Noncommutative Field Theory and the Dynamics of Quantum Hall Fluids

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ABSTRACT

We study the spectrum of density fluctuations of Fractional Hall Fluids in the context of the noncommutative hidrodynamical model of Susskind. We show that, within the weak-field expansion, the leading correction to the noncommutative Chern–Simons Lagrangian (a Maxwell term in the effective action,) destroys the incompressibility of the Hall fluid due to strong UV/IR effects at one loop. We speculate on possible relations of this instability with the transition to the Wigner crystal, and conclude that calculations within the weak-field expansion must be carried out with an explicit ultraviolet cutoff at the noncommutativity scale.

We point out that the noncommutative dipoles exactly match the spatial structure of the Halperin–Kallin quasiexcitons. Therefore, we propose that the noncommutative formalism must describe accurately the spectrum at very large momenta, provided no weak-field approximations are made. We further conjecture that the noncommutative open Wilson lines are ‘vertex operators’ for the quasiexcitons.

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1. Introduction

Noncommutative Geometry is relevant to the physics of the Quantum Hall Effect (QHE) in various guises. The most fundamental is the fact that projection to the lowest Landau level (LLL) yields non-commuting position operators for planar electrons in a magnetic field, i.e.

\[ [X, Y] = i \theta_B \quad (1.1) \]

where \( X, Y \) denote the projected operators and

\[ \theta_B = \frac{hc}{eB}. \quad (1.2) \]

The area of an elementary quantum of magnetic flux is given by \( 2\pi \theta_B \). Equation (1.1) largely determines the properties of free electrons in the LLL, and leads to the natural appearance of the Moyal algebra and non-relativistic noncommutative field theory (NCFT) with deformation parameter \( \theta_B \) (c.f. for example \cite{1}). Notice that this noncommutative geometry of the LLL arises dynamically at the quantum level, in particular \( \theta_B \to 0 \) in the classical limit.

Recently, Susskind has proposed a different application of NCFT (c.f. \cite{2}) in the context of an effective theory of the incompressible quantum Hall fluid. The proposal is based on the hidrodynamical model of the Hall liquid \cite{3,4}. In the Lagrangian point of view, one works with a comoving frame \( y^i \) that labels the particles in the fluid. The flow is represented by functions \( x^i(y, t) \) that embed the \( y \)-plane (the world-sheet parameters) into real space. A crucial point is that the statistical permutation group of identical particles is classically modelled by area-preserving mappings in \( y \)-space. Thus, we are naturally led to a gauge theory of the group of area-preserving diffeomorphisms \( \text{SDiff} \). The gauge-field picture is literal in terms of the ‘displacement field’

\[ \theta^{ij} a_j = x^i - y^i, \quad (1.3) \]

where \( \theta^{ij} = \theta \epsilon^{ij} \) and \( \theta \) is a constant with dimensions of area that fixes the normalization of the fluid gauge field. The \( \text{SDiff} \) group of gauge transformations acts on the gauge field as:

\[ \delta a_j = \partial_j \lambda + \{ a_j, \lambda \}, \quad (1.4) \]

where \( \{ , \} \) represents the Poisson bracket:

\[ \{ f, g \} = \theta^{ij} \partial_i f \partial_j g. \quad (1.5) \]

In more detail, let us start from an effective particle Lagrangian

\[ \mathcal{L}_{\text{particle}} = \sum_{\alpha} \frac{1}{2} m_\alpha (\dot{x}_\alpha^i)^2 - \frac{eB}{2c} \sum_{\alpha} \epsilon_{ij} \dot{x}_\alpha^i \dot{x}_\alpha^j - V_C, \quad (1.6) \]
with $\alpha$ denoting the many-particle label. In the microscopic theory, the Coulomb potential $V_C$ features both direct and exchange terms, of which only the first one has a standard hidrodynamical (classical) interpretation:

$$V_C^{\text{direct}} = \frac{e^2}{2\epsilon} \int d^2x d^2x' (\rho(x) - \rho_0) \frac{1}{|x - x'|} (\rho(x') - \rho_0),$$

where $\epsilon$ denotes the dielectric constant and we have normalized the energy to that of a fluid with ground-state uniform density $\rho_0$.

The basic dynamical hypothesis is made that the interplay between the Coulomb interactions and the fermionic statistics of the electrons results in an effective potential that, at least on long wavelength scales, can be approximated by an ultralocal harmonic term,

$$V_C \rightarrow \frac{\mu}{2} \int d^2y (\rho(x) - \rho_0)^2. \quad (1.8)$$

As part of this dynamical hypothesis, it is also assumed that an effective inertial parameter $m^*$ is generated as a result of the same underlying dynamics that leads to (1.8), c.f. [5]. Thus, we have an effective kinetic energy for the ‘dressed electrons’ of the fluid, even if the LLL projection has quenched the kinetic energy of the bare electrons.

Hence, $\mu$ and $m^*$ are phenomenological parameters encoding the Coulomb-force dynamics that generates the gap in the fractionally filled lowest Landau level. A derivation of these parameters in terms of the true microscopic parameters, such as the electron mass and charge, is highly non-trivial and should proceed along the lines of [4]. We now show that the fluid limit of (1.6) and (1.8) imply a gapped spectrum with the standard phenomenology of the QHE.

Passing to the fluid picture one has

$$\rho(x) = \rho_0 |\partial y/\partial x| = \frac{\rho_0}{1 + \theta^{ij} f_{ij}}, \quad (1.9)$$

where $f_{ij} = \partial_i a_j - \partial_j a_i + \{a_i, a_j\}$ is the Poisson field strenght. Thus, we have a highly non-linear action for the Poisson gauge theory in the temporal gauge $a_0 = 0$. We can restore covariant notation by interpreting the vortex-free flow conditions as a vacuum Gauss law. The resulting field $a_\mu$ becomes a $U(1)$ gauge field in the linearized approximation. Furthermore, in the limit of large magnetic field the leading terms in the action at long distances give a Maxwell–Chern–Simons Lagrangian

$$S_{\text{eff}} \rightarrow -\frac{1}{4g^2} \int dt d^2y \left( \frac{1}{c_s^2} |f_{0i}|^2 - |f_{ij}|^2 \right) - \frac{\hbar k}{4\pi} \int dt d^2y \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\nu, \quad (1.10)$$

where the velocity of sound is:

$$c_s^2 = \frac{\mu \rho_0}{m^*}, \quad (1.11)$$
and the coupling parameters are given by

\[ g^2 = \frac{1}{\mu \rho_0^2 \theta^2}, \quad k = \frac{2\pi e B \rho_0}{\hbar c} \theta^2. \]  \hspace{1cm} (1.12)

The standard phenomenology of the FQHE then follows for the particular choice

\[ \theta = \frac{1}{2\pi \rho_0} = k \theta_B, \]  \hspace{1cm} (1.13)

with \(1/k = \nu\), the filling fraction. It is remarkable that this determination of \(\theta\) is formally independent of \(\hbar\). Indeed, the Poisson structure (1.5) makes sense even as a purely classical description of the fluid.

