HITTING TIMES OF RARE EVENTS IN BOUNDARY DRIVEN
SYMMETRIC SIMPLE EXCLUSION PROCESSES

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Abstract. In the boundary driven symmetric simple exclusion process consider an open set $O$ of density profiles which does not contain the stationary density profile. We prove that the first time the empirical measure visits the set $O$ converges to an exponential distribution.

1. Introduction

It has long been observed that in finite-state, reversible Markov processes the hitting time of a rare event is approximately exponentially distributed [20, 13, 2, 3]. For non-reversible dynamics much less is known. By estimating the total variation distance between the stationary measure and the quasi-stationary measure, Aldous [1] proved that the distribution of the hitting time of a rare event is close to an exponential random variable when the mixing time is small compared to the stationary expectation of the hitting time. Fill and Lyzinski [23] proved that starting from the stationary distribution the hitting time of a configuration $\eta$ can be represented as an independent geometric sum of i.i.d. random variables if the probability of hitting this configuration $\eta$ at time $t$ starting from $\eta$, viz. $p_t(\eta, \eta)$, decreases in time. This representation permits to obtain bounds for the distance between the distribution of the hitting time and the distribution of an exponential random variable. Imbuzeiro [25] proved that the hitting time of a rare event $A$ is approximately exponential starting from a distribution $\nu$ if starting from $\nu$ the probability of hitting $A$ before the mixing time is small. Fernandez et al. [20] are presently working on this problem in the sequel of [6].

In this article we examine the hitting time of rare events in a well studied non-reversible dynamics, the boundary driven symmetric simple exclusion processes (BDSSEP). Beyond the complications arising from non-reversibility, this model presents a further difficulty in the lack of an explicit formula for the stationary measure. This obstacle is overcome by the use of a large deviations principle to estimate the measure of sets, but prevents us from obtaining bounds for the stationary expectation of the hitting time with errors sharper than exponential.

In the context of interacting particle systems the convergence of hitting times of rare events to exponential random variables has been abundantly investigated. Several results have been obtained for non-conservative dynamics, processes in which the local number of particles changes in time and which lose memory much faster than conservative ones. On the conservative side, which includes the dynamics examined here, Ferrari et al. [21] considered the case of a totally asymmetric one-dimensional zero-range process, and Ferrari et al. [22] examined the case of the

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one-dimensional symmetric simple exclusion process. This latter result was generalized to any dimension and extended to independent random walks by Asselah and Dai Pra \[4, 5\].

The article is organized as follows. In the next section we state the main result. In Section 3 we present a general method to derive the asymptotic exponentiality of the hitting time of a rare event for finite-state, non-reversible continuous-time Markov processes starting from a measure not too far from the stationary measure in the sense of Lemmas 3.7, 3.8 or 3.9. In Section 4 we estimate the expectation of the hitting time under the stationary state assuming a dynamical large deviations principle. In Section 5 we apply the results presented in the two previous sections to the BDSSEP.

2. Notation and Results

The one-dimensional boundary driven symmetric simple exclusion process (BDSSEP). For \( N \geq 1 \), let \( \Lambda_N = \{1, \ldots, N-1\} \). Fix \( 0 < \alpha \leq \beta < 1 \) and consider the Markov process \( \{\eta^N(t) : t \geq 0\} \) on \( \Omega_N = \{0,1\}^{\Lambda_N} \) whose generator \( L_N \) is given by

\[
(L_N f)(\eta) = \frac{1}{2} \sum_{x=1}^{N-2} \{f(\sigma^{x,x+1}\eta) - f(\eta)\} + \frac{1}{2} \left\{ \alpha [1 - \eta(1)] + (1 - \alpha) \eta(1) \right\} \{f(\sigma^1\eta) - f(\eta)\} + \frac{1}{2} \left\{ \beta [1 - \eta(N-1)] + (1 - \beta) \eta(N-1) \right\} \{f(\sigma^{N-1}\eta) - f(\eta)\}.
\]

In this formula, \( \eta = \{\eta(x), x \in \Lambda_N\} \) is a configuration of the state space \( \{0,1\}^{\Lambda_N} \) so that \( \eta(x) = 0 \) if and only if site \( x \) is vacant for \( \eta \); \( \sigma^{x,y}\eta \) is the configuration obtained from \( \eta \) by interchanging the occupation variables \( \eta(x), \eta(y) \):

\[
(\sigma^{x,y}\eta)(z) = \begin{cases} 
\eta(z) & \text{if } z \neq x, y, \\
\eta(y) & \text{if } z = x, \\
\eta(x) & \text{if } z = y;
\end{cases}
\]

and \( \sigma^x\eta \) is the configuration obtained from \( \eta \) by flipping the variable \( \eta(x) \):

\[
(\sigma^x\eta)(z) = \begin{cases} 
\eta(z) & \text{if } z \neq x, \\
1 - \eta(z) & \text{if } z = x.
\end{cases}
\]

Hence, at rate \( \alpha \) (resp. \( 1 - \alpha \)) a particle is created (resp. removed) at the boundary site 1 if this site is vacant (resp. occupied). The same phenomenon occurs at the boundary \( x = N - 1 \) with \( \beta \) in place of \( \alpha \).

Denote by \( D(\mathbb{R}_+, \Omega_N) \) the Skorohod space of paths from \( \mathbb{R}_+ \) to \( \Omega_N \). Let \( P^N_\eta \), \( \eta \in \Omega_N \), be the distribution of the Markov process \( \eta^N(t) \) when the initial configuration is \( \eta \). The probability measure \( P^N_\eta \) is thus a measure on the path space \( D(\mathbb{R}_+, \Omega_N) \) endowed with the Skorohod topology. Expectation with respect to \( P^N_\eta \) is denoted by \( E^N_\eta \).

The finite state Markov process \( \eta^N(t) \) is irreducible and has therefore a unique stationary measure, denoted by \( \nu^N_{\alpha, \beta} \). The process is reversible if and only if \( \alpha = \beta \), in which case the measure \( \nu^N_{\alpha, \alpha} \) is a product measure.
The empirical measure. Denote by $\langle \cdot , \cdot \rangle$ the inner product in $L_2([0,1])$ and set

$$\mathcal{M} := \{\rho \in L_\infty([0,1]) : 0 \leq \rho \leq 1\}$$

which we equip with the topology induced by the weak convergence of measures, namely a sequence $\{\rho^n : n \geq 1\} \subset \mathcal{M}$ converges to $\rho$ in $\mathcal{M}$ if and only if $\langle \rho^n , G \rangle \to \langle \rho , G \rangle$ for any continuous function $G : [0,1] \to \mathbb{R}$. Note that $\mathcal{M}$ is a compact Polish space that we consider endowed with the corresponding Borel $\sigma$-algebra.

Let $d$ be a distance in $\mathcal{M}$ compatible with the weak topology,

$$d(\gamma, \gamma') = \sum_{k \geq 1} \frac{1}{2k} \|\langle \gamma, F_k \rangle - \langle \gamma', F_k \rangle\|,$$

(2.1)

where the continuous test functions $F_k$ are absolutely bounded by 1.

The empirical density of a configuration $\eta \in \Omega_N$, denoted by $\pi^N(\eta) \in \mathcal{M}$, is defined as

$$\pi^N(\eta) := \sum_{x=1}^{N-1} \eta(x) \mathbf{1}\{\left\lfloor \frac{x}{2N} \right\rfloor - \frac{1}{2N}, \left\lfloor \frac{x}{2N} \right\rfloor + \frac{1}{2N}\} ,$$

where $\mathbf{1}\{A\}$ stands for the indicator function of the set $A$.

Denote by $\nabla$ the space derivative and by $\Delta$ the Laplacian. It has been proved in [16] that under the stationary state $\nu^{N}_{\alpha, \beta}$, the empirical measure $\pi^N$ converges in probability to the unique solution of the elliptic equation

$$\begin{cases}
\Delta \rho = 0 , \\
\rho(0) = \alpha , \quad \rho(1) = \beta .
\end{cases}$$

We denote the solution of this equation by $\bar{\rho} = \bar{\rho}_{\alpha, \beta}$.

The dynamical rate function. To state the main result of this article we need to introduce the rate functions of the dynamical and the static large deviations principle of the empirical measure. We start with the dynamical one.

For $T > 0$ and positive integers $m,n$, we denote by $C^{m,n}([0,T] \times [0,1])$ the space of functions $G : [0,T] \times [0,1] \to \mathbb{R}$ with $m$ derivatives in time, $n$ derivatives in space which are continuous up to the boundary. We improperly denote by $C^{m,n}_0([0,T] \times [0,1])$ the subset of $C^{m,n}([0,T] \times [0,1])$ of the functions which vanish at the endpoints of $[0,1]$, i.e. $G \in C^{m,n}([0,T] \times [0,1])$ belongs to $C^{m,n}_0([0,T] \times [0,1])$ if and only if $G(t,0) = G(t,1) = 0$, $t \in [0,T]$.

