An edge CLT for the log determinant of Gaussian ensembles

Iain M. Johnstone, Yegor Klochkov, Alexei Onatski,
and Damian Pavlyshyn

November 30, 2020

Abstract

We derive a Central Limit Theorem (CLT) for log \[ \left| \det \left( \frac{M_N}{\sqrt{N}} - 2\theta_N \right) \right|, \]
where \( M_N \) is from the Gaussian Unitary or Gaussian Orthogonal Ensemble (GUE and GOE), and \( 2\theta_N \) is local to the edge of the semicircle law. Precisely, \( 2\theta_N = 2 + N^{-2/3}\sigma_N \) with \( \sigma_N \) being either a constant (possibly negative), or a sequence of positive real numbers, slowly diverging to infinity so that \( \sigma_N \ll \log^2 N \). For slowly growing \( \sigma_N \), our proofs hold for general Gaussian \( \beta \)-ensembles. We also extend our CLT to cover spiked GUE and GOE.

1 Introduction

In this paper we derive a CLT for the log determinant

\[ \log |D_N| \equiv \log \left| \det \left( \frac{M_N}{\sqrt{N}} - 2\theta_N \right) \right|, \]  
(1)

where the \( N \)-dimensional matrix \( M_N \) belongs to the GUE or GOE. Dividing by \( \sqrt{N} \) makes the support of the limiting spectral distribution \([-2, 2]\). We consider the values of the shift parameter \( 2\theta_N \) that are local to the upper boundary of the support in the sense that

\[ 2\theta_N = 2 + N^{-2/3}\sigma_N, \]  
(2)

where \( \sigma_N \) is either an arbitrary constant or a sequence of positive real numbers, slowly diverging to infinity so that \( \sigma_N \ll \log^2 N \). Thus, we refer to (1) as the “log determinant” or “log statistic” at the edge, and to \( 2\theta_N \) as the “location of the singularity”.

Starting in the first half of the 20th century, much of the research on determinants of random matrices have been devoted to matrices with independent entries (see Nguyen et al. [2014] for a brief review). The interest in determinants of Hermitian matrices from GUE, GOE and other classical ensembles of Random Matrix Theory emerged in the 1960s from motivations in nuclear physics. The first published derivation\(^1\) of the joint distribution of the eigenvalues of GUE in Wigner [1965] was spurred by the problem of approximating the value of log \[ \left| \det \left( \frac{M_N}{\sqrt{N}} - s \right) \right|, \]
where \( s \in (-2, 2) \).

Our interest in such log determinants stems from earlier work of two of the present authors, Johnstone and Onatski [2020] and Dharmawansa et al. [2014], where such statistics play an important role in the asymptotic analysis of the likelihood ratios for testing problems in spiked high-dimensional statistical models. When the spikes are either below or above a critical level (situations studied in the two papers mentioned above), the singularity of the relevant log determinant lies away from an open set covering \([-2, 2]\). The CLT for log statistics with such singularity locations is well known (e.g. Johansson [1998]). However, for the critical regime, the singularity is local to 2 and the corresponding CLT has not been available.\(^2\) This paper derives such a CLT.

\(^1\)In a footnote on p. 447, Wigner [1965] points out that the joint distribution of the eigenvalues of GUE is “probably known to many readers”.

\(^2\)When this paper was close to completion, we learned about a recent work, Lambert and Paquette [2020a,b], that derives a related CLT. For a short discussion and comparison to our result, see Section 5.
Theorem 1. For the log statistic (1) with singularity (2) at the edge such that the local parameter \( \sigma_N \) is either fixed or slowly diverges to \( +\infty \) so that \( \sigma_N \ll \log^2 N \), we have

\[
\frac{\log |\mathcal{D}_N| - N/2 + \frac{\sigma_N}{\theta} \log N - \sigma_N N^{1/3} + \frac{3}{2} \sigma_N^{3/2}}{\sqrt{\frac{3}{2} \log N}} \to N(0, 1)
\]

where \( \alpha = 1 \) for \( M_N \) from the GUE and \( \alpha = 2 \) for \( M_N \) from the GOE.

Denoting the eigenvalues of \( M_N/\sqrt{N} \) as \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N \), we can represent \( \log |\mathcal{D}_N| \) in the form of the linear spectral statistic \( L_N(f) = \sum_{i=1}^{N} f(\lambda_i) = \sum_{i=1}^{N} \log |\lambda_i - 2\theta_N| \). In the typical CLT for linear spectral statistics, when \( f \) is analytic in a neighborhood of the bulk, which includes cases with \( 2\theta_N \) separated from the semicircle support, the asymptotic variance of \( L_N(f) \) is of constant order. Here, by contrast, the variance is of order \( \log N \).

Note that \( \sum_{i=1}^{N} \log |\lambda_i - 2\theta_N| \) can be interpreted as a deterministic shift of the statistic \( \sum_{i=1}^{N} \log (|\lambda_i - 2\theta_N|) \), where \( \{\eta_N\} \) is an arbitrary deterministic sequence. Statistics of the form \( \sum_{i=1}^{N} f \left((\lambda_i - 2\theta_N)/\eta_N\right) \) with \( N^{-2/3} \ll \eta_N \ll 1 \) and smooth \( f \) with rapid decay are called mesoscopic statistics, in this case at the spectral edge. Various CLT’s for such statistics are known. Basor and Widom [1999] is a relatively early contribution. For more recent work see Li et al. [2019] and references therein. Note however, that the logarithmic function has a singularity whose argument increases. In particular, the log statistic of interest is not local in the sense that both the “edge” and the “bulk” eigenvalues contribute non-trivially to its value. Hence, we cannot rely on the methods used in the derivations of mesoscopic CLTs.

Our proof of Theorem 1 is inspired by Tao and Vu [2012], who derive a CLT for the log determinant with singularity at zero via an elementary but ingenious approach. Their derivation is based on the equivalence [Trotter, 1984] of the joint distribution of the eigenvalues of \( M_N \) to that of

\[
\hat{M}_N = \begin{pmatrix} a_1 & b_1 & \cdots & \cdots \\ b_1 & a_2 & b_2 & \cdots \\ \cdots & \cdots & \cdots \cdots \\ b_{N-1} & a_{N-1} & b_{N-1} \\ b_{N-1} & b_{N-1} & a_N \end{pmatrix},
\]

where \( a_i \sim \mathcal{N}(0, \alpha) \) and \( b_i^2 \sim \chi^2(2i/\alpha)/(2/\alpha) \), \( i = 1, \ldots, N \), are jointly independent, and \( \alpha = 1 \) for GUE and \( \alpha = 2 \) for GOE. Here, by definition, \( \chi^2(d) \) has the density 

\[
\frac{1}{2^{d/2} \Gamma(d/2)} x^{d/2 - 1} e^{-x/2} 1_{x > 0}
\]

for arbitrary \( d > 0 \).

Denote the determinant of the \( i \)-th minor of \( \hat{M}_N \) as \( D_i \). Note that \( \mathcal{D}_N = N^{-N/2} \mathcal{D}_N \). Using the cofactor expansion yields

\[
D_i = a_i D_{i-1} - \left( i^2 - 1 \right) c_{i-1} D_{i-2},
\]

where \( c_i = (b_i^2 - i) / \sqrt{i} \), so that \( E c_i = 0 \) and \( \text{Var} (c_i) = \alpha \). To eliminate the \( i \)-factor, which makes the above dynamic equation explosive, Tao and Vu use the normalization \( E_i = D_i/\sqrt{i} \). Then, they study the dynamics of vector \((E_{2j}, E_{2j-1})\). We illustrate this dynamics in Figure 1 below.
Tao and Vu show that the logarithm of the length of vector \((E_{2j}, E_{2j-1})\) can be approximated by the sum of a martingale difference sequence, whereas the angle that the vector forms with the x-axis is uniformly distributed on \([0, 2\pi]\). These two results yield Tao and Vu’s CLT.

A naive extension of this method to non-zero locations of singularity does not work because the dynamics of \(E_i\) become unstable. Indeed, the analogue of (5) for \(M_{N} - 2\theta_N \sqrt{N}\) is

\[
D_i = (a_i - 2\theta_N \sqrt{N})D_{i-1} - \left(i - 1 + \sqrt{i - 1}c_{i-1}\right)D_{i-2},
\]

for \(i \geq 1\) with the conventions \(D_0 = 1, D_{-1} = 0\). A deterministic version of such a recursion \((a_i = c_{i-1} = 0)\) has explosive characteristic roots \(\rho_i^\pm\) with

\[
\rho_i^\pm = -\left(\theta_N \sqrt{N} \pm \sqrt{\theta_N^2 N - (i - 1)}\right),
\]

which are not eliminated by the scaling \(E_i = D_i/\sqrt{i!}\) because \(\theta_N \neq 0\).

We therefore propose a different normalization

\[
E_i = D_i/\prod_{j=1}^{i} |\rho_j^\pm|.
\]

This normalization achieves stability of the dynamics of vectors \((E_i, E_{i-1})\) as illustrated in Figure 2 for cases \(2\theta_N = 1\) (left panel) and \(2\theta_N = 2\) (right panel).
The elliptic ring cloud in the left panel corresponds to vectors $(E_i, E_{i-1})$ with $i > \theta^2_N N = N/4$. Note that the characteristic roots of the dynamic equation describing $E_i$ can be approximated by the pair $\pm \frac{\rho^+}{|\rho^+|}$. For $i > \theta^2_N N$, this is a complex-conjugated pair located on the unit circle in $\mathbb{C}$. In contrast, for $i \leq \theta^2_N N$, both characteristic roots are real, one equals minus one, and the other is moving from zero to minus one as $i$ goes from 1 to $\theta^2_N N$. The corresponding simulated points lie along the inward-pointing spikes clearly visible inside the ring. The right panel does not feature any elliptic ring because, for that panel, $\theta_N = 1$ so that there are no simulations with $i > \theta^2_N N$.

As pointed out by Fyodorov et al. [2016], a CLT for the GUE log statistic with singularity $2\theta_N$ from a compact subset of $(-2, 2)$, cf. Figure 2 left, can be easily obtained from Theorem 1 of Krasovsky [2007]. That theorem derives detailed asymptotics for the Laplace transform of the log statistic using Riemann-Hilbert machinery. Tao and Vu [2012] derive their CLT for $\theta_N = 0$ and for Wigner matrices with atom distributions that match the first four moments of the normal. Bourgade and Mody [2019] relax the conditions to require only matching of the first two moments. Duy [2017] describes a very elegant proof of such a CLT for GOE and GUE based on a representation of the corresponding log-determinants in the form of sums of independent random variables.

For log statistics at the edge, $\theta_N$ is local to one, so that $2\theta_N$ is not bound to a compact subset of $(-2, 2)$. This situation is illustrated by the right panel of Figure 2, and is the focus of our study. Qualitatively, for most $i$, $E_i$ and $E_{i-1}$ have opposite signs and are very similar in magnitude, so that $R_i = E_i / E_{i-1} + 1$ remains close to zero. However, for $i$ approaching $N$, $R_i$ starts to develop more excited dynamics, as illustrated by Figure 3.
Figure 3: Simulated $R_i = E_i/E_{i-1} + 1$, $i = 2, ..., 10000$. GUE case.

It may be of interest to note that $E_i/E_{i-1}$ can be interpreted as normalized Sturm ratio sequence of matrix $\hat{M}_N$. Sturm ratios play a useful role in the analysis of large random matrices (see e.g. Albrecht et al. [2009] or Section 1.9.3 in Forrester [2010]).

In Section 2, we show that as long as the local parameter of the singularity, $\sigma_N$, is slowly diverging to infinity so that $\sigma_N \gg (\log \log N)^2$, the dynamics of $R_i$, $i = 1, ..., N$, can be well approximated by a linear one. Then we use this linear approximation to obtain a CLT for the sums of the logarithms of the normalized Sturm ratios. This leads to Theorem 1 with $\sigma_N \gg (\log \log N)^2$. In fact, for such $\sigma_N$, our proof remains valid for matrices $\hat{M}_N$ from general Gaussian $\beta$-ensembles (with $\beta = 1/\alpha \in (0, \infty)$).

To extend the theorem to slower growing and constant $\sigma_N$, Section 3 derives simple asymptotic formulae for the Stieltjes transform of the empirical spectral distribution of $\hat{M}_N/\sqrt{N}$ and its derivative at the edge of the support $[-2, 2]$. These formulae and the Taylor expansion of the logarithm describe the asymptotic behavior of the log statistics at the edge with $\sigma_N \leq (\log \log N)^2$ in terms of that of the statistic with $\sigma_N \gg (\log \log N)^2$. Thus, we obtain Theorem 1 in its generality.

Our proof can be easily extended to cover spiked GUE and GOE matrices. Theorem 7 of Section 4 generalizes Theorem 1 for spiked GUE and GOE.

Before turning to the details of the derivations, we would like to point out that the log statistics (1) appears in an integral representation of the free energy of the spherical version of the Sherrington-Kirkpatrick model for a spin glass (Lemma 1.3 of Baik and Lee [2016]). Studying fluctuations of (1) with singularity local to 2 is essential for understanding the behavior of the free energy in the so-called critical temperature regime. Although the free energy fluctuations in the sub- and super-critical regime are known, the critical temperature transition is yet to be understood.

When this paper was near completion, we learned about related works by Lambert and Paquette [2020a,b]. These two articles obtain powerful asymptotic approximations to the logarithmic statistics for Gaussian $\beta$-ensembles, which respectively imply CLTs for the logarithmic statistic in two different regimes for the singularity. In our notation, these regimes correspond to constant $\sigma_N$ and $\sigma_N \geq C \log^{2/3} N$. Our results cover both regimes for GUE and GOE, and the
gap between them (though we do assume $\sigma_N \ll \log^2 N$). Our approach also uses two (different) regimes, described in Sections 2 and 3. The method we use is different from that of Lambert and Paquette, and is elementary in the region $(\log \log N)^2 \ll \sigma_N \ll (\log N)^2$ of Theorem 2. Some further discussion appears in Section 5.

## 2 The CLT slightly away from the edge

In this section we establish the following analogue of Theorem 1 for general Gaussian $\beta$-ensembles in cases where $(\log \log N)^2 \ll \sigma_N \ll \log^2 N$. Hence, the location of singularity $2\theta_N$ is slightly away from the edge in the sense that $(2\theta_N - 2)^{2/3}$ slowly diverges to infinity.

**Theorem 2.** Consider matrix $\hat{M}_N$ from a general Gaussian $\beta$-ensemble (4) with $\beta = 1/\alpha$. Let $D_N = \det(\hat{M}_N/\sqrt{N} - 2\theta_N)$, where $2\theta_N = 2 + N^{-2/3}\sigma_N$ with $(\log \log N)^2 \ll \sigma_N \ll (\log N)^2$. Then,

$$
\log |D_N| - N/2 + \frac{\alpha-1}{2} \log N - \frac{\sigma_N}{N^{1/3}} + \frac{3}{3} \frac{\sigma_N}{3/2} \sqrt{\frac{2}{\log \log N}} \xrightarrow{d} N(0, 1).
$$

**Remark 3.** The scaling in the above CLT naturally arises from the arguments in our proof. Note that

$$
\log \frac{\theta_N + \sqrt{\theta_N^2 - 1}}{2N} = \frac{1}{3} \log N - \frac{1}{2} \log \sigma_N - \log 2 + O(N^{-1/3}\sigma_N^{1/2}),
$$

so that the asymptotic variance of $\log |D_N|$ is $\frac{3}{2} \log N$, as in Theorem 1. However, our Monte Carlo experiments (which we do not report here) indicate that the scaling in Theorem 2 makes the standard normal approximation better in finite samples.

The proof of Theorem 2 is based entirely on the recurrence equation (6) with application of some well-known deviation and concentration inequalities for sums of independent random variables. In this section, we briefly outline the main steps of the proof. Details can be found in Appendix A.6.

Let

$$
r_i = 1 + \sqrt{1 - \frac{i - 1}{N\theta_N^2}}, \quad m_i = 1 - \sqrt{1 - \frac{i - 1}{N\theta_N^2}}.
$$

Then, from (6), the normalized determinants (7) follow the recurrence

$$
E_i = \left( \frac{a_i}{\sqrt{N\theta_N r_i}} - \frac{2}{r_i} \right) E_{i-1} - \left( \frac{i - 1 + \sqrt{i - 1} c_{i-1}}{N\theta_N^2 r_i r_{i-1}} \right) E_{i-2}.
$$

for $i \geq 1$, with the conventions $E_0 = 1, E_{-1} = 0$. Since $r_i + m_i = 2$ and $r_i m_i = (i - 1)/N\theta_N^2$, this becomes

$$
E_i = \left( \frac{a_i}{\sqrt{N\theta_N r_i}} - \frac{m_i}{r_i} \right) E_{i-1} - \left( \frac{m_i}{r_i} + \sqrt{\frac{m_i}{r_i} c_{i-1}} \right) E_{i-2}.
$$

Introducing new notation

$$
\alpha_i = \frac{a_i}{\sqrt{N\theta_N r_i}}, \quad \beta_i = \sqrt{\frac{m_i}{r_i} c_{i-1}}, \quad \gamma_i = \frac{m_i}{r_i}, \quad \delta_i = \frac{m_i}{r_i} - \frac{m_i}{r_{i-1}},
$$

we can rewrite (8) more compactly as

$$
E_i = (\alpha_i - \gamma_i - 1)E_{i-1} - (\gamma_i + \beta_i - \delta_i)E_{i-2}.
$$

Divide both sides of this equality by $E_{i-1}$, and define $R_i = E_i/E_{i-1} + 1$ for $i \geq 2$. This yields

$$
R_i = \alpha_i - \gamma_i + \frac{\gamma_i + \beta_i - \delta_i}{1 - R_{i-1}}.
$$

(9)
This can be written as

\[ R_i = \xi_i + \gamma_i R_{i-1} + \varepsilon_i, \tag{10} \]

which we use for \( i \geq 3 \), where

\[ \begin{align*}
\xi_i &= \alpha_i + \beta_i, \\
\varepsilon_i &= -\delta_i + (\beta_i - \delta_i) \frac{R_{i-1}}{1 - R_{i-1}} + \gamma_i \frac{R_{i-1}^2}{1 - R_{i-1}}.
\end{align*} \]

By dropping the non-linear term \( \varepsilon_i \) from (10), we now define a linear process \( \{L_i\}_{i=2}^{N} \) with \( L_2 = 0 \) and satisfying the recursion \( L_i = \xi_i + \gamma_i L_{i-1} \). Note that \( \{\xi_i\} \) are independent random variables, while \( \{\gamma_i\} \) are deterministic. Iterating this yields \( L_3 = \xi_3 \) and

\[ L_i = \xi_i + \gamma_i \xi_{i-1} + \cdots + \gamma_i \cdots \gamma_3 \xi_3, \quad i \geq 4, \tag{11} \]

and a similar iteration of the recursion for \( R_i \) yields \( R_3 = L_3 + \varepsilon_3 + \gamma_3 R_2 \) and

\[ R_i = L_i + \varepsilon_i + \gamma_i \varepsilon_{i-1} + \cdots + \gamma_i \cdots \gamma_3 \varepsilon_3 + \gamma_i \cdots \gamma_3 R_2 \quad i \geq 4. \tag{12} \]

To establish the CLT, we study the dynamics of \( L_i \) and \( R_i \). Our proof consists of the following three steps:

1. First, we show in the regime \( \sigma_N \gg (\log \log N)^2 \) that both \( \max_i |L_i| = o_P(N^{-1/3}) \) and \( \max_i |R_i| = o_P(N^{-1/3}) \). This allows us to use Taylor’s approximation for the logarithm, so that

\[ \log |E_N| = \sum_{j=3}^N \log |1 - R_j| + \log |E_2| = \sum_{j=3}^N (-R_j - R_j^2/2) + o_P(1). \]

We point out that all values \( R_i, i = 2, \ldots, N \), remain strictly smaller than 1 as long as the eigenvalues of the matrix \( M_N/\sqrt{N} \) remain smaller than \( 2\theta_N \), see Albrecht et al. [2009]. In this sense the condition \( \sigma_N \gg (\log \log N)^2 \) seems a bit excessive. On the other hand, we show that in this regime, the process \( R_i \) closely follows the linear process \( L_i \), which is obviously a lot easier to control.

