An Affine Analogue of Fulton’s Ideals for Matrix Schubert Varieties

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Abstract

We give an affine analogue of Fulton’s generators for ideals defining matrix Schubert varieties described in terms of essential sets, in the affine case by considering infinite periodic matrices. This provides a tool for computing with Schubert varieties in the affine flag variety of type $A$.

1 Introduction

Given a permutation $\pi \in S_n$, the matrix Schubert variety $X_\pi$ for $\pi$ is defined as the closure of the $B_- \times B$-orbit of the permutation matrix for $\pi$ in the space of matrices $\text{Mat}_{n \times n}(\mathbb{C})$. (Here $B_-$ denotes the group of invertible lower triangular matrices and $B$ denotes the group of invertible upper triangular matrices. The convention in this paper for permutation matrices is that the $p_{i,j}$th entry is 1 iff $\pi(p) = i$. For example the permutation matrix for $\pi = 231$ is $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.) It is the closure in $\text{Mat}_{n \times n}(\mathbb{C})$ of the preimage of the corresponding Schubert variety $B_- \pi B/B$ in the flag manifold $GL_n(\mathbb{C})/B$. In [Ful92], Fulton gave an explicit description for generators of the ideals defining matrix Schubert varieties, as stated in the definition and theorem below.

Definition 1 (Ful92). The essential set $\mathcal{E}_{\text{ss}}(\pi)$ of an $n \times n$ permutation matrix $\pi$ is defined as

$$\mathcal{E}_{\text{ss}} := \{(i, j) \in [1, n - 1] \times [1, n - 1] : \pi(j) > i, \pi^{-1}(i) > j, \pi(j + 1) \leq i, \pi^{-1}(i + 1) \leq j\}.$$ 

Pictorially, these are the matrix entries that are the southeast corners of the remaining connected regions obtained by crossing out entries to the east and the south of each 1 in the permutation matrix, including the 1 itself.

Theorem 1 (Ful92). For any permutation matrix $\pi \in S_n$ and a generic $n \times n$ matrix $A = (x_{ij})$, the radical ideal $I_\pi$ in the polynomial ring of the variables $x_{11}, \ldots, x_{nn}$ defining $X_\pi$ is generated by minors of $A$ of size $m(i, j) + 1$ in the northwest submatrix with southeast corner $(i, j)$, where $(i, j) \in \mathcal{E}_{\text{ss}}(\pi)$ and $m(i, j)$ counts the number of 1’s in that northwest submatrix of $\pi$. 
Our result shows an affine analogue of this statement. An element \( M \) in \( GL_n(\mathbb{C}((t^{-1}))) \) descends to an element in the affine flag variety \( GL_n(\mathbb{C}((t^{-1}))) \). We will define a map \( \phi \) that unfolds \( M \) into an \( \infty \times \infty \) matrix with entries in \( \mathbb{C} \), and get a nested chain of lattices representing an affine flag by taking for each column the span of all columns (weakly) to the left of this column. Furthermore, any element in the extended Weyl group \( \bar{W} = \{ w : \mathbb{Z} \to \mathbb{Z} \text{ bijection } | w(i+n) = w(i)+n \} \) defines a Schubert variety in \( GL_n(\mathbb{C}((t^{-1}))) \), and can be written as an infinite periodic permutation matrix. We use the direct analogue of Fulton’s definition of the essential set associated to a finite permutation to define the essential set \( \mathcal{E}ss(w) \) of an affine permutation \( w \in \bar{W} \). Our main theorem gives a necessary and sufficient condition on the unfoldings \( M \) for it to represent an affine flag that lies in a particular Schubert variety \( X_w \).

This provides a tool for explicitly computing with Schubert varieties in the affine flag variety of type \( A \). For example, given a parametrization of an opposite Schubert cell \( X_w \), we may use this result to pull back the explicitly equations defining \( X_w \) under this parametrization. Note that our result is set-theoretic and we leave the proof that these equations define a radical ideal in the coordinates of \( X_w \) for future work.

**Theorem.** Let \( M \in G_k \subset GL_n(\mathbb{C}((t^{-1}))) \) and \( w \in \bar{W} \) with index \( k \) (defined later). \([M] \in X_w \subset Fl_k(V)\) if and only if for all \((i,j) \in \mathcal{E}ss(w) \subset \mathbb{Z} \times \mathbb{Z}\), there exists \( l \geq l(i,j) \) such that all minors of size \( n(i,j,l)+1 \) in the submatrix (infinite in the north direction, bounded in the other three directions) of width \( l \) with rightmost column \( i \), bottom row \( j \) in the matrix \( \phi(M) \) vanish. Here \( n(i,j,l) \) counts the number of 1’s in this submatrix in the \( \mathbb{Z} \times \mathbb{Z} \) matrix of \( w \), and \( \mathcal{E}ss(w) \) is the set of (periodic) essential boxes of \( w \) in this matrix.

