PERSISTENCE PROPERTIES FOR THE DISPERSION GENERALIZED BO-ZK EQUATION IN WEIGHTED ANISOTROPIC SOBOLEV SPACES

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ABSTRACT. In this paper we study the initial-value problem associated with the dispersion generalized-Benjamin-Ono-Zakharov-Kuznetsov equation,

\[ u_t + D_{x}^{a+1} \partial_x u + u_{xyy} + uu_x = 0, \quad a \in (0, 1). \]

More specifically, we study the persistence property of the solution in the weighted anisotropic Sobolev spaces

\[ H^{(1+a)s, 2s}(\mathbb{R}^2) \cap L^2((x^{2r_1} + y^{2r_2})dx dy), \]

for appropriate \( s, r_1 \) and \( r_2 \). By establishing unique continuation properties we also show that our results are sharp with respect to the decay in the \( x \)-direction.

1. Introduction

This paper is concerned with the initial-value problem (IVP) associated with the two-dimensional dispersion generalized-Benjamin-Ono-Zakharov-Kuznetsov (gBO-ZK) equation,

\[
\begin{aligned}
    u_t &+ D_{x}^{a+1} \partial_x u + u_{xyy} + uu_x = 0, \quad (x,y) \in \mathbb{R}^2, \quad t > 0, \quad a \in [0,1], \\
    u(x,y,0) &= \phi(x,y),
\end{aligned}
\]  

(1.1)

where \( D_{x}^{a+1} \) stands for the fractional derivative of order \( a + 1 \) with respect to the variable \( x \) and is defined, via Fourier transform, as \( D_{x}^{a+1} f(x,y) = (|\xi|^{a+1} \hat{f})(x,y) \).

In the limiting case \( a = 1 \), equation in (1.1) becomes the Zakharov-Kuznetsov (ZK) equation

\[ u_t + \partial_x \Delta u + uu_x = 0, \]

(1.2)

while for \( a = 0 \) it reduces to the Benjamin-Ono-Zakharov-Kuznetsov (BO-ZK) equation

\[ u_t + \mathcal{H} D_{x}^{2} u + u_{xy} + uu_x = 0, \]

(1.3)

where \( \mathcal{H} \) denotes the Hilbert transform in the \( x \)-variable. Equations in (1.2) and (1.3) appear in physical application. Indeed, the ZK equation was first derived in [31] and it models the propagation of nonlinear ion-acoustic waves in magnetized plasma (see also [22] for a rigorous derivation in the long-wave limit of the Euler-Poisson system). On the other hand, the BO-ZK equation was introduced in [17] and [23] and it has applications to thin nanoconductors on a dielectric substrate.

From the mathematical viewpoint, equation in (1.1) may be seen as a two-dimensional extension of the dispersion generalized Benjamin-Ono equation,

\[ u_t + D_{x}^{a+1} \partial_x u + uu_x = 0, \]

(1.4)

in much the same way ZK and BO-ZK equations may be seen as two-dimensional versions of the well-known Korteweg-de Vries and Benjamin-Ono equations, respectively.

Both ZK and BO-ZK equations have been extensively studied in the last two decades. In the next paragraphs we recall some results concerning the well-posedness in weighted Sobolev spaces and which are close to the main issue of this manuscript. Here and throughout the paper by well-posedness we mean in Kato’s sense, that is, it includes existence, uniqueness, persistence (if the initial data belongs to some function space \( X \) then there exists a unique solution that also belong to \( X \)) and continuous dependence upon the initial data. In addition, if these properties hold in a small time interval we say the IVP is locally well-posed; on the other, if the properties hold for all \( t > 0 \) we say that the IVP is globally well-posed. Concerning the ZK equation, the IVP
in weighted spaces was studied in [1] and [9]. In [1] the authors proved the local well-posedness in the isotropic space \( H^s(\mathbb{R}^2) \cap L^2((1 + x^2 + y^2)^{s/2} dx dy), \ s > 3/4; \) in their proof they took the advantage of change of variables introduced in [10] in order to explore the symmetric form of \((1.2)\). On the other hand, in [9] the authors proved the local well-posedness in the anisotropic spaces \( H^s(\mathbb{R}^2) \cap L^2((1 + |x|^{2r_1} + |y|^{2r_2}) dx dy), \) where \( s > 3/4 \) and \( r_1, r_2 > 0 \) are such that \( \max\{r_1, r_2\} \leq s/2 \). Their proof is a little bit different from the one [1]; the main tool is a commutator estimate between weights and the linear group associated with \((1.2)\). In addition, their method also extends to the generalized nonlinearity \( u^k u_x, \ k \geq 2 \).

Concerning the BO-ZK equation, local well-posedness in weighted spaces was studied in [2] from several viewpoints. First the authors proved local well-posedness in \( H^s(\mathbb{R}^2) \cap L^2(w^2 dx dy), \ s > 2, \) provided \( w = w(x, y) \) is a weight with bounded derivatives up to order three. In addition, if \( r \in (1, 5/2) \) and \( s \geq 2r \) then local well-posedness holds in \( Z_{s,r} := H^s(\mathbb{R}^2) \cap L^2((1 + x^2 + y^2)^r dx dy). \) Also, if \( r \in [5/2, 7/2) \) then local well-posedness in \( Z_{s,r} \) holds provided the initial data \( \phi \) is such that \( \hat{\phi}(0, \eta) = 0, \) for any \( \eta \in \mathbb{R}, \) where the hat stands for the Fourier transform; in this case, as long as the solution exists it also satisfies \( \tilde{u}(0, \eta, t) = 0. \) These results were shown to be sharp in the sense that a sufficiently smooth nontrivial solution do not persist in \( L^2((1 + x^2 + y^2)^{7/2} dx dy). \) For recent results concerning local well-posedness in the standard Sobolev spaces we refer the reader to [3] and [25].

Another model that extends \((1.4)\) to a two-dimensional model is the so-called fractional Zakharov-Kuznetsov equation

\[
 u_t + D^{a+1} \partial_x u + uu_x = 0, \quad a \in [0, 1], \tag{1.5}
\]

where now \( D^{a+1} \) is the operator defined in Fourier variables as \( \hat{D^{a+1}f}(\xi, \eta) = (\xi^2 + \eta^2)^{(a+1)/2}\hat{f}(\xi, \eta), \) which has been studied very recently. By using the short-time Strichartz method introduced in [21] to deal with the Benjamin-Ono equation the authors in [11] considered \( a = 0 \) and established local well-posedness in \( H^s(\mathbb{R}^2), \ s > 5/3. \) They also proved an ill-posedness result in the sense that the data-to-solution map cannot be \( C^2 \)-differentiable from \( H^s(\mathbb{R}^2) \) to \( H^s(\mathbb{R}^2), \) for any \( s \in \mathbb{R}. \) The local well-posedness was extended to \( 0 \leq a < 1 \) in [29] where, by using transversality and localization of time to small frequency dependent time intervals, the author showed the local well-posedness in \( H^s(\mathbb{R}^2), \ s > 3/2 - a. \) In weighted spaces, local well-posedness was studied in [27] only for \( a = 0. \) In particular it was shown that local well-posedness in \( Z_{s,r} \) holds for \( s \geq r \) and \( r \in [0, 3) \) (with \( s > 5/3 \)); if \( r \in [3, 4) \) then local well-posedness in the same space holds provide the initial data also satisfies \( \hat{\phi}(0, 0) = 0. \) These results are sharp in the sense that no nontrivial solutions persist in \( Z_{4,4}. \)

The IVP \((1.4)\) in anisotropic Sobolev spaces \( H^{(1+a) \cdot 2s}(\mathbb{R}^2) \) was studied in [28]. For future references we quote their result in next theorem.

**Theorem A.** Let \( E^s = H^{(1+a) \cdot 2s}(\mathbb{R}^2). \)

(a) Assume \( a \in [0, 1] \) and \( s > \frac{3}{2+a} - \frac{3}{2}. \) Then \((1.4)\) is locally well-posed in \( E^s. \)

(b) Assume \( a \in (3/5, 1] \) and \( s = 1/2. \) Then \((1.4)\) is globally well-posed in \( E^{1/2}. \)

To prove part (a) in Theorem A the authors used the method introduced in [13], which combines the energy method with linear and nonlinear estimates in the short-time Bourgain spaces. Part (b) may be proved taking the advantage of the conservation of the quantities

\[
 \int_{\mathbb{R}^2} \left( |D^{a+1}_x u|^2 + u_x^2 \right) dx dy \quad \text{and} \quad \int_{\mathbb{R}^2} u^2 dx dy
\]

to obtain an or before bound for the local solution.

Let us now turn attention to the results in the present paper. Our purpose here is to extend the well-posedness results of Theorem A to anisotropic weighted spaces. Thus our main goal is to establish the persistence property in \( L^2((1 + |x|^{2r_1} + |y|^{2r_2}) dx dy) \) for appropriate \( r_1, r_2 \geq 0. \) As we pointed out above, the cases \( a = 0 \) and \( a = 1 \) have already been treated in the literature. So,
we will restrict our attention to the case \( a \in (0,1) \); to the best of our knowledge this case has not been treated.

Our first result reads as follows (see next section for the definition of the function spaces).

**Theorem 1.1.** Let \( a \in (0,1) \) and \( r_1, r_2 \geq 0 \). Assume

\[
s > \frac{2}{a + 1} - \frac{3}{4} \quad \text{and} \quad s \geq \frac{2r_2}{1 + a},
\]

The following statements are true.

1. If \( r_1 \in [0,1] \) and \( s \geq 1 \), then the IVP \((1.1)\) is locally well-posed in \( \dot{Z}_{r_1,r_2}^s \).
2. If \( r_1 \in (1,2] \) and \( s \geq r_1 + \frac{1}{1+a} \), then the IVP \((1.1)\) is locally well-posed in \( \dot{Z}_{r_1,r_2}^s \).
3. If \( r_1 \in (2,5/2 + a) \), \( s \geq r_1 + \frac{1}{1+a} \) and \( r_2 > 2 \), then the IVP \((1.1)\) is locally well-posed in \( \dot{Z}_{r_1,r_2}^s \).
4. If \( r_1 \in [5/2 + a, 7/2 + a) \), \( s \geq r_1 + \frac{1}{1+a} \) and \( r_2 > 3 \), then the IVP \((1.1)\) is locally well-posed in \( \dot{Z}_{r_1,r_2}^s \).

In addition the time interval where the solution exists is the same as in Theorem A.

Our arguments to prove Theorem 1.1 are inspired in the ones presented in [8], where the authors proved the well-posedness in weighted spaces for the dispersion generalized BO equation \((1.4)\). Since we are dealing with a two-dimensional model, the arguments do not follow directly from [8] and we need to deal with many additional terms in the necessary estimates. Due to the nonlocal operator \( D_x^{a+1} \), the most difficult part relies on the estimates when the weights are set in the \( x \)-direction and we need to play with several product and commutator estimates.

Some remarks concerning the assumptions of Theorem 1.1 are in order. First of all, the condition \( s > \frac{2}{a + 1} - \frac{3}{4} \) appears in order to have the local well-posedness according to Theorem A, in such a way we spend our efforts to show the persistence property in the weighted space. The conditions \( s \geq 1 \) and \( s \geq r_1 + \frac{1}{1+a} \) are used to bound the solution in the resolution space \( E^s \). Probably the regularity \( s \) may be pushed down to \((1 + a)s \geq 2 \max\{r_1, r_2\}\), which agrees with the case \( a = 1 \) as described above; however, our strategy do not allow us to achieve this index. In addition, since we use Sobolev’s embedding in Fourier variables to estimate some terms, this give rise to the assumptions \( r_2 > 2 \) in part 3) and \( r_2 > 3 \) in part 4).

Note that part 4) in Theorem 1.1 establishes the well-posedness in \( \dot{Z}_{r_1,r_2}^s \), which means that the initial data satisfies \( \hat{\phi}(0, \eta) = 0 \), for any \( \eta \in \mathbb{R} \). Next theorem shows that this is a necessary condition to have local well-posedness in the following sense: if a sufficiently smooth solution has a decay of order \( 5/2 + a \) in the \( x \)-direction then the initial data satisfies \( \hat{\phi}(0, \eta) = 0 \), for any \( \eta \in \mathbb{R} \).

**Theorem 1.2.** Let \( u \in C([0,T]; \dot{Z}_{r_1,r_2}^s) \) be a solution of the IVP \((1.1)\), where \( a \in (0,1) \), \( s \geq 2 \) and \( r_1, r_2 > 2 \). If there exist two different times \( t_1, t_2 \in [0,T] \) such that \( u(t_j) \in \dot{Z}_{5/2+a,r_2}^s \), \( j = 1,2 \), then

\[
\hat{u}(0, \eta, t) = 0,
\]

for any \( \eta \in \mathbb{R} \) and any \( t \in [0,T] \).

Having Theorem 1.1 in hand, a natural question is what happens if \( r_1 \geq 7/2 + a \). Next theorem establishes that a nontrivial sufficiently smooth solution cannot have such a decay in the \( x \)-direction. In particular local well-posedness is not expected in \( \dot{Z}_{r_1,r_2}^s \), for \( r_1 \geq 7/2 + a \).

**Theorem 1.3.** Let \( u \in C([0,T]; \dot{Z}_{r_1,r_2}^s) \) be a solution of the IVP \((1.1)\), where \( a \in (0,1) \), \( s \geq 4 \) and \( r_1, r_2 > 3 \). If there exist three different times \( t_1, t_2, t_3 \in [0,T] \) such that \( u(t_j) \in \dot{Z}_{7/2+a,r_2}^s \), \( j = 1,2,3 \), then

\[
u(t) \equiv 0,
\]

for any \( t \in [0,T] \).
Statements in Theorems 1.2 and 1.3 may be seen as unique continuation principles. The first results in this direction for nonlocal dispersive equations was put forward by R. Iorio in [14], [15] and [16], where the author studied the Benjamin-Ono equation in weighed spaces $L^2((1 + |x|^k)dx)$ with $k$ an integer number. Iorio’s results were extended to encompass non-integer values of $k$ in [7]. Then, similar results were established for the dispersion generalized Benjamin-Ono equation (1.3) in [8]. Our strategy to prove Theorems 1.2 and 1.3 are also inspired in [8].

The paper is organized as follows. In Section 2 we introduce some notation and give preliminary results. In particular we recall several product and commutator estimates. In Section 3 we prove (1.4) in [8]. Our strategy to prove Theorems 1.2 and 1.3 are also inspired in [8].

2. Notation and Preliminaries

Let us first introduce some notation. We use $c$ to denote various positive constants that may vary line by line; if necessary we use subscript to indicate dependence on parameters. Given positive numbers $A$ and $B$, we write $A \lesssim B$ to say that $A \leq cB$ for some positive constant $c$. By $\| \cdot \|_{L^p(\mathbb{R}^d)}$ we denote the usual $L^p(\mathbb{R}^d)$ norm. If no confusion is caused we will use $\| \cdot \|_p$ instead of $\| \cdot \|_{L^p(\mathbb{R}^d)}$. For short we denote the $L^2$ norm simply by $\| \cdot \|$. In particular, if $f = f(x,y)$ then $\|f\| = \|f(\cdot, y)\|_{L^2_y}\|\cdot\|_{L^2_y}$, where by $\| \cdot \|_{L^2_y}$ we mean the $L^2_y$ norm with respect to the variable $z$. The scalar product in $L^2$ will be then represented by $(\cdot, \cdot)$. For any $s \in \mathbb{R}$, $H^s := H^s(\mathbb{R}^d)$ represents the usual $L^2$-based Sobolev space endowed with the norm $\| \cdot \|_{H^s}$. The Fourier transform of $f$ is defined by

$$\hat{f}(\zeta) = \int_{\mathbb{R}^d} e^{-i\zeta \cdot x} dx, \quad \zeta = (\zeta_1, \ldots, \zeta_d) \in \mathbb{R}^d.$$ 

Given any complex number $z$ and a function $f$ defined on $\mathbb{R}^d$, let us define the Bessel and Riesz operators, via their Fourier transforms, as follows

$$\hat{J}^z f(\zeta) = (1 + |\zeta|^2)^{z/2} \hat{f}(\zeta), \quad \hat{D}^z f(\zeta) = |\zeta|^z \hat{f}(\zeta),$$

$$\hat{\tilde{J}}^x f(\zeta) = (1 + |\zeta|^2)^{z/2} \hat{f}(\zeta), \quad \hat{\tilde{D}}^x f(\zeta) = |\zeta|^z \hat{f}(\zeta).$$

Given $s_1, s_2 \in \mathbb{R}$, the anisotropic Sobolev space $H^{s_1, s_2} = H^{s_1, s_2}(\mathbb{R}^2)$ is the set of all tempered distributions $f = f(x, y)$ such that

$$\|f\|_{H^{s_1, s_2}} := \|f\|^2 + \|J_x f\|^2 + \|J_y f\|^2 < \infty.$$ 

We also define the Sobolev spaces in $x$- and $y$-directions, $H^{s_1}_x$ and $H^{s_2}_y$, respectively, as being the set of tempered distributions $f$ such that

$$\|f\|_{H^{s_1}_x} := \|J_x f\| < \infty \quad \text{and} \quad \|f\|_{H^{s_2}_y} := \|J_y f\| < \infty.$$ 

Let $r_1, r_2 \in \mathbb{R}$. We define $L^2_{r_1, r_2}$ to be the space all functions $f = f(x, y)$ satisfying

$$\|f\|^2_{L^2_{r_1, r_2}} := \int_{\mathbb{R}^2} (1 + x^{2r_1} + y^{2r_2})|f(x, y)|^2 dxdy < \infty.$$ 

Note that $L^2_{r_1, r_2} = L^2_{r_1, 0} \cap L^2_{0, r_2}$. For $s_1, s_2, r_1, r_2 \in \mathbb{R}$, we denote

$$Z^{s_1, s_2}_{r_1, r_2} := H^{s_1, s_2}(\mathbb{R}^2) \cap L^2_{r_1, r_2}(\mathbb{R}^2),$$

The norm in $Z^{s_1, s_2}_{r_1, r_2}$ is given by $\| \cdot \|_{Z^{s_1, s_2}_{r_1, r_2}} = \| \cdot \|_{H^{s_1, s_2}_{r_1, r_2}} + \| \cdot \|_{L^2_{r_1, r_2}}$. Also, the subspace $\hat{Z}^{s_1, s_2}_{r_1, r_2}$ of $Z^{s_1, s_2}_{r_1, r_2}$ is defined as

$$\hat{Z}^{s_1, s_2}_{r_1, r_2} := \{ f \in Z^{s_1, s_2}_{r_1, r_2} | \hat{f}(0, \eta) = 0, \quad \eta \in \mathbb{R} \}.$$ 

Finally, the spaces $Z^s_{r_1, r_2}$ and $\hat{Z}^s_{r_1, r_2}$ are defined as

$$Z^s_{r_1, r_2} := Z^{(1+\alpha)s, 2s}_{r_1, r_2} \quad \text{and} \quad \hat{Z}^s_{r_1, r_2} := \hat{Z}^{(1+\alpha)s, 2s}_{r_1, r_2}.$$

Suppose \( \phi \in \mathcal{Z}_{s_1, s_2}^{p_1, r_2} \) and let \( u \) be the corresponding local solution of (2.1). Assuming that \( u \) is sufficiently regular, we can integrate the equation with respect to \( x \) to obtain

\[
\int_{\mathbb{R}} u(x, y, t) dx = \int_{\mathbb{R}} \phi(x, y) dx, \quad y \in \mathbb{R}
\]  

(2.1)
as long as the solution exists. This implies that

\[
\hat{u}(0, \eta, t) = \hat{\phi}(0, \eta), \quad \eta \in \mathbb{R},
\]

(2.2)
for all \( t \) for which the solution exists. In particular, if \( \phi \in \mathcal{Z}_{s_1, s_2}^{p_1, r_2} \) then \( u(t) \in \mathcal{Z}_{s_1, s_2}^{p_1, r_2} \) for any \( t \) for which the solution exits.

Next, we introduce some preliminaries results which will be useful to prove our main results.

**Theorem 2.1.** For any \( p \in (1, \infty) \) and \( l, m \in \mathbb{Z}^+ \cup \{0\} \), with \( l + m \geq 1 \), there exists a constant \( c > 0 \), depending only on \( p, l, \) and \( m \) such that

\[
\| \partial_x^k [H; g] \partial_x^m f \|_{L^p(\mathbb{R})} \lesssim \| \partial_x^{l+m} g \|_{L^\infty(\mathbb{R})} \| f \|_{L^p(\mathbb{R})},
\]

where \( \partial_x^k \) denotes the derivative of order \( k \).