2. We then show that the sum \( \sum_{j=3}^N (-R_j - R_j^2/2) \) can be replaced with \( \sum_{j=3}^N -L_j \) at the cost of some \( o_P(\sqrt{\log N}) \) error term and with some explicit deterministic shift.

3. Finally, we derive the CLT for \( \sum_{j=3}^N L_j \) with the variance of exact order \( \log N \).

Achieving these objectives will show that the asymptotic behavior of \( \log |E_N| \) is the same as that of \( -\sum_{i=3}^N L_i \) up to \( O_P(1) \) and an explicit deterministic shift, and hence an appropriately centered \( \log |E_N| \) satisfies the same CLT as \( -\sum_{i=3}^N L_i \). After calculating the deterministic shift between \( \log |D_N| \) and \( \log |E_N| \), we derive the CLT for the log-determinant as required. Details of these derivations are given in Appendix A.

### 3 All the way to the edge

We now would like to extend Theorem 2 to cases where \( \sigma_N \) is either a constant (possibly negative), or a sequence slowly diverging to \( +\infty \) so that \( \sigma_N \ll \log^2 N \). In other words, we would like to allow for the location of the singularity of the logarithmic statistic to be arbitrarily close to the edge of the semicircle law, and even for the locations entering the support of the law, but staying within \( N^{-2/3} \)-order distance from the edge.

Note that Theorem 2 covers cases of diverging sequences \( (\log \log N)^2 \ll \sigma_N \ll \log^2 N \) and holds for any positive \( \alpha \). However, we will extend the theorem only for the cases \( \alpha = 1 \) and \( \alpha = 2 \).
because our proof will rely on the properties of GUE and GOE. Indeed, the main tool is uniform
asymptotics for the one-point function of GUE for regions containing the spectral edge. We rely
on results of Ercolani and McLaughlin [2003] which use Riemann-Hilbert methods, even though
we use only the GUE case (and extensions to GOE).

We outline the approach here, leaving the key estimates to Appendix B. Thus, we consider
now sequences $\sigma_N$ satisfying

$$C \leq \sigma_N \leq \bar{\sigma}_N := (\log \log N)^3.$$  \hspace{1cm} (13)

Of course the case $\sigma_N \equiv C$, possibly negative, is included. Let

$$S_N(\sigma_N) = \sum_{i=1}^{N} \log |2 + N^{-2/3} \sigma_N - \lambda_i| - N/2 + \frac{\alpha - 1}{6} \log N - \sigma_N N^{1/3} + \frac{2}{9} |\sigma_N|^{3/2}.$$  \hspace{1cm} (13)

The strategy is to show that

$$S_N(\bar{\sigma}_N) - S_N(\sigma_N) = o_P(\sqrt{\log N}),$$  \hspace{1cm} (14)

so that $S_N(\sigma_N)/\sqrt{\frac{2}{3} \log N}$ has the same limiting $N(0, 1)$ distribution as $S_N(\bar{\sigma}_N)/\sqrt{\frac{2}{3} \log N}$, the latter being given by Theorem 2.

Abbreviate $L_N = (\log \log N)^{5/2}$ and note that $|\sigma_N|^{3/2} \leq \bar{\sigma}_N^{3/2} = o(L_N)$. We can decompose

$$S_N(\bar{\sigma}_N) - S_N(\sigma_N) = \sum_{i=1}^{N} \left[ \log(2 + N^{-2/3} \bar{\sigma}_N - \lambda_i) - \log |2 + N^{-2/3} \sigma_N - \lambda_i| - (\sigma_N - \bar{\sigma}_N) N^{1/3} + o(L_N) \right]$$

$$= \sum_{i=1}^{N} d_i + \delta_N \left[ \sum_{i=1}^{N} \mu_i^{-1} - N \right] + o(L_N)$$  \hspace{1cm} (15)

where we have set

$$\mu_i = 2 + N^{-2/3} \bar{\sigma}_N - \lambda_i, \quad \delta_N = N^{-2/3}(\bar{\sigma}_N - \sigma_N),$$

$$d_i = \log(\mu_i + \delta_N) - \log |\mu_i| - \delta_N \mu_i^{-1}.$$

We will show that for each $\varepsilon > 0$, there exists $C_2(\varepsilon, C)$ such that with probability at least $1 - \varepsilon,$

$$\left| \sum_{i=1}^{N} d_i \right| \leq \delta_N^2 \sum_{i=1}^{N} \mu_i^{-2} + o(L_N).$$  \hspace{1cm} (16)

In Appendix B, we use the one-point function asymptotics to obtain the following estimates for $\sum \mu_i^{-1}$ and $\sum \mu_i^{-2}$ in (15) and (16).

**Proposition 4.** Suppose that $\alpha = 1$ or $\alpha = 2$ and let $\sigma_N$ satisfy condition (13). Then

$$\sum_{i=1}^{N} \mu_i^{-1} - N = O_P \left( (1 + |\sigma_N|^{1/2}) N^{2/3} \right)$$

and

$$\sum_{i=1}^{N} \mu_i^{-2} = O_P(N^{4/3}).$$

The bound (14) then follows directly from Proposition 4. Thus it remains to establish (16). For this we use some consequences of convergence to the Tracy-Widom law formulated in the following Lemma, also proved in Appendix B.
Lemma 5. Suppose that $M_N$ is either GUE or GOE. Then, for any constant $C > 0$, there exists $k = k(\epsilon, C)$ such that whenever $\sigma_N > -C$ and for large enough $N$,

$$P(\mu_k < 0) < \epsilon.$$ 

Moreover, there are constants $c_1 = c_1(\epsilon, C), C_1 = C_1(\epsilon, k)$, such that for large enough $N$,

$$P \left( \min_{i \leq N} |N^{2/3} \mu_i| < c_1 \right) < \epsilon, \quad P \left( \max_{i \leq k} |N^{2/3} \mu_i| > C_1 + |\sigma_N| \right) < \epsilon.$$ 

Turning to (16), our first goal is to establish probabilistic bounds on $|d_i|$ for $i = 1, \ldots, N$. On the event

$$E(k, c_1, C_1) = \left\{ \mu_k \geq 0, \min_{i=1,\ldots,N} |N^{2/3} \mu_i| \geq c_1, \max_{i=1,\ldots,k} |N^{2/3} \mu_i| \leq C_1 + |\sigma_N| \right\},$$

we have that for any $i \geq k$,

$$|d_i| \leq \delta_N^2 \mu_i^{-2}.$$ 

This follows from the second order Taylor expansion of log($\mu_i + \delta_N$) around log $|\mu_i|$. Further, still on $E$, for any $i < k$ and all sufficiently large $N$,

$$|d_i| = \left| \log \left( \frac{\tilde{\sigma}_N - \sigma_N + N^{2/3} \mu_i}{\mu_i} \right) - \log \left( N^{2/3} \mu_i \right) - \delta_N \mu_i^{-1} \right| \leq \log \left( 2 (\log \log N)^{3/2} \right) + \log(C_1 + |\sigma_N|) + c_1^{-1} (\log \log N)^3.$$ 

Hence, overall on $E$,

$$\sum_{i=1}^{N} |d_i| \leq \sum_{i=1}^{N} |d_i| \leq \delta_N^2 \sum_{i=1}^{N} \mu_i^{-2} + C_2 L_N$$

for some constant $C_2 \equiv C_2(\epsilon, C)$. Since $P(E) \geq 1 - \epsilon$, the latter inequality holds for all sufficiently large $N$, with probability at least $1 - \epsilon$. This yields (16) and completes the proof.

### 4 Spiked GUE and GOE

Let $M_{h,N} = \sqrt{Nh} \psi \psi^* + M_N$ be the spiked GUE or GOE with spike $h \in [0, \infty)$ along the direction $\psi \in \mathbb{C}^N$ for GUE and $\psi \in \mathbb{R}^N$ for GOE, and $|\psi| = 1$ in both cases. Since the joint distribution of the elements of $M_N$ is invariant with respect to transformations $M_N \to U M_N U^*$, where $U$ is any unitary matrix for GUE and any orthogonal matrix for GOE, the joint distribution of the eigenvalues of $M_{h,N}$ does not depend on the exact value of vector $\psi$. Therefore, without loss of generality, it will be convenient to set $\psi = (0, \ldots, 0, 1)^*$. 

For GUE and GOE, the tridiagonalization algorithm does not change the bottom right value, see proof of Proposition 7 in Tao and Vu [2012]. Hence, the analogue of (4) for $M_{h,N}$ is

$$\tilde{M}_{h,N} = \begin{pmatrix} a_1 & b_1 & & & \\ b_1 & \ddots & \ddots & & \\ & \ddots & a_{N-1} & b_{N-1} & \\ b_{N-1} & & a_N + \sqrt{Nh} \end{pmatrix} = M_N + \sqrt{Nh} \psi \psi^*.$$ 

Recall the definition of sequence $R_2, \ldots, R_N$ for the tridiagonal $M_N$. Since $\tilde{M}_{h,N}$ differs only in the lower right element, it will have the corresponding ratio sequence $R_2, \ldots, R_{N-1}, R_N^{(new)}$, with the only difference coming from the fact that the very last step of recursion (9) now becomes

$$R_N^{(new)} = \alpha_N - \gamma_N + \frac{\gamma_N + \beta_N - \delta_N}{1 - \theta \theta_{N-1}} + \frac{h}{\theta \theta_{N-1}} = R_N + \frac{h}{\theta \theta_{N-1}}.$$
Therefore, denoting $D_N^{(new)} = \det \left( \frac{1}{N} M_{h,N} - 2\theta_N \right)$, we get that

$$\log |D_N^{(new)}| = \log |D_N| + \log \left| 1 - \frac{R_N^{(new)}}{1 - R_N} \right| = \log |D_N| + \log \left| 1 - \frac{h/ (\theta_N \tau_N)}{1 - R_N} \right|. \quad (18)$$

If $h \neq 1$, so that the spike is either sub- or super-critical, we have

$$\log \left| 1 - \frac{h/ (\theta_N \tau_N)}{1 - R_N} \right| = \log |1 - h| + o_p(1),$$

and therefore, Theorem 2 continues to hold.

If $h = 1$, so that the spike is critical, then

$$\log \left| 1 - \frac{h/ (\theta_N \tau_N)}{1 - R_N} \right| = \log \left| 1 - \frac{1/ (\theta_N \tau_N)}{1 - R_N} \right| = \log \left| \frac{\sigma N^{1/2} N^{-1/3} - R_N + o(N^{-1/3})}{1 - R_N} \right|$$

By Lemma 16, $|R_N| = o_p \left( N^{-1/3} \right)$, so we have

$$\log \left| 1 - \frac{h/ (\theta_N \tau_N)}{1 - R_N} \right| = -\frac{1}{3} \log N + \frac{1}{2} \log \sigma N + o_p(1),$$

and hence it results in an extra shift $-\frac{1}{3} \log N$. Combining Theorem 2 and (18), we have the following theorem.

**Theorem 6.** Let $D_{h,N}$ be the determinant of $M_{h,N}/\sqrt{N} - 2\theta_N$, where $2\theta_N = 2 + N^{-2/3} \sigma_N$ with $(\log \log N)^2 \ll \sigma_N \ll (\log N)^2$. Then,

$$\log |D_{h,N}| - N/2 + \frac{\alpha - 1}{6} \log N - \sigma_N N^{1/3} + \frac{2}{3} \sigma_N^{3/2} + 1_{\{h=1\}} \frac{1}{2} \log N \xrightarrow{d} \mathcal{N}(0,1).$$

For sub-critical spikes, this theorem can be extended to situations where $\sigma_N$ is a constant or diverges to $+\infty$ slower than $(\log \log N)^3$ similarly to how Theorem 2 was extended to Theorem 1 above.

The key observation is that in the sub-critical case, any finite number of the largest eigenvalues of $M_{h,N}$ are asymptotically distributed according to the multivariate Tracy-Widom law (of type one for GOE and of type two for GUE). This fact is established in Theorems 1.5 and 1.6 of Féral and Pêché [2007]. It implies, in particular, that Lemma 5 holds for sub-critically spiked GUE and GOE. Combined with the interlacing theorem for eigenvalues of rank-one perturbations, it also implies that Lemma 4 still holds. Hence, the extension arguments used in the previous section go through, and we have the following result.

**Theorem 7.** Let $D_{h,N}$ be the determinant of $M_{h,N}/\sqrt{N} - 2\theta_N$, where $2\theta_N = 2 + N^{-2/3} \sigma_N$ with the local parameter $\sigma_N$, which is either fixed or slowly diverges to $+\infty$ so that $\sigma_N \ll \log^2 N$. Then, for sub-critical values of the spike, $h \in [0,1)$, we have that

$$\log |D_{h,N}| - N/2 + \frac{\alpha - 1}{6} \log N - \sigma_N N^{1/3} + \frac{2}{3} \sigma_N^{3/2} \xrightarrow{d} \mathcal{N}(0,1),$$

where $\alpha = 1$ for GUE and $\alpha = 2$ for GOE.
5 Discussion

In this section, we briefly compare our method with that of Lambert and Paquette [2020a,b] (LP in what follows). LP also start their analysis from recurrence (6), which in their notation, is written as (see Section 1.1 of Lambert and Paquette [2020a])

\[
\begin{bmatrix}
\Phi_n(z) \\
\Phi_{n-1}(z)
\end{bmatrix} = \begin{bmatrix}
\frac{\beta}{2\sqrt{N\beta}} \\
1
\end{bmatrix} \begin{bmatrix}
\frac{\beta}{2\sqrt{N\beta}} \\
0
\end{bmatrix} \begin{bmatrix}
\Phi_{n-1}(z) \\
\Phi_{n-2}(z)
\end{bmatrix} =: T_n(z) \begin{bmatrix}
\Phi_{n-1}(z) \\
\Phi_{n-2}(z)
\end{bmatrix}.
\]

A dictionary for translating this equation to our notation is presented in Table 1.

| JKOP | LP |
|-----------------|-----|
| \(2/\alpha\) | \(\beta\) |
| \(a_n\sqrt{2/\alpha}\) | \(b_n\) |
| \(\theta_{N}\) | \(z\) |
| \((-1)^n D_n/(2\sqrt{N})\) | \(\Phi_n(z)\) |

Table 1: Comparison of the key notation in our paper and in LP

Next, LP iterate the recurrence to obtain

\[
\begin{bmatrix}
\Phi_n(z) \\
\Phi_{n-1}(z)
\end{bmatrix} = T_n(z)...T_2(z) \begin{bmatrix}
\frac{\beta}{2\sqrt{N\beta}} \\
1
\end{bmatrix}.
\]

They point out that the deterministic version of (19) (where \(T_i(z)\) are replaced by \(\hat{T}_i(z) = \mathbf{E}T_i(z)\)) can be interpreted as the Hermite recurrence \(\begin{bmatrix}
\pi_n(z) \\
\pi_{n-1}(z)
\end{bmatrix} = \hat{T}_n(z)...\hat{T}_2(z) \begin{bmatrix}
\frac{\beta}{2\sqrt{N\beta}} \\
1
\end{bmatrix}\), generating the monic Hermite polynomials \(\{\pi_n\}\), and provide an illuminating discussion of the three different regimes: hyperbolic, parabolic, and elliptic. These regimes correspond, respectively, to situations where \(n \ll N\), \(n \approx N\), and \(n \gg N\).

One can interpret our Figure 2 as illustrating these regimes. On the left panel, the elliptic ring corresponds to the elliptic regime, and the inward-pointing spikes correspond to the hyperbolic and parabolic regimes. On the right panel, the hyperbolic regime corresponds to points located relatively close to zero, while the parabolic regime corresponds to more spread points.

Using this terminology, our analysis of the case of slowly diverging \(\sigma_N\) may be interpreted as an analysis of the hyperbolic regime. The extension of Theorem 2 to Theorem 1 can be understood as linking the hyperbolic and parabolic regimes, which we do using the 1-point correlation function asymptotics for GUE and relationships between GUE and GOE eigenvalues.

LP’s analysis of the hyperbolic and parabolic regimes takes a different route. Instead of scaling \(D_i \rightarrow E_i = D_i/\prod_{j=1}^i |\rho_j^+|\) and analysing the Sturm ratios \(E_i/E_{i-1}\) as we do, LP represent the matrix product \(\prod_{k=2}^n T_k = T_n(z)...T_2(z)\) from (19) in the form \(V_{n+1} (\prod_{k=2}^n V_{k-1}^{-1} T_k V_k) V_{n}^{-1}\), where \(V_k\) is a matrix of eigenvectors of \(\hat{T}_k\). In our notation, \(V_k\) is

\[
V_k = \begin{bmatrix}
-\rho_k^+/(2\sqrt{N}) & -\rho_k^-/(2\sqrt{N}) \\
1 & 1
\end{bmatrix}.
\]

Then they show that each matrix \(V_{k+1}^{-1} T_k V_k\) can be well approximated by a random scalar times a small random perturbation of matrix \(\begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}\). A detailed analysis of such an approximation leads to their CLT in the hyperbolic regime.

For the parabolic regime, LP scale \(\Phi_n(z)\) (with the choice of the scale guided by their result for the hyperbolic regime) and note that the scaled version satisfies a finite difference equation (their equation (8.1)), which can be interpreted as a discretisation of the stochastic Airy equation (their equation (1.4)). LP establish the stability of solutions to the stochastic Airy equation to show that the solution to its discretization is close to what they call the stochastic Airy function. This characterizes small-order deviations of the log statistic from the Gaussian limit, and leads to the CLT for the parabolic regime as a simple corollary.
Our CLT for the parabolic regime does not rely on the stochastic Airy equation machinery. Instead, we use asymptotics of 1-point correlation function for GUE to link hyperbolic and parabolic regimes. This approach is relatively simpler, but it works only for GUE and GOE, and does not deliver any results on the small-order deviations of the log statistic from the Gaussian limit.

