TRAVELING WAVE SOLUTIONS OF A REACTION-DIFFUSION PREDATOR-PREY MODEL

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Abstract. This paper is concerned with the dynamics of traveling wave solutions for a reaction-diffusion predator-prey model with a nonlocal delay. By using Schauder’s fixed point theorem, we establish the existence result of a traveling wave solution connecting two steady states by constructing a pair of upper-lower solutions which are easy to construct in practice. We also investigate the asymptotic behavior of traveling wave solutions by employing the standard asymptotic theory.

1. Introduction. Predator-prey model is an important tool that helps us to understand the ecological and biological systems surrounding us, and it is also one of the basic models between different species in nature [5]. Recently, traveling wave solutions of delayed reaction-diffusion equations have been widely studied due to the significant applications in mathematical theory and other practical fields (see, for example, Liang and Zhao [8], Ma [11], Schaaf [12], Huang and Zou [6], Lin et al. [9], Britton [1]).

In the traditional Lotka-Volterra type, we can see that the spatial content of environment has always been ignored by people. Traditionally, these models have been worked out in connection with the time evolution of uniform population distribution in the habitat and are strictly dominated by ordinary differential equation. Whereas, as argued in, the species taken into consideration in many ecological systems many disperse both in space and time. The spatial dispersal or diffusion results from the resource limitation which brings the trends of some species towards regions of lower population density. In recent years, great important has been attached on the influence of dispersion of a population in a bordered area, and in this case a system of reaction-diffusion equation are applied to describe the governing equations concerning the population density. The problem that interest both an ecological system and the mathematically research is to determine the condition under which the time-dependent solution converges to a positive steady-state solution, and the role of the influence of diffusion and time delays. It is argued that any delays should be spatially inhomogeneous in more realistic ecological models. That is to say that the delays influence not only temporal variables but also spatial variables. This is because the fact that any given individual in former times may not necessarily

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has been at the same spatial. Such delays are called nonlocal. Lately, the influence of nonlocal delays on the dynamics of ecological models has been taken into consideration (see [1, 14, 2, 17, 16, 15, 7, 4, 19, 13, 18]).

Gan et al. [4] discussed the following three-species food-chain model with spatial diffusion and time delays

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= D_1 \frac{\partial^2 u}{\partial x^2} + u(t,x)(r_1 - a_{11}u(t,x) - a_{12}v(t,x)), \\
\frac{\partial v}{\partial t} &= D_2 \frac{\partial^2 v}{\partial x^2} + v(t,x)(-r_2 + a_{21}u(t-x) - a_{22}w(t,x) - a_{23}w(t,x)), \\
\frac{\partial w}{\partial t} &= D_3 \frac{\partial^2 w}{\partial x^2} + w(t,x)(-r_3 + a_{32}v(t-x) - a_{33}w(t,x)),
\end{aligned}
\]

(1)

with initial conditions

\[
u(t,x) = \rho_1(t,x), v(t,x) = \rho_2(t,x), w(t,x) = \rho_3(t,x); \quad t \in [-\tau,0],
\]

where \(u(x,t), v(t,x)\) and \(w(t,x)\) represent the densities of the prey, predator and top predator at time \(t\) and location \(x\), respectively. The parameter \(a_{11}, a_{12}, a_{22}, a_{23}, a_{32}, a_{33}, \tau, D_i, r_i (i = 1, 2, 3)\) are positive constants. The existence of traveling wave solutions for the system (1) with initial conditions was obtained by using Schauder’s fixed point theorem, the cross iteration method and constructing a pair of upper-lower solutions [4].

Shang, Du and Lin [13] considered a \(n\)-dimensional diffusive system with delays

\[
\begin{aligned}
\frac{\partial u_i(x,t)}{\partial t} &= d_i \Delta u_i(x,t) + r_i u_i(x,t)[1 - u_i(x,t) - \sum_{j=1, j \neq i}^{n-1} a_{ij} u_j(x,t - \tau_{ij})], i \in I, \\
\frac{\partial u_n(x,t)}{\partial t} &= d_n \Delta u_n(x,t) + r_n u_n(x,t)[1 - u_n(x,t) + \sum_{j=1}^{n-1} a_{nj} u_j(x,t - \tau_{nj})],
\end{aligned}
\]

where \(I = \{i = 1, \ldots, n\}\), \(J = \{i = 1, \ldots, n - 1\}\), \(u_i(x,t) (i \in J)\) and \(u_n\) represent the densities of the prey and predator populations at location \(x\) and time \(t\), respectively. The parameters \(d_i (i \in J)\) and \(d_n\) are the diffusion rates of the prey and predator populations, respectively. The existence of traveling wave solutions for the \(n\)-dimensional delayed reaction-diffusion system was established by using Schauder’s fixed point theorem and constructing a pair of upper-lower solutions [13].

Zhang and Xu [18] were concerned with the following reaction-diffusion predator-prey model stage structure and nonlocal delay

\[
\begin{aligned}
\frac{\partial u_1}{\partial t} &= d_1 \frac{\partial^2 u_1}{\partial x^2} + au_2(x,t) - r_1 u_1(x,t) - a_{11}u_1^2(x,t) - bu_1(x,t) - a_{12}u_1(x,t)v(x,t), \\
\frac{\partial u_2}{\partial t} &= d_2 \frac{\partial^2 u_2}{\partial x^2} + bu_1(x,t) - r_2 u_2(x,t), \\
\frac{\partial v}{\partial t} &= d_2 \frac{\partial^2 v}{\partial x^2} + v(x,t)(-r + a_{21} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4d_1 \pi}} e^{-\frac{(x-y)^2}{4d_1}} u_1(y, t - \tau)dy - a_{22}v(x,t)),
\end{aligned}
\]

for \(t > 0, x \in (-\infty, +\infty), \) with initial conditions

\[
\begin{aligned}
&u_1(x, \theta) = \phi(x, \theta) \geq 0, u_2(x, \theta) = \varphi(x, \theta) \geq 0, (x, \theta) \in (-\infty, +\infty) \times (-\infty, 0], \\
v(x, \theta) = \psi(x, \theta) \geq 0.
\end{aligned}
\]

By applying the cross iteration method and Schauder’s fixed point theorem, the existence result of traveling wave solution for the above system with initial conditions was established in [18].

Lv and Wang [10] were concerned with the following diffusive and time-delayed integro-differential equation

\[
\frac{\partial u(x,t)}{\partial t} = d \Delta u(x,t) - au(x,t) + b[1 - u(x,t)] \int_{-\infty}^{t} \int_{\Omega} F(x,y,t,s)u(y,s)dyds,
\]
where the constants $d$, $a$ and $b$ are positive, $u(x, t)$ is normalized spatial density of an infections host at time $t$ and at point $x$, and $F(x, y, t, s)$ is the convolution kernel. The authors [10] obtained the asymptotic behavior and uniqueness of the traveling wave fronts by using the standard asymptotic theory and sliding method.

Xu and Ma [15] considered the following reaction-diffusion predator-prey model with a nonlocal delay:

$$
\begin{aligned}
\frac{\partial u_1}{\partial t} &= D_1 \frac{\partial^2 u_1}{\partial x^2} + u_1(x, t)(r_1 - a_{11} u_1(x, t) - a_{12} u_2(x, t)), \\
\frac{\partial u_2}{\partial t} &= D_2 \frac{\partial^2 u_2}{\partial x^2} - r_2 u_2(x, t) - a_{22} u_2^2(x, t) \\
&\quad + a_{21} \int_{-\infty}^{t} \int_{0}^{\pi} G(x, y, t-s) f(t-s) u_1(y-s) u_2(y-s) dy ds,
\end{aligned}
$$

for $t > 0, x \in (0, \pi)$, with homogeneous Neumann boundary conditions

$$
\frac{\partial u_1}{\partial x} = \frac{\partial u_2}{\partial x} = 0, t > 0, x = 0, \pi,
$$

and initial conditions

$$
u_i(x, \theta) = \phi_i(x, \theta) \geq 0 (i = 1, 2), (x, \theta) \in [0, \pi] \times (-\infty, 0],
$$

where

$$G(x, y, t) = \frac{1}{\pi} + \frac{2}{\pi} \sum_{n=1}^{\infty} e^{-D_2 n^2 t} \cos nx \cos ny, f(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}},
$$

is the solution of

$$
\frac{\partial G}{\partial t} = D \frac{\partial^2 G}{\partial y^2},
$$

subject to

$$
\frac{\partial G}{\partial y} = 0 \text{ at } y = 0, \pi \text{ and } G(x, y, 0) = \delta(x - y).
$$

And the function $f$ in (2) is called the delay kernel and satisfies $f(t) \geq 0$ for all $t \geq 0$ together with $\int_{0}^{\infty} f(t) dt = 1$.

