Norms of Minimal Projections

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Abstract

It is proved that the projection constants of two- and three-dimen-
sional spaces are bounded by $4/3$ and $(1 + \sqrt{5})/2$, respectively. These
bounds are attained precisely by the spaces whose unit balls are the
regular hexagon and dodecahedron. In fact, a general inequality for
the projection constant of a real or complex $n$-dimensional space is
obtained and the question of equality therein is discussed.

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1 Introduction and the main results

In this paper we prove results on upper estimates for the norms of minimal projections onto finite-dimensional subspaces of Banach spaces, which are optimal in general. By Kadec–Snobar [KS], onto $n$-dimensional spaces there are always projections of norm smaller than or equal to $\sqrt{n}$. General bounds for these so-called projection constants were further studied by various authors, including Chalmers, Garling, Gordon, Grünbaum, König, Lewis and Tomczak-Jaegermann ([GG], [G], [KLL], [KT], [L], [T]). Some other aspects of minimal projections, like the existence or norm estimates for concrete spaces, were investigated by many authors, among them e.g., Chalmers, Cheney, Franchetti ([CP], [IS], [FV]).

In [KT] a very tight formula for the projection constants of spaces with enough symmetries was shown. We now prove that this formula holds for arbitrary spaces, and study cases of equality.

The formula yields, in particular, that the projection constant of any real (resp. complex) 2-dimensional space is bounded by $4/3$ (resp. $(1 + \sqrt{3})/2$). Up to isometry, there is just one space (in each case) attaining the bound. The values for 3-dimensional spaces are $(1 + \sqrt{5})/2$ (resp. $5/3$). In the real case, the unique extremal spaces are those whose unit balls are the regular hexagon and the regular dodecahedron. The $4/3$-result solves a problem of Grünbaum [G]. A proof of this fact has also been announced by Chalmers et al. [CMSS]; it is our understanding that their argument is incomplete as of now.

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We use standard Banach space notation, see e.g., [T.2]. By $K$ we denote the scalar field, either the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$. The relative projection constant of a (closed) subspace $E$ of a Banach space $X$ is defined by

$$\lambda(E, X) := \{\|P\| \mid P : X \to E \subset X \text{ is a linear projection onto } E\},$$

the (absolute) projection constant of $E$ is given by

$$\lambda(E) := \{\lambda(E, X) \mid X \text{ is a Banach space containing } E \text{ as a subspace}\}.$$  \hfill (1.1)
Any separable Banach space $E$ can be embedded isometrically into $l_\infty$. For any such embedding, $\lambda(E) = \lambda(E, l_\infty)$, i.e., the supremum in (1.1) is attained. We can therefore restrict our attention to finite-dimensional subspaces $E \subset l_\infty$. Also note that $\lambda(l_n^\infty) = \lambda(l_\infty) = 1.$

Let $n \in \mathbb{N}$ be a positive integer, $\langle \cdot, \cdot \rangle$ denote the standard scalar product in $\mathbb{K}^n$ and let $\| \cdot \|_2 = \sqrt{\langle \cdot, \cdot \rangle}$. For $N \in \mathbb{N}$, vectors $x_1, \ldots, x_N \in \mathbb{K}^n$ spanning lines in $\mathbb{K}^n$ are called **equiangular** provided that there is $0 \leq \alpha < 1$ such that $\|x_i\|_2 = 1$ and $|\langle x_i, x_j \rangle| = \alpha$ for $i \neq j, i, j = 1, \ldots, N$.

Put

$$N(n) := \begin{cases} n(n+1)/2 & \text{if } \mathbb{K} = \mathbb{R} \\ n^2/2 & \text{if } \mathbb{K} = \mathbb{C}. \end{cases}$$  \hfill (1.2)

By Lemmens–Seidel [LS] and Gerzon, in $\mathbb{K}^n$ there are at most $N(n)$ equiangular vectors. (Indeed, the hermitian rank 1 operators $x_i \otimes x_i$ are linearly independent in a suitable real linear space of operators.) This bound is attained for $n = 2, 3, 7, 23$ if $\mathbb{K} = \mathbb{R}$ and for $n = 2, 3$ if $\mathbb{K} = \mathbb{C}$. If the bound is attained, necessarily $\alpha = 1/\sqrt{n+2}$ if $\mathbb{K} = \mathbb{R}$ and $\alpha = 1/\sqrt{n+1}$ if $\mathbb{K} = \mathbb{C}$.

Our main result is

**Theorem 1.1** (a) The projection constant of any $n$-dimensional normed space $E_n$ is bounded by

$$\lambda(E_n) \leq \begin{cases} (2 + (n-1)\sqrt{n+2})/(n+1) & \text{in the real case,} \\ (1 + (n-1)\sqrt{n+1})/n & \text{in the complex case}. \end{cases}$$  \hfill (1.3)

(b) Given $\mathbb{K}$ and $n \in \mathbb{N}$, there exist $n$-dimensional spaces $E_n$ for which the bound is attained if and only if there exist $N(n)$ equiangular vectors in $\mathbb{K}^n$. In this case, such a space $E_n$ can be realized as an isometric subspace of $l_\infty^{N(n)}$, and the orthogonal projection is a minimal projection onto $E_n$.

(c) For $\mathbb{K} = \mathbb{R}$ and $n = 2, 3, 7, 23$, there are unique spaces $E_n$ (up to isometry) attaining the bound (1.3); for $\mathbb{K} = \mathbb{C}$ and $n = 2, 3$ such spaces also exist. For $\mathbb{K} = \mathbb{R}$ and $n = 2, 3$ the unit balls of $E_n$ are the regular hexagon and the regular dodecahedron, respectively.

**Remarks** (i) The right hand side of (1.3) equals the bound $f(n,N(n))$ derived in [KLL] for the relative projection constant of an $n$-dimensional space in an $N(n)$-dimensional superspace.
(ii) The bounds in (1.3) are of the order $\sqrt{n} - 1/\sqrt{n} + 2/n$ if $K = R$ and $\sqrt{n} - 1/2\sqrt{n} + 1/n$ if $K = G$, for large $n \in N$.

To prove just (1.3) it would suffice (by approximation) to consider polyhedral spaces $E \subset l_\infty^N$ for an arbitrary $N \in N$ (the “finite” case); in which case the proofs of most of the results which follow can be simplified. For the examination of the equality in (1.3) and the uniqueness we need, however, the general (“infinite”) case of $E \subset l_\infty$ as well, even though the spaces attaining the bound (1.3) turn out in the end to be polyhedral.

To unify the notation in the finite and infinite case which we would like to discuss simultaneously, we set $T = \{1, \ldots, N\}$, for some $N \in N$, in the finite case and $T = N$, in the infinite case. In particular, $l_\infty(T)$ denotes $l_\infty^N$ in the former case and $l_\infty$ in the latter case.

If $\mu = (\mu_t)_{t \in T}$ is a probability measure on $T$, and $1 \leq p < \infty$, we let

$$l_p(T, \mu) := \{(\xi_t)_{t \in T} \mid \|\xi_t\|_{p, \mu} = (\sum_{t \in T} |\xi_t|^p \mu_t)^{1/p} < \infty\}.$$  

For a subspace $E \subset l_\infty(T)$, we denote by $E_{p, \mu}$ the same space $E$ considered as a subspace of $l_p(T, \mu)$, via the embedding $l_\infty(T) \to l_p(T, \mu)$.

Finally, for $N \in N$, by $R_N : l_p \to l_p^N$ we denote the projection onto the first $N$ coordinates, acting in an appropriate sequence space ($1 \leq p \leq \infty$).

Let $n \in N$. The set $\mathcal{F}_n$ of all $n$-dimensional spaces, equipped with the (logarithm of the) Banach–Mazur distance, is a compact metric space, cf. e.g., [T.2]. The projection constant $\lambda$, as a function $\lambda : \mathcal{F}_n \to R^+$, is continuous with respect to this metric, and hence the supremum $\sup_{E \in \mathcal{F}_n} \lambda(E)$ is attained: there is $F \in \mathcal{F}_n$ with

$$\lambda(F) = \sup\{\lambda(E) \mid E \in \mathcal{F}_n\}.  \quad (1.4)$$

The proof of the bound (1.3) is based upon an estimate in terms of orthonormal systems, which, in fact, is a characterization of the maximal projection constant, and it seems to be of independent interest.

**Theorem 1.2** Let $n \in N$. Then

$$\max_{E \in \mathcal{F}_n} \lambda(E) = \sup_{\mu} \sum_{f_j \in \mathcal{F}_n} \left| \sum_{s,t \in N} f_j(s)f_j(t) \mu_s \mu_t \right|,  \quad (1.5)$$
where the outside supremum runs over the set of all discrete probability measures $\mu = (\mu_t)_t$ on $\mathbb{N}$ and the inside supremum runs over all orthonormal systems $\{f_j\}$ in $l_2(\mathbb{N}, \mu)$. The double supremum in (1.3) is attained for some $\mu$ and $\{f_j\} \subseteq l_2(\mathbb{N}, \mu) \cap l_\infty$. In this case, the space $E = \text{span}\{f_1, \ldots, f_n\} \subseteq l_\infty$ has maximal projection constant. The square function $(\sum_{j=1}^n |f_j(s)|^2)^{1/2}$ is constant $\mu$-a.e. in the extremal case.

In the extremal case the support of $\mu$ can be finite; and in dimensions $n = 2, 3$ it is actually so. The upper estimate in (1.3) relies on an idea of Lewis [L]. To prove Theorem 1.1, we then have to find an upper estimate for the right hand side of (1.5). In certain dimensions ($n = 2, 3$, and in the real case additionally $n = 7, 23$), we find the exact value of (1.5); for other $n \in \mathbb{N}$, the expression in (1.3) might be possibly used to slightly improve (1.3).

