Classical understanding of the electron vortex beams in a uniform magnetic field

Yeong Deok Han

Department of Computer Science and Engineering, Woosuk University, Wanju, Cheonbuk, 565-701, Republic of Korea

Taeseung Choi

Division of Applied Food System, College of Natural Science, Seoul Women’s University, Seoul 139-774, Republic of Korea and School of Computational Sciences, Korea Institute for Advanced Study, Seoul 130-012, Korea

Abstract

Recently interesting observations on electron vortex beams, which have angular momentum about the center of the vortex beams, have been made. We have shown that the basic features of the electron vortex beams in a uniform magnetic field are understandable by using the classical motions of electrons. We have constructed a classical vortex-like motion by the collective motion of individual electrons in their cyclotron motions with a constant canonical angular momentum in the symmetric gauge, which models electron vortex beams, in a uniform magnetic field. With this model the various properties of circulating currents and the relation between energy and kinetic angular momentum in the electron vortex beams are well explained. We have also shown that the mismatch between the centers of the electron vortex beam and the classical cyclotron orbits naturally induces the parallel axis theorem and also the time-varying kinetic angular momentum of the electron vortex beam for certain distributions of classical electrons.

*tschoi@swu.ac.kr
Free electron vortex states have recently been predicted by considering semiclassical (paraxial) wave packet [1] and observed in electron microscopy [2-4]. The free electron vortex is practically equivalent to the optical vortex beam, however, in the presence of a magnetic field, the properties of electron vortex beam becomes different from those of its optical counterpart. The electron vortex beam with angular momentum has a magnetic moment and interacts with an external magnetic field, which gives interesting physics and applications [1, 5-12].

For the electron vortex beam, the interactions between electrons are assumed to be so small that can be neglected. Therefore, the problem of electron vortex beams in a magnetic field seems to be reduced to the problem of one electron in the same magnetic field. The physics of an electron in a uniform magnetic field is well understood both classically and quantum mechanically [13]. The motion of a classical electron in a uniform magnetic field is circular with constant speed, known as the cyclotron motion, hence it is expected that the kinetic angular momentum of such motion about its center is constant. However, one of interesting issues, which seems to be contrary to classical cyclotron motion, was pointed out by Greenshields et al. [14]. They showed that the “diamagnetic” angular momentum of the electron vortex beam in a uniform magnetic field is time-varying in general, which implies that the kinetic angular momentum of the electron vortex beam could also be time-dependent. They used the quantum vortex solutions to show the time-dependence of the kinetic angular momentum. This fact seems to be contradictory to the rotational symmetry of the system and they showed that the conservation of kinetic angular momentum is recovered by involving the angular momentum of electromagnetic field.

It is, however, still surprising how the rotationally symmetric vortex solutions can have time-varying radius in quantum vortex solutions, contrast to the fact that the radius of the rotationally symmetric classical cyclotron motion is constant. It was found that the average radial position of the electron vortex state expands and contracts [11]. Recently it was also shown that the orbital angular momentum of an electron vortex beam can be decomposed into separate angular momenta according to parallel axis theorem, which seems to be only meaningful for an extended probability distribution [15].

As we have seen above, the characteristics of the electron vortex beams are different from those of the classical cyclotron motion of one electron. In this paper, we will show that it is possible to explain the motion of the electron vortex beams as a collective motion of
electrons in their classical cyclotron motions. Using this picture, the basic features of the electron vortex beams are well explained. We will also show that the time-varying behavior of the kinetic angular momentum and the parallel axis theorem can be understood. This suggests that our classical picture is very instructive to explain the electron vortex beams intuitively.

I. ONE ELECTRON MOTION IN A UNIFORM MAGNETIC FIELD

The motion of one electron in a uniform magnetic field is a textbook problem, which is well understood both classically and quantum mechanically [13]. However, not much attention is paid to the time-dependence of angular momenta, especially in classical point of view. And the classical motion of one electron will be the building block of the collective motion to explain the electron vortex beams. Hence, we will briefly review the problem of the motion of one electron in a uniform magnetic field as we focus on the time dependence of angular momenta.

