FIXED SETS OF AUTOMORPHISMS OF COUNTABLE, ARITHMETICALLY SATURATED STRUCTURES

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Abstract. If an automorphism $f$ of a structure $M$ is such that $\text{fix}(f^k) = \text{fix}(f)$ for all positive $k$, then $M|\text{fix}(f)$ is a substructure of $M$. The possible isomorphism types of $M|\text{fix}(f)$ are characterized when $M$ is countable and arithmetically saturated.

The purpose of this note is to present the following theorem and its proof.

Theorem: If $M$ is a countable, arithmetically saturated structure and $D \subseteq M$ is algebraically closed, then there is $f \in \text{Aut}(M)$ such that $M|\text{fix}(f) \cong M|D$ (even elementarily isomorphic) and $\text{fix}(f) = \text{fix}(f^k)$ for all $k > 0$.

The origin of this theorem can be traced back to 1991 when Kaye, Kossak & Kotlarski [K3, Th. 5.3] proved that if $M$ is a countable, arithmetically saturated model of Peano Arithmetic ($\mathsf{PA}$), then there is an automorphism $f \in \text{Aut}(M)$ that moves every undefinable element of $M$ – that is, $\text{fix}(f) = \text{acl}(\emptyset)$, where $\text{fix}(f) = \{x \in M : f(x) = x\}$ and $\text{acl}(X)$ is the algebraic closure of the set $X \subseteq M$. For models of $\mathsf{PA}$, this means that $M|\text{fix}(f)$ is the prime elementary substructure of $M$. Furthermore, they showed that for any finite $X \subseteq M$, there is $f \in \text{Aut}(M)$ such that $\text{fix}(f) = \text{acl}(X)$. As a counterpoint to this theorem, Kossak [Ko97, Th. 2.6] proved that if $M$ is a countable, recursively saturated model of $\mathsf{PA}$ that is not arithmetically saturated, then $M|\text{fix}(f) \cong M$ for every $f \in \text{Aut}(M)$.

Although the proof of the K3 theorem apparently made good use of $\mathsf{PA}$, Körner [Ko98, Lemma 1.3] eliminated the need for $\mathsf{PA}$ by proving that every countable, arithmetically saturated structure $M$ has an automorphism $f$ such that $\text{fix}(f) \subseteq \text{acl}(\emptyset)$. Körner calls such an $f$, for any $M$, a maximal automorphism of $M$. Körner’s theorem was subsequently improved by Duby [Du03, Th. 8] who strengthened the...
conclusion by adding that $\text{fix}(f^k) = \text{acl}(\emptyset)$ for all $k > 0$. (This addition is automatic for models of $\text{PA}$ and and even for all linearly ordered $\mathcal{M}$, but not so for some other structures.) Duby refers to such an $f$ as an $\omega$-maximal automorphism of $\mathcal{M}$. It is readily seen that Duby’s theorem generalizes to: If $\mathcal{M}$ is countable and arithmetically saturated and $X \subseteq M$ is finite, then then there is $f \in \text{Aut}(\mathcal{M})$ such that $\text{fix}(f^k) = \text{acl}(X)$ for all $k > 0$.

The question was raised ([Ko97, Question 2.7] and [KS, Question 9, p. 291]) as to what other possibilities there are for the isomorphism types of $\mathcal{M}|_{\text{fix}(f)}$ when $\mathcal{M}$ is a countable, arithmetically saturated model of $\text{PA}$ and $f \in \text{Aut}(\mathcal{M})$. According to Enayat [En07, §4.2], I had earlier conjectured that if $\mathcal{M}$ is a countable, arithmetically saturated model of $\text{PA}$ and $\mathcal{N} \prec \mathcal{M}$, then there is $f \in \text{Aut}(\mathcal{M})$ such that $\mathcal{M}|_{\text{fix}(f)} \cong \mathcal{N}$. Kossak [Ko97, Th. 2.8] lent credence to this conjecture by observing that for every countable $\mathcal{N}$ (not necessarily a model of $\text{PA}$), there are a countable, arithmetically saturated $\mathcal{M} \succ \mathcal{N}$ and $f \in \text{Aut}(\mathcal{M})$ such that $\text{fix}(f) = \mathcal{N}$. Incidentally, Körner’s theorem easily implies that for every countable $\mathcal{N}$ (not necessarily a model of $\text{PA}$), there are a countable, arithmetically saturated $\mathcal{M} \succ \mathcal{N}$ and $f \in \text{Aut}(\mathcal{M})$ such that $\text{fix}(f) = \mathcal{N}$.

My conjecture was later confirmed by Enayat [En07, Th. 4.2.1] using iterated ultrapowers. The Theorem does for Enayat’s theorem what Körner’s and Duby’s theorems did for the K$^3$ theorem by eliminating $\text{PA}$.

We note that the Theorem is best possible in the sense that if $\mathcal{M}$ is any structure and $f \in \text{Aut}(\mathcal{M})$ is such that $\text{fix}(f) = \text{fix}(f^k)$ for all $k > 0$, then $\mathcal{M}|_{\text{fix}(f)}$ is algebraically closed.

