TWO DIMENSIONAL DILATON GRAVITY COUPLED TO AN ABELIAN GAUGE FIELD

by

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ABSTRACT
The most general two-dimensional dilaton gravity theory coupled to an Abelian gauge field is considered. It is shown that, up to spacetime diffeomorphisms and $U(1)$ gauge transformations, the field equations admit a two-parameter family of distinct, static solutions. For theories with black hole solutions, coordinate invariant expressions are found for the energy, charge, surface gravity, Hawking temperature and entropy of the black holes. The Hawking temperature is proportional to the surface gravity as expected, and both vanish in the case of extremal black holes in the generic theory. A Hamiltonian analysis of the general theory is performed, and a complete set of (global) Dirac physical observables is obtained. The theory is then quantized using the Dirac method in the WKB approximation. A connection between the black hole entropy and the imaginary part of the WKB phase of the Dirac quantum wave functional is found for arbitrary values of the mass and $U(1)$ charge. The imaginary part of the phase vanishes for extremal black holes and for eternal, non-extremal Reissner-Nordstrom black holes.
1 Introduction

Two dimensional models of gravity have been the subject of much research in the hope that the simplicity of these models might yield clues to theoretical problems associated with the evaporation of black holes via Hawking radiation\cite{1}. One question of particular current interest concerns the nature of the radiation for charged black holes near extremality where the standard thermal description is expected to break down\cite{3}. Recently, a unified description was presented of the most general 2-D vacuum dilaton gravity theory\cite{3}: the Killing vector for the generic theory was found, and in the case where black hole solutions were present, expressions for the surface gravity and entropy were derived. In the process, an intriguing relationship was uncovered between the black hole entropy and the phase of the Dirac quantized physical wave functionals for stationary states in the theory.

The purpose of the present paper is to extend the analysis of \cite{3} to include coupling to a $U(1)$ gauge field. The action functional we will consider depends on the two-dimensional metric tensor, the dilaton scalar and an Abelian gauge field. We consider the most general case for which the field equations are at most second order in derivatives. This allows two arbitrary functions of the dilaton field in the action.\cite{1} The first is the dilaton potential, while the second is effectively a dilaton dependent “electromagnetic coupling” in the Maxwell term for the gauge field. For specific choices of these two functions, we obtain various models of current interest. One such special case describes “dimensionally reduced” 3+1 spherically symmetric black holes\cite{6} with electric charge. Another interesting example is Jackiw-Teitelboim 2-D gravity\cite{7} with $U(1)$ gauge coupling, which provides a dimensionally reduced model\cite{8} for the spinning (axially symmetric) black holes of Bañados, Teitelboim and Zanelli\cite{9}. String inspired dilaton gravity coupled to an electromagnetic field, which was discussed by Frolov\cite{10}, is also contained as a special case.

The paper is organized as follows: Section 2 presents the action and field equations. Section 3 derives the most general solution in conformal gauge, identifying the two coordinate invariant parameters labelling inequivalent solutions. The Killing vector associated with each solution is also written down in covariant form, and used to

\footnote{The Dirac quantization of dilaton gravity coupled to a Yang-Mills field has recently been considered by Strobl\cite{4}, while the perturbative quantization of Dilaton-Maxwell theory has been examined by Elizalde and Odinstov\cite{5}.}
derive a necessary condition for the existence of black hole solutions. This condition in
effect provides the equation of state for the black hole, from which the thermodynamic
quantities can easily be derived. The Hamiltonian analysis is described in Section 4.
The reduced phase space is shown to be four dimensional and explicit expressions
are given for the corresponding (global) physical observables. Section 5 derives the
thermodynamical properties of black hole solutions in the generic theory, including the
surface gravity and entropy. It is shown that the surface gravity vanishes for extremal
black holes in the generic theory. The Dirac quantization of the generic theory is
presented in Section 6, and a relationship is found between the imaginary contribution
to the WKB phase of stationary states and the entropy of corresponding classical
black hole solutions. The imaginary contribution to the phase vanishes for extremal
black holes and for eternal, non-extremal Reissner-Nordstrom type black holes in the
general theory. In Section 7, specific cases of physical interest are described within
this formalism. Finally, Section 8 closes with conclusions and prospects for future
work.

2 The Model

The most general action functional depending on the metric tensor $g_{\mu\nu}$, scalar field
$\phi$ and vector potential $A_\mu$ in two spacetime dimensions that contains at most second
derivatives of the fields can be written\cite{5} \cite{11}:

$$S[g, \phi, A] = \int d^2x \sqrt{-g} \left[ \frac{1}{2G} \left( \frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + \frac{1}{l^2} V(\phi) + D(\phi) R(g) \right) - \frac{1}{4} W(\phi) F^{\mu\nu} F_{\mu\nu} \right].$$

(1)

where $R(g)$ is the Ricci curvature scalar and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the Abelian field
strength tensor. $V(\phi)$ and $W(\phi)$ are arbitrary functions of the dilaton field. It should
be noted that in the above, the fields $\phi$, $g_{\mu\nu}$ and $A_\mu$ are dimensionless, as is the
2-D Newton constant, $G$. This requires the inclusion of a coupling constant, $l$, of
dimension length in the potential term.

If $D(\phi)$ is a differentiable function of $\phi$ such that $D(\phi) \neq 0$ and $\frac{dD(\phi)}{d\phi} \neq 0$ for any
admissible value of $\phi$ then the kinetic term for the scalar field can be eliminated by
means of the (invertible) field redefinition\cite{12}:
\[ \bar{g}_{\mu\nu} = \Omega^2(\phi)g_{\mu\nu} \]  
\[ \bar{\phi} = D(\phi) \]  
\[ \bar{A}_\mu = A_\mu \]  

where
\[ \Omega^2(\phi) = \exp \left( \frac{1}{2} \int \frac{d\phi}{dD/d\phi} \right) \]  

In terms of the new fields, the action Eq.\( (1) \) takes the form:
\[ S = \int d^2x \sqrt{-g} \left[ \frac{1}{2G} \left( \bar{\phi} R(\bar{g}) + \frac{1}{l^2} \nabla^2(\bar{\phi}) \right) - \frac{1}{4} \bar{W}(\bar{\phi}) F_{\mu\nu} F^{\mu\nu} \right]. \]  

where \( \bar{V} \) and \( \bar{W} \) are defined as:
\[ \nabla(\bar{\phi}) = \frac{V(\phi(\bar{\phi}))}{\Omega^2(\phi(\bar{\phi}))} \]  
\[ \bar{W}(\bar{\phi}) = W(\phi(\bar{\phi}))\Omega^2(\phi(\bar{\phi})) \]  

For example in the Achuracco-Ortiz model\cite{8}, \( \nabla = \Lambda \bar{\phi} \) (\( \Lambda > 0 \)) and \( \bar{W} = \bar{\phi}^3 \), while in the string inspired model\cite{10}, \( \nabla = constant \) and \( \bar{W} = \bar{\phi}^2 \). For spherically symmetric gravity(SSG) with an electromagnetic field, \( \nabla = 1/\sqrt{2\phi} \) and \( \bar{W} = (2\bar{\phi})^{3/2} \). In the following we consider the action in the form Eq.\( (6) \), and henceforth drop the bars over the fields.

