EXACT RECONSTRUCTION OF SPATIALLY UNDERSAMPLED SIGNALS IN EVOLUTIONARY SYSTEMS

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ABSTRACT. We consider the problem of spatiotemporal sampling in which an initial state $f$ of an evolution process $f_t = A_t f$ is to be recovered from a combined set of coarse samples from varying time levels $\{t_1, \ldots, t_N\}$. This new way of sampling, which we call dynamical sampling, differs from standard sampling since at any fixed time $t_i$ there are not enough samples to recover the function $f$ or the state $f_{t_i}$. Although dynamical sampling is an inverse problem, it differs from the typical inverse problems in which $f$ is to be recovered from $A_T f$ for a single time $T$. In this paper, we consider signals that are modeled by $\ell^2(\mathbb{Z})$ or a shift invariant space $V \subset L^2(\mathbb{R})$.

1. Introduction.

In sampling theory we seek to reconstruct a function $f$ from its samples $\{f(x_j) : x_j \in X\}$ where $X \subset \mathbb{R}$ is a countable set. Perhaps the most well-known result is the Shannon Sampling Theorem [15]. Specifically, if a function $f \in L^2(\mathbb{R})$ is $T$-bandlimited, i.e., its Fourier transform

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x} dx, \quad \xi \in \mathbb{R},$$

has support contained in $[-T, T]$, then

$$(1.1) \quad f(x) = \sum_{n=-\infty}^{\infty} f\left(\frac{n}{2T}\right) \sin \frac{\pi (2Tx - n)}{\pi (2Tx - n)} = \sum_{n=-\infty}^{\infty} f\left(\frac{n}{2T}\right) \text{sinc}(2Tx - n), \quad x \in \mathbb{R},$$

where the series converges in $L^2(\mathbb{R})$ and uniformly on compact sets. Thus, in this situation, $f$ can be recovered from its samples $\{f\left(\frac{n}{2T}\right) : n \in \mathbb{Z}\}$.

However, there are many situations in which the sampling of a function $f$ is restricted. For example, assume that we wish to find a function $f$ in $[0, 1]$ at time $t = 0$ at a spatial resolution of 0.1, i.e., we need to know the values of $f$ on the set $\{0.1k : k = 1, \ldots, 10\}$. Practical considerations, however, dictate that we can only use two sampling devices, i.e., we can only sample at two locations $\{x_1, x_2\} \subset \{0.1k : k = 0, \ldots, 10\}$. Can we still recover $f$ at the spatial resolution of 0.1? The answer is that it may be possible to determine $f$ at the correct resolution if we know that $f$ is evolving in time under the action of a known

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operator, such as diffusion. This toy problem illustrates a new type of sampling problems that we call dynamical sampling.

**The dynamical sampling problem.** Assume that a function \( f \) on a domain \( D \) is an initial state of a physical process evolving in time under the action of a family of operators \( A_t \) indexed by \( t \geq 0 \). Can we recover \( f \) from the samples \( \{f(X), f_{t_1}(X), \ldots, f_{t_N}(X)\} \) of \( f \) on \( X \subset D \) and its various states \( f_t(X) := (A_t f)(X) \) at times \( \{t_1, \ldots, t_N\} \)?

Figure 1 gives an example of a spatio-temporal sampling set for a finite domain. More general problems encountered in applications can also be stated. For example, the operators \( A_t \) may be unknown or only partially known, and the sampling set \( X \) may be a function of \( t \) so that the samples are given by \( \{f(X), f_{t_1}(X_1), \ldots, f_{t_N}(X_N)\} \). This is a natural problem for wireless sensor networks (WSN), where a large number of sensor nodes are deployed over a physical region to monitor a physical phenomenon such as temperature, pollution concentration, or pressure. In [2], numerous examples of applications of WSN are given in military, environmental, health, and home and commercial areas. Although several approaches for the reconstruction of signals from WSN samples have emerged recently [3, 11, 12, 13, 14], most of them do not take into account the evolutionary nature of the sampled processes. A different approach that does exploit the
evolutionary aspect of these problems has been proposed and studied in [8, 9, 10] and inspired our current research.

Although dynamical sampling aims to recover a function from samples, it differs from standard sampling problems since it is not only the function $f$ that is sampled but also its various states at different times ($\{t_0, t_1, \ldots, t_N\}$). Moreover, it is assumed that at any fixed time $t_i$ there are not enough samples to recover the function $f$ or its state $f_{t_i}$. Although dynamical sampling is an inverse problem, it differs from the typical inverse problems in which $f_T(X) = (A_T f)(X)$ is known at a single time $T$.

In this paper we will concentrate on a few special cases of the general dynamical sampling problem. In particular, we will assume that the initial function $f$ that we want to recover belongs to $\ell^2(\mathbb{Z})$ or a shift invariant space $V(\phi)$ described below. Furthermore, we will assume that the family of operators $A_t$ acting on the initial state $f$ is spatially invariant, i.e., it is independent of (the absolute) position. This means that for each fixed $t$ we have $A_t f = a_t * f$, that is $A_t$ is a convolution operator. We also assume time invariance in the form $A_{t_1+t_2} = A_{t_1} A_{t_2}$. Additional assumptions on the sampling set $X$ will also be made. These assumptions allow us to use Fourier techniques and simplify some of the calculations.

1.1. Organization. The paper is organized as follows. Section 2 is devoted to necessary and sufficient conditions for solvability of the dynamical sampling problem and has three subsections. The first of these states dynamical sampling results in $\ell^2(\mathbb{Z})$ and the second states results in shift invariant spaces. The last subsection contains the proofs of the results. Section 3 deals with the estimates for the reconstruction error in the presence of additive white noise. As in the previous section, we first state the theorems and then provide the proofs in a separate subsection.

2. Dynamical sampling in $\ell^2(\mathbb{Z})$ and shift invariant spaces.

By $f \in \ell^2(\mathbb{Z})$ we model an unknown spatial signal at time $t = 0$. Let $a \in \ell^2(\mathbb{Z})$ represent the kernel of an evolution operator so that the signal at time $t = n$ is given by $a^n * x = (a * \ldots * a) * x$. By $S_m : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ we denote the operator of subsampling by a factor of $m$ so that $(S_m z)(k) = z(mk)$. The dynamical sampling problem under these assumptions can be stated as follows:

Under what conditions on $a$, $m$, and $N$ can a function $f \in \ell^2(\mathbb{Z})$ be recovered from the samples $\{S_m f, S_m(a * f), \ldots, S_m(a^N * f)\}$ of $f$, or, equivalently, from $\{f(X), (a * f)(X), \ldots, (a^N * f)(X)\}$, $X = m\mathbb{Z}$?

If we let $y_m = S_m(a^{n-1} * f)$, $n = 1, \ldots N$, we can rephrase the problem by writing it in the form:

$$y = Af,$$
where $A$ is the operator from $\ell^2(\mathbb{Z})$ to $(\ell^2(\mathbb{Z}))^N$, such that $y = (y_1, \ldots, y_N) = (S_m f, S_m(a \ast f), \ldots, S_m(a^{N-1} \ast f))$.

In order to stably recover $f$, the operator $A$ must have a bounded left inverse. This means that there must exist an operator $B$ from $(\ell^2(\mathbb{Z}))^N$ to $\ell^2(\mathbb{Z})$ such that $BA = I$. In particular $A$ must be injective and its range, $\text{ran} A$, must be closed. In the theorem below we provide necessary and sufficient conditions for such an inverse to exist in terms of the Fourier transform $\hat{a}$ of the filter $a \in \ell^2(\mathbb{Z})$. For $a \in \ell^1(\mathbb{Z})$ the Fourier transform is defined on the torus $\mathbb{T} \simeq [0, 1)$ by

$$\hat{a}(\xi) = \sum_{n \in \mathbb{Z}} a(n)e^{-2\pi in\xi}, \xi \in \mathbb{T}.$$  

**Theorem 2.1.** Assume that $\hat{a} \in L^\infty(\mathbb{T})$ and define

$$A_m(\xi) = \begin{pmatrix}
1 & 1 & \ldots & 1
\hat{a}(\xi) & \hat{a}(\xi + 1) & \ldots & \hat{a}(\xi + m - 1)
\vdots & \vdots & \ddots & \vdots
\hat{a}(m-1)(\xi) & \hat{a}(m-1)(\xi + 1) & \ldots & \hat{a}(m-1)(\xi + m - 1)
\end{pmatrix},$$

$\xi \in \mathbb{T}$. Then $A$ in (2.1) has a bounded left inverse for some $N \geq m - 1$ if and only if there exists $\alpha > 0$ such that the set $\{\xi : |\det A_m(\xi)| < \alpha\}$ has zero measure. Consequently, $A$ in (2.1) has a bounded left inverse for some $N \geq m - 1$ if and only if $A$ has a bounded left inverse for all $N \geq m - 1$.

