Euclidean hypersurfaces with a totally geodesic foliation of codimension one

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Abstract

We classify the hypersurfaces of Euclidean space that carry a totally geodesic foliation with complete leaves of codimension one. In particular, we show that rotation hypersurfaces with complete profiles of codimension one are characterized by their warped product structure. The local version of the problem is also considered.

A smooth foliation \( \mathcal{F} \) on an \( n \)-dimensional Riemannian manifold \( M^n \) is totally geodesic if all leaves of \( \mathcal{F} \) are totally geodesic submanifolds of \( M^n \), that is, if any geodesic of \( M^n \) that is tangent to \( \mathcal{F} \) at some point is contained in the leaf of \( \mathcal{F} \) through that point.

Several authors have investigated whether on a given Riemannian manifold \( M^n \) there exists a totally geodesic foliation of codimension one, that is, with \( (n-1) \)-dimensional leaves, as well as the inverse problem of determining whether one can find a Riemannian metric on a manifold \( M^n \) with respect to which a given smooth foliation of codimension one on \( M^n \) becomes totally geodesic. We refer to the work of Ghys [5] for a complete solution of the latter problem in the compact as well as some noncompact cases, where several other references can be found. See also [10] for further discussion on the subject as well as an updated list of references.

In this paper, we address the following related and rather basic extrinsic problem: What are all Euclidean hypersurfaces \( f: M^n \to \mathbb{R}^{n+1}, n \geq 3 \), that carry a foliation of codimension one with totally geodesic (complete or not) leaves?

First, we discuss several families of solutions to this problem, starting with some trivial ones.
Flat hypersurfaces. These are isometric immersions $f: U \to \mathbb{R}^{n+1}$ of open subsets $U \subset \mathbb{R}^n$, which admit foliations by (open subsets of) affine hyperplanes. Complete flat Euclidean hypersurfaces are well known to be cylinders over plane curves [6].

Surface-like hypersurfaces. For a surface $g: L^2 \to \mathbb{R}^3$, let $\mathcal{D}_0$ be the one-dimensional distribution on $L^2$ spanned by the tangent directions to a foliation of $L^2$ by geodesics. Set $M^n = L^2 \times \mathbb{R}^{n-2}$ and define an isometric immersion $f: M^n \to \mathbb{R}^{n+1}$ by $f = g \times \text{id}$, where $\text{id}: \mathbb{R}^{n-2} \to \mathbb{R}^{n-2}$ is the identity map. Then $\mathcal{D} = \mathcal{D}_0 \oplus \mathbb{R}^{n-2}$ is clearly a totally geodesic distribution on $M^n$ of codimension one, whose leaves are complete whenever the same holds for those of $\mathcal{D}_0$. We call $f$ a cylindrical surface-like hypersurface.

Similar examples, but with never complete leaves, can be constructed by starting with a surface in the sphere $g: L^2 \to S^3 \subset \mathbb{R}^4$, with $\mathcal{D}_0$ as before, and defining $M^n = L^2 \times \mathbb{R}^{n-3}$ and $f = C(g) \times \text{id}$, where $C(g): L^2 \times \mathbb{R}^+ \to \mathbb{R}^4$, given by $C(g)(x,t) = tg(x)$, is the cone over $g$ in $\mathbb{R}^4$. Then, the distribution $\mathcal{D} = \mathcal{D}_0 \oplus \mathbb{R} \oplus \mathbb{R}^{n-3}$ on $M^n$ is again totally geodesic of rank $n - 1$, that is, $\dim \mathcal{D}(x) = n - 1$ for any $x \in M^n$. In this case, we say that $f$ is a conical surface-like hypersurface.

Ruled hypersurfaces. Nonflat ruled hypersurfaces $f: M^n \to \mathbb{R}^{n+1}$ carry a smooth foliation of codimension one by (open subsets of) affine subspaces of $\mathbb{R}^{n+1}$, the rulings of $f$. Thus, the rulings are totally geodesic in $\mathbb{R}^{n+1}$, hence also in $M^n$. For complete examples see Example 1 below.

Partial tubes. Let $\gamma: I \subset \mathbb{R} \to \mathbb{R}^{n+1}$ be a unit speed curve. Consider a hypersurface $N^{n-1}$ of the (affine) normal space to $\gamma$ at some point and parallel transport $N^{n-1}$ along $\gamma$ with respect to the normal connection. Then, if $N^{n-1}$ lies in a suitable open subset of that normal space (as described in the next section), this generates an $n$-dimensional hypersurface $M^n$ of $\mathbb{R}^{n+1}$, called the partial tube over $\gamma$ with fiber $N^{n-1}$. It turns out that the parallel translates of $N^{n-1}$ give rise to a totally geodesic foliation of $M^n$ of codimension one, whose leaves are complete if so is $N^{n-1}$.

