Learning and Verifying Quantified Boolean Queries by Example

Azza Abouzied∗, Dana Angluin∗∗, Christos Papadimitriou∗∗∗, Joseph M. Hellerstein∗∗∗, Avi Silberschatz∗
∗Yale University, ∗∗University of California, Berkeley
azza@cs.yale.edu, angluin@cs.yale.edu, christos@cs.berkeley.edu, hellerstein@cs.berkeley.edu, avi@cs.yale.edu

ABSTRACT
To help a user specify and verify quantified queries — a class of database queries known to be very challenging for all but the most expert users — one can question the user on whether certain data objects are answers or non-answers to her intended query. In this paper, we analyze the number of questions needed to learn or verify qhorn queries, a special class of Boolean quantified queries whose underlying form is conjunctions of quantified Horn expressions. We provide optimal polynomial-question and polynomial-time learning and verification algorithms for two subclasses of the class qhorn with upper constant limits on a query’s causal density.

Categories and Subject Descriptors
H.2.3 [Database Management]: Languages—query languages; I.2.2 [Artificial Intelligence]: Automatic Programming—program synthesis, program verification; I.2.6 [Artificial Intelligence]: Learning—concept learning

Keywords
quantified boolean queries, qhorn, query learning, query verification, example-driven synthesis

1. INTRODUCTION
It’s a lovely morning, and you want to buy a box of chocolates for your research group. You walk into a chocolate store and ask for “a box with dark chocolates — some sugar-free with nuts or filling”. However, your server is a pedantic logician who expects first-order logic statements. In response to your informal query he places in front of you a hundred boxes! Despite your frustration, you are intrigued: you open the first box only to find one dark, sugar-free chocolate with nuts and many other varieties of white chocolates that you didn’t order. You push it aside, indicating your disapproval, and proceed to the second. Inside, you are wondering: Is there hope that I can communicate to this person my needs through a sequence of such interactions?

Everybody, we request things from each other using informal and incomplete query specifications. Our casual interactions facilitate such under-specified requests because we have developed questioning skills that help us clarify such requests. A typical interlocutor might ask you about corner cases, such as the presence of white chocolates in the box, to get to a precise query specification by example. As requesters, we prefer to begin with an outline of our query — the key properties of the chocolates — and then make our query precise using a few examples. As responders, we can build a precise query from the query outline and a few positive or negative examples — acceptable or unacceptable chocolate boxes.

Typical database query interfaces behave like our logician. SQL interfaces, for example, force us to formulate precise quantified queries from the get go. Users find quantified query specification extremely challenging [1]. Such queries evaluate propositions over sets of tuples rather than individual tuples, to determine whether a set as a whole satisfies the query. Inherent in these queries are (i) the grouping of tuples into sets, and (ii) the binding of query expressions with either existential or universal quantifiers. Existential quantifiers ensure that some tuple in the set satisfies the expression, while universal quantifiers ensure that all tuples in the set satisfy the expression.

To simplify the specification of quantified queries, we built DataPlay [1]. DataPlay tries to mimic casual human interactions: users first specify the simple propositions of a query. DataPlay then generates a simple quantified query that contains all the propositions. Since this query may be incorrect, users can label query results as answers or non-answers to their intended query. DataPlay uses this feedback on example tuple-sets to fix the incorrect query. Our evaluation of DataPlay shows that users prefer example-driven query specification techniques for specifying complex quantified queries [1]. Motivated by these findings, we set out to answer the question: How far can we push the example-driven query specification paradigm? This paper studies the theoretical limits of using examples to learn and to verify a special subclass of quantified queries, which we call qhorn, in the hope of eventually making query interfaces more human-like.

1.1 Our contributions
We formalize a query learning model where users specify propositions that form the building blocks of a Boolean quantified query. A learning algorithm then asks the users membership questions: each question is an example data object, which the user classifies as either an answer or a

This is the extended version of the paper that appears in PODS 2013, New York, NY, USA
non-answer. After a few questions, the learning algorithm terminates with the unique query that satisfies the user’s responses to the membership questions. The key challenge we address in this paper is how to design a learning algorithm that runs in polynomial time, asks as few questions as possible and exactly identifies the intended query.

We prove the following:

1. Learning quantified Boolean queries is intractable: A doubly exponential number of questions is required. Within a special class of quantified Boolean queries known as qhorn [2.1] we prove two subclasses are exactly and efficiently learnable: qhorn-1 [2.1.3] and its superset role-preserving qhorn [2.1.4] with constant limits on causal density (Def. 2.6).

2. We design an optimal algorithm to learn qhorn-1 queries using \( O(n \log n) \) questions where \( n \) is the number of positions in a query (§3.1).

3. We design an efficient algorithm to learn role-preserving qhorn queries using \( O(k n \log n + \theta^{d+1}) \) questions where \( k \) is query size (Def. 2.6) and \( \theta \) is causal density (§3.2).

We also formalize a query verification model where the user specifies an entire query within the role-preserving qhorn query class. A verification algorithm then asks the user a set of membership questions known as the verification set. Each query has a unique verification set. The verification algorithm classifies some questions in the set as answers and others as non-answers. The query is incorrect if the user disagrees with any of the query’s classification of questions in the verification set.

We design a verification algorithm that asks \( O(k) \) membership questions (§4).

2. PRELIMINARIES

Before we describe our query learning and verification algorithms, we first describe our data model — nested relations — and the qhorn query class.

**Definition 2.1.** Given the sets \( D_1, D_2, ..., D_m \), \( \mathcal{R} \) is a relation on these \( m \) sets if it is a set of \( m \)-tuples \((d_1, d_2, ..., d_m)\) such that \( d_i \in D_i \) for \( i = 1, ..., m \). \( D_1, ..., D_m \) are the the domains of \( \mathcal{R} \).

**Definition 2.2.** A nested relation \( \mathcal{R} \) has at least one domain \( D_i \) that is a set of subsets (powerset) of another relation \( \mathcal{R}_i \). This \( \mathcal{R}_i \) is said to be an embedded relation of \( \mathcal{R} \).

**Definition 2.3.** A relation \( \mathcal{R} \) is a flat relation if all its domains \( D_1, ..., D_m \) are not powersets of another relation.

For example, a flat relation of chocolates can have the following schema:

\[
\text{Chocolate(isDark, hasFilling, isSugarFree, hasNuts, origin)}
\]

A nested relation of boxes of chocolates can have the following schema:

\[
\text{Box(name, Chocolate(isDark, hasFilling, isSugarFree, hasNuts, origin))}
\]

In this paper, we analyze queries over a nested relation with single-level nesting, i.e. the embedded relation is flat. The Box relation satisfies single-level nesting as the Chocolate relation embedded in it is flat. To avoid confusion, we refer to elements of the nested relation as objects and elements of the embedded flat relation as tuples. So the boxes are objects and the individual chocolates are tuples.

**Definition 2.4.** A **Boolean query** maps objects into either answers or non-answers.

The atoms of a query are Boolean propositions such as:

\[
p_1 : c.\text{isDark}, p_2 : c.\text{hasFilling}, p_3 : c.\text{origin} = \text{Madagascar}
\]

A complete query statement assigns quantifiers to expressions on propositions over attributes of the embedded relation. For example:

\[
\forall c \in \text{Box.Chocolates} (p_1) \land \\
\exists c \in \text{Box.Chocolates} (p_2 \land p_3)
\]

A box of chocolates is an answer to this query if every chocolate in the box is dark and there is at least one chocolate in the box that has filling and comes from Madagascar.

Given a collection of propositions, we can construct an abstract Boolean representation for the tuples of the nested relation. For example, given propositions \( p_1, p_2, p_3 \), we can transform the chocolates from the data domain to the Boolean domain as seen in Figure 1.

![Figure 1: Transforming data from its domain into a Boolean domain.](image)

Thus, each proposition \( p_i \) is replaced with a Boolean variable \( x_i \). We rewrite the Boolean query (1) as follows:

\[
\forall t \in S (x_1) \land \\
\exists t \in S (x_2 \land x_3)
\]

where \( S \) is the set of Boolean tuples for an object. This Boolean representation allows us to create learning and verification algorithms independent of the data domain or of the actual propositions that the user writes.

To support this Boolean representation of tuples, however, we assume that (i) it is relatively efficient to construct an actual data tuple from a Boolean tuple and that (ii) the true/false assignment to one proposition does not interfere with the true/false assignments to other propositions. The propositions \( p_m : c.\text{origin} = \text{Madagascar} \) and \( p_b : c.\text{origin} = \text{Belgium} \) interfere with each other as a chocolate cannot be both from Madagascar and Belgium: \( p_m \to \neg p_b \) and \( p_b \to \neg p_m \).

With three propositions, we can construct \( 2^3 \) possible Boolean tuples, corresponding to the \( 2^3 \) possible true or false
assignments to the individual propositions, i.e. we can construct 8 different chocolate classes. With n propositions, we can construct $2^n$ Boolean tuples.

There are $2^{2^n}$ possible sets of Boolean tuples or unique objects. With our three chocolate propositions, we can construct 256 boxes of distinct mixes of the 8 chocolate classes. Since a Boolean query maps each possible object into an answer or a non-answer, it follows that there are $2^{2^n}$ distinguishable Boolean queries (for n = 3, about $10^{77}$). If our goal is to learn any query from n simple propositions by asking users to label objects as answers or non-answers, i.e. asking membership questions, then we would have to distinguish between $2^{2^n}$ queries using $\Omega(\lg(2^{2^n}))$ or $2^{2^n}$ questions.

Since this ambitious goal of learning any query with few membership questions is doomed to fail, we have to constrain the query space. We study the learnability of a special space of queries, which we refer to as qhorn.

### 2.1 Qhorn

Qhorn has the following properties:

1. It supports if-then query semantics via quantified Horn expressions: $\forall t \in S \ (x_1 \land x_2 \rightarrow x_3)$. A Horn expression has a conjunction of body variables that imply a single head variable. The degenerate headless Horn expression is simply a quantified conjunction of body variables ($\exists t \in S \ (x_1 \land x_2)$) and the degenerate bodyless Horn expression is simply a single quantified variable ($\forall t \in S (T \rightarrow x_1) \equiv \forall t \in S(x_1)$).

