Analysis of Some Singular Solutions in Fluid Dynamics

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Abstract

Studies on singular flows in which either the velocity fields or the vorticity fields change dramatically on small regions are of considerable interests in both the mathematical theory and applications. Important examples of such flows include supersonic shock waves, boundary layers, and motions of vortex sheets, whose studies pose many outstanding challenges in both theoretical and numerical analysis. The aim of this talk is to discuss some of the key issues in studying such flows and to present some recent progress. First we deal with a supersonic flow past a perturbed cone, and prove the global existence of a shock wave for the stationary supersonic gas flow past an infinite curved and symmetric cone. For a general perturbed cone, a local existence theory for both steady and unsteady is also established. We then present a result on global existence and uniqueness of weak solutions to the 2-D Prandtl’s system for unsteady boundary layers. Finally, we will discuss some new results on the analysis of the vortex sheets motions which include the existence of 2-D vortex sheets with reflection symmetry; and no energy concentration for steady 3-D axisymmetric vortex sheets.

2000 Mathematics Subject Classification: 35L70, 35L65, 76N15.
Keywords and Phrases: Singular flows, Shock waves, Vortox sheets, Prandtl’s system, Boundary layers.

1. Introduction

Many physically interesting phenomena involve evolutions of singular flows whose velocity fields or vorticity fields change dramatically. Shock waves, vortex sheets, and boundary layer are some of the well-known examples of such singular flows which provide better approximations for significant parts of the flow fields

*Supported in part by grants from the Research Grants Council of Hong Kong Special Administrative Region CUHK 4219/99D and CUHK 4279/00P.
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near the physical boundaries, in the mixing layers and trailing wakes, etc., at high Reynolds numbers. The better understanding of the dynamics of such singular flows is the key in the analysis of general fluid flows governed by the well-known Euler (or Navier-Stokes) systems or their variants for both compressible and incompressible fluids, and has been one of the main focuses for applied analysts for decades. Substantial progress has been made in the past in studying singular flows by either rigorous analysis, or numerical simulations, or asymptotic methods [18]. In particular, a rather complete theory exists for the 1-D shock wave problems, and both the theoretical understanding and numerical methods for 2-D smooth incompressible flows are quite satisfactory. Yet there are still many important issues to be settled, such as, the global existence of weak solutions and asymptotic structures of various approximate solutions generated by either physical considerations or numerical methods for such singular flows.

One of the fundamental problems in the mathematical theory of shock waves for hyperbolic conservation laws is the well-posedness of the multi-dimensional gas-dynamical shock waves, for which the celebrated Glimm’s method does not apply. Most of the previous studies along this line deal with either short time structural stability of basic fronts or asymptotic analysis and numerical simulations of dynamics of shock fronts. Due to the great complexity and the lack of understanding, it is reasonable for one to begin with some of the physically relevant wave patterns where a lot of experimental data, numerical simulations, and asymptotic results are available. One of the basic model for such studies is the supersonic flow past a pointed body [4], which is one of the fundamental problems in gas dynamics. We first study the stationary supersonic gas flow past an infinite curved and symmetric cone. The flow is governed by the potential equation as well as the boundary conditions on the shock and the surface of the body. This problem has been studied extensively by either physical experiments or numerical simulations. The rigorous analysis starts with the work of Courant and Friedrichs in [4], where they show that if a supersonic flow hits a circular cone with axis being parallel to the velocity of the upstream flow and the vertex angle being less than a critical value, then there appears a circular conical shock attached at the tip of the cone, and the flow field between the shock front and the surface of the body can be determined by solving a boundary value problem of a system of ordinary differential equations. The local existence of supersonic flow past a pointed body has been established recently [1]. Our interest is on the structure of the global solution to this problem. We show that the solution to this problem exists globally in the whole space with a pointed curved shock attached at the tip of the cone and tends to a self-similar solution [2]. Our analysis is based on a global uniform weighted energy estimate for the linearized problem. The method we developed in [2] seems to be quite effective for other multi-dimensional problems. Indeed, similar approach can be used to study unsteady supersonic flows past a curved body for both potential flows and the full Euler system, and we obtain the local existence of shock waves in these cases [3].