Let the energy $Q : D([0,T], \mathcal{M}) \to [0,\infty]$ be given by

$$Q(u) = \sup_G \left\{ \int_0^T dt \int_0^1 dx \ u(t,x) (\nabla G)(t,x) - \frac{1}{2} \int_0^T dt \int_0^1 dx \ G(t,x)^2 \chi(u(t,x)) \right\} ,$$

where $\chi : [0,1] \to \mathbb{R}_+$ is the mobility of the system, $\chi(a) = a(1-a)$, and where the supremum is carried over all smooth functions $G : [0,T] \times (0,1) \to \mathbb{R}$ with compact support. It has been shown in [10] that the energy $Q$ is convex and lower semicontinuous. Moreover, if $Q(u)$ is finite, $u$ has a generalized space derivative, $\nabla u$, and

$$Q(u) = \frac{1}{2} \int_0^T dt \int_0^1 dx \ \frac{(\nabla u(t))^2}{\chi(u(t))} .$$
Fix a function $\gamma \in \mathcal{M}$ which corresponds to the initial profile. For each $H$ in $C^{1,2}_{0}(\mathbb{R} \times [0,1])$, let $\hat{J}_H(\cdot|\gamma) = J_{T,H,\gamma} : D([0,T],\mathcal{M}) \longrightarrow \mathbb{R}$ be the functional given by

$$\hat{J}_H(u|\gamma) := \langle u_T, H_T \rangle - \langle \gamma, H_0 \rangle - \int_0^T dt \langle u_t, \partial_t H_t \rangle - \frac{1}{2} \int_0^T dt \langle u_t, \Delta H_t \rangle \quad \text{(2.2)}$$

$$+ \frac{\beta}{2} \int_0^T dt \nabla H_t(1) - \frac{\alpha}{2} \int_0^T dt \nabla H_t(0) - \frac{1}{2} \int_0^T dt \langle \chi(u_t), (\nabla H_t)^2 \rangle.$$  

Let $I_{[0,T]}(\cdot|\gamma) : D([0,T],\mathcal{M}) \longrightarrow [0,\infty]$ be the functional defined by

$$I_{[0,T]}(u|\gamma) := \sup_{H \in C^{1,2}_{0}(\mathbb{R} \times [0,1])} \hat{J}_H(u|\gamma).$$

The dynamical rate functional $I_{[0,T]}(\cdot|\gamma) : D([0,T],\mathcal{M}) \rightarrow [0,\infty]$ is given by

$$I_{[0,T]}(u|\gamma) = \left\{ \begin{array}{ll} I_{[0,T]}(u|\gamma) & \text{if } Q(u) < \infty, \\ \infty & \text{otherwise.} \end{array} \right. \quad \text{(2.3)}$$

**The static rate functional.** Denote by $V : \mathcal{M} \rightarrow \mathbb{R}_+$ the quasi-potential associated to the dynamical rate functions $I_{[0,T]}$:

$$V(\gamma) = \inf_{T > 0} \inf \left\{ I_{[-T,0]}(u|\rho) : u(-T) = \bar{\rho}, u(0) = \gamma \right\}. \quad \text{(2.4)}$$

It has been proved in [9] Theorems 2.2, 4.5 and A.1 that $V$ is bounded, convex and lower-semicontinuous, and that $V(\rho) > 0$ for all $\rho \neq \bar{\rho}$. We are now in a position to state the main result of this article.

**Theorem 2.1.** Fix an open subset $O$ of $\mathcal{M}$ such that $d(\bar{\rho}, O) > 0$ and let $H_O$ be the hitting time of the set $O$, $H_O = \inf\{t : \pi^N(\eta(t)) \in O\}$. Then, under $\nu_{\alpha,\beta}^N$, $H_O/E_{\nu_{\alpha,\beta}^N}[H_O]$ converges in distribution to a mean one exponential time. Moreover, if

$$\inf_{\gamma \in \overline{O}} V(\gamma) = \inf_{\gamma \in \overline{O}} V(\gamma),$$

where $\overline{O}$ represents the closure of $O$, we have that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \log E_{\nu_{\alpha,\beta}^N}[H_O] = \inf_{\gamma \in \overline{O}} V(\gamma).$$

Finally, consider a subset $\mathcal{B}$ of $\mathcal{M}$ such that

$$\inf_{\gamma \in \overline{\mathcal{B}}} V(\gamma) < \inf_{\gamma \in \overline{O}} V(\gamma) \quad \text{(2.5)}$$

where $\overline{\mathcal{B}}$ stands for the interior of $\mathcal{B}$. Let $B_N = (\pi^N)^{-1}(\mathcal{B}) = \{ \eta \in \Omega_N : \pi^N(\eta) \in \mathcal{B} \}$ and let $\mu_N$ be the probability measure on $\Omega_N$ defined by $\mu_N(\eta) = 1\{ \eta \in B_N \nu_{\alpha,\beta}^N(\eta)/\nu_{\alpha,\beta}^N(B_N) \}$. Then, under $\mu_N$, $H_O/E_{\nu_{\alpha,\beta}^N}[H_O]$ converges in distribution to a mean one exponential time.

This result holds in all dimensions, we restricted ourselves to dimension one for sake of simplicity.
3. Hitting times of rare events have exponential distributions

Consider a sequence of irreducible, continuous-time Markov processes \( \{\eta^N(t) : t \geq 0\} \), \( N \geq 1 \), taking values on a finite state space \( \Omega_N \). The points of \( \Omega_N \) are represented by the Greek letters \( \eta, \xi \). Denote by \( \nu_N \) the unique stationary state, by \( L_N \) the generator of the process, by \( \lambda_N(\eta), \eta \in \Omega_N \), the holding rates, by \( p_N(\eta, \xi), \xi \neq \eta \in \Omega_N \), the jump probabilities, and by \( R_N(\eta, \xi) = \lambda_N(\eta)p_N(\eta, \xi) \) the jump rates. In particular, for every function \( f : \Omega_N \to \mathbb{R} \),

\[
(L_N f)(\eta) = \sum_{\xi \in \Omega_N} R_N(\eta, \xi) [f(\xi) - f(\eta)] ,
\]

We often omit the superscript \( N \) of \( \eta^N(t) \).

For a subset \( A \) of \( \Omega_N \), denote by \( H_A \) (resp. \( H_A^+ \)) the hitting (resp. return) time of a set \( A \):

\[
H_A := \inf \{ s > 0 : \eta(s) \in A \} ,
\]

\[
H_A^+ := \inf \{ t > 0 : \eta(t) \in A, \eta(s) \neq \eta(0) \text{ for some } 0 < s < t \} .
\]

When the set \( A \) is a singleton \( \{\eta\} \), we denote \( H_{\{\eta\}}, H_{\{\eta\}}^+ \) by \( H_\eta, H_\eta^+ \), respectively.

Let \( D(\mathbb{R}_+, \Omega_N) \) be the space of \( \Omega_N \)-valued, right continuous paths with left limits endowed with the Skorohod topology. Denote by \( \mathbb{P}_\eta = \mathbb{P}_{\eta^N}^N, \eta \in \Omega_N \), the probability measure on \( D(\mathbb{R}_+, \Omega_N) \) induced by the Markov process \( \eta(t) \) starting from \( \eta \). Expectation with respect to \( \mathbb{P}_\eta \) is represented by \( \mathbb{E}_\eta \). For a probability measure \( \mu \) in \( \Omega_N \), \( \mathbb{E}_\mu[\cdot] = \sum_{\eta \in \Omega_N} \mu(\eta) \mathbb{P}_\eta[\cdot] \), with the same notation for expectations.

Let \( P_t(\eta, \xi), t \geq 0, \eta, \xi \in \Omega_N \), be the semigroup associated to \( \eta(t) \), \( P_t(\eta, \xi) = \mathbb{P}_\eta[\eta(t) = \xi] \). Denote by \( \|\mu - \nu\|_{\text{TV}} \) the total variation distance between two probability measures \( \mu \) and \( \nu \) defined on \( \Omega_N \). Let \( T_N^{\text{mix}} \) be the mixing time of the process \( \eta(t) \):

\[
T_N^{\text{mix}} = \inf \left\{ t > 0 : \max_{\eta \in \Omega_N} \| P_t(\eta, \cdot) - \nu_N \|_{\text{TV}} \leq \frac{1}{4} \right\} .
\]

Let \( A_N \) be a sequence of subsets of \( \Omega_N \) such that

\[
\lim_{N \to \infty} \nu_N(A_N) = 0 . \tag{3.1}
\]

Denote by \( H_N = H_{A_N} \) the hitting time of \( A_N \):

\[
H_N = \inf \{ t > 0 : \eta^N(t) \in A_N \} ,
\]

and by \( r_N(A_N^c, A_N) \) the average rate at which the process jumps from \( A_N^c \) to \( A_N \):

\[
r_N(A_N^c, A_N) = \frac{1}{\nu_N(A_N^c)} \sum_{\xi \in A_N^c} \nu_N(\xi) R_N(\xi, A_N) ,
\]

where \( R_N(\xi, A_N) = \sum_{\xi \in A_N} R_N(\xi, \xi) \).

Next statement is the main result of this section. It has to be compared with [1, Theorem 1.4]. Instead of requiring that the mixing time is small compared to the stationary expectation of the hitting time, we assume that the mixing time is small compared to the inverse of the averaged jump rate, \( r_N(A_N^c, A_N)^{-1} \), a quantity easily estimated. Moreover, by [11, Lemma 2.3], for reversible dynamics, \( r_N(A_N^c, A_N)^{-1} \) is bounded by the expected value of the hitting time of \( A_N^c \) starting from the quasi-stationary state.
Theorem 3.1. Let $A_N$ be a sequence of subsets of $\Omega_N$ satisfying $3.1$. Assume that $T_N^{\text{mix}} \leq r_N(A_N, A_N)^{-1}$. Then, under $\nu_N$, the sequence $H_N/\mathbb{E}_{\nu_N}[H_N]$ converges in distribution to a mean one exponential random variable.

Theorem 3.1 follows from Lemmas 3.4 and 3.5. We prove below in (3.6) and Lemma 3.5 that

$$\liminf_{N} r_N(A_N^c, A_N) \mathbb{E}_{\nu_N}[H_N] > 0.$$  

In fact, under some assumptions this product converges to 1. To state this hypothesis we need to introduce some notation.

For two disjoint subsets $A$, $B$ of the state space $\Omega_N$, denote by $\text{cap}(A, B)$ the capacity between $A$ and $B$:

$$\text{cap}(A, B) = \sum_{\eta \in A} \nu_N(\eta) \lambda_N(\eta) \mathbb{P}_\eta[H_B < H_A^+] .$$

When the set $A$ is a singleton, $A = \{\eta\}$, we write $\text{cap}(\eta, B)$ for $\text{cap}(\{\eta\}, B)$.

Denote by $\{\eta^*(t) : t \geq 0\}$ the stationary Markov process $\eta(t)$ reversed in time. We shall refer to $\eta^*(t)$ as the adjoint or the time reversed process. It is well known that $\eta^*(t)$ is a Markov process on $\Omega_N$ whose generator $L_N^*$ is the adjoint of $L_N$ in $L^2(\nu_N)$. The jump rates $R_N^*(\eta, \xi)$, $\eta \neq \xi \in \Omega_N$, of the adjoint process satisfy the balanced equations

$$\nu_N(\eta) R_N(\eta, \xi) = \nu_N(\xi) R_N^*(\xi, \eta) .$$

Denote by $\lambda^*(\eta) = \lambda(\eta)$, $\eta \in \Omega_N$, $p^*(\eta, \xi)$, $\eta \neq \xi \in \Omega_N$, the holding rates and the jump probabilities of the time reversed process $\eta^*(t)$.

