Dynamics of asymmetric kinetic Ising systems revisited

Haiping Huang and Yoshiyuki Kabashima

Department of Computational Intelligence and Systems Science, Tokyo Institute of Technology, Yokohama 226-8502, Japan
E-mail: physhuang@gmail.com and kaba@dis.titech.ac.jp

Received 25 January 2014
Accepted for publication 6 April 2014
Published 22 May 2014

Abstract. The dynamics of an asymmetric kinetic Ising model is studied. Two schemes for improving the existing mean-field description are proposed. In the first scheme, we derive the formulas for instantaneous magnetization, equal-time correlation, and time-delayed correlation, considering the correlation between different local fields. To derive the time-delayed correlation, we emphasize that the small-correlation assumption adopted in previous work (Mézard and Sakellariou, 2011 J. Stat. Mech. L07001) is in fact not required. To confirm the prediction efficiency of our method, we perform extensive simulations on single instances with either temporally constant external driving fields or sinusoidal external fields. In the second scheme, we develop an improved mean-field theory for instantaneous magnetization prediction utilizing the notion of the cavity system in conjunction with a perturbative expansion approach. Its efficiency is numerically confirmed by comparison with the existing mean-field theory when partially asymmetric couplings are present.

Keywords: disordered systems (theory), statistical inference, kinetic Ising models

ArXiv ePrint: 1310.5003
1. Introduction

The dynamics of asymmetric kinetic Ising systems has been intensively studied in the statistical physics community [1]–[5]. In equilibrium statistical physics, symmetry is assumed to construct couplings between spins, which leads to a simple stationary state described by the Gibbs–Boltzmann distribution [6]. However, a more realistic case is that couplings between spins are fully or partially asymmetric; an example has been observed in real neuronal systems [7], where two neurons do not simply affect each other in a symmetric way. In this case, the dynamics still has a stationary state but with a rather complicated form depending on the details of the model [5]. Therefore, the static macroscopic quantities of interest have to be computed in the long-time limit [1]. Further, studies of such nonequilibrium systems are relevant to model spatio-temporal statistics of various biological systems [8]–[13], in the sense that the time-dependent observables can be predicted at the current time point according solely to knowledge at the previous time point. Here, we focus on evaluating time-dependent magnetizations and equal-time and time-delayed correlations for different sites in a fully or partially asymmetric kinetic Ising system with parallel (synchronous) dynamics. In [4], these observables were already evaluated by assuming negligible correlations of local fields or correlations between spins at the same time step. We argue that such a small-correlation assumption is not necessary to derive a closed-form equation, and we improve the prediction accuracy of these time-dependent quantities by incorporating these correlations. Given the finite system size, we show that the improvement is much more significant, particularly in the low-temperature region, by comparing these two mean-field methods.

In general, there exist correlations between couplings; i.e., spins in the system are partially asymmetrically coupled. As a result, memory effects become increasingly important, and the theory developed for fully asymmetric networks [4] should be revised by considering the retarded self-interactions induced by the connection symmetry [14]. To this end, we propose an improved mean-field theory to capture the memory effects and...
thus improve the prediction accuracy of time-dependent observables, and we support this assertion by numerical simulations on single instances.

The rest of this paper is organized as follows. The asymmetric kinetic Ising model and the parallel dynamics are introduced in section 2. Closed-form equations for evaluating time-dependent quantities such as magnetization, equal-time correlation, and time-delayed correlation are derived in section 3. Extensive numerical simulations to confirm the efficiency of our method compared with the method introduced in [4] are performed and discussed. In section 4, we develop an improved mean-field theory to treat the memory effects arising in partially asymmetric connected networks. Its significance is supported by the numerical simulation presented in this section. The final section is devoted to a summary.

2. Asymmetric kinetic Ising model

The parallel dynamics of a kinetic Ising system is described by a Markov chain with the transition probability

$$ p(s(t)|s(t-1)) = \prod_{i=1}^{N} \frac{e^{\beta s_i(t) h_i(t)}}{2 \cosh(\beta h_i(t))}, $$

conditioned to the fact that the $N$-dimensional Ising spin configuration $s(t-1)$ at the $(t-1)$th time step is given. The inverse temperature $\beta$ serves as a measure of the degree of stochasticity. Parallel dynamics means that the transition probability for each $s_i(t)$ at time $t$ relies only on the state of its neighbors at time $t-1$. Therefore, we define the effective field as $h_i(t) \equiv \theta_i(t) + \sum_{j \in \partial i} J_{ij} s_j(t-1)$ [5]. $\partial i$ denotes the neighbors of spin $i$. In the current context, each spin is connected to other spins; i.e., the cardinality of spin $i$, $|\partial i| = N - 1$. $J_{ij}$ denotes the coupling strength for the directed edge $(ij)$ from spin $j$ to spin $i$. We assume completely uncorrelated (fully asymmetric) couplings in the sense that they are all drawn independently from a Gaussian distribution with zero mean and variance $1/N$. $\theta_i(t)$ refers to the time-dependent external field and it is chosen to be $\theta_0(t)$ or $-\theta_0(t)$ with equal probability for each spin $i$. Therefore, the parallel dynamics of the asymmetric kinetic Ising system at a discrete time step is described by the following Glauber rule for all spins $(i=1, \ldots, N)$ [1, 9]:

$$ s_i(t) = \begin{cases} -1 & \text{with Prob } 1 - g(h_i(t)), \\ +1 & \text{with Prob } g(h_i(t)), \end{cases} $$

where $g(h(t)) = (1 + e^{-2\beta h(t)})^{-1}$. In the parallel dynamics, all spins are updated according to equation (2) simultaneously at each discrete time step. In the presence of symmetric couplings, the dynamics will evolve to a simple equilibrium state; however, if the couplings are asymmetric, the dynamics still has a steady state but this state is unknown $a$ priori. Using features of this model, we will derive the mean-field equations for instantaneous macroscopic quantities in the following section and further demonstrate the difference from the derivation in [4]. We then relax the fully asymmetric assumption to a partially asymmetric one.
3. Prediction with correlations between different spins