A Proofs from Section 2

This appendix implements the three steps of the analysis, described in Section 2, that lead to Theorem 1. We start from the most straightforward step (step 3), which calls for deriving a CLT for the sum of the linear process $L_i$. First we introduce some basic notation.

Notation

For $p \geq 1$, denote by $\|X\|_p = \mathbf{E}^{1/p} |X|^p$ the $p$-norm of a random variable $X$. In addition, we say that a centred random variable $X$ belongs to the sub-gamma family $SG(u, v)$ for $u, v > 0$, if

$$
\mathbf{E} e^{tx} \leq e^{\frac{v t^2}{2 u}}, \quad \forall t : |t| < \frac{1}{u}.
$$

(20)

It is known that if $X \in SG(v_X, u_X)$ and $Y \in SG(v_Y, u_Y)$ are independent, then $X + Y \in SG(v_X + v_Y, u_X + u_Y)$. For arbitrary $c \in \mathbb{R}$, we have $cX \in SG(c^2 v_X, |c| u_X)$. If $X \sim N(0, 1)$, then $X \sim SG(1, 0)$, and if $X \sim \chi^2(d) - d$, then $X \sim SG(2d, 2)$. We refer to Chapter 2 of Boucheron et al. [2013] for proof of these results.

A.1 CLT for $\sum_{i=3}^N L_i$

Equation (11) yields

$$
\sum_{i=3}^N L_i = \sum_{i=3}^N g_{i+1} \xi_i,
$$

where $g_i = 1 + \gamma_1 + \cdots + \gamma_i \cdots \gamma_N$ and $\xi_i$ are independent, zero-mean random variables. Lyapunov’s CLT implies that

$$
\frac{\sum_{i=3}^N L_i}{\sqrt{\sum_{i=3}^N g_{i+1}^2 \mathbf{E} \xi_i^2}} \xrightarrow{d} N(0, 1)
$$

as long as

$$
\sum_{i=3}^N g_{i+1}^4 \mathbf{E} \xi_i^4 / \left( \sum_{i=3}^N g_{i+1}^2 \mathbf{E} \xi_i^2 \right)^2 \to 0.
$$

(21)

Let us establish (21), and find $\sum_{i=3}^N g_{i+1}^2 \mathbf{E} \xi_i^2$.

The proofs of the following two lemmas are given in Subsections A.6.3 and A.6.4.

Lemma 8. For any integer $q \geq 1$, there exists a constant $C_q > 0$, such that for all $3 \leq i \leq N$,

$$
\mathbf{E} \xi_i^{2q} \leq C_q \alpha^4 N^{-q} \quad \text{and} \quad \mathbf{E} \xi_i^2 \geq \alpha (2N)^{-1}.
$$

Lemma 9. Let $w_N = \sigma_N / 2$ so that $\theta_N = 1 + N^{-2/3} w_N$. Suppose that $(\log \log N)^2 \ll w_N \ll (\log N)^2$. Then, for all $3 \leq i \leq N - N^{1/3}$, any $k > 0$, and all sufficiently large $N$,

$$
g_i > \frac{r_i}{2(r_i - 1)} \left( 1 - \log^{-k} N \right).
$$

(22)

Further, for all $3 \leq i \leq N$ and all sufficiently large $N$,

$$
g_i < \frac{r_i}{2(r_i - 1)} \left( 1 + w_N^{-3/2} \right).
$$

(23)
These lemmas yield, for some $C > 0$ (whose value may change from one appearance to another) and all sufficiently large $N$,
\[
\sum_{i=3}^{N} g_{i+1}^{2} E_{i}^{2} > \sum_{i=3}^{N} \frac{g_{i+1}^{2} E_{i}^{2}}{N (r_{i} - 1)^{2}} > C \int_{1}^{N} x^{-1} \, dx = -C \log (1 - \theta_{N}^{-2}) > C \log N.
\]

Here we interpret the last sum in the first line as a Riemann sum approximation to the integral, where the approximation grid has increments $1/(N\theta_{N}^{2})$, and $(r_{i} - 1)^{2} = 1 - \frac{i - 1}{N\theta_{N}^{2}}$ is interpreted as the value of $x$ at a grid point. If the highest possible value of $i$ in the sum is $N - \lfloor N^{1/3} \rfloor - 1$, then the lower limit of the approximated integral should equal
\[
1 - \frac{N - \lfloor N^{1/3} \rfloor - 1}{N\theta_{N}^{2}} = 1 - \theta_{N}^{-2} + \theta_{N}^{-2} \left( \frac{N^{1/3}}{N} + 2 \right) < 2 \left( 1 - \theta_{N}^{-2} \right)
\]
for all sufficiently large $N$. This explains our use of $2 \left( 1 - \theta_{N}^{-2} \right)$ as the lower limit of integration in the second-last display.

Remark 10. The logarithmic growth of $\sum_{i=3}^{N} g_{i+1}^{2} E_{i}^{2}$ is a consequence of our choosing $\theta_{N}$ local to one. Had it been separated from one, the asymptotic variance of $\sum_{i=3}^{N} \xi_{i}$ would be constant. This agrees well with the fact that linear spectral statistics without singularities close to the edge of the semicircle law do not need scaling for the convergence to normality.

Similarly,
\[
\sum_{i=3}^{N} g_{i+1}^{4} E_{i}^{4} < \sum_{i=3}^{N} \frac{C}{N^{2} (r_{i} - 1)^{4}} < \frac{C}{N} \int_{1 - \theta_{N}^{-2}}^{\infty} x^{-2} \, dx < \frac{C}{N^{1/3} w_{N}}.
\]

Hence,
\[
\sum_{i=3}^{N} g_{i+1}^{4} E_{i}^{4} / \left( \sum_{i=3}^{N} g_{i+1}^{2} E_{i}^{2} \right)^{2} < \frac{C}{N^{1/3} w_{N} \log N} \rightarrow 0,
\]
which establishes the Lyapunov condition (21). Let us now approximate $\sum_{i=3}^{N} g_{i+1}^{2} E_{i}^{2}$.

Since, as we have just shown, $\sum_{i=3}^{N} g_{i+1}^{2} E_{i}^{2} > C \log N$, we will tolerate approximation errors of magnitude $o(\log N)$. The following lemma is established in Subsection A.6.5.

Lemma 11. For all $3 \leq i \leq N$,
\[
E_{i}^{2} = \frac{2\alpha}{N\theta_{N}^{2} r_{i}^{4}} (1 + \varepsilon_{i}),
\]
where $|\varepsilon_{i}| < \frac{C}{N(r_{i} - 1)} < CN^{-2/3}$ for some $C > 0$ and all sufficiently large $N$.

Combining this lemma with Lemma 9 yields that for all sufficiently large $N$,
\[
\sum_{i=3}^{N} g_{i+1}^{2} E_{i}^{2} < \sum_{i=3}^{N} \frac{\alpha}{2N\theta_{N}^{2} r_{i} (r_{i} - 1)^{2}} \left( 1 + 2w_{N}^{-3/2} \right) < \frac{1 + o(1)}{2} \int_{1 - \theta_{N}^{-2}}^{1} \frac{o(1)}{(1 + \sqrt{x}) x} dx = (1 + o(1)) \alpha \log \frac{\theta_{N} + \sqrt{\theta_{N}^{2} - 1}}{2\sqrt{\theta_{N}^{2} - 1}}.
\]
Similarly,
\[
\sum_{i=3}^{N} g_{i+1}^{2} E \xi_{i}^{2} > N^{-[N^{1/3}]^{-1}} \sum_{i=3}^{N} \frac{\alpha}{2N \theta_{N}^{2} r_{i} (r_{i} - 1)^{2}} \left( 1 - \frac{2}{\log^{2} N} \right) \\
> \frac{1 + o(1)}{2} \int_{1 - \theta_{N}^{-1} + N^{-2/3} \theta_{N}^{-2}}^{1} \frac{dx}{(1 + \sqrt{x})} \\
= (1 + o(1)) \log \left( \frac{\theta_{N} + \sqrt{\theta_{N}^{2} - 1} + N^{-2/3}}{2\sqrt{\theta_{N}^{2} - 1} + N^{-2/3}} \right) \\
= (1 + o(1)) \alpha \log \left( \frac{\theta_{N} + \sqrt{\theta_{N}^{2} - 1}}{2\sqrt{\theta_{N}^{2} - 1}} \right). 
\]

Hence, we conclude that
\[
\sum_{i=3}^{N} g_{i+1}^{2} E \xi_{i}^{2} = \alpha \log \left( \frac{\theta_{N} + \sqrt{\theta_{N}^{2} - 1}}{2\sqrt{\theta_{N}^{2} - 1}} \right) + o(\log N). 
\]

We have established the following theorem.

**Theorem 12.** Suppose that \( \theta_{N} = 1 + N^{-2/3} w_{N} \) with \((\log \log N)^{2} \ll w_{N} \ll (\log N)^{2}\). Then,
\[
\sum_{i=3}^{N} L_{i} \sim \frac{\alpha}{\sqrt{\alpha \log \left( \frac{\theta_{N} + \sqrt{\theta_{N}^{2} - 1}}{2\sqrt{\theta_{N}^{2} - 1}} \right)}} N(0, 1).
\]

We now turn to the second step of the analysis proposed in Section 2.

**A.2 Uniform bound on \( L_{i} \)**

Recall the definition (20). Straightforward calculations based on the definitions of \( \alpha_{i} \) and \( \beta_{i} \) show that
\[
\alpha_{i} \in SG \left( \frac{\alpha}{N \theta_{N}^{2} r_{i}^{2}}, \frac{0}{N \theta_{N}^{2} r_{i}^{2}} \right), \\
\beta_{i} \in SG \left( \frac{\alpha m_{i}}{N \theta_{N}^{2} r_{i} r_{i-1}^{2}}, \frac{\alpha}{N \theta_{N}^{2} r_{i} r_{i-1}^{2}} \right),
\]
where \( \alpha \) is the parameter introduced immediately after (4), and
\[
\xi_{i} = \alpha_{i} + \beta_{i} \in SG \left( \frac{\alpha}{N \theta_{N}^{2} r_{i}^{2}} + \frac{\alpha m_{i}}{N \theta_{N}^{2} r_{i} r_{i-1}^{2}}, \frac{\alpha}{N \theta_{N}^{2} r_{i} r_{i-1}^{2}} \right) \\
\in SG \left( \frac{2\alpha}{N \theta_{N}^{2} r_{i}^{2}}, \frac{\alpha}{N \theta_{N}^{2} r_{i}^{2}} \right) \equiv SG(v_{i}, u_{i}).
\]

The latter inclusion follows from the facts that \( r_{i-1} > r_{i} \) and \( r_{i} + m_{i} = 2 \). The proof of the following lemma is based on the identity (11) that expresses \( L_{i} \) as a weighted sum of \( \xi_{j} \). It is given in Subsection A.6.6.

**Lemma 13.** For any \( 3 \leq i \leq N \), \( L_{i} \in SG(v_{Li}, u_{Li}) \) with
\[
v_{Li} = \frac{\alpha}{2N \theta_{N}^{2} (r_{i} - 1)} \quad \text{and} \quad u_{Li} = \frac{\alpha}{N \theta_{N}^{2} r_{i}^{2}}.
\]

14
By Theorem 2.3 of Boucheron et al. [2013]
\[
P \left( |L_i| > \sqrt{2u_{Li}t + tu_{Li}} \right) \leq 2e^{-t}
\]
for all \( t \geq 0 \). Hence, by Lemma 13,
\[
P \left( |L_i| > \sqrt{\frac{\alpha t}{N\theta_N^2 (r_i - 1)} + \frac{\alpha t}{N\theta_N^2}} \right) \leq 2e^{-t}
\]
Changing the variable \( t \mapsto t + \log (2N) \), and taking the union bound yields
\[
P \left( |L_i| > \frac{\alpha (t + \log (2N))}{N\theta_N^2 (r_i - 1)} + \frac{\alpha (t + \log (2N))}{N\theta_N^2} \right) \text{ for all } i = 3, \ldots, N \leq e^{-t}.
\]
This bound implies \( |L_i| = o_P \left( N^{-1/3} \right) \) uniformly in \( i \leq N - N^{1/3} \log^{2+\eta} N \) for any fixed \( \eta > 0 \). This is so because for such \( i \), \( r_i - 1 \geq CN^{-1/3} \log^{1+\eta/2} N \), where \( C \) is some constant. To extend the uniform bound on \( |L_i| \) to \( i > N - N^{1/3} \log^{2+\eta} N \) one needs Kolmogorov’s maximum inequality.

**Lemma 14.** Suppose that \( \theta_N = 1 + N^{-2/3} w_N \) with \( (\log \log N)^2 \ll w_N \ll (\log N)^2 \). Then,
\[
\max_{3 \leq i \leq N} |L_i| = o_P \left( N^{-1/3} \right).
\]

**Proof.** We first bound products of \( \gamma \)'s from below. A proof of the following lemma is in Subsection A.6.7.

**Lemma 15.** Let \( 1 < \theta = 1 + N^{-2/3} \log^2 N \) let \( N \) be large enough. Then, for any \( i \geq N - (\theta^2 N)^{1/3} \log^3 N \) and \( i < j \leq i + (\theta^2 N)^{1/3} \log^{-2} N \),
\[
\gamma_{i+1} \gamma_j \geq 1/2.
\]

Write \( \tilde{L}_j \equiv \frac{L_j}{\gamma_1 \cdots \gamma_{i+1}} - L_i \). Then, \( \tilde{L}_i = 0 \) and
\[
\tilde{L}_j = \frac{\gamma_j L_j - \xi_j}{\gamma_1 \cdots \gamma_{i+1}} - L_i = \tilde{L}_{j-1} + \frac{\xi_j}{\gamma_1 \cdots \gamma_{i+1}}.
\]

Hence, \( \tilde{L}_j \) are partial sums of independent random variables \( \frac{\xi_j}{\gamma_1 \cdots \gamma_{i+1}} \). By Theorem 1 from Etemadi [1985], for any \( t > 0 \),
\[
P \left( \max_{i \leq j \leq N} |\tilde{L}_j| > 4t \right) \leq 4 \max_{i \leq j \leq N} P \left( |\tilde{L}_j| > t \right).
\]

Let \( i < j \) be such that \( i \geq N - (\theta^2 N)^{1/3} \log^3 N \) and \( j \leq i + (\theta^2 N)^{1/3} \log^{-2} N \), so that by Lemma 15, \( \gamma_j \cdots \gamma_{i+1} \geq 1/2 \). Then, by (24),
\[
\tilde{L}_j = \frac{\xi_{i+1}}{\gamma_{i+1}} + \cdots + \frac{\xi_j}{\gamma_j \cdots \gamma_{i+1}} \in SG \left( \frac{8\alpha}{(N\theta_N^2)^{2/3} \log^2 N}, \frac{2\alpha}{N\theta_N^2} \right).
\]

Hence, for some absolute constant \( C > 0 \) and all sufficiently large \( N \), by Theorem 2.3 of Boucheron et al. [2013],
\[
\max_{i \leq j \leq N} P \left( |\tilde{L}_j| > C \sqrt{\frac{\log \log N}{N^{1/3} \log N}} \right) \leq \frac{1}{\log^{10} N},
\]
where the power 10 is sufficiently large for our purposes, but could have been made arbitrarily large by choosing \( C \) sufficiently large. Therefore,

\[
P \left( \max_{i \leq j \leq k} |\tilde{L}_{ij}| > 4C \frac{\sqrt{\log \log N}}{N^{1/3} \log N} \right) \leq \frac{4}{\log^{10} N}. \tag{26}\]

In addition, by Lemma 13, for each \( 3 \leq i \leq N \) it holds (taking into account \( r_N - 1 \geq 1 - \theta_N^{-2} \geq N^{-1/3} w_N \) for sufficiently large \( N \)),

\[
P \left( |L_i| > C \frac{\sqrt{\log \log N}}{N^{1/3} \log N} \right) \leq \frac{2}{\log^{10} N}. \tag{27}\]

Now let us pick indices \( n_0 < n_1 < \cdots < n_K = N, K \leq 2 \log^5 N \), so that

\[
N - (\theta_N^2 N)^{1/3} \log^3 N \leq n_0 \leq N - N^{1/3} \log^3 N,
\]

\[
\frac{N^{1/3}}{2 \log^2 N} \leq n_k - n_{k-1} \leq \frac{N^{1/3}}{\log^2 N}.
\]

Taking the union bound of (27) for each \( i = n_{k-1} - 1, k = 1, \ldots, K \) and (26) for \( K \) pairs \((i, j) = (n_{k-1}, n_k), k = 1, \ldots, K \), we have, with probability at least \( 1 - 12 \log^{-5} N \), that for each \( k = 1, \ldots, K \),

\[
|L_{n_{k-1}}| \leq C \frac{\sqrt{\log \log N}}{N^{1/3} \log N^{1/2}}, \quad \max_{n_{k-1} \leq j \leq n_k} |\tilde{L}_{ij}^{n_k-1}| \leq 4C \frac{\sqrt{\log \log N}}{N^{1/3} \log N},
\]

which yield for \( j \in [n_{k-1}, n_k] \), by definition of \( \tilde{L}_{ij} \),

\[
|L_i| < |L_{n_{k-1}}| + |\tilde{L}_{ij}^{n_k-1}| \leq 5C \frac{\sqrt{\log \log N}}{N^{1/3} \log N^{1/2}} = o \left( N^{-1/3} \right).
\]

This covers all indices \( j \geq N - N^{1/3} \log^3 N \) and, as explained above, the rest is covered by Lemma 13. \( \square \)

### A.3 Uniform bound on \( R_i \)

Consider the following decomposition of (12),

\[
R_i = L_i + \gamma_1 \cdots \gamma_3 R_2 - T_{3i} + T_{1i} + T_{2i} + T_{3i}, \tag{28}\]

where

\[
T_{3i} = \delta_i + \gamma_i \delta_{i-1} + \cdots + \gamma_i \cdots \gamma_4 \delta_3,
\]

\[
T_{1i} = \beta_i R_i^{(1)} + \gamma_i \beta_{i-1} R_i^{(1)} + \cdots + \gamma_i \cdots \gamma_4 \beta_3 R_3^{(1)}, \quad R_i^{(1)} = \frac{R_i - 1 - R_i^{(1)}}{1 - R_i - 1},
\]

\[
T_{2i} = R_i^{(2)} + \gamma_i R_i^{(2)} + \cdots + \gamma_i \cdots \gamma_4 R_3^{(2)}, \quad R_i^{(2)} = \frac{\gamma_i R_i^{(2)} - \delta_i R_i^{(2)}}{1 - R_i - 1},
\]

\[
T_{3i} = R_i^{(3)} + \gamma_i R_i^{(3)} + \cdots + \gamma_i \cdots \gamma_4 R_3^{(3)}, \quad R_i^{(3)} = \gamma_i R_i^{(3)} - 1.
\]

The deterministic term \( T_{3i} \) could be easily bounded using (34) and the fact that both \( \delta_i \) and \( \gamma_i \) are increasing in \( i \),

\[
T_{3i} \leq \frac{\delta_i}{1 - \delta_i} \leq \frac{1}{2N \theta_N^2 (r_i - 1)^2} = o \left( N^{-1/3} \right). \tag{29}\]

Here we also use the fact that \( \theta_N^2 - 1 \gg N^{-2/3} \).