The model Lagrangian for an electron in a uniform magnetic field \( \mathbf{B} = B \hat{z} \), in a cylindrical coordinate \((\rho, \phi, z)\) and the symmetric gauge with \( \mathbf{A}_s = (B \rho / 2) \hat{\phi} \) is as follows

\[
\mathcal{L} = \frac{1}{2} m \mathbf{v}^2 + e \mathbf{v} \cdot \mathbf{A}_s, \\
= \frac{1}{2} m \dot{\rho}^2 + \frac{1}{2} m \rho^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2 + \frac{e}{2} B \rho^2 \dot{\phi},
\]

where \( e, m, \) and \( \mathbf{v} \) are the charge, the mass, and the velocity of the electron, respectively. Then the canonical conjugate momenta are defined as

\[
p_\rho = \frac{\partial \mathcal{L}}{\partial \dot{\rho}} = m \dot{\rho}, \quad p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = m \rho^2 \dot{\phi} + \frac{e}{2} B \rho^2.
\]

This Lagrangian has a rotational symmetry about \( z \)-axis, i.e., there is no \( \phi \)-dependence, so that the canonical conjugate momentum \( p_\phi \) is constant of motion. Note that \( p_\phi \) is the \( z \)-component of the canonical angular momentum, i.e., \( p_\phi \equiv L_z = (\mathbf{r} \times \mathbf{p})_z \). That is, the canonical angular momentum \( L_z \) of the electron in a uniform magnetic field is conserved in the symmetric gauge. The \( L_z \), however, is gauge dependent.

The gauge invariant kinetic angular momentum is defined as

\[
L_z^{\text{kin}} = \mathbf{r} \times (\mathbf{p} - e \mathbf{A}_s) = L_z + \frac{m \omega_c}{2} \rho^2,
\]
where \( \mathbf{r} = \rho \hat{\rho} + z \hat{z} \) and \( \omega_c = -eB/m \) is the classical cyclotron frequency of the electron. The second term in Eq. (3) is called the diamagnetic angular momentum. The conservation of the canonical angular momentum is guaranteed by the rotational symmetry of the Lagrangian, on the other hand, a kinetic angular momentum depends on the origin of the coordinate. This means that the conservation of a kinetic angular momentum in one coordinate system does not automatically guarantee the conservation of the kinetic angular momentum in another coordinate system with a different origin.

To study the effect of the choice of the origin of the coordinate, the Cartesian coordinate is convenient. The Hamiltonian \( \mathcal{H} = \mathbf{v} \cdot \mathbf{p} - \mathcal{L} \) is written in the Cartesian coordinate as

\[
\mathcal{H} = \frac{1}{2m} \mathbf{p}^2 + \frac{\omega_c}{2} L_z + \frac{m}{8} \omega_c^2 (x^2 + y^2),
\]

where \( \mathbf{p}^2 = p_x^2 + p_y^2 + p_z^2 \). The classical equation of motion for the electron becomes the usual Lorentz force equation,

\[
m \frac{d^2 \mathbf{r}}{dt^2} = e \mathbf{v} \times \mathbf{B}.
\]

The \( z \)-directional classical motion of the electron described by Eq. (5) is trivial because there is a translation symmetry along \( z \)-direction. Hence we focus on the classical motion of the electron in the \( xy \)-plane.

Eq. (5) describes the cyclotron motion in the \( xy \)-plane and the general solutions are

\[
x(t) = x_0 + R \cos (\omega_c t + \theta), \quad y(t) = y_0 + R \sin (\omega_c t + \theta),
\]

where \( (x_0, y_0) \) is the center of the cyclotron orbit with radius \( R \) and \( \theta \) is the phase shift from the \( +x \) direction. Then the squared 2-dimensional radial distance \( \rho^2 = x^2 + y^2 \) of the electron from the origin of the coordinate becomes

\[
\rho^2 = x_0^2 + y_0^2 + R^2
\]

\[+ 2x_0 R \cos (\omega_c t + \theta) + 2y_0 R \sin (\omega_c t + \theta),
\]

which shows time dependence. By direct calculation one obtains the following dynamical equation for the squared 2-dimensional radial distance as