## 2 Matrix Realization of the Affine Flag Variety in Type A

Our main references for this section are [PS86] and [Mag02]. Let \( \mathcal{K} \) denote the local field of Laurent series \( \mathbb{C}((t^{-1})) \), and \( \mathcal{O} = \mathbb{C}[[t^{-1}]] \) its ring of formal power series. For \( f \in \mathcal{K} \) where \( f(t) = \sum_{i \geq N} a_i t^{-i} \) with \( a_i \in \mathbb{C} \), we let \( \text{ord}(f) \) be the smallest integer for which \( a_i \neq 0 \) for \( f \neq 0 \). Fix a positive integer \( n \), let \( G(\mathcal{K}) = GL_n(\mathcal{K}) \) and \( G(\mathcal{O}) = GL_n(\mathcal{O}) \).

Let \( \{e_1, \ldots, e_n\} \) denote the standard \( \mathcal{K} \)-basis of \( V \), and for \( c \in \mathbb{Z} \), define \( e_{i+nc} := t^c e_i \). A lattice \( L \subset V \) is an \( \mathcal{O} \)-submodule \( L = \mathcal{O} v_1 + \cdots + \mathcal{O} v_n \), where \( \{v_1, \cdots, v_n\} \) is a \( \mathcal{K} \)-basis of \( V \). Consider the family of standard \( \mathcal{O} \)-lattices

\[
E_j := \text{Span}_\mathcal{O}\langle e_j, e_{j-1}, \cdots, e_{j-n+1} \rangle = \text{Span}_\mathcal{K}\langle e_i \rangle_{i \leq j}.
\]

Note that \( E_j = \sigma^j E_1 \), where \( \sigma \) is the shift operator defined as \( \sigma(e_i) = e_{i+1} \), or as a matrix

\[
\sigma = \begin{bmatrix}
0 & 0 & \cdots & 0 & t \\
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & 0 \\
\end{bmatrix}.
\]
Let $G_j := \{ g \in G : \text{ord det}(g) = -j \}$, so $G_j G_k = G_{j+k}$. Note that $\sigma \in G_1$ and $G_j = \sigma^j G_0 = G_0 \sigma^j$. The decomposition $G = \bigsqcup_{j \in \mathbb{Z}} G_j$ can be thought of as decomposing $G$ into connected components.

The **complete affine flag variety** $Fl(V)$ is the space of all chains of lattices $L_* = (\cdots \subset L_1 \subset L_2 \subset \cdots \subset L_n \subset \cdots)$ such that $L_{i+n} = t L_i$ (equivalently, $t^{-1} L_{i+n} = L_i$) and $\dim(L_i/L_{i-1}) = 1$ for all $i$. The **standard flag** is $E_* := (\cdots \subset E_1 \subset \cdots \subset E_n \subset \cdots)$, whose stabilizer $I$ is the subgroup of $G(O)$ which are upper-triangular modulo $t^{-1}$:

$$I = \{ b = (b_{ij}) \in G(O) : \text{ord}(b_{ij}) > 0 \forall i > j \}.$$ 

Therefore $Fl(V) \cong G(K)/I$, and $I$ is the **standard Iwahori subgroup** of $G(K)$. $Fl(V)$ decomposes as $\bigsqcup_j Fl_j(V)$ where $Fl_j(V) := G_j \cdot E_* \cong G_j/I$.

We define a map $\phi : G(K) \to \text{Mat}_{\infty \times \infty}(\mathbb{C})$ as follows. Given $M = \sum_{i \geq N} M_i t^{-i}$ where $M_i \in GL_n(\mathbb{C})$, we let $\phi(M)_{ni+c,nj+d} = (M_{i-j})_{c,d}$. Namely, $\phi(f)$ is the following matrix:

$$\begin{bmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & M_0 & M_{-1} & M_{-2} & M_{-3} \\
\vdots & M_1 & M_0 & M_{-1} & \cdots \\
\vdots & \cdots & M_1 & M_0 & M_{-1} \\
0 & M_N & \cdots & M_1 & 0 \\
0 & 0 & M_N & \cdots \\
\cdots & \vdots & \vdots & \vdots & \vdots 
\end{bmatrix}$$

This presentation of elements of $G(K)$ can be found in [LP20] where it is used to define and study total nonnegativity in loops groups.