Consider the function

\[
\phi_u (x) = \begin{cases} 
-u, & x < -u \\
x, & |x| < u \\
u, & x > u 
\end{cases} \tag{30}
\]
Lemma 16. Under the assumptions of Lemma 14, with probability $1 - o(1)$,

$$
\max_{3 \leq i \leq N} |R_i| = o_p \left( N^{-1/3} \right),
$$
as well as, uniformly for all $i \leq N$,

$$
|T_{1i}| \leq \frac{8e\alpha^{1/2} \left( \log N \right)^{3/2}}{N^{5/6} \theta_N (r_i - 1)^{1/2}}, \quad |T_{2i}| \leq \frac{2}{N (r_i - 1)}.
$$

Proof. Denote $\tilde{R}_i = \phi_{N^{-1/3}/2} (R_i)$, so that we always have $|\tilde{R}_i| \leq N^{-1/3}/2$. We will assume that $N$ is large enough to have $(1 - N^{-1/3}/2)^{-1} \leq 4/3$. Consider the alternative process

$$
\tilde{R}_i = L_i + \gamma_i \ldots \gamma_3 R_2 - T_{si} + \tilde{T}_{1i} + \tilde{T}_{2i} + \tilde{T}_{3i},
$$

where

$$
\tilde{T}_{1i} = \beta_i \tilde{R}_{i1} + \gamma_i \beta_{i-1} \tilde{R}_{i-1} + \ldots + \gamma_i \ldots \gamma_3 \beta_3 \tilde{R}_3^{(1)}, \quad \tilde{R}_{i1}^{(1)} = \frac{\tilde{R}_{i-1} - \tilde{R}_{i-1}}{1 - \tilde{R}_{i-1}},
$$

$$
\tilde{T}_{2i} = \tilde{R}_{i2} + \gamma_i \tilde{R}_{i-1} + \ldots + \gamma_i \ldots \gamma_3 \tilde{R}_3^{(2)}, \quad \tilde{R}_{i2}^{(2)} = \frac{\gamma_i \tilde{R}_{i-1} - \delta_i \tilde{R}_{i-1}}{1 - \tilde{R}_{i-1}},
$$

$$
\tilde{T}_{3i} = \tilde{R}_{i3} + \gamma_i \tilde{R}_{i-1}^{(3)} + \ldots + \gamma_i \ldots \gamma_3 \tilde{R}_3^{(3)}, \quad \tilde{R}_{i3}^{(3)} = \gamma_i \tilde{R}_{i-1}^2 - \tilde{R}_{i-1}.
$$

and $\tilde{R}_2 = R_2$. Obviously, on the event $\max |\tilde{R}_i| \leq N^{-1/3}/2$, we have $|R_2| = |\tilde{R}_2| \leq N^{-1/3}/2$ and hence, $\tilde{R}_2 = R_2$ and $\tilde{R}_3^{(j)} = R_3^{(j)}$, $j = 1, 2, 3$. Therefore, $\tilde{R}_3 = R_3$, which yields $\tilde{R}_4 = R_4$ and so on. That is, on the event $\max |\tilde{R}_i| \leq N^{-1/3}/2$ we have $\tilde{R}_i = R_i$, $i = 2, \ldots, N$.

Observe that $|\tilde{R}_i^{(1)}| \leq N^{-1/3}$ are $F_{i-1}$-measurable, where $F_i$ is the sigma algebra generated by $\alpha_1, \beta_1, \ldots, \alpha_i, \beta_i$. Therefore, by Theorem 2.1 of Rio [2000] (Marcinkiewicz-Zygmund-type inequality for martingales), using notations of that paper, for $p > 2$,

$$
\|\tilde{T}_{1i}\|_p^2 \leq (p - 1) N^{-2/3} (1 + \gamma_i^2 + \ldots + \gamma_i^{2(i-3)}) \max_{j \leq i} \|\beta_j\|_p^2.
$$

Since $\beta_j \in SG \left( \frac{\alpha}{N \theta_N}, \frac{\alpha}{N \theta_N} \right)$, we have $\|\beta_j\|_p^2 \leq \frac{8p^2}{N^{5/6} \theta_N}$ by Lemma 22. Using $\frac{1}{1 - \gamma_i} = \frac{r^2}{4(r_i - 1)} < \frac{1}{r_i - 1}$,

$$
\|\tilde{T}_{1i}\|_p \leq \sqrt{p} N^{-1/3} \left( \frac{8p^2}{(r_i - 1) N \theta_N^2} \right)^{1/2} = \frac{\alpha^{1/2} (2p)^{3/2}}{N^{5/6} \theta_N (r_i - 1)^{1/2}}.
$$

By the Markov inequality,

$$
P \left( |\tilde{T}_{1i}| > e \|\tilde{T}_{1i}\|_{2 \log N} \right) \leq \left( \frac{\|\tilde{T}_{1i}\|_{2 \log N}}{e \|\tilde{T}_{1i}\|_{2 \log N}} \right)^{2 \log N} = e^{-2 \log N} = N^{-2},
$$

so that taking the union bound over $i = 3, \ldots, N$ yields, with probability at least $1/N$,

$$
|\tilde{T}_{1i}| \leq \frac{e \alpha^{1/2} (4 \log N)^{3/2}}{N^{5/6} \theta_N (r_i - 1)^{1/2}} = o \left( N^{-1/3} \right), \text{ for all } i = 3, \ldots, N.
$$

Next, we have $|\tilde{R}_i^{(2)}| = \left| \frac{\gamma_i \tilde{R}_{i-1} - \delta_i \tilde{R}_{i-1}}{1 - \tilde{R}_{i-1}} \right| \leq \frac{4}{3} \left( (8N)^{-1} + \delta_i N^{-1/3} \right)$. Therefore, using (34),

$$
|\tilde{T}_{2i}| \leq \frac{N^{-1}}{6} \frac{1}{1 - \gamma_i} + \frac{2}{3} \frac{1}{N^{4/3} \theta_N^2 (r_i - 1)^2} \leq \frac{2}{N (r_i - 1)}.
$$
where in the last inequality we used \( \frac{1}{N^{1/3}} = \frac{o(N^{1/3})}{N^{1/3}} = o(1) \). In particular, we have \( \max_i |\tilde{T}_{2i}| = o(N^{2/3}) \). Finally, since \( |\tilde{R}_i^{(3)}| \leq \tilde{R}_{i-1}^2 \leq \frac{1}{4} N^{-2/3} \), we have
\[
|\tilde{T}_{3i}| \leq \frac{1}{4N^{2/3}} \leq o\left(N^{-1/3}\right).
\]

Taking into account the result of Lemma 14 (and using the fact that \( R_2 \) is of order \( N^{-1/2} \)), we see that on an event of probability at least \( 1 - o(1) \),
\[\max_{i \leq N} |\tilde{R}_i| = o\left(N^{-1/3}\right) < N^{-1/3}/2.\]

Hence, \( \tilde{R}_i = R_i \) on that event, and the stated bounds on \( T_{1i} \) and \( T_{2i} \) take place.

**Remark 17.** In both Lemma 14 and Lemma 16, we only guarantee the probability \( 1 - o(1) \) of the form \( 1 - O\left(\log^{-k} N\right) \), which is a rather slow convergence rate.

### A.4 Linear approximation for \( \log |E_N| \)

Recall that
\[\log |E_N| = \sum_{i=3}^{N} \log |1 - R_i| + \log |E_2| .\]

A direct calculation yields that \( \log |E_2| = O_P \left( N^{-1/2} \right) \). Thus, it is asymptotically negligible. Furthermore, since \( \max_i |R_i| = o_P(N^{-1/3}) \), we have a uniform Taylor’s approximation
\[\log |1 - R_i| = -R_i - R_i^2/2 + o(N^{-1}).\]

Summing up,
\[\log |E_N| = \sum_{i=3}^{N} (-R_i - R_i^2/2) + o_P(1). \quad (31)\]

In the remainder of this subsection, our goal is to show that we can replace each term \( -R_i - R_i^2/2 \) with the linear process \( L_i \), with inclusion of a deterministic shift. To be precise, we will show that
\[\sum_{i=3}^{N} (-R_i - R_i^2/2) + \sum_{i=3}^{N} L_i = \frac{1}{6} \log N + O_P(\log \log N). \quad (32)\]

An analysis similar to one we used to prove Lemma 16 leads to the following lemma. The proof is rather technical and is given in Subsection A.6.8.

**Lemma 18.** Under the assumptions of Lemma 14,
\[\sum_{i=3}^{N} (R_i - L_i - T_{3i}^* + T_{3i}) = O_P(1), \quad \text{and} \quad \sum_{i=3}^{N} R_i^2 = O_P(1),\]

where \( T_{3i}^* = \sum_{i=4}^{N} (\gamma_i L_{i-1}^2 + \gamma_i \gamma_{i+1} L_{i-2}^2 + \ldots + \gamma_{i\ldots4} L_3^2). \)

Since \( \sum_{i=3}^{N} R_i^2 = O_P(1) \), it is sufficient to show that \( \sum_{i=3}^{N} (T_{3i}^* - T_{3i}) = o_P(1) \) to ensure that (32) holds. First, let us show that the sum of \( T_{3i}^* - ET_{3i}^* \) is \( o_P(1) \). We have,
\[
\sum_{i=3}^{N} T_{3i}^* = \sum_{i=4}^{N} (\gamma_i L_{i-1}^2 + \ldots + \gamma_i \ldots 4 L_3^2) = \sum_{i=4}^{N} (\gamma_i + \gamma_i \gamma_{i+1} + \ldots + \gamma_i \ldots N L_{i-1}^2) = \sum_{i=4}^{N} (g_i - 1) L_{i-1}^2.
\]
where \( g_i = 1 + \gamma_i + \cdots + \gamma_i \cdots \gamma_N \). We thus have a quadratic form w.r.t. to the vector \( \xi = (\xi_3, \ldots, \xi_{N-1}) \) with independent coordinates. Indeed,

\[
\begin{pmatrix}
L_3 \\
L_4 \\
L_5 \\
\vdots \\
L_{N-1}
\end{pmatrix} = \begin{pmatrix}
1 & \gamma_4 & \gamma_5 & 1 \\
\gamma_4 & \gamma_5 & \gamma_6 & \cdots \\
\gamma_5 & \cdots & \gamma_{N-1} & \gamma_N \\
\gamma_6 & \cdots & \gamma_{N-1} & \gamma_N & \cdots \\
\vdots & \cdots & \cdots & \cdots & \cdots \\
\gamma_{N-1} & \cdots & \gamma_{N-1} & \gamma_N & \cdots & \gamma_N
\end{pmatrix} \begin{pmatrix}
\xi_3 \\
\xi_4 \\
\xi_5 \\
\vdots \\
\xi_{N-1}
\end{pmatrix}.
\]

Denoting the matrix in the last display by \( X \) and \( G = \text{diag}(g_4 - 1, \ldots, g_N - 1) \), we have

\[
\sum_{i=3}^{N} T_{3i}^* = \sum_{i=4}^{N} (g_i - 1) L_{i-1}^2 = \xi^T X^T G X \xi.
\]

Since \( \xi_i \in \text{SG}(2\alpha/N, \alpha/N) \) for all \( i \), we have by the Hanson-Wright inequality for sub-gamma random variables (see Proposition 1.1 from Götze et al. [2019]),

\[
\xi^T X^T G X \xi - E \xi^T X^T G X \xi = O_p \left( \frac{1}{N} \right) \|X^T G X\|_{\text{HS}}.
\]

Here \( \| \cdot \|_{\text{HS}} \) is the Hilbert-Schmidt norm. We also have, \( \|X^T G X\|_{\text{HS}} \leq \|X\|\|G X\|_{\text{HS}} \), where \( \| \cdot \| \) is the operator norm. Decomposing \( X \) into a sum of the (sub-)diagonal matrices, we have by the triangle inequality

\[
\|X\| \leq 1 + \gamma_N - \gamma_1 + \cdots + \gamma_{N-1} \gamma_4 \leq \frac{1}{1 - \gamma_1} = O(N^{-1/2}w_N^{-1/2}).
\]

We also have,

\[
\|G X\|_{\text{HS}}^2 = \sum_{i=3}^{N} (g_i - 1)^2 (1 + \gamma_i^2 - \cdots + \gamma_i^2) \leq \sum_{i=3}^{N} (g_i - 1)^2 \frac{1}{1 - \gamma_i^2}.
\]

By Lemma 9, for large enough \( N \), each \( g_i \geq 1 \) satisfies \( g_i \leq \frac{2}{\gamma_i} \). We therefore have,

\[
\frac{1}{N} \|G X\|_{\text{HS}}^2 < \sum_{i=3}^{N} \frac{4}{N(1 - \gamma_i)^3} < \sum_{i=3}^{N} \frac{4}{n(n-1)^3} < \int_0^1 \frac{4dx}{(1 - x/\theta_N)^3/2} = 8\theta_N^3((\theta_N^2 - 1)^{-1/2} - \theta_N^{-1}) = O(N^{-1/2}w_N^{-1/2}).
\]

Summing up, we get

\[
\frac{1}{N} \|X^T G X\|_{\text{HS}} \leq \frac{1}{\sqrt{N}} \|X\| \sqrt{\frac{1}{N} \|G X\|_{\text{HS}}^2} = O(w_N^{-3/4}).
\]

Therefore,

\[
\sum_{i=3}^{N} T_{3i}^* = \sum_{i=3}^{N} \mathbf{E} T_{3i}^* = O_p(w_N^{-3/4}) = o_p(1).
\]

To establish that \( \sum_{i=3}^{N} (R_i - L_i) = O_p(1) \) it remains to show that \( \mathbf{E} T_{3i}^* - T_{3i} \) sum to \( o_p(1) \). By definition,

\[
\mathbf{E} T_{3i}^* - T_{3i} = (\gamma_i \mathbf{E} L_{i-1}^2 - \delta_i) + \gamma_i(\gamma_{i-1} \mathbf{E} L_{i-2}^2 - \delta_{i-1}) + \cdots + \gamma_i \cdots \gamma_4(\gamma_3 \mathbf{E} L_2^2 - \delta_3),
\]

where \( \mathbf{E} L_2^2 = 0 \) because \( L_2 \equiv 0 \). The following lemma is established in Subsection A.6.9.
Lemma 19. There exists $C > 0$ such that for all $3 \leq i \leq N$,

$$\left| \frac{\gamma_i}{\alpha} - \delta_i \right| < \frac{C}{N^2 (r_i - 1)^4}.$$ 

The lemma implies that

$$\left| \sum_{i=3}^{N} (ET_{3i}^* - \alpha T_{3i}) \right| \leq \sum_{i=3}^{N} \frac{C}{N^2 (r_i - 1)^4} \frac{1}{1 - \gamma_i} \leq \sum_{i=3}^{N} \frac{C}{N^2 (r_i - 1)^5} < \int_1^1 \frac{C dx}{N(1 - x/\theta_N^2)^{5/2}} = \frac{3C}{2N} \theta_N^5 ((\theta_N^2 - 1)^{-3/2} - \theta_N^{-3}) = O(w_N^{-3/2}).$$

Remarkably, for $\alpha = 1$, this will be the end of the proof. When $\alpha \neq 1$, the remaining sum $(\alpha - 1) \sum_{i=3}^{N} T_{3i}$ results in an additional shift. The proof of the following lemma is postponed to Section A.7.

Lemma 20. It holds, for large enough $N$,

$$\sum_{i=3}^{N} T_{3i} = \frac{1}{6} \log N + O(\log \log N).$$

This completes the proof of the approximation (32). Indeed, we have that

$$\sum_{i=3}^{N} (-R_i - R_i^2/2) = \sum_{i=3}^{N} -R_i + \sum_{i=1}^{N} (R_i - L_i - ET_{3i}^* + T_{3i}) + O_P(1) = \sum_{i=3}^{N} -L_i + \sum_{i=3}^{N} (-ET_{3i}^* + \alpha T_{3i}) + (1 - \alpha) \sum_{i=3}^{N} T_{3i} + O_P(1) = \sum_{i=3}^{N} -L_i + \frac{1 - \alpha}{6} \log N + O_P(\log \log N).$$

Now, using Theorem 12 and the continuous mapping theorem, we get the CLT for $\log |E_N|$.

Corollary 21. Under the assumptions of Theorem 12,

$$\frac{\log |E_N| + \frac{\alpha - 1}{6} \log N}{\sqrt{\alpha \log \theta_N^2 + \sqrt{\theta_N^4 - 1}}} \xrightarrow{d} N(0, 1).$$

A.5 CLT for $\log |D_N|$ 

From the definitions of $D_N$ and $E_N$, we have

$$\log |D_N| = \log |E_N| + \frac{N}{2} \log (\theta_N^2) + \sum_{i=1}^{N} \log (1 + \sqrt{x_{i-1}}),$$

where $x_i = 1 - i/(N\theta_N^2)$. Note that the latter sum can be written in the form

$$N\theta_N^2 T_N + \frac{1}{2} \left( \log(1 + \sqrt{x_0}) - \log(1 + \sqrt{x_N}) \right),$$

where

$$T_N = \frac{1}{N\theta_N^2} \left( \frac{1}{2} \log(1 + \sqrt{x_0}) + \log(1 + \sqrt{x_1}) + ... + \log(1 + \sqrt{x_{N-1}}) + \frac{1}{2} \log(1 + \sqrt{x_N}) \right).$$
is a trapezoidal approximation to \( \int_{a}^{b} \log(1 + \sqrt{x}) \, dx \) with \( a = 1 - \theta_N^{-2} \) and \( b = 1 \). As is well known, the absolute value of the error of such an approximation is bounded from above by \( M(b-a)^3/(12N^2) \), where \( M \) is the maximum of the absolute value of the second derivative of the integrand over the integration area.

Therefore, 
\[
\left| T_N - \int_{1-\theta_N^{-2}}^{1} \log(1 + \sqrt{x}) \, dx \right| \leq (1 - \theta_N^{-2})^{-3/2}/(12N^2) = O(N^{-\frac{3}{2}}),
\]
and we have
\[
\sum_{i=1}^{N} \log \left( 1 + \sqrt{x_{i-1}} \right) = N\theta_N^2 \int_{1-\theta_N^{-2}}^{1} \log(1 + \sqrt{x}) \, dx + \frac{1}{2} \left( \log 2 - \log(1 + \sqrt{1 - \theta_N^{-2}}) \right) + O(w_N^{-3/2})
\]
\[
= \frac{N}{2} + w_N N^{1/3} - \frac{2}{3} \left( 2w_N \right)^{3/2} + \frac{1}{2} \log 2 + O(w_N^{-3/2}).
\]
Using this in (33) together with the fact that
\[
\frac{N}{2} \log \left( \theta_N^2 \right) = w_N N^{1/3} + O \left( w_N^{2} N^{-1/3} \right)
\]
yields
\[
\log |D_N| = \log |E_N| + N/2 + 2w_N N^{1/3} - \frac{2}{3} \left( 2w_N \right)^{3/2} + \frac{1}{2} \log 2 + O(w_N^{-3/2}).
\]
Recalling that \( 2w_N = \sigma_N \) and using Theorem 21, we obtain Theorem 2.