Another challenging problem in the mathematical theory of fluid-dynamics is the theoretical foundation of the Prandtl’s boundary layer theory [14]. In the presence of physical boundaries, the solutions to the inviscid Euler system cannot be
the uniform asymptotic ansatz of the corresponding Navier-Stokes system for large Reynolds number due to the discrepancies between the no-slip boundary conditions for the Navier-Stokes system and the slip boundary condition for the Euler system. Indeed, the physical boundaries will create vorticity and there is a thin layer (called boundary layer) in which the leading order approximation of the flow velocity is governed by the Prandtl’s system for the boundary layers (see (3.1) in Section 3). There are extensive literatures on the theoretical, numerical, and experimental aspects of the Prandtl’s boundary layer theory [14]. Yet very little rigorous theory exists for the dynamical boundary layer behavior of the Navier-Stokes solutions for both compressible and incompressible fluids. One of the main difficulties is the well-posedness theory in some standard Hölder or Sobolov spaces for the initial-boundary value problems for the Prandtl’s system for boundary layers which is a severely degenerate parabolic-elliptic system, for which the only known existence results are proved locally in the analytic class in [15], except the series of important works of Oleinik who dealt with a class of monotonic data [12]. Indeed, Oleinik considered a plane unsteady flow of viscous incompressible fluid in the presence of an arbitrary injection and removal of the fluid across the boundary. Under the monotonicity assumption (see (3.5) in Section 3), Oleinik proved the well-posedness of local classical solutions to the Prandtl’s system [12]. One of the open questions posed in [12] is to prove the global well-posedness of solution for the Prandtl’s system under suitable conditions. Recently, we establish a global existence and uniqueness of weak solutions to the 2-D Prandtl’s system for unsteady boundary layers in the class considered by Oleinik provided that the pressure is favorable. This is achieved by introducing a viscous splitting method and new weighted total variation estimate [16,17]. See Section 3 and [16,17] for more details.

Finally, we turn to the motion of vortex sheets, which corresponds to a singular inviscid flow where the vorticity field is zero except on lower-dimensional surfaces, the sheets, and can be characterized as inviscid flows with finite local energy and with vorticity fields being finite Radon measures. The study on the existence and structure of solutions for the inviscid Euler system for incompressible fluids with data in such class is of fundamental importance both physically and mathematically. Physically, vortex sheets can be used to model important flows such as high Reynolds number shear layers, and have many engineering applications. Mathematically, the evolution of a vortex sheets gives a classical example of ill-posed problem in the sense of Hadamard, a curvature singularity develops in finite time, and the nature of the solution past singularity formation is of great interest to know. This gives rise to many interesting yet challenging problems. Some of these are: Is there a well-posedness theory of classical weak solutions to the inviscid Euler system with general vortex sheets initial data? What are the structures of the approximate solutions to vortex sheets motions generated by either Navier-Stokes solutions or practical numerical methods (such as particle method)? Can vorticity concentration and energy defects occur dynamically? etc.. Despite the importance of these problems and past intensive effort in rigorous mathematical analysis, these problems are far from being solved. Better understanding has been achieved before singularity formation in the analytical setting, and studies on global weak solutions and their approxima-
tions start with the important works of Diperna-Majda [6]. Delort observes that no vorticity concentration implies that a weak limit in $L^2$ of an approximate solution sequences is in fact a classical weak solution to the two-dimensional Euler system, and thus proved the first existence of global (in time) classical weak solution to the 2-D incompressible Euler equations with vortex sheets initial data provided that the initial vorticity is of distinguished sign [5]. Similar ideas have been used to study the convergence of approximate solutions generated by either viscous regularization [11] or partical methods [8,9] for vortex sheets with one sign vorticity. In the case that vorticity may change sign, the vortex sheets motion becomes extremely complex after singularity formation. Indeed, many important features of irregular flows seem to be connected with interactions and intertwining of regions of both positive and negative vorticities. Here we consider a mirror-symmetric flow which allows interactions but excludes intertwining of regions of distinguish vorticity, prove that there is no vorticity concentrations for approximate solution of such flow, and thus give the first global (in time) existence of vortex sheets motion with two-sign vorticity. We also consider the 3-D axisymmetric vortex motions. It is well-known that for smooth flows, the analysis for the axisymmetric 3-D Euler system without swirls is almost identical to that of 2-D incompressible Euler system. However, this parallelness breaks down for the vortex sheets motions. Indeed, we will show that there exist no energy concentrations in approximate solutions to vortex sheets motion with one-sign vorticity for steady axisymmetric 3-D Euler system without swirls [7]. Some partial results on unsteady axisymmetric vortex sheets motion will also be discussed.

