MICROFORMAL GEOMETRY

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Abstract. We consider two formal categories whose morphisms are 'formal thickenings' of smooth maps of supermanifolds defined by formal canonical relations of a special form between the cotangent or anticotangent bundles (for the 'even' or 'odd' thickening, respectively). We came to such morphisms in arXiv:1409.6475 from construction of a nonlinear mapping of functions which generalizes the pullback w.r.t. a smooth map.

Such 'microformal' or 'thick' morphisms are analogs (in the formal setup) of the micro-symplectic morphisms introduced earlier by Cattaneo–Dherin–Weinstein from a different motivation. In our previous paper arXiv:1409.6475, we constructed for thick morphisms certain pullbacks of functions, which are nonlinear mappings of formal functional supermanifolds. In this paper, we establish functorial properties of such nonlinear pullbacks. We consider applications to vector bundles and algebroids. In particular, we introduce the adjoint for a nonlinear morphism of vector bundles, which is a thick morphism of the dual bundles such that for linear operators it reduces to the ordinary adjoint. By using this construction, we show how an \( L_\infty \)-morphism of \( L_\infty \)-algebroids induces an \( L_\infty \)-morphism of the homotopy Lie–Poisson or Lie–Schouten algebras of functions on the (anti)dual vector bundles. This has an application to higher Koszul brackets and triangular \( L_\infty \)-bialgebroids.

Introduction

In a remarkable series of papers [1, 2, 3], see also [4] and [26], A. Cattaneo, B. Dherin and A. Weinstein developed 'symplectic microgeometry' as an approach to Poisson geometry aimed at obtaining a functorial framework for Poisson manifold quantization and based on considering germs of symplectic manifolds near Lagrangian submanifolds. They call such germs, 'symplectic microfolds', and the corresponding germs of canonical relations, 'symplectic micromorphisms'. Crucial elements of this approach, namely, that one should consider germs of pairs (manifold, closed submanifold) and the emphasis on canonical relations as generalization of symplectomorphisms, can be traced back to Weinstein’s earlier works [20, 21, 22, 23, 24].

We arrived at a similar categorical structure in [19] from a quite different starting point, namely, we were searching for a natural construction which would give \( L_\infty \)-morphisms of homotopy Schouten or homotopy Poisson algebras of functions (in relation with our earlier study of higher Koszul brackets [7]). Such morphisms by definition should be nonlinear maps of the corresponding functional supermanifolds. In solving this problem, we discovered the notion of a ‘nonlinear pullback’ of functions\(^1\) with respect to a generalization of a smooth map of supermanifolds (called in this paper a ‘thick morphism’). It is a canonical relation of a particular type between the corresponding cotangent bundles [19]. In this

\(^1\)Functions have be of definite parity, even or odd, because of the nonlinearity; there are two parallel but different constructions for both cases.
way, behind our constructions we have found two formal categories that are very close to the category considered by Cattaneo–Dherin–Weinstein. Roughly speaking, they can be seen as its formal analogs (with an important change of viewpoint, see below).

In the present paper, we describe these two formal categories $\mathcal{E}\text{Thick}$ and $\mathcal{O}\text{Thick}$ more systematically and in greater detail than in [19] and give new applications to vector bundles and $L_\infty$-algebroids. We call morphisms in $\mathcal{E}\text{Thick}$ and $\mathcal{O}\text{Thick}$, respectively, even and odd, microformal morphisms, or thick morphisms. The usual smooth maps between supermanifolds form closed subspaces in the spaces of thick morphisms. Each formal category, $\mathcal{E}\text{Thick}$ or $\mathcal{O}\text{Thick}$, is a formal neighborhood of the category of supermanifolds $\mathcal{S}\text{Man}$ (more precisely, of the “affine” category obtained as the semidirect product of $\mathcal{S}\text{Man}$ with the spaces of even or odd functions). We use the name microformal geometry as a general name for the theory.

There are several important differences between this theory and the ‘symplectic microgeometry’ of Cattaneo–Dherin–Weinstein. First, we systematically work with supermanifolds. This may seem inessential, and, in fact, we have no doubt that the concepts of [1, 2, 3] should extend readily to the supercase. However, we worked with supermanifolds from the start because our motivation was to construct $L_\infty$-morphisms, and without that the results of [19] would not have been possible. Also, in the super setup one finds two parallel but distinct ‘thickenings’ of the category of (super)manifolds. Secondly, we consider formal neighborhoods or infinite jets rather than germs (as in [1, 2, 3]). Some of our constructions may extend to the germ setting, but we have not looked into that systematically. Moreover, working with formal objects has its own advantages. Finally, and most importantly: our principal novelty is the nonlinear pullbacks discovered in [19] and everything that has emerged from that. The categorical framework plays a more subordinate role and is introduced as an answer to the question, ‘A pullback... with respect to what?’, while in [1, 2, 3] the category is the central object of study. A substantial difference of viewpoints is that in [1, 2, 3] the theory is designed as a ‘micro’ version of symplectic geometry, while we look for a formal extension of the usual differential topology. Our ‘thick morphisms’ are generalized morphisms between supermanifolds themselves, not between their cotangent bundles.

The structure of this paper is as follows.

In Section 1 we define thick morphisms of supermanifolds and check that they form a category (in the even and odd versions). We recall nonlinear pullbacks introduced in [19] and establish their functorial properties (as stated in [19]).

In Sections 2 and 3 we consider applications to vector bundles and algebroids. In Section 2 we obtain an adjoint for a nonlinear mapping of vector bundles. It is a thick morphism of the dual bundles and reduces to the ordinary adjoint in the linear case. The construction uses the Mackenzie–Xu transformation $T^*E \cong T^*E$ and its odd analog $\Pi T^*E \cong \Pi T^*\Pi E$ from [14]. By using this construction, in Section 3 we show how an $L_\infty$-morphism of $L_\infty$-algebroids induces an $L_\infty$-morphism of the homotopy Lie–Poisson

\footnote{The first root in the adjective is the same as in ‘symplectic microgeometry’, as well as in microlocal analysis (i.e., local in the cotangent or jet directions) and in Milnor’s microbundles. The second root indicates our viewpoint that this theory is a formal extension of the familiar differential topology.

\footnote{So that the ‘micro’ approach makes it possible to overcome certain difficulties such as with the composition of canonical relations and the global existence of the symplectic groupoid.}
brackets of functions on the dual vector bundles. This has an application to higher Koszul brackets and triangular $L_\infty$-bialgebroids.

A remark about usage: to simplify the language, we speak about ‘manifolds’ meaning ‘supermanifolds’ unless we wish to emphasize that we consider the supercase.

Acknowledgements. I would like to thank all the listeners of the mini-course organized by the Steklov Mathematical Institute and the Skolkovo Institute of Science and Technology (Moscow) in October 2014, and the participants of the seminars at the Department of Mathematics of the Moscow State University and the Department of Quantum Field Theory of the Lebedev Physical Institute in Moscow, where this work was reported. In particular, I thank H. M. Khudaverdian, K. C. H. Mackenzie, M. A. Grigoriev and M. A. Vassiliev for very useful discussions.