**Proof.** This is a generalization of the Calderón commutator estimates [5]. See Lemma 3.1 in [6] or Theorem 6 in [7]. □

**Proposition 2.2.** Let \( g \in L^\infty(\mathbb{R}) \), with \( \partial_x^k g \in L^2(\mathbb{R}) \) for \( k = 1, 2 \). Then, for any \( \theta \in (0, 1) \), there exists a constant \( c > 0 \), depending only on \( \theta \), such that

\[
\| [J^\theta g] f \|_{L^2(\mathbb{R})} \leq c \| f \|_{L^2(\mathbb{R})}.
\]

(2.3)
In addition,

\[
\| J^\theta (g f) \|_{L^2(\mathbb{R})} \leq c \| J^\theta f \|_{L^2(\mathbb{R})}.
\]

(2.4)

**Proof.** See Propositions 2.4 and 2.5 in [8]. □

**Proposition 2.3.** Let \( 0 \leq \alpha < 1, 0 < \beta \leq 1 - \alpha, 1 < p < \infty \) and \( d \geq 1 \), then

\[
\| D^\alpha [D^\beta g] D^{1-(\alpha+\beta)} f \|_{L^p(\mathbb{R}^d)} \leq c \| \nabla g \|_{L^\infty(\mathbb{R}^d)} \| f \|_{L^p(\mathbb{R}^d)},
\]

where \( c \) depends on \( \alpha, \beta, p, \) and \( d \).

**Proof.** This result is a consequence of Proposition 3.10 in [18]. For a similar result in the one-dimensional case see Proposition 3.2 in [6]. See also Proposition 2.2 in [8]. □

**Proposition 2.4.** If \( f \in L^2(\mathbb{R}) \) and \( \Phi \in H^2(\mathbb{R}) \), then

\[
\| [D^\alpha \Phi] f \|_{L^2(\mathbb{R})} \lesssim \| \Phi \|_{H^2(\mathbb{R})} \| f \|_{L^2(\mathbb{R})},
\]

where \( \alpha \in (0, 1) \).

**Proof.** See Proposition 2.12 in [4]. □

In what follows, \( L^p_b \) denotes the Sobolev space defined as \( L^p_b := (1 - \Delta)^{-s/2} L^p(\mathbb{R}^d) \). Such spaces can be characterized by the Stein derivative of order \( b \) as follows.

**Theorem 2.5.** Let \( b \in (0, 1) \) and \( 2d/(d + 2b) < p < \infty \). Then \( f \in L^p_b(\mathbb{R}^d) \) if and only if

a) \( f \in L^p(\mathbb{R}^d) \),

b) \( D^b f(x) := \left( \int_{\mathbb{R}^d} \frac{|f(x) - f(y)|^2}{|x-y|^{d+2b}} dy \right)^{1/2} \in L^p(\mathbb{R}^d) \), with

\[
\| f \|_{b, p} := \| J^b f \|_p \approx \| f \|_p + \| D^b f \|_p \approx \| f \|_p + \| D^b f \|_p.
\]

(2.5)

**Proof.** See Theorem 1 in [26]. □
From the last equivalence in (2.3) we see that the $L^p$ norms of $D^b$ and $D^b$ are equivalent. The advantage in using $D^b$ is that it is suitable when dealing with pointwise estimates, as we will see below. In addition, from Fubini’s theorem we have the product estimate (see [24 Proposition 1])

$$
\|D^b(fg)\|_{L^2(\mathbb{R}^d)} \leq \|fD^b g\|_{L^2(\mathbb{R}^d)} + \|gD^b f\|_{L^2(\mathbb{R}^d)}.
$$

(2.6)

We also recall the following.

**Lemma 2.6.** Let $b \in (0,1)$ and $h$ be a measurable function on $\mathbb{R}$ such that $h, h' \in L^\infty(\mathbb{R})$. Then, for all $x \in \mathbb{R}$

$$
D^b h(x) \lesssim \|h\|_{L^\infty(\mathbb{R})} + \|h'\|_{L^\infty(\mathbb{R})}.
$$

(2.7)

Moreover,

$$
\|D^b(hf)\|_{L^2(\mathbb{R})} \leq \|D^b h\|_{L^\infty(\mathbb{R})} \|f\|_{L^2(\mathbb{R})} + \|h\|_{L^\infty(\mathbb{R})} \|D^b f\|_{L^2(\mathbb{R})}.
$$

(2.8)

**Proof.** For (2.7) see Lemma 2.7 in [19]. Note that (2.8) is a consequence of (2.7). \qed

Some pointwise estimates in terms of the Stein derivative is given below. We start by introducing a cut-off function

$$
\varphi \in C_0^\infty(\mathbb{R}) \text{ such that } \text{supp } \varphi \subset [-2, 2] \text{ and } \varphi \equiv 1 \text{ in } (-1,1).
$$

(2.9)

**Proposition 2.7.** For any $\theta \in (0,1)$ and $\alpha > 0$, the function $D^\theta(\lvert \xi \rvert^\alpha \varphi(\xi))(\cdot)$ is continuous in $\eta \in \mathbb{R} \setminus \{0\}$ with

$$
D^\theta(\lvert \xi \rvert^\alpha \varphi(\xi))(\eta) \sim \begin{cases} 
 c|\eta|^\alpha - \theta + c_1, & \alpha \neq \theta, |\eta| \ll 1, \\
 c(-\ln |\eta|)^{1/2}, & \alpha = \theta, |\eta| \ll 1,
\end{cases}
$$

in particular, one has that

$$
D^\theta(\lvert \xi \rvert^\alpha \varphi(\xi)) \in L^2(\mathbb{R}) \text{ if and only if } \theta < \alpha + 1/2.
$$

(2.10)

In a similar fashion

$$
D^\theta(\lvert \xi \rvert^\alpha \text{sgn}(\xi)\varphi(\xi)) \in L^2(\mathbb{R}) \text{ if and only if } \theta < \alpha + 1/2.
$$

(2.11)

**Proof.** See Proposition 2.9 in [8]. \qed

Note that in the above proposition we are always taking $\alpha > 0$. However, in the proof of our main results we also need $\alpha < 0$. This is the content of the next two results.

**Proposition 2.8.** If $\gamma \in [0,1/2)$ then

$$
D^\gamma(\lvert \xi \rvert^{\gamma-1/2} \varphi(\xi)) \notin L^2(\mathbb{R}),
$$

where by $D^0$ we mean the identity operator.

**Proof.** See Proposition 2.11 in [4]. \qed

**Proposition 2.9.** If $\gamma \in (0,1/2)$ and $0 < \epsilon < \gamma$ then

$$
D^{\gamma-\epsilon}(\lvert \xi \rvert^{\gamma-1/2} \varphi(\xi)) \in L^2(\mathbb{R}).
$$

(2.13)

**Proof.** Here we use the same approach as in the proof of Proposition 2.9 in [8]. By setting $\theta = \gamma - \epsilon$ and $\gamma_1 = \gamma - 1/2$, we see that for $\eta \neq 0$, $D^\theta(\lvert \xi \rvert^{\gamma_1} \varphi(\xi))(\eta)$ is continuous in $\delta < |\eta| < \frac{1}{\eta}$, for all $\delta > 0$. First, we assume $0 < \eta < 2/3$. Then

$$
[D^\theta(\lvert \xi \rvert^{\gamma_1} \varphi(\xi))(\eta)]^2 = \int \frac{|(\xi + \eta)^{\gamma_1} \varphi(\xi + \eta) - |\eta|^{\gamma_1} \varphi(\eta)|^2}{|\eta - \xi|^{1+2\theta}} d\xi
$$

$$
= \int \frac{|(\xi + \eta)^{\gamma_1} \varphi(\xi + \eta) - |\eta|^{\gamma_1} \varphi(\eta)|^2}{|\xi|^{1+2\theta}} d\xi
$$

$$
= \int_0^{\eta/2} + \int_{\eta/2}^{\infty} + \int_{-\eta/2}^{0} + \int_{-\eta/2}^{0}
$$

$$
= I_1 + I_2 + I_3 + I_4.
$$

(2.14)
Next, we deal with the first integral on the right-hand side of (2.14). In view of $0 < \eta < \xi + \eta < 3\eta/2 < 1$, it follows that $\varphi(\xi + \eta) = \varphi(\eta) = 1$. Hence, by the mean value theorem there exists $z \in (\eta, \xi + \eta)$ such that

$$
\eta^{\gamma_1} - (\xi + \eta)^{\gamma_1} = -\gamma_1 \eta^{\gamma_1 - 1} \xi \leq \xi \eta^{\gamma_1 - 1} \leq \xi \eta^{\gamma_1 - 1},
$$

where we used that $\gamma_1 < 0$ and $\eta < z$. Thus, from (2.14)

$$
I_1 \leq \int_0^{\eta/2} \frac{\xi^2 \eta^{2(\gamma_1 - 1)}}{\xi^{4+2\theta}} d\xi \leq \eta^{2(\gamma_1 - 1)} \int_0^{\eta/2} \xi^{1-2\theta} d\xi \leq \eta^{2(\gamma_1 - 1)} (\eta/2)^{2(1-\theta)} \leq \eta^{2\varepsilon - 1}.
$$

Also,

$$
I_2 \leq \int_{\eta/2}^{\infty} \frac{((\xi + \eta)^{\gamma_1} + \eta^{\gamma_1})^2}{\xi^{4+2\theta}} d\xi \leq \eta^{2\gamma_1} \int_{\eta/2}^{\infty} \xi^{1-2\theta} d\xi \leq \eta^{2\varepsilon - 1}.
$$

With respect to $I_4$ we see that $-\eta/2 < \xi < 0$ implies $\eta/2 < \xi + \eta < \eta < 2/3$. Using the mean value theorem again we obtain $(\xi + \eta)^{\gamma_1} - \eta^{\gamma_1} \leq \frac{2}{\gamma_1} |\xi|^{\gamma_1 - 1}$. Thus

$$
I_4 = \int_{-\eta/2}^{0} \frac{((\xi + \eta)^{\gamma_1} - \eta^{\gamma_1})^2}{\xi^{4+2\theta}} d\xi \leq \int_{-\eta/2}^{0} \frac{\xi^2 \eta^{2(\gamma_1 - 1)}}{\xi^{4+2\theta}} d\xi = \int_0^{\eta/2} \frac{\xi^2 \eta^{2(\gamma_1 - 1)}}{\xi^{4+2\theta}} d\xi \leq \eta^{2\varepsilon - 1}.
$$

Concerning $I_3$ we write

$$
I_3 = \int_{-\infty}^{-\eta/2} \frac{((\xi + \eta)^{\gamma_1} \varphi(\xi + \eta) - \eta^{\gamma_1})^2}{\xi^{4+2\theta}} d\xi = \int_{-\infty}^{-\eta/2} + \int_{-\eta/2}^{0} =: I_3^1 + I_3^2.
$$

In the first integral in (2.15) we have $\varphi(\xi + \eta) = 0$. Hence

$$
I_3^1 \leq \int_{-\infty}^{-\eta/2} \frac{\eta^{2\gamma_1}}{\xi^{4+2\theta}} d\xi = c \eta^{2\gamma_1} (2 + \eta)^{-2\theta} \leq c \eta^{2\gamma_1} (2 + \eta)^{-2\theta} = c \eta^{2\varepsilon - 1}.
$$

The second integral in (2.15) can be estimated as

$$
I_3^2 \leq \int_{-\eta/2}^{-\eta/2} \frac{\eta^{2\gamma_1}}{\xi^{4+2\theta}} d\xi + \int_{-\eta/2}^{0} \frac{\eta^{2\gamma_1}}{\xi^{4+2\theta}} d\xi =: I_3^{2,1} + I_3^{2,2}.
$$

Now we have

$$
I_3^{2,2} = \eta^{2\gamma_1} \int_{-\eta/2}^{0} \xi^{1-2\theta} d\xi = \eta^{2\gamma_1} \frac{\eta^{2\gamma_1}}{2\theta} \left[ (\eta/2)^{-2\theta} - (2 + \eta)^{-2\theta} \right] \leq \eta^{2\varepsilon - 1}.
$$

The first integral on the right-hand side of (2.16) can be decomposed as

$$
I_3^{2,1} = \int_{-\eta}^{-\eta/2} + \int_{-\eta/2}^{0} =: I + \hat{I}.
$$

To estimate $\hat{I}$, by using that $\eta/2 \leq |\xi|$ we deduce

$$
\hat{I} \leq \eta^{-1-2\theta} \int_{-\eta}^{-\eta/2} (\xi + \eta)^{2\gamma_1} d\xi \leq \eta^{-1-2\theta} \eta^{2\gamma_1 + 1} \leq \eta^{2\varepsilon - 1}.
$$

To deal with the integral $I$ we choose $p, q$ such that $1 < p < -\frac{1}{2\gamma_1}$ and $\frac{1}{p} + \frac{1}{q} = 1$. Hence, by Young’s inequality we obtain

$$
I \leq \eta^{-2\theta} \int_{-\eta}^{-\eta/2} \frac{(\xi + \eta)^{2\gamma_1}}{|\xi|} d\xi
$$

$$
\leq \eta^{-2\theta} \left( \int_{-\eta}^{-\eta/2} (\xi + \eta)^{2\gamma_1} d\xi + \int_{-\eta}^{-\eta/2} \frac{d\xi}{|\xi|^q} \right)
$$

$$
\leq \eta^{-2\theta} \left[ 1 + (2 + \eta)^{1-q} \right].
$$

This completes the proof if $0 < \eta < 2/3$. The case $-2/3 < \eta < 0$ may be treated similarly.
Next we suppose $\eta > 200$. Here,
\[
D^b(|\xi|^\alpha \varphi(\xi))(\eta)^2 = \int \frac{(\xi + \eta)^{2\gamma_1} \varphi(\xi + \eta)^2}{|\xi|^{1+2\theta}} \, d\xi
\]
\[
\lesssim \int_{-2-\eta}^{2-\eta} \frac{(\xi + \eta)^{2\gamma_1} \, d\xi}{|\xi|^{1+2\theta}}
\]
\[
\lesssim \frac{1}{(\eta - 2)^{1+2\theta}}.
\]

(2.17)

The case $\eta < -200$ may be treated in a similar fashion. The proof of the proposition is thus completed.

In the next two results we recall some pointwise estimates we need in the sequel.

**Lemma 2.10.** Let $b \in (0, 1)$. For any $t > 0$,
\[
D^b(e^{-it|x|^{1+a}}) \lesssim t^b/(2+a) + t^b|x|^{(1+a)b}.
\]

**Proof.** See Proposition 2.7 in [8].

**Lemma 2.11.** Let $b \in (0, 1)$, then for all $t > 0$ and $\eta \in \mathbb{R}$,
\[
D^b(e^{it\eta x^2}) \lesssim \eta^{2b}t^b.
\]

**Proof.** See Lemma 2.9 in [2].

Since we will be dealing with weighted spaces, let us introduce the truncated weights $\langle x \rangle_N$, $N \in \mathbb{Z}^+$, by letting
\[
\langle x \rangle_N := \begin{cases} 
\langle x \rangle & \text{if } |x| \leq N, \\
2N & \text{if } |x| \geq 3N,
\end{cases}
\]
where $\langle x \rangle = (1 + x^2)^{1/2}$. Also, we assume that $\langle x \rangle_N$ is smooth and non-decreasing in $|x|$ with $\langle x \rangle_N(x) \leq 1$, for any $x \geq 0$, and there exists a constant $c$ independent of $N$ such that $|\langle x \rangle_N^\prime(x)| \leq c\partial_x^2(x)$.

**Lemma 2.12.** Let $\alpha, b > 0$. Assume that $J_\alpha^b f(x_1, x_2) \in L^2(\mathbb{R}^2)$ and $\langle x_j \rangle^b f(x_1, x_2) = (1 + x_j^a)^{b/2} f(x_1, x_2) \in L^2(\mathbb{R}^2)$. Then, for any $\beta \in (0, 1)$,
\[
\|J_\alpha^{\beta}(\langle x_j \rangle^{(1-\beta)b})\|_{L^2_x} \leq c\|\langle x_j \rangle^{\beta} f\|_{L^2_x}^{1-\beta}\|J_\alpha^{\beta} f\|_{L^2_x}^{\beta}, \quad i, j = 1, 2.
\]

Moreover, inequality (2.18) is still valid with $\langle \cdot \rangle_N$ instead of $\langle \cdot \rangle$ with a constant $c$ independent of $N$.

**Proof.** For the case $i = j$ see Lemma 4 in [24]. However, the same proof holds with $i \neq j$. 

To establish some of our estimates in next sections we need the following computations. Set
\[
\psi(\xi, \eta, t) = e^{it\xi(\eta^2 - |\xi|^{1+a})}.
\]

Then
\[
\partial_\xi(\psi \hat{\phi}) = \psi [it(\eta^2 - (2 + a)|\xi|^{a+1}) \hat{\phi} + \partial_\xi \hat{\phi}],
\]
(2.19)
\[
\partial_\xi^2(\psi \hat{\phi}) = \psi [-t(2 + a)(1 + a)\text{sgn}(\xi)|\xi|^a + t(2 + a)^2|\xi|^{2(1+a)} - 2t(2 + a)|\xi|^{1+a}\eta^2 + t\eta^4] \hat{\phi} + 2it(\eta^2 - (2 + a)|\xi|^{1+a}) \partial_\xi \hat{\phi} + \partial_\xi^2 \hat{\phi}]
\]
(2.20)
\[
=: F_1 + \cdots + F_7,
\]
(2.21)
\[
\partial^3_\xi(\psi \hat{\phi}) = \psi \left\{ 3(2 + a)(1 + a)\text{sgn}(\xi)t^2\eta^2|\xi|^{a} - 3it^3(2 + a)^2\eta^2|\xi|^{2(1+a)} \\
+ it^3(2 + a)^3|\xi|^{3(1+a)} + 3it^3(2 + a)|\xi|^{1+a}\eta^4 - it^3\eta^6 - ita(2 + a)(1 + a)|\xi|^{a-1} \\
- 3it^2(2 + a)^2(1 + a)\text{sgn}(\xi)|\xi|^{1+2a} \hat{\phi} + \left[ -3it(2 + a)(1 + a)\text{sgn}(\xi)|\xi|^a \\
- 3it^2(2 + a)^2|\xi|^{2(1+a)} + 6t^2(2 + a)\eta^2|\xi|^{1+a} - 3t^2\eta^4 \right] \partial^2_\xi \hat{\phi} \\
+ \left[ 3t\eta^2 - 3it(2 + a)|\xi|^{1+a} \right] \partial^2_\xi \hat{\phi} + \partial^3_\xi \hat{\phi} \right\} \\
=: G_1 + \cdots + G_{14},
\]

(2.22)

\[
\partial^4_\xi(\psi \hat{\phi}) = \psi \left\{ 4a(2 + a)(1 + a)t^2\eta^2|\xi|^{a-1} - (7a + 3)(1 + a)(2 + a)^2t^3|\xi|^{2a} + \\
- 9(1 + a)(2 + a)^2t^3\text{sgn}(\xi)\eta^2|\xi|^{1+2a} - 6i(2 + a)^3(1 + a)t^3\text{sgn}(\xi)|\xi|^{2+3a} + \\
+ 6i(2 + a)(1 + a)t^3\text{sgn}(\xi)\eta^4|\xi|^a - ita(2 + a)(a^2 - 1)\text{sgn}(\xi)|\xi|^{a-2} + \\
+ 6t^4(2 + a)^2\eta^4|\xi|^{2(1+a)} - 4t^4(2 + a)^2\eta^6|\xi|^{1+a} + \\
+ t^4 \phi + \left[ -4ita(2 + a)(1 + a)|\xi|^{a-1} - 12t^2(2 + a)^2(1 + a)\text{sgn}(\xi)|\xi|^{1+2a} + \\
+ 12t^2(1 + a)(2 + a)\text{sgn}(\xi)\eta^2|\xi|^a + 12it^3(2 + a)\eta^4|\xi|^{1+a} + \\
- 12it^3(2 + a)^2\eta^4|\xi|^{2(1+a)} + 4it^3(2 + a)^3|\xi|^{3(1+a)} - 4it^3\eta^6 \right] \partial_\xi \hat{\phi} + \\
+ 6 \left[ -it(2 + a)(1 + a)\text{sgn}(\xi)|\xi|^{a} - t^2\eta^4 - t^2(2 + a)^2|\xi|^{2(1+a)} + 2t^2(2 + a)\eta^2|\xi|^{1+a} \right] \partial^2_\xi \hat{\phi} \\
+ \left[ 4it\eta^2 - 4it(2 + a)|\xi|^{1+a} \right] \partial^2_\xi \hat{\phi} + \partial^4_\xi \hat{\phi} \right\} \\
=: H_1 + \cdots + H_{25}.
\]

(2.23)

Note that \( F_j, G_j \) and \( H_j \) depends on \( \xi, \eta, t \) and \( \hat{\phi} \), that is, \( F_j = F_j(\xi, \eta, t, \hat{\phi}) \), \( G_j = G_j(\xi, \eta, t, \hat{\phi}) \) and \( H_j = H_j(\xi, \eta, t, \hat{\phi}) \).

