EQUIVARIANT DEGENERATIONS OF PLANE CURVE ORBITS

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Abstract. In a series of papers, Aluffi and Faber computed the degree of the $GL_3$-orbit closure of an arbitrary plane curve. We attempt to generalize this to the equivariant setting by studying how orbits degenerate under some natural specializations, yielding a fairly complete picture in the case of plane quartics.

1. Introduction

Let $V$ be an $(r + 1)$-dimensional vector space, and let $F \in \text{Sym}^d V^\vee$ be a non-zero degree $d$ homogeneous form on $V$. The form $F$ naturally produces two varieties, first the $GL(V)$-orbit closure

$$\text{Orb}(F) \subset \text{Sym}^d V^\vee,$$

and secondly its projectivization

$$\mathbb{P}\text{Orb}(F) \subset \mathbb{P}\text{Sym}^d V^\vee.$$ 

The relationship between the geometry of $\mathbb{P}\text{Orb}(F)$ and the geometry of the hypersurface $X$ defined by $\{F = 0\}$ remains mostly mysterious.

Consider, for example, the enumerative problem of computing the degree of $\mathbb{P}\text{Orb}(F)$. The analysis of the degree of $\mathbb{P}\text{Orb}(F)$ was carried out for the first two cases $r = 1, 2$ in a series of remarkable papers of Aluffi and Faber [AF93a, AF93b, AF00a, AF00c, AF00b, AF10a, AF10b]. Aluffi and Faber’s computation in the special case $r = 2, d = 4$ of quartic plane curves yields the highly non-trivial enumerative consequence: In a general 6-dimensional linear system of quartic curves, a general genus 3 curve arises $14280$ times.

Our starting point is to interpret the calculation of the degree of $\mathbb{P}\text{Orb}(F)$ as equivalent to the computation of the fundamental class $[\mathbb{P}\text{Orb}(F)] \in A^*(\mathbb{P}\text{Sym}^d V^\vee)$. Our extra contributions stem from one very simple observation. Since $\mathbb{P}\text{Orb}(F)$ is evidently preserved by the action of $GL(V)$, there is a natural equivariant extension of the problem: Compute the equivariant fundamental class

$$[\mathbb{P}\text{Orb}(F)]_{GL(V)} \in A^*_{GL(V)}(\mathbb{P}\text{Sym}^d V^\vee).$$

In elementary terms, beginning with a rank $r+1$ vector bundle $\mathcal{V}$ over a base $B$, the class $[\mathbb{P}\text{Orb}(F)]$ encodes the universal expressions in the chern classes $c_1 \mathcal{V}, ..., c_{r+1} \mathcal{V}$ appearing in the fundamental class of the relative orbit closure cycle $(\mathbb{P}\text{Orb}(F))_{\mathcal{V}} \subset \mathbb{P}\text{Sym}^d V^\vee$. 

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This larger equivariant setting encodes the solution to many more challenging enumerative problems. For instance, our analysis of the case of quartic plane curves yields the enumerative consequence: A general genus 3 curve appears 510720 times as a 2-plane slice of a fixed general quartic threefold.

To date, very few equivariant classes $[\mathbb{P} \text{Orb}(F)]_{GL(V)}$ have been determined. For quadric hypersurfaces ($d = 2$), the class $[\mathbb{P} \text{Orb}(F)]_{GL(V)}$ is determined by the rank of the quadric hypersurface $F = 0$ and its computation amounts to the Porteous formula for symmetric maps [HT84]. For another class of examples, the authors’ work with H. Spink in [LPST20] provides access to $[\mathbb{P} \text{Orb}(F)]_{GL(V)}$ when $F = 0$ defines a hyperplane arrangement. In this paper, we study the frontier case $r = 2$ of plane curves. As in [LPST20], our strategy will be to specialize the orbit $\mathbb{P} \text{Orb}(F)$ until it breaks into a union of other orbits $\mathbb{P} \text{Orb}(F_i)$ whose classes are more directly computable. To execute this strategy, we initiate a detailed study of how orbits of plane curves behave under some common degenerations. This is our main purpose in the paper.

We will summarize the results of this study in the next subsection, but we emphasize here that the overall picture in the case of quartic plane curves is quite neat and beautiful – we deduce clean relationships among different classes $[\mathbb{P} \text{Orb}(F)]_{GL(V)}$ for $F$ ranging over several types of quartic plane curves possessing special geometric properties.

Finally, because the computation of equivariant orbit classes does not have a strong presence in the literature, we have written an appendix containing the $r = 1$ case (points on a line) and the $r = 2, d = 3$ case of cubic plane curves. As a refreshing demonstration of alternate techniques, the case of points on a line is done in two independent ways: one by specializing the results in [LPST20] and the other by applying the Atiyah-Bott formula to the resolution of the orbit given in [AF93a].

1.1. Summary of Degenerations. When studying degenerations, it is more convenient to weight orbits $\mathbb{P} \text{Orb}(F)$ by the number of linear automorphisms of the hypersurface $X$ given by $F = 0$. We will write $\text{Orb}(F)$ and $\text{Orb}(X)$ interchangeably (and similarly for their projectivized versions). In this section, we will exclusively be concerned with the case of plane curves, $r = 2$.

If $C_t$ is a family of smooth plane curves specializing at $t = 0$ to a curve $C_0$ possessing nodes and cusps, we get a corresponding specialization of the (weighted) orbit closures $\mathbb{P} \text{Orb}(C_t)$ to some union of $GL(V)$-invariant varieties. Theorem 5.3 gives a complete description of the new orbits appearing in the flat limit as $t \to 0$. To illustrate this theorem, we will describe what happens in the special case where the curve acquires a single node or a single cusp. The general case is essentially a sum of contributions corresponding to each node or cusp.

1.1.1. Acquiring a node. If $C_t$ acquires a single node in the limit $C_0$, then as a limit of weighted orbits, one obtains the obvious weighted orbit $\mathbb{P} \text{Orb}(C_0)$ along with one other weighted orbit, $\mathbb{P} \text{Orb}(C_{BN})$, which occurs with multiplicity 2.
The curve $C_{BN}$ is the union of a nodal cubic with one of its two tangent lines (branches) at the node, the line taken with multiplicity $(d - 3)$.

1.1.2. **Acquiring a cusp.** If $C_t$ acquires a single cusp in the limit $C_0$, then as a limit of weighted orbits, one obtains the weighted orbit of the cuspidal curve $C_0$ along with another weighted orbit, $\mathbb{P} \text{Orb}(C_{\text{flex}})$.

The curve $C_{\text{flex}}$ is a smooth cubic (with general $j$-invariant) union a $(d - 3)$-fold flex line. We can degenerate the weighted orbit $\mathbb{P} \text{Orb}(C_{\text{flex}})$ even further to get the weighted orbit of $C_{BN}$ from before (with multiplicity 2) together with the weighted orbit of a new curve $C_{AN}$. $C_{AN}$ is the union of a nodal cubic with one of its flex lines (at a smooth point), with the line taken $(d - 3)$ times.

In either case, we deduce that the equivariant class $[\mathbb{P} \text{Orb}(C_t)]_{GL(V)}$, for $t$ general, is a particular combination of $[\mathbb{P} \text{Orb}(C_0)]_{GL(V)}$ and two specific classes $[\mathbb{P} \text{Orb}(C_{BN})]_{GL(V)}$, and $[\mathbb{P} \text{Orb}(C_{AN})]_{GL(V)}$.

1.1.3. **Splitting off a line.** The equivariant class of the orbit closure of a union of lines can be deduced using the results of [LPST20]. In light of this, it is natural to want to study the degeneration where a degree $d$ plane curve specializes to a union of $d$ general lines. We will show that if $C_t$ is a general family of curves such that $C_0$ is a general union of lines, then in the flat limit of orbit closures, beyond the obvious orbit closure $\mathbb{P} \text{Orb}(C_0)$ we find $d$ other orbits, each being the weighted orbit of a general irreducible plane curve possessing a multiplicity $d - 1$ point.
More generally, we also study specializations where $C_0$ is a union of a general degree $e$ curve along with $d - e$ general lines (see Proposition 4.3).

Remark 1.1. For any reader experienced in the art of degeneration, the fact that the curves with $(d - 1)$-fold points arise in the limit is not a surprise. The difficulty lies in showing that all new orbits are accounted for.

1.1.4. Degeneration to the Double Conic. Next suppose $C_t$ is a family of general plane quartics specializing to a double conic – a classic example in the study of moduli of curves. Since a double conic has a large stabilizer group under the action of $PGL(V)$, its orbit closure will not appear as an irreducible component of the $t \to 0$ flat limit of orbit closures. We will show (in Theorem 7.6) that in this situation, the weighted orbit of $C_t$ specializes to 8 times the weighted orbit of a particular rational quartic curve possessing an $A_6$ singularity.

It is striking that the limit has such a clean answer, consisting set-theoretically of the orbit closure of the irreducible quartic plane curve with an $A_6$ singularity. This fact is related to the question of which planar quartics account for a general hyperelliptic curve after applying semistable reduction. The multiplicity 8 we obtain corresponds to the 8 Weierstrass points on a genus 3 hyperelliptic curve. For literature relevant to this circle of ideas, see [Pin74, Has00, CML13, Fed14].

1.2. Orbit classes of quartic curves. In the specific setting of quartic curves, we find that the orbit class of an arbitrary smooth quartic can be deduced in a direct way from the orbit classes of special quartics with $A_6$ and $E_6$ singularities. We have already explained the relation with curves having an $A_6$ singularity above. By adapting an idea of Aluffi and
In Section 7.4 we specialize the orbit closure of a general quartic plane curve to the orbit of a smooth quartic plane curve possessing a hyperflex. In the limit, the orbit closure of a particular rational quartic (denoted $C_{E_6}$ in the text) possessing an $E_6$ singularity appears (with multiplicity twice the number of hyperflexes of the limiting smooth quartic).

Ultimately, we deduce that the orbit class $[\mathbb{P} \text{Orb}(F)]_{GL(V)}$ of an arbitrary smooth quartic $F = 0$ is expressible solely in terms of orbit classes of two specific rational curves having an $A_6$ or $E_6$ singularity. These are the curves $C_{A_6}$ and $C_{E_6}$ in the text. From here, we arrive at explicit formulas for orbit classes by invoking Kazarian’s work [Kaz03a] on counting $A_6$ and $E_6$ singularities in families of curves.

We also compute the equivariant classes of orbit closure for many singular quartics using the degenerations in Section 1.1. (For those readers curious about the lack of the presence of $D_6$ singularities, the curves with a $D_6$ singularity arise when specializing to a node as in Section 1.1.1.)

We summarize all computations in the table in Figure 1. Let us explain how to read the table. For a plane curve $C \subset \mathbb{P}V$ with an 8-dimensional $\text{PGL}(V)$ orbit, the expressions $p_C$ are defined to be the $\text{GL}(V)$-equivariant classes $[\text{Orb}(C)]_{GL(V)}$ multiplied by the number of $\text{PGL}(V)$-automorphisms of $C$. We note here that the classes $[\mathbb{P} \text{Orb}(C)]_{GL(V)}$ are related to $[\text{Orb}(C)]_{GL(V)}$ by a simple substitution –see Proposition 2.3

1.2.1. Application: plane sections of a quartic threefold. To see how to use the expressions $p_C$, we provide the calculation of 510720 mentioned earlier in the introduction.

Starting with a general smooth quartic threefold $X \subset \mathbb{P}^4$, one obtains a rational map

$$\Phi : \mathbb{G}(2, 4) \dashrightarrow \overline{\mathcal{M}_3}$$

sending a general 2-plane $\Lambda \subset \mathbb{P}^4$ to the moduli point of the plane quartic $C := X \cap \Lambda$. Our calculation of the equivariant class $p_C$ for $C$ a general quartic implies:

**Corollary 1.2.** If $X$ is general, the map $\Phi$ has degree 510720.

**Proof of Corollary 1.2.** We find the degree of $\Phi$ directly by choosing a general quartic plane curve $C \subset \mathbb{P}^2$ and counting the number of 2-planes $\Lambda$ such that $X \cap \Lambda$ is isomorphic to $C$.

Let $G$ be a quartic homogenous form cutting out a general quartic threefold $X \subset \mathbb{P}^4$, and let $\pi : \mathcal{S} \rightarrow \mathbb{G}(2, 4)$ denote the rank 3 tautological subbundle over the Grassmannian. The form $G$ defines a section of $\mathcal{O}_{\mathbb{P}(\mathcal{S})}(4)$ on $\mathbb{P}(\mathcal{S})$, which in turn induces a section $s : \mathbb{G}(2, 4) \rightarrow \text{Sym}^4 \mathcal{S}^\vee$. Let $(\text{Orb}(C))_\mathcal{S} \subset \text{Sym}^4 \mathcal{S}^\vee$ be the relative orbit. Since $G$ is general, the section $s$ will intersect $(\text{Orb}(C))_\mathcal{S}$ only in the interior of the relative orbit. Since $C$ is general
A smooth quartic with $n$ hyperflexes

$$C_{A_6} : (x^2 + yz)^2 + 2yz^3 = 0$$
$$C_{D_6} : z(xy + x^3 + z^3)$$
$$C_{E_6} : y^3z + x^4 + x^2y^2 = 0$$

$C_{AN}$: Nodal cubic union flex line

$C_{flex}$: smooth cubic union flex line

$Q$: Quadrilateral

$C_{D_4}$: a general curve with $D_4$ singularity

Two lines plus conic

A line plus a general cubic

Quartic with $\delta$ nodes and $\kappa$ cusps and no hyperflexes

A smooth quartic with $n$ hyperflexes

General smooth quartic

| Quartic Plane Curve $C$ | $\mathcal{P}_C(c_1, c_2, c_3)$ |
|-------------------------|----------------------------------|
| $C_{A_6} : (x^2 + yz)^2 + 2yz^3 = 0$ | $3 \cdot 112 (9c_1^3 + 12c_1c_2 - 11c_3)(2c_1^3 + c_1c_2 + c_3)$ |
| $C_{D_6} : z(xy + x^3 + z^3)$ | $3 \cdot 64 (18c_1^6 + 33c_1^4c_2 + 12c_1^2c_2^2 - 85c_3^3 - 11c_1c_2c_3 - 7c_3^2)$ |
| $C_{E_6} : y^3z + x^4 + x^2y^2 = 0$ | $2 \cdot 48 (2c_1^3 + c_1c_2 + c_3)(9c_1^3 - 6c_1c_2 + 7c_3)$ |

$C_{AN}$: Nodal cubic union flex line

$p_{C_{AN}} + 2p_{D_6}$

$C_{flex}$: smooth cubic union flex line

$p_{C_{AN}} + 2p_{C_{flex}}$

$Q$: Quadrilateral

$8p_{C_{AN}} - n p_{C_{E_6}}$

$8p_{C_{AN}}$

$\frac{1}{4} (8p_{C_{A_6}} - p_Q)$

$\mathcal{P}_Q + 3p_{C_{D_4}}$

$8p_{C_{A_6}} - 2\delta p_{C_{D_6}} - \kappa p_{C_{flex}}$

$8p_{C_{A_6}}$

$\mathcal{P}_Q + 2p_{C_{D_4}}$

$8p_{C_{A_6}}$

Figure 1. Equivariant classes of orbits of quartic plane curves

| Quartic Plane Curve $C$ | (# Aut$(C) \cdot$ # planar sections of general quartic threefold) |
|-------------------------|---------------------------------------------------------------------|
| $C_{A_6} : (x^2 + yz)^2 + 2yz^3 = 0$ | $3 \cdot 21280$ |
| $C_{D_6} : z(xy + x^3 + z^3)$ | $3 \cdot 7040$ |
| $C_{E_6} : y^3z + x^4 + x^2y^2 = 0$ | $2 \cdot 4800$ |
| $C_{AN}$: Nodal cubic union flex line | $2 \cdot 36480$ |
| $C_{flex}$: smooth cubic union flex line | $2 \cdot 57600$ |
| $Q$: Quadrilateral | $24 \cdot 5600$ |
| $C_{D_4}$: a general curve with $D_4$ singularity | $94080$ |
| Two lines plus conic | $322560$ |
| A line plus a general cubic | $416640$ |
| Quartic with $\delta$ nodes and $\kappa$ cusps and no hyperflexes | $510720 - 2\delta(3 \cdot 7040) - \kappa(2 \cdot 57600)$ |
| A smooth quartic with $n$ hyperflexes | $510720 - n(2 \cdot 4800)$ |
| General smooth quartic | $510720$ |

Figure 2. Number of times we see a particular curve as a planar section of a quartic threefold with specified moduli.
the intersection will consists of reduced points by generic reducedness in characteristic zero. In this paper, we will assume the characteristic is at least 7 (see Section 1.4), and a standard transversality argument which we omit shows that the intersection is also still reduced in this case. Therefore, \( \text{Image}(s) \) and \((\text{Orb}(C))_S\) are smooth at the reduced scheme \( \text{Image}(s) \cap (\text{Orb}(C))_S \), and \( \deg(\Phi) = \int_{G(2,4)} s^*[\text{Orb}(C)]_S \). Expanding the formula for \( p_C \), we get

\[
48384c_1(S)^6 + 88704c_1(S)^4c_2(S) + 32256c_1(S)^2c_2(S)^2 - 34944c_1(S)^3c_3(S) + 2688c_1(S)c_2(S)c_3(S) - 29568c_3(S)^2.
\]

By evaluating on the Grassmannian, we conclude:

\[
\deg \Phi = 48384 \cdot 5 + 88704 \cdot 3 + 32256 \cdot 2 - 34944 + 2688 - 29568 = 510720,
\]

as claimed. 

Beyond computing the number of times a general planar quartic appears on \( X \), our methods also apply to various singular quartics as well. The results are summarized in Figure 2.

**Example 1.3.** The number of tricuspidal curves arising as a section of a quartic threefold is 27520 by applying Kazarian’s theory of multisingularities. More precisely, the number can in principle be deduced from [Kaz03a, Section 8], but the formula for 2-planes in \( \mathbb{P}^4 \) meeting a degree \( d \) hypersurface in a curve with three cusps can be found on Kazarian’s website. From Figure 2 we get \( 510720 - 3 \cdot 2 \cdot 57600 = 6 \cdot 27520 \), accounting for the 6 automorphisms of the tricuspidal quartic. This agrees with Kazarian’s formula. However, for example, our Figure 2 also computes the number of 1-cuspidal and 2-cuspidal curves having prescribed moduli, which is not covered by the theory of multisingularities.

**Remark 1.4.** The class \( p_C \) for \( C \) a general quartic curve, as well as the number 510720 has also been verified independently by the authors using the SAGE Chow ring package [SL15] to implement the resolution used by Aluffi and Faber [AF93b] for smooth plane curves, though now in the relative setting. However, the computations were too cumbersome to verify by hand.

### 1.3. Related Work.

This paper was heavily influenced and inspired by Aluffi and Faber’s computation of degrees of orbit closures of plane curves of arbitrary degree. Zinger also computed the degree of the orbit closure of a general quartic as a special case of interpolating genus 3 plane curves with a fixed complex structure [Zin05].