1. ‘Even’ and ‘odd’ microformal categories

Consider supermanifolds $M_1$ and $M_2$. They can very well be ordinary manifolds and for a substantial part of the constructions below the super context is not really important. However, working with supermanifolds is essential for applications. Also, for some applications such as in physics, it may be important to have graded manifolds, i.e., to assume that besides parity there is an independent $\mathbb{Z}$-grading or ‘weight’ (see [14], also [17], [18], [16]). Weights can be systematically introduced when necessary.

Definition 1. A thick morphism (or microformal morphism) $\Phi: M_1 \rightarrow M_2$ is defined by a formal canonical relation (which we denote by the same letter) $\Phi \subset T^*M_2 \times (-T^*M_1)$ such that in local coordinates it has the following form:

$$\Phi = \left\{ (y^i, q_i; x^a, p_a) \mid y^i = (-1)^i \frac{\partial S}{\partial q_i}(x, q), p_a = \frac{\partial S}{\partial x^a}(x, q) \right\} \quad (1)$$

where $S = S(x, q)$ is a (locally defined) generating function. (The meaning of the word ‘formal’ is explained below.) We consider $S$ as part of the structure and call it the generating function of a thick morphism. (Note that $S$ is even.)

Here and throughout the paper we use the following notations. Local coordinates on $M_1$ are denoted by $x^a$ and the canonically conjugate momenta by $p_a$. Respectively, for $M_2$ we have $y^i$ and $q_i$. So the canonical symplectic forms on $T^*M_1$ and $T^*M_2$ are:

$$\omega_1 = dp_a dx^a = d(p_a dx^a) \quad \text{and} \quad \omega_2 = dq_i dy^i = d(q_i dy^i).$$

The minus sign in $T^*M_2 \times (-T^*M_1)$ means that the product $T^*M_2 \times T^*M_1$ is taken with the symplectic form

$$\omega = \omega_2 - \omega_1 = d(q_i dy^i - p_a dx^a).$$

That $\Phi \subset T^*M_2 \times (-T^*M_1)$ is a canonical relation between $T^*M_1$ and $T^*M_2$ (sometimes called a ‘canonical correspondence’, denoted $T^*M_1 \rightarrow T^*M_2$) means that it is a Lagrangian submanifold in $T^*M_2 \times T^*M_1$ w.r.t. the form $\omega = \omega_2 - \omega_1$. On such submanifolds we, in general, have

$$q_i dy^i - p_a dx^a = dF \quad (2)$$

We have changed notations in comparison with [19], where $T^*M_1 \times (-T^*M_2)$ was used. The order $T^*M_2 \times (-T^*M_1)$ is more traditional in symplectic geometry.
for some locally defined function $F$ on $\Phi$. An extra condition imposed on the submanifold $\Phi$ in Definition 1 is that the variables $x^a$ and $q_i$ are independent on $\Phi$ and can be taken as local coordinates there. Then the function $S(x, q)$ in equation (1) is related with the function $F$ in (2) by a kind of Legendre transform:

$$S = y^i q_i - F$$

(to clarify: $F$ is a genuine scalar function defined locally on the submanifold $\Phi$ and its definition does not depend on coordinate systems; in contrast with it, $S$ is a function of particular variables and is by construction coordinate-dependent). The relation $\Phi$ determines only the differentials $dF$ or $dS$, but we assume that constants of integration can be unambiguously chosen, which makes it possible to speak about the generating functions themselves. (For example, if the closed canonical 1-form $q_i dy^i - p_a dx^a$ on $\Phi$ is exact, we can choose a global $F$ and this gives a transformation law for $S$.)

What is a coordinate-free characterization of Lagrangian submanifolds considered in Definition 1? The condition that $x^a$ are independent on $\Phi$ is invariant and is equivalent to saying that the submanifold $\Phi$ projects on $M_1$ without degeneration (the projection has full rank). In contrast with it, the second condition that $q_i$ are independent on $\Phi$ is equivalent to saying that $\Phi$ “projects without degeneration on the fibers of $T^*M_2$”, but this does not have a well-defined meaning without a choice of a local trivialization. Consider, however, the differentials $dq_i$. We have $q_i = \partial y^{i'} / \partial y^i q_{i'}$, so we obtain

$$dq_i = d\left( \frac{\partial y^{i'}}{\partial y^i} \right) q_{i'} + (1)^{i' + r} \frac{\partial y^{i'}}{\partial y^{i'}} dq_{i'}. \tag{3}$$

We see that when $q_{i'}$ are small (i.e., we are near the zero section of $T^*M_2$), the linear independence of $dq_i$ implies the linear independence of $dq_{i'}$, and vice versa. Therefore we conclude that the condition that the variables $q_i$ are independent on $\Phi$ (or “$\Phi$ projects without degeneration on the fibers of $T^*M_2$”) has invariant meaning on an unspecified neighborhood of the zero section of $T^*M_2$, in particular, on the formal neighborhood of $M_2$ in $T^*M_2$. This is the explanation of the adjective ‘formal’ in Definition 1: the relation $\Phi$ is a formal canonical relation in the sense that it is a Lagrangian submanifold of the formal neighborhood of $M_2 \times M_1$ in $T^*M_2 \times (-T^*M_1)$.

In particular, the generating function $S(x, q)$ of a thick morphism $\Phi$: $M_1 \rightarrow M_2$ is a formal power series

$$S(x, q) = S_0(x) + S^i(x)q_i + \frac{1}{2} S^{ij}(x)q_iq_j + \frac{1}{3!} S^{ijk}(x)q_iq_jq_k + \ldots \tag{4}$$

In the sequel we shall frequently suppress the adjective ‘formal’ speaking about various objects.

Consider the algebras of smooth functions $C^\infty(M)$. We shall regard functions of particular parity on a supermanifold as points of a supermanifold in its own right (infinite-dimensional, of course). We use boldface to distinguish it from the $\mathbb{Z}_2$-graded vector space.

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5 Replacing the formal neighborhood of $M_2 \times M_1$ by the germ of the ambient manifold would give the notion, due to Cattaneo–Dherin–Weinstein, of a ‘symplectic micromorphism’ of a ‘symplectic microfold’ represented by the pair $(T^*M_1, M_1)$ to a ‘symplectic microfold’ represented by the pair $(T^*M_2, M_2)$. A difference of viewpoint is that we see ‘thick morphisms’ $\Phi$: $M_1 \rightarrow M_2$ as morphisms between $M_1$ and $M_2$, while the ‘symplectic micromorphisms’ are considered as morphisms of the objects of doubled dimensions.
Specifically, we use $\mathcal{C}^\infty(M)$ for the supermanifold of even functions on $M$ and $\mathbf{P}\mathcal{C}^\infty(M)$ for the supermanifold of odd functions on $M$. (A physicist would say that the points of $\mathcal{C}^\infty(M)$ are ‘bosonic fields’ on $M$, while the points of $\mathbf{P}\mathcal{C}^\infty(M)$ are ‘fermionic fields’ on $M$.)

