We establish two WDVV-style relations for the disk invariants of real symplectic fourfolds by implementing Georgieva’s suggestion to lift homology relations from the Deligne-Mumford moduli spaces of stable real curves. This is accomplished by lifting judiciously chosen bounding chains along with the relations. The resulting lifted relations lead to the recursions for Welshinger’s invariants announced by Solomon in 2007 and have the same structure as his WDVV-style relations, but differ by signs from the latter. Our topological approach provides a general framework for lifting relations via morphisms between not necessarily orientable spaces.

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1 Introduction

The WDVV relation [17, 20] for genus 0 Gromov-Witten invariants completely solves the classical problem of enumerating complex rational curves in the complex projective space \( \mathbb{P}^n \). Invariant counts of real rational \( J \)-holomorphic curves in compact real symplectic fourfolds, now known as Welschinger’s invariants, were defined in [27] and interpreted in terms of counts of \( J \)-holomorphic maps from the disk \( D^2 \) in [22]. J. Solomon announced two distinct WDVV-type relations for these counts in February 2007 and outlined an approach to their proof in the general spirit of the original proof of the complex WDVV relation in [20]. However, the outline of the proof described in [23] left a number of conceptual points mysterious and clearly required a major technical effort to implement.

The proof of the complex WDVV relation in [20] involves defining a count of \( J \)-holomorphic maps in a symplectic manifold \((X, \omega)\) for every cross-ratio \( \varpi \) of four points on \( \mathbb{P}^1 \) and showing that this count does not depend on \( \varpi \). The complex WDVV relation can alternatively be viewed as a direct consequence (at least conceptually) of two specific points, \( \varpi_1 \) and \( \varpi_2 \), of the Deligne-Mumford moduli space \( \overline{\mathcal{M}}_{0,4} \approx \mathbb{P}^1 \) of stable complex genus 0 curves with 4 marked points determining the same element of \( H_0(\overline{\mathcal{M}}_{0,4}) \). This perspective is suitable for lifting homology relations in any dimension and has proved instrumental to studying the structure of complex Gromov-Witten invariants as in [11, 25].

1.1 Lifting homology relations

In Spring 2014, P. Georgieva suggested that WDVV-type recursions for the real genus 0 invariants of [27] might be obtainable by lifting

(R1) a zero-dimensional homology relation on the moduli space \( \overline{\mathcal{M}}_{0,1,2} \approx \mathbb{R}P^2 \) of stable real genus 0 curves with 1 real marked point and 2 conjugate pairs of marked points and

(R2) the one-dimensional homology relation on the moduli space \( \overline{\mathcal{M}}_{0,0,3} \) of stable real genus 0 curves with 3 conjugate pairs of marked points discovered in [8]

to the moduli spaces \( \overline{\mathcal{M}}_{k,l}(B; J) \) of real rational \( J \)-holomorphic maps constructed in [7]. Unlike in the complex case, major conceptual issues arise in lifting relations from the Deligne-Mumford spaces of curves to the moduli spaces of \( J \)-holomorphic maps and in translating any lifted relations into invariant counts of curves because the moduli spaces in real Gromov-Witten theory are generally not orientable. The present paper deals with these issues by lifting homology relations along with boundary chains for them.

We first re-interpret the disk counts of [22] in the spirit of Steenrod homology [24] in terms of counts of real \( J \)-holomorphic maps with marked points decorated by signs as in [7]. We then lift homology relations, along with suitably chosen bounding chains \( \Upsilon \), from \( \overline{\mathcal{M}}_{0,1,2} \) and \( \overline{\mathcal{M}}_{0,0,3} \) to the bordered moduli spaces \( \overline{\mathcal{M}}_{k,l}(B; J) \) obtained by cutting \( \overline{\mathcal{M}}_{k,l}(B; J) \) along hypersurfaces that
obstruct the relative orientability of the forgetful morphisms

\[ f_{1,2} : \overline{\mathcal{M}}_{k,l}(B; J) \longrightarrow \mathbb{R} \overline{\mathcal{M}}_{0,1,2} \quad \text{and} \quad f_{0,3} : \overline{\mathcal{M}}_{k,l}(B; J) \longrightarrow \mathbb{R} \overline{\mathcal{M}}_{0,0,3}. \]  

The simple topological Lemma 3.5 expresses the wall-crossing effects on the lifted relations in \( \overline{\mathcal{M}}_{k,l}(B; J) \) in terms of the intersections of the boundary of \( \overline{\mathcal{M}}_{k,l}(B; J) \) with \( \Upsilon \). This allows us to obtain the two WDVV-type relations for the map counts depicted in Figure 1 on page 10, with the left-hand sides representing the initial relations in \( \mathbb{R} \overline{\mathcal{M}}_{0,1,2} \) and \( \mathbb{R} \overline{\mathcal{M}}_{0,0,3} \) and the right-hand sides representing the wall-crossing corrections. The first two relations of Theorem 1.1 are obtained by using the two relations of Figure 1 with the divisors \( H_1 \) and \( H_2 \) as the first two non-real constraints and points as the remaining constraints. The last relation of Theorem 1.1 is obtained by using the second relation of Figure 1 with the divisors \( H_1, H_2, \) and \( H_3 \) as the first two non-real constraints and points as the remaining constraints.

By the comparison between the curve counts of [27] and the map counts of [22] established in [5], the relations of Theorem 1.1 are equivalent to the relations for the former stated in [4, Theorem 1]. They completely determine Welschinger’s invariants of \( \mathbb{P}^2 \) and its blowups, as shown in [23] and [12], respectively. For the ease of use, these results are summarized in [4]; the low-degree numbers obtained from the three relations for Welschinger’s invariants and listed in [4] agree with [1, 2, 3, 13, 14, 15, 16, 29].

The relations of [4, Theorem 1] for Welschinger’s invariants are the same as implied by the statements of Theorem 8, Proposition 10, and Theorem 11 in [23]. The relations of Theorem 1.1 for the map invariants in the present paper involve the same terms as the difference between equations (6) and (7) in [23] and the symmetrization of equation (5) in [23], but different signs. The comparison between the curve counts of [27] and the disk counts of [22] established in [5] likewise differs by sign from the claim of [23, Thm. 11]. The two sign discrepancies, which do not appear to be due to the formulations of the definitions of the disk invariants in the present paper and [22, 23], precisely cancel out to yield the same recursions for Welschinger’s invariants.

This paper presents a general approach for pulling back a relation by a morphism \( f : \mathfrak{M} \longrightarrow \mathcal{M} \) between two spaces which is not necessarily relatively orientable. The lifted relation then acquires a correction which doubly covers a “non-orientability” hypersurface in \( \mathfrak{M} \). This approach should be applicable in many other settings. We apply it in [6] to study the behavior of real GW-invariants in higher dimensions, including those defined in [28, 22, 7].

1.2 Main theorem

Let \( (X, \omega, \phi) \) be a compact real symplectic manifold, i.e. \( \omega \) is a symplectic form on \( X \) so that \( \phi^* \omega = -\omega \). The fixed locus \( X^\phi \) of the anti-symplectic involution \( \phi \) on \( X \) is then a Lagrangian submanifold of \( (X, \omega) \). We denote by \( \mathcal{J}_\omega \) the space of \( \omega \)-compatible (or -tamed) almost complex structures \( J \) on \( X \) and by \( \mathcal{J}_\omega^\omega \subset \mathcal{J}_\omega \) the subspace of almost complex structures \( J \) such that \( \phi^* J = -J \). Let

\[ c_1(X, \omega) = c_1(TX, J) \in H^2(X) \]

be the first Chern class of \( TX \) with respect to some \( J \in \mathcal{J}_\omega ; \) it is independent of such a choice. For simplicity, we take the degrees of the curves in the quotient \( H_2(X) \) of \( H_2(X; \mathbb{Z}) \) modulo torsion,
but the formulas of Theorem 1.1 apply with finer notions of degree (e.g. as in \[2\]). For \(B \in H_2(X)\), define

\[\ell_\omega(B) = \langle c_1(X, \omega), B \rangle - 1 \in \mathbb{Z}, \quad \langle B \rangle_l = \begin{cases} 1, & \text{if } 2l = \ell_\omega(B) - 1; \\ 0, & \text{otherwise.} \end{cases}\]

For \(J \in \mathcal{J}_\omega\) and \(B \in H_2(X)\), a subset \(C \subset X\) is a genus 0 (or rational) irreducible \(J\)-holomorphic degree \(B\) curve if there exists a simple (not multiply covered) \(J\)-holomorphic map

\[u : \mathbb{P}^1 \to X \quad \text{s.t.} \quad C = u(\mathbb{P}^1), \quad u_*[\mathbb{P}^1] = B. \quad (1.2)\]

Such a subset \(C\) is called a real rational irreducible \(J\)-holomorphic degree \(B\) curve if in addition \(\phi(C) = C\).