We end this section with two important estimates that will be used several times in the proof of our main results.

**Lemma 2.13.** Let \( \psi \) be as in (2.19). For all \( \theta \in (0, 1) \) and \( t \in (0, \infty) \),

\[
\|D^\theta_\xi (\psi \hat{f})\| \lesssim \rho(t) \left( \|f\| + \|D^\theta_\eta f\| + \|D^{(1+a)\theta}_\xi f\| \right) + \|x|^{\theta} f\|
\]

where \( \rho(t) = 1 + t^\theta + t^\frac{\theta}{1+a} \).

**Proof.** Using (2.6) and Lemmas 2.11 and 2.10,

\[
\|D^\theta_\xi (\psi \hat{f})\| \lesssim \|D^\theta_\xi (e^{i\xi t\eta^2})e^{-i\xi t|\xi|^{1+a}\hat{f}}\| + \|e^{i\xi t\eta^2}D^\theta_\xi (e^{-i\xi t|\xi|^{1+a}\hat{f}})\|
\]

\[
\lesssim \rho(t) \|f\| + \|D^\theta_\xi (e^{-i\xi t|\xi|^{1+a}\hat{f}})\| + \|e^{-i\xi t|\xi|^{1+a}}D^\theta_\xi \hat{f}\|
\]

\[
\lesssim \rho(t) \|\eta^{2\theta} \hat{f}\| + \|(1+\frac{\theta}{1+a})\|\|\xi|^{(1+a)\theta} \hat{f}\| + \|D^\theta_\xi \hat{f}\|
\]

\[
\lesssim \rho(t) \left( \|\hat{f}\| + \|\eta^{2\theta} \hat{f}\| + \|\xi|^{(1+a)\theta} \hat{f}\| + \|D^\theta_\xi \hat{f}\| \right)
\]

Then, Plancherel’s identity gives us the desired result. \( \square \)
In addition, recalling that 

$$\chi(\xi, \eta) = \varphi(\xi) \varphi(\eta),$$

where \( \varphi \) is given by (2.9).

**Lemma 2.14.** For all \( \theta \in (0, 1) \), \( t \in [0, \infty) \), \( \sigma_1 \in \{0, 1\} \), \( \sigma_2 \geq 1 \) and \( \sigma_3 \geq 0 \), it follows that 

$$\|D_{\xi}^\theta (\chi(\xi, \eta) \psi \text{sgn}(\xi^{\sigma_1} |\xi^{\sigma_2} \eta^{\sigma_3} \hat{f}|)\| \lesssim \|f\| + \|x|^\theta f\|,$$

and 

$$\|D_{\xi}^\theta (\chi(\xi, \eta) \psi ^{\sigma_3} \hat{f})\| \lesssim \|f\| + \|x|^\theta f\|,$$

where the implicit constants depend on \( t \) and \( a \). Moreover, if \( 1/2 < a < 1 \) then 

$$\|D_{\xi}^\theta (\chi(\xi, \eta) \psi ^{\sigma_3} |\xi^{\sigma_3} \hat{f}|)\| \lesssim J_\sigma^2 f + \|x^2 f\| + \|y|^\sigma f\|,$$

where \( \sigma_3 > 1 \) is an arbitrary number.

This result still holds if we replace \( \chi(\xi, \eta) \) by \( \hat{\chi}(\xi, \eta) = \varphi(\xi)e^{-\eta^2} \).

**Proof.** We will give the proof of (2.25) with \( \sigma_1 = 1 \). The proof of the other cases are similar.

Setting \( h(\xi, \eta) = \chi(\xi, \eta) \psi \text{sgn}(\xi^{\sigma_1} |\xi^{\sigma_2} \eta^{\sigma_3} \hat{f}|) \) and noting that \( \text{sgn}(\xi^{\sigma_1} |\xi^{\sigma_2} \hat{f}|) = \xi \sigma_3^{-1} \), it is easy to see that \( h \) together with its derivative with respect to \( \xi \) are bounded. So, the result follows as an application of Lemma 2.6. The proof of (2.26) is similar.

Next we will establish (2.27). Using (2.28),

$$\|D_{\xi}^\theta (\chi(\xi, \eta) \psi ^{\sigma_3} |\xi^{\sigma_3} \hat{f}|)\| \lesssim \|\xi^{\sigma_3} \hat{f}\| + \|D_{\xi}^\theta (\|\xi^{\sigma_3} \hat{f}\|)\|$$

$$\lesssim \|\xi^{\sigma_3} \hat{f}\| + \|\partial_\xi (\|\xi^{\sigma_3} \hat{f}\|)\|$$

$$\lesssim \|\xi^{\sigma_3} \hat{f}\| + \|\xi^{\sigma_2 - 1} \hat{f}\| + \|\xi^{\sigma_3} \partial_\xi \hat{f}\|,$$

where we used the interpolation inequality \( \|D_{\xi}^\theta (\|\xi^{\sigma_3} \hat{f}\|)\| \lesssim_{\sigma, t} \|\xi^{\sigma_3} \hat{f}\|^{1-\theta} \|\partial_\xi (\|\xi^{\sigma_3} \hat{f}\|)\|^\theta \). Now, using Sobolev’s embedding

$$\|\xi^{\sigma_3 - 1} \hat{f}\| \lesssim \|\xi^{\sigma_3 - 1} \chi \hat{f}\| + \|\xi^{\sigma_3 - 1} (1 - \chi) \hat{f}\|$$

$$\lesssim \|\xi^{\sigma_3 - 1} \chi \hat{f}\| + \|\hat{f}\|_{L_{\xi\eta}^{\sigma_3}} + \left\| \frac{1 - \chi}{\xi} \right\|_{L_{\xi\eta}^{\sigma_3}} \|\xi^{\sigma_3} \hat{f}\|$$

$$\lesssim \|\hat{f}\|^2 + \|y|^\sigma f\| + \|J_\sigma^2 f\|.$$

In addition, from Lemma 2.12 and Plancherel’s identity

$$\|\xi^{\sigma_3} \partial_\xi \hat{f}\| \lesssim \|J_\sigma^2 \hat{f}\| + \|\xi^{\sigma_2} \hat{f}\| \lesssim \|\hat{f}\|^2 + \|J_\sigma^2 f\|.$$

Gathering together (2.28), (2.30) we establish (2.27). The proof of the lemma is thus completed. \( \square \)

### 3. LOCAL WELL POSEDNESS IN WEIGHTED SPACES

In this section, we prove Theorem 1.1. So, let us assume that \( \phi \in Z^2_{r_1, r_2} = E^\circ \cap L^2_{r_1, r_2} \). First of all, we note that the existence of a continuous local solution, say \( u : [0, T] \to E^\circ \), is given by Theorem A. Thus, we only need to establish the persistence property in \( L^2_{r_1, r_2} \). Moreover, once we obtain the persistence property in \( L^2_{r_1, r_2} \), the continuity of \( u : [0, T] \to L^2_{r_1, r_2} \) and the continuity of the map-data-solution follow as in [2, Theorem 1.3].

If \( r_1 = r_2 = 0 \), there is nothing to prove. Hence, we can always assume either \( r_1 > 0 \) or \( r_2 > 0 \). In addition, recalling that \( L^2_{r_1, r_2} = L^2_{r_1, 0} \cap L^2_{0, r_2} \) we see that it suffices to prove the persistence in \( L^2_{r_1, 0} \) and in \( L^2_{0, r_2} \).

**Part 1:** We will divide in two other cases.

**Case a) Weights in the \( y \)-direction: persistence in \( L^2_{0, r_2} \), \( r_2 > 0 \).**

Take \( \phi \in E^\circ \cap L^2_{0, r_2} \). We multiply the differential equation (1.1) by \( \langle y \rangle^{r_2} u \) and integrate on \( \mathbb{R}^2 \) to obtain

$$\frac{1}{2} \frac{d}{dt} \|\langle y \rangle^{r_2} u\|^2 + \left( \langle y \rangle^{r_2} u, \langle y \rangle^{r_2} D_x^{a+1} \partial_y u + \langle y \rangle^{r_2} u_{xyy} + \langle y \rangle^{r_2} uu_x \right) = 0.$$
Let
\[ M = \sup_{[0,T]} \| u(t) \|_{L^\infty}. \] (3.2)

Since \( (y)_N \) is independent of \( x \) we obtain \( (y)_N^2 D_x^{2+1} \partial_x u = D_x^{2+1} \partial_x (y)_N^2 u \). Therefore, taking into account that \( D_x^{2+1} \partial_x \) is antisymmetric, the contribution of the term \( (y)_N^2 u, (y)_N^2 D_x^{2+1} \partial_x u \) in (3.1) is null. In addition,
\[ \langle (y)_N^2 u, (y)_N^2 u \rangle = \frac{1}{3} \int \partial_x \langle (y)_N^2 u \rangle \partial_x = 0. \]

It remains to estimate the middle term in (3.1). To do that, let us first assume \( r_2 > 1/2 \). By Lemma 2.12 with \( \alpha = 2r_2 \), \( \beta = \frac{1}{2r_2} \), \( b = r_2 \) and by Young’s inequality we see that
\[ \| J_y (y)_N^{2-1/2} u \| \lesssim \| (y)_N^2 u \| + \| J_y^2 r_2 u \|. \] (3.3)

In a similar fashion,
\[ \| J_y (y)_N^{2-1/2} u \| \lesssim \| (y)_N^2 u \| + \| J_y^2 r_2 u \|. \] (3.4)

It is to be clear that to obtain (3.3) for Lemma 2.12 only in the \( y \)-direction. In fact, by writing \( \| J_y (y)_N^{2-1/2} u \| = \| J_y (y)_N^{2-1/2} u \|_{L^2_y} \), we first use Lemma 2.12 in the \( y \)-direction and then Hőlder’s inequality in the \( x \)-variable. An application of Young’s inequality then gives (3.3). This kind of argument will be used along the paper without additional comments.

Using integration by parts, the inequality \( \| \partial_y (y)_N^{2r_2} \| \lesssim (y)_N^{2r_2-1} \) and (3.3) and (3.4) we obtain
\[
\int (y)_N^{2r_2} u \partial_x \partial_y^2 u = - \int \partial_y (y)_N^{2r_2} u \partial_x \partial_y u - \underbrace{\int (y)_N^{2r_2} \partial_y u \partial_x \partial_y u}_{= 0}
\lesssim \| (y)_N^{2-1/2} \partial_x u \| \| (y)_N^{2-1/2} \partial_y u \|
\lesssim \| J_y (y)_N^{2-1/2} u \|^{2} + \| J_y (y)_N^{2-1/2} u \| + \| (y)_N^2 u \|^{2}
\lesssim \| (y)_N^2 u \|^{2} + \| u \|_{H^{2r_2}}^{2}
\lesssim \| (y)_N^2 u \|^{2} + M^2,
\] (3.5)

where we used that \( E^s \hookrightarrow H^{2r_2} \).

On the other hand, if \( r_2 \in (0,1/2) \), we have \( \| \partial_y (y)_N^{2r_2} \| \lesssim (y)_N^{2r_2-1} \lesssim 1 \). Hence, as in (3.5),
\[
\int (y)_N^{2r_2} u \partial_x \partial_y^2 u = - \int \partial_y (y)_N^{2r_2} u \partial_x \partial_y u = \int \partial_y (y)_N^{2r_2} \partial_x u \partial_y u
\lesssim \| \partial_x u \| \| \partial_y u \| \lesssim \| u \|_{H^1}^{2} \lesssim M^2,
\]

where now we used that \( E^s \hookrightarrow H^1 \). The implicit constants that appears here and in the rest of the proof will always be independent of \( N \).

From (3.1) and the above inequalities we find that
\[
\frac{d}{dt} \| (y)_N^2 u \|^{2} \leq c (1 + \| (y)_N^2 u \|^{2}).
\]

So, by the Gronwall lemma (see, for instance, [12] Theorem 12.3.3]),
\[
\| (y)_N^2 u \|^{2} \leq \| (y)_N^2 \phi \|^{2} + tc + c \int_{0}^{t} e^{c(t-t')} (\| (y)_N^2 \phi \|^{2} + t' c) dt'.
\]

By solving the above integral and using the monotone convergence theorem we get
\[
\| (y)_N^2 u \|^{2} \leq e^{ct} \| (y)_N^2 \phi \|^{2} + e^{ct} - 1.
\]

This proves the persistence property in \( L^2_{0,T} \).

So in what follows, we only consider weights in the \( x \)-direction. That is, it remains to show the persistence property in \( L^2_{r_1,0} \).
Case b). Weights in the $x$-direction: persistence in $L^2_{r_1,0}$, $r_1 > 0$.

Let $r_1 \in (0,1]$. Putting $r_1 = \theta$, multiplying $\langle x \rangle^\theta_N u$ by $\langle x \rangle^\theta_N u$ and integrating on $\mathbb{R}^2$, we obtain

\[ \frac{1}{2} \frac{d}{dt} \| \langle x \rangle^\theta_N u \|^2 + \left( \langle x \rangle^\theta_N u, \langle x \rangle^\theta_N D_x^{1+a} \partial_x u + \langle x \rangle^\theta_N u_{xxy} + \langle x \rangle^\theta_N u_{xx} \right) = 0. \]  

(3.6)

To start with, following the ideas contained in [2], we write

\[ \langle x \rangle^\theta_N D_x^{1+a} \partial_x u = D_x^\theta((\langle x \rangle^\theta_N D_x \partial_x u) - [D_x^\theta; \langle x \rangle^\theta_N] D_x \partial_x u =: A_1 + A_2. \]  

(3.7)

From Proposition 2.3 and the fact that $\| \partial_x (\langle x \rangle^\theta_N) \|_{L^\infty} \lesssim 1$, we obtain

\[ \| A_2 \| = \| D_x^\theta((\langle x \rangle^\theta_N D_x \partial_x u) - [D_x^\theta; \langle x \rangle^\theta_N] D_x \partial_x u \| \lesssim \| \partial_x (\langle x \rangle^\theta_N) \|_{L^\infty} \| D_x^\theta \partial_x u \| \lesssim \| D_x^\theta \partial_x u \|. \]  

(3.8)

Inequality (3.8) and the fact that $s \geq 1$ yield

\[ \| A_2 \| \lesssim \| J^a_{x+1} u \| \lesssim \| J^a_{(s+1)s} u \| \lesssim M, \]

where, as before, $M$ is given in (5.2). For $A_1$, we write

\[ A_1 = D_x^\theta(\langle x \rangle^\theta_N D_x \partial_x u) = D_x^\theta \partial_x(\langle x \rangle^\theta_N D_x u) - D_x^\theta(\langle x \rangle^\theta_N D_x u) =: B_1 + B_2. \]

Another application of Proposition 2.3 together with the fact that $|\partial_x^a (\langle x \rangle^\theta_N)| \lesssim 1$, $\alpha = 1, 2$, yield

\[ \| B_2 \|_{L^2} \leq \| D_x^\theta \partial_x(\langle x \rangle^\theta_N D_x u) \| + \| \partial_x (\langle x \rangle^\theta_N D_x^{1+a} u) \| \]

\[ = \| D_x^\theta \partial_x(\langle x \rangle^\theta_N D_x^{1-a} D_x^\theta u) \| + \| \partial_x (\langle x \rangle^\theta_N D_x^{1+a} u) \| \]

\[ \lesssim \| \partial_x^a (\langle x \rangle^\theta_N) \|_{L^\infty} \| D_x^\theta u \| + \| D_x^{1+a} u \| \]

\[ \lesssim \| J^a_{x+a} u \| \lesssim M. \]

Observe that $B_1$ reads as

\[ B_1 = D_x^\theta \partial_x(\langle x \rangle^\theta_N D_x u) = D_x^\theta \partial_x D_x(\langle x \rangle^\theta_N u) - D_x^\theta \partial_x[ D_x; \langle x \rangle^\theta_N ] u =: C_1 + C_2. \]

Inserting $C_1$ in (3.10), from the antisymmetry of operator $D_x^{1+a} \partial_x$, we see that its contribution is null. On the other hand, using that $D_x = \mathcal{H} \partial_x$, we get

\[ [D_x; \langle x \rangle^\theta_N] u = D_x(\langle x \rangle^\theta_N u) - \langle x \rangle^\theta_N D_x u \]

\[ = \mathcal{H} \partial_x(\langle x \rangle^\theta_N u) - \langle x \rangle^\theta_N \mathcal{H} \partial_x u \]

\[ = \mathcal{H}((\partial_x \langle x \rangle^\theta_N) u) + [\mathcal{H}; \langle x \rangle^\theta_N ] \partial_x u. \]

Therefore,

\[ C_2 = -D_x^\theta \partial_x \mathcal{H}((\partial_x \langle x \rangle^\theta_N) u) - D_x^\theta \partial_x[ \mathcal{H}; \langle x \rangle^\theta_N ] \partial_x u =: D_1 + D_2. \]

From the interpolation inequality $\| D_x^\theta u \|_{L^2(\mathbb{R})} \lesssim \| u \|_{L^2(\mathbb{R})} \| D_x^\theta u \|_{L^2(\mathbb{R})}$, Young’s inequality, and Theorem 2.1, we infer

\[ \| D_2 \| = \| D_x^\theta \partial_x \mathcal{H}; \langle x \rangle^\theta_N ] \partial_x u \|

\[ \lesssim \| \partial_x \mathcal{H}; \langle x \rangle^\theta_N ] \partial_x u \|^{1-a} \| D_x \partial_x \mathcal{H}; \langle x \rangle^\theta_N ] \partial_x u \|^{a}

\[ \lesssim \| \partial_x \mathcal{H}; \langle x \rangle^\theta_N ] \partial_x u \|^{1-a} \| D_x^2 \mathcal{H}; \langle x \rangle^\theta_N ] \partial_x u \|^{a}

\[ \lesssim (\| \partial_x^2 \langle x \rangle^\theta_N \|_{L^\infty} + \| \partial_x^2 \langle x \rangle^\theta_N \|_{L^\infty}) \| u \|

\[ \lesssim \| u \| \lesssim M, \]
where we used that $|\partial^\alpha_x \langle x \rangle_N^\theta| \lesssim 1$, $\alpha = 2, 3$. Similarly,
\[
\|D_1\| = \|\mathcal{H}D^a_x \partial_x((\partial_x \langle x \rangle_N^\theta)u)\|
\lesssim \|D^a_x((\partial^2_x \langle x \rangle_N^\theta)u)\| + \|D^a_x((\partial_x \langle x \rangle_N^\theta)\partial_x u)\|
\lesssim \|\partial^2_x \langle x \rangle_N^\theta\|u\| + \|D^a_x(D^2_x \partial_x \langle x \rangle_N^\theta)u\| + \|D^a_x(D_x \langle x \rangle_N^\theta)\partial_x u\|
\lesssim \|\partial^2_x \langle x \rangle_N^\theta\|u\| + \|\partial^3_x \langle x \rangle_N^\theta\|u\| + \|\partial^2_x \langle x \rangle_N^\theta\|\|\partial_x u\|
+ \|\partial^3_x \langle x \rangle_N^\theta\|\|\partial_x u\|
\lesssim (\|\partial^2_x \langle x \rangle_N^\theta\|u\| + \|\partial^3_x \langle x \rangle_N^\theta\|u\|) \|J_x u\| + \|\partial^2_x \langle x \rangle_N^\theta\|L_x \|J_x^{1+a} u\|
\lesssim \|J_x^{1+a} u\| \lesssim M.
\]
From the above inequalities and (3.7), we conclude
\[
\|\langle x \rangle_N^\theta \partial^a_x u\| \lesssim M. \tag{3.9}
\]

Next, using integration by parts and $|\partial_x \langle x \rangle_N^\theta| \lesssim \langle x \rangle_N^{2\theta-1}$, we obtain
\[
\int \langle x \rangle_N^{2\theta} u \partial_x \partial^2_y u = \frac{1}{2} \int \partial_x \langle x \rangle_N^{2\theta} (\partial_y u)^2 \lesssim \|\langle x \rangle_N^{-1/2+\theta} \partial_y u\|^2.
\]
If $\theta = r_1 \in (0, 1/2]$, we promptly see that
\[
\int \langle x \rangle_N^{2\theta} u \partial_x \partial^2_y u \lesssim \|\partial_y u\|^2 \lesssim \|u\|^2_{H^{1\theta}} \lesssim M^2.
\]
Also, if $\theta = r_1 \in (1/2, 1]$, Lemma 2.12 and Young’s inequality imply
\[
\int \langle x \rangle_N^{2\theta} u \partial_x \partial^2_y u \lesssim \|\langle x \rangle_N^{-1/2+\theta} \partial_y u\|^2 \lesssim \|J_y(\langle x \rangle_N^{-1/2+\theta} u)\|^2
\lesssim \|\langle x \rangle_N^\theta u\| + \|J_y^\theta u\| \lesssim \|\langle x \rangle_N^\theta u\|^2 + M^2,
\]
where we used that $E^s \hookrightarrow H_y^{2r_1}$. In both cases we get
\[
\int \langle x \rangle_N^{2\theta} u \partial_x \partial^2_y u \lesssim \|\langle x \rangle_N^\theta u\|^2 + M^2. \tag{3.10}
\]
Finally, since $E^s \hookrightarrow H^{(1+\alpha)\theta} \hookrightarrow L^\infty$ and $|\partial_x \langle x \rangle_N^\theta| \lesssim \langle x \rangle_N^{2\theta-1} \lesssim \langle x \rangle_N^\theta$, we deduce
\[
\left| \langle x \rangle_N^\theta u, \langle x \rangle_N^\theta uu_x \right| = \frac{1}{3} \int \langle x \rangle_N^{2\theta} u^3 \lesssim \|\langle x \rangle_N^\theta u\| \|u\|_\infty \lesssim M^4 + \|\langle x \rangle_N^\theta u\|^2 \tag{3.11}
\]
Combining estimates (3.9), (3.10), and (3.11) with (3.6), we deduce
\[
\frac{1}{2} \frac{d}{dt} \|\langle x \rangle_N^\theta u\|^2 \leq c(1 + \|\langle x \rangle_N^\theta u\|^2).
\]
By using Gronwall’s lemma and arguing as before, we finally obtain
\[
\|\langle x \rangle^{r_1} u\|^2 \leq e^{2ct} \|\langle x \rangle^{r_1} \phi\|^2 + e^{2ct} - 1.
\]
This proves Case b) and completes the proof of Part 1).