The field equations that follow from the action Eq.\( (6) \) are:
\[ R + \frac{1}{l^2} \frac{dV}{d\phi} - \frac{G}{2} \frac{dW}{d\phi} F^{\alpha\beta} F_{\alpha\beta} = 0 \]  
\[ \nabla_\mu \nabla_\nu \bar{\phi} - \frac{1}{2l^2} g_{\mu\nu} V(\phi) + G \left( \delta^\alpha_\mu \delta^\beta_\nu - \frac{3}{4} g_{\mu\nu} g^{\alpha\beta} \right) W(\phi) F_\alpha \gamma F_{\beta\gamma} = 0 \]  
\[ \nabla_\nu \left( W(\phi) F^{\mu\nu} \right) = 0 \]
3 Generalized Birkhoff Theorem

We now provide a simple proof of the two-dimensional version of Birkhoff’s theorem for the model, extending the results of [13]. Without loss of generality we choose a coordinate frame in which the metric is conformally flat, and introduce light cone coordinates \( z_+ = x + t \) and \( z_- = x - t \). In this frame we have:

\[
\begin{align*}
g_{++} &= g_{--} = 0 \\
g_{+-} &= g_{-+} = \frac{1}{2} e^{2\rho(z_+, z_-)}
\end{align*}
\] (12) (13) (14)

We also define for convenience \( F(z_+, z_-) := 2F_{++} = -2F_{-+} \). In conformal gauge, the field equations take the form:

\[
8e^{-2\rho} \frac{\partial^2 \rho}{\partial z_+ \partial z_-} - \frac{1}{l^2} \frac{dV}{d\phi} - Ge^{-4\rho} \frac{dW(\phi)}{d\phi} F^2 = 0
\] (15)

\[
\frac{\partial^2 \phi}{\partial z_+^2} + \frac{\partial^2 \phi}{\partial z_-^2} + 2 \frac{\partial^2 \phi}{\partial z_+ \partial z_-} - 2 \frac{\partial \rho}{\partial z_+} \frac{\partial \phi}{\partial z_-} - 2 \frac{\partial \rho}{\partial z_-} \frac{\partial \phi}{\partial z_+} + \frac{1}{2} e^{2\rho} \frac{V(\phi)}{l^2} + G \frac{e^{-2\rho} W(\phi)}{2} F^2 = 0
\] (16)

\[
\frac{\partial^2 \phi}{\partial z_+^2} + \frac{\partial^2 \phi}{\partial z_-^2} + 2 \frac{\partial^2 \phi}{\partial z_+ \partial z_-} - 2 \frac{\partial \rho}{\partial z_+} \frac{\partial \phi}{\partial z_-} - 2 \frac{\partial \rho}{\partial z_-} \frac{\partial \phi}{\partial z_+} - \frac{1}{2} e^{2\rho} \frac{V(\phi)}{l^2} - G e^{-2\rho} W(\phi) F^2 = 0
\] (17)

\[
\frac{\partial^2 \phi}{\partial z_+^2} - \frac{\partial^2 \phi}{\partial z_-^2} - 2 \frac{\partial \rho}{\partial z_+} \frac{\partial \phi}{\partial z_+} + 2 \frac{\partial \rho}{\partial z_-} \frac{\partial \phi}{\partial z_-} = 0
\] (18)

\[
\frac{\partial}{\partial z_+} \left( W(\phi)e^{-2\rho} F \right) = 0
\] (19)

\[
\frac{\partial}{\partial z_-} \left( W(\phi)e^{-2\rho} F \right) = 0
\] (20)
From Eqs(19,20) it follows that:

$$F = q \frac{e^{2\rho}}{W(\phi)}$$  \hspace{1cm} (21)

where $q$ is a constant.

Substituting the above solution for $F$ into (15-17) and then comparing the result with Eqs(16-19) of Ref.[13], we see that they are identical providing we replace

$$V(\phi) \rightarrow V(\phi) - \frac{Gq^2l^2}{W(\phi)}.$$  \hspace{1cm} (22)

The rest of the proof therefore proceeds exactly as in Ref.[13]. As long as

$$\frac{\partial \phi}{\partial z_+ \, \partial z_-} > 0,$$  \hspace{1cm} (23)

it is always possible to choose a local coordinate system in which $\partial \phi / \partial t = 0$, so that the field equations reduce to:

$$\frac{1}{2} \frac{d\phi}{dx} - \frac{1}{4} \frac{j(\phi)}{l^2} + \frac{1}{4} Gq^2k(\phi) = -\frac{1}{4} C$$  \hspace{1cm} (24)

$$e^{2\rho} = \frac{1}{2} \frac{d\phi}{dx} = \frac{1}{4} \left( -C + \frac{j(\phi)}{l^2} - Gq^2k(\phi) \right)$$  \hspace{1cm} (25)

where $C$ is another constant, and $j(\phi)$ and $k(\phi)$ are defined by:

$$\frac{dj(\phi)}{d\phi} = V(\phi)$$  \hspace{1cm} (26)

$$\frac{dk(\phi)}{d\phi} = \frac{1}{W(\phi)}$$  \hspace{1cm} (27)

Note that the constants of integration that result from the implicit definitions of $j$ and $k$ above can without loss of generality be absorbed into $C$. Also, since $\phi$ is independent of time, so is $\rho$ (cf. Eq.(25)) and $F$ (cf. Eq.(21)). Thus, we have shown that (up to spacetime diffeomorphisms) every solution is static, and depends on two independent parameters: $q$ and $C$. The solutions can be expressed in terms of the spatial coordinate, $x$, by inverting Eq.(24):

$$x = x(\phi) = -2 \int \frac{d\phi}{(C - \frac{j(\phi)}{l^2} + q^2Gk(\phi))}$$  \hspace{1cm} (28)
The constant of integration here corresponds to a trivial shift \( x \to x + \text{constant} \). The metric and electromagnetic field strength then take the simple form

\[
2g_{+-} = e^{2\rho} = -\frac{(C - \frac{j(\phi)}{l^2} + Gq^2k(\phi))}{4}
\]  

\[
F = -q\frac{(C - \frac{j(\phi)}{l^2} + q^2Gk(\phi))}{4W(\phi)}
\]  

It is worth noting that in the case \( \frac{\partial \phi}{\partial z_+} \frac{\partial \phi}{\partial z_-} < 0 \), one can repeat the argument above to show that there always exists a local coordinate system in which the solution is independent of the spatial coordinate\(^\text{[13]}\). In this case, the solution is identical in form to the one given above, but with the \( x \)-dependence replaced by \( t \)-dependence.

In order to verify that the constants of integration \( q \) and \( C \) are indeed independent of coordinate system and choice of \( U(1) \) gauge we observe that for a generic solution they can be written in the following invariant form:

\[
C = -g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} + \frac{j(\phi)}{l^2} - Gq^2k(\phi)
\]

\[
q = \frac{1}{\sqrt{-g}}W(\phi)F.
\]

Clearly \( q \) plays the role of electric charge in the theory, while as shown below, \( C \) determines the energy of the solution.

