SOME TIME-CHANGED FRACTIONAL POISSON PROCESSES

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Abstract. In this paper, we study the fractional Poisson process (FPP) time-changed by an independent Lévy subordinator and the inverse of the Lévy subordinator, which we call TCFPP-I and TCFPP-II, respectively. Various distributional properties of these processes are established. We show that, under certain conditions, the TCFPP-I has the long-range dependence property and also its law of iterated logarithm is proved. It is shown that the TCFPP-II is a renewal process and its waiting time distribution is identified. Its bivariate distributions and also the governing difference-differential equation are derived. Some specific examples for both the processes are discussed. Finally, we present the simulations of the sample paths of these processes.

1. Introduction

Recently, there has been a considerable interest in studying the fractional Poisson process (FPP) \( \{N_\beta(t, \lambda)\}_{t \geq 0} \). The early development of the FPP is due [32, 16, 21]. Later, a rich growth of the literature is contributed by [24, 25, 5, 6]. It is proved in [25] that the FPP can be seen as the subordination of the Poisson process by an independent inverse \( \beta \)-stable subordinator, that is,

\[
N_\beta(t, \lambda) = N(E_\beta(t), \lambda), \quad t \geq 0, \quad 0 < \beta < 1,
\]

where \( \{N(t, \lambda)\}_{t \geq 0} \) is the Poisson process with rate \( \lambda > 0 \) and \( \{E_\beta(t)\}_{t \geq 0} \) is the inverse \( \beta \)-stable subordinator. The relation between the inverse \( \beta \)-stable subordinator and the \( \beta \)-stable subordinator \( \{D_\beta(t)\}_{t \geq 0} \) is

\[
E_\beta(t) = \inf\{r \geq 0 : D_\beta(r) > t\}, \quad t \geq 0,
\]

where the Laplace transform (LT) of the \( \beta \)-stable subordinator is given by \( \mathbb{E}[e^{-sD_\beta(t)}] = e^{-ts^\beta}, \ s > 0 \). [18] studied the time-changed Poisson process subordinated with the inverse Gaussian, the first-exit time of the inverse Gaussian, the stable and the tempered stable subordinator. [28] studied the Poisson process subordinated by an independent \( \beta \)-stable subordinator \( \{D_\beta(t)\}_{t \geq 0} \), called the space fractional Poisson process. In [30], studied the Poisson process subordinated with independent Lévy subordinator and [36] studied the FPP subordinated with an independent gamma subordinator to obtain the fractional negative binomial process (FNBP). Observe that the Lévy subordinator covers most of the special subordinators (see [3, Theorem 1.3.15]) considered in the literature.

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The goal of the present work is to study the FPP \( \{N_\beta(t,\lambda)\}_{t \geq 0} \) time-changed by an independent Lévy subordinator (hereafter referred to as the subordinator) \( \{D_f(t)\}_{t \geq 0} \) with LT (see [3, Section 1.3.2])

\[
E[e^{-sD_f(t)}] = e^{-tf(s)},
\]

where

\[
f(s) = bs + \int_0^\infty (1 - e^{-sx})\nu(dx), \quad b \geq 0,
\]
is the Bernstein function. Here \( b \) is the drift coefficient and \( \nu \) is a non-negative Lévy measure on positive half-line such that

\[
\int_0^\infty (x \wedge 1)\nu(dx) < \infty.
\]

The assumption \( \nu(0,\infty) = \infty \) guarantees that the sample paths of \( D_f(t) \) are almost surely \((a.s.)\) strictly increasing. [30] studied the process \( \{N(D_f(t),\lambda)\}_{t \geq 0} \), where \( \{D_f(t)\}_{t \geq 0} \) is the subordinator with drift coefficient \( b = 0 \). We investigate the process

\[
\{Q^f_\beta(t,\lambda)\} = \{N_\beta(D_f(t),\lambda)\}, \quad t \geq 0,
\]

where the time variable \( t \) is replaced by an independent subordinator \( \{D_f(t)\}_{t \geq 0} \) and call the time-changed fractional Poisson process version one (TCFPP-I). The probability mass function (pmf) of TCFPP-I \( \{Q^f_\beta(t,\lambda)\}_{t \geq 0} \) is obtained and its mean and covariance functions are computed. We discuss the asymptotic behavior of the covariance function for large \( t \). Using these results, we prove the long-range dependence (LRD) property for the TCFPP-I process, under certain conditions. The law of iterated logarithm for the TCFPP-I is also proved.

The first-exit time of \( \{D_f(t)\}_{t \geq 0} \) is defined as

\[
E_f(t) = \inf\{r \geq 0 : D_f(r) > t\}, \quad t \geq 0,
\]

which is the right-continuous inverse of the subordinator \( \{D_f(t)\}_{t \geq 0} \). We consider also the time-changed fractional Poisson process version two (TCFPP-II) defined by

\[
\{W^f_\beta(t,\lambda)\} = \{N_\beta(E_f(t),\lambda)\}, \quad t \geq 0.
\]

The pmf, mean and covariance functions for the TCFPP-II are derived. We also discuss the asymptotic behavior of the mean and variance functions of the TCFPP-II. The bivariate distribution and the difference-differential equation governing the pmf of the TCFPP-II are derived. Lastly, we present the simulations for some special TCFPP-I and TCFPP-II processes.

The paper is organized as follows. In Section 2, we present some preliminary definitions and results. The TCFPP-I and the TCFPP-II processes are investigated in detail in Section 3 and 4, respectively. In Section 5, we present the simulations for some specific TCFPP-I and TCFPP-II processes.
2. Preliminaries

In this section, we present some preliminary results which are required later in the paper. The Mittag-Leffler function $L_\alpha(z)$ is defined as (see [27])

$$L_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha, z \in \mathbb{C} \text{ and } \Re(\alpha) > 0.$$  

(2.1)

The generalized Mittag-Leffler function $L^\gamma_{\alpha,\beta}(z)$ is defined as (see [31])

$$L^\gamma_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{\Gamma(\gamma + k)}{\Gamma(\gamma) \Gamma(\alpha k + \beta)} \frac{z^k}{k!}, \quad \alpha, \beta, \gamma, z \in \mathbb{C} \text{ and } \Re(\alpha), \Re(\beta), \Re(\gamma) > 0.$$  

(2.2)

Let $0 < \beta \leq 1$. The fractional Poisson process (FPP) \(\{N_\beta(t, \lambda)\}_{t \geq 0}\), which is a generalization of the Poisson process \(\{N(t, \lambda)\}_{t \geq 0}\), is defined to be a stochastic process for which $p_\beta(n|t, \lambda) = \mathbb{P}[N_\beta(t, \lambda) = n]$ satisfies (see [21, 24, 25])

$$\begin{align*}
\partial_t^\beta p_\beta(n|t, \lambda) &= -\lambda \left[p_\beta(n|t, \lambda) - p_\beta(n - 1|t, \lambda)\right], \quad \text{for } n \geq 1, \\
\partial_t^\beta p_\beta(0|t, \lambda) &= -\lambda p_\beta(0|t, \lambda),
\end{align*}$$

(2.3)

with $p_\beta(n|0, 0) = 1$ if $n = 0$ and is zero if $n \geq 1$. Here, $\partial_t^\beta$ denotes the Caputo-fractional derivative defined by

$$\partial_t^\beta f(t) = \begin{cases} 
\frac{1}{\Gamma(1 - \beta)} \int_0^t \frac{f'(s)}{(t - s)^\beta} ds, & 0 < \beta < 1, \\
\frac{d}{dt} f(t), & \beta = 1.
\end{cases}$$

The pmf $p_\beta(n|t, \lambda)$ of the FPP is given by (see [21, 25])

$$p_\beta(n|t, \lambda) = \frac{(\lambda t)^n}{n!} \sum_{k=0}^{\infty} \frac{(n + k)!}{k!} \frac{(-\lambda t)^k}{\Gamma(\beta(k + n) + 1)}.$$  

(2.4)

The mean, variance and covariance functions (see [21, 22]) of the FPP are given by

$$\begin{align*}
\mathbb{E}[N_\beta(t, \lambda)] &= qt^\beta; \quad \text{Var}[N_\beta(t, \lambda)] = qt^\beta + Rq^{2\beta}, \\
\mathbb{Cov}[N_\beta(s, \lambda), N_\beta(t, \lambda)] &= qs^\beta + ds^{2\beta} + q^2[\beta t^\beta B(\beta, 1 + \beta; s/t) - (st)^\beta], \quad 0 < s \leq t,
\end{align*}$$

(2.5)

where $q = \lambda / \Gamma(1 + \beta)$, $R = \frac{\lambda^2}{\beta} \left( \frac{1}{\Gamma(2\beta)} - \frac{1}{\beta \Gamma(1 + \beta)} \right) > 0$, $d = \beta q^2 B(\beta, 1 + \beta)$, and $B(a, b; x) = \int_0^x t^{a-1}(1 - t)^{b-1} dt$, $0 < x < 1$, is the incomplete beta function.

3. Time-changed fractional Poisson process-I

In this section, we consider the FPP time-changed by a subordinator $\{D_f(t)\}_{t \geq 0}$, defined in (1.2), for which the moments $\mathbb{E}[D_f^\rho(t)] < \infty$ for all $\rho > 0$. Note $\{D_f(t)\}_{t \geq 0}$ is an increasing process with $D_f(0) = 0$ a.s.
**Definition 3.1 (TCFPP-I).** The time-changed fractional Poisson process version one (TCFPP-I) is defined as

\[ \{Q^I_\beta(t, \lambda)\} = \{N_\beta(D_f(t), \lambda)\}, \quad t \geq 0, \]

where \(\{N_\beta(t, \lambda)\}_{t \geq 0}\) is the FPP and is independent of the subordinator \(\{D_f(t)\}_{t \geq 0}\).

We suppress the parameter \(\lambda\), unless the context requires, associated with the processes \(\{N_\beta(t, \lambda)\}_{t \geq 0}\) and \(\{Q^I_\beta(t, \lambda)\}_{t \geq 0}\).