Given a matrix in the image of $\phi$, one can construct a flag $L_*$ such that $L_i$ is the column span of the columns with indices $\leq i$. The standard flag is given by the $\infty \times \infty$ identity matrix.

The **extended affine Weyl group** $\widehat{W}$ for $G(K)$ can be realized as the subset $\{ \pi : \mathbb{Z} \to \mathbb{Z} \text{ bijection : } \pi \sigma^n = \sigma^n \pi \}$ of the set of bijections on $\mathbb{Z}$. Hence for any $w \in \widehat{W}$ and $i \in \mathbb{Z}$, $w(i+n) = w(i)+n$. Given any $w \in \widehat{W}$, we let $w$ act $K$-linearly on $V$ by $w(e_i) = e_{w(i)}$. This gives an embedding $\widehat{W} \subset G(K)$. The corresponding **affine permutation matrix** is the $\infty \times \infty$ matrix $(a_{ij})$ where $a_{ij} = 1$ iff $w(j) = i$. We define

$$W_j := \widehat{W} \cap G_j = \left\{ w \in \widehat{W} : \sum_{i=1}^n w(i) - i = nj \right\}.$$ 

For $0 \leq i \leq n-1$, let $s_i \in W_0$ be defined so that $s_i(i) = i+1$, $s_i(i+1) = i$, and $s_i(j) = j$ for all $j \neq i$ or $i + 1$ (mod $n$). $W_0$ is the subgroup of $\widehat{W}$ generated by $s_0, \cdots, s_{n-1}$. We have a semidirect product $\widehat{W} \cong \langle \sigma \rangle \ltimes W_0$, where $\sigma$ acts on $W_0$ via the outer automorphism $\sigma s_i \sigma^{-1} = s_{i+1}$. For $w \in \widehat{W}$, we say $w$ has **index** $j$ if $w \in W_j$. Equivalently, $w = w' \sigma^j$ for some $w' \in W_0$. 

3
$G(K)$ admits a Bruhat decomposition, $G(K) = \bigsqcup_{w \in \widehat{W}} TwI$. Hence we also have a Bruhat decomposition of the complete affine flag variety, $Fl(V) = \bigsqcup_{w \in \widehat{W}} X_w^u$ where $X_w^u := TwI/I = TwE_u$. Alternatively, we may define $X_w^u$ explicitly as follows:

$$X_w^u := \{ L_\bullet \in Fl(V) : \forall i, j, \text{dim}(E_j/(L_i \cap E_j)) = |Z_{\leq j} - wZ_{\leq i}| \}.$$ 

Here $wZ_{\leq i} = \{ w(k) : k \leq i \}$. Replacing the equality with $\leq$ we get conditions for defining the opposite Schubert variety $X_w^o$.

3 Proof of the Main Theorem

In this section, let $M \in G(K)$. $M$ defines an affine flag $L_\bullet = [M] \in G(K)/I$, where $L_j$ is the span of the columns of $\phi(M)$ with indices no greater than $j$. For $I, J \subset \mathbb{Z}$, we let $M_{I,J}$ denote the submatrix of $\phi(M)$ obtained by taking the entries whose row indices are in $I$ and column indices are in $J$. Furthermore, for $j \in \mathbb{Z}$, we let $M_{*,j}$ denote the $j$th column of $\phi(M)$.

**Definition 2.** Let $w \in \widehat{W}$. The essential set of $w$ is

$$Ess(w) := \{ (i, j) \in \mathbb{Z} \times \mathbb{Z} : w(j) > i, w^{-1}(i) > j, w(j+1) \leq i, w^{-1}(i+1) \leq j \}.$$ 

**Definition 3.** For any $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ and $w \in \widehat{W}$, let

$$l(i, j) := \max(0, j - \min_{k \geq i}(w^{-1}(k)) + 1)$$

and for any $l \geq l(i, j)$,

$$n(i, j, l) := |\{ j' : j - l < j' \leq j, w(j') \leq i \}|.$$ 

The pictures below show an example where $w = [261]$ (the window notation means that $n = 3$, and $w \in \widehat{W}$ is determined by $w(1) = 2$, $w(2) = 6$, and $w(3) = 1$).
The following lemma shows a correspondence between the dimension condition on each $E^i \cap L_j$ coming from $w$ and conditions on certain minors of $\phi(M)$.

