Quantum theory as a critical regime of language dynamics

Alexei Grinbaum

CEA-Saclay/IRFU/LARSIM, 91191 Gif-sur-Yvette, France

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Quantum mechanics relies on the cut between the observer and the quantum system, but it does not define the observer physically. We propose an informational definition based on bounded complexity of strings. Language dynamics then leads to an emergent continuous model in the critical regime. Restricting it to a subfamily of ‘quantum’ binary codes describing ‘bipartite systems’, we find strong evidence of an upper bound on bipartite correlations equal to \(2.82537\). This is measurably different from the Tsirelson bound of the CHSH inequality. If such a reconstruction of quantum theory is experimentally confirmed, it would show that the Hilbert space formalism is but an effective description of a fundamental ‘linguistic’ theory in the critical regime.

* alexei.grinbaum@cea.fr
John Wheeler emphasized that it is extraordinarily difficult to decide where “the community of observer-participants” begins and where it ends [22]. Quantum theory says nothing about the physical composition of observer; this term has no further theoretical description. One cannot infer from a set of quantum measurements if the observer is a human being, a machine, a stone, a Martian, or the whole Universe. Nevertheless quantum mechanics describes the observer’s information; the only requirement it makes is that information must be somehow registered. Hugh Everett argued that observers are characterized by their memory, i.e., “parts... whose states are in correspondence with past experience of the observers” [10]. It seems reasonable to assume that differently constituted observers with the same memory size will be equally able to register measurement results obtained on a quantum system. Material ‘hardware’ does not matter, because quantum mechanics uses abstract mathematics: it deals with observers possessing information about systems, not with stones or Martians. Now, can one use such abstract mathematics to further describe observers?

The choice of abstract mathematics to be used as a formalism of quantum theory has been a problem since the theory’s discovery in the 1920s. To give just one example, quantum logic was born as an attempt to propose an alternative formulation that would not be based on matrices or Hermitian operators [2]. A somewhat puzzling approach to this problem was attempted by Niels Bohr. As the historian Max Jammer put it, Bohr over time “became more and more convinced of the need of a symbolization if one wants to express the latest results of physics” [13]. Bohr had already written extensively on the problem of objective description, i.e. an expression shared among all observers. However, it was only late in his life that he connected it with the choice of mathematical formalism for quantum theory. In 1958 he wrote, “The use of mathematical symbols secures the unambiguity of definition required for objective description.” [4] What exactly Bohr meant here remains unclear. It is unlikely that his point was that quantum theory must rely on the Hilbert space, by then a standard tool. Had it been so, Bohr could have named the Hilbert space explicitly, yet he only vaguely referred to “mathematical symbols”. It is conceivable that Bohr’s view was that such mathematical symbols remained to be found. If so, their discovery would purportedly guarantee the unambiguity of communication and, consequently, secure the objectivity of description. It would then account to a “common-language” basis of physical theory, in line with Bohr’s well-known insistence on the role of classical concepts [3].

Any attempt to clarify the meaning of Bohr’s statement begs two questions. First, is there a mathematical framework that includes both ambiguous and unambiguous descriptions? In Section II we introduce such a framework based on the algebraic coding theory. This theory provides a general model of communication and deals mathematically with errors or ambiguity. Second, how is quantum theory different from all other unambiguous descriptions? We answer by providing two criteria of fundamental importance for the reconstruction of quantum theory: a condition of continuity and a bound on bipartite correlations.

Our reconstruction is based on the definition of an observer in information-theoretic terms. This definition involves a limit on the complexity of strings that — using a common-language expression — the observer can ‘store and handle’ (Section II). Strings satisfying these criteria contain the descriptions of states allowed by quantum theory but also much more: they may not refer to systems or bear any semantics in terms of preparations or measurements. Using the work of Manin and Marcolli, we show that symbolic dynamics on such strings leads to an emergent continuous model in the critical regime (Section III). Restricting this model to a subfamily of ‘quantum’ binary codes describing ‘bipartite systems’ (Section IV), we find strong evidence of an upper bound on bipartite correlations equal to $2\sqrt{2}$ of the CHSH inequality [4]. If confirmed, our analysis would show that the Hilbert space formalism is but an effective description of a fundamental discrete theory of quantum languages in the critical regime, somewhat similarly to the description of phase transitions by the Landau theory.

