Particle Number Fluctuations, Rényi and Symmetry-resolved Entanglement Entropy in Two-dimensional Fermi Gas from Multi-dimensional Bosonization

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In this paper, we revisit the computation of particle number fluctuations and the Rényi entanglement entropy of a two-dimensional Fermi gas using multi-dimensional bosonization. In particular, we compute these quantities for a circular Fermi surface and a circular entangling surface. Both quantities display a logarithmic violation of the area law, and the Rényi entropy agrees with the Widom conjecture. Lastly, we compute the symmetry-resolved entanglement entropy for the two-dimensional circular Fermi surface and find that, while the total entanglement entropy scales as \( R \log R \), the symmetry-resolved entanglement scales as \( \sqrt{R \log R} \), where \( R \) is the radius of the sub-region of our interest.

I. INTRODUCTION

In recent years, there has been a surge of interest in quantum entanglement and its various measures, in the condensed matter as well as the high energy physics communities [11, 12]. One of the most profound results pertaining to the entanglement entropy in many-body systems is the area law for ground states of gapped systems, where the entanglement entropy is known to be proportional to the area of a subregion \( A \). This area law underlies the simulatability of gapped ground states by matrix product states [10]. Intuitively, degrees of freedom in a system with local interactions are entangled only with their neighbours, so the entanglement entropy receives contributions primarily from the degrees of freedom situated close to the boundary.

Even though the appearance of the area law behaviour in ground states is ubiquitous, there are known exceptions where the area law is violated, typically with a logarithmic correction. Some well-known examples are conformal field theories in one spatial dimension which describe quantum critical points, and Fermi liquids in higher spatial dimensions with a Fermi surface [11, 12].

While the von Neumann entropy and related measures are important physical quantities, they are difficult to compute analytically for generic many-body systems. Conformal field theories in one spatial dimension are among the most analytically tractable systems since the replica technique can be applied there. In these cases, the computation of the entanglement entropy boils down to the evaluation of the correlation functions of twist operators. Another approach for one-dimensional systems would be to use the Fisher-Hartwig formula for free systems. There are, however, fewer analytical calculations done in spatial dimensions greater than one. Calculations for the entanglement entropy of a higher dimensional Fermi surface can either be done by applying the Widom conjecture or bosonization, or by simply dividing up the multi-dimensional Fermi surface into many one-dimensional pieces where one can use the known one-dimensional results [5, 13–15].

In this paper, we apply the multi-dimensional bosonization technique developed in [16, 17] to calculate the entanglement entropy and related quantities of a Fermi gas analytically and non-perturbatively. Firstly, we compute the particle number cumulants generating function. This quantity is then used to carefully derive the entanglement entropy for an isotropic Fermi gas, which is found to be in agreement with Widom’s conjecture. This implies that the leading term in the entanglement entropy of a Fermi gas comes primarily from the modes near the Fermi surface. Next, the particle number cumulants generating function is also used to compute the symmetry-resolved entanglement of a two-dimensional Fermi gas [13]. We find that each particle number sector contributes an entanglement of \( \sqrt{R \log R} \), while the total entanglement entropy scales as \( R \log R \), where \( R \) is the radius of the subregion of our interest.

II. REVIEW OF MULTI-DIMENSIONAL BOSONIZATION

Before proceeding with the calculations, we review a scheme of multi-dimensional bosonization developed in [16, 17]. Alternate formulations of multi-dimensional bosonization can be found in [19, 20]. Given a filled Fermi sea, we can create and annihilate particle-hole pairs with the following operators

\[
n_q(\vec{k}) = c_{\vec{k}}^\dagger - \frac{q}{2} c_{\vec{k} + \frac{\vec{q}}{2}}^\dagger c_{\vec{k} - \frac{\vec{q}}{2}},
\]

where \( c_{\vec{k}}^\dagger, c_{\vec{k}} \) are the electron annihilation/creation operators with momentum \( \vec{k} \). Because they are quadratic in the fermion operators, their commutators are almost bosonic. However, they do not annihilate the Fermi sea. We need to normal order the particle-hole operators relative to the Fermi sea, so [17] defined the following operators...
where $\Theta(x) = 1(-1)$ if $x > 0(<0)$ and $\Phi_A(|\vec{k} - \vec{k}_F|)$ is some dimensionless smearing function that keeps the vectors $\vec{k}$ close to the Fermi momentum $\vec{k}_F$. More precisely, it is defined as

$$\lim_{\Lambda \to 0} \Phi_A(|\vec{k} - \vec{k}_F|) = \delta_{\vec{k}, \vec{k}_F}$$