Thus, under the conditions of Theorem 2.1 a vector $f \in \ell^2(\mathbb{Z})$ satisfying (2.1) can be recovered in a stable way from the measurements $y_n, n = 1, \ldots, N$, for any $N \geq m - 1$. We shall see in the proof that in the case $N = m - 1$ the operator $A$ is, in fact, invertible and not just left invertible. For the case $N < m - 1$, the operator $A$ is not injective and hence no recovery of $f$ is possible. We also note that if a signal $f$ cannot be recovered from the dynamical samples in Theorem 2.1 then taking additional samples at the same spatial locations will not help. The same phenomenon was observed in [9].

In the special case when $\hat{a}$ is continuous on $\mathbb{T}$, $|\det A_m(\xi)|$ is a continuous function over the compact set $\mathbb{T}$. Therefore, an $\alpha$ in Theorem 2.1 exists if and only if $|\det A_m(\xi)| \neq 0$ for all $\xi \in \mathbb{T}$. We capture this fact in the corollary below.

**Corollary 2.2.** Suppose $\hat{a} \in C(\mathbb{T})$. Then $A$ in (2.1) has a bounded left inverse for some (and, hence, all) $N \geq m - 1$ if and only if $|\det A_m(\xi)| \neq 0$ for all $\xi \in \mathbb{T}$.

Although Theorem 2.1 gives necessary and sufficient conditions on convolution operators on $\ell^2(\mathbb{Z})$ for this special case of dynamical sampling problem to be solvable, many typical operators encountered in physical systems or in applications do not satisfy these conditions. For example, a typical convolution operator is such that $\hat{a}$ is real, symmetric, continuous, and strictly decreasing on $[0, \frac{1}{2}]$. The following corollary shows that the dynamical sampling problem cannot be solved in this case without additional samples.
Figure 2. An example of a stable sampling scheme in Theorem 2.4 with $m = 5$ and $n = 7$. The sampling locations are marked by crosses and the extra samples at $t = 0$ are marked as crosses inside squares.

**Proposition 2.3.** If $\hat{a}$ is real, symmetric, continuous, and strictly decreasing on $[0, \frac{1}{2}]$, then $A_m(\xi)$ is singular if and only if $\xi \in \{0, \frac{1}{2}\}$.

Because $A_m(0)$ and $A_m(\frac{1}{2})$ are not invertible, we cannot solve (2.1). To make reconstruction possible in this case, the sampling set needs to be modified or expanded. This can be done in the following way. Let $T_c$ be the operator that shifts a vector $z \in \ell^2(\mathbb{Z})$ to the right by $c$ units so that $T_c z(k) = z(k - c)$. Let also $S_{mn} T_c$ represent shifting by $c$ and then sampling by $mn$ for some $n \in \mathbb{N}$.

**Theorem 2.4.** Suppose $\hat{a}$ is real, symmetric, continuous, and strictly decreasing on $[0, \frac{1}{2}]$, $n$ is odd, and $\Omega = \{1, \ldots, \frac{m-1}{2}\}$. Then the extra sampling given by $\{S_{mn} T_c\}_{c \in \Omega}$ provides enough additional information to stably recover $f$.

Figure 2 illustrates a sampling set for stable reconstruction.

**Remark 2.1.** Note that the extra samples in Theorem 2.4 needed to recover $f$ are chosen as $S_{mn} T_c f$. However, it may be natural to also include the samples on $X_c = mn \mathbb{Z} + c$ at $t = 1, \ldots, N$ for each $c \in \Omega$. In fact, if we have the samples of $f, Af, \ldots, A^N f$ on $X \cup \bigcup_{c \in \Omega} X_c$, we can expect the recovery process to be more stable.

**Remark 2.2.** Theorems 2.1 and 2.4 parallel the finite dimensional results we obtained in [3]. For example, one can use more complicated choices for $\Omega \subset \{1, \ldots, mn - 1\}$ in Theorem 2.4 and the admissible choices are determined by the same equivalence relations as in the finite dimensional case (see [3] for more details). Moreover, some of the the methods we use for obtaining stability results in Section 3 are the same as in [3]. There are, however, subtle but important differences in the infinite dimensional case. For example, in Theorem 2.4, the dynamical samples without the samples in the extra sampling set $\Omega$ still form a uniqueness set (the operator $A$ has a trivial kernel). The latter was not the case in [3].
2.1. Dynamical Sampling in Shift-Invariant Spaces. Signals are not always modeled by $\ell^2(\mathbb{Z})$. For example, analog functions are often assumed to belong to a Shift-Invariant Space (SIS) $V(\phi)$, $\phi \in L^2(\mathbb{R})$, defined by

$$
V(\phi) = \left\{ \sum_{k \in \mathbb{Z}} c_k \phi(\cdot - k) : (c_k)_{k \in \mathbb{Z}} \in \ell^2(\mathbb{Z}) \right\}.
$$

The dynamical sampling problem in shift-invariant spaces is to reconstruct the function $f \in V(\phi)$ from the coarse samples $\{g_0 = S(\Omega_0)f, \ g_n = S_mA^{n-1}f, \ n = 1, \ldots, N\}$, where $\Omega_0$ is a “small” and possibly empty extra sampling set. Here $S_m g = g(m\cdot)$, $g \in L^2(\mathbb{R})$. Although all separable Hilbert spaces are isometrically isomorphic, the dynamical sampling problem in SIS is not always reducible to that in $\ell^2(\mathbb{Z})$. The reason for this phenomenon is that a convolution operator $A$ acting on a function $f \in V(\phi)$ does not necessarily result in a function $a \ast f$ that belongs to $V(\phi)$. For the case of $(\frac{1}{2})$-bandlimited functions $V(\text{sinc})$, we have that $a \ast f \in V(\text{sinc})$ for any $f \in V(\text{sinc})$, and, in this case, the dynamical sampling in $V(\text{sinc})$ does reduce to that in $\ell^2(\mathbb{Z})$. The reduction is done in the following way. Let $\hat{a} \in L^\infty(\mathbb{R})$ and define $\hat{b} = \hat{a}\chi_{[\frac{1}{2}, \frac{3}{2}]}$ and $x = f|_Z$. Then the maps $f \mapsto x$ and $a \ast f \mapsto b \ast x$ from $V(\text{sinc})$ to $\ell^2(\mathbb{Z})$ are isometric isomorphisms; $b \in \ell^2(\mathbb{Z})$ is the inverse discrete Fourier transform of $\hat{b} \in L^2(\mathbb{T})$. Thus, solving the dynamical sampling problem in $V(\text{sinc})$ with a convolution operator defined by a filter $a$ such that $\hat{a} \in L^\infty(\mathbb{R})$ is equivalent to solving the corresponding dynamical sampling problem for $x = f|_Z$ in $\ell^2(\mathbb{Z})$ with the convolution operator defined by the filter $b \in \ell^2(\mathbb{Z})$.