2. The supersonic flow past a pointed body

A projectile moving in the air with supersonic speed, is governed by the inviscid compressible Euler systems
\[
\begin{aligned}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) + \nabla p &= 0,
\end{aligned}
\]
where $\rho, u = (u_1, u_2, u_3)$ and $p$ stand for the density, the velocity and the pressure respectively. We will only treat the polytropic gases so that $p = p(\rho) = A \rho^\gamma$ with gas constant $A > 0$ and $1 < \gamma < 3$, $\gamma$ being the adiabatic exponent.

Suppose that there is a uniform supersonic flow $(u_1, u_2, u_3) = (0, 0, q_0)$ with constant density $\rho_0 > 0$ which comes from negative infinity. Then the flow can be described by the steady Euler system. If we assume further that the flow is irrotational, so that one can introduce a potential function $\Phi$ such that $u = \nabla \Phi$. Then the Bernoulli’s law implies that $\rho = h^{-1} \left( \frac{1}{2} q_0^2 + h(\rho_0) - \frac{1}{2} |\nabla \Phi|^2 \right) \equiv H(\nabla \Phi)$, where $h(\rho)$ is the specific enthalpy defined by $h'(\rho) = \frac{p'(\rho)}{\rho}$. In this case, (2.1) is reduced to a second order quasilinear equation
\[
\text{div}(H(\nabla \Phi) \nabla \Phi) = 0,
\]
which can be verified to be strictly hyperbolic with respect to $x_3$ if $\partial_3 \Phi > c$ with $c$ being the sound speed given by $c^2(\rho) = p'(\rho)$. The flow hits a point body, whose
surface is denoted by \( m(x_1, x_2, x_3) = 0 \). Since no flow can cross the boundary, the natural boundary condition is
\[
u \cdot \nabla m \equiv \nabla \Phi \cdot \nabla m = 0 \quad \text{on} \quad m(x_1, x_2, x_3) = 0.
\] (2.3)

If the vertex angle of the tangential cone of the pointed body is less than a critical value, it is then expected that a shock front is attached at the tip of the pointed body. Denote by \( \mu(x_1, x_2, x_3) = 0 \) the equation of the shock front, then the Rankine-Hugoniot conditions become
\[
\nabla \mu \cdot [H(\nabla \Phi) \nabla \Phi] = 0, \quad \text{on} \quad \mu(x_1, x_2, x_3) = 0.
\] (2.4)

Our aim is to find a solution to this free boundary value problem, (2.2)–(2.4). When the pointed body is small perturbation of a circular cone, the local existence of solution to the problem (2.2)–(2.4) has been established in [1]. Our main goal is to establish a global solution. However, such a global shock wave might not exist in general for arbitrary pointed body due to the possibility of development of new shock waves in the large. Thus we assume further that the pointed body is a curved and symmetric cone. In this case, it will be more convenient to rewrite the problem (2.2)–(2.4) in terms of polar coordinates \((r, \theta, z)\) with
\[
r = \sqrt{x_1^2 + x_2^2}, \quad z = x_3.
\]
Assume that the tip of the pointed body locates at the origin, the equation of the surface of the body is \( r = b(z) \) with \( b(0) = 0 \), and the equation of the shock front is \( r = S(z) \) with \( S(0) = 0 \). Set \( \Phi = q_0 z + \varphi(r, z) \). Then (2.2)–(2.4) become
\[
\left( (q_0 + \partial_z \varphi)^2 - c^2 \right) \partial_{zz} \varphi + \left( (\partial_r \varphi)^2 - c^2 \right) \partial_{rr} \varphi + 2 \partial_r \varphi (q_0 + \partial_z \varphi) \partial_{rz} \varphi - \frac{c^2}{r} \partial_r \varphi = 0,
\] (2.5)
\[
-(q_0 + \partial_z \varphi) b^\prime (z) + \partial_r \varphi = 0 \quad \text{on} \quad r = b(z),
\] (2.6)
\[
-[(q_0 + \partial_z \varphi) H] S^\prime (z) + [\partial_r \varphi H] = 0 \quad \text{on} \quad r = S(z).
\] (2.7)