As above, for each $\eta \in \Omega_N$, denote by $\mathbb{P}_\eta^*$ the probability measure on the path space $D(\mathbb{R}_+, \Omega_N)$ induced by the Markov process $\eta^*(t)$ starting from $\eta$. Expectation with respect to $\mathbb{P}_\eta^*$ is denoted by $\mathbb{E}_\eta^*$.

Lemma 3.2. Assume that there exists a sequence of subsets $B_N$, $B_N \subset A_N^c$, $\lim_{N \to \infty} \nu_N(B_N) = 1$, such that

$$\limsup_{N \to \infty} \sup_{\eta \in B_N} \mathbb{E}_{\eta}^*[H_{A_N} < H_{\eta}] = 0 , \quad \lim_{N \to \infty} \sum_{\eta \in A_N \cup B_N} \frac{\nu_N(\eta)}{\text{cap}(\eta, A_N)} = 0 . \quad (3.2)$$

Assume, furthermore, that

$$\limsup_{N \to \infty} r_N(A_N^c, A_N) < \infty . \quad (3.3)$$

Then,

$$\lim_{N \to \infty} r_N(A_N^c, A_N) \mathbb{E}_{\nu_N}[H_N] = 1 .$$

Proof. Fix $\eta \notin A_N$. By definition of the capacity, by equation (2.4) and Lemma 2.3 in [24], and by the Markov property,

$$\text{cap}(\eta, A_N) = \text{cap}^*(\eta, A_N) = \sum_{\xi \in A_N \cap A_N^c} \nu_N(\xi) R_N^*(\xi, \eta) \mathbb{P}_{\eta}^*[H_\eta < H_{A_N}] .$$

This sum is bounded above by $\nu_N(A_N^c) r_N(A_N^c, A_N) \leq r_N(A_N^c, A_N)$. On the other hand, if $\eta$ belongs to $B_N$, by assumption (3.2), the sum is bounded below by
\[(1 - \epsilon_N)\nu_N(A_N^c) r_N(A_N^c, A_N) \geq (1 - \epsilon_N) r_N(A_N^c, A_N),\] where \(\epsilon_N\) is a sequence which vanishes as \(N \uparrow \infty\), and which may change from line to line.

By [7, Proposition A.2],

\[E_{\nu_N}[H_N] = \sum_{\eta \notin A_N} \nu_N(\eta) \mathbb{E}[H_N] = \sum_{\eta \notin A_N} \nu_N(\eta) \frac{\sum_{\xi \notin A_N} \nu_N(\xi) \mathbb{P}^\xi[H_\eta < H_{A_N}]}{\text{cap}(\eta, A_N)}.\]

By the lower bound for the capacity obtained in the beginning of the proof and by (3.2), this expression is bounded above by

\[\sum_{\eta \in B_N} \frac{\nu_N(\eta)}{\text{cap}(\eta, A_N)} + \sum_{\eta \notin A_N \cup B_N} \frac{\nu_N(\eta)}{\text{cap}(\eta, A_N)} \leq (1 + \epsilon_N) r_N(A_N^c, A_N)^{-1} + \epsilon_N.\]

In view of (3.3), this proves that

\[\limsup_{N \to \infty} r_N(A_N^c, A_N) E_{\nu_N}[H_N] \leq 1.\]

By (3.2) and by the upper bound for the capacity obtained in the beginning of the proof, \(E_{\nu_N}[H_N]\) is bounded below by

\[\sum_{\eta \in B_N} \frac{\nu_N(\eta)}{\text{cap}(\eta, A_N)} \frac{\sum_{\xi \notin A_N} \nu_N(\xi) \mathbb{P}^\xi[H_\eta < H_{A_N}]}{\text{cap}(\eta, A_N)} \geq (1 - \epsilon_N) \sum_{\eta \in B_N} \frac{\nu_N(\eta)}{\text{cap}(\eta, A_N)} \geq (1 - \epsilon_N) r_N(A_N^c, A_N)^{-1}.\]

This concludes the proof of the lemma.

Denote by \(N_t, t \geq 0\), the number of jumps from \(A_N^c\) to \(A_N\) in the time interval \([0, t]\). \(N_t\) is a Poisson process and \(M_t\), defined by

\[M_t = N_t - \int_0^t R_N(\eta(s), A_N) 1\{\eta(s) \notin A_N\} ds,\]

is a martingale. In particular,

\[E_{\nu_N}[N_t] = t \nu_N(A_N^c) r_N(A_N^c, A_N).\]

Note that \(\{H_N \leq t\} = \{\eta(0) \in A_N\} \cup \{N_t \geq 1\}\). Define

\[X_t = 1\{\eta(0) \in A_N\} + N_t\]

so that \(\{H_N \leq t\} = \{X_t \geq 1\}\), and

\[P_{\nu_N}[H_N \leq t] = P_{\nu_N}[X_t \geq 1] \leq E_{\nu_N}[X_t] \leq \nu_N(A_N) + t \nu_N(A_N^c) r_N(A_N^c, A_N).\] (3.4)

**Lemma 3.3.** Assume that \(T_N^{\text{mix}} \ll r_N(A_N^c, A_N)^{-1}\). Let \(\gamma_N, \sigma_N\) be two sequences such that \(T_N^{\text{mix}} \ll \sigma_N \ll \min\{\gamma_N, r_N(A_N^c, A_N)^{-1}\}\). Then, for every \(t, s > 0\),

\[P_{\nu_N}[H_N > (t + s)\gamma_N] - P_{\nu_N}[H_N > s\gamma_N] P_{\nu_N}[H_N > t\gamma_N] \leq 2 \nu_N(A_N) + 2 \sigma_N r_N(A_N^c, A_N) + (1/2)^{\sigma_N/T_N^{\text{mix}}}.\]
Lemma 3.5. The sequence \( \theta_N \) satisfies
\[
\lim_{N \to \infty} \frac{\mathbb{E}_{\nu_N} [H_N]}{\theta_N} = 1.
\]

Proof. In view of the definition of \( X_t \), we have to estimate the difference
\[
\mathbb{P}_{\nu_N} [X_{(t+\gamma_N)} = 0] - \mathbb{P}_{\nu_N} [X_{t\gamma_N} = 0] \mathbb{P}_{\nu_N} [X_t = 0] .
\]

Let \( \sigma_N \) be a sequence such that \( T_N^{\text{mix}} \ll \sigma_N \ll r_N(A_N^c, A_N)^{-1} \). Clearly,
\[
\mathbb{P}_{\nu_N} [X_{t\gamma_N} = 0] - \mathbb{P}_{\nu_N} [X_{t\gamma_N} - X_{\sigma_N} = 0] \leq \mathbb{P}_{\nu_N} [X_{\sigma_N} \geq 1] ,
\]
and, by (3.4), this last probability is bounded by \( \nu_N(A_N) + \sigma_N r_N(A_N^c, A_N) \). By stationarity, a similar bound holds for the absolute value of the difference
\[
\mathbb{P}_{\nu_N} [X_{(t+s)\gamma_N} = 0] - \mathbb{P}_{\nu_N} [X_t = 0, X_{(t+s)\gamma_N} - X_t + \sigma_N = 0] .
\]

It remains to estimate the absolute value of the difference
\[
\mathbb{P}_{\nu_N} [X_{t\gamma_N} = 0, X_{(t+s)\gamma_N} - X_{t\gamma_N} + \sigma_N = 0] - \mathbb{P}_{\nu_N} [X_{(t+s)\gamma_N} - X_{\sigma_N} = 0] P_{\nu_N} [X_t = 0] .
\]

By the Markov property, this expression is equal to
\[
\mathbb{E}_{\nu_N} [1 \{X_t = 0\} \{\mathbb{P}_{\eta(t\gamma_N)} [X_{s\gamma_N} - X_{\tau_N} = 0] - \mathbb{P}_{\nu_N} [X_{s\gamma_N} - X_{\tau_N} = 0]\}] .
\]

This expectation is absolutely bounded by
\[
\sup_{\eta \in \Omega_N} \mathbb{E}_\eta \left[ \mathbb{P}_{\eta(\sigma_N)} [X_{s\gamma_N - \sigma_N} = 0] - \mathbb{P}_{\nu_N} [X_{s\gamma_N - \sigma_N} = 0] \right] \leq (1/2)^{\sigma_N/T_N^{\text{mix}}} ,
\]
where we used the definition of the mixing time in the last inequality. This concludes the proof of the lemma.

Let \( \theta_N \) be given by
\[
\theta_N = \inf \left\{ t > 0 : \mathbb{P}_{\nu_N} [H_N > t] < e^{-1} \right\} .
\] (3.5)

Note that \( \mathbb{P}_{\nu_N} [H_N > \theta_N] \leq e^{-1} \). Hence, by (3.3),
\[
1 - e^{-1} \leq \mathbb{P}_{\nu_N} [H_N \leq \theta_N] \leq \nu_N(A_N) + \theta_N \nu_N(A_N^c) r_N(A_N^c, A_N) .
\]

Since \( \nu_N(A_N) \) vanishes, we deduce from this inequality that
\[
\lim inf \frac{\theta_N r_N(A_N^c, A_N) > 0 .} {N}
\] (3.6)

In particular, \( \theta_N \gg T_N^{\text{mix}} \).

Lemma 3.4. Assume that \( T_N^{\text{mix}} \ll r_N(A_N^c, A_N)^{-1} \). Let \( \theta_N \) be the sequence defined by (3.5). Under \( \nu_N \), the sequence of random variables \( H_N/\theta_N \) converges in distribution to a mean one exponential random variable.