In system (2), $u_1(x, t)$ and $u_2(x, t)$ represent the densities of the prey and predator populations at location $x$ and time $t$, respectively. The parameters $D_1$ and $D_2$ are the diffusion rates of the prey and predator populations, respectively. The parameter $r_1$ is the intrinsic growth rate of the prey; the parameter $r_2$ is the death rate of the predator; $a_{11}$ is the capture rate of the predator; $a_{21}$ is the conversion rate of the predator by consuming prey; $a_{22}$ is the intra-specific competition rate of the predator. $\phi_1(x, \theta)$ and $\phi_2(x, \theta)$ are nonnegative and Hölder continuous and satisfy

$$
\frac{\partial \phi_1}{\partial x} = \frac{\partial \phi_2}{\partial x} = 0 \text{ for } t \in (-\infty, 0), x = 0, \pi.
$$

The term

$$
\int_{-\infty}^{t} \int_{0}^{\pi} G(x, y, t-s) f(t-s) u_1(y-s) u_2(y-s) dy ds,
$$

represents a time delay due to the gestation of the predator. In [15], the global dynamics of problem (2) - (4) was discussed by using upper-lower solutions and monotone iteration technique.

In this paper, we will further study the corresponding spatial-temporal patterns by traveling wave solutions of system (2). In other words, we will discuss the existence of traveling wave solutions for system (2) by using Schauder’s fixed point theorem and constructing a pair of upper-lower solutions in ([4 15]) and study the asymptotic behavior of the traveling wave by applying the standard asymptotic theory ([10]).
The rest of this paper is organized as follows. In Section 2, we introduce some preliminaries and several lemmas which will be essential to our proofs. In Section 3, it is devoted to establishing the existence of traveling wave solution for system (2). In Section 4, the asymptotic behavior of the traveling wave is obtained by employing the standard asymptotic theory.

2. Preliminaries. In order to establish the existence results of traveling wave solution for system (2), we introduce some basic notations and concepts.

Letting

\[ u_3(x,t) = \int_{-\infty}^{t} \int_{0}^{\pi} G(x,y,t-s)f(t-s)u_1(y-s)u_2(y-s)dyds. \]

Then the system with boundary and initial conditions (3), (4) is equivalent to the system

\[
\begin{align*}
\frac{\partial u_1}{\partial t} &= D_1 \frac{\partial^2 u_1}{\partial x^2} + u_1(x,t)(r_1 - a_{11}u_1(x,t) - a_{12}u_2(x,t)), \\
\frac{\partial u_2}{\partial t} &= D_2 \frac{\partial^2 u_2}{\partial x^2} + a_{21}u_3(x,t) - r_2u_2(x,t) - a_{22}u_1^2(x,t), \\
\frac{\partial u_3}{\partial t} &= D_2 \frac{\partial^2 u_3}{\partial x^2} + \frac{1}{2}(u_1(x,t)u_2(x,t) - u_3(x,t)),
\end{align*}
\]

for \( t > 0, x \in (0, \pi) \), with homogeneous Neumann boundary conditions

\[ \frac{\partial u_1}{\partial x} = \frac{\partial u_2}{\partial x} = \frac{\partial u_3}{\partial x}, \]

and initial conditions

\[
\begin{align*}
u_i(x,\theta) &= \phi_i(x,\theta) \geq 0 (i = 1,2) ,
\phi_3(x,\theta) &= \int_{-\infty}^{t} \int_{0}^{\pi} G(x,y,t-s)f(t-s)u_1(y-s)u_2(y-s)dyds, \\
&\phi_3(x,\theta) \in [0, \pi] \times (-\infty, 0),
\end{align*}
\]

First, we consider the general delayed reaction-diffusion system

\[
\begin{align*}
\frac{\partial u_1}{\partial t} &= D_1 \frac{\partial^2 u_1}{\partial x^2} + f_1(u_{11}(x) + u_{21}(x) + u_{31}(x)), \\
\frac{\partial u_2}{\partial t} &= D_2 \frac{\partial^2 u_2}{\partial x^2} + f_2(u_{12}(x) + u_{22}(x) + u_{32}(x)), \\
\frac{\partial u_3}{\partial t} &= D_3 \frac{\partial^2 u_3}{\partial x^2} + f_3(u_{13}(x) + u_{23}(x) + u_{33}(x)),
\end{align*}
\]

which satisfies the hypotheses:

\((A1)\) \( f_i(0,0,0) = f_i(k_1,k_2,k_3) = 0, i = 1,2,3. \)

\((A2)\) There exist three positive constants \( L_i > 0, (i = 1,2,3) \), such that

\[
\begin{align*}
&|f_1(\phi_1, \varphi_1, \psi_1) - f_1(\phi_2, \varphi_2, \psi_2)| \leq L_1 \| \Phi - \Psi \|, \\
&|f_2(\phi_1, \varphi_1, \psi_1) - f_2(\phi_2, \varphi_2, \psi_2)| \leq L_2 \| \Phi - \Psi \|, \\
&|f_3(\phi_1, \varphi_1, \psi_1) - f_3(\phi_2, \varphi_2, \psi_2)| \leq L_3 \| \Phi - \Psi \|,
\end{align*}
\]

for \( \Phi = (\phi_1, \varphi_1, \psi_1), \Psi = (\phi_2, \varphi_2, \psi_2) \in C([-\tau,0], R^3) \) with \( 0 \leq \varphi_2(s) \leq \varphi_1(s) \leq M_1, 0 \leq \varphi_2(s) \leq \varphi_1(s) \leq M_2, 0 \leq \varphi_3(s) \leq \psi_1(s) \leq M_3, \) \( s \in [-\tau,0], \) where \( M_i > k_i \) \( (i = 1,2,3) \) are the positive constants.

Then system (9) satisfies the following partial quasi-monotonicity conditions (PQM), i.e.,

\((PQM)\) There exist three positive constants \( \beta_1, \beta_2, \beta_3 > 0 \) so that

\[
\begin{align*}
f_{13}(\phi_1, \varphi_1, \psi_1) - f_{13}(\phi_2, \varphi_2, \psi_2) + \beta_1(\phi_1(0) - \phi_2(0)) \geq 0, \\
f_{23}(\phi_1, \varphi_1, \psi_1) - f_{23}(\phi_2, \varphi_2, \psi_2) + \beta_2(\varphi_1(0) - \varphi_2(0)) \geq 0, \\
f_{33}(\phi_1, \varphi_1, \psi_1) - f_{33}(\phi_2, \varphi_2, \psi_2) + \beta_3(\psi_1(0) - \psi_2(0)) \geq 0,
\end{align*}
\]
where \( \phi_i, \varphi_i, \psi_i \in C([\tau, 0], R), i = 1, 2, 3 \) with \( 0 \leq \phi_2(s) \leq \phi_1(s) \leq M_1, \) \( 0 \leq \varphi_2(s) \leq \varphi_1(s) \leq M_2, \) \( 0 \leq \psi_2(s) \leq \psi_1(s) \leq M_3, \) \( s \in [\tau, 0]. \)

**Definition 1.** A traveling wave solution of (9) is a special solution of the form 
\[ u(x, t) = \Phi(x + ct), \]
where \( u = (u_1(x, t), u_2(x, t), u_3(x, t)), \) \( \Phi = (\phi(x, t), \varphi(x, t), \psi(x, t)) \in C^2(R, R^3) \) is the wave profile that propagates through the one-dimension spatial domain \( R \) at the constant wave speed \( c > 0. \)

We denote the traveling wave coordinate \( x + ct \) by \( t \) and derive from (9) that
\[
\begin{align*}
D_1 \phi''(t) - c \phi'(t) + f_{c1}(\phi(t), \varphi(t), \psi(t)) &= 0, \\
D_2 \varphi''(t) - c \varphi'(t) + f_{c2}(\phi(t), \varphi(t), \psi(t)) &= 0, \\
D_3 \psi''(t) - c \psi'(t) + f_{c3}(\phi(t), \varphi(t), \psi(t)) &= 0,
\end{align*}
\]
with asymptotic boundary conditions
\[
\lim_{t \to -\infty} \phi(t) = \phi_-, \quad \lim_{t \to +\infty} \phi(t) = \phi_+; \quad \lim_{t \to -\infty} \varphi(t) = \varphi_-, \quad \lim_{t \to +\infty} \varphi(t) = \varphi_+; \quad \lim_{t \to -\infty} \psi(t) = \psi_-, \quad \lim_{t \to +\infty} \psi(t) = \psi_+;
\]
where \( f_{c1} : x_c = C([-c, 0], R^3) \to R, \) is given by \( f_{c1}(\phi, \varphi, \psi) = f_i(\phi^c, \varphi^c, \psi^c), \)
\( \phi^c(s) = \phi(cs), \varphi^c(s) = \varphi(cs), \psi^c(s) = \psi(cs), s \in [\tau, 0], i = 1, 2, 3. \)

Without loss of generality, we will assume that \( (\phi_-, \varphi_-, \psi_-) = (0, 0, 0) \) and \( (\phi_+, \varphi_+, \psi_+) = (k_1, k_2, k_3) \) are steady states of (9), and seek for the traveling wave solution connecting these two steady states.

