INVARIANT PROLONGATION AND DETOUR COMPLEXES

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Dedicated to the memory of Thomas P. Branson (1953 - 2006)

Abstract. In these expository notes we draw together and develop the ideas behind some recent progress in two directions: the treatment of finite type partial differential operators by prolongation, and a class of differential complexes known as detour complexes. This elaborates on a lecture given at the IMA Summer Programme “Symmetries and overdetermined systems of partial differential equations”.

1. Introduction

Differential complexes capture integrability conditions for linear partial differential operators. When they involve sequences of natural operators on geometric structures, such complexes typically encode deep geometric information about the structure. In these expository notes we survey some recent results concerning the construction of such complexes. A particular focus is the construction via prolonged differential systems and the links to the treatment of finite type differential operators. The broad issues arise in many contexts but we take our motivation here from conformal and Riemannian geometry.

Elliptic differential operators with good conformal behaviour play a special role in geometric analysis on Riemannian manifolds. The “Yamabe Problem” of finding, via conformal rescaling, constant scalar curvature metrics on compact manifolds is a case in point. This exploits heavily the conformal Laplacian, since it controls the conformal variation of the scalar curvature (see [39] for an overview and references). The higher order conformal Laplacian operators of Paneitz, Riegert, Graham et al. [34] (the GJMS operators) have been brought to bear on related problems by Branson, Chang, Yang and others [5, 19, 36]. However for many important tensor or spinor fields there is no natural conformally invariant elliptic operator (taking values in an irreducible bundle) available. This is true even in the conformally flat case, and indeed this claim follows easily from the classification of conformal differential operators on the sphere [3]. From this it is clear that for many bundles on the sphere the analogue, or replacement, for an elliptic operator is an elliptic complex of conformally invariant differential operators. Ignoring the issue of ellipticity, the requirement that a sequence

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of operators be both conformally invariant and form a complex is already severe, and so the construction of such complexes is a delicate matter.

On the conformal sphere there is a class of conformal complexes known as Bernstein-Gelfand-Gelfand (BGG) complexes. Equivalents of these were first constructed in the representation theory of (generalised) Verma modules (see [40] and references therein). From there we see, more generally, versions of these complexes for a large class of homogeneous structures in the category of so-called parabolic geometries, that is manifolds of the form $G/P$ where $G$ is a semisimple Lie group and $P$ a parabolic subgroup. Although these early constructions were motivated by geometric questions, the entry of the unifying picture of these complexes into the mainstream of Differential Geometry was pioneered by Eastwood, Baston and collaborators [22, 2]. In particular they developed techniques for constructing sequences of differential operators on general curved backgrounds that generalise the BGG sequences. Through work over the years the nature of many of the operators involved has become quite well developed [26, 14], and there has been spectacular recent progress in general approaches to constructing these complexes [18, 13]. However, as we shall discuss in Section 3.1 these sequences are in general not complexes on curved manifolds.

In the BGG sequences each operator may be viewed as an integrability condition for the preceding operator. The operators encode in part the geometry and it seems likely that in general, at least for conformal structures, one obtains a complex only in the conformally flat setting. In work with Tom Branson the ambient metric of Fefferman-Graham [23] and then variational ideas were used to give two constructions [11, 10] of what we have termed “detour complexes”. These are related to BGG complexes, but involve fewer operators, weaker integrability conditions and yield complexes in curved settings. The central results and ideas (with some extensions) are reviewed and developed in Sections 3.1, 3.2 and 3.4. The complexes obtained there have interesting geometric applications and interpretations, but the constructions involved do not suggest how a full theory of such complexes could be developed. We should note that considerably earlier, following related constructions of Calabi and Gasqui-Goldschmidt, Gasqui constructed [24] a three step elliptic sequence which forms a complex if and only if the manifold is Einstein. This was studied further in [4]. In our current language this is a detour complex, exactly along lines we discuss, and may be viewed as a Riemannian analogue of the conformal complex of [10] as in Theorem 3.5.

Beginning at Section 3.6 we start the development of a model for a rather general construction principle for a class of detour complexes on pseudo-Riemannian manifolds. This is based on recent work with Petr Šomberg and Vladimir Souček. We treat two examples, one in some detail, and the complexes we obtain are elliptic in the case that the background structure has Riemannian signature. In dimension 4 the complexes are conformally invariant. There are two main tools involved in the constructions. The first is general sequence which yields a detour complex for every Yang-Mills connection. The second ingredient is the use of prolonged differential systems.
The use of prolonged systems is related to the following problem: Given a suitable
(i.e. finite type, as explained below) linear partial differential operator $D$, can one
construct an equivalent first order prolonged system that is actually a vector bundle
connection? Here there are close connections with the recent work \cite{8} with Branson,
Čap and Eastwood (for this, and further discussion of BGG complexes, see also
the proceedings of A. Čap \cite{15}). In fact we want more. The connection should be
invariant in the sense that it should share the symmetry and invariance properties of
$D$.

2. Invariant tractor connections for finite type PDE

First we introduce some notation to simplify later presentations. We shall write $\Lambda^1$
for the cotangent bundle, and $\Lambda^p$ for its $p^{th}$ exterior power. Thus the trivial bundle
(with fibre $\mathbb{R}$) may be denoted $\Lambda^0$ although, for reasons that will later become obvious,
we will also write $E$ for this. For simplicity the $k^{th}$ symmetric powers of $\Lambda^1$ will be
denoted $S_k$ (with the metric trace-free part of this denoted $S^0_k$); albeit that this
introduces the redundancy that $S^1 = \Lambda^1$. In all cases we will use the same notation
for a bundle as for its section spaces. All structures will be assumed smooth and we
restrict to differential operators that take smooth sections to smooth sections.