### A.6 Proofs of technical lemmas

#### A.6.1 A moment bound for sub-gamma random variables

**Lemma 22.** Suppose, \( X \in SG(v,v) \) with \( v \leq 1/2 \). For any \( p > 2 \),
\[
\|X\|_p^2 \leq 8vp^2.
\]

**Proof.** First, closely following the proof of Theorem 2.3 of Boucheron et al. [2013], we obtain the following inequality
\[
\|X\|_p^p \leq p^{2p-1} \left( (2v)^{p/2} \Gamma \left( p/2 \right) + (2v)^p \Gamma \left( p \right) \right).
\]
Since for any \( x > 1 \), \( \Gamma(x) \leq x^{x-1} \) (see Anderson and Qiu [1997]), and \( (2v)^p \leq (2v)^{p/2} \), we get
\[
\|X\|_p^p \leq \left( 2^{p^2/2} + 2^{3p/2-1} p^p \right) \leq 2^{3p/2} v^{p/2} p^p.
\]

#### A.6.2 About \( \delta_i \) and \( R_{i2} \)

**Lemma 23.** For \( i = 3, \ldots, N \), we have
\[
\delta_i - \delta_{i-1} > \frac{1}{2N\theta_N^2 (r_i - 1)}.
\]
Furthermore,
\[
\delta_i = \delta_i + \frac{1}{2N\theta_N^2 (r_i - 1)} + \tilde{O} \left( \frac{1}{N^2 \theta_N^2 (r_i - 1)^3} \right),
\]
where \( \tilde{O}(z) \) is a quantity such that \( c|z| \leq |\tilde{O}(z)| \leq C|z| \) for some positive constants \( c \) and \( C \) that do not depend on \( i \) or \( N \).
Proof. Denote $1 - (i - 1) / (N\theta_N^2)$ as $x_i$. Then

$$r_{i-1} - r_i = \sqrt{x_i + \frac{1}{N\theta_N^2}} - \sqrt{x_i} = \frac{1}{N\theta_N^2} \sqrt{x_i + 1} + \sqrt{x_i}$$

$$= \frac{1}{\sqrt{x_i N\theta_N^2}} \frac{1}{\sqrt{1 + \frac{1}{N\theta_N^2 x_i}} + 1}$$

$$= \frac{1}{2N\theta_N^2 (r_i - 1)} + \frac{1}{N\theta_N^2 (r_i - 1)} \left( \frac{1}{\sqrt{1 + \frac{1}{N\theta_N^2 x_i}} + 1} - \frac{1}{2} \right).$$

Since $N\theta_N^2 x_i > 1$, this implies the first displayed inequality in the statement of the lemma. Further, denote $\frac{1}{N\theta_N^2 x_i}$ as $y_i$. We have

$$\frac{1}{\sqrt{1 + y_i} + 1} - \frac{1}{2} = \frac{1}{2} \frac{1 - \sqrt{1 + y_i}}{1 + \sqrt{1 + y_i}} = \frac{1}{2} \frac{y_i}{(1 + \sqrt{1 + y_i})^2}.$$

On the other hand, since $\frac{1}{N\theta_N^2} \leq x_i \leq 1$, we have $y_i \in (0, 1]$. Hence,

$$\frac{y_i}{6 + 4\sqrt{2}} \leq \frac{1}{2} \frac{y_i}{(1 + \sqrt{1 + y_i})^2} \leq \frac{y_i}{8}.$$

Since $y_i = \frac{1}{N\theta_N^2 (r_i - 1)}$, this shows that

$$\left| r_{i-1} - r_i - \frac{1}{2N\theta_N^2 (r_i - 1)} \right| = \tilde{O} \left( \frac{1}{N^2 \theta_N^4 (r_i - 1)^3} \right).$$

From the lemma, we have

$$\delta_i = \frac{m_i}{r_i} - \frac{m_i}{r_i - 1} = \frac{m_i}{2N\theta_N^2 r_i (r_i - 1)} + \tilde{O} \left( \frac{m_i}{N^2 (r_i - 1)^3} \right). \tag{34}$$

It is also worthwhile mentioning that

$$\delta_i > 0 \text{ and } \delta_i \text{ is increasing in } i.$$

The latter monotonicity property follows from the fact that function $(1 + \sqrt{1 - x})^{-1}$ is convex.

Finally, consider $R_2$. A direct calculation shows that

$$R_2 = \frac{\left(a_1 - 2\sqrt{N\theta_N}\right) \left(a_2 - 2\sqrt{N\theta_N}\right) - b_i^2}{r_2 \sqrt{N\theta_N} \left(a_1 - 2\sqrt{N\theta_N}\right)} + 1$$

$$= \frac{a_2}{r_2 \sqrt{N\theta_N}} + \frac{1}{r_2 \sqrt{N\theta_N} \left(2\sqrt{N\theta_N} - a_1\right)} - \gamma_2 = O_P \left(N^{-1/2}\right). \tag{35}$$

A.6.3 Proof of Lemma 8

The existence of $C_q$ follows from the fact that $\xi_i$ are sub-gamma random variables (see Definition 20 and equation (24)) and from equation (2.7) of Boucheron et al. [2013]. The lower bound on $\mathbb{E}x_i^2$ follows from the following identity

$$\mathbb{E}x_i^2 = \frac{\alpha}{N\theta_N^2 r_i^2} + \frac{\alpha m_i r_i}{N^2 \theta_N^2 r_i^2 r_{i-1}^2}. \tag{36}$$
A.6.4 Proof of Lemma 9

Since $\gamma_i$ is increasing with $i$, we have

$$g_i > 1 + \gamma_i + \ldots + \gamma_i^{N-i+1} = \frac{1 - \gamma_i^{N-i+2}}{1 - \gamma_i}.$$ 

On the other hand, for all sufficiently large $N$

$$\gamma_i < m_i \leq 1 - \sqrt{1 - \frac{N}{N\theta^2_N}} = 1 - \sqrt{1 - \frac{1}{N} - \frac{1}{N^{1/3}}} w_N^{1/2}. $$

Hence, for $i \leq N - N^{1/3}$, any $k > 0$, and all sufficiently large $N$,

$$\gamma_i^{N-i+2} \leq (1 - \frac{1}{N})^{N-i+2} < e^{-\frac{1}{N^{1/3}}} < e^{-\frac{1}{N^{1/3}}} < e^{-k \log \log N} = \log^{-k} N.$$ 

The lower bound (22) follows from this and the fact that $1 - \gamma_i = 2(r_i - 1)/r_i$.

The upper bound is based on the recursive relation $g_i \gamma_{i-1} + 1 = g_{i-1}$ and is carried out by induction. Consider,

$$n_i = (1 - \gamma_i)g_i - 1 - w_N^{-3/2}.$$ 

Since $g_n = 1 + \gamma_N$, we have that $n_N = -\gamma_N^2 - w_N^{-3/2} < 0$. Suppose, $n_i < 0$. Then, by the recurrence for $g_{i-1}$,

$$n_{i-1} = -1 - w_N^{-3/2} + (1 - \gamma_{i-1}) \left\{ \frac{\gamma_{i-1}}{1 - \gamma_i} (n_i + 1 + w_N^{-3/2}) + 1 \right\}$$

$$< -\gamma_{i-1} - w_N^{-3/2} \frac{\gamma_{i-1}}{1 - \gamma_i} \left( 1 + w_N^{-3/2} \right)$$

$$= \frac{\gamma_{i-1} + w_N^{-3/2}}{1 - \gamma_i} \left( \frac{\gamma_{i-1}}{1 - \gamma_i} \left( 1 + w_N^{-3/2} \right) \right)$$

$$\leq \frac{(\gamma_N + w_N^{-3/2} \gamma_i - \gamma_{i-1} - (1 - \gamma_N) w_N^{-3/2}}{1 - \gamma_i}.$$ 

Notice that $\gamma_i = f(\frac{1}{N})$ for $f(x) = \frac{\theta_N - \sqrt{\theta_N^2 - x}}{\theta_N + \sqrt{\theta_N^2 - x}}$. Since $f'(x) = \frac{\theta_N}{\sqrt{\theta_N^2 - x}(\theta_N + \sqrt{\theta_N^2 - x})^2}$ is increasing in $x$, we have that

$$\gamma_i - \gamma_{i-1} = \int_{\gamma_i}^{1} f'(x) dx < \frac{1}{N} f'(1) \leq \frac{1}{\theta_N N \sqrt{\theta_N^2 - 1}}.$$ 

Hence, using that $1 - \gamma_N > \sqrt{\theta_N^2 - 1}/\theta_N$ and $\gamma_N + w_N^{-3/2} < 2$ for large enough $N$,

$$n_{i-1} < \frac{2}{\theta_N N \sqrt{\theta_N^2 - 1} - 2 (\theta_N^2 - 1) w_N^{-3/2}} = \frac{2 \theta_N - 2 N (\theta_N^2 - 1) w_N^{-3/2}}{\theta_N N \sqrt{\theta_N^2 - 1} (1 - \gamma_i)}$$

$$< \frac{2 \theta_N - 2 \times 2^{3/2}}{\theta_N N \sqrt{\theta_N^2 - 1} (1 - \gamma_i)} < 0. $$

A.6.5 Proof of Lemma 11

We have

$$\frac{E x_i^2}{\alpha} = \frac{1}{N \theta_N^2 r_i} + \frac{m_i r_i}{N \theta_N^2 r_i^2 r_{i-1}^2} = \frac{r_i^2 \gamma_i + m_i - r_i - r_{i-1}}{N \theta_N^2 r_i^2 r_{i-1}^2} + \frac{1}{(N \theta_N^2)^2 r_i^2 r_{i-1}^2}$$

$$= \frac{2}{N \theta_N^2 r_i^2 r_{i-1}^2} + \frac{1}{(N \theta_N^2)^2 r_i^2 r_{i-1}^2} = \frac{2}{N \theta_N^2 r_i^2} + \frac{2 (r_i - r_{i-1})}{N \theta_N^2 r_i^2 r_{i-1}^2} + \frac{1}{(N \theta_N^2)^2 r_i^2 r_{i-1}^2}. $$

23
Here the first equality is from (36), the second uses the identity $m_i r_i - m_{i-1} r_{i-1} = 1/(N \theta_i^2)$. Using Lemma 23, we obtain
\[
\frac{E \xi_i^2}{\alpha} - \frac{2}{N \theta_i^2 r_i^3} = - \frac{1}{(N \theta_i^2)^2 (r_{i-1}) r_i^3} + \frac{1}{(N \theta_i^2)^2 r_i^2 r_{i-1}^2} + O\left( \frac{1}{(N \theta_i^2)^3 (r_{i-1})^3} \right).
\]
Hence, (again using (37))
\[
E \xi_i^2 = \frac{2 \alpha}{N \theta_i^2 r_i^3} (1 + \varepsilon_i),
\]
where $|\varepsilon_i| < \frac{C}{N(r_i-1)}$ for some $C > 0$ and all sufficiently large $N$. Moreover, we have the bound
\[
\frac{1}{r_i - 1} \leq \frac{1}{r_N - 1} < \frac{1}{\sqrt{1 - \theta^2}} < N^{1/3},
\]
so that $|\varepsilon_i| < \frac{C}{N^{2/3}}$.

**A.6.6 Proof of Lemma 13**

From (24), $\alpha_i + \beta_i \in SG(v_i, u_i)$ with $v_i = \frac{2 \alpha}{N \theta_i^2 r_i}$ and $u_i = \frac{\alpha}{N \theta_i^2 r_i}$. For any $i \leq N$ we have
\[
v_{Li} \leq v_i + \sum_{j=0}^{i-4} \gamma_i^2 \gamma_{i-j}^2 v_{i-j-1}.
\]
Since $\gamma_i$ is increasing in $i$, this yields
\[
v_{Li} \leq v_i (1 + \gamma_i^2 + \gamma_i^4 + \ldots) = \frac{v_i}{1 - \gamma_i^2}
\]
\[
= \frac{\alpha}{2N \theta_i^2 r_i} \frac{1}{r_{i-1}} \leq \frac{\alpha}{2N \theta_i^2 (r_i - 1)},
\]
where we used identities $1 - \gamma_i^2 = (1 - \gamma_i)(1 + \gamma_i) = \frac{r_i - m_i}{r_i} \frac{r_i + m_i}{r_i} = \frac{r_i - (2 - r_i)}{r_i} 2$ and, for the last inequality, the fact that $r_i > 1$. For $u_{Li}$, we have $u_{Li} \leq \max_{j \leq i} u_j = \alpha/(N \theta_i^2 r_i^2)$ because $r_i$ is decreasing in $i$.

**A.6.7 Proof of Lemma 15**

Since $\gamma_{i+1} < \ldots < \gamma_j < 1$, we have $\log (\gamma_{i+1} \ldots \gamma_j) \geq (j - i) \log \gamma_{i+1}$. On the other hand,
\[
\log \gamma_{i+1} = \log \frac{m_{i+1}}{r_{i+1}} \geq \log \left( 1 - \sqrt[3]{\frac{\theta_i^2 - 1}{\theta_i^2} + \frac{(\theta_i^2 N)^{1/3} \log^3 N}{\theta_i^2 N}} \right)
\]
\[
- \log \left( 1 + \sqrt[3]{\frac{\theta_i^2 - 1}{\theta_i^2} + \frac{(\theta_i^2 N)^{1/3} \log^3 N}{\theta_i^2 N}} \right).
\]
Note that $- \log (1 + x) \geq -x$ and $\log (1 - x) \geq -2x$ for $x \in (0, 1/2)$. Since $\theta_i^2 - 1 = o\left( N^{-2/3} \log^2 N \right)$, for sufficiently large $N$,
\[
\log \gamma_{i+1} \geq -o\left( N^{-1/3} \log^{3/2} N \right)
\]
and
\[
\log (\gamma_{i+1} \ldots \gamma_j) \geq \frac{(\theta_i^2 N)^{1/3}}{\log^2 N} o\left( N^{-1/3} \log^{3/2} N \right) = o\left( \log^{-1/2} N \right) \geq \log (1/2).
\]
A.6.8 Proof of Lemma 18

Recall (28)

\[ R_i = L_i + \gamma_i \ldots \gamma_3 R_2 - T_{3i} + T_{1i} + T_{2i} + T_{3i}. \]

Consider the alternative process

\[ \hat{R}_i = L_i + \gamma_i \ldots \gamma_3 \hat{R}_2 - T_{3i} + \hat{T}_{1i} + \phi \frac{2}{N_{(r_{i−1})}} \left( T_{2i} \right) + \hat{T}_{3i} \] (38)

with \( \hat{R}_2 = R_2 \phi_1 \left( \left| a_1 / \sqrt{N} - 2 \theta_N \right| \right) \) and

\[
\begin{align*}
\hat{T}_{1i} & = \beta_i \hat{R}^{(1)} + \gamma_i \beta_{i−1} \hat{R}^{(1)}_{i−1} + ... + \gamma_i \ldots \gamma_4 \beta_3 \hat{R}^{(1)}_3 , \quad \hat{R}^{(1)}_i = \frac{\hat{R}_{i−1}}{1 − \phi_1/2(R_{i−1})}, \\
\hat{T}_{3i} & = \hat{R}^{(3)}_i + \gamma_i \hat{R}^{(3)}_{i−1} + ... + \gamma_i \ldots \gamma_4 \hat{R}^{(3)}_3 , \quad \hat{R}^{(3)}_i = \gamma_i \phi_{N−1/3} \left( \hat{R}_{i−1} \right) \hat{R}_{i−1}.
\end{align*}
\]

By Lemma 16, on an event of probability \( 1 − o(1) \), \( |R_i| \leq N^{−1/3} \) and \( |T_{2i}| \leq \frac{2}{N_{(r_{i−1})}} \) for all \( i = 3, \ldots, N \), and \( \hat{R}_2 = R_2 \). Therefore, on this event, processes \( \{R_i\} \) and \( \{\hat{R}_i\} \) coincide for \( i = 2, \ldots, N \). First, we want to derive a bound on \( \|\hat{R}_i\|_4 \). For this, we obtain bounds on the norms of all terms on the right hand side of (38) as follows.