Partial tubes were introduced in [1] and [2] and will be discussed in more detail in the next section. We point out that, starting with a unit speed circle $\gamma: I \to \mathbb{R}^2 \subset \mathbb{R}^{n+1}$, the preceding construction yields a rotation hypersurface of $\mathbb{R}^{n+1}$ having $N^{n-1}$ as profile.

The classes of hypersurfaces just described are clearly not disjoint. For instance, the class of flat hypersurfaces is precisely the intersection of the
classes of ruled hypersurfaces and partial tubes over curves. In fact, flat hypersurfaces free of totally geodesic points correspond to partial tubes over curves with fiber a totally geodesic hypersurface $N^{n-1}$ of a fixed normal space to the curve. On the other hand, a surface-like hypersurface is also a partial tube if and only if the integral (geodesic) curves of $D_0$ are also lines of curvature of $g$, i.e., the surface $g$ is itself a partial tube over an orthogonal trajectory of $D_0$. Moreover, a surface-like hypersurface is flat (respectively, ruled) if and only if the surface $g$ has index of relative nullity one (respectively, is ruled).

In view of the discussion in the preceding paragraph, it is easy to construct examples, even complete ones, where different types of hypersurfaces are smoothly attached. This is illustrated by the following simple class of examples.

**Example 1.** Let $c: \mathbb{R} \to \mathbb{R}^{n+1}$ be a smooth curve parametrized by arc-length $s$ with curvatures $\kappa_1, \ldots, \kappa_n$ in a Frenet frame $e_1 = c', e_2, \ldots, e_{n+1}$. Assume that $\kappa_1 > 0$ along $c$ and that $\kappa_j = 0$ for $s \leq 0$ and $\kappa_j > 0$ for $s > 0$, $j \geq 2$. Then, the complete hypersurface $F: \mathbb{R}^n \to \mathbb{R}^{n+1}$ parametrized by

$$F(s, t_1, \ldots, t_{n-1}) = c(s) + \sum_{j=1}^{n-1} t_j e_{j+2}$$

is surface-like for $s \leq 0$ and ruled but not surface-like for $s > 0$. Of course, the ruled surface factor for $s \leq 0$ can be deformed or replaced by a non-ruled one. Notice that if $\kappa_j > 0$ on all of $\mathbb{R}$ then $F$ is ruled and complete.

Under an assumption of global nature, our main result shows that partial tubes over curves and ruled hypersurfaces are the only examples with complete leaves. Given a hypersurface $f: M^n \to \mathbb{R}^{n+1}$, $n \geq 3$, we say that $f(M)$ contains a surface-like strip if there exists an open subset $U \subset M^n$ isometric to a product $L^2 \times \mathbb{R}^{n-2}$ where $f$ splits as $f = g \times id$, with $g: L^2 \to \mathbb{R}^3$ an isometric immersion and $id: \mathbb{R}^{n-2} \to \mathbb{R}^{n-2}$ the identity map.

**Theorem 2.** Let $f: M^n \to \mathbb{R}^{n+1}$, $n \geq 3$, be an isometric immersion of a nowhere flat connected Riemannian manifold that carries a totally geodesic foliation of codimension one with complete leaves. If $f(M)$ does not contain any surface-like strip then it is either ruled or a partial tube over a curve.

If $f: M^n \to \mathbb{R}^{n+1}$ is an isometric immersion of a Riemannian manifold with positive sectional curvatures, then neither $f$ can be ruled on any open subset nor $f(M)$ can contain any surface-like strip. Thus we obtain the following immediate consequence of the preceding result.
Corollary 3. Let \( f: M^n \to \mathbb{R}^{n+1}, n \geq 3, \) be an isometric immersion of a Riemannian manifold with positive sectional curvatures that carries a totally geodesic foliation of codimension one with complete leaves. Then \( f(M) \) is a partial tube over a curve.

A Riemannian manifold \( M^n \) that carries a totally geodesic foliation of codimension one is locally (globally, if \( M^n \) is simply connected and the leaves of the foliation are complete) isometric to a product manifold \( N^{n-1} \times \mathbb{R} \), with a twisted product metric \( d\sigma^2 + \rho^2 dt^2 \), where \( d\sigma^2 \) is a fixed metric on \( N^{n-1} \) and \( \rho \in C^\infty(N^{n-1} \times \mathbb{R}) \) (see [9, Theorem 1]). Thus, an equivalent statement of Theorem 2 is that an isometric immersion \( f: M^n \to \mathbb{R}^{n+1}, n \geq 3, \) of a twisted product manifold \( M^n = N^{n-1} \times \rho I \), where \( I \subset \mathbb{R} \) is an open interval and \( N^{n-1} \) is a complete manifold free of flat points, is either ruled or a partial tube over a curve, as soon as \( f(M) \) does not contain any surface-like strip.