From now on, suppose that the (real) dimension of \(X\) is 4. The (tangent bundle of the) fixed locus \(X^\phi\) then admits a Pin\(^{-}\)-structure \(\mathfrak{p}\). Let \(B \in H_2(X)\) and \(l \in \mathbb{Z}_{\geq 0}\) be such that

\[k = \ell_\omega(B) - 2l \in \mathbb{Z}_{\geq 0}. \quad (1.3)\]

For a generic \(J \in \mathcal{J}_\omega\), there are then only finitely many real rational irreducible \(J\)-holomorphic degree \(B\) curves \(C \subset X\) intersecting a connected component \(\check{X}^\phi\) of \(X^\phi\) and passing through \(k\) points in \(\check{X}^\phi\) and \(l\) points in \(X - X^\phi\) in general position. According to \[22\] Thm. 1.3], the number of such curves counted with appropriate signs determined by \(\mathfrak{p}\) is independent of the choices of \(J\) and the points. We denote this signed count of genus 0 curves by \(N_{B,l}^{\phi,\mathfrak{p}}(\check{X}^\phi)\). If the number \(k\) in (1.3) is negative, we set \(N_{B,l}^{\phi,\mathfrak{p}}(\check{X}^\phi) = 0\). We denote by \(N_{B,l}^{\phi,\mathfrak{p}}\) the sum of the numbers \(N_{B,l}^{\phi,\mathfrak{p}}(\check{X}^\phi)\) over the connected components \(\check{X}^\phi\) of \(X^\phi\).

Suppose \(B \in H_2(X)\) and \(\ell_\omega(B) \geq 0\). For a generic \(J \in \mathcal{J}_\omega\), there are then only finitely many rational irreducible \(J\)-holomorphic degree \(B\) curves \(C \subset X\) passing through \(\ell_\omega(B)\) points in general position. The number of such curves counted with appropriate signs is independent of the choices of \(J\) and the points. This is the standard (complex) genus 0 degree \(B\) Gromov-Witten invariant of \((X, \omega)\) with \(\ell_\omega(B)\) point insertions; we denote it by \(N_B^X\). If \(\ell_\omega(B) < 0\), we set \(N_B^X = 0\).

For \(B, B' \in H_2(X)\), we denote by \(B \cdot_X B' \in \mathbb{Z}\) the homology intersection product of \(B\) with \(B'\) and by \(B^2 \in \mathbb{Z}\) the self-intersection number of \(B\). Define

\[\mathfrak{d} : H_2(X) \to H_2(X), \quad \mathfrak{d}(B) = B - \phi_*(B), \quad H^2(X)^\phi_\perp = \{ H \in H^2(X) : \phi^*H = -H \}.\]

**Theorem 1.1** Suppose \((X, \omega, \phi)\) is a compact real symplectic fourfold, \(\mathfrak{p}\) is a Pin\(^{-}\)-structure on \(X^\phi\), \(\check{X}^\phi\) is a connected component of \(X^\phi\), and \(H_1, H_2, H_3 \in H^2(X)^\phi_\perp\) are such that \(\langle H_1 H_2, X \rangle = 1\) and \(H_1 H_3 = 0\). Let \(l \in \mathbb{Z}_{\geq 0}\) and \(B \in H_2(X)\).
(RWDVV1) If \(l \geq 1\) and \(\ell_\omega(B) - 2l \geq 1\) (i.e. \(k \geq 1\)), then

\[
N_{B,l}^{\phi}(\tilde{X}) = -2^{l-3}\langle B, B_1, B_2 \rangle \sum_{B' \in H_2(X), B_0 + B' = B} N_{B'}^X \left( \langle H_1, B, B' \rangle \left( \frac{l-1}{\ell_\omega(B')} \right) N_{B_0,l-1-\ell_\omega(B')} N_{B_0,l-1-\ell_\omega(B')} (\tilde{X}) \right)
- \sum_{B_0, B' \in H_2(X), B_0 = B} 2^{\ell_\omega(B')} \langle B_0, B' \rangle \langle H_1, B, B' \rangle \left( \frac{l-1}{\ell_\omega(B')} \right) N_{B'}^X N_{B_0,l-1-\ell_\omega(B')} (\tilde{X})
+ \sum_{B_1, B_2 \in H_2(X), B_1 + B_2 = B} \langle H_1, B_1 \rangle \left( \frac{l-1}{\ell_\omega(B_1)} \right) \left( \frac{l-1}{\ell_\omega(B_2)} \right) N_{B_1,l_1}^{\phi} (\tilde{X}) N_{B_2,l_2}^{\phi} (\tilde{X})

(RWDVV2) If \(l \geq 2\), then

\[
N_{B,l}^{\phi}(\tilde{X}) = -2^{l-2}(B_0, B') \langle H_1, B' \rangle \left( \frac{l-2}{\ell_\omega(B')} - 2\langle H_2, B' \rangle \left( \frac{l-2}{\ell_\omega(B')} \right) \right) N_{B_0,l-1-\ell_\omega(B')} (\tilde{X})
+ \sum_{B_1, B_2 \in H_2(X), B_1 + B_2 = B} \langle H_1, B_1 \rangle \left( \frac{l-2}{\ell_\omega(B_1)} \right) \left( \frac{l-2}{\ell_\omega(B_2)} \right) N_{B_1,l_1}^{\phi} (\tilde{X}) N_{B_2,l_2}^{\phi} (\tilde{X})
- \langle H_1, B_1 \rangle \left( \frac{l-2}{\ell_\omega(B_1)} \right) N_{B_1,l_1}^{\phi} (\tilde{X}) N_{B_2,l_2}^{\phi} (\tilde{X})

(RWDVV3) If \(l \geq 1\), then

\[
\langle H_3, B \rangle N_{B,l}^{\phi}(\tilde{X}) = \sum_{B_0, B' \in H_2(X), B_0 + B' = B} 2^{\ell_\omega(B')} \langle B_0, B' \rangle \langle H_1, B \rangle \left( \frac{l-1}{\ell_\omega(B')} \right) N_{B_0,l-1-\ell_\omega(B')} (\tilde{X})
+ \sum_{B_1, B_2 \in H_2(X), B_1 + B_2 = B} \langle H_1, B_1 \rangle \langle H_3, B_1 \rangle \left( \frac{l-1}{\ell_\omega(B_1)} \right) \left( \frac{l-1}{\ell_\omega(B_2)} \right) N_{B_1,l_1}^{\phi} (\tilde{X}) N_{B_2,l_2}^{\phi} (\tilde{X})

Taking the difference between the relations of Theorem 1.1 for \(N_{B+B_\bullet, l}^{\phi}(\tilde{X})\) with \(\ell_\omega(B_\bullet) > 0\) small yields relations involving the invariants \(N_{B+B_\bullet, 0}^{\phi}(\tilde{X})\) without conjugate pairs of marked points. In some cases, the resulting relations determine these numbers; see [23] [12].
Remark 1.2. We define the invariants $N_{B,l}^{φ, p}(X^φ)$ via the moduli spaces of real maps constructed in [7]. This definition of $N_{B,l}^{φ, p}(X^φ)$ differs by a power of 2 from the definitions in [22, 23], but agrees in absolute value with the invariants $N_{B,l}^{φ}(X^φ)$ of [27].

1.3 Outline of the proof

In order to simplify the notation, we prove the analogue of Theorem 1.1 for the sums $N_{φ, B,l}^{p}$ of the numbers $N_{φ, B,l}^{p}$ over the connected components $X^φ$ of $X^φ$. It follows from the two relations for nodal map counts represented by Figure 1 and from Propositions 5.3 and 5.7. Theorem 1.1 itself is obtained by restricting all arguments to the open and closed subspaces of the map moduli spaces consisting of maps that take the fixed locus of the domain to $X^φ$.

We denote by $τ$ the standard conjugation on $P^1$, i.e.

$$τ : P^1 \longrightarrow P^1, \quad τ([z_0, z_1]) = [\overline{z_1}, \overline{z_0}].$$

For every real rational irreducible $J$-holomorphic degree $B$ curve contributing to $N_{B,l}^{φ, p}$, there exists a $J$-holomorphic map $u : P^1 \longrightarrow X$ as in (1.2) such that $u\circτ = φ\circ u$. Thus, the number $N_{B,l}^{φ, p}$ is a signed cardinality of the subset of the moduli space $\overline{M}_{k,l}(B; J)$ of real rational degree $B$ $J$-holomorphic maps sending the $k$ real marked points and the first points in the $l$ conjugate points to generic points in $X^φ$ and $X$, respectively.