I. CODES

Communication is based on encoding messages that are transmitted in suitable codes using an alphabet shared between communicating parties. An alphabet is a finite set $A$ of cardinality $q \geq 2$. A code is a subset $C \subset A^n$ consisting of some of the words of length $n \geq 1$. A language is an ensemble of codes of different lengths using the same alphabet. As an example, take the alphabet $A = F_q$, the finite field of $q$ elements. Linear subspaces of $F_q^n$ gives rise to codes called linear. Linearity provides such codes with extra structure. Another example is given by binary codes of length $n$ based on a two-letter alphabet, say, $\{0, 1\}$. Strings of zeros and ones of arbitrary length belong to a language formed by binary codes with different values of $n$.

In full generality, nothing can be stipulated about the semantics of the message, the material support of the encoding and decoding operations, or their practical efficiency. One can observe, however, that decoding a message is less prone to error if the number of words in the code is small. On the other hand, reducing the number of code words requires
the words to be longer. The number

$$R = \frac{\log_q \#C}{n}$$

is called the (transmission) rate of code $C$.

One can associate a fractal to any code in the following way [16, 17]. Define a rarified interval $(0, 1)_q = [0, 1] \setminus \{m/q^n | m, n \in \mathbb{Z}\}$. Points $x = (x_1, \ldots, x_n) \in (0, 1)_q^n$ can be identified with $(\infty \times n)$ matrices whose $k$-th column is the $q$-ary decomposition of $x_k$. The Sierpinski fractal set $S_C \subset (0, 1)_q^n$ of Hausdorff dimension $R$ has all rows of these matrices in $C$. The closure of $S_C$ inside the cube $[0, 1]^n$ includes the rational points with $q$-ary digits. This new fractal $\hat{S}_C$ is a metric space in the induced topology from $[0, 1]^n$. Now consider a family of codes $C_r$ of $k_r = \log_q \#C_r$ words of length $n_r$, with rate $R$:

$$\frac{k_r}{n_r} \nearrow R.$$ (2)

They define a fractal $S_R = \bigcup_r S_{C_r}$ of Hausdorff dimension $\dim_H(S_R) = R$.

II. BOUNDED COMPLEXITY

Any observer’s memory is limited in size. While their material constitution may be radically different, different observers with the same memory size should demonstrate similar performance in handling information. This intuition serves as a motivation for the following information-theoretic definition of observer.

**Definition II.1.** An observer is a subset of strings of bounded complexity, i.e., strings compressible below a certain threshold.

This limit can be viewed as the length of the observer’s memory. If a string has high complexity, it cannot describe an observer with a memory smaller that the minimal length required to store it; but it remains admissible for an observer with a larger memory.

Definition II.1 requires a notion of string complexity independent of the observer’s material organization. Kolmogorov complexity is a suitable candidate. It has already been used in fundamental physics, e.g. by Zurek who argued that physical entropy should be defined as a sum of Shannon entropy $H$ and algorithmic randomness of available information [26–28]. The latter was to be understood as information contained in a ‘binary image’ of the state of the system, defined as Kolmogorov complexity $K$ of the shortest program able to generate it. When the state of the system is known sufficiently well, $K$ supplies the main contribution to entropy. In particular, it is hard to extract work from a state with large $K$, which stands as a motivation for counting $K$ in physical entropy. Zurek argued that this algorithmic component of physical entropy can be made observer-independent by discretizing the system on a family of grids that are concisely describable by a universal computer. We reinterpret his ‘binary image’ as a string of symbols in a given alphabet used by an observer who ‘reads’ it as information about the state of the system. Such strings are information-theoretic primitives; taken together, they define the observer. Unlike in Zurek’s work, the strings do not necessarily describe states: they may not ‘refer’ to any system. They may not have any semantics at all. For a set of strings that are code words of code $C$ with rate $R$, the lower Kolmogorov complexity satisfies [17]:

$$\sup_{x \in S_C} \kappa(x) = R.$$ (3)

For all words $x \in S_R$ in a language formed by codes $C_r$, the lower Kolmogorov complexity is bounded by $\kappa(x) \leq R$. Hence the closure $\hat{S}_R$ of the fractal $S_R$ is a metric space that describes the handling of words of bounded Kolmogorov complexity. It is a ‘minimal’ geometric structure corresponding, in Zurek’s terms, to the space of ‘states’ assigned by the observer to a ‘system’.

