ELECTROMAGNETIC FIELDS
IN KERR-SHILD SPACE-TIMES

Vladimir V. Kassandrova

Institute of Gravitation and Cosmology, Peoples’ Friendship University of Russia, 6 Mikluho-Maklay St.,
Moscow 117198, Russia

Making use of twistor structures and the Kerr theorem for shear-free null geodesic congruences, an infinite family
of electromagnetic fields satisfying the homogeneous Maxwell equations in flat Minkowski and the associated curved
Kerr-Schild backgrounds is obtained for any such congruence in a purely algebraic way. Simple examples of invariant
axisymmetric Maxwell fields are presented.

1. Introduction

Rather unexpectedly, one observes a growing interest
in solutions of the most investigated, seemingly sim-
ple and important equations of classical field theory,
the Maxwell linear homogeneous equations in vacuum.
It occurs that these possess various kinds of solutions
with different topology of field lines [1, 2], with extended
singularities of complicated shape and temporal dynamics
[3, 4, 5], multi-valued complexified solutions etc. The
simplest example of the latter is provided by the electro-
statics [3, 4, 5], multi-valued complexified solutions etc. The
well known examples of such an invariance of
invariant fields introduced in the paper are new and can
invariant fields. In general, however, the in-
variant fields introduced in the paper are new and can
open a way to complicated electrovacuum solutions with
metrics of the Kerr-Schild type.

In Section 2, we describe a procedure which makes
use of twistor structures and allows us to associate a
whole family of electromagnetic fields with any shear-
free null congruence (SFNC) of rays in Minkowski space.
We demonstrate that the condition (3) holds identically
for all such fields if \( k_\mu \) is a null 4-vector field tangen-
to the SFNC under consideration. Consequently, any
SFNC gives rise to a family of Kerr-Shild metrics (with
different \( H(x) \) in (1) and a family of electromagnetic
fields identically satisfying Maxwell equations

\[
\partial_\nu (\sqrt{-g} F^{\mu \nu}) = 0
\]

in the corresponding curved background. In addition, it
is well known that the scalar “gravitational potential”
\( H(x) \) may be often fitted to ensure that the Kerr-Shild
metric satisfies the vacuum or electrovacuum Einstein
equations.

In Section 3, we present simple examples of invariant
electromagnetic fields associated with SFNC and inher-
itating the spherical or axial symmetry of the congruence.
In all such cases the newly found fields correspond to the
previously determined ones (which are known to possess
the invariance property). In general, however, the in-
variant fields introduced in the paper are new and can
open a way to complicated electrovacuum solutions with
metrics of the Kerr-Schild type.

2. Shear-free null congruences and the
associated Maxwell fields in flat and
curved background

According to the Kerr theorem [7], any SFNC in Minkowski
space may be obtained from the generating equation

\[
\Pi(G, \tau^1, \tau^2) \equiv \Pi(G, wG + u, vG + \bar{w}) = 0,
\]

where \( \Pi \) is an analytical function of three complex argu-
ments \( \{G, \tau^1, \tau^2\} \) representing the components of a pro-
jective twistor. The latter is associated with the coordi-
nates \( u, v = t \pm z, w, \bar{w} = x \mp iy \) of points in Minkowski
space through the Penrose’s incidence relation

\[
\tau^1 = wG + u, \quad \tau^2 = vG + \bar{w}.
\]
Resolving (5) with respect to the unknown \(G\), one comes to a multi-valued field \(G = G(u, v, w, \bar{w})\). It is easy to demonstrate (via differentiation of (5)) that any continuous branch of this multi-valued field satisfies the determining equations of a SFNC,

\[
\partial_u G = G \partial_u G, \quad \partial_v G = G \partial_v G, \quad (7)
\]

and therefore represents the principal spinor of the latter. Thus, one has a one-to-one correspondence between the SFNCs and the twistor functions \(\Pi(G, \tau^1, \tau^2)\) (precisely, the surfaces \(\Pi = 0\) in the \(\mathbb{CP}^3\) projective twistor space).

From (7), the complex eikonal equation follows immediately,

\[
|\nabla G|^2 := \partial_u G \partial_u G - \partial_v G \partial_v G = 0, \quad (8)
\]

and, as the integrability condition of (7), one obtains then the d’Alembert equation [8]

\[
\Box G := \partial_u \partial_v G - \partial_v \partial_u G = 0. \quad (9)
\]

One might thus suspect that any SFNC gives rise to a (complexified) Maxwell field, with \(G\) being a sort of super-potential. Indeed, in [3, 6, 8] such a field has been obtained, with its strength expressed through the second-order derivatives of \(G\). Electromagnetic fields of this type possess a number of the afore-mentioned peculiar properties. However, these fields do not preserve their form under the Kerr-Schild deformation of metric. To ensure such an invariance, we propose below another expression for Maxwell fields to be associated with arbitrary SFNC.