The domain and target of the evaluation morphism

$$ev : \overline{M}_{k,l}(B; J) \longrightarrow X_{k,l} = (X^φ)^k \times X^l \quad (1.4)$$

may not be relatively orientable, but it becomes relatively orientable after removing certain codimension 1 strata from the domain (i.e. the pull-back of the first Stiefel-Whitney class $w_1$ of the target is the $w_1$ of the domain). We cut $\overline{M}_{k,l}(B; J)$ along these codimension 1 strata to obtain a bordered manifold $\overline{M}_{k,l}(B; J)$ and give

$$ev : \overline{M}_{k,l}(B; J) \longrightarrow X_{k,l}$$

a relative orientation. The codimension 1 strata of $\overline{M}_{k,l}(B; J)$ consist of curves with two components and one real node.

The forgetful morphisms (1.1) we encounter take values in the subspaces $\overline{M}_{k', l'}$ of $\overline{M}_{0, k', l'}$ of real curves with non-empty fixed locus; $\overline{M}_{k', l'}$ is a proper subspace of $\overline{M}_{0, k', l'}$ if and only if $k' = 0$. We choose a bordered hypersurface $Γ$ in $\overline{M}_{k', l'}$ whose boundary consists of curves with three components and a conjugate pair of nodes and a relative orientation on the inclusion of $Γ$ into $\overline{M}_{k', l'}$. Let $C \subset X_{k,l}$ be a generic constraint consisting of divisors and points so that the maps

$$ev \times j_{k', l'} : \overline{M}_{k,l}(B; J) \longrightarrow X_{k,l} \times \overline{M}_{k', l'} \quad \text{and} \quad ι_{C, Γ} : C \times Γ \hookrightarrow X_{k,l} \times \overline{M}_{k', l'}$$

are transverse and

$$\dim \overline{M}_{k,l}(B; J) + \dim (C \times Γ) = \dim (X_{k,l} \times \overline{M}_{k', l'}) + 1.$$

With the relative orientations above, the signed counts of the intersection points of
The first count above decomposes into curve-counting invariants similarly to the complex case. The second count can also be decomposed, based on the following observations:

- most boundary strata of \( \tilde{\mathcal{M}}_{k,l}(B; J) \) get contracted by \( \text{ev} \times f_{k',l'} \) and thus do not contribute to \((C2)\);
- some boundary strata that do not get contracted do not intersect \( \Upsilon \) via \( f_{k,l} \) due to our choice of \( \Upsilon \subset \overline{\mathcal{M}}_{k,l} \) and thus do not contribute to \((C2)\) either;
- intersecting the remaining boundary strata with \( \Upsilon \) via \( f_{k',l'} \) has the effect of specifying the position of the node (relative to the marked points) on the component of the curve carrying the first conjugate pair of marked points.

These statements are explained in the proof of Corollary \(5.10\) at the end of Section \(5.3\) and in the proof of Proposition \(5.7\) in Section \(6.3\). The equality of the counts \((C1)\) and \((C2)\) then translates into \((R\ WDVV1)\) in the case \( l^* = p \) and into \((R\ WDVV2)\) and \((R\ WDVV3)\) in the case \( l^* = 0 \).

The paper is organized as follows. Section \(2\) is a detailed version of the above outline of the proof of Theorem \(1.1\). The notions of relative orientations, pseudocycles with relative orientations (called Steenrod pseudocycles), and intersection signs between them are defined in Section \(3\); this section also contains all relevant observations concerning these notions. Section \(4\) describes in detail the hypersurfaces \( \Upsilon \) in the Deligne-Mumford spaces \( \overline{\mathcal{M}}_{1,2} \) and \( \overline{\mathcal{M}}_{0,3} \) used in the proof of Theorem \(1.1\). Section \(5\) sets up the notation relevant to the map spaces \( \mathcal{M}_{k,l}(B; J) \), states the propositions that are among the main steps in the proof of Theorem \(1.1\), and deduces this theorem from them and the lemmas of Section \(4.4\). The (somewhat technical) proofs of these propositions are deferred to Section \(6\).

Acknowledgments. I would like to thank my thesis advisor Aleksey Zinger for introducing me to this subject and background material, suggesting the topic, a lot of guidance and discussions throughout the process of the work, very detailed help with exposition, and constant encouragements.
If \( k, l \in \mathbb{Z}_{\geq 0} \) and \( B \in H_2(X) \) satisfy \( (1.3) \), the path in \( \overline{\mathcal{M}}_{k,l}(B; J) \) determined by a generic path of collections of \( k \) points in \( X^\partial \) and \( l \) points in \( X - X^\partial \) and of almost complex structures \( J_1 \in J^0 \) does not cross the codimension 1 boundary strata \( S \) with \( \epsilon_0(S) \) congruent to 2 or 3 mod 4. This fundamental insight, formulated in terms of moduli spaces of disk maps in \( [22] \), along with the above orientation statements established the invariance of the counts \( N_{B,l}^{\phi,p} \) and has since been used to construct numerical invariants in some other settings.

The image of each codimension 1 stratum \( S \) with \( \epsilon_0(S) \) congruent to 2 or 3 mod 4 under \( (1.4) \) is of smaller dimension than \( S \). Along with the orientation statements above, this implies that the restriction of \( (1.4) \) to the complement \( \overline{\mathcal{M}}_{k,l,0}(B; J) \) of the closures \( \overline{S} \) of these strata is a codimension 0 Steenrod pseudocycle with respect to the orientation \( \alpha_{p,0} \); see Proposition 5.2. The number \( N_{B,l}^{\phi,p} \) is the degree \( \deg(\text{ev}, \alpha_{p,0}) \) of this pseudocycle.

The orientations on the restriction of \( (1.4) \) to \( \mathcal{M}_{k,l}(B; J) \) relevant to lifting relations from \( \overline{\mathcal{M}}_{1,2} \) and \( \overline{\mathcal{M}}_{0,3} \) to \( \overline{\mathcal{M}}_{k,l}(B; J) \) are the orientations \( \alpha_{p,l^*} \) of Lemma 5.1 with \( l^* = 2, 3 \), as we would like to apply the lifted relations with two and three divisor insertions. The relevant restriction of \( (1.4) \) shrinks the codimension 1 strata \( S \) with \( \epsilon_{l^*}(S) \) congruent to 2 or 3 mod 4, but not with \( \epsilon_{l^*}(S) = 2 \). In order to deal with this issue, we cut \( \overline{\mathcal{M}}_{k,l}(B; J) \) along the closures \( \overline{S} \) of the strata \( S \) with \( \epsilon_{l^*}(S) \) congruent to 2 or 3 mod 4. We obtain a moduli space \( \hat{\mathcal{M}}_{k,l,1^*}(B; J) \) with boundary consisting of double covers \( \hat{S} \) of these strata. The relative orientation \( \alpha_{p,l^*} \) extends to a relative orientation \( \hat{\alpha}_{p,l^*} \) of the total evaluation morphism

\[
\text{ev}: \hat{\mathcal{M}}_{k,l,1^*}(B; J) \rightarrow X_{k,l}
\]

induced by \( (1.4) \).

