CP violation from new quarks in the chiral limit

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Abstract

We characterize CP violation in the SU(2) × U(1) model due to an extra vector-like quark or sequential family, giving special emphasis to the chiral limit \( m_u,d,s = 0 \). In this limit, CP is conserved in the three generation Standard Model (SM), thus implying that all CP violation is due to the two new CP violating phases whose effects may manifest either at high energy in processes involving the new quark or as deviations from SM unitarity equalities among imaginary parts of invariant quartets (or, equivalently, areas of unitarity triangles). In our analysis we use an invariant formulation, independent of the choice of weak quark basis or the phase convention in the generalized Cabibbo-Kobayashi-Maskawa matrix. We identify the three weak-basis invariants, as well as the three imaginary parts of quartets \( B_{1-3} \) which, in the chiral limit, give the strength of CP violation beyond the SM. We find that for an extra vector-like quark \( |B_i| \leq 10^{-4} \), whereas for an extra sequential family \( |B_i| \leq 10^{-2} \).

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1 Introduction

In the Standard Model (SM) CP violation is parametrized by one CP violating phase in the Cabibbo-Kobayashi-Maskawa (CKM) matrix \([1]\). Although this phase accounts for the observed CP violation in the \( K^0-K^0 \) system \([2]\), there is no deep understanding of CP violation. Furthermore, it has been established that the amount of CP violation present in the SM is not sufficient to generate the baryon asymmetry of the universe \([3]\). This provides motivation for looking for new sources of CP violation which can lead to deviations from the SM predictions for CP asymmetries in \( B^0 \) decays and/or to new signals of CP violation observable at high energy, in future colliders.

In the SM the CKM matrix \( V_{CKM} \) is a 3 × 3 unitary matrix whose matrix elements \( V_{ij} \) are strongly constrained by unitarity which implies, for example, that the imaginary
parts of all invariant quartets, $\text{Im} \, V_{ij} V_{kl}^* V_{il}^* V_{aj}^* \ (i \neq k, \ j \neq l)$, are equal up to a sign. In particular, one has

\begin{align*}
T_1 & \equiv \text{Im} \, V^*_{ud} V^*_{cb} V^*_{ub} + \text{Im} \, V_{us} V_{cs} V_{cb} V_{ub}^* = 0, \\
T_2 & \equiv \text{Im} \, V^*_{ud} V^*_{tb} V_{ub} + \text{Im} \, V_{us} V_{ts} V_{tb} V_{ub}^* = 0, \\
T_3 & \equiv \text{Im} \, V^*_{cd} V^*_{tb} V_{cb} + \text{Im} \, V_{cs} V_{ts} V_{tb} V_{cb}^* = 0. \quad (1)
\end{align*}

In the SM one may have CP violation in the limit $m_{u,d} = 0$, but degeneracy of two quarks of the same charge does imply CP invariance. Hence in the chiral limit $m_{u,d,s} = 0$, with $d$ and $s$ quark masses degenerate, CP violation can only originate in physics beyond the SM. These CP properties of the SM are summarized through a necessary and sufficient condition for CP invariance, expressed in terms of a weak quark basis invariant \[4, 5\]

\[ I \equiv \det [M_u M_d^\dagger, M_d M_d^\dagger] = \frac{1}{3} \text{tr} [M_u M_u^\dagger, M_d M_d^\dagger]^3 \]

\[ = -2i(m_t^2 - m_c^2)(m_c^2 - m_u^2)(m_b^2 - m_s^2) \times (m_b^2 - m_d^2)(m_s^2 - m_d^2) \text{Im} \, V_{ud} V^*_{cd} V_{cs} V_{us}^* = 0, \quad (2) \]

where $M_u, M_d$ are the up and down quark mass matrices and $m_i$ the mass of the quark $i$. Although for three generations the two invariants of Eq. (2) are proportional to each other, $\text{tr} [M_u M_u^\dagger, M_d M_d^\dagger]^3 = 0$ has the advantage of being a nontrivial necessary condition for CP invariance in the SM, for an arbitrary number of generations \[5\].

Within the three generation SM, CP violation in the B system is not suppressed by the factor $(m_s - m_d)$ due to the fact that one is able to distinguish $B_d$ from $B_s$ in the initial state and kaons from pions in the final state. This flavour identification is crucial in order to detect CP violation effects arising from gauge interactions \[5, 6, 7, 8, 9\]. At high energy colliders, the natural asymptotic states are no longer hadronic states, but quark jets \[10\]. Now, at high energies, it will be very hard, if not impossible, to identify the flavour of light quark jets $(d, s, u)$. In this limit, CP violation can only originate in physics beyond the SM. Indeed many simple SM extensions incorporate new sources of CP violation which may be observable at future colliders in the chiral limit \[1\]. In this paper we concentrate on the case of extra quarks, with special emphasis on vector-like quarks, i. e. quarks whose left-handed and right-handed components are in the same type of multiplet. Vector-like quarks naturally arise in various extensions of the SM, for example in grand-unified theories based on $E_6$, as well as in other superstring inspired extensions of the SM. In most cases, these vector-like quarks are isosinglets and their mass could be of the order of the electroweak scale \[2\]. A new sequential quark family is also allowed \[16\], although precision electroweak data put an upper limit on their square mass difference \[2\]. We will show in this paper that with the adition of an extra quark, either sequential
or vector-like, there is CP violation even in the limit where $m_{u,d,s,c} = 0$. Therefore, in these simple extensions of the SM, CP violation effects can be seen even if only the flavour of heavy quarks ($b$, $t$, $b'$) is identified. In both cases, for an isosinglet quark and for a sequential extra family, CP violation mediated by gauge bosons is parametrized by a $4 \times 4$ unitary matrix $V$ defined up to quark mass eigenstate phase redefinitions. For a new down (up) vector-like quark the charged couplings are described by the CKM matrix $V_{\text{CKM}}$, the first 3 rows (columns) of $V$. But these 3 rows (columns) $V_{\text{CKM}}$ completely fix $V$. The neutral couplings are a function of the $V_{\text{CKM}}$ matrix elements and are not independent. In the case of an isosinglet quark, the neutral couplings are no longer diagonal, i.e. there are flavour changing neutral currents (FCNC). The $3 \times 3$ block of the CKM matrix connecting standard quarks is no longer unitary either, but deviations from unitarity are naturally suppressed by powers of $m/M$, where $m$ is a standard quark mass and $M$ denotes the mass of the isosinglet quark. The strength of FCNC among standard quarks is also suppressed by powers of $m/M$, since FCNC are proportional to deviations from $3 \times 3$ unitarity in $V_{\text{CKM}}$. For a new sequential family $V_{\text{CKM}} = V$ and the neutral couplings are diagonal and real. It turns out that in both cases there are three CP violating phases in $V_{\text{CKM}}$.

In our analysis we will adopt the following strategy: First, we identify a set of weak-basis invariants which completely specify the properties of the model considered in the sense that if any one of the invariants is nonzero there is CP violation, while if all the invariants of the set vanish there is CP invariance. These weak-basis invariants are physically meaningful quantities, and they are the analog of the invariant in Eq. (2) for the class of models we are considering. They can be expressed in terms of quark masses and the imaginary parts of various invariant products of CKM matrix elements. We then study the chiral limit $m_{u,d,s} = 0$, starting with the simpler case $m_{u,d,s,c} = 0$. Both are especially interesting since in these cases the three generation SM conserves CP, thus implying that in the chiral limit CP violation arises exclusively from physics beyond the SM. The chiral limit is not only physically natural for studying CP violation beyond the SM but phenomenologically relevant at high energy, where the light fermion masses are negligible. The above mentioned weak-basis invariants are especially useful in the analysis of the chiral limit, allowing one to readily identify which $\text{Im} \ V_{ij} V_{kj}^* V_{kl} V_{il}^*$ continue being physically meaningful and nonvanishing when taking these degenerate mass limits. In a $4 \times 4$ unitary matrix 9 of these imaginary products are independent, and all of them can be made to vanish if CP is conserved. In the chiral limit $m_{u,d,s} = 0$, there are two CP violating phases and three independent imaginary products physically relevant. If $m_c$ is also neglected compared to $m_t$, $m_{u,d,s,c} = 0$, there is one CP violating phase left and one independent imaginary product physically significant. These imaginary products

\[
B_1 \equiv \text{Im} \ V_{cb} V_{4b}^* V_{4b'} V_{cb'}^* = T_1 - T_3, \\
B_2 \equiv \text{Im} \ V_{tb} V_{4b}^* V_{4b'} V_{tb'}^* = T_2 + T_3,
\]
which survive in the chiral limit and which involve mixings in the heavy quark sector (t, b, c and the new quark(s)), can also be expressed in terms of the rephasing invariants $T_i$ defined in Eqs. (3). (We will use for the subindex of the fourth row ‘4’ and ‘$t'$’ when referring to the vector-like and sequential cases, respectively. When referring to both we will also use ‘4’.) These invariants $T_i$ only involve mixings among standard quarks and they vanish in the SM. Thus, the effects of physics beyond the SM may be seen measuring $B_i$ at high energy in processes involving new quarks or at low energy through the nonvanishing of $T_i$. We shall show that the present bounds on $|B_i|$ are $10^{-2}$ and $10^{-4}$ for a fourth family and a new vector-like quark, respectively. In our analysis, we will only take into account CP violating effects arising from gauge interactions (in some specific models with isosinglet quarks the Higgs sector is more involved than in the SM, leading to new CP violating contributions from scalar interactions) and we will neglect the Higgs contributions to CP violation.

It may be worth to emphasize that the invariant formulation of CP violation requires an educated use of symbolic programs [20]. However to go beyond the simplest cases is difficult for as explained in the Appendix the number of invariants needed to get a complete set grows very rapidly with the number of phases. We study the simplest cases of a new vector-like quark or an extra sequential family, deriving limits for observables involving known particles as final states. If there exist more vector-like or sequential quarks, larger CP violating effects than the ones studied here are possible but in observables involving several of these new quarks.