The above solutions have a Killing vector which (for \( e^{2\rho} > 0 \) is timelike or spacelike depending on the sign of

\[
\frac{\partial \phi}{\partial z_+} \frac{\partial \phi}{\partial z_-}.
\]

In fact one can verify that Lie derivation along the vector field:

\[
k^\mu = \frac{l}{\sqrt{-g}}\epsilon^{\mu\nu}\partial_\nu\phi
\]

leaves the metric, the scalar field and the electromagnetic field strength invariant as long as the field equations (9-11) are satisfied. Thus the Killing vector Eq.(33) represents a symmetry of every solution in the generic theory. The norm of the Killing vector is:

\[
|k|^2 = -l^2g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi
\]

\[
= l^2C - j(\phi) + l^2Gq^2k(\phi)
\]
where we have used Eq.(31) to obtain the second line of the above. Defining
\[ f(\phi; C, q) = -|k|^2 = -(l^2 C - j(\phi) + l^2 G q^2 k(\phi)) , \]
the necessary condition for a given theory to admit charged black hole configurations
is the existence of curves in spacetime given by \( \phi(x, t) = \phi_0 = \text{constant} \) such that
\[ f(\phi_0; C, q) := j(\phi_0) - l^2 q^2 G k(\phi_0) - l^2 C = 0 \] (36)
Note that for fixed \( C \) there may exist critical values \( q(C) \) for which
\[ \frac{df(\phi; C, q)}{\phi} \bigg|_{\phi_0} = 0. \] (37)
Thus \( \phi_0 \) may be a local extremum of the function \( f(\phi; C, q) \), or a point of inflection.
If it is an extremum, the norm of the Killing vector does not change sign as one moves
through the event horizon at \( \phi = \phi_0 \). Moreover, if \( q \) is varied away from its critical
value, the horizon in general will either disappear, or two event horizons (an inner and
outer horizon) will appear. In this case, the condition Eq.(37) signals the presence of
an extremal black hole. In the special case that one has a point of inflection, the norm
of the Killing vector does change sign, but as the parameters are varied away from
their critical values, one expects the formation of either one, or three horizons. These
model independent conditions for the presence of extremal horizons will be useful in
Section 5, when the thermodynamic properties are discussed.

We close this section on the space of solutions by writing down the most general
solution in a particular convenient gauge. We choose the \( x \)-coordinate to be \( x = l\phi \),
and \( g_{tx} = 0 \). In this case, the metric takes the “Reissner-Nordstrom” form:
\[ ds^2 = -f(x; C, q) dt^2 + f(x; C, q)^{-1} dx^2 \]
\[ F = \frac{q}{W(x)} \] (38) \hspace{1cm} (39)
where \( f \) is defined in Eq.(35). As we shall see in Section 5, this form of the solution
allows one to extract in a very simple way the thermodynamic properties of the black
hole. First, however, we will do a Hamiltonian analysis of the theory to prove that
\( C \) and \( q \) are indeed a complete set of physical configuration space variables, and that
the parameter \( C \) determines the energy of the solution.
4 Hamiltonian Analysis

Spacetime is assumed to be locally a direct product $R \times \Sigma$, where the spatial manifold, $\Sigma$, can at this stage be either open or closed. The metric can be parametrized as follows:

$$ds^2 = e^{2\rho} \left[-\sigma^2 dt^2 + (dx + M dt)^2\right].$$

where $x$ is a local coordinate for the spatial section $\Sigma$ and $\rho$, $\sigma$ and $M$ are functions of spacetime coordinates $(x, t)$. In terms of this parametrization, the action Eq.(6) takes the form (up to surface terms):

$$S = \int dtdx \left[ \frac{1}{G} \frac{\dot{\phi}}{\sigma} (M \rho' + M' - \dot{\rho}) + \frac{\dot{\phi}'}{\sigma} (\sigma \sigma' - MM' + M \dot{\rho} + \sigma^2 \rho' - M^2 \rho') + \frac{1}{2} \sigma e^{2\rho} V(\phi) l^2 \right] + \frac{1}{2} e^{-2\rho} W(\phi) (A_1 - A_0')^2$$

In the above dots and primes denote differentiation with respect to time and space, respectively. The canonical momenta for the fields $\{\phi, \rho, A_1\}$ are:

$$\Pi_{\phi} = \frac{1}{G \sigma} (M \rho' + M' - \dot{\rho})$$

$$\Pi_{\rho} = \frac{1}{G \sigma} (-\dot{\phi} + M \phi')$$

$$\Pi_{A_1} = \frac{e^{-2\rho}}{\sigma} W(\phi) (A_1 - A_0')$$

The momenta conjugate to $\sigma$, $M$ and $A_0$ vanish: these fields play the role of Lagrange multipliers that are needed to enforce the first class constraints associated with diffeomorphism and gauge invariance of the classical action. A straightforward calculation leads to the canonical Hamiltonian (up to surface terms which will be discussed below):

$$H_c = \int dx (M\mathcal{F} + \frac{\sigma}{2G} \mathcal{G} + A_0 \mathcal{J})$$

where

$$\mathcal{F} = \rho' \Pi_{\rho} + \phi' \Pi_{\phi} - \Pi'_{\rho} \sim 0$$

$$\mathcal{G} = 2\phi'' - 2\phi' \rho' - 2G^2 \Pi_{\rho} \Pi_{\phi} - e^{2\rho} \frac{V(\phi)}{l^2} + G \frac{e^{2\rho}}{W(\phi)} \Pi_{A_1}^2 \sim 0$$

$$\mathcal{J} = -\Pi'_{A_1} \sim 0$$
are secondary constraints. The notation \( \sim 0 \) denotes weakly vanishing in the Dirac sense\[^{[21]}\]. It is straightforward to verify that the above constraints are first class and that no further constraints appear in the Dirac algorithm. The constraints \( \mathcal{F} \) and \( \mathcal{G} \) generate spacetime diffeomorphisms. The remaining constraint \( \mathcal{J} \) is the analogue of Gauss’s law, and generates \( U(1) \) gauge transformations. Since there are three first class constraints and six phase space degrees of freedom at each point in space, there are no propagating modes in the theory. In fact, the reduced phase space is finite dimensional. As in the vacuum case\[^{[12]}\], the constraints can be explicitly solved for the momenta as a function of the fields:

\[
\Pi_\rho = Q[C, q; \rho, \phi] \\
\Pi_\phi = \frac{g[q; \rho, \phi]}{(2G)^2 Q[C, q; \rho, \phi]} \\
\Pi_{A_1} = q
\]

where we have defined:

\[
Q[C, q; \rho, \phi] \equiv \frac{1}{G} \sqrt{(\phi')^2 + e^{2\rho}(C - \frac{j(\phi)}{l^2} + Gq^2k(\phi))} \tag{52}
\]

\[
g[q; \rho, \phi] \equiv 4\phi'' - 4\phi' \rho' - 2e^{2\rho} \frac{V(\phi)}{l^2} + 2Gq^2 \frac{e^{2\rho}}{W(\phi)} \tag{53}
\]

In the above, \( C \) and \( q \) are arbitrary constants of integration which represent physical observables in the theory. Using Eq.(49) and Eq.(51), these observables can be written in terms of the original phase space variables as follows:

\[
C = e^{-2\rho}(G^2 \Pi_\rho^2 - (\phi')^2) + \frac{j(\phi)}{l^2} - Gk(\phi)\Pi_{A_1}^2 \tag{54}
\]

\[
q = \Pi_{A_1} \tag{55}
\]