Under the transition probability of equation (1), the joint probability of any spin trajectory \(s(0), s(1), \ldots, s(t)\) is given by

\[
p(s(0), s(1), \ldots, s(t)) = p(s(t)|s(t-1)) \prod_{\sigma=1}^{t-1} p(s(\sigma)|s(\sigma-1)) P(s(0)),
\]

due to the Markovian property. \(P(s(0))\) is the initial distribution. The instantaneous magnetization is defined as \(m_i(t) \equiv \langle s_i(t) \rangle\), where the average operation \(\langle \cdots \rangle\) is taken over the trajectory spin history (i.e., over the path probability equation (3)) [1]. We are also interested in the time evolution of the equal-time correlation and the time-delayed correlation. They are defined, respectively, as \(C_{ij}(t) \equiv \langle s_i(t)s_j(t) \rangle - m_i(t)m_j(t)\) and \(D_{ij}(t) \equiv \langle s_i(t+1)s_j(t) \rangle - m_i(t+1)m_j(t)\). Using equation (3), we re-write these macroscopic observables for the parallel dynamics as [15]:

\[
\begin{align*}
    m_i(t) &= \langle \tanh(\beta h_i(t)) \rangle, \\
    C_{ij}(t) &= \langle \tanh(\beta h_i(t)) \tanh(\beta h_j(t)) \rangle - m_i(t)m_j(t), \\
    D_{ij}(t) &= \langle s_j(t) \tanh(\beta h_i(t+1)) \rangle - \langle \tanh(\beta h_i(t+1)) \rangle m_j(t).
\end{align*}
\]

In the definition of the effective field, the sum of a large number of independent random variables can be assumed to follow a Gaussian distribution from the central limit theorem [4], because of the fully asymmetric and connected property of the model. As a result, the distribution of the local field \(\tilde{h}_i(t-1) \equiv \sum_{j \in \partial_i} J_{ij}s_j(t-1)\) is characterized by its mean and variance. The mean is given by \(a_i(t-1) = \sum_{j \in \partial_i} J_{ij}m_j(t-1)\) and the correlation between two local fields reads

\[
\langle \tilde{h}_i(t)\tilde{h}_j(t) \rangle - \langle \tilde{h}_i(t) \rangle \langle \tilde{h}_j(t) \rangle = \sum_{k,l} J_{ik}J_{jk}C_{kl}(t) = [JCJT]_{ij} \equiv \Delta_{ij}(t).
\]

With this Gaussian approximation, the trajectory history average in equation (4) can be transformed into an integral over the Gaussian distribution, resulting in the following magnetization and equal-time correlation:

\[
\begin{align*}
    m_i(t) &= \int Dz \tanh(\beta \theta_i(t) + a_i(t-1) + \sqrt{\Delta_{ii}} z), \\
    C_{ij}(t) &= \int Dz \int Dx \tanh(\beta \theta_i(t) + a_i(t-1) + \sqrt{\Delta_{ii} - \Delta_{ij}} x + \sqrt{\Delta_{ij}} z) \\
    &\quad \times \int Dy \tanh(\beta \theta_j(t) + a_j(t-1) + \sqrt{\Delta_{jj} - \Delta_{ij}} y + \sqrt{\Delta_{ij}} z) \\
    &\quad - m_i(t)m_j(t),
\end{align*}
\]

where \(Dz \equiv e^{-z^2/2}dz/\sqrt{2\pi}\) and we omit the time index \((t-1)\) for all field covariances. In equation (6b), if \(\Delta_{ij} < 0\), it should be replaced by \(-\Delta_{ij}\) and only \(z\) in the first \(\tanh(\cdot)\) is replaced by \(-z\) to retain correct covariance between local fields. Note that \(\Delta_{ii}\) in the above equations was treated as \(\sum_{j \in \partial_i} J_{ij}^2(1-m_j^2(t-1))\) in [4]. We call this simplified method MF (mean field). Here, we keep the entire knowledge of the equal-time correlation and expect
to improve the prediction, especially in the low-temperature region. Correspondingly, our method is called MFcorre (mean field with correlations). A similar idea was also proposed in a recent interesting work [16], where the covariance of local fields could be recursively determined. Here we use directly the entire knowledge of the equal-time correlation to compute the field covariance. Furthermore, the time-delayed correlation derived in [4] can be recovered without any small-correlation (of local fields) assumption. This is shown by the following derivation:

$$\sum_k J_{jk} D_{ik} = \langle \hat{h}_j(t) \tanh(\theta_i(t+1) + \hat{h}_i(t)) \rangle - a_j(t) \langle \tanh(\theta_i(t+1) + \hat{h}_i(t)) \rangle$$

$$= \langle \delta a_j(t) \beta(\theta_i(t+1) + a_i(t) + \delta a_i(t)) \rangle$$

$$= \int Dz \int Dx \int Dy \left( \sqrt{\Delta_{jj} - \Delta_{ij}y + \Delta_{ij}z} \right) \tanh \beta \left( \theta_i(t+1) + a_i(t) + \sqrt{\Delta_{ii} - \Delta_{ij}x + \Delta_{ij}z} \right)$$

$$= \beta \Delta_{ij} \int Dz \int Dz' \left( 1 - \tanh^2 \beta \left( \theta_i(t+1) + a_i(t) + \sqrt{\Delta_{ii} - \Delta_{ij}x + \Delta_{ij}z} \right) \right)$$