(3)

where $\Lambda$ is a momentum space cutoff. We have also defined the velocity of the particles as $\vec{v}_k = \vec{V}_\epsilon_k$, with $\epsilon_k$ being the spectrum of the one-particle states. The idea is to divide the Fermi surface into patches of radius $\Lambda$ centered about $\vec{k}_F$, and $\vec{q}$ is constrained to lie within the patch, so that $\vec{q} \ll \Lambda \ll \vec{k}_F$. By construction, $\partial_{\vec{q}}(\vec{k}_F)$ annihilates the Fermi sea $|F.S.\rangle$, $a_{\vec{q}}(\vec{k}_F)|F.S.\rangle = 0$.

For each patch, the local density of states is

$$N_A(\vec{k}_F) = \frac{1}{V} \sum_{\vec{k}} |\Phi_A(|\vec{k} - \vec{k}_F|)|^2 \delta(\mu - \epsilon_\vec{k})$$

(4)

where the chemical potential is $\mu = \epsilon_\vec{k}_F$ and the total system size is $V$. The total density of states is

$$N(0) = \frac{1}{V} \sum_{\vec{k}} \delta(\mu - \epsilon_\vec{k})$$

(5)

and they are related by $N_A(\vec{k}_F) = \frac{N(0)}{d}$ for an isotropic Fermi surface, where $S_d$ is the $d$-dimensional solid angle. For convenience, rescale the bosonic operators (2) as

$$b_q(\vec{k}_F) = \left[N_A(\vec{k}_F)V|\vec{q} \cdot \vec{v}_{\vec{k}_F}|\right]^{-1/2} a_q(\vec{k}_F).$$

(6)

These operators obey the usual bosonic algebra

$$[b_q(\vec{k}_F), b_{-q}^\dagger(\vec{k}_F')] = \delta_{\vec{k}_F, \vec{k}_F'} (\delta_{\vec{q}, \vec{q}'} + \delta_{\vec{q}' - \vec{q}, \vec{q}}).$$

(7)

For the restricted Hilbert space that contains excitations close to the Fermi surface, the non-interacting Hamiltonian is effectively given by

$$H_0 = \sum_{\vec{k}_F} \sum_{\vec{q}, \vec{q}'} |\vec{q} \cdot \vec{v}_{\vec{k}_F}| b_{\vec{q}}(\vec{k}_F) b_{-\vec{q}}^\dagger(\vec{k}_F).$$

(8)

We see that these bosons diagonalize the non-interacting low energy Hamiltonian. The electronic density is related to the bosons as follows

$$\rho(\vec{q}) = \sum_{\vec{k}_F} \left[N_A(\vec{k}_F)|\vec{q} \cdot \vec{v}_{\vec{k}_F}|\right]^{1/2} \left[b_{\vec{q}}^\dagger(-\vec{k}_F) + b_{\vec{q}}(\vec{k}_F)\right].$$

(9)

This is the multi-dimensional bosonization identity which relates the fermionic density with the bosonic modes.

For the rest of the paper, we will restrict ourselves to two spatial dimensions.

### III. Free Fermion Particle Number Cumulant Generating Functional

For a given subregion $A$, we define the generating function of particle number cumulants to be

$$\langle e^{i\lambda N_A} \rangle, \quad \lambda \in \mathbb{C},$$

(10)

where $\hat{N}_A$ is the number operator of subregion $A$. The generating function produces the cumulants of the particle number distribution in subregion $A$ via

$$V_A^{(m)} = (-i\partial_\lambda)^m \log \langle e^{i\lambda N_A} \rangle \bigg|_{\lambda = 0}. $$

(11)

In particular, the second cumulant (the variance) is

$$V_A^{(2)} = \langle (\hat{N}_A - \langle\hat{N}_A\rangle)^2 \rangle. $$

(12)

Without interactions, the Hamiltonian is given by (8) for low lying states, so the ground state is annihilated by $b_{\vec{q}}(\vec{k}_F)$. Defining $f(\vec{r}) = i\lambda \Theta(\vec{r} \in A)$, where $\Theta(\vec{r})$ is the two-dimensional step function, the generating function (10) can be written as

$$\langle e^{i\lambda \hat{N}_A} \rangle = \langle \exp \left[ \int d^d r \rho(\vec{r}) f(\vec{r}) \right] \rangle = \langle \exp \left[ \sum_{\vec{k}} \rho(\vec{k}) f(-\vec{k}) \right] \rangle, $$

(13)

where the momentum-space density operator is related to the bosonic modes via (9). Since the expectation value is computed in the ground state of (8), it can be simplified by the Baker-Campbell-Hausdorff formula. Let us further restrict ourselves to a circular Fermi surface for the rest of the paper. The generating function then simplifies to
Here, a momentum regulator $\alpha$ has been introduced so that only states with small excitation momenta normal to the Fermi surface are kept.