Suppose \( k, l \in \mathbb{Z}_{\geq 0} \) and \( B \in H_2(X) \) are as in \( (1.3) \), \( k' \leq k \), and \( l^* \leq l' \leq l + l^* - 1 \) so that there are well-defined forgetful morphisms

\[
\hat{f}_{k',l'}: \hat{\mathcal{M}}_{k,l+1^*,-1^*}(B; J) \rightarrow \overline{\mathcal{M}}_{k',l'} \quad \text{and} \quad \hat{f}_{k',l'}: \hat{\mathcal{M}}_{k,l+1^*,-1^*}(B; J) \rightarrow \overline{\mathcal{M}}_{k',l'}.
\]

An \( l^* \)-tuple \( h = (H_1, \ldots, H_{l^*}) \) of divisors in \( X \) cuts out the subspace

\[
\hat{Z}_{k,l+1^*,-1^*; h}(B; J) \subset \hat{\mathcal{M}}_{k,l+1^*,-1^*}(B; J) \times H_1 \times \ldots \times H_{l^*}
\]

of maps with the first \( l^* \) non-real marked points lying on \( H_1, \ldots, H_{l^*} \). The relative orientation \( \hat{\alpha}_{p,l^*} \) of \( (2.2) \) and the orientation \( \alpha_h \) on \( H_1 \times \ldots \times H_{l^*} \) induce a relative orientation \( \hat{\alpha}_{p,h} \) of the evaluation morphism

\[
\text{ev}_h: \hat{Z}_{k,l+1^*,-1^*; h}(B; J) \rightarrow X_{k,l-1}
\]

at the remaining marked points. A tuple \( p \) of points in \( X_{k,l-1} \) and a bordered compact real hypersurface \( \mathcal{Y} \subset \overline{\mathcal{M}}_{k',l'} \) determine an embedding

\[
f_{p,\mathcal{Y}}: \mathcal{Y} \rightarrow X_{k,l-1} \times \overline{\mathcal{M}}_{k',l'}.
\]

Under appropriate regularity assumptions, the fiber product \( M_{(\text{ev}_h, f_{k',l'}), f_{p,\mathcal{Y}}} \) of

\[
(\text{ev}_h, f_{k',l'}): \hat{Z}_{k,l+1^*,-1^*}(B; J) \rightarrow X_{k,l-1} \times \overline{\mathcal{M}}_{k',l'}
\]
The one-dimensional strata of to sign cancellations in \[23, p10\] do not appear in our approach at all. These curves arise from the last part of the boundary in (2.4) and thus contribute twice of the first row in Figure 1. We take Υ in of one real component and one conjugate pair of components; see the diagrams on the left-hand side

\[ \partial M_{(ev,h,l',p)} \cap (\partial \mathcal{Z})^\ast_{k,l+1} \times \mathcal{Z}^\ast_{k,l+1} \times f_{p-1} \mathcal{Z} \]

(2.4)

The relative orientation \(\hat{\partial}_{p,h} \) and a co-orientation \(\sigma_\mathcal{Z} \) on \(\mathcal{Z} \) determine signs of the points on the right-hand side of (2.4) so that

\[ |\hat{\mathcal{Z}}_{k,l+1}^\ast \times \mathcal{Z}^\ast_{k,l+1} \times f_{p-1} \mathcal{Z} |^\pm \]

(2.5)

where \(| \cdot |^\pm \) denotes the signed cardinality; see Lemma 3.5.

Since only the strata \(\hat{\mathcal{S}} \) of \(\hat{\mathcal{M}}_{k,l+1} \) with \(\epsilon_l(\hat{\mathcal{S}}) = 2 \) are not shrunk by (2.2), only the strata

\[ \hat{\mathcal{S}}_h \equiv (\hat{\mathcal{S}} \times H_1 \times \ldots \times H_{l*}) \cap \hat{\mathcal{Z}}_{k,l+1} \]

of \(\hat{\mathcal{Z}}_{k,l+1} \) with \(\epsilon_l(\hat{\mathcal{S}}) = 2 \) contribute to the right-hand side of (2.5). Since \(\hat{\mathcal{S}}_h \) is a double cover of the subspace

\[ \mathcal{S}_h \subset \mathcal{S} \times H_1 \times \ldots \times H_{l*} \]

of maps with the first \(l* \) non-real marked points lying on \(H_1, \ldots, H_{l*} \), we conclude that

\[ |\hat{\mathcal{Z}}_{k,l+1}^\ast \times \mathcal{Z}^\ast_{k,l+1} \times f_{p-1} \mathcal{Z} |^\pm = 2 \sum_{\epsilon_l(\hat{\mathcal{S}}) = 2} |(\mathcal{S}_h) \times f_{p-1} \mathcal{Z} |^\pm \]

The moduli space \(\mathcal{M}_{k',l'} \) contains codimension 1 strata \(S_i \) with \(i \leq l' \) parametrizing marked curves with two real components so that one of the components carries only the \(i\)-th conjugate pair of marked points. We establish Theorem 1.1 by applying (2.5) with certain bordered compact hypersurfaces \(\mathcal{Y} \) in \(\mathcal{M}_{1,2} \cong \mathbb{RP}^2 \) and in the three-dimensional orientable manifold \(\mathcal{M}_{0,3} \) so that \(\mathcal{Y} \) is disjoint from the closure \(\overline{S}_1 \) of \(S_1 \).

The moduli space \(\mathcal{M}_{1,2} \) contains two points \(P^\pm \) corresponding to the two marked curves consisting of one real component and one conjugate pair of components; see the diagrams on the left-hand side of the first row in Figure 1. We take \(\mathcal{Y} \) in \(\mathcal{M}_{1,2} \) to be a path from \(P^- \) to \(P^+ \) as in Lemma 1.4.

In this case, (2.5) is represented by the first row in Figure 1. The labels \(\epsilon_l(\mathcal{S}) = 2 \) under the diagrams on the right-hand side indicate that only some two-component curves contribute to this relation. These curves arise from the last part of the boundary in (2.4) and thus contribute twice each (with the same sign). The curves whose contributions are described as being insignificant due to sign cancellations in \[23, p10\] do not appear in our approach at all.

The one-dimensional strata of \(\mathcal{M}_{0,3} \) that parametrize marked curves consisting of one real component and one conjugate pair of components come in three pairs \(\Gamma_i^\pm \) with \(i = 1, 2, 3 \); see the diagrams on the left-hand side of the second row in Figure 1. The closures \(\overline{\Gamma_i^\pm} \) of these strata with \(i = 2, 3 \) bound a compact oriented surface \(\mathcal{Y} \) in \(\mathcal{M}_{0,3} \) as in Lemma 1.5.

In this case, (2.5) is represented
The relations on stable maps induced via (2.5) by lifting codimension 2 relations from \( \overline{\mathcal{M}}_{1,2} \) and \( \overline{\mathcal{M}}_{0,3} \); the curves on the right-hand sides of the two relations are constrained by the hypersurfaces \( \mathcal{Y} \) in \( \overline{\mathcal{M}}_{1,2} \) and \( \overline{\mathcal{M}}_{0,3} \).

The curves represented by the diagrams on the right-hand side in this relation again arise from the last part of the boundary in (2.4).

We apply the relations represented by Figure 1 with the divisors \( H_1, H_2 \) as the first two non-real insertions and points as the remaining insertions; we also apply the second relation with the divisors \( H_1, H_2, H_3 \) as the first three non-real insertions. The normal bundle to the strata of maps represented by the three-component curves in this figure is canonically oriented. Thus, the restriction of the total evaluation map (1.4) to these strata inherits a relative orientation from its restriction to \( \mathcal{M}_k,B(\mathcal{B}; J) \). The proof of [8, Prop. 4.2] readily applies to express the associated counts of nodal maps in terms of the real map counts \( N_{B,1}^{\phi, p} \) and the complex map counts \( N_B^X \); see Proposition 5.3.

The map counts represented by the two-component curves in Figure 1 are more elaborate. Each stratum \( \mathcal{S}_h \) of such maps is the fiber product of the evaluation morphisms

\[
ev_{\text{nd}}: Z = \mathbb{Z}_{k_1+1,l_1; h_1}(B_1; J) \longrightarrow X^\phi \quad \text{and} \quad \ev_{\text{nd}}: Z = \mathbb{Z}_{k_2+1,l_2; h_2}(B_2; J) \longrightarrow X^\phi
\]

at the nodal points from moduli spaces associated with the two components, for a split of \( h \) into an \( l_1 \)-tuple \( h_1 \) and an \( l_2 \)-tuple \( h_2 \). The condition \( \epsilon_{l*}(S) = 2 \) implies that each of the total evaluation morphisms

\[
\ev'_{h_1}: Z' = \mathbb{Z}_{k_1,l_1; h_1}(B_1; J) \longrightarrow X_{k_1,l_1-l_1^*} \quad \text{and} \quad \ev_{h_2} = (\ev'_{h_2}, \ev_{\text{nd}}): Z_2 \longrightarrow X_{k_2+1,l_2-l_2^*} \equiv X_{k_2,l_2-l_2^*} \times X^\phi
\]

at the nodal points from moduli spaces associated with the two components, for a split of \( h \) into an \( l_1 \)-tuple \( h_1 \) and an \( l_2 \)-tuple \( h_2 \). The condition \( \epsilon_{l*}(S) = 2 \) implies that each of the total evaluation morphisms