There are other SIS for which the dynamical sampling problem is reducible to that in $\ell^2(\mathbb{Z})$. Necessary and sufficient conditions for the simple reduction similar to the one described above are presented in [1]. In particular, if $\phi$ belongs to the Wiener amalgam space $W_0(L^1) = W(C, \ell^1)$ [4], $\{\phi(\cdot - k) : k \in \mathbb{Z}\}$ forms a Riesz basis for $V(\phi)$, and $\sum_k \hat{\phi}(\xi + k) \neq 0$, then any of the three equivalent conditions below are sufficient for the reduction to the $\ell^2(\mathbb{Z})$ case:

1. $a \ast \phi \in V(\phi)$;
2. $a \ast V(\phi) \subseteq V(\phi)$;
3. There exists a function $\hat{b} \in L^2[0, 1]$ such that for every $k \in \mathbb{Z}$,

$$
\hat{a}(\xi + k)\hat{\phi}(\xi + k) = \hat{b}(\xi)\hat{\phi}(\xi + k) \quad a.e. \in [0, 1].
$$

When the dynamical sampling problem is not reducible to the $\ell^2(\mathbb{Z})$ case, a similar approach can be followed. Let $f = \sum_k c_k \phi(\cdot - k) \in V(\phi)$, $a^j := a \ast a \ast \ldots \ast a$ ($j - 1$ convolutions), $\phi_j := a^j \ast \phi$, and $\Phi_j := \phi_j|_Z$. 


Letting \( (y_l)(k) := \left(S_m(a^* f)\right)(k), k \in \mathbb{Z}, l = 0, \ldots, m - 1, \) and using calculations similar to the ones described below for the case of \( \ell^2(\mathbb{Z}) \), we get

\[
\begin{pmatrix}
\hat{y}_0(\xi) \\
\hat{y}_1(\xi) \\
\vdots \\
\hat{y}_{m-1}(\xi)
\end{pmatrix} =
\begin{pmatrix}
\hat{\Phi}_0(\frac{\xi}{m}) & \hat{\Phi}_0(\frac{\xi+1}{m}) & \cdots & \hat{\Phi}_0(\frac{\xi+m-1}{m}) \\
\hat{\Phi}_1(\frac{\xi}{m}) & \hat{\Phi}_1(\frac{\xi+1}{m}) & \cdots & \hat{\Phi}_1(\frac{\xi+m-1}{m}) \\
\vdots & \vdots & \ddots & \vdots \\
\hat{\Phi}_{m-1}(\frac{\xi}{m}) & \hat{\Phi}_{m-1}(\frac{\xi+1}{m}) & \cdots & \hat{\Phi}_{m-1}(\frac{\xi+m-1}{m})
\end{pmatrix}
\begin{pmatrix}
\hat{c}(\frac{\xi}{m}) \\
\hat{c}(\frac{\xi+1}{m}) \\
\vdots \\
\hat{c}(\frac{\xi+m-1}{m})
\end{pmatrix}.
\]

In short notation, we have

\[(2.4) \quad \hat{y}(\xi) = \hat{A}_m(\xi)\hat{c}_m(\xi).\]

It is now easy to see how to get the results corresponding to Theorem 2.1 and Corollary 2.2 for the case of SIS. In particular, if \( \phi \in W_0(L^1) \) and \( a \in W(L^1) \), then \( \hat{\Phi}_j \in C(\mathbb{T}) \) for \( j = 1, \ldots, m \), and a vector \( f \in V(\phi) \) can be recovered in a stable way from the measurements \( y_n, n = 0, \ldots, m - 1 \), if and only if \( \det \hat{A}_m(\xi) \neq 0 \) for all \( \xi \in [0, 1) \). We refer to [1] for more details on the subject.

As in the \( \ell^2(\mathbb{Z}) \) case, there are many situations in practice for which the hypotheses of Theorem 2.1 are not satisfied and additional samples are needed. For example, when both \( \hat{a} \) and \( \hat{\phi} \) are real and symmetric, forcing \( \hat{A}_m(\xi) \) to be singular at \( \xi = 0, \frac{1}{2} \), as well as possibly other values of \( \xi \). In special cases, the number of additional samples and their locations may be determined from the Theorem 2.5 below.

As before, \( T_c, c \in \mathbb{Z} \), are the operators that shift a vector in \( \ell^2(\mathbb{Z}) \) to the right by \( c \) units so that \( T_c z(k) = z(k - c) \), and \( S_m T_c \) represent shifting by \( c \) followed by sampling on \( mn\mathbb{Z} \) for some positive integer \( n \).

**Theorem 2.5.** Suppose \( \hat{A}_m(\xi) \) is singular only when \( \xi \in \{\xi_i\}_{i \in I} \) with \( |I| < \infty \). Let \( n \) be a positive integer such that \( |\xi_i - \xi_j| \neq \frac{k}{n} \) for any \( i, j \in I \) and \( k \in \{1, \ldots, n-1\} \). Then the extra samples given by \( \{S_{mn}T_c\}_{c \in \{1,\ldots,m-1\}} \) provide enough additional information to stably recover any \( f \in V(\phi) \).

**Remark 2.3.** The finite nature of \( I \) guarantees the existence of an \( n \) satisfying the conditions of Theorem 2.5. The proof of Theorem 2.5 is similar but simpler than that of Theorem 2.4 and will be omitted (see also [1]).

### 2.2. Proofs for Section 2

The following Lemma is useful for proving Theorem 2.1.

**Lemma 2.6.** Suppose \( \mathcal{A} : (L^2(\mathbb{T}))^m \to (L^2(\mathbb{T}))^n \) is defined by \( (\mathcal{A}x)(\xi) = A(\xi)x(\xi) \) where the map \( \xi \mapsto A(\xi) \) from \( \mathbb{T} \) to the space of \( n \times m \) matrices \( \mathcal{M}_{nm} \) is measurable. Then \( \|\mathcal{A}\|_{op} = \text{ess sup}_{\xi} \|A(\xi)\|_{op} \).
Proof. The proof is standard.

Suppose \( \text{ess sup}_T \| A(\xi) \|_{op} = \alpha < \infty \) and let \( z \in (L^2(\mathbb{T}))^m \) be such that \( \| z \|_{(L^2(\mathbb{T}))^m} = 1 \). Then it is easy to see that \( \| \mathcal{A} z \|_{(L^2(\mathbb{T}))^m} \leq \alpha^2 \) and, therefore, \( \| \mathcal{A} \|_{op} = \sup_{\| z \|_{(L^2(\mathbb{T}))^m} = 1} \| \mathcal{A} z \|_{(L^2(\mathbb{T}))^m} \leq \alpha \).

To prove the opposite inequality, let \( \epsilon > 0 \) and \( B = \{ \xi : \| A(\xi) \|_{op} \geq \alpha - \epsilon \} \). Using the singular value decomposition we write \( A(\xi) = U(\xi)\Sigma(\xi)V^*(\xi) \) where \( U(\xi) \) is an \( n \times n \) unitary matrix, \( \Sigma(\xi) \) is an \( n \times m \) matrix with nonnegative, real entries on the diagonal, and \( V(\xi) \) is an \( m \times m \) unitary matrix. We assume that the diagonal entries, \( s_i(\xi) \) of \( \Sigma(\xi) \), called singular values of \( A(\xi) \), are listed in descending order. Then \( \| A(\xi) \|_{op} = \sqrt{s_1(\xi)} \). Let \( v_1(\xi) \) be the first column vector of \( V(\xi) \), and define

\[
(2.5) \quad z(\xi) = \frac{1}{\sqrt{|B|}} \chi_B(\xi)v_1(\xi),
\]

where \( \chi_B \) is the characteristic function of the set \( B \). Since the function \( z \) is measurable we get

\[
(2.6) \quad \| \mathcal{A} z \|^2 = \frac{1}{|B|} \int_B |A(\xi) z(\xi)|^2 d\xi = \frac{1}{|B|} \int_B |\sigma_1(\xi)|^2 d\xi \geq (\alpha - \epsilon)^2.
\]

Thus, \( \| \mathcal{A} \|_{op} \geq \alpha \).

