Spin-Statistics Theorem in Path Integral Formulation

Kazuo Fujikawa

Department of Physics, University of Tokyo
Bunkyo-ku, Tokyo 113, Japan

Abstract

We present a coherent proof of the spin-statistics theorem in path integral formulation. The local path integral measure and Lorentz invariant local Lagrangian, when combined with Green’s functions defined in terms of time ordered products, ensure causality regardless of statistics. The Feynman’s $m - i\epsilon$ prescription ensures the positive energy condition regardless of statistics, and the abnormal spin-statistics relation for both of spin-0 scalar particles and spin-1/2 Dirac particles is excluded if one imposes the positive norm condition in conjunction with Schwinger’s action principle. The minus commutation relation between one Bose and one Fermi field arises naturally in path integral. The Feynman’s $m - i\epsilon$ prescription also ensures a smooth continuation to Euclidean theory, for which the use of the Weyl anomaly is illustrated to exclude the abnormal statistics for the scalar and Dirac particles not only in 4-dimensional theory but also in 2-dimensional theory.

1 Introduction

The spin-statistics theorem is one of the basic theorems in theoretical physics. It has a long history\cite{1}-\cite{9}, and Pauli established the theorem in its standard form\cite{6}. In a Lorentz invariant local field theory, the theorem holds provided the following 3 conditions are satisfied:

1. The vacuum is the lowest energy state.
2. Field variables either commute or anti-commute at space-like separation.
3. Norm in the Hilbert space is positive definite.

A further refinement of the theorem, in particular, the logical independence of the spin-statistics theorem and CPT theorem, another fundamental theorem in local field theory, has been shown by Lüders and Zumino\cite{10} and Burgoyne\cite{11}. The formulation of Wightman\cite{12} plays an essential role
here. All these approaches are based on the operator formalism. A comprehensive account of the spin-statistics theorem has been given recently in Ref. [13].

Another formulation of quantum theory, namely, the Feynman path integral (and Schwinger’s action principle) is widely used in the applications of modern field theory. It is desirable to show the spin-statistics theorem in path integral approach not only for its logical completeness but also for a pedagogical purpose. Also, the particles with abnormal statistics, such as the Faddeev-Popov ghost [14] and the bosonic Dirac particle as the Pauli-Villars regulator (see, for example, [15]), are commonly used in path integral, but these particles do not violate the causality as one might naively expect. The aim of the present note is to show that we can now give a coherent proof of the spin-statistics theorem in modern path integral formulation which incorporates the Grassmann numbers as an integral part [16]. A salient feature of path integral formulation is that all the Green’s functions are defined in terms of time ordered products, and the notion such as the Wightman function [12] is not available, at least in a natural way.

From a technical viewpoint, the major difference of various approaches to the spin-statistics theorem lies in how to incorporate the positive energy condition (Condition 1 above). Pauli in his original paper [8] used essentially an explicit form of energy-momentum tensor. Lüders and Zumino [10] and also Burgoyne [11] greatly simplified the analysis by simply imposing the positive energy condition, which in turn leads to an analyticity of the Wightman function. The formulation of Wightman [12] enjoys a mathematical rigor and generality, but it is not quite accessible to everybody who is interested in the applications of field theory. Besides, an explicit construction of non-trivial models remains as a difficult issue in Wightman’s formulation. We here emphasize the familiar Feynman’s $m - i\epsilon$ prescription as a manifestation of positive energy condition, which simplifies the analysis. The Feynman’s $m - i\epsilon$ prescription is here assigned a more fundamental meaning than just representing a specific boundary condition to reproduce the result of operator formalism: In any fixed time slice of 4-dimensional space-time, positive energy particles propagate in forward time direction and negative energy particles propagate in backward time.
direction. By this way, the positive energy condition up to any finite order in perturbation theory is ensured regardless of statistics.

The plan of this note is as follows: We first briefly summarize the basic requirement of the path integral measure. The ordinary complex numbers and the Grassmann numbers satisfy the basic requirement to define a local path integral measure. In the classical level, the field variables are thus either totally commuting or totally anti-commuting. The complex (including real) numbers naturally give rise to Bose-Einstein statistics and the Grassmann numbers give rise to Fermi-Dirac statistics after performing path integral. The minus commutation relation between one Bose and one Fermi field arises naturally in path integral, since the complex numbers and Grassmann numbers commute in the classical level. The local path integral measure and Lorentz invariant local Lagrangian together with propagators, which are always defined in terms of time ordered products, are shown to ensure the causality (Condition 2 above) regardless of statistics. If one employs Feynman’s $m - i\epsilon$ prescription, which allows a smooth continuation from Euclidean theory, the positive energy condition is ensured regardless of statistics. The basic criterion to exclude abnormal spin-statistics relation is thus the positive norm in the Hilbert space or positive probability for scattering processes (Condition 3 above).

We thus examine what happens if we apply the Grassmann numbers to spin-0 particles, or the complex numbers to spin-1/2 particles in path integral. In conjunction with Schwinger’s action principle, it is shown that the indefinite metric appears for spin 0 particles if one uses Grassmann numbers, and the negative metric for the negative energy states appears if one uses complex numbers for Dirac particles. The Feynman’s $m - i\epsilon$ prescription also allows a smooth continuation to Euclidean theory, for which we illustrate the use of the Jacobian factor $[17]$ related to the Weyl transformation, which is sensitive to statistics, to exclude the abnormal statistics for the scalar and Dirac particles not only in 4-dimensional theory but also in 2-dimensional theory. The natural reasoning of our path integral analysis corresponds to that of Feynman$[7]$ and Pauli$[8]$. This is in contrast to the analyses on the basis of operator formalism in the standard textbooks$[18]$, which emphasize the acausal behavior for abnormal statistics. See also Ref.$[19]$ for a variation of the argument.
2 Schwinger’s action principle and path integral measure

The basic requirement of the path integral measure is that it is “translation” invariant in the functional space. By considering an infinitesimal quantity $\epsilon$ in the conventional integral, we have

$$\int_{-\infty}^{\infty} dx f(x) = \int_{-\infty}^{\infty} d(x + \epsilon) f(x + \epsilon) = \int_{-\infty}^{\infty} dx f(x + \epsilon)$$

$$= \int_{-\infty}^{\infty} dx f(x) + \epsilon \int_{-\infty}^{\infty} dx \frac{d}{dx} f(x)$$

namely,

$$\int_{-\infty}^{\infty} dx \frac{d}{dx} f(x) = 0$$

which states the vanishing integral of a derivative. The relation which appeared here

$$d(x + \epsilon) = dx$$

becomes the basic relation in path integral also.

To derive the basic requirement for the path integral measure, we start with a path integral for a real scalar particle

$$\langle 0 | 0 \rangle_J = \int D\phi \exp\{i \int d^4x [\frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) - \frac{1}{2} m^2 \phi(x)^2 + \phi(x) J(x)]\}$$

where we added the source term. For the moment, we deal with an abstract definiton of the path integral measure $D\phi$ without precise specification.