Part 2): The persistence in $L^2_{x,t}$ follows exactly as in Part 1). So we need only to prove the persistence in $L^2_{x,t}$. Here, instead of using the differential equation itself we will use the equivalent integral formulation
\[
u(t) = U(t)\phi - \frac{1}{2} \int_0^t U(t - \tau)\partial_x u^2(\tau)d\tau, \tag{3.12}
\]
where $U(t)\phi$ is the solution of the IVP associated with the linear gBO-ZK equation. This is necessary because we are not able to reiterate the process in Part 1). At this point our analysis diverges from that in [8].

We will divide into two other cases.
**Case a.** $r_1 \in (1, 2)$. Let us start by writing $r_1 = 1 + \theta$, $\theta \in (0, 1)$. Since
\[
\| |x|^{r_1} u(t)\| \leq \| |x|^{r_1} U(t) \phi \| + \int_0^t \| |x|^{r_1} U(t - \tau) z(\tau) \| d\tau, \quad z = \frac{1}{2} \partial_x u^2, \tag{3.13}
\]
we need to estimate each term on the right-hand side. Using (2.20)
\[
\| |x|^{1+\theta} U(t) \phi \| \lesssim \| D^1_{\xi} \hat{U}(\phi) \| = \| D^2_{\xi} \partial_\xi \psi \| \lesssim t \left( \| D^2_{\xi} (\psi \eta^2 \hat{\phi}) \| + \| D^2_{\xi} (\psi |\xi|^{1+\alpha} \hat{\phi}) \| \right) + \| D^2_{\xi} (\psi \partial_\xi \hat{\phi}) \|
\]
\[
=: A_1 + A_2 + A_3.
\]
Now, using Lemma 2.13 and Young’s inequality,
\[
A_2 \lesssim t \rho(t) \left( \| D^1_{x} \phi \| + \| D^{2\theta} \partial_\xi \phi \| + \| D_{x}^{(1+\alpha)\theta} D^1_{x} \phi \| \right) + \| |x|^\theta D^1_{x} \phi \| \quad \quad (3.14)
\]
\[
\lesssim t \rho(t) \left( \| \phi \| _{H^2(1+\theta)} + \| \phi \| _{H^1_{\xi}(1+\alpha+\theta)} \right) + A_2.1.
\]
Since $E^s \hookrightarrow H^{2r_1}_{y}$ and $E^s \hookrightarrow H^{(1+\alpha)r_1}_{x}$ the first two terms in (3.14) are finite. To estimate $A_2.1$ we use function $\varphi$ in (2.9) to write
\[
A_{2.1} = \| D^2_{\xi} (|\xi|^{1+\alpha} \hat{\phi}) \| \leq \| D^2_{\xi} (|\xi|^{1+\alpha} \varphi(\xi) \hat{\phi}) \| + \| D^2_{\xi} (|\xi|^{1+\alpha} (1 - \varphi(\xi)) \hat{\phi}) \| =: A_{2.1}^1 + A_{2.1}^2.
\]
From (2.6) we deduce
\[
A_{2.1}^1 \lesssim \| |\xi|^{1+\alpha} \varphi(\xi) D^2_{\xi} \hat{\phi} \| + \| \hat{\phi} D^2_{\xi} (|\xi|^{1+\alpha} \varphi(\xi)) \|
\]
\[
\lesssim \| |\xi|^{1+\alpha} \varphi(\xi) \| _{L^\infty} \| D^2_{\xi} \hat{\phi} \| + \| \hat{\phi} \| \| D^2_{\xi} (|\xi|^{1+\alpha} \varphi(\xi)) \| _{L^\infty}
\]
\[
\lesssim \| |x|^\theta \phi \| + \| \phi \|, \quad \quad \quad (3.15)
\]
where we used Proposition 2.7 to obtain that $\| D^2_{\xi} (|\xi|^{1+\alpha} \varphi(\xi)) \| _{L^\infty}$ is finite. Also, observing that the function $\xi \mapsto \frac{|\xi|^{1+\alpha} (1 - \varphi(\xi))}{(\xi)^{1+\alpha}}$ satisfies the assumptions in Proposition 2.7 from (2.41) we obtain
\[
A_{2.1}^2 = \| J^\theta_{\xi} \left( \frac{|\xi|^{1+\alpha} (1 - \varphi(\xi))}{(\xi)^{1+\alpha}} \right) (\xi)^{1+\alpha} \hat{\phi} \| \lesssim \| J^\theta_{\xi} (|\xi|^{1+\alpha} \hat{\phi}) \|. \quad \quad (3.16)
\]
An application of Lemma 2.13 gives
\[
A_{2.1}^2 \lesssim \| J^\theta_{x}(1+\alpha)(1+\theta) \hat{\phi} \| + \| (x)^{1+\theta} \phi \|
\]
and we deduce that
\[
A_{2.1} \lesssim \| J^\theta_{x}(1+\alpha)(1+\theta) \hat{\phi} \| + \| (x)^{1+\theta} \phi \|. \quad \quad (3.17)
\]
Let us now estimate $A_3$. By recalling that $\partial_\xi \hat{\phi} = -i \bar{x} \hat{\phi}$ we use Lemma 2.13 to write
\[
A_3 = \| D^2_{\xi} (\psi \bar{x} \hat{\phi}) \| \lesssim \rho(t) \left( \| x \phi \| + \| D^{2\theta} (x \phi) \| + \| D_{x}^{(1+\alpha)\theta} (x \phi) \| \right) + \| |x|^\theta x \phi \|
\]
\[
\lesssim \rho(t) \left( \| D^{2\theta} (x \phi) \| + B \right) + \| |x|^\theta + 1 \phi \|
\]
\[
\lesssim \rho(t) \left( \| D^{2\theta} (x \phi) \| + \| (x)^{1+\theta} \phi \| + B \right) + \| |x|^\theta + 1 \phi \|,
\]
where we also used Lemma 2.12 in the last inequality. Using Lemma 2.12 again, the term $B$ can be estimated as follows:
\[
B \leq \| J^\theta_{x}(1+\alpha) \phi \| \leq \| (\xi)^{(1+\alpha)} \partial_\xi \hat{\phi} \|
\]
\[
\lesssim \| (\xi)^{(1+\alpha)\theta - 1} \| + \| J^\theta_{\xi} (|\xi|^{1+\alpha} \hat{\phi}) \|
\]
\[
\lesssim \| J^\theta_{x}(1+\alpha) \phi \| + \| J^\theta_{\xi} \hat{\phi} \| + \| (\xi)^{(1+\theta)(1+\alpha)} \hat{\phi} \|
\]
\[
\lesssim \| J^\theta_{x}(1+\alpha) \phi \| + \| (x)^{1+\theta} \phi \|. \quad \quad (3.18)
\]
For $A_1$, using Lemma 2.13 and Young's inequality we have

$$A_1 = \|D^\theta_\xi (\psi \partial^2_y \phi)\|$$

$$\lesssim \rho(t) \left( \|\partial_y^2 \phi\| + \|D_y^{2\theta} \partial_x^2 \phi\| + \|D_x^{(1+a)} \partial_x^2 \phi\| \right) + \|x \| \partial^2_y \phi\|$$

$$\lesssim \rho(t) \left( \|J_y^{2(1+\theta)} \phi\| + \|J_x^{(1+a)(1+\theta)} \phi\| \right) + \|x \| \partial^2_y \phi\|. $$

The last term in the above inequality can be estimated using Lemma 2.12,

$$\|x \| \partial^2_y \phi\| \lesssim \|J_y^{2(1+\theta)} \phi\| \lesssim \|J_y^{2(1+\theta)} \phi\| + \|x \| \partial^2_y \phi\|,$$

from which we obtain

$$A_1 \lesssim \rho(t) \left( \|J_y^{2(1+\theta)} \phi\| + \|J_x^{(1+a)(1+\theta)} \phi\| \right) + \|x \| \partial^2_y \phi\|.$$ 

Gathering together the above estimates for $A_1$, $A_2$, and $A_3$, we then infer

$$\|x \| \partial^2_y \phi\| \lesssim \rho_1(t) \left( \|\phi\|_{H^2_y} + \|\phi\|_{H^2_x \theta} \right) + \|x \| \partial^2_y \phi\|,$$

where $\rho_1$ is a continuous increasing function on $t \in [0, T]$. Now using (3.18) in (3.13), we obtain for all $t \in [0, T]$,

$$\|x \| \partial^2_y \phi\| \lesssim \rho_1(t) \left( \|\phi\|_{H^2_y} + \|\phi\|_{H^2_x \theta} \right) + \|x \| \partial^2_y \phi\|.$$ 

Note that

$$\|\partial_x u^2\|_{H^2_x} \lesssim \|J_x^{(1+a)} r_1 + 1 \| u^2\| + \|J_x^{2r_1 + 1/2} \| u^2\| \lesssim \|u\|_{H^2_x}^{1/2} + 1 \| u^2\|,$$

where we used that $H^2_x \theta$ is a Banach algebra. Our assumption $s \geq r_1 + 1/2$ implies $E^s \Rightarrow H^2_x \theta$ and we deduce

$$\|\partial_x u^2\|_{H^2_x} \lesssim M^2.$$ 

A similar argument also show that

$$\|\partial_x u^2\|_{H^2_x \theta} \lesssim M^2.$$ 

and

$$\|x \| \partial_x u^2\| \lesssim \|\partial_x u\|_{L^\infty} \|x \| \partial^2_x \phi\| \lesssim M \|x \| \partial^2_x \phi\|.$$ 

Consequently, from (3.19) we deduce

$$\|x \| \partial^2_y \phi\| \lesssim \|x \| \partial^2_x \phi\| \lesssim M \|x \| \partial^2_x \phi\|.$$ 

An application of Gronwall's lemma gives $\sup_{t \in [0, T]} \|x \| \partial^2_x \phi\| < \infty$.

**Case b.** $r_1 = 2$. In view of (2.21),

$$\|x^2 U(t)\phi\| = \|D^\theta_\xi (\psi \phi)\|$$

$$\lesssim t \left( \|\psi \|_{L^\infty} \|x \| \partial^2_x \phi\| + \|\psi \|_{L^2} \|x \| \partial^2_x \phi\| \right) + \|\eta \| \partial^2_x \phi\| + \|\eta \| \partial^2_x \phi\|.$$ 

From Young's inequality and Lemma 2.12 it is not difficult to obtain

$$\|x^2 U(t)\phi\| \lesssim \|\phi\|_{H^{2(r_1 + a)}} + \|\phi\|_{H^2_y} + \|x^2 \phi\|, \quad t \in [0, T].$$
Using the same argument as in (3.19) and (3.20) we also deduce \( \sup_{t \in [0,T]} \|x^2 u(t)\| < \infty \). Part 2) is thus completed.

**Part 3.** As we already said it suffices to show the persistence in \( L^2_{r_1,0} \). So assume \( r_1 \in (2, 5/2 + a) \). Next we divide the proof into the cases \( 0 < a \leq 1/2 \) and \( 1/2 < a < 1 \).

**Case a.)** \( 0 < a \leq 1/2 \). Write \( r_1 = 2 + \theta \), where \( 0 < \theta < 1/2 + a \). In this case it is clear that \( \theta \in (0,1) \). Using (2.21),

\[
\|x|^{2+\theta} U(t) \phi \| \lesssim t \left( \|D^\theta \psi \| + \|D^\theta \psi \| + \|D^\theta \psi \| + \|D^\theta \psi \| + \|D^\theta \psi \| \right) + \|D^\theta \psi \| + \|D^\theta \psi \| + \|D^\theta \psi \| + \|D^\theta \psi \| + \|D^\theta \psi \| + \|D^\theta \psi \| + \|D^\theta \psi \|
\]

(3.21)

Let us estimate each one of the terms \( B_j, j = 1, \ldots, 7 \). By Lemma 2.13 and Young’s inequality

\[
B_i \lesssim t \rho(t) \left( \|D^\theta \psi \| + \|D^\theta \psi \| + \|D^\theta \psi \| + \|D^\theta \psi \| + \|D^\theta \psi \| \right) + \|x|^\theta D^\theta \psi
\]

To estimate \( K \), we make use of function \( \chi \) in (2.24) to write

\[
K = \|D^\theta (\|\xi|^{\alpha} \chi \phi) \| \lesssim \|D^\theta (\|\xi|^{\alpha} \chi \phi) \| + \|D^\theta (\|\xi|^{\alpha} \chi (1 - \chi \phi) \|
\]

Thus, in view of (2.8),

\[
K_1 \lesssim \|D^\theta (\|\xi|^{\alpha} \chi \phi) \| + \|D^\theta (\|\xi|^{\alpha} \chi \phi) \| + \|\phi\|_\infty \|D^\theta (\|\xi|^{\alpha} \chi \phi) \|
\]

(3.22)

Since \( \chi(\xi, \eta) = \phi(\xi) \phi(\eta) \) the term \( \|\|\xi|^{\alpha} \chi \phi \|_\infty \) is clearly finite. Also, an application of Proposition 2.7 gives that \( \|D^\theta (\|\xi|^{\alpha} \chi \phi) \| \) is finite. It is to be clear that at this point the assumption \( \theta < 1/2 + a \) is crucial. From Sobolev’s embedding we then obtain

\[
K_1 \lesssim \|x|^\theta \phi \| + \|x|^{1+\theta} \phi \| + \|x|^{1+\theta} \phi \|.
\]

For \( K_2 \) we use the inequality \( \|D^\theta L \| \leq \|L\|^{1-\theta} \|\xi\|_{L^\theta} \| + \|\xi\|_{L^\theta} \| \). The term \( \|L\| \) is clearly finite. In addition, since \( 1 - \chi \) vanishes around the origin,

\[
\|\xi\|_{L^\theta} \| = \left\| \frac{1 - \chi}{\|\xi\|_{L^\theta}} \right\| + \|\xi\|_{L^\theta} \| + \|\xi\|_{L^\theta} \|
\]

(3.23)

where \( \chi \) stands for the characteristic function of the set \( \Omega \). The first two terms in the above inequality are clearly finite. The last one may be estimated as follows:

\[
\|\xi\|_{L^\theta} \| \lesssim \|\xi\|_{L^\theta} \| + \|\xi\|_{L^\theta} \| \lesssim \|\phi\| + \|\xi\|_{L^\theta} \| + \|\xi\|_{L^\theta} \|
\]

where we used that \( \xi \) is bounded and Lemma 2.12. Hence, we obtain

\[
K_2 \lesssim \|\phi\|_{H^2} \| + \|x\|_{L^2} \|
\]

and consequently,

\[
B_1 \lesssim t \rho(t) \left( \|\phi\|_{H^2} + \|\phi\|_{H^2} + \|x\|_{L^2} \| + \|x\|_{L^2} \| + \|y\|_{L^2} \|
\]

(3.24)
For $B_2$ we use Lemma 2.13 to get
\[
B_2 \lesssim t \rho(t)\left(\|D_x^{1+a}\phi\| + \|D_y^{2a}D_x^{2(1+a)}\phi\| + \|D_x^{1+a}\theta D_x^{2(1+a)}\phi\| + \|x^\theta D_x^{2(1+a)}\phi\|\right).
\]

The first three terms on the right-hand side of the above inequality can be estimated by using the Young inequality. The last one may be estimated as the term $A_{2,1}$ in (3.14). Thus, we obtain
\[
B_2 \lesssim t \rho(t)\left(\|\phi\|_{H_y^{2(2+a)}} + \|\phi\|_{H_{x,y}^{1+a}(2+a)}\right) + \|x^{2+a}\phi\|.
\]

Terms $B_3$ and $B_4$ are estimated similarly. Indeed, Lemma 2.13, Young’s inequality, (3.17), and Lemma 2.12 yield
\[
B_3 \lesssim t \rho(t)\left(\|D_x^{1+a} \partial_y^2 \phi\| + \|D_y^{2a}D_x^{1+a} \partial_y^2 \phi\| + \|D_x^{1+a}\theta \partial_y^2 \phi\| + \|x^\theta D_x^{1+a} \partial_y^2 \phi\|ight)
\lesssim t \rho(t)\left(\|\phi\|_{H_y^{2(2+a)}} + \|\phi\|_{H_{x,y}^{1+a}(2+a)}\right) + \|J_x^{2(1+a)} \partial_y^2 \phi\| + \|x^{2+a} \partial_y^2 \phi\|
\lesssim t \rho(t)\left(\|\phi\|_{H_y^{2(2+a)}} + \|\phi\|_{H_{x,y}^{1+a}(2+a)}\right) + \|J_x^{2(1+a)} \partial_y^2 \phi\| + \|x^{2+a} \partial_y^2 \phi\|,
\]
and
\[
B_4 \lesssim t \rho(t)\left(\|D_x^{1+a} \partial_y \phi\| + \|D_y^{2a} \partial_y \phi\| + \|D_x^{1+a}\theta \partial_y \phi\| + \|x^\theta \partial_y \phi\|ight)
\lesssim t \rho(t)\left(\|\phi\|_{H_y^{2(2+a)}} + \|\phi\|_{H_{x,y}^{1+a}(2+a)}\right) + \|J_x^{2(2+a)} \partial_y \phi\| + \|x^{2+a} \partial_y \phi\|.
\]