**Definition 2 ([19]).** Let $\Phi: M_1 \rightleftharpoons M_2$ be a thick morphism of supermanifolds. The pullback $\Phi^*$ is a formal mapping of functional supermanifolds

$$\Phi^*: \mathcal{C}^\infty(M_2) \to \mathcal{C}^\infty(M_1)$$

(of even functions) defined by

$$\Phi^*[g](x) = g(y) + S(x, q) - y^i q_i ,$$

where $q_i$ and $y^i$ are defined from the equations

$$q_i = \frac{\partial g}{\partial y^i}(y)$$

and

$$y^i = (-1)^i \frac{\partial S}{\partial q_i}(x, q).$$

Here $g \in \mathcal{C}^\infty(M_2)$ and $f = \Phi^*[g]$ is its image in $\mathcal{C}^\infty(M_1)$.

Combining (5) with (7) gives the equation for $y^i$

$$y^i = (-1)^i \frac{\partial S}{\partial q_i}(x, q),$$

which can be solved by iterations. It defines a map $\varphi_g: M_1 \to M_2$ depending on $g \in \mathcal{C}^\infty(M_2)$ as a formal functional parameter:

$$\varphi_g = \varphi + \varphi_1 + \varphi_2 + \ldots ,$$

where $\varphi = \varphi_0: M_1 \to M_2$ is a smooth map which is independent of $g$ and defined only by the thick morphism $\Phi$:

$$\varphi^*(y^i) = S^i(x),$$

and the other terms $\varphi_k$ are higher corrections to $\varphi$ (linear, quadratic, etc., in $g$). (See details in [19].) Using this map $\varphi_g$, the pullback $\Phi^*[g]$ can be expressed as

$$\Phi^*[g](x) = g(\varphi_g(x)) + S(x, \frac{\partial g}{\partial y}(\varphi_g(x))) - \varphi_g^i(x) \frac{\partial g}{\partial y^i}(\varphi_g(x)),$$

which is not a very elucidating formula, but amply demonstrates the nonlinear dependence on $g$.

In general, the pullback $\Phi^*: \mathcal{C}^\infty(M_2) \to \mathcal{C}^\infty(M_1)$ is a nonlinear differential operator. It has the following functional power expansion [19]:

$$\Phi^*[g](x) = S_0(x) + g(\varphi(x)) + \frac{1}{2} S^{ij}(\varphi(x)) \partial_i g(\varphi(x)) \partial_j g(\varphi(x)) + \ldots .$$

From here we see the different roles of the terms of the expansion [14] of the generating function $S(x, q)$ of a thick morphism $\Phi: M_1 \rightleftharpoons M_2$. The term $S_0$, which is a genuine function on $M_1$, gives a shift of the origin in the space of functions under $\Phi^*$, the linear term $S^{i}(x)q_i$ defines an (ordinary) smooth map $\varphi: M_1 \to M_2$ with the corresponding ordinary pullback $\varphi^*$, and all the remaining terms are responsible for the higher corrections.
in $Φ^*$. Thus we may say that a thick morphism $Φ : M_1 → M_2$ is a formal perturbation of an ordinary smooth map $ϕ : M_1 → M_2$. (More precisely, to include shifts by $S_0$, we need to speak about morphisms in the semidirect product of the category of supermanifolds and their smooth maps with the algebras of functions on supermanifolds, see below.)

**Remark 1.** Pullbacks w.r.t. thick morphisms can be applied to functions on $M_2$ defined on an open domain $U ⊂ M_2$. Then the image of $Φ^*$ will be in $C^∞(ϕ^{-1}(U))$, where $ϕ : M_1 → M_2$ is the underlying ordinary map.

**Example 1.** If we apply $Φ^*$ to the function $g = y^i c_i$, where $y^i$ are local coordinates on $M_2$ and $c_i$ are some auxiliary variables, then we obtain $q_i = c_i$ from (7), and

$$Φ^*[y^i c_i] = y^i c_i - y^i q_i + S(x, q) = S(x, c).$$

In this way we recover the generating function $S = S(x, q)$.

We see that a thick morphism $Φ$ is determined by the pullback $Φ^*$ on functions, more precisely, by the action on the coordinate functions. Hence although $Φ^*$ is a nonlinear mapping and as such is far from being an algebra homomorphism unlike the familiar pullbacks w.r.t. ordinary maps, it still respects some algebraic properties of functions such as the role of local coordinates as 'free generators'.

Now we wish to establish the categorical properties of thick morphisms, which were only briefly outlined in [19].

Consider thick morphisms $Φ_{21} : M_1 → M_2$ and $Φ_{31} : M_2 → M_3$ with the generating functions $S_{21} = S_{21}(x, q)$ and $S_{32} = S_{32}(y, r)$, respectively. Here $z^μ$ are local coordinates on $M_3$ and by $r_μ$ we denoted the corresponding conjugate momenta.

**Theorem 1.** The composition $Φ_{32} ◦ Φ_{21}$ is well-defined as a thick morphism $Φ_{31} : M_1 → M_3$ with the generating function $S_{31} = S_{31}(x, r)$, where

$$S_{31}(x, r) = S_{32}(y, r) + S_{21}(x, q) - y^i q_i,$$

and $y^i$ and $q_i$ are expressed through $(x^μ, r_μ)$ from the system

$$q_i = \frac{∂S_{32}}{∂y^i}(y, r),$$

and

$$y^i = (-1)^i \frac{∂S_{21}}{∂q_i}(x, q),$$

which has a unique solution as a power series in $r_μ$ (and a functional power series in $S_{32}$).

**Proof.** Recall composition of relations. In our case, to find the composition of $Φ_{32}$ and $Φ_{21}$ as relations, $Φ_{32} ⊂ T^*M_3 × T^*M_2$ and $Φ_{21} ⊂ T^*M_2 × T^*M_1$, we need to consider all pairs $(z, r ; x, p) ∈ T^*M_3 × T^*M_1$ for which there exist $(y, q) ∈ T^*M_2$ such that $(z, r ; y, q) ∈ T^*M_3 × T^*M_2$.

