Generalized Quantum Arthur-Merlin Games

Hirotada Kobayashi* François Le Gall† Harumichi Nishimura‡

*Principles of Informatics Research Division
National Institute of Informatics
Tokyo, Japan

†Department of Computer Science
Graduate School of Information Science and Technology
The University of Tokyo
Tokyo, Japan

‡Department of Computer Science and Mathematical Informatics
Graduate School of Information Science
Nagoya University
Nagoya, Aichi, Japan

12 September 2014

Abstract

This paper investigates the role of interaction and coins in public-coin quantum interactive proof systems (also called quantum Arthur-Merlin games). While prior works focused on classical public coins even in the quantum setting, the present work introduces a generalized version of quantum Arthur-Merlin games where the public coins can be quantum as well: the verifier can send not only random bits, but also halves of EPR pairs. This generalization turns out to provide several novel characterizations of constant-turn interactive proof systems. First, it is proved that the class of two-turn quantum Arthur-Merlin games with quantum public coins, denoted $\text{qq-QAM}$ in this paper, does not change by adding a constant number of turns of classical interactions prior to the communications of the $\text{qq-QAM}$ proof systems. This can be viewed as a quantum analogue of the celebrated collapse theorem for AM due to Babai. To prove this collapse theorem, this paper provides a natural complete problem for $\text{qq-QAM}$: deciding whether the output of a given quantum circuit is close to a totally mixed state. This complete problem is on the very line of the previous studies investigating the hardness of checking the properties related to quantum circuits, and is of independent interest. It is further proved that the class $\text{qq-QAM}_1$ of two-turn quantum-public-coin quantum Arthur-Merlin proof systems with perfect completeness gives new bounds for standard well-studied classes of two-turn interactive proof systems. Finally, the collapse theorem above is extended to comprehensively classify the role of interaction and public coins in quantum Arthur-Merlin games: it is proved that, for any constant $m \geq 2$, the class of problems having an $m$-turn quantum Arthur-Merlin proof system is either equal to PSPACE or equal to the class of problems having a two-turn quantum Arthur-Merlin game of a specific type, which provides a complete set of quantum analogues of Babai’s collapse theorem.
1 Introduction

**Background and motivation.** Interactive proof systems [GMR89, Bab85] play a central role in computational complexity and has many applications such as probabilistic checkable proofs and zero-knowledge proofs. The aim of such a system is the verification of an assertion (e.g., verifying if an input is in a language) by a party implementing a polynomial-time probabilistic computation, called the verifier, interacting with another party with unlimited power, called the prover, in polynomially many turns. Two definitions are given on the secrecy of the coin which the verifier can flip: Goldwasser, Micali, and Rackoff [GMR89] defined private-coin proof systems, where the prover cannot see the outcomes of coin flips, while Babai [Bab85] defined public-coin proof systems, where the prover can see all the outcomes of coin flips. Public-coin interactive proof systems are often called Arthur-Merlin games or Arthur-Merlin proof systems, since in Ref. [Bab85] the verifier was called Arthur and the prover was called Merlin.

It is natural to expect that the power of interactive proof systems depends on the number of interaction turns. However, Babai [Bab85] showed that as long as the number of turns is a constant at least two, the number of turns does not affect the power of Arthur-Merlin proof systems, i.e., $AM(m) = AM(2)$ for any constant $m \geq 2$ (the collapse theorem), where $AM(m)$ is the class of problems having an $m$-turn Arthur-Merlin proof system. Goldwasser and Sipser [GS89] then showed that a private-coin interactive proof system can be simulated by an Arthur-Merlin proof system by adding two turns, and thus, these two types of interactive proof systems are computationally equivalent. By the above results, the class of problems having an interactive proof system of a constant number of turns is equal to $AM(2)$ (regardless of definitions with public coins or private coins), and this class is nowadays called AM. The class AM is believed to be much smaller than PSPACE, as it is contained in $\Pi^p_2$ in the second-level polynomial hierarchy [Lau83, Bab85]. On the contrary, the class of problems having a more general interactive proof system of polynomially many turns, called IP, does coincide with PSPACE [Pap85, LFKN92, Sha92] (again regardless of definitions with public coins or private coins [GS89, She92]).

Quantum interactive proof systems were introduced by Watrous [Wat03], and the class of problems having a quantum interactive proof system is called QIP. In the quantum world, the importance of the number of turns in interactive proof systems is drastically changed. The first paper on quantum interactive proofs [Wat03] already proved the surprising power of constant-turn quantum interactive proof systems, by showing that any problem in PSPACE has a three-turn quantum interactive proof system. Kitaev and Watrous [KW00] then proved that any quantum interactive proof system can be simulated by a three-turn quantum interactive proof system, namely, $QIP = QIP(3)$, where $QIP(m)$ denotes the class of problems having an $m$-turn quantum interactive proof system. Finally, the recent result $QIP = \text{PSPACE}$ by Jain, Ji, Upadhyay, and Watrous [JUWT1] completely characterized the computational power of quantum interactive proof systems with three turns or more. In contrast, despite of a number of intensive studies [Wat02b, Wch06, JUW09, HMW13], still very little is known on the class $QIP(2)$ corresponding to two-turn quantum interactive proof systems, and characterizing the computational power of two-turn quantum interactive proof systems is one of the main open problems in this field.

A public-coin version of quantum interactive proof systems was first introduced by Marriott and Watrous [MW05], named quantum Arthur-Merlin proof systems, where the messages from the verifier are restricted to classical strings consisting only of outcomes of polynomially many attempts of a fair coin flip. They then showed that three-turn quantum Arthur-Merlin proof systems can simulate three-turn standard quantum interactive proof systems, and hence the corresponding class, denoted QMAM, coincides with $QIP = \text{PSPACE}$. They also investigated the case of two-turn quantum Arthur-Merlin proof systems and showed that the corresponding class, denoted QAM, is included in $BP \cdot PP$, a subclass of PSPACE obtained by applying the BP operator to the class PP, which is still the only nontrivial upper bound known for QAM.

**Results and their meanings.** This paper introduces a “quantum public-coin” version of quantum interactive proof systems, which generalizes quantum Arthur-Merlin proof systems in Ref. [MW05]. In this generalized model, the verifier can send quantum messages, but these messages can be only used for sharing EPR pairs with
the prover, i.e., the verifier at his/her turn first generates polynomially many EPR pairs and then sends one half of each of them to the prover. The main interest in this model is again on the two-turn case, as allowing three or more turns in this model obviously hits the PSPACE ceiling. Let \( qqQAM \) be the class of problems having a two-turn “quantum public-coin” interactive proof system in which the first message from the verifier consists only of polynomially many halves of EPR pairs. Note that the only difference from the existing class QAM lies in the type of the message from the verifier: uniform random classical bits are replaced by halves of EPR pairs, which can be thought as a natural quantum version of classical public coins. The main goal of this paper is to investigate the computational power of this class \( qqQAM \) in order to figure out the advantages offered by quantum public-coins, and more generally, to make a step forward in the understanding of two-turn quantum interactive proof systems.

While the class \( qqQAM \) is the main target of investigation, this paper further studies the power of various models of quantum Arthur-Merlin proofs with quantum/classical public coins. For any constant \( m \geq 1 \) and any \( t_1, \ldots, t_m \) in \( \{c, q\} \), let \( t_m \cdots t_1QAM(m) \) be the class of problems that have an \( m \)-turn quantum interactive proof system with the following restrictions:

- For any odd \( j \), \( 1 \leq j \leq m \), the \((m - j + 1)\)st message (or the \(j\)th message counting from the last), which is the message from the prover sent at the \((m - j + 1)\)st turn, is a quantum message if \( t_j = q \), and is restricted to a classical message if \( t_j = c \).
- For any even \( j \), \( 1 \leq j \leq m \), at the \((m - j + 1)\)st turn, which is a turn for the verifier, the verifier first generates polynomially many EPR pairs and then sends halves of them if \( t_j = q \), while the verifier flips a fair coin polynomially many times and then sends their outcomes if \( t_j = c \).

The class \( t_m \cdots t_1QAM(m) \) may be simply written as \( t_m \cdots t_1QAM \) when there is no ambiguity in the number of turns: for instance, \( qqQAM(2) \) may be abbreviated to \( qqQAM \). Note that the classes QAM and QMAM defined in Ref. [MW05] are exactly the classes \( cqQAM \) and \( cqQAM \), respectively. The class \( ccQAM \) corresponds to two-turn public-coin quantum interactive proofs with classical communications: the verifier sends a question consisting only of outcomes of polynomially many attempts of a fair coin flip, then the prover responds with polynomially many classical bits, and the final verification is done by the verifier via polynomial-time quantum computation. By definition, \( AM \subseteq ccQAM \subseteq cqQAM \subseteq qqQAM \subseteq QIP(2) \).

As mentioned above, the main target in this paper is the class \( qqQAM \). First, it is proved that the power of \( qqQAM \) proof systems does not change by adding a constant number of turns of classical interactions prior to the communications of the \( qqQAM \) proof systems.

**Theorem 1.** For any constant \( m \geq 2 \), \( c \cdots ccqQAM(m) = qqQAM \).

In stark contrast to this, as mentioned before and will be stated clearly in Theorem 7, adding one turn of prior quantum interaction gives the \( qqQAM \) proof systems the full power of quantum interactive proof systems (i.e., the resulting class is PSPACE). Hence, Theorem 1 may be viewed as a quantum analogue of Babai’s collapse theorem [Bab85] for the class \( qqQAM \).

The proof of Theorem 1 comes in three parts: The first part proves that, for any constant \( m \geq 4 \), \( c \cdots ccqQAM(m) \) is necessarily included in \( ccqQAM \). The second part proves that \( ccqQAM \) is included in \( qqQAM \). Finally, the third part proves that \( ccqQAM \) is included in \( qqQAM \), by using the containment proved in the second part.

The first part is proved by carefully extending the argument in Babai’s collapse theorem. The core idea of Babai’s proof is that, by a probabilistic argument applied to a parallel repetition of the original proof system, the order of the verifier and the prover in the first three turns of the original system can be switched, which results in another proof system that has fewer number of turns. When proving the first part the messages of the first three turns of the original \( m \)-turn QAM proof system are classical, and thus, the argument in Babai’s collapse theorem still works.
The proof of the second part is one of the highlights in this paper. The main difficulty in proving this part (and the third part) is that the argument used in Babai’s collapse theorem fails when any of the first three turns is quantum in the starting proof system.

To overcome this difficulty, this paper first provides a natural complete promise problem for qq-QAM, namely, the CLOSE IMAGE TO TOTALLY MIXED (CITM) problem, which asks to check if the image of a given quantum circuit can be close to a totally mixed state, formally defined as follows.

**CLOSE IMAGE TO TOTALLY MIXED PROBLEM: CITM(𝑎, 𝑏)**

| Input: | A description of a quantum circuit 𝑄 acting on 𝑞_{all} qubits that has 𝑞_{in} specified input qubits and 𝑞_{out} specified output qubits. |
| Yes Instances: | There exists a quantum state 𝜌 of 𝑞_{in} qubits such that \(D(\Psi(\rho), (1/2)^{\otimes 𝑞_{out}}) \leq a\). |
| No Instances: | For any quantum state 𝜌 of 𝑞_{in} qubits, \(D(\Psi(\rho), (1/2)^{\otimes 𝑞_{out}}) \geq b\). |

Here, \(D(\cdot, \cdot)\) denotes the trace distance, \(\Psi(\rho)\) is the \(𝑞_{out}\)-qubit output state of \(𝑄\) when the input state was \(ρ\) (i.e., the reduced state obtained by tracing out the space corresponding to the \((𝑞_{all} − 𝑞_{out})\) non-output qubits after applying \(𝑄\) to \(ρ \otimes (|0\rangle⟨0|)^{\otimes (𝑞_{all} − 𝑞_{in})}\)), and \(I\) is the identity operator of dimension two (and thus, \((1/2)^{\otimes 𝑞_{out}}\) corresponds to the totally mixed state of 𝑞_{out} qubits). The following completeness result is proved.

**Theorem 2.** For any constants \(a\) and \(b\) in \((0, 1)\) such that \((1 − a)^2 > 1 − b^2\), CITM(𝑎, 𝑏) is qq-QAM-complete under polynomial-time many-one reduction.

Then the core idea for proving the second part is to use the structure of this complete problem that yes-instances are witnessed by the existence of a quantum state (i.e., the \(\exists\) quantifier appears in the first place), while no such witness quantum state exists for no-instances (i.e., the \(\forall\) quantifier appears in the first place). This makes it possible to incorporate the first turn of the cqq-QAM system into the input quantum state of the complete problem CITM (as the quantifier derived from the first turn of the cqq-QAM system matches the quantifier derived from the complete problem CITM), and thus, any problem in cqq-QAM can be reduced in polynomial time to the CITM problem with appropriate parameters, which is in qq-QAM.

Actually, for the proof, whether the image of a constructed quantum circuit can be close to a totally mixed state is partly evaluated by using the maximum output entropy of quantum channels, which shows implicitly the qq-QAM-completeness of another problem that asks to check whether the maximum output entropy of a quantum channel is larger than a given value or not. More formally, the following maximum output quantum entropy approximation (MaxOutQEA) problem is also qq-QAM-complete.

**MAXIMUM OUTPUT QUANTUM ENTROPY APPROXIMATION PROBLEM: MAXOUTQEA**

| Input: | A description of a quantum circuit that specifies a quantum channel \(Φ\), and a positive integer \(t\). |
| Yes Instances: | \(S_{\text{max}}(Φ) \geq t + 1\). |
| No Instances: | \(S_{\text{max}}(Φ) \leq t − 1\). |

Here, \(S_{\text{max}}(\cdot)\) denotes the maximum output von Neumann entropy. Namely, \(S_{\text{max}}(Φ) = \max_\rho S(Φ(\rho))\), where \(S(\cdot)\) denotes the von Neumann entropy and \(Φ(\rho)\) is the output quantum state of the quantum channel \(Φ\) when the input quantum state to it was \(ρ\).
Theorem 3. \textsc{MaxOutQea} is \textit{qq-QAM}-complete under polynomial-time many-one reduction.

Finally, the third part then can be proved by first providing a randomized reduction from a problem in \textit{ccqQAM} to a problem in \textit{cqQAM}, and then using the containment proved in the second part for the resulting problem in \textit{cqQAM}.

Besides its usefulness in proving Theorem 1, the complete problem \textsc{Citm} is of independent interest in the following sense. Recall that problems with formulations similar to \textsc{Citm} have already been studied, and were crucial to understand and characterize the computational power of several classes related to quantum interactive proof systems: testing closeness between the images of two given quantum circuits is \textit{QIP}-complete \cite{RaoW05} (and hence \textit{PsPace}-complete), testing closeness between a state produced by a given circuit and the image of another quantum circuit is \textit{QIP(2)}-complete \cite{Wat02}, testing closeness between two states produced by two given quantum circuits is \textit{QSZK}-complete \cite{Wat02, Wat09}, and testing closeness between the state produced by a quantum circuit and the totally mixed state is \textit{NIQSZK}-complete \cite{Kob03, CCKV08}. Theorem 2 shows that the class \textit{cqQAM}, besides its theoretical interest in the context of interactive proofs, is a very natural one that actually corresponds to a concrete computational problem that is on this line of studies investigating the hardness of checking the properties related to quantum circuits. Since \textsc{Citm} corresponds to the remaining pattern (image versus totally mixed state), Theorem 2 provides the last piece for characterizing the hardness of these kinds of computational problems.

It is further proved that the class \textit{cqQAM} (i.e., the standard \textit{QAM}) is necessarily contained in the one-sided bounded error version of \textit{qqQAM} of perfect completeness, denoted by \textit{qqQAM}_1 (throughout this paper, the perfect completeness version of each complexity class is indicated by adding the subscript “1”).

Theorem 4. \textit{cqQAM} \subseteq \textit{qqQAM}_1.

One useful property when proving this theorem is that the proof of Theorem 1 does not harm the perfect completeness property, i.e., it also holds that \( c \cdots \textit{cqQAM}_1(m) = \textit{qqQAM}_1 \), for any constant \( m \geq 2 \). Especially, the class \textit{ccqQAM}_1 is included in the class \textit{cqQAM}_1, and thus, one has only to prove that \textit{cqQAM} is included in \textit{ccqQAM}_1. This can be proved by combining the classical technique due to Cai \cite{Cai12} for proving AM = AM_1 (which itself originates in the proof of BPP \subseteq \Sigma^p_2 due to Lautemann \cite{Lau83}), and the recent result that any problem in QMA has a one-sided bounded error QMA system of perfect completeness in which Arthur and Merlin initially share a constant number of EPR pairs \cite{KLGN13} (which in particular implies that QMA is included in \textit{qqQAM}_1). Now the point is that, using two classical turns, the classical technique in Ref. \cite{Cai12} can be used to generate polynomially many instances of a (promise) QMA problem, all of which are QMA yes-instances if the input was a yes-instance, while at least one of which is a QMA no-instance with high probability if the input was a no-instance. Hence, by making use of the proof system in Ref. \cite{KLGN13} for each QMA instance, which essentially runs polynomially many attempts of a protocol of \textit{qqQAM} type in parallel to check that none of them results in rejection, one obtains a proof system of \textit{ccQAM} type with perfect completeness.

An immediate corollary of this theorem is the first nontrivial upper bound for \textit{QAM} in terms of quantum interactive proofs.

Corollary 5. \textit{QAM} \subseteq \textit{QIP}_1(2).

Here, \textit{QIP}_1(2) denotes the class of problems having a two-turn quantum interactive proof system of perfect completeness. This also improves the best known lower bound of \textit{QIP}_1(2) (from QMA shown in Ref. \cite{KLGN13} to \textit{QAM}). By using the fact MQA = MQA_1 (a.k.a., QCMA = QCMA_1) stating that classical-witness QMA systems can be made perfectly complete \cite{JKNN12}, a technique similar to the proof of Theorem 4 proves that perfect completeness is achievable in \textit{cc-QAM}.