III. CRITICAL LANGUAGE DYNAMICS

A study of the reconstructions of quantum theory drives home the importance of the assumptions of continuity [11]. While various toy or post-quantum models reproduce select properties of quantum mechanics [1, 20, 21], they invariably require either the existence of continuous reversible transformations between pure states [8, 12] or another similar axiom [14, 23]. Such an axiom must be present in any attempt at axiomatizing quantum theory.
A change of observer’s information can be represented via dynamical evolution on the fractal set $S_R$. As proposed in [17], this involves a change in the ‘occupation numbers’ $\lambda_a$ of words $a_i \in \bigcup_r C_r$ under Hamiltonian dynamics in the Fock space:

$$H_{a_1 \cdots a_m} = (\lambda_{a_1} + \cdots + \lambda_{a_m})\epsilon_{a_1 \cdots a_m},$$

with the Keane ‘ergodicity’ condition:

$$\sum_{a \in \bigcup_r C_r} e^{-R\lambda_a} = 1. \tag{5}$$

The Keane condition gives a meaning to the weights $\lambda_a$ as normalized logarithms of inverse probabilities that a is stored in the observer’s memory. This evolution has a partition function:

$$Z(\beta) = \frac{1}{1 - \sum_{a \in \bigcup_r C_r} e^{-\beta\lambda_a}}. \tag{6}$$

Manin and Marcolli have shown that at the critical temperature (equivalently, string complexity) $\beta = R$, the behaviour of this system is given by a KMS state on an algebra respecting unitarity [17]. Consider characteristic functions $\chi_{S_C(w)}$, where $w = w_1 \ldots w_m$ runs over finite words composed of $w_i \in C$ and $S_C(w)$ denotes the subset of infinite words $x \in S_C$ that begin with $w$. These functions can be identified with the range projections

$$P_w = T_wT_w^* = T_{w_1} \cdots T_{w_m}T_{w_m}^* \cdots T_{w_1}^*.$$

At the low temperature $\beta > R$ there exists a unique type $I_\infty$ KMS-state $\phi_R$ on the statistical system of codes, which is a Toeplitz-Cuntz algebra with time evolution:

$$\sigma_t(T_w) = q^{it\ln T_w}. \tag{8}$$

The partition function is:

$$Z_C(\beta) = \left(1 - q^{(R-\beta)n}\right)^{-1}. \tag{9}$$

However the isometries in the algebra do not add up to unity. Only at the critical temperature $\beta = R$, where a phase transition occurs for all codes $C_r$, is there a unique KMS state on the Cuntz algebra, i.e., an algebra such that isometries add up to unity: $\sum_a T_aT_a^* = 1$.

Critical behavior of the original discrete linguistic model is described at $\beta = R$ by a field theory on the metric space $S_R$, which obeys unitarity. Recall that, by construction, this fractal also has scaling symmetry. Consequently, we have a field theory respecting scaling and unitarity. While there has been some discussion of theories that are scale invariant but not conformal, we assume that, in agreement with Polyakov’s general conjecture [19], this field theory is conformal. Due to its property of continuity and to the geometric character of its state space, it is a tentative candidate for the reconstruction of quantum theory.

### IV. AMOUNT OF CORRELATIONS

The conformal model likely contains more than a description of ‘quantum’ languages. The $C^*$-algebraic reconstructions of quantum theory demonstrate that an extra condition is required to select quantum theory from other continuous models [5]. In this section we pick out a class of models corresponding to the critical regime of binary codes describing measurements on bipartite quantum systems in the usual 3-dimensional Euclidean space. First we define ‘bipartite’ informationally. In quantum theory, subsystems that are entangled can be materially different, but they are described by the same number of the entangled degrees of freedom. Their informational content is represented by strings of identical complexity. For example, measurements in a CHSH-type experiment will produce such strings for a choice of $\sigma_x, \sigma_y, \sigma_z$ measurements. The no-signalling condition implies that the probability of 0 on Alice’s side does not depend on the settings in Bob’s lab, and vice versa. The strings then resemble Bernoulli distributions, whose Kolmogorov complexity is equal to binary entropy of the probability of 0, but their complexity also contains a correction due to the existence of non-zero mutual information between Alice’s and Bob’s outputs. Since both sides enter symmetrically in the CHSH inequality, this correction to Kolmogorov complexity is a priori the same on Alice’s and Bob’s side. We now use this intuition to replace Eq. [4] with a class of Hamiltonians that describe a ‘bipartite system’ in the framework of codes.
The Kolmogorov order is an arrangement of words \(a_i \in \bigcup_r C_r\) in the increasing order of their complexities. It is not computable and it differs radically from any numbering of \(a_i\) based on the Hamming distance in the codes \(C_r\). However, words that are adjacent in the Kolmogorov order have the same complexity. We can choose an Ising-type Hamiltonian

\[
H = - \sum_{ij} a_i \times a_j,
\]

as a dynamical model on the language that describes bipartite quantum systems. The sum is taken over \(N\) neighbors in the Kolmogorov order, i.e. all strings of identical complexity. The result of multiplication on binary words is a new word with letters isomorphic to multiplication results in a two-element group \(\{\pm 1\}\).