It remains to perform the momentum sums and the spatial integrals. We first integrate over the excitation momentum $\vec{q}$. This can be done by decomposing $\vec{q}$ into two parts, $\vec{q} = \vec{q}_N + \vec{q}_T$ where $\vec{q}_N/T$ are the components normal/tangential to the Fermi surface. The subsequent spatial integral leads to

$$\langle e^{i\lambda \hat{N}_A} \rangle = \exp \left[ -\frac{\lambda^2}{2V} \sum_{\vec{k}_F} \int_{\vec{r} \in A} d^d r \int_{\vec{r}' \in A} d^d r' \sum_{\vec{q} \neq 0} N_A(\vec{k}_F) (\vec{q} \cdot \vec{v}_{\vec{k}_F}) e^{i\vec{q} \cdot (\vec{r}' - \vec{r})} e^{\alpha |\vec{q} \cdot \vec{v}_{\vec{k}_F}|} \right].$$

(14)

FIG. 1. The variance $V_A^{(2)} / R$ as a function of the radius $R$ of the subregion obtained from the numerical evaluation of (19) (dots) fitted against the analytical result (18) (curve). The fit gives $\alpha = 0.0603$. The system size is $L = 20$ and $k_F = \pi$.

where $A$ is the overlap matrix given in terms of the single particle eigenfunctions $\phi_n(\vec{r})$ as

$$\mathcal{A}_{nm} = \int_{\vec{r} \in A} d^d r \phi_n^*(\vec{r}) \phi_m(\vec{r}).$$

(20)

The overlap matrix $A$ is the continuum limit of the more familiar correlation function $C$ used in computing entanglement entropy for free fermions on a lattice [21], $\text{Tr} C^n = \text{Tr} \mathcal{A}^n$. (Working in the continuum model allows us to select a perfectly circular subregion, which is not possible in a lattice model.) For a total system of size $L \times L$ with periodic boundary conditions,

$$\mathcal{A}_{nm} = \frac{1}{L^2} \int_{\vec{r} \in A} d^d r e^{i(k_m - k_n) \cdot \vec{r}}$$

(21)

$$= \frac{1}{2\pi R} \frac{J_1(|k_m - k_n| R)}{|k_m - k_n| L^2},$$

where $\vec{k}$ lies within a circular Fermi surface of radius $k_F$ and $J_1(x)$ is a Bessel function of the first kind. The subregion $A$ is a disk of radius $R$ as before. The numerically obtained variance [19] with $L = 20$, $k_F = \pi$ is compared with the analytical result [18] in Fig. [1]. Here, we use the regulator $\alpha$ as a fitting parameter with $\alpha = 0.0603$.

IV. ENTANGLEMENT ENTROPY

In this section, we compute the Rényi entanglement entropy of a free Fermi gas in a circular subregion $A$ with a two-dimensional isotropic Fermi surface:

$$S_A^{(n)} = \frac{1}{1 - n} \log \text{Tr} (\rho_A^n).$$

(22)
As mentioned earlier, for a free Fermi gas, the quantum entanglement and the particle number variance for a given subregion are proportional to each other\textsuperscript{21–23},

\[
\frac{S^{(n)}_A}{V_A^{(2)}} = \frac{(1 + n^{-1})\pi^2}{6} + O(1), \tag{23}
\]

where $V_A^{(2)}$ is the second cumulant of the generating function we found earlier. Using our previously obtained analytical result for the particle number variance, we readily obtain

\[
S^{(n)}_A = (1 + n^{-1}) \frac{k_F}{2} R \ln \frac{R}{\alpha} + \ldots. \tag{24}
\]

The fact that we are able to obtain the leading term in the Rényi entropy by multi-dimensional bosonization implies that the leading contribution comes from the modes near the Fermi surface.