Theorem 6. \textit{ccQAM} = \textit{ccQAM}_1.
Finally, results similar to Theorem 1 can be derived for other complexity classes related to the generalized quantum Arthur-Merlin proof systems. Namely, the following complete characterization is proved on the power of constant-turn generalized quantum Arthur-Merlin proofs, which can be viewed as the complete set of quantum analogues of Babai’s collapse theorem.

**Theorem 1.** The following four properties hold:

(i) For any constant $m \geq 3$ and any $t_1, \ldots, t_m$ in $\{c, q\}$, if there exists an index $j \geq 3$ such that $t_j = q$, then $t_m \cdots t_1$-$QAM(m) = \text{PSPACE}$. 

(ii) For any constant $m \geq 2$ and any $t_1$ in $\{c, q\}$, $c \cdots cq$-$t_1$-$QAM(m) = \text{qQ-QAM}$. 

(iii) For any constant $m \geq 2$, $c \cdots cq$-$QAM(m) = \text{cc-QAM} (= \text{QAM})$. 

(iv) For any constant $m \geq 2$, $c \cdots c$-$QAM(m) = \text{cc-QAM}$. 

**Further related work.** There are several works in which relevant subclasses of qQ-QAM were treated. In Ref. [KLGN13], the class QMA$^{\text{const-EPR}}$ was introduced to give an upper bound of QMA by its one-sided bounded error subclass QMA$^{\text{const-EPR}}$ with perfect completeness. This QMA$^{\text{const-EPR}}$ is an obvious subclass of qQ-QAM with a restriction that the first message from the verifier consists of not polynomially many but a constant number of halves of EPR pairs. The class qQ-QAM may be called QMA$^{\text{poly-EPR}}$, following the notation in Ref. [KLGN13]. Another subclass of qQ-QAM is the class NIQSZK studied in Refs. [Kob03, CCKV08] that corresponds to non-interactive quantum statistical zero-knowledge proof systems, where the zero-knowledge property must also be satisfied.

**Organization of the paper.** Section 2 summarizes the notions and properties that are used throughout this paper, and gives formal definitions of generalized quantum Arthur-Merlin proof systems. Section 3 proves the qQ-QAM-completeness of the CITM problem. Section 4 then gives a proof of Theorem 1, the collapse theorem for qQ-QAM. This essentially proves the qQ-QAM-completeness of the MAXOUTQEA problem also. Section 5 treats the result that the standard QAM is contained in qQ-QAM, the perfect-completeness version of qQ-QAM. Section 6 presents the complete classification of the complexity classes derived from generalized quantum Arthur-Merlin proof systems. Finally, Section 7 concludes the paper with some open problems. For completeness, a rigorous proof of the qQ-QAM-completeness of the MAXOUTQEA problem (Theorem 1) is given in the Appendix.

## 2 Preliminaries

Throughout this paper, let $\mathbb{N}$ and $\mathbb{Z}^+$ denote the sets of positive and nonnegative integers, respectively, and let $\Sigma = \{0, 1\}$ denote the binary alphabet set. A function $f : \mathbb{Z}^+ \to \mathbb{N}$ is *polynomially bounded* if there exists a polynomial-time deterministic Turing machine that outputs $1^f(n)$ on input $1^n$. A function $f : \mathbb{Z}^+ \to [0, 1]$ is *negligible* if, for every polynomially bounded function $g : \mathbb{Z}^+ \to \mathbb{N}$, it holds that $f(n) < 1/g(n)$ for all but finitely many values of $n$.

### 2.1 Quantum Fundamentals

We assume the reader is familiar with the quantum formalism, including pure and mixed quantum states, density operators, measurements, trace norm, fidelity, as well as the quantum circuit model (see Refs. [NC00, KSV02], for instance). This subsection summarizes some notations and properties that are used in this paper.

For each $k$ in $\mathbb{N}$, let $\mathbb{C}^k$ denote the $2^k$-dimensional complex Hilbert space whose standard basis vectors are indexed by the elements in $\Sigma^k$. In this paper, all Hilbert spaces are complex and have dimension a power of two.
For a Hilbert space $\mathcal{H}$, let $I_{\mathcal{H}}$ denote the identity operator over $\mathcal{H}$, and let $\mathcal{D}(\mathcal{H})$ and $\mathcal{U}(\mathcal{H})$ be the sets of density and unitary operators over $\mathcal{H}$, respectively. For a quantum register $R$, let $|0\rangle_R$ denote the state in which all the qubits in $R$ are in state $|0\rangle$. As usual, denote the four two-qubit states in $\mathbb{C}(\Sigma^2)$ that form the Bell basis by

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \quad |\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle),$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \quad |\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle),$$

respectively. Let

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

denote the Pauli operators. For convenience, we may identify a unitary operator with the unitary transformation it induces. In particular, for a unitary operator $U$, the induced unitary transformation is also denoted by $U$.

For two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ and a quantum state $\rho$ in $\mathcal{D}(\mathcal{H} \otimes \mathcal{K})$, the state obtained from $\rho$ by tracing out $\mathcal{K}$ (i.e., discarding the qubits in the reference system corresponding to $\mathcal{K}$) is the reduced state in $\mathcal{D}(\mathcal{H})$ of $\rho$ denoted by $\text{tr}_\mathcal{K}\rho$. For two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, a pure quantum state $|\psi\rangle$ in $\mathcal{H} \otimes \mathcal{K}$ is a purification of a mixed quantum state $\rho$ in $\mathcal{D}(\mathcal{H})$ iff $\text{tr}_\mathcal{K}|\psi\rangle\langle\psi| = \rho$.

For a linear operator $A$, the trace norm of $A$ is defined by

$$\|A\|_\text{tr} = \text{tr} \sqrt{A^\dagger A}.$$ 

For two quantum states $\rho$ and $\sigma$, the trace distance between them is defined by

$$D(\rho, \sigma) = \frac{1}{2} \|\rho - \sigma\|_\text{tr}.$$ 

A special case of the trace distance is the statistical difference between two probability distributions $\mu$ and $\nu$, which is defined by

$$\text{SD}(\mu, \nu) = D(\mu, \nu)$$

by viewing probability distributions as special cases of quantum states with diagonal density operators. The following important property is well known on probability distributions derived from quantum states.

**Lemma 8.** Let $\mu_\rho$ and $\mu_\sigma$ be the probability distributions derived from two quantum states $\rho$ and $\sigma$, respectively, by performing an arbitrary identical measurement. Then,

$$\text{SD}(\mu_\rho, \mu_\sigma) \leq D(\rho, \sigma).$$

For two quantum states $\rho$ and $\sigma$, the fidelity between them is defined by

$$F(\rho, \sigma) = \text{tr} \sqrt{\sqrt{\rho} \sigma \sqrt{\rho}}.$$ 

In particular, for two pure states $|\phi\rangle$ and $|\psi\rangle$, the fidelity between them is given by $F(|\phi\rangle\langle\phi|, |\psi\rangle\langle\psi|) = |\langle\phi|\psi\rangle|$. The fidelity can also be represented as follows [Uhl76].

**Lemma 9 (Uhlmann’s theorem).** For any Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ satisfying $\dim \mathcal{K} \geq \dim \mathcal{H}$ and any quantum states $\rho$ and $\sigma$ in $\mathcal{D}(\mathcal{H})$, let $|\phi_\rho\rangle$ and $|\phi_\sigma\rangle$ in $\mathcal{H} \otimes \mathcal{K}$ be any purifications of $\rho$ and $\sigma$. Then,

$$F(\rho, \sigma) = \max \{|\langle\phi_\rho|(I_{\mathcal{H}} \otimes U)|\phi_\sigma\rangle| : U \in \mathcal{U}(\mathcal{K})\}.$$
The following inequalities relate the trace distance and fidelity [FvdG99].

**Lemma 10** (Fuchs-van-de-Graaf inequalities). For any Hilbert space \( \mathcal{H} \) and any quantum states \( \rho \) and \( \sigma \) in \( \mathcal{D}(\mathcal{H}) \),
\[ 1 - F(\rho, \sigma) \leq D(\rho, \sigma) \leq \sqrt{1 - (F(\rho, \sigma))^2}. \]

This paper also uses the following property.

**Lemma 11.** For any Hilbert space \( \mathcal{H} \), any quantum states \( \rho, \sigma, \) and \( \tau \) in \( \mathcal{D}(\mathcal{H}) \), and any constant \( p \) in \([0, 1]\),
\[ D((1 - p)\rho + p\sigma, \tau) \geq D(\rho, \tau) - p. \]

**Proof.** By the triangle inequality, \( \| (1 - p)\rho + p\sigma - \tau \|_\text{tr} \geq \| \rho - \tau \|_\text{tr} - p \| \rho - \sigma \|_\text{tr} \), and thus,
\[ D((1 - p)\rho + p\sigma, \tau) \geq D(\rho, \tau) - p D(\rho, \sigma) \geq D(\rho, \tau) - p, \]
as desired. \( \square \)

For Hilbert spaces \( \mathcal{H} \) and \( \mathcal{K} \), let \( \mathcal{L}(\mathcal{H}) \) denote the set of linear mappings from \( \mathcal{H} \) to itself, let \( \mathcal{T}(\mathcal{H}, \mathcal{K}) \) denote the set of linear mappings from \( \mathcal{L}(\mathcal{H}) \) to \( \mathcal{L}(\mathcal{K}) \) (i.e., the set of linear mappings from \( \mathcal{L}(\mathcal{H}) \) to \( \mathcal{L}(\mathcal{K}) \) that are completely positive and trace-preserving).

For a linear mapping \( \Phi \) in \( \mathcal{T}(\mathcal{H}, \mathcal{K}) \), the diamond norm of \( \Phi \) is defined by
\[ \| \Phi \|_\diamond = \max \{ \| (\Phi \otimes I_{\mathcal{L}(\mathcal{H})})(\rho) \|_\text{tr} : \rho \in \mathcal{D}(\mathcal{H}^\otimes 2) \}, \]
where \( I_{\mathcal{L}(\mathcal{H})} \) is the identity mapping over \( \mathcal{L}(\mathcal{H}) \).

For Hilbert spaces \( \mathcal{H} \) and \( \mathcal{K} \) and quantum channels \( \Phi \) and \( \Psi \) in \( \mathcal{C}(\mathcal{H}, \mathcal{K}) \), the minimum output trace distance between \( \Phi \) and \( \Psi \) is defined by
\[ D_{\min}(\Phi, \Psi) = \min \{ D(\Phi(\rho), \Psi(\sigma)) : \rho, \sigma \in \mathcal{D}(\mathcal{H}) \}, \]
and the maximum output fidelity between \( \Phi \) and \( \Psi \) is defined by
\[ F_{\max}(\Phi, \Psi) = \max \{ F(\Phi(\rho), \Psi(\sigma)) : \rho, \sigma \in \mathcal{D}(\mathcal{H}) \}. \]

The Fuchs-van-de-Graaf inequalities relate the minimum output trace distance and the maximum output fidelity as follows.

**Lemma 12.** For any Hilbert spaces \( \mathcal{H} \) and \( \mathcal{K} \) and any quantum channels \( \Phi \) and \( \Psi \) in \( \mathcal{C}(\mathcal{H}, \mathcal{K}) \),
\[ 1 - F_{\max}(\Phi, \Psi) \leq D_{\min}(\Phi, \Psi) \leq \sqrt{1 - (F_{\max}(\Phi, \Psi))^2}. \]

**Proof.** Let \( \rho_\ast \) and \( \sigma_\ast \) be the quantum states in \( \mathcal{D}(\mathcal{H}) \) that minimize the expression \( D(\Phi(\rho), \Psi(\sigma)) \). Then,
\[ 1 - F_{\max}(\Phi, \Psi) \leq 1 - F(\Phi(\rho_\ast), \Psi(\sigma_\ast)) \leq D(\Phi(\rho_\ast), \Psi(\sigma_\ast)) = D_{\min}(\Phi, \Psi), \]
and thus, the first inequality holds. Similarly, let \( \rho_\ast \) and \( \sigma_\ast \) be the quantum states in \( \mathcal{D}(\mathcal{H}) \) that maximize the expression \( F(\Phi(\rho), \Psi(\sigma)) \). Then,
\[ D_{\min}(\Phi, \Psi) \leq D(\Phi(\rho_\ast), \Psi(\sigma_\ast)) \leq \sqrt{1 - (F(\Phi(\rho_\ast), \Psi(\sigma_\ast)))^2} = \sqrt{1 - (F_{\max}(\Phi, \Psi))^2}, \]
and the second inequality holds. \( \square \)
The following property is implicit in Ref. [KW00], which can be proved by using the multiplicativity of the diamond norm (see Problem 11.10 of Ref. [KSV02] as well as Theorem 3.24 of Ref. [Ros09], for instance).

**Lemma 13.** For any Hilbert spaces $\mathcal{H}_1, \mathcal{K}_1, \mathcal{H}_2, \text{ and } \mathcal{K}_2,$ and any quantum channels $\Phi_1$ and $\Psi_1$ in $\mathcal{C}(\mathcal{H}_1, \mathcal{K}_1)$ and $\Phi_2$ and $\Psi_2$ in $\mathcal{C}(\mathcal{H}_2, \mathcal{K}_2)$,

$$F_{\text{max}}(\Phi_1 \otimes \Phi_2, \Psi_1 \otimes \Psi_2) = F_{\text{max}}(\Phi_1, \Psi_1) F_{\text{max}}(\Phi_2, \Psi_2).$$

From Lemmas 12 and 13 one can show the following.

**Lemma 14.** For any Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, any quantum channels $\Phi$ and $\Psi$ in $\mathcal{C}(\mathcal{H}, \mathcal{K})$, and any $k$ in $\mathbb{N}$,

$$1 - \left[1 - (D_{\text{min}}(\Phi, \Psi))^2\right]^\frac{k}{2} \leq D_{\text{min}}(\Phi^\otimes k, \Psi^\otimes k) \leq k D_{\text{min}}(\Phi, \Psi).$$

**Proof.** From Lemmas 12 and 13 it holds that

$$1 - \left[1 - (D_{\text{min}}(\Phi, \Psi))^2\right]^\frac{k}{2} \leq 1 - (F_{\text{max}}(\Phi, \Psi))^k = 1 - F_{\text{max}}(\Phi^\otimes k, \Psi^\otimes k) \leq D_{\text{min}}(\Phi^\otimes k, \Psi^\otimes k),$$

and the first inequality of the claim follows.

On the other hand, by the triangle inequality, for any quantum states $\rho$ and $\sigma$ in $\mathcal{D}(\mathcal{H}),$

$$D((\Phi(\rho))^\otimes k, (\Psi(\sigma))^\otimes k) \leq D((\Phi(\rho))^\otimes k, \Psi(\sigma) \otimes (\Phi(\rho))^\otimes (k-1)) + D(\Psi(\sigma) \otimes (\Phi(\rho))^\otimes (k-1), (\Psi(\sigma))^\otimes k)$$

$$= D(\Phi(\rho), \Psi(\sigma)) + D((\Phi(\rho))^\otimes (k-1), (\Psi(\sigma))^\otimes (k-1)).$$

By repeatedly applying this bound with $\rho_s$ and $\sigma_s$ in $\mathcal{D}(\mathcal{H})$ that minimize the expression $D(\Phi(\rho), \Psi(\sigma)),$ it holds that

$$D_{\text{min}}(\Phi^\otimes k, \Psi^\otimes k) \leq D((\Phi(\rho_s))^\otimes k, (\Psi(\sigma_s))^\otimes k) \leq k D(\Phi(\rho_s), \Psi(\sigma_s)) = k D_{\text{min}}(\Phi, \Psi),$$

and the second inequality of the claim follows. \hfill \Box

Finally, for any quantum state $\rho$, the **von Neumann entropy** of $\rho$ is defined by

$$S(\rho) = - \text{tr}(\rho \log \rho).$$

A special case of the von Neumann entropy is the **Shannon entropy** of a probability distribution $\mu$, which is defined by

$$H(\mu) = S(\mu)$$

by viewing probability distributions as special cases of quantum states with diagonal density operators.

For Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ and a quantum channel $\Phi$ in $\mathcal{C}(\mathcal{H}, \mathcal{K})$, the **maximum output von Neumann entropy** of $\Phi$ is defined by

$$S_{\text{max}}(\Phi) = \max \{ S(\Phi(\rho)) : \rho \in \mathcal{D}(\mathcal{H}) \}.$$ 

This paper uses the following two properties on von Neumann entropy.

The first lemma provides an upper bound on the von Neumann entropy of a mixture of quantum states [NC00, Theorem 11.10].

8
Lemma 15. For any Hilbert space $\mathcal{H}$ and any quantum state $\rho$ in $\mathcal{D}(\mathcal{H})$ such that $\rho = \sum_i \mu_i \rho_i$ for some probability distribution $\mu = \{\mu_i\}$ and quantum states $\rho_i$ in $\mathcal{D}(\mathcal{H})$,

$$S(\rho) \leq H(\mu) + \sum_i \mu_i S(\rho_i).$$

The second lemma describes relations between the von Neumann entropy of a quantum state and the trace distance between the state and the totally mixed state (a similar but slightly stronger statement appeared in Ref. [CCKV07] without a proof).

Lemma 16. For any quantum state $\rho$ of $n$ qubits, it holds that

$$(1 - D(\rho, (I/2)^\otimes n) - 2^{-n})n \leq S(\rho) \leq n - \log \frac{1}{1 - D(\rho, (I/2)^\otimes n)} + 2.$$

Proof. First we show the first inequality. By considering the spectral decomposition of $\rho$, one can write $\rho = \sum_{x \in \{0,1\}^n} \mu_x |\psi_x\rangle \langle \psi_x|$ for some probability distribution $\mu = \{\mu_x\}_{x \in \{0,1\}^n}$ over $\{0,1\}^n$ and orthonormal basis $\{|\psi_x\rangle\}_{x \in \{0,1\}^n}$. Note that $D(\rho, (I/2)^\otimes n) = SD(\mu, \nu)$ and $S(\rho) = H(\mu)$, where $\nu$ is the uniform distribution over $\{0,1\}^n$. Hence, it suffices to show that the inequality $H(\mu) \geq (1 - SD(\mu, \nu))n - \frac{n}{2^2}$ holds for any probability distribution $\mu$.

