On existence and smoothness of the incompressible
Navier-Stokes Equation and the Euler Equation

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Abstract: In this paper, we solve the existence and smoothness problem of the Navier-Stokes Equation, and it's also true for the Euler Equation. It's widely accepted that the Navier-Stokes Equation (Euler equation) has a local-in-time solution, but people failed to extend it to a global solution. In a previous study, we have suggested a power series solution, but could not prove its convergence. In this paper, the local-in-time solution and the series solution are combined to prove the convergence of the series, so that the series are the global smooth solution.

Key Word: the Navier-Stokes Equation; the Euler Equation; fluid dynamics; Millennium Problems

1. Introduction

Let $N \geq 2$. The Navier-Stokes equation and the Euler equation\cite{1} describe the motion of a fluid in $\mathbb{R}^N$. These equations are to be solved for an unknown velocity vector $v(x, t) = (v_1(x, t), \ldots, v_N(x, t)) \in \mathbb{R}^N$ and pressure $p(x, t) \in \mathbb{R}$, defined for position $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$ and time $t \geq 0$. For the incompressible fluids filling all of $\mathbb{R}^N$, the Navier-Stokes equations are given by:
\[
\frac{\partial v_i}{\partial t} + \sum_{j=1}^{N} v_j \frac{\partial v_i}{\partial x_j} = u \Delta v_i - \frac{\partial p}{\partial x_i}, \quad (i = 1, \ldots, N)
\] (1.1)

\[
\frac{\partial v_1}{\partial x_1} + \ldots + \frac{\partial v_N}{\partial x_N} = 0
\] (1.2)

\[
v(x, t)|_{t=0} = v_0(x)
\] (1.3)

Where \( v_0(x) = (v_{1,0}(x), \ldots, v_{N,0}(x)) \) is a given, \( C^\infty \) divergence-free vector field on \( \mathbb{R}^N \), it is the initial condition; \( \Delta = \sum \frac{\partial^2}{\partial x_j^2} \) is the Laplacian in the space variables; \( u > 0 \) is a positive coefficient (the viscosity), and the Euler equations are the above equations with \( u = 0 \).

For physically reasonable solutions, it is supposed that:

\[
(p, v) \in C^\infty[\mathbb{R}^N \times [0, \infty))
\] (1.4)

and \( v(x, t) \) doesn’t grow large as \( |x| \to \infty \). Hence, the initial condition \( v_0 \) are restricted to satisfy

\[
|\frac{\partial^\alpha v_0(x)}{\partial x_1^\alpha}| \leq C_{\alpha K} (1 + |x|)^{-K}
\] (1.5)

on \( \mathbb{R}^N \) for any multi-index \( \alpha \) and \( K > 0 \)

And there is a constant \( C \) such that:

\[
\int_{\mathbb{R}^N} |v(x, t)|^2 \, dx < C
\] (1.6)

If there is a \((p, v)\) satisfying (1.1)-(1.6), we say that the Navier-Stokes equation has a smooth solution; otherwise, there are blow-ups in the equation.

Solutions to the Navier–Stokes equations are used in many practical applications. However, theoretical understanding of the solutions to these equations is incomplete. In particular, solutions of the Navier–Stokes equations often
include turbulence, which remains one of the greatest unsolved problems in physics, despite its immense importance in science and engineering.

Even more basic (and seemingly intuitive) properties of the solutions to Navier–Stokes have never been proven. For the three-dimensional system of equations, and given some initial conditions, people have neither proved that smooth solutions always exist, nor found any counter-examples. This is called the Navier–Stokes existence and smoothness problem. The Clay Mathematics Institute in May 2000 made this problem one of its seven Millennium Prize problems in mathematics\(^1\).

By now, there are the following partial results:

1. In the case of \( N = 2 \), the Navier–Stokes problem was solved by the 1960s, there exist smooth and globally defined solutions\(^2\).

2. If the initial velocity \( v_0(x) \) is sufficiently small, then there are smooth and globally defined solutions to the Navier–Stokes equations\(^1\).

3. Given an initial velocity \( v_0(x) \), there exists a finite time \( T > 0 \), depending on \( v_0(x) \) such that the Navier–Stokes equations on \( (x, t) \in \mathbb{R}^3 \times [0, T) \) have smooth solutions. It is not known if the solutions exist beyond that "blowup time" \( T\)\(^1\).

This paper will be written in six chapters:

1. Introduction: A brief introduction to the equations and their current states.

2. Preliminary knowledge: Some elementary results about Sobolev Space that will be used in following chapters.

3. Local-in-Time Existence of Solutions: Description of the well-known local-in-time solutions.

4. The series form solution: To prove that the local-in-time solution is indeed a series form solution, and the series are convergent in the time interval \([0, T]\).
(5) Global existence and smoothness: To prove that the series form solution is convergent in \((x, t) \in \mathbb{R}^N \times [0, \infty)\), so that the global existence and smoothness problem are solved.

(6) Conclusions.

Chapter (2) and Chapter (3) are all well known contents, they are compiled from A. Bertozzi and A. Majda (Cambridge U. Press, 2002).

2. Preliminary knowledge

2.1. Calculus Inequalities for Sobolev Spaces

The Sobolev space \(H^m(\mathbb{R}^N), m \in \mathbb{Z}^+ \cup \{0\}\), consists of functions \(v \in L^2(\mathbb{R}^N)\) such that \(D^\alpha v \in L^2(\mathbb{R}^N), 0 \leq |\alpha| \leq m\), where \(D^\alpha\) is the distribution derivative. Let \(\|v\|_0 = (\int_{\mathbb{R}^N} v^2 d\xi)^{1/2}\), the \(H^m\) norm, denoted as \(\|\cdot\|_m\), is

\[
\|v\|_m = \left( \sum_{0 \leq |\alpha| \leq m} \|D^\alpha v\|_0^2 \right)^{1/2} \tag{2.1}
\]

The Sobolev space \(H^m\) generalizes to the case \(m = s \in \mathbb{R}\). Consider the functional

\(\|\cdot\|_s : \mathcal{S}(\mathbb{R}^N) \to \mathbb{R}^+ \cup \{0\}\)

defined by

\[
\|v\|_s = \left[ \int_{\mathbb{R}^N} (1 + |\xi|^2)^s |\hat{v}(\xi)|^2 d\xi \right]^{1/2} \tag{2.2}
\]

acting on the Schwarz space of rapidly decreasing smooth functions \(\mathcal{S}(\mathbb{R}^N)\). Here \(\hat{v}\) denotes the Fourier transform of \(v\). The Sobolev space \(H^s(\mathbb{R}^N), s \in \mathbb{R}\), is the completion of \(\mathcal{S}(\mathbb{R}^N)\) with respect to the norm \(\|\cdot\|_s\). For \(s = m\) the two norms are equivalent.
We review some properties of Sobolev spaces. First we state a well-known and important fact, that an $L^2$ bound on a higher derivative implies a pointwise bound on a lower derivative (see Folland, 1995).

