma(x) - \int_C v^\gamma(y) \mu(dy) \geq u^\gamma(x) - (v^C(x))^\gamma \geq v^\lambda^\gamma,
\]

so that by Chebyshev’s inequality

\[
|A_n(C)| \leq v^{-1} \lambda^{-\gamma} \int_C \int_C |u^C(z) - u^C(y)|^\gamma \mu(dz) \mu(dy).
\]

It follows by assumption (2.2) that

\[
\frac{|A_n(C)|}{|C|} \leq v^{-1} \lambda^{-\gamma} g^\gamma(x)
\]

for any \(x \in \Omega\). Since \(w\) is of \(\beta\)-type,

\[
\chi(A_n(C)) \leq N_{w, \beta} v^{-\beta} \lambda^{-\gamma^\beta} g^\gamma^\beta(x) \chi(C).
\]

Since this holds for any \(x \in C\),

\[
\chi(A_n(C)) \leq N_{w, \beta} v^{-\beta} \lambda^{-\gamma^\beta} \int_C g^\gamma^\beta(x) \chi(dx).
\]

Hence,

\[
\chi\{x : u(x) \geq \lambda\} \leq N_{w, \beta} v^{-\beta} \lambda^{-\gamma^\beta} \sum_{n \in \mathbb{Z}} \sum_{C \in \mathcal{F}_n} \int_C g^\gamma^\beta(x) \chi(dx)
\]

\[
= N_{w, \beta} v^{-\beta} \lambda^{-\gamma^\beta} \int_\Omega g^\gamma^\beta(1_{\tau<\infty}) \chi(dx).
\]

It only remains to observe that \(\{\tau < \infty\} = \{Mv > \alpha \lambda\}\). The lemma is proved. \(\square\)

**Corollary 2.9** Under the assumption of Lemma 2.8, for any \(p > \gamma \beta\),

\[
\int_\Omega |u|^p \chi(dx) \leq N \left(\int_\Omega |Mv|^p \chi(dx)\right)^{(p-\gamma \beta)/p} \left(\int_\Omega |g|^p \chi(dx)\right)^{\gamma \beta/p}.
\]

where \(N\) depends only on \(N_0, N_{w, \beta}, p, \beta, \) and \(\gamma\).

Indeed, by Lemma 2.8 and the Fubini theorem,

\[
\int_\Omega |u|^p \chi(dx) = p \int_0^\infty \chi\{x : |u(x)| \geq \lambda\} \lambda^{p-1} d\lambda
\geq p N_{w, \beta} v^{-\beta} \int_0^\infty \int_\Omega g^\gamma^\beta(x) I_{Mv(x) > \alpha \lambda} \lambda^{p-1-\gamma^\beta} \chi(dx) d\lambda
\geq p N_{w, \beta} v^{-\beta} / (p - \gamma \beta) \int_\Omega g^\gamma^\beta(x) (Mv(x) / \alpha)^{p-\gamma^\beta} \chi(dx).
\]

To get the desired inequality, it only remains to apply Hölder’s inequality.
For $m \in \mathbb{Z}$ introduce
\[
\begin{align*}
u^\#,_{\gamma,m}(x) &= \sup_{n \geq m} \sup_{C \in \mathbb{C}_n, C \ni x} \left( \int_C \int_C |u(z) - u(y)|^{\gamma} \mu(dz) \mu(dy) \right)^{1/\gamma}, \\
\mathcal{M}_m v &= \sup_{n \leq m} v_n.
\end{align*}
\]

**Corollary 2.10** Take $m \in \mathbb{Z}$. Assume that $|u|_n \to 0$ as $n \to -\infty$, and let $w$ be of $\beta$-type. Then for any $p > \gamma \beta$,
\[
\int_{\Omega^1} |u|_p \chi(dx) \leq NI^{(p-\gamma\beta)/p} J^{\gamma\beta/p},
\]
where
\[
\begin{align*}
I &= \int_{\Omega} |\mathcal{M}u|_p \chi(dx), \\
J &= \int_{\Omega} \left( \nu^\#,_{\gamma,m} + \mathcal{M}^{1/\gamma}_m (|u|^{\gamma}) \right)_p \chi(dx),
\end{align*}
\]
and the constant $N$ depends only on $N_0, N_\omega, \beta, p, \beta,$ and $\gamma$.

This obviously follows from Corollary 2.9 with $u^C = v = |u|$ since for $n \leq m$ the left-hand side of (2.2) is less that $2^{1/\gamma} \mathcal{M}^{1/\gamma}_m (v^{\gamma})$.

**3 Elliptic case**

In this section, we study fully nonlinear elliptic equations in weighted and mixed-norm Sobolev spaces. Set
\[
B_r(x) = \{ y \in \mathbb{R}^d : |x - y| < r \}, \quad B_r = B_r(0).
\]
Suppose that we are given a function $F(u'', x), u'' \in \mathbb{S}, x \in \mathbb{R}^d$. In our results we will impose some of the following assumptions with a parameter $\theta \in (0, 1)$ to be specified later.

**Assumption 3.1** ($\theta$)

(i) The function $F$ is Lipschitz continuous with respect to $u''$ with Lipschitz constant $K_F$ and $F(0, x) \equiv 0$.

There exist $R_0 \in (0, 1]$ and $\tau_0 \in [0, \infty)$ such that, if $r \in (0, R_0]$ and $z \in \mathbb{R}^d$, then one can find a convex function $\tilde{F}(u'') = \tilde{F}_{z,r}(u'')$ (independent of $x$) for which

(ii) We have $\tilde{F}(0) = 0$ and $D_{u''} \tilde{F} \in \mathbb{S}_{\hat{\delta}}$ at all points of differentiability of $\tilde{F}$;

(iii) For any $u'' \in \mathbb{S}$ with $|u''| = 1$ ($|u''| := \text{tr}^{1/2}(u''u'')$), we have
\[
\int_{B_r(z)} \sup_{\tau > \tau_0} \tau^{-1} |F(\tau u'', x) - \tilde{F}(\tau u'')| \, dx \leq \theta \, |B_r(z)|,
\]
where by $|A|$ we denote the volume of $A$ in $\mathbb{R}^d$.

**Assumption 3.2** The function $F$ is Lipschitz continuous with respect to $u''$, $F(0, x) \equiv 0$, and $D_{u''} F \in \mathbb{S}_{\hat{\delta}}$ at all points of differentiability of $F$.

**Remark 3.3** Assumption 3.2 implies that, for any $u'' \in \mathbb{S}$ and $x \in \mathbb{R}^d$, we have $F(u'', x) = a^{ij} u''_{ij}$, where $a = (a^{ij}) \in \mathbb{S}_{\hat{\delta}}$. 

\[\text{Springer}\]
Example 3.4  Let $d = 3$ and let $G \geq 0$ be a bounded measurable function on $\mathbb{R}^d$. It is not hard to see that for any $\theta > 0$ there exists $\tau_0 > 0$ such that Assumption 3.1 is satisfied for $(a \wedge b = \min(a, b))$

$$F(D^2 u, x) := \bar{G}(x) \wedge |D_{12}u| + \bar{G}(x) \wedge |D_{23}u| + \bar{G}(x) \wedge |D_{31}u| + \Delta u,$$  \hfill (3.1)

and $\bar{F}(u'') = \text{tr } u''$. Observe that $F$ in (3.1) is neither convex nor concave with respect to $D^2 u$. Also note that we can replace $\Delta u$ with $a^{ij}(x)D_{ij}u$ if $a(x) = (a^{ij}(x))$ is an $S_\delta$-valued VMO-function such that $a(x) \geq (\delta^{ij})$.

Example 3.5  For $\epsilon > 0$ take

$$F(u) = \left(1 + \epsilon \cos \sqrt{|\ln |u''||}\right) \text{tr } u'',$$

and choose $\epsilon$ so small that $D_{ij}u \in S_\delta$ for a $\delta \in (0, 1]$. Then again $F$ is neither convex nor concave with respect to $u''$ and our assumptions are satisfied perhaps with a further reduced $\epsilon$ and $\bar{F}(u'') = \text{tr } u''$. Also observe that one could take this $F(D^2 u)$ in place of $\Delta u$ in (3.1).

One more example is related to the stochastic differential games and the Isaacs equations.

Example 3.6  Let $A$ and $B$ be some countable sets and assume that for $\alpha \in A$, $\beta \in B$, and $x \in \mathbb{R}^d$ we are given functions $a^{\alpha}(x)$, $f^{\alpha \beta}(x)$ with values in $S_\delta, \mathbb{R}$, respectively. Assume that these functions are measurable in $x$, $f^{\alpha \beta}$ is bounded, and

$$\inf_{\beta \in B} \sup_{\alpha \in A} f^{\alpha \beta}(x) = 0.$$

Also assume that there is an $R_0 \in (0, \infty)$ such that for any $z \in \mathbb{R}^d$, and $r \in (0, R_0]$, one can find $\bar{a}^\alpha \in S_\delta$ (independent of $x$) such that

$$\int_{B_r(z)} \sup_{\alpha \in A} \left| a^{\alpha}(x) - \bar{a}^\alpha \right| \, dx \leq \theta.$$

Introduce

$$F(u, x) := \inf_{\beta \in B} \sup_{\alpha \in A} \left[ \sum_{i, j=1}^d a^{\alpha}_{ij}(x) u''_{ij} + f^{\alpha \beta}(x) \right].$$ \hfill (3.2)

The reader will easily check that Assumptions 3.1 and 3.2 are satisfied with

$$\bar{F}(u'', x) = \sup_{\alpha \in A} \sum_{i, j=1}^d \bar{a}^{\alpha}_{ij} u''_{ij}.$$

For functions $h$ on $\mathbb{R}^d$, $\rho > 0$, and $x \in \mathbb{R}^d$, introduce

$$h^\rho_{\gamma, \rho}(x) = \sup_{r \in (0, \rho], \rho, \gamma \ni x} \left( \int_{B_r(y)} \int_{B_r(x)} |h(x_1) - h(x_2)|^\gamma \, dx_1 \, dx_2 \right)^{1/\gamma},$$

$$\mathbb{M}h(x) = \sup_{r > 0, \rho, \gamma \ni x} \int_{B_r(x_0)} |h(y)| \, dy,$$
\[
\mathcal{M}_\rho h(x) = \sup_{r \in (\rho, \infty), \; B_r(x) \ni y} |h(y)| dy.
\]

We set \( \Omega = \mathbb{R}^d \) and for \( n \in \mathbb{Z} \) we take \( \mathbb{C}_n \) as the collection of \( x + [0, 2^{-n})^d, \; x \in 2^{-n}\mathbb{Z}^d \). We also set \( \mu \) to be Lebesgue measure and \( \mathbb{L} \) to be the set of continuous functions with compact support. Then observe that for a constant \( c = \sqrt{d}/2 \),

\[
h^\nu_{\gamma, m} \leq N(d, \gamma) h^\nu_{\gamma, c2^{-m}}, \quad \mathcal{M}_m h \leq N(d) \mathcal{M}_{c2^{-m}} h.
\]  

(3.3)

From Lemma 3.6 of [17] and the proof of Lemma 5.2 (related to estimates in bounded domains) of [17] one can easily obtain the following result.

**Lemma 3.7** Let \( u \in W^2_{d, \text{loc}}(\mathbb{R}^d), \; \mu \in (0, \infty), \; \nu \geq 2, \; \xi \in (1, \infty). \) Then there exists \( \theta = \theta(d, \xi, K_F, \mu, \xi) \in (0, 1) \) such that, if Assumption 3.1 (\( \theta \)) is satisfied, then one can find \( \gamma_0 = \gamma_0(d, \theta, \xi) \in (0, 1), \; \alpha = \alpha(d, \theta, \xi) \in (0, 1), \) such that for \( \gamma \in (0, \gamma_0), \; h = \mathcal{D}^2 u, \) and \( \rho = \rho_0/\nu, \) we have

\[
h^\nu_{\gamma, \rho} \leq N\nu^{d/\gamma} M^{1/d} \left( |F[u]|^d \right) + N\nu^{-1} \mathcal{M}^{1/d} \left( |\mathcal{D} u|^d \right) + N\rho^{-1} M^{1/d} \left( |u|^d \right),
\]

where the constants \( N \) depend only on \( d, \xi, K_F, \) and \( \gamma \).

We write \( w \in A_p(\mathbb{R}^d) \) if \( w \) is an \( A_p \)-weight on \( \mathbb{R}^d \).

**Lemma 3.8**

(i) There exists \( \gamma_0 = \gamma_0(d, \xi) \in (0, 1) \) such that under Assumption 3.2, for any \( u \in W^2_{d, \text{loc}}(\mathbb{R}^d), \; \rho > 0, \) and \( \gamma \in (0, \gamma_0) \) we have

\[
\mathcal{M}_{\rho}^{1/\gamma} (|\mathcal{D} u|^d) \leq N \mathcal{M}_{\rho}^{1/d} (|F[u]|^d) + N \rho^{-1} \mathcal{M}_{\rho}^{1/d} (|\mathcal{D} u|^d) + N \rho^{-2} \mathcal{M}_{\rho}^{1/d} (|u|^d),
\]

where the constants \( N \) depend only on \( d, \xi, K_F, \) and \( \gamma \).

(ii) For any \( \rho > 0, \; p \in [1, \infty), \) and \( u \in W^2_{p, \text{loc}}(\mathbb{R}^d), \) we have

\[
\mathcal{M}_{\rho} (|\mathcal{D} u|^p) \leq N \mathcal{M}_{\rho}^{1/2} (|\mathcal{D}^2 u|^p) \mathcal{M}_{\rho}^{1/2} (|u|^p) + N \rho^{-p} \mathcal{M}_{\rho} (|u|^p),
\]

where the constants \( N \) depend only on \( d \) and \( p \).