Assume now that \( \text{ess sup}_T \| A(\xi) \|_{op} = \infty \). Fix \( N > 0 \). Then the set \( B = \{ \xi : \| A(\xi) \|_{op} \geq N \} \) has positive measure. Repeating the process above, we find a function \( z_N \) of unit norm in \( (L^2(\mathbb{T}))^m \) such that \( \| \mathcal{A} z_N \|_{(L^2(\mathbb{T}))^m} \geq N \). Since \( N \) was arbitrary, we conclude that \( \| \mathcal{A} \|_{op} = \infty \). In particular, \( \mathcal{A} \) is a bounded operator if and only if \( \text{ess sup}_T \| A(\xi) \|_{op} < \infty \).

Assume now that the matrix \( A(\xi) \) in the theorem above has a bounded left inverse for almost every \( \xi \), denoted \( \mathcal{A}^\ell(\xi) \). Then a left inverse \( \mathcal{A}^\ell \) can be defined on the range of \( \mathcal{A} \) by \( (\mathcal{A}^\ell y)(\xi) = A^\ell(\xi)y(\xi) \). However, in this case, \( \mathcal{A}^\ell \) will be a bounded operator if and only if the range of \( \mathcal{A} \) is closed.

Proof of Theorem 2.1. Using the fact that

\[
\sum_{l=0}^{m-1} e^{i2\pi l j/m} = \begin{cases} m, & j = 0 \mod m \\ 0, & \text{otherwise} \end{cases},
\]

we get the Poisson summation formula

\[
(2.7) \quad (S_m z)^\wedge(\xi) = \frac{1}{m} \sum_{l=0}^{m-1} \hat{z}\left(\frac{\xi + l}{m}\right), \quad \xi \in \mathbb{T}, \quad z \in \ell^2(\mathbb{Z}).
\]

Let \( G : L^2(\mathbb{T}) \to (L^2(\mathbb{T}))^m \) be given by

\[
(2.8) \quad (G z)(\xi) = \frac{1}{\sqrt{m}} \left( z(\frac{\xi}{m}), z(\frac{\xi + 1}{m}), \ldots, z(\frac{\xi + m - 1}{m}) \right)^T.
\]
Taking the Fourier transform of \((2.1)\) we get
\[
(2.9) \quad m \begin{pmatrix}
\hat{y}_1(\xi) \\
\hat{y}_2(\xi) \\
\vdots \\
\hat{y}_N(\xi)
\end{pmatrix} = A_m(\xi) \begin{pmatrix}
\hat{f}(\frac{\xi}{m}) \\
\hat{f}(\frac{\xi+1}{m}) \\
\vdots \\
\hat{f}(\frac{\xi+m-1}{m})
\end{pmatrix},
\]
or, using a more compact notation,
\[
(2.10) \quad \bar{y}(\xi) = \frac{1}{m} A_m(\xi) \bar{x}(\xi),
\]
where \(\bar{x} = \sqrt{m} \hat{f}.\)

Define the operator \(A : (L^2(\mathbb{T}))^N \to (L^2(\mathbb{T}))^m\) by \((A \bar{x})(\xi) = A_m(\xi) \bar{x}(\xi),\) where
\[
A_m(\xi) = \begin{pmatrix}
1 & 1 & \ldots & 1 \\
\hat{a}(\frac{\xi}{m}) & \hat{a}(\frac{\xi+1}{m}) & \ldots & \hat{a}(\frac{\xi+m-1}{m}) \\
\vdots & \vdots & \ddots & \vdots \\
\hat{a}^{(N-1)}(\frac{\xi}{m}) & \hat{a}^{(N-1)}(\frac{\xi+1}{m}) & \ldots & \hat{a}^{(N-1)}(\frac{\xi+m-1}{m})
\end{pmatrix}.
\]

Since \(G\) is an isometric isomorphism, the signal \(f\) can be recovered from \(\bar{y}\) in a stable way if and only if the operator \(A\) has a bounded left inverse. Now it is easy to see that the operator \(A\) has a bounded left inverse for some \(N \geq m - 1,\) if and only it has a bounded (left) inverse for \(N = m - 1.\) The latter happens if and only if there exists \(\alpha > 0\) such that the set \(\{\xi : |\det A_m(\xi)| < \alpha\}\) has zero measure. □

**Proof of Proposition 2.3.** The Vandermonde matrix \(A_m(\xi)\) in \((2.2)\) is singular if and only if two of its columns coincide. Suppose the \(j\)-th and \(l\)-th columns coincide and \(j < l.\) Then \(\hat{a}(\frac{\xi}{m}) = \hat{a}(\frac{\xi+l}{m}).\) The symmetry and monotonicity conditions on \(\hat{a}\) imply that \(\frac{\xi+l}{m} = 1 - \frac{\xi+l}{m}.\) Then \(\xi = \frac{m-j-l}{2}.\) Observing that \(m-j-l \in \mathbb{Z}\) and \(\xi \in \mathbb{T},\) we conclude that \(\xi \in \{0, \frac{1}{2}\}.\) □

**Proof of Theorem 2.4.** We begin with a few useful formulas and notation. Combining the identity \((2.7)\) with the identity
\[
(2.12) \quad (T_c f)^\wedge(\xi) = e^{-i2\pi c \xi} \hat{f}(\xi)
\]
we get
\[
(S_{mn}T_c f)^\wedge(n\xi) = \frac{1}{mn} e^{-i2\pi \xi \frac{m}{m}} \sum_{l=0}^{m-1} e^{-i2\pi c \xi \frac{l}{m}} \hat{f}(\frac{\xi}{m} + \frac{l}{mn})
\]
\[
= \frac{1}{mn} e^{-i2\pi \xi \frac{m}{m}} \sum_{k=0}^{n-1} e^{-i2\pi c \xi \frac{k}{mn}} \sum_{j=0}^{m-1} e^{-i2\pi \xi \frac{j}{m}} \hat{f}(\frac{\xi+j}{m} + \frac{k}{mn}).
\]
2.3. Next, notice that for a fixed \( \xi \) rank for every \( \hat{\xi} \) is sufficient for the recovery of \( \bar{S} \) using the notation of (2.10) and defining the row vector
\[
\bar{u}(k) = e^{-i2\pi\xi k/m}(1, e^{-i2\pi\xi/m}, e^{-i4\pi\xi/m}, \ldots, e^{-i2\pi(\xi(m-1))/m}),
\]
we have
\[
(S_{nm}T_\xi f)^(n\xi) = \frac{1}{mn} e^{-i2\pi\xi n/m} \sum_{k=0}^{n-1} \bar{u}(k) \bar{x}(\xi + \frac{k}{n}),
\]
where \( \bar{x}(\xi) = \left( \hat{f}(\xi/m), \hat{f}(\xi+1/m), \ldots, \hat{f}(\xi+m-1/m) \right)^T = \sqrt{m} G \hat{f} \) as before.

We consider an initial extra sampling set \( \Omega = \{1, \ldots, \frac{m-1}{2}\} \). Combining the dynamical samples with the additional initial samples we have
\[
\begin{pmatrix}
ne^{i2\pi\xi/m}(S_{nm}T_1x)^(n\xi) \\
\vdots \\
ne^{i2\pi\xi(m-1)/2m}(S_{nm}T_{(m-1)x})^(n\xi) \\
\bar{y}(\xi) \\
\bar{y}(\xi + \frac{1}{n}) \\
\vdots \\
\bar{y}(\xi + \frac{n-1}{n})
\end{pmatrix} = A_\Omega(\xi)
\begin{pmatrix}
\bar{x}(\xi) \\
\bar{x}(\xi + \frac{1}{n}) \\
\vdots \\
\bar{x}(\xi + \frac{n-1}{n})
\end{pmatrix},
\]
where \( A_\Omega \) is given by
\[
A_\Omega(\xi) = \begin{pmatrix}
\bar{u}_1(0) & \bar{u}_1(1) & \ldots & \bar{u}_1(n-1) \\
\vdots & \vdots & \ddots & \vdots \\
\bar{u}_{m-1}(0) & \bar{u}_{m-1}(1) & \ldots & \bar{u}_{m-1}(n-1) \\
A_m(\xi) & 0 & \ldots & 0 \\
0 & A_m(\xi + \frac{1}{n}) & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & A_m(\xi + \frac{n-1}{n})
\end{pmatrix}.
\]

If \( A_\Omega(\xi) \) has full column rank, then it has a left inverse. By Lemma 2.6 and the fact that \( \hat{a} \) is continuous, it suffices to show that the matrix \( A(\xi) \) has full rank for every \( \xi \in [0, \frac{1}{n}] \); it is not difficult to see that solving (2.15) for \( \xi \in [0, \frac{1}{n}] \) is sufficient for the recovery of \( \bar{x}(\xi) \) for all \( \xi \in [0, 1] \).