2. It requires at least one positive instance for each Horn expression via a guarantee clause. Thus, we add the existential clause $\exists t \in S \ (x_1 \land x_2 \land x_3)$ to the expression $\forall t \in S \ (x_1 \land x_2 \rightarrow x_3)$ to get a complete query. Note that the expression $\exists t \in S \ (x_1 \land x_2 \rightarrow x_3)$ is implied by its guarantee clause $\exists t \in S \ (x_1 \land x_2 \land x_3)$.

3. It represents queries in a normalized form: conjunctions of quantified (Horn) expressions.

We use a shorthand notation for queries in qhorn. We drop the implicit ‘t’ $\in S$, the ‘∧’ symbol and the guarantee clause. Thus, we write the query

$\forall t \in S \ (x_1 \land x_2 \rightarrow x_3) \land \exists t \in S \ (x_1 \land x_2 \land x_3) \land \forall t \in S \ (x_4) \land \exists t \in S \ (x_5)$

as $\forall x_1 x_2 \rightarrow x_3 \ \forall x_4 \exists x_5$.

### 2.1.1 Qhorn’s Equivalence Rules

**R1** The query representation $\exists x_1 x_2 x_3 \ \exists x_1 x_2 \ \exists x_2 x_3$ is equivalent to $\exists x_1 x_2 x_3$. This is because if a set contains a tuple that satisfies $\exists x_1 x_2 x_3$, that tuple will also satisfy $\exists x_1 x_2$ and $\exists x_2 x_3$. An existential conjunction over a set of variables dominates any conjunction over a subset of those variables.

**R2** The query representation $\forall x_1 x_2 x_3 \rightarrow h \ \forall x_1 x_2 \rightarrow h \ \forall x_1 \rightarrow h$ is equivalent to $\forall x_1 \rightarrow h \ \exists x_1 x_2 x_3 \rightarrow h$. This is because h has to be true whenever $x_1$ is true regardless of the true/false assignment of $x_2, x_3$. Thus a universal Horn expression with body variables B and head variable h dominates any universal Horn expression with body variables $B'$ and head variable h where $B' \supseteq B$.

**R3** The query representation $\forall x_1 \rightarrow h \ \exists x_1 x_2 h$ is equivalent to $\forall x_1 \rightarrow h \ \exists x_1 x_2 h$. Again, this equivalence is because h has to be true whenever $x_1$ is true.

### 2.1.2 Learning with Membership Questions

A membership question is simply an object along with its nested data tuples. The user responds to such a question by classifying the object as an answer or a non-answer for their intended query.

Given a collection of n propositions on the nested relation, the learning algorithm constructs a membership question in the Boolean domain: a set of Boolean tuples on n Boolean variables $x_1, \ldots, x_n$ — a variable for each proposition. Such a set is transformed into an object in the data domain before presentation to the user.

For brevity, we describe a membership question in the Boolean domain only. As a notational shorthand, we use 1st to denote a Boolean tuple where all variables are true. We use lowercase letters for variables and uppercase letters for sets of variables.

The following definitions describe two structural properties of qhorn queries that influence its learnability:

**Definition 2.5. Query size, k, is the number of expressions in the query.**

**Definition 2.6. Causal Density, $\theta$, is the maximum number of distinct non-dominated universal Horn expressions for a given head variable h.**

Conceptually, universal Horn expressions represent causation: whenever the body variables are true, the head variable has to be true. If a head variable has many universal Horn expressions, it has many causes for it to be true and thus has a high causal density.

The following inequality between causal density, $\theta$, and query size $k$ holds: $0 \leq \theta \leq k$. We would expect users’ queries to be small in size $k = O(n)$ and to have low causal density $\theta$.

A query class is efficiently learnable if (i) the number of membership questions that a learning algorithm asks the user is polynomial in the number of propositions n and query size $k$ and (ii) the learning algorithm runs in time polynomial in n and $k$. Question generation needs to be in polynomial time to ensure interactive performance. This requirement entails that the number of Boolean tuples per question is polynomial in n and $k$. A query class is exactly learnable if we can learn the exact target query that satisfies the user’s responses to the membership questions.

Due to the following theorem, qhorn cannot be efficiently and exactly learned with a tractable number of questions (even when query size is polynomially bounded in the number of propositions ($k = n$) and causal density has an upper bound of one ($\theta = 1$)).
THEOREM 2.1. Learning qhorn queries where variables can repeat \( r \geq 2 \) times requires \( \Omega(2^n) \) questions.

Proof: Suppose we split our \( n \) variables into two disjoint subsets: \( X, Y \). Consider the query class:

\[
\phi = \text{Uni}(X) \land \text{Alias}(Y)
\]

\[
\text{Uni}(X) = \forall x_1 \forall x_2 \ldots \forall x_{|X|}
\]

\[
\text{Alias}(Y) = \forall y_1 \rightarrow y_2 \forall y_2 \ldots y_{|Y|} \forall y_{|Y|} \rightarrow y_1
\]

\( \phi \) is simply the class of qHorn queries where some variables, \( X \), are universally quantified and bodyless and the other variables \( Y \) form an alias i.e. all variables in \( Y \) are either all true or all false. An example instance from this class of queries over variables \( \{x_1, x_2, \ldots, x_6\} \) is:

\[
\text{Uni}(\{x_1, x_3, x_5\}) \land \text{Alias}(\{x_2, x_4, x_6\}) = \forall x_1 \forall x_3 \forall x_5 \land \forall x_2 \rightarrow x_4 \forall x_3 \rightarrow x_6 \forall x_6 \rightarrow x_2.
\]

Only two questions satisfy the example instance (i) a question with only the Boolean tuple \( 1^6 \) and (ii) a question with tuples: \( \{1^6, 101010\} \).

There are \( 2^n \) query instances in the class \( \phi \). If we construct a membership question with only \( 1^n \) tuples then for all instances in \( \phi \), the question is an answer and we cannot learn the target query. If we augment the question with one tuple where some variables are false then all such questions are non-answers unless exactly the false variables are the alias variables. Augmenting the question with two or more tuples will always be a non-answer even if one of the tuples has exactly and only the alias variables set to false. This is because the other tuples either have some universally-quantified bodyless variables that are false or some alias variables that are true and some alias variables that are false.

This leaves us with \( 2^n \) membership questions where each question satisfies exactly one target query. Consider an adversary who always responds ‘non-answer’. In the worst case we have to ask \( 2^n - 1 \) questions as each question eliminates exactly one query from consideration as the target query. □

Qhorn’s intractability does not mean that we cannot construct efficiently and exactly learnable qhorn subclasses. We describe two such sub-classes:

2.1.3 Qhorn-1

Qhorn-1 defines certain syntactic restrictions on qhorn. Not counting guarantee clauses, if a query has \( k \) distinct expressions (\( 1 \leq k \leq n \)) and each expression \( i \) has body variables \( B_i \) and a head variable \( h_i \), such that \( B = B_1 \cup \ldots \cup B_k \) is the collection of all body variables and \( H = \{h_1, \ldots, h_n\} \) is the set of all head variables then the following restrictions hold in qhorn-1:

1. \( B_i \cap B_j = \emptyset \lor B_i = B_j \) if \( i \neq j \)
2. \( h_i \neq h_j \) if \( i \neq j \)
3. \( B \cap H = \emptyset \)

The first restriction ensures that different head variables can either share the exact same set of body variables or have disjoint bodies. The second restriction ensures that a head variable has only one body. Finally, the third restriction ensures that a head variable does not reappear as a body variable. Effectively, qhorn-1 has no variable repetition: a variable can appear once in a set of body variables or as a head variable. The following diagram labels the different components of a qhorn-1 query.

Note that qhorn-1 queries have a maximum query size \( k \) of \( n \) and have a causal density \( \theta \) of at most one. From an information-theoretic perspective, \( \Omega(n \lg n) \) membership questions are required to learn a target query in qhorn-1. This is because qhorn-1 has \( 2^{6n}(n \lg n) \) queries. We can think of all head variables that share the same set of body variables to be one part of a partition of the \( n \) Boolean variables. If we can construct a unique qhorn-1 query from every partition of \( n \) variables, then a lower bound on the number of queries is the Bell Number \( B_n \), i.e. the number of ways we can partition a set into non-empty, non-overlapping subsets. One way to construct a unique query for every partition is as follows:

1. We universally quantify all variables that appear in a singleton part: \( \forall x_1 \).
2. For all other parts, we pick any one variable as the head of an existentially quantified Horn expression with the remaining variables as body variables.

For example, we construct the query \( \forall x_1 \forall x_2 \exists x_3 \rightarrow x_4 \exists x_5 x_6 \rightarrow x_7 \) from the partition \( x_1 \mid x_3 x_4 \mid x_5 x_6 \mid x_7 \). Since \( \ln(B_n) = \Theta(n \ln n) \), a lower bound estimate on the number of queries in qhorn-1 is \( 2^{n \lg n} \).

Note that for each part, we can have either an existentially or a universal quantifier and we can set a variable’s role as either a head or a body variable. Since, we can have most \( n \) parts, an upper bound estimate on the number of queries is \( 2^n \times 2^n = 2^{2n} \). Thus the size of qhorn-1 is \( 2^{6n}(n \lg n) \).

2.1.4 Role-preserving qhorn

In role-preserving qhorn queries, variables can repeat many times, but across universal Horn expressions head variables can only repeat as head variables and body variables can only repeat as body variables. For example, the following query is in role-preserving qhorn

\[
\forall x_1 x_4 \rightarrow x_5 \forall x_2 x_4 \rightarrow x_5 \forall x_2 x_4 \rightarrow x_6 \exists x_1 x_2 x_3 x_4 \forall x_1 x_2 x_3 x_5 x_6
\]

while the following query is not in role-preserving qhorn

\[
\forall x_1 x_4 \rightarrow x_5 \forall x_2 x_3 x_5 \rightarrow x_6
\]

because \( x_5 \) appears both as a head variable and a body variable in two universally quantified Horn expressions. Existential Horn expressions in role-preserving qhorn are rewritten as existential conjunctions and variables do not have roles in these conjunctions. Thus, existential conjunctions can contain one or more head variables (e.g. \( \exists x_1 x_2 x_3 x_5 \) in the first query). The following diagram labels the different components of a role-preserving qhorn query.