Proof. Since \( T_N^{\text{mix}} \ll r_N(A_N^c, A_N)^{-1} \), by (3.6), \( T_N^{\text{mix}} \ll \theta_N \). By Lemma 3.3 with \( \gamma_N = \theta_N \) and some sequence \( \sigma_N \), \( T_N^{\text{mix}} \ll \sigma_N \ll r_N(A_N^c, A_N)^{-1} \), we have that
\[
\lim \mathbb{P}_{\nu_N} [H_N \theta_N > t] = e^{-t}, \quad t > 0 .
\]

□

Lemma 3.5. The sequence \( \theta_N \) introduced in (3.5) satisfies
\[
\lim_{N \to \infty} \frac{\mathbb{E}_{\nu_N} [H_N]}{\theta_N} = 1 .
\]
Proof. Let
\[
\theta_N(\eta) := \inf \{t > 0 : \mathbb{P}_\eta[H_N > t] \leq e^{-1} \}, \quad \eta \in \Omega_N,
\]
and let \( \hat{\theta}_N = \max_{\eta \in \Omega_N} \theta_N(\eta) \). We first claim that
\[
\lim_{N \to \infty} \theta_N/\hat{\theta}_N = 1. \tag{3.7}
\]
It is clear that \( \theta_N \leq \hat{\theta}_N \). Indeed, if \( t > \hat{\theta}_N \), \( t > \theta_N(\eta) \) for all \( \eta \in \Omega_N \), so that
\[
\mathbb{P}_{\nu_N}[H_N > t] = \sum_{\eta \in \Omega_N} \nu_N(\eta) \mathbb{P}_\eta[H_N > t] \leq e^{-1}.
\]
Hence, \( \theta_N \leq t \) and \( \theta_N \leq \hat{\theta}_N \).

To prove the converse inequality, let \( \theta_N(a), a > 0 \), be given by
\[
\theta_N(a) := \inf \{t > 0 : \mathbb{P}_{\nu_N}[H_N > t] \leq e^{-a} \}.
\]
For any \( \eta \in \Omega_N \), \( \epsilon > 0 \), \( L \geq 1 \),
\[
\mathbb{P}_\eta[H_N > \theta_N(1+\epsilon) + LT_N^{mix}] \leq \mathbb{E}_\eta\left[\mathbb{P}_{\eta(LT_N^{mix})}[H_N > \theta_N(1+\epsilon)]\right]
\]
By definition of the mixing time and of \( \theta_N(1+\epsilon) \), the last expectation is bounded by
\[
2^{-L} + \mathbb{P}_{\nu_N}[H_N > \theta_N(1+\epsilon)] \leq 2^{-L} + e^{-(1+\epsilon)} \leq e^{-1}
\]
provided \( 2^{-L} \leq e^{-1}[1 - e^{-\epsilon}] \). Hence, \( \theta_N(\eta) \leq \theta_N(1+\epsilon) + LT_N^{mix} \) for all \( \eta \in \Omega_N \) so that \( \theta_N \leq \theta_N(1+\epsilon) + LT_N^{mix} \).

Denote by \( R_N \) the right hand side of the inequality appearing in the statement of Lemma 3.3 with \( \gamma_N \) replaced by \( \theta_N \). Iterating \( k - 1 \) times this estimate, we obtain that
\[
\mathbb{P}_{\nu_N}[H_N > \theta_N/k] \leq \left(\mathbb{P}_{\nu_N}[H_N > \theta_N] + kR_N\right)^{1/k}.
\]
Applying once more Lemma 3.3, we get that
\[
\mathbb{P}_{\nu_N}[H_N > (k+1)\theta_N/k] \leq \mathbb{P}_{\nu_N}[H_N > \theta_N]\mathbb{P}_{\nu_N}[H_N > \theta_N/k] + R_N,
\]
so that
\[
\mathbb{P}_{\nu_N}[H_N > (k+1)\theta_N/k] \leq e^{-1}(e^{-1} + kR_N)^{1/k} + R_N.
\]
Since \( R_N \) vanishes, if \( k > e^{-1} \) this expression is bounded by \( e^{-(1+\epsilon)} \) for \( N \) sufficiently large. Therefore, \( \theta_N(1+\epsilon) \leq (1+k^{-1})\theta_N \) for all \( N \) large enough if \( k > e^{-1} \). Taking \( k = \lceil e^{-1} \rceil + 1 \), where \([a]\) stands for the integer part of \( a \), we conclude from the previous two estimates that for \( N \) large enough
\[
\hat{\theta}_N \leq \left(1 + \frac{1}{\lceil e^{-1} \rceil + 1}\right)\theta_N + LT_N^{mix}
\]
provided \( 2^{-L} \leq e^{-1}[1 - e^{-\epsilon}] \). This proves that for every \( \epsilon > 0 \), \( \limsup_N(\hat{\theta}_N/\theta_N) \leq 1 + \left(\lceil e^{-1} \rceil + 1\right)^{-1} \), i.e., that \( \limsup_N(\hat{\theta}_N/\theta_N) \leq 1 \), proving claim (3.7).

It follows from Lemma 3.3 and (3.7) that \( H_N/\theta_N \) converges in distribution to a mean one exponential random variable. We claim that
\[
\lim_{N \to \infty} \frac{\mathbb{E}_{\nu_N}[H_N]}{\theta_N} = 1. \tag{3.8}
\]
To prove (3.8), we change variables to obtain that
\[
\hat{\theta}_N^{-1} \mathbb{E}_{\nu_N} [H_N] = \hat{\theta}_N^{-1} \int_0^\infty \mathbb{P}_{\nu_N} [H_N > t] dt = \int_0^\infty \mathbb{P}_{\nu_N} [H_N / \hat{\theta}_N > t] dt.
\]
It remains to obtain a bound to apply the dominated convergence theorem. By definition of \(\hat{\theta}_N\), \(\mathbb{P}_\eta [H_N > \hat{\theta}_N] \leq e^{-1}\) for all \(\eta \in \Omega_N\). By the Markov property, we obtain that \(\mathbb{P}_\eta [H_N > t\hat{\theta}_N] \leq \mathbb{P}_\eta [H_N > \hat{\theta}_N] \leq e^{-1}\) so that \(\mathbb{P}_{\nu_N} [H_N / \hat{\theta}_N > t] \leq e^{-t}\).

**Corollary 3.6.** Assume that the hypotheses of Theorem 3.1 are fulfilled. Let \(\{\mu_N : N \geq 1\}\) be a sequence of probability measures and suppose that there exists a sequence \(S_N, T^\text{mix}_N \ll S_N \ll \mathbb{E}_{\nu_N} [H_N]\), such that
\[
\lim_{N \to \infty} \mathbb{P}_{\mu_N} [H_N < S_N] = 0.
\]
Then, under \(\mu_N\), \(H_N / \mathbb{E}_{\nu_N} [H_N]\) converges in distribution to a mean one exponential random variable.

**Proof.** Let \(U_N = \mathbb{E}_{\nu_N} [H_N]\) and fix \(t > 0\). Clearly,
\[
\mathbb{P}_{\mu_N} [H_N \leq tU_N] = \mathbb{P}_{\mu_N} [S_N \leq H_N \leq tU_N] + \mathbb{P}_{\mu_N} [S_N > H_N, H_N \leq tU_N].
\]
By assumption, the second term on the right hand side vanishes as \(N \uparrow \infty\), while the first one, by the Markov property, is equal to
\[
\mathbb{E}_{\mu_N} \left[ \mathbb{1}_{\{S_N \leq H_N\}} \mathbb{P}_{\eta(S_N)} [H_N \leq tU_N - S_N] \right]
= \mathbb{E}_{\mu_N} \left[ \mathbb{P}_{\eta(S_N)} [H_N \leq tU_N - S_N]\right] - \mathbb{E}_{\mu_N} \left[ \mathbb{1}_{\{S_N > H_N\}} \mathbb{P}_{\eta(S_N)} [H_N \leq tU_N - S_N]\right].
\]
As before, the second term on right hand side vanishes as \(N \uparrow \infty\). The first one, since \(T^\text{mix}_N \ll S_N\) is equal to
\[
\mathbb{P}_{\nu_N} [H_N \leq tU_N - S_N] + R_N,
\]
where \(\lim_N R_N = 0\). Since \(S_N \ll U_N\), by Theorem 3.1 the first term in the previous displayed formula converges to \(1 - e^{-t}\), which proves the corollary.

To apply the previous corollary one needs among other things to estimate \(\mathbb{P}_{\mu_N} [H_N < S_N]\) and \(\mathbb{E}_{\nu_N} [H_N]\). In the next section we present a general method to estimate the latter sequence when a dynamical large deviations principle is available. There are several ways to bound \(\mathbb{P}_{\mu_N} [H_N < S_N]\). We present below three approaches. The first two uses the enlarged processes introduced by Bianchi and Gaudillière [11], the second and the third ones are taken from the martingale approach to metastability [8].

Consider a sequence \(\gamma_N\) of positive real numbers. Let \(\Omega_N^*\) be a copy of the set \(\Omega_N\) and recall from [8] Section 2.C the definition of the enlarged process associated to the sequence \(\gamma_N\), a Markov process, denoted by \(\eta^*(t)\), on \(\Omega_N \cup \Omega_N^*\), which jumps from a state \(\eta \in \Omega_N\) to its copy \(\eta^* \in \Omega_N^*\) at rate \(\gamma_N\). Denote by \(\nu_N^*\) the stationary measure of the enlarged process and recall that \(\nu_N^* (\eta) = \nu_N (\eta^*) = (1/2)\nu_N (\eta)\). Let \(\text{cap}_\gamma\) be the capacity with respect to the enlarged process, and for a subset \(B\) of \(\Omega_N\), denote by \(B^\gamma\) the copy of the set \(B\). Next result is Corollary 4.2 in [8].
Lemma 3.7. Let $\mu_N$ be a sequence of probability measures concentrated on $A_N^c$ and set $\gamma_N = S_N^{-1}$. Assume that
\[
\lim_{N \to \infty} S_N E_{\nu_N} \left[ \left( \frac{d\mu_N}{d\nu_N} \right)^2 \right] \text{cap}_*(A_N, (A_N^c)^*) = 0.
\]
Then, (3.9) holds.