The prolongations of a $k^{th}$-order semilinear differential operator $D : E \to F$,
between vector bundles, are constructed from its leading symbol $\sigma(D) : S^k \otimes E \to F$.
At a point of $M$, denoting by $K^0$ the kernel of $\sigma(D)$, the spaces $V_i = (S^i \otimes E) \cap
(S^{i-k} \otimes K^0)$, $i \geq k$, capture spaces of new variables to be introduced and the system
closes up and is said to be of finite type if $V_i = 0$ for sufficiently large $i$. For example,
on a Riemannian manifold $(M^n, g)$ a vector field $k$ is an infinitesimal isometry, a
so-called Killing vector field, if Lie differentiation along its flow preserves the metric
$g$, that is $L_k g = 0$. Rewriting this in terms of the Levi-Civita connection $\nabla$ (i.e. the
unique torsion free connection on $TM$ preserving the metric) the Killing equation is
seen to be a system of $n(n + 1)/2$ equations in $n$ unknowns, viz.

$$\nabla_a k_b + \nabla_b k_a = 0,$$

where we have used an obvious abstract index notation and also we have used the
metric to identify $k$ with the 1-form $g(k, \cdot )$. In this example $K^0$ is evidently $\Lambda^2$, the
space of 2-forms, and we may rewrite the system as $\nabla_a k_b = \mu_{ab}$ where $\mu \in \Lambda^2$ (that
is $\mu_{ab} = -\mu_{ba}$). Since $(S^2 \otimes \Lambda^1) \cap (\Lambda^1 \otimes \Lambda^2) = 0$, when we differentiate and compute
consequences the system closes up algebraically after just one step. We obtain a
prolonged system which is the equation of parallel transport for a connection:

$$\nabla^D_a \begin{pmatrix} k_b \\ \mu_{bc} \end{pmatrix} := \begin{pmatrix} \nabla_a k_b - \mu_{ab} \\ \nabla_a \mu_{bc} - R_{bcda} k_d \end{pmatrix} = 0,$$

where $R$ is the curvature of $\nabla$. We will regard this connection as being equivalent to
the original equation, since solutions of the original equation are in 1-1 correspondence
with sections of $T := \Lambda^1 \oplus \Lambda^2$ that are parallel for $\nabla^D$. It follows that the original
equation has a solution space of dimension at most $\text{rank}(T) = n(n + 1)/2$. The
curvature of $\nabla^D$ evidently obstructs solutions and, in particular, the maximal number of solutions is achieved only if the connection $\nabla^D$ is flat.
Although this example is very simple it already brings us to several of the essential issues. The original equation $L_k g = 0$ arose in a Riemannian setting. By the explicit formula given it is clear that the connection $\nabla^D$ is naturally invariant for Riemannian structures. However comparing with section 3 in [1], or [16], one identifies this connection as an obvious curvature modification of the normal tractor connection on projective structures. The Levi-Civita connection $\nabla$ in (2.1) may be viewed as an affine connection, so this makes sense. If $\nabla$ is any torsion-free connection on $\Lambda^1$ then note that the equation $\nabla_a k_b + \nabla_b k_a = 0$ is invariant under the transformations $k_b \mapsto \hat{k}_b = e^{2\omega} k_b$ and $\nabla \mapsto \hat{\nabla}$ where, as an operator on 1-forms,

$$\hat{\nabla}_a u_b = \nabla_a u_b - \Upsilon_a u_b - \Upsilon_b u_a \quad \text{with} \quad \Upsilon = d\omega.$$

These show that the equation is in fact projectively invariant, it is well defined on manifolds having only an equivalence class of torsion-free connections, where the equivalence relation is given by the transformations indicated, i.e. on projective manifolds. This leads us to question whether the tractor connection $\nabla^D$ in (2.1) shares this property. It does. It is readily verified directly that it is also projectively invariant; it differs from the invariant normal projective connection in [1, 16] by an action of the (projectively invariant) curvature of the normal connection. In this case it is straightforward to see that this was inevitable.

In general prolonged systems are more complicated. In [8] Kostant’s algebraic Hodge theory [38] led to an explicit and uniform treatment of prolongations for a large class of overdetermined PDE. One of the simplest classes of examples is the set of equations controlling conformal Killing forms and via explicit calculations in terms of the Levi-Civita connection, prolonged systems for these were earlier calculated by [43]. However in neither of these works was the issue of invariance discussed. The equation for a conformal Killing $p$-form $\kappa$ can be given as follows: for any tangent vector field $u$ we have

$$\nabla_u \kappa = \varepsilon(u) \tau + \iota(u) \rho$$

where, on the right-hand side $\tau$ is a $(p-1)$-form, $\rho$ is a $(p+1)$-form, and $\varepsilon(u)$ and $\iota(u)$ indicate, respectively, the exterior multiplication and (its formal adjoint) the interior multiplication of $g(u, \cdot)$. An important property of the conformal Killing equation is that it is conformally invariant (where we require the $p$-form $\kappa$ to have conformal weight $p + 1$). This can be phrased in similar terms to the projective invariance of the equation described above, but alternatively it simply means the equation descends to a well defined equation on manifolds where we do not have a metric, but rather only an equivalence class of conformally related metrics (this is a conformal structure). So it is natural to ask whether there is an equivalent prolonged system that may be realised as a conformally invariant connection. Using the framework of tractor calculus, this is answered in the affirmative in [32], where also the invariant connection is related to the normal conformal tractor connection.

The leading symbol determines whether or not an equation is of finite type; an operator $D : E \to F$ is of finite type if and only if its (complex) characteristic variety is empty [44]. One may ask whether any finite type overdetermined linear differential
operator is equivalent, in the sense of prolongations, to a connection on vector bundle where the connection is “as invariant” as the original operator, that is it shares the same symmetries. This is obviously an important question in its right. For example if for some PDE one finds a connection on prolonged system which depends only on operator \( D \) and no other choices, then it is invariant in this sense and its curvature is really a geometric invariant of the equation \( D \). In general this is probably more than one can hope for. It is probably only reasonable to hope to find a canonical connection after some well defined choices controlled by representation theory on finite dimensional vector spaces that are associated with the original equation. It turns out such questions are also important for the construction of certain natural differential complexes.

3. Detour complexes

We we will specialise our discussions here to the setting of oriented (pseudo-)Riemannian structures and conformal geometries. The restriction to oriented structures is just to simplify statements and is not otherwise required. It should also be pointed out that similar ideas apply in many other settings including, for example, CR geometry.