Both query size and causal density play a role in the behavior of learning and verification algorithms. Once we remove the syntactic restriction of variables appearing at most once, the size of a target query instance is no longer polynomially bounded in \( n \). Thus, the complexity of learning and verification algorithms for role-preserving qhorn queries is
parameterized by $k$, $\theta$ and $n$. We would expect user queries to have low causal densities and to be small in size. Provided that $\theta$ has a constant upper bound, then we can efficiently learn role-preserving queries.

3. QUERY LEARNING

3.1 Learning qhorn-1

**Theorem 3.1.** $O(n \lg n)$ questions are sufficient to learn qhorn-1 queries in polynomial time.

*Proof:* The learning algorithm breaks down query learning into a series of small tasks. First, it classifies all variables into either universal head variables or existential variables (Fig. 2 describes qhorn-1 terminology). Second, it learns the body variables (if any) for each universal head variable. Finally, it learns existential Horn expressions. We show that each task requires at most $O(n \lg n)$ membership questions (Section 3.1.1 Lemmas 3.2 and 3.3), thus proving that the learning algorithm asks $O(n \lg n)$ questions. \hfill $\Box$

### 3.1.1 Learning universal head variables

The simplest learning task is to determine whether a variable is a universal head variable. Suppose we have three variables: $x_1, x_2, x_3$. To determine if $x_1$ is the head of a universal Horn expression, we ask the user if the set \{111,011\} is an answer. By setting the other variables \{$x_2, x_3$\} to be always true, we are setting all potential body variables of $x_1$ to true. We are also neutralizing the effect of other unknown head variables on the outcome of a membership question. If the set \{111,011\} is an answer, then we are sure that $x_1$ is not a universal head variable because it can exist with a false value as long as at least one tuple has a true value for it. If the set is a non-answer, then we learn that $x_1$ is a universal head variable.

We need one question to determine whether a variable is a universal head variable and we need $O(n)$ time to generate each question — the time to construct a set with two tuples of size $n$. Thus, we learn which variables are universal head variables, $U$, and which variables are existential variables, $E$, in polynomial time.

### 3.1.2 Learning body variables of universal Horn expressions

**Definition 3.1.** Given a universal head variable $h$ and a subset of existential variables $V \subseteq E$, a universal dependence question on $h$ and $V$ is a membership question with two tuples: 1\(^{st}\) and a tuple where $h$ and $V$ are false and all other variables are true.

**Algorithm 1** Find bodies of universal head variable

$h$: The universal head variable
$E$: The set of existential variables
$B$: the set of bodies learned so far \{B\}_1, B_2, ...

$b \leftarrow \text{Find}(\text{UniversalDependence}(h, ?), \text{Non-Answer}, B_1 \cup B_2 \cup \ldots \cup B_{|B|})$

if $b \neq \emptyset$ then
for $B \in B$ do
if $b \in B$ then
return $B$
end if
end for
end if

$B \leftarrow \text{FindAll}(\text{UniversalDependence}(h, ?), \text{Non-Answer}, E)$

return $B$

If a universal dependence question on $h$ and $V$ is an answer, then we learn that a subset of $h$'s body variables is in $V$. This is because when the conjunction of body variables is not satisfied, the head variable can be false. We say that $h$ depends on some variables in $V$. If the question is a non-answer, then we learn that $h$'s body variables are a subset of $E - V$; $h$ has no body variables in $V$ because in qhorn-1, $h$ can have at most one body.

The most straightforward way to learn the body variables, $B$, of one universal variable is with $O(|E|) = O(n)$ universal dependence questions: we serially test if $h$ depends on each variable $e \in E$. This means we use $O(n^2)$ questions to determine the body variables for all universal variables. We can do better.

**Algorithm 2** Find

$Q$: The question to ask
$r$: The response on which we eliminate a set of variables from further consideration
$V$: The variables to apply binary search within

if $\text{Ask}(Q(D)) = r$ then
return $\emptyset$
else
if $|D| = 1$ then
return $D$
else
Split $D$ into $D_1$ (1\(^{st}\) half) and $D_2$ (2\(^{nd}\) half)
$x \leftarrow \text{Find}(Q, r, D_1)$
if $x = \emptyset$ then
return $\text{Find}(Q, r, D_2)$
else
return $x$
end if
end if
end if

We perform a binary search for $h$'s body variables in $E$. If $h$ has $B$ body variables, we ask $O(|B| \lg n)$ instead of $O(n)$ questions to determine $B$. Suppose we have four variables $x_1, x_2, x_3, x_4$ such that $x_1$ is a universal head variable and all other variables are existential variables. $x_2, x_3, x_4$ are potential body variables for $x_1$. If the set \{1\(^{st}\), 0\(^{th}\}\} is a non-answer then $x_1$ is independent of all other variables and it has no body. If the set is an answer, we divide and conquer the variables. We ask if $x_1$ universally depends on half the variables, \{\$x_2, x_3\}, with the set \{1\(^{st}\), 0001\}. If the set is a non-answer then we eliminate half the variables, \{\$x_2, x_3\}, from further consideration as body variables. We know that
Algorithm 3 FindAll

Q: The question to ask
r: The response on which we eliminate a set of variables from further consideration
V: The variables to apply binary search within

if Ask(Q(D)) = r then
  return ∅
else if |D| = 1 then
  return D
else
  Split D into D₁ (1st half) and D₂ (2nd half)
  return FindAll(Q, r, D₁) ∪ FindAll(Q, r, D₂)
end if
end if

a body variable has to exist in the remaining half and since, x₄ is the last remaining variable, we learn the expression ∀x₄ → x₁. If the set {1ⁿ, 0001} is an answer, then we know at least one body variable exists in {x₂, x₃} and we continue the search for body variables in {x₂, x₃}, making sure that we also search the other half {x₄} for body variables.

**Lemma 3.2.** O(n lg n) universal dependence questions are sufficient to learn the body variables of all universal head variables.

**Proof:** Suppose we partition all variables into m non-overlapping parts of sizes k₁, k₂, ..., kₘ such that \(\sum_{i=1}^{m} k_i = n\). Each part has at least one body variable and at least one universal head variable. Such a query class is in qhorn-1 as all body variables are disjoint across parts and head variables cannot reappear as head variables for other bodies or in the bodies of other head variables.

Given a head variable hᵢ, we can determine its body variables Bᵢ using the binary search strategy above: we ask \(O(|Bᵢ| lg n)\) questions (it takes \(O(lg n)\) questions to determine one body variable). For each additional head variable, hᵢ′, that shares Bᵢ, we require at most \(1 lg n\) questions: we only need to determine that hᵢ′ has one body variable in the set Bᵢ. Thus to determine all variables and their roles in a part of size kᵢ with |Bᵢ| body variables and |Hᵢ| head variables we need \(O(|Bᵢ| lg n + |Hᵢ| lg 1 lg n) = O(kᵢ lg n)\) questions. Since there are m parts, we ask a total of \(O(\sum_{i=1}^{m} kᵢ lg n) = O(n lg n)\) questions. □

Since universal dependence questions consist of two tuples we only need O(n) time to generate each question. Thus, the overall running time of this subtask is in polynomial time.

### 3.1.3 Learning existential Horn expressions

After learning universal Horn expressions, we have established some non-overlapping distinct bodies and their universal head variables. Each variable in the remaining set of existential variables, can either be (i) an existential head variable of one of the existing bodies or (ii) an existential head variable of a new body or (i) a body variable in the new body. We use existential independence questions to differentiate between these cases.

**Definition 3.2.** Given two disjoint subsets of existential variables \(X \subset E, Y \subset E, X \cap Y = \emptyset\), an existential independence question is a membership question with two tuples: (i) a tuple where all variables \(x \in X\) are false and all other variables are true and (ii) a tuple where all variables \(y \in Y\) are false and all other variables are true.

If an independence question between two existential variables \(x\) and \(y\) is an answer then either:
1. \(x\) and \(y\) are existential head variables of the same body
2. \(x\) and \(y\) are not in the same Horn expression.

We say that \(x\) and \(y\) are independent of each other. Two sets \(X\) and \(Y\) are independent of each other if all variables \(x \in X\) are independent of all variables \(y \in Y\). Conversely, if an independence question between \(x\) and \(y\) is a non-answer then either:
1. \(x\) and \(y\) are body variables in the same body or
2. \(y\) is an existential head variable and \(x\) is in its body or
3. \(x\) is an existential head variable and \(y\) is in its body.

We say that \(x\) and \(y\) depend on each other. If both \(X\) and \(Y\) depend on each other then at least one variable \(x \in X\) depends on one variable \(y \in Y\).

Given an existential variable \(e\), if we discover that \(e\) depends on a body variable \(b\) of a known set of body variables \(B\), then we learn that \(e\) is an existential head variable in the Horn expression \(\exists B \rightarrow e\).

Otherwise, we find all existential variables \(D\) that \(e\) depends on. We can find all such variables with \(O(|D| lg n)\) existential independence questions using the binary search strategy of Section 3.1.2.

Knowing that \(D\) depends on \(e\) only tell us that one of the following holds: (i) A subset \(H\) of \(D\) are existential head variables for the body of \(e \cup (D - H)\) or (ii) \(e\) is a
head variable and \( D \) is a body. To differentiate between the two possibilities we make use of the following rule: If two variables \( x, y \) depend on \( z \) but \( x \) and \( y \) are independent then \( z \) is a body variable and \( x, y \) are head variables. If we find a pair of independent variables \( h_1, h_2 \) in \( D \), we learn that \( z \) must be a body variable. If we do not find a pair of independent variables in \( D \) then we may assume that \( x \) is an existential head variable and all variables in \( D \) are body variables.

After finding head variables in \( D \), we can determine the roles of the remaining variables in \( D \) with \( |D| = O(n) \) independence questions between \( h_1 \) and each variable \( d \in D - h_1 \). If \( h_1 \) and \( d \) are independent then \( d \) is an existential head variable, otherwise \( d \) is a body variable.

Our goal, therefore, is to locate a definitive existential head variable in \( D \) by searching for an independent pair of variables.

**Definition 3.3.** An independence matrix question on \( D \) variables consists of \( |D| \) tuples. For each variable \( d \in D \), there is one tuple in the question where \( d \) is false and all other variables are true.