**Lemma 1.** For all $(i, j) \in \mathbb{Z} \times \mathbb{Z}$, $\dim(E^i \cap L_j) \geq |w\mathbb{Z}_{<j} \cap \mathbb{Z}_{>i}|$ if and only if there exists $l \geq l(i, j)$ such that all minors $\det(M_{i,j})$ where $I \in \mathbb{Z}_{<i}$, $J \subseteq [j - l + 1, j]$, and $|I| = |J| = n(i, j, l) + 1$ vanish.

**Proof.** Suppose $\dim(E^i \cap L_j) \geq |w\mathbb{Z}_{<j} \cap \mathbb{Z}_{>i}|$. In other words, there exists $l \geq l(i, j)$ such that $\dim(\text{span}\{M_{i,j-l+1}, \ldots , M_{i,j}\} \cap E^i) \geq |w\mathbb{Z}_{<j} \cap \mathbb{Z}_{>i}|$. Suppose the rank of the matrix $M_{(−x,i),[j−l+1,j]}$ is $r$. This means that we can perform column operations on $M_{(−x,i),[j−l+1,j]}$ so that there are exactly $l - r$ columns whose entries with row indices less than or equal to $i$ are 0. Therefore, this is equivalent to $\dim(\text{span}\{M_{i,j-l+1}, \ldots , M_{i,j}\} \cap E^i) = l - r \geq |w\mathbb{Z}_{<j} \cap \mathbb{Z}_{>i}|$. In other words, $r \leq l - |w\mathbb{Z}_{<j} \cap \mathbb{Z}_{>i}| = n(i, j, l)$. This is true if and only if all minors of size $n(i, j, l) + 1$ in the matrix $M_{(−x,i),[j−l+1,j]}$ vanish.

**Lemma 2.** Suppose $w \in \tilde{W}$ has index $k$. For any $j \in \mathbb{Z}$, there exists $N << j$ such that for all $i < N$, $|w\mathbb{Z}_{<j} \cap \mathbb{Z}_{>i}| = j + k - i$.

**Proof.** Since $w$ is an affine permutation, for any $j$ there exists $N << j$ such that for any $i < N$, $w^{-1}(i) < j$. Pick any $i < N$. In the permutation matrix of $w$, $|w\mathbb{Z}_{<j} \cap \mathbb{Z}_{>i}|$ counts the number of 1’s in the southwest quadrant with the northeast corner $(i + 1, j)$. Consider multiplying $w$ on the left by a simple reflection $s$, which swaps corresponding adjacent rows. Now notice the following for any $j' < j$:

- If $w(j') > i + 1$, then $sw(j') > i$. Therefore $w(j') \in w\mathbb{Z}_{<j} \cap \mathbb{Z}_{>i}$ and $sw(j') \in sw\mathbb{Z}_{<j} \cap \mathbb{Z}_{>i}$.
- If $w(j') < i$, then $sw(j') \leq i$. Therefore $w(j') \notin w\mathbb{Z}_{<j} \cap \mathbb{Z}_{>i}$ and $sw(j') \notin sw\mathbb{Z}_{<j} \cap \mathbb{Z}_{>i}$.
- If $w(j') = i + 1$ and $sw(j') = i + 2$, we have $w(j') \in w\mathbb{Z}_{<j} \cap \mathbb{Z}_{>i}$ and $sw(j') \in sw\mathbb{Z}_{<j} \cap \mathbb{Z}_{>i}$.
- If $w(j') = i$ and $sw(j') = i - 1$, we have $w(j') \notin w\mathbb{Z}_{<j} \cap \mathbb{Z}_{>i}$ and $sw(j') \notin sw\mathbb{Z}_{<j} \cap \mathbb{Z}_{>i}$.
- If $w(j') = i + 1$ and $sw(j') = i$, then $j' := w^{-1}(i) < j$ and $sw(j') = i + 1$. Namely, $w(j') \in w\mathbb{Z}_{<j} \cap \mathbb{Z}_{>i}$, $sw(j') \notin sw\mathbb{Z}_{<j} \cap \mathbb{Z}_{>i}$, $w(j') \notin w\mathbb{Z}_{<j} \cap \mathbb{Z}_{>i}$, and $sw(j') \in sw\mathbb{Z}_{<j} \cap \mathbb{Z}_{>i}$.
- If $w(j') = i$ and $sw(j') = i + 1$, then $j'' := w^{-1}(i + 1) < j$ and $sw(j'') = i$. Namely, $w(j') \notin w\mathbb{Z}_{<j} \cap \mathbb{Z}_{>i}$, $sw(j') \in sw\mathbb{Z}_{<j} \cap \mathbb{Z}_{>i}$, $w(j'') \in w\mathbb{Z}_{<j} \cap \mathbb{Z}_{>i}$, and $sw(j'') \notin sw\mathbb{Z}_{<j} \cap \mathbb{Z}_{>i}$.