It is straightforward to show that \( C \) and \( q \) as defined above have vanishing Poisson brackets with all the constraints: they are physical observables in the Dirac sense. Moreover, both \( C \) and \( q \) are independent of the spatial coordinate on the constraint surface. \( q' \sim 0 \) follows trivially from Gauss’ law, while from Eq.(54), Eq.(55) and (46 - 48) one can show that:

\[
\tilde{\mathcal{G}} := C' = -e^{-2\rho}(\phi' \mathcal{G} + 2G^2 \Pi_\rho \mathcal{F}) + 2Gk(\phi)\Pi_{A_1} \mathcal{J} \sim 0 \tag{56}
\]
Since the observables $C$ and $q$ have vanishing Poisson bracket with each other, the reduced phase space is four dimensional. The momenta conjugate to $C$ and $q$ are:

$$p_C = -\frac{1}{2} \int dx \frac{e^{2\rho} \Pi_{\rho}}{((G\Pi_{\rho})^2 - (\phi')^2)}$$

$$p_q = -\int dx \left( A_1 + \frac{Gk(\phi)e^{2\rho} \Pi_{\rho} \Pi_{A_1}}{((G\Pi_{\rho})^2 - (\phi')^2)} \right)$$

It is straightforward to verify the canonical Poisson bracket relations:

$$\{C, p_C\} = \{q, p_q\} = 1$$

$$\{C, q\} = \{C, p_q\} = \{q, p_C\} = \{p_C, p_q\} = 0$$

As in the case of pure dilaton gravity\cite{12}, the apparent discrepancy between the generalized Birkhoff theorem (which states the existence of a two parameter family of solutions up to spacetime diffeomorphisms and gauge transformations) and the dimension of the reduced phase space is explained by the following observation. The configuration space variables $C$ and $q$ are invariant under a general canonical transformation generated by:

$$\int dx \left( \xi \mathcal{F} + \eta \mathcal{G} + \chi \mathcal{J} \right),$$

independent of the boundary conditions on the test functions. The momenta, on the other hand, transform as follows:

$$\delta p_C = -\frac{1}{2} \int dx \left( \frac{e^{2\rho}(\xi \Pi_{\rho} - 2\eta \phi')}{{((G\Pi_{\rho})^2 - (\phi')^2)}} \right)'$$

$$\delta p_q = -\int dx \left( \frac{Gk(\phi)e^{2\rho} \Pi_{A_1}}{((G\Pi_{\rho})^2 - (\phi')^2)} (\xi \Pi_{\rho} - 2\eta \phi') + \chi \right)'$$

For non-compact spacetimes, the momenta are invariant only if the test functions vanish sufficiently rapidly at infinity. Since symmetry transformations in the canonical theory are defined precisely in terms of test functions that vanish at infinity, $p_C$ and $p_q$ are physical observables, as claimed above. However, in the proof of Birkhoff’s theorem, the global behaviour of the spacetime diffeomorphisms and gauge transformations is not restricted, and hence the momenta are not invariant in the covariant

\footnote{In practice, these were calculated by differentiation of the exact Hamilton-Jacobi functional derived in Section 6 with respect to the observables $C$ and $q$.}
sense: both $p_C$ and $p_q$ can be changed by “non-canonical”, large diffeomorphisms and/or U(1) gauge transformations. As shown for SSG by Kuchar [6], and for generic 2-D dilaton gravity[3], the momentum conjugate to $C$ has a physical interpretation as the time separation at infinity. $p_q$, on the other hand, is clearly related to the asymptotic choice of $U(1)$ gauge.

We also note that the transformation properties of $p_C$ and $p_q$ under large diffeomorphisms and U(1) gauge transformations play an important role in the quantum theory. In particular, they ensure that physical quantum wave functions have the right transformation properties under such large transformations. For example, states of definite charge $q$ should pick up a phase proportional to $q\delta p_q$. This will be discussed further in the Section on Dirac quantization.

We now address the question of surface terms and derive the ADM energy of the generic solution. The analysis is simplified by first using Eq.(56) and Eq.(42) to write the canonical Hamiltonian in the following form:

$$H_c = \int dx \left[ \frac{\dot{\phi}}{\phi} F - \frac{1}{2G} \left( \frac{\sigma e^{2\rho}}{\phi'} \right) \tilde{G} - A_0 q' \right]$$

(63)

For configurations that approach Eq.(38) asymptotically, so that $\dot{\phi} \to 0$ and

$$\frac{\sigma e^{2\rho}}{l\phi'} \to 1$$

(64)

at spatial infinity, the surface term that is required in order to ensure that the canonical Hamiltonian yields the correct equations of motion is:

$$H_{surface} = H_{ADM} + \int (A_0 q)'$$

(65)

where we have defined the ADM surface Hamiltonian:

$$H_{ADM} = \frac{l}{2G} \int dx \left( \frac{\sigma e^{2\rho}}{l\phi'} C \right)'$$

(66)

The resulting ADM energy, which is by definition equal to $H_{ADM}$ evaluated at spatial infinity is:

$$M = \frac{lC}{2G}$$

(67)

Thus $C$ is indeed related to the energy of the solution, as claimed above. Similarly, the “electric charge” $q$ can be seen to be the conserved charge associated with U(1) gauge
transformations that leave $A_0$ invariant at spatial infinity. Note that this derivation of the ADM energy does not require asymptotic flatness\textsuperscript{3} merely that the solutions approach the static Reissner-Nordstrom form given in the previous sections. This is important since the black hole solutions are not asymptotically flat in the generic case: for example in J-T gravity, the solution is asymptotically deSitter.

For completeness we now write down the Hamiltonian equations of motion that follow from the canonical Hamiltonian Eq.(63) when supplemented with the above surface term:

\[ \dot{\Pi}_\phi := \{\Pi_\phi, H_c\} = -\frac{\sigma''}{G} - \frac{1}{G} (\sigma \rho')' + \frac{\sigma}{2G} \frac{e^{2\rho}}{I^2} \frac{dV}{d\phi} + (M\Pi_\rho)' + \frac{\sigma e^{2\rho}}{2W^2} \frac{dW}{d\phi} \Pi_{A_1}^2 \]  

\[ \dot{\Pi}_\rho := \{\Pi_\rho, H_c\} = (M\Pi_\rho)' - \frac{1}{G} (\sigma \phi')' + \frac{\sigma}{2G} \frac{e^{2\rho}}{I^2} V(\phi) - \frac{\sigma e^{2\rho}}{W(\phi)} \Pi_{A_1}^2 \]  

\[ \dot{\Pi}_{A_1} = \{\phi_{A_1}, H_c\} = 0 \]  

The physical observables $C$ and $q$ commute with the canonical Hamiltonian, and are therefore time independent, as expected.