$^6$The algebra of $C^∞$ functions on a domain of $\mathbb{R}^{n|m}$ is not of course a free algebra in the abstract algebraic sense, but it behaves as such with respect to the homomorphisms induced by smooth maps, since they are defined by the images of the coordinate functions, and there are no constraints for the latter.
we obtain

\[ q = \text{the lowest order in } r \]

the generating function given by (14). We have

\[ S^\text{pullback} \]

Here the ‘parameter of smallness’ is \( q \). Suppose \( \Phi \) and \( y \) which has a unique solution \( y \in S \). Therefore, \( \Phi \) and now \( y \), \( r \) are free variables. Therefore we arrive at the system \([15], [16]\) where \( y \) and \( q \) are to be determined and the variables \( x, r \), \( \mu \) are free. By substituting \([15]\) into \([16]\), we obtain

\[
y^i = (-1)^i \frac{\partial S_{21}}{\partial q_i} (x, q) \]

and \( x^\alpha, q_i \) are free variables, and by the definition of \( \Phi_{32} \), we should have

\[
q_i = \frac{\partial S_{32}}{\partial y^i} (y, r)
\]

and now \( y^i, r \) are free variables. Therefore we obtain \( y^i = y^i(x, r) \) by iterations similarly to the construction of pullback. Here the ‘parameter of smallness’ is \( S_{32} \), more precisely its derivative in \( y^i \) in the lowest order in \( r \). The solution for \( y^i \) can be substituted back to \([16]\) to obtain \( q_i = q_i(x, r) \). It remains to show that this composition of relations is indeed specified by the generating function given by \([14]\). We have

\[
q_i dy^i - p_i dx^\alpha = d(y^i q_i - S_{21})
\]

and

\[
r^\mu dz^\mu - q_i dy^i = d(z^\mu r^\mu - S_{32})
\]

By adding, we obtain

\[
r^\mu dz^\mu - p_i dx^\alpha = d(z^\mu r^\mu - S_{32} + y^i q_i - S_{21}).
\]

Therefore, \( S_{31} := S_{32} - y^i q_i + S_{21} \) is as required. \( \square \)

**Example 2.** Let us compute the composition of thick morphisms in the lowest order. Suppose \( \Phi_{21} \) and \( \Phi_{32} \) are given by generating functions

\[
S_{21}(x, q) = f_{21}(x) + \varphi_{21}^i(x) q_i + \ldots, \quad (17)
\]

\[
S_{32}(y, r) = f_{32}(y) + \varphi_{32}^\mu(y) r^\mu + \ldots. \quad (18)
\]

We need to determine the generating function for the composition \( \Phi_{32} \circ \Phi_{21} \),

\[
S_{31}(x, r) = f_{31}(x) + \varphi_{31}^\mu(x) r^\mu + \ldots \quad (19)
\]

(Here dots stand for the terms of higher order in momenta.) We have, in the lowest order,

\[
S_{31}(x, r) = S_{32}(y, r) + S_{21}(x, q) - y^i q_i = f_{32}(y) + \varphi_{32}^\mu(y) r^\mu + f_{21}(x) + \varphi_{21}^i(x) q_i - y^i q_i + \ldots = f_{32}(y) + \varphi_{32}^\mu(y) r^\mu + f_{21}(x) + \ldots = f_{32}(\varphi_{21}(x)) + \varphi_{32}^\mu(\varphi_{21}(x)) r^\mu + f_{21}(x) + \ldots.
\]

Here we are calculating modulo \( J^2 \) where the ideal \( J \) is generated by the momenta and the zero-order terms such as \( f_{21} \). Note that \( y^i \) have to be determined only modulo \( J \), so from \([16]\), \( y^i = \varphi_{21}^i(x) \mod J \), and the terms \( \varphi_{21}^i(x) q_i \) and \( y^i q_i \) mutually cancel. Therefore we see that

\[
f_{31} = \varphi_{31}^i(f_{32}) + f_{21}, \quad (20)
\]

\[
\varphi_{31} = \varphi_{32} \circ \varphi_{21}. \quad (21)
\]
That mean that, in the lowest order, we obtain the composition in the semidirect product category $\text{SMan} \ltimes C^\infty$. The objects in this category are supermanifolds and the morphisms are pairs $(\varphi_{21}, f_{21})$, where $\varphi_{21} : M_1 \to M_2$ is a supermanifold map and $f_{21} \in C^\infty(M_1)$ is an even function, with the composition $(\varphi_{32}, f_{32}) \circ (\varphi_{21}, f_{21}) = (\varphi_{32} \circ \varphi_{21}, \varphi_{21}^* f_{32} + f_{21})$.

**Theorem 2.** The composition of thick morphisms is associative.

**Proof.** Consider the diagram

$$
\begin{array}{ccc}
M_1 & \xrightarrow{\Phi_{21}} & M_2 \\
& \xrightarrow{\Phi_{32}} & M_3 \\
& \xrightarrow{\Phi_{43}} & M_4 \\
\end{array}
$$

Denote $\Phi_{42} = \Phi_{43} \circ \Phi_{32}$ and $\Phi_{31} = \Phi_{32} \circ \Phi_{21}$. We need to check that $\Phi_{43} \circ \Phi_{31} = \Phi_{42} \circ \Phi_{21}$.

Consider the generating functions. For the LHS, we obtain $S_{43} + S_{31} - z^\mu r_\mu = S_{43} + S_{32} + S_{21} - y_i q_i - z^\mu r_\mu$. For the RHS, we obtain $S_{42} + S_{21} - y_i q_i = S_{43} + S_{32} - z^\mu r_\mu + S_{21} - y_i q_i$. So the RHS equals the LHS, and we have the associativity.

Since we obviously have the identity thick morphism for each supermanifold $M$, given by the generating function $S = x^\alpha q_\alpha$, we have established that thick morphisms form a category $\mathcal{E}\text{Thick}$ (with the same set of objects as the usual category of supermanifolds).

**Remark 2.** More precisely, $\mathcal{E}\text{Thick}$ is a formal category, in the sense that its composition law is given by (functional) power series. Notice that ‘formality’ enters our constructions in two related but different ways: as microformality, i.e., as power expansions in the cotangent directions, and as functional power series expressing the nonlinear pullbacks and the generating functions of compositions. The category $\text{SMan} \ltimes C^\infty$ is a closed subspace in the formal category $\mathcal{E}\text{Thick}$ and the whole $\mathcal{E}\text{Thick}$ is its formal neighborhood.

**Theorem 3.** For the nonlinear pullbacks defined by thick morphisms the identity

$$
(\Phi_{32} \circ \Phi_{21})^* = \Phi_{21}^* \circ \Phi_{32}^*
$$

holds.

**Proof.** (As mentioned in [19], this is similar to the associativity of composition of thick morphisms.) Consider $f_3 \in C^\infty(M_3)$. Then for $\Phi_{32}^*[f_3]$ we have

$$
\Phi_{32}^*[f_3] = f_3 + S_{32} - z^\mu r_\mu
$$

and for $(\Phi_{21}^* \circ \Phi_{32}^*)[f_3]$ we obtain

$$
(\Phi_{21}^* \circ \Phi_{32}^*)[f_3] = \Phi_{21}^*[\Phi_{32}^*[f_3]] = f_3 + S_{32} - z^\mu r_\mu + S_{21} - y^i q_i.
$$

This coincides with

$$
\Phi_{31}^*[f_3] = f_3 + S_{31} - z^\mu r_\mu = f_3 + S_{32} + S_{21} - y^i q_i - z^\mu r_\mu,
$$

by (14), where $\Phi_{31} = \Phi_{32} \circ \Phi_{21}$. 

So far we have dealt with even functions and what we have defined as $\mathcal{E}\text{Thick}$ is the *even microformal category*. Parallel constructions are based on the anticotangent bundles (see [19]) The notation used is as follows. For local coordinates $x^a$ on a supermanifold $M$, $x^*_a$ denote the conjugate antimomenta (the fiber coordinates on II$^*M$). The canonical odd symplectic form on II$^*M$ is

$$
\omega = d(dx^a x^*_a) = -(-1)^d dx^a dx^*_a = -(-1)^d dx^*_a dx^a.
$$
The notation $-\Pi T^* M$ means the same as above, i.e., that the form $\omega$ is taken with the negative sign.