2 Projection constants and trace duality

In this section, we prove the upper bound for $\max\{\lambda(E) \mid E \in \mathcal{F}_n\}$ in (1.3). The argument is based on trace duality. For the convenience of the general reader, we try to use only basic Banach space theory. The first lemma is similar to Lemma 1 of [KLL].

**Lemma 2.1** Let $E \subseteq l_\infty(T)$ be a finite-dimensional subspace, where $T = \{1, \ldots, N\}$, or $T = \mathbb{N}$. There exists a map $u : l_\infty(T) \to l_\infty(T)$ with $u(E) \subset E$ such that

$$
\lambda(E) = \text{tr}(u : E \to E) \quad \text{and} \quad \sum_{t \in T} \|ue_t\|_\infty = 1.
$$

Here $(e_t)_{t \in T}$ denotes the standard unit vector basis in $l_\infty(T)$.

In fact, for any map $u$ with $u(E) \subset E$ and $\sum_{t \in T} \|ue_t\|_\infty = 1$ one has $\text{tr}(u : E \to E) \leq \lambda(E)$, see (??) below.

**Proof** Since $E$ is finite-dimensional, there exists a minimal projection onto $E$, say $P_0 : l_\infty(T) \to E \subset l_\infty(T)$ with $\|P_0\| = \lambda = \lambda(E) < \infty$ (cf. [BC], [IS]). Let $\mathcal{F}(l_\infty, l_\infty)$ denote the space of finite-rank operators on $l_\infty = l_\infty(T)$, equipped with the operator norm. The sets

$$A = \{S \in \mathcal{F}(l_\infty, l_\infty) \mid \|S\| < \lambda\}$$

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and

\[ B = \{ P \in \mathcal{F}(l^\infty, l^\infty) \mid P = P_0 + \sum_{i=1}^{m} x_i^* \otimes x_i \} \]

for some \( x_1, \ldots, x_m \in E, \ x_1^*, \ldots, x_m^* \in E^\perp \subset l^*_\infty, \ m \in \mathbb{N} \)

are convex and disjoint, since \( B \) consists of projections onto \( E \) and \( \|P\| \geq \lambda \) for every projection \( P \). Since \( A \) is open, by the Hahn–Banach theorem there is a functional \( \varphi \in \mathcal{F}(l^\infty, l^\infty)^* \) of norm \( \|\varphi\| = 1 \) such that \( \varphi(P_0) \in \mathbb{R} \) and for \( S \in A \) and \( P \in B \) we have

\[ \Re \varphi(S) < \lambda \leq \Re \varphi(P). \]

By the trace duality, \( \varphi \) is represented by a map \( v \) defined on \( l^\infty(T) \). In the case \( T = \{1, \ldots, N\} \), the operator norm of \( w \in \mathcal{F}(l^\infty, l^\infty) \) is just \( \sup_{t \in T} \|w^e_t\|_1 \), so the dual norm is \( \sum_{t \in T} \|v^e_t\|_\infty \). If \( T = \mathbb{N} \), define a linear operator \( v : l^\infty \to l^*_\infty \) by \( \langle v(x), x^* \rangle = \varphi(x^* \otimes x) \) for \( x \in l^\infty \) and \( x^* \in l^*_\infty \). Writing any \( S \in \mathcal{F}(l^\infty, l^\infty) \) as \( S = \sum_{i=1}^{m} x^* \otimes x_i \), one finds that \( \varphi(S) = \text{tr}(vS) \), with the integral norm \( i(v) \) equal to

\[ i(v) = \sup_{S \in \mathcal{F}(l^\infty, l^\infty)} \text{tr}(vS)/\|S\| = 1. \]

Let \( x^* \in E^\perp, \ x \in E \). Then \( \lambda \leq \Re \varphi(P_0 + x^* \otimes x) = \lambda + \Re \text{tr}(v(x^* \otimes x)) \). Hence \( \Re \langle vx, x^* \rangle \geq 0 \) for all \( x^* \in E^\perp, \ x \in E \), which implies \( \langle vx, x^* \rangle = 0 \). Thus \( v(E) \subset E^\perp \subset E \subset l^\infty \), in view of \( \dim E < \infty \). Let \( Q : l^*_\infty \to l^\infty \) be the canonical projection onto \( l^\infty \) with \( \|Q\| = 1 \). Let \( u := Qv : l^\infty \to l^\infty \). Then \( u(E) \subset E \) and, since \( Qx = x \) for \( x \in E \), we have \( uP_0 = vP_0 \). Furthermore, \( i(u) \leq \|Q\| i(v) = 1 \) and

\[ \lambda(E) = \lambda = \varphi(P_0) = \text{tr}(uP_0) = \text{tr}(u : E \to E). \]

Let \( R_N : l^\infty \to l^N_\infty \) be the natural projection and let \( u_N := R_Nu : l^\infty \to l^N_\infty \). Then \( i(u_N) \leq 1 \) and, similar as in the case of \( T = \{1, \ldots, N\} \) discussed above, this norm, being dual to the operator norm on \( \mathcal{F}(l^N_\infty, l^\infty) \), is equal to \( i(u_N) = \sum_{i \in T} \|u_N^e_i\|_\infty \leq 1 \). Taking the limit as \( N \to \infty \) (first for finite sums in \( t \)) we get that \( \sum_{i \in T} \|u^e_i\|_\infty \leq 1 \). In fact we have the equality. \( \square \)

The following upper estimate is a consequence of Lemma 2.1 and relies essentially on an idea of Lewis [L].

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Proposition 2.2 Let $E \subset l_\infty(T)$ be an $n$-dimensional subspace, where $T = \{1, \ldots, N\}$, or $T = N$. There is a discrete probability measure $\mu = (\mu_t)_{t \in T}$ on $T$, $\|\mu\|_1 = 1$ such that for any orthonormal basis $(f_j)_{j=1}^n$ in $E_{2,\mu}$ we have

$$\lambda(E) \leq \sum_{s,t \in T} \left| \sum_{j=1}^n f_j(s)f_j(t) \right| \mu_s \mu_t.$$

Note that the double sum in this proposition is finite, since $f_j \in l_2(T, \mu)$.

Proof Let $u : l_\infty(T) \to l_\infty(T)$ be as in Lemma 2.1 and put $\mu_t = \|ue_t\|_\infty$, for $t \in T$. Then $\mu$ is a probability measure on $T$, $\sum_{t \in T} \mu_t = 1$. For every $N \in \mathbb{N}$, let $u_N = R_Nu : l_\infty \to l_\infty^N$. Then we have $\|u_N : l_1(T, \mu) \to l_\infty(T)\| \leq 1$.

Indeed, the extreme points of the unit ball in $l_1(T, \mu)$ are, up to a multiple of modulus 1, of the form $e_t/\mu_t$ for $t \in T$ and we have $\|u_N(e_t/\mu_t)\|_\infty = \|u_N(e_t)\|_\infty/\mu_t \leq 1$.

Now consider $E_{2,\mu} \subset l_2(T, \mu)$ and fix an arbitrary orthonormal basis $(f_j)_{j=1}^n$ in $E_{2,\mu}$. Then

$$\lambda(E) = \text{tr} (u : E \to E) = \text{tr} (u : E_{2,\mu} \to E_{2,\mu})$$

$$= \sum_{j=1}^n (uf_j, f_j)_{l_2(\mu)} = \lim_{N \to \infty} \sum_{j=1}^n (u_N f_j, f_j)_{l_2(\mu)}.$$

The second equality is purely algebraic; for the last one use the fact that $(R_N g, h)$ tends to $(g, h)$ as $N \to \infty$, for all $g, h \in l_2(T, \mu)$. Hence, by Lewis’ idea of how to use the bound for the norm of $u_N$ considered above, we have

$$\lambda(E) \leq \limsup_{N \to \infty} \sum_{t \in T} \left| \sum_{j=1}^n u_N f_j(t) \overline{f_j(t)} \right| \mu_t$$

$$\leq \limsup_{N \to \infty} \sum_{t \in T} \|u_N(\sum_{j=1}^n \overline{f_j(t)} f_j)\|_\infty \mu_t$$

$$\leq \sum_{s,t \in T} \left| \sum_{j=1}^n f_j(s)f_j(t) \right| \mu_s \mu_t,$$

as required. \qed
3 Square function in an extremal case

As a consequence of Proposition 2.2, given a space \( E \subset l_\infty(T) \), an upper bound for \( \lambda(E) \) would follow from an upper estimate for the quantity

\[
\phi(n, T) = \sup_{\mu \in \mathcal{M}} \left( \sum_{s,t \in T} |\sum_{j=1}^n f_j(s)f_j(t)| \mu_s \mu_t \right),
\]

(3.1)

where the outside supremum runs over the set \( \mathcal{M} \) of all discrete probability measures \( \mu \) on \( T \) and the inside supremum runs over all orthonormal bases \( \{f_j\} \) in \( E_{2,\mu} \subset l_2(T, \mu) \).

To estimate (3.1), we first show, using Lagrange multipliers, that the square function of an extremal system \( \{f_j\} \) is constant \( \mu \)-a.e.