The spectrum of density fluctuations at long wavelengths is gapped at the effective cyclotron frequency

\[ \omega_0 = \frac{g^2 c_s^2 \hbar k}{2\pi} = \frac{eB}{m^*_c}. \]  \hspace{1cm} (1.14)

Physically, this gap is mostly induced by Coulomb interactions, i.e. \(\hbar \omega_0 = \mathcal{O}(e^2/\epsilon \sqrt{\theta})\). The relation (1.14) determines phenomenologically the value of the effective inertial parameter \(m^*_c\). If the microscopic dynamics is such that \(m^*_c = \infty\), or \(k = 0\), the massless ‘photon’ is nothing but the phonon of the acoustic excitations in a superfluid phase. In the Hall phases there is no such acoustic branch of phonons due to the magnetically induced Chern–Simons mass. The dispersion relation following from (1.10),

\[ \omega(p) = \sqrt{\omega_0^2 + c_s^2 |p|^2}, \]  \hspace{1cm} (1.15)

is phonon-like, \(\omega(p) \approx c_s|p|\), at very large momenta, but this turns out to be an unphysical feature of (1.10), that should be analyzed in a low-energy expansion around zero momentum, corrected by a tower of non-renormalizable operators that are suppressed by powers of \(\theta\). In particular, such operators were discarded in obtaining (1.10) as part of the weak-field expansion. Thus, (1.10) approximates the physics only in the regime

\[ |p| \ll \min (1/\sqrt{\theta}, \omega_0/c_s). \]  \hspace{1cm} (1.16)

The proposal of ref. [4] consists in approximating the statistics group by a version of \(U(\infty)\) rather than SDiff, i.e. we replace the Poisson brackets by Moyal brackets:

\[ \{f, g\} \rightarrow -i [f, g] = -i (f \star g - g \star f), \]  \hspace{1cm} (1.17)

with the standard definition of the Moyal product:

\[ f(x) \star g(x) = \lim_{\eta, \xi \to 0} \exp \left( i \frac{\theta^{\alpha \beta}}{2} \frac{\partial}{\partial \eta^\alpha} \frac{\partial}{\partial \xi^\beta} \right) f(x + \eta) g(x + \xi). \]  \hspace{1cm} (1.18)

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The Moyal and Poisson brackets differ to second order in the derivative expansion in powers of $\theta^{ij} \partial_i \partial_j$. The standard $U(1)$ gauge symmetry is promoted to a noncommutative $U(1)$ gauge symmetry that completes the long-distance Chern–Simons effective Lagrangian with nonlinear terms:

$$S_{\text{NC-S}} = -\frac{\hbar k}{4\pi} \int dt d^2 y e^{\mu \nu \rho} \left( A_\mu \partial_\nu A_\rho + \frac{2i}{3} A_\mu \star A_\nu \star A_\rho \right).$$  (1.19)

The resulting model successfully accounts for the connection between the statistics of the electrons and the filling fraction \([4]\). It is worth stressing at this point the conceptual difference between the LLL noncommutativity parameter $\theta_B$, essentially a single-particle effect, and the noncommutativity of the Lagrangian coordinates $[y^i, y^j] = i\epsilon^{ij} \theta$, tied to the implementation of the quantum statistics of the many body system. Numerically, they are related via the filling fraction $\theta_B = \nu \theta$, so that it would be desirable to have a more physical explanation of the interplay between these two notions of noncommutativity.

A very important aspect of (1.19) is the fact that, when written in terms of the ‘electron’ coordinate field

$$X^i = y^i + \theta^{ij} A_j,$$  (1.20)

it is just a Chern–Simons matrix model where the $X^i$ are infinite-dimensional matrices whose eigenvalues characterize the individual electron positions. Therefore, the noncommutative model of the ‘fluid’ is in principle capable of capturing the ‘granularity’ of the electrons. More specific studies of this model have proceeded by truncation to a localized finite electron system (a droplet). In this case, the noncommutative Chern–Simons model collapses to a quantum mechanical $U(N)$ Chern–Simons matrix model for a system of $N$ electrons, together with extra ‘edge’ degrees of freedom in the fundamental representation \([7]\). Formally, this corresponds to substituting the statistical permutation group $S_N$ by the unitary group $U(N)$. Notice that the number of gauge-invariant degrees of freedom is the same, at least in the large-$N$ limit, since coordinates of electrons are promoted to hermitian matrices; in the unitary gauge one has the set of eigenvalues, and the residual Weyl group yields the standard statistics group. The effectiveness of the finite matrix models must be judged against the Laughlin wave functions \([8]\), since one is working directly with $N$ electron degrees of freedom (see \([4,7,10]\). Thus, it is a more microscopic approach in nature. In this note we will mostly concentrate on the fluid picture and seek a description of the physics in terms of continuum field theory.

Both the noncommutative fluid model (1.19) and its matrix approximations can be obtained as a Seiberg–Witten limit \([11]\) of certain bound states of D-branes \([12]\).
2. Beyond the Topological Dynamics

The non-linear completion of the standard Chern–Simons Lagrangian that is given in (1.19) is of little relevance for the bulk physics of the fluid, for it can be shown [13] that the Seiberg–Witten map to ordinary gauge fields cancels the non-linear terms and leaves out the standard Chern–Simons Lagrangian (1.10). This manipulation requires discarding total derivatives, i.e. working on $\mathbb{R}^2$ or a manifold without boundary.