This paper is organized as follows. In Section 2 we set our notation and for the case of an extra down (up) quark isosinglet we propose a complete set of weak-basis invariant conditions which are necessary and sufficient for CP invariance. We also give the explicit expressions of the invariants in terms of quark masses and imaginary parts of invariant products. The proof that these invariant conditions form a complete set is given in the Appendix. The corresponding set of invariants for the case of a sequential family was discussed in Ref. [19]. In Section 3 we use these invariants to study the simplest case of two degenerate (massless) up and down quark masses, $m_{u,d,s,c} = 0$, and the chiral limit, $m_{u,d,s} = 0$. In Section 4 we discuss the corresponding geometrical description of CP violation with triangles and quadrangles and the bounds on the CP violating effects of these new fermions commenting on the prospects to measure them (at large colliders). Section 5 is devoted to our conclusions.
2 Characterization of CP violation for extra quarks

Let us consider the SM with $N$ standard families plus $n_d$ down quark isosinglets (the case of $n_u$ up quark isosinglets is similar). In the weak eigenstate basis the gauge couplings to quarks and the mass terms are

$$\mathcal{L}_{\text{gauge}} = -\frac{g}{\sqrt{2}} \left[ \left( \bar{u}_L \gamma^\mu d_L^{(d)} \right) W_\mu^\dagger + \text{h.c.} \right] - \frac{g}{2c_W} \left( \bar{u}_L \gamma^\mu u_L^{(d)} - \bar{d}_L \gamma^\mu d_L^{(d)} - 2s_W^2 J_{\text{EM}}^\mu \right) Z_\mu - e J_{\text{EM}}^\mu A_\mu ,$$

$$\mathcal{L}_{\text{mass}} = - \left( \bar{u}_L M_u^{(s)} + \bar{d}_L M_d^{(s)} + \bar{s}^{(s)} L M_d^{(s)} \right) + h.c. ,$$

where $u_L^{(d)}$, $d_L^{(s)}$ are $N$ SU(2)$_L$ doublets, $d_L^{(s)}$ are $n_d$ SU(2)$_L$ singlets and $u_R^{(s)}$ and $d_R^{(s)}$ are $N$ and $N + n_d$ SU(2)$_L$ singlets, respectively, and $J_{\text{EM}}^\mu = \frac{2}{3} \bar{u}_L \gamma^\mu u - \frac{1}{3} \bar{d}_L \gamma^\mu d$. Hence, the up and down quark mass matrices are

$$M_u = M_u , \quad M_d = \left( \begin{array}{c} M_d \\ m_d \end{array} \right) ,$$

with $M_u$, $M_d$ and $m_d$ submatrices of dimension $N \times N$, $N \times (N + n_d)$ and $n_d \times (N + n_d)$, respectively. The weak quark basis can be transformed by unitary matrices without changing the physics. Under these unitary transformations

$$q_L^{(d)} \rightarrow U_L q_L^{(d)} , \quad d_L^{(s)} \rightarrow U_L d_L^{(s)} , \quad q_R^{(s)} \rightarrow U_R q_R^{(s)} ,$$

with $q = u, d$, the mass matrices transform as

$$M_q \rightarrow U_L M_u U_R^{\dagger} , \quad m_d \rightarrow U_L^\dagger m_d U_R^{\dagger} ,$$

whereas the gauge couplings remain unchanged. Then the mass matrices are defined up to the unitary transformations in Eq. \(8\).

A set of physical parameters can be defined using the mass eigenstate basis. We assume without loss of generality $M_u = D_u$ diagonal and $M_d = V D_d V^\dagger$, with $V$ unitary and $D_d$ diagonal (remember that we can always assume $M_u$ and $M_d$ hermitian with nonnegative eigenvalues by choosing $U_R^{\dagger}$ appropriately). We define the $N \times (N + n_d)$ matrix $V_{\text{CKM}}$ as the first $N$ rows of the $(N + n_d) \times (N + n_d)$ unitary matrix $V$, and the $(N + n_d) \times (N + n_d)$ matrix $X \equiv V_{\text{CKM}}^\dagger V_{\text{CKM}}$. Then the Lagrangian in Eqs. \(4,5\) reads in the quark mass eigenstate basis (where no superscripts are needed)

$$\mathcal{L}_{\text{gauge}} = -\frac{g}{\sqrt{2}} \left( \bar{u}_L \gamma^\mu V_{\text{CKM}} d_L W_\mu^\dagger + \text{h.c.} \right) - \frac{g}{2c_W} \left( \bar{u}_L \gamma^\mu u_L - \bar{d}_L \gamma^\mu X d_L - 2s_W^2 J_{\text{EM}}^\mu \right) Z_\mu - e J_{\text{EM}}^\mu A_\mu ,$$

$$\mathcal{L}_{\text{mass}} = - \left( \bar{u}_L D_u u_R + \bar{d}_L D_d d_R \right) + h.c. .$$
In this basis one can make the counting of CP violating phases. This is equal to the number of phases in $V_{CKM}$ minus the number of independent phase field redefinitions [14],

$$n_{CP} = N(N + n_d) - \frac{N(N - 1)}{2} - (2N + n_d - 1)$$

$$= (N - 1)n_d + \frac{1}{2}(N - 1)(N - 2).$$

(11)

CP is conserved if $V_{CKM}$ can be made real. This is the case if all $n_{CP}$ phases vanish. In the SM, $N = 3$, $n_d = 0$, there is only 1 CP violating phase. For one extra quark, $N = 4$, $n_d = 1$, there are already 3 CP violating phases, the same as for an extra family, $N = 4$, $n_d = 0$. The number of physical parameters, and in particular of CP violating phases, grows rapidly with the addition of more quarks. We will stick to these cases which incorporate many of the new features of the addition of new quark fields. For an extra down quark isosinglet, $V_{CKM}$ consists of the first 3 rows of a $4 \times 4$ unitary matrix which can be parametrized as (we explicit it for later use) [15]

$$V = \begin{pmatrix}
  c_1 & -s_1c_3 & s_1s_3 & s_1s_3c_5e^{i\delta_1} \\
  -s_1c_2 & c_1c_2c_3 + s_2s_3c_6e^{i\delta_2} & c_1c_2s_3c_5 - s_2c_3c_5c_6e^{i\delta_1} & c_1c_2s_3c_5 - s_2c_3c_5c_6e^{i\delta_1} + s_2s_3c_6e^{i(\delta_1 + \delta_2)} \\
  -s_1s_2c_4 & c_1s_2c_3c_4 - c_2s_3c_6e^{i\delta_1} & c_1s_2c_3c_4 - c_2s_3c_6e^{i\delta_1} - c_2s_4c_5c_6e^{i(\delta_1 + \delta_2)} & c_1s_2c_3c_4 - c_2s_3c_6e^{i\delta_1} - c_2s_4c_5c_6e^{i(\delta_1 + \delta_2)} + c_4s_5c_6e^{i\delta_2} \\
  -s_1s_2s_4 & c_1s_2c_3s_4 - c_2s_3c_6e^{i\delta_1} + s_3c_4s_6e^{i\delta_2} & c_1s_2s_3c_4c_5c_6e^{i\delta_1} - c_2s_4s_5c_6e^{i(\delta_1 + \delta_2)} & c_1s_2s_3c_4c_5c_6e^{i\delta_1} - c_2s_4s_5c_6e^{i(\delta_1 + \delta_2)} & -c_3s_4c_5c_6e^{i(\delta_2 + \delta_3)} \\
  +s_3c_4c_6e^{i\delta_2} & c_1s_2c_3s_4c_5c_6e^{i\delta_1} - c_2s_4s_5c_6e^{i(\delta_1 + \delta_2)} & -c_3c_4c_5c_6e^{i(\delta_2 + \delta_3)} & c_1s_2s_3c_4c_5c_6e^{i\delta_1} - c_2s_4s_5c_6e^{i(\delta_1 + \delta_2)} & -c_3c_4c_5c_6e^{i(\delta_2 + \delta_3)} + c_4c_5c_6e^{i(\delta_2 + \delta_3)} 
\end{pmatrix}.$$  

(12)

Note that the first 3 rows completely fix $V$. On the other hand, $V$ is the CKM matrix for 4 families. In the limit where the new quark does not mix with the standard quarks (i.e., $s_4 = s_5 = s_6 = 0$), the $3 \times 3$ block of $V$ becomes just the standard CKM matrix with only one CP violating phase $\delta_1$. The CP properties of the model with one isosinglet quark can be most conveniently studied by using weak-basis invariants. In the Appendix we present the general treatment and provide the proof that the vanishing of the following set of invariants is necessary and sufficient to have CP invariance:

$$I_1 = \text{Im} \text{ tr } H_uH_dh_dh_d^\dagger,$$

$$I_2 = \text{Im} \text{ tr } H_u^2H_dh_dh_d^\dagger,$$

$$I_3 = \text{Im} \text{ tr } (H_u^3H_dh_dh_d^\dagger - H_u^2H_dH_uh_dh_d^\dagger),$$

$$I_4 = \text{Im} \text{ tr } H_uH_d^2h_dh_d^\dagger,$$

$$I_5 = \text{Im} \text{ tr } H_u^2H_d^2h_dh_d^\dagger,$$
the usefulness of these invariant conditions, by considering some appropriate chiral limits.

is a very convenient method to describe CP violation. In the next Section we will illustrate

finding a complete set of invariants, but once this is accomplished, the invariant approach

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difficult task is, of course, to find a complete set of necessary and sufficient conditions for CP invariance consists of eight

invariants which have been given in Ref. [19]. In both cases these invariant conditions

up quark $i$ and $n_\alpha$ the mass of the down quark $\alpha$)

\begin{align}
I_0 &= \text{Im } \text{tr} \left( H_u^2 H_d^2 h_d h_d^\dagger - H_u^2 H_d^2 H_u h_d h_d^\dagger \right), \\
I_7 &= \text{Im } \text{tr} H_u^2 H_d H_u H_d^2 ,
\end{align}

with $H_u = M_u M_u^\dagger$, $H_d = M_d M_d^\dagger$, $h_d = M_d m_d^\dagger$ (see Eq. (3) ). These invariants, which

obviously do not depend on the choice of weak quark basis (see Eq. (8)), can be written in the quark mass eigenstate basis (in the equations below Greek indices run from 1 to 4, Latin indices run from 1 to 3 and a sum over all indices is implicit; $m_i$ is the mass of the

imaginary parts of invariant quartets. In the chiral limits there only appear

\begin{align}
I_1 &= m_i^2 n_\alpha n_\beta^4 \text{ Im } V_{i\alpha} X_{\alpha \beta} V_{i\beta}^*, \\
I_2 &= m_i^2 n_\alpha n_\beta^4 \text{ Im } V_{i\alpha} X_{\alpha \beta} V_{i\beta}^*, \\
I_3 &= m_i^2 n_\alpha n_\beta^4 \text{ Im } V_{i\alpha} X_{\alpha \beta} V_{i\beta}^* \\
&\quad - m_i^2 n_\alpha^2 n_\beta^2 \text{ Im } V_{i\alpha} V_{i\alpha}^* V_{i\beta}^* V_{i\beta}^*, \\
&\quad + m_i^2 n_\alpha^2 n_\beta^2 \text{ Im } V_{i\alpha} V_{i\alpha}^* V_{i\beta}^* V_{i\beta}^*, \\
I_4 &= m_i^2 n_\alpha n_\beta n_\rho^2 \text{ Im } V_{i\alpha} X_{\alpha \beta} V_{i\beta}^*, \\
I_5 &= m_i^2 n_\alpha n_\beta n_\rho^2 \text{ Im } V_{i\alpha} X_{\alpha \beta} V_{i\beta}^*, \\
I_6 &= m_i^2 n_\alpha n_\beta n_\rho^2 \text{ Im } V_{i\alpha} X_{\alpha \beta} V_{i\beta}^* \\
&\quad - m_i^2 n_\alpha^2 n_\beta n_\rho^2 V_{i\alpha} V_{i\alpha}^* V_{i\beta}^* V_{i\beta}^*, \\
I_7 &= m_i^2 n_\alpha n_\beta n_\rho^2 \text{ Im } V_{i\alpha} V_{i\alpha}^* V_{i\beta}^* V_{i\beta}^*,
\end{align}