**Theorem 3.1.** The one-dimensional distributions of the TCFPP-I is given by

\[ P[Q^I_\beta(t) = n] = \frac{\lambda^n}{n!} \sum_{k=0}^{\infty} \frac{(n+k)!}{k!} \frac{(-\lambda)^k}{\Gamma(\beta(k+n)+1)} E[D_f^{\beta(n+k)}(t)], \quad n \geq 0. \]  \hspace{1cm} (3.1)

**Proof.** Let \(g_f(x, t)\) be the pdf of \(D_f(t)\). Then, from [2,4],

\[ P[Q^I_\beta(t) = n] = \delta^I_\beta(n|t, \lambda) = \int_0^\infty p_\beta(n|y, \lambda) g_f(y, t) dy \]

\[ = \int_0^\infty (\lambda y^{\beta}) \sum_{k=0}^{\infty} \frac{(n+k)!}{k!} \frac{(-\lambda y^{\beta})^k}{\Gamma(\beta(k+n)+1)} g_f(y, t) dy \]

\[ = \frac{\lambda^n}{n!} \sum_{k=0}^{\infty} \frac{(n+k)!}{k!} \frac{(-\lambda)^k}{\Gamma(\beta(k+n)+1)} E[D_f^{\beta(n+k)}(t)]. \]

\[ \square \]

**Remark 3.1.** It can be seen that the pmf \(\delta^I_\beta(n|t, \lambda)\) satisfies the normalizing condition \(\sum_{n=0}^{\infty} \delta^I_\beta(n|t, \lambda) = 1\). We have

\[ \sum_{n=0}^{\infty} \delta^I_\beta(n|t, \lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \sum_{k=0}^{\infty} \frac{(n+k)!}{k!} \frac{(-\lambda)^k}{\Gamma(\beta(k+n)+1)} E[D_f^{\beta(n+k)}(t)] \]

\[ = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \sum_{r=n}^{\infty} \frac{r!}{(r-n)!} \frac{(-\lambda)^{r-n}}{\Gamma(\beta r+1)} E[D_f^{\beta r}(t)] \quad \text{(taking } r = k+n) \]

\[ = \sum_{r=0}^{\infty} \frac{\lambda^r}{\Gamma(\beta r+1)} \sum_{n=0}^{r} \frac{r!}{n!} \frac{(-\lambda)^{r-n}}{\Gamma(\beta r+1)} E[D_f^{\beta r}(t)] \]

\[ = \sum_{r=0}^{\infty} \frac{\lambda^r E[D_f^{\beta r}(t)]}{\Gamma(\beta r+1)} \sum_{n=0}^{\infty} \frac{r!}{n!} (-1)^{r-n} = \sum_{r=0}^{\infty} \frac{\lambda^r E[D_f^{\beta r}(t)]}{\Gamma(\beta r+1)} (1-1)^r \]

\[ = \frac{E[1]}{\Gamma(1)} = 1. \]

We next present some examples of the TCFPP-I processes.

**Example 3.1 (Fractional negative binomial process).** Let \(\{Y(t)\}_{t \geq 0}\) be the gamma subordinator, where \(Y(t) \sim G(\alpha, pt)\), the gamma distribution with density

\[ f(x|\alpha, pt) = \frac{\alpha^{pt}}{\Gamma(pt)} x^{\alpha-1} e^{-\alpha x}, \quad x > 0, \]
where both \( \alpha \) and \( p \) are positive. It is known that (see [3, p. 54])
\[
\mathbb{E}[e^{-sY(t)}] = \left(1 + \frac{s}{\alpha} \right)^{-pt} = \exp\left(-pt \log \left(1 + s/\alpha \right)\right).
\]
The fractional negative binomial process (FNBP), introduced and studied in detail in [36], is defined by time-changing the FPP by an independent tempered \( \alpha \)-stable subordinator, that is,
\[
\{Q^{(1)}_\beta(t)\} = \{N_\beta(Y(t))\}, \ t \geq 0.
\]
It is known that (see [36, eq. (4.4)])
\[
\mathbb{E}[Y^p(t)] = \frac{\Gamma(pt + \rho)}{\alpha^p \Gamma(pt)}, \ \rho > 0.
\]
From (3.1), the pmf of \( Q^{(1)}_\beta(t) \) is
\[
\mathbb{P}[Q^{(1)}_\beta(t) = n] = \left(\frac{\lambda}{\alpha^\beta}\right)^n \frac{1}{n! \Gamma(pt)} \sum_{k=0}^{\infty} \frac{(n+k)! \Gamma((n+k)\beta + pt)}{k! \Gamma(\beta(n+k) + 1)} \left(-\frac{\lambda}{\alpha^\beta}\right)^k, \ n \geq 0,
\]
which coincides with the pmf of the FNBP obtained in [36]. Also, it holds that
\[
\sum_{n=0}^{\infty} \mathbb{P}[Q^{(1)}_\beta(t) = n] = 1.
\]
**Example 3.2** (FPP subordinated by tempered \( \alpha \)-stable subordinator). Let \( \{D_\alpha^\mu(t)\}_{t \geq 0}, \mu > 0, 0 < \alpha < 1 \) be the tempered \( \alpha \)-stable subordinator with LT
\[
\mathbb{E}[e^{-sD_\alpha^\mu(t)}] = e^{-t((\mu+s)\alpha^\mu - \mu^\mu)}.
\]
The pdf of the tempered \( \alpha \)-stable subordinator is given by (see [2, eq. (2.2)])
\[
g_\mu(x,t) = e^{-\mu x + \mu^\beta t} g(x,t), \ x > 0,
\]
where \( g(x,t) \) is the pdf of the \( \alpha \)-stable subordinator \( \{D_\alpha(t)\}_{t \geq 0} \). The FPP time-changed by an independent tempered \( \alpha \)-stable subordinator is defined as
\[
\{Q^{(2)}_\beta(t)\} = \{N_\beta(D_\alpha^\mu(t))\}, \ t \geq 0.
\]
In this case, the pmf (3.1) reduces to
\[
\mathbb{P}[Q^{(2)}_\beta(t) = n] = \frac{\lambda^ne^{\mu^\beta t}}{n!} \sum_{k=0}^{\infty} \frac{(n+k)!}{k!} \frac{(-\lambda)^k}{\Gamma(\beta(k+n) + 1)} \mathbb{E}[(D_\alpha(t))^\beta(n+k)e^{-\mu D_\alpha(t)}], \ n \geq 0.
\]
It is easy to show that
\[
\sum_{n=0}^{\infty} \mathbb{P}[Q^{(2)}_\beta(t) = n] = 1.
\]
**Example 3.3** (FPP subordinated by inverse Gaussian subordinator). Let \( \{G(t)\}_{t \geq 0} \) be the inverse Gaussian subordinator with LT (see [3, Example 1.3.21])
\[
\mathbb{E}[e^{-sG(t)}] = e^{-t(\delta(\sqrt{2s+\gamma^2} - \gamma))}, \ \delta, \gamma > 0.
\]
The FPP time-changed by an independent inverse Gaussian subordinator is defined as
\[
\{Q^{(3)}_\beta(t)\} = \{N_\beta(G(t))\}, \ t \geq 0.
\]
It is known that (see [15, 18]) the moments of \( \{G(t)\}_{t \geq 0} \) are given by
\[
(3.2) \quad \mathbb{E}[G^q(t)] = \sqrt{\frac{2}{\pi}} \delta \left(\frac{\delta}{\gamma}\right)^{q-1/2} t^{q+1/2} e^{\delta t} K_{q-1/2}(\delta\gamma t), \ \delta, \gamma > 0, t \geq 0, \ q \in \mathbb{R},
\]
where $K_{\nu}(z), z > 0$ is the modified Bessel function of third kind with index $\nu \in \mathbb{R}$. We can substitute the moments of $\{G(t)\}_{t \geq 0}$ in (3.1) to obtain the pmf of $\{Q_{\beta}^{(3)}(t)\}_{t \geq 0}$. Moreover, it can be shown that $\sum_{n=0}^{\infty} \mathbb{P}[Q_{\beta}^{(3)}(t) = n] = 1$.

We next obtain the mean, the variance and the covariance functions of the TCFPPI.

**Theorem 3.2.** Let $0 < s \leq t < \infty$, $q = \lambda/\Gamma(1 + \beta)$ and \( d = \beta q^2 B(\beta, 1 + \beta) \). The distributional properties of the TCFPPI $\{Q_{\beta}^{(3)}(t, \lambda)\}_{t \geq 0}$ are as follows:

(i) $\mathbb{E}[Q_{\beta}^{(3)}(t)] = q \mathbb{E}[D_{f}(t)]$,

(ii) $\text{Var}[Q_{\beta}^{(3)}(t)] = q \mathbb{E}[D_{f}^2(t)]\left(1 - q \mathbb{E}[D_{f}(t)]\right) + 2d \mathbb{E}[D_{f}^{2\beta}(t)]$,

(iii) $\text{Cov}[Q_{\beta}(s), Q_{\beta}^{(3)}(t)] = q \mathbb{E}[D_{f}^{2\beta}(s)] + d \mathbb{E}[D_{f}^{2\beta}(s)] - q^2 \mathbb{E}[D_{f}^{\beta}(s)] \mathbb{E}[D_{f}^{\beta}(t)] + q^2 \beta \mathbb{E}\left[D_{f}^{2\beta}(t)B\left(\beta, 1 + \beta; \frac{D_{f}(s)}{D_{f}(t)}\right)\right]$.