Specifically, consider the following simple anzatz for the components of the spintensor \(\varphi_{A'B'}\), \(A', B' = 1', 2'\) of electromagnetic field strength:

\[
\varphi_{1'1'} = \partial_u \partial_u F, \quad \varphi_{1'2'} = \varphi_{2'1'} = \partial_u \partial_v F, \quad \varphi_{2'2'} = \partial_v \partial_v F, \quad (10)
\]

where \(F = F(G)\) is an arbitrary (holomorphic) function of the principal spinor component \(G(X)\).

Taking into account that \(X = \{X^{A A'}\} = \{u, w, \bar{w}, v\}\), it is easy to check that the field (10) in fact satisfies the homogeneous Maxwell equations

\[
\partial^{A A'} \varphi_{A'B'} = 0 \quad (11)
\]

provided the eikonal (8) and d’Alembert (9) equations hold both together for the function \(G\).

Now, let us prove that all fields (10) associated with a SFNC always obey the eigenvector condition (8). To do so, let us rewrite the latter in the equivalent 2-spinor form

\[
\varphi_{A'B'} \xi^{A'} \xi^{B'} = 0, \quad (12)
\]

where \(\xi_{A'}\) is the principal spinor of the SFNC in the gauge \(\xi_{1'} = 1\), \(\xi_{2'} = G\). Substituting the components (10) into the above condition (12) and transforming the l.h.s. of the latter, one obtains

\[
G \partial_u (F'M) - \partial_v (F'M) - (F'M) \partial_u G = 0, \quad (13)
\]

where \(F' := dF(G)/dG\), \(M := G \partial_u G - \partial_v G\). However, \(M \equiv 0\) on account of the first of the SFNC determining equations (7), so that the eigenvector condition (12) is identically satisfied for the fields (10). This means that there exists an infinite family of electromagnetic fields (10) associated with a SFNC of a general form, which all satisfy the homogeneous Maxwell equations in Minkowski space and in the corresponding curved Kerr-Schild space.

Some remarks are in order here.

1. In a number of papers E.T. Newman et al. and A.Ya. Burinskii (see, e.g., [9, 10]) had been working with electromagnetic fields generated by a pointlike charge moving in real or complexified Minkowski space. These fields indeed satisfy the invariance condition (8). However, even for this particular case, to obtain such fields, explicit integration of the Maxwell equations was required.

2. To simplify the exposition, nearly all the above constructions have been presented in a particular gauge. One understands, nonetheless, that the formalism can be easily transformed into a manifestly Lorentz invariant form.

3. For the newly found fields, the theorem on “quantization of the electric charge” for isolated singularities [5] remains valid and will be reproduced elsewhere (see also the examples below).

4. Remarkably, symmetries of the electromagnetic fields (10) for a general form of \(F(G)\) can be weaker than those of the SFNC which they are generated from. For example, the Kerr axisymmetric congruence can give rise to electromagnetic fields devoid of any symmetries at all. To preserve the axial symmetry of a SFNC, a particular form of the function \(F(G)\), namely \(F(G) = G^{-1}\), must be used (see below).

3. Some examples of invariant Maxwell fields associated with a SFNC

1. Consider first a static, spherically symmetric SFNC and the associated fields corresponding to the following form of generating equation (5):

\[
\Pi = G r^1 - r^2 = w G^2 - 2 z G + \bar{w} = 0, \quad z = (u - v)/2. \quad (14)
\]

Resolving (14), one obtains

\[
G = \frac{\bar{w}}{z + r}, \quad r := \sqrt{w u + z^2} \equiv \sqrt{x^2 + y^2 + z^2}, \quad (15)
\]

i.e. the stereographic projection \(S^2 \to \mathbb{C}\) from the South or North poles, respectively. Corresponding SFNC is the radial, spherically symmetric congruence with a point singularity.
One can easily verify that, from the whole family of associated electromagnetic fields, only the choice $F(G) = G^{-1}$ corresponds to a field inheriting the spherical symmetry of the congruence. Specifically, calculating the components of the spintensor $\varphi^{AB'}$ for $F(G) = G^{-1}$, one gets

\[ \varphi^{12} = \frac{w}{r^3}, \quad \varphi^{12'} = -\frac{z}{r^3}, \quad \varphi^{22'} = -\frac{\bar{w}}{r^3}, \quad \text{(16)} \]

with the constant $q = \mp 1/4$. The expression describes the pure Coulomb field with an electric charge necessarily fixed in value (in fact this is a minimum “elementary” charge, see [5, 6] for details). Of course, the Coulomb field is invariant under the Kerr-Schild type of deformation of flat metric, i.e., under transition to the Reissner-Nordström space-time.