For isometric immersions \( f: M^n \to \mathbb{R}^{n+1}, n \geq 3, \) of a warped product manifold \( M^n = N^{n-1} \times \rho I \), in which case \( \rho \) depends only on \( N^{n-1} \), we prove the following result.

Theorem 4. Let \( f: M^n \to \mathbb{R}^{n+1}, n \geq 3, \) be an isometric immersion of a warped product connected Riemannian manifold \( M^n = N^{n-1} \times \rho I \) where \( N^{n-1} \) is a complete manifold free of flat points, \( \rho \in C^\infty(N) \) and \( I \subset \mathbb{R} \) is an open interval. If \( f(M) \) does not contain any surface-like strip then it is a rotation hypersurface having \( N^{n-1} \) as profile.

Note that the assumption that \( f(M) \) does not contain any surface-like strip here means that \( N^{n-1} \) does not contain any open subset \( U \) isometric to \( \mathbb{R}^{n-2} \times J \), where \( J \subset \mathbb{R} \) is an open interval, such that \( f|_{U \times I} \) splits as \( f = id \times f_1 \), with \( f_1: J \times \rho I \to \mathbb{R}^3 \) an isometric immersion.

The case in which \( M^n \) is assumed to be compact in Theorem 4 was already considered in [7]. Isometric immersions \( f: M^n \to \mathbb{R}^{n+1}, n \geq 3, \) of a warped product connected Riemannian manifold free of flat points \( M^n = N^{n-k} \times \rho L^k \), with \( k \geq 2 \), were shown in [3] to be, even locally, either rotation hypersurfaces, products with \( \mathbb{R}^k \) of hypersurfaces of \( \mathbb{R}^{n-k+1} \) or products with \( \mathbb{R}^{k-1} \) of cones over hypersurfaces of \( S^{n-k+1} \subset \mathbb{R}^{n-k+2} \).

Next, we consider the local version of the problem stated in the beginning of the introduction. We prove that exactly one further class of examples may occur.

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Theorem 5. Let \( f: M^n \to \mathbb{R}^{n+1}, n \geq 3 \), be an isometric immersion of a Riemannian manifold that carries a totally geodesic foliation of codimension one. Then, there exists an open dense subset of \( M^n \) where \( f \) is locally either surface-like, ruled, a partial tube over a curve or an envelope of a one-parameter family of flat hypersurfaces.

A hypersurface \( f: M^n \to \mathbb{R}^{n+1} \) is called the envelope of a one-parameter family of hypersurfaces \( F_t: M^n_t \to \mathbb{R}^{n+1} \) if there exists an integrable smooth distribution \( D \) of rank \( n-1 \) on \( M^n \) for each leaf \( \sigma_t \) of which one has an embedding \( j_t: \sigma_t \to M^n_t \) such that \( F_t \circ j_t = f|_{\sigma_t} \) and \( F_t \ast T_{j_t(x)} M_t = f \ast T_x M \) for any \( x \in \sigma_t \).

Geometrically, the hypersurface \( F_t \) is tangent to \( f \) along the leaf \( \sigma_t \) of \( D \), called the characteristic of the one-parameter family \( \{F_t\} \) at level \( t \). If the one-parameter family of hypersurfaces \( F_t \) is locally defined by the equation \( G(t,x) = 0 \) where \( x = (x_1, \ldots, x_{n+1}) \), then the envelope of the family is locally given by

\[
\begin{align*}
G(t,x) &= 0 \\
G_t(t,x) &= 0,
\end{align*}
\]

where the subscript denotes partial derivative with respect to \( t \). The characteristic at level \( t = t_0 \) is then the set of solutions of the preceding system for \( t = t_0 \). In Theorem 5, the leaves of the totally geodesic foliation of \( M^n \) are precisely the characteristics of the one-parameter family of flat hypersurfaces that envelope \( f \).

1 Partial tubes

We first recall the precise definition of a partial tube, and then state a result from [11] that implies that partial tubes over curves are precisely the solutions to our problem for which the orthogonal trajectories to the totally geodesic foliation are lines of curvature of the hypersurface.