**Proof.** Using (2.5), we get

(3.3) \[ \mathbb{E}[Q_{\beta}^{(3)}(t)] = \mathbb{E}\left[\mathbb{E}[N_{\beta}(D_{f}(t))|D_{f}(t)]\right] = q \mathbb{E}[D_{f}^{\beta}(t)], \]

which proves Part (i). From (2.6) and (2.5),

\[ \mathbb{E}[N_{\beta}(s)N_{\beta}(t)] = q s^{\beta} + ds^{2\beta} + q^2 \beta \left[t^{2\beta} B(\beta, 1 + \beta; s/t)\right], \]

which leads to

(3.4) \[ \mathbb{E}[Q_{\beta}(s)Q_{\beta}^{(3)}(t)] = \mathbb{E}\left[\mathbb{E}[N_{\beta}(D_{f}(s))N_{\beta}(D_{f}(t))|(D_{f}(s), D_{f}(t))]|(D_{f}(s), D_{f}(t))\right] = q \mathbb{E}[D_{f}^{\beta}(s)] + d \mathbb{E}[D_{f}^{2\beta}(s)] + \beta q^2 \mathbb{E}\left[D_{f}^{2\beta}(t)B\left(\beta, 1 + \beta; \frac{D_{f}(s)}{D_{f}(t)}\right)\right]. \]

By (3.3) and (3.4), Part (iii) follows. Part (ii) follows from Part (iii) by putting $s = t$. \( \square \)

**Index of dispersion.** The index of dispersion for a counting process $\{X(t)\}_{t \geq 0}$ is defined by (see [12, p. 72])

\[ I(t) = \frac{\text{Var}[X(t)]}{\mathbb{E}[X(t)]}. \]

The stochastic process $\{X(t)\}_{t \geq 0}$ is said to be overdispersed if $I(t) > 1$ for all $t \geq 0$ (see [7, 36]). Since the mean of the TCFPPI $\{Q_{\beta}^{(3)}(t)\}_{t \geq 0}$ is nonnegative, it suffices to show that $\text{Var}[Q_{\beta}^{(3)}(t)] - \mathbb{E}[Q_{\beta}^{(3)}(t)] > 0$. From Theorem 3.2, we have that

\[ \text{Var}[Q_{\beta}^{(3)}(t)] - \mathbb{E}[Q_{\beta}^{(3)}(t)] = 2d \mathbb{E}[D_{f}^{2\beta}(t)] - \left(q \mathbb{E}[D_{f}^{\beta}(t)]\right)^2 \]

\[ = \frac{\lambda^2}{\beta} \left(\frac{\mathbb{E}[D_{f}^{2\beta}(t)]}{\Gamma(2\beta)} - \left(\frac{\mathbb{E}[D_{f}^{\beta}(t)]}{\beta T^2(\beta)}\right)^2\right) \]

\[ \geq R \left(\mathbb{E}[D_{f}^{\beta}(t)]\right)^2, \quad (\because \mathbb{E}[D_{f}^{2\beta}(t)] \geq (\mathbb{E}[D_{f}^{\beta}(t)])^2) \]

where $R = \frac{\lambda^2}{\beta} \left(\frac{1}{\Gamma(2\beta)} - \frac{1}{\beta T^2(\beta)}\right) > 0$ and $\lambda > 0$ for all $\beta \in (0, 1)$ (see [7, Section 3.1]). Hence, the TCFPPI exhibits overdispersion.
We next derive the asymptotic expansion for the covariance function of the TCFPP-I process. The following result generalizes Lemma 4.1 of [23] to the subordinator \(\{D_f(t)\}_{t \geq 0}\).

**Definition 3.2.** Let \(f(x)\) and \(g(x)\) be positive functions. We say that \(f(x)\) is asymptotically equal to \(g(x)\), written as \(f(x) \sim g(x)\), as \(x \to \infty\), if

\[
\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1.
\]

**Theorem 3.3.** Let \(0 < \beta < 1\), \(0 < s \leq t\) and \(s\) be fixed. Let \(\{D_f(t)\}_{t \geq 0}\) be the subordinator with \(\mathbb{E}[e^{-sD_f(t)}] = e^{-tf(s)}\), where \(f(s)\) is the Bernstein function defined in [1,3]. If \(\mathbb{E}[D_f^\beta(t)] \to \infty\) as \(t \to \infty\), then

(i) the asymptotic expansion of \(\mathbb{E} \left[ D_f^\beta(s)D_f^\beta(t) \right] \)

\[
\mathbb{E} \left[ D_f^\beta(s)D_f^\beta(t) \right] \sim \mathbb{E} \left[ D_f^\beta(s) \right] \mathbb{E} \left[ D_f^\beta(t - s) \right].
\]

(ii) the asymptotic expansion of \(\beta \mathbb{E} \left[ D_f^{2\beta}(t)B(\beta, 1 + \beta; D_f(s)/D_f(t)) \right] \)

\[
\beta \mathbb{E} \left[ D_f^{2\beta}(t)B(\beta, 1 + \beta; D_f(s)/D_f(t)) \right] \sim \mathbb{E} \left[ D_f^\beta(s) \right] \mathbb{E} \left[ D_f^\beta(t - s) \right].
\]

*Proof.* (i) Since the subordinator \(\{D_f(t)\}_{t \geq 0}\) has stationary and independent increments, it suffices to show that

\[
\lim_{t \to \infty} \frac{\mathbb{E} \left[ D_f^\beta(s)D_f^\beta(t) \right]}{\mathbb{E} \left[ D_f^\beta(s)(D_f(t) - D_f(s))^\beta \right]} = 1.
\]

Also, \(\{D_f(t)\}_{t \geq 0}\) is an increasing process with \(D_f(0) = 0\) a.s. so that

\[
D_f(t) - D_f(s) \leq D_f(t) \quad \text{a.s.}
\]

\[
\Rightarrow \quad \mathbb{E} \left[ D_f^\beta(s)(D_f(t) - D_f(s))^\beta \right] \leq \mathbb{E} \left[ D_f^\beta(s)D_f^\beta(t) \right]
\]

\[
\Rightarrow \quad \frac{\mathbb{E} \left[ D_f^\beta(s)D_f^\beta(t) \right]}{\mathbb{E} \left[ D_f^\beta(s)(D_f(t) - D_f(s))^\beta \right]} \geq 1.
\]

Now consider

\[
\frac{\mathbb{E} \left[ D_f^\beta(s)D_f^\beta(t) \right]}{\mathbb{E} \left[ D_f^\beta(s)(D_f(t) - D_f(s))^\beta \right]} = \frac{\mathbb{E} \left[ D_f^\beta(s) \right] \left\{ D_f^\beta(t) - (D_f(t) - D_f(s))^\beta \right\}}{\mathbb{E} \left[ D_f^\beta(s)(D_f(t) - D_f(s))^\beta \right]} + 1.
\]

Since \(0 \leq a^\beta - b^\beta \leq (a - b)^\beta\), for \(0 < \beta < 1\) and \(a \geq b \geq 0\),

\[
\frac{\mathbb{E} \left[ D_f^\beta(s)D_f^\beta(t) \right]}{\mathbb{E} \left[ D_f^\beta(s)(D_f(t) - D_f(s))^\beta \right]} \leq \frac{\mathbb{E} \left[ D_f^\beta(s) \right] \left\{ D_f^\beta(t) - (D_f^\beta(t) - D_f^\beta(s)) \right\}}{\mathbb{E} \left[ D_f^\beta(s)D_f^\beta(t - s) \right]} + 1
\]
\[ (3.6) \quad \frac{\mathbb{E} \left[ D_f^{2\beta}(s) \right]}{\mathbb{E} \left[ D_f^\beta(s)D_f^\beta(t - s) \right]} + 1. \]

From (3.5) and (3.6), we have that
\[ 1 \leq \frac{\mathbb{E} \left[ D_f^\beta(s)D_f^\beta(t) \right]}{\mathbb{E} \left[ D_f^\beta(s)(D_f(t) - D_f(s))^{\beta} \right]} \leq \frac{\mathbb{E} \left[ D_f^{2\beta}(s) \right]}{\mathbb{E} \left[ D_f^\beta(s)D_f^\beta(t - s) \right]} + 1. \]

Taking the limit as \( t \) tends to infinity in the above equation and using the fact that \( \{D_f(t)\}_{t \geq 0} \) has independent increments, we get
\[ 1 \leq \lim_{t \to \infty} \mathbb{E} \left[ D_f^\beta(s)D_f^\beta(t) \right] \leq 1 + \lim_{t \to \infty} \frac{\mathbb{E} \left[ D_f^{2\beta}(s) \right]}{\mathbb{E} \left[ D_f^\beta(s)D_f^\beta(t - s) \right]} \]
\[ 1 \leq \lim_{t \to \infty} \frac{\mathbb{E} \left[ D_f^\beta(s)(D_f(t) - D_f(s))^{\beta} \right]}{\mathbb{E} \left[ D_f^\beta(s)D_f^\beta(t - s) \right]} \leq 1, \quad \text{(since } \mathbb{E}[D_f^{\beta}(t)] \to \infty \text{ as } t \to \infty), \]

which proves Part (i).

To prove Part (ii), it suffices to show that, in view of Part (i),
\[ \lim_{t \to \infty} \frac{\beta \mathbb{E} \left[ D_f^{2\beta}(t)B(\beta, 1 + \beta; D_f(s)/D_f(t)) \right]}{\mathbb{E} \left[ D_f^\beta(s)D_f^\beta(t) \right]} = 1. \]

Note first that
\[ B \left( \beta, 1 + \beta; \frac{D_f(s)}{D_f(t)} \right) = \int_{0}^{\frac{D_f(s)}{D_f(t)}} u^{\beta - 1}(1 - u)^{\beta} du \]
\[ \leq \int_{0}^{\frac{D_f(s)}{D_f(t)}} u^{\beta - 1} du \quad \text{(since } (1 - u)^{\beta} \leq 1) \]
\[ = \frac{D_f^\beta(s)}{\beta D_f^\beta(t)}, \]

which leads to
\[ \beta D_f^{2\beta}(t)B \left( \beta, 1 + \beta; \frac{D_f(s)}{D_f(t)} \right) \leq D_f^\beta(s)D_f^\beta(t). \]

Hence,
\[ \frac{\beta \mathbb{E} \left[ D_f^{2\beta}(t)B(\beta, 1 + \beta; D_f(s)/D_f(t)) \right]}{\mathbb{E} \left[ D_f^\beta(s)D_f^\beta(t) \right]} \leq 1. \]

