Skyrmion Excitations in Quantum Hall Systems

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(Draft on January 13, 2022)

Using finite size calculations on the surface of a sphere we study the topological (skyrmion) excitation in quantum Hall system with spin degree of freedom at filling factors around $\nu = 1$. In the absence of Zeeman energy, we find, in systems with one quasi-particle or one quasi-hole, the lowest energy band consists of states with $L = S$, where $L$ and $S$ are the total orbital and spin angular momentum. These different spin states are almost degenerate in the thermodynamic limit and their symmetry-breaking ground state is the state with one skyrmion of infinite size. In the presence of Zeeman energy, the skyrmion size is determined by the interplay of the Zeeman energy and electron-electron interaction and the skyrmion shrinks to a spin texture of finite size. We have calculated the energy gap of the system at infinite wave vector limit as a function of the Zeeman energy and find there are kinks in the energy gap associated with the shrinking of the size of the skyrmion.

Our numerical diagonalization is done in the spherical geometry introduced by Haldane [10]. In this geometry electrons are placed on the surface of a sphere under the influence of an uniform radial magnetic field. The magnetic field is produced by a magnetic monopole of suitable strength placed at the center of the sphere. For a monopole of total magnetic flux $N_\phi = 2q$, the lowest Landau level (LL) degeneracy is also $2q + 1$. All single particle states in the lowest LL have fixed angular momentum value $q$.

At filling factor $\nu = 1$, corresponding to electron number $N = 2q + 1$, the ground state is spin polarized, independent of the strength of Zeeman splitting. Now let us study the case when one additional flux quantum is added or removed from the system. In Fig.1 we show the energy spectra versus total orbital angular momentum $L$ for $N = 10$ with Coulomb interactions in the absence of Zeeman splitting. All possible spin states from $S = 0$ to $S = 5$ are plotted in Fig.1. Fig.1(a) is for the case of one extra flux with $N_\phi = N + 1$ and Fig.1(b) is for one less flux with $N_\phi = N - 1$. The ground state in both cases has quantum numbers $L = S = 0$. There are two important features in Fig.1: (1) There exists an energy gap separating the lowest energy band from high energy states; (2) The lowest energy band consists of states of $L = S$ with all possible values of $S$ in the range $0 \leq S \leq N$. If short range interaction [10] where only the $V_0$ component of its Haldane pseudopotential is non-zero, then the states in the lowest energy band are degenerate. For the Coulomb interaction, the energy of a state in this band goes up like $L^2$ in the small $L$ limit.
The state with \( L = S = N/2 \) has the highest energy in the band. Its energy is higher than that of the ground state by a finite amount of \( \Delta_p \). As \( N \to \infty \), these states become almost degenerate. They can linear combine to form the macroscopic ground state of the system with one skyrmion of infinite size. This macroscopic ground state breaks the both \( L \) and \( S \) symmetries. In the following we will further demonstrate that the condition \( L = S \) is a manifestation of a classical skyrmion excitations in a quantum system.

\[
\hat{\phi}(\Omega) = \cos(g(\theta) - \theta)\hat{e}_r + \sin(g(\theta) - \theta)\hat{e}_\theta, \tag{1}
\]

where \( g(\theta) \) is a smooth and monotonically increasing from \( g(0) = 0 \) to \( g(\infty) = 0 \) as \( r \) increases. The connection between the skyrmion states in the two geometries is through the equation

\[
g(\theta) = f(2\cot(\theta/2)) \tag{3}
\]

We should point out that Eq.(1) is for the skyrmion state with the constraint \( |\hat{\phi}(\Omega)| = 1 \). If this constraint is relaxed, the generalized skyrmion state can be defined as

\[
\hat{\phi}(\Omega) = h_r(\theta)\hat{e}_r + h_\theta(\theta)\hat{e}_\theta, \tag{4}
\]

where \( h_r(\theta) \) does not change sign for \( 0 \leq \theta \leq \pi \) and the boundary conditions require \( h_\theta(0) = h_\theta(\pi) = 0 \). It turns out that this generalized skyrmion state is needed to write down the quantum many-body skyrmion states as we will discuss later.