Suppose we have four variables \( x_1, \ldots, x_4 \); \( D = \{x_2, x_3, x_4\} \) and \( D \) depends on \( x_1 \). \{1011, 1101, 1110\} is a matrix question on \( D \). If such a question is an answer then there is at least a pair of head variables in \( D \); the question will always contain a pair of tuples that ensure that each head and the body is true. For example if \( x_2 \), \( x_4 \) are head variables then tuples \{1011, 1110\} in the question satisfy the Horn expressions: \( x_3 \rightarrow x_2 \), \( \exists x_1 x_3 \rightarrow x_4 \). If at most one variable in \( D \) is a head variable, then there is no tuple in the matrix question where all body variables are true and the head variable is true and the question is a non-answer. For example, if only \( x_4 \) is a head variable, then the tuple, 1111 that satisfies the Horn expression \( \exists x_1 x_2 x_3 \rightarrow x_4 \) is absent from the question.

**Lemma 3.3.** Given an existential variable \( x \) and its dependents \( D \), we can find an existential head variable in \( D \) with \( O(|D| \log |D|) \) independence matrix questions of \( O(|D|) \) tuples each if at least two head variables exist in \( D \).

**Proof.** Consider the ‘GetHead’ procedure in Alg. 5 that finds an existential head variable in the set \( D \) of dependents of variable \( x \). The central idea behind the ‘GetHead’ procedure is if the user responds that a matrix question on \( D_1 \) \((D_1 \subseteq D)\) is an answer, then a pair of head variables must exist in \( D_1 \) and we can eliminate the remaining variables \( D - D_1 \) from further consideration. Otherwise, we know that at most one head variable exists in \( D_1 \) and another exists in \( D - D_1 \) so we can eliminate \( D_1 \) from further consideration and focus on finding the head variable in \( D - D_1 \).

Each membership question eliminates half the variables from further consideration as head variables. Thus, we require only \( O(|D| \log |D|) = O(|D| \log n) \) questions to pinpoint one head variable.

Then, we ask \( O(|D|) \) questions to differentiate head from body variables in \( D \). If we do not find head variables in \( |D| \) then we may assume that \( x \) is a head variable and all variables in \( D \) are body variables. Once we learn one existential Horn expression, we process the remaining existential variables in \( E \). If a variable depends on any one of the body variables, \( B \), of a learned existential Horn expression, it is a head variable to all body variables in \( B \).

**Algorithm 5 Get Head**

\[ x: \text{an existential variable} \]
\[ D: \text{the dependents of } x \text{, } |D| \geq 1 \]
\[ D_1 \leftarrow D, D_2 \leftarrow \emptyset, D_3 \leftarrow \emptyset \]
\[ \textbf{while } D_1 \neq \emptyset \textbf{ do} \]
\[ \text{isAnswer } \leftarrow \text{Ask}([\text{MatrixQuestion}(x, D_1)]) \]
\[ \text{if isAnswer then} \]
\[ \text{if } |D_1| = 2 \text{ and } D_2 = \emptyset \text{ then return } D_1 \]
\[ \text{else if } |D_1| > 2 \text{ and } D_3 = \emptyset \text{ then} \]
\[ \text{Split } D_1 \text{ into } D_1 \text{ (1st half)} \text{ and } D_3 \text{ (2nd half)} \]
\[ D_1 \leftarrow D_1 - D_3 \]
\[ \text{end if} \]
\[ \text{else} \]
\[ \text{if } D_3 = \emptyset \text{ then return } \emptyset \]
\[ \text{else if } |D_3| = 1 \text{ then return } D_3 \]
\[ \text{else} \]
\[ \text{Split } D_3 \text{ into } D_2 \text{ (1st half)} \text{ and } D_3 \text{ (2nd half)} \]
\[ D_1 \leftarrow D_1 \cup D_2 \]
\[ \text{end if} \]
\[ \text{end if} \]
\[ \text{end while} \]

Suppose a query has \( m \) distinct existential expressions with \( k_1, \ldots, k_m \) variables each, then \( \sum_{i=1}^{m} k_i < n \). The size of each set of dependendent variables for each expression \( i \) is \( k_i - 1 \). So the total number of questions we ask is \( \sum_{i=1}^{m} (O(k_i \log n) + O(k_i \log k_i) + O(k_i)) = O(n \log n) \) tuples of \( n \) variables each and therefore requires \( O(n^2) \) time to generate. If we limit the number of tuples per question to a constant number, then we increase the number of questions asked to \( O(n^2) \).

**Lemma 3.4.** \( \Omega(n^2) \) membership questions, with a constant number of tuples each, are required to learn existential expressions.

**Proof:** Consider the class of queries on \( n \) variables such that all variables in the set \( X - \{x_i, x_j\} \) are body variables and the pair \( x_i, x_j \) are head variables. Thus, the target class for our learning algorithm is the set of queries of this form:

\[ \exists C_{ij} \rightarrow x_i \land \exists C_{ij} \land \exists C_{ij} \rightarrow x_j \land \exists \exists C_{ij} \land x_j \]

where \( C_{ij} = X - \{x_i, x_j\} \) and \( \exists C_{ij} \land x_i, \exists C_{ij} \land x_j \) are guarantee clauses.

Any algorithm that learns such a query needs to determine exactly which of the possible \( \binom{n}{2} \) pairs of variables in \( n \) is the head variable pair. If we only have a constant number \( c \) of tuples to construct a question with, we need to choose tuples that provide us with the most information.

We can classify tuples as follows:

- **Class 1:** Tuples where all variables are true. Any question with such a tuple provides us with no information as the user will always respond that the set is an answer, regardless of the variable assignments in the other tuples.

- **Class 2:** Tuples where one variable is false. A question with one such tuple is always a non-answer as there are no tuples that satisfy \( \exists C_{ij} \land x_i \) \land \( \exists C_{ij} \land x_j \). We denote a tuple where \( x_i \) is false as \( T_i \). A question with two or more class-2 tuples is an answer if it has two tuples \( T_i, T_j \) such that \( x_i, x_j \) are the head variables. If such a question is a non-answer, then for all the tuples \( T_i, T_j \) in the question,
Class 3: Tuples where more than one variable is false. If a question has only class-3 tuples, then it will always be a non-answer. We denote a tuple where at least \( x_i, x_j \) are false as \( T_{ij} \). If \( x_i, x_j \) are head variables then we need two more tuples \( T_i \) and \( T_j \) for the question to be an answer. We cannot augment the question with more class-3 tuples to change it to an answer. Any other \( T_{ij} \) tuple, where \( T_{ij} \neq T_{ij} \) is bound to have at least one of its false \( x_i, x_j \) variables as body variables: this makes \( C_{ij} \) false, thus violating the clause \( \exists (C_{ij} \land x_i) \land \exists (C_{ij} \land x_j) \). So questions with only class-3 tuples will always be non-answers and we improve the information gain of a question with some class-3 tuples by replacing those tuples with class-2 tuples.

Therefore, with \( c \) tuples per question, we gain the most information from questions with only class-2 tuples. With \( c \geq 2 \), we construct a question with \( c \) tuples such that for each variable \( x_i \in H \), \( H \subset X \) and \( |H| = c \) the questions contains tuple \( T_i \). If the question is an answer, then we narrowed our search for the pair of head variables to the set \( H \). If the question is a non-answer, then we are sure that all pairs of \( x_i, x_j \) variables with tuples \( T_i, T_j \) in the question do not form the pair of head variables, so we eliminated \( \binom{c}{2} \) pairs from consideration as head variables. An adversary will always respond to such questions with ‘non-answer’. In the worst case, we have to ask \( \binom{c}{2} \approx \frac{n^2}{2} = \Omega(n^2) \) questions. 

3.2 Learning role-preserving qhorn

Since some queries are more complex than others within the role-preserving qhorn query class it is natural to allow our learning algorithm more time, more questions and more tuples per question to learn the more complex target queries. One can argue that such a powerful learning algorithm may not be practical or usable as it may ask many questions with many tuples each. If we assume that user queries tend to be simple (i.e., they are small in size \( k \) (Def. 2.5) and have low causal densities \( \theta \) (Def. 2.6)), then such an algorithm can be effective in the general case.

Role-preserving qhorn queries contain two types of expressions: universal Horn expressions \( \forall x_1, x_2, \ldots \rightarrow h \) and existential conjunctions \( \exists x_1, x_2, \ldots \) (Fig. 3 describes role-preserving qhorn terminology). In this section, we show that we can learn all universal Horn expressions with \( O(n^{d+1}) \) questions and all existential conjunctions with \( O(n \log n) \) questions. We show lower bounds of \( \Omega(n^{d-1}) \) for learning universal Horn expressions and \( \Omega(nk) \) for learning existential conjunctions. Since run-time is polynomial in the number of questions asked, our run-time is \( poly(nk) \) and \( poly(n^d) \) respectively. By setting a constant upper limit on the causal density of a head variable we can learn role-preserving qhorn queries in \( poly(nk) \) time.

We employ a Boolean lattice on the \( n \) variables of a query to learn the query’s expressions. Fig. 4 illustrates the Boolean lattice and its key properties. Each point in the lattice is a tuple of true or false assignments to the variables. A lattice has \( n+1 \) levels. Each level \( l \) starting from level 0 consists of tuples where exactly \( l \) variables are false. A tuple’s children are generated by setting exactly one of the true variables to false. Tuples at \( l \) have out-degree of \( n-l \), i.e., they have \( n-l \) children and in-degree of \( l \) or \( l \) parents.

A tuple has an upset and a downset. These are visually illustrated in Fig. 4. If a tuple is not in the upset or downset of another tuple, then these two tuples are incomparable.

Figure 4: The Boolean lattice on four variables.

The gist of our lattice-based learning algorithms is as follows:

1. We map each tuple in the lattice to a distinct expression. This mapping respects a certain generality ordering of expressions. For example, the lattice we use to learn existential conjunctions maps the top tuple in the lattice to the most specific conjunction \( \exists x_1, x_2, \ldots \) tuples in the level above the bottom of the lattice map to the more general conjunctions \( \exists x_1, \exists x_2, \ldots, \exists x_n \) (Def. 3.2.2). The exact details of this mapping for learning universal Horn expressions and learning existential conjunctions are described in the following section.

2. We search the lattice in a top-to-bottom fashion for the tuple that distinguishes or maps to the target query expression. The learning algorithm generates membership questions from the tuples of the lattice and the user’s responses to these questions either prune the lattice or guide the search.