We then have the basic condition on the path integral

$$\langle 0 | \delta^2 \hat{\phi}(x) + m^2 \hat{\phi}(x) - J(x) | 0 \rangle_J$$

$$= \int D\phi \{ \partial_\mu \phi(x) + m^2 \phi(x) - J(x) \}$$

$$\times \exp\{i \int d^4x [\frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) - \frac{1}{2} m^2 \phi(x)^2 + \phi(x) J(x)]\}$$

$$= i \int D\phi \frac{\delta}{\delta \phi(x)} \exp\{i \int d^4x [\frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) - \frac{1}{2} m^2 \phi(x)^2 + \phi(x) J(x)]\}$$

$$= 0$$

where the first equality in this relation is a result of Schwinger’s action principle

$$\frac{\delta}{\delta J(x)} \langle 0 | 0 \rangle_J = i \langle 0 | \hat{\phi}(x) | 0 \rangle_J.$$
and

\[ \langle 0 | \partial_{\mu}^2 \hat{\phi}(x) + m^2 \hat{\phi}(x) - J(x) | 0 \rangle_J = 0 \]  \hspace{1cm} (2.7) 

is the equation of motion in the language of operator formalism.

This basic relation is satisfied by the “translational” invariance of the path integral measure in functional space

\[ \mathcal{D}(\phi + \varepsilon) = \mathcal{D}\phi. \]  \hspace{1cm} (2.8) 

Here, \( \varepsilon(x) \) is an infinitesimal arbitrary function. This fact is understood by defining \( \phi'(x) = \phi + \varepsilon \) as follows:

\[
\int \mathcal{D}\phi \exp\{ i \int d^4x [\frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) - \frac{1}{2} m^2 \phi(x)^2 + \phi(x) J(x)] \} \\
= \int \mathcal{D}\phi' \exp\{ i \int d^4x [\frac{1}{2} \partial_\mu \phi'(x) \partial^\mu \phi'(x) - \frac{1}{2} m^2 \phi'(x)^2 + \phi'(x) J(x)] \} \\
= \int \mathcal{D}\phi \exp\{ i \int d^4x [\frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) - \frac{1}{2} m^2 \phi(x)^2 + \phi(x) J(x)] - i \int d^4x \varepsilon(x) [\partial_\mu^2 \phi(x) + m^2 \phi(x) - J(x)] \} \]  \hspace{1cm} (2.9)

where the first equality is the statement that the naming of integration variables does not change the integral itself. This relation gives in the order linear in \( \varepsilon(x) \)

\[
\int \mathcal{D}\phi i \int d^4y \varepsilon(y) [\partial_\mu^2 \phi(y) + m^2 \phi(y) - J(y)] \\
\times \exp\{ i \int d^4x [\frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) - \frac{1}{2} m^2 \phi(x)^2 + \phi(x) J(x)] \} = 0(2.10)
\]

If one chooses \( \varepsilon(y) \) as a \( \delta \)-functional one with a peak at \( x \), one satisfies the basic requirement of the action principle (2.5). This analysis is valid for interacting fields also.

We thus learn that the “translation” invariance of the path integral measure is equivalent to the equation of motion in operator formalism. At this moment, two viable definitions of the path integral measure are known:

1. The first is a generalization of the ordinary integral, which is “translation” invariant, by using the real or complex field variables

\[ \mathcal{D}\phi \equiv \prod_x d\phi(x) \]  \hspace{1cm} (2.11)
namely we integrate over the field variable \( d\phi(x) \), which is an ordinary number, at each space-time point. The ordinary complex or real numbers satisfy

\[
[\phi(x), \phi(y)] = 0
\] (2.12)

which leads to Bose-Einstein statistics after performing path integral.

2. The second choice is to regard the variable \( \psi(x) \) as Grassmann numbers defined at each point of space-time. (We here use \( \psi(x) \) for Grassmann variables, just for notational convenience.) The integral is then defined as the (left-)derivative with respect to the Grassmann numbers

\[
\mathcal{D}\psi \equiv \prod_x \delta \frac{\delta}{\delta \psi(x)}
\] (2.13)

which is also “translation” invariant in the functional space. The Grassmann numbers anti-commute with themselves

\[
\{\psi(x), \psi(y)\} = \psi(x)\psi(y) + \psi(y)\psi(x) = 0
\] (2.14)

and thus give rise to Fermi-Dirac statistics after performing path integral.

Field variables are either totally commuting or totally anti-commuting in the classical level, and the propagators we use in path integral are defined in terms of time ordered product. The appearance of the time ordered product is most easily understood if one formulates path integral starting with the evolution operator

\[
\langle f | \exp[-i\hat{H}(t_f - t_i)] | i \rangle
\] (2.15)

and time slicing. It is important to recognize that our path integral measure (2.11) or (2.13) is \textit{local} in the sense that path integral variables at each space-time point are allowed to change independently. The space-time correlation of field variables after path integral is thus what the Lorentz invariant local Lagrangian implies. It is shown later that the time ordered product combined with the local path integral measure and Lorentz invariant local Lagrangian ensure the basic requirement of causality (Condition 2 above) regardless of statistics.

We also emphasize that Grassmann numbers and ordinary complex numbers commute at the classical level, which naturally leads to commuting quantized variables after performing path integral. This property is not obvious in the operator formulation[10].
3 Fermi-Dirac statistics for spin 0

We now illustrate the difficulty if one uses Grassmann numbers for spin-0 real particles in path integral. The complex scalar field is written in terms of real scalar fields, and we first examine a real scalar field

$$\phi(x)^\dagger = \phi(x).$$ (3.1)

If one writes the ordinary classical Lagrangian in terms of Grassmann numbers, one obtains

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) - \frac{1}{2} m^2 \phi(x)^2 = 0$$ (3.2)

namely, one cannot define a meaningful theory. This is a path integral version of the standard argument against the fermionic interpretation of spin 0 particles in [1]-[6] and [10]-[13].

To avoid this difficulty, one may use a pair of real Grassmann fields $\xi(x)$ and $\eta(x)$

$$\mathcal{L} = \partial_\mu \xi(x) \partial^\mu \eta(x) - m^2 \xi(x) \eta(x)$$ (3.3)

where

$$\xi(x)^\dagger = \xi(x), \quad \eta(x)^\dagger = \eta(x).$$ (3.4)

The Faddeev-Popov ghost fields in gauge theory have this structure [14]. (One may equally consider a complex scalar defined by $\varphi(x) = (\xi(x) + i\eta(x))/\sqrt{2}$.) To ensure the unitarity of S-matrix, $SS^\dagger = S^\dagger S = 1$, the Lagrangian need to be hermitian. The above Lagrangian gives

$$\mathcal{L}^\dagger = \partial^\mu \eta(x) \partial_\mu \xi(x) - m^2 \eta(x) \xi(x)$$
$$= -\partial^\mu \xi(x) \partial_\mu \eta(x) + m^2 \xi(x) \eta(x)$$
$$= -\mathcal{L}$$ (3.5)

and thus we have to add an extra imaginary factor $i$

$$\mathcal{L} = i \partial_\mu \xi(x) \partial^\mu \eta(x) - im^2 \xi(x) \eta(x).$$ (3.6)

Path integral in this case is defined by

$$\int \mathcal{D}\xi \mathcal{D}\eta \exp\{i \int d^4x [i \partial_\mu \xi(x) \partial^\mu \eta(x) - im^2 \xi(x) \eta(x)]\}. \quad (3.7)$$
To derive the propagator, we add sources which are Grassmann numbers

\[ S_J = \int d^4x \{ i\partial_\mu \xi(x)\partial^\mu \eta(x) - i m^2 \xi(x)\eta(x) + \xi(x) J_\xi + J_\eta \eta(x) \} \]  

