Note: interpreting iterative methods convergence with diffusion point of view

Dohy Hong
Alcatel-Lucent Bell Labs
Route de Villejust
91620 Nozay, France
dohy.hong@alcatel-lucent.com

Abstract—In this paper, we explain the convergence speed of different iteration schemes with the fluid diffusion view when solving a linear fixed point problem. This interpretation allows one to better understand why power iteration or Jacobi iteration may converge faster or slower than Gauss-Seidel iteration.

Keywords—Iteration, Fixed point, Convergence, Diffusion approach.

I. INTRODUCTION

Based on the previous research results on the diffusion approach [1] to solve fixed point problem in linear algebra, we propose here a new analysis of the convergence speed of different iteration schemes with the fluid diffusion view when may converge faster or slower than Gauss-Seidel iteration.

In Section II, we define the iteration methods that are considered. Section III shows how to define the associated equivalent diffusion iteration. Section IV shows few examples to illustrate the application.

II. ALGORITHMS DESCRIPTION

A. Notations

We will use the following notations:
- \( \mathbf{P} \in \mathbb{R}^{N \times N} \) a real matrix;
- \( \mathbf{I} \in \mathbb{R}^{N \times N} \) the identity matrix;
- \( \mathbf{J} \), the matrix with all entries equal to zero except for the \( i \)-th diagonal term: \( (\mathbf{J})_{ii} = 1 \);
- \( \Omega = \{1, \ldots, N\} \);
- \( \sigma : \mathbb{R}^N \rightarrow \mathbb{R} \) defined by \( \sigma(X) = \sum_{i=1}^{N} x_i \);
- \( \mathcal{I} = \{i_1, i_2, \ldots, i_n, \ldots\} \) a sequence of coordinate: \( i_k \in \Omega \); a fair sequence is a sequence where all elements of \( \Omega \) appears infinitely often;
- \( e = (1/N, \ldots, 1/N)^T \).

B. Problem to solve

We will consider two types of linear fixed point problems:

\[
X = \mathbf{P}X
\]

\( X \in \mathbb{R}^N \) and:

\[
X = \mathbf{P}X + B
\]

\( B \in \mathbb{R}^N \).

C. Linear equation: \( X = \mathbf{P}X \)

1) Power iteration (PI): The power iteration \( \mathbf{P}I(\mathbf{P}, X_0) \) is defined by:

\[
X_n = \mathbf{P}X_{n-1}
\]

starting from \( X_0 \).

2) Gauss-Seidel iteration (GSI): Given a sequence of nodes for the update \( \mathcal{I} = \{i_1, i_2, \ldots, i_n, \ldots\} \), the Gauss-Seidel iteration \( GSI(\mathbf{P}, X_0, \mathcal{I}) \) is defined by:

\[
(X)_{i_n} = (\mathbf{P}X)_{i_n}
\]

starting from \( X_0 \), which simply means that the \( n \)-th update on \( X \) is on coordinate \( i_n \) based on the last vector \( X \) (each update modifying only one coordinate of \( X \)). We could equivalently write it as:

\[
X_n = X_{n-1} + \mathbf{J}_{i_n}(\mathbf{P} - \mathbf{I})X_{n-1}.
\]

D. Affine equation: \( X = \mathbf{P}X + B \)

1) Jacobi iteration (Jac): The Jacobi iteration \( \mathbf{J}(\mathbf{P}, B) \) is defined by:

\[
X_n = \mathbf{P}X_{n-1} + B
\]

starting from \( X_0 = (0, \ldots, 0)^T \).

2) Gauss-Seidel iteration (GSI): Given a sequence of nodes for the update \( \mathcal{I} = \{i_1, i_2, \ldots, i_n, \ldots\} \), the Gauss-Seidel iteration \( GSA(\mathbf{P}, B, \mathcal{I}) \) is defined by:

\[
(X)_{i_n} = (\mathbf{P}X_{n-1} + B)_{i_n}
\]

starting from \( X_0 = (0, \ldots, 0)^T \). We could equivalently write it as:

\[
X_n = (\mathbf{I} - \mathbf{J}_{i_n})X_{n-1} + \mathbf{J}_{i_n}(\mathbf{P}X_{n-1} + B).
\]

Remark 1: One could also consider a more general iteration scheme such as BiCGSTAB or GMRES, but they require a fluid injection method, which may be more complex. This is left for a future research.
III. DIFFUSION EQUATIONS

The diffusion equations are defined by two state vectors $F_n$ and $H_n$ associated to the affine equation $X = PX + B$ ($DI(P, B, I)$):

\[
\begin{align*}
F_n &= F_{n-1} + (P - I)J_n F_{n-1}, \\
H_n &= H_{n-1} + J_n F_{n-1}.
\end{align*}
\] (5)

The VLU adaptation of the above equation would be:

\[
\begin{align*}
F_n &= PF_{n-1}, \\
H_n &= H_{n-1} + F_{n-1}.
\end{align*}
\] (6)

which is equivalent to

\[
H_n = PH_{n-1} + B,
\]

which is a Jacobi iteration.