A binary language with \(N = 6\) using the Hamiltonian (10) gives rises to information dynamics equivalent to the dynamics of a 3-dim Ising model. The class of Hamiltonians that lead to such dynamics includes all weight distributions on the strings that describe bipartite systems with \(N = 6\), however its precise characteristics are uncomputable due to the properties of Kolmogorov complexity. Plainly, we do not know which binary codes give rise to the \(\nu = 3\) primary scalar described by a conformal field theory. Bipartite correlations in this regime are described by the lowest-dimension even primary scalar \(\epsilon = \sigma \times \sigma\) in the CFT. This field is symmetric under the switch between Alice and Bob. It provides a conformal description of ‘bipartiteness’, which is encoded in the Ising interaction. The operator dimension of \(\epsilon\) is

\[
\Delta_{\epsilon} = 3 - \frac{1}{\nu},
\]

where \(\nu\) is a well-known critical exponent describing the correlation length.

The 3-dim spatial analogy has its limitations since the true metric space of code evolution is the fractal \(\hat{S}_C\). Still it provides significant evidence that, if the Hamiltonian (11) is to be replaced by (10), there emerges a critical regime, whereby the symmetric bipartite correlations are described in the CFT by an analogue of the 3-dim Ising primary even scalar \(\epsilon\). Further, the exponential character of the mapping that links the fractal with a Euclidean model hints at the existence of a connection between their critical behaviour. The correlation length in the fractal representation of a language describes a logarithmic distance from which words are brought in clusters by the Kolmogorov reordering. It diverges in the critical regime, when string complexity reaches the value \(\beta = R\). However, in the Ising analogy the amount of correlations on the words of equal complexity remains limited by the scaling of the correlator of the lowest primary even field:

\[
\langle \epsilon(a)\epsilon(0) \rangle \sim a^{-2\Delta_{\epsilon}}.
\]

We conjecture that the corresponding correlations in the fractal are limited by the logarithm of the RHS of (13). Their maximum strength \(2\Delta_{\epsilon}\) can be computed based on the value \(\nu = 0.62999(5)\) in [8]:

\[
2\Delta_{\epsilon} = 2.82537(2).
\]

V. CONCLUSION

Freely interpreting Bohr’s dictum that unambiguous communication of measurement results requires a mathematical formulation, we suggested an information-theoretic definition of quantum observer based on the idea of limited string complexity. We then adopted a framework that leads to a linguistic model possessing two key features of quantum theory: continuity and the non-classical amount of correlations between entangled subsystems. The key result is a conjecture leading to a bound on the amount of bipartite correlations in the linguistic model. Its numerically calculated value is slightly different from the Tsirelson bound but consistent with the available experimental results \(S = \Delta_{\epsilon} + 2 \simeq 3.41267 \leq 3.426 \pm 0.016\) in [10] and \(2\Delta_{\epsilon} \simeq 2.82534 \leq 2.827 \pm 0.017\) in [8].

The crucial analogy with the Ising model relies on the assumption that the number of strings possessing the same complexity after uncomputable Kolmogorov reordering is \(N = 6\). This does not need to be so for all codes. Codes with \(N = 4\) correspond to 2-dim Ising interaction (\(\nu = 1\)) and give rise to the classical bound on bipartite correlations.
2\(\Delta_v = 2 \cdot 1 = 2\). Codes with \(N = 8\) correspond to 4-dim Ising interaction (\(\nu = 1/2\)) and give rise to the Popescu-Rohrlich maximum correlation 2\(\Delta_v = 2 \cdot (4 - \frac{1}{\sqrt{2}}) = 4\). It is not clear whether binary codes endowed with critical dynamics exist for other values of \(N\) and, if they do, what meaning they may have.

Although our model is highly speculative, we believe that it demonstrates the interest to explore information-theoretic approaches to observation, particularly those leading to experimentally testable predictions. As Wittgenstein said, “A particular method of symbolizing may be unimportant, but it is always important that this is a possible method of symbolizing. [This] possibility… reveals something about the nature of the world.” [24]

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