Pictorially, when $i$ is small enough, swapping any two rows does not change the number of 1’s in the southwest quadrant specified above. The above argument shows that there is a bijection between the sets $\{j : j' < j, w(j') > i\}$ and $\{j : j' < j, sw(j') > i\}$ for any simple reflection $s$. Therefore, we have $|w\mathbb{Z}_{<j} \cap \mathbb{Z}_{>i}| = |\sigma^k(\mathbb{Z}_{<j}) \cap \mathbb{Z}_{>i}| = |\mathbb{Z}_{<j+k} \cap \mathbb{Z}_{>i}| = j + k - i$, since by assumption $w = w's^k$ for some $w' \in W_0$. □
Lemma 3. Let $M \in G_k$ and $L_a = [M] \in Fl_k(V)$. Then for any $w \in \tilde{W}$ such that $w$ has index $k$, if for all $(i, j) \in \mathcal{E}ss(w)$, $\dim(E^i \cap L_j) \geq |wZ_{\leq j} \cap Z_{> i}|$, we have for all $(i, j) \in \mathbb{Z} \times \mathbb{Z}$, $\dim(E^i \cap L_j) \geq |wZ_{\leq j} \cap Z_{> i}|$.

Proof. Case 1 (when $(i, j)$ lies in a region that is not crossed out). Suppose $w^{-1}(i) > j$ and $w(j) > i$. Then $(i, j)$ lies in the same connected component as some $(i', j') \in \mathcal{E}ss(w)$ for which $i' \geq i$ and $j' \geq j$. Suppose $w(j + 1) > i$, then $|wZ_{\leq j+1} \cap Z_{> i}| = |wZ_{\leq j} \cap Z_{> i}| + 1$. Therefore we have the statement

$$\dim(E^i \cap L_{j+1}) \geq |wZ_{\leq j+1} \cap Z_{> i}| \implies \dim(E^i \cap L_j) \geq |wZ_{\leq j} \cap Z_{> i}|.$$ 

Similarly suppose $w^{-1}(i + 1) > j$, then $|wZ_{\leq j} \cap Z_{> i+1}| = |wZ_{\leq j} \cap Z_{> i}|$. Therefore we have the statement

$$\dim(E^{i+1} \cap L_j) \geq |wZ_{\leq j} \cap Z_{> i+1}| \implies \dim(E^i \cap L_j) \geq |wZ_{\leq j} \cap Z_{> i}|.$$ 

These two statements combined give that $\dim(E^i \cap L_j) \geq |wZ_{\leq j} \cap Z_{> i}|$ implies $\dim(E^i \cap L_j) \geq |wZ_{\leq j} \cap Z_{> i}|$. By picking the largest such $j'$ and (repeatedly) applying the statement above, we get $\dim(E^i \cap L_j) \geq |wZ_{\leq j} \cap Z_{> i}|$. Otherwise if $w(j') \leq i$ for all $j' < j$, all entries to the left of $(i, j)$ are crossed out (only) vertically. However in this case we know that there exists $j' << j$ sufficiently small so that $|wZ_{\leq j'} \cap Z_{> i}| = 0$. Using the statement above again we may deduce $\dim(E^i \cap L_j) \geq |wZ_{\leq j} \cap Z_{> i}|$ as desired.

Case 2 (when $(i, j)$ lies in a region that is crossed out by a vertical ray but not a horizontal ray). Suppose $w(j) \leq i$ and $w^{-1}(i) > j$. In this case either $w(j - 1) \leq i$ or $w(j - 1) > i$. In other words, if an entry is crossed out only vertically, the entry to the left is either crossed out only vertically, or not crossed out. Notice that $w(j) \leq i$ implies $|wZ_{\leq j} \cap Z_{> i}| = |wZ_{\leq j-1} \cap Z_{> i}|$. Therefore we have the statement

$$\dim(E^i \cap L_{j-1}) \geq |wZ_{\leq j-1} \cap Z_{> i}| \implies \dim(E^i \cap L_j) \geq |wZ_{\leq j} \cap Z_{> i}|.$$ 