Let \( \gamma: I \subset \mathbb{R} \to \mathbb{R}^N \) be a unit speed curve and let \( \{\xi_1, \ldots, \xi_m\} \) be an orthonormal set of parallel normal vector fields along \( \gamma \). Hence, the vector subbundle \( E = \text{span}\{\xi_1, \ldots, \xi_m\} \) of the normal bundle of \( \gamma \) is parallel and flat. Then the map \( \phi: I \times \mathbb{R}^m \to E \) given by

\[
\phi_s(y) = \phi(s,y) = \sum_{i=1}^m y_i \xi_i(s)
\]
for \( s \in I \) and \( y = (y_1, \ldots, y_m) \in \mathbb{R}^m \), is a parallel vector bundle isometry. Let \( f_0 : N^{n-1} \rightarrow \mathbb{R}^m \) be a substantial isometric immersion, i.e., an immersion whose codimension cannot be reduced. Denote \( M^n = N^{n-1} \times I \) and define a map \( f : M^n \rightarrow \mathbb{R}^N \) by
\[
f(p, s) = \gamma(s) + \phi_s(f_0(p)).
\]
One can check that \( f \) is an immersion whenever \( f_0(N) \subset \Omega(\gamma; \phi) \), where
\[
\Omega(\gamma; \phi) = \{ Y \in \mathbb{R}^m : \langle \gamma''(s), \phi_s(Y) \rangle \neq 1 \text{ for any } s \in I \}.
\]
In this case, we say that \( f(M) \) is the partial tube over \( \gamma \) with fiber \( f_0 \). Endowing \( M^n \) with the induced metric, the distribution on \( M^n \) given by the tangent spaces to the first factor is totally geodesic. Moreover, the second fundamental form of \( f \) satisfies
\[
\alpha_f(X, \partial/\partial s) = 0 \quad \text{for any } X \in T\mathbb{N},
\]
where \( \partial/\partial s \) is a unit vector field tangent to the factor \( I \).

If \( M^n \) is not simply connected and \( \pi : \tilde{M}^n \rightarrow M^n \) is its universal covering, then the map \( \tilde{f} = f \circ \pi \) satisfies \( \tilde{f}(\tilde{M}) = f(M) \). Therefore, in proving that \( f(M) \) is a partial tube there is no loss of generality in assuming that \( M^n \) is simply connected.

The next result is a direct consequence of Theorem 16 in [11].

Theorem 6. Let \( f : M^n \rightarrow \mathbb{R}^N \) be an isometric immersion of a twisted product \( M^n = N^{n-1} \times \rho I \), where \( I \subset \mathbb{R} \) is an open interval and \( \rho \in C^\infty(M) \). Assume that the second fundamental form of \( f \) satisfies
\[
\alpha_f(X, \partial/\partial s) = 0 \quad \text{for any } X \in T\mathbb{N}.
\]
Then \( f(M) \) is a partial tube over a curve.

In view of the discussion after Corollary 3, this yields the following result.

Corollary 7. Let \( f : M^n \rightarrow \mathbb{R}^{n+1} \) be an isometric immersion carrying a smooth totally geodesic foliation of rank \( n - 1 \) whose orthogonal trajectories are lines of curvature of \( f \). Then \( f(M) \) is locally a partial tube over a curve.

2 Hypersurfaces with a curvature invariant distribution

In this section we prove a preliminary result on oriented Euclidean hypersurfaces that carry a curvature invariant distribution.
That a smooth distribution $\mathcal{D}$ on a Riemannian manifold $M^n$ is \textit{curvature invariant} means that

$$R(X,Y)Z \in \mathcal{D} \quad \text{for all} \quad X, Y, Z \in \mathcal{D}, \quad (2)$$

where $R$ denotes the curvature tensor of $M^n$. By $\mathcal{D}$ being \textit{totally geodesic} we mean that

$$\nabla_X Y \in \mathcal{D} \quad \text{for all} \quad X, Y \in \mathcal{D}.$$ 

In particular, any totally geodesic distribution $\mathcal{D}$ in the above sense is curvature invariant and integrable, and its leaves are totally geodesic submanifolds of $M^n$. For results on curvature invariant distributions we refer to [12] and the references therein.

For an oriented Euclidean hypersurface, we always denote by $A$ its shape operator with respect to a globally defined smooth unit normal vector field, and by $\Delta = \ker A$ its \textit{relative nullity} distribution.

\textbf{Proposition 8.} \textit{Let} $f : M^n \to \mathbb{R}^{n+1}$ \textit{be an oriented hypersurface carrying a curvature invariant distribution} $\mathcal{D}$ \textit{of rank} $k > 1$. \textit{Then one of the following possibilities holds pointwise:}

(i) $A(\mathcal{D}) \subset \mathcal{D}^\perp$,
(ii) $A(\mathcal{D}) \subset \mathcal{D}$,
(iii) $\text{rank} \, \mathcal{D} \cap \Delta = \text{rank} \, \mathcal{D} - 1$.