5 Thermodynamic Properties

The surface gravity, $\kappa$ of a black hole is given in terms of the Killing vector by\textsuperscript{16}:

\[ \kappa^2 = -\frac{1}{2} \nabla^\mu k^\nu \nabla_\mu k_\nu \bigg|_{\phi_0} \]  

$\phi = \phi_0$ is a solution Eq.(36). Using the expression Eq.(33) for the Killing vector and the field equation Eq.(10) one obtains (after some tedious algebra):

\[ \kappa = \frac{V(\phi_0)}{2l} - \frac{lq^2 G}{2W(\phi_0)} \]  

\textsuperscript{3}The derivation of the ADM mass in generic dilaton gravity has recently been discussed in \textsuperscript{14} and \textsuperscript{15}
The Hawking temperature in the generic theory can be calculated by analytically continuing the solutions Eq.(38) to Euclidean spacetime and imposing periodicity in the imaginary time direction in order to allow non-singular solutions without boundary at the outer horizon. The Euclidean solutions take the general form:

\[ ds^2 = f(\frac{x}{l}; M, q)dt_E^2 + \frac{1}{f(\frac{x}{l}; M, q)}dx^2 \]  

(73)

where \( M = lC/2G \), and \( f(\frac{x}{l}; M, q) \) is the square of the norm of the Killing vector defined in Eq.(35). By defining a dimensionless time coordinate \( \alpha := t_E/a \) and a new radial coordinate, \( R \), such that

\[ R^2 = a^2 f(\frac{x}{l}; M, q) \]  

(74)

the metric can be placed in the form:

\[ ds^2 = R^2 d\alpha^2 + H(R; M, q)dR^2 \]  

(75)

where

\[ H(R; q, M) := \frac{4}{a^2(f'((\frac{x}{l}; M, q))^2} \]  

(76)

and the prime denotes differentiation with respect to \( x \). The solution will be regular at \( R = 0 \), if \( H(R) \to 1 \) as \( R \to 0 \) and the coordinate \( \alpha \) be periodic with period \( 2\pi \). The Hawking temperature is then the inverse of the period of the corresponding Euclidean time coordinate:

\[ T = (2\pi a)^{-1} = \frac{f'((x_0/l); M, q)}{4\pi} \]  

(77)

where \( x_0/l = \phi_0 \) is the location of the horizon. Using Eq.(35) and the fact that \( j' = V/l \) and \( k' = 1/(lW) \), we are able to verify the standard relationship between the Hawking temperature and the surface gravity in the generic theory:

\[ \kappa = 2\pi T \]  

(78)

Since Eq.(37) signals the presence of an extremal black hole at \( \phi = \phi_0 \), this shows that the surface gravity and Hawking temperature both vanish for extremal black holes in the generic theory.
From the form of Eq.(73) and Eq.(35) it is also possible to extract directly the
entropy of the black hole. In particular, if we vary the solution infinitesimally, but
stay on the event horizon $\phi_0$ which is a solution to $f(\phi_0; M, q) = 0$, we obtain the
condition:

$$0 = \delta f(\phi_0; M, q) = -2G_l \delta M + 2l \kappa \delta \phi - 2ql^2Gk(\phi_0)\delta q$$

(79)

where $\kappa$ is precisely the surface gravity defined in Eq.(72). This gives:

$$\delta M = \left( \frac{V(\phi_0)}{l} - \frac{lq^2G}{W(\phi_0)} \right) \frac{\delta \phi_0}{2G} - lj k(\phi_0)\delta q$$

(80)

which is of the desired form:

$$\delta M = \kappa \delta S - \mathcal{P} dq$$

(81)

with entropy

$$S = \frac{2\pi}{G} \phi_0$$

(82)

and generalized force $\mathcal{P} := qlk(\phi_0)$ associated with the charge $q$. Note that the
entropy Eq.(82) is proportional to the value of the scalar field at the event horizon.
Eq.(82) has precisely the same form as the expression obtained for general dilaton
gravity without $U(1)$ gauge field in [3]. The above analysis implies that in dilaton
gravity, Eq.(36) can be interpreted as the thermodynamic equation of state for the
black hole since it relates the energy $M$ and entropy $\frac{2\pi\phi_0}{G}$ to the extrinsic macroscopic
variable $q$.

Although Eq.(82) was obtained using a convenient choice of coordinates, it is in
fact coordinate invariant. We will now verify this by using Wald’s general method[17]
to derive the entropy of black hole solutions in the generic model. The application
of the method in 1+1 dimensions turns out to be simpler than in higher dimensions.
Here we use the language of tensor calculus but the analysis can easily be repeated
using forms as done by Wald. In general one looks at the variation of the action under
space-time diffeomorphisms of the form:

$$x^\mu \rightarrow x'^\mu = x^\mu + \delta x^\mu$$
$$\Phi^A(x) \rightarrow \Phi'^A(x') = \Phi^A(x) + \delta \Phi^A(x)$$

(83)

where for the moment we use a condensed notation in which the complete set of fields
(including the metric) is denoted by $\Phi^A(x)$, and $x$ is the spacetime coordinate. It can
be shown that, under such a general transformation, an action which is second order in derivatives of the fields has the following variation:

$$\delta I = \int d^2x \left( \frac{\delta I}{\delta \Phi} \delta \Phi + \frac{\partial j^\mu}{\partial x^\mu} \right)$$  \hspace{1cm} (84)$$

where $j^\mu$ is the associated Noether current. Diffeomorphism invariance of the action requires that the Noether current be divergence free when the classical field equations

$$\frac{\delta I}{\delta \Phi} = 0$$  \hspace{1cm} (85)$$

are satisfied.

For the action Eq. (5) the Noether current is:

$$j^\lambda = \left( \frac{1}{2G} (\phi R + \frac{V}{l^2}) - \frac{1}{4} WF^2 \right) \delta x^\lambda - \frac{1}{2G} \nabla_\sigma \phi (g^{\alpha\lambda} g^{\beta\sigma} - g^{\alpha\beta} g^{\lambda\sigma}) \delta g_{\alpha\beta}$$

$$+ \frac{1}{2G} \phi (g^{\alpha\sigma} g^{\beta\lambda} - g^{\alpha\beta} g^{\sigma\lambda}) \nabla_\sigma (\delta g_{\alpha\beta}) - WF^{\lambda\sigma} \delta A_\sigma$$  \hspace{1cm} (86)$$

where $\delta$ denotes variation of the corresponding field under Lie derivation along $\delta x^\mu$.

As shown in Section 2, the solutions have a single Killing vector, so we take:

$$\delta x^\lambda = -k^\lambda = -\frac{l}{\sqrt{-g}} \epsilon^{\lambda\sigma} \nabla_\sigma \phi$$  \hspace{1cm} (87)$$

The variations of the scalar field and metric vanish for such transformations, but the Lie derivative of the vector potential in the Killing direction can be written:

$$\delta A_\sigma = -k^\eta F_{\sigma\eta} + \nabla_\sigma (k^\eta A_\eta)$$  \hspace{1cm} (88)$$

Using this expression and the field equations, we write the Noether current as:

$$j^\lambda = \left[ \frac{1}{2G} \left( \frac{\phi}{l^2} \frac{dV}{d\phi} + \frac{V}{l^2} \right) + \frac{1}{4} (\phi \frac{dW}{d\phi} - W) F^2 \right] \frac{l}{\sqrt{-g}} \epsilon^{\lambda\sigma} \nabla_\sigma \phi$$

$$+ W F^{\lambda\sigma} F_{\sigma\eta} \frac{l}{\sqrt{-g}} \epsilon^{\eta\rho} \nabla_\rho \phi - \nabla_\sigma \left( \frac{l}{\sqrt{-g}} W F^{\sigma\lambda} \epsilon^{\eta\rho} \nabla_\rho \phi A_\eta \right)$$  \hspace{1cm} (89)$$

