A proof of selection rules for critical dense polymers

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Abstract
Among the lattice loop models defined by Pearce et al (2006 J. Stat. Mech. P11017), the model corresponding to critical dense polymers ($\beta = 0$) is the only one for which an inversion relation for the transfer matrix $D_N(u)$ was found by Pearce and Rasmussen (2007 J. Stat. Mech. P02015). From this result, they identified the set of possible eigenvalues for $D_N(u)$ and gave a conjecture for the degeneracies of its relevant eigenvalues in the link representation, in the sector with $d$ defects. In this paper, we set out to prove this conjecture, using the homomorphism of the $TL_N(\beta)$ algebra between the loop model link representation and that of the XXZ model for $\beta = -(q + q^{-1})$.

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(Some figures may appear in colour only in the online journal)

1. Introduction
This paper proves a recent conjecture by Pearce and Rasmussen [1] for the model of critical dense polymers on the strip, by using the relation between this model and the Heisenberg spin model. The Heisenberg model (or the XXZ model) is a long studied family of Hamiltonians of $N$ interacting spins on a chain. The models depend upon a spectral parameter $q$, which controls the $z$ interaction between neighboring spins. The Hamiltonian $H_{XXZ}$ acts on $(\mathbb{C}^2)^\otimes N$ (every spin is $\frac{1}{2}$) and commutes with $S^z$. The spectrum of the XXX Hamiltonian ($q = 1$) for the periodic chain was computed by Bethe [2] long ago and his method, the Bethe ansatz, has since allowed for solutions of the more general XXZ problem on various geometries [3, 4]. In this paper, we focus on the case where the chain is finite and the Hamiltonian has very particular boundary terms for which the model is invariant under $U_q(\mathfrak{sl}_2)$ [5]. This symmetry will play an important role. We will be particularly interested in the case $q = i$ for which the $z$ coupling in the Hamiltonian is absent (known as the XX model). Though the Bethe ansatz solution is known, the spectrum of this Hamiltonian can be found using the simpler technique of Jordan–Wigner transformation [6].
The loop models introduced in [7] are two-dimensional lattice models on the strip that obey Yang–Baxter relations and are, in this sense, integrable. The transfer matrix $D_N(\beta)$ and Hamiltonian $\mathcal{H}_N$ of the model are elements of the Temperley–Lieb algebra $TL_N(\beta)$ and depend on one free parameter, the fugacity $\beta$ of the loops. The action of $TL_N(\beta)$ connectivities on link states (i.e. on $V_N$, the space they generate) defines a representation $\rho$ of $TL_N(\beta)$. For a given connectivity $c$, the matrix $\rho(c)$ is upper triangular (the number of defects, $d$, is a nonincreasing quantity) and its spectrum $\rho(c)$ is the union of the spectra of the diagonal blocks, indexed by $d$, the number of defects. Moreover, the partition functions of Potts models and Fortuin–Kasteleyn models can be computed from the eigenvalues of $\rho(D_N(\beta))$ of the loop models for specific values of $\beta$ ([8–10]).

These models have attracted much interest because the $\rho$ representation of the Hamiltonian and transfer matrix exhibit nontrivial Jordan cells ([7, 10, 11]). The corresponding representations of the Virasoro algebra should then be indecomposable and the underlying conformal field theory, logarithmic [7]. On the finite lattice, the diagonal blocks $\rho(D_N)_{\mid_d}$ have been conjectured to be diagonalizable for $\beta \in [-2, 2]$ for all $d$. Nontrivial Jordan cells do occur, but they tie eigenvalues belonging to sectors with different numbers of defects. This structure appears for specific values of the fugacity $\beta = -(q + q^{-1})$ when $q$ is a root of unity.

The case $\beta = 0$ is somewhat special, as an inversion relation for the transfer matrix was found [1]: $D_N(\beta)D_N(\beta + \frac{\pi}{2})$ is a scalar multiple of the identity. From this, one can identify the set of all possible eigenvalues, and the degeneracies of each of these in a given sector $d$ was conjectured by Pearce and Rasmussen through selection rules [1].

The two models introduced previously are known to be related (for example in [12], [13] and [11]). Namely, there exists a $TL_N$-homomorphism $\rho^d_N$ from $V^d_N$ to $(C^2)^{\otimes N}|_{S = d/2}$ (the restriction of $(C^2)^{\otimes N}$ to spin configurations with $n = (N - d)/2$ down-spins). The Heisenberg Hamiltonians can be expressed in terms of some matrices $e_s$ that act on $\rho^d_N(V^d_N)$ in the same way the Temperley–Lieb generators $U_i$ act on $V^d_N$ for $\beta = -(q + q^{-1})$ (except that the number of defects is conserved). For any $q$ and $\beta$ satisfying this relation and any $c \in TL_N(\beta)$, the spectrum of $\rho(c)$ can be found in the spectrum of $X(c)$, the representation of $c$ in the XXZ model. We will use the homomorphism to compute the degeneracies of $\rho(\mathcal{H}_N)$ and show they are those predicted by Pearce and Rasmussen [1].

The outline of the paper is as follows. In section 2, we review the definition of Temperley–Lieb algebra and of the transfer matrix for critical dense polymers. We recall the selection rules conjectured in [1] and translate these in terms of eigenvalue degeneracies of the Hamiltonian. In section 3, we perform the Jordan–Wigner transformation on the XX Hamiltonian and write it in terms of creation and annihilation operators. For $N$ odd, we find $H_{XX}$ to be diagonalizable, but not for $N$ even, for which we provide its Jordan form (some technical details for $N$ even are given in appendix A). The Hamiltonian $H_{XX}$ is invariant under $U_u(sl_2)$ and, in section 4, we write down the generators of the $U_{gln}(sl_2)$ algebra in terms of the creation and annihilation operators of section 3. In section 5, we make explicit the homomorphism $\rho^d_N$, between $V^d_N$ and $(C^2)^{\otimes N}|_{S = d/2}$, the vector space generated by spin configurations with $d$ down spins. We show that $\rho^d_N$ sends link states to $(C^2)^{\otimes N}|_{S = d/2} \cap \ker(S^+)$. Because this homomorphism is injective, one can find the spectrum of any element of $TL_N(\beta)$ by looking at its representation in the Heisenberg problem. This is the goal of section 6: we find a set of eigenvectors that complement those in $\rho^d_N(V^d_N)$ and prove in appendix B that these states are indeed independent. From this we can identify degeneracies in the XX Hamiltonian of eigenvectors $\in \ker(S^+)$ and show that they reproduce the spectrum given by the selection rules in section 2.
2. Critical dense polymers and selection rules

2.1. The algebra $\mathcal{TL}_N(\beta)$ and the double-row matrix

We start this section by recalling known definitions and results for the Temperley–Lieb algebra and its transfer matrices. The Temperley–Lieb algebra $\mathcal{TL}_N(\beta)$ is a finite algebra, with generators $i, U_1, \ldots, U_{N-1}$ satisfying the relations

\begin{align}
U_i^2 &= \beta U_i, \\
U_i U_j &= U_j U_i & \text{for } |i - j| > 1, \\
U_i U_{i\pm1} U_i &= U_i & \text{when } i, i \pm 1 \in \{1, 2, \ldots, N-1\}.
\end{align}

The algebra $\mathcal{TL}_N(\beta)$ is sometimes referred to as a connectivity algebra. A connectivity is a diagram made of a rectangular box with $N$ marked points on the top segment and as many marked points on the bottom. Inside the box, the $2N$ points are connected pairwise by nonintersecting curves. With the generator $U_i$, we associate the connectivity

\begin{align*}
U_i &= \begin{array}{c}
1 \\
2 \\
\vdots \\
i-1 \\
i \\
i+1 \\
\vdots \\
N
\end{array}
\end{align*}

Diagrammatically, the product $U_i U_j$ amounts to gluing the diagram of $U_j$ over the diagram of $U_i$. The resulting connectivity is obtained by reading the connections between the top and bottom marked points. With this identification, the first equation of (1) is

\begin{align*}
U_i^2 &= \begin{array}{c}
\cdots \\
\cdots \\
\cdots \\
\cdots
\end{array} \\
&= \beta U_i,
\end{align*}

so that the free parameter $\beta$ is the weight given to loops closed in the process. The other two equations in (1) have similar interpretations. Any connectivity can be obtained by a product of the generators, and the product of any two connectivities $c_1$ and $c_2$ in $\mathcal{TL}_N(\beta)$ is given by the same concatenation rule. The algebra $\mathcal{TL}_N(\beta)$ is the algebra of connectivities endowed with the product just described and is of dimension $\frac{1}{\pi^2} \binom{2N}{N}$.

A useful representation is the representation $\rho$ on link states (or link patterns). A link pattern is a set of $N$ marked points on a horizontal segment. The points are connected pairwise, or to infinity, by nonintersecting curves that lay above the segment. Points connected to infinity are called defects. The set of link states of length $N$ with $d$ defects is denoted $B_N^d$ and their linear span by $V_N^d$, with $\dim(V_N^d) = \left(\begin{array}{c}N \\
(N-d)/2 \end{array}\right) - \left(\begin{array}{c}N \\
(N-d-2)/2 \end{array}\right)$. The set of all link states of size $N$ is denoted $B_N$ (and $V_N$ the corresponding vector space). Let $v \in B_N$ and $c$ a connectivity. The product $c v$ is obtained by connecting the marked points of $v$ to the top marked points of $c$, by reading the resulting link pattern given by the new connections at the bottom of $c$, and by adding a multiplicative factor of $\beta$ for each closed loop. An example is as follows:

\begin{align*}
\beta^2 \\
\end{align*}
The matrix representing $c$ in the link representation is denoted by $\rho(c)$. It is of size $\dim(V_N)$ and obtained by acting on $c$ with all the link patterns of $B_N$. We introduce the double-row matrix $D_N(u)$ as an element of $TL_N(\beta = 0)$. It is defined diagrammatically by

$$D_N(u) = \frac{1}{\sin 2u}$$

where each box is given by

$$u = \cos u + \sin u,$$

and $u \in [0, \frac{\pi}{2}]$ is the anisotropy parameter. (A definition of $D_N(u)$ for general $\beta$ exists; see [7].) From the definition, it can easily be shown that $D_N(u) = D_N(\pi/2 - u)$ and $D_N(0) = D_N(\pi/2) = id$ are satisfied, where $id$ is the unique connectivity connecting every point on top to the corresponding point on the bottom. In [1], it is also shown that $D_N(u)$ satisfies the following inversion identity:

$$D_N(u)D_N\left(u + \frac{\pi}{2}\right) = \left(\frac{\cos 2u - \sin 2u}{\cos^2 u - \sin^2 u}\right)^2 id,$$

from which is it possible to retrieve a closed expression for the eigenvalues of $D_N(u)$, which we denote as $d_N(u)$:

N odd:

$$d_N(u) = \frac{1}{2^{N-1}} \prod_{j=1}^{N-1} \left( \frac{1}{\sin \frac{j(j-1)\pi}{2N}} + \epsilon_j \sin 2u \right) \left( \frac{1}{\sin \frac{j(j-1)\pi}{2N}} + \mu_j \sin 2u \right),$$

(2)

N even:

$$d_N(u) = \frac{N}{2^{N-2}} \prod_{j=1}^{N-1} \left( \frac{1}{\sin \frac{j\pi}{N}} + \epsilon_j \sin 2u \right) \left( \frac{1}{\sin \frac{j\pi}{N}} + \mu_j \sin 2u \right),$$

(3)

where $\epsilon_j, \mu_j = \pm 1$ for every $j$. Fixing values for each $\epsilon_j$ and each $\mu_j$, the set of zeros of $d_N(u)$ is

$$\{u | d_N(u) = 0\} = \bigcup_{\epsilon, \mu} \bigcup_j \left\{ (2 + v_j)\frac{\pi}{4} \pm \frac{i}{2} \ln \tan \frac{t_j}{2} + \pi k, \quad k \in \mathbb{Z} \right\},$$

where

N odd: $t_j = (2j - 1)\pi$,

N even: $t_j = \frac{j\pi}{N},$

Given a fixed $d_N(u)$, every zero in the above set appears 0, 1 or 2 times, and the number of zeros with imaginary value $(i/2) \ln(t_j/2)$ is always 2. There are $N - 1$ zeros for $N$ odd and $N - 2$ for $N$ even, which results in a total of $2^{N-1}$ and $2^{N-2}$ choices, respectively, for the eigenvalues $d_N(u)$. The set of possible solutions for eigenvalues of $\rho(D_N(u))$ is too large and one must identify which ones are relevant. This will be the subject of the next section.
$D_N(u)$ can be developed in a Taylor series around the point $u = 0$, yielding

$$D_N(u) = id + 2a\mathcal{H}_N + o(u^2) \quad \text{with} \quad \mathcal{H}_N = \sum_{i=1}^{N-1} U_i. \quad (4)$$

To understand and prove the selection rules, we will calculate the eigenvalues of $\mathcal{H}_N$. Using the expansions of (2) and (3) around $u = 0$, and using $d_N(0) = 1$, i.e.

$$\frac{1}{2^{N-1}} \prod_{j=1}^{\frac{N-1}{2}} \sin^2 \left(\frac{j-1}{2N}\pi\right) = 1 \quad \text{and} \quad \frac{N}{2^{N-1}} \prod_{j=1}^{\frac{N}{2}} \frac{1}{\sin^2 \frac{\pi j}{N}} = 1$$

for $N$ odd and $N$ even, respectively, one finds that eigenvalues of $\mathcal{H}_N$, denoted $h_N$ are

- $N$ odd:
  $$h_N = \sum_{j=1}^{\frac{N+1}{2}} \cos \left(\frac{\pi j}{N}\right) (\epsilon_{\frac{N+1}{2} - j} + \mu_{\frac{N+1}{2} - j}), \quad (5)$$

- $N$ even:
  $$h_N = \sum_{j=1}^{\frac{N}{2}} \cos \left(\frac{\pi j}{N}\right) (\epsilon_{\frac{N}{2} - j} + \mu_{\frac{N}{2} - j}), \quad (6)$$

and the $\epsilon_j$s and $\mu_j$s are those of $d_N(u)$.