**Lemma 2.1.** Sobolev Inequality. The space $H^{s+k}(\mathbb{R}^N), s > N/2, k \in \mathbb{Z}^+ \cup \{0\}$, is continuously embedded in the space $C^k(\mathbb{R}^N)$. That is, there exists $c > 0$ such that

$$|v|_{c^k} \leq c \|v\|_{s+k} \forall v \in H^{s+k}(\mathbb{R}^N) \quad (2.3)$$

**Lemma 2.2.** Interpolation in Sobolev Spaces. (Adams, 1995). Given $s > 0$, there exists a constant $C_s$ so that for all $v \in H^s(\mathbb{R}^N)$ and $0 < s' < s$,

$$\|v\|_{s'} \leq C_s \|v\|_{0}^{1-s'/s} \|v\|_{s}^{s'/s} \quad (2.4)$$

**Lemma 2.3.** Calculus Inequalities in the Sobolev Spaces. (i) For all $m \in \mathbb{Z}^+ \cup \{0\}$, there exists $c > 0$ such that, for all $u, v \in L^\infty \cap H^m(\mathbb{R}^N)$,

$$\|uv\|_m \leq c \{\|u\|_{L^\infty} \|D^m v\|_0 + \|D^m u\|_0 |v|_{L^\infty}\} \quad (2.5)$$

$$\sum_{0 \leq |\alpha| \leq m} \|D^\alpha (uv) - uD^\alpha v\|_0 \leq c \{\|\nabla u\|_{L^\infty} \|D^{m-1} v\|_0 + \|D^m u\|_0 |v|_{L^\infty}\} \quad (2.6)$$

(ii) For all $s > (N/2), H^s(\mathbb{R}^N)$ is a Banach algebra. That is, there exists $c > 0$ such that, for all $u, v \in H^s(\mathbb{R}^N)$,

$$\|uv\|_s \leq c \|u\|_s \|v\|_s \quad (2.7)$$

### 2.2. Properties of Mollifiers

Given any radial function $\rho(x) \in C^\infty_0(\mathbb{R}^N), \rho \geq 0, \int_{\mathbb{R}^N} \rho \, dx = 1$ (2.8)

define the mollification $J_\epsilon v$ of functions $v \in L^p(\mathbb{R}^N), 1 \leq p \leq \infty$, by

$$(J_\epsilon v)(x) = \epsilon^{-N} \int_{\mathbb{R}^N} \rho \left(\frac{x-y}{\epsilon}\right) v(y) \, dy, \quad \epsilon > 0 \quad (2.9)$$

Mollifiers have several well-known properties (see, for example, Taylor, 1991).

**Lemma 2.4.** Properties of Mollifiers. Let $J_\epsilon$ be the mollifier defined in (2.8), then $J_\epsilon v$ is a $C^\infty$ function and
(i) For all $v \in C^0(\mathbb{R}^N), J_\varepsilon v \to v$ uniformly on any compact set $\Omega$ in $\mathbb{R}^N$ and

$$\|J_\varepsilon v\|_L^\infty \leq \|v\|_L^\infty$$

(2.10)

(ii) Mollifiers commute with distribution derivatives,

$$D^\alpha J_\varepsilon v = J_\varepsilon D^\alpha v, \quad \forall |\alpha| \leq m, \quad v \in H^m$$

(2.11)

(iii) For all $u \in L^p(\mathbb{R}^N), v \in L^q(\mathbb{R}^N), (1/p) + (1/q) = 1$,

$$\int_{\mathbb{R}^N} (J_\varepsilon u)v dx = \int_{\mathbb{R}^N} u(J_\varepsilon v)dx$$

(2.12)

(iv) For all $v \in H^s(\mathbb{R}^N), J_\varepsilon v$ converges to $v$ in $H^s$ and the rate of convergence in the $H^{s-1}$ norm is linear in $\varepsilon$:

$$\lim_{\varepsilon \to 0} \|J_\varepsilon v - v\|_s = 0,$$

$$\|J_\varepsilon v - v\|_{s-1} \leq C \varepsilon \|v\|_s.$$  

(2.13, 2.14)

(v) For all $v \in H^m(\mathbb{R}^N), k \in \mathbb{Z}^+ \cup \{0\}, \varepsilon > 0$,

$$\|J_\varepsilon v\|_{m+k} \leq \frac{c_{mk}}{\varepsilon^k} \|v\|_m,$$

$$\|J_\varepsilon D^k v\|_L^\infty \leq \frac{c_k}{\varepsilon^{N/2+k}} \|v\|_0.$$  

(2.15, 2.16)

Finally, we recall some properties of the Leray projection operator $P$ on the space of divergence-free functions in $H^m(\mathbb{R}^N)$.

2.3. The Hodge Decomposition in $H^m$

**Lemma 2.5** Every vector field $v \in H^m(\mathbb{R}^N), m \in \mathbb{Z}^+ \cup \{0\},$ has the unique orthogonal decomposition

$$v = w + \nabla \phi$$

such that the Leray's projection operator $Pv = \omega$ on the divergence-free functions satisfies

(i) $Pv, \nabla \phi \in H^m, \int_{\mathbb{R}^N} P \v \cdot \nabla \phi dx = 0, \text{div} \, P \v = 0$, and

$$\|Pv\|_m^2 + \|\nabla \phi\|_m^2 = \|v\|_m^2$$

(2.17)

(ii) $P$ commutes with the distribution derivatives,

$$PD^\alpha v = D^\alpha Pv, \quad \forall \alpha \in H^m, \quad |\alpha| \leq m$$

(2.18)

(iii) $P$ commutes with mollifiers $J_\varepsilon$,

$$P(J_\varepsilon v) = J_\varepsilon (Pv), \quad \forall \v \in H^m, \quad \varepsilon > 0$$

(2.19)

(iv) $P$ is symmetric,
(Pu,v)_m = (u,Pv)_m \quad (2.20)

2.4. The Picard Theorem

Theorem 2.1. Picard Theorem on a Banach Space. Let $O \subseteq B$ be an open subset of a Banach space $B$ and let $F: O \rightarrow B$ be a mapping that satisfies the following parameters:

(i) $F(X)$ maps $O$ to $B$.

(ii) $F$ is locally Lipschitz continuous, i.e., for any $X \in O$ there exists $L > 0$ and an open neighborhood $U_X \subset O$ of $X$ such that

$$\| F(X) - F(\bar{X}) \|_B \leq L \| X - \bar{X} \|_B \quad \text{for all} \quad X, \bar{X} \in U_X$$

Then for any $X_0 \in O$, there exists a time $T$ such that the ODE

$$\frac{dX}{dt} = F(X), \quad X|_{t=0} = X_0 \in O$$

has a unique (local) solution $X \in C^1([\![-T,T]\!]; O)$.

In the preceding theorem, $\| \cdot \|_B$ denotes a norm in the Banach space $B$, and $C^1([\![-T,T]\!]; B)$ denotes the space of $C^1$ functions $X(t)$ on the open interval $(-T,T)$ with values in $B$.

2.5. Uniqueness of solutions

Theorem 2.2. Let $v_1$, $v_2$ be two smooth solutions to equations (1.1),(1.2) and (1.3) in $(x,t) \in \mathbb{R}^N \times [0,T]$. Then $v_1 = v_2$.

3. Local-in-Time Existence of Solutions

It’s well known that there are local-in-time existence solutions for the Navier-Stokes equation and the Euler equation. In this chapter, we will discuss this result.

3.1 A regularization of the equations

The equations in (1.1),(1.2) and (1.3) can be rewritten as following:

$$v_t + v \cdot \nabla v = -\nabla p + v\Delta v$$

$${\text{div}} v = 0$$

$$v|_{t=0} = v_0 \quad (3.1)$$
The Picard theorem is a central tool in dealing with them. However, as the equations contain unbounded operators so that we cannot directly apply the Picard theorem for ODEs in a Banach space. A strategy is to design an approximation of the equations by regularization for which people can easily prove a global existence of solutions, and can also show an analogous energy estimate that is independent of the regularization parameter.