Let $\gamma = SD(\mu, \nu)$. By the concavity of the Shannon entropy, any probability distribution $\nu = \{\nu_x\}_{x \in \{0,1\}^n}$ over $\{0,1\}^n$ that minimizes $H(\nu)$ under the condition $SD(\nu, \nu) = \gamma$ can be expressed as follows: there exist $x_0, x_1, \ldots, x_k, x_{k+1}$ in $\{0,1\}^n$ such that

$$\nu_x = \begin{cases} 
\frac{1}{2^n} + \gamma & \text{if } x = x_0, \\
\frac{\epsilon}{2^n} & \text{if } x \in \{x_1, \ldots, x_k\}, \\
\frac{\epsilon}{2^n} & \text{if } x = x_{k+1}, \\
0 & \text{otherwise},
\end{cases}$$

where $k = [2^n(1 - \gamma)] - 1$ and $\epsilon = 2^n(1 - \gamma) - [2^n(1 - \gamma)]$ (in fact, any probability distribution with statistical distance $\gamma$ from the uniform distribution $\nu$ is necessarily a mixture of probability distributions of this type). It follows that

$$H(\nu) = \left(\frac{1}{2^n} + \gamma\right) \log \left(\frac{1}{\frac{1}{2^n} + \gamma}\right) + k \frac{n}{2^n} + \frac{\epsilon}{2^n} \log \frac{2^n}{\epsilon}$$

$$\geq \left(2^n(1 - \gamma) - 1\right) \frac{n}{2^n} + \frac{\epsilon}{2^n} \left(n + \log \frac{1}{\epsilon}\right)$$

$$= (1 - \gamma)n - \frac{n}{2^n} + \frac{\epsilon}{2^n} \log \frac{1}{\epsilon}$$

$$\geq (1 - \gamma)n - \frac{n}{2^n},$$

and thus, the inequality $H(\mu) \geq (1 - SD(\mu, \nu))n - \frac{n}{2^n}$ holds.

Now we show the second inequality. Similarly to the first inequality case, it suffices to show that the inequality $H(\mu) \leq n - \log \frac{1}{1 - SD(\mu, \nu)} + 2$ holds for any probability distribution $\mu$.

Again let $\gamma = SD(\mu, \nu)$. From the Vajda inequality [Vaj70] (see Theorem 4.8 of Ref. [Dru12] also), it holds that

$$D(\mu \parallel \nu) \geq \frac{1}{\ln 2} \left(\ln \frac{1}{1 - \gamma} - 1\right).$$
where $D(\cdot \| \cdot)$ denotes the relative entropy between two probability distributions. Since $D(\mu \| \nu) = n - H(\mu)$, it follows that
\[
H(\mu) \leq n - \frac{1}{\ln 2} \left( \ln \frac{1}{1 - \gamma} - 1 \right) = n - \log \frac{1}{1 - \gamma} + \frac{1}{\ln 2} \leq n - \log \frac{1}{1 - \gamma} + 2,
\]
as desired. \hfill \square

### 2.2 Polynomial-Time Uniformly Generated Families of Quantum Circuits

Following conventions, this paper defines quantum Arthur-Merlin proof systems in terms of quantum circuits. In particular, this paper uses the following notion of polynomial-time uniformly generated families of quantum circuits.

A family \(\{Q_x\}\) of quantum circuits is \textit{polynomial-time uniformly generated} if there exists a deterministic procedure that, on every input \(x\), outputs a description of \(Q_x\) and runs in time polynomial in \(|x|\). It is assumed that the circuits in such a family are composed of gates in some reasonable, universal, finite set of quantum gates. Furthermore, it is assumed that the number of gates in any circuit is not more than the length of the description of that circuit. Therefore \(Q_x\) must have size polynomial in \(|x|\). For convenience, we may identify a circuit \(Q_x\) with the unitary operator it induces.

For the results in which perfect completeness is concerned, this paper assumes a gate set with which the Hadamard and any classical reversible transformations can be exactly implemented. Note that this assumption is satisfied by many standard gate sets such as the Shor basis \cite{Sho96} consisting of the Hadamard, controlled-\(i\)-phase-shift, and Toffoli gates, and the gate set consisting of the Hadamard, Toffoli, and NOT gates \cite{Shi02,Aha03}. Moreover, as the Hadamard transformation in some sense can be viewed as a quantum analogue of the classical operation of flipping a fair coin, our assumption would be the most natural quantum correspondence to the tacit classical assumption in randomized complexity theory that fair coins and perfect logical gates are available. Hence we believe that our condition is very reasonable and not restrictive.

Since non-unitary and unitary quantum circuits are equivalent in computational power \cite{AKN98}, it is sufficient to treat only unitary quantum circuits, which justifies the above definition. Nevertheless, for readability, most procedures in this paper will be described using intermediate projective measurements and unitary operations conditioned on the outcome of the measurements. All of these intermediate measurements can be deferred to the end of the procedure by a standard technique so that the procedure becomes implementable with a unitary circuit.

### 2.3 Generalized Quantum Arthur-Merlin Proof Systems

A generalized quantum Arthur-Merlin (QAM) proof system consists of a polynomial-time quantum verifier and an all-powerful quantum prover.

For any constant \(m \geq 1\) and any \(t_j\) in \(\{c, q\}\) for each \(j\) in \(\{1, \ldots, m\}\), a generalized QAM proof system is of \(t_m \cdots t_1\)-QAM type if the message at the \((m - j + 1)\)st turn is quantum (resp. is restricted to classical) for each \(j\) such that \(t_j = q\) (resp. \(t_j = c\)).

Formally, an \(m\)-turn quantum verifier \(V\) for generalized quantum Arthur-Merlin proof systems is a polynomial-time computable mapping of the form \(V: \{0,1\}^* \rightarrow \{0,1\}^*\). For each \(x\) in \(\{0,1\}^*\), \(V(x)\) is interpreted as a description of a quantum circuit acting on \((q_V(|x|) + mq_M(|x|))\) qubits with a specification of a \(q_V(|x|)\)-qubit quantum register \(V\) and a \(q_M(|x|)\)-qubit quantum register \(M_j\) for each \(j\) in \(\{1, \ldots, m\}\), for some polynomially bounded functions \(q_V, q_M: \mathbb{Z}^+ \rightarrow \mathbb{N}\). One of the qubits in \(V\) is designated as an output qubit. At the \((m - j + 1)\)st turn for any even \(j\) such that \(2 \leq j \leq m - 1\), \(V\) receives a message from a prover, either classical or quantum, which is stored in the quantum register \(M_{m-j}\). If the system is of \(t_m \cdots t_1\)-QAM type, at the \((m - j + 1)\)st turn for any even \(j\) such that \(2 \leq j \leq m\), if \(t_j = c\), \(V\) flips a fair coin \(q_M(|x|)\) times to obtain a binary string \(r\) of length \(q_M(|x|)\), then sends \(r\) to a prover, and stores \(r\) in the quantum register \(M_{m-j+1}\), while if \(t_j = q\), \(V\) generates
functions

Lemma 20. Given a constant $m \in \mathbb{N}$ and any polynomially bounded function $p$, and any functions $c, s : \mathbb{Z}^+ \to [0, 1]$ satisfying $c - s \geq \frac{1}{m}$ for some polynomially bounded function $q$, 

\[ t_m \cdots t_1\text{-QAM}(m, c, s) \subseteq t_m \cdots t_1\text{-QAM}(m, 1 - 2^{-p}, 2^{-p}). \]
Lemma 21. For any constant \( m \in \mathbb{N} \), any \( t_1, \ldots, t_m \in \{c, q\} \), any polynomially bounded function \( p \), and any function \( s : \mathbb{Z}^+ \to [0, 1] \) satisfying \( 1 - s \geq \frac{1}{q} \) for some polynomially bounded function \( q \),
\[
t_m \cdots t_1 \cdot \text{QAM}(m, 1, s) \subseteq t_m \cdots t_1 \cdot \text{QAM}(m, 1, 2^{-p}).
\]

The proof of Lemma 20 uses the following lemma (the claim was proved in this form in Ref. [KMY09], but similar statements are also found in Refs. [ABD09, IUW09]).

Lemma 22. Let \( c, s : \mathbb{Z}^+ \to [0, 1] \) be any functions that satisfy \( c - s \geq \frac{1}{q} \) for some polynomially bounded function \( q \), and let \( \Pi \) be any proof system with completeness \( c \) and soundness \( s \). Fix any polynomially bounded function \( q' \), and consider another proof system \( \Pi' \) such that, for every input of length \( n \), \( \Pi' \) carries out \( N = 2q(n)(q(n))^2 \) attempts of \( \Pi \) in parallel, and accepts if and only if at least \( \frac{c(n)+s(n)}{2} \) fraction of these \( N \) attempts results in acceptance in \( \Pi \). Then \( \Pi' \) has completeness \( 1 - 2^{-q'} \) and soundness \( \frac{2c}{c+s} \leq 1 - \frac{c-s}{2} \leq 1 - \frac{1}{2q} \).

Now the amplification result for generalized quantum Arthur-Merlin proof systems follows from Lemma 22 and the perfect parallel repetition theorem for general quantum interactive proof systems [Gut09].

Proof of Lemma 20. First, the inclusion \( t_m \cdots t_1 \cdot \text{QAM}(m, c, s) \subseteq t_m \cdots t_1 \cdot \text{QAM}(m, 1 - \frac{2^{-p+1}}{|p/(c-s)|}, 1 - \frac{c-s}{2}) \) follows from Lemma 22 by taking \( q' \) in the statement of Lemma 22 as \( q' = p + \lceil \log_2 \left( \lceil \frac{p}{c-s} \rceil \right) \rceil + 1 \).

We show the inclusion \( t_m \cdots t_1 \cdot \text{QAM}(m, 1 - \frac{2^{-p+1}}{|p/(c-s)|}, 1 - \frac{c-s}{2}) \subseteq t_m \cdots t_1 \cdot \text{QAM}(m, 1 - 2^{-p}, 2^{-p}) \) to complete the proof.

Fix any protocol \( \Pi \) of \( t_m \cdots t_1 \cdot \text{QAM}(m) \) proof systems, and consider the \( k \)-fold repetition \( \Pi \otimes k \) of \( \Pi \), where Arthur runs \( k \) attempts of \( \Pi \) in parallel, and accepts if and only if all of the \( k \) attempts result in acceptance in the original \( \Pi \). We claim that the maximum acceptance probability in \( \Pi \otimes k \) is exactly \( a^k \) if the maximum acceptance probability in \( \Pi \) was \( a \). To show this claim, consider another protocol \( Q(\Pi) \) of \( m \)-turn (general) quantum interactive proof systems that exactly simulates \( \Pi \) as follows: the verifier in \( Q(\Pi) \) behaves exactly the same manner as Arthur in \( \Pi \) except that, upon receiving the \( j \)th message from a prover (resp. sending the \( j \)th message to a prover), if \( t_j = c \) in \( \Pi \), the verifier of \( Q(\Pi) \) first makes sure that the received message (resp. the sent message) is indeed classical by taking a copy of the message by CNOT operations (and the copied message will never be touched in the rest of the protocol). This clearly makes it useless for a malicious prover to send a quantum message, deviating the original protocol \( \Pi \), and thus, the maximum acceptance probability in \( Q(\Pi) \) obviously remains \( a \). Now from the perfect parallel repetition theorem for general quantum interactive proofs [Gut09], the \( k \)-fold parallel repetition \( (Q(\Pi)) \otimes k \) of \( Q(\Pi) \) has its maximum acceptance probability exactly \( a^k \). As the protocol \( (Q(\Pi)) \otimes k \) is identical to the protocol \( (Q(\Pi)) \otimes k \) of the \( m \)-turn (general) quantum interactive proof system that exactly simulates \( \Pi \otimes k \), the maximum acceptance probability in \( \Pi \otimes k \) is also \( a^k \). Hence, letting \( k = 2\lceil \frac{p}{c-s} \rceil \), the desired inclusion \( t_m \cdots t_1 \cdot \text{QAM}(m, 1 - \frac{2^{-p+1}}{|p/(c-s)|}, 1 - \frac{c-s}{2}) \subseteq t_m \cdots t_1 \cdot \text{QAM}(m, 1 - 2^{-p}, 2^{-p}) \) follows from the \( k \)-fold repetition.

Lemma 21 is proved in essentially the same manner as in Lemma 20 (Lemma 22 is not necessary in this case, which makes the proof slightly simpler).

3 qq-QAM-Completeness of CITM

This section proves Theorem 2 which states that the CLOSE IMAGE TO TOTALLY MIXED (CITM) problem is complete for the class qq-QAM.

First, it is proved that CITM(\( a, b \)) is in qq-QAM for appropriately chosen parameters \( a \) and \( b \). The proof is a special case of the proof of the CLOSE IMAGE problem being in QIP(2) [Wat02a, HMW12].
Verifier’s qq-QAM Protocol for CITM\((a, b)\)

1. Prepare \(q_{\text{out}}\) qubit registers \(S_1\) and \(S_2\), and generate \(q_{\text{out}}\) EPR pairs \(|\Phi^+\rangle^\otimes q_{\text{out}}\) in \((S_1, S_2)\) so that the \(j\)th qubit of \(S_1\) and that of \(S_2\) form an EPR pair, for every \(j \in \{1, \ldots, q_{\text{out}}\}\). Send \(S_2\) to the prover.

2. Receive a \((q_{\text{all}} - q_{\text{out}})\)-qubit quantum register \(R\) from the prover. Apply the unitary transformation \(U_{Q_x}^\dagger\) to \((R, S_1)\). Accept if all the qubits in \(A\) are in the \(|0\rangle\) state, and reject otherwise, where \(A\) is the quantum register consisting of the last \((q_{\text{all}} - q_{\text{in}})\) qubits of \((S_1, R)\) (i.e., the non-input qubits of \(Q_x\)).

---

**Lemma 23.** CITM\((a, b)\) is in qq-QAM for any constants \(a, b \in [0, 1]\) satisfying \((1 - a)^2 > 1 - b^2\).

**Proof.** Let \(Q_x\) be a quantum circuit of an instance \(x\) of CITM\((a, b)\) acting on \(q_{\text{all}}\) qubits with \(q_{\text{in}}\) specified input qubits and \(q_{\text{out}}\) specified output qubits. Without loss of generality, one can assume that the first \(q_{\text{in}}\) qubits correspond to the input qubits, and the last \(q_{\text{out}}\) qubits correspond to the output qubits. Let \(U_{Q_x}\) denote the unitary operator induced by \(Q_x\). We construct a verifier \(V\) of the qq-QAM proof system with completeness \((1 - a)^2\) and soundness \(1 - b^2\) as follows (recall that \(a\) and \(b\) are constants in the interval \([0, 1]\) such that \((1 - a)^2 > 1 - b^2\), and thus this qq-QAM proof system is sufficient for the claim).

Let \(S_1\) and \(S_2\) be quantum registers of \(q_{\text{out}}\) qubits. The verifier \(V\) first generates \(q_{\text{out}}\) EPR pairs \(|\Phi^+\rangle^\otimes q_{\text{out}}\) in \((S_1, S_2)\) so that the \(j\)th qubit of \(S_1\) and that of \(S_2\) form an EPR pair, for every \(j \in \{1, \ldots, q_{\text{out}}\}\). Then \(V\) sends \(S_2\) to the prover. Upon receiving a quantum register \(R\) of \((q_{\text{all}} - q_{\text{out}})\) qubits, \(V\) applies the unitary transformation \(U_{Q_x}^\dagger\) to \((R, S_1)\). Letting \(A\) be the quantum register consisting of the last \((q_{\text{all}} - q_{\text{in}})\) qubits of the register \((R, S_1)\) (i.e., corresponding to the non-input qubits of \(Q_x\)), \(V\) accepts \(x\) if and only if all the qubits in \(A\) are in the \(|0\rangle\) state. Figure 1 summarizes the protocol of the verifier \(V\).

Figure 1: Verifier’s qq-QAM protocol for CITM.

Let \(\mathcal{W}\) denote the Hilbert space corresponding to the \(q_{\text{in}}\) input qubits of \(Q_x\).