3.2.1 Learning universal Horn expressions

We first determine head variables of universal Horn expressions. We use the same algorithm of (Def. 3.1). The algorithm uses \( O(n) \) questions. We then determine bodyless head variables. To determine if \( h \) is bodyless, we construct a question with two tuples: \( 1^* \) and a tuple where \( h \) and all existential variables are false and all other variables are true. If the question is a non-answer then \( h \) is bodyless. If \( h \) is not bodyless then we utilize a special lattice (Fig. 5) to learn \( h \)’s different bodies. In this lattice, we neutralize the effect of other head variables by fixing their value to true and we fix the value of \( h \) to false.

**Definition 3.4.** A universal Horn expression for a given head variable \( h \) is distinguished by a tuple if the true variables of the tuple represent a complete body for \( h \).

Thus, each tuple in the lattice distinguishes a unique universal Horn expression. For example, consider the target query:

\[ \forall x_1, x_4 \rightarrow x_5 \forall x_3, x_4 \rightarrow x_5 \forall x_1, x_2 \rightarrow x_6 \exists x_1, x_3 \exists x_2, x_4 \exists x_1, x_2 \exists x_2, x_5 x_6 \]

In the target query, the head variable \( x_3 \) has two universal Horn expressions:

\[ \forall x_1, x_4 \rightarrow x_5 \forall x_2, x_4 \rightarrow x_5 \]
In Fig. 5, we marked the two tuples that distinguish $x_5$’s universal Horn expressions: 100101 and 001101. Notice that the universal Horn expressions are ordered from most to least specific. For example the top tuple of the lattice in Fig. 5 is the distinguishing tuple for the expression $\forall x_1 x_2 x_3 x_4 \rightarrow x_5$. While the bottom tuple is the distinguishing tuple for the expression $x_5$. Our learning algorithm searches for distinguishing tuples of only dominant universal Horn expressions.

A membership question with a distinguishing tuple and the all-true tuple (a tuple where all variables are true) is a non-answer for one reason only: it violates the universal Horn expression it distinguishes. This is because the all-true tuple satisfies all the other expressions in the target query and the distinguishing tuple sets a complete set of body variables to true but the head to false. More importantly, all such membership questions constructed from tuples in the upset of the distinguishing tuple are non-answers and all questions constructed from tuples in the downset of the distinguishing tuple are answers. Thus, the key idea behind the learning algorithm is to efficiently search the lattice to find a tuple where questions constructed from tuples in the upset are non-answers and questions constructed from tuples in the downset are answers.

Algorithm 6 Learn one body for head variable $h$

- $h$: the head variable we are learning a body for
- $N$: the set of all non-head variables
- $H$: the set of all head variables

1. $X \leftarrow \emptyset \quad \triangleright \quad X$ holds all variables that are not in the body.
2. For $x \in N$ do
   1. SetTuple$(t_0, N \cup H, T)$ \quad \triangleright \quad This is the all-true tuple.
   2. $t_1 \leftarrow t_0$
   3. SetTuple$(t_1, X \cup \{x, h\}, F)$
   4. isAnswer $\leftarrow \text{Ask}(\{t_0, t_1\})$
   5. if not isAnswer then
      1. $X \leftarrow X \cup x$

   end if

end for

return $N \setminus X$

Once we find a body, we can safely eliminate its upset. Any body in the upset is dominated (Rule 2) by the discovered distinguishing tuple. Looking at Fig. 5, we notice that the upset simply contains all tuples where all body variables of the distinguishing tuple are true. The remaining lattice structure is rooted at tuples where one of the body variables is false. Since two incomparable bodies need to differ on at least one body variable, we set one body variable to false and search the resulting sub-lattices for bodies.

Theorem 3.5. $O(n^\theta)$ membership questions, where $\theta$ is the causal density of the given head variable $h$, are sufficient to learn the $\theta$ universal Horn expressions of $h$.

Proof: Let $b_i$ denote the number of body variables for each distinguishing tuple $t_i$ found. Initially we set $b_0$ to $n$ and we search the entire lattice or the $n$ sub-lattices rooted at the tuples where exactly one Boolean variable is false. In Fig. 5 those are the tuples at level 1: {011101, 101101, 110101, 111001, 111101}.

If the first distinguishing tuple found has $|B_1|$ true variables, then we need to search $|B_1|$ sub-lattices for bodies. For example, after finding the distinguishing tuple 001101, we continue searching for more distinguishing tuples from $|B_1| = 2$ roots: {110101, 111001}.

Suppose we find a second distinguishing tuple: 100101 with $B_2$ body variables; then we need to search for more bodies in the sub-lattices rooted at tuples where one of each body variable from the distinct bodies are set to false. Our new $|B_1| \times |B_2|$ roots are: {010101, 011001, 101011, 111011}. These search roots are illustrated in Fig. 5.

In the worst case, we ask $O(n)$ questions to find a body. Thus to determine all $\theta$ expressions for a universal head variable, an upper bound on the number of questions, $Q$, is:

$$Q \leq \sum_{i=1}^{\theta} (|B_1| \times |B_2| \times \ldots \times |B_0|)$$

$$Q \leq n^\theta + \sum_{b=1}^{\theta} \prod_{i=1}^{b} |B_i| \leq n^\theta + \sum_{i=1}^{\theta} (n^i) = O(n^\theta) \quad \square$$

Since there are $O(n)$ head variables and for each head variable we ask $O(n^\theta)$ questions to determine its universal Horn expressions, we learn all universal Horn expression with $O(n \times n^\theta) = O(n^{\theta+1})$ questions.
**Theorem 3.6.** $\Omega\left(\left(\frac{n}{m}\right)^{\theta - 1}\right)$ membership questions, where $\theta$ is causal density of $h$, are required to learn the $\theta$ universal Horn expressions of $h$.

Consider the class of role-preserving queries with $\theta$ universal Horn expressions and $n$ body variables where each expression $C_{i}$ for $1 \leq i < \theta$ consists of $\frac{n}{n-1}$ body variables, $B_{i}$, that determine $h$. The expression $C_{0}$ consists of $n - \theta - 1$ body variables, $B_{\theta}$, such that $|B_{\theta} \cap B_{i}| = \frac{n}{n-1} - 1$. The following is an example instance with $n = 12$ body variables and $\theta = 4$ expressions:

$$\forall x_{1}, x_{3}, x_{5}, x_{9} \rightarrow h \forall x_{2}, x_{4}, x_{6}, x_{10} \rightarrow h \forall x_{7}, x_{8}, x_{11}, x_{12} \rightarrow h$$

If we construct a tuple where two or more variables from each body $B_{i}$ and the head variable are false, then all $\theta - 1$ bodies and $B_{\theta}$ are not satisfied and a question consisting of such a tuple and the all-true tuple $(1, \ldots, 1)$ will always be an answer.

Alternatively, if we set all of variables of one body $B_{i}$ to true and the head variable to false then a question with such a tuple will always be a non-answer. Therefore, we can only set exactly one variable from each body to false to learn $B_{\theta}$. There are $\left(\frac{n}{n-1}\right)^{\theta - 1}$ choices for the body variable to be false, leaving us with $\left(\frac{n}{n-1}\right)^{\theta - 1}$ possible questions. If a question is an answer, we eliminate only one combination of body variables for $B_{\theta}$ and if the question is a non-answer then $B_{\theta}$ consists of all body variables that are true. In the worst-case the user responds that each question is an answer forcing the algorithm to ask $\left(\frac{n}{n-1}\right)^{\theta - 1} - 1 = \Omega\left(\left(\frac{n}{m}\right)^{\theta - 1}\right)$ questions.

### 3.2.2 Learning existential conjunctions

To learn existential conjunctions of a query we use the full Boolean lattice on all $n$ variables of a query (including head variables).

**Definition 3.5.** An existential conjunction $C$ is distinguished by a tuple if the true variables of the tuple are the variables of the conjunction.

Thus, each tuple in the lattice distinguishes a unique existential conjunction. For example, consider the target query:

$$\forall x_{1}, x_{4} \rightarrow x_{3} \forall x_{1}, x_{4} \rightarrow x_{5} \forall x_{1}, x_{2} \rightarrow x_{6}$$

$$\exists x_{1}, x_{3}, x_{5}, x_{6}, x_{2}, x_{4}, x_{5}, x_{6}, x_{3}, x_{2}, x_{5}, x_{6}$$

The conjunction $\exists x_{2}, x_{3}, x_{5}, x_{6}$ is distinguished by the tuple 011011 in a six-variable Boolean lattice.

Existential conjunctions are ordered from most to least specific on the lattice. For example, the top tuple 111111 of a six-variable Boolean lattice is the distinguishing tuple for the expression $\exists x_{1}, x_{2}, x_{3}, x_{4}, x_{5}, x_{6}$; the tuples $\{00001, 000010, 000100, 001000, 010000, 100000\}$ at level five of the lattice are the distinguishing tuples for the expressions $\exists x_{6}, \exists x_{5}, \exists x_{4}, \exists x_{3}, \exists x_{2}, \exists x_{1}$ respectively.

Our learning algorithm searches for distinguishing tuples of a normalized target query. For example, the target query above is normalized to the following semantically equivalent query using **Rule 3**.

$$\forall x_{1}, x_{4} \rightarrow x_{3} \forall x_{1}, x_{4} \rightarrow x_{5} \forall x_{1}, x_{2} \rightarrow x_{6}$$

$$\exists x_{1}, x_{2}, x_{3}, x_{5}, x_{6}, x_{2}, x_{3}, x_{4}, x_{5}, x_{6}, x_{2}, x_{3}, x_{5}, x_{6}$$

This query has the following dominant conjunctions (which include guarantee clauses):

$$\exists x_{1}, x_{5}, \exists x_{1}, x_{2}, x_{6}, \exists x_{2}, x_{4}, x_{5}, \exists x_{2}, x_{4}, x_{6}, \exists x_{2}, x_{4}, x_{5}, x_{6}$$

A membership question with all dominant distinguishing tuples of a query is an answer: all existential conjunctions (including guarantee clauses) are satisfied. For example, a question with the tuples: $\{010011, 011001, 011010, 011010\}$ is an answer for the target query above.

Replacing a distinguishing tuple with its children results in a non-answer: the existential conjunction of that tuple is no longer satisfied. For example replacing 011011 with its children $\{001011, 010011, 011001, 010101\}$ results in a membership question where none of the tuples satisfy the expression $\exists x_{2}, x_{3}, x_{5}, x_{6}$.