An approximate equation to the Navier-Stokes equation has been created, in which the mollifier $\mathcal{J}_\epsilon$ defined in (2.9) is used to regularize the equations, so that the new equation can satisfy the condition of the Picard theorem.

\[
\begin{align*}
\dot{v}_\epsilon^f + J_\epsilon[(J_\epsilon v^f) \cdot \nabla (J_\epsilon v^f)] &= -\nabla p^\epsilon + \nu J_\epsilon (J_\epsilon \Delta v^f) \\
\text{div } v^\epsilon &= 0 \\
v^\epsilon|_{t=0} &= v_0
\end{align*}
\] (3.2)

Equations (3.2) explicitly contain the pressure $p^\epsilon$. Following Leray, we can eliminate $p^\epsilon$ and the incompressibility condition $\text{div } v^\epsilon = 0$ by projecting these equations onto the space of divergence-free functions:

\[V^s = \{ v \in H^s(\mathbb{R}^N) : \text{div } v = 0 \}. \] (3.3)

Because the Leray projection operator $P$ commutes with derivatives and mollifiers (Lemma 2.5) and $P v^\epsilon = v^\epsilon$, we have

\[
\dot{v}_\epsilon^f + P J_\epsilon[(J_\epsilon v^f) \cdot \nabla (J_\epsilon v^f)] = \nu J_\epsilon^2 \Delta v^f
\] (3.4)

The regularized Euler or Navier-Stokes equation in (3.2) reduces to an ODE in the Banach space $V^s$:

\[
\begin{align*}
\frac{dv^\epsilon}{dt} &= F_\epsilon(v^\epsilon) \\
v^\epsilon|_{t=0} &= v_0
\end{align*}
\] (3.5)

where

\[
F_\epsilon(v^\epsilon) = \nu J_\epsilon^2 \Delta v^\epsilon - P J_\epsilon[(J_\epsilon v^f) \cdot \nabla (J_\epsilon v^f)] = F_\epsilon^1(v^\epsilon) - F_\epsilon^2(v^\epsilon).
\] (3.6)

### 3.2 Global existence of regularized equations

**Theorem 3.1.** Local existence of solutions to the regularized equations.

Consider an initial condition $v_0 \in V^m$, $m \in \mathbb{Z}^+ \cup \{0\}$. Then
(i). for any $\epsilon > 0$ there exists the unique solution $v^\epsilon \in C^1([0, T_\epsilon); V^m)$ to the ODE in (3.5), where $T_\epsilon = T(\|v_0\|_m, \epsilon)$;

(ii). on any time interval $[0, T]$ on which this solution belongs to $C^1([0, T]; V^0)$,

$$\sup_{0 \leq t \leq T} \|v^\epsilon\|_0 \leq \|v_0\|_0$$

(3.7)

Proof.

(i). First, we will show that the function $F_\epsilon$ in (3.6) maps $V^m$ into $V^m$ and is locally Lipschitz continuous. Note that $F_\epsilon : V^m \rightarrow V^m$ because div $v^\epsilon = 0, P$ maps into divergence-free vector fields, and $J_\epsilon$ commutes with derivatives.

The definition of Sobolev spaces and estimate (2.15) for mollifiers implies that

$$\|F_\epsilon^1(v^1) - F_\epsilon^1(v^2)\|_m = \|\nabla J_\epsilon \Delta (v^1 - v^2)\|_m$$

$$\leq \|\nabla J_\epsilon^2 (v^1 - v^2)\|_{m+2}$$

$$\leq \frac{c}{\epsilon^2} \|v^1 - v^2\|_m.$$ 

Calculus inequality (2.5) and commutation property (2.19) of $P$ and $J_\epsilon$ imply that

$$\| F_\epsilon^2(v^1) - F_\epsilon^2(v^2)\|_m$$

$$\leq \| P J_\epsilon \{ J_\epsilon (v^1) \cdot \nabla J_\epsilon (v^1 - v^2) \} \|_m + \| P J_\epsilon \{ J_\epsilon (v^1 - v^2) \cdot \nabla J_\epsilon v^2 \} \|_m$$

$$\leq c \left\{ \| J_\epsilon v^1 \|_{L^\infty} \| D^m J_\epsilon \nabla (v^1 - v^2) \|_0 + \| D^m J_\epsilon v^1 \|_0 \| J_\epsilon \nabla (v^1 - v^2) \|_{L^\infty} \right\}$$

$$+ \| J_\epsilon (v^1 - v^2) \|_{L^\infty} \| D^m J_\epsilon \nabla v^2 \|_0 + \| D^m J_\epsilon (v^1 - v^2) \|_0 \| J_\epsilon \nabla v^2 \|_{L^\infty}$$

Mollifier properties (2.15) and (2.16) then give

$$\| F_\epsilon^2(v^1) - F_\epsilon^2(v^2)\|_m \leq \frac{c}{\epsilon^{N/2 + 1 + m}} \left( \| v^1 \|_0 + \| v^2 \|_0 \right) \| v^1 - v^2\|_m.$$ 

The final result is

$$\|F_\epsilon(v^1) - F_\epsilon(v^2)\|_m \leq \left( \frac{c}{\epsilon^2} + \frac{c}{\epsilon^{N/2 + 1 + m}} \left( \| v^1 \|_0 + \| v^2 \|_0 \right) \right) \| v^1 - v^2\|_m$$

(3.8)

so that $F_\epsilon$ is locally Lipschitz continuous on any open set:

$$O^M = \{ v \in V^m \mid \| v \|_m < M \}.$$ 

Taking $v^1 = v, v^2 = 0$, there is

$$\|F_\epsilon(v)\|_m \leq \left( \frac{c}{\epsilon^2} + \frac{c}{\epsilon^{N/2 + 1 + m}} \| v \|_0 \right) \| v \|_m$$

Let $M = 2 \| v_0 \|_m, T_\epsilon = (M - \| v_0 \|_m) / \left( \left( \frac{c}{\epsilon^2} + \frac{c}{\epsilon^{N/2 + 1 + m}} M \right) M \right)$
The Picard iteration can be built as:

\[ v_0^\epsilon = v_0 \]

\[ v_{n+1}^\epsilon = v_0 + \int_0^t F_\epsilon(v_n^\epsilon) ds, \quad (n \geq 0) \]

It is easy to see that \( \| v_0^\epsilon \|_m \leq M \), and when \( t \leq T_\epsilon \),

\[ \| v_{n+1}^\epsilon \|_m \leq \| v_0^\epsilon \|_m + \int_0^t \left( \frac{C v}{\epsilon^2} + \frac{c}{\epsilon^{N/2+1+m}} M \right) M ds \leq M, \quad (n \geq 0) \]

By the Picard theorem, \( v_n^\epsilon \) converges to a divergence free function \( v^\epsilon = \lim_{n \to \infty} v_n^\epsilon \), the unique solution to (3.2) in \( t \in [0, T_\epsilon] \).

(ii). Now we prove energy bound (3.7): if \( v^\epsilon \in C^1([0, T_\epsilon); V^m \cap O^M), m \in \mathbb{Z}_+ \cup \{0\} \).