Theorems 2.4 and 2.7 in [24] provide variational formulae for the capacity. The second theorem expresses the capacity as an infimum over flows. It permits, in particular, to obtain simple upper bounds. An elementary bound for the capacity is obtained as follows. By definition of the capacity and since $\nu^*(\eta) = (1/2) \sum_{\eta \in A_N} \nu(\eta) \{ R_N(\eta, \Omega_N) + \gamma_N \} \mathbb{P}_\eta^* [ H_{A_N^c}^\tau < H_{A_N}^+ ]$,
\[
\text{cap}_*(A_N, (A_N^c)^*) = (1/2) \sum_{\eta \in A_N} \nu(\eta) \{ R_N(\eta, \Omega_N) + \gamma_N \} \mathbb{P}_\eta^* [ H_{A_N^c}^\tau < H_{A_N}^+ ],
\]
where $\mathbb{P}_\eta^*$ represent the distribution of the enlarged process $\eta^*(t)$ starting from $\eta$. Therefore, (3.9) holds if
\[
\lim_{N \to \infty} S_N E_{\nu_N} \left[ \left( \frac{d\mu_N}{d\nu_N} \right)^2 \right] \{ S_N^{-1} + \max_{\eta \in A_N} R_N(\eta, \Omega_N) \} \nu_N(A_N) = 0. \tag{3.10}
\]

Lemma 3.8. Let $\mu_N$ be a sequence of probability measures on $\Omega_N$ and let $R'(\eta, A_N) = 1 \{ \eta \in A_N^c \} R_N(\eta, A_N)$. Assume that
\[
\lim_{N \to \infty} \{ \mu_N(A_N) + E_{\nu_N} [ R'(\eta, A_N) ] \sum_{\eta \in \Omega_N} \mu_N(\eta) \frac{1}{\text{cap}_*(\eta, \Omega_N^c)} \} = 0
\]
for some sequence $\gamma_N^{-1} \gg S_N$. Then, (3.9) holds.

In the reversible case, the Thomson principle permits to estimate from below the capacity. If $\gamma_N^{-1} \gg T_N^{\text{mix}}$, starting from any state in $\Omega_N$, the distribution of $\eta^*(H_{\Omega_N^c})$ is close to the stationary state $\nu_N$ lifted to $\Omega_N^c$. Since the capacity can be interpreted as the inverse of a distance, the sum on the right hand side measures the distance from $\mu_N$ to the stationary state $\nu_N$.

Proof of Lemma 3.8. We first replace the deterministic sequence $S_N$ in (3.9) by a sequence of exponential random variables independent of the Markov process $\eta(t)$. Denote by $\varepsilon_N$ a mean $\gamma_N^{-1}$ exponential time independent of the Markov process $\eta(t)$. Since $S_N \ll \gamma_N^{-1}$,
\[
\lim \sup_{N \to \infty} \mathbb{P}_{\mu_N} [ H_N < S_N ] \leq \lim \inf_{N \to \infty} \mathbb{P}_{\mu_N} [ H_N < \varepsilon_N ] . \tag{3.11}
\]

Repeating the steps which led to (3.4), we obtain that
\[
\mathbb{P}_{\mu_N} [ H_N < \varepsilon_N ] \leq \mu_N(A_N) + \mathbb{E}_{\mu_N} \left[ \int_0^{\varepsilon_N} R'_N(\eta(s), A_N) \, ds \right].
\]
In this step we used twice the monotone convergence theorem and we replaced $\varepsilon_N$ by $\varepsilon_N \wedge t$ to overcome the unboundedness of $\varepsilon_N$.

Clearly, starting from any configuration in $\Omega_N$, we may interpret $\varepsilon_N$ as the hitting time of $\Omega_N^c$ for the enlarged process so that
\[
\mathbb{E}_{\mu_N} \left[ \int_0^{\varepsilon_N} R'_N(\eta(s), A_N) \, ds \right] = \mathbb{E}_{\mu_N}^* \left[ \int_0^{H_{\Omega_N}} R'_N(\eta^*(s), A_N) \, ds \right].
\]
By \cite[Proposition A.2]{7}, since the equilibrium potential is bounded by 1 and since $\nu_N^*)(\eta) = (1/2)\nu_N(\eta)$, $\eta \in \Omega_N$, the previous expectation is equal to

$$\sum_{\eta \in \Omega_N} \mu_N(\eta) \mathbb{E}_\nu^* \left[ \int_0^{H_{\Omega_N}} R'_N(\eta^*(s), A_N) \, ds \right]$$

$$\leq E_{\nu_N} \left[ R'_N(\eta, A_N) \right] \sum_{\eta \in \Omega_N} \mu_N(\eta) \frac{1}{2\text{cap}_*(\eta, \Omega_N^*)},$$

which proves the lemma.

We conclude this section with a third set of sufficient conditions for \ref{3.9}. Denote by $T_N^\text{rel}$ the relaxation time, i.e. the inverse of the spectral gap of the symmetric part of the generator, and denote by $\| \cdot \|_p$ the norm of $L^p(\nu_N)$, $0 < p \leq \infty$.

**Lemma 3.9.** Let $S_N$ be an increasing sequence and let $\mu_N$ be a sequence of probability measures on $\Omega_N$. Assume that

$$\lim_{N \to \infty} \left\{ \mu_N(A_N) + S_N \nu_N(A_N) r_N(A_N, A_N) \right\} = 0,$$

$$\lim_{N \to \infty} \| R'_N(\cdot, A_N) \|_2 \left\| \frac{d\mu_N}{d\nu_N} \right\|_2 T_N^\text{rel} \left( 1 - e^{-S_N/T_N^\text{rel}} \right) = 0,$$

where $R'_N(\eta, A_N) = 1\{\eta \notin A_N\} R_N(\eta, A_N)$. Then, \ref{3.9} holds.

We may estimate $\| R'_N(\cdot, A_N) \|_2^2$ by $\| R'_N(\cdot, A_N) \|_\infty \| R'_N(\cdot, A_N) \|_1$ and recall that $\| R'_N(\cdot, A_N) \|_1 = \nu_N(A_N^c) r_N(A_N^c, A_N)$ which vanishes asymptotically.

**Proof of Lemma 3.9.** Repeating the steps which led to \ref{3.5}, we obtain that

$$\mathbb{P}_{\mu_N}[H_N < S_N] \leq \mu_N(A_N) + \mathbb{E}_{\mu_N} \left[ \int_0^{S_N} 1\{\eta(s) \notin A_N\} R_N(\eta(s), A_N) \, ds \right]$$

$$= \mu_N(A_N) + S_N \nu_N(A_N^c) r_N(A_N^c, A_N) + \int_0^{S_N} \mathbb{E}_{\mu_N} \left[ \hat{R}_N(\eta(s)) \right] \, ds,$$

where $\hat{R}_N(\eta)$ is the $\nu_N$-mean zero function $\hat{R}_N(\eta) = R'_N(\eta, A_N) - E_{\nu_N}[R'_N(\eta, A_N)]$, and $E_{\nu_N}[R'_N(\eta, A_N)] = \nu_N(A_N^c) r_N(A_N^c, A_N)$. We estimate the last term of the previous displayed equation. Let $f_\tau(\eta)$, $\eta \in \Omega_N$, $\tau \geq 0$, be the unique solution of

$$f_\tau(\eta) = \frac{\mu_N(\eta)}{\nu_N(\eta)} \frac{d}{d\tau} f_\tau = L_N^* f_\tau,$$

where $L_N^*$ stands for the adjoint of $L_N$ in $L^2(\nu_N)$. With this notation the integral in the penultimate displayed equation becomes

$$\int_0^{S_N} \langle \hat{R}_N, f_\tau \rangle_{\nu_N} \, d\tau \leq \| \hat{R}_N \|_2 \int_0^{S_N} (f_\tau; f_\tau)_{\nu_N}^{1/2} \, d\tau,$$

where $\langle \cdot, \cdot \rangle_{\nu_N}$ represents the scalar product in $L^2(\nu_N)$ and $(f_\tau; f_\tau)_{\nu_N}$ the variance of $f_\tau$. It is well known that $(f_\tau; f_\tau)_{\nu_N} \leq (f_0; f_0)_{\nu_N} e^{-2\tau/T_N^\text{rel}}$. The previous expression is thus bounded by

$$\| \hat{R}_N \|_2 (f_0; f_0)_{\nu_N}^{1/2} T_N^\text{rel} \left( 1 - e^{-S_N/T_N^\text{rel}} \right),$$

which proves the lemma by replacing variances by $L^2$ norms. \qed
4. Expectations of hitting times

We showed in the previous section that in the context of finite state Markov processes, the hitting time of rare events is asymptotically distributed according to an exponential law. We show in this section that the expectation under the stationary measure of these hitting times can be estimated if one is able to prove a dynamical large deviations principle. Instead of presenting this result in a general setting, we examine the case of the BDSSEP.

The dynamical large deviation principle. We recall a result first proved in [9], and then in [10] in the form presented below. We say that sequence of configurations \( \{\eta^N : N \geq 1\}, \eta^N \in \Omega_N \), is associated to the macroscopic density profile \( \rho \in \mathcal{M} \) if the sequence \( \pi^N(\eta^N) \) converges to \( \rho \) in \( \mathcal{M} \) as \( N \to \infty \).

Given \( T > 0 \), we denote by \( D([0,T];\mathcal{M}) \) the Skorohod space of paths from \([0,T]\) to \( \mathcal{M} \) equipped with its Borel \( \sigma \)-algebra. Elements of \( D([0,T],\mathcal{M}) \) will be denoted by \( u(t) \) and sometimes \( u_i \).

Fix a profile \( \gamma \in \mathcal{M} \) and consider a sequence \( \{\eta^N : N \geq 1\} \) associated to \( \gamma \). It has been proven in [17] following the work of [15] that as \( N \to \infty \) the sequence of random variables

\[
\pi^N(t) := \pi^N(\eta^N(tN^2)),
\]

which take values in \( D([0,T],\mathcal{M}) \), converges in probability to the unique weak solution \( u(t) \) of the heat equation with Dirichlet boundary conditions:

\[
\begin{align*}
\left\{ \begin{array}{ll}
\partial_t u - \frac{1}{2} \Delta u, & \\
u(t,0) = \alpha, & u(t,1) = \beta, & t \geq 0, \\
u(0,x) = \gamma(x), & 0 \leq x \leq 1.
\end{array} \right.
\]

Note that time has been speeded-up by \( N^2 \) in \( [0,1] \).

Recall the definition of the rate functional \( I_{[0,T]}(\cdot|\gamma) \) of the dynamical large deviations principle introduced in [23]. The next two results have been proven in [10].

**Lemma 4.1.** Fix \( \gamma \in \mathcal{M} \) and \( T > 0 \). The functional \( I_{[0,T]}(\cdot|\gamma) \) is lower semicontinuous and has compact level sets. Any path \( u \) with finite rate function, \( I_{[0,T]}(u|\gamma) < \infty \), is continuous in time and satisfies the boundary conditions \( u(0,\cdot) = \gamma(\cdot) \), \( u(\cdot,0) = \alpha, \ u(\cdot,1) = \beta \). Furthermore, any trajectory \( u \) with finite rate function can be approximated by a sequence of smooth trajectories \( \{u^n : n \geq 1\} \) in such a way that \( I_{[0,T]}(u^n|\gamma) \) converges to \( I_{[0,T]}(u|\gamma) \).

