Generalized Distribution Functions and an Alternative Approach to Generalized Planck Radiation Law

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(February 12, 2018)

In this study, recently introduced generalized distribution functions are summarized and by using one of these distribution functions, namely generalized Planck distribution, an alternative approach to the generalized Planck law for the blackbody radiation has been tackled. The expression obtained is compared with the expression given by C. Tsallis et al. [Phys. Rev. E52, (1995) 1447], and it is found that this approximate scheme provides bounds to the exact result, depending on the values of $q$ index.

I. INTRODUCTION

Although the necessity of nonextensive statistical thermodynamics has been very clear for a long time in many physical systems such as three-dimensional self-gravitating astrophysical objects [1], black holes and superstrings [2], Lévy-type random walks [3], vortex problem [4] and dark matter [5] where long-range microscopic interactions are present; an increasing tendency towards nonextensive formalisms keeps growing nowadays, along two lines: Quantum-Group-like approaches and Generalized Statistical Thermodynamics (GST).

GST is recently introduced by C. Tsallis [6,7] and then the formalism not only has been applied to numerous concepts of statistical thermodynamics [8-28], but also prosperous for the physical systems [29-33] where extensive Boltzmann-Gibbs statistics fails. The detailed

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reviews of the general aspects of the formalism can be found in [34], and some investigations of the subject from the mathematical point of view are now available in [35].

The formalism has been based upon two axioms; namely (i) the entropy of the system is given by

\[ S_q = -k \frac{1 - \sum_{i=1}^{W} p_i^q}{1 - q} \]  

(1)

where \( k \) is a conventional positive constant, \( p_i \) is the probability of the system to be in a microstate, \( W \) is the total number of configurations of the system and \( q \) is a new parameter which is especially called the Tsallis \( q \)-index; (ii) the \( q \)-expectation value of an observable \( O \) is given by

\[ \langle O \rangle_q = \sum_{i=1}^{W} p_i^q O_i \]  

(2)

On the other hand, GST contains Boltzmann-Gibbs Statistics as a special case when \( q = 1 \). Clearly \( S_q \) is extensive if and only if \( q = 1 \) and \( (1 - q) \) can be interpreted as the measure of the lack of extensivity of the system [24].

II. GENERALIZED DISTRIBUTION FUNCTIONS

Amongst the systems handled in the frame of GST, the classical and quantum gases have been investigated for the first time by two of the authors of the present study and the results have been presented in two separate manuscripts [13,14]. In the first paper, the fractal inspired entropies of the classical and quantum gases have been found by making use of the statistical weights and the generalized distribution functions have been obtained by introducing the constraints related to the statistical properties of the particles into the entropy [13]. In the second paper, the Tsallis entropy have been used for the same purpose and the same conclusions for the generalized distribution functions have been attained but in this case the constraints concerned with the statistical properties of the particles have been introduced in the calculation of the partition functions. However, the factorization
of the partition function which was used in this calculations has been the main disputable point of the approximation. We claim that this approximate procedure provides a bound to the exact results, depending on the values of $q$. In order to prove this, let us take a very simple physical system which could be in bounded or free states with the energy levels $A$ and $B$. For this system let us take the inequality

$$[1 + (1 - q)(A + B)]^{\frac{1}{1-q}} \neq [1 + (1 - q)A]^{\frac{1}{1-q}} [1 + (1 - q)B]^{\frac{1}{1-q}}. \quad (3)$$

It is easy to rewrite this in exponential form

$$\exp \left\{ \frac{1}{1-q} \ln [1 + (1 - q)(A + B)] \right\} \neq \exp \left\{ \frac{1}{1-q} \ln [1 + (1 - q)A] + \frac{1}{1-q} \ln [1 + (1 - q)B] \right\}. \quad (4)$$

After some algebra it is straightforward to find

$$\exp \left\{ \frac{1}{1-q} \ln [1 + (1 - q)A + (1 - q)B] \right\} \neq \exp \left\{ \frac{1}{1-q} \ln \left[ 1 + (1 - q)A + (1 - q)B + (1 - q)^2 AB \right] \right\}. \quad (5)$$

If the left-hand side and right-hand side of this inequality are called "exact" and "approximation" respectively, then it follows that exact $> \text{approximation}$ for $q > 1$ whereas exact $< \text{approximation}$ for $q < 1$. It must be emphasized that this situation is valid if and only if the product $A.B$ is positive. In photon case, which is the subject of this study, $A$ and $B$, namely energy levels of the system are defined to be $h\nu/kT$ that is always positive.