(3.8)

and consider the change of variables

\[ \xi(x) = \xi'(x) - J_\eta \frac{i}{\partial_\mu \partial^\mu + m^2 - i\epsilon} \eta(x), \]
\[ \eta(x) = \eta'(x) - \frac{i}{\partial_\mu \partial^\mu + m^2 - i\epsilon} J_\xi. \]  

(3.9)

The action is then written as

\[ S_J = \int d^4x \{ i\partial_\mu \xi'(x)\partial^\mu \eta'(x) - i m^2 \xi'(x)\eta'(x) - \frac{i}{\partial_\mu \partial^\mu + m^2 - i\epsilon} J_\xi \} \]  

(3.10)

and the path integral is written as

\[ Z(J) = \int D\xi D\eta \exp \left\{ i \int d^4x [i\partial_\mu \xi'(x)\partial^\mu \eta'(x) - i m^2 \xi'(x)\eta'(x) - \frac{i}{\partial_\mu \partial^\mu + m^2 - i\epsilon} J_\xi] \right\} \]  

(3.11)

by using the translational invariance of the measure \( D\xi D\eta = D\xi' D\eta' \). The propagator is then given by

\[ \langle 0| T^* \hat{\xi}(x)\hat{\eta}(y)|0 \rangle \]
\[ = \frac{1}{Z} \int D\xi D\eta \xi(x)\eta(y) \exp \left\{ i \int d^4x [i\partial_\mu \xi(x)\partial^\mu \eta(x) - i m^2 \xi(x)\eta(x)] \right\} \]
\[ = -\frac{1}{Z} \frac{\partial}{\partial J_\xi(x)} \frac{1}{\partial J_\eta(y)} \ln Z(J) \bigg|_{J=0} \]
\[ = (-i) \frac{i}{\partial_\mu \partial^\mu + m^2 - i\epsilon} \delta^4(x - y). \]  

(3.12)

where the operator expression in the left-hand side is a result of Schwinger’s action principle. This propagator reflects precisely the local structure of the Lagrangian, since the path integral measure is local. Note that we use the same Feynman’s \( m^2 - i\epsilon \) prescription as ordinary particles, which ensures
a smooth continuation from Euclidean theory. This $m^2 - i\epsilon$ prescription dictates the propagation of the negative energy solution in the negative time direction and ensures the positive energy condition (Condition 1).

One can first confirm the causality on the basis of Bjorken-Johnson-Low (BJL) prescription. By using the relation $\delta^4(x-y) = \int \frac{d^4k}{(2\pi)^4} \exp[-ik(x-y)]$, one can write the above propagator as

$$\langle 0 | T^* \hat{\xi}(x) \hat{\eta}(y) | 0 \rangle = (i) \int \frac{d^4k}{(2\pi)^4} \exp[-ik(x-y)] \frac{i}{k_\mu k^\mu - m^2 + i\epsilon}$$

(3.13)

or equivalently

$$\int d^4x \exp[ik(x-y)] \langle 0 | T^* \hat{\xi}(x) \hat{\eta}(y) | 0 \rangle = \frac{-1}{k_\mu k^\mu - m^2 + i\epsilon}.$$ \hspace{1cm} (3.14)

We now employ the BJL definition of $T$-product[20]: We can replace $T^*$-product by the conventional $T$-product if the following condition is satisfied

$$\lim_{k^0 \to \infty} \int d^4x \exp[ik(x-y)] \langle 0 | T^* \hat{\xi}(x) \hat{\eta}(y) | 0 \rangle = 0,$$ \hspace{1cm} (3.15)

namely, the equal time limit (i.e., $k^0 \to \infty$ limit) is well defined for the $T$-product. If the limit (3.15) does not vanish, the $T$-product is defined by subtracting it from $T^*$-product. This gives a general definition of $T$ product.

The above propagator (3.14) satisfies this condition, and we have

$$(-ik^0) \int d^4x \exp[ik(x-y)] \langle 0 | T^* \hat{\xi}(x) \hat{\eta}(y) | 0 \rangle = \int d^4x \exp[ik(x-y)]$$

$$\times \frac{\partial}{\partial x^0} [\theta(x^0 - y^0) \langle 0 | \hat{\xi}(x) \hat{\eta}(y) | 0 \rangle - \theta(y^0 - x^0) \langle 0 | \hat{\eta}(y) \hat{\xi}(x) | 0 \rangle]$$

$$= \int d^4x \exp[ik(x-y)] \delta(x^0 - y^0) \langle 0 | \{ \hat{\xi}(x), \hat{\eta}(y) \} | 0 \rangle$$

$$+ \int d^4x \exp[ik(x-y)] \langle 0 | T \partial_{x^0} \hat{\xi}(x) \hat{\eta}(y) | 0 \rangle$$

$$= \frac{ik^0}{k_\mu k^\mu - m^2 + i\epsilon}$$ \hspace{1cm} (3.16)

where we used the fact that Grassmann numbers are anti-commuting $\xi(x)\eta(y) = \eta(y)\xi(x)$. 
\(-\eta(y)\xi(x)\). By taking the limit \(k^0 \to \infty\) in this expression and remembering the definition of \(T\)-product, we conclude
\[
\int d^4 x \exp[i k(x - y)] \delta(x^0 - y^0) \langle 0|\{\hat{\xi}(x), \hat{\eta}(y)\}|0\rangle = 0,
\]
\[
\int d^4 x \exp[i k(x - y)] \langle 0|T \partial_{x^0} \hat{\xi}(x) \hat{\eta}(y)|0\rangle = \frac{ik^0}{k_\mu k^\mu - m^2 + i\epsilon}. \tag{3.17}
\]

Repeating the same procedure for the second expression in (3.17), we have
\[
(-ik^0) \int d^4 x \exp[i k(x - y)] \langle 0|T \partial_{x^0} \hat{\xi}(x) \hat{\eta}(y)|0\rangle
= \int d^4 x \exp[i k(x - y)] \delta(x^0 - y^0) \langle 0|\{\partial_{x^0} \hat{\xi}(x), \hat{\eta}(y)\}|0\rangle
+ \int d^4 x \exp[i k(x - y)] \langle 0|T \partial_{x^0}^2 \hat{\xi}(x) \hat{\eta}(y)|0\rangle
\]
\[
= \frac{(k^0)^2}{k_\mu k^\mu - m^2 + i\epsilon} \tag{3.18}
\]
and considering the limit \(k^0 \to \infty\), we conclude
\[
\int d^4 x \exp[i k(x - y)] \delta(x^0 - y^0) \langle 0|\{\partial_{x^0} \hat{\xi}(x), \hat{\eta}(y)\}|0\rangle = 1,
\]
\[
\int d^4 x \exp[i k(x - y)] \langle 0|T \partial_{x^0}^2 \hat{\xi}(x) \hat{\eta}(y)|0\rangle = \frac{\vec{k}^2 + m^2}{k_\mu k^\mu - m^2 + i\epsilon}. \tag{3.19}
\]