2.2. Two-column configurations

The selection rules given in [1] have been formulated in terms of column configurations. This section is a quick review of their definitions.

**Definition 2.1.** A one-column configuration of height $M$ is a configuration of $M$ sites disposed in a column and labeled from 1 to $M$, starting from the top. In a column configuration, every site is either occupied or unoccupied and we define its signature, $S = \{S_1, S_2, \ldots, S_m\}$, where the $S_i$s are the labels of the occupied sites in ascending order (and $m \leq M$ is their number and will be called the length of the signature). We identify unoccupied sites with white circles ‘‘’ and occupied sites with blue/dark circles ‘‘.’’

**Definition 2.2.** A two-column configuration of height $M$ is a pair of one-column configurations, both of height $M$, and is usually depicted as in figure 1. Its signature is $S = (L, R)$, where $L$ and $R$ are the respective signatures of the left and right column configurations and may have different lengths $m$ and $n$. A two-column configuration will be said to be admissible if $0 \leq m \leq n \leq M$ and $L_i \geq R_i$, for all $i = 1, \ldots, m$. We denote by $A_{m,n}^M$ the set of admissible two-column configurations of height $M$ and signature lengths $m$ and $n$. When $m$, $n$ and $M$ are such that the previous constraint is violated, $A_{m,n}^M \equiv \emptyset$.

The graphical interpretation of this last definition is simple. Fix a two-column configuration. To see if it is admissible, we draw on the two-column configuration segments connecting sites with label $L_i$ from the left column to sites with label $R_i$ from the left column, for $i = 1, \ldots, m$ (the remaining sites at positions $R_j$ with $m < j \leq n$ are not connected to any other site). If all the segments have positive or null slopes, the configuration is admissible.

**Definition 2.3.** The reduced set $\tilde{A}_{x,y}^{+}$ of admissible two-column configurations is the subset of configurations of $A_{x,y}^{+}$ that have one and only one excitation for every $j$.

Evaluating $|\tilde{A}_{x,y}^{+}|$ is simple, as there exist bijections between reduced configurations in $\tilde{A}_{x,y}^{+}$, Dyck paths $\tilde{x} \in DP_{x+y}^{++}$ (see definition 5.2) and link states in $V_{x+y}^{+}$.
\begin{align*}
\epsilon_1 &= +1 & \mu_1 &= -1 \\
\epsilon_2 &= -1 & \mu_2 &= -1 \\
\epsilon_3 &= -1 & \mu_3 &= +1 \\
\epsilon_4 &= +1 & \mu_4 &= +1 \\
\epsilon_5 &= -1 & \mu_5 &= -1 \\
\epsilon_6 &= +1 & \mu_6 &= -1 \\
\epsilon_7 &= +1 & \mu_7 &= -1 \\
\epsilon_8 &= -1 & \mu_8 &= -1
\end{align*}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1}
\caption{An admissible two-column configuration in $A_{8,6}$ with $L = (2,3,5,8)$ and $R = (1,2,5,6,7,8)$: blue/dark sites are occupied and white sites unoccupied. To its right are the corresponding values of the $\epsilon_j$ and $\mu_j$.}
\end{figure}

1. From an element of $\tilde{A}_{x+y}^{x}$, we set $\epsilon_j = +1$ if the site of the left one-column configuration at height $j$ is unoccupied, and $-1$ otherwise. $\vec{x} = (\epsilon_1, \ldots, \epsilon_{x+y})$ is a Dyck path of length $x+y$ as, from the definition of reduced admissible configurations, $\sum_{j=1}^{x+y} \epsilon_j \geq 0$ for every $k$ in $1, \ldots, x+y$. Since there are, in total, $y' + 1$’s and $x' - 1$’s, the endpoint of the Dyck path is at $y-x$. This transformation is bijective.

2. The bijection between Dyck paths and link states is given as follows. With each of the entries of the link state, we associate the integer $j$ in $1, \ldots, N$ from left to right and build pairings $(j', j)$ (the positions where the bubbles connect). Starting from the left, for every $x_j = -1$, we pair $j$ with the closest available $j'$ such that $x_j' = +1$ and $j > j'$. When every $j$ with $x_j = -1$ is paired, the remaining $y-x$ unpaired sites are chosen to be defects. The link state $v$ obtained from a given Dyck path $\vec{x}$ by the previous procedure will be denoted $v = B(\vec{x})$.

From this bijection,
\begin{equation}
|\tilde{A}_{x+y}^{x+y}| = \dim V_{x+y}^{x+y} = \left(\begin{array}{c} x+y \\ x \end{array}\right) - \left(\begin{array}{c} x+y \\ x-1 \end{array}\right).
\end{equation}

2.3. Conjectured degeneracies and selection rules

In this section, we state the conjecture of [1] and use the definitions of $A_{m,n}^M$ to translate it in terms of degeneracies in the spectrum of $\rho(H_N(u))$. To each two-column configuration corresponds a choice of $\epsilon_j$ and $\mu_j$. The rules are as follows:

- A white circle ‘‘’ corresponds to $+1$ and a blue/dark circle ‘’ to $-1$.
- The left column corresponds to $\epsilon$ excitations, and the right to $\mu$ excitations.
- As before, $j$ grows from top to bottom.

Pearce and Rasmussen [1] give the following conjecture.

Conjecture 2.1. In the sector with $d$ defects, the set of choices of the $\epsilon_j$ and $\mu_j$ belonging to
\begin{align}
N \text{ odd: } & \bigcup_{p=0}^{N/2} A_{p+p+1}^{p+1}, \\
N \text{ even: } & \bigcup_{p=0}^{N/2} \left(A_{p+p+1}^{p+1} \cup A_{p+p+2}^{p+2}\right),
\end{align}
forms the spectra of $\rho(D_N(u))$ and $\rho(H_N)$. 


Figure 2. A two-column admissible configuration in $\tilde{A}_8$ and, to the right, the corresponding Dyck path $\in DP_2$ and link state $\in V_2^A$.

Recall that when some indices of $A_{m,n}^M$ do not satisfy the constraint $0 \leq m \leq n \leq M$, the set $A_{m,n}^M$ is empty. In this sense, the case $d = 0$ is special, as the selection rule reduces to

$$\bigcup_{p=0}^{\frac{n-2}{2}} A_{p,p}^N .$$

(9)

**Definition 2.4.** The set of eigenvalues of $\rho(H_N)$ in the sector with $d$ defects, as given by the selection rules (8), will be denoted $H_d^N$. An eigenvalue $\lambda$ will be said to belong to $A_{m,n}^M$ if it can be obtained by a choice of $\epsilon, \beta$ and $\mu, \beta$ represented by an element of $A_{m,n}^M$. For $N$ even, we distinguish between $H_{d=0}^N$ and $H_{d=1}^N$, the sets of eigenvalues $\lambda$ obtained from admissible two-column configurations in $\bigcup_{p=0}^{\frac{n-2}{2}} A_{p,p}^N + \frac{n-2}{2}$ and $\bigcup_{p=0}^{\frac{n-2}{2}} A_{p,p+1}^N$, respectively.

In the following, the cases $N$ odd and $N$ even will often be treated separately. In preparation, we give the following two definitions.

**Definition 2.5.** Let $\delta = 0, 1$; we define the set $\Lambda^\delta_N$ of $\lambda$s given by

$$\lambda = 2 \sum_{i=1}^{m} \eta_i \cos \frac{\pi k_i}{N},$$

where

1. $\eta_i = \pm 1$ for all $i$;
2. $m$ may take all values satisfying both $0 \leq m \leq n$ and $n - m \equiv \delta \mod 2$;
3. $k_i \in \mathbb{N}, 1 \leq k_1 < k_2 < \cdots < k_m \leq F(N)$ with $F(N) = \{(N-1)/2, N/2\} \text{ odd, } (N-2)/2, N/2 \text{ even}$.

Let $\lambda \in \Lambda^\delta_N$. We also define

1. $K^+: the$ set of $k$s in $\{k_1, \ldots, k_m\}$ with $\eta_k = +1$,
2. $K^-: the$ set of $k$s in $\{k_1, \ldots, k_m\}$ with $\eta_k = -1$,
3. $K^c: the$ set of $k$s in $\{1, \ldots, F(N)\}$ that are in neither $K^+$ nor $K^-$.

With each $\lambda \in \Lambda^\delta_N$ we associate the smallest number $m$ such that $\lambda$ can be written as (10), ignoring accidental cancellations. For instance, with $N = 9$, $\lambda_1 = 0$ has $m = 0$ and $\lambda_2 = 2 \cos \pi/9 - 2 \cos 2\pi/9 - 2 \cos 4\pi/9$ has $m = 3$, even though $\lambda_2$ evaluates to 0. The accidental degeneracies like the one given previously will not be considered, as they are degeneracies of $\rho(H_N)$, but not of $\rho(D_N(u))$. 

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2.4. $N$ odd

**Proposition 2.2.** The two sets $H^d_N$ and $A^{(N-d)/2}_0$ are equal.

**Proof.** First, let $h \in H^d_N$. It is obvious that $h$ can be written as (10) for a certain $0 \leq m \leq (N - 1)/2$. The rules are as follows: if at level $j$

(a) there are two white circles, put $k_j$ in $K^+$;
(b) there are two blue/dark circles, put $k_j$ in $K^-$;
(c) there is one white and one blue/dark circle, put $k_j$ in $K^c$.

To prove that $h \in A^{(N-d)/2}_0$, one must show two things: that the top bound for $m$ can be lowered from $(N - 1)/2$ to $(N - d)/2$, and that $n - m = 0 \mod 2$. To do this, note first that if $h \in A^{(N-1)/2}_p$, the maximal number of elements in $K^-$ and $K^+$ are $p$ and $(N - d)/2 - p$, respectively (and these two events occur simultaneously). The maximal value of $m \equiv |K^+ \cup K^-|$ is $(N - d)/2$; it never goes beyond $n$. The values $m$ can take jumps of 2 and are $n, n - 2, n - 4, \ldots, 0; n - m = 0 \mod 2$ as expected.

Second, let $\lambda \in \Lambda^d_N$ with $m$ fixed. We show that $\lambda \in H^{N-2n}_N$. The rule is as follows:

(a) if $k_j \in K^+$, put two white circles at level $j$;
(b) if $k_j \in K^-$, put two blue/dark circles at level $j$;
(c) if $k_j \in K^c$, put one circle of each color at level $j$.

One must then choose carefully the position of the pairs of colored circles in (c) to ensure that the two-column configuration is admissible and that it is in $A^{(N-1)/2}_{p,p+(d-1)/2}$ for some $p$. Among all $k_j$ in $K^c$, one must put $a_1$ blue/dark circles in the left column and $a_2$ in the right column, and impose that $a_1 + a_2 = |K^c| = (N - 1)/2 - m$ and $a_2 - a_1 = (N - 1)/2 - n$. This is always possible, with the choice $a_1 = (n - m)/2$ and $a_2 = (N - n - m - 1)/2$ (note that $a_1$ and $a_2$ are integers). $\lambda$ is then contained in $A^{(N-1)/2}_{p,p+(d-1)/2}$ with $p = |K^c| + (n - m)/2$. \hfill \Box

From the previous proof, all the eigenvalues of $\rho(\mathcal{H}_N)$ are in $\Lambda^d_N$, and we need not worry about values in $\Lambda^1_N$. For a given element of $\Lambda^d_N$, we can now calculate its degeneracy in $\rho(\mathcal{H}_N)$ in the sector with $d$ defects, as given by the selection rules. The following statement is therefore equivalent to conjecture 2.1 for $N$ odd (omitting accidental degeneracies).

**Conjecture 2.3.** Let $\lambda \in \Lambda^d_N$ with a fixed value of $m$ (and $n - m = 0 \mod 2$). Its degeneracy in $\rho(\mathcal{H}_N)$ in the sector with $N - 2n$ defects, as conjectured in [1], is

$$\deg_N(\lambda) = \left(\frac{N - 1}{2} - m\right) \left(\frac{n - m}{2}\right) - \left(\frac{N - 1}{2} - m\right) \left(\frac{n - m - 2}{2}\right), \quad 0 \leq m \leq n. \quad (11)$$

**Proof.** In the second part of the previous proof, for every $k_j$ in $K^c$, there was a freedom in the choice of admissible configurations. To count the degeneracies, one has to count these possible choices, as a pair of occupied and unoccupied sites at height $j$ gives contribution 0 to eigenvalues of $\rho(\mathcal{H}_N)$, regardless of $j$. For a given two-column configuration, whether it is admissible does not depend on levels with two blue/dark circles or two white circles. These can be removed. The configuration resulting from this operation is in the reduced set $A^{(N-1)/2-m}_{(n-m)/2,(N-1-n-m)/2}$ whose dimension, given by (7), is the desired result (11). \hfill \Box

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2.5. \(N\) even

The case \(N\) even is analogous to the case \(N\) odd, though the selection rule is more complicated.