Recall that a conformal manifold of signature \((p, q)\) on \( M \) is a smooth ray subbundle \( Q \subset S^2T^*M \) whose fibre over \( x \) consists of conformally related signature-\((p, q)\) metrics at the point \( x \). Sections of \( Q \) are metrics \( g \) on \( M \). So we may equivalently view the conformal structure as the equivalence class \([g]\) of these conformally related metrics. The principal bundle \( \pi : Q \to M \) has structure group \( \mathbb{R}_+ \), and so each representation \( \mathbb{R}_+ \ni x \mapsto x^{-w/2} \in \text{End}(\mathbb{R}) \) induces a natural line bundle on \((M, [g])\) that we term the conformal density bundle and denote \( E[w] \). As usual we use the same notation for its section space.

We write \( g \) for the conformal metric, that is the tautological section of \( S^2[2] := S^2 \otimes E[2] \) determined by the conformal structure. This will be used to identify \( T \) with \( \Lambda^1[2] \). For many calculations we will use abstract indices in an obvious way. Given a choice of metric \( g \) from the conformal class, we write \( \nabla \) for the corresponding Levi-Civita connection. With these conventions the Laplacian \( \Delta \) is given by \( \Delta = g^{ab} \nabla_a \nabla_b = \nabla^b \nabla_b \). Note \( E[w] \) is trivialised by a choice of metric \( g \) from the conformal class, and we write \( \nabla \) for the connection corresponding to this trivialisation. It follows immediately that \( \nabla_a \) preserves the conformal metric.

Since the Levi-Civita connection is torsion-free, its curvature \( R_{abcd} \) (the Riemannian curvature) is given by \([\nabla_a, \nabla_b]v^c = R_{abcd} v^d \) \( ([\cdot, \cdot] \) indicates the commutator bracket). This can be decomposed into the totally trace-free Weyl curvature \( C_{abcd} \) and a remaining part described by the symmetric Schouten tensor \( P_{ab} \), according to \( R_{abcd} = C_{abcd} + 2g_{[c[a}P_{b]d} + 2g_{[a}P_{bc]} \), where \([\cdot, \cdot] \) indicates antisymmetrisation over the enclosed indices. The Schouten tensor is a trace modification of the Ricci tensor \( \text{Ric}_{ab} \) and vice versa: \( \text{Ric}_{ab} = (n-2)P_{ab} + Jg_{ab} \), where we write \( J \) for the trace \( P_a^a \) of \( P \). The Weyl curvature is conformally invariant.
3.1. **Detour sequences.** In conformal geometry the de Rham complex is a prototype for a class of sequences of bundles and conformally invariant differential operators, each of the form

$$B^0 \rightarrow B^1 \rightarrow \cdots \rightarrow B^n$$

where the vector bundles $B^i$ are irreducible tensor-spinor bundles. For example, on the sphere with its standard conformal structure we have the following: these are complexes and there is one such complex for each irreducible module $V$ for the group $G = SO(n+1,1)$ of conformal motions; the space of solutions of the first (finite type) conformal operator $B^0 \rightarrow B^1$ is isomorphic to $V$; and the complex gives a resolution of this space viewed as a sheaf. The geometry of the manifold is partly encoded in the coefficients of the differential operators in these sequences, and in the conformally flat setting of the sphere the first PDE operator $D_0 : B^0 \rightarrow B^1$ is fully integrable. From representation theory (see [3] and references therein) one can deduce that there is a prolonged system and connection $\nabla D_0$, equivalent to $D_0$ in the sense discussed in Section 2. In this setting the space of solutions $V$ is maximal in that $\dim V$ achieves the dimension bound for the maximal size of the solution space. Since the sequence is a differential complex, the next operator in the sequence $D_1$ gives an integrability condition for $D_0$ and so on. That the complex gives a resolution means that these integrability conditions are in a sense maximally severe.

It turns out that there are “curved analogues” of these sequences, these are the conformal cases of the (generalised) Bernstein-Gelfand-Gelfand (BGG) sequences, a class of sequences of differential operators that exist on any parabolic geometry [18, 21]. Unfortunately in the general curved setting these sequences are no longer complexes, which limits their applications.

As well as the operators $D_i : B^i \rightarrow B^{i+1}$ of the BGG sequence, in even dimensions there are conformally invariant “long operators” $B^k \rightarrow B^{n-k}$ for $k = 1, \ldots, n/2 - 1$ [3]. Thus there are sequences of the form

$$(3.1) \quad B^0 \xrightarrow{D_0} B^1 \xrightarrow{D_1} \cdots \xrightarrow{D_{k-1}} B^k \xrightarrow{L_k} B^{n-k} \xrightarrow{D_{n-k}} \cdots \xrightarrow{D_{n-1}} B^n.$$ 

and, following [11] (see also [7]), we term these detour sequences since, in comparison to the BGG sequence, the long operator here bypasses the middle of the BGG sequence and the operator $L_k$ takes us (or “detours”) directly from $B^k$ to $B^{n-k}$. Once again using the classification it follows immediately that these detour sequences are in fact complexes in the case that the structure is conformally flat. The operator $L_k$ is again an integrability condition for $D_{k-1}$ but since $L_k$ has higher order than the operator $D_k$ one does not expect that the detour complex is a resolution. In fact, by for example considering Taylor series expansions for solutions, it is straightforward to show that there is local cohomology at the bundle $B^k$. We do not need the details here, the main point is that $L_k$ is weaker, as an integrability condition, than $D_k$.

3.2. **Curved detour complexes.** An interesting direction is to try to find curved analogues of these complexes. That is detour sequences that are actually complexes on conformally curved structures. Remarkably this works in general in the de Rham case, that is the case where the bundles $B^i$ are exterior powers of the cotangent bundle, $\Lambda^i$. We write $\Lambda_k$ to denote the bundle of conformal density weighted $k$-forms.
λ^k[2k - n]; sections of this pair conformally with Λ^k in integrals. Then the formal adjoint of the exterior derivative d acts conformally δ : Λ_i → Λ_{i-1}.

**Theorem 3.3.** [11] In even dimensions there are conformally invariant differential operators

L_k : Λ^k → Λ_k

so that

Λ^0 \overset{d}{\to} \cdots \overset{d}{\to} Λ^{k-1} \overset{d}{\to} Λ^k \overset{L_k}{\to} Λ_{k-1} \overset{δ}{\to} \cdots \overset{δ}{\to} Λ^0

is a conformal complex. In Riemannian signature the complex is elliptic.