We will be mainly concerned with the situation when \( \phi \) really increases at the dimension \( n \),

\[
\phi(n_1, T_1) < \phi(n, T) \quad \text{whenever } n_1 < n \text{ and } T_1 \subset T.
\]

(3.2)

**Proposition 3.1** Let \( n \in \mathbb{N} \) and let \( T = \{1, \ldots, N\} \), or \( T = \mathbb{N} \) satisfy (3.2). Assume that \( \mu^0 \in \mathcal{M} \) and an orthonormal system \( \{f_j^0\}_{j=1}^n \) in \( l_2(T, \mu) \) attains the supremum

\[
\sum_{s,t \in T} |\sum_{j=1}^n f_j^0(s)f_j^0(t)| \mu^0_s \mu^0_t = \phi(n, T).
\]

(3.3)

Then the square function \( f^0 \) is constant \( \mu \)-a.e.,

\[
f^0(s) := \left( \sum_{j=1}^n |f_j^0(s)|^2 \right)^{1/2} = \begin{cases} \sqrt{n} & \text{if } \mu^0_s \neq 0 \\ 0 & \text{if } \mu^0_s = 0 \end{cases}
\]

First notice that if \( \mu^0_s = 0 \) for some \( s \in T \) then \( f_j^0(s) = 0 \) for \( j = 1, \ldots, n \), hence also \( f^0(s) = 0 \). Indeed, otherwise decreasing \( |f_j^0(s)| \) would allow us to multiply all the remaining \( |f_j^0(t)| \) for \( t \neq s \), by \( \xi > 1 \), thus increasing the \( v \) of the sum in (3.3).

Condition (3.2) implies that the matrix \( (f_j^0(s)) \) does not split into a non-trivial block diagonal sum of smaller submatrices.
Lemma 3.2 For all \( l, m = 1, \ldots, n \) we have
\[
\exists l = l_0, \ldots, l_\rho = m \quad \forall 1 \leq r \leq \rho \quad \exists s \in T, \mu_s^0 \neq 0 \quad f^0_{l_{r-1}}(s)f^0_{l_r}(s) \neq 0. \tag{3.4}
\]

Proof For \( 0 < \tau \leq 1 \), by \( \mathcal{M}_\tau \) denote the set of all discrete measures \( \mu \) on \( T \) such that \( \mu(T) = \tau \). By \( \phi(n,T,\tau) \) denote the corresponding supremum, analogous to \( \Phi_1 \), so that \( \phi(n,T) = \phi(n,T,1) \).

It is easy to check that \( \phi(n,T,\tau) = \tau \phi(n,T,1) \). Moreover, \( \phi(n_1,T_1,1) \leq \phi(n,T,1) \) if \( n_1 \leq n \) and \( T_1 \subset T \).

Let \( J_1 \subset \{1, \ldots, n\} \) be a maximal set such that \( \Phi_1 \) is satisfied for all \( l, m \in J_1 \) and let \( J_2 = \{1, \ldots, n\} \setminus J_1 \) be the complement of \( J_1 \). Clearly, \( J_1 \) is non-empty. Let \( T_1 \subset T \) be the set of all \( s \) such that \( f^0_j(s) \neq 0 \) for some \( j \in J_1 \), let \( T_2 = T \setminus T_1 \). By the maximality of \( J_1 \) and the definition of \( T_1 \) we have
\[
f^0_j(s) = 0 \quad \text{whenever} \quad (s,j) \in (T_2 \times J_1) \cup (T_1 \times J_2).
\]

Denote by \( \Phi \) the function whose supremum is taken in \( \Phi_1 \), and by \( \Phi_1 \) and \( \Phi_2 \) the functions given by the analogous formulas, with the summation extended over \( s, t \in T_1 \) and \( j \in J_1 \) for \( \Phi_1 \), and over \( s, t \in T_2 \) and \( j \in J_2 \) for \( \Phi_2 \). We have \( \Phi = \Phi_1 + \Phi_2 \). Moreover, as the functions \( \Phi_i \) involve only sets \( J_i \) and \( T_i \), then \( \Phi_i(z_{sj}, \lambda_s) \leq \phi(n_i, T_i, \tau_i) \), where \( n_i = |J_i| \) and \( \tau_i = \sum_{s \in T_i} \lambda_s^2 \), for \( i = 1, 2 \). Thus
\[
\phi(n,T,1) = F(z_{sj}, \lambda_s) = F_1(z_{sj}, \lambda_s) + F_2(z_{sj}, \lambda_s) \leq \phi(n_1, T_1, \tau_1) + \phi(n_2, T_2, \tau_2) = \tau_1 \phi(n_1, T_1, 1) + \tau_2 \phi(n_2, T_2, 1). \tag{3.5}
\]

Since \( \tau_1 + \tau_2 = 1 \) and \( n_1 > 0 \), the assumption \( \Phi_1 \) implies that the inequality in \( \Phi_1 \) is not possible unless \( \tau_2 = 0 \) and \( n_1 = n \). Thus \( J_1 = \{1, \ldots, n\} \) and hence \( \Phi_1 \) holds for all \( l \) and \( m \), as required. \( \square \)

To simplify the orthogonality conditions, we let
\[
Z_{sj} = f_j(s)\sqrt{\mu_s} \quad \text{and} \quad \lambda_s = \sqrt{\mu_s} \quad \text{for} \quad s \in T, \quad j = 1, \ldots, n. \tag{3.6}
\]

Given the matrix \((Z_{sj})_{s \in T,1 \leq j \leq n}\) we consider “short” vectors \( Z_s = (Z_{sj})_j \in K^n, \ s \in T \), and “long” vectors \( \bar{Z}_j = (Z_{sj})_{s \in T} \in l_2, \ j = 1, \ldots, n. \) The natural scalar product both in \( K^n \) and in \( l_2 \) will be denoted by \( \langle \cdot, \cdot \rangle. \)
We work with the function $F(Z_{sj}, \Lambda_s)$ defined by

$$F(Z_{sj}, \Lambda_s) = \sum_{s, t \in T} |\langle Z_s, Z_t \rangle| \Lambda_s \Lambda_t = \sum_{s, t \in T} |\sum_{j=1}^{n} Z_{sj} \overline{Z}_{tj}| \Lambda_s \Lambda_t. \quad (3.7)$$

Proof of Proposition 3.1 (a) First let $K = R$ and $T = \{1, \ldots, N\}$. We use Lagrange multipliers. Clearly, the supremum $\phi(n, N)$ described in (3.3) is equal to the maximum of $F$ on the surface given by the conditions

$$G_{lm}(Z_{sj}, \Lambda_s) := \langle \tilde{Z}_l, \tilde{Z}_m \rangle - \delta_{lm} = \sum_{s \in T} Z_{sl} \overline{Z}_{sm} - \delta_{lm} = 0 \quad \text{for } 1 \leq l \leq m \leq n \quad (3.8)$$

$$G_0(Z_{sj}, \Lambda_s) := \langle \tilde{\Lambda}, \tilde{\Lambda} \rangle - 1 = \sum_{s \in T} \Lambda_s^2 - 1 = 0 \quad \text{for } s \in T. \quad (3.9)$$

The supremum is attained for a sequence of non-negative $\Lambda_s$; if we set $z_{sj} := f^0_j(s) \sqrt{\mu_s}$ and $\lambda_s := \sqrt{\mu_s}$ for $s \in T$, $j = 1, \ldots, n$, then $F$ attains its maximum at $(z_{sj}, \lambda_s)$.

Consider the Lagrange function $L$ defined by

$$2L(Z_{sj}, \Lambda_s) = F(Z_{sj}, \Lambda_s) - \sum_{l \leq m} \tilde{\gamma}_{lm} G_{lm}(Z_{sj}, \Lambda_s) - \beta G_0(Z_{sj}, \Lambda_s).$$

Assume that $(z_{sj}, \lambda_s)$ is a point where $F$ attains a local maximum subject to (3.8) and (3.9). If for some $1 \leq s, t \leq N$ we had $\langle z_s, z_t \rangle = 0$, we would leave this term out from the sum defining $F$. This would lead to a new sum, defining the new function $F_1$. Clearly, $F_1 \leq F$ and max $F_1 = \text{max } F$. The maximum is attained at the same point $(z_{sj}, \lambda_s)$ and the function $F_1$ is $C^2$ in the neighborhood of this point. Moreover, by setting sgn 0 = 0, in the formulas for derivatives which follow we will still be able to extend the sums over all indices $s, t$.

To use standard necessary conditions for Lagrange multipliers we first check that the point $(z_{sj}, \lambda_s)$ is regular. This means that the gradients $\nabla G_0$ and $\nabla G_{lm}$ for $1 \leq l \leq m \leq n$, are linearly independent vectors in $R^{N(n+1)}$.

Denoting vectors $(z_{sj})_{s=1}^{N}$ by $\tilde{z}_j$ for $j = 1, \ldots, N$ and $(\lambda_s)_{s=1}^{N}$ by $\tilde{\lambda}$, by a straightforward differentiation with respect to $Z_{sj}$ and $\Lambda_s$ we get, for $1 \leq
\( l, m \leq n \) and \( l < m \),

\[
\nabla G_0 = \begin{pmatrix}
0 \\
\vdots \\
0 \\
2\lambda
\end{pmatrix}, \quad \nabla G_{lm} = \begin{pmatrix}
0 \\
\vdots \\
\tilde{z}_m \\
\vdots \\
\tilde{z}_l \\
0 \\
0
\end{pmatrix}, \quad \nabla G_{mm} = \begin{pmatrix}
0 \\
\vdots \\
2\tilde{z}_m \\
\vdots \\
0 \\
0
\end{pmatrix}. \quad (3.10)
\]

In the formula for \( \nabla G_{lm} \), with \( l < m \), \( \tilde{z}_m \) stays on the \( l \)th place and \( \tilde{z}_l \) stays on the \( m \)th place; and in the formula for \( \nabla G_{mm} \), \( 2\tilde{z}_m \) stays on the \( m \)th place. Since \( \lambda_s \neq 0 \) for \( s = 1, \ldots, N \), the linear independence of the gradient vectors (3.10) follows directly from the linear independence of the vectors \( \tilde{z}_l \in \mathbb{R}^N \), for \( l = 1, \ldots, n \); the latter fact is an immediate consequence of the orthogonality, hence linear independence, of the system \( \{f^0_l\}_{l=1}^n \).