(3.7)
On the other hand,
\[
B (\beta, 1 + \beta; D_f(s)/D_f(t)) = \int_0^{D_f(s)/D_f(t)} u^{\beta-1}(1 - u)^{3} du \\
\geq \int_0^{D_f(s)/D_f(t)} u^{\beta-1}(1 - u)^{3} du \quad \text{(since } (1 - u)^{3} \geq 1 - u^{3})
\]
\[
= \frac{1}{\beta} \left( \frac{D_f^3(s)}{D_f^3(t)} - \frac{D_f^{2\beta}(s)}{2D_f^{2\beta}(t)} \right).
\]
This leads to
\[
\frac{\beta E \left[ D_f^{2\beta}(t) B(\beta, 1 + \beta; D_f(s)/D_f(t)) \right]}{E \left[ D_f^{3}(s) D_f^{3}(t) \right]} \geq \frac{E \left[ D_f^{2\beta}(t) \left( \frac{D_f^3(s)}{D_f^3(t)} - \frac{D_f^{2\beta}(s)}{2D_f^{2\beta}(t)} \right) \right]}{E \left[ D_f^{3}(s) D_f^{3}(t) \right]} = 1 - \frac{E \left[ D_f^{3}(s) \right]}{2E \left[ D_f^{3}(s) \right] E \left[ D_f^{3}(t - s) \right]}.
\]
(3.8)

using Part (i). By (3.7) and (3.8), we have that
\[
1 - \frac{E \left[ D_f^{3}(s) \right]}{2E \left[ D_f^{3}(s) \right] E \left[ D_f^{3}(t - s) \right]} \leq \frac{\beta E \left[ D_f^{2\beta}(t) B(\beta, 1 + \beta; D_f(s)/D_f(t)) \right]}{E \left[ D_f^{3}(s) D_f^{3}(t) \right]} \leq 1.
\]
Now taking limit as \( t \) tends to infinity in the above inequality, we get
\[
1 \leq \lim_{t \to \infty} \frac{\beta E \left[ D_f^{2\beta}(t) B(\beta, 1 + \beta; D_f(s)/D_f(t)) \right]}{E \left[ D_f^{3}(s) D_f^{3}(t) \right]} \leq 1 \quad (\because \ E[D_f^{3}(t)] \to \infty, \ \text{as } t \to \infty).
\]
This completes the proof of Part (ii).
\[\square\]

Remark 3.2. The assumption, in Theorem 3.3, that \( E[D_f^{3}(t)] \to \infty \) as \( t \to \infty \), is satisfied for the following subordinators.

(a) Gamma subordinator: It is known (see [36]) that \( E[Y^{\beta}(t)] = \frac{\Gamma(p+\beta)}{\alpha^{p+\beta}} \sim \left( \frac{\alpha}{\epsilon} \right)^{\beta} \), which implies \( E[Y^{\beta}(t)] \to \infty \) as \( t \to \infty \).

(b) Inverse Gaussian subordinator: The moments of inverse Gaussian subordinator \( \{G(t)\}_{t \geq 0} \) are given in (3.2). The asymptotic expansion of \( K_{\nu}(z) \), for large \( z \), is (see [15] eq. (A.9))
\[
K_{\nu}(z) = \sqrt{\frac{\pi}{2}} z^{-1/2} e^{-z} \left( 1 + \frac{\mu-1}{8z} + \frac{\mu-1)(\mu-9)}{2!(8z)^2} + \frac{\mu-1)(\mu-9)(\mu-25)}{3!(8z)^3} + \cdots \right)
\]
has the correlation function $\text{Corr}[X_t]$ for large $t$.

Using Theorem 3.3 (ii), we get for large $t$

$$\lim_{t \to \infty} \frac{\text{Corr}[X(s), X(t)]}{t^{-d}} = c(s),$$

for some $c(s) > 0$ and $d > 0$. We say $\{X(t)\}_{t \geq 0}$ possesses the LRD property if $d \in (0, 1)$ and has the short-range dependence (SRD) property if $d \in (1, 2)$.

Note (3.10) implies that $\text{Corr}[X(s), X(t)]$ behaves like $t^{-d}$, for large $t$.

**Theorem 3.4.** Let $\{D_f(t)\}_{t \geq 0}$ be such that $\mathbb{E}[D^\beta_f(t)] \sim k_1 t^\rho$ and $\mathbb{E}[D^{2\beta}_f(t)] \sim k_2 t^{2\rho}$, for some $0 < \rho < 1$, and positive constants $k_1$ and $k_2$ with $k_2 \geq k_1^2$. Then the TCFPP-I $\{Q^I_\beta(t)\}_{t \geq 0}$ has the LRD property.

**Proof.** Consider the last term of $\text{Cov}[Q^I_\beta(s), Q^I_\beta(t)]$ given in Theorem 3.2 (iii), namely,

$$q^2 \mathbb{E}[D^{2\beta}_f(t)B \left( \beta, 1 + \beta; \frac{D_f(s)}{D_f(t)} \right)].$$

Using Theorem 3.3 (ii), we get for large $t$

$$q^2 \mathbb{E}[D^{2\beta}_f(t)B \left( \beta, 1 + \beta; \frac{D_f(s)}{D_f(t)} \right)] \sim q^2 \mathbb{E}[D^{\beta}_f(s)] \mathbb{E}[D^{\beta}_f(t-s)].$$

Using (3.3), Theorem 3.2 (iii) and $\mathbb{E}[D^{\beta}_f(t)] \sim k_1 t^\rho$, we get,

$$\text{Cov}[Q^I_\beta(s), Q^I_\beta(t)] \sim q \mathbb{E}[D^{\beta}_f(s)] + d \mathbb{E}[D^{2\beta}_f(s)] - q^2 \mathbb{E}[D^{\beta}_f(s)] k_1 t^\rho + q^2 \mathbb{E}[D^{\beta}_f(s)] k_1(t-s)^\rho$$

$$= q \mathbb{E}[D^{\beta}_f(s)] + d \mathbb{E}[D^{2\beta}_f(s)] - q^2 \mathbb{E}[D^{\beta}_f(s)] k_1(t^\rho - (t-s)^\rho)$$

$$\sim q \mathbb{E}[D^{\beta}_f(s)] + d \mathbb{E}[D^{2\beta}_f(s)] - q^2 k_1 s\rho \mathbb{E}[D^{\beta}_f(s)] t^\rho - 1,$$

since $t^\rho - (t-s)^\rho \sim \rho s t^{\rho - 1}$ for large $t$, and $\rho > 0$.

Similarly, from Theorem 3.2 (ii) and $\mathbb{E}[D^{2\beta}_f(t)] \sim k_2 t^{2\rho}$, we have that

$$\text{Var}[Q^I_\beta(t)] \sim q k_1 t^\rho - q^2 (k_1 t^\rho)^2 + 2d k_2 t^{2\rho}$$
\[
\sim 2dk_2t^{2\rho} - q^2k_1^2t^{2\rho} \quad \text{(see Definition 3.2)}
\]

\[
= \frac{\lambda^2}{\beta} \left( \frac{k_2}{\Gamma(2\beta)} - \frac{k_1^2}{\beta \Gamma^2(\beta)} \right) t^{2\rho}
\]

\[
d_1 t^{2\rho},
\]

where \( d_1 = \frac{\lambda^2}{\beta} \left( \frac{k_2}{\Gamma(2\beta)} - \frac{k_1^2}{\beta \Gamma^2(\beta)} \right) \). Thus, from (3.12) and (3.13), we have for large \( t > s \),

\[
\text{Corr}[Q_{\beta}(s), Q_{\beta}(t)] \sim \frac{qE[D_{\beta}(s)] + dE[D_{\beta}(s)] - q^2k_1sE[D_{\beta}(s)]t^{-\rho}}{\sqrt{\text{Var}[Q_{\beta}(s)\sqrt{d_1}t^{2\rho}]}}
\]

\[
= \left( \frac{qE[D_{\beta}(s)] + dE[D_{\beta}(s)]}{\sqrt{d_1 \text{Var}[Q_{\beta}(s)]}} \right) t^{-\rho} - \frac{q^2k_1sE[D_{\beta}(s)]}{\sqrt{d_1 \text{Var}[Q_{\beta}(s)]}} t^{-1}
\]

\[
\sim \left( \frac{qE[D_{\beta}(s)] + dE[D_{\beta}(s)]}{\sqrt{d_1 \text{Var}[Q_{\beta}(s)]}} \right) t^{-\rho},
\]

which decays like the power law \( t^{-\rho} \), \( 0 < \rho < 1 \). Hence, the TCFPP-I exhibits the LRD property.

**Remark 3.3.** From Remark 3.2 it can be seen that moments of the gamma subordinator has the asymptotic expansion \( \mathbb{E}[Y^\beta(t)] \sim (p/\alpha)^\beta t^\beta \) and \( \mathbb{E}[Y^{2\beta}(t)] \sim (p/\alpha)^{2\beta} t^{2\beta} \). Therefore, the FNBP \( \{Q_{\beta}(t)\}_{t \geq 0} \) exhibits the LRD property. Similarly, for the inverse Gaussian subordinator \( \{G(t)\}_{t \geq 0} \), we have the asymptotic expansion \( \mathbb{E}[G^\beta(t)] \sim (\delta/\gamma)^\beta t^\beta \) and \( \mathbb{E}[G^{2\beta}(t)] \sim (\delta/\gamma)^{2\beta} t^{2\beta} \). Hence, \( \{Q_{\beta}(t)\}_{t \geq 0} \) also has the LRD property.

**Definition 3.4.** We call a function \( l : (0, \infty) \to (0, \infty) \) regularly varying at 0+ with index \( \alpha \in \mathbb{R} \) (see [8]) if

\[
\lim_{x \to 0+} \frac{l(\lambda x)}{l(x)} = \lambda^\alpha, \quad \text{for} \ \lambda > 0.
\]

We first reproduce the following law of iterated logarithm (LIL) for the subordinator from [8] Chapter III, Theorem 14.