Take the \( L^2 \) inner product of (3.5) with \( v^\epsilon \) to obtain

\[ \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |v^\epsilon|^2 dx = \nu \int_{\mathbb{R}^3} v^\epsilon \mathcal{J}_\epsilon^2 \Delta v^\epsilon dx - \int_{\mathbb{R}^3} v^\epsilon \mathcal{J}_\epsilon \left( (\mathcal{J}_\epsilon v^\epsilon) \cdot \nabla (\mathcal{J}_\epsilon v^\epsilon) \right) dx \]

The properties of mollifiers and the operator \( P \) from Lemmas 2.4 and 2.5 imply, after integration by parts, that

\[ \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} (v^\epsilon)^2 dx = \nu \int_{\mathbb{R}^3} (\mathcal{J}_\epsilon v^\epsilon) \Delta (\mathcal{J}_\epsilon v^\epsilon) dx + \frac{1}{2} \int_{\mathbb{R}^3} (\mathcal{J}_\epsilon v^\epsilon) \cdot \nabla (\mathcal{J}_\epsilon v^\epsilon)^2 dx \]

\[ = -\nu \int_{\mathbb{R}^3} (\mathcal{J}_\epsilon \nabla v^\epsilon)^2 dx - \frac{1}{2} \int_{\mathbb{R}^3} (\text{div} (\mathcal{J}_\epsilon v^\epsilon)) (\mathcal{J}_\epsilon v^\epsilon)^2 dx \]

so that

\[ \frac{d}{dt} \| v^\epsilon \|^2_0 + 2\nu \| \nabla \mathcal{J}_\epsilon v^\epsilon \|^2_0 = 0 \]

Because \( \nu \geq 0 \), we obtain energy bound (3.7).

**Theorem 3.2.** Global Existence of Regularized Solutions.

Given an initial condition \( v_0 \in V^m, m \in \mathbb{Z}_+ \cup \{0\} \), for any \( \epsilon > 0 \) there exists for all time a unique solution \( v^\epsilon \in C^1([0, \infty); V^m) \) to regularized equation (3.5).
Proof: We will extend the local solution \( v^\varepsilon \) globally. Suppose \( T > 0 \) is finite, and it’s the maximal time that admit a local solution \( v^\varepsilon \) to the regularized equation (3.5) in \((x, t) \in \mathbb{R}^N \times [0, T] \). Firstly, we can calculate a priori bound on \( \|v^\varepsilon(\cdot, t)\|_m \).

By using (3.8), and let \( v^\varepsilon(\cdot, t) = v^1(x, t), v^2(x, t) = 0 \), there is:

\[
\frac{d}{dt} \|v^\varepsilon(\cdot, t)\|_m \leq \left( \frac{CV}{\varepsilon^2} + \frac{C}{N^{7/2+1+m}} \|v^\varepsilon\|_0 \right) \|v^\varepsilon\|_m \leq \left( \frac{CV}{\varepsilon^2} + \frac{C}{N^{7/2+1+m}} \|v^\varepsilon\|_0 \right) \|v^\varepsilon\|_m = C(\|v^\varepsilon\|_0, \varepsilon, N) \|v^\varepsilon\|_m
\]

Then, by the Grönwall’s lemma, there is a priori bound \( \|v^\varepsilon(\cdot, T)\|_m \leq e^{CT} \).

Now, let’s consider the regularized equation with \( v^\varepsilon(\cdot, T) \) as new initial condition:

\[
\frac{dv^\varepsilon}{dt} = F^\varepsilon(v^\varepsilon) \\
v^\varepsilon|_{t=0} = v^\varepsilon(\cdot, T)
\]

There will exist a \( T' > 0 \) and a local solution \( \tilde{v}^\varepsilon \) in \( t \in [0, T'] \). With this \( T' \) and \( \tilde{v}^\varepsilon \), \( v^\varepsilon \) can then extend to \((x, t) \in \mathbb{R}^N \times [0, T + T'] \), this contradicts the assumption.

So that, we have extended \( v^\varepsilon \) globally to \((x, t) \in \mathbb{R}^N \times [0, \infty) \).

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3.3. Local-In-Time Existence of Solutions to the Euler and the Navier-Stokes Equations

Definition 3.1. The space \( C^1([0, T]; H^s(\mathbb{R}^N)) \) denotes continuity on the interval \([0, T] \) with values in the weak topology of \( H^s \), that is, for any fixed \( \varphi \in H^s, [\varphi, u(t)] \) is a continuous scalar function on \([0, T] \), where

\[
(u, v)_s = \sum_{a \leq s} \int_{\mathbb{R}^3} D^a u \cdot D^a v \, dx
\]

Proposition 3.1. The \( H^m \) Energy Estimate. Let \( v_0 \in V^m \). Then the unique regularized solution \( v^\varepsilon \in C^1([0, \infty); V^m) \) to Equation (3.5) satisfies

\[
\frac{d}{dt} \frac{1}{2} \|v^\varepsilon\|_m^2 + \nu \|\nabla v^\varepsilon\|_2^2 \leq c_m \|\nabla J^\varepsilon v^\varepsilon\|_{L^\infty} \|v^\varepsilon\|_m^2 \tag{3.9}
\]

Proof. Let \( v^\varepsilon \) be a smooth solution to Equation (3.5):
\[ v_t^\epsilon = \nu J^2 \Delta v^\epsilon - P J^\epsilon \left( (J^\epsilon v^\epsilon) \cdot \nabla (J^\epsilon v^\epsilon) \right) \]

In particular, we used integration by parts by means of the divergence theorem. Following a similar procedure, we take the derivative \( D^\alpha, |\alpha| \leq m \) of this equation and then the \( L^2 \) inner product with \( D^\alpha v^\epsilon \):

\[
(D^\alpha v_t^\epsilon, D^\alpha v^\epsilon) = (D^\alpha J^2 \Delta v^\epsilon, D^\alpha v^\epsilon) - \left\{ D^\alpha P J^\epsilon \left( (J^\epsilon v^\epsilon) \cdot \nabla (J^\epsilon v^\epsilon) \right), D^\alpha v^\epsilon \right\} \\
= -\nu \| J^\epsilon D^\alpha \nabla v^\epsilon \|_0^2 - \left\{ P J^\epsilon \left[ (J^\epsilon v^\epsilon) \cdot \nabla (D^\alpha J^\epsilon v^\epsilon) \right], D^\alpha v^\epsilon \right\} \\
- \left\{ (D^\alpha P J^\epsilon \left( (J^\epsilon v^\epsilon) \cdot \nabla (J^\epsilon v^\epsilon) \right) - P J^\epsilon \left( (J^\epsilon v^\epsilon) \cdot \nabla (D^\alpha J^\epsilon v^\epsilon) \right) \right\}, D^\alpha v^\epsilon \right) \\
\]

Lemmas 2.4 and 2.5 and the divergence theorem imply that

\[
\left\{ P J^\epsilon \left[ (J^\epsilon v^\epsilon) \cdot \nabla (D^\alpha J^\epsilon v^\epsilon) \right], D^\alpha v^\epsilon \right\} = \frac{1}{2} \left[ J^\epsilon v^\epsilon, \nabla (J^\epsilon D^\alpha v^\epsilon)^2 \right] \\
= -\frac{1}{2} (\text{div} J^\epsilon v^\epsilon, |J^\epsilon D^\alpha v^\epsilon|^2) = 0.
\]