(iii) For any \( \rho > 0, \; K_0, \; p \in (1, \infty), \) \( w \in A_p(\mathbb{R}^d) \) with \( \|w\|_p \leq K_0, \) and \( u \in W^2_{p, w}(\mathbb{R}^d), \) we have

\[
\int_{\mathbb{R}^d} |\mathcal{D} u|^p w \, dx \leq \rho^p \int_{\mathbb{R}^d} |\mathcal{D}^2 u|^p w \, dx + N \rho^{-p} \int_{\mathbb{R}^d} |u|^p w \, dx,
\]

where \( N \) depends only on \( d, \; p, \) and \( K_0 \).

**Proof** First by Assumption 3.2, for a given \( u \) write \( F[u] = a^{ij} D_{ij} u \) and take \( r \geq \rho \) and a function \( \xi \in C_0^\infty (\mathbb{R}^d) \) such that \( \xi = 1 \) on \( B_r, \; \xi = 0 \) outside \( B_{2r}, \) and

\[
|D\xi| \leq N/r \leq N/\rho, \quad |D^2\xi| \leq N/r^2 \leq N/\rho^2.
\]

Then by a result of Lin [22],

\[
\int_{B_r} |\mathcal{D} u|^\nu \, dx \leq \int_{B_{2r}} |\mathcal{D} (\xi u)|^\nu \, dx \leq N \left( \int_{B_{2r}} |a^{ij} D_{ij}(\xi u)|^d \, dx \right)^{\nu/d} = N \left( \int_{B_{2r}} |\xi F[u] + a^{ij} 2 D_i \xi D_j u + au^{ij} D_{ij} \xi| \, dx \right)^{\nu/d}.
\]

This proves (3.5). \( \square \)
The fact that $\zeta = 1$ on $B_r$ and multiplicative inequalities show that
\[
\int_{B_r} |Du|^p \, dx \leq N \int_{B_{2r}} |D(\zeta u)|^p \, dx
\leq N \left( \int_{B_{2r}} |D^2(\zeta u)|^p \, dx \right)^{1/2} \left( \int_{B_{2r}} |u|^p \, dx \right)^{1/2},
\]
where for $r \geq \rho$,
\[
\int_{B_{2r}} |D^2(\zeta u)|^p \, dx \leq N M_\rho(|D^2 u|^p) + N \rho^{-p} M_\rho(|Du|^p) + N \rho^{-2p} M_\rho(|u|^p).
\]
Hence,
\[
M_\rho(|Du|^p) \leq N \left( M_\rho(|D^2 u|^p) + \rho^{-p} M_\rho(|Du|^p) \right)^{1/2} \left( M_\rho^{1/2}(|u|^p) + N \rho^{-p} M_\rho(|u|^p) \right),
\]
and (3.6) follows.

Finally, we prove (3.7). We take an integer $m$ such that $c2^{-m} \in (\rho/2, \rho]$. By Remark 2.7, Corollary 2.10 with $\gamma = 1$, (3.3), and the weighted Hardy–Littlewood maximal function theorem (see more about this in the proof of Theorem 3.13)
\[
\int_{\mathbb{R}^d} |Du|^p \, dx \leq N \int_{\mathbb{R}^d} ((Du)^{#}_{1,m} + M_m(|Du|)) \, dx
\leq N \int_{\mathbb{R}^d} ((Du)^{#}_{1,\rho} + M_{\rho/2}(|Du|)) \, dx.
\]
To apply Corollary 2.10 formally we need a certain condition on the averages of $|Du|$. However, we always can use cut-off functions and pass to the limit. By Poincaré’s inequality,
\[
(Du)^{#}_{1,\rho} \leq N \rho M(|D^2 u|).
\]
This together with (3.8) and (3.6) with $p = 1$ gives
\[
\int_{\mathbb{R}^d} |Du|^p \, dx \leq N \int_{\mathbb{R}^d} \left( \rho M(|D^2 u|) + \rho^{-1} M_{\rho/2}(|u|) \right)^{p} \, dx,
\]
which, by the weighted Hardy–Littlewood maximal function theorem, is bounded by the right-hand side of (3.7). The lemma is proved.

Estimate (3.7) admits the following localization.

**Lemma 3.9** For any $\rho \in (0, \infty)$, $\varepsilon \in (0, 1)$, $K_0$, $p \in (1, \infty)$, $w \in A_p(\mathbb{R}^d)$ with $[w]_p \leq K_0$, and $u \in W^2_{p,w}(B_\rho)$, we have
\[
\int_{B_{\rho/2}} |Du|^p \, dx \leq \varepsilon \rho^p \int_{B_\rho} |D^2 u|^p \, dx + \rho^{-1} K_0 \rho^{-p} \int_{B_\rho} |u|^p \, dx.
\]
where $N$ depends only on $d$, $p$, and $K_0$.

**Proof** By scaling and noting that $[w(\rho \cdot)]_p = [w]_p$, we may assume that $\rho = 1$. For $k = 1, 2, \ldots$, we take $\rho_k = 1 - 2^{-k}$, $B_k = B_{\rho_k}$, and $\xi_k \in C_0^\infty(B_k)$ such that $\xi_k = 1$ on $B_k$ and
\[
|D\xi_k| \leq N 2^k, \quad |D^2 \xi_k| \leq N 2^{2k}.
\]
It follows from (3.7) that for any $\varepsilon_0 \in (0, 1)$
\[
\int_{B_k} |Du|^p w \, dx \leq \int_{B_{k+1}} |D(\zeta^k u)|^p w \, dx \\
\leq \varepsilon_0 2^{-kp} \int_{\mathbb{R}^d} |D^2(\zeta^k u)|^p w \, dx + N \varepsilon_0^{-1} 2^{kp} \int_{\mathbb{R}^d} |\zeta^k u|^p w \, dx \\
\leq N \varepsilon_0 \int_{B_{k+1}} |Du|^p w \, dx + N \int_{B_{k+1}} (\varepsilon_0 2^{-kp} |D^2 u|^p + \varepsilon_0^{-1} 2^{kp} |u|^p) w \, dx.
\]

Now to get (3.9), it suffices to multiply both sides by $(N \varepsilon_0)^k$, sum in $k = 1, 2, \ldots$, and take a sufficiently small $\varepsilon_0$ according to $\varepsilon$. \qed

**Lemma 3.10** In Lemma 3.8 (iii) the condition $u \in W^2_{p,w}(\mathbb{R}^d)$ can be replaced with $u \in W^2_{p,w,\text{loc}}(\mathbb{R}^d)$. \qed

**Proof** We may assume that the right-hand side of (3.7) is finite. In this situation plug $B_1(x_0)$ and $B_2(x_0)$ in place of $B_{p/2}$ and $B_p$ into (3.9) with $\varepsilon = 1/2$ and integrate with respect to $x_0$ over $\mathbb{R}^d$. Then we will see that $Du \in L_{p,w}(\mathbb{R}^d)$. After that Lemma 3.8 (iii) yields the result. The lemma is proved. \qed

In the future we will use the following.

**Lemma 3.11** Let $0 < r < R < \infty$, $\varepsilon \in (0, 1)$, $K_0$, $p \in (1, \infty)$, $w \in A_p(\mathbb{R}^d)$ with $[w]_p \leq K_0$, and $u \in W^2_{p,w}(B_R)$. Then
\[
\int_{B_r} |Du|^p w \, dx \leq \varepsilon (R - r)^p \int_{B_R} |D^2 u|^p w \, dx + N \varepsilon^{-1}(R - r)^{-p} \int_{B_r} |u|^p w \, dx,
\]
where $N$ depends only on $d$, $p$, and $K_0$.

This lemma is a simple corollary of Lemma 3.9. Indeed, set $\rho = R - r$ in Lemma 3.9 and plug $B_{p/2}(x_0)$ and $B_p(x_0)$ into (3.9) in place of $B_{p/2}$ and $B_p$, respectively, with $x_0 \in B_r$. Then it will only remain to integrate the resulting inequality with respect to $x_0$ over $B_r$.

**Remark 3.12** Below we use a few times the fact that if $w$ is an $A_{p/d}(\mathbb{R}^d)$-weight for some $p \in (d, \infty)$, then by definition $w^{-1} \in L_{d/(p-d),\text{loc}}(\mathbb{R}^d)$. Hence by Hölder’s inequality, if $f \in L_{p,w,\text{loc}}(\mathbb{R}^d)$, then $|f|^d \in L_{1,\text{loc}}(\mathbb{R}^d)$. In particular, if $u \in W^2_{p,w,\text{loc}}(\mathbb{R}^d)$, then $|u|^d, |Du|^d, |D^2 u|^d \in L_{1,\text{loc}}(\mathbb{R}^d)$, which implies that $u \in W^2_{d,\text{loc}}(\mathbb{R}^d)$.

**Theorem 3.13** Take $R \in (0, \infty)$, $K_0 \in (1, \infty)$. Let $p > d$ and let $w \in A_{p/d}(\mathbb{R}^d)$ with $[w]_{p/d} \leq K_0$. Suppose that $D^2 u \in L_{p,w}(\mathbb{R}^d)$ and $u$ vanishes outside $B_R$. Then there exists $\theta = \theta(d, \delta, K_F, p, K_0) \in (0, 1)$ such that if Assumptions 3.1 (\(\theta\)) and 3.2 are satisfied, then
\[
\int_{\mathbb{R}^d} |D^2 u|^p w \, dx \leq N \int_{\mathbb{R}^d} |F[u]|^p w \, dx \\
+ N \int_{\mathbb{R}^d} |u|^p w \, dx + N \varepsilon_0^p \int_{\mathbb{R}^d} I_{B_{R+\delta K_0}} w \, dx, \tag{3.10}
\]
where $N$ is a constant depending only on $d$, $\delta$, $K_F$, $K_0$, $p$, and $R_0$.

**Proof** It is well known that the appropriately stated Hardy–Littlewood maximal function theorem holds for $A_p$-weights. Therefore, by Remark 2.7, Corollary 2.10, and (3.3) the left-hand side of (3.10) is less than a constant times
\[
\left( \int_{\mathbb{R}^d} |D^2 u|^p w \, dx \right)^{(p-\gamma\beta)/p} \left( \int_{\mathbb{R}^d} \left[ |D^2 u|^{\gamma m} + M^{1/\gamma}_m (|D^2 u|^{\gamma}) \right]^p w \, dx \right)^{\gamma\beta/p},
\]
where $\gamma \in (0, 1)$ is a constant depending only on $d$ and $\delta$ taken from Lemma 3.7. It follows that the left-hand side of (3.10) is less than a constant times

$$\int_{\mathbb{R}^d} \left((D^2 u)^\# \right)^\nu w \, dx + \int_{\mathbb{R}^d} M_m^{p/\gamma} \left(|D^2 u|^{\gamma} \right)^\nu w \, dx.$$  \hspace{1cm} (3.11)

By a reverse Hölder’s inequality (also called self-improving property of $A_p$ weights, see for instance, Corollary 9.2.6 of [13]), we can find $\xi \in (1, \infty)$ depending only on $d$, $p$, and $K_0$ such that $\rho > \xi' d$ ($\xi' = \xi / (\xi - 1)$), $w \in A_p/(\xi' d)$, and $[w]_{p/(\xi' d)} \leq N(d, K_0)$. This is the first step to specify $\theta$ which will be taken from Lemma 3.7 after we find an appropriate $\mu > 0$. To this end, take $\nu \geq 2$ to be specified later and for $m$ such that $2^m \sim R_0 / \nu$ use (3.3) and (3.4) to estimate the first integral in (3.11). Observe that $h_{\gamma, \rho}$ vanishes outside $B_{R+R_0}$ and therefore we only need to integrate the right-hand side of (3.4) over this ball. This gives the last term in (3.10) (after we fix $\nu$).

Then we again use the well-known properties of $A_p$-weights mentioned above and Lemma 3.7 to conclude that the first term in (3.11) is less than $v^{p d / \gamma}$ times the last term in (3.10) plus

$$N v^{p d / \gamma} \int_{\mathbb{R}^d} |F(u)|^{\nu} w \, dx + N(\mu v^{d / \gamma} + v^{-\mu}) \int_{\mathbb{R}^d} |D^2 u|^{\nu} w \, dx.$$  

We choose first large $\nu$ and then small $\mu$ to absorb the last expression, which is finite, into the left-hand side of (3.10). This shows how to choose $\mu$ and now we take $\theta$ from Lemma 3.7. After that it only remains to use Lemma 3.8 in order to estimate the second term in (3.11) first taking care of adjusting $\gamma = \gamma(d, \delta)$ to fit both Lemmas 3.7 and 3.8. The theorem is proved.

**Remark 3.14** Scalings show that the only constant $N$ in (3.10) depending on $R_0$ is the one in front of the integral of $|u|^\nu w$. This one equals $N(d, \delta, K_F, p, K_0) R_0^{-2p}$.