First, if \( \xi + \frac{k}{n} \notin \{0, \frac{1}{2}\} \), \( k = 0, \ldots, n-1 \), the solvability is implied by Proposition 2.3. Next, notice that for a fixed \( \xi \in [0, \frac{1}{n}] \), we have \( \xi + \frac{k}{n} \notin \{0, \frac{1}{2}\} \) for at most one \( k = 0, \ldots, n-1 \). This follows from the parity of \( n \) (\( n \) is assumed to be odd). Therefore, for any \( \xi \in [0, \frac{1}{n}] \) there is at most one singular block \( A_m \) in \( A_\Omega(\xi) \). This allows us to consider the singularities of \( A_m(0) \) and \( A_m(\frac{1}{2}) \) separately.

Because of the block diagonal structure of the lower portion of \( A_\Omega(\xi) \), we can focus only on showing that the additional samples eliminate any singularities created by \( A_m(0) \) and \( A_m(\frac{1}{2}) \).
For a fixed $k = 0, \ldots, \frac{m-1}{2}$, we define the $\left(\frac{m-1}{2}\right) \times m$ matrix

$$U_k = \begin{pmatrix} \bar{u}_1(k) \\ \vdots \\ \bar{u}_{\frac{m-1}{2}}(k) \end{pmatrix}.$$ 

Since

$$\langle U_k(c, \cdot), U_k(d, \cdot) \rangle = \sum_{j=0}^{m-1} e^{-\frac{2\pi i c j}{m} n} e^{-\frac{2\pi i d j}{m}} = e^{-\frac{2\pi i (c-d) k}{m n}} \sum_{j=0}^{m-1} e^{-\frac{2\pi i (c-d) j}{m}} = \begin{cases} me^{-\frac{2\pi i (c-d) k}{m n}}, & (c - d) = 0 \mod m \\ 0, & \text{otherwise} \end{cases},$$

the rows of the matrix $U_k$ form an orthogonal set, and we conclude that it has full rank. Next, we show that $\left( A_m(\xi + \frac{k}{n}) \right)$ has a trivial kernel and, hence, full rank. A vector is in $\ker \left( A_m(\xi + \frac{k}{n}) \right)$ if and only if it is in the kernels of both $A_m(\xi + \frac{k}{n})$ and $U_k$. Therefore, we only need to consider $\xi + \frac{k}{n} \in \{0, \frac{1}{2}\}$.

Under the conditions of Proposition 2.3, we can completely characterize the kernels of $A_m(0)$ and $A_m(\frac{1}{2})$. For simplicity, we assume $m$ is odd and begin indexing the columns of $A_m(\xi)$ at zero. When $\xi = 0$, the $l$-th column of the Vandermonde matrix $A_m(0)$ is found by evaluating $\hat{a}$ at $\frac{l}{m}$. By the symmetry and 1-periodicity of $\hat{a}$, we have $\hat{a}(\frac{l}{m}) = \hat{a}(\frac{m-l}{m})$. Therefore, the $j$-th and $(m-j)$-th columns of $A_m(0)$ coincide, and the kernel of $A_m(0)$ has dimension $\frac{m-1}{2}$. Similarly, the $j$-th and $(m-j-1)$-th columns of $A_m(\frac{1}{2})$ coincide for $j = 0, \ldots, \frac{m-3}{2}$ and and the kernel of $A_m(\frac{1}{2})$ also has dimension $\frac{m-1}{2}$.

The vector $\bar{v}_j$ with a 1 in the $j$-th position, a $(-1)$ in the $(m-j)$-th position, and zeros elsewhere is in the kernel of $A_m(0)$. Since there are exactly $\frac{m-1}{2}$ of such vectors, the kernel is their span:

$$\ker A_m(0) = \text{span}\{\bar{v}_j\}_{j=1}^{\frac{m-1}{2}} = \text{span}\{\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ -1 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 1 \\ -1 \\ 0 \end{pmatrix}\}.$$
Similarly, the $j$-th and $(m - j - 1)$-th columns of $A_m(\frac{1}{2})$ coincide for $j = 0, \ldots, \frac{m-3}{2}$, and the vector $\bar{w}_j$ with a 1 in the $j$-th position, a $(-1)$ in the $(m - j - 1)$-th position, and zeros elsewhere is in the kernel of $A_m(\frac{1}{2})$. Therefore,
\begin{equation}
\ker A_m(\frac{1}{2}) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \\ \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 1 \\ \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ -1 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ \end{pmatrix} \right\} = \text{span}\{\bar{w}_j\}_{j=0}^{\frac{m-3}{2}}.
\end{equation}

Suppose $\bar{x} \in \ker A_m(0)$. Then $\bar{x} = \sum_{j=1}^{\frac{m-1}{2}} \alpha_j \bar{v}_j$, where $\bar{v}_j$ is defined in (2.18). We want to know if the equation $0 = U_k \bar{x} = \sum_{j=1}^{\frac{m-1}{2}} \alpha_j U_k \bar{v}_j$ has a unique (trivial) solution. This happens if and only if the matrix $B = U_k \left( \bar{v}_1 \ldots \bar{v}_{\frac{m-1}{2}} \right)$ has full rank. Computing the $(c, j)$ entry of $B$, we have
\begin{equation}
B(c, j) = e^{-\frac{i2\pi ck}{m}} (e^{-\frac{i2\pi c}{m}j} - e^{-\frac{i2\pi c}{m}(m-j)})
= e^{-\frac{i2\pi c}{m}j} (e^{-\frac{i2\pi c}{m}j} - e^{\frac{i2\pi c}{m}j})
= 2e^{-\frac{i2\pi c}{m}j} \sin\left(\frac{2\pi}{m}cj\right).
\end{equation}

Note that $\{1, \cos\left(\frac{2\pi}{m}cj\right), \sin\left(\frac{2\pi}{m}cj\right) : c = 1, \ldots, \frac{m-1}{2}\}$ is the Fourier basis for $\mathbb{C}^m$.

Thus, $\{\sin\left(\frac{2\pi}{m}cj\right) : c = 0, \ldots, \frac{m-1}{2}\}$ are linearly independent in $\mathbb{C}^m$. Using the fact that $\{\sin\left(\frac{2\pi}{m}cj\right) : c = 0, \ldots, \frac{m-1}{2}\}$ are odd functions, it follows that $\{\sin\left(\frac{2\pi}{m}cj\right) : c = 0, \ldots, \frac{m-1}{2}\}$ form a basis of $\mathbb{C}^{\frac{m-1}{2}}$. Therefore, the $\frac{m-1}{2} \times \frac{m-1}{2}$ matrix $B$ does, indeed, have full rank.

Similarly, for the case $A_m(\frac{1}{2})$, we consider the matrix $D = U_k \left( \bar{w}_0 \ldots \bar{w}_{\frac{m-3}{2}} \right)$. Its entries are
\begin{equation}
D(c, j) = 2e^{-\frac{i2\pi c}{m}k} e^{-\frac{i\pi c}{m}} \sin\left(\frac{2\pi}{m}c(2j+1)\right),
\end{equation}

and, therefore, $D$ has full rank.