The operator L_{n/2-1} is the usual Maxwell operator δd, while L_0, included here (cf. above) as an extreme case, is the critical (i.e. dimension order) conformal Laplacian of GJMS [34]. For k ≠ 0 the complex may be viewed as a differential form analogue of the critical GJMS operator. For 0 ≤ k ≤ n/2 - 2 the L_k have the form δQ_{k+1}d where the Q_ℓ are operators on closed forms that generalise the Q-curvature; for example they give conformal pairings that descend to pairings on de Rham cohomology [9].

3.4. Variational constructions. On curved backgrounds, obtaining complexes for almost any other case of (3.1) seems, at first, to be hopeless. The point is this. One of the main routes to constructing BGG sequences is via some variant of the curved translation principle of Eastwood and others [22, 2, 18, 13]. In the first step of this, one replaces the de Rham complex with the sequence obtained by twisting this sequence with an appropriate bundle and connection. In fact the connections used are usually so called normal conformal tractor connections as in [17]. These are connections on prolonged systems, along the lines of the connection constructed explicitly in Section 2, but these are natural to the conformal structure and are normalised in a way that means that they are unique up to isomorphism. Appropriate differential splitting operators are then used to extract from the twisted de Rham sequence the sought BGG sequence. The full details are not needed here, the main point for our current discussion is as follows. If the structure is conformally flat then the tractor connection is flat and the twisted de Rham sequence sequence is still a complex. It follows from this that the BGG sequence is a complex. However in general the curvature of the tractor connection will obstruct the forming of a complex. This is evident even at the initial stages of the sequence: if we write Λ^i(\mathcal{V}) for the twistings of i-forms by some tractor bundle then, writing ∇ for the tractor connection, the composition

\begin{equation}
Λ^0(\mathcal{V}) \overset{d^∇}{\to} Λ^1(\mathcal{V}) \overset{d^∇}{\to} Λ^2(\mathcal{V})
\end{equation}

is simply the curvature of ∇ acting on \mathcal{V}.

Ignoring this difficulty for the moment, an interesting case from conformally flat structures is the conformal deformation complex of [25]. Some notation first. We write Λ^{k,ℓ} for the space of trace-free covariant (k + ℓ)-tensors t_{a_1...a_kb_1...b_ℓ} which are skew on the indices a_1...a_k and also on the set b_1...b_ℓ. Skewing over more than k indices annihilates t, as does symmetrising over any 3 indices. Thus for example
\( \Lambda^{1,1} = S^2_0 \), but in the case of this bundle we will postpone changing to this notation. In dimensions \( n \geq 5 \) the initial part of this BGG complex is,

\[
T \xrightarrow{K_0} S^2_0[2] \xrightarrow{\mathcal{C}} \Lambda^{2,2}[2] \xrightarrow{\mathcal{B}_i} \Lambda^{3,2}[2] \rightarrow \cdots
\]

where \( T \) is the tangent bundle. Here \( \mathcal{C} \) is the linearisation, at a conformally flat structure, of the Weyl curvature as an operator on conformal structure; \( \mathcal{B}_i \) is a conformal integrability condition arising from the Bianchi identity; the operator \( K_0 \) is the conformal Killing operator, viz the operator which takes infinitesimal diffeomorphisms to their action on conformal structure. Since infinitesimal conformal variations take values in \( S^2_0[2] \), it follows from these interpretations that the cohomology of the complex, at this bundle, may be interpreted as the formal tangent space to the moduli space of conformally flat structures.

From this picture, and also from the discussion around (3.2), we do not expect a curved generalisation of the complex (3.3). One way to potentially avoid the issues brought up with (3.2) is to consider detour sequences of the form (3.1) with \( k = 1 \); these will be termed short detour sequences. (In [31] there is some discussion and applications of detour complexes for this BGG in the conformally flat setting.)

It turns out that this idea is fruitful. One construction is based around the so-called obstruction tensor \( \mathcal{B}_{ab} \) of Fefferman and Graham [23] which generalises to even dimensions the Bach tensor of Bach’s gravity theory. For our current purposes we need only to know some key facts about this, and we shall introduce these as required. It is a trace-free conformal conformal 2-tensor with leading term \( \Delta^{n/2-2} C \) (where, recall, \( C \) denotes the Weyl curvature). Let us write \( K_0^* \) for the formal adjoint of \( K_0 \) and \( B \) for the linearisation, at the metric \( g \), of the operator which takes metrics \( g \) to \( \mathcal{B} \). By taking the Lie derivative of \( \mathcal{B}_{ab} \), and using the fact [32] that \( \mathcal{B}_{ab} \) is the total metric variation of an action (viz. \( \int Q \) where \( Q \) is Branson’s \( Q \) curvature [5]) we obtain the following (where to simplify notation we have omitted the conformal weights).

**Theorem 3.5.** [10] On even dimensional pseudo-Riemannian manifolds with the Fefferman-Graham obstruction tensor vanishing everywhere, the sequence of operators

\[
T \xrightarrow{K_0^*} S^2_0 \xrightarrow{B} S^2_0 \xrightarrow{\mathcal{B}_i} T
\]

is a formally self-adjoint complex of conformally invariant operators. In Riemannian signature the complex is elliptic.

As with the deformation complex (3.3) there is an immediate interpretation of the cohomology at \( S^2_0 \). By construction it is the formal tangent space to the moduli space of obstruction-flat structures. There are determinant quantities, with interesting conformal behaviour, for detour complexes [12, 7]; this detour torsion should be especially interesting for (3.4).

Leaving aside the potential applications it is already interesting that there is a differential complex along these lines in such a general setting. Although the obstruction tensor is rather mysterious, and at this point has not been fully explored,
it is known that it vanishes on conformally Einstein manifolds \cite{23, 31, 35}, certain products of Einstein manifolds \cite{29}, and on half-flat structures in dimension 4.

3.6. The Yang-Mills detour complex. The construction in Theorem 3.5 and subsequent observations suggest the possibility of a rich theory of (short) detour complexes. On a large class of manifolds, the formally self-adjoint operator \( B \) is evidently sufficiently “weak” as an integrability condition for \( K_0 \) that we obtain a complex. The construction may obviously be generalised by using different Lagrangian densities (cf. \( Q \)) in the first step. On the other hand, if we use directly the ideas from the proof of Theorem 3.5 then it seems that we may only obtain detour complexes with either the Killing operator on vector fields, \( k \mapsto L_k g \) (with \( g \) the metric), or its conformal analogue \( K_0 \), as the first operator \( B^0 \rightarrow B^1 \). This motivates a rather different approach.