Duby [Du03] proves, not only his theorem that we have been referring to, but also the theorem asserting that every uncountable saturated structure has an $\omega$-maximal automorphism. Our proof of the Theorem is influenced more by Duby’s proof of this latter theorem than by his proof of the former one.

Following this introduction there are two sections. The first presents preliminary material, including those definitions that are needed to understand the Theorem. The second section contains our proof of the Theorem.

§1. Preliminaries. All structures $\mathcal{M}$ considered here are for a finite language – that is, for each $\mathcal{M}$, there is a finite $\mathcal{L}$ such that $\mathcal{M}$ is an $\mathcal{L}$-structure. Typically, an $\mathcal{L}$-structure will be denoted by a script letter, such as $\mathcal{M}$, and then its universe is understood to be $M$. If $X \subseteq M$,

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\footnote{Assume $\mathcal{N}$ is infinite. Let $<$ be a linear order of $\mathcal{N}$ having ordertype $\omega$, and let $(\mathcal{M}, <) \succ (\mathcal{N}, <)$ be countable and arithmetically saturated. Apply Körner to $\mathcal{N}$.}
then \( \mathcal{L}(A) \) is the language \( \mathcal{L} \) adjoined with a constant symbol for each \( a \in A \).

As usual, \( \omega \) is the set of natural numbers (i.e. nonnegative integers) and \( \mathbb{Z} \) is the set of (negative and nonnegative) integers. Hence, we have that \( \omega \subseteq \mathbb{Z} \). We typically write \( n < \omega \) instead of \( n \in \omega \).

If \( x_0, x_1, \ldots, x_{n-1} \) are variables, where \( n < \omega \), we will often write \( \overline{x} \) instead of \( x_0, x_1, \ldots, x_{n-1} \). Similarly, if \( A \) is a set and \( a_0, a_1, \ldots, a_{n-1} \in A \), then we write \( \overline{a} \) instead of \( a_0, a_1, \ldots, a_{n-1} \) or for the \( n \)-tuple \( \langle a_0, a_1, \ldots, a_{n-1} \rangle \). We often write \( \overline{a} \in A \) instead of \( \overline{a} \in A^n \). If \( \mathcal{M} \) is a structure, \( \overline{a} \in \mathcal{M} \) and \( X \subseteq \mathcal{M} \), then \( \text{tp}_X(\overline{a}) \) is the set of all \( n \)-ary \( \mathcal{L}(X) \)-formulas \( \varphi(\overline{x}) \) such that \( \mathcal{M} \models \varphi(\overline{a}) \). As usual, \( \text{tp}(\overline{a}) = \text{tp}_\emptyset(\overline{a}) \).

The set of automorphisms of a structure \( \mathcal{M} \) is \( \text{Aut}(\mathcal{M}) \). For \( f \in \text{Aut}(\mathcal{M}) \), its fixed set is \( \text{fix}(f) = \{ x \in \mathcal{M} : f(x) = x \} \). If \( X \subseteq \mathcal{M} \), then we let \( \text{Aut}_X(\mathcal{M}) \) be the subgroup of \( \text{Aut}(\mathcal{M}) \) consisting of those \( f \in \text{Aut}(\mathcal{M}) \) such that \( \text{fix}(f) \supseteq X \). If \( n < \omega \), then \( D \subseteq \mathcal{M}^n \) is \( X \)-definable if it is definable in \( \mathcal{M} \) by an \( \mathcal{L}(X) \)-formula. We denote by \( \text{Def}(\mathcal{M}; X) \) the set of \( X \)-definable subsets of \( \mathcal{M} \), and then \( \text{Def}(\mathcal{M}) = \text{Def}(\mathcal{M}; \emptyset) \). The algebraic closure of \( X \), denoted by \( \text{acl}(X) \), is the union of all finite \( D \in \text{Def}(\mathcal{M}; X) \). A subset \( X \subseteq \mathcal{M} \) is algebraically closed if \( X = \text{acl}(X) \). If \( A \subseteq \mathcal{M} \), then we let \( \text{acl}_A(X) = \text{acl}(X \cup A) \). If \( \sigma : A \rightarrow \mathcal{M} \) and \( \overline{a} \in A \), then \( \sigma(\overline{a}) = (\sigma(a_0), \sigma(a_1), \ldots, \sigma(a_{n-1})) \). If \( A \subseteq \mathcal{M} \) and \( \sigma : A \rightarrow \mathcal{M} \), then \( \sigma \) is elementary if the following: whenever \( \varphi(x_0, x_1, \ldots, x_{n-1}) \) is an \( n \)-ary \( \mathcal{L} \)-formula and \( a_0, a_1, \ldots, a_{n-1} \in A \), then \( \mathcal{M} \models \varphi(\overline{a}) \) iff \( \mathcal{M} \models \varphi(\sigma(\overline{a})) \). If \( B \subseteq A \), then \( \sigma \) is elementary over \( B \) if the following: whenever \( \varphi(\overline{a}) \) is an \( n \)-ary \( \mathcal{L}(B) \)-formula and \( a_0, a_1, \ldots, a_{n-1} \in A \), then \( \mathcal{M} \models \varphi(\overline{a}) \) iff \( \mathcal{M} \models \varphi(\sigma(\overline{a})) \). For such a \( \sigma \), we let \( \text{fix}(\sigma) = \{ x \in A : \sigma(x) = x \} \).