Using the operator equation of motion \((\partial_\mu \partial^\mu + m^2) \hat{\xi}(x) = 0\), the last equation in (3.19) is written as
\[
\int d^4 x \exp[i k(x - y)] \langle 0|T (\partial_l \partial^l - m^2) \hat{\xi}(x) \hat{\eta}(y)|0\rangle
= - (\vec{k}^2 + m^2) \int d^4 x \exp[i k(x - y)] \langle 0|T \hat{\xi}(x) \hat{\eta}(y)|0\rangle
= - (\vec{k}^2 + m^2) \frac{-1}{k_\mu k^\mu - m^2 + i\epsilon} \tag{3.20}
\]
and coming full circle back to the original equation (3.14). We thus obtain the anti-commutation relations
\[
\delta(x^0 - y^0) \langle 0|\{\hat{\xi}(x), \hat{\eta}(y)\}|0\rangle = 0,
\]
\[
\delta(x^0 - y^0) \langle 0|\{\partial_{x^0} \hat{\xi}(x), \hat{\eta}(y)\}|0\rangle = \delta^4(x - y) \tag{3.21}
\]
and the causality (Condition 2) is ensured.
We here note that an $i\epsilon$ prescription different from (3.14) such as
\[
\int d^4x \exp[ik(x - y)] |0| T^* \hat{\xi}(x) \hat{\eta}(y) |0\rangle = \frac{-1}{(k_0 + i\epsilon)(k_0 + i\epsilon) - \vec{k}^2 - m^2}
\] (3.22)
still satisfies the causality condition, though this choice does not ensure positive energy condition nor positive norm condition. This fact shows that the causality is a condition independent from other two basic conditions. Incidentally, $T^*$-product and $T$-product do not always agree with each other. For example, if one considers
\[
\mathcal{L} = -im^2 \xi(x) \eta(x)
\] (3.23)
the path integral gives
\[
\int d^4x \exp[ik(x - y)] |0| T^* \hat{\xi}(x) \hat{\eta}(y) |0\rangle = \frac{1}{m^2 - i\epsilon}
\] (3.24)
and BJL prescription gives
\[
\int d^4x e^{ik(x-y)} |0| T \hat{\xi}(x) \hat{\eta}(y) |0\rangle = \int d^4x e^{ik(x-y)} |0| T^* \hat{\xi}(x) \hat{\eta}(y) |0\rangle - \lim_{k_0 \to \infty} \int d^4x e^{ik(x-y)} |0| T^* \hat{\xi}(x) \hat{\eta}(y) |0\rangle = 0.
\] (3.25)
We also note that if one imposes the positive norm condition for abnormal spin-statistics assignment in operator formalism\[18\], one can of course detect the acausal behavior of the time ordered product by BJL prescription.

If one integrates over the variable $k^0$ in (3.13), one can write the propagator as
\[
\langle 0| T \hat{\xi}(x) \hat{\eta}(y) |0\rangle
\]
\[= \theta(x^0 - y^0) \langle 0| \hat{\xi}(x) \hat{\eta}(y) |0\rangle - \theta(y^0 - x^0) \langle 0| \hat{\eta}(y) \hat{\xi}(x) |0\rangle
\]
\[= (i) \int \frac{d^4k}{(2\pi)^4} \exp[-ik(x - y)] \frac{1}{k_\mu k^\mu - m^2 + i\epsilon}
\]
\[= (i) \left\{ \theta(x^0 - y^0) \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega} \exp[-i\omega(x^0 - y^0) + i\vec{k}(\vec{x} - \vec{y})]
\]
\[+ \theta(y^0 - x^0) \int \frac{d^3k}{(2\pi)^3} \exp[-i\omega(y^0 - x^0) + i\vec{k}(\vec{y} - \vec{x})] \right\}
\] (3.26)
with $\omega = \sqrt{\vec{k}^2 + m^2}$. Note that both of the terms with $\theta(x^0 - y^0) \exp[-i\omega(x^0 - y^0)]$ and $\theta(y^0 - x^0) \exp[-i\omega(y^0 - x^0)]$ ensure the positive energy condition,
which is a result of Feynman’s $m - i\epsilon$ prescription. The presence of the extra imaginary factor $i$ in the right-hand side of (3.26) thus shows the indefinite inner product of $\xi$ and $\eta$ fields in the Hilbert space. We note that this $m^2 - i\epsilon$ prescription and the propagator (3.13), when applied to the ghosts in gauge theory, are consistent with the BRST cohomology. See, for example, Ref. [21].

If one uses the complex Grassmann variable $\varphi(x) = (\xi(x) + i\eta(x))/\sqrt{2}$ instead, (3.26) is replaced by

$$
\langle 0| T\hat{\varphi}(x)\hat{\varphi}^\dagger(y)|0 \rangle = \theta(x^0 - y^0)\langle 0| \hat{\varphi}(x)\hat{\varphi}^\dagger(y)|0 \rangle - \theta(y^0 - x^0)\langle 0| \hat{\varphi}^\dagger(y)\hat{\varphi}(x)|0 \rangle
$$

$$
= \left(-\frac{i}{2}\right)\left[\langle 0| T\hat{\xi}(x)\hat{\eta}(y)|0 \rangle + \langle 0| T\hat{\xi}(y)\hat{\eta}(x)|0 \rangle\right]
$$

and the negative norm in the sense of operator formalism appears in the second term of the time ordered product: Namely, if one expands

$$
\hat{\varphi}(x) = \int \frac{d^3k}{(2\pi)^32\omega} [\hat{a}_k e^{-ikx} + \hat{b}_k^\dagger e^{ikx}]
$$

one obtains $\langle 0| \hat{b}_k\hat{b}_k^\dagger|0 \rangle < 0$. (If one uses the ordinary complex numbers for $\hat{\varphi}(x)$, the right-hand side of (3.27) remains the same but the coefficient of the term $\theta(y^0 - x^0)\langle 0| \hat{\varphi}^\dagger(y)\hat{\varphi}(x)|0 \rangle$ changes sign and the positive norm condition is satisfied. The causality is also satisfied for a general choice of $i\epsilon$ prescription.) What this means is that the operator transcription induced by path integration

$$
\varphi(x) \rightarrow \hat{\varphi}(x),
$$

$$
\varphi^\dagger(x) \rightarrow \hat{\varphi}^\dagger(x)
$$

ensures the hermitian conjugation $\hat{\varphi}^\dagger(x) = (\hat{\varphi}(x))^\dagger$ in the operator sense for the complex numbers but not for the Grassmann numbers.
An attempt to quantize a spin-0 field by using Grassmann variables thus either leads to a vanishing action or an indefinite metric in Hilbert space.

In passing, we comment on the quantization of Maxwell field (and also general Yang-Mills fields). In the Feynman gauge, one deals with the Lagrangian
\[
L = -\frac{1}{2} \partial_\alpha A_\mu \partial^\alpha A^\mu
\] (3.30)
and thus the use of Grassmann variables for \(A_\mu\) leads to a trivial theory. Besides, a consistent description of all the classical electromagnetic phenomena is lost in such a case. The Maxwell field need to be quantized in terms of ordinary real numbers.