Notice that $X_{\alpha \beta} = V_{i\alpha} V_{i\beta}^*$, which implies $X = X^\dagger$ and $X_{\alpha \beta} X_{\beta \gamma} = X_{\alpha \gamma}$ (note however that in Eqs. (14) the sums include also mass factors). The imaginary parts involve invariant quartets and invariant sextets, which can be reduced also to products of moduli squared of $V_{ij}$ elements times imaginary parts of quartets. In the chiral limits there only appear imaginary parts of invariant quartets. For the case of four sequential families, the corresponding set of necessary and sufficient conditions for CP invariance consists of eight invariants which have been given in Ref. [19]. In both cases these invariant conditions completely characterize the CP properties of the model. If any of the invariants is nonvanishing, there is CP violation and the vanishing of the invariants implies CP invariance. The description of the CP properties of a model through invariants is especially useful when considering limiting cases where some of the quark masses can be considered as degenerate (massless). The invariant approach clearly identifies which ones of the Im $V_{ij} V_{ik}^* V_{kl} V_{il}^*$ can be nonvanishing in the various limiting cases one considers. The difficult task is, of course, finding a complete set of invariants, but once this is accomplished, the invariant approach is a very convenient method to describe CP violation. In the next Section we will illustrate the usefulness of these invariant conditions, by considering some appropriate chiral limits.
3 CP violation in the chiral limit

In the chiral limit, $m_{u,d,s} = 0$, CP is conserved within the SM. Hence all CP violating effects are due to new physics. Sizeable CP violation at high energy is expected to have its origin beyond the SM. Let us discuss in turn the simplest limit of $m_t \gg m_c \sim 0$, $m_{u,d,s} = 0$ and the chiral limit $m_{u,d,s} = 0$. We will consider the cases of an extra isosinglet quark and of a fourth sequential family.

3.1 $m_{u,d,s,c} = 0$ limit

In this limit there is only one CP violating phase for an extra down quark isosinglet $b'$ or for a fourth family $b', t'$. The best way to study this limit is to substitute $m_{u,d,s,c} = 0$ in the complete set of invariants characterizing CP. The 7 invariants in Eqs. (13) reduce to

$$
I_1 = m_t^2 m_b^2 (m_b^2 - m_b'^2) \text{Im} V_{tb} X_{bb'} V_{tb}'^*,
I_2 = m_t^2 I_1,
I_3 = m_t^4 I_1,
I_4 = (m_b^2 X_{bb'})^2 (m_b'^2 - m_b^2) I_1,
I_5 = m_t^2 I_4,
I_6 = m_t^4 I_4,
I_7 = 0.
$$

(15)

It is clear that CP is conserved if and only if $\text{Im} V_{tb} X_{bb'} V_{tb}'^* = 0$. Obviously, we have made the assumption that $b, b'$ are nondegenerate, with $m_{b'} > m_b$. Hence all CP violating effects are proportional to this imaginary product which gives the size of CP violation. Similarly, for 4 families the corresponding 8 invariants [19] reduce to (we use a prime to distinguish them)

$$
I_1' = -m_t^2 m_b^2 (m_{b'}^2 - m_t^2) m_b^2 (m_b^2 - m_b'^2) \text{Im} V_{tb} V_{tb'} V_{tb'}^*,
I_2' = (m_{b'}^2 + m_t^2) I_1',
I_3' = (m_{b'}^4 + m_t^4) I_1',
I_4' = (m_{b'}^6 + m_t^6) I_1',
I_5' = (m_{b'}^2 + m_t^2) I_1',
I_6' = (m_{b'}^4 + m_t^4) I_1',
I_7' = (m_{b'}^6 + m_t^6) I_1'.
$$

(16)
In this case CP conservation reduces to requiring $\text{Im} \ V_{tb} V_{tb}^* V_{t'b'} = 0$, with the implicit assumption $m_{t'} > m_t$ and $m_{t'} > m_t$. Not only the same comments as for an extra isosinglet apply but the observable for SM final states is the same. Thus using unitarity

$$B_2 = \text{Im} \ V_{tb} V_{tb}^* V_{t'b'}^* = -\text{Im} \ V_{tb} X_{t'b'} V_{t'b'}^* = -\text{Im} \ V_{tb} V_{t'b'} V_{t'b'}^*.$$  \hspace{1cm} (17)

It is clear that $B_2$ measures the strength of CP violation in high energy processes involving the new quark. On the other hand due to the unitarity constraints, $B_2$ is actually related to the invariants $T_i$ defined in Eqs. (11), which only depend on standard quark mixings. Indeed, one obtains

$$B_2 = -\text{Im} \ V_{tb} V_{t'b'}^* = -\text{Im} \ V_{tb} V_{tb}^* V_{t'b'} - \text{Im} \ V_{tb} V_{t'b'} V_{t'b'}^*$$

$$= \text{Im} \ V_{tb} V_{t'b'} V_{t'd} V_{t'd}^* + \text{Im} \ V_{tb} V_{t'b'} V_{t'd} V_{t'd}^* + \text{Im} \ V_{tb} V_{tb}^* V_{t'd} V_{t'd}^* + \text{Im} \ V_{tb} V_{tb}^* V_{t'd} V_{t'd}^* = T_2 + T_3,$$ \hspace{1cm} (18)

with $T_{2,3} = 0$ in the three generation SM as emphasized in the Introduction.

All this can also be proven using the CKM matrix, although the physics is less transparent. If $m_{u,c} = 0$, $m_{d,s} = 0$, the general $4 \times 4$ unitary matrix in Eq. (12) (up to quark field phase redefinitions) can be written

$$V = \begin{pmatrix}
    c_1 & s_1 c_3 & s_1 s_3 & 0 \\
    -s_1 c_2 & c_1 c_2 c_3 + s_2 s_3 c_6 e^{i \delta_1} & c_1 c_2 s_3 - s_2 c_3 c_6 e^{i \delta_1} & -s_2 s_6 e^{i \delta_1} \\
    -s_1 s_2 & c_1 s_2 c_3 - c_2 s_3 c_6 e^{i \delta_1} & c_1 s_2 s_3 + c_2 c_3 c_6 e^{i \delta_1} & c_2 s_6 e^{i \delta_1} \\
    0 & s_3 s_6 & -c_3 s_6 & c_6
\end{pmatrix},$$ \hspace{1cm} (19)

where we have used the freedom to make unitary transformations in the $(u,c)$ and $(d,s)$ spaces. This freedom results of course from the fact that $(u,c)$ and $(d,s)$ are degenerate in the limit we are considering. As expected, in this limit there is only one CP violating phase and the strength of CP violation is given by

$$B_2 = \text{Im} \ V_{tb} V_{t'b'} V_{t'b'}^* = c_1 c_2 s_2 c_3 s_3 c_6 s_6^2 \sin \delta_1.$$ \hspace{1cm} (20)

Unitarity allows to recover Eqs. (17). In this particular parametrization $B_2$ is in fact equal to $T_3$ for $T_{1,2} = 0$ as defined in Eqs. (11).

### 3.2 The chiral limit, $m_{u,d,s} = 0$

In this limit there is no CP violation in the SM, but for one extra quark isosinglet or a fourth sequential family there are two new CP violating phases which remain physical. In
the case of the model with an extra down quark isosinglet, CP conservation is equivalent to the vanishing of $I_{1-3}$ in Eqs. (13), because in this limit

$$
I_1 = m_1^2 I_t + m_c^2 I_c,
$$
$$
I_2 = m_1^4 I_t + m_c^4 I_c,
$$
$$
I_3 = m_1^6 I_t + m_c^6 I_c + m_1^2 m_c^2 (m_1^2 - m_c^2) m_1^2 (m_2^2 - m_2^2) \text{Im } V_{cb} V_{tb}^* V_{t'b'} V_{c'b'}^* + I_7,
$$
$$
I_4 = (m_1^2 X_{t'b'} + m_c^2 X_{bb}) I_1,
$$
$$
I_5 = (m_1^2 X_{t'b'} + m_c^2 X_{bb}) I_2,
$$
$$
I_6 = (m_1^2 X_{t'b'} + m_c^2 X_{bb}) (m_1^2 I_t + m_c^4 I_c)
$$
$$
-m_1^2 m_c^2 (m_1^2 - m_c^2) m_1^2 m_1^2 (m_2^2 - m_2^2) (X_{t'b'} - X_{bb}) \text{Im } V_{cb} V_{tb}^* V_{t'b'} V_{c'b'}^*
$$
$$
+(m_1^4 |V_{t'b'}|^2 - m_1^2 |V_{t'b'}|^2) \text{Im } V_{tb} X_{t'b'} V_{t'b'}^*
$$
$$
-(m_1^4 |V_{t'b'}|^2 - m_c^4 |V_{t'b'}|^2) \text{Im } V_{tb} X_{bb} V_{t'b'}^*,
$$
$$
I_7 = -m_1^2 m_c^2 (m_1^2 - m_c^2) m_1^2 m_1^2 (m_2^2 - m_2^2) (X_{t'b'} - X_{bb}) \text{Im } V_{cb} V_{tb}^* V_{t'b'} V_{c'b'}^*
$$
$$
+(m_1^4 |V_{t'b'}|^2 - m_1^2 |V_{t'b'}|^2) \text{Im } V_{tb} X_{t'b'} V_{t'b'}^*
$$
$$
-(m_1^4 |V_{t'b'}|^2 - m_c^4 |V_{t'b'}|^2) \text{Im } V_{tb} X_{bb} V_{t'b'}^*,
$$

with

$$
I_c = m_1^2 m_2^2 m_1^2 (m_1^2 - m_2^2) \text{Im } V_{cb} X_{bb} V_{c'b'}^*,
$$
$$
I_t = m_1^4 m_2^2 m_1^2 (m_1^2 - m_2^2) \text{Im } V_{tb} X_{t'b'} V_{t'b'}^*. 
$$

