ON THE 3-TORSION PART OF THE HOMOLOGY OF THE CHESSBOARD COMPLEX

JAKOB JONSSON

Abstract. Let $1 \leq m \leq n$. We prove various results about the chessboard complex $M_{m,n}$, which is the simplicial complex of matchings in the complete bipartite graph $K_{m,n}$. First, we demonstrate that there is nonvanishing 3-torsion in $\tilde{H}_d(M_{m,n}; \mathbb{Z})$ whenever $\frac{m+n-4}{3} \leq d \leq m-4$ and whenever $6 \leq m < n$ and $d = m-3$. Combining this result with theorems due to Friedman and Hanlon and to Shareshian and Wachs, we characterize all triples $(m, n, d)$ satisfying $\tilde{H}_d(M_{m,n}; \mathbb{Z}) \neq 0$. Second, for each $k \geq 0$, we show that there is a polynomial $f_k(a, b)$ of degree $3k$ such that the dimension of $\tilde{H}_{k+a+2b-2}(M_{k+a+3b-1,k+2a+3b-1}; \mathbb{Z}_3)$, viewed as a vector space over $\mathbb{Z}_3$, is at most $f_k(a, b)$ for all $a \geq 0$ and $b \geq k + 2$. Third, we give a computer-free proof that $\tilde{H}_2(M_{5,5}; \mathbb{Z}) \cong \mathbb{Z}_3$. Several proofs are based on a new long exact sequence relating the homology of a certain subcomplex of $M_{m,n}$ to the homology of $M_{m-2,n-1}$ and $M_{m-2,n-3}$.

1. Introduction

Given a family $\Delta$ of graphs on a fixed vertex set, we identify each member of $\Delta$ with its edge set. In particular, if $\Delta$ is closed under deletion of edges, then $\Delta$ is an abstract simplicial complex.

A matching in a simple graph $G$ is a subset $\sigma$ of the edge set of $G$ such that no vertex appears in more than one edge in $\sigma$. Let $M(G)$ be the family of matchings in $G$; $M(G)$ is a simplicial complex. We write $M_n = M(K_n)$ and $M_{m,n} = M(K_{m,n})$, where $K_n$ is the complete graph on the vertex set $[n] = \{1, \ldots, n\}$ and $K_{m,n}$ is the complete bipartite graph with block sizes $m$ and $n$. $M_n$ is the matching complex and $M_{m,n}$ is the chessboard complex.

The topology of $M_n$, $M_{m,n}$, and related complexes has been subject to analysis in a number of theses [1, 7, 11, 12, 15, 17] and papers.
Despite the simplicity of the definition, the homology of the matching complex $M_n$ and the chessboard complex $M_{m,n}$ turns out to have a complicated structure. The rational homology is well-understood and easy to describe thanks to beautiful results due to Bouc [5] and Friedman and Hanlon [10], but very little is known about the integral homology and the homology over finite fields.

In a previous paper [13], we proved a number of results about the integral homology of the matching complex $M_n$. The purpose of the present paper is to extend a few of these results to the chessboard complex $M_{m,n}$.

For $1 \leq m \leq n$, define

$$\nu_{m,n} = \min \{ m - 1, \left\lceil \frac{m+n-4}{3} \right\rceil \} = \left\{ \begin{array}{ll} \frac{m+n-4}{3} & \text{if } m \leq n \leq 2m + 1; \\ m-1 & \text{if } n \geq 2m - 1. \end{array} \right.$$  

Note that $\left\lceil \frac{m+n-4}{3} \right\rceil = m-1$ for $2m-1 \leq n \leq 2m+1$. By a theorem due to Shareshian and Wachs [20], $M_{m,n}$ contains nonvanishing homology in degree $\nu_{m,n}$ for all $m,n \geq 1$ except $(m,n) = (1,1)$. Previously, Friedman and Hanlon demonstrated that this bottom nonvanishing homology group is finite if and only if $m \leq n \leq 2m-5$ and $(m,n) \notin \{(6,6),(7,7),(8,9)\}$.

To settle their theorem, Shareshian and Wachs demonstrated that $\tilde{H}_{\nu_{m,n}}(M_{m,n}; \mathbb{Z})$ contains nonvanishing 3-torsion whenever the group is finite. One of our main results provides upper bounds on the rank of the 3-torsion part. Specifically, in Section 4.2, we prove the following:

**Theorem 1.** For each $k \geq 0$, $a \geq 0$, and $b \geq k+2$, we have that
dim $\tilde{H}_{\nu_{m,n}}(M_{m,n}; \mathbb{Z}_3)$ is bounded by a polynomial in $a$ and $b$ of degree $3k$.

An equivalent way of expressing Theorem 1 is to say that
\[
\dim \tilde{H}_d(M_{m,n}; \mathbb{Z}_3) \leq f_{3d-m-n+4}(n-m, m-d-1)
\]
whenever $m \leq n \leq 2m - 5$ and $\frac{m+n-4}{3} \leq d \leq \frac{2m+n-7}{4}$, where $f_k$ is a polynomial of degree $3k$ for each $k$. The bound in Theorem 1 remains true over any coefficient field.

Note that we express the theorem in terms of the following transformation:

$$\begin{align*}
(1) \quad & \left\{ \begin{array}{ll} k = -m - n + 3d + 4 \\
      a = -m + n \\
      b = m - d - 1 \end{array} \right. \iff \left\{ \begin{array}{ll} m = k + a + 3b - 1 \\
      n = k + 2a + 3b - 1 \\
      d = k + a + 2b - 2. \end{array} \right.
\end{align*}$$
Assuming that \( m \leq n \), each of the three new variables measures the difference between two important parameters:

- For \( m \leq n \leq 2m + 1 \), we have that \( k \) measures the difference between the degree \( d \) and the bottom degree in which \( M_{m,n} \) has nonvanishing homology: \( \frac{k}{3} = d - \frac{m+n-4}{3} \).
- \( a \) is the difference between the block sizes \( n \) and \( m \).
- \( b \) is the difference between \( \dim M_{m,n} = m - 1 \) and \( d \).

To establish Theorem 1, we introduce two new long exact sequences; see Sections 2.3 and 2.4. These two sequences involve the subcomplex \( \Gamma_{m,n} \) of \( M_{m,n} \) obtained by fixing a vertex in the block of size \( n \) and removing all but two of the edges that are incident to this vertex. Our first sequence is very simple and relates the homology of \( M_{m,n} \) to that of \( \Gamma_{m,n} \) and \( M_{m-1,n-1} \). Our second sequence is more complicated and relates \( \Gamma_{m,n} \) to \( M_{m-2,n-1} \) and \( M_{m-2,n-3} \). Combining the two sequences and “cancelling out” \( \Gamma_{m,n} \), we obtain a bound on the dimension of the \( \mathbb{Z}_3 \)-homology of \( M_{m,n} \) in terms of \( M_{m-1,n-1}, M_{m-2,n-1}, \) and \( M_{m-2,n-3} \). By an induction argument, we obtain Theorem 1.

For \( k = 0 \), Theorem 1 says that \( \dim \tilde{H}_{\nu}(M_{m,n}; \mathbb{Z}_3) \) is bounded by a constant whenever \( m \leq n \leq 2m - 5 \) and \( (m+n) \mod 3 = 1 \). Indeed, Shareshian and Wachs [20] proved that \( \tilde{H}_{\nu}(M_{m,n}; \mathbb{Z}_3) \cong \mathbb{Z}_3 \) for any \( m \) and \( n \) satisfying these equations. Their proof was by induction on \( m+n \) and relied on a computer calculation of \( \tilde{H}_2(M_{5,5}; \mathbb{Z}) \). In Section 3, we provide a computer-free proof that \( \tilde{H}_2(M_{5,5}; \mathbb{Z}) \cong \mathbb{Z}_3 \), again using the exact sequences from Sections 2.3 and 2.4.

In Section 4.1, we use results about the matching complex \( M_n \) from a previous paper [13] to extend Shareshian and Wachs’ 3-torsion result to higher-degree groups:

**Theorem 2.** For \( m+1 \leq n \leq 2m-5 \), there is 3-torsion in \( \tilde{H}_d(M_{m,n}; \mathbb{Z}) \) whenever \( \frac{m+n-4}{3} \leq d \leq m - 3 \). There is also 3-torsion in \( \tilde{H}_d(M_{m,m}; \mathbb{Z}) \) whenever \( \frac{2m-4}{3} \leq d \leq m - 4 \).