The dynamical large deviation principle can now be stated.

**Theorem 4.2.** Fix \( T > 0 \) and an initial profile \( \gamma \) in \( \mathcal{M} \). Consider a sequence \( \{\eta^N : N \geq 1\} \) of configurations associated to \( \gamma \). Then, the sequence of probability measures \( \{\mathbb{P}_{\eta^N} \circ (\pi^N(N^2 \cdot)^{-1} : N \geq 1\} \) on \( D([0,T],\mathcal{M}) \) satisfies a large deviation principle with speed \( N \) and good rate function \( I_{[0,T]}(\cdot|\gamma) \). Namely, \( I_{[0,T]}(.|\gamma) : D([0,T];\mathcal{M}) \to [0,\infty) \) has compact level sets and for each closed set \( \mathcal{E} \subset D([0,T],\mathcal{M}) \) and each open set \( \mathcal{D} \subset D([0,T],\mathcal{M}) \)

\[
\limsup_{N \to \infty} \frac{1}{N} \log \mathbb{P}_{\eta^N}(\pi^N(N^2 \cdot) \in \mathcal{E}) \leq - \inf_{u \in \mathcal{E}} I_{[0,T]}(u|\gamma)
\]

\[
\liminf_{N \to \infty} \frac{1}{N} \log \mathbb{P}_{\eta^N}(\pi^N(N^2 \cdot) \in \mathcal{D}) \geq - \inf_{u \in \mathcal{D}} I_{[0,T]}(u|\gamma).
\]
The static large deviation principle. The large deviations principle for the empirical measure under the stationary state $\nu_{\alpha,\beta}^N$, stated below, is taken from [12] [18].

**Theorem 4.3.** The sequence of probability measures $\{\nu_{\alpha,\beta}^N \circ (\pi^N)^{-1}: N \geq 1\}$ on $\mathcal{M}$ satisfies a large deviation principle with speed $N$ and good rate function $V$. Namely, $V : \mathcal{M} \to [0, \infty]$ has compact level sets and for each closed set $C \subset \mathcal{M}$ and each open set $O \subset \mathcal{M}$

$$\limsup_{N \to \infty} \frac{1}{N} \log \nu_{\alpha,\beta}^N (\pi^N \in C) \leq - \inf_{\gamma \in C} V(\gamma)$$

$$\liminf_{N \to \infty} \frac{1}{N} \log \nu_{\alpha,\beta}^N (\pi^N \in O) \geq - \inf_{\gamma \in O} V(\gamma).$$

Expectation of hitting times. The main result of this section can now be stated. Fix an open subset $O$ of $\mathcal{M}$ and let

$$A_N = (\pi^N)^{-1}(O) = \{ \eta \in \Omega_N : \pi^N(\eta) \in O \},$$

and let $H_N = H_{A_N}$ be the hitting time of the set $A_N$. Note that $H_N$ coincides with the hitting time $H_O$ introduced in Theorem 2.1.

**Theorem 4.4.** Fix an open subset $O$ of $\mathcal{M}$. Assume that $T^{\text{mix}}_N \ll \exp\{aN\}$ for all $a > 0$, that $H_N / \mathbb{E}_{\nu_{\alpha,\beta}^N}[H_N]$ converges in distribution to a mean one exponential random variable, and that $V(O) := \inf_{\gamma \in O} V(\gamma) = \inf_{\gamma \in O} V(\gamma)$.

Then, for every $\epsilon > 0$,

$$\liminf_{N \to \infty} \frac{\mathbb{E}_{\nu_{\alpha,\beta}^N}[H_N]}{e^{N(V(O)-\epsilon)}} > 0, \quad \limsup_{N \to \infty} \frac{\mathbb{E}_{\nu_{\alpha,\beta}^N}[H_N]}{e^{N(V(O)+\epsilon)}} < \infty.$$

In particular,

$$\lim_{N \to \infty} \frac{1}{N} \log \mathbb{E}_{\nu_{\alpha,\beta}^N}[H_N] = V(O).$$

To prove this result we first need a dynamical large deviations principle starting from the stationary measure.

**Theorem 4.5.** For each $T > 0$, each closed set $\mathcal{C} \subset D([0, T], \mathcal{M})$ and each open set $\mathcal{D} \subset D([0, T], \mathcal{M})$,

$$\limsup_{N \to \infty} \frac{1}{N} \log \mathbb{E}_{\nu_{\alpha,\beta}^N} \left( \pi^N(N^2 \cdot) \in \mathcal{C} \right) \leq - \inf_{u \in \mathcal{C}} \left\{ I_{[0,T]}(u|u_0) + V(u_0) \right\}$$

$$\liminf_{N \to \infty} \frac{1}{N} \log \mathbb{E}_{\nu_{\alpha,\beta}^N} \left( \pi^N(N^2 \cdot) \in \mathcal{D} \right) \geq - \inf_{u \in \mathcal{D}} \left\{ I_{[0,T]}(u|u_0) + V(u_0) \right\}.$$ 

**Proof.** In order to simplify the expressions, we will use the fact that concerning the SSEP process, as mentioned in [10] last part of section 2, the two dynamical rate functionals $I_{[0,T]}(u|\gamma)$ and $\hat{I}_{[0,T]}(u|\gamma)$ (see (2.3)) are the same.

We start with the proof of the upper bound. The arguments closely follow the ones presented in [10]. Theorem 4.3 is used afterwards to estimate the large deviations from the initial stationary distribution.
It is well known that using an exponential tightness argument, it is enough to prove the upper bound for compact sets. For any function \((t,x) \mapsto H_t(x) \in C^{1,2}_0([0,T] \times [0,1])\), we introduce the exponential martingale \(M_t^H\) defined by

\[
M_t^H = \exp \left\{ N \left[ \langle \pi_t^N, H_t \rangle - \langle \pi_0^N, H_0 \rangle - \frac{1}{N} \int_0^t e^{-N \langle \pi_s^N, H_s \rangle} (\partial_s e^{N \langle \pi_s^N, H_s \rangle} ds) \right] \right\}.
\]

Using a super-exponential estimate (Theorem 3.2), for any \(\delta > 0\) and \(\epsilon > 0\), there exists a set of configurations \(\eta \in B_{\delta,\epsilon}^H\) such that for any \(\delta > 0\)

\[
\lim_{\epsilon \to 0} \limsup_{N \to \infty} \frac{1}{N} \log \mathbb{P}^{N}_{\nu_{\alpha,\beta}} \left[ \{ \pi^N \in \mathcal{K} \} \right] = -\infty.
\]

and on which

\[
M_t^H = \exp N \left\{ J_H(\pi^N|\pi_0^N) + O_H(\epsilon) + O(\delta) \right\},
\]

where the functional \(J_H\) was defined in (22), \(O_H(\epsilon)\) (resp. \(O(\delta)\)) is an deterministic expression which vanishes as \(\epsilon \downarrow 0\) (resp. \(\delta \downarrow 0\)) and where, for any density \(\pi \in \mathcal{M}\),

\[
\pi^*(u) = \frac{1}{2\epsilon} \int_{[u-\epsilon, u+\epsilon]} \pi(u) du.
\]

Let \(\mathcal{K}\) be a compact subset of \(D([0,T], \mathcal{M})\), then

\[
\limsup_{\epsilon \downarrow 0} \limsup_{N \to \infty} \frac{1}{N} \log \mathbb{P}^{N}_{\nu_{\alpha,\beta}} \left[ \{ \pi^N \in \mathcal{K} \} \right] \leq \limsup_{\delta \downarrow 0} \limsup_{N \to \infty} \frac{1}{N} \log \mathbb{P}^{N}_{\nu_{\alpha,\beta}} \left[ \{ \pi^N \in \mathcal{K} \} \cap B_{\delta,\epsilon}^H \right]
\]

and we can write

\[
\mathbb{P}^{N}_{\nu_{\alpha,\beta}} \left[ \{ \pi^N \in \mathcal{K} \} \cap B_{\delta,\epsilon}^H \right] = \mathbb{P}^{N}_{\nu_{\alpha,\beta}} \left[ M_t^H (M_t^H)^{-1} \right] \left\{ \pi^N \in \mathcal{K} \cap B_{\delta,\epsilon}^H \right\}.
\]

Therefore,

\[
\frac{1}{N} \log \mathbb{P}^{N}_{\nu_{\alpha,\beta}} \left[ \{ \pi^N \in \mathcal{K} \} \cap B_{\delta,\epsilon}^H \right] \leq \frac{1}{N} \log \mathbb{E}^{N}_{\nu_{\alpha,\beta}} \left[ M_t^H \exp N \sup_{u \in \mathcal{K}} \left\{ - \frac{1}{N} J_H(u^*|\pi_0^N) \right\} \right] + O_H(\epsilon) + O(\delta)
\]

and since \(M_t^H\) is a mean 1 martingale, we get

\[
\limsup_{\epsilon \downarrow 0} \limsup_{N \to \infty} \frac{1}{N} \log \mathbb{P}^{N}_{\nu_{\alpha,\beta}} \left[ \{ \pi^N \in \mathcal{K} \} \right] \leq \limsup_{\epsilon \downarrow 0} \limsup_{N \to \infty} \frac{1}{N} \log \mathbb{E}^{N}_{\nu_{\alpha,\beta}} \left[ \exp N \sup_{u \in \mathcal{K}} \left\{ - \frac{1}{N} J_H(u^*|\pi^N) \right\} \right].
\]

We notice that the map \(\pi \mapsto \sup_{u \in \mathcal{K}} \left\{ - \frac{1}{N} J_H(u^*|\pi) \right\}\) is continuous on \(\mathcal{M}\), so we can apply Varadhan’s Lemma to the large deviation principle stated in Theorem 4.3

\[
\lim_{N \to \infty} \frac{1}{N} \log \mathbb{E}_{\nu_{\alpha,\beta}}^{N} \left[ \exp N \sup_{u \in \mathcal{K}} \left\{ - \frac{1}{N} J_H(u^*|\pi^N) \right\} \right] = \sup_{\gamma \in \mathcal{M}} \left\{ \sup_{u \in \mathcal{K}} \left\{ - J_H(u^*|\gamma) \right\} - V(\gamma) \right\} = - \inf_{\gamma \in \mathcal{M}, u \in \mathcal{K}} \{ J_H(u^*|\gamma) + V(\gamma) \}.
\]
Now, since $\mathcal{M} \times \mathcal{K}$ is compact, we can follow step by step the arguments of [10] section 3.3 and we get

$$\limsup_{N \to \infty} \frac{1}{N} \log \mathbb{P}_{\nu_{\pi,\delta}}^N \left[ \pi^N \in \mathcal{K} \right] \leq - \inf_{\gamma \in \mathcal{M}, u \in \mathcal{K}} \{ I_{[0,T]}(u, \gamma) + V(\gamma) \},$$

which is precisely the required upper bound since $I_{[0,T]}(u, \gamma) < +\infty$ implies that $u_0 = \gamma$.