Next, from Lemma 2.13 we get
\[
B_5 \lesssim t \rho(t)\left(\|D_x^{1+a}(x \phi)\| + \|D_y^{2a}D_x^{1+a}(x \phi)\| + \|D_x^{1+a}(1+a)(1+a)(x \phi)\|\right) + \|x^\theta D_x^{1+a}(x \phi)\|
\lesssim t \rho(t)\left(\|J_x^{2(1+a)}(x \phi)\| + \|J_x^{2(1+a)}(1+a)(x \phi)\|\right) + \|x^\theta D_x^{1+a}(x \phi)\|
\lesssim t \rho(t)(B_{5,1} + B_{5,2}) + B_{5,3}
\]
where we used Young’s inequality to obtain
\[
\|D_y^{2a}D_x^{1+a}(x \phi)\| \lesssim \|J_x^{2(1+a)}(x \phi)\| + \|J_x^{2(1+a)}(1+a)(x \phi)\|.
\]

But, from Lemma 2.12
\[
B_{5,1} \lesssim \|J_x^{2(\theta+1)}(x \phi)\| \lesssim \|J_x^{2(2+a)} \phi\| + \|x^{2+a} \phi\|.
\]

Also, Plancherel’s identity and Lemma 2.12 give
\[
B_{5,2} = \|\langle x \rangle^{1+a} \partial_y \phi\|
\lesssim \|\langle x \rangle^{1+a}(1+a) \partial_y \phi\| + \|J_x \langle x \rangle^{(1+a)}(1+a) \partial_y \phi\|
\lesssim \|J_x^{(1+a)}(1+a) \phi\| + \|J_x^{2+a} \partial_y \phi\| + \|\langle x \rangle^{(1+a)} \partial_y \phi\|
\lesssim \|\langle x \rangle^{2+a} \phi\| + \|J_x^{(1+a)}(2+a) \partial_y \phi\|.
\]

Note that $B_{5,3}$ is exactly term $A_{2,1}$ in (3.14) with $x \phi$ instead of $\phi$. Thus, from (3.17), we have
\[
B_{5,3} \lesssim \|J_x^{(1+a)}(1+a) \phi\| + \|\langle x \rangle^{1+a} \phi\| \lesssim \|\langle x \rangle^{2+a} \phi\| + B_{5,2} \lesssim \|\langle x \rangle^{2+a} \phi\| + \|J_x^{(1+a)}(2+a) \phi\|,
\]
and conclude that
\[
B_5 \lesssim t \rho(t)\left(\|\phi\|_{H_y^{2(2+a)}} + \|\phi\|_{H_{x,y}^{1+a}(2+a)} + \|\langle x \rangle^{2+a} \phi\|\right) + \|J_x^{(1+a)}(2+a) \phi\| + \|\langle x \rangle^{2+a} \phi\|.
\]

For $B_6$, Lemma 2.13 implies
\[
B_6 \lesssim t \rho(t)\left(\|\partial_y^2 (x \phi)\| + \|D_y^{2a} \partial_y^2 (x \phi)\| + \|D_x^{1+a} \theta \partial_y^2 (x \phi)\|\right) + \|x^\theta \partial_y^2 (x \phi)\|.
\]

From Lemma 2.12 the first two terms on the right-hand side of (3.26) may be estimated as
\[
\|\partial_y^2 (x \phi)\| + \|D_y^{2a} \partial_y^2 (x \phi)\| \lesssim \|J_x^{2(2+a)} \phi\| + \|\langle x \rangle^{2+a} \phi\|.
\]
For the third one we use Young’s inequality to obtain
\[
\|D_x^{(1+a)\theta} \partial_y^2 (x\phi)\| \leq \|J_y^{(1+\theta)}(x\phi)\| + \|J_x^{(1+a)(1+\theta)}(x\phi)\| \lesssim \|\langle x \rangle^{2+\theta}\phi\| + \|J_x^{(1+a)(2+\theta)}\phi\| + \|J_y^{(2+\theta)}\phi\|,
\]
where we used the estimates for \(B_{5,1}\) and \(B_{5,2}\) above. Finally, using similar arguments,
\[
\begin{align*}
B_7 &\lesssim t\rho(t) \left( \|x^\theta \phi\| + \|D_y^2(x^2 \phi)\| + \|D_x^{(1+a)\theta}(x^2 \phi)\| \right) + \|\|x\|^\theta x^2 \phi\|
\lesssim t\rho(t) \left( \|\langle x \rangle^{2+\theta}\phi\| + \|J_y^{2\theta}(\langle x \rangle^2 \phi)\| + \|J_x^{(1+a)(1+\theta)}(x\phi)\| + \|\langle x \rangle^{1+\theta} x \phi\| \right) + \|\|x\|^\theta x^2 \phi\|
\lesssim t\rho(t) \left( \|\langle x \rangle^{2+\theta}\phi\| + \|J_y^{(2+\theta)}\phi\| + B_{5,2} \right).
\end{align*}
\]
Gathering together all the above inequalities, we deduce
\[
\|\|x\|^\theta U(t)\phi\| \lesssim \rho_2(t) \left( \|\phi\|_{H_x^{(1+a)(2+\theta)}} + \|\phi\|_{H_y^{(2+\theta)}} + \|\|x\|^\theta \phi\| + \|\|y\|^\theta \phi\| \right),
\]
where \(\rho_2\) is a continuous increasing function on \(t \in [0, T]\).

Recalling that \(r_1 = 2 + \theta\), as in (3.13), we then get
\[
\begin{align*}
\|\|x\|^{r_1} u(t)\| &\leq \|\|x\|^{r_1} U(t)\phi\| + \int_0^t \|\|x\|^{r_1} U(t - \tau)z(\tau)\|d\tau \\
&\lesssim \rho_2(T) \left( \|\phi\|_{H_x^{r_1}} + \|\phi\|_{H_x^{(1+a)+r_1}} + \|\|x\|^{r_1} \phi\| \right) + \|\|y\|^{r_1-1} \phi\| \\
&\quad + \int_0^t \rho_2(t - \tau)(\|\partial_x u^2(\tau)\|_{H_x^{r_1}} + \|\partial_x u^2(\tau)\|_{H_y^{(1+a)+r_1}} + \|\|x\|^{r_1} \partial_x u^2(\tau)\| + \|\|y\|^{r_1-1} \partial_x u^2(\tau)\|)d\tau.
\end{align*}
\]
Note that
\[
\|\|y\|^{r_1-1} \partial_x u^2(\tau)\| \lesssim \|\partial_x u\|_{L^\infty} \|\|y\|^{r_1-1} u(\tau)\| \lesssim M \sup_{t \in [0,T]} \|\|y\|^{r_1-1} u(t)\|.
\]
The right-hand side of the above inequality is finite thanks to Case a) in Part 1). Thus, we have
\[
\|\|x\|^{r_1} u(t)\| \leq c + \int_0^t \rho_2(t - \tau)(\|\partial_x u^2(\tau)\|_{H_x^{r_1}} + \|\partial_x u^2(\tau)\|_{H_y^{(1+a)+r_1}} + \|\|x\|^{r_1} \partial_x u^2(\tau)\|)d\tau.
\]
This last inequality is similar to that in (3.10). Consequently one can proceed as in Part 2) to get the desired.

**Case b):** \(1/2 < a < 1\). If \(2 < r_1 < 3\), by writing \(r_1 = 2 + \theta\), we can use the same ideas as in Case a) to obtain the persistence. Note that in this case we also have \(\theta < 1 < 1/2 + a\) and so we can still apply Proposition 2.7 to deduce that the term \(\|D_x^\theta(\|\xi\|^a \text{sgn}(\xi)\chi)\|\) appearing in \(K_1\) (see (3.22)) is finite.

If \(r_1 = 3\), from (2.22),
\[
\|x^3 U(t)\phi\| = \|\partial_x^3(\psi \hat{\phi})\| \lesssim \sum_{j=1}^{14} \|G_j\|
\]
where the implicit constant depends continuously on \(t \in [0, T]\). After several applications of Young’s inequality and Lemma 2.12 it is not difficult to see that
\[
\sum_{j=1,j \neq 6}^{14} \|G_j\| \lesssim \|J_y^6 \phi\| + \|J_x^3(1+a)\phi\| + \|\langle x \rangle^3 \phi\|.
\]
Moreover, if \(\chi = \varphi(\xi)\varphi(\eta)\) denotes the function in (2.21),
\[
\|G_6\| \lesssim \||\xi|^{a-1} \chi \hat{\phi}\| + \||\xi|^{a-1}(1-\chi) \hat{\phi}\| \lesssim \||\xi|^{a-1} \chi\|\|\hat{\phi}\|_{L^\infty} + \||\xi|^{a-1}(1-\chi)\|L^\infty\|\hat{\phi}\|.
\]
Since \( \chi \equiv 1 \) near the origin, \( \| \xi^{a-1}(1-\chi) \|_{L^\infty} \) is finite. Also, since \( 1/2 < a < 1 \) the function \( |\xi|^{a-1}\varphi(\xi) \) belongs to \( L^2(\mathbb{R}) \), from which we deduce that \( \| \xi^{a-1}\chi \| \) is finite. Consequently, from Sobolev’s embedding,

\[
\|G_6\| \lesssim \|\hat{\phi}\|_{H^a_{1,q}} + \|\hat{\phi}\| \lesssim \|(x)^3\phi\| + \|\langle y\rangle^{r_2}\phi\|. \tag{3.29}
\]

From these estimates we obtain

\[
\|x^3U(t)\phi\| \lesssim_T \|J_y^6\phi\| + \|J_x^{3(1+a)}\phi\| + \|(x)^3\phi\| + \|\langle y\rangle^{r_2}\|,
\]

and we can proceed as before.

It remains to consider the case \( 3 < r_1 < 5/2 + a \). First we write \( r_1 = 3 + \theta \) with \( 1/2 + \theta < a \).

By using (2.22) now we may write

\[
\|x^{3+\theta}U(t)\phi\| \lesssim_t \|D_y^6(\psi\text{sgn}(\xi)r^2\xi^{a}\hat{\phi})\| + \|D_x^6(\psi\eta^2\xi^{2(1+a)}\hat{\phi})\| + \|D_x^6(\psi\xi^{(3(1+a))}\hat{\phi})\| + 
\]

\[
+ \|D_y^6(\psi\xi^{1+a}\eta^4\hat{\phi})\| + \|D_y^6(\psi\xi^{a-1}\hat{\phi})\| + \|D_x^6(\psi\xi^{1+a}\hat{\phi})\| + \|D_x^6(\psi\xi^{2(1+a)}\partial_\xi\hat{\phi})\| + 
\]

\[
+ \|D_y^6(\psi\eta^4\xi^{1+a}\partial_\xi\hat{\phi})\| + \|D_x^6(\psi\eta^4\partial_\xi\hat{\phi})\| + \|D_y^6(\psi\xi^{1+a}\partial_\xi^2\hat{\phi})\| + \|D_x^6(\psi\partial_\xi^2\hat{\phi})\|
\]

\[
=: C_1 + \cdots + C_{14}, \tag{3.30}
\]

where the implicit constant depends continuously on \( t \in [0,T] \). Using Young’s inequality and Lemmas 2.13 and 2.12 it is not difficult to deduce that

\[
C_j \lesssim_{\alpha,\theta,T} \|J_y^{3(1+a)}\phi\| + \|J_x^{(1+a)(3+\theta)}\phi\| + \|\langle x\rangle^{3+\theta}\phi\|, \quad j = 1, \ldots, 14 \text{ and } j \neq 6, 7. \tag{3.31}
\]

What is left is to estimate \( C_6 \) and \( C_7 \). Let us start with \( C_6 \). Lemma 2.13 implies that

\[
\|D_x^{a-1}\phi\| \lesssim \rho(t)(\|D_x^{a-1}\phi\| + \|D_y^{2b}\| + \|\langle x\rangle^{3+\theta}\|) \tag{3.32}
\]

The term \( \|D_x^{a-1}\phi\| \) may be estimated as in (3.29). Now, with \( \chi \) as in (2.23),

\[
\|D_y^{2b}\| \lesssim \|\chi|\eta|^{2b}\| + \|\langle 1 - \chi\rangle\eta^{2b}\| \lesssim L_1 + L_2.
\]

But

\[
L_1 \leq \|\chi|\eta|^{2b}\| \lesssim \|\psi(\xi)\|_{L^\infty} \lesssim \rho(t)\|\langle x\rangle^{3+\theta}\| \lesssim L_1 + L_2.
\]

The term \( \|D_x^{a-1}\phi\| \) belongs to \( L^2(\mathbb{R}) \). Also, from Young’s inequality,

\[
L_2 \leq \|\xi^a\|_{L^\infty} \leq \|\eta(\xi)\|_{L^\infty} \lesssim \rho(t)\|\langle x\rangle^{3+\theta}\| \lesssim L_1 + L_2.
\]

Thus, from Sobolev’s embedding,

\[
\|D_y^{2b}\| \lesssim \|\phi(\xi)\|_{L^\infty} \lesssim \|\langle x\rangle^{3+\theta}\| \lesssim \rho(t)\|\langle x\rangle^{3+\theta}\| \lesssim L_1 + L_2.
\]

Clearly we have \( \|D_x^{(1+a)(3+\theta)}\phi\| \leq \|J_x^{(1+a)(3+\theta)}\phi\|. \) For \( E \) in (3.32), we write

\[
E = \|D_x^6(\xi^{a-1}\phi)\| \leq \|D_x^6(\xi^{a-1}\phi)\| \leq \|D_x^6(\xi^{a-1}\phi)\| =: E_1 + E_2, \tag{3.33}
\]

and split

\[
E_1 \leq \|D_x^6(\xi^{a-1}\phi(\xi, \eta) - \phi(0, \eta))\| + \|D_x^6(\xi^{a-1}\phi(0, \eta))\| =: E_{1,1} + E_{1,2}.
\]
By using the inequality $\|D^a f\| \leq \|f\| + \|\partial \xi f\|$ and the mean value theorem, we deduce

$$E_{1,1} \lesssim \left\| |x|^{-1} \chi(\hat{\phi}(\xi, \eta) - \hat{\phi}(0, 0)) \right\| + \left\| \partial \xi \left( |x|^{-1} \chi(\hat{\phi}(\xi, \eta) - \hat{\phi}(0, 0)) \right) \right\| +$$

$$+ \left\| |x|^{a} \chi \frac{\hat{\phi}(\xi, \eta) - \hat{\phi}(0, 0)}{\xi} \right\| + \left\| |x|^{a-1} \chi \partial \xi \hat{\phi} \right\|$$

$$\lesssim \left\| |x|^{a} \chi \frac{\hat{\phi}(\xi, \eta) - \hat{\phi}(0, 0)}{\xi} \right\| + \left\| |x|^{a-1} \chi \partial \xi \hat{\phi} \right\|$$

Also,

$$E_{1,2} \leq \|\hat{\phi}\|_{L^\infty} \|D^a_x (|\xi|^{-1} \chi)\| \lesssim (\|\langle x \rangle^{\gamma_1} \phi\| + \|\langle y \rangle^{\gamma_2} \phi\|) \|D^a_x (|\xi|^{-1} \chi)\| \lesssim \|\langle x \rangle^{\gamma_1} \phi\| + \|\langle y \rangle^{\gamma_2} \phi\|,$$

where we used Sobolev’s embedding and Proposition \ref{prop:2.4} with $\gamma = a - 1/2$ and $\epsilon = a - 1/2 - \theta$, to see that $\|D^a_x (|\xi|^{-1} \chi)\|$ is finite.

Moreover, by setting $h(\xi, \eta) = |\xi|^{a-1}(1 - \chi(\xi, \eta))$ it follows that $h, \partial \xi h \in L^\infty$. Thus, from \ref{eq:2.6} and \ref{eq:2.7},

$$E_2 \lesssim \|D^a_x h\|_{L^\infty} \|\hat{\phi}\| + \|h\|_{L^\infty} \|D^a_x \hat{\phi}\| \lesssim \|\phi\| + \|\langle x \rangle^a \phi\|,$$

which then gives that

$$E \lesssim \|\langle x \rangle^{\gamma_1} \phi\| + \|\langle y \rangle^{\gamma_2} \phi\|.$$

Collecting the above estimates we finally conclude

$$C_6 \lesssim \|J_x^{(1-a)(3+\theta)} \phi\| + \|J_y^{(2+3\theta)} \phi\| + \|\langle x \rangle^{\gamma_1} \phi\| + \|\langle y \rangle^{\gamma_2} \phi\|.$$  \hspace{1cm} (3.34)

Next we estimate $C_7$. First we write

$$C_7 = \|D^a_x (\psi \text{sgn}(\xi)) |\xi|^{1+2a}\hat{\phi}\|$$

$$\leq \|D^a_x (\psi \text{sgn}(\xi)) |\xi|^{1+2a}\hat{\phi}\| + \|D^a_x ((1 - \chi) \psi \text{sgn}(\xi)) |\xi|^{1+2a}\hat{\phi}\|$$

$$= C_{7,1} + C_{7,2}.$$  \hspace{1cm} \ref{eq:2.14}

In view of Lemma \ref{lem:2.12} we promptly obtain

$$C_{7,1} \lesssim \|\phi\| + \|\langle x \rangle^a \phi\|.$$  \hspace{1cm} (3.35)

In addition, using interpolation and the definition of the function $\psi$, we have

$$C_{7,2} \lesssim \|\langle 1 - \chi \rangle \psi \text{sgn}(\xi) |\xi|^{1+2a}\hat{\phi}\| + \|\partial \xi ((1 - \chi) \psi \text{sgn}(\xi) |\xi|^{1+2a}\hat{\phi}\|$$

$$\lesssim \|D^a_x (\psi \text{sgn}(\xi)) |\xi|^{1+2a}\hat{\phi}\| + \|\partial \xi \psi \text{sgn}(\xi) |\xi|^{1+2a}\hat{\phi}\|$$

$$\lesssim \|J_x^{(1+2a)\phi}\| + \|\langle x \rangle^{3+\theta} \hat{\phi}\| + \|J_y^{(1+2a)\phi}\| + \|J_x^{(1+2a)\phi}\|.$$  \hspace{1cm} \ref{eq:2.12} and Young’s inequality,

$$\|J_x (\xi)^{1+2a}\phi\| \lesssim \|J_x^{(3+\theta)\phi}\| + \|\langle x \rangle^{3+\theta} \phi\| \lesssim \|J_x^{(1+a)(3+\theta)} \phi\| + \|\langle x \rangle^{3+\theta} \phi\|,$$

and

$$\|D^2_x D^a_x \phi\| \lesssim \|D^2_y (3+\theta) \phi\| + \|D^2_x (3+\theta) \phi\| \lesssim \|J_x^{(1+a)(3+\theta)} \phi\| + \|J_x^{(1+a)(3+\theta)} \phi\|,$$

which implies

$$C_{7,2} \lesssim \|J_x^{(3+\theta)\phi}\| + \|J_x^{(1+a)(3+\theta)} \phi\| + \|\langle x \rangle^{3+\theta} \phi\|.$$  \hspace{1cm} (3.36)

From \ref{eq:3.35} and \ref{eq:3.36}, we infer

$$C_7 \lesssim \|J_x^{(3+\theta)\phi}\| + \|J_x^{(1+a)(3+\theta)} \phi\| + \|\langle x \rangle^{3+\theta} \phi\|.$$  \hspace{1cm} (3.37)
Finally, from \((3.30), (3.31), (3.34),\) and \((3.37),\) we have
\[
\| |x|^{3+\theta} U(t) \phi \| \lesssim \| J^2_{\phi} |x|^{1+\theta} \phi \| + \| \mathcal{I}^{2+\theta} \phi \| + \| |x|^1 \phi \| + \| |y|^2 \phi \|.
\]
As in \((3.27),\) this last inequality is enough to apply Gronwall’s inequality and obtain the desired.

**Part 4**: \(r_1 \in [5/2 + a, 7/2 + a), \) \(r_2 > 3.\) Let us prove the persistence in \(L^2_{r_1, 0}.\) We will divide into the cases \(a \in (1/2, 1)\) and \(a \in (0, 1/2)\) again.