\[
\frac{d^2 \rho^2}{dt^2} = -\omega_c^2 \rho^2 - 2 \frac{\omega_c}{m} L_z + \frac{4}{m} E,
\]

where \( E \) is the 2-dimensional energy (classical Hamiltonian) \( \frac{1}{2} m (v_x^2 + v_y^2) = \frac{1}{2} m R^2 \omega_c^2 \). One can easily check that Eq. (8) is equivalent to quantum mechanical equation Eq. (11) for the squared radius in Heisenberg formalism of Ref. [14].
This suggests that the oscillations of the 2-dimensional squared radial distance and the resultant time-varying diamagnetic response could be understood as originated from the mismatch of the origin of the coordinate and the center of the cyclotron orbit. One can also check that the classical torque $\mathbf{r} \times (e \mathbf{v} \times \mathbf{B})$ is not zero for the cyclotron motion of the electron with centers different from the origin of the coordinate. This torque is responsible for the change of the kinetic angular momentum of the electron. If the origin of the coordinate and the center of the cyclotron orbit match, i.e., $x_0 = y_0 = 0$, $\rho^2 (= R^2)$ is a constant of motion and the kinetic angular momentum is also conserved as well as the canonical angular momentum. The time dependence of the kinetic angular momentum of the classical electron vortex beam will be discussed in sec. [II]

II. CLASSICAL MODEL OF AN ELECTRON VORTEX BEAM IN A UNIFORM MAGNETIC FIELD

In this section we will study the collective motions of classical electrons to show the basic features of the electron vortex beams. We will call the collective motions showing vortex-like motion a classical electron vortex.

We consider that electrons are moving with the same speed in a uniform magnetic field and there is no Coulomb repulsion between electrons, which is assumed in the usual electron vortex beams. Then all electrons rotate in their cyclotron orbits with the same radius. The cyclotron orbits with the same radius $R$ given by the solutions in Eq. (6) can be classified by the canonical angular momentum in the symmetric gauge, because the canonical angular momentum is constant of motion.

The canonical angular momentum $L_z$ of the electron on its cyclotron orbit, the $z$-component of $\mathbf{r} \times (m \mathbf{v} + e \mathbf{A})$, is calculated as

$$ L_z = \frac{m}{2} \omega_c (R^2 - R_{\text{cen}}^2), $$

where $R_{\text{cen}}$ is the distance from the origin of the coordinate to the center of the cyclotron orbit, i.e., $R_{\text{cen}}^2 = x_0^2 + y_0^2$. Hence there are three categories in the cyclotron motions of the electrons according to $L_z > 0$, $L_z = 0$, and $L_z < 0$ as in Fig. [I]. The canonical angular momentum $L_z$ for the cyclotron motion of the electrons with the same radius $R$ becomes positive for $R^2 > R_{\text{cen}}^2$, zero for $R^2 = R_{\text{cen}}^2$, and negative for $R^2 < R_{\text{cen}}^2$. 

5
FIG. 1: The cyclotron orbits for (a) $L_z > 0$, (b) $L_z = 0$, and (c) $L_z < 0$.

Here we construct the classical electron vortex by the collective motion of the electrons with the same canonical angular momentum $L_z$. The centers of the cyclotron orbits with the same $L_z$ have the same distance to the origin of the coordinate. We assume that the distribution of the cyclotron orbits are rotationally symmetric about the origin of the coordinate and the electrons are also uniformly distributed on these cyclotron orbits as shown in Fig 2 (a), (b), and (c).

As a result, the average current of the collective motions of the electrons is created, which is the rotational current around the center of the coordinate as shown in Fig 2 (d), (e), and (f). This apparent motion is the vortex-like motion with torus-like profile, which is a rotation around the center of the vortex as an average motion. However, the actual motion of individual electrons constructing the classical electron vortex is not rotation about the vortex center, but the cyclotron motion about its own center.

The classical electron vortex can be considered as a classical wave, in the similar sense to a water wave, in which the apparent motion of water wave is translational (passing to a certain direction), but each water molecule creating the water wave executes its own circular motion.