Summing over \( |\alpha| \leq m \), we find that calculus inequality (2.6) implies that

\[
\frac{1}{2} \frac{d}{dt} \| v^\epsilon \|_m^2 + \nu \| J^\epsilon \nabla v^\epsilon \|_m^2 \\
\leq \| v^\epsilon \|_m \sum_{|\alpha| \leq m} \| D^\alpha \left[ (J^\epsilon v^\epsilon) \cdot \nabla (J^\epsilon v^\epsilon) \right] - \left[ (J^\epsilon v^\epsilon) \cdot \nabla (D^\alpha J^\epsilon v^\epsilon) \right] \|_0 \\
\leq c_m \| v^\epsilon \|_m \left( \| \nabla J^\epsilon v^\epsilon \|_{L^\infty} \| D^{m-1} \nabla J^\epsilon v^\epsilon \|_0 + \| D^m J^\epsilon v^\epsilon \|_0 \| \nabla J^\epsilon v^\epsilon \|_{L^\infty} \right) \\
\leq c_m \| J^\epsilon \nabla v^\epsilon \|_{L^\infty} \| J^\epsilon v^\epsilon \|_m^2
\]

so that

\[
\frac{d}{dt} \frac{1}{2} \| v^\epsilon \|_m^2 + \nu \| J^\epsilon \nabla v^\epsilon \|_m^2 \leq c_m \| J^\epsilon \nabla v^\epsilon \|_{L^\infty}^2 \| v^\epsilon \|_m^2
\]

**Theorem 3.3.** Local-in-Time Existence of Solutions to the Euler and the Navier-Stokes equations.

Given an initial condition \( v_0 \in V^m, m \geq \left\lceil \frac{N}{2} \right\rceil + 2 \), then

(i) there exists a time \( T \) with the rough upper bound

\[
T \leq \frac{1}{c_m \| v_0 \|_m},
\]

such that for any viscosity \( 0 \leq \nu < \infty \) there exists the unique solution \( v^\nu \in C([0, T]; C^2(\mathbb{R}^N)) \cap C^1([0, T]; C(\mathbb{R}^N)) \) to the Euler or the Navier-Stokes equation. The solution \( v^\nu \) is the limit of a subsequence of approximate solutions, \( v^\epsilon \), of Equation
(3.5) and (3.6).

(ii) The approximate solutions \( v^\epsilon \) and the limit \( v^\nu \) satisfy the higher-order energy estimates

\[
\sup_{0 \leq t \leq T} \| v^\epsilon \|_m \leq \frac{\| v_0 \|_m}{1 - c_m T \| v_0 \|_m}, \\
\sup_{0 \leq t \leq T} \| v^\nu \|_m \leq \frac{\| v_0 \|_m}{1 - c_m T \| v_0 \|_m}.
\]

**Proof.** The strategy for the local-existence proof, is to first prove the bounds (3.10) in the high norm, then show that we actually have a contraction in the \( H^0 = L^2 \) norm. We then apply an interpolation inequality to prove convergence as \( \epsilon \to 0 \).

(1). First we show that the family \( (v^\epsilon) \) of regularized solutions is uniformly bounded in \( H^m \). Energy estimate (3.9) and Sobolev inequality (2.3) imply that the time derivative of \( \| v^\epsilon \|_m \) can be bounded by a quadratic function of \( \| v^\epsilon \|_m \) independent of \( \epsilon \), provided that \( m > N/2 + 1 \):

\[
\frac{d}{dt} \| v^\epsilon \|_m \leq c_m \| J_\epsilon \nabla v^\epsilon \|_{L^\infty} \| v^\epsilon \|_m \leq c_m \| v^\epsilon \|_m^2,
\]

and hence, for all \( \epsilon \),

\[
\sup_{0 \leq t \leq T} \| v^\epsilon \|_m \leq \frac{\| v_0 \|_m}{1 - c_m T \| v_0 \|_m} = \| v_0 \|_m + \frac{\| v_0 \|_m^2 c_m T}{1 - c_m T \| v_0 \|_m}.
\]

Thus the family \( (v^\epsilon) \) is uniformly bounded in \( C([0, T]; H^m) \), \( m > N/2 \), provided that \( T < (c_m \| v_0 \|_m)^{-1} \).

Furthermore, the family of time derivatives \( (dv^\epsilon/dt) \) is uniformly bounded in \( H^{m-2} \). Equation (3.5) implies that, for \( m > (N/2) + 2 \),

\[
\| \frac{dv^\epsilon}{dt} \|_{m-2} \leq v \| J_\epsilon^2 \Delta v^\epsilon \|_{m-2} + \| P J_\epsilon \left[ (J_\epsilon v^\epsilon) \cdot \nabla (J_\epsilon v^\epsilon) \right] \|_{m-2}
\leq cv \| v^\epsilon \|_m + c \| v^\epsilon \|_m^2.
\]

Hence the previous estimates yield that for a given \( 0 \leq v < \infty \) the family \( (dv^\epsilon/dt) \) is uniformly bounded in \( H^{m-2} \).
We now show that the solutions $v^\epsilon$ to regularized equation (3.5) form a contraction in the low norm $C_0, T; L^2(\mathbb{R}^N)$. To do so, we will prove that the family $v^\epsilon$ forms a Cauchy sequence in $C_0, T; L^2(\mathbb{R}^N)$. In particular, there exists a constant $C$ that depends on only $\|v_0\|_m$ and the time $T$ so that, for all $\epsilon$ and $\epsilon'$,

$$\sup_{0 < t < T} \|v^\epsilon - v^{\epsilon'}\|_0 \leq C \max(\epsilon, \epsilon')$$

Using (3.5), we have that

$$\frac{d}{dt} \frac{1}{2} \|v^\epsilon - v^{\epsilon'}\|_0^2 = v(J^2_e \Delta v^\epsilon - J^2_e \Delta v^{\epsilon'}, v^\epsilon - v^{\epsilon'}) - \{ P J_e (J_e v^\epsilon) \cdot \nabla (J_e v^\epsilon) \}$$

$$= -P J_e \left[ (J_e v^\epsilon) \cdot \nabla (J_e v^\epsilon) \right], v^\epsilon - v^{\epsilon'}$$

$$= T_1 + T_2$$

We can estimate the first term, $T_1$, by means of integration by parts and mollifier property (2.14):

$$\left( J^2_e \Delta v^\epsilon - J^2_e \Delta v^{\epsilon'}, v^\epsilon - v^{\epsilon'} \right) = \left[ (J^2_e - J^2_e) \Delta v^\epsilon, v^\epsilon - v^{\epsilon'} - \|J_e \cdot \nabla (v^\epsilon - v^{\epsilon'})\|_0^2. \right.$$

$$\leq C \max(\epsilon, \epsilon') \|v^\epsilon\|_4 \|v^{\epsilon'}\|_0.$$

We can estimate the second term, $T_2$, by also using the same tools and the fact that $v^\epsilon$ is divergence free:

$$\{ P J_e (J_e v^\epsilon) \cdot \nabla (J_e v^\epsilon) \} = \{ (J_e - J_e) J_e v^\epsilon \cdot \nabla (J_e v^\epsilon) \}, v^\epsilon - v^{\epsilon'}$$

$$= \{ J_e \left[ (J_e - J_e) v^\epsilon \cdot \nabla (J_e v^\epsilon) \right], v^\epsilon - v^{\epsilon'} \}$$

$$+ \{ J_e \left[ (J_e - J_e) v^\epsilon \cdot \nabla (J_e v^\epsilon) \right], v^\epsilon - v^{\epsilon'} \}$$

$$+ \{ J_e \left[ (J_e - J_e) v^\epsilon \cdot \nabla (J_e v^\epsilon) \right], v^\epsilon - v^{\epsilon'} \}$$

$$= R_1 + R_2 + R_3 + R_4 + R_5$$

Using calculus inequality (2.5) and Sobolev inequality (2.3), we get

$$|R_1| \leq C \max(\epsilon, \epsilon') \|v^\epsilon \cdot \nabla (v^\epsilon)\|_1 \|v^\epsilon - v^{\epsilon'}\|_0$$

$$\leq C \max(\epsilon, \epsilon') \left( \|v^\epsilon\|_{L^\infty} + \|v^{\epsilon'}\|_{L^\infty} \right) \|v^\epsilon\|_1 \|v^\epsilon - v^{\epsilon'}\|_0$$

$$\leq C \max(\epsilon, \epsilon') \|v^\epsilon\|_m^2 \|v^\epsilon - v^{\epsilon'}\|_0$$