Replacing a distinguishing tuple with any tuple in its upset that does not violate a universal Horn expression still results in an answer.

Thus, the learning algorithm searches level-by-level from top-to-bottom for distinguishing tuples by detecting a change in the user's response to a membership question from answer to non-answer. The efficiency of the learning algorithm stems from pruning: when we replace a tuple with its children, we prune those down to a minimal set of tuples that still dominate all the distinguishing tuples.

We describe the learning algorithm (Alg. 7) with an example and then prove that the learning algorithm runs in $O(\ell n \log n)$ time (Theorem 3.8). We also prove the algorithm’s correctness (??) and provide a lower bound of $O(nk)$ for learning existential conjunctions (Theorem 3.9).

**Algorithm 7** Find Existential Distinguishing Tuples

$T \leftarrow \{1^n\}$  \quad \triangleright \text{The top tuple.}

$D \leftarrow \{\}$  \quad \triangleright D \text{ is the set of discovered distinguishing tuples.}

while $T \neq \emptyset$ do

$T' \leftarrow \{\}$

for $t \in T$ do

$C \leftarrow \text{Children}(t)$

$C \leftarrow \text{RemoveUniversalHornViolations}(C)$

$T \leftarrow T - \{t\}$

isAnswer $\leftarrow \text{Ask}(D \cup T \cup C \cup T')$

if isAnswer then

$T' \leftarrow T' \cup \text{Prune}(C, T \cup D)$

else

$D \leftarrow D \cup \{t\}$

end if

end for

$T \leftarrow T'$

end while

return $D$

Suppose we wish to learn the existential conjunctions of the target query listed in [2]. We use the six-variable Boolean lattice with the following modification: we remove all tuples that violate a universal Horn expression. These are tuples where the body variables of a universal Horn expression are true and the head variable is false. For example, the tuple 111111 violates $\forall x_{1}, x_{2} \rightarrow x_{6}$ is therefore removed from the lattice.

**Level 1:** We start at the top of the lattice. Since the tuple 111111 will satisfy any query, we skip to level one. We now construct a membership question with all the tuples of level 1 (after removing the tuples that violate universal Horn expressions: 111110, 111101, 110111, 101111, 011111).
Algorithm 8 Prune

$T$: the tuples to prune
$O$: other tuples
$K \leftarrow \{\}$ \quad \triangleright \quad K$ is the set of tuples to keep.

Split $T$ into $T_1$ ($1^{st}$ half) and $T_2$ ($2^{nd}$ half).

while $T_1 \cup T_2 \neq \emptyset$ do

    if isAnswer $\leftarrow$ Ask($T_1 \cup K \cup O$)

        if isAnswer then

            Split $T_2$ into $T_1$ ($1^{st}$ half) and $T_2$ ($2^{nd}$ half).

        end if

    else

        if $|T_2| = 1$ then

            $K \leftarrow K \cup T_2$

        else

            Add $1^{st}$ half of $T_2$ to $T_1$. Set $T_2$ to $2^{nd}$ half of $T_2$.

        end if

    end if

end while

return $K$

If such a question is a non-answer, then the distinguishing tuple is one level above and the target query has one dominant existential conjunction: $\exists x_1 x_2 x_3 x_4 x_5 x_6$.

If the question is an answer, we need to search for tuples we can safely prune. So we remove one tuple from the question set and test its membership. Suppose we prune the tuple 110111, the question is still an answer since all conjunctions of the target query are still satisfied: the remaining set of tuples still dominate the distinguishing tuples of the target query.

We then prune 011111. This question is a non-answer since no tuple satisfies the clause $\exists x_1 x_2 x_3 x_4 x_5$. We put 011111 back in and continue searching at level one for tuples to prune. We are left with the tuples: 111101, 110111 and 011111. Note that we asked $O(n)$ questions to determine which tuples to safely prune. We can do better. In particular, we only need $O(\log n)$ questions for each tuple we need to keep if we use a binary search strategy.

Level 2: We replace one of the tuples, 111011, with its children on level 2: $\{011011, 101011, 110011, 111011\}$. Note, that we removed 111010 because it violates $\forall x_1 x_2 \rightarrow x_6$. As before we determine which tuples we can safely prune. We are left with $\{100111, 110011\}$.

Similarly we replace 011111 with its children on level 2: $\{001111, 100111, 101011, 110111\}$. We did not consider 101101 because it violates $\forall x_3 x_4 \rightarrow x_5$. We can safely prune the children down to one tuple: 101110. We then replace 011111 with its children on level 2 and prune those down to $\{011011, 011110\}$. At the end of processing level 2, we are left with the tuples: $\{110011, 111001, 101110, 011011, 011110\}$. We repeat this process again now replacing each tuple, with tuples from level 3.

Level 3: When we replace 011101 with its children $\{001011, 010011, 011011, 011101\}$, we can no longer satisfy $\exists x_2 x_3 x_4 x_5$. The question is a non-answer and we learn that 011011 is a distinguishing tuple and that $\exists x_2 x_4 x_5$ is a conjunction in the target query. Note that we did not consider the child tuple 011100 because it violates the universal Horn expression $\forall x_2 x_4 \rightarrow x_5$. We fix 011110 in all subsequent membership questions.

When we replace 011011 with its children $\{001011, 010011, 110001\}$, we can no longer satisfy $\exists x_2 x_3 x_5 x_6$. Thus, the question is a non-answer and we learn that 110100 is a distinguishing tuple and that $\exists x_2 x_3 x_5 x_6$ is a conjunction in the target query. We fix 011010 in all subsequent membership questions.

When we replace 110001 with its children $\{010001, 100001, 110000\}$, we can no longer satisfy $\exists x_3 x_5 x_6$. The question is a non-answer and we learn that 110100 is a distinguishing tuple and that $\exists x_3 x_5 x_6$ is a conjunction in the target query. Note that we did not consider the tuple 111000 because it violates $\forall x_3 x_4 \rightarrow x_5$. We fix 110110 in all subsequent membership questions.

We can replace 101110 with the children $\{001110, 100110, 101010\}$. Note that the child 101100 is removed because it violates $\forall x_1 x_4 \rightarrow x_5$. We can safely prune the children down to one tuple 100110.

When we replace 100110 with its children $\{001010, 110010, 110101\}$, we can no longer satisfy $\exists x_1 x_3 x_5 x_6$. Thus, the question is a non-answer and we learn that 110110 is a distinguishing tuple. Note that we did not consider the tuple 100110 because it violates $\forall x_1 x_3 \rightarrow x_5$. At this stage, we are left with the following tuples:

$\{110011, 100110, 111001, 011011, 011110\}$

At this point, we can continue searching for conjunctions in the downside of 100110 which is the distinguishing tuple for a known guarantee clause for the universal Horn expression: $\forall x_1 x_4 \rightarrow x_5$. As an optimization to the algorithm, we do not search the downside because all tuples in the downside are dominated by 100110.

The algorithm terminates with the following distinguishing tuples $\{110011, 100110, 111001, 011011, 011110\}$.

We can relax the requirement of guarantee clauses for universal Horn expressions and our learning algorithms will still function correctly if they are allowed to ask about the membership of an empty set.
which represent the expressions:
\[ \exists x_1, x_2, x_3 \exists x_1, x_3 \exists x_1, x_2, x_3 \exists x_2, x_3, x_6 \exists x_2, x_3, x_5 \exists x_2, x_3, x_4, x_5 \]

**Theorem 3.7.** The lattice-based learning algorithm finds the distinguishing tuples of all dominant existential conjunctions of a normalized target query.

The algorithm will find a dominant distinguishing tuple if there exists a path from at least one tuple in its current pruned set of tuples to the distinguishing tuple. A path between two tuples \( t_0, t_1 \) is simply the sequence of variables to set to false to get from \( t_0 \) to \( t_1 \). When pruning, we ensure that the pruned set of tuples dominates all the distinguishing tuples: so there always exists a path from at least one tuple in the pruned set of tuples to a distinguishing tuple provided our lattice is complete.

To see that removing tuples that violate universal Horn expressions will not affect the existence of a path, suppose a tuple \( t_d \) in the pruned set dominates a distinguishing tuple \( t_d \) and \( t_a \) does not violate any universal Horn expressions. We relabel all the non-head variables in \( t_d \) to \( e_0^d...e_m^d \) and all the head variables to \( h_0^d...h_m^d \). Similarly, we relabel the non-head variables in \( t_d \) with \( e_0^d...e_m^d \) and the head variables with \( h_0^d...h_m^d \). Consider the tuple \( t_d \) has which the values of \( e_0^d...e_m^d \) for its non-head variables and the values of \( h_0^d...h_m^d \) for its head variables. Clearly there exists a path from tuple \( t_a \) to tuple \( t_d \) that does not encounter any tuples that violate universal Horn expressions because the head variables in both \( t_a \) and \( t_d \) have the same values.

Similarly, there exists a path from \( t_0 \) to \( t_d \). Since \( t_0 \) is in the upset of \( t_d \), \( h_0^d...h_m^d \) dominates \( h_0^d...h_m^d \). In a normalized query, all existential conjunctions are expanded to include head variables that are implied by the variables of a conjunction [Rule 3]. Thus, \( t_d \) does not violate any universal Horn expression and the path from \( t_0 \) to \( t_d \) only sets head variables to false that do not violate any universal Horn expressions.

Suppose the learning algorithm favors another tuple \( t_e \) (instead of \( t_d \)) such that \( e_0^d...e_m^d \) is in the downset of \( e_0^e...e_m^e \) and in the upset of \( e_0^d...e_m^d \) and \( h_0^d...h_m^d \) is in the downset of \( h_0^e...h_m^e \) and in the upset of \( h_0^d...h_m^d \). If the learning algorithm reaches \( t_e \), then \( t_e \) does not violate any universal Horn expressions and by induction (let \( t_a = t_e \)) there also exists a path from \( t_a \) to \( t_d \).

**Theorem 3.8.** The lattice-based learning algorithm asks \( O(kn \lg n) \) membership questions where \( k \) is the number of existential conjunctions.