A simple proof of this important property can be given by using the exact solution of the Seiberg–Witten transform found in [14]. Since these exact solutions are simple for the field-strength operators, we extend the gauge bundle on $M_\theta \times \mathbb{R}$, with $M_\theta$ the noncommutative plane or the noncommutative torus, to a bundle on $M' = M_\theta \times \mathbb{R} \times [0,1]$ by $A(s) = s A$, where $s \in [0,1]$, a commutative interval. Then the Chern–Simons action on $M_\theta \times \mathbb{R}$ can be obtained as a boundary term from the four-dimensional topological action on the extended space,

$$S_{\text{NCCS}} = -\frac{\hbar k}{4\pi} \int_{M'} F \wedge F.$$  (2.1)

Now, the Seiberg–Witten map of the ordinary $U(1)$ Lagrangian density in momentum space is given by:

$$(f \wedge f)(p) = \int_{M'} L_* \left[ \sqrt{\det(1-\vartheta F)} \frac{F}{1-\vartheta F} \wedge F \frac{1}{1-\vartheta F} e^{ip(y+\vartheta A)} \right].$$  (2.2)

where $L_*$ is the instruction of path-ordering with respect to the Moyal product and matrix notation in Lorentz indices is understood. Since we are looking at the integrated Lagrangian density we just need the zero-momentum coupling that eliminates the path-ordering prescription. We take $\vartheta^{23} = \theta$ as the only nonvanishing entry of the noncommutativity matrix. By explicit matrix algebra one finds $(\vartheta F)^2 = -c(\vartheta F)$, where $c = \theta F_{23}$. So that

$$F \frac{1}{1-\vartheta F} = \frac{1}{1+c} (F+\delta F),$$

where $\delta F$ has only non-vanishing entries in the 01 plane:

$$(\delta F)_{01} = -(\delta F)_{10} = \theta \text{Pf} (F) = \frac{\theta}{4} \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}.$$  

Finally, using $\det(1-\vartheta F) = (1+c)^2$ and the property $\text{Pf} (F+\delta F) = (1+c)\text{Pf} (F)$, we obtain

$$\int_{M'} f \wedge f = \int_{M'} \text{Pf} (F) = \int_{M'} F \wedge F$$

for this particular noncommutativity matrix. Picking out the boundary terms this extends to the Chern–Simons actions in 2+1 dimensions.
The on-shell triviality of the noncommutative Chern–Simons action agrees with the results of [15] (see also [14]), where it was found that on a spatial torus the model is T-dual to an ordinary non-abelian (twisted) $U(N)$ model whose physical Hilbert space only samples the diagonal $U(1)$ subgroup. Therefore, the on-shell bulk properties of (1.19) are equivalent to those of the ordinary, topological Chern–Simons Lagrangian. In hindsight, the difficulties found in deriving out of (1.19) the conformal field theory of edge states [17], can be trivially solved if we decide to quantize the theory resulting after the Seiberg–Witten field redefinition. In principle, this is a valid option from the physical point of view, but it largely trivializes the ‘kinematical’ successes of (1.19).

Therefore, in order to judge the real impact of $U(\infty)$ as a ‘statistical’ symmetry group of the Hall fluid it would be desirable to go beyond the topological term. A natural option is to study the gapped density excitations at low momentum by keeping the propagating terms in the fluid effective action. The noncommutative gauge theory of the fluid follows from the general expressions above by substituting the Poisson gauge field $a_\mu$ by the noncommutative gauge field $A_\mu$ and the corresponding field strength $f_{\mu\nu}$ by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - iA_\mu \star A_\nu + iA_\nu \star A_\mu$. Expanding in powers of the field strength we obtain the noncommutative version of (1.10):

$$S_{\text{eff}}^{(2)} = -\frac{1}{4g^2} \int |F_{\mu\nu}|^2 - \frac{k}{4\pi} \int \epsilon^{\mu\nu\rho} \left( A_\mu \partial_\nu A_\rho + \frac{2i}{3} A_\mu \star A_\nu \star A_\rho \right).$$

(2.3)

where we have chosen units with $c_s = \hbar = 1$.

To the extent that (2.3) describes a smooth deformation of (1.10) (such as, for example, via the classical Seiberg–Witten map), we expect a gapped spectrum of density fluctuations with $\omega(p) \to \omega_0$ as $p \to 0$. On the other hand, we would like to find new $\theta$-dependent features of the excitation spectrum at intermediate momentum scales.

At the classical level, these expectations are borne out since interactions are analytic in $\theta$. The spectrum is gapped at zero momentum at the scale of the Chern–Simons mass $\omega_0 = M = g^2 k/2\pi$. However, it is well-known that gauge theories present strong UV/IR mixing at one-loop order, rendering the theory non-analytic around $\theta = 0$, at least in perturbation theory [18,19]. In the following, we study the effect on the collective excitations of the one-loop UV/IR mixing. After this standard analysis we offer a physical interpretation of the results.

2.1. Collective Excitations at One Loop

The dispersion of neutral collective excitations follows from the poles of the polarization tensor $\Pi^{\mu\nu}$ in the quadratic effective action

$$S_{\text{eff}}^{(2)} = \frac{1}{2} \int A_\mu \Pi^{\mu\nu} A_\nu.$$  

(2.4)
The constraints of linearized gauge symmetry and rotational symmetry on the plane constrain the tensor structure to be of the form:

\[
\Pi_{\mu\nu} = \Pi_e (p_\mu p_\nu - p^2 \eta_{\mu\nu}) + i M \Pi_o \epsilon_{\mu\nu\rho} p^\rho + \Pi_{nc} \tilde{p}_\mu \tilde{p}_\nu,
\]

(2.5)

where \( \tilde{p}^\mu = \theta^{\mu\nu} p_\nu \). In addition to the usual even and odd parity selfenergies \( \Pi_e, \Pi_o \), we have a new tensor structure \( \Pi_{nc} \) which is only contributed by non-planar diagrams. Notice that a term proportional to \( \tilde{p}_\mu \tilde{p}_\nu \) is perfectly compatible with linearized gauge invariance, since \( p_\mu \tilde{p}_\mu = 0 \). All scalar self-energies are functions of the independent variables \( p^2 \) and \( \tilde{p}^2 \).