A similar estimate holds for $R_2$ and $R_4$. 

$$|R3| \leq C \|v^\epsilon - v^\epsilon\|_0^2 \|v^\epsilon\|_m + C \|v^\epsilon - v^\epsilon\|_0 \max(\epsilon, \epsilon^\prime) \|v^\epsilon\|_m^2$$

and, finally, integration by parts and the fact that $v^\epsilon$ is divergence free shows that

$$R5 = \left\{ J_\epsilon \cdot v^\epsilon \cdot \nabla \left[ J_\epsilon \cdot (v^\epsilon - v^\epsilon) \right], J_\epsilon \cdot (v^\epsilon - v^\epsilon)^2 \right\}$$

$$= \frac{1}{2} \int_{\mathbb{R}^3} J_\epsilon \cdot v^\epsilon \cdot \nabla \left[ \|J_\epsilon \cdot (v^\epsilon - v^\epsilon)^2 \right] dx = 0$$

Putting this all together gives

$$\frac{d}{dt} \|v^\epsilon - v^\epsilon^\prime\|_0 \leq C(M) \left[ \max(\epsilon, \epsilon^\prime) + \|v^\epsilon - v^\epsilon\|_0 \right]$$

where $M$ is an upper bound, from relation (3.13) for the $\|v^\epsilon\|_m$ on $[0, T]$. Integrating this yields

$$\sup_{0 \leq t \leq T} \|v^\epsilon - v^\epsilon^\prime\|_0 \leq e^{C(M)T} \left[ \max(\epsilon, \epsilon^\prime) + \|v^\epsilon_0 - v^\epsilon_0\|_0 \right] - \max(\epsilon, \epsilon^\prime) \leq C(M, T) \max(\epsilon, \epsilon^\prime)$$

(3.13)

where we establish the final inequality by recalling that $v^\epsilon_0 = v^\epsilon_0^\prime$.

Thus $v^\epsilon$ is a Cauchy sequence in $C([0, T]; L^2(\mathbb{R}^N))$ so that it converges strongly to a value $v^\nu \in C([0, T]; L^2(\mathbb{R}^N))$.

We have just proved the existence of a $v$ such that

$$\sup_{0 \leq t \leq T} \|v^\epsilon - v\|_0 \leq C\epsilon$$

(3.14)

We now use the fact that the $v^\epsilon$ are uniformly bounded in a high norm to show that we have strong convergence in all the intermediate norms.

To do this, we apply interpolation lemma 2.4 to the difference $v^\epsilon - v$. Taking $s = m$ and using relations (3.13) and (3.14) gives

$$\sup_{0 \leq t \leq T} \|v^\epsilon - v\|_{m^\prime} \leq C(\|v_0\|_{m^\prime}, T) \epsilon^{1 - m^\prime/m}$$

Hence for all $m^\prime < m$ we have strong convergence in $C([0, T]; H^{m^\prime}(\mathbb{R}^N))$. With $0 < 7/2 < m^\prime < m$, this implies strong convergence in $C([0, T]; C^2(\mathbb{R}^N))$. Also, from the equation

$$v^\epsilon_t = v J_\epsilon^2 \Delta v^\epsilon - P J_\epsilon \left[ (J_\epsilon v^\epsilon) \cdot \nabla (J_\epsilon v^\epsilon) \right]$$
so that $v_\varepsilon^\varepsilon$ converges in $\{C[0,T],C(\mathbb{R}^N)\}$ to $\nu \Delta v - \nu (\nu \cdot \nabla \nu)$. Because $v^\varepsilon \to v$, the distribution limit of $v_\varepsilon^\varepsilon$ must be $v_\varepsilon$ so, in particular, $v$ is a classical solution of the Navier-Stokes (Euler) equations.

4. The series form solution

**Theorem 4.1.** The series form solution. The local-in-time solutions to the Navier-Stokes equation and Euler equation in Theorem 3.3 are indeed in series form $v(x, t) = \sum_{n=0}^{\infty} a_n(x)t^n$, $(x, t) \in \mathbb{R}^N \times [0, T]$ , where $a_0(x) = v_0(x)$, and $a_n(x) = \frac{1}{n}\nu \Delta a_{n-1}(x) - \frac{1}{n}P[\sum_{i=0}^{n-1} a_i(x) \cdot \nabla a_{n-1-i}(x)]$ are all known functions determined only by $v_0(x)$.

**Proof:** Suppose $v^\varepsilon$ is the unique solution to the regularized equation (3.5), and $v$ is the local-in-time solution in theorem 3.3. Let

$$\frac{\|v_0\|_m}{1 - c_m \varepsilon T} \|v_0\|_m = M_0$$

Then From (3.10), we have $\sup_{0 \leq t \leq T} \|v^\varepsilon\|_m \leq M_0$, $\sup_{0 \leq t \leq T} \|v\|_m \leq M_0$.

Let $M = 2M_0$, and $T_\varepsilon = (M - M_0)/\left[\frac{c\varepsilon^2}{\nu^2} + \frac{\varepsilon}{\nu^{2+1+m}M}\right]$. Since $v = \lim_{\varepsilon \to 0} v^\varepsilon$, $v^\varepsilon = \lim_{n \to \infty} v_n^\varepsilon$, and $v_n^\varepsilon$ is a Picard iteration, we will prove the theorem in following steps:

(1). To prove that the local solution $v^\varepsilon(x, t), t \in [0, T_\varepsilon]$ to the regularized equation (3.5) is in series form $v^\varepsilon(x, t) = \sum_{n=0}^{\infty} a_n^\varepsilon(x)t^n$.

Recalling the proving process of Theorem 3.1, the Picard iteration is $v_0^\varepsilon = v_0$, and
\[ v_{n+1}^\epsilon = v_0 + \int_0^1 F_\epsilon(v_n^\epsilon) ds, (n \geq 0) \]

Where \( \| v^\epsilon_n \|_m \leq M, (n \geq 0) \) is uniformly bounded. It can be checked that

\[ v_1^\epsilon = v_0 + \int_0^1 F_\epsilon(v_0^\epsilon) ds = v_0 + t\{v_0^\epsilon \Delta v_0 - P J_\epsilon [(J_\epsilon v_0) \cdot \nabla (J_\epsilon v_0)]\} = a_0(x) + a_1^\epsilon(x)t \]

Generally speaking, if \( v_n^\epsilon(x, t) = \sum a_i^\epsilon(x) t^i \) has been got, then \( v_{n+1}^\epsilon = v_0 + \int_0^1 F_\epsilon(v_n^\epsilon) ds = \int_0^1 \{v_0^\epsilon \Delta v_n^\epsilon - P J_\epsilon [(J_\epsilon v_n^\epsilon) \cdot \nabla (J_\epsilon v_n^\epsilon)]\} ds = \sum a_i^\epsilon(x) t^i \).