The simplest example of a conformal detour complex is the Maxwell detour complex in dimension 4
\[ \Lambda^0 \rightarrow \Lambda^1 \rightarrow \Lambda^1 \rightarrow \Lambda^0. \]
In other dimensions this is also a complex, but the middle operator is no longer conformally invariant. Let us temporarily relax the condition of conformal invariance and consider this complex on a pseudo-Riemannian \( n \)-manifold \((M, g)\) of signature \((p , q)\), with \( n \geq 2 \).

For any vector vector bundle \( V \), with connection \( \tilde{\nabla} \), we might consider twisting the Maxwell detour. Of course the result would not be a complex. However by a minor variation on this theme we make an interesting observation. Recall that we write \( d_{\tilde{\nabla}} \) for the connection-coupled exterior derivative operator \( d_{\tilde{\nabla}} : \Lambda^k(V) \rightarrow \Lambda^{k+1}(V) \). Of course we could equally consider the coupled exterior derivative operator \( d_{\tilde{\nabla}} : \Lambda^{k}(V^*) \rightarrow \Lambda^{k+1}(V^*) \) and for the formal adjoint of this we write \( \delta_{\tilde{\nabla}} : \Lambda_{k+1}(V) \rightarrow \Lambda_{k}(V) \).

Denote by \( F \) the curvature of \( \tilde{\nabla} \) and write \( F \cdot \) for the action of the curvature on the twisted 1-forms, \( F : \Lambda^1(V) \rightarrow \Lambda_{1}(V) \) given by
\[ (F \cdot \phi)_a := F^b_a \phi_b, \]
where we have indicated the abstract form indices explicitly, whereas the standard \( \text{End}(V) \) action of the curvature on the \( V \)-valued 1-form is implicit. Using this we construct a differential operator
\[ M_{\tilde{\nabla}} : \Lambda^1(V) \rightarrow \Lambda_{1}(V) \]
by
\[ M_{\tilde{\nabla}} \phi = \delta_{\tilde{\nabla}} d_{\tilde{\nabla}} \phi - F \cdot \phi. \]
By a direct calculation, the composition \( M_{\tilde{\nabla}} d_{\tilde{\nabla}} \) amounts to an exterior algebraic action by the “Yang-Mills current” \( \delta_{\tilde{\nabla}} F \), thus we have the first statement of the following result. The other claims are also easily verified.

**Theorem 3.7.** \cite{33} The sequence of operators,
\[ \Lambda^0(V) \rightarrow \Lambda^1(V) \rightarrow \Lambda_{1}(V) \rightarrow \Lambda_{0}(V) \]
is a complex if and only if the curvature $F$ of the connection $\tilde{\nabla}$ satisfies the (pure) Yang-Mills equation
$$\delta \tilde{\nabla} F = 0.$$ 

In addition:
(i) If $\tilde{\nabla}$ is an orthogonal or unitary connection then the sequence is formally self-adjoint.
(ii) In Riemannian signature the sequence is elliptic.
(iii) In dimension 4 the complex is conformally invariant.

This obviously yields a huge class of complexes. For example, taking $V$ to be any tensor (or spin) bundle on a (spin) manifold with harmonic curvature (i.e. a pseudo-Riemannian structure where the Riemannian curvature satisfies the Yang-Mills equations) yields a complex. Einstein metrics are harmonic so this is large class of structures. To be specific if we take, for example, $V$ to be the second exterior power of the tangent bundle $T^2$, and use $\nabla$ to denote the Levi-Civita connection then we obtain the complex
$$T^2 \xrightarrow{d^\nabla} \Lambda^1 \otimes T^2 \xrightarrow{M^\nabla} \Lambda^1 \otimes T^2 \xrightarrow{\delta^\nabla} T^2,$$
where $M^\nabla$ is given by
$$S_{b}^{\; cd} \mapsto -2\nabla^a \nabla_{[a} S_{b]}^{\; cd} - R_{ba}^{\; e} S^{aed} - R_{ba}^{\; d} S^{ace}.$$ 

This example reminds us that part (iii) of the theorem means that the complex (3.6) is conformally invariant in dimension 4, provided the connection $\tilde{\nabla}$ is conformally invariant. For applications in conformal geometry we need suitable conformal $(V, \tilde{\nabla})$.

3.8. Short detour complexes and tractor connections. We may use Theorem 3.7 to construct more differential complexes. Consider the following general situation. Suppose that there are vector bundles (or rather section spaces thereof) $B^0, B^1, B_1$ and $B_0$ and differential operators $L_0, L_1, L^1, L^0, D$ and $\overline{D}$ which act as indicated in the following diagram:

![Diagram](image)

The top sequence is (3.6) for a connection $\tilde{\nabla}$ with curvature $F$ and the operator $M^B : B^1 \to B_1$ is defined to be the composition $L^1 M^\nabla L_1$. Suppose that the squares at each end commute, in the sense that as operators $B^0 \to \Lambda^1(V)$ we have $d^\nabla L_0 = L_1 D$ and as operators $\Lambda_1(V) \to B_0$ we have $L^1 \delta^\nabla = \overline{D} L^1$. Then on $B^0$ we have
$$M^B D = L^1 M^\nabla L_1 D = L^1 M^\nabla d^\nabla L_0 = L^1 \epsilon(\delta^\nabla F) L_0,$$
and similarly $\overline{\nabla} M^8 = - L^0 i(\delta \nabla F) L_0$. Here $\epsilon(\delta \nabla F)$ and $i(\delta \nabla F)$ indicate exterior and interior actions of the Yang-Mills current $\delta \nabla F$. Thus if $\nabla$ is Yang-Mills then the lower sequence, viz.

$$B^0 \xrightarrow{D} B^1 \xrightarrow{M^8} B_1 \xrightarrow{\overline{\nabla}} B_0,$$

is a complex.