Definition 3. An odd thick morphism (odd microformal morphism) $\Psi : M_1 \Leftrightarrow M_2$ is specified by a formal odd generating function $S = S(x, y^*)$ (locally defined) and corresponds to the formal canonical relation $\Psi \subset \Pi T^* M_2 \times (-\Pi T^* M_1)$ (denoted by the same letter)

$$\Psi = \left\{ (y^i, y^*_i ; x^a, x^*_a) \mid y^i = \frac{\partial S}{\partial y^*_i}(x, y^*), \; x^*_a = \frac{\partial S}{\partial x^a}(x, y^*) \right\}. \quad (24)$$

On the submanifold $\Psi$ we have

$$dy^i y^*_i - dx^a x^*_a = d(y^i y^*_i - S). \quad (25)$$

The following theorems are completely analogous to the even versions above.

Theorem 4. There is a well-defined composition $\Psi_3 \circ \Psi_2$ of odd thick morphisms, which is an odd thick morphism $\Psi_3 : M_1 \Leftrightarrow M_3$ with the generating function $S_{31} = S_{31}(x, z^*)$, where

$$S_{31}(x, z^*) = S_{32}(y, z^*) + S_{21}(x, y^*) - y^i y^*_i \quad (26)$$

and $y^i$ and $y^*_i$ are expressed uniquely via $(x^a, z^*_\mu)$ from the system

$$y^*_i = \frac{\partial S_{32}}{\partial y^*_i}(y, z^*) \quad (27)$$

and

$$y^i = \frac{\partial S_{21}}{\partial y^*_i}(x, y^*) \quad (28)$$

as power series in $z^*_\mu$ and functional power series in $S_{32}$.

Theorem 5. The composition of odd thick morphisms is associative.

Odd thick morphisms form a formal category $O\Theta hick$. We call it, the odd microformal category. Similarly to $E\Theta hick$ being a formal neighborhood of the category $SMan \times C^\infty$, the formal category $O\Theta hick$ contains the category $SMan \times \Pi C^\infty$ and is its formal neighborhood. The affine action of $SMan \times \Pi C^\infty$ on odd functions extends to a nonlinear action of the whole formal category $O\Theta hick$.

Definition 4. The pullback $\Psi^*$ w.r.t. an odd thick morphism $\Psi : M_1 \Leftrightarrow M_2$ is a formal mapping of functional supermanifolds (of odd functions)

$$\Psi^* : \Pi C^\infty(M_2) \to \Pi C^\infty(M_1) \quad (29)$$

defined by $\Psi^*[\gamma] = \chi,$

$$\chi(x) = \gamma(y) + S(x, y^*) - y^i y^*_i, \quad (30)$$

where $y^*_i$ and $y^i$ are determined from the equations

$$y^*_i = \frac{\partial \gamma}{\partial y^i}(y) \quad (31)$$

and

$$y^i = \frac{\partial S}{\partial y^*_i}(x, y^*). \quad (32)$$

Here $\gamma \in \Pi C^\infty(M_2)$ and $\chi \in \Pi C^\infty(M_1)$ is its image under $\Psi^*$. 
As in the even case, the pullback $\Psi^*$ is a formal nonlinear differential operator, more precisely, it is given as a formal power series whose $k$th term, for every $k$, is a nonlinear differential operator of order $\leq k - 1$ [19].

**Theorem 6.** For odd thick morphisms,

$$\Psi_{32}^* \circ \Psi_{21}^* = \Psi_{21}^* \circ \Psi_{32}^*. \quad (33)$$

So we have a formal (anti)representation of $\mathcal{O}\text{T}hick$ in the category of formal supermanifolds.

To summarize: the category of supermanifolds (with ordinary smooth maps as morphisms) $\text{SM}an$ is embedded into the semidirect product “affine” category $\text{SM}an \rtimes \mathbb{C}^\infty$ or $\text{SM}an \rtimes \Pi \mathbb{C}^\infty$, and then there is a “thickening” to a formal category, $\mathcal{E}\text{T}hick$ or $\mathcal{O}\text{T}hick$, respectively. Each of these microformal categories acts, respectively, on even or odd functions on supermanifolds by formal nonlinear differential operators (nonlinear pullbacks).

### 2. Application to Vector Bundles: The Adjoint of a Nonlinear Map

Recall the familiar notion of the adjoint operator for linear maps between vector spaces or vector bundles. In this section, we shall show that a generalization of the adjoint exists for nonlinear maps of vector spaces or vector bundles — but now as a thick morphism rather than an ordinary mapping.

Our construction is based on the canonical diffeomorphism

$$T^*E \cong T^*E^* \quad (34)$$

discovered by Mackenzie and Xu [12] (see also [11], [11] Ch. 9, and [14] for the statement and proof in the supercase). We refer to it as to Mackenzie–Xu transformation. There is a parallel canonical diffeomorphism

$$\Pi T^*E \cong \Pi T^*(\Pi E) \quad (35)$$

introduced in [14]. Let us recall these natural diffeomorphisms in the form suitable for our purposes. For a vector bundle $E \to M$ denote local coordinates on the base by $x^a$ and linear coordinates in the fibers by $u^i$. We assume that the transformation law for $u^i$ has the form $u^i = u'^i T_{ii}^i$ (that means that $u^i$ are left coordinates, a distinction important in the supercase). Denote the fiber coordinates for the dual bundle $E^* \to M$ and the antidual bundle $\Pi E^* \to M$ by $u_i$ and $\eta_i$, respectively. We assume that the invariant bilinear forms are $u^i u_i$ and $u^i \eta_i$. (Algebraically this means that $u_i$ and $\eta_i$ are the right coordinates w.r.t. the basis which is ‘right dual’ to the basis in $E$.) Consider the cotangent and the anticotangent bundles for $E$. Denote the canonically conjugate momenta for $x^a, u^i$ by $p_a, p_i$, and the conjugate antimomenta, by $x^*_a, u^*_i$. A similar notation will be used for $E^*$ and $\Pi E^*$. The Mackenzie–Xu transformation $\kappa: T^*E \to T^*E^*$ is defined by

$$\kappa^*(x^a) = x^a, \quad \kappa^*(u_i) = p_i, \quad \kappa^*(p_a) = -p_a, \quad \kappa^*(p^i) = (-1)^i u^i. \quad (36)$$

It is well-defined (invariant under the changes of coordinates) and is an antisymplectomorphism, in the form as defined here. (There is a choice of sign involved in the definition. Here the choice agrees with that in [11] and is different from [14]. The choice used in [14] gives
a symplectomorphism.) The analogous diffeomorphism \( \kappa: \Pi T^*E \rightarrow \Pi T^*(\Pi E^*) \), which we denote by the same letter, is defined by

\[
\kappa^*(x^a) = x^a, \quad \kappa^*(\eta_i) = u^i, \quad \kappa^*(x^*_a) = -x^*_a, \quad \kappa^*(\eta^*_i) = u^i,
\]

(note the absence of signs depending on parities). It is also an antisymplectomorphism, w.r.t. the canonical odd symplectic structures.