Let us also mention that (23) is not the only way to relate the Rényi entropy with the particle number variance. The Rényi entropy can be written in terms of expectation values of the form (10) with particular choices of $\lambda$\textsuperscript{29}. This approach yields the same result as (23).

V. SYMMETRY-RESOLVED ENTANGLEMENT

Having computed the Rényi entropy, we turn our attention to the charged entanglement,

\[
S^{(n)}_A(c) = \text{Tr} \left( \rho^n_A e^{icN_A} \right), \quad c \in \mathbb{R}. \tag{25}
\]

This quantity has a nice interpretation of a replicated path integral with flux insertion\textsuperscript{18}, as will be demonstrated later. The charged entanglement entropy and its variants have been used to detect and distinguish symmetry-protected topological phases\textsuperscript{27–28}. It has also been studied holographically\textsuperscript{29–32}. Performing the Fourier transform, we obtain the symmetry-resolved entanglement\textsuperscript{18},

\[
S^{(n)}_A(N_A) = \int_{-\pi}^{\pi} dc S^{(n)}_A(c) e^{-icN_A} = \text{Tr} \left( \rho^n_A P_{N_A} \right), \tag{26}
\]

where $P_{N_A}$ is the projector onto the subspace with $N_A$ particles in region $A$. In other words, the symmetry-resolved entanglement is the contribution to the $n$-th Rényi entropy from states with $N_A$ particles in region $A$.

We begin by computing the charged entanglement. The following derivation generalizes the computation of the entanglement entropy in\textsuperscript{26} to compute the charged entanglement entropy. Let us consider the partial $U(1)$ rotation restricted to region $A$:

\[
M = e^{icN_A}. \tag{27}
\]

In the basis of fermionic coherent states,

\[
M = \int d\psi d\bar{\psi} d\chi d\bar{\chi} e^{-(\bar{\psi}\psi + \bar{\chi}\chi)} M(\bar{\psi}, \chi) |\psi\rangle \langle \chi|, \tag{28}
\]

where $\psi, \bar{\psi}, \chi, \bar{\chi}$ are Grassmann numbers. (These coherent states are constructed in the basis that diagonalizes the entanglement Hamiltonian.) We have suppressed the indices of the Grassmann variables for notational simplicity. We also absorb all normalization constants into the integration measure. Here,

\[
M(\bar{\psi}, \chi) = e^{\phi \bar{\psi} \chi}, \quad \phi = e^{ic}. \tag{29}
\]

Performing the Grassmann integrals\textsuperscript{33}, we obtain the following simple form for the charged Rényi entropy

\[
\text{Tr} \left( \rho^n_A e^{icN_A} \right) = \int \prod_i d\alpha_i d\bar{\alpha}_i \rho_A(\alpha_i, \bar{\alpha}_i) e^{\sum_{i,j} \bar{\alpha}_i T_{ij} \alpha_j}, \tag{30}
\]

where $\alpha_i, \bar{\alpha}_i$ are Grassmann variables, and the $T$ matrix in the replica space is given by

\[
T = \begin{bmatrix}
-1 & e^{-ic} \\
-1 & \ddots \\
& & -1
\end{bmatrix} \tag{31}
\]

with eigenvalues

\[
\lambda_k = e^{i\left(\frac{2\pi k}{n}\right)}, \quad k = -\frac{n-1}{2}, \ldots, -\frac{n-1}{2}. \tag{32}
\]

This matrix connects the fermions in each sheet of the replica path integral to the next and the phase factor in the upper right hand corner of the matrix corresponds to an Aharonov-Bohm phase that the fermions acquire if they pass through all the sheets of the replicated spacetime and go back to the original sheet. This is the reason why we can interpret the charged Rényi entropy as a replicated path integral with flux insertion. One can then factorize $S^{(n)}_A(c)$ as

\[
S^{(n)}_A(c) = \text{Tr} \left( \rho^n_A e^{icN_A} \right) = \prod_k Z_k, \tag{33}
\]

where $Z_k$ is a ground state expectation value

\[
Z_k = \langle \Psi | T_k | \Psi \rangle = \text{Tr} (\rho_A T_k),
\]

\[
T_k = \exp \left( i \Lambda_k \sum_j c^\dagger_j c_j \right) = \exp \left( i \Lambda_k \sum_\mu f^\dagger_\mu f_\mu \right). \tag{34}
\]