If \( D_1, D_2 \subseteq \mathcal{M} \), then \( D_1 \) is elementarily isomorphic to \( D_2 \) if there is an elementary surjection \( \sigma : D_1 \rightarrow D_2 \). The parenthetical part in the conclusion of the Theorem says that \( \text{fix}(f) \) and \( D \) are elementarily isomorphic.

Recall that \( \mathcal{M} \) is recursively saturated if whenever \( X \subseteq \mathcal{M} \) is finite, \( \Sigma(x) \) is a computable (i.e. recursive) set of 1-ary \( \mathcal{L}(X) \)-formulas that is finitely realizable in \( \mathcal{M} \), then \( \Sigma(x) \) is realizable in \( \mathcal{M} \). Alternatively, \( \mathcal{M} \) is recursively saturated iff whenever \( X \subseteq \mathcal{M} \) is finite, \( \Sigma(x) \) is a set of 1-ary \( \mathcal{L}(X) \)-formulas that is recursive in some \( \text{tp}(\overline{a}) \) for \( \overline{a} \in M \) and \( \Sigma(x) \) is finitely realizable in \( \mathcal{M} \), then \( \Sigma(x) \) is realized in \( \mathcal{M} \). Analogous with this latter characterization of recursive saturation, \( \mathcal{M} \) is arithmetically saturated if whenever \( X \subseteq \mathcal{M} \) is finite, \( \Sigma(x) \) is a set of 1-ary \( \mathcal{L}(X) \)-formulas that is arithmetic in some \( \text{tp}(\overline{a}) \) for \( \overline{a} \in M \) and \( \Sigma(x) \) is finitely realizable in \( \mathcal{M} \), then \( \Sigma(x) \) is realized in \( \mathcal{M} \). A structure \( \mathcal{M} \) is resplendent if whenever \( R \) is a new \( k \)-ary relation symbol,
A = \{a_0, a_1, \ldots, a_{n-1}\} \subseteq M \text{ and } \sigma \text{ is an } (\mathcal{L}(A) \cup \{R\})\text{-sentence that is modeled by an expansion of some } \mathcal{N} \succ M, \text{ then } \mathcal{M} \text{ has an expansion modeling } \sigma. \text{ Every resplendent structure has the following stronger property: Whenever } \Sigma \text{ is a set of } (\mathcal{L}(A) \cup \{R\})\text{-sentences that is computable in } \text{tp}(\overline{b}) \text{ for some } \overline{b} \in M \text{ and that is modeled by an expansion of some } \mathcal{N} \succ M, \text{ then } \mathcal{M} \text{ has an expansion modeling } \Sigma. \text{ Every resplendent structure is recursively saturated, and every countable, recursively saturated structure is resplendent. Moreover, as is well known, every countable, recursively saturated structure is \textbf{chronically} resplendent, meaning that not only does it have an expansion modeling } \sigma, \text{ but it has a resplendent expansion modeling } \sigma. \text{ Perhaps not so well known are the corresponding notions and results for arithmetic saturation.}

\text{We say that } \mathcal{M} \text{ is \textbf{arithmetically} resplendent if: Whenever } \Sigma \text{ is a set of } (\mathcal{L}(A) \cup \{R\})\text{-sentences that is arithmetic in } \text{tp}(\overline{b}) \text{ for some } \overline{b} \in M \text{ and that is modeled by an expansion of some } \mathcal{N} \succ M, \text{ then } \mathcal{M} \text{ has an expansion modeling } \Sigma. \text{ We say that } \mathcal{M} \text{ is \textbf{chronically} arithmetically resplendent if, furthermore, there is such an arithmetically resplendent expansion of } \mathcal{M}. \text{ We will make use of the following lemma.}

\textbf{Lemma 1.1:} \textit{Every countable, arithmetically saturated structure is chronically arithmetically resplendent.}}

\text{Because this lemma does not seem to be so well known, we sketch a proof of it. This proof may not be the most efficient one, but it has a certain appeal (especially to this author).}

\textbf{Proof.} \text{Let } \mathcal{M} \text{ be countable and arithmetically saturated. Let } \mathfrak{X}_0 \text{ be the set of all } A \subseteq \omega \text{ such that } A \text{ is computable in some } \text{tp}(\overline{\pi}) \text{ for } \overline{\pi} \in M. \text{ Let } \mathfrak{X} \text{ be the set of all } B \subseteq \omega \text{ that is arithmetic in some } A \in \mathfrak{X}_0. \text{ If } \mathfrak{X} = \mathfrak{X}_0, \text{ then it’s not hard to see that } \mathcal{M} \text{ is chronically arithmetically resplendent since } \mathcal{M} \text{ is chronically resplendent.}