**Remark:** Note that if the connection $\nabla$ preserves a Hermitian or metric structure on $V$ then we need only the single commuting square $d \nabla L_0 = L_1 D$ on $B^0$ to obtain such a complex; by taking formal adjoints we obtain a second commuting square $(L^0 \delta \nabla = \overline{\nabla} L^1) : B_1 \rightarrow B_0$ where $B_0$ and $B_1$ are appropriate density twistings of the bundles $B^0$ and $B^1$ respectively.

We are now in a position to link the various constructions and indicate the general prospects. Suppose that in the setting of Riemannian or conformal geometry, we have some finite type PDE operator $B : B^0 \rightarrow B^1$ that we wish to study. A question we raised in Section 2 is whether there is a corresponding connection $\nabla^D$ that is equivalent in the sense that solutions of $D$ are in 1-1 correspondence with parallel sections of $\nabla^D$. Consider the situation of the first square in the diagram. If $D\phi = 0$ then obviously $L_0 \phi$ is parallel for $\nabla$. Now suppose that the $L_i$ are in fact differential splitting operators, that is for $i = 0, 1$ these may be inverted by differential operators $L_i^{-1}$ in the sense that $L_i^{-1} \circ L_i = id_{B^i}$. Then $D = L_i^{-1}(L_1 D) = L_i^{-1}(\nabla L_0)$ and so solutions for $D$ are mapped by $L_0$ injectively to sections of $V$ that are parallel for $\nabla$. Finally if also $L_0^{-1}$ is well defined on general sections of $V$ and $L_0 \circ L_0^{-1}$ acts as the identity on the space of parallel sections of $V$, then $L_0^{-1}$ maps parallel sections for $\nabla$ to $D$-solutions. All these conditions are somewhat more than is strictly necessary according to ideas of section 2. Nevertheless, as we indicate below, these are satisfied for some examples, and it seems likely that these are the simplest cases of a large class.

3.9. **Examples.** Here we treat mainly a single example (with a hint of a second example), but it illustrates well the general idea. We return to the setting of conformal $n$-manifolds. Modulo the trace part, the conformal transformation of the Schouten tensor is controlled by the equation

$$(3.8) \quad D\sigma = 0 \quad \text{where} \quad D\sigma := TF(\nabla_a \nabla_b \sigma + P_{ab} \sigma),$$

which is written in terms of some metric $g$ from the conformal class. In particular a metric $\sigma^{-2}g$ is Einstein if and only if the scale $\sigma \in E[1]$ is non-vanishing and satisfies (3.8).

The conformal standard tractor bundle and connection arises from a prolongation of this overdetermined equation as follows, cf. [1, 16]. Recalling the notation developed in Section 3.4 we have $\Lambda^{1,1}$ as an alternative notation for $S^0_0$. Let us also write $\Lambda_{1,1} := \Lambda^{1,1} \otimes E[4 - n]$. As usual we use the same notation for the spaces of sections of these. The *standard tractor bundle* $T$ may be defined as the quotient of $J^2 E[1]$ by the image of $\Lambda^{1,1}[1]$ in $J^2 E[1]$ through the jet exact sequence at 2-jets. Note that there is a tautological operator $D : E[1] \rightarrow \Lambda^0(T)$ which is simply the composition of the universal 2-jet operator $j^2 : E[1] \rightarrow J^2 E[1]$ followed by the canonical projection.
$J^2\mathcal{E}[1] \to \Lambda^0(\mathcal{T})$. By construction both $\mathcal{T}$ and $\mathcal{D}$ are invariant, they depend only on the conformal structure and no other choices. Via a choice of metric $g$, and the Levi-Civita connection it determines, we obtain a differential operator

\begin{equation}
\mathcal{E}[1] \to \mathcal{E}[1] \oplus \Lambda^1[1] \oplus \mathcal{E}[-1] \quad \text{by} \quad \sigma \mapsto (\sigma, \nabla_a \sigma, -\frac{1}{n}(\Delta + J)\sigma). \tag{3.9}
\end{equation}

Since this is second order it factors through a linear map $J^2\mathcal{E}[1] \to \mathcal{E}[1] \oplus \Lambda^1[1] \oplus \mathcal{E}[-1]$. Considering Taylor series one sees that the kernel of this is a copy of $\Lambda^1$, and so the map determines an isomorphism

\begin{equation}
\mathcal{T} \overset{g}{\cong} \mathcal{E}[1] \oplus \Lambda^1[1] \oplus \mathcal{E}[-1]. \tag{3.10}
\end{equation}

In terms of this the formula at the right extreme of the display (3.9) then tautologically gives an explicit formula for $\mathcal{D}$. This is a differential splitting operator since through the jet projections there is conformally invariant surjection $X : \mathcal{E}(\mathcal{T}) \to \mathcal{E}[1]$ which inverts $\mathcal{D}$. In terms of the splitting (3.10) this is simply $(\sigma, \mu, \rho) \mapsto \sigma$.

Observe that if we change to a conformally related metric $\hat{g} = e^{2\omega}g$ then the Levi-Civita connection has a corresponding transformation, and so we obtain a different isomorphism. For $t \in \mathcal{T}$ then via (3.10) we have $[t]_g = (\sigma, \mu, \rho)$. In terms of the analogous isomorphism for $\hat{g}$ we have

\begin{equation}
[t]_{\hat{g}} = (\hat{\sigma}, \hat{\mu}, \hat{\rho}) = (\sigma, \mu_a + \sigma \Upsilon_a, \rho - g^{bc} \Upsilon_b \mu_c - \frac{1}{2}\sigma g^{bc} \Upsilon_b \Upsilon_c), \tag{3.11}
\end{equation}

where $\Upsilon = d\omega$.

Now let us define a connection on $\mathcal{E}[1] \oplus \Lambda^1[1] \oplus \mathcal{E}[-1]$ by the formula

\begin{equation}
\nabla_a \begin{pmatrix} \sigma \\ \mu_b \\ \rho \end{pmatrix} := \begin{pmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b + g_{ab} \rho + P_{ab} \sigma \\ \nabla_a \rho - P_{ab} \mu^b \end{pmatrix} \tag{3.12}
\end{equation}

where, on the right-hand-side $\nabla$ is the Levi-Civita connection for $g$. Obviously this determines a connection on $\mathcal{T}$ via the isomorphism (3.10). What is more surprising is that if we repeat this using a different metric $\hat{g}$, this induces the same connection on $\mathcal{T}$. Equivalently the connection in the display transforms according to (3.11). This canonical connection on $\mathcal{T}$ depends only on the conformal structure and is known as the (standard) tractor connection.