#### 1.3.1. Planar sections of a hypersurface which have fixed moduli.

Counting linear sections of a hypersurface with fixed moduli has been considered in the case of line sections of a quintic curve [CL08] and generalized to line sections of hypersurfaces of degree \( 2r + 1 \) hypersurfaces in \( \mathbb{P}^r \) [LPST20] by extending the computations in the \( r = 1 \) case (see [AF93a]) to the equivariant setting.
1.3.2. Counting curves with prescribed singularities. In addition to Kazarian’s work [Kaz03a], there have been independent efforts to enumerate curves with various types of punctual singularities, including [BM16, Ker06, Rus03].

1.4. Assumptions on the characteristic of the base field. We will assume that our ground field is the complex numbers \( \mathbb{C} \) has characteristic zero. However, all our techniques should work for perfect fields of finite characteristic \( \geq 7 \), though we have not paid special attention to the matter.

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2. Preliminary Definitions, Conventions, and Notation

In this section, we define equivariant generalizations of predegrees of orbits of hypersurfaces as defined and studied by Aluffi, Faber, and Tzigantchev [Tzi08, AF93a, AF93b, AF00b]. Although we will ultimately only deal with the case of points on a line, and plane cubics and quartics, we provide general definitions for clarity and to set the stage for future work.

2.1. Conventions. We will work over an arbitrary algebraically closed base field \( \mathbb{C} \). By a scheme we will mean a scheme of finite type over \( \mathbb{C} \). By a variety we will mean a reduced, though not necessarily irreducible, scheme. All varieties in the paper will be quasi-projective. If \( X \) is a variety, then a subvariety will be a closed subscheme of \( X \) which is also a variety. As a rule, the projectivization of a vector bundle parametrizes 1-dimensional subspaces, not quotients. If \( \mathcal{V} \to B \) is a vector bundle, and if \( H \) is the hyperplane class associated to the natural \( \mathcal{O}(1) \) on \( \mathcal{V} \), then \( H \) satisfies the Leray relation

\[
H^{r+1} + c_1(\mathcal{V}) \cdot H^r + \cdots + c_{r+1}(\mathcal{V}),
\]

where \( c_i \) are the Chern classes of \( \mathcal{V} \).

Until we explicitly specialize to the case \( r = 2 \), \( V \) will be a fixed \( (r+1) \)-dimensional \( \mathbb{C} \)-vector space. The action of \( GL(V) \) on the space \( \text{Sym}^d V^\vee \) is as follows: If \( F : V \to \mathbb{C} \) is a degree \( d \) form (i.e. an element of \( \text{Sym}^d V^\vee \)), and if \( g \in GL(V) \), then \( g \cdot F \) is defined to be the form \( F \circ g^{-1} : V \to \mathbb{C} \). This action descends to an action of \( PGL(V) \) on the projectivization \( \mathbb{P}\text{Sym}^d V^\vee \), which we will denote by \( g \cdot [F] \). If \( X \subset \mathbb{P}V \) is a degree \( d \) hypersurface, we will often write \( X \in \mathbb{P}\text{Sym}^d V^\vee \) to denote \( [F] \) where \( F \) is any defining equation for \( X \).

The action of \( GL(V) \) on \( \text{End}(V) \) by \textit{post-composition} is as follows: \( g \in GL(V) \) acts on \( \gamma \in \text{End}(V) \) via \( g \circ \gamma \). (This is to distinguish the action by \textit{pre-composition} where \( g \) acts by sending \( \gamma \) to \( \gamma \circ g^{-1} \).) Either action descends to an action of \( PGL(V) \) on \( \mathbb{P}\text{End}(V) \) in the obvious way.
2.2. GL(V)-equivariant Chow classes. Suppose \( Z \subset \text{Sym}^d V^\vee \) is a subvariety preserved by the action of \( GL(V) \), and let \( \mathbb{P}Z \subset \mathbb{P}(\text{Sym}^d V^\vee) \) denote its projectivization.

In the presence of a rank \( r+1 \) vector bundle \( V \to B \) over a base variety \( B \), \( Z \) determines a corresponding relative cycle \( Z_V \subset \text{Sym}^d V^\vee \) as follows: Let \( U \subset B \) denote an open set over which \( V \) can be trivialized. After choosing any particular isomorphism of \( V|_U \) with the product vector bundle \( U \times V \) we obtain a corresponding identification of \( \text{Sym}^d V^\vee|_U \) with \( U \times \text{Sym}^d V^\vee \). In the latter space, we can take \( U \times Z \), and consider it, via the chosen isomorphism, as a locally closed subset of \( \text{Sym}^d V^\vee \). Its closure is denoted

\[
Z_V \subset \text{Sym}^d V^\vee,
\]

and we call it the relative cycle corresponding to \( Z \). \( Z_V \) is well-defined precisely because \( Z \) was preserved by the action of \( GL(V) \). In a similar fashion, we define the projectivized relative cycle \( \mathbb{P}Z_V \subset \mathbb{P}\text{Sym}^d V^\vee \).

Although \( Z_V \) obviously depends on \( B \) and \( V \), its class in \( A^\bullet(B) \cong A^\bullet(\text{Sym}^d V^\vee) \) is a universal expression in chern classes of \( V \) – this is the fundamental input we need from the equivariant intersection theory of Edidin and Graham developed in [EG98a]. In the language of equivariant intersection theory, this universal class is the equivariant fundamental class \( [Z]_{GL(V)} \) in the equivariant Chow ring

\[
A^\bullet_{GL(V)}(\text{Sym}^d V^\vee).
\]

The latter ring is the free polynomial ring \( \mathbb{Z}[c_1, \ldots, c_{r+1}] \), where \( c_i \) are interpreted as the chern classes of the universal vector bundle \( V \) over the classifying stack \( \mathbb{B}GL(V) \). To summarize:

**Definition 2.1.** Given \( Z \) as above, define \( [Z]_{GL(V)} \) to be the polynomial in \( c_1, \ldots, c_{r+1} \) such that the class of \( Z_V \) is \( [Z]_{GL(V)} \) with the chern classes of \( V \) substituted for \( c_1, \ldots, c_{r+1} \). Equivalently, \( [Z]_{GL(V)} \) is the \( GL(V) \)-equivariant class of \( Z \) in \( A^\bullet_{GL(V)}(\text{Sym}^d V^\vee) \cong \mathbb{Z}[c_1, \ldots, c_{r+1}] \).

In a similar fashion to Definition 2.1, we can define an equivariant Chow class of \( \mathbb{P}Z \).

As before, there is a single formula in the chern classes \( c_i \) of \( V \) and the hyperplane class \( H \) on the universal projective bundle \( \mathbb{P}\text{Sym}^d V^\vee \) (\( H \) corresponds to the line bundle \( \mathcal{O}_{\mathbb{P}(\text{Sym}^d V^\vee)}(1) \)) which gives the class of \( [\mathbb{P}Z_V] \in A^\bullet(\mathbb{P}(\text{Sym}^d V^\vee)) \).

**Definition 2.2.** Given \( Z \) as above, define \( [\mathbb{P}Z]_{GL(V)} \) to be the polynomial in \( c_1, \ldots, c_{r+1} \) and \( H \) (of degree \( \leq \binom{d+r}{r} - 1 \) in \( H \)) such that the class of \( \mathbb{P}Z_V \) is \( [\mathbb{P}Z]_{GL(V)} \) with the chern classes of \( V \) substituted for \( c_1, \ldots, c_{r+1} \) and \( c_1 \mathcal{O}_{\mathbb{P}(\text{Sym}^d V^\vee)}(1) \) substituted for \( H \). In equivariant language, \( [\mathbb{P}Z]_{GL(V)} \) is the \( GL(V) \)-equivariant class of \( \mathbb{P}Z \) in

\[
A^\bullet_{GL(V)}(\mathbb{P}(\text{Sym}^d V^\vee)) \cong \mathbb{Z}[c_1, \ldots, c_{r+1}] [H]/(H^{\binom{d+r}{r}} + s_1 H^{\binom{d+r}{r}-1} + \cdots + s_{\binom{d+r}{r}}),
\]

Here \( s_i \) is the \( i \)-th chern class of \( \text{Sym}^d V^\vee \), expressed in terms of the chern classes \( c_i \).
It may seem like $\mathbb{P}Z_{GL(V)}$ contains more information than $Z_{GL(V)}$, but the two are related by a simple algebraic manipulation. Let $u_1, \ldots, u_{r+1}$ denote the formal chern roots of the universal vector bundle $V$ over $BGL(V)$ — in other words, $c_i$ is the $i$-th elementary symmetric expression in $u_1, u_2, \ldots$. Using the inclusion $\mathbb{Z}[c_1, \ldots, c_{r+1}] \hookrightarrow \mathbb{Z}[u_1, \ldots, u_{r+1}]$ where $c_i$ maps to the $i$-th elementary symmetric function, we can express $Z_{GL(V)}$ as a symmetric polynomial in $u_1, \ldots, u_{r+1}$ and similarly $\mathbb{P}Z_{GL(V)}$ as a polynomial in $u_1, \ldots, u_{r+1}$ and $H$ symmetric in the $u_i$’s. With this understood, we have:

**Proposition 2.3** ([FNR05, Theorem 6.1]). We have:

$$Z_{GL(V)}(u_1, \ldots, u_{r+1}) = \mathbb{P}Z_{GL(V)}(u_1, \ldots, u_{r+1}, 0)$$

$$\mathbb{P}Z_{GL(V)}(u_1, \ldots, u_{r+1}, H) = Z_{GL(V)}(u_1 - \frac{H}{d}, \ldots, u_{r+1} - \frac{H}{d}).$$

When $r = 1$ and $2$, we will use letters $(u, v)$ for $(u_1, u_2)$ and $(u, v, w)$ for $(u_1, u_2, u_3)$, respectively.

### 2.3. Weighting by automorphism groups.

**Definition 2.4.** Given $X \subset \mathbb{P}(V)$ a degree $d$ hypersurface, we define

$$\text{Orb}(X) \subset \text{Sym}^d V^\vee$$

to be the $GL(V)$-orbit closure of any defining equation $F$ of $X$. Furthermore, we define $\text{Aut}(X) \subset PGL(V)$ to be the subgroup consisting of those projective automorphisms preserving $X$. Finally, we say that $X$ has full dimensional orbit if the group $\text{Aut}(X)$ is finite.

In this paper, we will exclusively be concerned with hypersurfaces which have full dimensional orbit.

**Definition 2.5.** Let $F \in \text{Sym}^d V^\vee$ be a degree $d$ homogeneous form cutting out $X \subset \mathbb{P}(V)$. Then, define

$$p_X := \begin{cases} 
\# \text{Aut}(X)[\text{Orb}(X)]_{GL(V)} & \text{if } \# \text{Aut}(X) < \infty \\
0 & \text{if } \# \text{Aut}(X) = \infty.
\end{cases}$$

$$P_X := \begin{cases} 
\# \text{Aut}(X)[\mathbb{P}\text{Orb}(X)]_{GL(V)} & \text{if } \# \text{Aut}(X) < \infty \\
0 & \text{if } \# \text{Aut}(X) = \infty.
\end{cases}$$

The polynomials $P_X$ are equivariant generalizations of the notion of predegree, as defined by Aluffi and Faber [AF93b, Definition]:

**Definition 2.6.** The predegree of a hypersurface $X \subset \mathbb{P}V$ having full dimensional orbit is $\# \text{Aut}(X)$ times the degree of the orbit closure $\mathbb{P}\text{Orb}(X)$ in the projective space $\mathbb{P}\text{Sym}^d V^\vee$. If the orbit of $X$ is not full dimensional then we define its predegree to be zero.
**Remark 2.7.** The predegree of a hypersurface $X$ is the coefficient of $H^{\frac{(d+r)-(r+1)^2}{r}}$ in $P_X$. Thus, the equivariant classes contain much more enumerative data than the predegree, namely all the other coefficients. We will critically use the knowledge of the predegree in equivariant arguments.

**2.4. Notation for $GL(V)$-equivariant degenerations.** Our relations among $GL(V)$-equivariant orbit classes will be given by degenerating orbits. We now provide the basic setup for our degenerations.

Let $m \cdot Z, m_1 \cdot Z_1, \ldots, m_k \cdot Z^k \subset \text{Sym}^d V^\vee$ be cycles (i.e. irreducible, closed subvarieties with attached positive multiplicities $m, m_i \in \mathbb{Z}$), each preserved by the action of $GL(V)$. Then we write

$$Z \rightsquigarrow \sum_{i=1}^k Z^i$$

if there exists:

1. An open neighborhood $U \subset \mathbb{A}^1$ containing the point 0,
2. a closed subvariety $W \subset U \times \text{Sym}^d V^\vee$ flat over $U$ and invariant under the action of $GL(V)$ (acting on the second factor) with the property that
3. there exists a point $u \in U$ such that the fiber $W_u \subset \{u\} \times \text{Sym}^d V^\vee$ is equal to $Z$ with multiplicity $m$, and the fiber $W_0 \subset \{0\} \times \text{Sym}^d V^\vee$ (the scheme-theoretic fiber of $W \to U$ over 0) is the union of the $Z^i$’s with $Z^i$ having multiplicity $m_i$.

(The concept of $\rightsquigarrow$ affords its obvious projectivized version for $GL(V)$-equivariant cycles in $\mathbb{P}\text{Sym}^d V^\vee$.)

We will primarily be interested in the notion of $\rightsquigarrow$ in the context were $Z$ is an orbit closure $\text{Orb}(F)$. This is the subject of the next section.

**2.5. Families of orbits.** Our intention here is to gather the basic degeneration tools specific to orbit closures, tools we will repeatedly use throughout the paper. Every degeneration in the paper will implicitly use the framework described below.

Let $(U, 0)$ denote an open neighborhood of 0 in $\mathbb{A}^1$, and suppose

$$\alpha : U \to \mathbb{P}\text{Sym}^d V^\vee$$

is a map, inducing a corresponding family of degree $d$ hypersurfaces

$$\pi : \mathcal{X} \to U.$$

Next, let

$$\mathcal{Y} \subset U \times PGL(V) \times \mathbb{P}\text{Sym}^d V^\vee$$

denote the closed subset consisting of triples $(u, g, X)$ satisfying $X = g \cdot \alpha(u)$. In other words, $\mathcal{Y}$ is the graph of the action map $(u, g) \mapsto g \cdot \alpha(u)$.
Definition 2.8. Define
\[ \mathcal{Y} \subset U \times \mathbb{P} \text{End}(V) \times \mathbb{P} \text{Sym}^d V^\vee \] (1)
to be the closure of \( \hat{\mathcal{Y}} \) (using the natural open inclusion \( \text{PGL}(V) \subset \mathbb{P} \text{End}(V) \)).

Since \( \text{PGL}(V) \subset \mathbb{P} \text{End}(V) \) is open, and since graphs are closed, it follows that \( \hat{\mathcal{Y}} \subset \mathcal{Y} \) is an open set. The map sending \((u, g) \in U \times \text{PGL}(V)\) to \((u, g, g \cdot \alpha(u)) \in \mathcal{Y}\) clearly induces an isomorphism which we denote by \( \iota : U \times \text{PGL}(V) \to \hat{\mathcal{Y}} \).

We let \( \pi_1, \pi_2, \pi_3 \) denote projections of \( \mathcal{Y} \) to the respective factors \( U, \mathbb{P} \text{End}(V), \mathbb{P} \text{Sym}^d V^\vee \). For each \( u \in U \), we let \( \mathcal{Y}_u \) denote the scheme-theoretic fiber \( \pi_1^{-1}(u) \). Observe that, for a general point \( u \in U \), \( \mathcal{Y}_u := \pi_1^{-1}(u) \subset \mathcal{Y} \) is irreducible, and therefore \( \pi_3(\pi_1^{-1}(u)) \) is the orbit closure \( \mathbb{P} \text{Orb}(X_u) \).

Since \( \pi_1 : \mathcal{Y} \to U \) is flat (\( \mathcal{Y} \) is irreducible and \( U \) is a smooth curve), every fiber of \( \pi_1 \) has pure dimension equal to \( \dim \text{PGL}(V) \). The geometry of the special fiber \( \mathcal{Y}_0 \) will be of utmost importance to us.

We let \( A_0 \subset \mathcal{Y}_0 \) denote the irreducible component \( \iota((\{0\} \times \text{PGL}(V))) \). Then, by construction, the map \( \pi_3 : A_0 \to \mathbb{P} \text{Sym}^d V^\vee \) restricts on \( \{0\} \times \text{PGL}(V) \) to the action map \( g \mapsto g \cdot \alpha(0) \), and hence
\[ \pi_3(A_0) = \mathbb{P}\text{Orb}(X_0). \]
Furthermore, in terms of cycles,
\[ \pi_3^*[A_0] = \# \text{Aut}(X_0) \cdot [\mathbb{P}\text{Orb}(X_0)]. \]
Along with \( A_0 \), there may be several other components of the fiber \( \mathcal{Y}_0 := \pi_1^{-1}(0) \), and our aim is to gain a better understanding of these.

To that end, we must introduce the idea of twisting by 1-parameter families. Let \( U^\times = U \setminus \{0\} \), and suppose we have a morphism
\[ \gamma : U^\times \to \text{PGL}(V); \] (2)
any such \( \gamma \) will be called a 1-parameter family. We will abuse notation and let \( \gamma \) also denote its canonical extension \( U \to \mathbb{P} \text{End}(V) \) across \( 0 \in U \). In the presence of a 1-parameter family \( \gamma \), we obtain a new family of hypersurfaces,
\[ \alpha^\gamma : U^\times \to \mathbb{P} \text{Sym}^d V^\vee \] (3)
defined by \( \alpha^\gamma(u) := \gamma(\alpha(0)) \). We call \( \alpha^\gamma \) the twist of \( \alpha \) by \( \gamma \). Again, we will abuse notation and use \( \alpha^\gamma \) to also denote the natural extension to \( U \). Then, \( \alpha^\gamma \) induces a new family of hypersurfaces which we denote by \( \pi^\gamma : X^\gamma \to U \).
Now consider the map

$$\iota^\gamma : U \times PGL(V) \to \mathcal{Y}$$

defined by the formula

$$\iota^\gamma(u, g) = (u, g \circ \gamma(u), g \cdot \alpha^\gamma(u)).$$

Note that the appropriate restriction of $\iota^\gamma$ induces an isomorphism between $U \times PGL(V)$ and $\tilde{\mathcal{Y}}$, the latter being the open set $\pi^{-1}(U) \cap \tilde{\mathcal{Y}}$. Denote by $\iota^\gamma_0 : PGL(V) \to \mathcal{Y}_0$ for the restriction of $\iota^\gamma$ to the subset $\{0\} \times PGL(V)$; simply put,

$$\iota^\gamma_0(g) = (0, g \circ \gamma(0), g \cdot \alpha^\gamma(0)).$$

Under the action of $PGL(V)$ (post-composition on middle factor) on $\mathcal{Y}$, $\iota^\gamma_0$ is the action map for the point $(0, \gamma(0), \alpha^\gamma(0))$.

**Definition 2.9.** We say the 1-parameter family $\gamma$ is full for $\alpha$ if the $PGL(V)$ orbit of the point

$$(0, \gamma(0), \alpha^\gamma(0)) \in \mathcal{Y}$$

is full dimensional. Equivalently, $\gamma$ is full for $\alpha$ if $\iota^\gamma_0$ is quasi-finite onto its image.