Note that \( m + 1 \leq n \leq 2m - 5 \) and \( \frac{m+n-4}{3} \leq d \leq m - 3 \) if and only if \( k \geq 0, a \geq 1, \) and \( b \geq 2 \), where \( k, a, \) and \( b \) are defined as in (1). Moreover, \( m = n \) and \( \frac{2m-4}{3} \leq d \leq m - 4 \) if and only if \( k \geq 0, a = 0, \) and \( b \geq 3 \).

Our proof of Theorem 2 relies on properties of the top homology group of \( M_{k,k+1} \) for different values of \( k \); this group was of importance also in the work of Shareshian and Wachs [20].

Thanks to Theorem 2 and Friedman and Hanlon’s formula for the rational homology [10], we may characterize those \((d, m, n)\) satisfying \( \tilde{H}_d(M_{m,n}; \mathbb{Z}) \neq 0 \):
Theorem 3. For $1 \leq m \leq n$, we have that $\tilde{H}_d(M_{m,n}; \mathbb{Z})$ is nonzero if and only if either of the following is true:

- $\lceil \frac{m+n-4}{3} \rceil \leq d \leq m-2$. Equivalently, $k \geq 0$, $a \geq 0$, and $b \geq 1$.
- $d = m - 1$ and $n \geq m + 1$. Equivalently, $k \geq 2 - a$, $a \geq 1$, and $b = 0$.

Again, see Section 4.1 for details.

The problem of detecting $p$-torsion in the homology of $\tilde{M}_{m,n}$ for $p \neq 3$ remains open. In this context, we may mention that there is 5-torsion in the homology of the matching complex $\tilde{M}_{14}$.

1.1. Notation. We identify the two parts of the graph $K_{m,n}$ with the two sets $[m] = \{1, 2, \ldots, m\}$ and $[n] = \{1, 2, \ldots, n\}$. The latter set should be interpreted as a disjoint copy of $[n]$; hence each edge is of the form $ij$, where $i \in [m]$ and $j \in [n]$. Sometimes, it will be useful to view $M_{m,n}$ as a subcomplex of the matching complex $M_{m+n}$ on the complete graph $K_{m+n}$. In such situations, we identify the vertex $j$ in $K_{m,n}$ with the vertex $m+j$ in $K_{m+n}$ for each $j \in [n]$.

For finite sets $S$ and $T$, we let $M_{S,T}$ denote the matching complex on the complete bipartite graph with blocks $S$ and $T$, viewed as disjoint sets in the manner described above. In particular, $M_{[m],[n]} = M_{m,n}$. For integers $a \leq b$, we write $[a,b] = \{a, a+1, \ldots, b-1, b\}$.

The join of two families of sets $\Delta$ and $\Sigma$, assumed to be defined on disjoint ground sets, is the family $\Delta \ast \Sigma = \{\delta \cup \sigma : \delta \in \Delta, \sigma \in \Sigma\}$.

Whenever we discuss the homology of a simplicial complex or the relative homology of a pair of simplicial complexes, we mean reduced simplicial homology. For a simplicial complex $\Sigma$ and a coefficient ring $\mathbb{F}$, we denote the generator of $\tilde{C}_d(\Sigma; \mathbb{F})$ corresponding to a set $\{e_0, \ldots, e_d\} \in \Sigma$ as $e_0 \wedge \cdots \wedge e_d$. Given a cycle $z$ in a chain group $\tilde{C}_d(\Sigma; \mathbb{F})$, whenever we talk about $z$ as an element in the induced homology group $\tilde{H}_d(\Sigma; \mathbb{F})$, we really mean the homology class of $z$.

We will often consider pairs of complexes $(\Gamma, \Delta)$ such that $\Gamma \setminus \Delta$ is a union of families of the form

$$\Sigma = \{\sigma\} \ast M_{S,T},$$

where $\sigma = \{e_1, \ldots, e_s\}$ is a set of pairwise disjoint edges of the form $ij$ and $S$ and $T$ are subsets of $[m]$ and $[n]$, respectively, such that $S \cap e_i = \emptyset$. We may write the chain complex of $\Sigma$ as

$$\tilde{C}_d(\Sigma; \mathbb{F}) = (e_1 \wedge \cdots \wedge e_s) \mathbb{F} \otimes_{\mathbb{F}} \tilde{C}_d-|\sigma|(M_{S,T}; \mathbb{F}),$$

defining the boundary operator as

$$\partial(e_1 \wedge \cdots \wedge e_s \otimes_{\mathbb{F}} c) = (-1)^s e_1 \wedge \cdots \wedge e_s \otimes_{\mathbb{F}} \partial(c).$$
For simplicity, we will often suppress $\mathbb{F}$ from notation. For example, by some abuse of notation, we will write
\[ e_1 \wedge \cdots \wedge e_s \otimes \tilde{C}_{d-\rho}(M_{S,T}) = (e_1 \wedge \cdots \wedge e_s)\mathbb{F} \otimes \tilde{C}_{d-\rho}(M_{S,T};\mathbb{F}). \]

We say that a cycle $z$ in $\tilde{C}_{d-1}(M_{m,n};\mathbb{F})$ has type $[m_1,m_2] \wedge \cdots \wedge [m_s,n_s]$ if there are partitions $[m] = \bigcup_{i=1}^s S_i$ and $[n] = \bigcup_{i=1}^s T_i$ such that $|S_i| = m_i$ and $|T_i| = n_i$ and such that $z = z_1 \wedge \cdots \wedge z_s$, where $z_i$ is a cycle in $\tilde{C}_{d-1}(M_{S_i,T_i};\mathbb{F})$ for each $i$.

### 1.2. Two classical results.

Before proceeding, we list two classical results pertaining to the topology of the chessboard complex $M_{m,n}$.

**Theorem 1.1** (Björner et al. [4]). For $m,n \geq 1$, $M_{m,n}$ is $(\nu_{m,n} - 1)$-connected.

Indeed, the $\nu_{m,n}$-skeleton of $M_{m,n}$ is vertex decomposable [23]. Garst [11] settled the case $n \geq 2m - 1$ in Theorem 1.1. As already mentioned in the introduction, there is nonvanishing homology in degree $\nu_{m,n}$ for all $(m,n) \neq (1,1)$; see Section 3 for details.

The transformation (1) maps the set $\{(m,n,\nu_{m,n}) : 1 \leq m \leq n\}$ to the set of triples $(k,a,b)$ satisfying either of the following:

- $k \in \{0,1,2\}$, $a \geq 0$, and $b \geq 1$ (corresponding to $d = \lceil \frac{m+n-4}{3} \rceil$ and $m \leq n \leq 2m - 2$).
- $2 - a \leq k \leq 2$ and $b = 0$ (corresponding to $0 \leq d = m - 1$ and $n \geq 2m - 1$).

Friedman and Hanlon [10] established a formula for the rational homology of $M_{m,n}$; see Wachs [22] for an overview. For our purposes, the most important consequence is the following result:

**Theorem 1.2** (Friedman and Hanlon [10]). For $1 \leq m \leq n$, we have that $H_d(M_{m,n};\mathbb{Z})$ is infinite if and only if $(m-d-1)(n-d-1) \leq d+1$, $m \geq d+1$, and $n \geq d+2$. In particular, $H_{\nu_{m,n}}(M_{m,n};\mathbb{Z})$ is finite if and only if $n \leq 2m - 5$ and $(m,n) \notin \{(6,6),(7,7),(8,9)\}$.