**Definition 2.** A pair of continuous functions \( \bar{\rho} = (\bar{\phi}, \bar{\varphi}, \bar{\psi}) \) and \( \rho = (\phi, \varphi, \psi) \) are called an upper solution and a lower solution for system (9), respectively, if \( \bar{\rho} \) and \( \rho \) are twice differentiable almost everywhere in \( R \) and essentially bounded on \( R, \) and content
\[
\begin{align*}
D_1 \bar{\phi}''(t) - c \bar{\phi}'(t) + f_{c1}(\bar{\phi}(t), \bar{\varphi}(t), \bar{\psi}(t)) &\leq 0, \quad \text{a.e. in } R, \\
D_2 \bar{\varphi}''(t) - c \bar{\varphi}'(t) + f_{c2}(\bar{\phi}(t), \bar{\varphi}(t), \bar{\psi}(t)) &\leq 0, \quad \text{a.e. in } R, \\
D_3 \bar{\psi}''(t) - c \bar{\psi}'(t) + f_{c3}(\bar{\phi}(t), \bar{\varphi}(t), \bar{\psi}(t)) &\leq 0, \quad \text{a.e. in } R,
\end{align*}
\]
and
\[
\begin{align*}
D_1 \rho''(t) - c \rho'(t) + f_{c1}(\rho(t), \varphi(t), \psi(t)) &\geq 0, \quad \text{a.e. in } R, \\
D_2 \rho''(t) - c \rho'(t) + f_{c2}(\rho(t), \varphi(t), \psi(t)) &\geq 0, \quad \text{a.e. in } R, \\
D_3 \rho''(t) - c \rho'(t) + f_{c3}(\rho(t), \varphi(t), \psi(t)) &\geq 0, \quad \text{a.e. in } R.
\end{align*}
\]

We assume that a pair of upper and lower solutions \( (\bar{\phi}(t), \bar{\varphi}(t), \bar{\psi}(t)) \) and \( (\rho(t), \varphi(t), \psi(t)) \) are given such that

(P1) \( (0, 0, 0) \leq (\bar{\phi}, \bar{\varphi}, \bar{\psi}) \leq (\bar{\phi}, \bar{\varphi}, \bar{\psi}) \leq (M_1, M_2, M_3), t \in R. \)

(P2) \( \lim_{t \to -\infty} (\bar{\phi}(t), \bar{\varphi}(t), \bar{\psi}(t)) = (0, 0, 0), \quad \lim_{t \to +\infty} (\bar{\phi}(t), \bar{\varphi}(t), \bar{\psi}(t)) = (k_1, k_2, k_3). \)

We will look for traveling wave solutions of system (9) in the following profile set
\[
\Gamma((\phi, \varphi, \psi), (\bar{\phi}, \bar{\varphi}, \bar{\psi})) = \left\{ \begin{array}{ll}
(\phi(t) & \text{is nondecreasing in } R; \\
(\phi, \varphi, \psi)(t) & \leq (\phi, \varphi, \psi)(t) \leq (\bar{\phi}, \bar{\varphi}, \bar{\psi})(t).
\end{array} \right.
\]
Obviously, \( \Gamma((\phi, \varphi, \psi), (\bar{\phi}, \bar{\varphi}, \bar{\psi})) \) is non-empty, closed, convex and bound.

For the constants \( \beta_i > 0 \) in (10), define \( H : C(R, R^3) \to C(R, R^3) \) by
\[
\begin{align*}
H_1(\phi, \varphi, \psi)(t) &= f_{c1}(\phi(t), \varphi(t), \psi(t)) + \beta_1 \phi(t), \quad \phi, \varphi, \psi \in C(R, R), \\
H_2(\phi, \varphi, \psi)(t) &= f_{c2}(\phi(t), \varphi(t), \psi(t)) + \beta_2 \varphi(t), \quad \phi, \varphi, \psi \in C(R, R), \\
H_3(\phi, \varphi, \psi)(t) &= f_{c3}(\phi(t), \varphi(t), \psi(t)) + \beta_3 \psi(t), \quad \phi, \varphi, \psi \in C(R, R).
\end{align*}
\]
Let \( A1 \) and \( (10) \) hold, then for any \( \psi \) admit the following properties:

**Lemma 1.** Assume that \( (A1) \) and \( (10) \) hold, then for any \( \phi \), \( \varphi \), \( \psi \) with \( M \), \( N \), for any \( t \in R \) we have

The operators \( H_1, H_2, H_3 \) have the following properties:

**Lemma 2.** Assume that \( (A1) \) and \( (10) \) hold, then for any \( 0, 0, 0 \leq (\phi, \varphi, \psi) \leq (k_1, k_2, k_3) \), we have

(i) \( H_2(\phi, \varphi, \psi)(t) \geq 0 \) is nondecreasing for \( t \in R \),

(ii) \( H_2(\phi_2, \varphi_2, \psi_2)(t) \leq H_2(\phi_1, \varphi_1, \psi_1)(t) \) for \( t \in R \) with \( 0 \leq \phi_2(t) \leq \phi_1(t) \leq M_1 \), \( 0 \leq \varphi_2(t) \leq \varphi_1(t) \leq M_2 \), \( 0 \leq \psi_2(t) \leq \psi_1(t) \leq M_3 \).

From the definitions of \( H_1, H_2, H_3 \) in \( (19)-(21) \), system \( (11) \) can be rewritten as

\[
\begin{align*}
D_1 \phi''(t) - c \phi'(t) - \beta_1 \phi(t) + H_1(\phi, \varphi, \psi)(t) &= 0, \\
D_2 \varphi''(t) - c \varphi'(t) - \beta_2 \varphi(t) + H_2(\phi, \varphi, \psi)(t) &= 0, \\
D_3 \psi''(t) - c \psi'(t) - \beta_3 \psi(t) + H_3(\phi, \varphi, \psi)(t) &= 0.
\end{align*}
\]

We define

\[
\begin{align*}
\lambda_1 &= \frac{c - \sqrt{c^2 + 4\beta_1 D_1}}{2D_1}, \\
\lambda_2 &= \frac{c + \sqrt{c^2 + 4\beta_1 D_1}}{2D_1}, \\
\lambda_3 &= \frac{c - \sqrt{c^2 + 4\beta_2 D_2}}{2D_2}, \\
\lambda_4 &= \frac{c + \sqrt{c^2 + 4\beta_2 D_2}}{2D_2}, \\
\lambda_5 &= \frac{c - \sqrt{c^2 + 4\beta_3 D_2}}{2D_2}, \\
\lambda_6 &= \frac{c + \sqrt{c^2 + 4\beta_3 D_2}}{2D_2}.
\end{align*}
\]

Let

\[
C_k(R, R^3) = \{(\phi, \varphi, \psi) \in C(R, R^3) : (0, 0, 0) \leq (\phi, \varphi, \psi) \leq (M_1, M_2, M_3) \},
\]

and define \( F = (F_1, F_2, F_3) : C_k(R, R^3) \to C(R, R^3) \) by

\[
\begin{align*}
F_1(\phi, \varphi, \psi)(t) &= \frac{1}{D_1(\lambda_2 - \lambda_1)} \left[ \int_{-\infty}^{t} e^{\lambda_1(t-s)} H_1(\phi, \varphi, \psi)(s) ds \right. \\
&\quad \left. + \int_{t}^{+\infty} e^{\lambda_2(t-s)} H_1(\phi, \varphi, \psi)(s) ds \right], \\
F_2(\phi, \varphi, \psi)(t) &= \frac{1}{D_2(\lambda_4 - \lambda_3)} \left[ \int_{-\infty}^{t} e^{\lambda_3(t-s)} H_2(\phi, \varphi, \psi)(s) ds \right. \\
&\quad \left. + \int_{t}^{+\infty} e^{\lambda_4(t-s)} H_2(\phi, \varphi, \psi)(s) ds \right], \\
F_3(\phi, \varphi, \psi)(t) &= \frac{1}{D_3(\lambda_5 - \lambda_6)} \left[ \int_{-\infty}^{t} e^{\lambda_5(t-s)} H_3(\phi, \varphi, \psi)(s) ds \right. \\
&\quad \left. + \int_{t}^{+\infty} e^{\lambda_6(t-s)} H_3(\phi, \varphi, \psi)(s) ds \right].
\end{align*}
\]
From Lemma 1 - Lemma 5 we see that system \((11)\) satisfying \((P1)\) and \((P2)\), then system \((11)\) has a traveling wave solution.\[ F(\phi, \varphi, \psi)(t) = \frac{1}{D_2(\lambda_6 - \lambda_5)} \int_{-\infty}^{t} e^{\lambda_5(t-s)} H_3(\phi, \varphi, \psi)(s)ds \]

for \((\phi, \varphi, \psi) \in C_b(\mathbb{R}, \mathbb{R}^3)\). It is easy to see that \(F_i(\phi, \varphi, \psi), (i = 1, 2, 3)\) satisfy\[
\begin{align*}
D_1F_i''(\phi, \varphi, \psi) - cF_i'(\phi, \varphi, \psi) - \beta_1F_i(\phi, \varphi, \psi) + H_1(\phi, \varphi, \psi) &= 0, \\
D_2F_i''(\phi, \varphi, \psi) - cF_i'(\phi, \varphi, \psi) - \beta_2F_i(\phi, \varphi, \psi) + H_2(\phi, \varphi, \psi) &= 0, \\
D_3F_i''(\phi, \varphi, \psi) - cF_i'(\phi, \varphi, \psi) - \beta_3F_i(\phi, \varphi, \psi) + H_3(\phi, \varphi, \psi) &= 0.
\end{align*}
\]