Thus, the matrix $A_\Omega$ has a bounded left inverse for every $\xi \in \mathbb{T}$ and the theorem is proved. \hfill \Box

3. Stability in the Presence of Additive Noise

In this section, we assume that $\hat{a}$ and $\Omega$ satisfy the hypotheses of Theorem 2.4 and consider the recovery of the signal $f$ in the presence of additive noise. The minimal extra sampling set $\Omega$ in Theorem 2.4 allows us to stably recover any signal $f \in \ell^2(\mathbb{Z})$. In the presence of additive Gaussian white noise, however, any linear recovery method does not generally reproduce the original function.
Suppose $w$ worsen. The expected discrepancy, $\hat{f} - f$, between the recovered function $\hat{f}$ and the original function $f$ is controlled by the norm of the operator $A_1^\dagger : (L^2(\mathbb{T}))^{m+|\Omega|} \rightarrow (L^2(\mathbb{T}))^{m}$ defined by $(A_1^\dagger y)(\xi) = A_1^\dagger(\xi)y(\xi)$, where $A_1^\dagger(\xi)$ is the Moore-Penrose pseudoinverse of the matrix $A_1(\xi)$ in (2.2) below. An upper bound for $\|A_1^\dagger\|$ is given in the following theorem.

**Theorem 3.1.** If $\Omega = \{0, \ldots, m-1\}$ and $\hat{a}$ and $n$ satisfy the hypotheses of Theorem 2.4 then

$$\|A_1^\dagger\| \leq m\beta_1(1 + m\sqrt{n-1})$$

where $\beta_1 = \max\{n, \text{ess sup}_{\xi \in J} \|A_m^\dagger(\xi)\|\} < \infty$, $J = [\frac{1}{4n}, \frac{1}{2} - \frac{1}{2n}] \cup [\frac{1}{2} + \frac{1}{4n}, 1 - \frac{1}{4n}]$, and $A_m(\xi)$ is defined by (2.2).

In the following corollaries we give more explicit bounds for the value of $\beta_1$. There, without loss of generality, we assume that $\sup |\hat{a}(\xi)| \leq 1$.

**Corollary 3.2.** If $\Omega = \{0, \ldots, m-1\}$, $\hat{a}$ and $n$ satisfy the hypotheses of Theorem 2.4 and $\sup |\hat{a}(\xi)| \leq 1$ then

$$\|A_1^\dagger\| \leq m\beta_2(1 + m\sqrt{n-1})$$

where $\beta_2 = \max \left\{n, \left(\frac{2}{3}\right)^{m-1}\right\} < \infty$, $\delta = \min_{\xi \in J} |\hat{a}(\xi)|$, and $J = [\frac{1}{4n}, \frac{1}{2} - \frac{1}{4n}] \cup [\frac{1}{2} + \frac{1}{4n}, 1 - \frac{1}{4n}]$.

**Corollary 3.3.** If $\Omega = \{0, \ldots, m-1\}$, $\hat{a}$ and $n$ satisfy the hypotheses of Theorem 2.4, $\sup |\hat{a}(\xi)| \leq 1$, and, in addition, $\hat{a} \in C^1(0, \frac{1}{2})$ and the derivative $\hat{a}'$ of $\hat{a}$ is nonzero (and, hence, negative) on $(0, \frac{1}{2})$, then

$$\|A_1^\dagger\| \leq m\beta_3(1 + m\sqrt{n-1}),$$

where $\beta_3 = \max \left\{n, \left(\frac{4m}{7}\right)^{m-1}\right\}$, $\gamma = \min_M |\hat{a}'(\xi)|$, and $M = [\frac{1}{4m}, \frac{1}{2} - \frac{1}{4m}]$.

For a Gaussian i.i.d. additive noise $\mathcal{N}(0, \sigma^2)$ a reconstruction of $f$ using $A_1^\dagger$ will result in an error estimated by $\|f - \hat{f}\| \leq \|A_1^\dagger\|\sigma\gamma^{-\frac{1}{2}}$. The theorem above provides an upper bound for the operator norm $\|A_1^\dagger\|$. However, although the upper bound grows to infinity as $n$ or $m$ increases, it is not yet clear that $\|A_1^\dagger\|$ deteriorates in this case. The following two results show that, indeed, as $m$ or $n$ increases $\|A_1^\dagger\|$ is unbounded and the stability of reconstruction does in fact worsen.

**Theorem 3.4.** Suppose $\hat{a}$, $n$, and $\Omega$ satisfy the hypotheses of Theorem 2.4 with $|\Omega| = \frac{m-1}{2}$. Then $\|A_1^\dagger\| \geq m\|A_m^{-1}(\frac{1}{n})\|$.

**Corollary 3.5.** Suppose $\hat{a}$, $n$, and $\Omega$ satisfy the hypotheses of Theorem 2.4 with $|\Omega| = \frac{m-1}{2}$. Then $\|A_1^\dagger\| \rightarrow \infty$ as $n \rightarrow \infty$. 
Remark 3.1. The proof of the theorem shows that if \( \Omega \) is some larger set, that is \( |\Omega| > \frac{m-1}{2} \), then the growth of \( \|A^\dagger\| \) may be alleviated. It should also be noted that in practice sampling on \( \Omega \) will also likely to be performed at all times \( n = 0, \ldots, m - 1 \), rather than just when \( n = 0 \). This may also have the effect of decreasing \( \|A^\dagger\| \).

3.1. Proofs of Theorems. In the beginning, we provide two well-known lemmas that we use in the proofs.

Lemma 3.6. Let \( A \) be an \( m \times n \) matrix with \( m > n \) so that the Moore-Penrose left inverse is given by \( A^\dagger = (A^*A)^{-1}A^* \). If \( A^\ell \) is any other left inverse of \( A \), then \( \|A^\dagger\| \leq \|A^\ell\| \).

Lemma 3.7. Suppose \( A \) is an \( m \times n \) matrix with \( m > n \), the maps \( \sigma : \{1, \ldots, m\} \to \{1, \ldots, n\} \) and \( \eta : \{1, \ldots, n\} \to \{1, \ldots, m\} \) are permutations, and \( B \) is an \( n \times m \) matrix such that \( BA = I_n \). If the matrices \( \tilde{A} \) and \( \tilde{B} \) are given by \( \tilde{A}(i, j) := A(\sigma(i), \eta(j)) \) and \( \tilde{B}(j, i) := B(\eta(j), \sigma(i)) \), then \( \tilde{B}\tilde{A} = I_n \) and \( \|B\|_{op} = \|\tilde{B}\|_{op} \).

3.1.1. Proof of Theorem 3.7. Similar to the matrix (2.16), the matrix obtained for the additional sampling on \( \Omega = \{1, \ldots, m - 1\} \) is given by

\[
A_\Omega(\xi) = \begin{pmatrix}
\frac{1}{mn} \bar{u}_1(0) & \frac{1}{mn} \bar{u}_1(1) & \cdots & \frac{1}{mn} \bar{u}_1(n-1)
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{mn} \bar{u}_{m-1}(0) & \frac{1}{mn} \bar{u}_{m-1}(1) & \cdots & \frac{1}{mn} \bar{u}_{m-1}(n-1)
0 & 0 & \cdots & 0
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \frac{1}{m} A_m(\xi + \frac{n-1}{n})
\end{pmatrix}
\]

In light of Lemma 2.6, a uniform upper bound for \( \|A_\Omega(\xi)\| \), that is an upper bound independent of \( \xi \), provides an upper bound for \( \|A^\dagger\| \). We choose \( \Omega = \{0, 1, \ldots, m - 1\} \) and \( n, m \) to be odd.

We will rearrange the rows and columns of the matrix \( A_\Omega(\xi) \) to create a matrix \( \tilde{A}_\Omega(\xi) \) for which we can explicitly give a left inverse. By Lemmas 3.7 and 3.6, it suffices to find an upper bound for any left inverse of \( \tilde{A}_\Omega(\xi) \).