$$= \beta \Delta_{ij} \int D\hat{z} \left( 1 - \tanh^2 \beta \left( \theta_i(t+1) + a_i(t) + \sqrt{\Delta_{ii} \hat{z}} \right) \right),$$  \hspace{1cm} (7)

where $\delta a_i(t) \equiv \hat{h}_i(t) - a_i(t)$. Note that $\langle \delta a_i(t) \delta a_j(t) \rangle = \Delta_{ij}(t)$, and all field covariances in equation (7) have time index ($t$). From the third to fourth equality, we used the identity $\int Dz F(z) = \int Dz F'(z)$. When arriving at the final equality, we made the transformation $\hat{z} = \sqrt{\Delta_{ii} - \Delta_{ij}x + \Delta_{ij}z}$ (where $\hat{z}$ follows a Gaussian distribution with zero mean and variance $\Delta_{ii}$). Finally, we recover the formula for evaluating the time-delayed correlation as $D(t) = A(t)JC(t)$, which has been derived in [4] by discarding terms of order $O(\Delta_{ij}^2)$ when calculating the average. $A(t)$ is a diagonal matrix with diagonal terms $A_{ii} = \beta \int D\hat{z} (1 - \tanh^2 \beta(\theta_i(t+1) + a_i(t) + \sqrt{\Delta_{ii} \hat{z}}))$. We remark here that the only assumption we used is the Gaussian approximation, which is guaranteed by the fully asymmetric and connected properties of the kinetic Ising model under consideration. In this sense, the equations derived above for time-dependent macroscopic observables are exact even in the low-temperature region.

The fully asymmetric constraint can be relaxed to a partially asymmetric one by introducing correlations for couplings. In this case, the central limit theorem becomes invalid due to the presence of correlated couplings. Thus, the above-derived equations can only be used as a crude approximation. The effects of coupling asymmetry were studied in [17]. We applied the same construction as that in [1, 17], i.e., $J_{ij} = J^s_{ij} + k J^as_{ij}$, where $k \geq 0$ specifies the asymmetry degree of couplings. $J^s_{ij} = J^s_{ji}$ and $J^as_{ij} = -J^as_{ji}$, where they follow a Gaussian distribution with zero mean and variance $J^2/N(1+k^2)$. We choose $J = 1$ here. According to the construction, we have $\langle J_{ij} J_{ji} \rangle = \left( \frac{1-k^2}{1+k^2} \right) (J^2/N)$, such that $k = 0$ corresponds to a fully symmetric network, while a fully asymmetric network has $k = 1$. To evaluate the instantaneous equal-time correlation, equation (6b) may not be used directly, because $\Delta_{ii} - \Delta_{ij}$ or $\Delta_{ij} - \Delta_{ij}$ may become negative, which never happens.
when \( k = 1 \). Instead, for \( k \neq 1 \), one can use the following approximation

\[
\int Dx \tanh \beta \left( \theta_i(t) + a_i(t-1) + \sqrt{\Delta_{ii} - \Delta_{ij}} x + \sqrt{\Delta_{ij}} z \right)
\]

\[
\simeq \tanh \beta \left[ \theta_i(t) + a_i(t-1) + \sqrt{\Delta_{ij}} z - \beta m_i(t)(\Delta_{ii} - \Delta_{ij} + \Delta_{ij}z^2) \right]
\]

(8)
based on a small-coupling expansion [5]. A similar approximation can be applied to the \( y \)-term in equation (6b). Another possible way is to re-write the Gaussian random number dependent terms as

\[
C_{ij}(t) = \int Dz \int Dx \tanh \beta \left( \theta_j(t) + a_j(t-1) + v_1 x + vz \right)
\]

\[
\times \tanh \beta \left( \theta_j(t) + a_j(t-1) + v_2 x + vz \right) - m_i(t)m_j(t),
\]

(9)
where \( v_1 = (\Delta_{ii} - \Delta_{ij})/\sqrt{\Delta_{ii} + \Delta_{jj} - 2\Delta_{ij}} \), \( v_2 = -(\Delta_{jj} - \Delta_{ij})/\sqrt{\Delta_{ii} + \Delta_{jj} - 2\Delta_{ij}} \) and \( v = \sqrt{(\Delta_{ii}\Delta_{jj} - \Delta_{ij}^2)/(\Delta_{ii} + \Delta_{jj} - 2\Delta_{ij})} \). In our simulations, this expression caused no problems, keeping both \( \Delta_{ii} + \Delta_{jj} - 2\Delta_{ij} \) and \( \Delta_{ii}\Delta_{jj} - \Delta_{ij}^2 \) positive. As far as we investigated, equations (8) and (9) yielded similar prediction errors at all temperatures. When cross correlation starts to have significant contributions to the field covariance (this does happen in the low-temperature regime), prediction of MF, which incorporates only the autocorrelation, is supposed to have quite large errors whichever formula (equation (8) or (9)) is employed.