There are only three independent imaginary products, $\text{Im } V_{t'b'} X_{bb} V_{t'b'}^*$, $\text{Im } V_{cb} X_{bb} V_{c'b'}^*$ and $\text{Im } V_{cb} V_{tb}^* V_{t'b'} V_{c'b'}^*$, entering in $I_{1-7}$. Their vanishing guarantees CP conservation. The first one is the only one which survives for $m_c = m_a$ as proven in the previous Subsection. Analogously, in the case of a fourth family CP invariance is equivalent to the vanishing of $I_{1-3}$ in Ref. [19]. In this case the complete set of 8 invariants reduces to

$$
I'_1 = I'_c + I'_t + I'_{t'b'},
$$
$$
I'_2 = (m_t^2 + m_c^2) I'_c + (m_t^2 + m_c^2) I'_t + (m_t^2 + m_c^2) I'_{t'b'},
$$
$$
I'_3 = (m_t^4 + m_c^4) I'_c + (m_t^4 + m_c^4) I'_t + (m_t^4 + m_c^4) I'_{t'b'},
$$
$$
I'_4 = (m_t^6 + m_c^6) I'_c + (m_t^6 + m_c^6) I'_t + (m_t^6 + m_c^6) I'_{t'b'},
$$
$$
\quad -m_c^2 m_1^2 m_1^2 (m_1^2 - m_c^2) (m_2^2 - m_2^2) m_1^2 m_1^2
$$
$$
\quad \times [(|V_{t'b'}|^2 - |V_{t'b'}|^2 m_2^2) \text{Im } V_{cb} V_{tb}^* V_{t'b'} V_{c'b'}^*
$$
$$
\quad - (|V_{t'b'}|^2 m_2^2 - |V_{t'b'}|^2 m_1^2) \text{Im } V_{tb} V_{t'b'} V_{t'b'}^* V_{c'b'}^*]
$$
$$
\quad + (|V_{cb}|^2 m_2^2 - |V_{cb}|^2 m_1^2) \text{Im } V_{tb} V_{t'b'} V_{t'b'}^* V_{c'b'}^*],
$$
$$
I'_5 = (m_1^2 + m_2^2) I'_1.
$$
\[ I_6' = (m_{1'}^2 + m_{2'}^2)I_2', \]
\[ I_7' = (m_{1'}^4 + m_{2'}^4)I_1', \]
\[ I_8' = (m_{3'}^2 + m_{4'}^2)I_5', \]

(23)

with
\[ I_{ct}' = -m_t^2 m_c^2 (m_c^2 - m_t^2) m_b^2 m_{1'}^2 (m_{1'}^2 - m_b^2) \text{Im} V_{cb}V_{ub}^* V_{cb}^* V_{ub}^*, \]
\[ I_{ct'}' = -m_t^2 m_c^2 (m_c^2 - m_t^2) m_b^2 m_{2'}^2 (m_{2'}^2 - m_b^2) \text{Im} V_{cb}V_{ub}^* V_{cb}^* V_{ub}^*, \]
\[ I_{tt'}' = -m_t^2 m_t^2 (m_t^2 - m_t^2) m_b^2 m_{1'}^2 (m_{1'}^2 - m_b^2) \text{Im} V_{tb}V_{tb}^* V_{tb}^* V_{tb}^*. \]

(24)

For four families there are also three independent imaginary products, \( \text{Im} V_{tb}V_{tb}^* V_{tb}^* \), \( \text{Im} V_{cb}V_{cb}^* V_{cb}^* \), and \( \text{Im} V_{cb}V_{tb}^* V_{tb}^* V_{cb}^* \), entering in \( I_{1-8}' \). Similarly to the case of an extra isosinglet their vanishing guarantees CP conservation. There is an interesting connection between these rephasing invariants \( B_i \), which can see CP violation through high energy processes involving the new quark, and \( T_i \), which only involve mixings among SM quarks. Using unitarity one obtains
\[
B_2 = \text{Im} V_{tb}V_{tb}^* V_{tb}^* V_{tb}^* = -\text{Im} V_{tb}X_{bb}^* V_{tb}^* = T_2 + T_3, \\
B_1 = \text{Im} V_{cb}V_{tb}^* V_{tb}^* V_{cb}^* = -\text{Im} V_{cb}X_{bb}^* V_{cb}^* = T_1 - T_3, \\
B_3 = \text{Im} V_{cb}V_{tb}^* V_{tb}^* V_{cb}^* = T_3, \]

(25)

where \( T_{1-3} \) are defined in Eqs. (3) and vanish in the three generation SM.

These results can be also reproduced using explicitly the CKM matrix. Using the freedom one has in the chiral limit to make unitary transformations in the \((d,s)\) space, one may write the CKM matrix in the form
\[
V = \begin{pmatrix}
  c_1 & s_1 s_3 c_5 & s_1 s_3 s_5 \\
-s_1 c_2 & c_1 c_2 c_3 + s_2 s_3 c_6 e^{i \delta_1} & c_1 s_2 s_3 c_5 - s_2 c_3 s_5 c_6 e^{i \delta_1} + s_2 s_5 s_6 e^{i (\delta_1 + \delta_3)} \\
-s_1 s_2 & c_1 s_2 c_3 - c_2 s_3 c_6 e^{i \delta_1} & c_1 s_2 s_3 c_5 + c_2 c_3 s_5 c_6 e^{i \delta_1} - c_2 s_5 s_6 e^{i (\delta_1 + \delta_3)} + c_2 c_5 s_6 e^{i (\delta_1 + \delta_3)} \\
0 & s_3 s_6 & -c_3 c_5 s_6 - s_5 c_6 e^{i \delta_3} - c_3 s_5 s_6 + c_5 c_6 e^{i \delta_3}
\end{pmatrix}.
\]

(26)

The invariants \( B_i \) can then be expressed in terms of mixing angles and the two physical CP violating phases,

\[
B_2 = \text{Im} V_{tb}V_{tb}^* V_{tb}^* V_{tb}^* = T_2 + T_3 =
\]
\[
= c_1 s_2 c_3 s_5 c_6 (c_5^2 - s_5^2) \sin \delta_1 + s_2^2 c_3 s_5 c_6 (c_5^2 - s_5^2) \sin \delta_3
+ c_1 s_2 c_3 s_5 s_6 (c_6^2 - c_5^2) \sin (\delta_1 + \delta_3) + c_1 s_2 c_3 s_5 c_6 s_6 c_5^2 \sin (\delta_1 - \delta_3),
\]

11
\[ B_1 = \Im V_{cb} V_{ub}^* V_{td}^* V_{td} = T_1 - T_3 = \\
= c_1 s_2 s_3 c_3 s_6^2 c_6 (s_5^2 - c_5^2) \sin \delta_1 + s_3^2 c_3 s_5 c_6 (s_2^2 - c_2^2) \sin \delta_3 \\
+ c_1 s_2 c_3 s_5 s_6 (c_2^2 - s_2^2) \sin (\delta_1 + \delta_3) - c_1 s_2 c_3 s_5^2 c_6^2 \sin (\delta_1 - \delta_3) , \\
B_3 = \Im V_{cb} V_{tb}^* V_{td}^* = T_3 = \\
= c_1 s_2 c_3 s_3 c_6^2 c_6 (c_2^2 - s_2^2) \sin \delta_1 + c_1^2 s_2^2 c_3 s_5 c_6 (c_2^2 - s_2^2) \sin \delta_3 \\
+ c_1 s_2 c_3 s_5 s_6 (c_2^2 s_3^2 - s_2^2) \sin (\delta_1 + \delta_3) + c_1 s_2 c_3 s_5^2 c_6^2 \sin (\delta_1 - \delta_3) . \] 

The unitary matrices in Eqs. \((19,26)\) have been used to rederive the relevant imaginary parts of invariant quartets in the chiral limits. Obviously, they do not correspond to the actual CKM matrix in the physical situation where the quarks are distinguished.

### 4 Limits on CP violating effects from new quarks

In this Section we revise the three generation SM for which CP violation is summarized in a unitarity triangle, extending this description to the unitarity quadrangles for an extra vector-like (sequential) quark (family) and their restriction to subtriangles in the chiral limit. Then we estimate the experimental bounds on the three independent invariants \(B_i\) characterizing CP violation in this case and comment on the determination of the new CP violating effects.

#### 4.1 Triangles, quadrangles and the chiral limit

Before considering physics beyond the SM, it is worthwhile reviewing the main features of CP violation in the three generation SM. The information on CP violation is conventionally summarized in this case in terms of the unitarity triangle \([21]\). This triangle is the geometrical representation of the unitarity relations between any two different rows or columns of the \(3 \times 3\) CKM matrix. One can draw different triangles choosing different pairs of rows or columns, but for the three generation SM unitarity implies that all these triangles have the same area (which equals \(\Im V_{ij} V_{kl}^* V_{il} V_{kl}^* / 2\)) because all \(\Im V_{ij} V_{kl}^* V_{il} V_{kl}^*\) have the same modulus, and this modulus gives the strength of CP violation in the SM. The most interesting of the unitarity triangles is the one which results from the orthogonality of the first and third columns,

\[ V_{ud} V_{ub}^* + V_{cd} V_{cb}^* + V_{td} V_{tb}^* = 0 . \] 

The measurement of CP asymmetries in \(B^0\) decays offers the possibility of measuring the internal angles of this triangle. Note that these angles are rephasing invariant quantities since they are the arguments of invariant quartets. In the three generation SM,
$|\text{Im} \ V_{ij} V_{k\ell}^* V_{i\ell}^*|$, is necessarily small, due essentially to the fact that the third generation almost decouples from the other two (in the limit where the third generation decouples there is no CP violation in the SM). An upper limit on $|\text{Im} \ V_{ij} V_{k\ell}^* V_{i\ell}^*|$ is readily obtained since $|\text{Im} \ V_{us} V_{cb}^* V_{ub}^*| \leq |V_{us}| |V_{cs}| |V_{cb}| |V_{ub}| \leq 5 \times 10^{-5}$.