For a fixed \( \xi \in [0, \frac{1}{n}] \), let \( k_0 \) be such that \( \xi + \frac{k_0}{n} \) is the closest point of \( \{\xi + \frac{k}{n}\}_{k=0,\ldots,n-1} \) on the torus to a singularity of \( A_m \). Specifically, if \( \xi \in [0, \frac{1}{4n}] \), then \( k_0 = 0 \); if \( \xi \in [\frac{1}{4n}, \frac{3}{4n}] \), then \( k_0 = \frac{n-1}{2} \); and if \( \xi \in [\frac{3}{4n}, \frac{1}{n}] \), then \( k_0 = n - 1 \). We see that

\[
\min_{k=0,\ldots,n-1 \atop k \neq k_0} \{ \text{dist}(\xi + \frac{k}{n}, 0), \text{dist}(\xi + \frac{k}{n}, \frac{1}{2}), \text{dist}(\xi + \frac{k}{n}, 1) \} \geq \frac{1}{4n}.
\]
In other words, for \( k \neq k_0 \), and \( \xi \in [0, \frac{1}{n}] \), we have \( \xi + \frac{k}{n} \in J \) where \( J = J(n) \) is defined by

\[
J = \left[ \frac{1}{4n}, \frac{1}{2} - \frac{1}{4n} \right] \cup \left[ \frac{1}{2} + \frac{1}{4n}, 1 - \frac{1}{4n} \right].
\]

By rearranging the columns and rows of the matrix \( A_\Omega \) so that it has the form \( \tilde{A}_\Omega \) below, we are able to explicitly define a left inverse that is independent of \( A_m(\xi + \frac{k_0}{n}) \). We write

\[
(3.4) \quad \tilde{A}_\Omega(\xi) = \begin{pmatrix}
\frac{1}{mn} \tilde{u}_0(k_0) & \frac{1}{mn} \tilde{u}_0(k_1) & \ldots & \frac{1}{mn} \tilde{u}_0(k_{n-1}) \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{mn} \tilde{u}_{m-1}(k_0) & \frac{1}{mn} \tilde{u}_{m-1}(k_1) & \ldots & \frac{1}{mn} \tilde{u}_{m-1}(k_{n-1}) \\
\frac{1}{m} A_m(\xi + \frac{k_0}{n}) & 0 & \ldots & 0 \\
0 & \frac{1}{m} A_m(\xi + \frac{k_1}{n}) & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \frac{1}{m} A_m(\xi + \frac{k_{n-1}}{n})
\end{pmatrix},
\]

in the block form

\[
(3.5) \quad \tilde{A}_\Omega(\xi) = \frac{1}{m} \begin{pmatrix}
mU_{k_0} & mQ \\
A_m(\xi + \frac{k_0}{n}) & 0 \\
0 & D(\xi)
\end{pmatrix},
\]

where \( U_{k_0} = \begin{pmatrix}
\frac{1}{mn} \tilde{u}_0(k_0) \\
\frac{1}{mn} \tilde{u}_1(k_0) \\
\vdots \\
\frac{1}{mn} \tilde{u}_{m-1}(k_0)
\end{pmatrix} \), and \( D(\xi) \) is a \( m(n-1) \times m(n-1) \) block diagonal matrix with \( A_m(\xi + \frac{k}{n}), k = 0, \ldots, n-1, k \neq k_0 \), on the main diagonal. Then, a left inverse is given by

\[
(3.6) \quad A_\ell(\Omega)(\xi) = m \begin{pmatrix}
\frac{1}{m} U_{k_0} & 0 & -U_{k_0}^{-1}QD^{-1}(\xi) \\
0 & D^{-1}(\xi)
\end{pmatrix},
\]

and we easily compute that

\[
(3.7) \quad \|A_\ell(\Omega)(\xi)\| \leq m(\max\{\|\frac{1}{m} U_{k_0}^{-1}\|, \|D^{-1}(\xi)\|\} + \|U_{k_0}^{-1}\||\|D^{-1}(\xi)\||Q\|)).
\]

Since \( D \) is a block diagonal matrix, we have

\[
(3.8) \quad \|D^{-1}(\xi)\| = \max_{k \neq k_0} \left\{ \|A_m^{-1}(\xi + \frac{k}{n})\| \right\}.
\]

The submatrix \( Q \) is an \( m \times m(n-1) \) matrix with entries of norm \( \frac{1}{mn} \). Thus, we have

\[
(3.9) \quad \|Q\| \leq m\sqrt{n-1}\|Q\|_{\max} = \frac{\sqrt{n-1}}{n}.
\]
Observing that the columns of $U_k$ are orthogonal, and we have $\|U_k^{-1}\| = mn$. Our estimate (3.7) becomes,

$$
(3.10) \quad \|A_Ω^k(\xi)\| \leq m \max \left\{ n, \max_{k \neq k_0} \|A_m^{-1}(\xi + \frac{k}{n})\| \right\} (1 + m\sqrt{n - 1}).
$$

Taking the essential supremum over $\xi \in [0, \frac{1}{n}]$, and noting that for $k \neq k_0$, $\xi + \frac{k}{n} \in J$ as in (3.3), this last equation can be estimated by

$$
(3.11) \quad \|A_Ω^k\| \leq \text{ess sup}_{\xi \in [0, \frac{1}{n}]} \left( m \max \left\{ n, \max_{k \neq k_0} \|A_m^{-1}(\xi + \frac{k}{n})\| \right\} \right) (1 + m\sqrt{n - 1}) \leq m \max \left\{ n, \text{ess sup}_{\eta \in J} \|A_m^{-1}(\eta)\| \right\} (1 + m\sqrt{n - 1}).
$$

Since $A_m(\eta)$ is invertible for all $\eta \in J$ and $J$ is a compact set, it follows that $\text{ess sup}_{\eta \in J} \|A_m^{-1}(\eta)\|$ is finite, and Theorem 3.1 follows.

To find the more explicit bound in Corollary 3.2, we use the estimate for the norm of the inverse of a Vandermonde matrix [7]:

$$
(3.12) \quad \|A_m^{-1}(\xi)\| \leq \sqrt{m} \max_{0 \leq i \leq m-1} \prod_{j=0}^{m-1} \frac{1 + |\hat{a}(\frac{\xi+j}{m})|}{|\hat{a}(\frac{\xi+j}{m}) - \hat{a}(\frac{\xi+i}{m})|}.
$$

To prove Corollary 3.3 we find a uniform lower bound for $|\hat{a}(\frac{\xi+i}{m}) - \hat{a}(\frac{\xi+i}{m})|$. Note that when $\xi \in J$, we have $\frac{\xi+j}{m} \in \frac{1}{m} + \frac{1}{m} J$, $j = 0, \ldots, m - 1$. Then for $\xi \in J$ and any $j = 0, \ldots, m - 1$, we have

$$
\frac{\xi+j}{m} \in \bigcup_{j=1}^{m-1} \left\{ \frac{j}{m} + \frac{1}{m} J \right\} \subset \left[ \frac{1}{4mn}, \frac{1}{4mn} \right] \cup \left[ \frac{1}{2} + \frac{1}{4mn}, 1 - \frac{1}{4mn} \right].
$$

Thus, defining $M := [\frac{1}{4mn}, \frac{1}{2} - \frac{1}{4mn}]$, we have that $\frac{\xi+j}{m} \in M \cup (M + \frac{1}{2})$ for any $j = 0, \ldots, m - 1$.

Let $\gamma = \min_{\xi \in M} |\hat{a}'(\xi)| > 0$ where $\hat{a}'(\xi)$ denotes the first derivative of $\hat{a}(\xi)$. By the symmetry of $\hat{a}$, we also have $\gamma = \min_{\xi \in M + \frac{1}{2}} |\hat{a}'(\xi)|$. Without loss of generality, assume $\frac{\xi+j}{m} > \frac{\xi+i}{m}$. If the interval $[\frac{\xi+i}{m}, \frac{\xi+j}{m}]$ is contained in $M$ or in $M + \frac{1}{2}$, the Mean Value Theorem gives

$$
\left| \hat{a}(\frac{\xi+j}{m}) - \hat{a}(\frac{\xi+i}{m}) \right| \geq \gamma \left| \frac{\xi+i}{m} - \frac{\xi+j}{m} \right| \geq \gamma \frac{1}{m}.
$$

If $\frac{\xi+i}{m} \in M$ and $\frac{\xi+i}{m} \in M + \frac{1}{2}$, we exploit the symmetry of $\hat{a}$ and consider the interval between $1 - \frac{\xi+i}{m}$ and $\frac{\xi+i}{m}$, which is contained in $M$. Defining $l = m - i - j$ and using the Mean Value Theorem again, we have

$$
\left| \hat{a}(1 - \frac{\xi+j}{m}) - \hat{a}(\frac{\xi+i}{m}) \right| \geq \gamma \left| \frac{l}{m} - 2(\frac{\xi}{m}) \right| = \gamma \frac{2}{m} \left| \frac{1}{2} - \frac{\xi}{m} \right| \geq \gamma \frac{1}{2mn},
$$
where the last inequality follows from the fact that \( l \in \mathbb{Z} \) and \( \xi \in J \). This gives Corollary 3.3. Notice that if \( \hat{a}' \in C(T) \) then \( \gamma \to 0 \) as \( n \to \infty \), due to the fact that the minimum is taken over a larger interval getting closer to the zeros of \( \hat{a}' \).