In the numerical simulation, we predict the instantaneous macroscopic quantities at the current time point based on the knowledge (data) of the previous time point, using the equations derived in this section. To test the prediction performance, we compare the prediction result with that obtained by Monte Carlo simulations (denoted by exp), and the performance is evaluated using the root-mean-squared errors

\[
\Delta_m = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (m_i(t) - m_i^{\exp}(t))^2},
\]

(10a)

\[
\Delta_C = \sqrt{\frac{1}{N^2} \sum_{i,j}^{N} (C_{ij}(t) - C_{ij}^{\exp}(t))^2},
\]

(10b)

\[
\Delta_D = \sqrt{\frac{1}{N^2} \sum_{i,j}^{N} (D_{ij}(t) - D_{ij}^{\exp}(t))^2}.
\]

(10c)

We simulated asymmetric kinetic Ising systems of system size \( N = 100 \) by following the Glauber rule defined in equation (2). The initial distribution is chosen such that each spin is randomly independently assigned +1 or −1. The dynamics is run up to 31 time steps with \( 10^5 \) spin trajectories. Therefore, a total of \( 10^5 \) instantaneous spin configurations at \( t = 30 \) and \( t = 31 \) are collected, respectively, where the trajectory data at \( t = 30 \) are used to compute inputs (magnetizations and equal-time correlations at time \( t = 30 \)) for predicting the time-dependent macroscopic quantities at \( t = 31 \), and the trajectory data
Figure 1. Comparison of prediction performance of MFcorre and MF on fully asymmetric kinetic Ising systems of system size $N = 100$. Each data point is the average over ten random realizations. A total number of $10^5$ spin trajectories are collected up to 31 time steps and these trajectory data are used either to compute inputs for prediction equations or to compute the experimental values for comparison. We predict the instantaneous magnetization, equal-time correlation, and time-delayed correlation at $t = 31$ based on the data at $t = 30$. (a) Constant external fields with $\theta_0(t) = 0$. (b) Sinusoidal external fields with $\theta_0(t) = 0.1 \sin(2\pi t/t_0)$. The period ($t_0$) is chosen to be 10 time steps.

at $t = 31$ are used to compute the experimental values for comparison. The prediction error reported in figure 1 is averaged over ten different random realizations (coupling constructions and external fields). Temporally constant and time-varying external fields are applied. As observed in figure 1, MFcorre improves upon MF, especially in the low-
temperature region, where thermal fluctuations are not strong and pairwise correlation dominates the dynamic behavior. Moreover, the prediction error of MFcorre increases slowly with the inverse temperature, whereas the prediction error of MF increases rapidly as the temperature decreases. The result implies that, to achieve perfect accuracy of prediction, incorporating the correlation is necessary, especially for low temperatures and finite system size of order $O(10^2)$, for which the computational cost is tolerable. We also show the prediction performance of MFcorre on networks of different sizes in figure 2, which illustrates that the prediction error increases for smaller size network, especially at low temperatures.

The effects of coupling asymmetry on the prediction performance of the two mean-field methods are summarized in figure 3. The performance for both methods degrades as $k$ decreases; however, MFcorre still outperforms MF in the low-temperature region, suggesting that considering correlations can compensate for the prediction error induced by the partial asymmetry, even if the central limit theorem is applied to derive the prediction equations.

4. Improved mean-field theory

In section 3, we improved the dynamical prediction by incorporating the correlation between two different sites. However, it increases the necessary computational time for prediction particularly for large systems. Furthermore, in (6a) memory effects induced by connection symmetry have not been considered, which leads to the high prediction error already reported in [17]. In the case of sparsely coupled systems, one can keep track of the directed influence from neighbors by effectively modifying the external field of each
Figure 3. The same as figure 1, but for constant external fields and different asymmetry degrees $k$. (a) $k = 0.8$. (b) $k = 0.6$. (c) $k = 0.4$. 

doi:10.1088/1742-5468/2014/05/P05020
spin along the dynamics [18, 19]. However, the direct employment of the scheme to fully coupled systems requires significant computational cost and is practically infeasible with current standard computational resources.

To overcome such a situation, we develop an improved mean-field theory (IMF) applicable to fully coupled networks with connection correlations but retain the low complexity of the prediction algorithm. Here, we treat the memory effect explicitly by introducing an additional field that describes a backaction from the states at earlier time steps. In the following derivation, we assume that couplings are drawn with correlations specified by the asymmetry degree \( k \) introduced in section 3. Combining equations (1) and (3), we have the joint probability of spin trajectory given by

\[
p(s(0), s(1), \ldots, s(t)) = P(s(0)) \prod_{\sigma=1}^{t} \prod_{l \neq i}^{\infty} \frac{e^{\beta s_i(\sigma) h_i(\sigma)}}{2 \cosh(\beta h_i(\sigma))}.
\]

We then separate the spin-\( i \)-related term \( J_{li}s_i(\sigma - 1) \) in the local field \( h_i(\sigma) \) to consider its directed influence over its neighbors, and we make the following expansion:

\[
\exp \left[ -\ln 2 \cosh(\beta h_i^{(t)}(\sigma) + \beta J_{li}s_i(\sigma - 1)) \right] \simeq \frac{\exp \left[ -\beta \tanh(\beta h_i^{(t)}(\sigma)) J_{li}s_i(\sigma - 1) \right]}{2 \cosh(\beta h_i^{(t)}(\sigma))},
\]

where we are allowed to truncate the expansion with respect to \( \beta J_{li}s_i(\sigma - 1) \) up to the first order due to its weakness and the statistical independence among different indices of \( J_{li} \). \( h_i^{(t)}(t) \) defines the cavity local field as \( h_i^{(t)}(t) \equiv \theta_i(t) + \sum_{j \in \mathcal{V}_i \setminus i} J_{lj}s_j(t - 1) \), where \( \mathcal{V}_i \) indicates that node \( i \) is excluded. Note that this expansion puts a less stringent constraint on the strength of couplings than that used to derive the dynamical TAP equation in [3]–[5].