There are three kinds of collective motions as shown in Fig. 2 according to the three categories of the $L_z$ in Fig. 1. The collective motion of constituent electrons generates azimuthal current by averaging their individual motions. For $L_z > 0$, all azimuthal currents are counter-clockwise and the magnitudes of the inmost and the outmost currents are given by the same and the maximum average speed. For $L_z = 0$, the magnitude of the azimuthal currents reduces from the maximum value of the outmost current to zero at the origin of the coordinate, which is the center of the classical electron vortex. For $L_z < 0$, the azimuthal current changes its sign at the point between the inmost and the outmost edges.
Therefore, the azimuthal currents on the outer side and the inner sides are counter-clockwise and clockwise, respectively. The three different types of azimuthal currents are qualitatively equivalent to the types of the azimuthal currents in Fig. 5 of Ref. [17]. This shows that the basic rotating features of the electron vortex beams can be understood by the classical electron vortex created by individual cyclotron motions.

![Diagram of classical electron vortices](image)

**FIG. 2:** (Color online) The construction of the classical electron vortices by the collective motions of the constituent electrons in their own cyclotron orbits for (a) $L_z > 0$, (b) $L_z = 0$, and (c) $L_z < 0$ and the resultant classical electron vortices with torus-like profile for (d) $L_z > 0$, (e) $L_z = 0$, and (f) $L_z < 0$. The azimuthal frequencies of the circulating currents are $\omega_c$ ((a) and (d)), $\omega_c/2$ ((b) and (e)), and zero ((c) and (f)), respectively.

One of the intriguing result is that the azimuthal angular frequencies of the electron vortex beams are cyclotron, Larmor, and zero frequencies according to the three categories of the canonical angular momentum [12]. The expectation value of azimuthal angular frequency of the electron vortex beam is determined by $\langle \omega \rangle = \langle j/r \rangle$, where $j = \hbar/m[\text{Im}(\psi^*j\psi) - eA_s|\psi|^2]$ and $\psi$ is the vortex solution. Hence, the azimuthal angular frequency is related with the circulating current around the vortex center by $I_c = \langle \omega \rangle/(2\pi)$. The three circulating currents given by the three azimuthal angular frequencies can also be explained by the classical electron vortex.

The frequency of the azimuthal current of the classical electron vortex is determined by the angle through which each constituent electron rotates around the center of the vortex.
The cyclotron orbits of electrons with $L_z > 0$ encloses the origin of the coordinate. This implies that the electron circles around the origin of the coordinate every time the electron circles along its own cyclotron orbit. Hence the frequency of the circulating current of the electron vortex beam becomes the cyclotron frequency $\omega_c$. On the other hand, the cyclotron orbits of electrons with $L_z < 0$ does not enclose the origin of the coordinate. This means that the circulating current of the electron vortex beam is zero, i.e., the frequency of the circulating currents for the electron vortex beams is zero. The cyclotron orbits of electrons with $L_z = 0$ cuts the origin of the coordinate. In this case the change of the azimuthal angle of the electrons for one cyclotron orbit is $\pi$, hence the frequency of the circulating current of the electron vortex beam becomes half the cyclotron frequency $\omega_c/2$, i.e., Larmor frequency.

In summary, the average circulating current for one electron in the classical electron vortex becomes

$$I_c^> = \frac{1}{2\pi} \omega_c \text{ for } L_z > 0,$$

$$I_c^0 = \frac{1}{4\pi} \omega_c \text{ for } L_z = 0,$$

$$I_c^< = 0 \text{ for } L_z < 0.$$  (10a, 10b, 10c)

This relation is equivalent to the quantum relation in Eq. (4) of Ref. [12]. Thus the frequencies of the circulating current around the beam axis for the electron vortex beam is also well explained by the classical electron vortex.