\[
\ev'_{h_1}: Z' = \mathbb{Z}_{k_1,l_1; h_1}(B_1; J) \longrightarrow X_{k_1,l_1-l_1^*} \quad \text{and} \quad \ev_{h_2} = (\ev'_{h_2}, \ev_{\text{nd}}): Z_2 \longrightarrow X_{k_2+1,l_2-l_2^*} \equiv X_{k_2,l_2-l_2^*} \times X^\phi
\]
A crucial consequence of our choices of the hypersurfaces $\Upsilon \subset \overline{\mathcal{M}}_{0,3}$ is that the restriction of the first morphism in (2.3) to $\mathcal{S}_h$ factors through a morphism

$$f_1: \mathcal{Z}_1 \longrightarrow \overline{\mathcal{M}}'_{k',l'}$$

if $\ell_{l_{\ast}}(S) = 2$ and $S \cap f_{k',l'}^{-1}(\Upsilon) \neq \emptyset$; see Corollary 5.10. Thus,

$$(\mathcal{S}_h)_{(evh_{k',l'})} \times f_{p,\Upsilon} \ Upsilon = \left( (\mathcal{Z}_1)_{(ev_{h_1}^{l_1})} \times f_{p_1,\Upsilon} \ Upsilon \right) ev_{na} \times ev_{na} ev_{h_2}^{-1}(p_2); \tag{2.7}$$

for a split of $p$ into a $k_1$-tuple $p_1$ and a $k_2$-tuple $p_2$. The equality above holds set-theoretically; Lemma 3.3 compares the signs on the two sides. The morphism $ev_{h_1}^{l_1}$ on the right-hand side of this equality denotes the composition of $f_{p_1}$ with the natural projection

$$f: \mathcal{Z}_{h_1} \longrightarrow \mathcal{Z}_{h_1}^{'}$$

dropping the real marked point corresponding to the node. Thus,

$$(\mathcal{Z}_1)_{(ev_{h_1}^{l_1})} \times f_{p_1,\Upsilon} \ Upsilon = \{ u_1 \in \mathcal{Z}_1|_{ev_{h_1}^{l_1}(p_1)}: f_1(u_1) \in \Upsilon \};$$

Lemma 3.3 compares the signs on the two sides. Since this set is finite, (2.7) implies that

$$\left| (\mathcal{S}_h)_{(evh_{k',l'})} \times f_{p,\Upsilon} \ Upsilon \right|_{ev_{h_1}^{l_1}, ev_{h_2}^{l_2}} = \alpha(S, \Upsilon) \deg(ev_{h_1}^{l_1}, o_{p,h_1}) \deg(ev_{h_2}^{l_2}, o_{p,h_2})$$

for some $\alpha(S, \Upsilon) \in \mathbb{Z}$ determined by $S$ and $\Upsilon$. This leads to a decomposition of the nodal map counts associated with the two-component diagrams in Figure 11 into sums of pairwise products of the real map counts $N_{B_1,l_1}^{p,p}$; see Proposition 5.7.

### 3 Topological preliminaries

#### 3.1 Relative orientations

For a real vector space or vector bundle $V$, let

$$\lambda(V) \equiv \Lambda_{\mathbb{R}}^{top} V$$

be its top exterior power. For a manifold $M$, possibly with nonempty boundary $\partial M$, we denote by

$$\lambda(M) \equiv \lambda(TM) \equiv \Lambda_{\mathbb{R}}^{top} TM \longrightarrow M$$

its orientation line bundle. An orientation of $M$ is a homotopy class of trivializations of $\lambda(M)$. By definition, $\lambda(\text{pt}) = \mathbb{R}$. We identify the two orientations of any point with $\pm 1$ in the obvious way.
For submanifolds \( S' \subset S \subset M \), the short exact sequences
\[
0 \rightarrow TS \rightarrow TM|_S \rightarrow NS \equiv \frac{TM|_S}{TS} \rightarrow 0 \quad \text{and} \quad 0 \rightarrow N_S S' \equiv \frac{TS|_{S'}}{TS'} \rightarrow N_S' \equiv \frac{TM|_{S'}}{TS'} \rightarrow 0
\]
of vector spaces determine isomorphisms
\[
\lambda(M)|_S \approx \lambda(S) \otimes \lambda(NS) \quad \text{and} \quad \lambda(NS') \approx \lambda(N_S S') \otimes \lambda(NS)|_{S'}
\]
of line bundles over \( S \) and \( S' \), respectively. A co-orientation of \( S \) in \( M \) is an orientation of \( NS \). We define the canonical co-orientation \( o^c_M \) of \( S \) in \( M \) to be given by the outer normal direction.

For a fiber bundle \( f_M : \mathcal{M} \rightarrow \mathcal{M}' \), we denote by
\[
TM^v \equiv \ker df_M
\]
its vertical tangent bundle. The short exact sequence
\[
0 \rightarrow TM^v \rightarrow TM \xrightarrow{df_M} f_M^* TM' \rightarrow 0
\]
of vector bundles determines an isomorphism
\[
\lambda(\mathcal{M}) \approx f_M^* \lambda(\mathcal{M}') \otimes \lambda(TM^v)
\]
of line bundles over \( \mathcal{M} \). If \( f_{M'} : \mathcal{M}' \rightarrow \mathcal{M}'' \) is another fiber bundle, the short exact sequence
\[
0 \rightarrow TM^v \rightarrow \ker \{ f_M \circ f_M' \} \rightarrow f_M^* TM' \rightarrow 0
\]
of the vertical tangent bundles induces an isomorphism
\[
\lambda(\ker \{ f_M \circ f_M' \}) \approx f_M^* \lambda(\mathcal{M}'' \otimes \lambda(TM^v)).
\]
The switch of the ordering of the factors in (3.3) from (3.2) is motivated by Lemma 3.1(1) below and by the inductive construction of the orientations \( o_{k,l} \) on the real Deligne-Mumford moduli spaces \( \overline{M}_{k,l} \) in Section 4.2.

If \( f : Z \rightarrow Y \) is a continuous map between two smooth manifolds, possibly with boundary, let
\[
\lambda(f) \equiv f^* \lambda(Y) \otimes \lambda(Z) \rightarrow Z.
\]
A relative orientation of \( f \) is an orientation on the line bundle \( \lambda(f) \). For a relative orientation \( o \) of \( f \) and \( u \in Z \), we denote by \( o_u \) the associated homotopy class of trivializations of the fiber \( \lambda_u(f) \) over \( u \) and the associated homotopy class of isomorphisms \( \lambda_u(Z) \rightarrow \lambda_{f(u)}(Y) \). If in addition \( o' \) is a relative orientation of another continuous map \( g : Y \rightarrow Z \), we denote by \( o o' \) the relative orientation of \( g \circ f \) corresponding to the homotopy class of the compositions
\[
\lambda_u(Z) \rightarrow \lambda_{f(u)}(Y) \rightarrow \lambda_{g(f(u))}(Z)
\]
of isomorphisms in the homotopy classes \( o_u \) and \( o'_{f(u)} \) for each \( u \in Z \).
We identify an orientation \( \sigma \) on a manifold \( Z \) with a relative orientation of \( Z \to \text{pt} \) in the obvious way. For a submanifold \( S \subset Z \), we identify a co-orientation \( \sigma_S^c \) on \( S \) with a relative orientation of the inclusion \( \iota_S : S \to Z \) via the first isomorphism in (3.1). If \( S' \subset S \) is also a submanifold with a co-orientation \( \sigma_{S'}^c \) in \( S \), then the relative orientation \( \sigma_{S'}^c \sigma_S^c \) of the inclusion

\[
\iota_S : S' \to S \to Z
\]
corresponds to the co-orientation of \( S' \) in \( Z \) induced by the co-orientations \( \sigma_{S'}^c \) and \( \sigma_S^c \) via the second isomorphism in (3.1). If \( f_M : M \to M' \) is a fiber bundle, we similarly identify an orientation \( \sigma_M^v \) of \( TM^v \) with a relative orientation of \( f_M \) via (3.3). If \( f_{M'} : M' \to M'' \) is another fiber bundle with an orientation \( \sigma_{M'}^v \) on \( TM'^v \), then the relative orientation \( \sigma_M^v \sigma_{M'}^v \) of the fiber bundle

\[
f_{M'} \circ f_M : M \to M''
\]
corresponds to the orientation on \( \ker d \{ f_{M'} \circ f_M \} \) induced by the co-orientations \( \sigma_M^v \) and \( \sigma_{M'}^v \) via (3.4).