In addition to this, an inequality can be derived for the partition function within the frame of (3). The grand partition function reads:

$$Z_q = \sum_{\{r_i\}} \left[ 1 - \beta(1 - q) \sum_{r=1}^{m} (\epsilon_r - \mu) \right]^{\frac{1}{1-q}}, \quad (6)$$

where $\{r_i\}$ are the elements of the grand canonical ensemble. For the photon case ($\mu = 0$), the eq.(6) becomes

$$Z_q = \sum_{\{r_i\}} \left[ 1 - \beta(1 - q)\mathcal{H}_{\{r_i\}} \right]^{\frac{1}{1-q}} \quad (7)$$
where $H_{\{r_i\}} = \sum_{r=1}^{m} \epsilon_r$ is the Hamiltonian of the system. Employing the inequality (3), another inequality for the partition function can be written down as

$$
\sum_{\{r_i\}} [1 - \beta(1 - q) (\epsilon_1 + \cdots + \epsilon_m)]^{\frac{1}{1-q}} \neq \sum_{\{r_i\}} [1 - \beta(1 - q)\epsilon_1]^{\frac{1}{1-q}} \cdots [1 - \beta(1 - q)\epsilon_m]^{\frac{1}{1-q}}.
$$

(8)

Since the free energy is a monotonic function of $Z_q$, namely,

$$
F_q = -\frac{1}{\beta} \frac{Z_q^{1-q} - 1}{1 - q},
$$

(9)

it is straightforward to write an inequality for the free energy.

In the frame of the above simple mathematical analysis, it is evident that our approximation scheme provides a lower or upper bound to the exact results, depending on the $q$ values.

Within this approximation procedure the generalized distribution functions are given by

$$
\langle n_r \rangle_q^{MB} = \frac{1}{1 - (1 - q)\beta(\epsilon_r - \mu)]^{\frac{1}{1-q}}}
$$

(10)

$$
\langle n_r \rangle_q^{FD} = \frac{1}{[1 - (1 - q)\beta(\epsilon_r - \mu)]^{\frac{1}{1-q}} + 1}
$$

(11)

$$
\langle n_r \rangle_q^{BE} = \frac{1}{[1 - (1 - q)\beta(\epsilon_r - \mu)]^{\frac{1}{1-q}} - 1}
$$

(12)

where $MB$, $FD$ and $BE$ stand for Maxwell-Boltzmann, Fermi-Dirac and Bose-Einstein, respectively. In a very recent effort [36], quantum statistics has been studied by means of a kinetic approach and the distribution functions corresponding to the Tsallis probability are found to be the same with the eqs.(10)-(12). It is easy to show that the standard distribution functions are recovered in the $q \rightarrow 1$ limit. On the other hand, when $\mu$ is set equal to zero in eq.(12), one can obtain the generalized Planck distribution ($PD$)

$$
\langle n_r \rangle_q^{PD} = \frac{1}{[1 - (1 - q)\beta\epsilon_r]^{\frac{1}{q-1}} - 1}
$$

(13)

which is going to be used here for making an approximation to the derivation of the generalized Planck radiation law by following the well-known procedure available in many textbooks [37] of Statistical Physics.
III. THE GENERALIZATION OF THE PLANCK RADIATION LAW

Very recently, the generalization of the Planck radiation law is obtained by Tsallis et al. [24] in order to see whether present cosmic background radiation is (slightly) different from Planck radiation law due to long-range gravitational influence. In the course of their investigations, making use of the partition function given by

\[ Z_q \approx Z_1 \left\{ 1 - \frac{1}{2} (1 - q) \beta^2 \langle H^2 \rangle_1 \right\} \]