Now we will discuss the time dependence of the kinetic angular momentum of the classical electron vortex. It is surprising that the rotationally symmetric electron vortex beam has time-dependent kinetic angular momentum about its center contrast to the fact that the rotationally symmetric classical cyclotron motion has the constant kinetic angular momentum about its center. The expectation value of the kinetic angular momentum for the quantum mechanical vortex solution was shown to be time-dependent in general. The time-dependence of the quantum kinetic angular momentum is represented by [14]

$$\langle L_z^\text{kin} \rangle(t) = \tilde{L}_z^\text{kin} + \left( \langle L_z^\text{kin} \rangle(0) - \tilde{L}_z^\text{kin} \right) \cos(\omega_c t),$$  (11)

where $\langle L_z^\text{kin} \rangle(t)$ is the expectation value of the kinetic angular momentum at time $t$ and $\tilde{L}_z^\text{kin}$ corresponds to the classical value of the kinetic angular momentum for an electron in a rotational motion about the origin of the coordinate with the radius $\tilde{\rho} = \sqrt{R^2 + R_{cen}^2}$. The kinetic angular momentum in Eq. (11) is constant for the special case of $\langle L_z^\text{kin} \rangle(0) = \tilde{L}_z^\text{kin}$,
in which the vortex wave function is the Landau energy eigenstate (Landau state), however, it shows time-dependence for $\langle L_z^{\text{kin}} \rangle (0) \neq \tilde{L}_z^{\text{kin}}$, which describes a general electron vortex beams.

The kinetic angular momentum of the classical electron vortex is defined by the average of the kinetic angular momenta of constituent electrons. The time-dependence of the kinetic angular momentum of each electron is given by the time behavior of the squared radius of the electron as Eqs. [3] and [5]. The squared radius of the classical electron vortex $\rho_V^2$ is defined by the average of the squared radii of constituent electrons $\rho^2$. The $\rho_V^2$ is equal to the average of $\rho^2$ over one cyclotron orbit because of the azimuthal symmetry of the vortex. The squared radius of the classical electron vortex is given by

$$\rho_V^2 = \langle \rho^2 \rangle_c = \langle (R + R_{\text{cen}}) \cdot (R + R_{\text{cen}}) \rangle_c,$$

where $R$ is the displacement of the electron from the center of the cyclotron orbit, $R_{\text{cen}}$ is the position vector of the center of the cyclotron orbit from the center of the classical vortex, and $\langle \cdot \rangle_c$ means the average over one cyclotron orbit. For our classical electron vortex with uniform distributions of electrons on each cyclotron orbit, $\langle R \rangle_c = 0$, which implies that $\langle \rho \rangle_c$ is equal to $\bar{\rho}$ and the average kinetic angular momentum $\langle L_z^{\text{kin}} \rangle_c$ is the same as $\tilde{L}_z^{\text{kin}}$ so that the kinetic angular momentum remains constant.

Then the question is how one can create a classical electron vortex with time-dependent kinetic angular momentum. That classical electron vortex is created in the case that the distribution of the constituent electrons on each cyclotron orbit is not uniform and satisfies $\langle R \rangle_c \neq 0$. One simple example is the case that there is only one electron on each cyclotron orbit as shown in Fig. 3. We suppose that each electron is initially on the outmost point of each cyclotron orbit as in Fig. 3 (a). Then it is obvious that the radius of the classical electron vortex created by the cyclotron motions of these electrons becomes time-dependent as the constituent electrons rotate in their cyclotron orbits as shown in Fig. 3 (b) and (c). This makes the diamagnetic angular momentum time-dependent and as a result, the kinetic angular momentum becomes time-dependent. The time-dependence of the squared radial distance of one electron is governed by Eq. [8], which is equivalent to Eq. (11) of Ref. [14], because the vortex motion is created by the motions of the constituent electrons.

As another feature of the classical electron vortex, let us study the relation between the kinetic angular momentum and the energy of the electron vortex. It is expected that that
FIG. 3: (Color online) Simple example to show the oscillating behavior of the radius of the classical electron vortex for $L_z < 0$. Here (a) and (c) show the azimuthal currents at outmost and inmost positions, respectively, and (b) shows the instant, at which the azimuthal current is zero.

relation of the electron vortex per one electron is equal to that of one cyclotron orbit because the electron vortex is composed of the cyclotron motion of the constituent electrons.