**Proposition 2.4.** Let \(\delta = 0, 1\). Then, \(H_{N,\delta}^d = \Lambda^{(N-d)/2}_d\).

**Proof.** We start by showing that for \(\delta = 0, 1\), \(H_{N,\delta}^d \subset \Lambda^{(N-d)/2}_d\). The beginning of this proof is identical to that of proposition 2.2. The arguments for lowering the upper bound for \(m\) from \((N - 2)/2\) to \((N - d)/2\) and for the parity of \(n - m\) must be repeated. (Note that in the case \(d = 0\), it seems that this raises the upper bound, but since the selection rule is given in (9), this is not the case.) For \(\delta = 0\), \(K^-\) has at most \(p\) elements and \(K^+\), at most \((N - d)/2 - p\). Then, \(m = |K^+ \cup K^-|\) is at most \(n = (N - d)/2\) and \(m\) takes values \(n, n - 2, \ldots\); this is the case \(n - m = 0 \mod 2\). For \(\delta = 1\), \(m = \max|K^-| = p\), \(\max|K^+| = (N - d)/2 - p - 1\), \(\max(|K^+ \cup K^-|) = (N - d)/2 - 1 = n - 1\) and \(n - m = 1 \mod 2\).

In the other direction, we show \(\Lambda^{(N-d)/2}_d \subset H_{N,0}^d\). The rules are those of proposition 2.2. The positions of the pairs in \(K^+\) are as follows.

- If \(\lambda \in \Lambda^0_d\), the constraints are \(a_1 + a_2 = (N - 2)/2 - m\) and \(a_2 - a_1 = (N - 2)/2 - n\). Among the \(kj\)'s in \(K^+\), we put \(a_1 = (n - m)/2 \) excitations in the left column and \(a_2 = (N - n - m - 2)/2\) in the right column.
- If \(\lambda \in \Lambda^1_d\), the constraints are \(a_1 + a_2 = (N - 2)/2 - m\) and \(a_2 - a_1 = N/2 - n\). Among the \(kj\)'s in \(K^+\), we put \(a_1 = (n - m - 1)/2 \) excitations in the left column and \(a_2 = (N - n - m - 1)/2\) in the right column.

For \(N\) even, the following is the translation of conjecture 2.1.

**Conjecture 2.5.** The conjectured degeneracy of \(\lambda \in \Lambda^d_n\), \(m\) fixed and \(n - m \equiv \delta \mod 2\), in the sector \(d = N - 2n\), is given by

\[
\delta = 0: \quad \text{deg}_N(\lambda) = \begin{pmatrix} \frac{N - 2}{2} - m \\frac{n - m}{2} \end{pmatrix}, \quad 0 \leq m \leq n.
\]
\[ \delta = 1 : \quad \text{deg}_{\mathcal{H}_d}(\lambda) = \left( \frac{N - 2}{n - m - 1} - \frac{m}{n - m - 1} \right) - \left( \frac{N - 2}{n - m - 3} - \frac{m}{n - m - 3} \right), \quad 0 \leq m \leq n - 1. \quad (13) \]

This proof is identical to that of 2.3 and left to the reader. One can also verify that these formulae are valid for \( d = 0 \) and that \( \text{deg}_{\mathcal{H}_0}(\lambda) = 0 \) for \( \delta = 0 \), as expected. The results of conjectures 2.3 and 2.5 are statements equivalent to (8): they provide a conjecture for degeneracies of eigenvalues of \( \rho(\mathcal{H}_d) \) in the sector with \( d = N - 2n \) defects (in fact, the statement is not as strong because of the accidental degeneracies due to exceptional trigonometric identities, but these will be ignored). To prove the selection rules, we will show that the degeneracies of \( \rho(\mathcal{H}_d) \) are indeed given by equations (11), (12) and (13).

3. The XXZ Hamiltonian

On the finite (non-periodic) lattice, the well-studied [5] XXZ Hamiltonian for spin-\( \frac{1}{2} \) particles is

\[ H^q_{\text{XXZ}} = \frac{1}{2} \left( \sum_{j=1}^{N-1} \left( \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \frac{q + q^{-1}}{2} \sigma_j^z \sigma_{j+1}^z \right) - \frac{q - q^{-1}}{2} (\sigma_1^z - \sigma_N^z) \right), \quad (14) \]

where

\[ \sigma_j^a = \mathcal{I}^{d_2} \otimes \mathcal{I}^{d_2} \otimes \cdots \otimes \mathcal{I}^{d_2} \otimes \sigma^a \otimes \mathcal{I}^{d_2} \otimes \mathcal{I}^{d_2} \otimes \cdots \otimes \mathcal{I}^{d_2}. \]

This Hamiltonian acts on \((\mathbb{C}^2)^{\otimes N}\) and can also be written as

\[ H^q_{\text{XXZ}} = \sum_{j=1}^{N-1} \left( \frac{q + q^{-1}}{4} I + e_j \right), \]

where

\[ e_j = \frac{1}{2} \left( \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \frac{q + q^{-1}}{2} (\sigma_j^z \sigma_{j+1}^z - id) - \frac{q - q^{-1}}{2} (\sigma_j^z - \sigma_{j+1}^z) \right) \]

\[ = \mathcal{I}^{d_2} \otimes \mathcal{I}^{d_2} \otimes \cdots \otimes \mathcal{I}^{d_2} \otimes \tilde{e} \otimes \mathcal{I}^{d_2} \otimes \mathcal{I}^{d_2} \otimes \cdots \otimes \mathcal{I}^{d_2} \]

\[ \text{and} \quad \tilde{e} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -q & 1 & 0 \\ 0 & 1 & -q^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (15) \]

The matrices \( e_j \)s form a representation of \( TL_N(\beta) \) with \( \beta = -(q + q^{-1}) \). We will be interested in diagonalizing this Hamiltonian when \( q = i \). More precisely, we will show that \( H^q_{\text{XXZ}} \) can be diagonalized when \( N \) is odd, but not when \( N \) is even, in which case we will give its Jordan form. We start with

\[ H \equiv H^{q=i}_{\text{XXZ}} = \frac{1}{2} \left( \sum_{j=1}^{N-1} \left( \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y \right) - i (\sigma_j^z - \sigma_{j+1}^z) \right). \]
3.1. Free fermions

Ideas in this section are similar to those found in [14], [15] and [16]. $H$ can be transformed by writing $\sigma^+_j, \sigma^+_j$ and $\sigma^-_j$ in terms of $\sigma^+_j = (\sigma^+_j \pm i \sigma^-_j)/2$:

$$H = \sum_{j=1}^{N-1} (\sigma^+_j \sigma^-_{j+1} + \sigma^-_j \sigma^+_{j+1}) - i (\sigma^+_1 \sigma^-_1 - \sigma^+_N \sigma^-_N).$$

We perform the celebrated Jordan–Wigner transformation by passing to creation and annihilation operators $c_j$ and $c^+_j$,

$$c_j = \left( \prod_{k=1}^{j-1} (-\sigma^-_k) \right) \sigma^-_j, \quad \sigma^-_j = \left( \prod_{k=1}^{j-1} (-\sigma^-_k) \right) c_j,$$

$$c^+_j = \left( \prod_{k=1}^{j-1} (-\sigma^-_k) \right) \sigma^+_j, \quad \sigma^+_j = \left( \prod_{k=1}^{j-1} (-\sigma^-_k) \right) c^+_j,$$

which satisfy the usual anti-commutation relations for fermions,

$$\{c^+_j, c^+_j\} = \delta_{j,j'}, \quad \{c_j, c_j\} = \{c^+_j, c^+_j\} = 0.$$

The $c_j$ and $c^+_j$ are real matrices and are indeed conjugate to one another. With this transformation,

$$H = \sum_{j=1}^{N-1} (c^+_j c^-_{j+1} + c^-_j c^+_{j+1}) - i (c^+_1 c^-_1 - c^+_N c^-_N),$$

which can also be written as

$$H = \sum_{k_1, k_2} c^+_k c^-_k \mathcal{N}_{k_1, k_2}, \quad (17)$$

where

$$\mathcal{N} = \begin{pmatrix}
-i & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & \ldots & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & \ldots & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & \ldots & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & \ldots & 0 & 1 & i \\
\end{pmatrix}$$

is a symmetric matrix (but not a Hermitian matrix) of size $N$. We perform a Bogoliubov transformation

$$b_n = \sum_j f^+_j c^+_j, \quad a_n = \sum_j g^+_j c^-_j, \quad (18)$$

that will make $H$ as simple as possible in terms of these new operators. We also require that the $a_n$s and $b_n$s satisfy the fermionic anticommutation relations

$$\{b_n, a_{n'}\} = \delta_{n,n'}, \quad \{b_n, b_{n'}\} = \{a_n, a_{n'}\} = 0. \quad (19)$$
With this intent, we diagonalize \( \mathcal{N} \). Define the matrix \( \mathcal{K}_L \) of dimensions \( L \times L \):

\[
\mathcal{K}_L = \begin{pmatrix}
0 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & \ldots & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & \ldots & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & \ldots & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & \ldots & 0 & 1 & 0
\end{pmatrix}.
\]

Also let \( \tilde{\mathcal{N}} = \mathcal{N} - \xi \text{id}_L \) and \( \tilde{\mathcal{K}}_L = \mathcal{K}_L - \xi \text{id}_L \). The eigenvalues of \( \mathcal{N} \) are \( \xi \)'s for which \( \text{det}(\tilde{\mathcal{N}}) = 0 \). Summing over the first and last line, we find

\[
\text{det}(\tilde{\mathcal{N}}) = (\xi^2 + 1) \text{det}(\tilde{\mathcal{K}}_{N-2}) + 2\xi \text{det}(\tilde{\mathcal{K}}_{N-3}) + \text{det}(\tilde{\mathcal{K}}_{N-4})
\]

and, similarly,

\[
\text{det}(\tilde{\mathcal{K}}_L) = -\xi \text{det}(\tilde{\mathcal{K}}_{L-1}) - \text{det}(\tilde{\mathcal{K}}_{L-2})
\]

with initial conditions \( \text{det}(\tilde{\mathcal{K}}_1) = -\xi \) and \( \text{det}(\tilde{\mathcal{K}}_2) = \xi^2 - 1 \) (or, more simply, \( \text{det}(\tilde{\mathcal{K}}_0) = 1 \)). These are Chebyshev polynomials of the second type, with recursion relations

\[
U_{k+1}(x) = 2xU_k(x) - U_{k-1}(x)
\]

and initial conditions \( U_0 = 1 \) and \( U_1(x) = 2x \). They can be written in a simple closed form:

\[
U_k(x) = \sin((k + 1)\nu)/\sin\nu.
\]

With \( \xi = -2\cos\nu \), one finds \( \text{det}(\tilde{\mathcal{K}}_L) = \sin(L + 1)\nu/\sin\nu \) and

\[
\text{det}(\tilde{\mathcal{N}}) = \frac{(4\cos^2\nu + 1)\sin(N - 1)\nu}{\sin\nu} - \frac{4\cos\nu\sin(N - 2)\nu}{\sin\nu} + \frac{\sin(N - 3)\nu}{\sin\nu}
\]

\[
= \frac{2\cos\nu\sin N\nu}{\sin\nu}.
\]

(20)

Eigenvalues of \( \mathcal{N} \) satisfy one of the two conditions:

- \( \sin N\nu/\sin\nu = 0 \). Solutions for \( \xi \) are \( \xi_n = 2\cos\pi n/N \) with \( n = 1, \ldots, N - 1 \). (The minus sign has disappeared because we changed \( n \leftrightarrow N - n \).) The values \( n = 0 \) and \( n = N \) are absent because of the \( \sin\nu \) in the denominator of (20).
- \( \cos\nu = 0 \), with the solution \( \xi_{N/2} = 0 \) (even when \( N \) is not even).

When \( N \) is odd, \( \nu_n = \pi n/N \) is never \( \pi/2 \). All eigenvalues are distinct and \( \mathcal{N} \) is diagonalizable. When \( N \) is even, however, the eigenvalue \( \xi = 0 \) appears twice.

For a fixed value of \( n \) in the interval \( 1, \ldots, N - 1 \), we now look for \( u_n = (u_n^1, \ldots, u_n^N) \), the eigenvector of \( \mathcal{N} \) with eigenvalue \( \xi_n \). Its components satisfy the constraints

\[
u_n^j - \xi_n u_n^{j+1} + u_n^{j+2} = 0 \quad \text{for} \quad j = 1, \ldots, N - 2,
\]

\[
(-i - \xi_n)u_n^1 + u_n^2 = 0,
\]

\[
u_n^{N-1} + (i - \xi_n)u_n^N = 0.
\]

Let \( x_n \) be such that \( \xi_n = x_n + x_n^{-1} \) (and \( x_n = e^{i\pi n/N} \)). One can easily verify that the ansatz

\[
u_n^j = K_n(\alpha_n x_n^j + \gamma_n x_n^{-j}) \quad \text{with} \quad \alpha_n = -(1 + ix_n^{-1}) \quad \text{and} \quad \gamma_n = 1 + ix_n
\]

(21)
satisfies all three constraints. For reasons that will soon be clear, when \( n \neq N/2 \), we fix the constant \( K_n \) to \( (2\alpha_n^2 \gamma_n N)^{-1/2} \) ensuring that \( u_n^T u_n = 1 \). Indeed,

\[
u_n^T u_n = \sum_{j=1}^{N} (u_n^j)^2 = \frac{1}{2\alpha_n \gamma_n N} \sum_{j=1}^{N} (\alpha_n x_n^j + \gamma_n x_n^{-j})^2
\]

\[
= 1 + \frac{\alpha_n^2}{2\alpha_n \gamma_n N} x_n^2 (1 - x_n^{2N}) + \frac{\gamma_n^2}{2\alpha_n \gamma_n N} x_n^{-2} (1 - x_n^{-2N}) = 1,
\]

because \( x_n^{2N} = 1 \). Note that we have

\[
\alpha_n \gamma_n = -i(x_n + x_n^{-1}) = -i \xi_n.
\]

For the states with \( \xi = 0 \), the cases \( N \) odd and \( N \) have to be treated separately.