Thus, a fixed point of the operator \(F\) is a solution of \((22)\). On the other hand, a solution of \((22)\) is also a fixed point of the operator \(F\). Obviously, a bounded solution \(\Phi\) of \((22)\) or a bounded fixed point of the operator \(F\) allows bounded first and second derivatives.

Corresponding to Lemma 1 and Lemma 2, we have the following results for \(F\).

**Lemma 3.** \((4)\) Assume that \((A2)\) holds, then \(F = (F_1, F_2, F_3)\) is continuous with respect to the norm \(|\cdot|\).

**Lemma 4.** \((4)\) Assume that \((A2)\) and \((10)\) hold, then

\[ F(\Gamma((\phi, \varphi, \psi), (\overline{\phi}, \overline{\varphi}, \overline{\psi}))) \subset \Gamma((\phi, \varphi, \psi), (\overline{\phi}, \overline{\varphi}, \overline{\psi})). \]

**Lemma 5.** \((4)\) Assume the \((10)\) holds, then

\[ F : \Gamma((\phi, \varphi, \psi), (\overline{\phi}, \overline{\varphi}, \overline{\psi})) \rightarrow \Gamma((\phi, \varphi, \psi), (\overline{\phi}, \overline{\varphi}, \overline{\psi})), \]

is compact.

Now, we are in a position to state and prove the following theorem.

**Theorem 1.** Assume that \(f\) satisfies \((A1)\), \((A2)\) and \((10)\) conditions, and system \((11)\) has a pair of upper and lower solutions \(\bar{\Phi} = (\bar{\phi}, \bar{\varphi}, \bar{\psi})\), and \(\Psi = (\phi, \varphi, \psi)\) for \((11)\) satisfying \((P1)\) and \((P2)\), then system \((11)\) has a traveling wave solution.

**Proof.** From Lemma 1 - Lemma 3 we see that \(FT \subset \Gamma\) and \(F\) is compact. By Schauder’s fixed point theorem we look for a fixed point \((\hat{\phi}, \hat{\varphi}, \hat{\psi})\) of \(F\) in \(\Gamma((\bar{\phi}, \bar{\varphi}, \bar{\psi}), (\overline{\phi}, \overline{\varphi}, \overline{\psi}))\), which gives a solutions of \((11)\).

From \((P2)\) and inequality

\[ (0, 0, 0) \leq (\phi, \varphi, \psi) \leq (\hat{\phi}, \hat{\varphi}, \hat{\psi}) \leq (\overline{\phi}, \overline{\varphi}, \overline{\psi}) \leq (M_1, M_2, M_3), \]

we know that

\[ \lim_{t \rightarrow \infty} (\hat{\phi}, \hat{\varphi}, \hat{\psi}) = (0, 0, 0); \quad \lim_{t \rightarrow \infty} (\hat{\phi}, \hat{\varphi}, \hat{\psi}) = (k_1, k_2, k_3). \]

Therefore, the fixed point \((\hat{\phi}, \hat{\varphi}, \hat{\psi})\) satisfies the asymptotic boundary conditions \((12)\), then system \((11)\) has a traveling wave solutions. \(\Box\)

3. Existence of traveling wave solution for system \((2)\). In this section, we will discuss the existence of traveling wave solution for model \((2)\).

It is not difficult to prove that system \((2)\) always has a trivial steady-state solution \(E_0(0, 0, 0)\) and a semi-trivial steady-state solution \(E_1(\frac{r_1 a_{21}}{a_{11}}, 0, 0)\). If \(r_1 a_{21} > r_2 a_{11}\) holds, then system \((2)\) has a unique position steady state \(E^*(k_1, k_2, k_3)\), where

\[ k_1 = \frac{r_1 a_{22} + r_2 a_{12}}{a_{11} a_{22} + a_{12} a_{21}}; \quad k_2 = \frac{r_1 a_{21} - r_2 a_{11}}{a_{11} a_{22} + a_{12} a_{21}}; \]
The functions

\[ f \]

then

\[ f \]

denoting the traveling wave coordinate \( x + ct \) still by \( t \), we derive from \( \text{[2]} \) that

\[
\begin{align*}
D_1\phi''(t) - c\phi'(t) + \phi(t)(r_1 - a_{11}\phi(t) - a_{12}\phi(t)) &= 0, \\
D_2\phi''(t) - c\phi'(t) + a_{21}\psi(t) - r_2\phi(t) - a_{22}\phi^2(t) &= 0, \\
D_2\psi''(t) - c\psi'(t) + \frac{1}{\tau}(\phi(t)\phi(t) - \psi(t)) &= 0.
\end{align*}
\]

We are interested in the possibility of a transition between the equilibria \( E_0 \) and \( E_* \) in the form of a traveling wave solution.

**Lemma 6.** The functions \( f_{c1}(\phi_1, \varphi_1, \psi_1) \), \( f_{c2}(\phi_2, \varphi_1, \psi_1) \), \( f_{c3}(\phi_3, \varphi_1, \psi_1) \) of system \( \text{[3]} \) satisfy \( (PQM) \).

**Proof.** For \( \forall \phi_i, \varphi_i, \psi_i \in C([-\tau, 0], R), (i = 1, 2, 3) \), with \( 0 \leq \phi_2(s) \leq \phi_1(s) \leq M_1, 0 \leq \varphi_2(s) \leq \varphi_1(s) \leq M_2, 0 \leq \psi_2(s) \leq \psi_1(s) \leq M_3 \).

For \( f_{c1}(\phi_1, \varphi_1, \psi_1) \), we have

\[
\begin{align*}
f_{c1}(\phi_{1t}, \varphi_{1t}, \psi_{1t}) - f_{c1}(\phi_{2t}, \varphi_{1t}, \psi_{2t}) &= \phi_1(0)(r_1 - a_{11}\phi(0) - a_{12}\varphi(0)) - \phi_2(0)(r_2 - a_{11}\phi(0) - a_{12}\varphi(0)) \\
&= [r_1 - a_{11}(\phi_1(0) + \phi_2(0)) - a_{12}\varphi_1(0)][\phi_1(0) - \phi_2(0)] \\
&\geq (r_1 - 2a_{11}M_1 - a_{12}M_2)[\phi_1(0) - \phi_2(0)].
\end{align*}
\]

Choosing appropriate constants such that \( \beta_1 = -(r_1 - 2a_{11}M_1 - a_{12}M_2) > 0 \), then \( f_{c1}(\phi_1, \varphi_1, \psi_1) - f_{c1}(\phi_2, \varphi_1, \psi_2) + \beta_1[\phi_1(0) - \phi_2(0)] \geq 0 \).

Also

\[
\begin{align*}
f_{c1}(\phi_{1t}, \varphi_{1t}, \psi_{1t}) - f_{c1}(\phi_{1t}, \varphi_{2t}, \psi_{1t}) &= \phi_1(0)(r_1 - a_{11}\phi(0) - a_{12}\varphi(0)) - \phi_1(0)(r_1 - a_{11}\phi(0) - a_{12}\varphi(0)) \\
&= -a_{12}\phi_1(0)[\varphi_1(0) - \varphi_2(0)] \\
&\leq 0.
\end{align*}
\]

For \( f_{c2}(\phi_2, \varphi_1, \psi_1) \), we have

\[
\begin{align*}
f_{c2}(\phi_{1t}, \varphi_{1t}, \psi_{1t}) - f_{c2}(\phi_{2t}, \varphi_{2t}, \psi_{2t}) &= a_{21}\psi_1(0) - r_2\varphi_1(0) - a_{22}\varphi_1^2(0) - a_{21}\psi_2(0) + r_2\varphi_2(0) + a_{22}\varphi_2^2(0) \\
&= a_{21}[\psi_1(0) - \psi_2(0)] - r_2[\varphi_1(0) - \varphi_2(0)] - a_{22}[\varphi_1^2(0) - \varphi_2^2(0)] \\
&\geq -[r_2 + a_{22}(\varphi_1(0) + \varphi_2(0))][\varphi_1(0) - \varphi_2(0)] \\
&\geq -(r_2 + 2a_{22}M_2)[\varphi_1(0) - \varphi_2(0)].
\end{align*}
\]

Let \( \beta_2 = r_2 + 2a_{22}M_2 \), then \( f_{c2}(\phi_1, \varphi_1, \psi_1) - f_{c2}(\phi_2, \varphi_2, \psi_2) + \beta_2[\varphi_1(0) - \varphi_2(0)] \geq 0 \).