The dispersion relation can be found by inverting the polarization tensor. The physical pole is located at

\[
p^2 \Pi_e^2 - M^2 \Pi_o^2 - \tilde{p}^2 \Pi_e \Pi_{nc} = 0,
\]

(2.6)

which yields the dispersion relation (we work in the (+ − −) signature):

\[
\omega(p) = \sqrt{p^2 + M^2 \frac{\Pi_o^2}{\Pi_e^2} + \tilde{p}^2 \frac{\Pi_{nc}}{\Pi_e}}.
\]

(2.7)

In general, the planar part of the perturbation theory is isomorphic to that of a \( SU(N) \) Yang–Mills–Chern–Simons model in the formal limit \( N \to 1 \). Therefore, according to the analysis of [20], [21] the noncommutative theory will be ultraviolet-finite at one loop. Standard dimensional analysis shows that the effective expansion parameter of planar perturbation theory is given by \( g^2/|p| \) at high momenta \( |p| \gg M \), whereas the finite tree-level mass of the photon yields \( g^2/M \sim 1/k \) as the effective expansion parameter at low momenta \( |p| \ll M \). The careful analysis of [21] obtains the scaling expected from dimensional arguments after subtle cancellations of infrared effects in which the choice of Landau gauge is instrumental.

The planar parts of the self-energy tensors have a low energy expansion of the form, c.f. [21]:

\[
\Pi_{\text{planar}} = 1 + \mathcal{O}(g^2/M) + \mathcal{O}(g^2|p|/M^2) = 1 + \mathcal{O}(1/k) + \mathcal{O}(|p|/kM).
\]

(2.8)

The corrections of \( \mathcal{O}(1/k) \) are responsible for the effective shift of the Chern–Simons level [22]. The contribution of low loop momenta \( |q| < M \) to the non-planar part of the self-energies is similar, since Moyal phases are negligible in this range. On the other hand, the extreme ultraviolet contribution to the non-planar diagrams exhibits the well-known UV/IR mixing.

The UV/IR mixing is based on the fact that Moyal phases tied to loop momenta introduce an effective ultraviolet cutoff in non-planar diagrams given by \( \Lambda_{\text{eff}} = 1/|\tilde{p}| \).
Since the one-loop contribution to the scalar self-energies is proportional to $g^2 \sim M/k$, the maximal possible degree of divergence of $\Pi_e, \Pi_o$ is $O(\Lambda_{\text{eff}}^{-1})$. Therefore, the UV non-planar contributions to $\Pi_e, \Pi_o$ at low momenta are of $O(M|\vec{p}|/k)$.

The Lorentz-violating tensor structure $\Pi_{nc}$ is different. Since it has mass dimension four, the non-planar one-loop diagram can be of $O(\Lambda_{\text{eff}}^3)$. Thus, we expect $\Pi_{nc} \sim 1/|\vec{p}|^3$.

By explicit inspection, one finds that the most singular contribution to the diagrams is the integral

$$
\int \frac{d^3q}{(2\pi)^3} \frac{2 q^\mu q^\nu - \eta^{\mu\nu} q^2}{q^4} e^{i\vec{q}\cdot\vec{p}} \sim \frac{\vec{p}^\mu \vec{p}^\nu}{|\vec{p}|^3} + \text{less singular.} \tag{2.9}
$$

One finds (see Appendix A for a more detailed discussion):

$$
\Pi_{nc} = \frac{M}{2k} \left( \frac{1}{|\vec{p}|^3} + O(M^2/|\vec{p}|) \right). \tag{2.10}
$$

This term is the first in a power expansion of the full gauge-invariant effective action that must be written in terms of open Wilson lines, as in $[23]$. A very significative fact is that the leading term comes with a positive sign. Thus, plugging this back into the general expression for the dispersion relation, one finds that, at low momenta:

$$
\omega(p)^2 = p^2 + M^2 - \frac{M}{2k} \frac{1}{|p|} + \ldots \tag{2.11}
$$

The dots stand for neglected contributions of $O(1/k), O(|p|/kM)$ and $O(M^3 |p|/k)$. Defining the dimensionless quantity

$$
\Delta = \frac{1}{M^2 \theta}, \tag{2.12}
$$

we can determine the range of applicability of (2.11) to be $|p|/M \ll \Delta$.

Therefore, we obtain the expected result that noncommutative UV/IR effects turn the gauge theory unstable in the infrared, despite the presence of a gauge-invariant Chern–Simons mass. Imaginary frequencies occur for sufficiently small momenta. If this happens for $|p| \ll M$, the critical momenta for the onset of tachyons is

$$
\frac{|p|_c}{M} \approx \frac{1}{k} \Delta. \tag{2.13}
$$

The physical interpretation is that perturbation theory breaks down at these values of the momenta. In principle, the infrared dynamics that resolves these singularities could be non-perturbative.
3. Physical Interpretation

The one-loop instability of the noncommutative Maxwell–Chern–Simons theory is at odds with the naive expectations based on a local (in powers of momenta) quantization of the model. In particular, one does not obtain the zero-momentum gap of density fluctuations that is the landmark of the Hall fluids. There are basically two possibilities regarding the physical interpretation of the tachyonic singularity: either it signals a physical instability of the fluid, or it is an artifact of the approximations used and has no relation to the physics of the real Hall fluid.

In principle, we should expect some kind of instability in the noncommutative perturbation theory, since the effective expansion parameter is $1/k = \nu$, the filling fraction. For small values of the filling fraction the incompressible fluid is unstable towards condensation into a solid phase, the so-called Wigner crystal. If the electron density is too low, each elementary cyclotron orbit is well separated from the others and the lowest energy state is an hexagonal two-dimensional lattice held by the Coulomb repulsion. As any other crystal, it has gapless phonon excitations.

Therefore, it is tempting to identify the tachyonic instability at (2.13) as a result of the expected physics at small filling fraction, $\nu \ll 1$, i.e. we are forcing the fluid description in a region where the Wigner crystal sets in. If this interpretation is correct, the behaviour of the dispersion relation in the intermediate regime of momenta

$$\nu \Delta \ll \frac{|p|}{M} \ll \Delta$$

(3.1)

is interesting and could capture some qualitative features of the real excitation spectrum of the Hall fluid.

The dispersion of the lowest neutral excitations in a real Hall fluid has the qualitative form depicted in Fig. 1. The lowest energy mode occurs at a finite value of the momentum $|p|_{\text{min}} \sim 1/\sqrt{\theta}$, the so-called magnetoroton, in analogy with the analogous modes in superfluids [24]. As the filling fraction is reduced, the magnetoroton eventually becomes unstable and a Wigner crystal is formed with lattice size of $O(\sqrt{\theta})$. The post-magnetoroton regime at $|p| > 1/\sqrt{\theta}$ has $\omega(p) \sim 1/|p|$ and reflects peculiar physics that we will discuss in the next section.