Now, the first order condition for Lagrange multipliers states that there exist multipliers \( \tilde{\gamma}_{lm} \) and \( \beta \) such that after setting

\[
\frac{1}{2} \begin{cases}
\tilde{\gamma}_{lm} & \text{if } l < m \\
\tilde{\gamma}_{ml} & \text{if } m < l \\
2\tilde{\gamma}_{ll} & \text{if } m = l
\end{cases}
\]

we have

\[
\frac{\partial L}{\partial Z_{sl}} = \sum_{t \in T} \text{sgn} \langle z_s, z_t \rangle z_t \lambda_s \lambda_l - \sum_{m=1}^n \gamma_{lm} z_{sm} = 0 \\
\text{for } s \in T, l = 1, \ldots, n
\]

(3.11)

\[
\frac{\partial L}{\partial \Lambda_s} = \sum_{t \in T} |\langle z_s, z_t \rangle| \lambda_t - \beta \lambda_s = 0 \quad \text{for } s \in T.
\]

(3.12)

First we simplify (3.11) by a suitable orthogonal transformation. Define two \( N \times N \) matrices \( A \) and \( B \) by

\[
A = (\text{sgn} \langle z_s, z_t \rangle \lambda_s \lambda_t)_{s,t \in T}, \quad B = (|\langle z_s, z_t \rangle|)_{s,t \in T}.
\]

(3.13)
Then the conditions (3.11) and (3.12) can be rewritten as

\[ A \tilde{z}_l = \sum_{m=1}^{n} \gamma_{lm} \tilde{z}_m \quad \text{for} \quad l = 1, \ldots, n \]  

(3.14)

\[ B \tilde{\lambda} = \beta \tilde{\lambda}. \]  

(3.15)

Let \( g = (g_{kl})_{k,l} \) be an \( n \times n \) orthogonal matrix which diagonalizes the hermitian \( n \times n \) matrix \( \Gamma = (\gamma_{lm})_{l,m} \), that is, \( g \Gamma g^* = D_\alpha \) is a diagonal matrix with diagonal entries \( \alpha_1, \ldots, \alpha_n \).

For \( k = 1, \ldots, n \) set \( \tilde{z}'_k = \sum_{l=1}^{n} g_{kl} \tilde{z}_l \). Then \( \tilde{z}_m = \sum_{l=1}^{n} g_{lm} \tilde{z}'_l \), for \( m = 1, \ldots, n \). We have \( \langle z'_s, z'_t \rangle = \langle z_s, z_t \rangle \) for \( 1 \leq s, t \leq N \). Thus the function \( F \) and the matrices \( A \) and \( B \) do not change if we pass from variables induced by the \( \tilde{z}_m \)'s to the variables induced by the \( \tilde{z}'_k \)'s. Similarly, the \( \tilde{z}'_k \)'s satisfy the constraints (3.8) and (3.9). Thus the point \((z'_s, \lambda_s)\) again gives a local extremum of \( F \), but with a new set of multipliers.

Expressing (3.14) in terms of primed vectors \( \tilde{z}'_k \)'s we get the \( n \) eigenvalue equations

\[ A \tilde{z}'_k = \alpha_k \tilde{z}'_k \quad \text{for} \quad k = 1, \ldots, n. \]  

(3.16)

The last two conditions mean that the multipliers corresponding to \((z'_s, \lambda_s)\) are just \( \alpha_1, \ldots, \alpha_n, \beta \), with the off-diagonal ones equal to 0.

Notice that if \( \{f'_0\} \) is related to \( \{\tilde{z}'_k\} \) by (3.6), then \( \{f'_0\} \) is an orthonormal basis in \( \text{span} \{f'_m\} \); in particular the new square function \( f'^0 \) is equal to \( f^0 \). Thus, without loss of generality, we can and will work with these new “primed” vectors, rather than with the original ones; we will leave however the “primes” out, for clarity of notation. In other words, we will assume that \((z_s, \lambda_s)\) satisfies (3.8), (3.9) and (3.15), (3.16).

We want to show that all \( \alpha_k \)'s are equal. To do so, we use the well-known second order conditions for a relative maximum [H]: the Hessian matrix \( H \),

\[
H = \begin{pmatrix}
\frac{\partial^2 L}{\partial Z_{pq} \partial Z_{rk}} & \frac{\partial^2 L}{\partial Z_{pq} \partial \Lambda_l}
\end{pmatrix},
\]

evaluated at the point \((z_{pq}, \lambda_p)\), needs to be negative semi-definite on the tangent space to the surface of constraints at that point.
We have
\[
\frac{\partial^2 L}{\partial Z_{pj} \partial Z_{qk}} = \text{sgn} \langle z_p, z_q \rangle \lambda_p \lambda_q \delta_{jk} - \alpha_k \delta_{pq} \delta_{jk} \quad (3.17)
\]
\[
\frac{\partial^2 L}{\partial \Lambda_p \partial \Lambda_q} = |\langle z_p, z_q \rangle| - \beta \delta_{pq} \quad (3.18)
\]
\[
\frac{\partial^2 L}{\partial \Lambda_p \partial Z_{qk}} = \text{sgn} \langle z_p, z_q \rangle \lambda_q z_{pk} (1 + \delta_{pq}) \quad (3.19)
\]
So \(H\) is an \(N(n+1) \times N(n+1)\) matrix of the form
\[
H = \begin{pmatrix}
A - \alpha_1 I & \ldots & 0 & C_1^t \\
\vdots & \ddots & \vdots & \vdots \\
0 & \ldots & A - \alpha_n I & C_n^t \\
C_1 & \ldots & C_n & B - \beta I
\end{pmatrix}, \quad (3.20)
\]
where \(A\) and \(B\) are defined in (3.13), \(I\) is the identity matrix, and \(C_k\) is the \(N \times N\) matrix \(C_k = (\partial^2 L / \partial \Lambda_p \partial Z_{qk})_{p,q=1}^{N} \) for \(k = 1, \ldots, n\).

The tangent space \(T\) to the surface of constraints described by (3.8) and (3.9) consists of all vectors \(\tilde{w} \in \mathbb{R}^{N(n+1)}\),
\[
\tilde{w} = \begin{pmatrix}
w_{pj} \\
\nu_p
\end{pmatrix} = \begin{pmatrix}
\tilde{w}_1 \\
\vdots \\
\tilde{w}_n \\
\tilde{\nu}
\end{pmatrix} \quad (3.21)
\]
orthogonal to all gradients \(\nabla G_0\) and \(\nabla G_{lm}\) for \(1 \leq l \leq m \leq n\), evaluated at \((z_{pj}, \lambda_p)\). Then necessarily \(\langle H \tilde{w}, \tilde{w} \rangle \leq 0\) for all \(\tilde{w} \in T\).

From (3.10) it follows that \(\tilde{w}\) of the form (3.21) is in the tangent space \(T\) if and only if it satisfies the following equations:
\[
\langle \nabla_{(z_{pj}, \lambda_p)} G_{lm} \tilde{z}, \tilde{w} \rangle = \sum_{p \in T} (z_{pl} w_{pm} + z_{pm} w_{pl}) = \langle \tilde{z}_l, \tilde{w}_m \rangle + \langle \tilde{z}_m, \tilde{w}_l \rangle = 0
\]
for \(1 \leq l \leq m \leq n\) \quad (3.22)
\[
\langle \nabla_{(z_{pj}, \lambda_p)} G_0 \tilde{z}, \tilde{w} \rangle = 2 \sum_{p \in T} \lambda_p \nu_p = 2 \langle \tilde{\lambda}, \tilde{\nu} \rangle = 0. \quad (3.23)
\]

We will now show that for any \(1 \leq l \neq m \leq n\) and any \(s \in T\)
\[
(\alpha_l - \alpha_m) z_{sl} z_{sm} = 0. \quad (3.24)
\]
This will follow from the negative-definiteness of the matrix $H$, by evaluating \( \langle H \tilde{w}, \tilde{w} \rangle \) on suitable vectors $\tilde{w}$.

Consider the vector $\tilde{w}^0$ of the form (3.21) with $\tilde{w}_l = \tilde{z}_m$, $\tilde{w}_m = -\tilde{z}_l$ and $\tilde{w}_k = 0$ otherwise, and $\tilde{\nu} \in \mathbb{R}^N$ arbitrary satisfying (3.23). The orthogonality of the vectors $\tilde{z}_k$ ensured by (3.8) implies that $\tilde{w}^0 \in T$.

Let $\tilde{H}$ denote the $Nn \times Nn$ matrix which appears in the upper left corner of (3.20) and let $\tilde{C}$ be the $N \times Nn$ matrix from the bottom left corner.

A simple calculation using (3.16) shows that
\[
\langle \begin{pmatrix} \tilde{H}_0 & 0 \\ 0 & 0 \end{pmatrix} \tilde{w}^0, \tilde{w}^0 \rangle = \langle \begin{pmatrix} (\alpha_m - \alpha_l)\tilde{z}_l \\ (\alpha_m - \alpha_l)\tilde{z}_m \end{pmatrix}, \begin{pmatrix} -\tilde{z}_l \\ -\tilde{z}_m \end{pmatrix} \rangle = 0.
\]
Thus
\[
\langle H \tilde{w}^0, \tilde{w}^0 \rangle = \langle \begin{pmatrix} \tilde{H} & \tilde{C}^t \\ \tilde{C} & B - \beta I \end{pmatrix} \tilde{w}^0, \tilde{w}^0 \rangle = \langle (B - \beta I)\tilde{\nu}, \tilde{\nu} \rangle + \sum_{k=1}^n \langle C_k \tilde{w}_k, \tilde{\nu} \rangle + \sum_{k=1}^n \langle \tilde{w}_k, C_k^t \tilde{\nu} \rangle = \langle (B - \beta I)\tilde{\nu}, \tilde{\nu} \rangle + 2(C_l\tilde{z}_m - C_m\tilde{z}_l, \tilde{\nu}) \leq 0.
\]

Replacing $\tilde{\nu}$ by $\varepsilon \tilde{\nu}$ and taking $\varepsilon \to 0$ we get that the first term in the final sum (which is of the second order in $\varepsilon$) can be disregarded from the estimate. In the inequality obtained this way $\tilde{\nu}$ can be replaced by $-\tilde{\nu}$, hence $\langle C_l\tilde{z}_m - C_m\tilde{z}_l, \tilde{\nu} \rangle = 0$. Since this equality holds for an arbitrary $\tilde{\nu}$ orthogonal to $\tilde{\lambda}$, we conclude that there is a constant $\gamma$ such that $C_l\tilde{z}_m - C_m\tilde{z}_l = \gamma \tilde{\lambda}$.