With $k$, $a$, and $b$ defined as in (1), the conditions $1 \leq m \leq n$, $(m-d-1)(n-d-1) \leq d+1 \leq m$, and $n \geq d+2$ are equivalent to
\[ b(a+b) \leq k + a + 2b - 1 \iff (b-1)(a+b-1) \leq k, \]
a $\geq 0$, $b \geq 0$, $a+b \geq 1$, and $k+a+3b \geq 2$. Moreover, the conditions $d = \nu_{m,n}$, $m \leq n \leq 2m - 5$, and $(m,n) \notin \{(6,6),(7,7),(8,9)\}$ are equivalent to $k \in \{0,1,2\}$, $a \geq 0$, $b \geq 2$, and $(k,a,b) \notin \{(1,0,2),(2,0,2),(2,1,2)\}$.
2. Four long exact sequences

We present four long exact sequences relating different families of chessboard complexes. In this paper, we will only use the third and the fourth sequences; we list the other two sequences for reference. Throughout this section, we consider an arbitrary coefficient ring $\mathbb{F}$, which we suppress from notation for convenience.

2.1. Long exact sequence relating $M_{m,n}$, $M_{m,n-1}$, and $M_{m-1,n-1}$.

**Theorem 2.1.** Define

$$P_{d_{m-1,n-1}}^{m-1,n-1} = \bigoplus_{s=1}^{m} s \bar{1} \otimes \tilde{H}_d(M_{[m]\setminus\{s\},[2,n]}).$$

For each $m \geq 1$ and $n \geq 1$, we have a long exact sequence

$$\cdots \longrightarrow P_{d_{m-1,n-1}}^{m-1,n-1} \longrightarrow \tilde{H}_d(M_{m,n-1}) \longrightarrow \tilde{H}_d(M_{m,n}) \longrightarrow P_{d_{m-1,n-1}}^{m-1,n-1} \longrightarrow \cdots.$$

**Proof.** This is the long exact sequence for the pair $(M_{m,n}, M_{m,n-1})$. □

We refer to this sequence as the 00-01-11 sequence, thereby indicating that the sequence relates $M_{m-0,n-0}$, $M_{m-0,n-1}$, and $M_{m-1,n-1}$. Note that the sequence is asymmetric in $m$ and $n$; swapping the indices, we obtain an exact sequence relating $M_{m,n}$, $M_{m-1,n}$, and $M_{m-1,n-1}$.

2.2. Long exact sequence relating $M_{m,n}$, $M_{m-1,n-2}$, $M_{m-2,n-1}$, and $M_{m-2,n-2}$.

**Theorem 2.2** (Shareshian & Wachs [20]). Define

$$P_{d_{m-1,n-2}}^{m-1,n-2} = \bigoplus_{t=2}^{n} \bar{1} \otimes \tilde{H}_d(M_{[2,m],[2,n]\setminus\{t\}});$$

$$Q_{d_{m-2,n-1}}^{m-2,n-1} = \bigoplus_{s=2}^{m} s \bar{1} \otimes \tilde{H}_d(M_{[2,m]\setminus\{s\},[2,n]});$$

$$R_{d_{m-2,n-2}}^{m-2,n-2} = \bigoplus_{s=2}^{m} \bigoplus_{t=2}^{n} \bar{1} \wedge s \bar{1} \otimes \tilde{H}_d(M_{[2,m]\setminus\{s\},[2,n]\setminus\{t\}}).$$

For each $m \geq 2$ and $n \geq 2$, we have a long exact sequence

$$\cdots \longrightarrow P_{d_{m-1,n-2}}^{m-1,n-2} \oplus Q_{d_{m-2,n-1}}^{m-2,n-1} \longrightarrow \tilde{H}_d(M_{m,n}) \longrightarrow R_{d_{m-2,n-2}}^{m-2,n-2} \longrightarrow \cdots.$$
We refer to this sequence as the 00-12-21-22 sequence. The sequence played an important part in Shareshian and Wachs’ analysis [20] of the bottom nonvanishing homology of \( M_{m,n} \). Note that the sequence is symmetric in \( m \) and \( n \).

2.3. Long exact sequence relating \( M_{m,n}, \Gamma_{m,n}, \) and \( M_{m-1,n-1} \). The sequence in this section is very similar, but not identical, to the 00-01-11 sequence in Section 2.3. Define

\[
\Gamma_{m,n} = \{ \sigma \in M_{m,n} : s \bar{\tau} \notin \sigma \text{ for } s \in [3,m] \}.
\]

**Theorem 2.3.** Define

\[
P_{d-1}^{m-1,n-1} = \bigoplus_{s=3}^{m} s \bar{\tau} \otimes \tilde{H}_d(M_{[s],[2,n]}); \\
\]

note that this definition differs from that in Section 2.1. For each \( m \geq 1 \) and \( n \geq 1 \), we have a long exact sequence

\[
\cdots \rightarrow P_{d-1}^{m-1,n-1} \rightarrow \tilde{H}_d(\Gamma_{m,n}) \rightarrow \tilde{H}_d(M_{m,n}) \rightarrow P_{d-1}^{m-1,n-1} \rightarrow \cdots.
\]

**Proof.** This is the long exact sequence for the pair \((M_{m,n}, \Gamma_{m,n})\). \(\square\)

We refer to this sequence as the 00-\( \Gamma \)-11 sequence. Note that the sequence is asymmetric in \( m \) and \( n \).

2.4. Long exact sequence relating \( \Gamma_{m,n}, M_{m-2,n-1}, \) and \( M_{m-2,n-3} \). Recall the definition of \( \Gamma_{m,n} \) from (2).

**Theorem 2.4.** Write

\[
Q_d^{m-2,n-1} = (1 \bar{\tau} - 2 \bar{\tau}) \otimes \tilde{H}_d(M_{[3,m],[2,n]});
\]

\[
R_d^{m-2,n-3} = \bigoplus_{s \notin t \in [2,n]} 1 \bar{\tau} \wedge 2 \bar{\tau} \otimes \tilde{H}_d(M_{[s],[2,n] \setminus \{s,t\}}).
\]

For each \( m \geq 2 \) and \( n \geq 3 \), we have a long exact sequence

\[
\cdots \rightarrow Q_{d-1}^{m-2,n-1} \rightarrow \phi^* \inj \tilde{H}_d(\Gamma_{m,n}) \rightarrow \tilde{H}_d(M_{m,n}) \rightarrow Q_{d-2}^{m-2,n-1} \rightarrow \cdots,
\]

where \( \phi^* \) is induced by the map \( \phi \) defined by

\[
\phi(1 \bar{\tau} \wedge 2 \bar{\tau} \otimes x) = (1 \bar{\tau} - 2 \bar{\tau}) \otimes x.
\]

and \( \inj^* \) is induced by the natural map \( \inj((1 \bar{\tau} - 2 \bar{\tau}) \otimes x) = ((1 \bar{\tau} - 2 \bar{\tau}) \wedge x.\]
Proof. Define a filtration
\[ \Delta^0_{m,n} \subset \Delta^1_{m,n} \subset \Delta^2_{m,n} = \Gamma_{m,n} \]
as follows:
- \( \Delta^2_{m,n} = \Gamma_{m,n} \).
- \( \Delta^1_{m,n} \) is the subcomplex of \( \Delta^2_{m,n} \) obtained by removing all faces containing \( \{1s, 2t\} \) for some \( s, t \in [2, n] \).
- \( \Delta^0_{m,n} \) is the subcomplex of \( \Delta^1_{m,n} \) obtained by removing the elements \( 11, \ldots, 1n \) and \( 22, \ldots, 2n \).