\textit{Proof.} By the Gauss equation

$$R(X,Y)Z = \langle AY, Z \rangle AX - \langle AX, Z \rangle AY$$

we have that (2) on a hypersurface is equivalent to

$$\langle AY, Z \rangle AX - \langle AX, Z \rangle AY \in \mathcal{D} \quad \text{for all} \quad X, Y, Z \in \mathcal{D}. \quad (3)$$

At a point, let $X_1, \ldots, X_k$ be a local orthonormal base of $\mathcal{D}$ such that

$$AX_j = \lambda_j X_j + V_j$$

where $V_j \in \mathcal{D}^\perp$ and $\lambda_j \in \mathbb{R}$. If $\lambda_j = 0$ for $1 \leq j \leq k$ then (i) holds. Otherwise, we may assume that $\lambda_1 \neq 0$. Applying (3) for $Y = Z = X_1$ and $X = X_j$, $j \geq 2$, we obtain

$$\lambda_1 (\lambda_j X_j + V_j) \in \mathcal{D},$$
hence $V_j = 0$ for $2 \leq j \leq k$. On the other hand, we have from (3) for $Y = Z = X_j, j \geq 2$, and $X = X_1$ that

$$\lambda_j (\lambda_1 X_1 + V_1) \in \mathcal{D}.$$ 

We conclude that either $V_1 = 0$ or $\lambda_j = 0$ for $2 \leq j \leq k$, which correspond to cases (ii) and (iii), respectively. \)

\section{The global case}

We first construct a suitable tangent orthonormal frame on hypersurfaces that carry a totally geodesic distribution of codimension one and satisfy condition (iii) in Proposition \[8\].

\textbf{Lemma 9.} Let $f: M^n \to \mathbb{R}^{n+1}$ be an isometric immersion of a Riemannian manifold that carries a totally geodesic distribution $\mathcal{D}$ of rank $n - 1$ such that condition (iii) in Proposition \[3\] holds everywhere. Then, there exist locally a smooth orthonormal tangent frame $\{Y, X, T_1, \ldots, T_{n-2}\}$ and smooth functions $\beta, \mu, \rho$ and $\lambda_j, 1 \leq j \leq n - 2$, such that

\begin{align*}
AY &= \beta Y + \mu X, \quad AX = \mu Y + \rho X, \quad AT_j = 0, \quad (4) \\
\nabla_{T_j} T_i &= \nabla_{T_j} X = \nabla_{T_j} Y = \nabla_X Y = 0 \quad (5)
\end{align*}

and

\begin{align*}
\nabla_X T_j &= \lambda_j X, \quad 1 \leq i, j \leq n - 2. \quad (6)
\end{align*}

Moreover, condition (i) (respectively, (ii)) in Proposition \[8\] holds at $x \in M^n$ if and only if $\rho$ (respectively, $\mu$) vanishes at $x$.

\textbf{Proof.} Choose unit vector fields $Y \in \mathcal{D}^\perp$ and $X \in \mathcal{D}$ orthogonal to $\Delta$. Then there exist smooth functions $\beta, \mu, \rho$ such that the first two equations in (4) are satisfied. Since $\mathcal{D}$ is totally geodesic, the last equation in (5) holds.

Let $\gamma: J \to M^n$ be the integral curve of $Y$ through an arbitrary given point in $M^n$, and define $\psi: I \times J \to M^n$ by requiring that, for any fixed $y \in J$, the map $x \mapsto \psi_{\gamma(y)}(x) = \psi(x, \gamma(y))$ is the integral curve of $X$ such that $\psi_{\gamma(y)}(0) = \gamma(y)$.

The normal space in $\mathbb{R}^{n+1}$ of the restriction $f|_{\sigma}$ of $f$ to a leaf $\sigma$ of $\mathcal{D}$ at each point $x \in \sigma$ is spanned by $f_* Y(x)$ and $\eta(x)$, where $\eta$ is a smooth unit normal vector field to $f$. Since $\sigma$ is totally geodesic in $M^n$, the shape
operator of $f|_\sigma$ at $x$ with respect to $f_*Y(x)$ is identically zero. On the other hand, since condition (iii) in Proposition \(8\) holds at $x$ by assumption, the shape operator of $f|_\sigma$ at $x$ with respect to $\eta$ has rank one. It follows from the Gauss equations for $f|_\sigma$ that $\sigma$ is flat.

To construct the desired smooth orthonormal tangent frame, start with any smooth orthonormal frame \(\{T^1, \ldots, T^{n-2}\}\) spanning $\Delta$ along $\gamma$. Then, for each integral curve $x \mapsto \psi_{\gamma(y)}(x)$ of $X$, extend $\{T^1, \ldots, T^{n-2}\}$ to an orthonormal frame spanning $\Delta$ along $\psi_{\gamma(y)}$ by parallel translation with respect to the normal connection of $\psi_{\gamma(y)}$ as a curve in the leaf of $D$ containing $\psi_{\gamma(y)}(f)$). Finally, parallel translate each $T^j(\psi_{\gamma(y)}(x))$, $1 \leq j \leq n-2$, along the leaf of $\Delta$ through $\psi_{\gamma(y)}(x)$. Since $\Delta$ and $D$ are both totally geodesic, it follows that the orthonormal frame $\{T^1, \ldots, T^{n-2}\}$ constructed in this way satisfies the last equation in (4), the first three equations in (5) as well as (6). The last assertion is clear.