For the completeness, suppose that there exists a quantum state \(\rho \in \mathcal{D}(\mathcal{W})\) such that \(D(Q_x(\rho), (I/2)^\otimes q_{\text{out}}) \leq a\). By Lemma 10 (the Fuchs-van-de-Graaf inequalities), it holds that \(F(Q_x(\rho), (I/2)^\otimes q_{\text{out}}) \geq 1 - a\). Consider a \(2q_{\text{in}}\)-qubit pure state \(|\phi_\rho\rangle\) that is a purification of \(\rho\) such that \(\rho\) is the reduced state obtained by tracing out the first \(q_{\text{in}}\) qubits of \(|\phi_\rho\rangle\) (such a purification always exists). Then, the \((q_{\text{all}} + q_{\text{in}})\)-qubit state

\[
|\psi_\rho\rangle = (I^\otimes q_{\text{in}} \otimes U_{Q_x})(|\phi_\rho\rangle \otimes |0\rangle^\otimes (q_{\text{all}} - q_{\text{in}}))
\]

is necessarily a purification of \(Q_x(\rho)\), and thus, the \((q_{\text{all}} + q_{\text{in}} + q_{\text{out}})\)-qubit state \(|\psi_\rho'\rangle = |0\rangle^\otimes q_{\text{out}} \otimes |\psi_\rho\rangle\) is also a purification of \(Q_x(\rho)\). On the other hand, an obvious purification of the \(q_{\text{out}}\)-qubit totally mixed state \((I/2)^\otimes q_{\text{out}}\) is the \(2q_{\text{out}}\)-qubit state \(|\xi\rangle\) that is obtained by rearranging the qubits of \(|\Phi^+\rangle^\otimes q_{\text{out}}\) so that the \(j\)th qubit and the \((q_{\text{all}} + j)\)th qubit form an EPR pair for every \(j \in \{1, \ldots, q_{\text{out}}\}\). Hence, the \((q_{\text{all}} + q_{\text{in}} + q_{\text{out}})\)-qubit state \(|\xi'\rangle = |0\rangle^\otimes (q_{\text{all}} + q_{\text{in}}) \otimes |\xi\rangle\) is also a purification of \((I/2)^\otimes q_{\text{out}}\). As the reduced state consisting of the last \(q_{\text{out}}\) qubits of \(|\psi_\rho'\rangle\) is exactly \(Q_x(\rho)\), while the reduced state consisting of the last \(q_{\text{out}}\) qubits of \(|\xi'\rangle\) is exactly \((I/2)^\otimes q_{\text{out}}\), it follows from Lemma 9 (Uhlmann’s theorem) that

\[
F(Q_x(\rho), (I/2)^\otimes q_{\text{out}}) = \max_U |\langle \psi_\rho'(U \otimes I^\otimes q_{\text{out}})|\xi'\rangle| = F(Q_x(\rho), (I/2)^\otimes q_{\text{out}}) \geq 1 - a.
\]
Thus, if a prover prepares $|0\rangle^{\otimes(q_{all}+q_{in}-q_{out})}$ in his/her private quantum register $P$ of $(q_{all} + q_{in} - q_{out})$ qubits, applies $U_P$ to $(P, S_2)$ after having received $S_2$, and sends the last $(q_{all} - q_{out})$ qubits of $(P, S_2)$ back to the verifier, the probability of acceptance is
\[
\left\| \left( I^{\otimes(2q_{in}+q_{out})} \otimes (|0\rangle \langle 0|)^{\otimes(q_{all}-q_{in})} \right) \left( I^{\otimes(q_{in}+q_{out})} \otimes U_{Q_x}^\dagger \right) \left( U_P \otimes I^{\otimes(q_{out})} \right) |\xi\rangle \right\|^2 \\
\geq \left\| \left( (|0\rangle \langle 0|)^{\otimes q_{out}} \otimes |\phi\rangle \langle \phi| \otimes (|0\rangle \langle 0|)^{\otimes(q_{all}-q_{in})} \right) \left( I^{\otimes(q_{in}+q_{out})} \otimes U_{Q_x}^\dagger \right) \left( U_P \otimes I^{\otimes(q_{out})} \right) |\xi\rangle \right\|^2 \\
= \left| \langle \psi' | (U_P \otimes I^{\otimes(q_{out})}) |\xi\rangle \right|^2 \\
\geq (1-a)^2,
\]
where the first inequality follows from the fact that $(|0\rangle \langle 0|)^{\otimes q_{out}} \otimes |\phi\rangle \langle \phi| \otimes I^{\otimes(q_{all}-q_{in})}$ is a projection operator. This implies the completeness $(1-a)^2$ of the constructed proof system.

For the soundness, suppose that for any quantum state $\rho \in D(W)$, it holds that $D(Q_x(\rho), (I/2)^{\otimes q_{out}}) \geq b$. Let $P'$ be any prover who uses his/her private quantum register $P'$ of $q$ qubits, for arbitrarily large integer $q$. Without loss of generality, one can assume that all the qubits in $P'$ are in the $|0\rangle$ state at the beginning of the protocol. Let $U_{P'}$ be the unitary operator acting on $(q + q_{out})$ qubits which $P'$ applies to $(P', S_2)$ after having received $S_2$, and let $|\phi\rangle$ be the $(q + 2q_{out})$-qubit state defined by
\[
|\phi\rangle = \left( I^{\otimes(q_{all}+2q_{out})} \otimes U_{Q_x}^\dagger \right) \left( U_{P'} \otimes I^{\otimes q_{out}} \right) |\xi''\rangle,
\]
where $|\xi''\rangle$ is the $(q + 2q_{out})$-qubit state defined as $|\xi''\rangle = |0\rangle^{\otimes q} \otimes |\xi\rangle$. Define the projection operator $\Pi_{acc}$ by $\Pi_{acc} = \Pi_{acc} = I^{\otimes(q_{all}+2q_{out})} \otimes (|0\rangle \langle 0|)^{\otimes(q_{all}-q_{in})}$. Then, the $(q + 2q_{out})$-qubit state $|\psi\rangle$ defined by $|\psi\rangle = \frac{1}{\|\Pi_{acc}|\phi\rangle\|} \Pi_{acc} |\phi\rangle$ must be written as $|\psi\rangle = |\psi'\rangle \otimes |0\rangle^{\otimes(q_{all}-q_{in})}$ for some $(q - q_{all} + q_{in} + 2q_{out})$-qubit state $|\psi'\rangle$, as $\Pi_{acc} |\psi'\rangle = |\psi'\rangle$ holds.

As $D(Q_x(\rho), (I/2)^{\otimes q_{out}}) \geq b$ for any quantum state $\rho \in D(W)$, from Lemma 11 (the Fuchs-van-de-Graaf inequalities), it holds that $F(Q_x(\rho), (I/2)^{\otimes q_{out}}) \leq \sqrt{1-b^2}$ for any quantum state $\rho \in D(W)$. This in particular implies that
\[
|\langle \psi | \phi \rangle | = \left| \langle \psi' | \otimes (0)^{\otimes(q_{all}-q_{in})} \right) \left( I^{\otimes(q_{all}+2q_{out})} \otimes U_{Q_x}^\dagger \right) \left( U_{P'} \otimes I^{\otimes q_{out}} \right) |\xi''\rangle \right| \\
\leq F(Q_x(\rho_{\psi}), (I/2)^{\otimes q_{out}}) \leq \sqrt{1-b^2},
\]
where $\rho_{\psi} \in D(W)$ is the reduced state of $|\psi'\rangle$ obtained by tracing out all but the last $q_{in}$ qubits, and we have used the fact that the reduced state consisting of the last $q_{out}$ qubits of $|\xi''\rangle$ is exactly $(I/2)^{\otimes q_{out}}$ on which $U_{P'}$ never acts. As the acceptance probability $p_{P'}$ with this prover $P'$ is exactly $\|\Pi_{acc} |\phi\rangle\|^2$, while $\|\Pi_{acc} |\phi\rangle\| = \frac{1}{\|\Pi_{acc} |\phi\rangle\|} |\langle \phi | \Pi_{acc} |\phi\rangle| = |\langle \psi | \phi \rangle|$, it holds from Eq. 11 that $p_{P'} \leq 1 - b^2$, and the soundness follows.

Now the CITM problem is proved to be hard for qq-QAM.

**Lemma 24.** For any constants $a$ and $b$ such that $0 < a < b < 1$, CITM$(a,b)$ is hard for qq-QAM under polynomial-time many-one reduction.

**Proof.** Let $A = (A_{yes}, A_{no})$ be a problem in qq-QAM. Then $A$ has a qq-QAM proof system with completeness $c$ and soundness $s$ for some constants $c$ and $s$ chosen later satisfying $0 < s < c < 1$. Let $V$ be the quantum verifier witnessing this proof system. Fix an input $x$, and let $V$ and $M$ be quantum registers consisting of $q_V$ and $q_M$ qubits, respectively, where $V$ corresponds to the private qubits of $V$ and $M$ corresponds to the message qubits $V$ would receive on input $x$. Without loss of generality, one can assume that the first qubit of $V$ is the output qubit of $V$, and the last $q_S$ qubits of $V$ form the quantum register $S$ corresponding to the halves of the EPR pairs $V$ would keep until the final verification procedure is performed. Let $\bar{S}$ be the quantum register of $(q_V - q_S)$ qubits consisting of
Algorithm Corresponding to Quantum Circuit $Q_x$

1. Prepare the quantum registers $V$ and $M$, each of $q_V$ and $q_M$ qubits, respectively. Denote by $S$ and $\overline{S}$ the quantum registers consisting of the last $q_S$ and first $(q_V - q_S)$ qubits of $V$, respectively. The last $(q_S + q_M)$ qubits in $(V, M) = (\overline{S}, S, M)$ (i.e., all the qubits in $(S, M)$) are designated as the input qubits, while the last $q_S$ qubits of $V = (\overline{S}, S)$ (i.e., all the qubits in $S$) are designated as the output qubits.

2. Flip a fair coin, and proceed to Step 2.1 if it results in “Heads”, and proceed to Step 2.2 if it results in “Tails”.

   2.1 Output all the qubits in $S$.

   2.2 Perform $V_x$ over $(V, M) = (\overline{S}, S, M)$. If the first qubit of $V$ is in state $|1\rangle$, output the totally mixed state $(I/2)^{\otimes q_S}$ (by first generating the totally mixed state using fresh ancillae, and then swapping the qubits in $S$ with the generated totally mixed state), and output $|0\rangle^{\otimes q_S}$ otherwise (by swapping the qubits in $S$ with $q_S$ fresh ancillae).

Figure 2: The construction of the quantum circuit $Q_x$.

the first $(q_V - q_S)$ qubits of $V$ (i.e., all the private qubits of $V$ but those belonging to $S$). Denote by $V_x$ the unitary operator induced by this $V$ on input $x$.

We construct a quantum circuit $Q_x$ that exactly implements the following algorithm. The circuit $Q_x$ expects to receive a $(q_S + q_M)$-qubit state as its input, and prepares the quantum registers $V = (\overline{S}, S)$ and $M$, where the input state is expected to be stored in $(S, M)$. Then with probability one-half, $Q_x$ just outputs the state in the register $S$. Otherwise $Q_x$ performs $V_x$ over $(V, M) = (\overline{S}, S, M)$, and outputs the totally mixed state $(I/2)^{\otimes q_S}$ if the first qubit of $V$ is in state $|1\rangle$ (i.e., if the system is in an accepting state of the original verifier $V$), and outputs $|0\rangle^{\otimes q_S}$ if the first qubit of $V$ is in state $|0\rangle$ (i.e., if the system is in a rejecting state of the original verifier $V$). Figure 2 summarizes the construction of the circuit $Q_x$.

First suppose that $x$ is in $A_{\text{yes}}$. Then there exists a quantum prover $P$ who makes $V$ accept with probability at least $c$. Let $\rho_x$ be the $(q_S + q_M)$-qubit state in $(S, M)$ just after $V$ has received a response from $P$ on input $x$. Note that the reduced state in $S$ of $\rho_x$ when tracing out all the qubits in $M$ is exactly $(I/2)^{\otimes q_S}$, as $P$ has never touched the qubits in $V = (\overline{S}, S)$. Let $\rho'_x$ be the $(q_V + q_M)$-qubit state in $(V, M) = (\overline{S}, S, M)$ defined by $\rho'_x = |0\rangle \langle 0|^{\otimes q_V} \otimes \rho_x$, and let $\Pi_{\text{acc}}$ be the projection operator defined by $\Pi_{\text{acc}} = |1\rangle \langle 1| \otimes I^{\otimes (q_V + q_M - 1)}$.

Then $p_{\text{acc}} = \text{tr} \Pi_{\text{acc}} V_x \rho'_x V_x^\dagger$ is exactly the acceptance probability with this prover $P$, which is at least $c$, and $Q_x$ outputs the state

$$\xi = p_{\text{acc}}(I/2)^{\otimes q_S} + (1 - p_{\text{acc}})(|0\rangle \langle 0|)^{\otimes q_S}$$

in Step 2.2, when $\rho_x$ is given as an input to $Q_x$. On the other hand, $Q_x$ clearly outputs the totally mixed state $(I/2)^{\otimes q_S}$ in Step 2.1, when $\rho_x$ is given as an input to $Q_x$. Hence, given the input state $\rho_x$, the circuit $Q_x$ outputs the state

$$Q_x(\rho_x) = \frac{1}{2}(I/2)^{\otimes q_S} + \frac{1}{2} \xi = \frac{1}{2}(1 + p_{\text{acc}})(I/2)^{\otimes q_S} + \frac{1}{2}(1 - p_{\text{acc}})(|0\rangle \langle 0|)^{\otimes q_S}.$$

Therefore,

$$\|Q_x(\rho_x) - (I/2)^{\otimes q_S}\|_{\text{tr}} = \frac{1}{2}(1 - p_{\text{acc}})\|(|0\rangle \langle 0|)^{\otimes q_S} - (I/2)^{\otimes q_S}\|_{\text{tr}},$$

which implies that

$$D(Q_x(\rho_x), (I/2)^{\otimes q_S}) = \frac{1}{2}(1 - p_{\text{acc}})D(|0\rangle \langle 0|)^{\otimes q_S}, (I/2)^{\otimes q_S}) \leq \frac{1}{2}(1 - p_{\text{acc}}) \leq \frac{1}{2}(1 - c).$$

Hence, choosing $c \geq 1 - 2\alpha$, the inequality $D(Q_x(\rho_x), (I/2)^{\otimes q_S}) \leq \alpha$ holds.
Now suppose that $x$ is in $A_{\text{no}}$. Then $V$ accepts with probability at most $s$ no matter which quantum prover he communicates with. Let $\rho$ be any $(q_S + q_M)$-qubit state in $(S, M)$, and consider the reduced state $\rho'$ in $S$ of $\rho$. As before, let $\Pi_{\text{acc}}$ be the projection operator defined by $\Pi_{\text{acc}} = |1\rangle\langle 1| \otimes I^\otimes (q_V + q_M - 1)$. The state $Q_x(\rho)$ that $Q_x$ outputs when the input state was $\rho$ is given by

$$Q_x(\rho) = \frac{1}{2}p' + \frac{1}{2}[p'_{\text{acc}}(I/2)^{\otimes q_S} + (1 - p'_{\text{acc}})(|0\rangle \langle 0|)^{\otimes q_S}],$$

where $p'_{\text{acc}} = \text{tr} \Pi_{\text{acc}} V_x ((|0\rangle \langle 0|)^{\otimes (q_V - q_S)} \otimes \rho) V_x^\dagger$ is the probability that $Q_x$ outputs the totally mixed state in Step 2.2, when given the input state $\rho$.

If $D(\rho', (I/2)^{\otimes q_S}) \geq 1 - \frac{1}{\sqrt{5}}$, by Lemma [11] the state $Q_x(\rho)$ that $Q_x$ outputs when the input state was $\rho$ satisfies that

$$D(Q_x(\rho), (I/2)^{\otimes q_S}) \geq D(\rho', (I/2)^{\otimes q_S}) - \frac{1}{2} \geq \frac{1}{2} - \frac{1}{\sqrt{5}}.$$

On the other hand, if $D(\rho', (I/2)^{\otimes q_S}) < 1 - \frac{1}{\sqrt{5}}$, consider any purification $|\phi_{\rho}\rangle$ in $(S, M, P)$ of $\rho$, where $P$ is a quantum register sufficiently large for the purification. Note that $|\phi_{\rho}\rangle$ is also a purification of the reduced state $\rho'$ of $\rho$, and thus, by Lemma [9] (Uhlmann’s theorem), there should be a purification $|\phi_{\text{legal}}\rangle$ in $(S, M, P)$ of the totally mixed state $(I/2)^{\otimes q_S}$ such that

$$F(|\phi_{\rho}\rangle \langle \phi_{\rho}|, |\phi_{\text{legal}}\rangle \langle \phi_{\text{legal}}|) = F(\rho', (I/2)^{\otimes q_S}).$$

Therefore, the reduced state $\rho_{\text{legal}}$ in $(V, M) = (S, S, M)$ of the state $(|0\rangle \langle 0|)^{\otimes (q_V - q_S)} \otimes |\phi_{\text{legal}}\rangle \langle \phi_{\text{legal}}|$ must satisfy that

$$F((|0\rangle \langle 0|)^{\otimes (q_V - q_S)} \otimes \rho, \rho_{\text{legal}}) = F(|\phi_{\rho}\rangle \langle \phi_{\rho}|, |\phi_{\text{legal}}\rangle \langle \phi_{\text{legal}}|) = F(\rho', (I/2)^{\otimes q_S}),$$

and thus, Lemma [10] (the Fuchs-van-de-Graaf inequalities) implies that

$$D((|0\rangle \langle 0|)^{\otimes (q_V - q_S)} \otimes \rho, \rho_{\text{legal}}) \leq \sqrt{1 - F((|0\rangle \langle 0|)^{\otimes (q_V - q_S)} \otimes \rho, \rho_{\text{legal}})^2}$$

$$= \sqrt{1 - F(\rho', (I/2)^{\otimes q_S})^2} \leq \sqrt{1 - (1 - D(\rho', (I/2)^{\otimes q_S}))^2} < \frac{2}{\sqrt{5}}, \quad (2)$$

As $\rho_{\text{legal}}$ is a legal state that can appear in $(V, M) = (S, S, M)$ of the starting $q_S$-QAM system just before the final verification procedure of $V$, from the soundness property of the system, it holds that $\text{tr} \Pi_{\text{acc}} V_x \rho_{\text{legal}} V_x^\dagger \leq s$. Thus, from Lemma [3] together with Eq. (2), the probability $p'_{\text{acc}}$ that $Q_x$ outputs the totally mixed state in Step 2.2, when given the input state $\rho$, is bounded from above by

$$p'_{\text{acc}} \leq s + D((|0\rangle \langle 0|)^{\otimes (q_V - q_S)} \otimes \rho, \rho_{\text{legal}}) < s + \frac{2}{\sqrt{5}}.$$

This implies that, when the input state was $\rho$, the probability $p'_0$ that $Q_x$ outputs the state $(|0\rangle \langle 0|)^{\otimes q_S}$ is bounded by

$$p'_0 = \frac{1}{2}(1 - p'_{\text{acc}}) > \frac{1}{2}\left(1 - s - \frac{2}{\sqrt{5}}\right),$$

and thus, by Lemma [11] the state $Q_x(\rho)$ that $Q_x$ outputs when the input state was $\rho$ satisfies that

$$D(Q_x(\rho), (I/2)^{\otimes q_S}) \geq D((|0\rangle \langle 0|)^{\otimes q_S}, (I/2)^{\otimes q_S}) - (1 - p'_0)$$

$$> (1 - 2^{-q_S}) - \left[1 - \frac{1}{2}\left(1 - s - \frac{2}{\sqrt{5}}\right)\right] = \frac{1}{2} - \frac{1}{\sqrt{5}} - \frac{s}{2} - 2^{-q_S}.$$
Hence, no matter which state $\rho$ given as input, it holds that

$$D(Q_x(\rho), (I/2)^{\otimes q_S}) > \min \left\{ \frac{1}{2} - \frac{1}{\sqrt{5}}, \frac{1}{2} - \frac{1}{\sqrt{5}} - \frac{s}{2} - 2^{-q_S} \right\} = \frac{1}{2} - \frac{1}{\sqrt{5}} - \frac{s}{2} - 2^{-q_S}. $$

Without loss of generality, one can assume that $q_S \geq 10$, and thus, by choosing $s \leq 2^{-9}$, the inequality $D(Q_x(\rho), (I/2)^{\otimes q_S}) > 1/20$ holds for any $\rho$.