We can formally regularize the infrared singularity by artificially turning off the UV/IR effects. This can be achieved by introducing an explicit ultraviolet cutoff $\Lambda$ in the loop integrals. The effective cutoff of the non-planar diagrams then becomes

$$\Lambda_{\text{eff}}^2 = \frac{1}{|\vec{p}|^2 + 1/\Lambda^2}.$$ 

(3.2)
The excitation of lowest frequency is the magnetoroton that occurs at the scale of the inter-particle separation $|p_W| \sim 1/\sqrt{\theta}$ and presages the Wigner crystal. The post-roton regime at larger momenta describes the so-called ‘quasiexcitons’ and scales as $1/|p|$.

The effect of this modification in the expression (2.11) is to turn the dispersion curve back up at momenta of order $|p| \sim 1/\Lambda \theta$. Thus the effect of the ultraviolet cutoff is to mimic the magnetoroton minimum!

In practice, one must ensure that the cutoff procedure respects gauge invariance and does not introduce longitudinal terms in the polarization tensor. Such terms would give $\Lambda$-dependent additive renormalizations of the frequency at zero momentum. On general grounds, we know that a cutoff procedure that respects gauge invariance must restore Lorentz-invariance asymptotically at momenta of order $|p| \ll 1/\Lambda \theta$. Therefore, the ‘non-commutative’ part of the polarization tensor must vanish at zero momentum and the long-distance gap is governed by the Chern–Simons mass $M$. We include in Appendix B
a proof of the stability of the zero-momentum gap by the addition of a gauge-invariant Pauli–Villars regulator.

If the ultraviolet cutoff is chosen at a value of order \( \Lambda \sim M k \), the turn-over coincides with the onset of non-perturbative effects, i.e. \(|p| \sim 1/\Lambda \theta \sim \nu M \Delta\). Then, the dispersion curve in the immediate post-roton regime \( \nu \Delta < |p|/M < \Delta \) is approximately given by

\[
\omega(p) \approx M \sqrt{1 - \frac{\nu}{2M \theta |p|}} \approx M - \frac{\nu}{4\theta |p|},
\tag{3.3}
\]

the correct qualitative behaviour. Therefore, it is tempting to regard an ultraviolet cutoff at \( \Lambda \sim M k \) as a simulation of the magnetoroton minimum and the one-loop induced term in (3.3) as a ‘holographic’ calculation of the post-roton dispersion curve.

Despite its qualitative appeal, the previous picture is probably unphysical for a variety of reasons. First of all, the scales do not come out exactly right. If the magnetoroton minimum is to be associated to perturbative UV/IR effects, one should have \( \sqrt{\theta} M > 1 \), since the dispersion curve shows an unphysical linear form for \(|p| > M\). This constraint implies that \( \Delta < 1 \). But then we have \( \nu \Delta \ll \Delta < \sqrt{\Delta} = 1/M \sqrt{\theta} \). Therefore the \( \Lambda \)-induced minimum at \(|p| \sim \nu \Delta M\) always occurs at a much lower momentum than the expected magnetoroton scale \(|p_W| \sim 1/\sqrt{\theta}\). It is also clear that the formula (3.3) always applies at lower momenta than the expected magnetoroton momentum.

A more serious problem has to do with the physical specification of the ultraviolet cutoff. It is plain that the behaviour (3.3) is induced by the integration over virtual modes of momentum in the range \( M < |p| < M k \). However, this is precisely the regime where the dispersion relation is approximately linear and thus unphysical! A more reasonable choice for the scale of the ultraviolet cutoff, such as \( \Lambda \sim M \), eliminates the appealing qualitative behaviour (3.3) from the dispersion curve.

In fact, since we are neglecting higher powers of \( F_{ij} \) in a weak-field expansion, the natural Wilsonian cutoff for calculations with the action (2.3) would be \( \Lambda \sim 1/\sqrt{\theta} \), since the weak-field expansion is organized in powers of \( \theta^{ij} \partial_i \partial_j \). However, in this case the strong UV/IR effects completely disappear, since these come from virtual loop momenta in the range \( 1/\sqrt{\theta} < |p| < \Lambda \). If we take this more physical point of view, the quantum corrections at low momentum calculated with (2.3), equipped with a cutoff at \( \Lambda \sim 1/\sqrt{\theta} \), become qualitatively indistinguishable from those obtained by calculation in the commutative Maxwell–Chern–Simons model. We have summarized this situation in Fig. 2.
4. Noncommutative Dipoles and the Halperin–Kallin Quasiexciton

If the noncommutative description of density fluctuations is to capture some qualitative features of the real system, it would be good to identify the basic degrees of freedom via some simple physical argument. We now come to the basic observation of this paper by providing such an intuitive link.
Physically, the main feature of noncommutative theories that explains the peculiarities of their non-local dynamics, including the UV/IR effects, is the dipolar nature of the elementary excitations. As far as their interactions are concerned, noncommutative quanta of momentum $p^\mu$ can be thought of as rigid dipoles of orientation vector $L^\mu = \tilde{p}^\mu = \theta^{\mu\nu} p_\nu \[25\]$. This picture is very intuitive in the stringy regularization of NCFT, in which one starts with open strings. The Seiberg–Witten low energy limit decouples the normal vibrational modes of the open strings and the large magnetic field prevents them from shrinking to a point-like object.

In open string theory, the endpoints are formally oppositely charged sources for the massless photon excitation of the open string. In the fluid description of the QHE, the charged sources for the fluid gauge field $A_\mu$ are Laughlin’s quasiholes and quasielectrons. Thus, the noncommutative photons could be thought of (at least in certain regime) as a description of quasiparticle bound states. This intuitive correspondence is incorporated in the D-brane models of the QHE \[12\], where one explicitly assigns Laughlin quasiparticles to end-points of open strings on certain D-branes.