Thus if there exists $j' < j$ such that $w(j') > i$, by Case 1 we know $\dim(E^i \cap L_{j'}) \geq |wZ_{\leq j'} \cap Z_{> i}|$. By picking the largest such $j'$ and (repeatedly) applying the statement above, we get $\dim(E^i \cap L_{j'}) \geq |wZ_{\leq j'} \cap Z_{> i}|$. Otherwise if $w(j') \leq i$ for all $j' < j$, all entries are crossed out (only) vertically. However in this case we know that there exists $j' << j$ sufficiently small so that $|wZ_{\leq j'} \cap Z_{> i}| = 0$. Using the statement above again we may deduce $\dim(E^i \cap L_j) \geq |wZ_{\leq j} \cap Z_{> i}|$ as desired.

Case 3 (when $(i, j)$ lies in a region that is crossed out by a horizontal ray but not a vertical ray). Suppose $w(j) > i$ and $w^{-1}(i) \leq j$. In this case $w^{-1}(i - 1) \leq i$ or $w^{-1}(i - 1) > j$. In other words, if an entry is crossed out only horizontally, the entry immediately above is either also crossed out only horizontally, or not crossed out. Notice that $w^{-1}(i) \leq j$ implies $|wZ_{\leq j} \cap Z_{> i-1}| = |wZ_{\leq j} \cap Z_{> i}| + 1$. Therefore we have the statement

$$\dim(E^{-i} \cap L_j) \geq |wZ_{\leq j} \cap Z_{> i-1}| \implies \dim(E^i \cap L_j) \geq |wZ_{\leq j} \cap Z_{> i}|.$$ 

Thus if there exists $i' < i$ such that $w^{-1}(i') > j$, by Case 1 we know that $\dim(E^{i'} \cap L_j) \geq |wZ_{\leq j} \cap Z_{> i'}|$. By picking the largest such $i'$ and (repeatedly) applying the statement above, we get $\dim(E^{i'} \cap L_j) \geq |wZ_{\leq j} \cap Z_{> i'}|$. Otherwise if $w^{-1}(i') \leq j$ for all $i' < i$, all entries directly above $(i, j)$ are crossed out only horizontally. We know by Lemma 2 that there exists $i' << i$ sufficiently small such that $|wZ_{\leq j} \cap Z_{> i'}| = j + k - i' = \dim(E^{i'} \cap \sigma^k(E_j))$. Since $(L_a) \in Fl_k(V)$, we know $\dim(E^i \cap L_j) \geq \dim(E^{i'} \cap \sigma^k(E_j))$, so
\[ \dim(E^i \cap L_j) \geq \left| w \mathbb{Z}_{\leq j} \cap \mathbb{Z}_{>l} \right|. \] We may then deduce that \( \dim(E^i \cap L_j) \geq \left| w \mathbb{Z}_{\leq j} \cap \mathbb{Z}_{>l} \right| \) by the statement above.

**Case 4** (when \((i, j)\) is crossed out by both a vertical and a horizontal ray). Suppose \(w(j) \leq i\) and \(w^{-1}(i) \leq j\). Similar to the argument in Case 2, we have

\[ \dim(E^i \cap L_{j-1}) \geq \left| w \mathbb{Z}_{\leq j-1} \cap \mathbb{Z}_{\geq l} \right| \implies \dim(E^i \cap L_j) \geq \left| w \mathbb{Z}_{\leq j} \cap \mathbb{Z}_{>l} \right|. \]

Let \(j' = w^{-1}(i) - 1\). Then \((i, j')\) is either not crossed out as in Case 1 or is only crossed out vertically as in Case 2. These together with the statements above give the desired result.

**Theorem 2.** Let \(M \in G_k\) and \(w \in \hat{W}\) with index \(k\). \([M] \in \mathcal{X}_w \subset \text{Fl}_k(V)\) if and only if for all \((i, j) \in \mathcal{E}ss(w) \subset \mathbb{Z} \times \mathbb{Z}\), there exists \(l \geq l(i, j)\) such that such that all minors of size \(n(i, j, l) + 1\) in the submatrix \(M_{(-\infty, i), [j-l+1, l]}\) of \(\phi(M)\) vanish.

**Proof.** By Lemma 3, we know the dimension conditions coming from the \(\mathcal{E}ss(w)\) imply dimension conditions for all \((i, j)\), and Lemma 1 translates the dimension conditions to conditions on the matrix \(\phi(M)\). Thus our theorem is a direct consequence of these two lemmas.

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