From Eq.(1) one can write down the basis vectors in the spin space \((\hat{s}_r, \hat{s}_\theta, \hat{s}_\phi)\) in terms of the basis vectors in real space \((\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi)\) as following:

\[
\begin{align*}
\hat{s}_r &= \cos(g(\theta) - \theta)\hat{e}_r + \sin(g(\theta) - \theta)\hat{e}_\theta \\
\hat{s}_\theta &= -\sin(g(\theta) - \theta)\hat{e}_r + \cos(g(\theta) - \theta)\hat{e}_\theta \\
\hat{s}_\phi &= \hat{e}_\phi
\end{align*} \tag{5}
\]

Therefore, a rotation in real space is equivalent as another rotation in spin space. Now let us look at the quantum mechanical consequences of the above statement. If we label \( |\Phi_{sky}\rangle \) as a quantum state for skyrmion, then a rotation in \( \alpha \) direction in real space is identical as another rotation \( \beta \) in spin space,

\[
|\Phi_{sky}\rangle = e^{-\frac{\beta}{2}L}e^{-\frac{\alpha}{2}S}|\Phi_{sky}\rangle. \tag{6}
\]

where \( L \) (or \( S \)) is the total orbital (or spin) angular momentum operator. Since \( \alpha \) can be in any direction, one must have \( L = S \) where \( L \) and \( S \) are the eigenvalues of \( \hat{L} \) and \( \hat{S} \). Thus, the lowest bands in Fig.1 are quantum manifestations of classical skyrmion states. In the thermodynamic limit, the lowest band is almost degenerate, their linear combination gives rise to a skyrmion state. In this sense, the skyrmion state is spontaneous symmetry-breaking state, where \( \hat{S} \) and \( \hat{L} \) symmetries are broken in the thermodynamic limit. To see this point further, we can study the operator

\[
\hat{\sigma} \cdot \hat{s}_r = \begin{pmatrix} \cos(g(\theta)) & \sin(g(\theta))e^{-i\phi} \\ \sin(g(\theta))e^{i\phi} & \cos(g(\theta)) \end{pmatrix}, \tag{7}
\]

and its eigenvector is

\[
\xi = \begin{pmatrix} \sin(g(\theta)/2)e^{-i\phi/2} \\ -\cos(g(\theta)/2)e^{i\phi/2} \end{pmatrix}. \tag{8}
\]

If we label \((u, v) = (\cos(\theta/2)e^{i\phi/2}, \sin(\theta/2)e^{-i\phi/2})\), Then the quantum state for the classical skyrmion \( \hat{\phi}(\Omega) \) can be written as

\[
\phi(\hat{r}) = (\hat{r} \sin(f(r)), \cos(f(r))), \tag{2}
\]
\[ |\psi\rangle = C \cdot \hat{P} \prod_k \left( \frac{\sin(g(\theta_k)/2)e^{-i\phi_k/2}}{-\cos(g(\theta_k)/2)e^{i\phi_k/2}} \right) |\psi_{\nu=1}\rangle, \quad (9) \]

where \( C \) is the normalization constant and \( \hat{P} \) is the projection operator to the lowest Landau level.

\[ |\psi_{\nu=1}\rangle = \prod_{i,j} (v_i v_j - v_i u_j) \quad (10) \]

is the spin polarized ground state at \( \nu = 1 \). The state for the hedgehog skyrmion with \( g(\theta) = \theta \) has been given by Moon et al.\[7\]. The above skyrmion state can also be written as

\[ |\psi\rangle = C' \prod_k \left( \frac{\psi_k}{-\alpha u_k} \right) |\psi_{\nu=1}\rangle, \quad (11) \]

where \( 0 \leq \alpha \leq 1 \) is determined from \( g(\theta) \) and it controls the size of a skyrmion. To see the size dependence on \( \alpha \), one can calculate the spin expectation value in the state \( \xi^+ = (v, -\alpha u)^+ \),

\[ \langle \xi | \hat{\sigma} | \xi \rangle = h_r(\theta) \hat{c}_r + h_\theta(\theta) \hat{c}_\theta, \quad (12) \]

where

\[ h_r(\theta) = -\alpha + (1 - \alpha) \cos \theta \left( \alpha \cos^2(\theta/2) + \sin^2(\theta/2) \right), \]

\[ h_\theta(\theta) = -(1 - \alpha) \sin \theta \left( \alpha \cos^2(\theta/2) + \sin^2(\theta/2) \right). \quad (13) \]

This expectation value corresponds to the generalized classical skyrmion state as in Eq.(4). Naturally, the size of the skyrmion can be defined by the position where spin direction is perpendicular to the radial (and spin) direction at either pole. With this convention, the skyrmion size is

\[ \theta_0 = 2 \arctan \alpha, \quad (14) \]

which equals \( \pi/2 \) for the hedgehog skyrmion with \( \alpha = 1 \).

From the above discussions, one can see that \( |\psi\rangle \) describes a many-body state with all electrons in the lowest Landau level and its maximum orbital angular momentum \( L \) is equal to \( N/2 \). Furthermore, \( |\psi\rangle \) is not eigenstate of either \( \hat{L} \) or \( \hat{S} \). Thus, it is consistent to consider this skyrmion state as a linear combination of the states in the lowest energy band shown in Fig.1.