If \( f, \sigma, S, \) and \( \sigma_S^c \) are as above, we denote by \( \sigma|_S \) the restriction of the trivialization of \( \lambda(f) \) determined by \( \sigma \) to \( S \) and define

\[
\sigma_S = \sigma_S^c \sigma
\]  
(3.5)
to be the relative orientation of \( \lambda(f|_S) \) induced by \( \sigma \) and \( \sigma_S^c \). If \( Z \) is a manifold with boundary, let

\[
\partial(Z, \sigma) = (\partial Z, \partial \sigma) = (\partial Z, \sigma_S^c \partial Z).
\]  
(3.6)
If \( Y \) is a point and so \( \sigma \) and \( \partial \sigma \) are orientations on \( Z \) and \( \partial Z \), respectively), this convention agrees with [26 p146] if and only if the dimension of \( M \) is odd. If \( S = \{ P \} \) is also a point, then the projection isomorphism \( T_P Z \to NS \) is orientation-preserving with respect to \( \sigma \) and \( \sigma_S^c \) if and only if

\[
\sigma_S^c \sigma = +1;
\]
this is the \( M', \emptyset = \{ \text{pt} \} \) case of Lemma 3.1(1) below.

If \( \sigma \) is a relative orientation of \( f : Z \to Y \) and \( u \in Z \) is such that \( d_u f \) is an isomorphism, we define

\[
s_u(\sigma) = \begin{cases} 
+1, & \text{if } d_u f \in \sigma_u; \\
-1, & \text{if } d_u f \notin \sigma_u.
\end{cases}
\]
If \( g : Y \to Z \) and \( \sigma' \) are also as above and \( d_{g(u)} g \) is an isomorphism as well, then

\[
s_u(\sigma \sigma') = s_u(\sigma)s_{f(u)}(\sigma')
\]  
(3.7)
If \( y \in Y \) is a regular value of \( f \) and the set \( f^{-1}(y) \) is finite, we define

\[
|f^{-1}(y)|_{\sigma} = \sum_{u \in f^{-1}(y)} s_u(\sigma).
\]
Let \( f_M : M \to M' \) be a fiber bundle. If \( \emptyset \subset M \) is a submanifold and \( P \in \emptyset \), then the differential \( d_P(f_M|_{\emptyset}) \) is an isomorphism if and only if the composition

\[
T_P M'' \equiv \ker d_P f_M \to T_P M \to \frac{T_P M}{T_P Y} = N_P Y
\]  
(3.8)
is. If \( \mathcal{M}_2 \) is another manifold, then
\[
\tilde{f}_\mathcal{M} \times \text{id}_{\mathcal{M}_2} : \mathcal{M} \times \mathcal{M}_2 \rightarrow \mathcal{M}' \times \mathcal{M}_2
\]
is also a fiber bundle and \( \Upsilon \times \mathcal{M}_2 \subset \mathcal{M}_1 \times \mathcal{M}_2 \) is a submanifold; see the first diagram in Figure 2.

The differential of
\[
\pi_1 : \mathcal{M} \times \mathcal{M}_2 \rightarrow \mathcal{M}
\]
induces a commutative diagram
\[
\begin{array}{ccc}
\ker\{\tilde{f}_\mathcal{M} \times \text{id}_{\mathcal{M}_2}\}^v & \xrightarrow{d\pi_1} & \mathcal{N}(\Upsilon \times \mathcal{M}_2) \\
\pi_1^* \mathcal{T}\mathcal{M}^v & \xrightarrow{d\pi_1} & \pi_1^* \mathcal{N}\Upsilon
\end{array}
\]
of vector bundle homomorphisms. Since the vertical arrows above are isomorphisms, they pull back a vertical orientation \( \sigma^v_\mathcal{M} \) of \( \tilde{f}_\mathcal{M} \) to a vertical orientation \( \pi_1^* \sigma^v_\mathcal{M} \) of \( \tilde{f}_\mathcal{M} \times \text{id}_{\mathcal{M}_2} \) and a co-orientation \( \sigma^c_\Upsilon \) of \( \Upsilon \) to a co-orientation \( \pi_1^* \sigma^c_\Upsilon \) of \( \Upsilon \times \mathcal{M}_2 \). We note the following.

**Lemma 3.1** Suppose \( \tilde{f}_\mathcal{M} : \mathcal{M} \rightarrow \mathcal{M}' \) is a fiber bundle with an orientation \( \sigma^v_\mathcal{M} \) on \( \mathcal{T}\mathcal{M}^v \), \( \Upsilon \subset \mathcal{M} \) is a submanifold with a co-orientation \( \sigma^c_\Upsilon \), and \( P \in \Upsilon \) is such that \( d\tilde{f}_\mathcal{M}(P|\Upsilon) \) is an isomorphism.

1. The isomorphism (3.8) is orientation-preserving with respect to \( \sigma^v_\mathcal{M} \) and \( \sigma^c_\Upsilon \) if and only if
\[
s_P(\sigma^c_\Upsilon \sigma^v_\mathcal{M}) = +1.
\]
2. If \( \mathcal{M}_2 \) is another manifold, \( \pi_1 \) is as in (3.9), and \( P_2 \in \mathcal{M}_2 \), then
\[
s_{(P,P_2)}((\pi_1^* \sigma^c_\Upsilon)(\pi_1^* \sigma^v_\mathcal{M})) = s_P(\sigma^c_\Upsilon \sigma^v_\mathcal{M}).
\]

Suppose that \( \tilde{f}_\mathcal{Z} : \mathcal{Z} \rightarrow \mathcal{Z}' \) is another fiber bundle, \( f, f' \) are maps as in the second diagram in Figure 2 so that it commutes, and \( \sigma^v_\mathcal{Z} \) and \( \sigma^v_\mathcal{M} \) are orientations on \( \mathcal{T}\mathcal{Z}^v \) and \( \mathcal{T}\mathcal{M}^v \), respectively. If \( u \in \mathcal{Z} \) is such that the restriction
\[
d_u f : T_u \mathcal{Z}^v = \ker d_u \tilde{f}_\mathcal{Z} \rightarrow T_{f(u)} \mathcal{M}'^v
\]
is an isomorphism, we define \( s_u(f, \sigma^v_\mathcal{Z}, \sigma^v_\mathcal{M}) \) to be +1 if this isomorphism is orientation-preserving with respect to the orientations \( \sigma^v_\mathcal{Z} \) and \( \sigma^v_\mathcal{M} \) and to be −1 otherwise. If \( P \in \Upsilon \) are as above, \( u \in f^{-1}(P) \), and the homomorphisms (3.8) and (3.10) are isomorphisms, then \( f \) is transverse to \( \Upsilon \) at \( u \), \( f^{-1}(\Upsilon) \subset \mathcal{Z} \) is a smooth submanifold near \( u \), the composition
\[
T_u \mathcal{Z}^v = \ker d_u \tilde{f}_\mathcal{Z} \rightarrow T_u \mathcal{Z} \rightarrow \frac{T_u \mathcal{Z}}{T_u f^{-1}(\Upsilon)} = \mathcal{N}_u f^{-1}(\Upsilon)
\]
is an isomorphism, and \( d_ug \) descends to an isomorphism
\[
d_ug: N uf^{-1}(Y) \cong \frac{T_uZ}{T uf^{-1}(Y)} \rightarrow \frac{T_PM}{T Pf^{-1}(Y)} \cong N Pf^{-1}(Y)
\]
and thus pulls back a co-orientation \( \sigma_Y^c \) on \( Y \subset M \) to a co-orientation \( f^*\sigma_Y^c \) on \( f^{-1}(Y) \subset Z \) near \( u \).