The proof of the lower bound is easier. Indeed recalling the definition of the rate function $V$, we only have to show that for any $u \in D([0,T], \mathcal{M})$, any $S > 0$, any $\pi \in D([-S,0], \mathcal{M})$ such that $\pi_{-S} = \tilde{\rho}$ and $\pi_0 = u_0$, and for any $\delta > 0$,

$$\liminf_{N \to \infty} \frac{1}{N} \log \mathbb{P}_{\nu_{\pi,\delta}}^N \left[ \pi^N \in B_{[0,T]}(u, \delta) \right] \geq -I_{[-S,0]}(\pi|\tilde{\rho}) - I_{[0,T]}(u|u_0),$$

where $B_{[0,T]}(u, \delta)$ is the ball centered at $u$ with radius $\delta$ for the Skorohod topology on $D([0,T], \mathcal{M})$. If we denote by $\tilde{u}$ the density path given by $\pi$ on $[-S,0]$ and $u$ on $[0,T]$, then $\tilde{u} \in D([-S,T], \mathcal{M})$ and $I_{[-S,T]}(\tilde{u}) = I_{[-S,0]}(\pi|\tilde{\rho}) + I_{[0,T]}(u|u_0)$. Therefore, since $\nu_{\pi,\delta}$ is a stationary distribution, we have

$$\mathbb{P}_{\nu_{\pi,\delta}}^N \left[ \pi^N \in B_{[0,T]}(u, \delta) \right] \geq \mathbb{P}_{\nu_{\pi,\delta}}^N \left[ \pi^N \in B_{[-S,T]}(\tilde{u}, \delta) \right].$$

As under $\nu_{\pi,\delta}$, the initial empirical density $\pi^N$ converges to the stationary density $\tilde{\rho}$, the lower bound proved in [10] applies here and we get

$$\liminf_{N \to \infty} \frac{1}{N} \log \mathbb{P}_{\nu_{\pi,\delta}}^N \left[ \pi^N \in B_{[-S,T]}(u, \delta) \right] \geq -I_{[-S,T]}(\tilde{u}|\tilde{\rho}).$$

Next lemma is also needed in the proof of Theorem 4.4.

**Lemma 4.6.** Fix a subset $\mathcal{B}$ of $\mathcal{M}$ and $T > 0$. Let $\mathfrak{A} = \{ u \in C([0,T], \mathcal{M}) : u(t) \in \mathcal{B} \text{ for some } 0 \leq t \leq T \}$. Then,

$$\inf_{u \in \mathfrak{A}} \{ I_{[0,T]}(u|u_0) + V(u_0) \} \geq \inf_{\rho \in \mathcal{B}} V(\rho).$$

**Proof.** Fix $\epsilon > 0$ and $u \in \mathfrak{A}$. Assume that $u(t_0) \in \mathcal{B}$, $0 \leq t_0 \leq T$. By (2.4), there exists $T_0 > 0$ and a path $v \in C([-T_0,0], \mathcal{M})$ such that $v(-T_0) = \tilde{\rho}$, $v(0) = u(0) = u_0$, $I_{[-T_0,0]}(v|\tilde{\rho}) \leq V(u_0) + \epsilon$. Defining the path $w$ in $C([-T_0,T_0], \mathcal{M})$ by $w(t) = v(t)$, $-T_0 \leq t \leq 0$, $w(t) = u(t)$, $0 \leq t \leq t_0$, we obtain a path connecting $\tilde{\rho}$ to $u(t_0) \in \mathcal{B}$. By (2.4), $I_{[-T_0,T_0]}(w|\tilde{\rho}) \geq \inf_{\rho \in \mathcal{B}} V(\rho)$. It follows from the estimates just obtained that

$$I_{[0,T]}(u|u_0) + V(u_0) \geq I_{[0,T_0]}(u|u_0) + V(u_0) = I_{[-T_0,0]}(v|\tilde{\rho}) + V(u_0) - I_{[-T_0,0]}(v|\tilde{\rho}) \geq \inf_{\rho \in \mathcal{B}} V(\rho) - \epsilon,$$

which proves the lemma. \hfill \Box

**Proof of Theorem 4.4.** Fix $\epsilon > 0$. There exists $\gamma \in \mathcal{O}$ such that

$$V(\gamma) < \inf_{\rho \in \mathcal{O}} V(\rho) + (\epsilon/2),$$

and there exists $\delta$ such that $B_\delta(\gamma) \subset \mathcal{O}$. By (2.4) and by translation invariance of the dynamical rate function, there exist $T_\epsilon > 0$ and a path $u(\epsilon)(t)$, $0 \leq t \leq T_\epsilon$, $u_0(\epsilon) = \tilde{\rho}$, $u(\epsilon)(T_\epsilon) = \gamma$ such that

$$I_{[0,T_\epsilon]}(u(\epsilon)|\tilde{\rho}) < \inf_{\rho \in \mathcal{O}} V(\rho) + \epsilon.$$  (4.3)
For $\varphi > 0$, $T > 0$ and a path $u \in D([0, T], \mathcal{M})$ denote by $\mathbb{B}_{\varphi, T}(u)$ the open ball in $D([0, T], \mathcal{M})$ of radius $\varphi$ centered around $u$. Let $G = \mathbb{B}_{\varphi, T}(u^{(t)})$, $G_L = \{u \in D([0, L T_N^{mix}/N^2 + T], \mathcal{M}) : u(L T_N^{mix}/N^2 + \cdot) \in G\}$ It is clear from the definition of $G_L \subset \{H_N \leq L T_N^{mix} + T, N^2\}$. Hence, for any configuration $\xi \in \Omega_N$,

$$P_\xi \left[ H_N \leq L T_N^{mix} + T, N^2 \right] \geq \sum_{\xi \in \Omega_N} P_{L T_N^{mix}}(\xi, \xi) P_\xi \left[ \pi N \in G_L \right],$$

where $P_t(\eta, \xi)$, $t > 0$, stands for the transition probability of the BDSSEP. By definition of the mixing time, the previous expression is bounded below by

$$P_{\nu_\alpha, \beta}^N \left[ \pi N \in G \right] - 2^{-L}.$$

Therefore, for every $L \geq 1$,

$$\inf_{\xi \in \Omega_N} P_\xi \left[ H_N \leq L T_N^{mix} + T, N^2 \right] \geq P_{\nu_\alpha, \beta}^N \left[ \pi N \in G \right] - 2^{-L}. \quad (4.4)$$

By Theorem 3.3, by definition of $G$, by (4.3) and since $u_0^{(t)} = \bar{\rho}, V(\bar{\rho}) = 0$,

$$\liminf_{N \to \infty} \frac{1}{N} \log \mathbb{P}_{\nu_\alpha, \beta} \left[ \pi N \in G \right] \geq - \inf_{u \in G} \left\{ I_{\{0, T\}}(u|u_0) + V(u_0) \right\} \geq - \left\{ I_{\{0, T\}}(u^{(t)}|u_0^{(t)}) + V(u_0^{(t)}) \right\} \geq -(V(0) + \varepsilon).$$

Hence, there exists $N_0 = N_0(\varepsilon, \delta)$ such that for all $N \geq N_0$,

$$\mathbb{P}_{\nu_\alpha, \beta}^N \left[ \pi N \in G \right] \geq \exp \left\{ -N \{V(0) + 2\varepsilon\} \right\}.$$

The previous estimate together with (4.4) for $L = \ell N$ gives that for all $N \geq N_0$,

$$\max_{\xi \in \Omega_N} P_\xi \left[ H_N > \ell N T_N^{mix} + T, N^2 \right] \leq 1 - e^{-N[V(0)+2\varepsilon]} + 2^{-\ell N}.$$

Iterating this estimate $M$ times, gives by the Markov property that

$$\max_{\xi \in \Omega_N} \mathbb{P}_\xi \left[ H_N > M \{ \ell N T_N^{mix} + T, N^2 \} \right] \leq \left( 1 - e^{-N[V(0)+2\varepsilon]} + 2^{-\ell N} \right)^M.$$

Taking $\ell$ large enough and setting $M = \exp\{N[V(0) + 2\varepsilon]\}$, we conclude that

$$\limsup_{N \to \infty} \max_{\xi \in \Omega_N} \mathbb{P}_\xi \left[ H_N > e^{N[V(0)+2\varepsilon]} \{ \ell N T_N^{mix} + T, N^2 \} \right] < 1.$$  

Since, by assumption, $N T_N^{mix} < \exp\{\varepsilon N\}$ for $N$ sufficiently large and since we assumed that $H_N/\mathbb{E}_{\nu_\alpha, \beta} \left[ H_N \right]$ converges to a mean one exponential random variable, we have that

$$\liminf_{N \to \infty} \frac{e^{N[V(0)+3\varepsilon]}}{\mathbb{E}_{\nu_\alpha, \beta} \left[ H_N \right]} > 0.$$

Conversely, for $k \geq 0$, let $\mathcal{A}_k = \{u \in D([k T_e, (k + 1) T_e], \mathcal{M}) : u(t) \in \emptyset \text{ for some } k T_e \leq t \leq (k + 1) T_e\}$. By definition, for every $L \geq 1$

$$\mathbb{P}_{\nu_\alpha, \beta} \left[ H_N \leq L N^2 T_e \right] \leq \sum_{k=0}^{L-1} \mathbb{P}_{\nu_\alpha, \beta} \left[ \pi N \in \mathcal{A}_k \right] \leq L \mathbb{P}_{\nu_\alpha, \beta} \left[ \pi N \in \overline{\mathcal{A}} \right],$$

where $\mathcal{A} = \mathcal{A}_0$ and $\overline{\mathcal{A}}$ stands for the closure of $\mathcal{A}$.