Equivalently, the definition of $C_l$ and (3.19) yield that for $p \in T$,
\[
\gamma \lambda_p = z_{pl} \sum_{q \in T} \text{sgn} \langle z_p, z_q \rangle \lambda_q z_{qm} (1 + \delta_{pq}) - z_{pm} \sum_{q \in T} \text{sgn} \langle z_p, z_q \rangle \lambda_q z_{ql} (1 + \delta_{pq}) = z_{pl} \sum_{q \in T} \text{sgn} \langle z_p, z_q \rangle \lambda_q z_{qm} - z_{pm} \sum_{q \in T} \text{sgn} \langle z_p, z_q \rangle \lambda_q z_{ql}.
\]

Observe that by (3.13) and (3.16) we have, for $p \in T$,
\[
\sum_{q \in T} \text{sgn} \langle z_p, z_q \rangle \lambda_p \lambda_q z_{qm} = (A \tilde{z}_m)_p = \alpha_m z_{pm}
\]
and an analogous equality holds for the second term in (3.26). This implies that
\[
\gamma \lambda_p^2 = (\alpha_m - \alpha_l)z_{pm}z_{pl} \quad \text{for} \quad p \in T.
\]
Summation over $p$ yields by (3.8) and (3.9) that
\[ \gamma = (\alpha_m - \alpha_l)(\bar{z}_m, \bar{z}_l) = 0. \]
Hence (3.24) holds. It obviously follows from Lemma 3.2 that
\[ \alpha_m = \alpha_l =: \alpha \quad \text{for all } 1 \leq l, m \leq n. \]  
(3.28)

Expressing (3.16) coordinatewise we have
\[ \sum_{t \in T} \text{sgn} \langle z_s, z_t \rangle z_t \lambda_s - \alpha z_s l = 0 \quad \text{for } l = 1, \ldots, n, \ s \in T. \]

Multiplying by $z_s l$, summing up over $l$ and using (3.15), we find, for $s \in T$,
\[ 0 = \sum_{t \in T} |\langle z_s, z_t \rangle| \lambda_s \lambda_t - \alpha \sum_{l=1}^n z_s^2 = \beta \lambda_s^2 - \alpha \sum_{l=1}^n z_s^2. \]

In terms of $\mu_s$ and $f^0$ this means that $\beta \mu_s = \alpha f^0(s) \mu_s$, i.e., $f^0(s) = \beta / \alpha$ is constant for all $s \in T$ with $\mu_s \neq 0$. If $\mu_s = 0$, then $f^0(s) = 0$, as mentioned already. Since $\sum_{s \in T} |f^0(s)|^2 \mu_s = n = (\beta / \alpha)^2$, we conclude that $\beta / \alpha = \sqrt{n}$, completing the proof in the case (a).

(b) We now consider the infinite case $T = N$ for $K = \mathbb{R}$. Assume that the function $F$ given by (3.7) attains a relative maximum subject to the constraints (3.8) and (3.9) at the point $(z_{sj}, \lambda_s)$, where $z_{sj} = f^0_j(s) \sqrt{\mu_s^0}$ and $\lambda_s = \sqrt{\mu_s^0}$ for $s \in T$, $j = 1, \ldots, n$. Note that $\bar{z}_j = (z_{sj})_{s \in T} \in l_2$, for $j = 1, \ldots, n$, and $\bar{\lambda} = (\lambda_s)_{s \in T} \in l_2$. For $N \in \mathbb{N}$ and an arbitrary vector $\tilde{z} \in l_2$ set $\tilde{z}^N = R_N \tilde{z} \in l_2^N$.

Fix $N \in \mathbb{N}$ sufficiently large so that $\tilde{z}_1^N, \ldots, \tilde{z}_n^N$ are linearly independent. In (3.7)–(3.9) fix the variables for $s > N$ by putting
\[ Z_{sj} = z_{sj}, \ A_s = \lambda_s \quad \text{for } s > N, j = 1, \ldots, n. \]

Relative to the new constraints, $F$ as a function in the variables $(Z_{sj}, A_s)$, with $s = 1, \ldots, N, j = 1, \ldots, n$, attains a relative maximum at $(z_{sj}, \lambda_s)$, with $s = 1, \ldots, N, j = 1, \ldots, n$. The first order Lagrange multiplier conditions (3.11) and (3.12) now take the form
\[ \sum_{t \in T} \text{sgn} \langle z_s, z_t \rangle z_t \lambda_s \lambda_t - \sum_{m=1}^n \gamma_{tm} z_{sm} = 0 \]
\[ \text{for } s = 1, \ldots, N, l = 1, \ldots, n \]  
(3.29)
\[ \sum_{t \in T} |\langle z_s, z_t \rangle| \lambda_t - \beta \lambda_s = 0 \quad \text{for } s = 1, \ldots, N. \]  
(3.30)
For a fixed \( s = 1, \ldots, N \), the first sum in (3.29) is independent of \( N \). Since \( \tilde{z}_1^N, \ldots, \tilde{z}_n^N \) are linearly independent, this uniquely determines the matrix \( \Gamma = (\gamma_{lm})_{l,m=1}^n \), which is then independent of \( N \). Thus (3.29) and (3.30) hold for all \( s \in I N \). Again, we diagonalize the \( n \times n \) matrix \( \Gamma \) and we introduce \( \tilde{z}_1', \ldots, \tilde{z}_n' \) satisfying the eigenvalue equations \( A\tilde{z}_k' = \alpha_k\tilde{z}_k' \) for \( k = 1, \ldots, n \). Moreover, \( B\lambda = \beta\lambda \). Note that \( A \) and \( B \), formally given by (3.13), are now infinite Hilbert–Schmidt matrices. Again, in what follows, we leave the “primes” out and write simply \( \tilde{z}_k \).

Now let \( T \) be the space of (infinite) vectors \( \tilde{w} \) in the direct sum \( \bigoplus l_2 \) of \( n + 1 \) copies of \( l_2 \), which are of the form (3.21) and satisfy (3.22) and (3.23) (with \( T = N \)). Let \( H \) be the (infinite) matrix of the form (3.20), with \( A, B \) and \( C_k \) being infinite as well. To conclude the same proof as in part (a), it suffices to show that \( \langle H\tilde{w}, \tilde{w} \rangle \leq 0 \) for all \( \tilde{w} \in T \).

To this end, denote by \( H^N \) and \( A^N, B^N, C_k^N \) the restricted matrices of order \( N(n+1) \times N(n+1) \) and \( N \times N \) respectively.

The constraints for the restricted problem in the variables \((Z_{sj}, \Lambda_s)\), with \( s = 1, \ldots, N, j = 1, \ldots, n \), are still of the form

\[
G^N_{lm}(Z_{sj}, \Lambda_s) = \sum_{s=1}^{N} Z_{sl}Z_{sm} - d_{lm} = 0
\]

\[
G^N_0(Z_{sj}, \Lambda_s) = \sum_{s=1}^{N} \Lambda_s^2 - d = 0,
\]

for some \( d, d_{lm} \in \mathbb{R} \). This implies that the corresponding tangent space \( T^N \subset \mathbb{R}^{N(n+1)} \) of vectors \( \tilde{w}_N \) of the form (3.21) is defined by the equations

\[
\sum_{p=1}^{N} (z_{pl}w_{pm} + z_{pm}w_{pl}) = 0 \quad \text{for } 1 \leq l \leq m \leq n \quad (3.31)
\]

\[
\sum_{p=1}^{N} \lambda_p \nu_p = 0. \quad (3.32)
\]

Hence, in general, the projection \( \tilde{w}_N^N = R_{N(n+1)} \tilde{w} \) of \( \tilde{w} \) onto \( \mathbb{R}^{N(n+1)} \) is not in \( T^N \), since \( \langle \tilde{z}_l, \tilde{w}_m \rangle + \langle \tilde{z}_m, \tilde{w}_l \rangle = 0 \) for \( \leq l \leq m \leq n \) does not imply \( \langle \tilde{z}_l^N, \tilde{w}_m^N \rangle + \langle \tilde{z}_m^N, \tilde{w}_l^N \rangle = 0 \) for \( \leq l \leq m \leq n \). However, since the limit of (3.31) and (3.32), as \( N \rightarrow \infty \), coincides with (3.24) and (3.23) (for \( T = N \)), it is
clear that for any \( \tilde{w} \in \mathcal{T} \), there is a sequence \((\tilde{w}_N)^\infty_{N=1}\), with

\[
\tilde{w}_N = \left( \begin{array}{c} (\tilde{w}_N)_1 \\ \vdots \\ (\tilde{w}_N)_l \\ \vdots \\ (\tilde{w}_N)_N \end{array} \right) \in \mathcal{T}^N,
\]

such that \( \tilde{w}_N \to \tilde{w} \) in the \( \bigoplus \ell_2 \)-norm.