\text{So, assume that } \mathfrak{X} \neq \mathfrak{X}_0. \text{ Clearly, } \mathfrak{X} \text{ is countable. By Scott’s Theorem, there is a countable, model } \mathcal{N} \text{ of } \text{PA such that } \text{SSy}(\mathcal{N}) = \mathfrak{X}, \text{ and, by adjoining the theory of an inductive satisfaction class, we can get that } \mathcal{N} \text{ is also recursively saturated. But then } \mathcal{N} \text{ is arithmetically saturated. Since } \text{Th}(\mathcal{M}) \in \mathfrak{X}, \text{ the completeness theorem as formalized in } \mathcal{N} \text{ says there is } \mathcal{M}' \models \text{Th}(\mathcal{M}) \text{ that is definable in } \mathcal{N}. \text{ Then } \mathcal{M}' \text{ is also arithmetically saturated and } \mathcal{M}' \cong \mathcal{M}, \text{ so we can assume that } \mathcal{M}' = \mathcal{M}. \text{ Then, } \mathfrak{X} \text{ is the set of all } A \subseteq \omega \text{ such that } A \text{ is computable in some } \text{tp}^\mathcal{N}(\overline{\pi}) \text{ for } \overline{\pi} \in \mathcal{N}. \text{ Thus, as was previously noted, } \mathcal{N} \text{ is chronically arithmetically resplendent, so that } \mathcal{M} \text{ also is.} \quad \Box
§2. Proving the Theorem. The proof of the Theorem is given in this section. We begin this section with some preparatory material.

Duby introduces and repeatedly uses what he calls the acl-descent argument. However, he fails to state explicitly what it actually is. The following lemma, which, undoubtedly, is a version of what Duby had in mind, will be adequate for our purposes. For completeness, we include a proof.

**Lemma 2.1:** Suppose that $\mathcal{M}$ is any structure and $X \subseteq M$. Let $f \in \text{Aut}_X(\mathcal{M})$ and $b \in M$. For $i \in \mathbb{Z}$, let $b_i = f^i(b)$ and then let $B = \{b_i : i \in \mathbb{Z}\}$. Suppose that

$$\text{acl}_X(\{b_0, b_1, \ldots, b_{i-1}\}) \cap \text{acl}_X(\{b_1, b_2, \ldots, b_i\}) = \text{acl}_X(\{b_1, b_2, \ldots, b_{i-1}\})$$

for all $i < \omega$. If $u \in \text{acl}_X(B)$, $0 \neq n \in \mathbb{Z}$ and $f^n(u) = u$, then $u \in \text{acl}(X)$.

**Proof.** For $j, k \in \mathbb{Z}$, let $B_{j,k} = \{b_i : j \leq i \leq k\}$. Since $f \in \text{Aut}_X(\mathcal{M})$, then

$$\text{acl}_X(B_{j,k-1}) \cap \text{acl}_X(B_{j+1,k}) = \text{acl}_X(B_{j+1,k-1})$$

for each $j, k \in \mathbb{Z}$.

Now, suppose that $u \in \text{acl}_X(B)$, $0 \neq n \in \mathbb{Z}$ and $f^n(u) = u$. We can assume that $0 < n < \omega$. Since $u \in \text{acl}_X(B)$, there are $j < k$ in $\mathbb{Z}$ such that $u \in \text{acl}_X(B_{j,k})$. Then, also $u \in \text{acl}_X(B_{j,k+n})$. Let $m = k + n + 1 - j$, so that $u \in \text{acl}_X(B_{j,j+m+1})$. Moreover, for every $i \in \mathbb{Z}$, $u \in \text{acl}_X(B_{i,i+m+1})$. Now let $\ell < \omega$ be the least such that for every $i \in \mathbb{Z}$, $u \in \text{acl}_X(B_{i,i+\ell - 1})$. It must be that $\ell = 0$. For, if $\ell \geq 1$, then for every $i \in \mathbb{Z}$, $u \in \text{acl}_X(B_{i,i+\ell - 1}) \cap \text{acl}_X(B_{i+1,i+\ell})$, so that $u \in \text{acl}_X(B_{i+1,i+\ell - 1})$, contradicting the minimality of $\ell$. Thus, $u \in \text{acl}_X(B_{0,-1}) = \text{acl}_X(\emptyset) = \text{acl}(X)$. \qed

The next definition is influenced by Duby’s definition \cite[p. 437]{Du03} of what he refers to as $p(\varphi(A)) \cup \{(\text{acl}(\varphi(A)) \setminus \text{acl}(A)) \cap B = \emptyset\}$.

**Definition 2.2:** Suppose that $\mathcal{M}$ is a structure, $N \in \text{Def}(\mathcal{M}; \emptyset)$, $n < \omega$, $A = \{a_0, a_1, \ldots, a_{n-1}\} \subseteq M$, $b \in M$ and $\sigma : A \rightarrow M$ is elementary over $N$. We let $\Delta(N, \varphi, b, \sigma)$ be a set of 1-ary $\mathcal{L}(M)$-formulas which are of either of two types. The first type consists of those formulas

(A) \quad $\forall \overline{y} \in N[\varphi(\overline{y}, \sigma(\overline{a}), x) \leftrightarrow \varphi(\overline{y}, \overline{a}, b)]$,

where $\varphi(\overline{y}, \overline{z}, x)$ is an $\mathcal{L}$-formula. The second type consists of those formulas

(B) \quad $\forall u \forall \overline{y} \in N[\theta(u, \overline{y}, \overline{a}, b) \rightarrow -\theta(u, \overline{y}, \sigma(\overline{a}), x)]$.
where $\theta(u, \overline{y}, \overline{z}, x)$ is an $\mathcal{L}$-formula such that the following:

(i) for some $n < \omega$,

$$\mathcal{M} \models \forall \overline{y} \in N \exists^{\leq n} u \theta(u, \overline{y}, \overline{a}, b);$$

(ii) whenever $m < \omega$ and $\theta'(u, \overline{v}, \overline{z})$ is an $\mathcal{L}$-formula such that

$$\mathcal{M} \models \forall \overline{v} \in N \exists^{\leq m} u \theta'(u, \overline{y}, \overline{a}),$$

then

$$\mathcal{M} \models \forall u \exists \overline{y} \in N \theta(u, \overline{y}, \overline{a}, b) \rightarrow \neg \exists \overline{v} \in N \theta'(u, \overline{v}, \sigma(\overline{a})).$$

The next two propositions indicate the significance of the previous definition.

**Proposition 2.3:** Suppose that $\mathcal{M}$ is a structure, $N \in \text{Def}(\mathcal{M}; \emptyset)$, $n < \omega$, $A = \{a_0, a_1, \ldots, a_{n-1}\} \subseteq M$ and $\sigma : A \rightarrow M$ is elementary over $N$. Then $\Delta(N, \overline{\pi}, b, \sigma)$ is arithmetic in $\text{tp}(\pi, \sigma(\overline{\pi}), b)$.

**Proof:** The set $\Delta(N, \overline{\pi}, b, \sigma)$ is visibly arithmetic in $\text{tp}(\pi, \sigma(\overline{\pi}), b)$. $\square$

**Proposition 2.4:** Suppose that $\mathcal{M}$ is a structure, $N \in \text{Def}(\mathcal{M}; \emptyset)$, $n < \omega$, $A = \{a_0, a_1, \ldots, a_{n-1}\} \subseteq M$ and $\sigma : A \rightarrow M$ is elementary over $N$. If $b' \in M$, then the following are equivalent:

1. $\text{acl}_N(A \cup \{b\}) \cap \text{acl}_N(\sigma[A] \cup \{b\}) = \text{acl}_N(A) \cap \text{acl}_N(\sigma[A])$ and $\sigma \cup \{b, b'\}$ is elementary over $N$.
2. $\mathcal{M} \models \psi(b')$ for all $\psi(x) \in \Delta(N, \overline{\pi}, b, \sigma)$.

**Proof:** Assume all the hypotheses including that $b' \in M$. It is clear that $\sigma \cup \{\langle b, b' \rangle \}$ is elementary over $N$ iff $\mathcal{M} \models \psi(b')$ for all $\psi(x) \in \Delta(N, \overline{\pi}, b, \sigma)$ of type (A). Therefore, to complete the proof, it suffices to prove the equivalence of:

1. $\text{acl}_N(A \cup \{b\}) \cap \text{acl}_N(\sigma[A] \cup \{b\}) = \text{acl}_N(A) \cap \text{acl}_N(\sigma[A]).$
2. $\mathcal{M} \models \psi(b')$ for all $\psi(x) \in \Delta(N, \overline{\pi}, b, \sigma)$ of type (B).

Before proving the equivalence $(1') \iff (2')$, we make a comment about formulas $\psi(x)$ of type (B). Let $\theta(u, \overline{y}, \overline{z}, x)$ be a formula from which $\psi(x)$ is defined; that is $\theta(u, \overline{y}, \overline{z}, x)$ satisfies both (i) and (ii) in Definition 2.2. For each $\overline{y} \in N$, let $C(\overline{y})$ be defined by $\theta(u, \overline{y}, \overline{z}, x)$. From (i) we get that each $C(\overline{y})$ is finite (even $|C(\overline{y})| \leq n$). From (ii) we get that each $C(\overline{y}) \cap \text{acl}_N(A) = \emptyset$. Conversely, if $\theta(u, \overline{y}, \overline{z}, x)$ has each of these properties, then $\theta(u, \overline{y}, \overline{z}, x)$ is as in (B).

$(1') \Rightarrow (2')$: Suppose that $\psi(x)$ and $\theta(u, \overline{y}, \overline{z}, x)$ are as in (B). Consider $\overline{d} \in N$, $C(\overline{d})$ and $c \in C(\overline{d})$. Let $\theta'(u, \overline{d}, \sigma(\overline{a}), b')$. 


We wish to show that \( c \notin C''(\bar{d}) \). If it were, then, by \((1')\), \( c \in \text{acl}_N(A) \) contradicting that \( C(\bar{d}) \cap \text{acl}_N((A) \neq \emptyset \).