**Case a** \(a \in (1/2, 1).\) Let us first suppose \(3 < r_1 < 4\) and write \(r_1 = 3 + \theta,\) where \(\theta \in [a - 1/2, 1).\) By using \((2.22)\) we obtain inequality \((3.30).\) Except for \(C_6\) all other terms are estimated as in Part 3). So, what is left is to estimate \(C_6.\) At this point the assumption \(\tilde{\phi}(0, \eta) = 0\) plays a crucial role. Indeed, Lemma \((2.13)\) implies that
\[
C_6 \lesssim \rho(t) \left( \| D_a^{-1} \phi \| \right) \left( \| D^{2\theta}_a D_a^{-1} \phi \| \right) \left( \| D(1+\theta) D_a^{-1} \phi \| \right) + \| |x|^\theta D_a^{-1} \phi \|.
\]
(3.38)
By the following strategy as in Case b) of Part 3) we only need to estimate the term \(E.\) We split
\[
E = \| D_a^\theta (|\xi|^{-1} \tilde{\phi}) \| \leq \| D_a^\theta (|\xi|^{-1} \tilde{\phi}) \| + \| D_a^\theta (|\xi|^{-1} (1 - \xi) \tilde{\phi}) \| =: E_1 + E_2.
\]
(3.39)
For \(E_2\) we follow the ideas above to conclude that \(E_2 \lesssim \| \phi \| + \| |x|^\theta \phi \|.\) So we only need to take care of \(E_1.\) Here we cannot use the same strategy as in Case b) of Part 3) because in that case we strongly used that \(\theta < a - 1/2.\) The idea here is to use the assumption \(\tilde{\phi}(0, \eta) = 0\) and Taylor’s theorem with integral remainder to write
\[
\tilde{\phi}(\xi, \eta) = \xi \partial_\xi \tilde{\phi}(0, \eta) + \int_0^\xi (\xi - \zeta) \partial_\xi^2 \phi(\xi, \eta) d\zeta.
\]
(3.40)
Thus
\[
E_1 \leq \| D_a^\theta (|\xi| \chi \partial_\xi \tilde{\phi}(0, \eta)) \| + \left| D_a^\theta \left( |\xi|^{-1} \chi \int_0^\xi (\xi - \zeta) \partial_\xi^2 \phi(\xi, \eta) d\zeta \right) \right|
\]
\[
\leq \| \partial_\xi \tilde{\phi} \|_{L^{\infty}_{\xi\eta}} \| D_a^\theta (|\xi| \chi \partial_\xi \tilde{\phi}(0, \eta)) \| + \| D_a^\theta N \|.
\]
Since \(\theta < a + 1/2,\) Proposition \((2.7)\) and Stein derivative give that \(\| D_a^\theta (|\xi| \chi \partial_\xi \tilde{\phi}(0, \eta)) \| \) is finite. By using the interpolation estimate \(\| D_a^\theta N \| \leq \| N \|^{1-\theta} \| \partial_\xi N \|^\theta,\) we estimate
\[
\| \partial_\xi N \| \leq \| \partial_\xi \chi |\xi|^{a-1} \int_0^\xi (\xi - \zeta) \partial_\xi^2 \phi(\xi, \eta) d\zeta \| + \| \chi \mathrm{sgn}(\xi) |\xi|^{a-2} \int_0^\xi (\xi - \zeta) \partial_\xi^2 \phi(\xi, \eta) d\zeta \|
\]
\[
\leq \left( \| \partial_\xi \chi |\xi|^{a-1} \xi^2 \| + \| \chi \mathrm{sgn}(\xi) |\xi|^{a-2} \xi^2 \| + \| \chi |\xi|^{a-1} \| \right) \| \partial_\xi^2 \phi \|_{L^{\infty}_{\xi\eta}}
\]
\[
\lesssim \| \partial_\xi^2 \phi \|_{L^{\infty}_{\xi\eta}} \lesssim \| \langle x \rangle^{a} \phi \| + \| \langle y \rangle^{a} \phi \|.
\]
Consequently,

\[ E \lesssim \| \langle x \rangle^r \phi \| + \| \langle y \rangle^{r_2} \phi \|. \]

and

\[ C_6 \lesssim \| J_x^{(1+\theta)(3+\theta)} \phi \| + \| J_y^{(3+\theta)} \phi \| + \| \langle x \rangle^{r_1} \phi \| + \| \langle y \rangle^{r_2} \phi \|. \]

Therefore, also here we obtain the estimate

\[ \| |x|^{3+\theta} U(t) \phi \| \lesssim T \| J_y^{(3+\theta)} \phi \| + \| J_x^{(1+\theta)(3+\theta)} \phi \| + \| \langle x \rangle^{3+\theta} \phi \| + \| \langle y \rangle^{r_2} \phi \|, \]

which is enough to conclude the desired.

Next we consider the case \( r_1 = 4 \). In this case we get the inequality

\[ \| x^4 U(t) \phi \| \lesssim_{a, \theta, T} \| J_y^a \phi \| + \| J_x^{(1+\theta)} \phi \| + \| \langle x \rangle^{4} \phi \| + \| \langle y \rangle^{r_2} \phi \|. \]  

Indeed, to obtain (3.41) we use identity (2.23). We will present the estimate only for the terms \( H_1 \), \( H_6 \) and \( H_{12} \) in (2.23). To deal with the terms \( H_j \), \( j \neq 1, 6, 12 \) it is enough to use Plancherel’s identity, Young’s inequality and Lemma 2.12. Using function \( \chi \) we write

\[ \| J_\xi \| \lesssim \| \langle \xi \rangle^{a-1} \phi \| \]

\[ \lesssim \| \chi \eta^2 |\xi|^{a-1} \phi \| + \| (1 - \chi) \eta^2 |\xi|^{a-1} \phi \| \]

\[ \lesssim \| \chi \eta^2 |\xi|^{a-1} \| \| \hat{\phi} \|_{L_\xi^\infty} + \left\| \frac{1 - \chi}{\xi} \right\|_{L_\xi^\infty} \| \eta^2 |\xi|^{a} \phi \| \]  

\[ \lesssim \| \hat{\phi} \|_{L_{11, r_2}^2} + \| \eta^4 \phi \| + \| \xi \|^{2a} \phi \| \]

\[ \lesssim \| J_y^a \phi \| + \| J_x^{(1+\theta)} \phi \| + \| \langle x \rangle^{4} \phi \| + \| \langle y \rangle^{r_2} \phi \|, \]

where we used the Sobolev’s embedding and assumption \( a > 1/2 \) to conclude that \( \chi \eta^2 |\xi|^{a-1} \in L^2(\mathbb{R}^2) \). Also using Taylor’s formula (3.40), we similarly obtain

\[ \| J_\xi \| \lesssim \| \langle |\xi| \rangle^{-2} \phi \| \]

\[ \lesssim \| \langle |\xi| \rangle^{-2} \hat{\phi} \| + \| \xi \|^{-2} (1 - \chi) \hat{\phi} \| \]

\[ \lesssim \| \chi |\xi|^{a-1} \partial_\xi \hat{\phi}(0, \eta) \| + \| \xi |\xi|^{-2} \chi \int_0^\xi (\xi - \zeta) \partial_\xi \hat{\phi}(\xi, \eta) d\zeta \| + \left\| \frac{1 - \chi}{\xi^2} \right\|_{L_\xi^\infty} \| |\xi| \| \phi \| \]  

\[ \lesssim \| \phi \|_{L_{11, r_2}^2} + \| \phi \|_{L_{11, r_2}^2} + \| J_x^a \phi \|. \]

The term \( H_{12} \) can be estimated as

\[ \| H_{12} \| \lesssim \| \langle |\xi| \rangle^{-1} \chi \partial_\xi \hat{\phi} \| + \| \xi \|^{-1} (1 - \chi) \partial_\xi \hat{\phi} \| \]

\[ \lesssim \| \chi \| \| \partial_\xi \hat{\phi} \|_{L_{11, r_2}^\infty} + \| \xi \| \| \partial^2_\xi \hat{\phi} \|_{L_{11, r_2}^\infty} + \| D_x^a \phi \| \]

\[ \lesssim \| \phi \|_{L_{11, r_2}^2} + \| J_x^a \phi \|. \]

Next we consider the case \( 4 < r_1 < 7/2 + a \). Here we write \( r_1 = 4 + \theta \) with \( \theta < a - 1/2 \). Thus, from (2.23) and Lemma 2.13, we can use the ideas employed above to estimate \( \| x |^{1+\theta} U(t) \phi \| \). Since all estimates demand too many calculation involving Plancherel’s identity, Young’s inequality and Lemma 2.12 we will estimate only the terms \( \| D_x^a (\psi |\xi|^{a-1} \partial_\xi \phi) \| \) and \( \| D_x^a (\psi \chi (\xi |\xi|^{-2} \phi) \| \), which present estimates slightly different and whose counterparts in (2.23) are given by \( H_{12} \) and \( H_6 \), respectively.
Using Lemma 2.13, we have
\[
\|D^\theta_\xi(\psi|^{a-1}\partial_\xi \hat{\phi})\| \lesssim \rho(t) \left(\|D^{a-1}_x(x\phi)\| + \|D^{|a-1|}_y D^a_x(x\phi)\| + \|D^{(1+a)\theta}_x D^{a-1}_x(x\phi)\| \right)
\]
\[+ \|x^\theta \| D^{a-1}_x(x\phi)\| . \tag{3.45}\]

The only term that brings extra difficulties in (3.45) is \(D\). Using function \(\chi\), we split
\[
D \lesssim \|D^\theta_\xi(|\xi|^{a-1}\partial_\xi \phi)\| + \|D^\theta_\xi(|\xi|^{a-1}(1-\chi)\partial_\xi \phi)\| =: D_{1.1} + D_{1.2}.
\]

The estimate for \(D_{1.2}\) is similar to that of \(E_2\) in (3.33). For \(D_{1.1}\) we write
\[
D_{1.1} \lesssim \|D^\theta_\xi(|\xi|^{a-1}\chi(\partial_\xi \hat{\phi}(\xi, \eta) - \partial_\xi \hat{\phi}(0, \eta)))\| + \|D^\theta_\xi(|\xi|^{a-1}\chi \partial_\xi \hat{\phi}(0, \eta))\| =: D^1_{1.1} + D^2_{1.1}.
\]

For \(D^1_{1.1}\), we will use the interpolation inequality \(\|D^\theta_\xi R\| \leq \|R\|^{1-\theta}\|\partial_\xi R\|^\theta\). But, from the mean value theorem and Sobolev’s embedding we infer
\[
\|R\| \lesssim \|\partial^2_\xi \hat{\phi}\|_{L^\infty_{\xi}} \|\xi^a \phi\| \lesssim \langle x \rangle^{\gamma_1} \phi + \langle y \rangle^{\gamma_2} \phi \tag{3.46}
\]
and
\[
\|\partial_\xi R\| \lesssim \|\xi|^{a-1}\chi \partial_\xi \hat{\phi}(\xi, \eta) - \partial_\xi \hat{\phi}(0, \eta)\| + \|\xi|^{a-1}\partial_\xi \chi(\partial_\xi \hat{\phi}(\xi, \eta) - \partial_\xi \hat{\phi}(0, \eta))\|
\]
\[+ \|\xi|^{a-1}\partial^2_\xi \hat{\phi}\|
\]
\[\lesssim \|\xi|^{a-1}\chi \|\|\partial_\xi \hat{\phi}\|_{L^\infty_{\xi}} + \|\partial^2_\xi \hat{\phi}\|_{L^\infty_{\xi}}\| + \|\xi|^{a-1}\partial_\xi \chi\| \|\partial_\xi \hat{\phi}\|_{L^\infty_{\xi}}
\]
\[\lesssim \langle x \rangle^{\gamma_1} \phi + \|\langle y \rangle^{\gamma_2} \phi\| . \tag{3.47}\]

From (3.46) and (3.47), we obtain \(D^1_{1.1} \lesssim \langle x \rangle^{\gamma_1} \phi + \|\langle y \rangle^{\gamma_2} \phi\|.\) Moreover, since by Proposition 2.9 the quantity \(\|D^\theta_\xi(|\xi|^{a-1}\chi)\|\) is finite, we have
\[
D^2_{1.1} \lesssim \|\partial_\xi \hat{\phi}\|_{L^\infty_{\xi}} \|D^\theta_\xi(|\xi|^{a-1}\chi)\| \lesssim \|\partial_\xi \hat{\phi}\|_{L^\infty_{\xi}} \lesssim \langle x \rangle^{\gamma_1} \phi + \|\langle y \rangle^{\gamma_2} \phi\| .
\]

Next we estimate the term \(\|D^\theta_\xi(\psi \text{sgn}(\xi)|\xi|^{a-2} \hat{\phi})\|\). From Lemma 2.13,
\[
\|D^\theta_\xi(\psi \text{sgn}(\xi)|\xi|^{a-2} \hat{\phi})\| \lesssim \rho(t)(\|D^{|a-1|}_y D^{a-2}_x \phi\| + \|D^{|a|}_y D^{a-2}_x \phi\| + \|D^{(1+a)\theta}_x D^{a-2}_x \phi\| + \|x^\theta \| D^{a-2}_x H\phi\|).
\]

Let us estimate the last term by writing
\[
\|x^\theta \| D^{a-2}_x H\phi\| \lesssim \|D^\theta_\xi(\psi \text{sgn}(\xi)|\xi|^{a-2} \hat{\phi})\| + \|D^\theta_\xi(\psi \text{sgn}(\xi)|\xi|^{a-2}(1-\chi) \hat{\phi})\| =: E_1 + E_2.
\]

Estimate for \(E_1\) may be performed by using Taylor’s formula (3.40) and proceeding as above. For \(E_2\), we use interpolation to obtain
\[
E_2 \lesssim \|\xi|^{a-2}(1-\chi) \hat{\phi}\| + \|\partial_\xi \text{sgn}(\xi)|\xi|^{a-2}(1-\chi) \hat{\phi}\|
\]
\[\lesssim \frac{1}{\xi^2} \|\xi|^{a-2} \hat{\phi}\|_{L^\infty_{\xi}} + \|\xi|^{a-2} \hat{\phi}\|_{L^\infty_{\xi}} \|\xi|^{a-2} \hat{\phi}\| + \]
\[+ \|\xi|^{a-2} \partial_\xi \hat{\phi}\|_{L^\infty_{\xi}} \phi \| + \|\xi|^{a-2} \partial_\xi \hat{\phi}\|_{L^\infty_{\xi}} \|\xi|^{a-2} \hat{\phi}\|
\]
\[\lesssim \|\phi\| + \|D^2_\phi \| + \|D^2_x  \phi\| \lesssim \|\phi\| + \|\langle x \rangle^{\gamma_2} \phi\|.
\]

After all estimates we arrive to the inequality
\[
\|x^\theta \| U(t) \phi\| \lesssim \phi + \|J^2_x \| + \|J^2_y \| + \|\langle x \rangle^{4\theta} \phi\| + \|\langle y \rangle^{\gamma_2} \phi\|,
\]
which is enough to our purpose again. This completes the proof in Case a).
**Case b.** \(0 < a \leq 1/2\). Assume first \(a = 1/2\). In this case we must have \(r_1 \in [3, 4)\). The case \(3 < r_1 < 4\) was already treated in Case a). So we may assume \(r_1 = 3\). Here the persistence follows from the inequality

\[
\|x^3 U(t)\phi\| \lesssim_{a, \theta, T} \|J_y^0 \phi\| + \|J_x^{3(1+a)} \phi\| + \|\langle x \rangle^3 \phi\| + \|\langle y \rangle^r \phi\|. \tag{3.48}
\]

To obtain (3.48) we use identity (2.22). Estimates for the terms \(G_j, j \neq 6\), follows as an application of Young’s inequality, Plancherel’s identity and Lemma 2.12. The term \(G_6\) is the only one that has a slightly different estimate. In fact, since \(\hat{\phi}(0, \eta) = 0\), for all \(\eta \in \mathbb{R}\), from the mean value theorem we obtain

\[
\|G_6\| \lesssim \|\xi|^{-1/2} \hat{\phi}\| \lesssim \left\| \frac{\hat{\phi}(\xi, \eta)}{\xi} \right\|_{L^\infty_{x, \xi}} \|\hat{\phi}\|_{L^1_{x, \xi}} \lesssim \|\partial_\xi \hat{\phi}\|_{L^1_{x, \xi}} \|\hat{\phi}\|_{L^2_{x, \xi}}. \tag{3.49}
\]

Since \(\|\hat{\phi}\|_{L^1_{x, \xi}} \lesssim \|\phi\|_{H^s}\), where sigma \(\sigma > 1\) is arbitrary, we can use Sobolev’s embedding and Young’s inequality to obtain

\[
\|G_6\| \lesssim \|\langle x \rangle^r \phi\| + \|\langle y \rangle^r \phi\| + \|\phi\|_{H^s}.
\]

Thus, after all calculations we obtain a similar term as in (3.28) with the additional term \(\int_0^T \|\partial_x u^2\|_{H^s} d\tau\). However, by choosing \(\sigma > 1\) satisfying \(1 + \sigma \leq (1 + a)s\) (this is always possible because \(s > r_1 > 5/2\)) we obtain

\[
\int_0^T \|\partial_x u^2\|_{H^s} d\tau \lesssim \int_0^T \|u\|_{H^s} \|u\|_{H^{1+s}} d\tau \lesssim M^2,
\]

where we used that \(E^s \hookrightarrow H^{1+s}\). Thus we still may apply Gronwall’s lemma to conclude the result.

Assume now \(0 < a < 1/2\). In this case, \(r_1\) must range the interval \((5/2, 4)\). The case \(3 \leq r_1 < 4\) has already been treated above. So, we may assume \(5/2 < a < r_1 < 3\). The proof runs as in Part 2)(Case b). In fact, by setting \(r_1 = 2 + \theta\), with \(1/2 < a < \theta\), all terms in (3.21) can be estimated as above, except the term \(K_1 = \|D_\xi^a (\xi^a \text{sgn}(\xi) \chi \hat{\phi})\|\) in the decomposition of \(B_1\). Here it can be estimated as follows

\[
K_1 = \|D_\xi^a (\xi^a \text{sgn}(\xi) \chi \hat{\phi}(\xi, \eta))\|
\]

\[
\lesssim \|\xi^a \text{sgn}(\xi) \chi \hat{\phi}(\xi, \eta)\| + \|\partial_\xi (\xi^a \text{sgn}(\xi) \chi \hat{\phi}(\xi, \eta))\|
\]

\[
\lesssim \|\phi\| + \|\xi^a \chi \hat{\phi}\| + \|\xi^a \partial_\xi \hat{\phi}\|
\]

\[
\lesssim \|\phi\| + \|\xi^a \chi\| + \|\partial_\xi \hat{\phi}\| + \|\xi^a \partial_\xi \chi\| + \|\xi^a \partial_\xi \hat{\phi}\|
\]

\[
\lesssim \|\phi\| + \|\xi^a \chi\| + \|\partial_\xi \hat{\phi}\| + \|\xi^a \partial_\xi \chi\| + \|\xi^a \partial_\xi \hat{\phi}\|
\]

\[
\lesssim \|\phi\| + \|\xi^a \chi\| + \|\partial_\xi \hat{\phi}\| + \|\xi^a \partial_\xi \chi\| + \|\xi^a \partial_\xi \hat{\phi}\|
\]

\[
\lesssim \langle \delta_\xi, f \rangle = \int f(0, \eta) d\eta, \text{ for all } f \in \mathcal{S}(\mathbb{R}^2).
\]

The proof of Theorem (1.1) is thus completed.

### 4. Unique continuation principles

This section is devoted to establish Theorems 1.2 and 1.3. As we already said, we follow closely the arguments in [3], where the authors proved a similar result for the dispersion generalized BO equation. The main idea is to explore the behavior of the gBO-ZK in the x-direction, which, in some sense, is similar to one presented by the dispersion generalized BO equation.
Claim 4.2. For any $z$ with $\tilde{z}$ in a similar way. From (2.5) and (2.25), we obtain This finishes the proof of Claim 4.1.

Claim 4.1. For all $\phi \in Z_0$, we have $A, B_j \in L^2$, where $j = 2, \ldots, 7$.

Indeed, using Proposition 2.4 with $\Phi = \phi$ and identity (2.21) we obtain

$$
\|A\| = \||\varphi; D_\xi^a\partial_\xi^2(e^{-\eta^2}\tilde{\psi}(\xi, \eta, t)\dot{\phi})\|_{L_\eta^2} \lesssim \||\xi|^a\phi\| + \||\xi|^{2(1+a)}\dot{\phi}\| + \|\eta^2 e^{-\eta^2}\tilde{\psi}(\xi, \eta, t)\dot{\phi}\| + \||\xi|^{1+a}\partial_\xi\phi\| + \|\partial_\xi^2\phi\|
$$

(4.3)

where we also used that $\|\eta^{2k}e^{-\eta^2}\|_{L_\eta^\infty} \lesssim 1, k = 1, 2$. The right-hand side of (4.3) is finite because $\phi \in Z_0$. Here, and in the inequalities to follow, the implicit constant may depend on $t$.