4 Statistics for Dirac particles

To analyze the statistics for Dirac particles, we examine the QED-type Lagrangian
\[
L = \bar{\psi} i \not{D} \psi - m \bar{\psi} \psi
\] (4.1)
where
\[
\not{D} = \gamma^\mu (\partial_\mu - ie_0 A_\mu).
\] (4.2)
Our metric convention is \(g_{\mu\nu} = (1, -1, -1, -1)\), and the \(4 \times 4\) \(\gamma^\mu\) matrices satisfy the relation \((\gamma^\mu)\dagger = \gamma_\mu\), namely, the spatial components \(\gamma^k\) are anti-hermitian.

The path integral is defined by
\[
\int D\bar{\psi}D\psi \exp\{i \int d^4x [\bar{\psi} i \not{D} \psi - m \bar{\psi} \psi]\}. \tag{4.3}
\]
As for the propagator, we have
\[
\langle 0| T^* \hat{\psi}(x) \hat{\psi}(y) |0\rangle = \frac{i}{\not{D} - m + i\epsilon} \delta^4(x - y) \tag{4.4}
\]
which does not distinguish the Grassmann or ordinary complex numbers for the field variables. Here we used Feynman’s \(m - i\epsilon\) prescription, which ensures a smooth continuation from Euclidean theory.

The derivation of this propagator is important and we give a detailed account. We start with the path integral with source terms added
\[
Z(\eta, \bar{\eta}) = \int D\bar{\psi}D\psi \exp\{i \int d^4x L_\eta\}. \tag{4.5}
\]
with
\[ L_{\eta} = \bar{\psi} i \not\!D \psi - m \bar{\psi} \psi + \bar{\eta} \psi + \bar{\psi} \eta. \] (4.6)

We make the following change of variables in this \( L_{\eta} \)
\[ \psi(x) \rightarrow \psi'(x) - \frac{1}{i \not\!D - m + i\epsilon} \eta, \]
\[ \bar{\psi}(x) \rightarrow \bar{\psi}'(x) - \bar{\eta} \frac{1}{i \not\!D - m + i\epsilon} \]
and we obtain
\[ L_{\eta} = \bar{\psi}'[i \not\!D - m] \psi' - \bar{\eta} \frac{1}{i \not\!D - m + i\epsilon} \eta. \] (4.8)

We thus have
\[ Z(\eta, \bar{\eta}) = \int \mathcal{D} \bar{\psi} \mathcal{D} \psi \exp \{ i \int d^4 x [\bar{\psi}'[i \not\!D - m] \psi' - \bar{\eta} \frac{1}{i \not\!D - m + i\epsilon} \eta] \} \]
\[ = \int \mathcal{D} \bar{\psi}' \mathcal{D} \psi' \exp \{ i \int d^4 x [\bar{\psi}'[i \not\!D - m] \psi' - \bar{\eta} \frac{1}{i \not\!D - m + i\epsilon} \eta] \} \] \[ = \mathcal{D} \bar{\psi} \mathcal{D} \psi \exp \{ i \int d^4 x [\bar{\psi} i \not\!D \psi - m \bar{\psi} \psi + \bar{\eta} \psi + \bar{\psi} \eta] \} \] (4.9)

where we used the translational invariance of the measure \( \mathcal{D} \bar{\psi} \mathcal{D} \psi = \mathcal{D} \bar{\psi}' \mathcal{D} \psi' \). The propagator is then calculated for the Grassmann variables as (for which \( \eta \) and \( \bar{\eta} \) are also Grassmann numbers)
\[ \langle 0 | T^{*} \hat{\psi}(x) \hat{\bar{\psi}}(y) | 0 \rangle = \frac{1}{Z} \int \mathcal{D} \psi \mathcal{D} \bar{\psi} \psi(x) \bar{\psi}(y) \exp \{ i \int d^4 x [\bar{\psi} i \not\!D \psi - m \bar{\psi} \psi] \} \]
\[ = \frac{1}{Z} \frac{i}{\delta \bar{\eta}(x) \delta\eta(y)} \ln Z |_{\eta=0} \]
\[ = \frac{i}{i \not\!D - m + i\epsilon} \delta^4(x - y) \] (4.10)

where the operator expression in the left-hand side follows from Schwinger’s action principle. Similarly, for complex numbered variables (for which \( \eta \) and \( \bar{\eta} \) are also complex numbers) we have
\[ \langle 0 | T^{*} \hat{\psi}(x) \hat{\bar{\psi}}(y) | 0 \rangle = \frac{1}{Z} \frac{i}{\delta \bar{\eta}(x) \delta\eta(y)} \ln Z |_{\eta=0} \]
\[ = \frac{i}{i \not\!D - m + i\epsilon} \delta^4(x - y). \] (4.11)
These propagators exhibit precisely the local structure of the Lagrangian, since the path integral measure is local. The propagator specifies the norm of single particle states, and Feynman’s $m - i\epsilon$ prescription dictates the propagation of negative energy states in the negative time direction and thus ensures the positive energy condition. These relations, (4.10) and (4.11), show that we cannot identify the statistics for Dirac particles by just looking at the right-hand side of propagators only.

There are thus two alternative paths to analyze the spin-statistics theorem in path integral formalism. The first path, which corresponds to the path taken by Pauli\[8\], is to combine Schwinger’s action principle with path integral. The second path taken by Feynman\[7\] will be discussed later.

We first establish the causality by considering the Fourier transform of the above Feynman propagator defined by time ordered product. By using $\delta^4(x - y) = \int \frac{d^4k}{(2\pi)^4} \exp[-ik(x - y)]$ we obtain the identical expression for these two cases in the right-hand side for a free field propagator (with spinor indices explicitly written)

$$\langle 0|T^x\hat{\psi}_\alpha(x)\hat{\psi}_\beta(y)|0\rangle = \int \frac{d^4k}{(2\pi)^4} \exp[-ik(x - y)](\frac{i}{k' - m + i\epsilon})_{\alpha\beta}. \quad (4.12)$$

By employing BJL prescription as we did for a scalar particle, we find in the present case as

$$\delta(x^0 - y^0)\langle 0|[\hat{\psi}_\alpha(x), \hat{\psi}_\beta(y)]_\pm|0\rangle = \gamma_0^{0\alpha\beta}\delta^4(x - y), \quad (4.13)$$

$$\delta(x^0 - y^0)\langle 0|[\hat{\psi}_\alpha(x), \hat{\psi}_\beta(y)]_\pm|0\rangle = \delta(x^0 - y^0)\langle 0|[\hat{\psi}_\alpha(x), \hat{\psi}_\beta(y)]_\pm|0\rangle = 0$$

where $\pm$ correspond to the Grassmann or complex numbers, respectively. The first relation in (4.13) is obtained from

$$(-ik^0) \int d^4x \exp[ik(x - y)]\langle 0|T\hat{\psi}_\alpha(x)\hat{\psi}_\beta(y)|0\rangle$$

$$= \int d^4x \exp[ik(x - y)]$$

$$\times \frac{\partial}{\partial x^0}[\theta(x^0 - y^0)\langle 0|\hat{\psi}_\alpha(x)\hat{\psi}_\beta(y)|0\rangle \mp \theta(y^0 - x^0)\langle 0|\hat{\psi}_\beta(y)\hat{\psi}_\alpha(x)|0\rangle]$$

$$= \int d^4x \exp[ik(x - y)]\delta(x^0 - y^0)\langle 0|[\hat{\psi}_\alpha(x), \hat{\psi}_\beta(y)]_\pm|0\rangle$$

$$+ \int d^4x \exp[ik(x - y)]\langle 0|T\partial_\alpha\hat{\psi}_\alpha(x)\hat{\psi}_\beta(y)|0\rangle$$

$$= k^0(\frac{1}{k' - m + i\epsilon})_{\alpha\beta}. \quad (4.14)$$
and considering the limit $k^0 \to \infty$. The second relations in (4.13) arise since there are no propagators for , for example, $\langle 0| T^\dagger \hat{\psi}_\alpha(x) \hat{\psi}_\beta(y)|0 \rangle = 0$. These expressions (4.13) show that the causality (Condition 2) is always ensured for either choice of variables. We note that the causality condition is satisfied for other choices of $i\epsilon$ prescription, though such choices violate the positive energy condition: See (3.22).