For \( f_{c3}(\phi_1, \varphi_1, \psi_1) \), we have

\[
\begin{align*}
f_{c3}(\phi_{1t}, \varphi_{1t}, \psi_{1t}) - f_{c3}(\phi_{1t}, \varphi_{1t}, \psi_{2t}) &= \frac{1}{\tau}[\phi_1(0)\varphi_1(0) - \frac{1}{\tau}\psi_1(0) - \frac{1}{\tau}\phi_1(0)\varphi_1(0) + \frac{1}{\tau}\psi_2(0) \\
&= \frac{1}{\tau}[\psi_1(0) - \psi_2(0)].
\end{align*}
\]
Let $\beta_3 = \frac{1}{\tau}$, then
$$f_{c3}(\phi_1, \varphi_1, \psi_1) - f_{c3}(\phi_1, \varphi_1, \psi_2) + \beta_3[\psi_1(0) - \psi_2(0)] \geq 0,$$
and
$$f_{c3}(\phi_{11}, \varphi_{11}, \psi_{1k}) - f_{c3}(\phi_{21}, \varphi_{21}, \psi_{2k})$$
$$= \frac{1}{\tau} \phi_1(0) \varphi_1(0) - \frac{1}{\tau} \psi_1(0) - \frac{1}{\tau} \phi_2(0) \varphi_2(0) + \frac{1}{\tau} \psi_1(0)$$
$$= - \frac{1}{\tau} [\phi_1(0) \varphi_1(0) - \phi_2(0) \varphi_2(0)]$$
$$\geq 0.$$

Then we complete the proof. \(\square\)

Let $c > c^* = \max\{2\sqrt{D_1r_1}, 2\sqrt{D_2(a_{21}M_3 - r_2)}\}$. There exist $\lambda_i > 0$, ($i = 1, 2, 3$) so that
$$D_1 \lambda_1^2 - c \lambda_1 + r_1 = 0, \; D_2 \lambda_2^2 - c \lambda_2 + a_{21}M_3 - r_2 = 0, \; D_2 \lambda_3^2 - c \lambda_3 - \frac{1}{\tau} = 0.$$

We find that there exist $\varepsilon_i > 0$ ($i = 0, 1, 2, 3, 4, 5, 6$) satisfying
$$\begin{cases}
a_{11}s_1 - r_1 + a_{11}k_1 > \varepsilon_0, \\
r_2s_2 - a_{21}M_3 + r_2k_2 + a_{22}k_2^2 > \varepsilon_0, \\
\varepsilon_3 - (M_1M_2 - k_3) > \varepsilon_0, \\
a_{11}s_4 - a_{12}s_2 > \varepsilon_0, \\
a_{22}s_5 - r_2 - a_{22}k_2 > \varepsilon_0.
\end{cases}
$$

(25)

For the above constants and suitable constants $t_i > 0$ ($i = 1, 2, 3, 4, 5, 6$) and $t_6 > t_5 > t_4 > 0$, we define the continuous functions $\Phi = (\phi_1(t), \varphi_1(t), \psi_1(t))$, and $\Psi = (\phi_2(t), \varphi_2(t), \psi_2(t))$ as follows
$$\phi_1(t) = \begin{cases} e^{\lambda_1 t} & t \leq t_1, \\
 k_1 + \varepsilon_1 e^{-\lambda_1 t} & t > t_1,
\end{cases} \quad \phi_2(t) = \begin{cases} 0 & t \leq t_4, \\
 k_1 - \varepsilon_4 e^{-\lambda_1 t} & t > t_4,
\end{cases}$$
$$\varphi_1(t) = \begin{cases} e^{\lambda_2 t} & t \leq t_2, \\
 k_2 + \varepsilon_2 e^{-\lambda_2 t} & t > t_2,
\end{cases} \quad \varphi_2(t) = \begin{cases} 0 & t \leq t_5, \\
 k_2 - \varepsilon_5 e^{-\lambda_2 t} & t > t_5,
\end{cases}$$
$$\psi_1(t) = \begin{cases} e^{\lambda_3 t} & t \leq t_3, \\
 k_3 + \varepsilon_3 e^{-\lambda_3 t} & t > t_3,
\end{cases} \quad \psi_2(t) = \begin{cases} 0 & t \leq t_6, \\
 k_3 - \varepsilon_6 e^{-\lambda_3 t} & t > t_6,
\end{cases}$$

where $\lambda > 0$ is a constant to be chosen later.

**Lemma 7.** Assume that (25) holds, then $\Phi(t) = (\phi_1(t), \varphi_1(t), \psi_1(t))$ is an upper solution of system [24].

**Proof.** For $\phi_1(t)$, we shall prove that
$$D_1 \phi_1'(t) - c \phi_1'(t) + \phi_1(t) (r_1 - a_{11} \phi_1(t) - a_{12} \varphi_2(t)) \leq 0.$$

When $t \leq t_1$, $\phi_1(t) = e^{\lambda_1 t}$. We have
$$D_1 \phi_1''(t) - c \phi_1'(t) + \phi_1(t) (r_1 - a_{11} \phi_1(t) - a_{12} \varphi_2(t))$$
$$= D_1 \lambda_1^2 e^{\lambda_1 t} - c \lambda_1 e^{\lambda_1 t} + e^{\lambda_1 t} (r_1 - a_{11} e^{\lambda_1 t} - a_{12} \varphi_2(t))$$
$$\leq D_1 \lambda_1^2 e^{\lambda_1 t} - c \lambda_1 e^{\lambda_1 t} + e^{\lambda_1 t} r_1$$
$$= (D_1 \lambda_1^2 - c \lambda_1 + r_1) e^{\lambda_1 t}$$
$$= 0.$$
When \( t > t_1 \), \( \phi_1(t) = k_1 + \varepsilon_1 e^{-\lambda t} \). One has
\[
D_1 \phi_1''(t) - c \phi_1'(t) + \phi_1(t)(r_1 - a_{11} \phi_1(t) - a_{12} \varphi_2(t))
\]
\[
= \varepsilon_1 D_1 \lambda^2 e^{-\lambda t} + c \lambda \varepsilon_1 e^{-\lambda t} + (k_1 + \varepsilon_1 e^{-\lambda t})[(r_1 - a_{11} k_1) e^{-\lambda t} - a_{11} \varepsilon_1].
\]

Let
\[
I_1(\lambda) = [D_1 \varepsilon_1 \lambda^2 + c \lambda \varepsilon_1 + (k_1 + \varepsilon_1 e^{-\lambda t})][(r_1 - a_{11} k_1) e^{-\lambda t} - a_{11} \varepsilon_1].
\]

Then \( a_{11} \varepsilon_1 - r_1 + a_{11} k_1 > \varepsilon_0 \) implies that \( I_1(0) < 0 \) and there exists \( \lambda_1^* \) so that
\[
D_1 \phi_1''(t) - c \phi_1'(t) + \phi_1(t)(r_1 - a_{11} \phi_1(t) - a_{12} \varphi_2(t)) \leq 0 \text{ for all } \lambda \in (0, \lambda_1^*).
\]

For \( \varphi_1(t) \), we shall prove that
\[
D_2 \varphi_1''(t) - c \varphi_1'(t) + a_{21} \psi(t) - r_2 \varphi_1(t) - a_{22} \varphi_1^2(t) \leq 0.
\]
When \( t \leq t_2 \), \( \varphi_1(t) = e^{\lambda_2 t} \). We have
\[
D_2 \varphi_1''(t) - c \varphi_1'(t) + a_{21} \psi(t) - r_2 \varphi_1(t) - a_{22} \varphi_1^2(t)
\]
\[
= D_2 \lambda_2^2 e^{\lambda_2 t} - c \lambda_2 \varepsilon_2 e^{-\lambda_2 t} + a_{21} \psi(t) - r_2 e^{\lambda_2 t} - a_{22} e^{2 \lambda_2 t}
\]
\[
\leq e^{\lambda_2 t}(D_2 \lambda_2^2 - c \lambda_2 + a_{21} M_3 - r_2)
\]
\[
= 0.
\]
When \( t > t_2 \), \( \varphi_1(t) = k_2 + \varepsilon_2 e^{-\lambda t} \). We obtain
\[
D_2 \varphi_1''(t) - c \varphi_1'(t) + a_{21} \psi(t) - r_2 \varphi_1(t) - a_{22} \varphi_1^2(t)
\]
\[
= D_2 \varepsilon_2 \lambda^2 e^{-\lambda t} + c \varepsilon_2 e^{-\lambda t} + a_{21} \psi(t) - r_2 (k_2 + \varepsilon_2 e^{-\lambda t}) - a_{22} (k_2 + \varepsilon_2 e^{-\lambda t})^2
\]
\[
\leq e^{-\lambda t}[D_2 \lambda^2 + c \lambda \varepsilon_2 - r_2 \varepsilon_2 + (a_{21} M_3 - r_2 k_2 - a_{22} k_2^2)e^{\lambda t}].
\]