**Lemma 3.1.** Let \( X(t) \) be a subordinator with \( \mathbb{E}[e^{-sX}(t)] = e^{-tf(s)} \), where \( f(s) \) is regularly varying at 0+ with index \( \alpha \in (0, 1) \). Let \( h \) be the inverse function of \( f \) and

\[
g(t) = \frac{\log \log t}{h(t^{-1} \log \log t)} , \quad (e < t).
\]

Then

\[
\liminf_{t \to \infty} \frac{X(t)}{g(t)} = \alpha(1 - \alpha)^{(1-\alpha)/\alpha}, \quad \text{a.s.}
\]

We next prove the LIL for the TCFPP-I.
Theorem 3.5 (Law of iterated logarithm). Let the Laplace exponent $f(s)$ of the subordinator $\{D_f(t)\}_{t \geq 0}$ be regularly varying at $0+$ with index $\alpha \in (0, 1)$. Then, for $0 < \beta < 1$,
\[
\liminf_{t \to \infty} \frac{Q^f_\beta(t)}{(g(t))^{\beta}} = \frac{\lambda E_\beta(1)\alpha^\beta (1 - \alpha)^{\beta(1-\alpha)/\alpha}}{a.s.,}
\]
where
\[
g(t) = \frac{\log \log t}{f^{-1}(t^{-1}\log \log t)} \quad (t > e),
\]
and $E_\beta(t)$ is the inverse $\beta$-stable subordinator.

**Proof.** Since $E_\beta(t) \overset{d}{=} t^\beta E_\beta(1)$ (see [23, Proposition 3.1]), we have
\[
Q^f_\beta(t) = N(E_\beta(D_f(t))) \overset{d}{=} N((D_f(t))^{\beta}E_\beta(1)).
\]
The law of large numbers for the Poisson process implies
\[
\lim_{t \to \infty} \frac{N(t)}{t} = \lambda, \text{ a.s.}
\]
Note that $D_f(t) \to \infty$, a.s. as $t \to \infty$ (see [8, page 73]). Consider now,
\[
\liminf_{t \to \infty} \frac{Q^f_\beta(t)}{(g(t))^{\beta}} = \liminf_{t \to \infty} \frac{N(E_\beta(D_f(t)))}{(g(t))^{\beta}} = \liminf_{t \to \infty} \frac{N(D_f(t)^{\beta}E_\beta(1))}{(g(t))^{\beta}}
\]
\[
= \liminf_{t \to \infty} \frac{N(D_f(t)^{\beta})}{D_f(t)^{\beta}E_\beta(1)} \frac{D_f(t)^{\beta}E_\beta(1)}{(g(t))^{\beta}}
\]
\[
= \lambda E_\beta(1) \liminf_{t \to \infty} \frac{D_f(t)^{\beta}}{(g(t))^{\beta}}, \text{ a.s.}
\]
\[
= \lambda E_\beta(1) \left( \liminf_{t \to \infty} \frac{D_f(t)}{g(t)} \right)^{\beta}, \text{ a.s.}
\]
\[
= \lambda E_\beta(1)\alpha^\beta (1 - \alpha)^{\beta(1-\alpha)/\alpha} \text{ a.s.,}
\]
where the last step follows from (3.14). $\square$

When $\beta = 1$, the LIL for the time-changed Poisson process $\{Q_f^1(t)\} \overset{d}{=} \{N_1(D_f(t))\}, \ t \geq 0$, (discussed in [30]) can be proved in a similar way and is stated below.

**Corollary 3.1.** Let the Laplace exponent $f(s)$ of the subordinator $\{D_f(t)\}_{t \geq 0}$ be regularly varying at $0+$ with index $\alpha \in (0, 1)$. Then
\[
\liminf_{t \to \infty} \frac{Q^1(t)}{g(t)} = \frac{\lambda \alpha(1 - \alpha)^{(1-\alpha)/\alpha}}{a.s.}
\]
where
\[
g(t) = \frac{\log \log t}{f^{-1}(t^{-1}\log \log t)} \quad (t > e).
\]
Example 3.4. The space fractional Poisson process, introduced in [28], defined by time changing the Poisson process by an independent \( \alpha \)-stable subordinator, that is,
\[
\bar{N}_\alpha(t) = N(D_\alpha(t)), \quad t \geq 0, \quad 0 < \alpha < 1,
\]
where \( \{D_\alpha(t)\}_{t \geq 0} \) is the \( \alpha \)-stable subordinator with LT \( \mathbb{E}[e^{-sD_\alpha(t)}] = e^{-ts^\alpha} \). Here, the corresponding Bernstein function \( f(s) = s^\alpha \) is regularly varying with index \( \alpha \in (0, 1) \). Therefore, by Corollary 3.1 we have the LIL for the space fractional Poisson process with
\[
g(t) = \frac{\log \log t}{(t^{-1} \log \log t)^{1/\alpha}}, \quad (t > e).
\]

4. Time-changed fractional Poisson process-II

The first-exit time of the subordinator \( \{D_f(t)\}_{t \geq 0} \) is its right-continuous inverse, defined by
\[
E_f(t) = \inf\{r \geq 0 : D_f(r) > t\}, \quad t \geq 0,
\]
and is called an inverse subordinator (see [8]). Note that for any \( \rho > 0 \), \( \mathbb{E}[E_f^\rho(t)] < \infty \) (see [1] Section 2.1). We now consider the FPP time-changed by an inverse subordinator.

Definition 4.1 (TCFPP-II). The time-changed fractional Poisson process version two (TCFPP-II) is defined as
\[
\{W^f_\beta(t)\} = \{N_\beta(E_f(t))\}, \quad t \geq 0,
\]
where \( \{N_\beta(t)\}_{t \geq 0} \) is the FPP and is independent of the inverse subordinator \( \{E_f(t)\}_{t \geq 0} \).

We now present some results and distributional properties of the TCFPP-II. The proofs of some of them are shortened or omitted to avoid repetition from the previous section.

The one-dimensional distributions of the TCFPP-II can be written as
\[
\eta^f_\beta(n|t, \lambda) = \mathbb{P}[W^f_\beta(t) = n] = \frac{\lambda^n}{n!} \sum_{k=0}^{\infty} \frac{(n+k)!}{k!} \frac{(-\lambda)^k}{\Gamma(\beta(k+n)+1)} \mathbb{E}[E_f^{2\beta(n+k)}(t)],
\]
which follows from [24].

Theorem 4.1. Let \( 0 < s \leq t \). Then
(i) \( \mathbb{E}[W^f_\beta(t)] = q\mathbb{E}[E_f^\beta(t)] \),
(ii) \( \text{Var}[W^f_\beta(t)] = q\mathbb{E}[E_f^\beta(t)] \left( 1 - q\mathbb{E}[E_f^\beta(t)] \right) + 2d\mathbb{E}[E_f^{2\beta}(t)] \),
(iii) \( \text{Cov}[W^f_\beta(s), W^f_\beta(t)] = q\mathbb{E}[E_f^\beta(s)] + d\mathbb{E}[E_f^{2\beta}(s)] - q^2\mathbb{E}[E_f^\beta(s)]\mathbb{E}[E_f^\beta(t)] + q^2\beta\mathbb{E}\left[ E_f^{2\beta}(t) B \left( \beta, 1 + \beta; \frac{E_f(s)}{E_f(t)} \right) \right] \).

Proof. The proof is similar to the proof of Theorem 3.2 and hence is omitted. \( \square \)

We next discuss the asymptotic behavior of moments of the TCFPP-II. The mean and variance functions contain the term of the form \( \mathbb{E}[E_f^\beta(t)] \). Therefore, we study the asymptotic behavior of \( \mathbb{E}[E_f^\beta(t)] \). It will be studied using the Tauberian theorem (see [34] Theorem 4.1 and [8] p. 10), which we reproduce here. Recall that a function \( \ell(t), t > 0 \),
Let $M$ be equivalent: at $0$ (respectively $\ell$). Let $\Phi: (0, \infty) \rightarrow (0, \infty)$ be a slowly varying function at $0$ (respectively $\infty$) and let $\rho \geq 0$. Then for a function $U: (0, \infty) \rightarrow (0, \infty)$, the following are equivalent:

(i) $U(x) \sim x^\rho \ell(x)/\Gamma(1 + \rho)$, $x \rightarrow 0$ (respectively $x \rightarrow \infty$).

(ii) $\hat{U}(s) \sim s^{-\rho-1} \ell(1/s)$, $s \rightarrow \infty$ (respectively $s \rightarrow 0$),

where $\hat{U}(s)$ is the LT of $U(x)$.

Let $M_p(t) = \mathbb{E}[W_1^p(t)]$, $p > 0$. The LT of the $p$-th moment of $W_1(t)$ is given by (see [19])

\begin{equation}
\hat{M}_p(s) = \frac{\Gamma(1+p)}{s^{(f(s))p}},
\end{equation}

where $f(s)$ is the Bernstein function associated with $\{D_f(t)\}_{t \geq 0}$. The asymptotic moments can be specifically computed for special cases, which also serves examples of the TCFPP-II processes.

Example 4.1 (FPP subordinated with inverse gamma subordinator). We study the FPP time-changed by the inverse of the gamma subordinator $\{Y(t)\}_{t \geq 0}$, with corresponding Bernstein function $f(s) = p \log(1 + s/\alpha)$. The right-continuous inverse of the gamma subordinator $\{Y(t)\}_{t \geq 0}$ is defined as

$E_Y(t) = \inf\{r \geq 0 : Y(r) > t\}$, $t \geq 0$.

We study the asymptotic behavior of the mean of $\{W_1^{(1)}(t)\}_{t \geq 0} = \{N_\beta(E_Y(t))\}_{t \geq 0}$, that is, the function $\mathbb{E}[W_1^{(1)}(t)] = q\mathbb{E}[E_Y^\beta(t)]$. The LT of $\mathbb{E}[E_Y^\beta(t)]$ is given by

$\hat{M}_\beta(s) = \mathcal{L} \left[ \mathbb{E}[E_Y^\beta(t)] \right] = \frac{\Gamma(1 + \beta)}{s^{(p \log(1 + s/\alpha))^{\beta}}}$.