Applying equation (12) in equation (11) completely decouples the contribution of the trajectory of spin \( i \) from the joint distribution of the cavity system as

\[
p(s(0), s(1), \ldots, s(t)) \simeq P(s(0)) \prod_{\sigma=1}^{t} \prod_{l \neq i} \frac{e^{\beta s_i(\sigma) h_i^{(t)}(\sigma)}}{2 \cosh(\beta h_i^{(t)}(\sigma))} \times \prod_{\sigma=1}^{t} \frac{e^{\beta s_i(\sigma) h_i(\sigma)}}{2 \cosh(\beta h_i(\sigma))}
\]

where the additional field \( \phi_i(t - 1) \equiv \sum_{l \neq i} J_{li}(s_l(t) - \tanh(\beta h_l^{(t)}(t))) \) was introduced. Let us denote \( s_i^{(t)}(\sigma) \) as the set of spins at time \( \sigma \) except for \( s_i(\sigma) \). Due to the nature of the current model, both \( J_{li} \) and \( J_{li}^{(t)} \) are statistically independent of the cavity distribution \( P^{(i)}(s_i^{(0)}(0), s_i^{(0)}(1), \ldots, s_i^{(0)}(t)) = P(s_i^{(0)}(0)) \prod_{\sigma=1}^{t} (e^{\beta s_i(\sigma) h_i^{(t)}(\sigma)}/2 \cosh(\beta h_i^{(t)}(\sigma))), \) where \( P^{(i)}(s_i^{(0)}(0)) \) denotes the joint distribution of the cavity system at \( t = 0 \). Hereafter, we assume that the initial state is described by a factorized distribution \( P(s(0)) = \prod_{i=1}^{N} P_i(s_i(0)) \), so that the joint cavity distribution is given as \( P^{(i)}(s_i^{(0)}(0)) = \prod_{j \neq i} P_j(s_j(0)) \).

This, in conjunction with the central limit theorem, enables us to handle the field
distribution

$$\Psi(\phi_i, h_i) = \sum_{s^{(i)}(0), s^{(i)}(1), \ldots, s^{(i)}(t)} p^{(i)}(s^{(i)}(0), s^{(i)}(1), \ldots, s^{(i)}(t)) \times \prod_{\sigma=1}^{t} \delta \left( h_i(\sigma) - \theta_i(\sigma) - \sum_{j \neq i} J_{ij} s_j(\sigma - 1) \right) \times \prod_{\sigma=0}^{t-1} \delta \left( \phi_i(\sigma) - \sum_{l \neq i} J_{il} (s_l(\sigma + 1) - \tanh(\beta h^{l(i)}_i(\sigma + 1))) \right), \quad (14)$$

as of the Gaussian form with the property that the original local field $$h_i = (h_i(1), h_i(2), \ldots, h_i(t))$$ and the additional backaction field $$\phi_i = (\phi_i(0), \phi_i(1), \ldots, \phi_i(t-1))$$ have correlations. This, in conjunction with the last product in equation $$(13)$$, incorporates the memory effect induced by retarded self-interaction via the cavity system to the $$i$$th spin, which can be understood by the fact that the dynamics of spin $$i$$ at earlier time steps will affect the current state of its neighbors. Equations $$(13)$$ and $$(14)$$ mean that the marginal distribution of the trajectory of spin $$i$$ can be written as

$$p(s_i(0), s_i(1), \ldots, s_i(t)) = \sum_{s^{(i)}(0), s^{(i)}(1), \ldots, s^{(i)}(t)} p(s(0), s(1), \ldots, s(t)) = \mathcal{N}^{-1} \int d\phi_i \, dh_i \, \Psi(\phi_i, h_i) P_i(s_i(0)) \times \exp \left[ \sum_{\sigma=1}^{t} \beta s_i(\sigma) h_i(\sigma) + \sum_{\sigma=1}^{t} \beta s_i(\sigma - 1) \phi_i(\sigma - 1) \right] - \sum_{\sigma=1}^{t} \ln 2 \cosh(\beta h_i(\sigma)) \right], \quad (15)$$

where $$\mathcal{N}$$ is a normalization constant. Writing equation $$(15)$$ has the advantage that we can directly take into account the contribution of the backaction field $$\phi_i$$ in deriving time-dependent quantities of interest.

To consider the memory effect, we should have data at least up to two time steps earlier (e.g., $$m(t-2)$$). This was also observed in the dynamical inference in a diluted partially asymmetric Ising system for which the dynamic cavity method $$[5, 19]$$ is computationally feasible. For the following derivation, we define $$\eta_i(t) \equiv h_i(t) - \langle h_i(t) \rangle^{(i)}$$, where the superscript $$(i)$$ means the average is taken without the backaction of spin $$i$$. $$\langle h_i(t) \rangle^{(i)}$$ can be calculated indirectly, as we shall show. For brevity, the time index for the field is neglected as $$\eta_i \equiv \eta_i(t)$$ and $$\phi_i \equiv \phi_i(t-2)$$. As a first approximation, we here consider the field correlations only for this time difference. This is reasonable because both fields are determined by the state of the cavity network at the same time slice $$t-1$$. Improving the approximation level by considering more time steps is also possible, although the necessary treatment would become more complicated technically.

The approximation is constructed by handling the state of the $$t-2$$th step as if it were the initial state in equation $$(15)$$. This allows us to carry out the summation over $$s_i(t-1)$$

\[\text{doi:10.1088/1742-5468/2014/05/P05020} \]
and integration over $\phi_i(t-1)$ independently of the other relevant variables, which yields an expression

$$p(s_i(t-2), s_i(t)) = \sum_{s_i(t-1), s^{(i)}(t-2), s^{(i)}(t-1), s^{(i)}(t)} p(s(t-2), s(t-1), s(t))$$

$$\simeq N^{-1} \int d\eta_i \ d\phi_i \ \Psi(\phi_i, \eta_i)p(s_i(t-2))$$

$$\times \exp \left[ \beta(\eta_i + \langle h_i \rangle^{(i)} s_i(t) + \beta \phi_i s_i(t-2) - \ln 2 \cosh(\eta_i + \langle h_i \rangle^{(i)}) \right]. \quad (16)$$