Since \( \langle H_N \tilde{w}_N, \tilde{w}_N \rangle \leq 0 \) for all \( N \in \mathbb{N} \), it suffices to show that

\[
\lim_{N \to \infty} \langle H_N \tilde{w}_N, \tilde{w}_N \rangle = \langle H \tilde{w}, \tilde{w} \rangle.
\]

This is shown term by term. A typical case is

\[
\lim_{N \to \infty} \langle (A^N - \alpha_l I)(\tilde{w}_N)_l, (\tilde{w}_N)_l \rangle = \langle A\tilde{w}_l - \alpha_l \tilde{w}_l, \tilde{w}_l \rangle,
\]

which reduces to

\[
\langle (\tilde{w}_N)_l, (\tilde{w}_N)_l \rangle \to \langle \tilde{w}_l, \tilde{w}_l \rangle \text{ and } \langle A^N(\tilde{w}_N)_l, (\tilde{w}_N)_l \rangle \to \langle A\tilde{w}_l, \tilde{w}_l \rangle. \tag{3.33}
\]

But (3.33) follows from \( \lim_{N \to \infty} (\tilde{w}_N)_l = \tilde{w}_l \) in the \( \ell_2 \)-norm and the fact that matrices \( A^N \) converge to \( A \) in the Hilbert–Schmidt norm.

As before, we find that \( \alpha_1 = \ldots = \alpha_n =: \alpha \) and \( \beta \mu_s = \alpha f^0(s) \mu_s \). We then complete the proof as in case (a).

\[(c)\] Finally, we indicate the necessary changes in the proof of the complex case, \( \mathcal{K} = \mathcal{C} \). We assume for simplicity that \( T = \{1, \ldots, N\} \). The function \( F \), as defined by (3.7), is now a function of the complex variables \( Z_{sj} = X_{sj} + iY_{sj} \) and the real variables \( \Lambda_s \). We consider \( F \) as a function of real variables \( (X_{sj}, Y_{sj}, \Lambda_s) \). There are now \( n^2 \) real constraints for \( 1 \leq l \leq m \leq n \),

\[
G^{(1)}_{lm}(X_{sj}, Y_{sj}, \Lambda_s) := \text{Re} G_{lm}(Z_{sj}, \Lambda_s) = 0 \quad \text{for } l < m
\]

\[
G^{(2)}_{lm}(X_{sj}, Y_{sj}, \Lambda_s) := \text{Im} G_{lm}(Z_{sj}, \Lambda_s) = 0 \quad \text{for } l < m
\]

\[
G_{ll}(X_{sj}, Y_{sj}, \Lambda_s) := G_{ll}(Z_{sj}, \Lambda_s) = 0 \quad \text{for } l = m, \tag{3.34}
\]

as well as (3.9).

Consider the Lagrange function \( L \) defined by

\[
2L := F - \sum_{l<m} (\tilde{\gamma}^{(1)}_{lm} G^{(1)}_{lm} - \tilde{\gamma}^{(2)}_{lm} G^{(2)}_{lm}) - \sum_l \tilde{\gamma}_{ll} G_{ll} - \beta G_0.
\]
If $F$ attains the extremum subject to conditions (3.9) and (3.34) at $(z_{sj} = x_{sj} + iy_{sj}, \lambda_s)$, then a calculation shows that the first order conditions can be written in the following complex form

$$\frac{\partial L}{\partial X_{sl}} + i \frac{\partial L}{\partial Y_{sl}} = \sum_{t \in T} \text{sgn}(z_s, z_t)z_t\lambda_s\lambda_t - \sum_{m=1}^{n} \gamma_{tm}z_sm = 0,$$

(3.35)

for $s \in T, l = 1, \ldots, n$. Here $\text{sgn} w = w/|w|$ for $w \in \mathbb{C}, w \neq 0$ and $\text{sgn} 0 = 0$. Moreover,

$$\gamma_{lm} = \frac{1}{2} \left\{ \begin{array}{ll}
\tilde{\gamma}_{lm}^{(1)} - i\tilde{\gamma}_{lm}^{(2)} & \text{if } l < m \\
\tilde{\gamma}_{ml}^{(1)} + i\tilde{\gamma}_{ml}^{(2)} & \text{if } m < l \\
2\tilde{\gamma}_{ll} & \text{if } m = l
\end{array} \right.$$ define an $n \times n$ hermitian complex matrix $\Gamma$. Thus we can again diagonalize $\Gamma$ and rewrite (3.33) and (3.12) as

$$A\tilde{z}_k = \alpha_k \tilde{z}_k \quad \text{for } k = 1, \ldots, n \quad \text{and} \quad B\tilde{\lambda} = \beta\tilde{\lambda},$$

(3.36)

where $A$ and $B$ are formally defined as in (3.13).

The tangent space $T$ to the surface of constraints (3.34) and (3.9) now consists of vectors $\tilde{w} \in \mathbb{C}^{N(n+1)}$, whose complex form is formally described by (3.21) and whose real form is

$$\tilde{w} = \begin{pmatrix}
\tilde{u}_1 \\
\tilde{v}_1 \\
\vdots \\
\tilde{u}_n \\
\tilde{v}_n
\end{pmatrix} \in \mathbb{R}^{N(2n+1)}$$

(3.37)

where $\tilde{w}_l = \tilde{u}_l + \tilde{v}_l$ for $l = 1, \ldots, n$. The equations defining $T$ can be written in the following (complex) form

$$\sum_{p \in T} (z_p \tilde{w}_m + w_p \tilde{z}_p) = (\tilde{z}_l, \tilde{w}_m) + (\tilde{w}_l, \tilde{z}_m) = 0$$

for $1 \leq l \leq m \leq n$, \quad (3.38)

$$\langle \tilde{\lambda}, \tilde{\nu} \rangle = 0.$$

(3.39)
The Hessian matrix $H$ in the real form has now size $N(2n+1) \times N(2n+1)$. In particular, the matrix $C$ in (3.25) consists of $2^n$ real matrices

$$C^{(1)}_l = \left( \frac{\partial^2 L}{\partial \Lambda_p \partial X_{ql}} \right)_{p,q=1}^N, \quad C^{(2)}_l = \left( \frac{\partial^2 L}{\partial \Lambda_p \partial Y_{ql}} \right)_{p,q=1}^N$$

for $l = 1, \ldots, n$ of size $N \times N$, evaluated at $(x_{sj}, y_{sj}, \lambda_s)$. The condition $\langle H\tilde{w}, \tilde{w} \rangle \leq 0$ translates into

$$\sum_{l=1}^n \langle C^{(1)}_l \tilde{u}_l + C^{(2)}_l \tilde{v}_l, \tilde{v} \rangle = 0, \quad (3.40)$$

for all $\tilde{v}$ satisfying (3.39). Thus $\sum_{l=1}^n (C^{(1)}_l \tilde{u}_l + C^{(2)}_l \tilde{v}_l)$ is a multiple of $\tilde{\lambda}$, for all $\tilde{w} \in T$ of the (complex) form (3.21) satisfying (3.38) and (3.39).

For $1 \leq l \neq m \leq n$ we pick two different types of vectors in $T$. In the complex form (3.21), the first vector $\tilde{w}$ looks as before, that is, $\tilde{w}_m = \tilde{z}_m$, $\tilde{w}_m = -\tilde{z}_l$ and $\tilde{w}_k = 0$ otherwise, and $\tilde{v} \in \mathbb{R}^N$ arbitrary satisfying (3.39). The second type, $\tilde{w}'$, is defined similarly by setting $\tilde{w}'_l = i\tilde{z}_m$, $\tilde{w}'_m = i\tilde{z}_l$ and $\tilde{w}'_k = 0$ otherwise. The real form of these vectors is the following, writing the non-zero terms only,

$$\tilde{w} = \begin{pmatrix} \bar{x}_m \\ \bar{y}_m \\ -\bar{x}_l \\ -\bar{y}_l \\ \bar{v} \end{pmatrix} \quad \text{and} \quad \tilde{w}' = \begin{pmatrix} -\bar{y}_m \\ \bar{x}_m \\ -\bar{y}_l \\ \bar{x}_l \\ \bar{v} \end{pmatrix}.$$

Both vectors satisfy (3.38) and (3.34). Calculating (3.40) for $\tilde{w}$ and $\tilde{w}'$ and using the eigenvalue equations (3.36) we find, in an analogous way as we obtained (3.27) in case (a), that there are $\gamma_1$ and $\gamma_2$ such that

$$\lambda_p \left( (C,0) \tilde{w} \right)_p = \text{Re} (\alpha_m - \alpha_l) \bar{z}_{pm} z_{pl} = \gamma_1 \lambda_p^2$$

$$\lambda_p \left( (C,0) \tilde{w}' \right)_p = \text{Im} (\alpha_m - \alpha_l) \bar{z}_{pm} z_{pl} = \gamma_2 \lambda_p^2$$

for all $p \in T$. Summing over $p$ and using (3.34) and (3.3) we infer that $\gamma_1 = \gamma_2 = 0$, thus

$$(\alpha_m - \alpha_l) \bar{z}_{pm} z_{pl} = 0.$$

Just as in case (a), the last equality implies $\alpha_1 = \ldots = \alpha_n =: \alpha$ and $\beta \mu_s = \alpha \tilde{f}^0(s) \mu_s$ for all $s \in T$, which completes the proof of (c).
The estimate for the projection constant

We start by a simple but useful lemma of Sidelnikov [Si] and Goethals and Seidel [GS.1]. It gives a lower bound for expressions related to those appearing in the definition (3.1) of $\phi$. Since the bound is essential for our estimate, we include its proof.