Writing \( \Delta^{-1}_{m,n} = \emptyset \), let us examine \( \Delta^i_{m,n} \setminus \Delta^{i-1}_{m,n} \) for \( i = 0, 1, 2 \).
- \( i = 0 \). Note that
  \[ \Delta^0_{m,n} = M_{2,1} \ast M_{[3,m],[2,n]} \cong M_{2,1} \ast M_{m-2,n-1} \]
As a consequence,
  \[ \tilde{H}_d(\Delta^0_{m,n}) \cong (1\bar{T} - 2\bar{T}) \otimes \tilde{H}_{d-1}(M_{[3,m],[2,n]}) = Q_{d-1}^{m-2,n-1} \).
- \( i = 1 \). Observe that
  \[ \Delta^1_{m,n} \setminus \Delta^0_{m,n} = \bigcup_{a=1}^{2} \bigcup_{u=2}^{n} \{\{a\bar{u}\}\} \ast M_{\{3-a\},\{1\}} \ast M_{[3,m],[2,n]\{u\}} \]
It follows that
  \[ \tilde{H}_d(\Delta^1_{m,n}, \Delta^0_{m,n}; F) = \bigoplus_{a,u} a\bar{u} \otimes \tilde{H}_{d-1}(M_{\{3-a\},\{1\}} \ast M_{[3,m],[2,n]\{u\}}; F) = 0; \]
  \( M_{\{3-a\},\{1\}} \cong M_{1,1} \) is a point.
- \( i = 2 \). We have that
  \[ \Delta^2_{m,n} \setminus \Delta^1_{m,n} = \bigcup_{s,t \in [2, n]} \{\{1s, 2t\}\} \ast M_{[3,m],[2,n]\{s,t\}} \]
we may hence conclude that
  \[ \tilde{H}_d(\Delta^2_{m,n}, \Delta^1_{m,n}; F) = \bigoplus_{s,t} 1s \wedge 2t \otimes \tilde{H}_{d-2}(M_{[3,m],[2,n]\{s,t\}}; F) = R_{d-1}^{m-2,n-3}. \]
By the long exact sequence for the pair \( (\Delta^2_{m,n}, \Delta^1_{m,n}) \), it remains to prove that the induced map \( \varphi^* \) has properties as stated in the theorem. Now, in the long exact sequence for \( (\Delta^2_{m,n}, \Delta^1_{m,n}) \), the induced boundary map from \( \tilde{H}_{d+1}(\Delta^2_{m,n}, \Delta^1_{m,n}) \) to \( \tilde{H}_d(\Delta^1_{m,n}) \) maps the element \( 1s \wedge 2t \otimes z \) to \( (2t - 1s) \otimes z \). Since
  \[ (2t - 1s) \otimes z - \partial((1T \wedge 2\bar{T} + 1\bar{s} \wedge 2T) \otimes z) = (1T - 2\bar{T}) \otimes z, \]
we are done. \( \square \)
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We refer to this sequence as the $\Gamma$-21-23 sequence. Note that the sequence is asymmetric in $m$ and $n$.

3. BOTTOM NONVANISHING HOMOLOGY

Using the long exact sequences in Sections 2.3 and 2.4, we give a computer-free proof that $\tilde{H}_2(M_{5,5};\mathbb{Z})$ is a group of size three. While the proof is complicated, our hope is that it may provide at least some insight into the structure of $M_{5,5}$ and related chessboard complexes.

Theorem 3.1. We have that $\tilde{H}_2(M_{5,5};\mathbb{Z}) \cong \mathbb{Z}_3$.

Proof. First, we examine $M_{3,4}$; for alignment with later parts of the proof, we consider $M_{[3,5],[2,5]}$, thereby shifting the first index two steps and the second index one step. The long exact 00-Γ-11 sequence from $\iota$ isomorphism. As a consequence, the map $\iota^*$ induced by the natural inclusion map is an isomorphism. The long exact $\Gamma$-21-23 sequence for $\Gamma_{[3,5],[2,5]}$ becomes

\[
0 \longrightarrow \tilde{H}_2(\Gamma_{[3,5],[2,5]}) \longrightarrow \tilde{H}_2(M_{[3,5],[2,5]}) \overset{\omega^*}{\longrightarrow} 5\mathbb{Z} \otimes \tilde{H}_1(M_{[3,4],[3,5]}) \longrightarrow \tilde{H}_1(\Gamma_{[3,5],[2,5]}) \longrightarrow 0.
\]

Now, $M_{m,m+1}$ is well-known to be an orientable pseudomanifold for all $m \geq 1$ [20]; hence $\tilde{H}_2(\Gamma_{[3,5],[2,5]}) = 0$ and the map $\omega^*$ is necessarily an isomorphism. As a consequence, the map $\iota^*$ induced by the natural inclusion map is an isomorphism. The long exact $\Gamma$-21-23 sequence for $\Gamma_{[3,5],[2,5]}$ from Section 2.4 becomes

\[
0 \longrightarrow (3\mathbb{Z} - 4\mathbb{Z}) \otimes \tilde{H}_0(M_{[3,5],[3,5]}) \overset{\iota^*}{\longrightarrow} \tilde{H}_1(\Gamma_{[3,5],[2,5]}) \longrightarrow 0,
\]

which yields that each of $\tilde{H}_1(\Gamma_{[3,5],[2,5]})$ and $\tilde{H}_1(M_{[3,5],[2,5]})$ is generated by $e_i = (3\mathbb{Z} - 4\mathbb{Z}) \wedge (5\mathbb{Z} - 5\mathbb{Z})$ for $i \in \{4, 5\}$.

Now, consider $M_{5,5}$. The tail end of the $\Gamma$-21-23 sequence is

\[
\bigoplus_{s,t} 15 \wedge 2\mathbb{I} \otimes \tilde{H}_1(M_{[3,5],[2,5]}) \overset{\varphi^*}{\longrightarrow} (1\mathbb{I} - 2\mathbb{I}) \otimes \tilde{H}_1(M_{[3,5],[2,5]}) \overset{\iota^*}{\longrightarrow} \tilde{H}_2(\Gamma_{5,5}) \rightarrow 0,
\]

where the first sum ranges over all pairs of distinct elements $s, t \in [2, 5]$. Writing $\{s, t, u, v\} = [2, 5]$, we note that $\tilde{H}_1(M_{[3,5],[2,5]}) = \tilde{H}_1(M_{[3,5],[2,5]})$ is generated by the cycle

\[
z_{uv} = 3\mathbb{I} \wedge 4\mathbb{I} \wedge 4\mathbb{I} \wedge 5\mathbb{I} + 5\mathbb{I} \wedge 5\mathbb{I} + 3\mathbb{I} \wedge 5\mathbb{I} + 4\mathbb{I} \wedge 5\mathbb{I} + 5\mathbb{I} + 5\mathbb{I} \wedge 3\mathbb{I}.
\]

By Theorem 2.4, $\varphi^*$ maps $15 \wedge 2\mathbb{I} \otimes z_{uv}$ to $(1\mathbb{I} - 2\mathbb{I}) \otimes z_{uv}$. Since $z_{uv} = z_{vu}$, we conclude that the image under $\varphi^*$ is generated by the six cycles $z_{23}, z_{24}, z_{25}, z_{34}, z_{35}, z_{45}$. 
In $\tilde{H}_1(M_{[3,5],[2,5]})$, we have that $z_{st} = z_{uv}$, because $z_{st} - z_{uv}$ equals the boundary of

$$\gamma = 3\pi \land 5\pi \land 4\pi - 5\pi \land 3\pi \land 5\pi \land 3\pi \land 3\pi \land 5\pi + 4\pi \land 5\pi \land 3\pi \land 3\pi \land 5\pi + 4\pi \land 4\pi \land 3\pi \land 3\pi \land 5\pi
+ 5\pi \land 3\pi \land 5\pi - 3\pi \land 5\pi \land 3\pi \land 3\pi \land 5\pi
+ 4\pi \land 3\pi \land 5\pi - 3\pi \land 5\pi \land 3\pi \land 5\pi \land 3\pi \land 5\pi.$$

Namely, $\gamma$ is of the form $a_1 \land a_2 \land a_3 - a_2 \land a_3 \land a_4 + \cdots - a_{12} \land a_1 \land a_2$, which yields the boundary $-a_1 \land a_3 + a_2 \land a_4 - \cdots + a_{12} \land a_2$. As a consequence, the image under $\varphi^*$ is generated by the three cycles $z_{34}, z_{35}, z_{45}$.

Assume that $s = 2$ and $\{t, u, v\} = \{3, 4, 5\}$ and write

$$w_{uv} = 5\pi \land 4\pi \land 3\pi - 4\pi \land 3\pi \land 5\pi + 3\pi \land 5\pi \land 4\pi
- 5\pi \land 4\pi \land 3\pi + 4\pi \land 3\pi \land 5\pi.$$

We obtain that

$$\partial(w_{uv} + w_{vu}) = (5\pi \land 4\pi - 5\pi \land 3\pi + 4\pi \land 5\pi - 3\pi \land 4\pi + 5\pi \land 3\pi
- 4\pi \land 5\pi + 3\pi \land 5\pi) + (5\pi \land 4\pi - 5\pi \land 3\pi + 4\pi \land 5\pi
- 3\pi \land 4\pi + 5\pi \land 3\pi - 4\pi \land 5\pi + 3\pi \land 5\pi)
= (4\pi - 3\pi) \land (2 \cdot 5\pi - 5\pi - 5\pi) - z_{uv}.$$

Since $s = 2$, it follows that $z_{uv}$ is equal to either $-e_4 - e_5, 2e_4 - e_5,$ or $-e_4 + 2e_5$ in $\tilde{H}_1(M_{[3,5],[2,5]})$ depending on the values of $t, u,$ and $v$.