### 3.2. \( N \) odd

For the eigenvector with \( \xi = 0 \), the ansatz (21) still works with \( x = i \). Then, \( \gamma_n = 0, \alpha_n = -2 \) and we can write \( u_{n/2} = \mathcal{K}_{n/2}^{-1} \):

\[
u_{n/2}^T u_{n/2} = (\mathcal{K}_{n/2}^{-1})^2 \sum_{j=1}^{N} (-1)^j = -(\mathcal{K}_{n/2})^2
\]

and \( \mathcal{K}_{n/2} = i \) is the correct choice. When \( N \) is odd, \( \mathcal{N} \) is diagonalizable and from (18), \( H \) can be written as \( H = \sum_{l=0}^{N-1} \Lambda_l b_l a_l \), and

\[
[H, \ a_m] = \sum_{l=0}^{N-1} \Lambda_l [b_l a_m, \ a_m] = -\Lambda_m a_m, \quad \{c_i^\dagger, [H, \ a_m]\} = -\Lambda_m \sum_j g_m^j \{c_i^\dagger, c_j\} = -\Lambda_m g_m^j,
\]

but because of (17), we also have

\[
[H, \ a_m] = \sum_{k_1, k_2} \sum_f \mathcal{N}_{k_1, k_2} g_m^f [c_{k_1}^\dagger, c_{k_2}] = -\sum_{k_1, k_2} \mathcal{N}_{k_1, k_2} g_m^f c_{k_2},
\]

\[
\{c_i^\dagger, [H, \ a_m]\} = -\sum_{k_1, k_2} \mathcal{N}_{k_1, k_2} g_m^f c_{k_2} = -\sum_{k_1} \mathcal{N}_{k_1} g_m^{k_1},
\]

where we used \( \mathcal{N}_{i,j} = \mathcal{N}_{j,i} \). We can write

\[
\mathcal{N}_{\tilde{g}_m} = \Lambda_m \tilde{g}_m \quad \text{where} \quad \tilde{g}_m = \begin{pmatrix} g_m^1 \\ g_m^2 \\ \vdots \\ g_m^N \end{pmatrix}, \quad (22)
\]

The \( g_m^j \)'s are the components of the eigenvectors of \( \mathcal{N} \) and the \( \Lambda_m \)'s, its eigenvalues. The same process can be carried out for the \( b_m \)'s, yielding

\[
\mathcal{N}_{\tilde{f}_m} = \Lambda_m \tilde{f}_m, \quad \text{where} \quad \tilde{f}_m = \begin{pmatrix} f_m^1 \\ f_m^2 \\ \vdots \\ f_m^N \end{pmatrix}, \quad (23)
\]

The labeling of the \( a_s \) and \( b_s \) is as follows.
For $n = 1, \ldots, N - 1$, we choose $\Lambda_n = \xi_n$ and $f^j_n = g^j_n = u^j_n$. This gives

$$a_n = K_n \sum_{j=1}^{N} (\alpha_n s^j_n + \gamma_n s_n^j) c_j, \quad b_n = K_n \sum_{j=1}^{N} (\alpha_n s^j_n + \gamma_n s_n^j) c_j^\dagger,$$

with $K_n, \alpha_n$ and $\gamma_n$ given previously.

For the eigenvector with eigenvalue zero, $\Lambda_0 = \xi_{N/2} = 0$, $f^j_0 = g^j_0 = u^j_{N/2} = i^{j+1}$ and

$$a_0 = \sum_{j=1}^{N} i^{j+1} c_j, \quad b_0 = \sum_{j=1}^{N} i^{j+1} c_j^\dagger.$$

Because $f^j_k$ has a nonzero imaginary part and $f^j_k = g^j_k$, $b_n \neq a_n^T$. Instead, $c^j = c^T$ gives

$$b_n = a_n^T c^j = c^T a_n^T = g^T \vec{g}^T.$$

When $n \neq n'$, this is trivial because

$$0 = \vec{g}^T (N - N^T) \vec{g}^T = \vec{g}^T \vec{g}^T (\xi_{n'} - \xi_n) \quad \text{and} \quad \xi_n \neq \xi_{n'}.$$

However, when $n = n'$, $\vec{g}^T \vec{g}^T = 1$ explains our previous choice for the $K_n$s. Finally, one finds that $H$ can be written as $H = 2 \sum_{i=1}^{N-1} \cos (\pi k_i / N) b_i a_i$. If we denote by $|0\rangle$ the state $|\uparrow \uparrow \ldots \uparrow \rangle$ with all spins up, then eigenvectors of $H$ in the sector $S_z = N/2 - n$ are

$$|\gamma \rangle = a_0 a_1 a_2 \ldots a_n |0\rangle,$$

(26)

where the $k_1, \ldots, k_n$ are in the interval $0, \ldots, N - 1$ and appear at most once. When the $a_0$ excitation is present, we decide to set it at the end, $k_n = 0$. With this convention, the eigenvalue of $|\gamma \rangle$ is

$$\gamma = \begin{cases} 2 \sum_{i=1}^{n} \cos (\pi k_i / N), & \text{if} \quad k_n \neq 0, \\ 2 \sum_{i=1}^{n-1} \cos (\pi k_i / N), & \text{if} \quad k_n = 0. \end{cases}$$

### 3.3. $N$ even

For $N$ even, the eigenvalue 0 appears twice and $N$ is not diagonalizable. To show this, we study $N^2$:

$$N^2 = \begin{pmatrix} 0 & -i & 1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 \\ -i & 2 & 0 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$
and check that
\[ w_1' = i^1 \left( N - j - \frac{1}{2} \right) \quad \text{and} \quad w_2' = i^1 \left( N - j + \frac{1}{2} \right) \]
are two independent eigenvectors of \( \mathcal{N}^2 \) with eigenvalue 0. The eigenvector \( u_{N/2}' = K_{N/2}' i^1 \) of \( \mathcal{N} \) is given by the linear combination \( u_1' - w_1' = i^1 \) (though the constant \( K_{N/2}' \) will be different from the \( N \) odd constant). Also

\[ (\mathcal{N} w_1)' = i^{j-1}, \quad (\mathcal{N} w_2)' = i^{j-1}, \]

and any linear combination \( w = \beta_1 w_1 + \beta_2 w_2 \) satisfies \( \mathcal{N} w \propto u_{N/2} \); \( \mathcal{N} \) is therefore not diagonalizable. Nevertheless, it is possible to write \( H \) in the following manner:

\[ H = b_0 a_{-1} + \sum_{n=N/2}^{N-1} \lambda_n b_n a_n, \tag{28} \]

where all the \( a \)s and \( b \)s obey (19). The identification for \( N \) even is slightly modified.

- For the \( N = 2 \) eigenvectors with \( \xi \neq 0 \), (22) and (23) stay valid and the same identification is made: \( \lambda_n = \xi_n = 2 \cos \pi n/N \) and \( f_n' = g_n' = u_n' \) (for \( n = 1, 2, \ldots, N - 1 \), except \( n = N/2 \)).

  The operators \( a \) and \( b \) are then given by solution (24).

- For the two remaining modes, a new feature appears:

\[
0 = -[H, a_{-1}] = \sum_{k_1, k_2} g_{k_1, k_2}^N N_{N_{k_1}} N_{k_2} c_{k_1} \Rightarrow \mathcal{N} g_{-1} = 0, \\
0 = [H, b_0] = \sum_{k_1, k_2} g_{k_1, k_2}^0 N_{N_{k_1}} N_{k_2} c_{k_1} \Rightarrow \mathcal{N} g_0 = 0, \\
b_0 = [H, b_{-1}] = \sum_{k_1, k_2} f_{k_1, k_2}^N N_{N_{k_1}} N_{k_2} c_{k_1} \Rightarrow \mathcal{N} f_{-1} = 0,
\]

where the equations on the right are obtained by anti-commuting the equations on the left with \( c_i \) and writing the result as matrix products. The result is \( f_0' = g_{-1}' = u_{N/2}' = K_{N/2}' i^1 \) and \( f_{-1}' = g_0' = w_1' = \beta_1 w_1' + \beta_2 w_2' \), where \( K_{N/2}' \), \( \beta_1 \) and \( \beta_2 \) are constants that remain to be fixed. The relation \( \mathcal{N} w = u_{N/2} \), along with the commutation relations (19), fixes these constants (this is done in appendix A). The final result is

\[
a_0 = \sum_{j=1}^{N} (\beta_1 w_1^j + \beta_2 w_2^j) c_j, \quad \quad b_0 = K_{N/2}' \sum_{j=1}^{N} i^1 c_\bar{j}^j, \\
a_{-1} = K_{N/2}' \sum_{j=1}^{N} i^1 c_j, \quad \quad b_{-1} = \sum_{j=1}^{N} (\beta_1 w_1^j + \beta_2 w_2^j) c_\bar{j}^j,
\]

with \( K_{N/2}' = (2i/N)^{1/2} \), \( \beta_1 = \frac{-1}{2k_{N/2}} \) and \( \beta_2 = -\frac{N-1}{N} \beta_1 \). The new feature here is the pairing \( a_0^\dagger = b_{-1} \) and \( a_{-1}^\dagger = b_0 \).

Finally, the canonical expression for the Hamiltonian is

\[ H = b_0 a_{-1} + 2 \sum_{k=N/2}^{N-1} \cos(\pi k/N) b_k a_k. \]
In the sector $S^i = N/2 - n$, the states $|\gamma\rangle$ given in equation (26) are tied to the eigenvalues
\[
\gamma = \begin{cases} 
2 \sum_{i=1}^{n} \cos(\pi k_i/N) & \text{if } a_0 \text{ and } a_{-1} \text{ are absent,} \\
2 \sum_{i=1}^{n} \cos(\pi k_i/N) & \text{if only one of } a_0 \text{ or } a_{-1} \text{ is present,} \\
2 \sum_{i=1}^{n} \cos(\pi k_i/N) & \text{if both } a_0 \text{ and } a_{-1} \text{ are present.}
\end{cases}
\]

All the $k_i$s are in the set $\{-1, 0, \ldots, N - 1\} \setminus \{N/2\}$ and, as in the $N$ odd case, the $a_0$ and $a_{-1}$ are always set to the last $k_i$s, when present. Not all the states $|\gamma\rangle$ are eigenstates of $H$. The generalized eigenvectors are those with the $a_0$ excitation, but not $a_{-1}$. In total, there are $2^{N-2}$ such states, while all others are eigenvectors.

4. The algebra $U_q(sl_2)$

The algebra $U_q(sl_2)$ is generated by the three generators $q^S$, $S^+$ and $S^-$ that satisfy the relations
\[
q^S S^\pm - q^{\mp S} = q^\pm S^\pm \quad \text{and} \quad [S^+, S^-] = \frac{q^{2S} - q^{-2S}}{q - q^{-1}}.
\]

Proposition 4.1. The representation
\[
q^S = q^{\sigma^1/2} \otimes q^{\sigma^2/2} \otimes \ldots \otimes q^{\sigma^N/2} = \prod_{j=1}^{N} q^{\sigma_j/2},
\]

\[
S^\pm = \sum_{j=1}^{N} \sigma_j^\pm / 2,
\]

\[
S^\pm = \sum_{j=1}^{N} S_j^\pm = \sum_{j=1}^{N} q^{-\sigma_j/2} \otimes \ldots \otimes q^{-\sigma_j/2} \otimes \sigma_j^\pm \otimes q^{\sigma_j/2} \otimes \ldots \otimes q^{\sigma_j/2}
\]
\[
= \sum_{j=1}^{N} \left( \prod_{k=1}^{j-1} q^{-\sigma_k/2} \right) \sigma_j^\pm \left( \prod_{k=j+1}^{N} q^{\sigma_k/2} \right)
\]

of $U_q(sl_2)$ commutes with the $e_i$ matrices given in (15).

Proof. The commutation of $q^S, S^+$ and $S^-$ with $e_i$ arises from the relations
\[
\tilde{e}, q^{\sigma^1/2} \otimes q^{\sigma^2/2} = 0 \quad \text{and} \quad \tilde{e}, q^{-\sigma^1/2} \otimes \sigma^\pm + \sigma^\pm \otimes q^{\sigma^1/2} = 0,
\]

where $\tilde{e}$ is the $4 \times 4$ matrix given in (16).

This property, first noticed in [5], will be used thoroughly. Note also that $S^- = (S^+)^T$.

Some particularities occur when $q^{2p} = 1$. Let $q_c$ be a $2P$th root of unity. Then, $\langle S^\pm \rangle^P_{\tilde{q} = q_c} = 0$.