**Remark 3.** The existence of the Mackenzie–Xu transformation and its odd analog is a deep fact. In particular, in the coordinate language used above, the invariance of formulas (36), (37) is nontrivial and follows from the analysis of \( T^*E \) and \( \Pi T^*E \) as double vector bundles over \( M \). On the other hand, from the coordinate formulas (36) and (37), it is obvious that \( \kappa^*\omega = -\omega \) for the canonical symplectic structures. Moreover, one can immediately see that

\[
\kappa^*(dx^a p_a + du^i p^i) = -(dx^a p_a + du^i p^i) + d(u^i p_i)
\]

and

\[
\kappa^*(dx^a x^*_a + d\eta_i \eta^*_i) = -(dx^a x^*_a + d\eta^*_i \eta^*_i) + d(u^i u^*_i),
\]

for the canonical 1-forms on the cotangent and anticotangent bundles.

Now we proceed to constructing generalized adjoints. Let \( E_1 \) and \( E_2 \) be vector bundles over a fixed base \( M \). Consider a fiberwise map over \( M \),

\[
\Phi: E_1 \rightarrow E_2,
\]

which not necessarily fiberwise linear. In local coordinates, it is given by

\[
\Phi^*(y^a) = x^a, \quad \Phi^*(w^a) = \Phi^a(x, u),
\]

for some functions \( \Phi^a(x, u) \), where \( u^i \) and \( w^a \), respectively, are linear coordinates on the fibers of \( E_1 \) and \( E_2 \). For the fiber coordinates on the dual bundles we use the letters with the lower indices and the forms \( u^i u_i \) and \( w^a w_a \) are invariant.

Note that it makes sense to speak about fiberwise thick morphisms (even and odd) in the same way as for ordinary maps.

**Theorem 7.** 1. There is a fiberwise even thick morphism

\[
\Phi^*: E_2^* \rightarrow E_1^*,
\]

and a fiberwise odd thick morphism

\[
\Phi^{\Pi^*}: \Pi E_2^* \rightarrow \Pi E_1^*,
\]

such that when the map \( \Phi: E_1 \rightarrow E_2 \) is fiberwise linear, i.e., a vector bundle homomorphism, both \( \Phi^* \) and \( \Phi^{\Pi^*} \) are ordinary maps and are the familiar adjoint homomorphisms (with the parity reversion in the second case).

2. For the composition of fiberwise maps of vector bundles over \( M \),

\[
\Phi_{32} \circ \Phi_{21} : E_1 \rightarrow E_2 \rightarrow E_3,
\]

we have

\[
(\Phi_{32} \circ \Phi_{21})^* = \Phi_{21}^* \circ \Phi_{32}^*,
\]

as an even thick morphism \( E_3^* \rightarrow E_1^* \), and

\[
(\Phi_{32} \circ \Phi_{21})^{\Pi^*} = \Phi_{21}^{\Pi^*} \circ \Phi_{32}^{\Pi^*},
\]

as an odd thick morphism \( E_3^* \rightarrow E_1^* \).
as an odd thick morphism $\Pi E_2^* \leftrightarrow \Pi E_1^*$.

Proof. 1. Consider a fiberwise (in general, nonlinear) map $\Phi: E_1 \to E_2$. For clarity, we use the physicists’ notation with indices in the arguments:

$$\Phi: (x^a, u^i) \mapsto (y^a = x^a, w^a = \Phi(x^a, u^i)).$$

To it corresponds the canonical relation $R_\Phi \subset T^*E_2 \times (-T^*E_1)$,

$$R_\Phi = \{(y^a, w^a, q_i, q_\alpha ; x^a, u^i, p_a, p_i) \mid (-1)^i dq_i y^a + (-1)^\alpha dq_\alpha w^a + dx^a p_a + du^i p_i = dS\},$$

with the generating function

$$S = x^aq_\alpha + \Phi^a(x^a, u^i)q_\alpha.$$

Define a thick morphism $\Phi^*: E_2^* \mapsto E_1^*$ by a generating function $S^* = S^*(y^a, w_\alpha, p_a, p^i)$, where

$$S^* = y^ap_a + \Phi^a(y^a, (-1)^i u^i)w_\alpha.$$

The corresponding canonical relation $\Phi^* \subset T^*E_1^* \times (-T^*E_2^*)$ is given by the equation

$$(-1)^i dp_i x^a + (-1)^\alpha dp_\alpha q_\alpha + dw_\alpha q^\alpha = dS^*$$

or more explicitly

$$x^a = y^a, \quad u_i = \frac{\partial \Phi^a}{\partial u^i}(y, (-1)^i p^i)w_\alpha, \quad q_\alpha = p_a + \frac{\partial \Phi^a}{\partial x^a}(y, (-1)^i p^i)w_\alpha, \quad q^\alpha = (-1)^\alpha \Phi^a(y, (-1)^i p^i).$$

One can see that it is obtained from the canonical relation $R_\Phi \subset T^*E_2 \times (-T^*E_1)$ as follows. We first apply the Mackenzie–Xu transformation. Since it is an antisymplectomorphism, this gives a Lagrangian submanifold in $-T^*E_2^* \times T^*E_1^*$. Then we take the opposite relation, i.e., swap the factors:

$$\Phi^* = ((\kappa \times \kappa)R_\varphi)^{op} \subset T^*E_1^* \times (-T^*E_2^*),$$

and this is the canonical relation $\Phi^*$. One can see that the thick morphism $\Phi^*: E_2^* \mapsto E_1^*$ is fiberwise. Let us check now that $\Phi^*$ is the ordinary adjoint when $\Phi: E_1 \to E_2$ is linear on fibers. Indeed, in such a case we have

$$\Phi(x^a, u^i) = u^i\Phi_i^a(x),$$

hence

$$x^a = y^a, \quad u_i = \Phi_i^a(y)w_\alpha.$$

The odd thick morphism $\Phi^{\Pi^*}: \Pi E_2^* \leftrightarrow \Pi E_1^*$ is built in a similar way: we take the canonical relation $R_\Phi \subset \Pi T^*E_2 \times (-\Pi T^*E_1)$ corresponding to a map $\Phi: E_1 \to E_2$, apply the odd version of the Mackenzie–Xu transformation and take the opposite.

2. To obtain equations (43) and (44), notice that to a composition of maps (42) corresponds the composition of the canonical relations between the cotangent bundles, in the same order. This is preserved by the Mackenzie–Xu transformation. Then taking the opposite relations reverses the order, hence (43) and (44).
Corollary 1. On functions on the dual bundles, the pullback w.r.t. the thick morphism $\Phi^*$ induces a nonlinear ‘pushforward map’

$$\Phi_* := (\Phi^*)^*: \mathcal{C}^\infty(E^*_1) \to \mathcal{C}^\infty(E^*_2).$$  \hspace{1cm} (45)

The restriction of $\Phi_*$ on the space of even sections $\mathcal{C}^\infty(M, E_1)$ regarded as the subspace in $\mathcal{C}^\infty(E^*_1)$ consisting of the fiberwise linear functions maps it to the subspace $\mathcal{C}^\infty(M, E_2)$ in $\mathcal{C}^\infty(E^*_2)$ and coincides with the standard pushforward of sections $\Phi_*(v) = \Phi \circ v$.