We conclude that the set $\{\varphi^*(1\pi \land 2\xi \otimes z_{uv}) : \{s, t, u, v\} = \{2, 5\}\}$ generates the subgroup $\{(1\xi - 2\xi) \otimes (ae_4 + be_5) : a - b \equiv 0 \mod 3\}$ of $(1\xi - 2\xi) \otimes \tilde{H}_1(M_{[3,5],[2,5]})$. As a consequence, $\tilde{H}_2(\Gamma_{5,5}) \cong \mathbb{Z}_3$ and

$$\rho = (1\xi - 2\xi) \land (3\xi - 4\xi) \land (5\xi - 5\xi)$$

is a generator for this group. Swapping $3$ and $4$, we obtain $-\rho$; we obtain the same result if we swap 3 and 4 or if we swap 1 and 2. Hence, by symmetry, the group

$$T = \mathcal{S}_{\{1,2\}} \times \mathcal{S}_{\{3,4,5\}} \times \mathcal{S}_{\{2,3,5\}}$$

acts on $\tilde{H}_2(\Gamma_{5,5}) \cong \mathbb{Z}_3$ by $\pi(\rho) = \text{sgn}(\pi) \cdot \rho$.

It remains to prove that $\tilde{H}_2(\Gamma_{5,5}) \cong \tilde{H}_2(M_{5,5})$. For this, consider the tail end of the 00-00-11 sequence from Section 2.3:

$$\bigoplus_{x=3}^5 x\xi \otimes \tilde{H}_2(M_{[5] \setminus \{x\},[2,5]}) \xrightarrow{\varphi^*} \tilde{H}_2(\Gamma_{5,5}) \longrightarrow \tilde{H}_2(M_{5,5}) \rightarrow 0$$

By a result due to Shareshian and Wachs [20, Lemma 5.9], we have that $\tilde{H}_2(M_{[5] \setminus \{x\},[2,5]}) \cong \tilde{H}_2(M_{4,4})$ is generated by cycles of type $[\frac{3}{2}] \land [\frac{1}{2}]$
and cycles of type \([\frac{2}{3}] \wedge [\frac{2}{1}]\); recall notation from Section 1.1. By properties of \(\psi^*\), we need only prove that any such cycle vanishes in \(H_2(\Gamma_{5,5})\) whenever \(x \in [3, 5]\).

- A cycle of the first type is of the form \(z = \lambda \cdot \gamma \wedge (d\bar{u} - d\bar{v})\), where \(\lambda\) is a constant,
  \[\gamma = a\bar{s} \wedge b\bar{t} + b\bar{t} \wedge c\bar{s} + c\bar{s} \wedge a\bar{t} + a\bar{t} \wedge b\bar{s} + b\bar{s} \wedge c\bar{t} + c\bar{t} \wedge a\bar{s},\]
  \(\{a, b, c, d\} = [5] \setminus \{x\}\), and \(\{s, t, u, v\} = [2, 5]\). By the above discussion, swapping \(\bar{s}\) and \(\bar{t}\) in \(z\) should yield \(-z\), but obviously the same swap in \(\gamma\) again yields \(\gamma\), which implies that \(z = -z\); hence \(z = 0\).

- A cycle of the second type is of the form \(z = \lambda \cdot \gamma \wedge (c\bar{u} - d\bar{v})\), where \(\lambda\) is a constant, say \(\lambda = 1\), and
  \[\gamma = a\bar{s} \wedge b\bar{t} + b\bar{t} \wedge a\bar{u} + a\bar{u} \wedge b\bar{s} + b\bar{s} \wedge a\bar{t} + a\bar{t} \wedge b\bar{u} + b\bar{u} \wedge a\bar{s};\]
  again \(\{a, b, c, d\} = [5] \setminus \{x\}\) and \(\{s, t, u, v\} = [2, 5]\). If \(\{a, b\} \subset [3, 5]\), then we may swap \(a\) and \(b\) and again conclude that \(z = -z\); the same argument applies if \(\{a, b\} = \{1, 2\}\). For the remaining case, we may assume that \(c \in [1, 2]\) and \(d \in [3, 5]\). Swapping \(d\) and \(x\) yields \(-z = \gamma \wedge (c\bar{v} - x\bar{u})\); recall that \(x \in [3, 5]\). As a consequence,
  \[2z = z - (-z) = \gamma \wedge (x\bar{v} - d\bar{u}) = \partial(c\bar{t} \wedge \gamma \wedge (x\bar{v} - d\bar{u}));\]
  hence \(z\) is again zero. Namely, since \(c \in [1, 2]\), we have that \(c\bar{t}\) is an element in \(\Gamma_{5,5}\). As a consequence, \(\psi^*\) is the zero map as desired. □

By Theorems 1.1 and 1.2, the connectivity degree of \(M_{m,n}\) is exactly \(\nu_{m,n} - 1\) whenever \(n \geq 2m - 4\) or \((m, n) \in \{(6, 6), (7, 7), (8, 9)\}\). As mentioned in the introduction, Shareshian and Wachs [20] extended this result to all \((m, n) \neq (1, 1)\), thereby settling a conjecture due to Björner et al. [4]:

**Theorem 3.2** (Shareshian & Wachs [20]). If \(m \leq n \leq 2m - 5\) and \((m, n) \neq (8, 9)\), then there is nonvanishing 3-torsion in \(\hat{H}_{\nu_{m,n}}(\nu_{m,n}; \mathbb{Z})\). If in addition \((m + n) \mod 3 = 1\), then \(\hat{H}_{\nu_{m,n}}(\nu_{m,n}; \mathbb{Z}) \cong \mathbb{Z}_3\).

By Theorem 4.4 in Section 4.1, there is nonvanishing 3-torsion also in \(\hat{H}_{\nu_{m,n}}(M_{m,n}; \mathbb{Z})\); in that theorem, choose \((k, a, b) = (2, 1, 2)\).

**[Table 1]**

In fact, Shareshian and Wachs provided much more specific information about the exponent of \(\hat{H}_{\nu_{m,n}}(M_{m,n}; \mathbb{Z})\); see Table 1.

**Conjecture 3.3** (Shareshian & Wachs [20]). The group \(\hat{H}_{\nu_{m,n}}(M_{m,n}; \mathbb{Z})\) is torsion-free if and only if \(n \geq 2m - 4\).
The conjecture is known to be true in all cases but \( n = 2m - 4 \) and \( n = 2m - 3 \); Shareshian and Wachs [20] settled the case \( n = 2m - 2 \).

**Corollary 3.4** (Shareshian & Wachs [20]). For all \((m, n) \neq (1, 1)\), we have that \( \tilde{H}_{\nu_{m,n}}(M_{m,n}; \mathbb{Z}) \) is nonzero.

### 4. Higher-degree homology

In Section 4.1, we detect 3-torsion in higher-degree homology groups of \( M_{m,n} \). In Section 4.2, we proceed with upper bounds on the dimension of the homology over \( \mathbb{Z}_3 \).

#### 4.1. 3-torsion in higher-degree homology groups.

This section builds on work previously published in the author’s thesis [12]. Fix \( n, d \geq 0 \) and let \( \gamma \) be an element in \( \tilde{H}_{d-1}(M_n; \mathbb{Z}) \); note that we consider the matching complex \( M_n \). For each \( k \geq 0 \), define a map

\[
\begin{align*}
\theta_k : \tilde{H}_{k-1}(M_{k,k+1}; \mathbb{Z}) &\rightarrow \tilde{H}_{k-1+d}(M_{2k+1,n}; \mathbb{Z}) \\
\theta_k(z) &= z \wedge \gamma^{(2k+1)},
\end{align*}
\]

where we obtain \( \gamma^{(2k+1)} \) from \( \gamma \) by replacing each occurrence of the vertex \( i \) with \( i + 2k + 1 \) for every \( i \in [n] \).