\((2') \Rightarrow (1')\): Since it is obvious that \((1')\) holds with \( \supseteq \) replacing \( = \), it suffices to prove only the reverse inclusion. So, suppose that \( c \in \text{acl}_N(A \cup \{b\}) \cap \text{acl}_N(\sigma[A] \cup \{b\}) \). Thus, there are \( \theta_1(u, \bar{y}, \bar{z}, x) \) and \( \theta_2(u, \bar{w}, \bar{z}, x) \) such that for some \( \bar{d} \in N \) and \( \bar{e} \in N \),

\[ \mathcal{M} \models \theta(c, \bar{d}, \sigma(\bar{e}), b') \land \theta_1(c, \bar{e}, \bar{a}, b) \]

We can assume that the sets defined by \( \theta(u, \bar{d}, \sigma(\bar{a}), b') \) and \( \theta_1(u, \bar{e}, \bar{a}, b) \) are as small as possible. We can also assume that \( \theta(u, \bar{d}, \sigma(\bar{a}), b') = \theta_1(u, \bar{e}, \bar{a}, b) \). It follows from the minimality of \( C(\bar{d}) \), the set defined by \( \theta(u, \bar{e}, \bar{a}, b) \), that \( \theta(u, \bar{y}, \bar{z}, x) \) satisfies \((i)\) and \((ii)\). Thus,

\[ \mathcal{M} \models \neg \theta(c, \bar{d}, \sigma(\bar{e}), b') \]

so that \( c \in \text{acl}_N(\sigma[A]) \) and also \( c \in \text{acl}_N(A) \). This proves \((1')\). \( \square \)

The proof of the next lemma can easily be inferred from the proof of [Dun03, Lemma 3], so we omit a proof.

**Lemma 2.5:** Suppose that \( \mathcal{M} \) is uncountable and saturated and that \( N \prec \mathcal{M} \) is countable. Let \( A \subseteq M \) be countable, \( \sigma : A \rightarrow M \) be elementary over \( N \) and \( b \in M \). Then there is \( b' \in M \) that satisfies \( \Delta(N, \bar{a}, b, \sigma) \).

\[ \square \]

**Lemma 2.6:** Suppose that \( N' \prec \mathcal{M} \), \( (\mathcal{M}, N') \) is uncountable and saturated, \( A \subseteq M \) is countable and \( A \cap \text{acl}(\emptyset) = \emptyset \). Then there is countable \( N \prec N' \) such that:

- \( \text{acl}_N(A) \cap N' = N' \);
- \( A \cap N = \emptyset \).

**Proof.** We get \( N \) as the union of an increasing sequence \( \langle N_n : n < \omega \rangle \) of countable, algebraically closed subsets of \( M \). Each \( N_n \) will be such that

\[ (\ast_n) \quad \text{acl}_{N_n}(A) \setminus N_n \subseteq \text{acl}(A) \].

Let \( N_0 = \text{acl}(\emptyset) \), so that \((\ast_0)\) holds trivially.

Suppose that we have countable, algebraically closed \( N_n \) satisfying \((\ast_n)\). If \( \mathcal{N}_n = \mathcal{M}|N_n \prec \mathcal{M} \), then we let \( N_{n+1} = N_n \). Otherwise, consider some \( L(N_n)\)-formula \( \varphi(x) \) such that, if \( B = \{ x \in M : \mathcal{M} \models \varphi(b) \} \), then \( B \neq \emptyset = B \cap N_n \). Then \( B \) is uncountable and \( B \in \text{Def}(\mathcal{M}) \). Since \((\ast_n)\), there is \( b \in B \) such that \( \text{acl}_{N_n \cup \{b\}} \cup \{\text{acl}(N_n \cup \{b\}) \cap \text{acl}(A) = \text{acl}(A) \). For such \( b \), let \( N_{n+1} = \text{acl}(N_n \cup \{b\}) \).
By being careful about the choice \( \varphi(x) \) in obtaining \( N_{n+1} \), we can arrange that \( N = \bigcup_n N_n \) is as required. \( \square \)

Having this preliminary material, we now present the proof of the Theorem.

**Proof of the Theorem.** Fix \( \mathcal{M} \) and \( D \) to be as in the Theorem; that is, \( \mathcal{M} \) is a countable, arithmetically saturated structure and \( D \subseteq M \) is algebraically closed. When we say that \( \mathcal{M} \) is countable, we allow the possibility that \( \mathcal{M} \) is finite. But, if \( \mathcal{M} \) is finite, then \( D = M \) so that the identity function is as required. Therefore, we assume that \( \mathcal{M} \) is infinite.

But we can assume even more: that \( D \) is infinite. For, if \( D \) is finite, then \( D = \text{acl}(D) \), so from the generalization of Duby’s theorem as noted in the introduction, there is an appropriate automorphism. The reader can check that this reliance on Duby’s theorem can easily be avoided by making only very small changes to the proof presented in this section.

**Definition 2.7:** We say that the 5-tuple \( \langle N, f, A, C, \delta \rangle \) is \( n \text{-good} \) if \( n < \omega \) and each of the following is satisfied:

1. \( N \preceq \mathcal{M} \);
2. \( f \in \text{Aut}_N(\mathcal{M}) \);
3. \( C \subseteq N \) is finite and \( \delta : C \rightarrow D \) is elementary;
4. \( A \) is the union of exactly \( n \) infinite orbits of \( f \);
5. if \( u \in \text{acl}_N(A) \), \( 0 < k < \omega \) and \( f^k(u) = u \), then \( u \in N \);
6. \( (\mathcal{M}, N, f) \) is arithmetically saturated.

If \( \langle N, f, A, C, \delta \rangle \) is \( n \)-good for some \( n < \omega \), then we say simply that it is \textit{good}.