**Definition 2.10.** Suppose $\gamma$ is full for $\alpha$. Define

$$A_\gamma \subset \mathcal{Y}_0$$

to be the closure of the orbit of $(0, \gamma(0), \alpha^\gamma(0))$.

By fullness of $\gamma$, $A_\gamma$ is an irreducible component of $\mathcal{Y}_0$ which contains a dense $PGL(V)$-orbit. The next lemma will be used to help us distinguish among irreducible components of $\mathcal{Y}_0$.

**Lemma 2.11.** Suppose that $\gamma_{1,2} : U \to PGL(V)$ are two 1-parameter families which are full for $\alpha$, and let $A_{\gamma_1}$ (resp. $A_{\gamma_2}$) denote the corresponding irreducible component of $\mathcal{Y}_0$ as in Definition 2.10. Further suppose that $\gamma_1(0) \in \mathbb{P}End(V)$ and $\gamma_2(0) \in \mathbb{P}End(V)$ are not translates under the action of $PGL(V)$ on $\mathbb{P}End(V)$ via post-composition.

Then

$$A_{\gamma_1} \neq A_{\gamma_2}.$$ 

**Proof.** We will argue the contrapositive.

Suppose $A_{\gamma_1} = A_{\gamma_2} = A$. Then both points $(0, \gamma_i(0), \alpha^\gamma_i(0))$ are in $A$, and both have full orbits. As $A$ is irreducible and has a dense orbit, it follows that both of the above points are in the same $PGL(V)$ orbit.

Since $\pi_2 : \mathcal{Y}_0 \to \mathbb{P}End(V)$ is $PGL(V)$-equivariant, it follows that $\gamma_1(0) = \gamma_2(0)$ are also in the same $PGL(V)$-orbit in $\mathbb{P}End(V)$, which is what we needed to show. \qed
Remark 2.12. Observe that the condition “$\gamma_1(0)$ and $\gamma_2(0)$ are not $PGL(V)$ translates” is equivalent to the condition that the projective endomorphisms $\gamma_i(0)$ do not have the same kernel subspaces in $\mathbb{P}V$ in the case of the action by post-composition. We will use this observation repeatedly.

Principle 2.13. Suppose $\gamma_i, 1 \leq i \leq n$ are 1-parameter families, full for $\alpha$, and suppose each pair $\gamma_i, \gamma_j$ satisfy the hypothesis in Lemma 2.11. Then, the equivariant class

$$p_{X_u} - \sum_{i=1}^{n} p_{X_{0}^{\gamma_i}}$$

can be represented by a nonnegative sum of equivariant fundamental classes of effective cycles. If, in addition, the predegrees of the hypersurfaces $X_{0}^{\gamma_i}, \ldots, X_{0}^{\gamma_n}$ add up to the predegree of $X_u$, then,

$$p_{X_u} = \sum_{i=1}^{n} p_{X_{0}^{\gamma_i}}.$$

Proof. Recall the variety $\mathcal{Y}$ from Definition 2.8 and $\mathcal{Y} \subset U \times \mathbb{P}Sym^d V^\vee$ its image under projection onto first and third factors ($\pi_1, \pi_3$). In light of the hypotheses in Lemma 2.11 we obtain pairwise distinct irreducible components $A_i \subset Y_0$ of $Y_0, i = 1, \ldots, n$. By their constructions, the composite

$$\pi_3 \circ \iota^\gamma : PGL(V) \to \mathbb{P}Sym^d V^\vee$$

the action map corresponding to each hypersurface $X_{0}^{\gamma_i}$. Therefore, in the cycle $\pi_3|_{Y_0}[\mathcal{Y}_0]$, the coefficient of $[\mathbb{P}Orb(X_{0}^{\gamma_i})]$ is $\text{deg}(PGL(V) \to \mathbb{P}Orb(X_{0}^{\gamma_i}))$, i.e. $\# \text{Aut}(X_{0}^{\gamma_i})$. Since $\pi_3|_{Y_0}[\mathcal{Y}_0] = \# \text{Aut}(X_u)[\mathbb{P}Orb(X_u)]$, we deduce that the cycle

$$\# \text{Aut}(X_u)[\mathbb{P}Orb(X_u)] - \sum_{i=1}^{n} \# \text{Aut}(X_{0}^{\gamma_i})[\mathbb{P}Orb(X_{0}^{\gamma_i})]$$

has an effective representative supported on the irreducible components of $\pi_3(\mathcal{Y}_0)$ not accounted for by the orbits of the hypersurfaces $X^{\gamma_i}$. The effectivity statement of

$$p_{X_u} - \sum_{i=1}^{n} p_{X_{0}^{\gamma_i}}$$

now follows.

Finally, assume that the predegrees of $X_{0}^{\gamma_i}$ add to the predegree of $X_u$. Let $\Lambda \subset \mathbb{P}Sym^d V^\vee$ be a general linear space of codimension $\dim PGL(V)$ intersecting $\mathbb{P}Orb(X_u)$ and $\mathbb{P}Orb(X_{0}^{\gamma_i})$ transversely for all $i$. Then, since $\text{deg} \pi_3^{-1}(\Lambda)$ is the sum of predegrees of the $X_{0}^{\gamma_i}$, and since this degree is constant over $U$, it follows that there are no irreducible
components $B \subset Y_0$ other than the $A_i$ having the property that $\pi_3(B)$ is an irreducible component of $\pi_3(Y)$. Thus, 

$$(\pi_1, \pi_3)(Y) \subset U \times \mathbb{P} \text{Sym}^d V^\vee$$

induces

$$\# \text{Aut}(X_u)[\mathbb{P} \text{Orb}(X_u)] \sim \sum_i \# \text{Aut}(X_0^n)[\mathbb{P} \text{Orb}(X_0^n)],$$

and the equality

$$p_{X_u} = \sum_{i=1}^n p_{X_0^{\gamma_i}}$$

follows.

For many of our applications, Principle 2.13 will suffice. But in Section 5.2, we will find ourselves in a situation that doesn’t quite obey the hypotheses of Principle 2.13, yet the conclusion will still follow. Our adjustment is to replace the compactification $PGL(V) \subset \mathbb{P} \text{End}(V)$ by another variety.

**Definition 2.14.** Define

$$\text{Inv}(V) \subset \mathbb{P} \text{End}(V) \times \mathbb{P} \text{End}(V)$$

to be the closure of the graph of the inversion morphism $i : PGL(V) \to PGL(V)$. In other words, $\text{Inv}(V)$ is the closure of the set of pairs $(g, g^{-1})$, where $g \in PGL(V)$.

The group $PGL(V)$ acts on $\text{Inv}(V)$ by post-composition on the first factor and pre-composition on the second factor, i.e. $h \in PGL(V)$ acts on a pair $(e_1, e_2) \in \text{Inv}(V)$ to give $(h \circ e_1, e_2 \circ h^{-1})$.

The varieties $Y$, $A_\gamma$, and Lemma 2.11 have their immediate generalizations to the $\text{Inv}(V)$ setting, and we will abuse notation and use the same letters whenever we are in this analogous setting.

**Principle 2.15.** Let $\gamma_1, \ldots, \gamma_n$ be 1-parameter families $\gamma_i : U^\times \to PGL(V)$ which are full for $\alpha$ and continue to let $\gamma_i : U \to \text{Inv}(V)$ denote the unique extension over $U$. Suppose furthermore that:

- The points $\gamma_i(0) \in \text{Inv}(V)$ are in distinct $PGL(V)$-orbits of $\text{Inv}(V)$.

Then, the equivariant class $p_{X_u} - \sum_{i=1}^n p_{X_0^{\gamma_i}}$ can be represented by a nonnegative sum of effective cycles. If additionally the predegrees of $X_0^{\gamma_i}$, $i = 1, \ldots, n$ sum up to the predegree of $X_u$, then

$$p_{X_u} = \sum_{i=1}^n p_{X_0^{\gamma_i}}.$$ 

We omit the proof, as it is similar to the proof of Principle 2.13.
2.6. **Specialize to** \( r = 2 \). From here onward, unless specified otherwise, we assume \( V \) is a 3-dimensional vector space. We will use the letter \( C \) with subscripts and superscripts to indicate a plane curve. Apart from the specific plane curves with particular subscripts (e.g. “\( C_{BN} \)”), the simple letter \( C \) may represent different curves in different sections. We hope its meaning will be clear in context.

3. **Known orbit classes of special quartic curves**

Our intention in the section is to get a few critical orbit classes \( p_C \) of curves in our hands without using degeneration. We use Kazarian’s formulas for counting curves with \( A_6, D_6 \) and \( E_6 \) singularities. It is known that in the space of quartic curves, the set of curves with such singularities form three respective full (i.e. 8-) dimensional orbits (Proposition 3.2). Kazarian’s formulas then directly yield the equivariant orbit classes of these three orbits (Corollary 3.4). We also record the computation of \( p_Q \) where \( Q \subset \mathbb{P} \text{Sym}^4 V^\vee \) is a complete quadrilateral, i.e. the union of four general lines.

We begin with the following calculation of Kazarian [Kaz03a, Theorem 1]:

**Proposition 3.1.** Let \( \mathcal{S} \rightarrow B \) be a smooth morphism of varieties whose fibers are smooth surfaces. Let \( L \) be a line bundle on \( \mathcal{S} \) and \( \sigma \) be a section of \( L \) cutting out a family of curves \( \mathcal{C} \subset \mathcal{S} \). The virtual classes \( [Z_{A_6}] \) (respectively \( [Z_{D_6}] \) and \( [Z_{E_6}] \)) supported on points \( p \in \mathcal{S} \) where the fiber of \( \mathcal{C} \rightarrow B \) has an \( A_6 \) (respectively \( D_6 \) and \( E_6 \)) singularity at \( p \) are given by:

\[
[Z_{A_6}] = u(-c_1 + u)(c_2 - c_1u + u^2)(720c_1^4 - 1248c_1^2c_2 + 156c_2^2 - 1500c_1^3u
+ 1514c_1c_2u + 1236c_1^2u^2 - 485c_2u^2 - 487c_1u^3 + 79u^4)
\]

\[
[Z_{D_6}] = 2u(-c_1 + u)(4c_2 - 2c_1u + u^2)(c_2 - c_1u + u^2)(12c_1^2 - 6c_2 - 13c_1u + 4u^2)
\]

\[
[Z_{E_6}] = 3u(-c_1 + u)(2c_1^2 + c_2 - 3c_1u + u^2)(4c_2 - 2c_1u + u^2)(c_2 - c_1u + u^2)
\]

where \( c_i := c_i(T_{\mathcal{S}/B}) \) and \( u = c_1(L) \).

**Proposition 3.2.** The set of irreducible quartic plane curves with an \( A_6 \) (respectively \( D_6 \) and \( E_6 \)) singularity having full dimensional orbit consists of a single \( \text{PGL}(V) \) orbit.

**Proof.** The case of \( D_6 \) singularities is clear, since one of the branches of the singularity must be a line, and therefore \( D_6 \) is the union of a nodal cubic along with one of the branch lines at the node. Hence such a curve must be the union of a nodal cubic with a tangent branch line, constituting a single orbit.

The fact that irreducible plane quartics with an \( A_6 \) or \( E_6 \) singularity form an irreducible subvariety of codimension 6 in the projective space \( \mathbb{P} \text{Sym}^4 V^\vee \) of all quartics follows from explicit classification, for example [NST1 Section 3.4].

That an orbit of a general curve with such a singularity is 8-dimensional can be checked by the formulas for their pre-degrees as found in [AF00b, Examples 5.2 and 5.4], which gives a nonzero result. This proves the proposition. \( \Box \)
Corollary 3.4. Let $C_{A_6}$ and $C_{E_6}$ denote the rational quartic curves with an $A_6$ and $E_6$ singularity, respectively, which have full dimensional orbits. By Proposition 3.2, this definition is well-defined up to projective equivalence.

There are explicit equations for $C_{A_6}$ and $C_{E_6}$ (see for example [NS11, Section 3.4]):

\begin{align}
C_{A_6} : (X^2 + YZ)^2 + 2YZ^3 &= 0 \\
C_{D_6} : Z(XYZ + X^3 + Z^3) &= 0 \\
C_{E_6} : Y^3Z + X^4 + X^2Y^2 &= 0.
\end{align}

Corollary 3.4. We have

\begin{align}
\rho_{C_{A_6}} &= 3 \cdot 112(9c_1^3 + 12c_1c_2 - 11c_3)(2c_1^3 + c_1c_2 + c_3) \\
\rho_{C_{D_6}} &= 3 \cdot 64(18c_1^6 + 33c_1^4c_2 + 12c_1^2c_2^2 - 85c_1^3c_3 - 11c_1c_2c_3 - 7c_3^2) \\
\rho_{C_{E_6}} &= 2 \cdot 48(2c_1^3 + c_1c_2 + c_3)(9c_1^3 - 6c_1c_2 + 7c_3),
\end{align}

where $\# \text{Aut}(C_{A_6}) = \# \text{Aut}(C_{D_6}) = 3$ and $\# \text{Aut}(C_{E_6}) = 2$.

We will also verify the result for $\rho_{C_{D_6}}$ independently in Section 6.

Proof. We apply Proposition 3.1 to the case where $B$ is an arbitrary base variety and $V \to B$ is an arbitrary rank 3 sub-bundle. Let $T$ be the relative tangent bundle of $\mathbb{P}(V) \to B$. By the splitting principle and the relative Euler exact sequence for projective bundles, we get:

\begin{align}
c_1(T) &= c_1(V) + 3c_1(\mathcal{O}_{\mathbb{P}(V)}(1)) \\
c_2(T) &= c_2(V) + 2c_1(V)c_1(\mathcal{O}_{\mathbb{P}(V)}(1)) + 3c_1(\mathcal{O}_{\mathbb{P}(V)}(1))^2.
\end{align}

Now, we substitute $u = 4c_1(\mathcal{O}_{\mathbb{P}(V)}(1))$ in the formulas for $[Z_{A_6}]$, $[Z_{D_6}]$ and $[Z_{E_6}]$ Proposition 3.1 and apply push-forward along the projection $\mathbb{P}(V) \xrightarrow{\pi} B$. This yields

\begin{align}
\pi_*[Z_{A_6}] &= 112(9c_1^3(V) + 12c_1(V)c_2(V) - 11c_3(V))(2c_1(V)^3 + c_1(V)c_2(V) + c_3(V)) \\
\pi_*[Z_{D_6}] &= 64(18c_1(V)^6 + 33c_1(V)^4c_2(V) + 12c_1(V)^2c_2(V)^2 - 85c_1(V)^3c_3(V) \\
&\quad - 11c_1(V)c_2(V)c_3(V) - 7c_3(V)^2) \\
\pi_*[Z_{E_6}] &= 48(2c_1(V)^3 + c_1(V)c_2(V) + c_3(V))(9c_1(V)^3 - 6c_1(V)c_2(V) + 7c_3(V)).
\end{align}

Now, $\pi_*[Z_{A_6}]$, $\pi_*[Z_{D_6}]$, and $\pi_*[Z_{E_6}]$ respectively give the formulas for $[\text{Orb}(C_{A_6})]_{\text{GL}(V)}$, $[\text{Orb}(C_{D_6})]_{\text{GL}(V)}$, $[\text{Orb}(C_{E_6})]_{\text{GL}(V)}$, as they are also the result of pulling back the relative cycles $(\text{Orb}(C_{A_6}))_V$, $(\text{Orb}(C_{D_6}))_V$, and $(\text{Orb}(C_{E_6}))_V$ under a generic section $B \to \text{Sym}^4 V^\vee$.

The statement on the automorphisms of $C_{A_6}$ and $C_{E_6}$ come from a direct analysis of equations. Alternatively, one could compare the predegrees of $C_{A_6}$ and $C_{E_6}$ with the projective versions of $[Z_{A_6}]$ and $[Z_{E_6}]$ using [AF00b, Examples 5.2 and 5.4] and Proposition 2.3. \qed
In order to calculate the orbit class of a general quartic with a triple point, we will need to know \( p_Q \) in the case where \( Q \) is the union of four lines with no three concurrent, i.e. a complete quadrilateral. The method is simply to “present” the orbit closure \( \mathbb{P} \text{Orb}(Q) \) by a more accessible variety.

**Proposition 3.5.** Let \( Q \) be the union of four lines, no three concurrent. Then,

\[
p_Q = 24 \cdot 16(18c_1^6 + 33c_1^4c_2 + 12c_1^2c_2^2 + 131c_1^2c_3 + 153c_1c_2c_3 - 147c_3^2).
\]

Here, \( \# \text{Aut}(Q) = 24 \).

**Proof.** We will closely follow the ideas in [FNR06, Theorem 3.1]. Let \( \mathcal{V} \to B \) be an arbitrary rank 3 vector bundle. Consider the map

\[
\phi : \mathbb{P}(\mathcal{V}^\vee)^4 \to \mathbb{P} \text{(Sym}^4 \mathcal{V}^\vee),
\]

which, fiber by fiber over \( B \), restricts to the map induced by sending a tuple of linear forms \((L_1, \ldots, L_4)\) to the quartic form \( L_1 \cdot L_2 \cdot L_3 \cdot L_4 \). Then, \( \phi \) maps 4! to 1 onto \( \mathbb{P} \text{Orb}(Q)_\mathcal{V} \), so \( \mathbb{P} \text{Orb}(Q) \big|_\mathcal{V} = \frac{1}{24} \phi_* (1) \).

Let \( H = c_1 \mathcal{O}_{\mathbb{P} \text{(Sym}^4 \mathcal{V}^\vee)}(1) \) and set

\[
\alpha := H^{14} + c_1(\text{Sym}^4 \mathcal{V}^\vee)H^{13} + \cdots + c_{14}(\text{Sym}^4 \mathcal{V}^\vee).
\]

The Leray relation states that \( \alpha H + c_{15}(\text{Sym}^4 \mathcal{V}^\vee) = 0 \), and it follows from this that the integral

\[
\int_{\mathbb{P}(\text{Sym}^4 \mathcal{V}^\vee) \to B} \alpha \cdot \beta
\]

returns the “constant term” (with respect to \( H \)) of any class \( \beta \). By this, we mean that any class \( \beta \in A^\bullet(\mathbb{P}(\text{Sym}^4 \mathcal{V}^\vee)) \) can be written as a polynomial in \( H \) of degree at most 14, with coefficients being pullbacks of classes of \( A^\bullet(B) \) and that integrating against \( \alpha \) and pushing forward to \( B \) extracts the \( H^0 \) or constant term of \( \beta \). (We use the notation \( \int_X \to Y \) to denote pushforward of classes along a map \( X \to Y \).)

To finish, we choose \( \beta := \frac{1}{24} \phi_* (1) \) and apply the push-pull formula to reduce our problem to the evaluation of

\[
\frac{1}{24} \int_{\mathbb{P}(\mathcal{V}^\vee)^4 \to B} \phi^*(\alpha).
\]

This evaluation is now standard (and we leave it to the reader), given that \( \phi^* H = h_1 + h_2 + h_3 + h_4 \), where \( h_i \) is the pullback of the relative hyperplane class under the projection \( p_i : \mathbb{P}(\mathcal{V}^\vee)^4 \to \mathbb{P}(\mathcal{V}^\vee) \) onto the \( i \)-th factor. The end result yields \( p_Q \) as stated in the proposition, in light of Proposition 2.3. \( \square \)
4. Degeneration I: Splitting off a line

In this section, we analyze our first degeneration: We investigate how the orbit closure specializes as a degree $d$ smooth curve specializes to a general degree $e$ smooth curve together with $d-e$ general lines. Let $U = \mathbb{A}^1$, with coordinate $t$ vanishing at 0. We will follow the terminology and framework of Section 2.5.