By Lemma 13 and Theorem 2.3 of Boucheron et al. [2013],

\[
\|L_i\|_4 \leq 2 \left( \frac{4\alpha}{N\theta_N^2(r_i−1)} \right)^2 + 4! \left( \frac{4\alpha}{N\theta_N^2 r_i^2} \right)^4 \leq \frac{33\alpha^2}{N^2\theta_N^4 (r_i−1)^2}
\]

for all sufficiently large \( N \). Hence

\[
\|L_i\|_4 \leq \frac{3\alpha^{1/2}}{\theta_N \sqrt{N(r_i−1)}}.
\]

Next, the definition of \( \hat{R}_2 \) and (35) yield

\[
\|\hat{R}_2\|_4 \leq (\|a_2\|_4 + \|b_i^\perp\|_4)/\sqrt{N} + \gamma_2 = O(N^{−1/2}).
\]

Hence,

\[
\|\gamma_i \ldots \gamma_3 \hat{R}_2\|_4 \leq \gamma_3 \|\hat{R}_2\|_4 = O \left( N^{−3/2} \right).
\]

Further, by (29),

\[
T_{3i} \leq \frac{1}{2N\theta_N^2 (r_i−1)^2}.
\]

To bound the norm of \( \hat{T}_{1i} \), observe that \( \|\hat{R}^{(1)}_i\|_p \leq 2\|\hat{R}_{i−1}\|_p \). Since \( \beta_i \in SG \left( \frac{\alpha}{\theta_N}, \frac{\alpha}{\theta_N} \right) \), we have \( \|\hat{R}^{(1)}_i\|_p \leq \frac{8\theta_N^2}{N\theta_N} \) by Lemma 22. Since \( \beta_i \) and \( \hat{R}^{(1)}_i \) are independent,

\[
\|\beta_i \hat{R}^{(1)}_i\|_p = \|\beta_i\|_p \|\hat{R}^{(1)}_i\|_p \leq \frac{\theta_N^2 \alpha^{1/2} \|\hat{R}_{i−1}\|_p}{\sqrt{N}} \|\hat{R}_{i−1}\|_p.
\]

By the Marcinkiewicz-Zygmund type inequality (see Theorem 2.1 from Rio (2009)), and using \( (1 − \gamma_i)^{−1} \leq (r_i−1)^{−1} = o \left( N^{1/3} \right) \),

\[
\|\hat{T}_{1i}\|_4 \leq \frac{2\theta_N^2 \alpha^{1/2}}{\theta_N \sqrt{N(1−\gamma_i)}^{1/2}} \max_{j \leq i−1} \|\hat{R}_{i−1}\|_4 = o \left( N^{−1/3} \right) \max_{j \leq i−1} \|\hat{R}_j\|_4.
\] (39)
Next, by definition,
\[ \left\| \phi_{N(r_i-1)}^2 (T_{2i}) \right\|_4 \leq \frac{2}{N(r_i - 1)}. \]

And finally, we have \( \| \hat{R}_i^{(3)} \|_4 \leq N^{-1/3} \| \hat{R}_{i-1} \|_4 \). Therefore, by the triangle inequality,
\[ \| \hat{T}_{3i} \|_4 \leq N^{-1/3} \frac{1}{1 - \gamma_i} \max_{j \leq i-1} \| \hat{R}_j \|_4 = o(1) \max_{j \leq i-1} \| \hat{R}_j \|_4. \]

All this sums up to
\[ \| \hat{R}_i \|_4 \leq \frac{3\alpha^{1/2}}{\sqrt{N(r_i - 1)}} + O\left(N^{-3/2}\right) + \frac{1}{2N(r_i - 1)^2} + o(1) \max_{j \leq i-1} \| \hat{R}_j \|_4, \]

which implies
\[ \| \hat{R}_i \|_4 \leq \frac{1}{\sqrt{N(r_i - 1)}} (3\alpha^{1/2} + o(1)) + o(1) \max_{j \leq i-1} \| \hat{R}_j \|_4. \]

On the other hand, as we have seen above, \( \| \hat{R}_2 \|_4 = O(N^{-1/2}) \). Hence, by induction (and for sufficiently large \( N \)),
\[ \| \hat{R}_i \|_4 \leq \frac{4\alpha^{1/2}}{\sqrt{N(r_i - 1)}} \text{ for } i = 3, 4, \ldots, N. \quad (40) \]

Since \( \| \hat{R}_i^{(3)} \|_2 \leq \| \hat{R}_i^2 \|_2 = \| \hat{R}_i \|_4^2 \), we have
\[ \| \hat{T}_{3i} \|_2 \leq \| \hat{R}_i^{(3)} \|_2 + \gamma_i \| \hat{R}_i^{(3)} \|_2 + \ldots + \gamma_i \ldots \gamma_4 \| \hat{R}_3^{(3)} \|_2 \leq \frac{16\alpha}{N(r_i - 1)} \frac{1}{1 - \gamma_i} \leq \frac{16\alpha}{N(r_i - 1)^2}. \]

Further,
\[ \| \hat{T}_{1i} \|_2 \leq 2 \left( ||\beta||_2 \| \hat{R}_i \|_2 + \gamma_i \| \beta_i - 1 \|_2 \| \hat{R}_{i-1} \|_2 + \ldots + \gamma_i \ldots \gamma_4 \| \beta_3 \|_2 \| \hat{R}_{3} \|_2 \right) \]
\[ \leq 2 \left( ||\beta||_2 \| \hat{R}_i \|_4 + \gamma_i \| \beta_i - 1 \|_2 \| \hat{R}_{i-1} \|_4 + \ldots + \gamma_i \ldots \gamma_4 \| \beta_3 \|_2 \| \hat{R}_{3} \|_4 \right) \]
\[ \leq \frac{4\sqrt{80}}{\theta N \sqrt{N}} \frac{4\alpha^{1/2}}{\theta N \sqrt{N}} \frac{1}{\sqrt{N(r_i - 1)}} \frac{1}{1 - \gamma_i} \leq \frac{48\alpha}{N(r_i - 1)^{3/2}}, \]
\[ \left\| \phi_{N(r_i-1)}^2 (T_{2i}) \right\|_2 \leq \frac{2}{N(r_i - 1)^3}, \text{ and } T_{3i} \leq \frac{1}{2N(r_i - 1)^2}. \] Finally,
\[ ||\gamma_i \ldots \gamma_3 \hat{R}_2 \|_4 \leq ||\gamma_3 \| \hat{R}_2 \|_4 \leq \frac{4\alpha^{1/2}}{N^{3/2}(r_2 - 1)^{1/2}} \]
for sufficiently large \( N \). Overall, by the triangle inequality
\[ \| \hat{R}_i - L_i \|_2 \leq ||\gamma_i \ldots \gamma_3 \hat{R}_2 \|_2 + \| T_{3i} \|_2 + \| \hat{T}_{1i} \|_2 + \left\| \phi_{N(r_i-1)}^2 (T_{2i}) \right\|_2 + \| \hat{T}_{3i} \|_2 \]
\[ \leq \frac{4\alpha^{1/2}}{N^{3/2}(r_2 - 1)^{1/2}} + \frac{1}{2N} \frac{1}{(r_i - 1)^2} + \frac{48\alpha}{N} \frac{1}{(r_i - 1)^{3/2}} \]
\[ + \frac{2}{16\alpha} \frac{1}{N} + \frac{N}{(r_i - 1)^2} \]
\[ \leq \frac{100\alpha}{N} \frac{1}{(r_i - 1)^2}, \text{ for } i = 2, \ldots, N. \]
Now we consider the decomposition
\[ \hat{R}_i - L_i - T_{3i} + T_{\delta i} + \Delta_{3i} = \gamma_i \hat{R}_2 + \hat{T}_{1i} + \frac{\phi}{\sqrt{N (r_i - 1)}} (T_{2i}) + \hat{T}_{4i}, \]
where
\[ \Delta_{3i} = \gamma_i \hat{R}_2 - T_{3i}, \]
\[ \hat{T}_{4i} = \gamma_i (\hat{R}_4^2 - L_i^2 - T_{\delta i}^2 - \hat{T}_{3i}). \]

so that with probability 1 - o(1), \( \Delta_{3i} = 0 \). Then,
\[ \| \hat{T}_{4i} \|_1 \leq \frac{\gamma_i}{1 - \gamma_i} \max_{j \leq i - 1} \| \hat{R}_j - L_j^2 \|_1. \]

We have
\[ \| \hat{R}_i^2 - L_i^2 \|_1 \leq \| \hat{R}_i - L_i \|_2 \| \hat{R}_i + L_i \|_2 \leq \frac{100 \alpha}{N (r_i - 1)^2} \left( \frac{4 \alpha^{1/2}}{\sqrt{N (r_i - 1)}} + \frac{3 \alpha^{1/2}}{\theta_N \sqrt{N (r_i - 1)}} \right) \leq 700 \alpha^{3/2} N^{3/2} (r_i - 1)^{5/2}. \]

Hence, \( \| \hat{T}_{4i} \|_1 \leq \frac{700 \alpha^{3/2}}{N^{3/2} (r_i - 1)^{5/2}} \) and for sufficiently large \( N \),
\[ \sum_{i=3}^N \| \hat{T}_{4i} \|_1 \leq \frac{700 \alpha^{3/2} \theta_N^2}{N^{1/2}} \sum_{i=3}^N \frac{1}{\theta_N^2 (r_i - 1)^{7/2}} \leq \frac{700 \alpha^{3/2} \theta_N^2}{N^{1/2}} \int_{1 - \theta_N^{-2}}^1 \frac{dx}{x^{7/4}} \leq -\frac{1000 \alpha^{3/2} \theta_N^2}{\sqrt{N}} \left( 1 - (1 - \theta_N^{-2})^{-3/4} \right) \leq \frac{1000 \alpha^{3/2} \theta_N^2}{\sqrt{N}} \sqrt{N} w_N^{3/4} = o(1). \]

Further,
\[ \sum_{i=3}^N \left| \phi \frac{\phi}{\sqrt{N} (r_i - 1)} (T_{2i}) \right| \leq \sum_{i=3}^N \frac{2}{N (r_i - 1)} = O(1). \]

Since \( \| T_{4i} \|_1 \leq \| T_{4i} \|_4 \), we have by (39) and (40),
\[ \sum_{i=3}^N \| T_{1i} \|_1 \leq \sum_{i=3}^N \frac{3^{1/2} 2^9 / 4 \alpha}{\theta_N N (r_i - 1)} = O(1). \]

Finally,
\[ \sum_{i=3}^N \gamma_i \cdots \gamma_3 \| \hat{R}_2 \|_1 \leq (\gamma_3 + N \gamma_3 \gamma_4) \| \hat{R}_2 \|_1 = O(N^{-1}). \]

We therefore conclude that
\[ \sum_{i=3}^N \left( \hat{R}_i - L_i - T_{3i}^* + T_{\delta i} + \Delta_{3i} \right) = O(1), \]

and since we have \( \hat{R}_i = R_i \) and \( \Delta_{3i} = 0 \) for all \( i = 3, \ldots, N \) on an event of probability \( 1 - o(1) \),
the first inequality of the lemma follows.

To see that the second inequality also holds, observe that by (40),
\[ \sum_{i=3}^N \| \hat{R}_i^2 \|_1 = \sum_{i=3}^N \| \hat{R}_i \|_2^2 \leq \sum_{i=3}^N \| \hat{R}_i \|_4^2 \leq \sum_{i=3}^N \frac{16 \alpha}{N (r_i - 1)} = O(1). \]

Therefore, \( \sum_{i=3}^N \hat{R}_i^2 = O(1) \) and so is the sum of \( R_i^2 \).
A.6.9 Proof of Lemma 19

We have due to the independence of $\xi_i$,
\[
E L_i^2 = E \xi_i^2 + \gamma_i^2 E \xi_{i-1}^2 + \cdots + \gamma_i^2 \cdots \gamma_4^2 E \xi_4^2.
\]

By Lemma 11,
\[
\frac{E \xi_i^2}{\alpha} = \frac{2}{N \theta_N^2 r_i^4} + O \left( \frac{1}{N^2 (r_i - 1)} \right).
\]

We want to write it in a form such that the sum corresponding to $E L_i^2$ cancels out as a telescopic sum. Observe that
\[
\frac{2}{r_i^4} = \frac{4(r_i - 1)}{2r_i (r_i - 1)} = \frac{1 - \gamma_i^2}{2r_i (r_i - 1)} = \frac{1}{2r_i (r_i - 1)} - \frac{\gamma_i^2}{2r_i (r_i - 1)}.
\]

Using Lemma 23, we write
\[
\frac{N \theta_N^2}{\alpha} E \xi_i^2 = \frac{1}{2r_i (r_i - 1)} - \frac{\gamma_i^2}{2r_i (r_i - 1)} + \frac{\gamma_i^2}{2r_i (r_i - 1)} (r_i - r_{i-1}) + O \left( \frac{1}{N (r_i - 1)} \right)
\]
\[
= \frac{1}{2r_i (r_i - 1)} - \frac{\gamma_i^2}{2r_i (r_i - 1)} + O \left( \frac{1}{N (r_i - 1)^3} \right).
\]

Since $1 + \gamma_i^2 + \cdots + \gamma_i^2 \cdots \gamma_4^2 \leq \frac{1}{1 - \gamma_i} \leq \frac{1}{r_i - 1}$, we get that
\[
\frac{N \theta_N^2}{\alpha} E L_i^2 = \left( \frac{1}{2r_i (r_i - 1)} - \frac{\gamma_i^2}{2r_i (r_i - 1)} \right) + \gamma_i^2 \left( \frac{1}{2r_i (r_i - 1)} - \frac{\gamma_i^2}{2r_i (r_i - 1)} \right)
\]
\[
+ \cdots + \gamma_i^2 \cdots \gamma_4 \left( \frac{1}{2r_i (r_i - 1)} - \frac{\gamma_i^2}{2r_i (r_i - 1)} \right) + O \left( \frac{1}{N (r_i - 1)^4} \right)
\]
\[
= \frac{1}{2r_i (r_i - 1)} + O(\gamma_i^2) + O \left( \frac{1}{N (r_i - 1)^4} \right)
\]
\[
= \frac{1}{2r_i (r_i - 1)} + O \left( \frac{1}{N (r_i - 1)^3} \right),
\]

where in the last line we used the fact that $\gamma_3 = O(1/N)$. Finally, we write
\[
\frac{\gamma_i}{2N \theta_N^2 r_{i-1} (r_i - 1)} - \delta_i = \frac{m_i}{N \theta_N^2} \left( \frac{1}{2r_i (r_i - 1)} - \frac{r_{i-1} - r_i}{r_i r_{i-1}} \right)
\]
\[
= \frac{m_i}{N \theta_N^2 r_i r_{i-1}} \left( \frac{1}{2(r_i - 1)} - \frac{1}{r_i + r_{i-1} - 1} \right)
\]
\[
= O \left( \frac{r_{i-1} - r_i}{(r_i - 1)^2} \right) = O \left( \frac{1}{N^2 (r_i - 1)^2} \right).
\]

Putting all together, we obtain,
\[
\frac{\gamma_i E L_{i-1}^2}{\alpha} - \delta_i = 2N \theta_N^2 r_{i-1} (r_i - 1) - \delta_i + O \left( \frac{1}{N^2 (r_i - 1)^4} \right) = O \left( \frac{1}{N^2 (r_i - 1)^4} \right).
\]

A.7 Proof of Lemma 20

For the upper bound, we use that by (29), $T \delta_i \leq \frac{\delta_i}{1 - \gamma_i}$. Recall that $\frac{1}{1 - \gamma_i} = \frac{r_i}{2(r_i - 1)}$. Moreover,
\[
\delta_i = \frac{m_i}{r_i r_{i-1}} (r_{i-1} - r_i) = \frac{m_i}{r_i r_{i-1}} \theta_N^2 N (r_i + r_{i-1} - 2) \leq \frac{m_i}{2r_i r_{i-1}} \theta_N^2 N (r_i - 1).
\]
Hence,
\[
\sum \frac{\delta_i}{1 - \gamma_i} \leq \sum \frac{m_i}{4N\theta_N^2(r_i - 1)^2} \leq \sum \frac{1}{4N\theta_N^2(r_i - 1)^2} \\
\leq \frac{1}{4} \int_0^1 \frac{dx}{\theta_N^2 - x} = \frac{1}{4} \log \frac{\theta_N^2}{\theta_N^2 - 1} \\
= \frac{1}{6} \log N + O(\log \log N).
\]

For the lower bound, observe that
\[
\sum_{i=3}^N T_{\delta_i} = \sum_{i=3}^N g_i \delta_i > \sum_{i=3}^N g_i \delta_i.
\]

First, we similarly have that
\[
\delta_i \geq \frac{1}{20^2 N r_i - 1(r_i - 1)}.
\]

Secondly, by (22), we have that \( g_i > \frac{1}{2(r_i - 1)(1 - \log^{-2} N) \text{ for all } i = 1, \ldots, N - N^{1/3}. \) Hence,
\[
(1 - \log^{-2} N)^{-1} \sum_{i=3}^N T_{\delta_i} > \sum_{i=3}^{N-N^{1/3}} \frac{1}{40^2 N r_i - 1(r_i - 1)^2} = \sum_{i=3}^{N-N^{1/3}} \frac{r_i - 1(r_i - 1)}{40^2 N r_i - 1(r_i - 1)^2} \\
= \sum_{i=3}^{N-N^{1/3}} \frac{1}{40^2 N (r_i - 1)^2} + O(1) \\
\geq \frac{1}{4} \int_{3/N}^{1-2N^{-2/3}} \frac{dx}{\theta_N^2 - x} = \frac{1}{4} \log \frac{\theta_N^2 - 3/N}{\theta_N^2 - 1 + 2N^{-2/3}} \\
= \frac{1}{6} \log N + O(\log \log N).
\]

\section{Proofs from Section 3}

\subsection{Proof of Lemma 5}

First we notice that the last part of the lemma follows from the Tracy-Widom law for \( \lambda_1, \lambda_k \), provided that \( k \) is a constant.

Now we prove the first inequality for the GUE case. Let \( \rho_{1,N}(x) \) denote the 1-point correlation function. By Johnstone and Ma [2012], for any \( u_0 \in \mathbb{R} \) there is \( C = C(u_0) \) such that for large enough \( N \),
\[
\rho_{1,N}(2 + N^{-2/3} u) \leq C N^{2/3} e^{-2u},
\]
holds uniformly over \( u \geq u_0 \). This bound yields that
\[
\mathbb{E}\#\{\lambda_i > 2 - CN^{-2/3}\} = \int_{2-CN^{-2/3}}^{\infty} \rho_{1,N}(x)dx \\
= \int_{-C}^{\infty} N^{-2/3} \rho_{1,N}(2 + uN^{-2/3})du \\
\leq C' \int_{-C}^{\infty} e^{-2u} du = C''.
\]

We now have,
\[
P(\lambda_k > 2 - CN^{-2/3}) = P(\#\{\lambda_i > 2 - CN^{-2/3}\} > k - 1) \leq \frac{C''}{k-1}.
\]
Taking \( k(\epsilon, C) = C''/\epsilon + 1 \) we get the first part of the lemma. The GOE case follows due to the relation between the GOE and the GUE distribution, namely,

\[
P^{\text{GOE}}(\lambda_{2k} > 2 - CN^{-2/3}) \leq P^{\text{GUE}}(\lambda_k > 2 - CN^{-2/3}).
\]

Hence it is sufficient to take \( k \) twice as large.

To show the second inequality, take \( k \) such that

\[
P(\lambda_k > 2 - 2CN^{-2/3}) \leq \epsilon/2.
\]

Then, by the Tracy-Widom law, take \( c_1 < C \) such that for each \( i = 1, \ldots, k \),

\[
P \left( \lambda_i \in [2 + N^{-2/3}(\tilde{\sigma}_n - c_1); 2 + N^{-2/3}(\tilde{\sigma}_n + c_1)] \right) \leq \epsilon/(2k),
\]

so that

\[
P \left( \lambda_1, \ldots, \lambda_k \notin [2 + N^{-2/3}(\tilde{\sigma}_n - c_1); 2 + N^{-2/3}(\tilde{\sigma}_n + c_1)] \right) \geq 1 - \epsilon/2,
\]

and since, with probability \( 1 - \epsilon/2 \) only \( \lambda_1, \ldots, \lambda_k \) can possibly fall into this interval, we get that

\[
P \left( \lambda_1, \ldots, \lambda_n \notin [2 + N^{-2/3}(\tilde{\sigma}_n - c_1); 2 + N^{-2/3}(\tilde{\sigma}_n + c_1)] \right) \geq 1 - \epsilon.
\]

### B.2 Proof of Proposition 4

First, we prove the proposition for GUE case \( \alpha = 1 \). Our proof is based on an asymptotic approximation for a one-point correlation function of GUE.

Let \( P_N(x_1, \ldots, x_N) \) be a joint density of unordered eigenvalues \( l_1, \ldots, l_N \) of scaled GUE (so that max \( l_i \) is close to 2 for large \( N \)). Following Tracy and Widom [1998], the \( k \)-point correlation function is defined as

\[
R_k(x_1, \ldots, x_k) = \frac{N!}{(N-k)!} \int \cdots \int P_N(x_1, \ldots, x_N)dx_1 \cdots dx_N.
\]

Note that this is not a probability density: it has total integral \( N!/(N-k)! \).