Where \( a_{n+1}^\epsilon(x) = \frac{1}{n+1} v_0^\epsilon J_\epsilon (\Delta a_n^\epsilon(x)) - \frac{1}{n+1} P J_\epsilon \left[ \sum_{i=0}^n (J_\epsilon a_i^\epsilon(x) \cdot \nabla J_\epsilon a_n^\epsilon(x)) \right] \)

Hence, \( \nu = \lim_{n \to \infty} v_n^\epsilon = \sum_{i=0}^\infty a_i^\epsilon(x) t^i = \sum_{n=0}^\infty a_n^\epsilon(x) t^n, t \in [0, T_\epsilon] \).

The rate of convergence can also be evaluated. To do so, let \( C_\epsilon = \left( \frac{c v}{\epsilon^2} + \frac{c}{\epsilon^{N/2+1+m}} M \right) \) and to rewrite \( v^\epsilon \):

\[ v^\epsilon = v_0^\epsilon + \sum_{i=1}^n (v_i^\epsilon - v_{i-1}^\epsilon) + \sum_{i=n+1}^\infty (v_i^\epsilon - v_{i-1}^\epsilon) = v_n^\epsilon + R_n^\epsilon \]

It can be calculated that \( \| v_0^\epsilon \|_m \leq M_0 \)

\[ \| v_1^\epsilon - v_0^\epsilon \|_m = \| \int_0^1 F_\epsilon(v_0^\epsilon) ds \|_m \leq C_\epsilon M_0 t \leq 2C_\epsilon M_0 t \]

\[ \| v_2^\epsilon - v_1^\epsilon \|_m \leq \int_0^1 \| F_\epsilon(v_1^\epsilon) - F_\epsilon(v_0^\epsilon) \|_m ds \leq \left( \frac{c v}{\epsilon^2} + \frac{2c M}{\epsilon^{N/2+1+m}} \right) \int_0^1 \| v_1^\epsilon - v_0^\epsilon \|_m ds \leq \frac{1}{2!} M_0 C_\epsilon^2 2^2 t^2. \]

... 

\[ \| v_n^\epsilon - v_{n-1}^\epsilon \|_m \leq \frac{1}{n!} M_0 C_\epsilon^n 2^n t^n \]
Hence, \( \|v_n\|_m = \|v_0\|_m + \sum_{i=1}^{n} (v_i - v_{i-1}) \|_m \leq \|v_0\|_m + \sum_{i=1}^{n} \|v_i - v_{i-1}\|_m \leq M_0 \sum_{i=0}^{n} \frac{1}{i!} C_i 2^i t^i \leq M_0 e^{2tC}.

It can also be calculated that:

\[
\|R_n\|_m \leq \sum_{i=n+1}^{\infty} \|v_i - v_{i-1}\|_m \leq M_0 \frac{e^{2tC}}{(n+1)!} 2^{n+1} t^{n+1} \leq M_0 \frac{e^{2tC}}{(n+1)!} C^{n+1} 2^{n+1} t^{n+1}
\]

Hence \( \sup_{t \in [0, T_\varepsilon]} \|R_n\|_m \leq \frac{e M_0}{(n+1)!} \).

(2). To extend the series from \( t \in 0, T_\varepsilon \) to \( t \in 0, T \).

Let \( k_\varepsilon = \lceil T/T_\varepsilon \rceil \). If \( k_\varepsilon > 1 \), then for the interval \( t \in [T_\varepsilon, 2T_\varepsilon] \), consider the regularized equation

\[
\frac{dv^\varepsilon}{dt} = F_\varepsilon(v^\varepsilon)
\]

\[v^\varepsilon|_{t=T_\varepsilon} = v^\varepsilon(., T_\varepsilon)\]

And build another Picard iteration \( \tilde{v}_0 = v^\varepsilon(., T_\varepsilon) \), \( \tilde{v}_{n+1} = v^\varepsilon(., T_\varepsilon) + \int_{T_\varepsilon}^{t} F_\varepsilon(\tilde{v}_n)ds \), \( (n \geq 0) \). It can also be checked that \( \|\tilde{v}_n\|_m \leq M, (n \geq 0) \) are uniformly bounded, and there is a solution \( \tilde{v}^\varepsilon = \lim_{n \to \infty} \tilde{v}_n = \sum_{i=0}^{\infty} b_i(\varepsilon)(t - T_\varepsilon)^n = \sum_{i=0}^{\infty} c_i(\varepsilon)t^n, t \in [T_\varepsilon, 2T_\varepsilon] \). Since \( v^\varepsilon \) is equal to \( \tilde{v}^\varepsilon \) in the vicinity near the point \( t = T_\varepsilon \), hence \( c_i(\varepsilon) = a_i(\varepsilon), (i \geq 0) \), so that the series has been extended to \( t \in [T_\varepsilon, 2T_\varepsilon] \).

Through \( v_n^\varepsilon \), it can also be calculated that \( \sup_{t \in [T_\varepsilon, 2T_\varepsilon]} \|R_n\|_m \leq \frac{e M_0}{(n+1)!} \).

Repeating this process for \( t \in [T_\varepsilon, 2T_\varepsilon] \), ... \( t \in [(k_\varepsilon - 1)T_\varepsilon, k_\varepsilon T_\varepsilon] \) and \( t \in [k_\varepsilon T_\varepsilon, T] \) respectively, the series can then be extended to \( t \in [0, T] \).

\[
v^\varepsilon = \sum_{n=0}^{\infty} a_n(\varepsilon)t^n, t \in [0, T] \quad (4.1)
\]

\[
v^\varepsilon = v_n^\varepsilon + R_n^\varepsilon \quad (4.2)
\]
To compute the local-in-time solution \( v \) to the Navier-Stokes equation and the Euler equation.

Using the sobolev inequality (2.3), we knew that \( |R^\epsilon_n| \leq \frac{c e M_0}{(n+1)!} \).

In (4.2), let \( \epsilon \to 0 \), we have

\[
v = \sum_{i=0}^{n} a_i(x) t^i + r_n(t), \quad t \in [0, T]
\]

(4.4)

Where \( r_n(t) = \lim_{\epsilon \to 0} R^\epsilon_n \), hence \( |r_n(t)| \leq \frac{c e M_0}{(n+1)!} \).

So that, we can take \( n \to \infty \) in (4.4), and got

\[
v = \sum_{i=0}^{\infty} a_i(x) t^i = \sum_{n=0}^{\infty} a_n(x) t^n, \quad t \in [0, T]
\]

(4.5)

Where \( a_0(x) = v_0(x) \), and \( a_n(x) = \frac{1}{n} \nu \Delta a_{n-1}(x) - \frac{1}{n} P \left[ \sum_{i=0}^{n-1} \left( a_i(x) \cdot \nabla a_{n-1-i}(x) \right) \right] \) are all known functions determined only by \( v_0(x) \).

Thus, the local-in-time solution \( v \) is in series form.

Since the existence of \( v^\epsilon(x, t) \) in \( t \in [0, T_\epsilon] \) and \( v \) in \( t \in [0, T] \) has been proved in the previous chapter, all computations in proving theorem 4.1 are reasonable.