There are several ways we may understand the connection (3.12) and its invariance. On the one hand this may be viewed as a special case of a normal tractor connection for parabolic geometries. This construction is treated from that point of view in [16]. Such connections determine the normal Cartan connections on the corresponding adapted frame bundles for the given tractor bundle $\mathcal{T}$, [17]. On the other hand one may start directly with the operator $D$ in (3.8). It is readily verified that, for $n \geq 3$, this is of finite type and so we may attempt to construct a prolonged system and connection $\nabla^D$ for the operator along the lines of the treatment of Killing operator in Section 2. This is done in detail in [1] and one obtains exactly (3.12). So the connection (3.12) is the sought $\nabla^D$. Some key points are as follows: since the operator in (3.8) acts $D : \mathcal{E}[1] \to \Lambda^{1,1}[1]$ and is second order, it factors through a linear bundle
map $D^{(0)} : J^2\mathcal{E}[1] \to \Lambda^{1,1}[1]$; from the formula for $D$ it is clear that this is a splitting of the exact sequence

$$0 \to \Lambda^{1,1}[1] \to J^2\mathcal{E}[1] \to \mathcal{T} \to 0$$

deﬁnes $\mathcal{T}$. It follows that the prolonged system will include $\mathcal{T}$. In this case the key to the closing up of the prolonged system is that any completely symmetric covariant 3 tensor that is also pure trace on some pair of indices is necessarily zero. From this it follows that the full prolonged system may be expressed in terms of $\mathcal{T}$.

There is also a differential splitting operator

$$E : \Lambda^{1,1}[1] \to \Lambda^1(\mathcal{T}) \quad \psi_{ab} \mapsto (0, \psi_{ab}, -(n-1)^{-1} \nabla^b \psi_{ab})$$

(cf. [20]). An easy calculation veriﬁes this is also conformally invariant, and from the formula it is manifest that we have an operator $E^{-1}$ (in fact bundle map) that inverts $E$ from the left. Crucially, we have the following.

**Proposition 3.10.** [33] As differential operators on $\mathcal{E}[1]$, we have

$$\nabla^D \mathbb{D} = ED.$$  

For $\sigma \in \mathcal{E}[1]$, $\mathbb{D}\sigma$ is parallel if and only if $D\sigma = 0$.

**Proof:** The second statement is immediate from the ﬁrst. A straightforward calculation veriﬁes that either composition applied to $\sigma \in \mathcal{E}[1]$ yields

$$\begin{pmatrix}
0 \\
TF(\nabla_a \nabla_b \sigma + P_{ab} \sigma) \\
-\frac{1}{n} \nabla_a (\Delta \sigma + J\sigma) - P_{ac} \nabla_c \sigma
\end{pmatrix}$$

For our present purposes the main point of Proposition 3.10 is that it gives the ﬁrst step in constructing a special case of a commutative detour diagram $[D]$. First note that we have exactly the situation discussed in the last part of Section 3.8: here $\mathbb{D}$ and $E$ play the roles of, respectively, $L_0$ and $L_1$. They are differential splitting operators and have inverses exactly as discussed there. It is an easy exercise to verify that if $I$ is a parallel section of $\mathcal{T}$ then $I = D\sigma$ for some section $\sigma$ of $\mathcal{E}[1]$, as observed in [30] (and so $X$ as an inverse to $\mathbb{D}$ maps parallel tractors to solutions of $D$). A useful consequence of this last result is that a conformal manifold with a parallel tractor is almost Einstein in the sense that it has a section of $\mathcal{E}[1]$ that gives an an Einstein scale on an open dense subset (see [27] for further details).

Next we observe that there is a conformally invariant tractor metric $h$ on $\mathcal{T}$ given (as a quadratic form) by $(\sigma, \mu, \rho) \mapsto g^{-1}(\mu, \mu) + 2\sigma\rho$. This has signature $(p+1, q+1)$ (corresponding to $g$ of signature $(p, q)$) and is preserved by the tractor connection. In view of this, and using our general observations in Section 3.8 the formal adjoints of the operators above give the other required commutative square of operators. That
is with
\[ D^* : \Lambda_1[-1] \to \Lambda_0[-1] \quad \theta_{ab} \mapsto \nabla^a \nabla^b \theta_{ab} + P^{ab} \theta_{ab} \]
\[ E^* : \Lambda_1(T) \to \Lambda_1[-1] \quad (\alpha_a, \nu_{ab}, \tau_a) \mapsto \nu_{(ab)} + \int \nabla_{(a} \alpha_{b)} \]
\[ \mathbb{D}^* : \Lambda_0(T) \to \Lambda_0[-1] \quad (\sigma, \mu_b, \rho) \mapsto \nabla_{\lambda} \sigma + \rho \]
\[ \delta \nabla^D : \Lambda_1(T) \to \Lambda_0(T) \quad \Phi_{aB} \mapsto -\nabla_a \Phi_{aB} \]
we have \( \mathbb{D}^* \delta \nabla^D = D^* E^* \) on \( \Lambda_1(T) \).

We now want to consider what it means for the tractor connection to satisfy the Yang-Mills equations. Using the explicit presentation (3.12) of the tractor connection at a metric, it is straightforward to calculate its curvature. This is
\[ \Omega_{ab}^{CD} = \begin{pmatrix} 0 & 0 & 0 \\ A_{ab}^c & C_{ab}^{cd} & 0 \\ 0 & -A_{dc}^a & 0 \end{pmatrix} \]
where \( A \) is the Cotton tensor, \( A_{abc} := 2 \nabla[a P_{bc}] \). Taking the required divergence we obtain \( -\delta \nabla^C \Omega \) (see e.g. [30] for further details),
\[ \nabla^D \Omega_{ab}^{CD} = \begin{pmatrix} 0 & 0 & 0 \\ B_{ab}^c & (n - 4) A_{bc}^e & 0 \\ 0 & -B_{eb}^c & 0 \end{pmatrix} \]
where, on the left-hand side, \( \nabla^D \) is really the Levi-Civita connection coupled with the tractor connection on \( \text{End}(T) \) induced from \( \nabla^D \). Here \( B_{ab} \) is the Bach tensor \( B_{ab} := \nabla^c A_{acb} + P^{dc} C_{dcb} \).