In the presence of an extra quark, the unitarity relation corresponding to Eq. (28) becomes

$$V_{ud} V_{ub}^* + V_{cd} V_{cb}^* + V_{td} V_{tb}^* + V_{4d} V_{4b}^* = 0.$$  (29)

In this case, three quadrangles are required to summarize the information on CP violation. The other two quadrangles can be chosen to be the ones obtained multiplying the first and second columns and the second and third columns, respectively. The vanishing of their areas is a sufficient condition for CP conservation, and in the case of nondegenerate quark masses it is also necessary. Alternatively, one could choose a set of three quadrangles arising from the orthogonality between the rows of $V$. A relevant observation is that some of the sides of these quadrangles cannot be measured separately in the case of degenerate quark masses. For example, in the limit where $m_{u,c} = 0$ the quantities $V_{ud} V_{ub}^*$ and $V_{cd} V_{cb}^*$ in Eq. (29) cannot be separately measured, due to the freedom to redefine degenerate quark fields. In this case, however, the subtriangle with sides $V_{ud} V_{ub}^* + V_{cd} V_{cb}^*$, $V_{td} V_{tb}^*$, and $V_{4d} V_{4b}^*$ is well-defined. In the chiral limits previously considered, CP violation can be geometrically described as follows.

In the $m_{u,d,s,c} = 0$ limit, we can define the subtriangle obtained multiplying the third and fourth columns of $V$ (shadowed region in Fig. 1),

$$V_{ub} V_{ub}^* + V_{cb} V_{cb}^* + V_{tb} V_{tb}^* + V_{4b} V_{4b}^* = 0,$$  (30)

and considering the sides $V_{ub} V_{ub}^* + V_{cb} V_{cb}^*$, $V_{tb} V_{tb}^*$, $V_{4b} V_{4b}^*$, with angles $\phi_{1-3}$,

$$\sin \phi_1 = |\text{sin arg} (V_{ub} V_{ub}^* V_{ub}^* + V_{cb} V_{cb}^* V_{cb}^*)|,$$

$$\sin \phi_2 = |\text{sin arg} V_{tb} V_{tb}^* V_{tb}^*|,$$

$$\sin \phi_3 = |\text{sin arg} (V_{ub} V_{ub}^* V_{ub}^* + V_{cb} V_{cb}^* V_{cb}^*)|.$$  (31)

The area of this triangle represents the strength of CP violation in this limit. This area is given by

$$A_{bb'} = \frac{1}{2} |B_2| = \frac{1}{2} |\text{Im} \ V_{tb} V_{tb}^* V_{tb}^*|$$  (32)

and vanishes if and only if CP is conserved (see Eqs. (13,14)). Note that under allowed quark mass eigenstate transformations, including those mixing $u$ and $c$, the length of the sides of the triangle remains constant and therefore they are measurable quantities even in the limit $m_{u,c} = 0$, where $u$ and $c$ are indistinguishable. Thus this triangle provides a good
Figure 1: The shadowed triangle describes CP violation in the $m_{u,d,s,c} = 0$ limit, whereas the complete quadrangle does it in the chiral limit $m_{u,d,s} = 0$. We use the same notation $\phi_{1,3}$ for the angles of the shadowed triangle as for the quadrangle, although for the (convex) quadrangle they are larger.

description of CP violation in the $m_{u,d,s,c} = 0$ limit. The subtriangle obtained multiplying the third and fourth rows,

$$V_{td} V_{4d}^* + V_{ts} V_{4s}^* + V_{tb} V_{4b}^* + V_{tb} V_{4b}^* = 0,$$  \hspace{0.8cm} (33)

with sides $V_{td} V_{4d}^* + V_{ts} V_{4s}^*, V_{tb} V_{4b}^*, V_{tb} V_{4b}^*$ has the same area $A_{4l} = |\text{Im } V_{tb} V_{4b} V_{4b}^*|/2$ and provides an equivalent description of CP violation in this limit.

In the chiral limit, $m_{u,d,s} = 0$, we have to consider the complete quadrangle in Eq. (30), with sides $V_{ub} V_{ub'}^*, V_{cb} V_{cb'}^*, V_{tb} V_{tb'}^*, V_{4b} V_{4b'}^*$ (see Fig. 1) and angles

$$\sin \phi_1 = |\text{sin arg } V_{cb} V_{cb'} V_{cb'}|,$$

$$\sin \phi_2 = |\text{sin arg } V_{tb} V_{4b} V_{4b'}|,$$

$$\sin \phi_3 = |\text{sin arg } V_{ub} V_{4b} V_{4b'}|,$$

$$\sin \phi_4 = |\text{sin arg } V_{ub} V_{cb} V_{cb'}|.$$

(34)

The area of this quadrangle is

$$A_{4b'} = \frac{1}{4} \left\{ |\text{Im } V_{ub} V_{cb} V_{cb'} V_{cb'}^*| + |\text{Im } V_{cb} V_{cb}^* V_{cb'}| \\
+ |\text{Im } V_{tb} V_{4b} V_{4b'} V_{4b'}^*| + |\text{Im } V_{ub} V_{4b} V_{4b'} V_{4b'}^*| \right\}
\quad + |B_1 + B_2| + |B_3| + |B_2| + |B_1 + B_2|.$$

(35)
We use the same notation as for the angles and area of the triangle in Eqs. \((31)\) and \((32)\) because this quadrangle reduces to that subtriangle in the appropriate limit. It is clear that the vanishing of \(A_{bb'}\) in Eq. \((35)\) is a necessary and sufficient condition for CP conservation. Alternatively, one can consider adding to the triangle in Eq. \((33)\) the analogous triangles obtained multiplying the second and third rows and the second and fourth rows respectively, with areas \(A_{ce} = |B_3| = |\text{Im } V_{cb} V_{tb}^* V_{cb'} V_{cb''}'|/2\) and \(A_{ce} = |B_1| = |\text{Im } V_{cb} V_{tb}^* V_{cb'} V_{cb''}'|/2\). The vanishing of \(A_{ce,t4}\) is also a necessary and sufficient condition for CP conservation.

### 4.2 Bounds on \(B_1\), \(B_2\) and \(B_3\)

We turn now to the important question of estimating the possible size of the new CP violating effects when new quarks are added to the SM. In order to establish upper bounds on the size of these effects one has to distinguish between the case of an extra isosinglet quark and the case of a sequential fourth family. In both cases, we use the experimental model-independent measurements \([2]\) \(|V_{ud}| = 0.9736 \pm 0.0010\), \(|V_{us}| = 0.2205 \pm 0.0018\), \(|V_{cd}| = 0.224 \pm 0.016\), \(|V_{cs}| = 1.01 \pm 0.18\), \(|V_{ub}/V_{cb}| = 0.08 \pm 0.02\), \(|V_{cb}| = 0.041 \pm 0.003\). These, together with the unitarity of the \(4 \times 4\) matrix \(V\), give \(|V_{ub}| \leq 0.079\), \(|V_{cb}| \leq 0.516\), \(|V_{td,4d}| \leq 0.104\), \(|V_{ts,4s}| \leq 0.513\), where we have used the measured lower bounds to obtain these upper limits. Our strategy will be to obtain rigorous upper bounds on \(|B_i|\) using the previous limits and to check afterwards that they are almost saturated in particular cases fulfilling all present experimental constraints. The former upper bounds imply \(|\text{Im } V_{ub} V_{cb}^* V_{cb'} V_{cb''}'| \leq |V_{ub}| |V_{cb}| |V_{cb'}| |V_{cb''}'| \leq 7.87 \times 10^{-6}\), \(|\text{Im } V_{ub} V_{tb}^* V_{cb'} V_{cb''}'| \leq |V_{ub}| |V_{tb}| |V_{cb'}| |V_{cb''}'| \leq 1.73 \times 10^{-4}\), \(|\text{Im } V_{cb} V_{tb}^* V_{tb'} V_{cb''}'| \leq |V_{cb}| |V_{tb}| |V_{tb'}| |V_{cb''}'| \leq 1.11 \times 10^{-2}\). Then, using unitarity and the triangular inequality for the absolute values, these limits translate into

\[
\begin{align*}
|B_1| &= |\text{Im } V_{ub} V_{cb}^* V_{cb'} V_{cb''}' - \text{Im } V_{cb} V_{tb}^* V_{tb'} V_{cb''}'| \leq 1.11 \times 10^{-2}, \\
|B_2| &= |\text{Im } V_{ub} V_{tb}^* V_{tb'} V_{cb''}' + \text{Im } V_{cb} V_{tb}^* V_{tb'} V_{cb''}'| \leq 1.12 \times 10^{-2}, \\
|B_3| &= |\text{Im } V_{cb} V_{tb}^* V_{tb'} V_{cb''}'| \leq 1.11 \times 10^{-2},
\end{align*}
\]

which are rigorous bounds, in particular for a fourth family. These bounds are mostly saturated for instance by the \(4 \times 4\) unitary matrix

\[
|V| = \begin{pmatrix}
0.973 & 0.220 & 0.0035 & 0.070 \\
0.230 & 0.918 & 0.041 & 0.321 \\
0.082 & 0.254 & 0.655 & 0.712 \\
0.082 & 0.212 & 0.755 & 0.621
\end{pmatrix},
\]
\[ \text{arg} \, V = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \pi & 0.007 & -1.08 & -0.057 \\ \pi & 0.095 & 0.873 & -3.00 \\ \pi & -1.31 & 2.38 & 1.62 \end{pmatrix} , \] 

(38)

for which \(|B_1| = 6.1 \times 10^{-3}, |B_2| = 6.3 \times 10^{-3}, |B_3| = 6.1 \times 10^{-3}\). We have also required in these matrices that the imaginary parts of the quartets involving the first two columns and entering in the calculation of \(\epsilon_K\) for four sequential families are \(\sim 10^{-4}\) not to rely on large cancellations. Without this requirement the bounds in Eqs. (36) can be almost completely saturated. Hence \(|B_3| \leq 10^{-2}\) for four families.