Let us denote $p(s_i(\sigma)) = (1 + m_i(\sigma)s_i(\sigma))/2$. In addition, we re-write the joint distribution of the fields as $\Psi(\phi_i, \eta_i) = \Psi(\eta_i | \phi_i)\Psi(\phi_i)$, where

$$\Psi(\eta_i | \phi_i) = \frac{1}{\sqrt{2\pi V_{\eta_i|\phi_i}}} \exp \left[-\frac{1}{2V_{\eta_i|\phi_i}} \left( \eta_i - \frac{V_{\eta_i \phi_i}}{V_{\phi_i \phi_i}} \phi_i \right)^2 \right], \quad (17)$$

$$\Psi(\phi_i) = \frac{1}{\sqrt{2\pi V_{\phi_i \phi_i}}} \exp \left[-\frac{1}{2V_{\phi_i \phi_i}} \phi_i^2 \right]. \quad (18)$$

$V_{\phi_i \phi_i}, V_{\eta_i \phi_i}$ and $V_{\eta_i|\phi_i}$ parameterize the variance of $\phi_i$, the covariance between $\phi_i$ and $\eta_i$, and the conditional variance of $\eta_i$ given $\phi_i$, respectively. By using these, the variance of $\eta_i$, $V_{\eta_i \eta_i}$, is given as $V_{\eta_i \eta_i} = V_{\eta_i|\phi_i} + V_{\phi_i \phi_i}^2 / V_{\phi_i \phi_i}$. Equations (16)–(18) provide the expression of instantaneous magnetization as

$$m_i(t) = \sum_{s_i(t), s_i(t-2)} \int d\phi_i \ d\eta_i \ \frac{e^{\beta(\eta_i + \langle h_i \rangle^{(i)} s_i(t))}}{2 \cosh(\beta(\eta_i + \langle h_i \rangle))} s_i(t) \Psi(\eta_i | \phi_i) \Psi(\phi_i | s_i(t-2)) p(s_i(t-2))$$

$$= \sum_{s_i(t-2)} \int Dz \ \tanh \beta \left[ \langle h_i \rangle^{(i)} + \beta V_{\eta_i \phi_i} s_i(t-2) + \sqrt{V_{\eta_i \eta_i}} z \right]$$

$$\times \frac{1 + m_i(t-2)s_i(t-2)}{2}$$

$$= \sum_{s_i(t-2)} \frac{1 + m_i(t-2)s_i(t-2)}{2} \int Dz \ \tanh \beta \Xi(z, s_i(t-2)), \quad (19)$$

where we defined the conditional distribution of $\phi_i$ given $s_i(t-2)$ as $\Psi(\phi_i | s_i(t-2)) \propto \Psi(\phi_i) \exp(\beta \phi_i s_i(t-2))$, and $\Xi(z, s_i(t-2)) = \theta_i(t) + \sum_{j \in \partial i} J_{ij} m_j(t-1) - \beta V_{\eta_i \phi_i} (m_i(t-2) - s_i(t-2)) + \sqrt{V_{\eta_i \eta_i}} z$. Note that to get the final expression, an equation to evaluate the cavity average from the full averages

$$\langle h_i \rangle^{(i)} = \theta_i(t) + \sum_{j \in \partial i} J_{ij} m_j(t-1) - \beta V_{\eta_i \phi_i} m_i(t-2) \quad (20)$$

was employed. This equation is derived by combining two relations $\langle h_i \rangle = \theta_i(t) + \sum_{j \in \partial i} J_{ij} m_j(t-1) = \langle h_i \rangle^{(i)} + \langle \eta_i \rangle$ and $\langle \eta_i \rangle = \sum_{s_i(t-2)} \int d\phi_i \ d\eta_i \ \Psi(\eta_i | \phi_i) \Psi(\phi_i | s_i(t-2)) p(s_i(t-2)) = \beta V_{\eta_i \phi_i} m_i(t-2)$. The last term of equation (20) indicates subtraction of the retarded self-interaction effect.
Equations (19) and (20) indicate that assessing the second moments of the cavity fields $V_{n,n}$ and $V_{n,\phi}$ is necessary for the evaluation of $m_i(t)$. Following earlier studies [16, 20], we approximately replace these with those of the full distribution as

$$V_{n,n} \simeq \left\langle \left( \sum_{l \neq i} J_{il} (s_l(t-1) - \langle s_l(t-1) \rangle) \right)^2 \right\rangle$$

$$\simeq \sum_{l \neq i} J_{il}^2 \left( 1 - \langle s_l(t-1) \rangle \right)^2 \simeq \frac{J^2}{N} \sum_{l=1}^{N} (1 - m_l^2(t-1)) \equiv V_{\eta\eta}(t), \quad (21a)$$

$$V_{n,\phi} \simeq \left\langle \left( \sum_{j \neq i} J_{ij} (s_j(t-1) - \langle s_j(t-1) \rangle) \right) \left( \sum_{l \neq i} J_{il} (s_l(t-1) - \tanh(\beta h_i^{(l)}(t-1))) \right) \right\rangle$$

$$\simeq \sum_{l \neq i} J_{il} J_{li} \left( 1 - \langle \tanh^2(\beta h_i(t-1)) \rangle \right) \approx \left( \frac{1 - k^2}{1 + k^2} \right) \frac{J^2}{N} \sum_{l=1}^{N} (1 - \hat{m}_l(t-1))$$

$$\equiv V_{\eta\phi}(t), \quad (21b)$$

where $\hat{m}_i(t-1) \equiv \langle \tanh^2(\beta h_i(t-1)) \rangle$ is evaluated using the update rule

$$\hat{m}_i(t) = \sum_{s_i(t-2)} \frac{1 + m_i(t-2) s_i(t-2)}{2} \int Dz \tanh^2(\beta \Xi(z, s_i(t-2))) \quad (22)$$

for the $t-1$th step.