4.1. Framework. Let $F(X,Y,Z)$ and $G(X,Y,Z)$ be forms of degrees $d-1$ and $d$ respectively, and assume $G$ does not vanish identically on the line $\{X = 0\}$. Let

$$\alpha : U \to \mathbb{P} \text{Sym}^d V^\vee$$

be given by the formula $t \mapsto X F + t G$. Finally, let

$$\gamma : U^\times \to \text{PGL}(V)$$

be the 1-parameter family of matrices

$$\begin{pmatrix} t^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We maintain this framework throughout the rest of the section.

**Lemma 4.1.** In the setting of Section 4.1, the curve $\alpha \gamma(0) \in \mathbb{P} \text{Sym}^d V^\vee$ has equation $X \cdot F(0,Y,Z) + G(0,Y,Z) = 0$.

**Proof.** Unraveling the definition of the family of curves $\alpha \gamma : U \to \mathbb{P} \text{Sym}^d V^\vee$, it suffices to prove

$$\lim_{t \to 0} t^{-1}((tX)F(tX,Y,Z) + tG(tX,Y,Z)) = X \cdot F(0,Y,Z) + G(0,Y,Z),$$

which is immediate. \qed

Let us interpret Lemma 4.1 in geometric terms, under the further condition that $F$ and $G$ are suitably general. The original family $\alpha$ represents a general degree $d$ curve $G = 0$ specializing to a reducible curve containing a line, $X \cdot F = 0$. Upon twisting by $\gamma$, the new family $\alpha \gamma$ now specializes the same general curve to the curve $X \cdot F(0,Y,Z) + G(0,Y,Z)$, which is a general curve among those possessing a multiplicity $(d-1)$ singular point.

**Remark 4.2.** Notice that the limiting endomorphism $\gamma(0) \in \mathbb{P} \text{End}(V)$ is given by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and therefore the kernel space of $\gamma(0)$ is the line $X = 0$ which was a component of the curve $\alpha(0)$. 

We can slightly generalize the analysis in Lemma 4.1 to the situation where a general degree \( d \) curve \( C \) specializes to a curve \( D \) (via a map \( \beta : U \to \mathbb{P} \text{Sym}^d V^\vee \)), where \( D \) is the union of a general degree \( e \geq 1 \) curve along with \( d - e \) generally chosen lines \( L_1, \ldots, L_{d-e} \). By adapting the family of matrices \( \gamma \) to this more general situation, we obtain \( \gamma_i : U^\times \to \text{PGL}(V) \), one per each line \( L_i \), satisfying:

1. \( \beta \gamma_i(0) \) is a degree \( d \) curve general among those with a multiplicity \( d-1 \) point, and
2. the kernel of the endomorphism \( \gamma_i \) is the line \( L_i \subset \mathbb{P}V \). (see Remark 4.2)

**Proposition 4.3.** Assume \( d \geq 4 \) and let \( C \) be a general degree \( d \) curve, \( C_{d-1} \) a general degree \( d \) curve possessing a point of multiplicity \( d-1 \), and \( D \) the union of a general degree \( e \geq 0 \) curve together with \( d - e \) general lines. Then,

\[
p_C = (d-e)p_{C_{d-1}} + p_D.
\]

**Proof.** This will be a direct application of Principle 2.13. Let \( \beta : U \to \mathbb{P} \text{Sym}^d V^\vee \) and \( \gamma_i \) be as in the generalized setup immediately prior to the statement of the proposition.

The hypotheses of Principle 2.13 are met:

1. \( C_{d-1} \) has full dimensional orbit, as does \( D \), (thanks to the assumption \( d \geq 4 \)).
2. The kernels of \( \gamma_i(0) \) are distinct, namely the lines \( L_i \).
3. It remains to check the compatibility of predegrees. For this, (by further specializing \( D \) to a union of \( d \) general lines) it suffices to consider the case \( e = 0 \), where now the task remaining is to check that the predegree of a general degree \( d \) curve \( C \) is \( d \) times the predegree of \( C_{d-1} \), a general degree \( d \) curve with a point of multiplicity \( d - 1 \) plus the predegree of the union of \( d \) general lines. Finally, this last check is accomplished by plugging into the formulas in [AF00b, Examples 3.1, 4.2] and [AF93b].

The proposition follows from Principle 2.13.

4.2. **Summary.** The proof of Proposition 4.3 provides a comprehensive understanding of the \( t \to 0 \) flat limit of orbit closures \( \mathbb{P}\text{Orb}(C_t) \), \( t \in U \) if \( C_t \) is a family of general curves specializing to a curve \( D \) general among those which contain \( d - e \) lines as components. Apart from \( \mathbb{P}\text{Orb}(D) \), we find \( d - e \) other orbits \( \mathbb{P}\text{Orb}(C_i) \), where \( C_i \) are general among curves possessing a \( d - 1 \)-fold point.

5. **Degeneration II: Acquiring nodes and cusps**

In this section, we establish the effect of acquiring an ordinary node or cusp (a cusp is a singularity with analytic equation \( y^2 = x^3 \)) on the polynomial \( p_C \), in the case of arbitrary \( d \geq 4 \). A node singularity \( p \) of a plane curve \( C \) is called *ordinary* if both tangent lines intersect \( C \) with multiplicity 3 at \( p \). Similarly, we call a cusp singularity *ordinary* if no line meets it with multiplicity \( \geq 4 \). Throughout, \( U \) will denote an appropriate open
neighborhood of 0 in $\mathbb{A}^1$ and $t$ will denote a coordinate around 0. We let $R$ denote the coordinate ring of $U$, and we let $v : R \to \mathbb{N}$ denote the valuation corresponding to the point $0 \in U$.

5.1. Framework and summary of main results. We assume 

$$\alpha : U \to \mathbb{P} \text{Sym}^d V^\vee$$

induces a family of curves $\pi : C \to U$ with the following properties:

1. The curve $C_u := \pi^{-1}(u)$, $u \in U$ general, is a smooth curve with no hyperflexes, and
2. the curve $C_0 := \pi^{-1}(0)$ has $\delta$ ordinary nodes and $\kappa$ ordinary cusps, and
3. $C$ is a smooth surface, and $C_0$ has no hyperflexes.

Certain curves of special significance will arise, so we collect their definitions here.

**Definition 5.1.** Define the curves $C_{BN}$, $C_{AN}$, $C_{\text{flex}}[j]$, $j \in \mathbb{C}$ as:

1. $C_{BN} : Z^{d-3}(XYZ + X^3 + Z^3) = 0$
2. $C_{AN} : Z^{d-3}(Y^2Z - X^3 + X^2Z) = 0$
3. $C_{\text{flex}}[j]$: This is the union of a smooth cubic curve with $j$-invariant $j$ along with one of its flex lines, the line taken with multiplicity $(d - 3)$.

Although these curves depend on $d$, we have suppressed it from the notation, and hope $d$ is clear from context. In geometric terms, $C_{BN}$ is a nodal cubic union a $d - 3$ line tangent to one of the branches at the node, taken with multiplicity $d - 3$. $C_{AN}$ is similarly a nodal cubic curve along with one of its three flex lines (at a smooth point), the line taken with multiplicity $(d - 3)$.

In fact, it will turn out that $p_{C_{\text{flex}}[j]}$ is independent of $j \in \mathbb{C}$. This follows from the following proposition, which is proven in Section 5.5 below.

**Proposition 5.2.** Keep the setting above. For every $j \in \mathbb{C}$,

$$\# \text{Aut}(C_{\text{flex}}[j]) \cdot [\mathbb{P} \text{Orb}(C_{\text{flex}}[j])] \sim \# \text{Aut}(C_{AN})[\mathbb{P} \text{Orb}(C_{AN})] + \# \text{Aut}(C_{BN})[\mathbb{P} \text{Orb}(C_{BN})]$$

and therefore

$$p_{C_{\text{flex}}[j]} = p_{C_{AN}} + 2p_{C_{BN}}.$$

The main result of this section, however, is to prove:

**Theorem 5.3.** Assume the setting above. Then

$$p_{C_u} = p_{C_0} + 2\delta \cdot p_{C_{BN}} + \kappa \cdot p_{C_{\text{flex}}[j]}.$$
5.2. **Degeneration to a node.** In this subsection, we assume

$$\alpha_{\text{node}} : U \to \mathbb{P} \text{Sym}^d V^\vee$$

is a family of curves cut out by a degree $d$ form $F(X,Y,Z)$ with coefficients in $R$ the coordinate ring of $U$, with the properties:

1. The curve $C_0 := \alpha_{\text{node}}(0)$ has an ordinary node at $[0 : 0 : 1] \in \mathbb{P}V$ with branch lines $X = 0$ and $Y = 0$, and
2. The total space $\mathcal{C}$ of the family of curves is smooth at the node of $C_0$.

In what follows, to ease exposition, rather than making a base changes $s^n = t$, we will abuse notation and work with the fractional powers $t^{1/n}$. We also extend the valuation $v$ to such fractional expressions in the obvious way.

**Lemma 5.4.** Maintain the setup immediately prior, and let $\gamma_1(t)$ denote the family of matrices

$$
\begin{pmatrix}
t^{-1/3} & 0 & 0 \\
0 & t^{-2/3} & 0 \\
0 & 0 & 1
\end{pmatrix}.
$$

Then, the limiting plane curve $\alpha_{\text{node}}^{\gamma_1}(0) \in \mathbb{P} \text{Sym}^d V^\vee$ has equation

$$\lim_{t \to 0} t^{-1}(F(t^{1/3}X, t^{2/3}Y, Z)).$$

This curve is projectively equivalent to $C_{BN}$.

Similarly, if $\gamma_2(t)$ is the family of matrices

$$
\begin{pmatrix}
t^{-2/3} & 0 & 0 \\
0 & t^{-1/3} & 0 \\
0 & 0 & 1
\end{pmatrix}
$$

then $\alpha_{\text{node}}^{\gamma_2}(0)$ is the curve defined by

$$\lim_{t \to 0} t^{-1}(F(t^{2/3}X, t^{1/3}Y, Z))$$

and is also projectively equivalent to $C_{BN}$.

**Proof.** It suffices to just do one of the two cases – we will prove the $\gamma_2$ case. We know the following:

1. The coefficient $a_{ij}$ of each monomial $X^iY^jZ^{d-i-j}$ occurring in $F(X,Y,Z)$ is an element of $R$, and therefore have non-negative integer valuations.
2. The assumptions on $C_0$ in the description of $\alpha_{\text{node}}$ imply:

   $$v(a_{0,0}), v(a_{1,0}), v(a_{0,1}), v(a_{2,0}), v(a_{0,2}) \geq 1.$$ 

3. Since the node singularity of $C_0$ is assumed to be ordinary, $v(a_{3,0}) = v(a_{0,3}) = 0$.
4. Since $\mathcal{C}$ is smooth at the node, it follows that $v(a_{0,0}) = 1$. 

Given these constraints, a direct check now shows that \( \frac{2}{3}i + \frac{1}{3}j - 1 + v(a_{i,j}) \) is zero if and only if \((i,j) \in \{(0,3), (1,1), (0,0)\}\) and is strictly positive otherwise. Therefore the result of substituting \( t = 0 \) into the expression in the limit produces a nodal cubic with \((d - 3)\)-fold branch line, as claimed. \( \square \)

**Remark 5.5.** In the case \( d = 4 \), any curve \( C_{D_6} \) (Definition 3.3) is equal to the curve \( C_{BN} \).

**Remark 5.6.** In the context of Lemma 5.4, observe that the limiting endomorphism \( \gamma_i(0) \) has kernel given by the line \( Y = 0 \), which is one of the branches of the node in \( C_0 \) while the limiting endomorphism \( \gamma_2(0) \) has kernel given by the line \( X = 0 \), which is the other branch. Meanwhile, the \( t \to 0 \) limit of both \( \gamma_i^{-1} \) have images equal to the node point \([0 : 0 : 1]\).

### 5.3. The degree of \( \mathbb{P} \text{Orb}(C_{BN}) \).

In light of Theorem 5.3, in order to employ the strategy implicit in Principle 2.13 we will need to compute the degree of the orbit closure \( \mathbb{P} \text{Orb}(C_{BN}) \). In principle, this can be deduced by applying the algorithm of Aluffi and Faber in [AF00b]. We provide an independent calculation in this section, as an extra check.

**Proposition 5.7.** Let \( d \geq 4 \). As a function of the degree \( d \), the degree of \( \mathbb{P} \text{Orb}(C_{BN}) \) is the quadratic polynomial 

\[
24 + 144 \cdot (d - 3) + 140 \cdot (d - 3)^2.
\]

The predegree of \( C_{BN} \) is 

\[
3(24 + 144 \cdot (d - 3) + 140 \cdot (d - 3)^2).
\]

We will prove Proposition 5.7 in pieces below. Observe that the calculation of the degree of \( \mathbb{P} \text{Orb}(C_{BN}) \) implies the assertion on the predegree thanks to the fact, left to the reader to check, that the curve \( C_{BN} \) has order 3 automorphism group.

First, we show that such a quadratic expression in \( d \) exists in the first place.

**Lemma 5.8.** Let \( d \geq 4 \). As a function of the degree \( d \), the degree of the orbit closure \( \mathbb{P} \text{Orb}(C_{BN}) \subset \mathbb{P} \text{Sym}^d V^\vee \) is a quadratic polynomial

\[
a + b \cdot (d - 3) + c \cdot (d - 3)^2
\]

with \( a, b, c \geq 0 \).

Explicitly \( a, b, c \) are the answers to the following enumerative problems:

1. \( a \) is twice the number of singular cubics through 8 general points in \( \mathbb{P}V \), i.e. \( a = 24 \).
2. \( b \) is \( \binom{8}{2} \) times the number of nodal cubics through 7 general points in \( \mathbb{P}V \) with a nodal branch line containing a fixed 8-th general point.
3. \( c \) is \( \binom{8}{2} \) times the number of nodal cubics through 6 general points having a specified line as a branch of the node.

**Proof.** Let

\[
\Delta \subset \mathbb{P} \text{Sym}^3 V^\vee \times \mathbb{P} V^\vee \times \mathbb{P} V
\]
denote the 8-dimensional variety which is the closure of the set of triples \((C, L, p)\) where \(C\) is a nodal cubic curve singular at the point \(p \in \mathbb{P}V\) and \(L \subset \mathbb{P}V\) is a line containing \(p\) whose intersection multiplicity with \(C\) is strictly greater than 2.

The variety \(\Delta\) has three natural projection maps \(p_1, p_2, p_3\) to the three respective factors of \(\mathbb{P} \text{Sym}^3 V^\vee \times \mathbb{P}V^\vee \times \mathbb{P}V\). Let \(H\) denote the divisor class on \(\Delta\) corresponding to \(p_1^* \mathcal{O}(1)\). Similarly, let \(h\) denote the divisor class \(p_2^* (\mathcal{O}(1))\).

Let 
\[
\nu : \mathbb{P} \text{Sym}^3 V^\vee \times \mathbb{P}V^\vee \to \mathbb{P} \text{Sym}^d V^\vee
\]
denote the map which sends a pair \((C, L)\) to the degree \(d\) curve \(C \cup (d - 3) \cdot L\). Then the composite map
\[
\nu \circ (p_1, p_2) : \Delta \to \mathbb{P} \text{Sym}^d V^\vee
\]
is such that the divisor class corresponding to \([\nu \circ (p_1, p_2)]^* \mathcal{O}(1)\) is \(H + (d - 3)h\). Furthermore, \(\nu \circ (p_1, p_2)\) is birational onto its image, and its image is precisely \(\mathbb{P} \text{Orb}(C_{BN})\).

Therefore, we conclude that the degree of \(\mathbb{P} \text{Orb}(C_{BN})\) is given by the intersection number \(\int_\Delta (H + (d - 3)h)^8\) on \(\Delta\).

Since \(h^3 = 0\) on \(\Delta\), this latter intersection number is equal to
\[
\int_\Delta H^8 + 8(d - 3)H^7h + \binom{8}{2}(d - 3)^2H^6h^2.
\]

The numbers \(a, b, c\) appearing in the statement of the lemma correspond to the monomials \(H^8, H^7h, H^6h^2\) evaluated on \(\Delta\). Each monomial is straightforwardly interpreted as the solution to certain enumerative problems:
\[
H^8 = 2\# \{\text{singular cubics through 8 points}\},
\]
where the coefficient of 2 arises because \(p_1 : \Delta \to \mathbb{P} \text{Sym}^3 V^\vee\) is 2 : 1 onto its image. Furthermore,
\[
H^7h = \# \{\text{nodal cubics through 7 points with a nodal branch line containing a fixed 8th point}\}
\]
\[
H^6h^2 = \# \{\text{nodal cubics through 6 points with specified nodal branch line}\}.
\]

This proof of the lemma is complete, after noting that the exact value of \(a\) comes from the fact that there are twelve nodal cubics in a general pencil of cubics [Wri08]. □

Obviously, our task is now to establish the numbers \(b, c\) in Lemma 5.8.

Lemma 5.9. The sum \(a + b + c\) in Lemma 5.8 is 308.

Proof. To compute \(a + b + c\), we must specialize the general formula in Lemma 5.8 to the case \(d = 4\). In this case, \(C_{BN}\) is also the curve \(C_{D_6}\). To calculate the degree of \(\mathbb{P} \text{Orb}(C_{D_6})\) in the space \(\mathbb{P} \text{Sym}^4 V^\vee\) of quartic plane curves, we apply Corollary 3.4 together with Proposition 2.3. Explicitly, we take
\[
\frac{1}{\# \text{Aut}(C_{D_6})} p_{C_{D_6}} = 64(18c_1^6 + 33c_4^2c_2 + 12c_1^2c_2^2 - 85c_3^3c_1 - 11c_1c_2c_3 - 7c_3^2)
\]
from Corollary 3.4, then make the substitution $c_1 \mapsto u + v + w$, $c_2 \mapsto uv + uw + vw$, $c_3 \mapsto uvw$ followed by the substitutions $u \mapsto u - \frac{H}{4}$, $v \mapsto v - \frac{H}{4}$, $w \mapsto w - \frac{H}{4}$, and then extract the coefficient of $H^6$. The end result is 308.

\[\square\]

**Proposition 5.10.** The coefficient $c$ in Lemma 5.8 is $5 \cdot \left(\binom{8}{2}\right)$.

**Proof.** By Lemma 5.8 $c$ amounts to the following enumerative fact: Fix 6 general points $p_1, \ldots, p_6$ in $\mathbb{P}V$ and fix a general line $L \subset \mathbb{P}V$. Then there are 5 cubics containing the points $p_i$ which are singular at a point on $L$ and meeting $L$ with multiplicity $\geq 3$ at the singular point.