In the case of an extra isosinglet quark the size of the CKM matrix elements is further constrained by existing bounds on FCNC [22]. The constraint on \(|V_{us}|\) for which \(|X_{ds}| \leq 4.08 \times 10^{-5}\). The limits on \(|X_{db}| = |V_{ud}| |V_{tb}|, |X_{sb}| = |V_{ts}| |V_{tb}|\) arise from the experimental bound [3]

\[
\text{Br} (K^+ \to \pi^+ \nu \bar{\nu}) = \frac{\Gamma (K^+ \to \pi^+ \nu \bar{\nu})}{\Gamma (K^+ \to \text{all})} < 2.4 \times 10^{-9}.
\]

(39)

Comparison with the process \(K^+ \to \pi^0 e^+ \nu\) leads to

\[
\frac{\text{Br} (K^+ \to \pi^+ \nu \bar{\nu})}{\text{Br} (K^+ \to \pi^0 e^+ \nu)} = \frac{|X_{ds}|^2}{2 |V_{us}|^2} \times 3 ,
\]

(40)

where the factor 3 takes into account the three different \(\nu \bar{\nu}\) pairs. Then the observed \(\text{Br} (K^+ \to \pi^0 e^+ \nu) = (4.82 \pm 0.06)\%\) [2] gives \(|X_{ds}| < 4.08 \times 10^{-5}\). The limits on \(|X_{db}| = |V_{ud}| |V_{tb}|, |X_{sb}| = |V_{ts}| |V_{tb}|\) arise from the experimental bound [4]

\[
\frac{\Gamma (B \to \mu^+ \mu^- X)}{\Gamma (B \to \mu \bar{\mu} X)} < 4.6 \times 10^{-4} ,
\]

(41)

which leads to [22]

\[
\frac{|X_{db}|^2 + |X_{sb}|^2}{|V_{tb}|^2 + R |V_{cb}|^2} < 3.67 \times 10^{-3} ,
\]

(42)

where \(R \simeq 0.5\) is a phase space factor, giving \(|X_{db, sb}| \leq 1.91 \times 10^{-3}\). Using this limit, the experimental bounds above and the triangular inequality for the absolute values we obtain

\[
|\text{Im} \, V_{ub} V^*_{db} V_{4b} V^*_{ub'}| = |\text{Im} \, V_{ud} V^*_{4d} V_{4b} V^*_{ub} + \text{Im} \, V_{us} V^*_{4s} V_{4b} V^*_{ub} | \leq 1.01 \times 10^{-5} ,
\]

\[
|\text{Im} \, V_{cb} V^*_{4b} V_{4b} V^*_{cb'}| = |\text{Im} \, V_{cd} V^*_{4d} V_{4b} V^*_{cb} + \text{Im} \, V_{cs} V^*_{4s} V_{4b} V^*_{cb} | \leq 1.02 \times 10^{-4} ,
\]

(43)

which together with the general bound \(|\text{Im} \, V_{ub} V^*_{cb} V_{cb'} V^*_{ub'}| \leq 7.87 \times 10^{-6}\) translate into

\[
|B_1| = |\text{Im} \, V_{cb} V^*_{4b} V_{4b} V^*_{cb'} | \leq 1.02 \times 10^{-4} ,
\]

\[
|B_2| = |\text{Im} \, V_{ub} V^*_{4b} V_{4b} V^*_{ub'} + \text{Im} \, V_{cb} V^*_{4b} V_{4b} V^*_{cb'} | \leq 1.12 \times 10^{-4} ,
\]

\[
|B_3| = |\text{Im} \, V_{ub} V^*_{cb} V_{cb'} V^*_{ub'} + \text{Im} \, V_{cb} V^*_{4b} V_{4b} V^*_{cb'} | \leq 1.11 \times 10^{-4} .
\]

(44)
These are the rigorous limits for an extra down quark isosinglet. (We have not made explicit use of the $|X_{ds}|$ bound.) The $4 \times 4$ unitary matrix

$$|V| = \begin{pmatrix} 0.975 & 0.222 & 0.0033 & 0.0007 \\ 0.222 & 0.974 & 0.039 & 0.013 \\ 0.011 & 0.039 & 0.978 & 0.204 \\ 0.0022 & 0.0093 & 0.204 & 0.979 \end{pmatrix},$$

(45)

$$\arg V = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \pi & 0.0006 & -1.97 & -2.09 \\ \pi & -0.300 & 0.832 & 1.59 \\ \pi & -0.234 & 2.56 & 0.142 \end{pmatrix},$$

(46)

gives $|B_1| = 7.6 \times 10^{-5}$, $|B_2| = 7.7 \times 10^{-5}$, $|B_3| = 7.6 \times 10^{-5}$ which are near the upper bounds in Eqs. (44). We also require that the dominant contributions to $\epsilon_K$ are the same as in the three generation SM without large cancellations, and not mediated by $Z$ tree level diagrams [13, 22].

### 4.3 CP violation from new quarks

Vector-like and sequential quark contributions to CP violating observables not distinguishing between $d$ and $s$ quarks are proportional to $B_i$. We have shown that for vector-like and sequential quarks $|B_i| \leq 10^{-4}$ and $|B_i| \leq 10^{-2}$, respectively. These values are relatively large. For instance, the maximum of $|\text{Im } V_{ij}V_{kj}^*V_{il}V_{li}^*|$ for an arbitrary $4 \times 4$ unitary matrix is $1/6\sqrt{3} \approx 0.096$, which is the same as for a $3 \times 3$ unitary matrix [23]. On the other hand, $|\text{Im } V_{ij}V_{kj}^*V_{lk}V_{li}^*| \leq 5 \times 10^{-5}$ in the three generation SM. In spite of the relatively large values allowed for $B_i$, it is clear that observing direct CP violation from gauge couplings of new quarks will not be an easy task. It is worth emphasizing that $B_i$ can also be obtained indirectly, by measuring $T_i$ and using Eqs. (25) which give $B_i$ as functions of $T_i$. The failure of the three generation SM unitarity relations would point out to new (CP violating) physics, in particular to new quarks if $B_i$ in Eqs. (25) are of the correct size. The study of CP violation at high energies would thus complement the information of CP asymmetries in B meson decays, at B factories. The effects of vector-like or sequential quarks on the CP asymmetries in $B^0$ decays have been extensively studied in the literature [22]. In the case of vector-like quarks, the most important effect results from a new contribution to $B_d - \bar{B}_d$ and $B_s - \bar{B}_s$ mixings, arising from tree level FCNC $Z$ exchange diagrams. It has been shown [23] that even for relatively small FCNC couplings, the prediction for the CP asymmetries in $B^0_d \rightarrow J/\psi K_0$ and $B^0_d \rightarrow \pi^+ \pi^-$ decays can differ significantly from the predictions of the SM. At this point, it should be emphasized that although the observation of CP asymmetries at B factories may lead to unambiguous evidence for new physics,
it will not be easy to identify the origin of the new physics by studying CP asymmetries alone \cite{17}. The study of CP violation at high energies, together with the study of rare B decays, will play an important rôle in identifying the origin of the new sources of CP violation.

5 Conclusions

Understanding the origin of CP violation will probably require obtaining new experimental information on CP violating observables outside the kaon system and the possible identification of new sources of CP violation. One of the simplest ways of obtaining new sources of CP violation consists of adding extra quarks to the SM. The addition of extra vector-like fermions is specially attractive, since they naturally arise in grand-unified theories, like $E_6$. We have derived a complete set of weak-basis invariants which constitute necessary and sufficient conditions for CP invariance. These weak-basis invariants are physical quantities which can be expressed in terms of quark masses and imaginary parts of rephasing invariant quartets. For simplicity we have restricted ourselves to the case of one additional isosinglet quark, since it is sufficient to illustrate the implications of new quarks for observables involving only known fermions.

At this point, it is worth emphasizing the usefulness of weak basis invariants in the study of CP violation:

1. The invariant approach can be very useful in model building. At the moment, there is no standard theory of flavour and for example in the SM the Yukawa couplings are arbitrary free parameters. As a result, one does not have in the SM any insight into the pattern of fermion masses and mixings. In the literature, there have been various attempts of introducing additional family symmetries in the Lagrangian leading to Yukawa couplings which are no longer arbitrary but are expressed in terms of a fewer number of parameters, with the Yukawa couplings exhibiting some texture zeros \cite{24}. Of course the quark mass matrices are no longer arbitrary, being constrained by the family symmetries. One has to check whether in spite of the additional family symmetries the model leads to genuine CP violation mediated by $W$ interactions. The usual method of diagonalizing the quark mass matrices becomes rather inadequate, specially in models with vector-like quarks, where the CKM matrix is no longer a unitary matrix. The simplest way of checking whether CP violation occurs in models with additional family symmetries consists of directly evaluating the weak-basis invariants which constitute the necessary and sufficient conditions for CP invariance in the model considered. If any of these invariants is non-vanishing one is sure to have CP violation.
2. Weak-basis invariants are also very useful for studying CP violation in various physical limits, especially those involving degenerate and vanishing masses. Here we discuss two chiral limits, the extreme one $m_t \gg m_c \sim 0$, $m_{u,d,s} = 0$ and the standard chiral limit $m_{u,d,s} = 0$. These limits are specially relevant at high energy colliders, where the natural asymptotic states are quark jets. In this chiral limit, $d$ and $s$ quark jets are either very difficult or impossible to distinguish from each other and there are no CP violation effects in the three generation SM. In the case of one extra vector-like quark or an extra sequential family we have shown, using weak-basis invariants, that there is CP violation even in this chiral limit. We have shown that CP violation can be characterized by two CP violating phases which are proportional to $B_1 = \text{Im} V_{cb} V_{4b}^* V_{4b}^* V_{cb}^*$, $B_2 = \text{Im} V_{tb} V_{4b}^* V_{4b}^* V_{tb}^*$ and $B_3 = \text{Im} V_{cb} V_{4b}^* V_{4b}^* V_{cb}^*$. In the extreme chiral limit ($m_{u,d,s,c} = 0$) we have shown that there is one CP violating phase and one weak-basis invariant which controls the strength of CP violation in this limit and is proportional to $B_2$.

In conclusion, extra quarks lead to new sources of CP violation which can manifest themselves in various phenomena, including CP asymmetries in $B^0$ decays, rare B decays as well as in CP violating observables at high energy. Weak-basis invariants, together with the imaginary part of rephasing invariant quartets, like $B_i$ and $T_i$ which we have introduced, are useful tools to study CP violation in these minimal extensions of the SM.

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A Appendix

The Lagrangian $\mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{mass}}$ in Eqs. (4,5) is invariant under CP if and only if there exist unitary transformations $U_L, U_{L}^d, U_{R}^{u,d}$ such that

$$
q_L^{(d)} \rightarrow U_{L} C q_L^{(d)*}, \quad d_L^{(s)} \rightarrow U_{L} C d_L^{(s)*}, \quad q_R^{(s)} \rightarrow U_{R} C q_R^{(s)*},
$$

with $C$ the Dirac charge-conjugation matrix, satisfying

$$
U_{L}^\dagger M q U_{R}^q = M^*, \quad U_{L}^d m_d U_{R}^d = m_d^*.
$$

19
As $U_R^{u,d}$ are unobservable (if Higgs mediated interactions are neglected) we can assume $\mathcal{M}_u$ and $\mathcal{M}_d$ in Eq. (3) hermitian with nonnegative eigenvalues. Then,

$$ U_L^\dagger H_u U_L = H_u^*, \quad U_L^\dagger H_d U_L = H_d^*, \quad U_L^\dagger h_d U_L^\dagger = h_d^*, \quad U_L^\dagger h_d U_L^\dagger = h_d^*, $$

with $h_d' = m_d m_d^\dagger$, are equivalent to Eqs. (48) and are also necessary and sufficient conditions for CP conservation. (The condition $U_d^\dagger h_d U_d^\dagger = h_d^T$ follows trivially from Eq. (49).)