**Lemma 3.2** Let \( f_M, Y, P, \sigma_M^u \), and \( \sigma_Y^c \) be as in Lemma 3.2. Suppose in addition that \( \bar{f}_Z: Z \rightarrow Z' \) is another fiber bundle with an orientation \( \sigma_Z^u \) on \( T^*Z \), \( f, f' \) are maps so that the second diagram in Figure 2 commutes, and \( u \in f^{-1}(P) \). If the homomorphism (3.10) is an isomorphism, then
\[
s_u((f^*\sigma_Y^c)\sigma_Z^u) = s_u(f, \sigma_Z^u, \sigma_M^u)_{\bar{f}}(\sigma_Y^c, \sigma_M^u).
\]

### 3.2 Intersection signs

For continuous maps \( f: Z \rightarrow Y \) and \( g: Y \rightarrow Y \) between manifolds with boundary, define
\[
M_{f,g} \equiv Z \times g Y = \{(u, P) \in Z \times Y - (\partial Z) \times (\partial Y) : f(u) = g(P)\},
\]
\[
f \times g Y: M_{f,g} \rightarrow Y, \quad f \times g (u, P) = f(u) = g(P).
\]

We call two such maps \( f \) and \( g \) strongly transverse if they are smooth and the maps \( f \) and \( f|_{\partial Z} \) are transverse to the maps \( g \) and \( g|_{\partial Y} \). The space \( M_{f,g} \) is then a smooth manifold and
\[
\dim M_{f,g} + \dim Y = \dim Z + \dim Y,
\]
\[
\partial M_{f,g} = (Z - \partial Z) \times g Y(\partial Y) \sqcup (\partial Z) \times g Y(\partial Y) \quad (3.11)
\]

Suppose in addition that
\[
f = (f_1, f_2): Z \rightarrow Y \equiv X \times M, \quad g = (g_1, g_2): Y \rightarrow X \times M, \quad (3.12)
\]
\( \sigma_1 \) is a relative orientation of \( f_1 \), and \( \sigma_2 \) is a relative orientation of \( g_2 \); see the second rows in the diagrams of Figure 3. For \((u, P) \in M_{f,g}\) such that the homomorphism
\[
T_uZ \oplus T_PY \rightarrow T_{f(u)Y} = T_{g(P)Y}, \quad (v, w) \rightarrow d_uf(v) + dPg(w), \quad (3.13)
\]
is an isomorphism, we define \((f, \sigma_1)_{u,P}(g, \sigma_2)\) to be +1 if the top exterior power \( \Lambda^u_Z \) of this isomorphism lies in the homotopy class determined by \((\sigma_1)_{u,P}\) and to be −1 otherwise. If \( M_{f,g} \) is also finite, let
\[
|M_{f,g}|_{\sigma_1, \sigma_2}^{\pm} = \sum (f, \sigma_1)_{u,P}(g, \sigma_2).
\]

Suppose \( Y \subset M \) is a bordered submanifold with co-orientation \( \sigma_Y^c \), \( g_1: Y \rightarrow X \) is a constant map, and \( g_2: Y \rightarrow M \) is the inclusion. Then,
\[
M_{f,g} = \{(u, f_2(u)) : u \in f_2^{-1}(Y) \cap f_1^{-1}(g_1(Y)) - (\partial Z) \cap f_2^{-1}(\partial Y)\}.
\]

If in addition \( f \) is strongly transverse to \( g \), then \( f_2^{-1}(Y) \subset Z \) is a smooth submanifold with co-orientation \( f^*\sigma_Y^c \) and the restriction
\[
f_1: f_2^{-1}(Y) \rightarrow X
\]
Figure 3: The maps of Lemma 3.3 with $g = (x, g_2)$ for some $x \in X$.

is a submersion. Along with the relative orientation $\sigma_1$ on $f_1$, $f_2^*\sigma_1^T$ induces a relative orientation $(f_2^*\sigma_1^T)\sigma_1$ on this restriction as in (3.5). If in addition $S \subset \mathcal{M}$ is another submanifold strongly transverse to $\Upsilon$, then the homomorphism

$$\mathcal{N}_S(\Upsilon \cap S) \equiv \frac{T|_{\Upsilon \cap S}}{T(\Upsilon \cap S)} \to \frac{\mathcal{T}M|_{\Upsilon \cap S}}{\mathcal{T}\Upsilon|_{\Upsilon \cap S}} \equiv \mathcal{N}_{\Upsilon|_{\Upsilon \cap S}}$$

of vector bundles is an isomorphism. The co-orientation $\sigma_1^T$ of $\Upsilon$ in $\mathcal{M}$, then restricts to a co-orientation $\sigma_1^T|_{\Upsilon \cap S}$. In such a case,

$$M_{f,g} = M_{f,g|_{\Upsilon \cap S}}.$$

The following observations are straightforward.

**Lemma 3.3** Suppose $Z, \Upsilon, X, \mathcal{M}, f, g, f_1, g_1, \sigma_1, \sigma_1^T$ are as above (with $\Upsilon \subset \mathcal{M}$ and $g_1$ constant),

$$\dim Z + \dim \Upsilon = \dim X + \dim \mathcal{M},$$

and $f$ is strongly transverse to $g$.

1. For every $u \in f_2^{-1}(\Upsilon) \cap f_1^{-1}(g_1(\Upsilon))$,

$$\langle f, \sigma_1 \rangle u \cdot f_2 (u) \left( g, \sigma_1^T \right) = (-1)^{\left( \dim \Upsilon \right) \left( \dim \mathcal{M} \right)} x_u \left( (f_2^* \sigma_1^T \sigma_1) \sigma_1 \right).$$

2. Suppose $S \subset \mathcal{M}$ is another submanifold strongly transverse to $\Upsilon$ and $f_2(Z) \subset S$. For every $(u, P) \in M_{f,g|_{\Upsilon \cap S}}$,

$$\langle f, \sigma_1 \rangle u \cdot p \left( g, \sigma_1^T \right) = (-1)^{\left( \dim \mathcal{M} \right) \left( \dim \Upsilon \right)} x_u \left( \left( f^* \sigma_1^T \sigma_1 \right) \sigma_1 \right)$$

with the intersection on the right-hand side above taken in $X \times S$.

3. Suppose $\mathcal{M}_2$, $\pi_1$, and $\pi_1^* \sigma_1^T$ are as in Lemma 3.1(2) and the second diagram in Figure 3 commutes. For every $(u, P) \in M_{f, g \times \text{id}_{\mathcal{M}_2}}$,

$$\langle f, \sigma_1 \rangle u \cdot \tilde{p} \left( g \times \text{id}_{\mathcal{M}_2}, \pi_1^* \sigma_1^T \right) = (-1)^{\left( \dim \mathcal{M}_2 \right) \left( \dim \Upsilon \right)} x_u \left( \left( f^* \sigma_1^T \sigma_1 \right) \sigma_1 \right).$$

Let $e_1: Z_1 \to X'$ and $e_2: Z_2 \to X'$ be strongly transverse maps so that

$$Z = M_{e_1, e_2} \equiv \{(u_1, u_2) \in Z_1 \times Z_2 - (\partial Z_1) \times (\partial Z_2): e_1(u_1) = e_2(u_2)\} \subset Z_1 \times Z_2.$$
is a smooth submanifold. For each \( u \equiv (u_1, u_2) \in \mathcal{Z} \), the short exact sequence
\[
0 \rightarrow T_u \mathcal{Z} \rightarrow T_{u_1} \mathcal{Z} \oplus T_{u_2} \mathcal{Z} \rightarrow T_{e_1(u_1)} \mathcal{X} = T_{e_2(u_2)} \mathcal{X}' \rightarrow 0,
\]
\[
(v_1, v_2) \mapsto d_{u_2} e_2(v_2) - d_{u_1} e_1(v_1),
\]
of vector spaces induces an isomorphism
\[
\lambda_u(\mathcal{Z}) \otimes \lambda(T_{e_2(u_2)} \mathcal{X}') \cong \lambda_u(\mathcal{Z}_1) \otimes \lambda_u(\mathcal{Z}_2).
\]
Combined with relative orientations \( \varphi_{11} \) and \( \varphi_{12} \) of
\[
f_{11}: \mathcal{Z}_1 \rightarrow \mathcal{X}_1 \quad \text{and} \quad (e_2, f_{12}): \mathcal{Z}_2 \rightarrow \mathcal{X}' \times \mathcal{X}_2,
\]
this isomorphism determines a homotopy class of isomorphisms
\[
\lambda_u(\mathcal{Z}) \otimes \lambda_{e_2(u_2)}(TX') \rightarrow \lambda_{f_{11}(u_1)}(\mathcal{X}_1) \otimes \lambda_{(e_2(u_2), f_{12}(u_2))}(X' \times \mathcal{X}_2)
\]
\[
\rightarrow \lambda_{f_{11}(u_1)}(\mathcal{X}_1) \otimes \lambda_{f_{12}(u_2)}(\mathcal{X}_2) \otimes \lambda(T_{e_2(u_2)} X').
\]
The homotopy class of trivializations in (3.14) corresponds to a relative orientation \((\varphi_{11})_{e_1 \times e_2} \varphi_{12}\) of the restriction
\[
f_1 \equiv (f_{11} \times f_{12})|_{\mathcal{Z}_1}: \mathcal{Z}_1 \rightarrow \mathcal{X}_1 \times \mathcal{X}_2.
\]
For a map \( f_2: \mathcal{Z}_1 \rightarrow \mathcal{M} \), let \( f_2: \mathcal{Z} \rightarrow \mathcal{M} \) also denote the composition of \( f_2 \) with the projection to \( \mathcal{Z}_1 \). If in addition \( g_2: \mathcal{Y} \rightarrow \mathcal{M} \) is the embedding of a (possibly bordered) submanifold, \( g_1, \mathcal{Y} \rightarrow \mathcal{X}_1 \) are constant maps with values \( x_1 \) and \( x_2 \), respectively, then
\[
f \equiv (f_{11}, f_{12}, f_2): \mathcal{Z} \rightarrow \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{M}, \quad \text{and} \quad g \equiv (g_{11}, g_{12}, g_2): \mathcal{Y} \rightarrow \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{M},
\]
\[
M_{f,g} = \{(u_1, u_2, P): (u_1, P), u_2) \in M_{\varphi_{11}(M_{f_{11}, f_{12}}, g_{11}, g_2), \{e_2|_{f_{12}^{-1}(u_2)}\}}\};
\]
see the diagram in Figure 4.