To prove this fact, we re-express the statement as the degree of the degeneracy locus of a map between two rank 4 vector bundles $e : A \to B$ on the line $L$. We omit the transversality argument implicit in this re-expression, and leave it to the reader.

The vector bundle $A$ is simply the trivial vector bundle whose fiber at any point $p \in L$ is the vector space $H^0(\mathbb{P}V, \mathcal{I}_{p_1, \ldots, p_6}(3))$ of cubic curves containing the 6 points $p_1, \ldots, p_6$.

We next describe the second vector bundle $B$ as a certain bundle of jets. For each point $p \in L$, let

\[\mathcal{J}_p \subset \mathcal{O}_{\mathbb{P}V}\]

denote the ideal defining the divisor $3p$ in $L$, and let

\[\mathfrak{m}_p^2 \subset \mathcal{O}_{\mathbb{P}V}\]

denote the square of the ideal sheaf of $p$. Let

\[W_p \subset Z_p\]

denote the subschemes defined by $\mathcal{J}_p$ and $\mathcal{J}_p \cap \mathfrak{m}_p^2$ respectively.

We first define $B'$ to be the rank 3 vector bundle on $L$ whose fiber at a point $p$ is given by:

\[B'\big|_p = H^0(\mathbb{P}V, \mathcal{O}(3))/H^0(\mathbb{P}V, \mathcal{J}_p(3)),\]

Finally, we define $B$ to be the rank 4 vector bundle on $L$ whose fiber at a given point $p$ is

\[B\big|_p = H^0(\mathbb{P}V, \mathcal{O}(3))/H^0(\mathbb{P}V, (\mathcal{J}_p \cap \mathfrak{m}_p^2)(3)).\]

Our next task is to compute the degree of $B$. The quotient $\mathcal{J}_p/((\mathcal{J}_p \cap \mathfrak{m}_p^2)$ can naturally be identified with the restriction of the conormal space $(\mathcal{I}_L/\mathcal{I}_L^2)|_p$: In local affine coordinates $(x, y)$, if $L$ is the line $x = 0$ and $p$ is the origin, then $\mathcal{J}_p = (x, y^3), \mathcal{J}_p \cap \mathfrak{m}_p^2 = (x^2, xy, y^3)$, and $\mathcal{J}_p/\mathcal{J}_p \cap \mathfrak{m}_p^2$ is generated by $\tilde{x}$, the local generator for $(\mathcal{I}_L/\mathcal{I}_L^2)|_p$. 
Putting these observations together, we obtain a short exact sequence of vector bundles:

$$0 \rightarrow \mathcal{I}_L / \mathcal{I}_L^2 \otimes \mathcal{O}_L(3) \rightarrow \mathcal{B} \rightarrow \mathcal{B}' \rightarrow 0. \quad (9)$$

Therefore, the degree of $\mathcal{B}$ (as vector bundle on $L$) is equal to the degree of $\mathcal{B}'$ plus the degree of the line bundle $\mathcal{I}_L / \mathcal{I}_L^2 \otimes \mathcal{O}_L(3)$. The latter clearly has degree 2. $\mathcal{B}'$ is the standard second order jet bundle for the line bundle $\mathcal{O}_L(3)$, and is easily seen to have degree 3. Therefore, the degree of $\mathcal{B}$ is 5.

Furthermore, in reference to the affine coordinates above, notice that a general cubic polynomial in the ideal $(x^2, xy, y^3)$ is precisely a cubic singular at $(0, 0)$ having one branch being the line $x = 0$.

At last, we take the map $e : A \rightarrow B$ induced by the natural quotient map over each point $p$. By the observation in the previous paragraph, the locus of points where $e$ drops rank is precisely the number $c$. Since $A$ is trivial, the number of points where $e$ drops rank is the degree of $B$, which is 5. The lemma now follows. □

**Proof of Proposition 5.7.** Since $a = 24$ and $c = 5 \cdot 28 = 140$ (from Proposition 5.10), it follows that $b = 144$ thanks to Lemma 5.9, and the proposition is proved. □

5.4. **Degeneration to a cusp.** Recall that a cusp singularity on a plane curve is called ordinary if no line meets it with multiplicity $\geq 4$. Let $U \subset \mathbb{A}^1$ be a suitable neighborhood of 0 with coordinate $t$. As in Section 5.2, we will write $R$ for the coordinate ring of $U$, with valuation $v$ corresponding to 0, and we abuse notation and use fractional powers of $t$ to indicate base changes $s^n = t$.

Throughout this subsection, we assume

$$ \alpha_{\text{cusp}} : U \rightarrow \mathbb{P} \text{Sym}^d V^\vee $$

is a family of curves defined by a degree $d$ homogeneous form $F(X, Y, Z)$ with coefficients in $R$, obeying:

1. The curve $C_0 := \alpha_{\text{cusp}}(0)$ has an ordinary cusp singularity at $[0 : 0 : 1]$ meeting the line $Y = 0$ to order 3,
2. the total space $\mathcal{C}$ of the family is smooth at the cusp point on $C_0$.

**Lemma 5.11.** Maintain the setting above, and let $\gamma(t)$ denote the family of matrices

$$ \begin{pmatrix} t^{-1/3} & 0 & 0 \\ 0 & t^{-1/2} & 0 \\ 0 & 0 & 1 \end{pmatrix} $$

Then the curve $\alpha_{\text{cusp}}^\gamma(0)$ is defined by the equation

$$ \lim_{t \to 0} t^{-1}(F(t^{1/3}X, t^{1/2}Y, Z)) $$

and is projectively equivalent to the curve $C_{\text{flex}}[0] : Z^{d-3}(X^3 + Y^2Z + Z^3) = 0$. (See Definition 5.4.)
Proof. The proof runs exactly parallel to the proof of Lemma 5.4 so we omit it. □

Remark 5.12. Notice that the limit endomorphism \( \gamma(0) \) has as kernel space the line \( Y = 0 \), i.e. the cuspidal tangent of the cusp of \( C_0 \). Meanwhile, the \( t = 0 \) limit of \( \gamma^{-1} \) has as its image the cusp point \([0 : 0 : 1] \).

5.5. Degenerating the orbit of \( C_{\text{flex}}[j] \). In this subsection, we will study how \( \mathbb{P}\text{Orb}(C_{\text{flex}}[j]) \) specializes as we vary the \( j \)-invariant to \( \infty \). The outcome will be the proof of Proposition 5.2. In order to proceed, we introduce and study particular variety which will induce the specialization implicit in Proposition 5.2.

Definition 5.13. Define

\[ \mathcal{W} \subset \mathbb{P}\text{Sym}^3(V) \times \mathbb{P}V \times \mathbb{P}V \]

to be the closed subvariety parametrizing triples \((C, L, p)\) where \( L \) is a line meeting the cubic curve \( C \) at the point \( p \) with multiplicity \( \geq 3 \). Let \( p_1, p_2, p_3 \) denote the projections of \( \mathcal{W} \) to the respective factors. Finally, we let \( H_{\text{curve}}, H_{\text{line}} \) and \( H_{\text{point}} \) denote the pullbacks of hyperplane classes under the maps \( p_1, p_2, p_3 \) respectively.

It is easy to see that \( \mathcal{W} \) is smooth: Indeed, \( \mathcal{W} \) is a projective bundle over the smooth 3-dimensional incidence variety parametrizing pairs \((L, p)\) with \( p \in L \).

Definition 5.14. We let

\[ J : \mathcal{W} \longrightarrow \mathbb{P}^1 \]

denote the rational map which sends a triple \((C, L, p)\) to the \( j \)-invariant \( j(C) \).

The \( j \)-invariant is well-defined (and finite) for smooth cubics. It takes on the value \( \infty \) for a cubic with a single node, for a cubic with two nodes (i.e. a smooth conic union a line), and for a cubic with three nodes (i.e. a union of three non-concurrent lines). For any other cubic curve, the \( j \)-invariant is not defined.

For each \( j \in \mathbb{P}^1 \) let

\[ \mathcal{W}_j \subset \mathcal{W} \]

(10)
denote the divisor (possibly non-reduced) given by \( J^{-1}(j) \). This is well-defined, as the locus of indeterminacy of \( J \) has codimension at least 2. \( J \) determines a rational equivalence of cycles:

\[ [\mathcal{W}_j] \sim [\mathcal{W}_\infty] \]

Our short-term objective is to understand the irreducible components of \( \mathcal{W}_\infty \). We leave it to the reader to check that \( \mathcal{W}_\infty \) has precisely three irreducible components:

\( \mathcal{W}_{BN,\infty} \) : This is the closure of the locus of triples \((C, L, p)\) where \( C \) has a unique node at \( p \) and \( L \) is one of the branches at the node,

\( \mathcal{W}_{AN,\infty} \) : This is the closure of the locus of those triples \((C, L, p)\) such that \( C \) has a unique node, \( p \) is a (smooth) flex point of \( C \), and \( L \) is the flex line at \( p \).
\(W_{\text{conic}}\) : This is the closure of the locus of those triples \((C, L, p)\) such that \(C = Q \cup L\), where \(Q\) is a smooth conic, and \(p \in L\) is a general point.

Thus, for any finite \(j \in \mathbb{P}^1\), we \(J\) provides a \(GL(V)\)-equivariant rational equivalence:

\[
[W_j] \sim t_{BN} \cdot [W_{BN, \infty}] + t_{AN} \cdot [W_{AN, \infty}] + t_{\text{conic}} \cdot [W_{\text{conic}}].
\]

(11)

We must determine the two multiplicities \(t_{BN}, t_{AN}\) in particular, because \([W_{\text{conic}}]\) will not matter in the final analysis.

**Proposition 5.15.** In (11), \(t_{BN} = 3\) and \(t_{AN} = 1\).

**Proof.** Recall from Definition 5.13 the class \(H_{\text{curve}}\) on \(W\) corresponding to the line bundle \(p_1^*(O(1))\). Upon intersecting both sides of (11) by \(H^8_{\text{curve}}\) we get:

\[
\int_W H^8_{\text{curve}} \cdot [W_j] = t_{BN} \int_W H^8_{\text{curve}} \cdot [W_{BN, \infty}] + t_{AN} \int_W H^8_{\text{curve}} \cdot [W_{AN, \infty}] + 0.
\]

(12)

The 0 arises because the cycle \(W_{\text{conic}}\) gets contracted under the projection \(p_1\). Now, we simply determine the three integrals above.

1. \(\int_W H^8_{\text{curve}} \cdot [W_j] = 12 \times 9\) because, in a general pencil of cubic curves, a given \(j\)-value arises 12 times, and each smooth cubic has 9 flexes.
2. \(\int_W H^8_{\text{curve}} \cdot [W_{BN, \infty}] = 12 \times 2\) because, in a general pencil of cubics, there are 12 singular cubics and each one has two branch lines,
3. \(\int_W H^8_{\text{curve}} \cdot [W_{AN, \infty}] = 12 \times 3\) because, a general pencil of cubics has 12 singular members, and each has 3 (smooth) flex points.

Therefore, we conclude that

\[
108 = 24t_{BN} + 36t_{AN}.
\]

The only positive integer solution to this latter equation is \(t_{BN} = 3, t_{AN} = 1\), and the proposition is proved.

**Proof of Proposition 5.2.** The rational equivalence induced by \(J\) in (11) is evidently \(GL(V)\)-equivariant. We push forward this rational equivalence to \(\mathbb{P} \text{Sym}^d V^\vee\) via the map \(\varphi : W \to \text{Sym}^d V^\vee\) sending \((C, L, p)\) to the curve \(C \cup (d - 3)L\), and after using Proposition 5.15, the proposition follows.

5.6. The degree of \(\mathbb{P} \text{Orb}(C_{\text{flex}}[j])\). Next, we compute the degree of the orbit closure \(\mathbb{P} \text{Orb}(C_{\text{flex}}[j]) \subset \mathbb{P} \text{Sym}^d V^\vee\), where \(j\) is general. Again, although this can be computed in principle using the algorithm of Aluffi and Faber [AF00b], we have decided to proceed independently, providing a further check.

First, we record the analogue to Lemma 5.8.
Lemma 5.16. As a function of $d$, the degree of the orbit closure $\mathbb{P}\text{Orb}(C_{\text{flex}}[j])$, $j$ general, is a quadratic polynomial $a + b \cdot (d - 3) + c \cdot (d - 3)^2$, where the coefficients $a, b, c$ are the answers to the following enumerative problems:

- $a = 9 \cdot 12 \# \{\text{Cubics through 9 points}\} = 108$
- $b = 12 \cdot \left(\begin{array}{c}8 \\ 1\end{array}\right) \# \{\text{Cubics through 8 points with flex line containing a fixed 9th point}\}$
- $c = 12 \cdot \left(\begin{array}{c}8 \\ 2\end{array}\right) \# \{\text{Cubics through 7 points flexed at a specified line}\}$

Proof. Recall the schemes $\mathcal{W}$ and $\mathcal{W}_j$ from (10), and the classes $H_{\text{curve}}$ and $H_{\text{line}}$ from Definition 5.13. The variety $\mathcal{W}$ affords the natural map

$$\varphi : \mathcal{W} \rightarrow \mathbb{P}\text{Sym}^d V^\vee$$

defined by sending $(C, L, p)$ to the curve $C \cup (d - 3)L$, and the pullback of a hyperplane under this map has class $H_{\text{curve}} + (d - 3)H_{\text{line}}$.

Our objective is to calculate the degree of the image $\varphi(\mathcal{W}_j)$, as this is precisely the orbit closure of $C_{\text{flex}}[j]$. Thus, it suffices to compute $\int_{\mathcal{W}} (H_{\text{curve}} + (d - 3)H_{\text{line}})^8 \cdot [\mathcal{W}_j]$ in the Chow ring of $\mathcal{W}$. Since $H_{\text{line}}^3 = 0$, we get that the degree of the orbit closure of $C_{\text{flex}}[j]$ is:

$$\int_{\mathcal{W}} \left( H_{\text{curve}}^8 \cdot [\mathcal{W}_j] + 8(d - 3)H_{\text{curve}}^7H_{\text{line}} \cdot [\mathcal{W}_j] + \left(\begin{array}{c}8 \\ 2\end{array}\right) (d - 3)^2 H_{\text{curve}}^6H_{\text{line}}^2 \cdot [\mathcal{W}_j] \right). \quad (13)$$

Next, we observe that the divisor class of $\mathcal{W}_j$ is $12H_{\text{curve}}$, because the degree of the divisorial locus in $\mathbb{P}\text{Sym}^3 V^\vee$ consisting of the closure of plane cubics with prescribed generic $j$-invariant is 12. Therefore, the degree of the orbit closure of $C_{\text{flex}}[j]$ is

$$12 \int_{\mathcal{W}} \left( H_{\text{curve}}^9 + 8(d - 3)H_{\text{curve}}^8H_{\text{line}} + \left(\begin{array}{c}8 \\ 2\end{array}\right) (d - 3)^2 H_{\text{curve}}^6H_{\text{line}}^2 \right).$$

The lemma now follows by interpreting the three intersection numbers

$$\int_{\mathcal{W}} H_{\text{curve}}^9, \int_{\mathcal{W}} H_{\text{curve}}^8H_{\text{line}}, \int_{\mathcal{W}} H_{\text{curve}}^7H_{\text{line}}^2$$

as the quantities appearing in the descriptions of $a, b$ and $c$ in the statement of the lemma. $\square$

Our next task is to determine the coefficients $a, b, c$ in Lemma 5.16

Lemma 5.17. There are 9 cubic curves passing through eight general points and having a flex line containing a general fixed ninth point, i.e.

$$\int_{\mathcal{W}} H_{\text{curve}}^8H_{\text{line}} = 9.$$
Proof. Let $\Lambda \subset \mathbb{P} \text{Sym}^3 V^\vee$ denote the Hesse pencil

$$s(X^3 + Y^3 + Z^3) + tXYZ = 0, \ [s : t] \in \mathbb{P}^1.$$  

Recall that the 9 base points of the Hesse pencil consist of the 9 flexes of every smooth cubic in $\Lambda$. At each base point $p$ of the pencil, the flex lines of the cubic curves in the pencil in sweep out a pencil of lines in $\mathbb{P}V^\vee$. Therefore, a general point $x$ in $\mathbb{P}^2$ is contained in exactly 9 flex lines of members of the Hesse pencil, one per basepoint.

Thus, if we use the pullback of the Hesse pencil $\Lambda$ to represent the curve class $H^8_{\text{curve}}$ on $W$, we deduce the lemma. \qed

Lemma 5.18. There are 3 cubic curves passing through 7 general points and possessing a particular line as flex line, i.e.

$$\int_W H^7_{\text{curve}} H^2_{\text{line}} = 3.$$  

Proof. Let $L \subset \mathbb{P}V$ be a fixed line, and suppose $p_1, ..., p_7 \in \mathbb{P}V$ are general points. Then the net of cubic curves containing the points $p_i$ restricts to a general net in the linear system $|O_L(3)|$. A general such net maps $L$ to a nodal cubic in $\mathbb{P}^2$. This nodal cubic has exactly 3 (smooth) flex points. The points on $L$ corresponding to these flex points, in turn, provide the solutions to the enumerative problem in the statement of the lemma. \qed

Corollary 5.19. The degree of $\mathbb{P} \text{Orb}(C_{\text{flex}}[j]), j$-general, is $12 (9 + 72(d - 3) + 84(d - 3)^2)$. The predegree of $C_{\text{flex}}[j]$ is $24 (9 + 72(d - 3) + 84(d - 3)^2)$.

Proof. Combine Lemma 5.16, Lemma 5.17, Lemma 5.18. The second statement on the predegree follows from the fact that the curve $C_{\text{flex}}[j]$ has an order 2 automorphism group, since the generic elliptic curve has an order 2 automorphism group. \qed

5.7. Proof of Theorem 5.3. We now have all ingredients for the proof of Theorem 5.3. Recall the setup in Section 5.1.

Proof of Theorem 5.3. Let $p_i \in C_0, \ i = 1, \ldots, \delta$ and $q_j \in C_0, \ j = 1, \ldots, \kappa$ denote the ordinary nodes and cusps, respectively. For each $i = 1, \ldots, \delta$ or $j = 1, \ldots, \kappa$ separately, upon conjugating by a suitable invertible change of coordinates in $PGL(V)$, we obtain 1-parameter families $\gamma_{i,1}(t), \gamma_{i,2}(t)$ and $\gamma_j(t)$ appearing Lemma 5.4 and Lemma 5.11 accordingly.

Recall the variety $Y \subset U \times \text{Inv}(V) \times \mathbb{P} \text{Sym}^d V^\vee$ attached to the family $\alpha$. (See Definition 2.14 for the definition of $\text{Inv}(V)$.) Each $\gamma_{i,1}(t), \gamma_{i,2}(t)$ and $\gamma_j(t)$ is full for $\alpha$ due to Lemma 5.4 and Lemma 5.11 — the curves $C_{BN}$ and $C_{\text{flex}}[0]$ have finitely many automorphisms. Thus, by Definition 2.10 we get corresponding irreducible components $A_{i,1}, A_{i,2}, A_j$ in the fiber $Y_0$. 