This completes the proof of the qq-QAM-hardness of CITM($a, 1/20$) for any positive constant $a < 1/20$.

The qq-QAM-hardness of CITM($a, b$) for any constants $a$ and $b$ satisfying $0 < a < b < 1$ follows by first creating an instance $Q_x$ of CITM($a/k, 1/20$) for some constant $k \in \mathbb{N}$ according to the construction above, and then constructing another circuit $Q'_x$ that places $k$ copies of $Q_x$ in parallel. Indeed, Lemma 14 ensures that $Q'_x$ is an instance of CITM($a, b$), by taking $k = \left\lceil 2^{(b/a)/\ln(400)} \right\rceil$ and considering the transformation $\Phi$ induced by $Q_x$ and the transformation $\Psi$ that receives an input state of $(q_S + q_M)$ qubits and always outputs the totally mixed state $(I/2)^{\otimes q_S}$ regardless of the input.

From Lemmas 23 and 24, Theorem 2 follows.

Note that, with essentially the same proofs as those for Lemmas 23 and 24, one can show that for any $b$ in $(0, 1)$, CITM($0, b$) is in qq-QAM$_1$ and is hard for qq-QAM$_1$, and thus, the following corollary holds.

**Corollary 25.** For any constant $b$ in $(0, 1)$, CITM($0, b$) is qq-QAM$_1$-complete under polynomial-time many-one reduction.

**Remark.** The proofs of this section actually also show that the variant of the CITM problem where the number of output qubits of the circuit is a fixed constant independent of instances is complete for the class QMA$_{\text{const-EPR}}$ introduced in Ref. [KLGN13], and thus, it is QMA-complete since QMA$_{\text{const-EPR}} = \text{QMA}$ [BSW11].

## 4 Collapse Theorem for qq-QAM

This section proves Theorem 1 the quantum analogue of Babai’s collapse theorem [Bab85] stating that $c \cdot \text{cqQ-AM}(m) = \text{qQ-AM}$ for any constant $m \geq 2$.

First, it is proved that for any constant $m \geq 4$, $c \cdot \text{cqQ-AM}(m) \subseteq \text{cqQ-AM}$ holds, meaning that the first $(m-4)$ classical turns can be removed. The proof essentially relies on the observation that the techniques used in the classical result by Babai [Bab85] can be applied in the quantum setting as well.

**Lemma 26.** For any constant $m \geq 4$, $c \cdot \text{cqQ-AM}(m) \subseteq \text{cqQ-AM}$.

**Proof.** It suffices to show that for any odd constant $m \geq 5$, $c \cdot \text{cqQ-AM}(m) \subseteq c \cdot \text{cqQ-AM}(m-1)$, and for any even constant $m \geq 6$, $c \cdot \text{cqQ-AM}(m) \subseteq c \cdot \text{cqQ-AM}(m-2)$.

Let $A = (A_{\text{yes}}, A_{\text{no}})$ be a problem in $c \cdot \text{cqQ-AM}(m)$. By Lemma 20, $A$ has an $m$-turn $c \cdot \text{cqQ-AM}$ proof system $\Pi$ with completeness $1 - 2^{-8}$ and soundness $2^{-8}$. Without loss of generality, one can assume that, for every input of length $n$, every classical message exchanged consists of $l(n)$ bits for some polynomially bounded function $l$.

First consider the case with odd $m$, where the first turn is for the prover. Fix an input $x$, and let $w_x(y, r)$ be the maximum of the probability that a prover can make the verifier accept, under the condition that the first message from the prover is $y \in \{0, 1\}^{l(x)}$ and the second message from the verifier is $r \in \{0, 1\}^{l(x)}$. Then, the maximum acceptance probability in the system $\Pi$ is given by $p_x = \max_y \{E[w_x(y, r)]\}$, where the expectation is taken over the uniform distribution with respect to $r \in \{0, 1\}^{l(x)}$. Note that $p_x \geq 1 - 2^{-8}$ if $x$ is in $A_{\text{yes}}$, and $p_x \leq 2^{-8}$ if $x$ is in $A_{\text{no}}$.

---

1 In Ref. [BM88], the journal version of Ref. [Bab85], a more efficient protocol (the speedup theorem) is given to reduce the number of turns, but it is more complicated, and not necessary for our purpose.
Verifier’s Protocol for Reducing the Number of Turns by One (for Odd \( m \))

1. Send \( k(|x|) \) strings \( r_1, \ldots, r_{k(|x|)} \), each chosen uniformly at random from \( \{0, 1\}^{l(|x|)} \), to the prover, for some polynomially bounded function \( k \).

2. Receive a pair of strings \( y, z \) in \( \{0, 1\}^{l(|x|)} \) from the prover. Run in parallel \( k(|x|) \) attempts of the \((m - 3)\)-turn protocol that simulates the last \((m - 3)\) turns of communications of the original \( m \)-turn \( c \cdots cqq\)-QAM proof system \( \Pi \) on input \( x \), where the \( j \)th attempt assumes that the first three messages in the original \( \Pi \) were \( y, r_j \), and \( z \), respectively, for each \( j \in \{1, \ldots, k(|x|)\} \). Accept if more than \( k(|x|)/2 \) attempts result in acceptance in the simulations of \( \Pi \), and reject otherwise.

---

Consider the \((m - 1)\)-turn \( c \cdots cqq\)-QAM proof system \( \Pi' \) specified by the following protocol of the verifier. At the first turn, the verifier sends \( k(|x|) \) strings \( r_1, \ldots, r_{k(|x|)} \) chosen uniformly at random from \( \{0, 1\}^{l(|x|)} \), for some polynomially bounded function \( k \). Upon receiving a pair of strings \( y, z \) in \( \{0, 1\}^{l(|x|)} \) at the third turn, the verifier enters the simulations of the last \((m - 3)\) turns of communications of \( \Pi \), by running in parallel \( k(|x|) \) attempts of such simulations, where the \( j \)th attempt assumes that the first three messages in the original \( \Pi \) were \( y, r_j \), and \( z \), respectively, for each \( j \in \{1, \ldots, k(|x|)\} \). The verifier accepts if and only if more than \( k(|x|)/2 \) attempts result in acceptance in the original \( \Pi \). Figure 3 summarizes the protocol of this verifier in \( \Pi' \).

In fact, the construction of this proof system \( \Pi' \) is exactly the same as in Ref. [Bab85] except that the last two messages exchanged are quantum and the final verification of the verifier is a polynomial-time quantum computation in the present case. The analysis in Ref. [Bab85] works also in the present case, since it only relies on the fact that \( w(x, r) \) gives the conditional probability defined above, and the perfect parallel repetition theorem holds for general quantum interactive proof systems [Gut09]. In particular, the following property holds also in the present case (see Lemmas 3.3 and 3.4 of Ref. [Bab85]).

**Claim 1.** The maximum acceptance probability \( p'_x \) in \( \Pi' \) satisfies that

\[
1 - 2^{k(|x|)}(1 - p_x)^{k(|x|)/2} \leq p'_x \leq 2^{k(|x|) + l(|x|)}p_x^{k(|x|)/2},
\]

Now let \( k = \left\lceil \frac{m+1}{2} \right\rceil \). If \( x \) is in \( A_{\text{yes}} \), then the maximum acceptance probability \( p' \) in \( \Pi' \) is at least

\[
1 - 2^{k(|x|)}(1 - p_x)^{k(|x|)/2} \geq 1 - 2^{k(|x|)}(2^{-8})^{k(|x|)/2} \geq 1 - \frac{1}{2^{l(|x|)+2}} \geq \frac{3}{4},
\]

while if \( x \) is in \( A_{\text{no}} \), then the maximum acceptance probability \( p' \) in \( \Pi' \) is at most

\[
2^{k(|x|) + l(|x|)}p_x^{k(|x|)/2} \leq 2^{k(|x|) + l(|x|)}(2^{-8})^{k(|x|)/2} \leq \frac{1}{4},
\]

which completes the proof for the case with odd \( m \).

Next consider the case with even \( m \), where the first message is a random string from a verifier. Let \( \Pi'(-1) \) be the \((m - 1)\)-turn \( c \cdots cqq\)-QAM proof system that on input \((x, r)\) simulates the last \( m - 1 \) turns of \( \Pi \) on \( x \) under the condition that the first message from the verifier was \( r \) in \( \Pi \). Let \( B = (B_{\text{yes}}, B_{\text{no}}) \) be the following promise problem in \( c \cdots cqq\)-QAM\((m - 1)\):

\[
B_{\text{yes}} = \{(x, r) : \text{the maximum acceptance probability in } \Pi'(-1) \text{ on input } (x, r) \text{ is at least 2/3}\},
\]

\[
B_{\text{no}} = \{(x, r) : \text{the maximum acceptance probability in } \Pi'(-1) \text{ on input } (x, r) \text{ is at most 1/3}\}.
\]
Note that, if \( x \) is in \( A_{\text{yes}} \), then \((x, r)\) is in \( B_{\text{yes}} \) for at least \((1 - 3 \cdot 2^{-8})\)-fraction of the choices of \( r \). Similarly, if \( x \) is in \( A_{\text{no}} \), then \((x, r)\) is in \( B_{\text{no}} \) for at least \((1 - 3 \cdot 2^{-8})\)-fraction of the choices of \( r \). By the result for the case with odd \( m \) above, it holds that \( B \) is in \( c \cdot \cdots \cdot cqq\text{-QAM}((m - 2)) \). Thus, there exists an \((m - 2)\)-turn \( c \cdot \cdots \cdot cqq\text{-QAM} \) proof system \( \Pi^{(-2)} \) for \( B \) such that if \((x, r)\) is in \( B_{\text{yes}} \), the maximum acceptance probability in \( \Pi^{(-2)} \) is at least \( 2/3 \), while if \((x, r)\) is in \( B_{\text{no}} \), the maximum acceptance probability in \( \Pi^{(-2)} \) is at most \( 1/3 \). Note that the first turn of \( \Pi^{(-2)} \) is a turn for the verifier, and thus, one can merge the turn for sending \( r \) with the first turn of \( \Pi^{(-2)} \). This results in an \((m - 2)\)-turn \( c \cdot \cdots \cdot cqq\text{-QAM} \) proof system \( \Pi'' \) for \( A \) in which at the first turn the new verifier sends a string \( r \in \{0, 1\}^{l(|x|)} \) chosen uniformly at random in addition to the original first message of the verifier in \( \Pi^{(-2)} \) on input \((x, r)\), and then behaves exactly in the same manner as the verifier in \( \Pi^{(-2)} \) on input \((x, r)\) in the rest of the protocol. If \( x \) is in \( A_{\text{yes}} \), the maximum acceptance probability in this \( \Pi'' \) is at least \((1 - 3 \cdot 2^{-8}) \cdot (2/3) > 5/8 \), while if \( x \) is in \( A_{\text{no}} \), the maximum acceptance probability in \( \Pi'' \) is at most \( 3 \cdot 2^{-8} + (1 - 3 \cdot 2^{-8}) \cdot (1/3) < 3/8 \), which is sufficient for the claim, due to Lemma 20.

Second, using the fact that CITM is \( cqq\text{-QAM}\)-complete, it is proved that \( cqq\text{-QAM} \subseteq cqq\text{-QAM} \).

**Lemma 27.** \( cqq\text{-QAM} \subseteq cqq\text{-QAM} \).

**Proof.** Let \( A = (A_{\text{yes}}, A_{\text{no}}) \) be a problem in \( cqq\text{-QAM} \). Then, \( A \) has a \( cqq\text{-QAM} \) proof system \( \Pi \) with completeness \( 2/3 \) and soundness \( 1/3 \). Let \( l \) be the polynomially bounded function that specifies the length of the first message in \( \Pi \). Consider the \( cqq\text{-QAM} \) proof system \( \Pi^{qq} \) that on input \((x, w)\) simulates the last two turns of \( \Pi \) on \( x \) under the condition that the first message in \( \Pi \) was \( w \in \{0, 1\}^{l(|x|)} \). Let \( B = (B_{\text{yes}}, B_{\text{no}}) \) be the following promise problem in \( cqq\text{-QAM} \):

\[
\begin{align*}
B_{\text{yes}} &= \{(x, w)\} : \text{the maximum acceptance probability in } \Pi^{qq} \text{ on input } (x, w) \text{ is at least } 2/3, \\
B_{\text{no}} &= \{(x, w)\} : \text{the maximum acceptance probability in } \Pi^{qq} \text{ on input } (x, w) \text{ is at most } 1/3.
\end{align*}
\]

Note that for any \( x \), if \( x \) is in \( A_{\text{yes}} \), there exists a string \( w \) in \( \{0, 1\}^{l(|x|)} \) such that \((x, w)\) is in \( B_{\text{yes}} \), and if \( x \) is in \( A_{\text{no}} \), for every string \( w \) in \( \{0, 1\}^{l(|x|)} \), \((x, w)\) is in \( B_{\text{no}} \).

Let \( p : \mathbb{Z}^+ \rightarrow \mathbb{N} \) be a non-decreasing polynomially bounded function, which will be fixed later. First notice that \( B \) has a \( cqq\text{-QAM} \) proof system that satisfies completeness \( 1 - 2^{-p} \) and soundness \( 2^{-p} \) (the existence of such a proof system is ensured by Lemma 20). Starting from this \( cqq\text{-QAM} \) proof system, the proof of Lemma 24 implies the existence of a polynomial-time algorithm that, given \((x, w)\), computes a description of a quantum circuit \( Q_{x,w} \) of \( q_{\text{in}}(|x|) \) input qubits and \( q_{\text{out}}(|x|) \) output qubits with the following properties:

(i) if \((x, w)\) is in \( B_{\text{yes}} \), there exists a quantum state \( \rho \) consisting of \( q_{\text{in}}(|x|) \) qubits such that
\[
D(Q_{x,w} (\rho), (I/2)^{\otimes q_{\text{out}}(|x|)}) \leq 2^{-p(|x|+|w|)-1} < 2^{-p(|x|)},
\]
and

(ii) if \((x, w)\) is in \( B_{\text{no}} \), for any quantum state \( \rho \) consisting of \( q_{\text{in}}(|x|) \) qubits, \( D(Q_{x,w} (\rho), (I/2)^{\otimes q_{\text{out}}(|x|)}) > 1/20 \).

Let \( q \) be another non-decreasing polynomially bounded function satisfying \( q(n) \geq \max\{l(n) + 4, n\} \) for any \( n \) in \( \mathbb{Z}^+ \). Considering the quantum circuit \( Q'_{x,w} \), that runs \( k(|x|) \) copies of \( Q_{x,w} \) in parallel for the polynomially bounded function \( k = \left\lceil \frac{2 \ln 2}{\ln(400/399)} q \right\rceil \) and taking \( p = q + \left\lceil \log k \right\rceil \), it follows from Lemma 14 (with \( \Phi \) being the transformation induced by \( Q_{x,w} \) and \( \Psi \) being the transformation that receives an input state of \( q_{\text{in}}(|x|) \) qubits and always outputs the totally mixed state \((I/2)^{\otimes q_{\text{out}}(|x|)}\) regardless of the input) that

(i) if \( x \) is in \( A_{\text{yes}} \), there exist a string \( w \) in \( \{0, 1\}^{l(|x|)} \) and a quantum state \( \rho' \) consisting of \( q'_{\text{in}}(|x|) \) qubits such that
\[
D(Q'_{x,w} (\rho'), (I/2)^{\otimes q'_{\text{out}}(|x|)}) < 2^{-q(|x|)},
\]
and

(ii) if \( x \) is in \( A_{\text{no}} \), for any string \( w \) in \( \{0, 1\}^{l(|x|)} \) and any quantum state \( \rho' \) consisting of \( q'_{\text{in}}(|x|) \) qubits,
\[
D(Q'_{x,w} (\rho'), (I/2)^{\otimes q'_{\text{out}}(|x|)}) > 1 - 2^{-q(|x|)},
\]
where \( q'_{in} = k q_{in} \) and \( q'_{out} = k q_{out} \).

Now consider the quantum circuit \( R_x \) of \( l(|x|) + q'_{in}(|x|) \) input qubits and \( q'_{out}(|x|) \) output qubits that corresponds to the following algorithm:

1. Measure all the \( l(|x|) \) qubits in the quantum register \( W \) in computational basis to obtain a classical string \( w \) in \( \{0, 1\}^{l(|x|)} \), where \( W \) corresponds to the first \( l(|x|) \) qubits of the input qubits.

2. Compute from \((x, w)\) a description of the quantum circuit \( Q'_{x,w} \). Perform the circuit \( Q'_{x,w} \) with qubits in the quantum register \( R \) as its input qubits, where \( R \) corresponds to the last \( q'_{in}(|x|) \) qubits of the input qubits of \( R_x \). Output the qubits corresponding to the output qubits of \( Q'_{x,w} \).

We claim that the circuit \( R_x \) satisfies the following two properties:

(i) if \( x \) is in \( A_{yes} \), there exists a quantum state \( \sigma \) consisting of \( l(|x|) + q'_{in}(|x|) \) qubits such that \( D(R_x(\sigma), (I/2)^{\otimes q'_{out}(|x|)}) < 2^{-q(|x|)} \), and

(ii) if \( x \) is in \( A_{no} \), for any quantum state \( \sigma' \) consisting of \( q'_{in}(|x|) \) qubits, \( D(R_x(\sigma), (I/2)^{\otimes q'_{out}(|x|)}) > 1/q'_{out}(|x|) \).