From these equalities new constraints for CP invariance can be derived which are independent of the choice of weak basis, with no reference to unitary matrices as in Eqs. (47, 48, 49). They result from the observation that any combination of products of $H_u$, $H_d$, $h_d h_d h_d^\dagger$, with $p$ arbitrary, has invariant trace and determinant. Then CP invariance, Eq. (49), requires that their imaginary part vanishes. This also holds for any combination of products of $h_d'$ and $h_d^T H h_d$, with $H$ any of the former combinations, but there is no need to consider these combinations because they do not give new constraints. Which subsets of these constraints are also sufficient has to be determined case by case. The explicit proof can be done in a simple, convenient basis. For the search of necessary and sufficient constraints and in general for parametrizing the model, we find convenient to consider the basis where $\mathcal{M}_u = M_u$ is diagonal with nonnegative eigenvalues and

$$ \mathcal{M}_d = \begin{pmatrix} M_d & N_d \\ m_d & \tilde{m}_d \end{pmatrix} = \begin{pmatrix} \tilde{M}_d & N_d \\ 0 & \tilde{m}_d \end{pmatrix}, $$

(50)

with $\tilde{M}_d$ upper triangular with real, nonnegative diagonal elements, $\tilde{m}_d$ diagonal with nonnegative eigenvalues and $N_d$ arbitrary ($\tilde{M}_d$ could also be chosen to be hermitian).

The proliferation of invariant constraints required to guarantee CP invariance makes necessary the use of a symbolic algebraic program to write down the expressions and to solve explicitly the constraints. This is done with *Mathematica* [25] and a set of routines analogous to those in Ref. [20].

$N = 3$, $n_d = 1$. In this case we shall show that $I_{1-7} = 0$ in Eqs. (13) is a set of necessary and sufficient conditions for CP conservation. In the proof we assume $M_u$ diagonal with $(M_u)_{ij} = m_i \delta_{ij}$ and $\mathcal{M}_d$ upper triangular with matrix elements $(\mathcal{M}_d)_{i<j} = n_{ij}$. We consider the products of $H_{u,d}$ and $h_d$ in Eq. (49), ordering them by increasing number of factors. Then the imaginary part of such products give the invariant conditions we look for.

The lowest order invariant not identically zero $I_1$ has 8 mass submatrix factors and gives in the convenient basis of Eq. (50) the condition

$$ I_1 = (m_1^2 - m_2^2)|n_{44}|^2(\text{Im} n_{12} n_{22}^* n_{24} n_{41}^* + \text{Im} n_{13} n_{23}^* n_{24} n_{41}^*) $$

$$ + (m_1^2 - m_3^2)|n_{44}|^2\text{Im} n_{13} n_{33}^* n_{34} n_{41}^* + (m_2^2 - m_3^2)|n_{44}|^2\text{Im} n_{23} n_{33}^* n_{34} n_{24}^* = 0 $$

(51)
The expression of $I_1$ suggests that we consider products with higher powers of $H_u$, to obtain independent linear combinations of the imaginary factors. In this way we find

\[
I_2 = (m_1^2 + m_2^2)(m_1^2 - m_2^2)\vert n_{44}\vert^2(\text{Im} n_{12}n_{22}^*n_{24}n_{14}^* + \text{Im} n_{13}n_{23}^*n_{24}n_{14}^*)
\]

\[
+ (m_1^2 + m_3^2)(m_1^2 - m_2^2)\vert n_{44}\vert^2\text{Im} n_{13}n_{33}^*n_{34}n_{14}^*
\]

\[
+ (m_2^2 + m_3^2)(m_2^2 - m_3^2)\vert n_{44}\vert^2\text{Im} n_{23}n_{33}^*n_{34}n_{24}^* = 0,
\]

\[
I_3 = (m_1^2 + m_2^2)(m_1^2 - m_2^2)\vert n_{44}\vert^2(\text{Im} n_{12}n_{22}^*n_{24}n_{14}^* + \text{Im} n_{13}n_{23}^*n_{24}n_{14}^*)
\]

\[
+ (m_1^4 + m_3^4)(m_1^2 - m_2^2)\vert n_{44}\vert^2\text{Im} n_{13}n_{33}^*n_{34}n_{14}^*
\]

\[
+ (m_2^4 + m_3^4)(m_2^2 - m_3^2)\vert n_{44}\vert^2\text{Im} n_{23}n_{33}^*n_{34}n_{24}^* = 0.
\] (52)

We will assume for the moment $n_{44} \neq 0$ and nondegenerate masses. Then Eqs. (51,52) imply

\[
\text{Im} n_{12}n_{22}^*n_{24}n_{14}^* + \text{Im} n_{13}n_{23}^*n_{24}n_{14}^* = 0,
\]

\[
\text{Im} n_{13}n_{33}^*n_{34}n_{14}^* = \text{Im} n_{23}n_{33}^*n_{34}n_{24}^* = 0.
\] (53)

These conditions do not guarantee CP conservation, hence we go on considering the products with increasing number of factors and giving independent conditions. The next lowest order invariant $I_4$ has 10 mass submatrix factors and after substituting (53) it can be written

\[
I_4 = (m_1^2 - m_2^2)\vert n_{33}\vert^2\vert n_{44}\vert^2\text{Im} n_{12}n_{22}^*n_{24}n_{14}^*
\]

\[
+ (m_3^2 - m_2^2)\vert n_{44}\vert^2\text{Im} n_{12}n_{22}^*n_{24}n_{33}^*n_{13}^*
\]

\[
+ (m_3^2 - m_2^2)\vert n_{44}\vert^2\text{Im} n_{12}n_{22}^*n_{23}n_{33}^*n_{14}^* = 0.
\] (54)

We again look to products with higher powers of $H_u$ to obtain independent linear combinations of the imaginary factors, finding

\[
I_5 = (m_1^2 + m_2^2)(m_2^2 - m_1^2)\vert n_{33}\vert^2\vert n_{44}\vert^2\text{Im} n_{12}n_{22}^*n_{24}n_{14}^*
\]

\[
+ (m_2^2 + m_3^2)(m_2^2 - m_3^2)\vert n_{44}\vert^2\text{Im} n_{12}n_{22}^*n_{24}n_{33}^*n_{13}^*
\]

\[
+ (m_1^2 + m_3^2)(m_1^2 - m_3^2)\vert n_{44}\vert^2\text{Im} n_{12}n_{22}^*n_{23}n_{33}^*n_{14}^* = 0,
\]

\[
I_6 = (m_1^4 + m_2^4)(m_2^2 - m_1^2)\vert n_{33}\vert^2\vert n_{44}\vert^2\text{Im} n_{12}n_{22}^*n_{24}n_{14}^*
\]

\[
+ (m_3^2 + m_2^4)(m_3^2 - m_2^2)\vert n_{44}\vert^2\text{Im} n_{12}n_{22}^*n_{24}n_{33}^*n_{13}^*
\]

\[
+ (m_3^4 + m_2^4)(m_3^4 - m_2^4)\vert n_{44}\vert^2\text{Im} n_{12}n_{22}^*n_{23}n_{33}^*n_{14}^* = 0.
\] (55)

These equations imply for $n_{44} \neq 0$ and nondegenerate masses

\[
\vert n_{33}\vert^2\text{Im} n_{12}n_{22}^*n_{24}n_{14}^* = \text{Im} n_{12}n_{22}^*n_{24}n_{34}^*n_{13}^* = \text{Im} n_{12}n_{22}^*n_{23}^*n_{33}^*n_{34}^*n_{14}^* = 0.
\] (56)
A tedious calculation shows that Eqs. (53,56) do imply CP conservation. First we find all the solutions to Eqs. (53,56) with all \( n_{ij} \neq 0 \), then with one \( n_{ij} = 0 \), with two, etc. In all cases we can redefine the quark eigenstate phases to make \( \mathcal{M}_d \) real.

When two up quark masses are degenerate, say \( m_1 = m_2 \), we can assume without loss of generality \( n_{12} = 0 \). Then, \( I_{2,3} \) are proportional to \( I_1 \) and \( I_{5,6} \) to \( I_4 \). Whereas

\[
I_1 = (m_1^2 - m_3^2)|n_{44}|^2(\text{Im} \ n_{13}n_{33}^*n_{34}n_{14}^* + \text{Im} \ n_{23}n_{33}^*n_{34}n_{24}^*) = 0,
I_4 = (m_1^2 - m_3^2)|n_{44}|^2(|n_{11}|^2\text{Im} \ n_{13}n_{33}^*n_{34}n_{14}^* + |n_{22}|^2\text{Im} \ n_{23}n_{33}^*n_{34}n_{24}^*) = 0. \quad (57)
\]

If \( |n_{11}| \neq |n_{22}| \), these equations are independent and

\[
\text{Im} \ n_{13}n_{33}^*n_{34}n_{14}^* = \text{Im} \ n_{23}n_{33}^*n_{34}n_{24}^* = 0. \quad (58)
\]

If \( |n_{11}| = |n_{22}| \), we can assume \( n_{13} = 0 \) and Eqs. (58) still hold. A long and tedious calculation shows that Eqs. (58) imply CP conservation. We look for all their solutions and check that we can redefine the weak quark basis conveniently and make \( \mathcal{M}_d \) real for each solution. (In most cases it is only necessary to redefine the phases of the eigenstates.) If the three up quark masses are degenerate, CP is conserved.

When \( n_{44} = 0 \), \( I_{1-6} = 0 \) and we need to introduce more constraints on the mass matrices to ensure CP conservation. In this case we can assume \( n_{11} = n_{22} = n_{33} = 0 \) by properly choosing the weak basis. There is only one CP violating phase, and the vanishing of the generalization of the SM invariant,

\[
I_7 = (m_2^2 - m_1^2)(m_1^2 - m_3^2)(m_3^2 - m_2^2)|n_{34}|^2\text{Im} \ n_{13}n_{23}^*n_{24}n_{14}^* = 0 \quad (59)
\]
does ensure CP conservation. What completes the proof.