By the Inv\((V)\)-adaptation of Lemma 2.11 and by the statements about kernels and images found in Remark 5.6 and Remark 5.12, it follows that the irreducible components \(A_{i,1}, A_{i,2}, A_j\) are all pairwise distinct. Therefore, the hypotheses of Principle 2.15 are met, and the statement follows. (The coefficient of 2 in \(-2\delta \cdot p_{C_{BN}}\) arises because there are two components \(A_{i,1}, A_{i,2}\) corresponding to each node.)

It remains to establish compatibility of predegrees. For this, we once again reference Aluffi and Faber, specifically [AF00b, Example 4.1 and Example 5.2] where the defect on the predegree caused by an ordinary node and cusp are shown to be, respectively

\[
24(35d^2 - 174d + 213) = 2 \cdot 3(24 + 144(d - 3) + 140(d - 3)^2)
\]
\[
72(28d^2 - 144d + 183) = 24(9 + 72(d - 3) + 84(d - 3)^2)
\]

These two numbers, fortuitously, happen to be twice the predegree of \(C_{BN}\) and the predegree of \(C_{flex}[j]\) respectively by Proposition 5.7 and Corollary 5.19. The theorem follows from Principle 2.15.

6. Computation of \(p_{C_{AN}}\) and \(p_{C_{BN}}\)

In light of the main results of Section 5, it is natural to seek explicit formulas for \(p_{C_{AN}}\) and \(p_{C_{BN}}\). In this section we provide a method for this computation, and apply it to the specific case \(d = 4\). Recall that when \(d = 4\), \(C_{BN}\) is equal to \(C_{D_6}\) – our computation yields the same result as the computation of \(p_{C_{D_6}}\) in Corollary 3.4, thankfully.

**Proposition 6.1.** When \(d = 4\),

\[
p_{C_{BN}} = 3 \cdot 64(18c_1^6 + 33c_1^4c_2 + 12c_1^2c_2^2 - 85c_1^3c_3 - 11c_1c_2c_3 - 7c_3^2)
\]
\[
p_{C_{AN}} = 2 \cdot 192(18c_1^6 + 33c_1^4c_2 + 12c_1^2c_2^2 + 19c_1^3c_3 - 7c_1c_2c_3 - 35c_3^2).
\]

**Proof.** In the interest of brevity, we will indicate how to arrive at the result, by breaking the calculation into several steps and explaining how each step is executed.

First, the factor of 3 in \(p_{C_{BN}}\) and 2 in \(p_{C_{AN}}\) come from automorphisms. What we will actually describe, and what is equivalent by Proposition 2.3, is the calculation of the classes \([\text{Orb}(C_{BN})]_{GL(V)}\) and \([\text{Orb}(C_{AN})]_{GL(V)}\), essentially using the same overall strategy of “presentation” used in the proof of Proposition 3.5.

We begin with the variety \(W\) defined in Definition 5.13 with its attendant divisor classes \(H_{\text{curve}}, H_{\text{line}}, H_{\text{point}}\). We can regard \(W\) as an iterated projective bundle, by first forgetting \(C\), then forgetting \(p\), and then forgetting \(L\) (to map to a point). Each of these projective bundles are given by the projectivization of a vector bundle over their associated base spaces, and so the Chow ring of \(W\) is determined by the chern classes of these vector bundles in a standard way. Furthermore, the Picard group of \(W\) is freely generated by the three divisor classes \(H_\bullet\).

Let

\[
\text{Inc} \subset \mathbb{P}V^\vee \times \mathbb{P}V
\]
denote the standard incidence variety parametrizing pairs \((L, p)\) where \(p \in L\). Observe that \(W\) is a projective bundle over \(\text{Inc}\).

We let
\[
\phi : W \to \mathbb{P} \text{Sym}^4 V^\vee
\]
denote the map sending a triple \((C, L, p)\) to the quartic curve \(C \cup L\), and let \(H\) denote the pullback of the hyperplane class via \(\phi\). Observe that \(H = H_{\text{curve}} + H_{\text{line}}\).

Finally, recall the irreducible divisors (specified a few lines after (10)) \(W_{\infty, BN}, W_{\infty, AN}, W_{\text{conic}}\) in \(W\). We now provide the stages of the calculation.

**Steps of the calculation:**

**Step 1:** The divisor class \([W_{\text{conic}}]\) is straightforward to compute because \(W_{\text{conic}}\) has a simple description. Indeed, \(W_{\text{conic}}\) is a projective sub-bundle of the projective bundle \(W \to \text{Inc}\). It is straightforward to access this sub-bundle, and the conclusion is:

\[
[W_{\text{conic}}] = H_{\text{curve}} - 3H_{\text{point}} + 3H_{\text{line}}.
\]  

(14)

**Step 2:** Let
\[
\mathcal{R} \subset W
\]
denote the ramification divisor of \(p_1\) – observe that
\[
p_1 : W \to \mathbb{P} \text{Sym}^3 V^\vee
\]
is unramified over the locus parametrizing smooth cubic curves, and is also unramified along \(W_{AN, \infty}\). Applying the Riemann-Hurwitz formula (i.e. subtracting canonical classes) to \(p_1\) yields the class of the ramification divisor:

\[
[\mathcal{R}] = 3H_{\text{curve}} + H_{\text{line}} + H_{\text{point}}.
\]  

(15)

\(\mathcal{R}\) is supported on \(W_{BN, \infty}\) and \(W_{\text{conic}}\). Using the classical fact that a branch line of a node is the limit of three flexes, \(W_{BN, \infty}\) appears with multiplicity 2 in the ramification divisor \(\mathcal{R}\). In light of this, by concentrating on the \(H_{\text{curve}}\) coefficients in \([\mathcal{R}]\) and \([W_{\text{conic}}]\), there is only one option for the class \([W_{BN, \infty}]\):

\[
[W_{BN, \infty}] = H_{\text{curve}} - H_{\text{line}} + 2H_{\text{point}}.
\]  

(16)

**Step 3:** Next, to access \(W_{AN, \infty}\) we pull back the discriminant hypersurface
\[
\text{Disc} \subset \mathbb{P} \text{Sym}^3 V^\vee
\]
(parametrizing singular cubics) via \(p_1\). The divisor \(p_1^*(\text{Disc})\) is clearly supported on \(W_{BN, \infty}, W_{AN, \infty}, W_{\text{conic}}\). Since \(p_1\) is locally \(3 : 1\) near a general point of \(W_{BN, \infty}\) and unramified at a general point of \(W_{AN, \infty}\) we conclude that \(p_1^*(\text{Disc}) = 3[W_{BN, \infty}] + [W_{AN, \infty}] + z \cdot [W_{\text{conic}}]\) for some positive integer \(z\).
Step 4: The integer $z$ above is 2. This follows from the local calculation that a general pointed curve (spectrum of a DVR) $(T,0) \subset W$ intersecting $W_{\text{conic}}$ transversely (at $0 \in T$) at a general point $(Q \cup L, L, p)$ will produce a curve in $\mathbb{P}\text{Sym}^3 V^\vee$ meeting Disc with multiplicity 2 at 0. Here $Q$ and $L$ are a general conic and line, respectively. The multiplicity 2 comes from the fact that the total space of the family of curves parametrized by $T$ is smooth at the two nodes $Q \cap L$ of the curve $Q \cup L$.

Step 5: Combining the previous steps, we conclude:

$$[W_{AN,\infty}] = 7H_{\text{curve}} - 3H_{\text{line}}$$ (17)

Step 6: We now observe that all objects and constructions in the previous step were $GL(V)$-equivariant. This means it is possible to do exactly the same calculations in the relative setting: $\mathcal{V} \to B$ is now an arbitrary rank 3 vector bundle over a variety $B$. $W$ is replaced by

$$W_{\mathcal{V}} \subset \mathbb{P}\text{Sym}^3 \mathcal{V}^\vee \times \mathbb{P}\mathcal{V}^\vee \times \mathbb{P}\mathcal{V},$$

the divisor classes $H_*$ relativize as the pullbacks of the $\mathcal{O}(1)$ classes on each of the bundles $\text{Sym}^3 \mathcal{V}^\vee, \mathbb{P}\mathcal{V}^\vee, \mathbb{P}\mathcal{V}^\vee$, etc...

Let $c_1, c_2, c_3$ denote the chern classes of $\mathcal{V}$. By performing the calculations now in the relative setting, the analogous divisor classes are:

$$H = H_{\text{curve}} + H_{\text{line}},$$

$$[W_{BN,\infty}]_{\mathcal{V}} = H_{\text{curve}} - H_{\text{line}} + 2H_{\text{point}}$$ (19)

$$[W_{AN,\infty}]_{\mathcal{V}} = 7H_{\text{curve}} - 3H_{\text{line}} - 6c_1$$ (20)

These are elements in the Chow ring $A^* (W_{\mathcal{V}})$.

Step 7: Recall that our objective is to compute the classes $[\text{Orb}(C_{BN})]_{GL(V)}$ and $[\text{Orb}(C_{AN})]_{GL(V)}$. To get there from the previous step, we use the exact same trick as in Proposition 3.5 and [FNR06, Theorem 3.1]. Letting

$$\phi : W_{\mathcal{V}} \to \mathbb{P}\text{Sym}^4 \mathcal{V}^\vee$$

denote the relative version of the map sending $(C, L, p)$ to $C \cup L$, we must pull back the class

$$\alpha = H^{14} + c_1(\text{Sym}^4 \mathcal{V}^\vee)H^{13} + \cdots + c_{14}(\text{Sym}^4 \mathcal{V}^\vee)$$

via $\phi$, intersect with $[W_{BN,\infty}]_{\mathcal{V}}$ (or $[W_{AN,\infty}]_{\mathcal{V}}$ respectively) and push-forward to $B$ (using the Leray relation for projective bundles). This yields $[\text{Orb}(C_{BN})]$ and $[\text{Orb}(C_{AN})]$, and concludes the calculation, after applying the change of variables Proposition 2.3.

□
Remark 6.2. The only place \( d = 4 \) was used in the proof of Proposition 6.1 was the definition of class \( \alpha \) and the pullback map \( A^*(\mathbb{P}(\text{Sym}^4 V^\vee)) \to A^*(W_V) \), meaning that the above provides a recipe to get the formulas for \( \rho_{\text{CAN}} \) and \( \rho_{\text{CBN}} \) for all \( d \), but we have not tried to use it to arrive at a closed expression.

7. Degenerations III: Quartic Plane Curves

In this section, we finally specialize all the way to the setting \( r = 2, d = 4 \) of quartic curves. We thoroughly study the classical degeneration to a double conic. Then we investigate the effect of acquiring a hyperflex on \( \rho_C \). (We believe the hyperflex specialization can be analyzed in arbitrary degree, but the algorithm in [AF00] is too complicated for us to apply with sufficient confidence.)

7.1. Degeneration to the double conic. In this section, we study how \( \rho_C \) changes as a general smooth quartic \( C \) specializes to a double conic.

7.2. Preliminary lemmas.

Lemma 7.1. Let \( Q \subset \mathbb{P}V \) be a smooth conic and \( p \in Q \) a point. Let \( p_1 \) and \( p_2 \) (respectively \( p_1 \)) be points of \( \mathbb{P}V \) so that \( p_1, p_2 \) and \( p \) are not collinear (respectively not lying on the tangent line to \( Q \) at \( p \)). Then, there exists a unique smooth conic \( Q' \subset \mathbb{P}V \) meeting \( Q \) at \( p \) with multiplicity 3 (respectively 4) and containing \( p_1 \) and \( p_2 \) (respectively \( p_1 \)).

Proof. Let \( Z \subset Q \) be the scheme of length 3 (respectively 4) supported at \( p \in Q \). By counting conditions, for both \( n = 3, 4 \) we see that there is a conic \( Q' \) containing \( Z, p_1, \ldots, p_{5-n} \). If \( n = 3 \), then \( Q' \) cannot be a double line (since \( Z, p_1, p_2 \) are not contained in a line), nor can it be the union of two distinct lines (since \( Z \) is not contained in a line, and also \( p, p_1, p_2 \) are not collinear). Therefore the conic \( Q' \) is smooth. It is unique because of Bezout’s theorem.

If \( n = 4 \), then \( Q' \) cannot be a double line since the underlying line would then have to be tangent to \( Q \) at \( p \), but the tangent line does not pass through \( p_1 \) by assumption. We also cannot have \( Q' \) be the union of two distinct lines or else \( Q' \) can only meet \( Q \) at \( p \) with maximum multiplicity 3. Therefore \( Q' \) is smooth, and again Bezout’s theorem provides uniqueness. \( \square \)

Lemma 7.2. Let \( n \) be such that \( 3 \leq n \leq 7 \) and let \( C \subset \mathbb{P}V \) be a general quartic curve with an \( A_n \) singular point. Then, there exists a smooth conic meeting \( C \) at \( C \)'s singular point with multiplicity \( n + 1 \) and meeting \( C \) transversely at \( 7 - n \) other points.

Proof. We will do this case by case. Let \( p \in C \) be the singular point.

Case \( n = 3 \): For the case of a tacnode \( n = 3 \), the desired conic is required to pass through \( p \) with a specified tangent direction and otherwise intersect \( C \) transversely. There is a 3-dimensional linear system

\[
\Lambda \subset \mathbb{P}\text{Sym}^2 V^\vee
\]
of conics passing through \( p \) with a specified tangent direction. In \( \Lambda \), the set of conics that intersect \( C \) at 4 distinct points, apart from \( p \), is a nonempty open set, because the union of the unique line passing through \( p \) in the specified tangent direction together with a line intersecting \( C \) transversely provides an example.

Since the set of smooth conics in \( \Lambda \) is also nonempty, there exists a smooth conic passing through \( p \) in the specified tangent direction and intersecting \( C \) at four other points transversely, finishing this case.

Case \( n = 4 \): For the case \( n = 4 \), (a ramphoid cusp) we need to resort to actual equations. The space of conics meeting \( C \) at \( p \) to order 5 is the same as the space of conics containing a particular length 3 curvilinear scheme \( Z \), and we can assume \( p = [0 : 0 : 1] \) and \( Z \) is given by the length 3 neighborhood of \( X^2 + YZ \) around \( p \). Since the condition we are trying to realize is open, we can specialize \( C \) while preserving \( p \) and \( Z \), and it suffices to prove the result for a specialization of \( C \). To this end, consider the particular rational quartic curve \( C_0 \) given by

\[
(X^2 + YZ)^2 + X^3Y = 0,
\]

which has a rhamphoid cusp \((A_4 \text{ singularity})\) at \([0 : 0 : 1]\) and an ordinary cusp at \([0 : 1 : 0]\).

Consider the conic \( Q_{a,b} \) given by

\[
X^2 + YZ + aXY + bY^2 = 0.
\]

If we reduce the equation of \( C_0 \) modulo the equation of \( Q_{a,b} \), then we get the same result as reducing the equation \((X^3 + a^2X^2Y + 2abXY^2 + b^2Y^3)Y\). Therefore, the intersection of \( C_0 \) with \( Q \) (as a scheme) is also given by the intersection of the union of 4 lines through \( p = [0 : 0 : 1] \) with \( Q_{a,b} \). One of these lines, the line given by \( Y = 0 \), is tangent to \( Q_{a,b} \) at \( p \), so it suffices to check the remaining three lines are distinct for general \( a, b \). This can be shown by noting that the discriminant of the cubic polynomial \( X^3 + a^2X^2Y + 2abXY^2 + b^2Y^3 \) does not vanish identically (indeed it is not even homogenous). Thus, for general \( a, b \), \( Q_{a,b} \) furnishes the conic we want.

Cases \( n = 5,6 \): For the cases \( n = 5,6 \), we use Lemma 7.1. In both cases, we have a curvilinear scheme \( Z \) of length \( n-2 \) contained in a conic, and we need to find a smooth conic containing \( Z \) and passing through \( 7-n \) distinct other points of \( C \). If \( n = 5 \), then it suffices to pick the remaining 2 points \( p_1, p_2 \) of \( C \) so that \( p, p_1, \) and \( p_2 \) do not all lie on a line. If \( n = 6 \), it suffices to pick the remaining point \( p_1 \) to not be contained in the tangent line to \( Z \).

\[ \square \]

7.3. How \( C_{A_6} \) arises. In this subsection, let \( F(X,Y,Z) \in \operatorname{Sym}^4 V^\vee \) define a smooth quartic containing the point \([0 : 0 : 1]\) and meeting the conic \( Q(X,Y,Z) = X^2 + YZ \)
transversely there. Next, let

\[ \alpha : U \to \mathbb{P} \text{Sym}^4 V \]

be the family defined by

\[ t^3 F + Q^2 \]

where \( t \) is a coordinate on \( U \subset \mathbb{A}^1 \) a neighborhood of 0.

Finally, define the 1-parameter family

\[ \gamma : U^\times \to PGL(V) \]

given by

\[
\begin{pmatrix}
  t^{-1} & 0 & 0 \\
  0 & t^{-2} & 0 \\
  0 & 0 & 1
\end{pmatrix}.
\]

**Lemma 7.3.** Let \( F(X, Y, Z) \) cut out a quartic plane curve and let \( Q(X, Y, Z) = X^2 + YZ \). Suppose \( F \) and \( Q \) meet transversely at \([0 : 0 : 1]\). Then the curve \( \alpha \gamma(0) \) is defined by

\[
\lim_{t \to 0} t^{-4} (t^3 F(tX, t^2 Y, Z) + Q(tX, t^2 Y, Z))
\]

and is projectively equivalent to \( C_{A_6} \) (from Definition 3.3.)

**Proof.** Note \( Q(tX, t^2 Y, Z)^2 = t^4 Q(X, Y, Z) \). Also, the only coefficients of \( t^3 F(tX, t^2 Y, Z) \) whose vanishing order with respect to \( t \) is at most 4 are the coefficients of \( Z^4 \) and \( Z^3 X \). Since \( F \) vanishes at \( p = [0 : 0 : 1] \) by assumption, the coefficient of \( Z^4 \) is zero.

The tangent line to \( \{Q = 0\} \) at \( p \) is given by \( Y = 0 \). Since \( \{F = 0\} \) is transverse to \( \{Q = 0\} \) at \( p \), the coefficient of \( Z^3 X \) is nonzero. Therefore,

\[
\lim_{t \to 0} t^{-4} (t^3 F(t^2 X, tY, Z) + Q(t^2 X, tY, Z)^2) = (X^2 + YZ)^2 + aZ^3 X
\]

for some constant \( a \neq 0 \in \mathbb{C} \), which is the unique, up to projective equivalence, rational curve with an \( A_6 \) singularity having a full dimensional orbit, as found in [AF00b, Example 5.4]. This is the curve \( C_{A_6} \). \( \square \)

**Remark 7.4.** Notice that the image of the \( t \to 0 \) limit endomorphism \( \gamma^{-1} \) is the point \( p \).

**Corollary 7.5.** For a general quartic plane curve \( C \),

\[ \mathbb{P} \text{Orb}(C) \sim 8 \cdot 3 \cdot \mathbb{P} \text{Orb}(C_{A_6}) \]

and therefore

\[ p_C = 8p_{C_{A_6}}. \]

**Proof.** Let \( F(X, Y, Z) \) cut out \( C \). Pick a conic intersecting \( C \) transversely in 8 points and let \( Q(X, Y, Z) \) cut out the conic. Then, consider the family of curves over \( \mathbb{A}^1 \) given by

\[ t^3 F(X, Y, Z) + Q(X, Y, Z)^2. \]
Applying Lemma 7.3 (after conjugating by appropriate elements in $PGL(V)$) yields eight 1-parameter families $\gamma_i : U \to PGL(V)$, $1 \leq i \leq 8$, to be used in Principle 2.15. The principle applies, thanks to the Inv($V$)-variant of Lemma 2.11 and the observation in Remark 7.4.