Note that the last term is gauge dependent, but has identically vanishing divergence. The Noether current is not uniquely defined in general since one can always add to
it an arbitrary divergence free term. As shown by Wald, however there is a unique
diffeomorphism covariant current which contains at most first derivatives of the fields.
In the present case we must also add the condition of gauge invariance, in which case,
we find the following, unique Noether current:
\[
J^\lambda = - \left[ \frac{1}{2G} \left( - \phi \frac{dV}{l^2} \frac{d\phi}{l^2} + V \right) - \frac{1}{2} q^2 \left( \phi \frac{dW}{W^2} \frac{d\phi}{l^2} + \frac{1}{W} \right) \right] \frac{1}{\sqrt{-g}} \epsilon^{\lambda \sigma} \nabla_\sigma \phi
\] (90)

Following Wald, we can define the associated one form:
\[
J_\mu := \sqrt{-g} \epsilon_{\mu \lambda} J^\lambda
\] (91)
\[
= - \left[ \frac{1}{2Gl} \left( V - \phi \frac{dV}{d\phi} \right) - \frac{1}{2} q^2 \left( \frac{\phi}{W^2} \frac{dW}{d\phi} + \frac{1}{W} \right) \right] \nabla_\mu \phi
\] (92)

Since the exterior derivative of \(J\) vanishes identically we can at least locally write it
as the derivative of a zero form:
\[
J_\mu = \frac{\partial Q}{\partial x^\mu}
\] (93)

where
\[
Q = \frac{1}{2Gl}(V - \frac{Gl^2 q^2}{W}) \phi - \frac{1}{G} g^{\alpha \beta} \nabla_\alpha \phi \nabla_\beta \phi.
\] (94)

According to Wald’s prescription, the value of \(Q\) at the event horizon should be
proportional to the entropy. Indeed, since on the event horizon \(|\nabla \phi|^2 = 0\), we find:
\[
Q|_{\text{horizon}} = \left. \frac{1}{2Gl}(V - \frac{Gl^2 q^2}{W}) \phi \right|_{\text{horizon}}
\] (95)
\[
= \frac{\kappa}{2\pi} S
\] (96)

where \(\kappa\) is the surface gravity and \(S\) is the entropy defined in Eq.(82) above.

6 Dirac Quantization

We will now quantize the generic theory in the functional Schrodinger representation,
using techniques first developed by Henneaux[18] to quantize Jackiw-Teitelboim grav-
ity, and later extended to the generic dilaton gravity theory in [12]. Similar techniques
have recently been applied by Strobl\cite{4} to the quantization of dilaton gravity coupled to a Yang-Mills field in a first order formalism. For the pure dilaton-gravity sector of String Inspired Dilaton gravity, these techniques have recently been shown\cite{19} to yield quantum theories equivalent to those obtained within the gauge theoretical formulation\cite{20}.

Following the standard Dirac\cite{21} prescription we first define a Hilbert space of states described by wave functionals \( \psi[\rho, \phi, A_1] \) defined on the unreduced configuration space. Given a Hilbert space scalar product of the form:

\[
< \psi | \psi > = \int \prod_x d\rho(x) d\phi(x) dA_1(x) \mu[\rho, \phi, A_1] \psi^* \psi \tag{97}
\]

we can define Hermitian operators for the momenta canonically conjugate to \( \rho, \phi \) and \( A_1 \) as follows:

\[
\hat{\Pi}_\rho = -i\hbar \frac{\delta}{\delta \rho(x)} + \frac{i\hbar}{2} \frac{\delta \ln(\mu[\alpha, \tau])}{\delta \rho} \tag{98}
\]

\[
\hat{\Pi}_\phi = -i\hbar \frac{\delta}{\delta \phi(x)} + \frac{i\hbar}{2} \frac{\delta \ln(\mu[\alpha, \tau])}{\delta \phi} \tag{99}
\]

\[
\hat{\Pi}_{A_1} = -i\hbar \frac{\delta}{\delta A_1(x)} + \frac{i\hbar}{2} \frac{\delta \ln(\mu[\alpha, \tau])}{\delta A_1} \tag{100}
\]

Physical quantum wave functionals \( \Psi[\rho, \phi, A_1] \) are defined to be those states annihilated by the quantized constraints:

\[
\hat{\mathcal{F}} \Psi = 0 \tag{101}
\]

\[
\hat{\mathcal{G}} \Psi = 0 \tag{102}
\]

\[
\hat{\mathcal{J}} \Psi = 0 \tag{103}
\]

The first and last constraints are linear in the momenta and the standard factor ordering with all the momenta on the right is self-adjoint providing that the functional measure \( \mu \) is invariant under the corresponding gauge transformation. The factor ordering for the Hamiltonian constraint \( \hat{\mathcal{G}} \) on the other hand is more subtle. Here we will be concerned only with the lowest order, or WKB approximation, and hence we can safely defer questions concerning the choice of factor ordering and measure.

To proceed we look for the analogue of stationary states in the theory. In particular, since the observable \( C \) defined in Eq.\((54)\) determines the energy of a configuration,
we look for physical states that are also eigenstates of the operator \( \hat{C} \)

\[ \hat{C} \Psi = C \Psi \quad (104) \]

where \( C \) is a constant. Note the Eq.\( (104) \) automatically guarantees that the Hamiltonian constraint is also satisfied, since:

\[ \tilde{\hat{G}} \Psi = C' \Psi = 0 \quad (105) \]

Analogously, the \( U(1) \) gauge constraint can be satisfied by looking for eigenstates of the charge operator:

\[ \hat{q} \Psi = \hat{\Pi}_{A_1} \Psi = q \Psi \quad (106) \]

Thus we are looking for (spatial) diffeomorphism invariant energy and charge eigenstates. Since \( C \) and \( q \) are a complete set of variables, one can construct all states by taking linear combinations of such states.

The WKB approximation for gauge theories in general and quantum gravity in particular has been discussed extensively by Barvinsky \cite{22} and more recently by Lifschytz \textit{et al} \cite{23}. We expand the phase of the wave functional as follows:

\[ \psi[^4\rho, \phi, A_1] = \exp \frac{i}{\hbar} [S_0[^4\rho, \phi, A_1] + \hbar S_1[^4\rho, \phi, A_1] + ...] \quad (107) \]

To lowest order in \( \hbar \) we find that \( S_0 \) must obey the following equations:

\[ e^{-2\rho} \left( G^2 \left( \frac{\delta S_0}{\delta \rho} \right)^2 - (\phi')^2 \right) + \frac{j(\phi)}{l^2} - Gk(\phi) \left( \frac{\delta S_0}{\delta A_1} \right)^2 \right] = C \quad (108) \]

\[ \rho \frac{\delta S_0}{\delta \rho} + \phi \frac{\delta S_0}{\delta \phi} - \left( \frac{\delta S_0}{\delta \rho} \right)' = 0 \quad (109) \]

\[ \frac{\delta S_0}{\delta A_1} = q \quad (110) \]