We now examine the norm in Hilbert space by performing integral over $k^0$ in the propagator as

$$\langle 0| T \hat{\psi}_\alpha(x) \hat{\psi}_\beta(y)|0 \rangle$$

$$= \theta(x^0 - y^0) \int \frac{d^3k}{(2\pi)^3 2\omega} e^{-i\omega(x^0 - y^0) + i\vec{k}(\vec{x} - \vec{y})} \sum_s u_\alpha(k, s) \bar{u}_\beta(k, s)$$

$$- \theta(y^0 - x^0) \int \frac{d^3k}{(2\pi)^3 2\omega} e^{-i\omega(y^0 - x^0) + i\vec{k}(\vec{y} - \vec{x})} \sum_s v_\alpha(k, s) \bar{v}_\beta(k, s)$$

where we defined $k^0$ anew as $k^0 = \omega = \sqrt{k^2 + m^2}$. Note that both of the terms with $\theta(x^0 - y^0) \exp[-i\omega(x^0 - y^0)]$ and $\theta(y^0 - x^0) \exp[-i\omega(y^0 - x^0)]$ ensure the positive energy condition, which is a result of Feynman’s $m - i\epsilon$ prescription. We normalize the positive energy $u_\alpha(k, s)$ and negative energy $v_\alpha(k, s)$ spinor solutions as

$$\sum_s u_\alpha(k, s) \bar{u}_\beta(k, s) = (k' + m)_{\alpha\beta},$$

$$\sum_s v_\alpha(k, s) \bar{v}_\beta(k, s) = (k' - m)_{\alpha\beta}. \quad (4.16)$$

On the other hand, in conjunction with Schwinger’s action principle, we have a different time ordering property for the Grassmann and ordinary numbers in the left-hand side:

$$\langle 0| T \hat{\psi}_\alpha(x) \hat{\psi}_\beta(y)|0 \rangle$$

$$= \theta(x^0 - y^0) \langle 0| \hat{\psi}_\alpha(x) \hat{\psi}_\beta(y)|0 \rangle \mp \theta(y^0 - x^0) \langle 0| \hat{\psi}_\beta(y) \hat{\psi}_\alpha(x)|0 \rangle$$

where the first minus sign corresponds to the Grassmann number and the second plus sign corresponds to the ordinary complex number. We know that the Grassmann choice gives rise to the positive normed inner product for both of the electron and the positron. This is understood by considering $\langle 0| T \hat{\psi}_\alpha(x) \hat{\psi}_\beta(y)|0 \rangle$ instead of $\langle 0| T \hat{\psi}_\alpha(x) \hat{\psi}_\beta(y)|0 \rangle$ in (4.15) and (4.17).
The Fourier transform of the right-hand side is identical for both choices, but the second term in the time ordering changes sign for the ordinary complex number. This means that the second term in the time ordering, which corresponds to the positron, acquires negative norm in the sense of conventional operator formalism\[8\] if one uses complex numbers: Namely, if one expands

$$\hat{\psi}_\alpha(x) = \sum_s \int \frac{d^3k}{\sqrt{(2\pi)^32\omega}} [\hat{a}_k(s)e^{-ikx}u_\alpha(k,s) + \hat{b}_k^\dagger(s)e^{ikx}v_\alpha(k,s)]$$  \hspace{1cm} (4.18)

one obtains $\langle 0|\hat{b}_k(s)\hat{b}_k^\dagger(s)|0\rangle < 0$.

What this means is that the operator transcription induced by path integration

$$\psi(x) \rightarrow \hat{\psi}(x),$$  
$$\psi^\dagger(x) \rightarrow \hat{\psi}^\dagger(x)$$

(4.19)

ensures the hermitian conjugation $\hat{\psi}^\dagger(x) = (\hat{\psi}(x))^\dagger$ in the operator sense for the Grassmann numbers but not for the complex numbers. The use of ordinary complex numbers for Dirac particles thus violates the positive norm condition (Condition 3).

5 Euclidean analysis

Feynman in his analysis of the spin-statistics theorem\[7\] took a path, which is more in line with the spirit of path integral formalism, and looked for other inconsistencies if one applies the abnormal statistics. The Feynman’s $i\epsilon$ prescription allows a smooth continuation to Euclidean theory by Wick rotation $x^0 \rightarrow -ix^4$ with real $x^4$, and the actual calculations in path integral are usually performed in Euclidean setting. The Euclidean theory thus obtained is regarded to satisfy the positive energy condition, since we can at any time rotate the metric back to Minkowski one up to any finite order in perturbation. Similarly, the time ordered product in Minkowski theory is readily recovered from Euclidean theory and thus the causality is regarded to be ensured. The basic criterion is thus the positive norm condition.
We now follow the path of Feynman, though our analysis is not quite identical to that of Feynman, and look for other inconsistencies if one applies the abnormal statistics in Euclidean theory

\[
\int \mathcal{D}\bar{\psi}\mathcal{D}\psi[\mathcal{D}A_\mu] \exp\{ \int d^4x[\bar{\psi}i\not{D}\psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}] \}. \tag{5.1}
\]

Here a suitable gauge fixing is included in \([\mathcal{D}A_\mu]\). As a phenomenon which is sensitive to the choice of path integral variables, we illustrate the use of the Weyl anomaly. The Weyl anomaly is basically a one (or higher)-loop effect, but it is treated as if it were a tree-level effect in path integral\(^{[17]}\).

The Weyl transformation is defined by \(g_{\mu\nu}(x) \rightarrow \exp[-2\alpha(x)]g_{\mu\nu}(x)\) and the Dirac fields are transformed as

\[
\bar{\psi}(x) \rightarrow \bar{\psi}'(x) = \exp[-\frac{1}{2}\alpha(x)]\bar{\psi}(x),
\]

\[
\psi(x) \rightarrow \psi'(x) = \exp[-\frac{1}{2}\alpha(x)]\psi(x). \tag{5.2}
\]

This is the Weyl transformation for weight \(1/2\) variables \(\tilde{\psi}(x) = (g)^{1/4}\psi(x)\) and \(\tilde{\bar{\psi}}(x) = (g)^{1/4}\bar{\psi}(x)\) in the Euclidean flat space-time limit, which differs from the naive Weyl transformation, for example, \(\psi(x) \rightarrow \psi'(x) = \exp[\frac{3}{2}\alpha(x)]\psi(x)\). We then obtain a Jacobian factor (for the general coordinate invariant measure \(\mathcal{D}\bar{\psi}\mathcal{D}\bar{\bar{\psi}}\) in the Euclidean flat space-time limit)\(^{[17]}\)

\[
\mathcal{D}\bar{\bar{\psi}}'\mathcal{D}\bar{\psi}' = J(\alpha)\mathcal{D}\bar{\bar{\psi}}\mathcal{D}\bar{\psi}, \tag{5.3}
\]

where

\[
J(\alpha) = \exp[\pm \int d^4x\alpha(x)\frac{e_0^2}{24\pi^2}F^{\mu\nu}F_{\mu\nu}]. \tag{5.4}
\]

The coefficients of this Weyl anomaly, in particular, \(\pm\) signs correspond to the Grassmann number or the ordinary complex number, respectively. The sign difference appears from the fundamental property of the path integral measure, namely, the measure is defined by (left-)derivatives for Grassmann variables.