\( N = 3, \ n_d > 1 \). In these cases the number of necessary and sufficient invariant conditions for CP invariance is too large to be in general manageable. Let us argue the fast growth of the number of these constraints by deriving lower bounds for \( n_d = 2 \) and \( n_d = 3 \). These bounds are general and based on cycle counting. A k-cycle is a product of \( k \) matrix elements \( n_{ij} \) of \( \mathcal{M}_d \): \( C(i_1, \ldots, i_k) = \bar{n}_{i_1i_2}\bar{n}_{i_2i_3}\ldots\bar{n}_{i_{k-1}i_k} \), where the indices \( i_j \) are all different and \( \bar{n}_{ij} \) can be \( n_{ij} \) or \( n_{ji}^* \). The number \( p_\text{min} \) of invariant constraints obtained by this method is smaller than the actual number \( p \) for (i) we consider only in this counting nondegenerate up masses, and (ii) we assume that only one invariant is needed to ensure the reality of a cycle (although we know that often this is not the case, see Refs. [13, 20] for examples). Then comparing with the exact result for the simplest case we expect \( p_\text{min} < p \sim 2p_\text{min} \).

For \( n_d = 2 \) there are seven 3-cycles \( C(1,2,3), C(1,2,4), C(1,2,5), C(1,3,4), C(1,3,5), C(2,3,5), C(2,3,4) \). We work in a convenient basis where \( n_{45} = n_{54} = 0 \), so the cycles with \( \bar{n}_{45} \) are zero (for instance, the 3-cycles \( C(1,4,5), C(2,4,5) \) and \( C(3,4,5) \) ). To ensure the
reality of these seven cycles we need seven constraints. In addition, there are situations in which all the 3-cycles are real but not necessarily the 4-cycles. This happens when some \( M_d \) matrix elements vanish. The maximum number of nonreal 4-cycles is achieved for instance if \( n_{12} = n_{13} = n_{23} = 0 \). We have in this case three 4-cycles not necessarily real \( C(1, 4, 2, 5), C(1, 4, 3, 5), C(2, 4, 3, 5) \), and to ensure their reality we need three more constraints. Their reality then implies the reality of the 5-cycles. Thus, \( p_{\text{min}} = 10 \) for \( n_d = 2 \). The analogous computation gives \( p_{\text{min}} = 20 \) for \( n_d = 3 \). Finally if we perform the computation for \( n_d = 1 \) we find \( p_{\text{min}} = 4 \) and we have shown that \( p = 7 \). It must be noted that for \( n_d = 1 \) the 3-cycle \( C(1, 2, 3) \) does not appear in the expressions of the invariants in this Appendix. It should appear in \( I_7 \) but due to the basis redefinition it is replaced by \( C(1, 3, 2, 4) \).

**References**

[1] N. Cabibbo, Phys. Rev. Lett. **10**, 531 (1963); M. Kobayashi and T. Maskawa, Prog. Theor. Phys. **49**, 652 (1973).

[2] Particle Data Group, R. M. Barnett et al., Phys. Rev. D **54**, 1 (1996).

[3] See for a review A. G. Cohen, D. B. Kaplan and A. E. Nelson, Annu. Rev. Nucl. Part. Phys. **43**, 27 (1993).

[4] C. Jarlskog, Phys. Rev. Lett. **55**, 1039 (1985), Z. Phys. **C29**, 491 (1985).

[5] J. Bernabéu, G. C. Branco and M. Gronau, Phys. Lett. **169B**, 243 (1986).

[6] H.-Y. Cheng, Phys. Rev. **D26**, 143 (1982); A. J. Buras, W. Slominski and H. Steger, Nucl. Phys. **B245**, 369 (1984); I. Dunietz and J. L. Rosner, Phys. Rev. **D34**, 1404 (1986); I. I. Bigi and A. I. Sanda, Nucl. Phys. **B281**, 41 (1987); P. Krawczyk, D. London, R. D. Peccei and H. Steger, Nucl. Phys. **B307**, 19 (1988); C. Dib, I. Dunietz, F. J. Gilman and Y. Nir, Phys. Rev. **D41**, 1522 (1990).

[7] Y. Nir and D. Silverman, Nucl. Phys. **B345**, 301 (1990); Y. Nir and H. R. Quinn, Phys. Rev. **D42**, 1473 (1990).

[8] J. P. Silva and L. Wolfenstein, Phys. Rev. **D55**, 5331 (1997); see also R. Aleskan, B. Kayser and D. London, Phys. Rev. Lett. **73**, 18 (1994).

[9] For recent reviews see A. Ali and D. London, in ‘Future Physics at HERA’, edited by G. Ingelman, A. De Roeck and R. Klanner, Hamburg 1996, p. 432; A. J. Buras, in ‘Phenomenology of Unification from Present to Future’, Rome 1994, [hep-ph/9406272].
M. Gronau, Nucl. Instrum. Meth. A384, 1 (1996); Y. Grossman, Y. Nir and R. Ratazzi, to appear in ‘Heavy Flavours II’, edited by A. J. Buras and M. Lindner, World Scientific, Singapore, [hep-ph/9701231]; R. Fleischer, Int. J. Mod. Phys. A12, 2459 (1997)

[10] J. Bernabéu, A. Santamaría and M. B. Gavela, Phys. Rev. Lett. 57, 1514 (1986); J. F. Donoghue and G. Valencia, Phys. Rev. Lett. 58, 451 (1987)

[11] C. R. Schmidt and M. E. Peskin, Phys. Rev. Lett. 69, 410 (1992); B. Grządkowski and F. Gunion, Phys. Lett. B287, 237 (1992); S. Bar-Shalom, D. Atwood, G. Eilam, R. R. Mendel and A. Soni, Phys. Rev. D53, 1162 (1996); D. Atwood, S. Bar-Shalom, G. Eilam and A. Soni, Phys. Rev. D54, 5412 (1996); D. Atwood and A. Soni, [hep-ph/9607481]; A. Bartl, E. Christova and W. Majerotto, Nucl. Phys. B460, 235 (1996); B. Grządkowski and Z. Hioki, in ‘Workshop on High-Energy Spin Physics’, Kobe, Japan, 1996, [hep-ph/9610306] see however C. J.-C. Im, G. L. Kane and P. J. Malde, Phys. Lett. B317, 454 (1993)

[12] F. del Aguila and M. J. Bowick, Nucl. Phys. B224, 107 (1983); P. Langacker and D. London, Phys. Rev. D38, 886 (1988); F. del Aguila, L. Ametller, G. L. Kane and J. Vidal, Nucl. Phys. B334, 1 (1990)

[13] F. del Aguila and J. Cortés, Phys. Lett. 156B, 243 (1985)

[14] G. C. Branco and L. Lavoura, Nucl. Phys. B278, 738 (1986); L. Bento, G. C. Branco and P. Parada, Phys. Lett. B267, 95 (1991); G. C. Branco, T. Morozumi, P. A. Parada and M. N. Rebelo, Phys. Rev. D48, 1167 (1993)

[15] I. Gürsey, P. Ramond and P. Sikivie, Phys. Lett. B60, 177 (1976); Y. Achiman and B. Stech, Phys. Lett. B77, 389 (1978); Q. Shafi, Phys. Lett. B79, 301 (1979); H. Ruegg and T. Schücker, Nucl. Phys. B161, 388 (1979); R. Barbieri and D. V. Nanopoulos, Phys. Lett. B91, 369 (1980); for a review on string theory, see M. Green, J. Schwarz and E. Witten, ‘Superstring theory’ (Cambridge U. P., Cambridge, 1987)

[16] C. Hamzaoui, A. I. Sanda and A. Soni, Phys. Rev. Lett. 63, 128 (1989); see also M. Gronau and J. Schechter, Phys. Rev. D31, 1668 (1985); X.-G. He and S. Pakvasa, Nucl. Phys. B278, 905 (1986)

[17] M. Gronau and D. London, Phys. Rev. D55, 2895 (1997)

[18] V. Barger, K. Whisnant and R. J. N. Phillips, Phys. Rev. D23, 2773 (1981); for other parametrizations see for instance F. J. Botella and L.-L. Chau, Phys. Lett. 168B, 97 (1986)
[19] F. del Aguila and J. A. Aguilar–Saavedra, Phys. Lett. B\textbf{386}, 241 (1996)

[20] F. del Aguila, J. A. Aguilar–Saavedra and M. Zralek, Comp. Phys. Comm. \textbf{100}, 231 (1997)

[21] L.-L. Chau and W.-Y. Keung, Phys. Rev. Lett. \textbf{53}, 1802 (1984); G. C. Branco and L. Lavoura, Phys. Lett. \textbf{208B}, 123 (1988); C. Jarlskog and R. Stora, Phys. Lett. \textbf{208B}, 268 (1988)

[22] Y. Nir and D. Silverman, Phys. Rev. D\textbf{42}, 1477 (1990); D. Silverman, Phys. Rev. D\textbf{45}, 1800 (1992); G. C. Branco, T. Morozumi, P. A. Parada and M. N. Rebelo, Ref. [14]; P. A. Parada, Ph. D. thesis, IST/CFIF (1987); G. Barenboim and F. J. Botella, \texttt{hep-ph/9708209}, to appear in Phys. Lett. B; G. Barenboim, F. J. Botella, G. C. Branco and O. Vives, \texttt{hep-ph/9709369}

[23] I. Dunietz, O. W. Greenberg and Dan-di Wu, Phys. Rev. Lett. \textbf{55}, 2935 (1985)

[24] H. Fritzsch, Phys. Lett. \textbf{73B}, 317 (1978); L. F. Li, ibid \textbf{84B}, 461 (1979); H. Georgi and D. V. Nanopoulos, Nucl. Phys. \textbf{B155}, 52 (1979); S. Dimopoulos, L. Hall and S. Raby, Phys. Rev D\textbf{45}, 4195(1992); G. F. Giudice, Mod. Phys. Lett A\textbf{7}, 2429 (1992); P. Ramond, R. G. Roberts and G. G. Ross, Nucl. Phys. B\textbf{406}, 19 (1993); G. C. Branco and J. I. Silva-Marcos, Phys. Lett. B\textbf{331}, 390 (1994); G. C. Branco, D. Emmanuel-Costa and J. I. Silva-Marcos, Phys. Rev D\textbf{56}, 115 (1997)

[25] S. Wolfram, Mathematica, a System for Doing Mathematics by Computer (Addison-Wesley Publishing Company, Redwood City, California, 1988)