Equations (19)–(22) constitute our improved mean-field theory. In practice, this is carried out as follows:

- Expectations for $t = 0$ and 1 are evaluated exceptionally as $m_i(0) = \sum s_i(0) s_i(0)$ $P_i(s_i(0))$, $m_i(1) = \int Dz \tanh(\beta \Xi_i(z, 1))$ and $\hat{m}_i(1) = \int Dz \tanh^2(\beta \Xi_i(z, 1))$ for $i = 1, 2, \ldots, N$, where $\Xi_i(z, t) \equiv \theta_i(t) + \sum_{j \in \partial^+ i} J_{ij} m_j(t-1) + \sqrt{V_{\eta\eta}(t)} z$. These provide the initial condition for the subsequent dynamics.

- For $t \geq 2$, $V_{\eta\eta}(t)$ and $V_{\eta\phi}(t)$ are computed first from $\{m_i(t-1)\}$ and $\{\hat{m}_i(t-1)\}$ by (21a) and (21b), respectively. Then, $m_i(t)$ and $\hat{m}_i(t)$ are assessed from $m_i(t-1)$ and $m_i(t-2)$ with the use of $V_{\eta\eta}(t)$ and $V_{\eta\phi}(t)$ following equations (19) and (22).

In the above treatment, we dropped all terms negligible for $N \to \infty$. Keeping the site dependence in equations (21a) and (21b) and/or considering the contributions from the off-diagonal correlations as developed in the previous section may improve the approximation accuracy for relatively small systems. Note that, by applying the above procedure starting from $m(0)$, we can only capture the short-time trend of dynamics (measured by the evolution of the global magnetization (data not shown)). This suggests that we should improve the approximation by considering correlations at more time steps. However, under the assumption of stationarity, $V_{\eta\phi}(t) = V_{\eta\phi}(t-1)$, $V_{\eta\phi}(t)$ can be determined self-consistently, which is effective in practical prediction, as we shall show subsequently.
We remark here that $V_{\eta\phi}$ (equation (21b)) vanishes in the fully asymmetric network and equation (19) gives back equation (6a), which is exact when the network is fully asymmetric. However, even if the asymmetry degree $k \neq 1$, our theory is expected to have a good prediction performance as the contributions from the backaction field $\phi_i$ are explicitly considered. To examine this point clearly, we compared the prediction error of instantaneous magnetization by using equations (19) and (6a) based on the numerically collected data, which is shown in figure 4. To keep the same low complexity as in [4, 17], we adopt equation (21a) by assuming the nondiagonal correlations to be negligible.

In the prediction, we kept the site dependence in equations (21a) and (21b), and we determined $V_{\eta\phi_i}(t)$ on the basis of the data of $m_i(t-1)$ and $m_i(t-2)$ in a self-consistent manner\(^\dagger\) assuming that the dynamics reaches the stationary state, so that $V_{\eta\phi_i}(t) = V_{\eta\phi_i}(t-1)$ holds. As seen in figure 4(a), IMF definitely outperforms MF, especially for $k$ close to zero with strong coupling correlations. The improvement becomes more apparent in the low-temperature region. As $k \to 1$, the prediction error of both methods becomes indistinguishable, as expected from the above theoretical derivation. From the scatter plot in figure 4(b), one can conclude that IMF predicts a value of magnetization closer to the true value, compared to MF. Figure 5 explores the time dependence of the prediction performance, which shows that IMF always yields a better performance than MF, and the prediction error saturates at large time for both methods. Figure 5 also implies that, even at short time, IMF still well predicts the experimental results.

5. Summary

In this paper, we proposed two schemes for improving the existing mean-field description of the dynamics of a kinetic Ising spin model. In the first scheme, we showed that the formula for the time-delayed correlation can be recovered without the small-correlation (of local fields) assumption. In addition, we developed formulas for improving the prediction accuracy of magnetizations, the same- and delayed-time correlations by incorporating the pairwise correlations of local fields, which are particularly effective in the low-temperature region.

In the second scheme, we focused on considering the influence of statistical correlations between couplings of two opposite directions for each pair of spins. When statistical correlations exist for the coupling pairs, the central limit theorem assumed in the existing mean-field theory, which was developed by supposing a fully asymmetric network, does not hold. Local fields of different spins correlate with one another in a complex way, and furthermore, the instantaneous value of spin is not independent of the couplings. To properly treat this significant memory effect present in a general system, we developed an improved mean-field theory utilizing the notion of the cavity system in conjunction with a perturbative expansion approach. Its efficiency was numerically confirmed by comparison with the existing mean-field theory.

Note that the first scheme applies a similar idea to the recent work by Mahmoudi and Saad [16], but in their work, the (auto-) field covariances are calculated recursively, which

---

\(^\dagger\) One method to get covariance $V_{\eta\phi_i}(t)$ is to iteratively update $\{\hat{m}_i(t-1)\}_{i=1}^{N}$ until they converge within some prescribed numerical precision. Note that when $\hat{m}_i$ is updated, all $V_{\eta\phi_i}(i \neq l)$ should also be updated as $V_{\eta\phi_i} = V_{\eta\phi_i} - J_{ij}J_{ji} \Delta \hat{m}_j$, where $t$ denotes the iteration step and $\Delta \hat{m}_i$ denotes the change of $\hat{m}_i$. 