**Lemma 3.15** Let $u \in W^2_{d, \text{loc}}(\mathbb{R}^d)$ be bounded and let $p > d$. Then for any $S_\delta$-valued function $a$ on $\mathbb{R}^d$

$$|u|^p \leq N(\delta, d, p) \mathbb{M}(|a^{ij} D_{ij} u - u|^p).$$

In particular, under Assumption 3.2

$$|u|^p \leq N(\delta, d, p) \mathbb{M}(|F(u) - u|^p).$$

**Proof** First observe that the second estimate follows from the first one since $F(u) = a^{ij} D_{ij} u$, where $(a^{ij})$ is an appropriate $S_\delta$-valued function. To prove the first estimate, let $G(x, y)$ be a Green’s function of $L := a^{ij} D_{ij} - 1$ in $\mathbb{R}^{d+1}$ and $f = -Lu$. Then we have

$$u(0) = \int_{\mathbb{R}^d} G(0, y) f(y) \, dy.$$  

Hence,

$$|u(0)| \leq \int_{\mathbb{R}^d} G(0, y) |f(y)| \, dy.$$  

We are going to use the following estimate easily obtained, say, by probabilistic arguments: for any $\beta \geq 0$

$$\int_{\mathbb{R}^d} G(0, y) |y|^{\beta} \, dy \leq N(\alpha, d, \delta).$$  \hspace{1cm} (3.12)
Observe that for any \( h(y) \geq 0 \) and \( \alpha > 0 \)
\[
\int_{1}^{\infty} r^{-\alpha - 1} \left( \int_{B_r} h(y) (|y|^\alpha \vee 1) \, dy \right) \, dr
= \int_{\mathbb{R}^d} h(y) (|y|^\alpha \vee 1) \left( \int_{|y| \leq 1} r^{-\alpha - 1} \, dr \right) \, dy = \frac{1}{\alpha} \int_{\mathbb{R}^d} h \, dy.
\]
Furthermore, by using (3.12), Hölder’s inequality, and the Aleksandrov estimate, for \( q = p/d > 1 \), we get
\[
\int_{B_r} G(0, y) |f(y)| (|y|^\alpha \vee 1) \, dy \leq N \left( \int_{B_r} G(0, y) |f(y)|^q \, dy \right)^{1/q}
\leq N \left( \int_{B_r} |f(y)|^p \, dy \right)^{1/p} \leq N r^{d/p} (\mathbb{M}(|f|^p)(0))^{1/p}.
\]
For \( d/p - \alpha - 1 < -1 \) we get the desired result by integrating in \( r \in [1, \infty) \) and collecting the above estimates. The lemma is proved. \( \Box \)

Thanks to the properties of \( A_p \)-weights mentioned at the beginning of the proof of Theorem 3.13, we have the following.

**Corollary 3.16** Under the assumption of Lemma 3.15 take \( K_0 \in (1, \infty) \) and let \( w \) be an \( A_{p/d} \)-weight with \( [w]_{p/d} \leq K_0 \). Then there exists a constant \( N = N(\delta, d, p, K_0) \) such that
\[
\int_{\mathbb{R}^d} |u|^p w \, dx \leq N \int_{\mathbb{R}^d} |a^{ij} D_{ij} u - u|^p w \, dx,
\]
in particular, under Assumption 3.2
\[
\int_{\mathbb{R}^d} |u|^p w \, dx \leq N \int_{\mathbb{R}^d} |F[u] - u|^p w \, dx.
\]

Here is a generalization for fully nonlinear operators with almost VMO dependence on \( x \) considered in Sobolev spaces with \( A_p \)-weights of the classical Sobolev estimates, which are known for linear operators with continuous coefficients.

**Theorem 3.17** Take \( K_0 \in (1, \infty) \). Let \( p > d \) and let \( w \in A_{p/d}(\mathbb{R}^d) \) with \( [w]_{p/d} \leq K_0 \). Let \( u \in W_{p,w}^2(\mathbb{R}^d) \).

Then there exists \( \theta = \theta(d, \delta, K_F, p, K_0) \in (0, 1) \) such that, if Assumption 3.1 (\( \theta \)) is satisfied with \( \tau_0 = 0 \), then
\[
\int_{\mathbb{R}^d} (|D^2 u|^p + |Du|^p) w \, dx \leq N \int_{\mathbb{R}^d} |F[u]|^p w \, dx + N \int_{\mathbb{R}^d} |u|^p w \, dx,
\]
and if, in addition, \( u \) is bounded and Assumption 3.2 is satisfied then
\[
\int_{\mathbb{R}^d} (|D^2 u|^p + |Du|^p + |u|^p) w \, dx \leq N \int_{\mathbb{R}^d} |F[u] - u|^p w \, dx,
\]
where the constants \( N \) depend only on \( d, \delta, K_F, K_0, p, \) and \( R_0 \).

**Proof** To prove (3.15), thanks to Lemma 3.8 (iii), it suffices to estimate \( |D^2 u| \).

To this end introduce \( \zeta \in C_0^\infty(\mathbb{R}^d) \) such that \( \zeta(0) = 1 \) and plug \( u_n := u \zeta_n \), where \( \zeta_n(x) = \zeta(x/n) \), into (3.10) (just in case remember that (3.10) is proved for functions with
compact support). Then the result follows by the dominated convergence theorem from the fact that
\[
|F[u] - F[u_n]| \leq K_F|D^2u - D^2u_n| \\
\leq N|1 - \xi_n| |D^2u| + Nn^{-1}|Du| + Nn^{-2}|u|.
\]

To prove (3.16) it suffices to use Corollary 3.16. The theorem is proved. □

**Theorem 3.18** Let \( p_i > d, i = 1, 2, \ldots, d \). Assume that \( u \in W^2_{1,\text{loc}}(\mathbb{R}^d) \) and Assumption 3.2 is satisfied. Then there exists \( \theta = \theta(d, \delta, d, p_1, \ldots, p_d) \in (0, 1) \) such that, if Assumptions 3.1 (\( \theta \)) and 3.2 are satisfied with \( \tau_0 = 0 \), then
\[
\|D^2u, Du, u\|_{L^{p_1,\ldots,p_d}(\mathbb{R}^d)} \leq N\|F[u] - u\|_{L^{p_1,\ldots,p_d}(\mathbb{R}^d)}
\]
provided that the left-hand side is finite, where
\[
\|f\|_{L^{p_1,\ldots,p_d}(\mathbb{R}^d)}^{p_d} := \int_{\mathbb{R}^d} \left( \cdots \left( \int_{\mathbb{R}} \left| f \right|^{p_1} \, dx_1 \right)^{p_2/p_1} \, dx_2 \right)^{p_3/p_2} \cdots dx_d,
\]
and the constant \( N \) depends only on \( d, \delta, p_1, \ldots, p_d \), and \( R_0 \).

**Proof** First we assume that \( u \) is smooth and has compact support. Then (3.17) follows from Theorems 3.17 and 7.11.

If \( u \) is just smooth (and the left-hand side of (3.17) is finite), one can use the same approximation of \( u \) as in the proof of (3.15).

Finally, in the general case introduce \( u^{(\varepsilon)} \) as the mollified \( u \). By the Minkowski inequality (the norm of a sum is less than the sum of norms) the left-hand side of (3.17) with \( u^{(\varepsilon)} \) in place of \( u \) is less than its original. After that writing (3.17) with \( u^{(\varepsilon)} \) in place of \( u \), using the Lipschitz continuity of \( F(u'', x) \) with respect to \( u'' \), noting \( D^2u^{(\varepsilon)} \to D^2u \) in the above mixed norm (see, for instance, [1]), and letting \( \varepsilon \downarrow 0 \), we easily finish the proof. The theorem is proved. □

Sometimes in the sequel we consider \( F \)’s that are positive homogeneous in \( u'' \). In that case we impose the following.

**Assumption 3.19 (\( \theta \))**

(i) The function \( F \) is Lipschitz continuous with respect to \( u'' \) with Lipschitz constant \( K_F \) and is positive homogeneous of degree one with respect to \( u'' \).
There exists \( R_0 \in (0, 1) \) such that, if \( r \in (0, R_0] \) and \( z \in \mathbb{R}^d \), then one can find a convex function \( \bar{F}(u'') = \bar{F}_z,r(u'') \) (independent of \( x \)) positive homogeneous of degree one, for which

(ii) We have \( D_{u''} \bar{F} \in S_\delta \) at all points of differentiability of \( \bar{F} \);
(iii) For any \( u'' \in S \) with \( |u''| = 1 \), we have
\[
\int_{B_r(z)} |F(u'', x) - \bar{F}(u'')| \, dx \leq \theta |B_r(z)|.
\]

**Remark 3.20** It is worth noting that if \( F \) is positive homogeneous of degree one with respect to \( u'' \) and satisfies Assumption 3.1, then it also satisfies Assumption 3.19. Indeed, let \( \bar{F} \) be the function from Assumption 3.1. Then the function \( \lim_{\lambda \to \infty} \lambda^{-1} \bar{F}(\lambda u'') \) is convex, positive homogeneous of degree one, and satisfies Assumption 3.19 (ii) and (iii).

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**Remark 3.21** A typical example of $F$ satisfying Assumption 3.19 is obtained from Example 3.6 by taking there $f^{a\beta} \equiv 0$ and dropping the inf in (3.2).

Sometimes the following result is useful.

**Lemma 3.22** Take $R \in (0, \infty)$, $K_0 \in (1, \infty)$, $p > d$ and let $w \in A_{p/d}(\mathbb{R}^d)$ with $[w]_{p/d} \leq K_0$. Suppose that

$$u \in W_{p,w}^2(B_R).$$

(3.19)

Then there exists $\theta = \theta(d, \delta, K_F, p, K_0) \in (0, 1)$ such that, if Assumptions 3.19 ($\theta$) and 3.2 are satisfied, then for any $r \in (0, R)$

$$\int_{B_r} |D^2 u|^p w \, dx \leq N \int_{B_R} |F[u]|^p w \, dx$$

$$+ N \int_{B_R} ((R - r)^{-1} |Du| + ((R - r)^{-2} + 1)|u|)^p w \, dx,$$

where $N$ is a constant depending only on $d$, $\delta$, $K_F$, $K_0$, $p$, and $R_0$.

**Proof** Take a nonnegative $\zeta \in C_0^\infty(B_R)$ such that $\zeta = 1$ on $B_r$ and $|D\zeta| \leq N(R - r)^{-1}$, $|D^2 \zeta| \leq N(R - r)^{-2}$. It follows from (3.19) that $\zeta u \in W_{p,w}^2(B_R)$. Then apply Theorem 3.13 to $\zeta u$ after observing that due to the homogeneity of $F$ we have

$$|\zeta F[u] - F[\zeta u]| \leq N(|D\zeta| |Du| + |D^2 \zeta||u|).$$

This yields the result. \(\Box\)

The following result and Lemma 3.15 easily imply Theorem 3.17 once more, however in Theorem 3.17 we do not assume that $F$ is positive homogeneous.

**Lemma 3.23** Let the assumptions of Lemma 3.22 be satisfied and take $\theta$ from that lemma. Then there is a constant $N$ depending only on $d$, $\delta$, $K_F$, $K_0$, $p$, and $R_0$, such that for any $r \in [R - 1, R)$, $r > 0$,

$$\int_{B_r} |D^2 u|^p w \, dx \leq NF + N(R - r)^{-2}p U,$$

(3.20)

$$\int_{B_r} |Du|^p w \, dx \leq N(R - r)^p F + N(R - r)^{-p} U,$$

(3.21)

where

$$F = \int_{B_R} |F[u]|^p w \, dx, \quad U = \int_{B_R} |u|^p w \, dx.$$

**Proof** For $k = 0, 1, \ldots$ set $\rho_k = R - 2^{-k}(R - r)$, $B^k = B_{\rho_k}$ and find $\xi_k \in C_0^\infty(B^{k+1})$ such that $\xi_k = 1$ on $B^k$ and $|D\xi_k| \leq N(R - r)^{-1}2^k$, $|D^2 \xi_k| \leq N(R - r)^{-2}2^k$, where $N = N(d)$. By Lemma 3.22 we have

$$D''_k := \int_{B^k} |D^2 u|^p w \, dx \leq NF + N(R - r)^{-p}2^{kp}D'_{k+1} + N(R - r)^{-2}2^{2kp} U,$$

where

$$D'_k = \int_{B^k} |Du|^p w \, dx.$$
By Lemma 3.11 for $\varepsilon \in (0, 1)$
\[
D'_k \leq \varepsilon 2^{-kp} (R - r)^p \, D''_{k+1} + N \varepsilon^{-1} (R - r)^{-p} 2^{kp} U. \tag{3.22}
\]
It follows that
\[
D''_k \leq N F + \varepsilon N_1 D''_{k+2} + N \varepsilon^{-1} (R - r)^{-2p} 2^{kp} U .
\]
We choose $\varepsilon$ so that $\varepsilon N_1 \leq 2^{-6p}$, multiply this inequality by $2^{-3kp}$ and sum up with respect to even $k$ from 0 to $\infty$. Then we cancel like terms (which are finite since $D^2 u \in L_{p,w}(B_R)$) and come to (3.20).

After that (3.21) follows from (3.20) and (3.22). The lemma is proved.

We choose $\varepsilon > 0$ for small $\varepsilon$, then $\varepsilon \in (0, 1)$ and $\varepsilon \in (0, \Lambda_0)$, where the constants $\Lambda_0$ depend only on $r, R, d, \Lambda_1$.

By substituting $B_r(x_0)$ and $B_R(x_0)$ in place of $B_r$ and $B_R$, respectively, then taking $R = 2r = 1$ and integrating with respect to $x_0$ over $\mathbb{R}^d$ we obtain the following.