**Lemma 4.1** Let $T = \{1, \ldots, N\}$, or $T = N$ and let $(\mu_s)_{s \in T}$ be a probability measure on $T$. Let $(z_s)_{s \in T} \in K^n$ with $\|z_s\|_2 = 1$. Let $\omega$ be the normalized rotation-invariant measure on $S^{n-1} = S^{n-1}(K)$. Then for every even natural number $k \in 2N$,

$$\sum_{s,t \in T} |\langle z_s, z_t \rangle|^k \mu_s \mu_t \geq \int_{S^{n-1}} \int_{S^{n-1}} |\langle z, w \rangle|^k d\omega(z) d\omega(w). \quad (4.1)$$

(In the complex case, express the integrand in the real variables and integrate over $S^{n-1}(C) = S^{2n-1}(R)$.)

**Proof** Let $n \in N$ and $k = 2m \in 2N$. For $z \in K^n$, let $z^\otimes j = z \otimes \ldots \otimes z \in K^{nj}$ denote the $j$-fold tensor product of $z$ with itself, for $j = 1, 2, \ldots$. Scalar products in $K^{nj}$ will be denoted by $\langle \cdot, \cdot \rangle_j$, and for $j = 1$ just by $\langle \cdot, \cdot \rangle$. Then for any $z, w \in K^n$ and $j = 1, 2, \ldots$ we have

$$\langle z^\otimes j, w^\otimes j \rangle_j = \langle z, w \rangle^j,$$

and

$$\langle z^\otimes m \otimes z^\otimes m, w^\otimes m \otimes w^\otimes m \rangle_k = \langle z, w \rangle^m \langle z, w \rangle^m = |\langle z, w \rangle|^k.$$

Consider

$$\xi := \sum_{s \in T} (z_s^\otimes m \otimes z_s^\otimes m) \mu_s - \int_{S^{n-1}} (z^\otimes m \otimes z^\otimes m) d\omega(z) \in K^{nh}.$$ 

By the rotation invariance of $\omega$, integrals of the form $\int_{S^{n-1}} |\langle e, w \rangle|^k d\omega(w)$ do not depend on $e \in S^{n-1}$. This allows to evaluate $\langle \xi, \xi \rangle_k$ as follows:

$$0 \leq \langle \xi, \xi \rangle_k = \sum_{s,t \in T} |\langle z_s, z_t \rangle|^k \mu_s \mu_t + \int_{S^{n-1}} \int_{S^{n-1}} |\langle z, w \rangle|^k d\omega(z) d\omega(w)$$

$$-2 \sum_{s \in T} \mu_s \int_{S^{n-1}} |\langle z_s, w \rangle|^k d\omega(w)$$

$$= \sum_{s,t \in T} |\langle z_s, z_t \rangle|^k \mu_s \mu_t - \int_{S^{n-1}} \int_{S^{n-1}} |\langle z, w \rangle|^k d\omega(z) d\omega(w),$$

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which proves the lemma. □

Now we are ready for the proof of Theorem 1.1.

**Proof of Theorem 1.1 (a)** Let \( n \in \mathbb{N} \) and let \( G(n) \) denote the right hand side of (1.3). We have to show that for any \( n \)-dimensional space \( E \) we have \( \lambda(E) \leq G(n) \). By Proposition 2.2, \( \lambda(E) \leq \phi(n, T) \), where \( T = \{1, \ldots, N\} \), if \( E \subset l^N_N \), and \( T = \mathbb{N} \), if \( E \subset l_\infty \). By Lemma ?? it suffices to show that

\[
\phi(n, T) \leq G(n) \quad \text{for } T = \{1, \ldots, N\}. \tag{4.2}
\]

Given \( n \) and \( T \) we may assume that \( \phi(n_1, T_1) < \phi(n, T) \) for all \( n_1 < n \) and all \( T_1 \subset T \); otherwise the proof which follows would be applied to the minimal \( n_1 \) with \( \phi(n_1, T) = \phi(n, T) \), to show that \( \phi(n, T) \leq G(n_1) < G(n) \).

The double supremum in the definition (3.1) of \( \phi(n, T) \) is attained for some probability measure \( \mu = (\mu_s)_{s \in T} \) on \( T \) and some orthonormal system \( f_j = (f_j(s))_{s \in T} \in l^2(T, \mu), \ j = 1, \ldots, n \). By Proposition 3.1, the square function \( f := (\sum_{j=1}^n |f_j|^2)^{1/2} \) equals \( \sqrt{n} \) for all \( s \) where \( \mu_s \neq 0 \), and equals to 0 otherwise. The span of the \( f_j \)'s is therefore supported by \( S := \text{supp} \mu \subset T \). For \( s \in S \), let \( z_s := n^{-1/2}(f_j(s))_{j=1}^n \in l^2_n \). Then \( \|z_s\|_2 = 1 \) and

\[
\phi(n, T) = n \sum_{s,t \in S} |\langle z_s, z_t \rangle| \mu_s \mu_t. \tag{4.3}
\]

Define \( \alpha \) and \( \beta \) by

\[
\alpha = \begin{cases} 
1/\sqrt{n+2} & K = \mathbb{R} \\
1/\sqrt{n+1} & K = \mathbb{C}
\end{cases}, \quad \beta = \begin{cases} 
3/(n+2) & K = \mathbb{R} \\
2/(n+1) & K = \mathbb{C}
\end{cases}.
\]

Then for \( u \in [-1, 1] \) we have

\[
(|u| - \alpha)^2 = \left((u^2 - \alpha^2)/(|u| + \alpha)\right)^2 \geq (u^2 - \alpha^2)^2/(1 + \alpha)^2.
\]

This implies

\[
|u| \leq \gamma_0 + \gamma_2 u^2 - \gamma_4 u^4 \quad \text{for } u \in [-1, 1], \tag{4.4}
\]

where

\[
\gamma_0 = \frac{\alpha}{2} - \frac{\alpha^3}{2(1+\alpha)^2}, \quad \gamma_2 = \frac{1}{2\alpha} + \frac{\alpha}{(1+\alpha)^2}, \quad \gamma_4 = \frac{1}{2\alpha(1+\alpha)^2}. \tag{4.5}
\]
are non-negative. Equality in (4.4) occurs for \( u \in [-1, 1] \) if and only if \(|u|\) equals to 1 or \( \alpha \). (The right hand side of (4.4) touches \(|u|\) at \( \pm \alpha \) and intersects \(|u|\) at \( \pm 1 \).)

Using (4.4) and (4.1) we can estimate (4.3).

\[
\phi(n, T) \leq n \sum_{s, t \in T} (\gamma_0 + \gamma_2 |\langle z_s, z_t \rangle|^2 - \gamma_4 |\langle z_s, z_t \rangle|^4) \mu_s \mu_t
\]

\[
\leq n \left( \gamma_0 + \gamma_2 /n - \gamma_4 \int_{S_n-1} \int_{S_n-1} |\langle z, w \rangle|^4 d\omega(z) d\omega(w) \right)
= n \gamma_0 + \gamma_2 - \gamma_4 \beta = G(n).
\]

(4.6)

Here we used the orthonormality of the \( f_j \)'s to evaluate the double sum

\[
\sum_{s, t \in T} |\langle z_s, z_t \rangle|^2 \mu_s \mu_t = 1/n
\]

and the fact that for any \( e \in S^{n-1} \),

\[
I := \int_{S_n-1} |\langle e, w \rangle|^4 d\omega(w) = \beta/n,
\]

since \( e.g., \) in the real case,

\[
I = \int_{-1}^{1} t^4 (1 - t^2)^{(n-3)/2} dt / \int_{-1}^{1} (1 - t^2)^{(n-3)/2} dt = 3/(n(n+2));
\]

in the complex case the calculation yields \( I = 2/(n(n+1)) \).

The last equality in (4.6) is established by a direct calculation using (4.3).

(b) and (c) We now assume that \( E \) is an \( n \)-dimensional space attaining the extremal bound, \( \lambda(E) = G(n) \). By Proposition 2.2, there is \( T = \{1, \ldots, N\} \) or \( T = \mathbb{N} \), a probability measure \( \mu = (\mu_s)_{s \in T} \) on \( T \) and an orthonormal basis \( (f_j)_{j=1}^n \) in \( E_{2, \mu} \) such that

\[
\lambda(E) \leq \sum_{s, t \in T} |\sum_{j=1}^n f_j(s) \overline{f_j(t)}| \mu_s \mu_t. \quad (4.7)
\]

For all \( n_1 < n \) and all \( T_1 \subset T \) we have \( \phi(n_1, T_1) < \phi(n, T) \); otherwise, for some \( n_1 < n \) and some \( T_1 \subset T \) we would have, by part (a),

\[
G(n) = \lambda(E) \leq \phi(n, T) \leq \phi(n_1, T_1) \leq G(n_1) < G(n).
\]
Therefore, by Proposition 3.1 on the support $S \subseteq T$ of $\mu$, the square function $f = (\sum_{j=1}^n |f_j|^2)^{1/2}$ is equal to $\sqrt{n}$. For $s \in S$ consider again the short vectors $z_s = (f_j(s))_{j=1}^n/\sqrt{n}$. Hence $\|z_s\|_2 = 1$ for $s \in S$. We may and will further assume that $S$ is minimal in the sense that for $s \neq t$ we have $z_s \neq \theta z_t$, with $|\theta| = 1$. Otherwise, we could replace the short vectors $z_s$ and $z_t$ by one vector $z_s$, assigning to it the measure $\mu_s + \mu_t$; the orthogonality and the normalization of the corresponding long vectors and the double sum in (4.7) would remain unchanged. Let $N := |S|$. We have to show that $N$ is finite, and, in fact, bounded by $N(n)$ as defined in (1.2).