For any prime \( p \), we have that \( \theta_k \) induces a homomorphism

\[
\theta_k \otimes \mathbb{Z}_p : \tilde{H}_{k-1}(M_{k,k+1}; \mathbb{Z}) \otimes \mathbb{Z}_p \rightarrow \tilde{H}_{k-1+d}(M_{2k+1,n}; \mathbb{Z}) \otimes \mathbb{Z}_p,
\]

where \( \iota_p : \mathbb{Z}_p \rightarrow \mathbb{Z}_p \) is the identity. The following result about the matching complex is a special case of a more general result from a previous paper [13].

**Theorem 4.1** (Jonsson [13]). Fix \( k_0 \geq 0 \). With notation and assumptions as above, if \( \theta_{k_0} \otimes \mathbb{Z}_p \) is a monomorphism, then \( \theta_k \otimes \mathbb{Z}_p \) is a monomorphism for each \( k \geq k_0 \).

As already mentioned in the proof of Theorem 3.1 in Section 3, we have that \( M_{k,k+1} \) is an orientable pseudomanifold of dimension \( k - 1 \); hence \( \tilde{H}_{k-1}(M_{k,k+1}; \mathbb{Z}) \cong \mathbb{Z} \). Shareshian and Wachs [20] observed that this group is generated by the cycle

\[
z_{k,k+1} = \sum_{\pi \in \mathcal{S}_{k+1}} \text{sgn}(\pi) \cdot 1\pi(1) \wedge \cdots \wedge k\pi(k).
\]

Note that the sum is over all permutations on \( k + 1 \) elements. Theorem 4.1 implies the following result.

**Corollary 4.2.** With notation and assumptions as in Theorem 4.1, if \( (z_{k_0,k_0+1} \wedge \gamma^{(2k_0+1)}) \otimes 1 \) is nonzero in \( \tilde{H}_{k_0-1+d}(M_{2k_0+1,n}; \mathbb{Z}) \otimes \mathbb{Z}_p \), then \( (z_{k,k+1} \wedge \gamma^{(2k+1)}) \otimes 1 \) is nonzero in \( \tilde{H}_{k-1+d}(M_{2k+1,n}; \mathbb{Z}) \otimes \mathbb{Z}_p \) for all \( k \geq k_0 \).
We will also need a result about the bottom nonvanishing homology of the matching complex. Define
\[ \gamma_{3r} = (12 - 23) \wedge (45 - 56) \wedge (78 - 89) \wedge \cdots \wedge ((3r - 2)(3r - 1) - (3r - 1)(3r)); \]
this is a cycle in both \( \tilde{C}_{r-1}(M_{3r}; \mathbb{Z}) \) and \( \tilde{C}_{r-1}(M_{3r+1}; \mathbb{Z}) \).

**Theorem 4.3** (Bouc [5]). For \( r \geq 2 \), we have that \( \tilde{H}_{r-1}(M_{3r+1}; \mathbb{Z}) \cong \mathbb{Z}_3 \). Moreover, this group is generated by \( \gamma_{3r} \) and hence by any element obtained from \( \gamma_{3r} \) by permuting the underlying vertex set.

Assume that \( (m + n) \mod 3 = 0 \) and \( m \leq n \leq 2m \). Define the cycle \( \gamma_{m,n} \) in \( \tilde{H}_{\nu_{m,n}}(M_{m,n}; \mathbb{Z}) \) recursively as follows, the base case being \( \gamma_{1,2} = 1 \tilde{1} - 1 \tilde{2} \):
\[ \gamma_{m,n} = \begin{cases} 
\gamma_{m-1,n-2} \wedge (m(n - 1) - m\overline{n}) & \text{if } m < n; \\
\gamma_{m-2,n-1} \wedge ((m - 1)\overline{n} - m\overline{n}) & \text{if } m = n.
\end{cases} \]

For \( n > m \), we define \( \gamma_{n,m} \) by replacing \( i \tilde{j} \) with \( j \tilde{i} \) in \( \gamma_{m,n} \) for each \( i \in [m] \) and \( j \in [n] \).

Recall that \( \nu_{m,n} = \frac{m+n-4}{3} \) whenever \( m \leq n \leq 2m - 2 \).

**Theorem 4.4.** There is 3-torsion in \( \tilde{H}_{d}(M_{m,n}; \mathbb{Z}) \) whenever
\[ \begin{cases} 
m + 1 \leq n \leq 2m - 5 \\
\left\lfloor \frac{m+n-4}{3} \right\rfloor \leq d \leq m - 3
\end{cases} \iff \begin{cases} 
k \geq 0 \\
a \geq 1 \\
b \geq 2,
\end{cases} \]
where \( k, a, \) and \( b \) are defined as in (1). Moreover, there is 3-torsion in \( \tilde{H}_{d}(M_{m,m}; \mathbb{Z}) \) whenever
\[ \left\lfloor \frac{2m - 4}{3} \right\rfloor \leq d \leq m - 4 \iff \begin{cases} 
k \geq 0 \\
a = 0 \\
b \geq 3.
\end{cases} \]

**Proof.** Assume that \( k \geq 0, a \geq 1, \) and \( b \geq 2 \). Writing \( m_0 = a + 3b - 2 \) and \( n_0 = 2a + 3b - 3 \), we have the inequalities
\[ a + 3b - 2 \leq 2a + 3b - 3 \leq 2a + 6b - 9 \iff m_0 \leq n_0 \leq 2m_0 - 5. \]

Note that \( m_0 + n_0 = 3a + 6b - 5 \equiv 1 \pmod{3} \). Define
\[ w_{k+1} = z_{k+1,k+2} \wedge \gamma_{m_0,n_0-1}^{(k+1,k+2)}, \]
where we obtain \( \gamma_{m_0,n_0-1}^{(k+1,k+2)} \) from the cycle \( \gamma_{m_0,n_0-1} \) defined in (4) by replacing \( i \tilde{j} \) with \( (i + k + 1) \tilde{j} + (k + 2) \). View \( \gamma_{m_0,n_0-1} \) as an element in the homology of \( M_{m_0,n_0} \). Since \( z_{k+1,k+2} \) has type \( \left[ \frac{k+1,k+2}{k+1} \right] \) and since
Clearly, The group Corollary 4.5 implies that the same must be true for \( w \) have exponent three in \( \gamma \). It follows that the exponent of \( m \) for every \( k \) viewed as an element in \( \gamma \) hence we may view \( w_{k+1} \) as an element in \( \tilde{H}_{d}(M_{m,n};\mathbb{Z}) \).

Choosing \( k = 0 \), we obtain that

\[
\begin{array}{c}
\gamma_{m_0,n_0-1} \text{ has type } \left[ \frac{a+3b-2,2a+3b-3}{a+2b-2} \right] \quad \text{or rather } \left[ \frac{a+3b-2,2a+3b-4}{a+2b-2} \right] \wedge \left[ \frac{0,1}{0,1} \right],
\end{array}
\]

we obtain that \( w_{k+1} \) has type

\[
\begin{bmatrix}
k + 1 + a + 3b - 2, k + 2 + 2a + 3b - 3 \\
k + 1 + a + 2b - 2
\end{bmatrix} = \begin{bmatrix} m, n \\ d + 1 \end{bmatrix};
\]

hence we may view \( w_{k+1} \) as an element in \( \tilde{H}_{d}(M_{m,n};\mathbb{Z}) \).