For any integrable function \( F(x_1, \ldots, x_k) \), we have

\[
\mathbf{E} F(l_1, \ldots, l_k) = \frac{(N-k)!}{N!} \int \cdots \int F(x_1, \ldots, x_k)R_k(x_1, \ldots, x_k)dx_1 \cdots dx_k.
\]

With determinantal structure (such as GUE), we have [Tracy and Widom [1998], (1.2)]

\[
R_k(x_1, \ldots, x_k) = \det(K_N(x_i, x_j))_{i,j=1,\ldots,k},
\]

where the kernel \( K_N(x, y) \) has the representation

\[
K_N(x, y) = \sum_{k=0}^{N-1} \phi_k(x)\phi_k(y)
\]

with \( \{\phi_k(x)\} \) obtained by orthonormalizing \( \{x^k e^{-Nx^2/4}\} \). Note that for unscaled GUE, one would orthonormalize functions \( \{x^k e^{-x^2/4}\} \).

Let \( s_N(x) = K_N(x, x) \). We then have in particular

\[
R_1(x) = s_N(x), \quad R_2(x, y) = s_N(x)s_N(y) - K_N^2(x, y).
\]

---

30
Using (41), we can interpret the normalized one-point correlation function, $\rho_N(\lambda) \equiv N^{-1} R_1(\lambda)$ as the “mean density” of the eigenvalues, and the expected value of linear spectral statistics $N^{-1} \sum_{i=1}^{N} f(l_i)$ can be written as
\[
\mathbb{E} \left[ N^{-1} \sum_{i=1}^{N} f(l_i) \right] = \int f(\lambda) \rho_N(\lambda) d\lambda.
\]

Furthermore, let $J = \mathbb{E} \left[ N^{-1} \sum_{i=1}^{N} f(l_i) \right]^2$. Expanding, we have
\[
J = N^{-2} \mathbb{E} \left[ \sum_{i=1}^{N} f^2(l_i) \right] + N^{-2} \mathbb{E} \left[ \sum_{i \neq j} f(l_i) f(l_j) \right]
\]
\[
= N^{-1} \mathbb{E} f^2(l_i) + N^{-2} \times N(N-1) \mathbb{E} [f(l_1) f(l_2)].
\]

Now apply (41) and then (42) to get
\[
J = N^{-2} \int f^2(x) R_1(x) dx + N^{-2} \int \int f(x) f(y) R_2(x, y) dy dx
\]
\[
= N^{-1} \int f^2(x) \rho_N(x) dx + \left[ \int f(x) \rho_N(x) dx \right]^2 - N^{-2} \int \int f(x) f(y) K_N^2(x, y) dy dx.
\]

Therefore,
\[
\mathbb{E} \left[ N^{-1} \sum_{i=1}^{N} f(l_i) \right]^2 \leq N^{-1} \int f^2(x) \rho_N(x) dx + \left[ \int f(x) \rho_N(x) dx \right]^2,
\]
which, since $\mathbb{E} N^{-1} \sum_{i=1}^{N} f(l_i) = \int f(x) \rho_N(x) dx$, may also be written as
\[
\text{Var} \left[ N^{-1} \sum_{i=1}^{N} f(l_i) \right] \leq N^{-1} \int f^2(x) \rho_N(x) dx.
\]

**Remark 24.** For the usual linear statistic, where $f$ does not depend on $N$, and is analytic in a neighborhood of $[-2, 2]$, this is a terrible bound, since then $\text{Var} \left[ \sum_{i=1}^{N} f(l_i) \right] = O(1)$. But in our critical case settings, it seems to give the right order, and will become useful below.

Ercolani and McLaughlin [2003] (EM in what follows) use the Riemann-Hilbert machinery to derive uniform asymptotic approximations for $\rho_N(\lambda)$. Their results imply the following lemma.

In preparation for the lemma, define
\[
\Phi_+(\lambda) = \begin{cases} 
- \left( \frac{3N}{4} \int_{\lambda}^{1} \sqrt{4-x^2} dx \right)^{2/3} & \text{for } \lambda \leq 2 \\
\left( \frac{3N}{4} \int_{2}^{\lambda} \sqrt{x^2 - 4} dx \right)^{2/3} & \text{for } \lambda > 2,
\end{cases}
\]
and similarly,
\[
\Phi_-(\lambda) = \begin{cases} 
- \left( \frac{3N}{4} \int_{-\lambda}^{2} \sqrt{4-x^2} dx \right)^{2/3} & \text{for } \lambda \geq -2 \\
\left( \frac{3N}{4} \int_{-2}^{-\lambda} \sqrt{x^2 - 4} dx \right)^{2/3} & \text{for } \lambda < -2,
\end{cases}
\]
We will only need $\Phi_+(\lambda)$ for $\lambda > -2$ and $\Phi_-(\lambda)$ for $\lambda < 2$, so that all square roots in the above definition are positive real numbers. Further, let
\[
\gamma(\lambda) = \left( \frac{\lambda - 2}{\lambda + 2} \right)^{1/4},
\]
where the latter expression is evaluated taking the limit from the upper half complex plane and using the principal branch of the root. That is,
\[
\gamma(\lambda) = \left| \frac{\lambda - 2}{\lambda + 2} \right|^{1/4} \times \begin{cases} 
1 & \text{for } \lambda \notin [-2, 2] \\
e^{i\pi/4} & \text{for } \lambda \in [-2, 2],
\end{cases}
\]

31
Finally, let \( \text{Ai}(x) \) denote the Airy function on the real line. Recall that \( \text{Ai}(x) \) is exponentially small for \( x > 0 \), and

\[
\text{Ai}(-x) = \pi^{-1/2} x^{-1/4} \left\{ \cos \left( \frac{2}{3} x^{3/2} - \frac{\pi}{4} \right) + O \left( x^{-3/2} \right) \right\}, \\
\text{Ai}'(-x) = \pi^{-1/2} x^{1/4} \left\{ \sin \left( \frac{2}{3} x^{3/2} - \frac{\pi}{4} \right) + O \left( x^{-3/2} \right) \right\}.
\]

(44)

(45)

**Lemma 25.** (Ercolani and McLaughlin [2003]) As \( N \to \infty \),

\[
N \rho_N(\lambda) = \left( \Phi'_+ (\lambda) \right) - \frac{\gamma' (\lambda)}{\gamma (\lambda)} \right) \left[ 2 \text{Ai} (\Phi_+ (\lambda)) \text{Ai}' (\Phi_+ (\lambda)) \right] + \Phi'_+ (\lambda) \left[ (\text{Ai}' (\Phi_+ (\lambda)))^2 - \Phi_+ (\lambda) (\text{Ai} (\Phi_+ (\lambda)))^2 \right] + O \left( \frac{1}{N} \right),
\]

where \( O \left( \frac{1}{N} \right) \) is uniform over \( \lambda \in [-\delta, 2 + \delta] \) for some small \( \delta > 0 \). Similarly,

\[
N \rho_N(\lambda) = - \left( \Phi'_- (\lambda) \right) + \frac{\gamma' (\lambda)}{\gamma (\lambda)} \right) \left[ 2 \text{Ai} (\Phi_- (\lambda)) \text{Ai}' (\Phi_- (\lambda)) \right] - \Phi'_- (\lambda) \left[ (\text{Ai}' (\Phi_- (\lambda)))^2 - \Phi_- (\lambda) (\text{Ai} (\Phi_- (\lambda)))^2 \right] + O \left( \frac{1}{N} \right),
\]

where \( O \left( \frac{1}{N} \right) \) is uniform over \( \lambda \in [-2 - \delta, \delta] \).

**Proof.** For \( \lambda \in [-\delta, 2 + \delta] \), the lemma follows from EM eq. (4.4) and Lemma 4.6. For \( \lambda \in [-2 - \delta, \delta] \) it follows from eq. (4.19) and Lemma 4.7.

**Remark 26.** The first asymptotic formula described in the above lemma also appears in Gustavsson [2005] in the first display of his proof of Lemma 2.1. However, instead of the term \( O \left( \frac{1}{N} \right) \) Gustavsson uses a more conservative \( O \left( \frac{1}{N \sqrt{2 - \lambda}} \right) \). We believe that such a conservatism is not necessary because the asymptotic terms of order \( 1/N \) in Lemma 4.6 of EM have several square-root-type singularities at \( \lambda = 2 \), which cancel each other. More specifically, those asymptotic terms depend on the ratio \( \sqrt{\Phi_+ (\lambda)/\gamma^2 (\lambda)} \), where both the numerator and the denominator behave similarly to \( \sqrt{2 - \lambda} \) in the neighborhood of \( \lambda = 2 \).

**Corollary 27.** As \( N \to \infty \),

\[
N \rho_N(\lambda) = \pm \Phi'_+ (\lambda) F_0 (\Phi_+ (\lambda)) + O (1),
\]

where \( F_0(\zeta) = (\text{Ai}'(\zeta))^2 - \zeta (\text{Ai}(\zeta))^2 \), and \( O (1) \) is uniform over \( \lambda \in [-\delta, 2 + \delta] \) for some small \( \delta > 0 \) when the sign is +, and over \( \lambda \in [-2 - \delta, \delta] \) when the sign is −.

**Proof.** Function \( \text{Ai}(x) \text{Ai}'(x) \) is continuous and bounded on \( \mathbb{R} \). Further,

\[
\frac{\Phi'_+ (\lambda)}{4 \Phi_+ (\lambda)} - \frac{\gamma' (\lambda)}{\gamma (\lambda)} = \left( \log \frac{\Phi_+^{1/4} (\lambda)}{\gamma (\lambda)} \right)'
\]

is continuous and bounded on \( \lambda \in [-\delta, 2 + \delta] \) because \( \log \frac{\Phi_+^{1/4} (\lambda)}{\gamma (\lambda)} \) is analytic for \( \lambda \in [-\delta, 2 + \delta] \) (the singularity at \( \lambda = 2 \) in the denominator cancels out with that in the numerator). Hence, \( \left( \frac{\Phi'_+ (\lambda)}{4 \Phi_+ (\lambda)} - \frac{\gamma' (\lambda)}{\gamma (\lambda)} \right) \left[ 2 \text{Ai} (\Phi_+ (\lambda)) \text{Ai}' (\Phi_+ (\lambda)) \right] \) is bounded on \( \lambda \in [-\delta, 2 + \delta] \), and we get the statement of the corollary for the ‘plus’ sign. For the ‘minus’ sign the statement is established similarly.
Now we are ready to prove Proposition 4 for the GUE case $\alpha = 1$. Consider the following function
\[
\phi(x) = 1 - 1(|x| < 1) \exp \left( -\frac{x^2}{1-x^2} \right),
\]
and let $f_\eta(x) = x^{-1}\phi(\eta^{-1}x)$. Then $f_\eta(x) = 1/x$ for all $|x| \geq \eta$. Moreover, since $\phi''(x)$ is bounded and $\phi'(0)$, it holds that $\phi(x) \leq Cx^2$ for all $x$ and some constant $C > 0$. Thus, we have
\[
\sup_{|x| \leq \eta} f_\eta(x) \leq \sup_{|x| \leq \eta} \left[ \frac{1}{x} \times C \left( \frac{x}{\eta} \right)^2 \right] = \frac{C}{\eta}.
\]
In addition, $\phi''(x)$ is bounded and $\phi''(0) = 2$ so that $|\phi'(x) - 2x| \leq Cx^2$ for all $x$. Then, for $|x| \leq \eta$,
\[
|f'_\eta(x)| \leq \frac{1}{|x|} \left( \frac{2|x|}{\eta} + C\frac{x^2}{\eta^2} \right) + \frac{1}{x^2} C\frac{x^2}{\eta^2} \leq C\eta^{-2},
\]
whereas $f''_\eta(x) = -1/x^2$ for $|x| > \eta$.

**Lemma 28.** Let $cN^{-2/3} < \eta < CN^{-2/3}$ for some positive constants $c, C$, and let $\tilde{\sigma}_N$ be either a constant or a sequence slowly diverging to $+\infty$ so that $\tilde{\sigma}_N \ll \sigma_N \equiv (\log \log N)^3$. Then
\[
\mathbb{E} \frac{1}{N} \sum_{i=1}^{N} f_\eta(2 + N^{-2/3}\tilde{\sigma}_N - \lambda_i) = \int f_\eta(2 + N^{-2/3}\tilde{\sigma}_N - \lambda) \rho_N(\lambda) d\lambda = 1 + O \left( (1 + |\tilde{\sigma}_N|^{1/2})N^{-1/3} \right).
\]

**Proof.** Let $\delta$ be a fixed small positive number, and $\chi_{in}(\lambda)$ be an infinitely differentiable ($C^\infty$) function supported on a compact subset of $(-2-\delta, 2+\delta)$ so that $\chi_{in}(\lambda) \equiv 1$ for $\lambda \in [-2-\delta, 2+\delta]$.

Since, as is well known, $\rho_N(\lambda)$ is asymptotically exponentially small in $\mathbb{R} \setminus [-2-\delta, 2+\delta]$, it is sufficient to show that
\[
\int g(\lambda) \rho_N(\lambda) d\lambda = 1 + O \left( (1 + |\tilde{\sigma}_N|^{1/2})N^{-1/3} \right),
\]
where $g(\lambda) = \chi_{in}(\lambda) f_\eta(2 + N^{-2/3}\tilde{\sigma}_N - \lambda)$.

Following EM p. 799, we consider a partition of unity $\{\chi_-(\lambda), \chi_+(\lambda)\}$ with $\chi_+(\lambda) \in C^\infty$ such that $0 \leq \chi_+(\lambda) \leq 1$, $\sup_{\lambda > \delta} \chi_+ \subset (-\delta, +\infty)$, and $\chi_+(\lambda) \equiv 1$ for $\lambda > \delta$. Then,
\[
\int g(\lambda) \rho_N(\lambda) d\lambda = \int_{-\delta}^{\delta} g_-(\lambda) \rho_N(\lambda) d\lambda + \int_{-\delta}^{\delta} g_+(\lambda) \rho_N(\lambda) d\lambda, \tag{46}
\]
where $g_- \equiv \chi_- g$ and $g_+ \equiv \chi_+ g$. By Corollary 27,
\[
\int_{-\delta}^{\delta} g_+(\lambda) \rho_N(\lambda) d\lambda = \frac{1}{N} \int_{-\delta}^{\delta} g_+(\lambda) \Phi_+(\lambda) F_0(\Phi_+(\lambda)) d\lambda + O(N^{-1}) \int_{-\delta}^{\delta} |g(\lambda)| d\lambda.
\]
The latter integral is $O(\log N)$ because $g(\lambda)$ is bounded by $C\eta^{-1}$ in the interval $2 + \tilde{\sigma}_N N^{-2/3} \pm \eta$, and the integral outside is at most $2 \int_{\eta}^{3} x^{-1} dx = 2 \log(3/\eta) = O(\log N)$. Further, EM eqs. (5.23), and (5.26) imply that
\[
\frac{1}{N} \int_{-\delta}^{\delta} g_+(\lambda) \Phi_+(\lambda) F_0(\Phi_+(\lambda)) d\lambda = \frac{c_0}{2} \int_{-\delta}^{\delta} g_+(\lambda) \sqrt{4 - \lambda^2} d\lambda + A,
\]
where (see EM eq. (5.12) and the asymptotic formulae (44-45) for the Airy function and its derivative) $c_0 = 1/\pi$, and
\[
A = -\frac{1}{N} \int_{-\delta}^{\delta} g_+(\lambda) G_1(\Phi_+(\lambda)) d\lambda - \frac{1}{N} \int_{2}^{2+\delta} g_+(\lambda) F_1(\Phi_+(\lambda)) d\lambda = O(N^{-1}) \int_{-\delta}^{\delta} |g_+(\lambda)| d\lambda
\]
33
with uniformly bounded $G_1(\cdot)$ and $F_1(\cdot)$. Since $\chi_{in}, \chi_+ \in C^\infty$, $|g'| \leq |f'| + C|f_\eta|$. Similarly to the integral of $|g|$, the integral of $C|f_\eta|$ is $O(\log N)$. On the other hand, in the $\eta$-neighborhood of $2 + \tilde{\sigma} N^{-2/3}$, $|f'|$ is bounded by $C\eta^{-2}$, whereas outside this neighborhood, it is at most $2 \int_\eta^3 x^{-2} dx = O(\eta^{-1})$. Hence, we get $A = O(N^{-1/3})$, and thus

$$
\int_{-\delta}^{2+\delta} g_+(\lambda) \rho_N(\lambda) d\lambda = \frac{1}{2\pi} \int_{-\delta}^{2} g_+(\lambda) \sqrt{4 - \lambda^2} d\lambda + O(N^{-1/3}).
$$

Similarly, we get

$$
\int_{-\delta}^{\delta} g_-(\lambda) \rho_N(\lambda) d\lambda = \frac{1}{2\pi} \int_{-\delta}^{\delta} g_-(\lambda) \sqrt{4 - \lambda^2} d\lambda + O(N^{-1/3}).
$$

Summing up the latter two displays gives

$$
\int g_+(\lambda) \rho_N(\lambda) d\lambda = \frac{1}{2\pi} \int_{-2}^{2} g(\lambda) \sqrt{4 - \lambda^2} d\lambda + O(N^{-1/3}).
$$

It remains to show that $\frac{1}{2\pi} \int_{2}^{\delta} g(\lambda) \sqrt{4 - \lambda^2} d\lambda = 1 + O\left((1 + |\tilde{\sigma} N|^{1/2}) N^{-1/3}\right)$.