In fact, the item (i) is obvious from the construction of \( R_x \).

For the item (ii), suppose that \( x \) is in \( A_{no} \). Then, for any string \( w \) in \( \{0, 1\}^{l(|x|)} \) and any quantum state \( \rho' \) consisting of \( q'_{in}(|x|) \) qubits, it holds that \( D(Q'_{x,w}(\rho'), (I/2)^{\otimes q'_{out}(|x|)}) > 1 - 2^{-q(|x|)} \). From Lemma 15 and the second inequality of Lemma 16 it follows that

\[
S(R_x(\sigma)) < l(|x|) + q'_{out}(|x|) - q(|x|) + 2 \leq q'_{out}(|x|) - 2 \leq \left( 1 - \frac{1}{q'_{out}(|x|)} - 2^{-q(|x|)} \right) q'_{out}(|x|).
\]

Hence, the first inequality of Lemma 16 ensures that \( D(R_x(\sigma), (I/2)^{\otimes q'_{out}(|x|)}) > 1/q'_{out}(|x|) \).

Finally, consider the quantum circuit \( R'_x \) that runs \( k'(|x|) \) copies of \( R_x \) in parallel for a polynomially bounded function \( k' = \left\lfloor \frac{2 \ln(1/2)}{\ln(1 - (1/2)^{q'_{out}(|x|)})} \right\rfloor \) \( \leq 2(q'_{out}(|x|))^2 \). Assuming that \( q'_{out}(|x|) \geq 2^{q(|x|)-4} \) (otherwise \(|x|\) is at most some fixed constant, as \( q'_{out} \) is a polynomially bounded function and \( q(|x|) \geq |x| \), and thus, it can be checked trivially whether \( x \) is in \( A_{yes} \) or in \( A_{no} \)), it follows from Lemma 14 that

(i) if \( x \) is in \( A_{yes} \), there exists a quantum state \( \sigma \) consisting of \( q''_{in}(|x|) \) qubits such that \( D(R'_x(\sigma), (I/2)^{\otimes q''_{out}(|x|)}) < 1/8 \), and

(ii) if \( x \) is in \( A_{no} \), for any quantum state \( \sigma' \) consisting of \( q''_{in}(|x|) \) qubits, \( D(R'_x(\sigma), (I/2)^{\otimes q''_{out}(|x|)}) > 1/2 \),

where \( q''_{in} = k'(l + q'_{in}) \) and \( q''_{out} = k'(l + q'_{out}) \). Therefore, \( R'_x \) is a yes-instance of \( \text{CITM}(1/8, 1/2) \) if \( x \) is in \( A_{yes} \), while \( R'_x \) is a no-instance of \( \text{CITM}(1/8, 1/2) \) if \( x \) is in \( A_{no} \). This implies that any problem \( A \) in \( cqQ-QAM \) is reducible to \( \text{CITM}(1/8, 1/2) \) in polynomial time, and thus in \( \text{cqQ-QAM} \) by Lemma 23 which completes the proof.

Remark. The proof of Lemma 27 essentially shows the \( cqQ-QAM \)-hardness of the \textsc{Maximum Output Quantum Entropy Approximation} (MAXOUTQEA) problem. On the other hand, the fact that MAXOUTQEA is in \( cqQ-QAM \) is easily proved by an almost straightforward modification of the arguments in Refs. [BASTS10, CCKV08] used to show that the \textsc{Quantum Entropy Approximation} (QEA) problem is in \( \text{NIQSZK} \). Hence, the MAXOUTQEA problem is also \( cqQ-QAM \)-complete, proving Theorem 3. A rigorous proof of Theorem 3 will be presented in the appendix.

Finally, using Lemma 27 it is proved that \( ccqQ-QAM \subseteq cqQ-QAM \).

**Lemma 28.** \( ccqQ-QAM \subseteq cqQ-QAM \).
Proof. Let \( A = (A_{\text{yes}}, A_{\text{no}}) \) be a problem in ccqq-QAM. By Lemma \[20\] one can assume that \( A \) has a ccqq-QAM proof system \( \Pi \) with completeness \( 1 - 2^{-8} \) and soundness \( 2^{-8} \). Let \( \Pi^{(-1)} \) be the ccqq-QAM proof system that on input \((x, r)\) simulates the last three turns of \( \Pi \) on input \(x\) assuming that the first message in \( \Pi \) from the verifier was \( r \). Let \( B = (B_{\text{yes}}, B_{\text{no}}) \) be the following promise problem in ccqq-QAM:

\[
B_{\text{yes}} = \{(x, r) : \text{the maximum acceptance probability in } \Pi^{(-1)} \text{ on input } (x, r) \text{ is at least } 2/3\},
\]

\[
B_{\text{no}} = \{(x, r) : \text{the maximum acceptance probability in } \Pi^{(-1)} \text{ on input } (x, r) \text{ is at most } 1/3\}.
\]

Note that, if \( x \) is in \( A_{\text{yes}} \), then \((x, r)\) is in \( B_{\text{yes}} \) for at least \((1 - 3 \cdot 2^{-8})\)-fraction of the choices of \( r \), while if \( x \) is in \( A_{\text{no}} \), then \((x, r)\) is in \( B_{\text{no}} \) for at least \((1 - 3 \cdot 2^{-8})\)-fraction of the choices of \( r \). By Lemma \[27\] it holds that \( B \) is in \( \text{qQ-QAM} \). Thus, there exists a \( \text{qQ-QAM} \) proof system \( \Pi' \) for \( B \) such that, if \((x, r)\) is in \( B_{\text{yes}} \), the maximum acceptance probability in \( \Pi' \) is at least \( 2/3 \), and if \((x, r)\) is in \( B_{\text{no}} \), the maximum acceptance probability in \( \Pi' \) is at most \( 1/3 \). Here, the first turn of \( \Pi' \) is a turn for the verifier, and thus, one can merge the turn for sending \( r \) with the first turn of \( \Pi' \). This results in another \( \text{qQ-QAM} \) proof system \( \Pi'' \) for \( A \) in which at the first turn the new verifier sends a string \( r \in \{0, 1\}^{l(|x|)} \) chosen uniformly at random in addition to the original first message of the verifier in \( \Pi' \) on input \((x, r)\), and then behaves exactly in the same manner as the verifier in \( \Pi' \) on input \((x, r)\) in the rest of the protocol. Notice that sending a random string \( r \) of length \( l(|x|) \) can be exactly simulated by sending the halves of \( l(|x|) \) EPR pairs and measuring in the computational basis all the remaining halves of them that the verifier possesses. If \( x \) is in \( A_{\text{yes}} \), the maximum acceptance probability in this \( \Pi'' \) is at least \((1 - 3 \cdot 2^{-8}) \cdot (2/3) > 5/8 \), while if \( x \) is in \( A_{\text{no}} \), the maximum acceptance probability in \( \Pi'' \) is at most \( 3 \cdot 2^{-8} + (1 - 3 \cdot 2^{-8}) \cdot (1/3) < 3/8 \), which is sufficient for the claim, due to Lemma \[20\]. \( \square \)

Now one inclusion of Theorem \[1\] is immediate from Lemmas \[26\] and \[28\], and the other inclusion is trivial, which completes the proof of Theorem \[1\].

Notice that all the proofs of Lemmas \[26\], \[27\] and \[28\] can be easily modified to preserve the perfect completeness property. Indeed, the proof of Lemma \[26\] can be modified to preserve the perfect completeness property by taking \( B_{\text{yes}} \) to be the set of \((x, r)\)'s such that the maximum acceptance probability in \( \Pi^{(-1)} \) on input \((x, r)\) is one, and using Lemma \[27\] instead of Lemma \[20\]. With a similar modification to the set \( B_{\text{yes}} \) as well as using Corollary \[25\] instead of Theorem \[2\] the proof of Lemma \[27\] can be modified to present a reduction from any problem in ccqq-QAM\(_1\) to CITM\((0, b)\), which shows the inclusion ccqq-QAM\(_1 \subseteq \text{qQ-QAM} \). Using this inclusion instead of Lemma \[27\] and again with a similar modification to \( B_{\text{yes}} \) and a replacement of Lemma \[20\] by Lemma \[21\] the proof of Lemma \[27\] can be modified so that it shows the inclusion ccqq-QAM\(_1 \subseteq \text{qQ-QAM} \). Hence, the following corollary holds.

**Corollary 29.** For any constant \( m \geq 2 \), \( c \cdot \text{ccqq-QAM}_1(m) = \text{qQ-QAM}_1 \).

5 QAM versus One-Sided Error \( \text{qQ-QAM} \)

This section shows that \( \text{qQ-QAM} \) proof systems of perfect-completeness are already as powerful as the standard QAM proof systems of two-sided bounded error (Theorem \[4\]). As mentioned at the end of Section 4 the collapse theorem for \( \text{qQ-QAM} \) holds even for the perfect-completeness variants. In particular, the inclusion ccqq-QAM\(_1 \subseteq \text{qQ-QAM} \) holds. Hence, for the proof of Theorem \[4\] it suffices to show that any problem in ccq-QAM (= QAM) is necessarily in the class ccqq-QAM\(_1\). As mentioned earlier, this can be shown by combining the classical technique in Ref. \[Cai12\] for proving AM = AM\(_1\), which originates in the proof of BPP \( \subseteq \Sigma_2^p \) due to Lautemann \[Lau83\], and the recent result that sharing a constant number of EPR pairs can make QMA proofs perfectly complete \[KLG11\].

Intuitively, with two classical turns of communications, the classical technique in Ref. \[Cai12\] can be used to generate polynomially many instances of a (promise) QMA problem such that all these instances are QMA yes-instances if the input was a yes-instance, while at least one of these instances is a QMA no-instance with high
probability if the input was a no-instance (some of the QMA instances may violate the promise if the input was a no-instance, but this does not matter, as the important point is that at least one instance is a no-instance in this case). Now one makes use of the QMA_{\text{const-EPR}} proof system in Ref. [KLN+13] for each QMA instance, by running polynomially many attempts of such a system in parallel to see that none of them results in rejection. The resulting proof system is thus of cqc-QAM type, as QMA_{\text{const-EPR}} proof systems are special cases of cqc-QAM proof systems. The perfect completeness of this proof system follows from the fact that all the QMA instances generated from an input of yes-instance are QMA yes-instances, and all of them are accepted without error in the attempts of the QMA_{\text{const-EPR}} system due to the perfect completeness property of the system. The soundness of this proof system follows from the fact that at least one QMA instance generated from an input of no-instance is a QMA no-instance with high probability, for which the QMA_{\text{const-EPR}} proof system results in rejection with reasonably high probability, due to the soundness property of it.

The rigorous proof will use the following notion of fat and thin subsets of \{0, 1\}. A subset \(S\) of \{0, 1\} is fat if \(|S|/2^l \geq 1 - 1/\ell\), and is thin if \(|S|/2^l \leq 1/\ell\). For any \(S \subseteq \{0, 1\}^l\) and \(r \in \{0, 1\}^l\), let \(S \oplus r = \{x \oplus r : x \in S\}\), where for any \(x\) and \(y\) in \{0, 1\}^l, \(x \oplus y\) denotes a string in \{0, 1\}^l obtained by taking the bitwise exclusive-OR of \(x\) and \(y\). The following property holds (see Lemma 5.15 of Ref. [Cai12]).

Lemma 30. For any positive integer \(l\) and any subset \(S\) of \{0, 1\}^l,

(i) if \(S\) is fat, for any positive integers \(k\) and \(l\) such that \(k < l\), \(\Pr_{r_1,\ldots,r_k \in \{0,1\}^l} \left[ \bigcap_{j=1}^k (S \oplus r_j) \neq \emptyset \right] = 1\), and

(ii) if \(S\) is thin, for any positive integer \(k\), \(\Pr_{r_1,\ldots,r_k \in \{0,1\}^l} \left[ \bigcap_{j=1}^k (S \oplus r_j) = \emptyset \right] \geq 1 - 2^k/\ell^k\).

Using this lemma, Theorem 4 is proved as follows.

Proof of Theorem 4 Let \(A = (A_{\text{yes}}, A_{\text{no}})\) be a problem in cqc-QAM (= QAM). By Lemma 20, \(A\) has a cqc-QAM proof system \(\Pi\) with completeness \(1 - \frac{1}{\ell^3}\) and soundness \(\frac{1}{\ell}\), where \(l\) is the polynomially bounded function that specifies the length of the random string sent by the verifier at the first turn (such a proof system indeed exists, as one can achieve exponentially small completeness and soundness errors if one likes, while the message length remain polynomially bounded even in such cases). Let \(V\) denote the verifier in this system \(\Pi\). Without loss of generality, one can assume that \(l \geq 4\), and \(l\) also specifies the number of qubits \(V\) would receive at the last turn in \(\Pi\). Consider the QMA proof system \(\Pi^{QMA}\) that on input \((x, r)\) simulates the last turn of \(\Pi\) on \(x\) assuming that the first message in \(\Pi\) from the verifier was \(r\) (i.e., on input \((x, r)\), the verifier in \(\Pi^{QMA}\) first receives a quantum witness of \(l(|x|)\) qubits, and then simulates the final verification procedure of \(V\) in \(\Pi\) on input \(x\) conditioned that \(V\) sent \(r\) as his/her question at the first turn). Let \(B = (B_{\text{yes}}, B_{\text{no}})\) be the following promise problem in QMA:

- \(B_{\text{yes}} = \{(x, r) : \text{the maximum acceptance probability in } \Pi^{QMA} \text{ on input } (x, r) \text{ is at least } 2/3\}\),
- \(B_{\text{no}} = \{(x, r) : \text{the maximum acceptance probability in } \Pi^{QMA} \text{ on input } (x, r) \text{ is at most } 1/3\}\).

Note that, if \(x\) is in \(A_{\text{yes}}\), then \((x, r)\) is in \(B_{\text{yes}}\) for at least \((1 - \frac{1}{\ell^3})\)-fraction of the choices of \(r \in \{0, 1\}^{l(|x|)}\), while if \(x\) is in \(A_{\text{no}}\), then \((x, r)\) is in \(B_{\text{no}}\) for at least \((1 - \frac{1}{\ell^3})\)-fraction of the choices of \(r \in \{0, 1\}^{l(|x|)}\).

Consider another cqc-QAM proof system \(\Pi'\) specified by the following protocol of the verifier on input \(x\):

1. Send \((l(|x|) - 1)\) strings \(r_1, \ldots, r_{l(|x|) - 1}\), each chosen uniformly at random from \{0, 1\}^{l(|x|)}.
2. Upon receiving a string \(r\) in \{0, 1\}^{l(|x|)} as well as \((l(|x|) - 1)\) quantum registers \(M_1, \ldots, M_{l(|x|) - 1}\) of \(l(|x|)\) qubits, simulate the final verification procedure of \(V\) in the original system \(\Pi\) on input \(x\) with the question \(r \oplus r_j\) and the quantum state in \(M_j\) for each \(j\) in \{1, \ldots, l(|x|) - 1\} (i.e., for each \(j\), simulate the QMA proof system \(\Pi^{QMA}\) on instance \((x, r \oplus r_j)\) with the quantum state in \(M_j\) as its quantum witness). Accept if and only if all the \((l(|x|) - 1)\) simulations result in the acceptance.
Verifier’s ccqq-QAM$_1$ Protocol for QAM

1. Send $(l(|x|) - 1)$ strings $r_1, \ldots, r_{l(|x|) - 1}$, each chosen uniformly at random from $\{0, 1\}^{l(|x|)}$, to the prover.

2. Receive a string $r$ in $\{0, 1\}^{l(|x|)}$ from the prover. Prepare $N(l(|x|) - 1)$ pairs of single-qubit registers $(S_{j,k}, S'_{j,k})$ for each $j$ in $\{1, \ldots, l(|x|) - 1\}$ and $k$ in $\{1, \ldots, N\}$, and generate an EPR pair in each of $(S_{j,k}, S'_{j,k})$, where $N$ is the constant such that $N$ shared EPR pairs can make any QMA proof system perfectly complete in the construction of Ref. [KLGN13]. Send each $S'_{j,k}$ to the prover.

3. Receive $M_j$ and $S'_{j,1}, \ldots, S'_{j,N}$ from the prover, for each $j$ in $\{1, \ldots, l(|x|) - 1\}$. Perform the verification procedure in the construction of Ref. [KLGN13] for each QMA instance $(x, r \oplus r_j)$, $j \in \{1, \ldots, l(|x|) - 1\}$, using $M_j$ and $(S_{j,1}, S'_{j,1}), \ldots, (S_{j,N}, S'_{j,N})$. Accept if all the verification procedures result in acceptance, and reject otherwise.

Figure 4: Verifier’s ccqq-QAM protocol for achieving perfect completeness for the problems in QAM.

The key point is that, if $x$ is in $A_{\text{yes}}$, for any choice of $(r_1, \ldots, r_{l(|x|) - 1})$, there always exists an $r$ in $\{0, 1\}^{l(|x|)}$ such that the pair $(x, r \oplus r_j)$ is in $B_{\text{yes}}$ for all $j$ in $\{1, \ldots, l(|x|) - 1\}$. Indeed, if $x$ is in $A_{\text{yes}}$, the set $S^\text{yes}_x$ defined by

$$S^\text{yes}_x = \{r \in \{0, 1\}^{l(|x|)} : (x, r) \in B_{\text{yes}}\}$$

is fat, and hence by Lemma 30 for any $r_1, \ldots, r_{l(|x|) - 1}$ in $\{0, 1\}^{l(|x|)}$, there exists an $r$ in $\{0, 1\}^{l(|x|)}$ such that, for every $j$ in $\{1, \ldots, l(|x|) - 1\}$, the pair $(x, r \oplus r_j)$ is in $B_{\text{yes}}$.