For the values $q_c$, the generators $(S^\pm)^P$ can be replaced by [17, 5]:
\[
S^\pm(P) = \lim_{q \to q_c} \frac{(S^\pm)^P}{[P]_q}, \quad \text{where} \quad [n]_q! = \prod_{k=1}^{n} [n]_q \quad \text{and} \quad [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.
\]
For $q = q_i$, a root or unity, $S^{\pm(P)}$ is nonzero and commutes with $e_i$, because

$$[S^{\pm(P)}, e_i] = \lim_{q \to q_i} \frac{[S^{\pm(P)}], e_i] = \lim_{q \to q_i} \frac{0}{[P]_q} = 0.$$ 

We are interested in the case $q_i = i$, $P = 2$, and calculate $S^{\pm(2)}$. The square of $S^\pm$ is

$$(S^\pm)^2 = \sum_{j_1, j_2} S^\pm_{j_1} S^\pm_{j_2} = \left( \sum_{j_1 < j_2} + \sum_{j_2 < j_1} \right) S^\pm_{j_1} S^\pm_{j_2} = \sum_{j_1 < j_2} (S^\pm_{j_1} S^\pm_{j_2} + S^\pm_{j_2} S^\pm_{j_1}).$$

When $j_1 < j_2$,

$$S^\pm_{j_1} S^\pm_{j_2} = \left( \prod_{k=1}^{j_1} q^{-\sigma_{k\downarrow}} \right) \sigma_{j_1}^\pm q^{-\sigma_{j_2\downarrow}/2} q^{\sigma_{j_2\uparrow}/2} \sigma_{j_2}^\pm \left( \prod_{k=j_2+1}^{N} q^{\sigma_{k\uparrow}} \right)$$

$$= q^{\pm 1} \left( \prod_{k=1}^{j_1} q^{-\sigma_{k\downarrow}} \right) \sigma_{j_1}^\pm \sigma_{j_2}^\pm \left( \prod_{k=j_2+1}^{N} q^{\sigma_{k\uparrow}} \right),$$

but

$$S^\pm_{j_2} S^\pm_{j_1} = q^{\mp 1} \left( \prod_{k=1}^{j_1} q^{-\sigma_{k\downarrow}} \right) \sigma_{j_1}^\pm \sigma_{j_2}^\pm \left( \prod_{k=j_2+1}^{N} q^{\sigma_{k\uparrow}} \right)$$

and finally,

$$\frac{(S^\pm)^2}{[2]_q} = \sum_{j_1 < j_2} \left( \prod_{k=1}^{j_1} q^{-\sigma_{k\downarrow}} \right) \sigma_{j_1}^\pm \sigma_{j_2}^\pm \left( \prod_{k=j_2+1}^{N} q^{\sigma_{k\uparrow}} \right).$$

### 4.1. $S^\pm$ and $S^{\pm(2)}$ for free fermions

The next step is to write $S^\pm$ and $S^{\pm(2)}$ first in terms of operators $c_j$ and $c_j^\dagger$, and then of the $a_k$s and $b_{kl}s$ calculated in section 3 (Deguchi et al did this for the periodic case [15]). We start with $S^+$ and $S^-$:

$$S^\pm = \left( \prod_{k=1}^{N} q^{\sigma_{k\uparrow}/2} \right) \sum_{j=1}^{N} \left( \prod_{k=1}^{j-1} q^{-\sigma_{k\downarrow}} \right) q^{-\sigma_{j\downarrow}/2} \sigma_{j}^\pm$$

$$= q^{\pm 1/2} \sum_{j=1}^{N} \left( \prod_{k=1}^{j-1} q^{-\sigma_{k\downarrow}} \right) \sigma_{j}^\pm$$

$$= i^{-1/2} \sum_{j=1}^{N} \left( \prod_{k=1}^{j-1} -i\sigma_{k\downarrow}^\pm \right) \sigma_{j}^\pm$$

$$= i^{-1/2} \sum_{j=1}^{N} i^l \left( \prod_{k=1}^{j-1} -\sigma_{k\downarrow}^\pm \right) \sigma_{j}^\pm$$

$$= \sum_{j=1}^{N} i^l \left( \prod_{k=1}^{j-1} \sigma_{k\downarrow}^\pm \right) \sigma_{j}^\pm$$

$$= \sum_{j=1}^{N} i^l \left( \prod_{k=1}^{j-1} \sigma_{k\downarrow}^\pm \right) \sigma_{j}^\pm.$$
and this yields

\[ S^+ = i^{s-3/2} \sum_{j=1}^{N} i^j c_j^+ = i^{s-3/2} \sum_{j=1}^{N} \frac{u_{N/2}}{K_N} c_j^+ \]  \\
\[ S^- = i^{s-1/2} \sum_{j=1}^{N} i^j c_j = i^{s-1/2} \sum_{j=1}^{N} u_{N/2} c_j. \]  

(30)

We can repeat the computation for \( S^{(2)} \) and \( S^{(2)} \):

\[ S^{(2)} = i^{-1} (-1)^s \sum_{j < k} i^{j+k} c_j^+ c_k^+ \]  \\
\[ S^{(2)} = -i^{-1} (-1)^s \sum_{j < k} i^{j+k} c_j c_k. \]

Though it is less apparent than before, both \( S^- = (S^+)^T \) and \( S^{(2)} = (S^{(2)})^T \) still hold. Our ultimate goal is to write \( S^{(2)} \) and \( S^{(2)} \) as

\[ S^{(2)} = \sum_{k_1, k_2} A(k_1, k_2) b_{k_1} b_{k_2}, \quad S^{(2)} = -\sum_{k_1, k_2} A(k_1, k_2) b_{k_1}^\dagger b_{k_2}^\dagger, \]

where \( A(k_1, k_2) = -A(k_2, k_1) \). To do this calculation, we need to find the inverse formula

\[ c_j^+ = \sum_k d_k^j b_k, \quad c_j = \sum_k e_k^j a_k. \]

To do so, we calculate \([c_j, a_k]\) and \([c_j, b_k]\) in the two possible ways to find \( d_k^j = e_k^j = g_k^j \). This allows us to pursue the computation

\[ S^{(2)} = i^{-1} (-1)^s \sum_{j < k} i^{j+k} (c_j^+ c_k^+ - c_j c_k^+), \]

and \( B(k_1, k_2) \) can be calculated directly. For any \( k_1, k_2 \) with \( \xi \neq 0 \),

\[ B(k_1, k_2) = K_1 K_2 \sum_{j < k} i^{j+k} (\alpha_1 \alpha_2 \gamma_1 \gamma_2 (x_{k_1}^j x_{k_2}^j - x_{k_1}^j x_{k_2}^j) + \gamma_1 \gamma_2 (x_{k_1}^j x_{k_2}^j - x_{k_1}^j x_{k_2}^j)) \]

\[ + \alpha_1 \alpha_2 \gamma_1 \gamma_2 (x_{k_1}^j x_{k_2}^j - x_{k_1}^j x_{k_2}^j) + \alpha_1 \alpha_2 \gamma_1 \gamma_2 (x_{k_1}^j x_{k_2}^j - x_{k_1}^j x_{k_2}^j)) \]

\[ = K_1 K_2 g(x_{k_1}, x_{k_2}) + g(x_{k_1}, x_{k_2}) - g(x_{k_1}, x_{k_2}) - g(x_{k_1}, x_{k_2}). \]

where \( g(z, w) = (f(z, w) - f(w, z))(1 + iz^{-1})(1 + iw^{-1}) \) and \( f(z, w) = \sum_{j < k} (iz)^j (iw)^k \).

After simplification, one finds

\[ g(z, w) = ((iz)^N - (iw)^N) + (iw - iz)(1 - (-zw)^N) \]

and

\[ B(k_1, k_2) = \frac{i(-1)^N K_1 K_2}{(x_{k_1} + x_{k_2})(1 + x_{k_1} x_{k_2})(x_{k_1} x_{k_2})^N} \]

\[ \times \left( (x_{k_1}^{2N} - x_{k_2}^{2N})(1 + x_{k_1} x_{k_2}) + (1 - x_{k_1} x_{k_2})(x_{k_1}^{2N} - x_{k_2}^{2N}) \right) \]

(31)

because \( x_{k_1} = e^{i\pi k_1/N}, x_{k_2}^{2N} = 1 \) and \( B(k_1, k_2) = 0 \) in general. There is an exception when \( k_1 + k_2 = N \). \( B(k_1, N - k_1) \) is calculated by taking the limit

\[ B(k_1, N - k_1) = \lim_{x_{k_1} \to -1/x_{k_1}} B(k_1, k_2). \]
The first term is zero, but not the second:

\[ B(k, N - k) = -2N i K_k N - k (x_k + x_k^{-1}) = -2N i K_k N - k \xi_k. \]

This simplifies even more, because when \( k < N/2 \),

\[ K_k N - k = \frac{1}{2N (\alpha_j \gamma_j \alpha_{N - k} \gamma_{N - k})^{1/2}} = \frac{1}{2N (-\xi_k \xi_{N - k})^{1/2}} = \frac{1}{2N \xi_k}, \]

and finally,

\[ B(k_1, k_2) = \begin{cases} -i \delta_{k_1 + k_2, N} & k_1 < N/2, \\ i \delta_{k_1 + k_2, N} & k_1 > N/2. \end{cases} \tag{32} \]

### 4.2. N odd

From (25) and (30), one finds directly

\[ S^+ = i S^{+3/2} b_0 \quad \text{and} \quad S^- = i S^{-3/2} a_0. \tag{33} \]

For \( S^{(2)} \), \( B(k_1, k_2) \) has been calculated except when \( k_1 = 0 \). The result (31) for \( B(k_1, k_2) \) is also valid for \( k_1 = 0 \) (as the eigenstate is still given by (21)); replacing \( x_k = i \) gives \( B(0, k) = 0 \) for all values of \( k \) in \( 1, \ldots, N - 1 \), and

\[ S^{(2)} = \frac{(-1)^{S+1}}{2} \left( \sum_{k=1}^{(N-1)/2} b_k b_{N-k} - \sum_{k=(N+1)/2}^{N-1} b_k b_{N-k} \right) = (-1)^{S+1} \sum_{k=1}^{(N-1)/2} b_k b_{N-k}, \tag{34} \]

\[ S^{(2)} = \frac{(-1)^{S-1}}{2} \left( \sum_{k=1}^{(N-1)/2} a_k a_{N-k} - \sum_{k=(N+1)/2}^{N-1} a_k a_{N-k} \right) = (-1)^{S} \sum_{k=1}^{(N-1)/2} a_k a_{N-k}. \]

Because the operators \( b_k b_{N-k} \) and \( a_k a_{N-k} \) commute with \( H, S^{(2)} \) and \( S^{(2)} \) also do, as expected.

### 4.3. N even

The case \( N \) even is again different because of the Jordan cell of size 2 in \( \mathcal{N} \) related to the eigenvalue 0. From (29) and (30),

\[ S^+ = \frac{i S^{-3/2}}{K_N^{1/2}} b_0, \quad S^- = \frac{i S^{-1/2}}{K_N^{1/2}} a_{-1}. \tag{35} \]

For \( S^{(2)} \) and \( S^{(2)} \), \( B(k_1, k_2) \) has been calculated for \( k_1, k_2 \) in \( \{1, \ldots, N - 1\} \setminus \{N/2\} \) in (32). When \( k = 0 \) or \(-1 \) and \( k' \in \{1, \ldots, N - 1\} \setminus \{N/2\} \), as before we can show that \( B(0, k') = B(-1, k') = 0 \). A quick argument consists in noticing that operators \( b_0 b_n \) and \( b_{-1} b_{k'} \) do not commute with \( H \) and that \( S^{(2)} \) could not have a component along these operators. But there is a component \( b_0 b_{-1} \):

\[ B(0, -1) = K_N^{1/2} \sum_{j_1 \in J_2} (-1)^{J_1} \left( \left[ \frac{N - j_1 - 1}{2} \right] - \frac{N - 4}{N} \left[ \frac{N - j_1 + 1}{2} \right] \right) \]
\[ - \left[ \frac{N - j_2 - 1}{2} \right] + \frac{N - 4}{N} \left[ \frac{N - j_2 + 1}{2} \right]. \]
To evaluate these sums (for $N$ even), note that

$$
\sum_{j_1 < j_2} (-1)^{j_1 + j_2} \left[ \frac{N - j_1 - 1}{2} \right] = \sum_{j_1=1}^{N} (-1)^{j_1} \left[ \frac{N - j_1 - 1}{2} \right] \sum_{j_2=1}^{N} (-1)^{j_2} = \sum_{j_1=1}^{N} (-1)^{j_1} \left[ \frac{N - j_1 - 1}{2} \right] \delta_{1,j_1 \mod 2} = - \frac{N/2}{2} = - \frac{N(N - 2)}{8}
$$

and in a similar fashion,

$$
\sum_{j_1 < j_2} (-1)^{j_1 + j_2} \left[ \frac{N - j_1 + 1}{2} \right] = - \frac{N(N - 2)}{8} - \frac{N}{2},
$$

$$
\sum_{j_1 < j_2} (-1)^{j_1 + j_2} \left[ \frac{N - j_2 - 1}{2} \right] = - \frac{N(N - 2)}{8} + \frac{N}{2},
$$

$$
\sum_{j_1 < j_2} (-1)^{j_1 + j_2} \left[ \frac{N - j_2 + 1}{2} \right] = - \frac{N(N - 2)}{8}.
$$

After simplification, $B(0, -1) = -2K_{N/2}b_1 = 1$, and

$$
S^{+(2)} = \frac{(-1)^{S}}{2} \left( i(b_{N} - b_{0}) - \sum_{k=1}^{(N-2)/2} b_{k}b_{N-k} + \sum_{k=(N+2)/2}^{N-1} b_{k}b_{N-k} \right) = (-1)^{S} \left( ib_{N} - \sum_{k=1}^{(N-2)/2} b_{k}b_{N-k} \right),
$$

$$
S^{-(2)} = (-1)^{S} \left( ia_{N} + \sum_{k=1}^{(N-1)/2} a_{k}a_{N-k} \right).
$$

5. The relation between $H$ and $\mathcal{H}_N$

In this section, we make explicit the relation between the XXZ model and the loop model. The results in this section hold for all $q$.

5.1. The homomorphism

We start by introducing a notation for link states. Let $v$ be a link state in $\mathcal{B}^d_N$ with $n = (N - d)/2$ bubbles and let $\psi(v) = \{(p_1, q_1), (p_2, q_2), \ldots, (p_n, q_n)\}$, where the $p_i$s are the positions where the bubbles of $v$ start and the $q_i$s the positions where they end. In $\psi(v)$, the $(p_i, q_i)$ pairs are ordered in ascending order of $p_i$, though this choice will play no role.