**Lemma 10.** Let $f: M^n \to \mathbb{R}^{n+1}$ be an isometric immersion of a nowhere flat Riemannian manifold that carries a totally geodesic distribution $D$ of rank $n-1$ with complete leaves. Let $U$ be the open subset of $M^n$ where neither of conditions (i) or (ii) in Proposition \(8\) occur. Then each connected component of $U$ is isometric to a product $W = L^2 \times \mathbb{R}^{n-2}$ and $f|_W = g \times \text{id}$, where $g: L^2 \to \mathbb{R}^3$ is an isometric immersion.

**Proof.** Since any totally geodesic distribution is curvature invariant, it follows from Proposition \(8\) and the assumptions that condition (iii) holds everywhere on $U$. Let $\{Y, X, T^1, \ldots, T^{n-2}\}$ be the frame on $U$ given by Lemma \(9\). Straightforward computations using the Codazzi equations yield

\[
T^i(\rho) = \rho \langle \nabla_X X, T^i \rangle,
\]

\[
T^i(\mu) = \mu \langle \nabla_X X, T^i \rangle,
\]

\[
T^i(\mu) = \rho \langle \nabla_Y X, T^i \rangle + \mu \langle \nabla_Y Y, T^i \rangle
\]

whereas the Gauss equations give

\[
X(\langle \nabla_Y Y, T^i \rangle) = \langle \nabla_Y Y, X \rangle \left( \langle \nabla_Y Y, T^i \rangle - \langle \nabla_X X, T^i \rangle \right),
\]

\[
T^i(\langle \nabla_X X, T^j \rangle) = \langle \nabla_X X, T^i \rangle \langle \nabla_X X, T^j \rangle,
\]

\[
T^i(\langle \nabla_Y Y, T^j \rangle) = \langle \nabla_Y Y, T^i \rangle \langle \nabla_Y Y, T^j \rangle.
\]

Since neither of conditions (i) or (ii) in Proposition \(8\) is satisfied at any point of $U$, we have that $\mu \rho \neq 0$ everywhere on $U$. On the other hand, from
equations (7a) and (7b) we obtain that
\[ \mu = \varphi \rho, \]
where \( T_j(\varphi) = 0 \) for \( 1 \leq j \leq n - 2 \).

Consider a unit speed geodesic \( \gamma \) starting at a point of \( U \) and tangent to some \( T_j \). Since the leaves of \( D \) are assumed to be complete, \( \gamma \) is defined at any value of the parameter. We claim that it remains indefinitely in \( U \). Otherwise \( \gamma \) would reach a point \( y \) of the boundary of \( U \). Since \( y \) is the limit of a sequence of points where either of conditions (i) or (ii) in Proposition 8 holds, then either \( \mu \) or \( \rho \) must vanish at \( y \). But then both \( \mu \) and \( \rho \) vanish at \( y \), in view of (9). Hence \( y \) is a flat point of \( M^n \), contradicting our assumption and proving our claim.

It follows that the leaves of \( \Delta \) through points of \( U \) are complete and that condition (iii) remains valid along them. In view of (8b) and (8c) for \( i = j \), we conclude that the functions
\[ \lambda_j = -\langle \nabla_X X, T^j \rangle \quad \text{and} \quad \theta_j = \langle \nabla_Y Y, T^j \rangle, \quad 1 \leq j \leq n - 2, \]
must be everywhere vanishing along such leaves. In general, we obtain from (7b) and (7c) that
\[ \mu(\lambda_j + \theta_j) + \rho \langle \nabla_Y X, T^j \rangle = 0 \quad \text{for all} \quad 1 \leq j \leq n - 2. \]
Since now \( \lambda_j = 0 = \theta_j \) and \( \rho \neq 0 \), we must have that \( \langle \nabla_Y X, T^j \rangle = 0 \). In particular, this implies that \( \Delta^\perp \) is totally geodesic. Moreover, from \( T^j \in \Delta \) we obtain that \( \nabla_X f_* T^j = 0 \) and \( \nabla_Y f_* T^j \in f_* \Delta \), where \( \nabla \) stands for the connection in the Euclidean ambient space. It follows that \( f_* \Delta \) is a parallel subbundle of \( f^* T\mathbb{R}^{n+1} \), and the result follows by choosing \( L^2 \) in each connected component of \( U \) as a maximal integral leaf of \( \Delta^\perp \).