**Definition 2.8:** If \( \langle N, f, A, C, \delta \rangle \) and \( \langle N', f', A', C', \delta' \rangle \) are good, then \( \langle N', f', A', C', \delta' \rangle \) \textbf{extends} \( \langle N, f, A, C, \delta \rangle \) if the following:

1. \( N' \preceq N \);
2. \( A \subseteq A' \);
3. \( C \subseteq C' \);
4. \( \delta = \delta'|C \);
5. \( f' \supseteq f|A \).

The relation of extending is a quasi-order (i.e., it is reflexive and transitive) on the set of good 5-tuples.

The following is the key lemma for proving the Theorem.
LEMMA 2.9: Suppose that $\langle N, f, A, C, \delta \rangle$ is $n$-good. If $b \in M$ and $b \not\in \text{acl}(C) \cup A$, then there is an $(n+1)$-good $\langle N', f', A', C, \delta \rangle$ extending $\langle N, f, A, C, \delta \rangle$ such that $b \in A'$.

Before proving this lemma, we will see how it is used to prove the Theorem.

The desired automorphism of $M$ will be obtained with a construction of $\omega$ steps. At the $n$-th step of the construction, we will obtain an $n$-good $\langle N_n, f_n, A_n, C_n, \delta_n \rangle$ extending all the previously constructed good 5-tuples. Having completed this construction, we will let $N = \bigcap_{n<\omega} N_n$, $A = \bigcup_{n<\omega} A_n$, $C = \bigcup_{n<\omega} C_n$ and $\delta = \bigcup_{n<\omega} \delta_n$, which are all natural definitions given (7) – (10). It easily follows that $C \subseteq N \subseteq M \setminus A$ and that $\delta : C \rightarrow D$ is elementary. In fact, the construction will be such that the following are also satisfied:

(12) $C = N = M \setminus A$;
(13) $\delta : C \rightarrow D$ is onto.

Finally, we let $f = \bigcup_{n<\omega} f_n|(A_n \cup C_n)$. Since $M = A \cup C = \bigcup_{n<\omega} (A_n \cup C_n)$ and each $f_n \in \text{Aut}(\mathcal{M})$, then $f \in \text{Aut}(\mathcal{M})$. In fact, this $f$ is the desired automorphism. From (2), (3) and (4), we get that $\text{fix}(f) = C$. From (3), (9) and (13) we get that $\delta$ demonstrates that $\text{fix}(f)$ and $D$ are elementarily isomorphic. And from (4), (8) and (12) we get that $\text{fix}(f^k) = C$ whenever $0 < k < \omega$.

We now proceed with the construction, using Lemma 2.9. To help with this construction, we let $< \in M$ be an ordering of $M$ having ordertype $\omega$. Whenever we refer to a least element with a certain property, it will be with reference to this ordering.

Step 0: Let $\mathcal{N}_0 = \mathcal{M}$, $f_0 \in \text{Aut}(\mathcal{M})$ be the identity function and $A_0 = C_0 = \delta_0 = \emptyset$. Clearly, $\langle \mathcal{N}_0, f_0, A_0, C_0, \delta_0 \rangle$ is 0-good.

Step $n + 1$: Thus, going into this step, we already have an $n$-good $\langle \mathcal{N}_n, f_n, A_n, C_n, \delta_n \rangle$. Since $D$ is infinite and $C_n$ is finite, we can let $d \in D$ be the least that is not in the range of $\delta_n$. (This choice of $d$ will lead to (13) being satisfied.) Let $c \in N_n$ be such that $\delta_n \cup \{\langle c, d \rangle\}$ is an elementary function. (There is such a $c$ since $\mathcal{N}_n \cong \mathcal{M}$ and $\mathcal{M}$ is recursively saturated.) Let $C_{n+1} = C_n \cup \{c\}$ and $\delta_{n+1} = \delta_n \cup \{\langle c, d \rangle\}$. Clearly, $\langle \mathcal{N}_n, f_n, A_n, C_{n+1}, \delta_{n+1} \rangle$ is $n$-good and extends $\langle \mathcal{N}_n, f_n, A_n, C_n, \delta_n \rangle$.

Let $a_0, a_1, \ldots, a_{n-1} \in A_n$ be in distinct orbits of $f_n$. Thus, $A_n = \{f_n^j(a_i) : j < n, i \in \mathbb{Z}\}$. Since $(\mathcal{M}, N_n, f_n, \overline{a})$ is recursively saturated and $C_{n+1}$ is finite, then $M \not= \text{acl}(C_{n+1}) \cup A_n$, so we let $b$ be the least element of $M$ that is not in $\text{acl}(C_{n+1}) \cup A_n$. (This choice of $b$ will lead to (12) being satisfied.) We next invoke Lemma 2.9 to get an $(n+1)$-good
\( \langle N_{n+1}, f_{n+1}, A_{n+1}, C_{n+1}, \delta_{n+1} \rangle \) extending \( \langle N_n, f_n, A_n, C_{n+1}, \delta_{n+1} \rangle \) such that \( b \in A_{n+1} \).

This completes the construction. As previously alluded to, both (12) and (13) are satisfied. Thus, all that remains is to prove Lemma 2.9.