---

doi:10.1088/1742-5468/2014/05/P05020
Figure 4. Comparison of prediction performance of IMF and MF on partially asymmetric kinetic Ising systems of system size $N = 100$. Each data point is the average over ten random realizations. A total number of $10^5$ spin trajectories are collected up to 31 time steps and these trajectory data are used either to compute inputs for prediction equations or to compute the experimental values for comparison. We predict the instantaneous magnetization at $t = 31$ based on the data at $t = 30$ and $t = 29$. (a) Constant external fields with $\theta_0(t) = 0.1$. The asymmetry degree $k$ is varied and the result for two different temperatures is shown. (b) Scatter plot for a typical example with $k = 0.3$ in (a). The full line indicates equality.

may demand expensive computational cost, like the case of MFcorre whose computational cost is of the order $N^3$. However, in the second scheme, by introducing additional backaction fields (on top of the original local fields), IMF provides efficient predictions.
Dynamics of asymmetric kinetic Ising systems revisited

Figure 5. Time dependence of the prediction performance of IMF and MF on partially asymmetric kinetic Ising systems of system size $N = 100$ and asymmetry degree $k = 0.3$. Each data point is the average over ten random realizations with constant external fields $\theta_0(t) = 0.1$.

with low complexity ($\sim O(N^2)$, the same as that of MF), while the usual Monte Carlo simulation takes a computer time proportional to $N^2tP_t$, where $t$ denotes the length of one trajectory and $P_t$ is the total number of trajectories. $P_t$ usually takes a large value (e.g., $10^5$) to ensure numerical accuracy.

Studies of such nonequilibrium behavior of asymmetric kinetic Ising systems could provide insights into nonequilibrium network reconstruction, which has received considerable interest in recent years [4, 21], for example, for improving the coupling and field inference in the context of dynamical inference. The two schemes proposed in this paper should prove promising for developing an inverse mean-field algorithm to construct asymmetric couplings between elements in a network based on time-series data.

Acknowledgments

This work was partially supported by the JSPS Fellowship for Foreign Researchers (Grant No. 24.02049) (HH) and JSPS/MEXT KAKENHI Grant Nos 22300003, 22300098, and 25120013 (YK).

References

[1] Crisanti A and Sompolinsky H, Dynamics of spin systems with randomly asymmetric bonds: Ising spins and Glauber dynamics, 1988 Phys. Rev. A 37 4865
[2] Coolen A C C, Laughton S N and Sherrington D, Dynamical replica theory for disordered spin systems, 1996 Phys. Rev. B 53 8184
[3] Roudi Y and Hertz J, Dynamical tap equations for non-equilibrium Ising spin glasses, 2011 J. Stat. Mech. P03031

doi:10.1088/1742-5468/2014/05/P05020
Dynamics of asymmetric kinetic Ising systems revisited

[4] Mézard M and Sakellariou J, Exact mean-field inference in asymmetric kinetic Ising systems, 2011 J. Stat. Mech. L07001

[5] Aurell E and Mahmoudi H, Dynamic mean-field and cavity methods for diluted Ising systems, 2012 Phys. Rev. E 85 031119

[6] Saad D and Mozeika A, Emergence of equilibrumlike domains within nonequilibrium Ising spin systems, 2013 Phys. Rev. E 87 032131

[7] Ko H, Hofer S B, Pichler B, Buchanan K A, Jesper Sjöström P and Mrsic-Flogel T D, Functional specificity of local synaptic connections in neocortical networks, 2011 Nature 473 87

[8] Parisi G, Asymmetric neural networks and the process of learning, 1986 J. Phys. A: Math. Gen. 19 L675

[9] Derrida B, Gardner E and Zippelius A, An exactly solvable asymmetric neural network model, 1987 Europhys. Lett. 4 167

[10] Marre O, El Boustani S, Frégnac Y and Destexhe A, Prediction of spatiotemporal patterns of neural activity from pairwise correlations, 2009 Phys. Rev. Lett. 102 138101

[11] Pillow J W, Shlens J, Paninski L, Sher A, Litke A M, Chichilnisky E J and Simoncelli E P, Spatio-temporal correlations and visual signalling in a complete neuronal population, 2008 Nature 454 995

[12] Bar-Joseph Z, Gitter A and Simon I, Studying and modelling dynamic biological processes using time-series gene expression data., 2012 Nature Rev. Genet. 13 552

[13] Tyrcha J, Roudi Y, Marsili M and Hertz J, The effect of nonstationarity on models inferred from neural data, 2013 J. Stat. Mech. P03005

[14] Hatchett J P L, Wemmenhove B, Pérez Castillo I, Nikoletopoulos T, Skantzos N S and Coolen A C C, Parallel dynamics of disordered Ising spin systems on finitely connected random graphs, 2004 J. Phys. A: Math. Gen. 37 6201

[15] Kappen H J and Spanjers J J, Mean field theory for asymmetric neural networks, 2000 Phys. Rev. E 61 5658

[16] Mahmoudi H and Saad D, Generalized mean field approximation for parallel dynamics of the Ising model, 2013 arXiv:1310.5460

[17] Sakellariou J, Roudi Y, Mézard M and Hertz J, Effect of coupling asymmetry on mean-field solutions of the direct and inverse Sherrington–Kirkpatrick model, 2012 Phil. Mag. 92 272

[18] Neri I and Bollé D, The cavity approach to parallel dynamics of Ising spins on a graph, 2009 J. Stat. Mech. P08009

[19] Aurell E and Mahmoudi H, A message-passing scheme for non-equilibrium stationary states, 2011 J. Stat. Mech. P04014

[20] Opper M and Winther O, Adaptive and self-averaging Thouless-Anderson-Palmer mean-field theory for probabilistic modeling, 2001 Phys. Rev. E 64 056131

[21] Roudi Y and Hertz J, Mean field theory for nonequilibrium network reconstruction, 2011 Phys. Rev. Lett. 106 048702

doi:10.1088/1742-5468/2014/05/P05020