A. Linear equation: $X = PX$

1) Power iteration (PI): We define: $F_0 = PX_0 - X_0$, $H_0 = 0$ and the iterative diffusion equation with VLU:

\[
\begin{align*}
F_n &= PF_{n-1}, \\
H_n &= H_{n-1} + F_{n-1}.
\end{align*}
\]

Then, we have the equalities:

\[
\begin{align*}
F_n &= X_{n+1} - X_n, \\
H_n &= X_n - X_0.
\end{align*}
\]

2) Gauss-Seidel iteration (GSI): We set $F_0 = PX_0 - X_0$, $H_0 = 0$ and $F_n$, $H_n$ defined by Equations (5), then we have $X_n = H_n + X_0$.

B. Affine equation: $X = PX + B$

1) Jacobi iteration (Jac): The Jacobi iteration is equivalent to the VLU of diffusion equations [6].

2) Gauss-Seidel iteration (GSa): The Gauss-Seidel iteration is equivalent to the (CLU of) diffusion equations [5]

C. Summary of results on the equivalent iterations

IV. EXAMPLE OF CONVERGENCE SPEED COMPARISON

For comparison analysis, we consider a PageRank equation [4]:

\[
X = PX = (dQ + (1 - d)/NJ)X = dQX + (1 - d)e
\]

assuming $\sigma(X) = 1$. Then we have:

- $PI(P, X_0) = DI(dQ, dQX_0 + (1 - d)e - X_0) + X_0$;
- $Jac(dQ, (1 - d)e) = DI(dQ, (1 - d)e)$;
- $GSa(dQ, (1 - d)e, I) = DI(dQ, (1 - d)e, I)$;
- $GSI(P, X_0 = e, I) = DI(dQ, d(Qe - e), I)$.

Note that $GSI(P, X_0, I)$ has no reason to converge in general. II will converge if $I$ is a negative or positive fair sequence cf. [3]. Here, the decomposition of $P$ guarantees that it always converges for any fair sequence $I$. Note also that $Jac(dQ, (1 - d)e)$ and $GSa(dQ, (1 - d)e, I)$ define a non-decreasing vector $X_n$ (positive fluid diffusion).

Below, we compare PI, Jac and GS in simple scenarios.

A. Case 1

Note that the convergence is below measured in residual fluid using the $L_1$ norm of $F_n$ of the equivalent diffusion equation. For an easy comparison, the x-axis shows the number of iterations: for VLU approaches, it is exactly the index $n$ of $F_n$; for CLU approaches, the $L_1$ norm of $F_{5 \times n}$ is shown. We take $d = 0.85$ and

\[
Q = \begin{pmatrix}
0 & 0 & 0 & 0.5 \\
1 & 0 & 0 & 0.5 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 
\end{pmatrix}
\]

Figure 1. Convergence.

Jacobi iteration converges exactly as $d^n$. PI does roughly the same. GSa is much faster because we follow the graph path and send cumulated fluid: it is roughly 4-5 times faster as expected. GSI has the convergence slope of GSa but with a larger jump at the first iteration.

B. Case 2

We take $d = 0.85$ and

\[
Q = \begin{pmatrix}
0 & 0 & 0.5 & 0.5 \\
1 & 0 & 0 & 0.5 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0.5 & 0 \\
0 & 0 & 1 & 0 
\end{pmatrix}
\]

Figure 2. Jacobi iteration converges still as $d^n$. PI and GSa have similar convergence slope. GSI still shows the first jump.
C. Case 3

We take $d = 0.85$ and

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 0.5 \\ 1 & 0 & 0 & 0 & 0.5 \\ 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Figure 3: Jacobi iteration converges always as $d^n$. PI starts to converge faster than GSa when we add more links. GS has the first jump.

To explain, this convergence speed difference between the four methods, we consider the case 4 below.

D. Case 4

We take $d = 0.85$ and

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 0.01 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0.99 \end{pmatrix}$$

Figure 4 with GS, after one iteration (5 updates), we have a big jump due to the cumulated negative fluid that meets the positive fluid (node 5) at 5-th update. Then, from 6-th update, we only move positive fluid. And only a fraction of 1% at node 5 are moved: this explains the convergence at $d^n$ after iteration 1 for GSa and GS. For cases 1-3, the whole fluid (no self loop) from each node is moved to children nodes and this explains the gain factor (merging fluid before moving).

Now the good performance of PI can be explained by the fact that doing partial diffusion we create more fluid cancellation and make the convergence faster than $d^n$. To confirm this explanation, we plot in Figure 5 the amount of fluid that has been canceled at each iteration. We see that this phenomenon is driving the convergence speed (the difference to the residual fluid is due to the contracting factor that eliminates at each diffusion a fraction $1 - d$ (15% here). In this case, the fluid disappeared after 5-th update 84% due to fluid cancellation and 16% due to the contracting factor.

V. Conclusion

In this paper, we described the equivalent equations of the diffusion iteration associated to power iteration, Jacobi and Gauss-Seidel iteration and showed how they can explain the convergence speed of each method.
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