Proof of Lemma 2.9. By replacing \( \mathcal{M} \) with the structure \( (\mathcal{M}, c)_{c \in C} \), we can ignore \( C \) and \( \delta \) by letting \( C = \delta = \emptyset \). Thus, we have an \( n \)-good \( \langle \mathcal{N'}, f, A, \emptyset, \emptyset \rangle \) and want an \( (n+1) \)-good \( \langle \mathcal{N'}, f', A', \emptyset, \emptyset \rangle \) extending \( \langle \mathcal{N}, f, A, \emptyset, \emptyset \rangle \) such that \( b \in A' \).

In this proof, whenever \( \langle N_0, f_0, A_0, C_0, \delta_0 \rangle \) is good, it will be that \( C_0 = \delta_0 = \emptyset \). So, in this proof, we will say that \( \langle N_0, f_0, A_0 \rangle \) is \( m \)-good (or good) whenever \( \langle N_0, f_0, A_0, \emptyset, \emptyset \rangle \) is \( m \)-good (or good).

Thus, we have an \( n \)-good \( \langle \mathcal{N}, f, A \rangle \) and \( b \in M \setminus (\text{acl}(\emptyset) \cup A) \). Let \( a_0, a_1, \ldots, a_{n-1} \in A \) be in pairwise distinct orbits of \( f \) so that
\[
A = \{ f^j(a_i) : i < n, \ j \in \mathbb{Z} \}.
\]

Let \( \sigma = f \restriction A \). Let \( (\mathcal{M}_*, N_*, f_*) \succ (\mathcal{M}, N, f) \) be uncountable and saturated. Using Lemma 2.6, let \( N_0 \prec N_* \) be such that:
- \( N_0 \) is countable;
- \( b \notin N_0 \);
- \( \text{acl}_{N_0}^\mathcal{M}_*(A) \cap N_* = N_0 \).

Next, we use Lemma 2.5 \( \omega \) times to get \( b_0^*, b_1^*, b_2^* \ldots \in M_* \) such that:
- \( b_0^* = b \);
- \( \sigma \cup \{ (b_i^*, b_{i+1}^*) : i < \omega \} \) is elementary over \( N_0 \);
- \( \text{acl}_{N_0}(A \cup \{ b_0^*, b_1^*, \ldots, b_j^* \}) \cap \text{acl}_{N_0}(A \cup \{ b_0^*, b_1^*, \ldots, b_{j+1}^* \}) = \text{acl}_{N_0}(A \cup \{ b_1^*, b_2^*, \ldots, b_{j+1}^* \}) \) for each \( j < \omega \).

Because \( (\mathcal{M}_*, N_*, f_*) \) is uncountable and saturated, we can let \( h \in \text{Aut}_{N_0}(\mathcal{M}_*) \) be such that \( h \supseteq \sigma \) and \( h(b_i^*) = b_{i+1}^* \) for \( i < \omega \). For each \( j < \omega \), let \( \overline{a}_j \) be the \( (n+j) \)-tuple \( \overline{a}, b_0^*, b_1^*, \ldots, b_{j-1}^* \) and let \( \sigma_j^* = h \restriction (A_n \cup \{ b_0^*, b_1^*, \ldots, b_{j-1}^* \}) \). Then, each \( b_{j+1}^* \) satisfies \( \Delta(N_0, \overline{a}_j, b_j^*, \sigma_j^*) \) in \( (\mathcal{M}_*, N_0) \).

Since \( (\mathcal{M}, N, f) \) is chronically arithmetically resplendent (Lemma 1.1), there are \( N' \prec N \) and \( f' \in \text{Aut}_{N'}(\mathcal{M}) \) such that if we let \( b_j = (f')^j(b) \) for \( j \in \mathbb{Z} \) and \( \overline{a}_j \) be the \( (n+j) \)-tuple \( \overline{a}, b_0, b_1, \ldots, b_{j-1} \) and \( \sigma_j = f' \restriction (A_n \cup \{ b_0, b_1, \ldots, b_{j-1} \}) \) for \( j < \omega \), then each \( b_{j+1} \) satisfies \( \Delta(N', f', \sigma_j) \). In addition, we can get \( (\mathcal{M}, N', f') \) to be arithmetically saturated. Let \( A' = A \cup \{ b_j : j \in \mathbb{Z} \} \).

We claim that \( (N', f', A') \) is as required; that is, \( (N', f', A') \) is \( (n+1) \)-good, \( (N', f', A') \) extends \( (N, f, A) \) and \( b \in A' \). It is clear that \( (N', f', A') \) is as required, with the possible exception of the following, corresponding to (5) in Definition 2.7:
(5') If \( u \in \text{acl}_{N'}(A') \), \( 0 < k < \omega \) and \( (f')^k(u) = u \), then \( u \in N' \).

To prove (5'), suppose that \( u \) and \( k \) are as in its hypothesis. By Lemma 2.1, \( u \in \text{acl}_{N'}(A) \). Since \( f' \supseteq f \upharpoonright \text{acl}_n'(A) \), then \( f^k(u) = u \). Since \( \langle N', f, A \rangle \) is good, then \( u \in N \). But \( \text{acl}_{N'}(A) \cap N = N' \), so \( u \in N' \), completing the proofs of (5'), Lemma 2.9 and the Theorem.

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