If $x$ is in $A_{\text{no}}$, on the other hand, it happens with very small probability that there exists an $r$ such that, for all $j$, the QMA instance $(x, r \oplus r_j)$ has maximum acceptance probability greater than $1/3$ (here one must be a bit careful, because there may be QMA instances breaking the promise, which is why the condition “greater than $1/3$” is used instead of “at least $2/3$”). This means that, if $x$ is in $A_{\text{no}}$, with very high probability over the choices of $(r_1, \ldots, r_{l(|x|) - 1})$, for any $r$ given, there exists at least one $j$ such that $(x, r \oplus r_j)$ is in $B_{\text{no}}$. Indeed, if $x$ is in $A_{\text{no}}$, the set $S^\text{no}_x$ defined by

$$S^\text{no}_x = \{r \in \{0, 1\}^{l(|x|)} : (x, r) \not\in B_{\text{no}}\}$$

is thin, and hence by Lemma 30, the probability over the choices of $(r_1, \ldots, r_{l(|x|) - 1})$ that for every $r \in \{0, 1\}^{l(|x|)}$ there exists an index $j$ in $\{1, \ldots, l(|x|) - 1\}$ such that the pair $(x, r \oplus r_j)$ is in $B_{\text{no}}$ is at least $1 - \frac{2^{l(|x|)}}{l(|x|)^{l(|x|) - 1}} \geq 1 - 2^{-\frac{2l(|x|)}{2^{l(|x|)} - 1}}$.

Finally, consider the following ccqq-QAM proof system $\Pi''$ that plugs in the idea of Ref. [KLGN13] into each instance $(x, r \oplus r_j)$ of the (promise) QMA problem: The verifier basically simulates $\Pi'$, except that now, instead of $\Pi^\text{QMA}$, he/she performs the QMA$_1^\text{const-EPR}$ protocol (Fig. 6 in Ref. [KLGN13]) for each QMA instances. For this, in addition to $r$ and $M_1, \ldots, M_{l(|x|) - 1}$, the verifier receives polynomially many single-qubit registers, assuming that the verifier and prover share that polynomially many number of EPR pairs beforehand – these EPR pairs can be shared by adding a quantum turn for the verifier after having received the response $r$ from the prover. Here note that one needs only a constant number of EPR pairs for each instance $(x, r \oplus r_j)$, but one needs them for all $(l(|x|) - 1)$ instances $(x, r \oplus r_j)$, which results in polynomially many EPR pairs in total. Figure 4 presents a more precise description of the protocol for the verifier in the ccqq-QAM proof system $\Pi''$.

This proves that $A$ is in ccqq-QAM$_1$: If $x$ is in $A_{\text{yes}}$, for every choice of $(r_1, \ldots, r_{l(|x|) - 1})$, the verifier of $\Pi''$ always accepts due to the perfect completeness of the QMA$_1^\text{const-EPR}$ proof system. If $x$ is in $A_{\text{no}}$, the verifier can reject with reasonably high probability, since it is guaranteed by the soundness of the QMA$_1^\text{const-EPR}$ proof system that the verifier of $\Pi''$ can detect a no-instance $(x, r \oplus r_j)$ of the QMA problem with reasonably high
probability, and at least one such no-instance exists with probability at least \( 1 - 2^{-l(|x|) + 2} \) over the choices of \((r_1, \ldots, r_{l(|x|) - 1})\). As Corollary 29 in particular ensures that qcc\text{-QAM}_1 \subseteq \text{qq-QAM}_1\), it follows that \( A \) is in qcc\text{-QAM}_1, as claimed. \( \square \)

The fact that perfect completeness is achievable in cc\text{-QAM} (Theorem 6) can be proved in a similar fashion, except that now one uses the fact MQA = MQA_1 (a.k.a., QCMA = QCMA_1) that any classical-witness QMA proofs can be made perfectly complete shown in Ref. [JKNN12] instead of the inclusion QMA \subseteq QMA^{\text{const-EPR}}. Each QMA instance in the argument above are replaced by an MQA (QCMA) instance in this case. Notice that no additional turn is necessary in this case, and the resulting proof system corresponding to \( \Pi'' \) is immediately a cc\text{-QAM} proof system of perfect completeness.

6 Collapse Theorem for General Quantum Arthur-Merlin Proof Systems

Before the proof of Theorem 7, first observe the simple fact that one can always replace classical turns by quantum ones without diminishing the verification power, which can be shown as in the proof of Lemma 20 by letting the verifier simulate classical turns by quantum turns via CNOT applications.

**Proposition 31.** For any constant \( m \in \mathbb{N} \), any \( j \in \{1, \ldots, m\} \), and any \( t_1, \ldots, t_m \) in \( \{c, q\} \),

\[
t_m \cdots t_{j+1} t_j t_{j-1} \cdots t_1\text{-QAM}(m) \subseteq t_m \cdots t_{j+1} q t_{j-1} \cdots t_1\text{-QAM}(m).
\]

As generalized quantum Arthur-Merlin proofs are nothing but a special case of general quantum interactive proofs, it is obvious that for any constant \( m \) and any \( t_1, \ldots, t_m \) in \( \{c, q\} \), \( t_m \cdots t_1\text{-QAM}(m) \) is contained in QIP = \text{PSPACE} [JUW11]. As mentioned in Section 1, Marriott and Watrous [MW05] proved that qcq\text{-QAM} (= QMAM) already hits the ceiling, i.e., coincides with QIP. Next lemma (Lemma 32) states that one can slightly improve this and even the third message is not necessary to be quantum to have the whole power of general quantum interactive proofs. The proof is based on a simulation of the original qcq\text{-QAM} system by a qcc\text{-QAM} system using quantum teleportation.

**Lemma 32.** \( \text{qcq-QAM} \subseteq \text{qcc-QAM} \).

**Proof.** Let \( A = (A_{\text{yes}}, A_{\text{no}}) \) be a problem in \( \text{qcq-QAM} \), meaning that \( A \) has a qcq\text{-QAM} proof system \( \Pi \) with completeness 2/3 and soundness 1/3 that is specified by the protocol of the verifier of the following form for every input \( x \):

1. Receive a quantum register \( M_1 \) from the prover, and then send a random string \( r \) to the prover.

2. Receive a quantum register \( M_2 \) from the prover. Prepare a private quantum register \( V \), and perform the final verification procedure over \((M_1, M_2, V)\).

Let \( l \) be the polynomially bounded function that specifies the number of qubits in \( M_2 \). Consider the teleportation-based simulation of \( \Pi \) by the qcc\text{-QAM} proof system \( \overline{\Pi} \) that is specified by the protocol of the verifier of the following form for every input \( x \):

1. Receive a quantum register \( S_1 \) of \( l(|x|) \) qubits, in addition to the quantum register \( M_1 \), from the prover. Send a random string \( r \) to the prover as would be done in \( \Pi \).

2. Receive a binary string \( b \) of length \( 2l(|x|) \) from the prover. Apply \( X^{b_{j,1}} Z^{b_{j,2}} \) to the \( j \)th qubit of \( S_1 \), for each \( j \) in \( \{1, \ldots, l(|x|)\} \), where \( b_{j,1} \) and \( b_{j,2} \) denote the \((2j - 1)\)st and \((2j)\)th bits of \( b \), respectively. Finally, prepare his/her private quantum register \( V \) as in \( \Pi \), and simulate the final verification procedure of the verifier in \( \Pi \) with \((M_1, S_1, V)\).

24
For the completeness, suppose that \( x \) is in \( A_{yes} \). Then there exists a prover \( P \) who makes the verifier accept with probability \( p \geq 2/3 \) in the original \( \text{qcq-QAM} \) system \( \Pi \). Without loss of generality, one can assume that \( P \) has quantum registers \( M_1, M_2, \) and \( P \) at the beginning of the protocol, where \( P \) is the private quantum register of \( P \). Let \( \rho_z \) be the quantum state \( P \) prepares in \( (M_1, M_2, P) \) at the first turn in \( \Pi \), and let \( P_{x,r} \) be the unitary transformation \( P \) applies to \( (M_2, P) \) at the third turn in \( \Pi \) when \( P \) has received \( r \).

In the \( \text{qcq-QAM} \) system \( \Pi \), let the prover \( \tilde{P} \) behave as follows: On input \( x \), \( \tilde{P} \) prepares quantum registers \( S_1 \) and \( S_2 \), each of \( l(|x|) \) qubits, in addition to \( M_1, M_2, \) and \( P \). \( \tilde{P} \) generates \( \rho_z \) in \( (M_1, M_2, P) \), and also generates \( |\Phi^+\rangle^{\otimes l(|x|)} \) in \( (S_1, S_2) \) so that the \( j \)th qubit of \( S_1 \) and that of \( S_2 \) form an EPR pair, for every \( j \) in \( \{1, \ldots, l(|x|)\} \). \( \tilde{P} \) then sends \( M_1 \) and \( S_1 \) to the verifier at the first turn. Upon receiving \( r \), \( \tilde{P} \) first applies \( P_{x,r} \) to \( (M_2, P) \) as \( P \) would do, and then measures the \( j \)th pair of qubits in \( (S_2, M_2) \) in the Bell basis to obtain a two-bit outcome \( b_j \), for every \( j \) in \( \{1, \ldots, l(|x|)\} \), where \( b_j \) equals 00, 01, 10, and 11 if the measurement results in \( |\Phi^+\rangle \), \( |\Phi^-\rangle \), \( |\Psi^+\rangle \), and \( |\Psi^-\rangle \), respectively. \( \tilde{P} \) sends a binary string \( b \) of length \( 2l(|x|) \) such that the pair of the \( (2j-1) \)st and \( (2j) \)th bits is exactly \( b_j \), for every \( j \) in \( \{1, \ldots, l(|x|)\} \). This makes the quantum state in \( M_2 \) be teleported to that in \( S_1 \), as the application of the Pauli operators in the final step of the verifier in \( \tilde{\Pi} \) correctly removes the phase and/or bit errors if exist. Hence the verifier accepts in \( \tilde{\Pi} \) with exactly the same probability \( p \) as in \( \Pi \), which ensures the completeness of \( \tilde{\Pi} \).

For the soundness, suppose that \( x \) is in \( A_{no} \). Let \( \tilde{P}' \) be any prover in \( \tilde{\Pi} \). Without loss of generality, one can assume that \( \tilde{P}' \) has quantum registers \( M_1, S_1, \) and \( \tilde{P}' \) at the beginning of the protocol, where \( \tilde{P}' \) is the private quantum register of \( \tilde{P}' \). Let \( \rho \) be the quantum state \( \tilde{P}' \) prepares in \( (M_1, S_1, \tilde{P}') \) at the first turn in \( \tilde{\Pi} \), and let \( \{\tilde{P}^b_{x,r}\}_{b \in \{0,1\}^{2l(|x|)}} \) be the \( 2l(|x|) \)-bit outcome measurement that \( \tilde{P}' \) performs over \( \tilde{P}' \) at the third turn in \( \tilde{\Pi} \), when \( \tilde{P}' \) has received \( r \).

In the \( \text{qcq-QAM} \) system \( \Pi \), let the prover \( P' \) behave as follows: On input \( x \), \( P' \) prepares quantum registers \( M_1, S_1, \) and \( \tilde{P}' \), and generates \( \rho \) in \( (M_1, S_1, \tilde{P}') \), as \( P' \) would do in \( \Pi \). \( P' \) then sends \( M_1 \) to the verifier at the first turn in \( \Pi \). Upon receiving \( r \), \( P' \) first performs the \( 2l(|x|) \)-bit outcome measurement \( \{\tilde{P}^b_{x,r}\}_{b \in \{0,1\}^{2l(|x|)}} \) over \( \tilde{P}' \) to obtain a \( 2l(|x|) \)-bit outcome \( b' \). Let \( b'_{j,1} \) and \( b'_{j,2} \) be the \( (2j-1) \)st and \( (2j) \)th bits of \( b' \), respectively, for each \( j \) in \( \{1, \ldots, l(|x|)\} \). \( P' \) then applies \( X_{b'_{j,1}}Z_{b'_{j,2}} \) to the \( j \)th qubit of \( S_1 \) for each \( j \) in \( \{1, \ldots, l(|x|)\} \), as the verifier in \( \Pi \) would do, and sends \( S_1 \) to the verifier as the quantum register \( M_2 \). From the construction, it is obvious that this \( P' \) can make the verifier accept in \( \Pi \) with exactly the same probability as \( \tilde{P}' \) could in \( \tilde{\Pi} \), which must be at most \( 1/3 \) from the soundness property of \( \Pi \), and the soundness of \( \tilde{\Pi} \) follows.

With Lemma 32 in hand, Theorem 7 is proved as follows.

**Proof of Theorem 7** For the item (i), first notice that the inclusion \( \text{qcq-QAM} \subseteq \text{qcq-QAM} \) can be proved in a manner very similar to the proof of Lemma 32, with not the honest prover but the verifier preparing the EPR pairs. As \( \text{qcq-QAM} = \text{QAM} = \text{QIP} = \text{PSPACE} \), together with Lemma 32, this implies that \( \text{qcq-QAM} = \text{qcq-QAM} = \text{PSPACE} \). As adding more turns to \( t_3t_2t_1 \)-QAM and \( t_2t_1 \)-QAM proof systems does not diminish the verification power for any \( t_1, t_2, \) and \( t_3 \) in \( \{q, c\} \), this establishes the claim in the item (i).

For the item (ii), again with a similar argument to the proof of Lemma 32, it holds that, for any constant \( m \geq 2 \), \( c \cdots \text{qcq-QAM}(m) \subseteq c \cdots \text{qcq-QAM}(m) \), and thus, combined with Theorem 1 and Proposition 31, the claim follows.

For the item (iii), it suffices to show that for any constant \( m \geq 3 \), \( c \cdots \text{qcq-QAM}(m) \subseteq c \cdots \text{qcq-QAM}(m-1) \). The case with \( m \geq 5 \) is proved with an argument similar to that in the proof of Lemma 26 since the first three (resp. four) turns of the \( m \)-turn \( c \cdots \text{qcq-QAM} \) proof systems are classical when \( m \) is odd (resp. when \( m \) is even). In the case where \( m = 3 \), one modifies the construction of \( \Pi' \) in the proof of Lemma 26 so that the message from the prover at the second turn (corresponding to Step 2 of \( \Pi' \)) is quantum, consisting of two parts: the \( Y \) part and \( Z \) part, each corresponding to \( y \) and \( z \) in Step 2 of \( \Pi' \). In order to force the content in the \( Y \) part to be classical, the verifier simply measures each qubit in the \( Y \) part in the computational basis. The analysis in the proof of Lemma 26 then works with the case where \( m = 3 \), i.e., the case where a \( \text{qcq-QAM} \) system is simulated by a \( \text{qcq-QAM} \) system.
The case where \( m = 4 \) can then be proved using this result with \( m = 3 \), with the same argument as in the proof of Lemma 26.

Finally, for the item (iv), it suffices to show that the inclusion \( c \cdots c \text{-QAM}(m) \subseteq c \cdots c \text{-QAM}(m - 1) \) holds for any constant \( m \geq 3 \), which easily follows from an argument similar to that in the proof of Lemma 26 since all the messages are classical. \( \square \)

## 7 Conclusion

This paper has introduced the generalized model of quantum Arthur-Merlin proof systems to provide some new insights on the power of two-turn quantum interactive proofs. A number of open problems are listed below concerning generalized quantum Arthur-Merlin proof systems and other related topics:

- Is there any natural problem, other than CITM and \textsc{MaxOutQEA}, in \( q_q\text{-QAM} \) that is not known to be in the standard QAM? Or is \( q_q\text{-QAM} \) equal to QAM?

- Currently no upper-bound is known for \( q_q\text{-QAM} \) other than \( \text{QIP}(2) \). Can a better upper-bound be placed on \( q_q\text{-QAM} \)? Is \( q_q\text{-QAM} \) contained in \( \text{BP} \cdot \text{PP} \)?

- Does \( q_q\text{-QAM} = q_q\text{-QAM} \)? In other words, is perfect completeness achievable in \( q_q\text{-QAM} \)? Similar questions remain open even for \( \text{QIP}(2) \) and QAM.

- What happens if some of the messages are restricted to be classical in the standard quantum interactive proof systems? Does a collapse theorem similar to the \( q_q\text{-QAM} \) case hold even with the \( \text{QIP}(2) \) case? More precisely, is the power of \( m \)-turn quantum interactive proof systems equivalent to \( \text{QIP}(2) \) for any constant \( m \geq 2 \), when the first \( (m - 2) \) turns are restricted to exchange only classical messages?

For the last question above, note that one might be able to show a similar collapse theorem even with \( \text{QIP}(2) \) when the verifier \textit{cannot} use quantum operations at all during the first \( (m - 2) \) turns (by extending the argument due to Goldwasser and Sipser \cite{GS89} to replace the classical interaction of the first \( (m - 2) \) turns by an \( m \)-turn classical public-coin interaction, and then applying arguments similar to those in this paper, using some appropriate \( \text{QIP}(2) \)-complete problem like the \textsc{Close Image} problem \cite{Wat02a, HMW12}). A more difficult, but more natural and interesting case is where the verifier can use quantum operations to generate his/her classical messages even for the first \( (m - 2) \) turns, to which the Goldwasser-Sipser technique does not seem to apply any longer. A collapse theorem for such a case, if provable, would be very helpful when trying to put more problems in \( \text{QIP}(2) \) and more generally investigating the properties of two-turn quantum interactive proof systems.

### Acknowledgements

The authors are grateful to Francesco Buscemi and Richard Cleve for very useful discussions. This work is supported by the Grant-in-Aid for Scientific Research (A) No. 24240001 of the Japan Society for the Promotion of Science and the Grant-in-Aid for Scientific Research on Innovative Areas No. 24106009 of the Ministry of Education, Culture, Sports, Science and Technology in Japan. HN also acknowledges support from the Grant-in-Aids for Scientific Research (A) Nos. 21244007 and 23246071 and (C) No. 25330012 of the Japan Society for the Promotion of Science.

### References

[ABD+09] Scott Aaronson, Salman Beigi, Andrew Drucker, Bill Fefferman, and Peter Shor. The power of unentanglement. Theory of Computing, 5:1–42 (Article 1), 2009.
A simple proof that Toffoli and Hadamard are quantum universal. arXiv.org e-Print archive, arXiv:quant-ph/0301040, 2003.