By Theorem 3.3,

$$\limsup_{N \to \infty} \frac{1}{N} \log \mathbb{P}_{\nu_\alpha, \beta} \left[ \pi N \in \overline{\mathcal{A}} \right] \leq - \inf_{u \in \overline{\mathcal{A}}} \left\{ I_{\{0, T\}}(u|u_0) + V(u_0) \right\}. $$
By Lemma 4.1, we may restrict the supremum to paths $u$ in $C([0,T_\ast],\mathcal{M})$. In this case, $\overline{\mathfrak{S}}$ is contained on the closed set $\mathfrak{S}' = \{u \in C([0,T_\ast],\mathcal{M}) : u(t) \in \overline{\Theta} \text{ for some } 0 \leq t \leq T_\ast\}$, so that

$$
\limsup_{N \to \infty} \frac{1}{N} \log \mathbb{P}_{\nu_{\alpha,\beta}^N} [\pi^N \in \overline{\mathfrak{S}}] \leq - \inf_{u \in \mathfrak{S}'} \left\{ I_{[0,T_\ast]}(u|u_0) + V(u_0) \right\}.
$$

By Lemma 4.10, $\inf_{u \in \mathfrak{S}'} \left\{ I_{[0,T_\ast]}(u|u_0) + V(u_0) \right\} \geq \inf_{\rho \in \Theta} V(\rho)$ and this latter quantity is by assumption equal to $\inf_{\rho \in \Theta} V(\rho)$. Hence, there exists $N_0$ such that for all $N \geq N_0$,

$$
\mathbb{P}_{\nu_{\alpha,\beta}^N} [\pi^N \in \overline{\mathfrak{S}}] \leq \exp - N \left\{ \inf_{\rho \in \Theta} V(\rho) - \epsilon \right\}.
$$

Taking $L = (1/2) \exp N \left\{ V(0) - \epsilon \right\}$ we deduce from the previous estimates that

$$
\mathbb{P}_{\nu_{\alpha,\beta}^N} [H_N \leq (1/2)e^{N(V(0)-\epsilon)} N^2 T_\ast] \leq 1/2
$$

for $N$ sufficiently large. Since, by assumption, $H_N/E_{\nu_{\alpha,\beta}^N} [H_N]$ converges in distribution to a mean one exponential random variable, we conclude from this inequality that

$$
\limsup_{N \to \infty} \frac{e^{N(V(0)-\epsilon)}}{E_{\nu_{\alpha,\beta}^N} [H_N]} < \infty.
$$

5. Hitting times of rare events in BDSSEP

We prove in this section Theorem 2.1. Denote by $R_N(\eta, \xi)$ the rate at which the BDSSEP $\eta(t)$ jumps from $\eta$ to $\xi$. Recall from (2.1) the distance $d$ introduced in $\mathcal{M}$. With this choice, by Schwarz inequality,

$$
d(\gamma, \gamma') \leq \|\gamma - \gamma'\|_2,
$$

where $\| \cdot \|_2$ stands for the $L_2$ norm.

**Lemma 5.1.** Fix an open subset $\Theta$ of $\mathcal{M}$ such that $d(\rho, 0) > 0$. Denote by $A_N$ the set of configurations in $\Omega_N$ for which $\pi^N(\eta)$ belongs to $\Theta$: $A_N = \{\eta \in \Omega_N : \pi^N(\eta) \in \Theta\}$. Then, there exists $a > 0$ such that

$$
r_N(A_N, A_N) \leq e^{-aN} \text{ and } \nu_{\alpha,\beta}^N(A_N) \leq e^{-aN}
$$

for $N$ sufficiently large.

**Proof.** Let $\Theta_\delta$ is the closed set defined by $\Theta_\delta = \{\gamma \in \mathcal{M} : d(\gamma, \overline{\Theta}) \leq \delta\}$, $\delta > 0$. We claim that there exists $\delta > 0$ such that

$$
\inf_{\gamma \in \Theta_\delta} V(\gamma) > 0.
$$

Indeed, let $2\delta = d(\rho, 0) > 0$. It is clear from the definition of $\Theta_\delta$ that $d(\rho, \gamma) \geq \delta$ for all $\gamma \in \Theta_\delta$. On the other hand, by [9] Theorem A.1,

$$
V(\rho) \geq \int_0^1 \left\{ \rho(x) \log \frac{\rho(x)}{\bar{\rho}(x)} + [1 - \rho(x)] \log \frac{1 - \rho(x)}{1 - \bar{\rho}(x)} \right\} \, dx.
$$

Therefore, since $0 < \alpha \leq \bar{\rho}(x) \leq \beta < 1$ and in view of (5.1), there exists $c_0 > 0$ such that for all $\gamma \in \Theta_\delta$,

$$
V(\gamma) \geq c_0 \int_0^1 \left\{ \gamma(x) - \bar{\rho}(x) \right\}^2 \, dx \geq c_0 d(\gamma, \bar{\rho})^2 \geq c_0 \delta^2.
$$
Denote by $\partial A_N$ the outer boundary of $A_N$:
\[
\partial A_N = \bigcup_{x=1}^{N-2} \{ \xi \not\in A_N : \sigma^{x,x+1}\xi \in A_N \} \cup \bigcup_{z=1,N-1} \{ \xi \not\in A_N : \sigma^z\xi \in A_N \} .
\]
Since $\sum_{\xi \in \Omega_N} R_N(\eta,\xi) \leq N$, by definition of the average rate $r_N(A_N^c,A_N)$,
\[
r_N(A_N^c,A_N) \leq \frac{1}{\nu_{\alpha,\beta}(A_N^c)} N \nu_{\alpha,\beta}(\partial A_N) .
\]
It is clear that for each $\delta > 0$, $\partial A_N \subset \{ \eta \in \Omega_N : \pi_N(\eta) \in \mathcal{O}_\delta \}$ for $N$ large enough. Hence, by Theorem 4.3 and by (5.2), there exists $a > 0$ such that
\[
\nu_{\alpha,\beta}(\partial A_N) \leq \nu_{\alpha,\beta}(\pi_N \in \mathcal{O}_\delta) \leq e^{-aN}
\]
for $N$ sufficiently large. The same bound holds for $A_N$, which proves the first part of the lemma.

**Estimation of the mixing time in the BDSSEP.** We show in this subsection by a coupling argument that
\[
T_{\text{mix}}^N \leq (1/2)N^3 .
\]
This bound is not sharp but sufficient for our purposes.

Assume that a coupling $(\eta_t, \xi_t)$ has been defined in the product space $\Omega_N \times \Omega_N$. This means that both coordinates evolve has the original BDSSEP and that the pair does not leave the diagonal once it reaches it. We denote by $P_{\eta,\xi}$ the distribution of the coupling when the initial configuration is $(\eta, \xi)$. Denote by $H_D$ the coupling time, the time the process reaches the diagonal. It is well known that
\[
T_{\text{mix}}^N \leq \inf \{ t : \max_{\eta,\xi \in \Omega_N} P_{\eta,\xi}[H_D \geq t] \leq 1/4 \} .
\]
The coupling of two copies of the BDSSEP is defined as follows. Fix two configurations $\eta, \xi$ in $\Omega_N$. We assume that the particles evolve according to a stirring dynamics and that particles are created simultaneously in both coordinates at the boundary. In particular, the coupled process has reached the diagonal when all initial particles have left the system. Denote by $H_j$ the time the particle initially at $j \in \Lambda_N$ leaves the system. If there are no particles at $j$ set $H_j = 0$ and note that if $j$ is occupied by an $\eta$-particle and a $\xi$-particle they both leave the system at the same time due to the stirring dynamics. With this notation, $H_D \leq \max_j H_j$ and for all $t > 0$
\[
P_{\eta,\xi}[H_D \geq t] \leq \sum_{j \in \Lambda_N} P_{\eta,\xi}[H_j \geq t] .
\]
Under the stirring dynamics, the particle at $j$ performs a symmetric random walk until it reaches the boundary. If we denote by $H_i$ the hitting time of the boundary, it is known that $E_j[H_i] = (1/2)j(N-j) \leq N^2/8$. The previous sum is thus bounded by $N^3/8t$, which proves claim (5.3).

**Proof of Theorem 2.1.** The first assertion of the proposition follows from Lemma 5.1, (5.3) and Theorem 3.1. The second one follows from Theorem 4.4.

To prove the third assertion, let $\gamma_N = N^4$ and consider the enlarged process associated to this sequence. By (5.3) and by the second assertion of the theorem, $T_{\text{mix}}^N \ll \gamma_N^{-1} \ll E_{\nu_{\alpha,\beta}}[H_D]$. 

Since \( d\mu_N / d\nu^{N}_{\alpha,\beta} = 1 \{ \eta \in B_N \} \nu^N_{\alpha,\beta}(B_N)^{-1} \), \( E_{\nu^{N}_{\alpha,\beta}}[(d\mu_N / d\nu^{N}_{\alpha,\beta})^2] = \nu^N_{\alpha,\beta}(B_N)^{-1} \).

Hence, as \( R_N(\eta, \Omega_N) \leq N \), the expression appearing on the left hand side of (3.10) is bounded by \( N^5 \nu^N_{\alpha,\beta}(A_N)/\nu^N_{\alpha,\beta}(B_N) \). By the static large deviation principle,

\[
\limsup \frac{1}{N} \log \nu^N_{\alpha,\beta}(A_N) \leq \limsup \frac{1}{N} \log \nu^N_{\alpha,\beta}(\pi^N \in B) \leq - \inf_{\gamma \in B} V(\gamma),
\]

\[
\liminf \frac{1}{N} \log \nu^N_{\alpha,\beta}(B_N) \geq \liminf \frac{1}{N} \log \nu^N_{\alpha,\beta}(\pi^N \in B^o) \geq - \inf_{\gamma \in B^o} V(\gamma).
\]

Therefore, by assumption (2.5), \( N^5 \nu^N_{\alpha,\beta}(A_N)/\nu^N_{\alpha,\beta}(B_N) \) vanishes as \( N \to \infty \). By remark (3.10), condition (3.9) is fulfilled. By Lemma 5.1 and by (5.3), the assumptions of Theorem 3.1 are in force. The third assertion of the theorem follows therefore from Corollary 3.6. 

\[\square\]

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