An intuitive picture of density fluctuations of the Hall fluid was developed by Halperin and Kallin \[26\] precisely in terms of quasiparticle bound states (see also \[27\].) The basic idea is that, since the elementary gapped excitations about the Laughlin ground state are the quasiparticles, a neutral density fluctuation can be thought of as a quantization of quasihole-quasielectron pairs. If these quasiparticle pairs are well-separated compared to their size, a simple argument gives the linear extension of the dipole: in a stationary configuration the Lorentz force must be balanced against the Coulomb attraction of the pair,

$$ (e\nu) \frac{v}{c} B = \frac{\partial V_C}{\partial \ell}, \quad (4.1) $$

where $\ell$ is the relative separation, $v$ is the velocity in the orthogonal direction, and $e\nu$ is the electric charge of the quasiparticles in the Laughlin fluids with $1/\nu = \text{odd}$. The velocity is related to the momentum vector by

$$ v = \frac{\partial E(\mathbf{p})}{h \partial \mathbf{p}}. \quad (4.2) $$

Now, in the microscopic system, the LLL projection means that all the energy of the fluid is well approximated by the Coulomb energy, $E \approx V_C$. Therefore, we have

$$ \ell \approx \frac{\hbar c}{e \nu B} |\mathbf{p}| = \frac{\theta \|\mathbf{p}\|}{\nu} = \theta |\mathbf{p}|. \quad (4.3) $$

We have found that the dipolar structure of the Halperin–Kallin quasiexciton exactly matches the noncommutative dipole! We think that this is not a coincidence and actually provides a strong indication that the basic insight of \[4\] might be correct.

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Based on the quasiexciton picture, one can argue that the large-\(|p|\) behaviour of the dispersion relation is saturated by the Coulomb potential between the quasiparticles separated by a distance \(\ell \approx \theta|p|\):

\[
\omega(p) \to \omega_C(p) = E_{\text{pair}} - \frac{e^2 \nu^2}{\epsilon \theta|p|},
\]

(4.4)

where \(E_{\text{pair}}\) is the energy to create a separate pair of quasiparticles. Notice that the surprising match of the quasiexciton dipole with the noncommutative dipole does not depend on the particular value of the interaction potential, in this case the Coulomb potential. In particular, the Halperin–Kallin argument provides the analog of the Seiberg–Witten limit for this particular system. The Hamiltonian of the free dipoles is therefore given by

\[
H_{\text{free}} \approx \int d^2p \, \omega_C(p) \, W_p^\dagger \, W_p,
\]

(4.5)

where \(W_p^\dagger\) creates a dipole of momentum \(p\). Clearly, the applicability of noncommutative perturbation theory to the Hall system depends on whether the interactions between the quasiexcitons are well-modelled by local splitting-joining of dipole endpoints, i.e. by quasiparticle pair creation-annihilation. In this context, the issue of UV/IR effects must be reexamined with the new propagator based on (4.7).

It is tempting to associate the operators \(W_p\) to modes of gauge-invariant open Wilson lines. If this conjecture is true, there must be a very non-trivial representation of soliton-antisoliton bound states in terms of open Wilson line ‘vertex operators’.

For momenta \(|p| \sim 1/\sqrt{\theta}\), corresponding to the average inter-electron distance, the picture of well-separated quasiparticles breaks down, and the screening of charge causes the dispersion curve to turn up. The minimum is the so-called magnetoroton excitation that sits around \(|p_{\text{min}}| \sim 1/\sqrt{\theta}\) and presages the instability towards the Wigner crystal at low values of the filling fraction. Therefore, to the extent that the noncommutative dipole can be associated to the Halperin–Kallin quasiexciton, the noncommutative formalism is less and less relevant for practical calculations as we consider larger wavelengths for the density fluctuations.

5. Conclusions

We have analyzed the prospects for using techniques from NCFT, notably perturbation theory with Moyal products, as a calculational technique for the physics of density fluctuations of fractional Hall fluids. The basic hypothesis of [1] is that promoting the electrons to ‘D-brane’ objects, i.e. enlarging the statistics group from \(S_N\) to \(U(N)\), is a
new and interesting way of performing the ‘flux-attachment’ transformation that is at the basis of most effective field theory treatments of the QHE \cite{28}. At the same time, the noncommutative fluid is ‘granular’ in character, with length scale $\sqrt{\theta}$, and we expect to see characteristic effects of this granularity at wavelengths of $\mathcal{O}(\sqrt{\theta})$.

We have seen that a straightforward application of the weak-field expansion and noncommutative perturbation theory fails to capture the physics due to strong UV/IR mixing, yielding infrared divergences that destroy the incompressibility of the Hall fluid.

We consider unlikely the possibility that the UV/IR divergences should be interpreted in terms of the Wigner-crystal phase transition. A standard Wilsonian picture of the effective action suggests that the UV/IR divergences are artifacts of the weak-field expansion and such simple effective actions should be defined with an explicit ultraviolet cutoff at the scale $\Lambda \sim 1/\sqrt{\theta}$. Under these conditions, the NCFT approach to the spectrum of density fluctuations at very long wavelengths is not qualitatively different from that of ordinary hydrodynamics.

Therefore, in order to test the hypothesis of \cite{4} at the level of the bulk dynamics we must work at wavelengths of $\mathcal{O}(\sqrt{\theta})$ or smaller, and go beyond the weak-field expansion. In this context, we have noticed that the geometrical structure of the Halperin–Kallin quasiexciton exactly matches that of a noncommutative dipole of length $\theta|\mathbf{p}|$. We regard this as really convincing evidence that the basic idea of \cite{4} is actually correct. Thus, it is tempting to conjecture that the open Wilson lines are interpolating fields for the quasiexcitons, a kind of non-local ‘vertex operators’ for bound states of quasiparticle solitons. In proving such a conjecture, it is likely that the direct Coulomb interaction (1.7) must be kept exactly, without any further dynamical approximations.

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**Appendix A.**

In the evaluation of the one-loop contribution to the gauge boson self-energy in noncommutative Maxwell–Chern–Simons theory, one finds the following expression (notice
that we are working on euclidean space):

\[ \Pi^{(1)}_{\mu \nu} = \frac{M}{k} \int \frac{d^3q}{(2\pi)^3} \frac{P_{\mu \nu}(q, p, M)}{q^2(q - p)^2(q^2 + M^2)((q - p)^2 + M^2)} \sin^2 \left[ \frac{\tilde{p}q}{2} \right], \quad (A.1) \]

where \( P_{\mu \nu} \) is a complicated polynomial of momentum dimension six, which comes from the sum of the three contributing one-loop diagrams, including the one with the ghost loop (for more details, see [21] and [22].) Because of gauge invariance, (A.1) must be of the form (2.5).