Moreover, the potential \( \varphi(r, z) \) is continuous on the shock, so it should satisfy \( \varphi(S(z), z) = 0 \). Then in [2], we have shown that the problem (2.5)–(2.7) has a globally defined solution as summarized in the following theorem:

**Theorem 2.1** Assume that a curved and symmetric cone is given such that \( b(0) = 0 \), \( b(0) = b_0 \), \( b(k) (0) = 0 \), \( 2 \leq k \leq k_1 \), and
\[
|z^k \frac{d^k}{dz^k} (b(z) - b_0 z) - \varepsilon_0 | \leq \varepsilon_0 \quad \text{for} \quad 0 \leq k \leq k_2, \quad z > 0
\] (2.8)
with \( k_1 \) and \( k_2 \) being some suitable integers. Suppose that a supersonic polytropic flow parallel to the \( z \)-axis comes from negative infinite with velocity \( q_0 \), and density \( \rho_0 > 0 \). Then for suitably small \( \varepsilon_0 \), \( b_0 \) and \( q_0^{-1} \), the boundary value problem (2.5)–(2.7) admits a global weak entropy solution with a pointed shock front attached at the origin. Moreover, the location of the shock front and the flow field between the shock and the surface of the body tend to the corresponding ones for the flow past the unperturbed circular cone \( r = b_0 z \) with the rate \( z^{-\frac{1}{4}} \).
It should be noted that there are no other discontinuities in our solution besides the main shock. Since the deviation of the surface of the body from that of a circular cone is sufficiently small (see (2.8)), any possible compression of the flow will be absorbed by the main shock. This is the mechanism to prevent the formation of any new shocks inside the flow field caused by the perturbation of the body. In particular, our results demonstrate that self-similar solution with a strong shock is structurally stable in a global sense. Indeed, the key element in the proof of Theorem 2.1 is to establish some global uniform weighted energy estimates for the linearized problem of (2.5)–(2.7) around the self-similar solution with a strong shock obtained when the pointed body is a circular cone. This is achieved by a deliberate choice of multipliers which must satisfy a system of ordinary differential inequalities with complicated coefficients due to the structure of the background self-similar solution and the requirement of obtaining global estimates independent of $z$ for the potential function and its derivatives on the boundary as well as its interior of a domain. When the projectile changes its speed, or it confronts some airstream, then the flow around the projectile will be time-dependent. Thus, we also consider the unsteady supersonic flow past a pointed body. Although our analysis applies to more general case [3], here we will only present the result for two-dimensional polytropic, unsteady, and irrotational flow past a curved wedge. For simplicity in presentation, we also assume that both the wedge and the perturbed incoming flow from infinity are symmetric about $x_1$-axis. Let $x_2 = b(x_1)$ with $b(0) = 0$ be the equation of the wedge, and $x_2 = S(t, x_1)$ with $S(t, x_1 = 0) = 0$ be the equation of the shock front. Then in terms of the velocity potential function $\phi$ (so that $u_1 = \partial_1 \phi$ and $u_2 = \partial_2 \phi$), we are looking for solutions to the following initial boundary value problem

$$
\partial_t \left(H(-\phi_t + \frac{1}{2}|\nabla \phi|^2)\right) + \sum_{i=1}^2 \partial_{x_i} \left(\partial_{x_i} \phi H(-\partial_t \phi - \frac{1}{2}|\nabla \phi|^2)\right) = 0,
$$

$$
t > 0, x_2 > b_1(x_1),
$$

$$
\nabla \phi \cdot (b(x_1), -1) = 0 \quad \text{on} \quad x_2 = b(x_1),
$$

$$
[H] \partial_t S + [H \partial_{x_1} \varphi] \partial_{x_1} S - [H \partial_{x_2} \phi] = 0 \quad \text{on} \quad x_2 = S(t, x_1),
$$

$$
S(0, x_1) = S_0(x_1), \quad \varphi(0, x_1, x_2) = \phi_0(x_1, x_2), \quad \partial_t \phi(0, x_1, x_2) = 0,
$$

where $S_0(x_1)$ and $\phi_0(x_1, x_2)$ are suitably small perturbations of the corresponding shock location and potential function respectively for the steady flow. Then we have the following local existence result [3].