**Proof of Theorem 2** : In view of the assumption that \( f(M) \) does not contain any surface-like strip, it follows from Lemma 10 that either of conditions (i) or (ii) must hold at any point of \( M^n \). Let \( S_1 \) (respectively, \( S_2 \)) be the subset of \( M^n \) where condition (i) (respectively, condition (ii)) is satisfied. Since both \( S_1 \) and \( S_2 \) are closed and \( M^n = S_1 \cup S_2 \), any point on \( \partial S_1 \) belongs to \( S_1 \cap S_2 \), and hence is a flat point. It follows from our assumption that either \( M^n = S_1 \) or \( M^n = S_2 \). In the first case \( f \) is a ruled hypersurface. In the latter, as pointed out before the statement of Theorem 6, there is no loss of generality in assuming that \( M^n \) is simply connected. In this case, by the assumption that the leaves of the totally geodesic foliation are complete,
it follows from [9, Theorem 1] that \( M^n \) is isometric to a twisted product \( N^{n-1} \times _\rho \mathbb{R} \), with \( \rho \in C^\infty(\mathbb{R}^{n-1} \times \mathbb{R}) \), and the fact that \( M^n = S_2 \) means that the second fundamental form of \( f \) satisfies \( \mathcal{H} \). Thus \( f(M) \) is a partial tube over a curve by Theorem 6.

**Proof of Theorem 4.** From the equivalent form of Theorem 2 discussed right after the statement of Corollary 3, it follows that \( f(M) \) must be either ruled or a partial tube over a curve.

We now show that, if \( M^n \) is a warped product, then the first possibility can not occur under our global assumptions. In fact, the warped product structure on \( M^n \) implies that the distribution on \( M^n \) given by the tangent spaces to the fibers corresponding to the second factor \( I \) is spherical. This means that the integral curves of a unit vector field \( Y \) tangent to \( I \) are extrinsic circles in \( M^n \), that is, \( \nabla_Y \nabla_Y Y \) is everywhere a multiple of \( Y \), or equivalently,

\[
\langle \nabla_Y \nabla_Y Y, X \rangle = 0 = \langle \nabla_Y \nabla_Y Y, T^j \rangle, \quad 1 \leq j \leq n-2,
\]

where \( \{Y, X, T^1, \ldots, T^{n-2}\} \) is the frame given by Lemma 9.

As in the proof of Lemma 10, completeness of \( N^{n-1} \) implies that the functions \( \theta_j \) in (10) must vanish everywhere for \( 1 \leq j \leq n-2 \). From

\[
\langle \nabla_Y \nabla_Y Y, T^j \rangle = 0, \quad 1 \leq j \leq n-2,
\]

we obtain

\[
0 = -\langle \nabla_Y Y, \nabla_Y T^j \rangle = \langle \nabla_Y Y, X \rangle \langle \nabla_Y X, T^j \rangle, \quad 1 \leq j \leq n-2.
\]

On the other hand, using that \( f \) is ruled we obtain from the Gauss equation for \( f \) that

\[
X \langle \nabla_Y Y, X \rangle = -\mu^2 + \langle \nabla_Y X, X \rangle^2,
\]

hence \( \langle \nabla_Y Y, X \rangle \) can not vanish on any open subset of \( M^n \), because \( \mu \) is nowhere vanishing. We conclude from (12) that

\[
\langle \nabla_Y X, T^j \rangle = 0, \quad 1 \leq j \leq n-2.
\]

But, as in the proof of Lemma 10 this implies that \( f \) is a cylindrical surface-like hypersurface, contradicting our assumption.

We conclude that \( f \) can not be ruled, hence it is a partial tube over a curve. Since \( M^n \) is a warped product, it follows from the main result in
In this section we prove Theorem 5 in the introduction.

4 The local case

[8] (see also [11, Theorem 30]), that \( f \) is either a cylinder over a plane curve, the product with an Euclidean factor \( \mathbb{R}^{n-2} \) of a cone over a curve in \( S^2 \subset \mathbb{R}^3 \) or a rotation hypersurface. The first two possibilities are ruled out by our assumptions. \( \Box \)

We now show that \( f \) is locally the envelope of a one-parameter family of flat hypersurfaces on \( U \). More precisely, we prove that for each leaf of \( \mathcal{D} \) there exist a flat hypersurface \( F_t: V_t^n \to \mathbb{R}^{n+1} \) and an embedding \( j_t: \sigma_t \to V_t^n \) such that \( F_t \circ j_t = f|_{\sigma_t} \) and \( F_t* T_{j_t(x)} V_t = f_* T_x M \) for any \( x \in \sigma_t \).