For $\tilde{\sigma} N$ that slowly diverges to $\pm\infty$ and all sufficiently large $N$, the latter integral equals the continuous extension of the negative of the Stieltjes transform of the semicircle law to the point $z = 2 + \tilde{\sigma} N^{-2/3} \in \mathbb{R}$. Hence, for such $\tilde{\sigma} N$,

$$
\frac{1}{2\pi} \int_{-2}^{2} g(\lambda) \sqrt{4 - \lambda^2} d\lambda = \frac{z - \sqrt{z^2 - 4}}{2} = 1 + O(\tilde{\sigma} N^{1/2} N^{-1/3}).
$$

For constant $\tilde{\sigma} N$, set $a = \max\{0, \eta N^{2/3} - \tilde{\sigma} N\}$. If $a = 0$, the latter display remains valid. If $a > 0$, we split the integral over $[-2, 2]$ into two parts. On the interval $[2 - aN^{-2/3}, 2]$, $|g(\lambda)| = O(N^{2/3})$. Therefore,

$$
\int_{2-aN^{-2/3}}^{2} g_+(\lambda) \rho_N(\lambda) d\lambda = O(N^{2/3}) \times O((aN^{-2/3})^{3/2}) = O(N^{-1/3}).
$$

For $\lambda \in [-2, 2 - aN^{-2/3}]$, we have $2 + \tilde{\sigma} N^{-2/3} - \lambda \geq 0$. Therefore,

$$
\int_{-2}^{2-aN^{-2/3}} \left| \frac{1}{2 + \tilde{\sigma} N - \lambda} - \frac{1}{2 - \lambda} \right| \sqrt{4 - \lambda^2} d\lambda
\leq 2|\tilde{\sigma} N|^{-2/3} \int_{-2}^{2-aN^{-2/3}} \left( 2 + \min\{\tilde{\sigma} N, 0\} N^{-2/3} - \lambda \right)^{-3/2} d\lambda = O(N^{-1/3}).
$$

On the other hand,

$$
\int_{-2}^{2-aN^{-2/3}} \frac{\sqrt{4 - \lambda^2}}{2 - \lambda} d\lambda = 1 + O(N^{-1/3}).
$$

Summing up,

$$
\frac{1}{2\pi} \int_{-2}^{2} g(\lambda) \sqrt{4 - \lambda^2} d\lambda = 1 + O\left((1 + |\tilde{\sigma} N|^{1/2}) N^{-1/3}\right).
$$

\[\square\]

**Lemma 29.** Under assumptions of Lemma 28,

$$
\int (f_\eta(2 + N^{-2/3} \tilde{\sigma} N - \lambda))^2 \rho_N(\lambda) d\lambda = O\left(N^{1/3}\right).
$$

34
Proof. Using arguments very similar to those that we used in the proof of Lemma 28, we reduce the problem to showing that
\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} (g(\lambda))^2 \sqrt{4 - \lambda^2} d\lambda = O(N^{1/3}).
\]

For \( \tilde{\sigma}_N \) that slowly diverges to \(-\infty\) and all sufficiently large \( N \), the latter integral equals the continuous extension of the derivative of the Stieltjes transform of the semicircle law to the point \( z = 2 + \tilde{\sigma}_N N^{-2/3} \in \mathbb{R} \). Hence, for such \( \tilde{\sigma}_N \),
\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} (g(\lambda))^2 \sqrt{4 - \lambda^2} d\lambda = -\frac{1}{2} + \frac{\sqrt{2}}{2\sqrt{\lambda^2 - 4}} = O(\tilde{\sigma}_N^{-1/2} N^{1/3}) = o\left(N^{1/3}\right).
\]

For constant \( \tilde{\sigma}_N \), set \( a = \max\{0, \eta N^{2/3} - \tilde{\sigma}_N\} \). If \( a = 0 \), the latter display remains valid. If \( a > 0 \), we split the integral over \([-2, 2]\) into two parts. On the interval \([2 - aN^{-2/3}, 2]\), \( |g(\lambda)| = O(N^{2/3}) \). Therefore,
\[
\int_{2 - aN^{-2/3}}^{2} (g(\lambda))^2 \sqrt{4 - \lambda^2} d\lambda = O(N^{4/3}) \times O((aN^{-2/3})^{3/2}) = O(N^{1/3}).
\]

For \( \lambda \in [-2, 2 - aN^{-2/3}] \), \( (2 - \lambda)^2 g(\lambda)^2 = \left(\frac{2 - \lambda}{2 + \tilde{\sigma}_N N^{-2/3} - \lambda}\right)^2 \) is bounded. Therefore,
\[
\int_{-2}^{2 - aN^{-2/3}} g(\lambda)^2 \sqrt{4 - \lambda^2} d\lambda \leq O(1) \int_{-2}^{2 - aN^{-2/3}} (2 - \lambda)^{-3/2} d\lambda = O(N^{1/3}).
\]

Combining Lemma 28 with inequality (43), we obtain the following corollary.

Corollary 30. Under assumptions of Lemma 28,
\[
\text{Var}\left(\frac{1}{N} \sum_{i=1}^{N} f_{\eta}(2 + \tilde{\sigma}_N N^{-2/3} - \lambda_i)\right) = O(N^{-2/3}).
\]

Finally, let \( \epsilon \) be an arbitrarily small positive constant. By Theorem 3.1.4 from Anderson et al. [2010], there exist \( c_\epsilon, n_\epsilon \) such that for all \( N > n_\epsilon \)
\[
P\left(\lambda_1, \ldots, \lambda_N \notin (2 + \tilde{\sigma}_N N^{-2/3} - c_\epsilon N^{-2/3}, 2 + \tilde{\sigma}_N N^{-2/3} + c_\epsilon N^{-2/3})\right) \geq 1 - \epsilon/2.
\]

Take \( \eta = c_\epsilon N^{-2/3} \). Then, by definition of \( f_{\eta} \) and \( \mu_i \), with probability \( 1 - \epsilon/2 \) and all sufficiently large \( N \),
\[
\sum_{i=1}^{N} f_{\eta}(2 + \tilde{\sigma}_N N^{-2/3} - \lambda_i) = \sum_{i=1}^{N} \mu_i^{-1} \quad \text{and} \quad \sum_{i=1}^{N} f_{\eta}^2(2 + \tilde{\sigma}_N N^{-2/3} - \lambda_i) = \sum_{i=1}^{N} \mu_i^{-2}.
\]

On the other hand, by Lemmas 28-29 and Corollary 30, with probability \( 1 - \epsilon/2 \) and all sufficiently large \( N \), we have
\[
\left|\sum_{i=1}^{N} f_{\eta}(2 + \tilde{\sigma}_N N^{-2/3} - \lambda_i) - N\right| \leq CN^{2/3} \quad \text{and} \quad \sum_{i=1}^{N} f_{\eta}^2(2 + \tilde{\sigma}_N N^{-2/3} - \lambda_i) \leq CN^{4/3},
\]
for some constant \( C \). Since \( \epsilon \) can be arbitrarily small, we conclude that
\[
\sum_{i=1}^{N} \mu_i^{-1} - N = O_P \left((1 + |\tilde{\sigma}_N|^{1/2})N^{2/3}\right) \quad \text{and} \quad \sum_{i=1}^{N} \mu_i^{-2} = O_P(N^{4/3}).
\]

This finishes our proof of Proposition 4 for the GUE case.

For the GOE case, the proposition follows from the following theorem. A proof of this theorem can be found in Section B.3 below.
**Theorem 31.** Let $M_N^C$ and $M_N^R$ be $N \times N$ (unscaled) GUE and GOE matrices, respectively. Suppose that $f_N$ is a series of functions such that

$$f_N(M_N^C) = a_N + O_p(b_N),$$

for some sequences $a_N$ and $b_N$. Then,

$$f_N(M_N^R) = a_N + O_p(b_N + TV(f_N)),$$

where $TV(f_N)$ is the total variation of $f_N$.

Indeed, the theorem and the fact that scaling of the argument does not change the total variation of a function yield the equivalents of Lemmas 28–29 and Corollary 30 for GOE. These equivalents, combined with the following lemma, imply Proposition 4 for GOE.

**Lemma 32.** Let $M_N$ be an $N \times N$ GOE, and let $\{\lambda_i\}$ be the eigenvalues of $M_N/\sqrt{N}$. For any $\varepsilon > 0$, there exist $c_\varepsilon, n_\varepsilon$ such that, for all $N > n_\varepsilon$,

$$P \left( \lambda_1, \ldots, \lambda_N \notin (2 + \tilde{\sigma}_N N^{-2/3} - c_\varepsilon N^{-2/3}, 2 + \tilde{\sigma}_N N^{-2/3} + c_\varepsilon N^{-2/3}) \right) \geq 1 - \varepsilon/2.$$

**Proof.** The lemma follows e.g. from eqs. (2.4) and (2.19 a) of Bornemann [2010].

**B.3 Proof of Theorem 31**

The main engine of this result is an identity stated in Forrester and Rains [2001], which relates the eigenvalues of a GUE to the eigenvalues of two independent GOEs. In particular, we use it in the following lemma.

**Lemma 33.** Let $M_N^C$ be an $N \times N$ GUE, and let $f$ be a function of bounded variation with total variation $TV(f)$. If $M_N^R, \tilde{M}_N^R$ are two independent GOEs, then

$$f(M_N^C) \overset{d}{=} \frac{1}{2} \left( f(M_N^R) + f(\tilde{M}_N^R) \right) + X_N,$$

where $|X_N| \leq TV(f)$, and $\overset{d}{=} \text{denotes equality in distribution.}$

**Proof.** Let $M_N^R, \tilde{M}_N^R$ be independent $N \times N$ and $(N + 1) \times (N + 1)$ GOEs. Call the eigenvalues of $M_N^R$ and $\tilde{M}_N^R + 1$ be $\{\lambda_i\}_{i=1}^N$ and $\{\tilde{\lambda}_i\}_{i=1}^{N+1}$, respectively. Further, denote the combined set of eigenvalues $\{\lambda_i\}_{i=1}^N \cup \{\tilde{\lambda}_i\}_{i=1}^{N+1}$ as $\Lambda^+$, and enumerate its elements in decreasing order

$$\Lambda^+ = \{\lambda_1^+ \geq \ldots \geq \lambda_{2N+1}^+\}.$$

Theorem 5.2 of Forrester and Rains [2001] implies that the even elements of this set are equal in distribution to the eigenvalues of an $N \times N$ GUE.

Thus, if $M_N^C$ is an $N \times N$ GUE, we have

$$f(M_N^C) \overset{d}{=} \sum_{i=1}^{N} f(\lambda_{2i}^+),$$

$$= \frac{1}{2} \left( \sum_{j=1}^{2N+1} f(\lambda_j^+) + \sum_{i=1}^{N} [f(\lambda_{2i}^+) - f(\lambda_{2i-1}^+)] - f(\lambda_{2N+1}^+) \right),$$

$$= \frac{1}{2} \left( f(W_N^R) + f(W_{N+1}^R) - f(\lambda_{2N+1}^+) + \sum_{i=1}^{N} [f(\lambda_{2i}^+) - f(\lambda_{2i-1}^+)] \right).$$
Notice that, since \( \lambda_j^+ \) are ordered, we have
\[
\left| \sum_{i=1}^{N} [f(\lambda^+_{2i}) - f(\lambda^+_{2i-1})] \right| \leq \text{TV}(f).
\]
Further, let \( \tilde{M}^R_N \) be the principal submatrix of \( \tilde{M}^R_{N+1} \), which is thus independent and equal in distribution to \( \tilde{M}^R_N \). If we let \( \tilde{\mu}_1, \ldots, \tilde{\mu}_N \) be the eigenvalues of \( \tilde{M}^R_N \), then Cauchy’s interlacing theorem yields
\[
\tilde{\lambda}_1 \geq \tilde{\mu}_1 \geq \tilde{\lambda}_2 \geq \ldots \geq \tilde{\lambda}_N \geq \tilde{\mu}_N \geq \tilde{\lambda}_{N+1},
\]
and so we have
\[
\left| f(\tilde{M}^R_{N+1}) - f(\lambda^+_{2N+1}) - f(\tilde{M}^R_N) \right| = \sum_{i=1}^{N} |f(\tilde{\lambda}_i) - f(\tilde{\mu}_i)| + |f(\tilde{\lambda}_{N+1}) - f(\lambda^+_{2N+1})| \leq \text{TV}(f).
\]
We conclude that (47) holds. \( \square \)

An immediate useful corollary is as follows.

**Corollary 34.** Under assumptions of Lemma 32,
\[
\mathbf{E}f(M^R_N) = \mathbf{E}f(M^C_N) + O(\text{TV}(f)),
\]
\[
\text{Var}f(M^R_N) \leq 2\text{Var}f(M^C_N) + 2\text{TV}^2(f).
\]

**Remark 35.** Notice that corollary 34 also holds for scaled Gaussian matrices \( M^R_N/\sqrt{N} \), since \( f(W^R_N) = g(M^R_N) \) for \( g(\lambda) = f(\lambda/\sqrt{N}) \), which satisfy \( \text{TV}(f) = \text{TV}(g) \).

However, to finish proving Theorem 31 in its generality, we require the following technical lemma about tightness.

**Lemma 36.** Let \( X_N, Y_N \) be iid sequences of random variables such that \( X_N + Y_N \) is tight. Then \( X_N \) (and thus also \( Y_N \)) is tight.

**Proof.** For any constant \( K \), we have
\[
P(X_N > K) = P(X_N > K, Y_N > K)^{1/2} \leq P(\{|X_N + Y_N| > K\}^{1/2},
\]
and similarly,
\[
P(X_N < -K) \leq P(\{|X_N + Y_N| > K\}^{1/2},
\]
which yield
\[
\sup_N P(\{|X_N| > K\} \leq 2\sup_N P(\{|X_N + Y_N| > K\}^{1/2}.
\]
The right hand side of the latter inequality can be made arbitrarily small, by the tightness of \( X_N + Y_N \). \( \square \)

With all these results in hand, we are ready to complete the proof of Theorem 31. We have
\[
\left| \frac{f_N(W^R_N) - a_N}{b_N + \text{TV}(f_N)} + \frac{f_N(W^C_N) - a_N}{b_N + \text{TV}(f_N)} \right| = 2 \left| \frac{(f_N(W^R_N) + f_N(W^C_N))/2 - a_N}{b_N + \text{TV}(f_N)} \right| \leq 2 \left| \frac{(f_N(W^R_N) + f_N(W^C_N))/2 + X_N - a_N}{b_N} \right| + 2 \left| \frac{X_N}{\text{TV}(f_N)} \right| = 2 \left| \frac{f_N(W^C_N) - a_N}{b_N} \right| + 2 \left| \frac{X_N}{\text{TV}(f_N)} \right|.
\]
The first term in the latter sum is tight by assumption, whereas the second term is no larger than 2. But since the two terms on the left hand side of (48) are iid, Lemma 36 yields that they must be tight, and so

\[ f_N(W^R_N) = a_N + O_P(b_N + TV(f_N)). \]

References

James T. Albrecht, Cy P. Chan, and Alan Edelman. Sturm sequences and random eigenvalue distributions. *Foundations of Computational Mathematics*, 9(4):461–483, 2009.

G. Anderson and S.-L. Qiu. A monotoneity property of the gamma function. *Proceedings of the American Mathematical Society*, 125(11):3355–3362, 1997.

Greg W. Anderson, Alice Guionnet, and Ofer Zeitouni. *An Introduction to Random Matrices*. Cambridge university press, 2010.

Jinho Baik and Ji Oon Lee. Fluctuations of the free energy of the spherical Sherrington–Kirkpatrick model. *Journal of Statistical Physics*, 165(2):185–224, 2016.

E. L. Basor and H. Widom. Determinants of Airy operators and applications to random matrices. *Journal of Statistical Physics*, 96:1–20, 1999.

Folkmar Bornemann. On the numerical evaluation of distributions in random matrix theory: a review. *Markov Processes and Related Fields*, 16:803–866, 2010.

Stéphane Boucheron, Gábor Lugosi, and Pascal Massart. *Concentration inequalities: A nonasymptotic theory of independence*. Oxford university press, 2013.

P. Bourgade and K. Mody. Gaussian fluctuations of the determinant of Wigner matrices. *Electronic Journal of Probability*, 24(96):1–28, 2019.

P. Dharmawansa, I. M. Johnstone, and A. Onatski. Local asymptotic normality of the spectrum of high-dimensional spiked F-ratios. *arXiv:1411.3875*, 2014.

T. K. Duy. Distributions of the determinants of Gaussian beta ensembles. In *2023 Spectral and Scattering Theory and Related Topics*, pages 77–85. RIMS Kokyuroku, 2017.

N. M. Ercolani and K. D. T.-R. McLaughlin. Asymptotics of the partition function for random matrices via Riemann-Hilbert techniques and applications to graphical enumeration. *International Mathematics Research Notices*, (14):755–820, 2003.

Nasrollah Etemadi. On some classical results in probability theory. *Sankhyā: The Indian Journal of Statistics, Series A*, pages 215–221, 1985.

D. Féral and S. Péché. The Largest Eigenvalue of Rank One Deformation of Large Wigner Matrices. *Communications in Mathematical Physics*, 272:185–228, 2007.

P. J. Forrester. *Log-Gases and Random Matrices*. Princeton University Press, 2010.

P. J. Forrester and E. M. Rains. Inter-relationships between orthogonal, unitary and symplectic matrix ensembles. In *Random Matrix Models and Their Applications*. Mathematical Sciences Research Institute publications, volume 40, pages 171–207. Cambridge University Press, 2001.

Y. V. Fyodorov, B. A. Khoruzhenko, N. J. Simm, et al. Fractional brownian motion with hurst index $h = 0$ and the Gaussian Unitary Ensemble. *The Annals of Probability*, 44(4):2980–3031, 2016.

Friedrich Götze, Holger Sambale, and Arthur Sinulis. Concentration inequalities for polynomials in $\alpha$-sub-exponential random variables. *arXiv preprint arXiv:1903.05964*, 2019.
J. Gustavsson. Gaussian fluctuations of eigenvalues in the GUE. *Annales de l’Institute Henry Poincare*, 41:151–178, 2005.

Kurt Johansson. On fluctuations of eigenvalues of random hermitian matrices. *Duke Mathematical Journal*, 91(1):151–204, 1998.

Iain M. Johnstone and Zongming Ma. Fast approach to the Tracy-Widom law at the edge of GOE and GUE. *The Annals of Applied Probability*, 22(5):1962–1988, 2012.

Iain M. Johnstone and Alexei Onatski. Testing in high-dimensional spiked models. *Annals of Statistics*, 48(3):1231–1254, 2020.

I. V. Krasovsky. Correlations of the characteristic polynomials in the Gaussian Unitary Ensemble or a singular Hankel determinant. *Duke Mathematical Journal*, 139(3):581–619, 2007.

Gaultier Lambert and Elliot Paquette. Strong approximation of Gaussian $\beta$-ensemble characteristic polynomials: the hyperbolic regime. *arXiv preprint arXiv:2001.09042*, 2020a.

Gaultier Lambert and Elliot Paquette. Strong approximation of Gaussian $\beta$-ensemble characteristic polynomials: the edge regime and the stochastic Airy function. *arXiv preprint arXiv:2009.05003*, 2020b.

Yiting Li, Kevin Schnelli, and Yuanyuan Xu. Central limit theorem for mesoscopic eigenvalue statistics of deformed Wigner matrices and sample covariance matrices. *arXiv preprint arXiv:1909.12821v2*, 2019.

Hoi H. Nguyen, Van Vu, et al. Random matrices: Law of the determinant. *The Annals of Probability*, 42(1):146–167, 2014.

Emmanuel Rio. Moment inequalities for sums of dependent random variables under projective conditions. *Journal of Theoretical Probability*, 22(1):146–163, 2009.

Terence Tao and Van Vu. A central limit theorem for the determinant of a Wigner matrix. *Advances in Mathematics*, 231(1):74–101, 2012.

Craig A. Tracy and Harold Widom. Correlation functions, cluster functions, and spacing distributions for random matrices. *Journal of Statistical Physics*, 92:809–835, 1998.

Hale F. Trotter. Eigenvalue distributions of large Hermitian matrices; Wigner’s semi-circle law and a theorem of Kac, Murdock, and Szegö. *Advances in mathematics*, 54(1):67–82, 1984.

Eugene P. Wigner. Distribution laws for the roots of a random Hermitian matrix. *Statistical Theories of Spectra: Fluctuations*, pages 446–461, 1965.