The coefficient of the Weyl anomaly is thus a good indicator of the statistics of particles. The coefficient of the Weyl (or trace) anomaly, which is related to scale transformation\(^{[22]}\), gives the (lowest order) \(\beta\) function
of the renormalization group\cite{23}, $\beta(e) = \pm \frac{e^3}{12\pi^2}$, or if one treats $e^2$ as the coupling constant
\begin{equation}
\beta(e^2) = \pm \frac{e^4}{6\pi^2}.
\end{equation}
The Grassmann number gives rise to the positive signature and asymptotically non-free theory, and the complex number gives rise to the negative signature and asymptotically free theory. It is known that the positive norm in the Hilbert space gives rise to asymptotically non-free theory for QED\cite{24}: A formal argument for this is based on the relation
\begin{equation}
e^2 = Z_1^{-2}Z_2^2Z_3^2e_0^2
\end{equation}
where $e_0$ stands for the bare charge. The Ward identity gives $Z_1 = Z_2$, and the Källen-Lehmann bound\cite{25} gives $0 \leq Z_3 \leq 1$ for the photon wave function renormalization factor, which is a result of the positive norm condition. We emphasize that the actual calculation of the wave function renormalization factor is performed in Euclidean momentum space. We thus expect for small $e$
\begin{equation}
Z_3 = 1 - ae_0^2\ln(\Lambda/m_0) + \ldots.
\end{equation}
with a positive constant $a$, and we have
\begin{equation}
\beta(e^2) = m_0 \frac{\partial e^2}{\partial m_0}|_{\Lambda,e_0^2} = ae^4 > 0
\end{equation}
to this order. The use of ordinary complex numbers for Dirac particles thus contradicts the positive norm condition $0 \leq Z_3 \leq 1$. This analysis is valid up to any finite order in perturbation theory for a sufficiently small coupling constant.

If one analyzes a complex scalar field theory defined by
\begin{equation}
\mathcal{L} = [(\partial_{\mu} - ie_0A_{\mu})\varphi(x)]^\dagger[(\partial^\mu - ie_0A^\mu)\varphi(x)] - m_0^2\varphi(x)^\dagger\varphi(x) - \delta \lambda(\varphi(x)^\dagger\varphi(x))^2
\end{equation}
and the path integral
\begin{equation}
\int \mathcal{D}\varphi\mathcal{D}\varphi^\dagger[\mathcal{D}A_{\mu}] \exp\left\{ \int d^4x[\mathcal{L} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}]\right\},
\end{equation}
where the term with $\delta\lambda = O(e_0^4)$ is a counter term to eliminate the induced $(\varphi(x)\varphi(x))^2$ coupling, the Weyl transformation of a weight 1/2 variable $\tilde{\varphi}(x) = (g)^{1/4}\varphi(x)$ is given by

$$
\tilde{\varphi}(x) \rightarrow \tilde{\varphi}'(x) = \exp[-\alpha(x)]\tilde{\varphi}(x),
\tilde{\varphi}^\dagger(x) \rightarrow (\tilde{\varphi}^\dagger)'(x) = \exp[-\alpha(x)]\tilde{\varphi}^\dagger(x). \tag{5.11}
$$

The Jacobian factor for the general coordinate invariant measure in the Euclidean flat space-time limit is then calculated as a straightforward generalization of the calculation in Ref.[17] as

$$
\mathcal{D}\varphi^\dagger\mathcal{D}\varphi' = \exp[\pm \int d^4x \alpha(x)\frac{e_0^2}{96\pi^2}F^{\mu\nu}F_{\mu\nu}]\mathcal{D}\varphi^\dagger\mathcal{D}\varphi \tag{5.12}
$$

where $\pm$ signatures correspond to the complex number or the Grassmann number, respectively. If one combines this result with an analysis of the $\beta$-function and the K"allen-Lehmann bound in the lowest order in $e_0^2$, one can exclude the Grassmann variables for scalar particles.

We can thus analyze the spin-statistical theorem in Euclidean theory without referring to the norm of on-shell states directly. Even if one does not know the precise coefficient of the Weyl anomaly, one can readily recognize the change of the signature of the Weyl anomaly for abnormal statistics in path integral and thus the appearance of inconsistency. As for the analysis of the spin-statistics theorem in Euclidean theory, see also Schwinger[9].

Alternatively if one uses a perturbative language for a Dirac particle in Minkowski space-time, which is close to the original analysis of Feynman[7], the vacuum polarization tensor changes sign if one uses the complex number. Consequently, its absorptive part which gives the decay probability of a virtual time-like photon

$$
\gamma \rightarrow e\bar{e} \tag{5.13}
$$

changes sign, and we obtain a negative decay probability. For Feynman, the spin-statistics theorem meant a clear understanding of the origin of this minus sign[13] [26]. What we have shown above is that this change of sign has a root in the very definition of the measure in path integral formulation.

We next comment on the statistics of the scalar and Dirac particles in two-dimensional (one space and one time) theory. The so-called bosonization [27] is well known in two-dimensional theory. The bosonization does
not imply that one can use a complex number for a Dirac particle, for example. Rather it implies an equivalent description either by using a real scalar particle or by using a Dirac particle in two-dimensional theory. In this respect, we note that the (gravitational) Weyl anomaly induced by a massless real scalar, which is described by a real number $\phi(x)$, and a massless Dirac particle, which is described by Grassmann numbers $\psi(x)$ and $\bar{\psi}(x)$, is identical. The Weyl transformation of relevant path integral variables in two-dimensional theory is defined by

$$\tilde{\phi}(x) = (g)^{1/4}\phi(x) \rightarrow \tilde{\phi}'(x) = \exp[-\alpha(x)]\tilde{\phi}(x) \quad (5.14)$$

and

$$\tilde{\psi}(x) = (g)^{1/4}\tilde{\psi}(x) \rightarrow \tilde{\psi}'(x) = \exp[-\frac{1}{2}\alpha(x)]\tilde{\psi}(x),$$

$$\tilde{\psi}(x) = (g)^{1/4}\psi(x) \rightarrow \tilde{\psi}'(x) = \exp[-\frac{1}{2}\alpha(x)]\tilde{\psi}(x). \quad (5.15)$$

The path integral measure for both cases (i.e., free particles in two-dimensional curved space-time) changes under the Weyl transformation as

$$d\mu \rightarrow d\mu' = \exp[-\int d^2x\alpha(x)\frac{1}{24\pi}\sqrt{gR}]d\mu \quad (5.16)$$

where $d\mu = D\tilde{\phi}$ or $d\mu = D\tilde{\psi}D\tilde{\psi}$. Since gravitational field couples to all the matter fields universally, this agreement of the Weyl anomaly, which specifies the central charge ($c = 1$) of the Virasoro algebra, suggests (though does not prove) the equivalence of a real scalar particle and a Dirac particle in two-dimensional theory. The abnormal statistics changes the signature of the Weyl anomaly and thus the signature of the central charge, which spoils the positive norm condition in the representation of the Virasoro algebra[29]. In this sense, one can exclude the abnormal assignment of statistics for the scalar and Dirac particles in two-dimensional theory also.