**Theorem 3.24** Theorem 3.17 remains true if condition (3.14) is replaced with $u \in W_{p,w,loc}(\mathbb{R}^d)$ provided, additionally, that $F$ is positive homogeneous of degree one with respect to $u''$. In particular, if $u$ is bounded and the right-hand side of (3.16) is finite, then $u \in W_{p,w}(\mathbb{R}^d)$.

**Remark 3.25** Generally, (3.16) may fail if $u$ is unbounded. Indeed, if $d = 1$ and $F[u] = u''$, the function $\varepsilon^u$ satisfies $F[u] - u = 0$ and is nonzero.

**Remark 3.26** Condition (3.19) is not well suited for application of the extrapolation theorem of Rubio de Francia [24]. In this connection it is useful to know that, for any $K_0, \, p \in (1, \infty)$ and $w \in A_p(\mathbb{R}^d)$ with $[w]_p \leq K_0$ there exists $q = q(d, K_0, p) \in (1, \infty)$ such that $W^2_q(B_R) \subset W^2_{p,w}(B_R)$ for any $R < \infty$. This follows from the fact that (see, for instance, Corollary 9.2.4 of [13]) $w$ is in $L_{r,loc}(\mathbb{R}^d)$ for an appropriate $r > 1$ depending only on $d, p$, and $K_0$.

**Theorem 3.27** Take $R \in (0, \infty), \, r \in (0, R), \, p > d$ and take $p_i > d, \, i = 1, 2, \ldots, d$. Assume that $u \in W^2_p(B_R)$. Then there exists \[ \theta = \theta(d, \delta, p, p_1, \ldots, p_d) \in (0, 1) \] such that, if Assumptions 3.19 $(\theta)$ and 3.2 are satisfied, then
\[
\| IB_r \, D^2 u, \, IB_r \, Du \|_{L_{p_1,\ldots,p_d}(\mathbb{R}^d)} \leq N \| IB_r \, F[u] \|_{L_{p_1,\ldots,p_d}(\mathbb{R}^d)} + N \| IB_r u \|_{L_{p_1,\ldots,p_d}(\mathbb{R}^d)}, \tag{3.23}
\]
where the constants $N$ depend only on $r, R, d, \delta, p_1, \ldots, p_d$, and $R_0$.

**Proof** In Theorem 7.11 take $m = d, \, K_0 = 1, \, k(1) = \ldots = k(d) = 1$ and take $\Lambda_0$ from there which now depends only on $d$ and $p_1, \ldots, p_d$. Then take $q = q(d, \Lambda_0, p_1)$ from Remark 3.26 and assume that $u \in W^2_q(B_R)$. In that case in light of Remark 3.26 estimate (3.23) follows from Lemma 3.23 and Theorems 7.11.

In the general case, we may assume that the right-hand side of (3.23) is finite and introduce $f = F[u]$ and $f(\varepsilon)$ as the mollified $f IB_R$. By Minkowski’s inequality (the norm of a sum is less than the sum of norms) the above mixed norm of $f(\varepsilon)$ is less than that of $f IB_R$. Then for small $\varepsilon > 0$ define smooth $u_\varepsilon$ so that they converge to $u$ uniformly on $\partial B_R$ and define $u_\varepsilon$ as unique $W^2_p(B_R)$-solutions of $F[u_\varepsilon] = f(\varepsilon)$ in $B_R$ with boundary condition $u_\varepsilon = u_\varepsilon$ on $\partial B_R$. Such solutions exist and belongs to $W^2_q(B_R)$ thanks to Theorem 2.1 of [17] (provided that an appropriate choice of $\theta$ is made).
Owing to the Aleksandrov estimate, \( u^\varepsilon \to u \) uniformly on \( B_R \) as \( \varepsilon \downarrow 0 \). In light of (3.23) the mixed norms of \( D^2 u^\varepsilon \) and \( Du^\varepsilon \) are bounded and since \( u^\varepsilon \to u \), they weakly converge in the space with mixed norm to \( D^2 u \) and \( Du \). The norm of the weak limit is less than the limit of norms and this proves (3.23) and the theorem. \( \square \)

4 Elliptic equations in half spaces: first approach

Here we consider elliptic equations in the half-space

\[
\mathbb{R}_+^d := \left\{ x = (x_1, x') : x_1 \geq 0, x' \in \mathbb{R}^{d-1} \right\}
\]

without boundary conditions, and prove estimates near the boundary with \( A_p \)-weights on \( \mathbb{R}_+^d \). A typical and probably the most interesting example of \( A_p \)-weights on \( \mathbb{R}_+^d \) is the distance to the boundary to some power, i.e., \( w(x) = x_1^q \). It is easy to see that \( w \in A_p(\mathbb{R}_+^d) \) (that is, \( w \) is an \( A_p \)-weight on \( \mathbb{R}_+^d \)) if and only if \( q \in (-1, p - 1) \). The way to build our estimates is taken from [20].

Our underlying \( \Omega \) is \( \mathbb{R}_+^d \) and \( C_n \) are the cubes from Sect. 3 only lying in \( \mathbb{R}_+^d \). Naturally, \( L \) is the set of continuous functions on \( \mathbb{R}_+^d \) with compact support.

For \( n \in \mathbb{Z}, R > 0 \) introduce

\[
S_n = [2^{-n}, 2^{-n+1}] \times \mathbb{R}^{d-1}, \quad T_n = [2^{-n-1}, 2^{-n+2}] \times \mathbb{R}^{d-1}, \quad B_R^+ = B_R \cap \mathbb{R}_+^d.
\]

In this section we consider a function \( F(u'', x), u'' \in S, x \in \mathbb{R}^d \), that is positive homogeneous of degree one with respect to \( u'' \).

Lemma 4.1 Take \( K_0 \in (1, \infty), p > d \), and let \( w \in A_{p/d}(\mathbb{R}_+^d) \) with \( |w|_{p/d} \leq K_0 \). Let \( u \) be a bounded function on \( \mathbb{R}_+^d \) such that

\[
u \in W^{2, w}(\mathbb{R}_+^d).
\]

Then there exists \( \theta = \theta(d, \delta, p, K_0) \in (0, 1) \) such that if Assumptions 3.19 (\( \theta \)) and 3.2 are satisfied, then there is a constant \( N \), depending only on \( d, \delta, K_0, p, \) and \( R_0 \), such that for any \( n \in \mathbb{Z} \) and any \( \varepsilon \in (0, 1) \) we have

\[
\int_{S_n} |D^2 u|^p w \, dx \leq N \int_{T_n} |F[u] - u|^p w \, dx
\]

\[
+ N 2^{pn} \int_{T_n} |D u|^p w \, dx + N(2^{2pn} + 1) \int_{T_n} |u|^p w \, dx, \tag{4.1}
\]

\[
\int_{S_n} |D u|^p w \, dx \leq N \varepsilon 2^{-pn} \int_{T_n} |D^2 u|^p w \, dx
\]

\[
+ N \varepsilon \int_{T_n} |D u|^p w \, dx + N \varepsilon^{-1} 2^{pn} \int_{T_n} |u|^p w \, dx. \tag{4.2}
\]

Furthermore, for any \( \varepsilon \in (0, 1) \)

\[
\int_{\mathbb{R}^d, x_1 \geq 2} |D^2 u|^p w \, dx \leq N \int_{\mathbb{R}^d, x_1 \geq 1} |F[u] - u|^p w \, dx
\]

\[
+ N \int_{\mathbb{R}^d, x_1 \geq 1} |D u|^p w \, dx + N \int_{\mathbb{R}^d, x_1 \geq 1} |u|^p w \, dx, \tag{4.3}
\]

\( \square \) Springer
\[
\int_{\mathbb{R}^d, x_1 \geq 2} |Du|^p w \, dx \leq N \varepsilon \int_{\mathbb{R}^d, x_1 \geq 1} |D^2 u|^p w \, dx \\
+ N \varepsilon \int_{\mathbb{R}^d, x_1 \geq 1} |Du|^p w \, dx + N \varepsilon^{-1} \int_{\mathbb{R}^d, x_1 \geq 1} |u|^p w \, dx.
\]

\[(4.4)\]

**Proof** To prove (4.1) we use the fact that there is a nonnegative \( \zeta \in C^\infty_c(\mathbb{R}) \) such that \( \zeta = 1 \) on \([2^{-n}, 2^{-n+1}]\), \( \zeta = 0 \) outside \([2^{-n-1}, 2^{-n+2}]\) and \( 2^{-n} |\zeta'|, 2^{-2n} |\zeta''| \leq N \), where \( N \) is an absolute constant. Then we apply Theorem 3.17 to \( u \zeta \) and, after observing that, due to the positive homogeneity and the Lipschitz continuity of \( F \), we have

\[
|\zeta F[u] - F[\zeta u]| \leq N(|D\zeta| |Du| + |D^2\zeta| |u|),
\]

immediately arrive at (4.1). Of course, since we used a result in which \( w \) is an \( A_{p/d} \)-weight on \( \mathbb{R}^d \) rather than on \( \mathbb{R}^d_+ \), we first extend \( w \) in an even way across \( \{x_1 = 0\} \) with its \( A_{p/d} \)-constant controlled by \( K_0 \). To prove (4.2) we use the same substitution but into (3.7) and choose \( \rho^p = \varepsilon 2^{-p n} \). Similarly (4.3) and (4.4) are obtained. The lemma is proved. \( \square \)

**Theorem 4.2** Let \( q \in \mathbb{R} \). Under the assumptions of Lemma 4.1 and for \( \theta \) from that lemma, if Assumptions 3.19 (\( \theta \)) and 3.2 are satisfied, then

\[
\int_{\mathbb{R}^d_+} \hat{x}_1^q |\hat{x}_1 D^2 u|^p w \, dx + \int_{\mathbb{R}^d_+} \hat{x}_1^q |Du|^p w \, dx \leq N \int_{\mathbb{R}^d_+} \hat{x}_1^q |\hat{x}_1 (F[u] - u)|^p w \, dx + N \int_{\mathbb{R}^d_+} \hat{x}_1^q |\hat{x}_1^{-1} u|^p w \, dx,
\]

\[(4.5)\]

where \( \hat{x}_1 = \min\{x_1, 1\} \), provided that the left-hand side is finite, where the \( N \)'s depend only on \( d, \delta, K_0, p, q, \) and \( R_0 \).

**Proof** Multiply both parts of (4.1) by \( 2^{-qn} - p n \), sum up over \( n \geq 0 \), and use the fact that \( 2^{-qn} - p n \sim x_1^{q+p} \) on \( S_n \) and \( T_n \). Then we get

\[
\int_{\mathbb{R}^d, x_1 \leq 2} x_1^q |x_1 D^2 u|^p w \, dx \leq N \int_{\mathbb{R}^d, x_1 \leq 4} x_1^q |x_1 (F[u] - u)|^p w \, dx \\
+ N \int_{\mathbb{R}^d, x_1 \leq 4} x_1^q |Du|^p w \, dx + N \int_{\mathbb{R}^d, x_1 \leq 4} x_1^q |x_1^{-1} u|^p w \, dx.
\]

Multiplying (4.2) by \( 2^{-qn} \) and summing up yields

\[
\int_{\mathbb{R}^d_+, x_1 \leq 2} x_1^q |Du|^p w \, dx \leq N \varepsilon \int_{\mathbb{R}^d_+, x_1 \leq 4} x_1^q |x_1 D^2 u|^p w \, dx \\
+ N \varepsilon \int_{\mathbb{R}^d_+, x_1 \leq 4} x_1^q |Du|^p w \, dx + N \varepsilon^{-1} \int_{\mathbb{R}^d_+, x_1 \leq 4} x_1^q |x_1^{-1} u|^p w \, dx.
\]

By combining these estimates with (4.3) and (4.4) we see that for any \( \varepsilon \in (0, 1] \)

\[
\int_{\mathbb{R}^d_+} \hat{x}_1^q |\hat{x}_1 D^2 u|^p w \, dx \leq N \int_{\mathbb{R}^d_+} \hat{x}_1^q |\hat{x}_1 (F[u] - u)|^p w \, dx \\
+ N \int_{\mathbb{R}^d_+} \hat{x}_1^q |Du|^p w \, dx + N \int_{\mathbb{R}^d_+} \hat{x}_1^q |\hat{x}_1^{-1} u|^p w \, dx,
\]
\[
\int_{\mathbb{R}^d_+} \hat{x}^q |Du|^p w \, dx \leq N \varepsilon \int_{\mathbb{R}^d_+} \hat{x}^q_1 |\hat{x}^q_1 D^2u|^p w \, dx \\
+N \varepsilon \int_{\mathbb{R}^d_+} \hat{x}^q_1 |Du|^p w \, dx + N \varepsilon^{-1} \int_{\mathbb{R}^d_+} \hat{x}^q_1 |\hat{x}^q_1 u|^p w \, dx.
\]

By choosing \( \varepsilon \) in an obvious way, we arrive at (4.5). The theorem is proved. \( \square \)

The next theorem follows from Theorems 4.2 and 7.11.