Definition 5.1. The linear transformation $\tilde{\sigma}^v_{N} : V^d_N \to (\mathbb{C}^2)^{\otimes N}_{S^d=N/2}$ (the subset of $(\mathbb{C}^2)^{\otimes N}$ of spin configurations with $n = (N - d)/2$ down-spins) is defined by its action on the basis elements of $\mathcal{B}^d_N$.

$$
\tilde{\sigma}^v_{N}(v) = \left( \prod_{(i,j) \in \psi(v)} T_{i,j} \right)|0\rangle, \quad \text{where} \quad T_{i,j} = w\sigma^+_{j} + w^{-1}\sigma^-_{i}.
$$
This definition can seem complex, but its graphical interpretation is not. In the simplest cases,
\[
\begin{align*}
i_2^0 \left( \begin{array}{c}
\hline
\end{array} \right) &= w \uparrow \downarrow + w^{-1} \downarrow \uparrow, \\
i_1^1 \left( \begin{array}{c}
\hline
\end{array} \right) &= | \uparrow \rangle,
\end{align*}
\]
and when a link state \( v \) has more than one bubble or more than one defect, they are replaced recursively by rule (38). For instance,
\[
\begin{align*}
i_6^0 \left( \begin{array}{c}
\hline
\hline
\hline
\hline
\end{array} \right) &= w^2 \uparrow \downarrow \uparrow \downarrow \downarrow \uparrow + w^{-2} \downarrow \uparrow \uparrow \downarrow \uparrow \uparrow, \\
i_6^0 \left( \begin{array}{c}
\hline
\hline
\hline
\end{array} \right) &= w^2 \uparrow \downarrow \uparrow \downarrow \uparrow \downarrow \uparrow + w^{-2} \downarrow \uparrow \uparrow \downarrow \uparrow \uparrow \uparrow.
\end{align*}
\]

The order of pairs \((i, j)\) in \( \psi(v) \) is unimportant, as indices in the product (37) are never repeated and \([T_{i,j}, T_{k,l}] = 0 \) when \( i, j, k, l \) are all different.

**Proposition 5.1.** For any \( c \in TL_N((q + q^{-1})) \) and any \( v \in V_d^N \), \( i_d^N(cv|d) = X(c)i_d^N(v) \) where \( X(c) \) is the matrix of \( c \) in the representation on \((\mathbb{C}^2)^{\otimes N}\) as given in (15), and where \( |d \rangle \) means that all components with less than \( d \) defects are set to 0.

**Proof.** To prove the proposition, one must show that the action of the matrix \( e_i \) on \( i_d^N(v) \) is the same as the action of the generators \( U_i \) on link states, except that annihilated defects always give 0. (We can restrict to the \( U_i \)s and \( e_i \)s only, as other connectivities are products of these.) More precisely, let \( Y(v) = \prod_{(m, n) \in \psi(v)} T_{m,n} \) and \( \psi(v) \) be the subset of \( \psi(v) \) that only contains positions of bubbles in \( v \) that do not touch \( i, i + 1, j \) and \( k \). We first give a list of properties sufficient to prove \( i_d^N(cv|d) = X(c)i_d^N(v) \) for any \( v \). For each entry of the list, we give a diagrammatic property followed by the algebraic identity that needs to be checked.

\[
\begin{align*}
(1) \quad & X \left( \begin{array}{c}
\hline
\end{array} \right) \ i_d^N\left( \begin{array}{c}
\hline
\end{array} \right) = X \left( \begin{array}{c}
\hline
\hline
\end{array} \right) \rightarrow e_i Y(v)|0\rangle = 0, \\
(2) \quad & X \left( \begin{array}{c}
\hline
\hline
\end{array} \right) \ i_d^N\left( \begin{array}{c}
\hline
\end{array} \right) = -(q + q^{-1})i_d^N\left( \begin{array}{c}
\hline
\end{array} \right) \\
& \quad \rightarrow e_i T_{i,i+1} Y(v)|0\rangle = -(q + q^{-1})T_{i,i+1} Y(v)|0\rangle, \\
(3) \quad & X \left( \begin{array}{c}
\hline
\hline
\end{array} \right) \ i_d^N\left( \begin{array}{c}
\hline
\hline
\end{array} \right) = i_d^N\left( \begin{array}{c}
\hline
\hline
\end{array} \right) \\
& \quad \rightarrow e_i T_{i,j} Y(v)|0\rangle = T_{i,j} Y(v)|0\rangle, \\
(4) \quad & X \left( \begin{array}{c}
\hline
\end{array} \right) \ i_d^N\left( \begin{array}{c}
\hline
\hline
\end{array} \right) = i_d^N\left( \begin{array}{c}
\hline
\hline
\end{array} \right) \\
& \quad \rightarrow e_i T_{j,i} Y(v)|0\rangle = T_{j,i} Y(v)|0\rangle,
\end{align*}
\]
Definition 5.2. Path, Dyck path and order.

(a) The set of paths with endpoint distance \( y \), \( P^N_y \), is the set of \( \vec{x} = \{x_1, x_2, \ldots, x_N\} \), where \( x_i = \pm 1 \forall i \) and \( \sum_{i=1}^{N} x_i = y \).

(b) The set of Dyck paths with endpoint distance \( y \), \( DP^N_y \), is the subset of \( \vec{x} \) in \( P^N_y \) satisfying \( \sum_{i=1}^{N} x_i \geq 0 \) for all \( k \) in \( \{1, \ldots, N\} \).

(c) We define an order for elements of \( \vec{x} \in P^N_y \): \( \vec{x}_1 < \vec{x}_2 \) if \( \mathcal{O}(\vec{x}_1) < \mathcal{O}(\vec{x}_2) \), with \( \mathcal{O}(\vec{x}) = \sum_{i=1}^{N} 2^i \delta_{x_i, -1} \).

We now verify that each algebraic identity holds. Since \( Y(v) \) commutes with \( e_i \) and with \( T_{i,j}, T_{j,i}, \ldots \), we can ignore it in our calculations. Because of (15), one can write

\[
e_i = (q + q^{-1}) \sigma_i^+ \sigma_i^- + q^2 \sigma_i^- \sigma_i^+ - q \sigma_i^+ \sigma_i^- \sigma_i^+ - q^{-1} \sigma_i^- \sigma_i^+ \sigma_i^- - \sigma_i^+ \sigma_i^- - \sigma_i^- \sigma_i^+
\]

Since \( \sigma_i^+ \sigma_i^- = 0 \) and \( \sigma_i^- \sigma_i^+ = 0 \), it is obvious that (1) is satisfied. As opposed to the \( \rho \) representation, here the number of defects is conserved, which explains the restriction to the XXZ model. The proofs of (4), (6) and (7) do not require any new ideas and are left to the reader.

The only difference between the action of the Temperley–Lieb algebra element \( c \) on \( V^d_N \) and that of the matrix \( X(c) \) on \( \mathcal{D}_N^d \) is that connected defects always give 0 in the second case. Nevertheless, for any connectivity \( c \), the diagonal blocks of \( \rho(c) \) can be calculated from those of \( X(c) \). Any information in nondiagonal blocks in the loop model is lost in the XXZ model.

5.2. The injectivity of \( \mathcal{D}_N^d \)
Prove. Let the \( v \)s be elements of \( B^d = B^d_{N/2 - n} \) and the \( |\alpha\rangle \)s as before. To show that \( \partial_N^d \) is injective, we must show that
\[
P_{a,v} = (|\alpha\rangle|\alpha\rangle^d(v)),
\]
a rectangular matrix of dimensions \( \binom{N}{d} \) by \( \binom{N}{d-1} \) (and again \( n = (N-d)/2 \) is the number of bubbles) is of maximal rank. For this, we study a square matrix \( \tilde{P}_{a,v} \), of size \( \binom{N}{d} \) by \( \binom{N}{d-1} \), with the same definition as \( P_{a,v} \), except making a restriction on the spin configurations \( |\alpha\rangle \). We will show that this matrix is of maximal rank. With this intent, we will order the \( v \)s of the link basis in decreasing order of their corresponding Dyck path, \( O(B^{-1}(v)) \) (\( B \) has been introduced in definition 2.3). For the \( |\alpha\rangle \)s, we choose the subset of spin configurations \( |\alpha\rangle = C(\bar{x}) \) for \( \bar{x} \) in \( DP^N \), and order them, again, in decreasing order of \( O(\bar{x}) \).

For a given \( v \in B^d \), \( C(B^{-1}(v)) \) is the state in \( (C^2)^N \) whose component in \( |\alpha\rangle^d(v) \) has the biggest power of \( w \): \( n \). Indeed, \( C(B^{-1}(v)) \) is the configuration obtained by replacing every bubble of \( v \) by \( w \uparrow \downarrow \). All other components of \( |\alpha\rangle^d(v) \) are obtained from the first by replacing certain pairs \( \uparrow \downarrow \) by \( \downarrow \uparrow \) and by diminishing the power of \( w \) by 2 for each pair changed. We conclude that in \( \tilde{P}_{a,v} \), every element on the diagonal is \( w^n \) and is nonzero (except for \( w = 0 \), which is an unphysical case). Every component \( |\alpha\rangle \) of \( |\alpha\rangle^d(v) \) has a \( O(C^{-1}(|\alpha\rangle)) \) smaller or equal to \( O(B^{-1}(v)) \), and the \( \tilde{P}_{a,v} \) matrix is therefore lower triangular. From the previous remark, the rank of \( \tilde{P}_{a,v} \), and therefore of \( P_{a,v} \), is maximal. □

An example is as follows, with \( N = 5, n = 2, d = 1 \):
\[
\tilde{x} \in DP^5, \\
O(\tilde{x}) = 2^4 + 2^5, 2^3 + 2^5, 2^2 + 2^5, 2^3 + 2^4, 2^2 + 2^4 \\
B(\tilde{x}) \in V^1, \\
\mathcal{C}(\tilde{x}) \in (C^2)^{\otimes 5}:
\]
\[
\tilde{P}_{a,v} = \begin{pmatrix}
w^2 & 0 & 0 & 0 & 0 \\
1 & w^2 & 0 & 0 & 0 \\
0 & 1 & w^2 & 0 & 0 \\
0 & 1 & 0 & w^2 & 0 \\
1 & w^{-2} & 1 & 1 & w^2 \\
\end{pmatrix}.
\]
From propositions 5.1 and 5.2, $\mathfrak{h}_N^d(V^d_N)$ is a subspace of $\dim V^d_N$ of $(\mathbb{C}^2)^{\otimes N}|_{S^d/2}$, invariant under the action of the $e_i$'s of XXZ. The eigenvectors of $\rho(\mathcal{H}_N)$ (for any $\beta$), restricted to the sector with $d$ defects, are in correspondence with eigenvectors of $H_{XXZ}$ in the $S^d = / d$ sector.

5.3. The relation between $U_q(sl_2)$ and $\mathfrak{h}_N^d(V^d_N)$

In this section, we establish the relation between the homomorphism $\beta_N^d$ and the algebra $U_q(sl_2)$.

**Proposition 5.3.** For all $v \in V^d_N$, $\beta_N^d(v) \in \ker S^+$. 

**Proof.** We start by restricting the proof to link patterns with only simple bubbles, i.e., to vs for which every $(i, j) \in \psi(v)$ is of the form $(i, i + 1)$. We note that, in general, whenever $k$ does not appear in any of the pairs $(i, j)$ in $\psi(v)$, $S^+_k \beta_N^d(v) = 0$. Indeed, when $i \neq j$,

$$S^+_k \sigma_j^- = q^{v_i} \sigma_j^- S^+_i$$

where $s_{i,j} = \begin{cases} +1, & i > j, \\ -1, & i < j. \end{cases}$

and, when $k$ is not in any of the pairs of $\psi(v)$,

$$S^+_k \beta_N^d(v) = S^+_k \left( \prod_{(i,j+1) \in \psi(v)} (w \sigma_{i+1}^- + w^{-1} \sigma_i^-) \right) |0\rangle = 0.$$

All that is left to calculate is $S^+_k \beta_N^d(v) = \sum_{(k,k+1) \in \psi(v)} (S^+_k + S^+_k) \beta_N^d(v)$,

$$(S^+_k + S^+_k) \beta_N^d(v) = \prod_{(i,j+1) \in \psi(v)} \left( q^{v_{i+1}} w \sigma_{i+1}^- + q^{v_i} w^{-1} \sigma_i^- \right) \left( w \sigma_{k+1}^- + w^{-1} \sigma_k^- \right) |0\rangle = 0.$$

When $v$ has only simple bubbles, $s_{k,j} = s_{k+1,j} = s_{k,j+1} = s_{k+1,j+1}$. This has been used at the last equality. Finally,

$$(S^+_k + S^+_k) \left( w \sigma_{k+1}^- + w^{-1} \sigma_k^- \right) |0\rangle = w^{-1} S^+_k \sigma_k^- |0\rangle + w S^+_k \sigma_{k+1}^- |0\rangle = w^{-1} \left( \prod_{i=1}^{k-1} q^{-\sigma_i/2} \sigma_k^- \sigma_{k+1}^- \prod_{j=k+1}^{N} q^{\sigma_j/2} \right) |0\rangle + w \left( \prod_{i=1}^{k} q^{-\sigma_i/2} \sigma_k^+ \sigma_{k+1}^- \prod_{j=k+2}^{N} q^{\sigma_j/2} \right) |0\rangle = w^{-1} \left( q^{N-2k+1}/2 + w^2 q^{N-2k+1}/2 \right) |0\rangle = 0.$$
For \( w \in B^d_N \) with bubbles that are not simple, from proposition 5.1, one can write \( \hat{r}_N^d(w) = (\prod_{j \in J} e_j) \rho^d_N(v) \) for some set \( J \) and for \( v \) a link state with only simple bubbles. Since \([S^+, e_j]=0\) by proposition 4.1, \( S^+ \rho^d_N(w) = 0 \) for all \( w \in B^d_N \).