With respect to $B_j$ we only deal, for instance, with $B_2$ and $B_7$. The other terms can be estimated in a similar way. From (2.24) and (2.25), we obtain

$$
\|B_2\| = \||D_\xi^\alpha(\tilde{\psi}\phi\partial_\xi^2\phi)\| \lesssim \|\phi\| + \||x|^\alpha\phi\|
$$

and

$$
\|B_7\| = \||D_\xi^\alpha(\tilde{\psi}\phi\partial_\xi^2\phi)\| \lesssim \|x^2\phi\| + \||x|^{2+a}\phi\| \lesssim \|(x)^{5/2+a}\phi\|
$$

This finishes the proof of Claim 4.1.

Using (2.21) again, the integral part in (4.2) can be write as

$$
\int_0^t \left\{ [\varphi; D_\xi^a\partial_\xi^2(e^{-\eta^2}\tilde{\psi}(\xi, \eta, t-\tau)\dot{\phi}) + \sum_{j=1}^7 D_\xi^a \left( \tilde{\chi}(\xi, \eta) F_j(\xi, \eta, t-\tau, \dot{\phi}) \right) \right\} d\tau
$$

(4.4)

Claim 4.2. For any $t \in [0, T]$, we have $A, B_j \in L^2$, for $j = 1, \ldots, 7$.

In fact, we can proceed as in the proof of Claim 4.1. To estimate $A$ it is enough to follow (4.3), with $z$ instead of $\phi$ to obtain

$$
||\varphi; D_\xi^\alpha\partial_\xi^2(e^{-\eta^2}\tilde{\psi}(\xi, \eta, t-\tau)\dot{\phi})|| \lesssim ||x^{2(1+a)}(wu_x)|| + \|(x)^2u_x\|
$$

(4.4)
From fractional Kato-Ponce’s inequality (see Remark 1.5 in [18]) and Sobolev’s embedding,

\[ \| J_x^{2(1+a)}(uu_x) \| \lesssim \| uu_x \| + \| D_x^{2(1+a)}(uu_x) \| \]
\[ \lesssim \| u \|_{L^\infty} \| u_x \| + \| u_x \|_{L^\infty} \| D_x^{2+2a}u_x \| + \| u_x \|_{L^\infty} \| D_x^{2(1+a)}u \| \]
\[ \lesssim \| J_x^{2(1+a)}u \|^2 \]
\[ \lesssim \| u \|^2_{E^{s}}. \]

Also, from Holder’s inequality and Sobolev’s embedding,

\[ \| \langle x \rangle^{2} uu_x \| \lesssim \| \langle x \rangle^{2} u \| \| u_x \|_{L^\infty} \lesssim \| \langle x \rangle^{2} u \| \| u \|_{E^{s}}. \] \tag{4.6}

Thus, from (4.4)-(4.6), we obtain

\[ \| A \| \lesssim \int_0^t \| u(\tau) \|_{Z_{2,0}^{s}}^2 \, d\tau \lesssim_T \sup_{[0,T]} \| u \|_{Z_{2,0}^{s}}^2. \] \tag{4.7}

The right-hand side of (4.7) is finite taking into account that \( u \in C([0,T]; Z_{r_s,\epsilon_s}^{s}). \)

Concerning the terms \( B_j \)'s, we only deal with \( B_7 \). The other terms can be estimated in an easier way. First note that from Sobolev’s embedding and Lemma 2.12,

\[ \| \langle x \rangle^{\alpha} u \|_{L^\infty} \lesssim \| J_x^{1+a/2}(\langle x \rangle^{\alpha} u) \| + \| J_x^{\alpha}(\langle x \rangle^{\alpha} u) \| \]
\[ \lesssim \| J_x^{1+a/2}u \| + \| J_x^{2}u \| + \| \langle x \rangle^{2\alpha} u \| \]
\[ \lesssim \| u \|_{Z_{2,0}^{s}}. \] \tag{4.8}

From (2.25) and (4.8) we get

\[ \| B_7 \| \lesssim \int_0^t \left( \| D_\xi^\alpha(\hat{\chi}\psi(\xi, \eta, t - \tau)\hat{\partial}_\xi \hat{u}^2) \| + \| D_\xi^{\alpha}(\hat{\chi}\psi(\xi, \eta, t - \tau)\hat{\xi}\hat{\partial}_\xi \hat{u}^2) \| \right) \, d\tau \]
\[ \lesssim \int_0^t \left( \| xu^2 \| + \| \langle x \rangle^{\alpha} xu^2 \| + \| x^2 u^2 \| + \| \langle x \rangle^{2\alpha} u^2 \| \right) \, d\tau \]
\[ \lesssim \int_0^t \| \langle x \rangle^{2+a} u^2 \| \, d\tau \]
\[ \lesssim \int_0^t \| \langle x \rangle^{2} u \| \| \langle x \rangle^{a} u \|_{L^\infty} \, d\tau \]
\[ \lesssim \int_0^t \| u \|_{Z_{2,0}^{s}}^2 \, d\tau. \]

As in (4.7) we obtain the desired. This finishes the proof of Claim 4.2.

Note that in Claim 4.1 we do not estimate the term \( \tilde{B}_1 \). Actually, this term allow us to obtain the result. First note we can write

\[ \hat{\chi} F_1 = tc_a |\xi|^a \text{sgn}(\xi) \psi(\xi, \eta, t) \hat{\chi}(\hat{\phi}(\xi, \eta) - \hat{\phi}(0, \eta)) + tc_a |\xi|^a \text{sgn}(\xi) \psi(\xi, \eta, t) \hat{\chi}(0, \eta) \]
\[ = F_{1,1} + F_{1,2}, \] \tag{4.9}

with \( c_a = -i(1+a)(2+a) \). We claim that \( \| D_\xi^\alpha(F_{1,1}) \| \) is finite. Note that interpolation (in the \( \xi \)-variable) and Young’s inequality give

\[ \| D_\xi^\alpha(F_{1,1}) \| = \| D_\xi^\alpha(F_{1,1}) \|_{L^\infty_\xi} \lesssim \| D_\xi^\alpha F_{1,1} \|_{L^2_\xi}^{1-a} \| \partial_\xi F_{1,1} \|_{L^2_\xi}^a \lesssim \| F_{1,1} \| + \| \partial_\xi F_{1,1} \|. \] \tag{4.10}
Thus, it suffices to show that the right-hand side of the last inequality is finite. It is easy to check that $F_{1,1} \in L^2$. In addition, by using the mean value theorem and Sobolev’s embedding, we deduce
\[
\|\partial_{\xi} F_{1,1}\| \lesssim \left\|\xi^a \hat{\chi} \left(\hat{\phi}(\xi, \eta) - \hat{\phi}(0, \eta)\right)\right\| + \left\|\xi^a \partial_{\xi} \hat{\chi} \left(\hat{\phi}(\xi, \eta) - \hat{\phi}(0, \eta)\right)\right\| + \left\|\xi^a \hat{\chi} \partial_{\eta} \hat{\phi}\right\| \\
+ \left\|\xi^a \partial_{\xi} \hat{\chi} \left(\hat{\phi}(\xi, \eta) - \hat{\phi}(0, \eta)\right)\right\| \\
\lesssim \left\|\xi^a \hat{\chi}\right\|_{L^\infty_{\xi,\eta}} + \left\|\xi^a \partial_{\xi} \hat{\chi}\right\|_{L^1_{\xi,\eta}} + \left\|\xi^a \hat{\chi}\right\|_{L^1_{\xi,\eta}} + \left\|\hat{\chi}\right\|_{L^\infty_{\xi,\eta}} \left\|\partial_{\xi} \hat{\phi}\right\|_{L^1_{\xi,\eta}} \\
\lesssim \left\|\partial_{\xi} \hat{\phi}\right\|_{L^1_{\xi,\eta}} + \left\|\hat{\phi}\right\|_{L^1_{\xi,\eta}} \\
\lesssim \left\|\hat{\phi}\right\|_{L^1_{\xi,\eta}} + \left\|\chi(t)\right\|_{L^1_{\xi,\eta}}.
\]

Next, we write
\[
F_{1,2} = tc_a(\chi(t, \eta, \tau) - 1)\xi^a |\chi|\hat{\phi}(0, \eta)\hat{\chi} + tc_a |\chi|\hat{\phi}(0, \eta)\hat{\chi} \\
= F_{1,2}^1 + F_{1,2}^2.
\]

As above it is easy to check that $\|F_{1,2}\|$ is finite. Therefore, putting $t = t_2$, from Claims 4.1 and 4.2 and our assumptions it must be the case that
\[
D^2_{\xi}(F_{1,2}^2) = D^2_{\xi}(tc_a \hat{\phi}(0, \eta)e^{-\eta^2}(|\chi|\hat{\phi}(0, \eta)\phi(\xi)) \in L^2(\mathbb{R}^2).
\]

Fubini’s theorem and Theorem 2.5 imply that
\[
tc_a c_{\xi} e^{-\eta^2} \hat{\phi}(0, \eta) D^2_{\xi}(|\chi|\hat{\phi}(0, \eta)\phi(\xi)) \in L^2_{\xi}(\mathbb{R}), \quad \text{a.e. } \eta \in \mathbb{R}.
\]

Taking into account that $\alpha = a + 1/2$, an application of Proposition 2.7 yields
\[
\hat{\phi}(0, \eta) = 0, \quad \text{a.e. } \eta \in \mathbb{R}.
\]

In view of (4.2) the proof of the theorem is completed in this case.

The case $a \in [1/2, 1)$ follows by writing $5/2 + a = 3 + \alpha$, where $a = a - 1/2$ and applying similar ideas as above. In this case, instead of (2.21) and Proposition 2.7 identity (2.22) and Proposition 2.8 must be used. This completes the proof of the theorem.

**Proof of Theorem 1.3** First we deal with the case $a \in (1/2, 1)$. Without loss of generality we assume $t_1 = 0 < t_2 < t_3$. By setting $\alpha = a - 1/2$ it is seen that $4 + \alpha = 7/2 + a$ with $\alpha \in (0, 1/2)$. In addition, for any $r_1 < 7/2 + a$ it follows that $u \in C([0, T]; Z_{r_1, r_2}^a)$.

Now multiplying (4.1) by $|x|^{7/2 + a}$ and using Fourier transform we may write
\[
D^2_{\xi} \partial^3_{\xi}(u(t)) = D^2_{\xi} \partial^3_{\xi}(\psi(\xi, \eta, t)\phi) - \int_0^t D^2_{\xi} \partial^3_{\xi}(\psi(\xi, \eta, t - \tau)z(\tau))d\tau,
\]

where, as before, $z = \frac{1}{2} \partial_x u^2$. If $\chi$ is as in (2.24), then in view of (2.23) we write the linear part of (4.13) as
\[
\chi D^2_{\xi} \partial^3_{\xi}(\psi(\xi, \eta, t)\phi) = [\phi(\xi); D^2_{\xi} \partial^3_{\xi}(\psi(\xi, \eta)\phi(\eta)) + D^2_{\xi}(\phi(\xi) \partial^3_{\xi}(\psi(\xi, \eta)\phi(\eta))\phi)] \\
= C + D_1 + \cdots + D_{25},
\]

where $D_j := D^2_{\xi}(\chi(\xi, \eta)H_j(\xi, \eta, t))$.

**Claim 4.3.** For all $t \in [0, T]$ we have $C, D_j \in L^2$, where $j \in \{1, \ldots, 25\}$ and $j \neq 6, 12$. 


To prove the claim, in view of (2.23) and Proposition 2.4 we infer

\[
\|C\| = \|\|\|\| (\varphi; \eta, \psi, \eta, \varphi)\|_{L^2} \|
\leq \|\eta^2 \varphi(\eta)\|_{\mathcal{L}^2} \|\xi^{a-1} \hat{\varphi}\| + \|\xi^{2a} \hat{\varphi}\| + \|\eta^2 \varphi(\eta)\|_{\mathcal{L}^2} \|\xi^{1+2a} \hat{\varphi}\| + \|\eta^2 \varphi(\eta)\|_{\mathcal{L}^2} \|\xi^{1+3a} \hat{\varphi}\| + \|\eta^2 \varphi(\eta)\|_{\mathcal{L}^2} \|\xi^{1+4a} \hat{\varphi}\| + \|\eta^2 \varphi(\eta)\|_{\mathcal{L}^2} \|\xi^{1+5a} \hat{\varphi}\| + \|\eta^2 \varphi(\eta)\|_{\mathcal{L}^2} \|\xi^{1+6a} \hat{\varphi}\| + \|\eta^2 \varphi(\eta)\|_{\mathcal{L}^2} \|\xi^{1+7a} \hat{\varphi}\|
\]

\[
\lesssim \|\xi^{a-1} \hat{\varphi}\| + \|\xi^{a-2} \hat{\varphi}\| + \|\xi^{1+4a} \hat{\varphi}\| + \|\xi^{1+5a} \hat{\varphi}\| + \|\xi^{1+6a} \hat{\varphi}\| + \|\xi^{1+7a} \hat{\varphi}\|
\]

(4.15)

where we also used \(\|\eta^2 \varphi(\eta)\|_{\mathcal{L}^2} \lesssim 1\), for \(k = 1, 2, 3, 4\), and Lemma 2.12.

To deal with terms \(I, J\) and \(L\) we may proceed as in (3.12), (3.13) and (3.14), respectively, to obtain

\[
I, J, L \lesssim \|\phi\|_{L^2_{1, r_2}} + \|J^2_\alpha \phi\|.
\]

(4.16)

Thus, by (4.15) and (4.16),

\[
\|C\| \lesssim \|J^{4+\alpha}_x \phi\| + \|\langle x \rangle^4 \phi\| + \|\langle y \rangle^{2+\alpha} \phi\|.
\]

(4.17)

Since \(\phi \in Z^{1/2+\alpha, r_2}_1\), we see that right-hand side of (4.17) is finite. Next we deal with terms \(D_j\).

First, note that Lemma 2.14 implies

\[
\|D_{25}\| = \|D^2_{\xi} (\chi \psi \partial^2_\xi \hat{\phi})\| \lesssim \|x^2 \phi\| + \|\langle x \rangle^{4+\alpha} \phi\| \lesssim \|\langle x \rangle^{7/2+\alpha} \phi\|.
\]

(4.18)

For the terms \(D_j\), \(j \neq 1, 5, 14, 19\), it is sufficient to follow an argument as in (4.14). For \(D_5\), using (2.27) (with \(\sigma_4 = r_2\)) we obtain

\[
D_5 \lesssim \|D^2_{\xi} (\chi \psi \text{sgn}(\xi) \eta^4 |\xi|^a \hat{\phi})\| \lesssim \|J^{2+\alpha}_x \phi\| + \|\langle x \rangle^2 \phi\| + \|\langle y \rangle^{2+\alpha} \phi\|.
\]

To estimate \(D_{14}\) and \(D_{22}\) we use Lemma 2.14, Plancherel’s identity and Lemma 2.12. In fact, by (2.27) (with \(\sigma_4 = 2\)),

\[
D_{14} \lesssim \|D^2_{\xi} (\chi \psi \eta^2 \text{sgn}(\xi) |\xi|^a \partial_\xi \hat{\phi})\|
\]

\[
\lesssim \|J^{2+\alpha}_x (\chi \psi \eta^2 \phi)\| + \|\langle x \rangle^2 \phi\| + \|\langle y \rangle^2 \phi\|
\]

\[
\lesssim \|J_{\xi}(\chi \psi \eta^2 \phi)\| + \|\langle x \rangle^3 \phi\| + \|\langle y \rangle^3 \phi\|
\]

\[
\lesssim \|J^2_\xi \phi\| + \|\langle x \rangle^{4+\alpha} \phi\| + \|\langle x \rangle^3 \phi\| + \|\langle y \rangle^3 \phi\|
\]

\[
\lesssim \|J^4_\xi \phi\| + \|\langle x \rangle^3 \phi\| + \|\langle y \rangle^3 \phi\|.
\]

Also, by (2.27) (with \(\sigma_4 = 3/2\)),

\[
D_{19} \lesssim \|D^2_{\xi} (\chi \psi \text{sgn}(\xi) |\xi|^a \partial^2_\xi \hat{\phi})\|
\]

\[
\lesssim \|J^{2+\alpha}_x (\chi \psi \eta^2 \phi)\| + \|\langle x \rangle^2 \phi\| + \|\langle y \rangle^{3/2} \phi\|
\]

\[
\lesssim \|J^2_\xi \phi\| + \|\langle x \rangle^{4+\alpha} \phi\| + \|\langle x \rangle^4 \phi\| + \|\langle y \rangle^3 \phi\|
\]

\[
\lesssim \|J^4_\xi \phi\| + \|\langle x \rangle^4 \phi\| + \|\langle y \rangle^3 \phi\|.
\]
Finally, for $D_1$, our assumption and Theorem 1.2 imply that $\hat{\phi}(0, \eta) = 0$. So, using (3.40) we obtain

$$D_1 = c_1 D_\xi^a (\eta^2 \text{sgn}(\xi) |\xi|^{a-1} \hat{\phi} \psi)$$

$$= c_1 D_\xi^a \left( \eta^2 |\xi|^a \partial_\xi \hat{\phi}(0, \eta) \chi \psi \right) + c_1 D_\xi^a \left( \eta^2 |\xi|^a \xi^{-1} \chi \psi \int_0^\xi (\xi - \zeta) \partial_\xi^2 \hat{\phi}(\zeta, \eta) d\zeta \right)$$

(4.19)

$$=: D_{1,1} + D_{1,2}.$$

where $c_1 = 4a(2 + a)(1 + a)t^2$. Now we write

$$D_{1,1} = c_1 D_\xi^a \left( \eta^2 |\xi|^a \partial_\xi \hat{\phi}(0, \eta) \chi(\psi - 1) \right) + c_1 D_\xi^a (\eta^2 |\xi|^a \partial_\xi \hat{\phi}(0, \eta) \chi)$$

$$=: D_{1,1}^1 + D_{1,1}^2.$$

Recalling the standard inequality $|e^{ir} - 1| \leq |r|$, for any $r \in \mathbb{R}$, we see that

$$|\psi - 1| \leq |\xi| (\eta^2 - |\xi|^{1+a}).$$

(4.20)

Thus using (4.20) and Sobolev’s embedding

$$\|L\| \lesssim \|\eta^2 |\xi|^a \partial_\xi \hat{\phi}(0, \eta) \chi(\psi - 1)\| \lesssim \|\eta^2 |\xi|^a \partial_\xi \hat{\phi}(0, \eta) \chi(\psi - 1)\| \lesssim \|\partial_\xi \hat{\phi}\|_{L_\xi^\infty} \lesssim \|\phi\|_{L_{1, r_2}},$$

and

$$\|\partial_\xi L\| \lesssim \|\eta^2 |\xi|^a \text{sgn}(\xi) \partial_\xi \hat{\phi}(0, \eta) \chi(\psi - 1)\| \lesssim \|\eta^2 |\xi|^a \partial_\xi \hat{\phi}(\chi(\psi - 1))\|$$

$$\lesssim \|\partial_\xi \hat{\phi}\|_{L_\xi^\infty} \lesssim \|\phi\|_{L_{1, r_2}}.$$

Consequently, by using interpolation (see (4.10)) we deduce that $D_{1,1}^1 \in L^2$. On the other hand, using (2.10),

$$\|D_{1,1}^2\| \lesssim \|\partial_\xi \hat{\phi}\|_{L_\xi^\infty} \|\eta^2 \varphi(\eta)\|_{L_\eta^\infty} \|D_\xi^a (\text{sgn}(\xi) \varphi(\xi))\|_{L_\xi^2} \lesssim \|\partial_\xi \hat{\phi}\|_{L_\xi^\infty} \|\phi\|_{L_{1, r_2}}.$$

This shows that $D_{1,1} \in L^2$. To see that $D_{1,2}$ also belongs to $L^2$, we note that

$$\|Q\| \leq \|\partial_\xi^2 \hat{\phi}\|_{L_\xi^\infty} \|\eta^2 |\xi|^a \xi^{-1} \chi \psi \int_0^\xi (\xi - \zeta) d\zeta\|$$

$$\leq \|\partial_\xi^2 \hat{\phi}\|_{L_\xi^\infty} \|\eta^2 |\xi|^a \xi\|$$

(4.21)

$$\lesssim \|\partial_\xi^2 \hat{\phi}\|_{L_\xi^\infty} \|\eta^2 |\xi|^a \xi\|$$

$$\lesssim \|\phi\|_{L_{1, r_2}},$$

and

$$\|\partial_\xi Q\| \leq \|\partial_\xi \chi \psi |\xi|^{a-1} \int_0^\xi (\xi - \zeta) \partial_\xi^2 \hat{\phi}(\zeta, \eta) d\zeta\| + \|\chi \psi \text{sgn}(\xi) |\xi|^{a-2} \int_0^\xi (\xi - \zeta) \partial_\xi^2 \hat{\phi}(\zeta, \eta) d\zeta\| +$$

$$+ \|\chi \psi |\xi|^{a-1} \int_0^\xi \partial_\xi^2 \hat{\phi}(\zeta, \eta) d\zeta\| + \|\chi \psi \text{sgn}(\xi) |\xi|^{a-2} \int_0^\xi (\xi - \zeta) \partial_\xi^2 \hat{\phi}(\zeta, \eta) d\zeta\|$$

$$\leq \left( \|\partial_\xi \chi |\xi|^{a-2} \xi^2\| + \|\chi \text{sgn}(\xi) |\xi|^{a-2} \xi^2\| + \|\chi |\xi|^{a-1} \xi\| + \|\chi \psi \text{sgn}(\xi) |\xi|^{a-1} \xi\| + \|\chi \psi |\xi|^{a-1} \xi\| \right) \|\partial_\xi^2 \hat{\phi}\|_{L_{1, r_2}}$$

$$\lesssim \|\partial_\xi^2 \hat{\phi}\|_{L_\xi^\infty} \|\phi\|_{L_{1, r_2}}.$$
Interpolation then gives $D_{1,2} \in L^2$. Therefore $D_1 \in L^2$ and the proof of Claim 4.3 is completed.