Note that $p \log(1 + s/\alpha) \sim ps/\alpha$, as $s \rightarrow 0$. Now using Theorem 4.2, we get (see also [20, Proposition 4.1])

$\mathbb{E}[W_1^{(1)}(t)] = q\mathbb{E}[E_Y^\beta(t)] \sim q(t\alpha/p)^\beta$, as $t \rightarrow \infty$.

The asymptotic behavior of variance function of $\{W_1^{(1)}(t)\}$ can also be computed using above expression.

Example 4.2 (FPP subordinated with the inverse tempered $\alpha$-stable subordinator). Consider the FPP subordinated with the inverse tempered $\alpha$-stable subordinator $\{E_\alpha^\mu(t)\}_{t \geq 0}$. The inverse tempered $\alpha$-stable subordinator is introduced by [19] and they studied its asymptotic behavior of moments. The $p$-th moment of $E_\alpha^\mu(t)$ satisfies (see [19, Proposition 3.1])

$\mathbb{E}[(E_\alpha^\mu(t))^p] \sim \begin{cases} \Gamma(1+p)/(1 + p\alpha)^p, & \text{as } t \rightarrow 0, \\ \lambda^{p(1-\alpha)} \alpha^\mu t^p, & \text{as } t \rightarrow \infty. \end{cases}$
Therefore, we have that
\[
\mathbb{E}[W^{(2)}_{\beta}(t)] = q\mathbb{E}[(E^\mu_{\alpha}(t))^{\beta}] \sim \begin{cases}
q\frac{\Gamma(1 + \beta)}{\Gamma(1 + \beta\alpha)} t^{\beta\alpha}, & \text{as } t \to 0, \\
q\frac{\lambda^{\beta(1-\alpha)}}{\alpha^{\beta}} t^\beta, & \text{as } t \to \infty.
\end{cases}
\]

Example 4.3 (FPP subordinated with inverse of the inverse Gaussian subordinator).
The right-continuous inverse of the inverse Gaussian subordinator \( \{G(t)\}_{t \geq 0} \), with corresponding Bernstein function \( f(s) = \delta \left( \sqrt{2s + \gamma^2} - \gamma \right) \), denoted by \( \{E_G(t)\}_{t \geq 0} \), defined as (see [35])
\[
E_G(t) = \inf\{r \geq 0 : G(r) > t\}, \ t \geq 0.
\]
Hence from (4.2)
\[
\tilde{M}_p(s) = \mathcal{L} [\mathbb{E}^p_{E_G(t)}] = \frac{\Gamma(1 + p)}{s \left( \frac{\Delta}{\sqrt{2}} + \sqrt{2s + \gamma^2 - \gamma} \right)^{p}},
\]
where \( p > 0 \). This gives
\[
\tilde{M}_p(s) \sim \begin{cases}
\frac{\Gamma(1 + p)}{(\delta/\gamma)^p} s^{-1-p}, & \text{as } s \to 0, \\
\frac{\Gamma(1 + p)}{(\delta\sqrt{2})^p} s^{-1-p/2}, & \text{as } s \to \infty.
\end{cases}
\]

Using above result and Theorem 4.2 we get
\[
\mathbb{E}[(E_G(t))^p] \sim \begin{cases}
\frac{\Gamma(1 + p)t^{p/2}}{(1 + p/2)(\delta\sqrt{2})^p}, & \text{as } t \to 0, \\
\left( \frac{\gamma}{\delta} \right)^p t^p, & \text{as } t \to \infty.
\end{cases}
\]

We finally get the asymptotic moments as
\[
\mathbb{E}[W^{(3)}_{\beta}(t)] = q\mathbb{E}[(E_G(t))^{\beta}] \sim \begin{cases}
q\frac{\Gamma(1 + \beta)}{(1 + \beta/2)(\delta\sqrt{2})^{\beta}} t^{\beta/2}, & \text{as } t \to 0, \\
q \left( \frac{\gamma}{\delta} \right)^{\beta} t^\beta, & \text{as } t \to \infty.
\end{cases}
\]

We next show that the TCFPP-II \( \{W^{f}_{\beta}(t)\}_{t \geq 0} \) is a renewal process. We begin with the following lemma.
Let \( \{D_f(t)\}_{t \geq 0} \) be a subordinator with the associated Bernstein function \( f(s) \). Let \( \{E_f(t)\}_{t \geq 0} \) be the right-continuous inverse of \( \{D_f(t)\}_{t \geq 0} \). We call, rather loosely, \( E_f(t) \) the inverse subordinator corresponding to \( f(s) \).

Lemma 4.1. Let \( \{E_{f_1}(t)\}_{t \geq 0} \) and \( \{E_{f_2}(t)\}_{t \geq 0} \) be two independent inverse subordinators corresponding to Bernstein functions \( f_1(s) \) and \( f_2(s) \), respectively. Then
\[
\{E_{f_1}(E_{f_2}(t))\} \overset{d}{=} \{E_{f_1 \circ f_2}(t)\}, \ t \geq 0,
\]
where \( (f_1 \circ f_2)(s) = f_1(f_2(s)) \).
Consider two independent subordinators \( \{D_{f_1}(t)\}_{t \geq 0} \) and \( \{D_{f_2}(t)\}_{t \geq 0} \) with
\[
\mathbb{E}[e^{-sD_{f_1}(t)}] = e^{-tf_1(s)} \quad \text{and} \quad \mathbb{E}[e^{-sD_{f_2}(t)}] = e^{-tf_2(s)},
\]
where \( f_1(s) \) and \( f_2(s) \) are the associated Bernstein functions. We claim that
\[
\{D_{f_2}(D_{f_1}(t))\} \overset{d}{=} \{D_{f_1 \circ f_2}(t)\}, \quad t \geq 0,
\]
where \( \circ \) denotes the composition of functions. To see this, let us compute the LT of the left-hand side
\[
\mathbb{E} \left[ e^{-sD_{f_2}(D_{f_1}(t))} \right] = \mathbb{E} \left[ \mathbb{E} \left[ e^{-sD_{f_2}(D_{f_1}(t))} | D_{f_1}(t) \right] \right] = \mathbb{E} \left[ e^{-f_2(s)D_{f_1}(t)} \right] = e^{-tf_1(f_2(s))}, \quad s > 0.
\]
Since \( f(s) = (f_1 \circ f_2)(s) \) is again a Bernstein function (see [33, Remark 5.28 (ii)]) and \( \{D_{f_2}(D_{f_1}(t))\}_{t \geq 0} \) is a Lévy process (see [3, Theorem 1.3.25]), it follows that \( \{D_f(t)\}_{t \geq 0} \) is a subordinator with associated Bernstein function \( f(s) = (f_1 \circ f_2)(s) \).

Consider next the inverse subordinators defined by
\[
E_{f_1}(t) = \inf \{ r \geq 0 : D_{f_1}(r) > t \} \quad \text{and} \quad E_{f_2}(t) = \inf \{ r \geq 0 : D_{f_2}(r) > t \}, \quad t \geq 0.
\]

Then the process
\[
E_{f_1 \circ f_2}(t) = \inf \{ r \geq 0 : D_{f_1 \circ f_2}(r) > t \}
= \inf \{ r \geq 0 : D_{f_2}(D_{f_1}(r)) > t \} \quad \text{(using (4.4)).}
\]

By the property of right-continuous inverse, we have that \( \{D_{f_2}(D_{f_1}(r)) > t\} = \{E_{f_2}(t) < D_{f_1}(r)\} \), and hence
\[
E_{f_1 \circ f_2}(t) = \inf \{ r \geq 0 : D_{f_1}(r) > E_{f_2}(t) \} = E_{f_1}(E_{f_2}(t)),
\]
which completes the proof.

**Corollary 4.1.** Let \( \{E_{\beta}(t)\}_{t \geq 0} \) be inverse \( \beta \)-stable subordinator corresponding to \( f_1(s) = s^\beta \), and \( \{E_f(t)\}_{t \geq 0} \) be an inverse subordinator corresponding to \( f_2(s) = f(s) \). Then from (4.3),
\[
\{E_{\beta}(E_f(t))\}_{t \geq 0} \overset{d}{=} \{E_{\phi}(t)\}_{t \geq 0},
\]
where \( \phi(s) = (f(s))^\beta \).

**Remark 4.1.** One can further generalize the TCFPP-I process \( \{Q^I_{\beta}(t)\}_{t \geq 0} \) and TCFPP-II process \( \{W^I_{\beta}(t)\}_{t \geq 0} \), by subordinating it again with a subordinator and an inverse subordinator, respectively. As it clearly shown in (4.4) and (4.3), the subordination of subordinator and inverse subordinator yields again a subordinator and an inverse subordinator, respectively. Hence, further subordination leads again to the processes of type TCFPP-I \( \{Q^I_{\beta}(t)\}_{t \geq 0} \) and TCFPP-II \( \{W^I_{\beta}(t)\}_{t \geq 0} \). This is also valid for \( n \)-iterated subordination.

**Theorem 4.3.** The TCFPP-II \( \{W^I_{\beta}(t)\}_{t \geq 0} \) is a renewal process with iid waiting times \( \{J_n\}_{n \geq 1} \) with distribution
\[
\mathbb{P}[J_n > t] = \mathbb{E}[e^{-\lambda E_{\phi}(t)}],
\]
where \( E_{\phi}(t) \) is the inverse subordinator corresponding to \( \phi(s) = (f(s))^\beta \).
Proof. Using (1.1) and Corollary 4.1 we have
\[ \{W^{f}_{\beta}(t)\}_{t \geq 0} = \{N_{\beta}(E_{f}(t))\}_{t \geq 0} \overset{d}{=} \{N(E_{\beta}(t))\}_{t \geq 0}, \]
where \( \phi(s) = (f(s))^\beta \). Therefore, the TCFPP-II \( \{W^{f}_{\beta}(t)\}_{t \geq 0} \) is a Poisson process time-changed by an inverse subordinator \( \{E_{\beta}(t)\}_{t \geq 0} \) corresponding to Bernstein function \( \phi(s) = (f(s))^\beta \). From [25, Theorem 4.1], we deduce that the time-changed Poisson process
\[ \{W^{f}_{\beta}(t)\}_{t \geq 0} \overset{d}{=} \{N(E_{\phi}(t))\}_{t \geq 0} \]
is a renewal process with iid waiting times \( \{J_n\}_{n \geq 1} \) having the distribution (4.6). \( \square \)

Remark 4.2. By [25, Remark 5.4], the pmf \( \eta_{\phi}^{f}(n|t, \lambda) \), given in (4.1), of the TCFPP-II \( \{W^{f}_{\beta}(t)\}_{t \geq 0} \) satisfies
\[ \phi(\partial_{1})\eta_{\phi}^{f}(n|t, \lambda) = -\lambda \left( \eta_{\phi}^{f}(n|t, \lambda) - \eta_{\phi}^{f}(n-1|t, \lambda) \right) + H(x)\psi_{\phi}(t, \infty) \]
in the mild sense, where \( \phi(s) = (f(s))^\beta, \psi_{\phi}(\cdot) \) is the Lévy measure associated to Bernstein function \( \phi(s) \) and \( H(x) = I(x \geq 0) \) is the Heaviside function.