**Theorem 2.2** There exist positive constant $\delta_1$ and $\delta_2$ and functions $S(t, x_1)$ and $\phi(t, x_1, x_2)$ defined on the regions $\{(t, x_1)| 0 < t < \delta_1, 0 < x_1 < \delta_2\}$ and $\{(t, x_1, x_2)| 0 < t < \delta_1, 0 < x_1 < \delta_2, b(x_1) < x_2 < S(t, x_1)\}$ respectively such that $(S(t, x_1),$ $\varphi(t, x_1, x_2))$ solves the problem (2.9)–(2.12).
3. Prandtl’s system for boundary layers

Consider a plane unsteady flow of viscous incompressible fluid in the presence of an arbitrary injection and removal of the fluid across the boundaries. In this case, the corresponding Prandtl’s system takes the form

\[
\partial_t u + u \partial_x u + v \partial_y u + \partial_x p = \nu \partial_{yy}^2 u, \quad \partial_x u + \partial_y v = 0,
\]

(3.1)
in the region, \( R = \{(x, y, t)|0 \leq x \leq L, \ 0 \leq y < +\infty, \ 0 < t < T\} \), where \( \nu, L \) and \( T \) are positive constants. The initial and boundary conditions can be imposed as

\[
\begin{align*}
&u|_{t=0} = u_0(x, y), \ u|_{x=0} = u_1(y, t), \ u|_{y=0} = 0, \ v|_{y=0} = v_0(x, t), \\
&\lim_{y \to +\infty} u(x, y, t) = U(x, t),
\end{align*}
\]

(3.2)

with \( U = U(x, t) \) given as is determined by the corresponding Euler flow. The pressure \( p = p(x, t) \) in (3.1) is determined by the Bernoulli’s law: \( \partial_t U + U \partial_x U + \partial_x p = 0 \).

It follows from the physical ground that one may assume that

\[
U(x, t) > 0, \ u_0(x, y) > 0, \ u_1(y, t) > 0, \text{ and } v_0(x, t) \leq 0.
\]

(3.3)

Due to the degeneracy in the Prandtl’s system (3.1), the problem of well-posedness theory of solutions to the problem (3.1)–(3.3) in the standard Hölder space or Sobolov space is quite difficult. In a series of important works by Oleinik and her coauthors [12], they studied this problem under the additional assumption that the data are monotonic in the sense that

\[
\partial_y u_0(x, y) > 0, \text{ and } \partial_y u_1(y, t) > 0,
\]

(3.4)

and prove that there exists a unique local classical smooth solution to the initial-boundary value problem (3.1)–(3.3) provided that the data are monotonic in the sense of (3.4). Here by local we mean that \( T \) is small if \( L \) is given and fixed, and \( T \) is arbitrary if \( L \) is small. One of the open problem in [12] is: What are the conditions ensuring the global in time existence and uniqueness of solutions to the problem (3.1)–(3.4) for arbitrarily given \( L \)? In [16,17], we study such problem and establish the global (in time) existence and uniqueness of weak solutions to the initial-boundary value problem (3.1)–(3.4) in the case that the pressure is favorable, i.e.,

\[
\partial_x p(x, t) \leq 0, \ t > 0, \ 0 < x < L.
\]

(3.5)

More precisely, we have ([16,17]):

**Theorem 3.1** Consider the initial-boundary value problem for the 2-D Prandtl’s system, (3.1)–(3.2). Assume that the initial and boundary data satisfy the constraints (3.3), (3.4) and (3.5). Then there exists a unique global bounded weak solutions to the initial-boundary value problem (3.1)–(3.2). Furthermore, these solutions are Lipschitz continuous in both space and time.
We remark here that the condition (3.5), that the pressure is favorable, is exactly what fluid-dynamists believe for the stability of a laminar boundary layer. This is also consistent with the case of stationary flows [12]. In the case of pressure adverse, i.e., \( \frac{\partial p}{\partial x} > 0 \), separation of boundary layer may occur, so one would not expect the long time existence of solution to (3.1)–(3.3). Finally, we remark that in the case that (3.5) fails as in many practical physical situations, short time existence of regular solution is still expected, which has not been established yet.