From this proposition, it follows that for \( q = q_e \) and \( (q_e)^{2p} = 1 \), \( \rho^d_N(V^d_N) \) is also \( \subset \ker S^{+(P)} \).

\[
S^{+(P)} \rho^d_N(v) = \lim_{q \to q_e} \frac{(S^{+(P)}) \rho^d_N(v)}{|P|_{ik}} = \lim_{q \to q_e} \frac{0}{|P|_{ik}} = 0.
\]

6. The reduction of state space and the degeneracies

In the last sections, we found that the set of eigenvalues of \( \rho(H_N) \) in the sector with \( n \) bubbles was a subset of the eigenvalues of \( H \) in the sector \( S^f = N/2 - n \). For \( \beta = 0 \), this will allow us to prove the selection rules of section 2: we will calculate the degeneracy of every eigenvalue in \( H \), remove those that are tied to eigenvectors not in \( \rho^d_N(V^d_N) \) and show that the degeneracies obtained match those of the loop model, given by equations (11), (12) and (13). The two corresponding vector spaces \((C^2)^{\otimes N}|_{S^f=N/2−n}\) and \(V^N_{2n}\) have respective dimensions \( \binom{N}{n} \) and \( \binom{\eta}{n} \). To get only states in \( \rho^d_N(V^N_{2n}) \), we will need to remove \( \binom{N}{n} \) independent states from \((C^2)^{\otimes N}|_{S^f=N/2−n}\).

**Definition 6.1.** Let \( O = \sum_i \alpha_i O_i \) with \( \vec{i} = (i_1, i_2, \ldots, i_{\eta}) \), where \( \alpha_i \in \mathbb{C} \) and \( O_i \) is the product of some annihilation operators: \( O_i = \prod_{i=1}^{\eta} b_{i} \). We define \( O^\prime \) with the following two rules:

- \( O^\prime = \sum_i \alpha_i^* O_i^\prime \)
- \( O_{i^\prime} = \prod_{i=1}^{\eta} d_{i^\prime} \delta_{i^\prime, i + \epsilon^\prime} \)

where the product of non-commuting elements is always taken from left to right.

The sum over \( \vec{i} \) is a sum over multi-indices that could potentially have different lengths, but the only \( O_i \)s we will need have \( O_i \) with a unique fixed length. Examples are as follows:

\[
(b_{i_1} b_{i_2} b_{i_3})^\prime = a_{i_1} a_{i_2} a_{i_3},
\]

\[
(3i b_{i_2} + (5i + 1) b_{i_7} b_{i_4} + 12 i b_{i_3} b_{i_1} b_{i_2})^\prime = -3 i a_{i_2} + (-5i + 1) a_{i_4} a_{i_7} + 12 a_{i_3} a_{i_1} a_{i_2} a_{i_0}.
\]

**Proposition 6.1.** Let an operator \( O \neq 0 \) that satisfies \( \rho^d_N(v) = 0 \) for all \( v \in V^d_N \). Then, \( O^\prime |0\rangle \neq \rho^d_N(V^d_N) \).

**Proof.** There does not exist a set of constants \( \gamma_i \)s such that

\[
O^\prime |0\rangle + \sum_{v \in V^d_N} \gamma_i \rho^d_N(v) = 0.
\]

Indeed, multiplying this equation from the left with \( O \), the second term is zero by hypothesis:

\[
OO^\prime |0\rangle = \sum_i \alpha_i^2 |0\rangle = 0,
\]

which contradicts the hypothesis \( O \neq 0 \). □

By proposition 5.3, the operators \( S^+ \) and \( S^{+(2)} \) are two such operators \( O \) satisfying \( \rho^d_N(v) = 0, \forall v \in V^d_N \). To find eigenvectors of \( H \) not in \( \rho^d_N(V^d_N) \) and that we will have to remove from all the states of the form \( a_{i_1} a_{i_2} \cdots a_{i_{\eta}} |0\rangle \) (with \( n = (N - d)/2 \)), we look for operators \( O = \sum_i \alpha_i O_i \) for which every \( O_i \)s is a product of \( n \) annihilation operators. They are:
\( \mathcal{O} = S^+ b_j b_{j+1} \cdots b_{j_n} \), where \( j_n \neq 0 \) for \( k = 1, \ldots, n-1 \) \((b_j \) is the generator corresponding to \( S^+ \), see equations (33) and (35), and \( \mathcal{O} \) must be nonzero). Because \( \{b_0, b_j\} = 0 \) for all \( j \), \( \mathcal{O}_n^\delta(v) = 0 \) for all \( v \). All the states \( a_{j_1} a_{j_2} \cdots a_{j_n} |0\rangle \) must be removed. They will be referred to as states of the first kind. There are \( \binom{N-1}{n-1} \) such states.

- \( \mathcal{O} = S^{+(2)} b_j b_{j+1} \cdots b_{j_n} \) and \( \mathcal{O}_n^\delta(v) = 0 \) for all \( v \) by the same argument. The states to be removed are of the form \( a_{k_1} a_{k_2} \cdots a_{k_n} (S^{+(2)})' |0\rangle \), where the \( a_{k_i} \)s can be any of the \( N-1 \) remaining operators (not \( a_0 \), as we want to avoid any overlap with states of the first kind).

They will be referred to as states of the second kind. There are \( \binom{N-1}{n-1} \) such states.

Of course, \( \binom{N-1}{n-1} + \binom{N-1}{n-2} = \binom{N}{n-1} \), precisely the number of states we need to remove.

The fact that all these states are independent is nontrivial and shown in appendix B. Having succeeded in finding a rule that removes all eigenstates of \( H \) not in \( \mathcal{O}_N^\delta(V_N^\delta) \), we can now calculate the degeneracies.

### 6.1. \( N \) odd

As seen in section 3.2, when \( N \) is odd, the eigenvectors of \( H \), restricted to the \( S^z = N/2 - n \) sector, are of the form

\[
|\gamma\rangle = \left( \prod_{i=1}^{n} a_{k_i} \right) |0\rangle
\]

for \( k_i \in \{0, 1, \ldots, N-1\} \). If one of the \( k_i \)s is 0, we put it at the end and set \( a_{k_0} = a_0 \). The eigenvalues are

- \( \gamma = 2 \sum_{i=1}^{n} \cos (\pi k_i/N) \), if no \( k_i \) is 0,
- \( \gamma = 2 \sum_{i=1}^{n} \cos (\pi k_i/N) \), if some \( k_i \) is 0.

We call \( \Gamma_0^\delta \) and \( \Gamma_1^\delta \) respectively the set of all \( \gamma \)s for (a) and (b).

**Proposition 6.2.** \( \Lambda_0^\delta = \Gamma_0^\delta \) for both \( \delta = 0 \) and 1.

**Proof.** Let \( \gamma \in \Gamma_0^\delta \). To show that \( \gamma \in \Lambda_0^\delta \), we construct the three subsets \( K^+, K^- \) and \( K^c \). For all \( k \in \{1, \ldots, (N-1)/2\} \),

(i) if \( k \in \{k_1, \ldots, k_n\} \) and \( N - k \notin \{k_1, \ldots, k_n\} \), we put \( k \) in \( K^+ \);

(ii) if \( k \notin \{k_1, \ldots, k_n\} \) and \( N - k \in \{k_1, \ldots, k_n\} \), we put \( k \) in \( K^- \);

(iii) if \( k \in \{k_1, \ldots, k_n\} \) and \( N - k \notin \{k_1, \ldots, k_n\} \), we put \( k \) in \( K^c \);

(iv) if \( k \notin \{k_1, \ldots, k_n\} \) and \( N - k \notin \{k_1, \ldots, k_n\} \), we put \( k \) in \( K^c \).

We stress that when \( k_0 = 0, 0 \) is not in any of \( K^+, K^- \) or \( K^c \), but for fixed \( n \), its presence or absence changes the number of elements in \( K^+ \cup K^- \). The case \( \delta = 0 \) is when the \( a_0 \) creation operator is absent: \( n - m = n - |K^+ \cup K^-| \) counts the number of elements in (iii) and is even. When \( \delta = 1 \), the last momentum is \( k_n = 0 \) and the number of elements in (iii) is still even, but now given by \( n - 1 - m \), so \( n - m \) is odd.

Now, let \( \lambda \in \Lambda_0^\delta \) with a fixed \( m \). To see it is also in \( \Gamma_0^\delta \), we construct the set of momenta as follows:

(i) if \( k \) is in \( K^+ \), we put \( k \) in \( \{k_1, \ldots, k_n\} \), but not \( N - k \);

(ii) if \( k \) is in \( K^- \), we put \( N - k \) in \( \{k_1, \ldots, k_n\} \), but not \( k \);

(iii) if \( \delta = 1 \), we set \( k_n = 0 \).
(iv) for all the $k$s that are in $K^\prime$, we choose $(n - m - \delta)/2$ among the $(N - 1)/2 - m$ remaining values and put, for each, $k$ and $N - k$ in $\{k_1, \ldots, k_n\}$.

From the previous construction, an eigenvalue $\lambda$ of $H$ has the eigenvector

$$\left( \prod_i a_{N-i}a_i \right) \left( \prod_{j \in K^-} a_{N-j} \right) \left( \prod_{k \in K^+} a_k \right) |0\rangle,$$

if $\delta = 0$,

$$\left( \prod_i a_{N-i}a_i \right) \left( \prod_{j \in K^-} a_{N-j} \right) \left( \prod_{k \in K^+} a_k \right) a_0 |0\rangle,$$

where the product on $i$ has $(n - m - \delta)/2$ terms, all different, with $i \in K^\prime$. The degeneracy comes from all the possibilities for the product on $i$, and is given by

$$\deg_H(\lambda) = \left( \frac{N - 1}{2} - \frac{m - \delta}{2} \right).$$

To obtain the degeneracies of these eigenvalues in $\rho(H_N)$, we remove the states of (41) (they are all of the first kind) and from (40), all the states of the second kind:

$$\left( \prod_i a_{N-i}a_i \right) \left( \prod_{j \in K^-} a_{N-j} \right) \left( \prod_{k \in K^+} a_k \right) \left( \sum_{l=1}^{(N-1)/2} a_la_{N-l} \right) |0\rangle,$$

where the product on $i'$ has $(n - m - 2)/2$ terms and where the constant $(-1)^S$ of (34) has been dropped for convenience. For some $\lambda$, with a fixed value of $m$, there are $\binom{N-1}{m}$ such possible choices, each corresponding to an eigenvector. The set of corresponding eigenvectors is linearly independent (see appendix B) and the result is

- for $\lambda \in \Lambda^0_n$, $\deg_H(\lambda) = \left( \frac{N-1}{2} - \frac{m-\delta}{2} \right)$,
- for $\lambda \in \Lambda^1_n$, $\deg_H(\lambda) = 0$.

This is precisely the content of conjecture 2.3, which is now proved.

6.2. N even

As in section 3.2, eigenvectors and generalized eigenvectors of $H$, for $S^z = N/2 - n$, are given in (39), but with $k_i \in \{-1, 0, 1, \ldots, N - 1\} \setminus \{N/2\}$. When the $a_0$ and/or $a_{-1}$ excitations are present, we set them to the last $k_s$ ($k_a$ and $k_{n-1}$, when both are present). Eigenvectors are

(a) $\gamma = 2 \sum_{i=1}^{n} \cos(\pi k_i/N)$ if $a_0, a_{-1}$ are not in the $a_i$s;
(b) $\gamma = 2 \sum_{i=1}^{n} \cos(\pi k_i/N)$ if
   (i) $a_{-1}$ is not in the $a_i$s, but $a_0$ is;
   (ii) $a_0$ is in the $a_i$s, but $a_{-1}$ is;
(c) $\gamma = 2 \sum_{i=1}^{n} \cos(\pi k_i/N)$ if $a_0$ and $a_{-1}$ are both among the $a_i$s.

We refer to the sets of eigenvalues in the cases (a), (b) and (c) as $\Gamma^a_n$, $\Gamma^b_n$ and $\Gamma^c_n$.

**Proposition 6.3.** Based on the definition of 2.5 for $\Lambda^0_n$ and $\Lambda^1_n$, $\Gamma^a_n = \Lambda^0_n$, $\Gamma^b_n = \Lambda^1_n$ and $\Gamma^c_n \subset \Lambda^2_n$.
The proof is identical to the proof of proposition 6.2, with a few subtleties. The first is that whenever \( \gamma \) has the \( a_{-1} \) excitation, the \( a_0 \) excitation or both, their momenta are not in either \( K^+ \), \( K^- \) or \( K^c \), but their absence changes the number of elements in \( K^+ \cup K^− \). The second concerns the fact that \( \Gamma^p \) is only a subset of \( \Lambda^0_n \). Indeed, the elements of \( \Lambda^0_n \) with \( m = n \) are not contained in \( \Gamma^p_n \). The rest of the proof is not repeated. Note that the number of pairs \((k, N − k)\) to be fixed (among the \((N-2)/2 - m \) possible choices) and the degeneracies of the eigenvalues are different for the three cases (a), (b) and (c):

(a) \( (n − m)/2 \) pairs to be fixed and \( \text{deg}_{\mathcal{H}}(\lambda) = \left( \frac{n^2 - m}{2} \right) \);

(b) \( (n − m − 1)/2 \) pairs to be fixed and \( \text{deg}_{\mathcal{H}}(\lambda) = \left( \frac{n^2 - m}{2} \right) \);

(c) \( (n − m − 2)/2 \) pairs to be fixed and \( \text{deg}_{\mathcal{H}}(\lambda) = \left( \frac{n^2 - m}{2} \right) \).