FIG. 2. Energy spectra versus total orbital angular momentum \( L \) for \( N = 10 \) electrons and magnetic flux \( N_q = 2q + 1 = 12 \). (a) The total spin of the system is \( S = 4 \); (b) The total spin of the system is \( S = 3 \).

At exactly filling factor \( \nu = 1 \) corresponding \( n = 2q+1 \), the ground state is ferromagnetic and gives rise to the topological charge \[3\] \( Q = 0 \). When one additional flux is added or removed from the system, the ground state is antiskyrmion or skyrmion with \( Q = \pm 1 \). Next, we would like to study the case when two flux quanta are added to the system. In this case, the lowest energy band corresponds to \( Q = 2 \) topological excitations. Our numerical studies are with particle number \( N = 10 \). At maximum total spin \( S = 5 \), the lowest energy band consists of two Laughlin quasiholes and is well understood \[14\]. Fig.2 shows the energy spectra \( S = 4, 3 \). Again, there is gap separating the low energy bands with high ones. The basic question is following: Can the low bands here be understood from the lowest excitations of \( Q = 1 \)? The same idea has been applied in the spin polarized (or spinless) case where one finds that all the low energy bands can be understood from the combinations of quasihole (or quasiparticle) excitations if their fractional statistics are properly incorporated \[14\]. Because of the statistics transmutation in the two dimensional systems, we consider the excitations at \( Q = 1 \) as bosons. The excess spins for the excitations are \( \Delta S = 5, 4, 3, 2, 1, 0 \), where \( \Delta S \) is the spin difference from the ferromagnetic state at \( \nu = 1 \). Fig.2(a) for \( S = 4 \) corresponding to \( \Delta S = 1 \) should be consisted of \( \Delta S_1 = 0 \) and \( \Delta S_2 = 1 \) states of Fig.1(a). Thus, \( \hat{L} = \hat{L}_1 + \hat{L}_2 \) with \( L_1 = 5 \) and \( L_2 = 4 \), so the possible values for \( L \) are from 1 to 9 as shown in Fig.2(a). Fig.2(b) for \( S = 3 \) (or \( \Delta S = 2 \)) consists of two low energy branches: (i) \( \Delta S_1 = 1 \), \( \Delta S_2 = 1 \) and \( L_1 = 4 \), \( L_2 = 4 \); (ii) \( \Delta S_1 = 0 \), \( \Delta S_2 = 2 \) and \( L_1 = 5 \), \( L_2 = 3 \). The second branch has \( L = 2 - 8 \) as shown in Fig.2(b). Because of the bosonic nature, the first branch should have \( L = 0, 2, 4, 6, 8 \), however, \( L = 8 \) state is clearly missing in Fig.2(b). Generally speaking, we have found in almost all cases there are missing low energy states. From the experience of early studies of spinless case, \[13\] the missing states there are caused by hard-core boundary constraint on the anyonic quasiparticles which pushes some states
up in energy. This implies there exists additional physical constraint for the spin systems which is elusive to us at the present time.

Next, we would like to address the experimental aspects of skyrmion excitations, in particular, what is the signature of skyrmion excitations in terms of the energy gap measurement. In Fig.3 we show the energy gaps versus Zeeman energy at filling factor $\nu = 1$ for $N = 12$ with sample thickness $b = 2l$ and $b = 4l$. This plot corresponds to a realistic experiment with tilted magnetic field. In the tilted field experiment, by keeping the perpendicular field constant one can fix the filling factor and changing the total field will cause the changes in Zeeman energy. The energy gap is determined by the interplay of the Zeeman energy and electron-electron interaction. Electron-electron interaction favors $S = 0$ as excitations shown in Fig.1, while Zeeman energy favors maximum spin. At large Zeeman energy, all electron spins are aligned in the same direction. As the Zeeman energy decreases, electron spins may flip over to give rise to the kink structures in Fig.3. As shown in Fig.1, for the skyrmion state the spin angular momentum and orbital angular momentum $L$ (corresponding to wave vector $q$ in the planar geometry) is related to each other, and later is related to skyrmion size. Therefore, the kinks in the energy gap for increasing Zeeman energy are associated with the shrinkages of the size of a skyrmion.

In conclusion, we have studied skyrmion excitations at $\nu = 1$ using finite size calculation. The low energy excitations consist of states with $L = S$ which is shown to be the quantum manifestation of the classical skyrmion state. We also performed the energy gap calculation as a function of Zeeman energy and found kinks in the energy gap associated with the shrinking of the skyrmion size.

One of us (X.C. Xie) thanks Professors Y.S. Wu and J.H.H. Perk for helpful discussions.

![FIG. 3. Energy gaps versus Zeeman energy at filling $\nu = 1$ for sample thickness $b = 2l$ and $b = 4l$. Both energies are in the unit of $e^2/\epsilon l$.](image-url)