4. Vortex sheets motions

Two-dimensional vortex sheets motion corresponds to an inviscid flow whose vorticity is zero except on one-dimensional curves, the sheets. Thus, it is governed by the following Cauchy problem

\[
\begin{dcases}
\partial_t w + u \cdot \nabla w = 0, & \ u = K * w, \\
\ w(t = 0, x) = w_0(x) \in \mathcal{M}(\mathbb{R}^2) \cap H^{-1}_\text{loc}(\mathbb{R}^2),
\end{dcases}
\]  

where \( u = (u_1, u_2) \) is the velocity field, \( \omega = \nabla^\perp \cdot u \) is the vorticity, and \( K \) is the Biot-Sawart kernel, and \( \mathcal{M}(\mathbb{R}^2) \) denotes the space of finite Radon measures in \( \mathbb{R}^2 \). Let \( (w^\varepsilon, w^\varepsilon) \) be a sequence of approximate solution to (4.1) with the following bounds

\[
\sup_{0 \leq t \leq T} \int_{\mathbb{R}^2} |w^\varepsilon(x, t)|^2 dx \leq C_1(T), \quad \sup_{0 \leq t \leq T} \int_{|x| \leq R} |w^\varepsilon(x, t)|^2 dx \leq C_2(T, R). \tag{4.2}
\]

Then there exist \( u \in L^\infty((0, T); L^2_\text{loc}(\mathbb{R}^2)) \) and \( w \in L^\infty((0, T); \mathcal{M}(\mathbb{R}^2)) \) with \( w = \nabla^\perp \cdot u \) such that

\[
w^\varepsilon \rightharpoonup w \quad \text{in} \quad \mathcal{M}([0, T] \times \mathbb{R}^2), \quad w^\varepsilon \rightharpoonup u \quad \text{in} \quad L^2_\text{loc}([0, T] \times \mathbb{R}^2). \tag{4.3}
\]

Two main questions arise: Is \((u, w)\) a classical weak solution to problem (4.1)? Does either vorticity concentration (i.e. \( \lim_{\varepsilon, r \to 0^+} \int_{B_r} |w^\varepsilon(x, t)| dx > 0 \)) or energy defects (\( \lim_{\varepsilon \to 0} \int_{|x| \leq R} |w^\varepsilon(x, t)|^2 dx > \int_{|x| \leq R} |u(x, t)|^2 dx \) for some \( t > 0, \ R > 0 \)) occur? It is clear from the structure of the 2-D Euler system that no energy defects implies strong \( L^2 \)-convergence of the velocity field and thus the existence of the classical weak solution to (4.1). A less obvious fact is that no vorticity concentration also implies the weak limit \((u, w)\) being a classical solution to (4.1), which follows from the vorticity formulation of (4.1) as observed by Delort [5]. Then the convergence to a classical weak solution to the Cauchy problem (4.1) is proved for approximate solutions generated by either regularizing the initial data [5], or Navier-Stokes approximation [11], or vortex blob methods [8], or point vortex methods [9] provided that the initial vorticity is of distinguished sign. To study the corresponding issues for flows where interactions of regions of both positive and negative vorticities are allowed, in [10], we study a 2-D mirror-symmetric vortex sheets motion whose vorticity is an integrable perturbation of a non-negative mirror-symmetric radon measure. Here a Radon measure \( \mu \) is said to be non-negative...
mirror-symmetric (NMS) if \( \mu|_{2z} \geq 0 \) and \( \mu \) is odd with respect to \( x_1 = 0 \). Then the main results in [10] can be summarized as

**Theorem 4.1** Assume that \( w_0 \equiv \mu_1 + \mu_2 \) such that \( \mu_1 \in \mathcal{M}_c(\mathbb{R}^2) \cap H^{-1}_c(\mathbb{R}^2) \cap \) NMS and \( \mu_2 \in L^1_c(\mathbb{R}^2) \). Then there exists a global (in time) classical solution \( (u, w) \in L^\infty (0, T, L^2(\mathbb{R}^2)) \otimes L^\infty (0, T, \mathcal{M}(\mathbb{R}^2)) \) to the initial value problem (4.1). Furthermore, this weak solution can be obtained as a limit of either a sequence of smooth inviscid solutions or a sequence of solutions to the Navier-Stokes system.