The planar and nonplanar contributions can be separated performing the usual substitution:

\[ \sin^2 \left[ \frac{\tilde{p}q}{2} \right] = \frac{1}{2} (1 - \cos \tilde{p}q). \quad (A.2) \]

Since the physical region is \( |p| \ll M \), we are interested in a low momentum expansion in powers of \( |p|/M \). Let us first concentrate on the infrared part of the integral. At first sight, it seems that it could be divergent as \( |p| \to 0 \). Nevertheless, it was rigorously proven by Pisarski and Rao that the infrared divergences cancel out. In fact, they exactly calculated the planar part of this integral, obtaining \( \Pi^{(1)}_{\mu \nu} \) as analytic functions of \( |p|/M \).

This analyticity cannot be spoiled in the non-planar integral, as in this region the phase is slowly varying and it can be expanded in powers of \( \tilde{p}q \). With each power, the small \( p \) contribution decreases.

For the purpose of calculating the leading terms of the ultraviolet contribution to the nonplanar term of (A.1), we can set \( |p| = 0 \), while keeping \( \tilde{p} \) on the phase factor, which can lead to the UV/IR mixing, as it is well known. Then, taking the explicit form of \( P_{\mu \nu} \)

in this limit, we get:

\[ 2\pi \frac{M}{k} \int \frac{d^3q}{(2\pi)^3} \frac{2q_\mu q_\nu - \delta_{\mu \nu} M^2 - \delta_{\mu \nu} q^2}{(q^2 + M^2)^2} e^{i\tilde{p}q}. \quad (A.3) \]

Notice that the terms that would appear in (A.3) if we had not taken \( p = 0 \) in the numerator are negligible at small \( p \), as they would increase the powers of \( p \) and decrease the powers of \( q \) (thereby decreasing the powers of \( 1/|\tilde{p}| \) in the final result.)

The integral (A.3) can be easily computed taking derivatives with respect to the components of \( \tilde{p} \) in the equality:

\[ \int \frac{d^3q}{(2\pi)^3} \frac{e^{i\tilde{p}q}}{(q^2 + M^2)^2} = \frac{e^{-|\tilde{p}|M}}{8\pi M}. \quad (A.4) \]

After a short calculation one obtains

\[ \Pi^{(1)}_{\mu \nu}(p \to 0)_{\text{nonplanar}} = -\frac{M}{2k} \frac{\tilde{p}_\mu \tilde{p}_\nu}{|\tilde{p}|^3} (1 + |\tilde{p}|M) e^{-|\tilde{p}|M} = -\frac{M}{2k} \frac{\tilde{p}_\mu \tilde{p}_\nu}{|\tilde{p}|^3} \left( 1 - \frac{(|\tilde{p}|M)^2}{2} + \ldots \right) \quad (A.5) \]
Despite the existence of the Chern–Simons mass, the UV/IR effect renders the dispersion relation singular in the infrared. At small $p$, we get the following dependences of the one-loop correction to the quantities appearing in the dispersion relation (2.7):

$$\Pi^{(1)}_o \propto \frac{1}{k} |p|^0, \quad \Pi^{(1)}_e \propto \frac{1}{k} |p|^0, \quad |\tilde{p}|^2 \Pi^{(1)}_{nc} \propto \frac{M}{2k} \frac{1}{|\tilde{p}|},$$

where we have returned to the Lorentzian signature $(+−−)$ used in the main text.

Appendix B.

It is known that a diagram with a fermion loop (with the fermion transforming in the adjoint representation) leads to an UV/IR divergence of the same type as a vector, but with opposite sign in the photon self-energy. The coefficient of the divergent term is independent of the fermion mass $m_f$, which works like a gauge-invariant Pauli–Villars regulator. Thus, if a Majorana fermion is included in the theory so the number of fermionic degrees of freedom equals the number of bosonic ones, the cutoff will eliminate the divergence in the dispersion relation (2.7).

Our goal in this appendix is to show explicitly that the gap in the dispersion relation at $|p| = 0$ is recovered (does not depend substantially on $m_f$), as was stated by general arguments in the main text. From the fermion diagram, one gets a one-loop contribution:

$$\Pi^{f}_{\mu\nu} \sim \frac{M}{k} \int \frac{d^3q}{(2\pi)^3} \text{tr} \left[ \gamma^\mu \frac{\not{q} - \not{p} - m_f}{(q-p)^2 + m_f^2} \gamma^\nu \frac{\not{q} - m_f}{q^2 + m_f^2} \right] \sin^2 \left[ \frac{\tilde{p}q}{2} \right].$$

It was proved in [29] that the planar part of the integral just induces a Chern–Simons term, renormalizing the original Chern–Simons coefficient $k \rightarrow k \pm \frac{1}{2}$. Thus, let us concentrate in the nonplanar part. All the integrals needed to compute it can be obtained by differentiation from:

$$I = \int \frac{d^3q}{(2\pi)^3} \frac{e^{i\tilde{p}q}}{((q-p)^2 + m^2_f)(q^2 + m^2_f)}. \quad \text{B.2}$$

Introducing Feynman parameters and defining:

$$\Omega = -x(x-1)p^2 + m^2_f, \quad \text{B.3}$$

we have:

$$I = \int_0^1 dx e^{i\tilde{p}p} \int \frac{d^3q}{(2\pi)^3} \frac{e^{i\tilde{p}q}}{(q^2 + \Omega)^2} = \int_0^1 dx e^{i\tilde{p}p} \int_0^\infty d\alpha \alpha e^{-\alpha \Omega} \int \frac{d^3q}{(2\pi)^3} e^{i\tilde{p}q - \alpha q^2} \int_0^1 dx e^{i\tilde{p}p} \int \frac{d^3q}{(2\pi)^3} e^{-\frac{|\tilde{p}|^2}{4\alpha}} = \frac{1}{8\pi^2} \int_0^1 dx e^{i\tilde{p}p} \int_0^\infty d\alpha \frac{1}{\sqrt{\alpha}} e^{-\alpha \Omega - \frac{|\tilde{p}|^2}{4\alpha}} = \frac{1}{8\pi} \int_0^1 dx e^{i\tilde{p}p} \int_0^\infty d\alpha \frac{e^{-|\tilde{p}|\sqrt{\Omega}}}{\sqrt{\Omega}}. \quad \text{B.4}$$

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Notice that the condition \( \tilde{p} \cdot p = 0 \) cannot be imposed until no more derivatives are to be done, since we must consider \( p, \tilde{p} \) to be independent parameters.