Note that Eq.\( (108) \) is the Hamilton-Jacobi equation for the theory. Remarkably, the above functional differential equations can be solved exactly. After some algebra, Eqs.\( (108) \) and \( (109) \) reduced to the following first order functional differential
equations for $S_0$:

\[
\frac{\delta S_0}{\delta \rho} = Q \tag{111}
\]

\[
\frac{\delta S_0}{\delta \phi} = \frac{g}{(2G)^2 Q} \tag{112}
\]

where $g$ and $Q$ are defined in Eqs.\[52\] and \[53\]. The closure of the constraint algebra guarantees that these functional differential equations are integrable. The resulting exact solution for the Hamilton principal functional, $S_0$, is:

\[
S_0[C, q; \rho, \phi, A_1] = \int dx \left[ Q + \frac{\phi'}{2G} \ln \left( \frac{\phi' - GQ}{\phi' + GQ} \right) + qA_1 \right] \tag{113}
\]

As expected, the partial derivatives of $S_0$ with respect to $C$ and $q$ are proportional (on the constraint surface) to the corresponding conjugate momenta, $p_C$ and $p_q$, as given in equations Eq.(57) and Eq.(58). This ensures (at least to the WKB order considered here) that the quantum operators $\hat{p}_C$ and $\hat{p}_q$ have the correct action on eigenstates of $\hat{C}$ and $\hat{q}$. i.e. $\hat{p}_C|C > = i\hbar (\partial / \partial C)|C >$, etc. Moreover, it is straightforward to verify that under a large $U(1)$ gauge transformation, the change in the WKB phase is precisely equal to $q\delta p_q$ as expected from the discussion in Section 4. The action of large spacetime diffeomorphisms is more complicated since it is not generated by constraints linear and homogeneous in the momenta. However, as long as the quantum operators $\hat{C}$ and $\hat{p}_C$ are correctly represented, energy eigenstates will necessarily change by a phase under a change in $p_C$, which classically is generated by large spacetime diffeomorphisms.

We also note that one can also interpret the WKB wave-functional $\Psi_0 = \exp(i/\hbar S_0)$ in a different way. If one chooses to first solve the constraints in the theory classically as in Eqs.\[49-51\] and then quantize, one obtains the complete set of quantum constraints on physical states:

\[
\left( \hat{\Pi}_\rho - Q \right) \Psi_0 = 0 \tag{114}
\]

\[
\left( \hat{\Pi}_\phi - \frac{g}{(2G)^2 Q} \right) \Psi_0 = 0 \tag{115}
\]

\[
\left( \hat{\Pi}_{A_1} - q \right) \Psi = 0 \tag{116}
\]

In this case, $\Psi_0 := \exp i S_0 / \hbar$ provides an exact solution to the constraints, providing that the quantum operators are defined to be Hermitian with respect to a trivial
functional measure \( \mu = 1 \). Thus, \( \Psi_0 \) can be interpreted \([18],[12]\) as an exact physical wave functional for the theory in which the original constraints are replaced by their classical solutions \( \text{Eqs.}[49-51] \). The quantum theories are equivalent classically and differ by factor ordering.

Finally we comment on the relationship between the analytic structure in the WKB phase and the existence of event horizons. The logarithm in Eq.\((113)\) appears to develop a singularity when

\[
(\phi')^2 - G^2Q^2 = 0
\]

which, as can be seen from Eq.\((34)\) and Eq.\((52)\) occurs at the event horizon \(|k|^2 = 0\) in any non-singular coordinate system \((e^{2\rho} \neq 0)\). Depending on the nature of the analytic continuation in defining the quantum wave function, the phase can pick up an imaginary contribution for values of \(x\) for which \((\phi')^2 - G^2Q^2 < 0\). As can be seen from Eq.\((34)\) and Eq.\((52)\), if one chooses nonsingular coordinates along a spacelike slice (for which \(e^{2\rho} > 0\)), this corresponds to the regions in spacetime for which the Killing vector is spacelike. For example, for non-extremal black holes in the generic theory it is possible to go to Kruskal-like coordinates \((X,t_K)\), in which case:

\[
\text{Im}S_0 = \frac{1}{2G} \text{Im} \int dX \frac{\partial \phi}{\partial X} \ln \left( \frac{X - t_K}{X + t_k} \right)
\]

If one chooses a spacelike slice that cuts across both horizons of an eternal black hole this yields:

\[
\text{Im}S_0 = \frac{i\pi}{2G} (\phi_+ - \phi_-) = 0
\]

where \(\phi_\pm\) corresponds to the value of the scalar field at the horizon \(X \pm t_K\) in the left and right asymptotic regions, respectively.

On the other hand, if one chooses a slicing that only cuts through the right horizon, say, as is appropriate in the case where the event horizon is formed by collapsing matter, then the imaginary part of the WKB phase is not necessarily zero. It will have a term proportional to \(\phi\) evaluated at the horizon, i.e. the entropy of the black hole. This analysis suggests that there is an intriguing connection between the entropy of the black hole, and the imaginary part of its WKB phase. Such a connection is in some sense natural, since the imaginary part of the WKB phase is generally related to quantum mechanical tunnelling, whereas Hawking radiation can heuristically be thought of as a quantum mechanical instability. In light of this interpretation, Eq.\((119)\) is also consistent with the recent result of Martinez\([25]\) who used
semi-classical partition function techniques to prove that the entropy of an eternal black hole vanishes due to a cancellation between contributions from the left and right hand asymptotic regions of the Kruskal diagram. It is also important to note that in the case of extremal black holes, the Killing vector is never spacelike, so that the imaginary part of the WKB phase is identically zero. This result is consistent with recent work\cite{26} which argues that the entropy of extreme black holes is zero. It is also consistent with the analysis of Kraus and Wilczeck\cite{27}, who showed semi-classically that gravitational back reaction corrections yield a vanishing amplitude for radiative processes at extremality.

7 Examples

7.1 Spherically Symmetric Gravity

We start with the Einstein-Maxwell action in 3+1 dimensions:

\[
I^{(4)} = \frac{1}{16\pi G^{(4)}} \int d^4 x \sqrt{-g^{(4)}} \left( R(g^{(4)}) - (F^{(4)})^2 \right)
\]  

(120)

We impose spherical symmetry on the electromagnetic field, and on the metric, as follows:

\[
ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu + \frac{l^2 \phi^2}{2} d\Omega^2,
\]  

(121)

where the \(x^\mu\) are coordinates on a two dimensional spacetime \(M_2\) with metric \(g_{\mu\nu}(x)\) and \(d\Omega^2\) is the line element of the 2-sphere with area \(4\pi\). The corresponding spherically symmetric solutions to the Einstein Maxwell equations are the stationary points of the dimensionally reduced action functional\cite{24}:

\[
\bar{I}[g, \phi] = \int_{M_2} d^2 x \sqrt{-g} \left[ \frac{1}{2} \left( \frac{\phi^2}{4} R(g) + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{l^2}{4} \right) - \frac{1}{8} \phi^2 F^\mu_\nu F^\nu_\mu \right],
\]  

(122)

where \(l^2 = G^{(4)}\) is the square of the 3+1 dimensional Planck length. This action is of the same form as Eq.(11) above. It can be verified that after the appropriate reparametrization as described in Section 2, the reduced action takes the form of
Eq. (6), with the corresponding values for the potential $V(\phi)$ and the function $W(\phi)$ are respectively:

$$V(\phi) = \frac{1}{\sqrt{2}\phi}$$  \hspace{1cm} (123)  

$$W(\phi) = (2\phi)^{3/2}$$  \hspace{1cm} (124)  