It should be noted that this is the only result of existence of classical weak solution to (4.1) involving vorticities with different signs. This is proved by showing

\[
\int_0^T \sup_{x_0 \in \mathbb{R}^2} \int_{B(x_0, \delta)} |w^\varepsilon(y, t)| dy \, dt \to 0 \quad \text{as} \quad \delta \to 0^+ \quad \text{uniformly in} \quad \varepsilon, \quad \text{i.e., no vorticity concentration occur any where.}
\]

Theorem 4.1 also indicates that interactions of regions of positive and negative vorticities without interwining may not cause concentration in vorticity. There remain many important open problems for the 2-D vortex sheets motion such as the existence of classical weak solution to (4.1) for general vortex sheets initial data, and whether energy defects occur dynamically even in the case of one-sign vorticity.

Finally, we consider 3-D axisymmetric vortex sheets motions. In cylindrical coordinate, \((r, \theta, z)\), axisymmetric solutions of 3-D Euler system have the form

\[
u(x, t) = u^\varepsilon(r, z, t) e_r + u^\theta(r, z, t) e_\theta + u^z(r, z, t) e_z, \quad p(x, t) = p(r, z, t) \quad (4.4)
\]

where \( e_r = (\cos \theta, \sin \theta, 0) \), \( e_\theta = (-\sin \theta, \cos \theta, 0) \), and \( e_z = (0, 0, 1) \). The axisymmetric flow is said without swirls if \( u^\theta \equiv 0 \). In this case, the vorticity field is given by \( w = \nabla \times u = w^\theta e^\theta \) with \( w^\theta = \partial_r u^z - \partial_z u^r \), and \( \partial_t (r^{-1} u^\theta) = 0 \) with \( \partial_t = \partial_r + u^r \partial_r + u^z \partial_z \). Thus, for smooth data, the theory for 3-D axisymmetric Euler system without swirls is almost parallel to that of 2-D Euler equation. However, this similarity breaks down for vortex sheets motions. Indeed, in sharp contrast to the 2-D case, we show in [7] that there are no energy defects for a suitable sequence of approximate solutions for 3-D steady axisymmetric vortex sheets motion with one-signed vorticity. Precisely, we have

**Theorem 4.2** Let \((u^\varepsilon, p^\varepsilon)\) be smooth axisymmetric solution to the 3-D steady Euler system: \((u^\varepsilon \cdot \nabla)u^\varepsilon + \nabla p^\varepsilon = f^\varepsilon, \quad \text{div} \, u^\varepsilon = 0, \quad x \in \mathbb{R}^3\), for some given axisymmetric function \( f^\varepsilon \) with \( f^\varepsilon \to f \) weakly in \( L^1(\mathbb{R}^3) \). Suppose further that

\[
(w^\varepsilon)^\theta \geq 0, \quad \sup_{\varepsilon} \int_{\mathbb{R}^3} |w^\varepsilon| \, dx < +\infty, \quad \sup_{\varepsilon} \int_{\mathbb{R}^3} |u^\varepsilon|^2 \, dx < +\infty \quad (4.5)
\]

where \( w^\varepsilon = \nabla \times u^\varepsilon = (w^\varepsilon)^\theta e_\theta \). Let \( u \) be the weak limit of \( u^\varepsilon \) in \( L^2(\mathbb{R}^3) \). Then \( u \) is a classical weak solution to \((u \cdot \nabla)u + \nabla p = f, \quad \text{div} \, u = 0, \quad x \in \mathbb{R}^3\). Moreover, there exists a subsequence \( \{u^\varepsilon_j\} \) of \( \{u^\varepsilon\} \) such that \( u^\varepsilon_j \) converges to \( u \) strongly in \( L^2(\mathbb{R}^3) \).

The proof of this theorem is based on a shielding method and the following fact which is valid for both steady and unsteady flows [7].
Theorem 4.3 Let \( \{u^\varepsilon\} \) be a sequence of approximate solutions for 3-D axisymmetric Euler system with general vortex-sheets data generated by either smoothing the initial data or Navier-Stokes approximations. Let \( A \) be the support of the defect measure associated with \( u^\varepsilon \in L^2([0,T] \times \mathbb{R}^3) \). If \( A \neq \emptyset \), then \( A \cap \{(x,t) \in \mathbb{R}^3 \times [0,T] \mid r > 0\} \neq \emptyset \).

Theorem 4.3 implies that if there are no energy defects away from the symmetry axis, then strong \( L^2 \)-convergence takes place. It remains to study whether energy defects occur for unsteady axisymmetric 3-D Euler equations.

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