2. The Kerr congruence with twist and a ring-like singularity of radius $a$ is known to arise from the radial one through a complex shift, say, $z \mapsto z + \lambda a$. Since all the above-presented constructions deal with complex holomorphic structures, the associated Maxwell field also follows from (16) through such a shift and coincides with the electromagnetic field of the Kerr-Neu- man electrovacuum solution. It is important, nonetheless, that in the same Kerr-Newman metric background, making use of different $F(G)$, a lot of other fields obeying the Maxwell equations could be introduced.

3. The Kerr congruence can be naturally generalized to a nonstationary form described in [11]. To do that, one transforms the principal spinor of the radial SFNC through a complex boost and obtains the following form of it:

\[ G = (1 + Iu) \frac{x + Iy}{(z - z_t) \pm \hat{r}}, \quad z_t := -Ia + Iut, \quad \text{(17)} \]

where $Iu \in \mathbb{C}$ is the “imaginary velocity” parameter of a “complex boost”, and “complex distance” from the point singularity to an observation point is

\[ \hat{r} = \sqrt{(z - z_t)^2 + w\bar{w}(1 + u^2)}. \quad \text{(18)} \]

The singular locus (caustic) of the SFNC determined by (17) coincides with the branching points $\hat{r} = 0$ of the principal spinor and represents a Kerr-like ring collapsing/expanding with the physical velocity $V = u/\sqrt{1 + u^2} < 1$. The invariant (complex-valued) electromagnetic field associated with such a congruence through the expression (10) under the assumption $F(G) = G^{-1}$ again reproduces the previously determined one (see Eq.(40) of the paper [11]) and manifests a number of interesting properties (including its vortex-like structure and concentration of the field along the symmetry axis).

4. We thus see that, for the simplest SFNCs, the invariant electromagnetic fields contain a distinguished one, inheriting the axial symmetry of the congruence and reproducing the fields associated with the latter through the old prescription. Generally, however (in particular, for the case of a bisingular SFNC [3, 8]), the newly obtained (and necessarily invariant) Maxwell fields differ from the previously considered (and generally noninvariant) ones. One can hope thus that discovery of a family of fields automatically satisfying the Maxwell equations in any given Kerr-Schild background will open the way to construction of extremely complicated electrovacuum solutions corresponding to arbitrary SFNCs.

References

[1] A.F. Ranada, J. Phys. A: Math. Gen., 25, 1621 (1992); A.F. Ranada and J.L. Trueba, Phys. Lett., A202, 337 (1995).
[2] R.M. Kiehn, Int. J. Theor. Phys., 5, 1779 (1991); “Electromagnetic Waves in the Vacuum with Torsion and Spin”, gr-qc/9802033.
[3] V.V. Kassandrov and J.A. Rizkallah, “Twistor and “Weak” Gauge Structures in the Framework of Quaternionic Analysis”, gr-qc/0112109.
[4] V.V. Kassandrov, in: “Space-Time Structure. Algebra and Geometry”, ed. D.G. Pavlov et al., Lilia-Print, Moscow, 2007, p. 441; arXiv: 0710.2895.
[5] V.V. Kassandrov and V.N. Trishin, Grav. Cosmol., 3, 216 (1995); gr-qc/0007026.
[6] R. Penrose and W. Rindler, “Spinors and Space-Time. Vol.II”, Cambridge Univ. Press, Cambridge, 1986.
[7] V.V. Kassandrov and J.A. Rizkallah, in: “Recent Problems in Field Theory”, ed. A.V. Aminova, Kasan Univ. Press, Kasan, 1998, p. 176; gr-qc/9809078.
[8] R.W. Lind and E.T. Newman, J. Math. Phys. 15, 1103 (1974); E.T. Newman, Phys. Rev. D65, 104005 (2002); gr-qc/0201055.
[9] A.Ya. Burinskii, J. Phys. A: Math. Theor. 41, 164069 (2008); arXiv: 0710.4249; “Aligned Electromagnetic Exitations of the Kerr-Schild Solutions”, gr-qc/0612186.
[10] V.V. Kassandrov, Grav. Cosmol. 15, 213 (2009).