In order to obtain the zero-momentum contribution to the mass gap,

\[
e^{-|\tilde{p}|m_f} \frac{\sqrt{\Omega}}{\sqrt{\Omega}}
\]

can be expanded in powers of \( p^2/m^2_f \):

\[
I = \frac{e^{-|\tilde{p}|m_f}}{16\pi m_f (\tilde{p}p)^3} \left[ -2i(-1 + e^{i\tilde{p}p}(\tilde{p}p)^2
+ (1 + |\tilde{p}|m_f) (-2i + (\tilde{p}p) + e^{i\tilde{p}p}(2i + \tilde{p}p)) \frac{p^2}{m^2_f} + O(p^4/m^4_f) \right].
\]

(B.5)

Differentiating this expression and inserting the results in (B.1), one gets, after some calculation (and rotating back to Lorentzian signature):

\[
\Pi^f_\phi(p \to 0) \propto M \frac{1}{k} \frac{1}{m_f} e^{-|\tilde{p}|m_f},
\]

\[
\Pi^f_\epsilon(p \to 0) \propto M \frac{1}{k} \frac{1}{m_f} e^{-|\tilde{p}|m_f},
\]

\[
|\tilde{p}|^2 \Pi^f_{nc}(p \to 0) = -\frac{M}{k} \frac{1}{2|\tilde{p}|} e^{-|\tilde{p}|m_f} (1 + |\tilde{p}|m_f) = -\frac{M}{k} \frac{1}{2|\tilde{p}|} + O(|\tilde{p}|),
\]

(B.6)

while the gauge non-invariant terms naturally vanish. So (B.6) cancels out the divergence (A.6), while the vanishing of the coefficient of \(|\tilde{p}|^0\) in the Taylor expansion of (B.6) makes stable the value of the gap at \(|p| = 0\) when increasing the mass of the fermion.

As the mass of the fermion goes to infinity, the dispersion relation with cutoff must approach the one without it, so we must have curves that are qualitatively like the ones showed in Fig. 2.
References

[1] Z.F. Ezawa, "Quantum Hall Effects", World Scientific (2000).
[2] M.R. Douglas and N.A. Nekrasov, hep-th/0106048; R.J. Szabo, hep-th/0109162.
[3] S. Bahcall and L. Susskind, Int. J. Mod. Phys. B5 (1991) 2735.
[4] L. Susskind, hep-th/0101029.
[5] M. Stone, Phys. Rev. B42 (1990) 212.
[6] N. Read, Phys. Rev. Lett. 62 (1989) 86.
[7] A.P. Polychronakos, J. High Energy Phys. 04 (2001) 011 hep-th/0103013, J. High Energy Phys. 06 (2001) 070 hep-th/0106011.
[8] R.B. Laughlin, Phys. Rev. Lett. 50 (1983) 1395.
[9] S. Hellerman and M. Van Raamsdonk, J. High Energy Phys. 10 (2001) 039 hep-th/0103179.
[10] D. Karabali and B. Sakita, hep-th/0106016.
[11] N. Seiberg and E. Witten, J. High Energy Phys. 09 (1999) 032, hep-th/9908142.
[12] O. Bergman, Y. Okawa and J. Brodie, J. High Energy Phys. 11 (2001) 019 hep-th/0107178; S. Hellerman and L. Susskind, hep-th/0107200.
[13] N. Grandi and G.A. Silva, Phys. Lett. B507 (2001) 345 hep-th/0010113.
[14] H. Liu, Nucl. Phys. B614 (2001) 305 hep-th/0011125; Y. Okawa and H. Ooguri, Phys. Rev. D64 (2001) 046009 hep-th/0104036; S. Mukhi and N.V. Suryanarayana, J. High Energy Phys. 05 (2001) 023 hep-th/0104045; H. Liu and J. Michelson, Phys. Lett. B518 (2001) 143 hep-th/0104139.
[15] L. Alvarez-Gaumé and J.L.F. Barbón, hep-th/0109176.
[16] A. Gorsky, I.I. Kogan and C. Korthals-Altes, hep-th/0111013.
[17] A. Pinzul and A. Stern, J. High Energy Phys. 11 (2001) 023 hep-th/0107179; A.R. Lugo, hep-th/0111064.
[18] S. Minwalla, M. Van Raamsdonk and N. Seiberg, J. High Energy Phys. 02 (2000) 020 hep-th/9912072.
[19] M. Hayakawa, Phys. Lett. B478 (2000) 394 hep-th/9912094, hep-th/9912167; A. Matusis, L. Susskind and N. Toumbas, J. High Energy Phys. 12 (2000) 002 hep-th/0002075;
I.Ya. Aref’eva, D.M. Belov and A.S. Koshelev, *Nucl. Phys. Proc. Suppl.* 102 (2001) 11 hep-th/0003176.

[20] S. Deser, R. Jackiw and S. Templeton, *Ann. Phys. (N.Y.)* 140 (1982) 372.

[21] R.D. Pisarski and S. Rao, *Phys. Rev.* D32 (1985) 2081.

[22] G.-H. Chen and Y.-S. Wu, *Nucl. Phys.* B593 (2001) 562 hep-th/0006114.

[23] M. van Raamsdonk, *J. High Energy Phys.* 11 (2001) 006 hep-th/0110093; A. Armoni and E. López, hep-th/0110113.

[24] S.M. Girvin, A.H. Mac-Donald, P.M. Platzman, *Phys. Rev. Lett.* 54 (1981) 581.

[25] C.-S. Chu and P.-M. Ho, *Nucl. Phys.* B550 (1999) 151 hep-th/9812219; M.M. Sheikh-Jabbari, *Phys. Lett.* B455 (1999) 129 hep-th/9901080; D. Bigatti and L. Susskind, *Phys. Rev.* D62 (2000) 066004 hep-th/9908056.

[26] C. Kallin and B.I. Halperin, *Phys. Rev.* B30 (1984) 5655.

[27] D.-H. Lee and S.-C. Zhang, *Phys. Rev. Lett.* 66 (1991) 1220.

[28] S.-C. Zhang, T.H. Hansson and S. Kivelson, *Phys. Rev. Lett.* 62 (1988) 82.

[29] C.-S. Chu, *Nucl. Phys.* B580 (2000) 352 hep-th/0003007.