Consider a smooth orthonormal frame \( \{Y, X, T^1, \ldots, T^{n-2}\} \) and smooth functions \( \beta, \mu, \rho \) and \( \lambda_j \), \( 1 \leq j \leq n-2 \), given locally in \( U \) by Lemma [9]. Let \( \sigma_t \) be a leaf of \( \mathcal{D} \) on \( U \) and set \( f_t = f|_{\sigma_t} \). Then, the normal space of \( f_t \) at each \( x \in \sigma_t \) is spanned by \( f_* Y(x) \) and \( \eta(x) \), where \( \eta(x) \) is a unit normal vector to \( f \) at \( x \). Let \( \pi_t: \Lambda_t \to \sigma_t \) be the line subbundle of the normal bundle of \( f_t \) that is spanned by the (restriction to \( \sigma_t \) of the) vector field \( Z = \rho f_* Y - \mu f_* X \), and define \( F_t: V_t \to \mathbb{R}^{n+1} \) as the restriction of the map

\[
\lambda \in \Lambda_t \mapsto f(\pi_t(\lambda)) + \lambda
\]

to a tubular neighborhood \( V_t \) of the 0-section \( j_t: \sigma_t \to \Lambda_t \) of \( \Lambda_t \). We prove below that \( F_t \) defines a flat hypersurface by showing that the subspaces \( F_t* T_{\lambda} V_t \) are constant along the distribution \( \hat{\mathcal{D}} \) of rank \( n-1 \) on \( V_t \) given as follows. For each \( \lambda \in V_t \), we have \( T_{\lambda} V_t = T_{\pi_t(\lambda)} \sigma_t \oplus \Lambda_t(\pi_t(\lambda)) \) where \( T_{\pi_t(\lambda)} \sigma_t \) is identified with its horizontal lift and \( \Lambda_t(\pi_t(\lambda)) \) with the vertical subspace at \( \lambda \). We prove below that \( F_t \) defines a flat hypersurface by showing that the subspaces \( F_t* T_{\lambda} V_t \) are constant along the distribution \( \hat{\mathcal{D}} \) of rank \( n-1 \) on \( V_t \) given as follows. For each \( \lambda \in V_t \), let \( \Delta(\pi_t(\lambda)) \) be the relative nullity subspace of \( f_t \) at \( \pi_t(\lambda) \) and define \( \hat{\mathcal{D}}(\lambda) = \Delta(\pi_t(\lambda)) \oplus \Lambda_t(\pi_t(\lambda)) \), where \( \Delta(\pi_t(\lambda)) \) is identified with its horizontal lift.
Given $\lambda = sZ/|Z| \in \Lambda_t$ and $W \in T_{\pi_t(\lambda)}\sigma_t$, we have
\[
F_t sW = f_s W + \tilde{\nabla}_W sZ/|Z| = f_s W + sW(|Z|^{-1})Z + s|Z|^{-1}\tilde{\nabla}_W Z. \quad (13)
\]
Since
\[
\langle \tilde{\nabla}_X Z, \eta \rangle = -\langle Z, \tilde{\nabla}_X \eta \rangle = 0,
\]
for $\tilde{\nabla}_X \eta = -f_s A_\eta X = -\mu f_s Y - \rho f_s X$, we obtain that
\[
\langle F_t sX, \eta \rangle = 0. \quad (14)
\]
On the other hand, from (5) we have that $\nabla T^j f_\epsilon = 0 = \nabla T^j X$, whereas equations (7a) and (7b) yield
\[
T^j (\rho) \rho = T^j (\mu) \mu .
\]
Using that $\Delta = \text{span}\{T^1, \ldots, T^{n-2}\}$ is the relative nullity distribution of $f$ we obtain
\[
\tilde{\nabla}_{T^j} Z = T^j (\rho) f_s Y + \rho \tilde{\nabla}_{T^j} f_s Y - T^j (\mu) f_s X - \mu \tilde{\nabla}_{T^j} f_s X
\]
\[
= \frac{T^j (\mu)}{\mu} Z.
\]
In particular,
\[
T^j(|Z|^2) = 2\langle \tilde{\nabla}_{T^j} Z, Z \rangle
\]
\[
= \frac{2T^j (\mu)|Z|^2}{\mu},
\]
and hence
\[
T^j(|Z|^{-1})Z + |Z|^{-1}\tilde{\nabla}_{T^j} Z = 0.
\]
It follows from (13) that $F_t T^j = f_s T^j$ for $1 \leq j \leq n - 2$, and thus
\[
F_t s \Delta(\pi_t(\lambda)) = f_s \Delta(\pi_t(\lambda)). \quad (15)
\]
We obtain from (14) and (15), together with the fact that $F_t s \Delta(\pi_t(\lambda))$ is spanned by $f_s Z$, that $\eta(\pi_t(\lambda))$ is normal to $F_t s T^j V_t$ along $\tilde{\Delta}$. In particular, $F_t s T^j_{j(x)} V_t = f_s T^j_x M$ for any $x \in \sigma_t$. \hfill \Box

We point out that the construction of the flat hypersurface $F_t : V_t \rightarrow \mathbb{R}^{n+1}$ extending $f_t$ in the proof of Theorem 5 is a special case of the ruled extension of submanifolds given in [4, Proposition 8].
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