Here, $c_j$ are the real space fermions while $f_\mu$ are the fermions that diagonalize the entanglement Hamiltonian, and they are related by a unitary transformation. $\Lambda_k$ is related to $\lambda_k$ as $\Lambda_k = \frac{e^{-i2\pi k}}{n}$. We can now utilize
our previous result for the generating function of particle number cumulants \[17\] with \( \lambda = \Lambda_k \). We thus arrive at

\[
S_A^{(n)}(c) = \exp \left[ -\frac{k_F I(R)}{(2\pi)^2} \sum_{k = -\infty}^{\infty} \Lambda_k^n \right] = e^{-\frac{n^2}{2\pi^2} k_F I(R) c} e^{-\frac{1}{4n} k_F I(R)}. \tag{35}
\]

The Fourier transform of \( S_A^{(n)}(c) \) gives the Rényi entropy for particle number \( N_A \)

\[
S_A^{(n)}(N_A) = S_A^{(n)}(c = 0) \int_{-\pi}^{\pi} \frac{dc}{2\pi} e^{-\frac{k_F I(R)}{N} \left( \frac{c}{2\pi} \right)^{2-icN_A}}. \tag{36}
\]

Assuming \( R \log R \gg 1 \), the integrand is negligible for large \( c \), so we might as well extend the integration region to \( \mathbb{R} \) and get a Gaussian integral, leading to

\[
S_A^{(n)}(N_A) = \frac{\pi n}{k_F I(R)} e^{-\frac{k_F I(R)}{N} \frac{\alpha^2 - 1}{4} - \frac{\alpha^2 N_A^2}{2 \pi^2 I(R)}.} \tag{37}
\]

Finally, the entanglement entropy for a given particle number \( N_A \) is

\[
S_A(N_A) = \frac{1}{6} \sqrt{2\pi k_F R \log R \alpha} e^{-\frac{\pi^2 N_A^2}{2 \pi^2 k_F R \log R}} \tag{38}
\]

where we kept only the leading order term in the final expression.

VI. CONCLUSION

In conclusion, we applied multi-dimensional bosonization to compute the generating function of the particle number cumulants of a circular subregion \( A \) for a two-dimensional Fermi gas. This generating function is then used to compute the Rényi entropy of the Fermi gas, which agrees with known results. These quantities show a logarithmic violation of the area law. We then proceed to compute the symmetry-resolved entanglement of the 2d Fermi gas, extending the results in \[18\] to two spatial dimensions. Each charge sector is observed to give a \( \sqrt{R \log R} \) contribution to the total von Neumann entanglement entropy, which scales as \( R \log R \).

The success of multi-dimensional bosonization in computing these quantities suggests that one could try to apply multi-dimensional bosonization to compute other quantities in a non-perturbative ab initio approach.

While we focused on a non-interacting Fermi gas, the power of bosonization lies in its ability to treat Fermi liquid interactions. Before closing, we give a brief comment on the effects of Fermi liquid interactions on the particle number cumulants. Utilizing the formalism of \[17\], we computed the particle number cumulants generating function with an isotropic contact interaction (with a spherical Fermi surface and a spherical entangling surface, as in the case of the free fermi gas). In this computation, the effects of interactions can be incorporated by a Bogoliubov transformation, which relates the modes that diagonalize the interacting Hamiltonian to the non-interacting modes, thereby realizing Landau’s adiabatic principle. Unfortunately, we found that the calculation is plagued by an infrared (IR) divergence. The best way to deal with the IR divergence is at this moment unclear, and left for future investigation. We suspect that the IR divergence might be cured with an improved treatment of collective modes within the bosonization framework.

Heuristically, dropping the IR divergence by hand, we found that the coefficient of the leading logarithmic term decreases as we turn on interactions:

\[
\langle e^{i\Lambda A} \rangle = \exp \left[ -\frac{\lambda^2 k_F}{2\pi^2} \left( 1 - \frac{1}{2} \left( \frac{gN}{1 + gN} \right)^2 \right) R \log R + \cdots \right],
\tag{39}
\]

where \( g \) is the dimensionless coupling constant. In particular, it approaches half of the non-interacting value in the limit of infinite coupling. This decrease is consistent with the known result in one-dimensional Tomonaga-Luttinger liquids \[22\].

It would be interesting to calculate the particle number cumulants numerically and compare them with the above findings.

\textit{Note added}: While this draft was being prepared, \[34\] appeared on arXiv, where the symmetry-resolved entanglement for higher-dimensional Fermi gases was computed using Widom's conjecture.

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