In one electron case, the kinetic angular momentum of an electron in its cyclotron motion becomes a constant of motion for the coordinate whose origin is the center of the motion, i.e., $x_0 = y_0 = 0$ in Eq. (6). Then the diamagnetic angular momentum is the same as the canonical angular momentum such that the kinetic angular momentum is $2L_z = m\omega_cR^2$. Therefore, the rotational kinetic energy $E$ of the electron is determined by the kinetic angular momentum as

$$E = \frac{|e|B}{2m}L_z^{\text{kin}} = \frac{1}{2}\omega_cL_z^{\text{kin}}. \quad (13)$$

In the classical electron vortex, even though the centers of the constituent cyclotron motions constructing the classical electron vortex are not the origin of the coordinate, the kinetic angular momentum can remain constant as we have seen for the classical electron vortex with $\langle R \rangle_e = 0$. In this classical electron vortex the average kinetic angular momentum of one electron, $\langle L_z^{\text{kin}} \rangle_e$, is calculated as

$$\langle L_z^{\text{kin}} \rangle_e = \langle \mathbf{r} \times m\mathbf{v} \rangle_e = \langle \mathbf{R} \times m\mathbf{v} \rangle_e = mR^2\omega_c, \quad (14)$$

using the uniform distribution of electrons in one cyclotron orbit, because $\langle m\mathbf{v} \rangle_e$ is zero, where $\langle \cdot \rangle_e$ means the classical average of $\cdot$ per one electron. Hence the energy per electron $E_e$ is given by the following relation

$$E_e = \frac{1}{2}\omega_c\langle L_z^{\text{kin}} \rangle_e. \quad (15)$$
For the Landau state, i.e., \( \langle L^\text{kin}_z \rangle(0) = \tilde{L}^\text{kin}_z \), the kinetic angular momentum of the electron vortex beams is also time-independent and the energy eigenvalue is represented by the expectation value of the kinetic angular momentum as \([18]\)

\[
E_{n,l} = \left( n + \frac{|l|}{2} + \frac{m}{2} + \frac{1}{2} \right) \hbar \omega_c = \frac{1}{2} \omega_c \langle L^\text{kin}_z \rangle(0)
\]

(16)

This relation is equivalent to the relation in Eq. \([15]\) and shows that the energy and kinetic angular momentum relations are equivalent in classical and quantum electron vortices.

For the generic time-dependent case, the time-average of the kinetic angular momentum must be used, which becomes the kinetic angular momentum of the cyclotron motion with the center at the origin of the coordinate for both quantum and classical electron vortices. This is because the time average of the cyclotron motion of one electron is equal to the average over the uniformly distributed electrons in their cyclotron motions. Hence the energy and kinetic angular momentum relation in Eqs. \([15]\) and \([16]\) are satisfied for all electron vortices.

The rotational motion of the classical electron vortex naturally leads to the parallel axis theorem with the observation that the moment of inertia \( I \) about the propagating direction of the classical electron vortex (\( z \)-axis) is

\[
I = m \rho^2 = m(R^2 + R_{\text{cen}}^2)
\]

(17)

For the uniform distribution of the constituent electrons, the \( R_{\text{cen}} \) becomes the center of mass of the electrons on each cyclotron orbit. In this case, the equality in Eq. \([17]\) represents the usual parallel axis theorem, in which the moment of inertia about any axis is the sum of the moment of inertia of one particle with the total mass of the system about the same axis and the moment of inertia about the parallel axis through the center of mass. When the distribution of the constituent electrons are not uniform, the center of mass is not \( R_{\text{cen}} \) and time-dependent, so that the parallel axis theorem becomes

\[
I = m \rho^2 = m(\langle \tilde{R} \rangle^2 + R_{\text{C.M.}}^2),
\]

(18)

where \( R_{\text{C.M.}} \) is the radial distance from the origin of the coordinate to the center of mass and \( \langle \tilde{R} \rangle^2 = \int \tilde{\mathbf{R}} \cdot \tilde{\mathbf{R}} dm/m \), where \( \tilde{\mathbf{R}} \) is the radial vector from the center of mass to the position of the infinitesimal mass \( dm \). The \( R_{\text{C.M.}} \) and \( \tilde{\mathbf{R}} \) can be definitely time-dependent and so is \( I \). This parallel axis theorem shows the same feature in Ref. \([16]\).
III. CONCLUSION