**Proof:** Consider the cost of learning one distinguishing tuple \( t_l \) at level \( l \). From the top of the Boolean lattice to \( t_l \), there is at least one tuple \( t_i \) at each level \( l \) that we did not prune and we traversed down from to get to \( t_i \). Let \( N_i \) be the set of \( t_i \)'s siblings. At each level \( i \), we ask at most \( \log |N_i| \) questions, \( |N_i| = n - (i - 1) \) or the out-degree of \( N_i \)'s parent. In the worst-case, \( l = n \), and the cost of learning \( t_l \) is \( \sum_{i=1}^{n} \log(n - (i - 1)) \leq \sum_{i=1}^{n} \log n = O(n \lg n) \). With \( k \) distinguishing tuples we ask at most \( O(kn \lg n) \) questions.

**Theorem 3.9.** \( \Omega(nk) \) questions are required to learn existential conjunctions.

From an information theoretic perspective, \( \Omega(nk) \) questions is a lower bound on the number of questions needed to learn existential expressions. Consider level \( n/2 \) of the Boolean lattice, which holds the maximum number of non-dominated distinguishing tuples. There are \( \binom{n}{n/2} \) tuples at this level. Suppose we wish to learn \( k \) existential expressions at this level. There are \( \binom{n/2}{k} \) possible \( k \)-expressions. Since each question provides a bit of information, a lower bound on the number of questions needed is \( \lg \binom{n/2}{k} \).

\[
\lg \binom{n/2}{k} \geq \lg \binom{n^2/2}{k} \geq \lg \binom{n^2/2}{k} - k \lg n = \frac{nk}{2} - k \lg k
\]

4. **QUERY VERIFICATION**

A query verifier constructs a set of membership questions to determine whether a given query is correct. The verifier will not find an alternate query if the query is incorrect. Thus, while query learning is a search problem — a learner searches for the one correct query that satisfies the user’s responses to membership questions; query verification is the decision problem — a verifier decides if a given query is correct or incorrect given the user’s responses to membership questions.

Our approach to query verification is straightforward: for a given role-preserving qhorn query \( q_0 \), we generate a verification set of \( O(k) \) membership questions, where \( k \) is the number of expressions in \( q_0 \). Note that our learning algorithm for role-preserving qhorn queries asks \( O(n^{k+1} + kn \lg n) \) questions. If the user’s intended query \( q_i \) is semantically different from the given query \( q_0 \), then for at least one of the membership questions \( M \) in the verification set \( q_0(M) \neq q_i(M) \).

**Proposition 4.1.** A user’s intended query \( q_i \) is semantically different from a given query \( q_0 \) iff \( q_i \) and \( q_0 \) have distinct sets of existential [Def. 3.5] and universal [Def. 3.4] distinguishing tuples.

Suppose we try to learn the two role-preserving qhorn queries \( q_i \) and \( q_0 \). If \( q_i \) and \( q_0 \) are semantically different, then our learning algorithm will terminate with distinct sets of existential [Def. 3.5] and universal [Def. 3.4] distinguishing tuples for each query. The verification set consists of membership questions that detect semantic differences between two queries by detecting differences in their respective sets of distinguishing tuples. Fig. 6 lists six types of membership questions from which the verification algorithm constructs a verification set for a given query.

We explain how to construct each question for an example query in §4.2.

4.1 **Normalizing User-specified Queries**

All membership questions are constructed from dominant existential [Rule 1] or universal [Rule 2] expressions. A user-specified query \( q_0 \) may contain redundant or dominated expressions. For example \( \exists x_1, x_2 \exists x_3 \) is dominated by \( \exists x_1, x_2, x_3 \) [Rule 1] and is therefore redundant.

4.1.1 **Dominant Existential Distinguishing Tuples**

To find the dominant existential expressions a simple routine orders all existential expressions by the number of par-

---

3Since qhorn-1 is a sub-class of role-preserving qhorn, our verification approach works for both query classes.
A4 (ii) A tuple where all variables are true
         (i) A tuple for each non-head variable s such that s is false and all other variables are true

\[ \exists r_1 r_2 h \forall \alpha_1 \rightarrow h \text{(Rule 3)} \]

4.1.2 Dominant Universal Distinguishing Tuples

A user-specified query \( q \) may also contain redundant universal Horn expressions. For example \( \forall x_1 x_2 \rightarrow x_3 \) is dominated by \( \forall x_1 \rightarrow x_3 \) (Rule 2). To find the dominant universal distinguishing expressions, a simple routine orders all universal Horn expressions by the number of the participating variables from smallest to largest. For each expression in the ordered list, we remove all other expressions in the list whose participating variables are a superset of the variables of the current expression. This leaves us the set of dominant universal Horn expressions.

To construct a universal distinguishing tuple from a universal Horn expression, we set the head variable to false and all body variables of the expression to true. The remaining head variables are set to true and the remaining body variables are set to false.

4.2 Example Verification Set

We demonstrate the construction of a verification set on a role-preserving qhorn query with six Boolean variables \( x_1, \ldots, x_6 \). This is the same query that we previously learned in \([3.2.2]\):

\[ \forall \alpha_1 x_4 \rightarrow x_5 \forall \alpha_1 x_1 \rightarrow x_6 \forall \alpha_3 x_4 \rightarrow x_5 \exists r_1 x_2 x_3 \exists r_2 x_2 x_4 \exists r_1 x_2 x_5 \exists r_2 x_3 x_5 x_6 \]

[A1] For each existential expression and guarantee clause, we construct the following distinguishing tuples.

1. \( \exists x_1 x_2 x_3 \Rightarrow 110001 \text{ Do not violate } \forall x_1 x_2 \rightarrow x_6 \)
2. \( \exists x_2 x_3 x_4 \Rightarrow 011110 \text{ Do not violate } \forall x_2 x_4 \rightarrow x_5 \)
3. \( \exists x_2 x_5 \Rightarrow 110011 \text{ Do not violate } \forall x_2 x_5 \rightarrow x_6 \)
4. \( \exists x_2 x_3 x_5 x_6 \Rightarrow 011110 \text{ The guarantee clause of: } \)
5. \( \exists r_1 x_2 x_3 (x_6) \exists r_2 x_2 x_4 (x_5) \exists r_1 x_2 x_5 (x_6) \exists r_2 x_3 x_5 x_6 \)

\[
\begin{array}{cccc}
110001 & 111001 & 111001 & 111001 \\
101001 & 011110 & 011110 & 011110 \\
011001 & 010110 & 110001 & 011010 \\
110011 & 010110 & 011010 & 011010 \\
011011 & 011011 & 011011 & 011011 \\
100110 & 100110 & 100110 & 100110 \\
\end{array}
\]

[A2] For each dominant universal Horn expression, we construct the following distinguishing tuples:

\[
\begin{align*}
\forall x_2 x_4 & \rightarrow x_5 \Rightarrow 100101 \\
\forall x_2 x_4 & \rightarrow x_5 \Rightarrow 001101 \\
\forall x_1 x_2 & \rightarrow x_6 \Rightarrow 110010 \\
\end{align*}
\]

A2 questions consist of children of the universal distin-
guishing tuples:
\[
\begin{align*}
\forall x_1x_4 \rightarrow x_5 & \quad \forall x_3x_4 \rightarrow x_5 & \quad \forall x_1x_2 \rightarrow x_6 \\
111111 & \quad 111111 & \quad 111111 \\
100001 & \quad 001001 & \quad 100010 \\
001010 & \quad 000110 & \quad 010010
\end{align*}
\]

[N2] For each universal distinguishing tuple, we construct the following questions
\[
\begin{align*}
\forall x_1x_4 \rightarrow x_5 & \quad \forall x_3x_4 \rightarrow x_5 & \quad \forall x_1x_2 \rightarrow x_6 \\
111111 & \quad 111111 & \quad 111111 \\
100101 & \quad 001101 & \quad 110010 \\
011001 & \quad 100001 & \quad 001001 \\
110011 & \quad 100010 & \quad 001001
\end{align*}
\]

[A3] First, we find existential expressions that dominate the guarantee clauses of a universal Horn expression. \(\exists x_2x_3x_4x_5\) dominates the guarantee clause \(\exists x_3x_4x_5\) of \(\forall x_1x_4 \rightarrow x_5\). Note that \(\exists x_2x_3x_4x_5\) is implied by the expressions \(\exists x_3x_4 \forall x_2x_4 \rightarrow x_5\) in the query [Rule 3]. Second, we generate the question by generating search roots for the body \(x_3x_4\) within the sub-lattice rooted at 011101.

\[
\begin{align*}
\forall x_1x_4 \rightarrow x_5 & \quad \forall x_3x_4 \rightarrow x_5 & \quad \forall x_1x_2 \rightarrow x_6 \\
111111 & \quad 111111 & \quad 111111 \\
100101 & \quad 001101 & \quad 110010 \\
011001 & \quad 100001 & \quad 001001 \\
110011 & \quad 100010 & \quad 001001
\end{align*}
\]

If in the intended query \(x_5\) has another body in \(x_2x_3x_4\) that is incomparable with \(x_2x_4\) the above question will be a non-answer.

[A4] The query has four non-head variables \(\{x_1, x_2, x_3, x_4\}\).
So we construct the following question.

\[
\begin{align*}
\forall x_1x_4 \rightarrow x_5 & \quad \forall x_3x_4 \rightarrow x_5 & \quad \forall x_1x_2 \rightarrow x_6 \\
111111 & \quad 111111 & \quad 111111 \\
100101 & \quad 001101 & \quad 110010 \\
011001 & \quad 100001 & \quad 001001 \\
110011 & \quad 100010 & \quad 001001
\end{align*}
\]

4.3 Completeness of a Verification Set

![Fig. 7: Verification sets for each role-preserving qhorn query on two variables](image)

Fig. 7 illustrates the verification sets of all role-preserving queries on two variables. Fig. 8 illustrates which membership questions in the verification sets enable the user to detect a discrepancy between the given query and the query they actually intended. Note that with only two variables, there is no need to generate A3 questions. These examples serve to demonstrate the completeness of the verification sets at least for role-preserving queries on two variables.

![Fig. 8: Membership questions that detect a difference between two specific role-preserving qhorn queries on two-variables](image)

4.3 Completeness of a Verification Set

Given query \(q_0\) and the intended query \(q_i\) by surfacing differences between the sets of distinguishing tuples of \(q_0\) and \(q_i\).

Proof: Case 1: \(q_i\) and \(q_0\) have different sets of dominant existential distinguishing tuples then by Lemma 4.3 questions A1 and N1 surface differences in the sets of dominant existential distinguishing tuples of \(q_0\) and \(q_i\).