**Theorem 4.3** Let \( p_1, p_2 > d, q \in \mathbb{R} \), and let \( u \in C^\infty_0(\mathbb{R}^d_+) \) have (closed) support in \( \{x_1 > 0\} \). Then there exists \( \theta = \theta(d, \delta, q, p_1, p_2) \in (0, 1) \) such that if Assumptions 3.19 (\( \theta \)) and 3.2 are satisfied, then there is a constant \( N \), depending only on \( d, \delta, q, p_1, p_2, \) and \( R_0 \), such that

\[
\int_0^\infty \hat{x}^q_1 \left( \int_{\mathbb{R}^d_{d-1}} \left[ |\hat{x}^q_1 D^2u| + |Du|^{p_1} \right] \, dx \right)^{p_2/p_1} \, dx_1 \leq N \int_0^\infty \hat{x}^q_1 \left( \int_{\mathbb{R}^d_{d-1}} |\hat{x}^q_1 (F[u] - u)|^{p_1} \, dx \right)^{p_2/p_1} \, dx_1 \\
+N \int_0^\infty \hat{x}^q_1 \left( \int_{\mathbb{R}^d_{d-1}} |\hat{x}^q_1 u|^{p_1} \, dx \right)^{p_2/p_1} \, dx_1.
\]

The reader understands that similar estimate holds for mixed norms when we integrate with respect to \( x_1 \) first.

**Remark 4.4** Introduce a Banach space of functions on \( \mathbb{R}^d_+ \) having finite norm defined by

\[
\|u\|^{p_2} = \int_0^\infty \hat{x}^q_1 \left( \int_{\mathbb{R}^d_{d-1}} \left[ |\hat{x}^q_1 D^2u| + |Du| + \hat{x}^q_1 |u| \right]^{p_1} \, dx \right)^{p_2/p_1} \, dx_1.
\]

It turns out that the set of \( u \in C^\infty_0(\mathbb{R}^d_+) \) that have (closed) support lying in \( \{x_1 > 0\} \) is everywhere dense in this space, so that estimate (4.6) automatically extends to all functions in this space.

To prove this, first take a smooth function \( \eta(r) \) such that \( \eta(r) = 0 \) for \( r < -1 \) and \( \eta(r) = 1 \) for \( r > 0 \) introduce \( \eta_k(x) = \eta(k^{-1} \ln x_1) \), \( u_k = u \eta_k \) and by using the dominated convergence theorem prove that, if \( \|u\| < \infty \), then \( \|u - u_k\| \to 0 \) as \( k \to \infty \). After that it only remains to apply usual tools to approximate \( u_k \) by smooth functions which have (closed) support lying in \( \{x_1 > 0\} \).

In the next section we show that for some values of \( q \) it is possible to eliminate the last term in (4.6).

### 5 Elliptic equations in half spaces: second approach

We use the setting and the notation from the beginning of Sect. 4 and in this section we deal with a function \( F(u'', x) \) given for \( u'' \in S \) and \( x \in \mathbb{R}^d_+ \) and satisfying one of the following assumptions before which we introduce

\[
B^+_{r}(x) = B_r(x) \cap \mathbb{R}^d_+.
\]

**Assumption 5.1** (\( \theta \)) Assumption 3.1 (\( \theta \)) is satisfied if we replace there \( \mathbb{R}^d \) and \( B_r(z) \) with \( \mathbb{R}^d_+ \) and \( B^+_{r}(z) \), respectively.
Assumption 5.2 (θ) Assumption 3.19 (θ) is satisfied if we replace \( \mathbb{R}^d \) and \( B_r(z) \) with \( \mathbb{R}^d_+ \) and \( B_r^+(z) \), respectively.

Similarly, we introduce
\[
h_{y,p}(x), \quad M_p h(x), \quad \text{and } M h(x)
\]
on \( \mathbb{R}^d_+ \) (by taking \( B_r^+(y) \subset \mathbb{R}^{d+1}_+, y \in \mathbb{R}^d_+ \)).

From Lemma 4.2 of [17] and the proof of Lemma 5.2 of [17], we can easily obtain a boundary analog of Lemma 3.7. This together with a boundary analog of Lemma 3.8 allows us to apply Corollary 2.10 and yields the following boundary estimate corresponding to Theorem 3.13 above.

Theorem 5.3 Take \( R \in (0,\infty) \) and \( K_0 \in (1,\infty) \). Let \( p > d \) and let \( w \) be an \( A_{p/d} \)-weight on \( \mathbb{R}^d_+ \) with \( [w]_{p/d} \leq K_0 \). Suppose that \( D^2 u \in L_{p,w}(\mathbb{R}^d_+) \) and \( u \) vanishes on \( \{x_1 = 0\} \) and on \( \mathbb{R}^d_+ \setminus B_R \). Then there exists \( \theta = \theta(d, \delta, K_F, p, K_0) \in (0,1) \) such that if Assumptions 5.1 (θ) and 3.2 are satisfied, then
\[
\int_{\mathbb{R}^d_+} |D^2 u|^p w \, dx \leq N \int_{\mathbb{R}^d_+} |F[u]|^p w \, dx \\
+ N \int_{\mathbb{R}^d_+} |u|^p w \, dx + N \tau_0^p \int_{\mathbb{R}^d_+} I_{B^+_R} w \, dx,
\]
where \( N \) is a constant depending only on \( d, \delta, K_F, K_0, p, \) and \( R_0 \).

Theorem 5.4 Take \( K_0 \in (1,\infty) \), \( p > d \), and let \( w \) be an \( A_{p/d} \)-weight on \( \mathbb{R}^d_+ \) with \( [w]_{p/d} \leq K_0 \). Let
\[
u \in W^2_{p,w}(\mathbb{R}^d_+)
\]
and \( u = 0 \) on \( \{x_1 = 0\} \). Then there exists \( \theta = \theta(d, \delta, K_F, p, K_0) \in (0,1) \) such that if Assumptions 5.1 (θ) and 3.2 are satisfied with \( \tau_0 = 0 \), then
\[
\int_{\mathbb{R}^d_+} (|D^2 u|^p + |Du|^p) w \, dx \leq N \int_{\mathbb{R}^d_+} |F[u]|^p w \, dx + N \int_{\mathbb{R}^d_+} |u|^p w \, dx,
\]
and if in addition \( u \) is bounded and \( F \) satisfies Assumption 3.2 then
\[
\int_{\mathbb{R}^d_+} (|D^2 u|^p + |Du|^p + |u|^p) w \, dx \leq N \int_{\mathbb{R}^d_+} |F[u] - u|^p w \, dx,
\]
where the constants \( N \) depend only on \( d, \delta, K_F, K_0, p, \) and \( R_0 \).

Proof Lemma 3.8 has a natural half space analog and as in the case of (3.15) it suffices to estimate \( |D^2 u| \). We prove this estimate in the same way as in the case of (3.15) by taking the same function \( u_h \), but substituting it into (5.1) instead of (3.10).

To prove (5.2), it suffices to apply (3.13) to the odd extension of \( u \) and the even extension of \( w \) across \( \{x_1 = 0\} \) and use the fact that so extended \( w \) is in \( A_{p/d}(\mathbb{R}^d) \) with its \( A_{p/d} \)-constant controlled by \( K_0 \). The theorem is proved.

The following theorem is proved in the same way as Theorem 4.3, by taking into account that \( \tilde{\chi}_1^q \) are \( A_p \)-weights on \( \mathbb{R}^d_+ \) for \( q \in (-1, p - 1) \).

Theorem 5.5 Let \( p_1, p_2 > d \), \( q \in (-1, p_2/d - 1) \), and let \( u \in C^{1,1}(\mathbb{R}^d_+) \) have bounded support and \( u = 0 \) on \( \{x_1 = 0\} \). Let \( F \) satisfy Assumption 3.2. Then there exists \( \theta = \)
\( \theta(d, \delta, q, p_1, p_2) \in (0, 1) \) such that if Assumption 3.1 (\( \theta \)) is satisfied with \( \tau_0 = 0 \), then there is a constant \( N \), depending only on \( d, \delta, q, p_1, p_2, \) and \( R_0 \), such that

\[
\int_0^\infty x_1^q \left( \int_{\mathbb{R}^{d-1}} \left[ |D^2 u| + |Du| + |u| \right]^{p_1} dx' \right)^{p_2/p_1} \, dx_1 \leq N \int_0^\infty x_1^q \left( \int_{\mathbb{R}^{d-1}} |F[u] - u|^{p_1} dx' \right)^{p_2/p_1} \, dx_1. \tag{5.3}
\]

**Remark 5.6** Estimate (5.3) also holds with \( x_1 \) in place of \( \hat{x}_1 \). In such a situation assume that \( F(u'', x) \) is independent of \( x \) and is positive homogeneous of degree one with respect to \( u'' \). Then scalings: \( x \to cx \), immediately leads to

\[
\int_0^\infty x_1^q \left( \int_{\mathbb{R}^{d-1}} |D^2 u|^{p_1} dx' \right)^{p_2/p_1} \, dx_1 \leq N \int_0^\infty x_1^q \left( \int_{\mathbb{R}^{d-1}} |F[u]|^{p_1} dx' \right)^{p_2/p_1} \, dx_1
\]

for any \( q \in (-1, p_2/d - 1) \) and \( u \in C^{1,1}(\mathbb{R}^d) \) with bounded support vanishing on \( \{ x_1 = 0 \} \).

As before, by using a localization argument, we obtain the following estimate.

**Theorem 5.7** Take \( x_0 \in \mathbb{R}^d, R \in (0, \infty), K_0 \in (1, \infty), p > d \) and let \( w \) be an \( A_{p/d} \)-weight with \( [w]_{p/d} \leq K_0 \). Suppose that \( D^2 u \in L_{p,w}(B_R^+(x_0)) \) and \( u \) vanishes on \( \{ x_1 = 0 \} \cap B_R^+(x_0) \) if this set is nonempty. Then there exists \( \theta = \theta(d, \delta, K_F, p, K_0) \in (0, 1) \) such that, if Assumptions 5.2 (\( \theta \)) and 3.2 are satisfied, then for any \( r \in (0, R) \)

\[
\int_{B_r^+(x_0)} |D^2 u|^{p_1} w \, dx \leq N \int_{B_R^+(x_0)} |F[u]|^{p_1} w \, dx + N \int_{B_R^+(x_0)} ((R - r)^{-1}|Du| + (R - r)^{-2} + 1)|u|^{p_1} w \, dx, \tag{5.4}
\]

where \( N \) is a constant depending only on \( d, \delta, K_F, K_0, p, \) and \( R_0 \).

The proofs of the next two theorems are obtained by closely following the proof of Theorem 3.27 (with the lemmas proceeding it) with one distinction that, since we do not have global solvability in Sobolev spaces for equations in \( B_R^+(x_0) \) if \( B_R^+(x_0) \not\subset \mathbb{R}^d_+ \), we take a smooth subdomain of \( B_R^+(x_0) \) containing \( B_R^+(x_0) \) and conduct the corresponding argument in the proof of Theorem 3.27 with this subdomain in place of \( B_R \).

**Theorem 5.8** Take \( R \in (0, \infty), r \in (0, R), p > d, p_1, p_2 > d, \) and \( u \in W^2_p(B_R^+) \). Suppose that \( u \) vanishes on \( \{ x_1 = 0 \} \). Finally, take \( q \in (-1, p_2/d - 1) \) and let \( F \) satisfy Assumption 3.2. Then there exists \( \theta = \theta(d, \delta, p, q, p_1) \in (0, 1) \) such that, if Assumption 5.2 (\( \theta \)) is satisfied, then

\[
\int_0^\infty x_1^q \left( \int_{\mathbb{R}^{d-1}} I_{B_r^+}(|D^2 u|^{p_1} + |Du|^{p_1}) \, dx' \right)^{p_2/p_1} \, dx_1 \leq N \int_0^\infty x_1^q \left( \int_{\mathbb{R}^{d-1}} I_{B_R^+} |F[u]|^{p_1} \, dx' \right)^{p_2/p_1} \, dx_1
\]

\[
+ N \int_0^\infty x_1^q \left( \int_{\mathbb{R}^{d-1}} I_{B_R^+} |u|^{p_1} \, dx' \right)^{p_2/p_1} \, dx_1,
\]

where the constants \( N \) depend only on \( r, R, d, \delta, p, p_1, p_2, q, \) and \( R_0 \).
Take \( x_0 \in \mathbb{R}^d \), \( R \in (0, \infty) \), \( r \in (0, R) \), \( p > d \) and take \( p_i > d \), \( i = 1, 2, \ldots, d \). Assume that \( u \in W^2_p(B^+_R(x_0)) \) and \( u \) vanishes on \( \{x_1 = 0\} \). Then there exists \( \theta = \theta(d, \delta, p, p_1, \ldots, p_d) \in (0, 1) \) such that, if Assumption 5.2 \((\theta)\) is satisfied and Assumption 3.2 is satisfied as well, then

\[
\| I_{B^+_R(x_0)} D^2 u, I_{B^+_R(x_0)} Du \|_{L_p,\ldots, p_d} \leq N \| I_{B^+_R(x_0)} F[u] \|_{L_p,\ldots, p_d},
\]

where the constants \( N \) depend only on \( r, R, d, \delta, p_1, \ldots, p_d, \) and \( R_0 \).

6 Parabolic case

We concentrate our attention here on

\[
\mathbb{R}^d = \{ (t, x) : t \geq 0, x \in \mathbb{R}^d \},
\]

and on functions defined on it.

For \( (t, x) \in \mathbb{R}^d \) introduce

\[
C_r(t, x) = [t, t + r^2] \times B_r(x), \quad C_r = C_r(0, 0).
\]

We consider a function \( F(u'', t, x), u'' \in \mathbb{S}, (t, x) \in \mathbb{R}^d \), on which we will impose some of the following assumptions.

**Assumption 6.1 \((\theta)\)** Assumption 3.1 \((\theta)\) is satisfied if we replace there \( x, \mathbb{R}^d, B_r(z) \) with \( (t, x), \mathbb{R}^d, C_r(z) \), respectively.