Let
\[
I_2(\lambda) = D_2 \varepsilon_2 \lambda^2 + c \varepsilon_2 \lambda - r_2 \varepsilon_2 + (a_{21} M_3 - r_2 k_2 - a_{22} k_2^2)e^{\lambda t}.
\]
Then \( r_2 \varepsilon_2 - a_{21} M_3 + r_2 k_2 + a_{22} k_2^2 > \varepsilon_0 \) implies that \( I_2(0) < 0 \) and there exists \( \lambda_2^* \) so that
\[
D_2 \varphi_1''(t) - c \varphi_1'(t) + a_{21} \psi(t) - r_2 \varphi_1(t) - a_{22} \varphi_1^2(t) \leq 0 \text{ for all } \lambda \in (0, \lambda_2^*).
\]

For \( \psi_1(t) \), we shall prove that
\[
D_2 \psi_1''(t) - c \psi_1'(t) + \frac{1}{\tau} \phi_2(t) \varphi_2(t) - \frac{1}{\tau} \psi_1(t) \leq 0.
\]
When \( t \leq t_3 \), \( \psi_1(t) = e^{\lambda_3 t} \). We have
\[
D_2 \psi_1''(t) - c \psi_1'(t) + \frac{1}{\tau} \phi_2(t) \varphi_2(t) - \frac{1}{\tau} \psi_1(t)
\]
\[
= D_2 \lambda_3^2 e^{\lambda_3 t} - c \lambda_3 e^{\lambda_3 t} - \frac{1}{\tau} e^{\lambda_3 t}
\]
\[
= e^{\lambda_3 t}[D_2 \lambda_3^2 - c \lambda_3 - \frac{1}{\tau}]
\]
\[
= 0.
\]
When \( t > t_3 \), \( \psi_1(t) = k_3 + \varepsilon_3 e^{-\lambda t} \). We get
\[
D_2 \psi_1''(t) - c \psi_1'(t) + \frac{1}{\tau} \phi_2(t) \varphi_2(t) - \frac{1}{\tau} \psi_1(t)
\]
\[
\leq D_2 \varepsilon_3 \lambda^2 e^{-\lambda t} + c \varepsilon_3 \lambda e^{-\lambda t} + \frac{1}{\tau} M_1 M_2 - \frac{1}{\tau} (k_3 + \varepsilon_3 e^{-\lambda t})
\]
\[
= e^{-\lambda t}[D_2 \varepsilon_3 \lambda^2 + c \varepsilon_3 \lambda + \frac{1}{\tau} (M_1 M_2 - k_3)e^{\lambda t} - \frac{\varepsilon_3}{\tau}].
\]
Let 
\[ I_3(\lambda) = D_2 \varepsilon_3 \lambda^2 + c \varepsilon_3 \lambda + \frac{1}{\tau} (M_1 M_2 - k_3) e^{\lambda \tau} - \frac{\varepsilon_3}{\tau}. \]

Then \( \frac{\varepsilon_3}{\tau} - \frac{1}{\tau} (M_1 M_2 - k_3) > \varepsilon_0 \) implies that \( I_3(0) < 0 \) and there exists \( \lambda_* \) so that 
\[ D_2 \psi_1''(t) - c \psi_1'(t) + \frac{1}{\tau} \phi_2(t) \varphi_2(t) - \psi_1(t) \leq 0 \]
for all \( \lambda \in (0, \lambda_*^0) \).

We let \( \lambda = (0, \min\{\lambda_*^1, \lambda_*^2, \lambda_*^3\}) \), then the conclusion is true. This completes the proof. \( \square \)

**Lemma 8.** Assume that \( [25] \) holds, then \( \Psi(t) = (\phi_2(t), \varphi_2(t), \psi_2(t)) \) is a lower solution of system \( [24] \).

**Proof.** For \( \phi_2(t) \), we shall prove that 
\[ D_1 \phi_2''(t) - c \phi_2'(t) + \phi_2(t)(r_1 - a_{11} \phi_2(t) - a_{12} \varphi_1(t)) \geq 0. \]
When \( t \leq t_4 \), \( \phi_2(t) = 0 \). We have
\[ D_1 \phi_2''(t) - c \phi_2'(t) + \phi_2(t)(r_1 - a_{11} \phi_2(t) - a_{12} \varphi_1(t)) = 0. \]
When \( t > t_4 \), \( \phi_2(t) = k_1 - \varepsilon_4 e^{-\lambda t} \). We have
\[ D_1 \phi_2''(t) - c \phi_2'(t) + \phi_2(t)(r_1 - a_{11} \phi_2(t) - a_{12} \varphi_1(t)) \]
\[ = -D_1 \varepsilon_4 \lambda^2 e^{-\lambda t} - c \varepsilon_4 e^{-\lambda t} + (k_1 - \varepsilon_4 e^{-\lambda t})(r_1 - a_{11} - \varepsilon_4 e^{-\lambda t} - a_{12} + \varepsilon_2 e^{-\lambda t})] \]
\[ = -D_1 \varepsilon_4 \lambda^2 e^{-\lambda t} - c \varepsilon_4 e^{-\lambda t} + (k_1 - \varepsilon_4 e^{-\lambda t})(a_{11} \varepsilon_4 - a_{12} \varepsilon_2) e^{-\lambda t} \]
\[ = e^{-\lambda t}[-D_1 \varepsilon_4 \lambda^2 - c \varepsilon_4 + k_1(a_{11} \varepsilon_4 - a_{12} \varepsilon_2) - \varepsilon_4(a_{11} \varepsilon_4 - a_{12} \varepsilon_2) e^{-\lambda t}]. \]
Let
\[ I_4(\lambda) = -D_1 \varepsilon_4 \lambda^2 - c \varepsilon_4 + k_1(a_{11} \varepsilon_4 - a_{12} \varepsilon_2) - \varepsilon_4(a_{11} \varepsilon_4 - a_{12} \varepsilon_2) e^{-\lambda t}. \]
Then \( a_{11} \varepsilon_4 - a_{12} \varepsilon_2 > \varepsilon_0 \) implies that \( I_4(0) > 0 \) and there exists \( \lambda_*^1 \) so that 
\[ D_1 \phi_2''(t) - c \phi_2'(t) + \phi_2(t)(r_1 - a_{11} \phi_2(t) - a_{12} \varphi_1(t)) \leq 0 \]
for all \( \lambda \in (0, \lambda_*^1) \).

For \( \varphi_2(t) \), we shall prove that 
\[ D_2 \varphi_1''(t) - c \varphi_1'(t) + a_{21} \psi(t) - r_2 \varphi_1(t) - a_{22} \varphi_2^2(t) \leq 0. \]
When \( t \leq t_5 \), \( \varphi_2(t) = 0 \). One has
\[ D_2 \varphi_2''(t) - c \varphi_2'(t) + a_{21} \psi(t) - r_2 \varphi_2(t) - a_{22} \varphi_2^2(t) = a_{21} \psi(t) > 0. \]
When \( t > t_5 \), \( \varphi_2(t) = k_2 - \varepsilon_5 e^{-\lambda t} \). We get
\[ D_2 \varphi_2''(t) - c \varphi_2'(t) + a_{21} \psi(t) - r_2 \varphi_2(t) - a_{22} \varphi_2^2(t) \]
\[ = -D_2 \varepsilon_5 \lambda^2 e^{-\lambda t} - c \varepsilon_5 e^{-\lambda t} + a_{21} \psi(t) + \varphi_2(t)(-r_2 - a_{22} k_2 - \varepsilon_5 e^{-\lambda t}) \]
\[ \geq e^{-\lambda t}[-D_2 \varepsilon_5 \lambda^2 - c \varepsilon_5 \lambda + \varphi_2(t)(-r_2 - a_{22} k_2) e^{\lambda t} + a_{22} \varepsilon_5]. \]
Let
\[ I_5(\lambda) = -D_2 \varepsilon_5 \lambda^2 - c \varepsilon_5 \lambda + \varphi_2(t)(-r_2 - a_{22} k_2) e^{\lambda t} + a_{22} \varepsilon_5. \]
Then \( a_{22} \varepsilon_5 - r_2 - a_{22} k_2 > \varepsilon_0 \) and \( \varphi_2(t) > 0 \) implies that \( I_5(0) > 0 \) and there exists \( \lambda_*^2 \) so that 
\[ D_2 \varphi_1''(t) - c \varphi_1'(t) + a_{21} \psi(t) - r_2 \varphi_1(t) - a_{22} \varphi_1^2(t) \leq 0 \]
for all \( \lambda \in (0, \lambda_*^2) \).