We next present the bivariate distributions of the TCFPP-II, which generalizes a result by [29, Theorem 2.1]. Let \( F(t) \) be the distribution function of the waiting time \( J_{n} \) and \( S_{n} = J_{1} + \ldots + J_{n} \) be the time of \( n \)-th jump. Since \( J_{n} \)'s are iid, we have that \( \mathbb{P}[S_{n} \leq t] = F^{*n}(t) \), where \( F^{*n}(t) \) denotes the \( n \)-fold convolution of \( F(t) \). For \( n, k \geq 1 \), define \( \tau_{n}^{(k)} = S_{n+k} - S_{n} \overset{d}{=} S_{k} \), where \( \tau_{n}^{(k)} \) is the time elapsed between \( n \)-th and \( (n+k) \)-th jump. Clearly, \( \mathbb{P}[\tau_{n}^{(1)} \in dt] = dF(t) \) and \( \mathbb{P}[\tau_{n}^{(k)} \in dt] = dF^{*k}(t), \) for \( k \geq 1 \).

Theorem 4.4. Let \( 0 \leq s < t \) and \( 0 \leq m \leq n \) be nonnegative integers. Let \( \{E_{\phi}(t)\}_{t \geq 0} \) be the inverse subordinator corresponding to \( \phi(s) = (f(s))^\beta \). The TCFPP-II \( \{W^{f}_{\beta}(t)\}_{t \geq 0} \) has the bivariate distributions given by,
\[ \mathbb{P}[W_{\beta}^{f}(s) = m, W_{\beta}^{f}(t) = n] = \begin{cases} \int_{0}^{s} \mathbb{E}[e^{-\lambda E_{\phi}(t-u)}]dF^{*m}(u), & \text{if } n = m \geq 0, \\ \int_{0}^{s} dF^{*m}(u) \int_{s-u}^{t-u} dF(v) \int_{0}^{t-(u+v)} \mathbb{E}[e^{-\lambda E_{\phi}(t-u-v-x)}]dF^{*n-m-1}(x), & \text{if } n \geq m + 1, \end{cases} \]
where \( F(t) = 1 - \mathbb{E}[e^{-\lambda E_{\phi}(t)}] \) and \( dF^{*n}(t) \) is the \( n \)-fold convolution of \( dF(t), \) \( n \geq 1, \) with \( dF^{*0}(t) = \delta_{0}(t), \) the Dirac delta function at zero.

Proof. Case 1: When \( n = m \), we have (see Figure 1)
\[ \mathbb{P}[W_{\beta}^{f}(s) = m, W_{\beta}^{f}(t) = m] = \mathbb{P}[0 < S_{m} \leq s; t < S_{m+1}] = \mathbb{P}[0 < S_{m} \leq s; t < S_{m} + \tau_{m}^{(1)}] \]
\[ = \mathbb{P}[0 < S_{m} \leq s; \tau_{m}^{(1)} > t - S_{m}] \]
\[ = \int_{0}^{s} dF^{*m}(u)\mathbb{P}[\tau_{m}^{(1)} > t - u] \] (since \( S_{m} \) and \( \tau_{m}^{(1)} \) are independent)
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Figure 1. Waiting times between events for $m = n$

$$= \int_{0}^{s} \mathbb{E}[e^{-\lambda E_\phi(t-u)}]dF^s(u).$$

Case 2: When $n \geq m + 1$, it follows that (see Figure 2)

$$\begin{align*}
\mathbb{P}[W^f_\beta(s) = m, W^f_\beta(t) = n] &= \mathbb{P}[0 < S_m \leq s; \tau_m^{(1)} > s - S_m; \tau_m^{(1)} < t - S_m; 0 < \tau_{m+1}^{(n-m-1)} < t - S_{m+1}; \tau_m^{(1)} > t - S_n] \\
&= \mathbb{P}[0 < S_m \leq s; s - S_m < \tau_m^{(1)} < t - S_m; 0 < \tau_{m+1}^{(n-m-1)} < t - S_m - \tau_m^{(1)}; \tau_n^{(1)} > t - S_m - \tau_m^{(1)} - \tau_{m+1}^{(n-m-1)}].
\end{align*}$$

Since the waiting times between events are iid, we have that

$$\begin{align*}
\mathbb{P}[W^f_\beta(s) = m, W^f_\beta(t) = n] &= \int_{0}^{s} \mathbb{P}[S_m \in du] \int_{s-u}^{t-u} \mathbb{P}[\tau_m^{(1)} \in dv] \int_{0}^{t-u+v} \mathbb{P}[\tau_{m+1}^{(n-m-1)} \in dw] \int_{t-(u+v+w)}^{\infty} \mathbb{P}[\tau_n^{(1)} \in dx] \\
&= \int_{0}^{s} dF^s(u) \int_{s-u}^{t-u} dF(v) \int_{0}^{t-u+v} dF(s(n-m-1))(w) \mathbb{P}[\tau_n^{(1)} > (t - u - v - w)] \\
&= \int_{0}^{s} dF^s(u) \int_{s-u}^{t-u} dF(v) \int_{0}^{t-u+v} \mathbb{E}[e^{-\lambda E_\phi(t-u-v-w)}]dF(s(n-m-1))(w),
\end{align*}$$

which completes the proof. \qed

Let us examine a special case of Theorem 4.4 for the FPP.

Remark 4.3. It is known (see [25]) that the FPP $\{N_\beta(t)\}_{t \geq 0} = \{N(E_\beta(t))\}_{t \geq 0}$ is a renewal process whose inter-arrival times follow the Mittag-Leffler distribution, that is,

$$\mathbb{P}[J_n \leq t] = F(t) = 1 - L_\beta(-\lambda t^\beta), \quad 0 < \beta < 1,$$
where \( L_{\beta}(z) \) is the Mittag-Leffler function defined in (2.1). Let us define \( L_{\beta,0}^0(-\lambda t^\beta) := t\delta_0(t) \), where \( \delta_0(t) \) is the Dirac delta function at zero. This implies for \( t \geq 0 \) (see [29] and references therein),

\[
(4.7) \quad \mathbb{P}[S_m \in dt] = \mathbb{P}[\tau_m^{(n)} \in dt] = dF^{*m}(t) = \lambda^m t^{m\beta-1} L_{\beta,m\beta}^m(-\lambda t^\beta)dt, \ m \geq 0,
\]

where \( L_{\beta,0}^0(z) \) is the generalized Mittag-Leffler function defined in (2.2). The LT of the inverse \( \beta \)-stable subordinator is given by (see [9, eq. (16)])

\[
(4.8) \quad \mathbb{E}[e^{-\lambda F_\beta(t)}] = L_{\beta}(-\lambda t^\beta).
\]

Using (4.7), (4.8) and Theorem 4.4 the bivariate distribution of the FPP, when \( n = m \geq 0 \), is

\[
\mathbb{P}[N_{\beta}(s) = m, N_{\beta}(t) = m] = \lambda^m \int_0^s u^{m\beta-1} L_{\beta,m\beta}^m(-\lambda u^\beta)L_{\beta}(-\lambda(t-u)^\beta)du, \ m \geq 0.
\]

For \( n \geq m + 1 \),

\[
\mathbb{P}[N_{\beta}(s) = m, N_{\beta}(t) = n] = \lambda^n \int_0^s u^{m\beta-1} L_{\beta,m\beta}^m(-\lambda u^\beta) \int_{s-u}^{t-u} v^{\beta-1} L_{\beta,\beta}^1(-\lambda v^\beta) \int_0^{t-(u+v)} x^{\beta(n-m-1)-1} dx \\
	imes L_{\beta,\beta(n-m-1)}^{n-m-1}(-\lambda x^\beta)L_{\beta}(-\lambda(t-u-v-x)^\beta)]dudvdx,
\]

which coincides (2.9) of [29]. Indeed, it is shown in [29, eq. (2.6)] that

\[
\mathbb{P}[N_{\beta}(s) = m, N_{\beta}(t) = n] = \lambda^n \int_0^s u^{m\beta-1} L_{\beta,m\beta}^m(-\lambda u^\beta)du \int_{s-u}^{t-u} v^{\beta-1} L_{\beta,\beta}^1(-\lambda v^\beta) \\
	imes (t-u-v)^{\beta(n-m-1)} L_{\beta,\beta(n-m-1)+1}^{n-m}(-\lambda(t-u-v)^\beta) dv, \ n \geq m + 1.
\]

When \( \beta = 1 \), \( L_{1,1}^1(x) = e^x \), and \( L_{1,m}^1(x) = e^x/(m-1)! \) and

\[
\mathbb{P}[N(s) = m, N(t) = n] = \begin{cases} 
\frac{\lambda^m s^m}{m!} e^{-\lambda s}, & \text{if } n = m, \\
\frac{\lambda^n s^m (t-s)^{n-m}}{n!} \binom{n}{m} e^{-\lambda t}, & \text{if } n \geq m + 1,
\end{cases}
\]

the bivariate distribution of the Poisson process, as expected.