Proof. The nonlinear pushforward $\Phi_*: \mathcal{C}^\infty(E^*_1) \to \mathcal{C}^\infty(E^*_2)$ is defined as the pullback w.r.t. the thick morphism $\Phi^*: E^*_2 \to E^*_1$. After a simplification, we arrive at the following. To an even function $f = f(x^a, u_i)$ the map $\Phi_*$ assigns the even function $g = \Phi_*[f]$, where $g(x^a, w_\alpha)$ is given by:

$$g(x, w_\alpha) = f(x, u_i) + \Phi^a(x, (-1)^i p^i) w_\alpha - u_i p^i,$$

and $u_i, p^i$ are found from the equations

$$p^i = \frac{\partial f}{\partial u_i}(x, u_i)$$ \hspace{1cm} (47)

and

$$u_i = \frac{\partial \Phi^a}{\partial u^i}(x, (-1)^i \frac{\partial f}{\partial u_i}(x, u_i)) w_\alpha,$$ \hspace{1cm} (48)

the latter equation solvable by iterations (recall the general case analyzed in the previous section). Now let the function $f$ on $E^*_1$ have the form $f(x, u_i) = v^i(x) u_i$, which corresponds to an even section $v = v^i(x) e_i$ of the bundle $E_1$. Then for it

$$p^i = (-1)^i v^i(x),$$ \hspace{1cm} (49)

hence

$$\Phi_*[f] = v^i(x) u_i + \Phi^a(x, v^i(x)) w_\alpha - u_i (-1)^i v^i(x) = \Phi^a(x, v^i(x)) w_\alpha,$$ \hspace{1cm} (50)

which is the fiberwise linear function on $E^*_2$ corresponding to the even section $\Phi \circ v$. \hspace{1cm} $\Box$

The similar statement holds for the odd case: there is a nonlinear pushforward $\Phi_* := (\Phi^*)^*$,

$$\Phi_*: \Pi \mathcal{C}^\infty(\Pi E^*_1) \to \Pi \mathcal{C}^\infty(\Pi E^*_2).$$ \hspace{1cm} (51)

On (event!) sections $\mathcal{C}^\infty(M, E_1)$ regarded as a subspace $\mathcal{C}^\infty(M, E_1) \subset \Pi \mathcal{C}^\infty(\Pi E^*_1)$ of the odd fiberwise linear functions on $\Pi E^*_1$ it again coincides with $\Phi_*(v) = \Phi \circ v$.

The algebra of functions on the dual bundle $E^*$ is freely generated by the sections of $E$ (at least the fiberwise polynomial functions). So for the fiberwise linear maps $E_1 \to E_2$ the pushforward of functions $\mathcal{C}^\infty(E^*_1) \to \mathcal{C}^\infty(E^*_2)$ is an algebra homomorphism extending a linear map of the generators. Comparing that with our case, we see that there is no nonlinear map of bundles and instead of an algebra homomorphism of functions we have a nonlinear formal differential operator. Nevertheless, as seen from Corollary 4 the nonlinear pushforward map $\Phi_*: \mathcal{C}^\infty(E^*_1) \to \mathcal{C}^\infty(E^*_2)$ can still be regarded as a ‘nonlinear extension’ of the map $\Phi$ on sections.

Remark 4. If the base $M$ is a point, we have a nonlinear map of vector spaces $\Phi: V \to W$. Replacing it by the Taylor expansion gives a sequence of linear maps $\Phi_k: S^k V \to W$. The functions on the dual spaces can themselves be seen as elements of the symmetric powers. By expanding the pushforward $\Phi_*$ into a Taylor series, we arrive at linear maps of the form
$S^n(⊕S^pV) \to ⊕S^qW$. It would be interesting to give a precise formulation and a purely algebraic construction for that. (Consider for simplicity the purely even case.) Such maps might have occurred before in an algebraic context.

**Remark 5.** From the proof of Theorem 7 it is clear that instead of an ordinary map one can start from a fiberwise even thick morphism $E_1 \Rightarrow E_2$ and construct the adjoint $E_2^* \Rightarrow E_1^*$ or start from a fiberwise odd thick morphism $E_1 \Leftrightarrow E_2$ and construct the ‘antiadjoint’ $\Pi E_2^* \Leftrightarrow \Pi E_1^*$. This would make the constructions completely symmetric.

**Remark 6.** Theorem 7 can also be generalized to vector bundles over different bases. This would require to use the concept of comorphisms of Higgins–Mackenzie [6]. Let $E_1 \to M_1$ and $E_2 \to M_2$ be vector bundles (more generally, arbitrary fiber bundles) over bases $M_1$ and $M_2$. Then a ‘morphism’ $\Phi: E_1 \to E_2$ can be defined as a fiberwise map over a fixed base $E_1 \to \varphi^* E_2$ and a ‘comorphism’ $\Phi: E_1 \to E_2$ can be defined as a fiberwise map over a fixed base $\varphi^* E_1 \to E_2$. In both cases a map of the bases is needed, but for ‘morphisms’ it is $\varphi: M_1 \to M_2$ (pointing in the same direction as the morphism) and for ‘comorphisms’ it is $\varphi: M_1 \leftarrow M_2$ (pointing in the opposite direction to the comorphism). For bundles over the same base, morphisms and comorphisms coincide, and for manifolds regarded as ‘zero bundles’, morphisms are the usual maps while comorphisms are the arrows in the opposite category. As shown in [6] for vector bundles (assuming the fiberwise linearity for maps over a fixed base), the adjoint of a morphism $E_1 \to E_2$ is a comorphism $E_2^* \to E_1^*$ and vice versa; so this gives an anti-isomorphism of the two categories of vector bundles. To generalize this to our setup, one may wish to keep a map between the bases as an ordinary map and use fiberwise thick morphisms over a fixed base. This incorporates the possible nonlinearity of morphisms. In this way, one obtains base-changing ‘thick morphisms’ and ‘thick comorphisms’ of vector bundles to which the duality theory extends.

### 3. Application to algebroids and homotopy Poisson brackets

It is well known that a linear map of the underlying vector spaces $\Phi: g_1 \to g_2$ for Lie algebras is a Lie algebra homomorphism if the adjoint map of the dual spaces $\Phi^*: g_2^* \to g_1^*$ is Poisson w.r.t. the induced Lie–Poisson brackets (also known as the Berezin–Kirillov brackets). The same holds true for Lie algebroids (see [11, Ch. 10]; in full generality, for base-changing morphisms, the statement is due to [6]). Here we shall extend it to the $L_\infty$-algebroids. For simplicity we confine ourselves to the case of a fixed base and we do not consider the ‘if and only if’ form of the statement. Our main theorem here is as follows.