For \( \psi_2(t) \), we shall prove that 
\[ D_2 \psi_2''(t) - c \psi_2'(t) + \frac{1}{\tau} \phi_1(t) \varphi_1(t) - \frac{1}{\tau} \psi_2(t) \geq 0. \]
When \( t \leq t_6 \), \( \psi_1(t) = 0 \). We obtain
\[ D_2 \psi_2''(t) - c \psi_2'(t) + \frac{1}{\tau} \phi_1(t) \varphi_1(t) - \frac{1}{\tau} \psi_2(t) = \frac{1}{\tau} \phi_1(t) \varphi_1(t) > 0. \]
Theorem 3. Assume that $E$ has asymptotic behavior for system (2).

Theorem 2. When $t > t_0$, $\psi(t) = k_3 - \varepsilon_6 e^{-\lambda t}$. We have

\[
D_2 \psi_2'(t) - c \psi_2(t) + \frac{1}{\tau} \phi_1(t) \varphi_1(t) - \frac{1}{\tau} \psi_2(t)
\]

\[= -D_2 \varepsilon_6 e^{-\lambda t} \lambda^2 - c \varepsilon_6 e^{-\lambda t} \lambda + \frac{1}{\tau}(k_1 k_2 + k_1 \varepsilon_2 e^{-\lambda t} + k_2 \varepsilon_1 e^{-\lambda t} + \varepsilon_2 \varepsilon_1 e^{-2\lambda t})
\]

\[= \frac{1}{\tau}(k_3 - \varepsilon_6 e^{-\lambda t})
\]

\[= e^{-\lambda t}[-D_2 \varepsilon_6 \lambda^2 - c \varepsilon_6 \lambda + \frac{1}{\tau}(k_1 k_2 - k_3) + \frac{1}{\tau}(k_1 \varepsilon_2 + k_2 \varepsilon_1) e^{-\lambda t} + \varepsilon_6].
\]

Let

\[I_0(\lambda) = -D_2 \varepsilon_6 \lambda^2 - c \varepsilon_6 \lambda + \frac{1}{\tau}(k_1 k_2 - k_3) + \frac{1}{\tau}(k_1 \varepsilon_2 + k_2 \varepsilon_1) e^{-\lambda t} + \varepsilon_6.
\]

It is easy to imply that $I_0(0) > 0$ and there exists $\lambda_0^\prime$ so that $D_2 \psi_2'(t) - c \psi_2(t) + \frac{1}{\tau} \phi_1(t) \varphi_1(t) - \frac{1}{\tau} \psi_2(t) \geq 0$. We let $\lambda \in (0, \min\{\lambda_1^\prime, \lambda_2^\prime, \lambda_3^\prime\})$, then the conclusion is true. This completes the proof. \(\square\)

By applying lemmas 6-8 and Theorem 1, we have the main results.

Theorem 2. If $r_1 a_{21} > r_2 a_{11}$ holds for every $c > c' = \max\{2\sqrt{D_1 r_1}, 2\sqrt{D_2 (a_{21} M_3 - r_2)}\}$, system (2) has a traveling wave solution connecting the trivial steady-state solution $E_0(0,0,0)$ and the position steady state $E^*(k_1, k_2, k_3)$.

4. Asymptotic behavior for system (2). In the section, we will prove the asymptotic behavior of the traveling wave by using the standard asymptotic theory.

Theorem 3. Assume that $r_1 a_{21} > r_2 a_{11}$, $a_{21} a_{12} < a_{11} a_{22}$, $2 a_{12} > a_{22}$ and $D_1 = D_2 = 1$, for any sufficiently small $\tau > 0$, there exist positive constants $P_i$ and $Q_i$ $(i = 1, 2, 3)$, so that system (24) has a traveling wave front $\Phi$ with the following asymptotic properties

\[
\Phi(\xi) = \begin{pmatrix} (p_1 + o(1)) e^{\lambda \xi} \\ (p_2 + o(1)) e^{\lambda \xi} \\ (p_3 + o(1)) e^{\lambda \xi} \end{pmatrix},
\]

as $\xi \to -\infty$, where $\lambda$ may be $\lambda_1$, $\lambda_2$, or $\lambda_3$, and

\[
\Phi(\xi) = \begin{pmatrix} k_1 - (Q_1 + o(1)) e^{\Lambda \xi} \\ k_2 - (Q_2 + o(1)) e^{\Lambda \xi} \\ k_3 - (Q_3 + o(1)) e^{\Lambda \xi} \end{pmatrix},
\]

as $\xi \to +\infty$, where $\Lambda$ may be $\Lambda_2$, $\Lambda_3$, where $\lambda_1$, $\lambda_2$ or $\lambda_3$ and $\Lambda_2$, $\Lambda_3$ are defined in the following proof.

Proof. Let $\xi = x + ct$ and $\Phi(\xi) = (\phi_0(\xi), \varphi_0(\xi), \psi_0(\xi))^T$ be the traveling wave solution of system (24), we differentiate (24) with respect to $\xi$, and denote $\Phi'(\xi) = (\phi_1(\xi), \varphi_1(\xi), \psi_1(\xi))^T$, then one has

\[\phi_1'(\xi) - c \phi_1'(\xi) + r_1 \phi_1(\xi) - 2 a_{11} \phi_1(\xi) \phi_0(\xi) - a_{12} \varphi_1(\xi) \phi_0(\xi) + a_{12} \varphi_0(\xi) \phi_1(\xi) = 0,
\]

\[\varphi_1'(\xi) - c \varphi_1'(\xi) + a_{21} \psi_1(\xi) - r_2 \psi_1(\xi) - 2 a_{22} \varphi_1(\xi) \phi_0(\xi) = 0,
\]

\[\psi_1'(\xi) - c \psi_1'(\xi) + \frac{1}{\tau} \phi_1(\xi) \varphi_1(\xi) + \frac{1}{\tau} \phi_0(\xi) \varphi_1(\xi) - \frac{1}{\tau} \psi_1(\xi) = 0.
\]

(28)
When $\xi \to -\infty$, the limiting system for (28) is
\begin{align*}
\begin{cases}
\phi_1''(\xi) - c\psi_1''(\xi) + r_1\phi_1(\xi) = 0, \\
\phi_2''(\xi) - c\psi_2''(\xi) + a_3\psi_1(\xi) - r_2\psi_1(\xi) = 0, \\
\phi_3''(\xi) - c\psi_3''(\xi) - \frac{1}{\tau}\psi_1(\xi) = 0.
\end{cases}
\tag{29}
\end{align*}

By setting $\phi'_1 = \phi_1 - \psi_1, \phi'_2 = \phi_2 - \psi_2, \phi'_3 = \phi_3 - \psi_3$, we can write system (29) as a first order system of ordinary differential equation in the six components $(\phi_1, \phi_2, \phi_3, \psi_1, \psi_2, \psi_3)^T$ with a constant $6 \times 6$ coefficient matrix. Then the system (29) becomes
\begin{equation}
Z' = PZ, \quad Z = (\phi_1, \phi_2, \phi_3, \psi_1, \psi_2, \psi_3)^T, \tag{30}
\end{equation}
where
\begin{equation*}
P = 
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
-r_1 & c & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & r_2 & c & -a_{21} & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & \frac{1}{\tau} & c
\end{bmatrix}.\end{equation*}
The linear equation (30) admits non-trivial solutions with the form
\begin{equation}
(\phi_1, \phi_2, \phi_3, \psi_1, \psi_2, \psi_3)^T = \sum_{i=1}^{6} c_i h_i e^{\lambda_i \xi}, \tag{31}
\end{equation}
where
\begin{align*}
\lambda_1 &= \frac{c + \sqrt{c^2 - 4r_1}}{2}, \\
\lambda_2 &= \frac{c - \sqrt{c^2 - 4r_1}}{2}, \\
\lambda_3 &= \frac{c + \sqrt{c^2 + 4r_2}}{2}, \\
\lambda_4 &= \frac{c - \sqrt{c^2 + 4r_2}}{2}, \\
\lambda_5 &= \frac{c + \sqrt{c^2 - 4\frac{1}{\tau}}}{2}, \\
\lambda_6 &= \frac{c - \sqrt{c^2 - 4\frac{1}{\tau}}}{2},
\end{align*}
h_i(i = 1, 2, 3, 4) are eigenvectors of the matrix $P$ with $\lambda_i$ as corresponding eigenvalues, and $c_i$ are arbitrary constants. Since $(\phi_1, \phi_2, \phi_3, \psi_1, \psi_2, \psi_3)^T$ as $\xi \to -\infty$, we obtain from (31) that $c_4 = c_5 = c_6 = 0$, and
\begin{equation*}
(\phi_1, \phi_2, \phi_3, \psi_1, \psi_2, \psi_3)^T = c_1 h_1 e^{\lambda_1 \xi} + c_2 h_2 e^{\lambda_2 \xi} + c_3 h_3 e^{\lambda_3 \xi}.
\end{equation*}