We have proposed the classical electron vortex model for the electron vortex beams in a uniform magnetic field. The classical electron vortex is constructed by the collective motion of the constituent electrons in their cyclotron motions under the uniform magnetic field with constant canonical angular momentum in the symmetric gauge. The basic features of the electron vortex beam were understandable by the classical electron vortex. The canonical angular momentum of one electron in a uniform magnetic field has three categories, positive, zero, and negative, according to the distance of the center of the cyclotron orbit from the center of the classical electron vortex. Then it is shown that the three kinds of the classical electron vortex, which is qualitatively equivalent to quantum ones, exist according to the three categories of the canonical angular momentum. The energy of the classical electron vortex, per one constituent electron, is shown to be the energy of the classical cyclotron motion of one electron as expected in the classical physics, independent on the value of the canonical angular momentum.

The surprising time-dependence of the kinetic angular momentum of the electron vortex beams was explained by the mismatch between the centers of the cyclotron orbits and the classical electron vortex for the distribution of electrons in which the average of the displacements of the electrons from the center of the cyclotron orbit is not the center of the cyclotron orbit. This mismatch also naturally induced the parallel axis theorem of the kinetic angular momenta of the electron vortex beams. Additionally, the three categories of the angular frequencies of the azimuthal currents of the electron vortex beams are also explained in the classical electron vortex by using the corresponding circulating currents of one electron.

The results in this paper suggest that the physics of the electron vortex can be understood by the classical motion of electrons. We hope that the classical electron vortex model would help to understand the abundant new physics intuitively and to differentiate classical and quantum physics in electron vortex beams.
ACKNOWLEDGEMENTS

T. Choi was supported by a research grant from Seoul Women’s University(2015) and Y. D. Han was supported by a research grant from Woosuk University.

[1] K. Y. Bliokh et al., Phys. Rev. Lett. 99, 190404 (2007).
[2] M. Uchida and A. Tonomura, Nature (London) 464, 737 (2010).
[3] B. J. McMorran, A. Agrawal, I. M. Anderson, A. A. Herzing, H. J. Lezec, J. J. McClelland, and J. Unguris, Science 331, 192 (2011).
[4] J. Verbeeck, H. Tian, and P. Schattschneider, Nature (London) 467, 301 (2010).
[5] P. Schattschneider and J. Verbeeck, Ultramicroscopy 111, 1461 (2011).
[6] J. Verbeeck, P. Schattschneider, S. Lazar, M. Stöger-Pollach, S. Löfler, A. Steiger-Thirsfeld, and G. Van Tendeloo, Appl. Phys. Lett. 99, 203109 (2011).
[7] K. Y. Bliokh, M. R. Dennis, and F. Nori, Phys. Rev. Lett. 107, 174802 (2011).
[8] G. Guzzinati, P. Schattschneider, K. Y. Bliokh, F. Nori, and J. Verbeeck, Phys. Rev. Lett. 110, 093601 (2013).
[9] S. Lloyd, M. Babiker, and J. Yuan, Phys. Rev. Lett. 108, 044801 (2012).
[10] E. Karimi, L. Marrucci, V. Grillo, and E. Santamato, Phys. Rev. Lett. 108, 074802 (2012).
[11] G. M. Gallatin and B. McMorran, Phys. Rev. A 86, 012701 (2012).
[12] P. Schattschneider, Th. Schachinger, M. Stöger-Pollach, S. Löfler, A. Steiger-Thirsfeld, K. Y. Bliokh, and F. Nori, Nat. Comm. 5, 4586 (2014).
[13] L. E. Ballentine, Quantum Mechanics : A Modern Development World Scientific Co. Pte. Ltd. Singapore 1988.
[14] C. R. Greenshields, R. L. Stamps, S. Franke-Arnold, and S. M. Barnett, Phys. Rev. Lett. 113, 240404 (2014).
[15] C. R. Greenshields, S. Franke-Arnold, and R. L. Stamps, New J. Phys. 17, 093015 (2015).
[16] C. R. Greenshields, R. L. Stamps, and S. Franke-Arnold, New J. Phys. 14, 103040 (2012).
[17] K. Y. Bliokh, P. Schattschneider, J. Verbeeck, and F. Nori, Phys. Rev. X 2, 041011 (2012).
[18] C-F. Li and Q. Wang, Physica B 269, 22 (1999).