By (4.7), (1.4) and (1.6) we have
\[
\lambda(E) = n \sum_{s,t \in S} |\langle z_s, z_t \rangle| \mu_s \mu_t
\leq n \sum_{s,t \in S} \left( \gamma_0 + \gamma_2 |\langle z_s, z_t \rangle|^2 - \gamma_4 |\langle z_s, z_t \rangle|^4 \right) \mu_s \mu_t
\leq G(n).
\]

Thus, the assumption $\lambda(E) = G(n)$ implies the equality of all terms. The equality in the first inequality requires that $|\langle z_s, z_t \rangle| = \alpha$ or 1 for all $s, t \in S$ (note that $\mu_s \neq 0$ for $s \in S$). For $s \neq t$, $z_s \neq \theta z_t$, hence $|\langle z_s, z_t \rangle| = \alpha$. Recall that $\alpha = 1/\sqrt{n + 2}$ in the real case, and $\alpha = 1/\sqrt{n + 1}$ in the complex case. We thus proved that the vectors $(z_s)_{s \in S} \subseteq S^{n-1}(K)$ are equiangular. Since in $K^n$ there are at most $N(n)$ equiangular vectors, it follows that $N = |S| \leq N(n)$. Using the Cauchy–Schwartz inequality, we get another chain of inequalities which become equalities,
\[
G(n) = n \left( \sum_{s,t \in S} \mu_s \mu_t \alpha + \sum_{s \in S} \mu_s^2 (1 - \alpha) \right)
\geq n\alpha + (n/N)(1 - \alpha) \geq n\alpha + (n/N(n))(1 - \alpha) = G(n),
\]
where the last equality follows by a direct calculation, inserting the value of $\alpha$. The equality implies, in particular, that $N = N(n)$. Also, all values of $\mu_s$ have to be equal ($\mu_s = 1/N(n)$). Since the vectors $f_j$ are all supported by $S$, it follows that $E$ is isometric to a subspace of $l^\infty(N(n))$. The orthogonal projection, given by the matrix $(n/N(n))((z_s, z_t))_{s,t}$, is a minimal projection.

Conversely, if in $K^n$ there exist $N(n)$ equiangular vectors $(z_s)$, we may construct $E = \text{span}[f_1, \ldots, f_n] \subseteq l^\infty(N(n))$ by letting $f_j(s) = \sqrt{n} z_{sj}$ ($j = 1, \ldots, n$, $s = 1, \ldots, N(n)$). By [K], the projection constant of $E$ is equal
to $G(n)$, and the $f_j$’s are orthonormal with respect to the equidistributed probability measure $\mu$ on \{1, \ldots, $N(n)\}. Moreover, $P$, given by the matrix $(n/N(n))\langle z_s, z_t \rangle_{s,t=1}^{N(n)}$, and acting as an operator from $l_\infty^{N(n)}$ to $l_\infty^{N(n)}$, is a minimal (and orthogonal) projection onto $E$ with norm $G(n)$.

Either way, the norm of a vector $\sum_{j=1}^n \alpha_j f_j$ in $l_\infty^{N(n)}$ is given by

$$\|\sum_{j=1}^n \alpha_j f_j\|_\infty = \sup_{1 \leq s \leq N(n)} \sqrt{n} |\langle \alpha, z_s \rangle|.$$  

Thus, given $N(n)$ equiangular vectors $(z_s)$ in $K^n$, we get an $n$-dimensional normed space with the maximal projection constant by setting

$$\|\langle \alpha \rangle_{j=1}^n\| := \sup_{1 \leq s \leq N(n)} |\langle \alpha, z_s \rangle|.$$  \hspace{1cm} (4.8)

In the real case, $N(n) = n(n+1)/2$ equiangular vectors exist in $R^n$ for $n = 2, 3, 7, 23$ and these systems are unique up to orthogonal transformations. Hence the real spaces with projection constant $G(n)$ are unique up to isometry if $n = 2, 3, 7, 23$. For $n = 2$, the uniqueness (up to orthogonal transformations) of the three vectors at angle $2\pi/3$ each, is trivial. For $n = 3$ one considers the $6 \times 6$ Gram matrix $(\langle z_s, z_t \rangle)$, with $|\langle z_s, z_t \rangle| = 1/\sqrt{5}$ for $s \neq t$. It is easy to see that up to permutations and multiplications of the $z_s$’s by $-1$, the sign pattern is uniquely determined. The standard paper on the subject is Lemmens, Seidel [LS]; for the uniqueness for $n = 7, 23$ we refer to Goethals, Seidel [GS.2] and Seidel [S]. For $n = 2$, (4.8) yields the norm with the (regular) hexagonal unit ball; for $n = 3$, the extremal ball defined via (4.8) is the (regular) dodecahedron, since the 6 equiangular vectors in $R^3$ are the diagonals of the icosahedron.

In the complex case, $N(n) = n^2$ equiangular vectors exist in $C^n$ at least for $n = 2, 3$. For $n = 2$, the system and the extremal space are again unique up to isometry. For $n = 3$, the system of 9 vectors in $C^3$ is connected to the Hessian polyhedron, cf. Coxeter [C].

\[\]  

**Remark** Part (a) of the previous proof also shows that Theorem 1.2 can be restated as

$$\max_{E \in F_n} \lambda(E) = \max_{\mu} \max_{z_s} \sum_{s,t \in N} |\langle z_s, z_t \rangle| \mu_s \mu_t.$$  \hspace{1cm} (4.9)
where the double maximum is taken over all discrete probability measures \( \mu = (\mu_s)_{s \in \mathbb{N}} \) and all sets of unit vectors \((z_s)_{s \in \mathbb{N}} \subset S^{n-1}(\mathbb{K})\) such that

\[
Id_{\mathbb{K}^n} = n \sum_{s \in \mathbb{N}} \mu_s z_s \otimes z_s.
\]

**Example** In \( \mathbb{R}^4 \), consider the following 10 vectors of the form

\[
x_s = \frac{1}{\sqrt{12}} \begin{pmatrix} 3 \\ -1 \\ -1 \\ -1 \end{pmatrix}, \quad x_s = \frac{1}{\sqrt{2}} \begin{pmatrix} \sin \alpha \\ \sin \alpha \\ -\cos \alpha \\ -\cos \alpha \end{pmatrix},
\]

permuting the 3 to all places in the first type of vectors \((1 \leq s \leq 4)\) and permuting the two \(-\cos \alpha\) in the second type of vectors \((5 \leq s \leq 10)\). Set \( a = -\sin 2\alpha + 1/2 \). One checks that

\[
4 \left( \sum_{s=1}^{4} (a/2(1 + 2a)) x_s \otimes x_s + \sum_{s=5}^{10} (1/6(1 + 2a)) x_s \otimes x_s \right) = Id_{\mathbb{R}^4}.
\]

Hence, letting \( \mu_s = a/2(1 + 2a) \) for \( 1 \leq s \leq 4 \) and \( \mu_s = 1/6(1 + 2a) \) for \( 5 \leq s \leq 10 \), we see that the \( x_s \)'s and \( \mu_s \)'s satisfy the constraints in (4.9). The scalar products \( |\langle x_s, x_t \rangle|\) satisfy the following: for \( 1 \leq s \neq t \leq 4 \) they are equal to \( 1/3 \); for \( 5 \leq s \neq t \leq 10 \) they take two values, \( (1 - \sin 2\alpha)/2 \) appears 24 times and \( |\sin 2\alpha|/2 \) appears 6 times; for \( 1 \leq s \leq 4 \) and \( 5 \leq t \leq 10 \), they are equal to \( (1/\sqrt{6})|\sin \alpha + \cos \alpha|\).

The maximum of the function

\[
\sum_{s,t=1}^{10} |\langle x_s, x_t \rangle| \mu_s \mu_t
\]

is equal to 1.8494 and it is attained for \( \alpha = 1.4592 \).

In \( \mathbb{R}^4 \), 10 equiangular vectors do not exist. By Theorem 1.1 and (4.9), the maximal projection constant \( \lambda = \sup \lambda(E_4) \), for 4-dimensional real spaces \( E_4 \) satisfies

\[
1.8494 \leq \lambda < (2 + 3\sqrt{6})/5 \sim 1.8697.
\]

The known explicit examples of equiangular lines allow to write down the extremal norms in the cases mentioned above, using (4.8).
| $K^n$ | $|| (\alpha_j)^2 ||$ | $\lambda(X)$ |
|------|-----------------|-------------|
| $\mathbb{R}^2$ | $\max(|2\alpha_1|, |\alpha_1 - \sqrt{3}\alpha_2|, |\alpha_1 + \sqrt{3}\alpha_2|)$ | $4/3$ hexagon |
| $\mathbb{R}^3$ | $\max_+ (|\tau \alpha_1 \pm \sigma \alpha_2|, |\tau \alpha_2 \pm \sigma \alpha_3|, |\tau \alpha_3 \pm \sigma \alpha_1|)$ where $\tau := \sqrt{(\sqrt{5} + 1)/2}$, $\sigma := \sqrt{(\sqrt{5} - 1)/2}$ | $\sqrt{5} + 1/2$ dodecahedron |
| $\mathbb{R}^7$ | $\max \left( \max_{1 \leq i < j \leq 7} |\alpha_i + \alpha_j|, \max_{1 \leq j \leq 7} \left| \sum_{i=1, i \neq j}^7 \alpha_i \right| \right)$ | $5/2$ |
| $\mathbb{R}^{23}$ | the norm is connected to points in the Leech lattice | $14/3$ |
| $\mathbb{C}^2$ | $\max( |\sqrt{3}\alpha_1 + \alpha_2|, |\alpha_1 + \sqrt{3}\alpha_2|, |\alpha_1 + i\alpha_2|, |\alpha_1 - i\alpha_2|)$ | $1 + \sqrt{3}/2$ |
| $\mathbb{C}^3$ | $\max_{j=1,2,3} (|\alpha_j - \alpha_{j+1}|, |\alpha_j - \omega \alpha_{j+1}|, |\alpha_j - \omega^2 \alpha_{j+1}|)$ where $\omega := \exp(2\pi/3)$ and $\alpha_4 := \alpha_1$ | $5/3$ |

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