We can define the following asymptotic behavior as $\xi \to -\infty$,
\begin{equation}
\begin{bmatrix}
\phi_1(\xi) \\
\phi_2(\xi) \\
\phi_3(\xi)
\end{bmatrix} = 
\begin{bmatrix}
\alpha_1(m_1 + o(1))e^{\lambda_1 \xi} + \alpha_2(m_2 + o(1))e^{\lambda_2 \xi} + \alpha_3(m_3 + o(1))e^{\lambda_3 \xi} \\
\alpha_1(n_1 + o(1))e^{\lambda_1 \xi} + \alpha_2(n_2 + o(1))e^{\lambda_2 \xi} + \alpha_3(n_3 + o(1))e^{\lambda_3 \xi} \\
\alpha_1(l_1 + o(1))e^{\lambda_1 \xi} + \alpha_2(l_2 + o(1))e^{\lambda_2 \xi} + \alpha_3(l_3 + o(1))e^{\lambda_3 \xi}
\end{bmatrix},
\tag{32}
\end{equation}
where $m_i, n_i, l_i (i = 1, 2, 3, 4)$ are constant and $\alpha_i (i = 1, 2, 3, 4)$ can not be zero simultaneously (see [3]). Then we claim that $m_i \neq 0, n_i \neq 0$ and $l_i \neq 0, (i = 1, 2, 3, 4)$. If one of the first third and fifth components of eigenvector $h_i$ is zero, the matrix $P$ implies that the other components are zero. So we can obtain that $m_i \neq 0, n_i \neq 0$ and $l_i \neq 0, (i = 1, 2, 3, 4)$.

Similarly, when $\xi \to +\infty$, the system (28) becomes
\begin{equation}
\begin{align*}
\begin{cases}
\phi_1''(\xi) - c\phi_1''(\xi) + r_1\phi_1(\xi) - 2a_{11}k_1\phi_1(\xi) - a_{12}k_1\phi_1(\xi) + a_{12}k_2\phi_1(\xi) = 0, \\
\phi_2''(\xi) - c\phi_2''(\xi) + a_{21}\psi_1(\xi) - r_2\phi_1(\xi) - 2a_{22}k_2\phi_1(\xi) = 0, \\
\phi_3''(\xi) - c\phi_3''(\xi) + \beta k_1\phi_1(\xi) + \frac{1}{\tau}k_1\phi_1(\xi) - \frac{1}{\tau}\phi_1(\xi) = 0.
\end{cases}
\tag{33}
\end{align*}
\end{equation}
By setting $\phi_1' = \overline{\phi}_1$, $\psi_1' = \overline{\psi}_1$, we can write system (33) as a first order system of ordinary differential equation in the six components $(\phi_1^+, \overline{\phi}_1^+, \varphi_1^+, \overline{\varphi}_1^+, \psi_1^+, \overline{\psi}_1^+)^T$ with a constant $6 \times 6$ coefficient matrix. Then the system (33) becomes

$$Z' = QZ, \quad Z = (\phi_1^+, \overline{\phi}_1^+, \varphi_1^+, \overline{\varphi}_1^+, \psi_1^+, \overline{\psi}_1^+)^T,$$

where

$$Q = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
2a_{11} - r_1 - a_{12}k_2 & c & a_{12}k_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & c & -a_{21} \\
-\frac{1}{2}k_2 & 0 & -\frac{1}{r}k_1 & 0 & \frac{1}{c} & \tau \\
\end{bmatrix}.$$ 

Let

$$B = r_1 + a_{21}k_2 - 2a_{11}k_1 - \frac{1}{\tau} - (r_2 + 2a_{22}k_2),$$

$$C = \frac{1}{\tau}(r_2 + 2a_{22}k_2) - \frac{1}{\tau}a_{21}k_1 - \frac{1}{\tau}(r_1 + a_{12}k_2 - 2a_{11}k_1) - (r_1 + a_{12}k_2 - 2a_{11}k_1)(r_2 + 2a_{22}k_2),$$

$$D = \frac{1}{\tau}[(r_1 + a_{12}k_2 - 2a_{11}k_1)(r_2 + 2a_{22}k_2) - a_{21}k_1(r_1 + a_{12}k_2 - 2a_{11}k_1) + a_{21}a_{12}k_1k_2],$$

denote

$$p = (3C - B^2)/3, q = (2B^3 - 9BC + 27D)/27.$$ 

We can obtain that the general solution of the system (34) has the following form

$$(\phi_1^+, \overline{\phi}_1^+, \varphi_1^+, \overline{\varphi}_1^+, \psi_1^+, \overline{\psi}_1^+)^T = \sum_{i=1}^{4} d_i f_i e^{\Lambda_i \xi},$$

where

$$\Lambda_1 = \frac{c + \sqrt{c^2 + 4\bar{v}_1}}{2}, \quad \Lambda_2 = \frac{c - \sqrt{c^2 + 4\bar{v}_1}}{2},$$

$$\Lambda_3 = \frac{c + \sqrt{c^2 + 4\bar{v}_2}}{2}, \quad \Lambda_4 = \frac{c - \sqrt{c^2 + 4\bar{v}_2}}{2},$$

$$\bar{v}_1 = \frac{z_1^2 - Bz_1 - p}{3z_1}, \quad \bar{v}_2 = \frac{z_2^2 - Bz_2 - p}{3z_2},$$

$$z_1 = \sqrt[3]{\frac{-q + \sqrt{q^2 + 4(\xi, e^3)}}{2}}, \quad z_2 = \sqrt[3]{\frac{-q - \sqrt{q^2 + 4(\xi, e^3)}}{2}},$$

$$f_i (i = 1, 2, 3, 4)$$

are eigenvectors of the matrix $Q$ with $\Lambda_i$ as corresponding eigenvalues, and $d_i$ are arbitrary constants. Since $(\phi_1^+, \overline{\phi}_1^+, \varphi_1^+, \overline{\varphi}_1^+, \psi_1^+, \overline{\psi}_1^+)^T \rightarrow (k_1, 0, k_2, 0, k_3, 0)^T$ as $\xi \rightarrow +\infty$, we obtain from (36) and

$$(\phi_1^+, \overline{\phi}_1^+, \varphi_1^+, \overline{\varphi}_1^+, \psi_1^+, \overline{\psi}_1^+)^T = d_2 f_2 e^{\Lambda_2 \xi} + d_4 f_4 e^{\Lambda_4 \xi}.$$ 

We can define the following asymptotic behavior as $\xi \rightarrow +\infty$,

$$\begin{bmatrix}
\phi_1(\xi) \\
\phi_2(\xi) \\
\phi_3(\xi)
\end{bmatrix} = \begin{bmatrix}
\beta_1 (p_1 + o(1)) e^{\Lambda_2 \xi} + \beta_2 (p_2 + o(1)) e^{\Lambda_4 \xi} \\
\beta_1 (q_1 + o(1)) e^{\Lambda_2 \xi} + \beta_2 (q_2 + o(1)) e^{\Lambda_4 \xi} \\
\beta_1 (t_1 + o(1)) e^{\Lambda_2 \xi} + \beta_2 (t_2 + o(1)) e^{\Lambda_4 \xi}
\end{bmatrix},$$

where $p_i, q_i, t_i (i = 1, 2)$ are constant and $\beta_1, \beta_2$ can not be zero simultaneously. Similarly to the case $\xi \rightarrow -\infty$, we can prove that $p_i \neq 0, q_i \neq 0$ and $t_i \neq 0, i = 1, 2$.

Through the above discussion, it is easy to establish the asymptotic behavior of the traveling waves. 

$\square$
Remark 1. In [15], the global stability of the positive steady state and semi-trivial steady state of the proposed problem were derived by using the method of upper-lower solutions and its associated monotone iteration scheme. In this paper, we obtain the existence of the system by combing the upper and lower solutions with the Schauder’s fixed point theorem. Our results supplement some work of ref. [15].

Remark 2. In [15] and [4], the authors considered the existence of traveling wave solution, but they don’t investigate the asymptotic behavior of traveling wave solutions. In this paper, we also consider the asymptotic behavior for system [2].

Remark 3. In theorem 3, in order to facilitate the calculation, we assume that \( D_1 = D_2 = 1 \). In fact, \( D_1, D_2 \) can be the positive integers.

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