The most general solution to the field equations in our parametrization can be written in the form:

$$ds^2 = -\frac{r}{l} \left(1 - \frac{2Ml^2}{r} + \frac{l^4Q^2}{r^2}\right) dt^2 + \frac{r}{l} \left(1 - \frac{2Ml^2}{r} + \frac{l^4Q^2}{r^2}\right) dr^2$$  \hspace{1cm} (125)  

$$\phi = \frac{r^2}{2l^2}$$  \hspace{1cm} (126)  

$$F = \frac{l^2Q}{r^2}$$  \hspace{1cm} (127)  

The physical observables Eq. (54) and Eq. (55) are $C = 2M/l$ and $q = Q$, as expected. Note that the above metric corresponds to the usual Reissner-Nordstrom metric up to the conformal reparametrization:

$$g_{\mu\nu} = \sqrt{2\phi}g_{RN}$$  \hspace{1cm} (128)  

As is well known, there are two event horizons, located at $r_{\pm} = l^2(M \pm \sqrt{M^2 - Q^2})$. From the expressions Eq. (72) and Eq. (82) we can calculate the standard expressions\cite{16} for the surface gravity, and entropy, respectively, for this model:

$$\kappa = \frac{\sqrt{M^2 - Q^2}}{l^2(M + \sqrt{M^2 - Q^2})^2}$$  \hspace{1cm} (129)  

$$S = \frac{A_+}{4l^2}$$  \hspace{1cm} (130)  

where $A_+$ is the area of the outer horizon:

$$A_+ := 4\pi l^2(M + \sqrt{M^2 - Q^2})^2$$  \hspace{1cm} (131)  

The first law then takes the expected form:

$$\delta M = \frac{\kappa}{2\pi}\delta S + \varphi\delta q$$  \hspace{1cm} (132)  

where $\varphi$ is the electrostatic potential at the horizon.

\footnote{In this section we set $G = 1$ for simplicity}
7.2 Achucarro-Ortiz Black Hole

This black hole is a solution to the field equations for the Jackiw-Teitelboim theory of gravity, which can be obtained by imposing axial symmetry in 2+1 gravity[8]. It is therefore the projection of the BTZ black hole[9]. In particular, one starts with the Einstein action in 2+1 dimensions, with cosmological constant $\Lambda$:

$$I^{(3)} = \int d^3x \sqrt{-g^{(3)}}(R^{(3)} + \Lambda)$$ (133)

and then restricts consideration to the most general axially symmetric metrics:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + \phi^2(x)[d\theta + A_\mu(x)dx^\mu]^2$$ (134)

where in this case $x^\mu = \{t, \rho\}$ refer to cylindrical coordinates in the 2+1 spacetime, with angular coordinate $\theta$. One then finds a reduced two-dimensional action of the form:

$$I = \int d^2x \sqrt{-g}\phi \left( R + \Lambda - \frac{1}{4}\phi^3 F_{\mu\nu} F^{\mu\nu} \right).$$ (135)

In the above, $F_{\mu\nu}$ is the field strength of the one form $A_\mu$ that appears in the parametrization of the axially symmetric metric above. If we define $l^2 := 1/\Lambda$, then this is again of the desired form with $V = \phi$ and $W = \phi^3$. The most general solution corresponds precisely to the dimensional reduction of the BTZ axially symmetric black hole:

$$\phi = \sqrt{\Lambda} r$$ (136)

$$ds^2 = -(\frac{\Lambda^2 r^2}{2} - \sqrt{\Lambda} M + \frac{J^2}{4\Lambda r^2})dt^2 + \frac{dr^2}{(\frac{\Lambda^2 r^2}{2} - \sqrt{\Lambda} M + \frac{J^2}{4\Lambda r^2})}$$ (137)

$$F = \frac{1}{\Lambda^{3/4} \, r^3} J$$ (138)

In this case, the phase space observables are $C = M$ and $q = -J$. Note that the $U(1)$ charge is related to the angular momentum of the BTZ black hole. There are again two event horizons, with locations:

$$r_\pm = \frac{1}{\Lambda^{3/4}} \left( M \pm \sqrt{M^2 - \frac{J^2}{2}} \right)^{\frac{1}{2}}$$ (139)
One can again compute the surface gravity and entropy associated with the outer horizon from the expressions given above:

\begin{align*}
k &= \frac{\Lambda}{2} \frac{(r_+^2 - r_-^2)}{r_+} \tag{140} \\
S &= 4\pi\sqrt{\Lambda}r_+ \tag{141}
\end{align*}

### 7.3 String Inspired Model

The string inspired model with an electromagnetic interaction was studied in detail in [10]. In our parametrization, the string inspired action (without tachyon fields) is of the form:

\[ S = \frac{1}{2} \int d^2x \sqrt{-g} \left[ \phi R + \lambda^2 - \frac{1}{2} g^{\alpha\beta} g^{\gamma\delta} F_{\alpha\gamma} F_{\beta\delta} \right] \tag{142} \]

Thus, if \( l^2 := \frac{1}{\lambda^2} \), then \( V(\phi) = 1 \) and \( W(\phi) = \phi^2 \). The most general solution in this parametrization can be written [10]:

\begin{align*}
\frac{ds^2}{d^2x} &= - \left( \lambda r - \frac{2M}{\lambda} + \frac{Q^2}{\lambda^3 r} \right) dt^2 + \left( \lambda r - \frac{2M}{\lambda} + \frac{Q^2}{\lambda^3 r} \right)^{-1} dx^2 \tag{143} \\
\phi &= \lambda r \tag{144} \\
F &= \frac{Q}{\lambda^2 r^2} \tag{145}
\end{align*}

These solutions describe black holes with mass \( M \) and charge \( Q \). The event horizons occur at

\[ r_{\pm} := \frac{M}{\lambda^2} \pm \frac{1}{\lambda^2} \sqrt{M^2 - Q^2} \tag{146} \]

Using formulae Eq.(72) and Eq.(82), respectively, we obtain the surface gravity and entropy:

\begin{align*}
k &= \frac{1}{2} \left( \lambda - \frac{Q^2}{\lambda^3 r_+^2} \right) = \frac{\lambda \sqrt{M^2 - Q^2}}{M + \sqrt{M^2 - Q^2}} \tag{147} \\
S &= 2\pi r_+ = \frac{2\pi}{\lambda} (M + \sqrt{M^2 - Q^2}) \tag{148}
\end{align*}
These expressions coincide with the ones found in Ref. [10], and one can verify that the first law:

$$\delta M = \frac{k}{2\pi} \delta S + \frac{Q}{r_+} \delta Q$$  \hspace{1cm} (149)

is obeyed.

\section{8 Conclusions}

We have analyzed in detail the space of solutions, physical phase space, classical thermodynamics and quantum mechanics of the most general dilaton gravity theory coupled to a $U(1)$ gauge potential. It was shown that the surface gravity, Hawking temperature and the imaginary part of the WKB phase all vanish for extremal black holes. In a future publication [29] we will examine in more detail the properties of the WKB wave functionals in the generic theory, both with and without $U(1)$ charge. We also hope to extend the analysis to more general matter couplings and to go beyond the semi-classical approximation in the quantum theory. Since SIG coupled to matter has been shown to be anomalous [20], it is of interest to determine whether such anomalies appear in the generic theory as well. Finally it is important to understand the apparent relationship between black hole entropy and the imaginary part of the WKB phase.

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