Case 2: \(q_i\) and \(q_0\) have different sets of dominant universal distinguishing tuples then

1. Both \(q_i\) and \(q_0\) classify \(h\) as a head variable. \(q_i\) has a dominant universal Horn expression \(C_i : \forall B_i \rightarrow h\) \((B_i\) is a set of body variables) and \(q_0\) has dominant universal Horn expressions of the form \(\forall B_0 \rightarrow h\).
(a) If for any \(B_i\) in \(q_0\), \(B_i \subset B_0\) or \(B_i \supset B_0\) then by Lemmas 4.4 and 4.5 questions A2 and N2 will surface this difference.
(b) If for all \(B_0\) in \(q_0\), \(B_i\) and \(B_0\) are incomparable then either (i) \(C_i\)’s guarantee clause dominates \(q_0\)’s existential expressions and \(q_i\)’s set of existential distinguishing tuples does not have the distinguishing tuple for \(C_i\)’s guarantee clause (See Case 1) or (ii) \(C_i\)’s guarantee clause is dominated by an existential expression in \(q_0\) and by Lemma 4.6 question A3 surfaces the difference.

2. \(h\) is a head variable in \(q_i\) but is a non-head variable in \(q_0\) then by Lemma 4.7 question A4 surfaces the difference.

Lemma 4.3. Let \(D_i\) be the set of \(q_i\)’s dominant existential distinguishing tuples and let \(D_0\) be the set of \(q_0\)’s dominant existential distinguishing tuples; membership questions A1 and N1 surface \(D_i \neq D_0\).

Proof: An existential distinguishing tuple represents an inflection point: all questions constructed with tuples in the distinguishing tuple’s upset are answers and all questions constructed with only tuples in the rest of the lattice are non-answers. We use this feature to detect if \(D_i \neq D_0\).

First, we define the following order relations over \(D_i\) and \(D_0\):

1. \(D_0 \leq D_i\) if for every tuple \(t_i \in D_i\), there exists a tuple \(t_0 \in D_0\) such that \(t_0\) is in the upset of \(t_i\).
Lemma 4.6. If
\- \( h \) is a head variable in \( q_1 \) and \( q_2 \).
\- \( q_1 \) has a dominant universal Horn expression \( \forall M \rightarrow h \) which \( q_2 \) does not have.
\- \( q_2 \) has universal Horn expressions \( \forall B_1 \rightarrow h \ldots \forall B_n \rightarrow h \).
\- \( B_i \mid M \) for \( i = 1 \ldots \theta \)
\- \( q_2 \) has an existential expression on \( C \) variables \( \exists C \) such that \( C \supseteq M \) and \( C \supseteq B_i \) for \( i = 1 \ldots \theta \).

then \( A3 \) surfaces a missing universal Horn expression \( \forall M \rightarrow h \) from \( q_2 \).

Proof: Consider \( q_2 \)'s universal Horn expressions whose guarantee clauses are dominated by \( \exists C \):
\[ \forall B_1 \rightarrow h \ldots \forall B_n \rightarrow h \]
\[ \text{such that } B_i \subseteq C \text{ for } i = 1 \ldots \theta \]
To build \( A3 \), we set one body variable from each of \( B_1, \ldots, B_n \) to false, the remaining variables in \( C \) to true and \( h \) to false. There are \( |B_1| \times |B_2| \times \ldots \times |B_n| = O(n^\theta) \) such tuples. \( A3 \) now consists of all such tuples and the all-true tuple.

\( A3 \) acts like the search phase of the learning algorithm that looks for new universal Horn expressions \( \text{[(3.2.1)]} \) \( A3 \) is a non-answer for \( q_2 \), as at least one of the tuples has all variables in \( M \) set to true (because \( M \mid B_i \) for \( i = 1 \ldots \theta \)) and \( h \) to false, thus violating \( \forall M \rightarrow h \).

Lemma 4.7. If \( h \) is a head variable in \( q_1 \), but not in \( q_2 \) then question \( A4 \) surfaces the difference.

Proof: The all-true tuple satisfies all existential expressions in \( q_2 \). For each body variable \( x \) in \( q_2 \), \( A4 \) has a tuple where \( x \) is false and all other variables are true. If \( x \) is a head variable in \( q_1 \), then \( A4 \) should be a non-answer.

This concludes the proof of Theorem 4.2.

5. RELATED WORK

Learning & Verifying Boolean Formula: Our work is influenced by the field of computational learning theory. Using membership questions to learn Boolean formulas was introduced in 1988 [2]. Angluin et al. demonstrated the polynomial learnability of conjunctions of (non-quantified) Horn clauses using membership questions and a more powerful class of questions known as equivalence questions [3].

The learning algorithm runs in time \( O(k^2n^2) \) where \( n \) is the number of variables and \( k \) is the number of clauses. Interestingly, Angluin proved that there is no PTIME algorithm for learning conjunctions of Horn clauses that only uses membership questions. Angluin et al.’s algorithm for learning conjunctions of Horn formula was extended to learn first-order Horn expressions [11, 9]. First-order Horn expressions contain quantifiers. We differ from this prior work in that in horn we quantify over tuples of an object’s nested relation; we do not quantify over the values of variables. Our syntactic restrictions on horn have counterparts in Boolean formulas. Both horn-l and read-once Boolean formulas [4] allow variables to occur at most once. Both role-preserving horn queries and depth-1 acyclic Horn formulas [5] do not allow variables to be both head and body variables.

Verification sets are analogous to the teaching sequences of Goldman and Kearns [2]. A teaching sequence is the smallest sequence of classifications examples a teacher must reveal to a learner to help it uniquely identify a target concept from
a concept class. Prior work provides algorithms to determine the teaching sequences for several classes of Boolean formula \[3, 7, 16\] but not for our class of qhorn queries.

**Learning in the Database Domain:** Two recent works on example-driven database query learning techniques — Query by Output (QBO) \[19\] and Synthesizing View Definitions (SVD) \[6\] — focus on the problem of learning a query \(Q\) from a given input database \(D\), and an output view \(V\). There are several key differences between this body of work and ours. First, QBO and SVD perform as decision trees; they infer a query’s propositions so as to split \(D\) into tuples in \(V\) and tuples not in \(V\). We assume that users can provide with us the propositions, so we focus on learning the structure of the query instead. Second, we work on a different subset of queries: QBO infers select-project-join queries and SVD infers unions of conjunctive queries. Learning unions of conjunctive queries is equivalent to learning \(k\)-term Disjunctive Normal Form (DNF) Boolean formulae \[10\]. We learn conjunctions of quantified Horn formulae. Since our target queries operate over objects with nested-sets of tuples instead of flat tuples, we learn queries in an exponentially larger query and data space. Finally, QBO and SVD work with a complete mapping from input tuples to output tuples.

Our goal, however, is to learn queries from the smallest possible mapping of input to output objects, as it is generally impractical for users to label an entire database of objects as answers or non-answers. We point out that we synthesize our input when constructing membership questions, thus we can learn queries independent of the peculiarities of a particular input database \(D\).

Using membership (and more powerful) questions to learn concepts within the database domain is not novel. For example, Cate, Dalmau and Kolaitis use membership and equivalence questions to learn schema mappings \[18\]. A schema mapping is a collection of first-order statements that specify the relationship between the attributes of a source and a target schema. Another example is Staworko’s and Wieczorek’s work on using example XML documents given by the user to infer XML queries \[17\]. In both these works, the concept class learned is quite different from the qhorn query class.

**The Efficacy of Membership Questions** Learning with membership questions is also known as Active Learning. Active learning elicits several criticisms due to mixed or negative results in some learning problems. We wish to address two of the main criticisms:

1. **Arbitrary Examples.** Early work by Lang and Baum \[13\] used membership questions to train a neural network to recognize hand-written digits. They discovered that users couldn’t reliably respond to the questions — images of artificially synthesized hybrids of two digits. This canonical negative result does not apply to our work. We synthesize examples from the actual data domain. Moreover, if we have a rich database, we can select instances from the database that match our synthesized Boolean tuples instead of synthesizing the data tuples.

2. **Noisy Users.** The criticism made here is that (i) users may not know have a clear idea of what constitutes a positive (answer) or negative (non-answer) example or (ii) users make mistakes. In query specification tasks, users typically have a clear idea of what they are looking for. This contrasts with data exploration tasks, where users search the database without a well-defined selection criteria. A good user-interface can ameliorate the second issue. For example, if we provide users with a history of all their responses to the different membership questions, users can double-check their responses and change an incorrect response. This triggers the query learning algorithm to restart query learning from the point of error.

A survey by Settles discusses recent advances and challenges in active learning \[15\].

**6. CONCLUSION & FUTURE WORK**

In this paper, we have studied the learnability of a special class of Boolean database queries — qhorn. We believe that other quantified-query classes (other than conjunctions of quantified Horn expressions) may exhibit different learnability properties. Mapping out the properties of different query classes will help us better understand the limits of example-driven querying. In our learning/verification model, we made the following assumptions: (i) the user’s intended query is either in qhorn-1 or role-preserving qhorn, (ii) the data has at most one level nesting. We plan to design algorithms to verify that the user’s query is indeed in qhorn-1 or role-preserving qhorn. We have yet to analyze the complexity of learning queries over data with multiple levels of nesting. In such queries, a single expression can have several quantifiers.

We plan to investigate *Probably Approximately Correct* learning: we use randomly-generated membership questions to learn a query with a certain probability of error \[20\]. We note that membership questions provide only one bit of information — a response to membership question is either ‘answer’ (1) or ‘non-answer’ (0). We plan to examine the plausibility of constructing other types of questions that provide more information bits but still maintain interface usability. One possibility is to ask questions to directly determine how propositions interact such as: “do you think \(p_1\) and \(p_2\) both have to be satisfied by at least one tuple?” or “when does \(p_1\) have to be satisfied?”

Finally, we see an opportunity to create efficient *query revision* algorithms. Given a query which is *close* to the user’s intended query, our goal is to determine the intended query through few membership questions — polynomial in the distance between the given query and the intended query. Efficient revision algorithms exist for (non-quantified) role-preserving Horn formula \[8\]. The Boolean-lattice provides us with a natural way to measure how close two queries are: the distance between the distinguishing tuples of the given and intended queries.

**Acknowledgments** Partial funding provided by NSF Grants CCF-0963922, CCF-0916389, CC-0964033 and a Google University Research Award.

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