This result is very good, because we have got an explicit solution to the Navier-Stokes equation and the Euler equation so long as the initial condition \( v_0 \) is given.

5. Global existence and smoothness

In the study of solutions, a special property of power series is used, it can be described in Lemma 5.1.
**Lemma 5.1.** If $\sum_{n=0}^{\infty} a_n t^n$ is a series, $\tau$ is a constant, $\tau > 0$ or $\tau < 0$, and $\sum_{n=0}^{\infty} a_n \tau^n$ is convergent, then for all $|t| < |\tau|$, $\sum_{n=0}^{\infty} a_n t^n$ is convergent.

**Proof:** From the convergence of $\sum_{n=0}^{\infty} a_n \tau^n$, we know that $a_n \tau^n \to 0$, ($n \to \infty$), such that $\exists M > 0, |a_n \tau^n| \leq M$, ($n \geq 0$). Hence, if $|t| < |\tau|$, then,

$$|\sum_{n=0}^{\infty} a_n t^n| \leq \sum_{n=0}^{\infty} |a_n t^n| \leq M \sum_{n=0}^{\infty} (|t|/|\tau|)^n = \frac{M}{1-|t/\tau|}.$$ 

#

**Theorem 5.1.** The local-in-time series solution $v(x,t)$ in Theorem 4.1 is indeed a globally smooth solution.

$$v(x,t) = \sum_{n=0}^{\infty} a_n(x) t^n$$

Where $a_0(x) = v_0(x)$, and $a_n(x) = \frac{1}{n} \nu \Delta a_{n-1}(x) - \frac{1}{n} P \left[ \sum_{i=0}^{n-1} (a_i(x) \cdot \nabla a_{n-1-i}(x)) \right]$, ($n \geq 1$).

**Proof:** From Theorem 4.1, there exists the local-in-time series solution $v(x,t)$ in $t \in [0,T]$.

$$v(x,t) = \sum_{n=0}^{\infty} a_n(x) t^n$$

To prove its global existence and smoothness, we need only to prove that 

$$|v(x,t)| < \infty, \quad (x,t) \in \mathbb{R}^N \times [0,\infty)$$

Assume that there is a $(x_0, t_0)$ such that $|v(x_0, t_0)| = \infty$. It’s obvious that $t_0 > T$, and $t_0$ can be chosen to be the minimal, so that $|v(x_0, t_0)| = \infty$, $|v(x_0, t)| < \infty$, ($0 \leq t < t_0$). Since $|v(x_0, (t_0 - T/4))| < \infty$, $|v(x_0, t) - (t_0 - T/2)| < t_0 - T/4$, hence $|v(x_0, -(t_0 - T/2))| < \infty$.

Replacing $v_0(x)$ with $v(x, T/2)$, the Navier-Stokes equation will become:

$$\frac{\partial v_i}{\partial t} + \sum_{j=1}^{N} \nu_j \frac{\partial v_i}{\partial x_j} = u \Delta v_i - \frac{\partial p}{\partial x_i}, \quad (i = 1, \ldots, N) \quad (5.1)$$
\[
\frac{\partial v_1}{\partial x_1} + \ldots + \frac{\partial v_N}{\partial x_N} = 0 \tag{5.2}
\]

\[v(x, t)|_{t=0} = v(x, T/2) \tag{5.3}\]

From Theorem 4.1, equation (5.1), (5.2) and (5.3) have a series form solution \(\bar{v}(x,t)\) in \(t \in [0, T_0], T_0 > 0\).

\[
\bar{v}(x,t) = \sum_{n=0}^{\infty} \bar{a}_n(x)t^n
\]

Noting that \(|\bar{v}(x_0, -(t_0 - T/2) - T/2)| = |v(x_0, -(t_0 - T/2))| < \infty\). Since \(- (t_0 - T/2) - T/2 > t_0 - T/2\). Hence, by Lemma 5.1, \(|\bar{v}(x_0, t_0 - T/2)| < \infty\).

However, on the other hand, there is \(|\bar{v}(x_0, t_0 - T/2)| = |v(x_0, t_0)| = \infty\).

Thus, a contradiction has happened.

So, there should be

\[|v(x,t)| < \infty, \ (x,t) \in \mathbb{R}^N \times [0, \infty).\]

This means that the Navier-Stokes Equation (1.1), (1.2) and (1.3) has a global smooth solution \(v(x,t)\) in \((x,t) \in \mathbb{R}^N \times [0, \infty)\). Similarly, the Euler Equation also has a global smooth solution \(v(x,t)\) in \((x,t) \in \mathbb{R}^N \times [0, \infty)\).

The technique of replacing \(v_0(x)\) with \(v(x, T/2)\) can be called “Time Shifting”, it has just been used to show the complex relationship between time and space residing in the Navier-Stokes equation and the Euler equation.

6. Conclusions

We have solved the existence and smoothness of the Navier-Stokes equation and the Euler equation for \((x,t) \in \mathbb{R}^N \times [0, \infty), (N \geq 2)\). Moreover, we have obtained explicit solutions to the Navier-Stokes equation and the Euler equation. By the way,
we have strong confidence that our methodology in this paper can also solve the Navier-Stokes Equation and the Euler Equation with an external force $f(x, t)$.

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**References:**

[1]. Charles L. Fefferman, EXISTENCE & SMOOTHNESS OF THE NAVIER–STOKES EQUATION, Official statement of the problem, Clay Mathematics Institute.

[2]. A. Bertozzi and A. Majda, Vorticity and Incompressible Flows, Cambridge U. Press, 2002.

[3]. Peter J. Olver, Introduction to Partial Differential Equations (Undergraduate Texts in Mathematics), springer, 2014.

[4]. L. Caffiffarelli, R. Kohn, and L. Nirenberg, Partial regularity of suitable weak solutions of the Navier–Stokes equations, Comm. Pure & Appl. Math. 35 (1982), 771–831.
[5]. O. Ladyzhenskaya, The Mathematical Theory of Viscous Incompressible Flows (2nd edition), Gordon and Breach, 1969.

[6]. F.–H. Lin, A new proof of the Caffarelli–Kohn–Nirenberg theorem, Comm. Pure. & Appl. Math. 51 (1998), 241–257.

[7]. A. Shnirelman, On the nonuniqueness of weak solutions of the Euler equation, Comm. Pure & Appl. Math.50 (1997), 1260–1286.

[8]. Hartman, P., Ordinary Differential Equations. Birkhäuser, Boston, 1982.

[9]. Royden, H. L., Real Analysis, Macmillan, New York, 1968.

[10].Taylor , M. E., Partial Differential Equations, I, Basic Theory, 2nd Edition, Vol. 115 of Applied Mathematical Sciences Series, Springer New York Dordrecht Heidelberg London.

[11]. Taylor, M. E., Partial Differential Equations II, 2nd Edition, Qualitative Studies of Linear Equations, Vol. 116 of Applied Mathematical Sciences Series, Springer New York Dordrecht Heidelberg London.

[12]. Taylor, M. E., Partial Differential Equations III, 2ND Edition, Nonlinear Equations, Vol. 117 of Applied Mathematical Sciences Series, Springer New York Dordrecht Heidelberg London.

[13]. G. K. Batchelor, An Introduction to Fluid Dynamics, Cambridge University Press (2000).

[14]. Pierre Gilles Lemarié-Rieusset, The Navier-Stokes Problem in the 21st Century, Taylor & Francis, 2016.