**Assumption 6.2** Assumption 3.2 is satisfied if we replace there \( F(\cdot, x) \) with \( F(\cdot, t, x) \).

**Assumption 6.3 \((\theta)\)** Assumption 3.19 \((\theta)\) is satisfied if we replace there \( x, \mathbb{R}^d, B_r(z) \) with \( (t, x), \mathbb{R}^d, C_r(z) \), respectively.

By using similar natural substitutions, we introduce

\[
h^2_{\nu, \rho}(t, x), \quad \mathbb{M}_\rho h(t, x), \quad \text{and} \quad \mathbb{M} h(t, x)
\]

on \( \mathbb{R}^d \) (taking only \( C_r(s, y) \subset \mathbb{R}^d, (s, y) \in \mathbb{R}^d \)).

Here we set \( \Omega = \mathbb{R}^d \) and for \( n \in \mathbb{Z} \) we take \( C_n \) as the collection of \( (t, x) + [0, 4^{-n}] \times [0, 2^{-n}] \), \( t \in 4^{-n}[0, 1], x \in 2^{-n} \mathbb{Z}^d \). We also set \( \mu \) to be Lebesgue measure on \( \mathbb{R}^d \) and \( \omega \) to be the set of continuous functions on \( \mathbb{R}^d \) with compact support. Then observe that relations (3.3) hold again for a constant \( c = c(d, \infty) \).

In what follows in this section by \( A_p \)-weights we mean weights on \( \mathbb{R}^d \) relative to the parabolic distance.

The following analog of Lemma 3.7 is an obvious corollary of Lemma 3.3 of [18].

**Lemma 6.4** Let \( u \in W^{1,2}_{p,\text{loc}}(\mathbb{R}^d), \mu (0, \infty), v \geq 2, \xi, \in (1, \infty). \) Then there exists \( \theta = \theta(d, \delta, K_F, \mu, \xi, \in (0, 1) \) such that, if Assumption 6.1 \((\theta)\) is satisfied, then one can find \( \gamma_0 = \gamma_0(d, \delta) \in (0, 1), \alpha = \alpha(d, \delta) \in (0, 1), \) such that, for \( \gamma \in (0, \gamma_0), h = D^2 u, \) and \( \rho = R_0 / \nu, \) we have

\[
h^2_{\nu, \rho} \leq N v^{(d+2)/\nu} M_{\nu, \rho}^{(d+2)/\nu} \left[ |\partial u + F[u]|^{d+1} \right] + N \tau_0 v^{(d+2)/\nu} + N (\mu v^{(d+2)/\nu} + v^{-\alpha}) M_{\nu, \rho}^{(d+2)/\nu} \left[ |h|^{d+1} \right],
\]

where \( \xi = \xi / (\xi - 1) \) and the constants \( N \) depend only on \( d, K_F, \) and \( \delta. \)
Here is a parabolic analog of Lemma 3.8.

**Lemma 6.5**

(i) There exists a constant $\gamma_0 = \gamma_0(d, \delta) \in (0, 1)$ such that under Assumption 6.2, for any $\gamma \in (0, \gamma_0)$, $\rho > 0$, and $u \in W^{1,2}_{d+1, \text{loc}}(\mathbb{R}^{d+1})$, we have

$$\mathbb{M}_{\rho}^{1/\gamma}(|D^2 u|^\gamma) \leq N \mathbb{M}_{\rho}^{1/(d+1)}(|\partial_t u + F[u]|^{d+1})$$

$$+ N \rho^{-1} \mathbb{M}_{\rho}^{1/(d+1)}(|Du|^{d+1}) + N \rho^{-2} \mathbb{M}_{\rho}^{1/(d+1)}(|u|^{d+1}),$$

where the constants $N$ depend only on $d$, $\delta$, and $K_F$.

(ii) For any $\rho > 0$, $p \in [1, \infty)$, and $u \in W^{1,2}_{p, \text{loc}}(\mathbb{R}^{d+1})$, we have

$$\mathbb{M}_{\rho}(|Du|^p) \leq N \mathbb{M}_{\rho}^{1/2}(|D^2 u|^p) \mathbb{M}_{\rho}^{1/2}(|u|^p) + N \rho^{-p} \mathbb{M}_{\rho}(|u|^p),$$

where the constants $N$ depend only on $d$ and $p$.

(iii) For any $\rho > 0$, $K_0$, $p \in (1, \infty)$, $w \in A_p$ with $[w]_p \leq K_0$, and $u \in W^{1,2}_{p, \text{loc}}(\mathbb{R}^{d+1})$, we have

$$\int_{\mathbb{R}^{d+1}_+} |Du|^p w \, dxdt \leq \rho^p \int_{\mathbb{R}^{d+1}_+} |D^2 u|^p w \, dxdt + N \rho^{-p} \int_{\mathbb{R}^{d+1}_+} |u|^p w \, dxdt,$$

where $N$ depends only on $d$, $p$, and $K_0$.

**Proof**

First write $F[u] = a^{ij} D_{ij} u$ and take $r \geq \rho$ and a function $\zeta \in C^\infty_0(\mathbb{R}^{d+1})$ such that $\zeta = 1$ on $C_r$, $\zeta = 0$ on $\partial^c C_{2r}$, and

$$|D\zeta| \leq N/r \leq N/\rho, \quad |\zeta| + |D^2 \zeta| \leq N/r^2 \leq N/\rho^2.$$

Then by Lemma 5.5 of [12]

$$\int_{C_r} |D^2 u|^\gamma \, dxdt \leq \int_{C_{2r}} |D^2(\zeta u)|^\gamma \, dxdt$$

$$\leq N \left( \int_{C_{2r}} |\partial_t(\zeta u) + \zeta F[u] + a^{ij} 2 D_t \zeta D_{ij} u + u a^{ij} (D_{ij} \zeta)|^{d+1} \, dxdt \right)^{\gamma/(d+1)}.$$

The rest is identical to the proof of Lemma 3.8. The lemma is proved.

**Theorem 6.6**

Take $R \in (0, \infty)$, $K_0 \in (1, \infty)$, $p > d + 1$ and let $w$ be an $A_{p/(d+1)}$-weight with $[w]_{p/(d+1)} \leq K_0$. Suppose that $D^2 u \in L_{p,w}(\mathbb{R}^{d+1})$, and that $u$ vanishes in $\mathbb{R}^{d+1} \setminus C_R$. Then there exists $\theta = \theta(d, \delta, K_F, p, K_0) \in (0, 1)$ such that, if Assumptions 6.1 ($\theta$) and 6.2 are satisfied, then

$$\int_{\mathbb{R}^{d+1}_+} |D^2 u|^p w \, dxdt \leq N \int_{\mathbb{R}^{d+1}_+} |\partial_t u + F[u]|^p w \, dxdt$$

$$+ N \int_{\mathbb{R}^{d+1}_+} |u|^p w \, dxdt + N \tau_0^p \int_{\mathbb{R}^{d+1}_+} I_{C_R+R_0} w \, dxdt,$$

where $N$ depends only on $d$, $\delta$, $K_F$, $K_0$, $p$, and $R_0$.

The proof of this theorem is practically the same as that of Theorem 3.13.

To prove a parabolic analog of Theorem 3.17 we need the following analog of Lemma 3.15.
Lemma 6.7 Let \( u \in W^{1,2}_{d+1, loc}(\mathbb{R}^{d+1}_+) \) be a bounded function and \( a = (a^{ij}(t, x)) \) be an \( S_{\delta} \)-valued function on \( \mathbb{R}^{d+1}_+ \). Also let \( p > d + 1 \). Then

\[
|u(0)| \leq N \left( M_\delta(|\partial_t u + a^{ij} D_{ij} u - u|^p)(0) \right)^{1/p},
\]

where \( N = N(d, \delta, p) \).

**Proof** Let \( G(x, t, x, y) \) be a Green’s function of \( L := \partial_t + a^{ij} D_{ij} - 1 \) in \( \mathbb{R}^{d+1}_+ \) and introduce \( f = -Lu \). Then for \( G(t, y) := G(0, t, 0, y) \) we have

\[
u(0) = \int_0^\infty \int_{\mathbb{R}_+^d} G(t, y) f(t, y) \, dy \, dt.
\]

Hence,

\[
|u(0)| \leq \int_0^\infty \int_{\mathbb{R}_+^d} G(t, y) |f(t, y)| \, dy \, dt.
\]

We are going to use the following estimate easily obtained, say, by probabilistic arguments: for any \( \alpha \geq 0 \)

\[
\int_{\mathbb{R}^{d+1}_+} G(t, y)(t^{2\alpha} + |y|^\alpha) \, dy \, dt \leq N(\alpha, d, \delta). \tag{6.1}
\]

Observe that for any \( h(t, y) \geq 0 \) and \( \alpha > 0 \)

\[
\int_1^\infty r^{-\alpha-1} \left( \int_{C_r} h(t^{2\alpha} \vee |y|^\alpha \vee 1) \, dy \right) \, dr
\]

\[
= \int_{\mathbb{R}_+^{d+1}} h(t^{2\alpha} \vee |y|^\alpha \vee 1) \left( \int_{t^2 \vee |y| \vee 1}^\infty r^{-\alpha-1} \, dr \right) \, dy \, dt = \frac{1}{\alpha} \int_{\mathbb{R}_+^{d+1}} h \, dy \, dt.
\]

Furthermore, by using (6.1), Hölder’s inequality, and the parabolic Aleksandrov estimate, for \( q = p/(d + 1) > 1 \), we get

\[
\int_{C_r} G(t, y)|f(t, y)|(t^{2\alpha} \vee |y|^\alpha \vee 1) \, dy \, dt \leq N \left( \int_{C_r} G(t, y)|f(t, y)|^q \, dy \, dt \right)^{1/q}
\]

\[
\leq N \left( \int_{C_r} |f(t, y)|^p \, dy \, dt \right)^{1/p} \leq Nr^{(d+2)/p} \left( M_\delta(|f|^p)(0) \right)^{1/p}.
\]

For \( (d + 2)/p - \alpha - 1 < -1 \), we get the desired result by integrating in \( r \in [1, \infty) \) and collecting the above estimates. The lemma is proved. \( \Box \)

Now by combining Theorem 6.6 and Lemmas 6.7 and 6.5 we get the following in the same way as Theorem 3.17.

**Theorem 6.8** Let \( \tau_0 = 0 \) and take \( K_0 \in (1, \infty) \). Let \( p > d + 1 \) and let \( w \) be an \( A_w \)-weight with \( [w]_{p/(d+1)} \leq K_0 \). Let

\[
u \in W^{1,2}_{p, w}(\mathbb{R}^{d+1}_+).
\]

Then there exists \( \theta = \theta(d, \delta, K_F, p, K_0) \in (0, 1) \) such that if Assumption 6.1 (\( \theta \)) is satisfied, then
\[ \int_{\mathbb{R}^{d+1}_+} (|D^2 u|^p + |Du|^p) w \, dx \, dt \]

\leq N \int_{\mathbb{R}^{d+1}_+} |\partial_t u + F[u]|^p w \, dx \, dt + N \int_{\mathbb{R}^{d+1}_+} |u|^p w \, dx \, dt, \]

and if, in addition, \( u \) is bounded and Assumption 6.2 is satisfied then

\[ \int_{\mathbb{R}^{d+1}_+} (|D^2 u|^p + |Du|^p + |u|^p) w \, dx \, dt \leq N \int_{\mathbb{R}^{d+1}_+} |\partial_t u + F[u] - u|^p w \, dx \, dt, \quad (6.3) \]

where the constants \( N \) depend only on \( d, \delta, K_F, K_0, p, \) and \( R_0 \).

To state an analog of Theorem 3.18 order the set of coordinates \((t, x) = (t, x_1, \ldots, x_d)\) arbitrarily as \((\tilde{x}_0, \ldots, \tilde{x}_d)\). Then we have the following result.

**Theorem 6.9** Let \( \tau_0 = 0 \) and take \( p_i > d+1, i = 0, 1, \ldots, d \). Assume that \( u \in W_{1, \text{loc}}^{1, 2}(\mathbb{R}^{d+1}_+) \) and Assumption 6.2 is satisfied. Then there exists \( \theta = \theta(d, \delta, p_0, \ldots, p_d) \in (0, 1) \) such that, if Assumptions 6.1 (\( \theta \)) and 6.2 are satisfied, then

\[ \|D^2 u, Du, u\|_{L_{p_0, \ldots, p_d}(\mathbb{R}^{d+1}_+)} \leq N\|\partial_t u + F[u] - u\|_{L_{p_0, \ldots, p_d}(\mathbb{R}^{d+1}_+)} \]

provided that the left-hand side is finite, where

\[ \|f\|_{L_{p_0, \ldots, p_d}(\mathbb{R}^{d+1}_+)}^{d} := \int_{\mathbb{R}} \left( \cdots \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f I_{p_d+1}^{0, d} d\tilde{x}_0|^{p_0/p_0} d\tilde{x}_1 \right)^{p_1/p_1} \right) \cdots \right) d\tilde{x}_d, \quad (6.4) \]

and the constant \( N \) depends only on \( d, \delta, p_1, \ldots, p_d, \) and \( R_0 \).

One proves this result in the same way as Theorem 3.18 taking care of defining the mollified functions \( u(t, x) \) by averaging the values of \( u \) with higher values of \( t \) in order not to bother about the fact that \( u \) may not be defined for negative \( t \).

Then one derives an obvious analogs of Lemmas 2.8 and 2.33 and, by using Theorem 1.9 of [18] (see also Remark 1.11 there) in place of Theorem 2.1 of [17], one arrives at an analog of Theorem 3.24.