6 Discussion

The local path integral measure and Lorentz invariant local Lagrangian together with time ordered products ensure causality regardless of statistics. The Feynman’s $m - i\epsilon$ prescription ensures positive energy condition
regardless of statistics. We find the indefinite metric for spin 0 particles if one uses Grassmann variables and the negative norm for negative energy states if one uses complex numbers for Dirac particles. This is in accord with the operator analysis of Pauli[8], in response to the Feynman’s analysis of the spin-statistics theorem[7]. In the framework of path integral proper, one need to go one more step further to recognize the negative norm for the abnormal case. By this way, we naturally arrive at the original treatment of Feynman in his analysis of the spin-statistics theorem[7]. We here illustrated the use of the coefficient of Weyl anomaly as a characteristic indicator of statistics, which is a direct consequence of the definition of path integral measure and works not only for 4-dimensional theory but also for 2-dimensional theory. (As for the Lorentz invariance of path integral measure, one can examine it precisely by analyzing the Jacobian for local Lorentz transformation[30].)

In passing, we note that the bosonic Dirac particle is practically used as the Pauli-Villars regulator for a fermionic Dirac particle in path integral formulation[15], which cancels all the possible anomalous Jacobians. In the regularization of continuum path integral, the cancellation of the anomalous Jacobian factor is essential to justify naive manipulations.

As for the generality of our arguments, the analysis of free propagators is applicable to all the cases. The analysis of free propagators, when combined with the notion of Feynman diagrams, is extended to any finite order in perturbation theory. As for the analyses of Weyl anomaly or the positivity of scattering processes, we note that all the known elementary particles with spin 1/2 couple to gauge fields in 4-dimensional theory. One can thus choose a suitable $U(1)$ gauge field associated with a Cartan subalgebra to establish the spin-statistics theorem for these elementary spinors.

In conclusion, we have shown a simple and coherent proof of the spin-statistics theorem in the framework of modern path integral, which incorporates the Grassmann numbers as an integral part[16]. The minus commutation relation between one Bose and one Fermi field arises naturally in this framework, since the complex numbers and Grassmann numbers commute in the classical level.

I thank Y. Matsuo for a helpful comment on the Virasoro algebra.
References

[1] W. Pauli, Ann. de l’Inst. Poincare 6,137 (1936).
[2] D. Iwanenko and A. Socolow, Phys.Zeit.Sow. 11,590 (1937).
[3] M. Fierz, Helv. Phys. Acta 12,3 (1939).
[4] F.J. Belinfante, Physica VI 9,870 (1939).
   W. Pauli and F.J. Belinfante, VII 3,177 (1940).
[5] J.S. deWet, Phys. Rev. 57,646 (1940).
[6] W. Pauli, Phys. Rev. 58,716 (1940).
[7] R. Feynman, Phys. Rev. 76,749 (1949); 76,769 (1949).
[8] W. Pauli, Prog. Theor. Phys. 5,526 (1950).
[9] J. Schwinger, Phys. Rev. 82,914 (1951).
   J. Schwinger, Phys. Rev. 115,721 (1959).
   L.S. Brown and J. Schwinger, Prog. Theor. Phys. 26,917 (1961).
[10] G. Lüders and B. Zumino, Phys. Rev. 110,1450 (1958).
[11] N. Burgoyne, Nuovo Cim. 8,607 (1958).
[12] D. Hall and A.S. Wightman, Mat.Fys.Dan.Vid.Selsk.31,31 (1957).
   R.F. Streater and A.S. Wightman, PCT, Spin and Statistics, and All
   That (W.A. Benjamin, New York, 1964).
[13] I. Duck and E.C.G. Sudarshan, Pauli and the Spin-Statistics Theorem,
    (World Scientific,Singapore,1997).
[14] L.D. Faddeev and N. Popov, Phys. Lett.25B,29 (1967).
[15] K. Fujikawa, Phys. Rev. D21,2848 (1980); D22,1499 (1980)(E).
[16] F.A. Berezin, The Method of Second Quantization, (Academic Press,
    New York,1966).
[17] K. Fujikawa, Phys. Rev, Lett.44,1733 (1980); Phys. Rev. 23 ,2262
    (1981).
[18] J.D. Bjorken and S.D. Drell, *Relativistic Quantum Fields*, (McGraw-Hill, New York, 1965).
S. Weinberg, *The Quantum Theory of Fields I*, (Cambridge Univ. Press, Cambridge, 1995).

[19] M.E. Peskin and D.V. Schroeder, *An introduction to Quantum Field Theory* (Addison-Wesley, Reading, Mass., 1995).

[20] J.D. Bjorken, Phys. Rev. **148**, 1467 (1966).
K. Johnson and F. Low, Prog. Theor. Phys. Suppl. **37-38**, 74 (1966).

[21] K. Fujikawa, Prog. Theor. Phys. **63**, 1364 (1980), and references therein.

[22] S. Coleman and R. Jackiw, Ann. Phys. **67**, 552 (1971).
R.J. Crewther, Phys. Rev. Lett. **28**, 1421 (1972).
M. Chanowitz and J. Ellis, Phys. Lett. **40B**, 397 (1972).

[23] S.L. Adler, J.C. Collins and A. Duncan, Phys. Rev. **D15**, 1712 (1977).
N.K. Nielsen, Nucl. Phys. **B120**, 212 (1977).

[24] A. Zee, Phys. Rev. **D7**, 3630 (1973).

[25] G. Källen, Helv. Phys. Acta **25**, 417 (1952).
H. Lehmann, Nuovo Cim. **11**, 342 (1954).
H. Umezawa and S. Kamefuchi, Prog. Theor. Phys. **4**, 543 (1951).

[26] R. Feynman, *Elementary Particles and the Laws of Physics. The 1986 Dirac Memorial Lecture*, (Cambridge, New York, 1987), pp.2-59.

[27] S. Coleman, Phys. Rev. **D11**, 2088 (1975).

[28] A.M. Polyakov, Phys. Lett. **103B**, 207 (1981); **103B**, 211 (1981).
K. Fujikawa, Phys. Rev. **D25**, 2584 (1982).
O. Alvarez, Nucl. Phys. **B216**, 125 (1983).

[29] P. Ginsparg, *Applied Conformal Field Theory*, in *Fields, Strings and Critical Phenomena*, edited by E. Brezin and J. Jinn-Justin, (Elsevier, Amsterdam, 1989).

[30] L. Alvarez-Gaume and E. Witten, Nucl. Phys. **B234**, 269 (1983).