To conclude, we use either [AF00b, Example 5.4] or Corollary 3.4 to see that the predegree of the rational quartic $C_{A_6}$ is 1785, and $1785 \cdot 8 = 14280$, which is the predegree of the orbit of a general quartic curve [AF93b]. Finally, the factor of 3 in front of $P\operatorname{Orb}(C_{A_6})$ arises because $\# \operatorname{Aut}(C_{A_6}) = 3$.

\[ \square \]

\textbf{Theorem 7.6.} For any smooth quartic plane curve $C$ with no hyperflexes,

\[ p_C = 8p_{C_{A_6}}. \]

\textit{Proof.} The result follows from combining Principle 2.13, Corollary 7.5, and the fact that the predegree of a general plane quartic is the same as the predegree of any smooth quartic not possessing a hyperflex ([AF93b]). \[ \square \]

\textbf{Remark 7.7.} We remark that our usage of the predegree computations of Aluffi and Faber in [AF93b] can in principle be replaced by the explicit description of the semistable reduction of an $A_n$ singularity given in [CML13].

\textbf{Theorem 7.8.} Let $C_{A_n}$ be a general curve with an $A_n$ singularity, where $3 \leq n \leq 6$. Then,

\[ \# \operatorname{Aut}(C_{A_n})P\operatorname{Orb}(C_{A_n}) \sim (7 - n) \cdot 3 \cdot P\operatorname{Orb}(C_{A_6}) \]

and therefore

\[ p_{C_{A_n}} = (7 - n)p_{C_{A_6}}. \]

\textit{Proof.} By Lemma 7.2 we can find a smooth conic that meets $C_{A_n}$ at its singular point to order $n + 1$ and meets $C$ transversely at $7 - n$ other points $p_1, \ldots, p_{7 - n}$. Let $F(X, Y, Z)$ cut out $C_{A_n}$ and $Q(X, Y, Z)$ cut out the conic.

Consider the family of quartic curves given by

\[ t^3F(X, Y, Z) + Q(X, Y, Z)^2. \]

Note, critically, that for general $t$, we get a curve with an $A_n$ singularity. From Lemma 7.3 we get, upon conjugating by appropriate elements of $PGL(V)$, $7 - n$ 1-parameter families $\gamma_i : U^\times \setminus \{0\} \to PGL_3$ to use in Principle 2.15. The rest of the argument is just as in the proof of Corollary 7.5.

Applying [AF00b, Example 5.4], we find that the predegree of $C_{A_n}$ is $(7 - n)$ times the predegree of $C_{A_6}$ if $n \geq 3$, and so Principle 2.15 gives the result. \[ \square \]
When a quartic acquires hyperflexes. Aluffi and Faber studied the situation of a smooth plane curve with no hyperflexes specializing to a smooth curve possessing a hyperflex in [AF91, Theorem IV(2)]. We will analyze this degeneration, and instead of using explicit equations as in [AF91] and as in our previous degenerations, we found the need to proceed more conceptually. The ideas in this section will be familiar to those readers having experience in the theory of limit linear series, though we won’t use any substantial part of that theory.

First, we need the following lemma characterizing $C_{E_6}$.

**Lemma 7.9.** Let $C \subset \mathbb{P}^2$ be a rational quartic possessing one $E_6$ singularity and two distinct flex points. Then, $C$ has a full dimensional orbit, and is therefore projectively equivalent to $C_{E_6}$.

**Proof.** We will show $\text{Aut}(C)$ is finite by showing that only a finite subgroup of $PGL_3$ preserves the flexes and the unique tangent branch at the $E_6$ singularity. Let $G$ be the component of $\text{Aut}(C)$ containing the identity.

Without loss of generality, we can assume the $E_6$ singularity is at $[0 : 0 : 1]$ and the two flex points are at $[0 : 1 : 0]$ and $[1 : 0 : 0]$. The group $G$ fixes these three points, so $G$ is a subgroup of the torus

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \subset PGL(V).$$

In addition, $G$ fixes the tangent line $L$ to the singularity. $L$ meets the curve to order 4 at $[0 : 0 : 1]$, so it cannot intersect $[0 : 1 : 0]$ or $[1 : 0 : 0]$. Since $G$ must preserve $L$, it follows that $a = b$.

Let $L_1$ be the flex line of $C$ at $[0 : 1 : 0]$ and $L_2$ be the analogue at $[1 : 0 : 0]$. By Bezout’s theorem we know neither line $L_i$ contains $[0 : 0 : 1]$, and also $L_1 \neq L_2$. Therefore, either $L_1$ does not pass through $[0 : 1 : 0]$ or $L_2$ does not pass through $[1 : 0 : 0]$. In the first case, we find $a = c$ and in the second case we find $b = c$. In any case, $G$ is a trivial group and we are done.

**Remark 7.10.** Note that in Lemma 7.9 we don’t assume the two flexes are ordinary flexes, i.e. not hyperflexes. The conclusion ends up forcing this to be the case, however.

Now, let us assume we have a family

$$\alpha_{h, \text{flex}} : U \rightarrow \mathbb{P} \text{Sym}^4 V^\vee$$

which obeys the following properties:

1. The family of curves $\pi : C \rightarrow U$ is a smooth morphism,
2. the curve $C_u$, $u \in U$ general, does not have any hyperflex,
3. the curve $C_0$ has a hyperflex at $p$. 


(4) the family $\pi$ has two sections $\sigma_{1,2} : U \to C$ meeting transversely at $p = \sigma_1(0) = \sigma_2(0)$ and such that $\sigma_i(u)$ is a flex point of $C_u$ for $u \in U$ general.

In order to extract the suitable 1-parameter family $\gamma$, we will essentially perform an “elementary transformation,” $\mathbb{P}V \to \mathbb{P}V$, though the reader need not be familiar with this concept.

Let $$\varphi : C \to \mathbb{P}V$$

 denote the natural map, and let $L$ denote the line bundle $\varphi^*O(1)$. Suppose we have selected homogeneous variables so that the homogeneous coordinate $X$ vanishes with order 4 at $p$ along $C_0$, the homogeneous coordinate $Y$ vanishes with multiplicity 1 at $p$ (i.e. $Y = 0$ is transverse to $C_0$ at $p$) and the homogeneous coordinate $Z$ does not vanish at $p$. (Thus, $p = [0 : 0 : 1]$, and $X = 0$ is the hyperflex line of $C_0$.) Abusing notation, we let $X, Y, Z$ denote the corresponding pulled back sections of $L$ as well.

Now, let $C'$ denote the blow up of $C$ at the point $p$, and let $$\pi_1 : C' \to C$$

 be the blow-down map with exceptional curve $E$. $E$ is a rational curve, and the family of curves $C' \to U$ has a two-component fiber over 0, namely $C_0 \cup E$, with the two curves intersecting transversely the point $p = C_0 \cap E$.

Consider the line bundle $$L' := \pi_1^*L(-4E).$$

Then $\pi_1^*X, \pi_1^*Y,$ and $\pi_1^*Z$ are rational sections of $L'$ with pole orders 0, 3, and 4, respectively along $E$. Therefore, the sections $$\pi_1^*X, t^3\pi_1^*Y, t^4\pi_1^*Z$$

are regular sections of $L'$ on $\widetilde{C}$, and they vanish with multiplicity 0, 3, 4 respectively along $C_0$. None of them vanish entirely on $E$, and therefore, on $E$ they vanish with orders 0, 3, and 4 at the point $p \in E$.

Now consider the (regular) map $$\varphi' : \widetilde{C} \to \mathbb{P}^2$$

(21)

given by $[\pi_1^*X : t^3\pi_1^*Y : t^4\pi_1^*Z]$. By what we have just said,

1. $\varphi'(C_0) = [1 : 0 : 0]$

(2) the degree 4 map $\varphi|_E : E \to \mathbb{P}^2$ is birational onto its image $\overline{E}$: this is because the section with vanishing order 3 at $p$ could not exist otherwise, as 3 is relatively prime to 4.

3. $\overline{E}$ has a uni-branched triple point at the point $\varphi'|_E(p)$. (No other point $q \neq p$ can map to this point because the section $t^4\pi_1^*Z$ would vanish at a scheme of length $\geq 5$ on $E$, meaning it would be identically zero, which it isn’t.)
Therefore, the curve $\overline{E} \subset \mathbb{P}^2$ is a quartic with an $E_6$ singularity at $\varphi'(p)$. $\overline{E}$ can have no other singularity, otherwise Bezout’s theorem is violated.

If $\sigma'_i : U \to \tilde{C}$, $i = 1, 2$ denote the proper transforms of the sections $\sigma_i$, we note that by our assumptions on the family $\alpha_{h, \text{flex}}$, $p_1 := \sigma'_1(0)$ and $p_2 := \sigma'_2(0)$ are distinct points of $E$, both distinct from $p$, and furthermore their corresponding points $\overline{p}_1, \overline{p}_2 \in E$ are flex points. Therefore, by Lemma 7.9, we conclude that $\overline{E}$ is projectively equivalent to the curve $C_{E_6}$.

**Lemma 7.11.** Maintain the setting in the discussion immediately prior. If

$$\gamma : U^\times \to \text{PGL}(V)$$

is the 1-parameter family defined by

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & t^3 & 0 \\
0 & 0 & t^4
\end{pmatrix}
$$

then $\gamma$ is full for $\alpha_{h, \text{flex}}$ and $\alpha_{h, \text{flex}}(0)$ is a curve projectively equivalent to $C_{E_6}$.

**Proof.** The map $\varphi' : C' \to \mathbb{P}^2$ from (21) resolves the composite

$$C \to \mathbb{P}V \dashrightarrow \mathbb{P}V$$

where the first map is $\varphi$ and the second rational map sends a point $[x : y : z]$ to $[x : t^3y : t^4z]$. Therefore, with $\gamma$ as stated in the lemma, the fact that $\overline{E} = C_{E_6}$ completes the proof. \qed

**Remark 7.12.** Not that the limiting endomorphism $\gamma(0)$ has kernel space equal to the flex line $X = 0$ of $C_0$.

**Remark 7.13.** Condition (4) imposed on our family $\alpha_{h, \text{flex}}$ is not strictly necessary. Possibly after performing a base change, and then blowing up $C$ iteratively until the sections $\sigma_i$ become separated, we arrive at a specialization to the curve $C_{E_6}$. We omitted this for brevity.

**Theorem 7.14.** Let $C$ be a smooth quartic plane curve with $n$ hyperflexes. Then,

$$p_C = 8p_{C_{A_6}} - np_{C_{E_6}}.$$

**Proof.** We consider a family of smooth quartic curves, where the general member $C'$ has no hyperflexes, and with $C$ as the special fiber.

Applying Lemma 7.11 once per hyperflex, (also see Remark 7.13) and using Principle 2.13 (Remark 7.12 shows that the hypotheses are met), we see that it suffices to check that the predegree of $C'$ is the predegree of $C$ plus $n$ times the predegree of $C_{E_6}$. The predegree of $C$ is $294n$ less than the predegree of $C'$ [AF93b, Section 3.6]. Also, the predegree of $C_{E_6}$ is $294$ from [AF00b, bottom of page 36] or Corollary 3.4 (noting $\# \text{Aut}(C_{E_6}) = 2$).

Finally, we use $p_{C'} = 8p_{C_{A_6}}$ from Theorem 7.6. \qed
APPENDIX A. POINTS ON \( \mathbb{P}^1 \)

This appendix will showcase two computations of \( p_X \) in the case \( X \) is a hypersurface in \( \mathbb{P}^1 \), i.e. a set of points with multiplicities. In the case \( X \subset \mathbb{P}^1 \) is supported on at most three points, these are strata of coincident root loci, which were first computed in \cite{FN2006} and generalized to \( PGL_2 \)-equivariant cohomology in \cite{ST2022}. Therefore, we only have to deal with the case where \( X \) is supported on at least four points, and we give two separate proofs. For formulas with fewer signs, and also because our conventions are opposite to the conventions in \cite{LPST2020}, we will actually compute the class \( p_X(-u,-v) \) instead of \( p_X(u,v) \). Here \( u,v \) are the chern roots of the universal rank 2 vector bundle \( V \) over \( \mathbb{P}GL(V) \).

**Theorem A.1.** Let \( X \subset \mathbb{P}^1 \) be a subscheme of length \( d = m_1 + \ldots + m_n \) supported on points \( p_1, \ldots, p_n \) with multiplicities \( m_1, \ldots, m_n \) and with \( n \geq 3 \). Then,

\[
p_X(-u,-v) = \prod_{i=0}^d (iu + (d-i)v) \left( \frac{n-2}{u-v} + \sum_{i=1}^n \frac{2m_i - d}{m_i u + (d-m_i)v} \right)
\]

We give a proof of Theorem A.1 using the resolution given by Aluffi and Faber \cite{AF1993}, together with the Atiyah-Bott formula \cite{EG1998} in Appendix B. This proof is self-contained and direct. The second proof we give is from the machinery developed in \cite{LPST2020} that apply to arbitrary hyperplane arrangements. A computation is required to specialize the results from the case of ordered points on \( \mathbb{P}^1 \) to unordered points on \( \mathbb{P}^1 \), we do this now.

**Proof using \cite{LPST2020}**. We use the same argument in \cite{LPST2020} Theorem 12.5], so we only describe the computation, and refer the motivation and proof of correctness to \cite{LPST2020}. Because our sign convention is opposite that of \cite{LPST2020}, we will actually compute \( p_X(-u,-v) \). Let \( d = \sum_{i=1}^n m_i \) and \( G(z) = \prod_{i=0}^d (H + iz + (d-i)z) \in \mathbb{Z}[u,v][z] \). Let \( L(z) = \frac{G(z)}{z} \). Let \( L(H_1, \ldots, H_n) \) be the result of reducing \( L(m_1H_1 + \ldots + m_nH_n) \) modulo \((H_1 + u)(H_1 + v)\) for each \( i \). Now, we carry out the three steps in the proof of \cite{LPST2020} Theorem 12.5].

**Step 1** By Lagrange interpolation,

\[
L(m_1H_1 + \ldots + m_nH_n) = \frac{G(m_1H_1 + \ldots + m_nH_n) - L(0)}{m_1H_1 + \ldots + m_nH_n}.
\]

**Step 2** Substituting \( z \) for each \( H_i \) yields

\[
L(z, \ldots, z) = G(0) \sum_{T \subset \{1, \ldots, n\}} \frac{1}{\sum_{i \in T} m_i v + \sum_{i \not\in T} m_i u} \frac{(z+u)^{d-\#T}}{\prod_{i \in T} (-v+u) \prod_{i \not\in T} (-u+v)}. \quad (22)
\]
Step 3 Let \( F(z) = (z + u)(z + v) \). All terms of (22) are divisible by \( F(z)^2 \) unless \#T \in \{0, 1, n - 1, n\}. Thus, \([z^1][F(z)^1]L(z, \ldots, z)\) is

\[
\frac{G(0)}{(u - v)^n}[z^1][F(z)^1] \left( \frac{(-1)^n(z + v)^n}{du} + \frac{(z + u)^n}{dv} + \sum_{i=1}^n \frac{(-1)^{n-1}F(z)(z + v)^{n-2}}{m_iv + (d - m_i)u} + \sum_{i=1}^n \frac{(-1)F(z)(z + u)^{n-2}}{m_iu + (d - m_i)v} \right)
\]

As in the proof of [LPST20 Theorem 12.5],

\[
[z^1][F(z)^1]F(z)(z + u)^k = (u - v)^{k-1}k \quad \text{and} \quad [z^1][F(z)^1](z + v)^k = (v - u)^{k-1}k,
\]

so \([z^1][F(z)^1]L(z, \ldots, z)\) simplifies to

\[
\frac{G(0)}{(u - v)^n} \left( \frac{(-1)^n(n - 2)(v - u)^{n-3}}{du} + \frac{(n - 2)(u - v)^{n-3}}{dv} + \sum_{i=1}^n \frac{(-1)^{n-1}(v - u)^{n-3}}{m_iv + (d - m_i)u} + \sum_{i=1}^n \frac{(-1)(v - u)^{n-3}}{m_iu + (d - m_i)v} \right)
\]

\[
\frac{G(0)}{(u - v)^n} \left( \frac{(-1)(n - 2)(u - v)^{n-3}}{du} + \frac{(n - 2)(u - v)^{n-3}}{dv} + \sum_{i=1}^n \frac{(u - v)^{n-3}}{m_iv + (d - m_i)u} + \frac{(-1)(u - v)^{n-3}}{m_iu + (d - m_i)v} \right)
\]

\[
\frac{G(0)}{(u - v)^n} \left( \frac{(n - 2)(u - v)^{n-2}}{d(v - u)^2} + \sum_{i=1}^n \frac{(2m_i - d)(u - v)^{n-2}}{m_iu + (d - m_i)v} \right)
\]

\[
\frac{G(0)}{(u - v)^2} \left( \frac{2m_i - d}{du} + \sum_{i=1}^n \frac{2m_i - d}{m_iu + (d - m_i)v} \right)
\]

\[
\square
\]

In the case all the multiplicities are all one, the formula in Theorem A.1 simplifies. We will also give a direct proof by slow projection immediately after this corollary.

Corollary A.2. In the setting of Theorem A.1 if each \( m_i = 1 \), then \( n = d \), and

\[
P_X(-u, -v) = n(n - 1)(n - 2)\prod_{j=2}^{n-2}(H + (ju + (n - j)v)).
\]

Proof using Theorem A.1 Applying Theorem A.1 we find \( p_X(-u, -v) \) is

\[
\frac{1}{(u - v)^2} \prod_{i=0}^n (iu + (n - i)v) \left( \frac{n - 2}{nuv} - \frac{(-2 + n)n}{((n - 1)u + v)((n - 1)v + u)} \right) = \frac{n(n - 2)}{(u - v)^2} \prod_{i=0}^n (iu + (n - i)v) \left( \frac{1}{(nu)(nv)} - \frac{1}{((n - 1)u + v)((n - 1)v + u)} \right) = \frac{n(n - 2)}{(u - v)^2} \prod_{i=0}^n (iu + (n - i)v) \left( \frac{n - 1}{(nu)((n - 1)u + v)((n - 1)v + u)(nv)} \right).
\]

Applying [FNR05 Theorem 6.1] yields the result. \( \square \)
**Proof by “slow projection”.** Let $V$ be a 2-dimensional vector space, $v_1,\ldots,v_n$ pairwise linearly independent vectors of $V$, $X \subset \mathbb{P}^1$ the corresponding point configuration supported on $p_1,\ldots,p_n$, and $Z \subset \mathbb{P}^{\text{Sym}^n V}$ the orbit closure. The key fact we will use is

**Claim A.3.** Every point in the boundary of $Z$ corresponds to a point configuration in $\mathbb{P}^1$ supported on two points with multiplicities $n-1$ and 1 or one point with multiplicity $n$. 