Applying Corollary 4.2, we conclude that \( w_{k+1} \otimes 1 \) is a nonzero element in the group \( \tilde{H}_{k+a+2b-2}(M_{2k+3a+6b-2};\mathbb{Z}) \otimes \mathbb{Z}_3 = \tilde{H}_{d}(M_{m+n};\mathbb{Z}) \otimes \mathbb{Z}_3 \) for every \( k \geq 0 \). As a consequence, \( w_{k+1} \otimes 1 \) is nonzero also in

\[
\tilde{H}_{k+a+2b-2}(M_{k+a+3b-1,k+2a+3b-1};\mathbb{Z}) \otimes \mathbb{Z}_3 = \tilde{H}_{d}(M_{m,n};\mathbb{Z}) \otimes \mathbb{Z}_3
\]

for every \( k \geq 1 \). Since \( \tilde{H}_{a+b-3}(M_{m_0,n_0};\mathbb{Z}) \) is an elementary 3-group by Theorem 3.2 and (5), the exponent of \( \gamma_{m_0,n_0-1} \) in \( \tilde{H}_{r}(M_{m_0,n_0};\mathbb{Z}) \) is three. It follows that the exponent of \( w_{k+1} \) in \( \tilde{H}_{d}(M_{m,n};\mathbb{Z}) \) is three as well.

The remaining case is \( m = n \), in which case the upper bound on \( d \) is \( m - 4 \) rather than \( m - 3 \). Since \( a = 0 \), we get

\[
\begin{cases}
k = -2m + 3d + 4 \\
b = m - d - 1
\end{cases} \iff \begin{cases}
m = k + 3b - 1 \\
d = k + 2b - 2.
\end{cases}
\]

Clearly, \( k \geq 0 \) and \( b \geq 3 \).

Consider the cycle \( w_{k+1} = z_{k+1,k+2} \wedge \gamma_{3b-3}^{(k+1,k+2)} \). By Corollary 4.2, \( w_{k+1} \otimes 1 \) is nonzero in \( \tilde{H}_{k+2b-2}(M_{2b+6b-2};\mathbb{Z}) \otimes \mathbb{Z}_3 \). Namely, \( w_1 \) is isomorphic to \( \gamma_{6b-3} \) in (4), which is a nonzero element with exponent three in \( \tilde{H}_{2b-3}(M_{6b-2};\mathbb{Z}) \) by Theorem 4.3; \( b \geq 3 \). We conclude that \( w_{k+1} \otimes 1 \) is a nonzero element in \( \tilde{H}_{k+2b-2}(M_{k+3b-1,k+3b-1};\mathbb{Z}) \otimes \mathbb{Z}_4 = \tilde{H}_{d}(M_{m,m};\mathbb{Z}) \otimes \mathbb{Z}_3 \). Since \( 3b - 3 \geq 6 \), we have that \( \gamma_{3b-3}^{(k,k+2)} \) must have exponent three in \( \tilde{H}_{2b-3}(M_{3b-2,3b-3};\mathbb{Z}) \); apply Theorem 3.2. This implies that the same must be true for \( w_{k+1} \) in \( \tilde{H}_{d}(M_{m,m};\mathbb{Z}) \).

**Corollary 4.5.** The group \( \tilde{H}_5(M_{8,9};\mathbb{Z}) = \tilde{H}_9(M_{8,9};\mathbb{Z}) \) contains nonvanishing 3-torsion. As a consequence, there is nonvanishing 3-torsion in \( \tilde{H}_{\nu_n}(M_{m,n};\mathbb{Z}) \) whenever \( m \leq n \leq 2m - 5 \).
Proof. The first statement is a consequence of Theorem 4.4; choose $k = 2$, $a = 1$, and $b = 2$. For the second statement, apply Theorem 3.2. □

**Theorem 4.6.** For $1 \leq m \leq n$, the group $\tilde{H}_d(M_{m,n}; \mathbb{Z})$ is nonzero if and only if either

$$\left\lceil \frac{m+n-4}{3} \right\rceil \leq d \leq m-2 \iff \begin{cases} k \geq 0 \\ a \geq 0 \\ b \geq 1 \end{cases}$$

or

$$\begin{cases} m \geq 1 \\ n \geq m+1 \iff a \geq 1 \\ d = m-1 \iff b = 0, \end{cases}$$

where $k$, $a$, and $b$ are defined as in (1).

Proof. For homology to exist, we certainly must have that $b \geq 0$, and we restrict to $a \geq 0$ by assumption. Moreover, $b = 0$ means that $d = m - 1$, in which case there is homology only if $m \leq n - 1$, hence $a \geq 1$ and $k + a \geq 2$; for the latter inequality, recall that we restrict our attention to $m \geq 1$. Finally, $k < 0$ reduces to the case $b = 0$, because we then have homology only if $n \geq 2m + 2$ and $d = m - 1$; apply Theorem 1.1.

For the other direction, Theorem 4.4 yields that we only need to consider the following cases:

- $k \geq 0$, $a = 0$, and $b = 2$. By Theorem 1.2, we have infinite homology for $a = 0$ and $b = 2$ if and only if $k \geq (b - 1)(a + b - 1) = a + 1 = 1$. The remaining case is $(k,a,b) = (0,0,2) \iff (m,n,d) = (5,5,2)$, in which case we have nonzero homology by Theorem 3.1.
- $k \geq 0$, $a \geq 0$, and $b = 1$. This time, Theorem 1.2 yields infinite homology for $a \geq 0$ and $b = 1$ as soon as $k \geq 0$.
- $k \geq 2 - a$, $a \geq 1$, and $b = 0$. By yet another application of Theorem 1.2, we have infinite homology for $b = 0$ whenever $a \geq 1$, $k \geq 1 - a$, and $k + a \geq 2$. Since the third inequality implies the second, we are done. □

**Conjecture 4.7** (Shareshian & Wachs [20]). For $1 \leq m \leq n$, the group $\tilde{H}_d(M_{m,n}; \mathbb{Z})$ contains 3-torsion if and only if

$$\begin{cases} m \leq n \leq 2m - 5 \\ \left\lceil \frac{m+n-4}{3} \right\rceil \leq d \leq m-3 \end{cases} \iff \begin{cases} k \geq 0 \\ a \geq 0 \\ b \geq 2. \end{cases}$$

Note that Conjecture 4.7 implies Conjecture 3.3. Conjecture 4.7 remains unsettled in the following cases:
\begin{itemize}
    \item $d = m - 2$: $9 \leq m + 2 \leq n \leq 2m - 3$. Equivalently, $k \geq 1$, $a \geq 2$, and $b = 1$. Conjecture: There is no 3-torsion.
    \item $d = m - 3$: $8 \leq m = n$. Equivalently, $k \geq 3$, $a = 0$, and $b = 2$. Conjecture: There is 3-torsion.
\end{itemize}

The conjecture is fully settled for $n = m + 1$ and $n \geq 2m - 2$; see Shareshian and Wachs [20] for the case $n = 2m - 2$ and use Theorem 1.1 for the case $n \geq 2m - 1$. For the case $n = m + 1$, we have that $\tilde{H}_{m-2}(M_{m,m+1}; \mathbb{Z})$ is torsion-free, because $M_{m,m+1}$ is an orientable pseudomanifold; see Spanier [21, Ex. 4.E.2].

4.2. Bounds on the homology over $\mathbb{Z}_3$. Fix a field $\mathbb{F}$ and let
\[
\beta^m_{d,n} = \dim \mathbb{F} \tilde{H}_d(M_{m,n}; \mathbb{F}); \quad \alpha^m_{d,n} = \dim \mathbb{F} \tilde{H}_d(\Gamma_{m,n}; \mathbb{F});
\]
$\Gamma_{m,n}$ is defined as in (2).

**Lemma 4.8.** For each $m \geq 2$ and $n \geq 3$, we have that
\[
\beta^m_{d,n} \leq \beta^{m-2,n-1}_{d-1} + (m-2)\beta^{m-1,n-1}_{d-1} + 2\binom{n-1}{2} \beta^{m-2,n-3}_{d-2}.
\]
Thus, by symmetry,
\[
\beta^m_{d,n} \leq \beta^{m-1,n-2}_{d-1} + (n-2)\beta^{m-1,n-1}_{d-1} + 2\binom{m-1}{2} \beta^{m-3,n-2}_{d-2},
\]
whenever $m \geq 3$ and $n \geq 2$.