(where \( Z_1 \) and \( \langle H^2 \rangle_1 \) stand for the values of these quantities in standard Boltzmann-Gibbs statistics) in the \( \beta(1 - q) \to 0 \) limit, Tsallis et al. have achieved the generalization of the Planck law given in the following expression for \( q \approx 1 \) [24]

\[
\frac{D_q(\nu) h^2 c^3}{8\pi (kT)^3} \approx \frac{x^3}{e^x - 1} \left( 1 - e^{-x} \right)^{q-1} \left\{ 1 + (1 - q) x \left[ \frac{1 + e^{-x}}{1 - e^{-x}} - \frac{x}{2} \frac{1 + 3e^{-x}}{(1 - e^{-x})^2} \right] \right\}
\]

(14)

where \( D_q(\nu) \) is the photon energy density per unit volume, \( \nu \) is the photon frequency and \( x \equiv h\nu/kT \). Then they applied this expression to the cosmic microwave background radiation to test for deviations from Planck radiation law and found a 95\% confidence limit of \( |q - 1| < 3.6 \times 10^{-5} \) from the data obtained via Cosmic Background Explorer Satellite by Mather et al. [38]. In the present work we’re trying to generalize the Planck law by using an approximate scheme which seems simpler and more general (not necessarily \( q \approx 1 \)).

The distribution of photons among the various quantum states with definite values of the momentum \( h\vec{k}/2\pi \) and energies \( h\nu \) can be given by generalized Planck distribution (eq.(13))

\[
\langle n_r \rangle^P D_q = \frac{1}{\left[ 1 - (1 - q) \frac{h\nu}{kT} \right]^{\frac{1}{q-1}} - 1} .
\]

(15)

On the other hand, the number of quantum states of photons with frequencies between \( \nu \) and \( \nu + d\nu \) is \( 8\pi V \nu^2 d\nu/c^3 \) (\( V \) being the volume of the photon gas). It is clear that multiplying this quantity by eq.(15), the number of photons in this frequency interval can be determined:

\[
dN = \frac{8\pi V}{c^3} \frac{\nu^2 d\nu}{\left[ 1 - (1 - q) \frac{h\nu}{kT} \right]^{\frac{1}{q-1}} - 1} .
\]

(16)

Therefore the photon energy in this interval is given by

\[
dE = \frac{8\pi hV}{c^3} \frac{\nu^3 d\nu}{\left[ 1 - (1 - q) \frac{h\nu}{kT} \right]^{\frac{1}{q-1}} - 1} ,
\]

(17)
and finally the photon energy density per unit volume is

\[ D_q(\nu) = \frac{8\pi h}{c^3} \frac{\nu^3}{\left[1 - (1 - q) \frac{h\nu}{kT}\right]^{\frac{1}{q-1}} - 1}, \quad (18) \]

which generalizes the Planck’s law. The comparison of this expression with that given by Tsallis et al. (eq.(5) of their paper) is illustrated in Figure. It is observed in the figure that, at low frequencies the two \( D_q(\nu) \) plots completely fit into one another. On the other hand, towards the frequency values where the maxima of the curves occur, the plots diverge from one another by a certain amount. This is an expected result since the implication of our approximation scheme shows itself as an upper (lower) bound to the exact result when \( q > 1 \) \((q < 1)\). It is worthwhile to imply here that the bound is exactly same as the one that has appeared in ref.[39] where an application of the generalized distribution functions has been discussed (see Fig.1 and Fig.3 of this reference).

Moreover, in the \( x \equiv h\nu/kT << 1 \) case, we verify that eq.(18) becomes

\[ D_q(\nu) = \frac{8\pi (kT)^3}{h^2 c^3} x + \frac{x^3}{2!} + \frac{(2-q)x^2}{3!} + \frac{(2-q)(3-2q)x^3}{3!} + \ldots, \quad (19) \]

which generalizes the Rayleigh-Jeans law. As it is expected, in the \( q \to 1 \) limit this expression transforms to \( D_1(\nu) \propto x^2 \propto \nu^2 \) which corresponds to the standard Rayleigh-Jeans law.