Let us say that a pseudo-Riemannian manifold is semi-harmonic if its tractor curvature is Yang-Mills, that is \( \nabla^a \Omega_{ab}^{CD} = 0 \). Note that in dimensions \( n \neq 4 \) this is not a conformally invariant condition and a semi-harmonic space is a Cotton space that is also Bach-flat. From our observations above, the semi-harmonic condition is conformally invariant in dimension 4 and according to the last display we have the following result (which in one form or another has been known for many years e.g. [41, 37]).

**Proposition 3.11.** In dimension 4 the conformal tractor connection is a Yang-Mills connection if and only if the structure is Bach-flat.

Writing \( M^T \) for the composition \( E^* M \nabla E \) from the construction in section 3.8 above, we have the following (except for the claim of ellipticity, which is straightforward).

**Theorem 3.12.** [33] The sequence
\[ \Lambda^0[1] \overset{P}{\to} \Lambda^{1,1}[1] \overset{M^T}{\to} \Lambda_{1,1}[-1] \overset{P^*}{\to} \Lambda_0[-1] \]
has the following properties.
(i) It is a formally self-adjoint sequence of differential operators and, for \( \sigma \in \mathcal{E}[1] \)
\[ (M^T D \sigma)_{ab} = -TFS(B_{ab} \sigma - (n - 4) A_{abc} \nabla^c \sigma), \]
where \( TFS(\cdots) \) indicates the trace-free symmetric part of the tensor concerned. In particular it is a complex on semi-harmonic manifolds.
(ii) In the case of Riemannian signature the complex is elliptic.

(iii) In dimension 4, \((3.17)\) is sequence of conformally invariant operators and it is a complex if and only if the conformal structure is Bach-flat.

**Corollary 3.13.** Einstein 4-manifolds are Bach-flat.

**Proof:** If a non-vanishing density \(\sigma\) is an Einstein scale then, calculating in that scale, we have \(M^T D\sigma = -B\sigma\), where \(B\) is the Bach tensor. On the other hand if \(\sigma\) is an Einstein scale then \(D\sigma = 0\) (see \((3.8)\)). \(\blacksquare\)

In fact, the result in the Corollary, and more general results, have been known by other means for some time (see e.g. [31, 35] and references therein). Nevertheless it seems the detour complex gives an interesting route to this.

We conclude here with just the statements of another example from [33]. This is also obtained from the diagram [D] and an appropriate tractor connection \(\nabla\) for an operator \(D\), in this case the operator \(D\) is the twistor operator. Thus we assume here that we have a conformal spin structure. Following [6] we write \(\mathbb{S}\) for the basic spinor bundle and \(\mathbb{S} = S[\!\!\!-n]\) (i.e. the bundle that pairs globally in an invariant way with \(S\) on conformal \(n\)-manifolds). The weight conventions here give \(\mathbb{S}\) a “neutral weight”. In terms of, for example, the Penrose weight conventions \(\mathbb{S} = E^\lambda[\!\!\!-\frac{1}{2}] = E_\lambda[\!\!\!\frac{1}{2}]\), where \(E^\lambda\) denotes the basic contravariant spinor bundle in [42].

We write \(\text{Tw}\) for the so-called twistor bundle, that is the subbundle of \(\Lambda^1 \otimes \mathbb{S}[1/2]\) consisting of form spinors \(u_a\) such that \(\gamma^a u_a = 0\), where \(\gamma_a\) is the usual Clifford symbol. We use \(\mathbb{S}\) and \(\text{Tw}\) also for the section spaces of these bundles. The **twistor operator** is the conformally invariant Stein-Weiss gradient

\[
T : \mathbb{S}[1/2] \rightarrow \text{Tw}
\]

given explicitly by

\[
\psi \mapsto \nabla_a \psi + \frac{1}{n} \gamma_a \gamma^b \nabla_b \psi.
\]

This completes to a differential complex as follows.

**Theorem 3.14.** [33] On semi-harmonic pseudo-Riemannian \(n\)-manifolds \(n \geq 4\) we have a differential complex

\[
(3.16) \quad \mathbb{S}[1/2] \xrightarrow{T} \text{Tw} \xrightarrow{N} \text{Tw} \xrightarrow{T^*} \mathbb{S}[\!\!\!-1/2],
\]

where \(T\) is the usual twistor operator, \(T^*\) its formal adjoint, and \(N\) is third order. The sequence is formally self-adjoint and in the case of Riemannian signature the complex is elliptic.

In dimension 4 the sequence \((3.16)\) is conformally invariant and it is a complex if and only if the conformal structure is Bach-flat.

The operator \(N\) is a third order analogue of a Rarita-Schwinger operator. Of course on a fixed pseudo-Riemannian manifold we may ignore the conformal weights.
3.15. **Outlook.** BGG complexes do not generally have curved analogues. The weaker integrability conditions involved with detour complexes suggest the possibility of large classes of such objects.

In the terminology of physics, the formally self-adjoint short detour complexes \( B^0 \to B^1 \to B_1 \to B_0 \) give “classically consistent systems”. The constraint equations, \( B_1 \to B_0 \) which give an integrability condition on the middle operator, are suitably dual to, the gauge transformations of \( B^1 \) given by \( B^0 \to B^1 \). Thus if we take \( B^1 \to B_1 \) as the field equations then the first cohomology of the complex gives “observable” field quantities. It seems that there is some scope for quantising this picture [28].

As we see in Corollary 3.13 the first operator \( B^0 \to B^1 \) can itself have an important geometric or physical interpretation and the complexes provide a tool for studying these. The first cohomology of the complex gives a global invariant.

Generalising the examples given above requires two main steps. The first is to develop the theory of prolonged differential systems in a way which leads to prolonged systems that share the invariance properties of the original equation (this is currently the subject of joint work with M.G. Eastwood). The second part is to understand what operators may be used to replace \( M^D \), so that, for example, we may have conformally invariant examples in higher dimensions.

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