5. Simulation

In this section, we present simulated sample paths for some TCFPP-I and TCFPP-II processes. The sample paths for the FNBP, the FPP subordinated with tempered \( \alpha \)-stable subordinator (FPP-TSS) and the FPP subordinated with inverse Gaussian subordinator (FPP-IGN) are presented for a chosen set of parameters. The simulations of the corresponding TCFPP-II process of the FPP subordinated with inverse gamma subordinator (FPP-IG), the FPP subordinated with inverse tempered \( \alpha \)-stable subordinator (FPP-ITSS), and the FPP subordinated with inverse of inverse Gaussian subordinator (FPP-IIGN) are also given in this section. We first present the algorithm for simulation of the FPP.
Algorithm 1 (Simulation of the FPP). This algorithm (see [10]) gives the number of events \( N_\beta(t) \), \( 0 < \beta < 1 \) of the FPP up to a fixed time \( T \).
(a) Fix the parameters \( \lambda > 0 \) and \( 0 < \beta < 1 \) for the FPP.
(b) Set \( n = 0 \) and \( t = 0 \).
(c) Repeat while \( t < T \)
   - Generate three independent uniform random variables \( U_i \sim U(0,1), i = 1, 2, 3 \).
   - Compute (see [17])
     \[
     dt = \frac{|\ln U_1|^{1/\beta} \sin(\beta \pi U_2)[\sin(1-\beta)\pi U_2]^{1/\beta-1}}{\lambda^{1/\beta} [\sin(\pi U_2)]^{1/\beta} |\ln U_3|^{1/\beta-1}}.
     \]
     \( t = t + dt \) and \( n = n + 1 \).
(d) Next \( t \).
Then \( n \) denotes the number of events \( N_\beta(t) \) occurred up to time \( T \).

We next present the algorithms for the simulation of the gamma subordinator, the tempered \( \alpha \)-stable subordinator and the inverse Gaussian subordinator. The generated sample paths from these algorithms will then be used to simulate the inverse subordinator and the TCFPP-I.

Algorithm 2 (Simulation for the gamma subordinator).
(a) Fix the parameters \( \alpha \) and \( p \) for gamma subordinator.
(b) Choose an interval \([0, T]\). Choose \( n+1 \) uniformly spaced time points \( 0 = t_0, t_1, \ldots, t_n = T \) with \( h = t_2 - t_1 \).
(c) Simulate \( n \) independent gamma random variables \( Q_i \sim G(\alpha, ph), 1 \leq i \leq n \), using GSS algorithm (see [4, p. 321]).
(d) The discretized sample path of \( Y(t) \) at \( t_i \) is \( Y(ih) = Y(t_i) = \sum_{j=1}^{i} Q_j, 1 \leq i \leq n \) with \( Q_0 = 0 \).

Algorithm 3 (Simulation for the TSS).
(a) Choose the parameters \( \mu > 0 \) and \( 0 < \alpha < 1 \).
(b) Choose an interval \([0, T]\). Choose \( n+1 \) time points \( 0 = t_0, t_1, \ldots, t_n = T \).
(c) Simulate \( D_\alpha^{(t_i-t_{i-1})^{1/\alpha}}(1) \) for \( 1 \leq i \leq n \) from the Algorithm 3.2 of [14].
(d) Compute the increments
   \[
   \Delta D_\alpha^{(i)} = D_\alpha^{(i)}(t_i) - D_\alpha^{(i)}(t_{i-1}) = (t_i - t_{i-1})^{1/\alpha} D_\alpha^{(t_i-t_{i-1})^{1/\alpha}}(1), 1 \leq i \leq n,
   \]
   with \( D_\alpha^{(0)}(0) = 0 \).
(e) The discretized sample path of \( D_\alpha^{(i)}(t) \) at \( t_i \) is \( D_\alpha^{(i)}(t_i) = \sum_{j=1}^{i} \Delta D_\alpha^{(j)}(1), 1 \leq i \leq n \).

Algorithm 4 (Simulation of the IGN subordinator). The algorithm to generate the IGN random variables is given in [11, p. 183].
(a) Choose an interval \([0, T]\). Choose \( n+1 \) uniformly spaced time points \( 0 = t_0, t_1, \ldots, t_n = T \) with \( h = t_2 - t_1 \).
(b) Since IGN subordinator \( \{G(t)\}_{t \geq 0} \) has independent and stationary increments, \( F_i (G(t_i) - G(t_{i-1}) \overset{d}{=} G(h) \sim \text{IGN}(h, 1) \) for \( 1 \leq i \leq n \) and \( h = T/n \). Now generate \( n \) iid IGN variables \( F_i \)'s as follows (see [11, p. 183], therein substituted \( \delta = 1 = \gamma \)):
   - Generate a standard normal random variable \( N \).
Assign $X = N^2$.
Assign $Y = h + \frac{X}{2} - \frac{1}{2} \sqrt{4hX + X^2}$.
Generate a uniform $[0, 1]$ random variable $U$.
If $U \leq \frac{h}{h + Y}$, return $Y$; else return $\frac{h^2}{Y}$.

(c) Assign $G(t_0) = 0$. The discretized sample path of $G(t)$ at $t_i$ is $G(t_i) = \sum_{j=1}^{i} F_j$, $1 \leq i \leq n$.

Consider next the algorithm to simulate the inverse subordinator $\{E_f(t)\}_{t \geq 0}$. We first define $E_f^\delta(t)$ with the step length $\delta$ as (see [20])

$$E_f^\delta(t) = (\min\{n \in \mathbb{N} : D_f(\delta n) > t \} - 1)\delta, \quad n = 1, 2, ... ,$$

where $D_f(\delta n)$ is the value of the subordinator $D_f(t)$ evaluated at $\delta n$, which can be simulated by using the method presented above. Observe that trajectory of $E_f^\delta(t)$ has increments of length $\delta$ at random time instants governed by process $D_f(t)$ and therefore $E_f^\delta(t)$ is the approximation of operational time.

**Algorithm 5 (Simulation of the inverse subordinator).**
(a) Fix the parameters for the inverse subordinator, whichever under consideration.
(b) Choose $n$ uniformly spaced time points $0 = t_1, t_2, \ldots, t_n = T$ with $h = t_2 - t_1$.
(c) Let $i = 0$ and $t = 0$.
(d) Repeat while $t < T$.
   Generate an independent $D_f(t)$ random variables with $Q_i \sim D_f(h)$.
   Set $W(\lceil t/h \rceil + 1) := h \ast i, \ldots, W(\lceil (t + Q_i)/h \rceil + 1) := h \ast i$.
   $i = i + 1, \quad t = t + Q_i$.
   Next $t$.
(e) The discretized sample path of $E_f(t)$ at $t_i$ is $W_i, 1 \leq i \leq n$ with $W_0 = 0$.

Note that the simulations for the inverse of gamma subordinator, the inverse of tempered $\alpha$-stable subordinator and the inverse of inverse Gaussian subordinator can be done using the above algorithm by replacing the special case for the subordinator.

We next present a general algorithm to simulate the TCFP-I, namely the FNBP, the FPP-TSS and the FPP-IGN processes. The same algorithm can be used to simulate the TCFPP-II, namely the FPP-IG, the FPP-ITSS and the FPP-IIGN processes.

**Algorithm 6 (Simulation of the TCFPP-I and the TCFPP-II).**
(a) Fix the parameters for the subordinator (inverse subordinator), under consideration.
   Choose the fractional index $\beta$ ($0 < \beta < 1$) and rate parameter $\lambda > 0$ for the FPP.
(b) Fix the time $T$ for the time interval $[0, T]$ and choose $n + 1$ uniformly spaced time points $0 = t_0, t_1, \ldots, t_n = T$ with $h = t_2 - t_1$.
(c) Simulate the values $X(t_i), 1 \leq i \leq n$, of the subordinator (inverse subordinator) at $t_1, \ldots, t_n$, using the algorithm for respective subordinator (inverse subordinator).
(d) Using the values $X(t_i), 1 \leq i \leq n$, generated in Step (c), as time points, compute the number of events of the FPP $N_\beta(X(t_i)), 1 \leq i \leq n$, using Algorithm [1].
(A) Sample paths of the FNBP for $\beta = 0.6, \alpha = 3.0, p = 4.0$ and $\lambda = 1.5$.

(B) Sample paths of the FNBP for $\beta = 0.90, \alpha = 3.0, p = 4.0$ and $\lambda = 2.0$.

Figure 3. Sample paths of the FNBP process

(A) Sample paths of the FPP-IG for $\beta = 0.6, \alpha = 3.0, p = 4.0$ and $\lambda = 1.5$.

(B) Sample paths of the FPP-IG for $\beta = 0.90, \alpha = 3.0, p = 4.0$ and $\lambda = 2.0$.

Figure 4. Sample paths of the FPP-IG process
(A) Sample paths of the FPP-TSS process for $\beta = 0.6, \mu = 2.0, \alpha = 0.5$ and $\lambda = 1.5$. 

(B) Sample paths of the FPP-TSS process for $\beta = 0.9, \mu = 2.0, \alpha = 0.7$ and $\lambda = 2.0$.

**Figure 5.** Sample paths of the FPP-TSS process

(A) Sample paths of the FPP-ITSS process for $\beta = 0.6, \mu = 2.0, \alpha = 0.5$ and $\lambda = 1.5$. 

(B) Sample paths of the FPP-ITSS process for $\beta = 0.9, \mu = 2.0, \alpha = 0.7$ and $\lambda = 2.0$.

**Figure 6.** Sample paths of the FPP-ITSS process

(A) Sample paths of the FPP-IIGN process for $\beta = 0.6, \delta = 1 = \gamma$ and $\lambda = 1.5$. 

(B) Sample paths of the FPP-IIGN process for $\beta = 0.9, \delta = 1 = \gamma$ and $\lambda = 2.0$.

**Figure 8.** Sample paths of the FPP-IIGN process
(A) Sample paths of the FPP-IGN process for $\beta = 0.6, \delta = 1 = \gamma$ and $\lambda = 1.5$.  

(B) Sample paths of the FPP-IGN process for $\beta = 0.9, \delta = 1 = \gamma$ and $\lambda = 2.0$.

Figure 7. Sample paths of the FPP-IGN process

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