In addition to this, it is straightforward to generalize Stefan-Boltzmann law and show that it remains the same, i.e. it is still proportional to \( T^4 \), but with a \( q \)-dependent constant \((\sigma_q)\). To see this, let us write the total emitted power per unit surface,

\[ P_q = \int_0^\infty d\nu D_q(\nu). \quad (20) \]

If we use here eq.(18) with the dimensionless variable \( x \),

\[ P_q = \frac{8\pi k^4}{h^3 c^3} T^4 \int_0^\infty \frac{x dx}{[1 - (1 - q)x]^{\frac{1}{q-1}} - 1} \quad (21) \]

can be obtained. Since the integral term is independent of \( T \), it can be written down as

\[ P_q = \sigma_q T^4, \quad (22) \]
which is the generalized Stefan-Boltzmann law, with a \( q \)-dependent prefactor.

It is also possible to obtain the generalized Wien shift law. By making eq.(18) maximum, \( \nu_m \) (the frequency value which makes \( D_q(\nu) \) maximum) yields a nonlinear equation:

\[
\left[ \frac{h\nu_m}{kT}(3q - 4) + 3 \right] \left[ 1 - (1 - q) \frac{h\nu_m}{kT} \right]^{\frac{1}{q - 1}} + \frac{h\nu_m}{kT}3(1 - q) - 3 = 0 .
\]

Although it is very difficult to find an analytical solution for this equation, the graphical solution is adequate for our purpose. From the graphical solution, for \( q = 0.95 \) and \( q = 1.05 \) we have found \( h\nu_m/kT = 2.444 \) and \( 3.347 \), respectively. For the same values of Tsallis \( q \)-index, eq.(19) in ref.[24] gives \( h\nu_m/kT = 2.563 \) and \( 3.08 \). Once again it is possible to see that our results provide bounds to the exact ones. Moreover, the result \( \nu_m(q > 1) > \nu_m(q = 1) > \nu_m(q < 1) \), which appears in ref.[24], is also valid in the present investigation.

The important diversity between this result and that of the \( q_G \)-deformed quantum groups investigations namely \( \nu_m(g_G) = \nu_m(1/g_G) \) and \( \nu_m(g_G < 1) < \nu_m(g_G = 1) \) is readily observed.

**IV. CONCLUSIONS**

Although the generalized distribution functions are first introduced in 1993, there has been no attempt to apply them to the physical systems until a very recent effort performed by Pennini et al. [39] where two single particles are considered. In this letter, our goal is to apply one of the generalized distribution functions (generalized Planck distribution) to another physical system (Planck law for the blackbody radiation). On the other hand, the approximate procedure used here to find a bound to the generalized Planck law is simpler than that of Tsallis et al. [24] (since the procedure is completely the same with that followed in standard textbooks of Statistical Physics for the derivation of standard Planck law) and also the bound seems to be more general, since not necessarily \( q \approx 1 \) (it will have a meaning, of course, if a physical system requiring this condition exists). Lastly, it is worthwhile to imply here that all the generalized laws derived here (i.e. generalized Planck law, generalized Rayleigh-Jeans law, Stefan-Boltzmann law and Wien law) transform to corresponding standard well-known laws in the \( q \to 1 \) limit.
ACKNOWLEDGMENTS

The authors acknowledge TUBITAK and Ege University for making Prof. C. Tsallis’ visit to Izmir possible. We are very indebted to Prof. C. Tsallis for the helpful discussions as well as kindly supplying to us some of the references therein.
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Blackbody photon energy density per unit volume versus $h\nu/kT$ in the frame of this work and ref.[24].
\[ D_q(hv/kT)^2 e^{3/8\pi kT^3} \]

- \( q = 1.05 \) (This Work)
- \( q = 1.05 \) (ref.[24])
- \( q = 1 \)
- \( q = 0.95 \) (ref.[24])
- \( q = 0.95 \) (This Work)