**Theorem 8.** An $L_\infty$-morphism of $L_\infty$-algebroids over a base $M$ induces $L_\infty$-morphisms of the homotopy Poisson and homotopy Schouten algebras of functions on the (anti)dual bundles.

Recall that an $L_\infty$-algebroid is a vector bundle $E \to M$ endowed with a sequence of $n$-ary brackets of sections and a sequence of $n$-ary anchors $E \times_M \ldots \times_M E \to TM$ so that it defines an $L_\infty$-algebra structure on sections and w.r.t. the module structure over functions on the base.
base there are Leibniz identities for the brackets involving the anchors (see, e.g., [7]). To be more precise, here we shall follow the convention of Lada–Stasheff [9] that the brackets in an $L_\infty$-algebra should be antisymmetric and of alternating parities (the alternative convention being symmetric odd brackets; the equivalence is by the parity reversion, see a discussion in [15]). Then an $L_\infty$-algebroid structure on $E \to M$ is equivalent to a (formal) homological vector field on the supermanifold $\Pi E$. An $L_\infty$-morphism of $L_\infty$-algebroids $\Phi: E_1 \to E_2$ is effectively specified by a map (in general, nonlinear) $\Phi: \Pi E_1 \to \Pi E_2$ such that the homological vector fields $Q_1 = Q_{E_1}$ on $\Pi E_1$ and $Q_2 = Q_{E_2}$ on $\Pi E_2$ are $\Phi$-related.\footnote{Warning: in spite of the notation, there is no single map from $E_1$ to $E_2$. With some abuse of language, it is convenient to call the map $\Pi E_1 \to \Pi E_2$ itself an $L_\infty$-morphism. The same applies to morphisms of $L_\infty$-algebras.} Such a definition combines the $L_\infty$-morphisms of $L_\infty$-algebras and the Lie algebroid morphisms.

An $L_\infty$-algebroid structure on $E \to M$ induces a homotopy Poisson structure on $E^*$ and a homotopy Schouten structure on $\Pi E^*$. That means sequences of brackets turning $C^\infty(E^*)$ into an $L_\infty$-algebra in the Lada–Stasheff ("antisymmetric") convention and $C^\infty(\Pi E^*)$ into an $L_\infty$-algebra in the alternative ("symmetric") convention. (Each bracket must also be a derivation in each argument.) All these structures are mutually equivalent and can be seen as different manifestations of the same structure, similarly with the familiar Lie algebroid case [14], [15].

**Proof of Theorem**\footnote{\begin{itemize}
\item[8] Consider for example the Lie–Schouten brackets on functions on $\Pi E^*$. This sequence of higher brackets is defined as the higher derived brackets for an odd 'master Hamiltonian' $H_{E^*}$ satisfying $(H_{E^*}, H_{E^*}) = 0$, where $(\cdot, \cdot)$ denotes the canonical Poisson bracket on the cotangent bundle. For an $L_\infty$-algebroid $E$, the master Hamiltonian $H_{E^*}$ specifying the Lie–Schouten brackets $\Pi E^*$ is a function on $T^*(\Pi E^*)$ defined by the Mackenzie–Xu transformation of the 'linear' Hamiltonian $H_E = Q_E \cdot p$ on $T^*(\Pi E)$. Suppose there is an $L_\infty$-morphism of $L_\infty$-algebroids $E_1 \to E_2$, i.e., a map $\Phi: \Pi E_1 \to \Pi E_2$ over $M$ such that the vector fields $Q_1$ and $Q_2$ are $\Phi$-related. This is equivalent to the Hamiltonians $H_1 = H_{E_1}$ and $H_2 = H_{E_2}$ being $R_\Phi$-related. By applying the Mackenzie–Xu transformations and swapping the factors, we deduce that the Hamiltonians $H_2^* = H_{E_2^*}$ and $H_1^* = H_{E_1^*}$ are $\Phi^*$-related, where $\Phi^*: \Pi E_2^* \to \Pi E_1^*$ is the adjoint thick morphism. One of the main results of [19] (Corollary from Theorem 4) was that if the master Hamiltonians are related by a thick morphism, then the pullback w.r.t. it is an $L_\infty$-morphism of the homotopy Schouten algebras of functions. So we may conclude that here the pullback w.r.t. the adjoint $\Phi^*$, which is the pushforward map $\Phi_*: C^\infty(\Pi E_1^*) \to C^\infty(\Pi E_2^*)$ is an $L_\infty$-morphism, as claimed. The case of the homotopy Lie–Poisson brackets for $E_1^*$ and $E_2^*$ is considered similarly.\end{itemize}}

With suitable modifications, the statement should hold for base-changing morphisms.

The following lemma should be known. We give for completeness (compare the corresponding statement for higher Lie algebroids [16], [17]).

**Lemma 1.** The higher anchors for an $L_\infty$-algebroid $E \to M$ assemble to an $L_\infty$-morphism $E \to TM$ (where $TM$ is considered with the canonical Lie algebroid structure), to which we also refer as anchor.
Proof. The sequence of anchors assemble into a single map \( a : \Pi E \to \Pi TM \), which is given by \( a = \Pi T p \circ Q \), where \( Q = Q_E \) and \( \Pi T p \) is the differential for the bundle projection \( \Pi E \to M \). For an arbitrary \( Q \)-manifold \( N \), the map \( Q : N \to \Pi TN \) is tautologically a \( Q \)-morphism, i.e., the vector fields \( d \) on \( \Pi TN \) and \( Q \) on \( N \) are \( Q \)-related. Also, for any map its differential is a \( Q \)-morphism. Hence \( a : \Pi E \to \Pi TM \) is a \( Q \)-morphism as the composition of \( Q \)-morphisms. \( \Box \)

Corollary 2 (from Theorem \[8\]). The anchor for an \( L_\infty \)-algebroid \( E \to M \) induces an \( L_\infty \)-morphism

\[
a_\ast : C^\infty(\Pi E^*) \to C^\infty(\Pi T^*M) \quad (52)
\]

for the homotopy Lie–Schouten brackets, and an \( L_\infty \)-morphism

\[
\Pi C^\infty(E^*) \to \Pi C^\infty(T^*M) \quad . \quad (53)
\]

for the homotopy Lie–Poisson brackets. (At the right hand sides, the functions on \( \Pi T^*M \) and \( T^*M \) are considered with the canonical Schouten and Poisson brackets, respectively.)

Corollary 3 (from Corollary \[2\]). There is an \( L_\infty \)-morphism for the higher Koszul brackets on a homotopy Poisson manifold:

\[
C^\infty(\Pi TM) \to C^\infty(\Pi T^*M) \quad . \quad (54)
\]

Corollary 4 (generalization of Corollary \[3\]). There is an \( L_\infty \)-morphism

\[
C^\infty(\Pi E) \to C^\infty(\Pi E^*) \quad . \quad (55)
\]

for ‘triangular \( L_\infty \)-bialgebroids’.

(We hope to elaborate the latter statement in a separate paper \[8\].)

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