3.1.2. **Proof of Theorem 3.4.** Recall that \( \|A_1'(\xi)\| \) is equal to the reciprocal of the smallest singular value of \( A(\xi) \), denoted \( s_{\min}(A(\xi)) \). We choose an extra sampling set \( \Omega \) according to Theorem 2.4. We claim that

Claim 1: There exists an interval \([0, r]) \subset [0, \frac{1}{4\varrho}]\), such that the smallest singular value of \( s_{\min}(A_\Omega(\xi)) \) is bounded above on \([0, r]\), by

\[
0 \leq s_{\min}(A_\Omega(\xi)) \leq \frac{1}{m\|A_{m}^{-1}(\xi + \frac{1}{n})\|} < \infty, \quad \xi \in [0, r].
\]

Using the claim, the theorem follows from

\[
(3.13) \quad m\|A_{m}^{-1}(\frac{1}{n})\| \leq m \cdot \text{ess sup}_{\xi \in [0, r]} \|A_{m}^{-1}(\xi + \frac{1}{n})\|
\]

\[
\leq \text{ess sup}_{\xi \in [0, r]} \frac{1}{s_{\min}(A_\Omega(\xi))}
\]

\[
\leq \text{ess sup}_{\xi \in [0, \frac{1}{n}]} \frac{1}{s_{\min}(A_\Omega(\xi))}
\]

\[
= \|A_{\Omega}^*\|.
\]

**Proof of Claim 1.** We first show that \( s_{\min}^2(A_\Omega(\xi)) \) is equal to the \( mn \)-th largest eigenvalue \( \lambda_{mn}(A_\Omega(\xi)A_\Omega^*(\xi)) \) of \( A_\Omega(\xi)A_\Omega^*(\xi) \):

\[
(3.14) \quad A_\Omega(\xi)A_\Omega^*(\xi) = \frac{1}{m^2} \begin{pmatrix}
* & A_m(\xi)A_m^*(\xi) & * \\
* & 0 & D(\xi)D^*(\xi)
\end{pmatrix},
\]

where the matrices in the first row have \(|\Omega| \) rows and \( D(\xi)D^*(\xi) \) is the block diagonal matrix with blocks \( A_m(\xi + \frac{k}{n})A_m^*(\xi + \frac{k}{n}), k \neq 0 \), as entries. The rank of the \((mn + |\Omega|) \times (mn + |\Omega|)\) matrix \( A_\Omega(\xi)A_\Omega^*(\xi) \) is equal to the rank of \( A_\Omega(\xi) \), which is \( mn \). Thus, the smallest positive eigenvalue of \( A_\Omega(\xi)A_\Omega^*(\xi) \) is the \( mn \)-th largest eigenvalue \( \lambda_{mn}(A_\Omega(\xi)A_\Omega^*(\xi)) \), and it is equal to \( s_{\min}^2(A_\Omega(\xi)) \). Thus, to estimate \( s_{\min}^2(A_\Omega(\xi)) \) from above, we need to estimate \( \lambda_{mn}(A_\Omega(\xi)A_\Omega^*(\xi)) \).

In turn, the \( mn \)-th largest eigenvalue \( \lambda_{mn}(A_\Omega(\xi)A_\Omega^*(\xi)) \) can be estimated above using the eigenvalues of the \( mn \times mn \) principal submatrix \( B(\xi) \)

\[
(3.15) \quad B(\xi) = \begin{pmatrix}
A_m(\xi + \frac{k}{n})A_m^*(\xi + \frac{k}{n}) & 0 \\
0 & D(\xi)D^*(\xi)
\end{pmatrix},
\]

via the Cauchy Interlacing Theorem [6]:

\[
(3.16) \quad s_{\min}^2(A_\Omega(\xi)) = \lambda_{mn}(A_\Omega(\xi)A_\Omega^*(\xi)) \leq \frac{1}{m^2} \lambda_{mn-|\Omega|}(B(\xi)),
\]

where we use \( \lambda_j(M) \) to denote the \( j \)-th largest eigenvalue of the a matrix \( M \) counting the multiplicity.
We chose $\Omega$ to be a minimal extra sampling set so that $|\Omega| = \frac{m-1}{2}$. Observing that $B(\xi)$ is block diagonal so that the eigenvalues of $B(\xi)$ are the eigenvalues of $A_m(\xi + \frac{k}{n})A_n^*(\xi + \frac{k}{n})$, and using a continuity argument below we show that there exists $r$ with $0 < r < \frac{1}{4m}$ such that for all $\xi \in [0, r]$

$\lambda_{mn-\frac{m-1}{2}}(B(\xi)) = \min_{k \neq 0} \{ \lambda_m(A_m(\xi + \frac{k}{n})A_n^*(\xi + \frac{k}{n})) \}$

$\leq \lambda_m(A_m(\xi + \frac{1}{n})A_n^*(\xi + \frac{1}{n}))$

$= \frac{1}{\|A_m^{-1}(\xi + \frac{1}{n})\|^2}$.

In the last equality above, we used the relation between the minimum singular values of a matrix $M$ and the norm of its inverse: $s_{\min}^{-1}(M) = \|M^{-1}\|$. Claim 1 then follows from (3.16) and (3.17).

We now use the continuity argument to prove (3.17). Let

$$\alpha := \inf_{[0, \frac{1}{4n}]} \lambda_{m(n-1)}(D(\xi)D^*(\xi)) > 0.$$ 

Since $\lambda_{m-j}(A_m(0)A_m^*(0)) = 0$ for $j = 0, \ldots, (\frac{m-1}{2} - 1)$, continuity in $\xi$ implies that there exists $r$ with $0 < r < \frac{1}{4m}$ such that

$$\lambda_{m-j}(A_m(\xi)A_m^*(\xi)) < \alpha \quad \text{for} \quad \xi \in [0, r].$$

Thus, when $\xi \in [0, r]$, the smallest $\frac{m-1}{2}$ eigenvalues of $B(\xi)$ are precisely the smallest $\frac{m-1}{2}$ eigenvalues of $A_m(\xi)A_m^*(\xi)$, i.e., $\lambda_{mn-\frac{m-1}{2}}(B(\xi)) = \lambda_{mn-\frac{m-1}{2}}(A_m(\xi)A_m^*(\xi))$ for $j = 0, \ldots, (\frac{m-1}{2} - 1)$ and

$$\lambda_{mn-\frac{m-1}{2}}(B(\xi)) = \min \left\{ \lambda_{m-\frac{m-1}{2}}(A_m(\xi)A_n^*(\xi)) : \lambda_{m(n-1)}(D(\xi)D^*(\xi)) \right\}$$

$\leq \lambda_{m(n-1)}(D(\xi)D^*(\xi))$

$= \inf_{k \neq 0} \{ \lambda_m(A_m(\xi + \frac{k}{n})A_n^*(\xi + \frac{k}{n})) \}$

$\leq \lambda_m(A_m(\xi + \frac{1}{n})A_n^*(\xi + \frac{1}{n}))$

$= \frac{1}{\|A_m^{-1}(\xi + \frac{1}{n})\|^2}$,

which is (3.17).
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