**Theorem 6.10** Theorem 6.8 remains true if condition (6.2) is replaced with \( u \in \bigcap_{R > 0} W_{p, w}^{1, 2} (C_R) \) provided, additionally, that \( F \) is positive homogeneous of degree one with respect to \( u'' \).

In particular, if \( u \) is bounded and the right-hand side of (6.3) is finite, then \( u \in W_{p, w}^{1, 2} (\mathbb{R}^{d+1}_+) \).

**Remark 6.11** Generally, (6.3) may fail if \( u \) is unbounded. Indeed, if \( d = 1 \) and \( F[u] = u'' \), the function \( e^t \) satisfies \( \partial_t u + F[u] - u = 0 \) and is nonzero.

Then from an analog of Lemma 3.23 one derives the following analog of Theorem 3.27. The only difference in the proofs worth noting is that one should use the existence Theorem 1.9 of [18] in place of Theorem 2.1 of [17].

**Theorem 6.12** Take \( R \in (0, \infty), r \in (0, R), p > d + 1, \) and \( p_i > d + 1 \) for \( i = 0, 1, \ldots, d \). Assume that \( u \in W_{p, w}^{1, 2}(C_R) \). Then there exists \( \theta = \theta(d, \delta, p_0, \ldots, p_d) \in (0, 1) \) such that, if Assumptions 6.3 (\( \theta \)) and 6.2 are satisfied, then

\[ \|IC, D^2 u, IC, Du\|_{L_{p_0, \ldots, p_d}(\mathbb{R}^{d+1}_+)} \leq N\|IC_R F[u]\|_{L_{p_0, \ldots, p_d}(\mathbb{R}^{d+1}_+)} + N\|IC_R u\|_{L_{p_0, \ldots, p_d}(\mathbb{R}^{d+1}_+)}, \]

where the constants \( N \) depend only on \( r, R, d, \delta, p_0, \ldots, p_d, \) and \( R_0 \).
7 Parabolic case in a half-space

Here we consider functions on
\[ \mathbb{R}^{d+1}_{+,+} = \{(t, x) : t \geq 0, x_1 \geq 0, x' \in \mathbb{R}^{d-1}\}. \]

We concentrate on parabolic equations in \( \mathbb{R}^{d+1}_{+,+} \) with zero Dirichlet boundary condition and prove boundary estimates with \( A_p \)-weights.

For \( (t, x) \in \mathbb{R}^{d+1}_{+,+} \) and \( r > 0 \) denote
\[ C^+_r(t, x) = [t, t + r^2) \times B^+_r(x), \]
and consider a function \( F(u'', t, x) \) given for \( (t, x) \in \mathbb{R}^{d+1}_{+,+} \) and \( u'' \in \mathbb{S} \).

We use the following assumptions.

**Assumption 7.1** (\( \theta \)) Assumption 3.1 (\( \theta \)) is satisfied if we replace there \( x, \mathbb{R}^{d}, B_r(z) \) with \( (t, x), \mathbb{R}^{d+1}_{+,+}, C^+_r(z), \) respectively.

**Assumption 7.2** (\( \theta \)) Assumption 3.19 (\( \theta \)) is satisfied if we replace there \( x, \mathbb{R}^{d}, B_r(z) \) with \( (t, x), \mathbb{R}^{d+1}_{+,+}, C^+_r(z), \) respectively.

Accordingly, we introduce
\[ h_{*,p}^\varepsilon(t, x), \quad M_p h(t, x), \quad \text{and} \quad M h(t, x) \]
on \( \mathbb{R}^{d+1}_{+,+} \) (by taking \( C^+_r(s, y) \subset \mathbb{R}^{d+1}_{+,+}, (s, y) \in \mathbb{R}^{d+1}_{+,+} \)). Here the underlying set \( \Omega \) is taken to be \( \mathbb{R}^{d+1}_{+,+} \) and the \( C_n \)'s are the parts of the \( C_n \)'s from the beginning of Sect. 6 which belong to that \( \Omega \).

In what follows by \( A_p \)-weights we mean weights on \( \mathbb{R}^{d+1}_{+,+} \) relative to the parabolic distance.

From Lemma 4.1 of [18] and the proof of Lemma 3.3 of [18], we can easily obtain a boundary analog of Lemma 3.7. This together with a boundary analog of Lemma 3.8, by relying on Corollary 2.10, gives the following boundary estimate corresponding to Theorems 3.13 and 6.6.

**Theorem 7.3** Take \( R \in (0, \infty), K_0 \in (1, \infty) \). Let \( p > d + 1 \) and let \( w \) be an \( A_{p/(d+1)} \)-weight on \( \mathbb{R}^{d+1}_{+,+} \) with \( [w]_{p/(d+1)} \leq K_0 \). Suppose that
\[ D^2 u \in L_p(w(\mathbb{R}^{d+1}_{+,+})) \]
and \( u \) vanishes on \( \{x_1 = 0\} \) and on \( \mathbb{R}^{d+1}_{+,+} \setminus C^+_{K} \). Then there exists \( \theta = \theta(d, \delta, K_F, p, K_0) \in (0, 1) \) such that if Assumptions 7.1 (\( \theta \)) and 6.2 are satisfied, then
\[ \int_{\mathbb{R}^{d+1}_{+,+}} |D^2 u|^p w \, dx \, dt \leq N \int_{\mathbb{R}^{d+1}_{+,+}} |\partial_t u + F[u]|^p w \, dx \, dt + N^P \int_{\mathbb{R}^{d+1}_{+,+}} I_{C^+_{K}} w \, dx \, dt, \]
where \( N \) is a constant depending only on \( d, \delta, K_F, K_0, p, \) and \( R_0 \).

By taking into account what was said before Theorems 5.8 and 5.9 and using the solvability of \( \partial_t u + F[u] = f \) in smooth cylinders (see Theorem 1.9 and Remark 1.11 of [18]), we have the following boundary estimates in mixed-norm spaces.
Theorem 7.4 Take $R \in (0, \infty)$, $r \in (0, R)$, $p > d + 1$, $p_1, p_2 > d + 1$, and $u \in W^{1,2}_p(C^+_R)$. Suppose that $u$ vanishes on $\{x_1 = 0\}$. Finally, take $q \in (-1, p_1/(d + 1) - 1)$ and let $F$ satisfy Assumption 6.2. Then there exists $\theta = \theta(d, \delta, p, q) \in (0, 1)$ such that, if Assumption 7.2 ($\theta$) is satisfied, then

$$
\int_0^\infty \left( \int_{\mathbb{R}^d_+} I_{C^+_R} x_1^q (|D^2 u|^p_1 + |Du|^{p_1}) \, dx \right)^{p_2/p_1} \, dt \\
\leq N \int_0^\infty \left( \int_{\mathbb{R}^d_+} I_{C^+_R} x_1^q |\partial_t u + F[u]|^{p_1} \, dx \right)^{p_2/p_1} \, dt \\
+ N \int_0^\infty \left( \int_{\mathbb{R}^d_+} I_{C^+_R} x_1^q |u|^{p_1} \, dx \right)^{p_2/p_1} \, dt,
$$

where the constants $N$ depend only on $r$, $R$, $d$, $\delta$, $\theta$, $p$, $p_1$, $p_2$, $q$, and $R_0$.

Theorem 7.5 Take $(t_0, x_0) \in \mathbb{R}^{d+1}_{d+1}$, $R \in (0, \infty)$, $r \in (0, R)$, $p > d + 1$ and take $p_i > d + 1$, $i = 0, 1, \ldots, d$. Assume that $u \in W^{1,2}_p(C^+_R(t_0, x_0))$ and $u$ vanishes on $\{x_1 = 0\}$. Then there exists $\theta = \theta(d, \delta, p, p_0, \ldots, p_d) \in (0, 1)$ such that if Assumptions 7.2 ($\theta$) and 6.2 are satisfied, then (the mixed norms below are taken from (6.4))

$$
\| I_{C^+_R(t_0, x_0)} D^2 u \|_{L^{p_0, \ldots, p_d}(\mathbb{R}^{d+1})} \leq N \| I_{C^+_R(t_0, x_0)} (\partial_t u + F[u]) \|_{L^{p_0, \ldots, p_d}(\mathbb{R}^{d+1})} \\
+ N \| I_{C^+_R(t_0, x_0)} u \|_{L^{p_0, \ldots, p_d}(\mathbb{R}^{d+1})},
$$

where the constants $N$ depend only on $r$, $R$, $d$, $\delta$, $p$, $p_0, \ldots, p_d$, and $R_0$.

To further estimate the lower-order terms on the right-hand sides of the estimates above, we need the following fact.

By using the odd extension of $u$ and the even extension of $w$ across $\{x_1 = 0\}$ and using the fact that so extended $w$ is in $A_{p/(d+1)}(\mathbb{R}^{d+1})$ with its $A_{p/(d+1)}$-constant controlled by $K_0$, from Lemmas 6.7 we get the following corollary in which

$$
W^{1,2}_{d+1, \text{loc}}(\mathbb{R}^{d+1}_{d+1}) = \bigcap_{R > 0} W^{1,2}_{d+1}(C^+_R).
$$

Corollary 7.6 Let $K_0 \in (1, \infty)$, $p > d + 1$ and let $u \in W^{1,2}_{d+1, \text{loc}}(\mathbb{R}^{d+1}_{d+1})$ be a bounded function and $a$ be an $\mathcal{S}_d$-valued function on $\mathbb{R}^{d+1}_{d+1}$. Let $w \in A_{p/(d+1)}(\mathbb{R}^{d+1}_{d+1})$ with $|w|_{p/(d+1)} \leq K_0$ and let $u = 0$ for $x_1 = 0$. Then

$$
\int_{\mathbb{R}^{d+1}_{d+1}} |u|^p w \, dx dt \leq N \int_{\mathbb{R}^{d+1}_{d+1}} |\partial_i u + a^{ij} D_{ij} u - u|^p w \, dx dt,
$$

where $N = N(d, \delta, p, K_0)$. Springer
We are now ready to prove the following theorem.

**Theorem 7.7** Let $K_0 \in (1, \infty)$, $p > d + 1$, $w \in A_{p/(d+1)}$ on $\mathbb{R}^{d+1\times +}$ with $[w]_{p/(d+1)} \leq K_0$, and $u \in W^{1,2}_{p,w}(\mathbb{R}^{d+1\times +})$ vanishing on $\{x_1 = 0\}$. Let $F$ satisfy Assumption 6.2. Then there exists $\theta = \theta(d, \delta, p, K_0) \in (0, 1)$ such that if Assumption 7.2 (\(\theta\)) is satisfied, then

$$
\int_{\mathbb{R}^{d+1\times +}} (|D^2 u|^p + |Du|^p + |u|^p) w \, dx \, dt \leq NI,
$$

where

$$
I = \int_{\mathbb{R}^{d+1\times +}} |\partial_t u + F[u] - u|^p w \, dx \, dt
$$

and $N$ depends only on $d$, $\delta$, $K_0$, $p$, and $R_0$.

**Proof** Observe that the following is a parabolic analog of (5.4) for $\mathbb{R}^{d\times +}$:

$$
\int_{C^+_1(t_0, x_0)} |D^2 u|^p \, w \, dx \, dt \leq N \int_{C^+_2(t_0, x_0)} |\partial_t u + F[u]|^p \, w \, dx \, dt
$$

$$
+ N \int_{C^+_2(t_0, x_0)} (|Du| + |u|)^p \, w \, dx \, dt. \tag{7.1}
$$

The way to obtain it from Theorem 7.3 is described in the proof of Theorem 5.7 and could be easily mimicked in the parabolic setting.

By integrating both sides of (7.1) with respect to $(t_0, x_0) \in \mathbb{R}^{d+1\times +}$ we get

$$
\int_{\mathbb{R}^{d+1\times +}} |D^2 u|^p \, w \, dx \, dt \leq N \int_{\mathbb{R}^{d+1\times +}} |\partial_t u + F[u]|^p \, w \, dx \, dt
$$

$$
+ N \int_{\mathbb{R}^{d+1\times +}} (|Du| + |u|)^p \, w \, dx \, dt
$$

$$
\leq NI + N \int_{\mathbb{R}^{d+1\times +}} (|Du| + |u|)^p \, w \, dx \, dt.
$$

By using Corollary 7.6 and a boundary parabolic analog of Lemma 3.8 (iii), we arrive at

$$
\int_{\mathbb{R}^{d+1\times +}} (|D^2 u|^p + |Du|^p + |u|^p) \, w \, dx \, dt \leq N \rho^{-p} I + N \rho^p \int_{\mathbb{R}^{d+1\times +}} |D^2 u|^p \, w \, dx \, dt
$$

for any $\rho \in (0, 1)$. The desired estimate follows by taking $\rho$ sufficiently small. The theorem is proved. \(\square\)

Theorems 7.7 and 7.11 and the way Theorem 3.18 is derived immediately lead to the following.

**Theorem 7.8** Let $p_1, p_2, p_3 > d + 1$, and $u \in W^{1,2}_{1,loc}(\mathbb{R}^{d+1\times +})$. Suppose that $u$ vanishes on $\{x_1 = 0\}$. Finally, take $q \in (-1, 1/(d+1) - 1)$ and let $F$ satisfy Assumption 6.2. Then there exists $\theta = \theta(d, \delta, p_1, p_2, p_3, q) \in (0, 1)$ such that, if Assumption 7.2 (\(\theta\)) is satisfied, then
\[
\int_0^\infty \left( \int_{\mathbb{R}^{d-1}} \left( \int_0^\infty x_1^q \left[ |D^2 u| + |Du| + |u|^p_1 \right] dx_1 \right)^{p_2/p_1} dx' \right)^{p_3/p_2} dt \\
\leq N \int_0^\infty \left( \int_{\mathbb{R}^{d-1}} \left( \int_0^\infty x_1^q \left| \partial_1 u + F[u] - u \right|^{p_1} dx_1 \right)^{p_2/p_1} dx' \right)^{p_3/p_2} dt,
\]
provided that the left-hand side is finite, where \( N \) depends only on \( d, \delta, p_1, p_2, p_3, q, \) and \( R_0 \).