Next we analyze the integral part of (4.13). By using (2.23) we see that it can be written as

$$
\int_0^t \left\{ \left[ x; D_\xi^1 \partial_\xi^1 (\psi(\xi, \eta, t - \tau)) + D_\xi^2 (\chi(\partial_\xi^2 (\psi(\xi, \eta, t - \tau))) \right] \right\} d\tau
$$

$$
= \int_0^t [x; D_\xi^1 \partial_\xi^1 (\psi(\xi, \eta, t - \tau))] d\tau + \sum_{j=1}^{28} \int_0^t D_\xi^2 (\chi H_j (\xi, \eta, t - \tau)) d\tau
$$

$$
= C + D_1 + \cdots + D_{25}.
$$

Claim 4.4. For any $t \in [0, T]$, we have $C, D_j \in L^2$, for $j \in \{1, ..., 25\}$ and $j \neq 6, 12$.

The idea to prove the claim is similar to that in Claim 4.2. In fact, as in (4.15), with $z = \frac{1}{2} \partial_x u^2$ instead of $\phi$,

$$
\| \varphi; D_\xi^1 \partial_\xi^1 (\psi(\xi, \eta, t - \tau)) \| \lesssim \| J_x^{4(1 + \sigma)} (u u_x) \| + \| \langle x \rangle^4 u_x \| + \| \langle y \rangle^{r2} u_x \|.
$$

(4.24)

By using Remark 1.5 in [18] again, we deduce

$$
\| J_x^{4(1 + \sigma)} (u u_x) \| \leq \| u \| \| L_{xy}^\infty \| u_x \| + \| u_x \| \| D_x^{3 + 4\sigma} u_x \| + \| u_x \| \| L_{xy}^\infty \| D_x^{4(1 + \sigma)} u \|
$$

$$
\lesssim \| J_x^{4(1 + \sigma)} u \|^2
$$

(4.25)

From Holder’s inequality and Sobolev’s embedding

$$
\| \langle x \rangle^4 u_x \| \leq \| \langle x \rangle^4 u \| \| u_x \| \| L_{xy}^\infty \| \leq \| \langle x \rangle^4 u \| \| u \| \| E^r \|.
$$

(4.26)

and

$$
\| \langle y \rangle^{r2} u_x \| \leq \| \langle y \rangle^{r2} u \| \| u_x \| \| L_{xy}^\infty \| \leq \| \langle y \rangle^{r2} u \| \| u \| \| E^r \|.
$$

(4.27)

Then by (4.24) - (4.27)

$$
\| C \| \lesssim \int_0^t \| u(\tau) \|_2 \| E^r \| \sup_{[0, T]} \| u \|_2.
$$

With respect to $D_j$’s we will only estimate $D_1$ and $D_{25}$. The other terms can be treated as in Claim 4.3. In view of (2.27) (with $\sigma_4 = 2$),

$$
\| D_\xi^0 (\chi \psi(\xi, \eta, t - \tau) \sgn(\xi) \eta^2 |\xi|^{\sigma-1} \xi u^2) \| \lesssim \| D_\xi^0 (\chi \psi(\xi, \eta, t - \tau) \eta^2 |\xi|^{\sigma} \xi u^2) \|
$$

$$
\lesssim \| J_x^{2(1 + \sigma)} u^2 \| + \| \langle x \rangle^2 u^2 \| + \| \langle y \rangle^2 u^2 \|
$$

$$
\lesssim (\| J_x^{2(1 + \sigma)} u \| + \| \langle x \rangle^2 u \| + \| \langle y \rangle^2 u \|) \| u \| \| L_{xy}^\infty \|
$$

$$
\lesssim \| u \|_2 \| E^r \|.
$$

where we also used the product estimate $\| J_x^\alpha (f g) \| \lesssim \| f \|_\infty \| J_x^\alpha g \| + \| g \|_\infty \| J_x^\alpha f \|$, $\sigma > 0$ (see, for instance Lemma X4 in [20] or Proposition 1.1 (page 105) in [30]).

Hence,

$$
\| D_1 \| \lesssim \sup_{[0, T]} \| u \|_2 \| E^r \|.
$$
Also, from (2.25) and (4.8),

\[ \|D_\xi^\alpha (\chi \psi \partial_\xi^4 \hat{\zeta})\| \leq \|D_\xi^\alpha (\chi \psi \partial_\xi^2 \hat{\zeta}^2)\| + \|D_\xi^\alpha (\chi \psi \partial_\xi^4 \hat{\zeta}^2)\| \]

\[ \leq \|x^2 u\| + \|x^2 u\| + \|x^3 u\| + \|x^3 u\| + \|x^4 u\| + \|x^4 u\| + \|x^4 u\| \]

\[ \leq \|\langle x\rangle^4 u\| \]

\[ \leq \|\langle x\rangle u\| \]

\[ \leq \|u\|_{L^2_{4,0}}, \]

implying that

\[ \|D_{25}\| \leq \sup_{[0,T]} \|u\|_{L^2_{4,0}}. \]

This finishes the proof of Claim 4.4.

Next we will deal with terms \(D_6, D_{12}, D_6\) and \(D_{12}\). First, for \(c_6 = -ia(a^2 - 1)(a + 2)\), using (3.40) we write

\[ D_6 = c_6 tD_\xi^\alpha \left( \text{sgn}(\xi)|\xi|^{a-2} \hat{\phi}(\chi \psi) \right) \]

\[ = c_6 \int_0^\xi \left( |\xi|^{-1} \chi \psi \partial_\xi \hat{\phi}(0, \eta) \right) + c_6 \int_0^\xi \left( |\xi|^{-2} \chi \psi \partial_\xi \hat{\phi}(\xi, \eta) d\eta \right) \]

\[ =: D_{6,1} + D_{6,2}, \]

and decompose

\[ D_{6,1} = t c_6 D_\xi^\alpha \left( |\xi|^{-1} \partial_\xi \hat{\phi}(0, \eta) \chi(\psi - 1) \right) + t c_6 D_\xi^\alpha \left( |\xi|^{-1} \partial_\xi \hat{\phi}(0, \eta) \chi \right) \]

\[ =: D_{6,1}^1 + D_{6,2}^2. \]

Now, using (4.20) and Sobolev's embedding we obtain

\[ \|S\| \leq \|t^2 |||\alpha| \chi(\eta^2 - |\alpha|^{1+a})||_{L^\infty_{\xi\eta}} \|\partial_\xi \hat{\phi}\| \leq \|x\|, \]

and

\[ \|\partial_\xi S\| \leq \||\alpha|^{-2} \partial_\xi \hat{\phi}(0, \eta) \chi(\psi - 1)\| + \|\alpha|^{-1} \partial_\xi \hat{\phi}(0, \eta) \partial_\xi \chi(\psi - 1)\| + \|\alpha|^{-1} \partial_\xi \hat{\phi}(0, \eta) \partial_\xi \psi\| \]

\[ \leq \left( |||\alpha|^{-1} \chi(\eta^2 - |\alpha|^{1+a})|| + ||\alpha|^{a-2} \partial_\xi \hat{\phi}(0, \eta) \partial_\xi \chi|| \right) \|\partial_\xi \hat{\phi}\| \]

\[ \leq \|\partial_\xi \hat{\phi}\| \]

\[ \leq \|\hat{\phi}\| \[L^\infty_{\xi\eta} \)

\[ \leq \|\hat{\phi}\| \[L^2_{1,2} \]

Hence interpolation gives that \(D_{6,1}^1 \in L^2\). By using similar arguments we obtain

\[ \|R\| \leq \|\alpha|^{-2} \psi \chi \int_0^{\xi} \text{sgn}(\xi) \partial_\xi^2 \hat{\phi}(\xi, \eta) d\eta \| \leq \|\partial_\xi^2 \hat{\phi}\| \[L^\infty_{\xi\eta} \]

\[ \|\alpha|^{-2} \psi \chi \| \leq \|\hat{\phi}\| \[L^2_{1,2} \], \]

(4.30)
and

\[ \| \partial_k R \| \leq \left\| \psi |^{a-3} \chi \int_0^\xi (\xi - \zeta) \partial_\zeta^2 \hat{\phi}(\zeta, \eta) d\zeta \right\| + \left\| |^{a-2} \chi \int_0^\xi \partial_\zeta^2 \hat{\phi}(\zeta, \eta) d\zeta \right\| + \\
+ \left\| |^{a-2} \partial_\zeta \psi \int_0^\xi (\xi - \zeta) \partial_\zeta^2 \hat{\phi}(\zeta, \eta) d\zeta \right\| + \left\| |^{a-2} \partial_\zeta \chi \psi \int_0^\xi (\xi - \zeta) \partial_\zeta^2 \hat{\phi}(\zeta, \eta) d\zeta \right\| \]

\[ \leq (|^{a-3} \chi^2 \| + |^{a} \chi \| + |^{a} \chi \|) \| \partial_\zeta^2 \hat{\phi} \|_{L^\infty_\xi} \]

\[ \lesssim \| \partial_\zeta^2 \hat{\phi} \|_{L^\infty_\xi} \]

\[ \lesssim \| \phi \|_{L^2_{1, r_2}}. \]

from which we also obtain \( D_{6, 2} \in L^2. \)

For \( D_{12} \), we first note that

\[ D_{12} = tc_{12} D^\xi_\zeta (|^{a-1} \partial_\zeta \hat{\phi} \chi) \]

\[ = tc_{12} D^\zeta_\chi (|^{a-1} \partial_\zeta \hat{\phi} (\psi - 1)) + tc_{12} D^\chi_\xi (|^{a-1} \partial_\zeta \hat{\phi} (\psi - 1)) \]

\[ =: D^1_{12} + D^2_{12}, \]

where \( c_{12} = -4ia(2 + a)(1 + a) \). But using (4.20)

\[ \| W \| \lesssim \| |^{a-1} \partial_\zeta \hat{\phi} \chi (\eta^2 - |^{1+a}) \| \lesssim \| |^{a} \chi (\eta^2 - |^{1+a}) \|_{L^\infty_\xi} \| \partial_\zeta \hat{\phi} \| \lesssim \| x \phi \| \]

and

\[ \| \partial_\zeta W \| \lesssim \| |^{a-1} \partial_\zeta \hat{\phi} (\eta^2 - |^{1+a}) \| + \| |^{a} \partial_\zeta^2 \hat{\phi} (\eta^2 - |^{1+a}) \| + \\
+ \| |^{a-1} \partial_\zeta \hat{\phi} (\psi - 1) \partial_\zeta \chi \|
\]

\[ \lesssim \| \partial_\zeta \hat{\phi} \|_{L^\infty_\xi} + \| x^2 \phi \| \]

\[ \lesssim \phi \|_{L^2_{1, r_2}}. \]

where we used that \( a \in (1/2, 1) \) to see that \( |^{a-1} \in L^2_\xi \). Hence \( D^1_{12} \in L^2 \).

We may also write

\[ D^2_{12} = tc_{12} \left( D^\zeta_\chi \left( |^{a-1} \partial_\zeta \hat{\phi} (\xi, \eta) - \partial_\zeta \hat{\phi} (0, \eta) \chi \right) + D^\chi_\xi \left( |^{a-1} \partial_\zeta \hat{\phi} (0, \eta) \chi \right) \right) \]

\[ =: D^1_{12} + D^2_{12}. \]

Then, following the arguments above,

\[ \| U \| \leq \left\| |^{a} \partial_\zeta \hat{\phi} (\xi, \eta) - \partial_\zeta \hat{\phi} (0, \eta) \| \right\| \lesssim \| \partial_\zeta^2 \hat{\phi} \|_{L^\infty_\xi} \| \xi \| \| \xi \| \lesssim \| \partial_\zeta^2 \hat{\phi} \| \| L^\infty_\xi \| \lesssim \| \phi \|_{L^2_{1, r_2}}. \]

and

\[ \| \partial_\zeta U \| \leq \left\| |^{a-1} \partial_\zeta \hat{\phi} (\xi, \eta) - \partial_\zeta \hat{\phi} (0, \eta) \| \right\| + \left\| |^{a-1} \partial_\zeta^2 \hat{\phi} (\xi, \eta) \| \right\| + \\
+ \left\| |^{a-1} \partial_\zeta \hat{\phi} (\xi, \eta) - \partial_\zeta \hat{\phi} (0, \eta) \| \| \partial_\zeta \chi \| \right\|
\]

\[ \leq \left( \left( ||^{a-1} \chi \| + ||^{a} \chi \| \right) \| \partial_\zeta^2 \hat{\phi} \| \| L^\infty_\xi \| \right. \]

\[ \lesssim \| \partial_\zeta \hat{\phi} \|_{L^\infty_\xi}, \]

\[ \lesssim \phi \|_{L^2_{1, r_2}}. \]

Thus from (4.33) and interpolation, it follows that \( D^2_{12} \in L^2 \).
By combining (4.37) and (4.38) we can write
\[ D_6 = D_{6,1} + D_{6,2} + D_{6,3} \quad \text{and} \quad D_{12} = D_{12,1} + D_{12,2} + D_{12,3}. \]

Also, by using the above arguments, with \( z \) instead of \( \phi \) it is not difficult to conclude that
\[ D_{6,1}, D_{6,2}, D_{12,1}, D_{12,2} \in L^2. \]

Hence, putting \( t = t_2 \) and setting \( \hat{D} = D_{6,1}^2 - D_{6,1}^2 - D_{12,2}^2 \), from (4.13), (4.14), (4.23), Claims 4.3 and 4.4, and gathering the information above, we obtain that
\[ D_2 D_\alpha \hat{u}(\cdot, t_2) \in L^2(\mathbb{R}^2) \]
if and only if
\[
\hat{D} = c_6 \left( t_2 D_\xi^2 \left( |\xi|^{a_1} \chi \partial_\xi \phi(0, \eta) - |\xi|^{a_1} \chi \int_0^{t_2} (t_2 - \tau) \partial_\xi \zeta(0, \eta, \tau) d\tau \right) \right)
+ c_{12} \left( t_2 D_\xi^2 \left( |\xi|^{a_1} \chi \partial_\xi \phi(0, \eta) - |\xi|^{a_1} \chi \int_0^{t_2} (t_2 - \tau) \partial_\xi \zeta(0, \eta, \tau) d\tau \right) \right)
= (c_6 + c_{12}) D_\xi^2 \left( |\xi|^{a_1} \chi \left( t_2 \partial_\xi \phi(0, \eta) - \int_0^{t_2} (t_2 - \tau) \partial_\xi \zeta(0, \eta, \tau) d\tau \right) \right) \in L^2(\mathbb{R}^2). \]

Now by using the definition of the Fourier transform and integration by parts we deduce
\[ \partial_\xi \zeta(0, \eta, \tau) = \frac{i}{2} \int e^{-i\eta y} u^2(x, y, \tau) dx dy. \] (4.37)

Also, from (1.1), it is easily seen that
\[ \frac{d}{d\tau} \int x e^{-i\eta y} u(x, y, \tau) dx dy = \frac{1}{2} \int x e^{-i\eta y} u^2(x, y, \tau) dx dy, \quad \eta \in \mathbb{R}. \] (4.38)

By combining (4.37) and (4.38)
\[ \partial_\xi \hat{\phi}(0, \eta) = -i \int x e^{-i\eta y} \phi(x, y) dx dy, \quad \text{for all} \quad \eta \in \mathbb{R}. \] (4.40)

Then, using (4.39), (4.40) and integrating by parts
\[
t_2 \partial_\xi \hat{\phi}(0, \eta) - \int_0^{t_2} (t_2 - \tau) \partial_\xi \hat{\phi}(0, \eta, \tau) d\tau = t_2 \partial_\xi \hat{\phi}(0, \eta) - i \int_0^{t_2} (t_2 - \tau) \frac{d}{d\tau} \int x e^{-i\eta y} u(x, y, \tau) dx dy d\tau
- i \int_0^{t_2} \int x e^{-i\eta y} u(x, y, \tau) dx dy d\tau.
\]

By replacing the last identity in (4.36) we obtain
\[
D_\xi^2 \left( |\xi|^{a_1} \chi \right) \int_0^{t_2} \int x e^{-i\eta y} u(x, y, \tau) dx dy d\tau \in L^2(\mathbb{R}^2).
\]

Therefore from Fubini’s theorem and (2.28) (recall that \( a - 1 = \alpha - 1 \))
\[
D_\xi^\alpha \left( |\xi|^{a_1/2} \phi \right) \int_0^{t_2} \int x e^{-i\eta y} u(x, y, \tau) dx dy d\tau \in L_2^\alpha, \quad \text{a.e.} \quad \eta \in \mathbb{R}.
\]

Thus from Proposition 2.28 we obtain
\[ \int_0^{t_2} \int x e^{-i\eta y} u(x, y, \tau) dx dy d\tau = 0, \quad \text{a.e.} \quad \eta \in \mathbb{R}. \] (4.41)
This last identity allows us to obtain \( \tau_1 \in (0, t_2) \) such that

\[
\int xu(x, y, \tau_1) \, dx \, dy = 0 \tag{4.42}
\]

Performing a similar analysis we may also find \( \tau_2 \in (t_2, t_3) \) such that

\[
\int xu(x, y, \tau_2) \, dx \, dy = 0 \tag{4.43}
\]

Using (4.38) (with \( \eta = 0 \)), (4.42), (4.43) and the fact that the \( L^2 \) norm is a conserved quantity for (1.1) we conclude that \( \| \phi \| = 0 \), implying the desired. This finishes the proof of the theorem 1.3 in the case \( a \in (1/2, 1) \).

If \( a = 1/2 \) then \( 7/2 + a = 4 \). Hence, using (2.23) and following the same strategy as above we arrive to

\[
|\xi|^{-1/2} \varphi(\xi) \int_0^{t_2} \int xe^{-i\eta y} u(x, y, \tau) \, dx \, dy \, d\tau \in L^2_\xi, \quad \text{a.e. } \eta \in \mathbb{R}.
\]

Since \( |\cdot|^{-1/2} \varphi(\cdot) \notin L^2 \), we also obtain (4.41).

Finally, if \( a \in (0, 1/2) \) we write \( 7/2 + a = 3 + \alpha \) and use (2.22) to obtain an expression similar to (4.13). After some calculations and the help of Proposition 2.7 we may also obtain (4.41). Since it demands too many calculations following the arguments above we will omit the details. The proof of Theorem 1.3 is thus completed. \( \square \)

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