States to be removed are those of the first kind, see (41), and those of the second kind,

\[
\left( \prod_{i} a_{N - i} a_{i} \right) \left( \prod_{j \in K^-} a_{N - j} \right) \left( \prod_{k \in K^+} a_{k} \right) \left( \sum_{l = 1}^{(N-2)/2} a_{l} a_{N - l} \right) |0\rangle,
\]

and the product on \( i \) has \( (n − m − 2)/2 \) terms. The \( a_0 a_{-1} \) contribution from \((S^+)^2\)' has been removed because this caused an overlap with states of the first kind. The degeneracies are

(a) \( \text{deg}_{\mathcal{H}} = \left( \frac{n^2 - m}{2} \right) - \left( \frac{n^2 - m}{2} \right) \),

(b) (i) \( \text{deg}_{\mathcal{H}} = 0 \),

(ii) \( \text{deg}_{\mathcal{H}} = \left( \frac{n^2 - m}{2} \right) - \left( \frac{n^2 - m}{2} \right) \),

(c) \( \text{deg}_{\mathcal{H}} = 0 \).

The cases (a) and (c) correspond to \( \Lambda^0_n \), while (b)(i) and (b)(ii) correspond to \( \Lambda^1_n \). This is the result of conjecture 2.5 and concludes the proof of the selection rules.

Note that Jordan partners were the states of (b)(i). Since they have all been removed, \( \rho(\mathcal{H}_N) \) is diagonalizable.

7. Conclusion

In this paper, we proved that the degeneracies of the eigenvalues of \( \rho(\mathcal{H}_N) \), as given by the selection rules, are correct. We must stress however that the proof ignored the problem of accidental degeneracies resulting from accidental trigonometric identities. Another problem is the case of loop models on other geometries. A recent paper [18] solved the model of critical dense polymers on the cylinder. An inversion relation was computed, eigenvalues were found and degeneracies conjectured by different selection rules from the ones here. The method proposed here might also lead to a proof of these conjectures.

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Appendix A. The computation of $K'_{N/2}$, $\beta_1$ and $\beta_2$ (for $N$ even)

The goal of this section is to calculate the three constants $K'_{N/2}$, $\beta_1$ and $\beta_2$ that fix the two states (the eigenstate and its Jordan partner) tied to the eigenvalues $\xi = 0$ of $N'$. The anticommutation relations, in terms of $u_{N/2}$ and $w$, are rewritten as

$$\{b_{-1}, a_{-1}\} = f^T_{-1}\vec{g}_{-1} = \sum_{j=1}^{N} u_{N/2}^{j} w_{j} = 1,$$

$$\{b_{0}, a_{-1}\} = f^T_{0}\vec{g}_{-1} = \sum_{j=1}^{N} (u_{N/2}^{j})^2 = 0,$$

$$\{b_{-1}, a_{0}\} = f^T_{-1}\vec{g}_{0} = \sum_{j=1}^{N} (w_{j})^2 = 0,$$

$$\{b_{0}, a_{0}\} = f^T_{0}\vec{g}_{0} = \sum_{j=1}^{N} w_{j} u_{N/2}^{j} = 1.$$

The second relation is trivially satisfied, since $\sum_{j=1}^{N} (-1)^j = 0$ for $N$ even. The third constraint reads

$$\beta_2^2 w_{1}^{T} w_{1} + \beta_2^2 w_{2}^{T} w_{2} + 2\beta_1\beta_2 w_{1}^{T} w_{2} = 0. \quad \text{(A.1)}$$

To calculate $w_{1}^{T} w_{1},$

$$w_{1}^{T} w_{1} = \sum_{j=1}^{N} (-1)^j \left[ \frac{N - j - 1}{2} \right]^2 = \sum_{k=0}^{N/2} \left( (-1)^{2k} \left[ \frac{N - 2k - 1}{2} \right]^2 + (-1)^{2k-1} \left[ \frac{N - 2k}{2} \right]^2 \right)$$

$$= \sum_{k=0}^{N/2} (N/2 - k - 1)^2 - \sum_{k=0}^{N/2} (N/2 - k)^2 = -N(N - 4)/4,$$

and one can also find $w_{1}^{T} w_{2} = -N^2/4$, $w_{1}^{T} w_{2} = -N(N - 2)/4$ and, from (A.1), $\beta_2/\beta_1 = -(N - 4)/N$ (the second solution, $\beta_1/\beta_2 = -1$, is not retained, because it corresponds to the eigenvector $u_{N/2}^{1} = K'_{N/2} (w_{1}^{N/2} - w_{1}^{1}) = K'_{N/2} w_{1}$ that we already found). It only remains to fulfill the first constraint (the fourth one is identical):

$$1 = \sum_{j=1}^{N} u_{N/2}^{j} w_{j} = K'_{N/2}\beta_1 \left( w_{1}^{T} w_{1} - \frac{N - 4}{N} w_{1}^{T} w_{2} + \left( \frac{N - 4}{N} + 1 \right) w_{2}^{T} w_{1} \right)$$

$$= K'_{N/2}\beta_1 \left( \frac{N(N - 4)}{4} + \frac{N(N - 4)}{4} - \left( \frac{N - 4}{N} + 1 \right) \frac{N(N - 2)}{4} \right) = -2K'_{N/2}\beta_1$$

which gives $K'_{N/2}\beta_1 = -1/2$. Finally, a last constraint is obtained from $N\vec{g}_0 = \vec{g}_{-1}$, which is equivalent to imposing that the coefficient in front of $b_{0}a_{-1}$ is 1 in equation (28):

$$K'_{N/2}\beta_1 = K'_{N/2} (N\vec{g}_0) = \beta_1 (N w_{1} - w_{2} (N - 4)/N) = \beta_1 \beta_1^{-1} (1 - (N - 4)/N) = \beta_1 (4i/N),$$

where equation (27) has been used at the fourth equality. This gives $K'_{N/2}/\beta_1 = -4i/N$ and the calculation of the three constants is complete.
Appendix B. Independence of states not in $\rho_N(V_N^d)$

In section 6, we have identified states to be removed from $(\mathbb{C}^2)^{\otimes N|S_{2\text{nd}}^d/2}$ and that should form a basis for the complement of $\rho_N(V_N^d)$. In this section, we show these states are nonzero and independent.

**Definition B.1.** Let $|v_1\rangle$ and $|v_2\rangle$ be any vector that can be written as $O_1|0\rangle$ and $O_2|0\rangle$, where $O_1$ and $O_2$ are multi-indices as in definition 6.1. We introduce a scalar product between such states by defining $\langle |v_1\rangle, |v_2\rangle \rangle = \langle 0\rangle O_1^\dagger O_2|0\rangle$. We will denote this scalar product by $\langle |v_1|v_2\rangle$.

The fact that states of the first kind $|w\rangle = a_{j_\text{th},1}a_{j_\text{th},2}\cdots a_{j_\text{th},n}|0\rangle$ (with $j_1 < j_2 < \cdots < j_{n-1}$) are independent and nonzero is trivial, as the scalar product restricted to such states is just $\langle w_1|w_2\rangle = \delta_{w_1,w_2}$: they all have length 1 and are mutually orthogonal. There are $\binom{N-1}{n-1}$ such vectors.

The proof for vectors of the second kind is more involved. It requires the following definition [19, 20].

**Definition B.2.** Let $v$ and $k$ be positive integers, with $v > k$. The Johnson graph $J(v, k)$ is as follows:

- its vertices $\theta$ are the subsets of length $k$ of $\{1, 2, \ldots, v\}$, their number is $\binom{v}{k}$;
- two vertices $\theta_1$ and $\theta_2$ are connected by an edge if and only if $|\theta_1 \cap \theta_2| = k - 1$.

The adjacency matrix $A(v, k)$ of the Johnson graph $J(v, k)$ is the matrix with entries

$$A(v, k)_{\theta_1, \theta_2} = \begin{cases} 1 & \text{if } \theta_1 \text{ and } \theta_2 \text{ are connected by an edge}, \\ 0 & \text{otherwise (even if } \theta_1 = \theta_2). \end{cases}$$

Johnson graphs have been thoroughly studied [19–21]. In particular, the eigenvalues of $A(v, k)$ are $k(v-k) - j(v-j+1)$ with $j = 0, \ldots, k$ with degeneracy $\binom{v}{j} - \binom{v}{j-1}$ [21]. Some pathologies occur when $v \leq 2k - 1$, as some of the degeneracies become negative or zero. We will see that in our cases, $v$ will always be larger than $2k - 1$.

For $N$ odd, we write in full generality the states of the second kind as

$$|w\rangle = \prod_{i \in \omega} a_{\omega_{N-i}} \prod_{j_1 \in J_+^\omega} a_{j_1} \prod_{j_2 \in J_-^\omega} a_{N-j_2} \sum_{k \in K^\omega} a_k a_{N-k}|0\rangle = \sum_{k \in K^\omega} |w_k\rangle. \quad (B.1)$$

In the previous formula, $I^\omega$ is the set of integers $i$ in the interval $1, \ldots, (N-1)/2$ such that $w$ contains both the $a_i$ and the $a_{N-i}$ excitation. $J_+^\omega$ ($J_-^\omega$) is the set of integers $j_1$ ($j_2$), also in the interval $1, \ldots, (N-1)/2$, such that the $a_{j_1}$ ($a_{N-j_2}$) excitation is present but the $a_{N-j_2}$ ($a_{j_1}$) is not (in fact, the sets $J^\pm$ are just the sets $K^\pm$ in definition 2.5). The sets $I^\omega$, $J^\omega_+$ and $J^\omega_-$ are all disjoint. Finally, the sum over $k$, in (42), was over all integers in $1, \ldots, (N-1)/2$, but since the square of any of the $a$s is zero, the sum really is on $K^\omega = \{1, \ldots, (N-1)/2\} \setminus (I^\omega \cup J^\omega_+ \cup J^\omega_-)$.

We also define $L^\omega = \{1, \ldots, (N-1)/2\} \setminus (J^\omega_+ \cup J^\omega_-)$.

Not all states (B.1) are nonzero. In fact, because $\langle w_k|w_{k'}\rangle = \delta_{k,k'}$, $\langle w|w\rangle = |K^\omega|$. If $I^\omega \cup J^\omega_+ \cup J^\omega_- = \{1, \ldots, (N-1)/2\}$, $K^\omega$ is empty and the state is zero, as can be seen more easily on (42). Recall that we are interested in states that have at most $(N-1)/2$ excitations for $N$ odd, as the number of excitations corresponds to the number of bubbles in the link states, $n$. This imposes that $2|I^\omega| + |J^\omega_+| + |J^\omega_-| + 2 = n \leq (N-1)/2$ (or equivalently, $|L| - 2|I| \geq 2$) and $I^\omega \cup J^\omega_+ \cup J^\omega_- \neq \{1, \ldots, (N-1)/2\}$.

Two states $|w_1\rangle$ and $|w_2\rangle$ are orthogonal unless $J^\omega_+ = J^\omega_-$. We can restrict the study of independence to sets of states with $J^\omega_+ = J^\omega_- \equiv J_\pm$ and $|I^\omega_+| = |I^\omega_-| \equiv |I|$ (and so
Because \( |−1| \) and \( |I| \) are on the boundaries; they are \( \{ |−1|, \ldots, |N| \} \), and \( |−1| \) has a \( b_{|−1|} \) contribution which can be ignored. We therefore consider vectors like (B.1), but with \( I^w \cap J^u = \emptyset \), and the possibility of having the \( a_{|−1|} \) excitation. \( L^w = \{ i_1, i_2, \ldots, i_{|w|} \} \), is represented by a subset of length \(|w|\) of \( L \) and is identified with a vertex of the Johnson graph \( J(|L|, |w|) \). The eigenvalues are given by

\[
\langle L_j | = | |w| (|L| - |I|) - j(|L| + 1 - j), \quad j = 0, \ldots, |I|.}
\]

Because \( -j(|L| + 1 - j) \) is a strictly decreasing function of \( j \) on the interval \([0, |I|]\), the extrema are on the boundaries; they are \( (1 + |I|)(|L| - |I|) \) and \(|L| - 2|I|\), both positive. Also because \(|L| - 2|I| > 1\), every degeneracy is positive. As all the eigenvalues are positive, there are no null states, and the independence is proved.

For \( N \) even, the proof requires a few subtleties. \( S^{(2)} \) has a \( b_{|−1|} \) contribution which can be ignored. We therefore consider vectors like (B.1), but with \( I^w \cap J^u \cap K^w = \emptyset \), and the possibility of having the \( a_{|−1|} \) excitation. \( L^w \) is then defined as \( \{ i_1, \ldots, (N - 2)/2 \} \setminus (J^u \cup J^v) \). The sets of states with and without this excitation, say \( S_1 \) and \( S_2 \), can be treated separately because, for any \( w_1 \in S_1 \) and \( w_2 \in S_2 \), \( \langle w_1 |w_2 \rangle = 0 \). For \( S_1 \), \( |L| - 2|I| > 1 \) and for \( S_2 \), \( |L| - 2|I| > 2 \). In both cases, all eigenvalues are positive.

The case \( d = 0 \) is particular. States of the second kind in \( S_2 \) are

\[
|w\rangle = \prod_i a_i \sum_{k \in K^w} a_k a_{N-k} |0\rangle,
\]

and the product on \( i \) has \( N/2 - 2 \) terms, all in \([1, \ldots, N - 1]\setminus[N/2] \). Their number is \( \binom{N/2}{N/2-2} \).

These are removed from the states

\[
|w\rangle = \prod_{i} a_i |0\rangle,
\]

where the product on \( i' \) has \( N/2 \) terms, also in \([1, \ldots, N - 1]\setminus[N/2] \). Their number is \( \binom{N/2}{N/2} \).

But these two numbers are equal and all states from (B.3) are independent from the previous argument. In other words, all the states (B.4) are removed, leaving no degeneracy in \( \rho(H_\lambda) \).

This is the result of proposition 2.5.

\[\square\]

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