The one-dimensional example of \( F[u] = D^2 u \) and \( u(t, x) = \sin x \) shows that (7.2) is wrong without the additional assumption on its left-hand side.

**Remark 7.9** The reader understands that one has similar estimates for the integrals with respect to \( x_1, x' \), and \( t \) mixed in any other order.

**Remark 7.10** In [14] the authors consider linear \( F \) with coefficients depending only on time in a measurable way and prove a priori estimates similar to the one in Theorem 7.8, however, for any \( p_1 = p_2, p_3 > 1 \) and \( q \in (-1, 2p_1 - 1) \). The latter range is much wider than ours \((-1, p_1/(d + 1) - 1)\), but our operators are much more general and we have three integrals.

It is worth noting that the range \((p_1 - 1, 2p_1 - 1)\) was used in [15] to build the solvability theory of parabolic equations in Sobolev spaces with weights with the highest order of derivatives being an arbitrary given number: positive, negative, integral or fractional.

**Appendix**

Here we take \( \Omega = \Omega^1 \times \cdots \times \Omega^d \), where \( \Omega^j = \mathbb{R} \) or \( \mathbb{R}_+ \), \( j = 1, \ldots, d \) and let \( \mu \) to be the Lebesgue measure on \( \Omega \). We take integers \( 0 = l_0 < l_1 < \ldots < l_m = d \) and express points in \( \Omega \) as

\[
x = (x_1, \ldots, x_d) = (\tilde{x}_1, \ldots, \tilde{x}_m),
\]
where \( \tilde{x}_i = (x_{l_{i-1}+1}, \ldots, x_{l_i}) \) and set

\[
\hat{\Omega}^j = \Omega^{l_{j-1}+1} \times \cdots \times \Omega^{l_j}, \quad \hat{\Omega}^j = \Omega^{l_{j-1}+1} \times \cdots \times \Omega^d,
\]

\[
\hat{x}_i = (x_{l_i+1}, \ldots, x_d). \text{ Take } k(1), \ldots, k(d) \in \{1, 2, \ldots \} \text{ and, for } n \in \mathbb{Z}, \text{ let}
\]

\[
\hat{C}_n = [0, 2^{-nk(l_{i-1}+1)}) \times \cdots \times [0, 2^{-nk(l_i)})
\]

be a subset of \( \hat{\Omega}^j \) and \( C_n = \hat{C}_n \times \cdots \times \hat{C}_m \). By \( A_p \)-weights on \( \hat{\Omega}^j \) we mean the \( A_p \)-weights relative to all translates of \( \hat{C}_n \), \( n \in \mathbb{Z} \), belonging to \( \hat{\Omega}^j \), and, naturally, \( A_p \)-weights on \( \Omega \) are defined using all translates of \( C_n \), \( n \in \mathbb{Z} \), belonging to \( \Omega \).

**Theorem 7.11** Let \( K_0, p_k \in (1, \infty), w^k \in A_{p_k}(\hat{\Omega}^k), [w^k]_{p_k} \leq K_0, k = 1, \ldots, m, \) and \( u, g \) be measurable functions on \( \Omega \). Then there exists a constant \( \Lambda_0 = \Lambda_0(d, p_1, \ldots, p_m, k(1), \ldots, k(d), K_0) \geq 1 \) such that if

\[
\|u\|_{L_{p_1}(w^1 d\mu)} \leq N_0 \|g\|_{L_{p_1}(w^1 d\mu)}
\]

for some \( N_0 \in (0, \infty) \) and for every \( w \in A_{p_1}(\Omega) \) with \([w]_{p_1} \leq \Lambda_0\), then we have

\[
\|u\|_{L_{p_1,\ldots,p_m}(w^1,\ldots,w^m)} \leq N \|g\|_{L_{p_1,\ldots,p_m}(w^1,\ldots,w^m)},
\]

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where the norms are defined as in (3.18) replacing $dx_i$ by $w^i(\hat{x}_i)\,d\hat{x}_i$, the constant $N$ depends only on $d$, $p_1, \ldots, p_m, k(1), \ldots, k(d)$, $K_0$, and $N_0$.

**Proof** We follow the proof of Corollary 2.7 in [11]. Recall the extrapolation theorem of Rubio de Francia [24] which says that for any constant $\Lambda_j \in (1, \infty)$, $j = 1, \ldots, m$, there exists a constant $\Lambda_{j-1} = \Lambda_{j-1}(d - j, p, p_{j+1}, K_0\Lambda_j) \in (1, \infty)$ (we drop its dependence on the $k(i)'s$) such that, if

(a) for two nonnegative functions $U_j$ and $G_j$ on $\hat{\Omega}^{j+1}$,

$$\int_{\hat{\Omega}^{j+1}} U_j^{p_j} w(\hat{x}_{j+1}) \, d\hat{x}_{j+1} \leq N_j \int_{\hat{\Omega}^{j+1}} G_j^{p_j} w(\hat{x}_{j+1}) \, d\hat{x}_{j+1}$$

for some $N_j \in (0, \infty)$ and for every $w \in A_{p_j}(\hat{\Omega}^{j+1})$ with $[w]_{p_j} \leq \Lambda_{j-1}$, then

(b) we have

$$\int_{\hat{\Omega}^{j+1}} U_j^{p_{j+1}} w(\hat{x}_{j+1}) \, d\hat{x}_{j+1} \leq N_{j+1} \int_{\hat{\Omega}^{j+1}} G_j^{p_{j+1}} w(\hat{x}_{j+1}) \, d\hat{x}_{j+1} \quad (7.3)$$

for some $N_{j+1} \in (0, \infty)$, depending only on $d, j, K_0\Lambda_j, p, p_{j+1}$, and $N_j$, and for every $w \in A_{p_{j+1}}(\hat{\Omega}^{j+1})$ with $[w]_{p_{j+1}} \leq K_0\Lambda_j$.

In this form the theorem is proved in [11]. We define $\Lambda_{m-1} = 1$ and find all $\Lambda_j, j = 0, 1, \ldots, m - 1$. Then assume that $m \geq 2$ and define $U_0(x) = u(x)$,

$$U_j(\hat{x}_{j+1}) = \left( \int_{\hat{\Omega}^{j}} U_j^{p_j} (\hat{x}_j) w^j(\hat{x}_j) \, d\hat{x}_j \right)^{1/p_j}, \quad 1 \leq j \leq m - 1,$$

and similarly we introduce $G_j$’s by taking $g$ in place of $u$. To prove the theorem, it suffices to prove that (b) holds for $j = m - 1$ because $w^m \in A_{p_m}(\hat{\Omega}^m)$ and $[w^m]_{p_m} \leq K_0 = K_0\Lambda_{m-1}$. We are going to use the induction on $j = 0, 1, \ldots, m - 1$.

Observe that (b) holds for $j = 0$ by assumption. Suppose that it holds for a $j \in \{0, 1, \ldots, m - 2\}$. Then (7.3) also holds for $w(\hat{x}_{j+1}) := w^{j+1}(\hat{x}_{j+1})w(\hat{x}_{j+2})$ if $w^{j+1} \in A_{p_{j+1}}(\hat{\Omega}^{j+1})$ and $w(\hat{x}_{j+2}) \in A_{p_{j+1}}(\hat{\Omega}^{j+2})$ with $[w^{j+1}]_{p_{j+1}} \leq K_0, [w(\hat{x}_{j+2})]_{p_{j+1}} \leq \Lambda_j$ because then $[w(\hat{x}_{j+1})]_{p_{j+1}} \leq K_0\Lambda_j$. Remarkably, this implies that (a) holds with $j + 1$ in place of $j$. Then (b) also holds with $j + 1$ in place of $j$. This justifies the induction and proves the theorem.

**References**

1. Benedek, A., Panzone, R.: The space $L^p$, with mixed norm. Duke Math. J. 28, 301–324 (1961)
2. Bramanti, M., Cerutti, M.C.: $W^{1,2}$ solvability for the Cauchy–Dirichlet problem for parabolic equations with VMO coefficients. Commun. Partial Differ. Equ. 18(9–10), 1735–1763 (1993)
3. Byun, S.-S., Lee, M., Palagachev, D.K.: Hessian estimates in weighted Lebesgue spaces for fully nonlinear elliptic equations. J. Differ. Equ. 260(5), 4550–4571 (2016)
4. Byun, S.-S., Jehan, O., Wang, L.: $W^{2,p}$ estimates for solutions to asymptotically elliptic equations in nondivergence form. J. Differ. Equ. 260(11), 7965–7981 (2016)
5. Caffarelli, L.A., Cabré, X.: Fully Nonlinear Elliptic Equations. American Mathematical Society Colloquium Publications, vol. 43. American Mathematical Society, Providence (1995)
6. Cejas, M.E., Durán, R.G.: Weighted a priori estimates for elliptic equations. Stud. Math. 243(1), 13–24 (2018)
7. Chiarenza, F., Frasca, M., Longo, P.: Interior $W^{2,p}$ estimates for nondivergence elliptic equations with discontinuous coefficients. Ric. Mat. 40(1), 149–168 (1991)
8. Chiarenza, F., Frasca, M., Longo, P.: $W^{2,p}$-solvability of the Dirichlet problem for nondivergence elliptic equations with VMO coefficients. Trans. Am. Math. Soc. 336(2), 841–853 (1993)
9. Crandall, M.G., Kocan, M., Święch, A.: $L^p$-theory for fully nonlinear uniformly parabolic equations. Commun. Partial Differ. Equ. 25(11–12), 1997–2053 (2000)
10. Dong, H., Gallarati, C.: Higher-order parabolic equations with vmo assumptions and general boundary conditions with variable leading coefficients. Int. Math. Res. Not. p. rny084 (2018). https://doi.org/10.1093/imrn/rny084
11. Dong, H., Kim, D.: On $L_p$-estimates for elliptic and parabolic equations with $A_p$ weights. Trans. Am. Math. Soc. 370(7), 5081–5130 (2018)
12. Dong, H., Krylov, N.V., Li, X.: On fully nonlinear elliptic and parabolic equations with VMO coefficients in domains. Algebra Anal. 24(1), 53–94 (2012)
13. Grafakos, L.: Modern Fourier Analysis. Graduate Texts in Mathematics, vol. 250, Second edn. Springer, New York (2009)
14. Kozlov, V., Nazarov, A.: The Dirichlet problem for non-divergence parabolic equations with discontinuous in time coefficients. Math. Nachr. 282(9), 1220–1241 (2009)
15. Krylov, N.V.: The heat equation in $L^q((0,T),L_p)$-spaces with weights. SIAM J. Math. Anal. 32(5), 1117–1141 (2001)
16. Krylov, N.V.: Lectures on Elliptic and Parabolic Equations in Sobolev Spaces. Graduate Studies in Mathematics, vol. 96. American Mathematical Society, Providence (2008)
17. Krylov, N.V.: On the existence of $W^{2,p}_p$ solutions for fully nonlinear elliptic equations under relaxed convexity assumptions. Commun. Partial Differ. Equ. 38(4), 687–710 (2013)
18. Krylov, N.V.: On the existence of $W^{1,2}_p$ solutions for fully nonlinear parabolic equations under either relaxed or no convexity assumptions. Nonlinear Analysis in Geometry and Applied Mathematics. Part 2. Harvard University, Center of Mathematical Sciences and Applications, Series in Mathematics, pp. 103–133, International Press, Somerville (2018)
19. Krylov, N.V.: Sobolev and Viscosity Solutions for Fully Nonlinear Elliptic and Parabolic Equations. Mathematical Surveys and Monographs, vol. 233. American Mathematical Society, Providence (2018)
20. Krylov, N.V.: On parabolic PDEs and SPDEs in Sobolev spaces $W^p_q$ without and with weights. Topics in Stochastic Analysis and Nonparametric Estimation. The IMA Volumes in Mathematics and its Applications, vol. 145, pp. 151–197. Springer, New York (2008)
21. Krylov, N.V.: On Bellman’s equations with VMO coefficients. Methods Appl. Anal. 17(1), 105–121 (2010)
22. Lin, F.-H.: Second derivative $L_p$-estimates for elliptic equations of nondivergent type. Proc. Am. Math. Soc. 96(3), 447–451 (1986)
23. Maugeri, A., Palagachev, D.K., Softova, L.G.: Elliptic and Parabolic Equations with Discontinuous Coefficients. Mathematical Research, vol. 109. Wiley, Berlin (2000)
24. Rubio de Francia, J.L.: Factorization theory and $A_p$ weights. Am. J. Math. 106(3), 533–547 (1984)
25. Winter, N.: $W^{2,p}$ and $W^{1,p}$-estimates at the boundary for solutions of fully nonlinear, uniformly elliptic equations. Z. Anal. Anwend. 28(2), 129–164 (2009)
26. Zhang, J., Zheng, S.: Lorentz estimates for asymptotically regular fully nonlinear parabolic equations. Math. Nachr. 291(5–6), 996–1008 (2018)

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