Dorit Aharonov, Alexei Kitaev, and Noam Nisan. Quantum circuits with mixed states. In Proceedings of the Thirtieth Annual ACM Symposium on Theory of Computing, pages 20–30, 1998.

László Babai. Trading group theory for randomness. In Proceedings of the Seventeenth Annual ACM Symposium on Theory of Computing, pages 421–429, 1985.

Avraham Ben-Aroya, Oded Schwartz, and Amnon Ta-Shma. Quantum expanders: Motivation and constructions. Theory of Computing, 6:47–79 (Article 3), 2010.

László Babai and Shlomo Moran. Arthur-Merlin games: A randomized proof system, and a hierarchy of complexity classes. Journal of Computer and System Sciences, 36(2):254–276, 1988.

Salman Beigi, Peter Shor, and John Watrous. Quantum interactive proofs with short messages. Theory of Computing, 7:101–117 (Article 7), 2011.

Jin-Yi Cai. Lectures in computational complexity, August 2012. Available at http://www.cs.wisc.edu/~jyc/710/book.pdf.

André Chailloux, Dragos Florin Ciocan, Iordanis Kerenidis, and Salil Vadhan. Interactive and noninteractive zero knowledge are equivalent in the help model. Cryptology ePrint Archive, Report 2007/467, 2007.

André Chailloux, Dragos Florin Ciocan, Iordanis Kerenidis, and Salil Vadhan. Interactive and noninteractive zero knowledge are equivalent in the help model. In Theory of Cryptography, Fifth Theory of Cryptography Conference, TCC 2008, volume 4948 of Lecture Notes in Computer Science, pages 501–534, 2008.

Andrew Drucker. New limits to classical and quantum instance compression. Electronic Colloquium on Computational Complexity, Report No. 112, 2012.

Christopher A. Fuchs and Jeroen van de Graaf. Cryptographic distinguishability measures for quantum-mechanical states. IEEE Transactions on Information Theory, 45(4):1216–1227, 1999.

Shafi Goldwasser, Silvio Micali, and Charles Rackoff. The knowledge complexity of interactive proof systems. SIAM Journal on Computing, 18(1):186–208, 1989.

Shafi Goldwasser and Michael Sipser. Private coins versus public coins in interactive proof systems. In Silvio Micali, editor, Randomness and Computation, volume 5 of Advances in Computing Research, pages 73–90. JAI Press, 1989.

Gustav Gutoski. Quantum Strategies and Local Operations. PhD thesis, David R. Cheriton School of Computer Science, University of Waterloo, 2009. arXiv:1003.0038 [quant-ph].

Patrick Hayden, Kevin Milner, and Mark M. Wilde. Two-message quantum interactive proofs and the quantum separability problem. arXiv.org e-Print archive, arXiv:1211.6120 [quant-ph], 2012.

Patrick Hayden, Kevin Milner, and Mark M. Wilde. Two-message quantum interactive proofs and the quantum separability problem. In CCC 2013, 2013 IEEE Conference on Computational Complexity, pages 156–167, 2013.
[JJUW11] Rahul Jain, Zhengfeng Ji, Sarvagya Upadhyay, and John Watrous. QIP = PSPACE. *Journal of the ACM*, 58(6):Article 30, 2011.

[JKNN12] Stephen P. Jordan, Hirotada Kobayashi, Daniel Nagaj, and Harumichi Nishimura. Achieving perfect completeness in classical-witness quantum Merlin-Arthur proof systems. *Quantum Information and Computation*, 12(5–6):0461–0471, 2012.

[JJUW09] Rahul Jain, Sarvagya Upadhyay, and John Watrous. Two-message quantum interactive proofs are in PSPACE. In *50th Annual Symposium on Foundations of Computer Science*, pages 534–543, 2009.

[KLGN13] Hirotada Kobayashi, François Le Gall, and Harumichi Nishimura. Stronger methods of making quantum interactive proofs perfectly complete. In *ITCS ’13, Proceedings of the 2013 ACM Conference on Innovations in Theoretical Computer Science*, pages 329–352, 2013.

[KMY09] Hirotada Kobayashi, Keiji Matsumoto, and Tomoyuki Yamakami. Quantum Merlin-Arthur proof systems: Are multiple Merlins more helpful to Arthur? *Chicago Journal of Theoretical Computer Science*, 2009:Article 3, 2009.

[Kob03] Hirotada Kobayashi. Non-interactive quantum perfect and statistical zero-knowledge. In *Algorithms and Computation, 14th International Symposium, ISAAC 2003*, volume 2906 of *Lecture Notes in Computer Science*, pages 178–188, 2003.

[KSV02] Alexei Yu. Kitaev, Alexander H. Shen, and Mikhail N. Vyalyi. *Classical and Quantum Computation*, volume 47 of *Graduate Studies in Mathematics*. American Mathematical Society, 2002.

[KW00] Alexei Kitaev and John Watrous. Parallelization, amplification, and exponential time simulation of quantum interactive proof systems. In *Proceedings of the Thirty-Second Annual ACM Symposium on Theory of Computing*, pages 608–617, 2000.

[Lau83] Clemens Lautemann. BPP and the polynomial hierarchy. *Information Processing Letters*, 17(4):215–217, 1983.

[LFKN92] Carsten Lund, Lance Fortnow, Howard Karloff, and Noam Nisan. Algebraic methods for interactive proof systems. *Journal of the ACM*, 39(4):859–868, 1992.

[MW05] Chris Marriott and John Watrous. Quantum Arthur-Merlin games. *Computational Complexity*, 14(2):122–152, 2005.

[NC00] Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information*. Cambridge University Press, 2000.

[Pap85] Christos H. Papadimitriou. Games against nature. *Journal of Computer and System Sciences*, 31(2):288–301, 1985.

[Ros09] William Rosgen. *Computational Distinguishability of Quantum Channels*. PhD thesis, David R. Cheriton School of Computer Science, University of Waterloo, 2009. arXiv:0909.3930 [quant-ph]

[RW05] Bill Rosgen and John Watrous. On the hardness of distinguishing mixed-state quantum computations. In *Twentieth Annual IEEE Conference on Computational Complexity*, pages 344–354, 2005.

[Sha92] Adi Shamir. IP = PSPACE. *Journal of the ACM*, 39(4):869–877, 1992.

[She92] Alexander Shen. IP = PSPACE: Simplified proof. *Journal of the ACM*, 39(4):878–880, 1992.
A qq-QAM-Completeness of MAXOUTQEA

This section gives a rigorous proof of Theorem 3 that states that the MAXOUTQEA problem is qq-QAM-complete. First, it is proved that MAXOUTQEA is in qq-QAM.

Lemma 33. MAXOUTQEA is in qq-QAM.

Proof. We present a reduction from the MAXOUTQEA problem to the CITM problem (with some appropriate parameters), by modifying the reduction from the QEA problem to the QUANTUM STATE CLOSENESS TO TOTALLY MIXED (QSCTM) problem presented in Ref. [CCKV07], which relies on the analysis found in Section 5.3 of Ref. [BASTS10].

Let \( x = (Q, t) \) be an instance of MAXOUTQEA, where \( Q \) is a description of a quantum circuit that specifies a quantum channel \( \Phi \), and \( t \) is a positive integer. For simplicity, in what follows, we identify the description \( Q \) and the quantum circuit it induces. Suppose that \( Q \) acts on \( m_{\text{in}} \) input qubits and \( m_{\text{out}} \) output qubits. Let \( q \) and \( \varepsilon \) be two functions that appear in Eqs. (5.1) and (5.2) of Ref. [BASTS10] to be specified later. We consider the quantum circuit \( Q^{\otimes q(|x|)} \) that runs \( q(|x|) \) copies of \( Q \) in parallel, and the \( (qt, d, \varepsilon) \)-quantum extractor \( E \) on \( q(|x|)m_{\text{out}} \) qubits given in Ref. [BASTS10], Section 5.3], which is written as \( E = \frac{1}{2^d} \sum_{i=1}^{2^d} E_i \), where \( E_i(\rho) = U_i \rho U_i^{\dagger} \) for unitary operators \( U_i \). Let \( R \) be the quantum circuit that runs \( Q^{\otimes q(|x|)} \) and then applies \( E \) to the output state of \( q(|x|)m_{\text{out}} \) qubits. By following the analysis in Ref. [BASTS10], one can show that

\footnote{Rigorously speaking, \( q \) in the present case corresponds to \( \frac{q}{2} \) in the left-hand sides of Eqs. (5.1) and (5.2) in Ref. [BASTS10]. This is due to the fact that the MAXOUTQEA problem in this paper is defined using threshold values \( t + 1 \) and \( t - 1 \), while the QEA problem in Ref. [BASTS10] is defined using threshold values \( t + \frac{1}{2} \) and \( t - \frac{1}{2} \).}
(i) if \( x = (Q, t) \) is a yes-instance of MAXOUTQEA, there exists a quantum state \( \rho \) of \( q(|x|)m_{\text{in}} \) qubits such that 
\[
D\left( R(\rho), (I/2)^{\otimes q(|x|)m_{\text{out}}} \right) \leq \frac{3}{2} \varepsilon, 
\]
and

(ii) if \( x = (Q, t) \) is a no-instance of MAXOUTQEA, for any quantum state \( \rho \) of \( q(|x|)m_{\text{in}} \) qubits,
\[
D\left( R(\rho), (I/2)^{\otimes q(|x|)m_{\text{out}}} \right) \geq \frac{1}{4q(|x|)m_{\text{out}}}. 
\]

In fact, the item (i) follows from exactly the same analysis as in Ref. \[BASTS10\], by taking \( \rho = \sigma^{\otimes q(|x|)} \) with \( \sigma \) being a quantum state of \( m_{\text{in}} \) qubits such that \( S(Q(\sigma)) \geq t + 1 \) (the condition \( S_{\text{max}}(\Phi) \geq t + 1 \) ensures the existence of such a state \( \sigma \)).

To prove the item (ii), first notice that, if \( x = (Q, t) \) is a no-instance of MAXOUTQEA, it holds that 
\( S(Q(\sigma)) \leq S_{\text{max}}(\Phi) \leq t - 1 \) for any quantum state \( \sigma \) of \( m_{\text{in}} \) qubits. Take an arbitrary quantum state \( \rho \) of \( q(|x|)m_{\text{in}} \) qubits. By Lemma \[15\] it holds that
\[
S\left( R(\rho) \right) = S\left( \frac{1}{2^d} \sum_{i=1}^{2^d} U_i Q^{\otimes q(|x|)}(\rho) U_i^\dagger \right) \leq S\left( Q^{\otimes q(|x|)}(\rho) \right) + d. 
\]

For each \( i \) in \( \{1, \ldots, q(|x|)\} \), let \( R_i \) be the output quantum register of the \( i \)th copy of \( Q \) (hence, the whole output state \( Q^{\otimes q(|x|)}(\rho) \) of \( Q^{\otimes q(|x|)} \) is in \( R_1, \ldots, R_{q(|x|)} \)), and let \( \sigma_{R_i} \) be the reduced state of \( Q^{\otimes q(|x|)}(\rho) \) of \( m_{\text{out}} \) qubits obtained by tracing out all the qubits except those in \( R_i \). By the subadditivity of von Neumann entropy, it follows that
\[
S\left( Q^{\otimes q(|x|)}(\rho) \right) \leq \sum_{i=1}^{q(|x|)} S\left( \sigma_{R_i} \right) \leq \sum_{i=1}^{q(|x|)} \max_{\sigma} S\left( Q(\sigma) \right) \leq (t - 1)q(|x|), 
\]
which implies that
\[
S\left( R(\rho) \right) \leq (t - 1)q(|x|) + d. 
\]

Now the item (ii) follows from exactly the same analysis as in Ref. \[BASTS10\].

To complete the reduction, similarly to Ref. \[CCKV07\], one takes \( \varepsilon = 1/2^k \) for a polynomially bounded function \( k \) such that \( k(n) \geq n \) for any \( n \) in \( \mathbb{Z}^+ \) and \( k(n) \in O(n) \), and a polynomially bounded function \( q \) such that \( q(n) \in \Theta(n^4) \) so that Eqs. (5.1) and (5.2) are satisfied in Ref. \[BASTS10\]. Consider the quantum circuit \( R' \) that runs \( r(|x|) \) copies of \( R \) in parallel for a polynomially bounded function \( r \) such that \( r(n) = \left\lfloor \frac{2\ln(1/\varepsilon)}{\ln(1/(1-2q(n)m_{\text{out}}^2))} \right\rfloor \leq 2(2q(n)m_{\text{out}})^2 \) for all \( n \) in \( \mathbb{Z}^+ \). Assuming that \( r(|x|) \leq 2|x|/12 \) (otherwise \(|x|\) is at most some fixed constant, as \( r \) is a polynomially bounded function, and thus, it can be checked trivially whether \( x = (Q, t) \) is a yes-instance or a no-instance), it follows from Lemma \[14\] that

(i) if \( x = (Q, t) \) is a yes-instance, there exists a quantum state \( \sigma \) of \( r(|x|)q(|x|)m_{\text{in}} \) qubits such that 
\[
D\left( R'(\sigma), (I/2)^{\otimes r(|x|)q(|x|)m_{\text{out}}} \right) \leq 1/8, 
\]
and

(ii) if \( x = (Q, t) \) is a no-instance, for any quantum state \( \sigma \) of \( r(|x|)q(|x|)m_{\text{in}} \) qubits,
\[
D\left( R'(\sigma), (I/2)^{\otimes r(|x|)q(|x|)m_{\text{out}}} \right) \geq 1/2. 
\]

Hence, MAXOUTQEA is reducible to CITM(1/8, 1/2) in polynomial time, and thus in qQ-QAM by Lemma \[23\] \( \square \)

Second, it is proved that the MAXOUTQEA problem is qQ-QAM-hard.

**Lemma 34.** MAXOUTQEA is hard for qQ-QAM under polynomial-time many-one reduction.

**Proof.** The claim is proved by modifying a part of the proof of Lemma \[27\]

Let \( A = (A_{\text{yes}}, A_{\text{no}}) \) be a problem in qQ-QAM, and let \( p: \mathbb{Z}^+ \to \mathbb{N} \) be a non-decreasing polynomially bounded function to be specified later. First notice that \( A \) has a qQ-QAM proof system with completeness \( 1 - 2^{-p} \) and
soundness $2^{-p}$. Starting from this qql-QAM proof system, the proof of Lemma \[\text{24}\] implies the existence of a polynomial-time algorithm that, given $x$, computes a description of a quantum circuit $Q_x$ of $q_{\text{in}}(|x|)$ input qubits and $q_{\text{out}}(|x|)$ output qubits with the following properties:

(i) if $x$ is in $A_{\text{yes}}$, there exists a quantum state $\rho$ consisting of $q_{\text{in}}(|x|)$ qubits such that $D(Q_x(\rho), (I/2)^{\otimes q_{\text{out}}(|x|)}) \leq 2^{-p(|x|)-1} < 2^{-p(|x|)}$, and

(ii) if $x$ is in $A_{\text{no}}$, for any quantum state $\rho$ consisting of $q_{\text{in}}(|x|)$ qubits, $D(Q_x(\rho), (I/2)^{\otimes q_{\text{out}}(|x|)}) > 1/20$.

Let $q$ be another non-decreasing polynomially bounded function satisfying $q(n) \geq \max\{6, n\}$ for any $n$ in $\mathbb{Z}^+$. Considering the quantum circuit $Q'_x$ that runs $k(|x|)$ copies of $Q_x$ in parallel for a polynomially bounded function $k = \lceil \frac{2 \ln 2}{\ln(400/399)} \rceil$ and taking $p = q + \lceil \log k \rceil$, it follows from Lemma \[\text{14}\] that

(i) if $x$ is in $A_{\text{yes}}$, there exists a quantum state $\rho'$ consisting of $q'_{\text{in}}(|x|)$ qubits such that $D(Q'_x(\rho'), (I/2)^{\otimes q'_{\text{out}}(|x|)}) < 2^{-q(|x|)}$, and

(ii) if $x$ is in $A_{\text{no}}$, for any quantum state $\rho'$ consisting of $q'_{\text{in}}(|x|)$ qubits, $D(Q'_x(\rho'), (I/2)^{\otimes q'_{\text{out}}(|x|)}) > 1 - 2^{-q(|x|)}$,

where $q'_{\text{in}} = kq_{\text{in}}$ and $q'_{\text{out}} = kq_{\text{out}}$.

In what follows, it is assumed that the inequality $q'_{\text{out}}(|x|) \leq 2^q(|x|)$ holds (otherwise $|x|$ is at most some fixed constant, as $q'_{\text{out}}$ is a polynomially bounded function and $q(|x|) \geq |x|$, and thus, it can be checked trivially whether $x$ is in $A_{\text{yes}}$ or in $A_{\text{no}}$). By the second inequality of Lemma \[\text{14}\] the circuit $Q'_x$ satisfies the following properties:

(i) if $x$ is in $A_{\text{yes}}$, there exists a quantum state $\sigma$ consisting of $q'_{\text{in}}(|x|)$ qubits such that $S(Q'_x(\sigma)) > (1 - 2^{-q(|x|)})q'_{\text{out}}(|x|) - 1 \geq q_{\text{out}}(|x|) - 2$, and

(ii) if $x$ is in $A_{\text{no}}$, for any quantum state $\sigma$ consisting of $q'_{\text{in}}(|x|)$ qubits, $S(Q'_x(\sigma)) < q'_{\text{out}}(|x|) - q(|x|) + 2 \leq q_{\text{out}}(|x|) - 4$.

Thus, $(Q'_x, q_{\text{out}}(|x|) - 3)$ is a yes-instance of MAXOUTQEA if $x$ is in $A_{\text{yes}}$, while it is a no-instance of MAXOUTQEA if $x$ is in $A_{\text{no}}$. This implies that any problem $A$ in qql-QAM is reducible to MAXOUTQEA in polynomial time, and the claim follows. \hfill $\square$

Now Theorem \[\text{3}\] follows from Lemmas \[\text{33}\] and \[\text{34}\].

31