**Proof.** By the long exact 00-G-11 sequence in Section 2.3, we have that
\[
\beta^m_{d,n} \leq \alpha^m_{d,n} + (m-2)\beta^{m-1,n-1}_{d-1}.
\]
Moreover, the long exact $\Gamma$-21-23 sequence in Section 2.4 yields the inequality
\[
\alpha^m_{d,n} \leq \beta^{m-2,n-1}_{d-1} + 2\binom{n-1}{2} \beta^{m-2,n-3}_{d-2}.
\]
Summing, we obtain the desired inequality. \hfill \Box

Define $\hat{\beta}_{k,a,b}^m = \beta_{d,n}^m$, where $k$, $a$, and $b$ are defined as in (1). We may rewrite the second inequality in Lemma 4.8 as follows:

**Corollary 4.9.** We have that
\[
\hat{\beta}_{k,a,b}^m \leq \hat{\beta}_{k+1,a+b-3}^{n-1} + (k+2a+3b-3)\hat{\beta}_{k-1}^{n-1} + 2\binom{k+a+3b-2}{2} \hat{\beta}_{k-1}^{n-1}.
\]
for $k \geq 0$, $a \geq 0$, and $b \geq 2$.

**Theorem 4.10.** With $\mathbb{F} = \mathbb{Z}_3$ and $d = \nu_{m,n}$, the second bound in Lemma 4.8 is sharp whenever $m \leq n \leq 2m - 5$, $m + n \equiv 1 \pmod{3}$, and $(m,n) \neq (5,5)$. Equivalently, $k = 0$, $a \geq 0$, $b \geq 2$, and $(k,a,b) \neq (0,0,2)$, where $k$, $a$, and $b$ are defined as in (1).
Proof. Since $\hat{\beta}_0^{a,b} = 1$ for $a \geq 0$ and $b \geq 2$ by Theorem 3.2, it suffices to prove that

$$\hat{\beta}_0^{a-1,b} + (2a + 3b - 3)\hat{\beta}_1^{a,b} + 2\left(a + 3b - 2\right)\hat{\beta}_1^{a+1,b-1} = 1$$

for all $a$ and $b$ as in the theorem; apply Corollary 4.9. Clearly, $\hat{\beta}_0^{a-1,b} = 1$; when $a = 0$, use the fact that $\hat{\beta}_0^{a-1,b} = \hat{\beta}_1^{a,b-1}$. Moreover, Theorem 1.1 yields that $\hat{\beta}_1^{a,b} = 0$ whenever $a \geq 0$ and $b \geq 1$. As a consequence, we are done. □

Theorem 4.11. For each $k \geq 0$, there is a polynomial $f_k(a, b)$ of degree $3k$ such that $\hat{\beta}_k^{a,b} \leq f_k(a, b)$ whenever $a \geq 0$ and $b \geq k + 2$ and such that

$$f_k(a, b) = \frac{1}{3^k k!} \left( (a + 3b)^3 - 9b^3 \right)^k + \epsilon_k(a, b)$$

for some polynomial $\epsilon_k(a, b)$ of degree at most $3k - 1$. Equivalently,

$$\beta_m^{m,n} \leq f_{3d-m-n+4}(n - m, m - d - 1)$$

for $m \leq n \leq 2m - 5$ and $\frac{m+n-4}{3} \leq d \leq \frac{2m+n-7}{4}$.

Proof. The case $k = 0$ is a consequence of Theorem 3.2. Assume that $k \geq 1$ and $b > k + 2$.

First, assume that $a > 0$. Induction and Corollary 4.9 yield that

$$\hat{\beta}_k^{a,b} - \hat{\beta}_k^{a-1,b} \leq (k + 2a + 3b - 3)f_{k-1}(a, b) + 2\left(k+a+3b-2\right)f_{k-1}(a+1, b-1),$$

where $f_{k-1}$ is a polynomial with properties as in the theorem. The right-hand side is of the form

$$g_k(a, b) = \frac{1}{3^{k-1}(k-1)!} \left( (a + 3b)^3 - 9b^3 \right)^{k-1} (a + 3b)^2 + h_k(a, b),$$

where $h_k(a, b)$ is a polynomial of degree at most $3k - 2$. Now,

$$= \frac{1}{3^{k-1}(k-1)!} \sum_{\ell=0}^{k-1} \binom{k-1}{\ell} (a + 3b)^{3k-3\ell-1}(-9b^3)^\ell.$$

Summing over $a$, we obtain that

$$\hat{\beta}_k^{a,b} \leq \hat{\beta}_k^{0,b} + \sum_{i=1}^{a} g_k(i, b).$$
The right-hand side is a polynomial in $a$ and $b$ with dominating term
\[
\frac{1}{3^{k-1}(k-1)!} \sum_{\ell=0}^{k-1} \binom{k-1-1}{\ell} \frac{(a+3b)^{3k-3\ell} - (3b)^{3k-3\ell}}{3k-3\ell} (-9b^3)^\ell
\]

\[= \frac{1}{3^k k!} \sum_{\ell=0}^{k} \binom{k}{\ell} \left((a+b)^3\right)^{k-\ell} \left((27b^3)^{k-\ell}\right) (-9b^3)^\ell
\]

(8) \[= \frac{1}{3^k k!} \left((a+b)^3 - 9b^3\right)^k - \frac{1}{3^k k!} (18b^3)^k.
\]

Proceeding with $\beta_k^{0, b}$ for $b \geq k + 3$, note that $\beta_{k-1}^{0, b} = \beta_{k-1}^{1, b-1}$. As a consequence,
\[
\beta_k^{0, b} \leq \beta_{k-1}^{1, b-1} + (k + 3b - 3)\beta_{k-1}^{0, b} + 2\left(\frac{k+3b-2}{2}\right)\beta^{1, b-1}_{k-1}
\]
\[\leq \beta_k^{0, b} + (k + 3b - 4)\beta_{k-1}^{1, b-1} + 2\left(\frac{k+3b-4}{2}\right)\beta_k^{2, b-2}
\]
\[+ (k + 3b - 3)\beta_k^{0, b} + 2\left(\frac{k+3b-2}{2}\right)\beta^{1, b-1}_{k-1}.
\]

Using induction, we conclude that
\[
\beta_k^{0, b} \leq \beta_k^{0, b-1} + 9b^2 f_{k-1}(2, b - 2) + 9b^2 f_{k-1}(1, b - 1) + O(b^{3k-2})
\]
\[= 18b^2 \left(\frac{18b^3)^k-1}{3k-1(k-1)!}\right) + O(b^{3k-2}) = \frac{18k b^{3k-1}}{3k-1(k-1)!} + O(b^{3k-2}),
\]
where $f_{k-1}$ is a polynomial with properties as in the theorem. Summing over $b$, we may conclude that $\beta_k^{0, b}$ is bounded by a polynomial in $b$ with dominating term $\frac{18k b^{3k}}{3k-1 k!}$. Combined with (8), this yields the theorem.

\[\square\]

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Table 1. The exponent $\epsilon_{m,n}$ of $\hat{H}_{m,n}(M_{m,n}; \mathbb{Z})$ for $m \leq n \leq 2m - 5$ and $(m, n) \not\in \{(6, 6), (7, 7), (8, 9)\}$. On the right we give the values $k$, $a$, and $b$ defined as in (1).

| $2m - n$ | Restriction | $\epsilon_{m,n}$ | $k$ | $a$ | $b$ |
|----------|-------------|-----------------|-----|-----|-----|
| 5        |             | 3               | 0   | $\geq 0$ | 2   |
| 6        | $m \geq 7$ | divides $\epsilon_{7,8}$ | 1   | $\geq 1$ |     |
| 7        | $m \geq 9$ | divides $\epsilon_{9,11}$ | 2   | $\geq 2$ |     |
| 8        |             | 3               | 0   | $\geq 0$ | 3   |
| 9        |             | divides $\gcd(9, \epsilon_{9,9})$ | 1   | $\geq 0$ |     |
| 10       | $m = 10$   | multiple of 3   | 2   | $= 0$    |     |
|          | $m \geq 11$| divides $\epsilon_{7,8}$ |     | $\geq 1$ |     |
| 11 + 3t  | $t \geq 0$ | 3               | 0   | $\geq 0$ | 4 + $t$ |
| 12 + 3t  |             | divides $\gcd(9, \epsilon_{9,9})$ | 1   |     | 2   |
| 13 + 3t  |             |                 |     |     |     |

Department of Mathematics, KTH, 10044 Stockholm, Sweden
E-mail address: jakobj@math.kth.se