**Proof of Claim A.3.** Let $A(t)$ be a 1-parameter family of matrices, or more precisely a map from the spectrum of a discrete valuation ring to $\text{End}(V)$ where the generic point maps to an element of $GL(V)$. We want to show that the multiset $S = \{\lim_{t \to 0} A(t)p_i \mid 1 \leq i \leq n\}$ does not have two copies each of two distinct points. First, we can assume the rank of $A(0) = 1$. If the rank of $A(0)$ is 2, then $S$ consists of distinct points. If $A(0) = 0$, we can divide out by a power of the uniformizing parameter so that $A(0) \neq 0$. Then, $\{\lim_{t \to 0} A(t)p_i \mid 1 \leq i \leq n\}$ is the point in $\mathbb{P}(V)$ corresponding to the 1-dimensional image of $A(0)$ if $v_i$ is not in the kernel of $A(0)$. Otherwise, there is at most one $v_i$ in the kernel of $A(0)$ and $\{\lim_{t \to 0} A(t)v_i \mid 1 \leq i \leq n\}$ is otherwise unrestricted. 

Let $x,y$ be a basis for $V$. Then, a basis of $\text{Sym}^n V$ is $x^n,x^{n-1}y,\ldots,y^n$. Let $T \subset GL(V)$ be the maximal torus corresponding to the basis $x,y$. Since $A_{GL(V)}^T(\mathbb{P}(\text{Sym}^n V)) \to A_T^* (\mathbb{P}(\text{Sym}^n V))$ is injective, we can use a $T$-equivariant degeneration and compute the $T$-equivariant class. Our $T$-equivariant degeneration will be to scale the coordinates corresponding to $x^{n-2}y^2,\ldots,x^2y^{n-2}$ to zero.

By Claim A.3 $Z$ is disjoint from the source of this “slow projection,” so the $T$-equivariant class of $Z$ is a multiple of the class of the 3-plane in $\mathbb{P}(\text{Sym}^n V)$ given by the vanishing of the coordinates corresponding to $x^{n-2}y^2,\ldots,x^2y^{n-2}$. The class of that 3-plane is $\prod_{i=2}^{n-2} (iu + (n-i)v)$. The multiple we need is the degree of $Z$ as a projective variety, which is $n(n-1)(n-2)$ by the combinatorial argument given in [AF93a, Introduction].

**Remark A.4.** Corollary A.2 can be generalized in a different direction. Suppose we fix $d$ general points $p_1,\ldots,p_d \in \mathbb{P}^1$ and consider all configurations of $d$ points given by mapping $p_1,\ldots,p_d$ via a linear rational map $\mathbb{P}^r \to \mathbb{P}^1$. Let the closure of these configurations in $\text{Sym}^d \mathbb{P}^1$ be $Z_{r,d}$. The same proof of Claim A.3 using slow projection shows the equivariant class of $Z_{r,d}$ in $A_{GL(V)}^T(\mathbb{P}(\text{Sym}^d \mathbb{P}^1)) = \mathbb{Z}[u,v][H]/(\prod_{i=0}^n H + iu + (n-i)v)$ has constant term

$$\prod_{i=0}^{2r+1} (d-i) \prod_{i=r+1}^{d-r-1} (iu + (d-i)r),$$

and the full class is given by substituting $u \to u + \frac{H}{d}, v \to v + \frac{H}{d}$ into the constant term [FNR05, Theorem 6.1]. These are examples of generalized matrix orbits defined in [LPST20]. Also see [Tse21, Example 1.3].
Appendix B. Points on $\mathbb{P}^1$ via Atiyah-Bott

The method in Appendix A was closer to the theme of equivariant degeneration explored in this paper. We note that there is self-contained proof given by the Atiyah-Bott formula, or equivalently resolution and integral via localization [FR06, Section 4]. The authors attempted to perform the same method for smooth plane curves using the resolution given by [AF93b], but the computation of the normal bundles quickly became intractable.

B.1. General setup. Let $V$ be a 2-dimensional vector space with $T = (\mathbb{C}^*)^2$ acting by scaling. Then, we have $T$-action on $\mathbb{P}^3 \cong \mathbb{P}\text{Hom}(V, \mathbb{C}^2)$. Given a point configuration of $d$-points in $\mathbb{P}^1$ (a central hyperplane configuration in $\mathbb{C}^2$), we have a rational map

$$\mathbb{P}\text{Hom}(V, \mathbb{C}^2) \to \mathbb{P}(\text{Sym}^d V)$$

Suppose our point configuration consists of $n$ distinct points $p_1, \ldots, p_n$ with multiplicities $m_1, \ldots, m_n$ summing to $d$. Then, the base locus is $n$ disjoint lines, given by matrices with image contained in each $p_1, \ldots, p_n$. Picking a basis, we find $\mathbb{P}\text{Hom}(V, \mathbb{C}^2)$ is given by 2 by 2 matrices $\begin{pmatrix} \ast & \ast \\ \ast & \ast \end{pmatrix}$ up to scaling. The $T$ action is by scaling the columns, and each of the base loci (after base change via a left $GL_2$-action) looks like $\begin{pmatrix} \ast & \ast \\ 0 & 0 \end{pmatrix}$.

Let $X$ be the blow up of $\mathbb{P}^3$ along these base loci $R_1, \ldots, R_n$. This resolves the rational map above [AF93b, Proposition 1.2].

B.2. Normal bundle to a proper transform.

Lemma B.1. Let $Z \subset Y$ be an inclusion of smooth varieties. Let $W \subset Y$ be a smooth subvariety and $\tilde{W} \subset \text{Bl}_Z Y$ be the proper transform of $W$. If $\pi : \text{Bl}_Z Y \to Y$ is the blowup map, then we have the short exact sequence

$$0 \to \text{coker}(\pi^* N^\vee_{W/Y} \to N^\vee_{\tilde{W}/\text{Bl}_Z Y}) \to \Omega_{\text{Bl}_Z Y/Y}|_{\tilde{W}} \to \Omega_{\tilde{W}/W} \to 0.$$ 

Proof. Consider the following diagram
The bottom two rows are exact by the relative cotangent sequence for a generically separable morphism of integral smooth varieties [Liu02, Remark 4.17]. The lemma follows from the nine lemma.

\[ \begin{array}{ccc} 0 & 0 \\ \downarrow & \downarrow \\ \pi^*N^\vee_{W/Y} & \rightarrow & N^\vee_{\tilde{W}/Bl_1Z} \\ \downarrow & \downarrow \\ 0 & \rightarrow & \pi^*\Omega_Y|\tilde{W} & \rightarrow & \Omega_{Bl_1Z/Y}|\tilde{W} & \rightarrow & 0 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & \rightarrow & \pi^*\Omega_W & \rightarrow & \Omega_{\tilde{W}/W} & \rightarrow & \Omega_{\tilde{W}/W} & \rightarrow & 0 \end{array} \]

\[ \begin{array}{ccc} 0 & 0 \\ \downarrow & \downarrow \\ 0 & 0 & 0 \end{array} \]

B.3. Setup for Atiyah-Bott integration. In order to apply Atiyah-Bott integration to \( X \), we need to identify the fixed loci, their normal bundles, and how classes restrict from \( X \) to the fixed point loci.

B.4. Fixed point loci. First, we note that the fixed-point loci of \( \mathbb{P}^3 \) under the action of \( T \) consists of two disjoint \( \mathbb{P}^1 \)'s which we will call \( C_1, C_2 \).

\[ \begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix} \]

The fixed-point loci of \( X \) under the action of \( T \) must lie over the fixed-point loci of \( \mathbb{P}^3 \) under \( T \). Therefore, we conclude that the fixed loci consist of the following \( 2n+2 \) components:

1. 2 fixed point loci corresponding to \( \mathbb{P}^1 \)'s that are the proper transforms \( \tilde{C}_1 \) and \( \tilde{C}_2 \) of \( C_1 \) and \( C_2 \). If we suppose \( R_1 \) is the \( \mathbb{P}^1 \) consisting of the matrices \( \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix} \), then the point of the proper transform lying above \( C_1 \cap R_1 \) is given by the limiting point of

\[ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + t \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \]

as \( t \rightarrow 0 \).

2. \( 2n \) isolated points that lie over the \( 2n \) pairwise intersections of \( C_1, C_2 \) with \( R_1, \ldots, R_n \).

If we suppose \( R_1 \) is the \( \mathbb{P}^1 \) consisting of the matrices \( \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix} \), then the isolated
torus-fixed point lying above $C_1 \cap R_1$ is given by the limiting point of
\[
\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + t \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\]
as $t \to 0$.

**B.4.1. Normal bundles and restriction of proper transforms.** Let $H$ be the $c_1(\mathcal{O}_{\mathbb{P}^3}(1))$ on $\mathbb{P}^3$ pulled back to $X$ and $E$ an exceptional divisor of $X \to \mathbb{P}^3$. We have $\tilde{C}_1$ is $\mathbb{P}^1$ with a trivial $T$-action. Therefore, $A^\bullet(\tilde{C}_1) \cong \mathbb{Z}[z][u,v]/(z^2)$. We have the following restrictions:

$H \mapsto H = z - u$

$E \mapsto z = H + u$.

Here, $E$ is any of the $n$ exceptional divisors. (We are thinking of $\tilde{C}_1$ as the $\mathbb{P}^1$ embedded as the first column of 2 by 2 matrices $\mathbb{P}^3$ up scaling. Therefore, it’s actually natural to think of it as the projectivization of a vector bundle with a nontrivial $T$-action, so it is a projective bundle over a point that is trivial, but $\mathcal{O}(1) = H$ is twisted. The Leray relation in this case is $(H + u)^2 = z^2$.)

We need to compute the normal bundle to the proper transform of $C_1$. The normal bundle of $C_1$ in $\mathbb{P}^3$ is

$\frac{c(\mathbb{P}^3)}{c(C_1)} = \frac{(1 + u + H)^2(1 + v + H)^2}{(1 + u + H)^2} = (1 + v + H)^2$.

Note that this also makes sense as $C_1$ is a complete intersection cut out by $(v + H)^2$. Applying Lemma [B.1] yields

$0 \to \pi^* N^\vee_{C_1/\mathbb{P}^3} \to N^\vee_{\tilde{C}_1/X} \to \Omega_{X/\mathbb{P}^3}|_{\tilde{C}_1} \to 0$.

The term on the right is a skyscraper sheaf supported on the intersection of $\tilde{C}_1 \cong \mathbb{P}^1$ with the exceptional locus. We need to find the torus action on the bundle $T_{X/\mathbb{P}^1}|_{\tilde{C}_1}$ supported on $E$ at the intersection $E \cap \tilde{C}_1$. There is an affine neighborhood of $E \cap \tilde{C}_1$ in $X$ of the form

\[
\begin{pmatrix} 1 & a_{01} \\ a_{10} & a_{00} \end{pmatrix} + t \begin{pmatrix} 0 & 0 \\ 1 & A_{11} \end{pmatrix}
\]

with coordinates given by $a_{01}$, $a_{10}$, $A_{11}$ as $A_{11}a_{10} = a_{11}$. We have the short exact sequence

$0 \to \tilde{C}_1(-z) \otimes \mathcal{O}_{v-u} \to \tilde{C}_1 \otimes \mathcal{O}_{v-u} \to \mathbb{C}_{v-u}|\tilde{C} \cap \pi^{-1}(R_i) \to 0$,

where $\mathbb{C}_{v-u}$ is the nonequivariantly trivial line bundle with an action of $T$ by the character $v - u$. The torus action has character $v - u$ on the coordinate $A_{11}$, so the term on the
right has chern class \( \frac{1+v-u}{1-z+v-u} \). We apply this for each \( i \) to find

\[
c(N_{\tilde{C}_1/X}) = (1 + H + v)^2 \frac{(1-z+v-u)^n}{(1+v-u)^n}
\]

\[
= (1 + v - u)^2 \left( 1 + \frac{z}{1+v-u} \right)^2 \left( 1 - \frac{z}{1+v-u} \right)^n
\]

\[
= (1 + v - u)^2 \left( 1 + \frac{(2-n)z}{1+v-u} \right)
\]

\[
= (1 + z + v - u)(1 + (1-n)z + v - u).
\]

\[B.4.2.\  \text{Restriction to isolated points.} \]

Suppose one of our exceptional divisors is the blow up of the locus \( R \) consisting of matrices \( \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix} \) with image contained in \([1 : 0] \in \mathbb{P}^1\). Then, as described in Section B.4, there is an isolated torus-fixed point in the exceptional \( E \) lying over the locus \( R \) given as the limit as \( t \to 0 \) of the one-parameter family

\[
\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + t \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Then, we have the restrictions

\[
H \mapsto -u
\]

\[
E \mapsto v - u.
\]

Here, \( E \) is the exceptional divisor containing \( p \). The first one is by restricting the tautological line bundle and considering the torus action. To see the restriction of \( E \), we note that the restriction of \( E \) to itself is \( \mathcal{O}_{\mathbb{P}(N_{R_1/p^3})}(-1) \). Then, we take the local chart around \( \pi(p) \) consisting of

\[
\begin{pmatrix}
1 & \frac{a_{01}}{a_{00}} \\
\frac{a_{10}}{a_{00}} & \frac{a_{11}}{a_{00}}
\end{pmatrix}
\]

and find the action on the coordinate \( \frac{a_{11}}{a_{00}} \) is \( v - u \). Also the normal bundle to \( p \) in \( X \) has chern class

\[
(1 + v - u)^2 (1 + u - v).
\]

To see this, consider the local chart around \( p \)

\[
\begin{pmatrix}
1 & \frac{a_{01}}{a_{00}} \\
0 & \frac{a_{11}}{a_{00}}
\end{pmatrix} + t \begin{pmatrix} 0 & 0 \\ \frac{A_{10}}{A_{11}} & 1 \end{pmatrix}
\]

which has local coordinates \( \frac{a_{01}}{a_{00}}, \frac{a_{11}}{a_{00}}, \frac{A_{10}}{A_{11}} \) on which \( T \) acts by characters \( v - u, v - u \) and \( u - v \) respectively.
B.5. Application of Atiyah-Bott.

Proof of Theorem [A.1] As before, we compute $p_X(-u, -v)$ due to our sign conventions. Let

$$\phi(H) = \frac{(H + du)(H + (d - 1)u + v) \cdots (H + dv) - (du) \cdots (dv)}{H}.$$ 

We want to pull $\phi(H)$ back to $X$ and integrate using Atiyah-Bott. We first integrate over $\tilde{C}_1$. Since $H$ pulls back to $dH - \sum_{i=1}^{n} m_i E_i = d(z - u) - dz = -du$, this is

$$[z] \frac{1}{(z + v - u)((1 - n)z + v - u)} \phi(-du) =$$

$$[z] \frac{1}{(v-u)^2} - \prod_{i=0}^{d} (iu + (d - i)v) \frac{1}{-du} =$$

$$\frac{(n - 2) \prod_{i=1}^{d} (iu + (d - i)v)}{(v - u)^3}.$$ 

Adding this to the contribution of $\tilde{C}_2$ yields

$$(n - 2) \prod_{i=0}^{d} (iu + (d - i)v) \frac{1}{(v - u)^3} \left( \frac{1}{du} - \frac{1}{dv} \right) =$$

$$(n - 2) \prod_{i=0}^{d} (iu + (d - i)v) \frac{1}{(v - u)^2} \frac{1}{duv}.$$ 

For each $1 \leq i \leq n$, we have a point in the configuration of multiplicity $m_i$. We have two isolated fixed points corresponding to $i$ lying above $R_i \cap C_1$ and $R_i \cap C_2$. For the point lying above $R_i \cap C_1$, $H$ pulls back to $dH - m_i E$, where $E$ is the exceptional divisor lying above $R_i$. This restricts to

$$-du - n(v - u) = (-d + m_i)u - m_i v$$

at the fixed point. The contribution to Atiyah Bott is

$$\frac{1}{(u - v)^3} \phi((-d + m_i)u - m_i v) =$$

$$\frac{1}{(u - v)^3} - \prod_{j=0}^{d} (ju + (d - j)v) \frac{1}{(-d + m_i)u - m_i v}.$$
Adding this to the contribution of the fixed point lying above \( R_i \cap C_2 \), we get

\[
\frac{1}{(u-v)^3} \prod_{j=0}^{d} (ju + (d-j)v) \left( \frac{1}{(d-m_i)u + m_i v} - \frac{1}{(d-m_i)v + m_i u} \right) =
\]

\[
- \frac{1}{(u-v)^2} \prod_{j=0}^{d} (ju + (d-j)v) \frac{d-2m_i}{((d-m_i)u + m_i v)((d-m_i)v + m_i u)}.
\]

Adding the contributions up yields the desired result. \( \square \)

**Appendix C. Cubic plane curves**

The computations of \( p_C \) for cubic plane curves \( C \) are elementary, but we provide them here for the sake of completeness.

The following table provides a complete list of polynomials classes \([\text{Orb}(C)]_{GL(V)}\) for all cubics. When the automorphism group is infinite, the entries are simply \([\text{Orb}(C)]_{GL(V)}\). When the automorphism group is finite, the entries are \( p_C \).

| Cubic Curve \( C \) | \([\text{Orb}(C)]_{GL(V)} \) or \( p_C \) | \# Aut |
|---------------------|-------------------------|-------|
| Triple Line         | \(-(72c_1^4c_2^2 + 36c_1c_3^2 + 36c_1^4c_3 + 162c_1^2c_2c_3 + 243c_1^2c_3^2)\) | \( \infty \) |
| Double Line plus Line | \(-(72c_1^2c_2 + 36c_1c_3^2 - 108c_1^2c_3)\) | \( \infty \) |
| Three concurrent lines | \(12c_1^4 + 6c_2^2c_3 + 27c_1c_3\) | \( \infty \) |
| Conic plus tangent line | \(-36c_1^3 - 18c_1c_2\) | \( \infty \) |
| Triangle            | \(-12c_1^2 + 6c_1c_2 + 27c_3\) | \( \infty \) |
| Conic plus line     | \(18c_1^2 + 9c_2\) | \( \infty \) |
| Cuspidal cubic      | \(24c_1^2\) | \( \infty \) |
| Irreducible nodal cubic | \((-12c_1)6\) | 6    |
| Smooth cubic \( j \neq 0, 1728 \) | \((-12c_1)18\) | 18   |
| Smooth cubic with \( j = 1728 \) | \((-6c_1)36\) | 36   |
| Smooth cubic with \( j = 0 \) | \((-4c_1)54\) | 54   |

Let us only indicate the methods of calculation, leaving details to the reader. The formulas for a triple line, double line plus line, conic plus line, and triangle can all be obtained via presentation and integration along the lines of [FNR06, Theorem 3.1] and Proposition 3.5. This is the method of resolution and integration [FR06, Section 3].

The formula for three concurrent lines, conic plus tangent line and cuspidal cubic can be gotten by applying Kazarian’s formula [Kaz03a, Theorem 1] for counting \( D_4 \), \( A_3 \), and \( A_2 \) singularities respectively. This was carried out for the case of quartic plane curves for \( A_6 \), \( D_6 \), and \( E_6 \) in the proof of Corollary 3.4. The formula for smooth and nodal cubics can be obtained by their predegree formulas [AF93b, Section 3.6] and Proposition 2.3.
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