**Lemma 3.4** Suppose \( \mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}, \mathcal{X}', \mathcal{X}_1, \mathcal{X}_2, \mathcal{M}, \mathcal{Y}, e_i, f_{11}, f_2, f, g_{11}, g_2, g, \varphi_{11} \) are as above, \( \varphi_{12} \) is a co-orientation on \( \mathcal{Y} \),
\[
\dim \mathcal{Z}_1 + \dim \mathcal{Y} = \dim \mathcal{X}_1 + \dim \mathcal{M}, \quad \dim \mathcal{Z}_2 = \dim \mathcal{X}' + \dim \mathcal{X}_2,
\]
the maps \( e_1 \) and \( e_2 \) are strongly transverse, and the maps \( f \) and \( g \) are strongly transverse. If
\[
((u_1, P), u_2) \in (\mathcal{Z}_1)_{(f_{11}, f_{12})} \times (g_{11}, g_2)(\mathcal{Y}) \quad e_1 \times e_2 f_{12}^{-1}(g_{12}(\mathcal{Y})),
\]
then
\[
(f, (\varphi_{11})_{e_1 \times e_2} \varphi_{12})(u_1, u_2, f_2(u_1))(g, \varphi_{12}) = (-1)^{\dim \mathcal{Z}_2}(\text{codim} \mathcal{Y} + \dim \mathcal{X}') \cdot \left(\left((f_{11}, f_{12}), \varphi_{11}\right) u_1; f_2(u_1) \left((g_{11}, g_2), \varphi_{12}\right) \right) \delta_{u_2}(\varphi_{12}).
\]
3.3 Steenrod pseudocycles

Let $Y$ be a smooth manifold, possibly with boundary. For a continuous map $f: Z \rightarrow Y$, let

$${\text{Bd}} f = \bigcap_{K \in Z} \text{cmpt} f(Z - K)$$

be the boundary of $f$. A $\mathbb{Z}_2$-pseudocycle into $Y$ is a continuous map $f: Z \rightarrow Y$ from a manifold, possibly with boundary, so that the closure of $f(Z)$ in $Y$ is compact and there exists a smooth map $h: Z' \rightarrow Y$ such that

$$\dim Z' \leq \dim Z - 2, \quad \text{Bd} f \subset h(Z'), \quad f(\partial Z) \subset (\partial Y) \cup h(Z').$$

The codimension of such a Steenrod pseudocycle is $\dim Y - \dim Z$. A continuous map $\tilde{f}: \tilde{Z} \rightarrow Y$ is bordered $\mathbb{Z}_2$-pseudocycle with boundary $f$ as above if the closure of $\tilde{f}(\tilde{Z})$ in $Y$ is compact,

$$Z \subset \partial \tilde{Z}, \quad \tilde{f}|_Z = f,$$

and there exists a smooth map $\tilde{h}: \tilde{Z}' \rightarrow Y$ such that

$$\dim Z' \leq \dim \tilde{Z} - 2, \quad \text{Bd} \tilde{f} \subset \tilde{h}(\tilde{Z}'), \quad \tilde{f}(\partial \tilde{Z} - Z) \subset (\partial Y) \cup \tilde{h}(Z').$$

If $\tilde{Z}$ is one-dimensional, then $\tilde{Z}$ is compact and $\tilde{f}(\partial \tilde{Z} - Z) \subset \partial Y$.

Two bordered $\mathbb{Z}_2$-pseudocycles $\tilde{f}_1: \tilde{Z}_1 \rightarrow Y$ and $\tilde{f}_2: \tilde{Z}_2 \rightarrow Y$ as above are transverse if

- the maps $\tilde{f}_1$ and $\tilde{f}_2$ are strongly transverse and
- there exist smooth maps $\tilde{h}_1: \tilde{Z}_1' \rightarrow Y$ and $\tilde{h}_2: \tilde{Z}_2' \rightarrow Y$ such that $\tilde{h}_1$ is transverse to $\tilde{f}_2$ and $\tilde{f}_2|_{\partial \tilde{Z}_2}$, $\tilde{h}_2$ is transverse to $\tilde{f}_1$ and $\tilde{f}_1|_{\partial \tilde{Z}_1}$, and

$$\dim \tilde{Z}'_1 \leq \dim \tilde{Z}_1 - 2, \quad \dim \tilde{Z}'_2 \leq \dim \tilde{Z}_2 - 2, \quad \text{Bd} \tilde{f}_1 \subset \tilde{h}_1(\tilde{Z}'_1), \quad \text{Bd} \tilde{f}_2 \subset \tilde{h}_2(\tilde{Z}'_2).$$

In such a case,

$$\tilde{f}_1 \times \tilde{f}_2: M_{\tilde{f}_1, \tilde{f}_2} \rightarrow Y$$

is a bordered $\mathbb{Z}_2$-pseudocycle with boundary (3.11).

A Steenrod pseudocycle into $Y$ is a $\mathbb{Z}_2$-pseudocycle $f: Z \rightarrow Y$ along with a relative orientation $\theta$ of $f$. A bordered $\mathbb{Z}_2$-pseudocycle $\tilde{f}: \tilde{Z} \rightarrow Y$ with boundary $f$ and a relative orientation $\tilde{\theta}$ of $\tilde{f}$ is a bordered Steenrod pseudocycle with boundary $(f, \theta)$ if $\partial \tilde{\theta} = \theta$. If $(f, \theta)$ is a codimension 0 Steenrod pseudocycle, then the number

$$\deg(f, \theta) = \sum_{u \in f^{-1}(y)} s_u(\theta) \in \mathbb{Z} \quad (3.15)$$

is well-defined for a generic choice of $y \in Y$ and is independent of such a choice. We call this number the degree of $(f, \theta)$. It vanishes if $(f, \theta)$ bounds a bordered Steenrod pseudocycle $(\tilde{f}, \tilde{\theta})$.

Lemma 3.5 Suppose $Z, \mathcal{Y}, X, \mathcal{M}, Y, f, g, f_1, f_2, g_1, g_2$ are as in (3.12) and just below and such that

$$\dim Z + \dim \mathcal{Y} = \dim Y + 1.$$

If $f$ and $g$ are transverse bordered $\mathbb{Z}_2$-pseudocycles, then $Z_f \times g(\partial \mathcal{Y})$ and $(\partial Z)_f \times g \mathcal{Y}$ are finite sets and

$$|Z_f \times g(\partial \mathcal{Y})|^{\pm}_{\theta_1, \theta_2} = (-1)^{\dim \mathcal{Y}} |(\partial Z)_f \times g \mathcal{Y}|^{\pm}_{\theta_1, \theta_2} \quad (3.16).$$
Proof. By the transversality and dimension assumptions, $M_{f,g}$ is a compact one-dimensional manifold and

$$\partial M_{f,g} = \mathcal{Z} \times g(\partial \mathcal{Y}) \sqcup (\partial \mathcal{Z}) \times g(\partial \mathcal{Y}).$$

In particular, the two sets on the right-hand side above are finite. By a direct computation, this equality respects the orientations with the orientation on the last fiber product modified by $(-1)^{\text{codim} \mathcal{Z}}$ and for a suitably chosen orientation on the left-hand side. Alternatively, (3.16) is equivalent to

$$|\mathcal{Z} \times g(\partial \mathcal{Y})|^\pm_{\partial_1, \partial_2} = (-1)^{\dim \mathcal{Z}(\dim \mathcal{Y})}|\mathcal{Y} \times f(\partial \mathcal{Z})|^\pm_{\partial_2, \partial_1}.$$ 

The sign in this statement must be symmetric in $\dim \mathcal{Z}$ and $\dim \mathcal{Y}$, depend only on their parity, be +1 if both dimensions or codimensions are even, and be −1 for linear maps from intervals to $\mathbb{R}$. □

4 Moduli spaces of stable curves

4.1 Main stratum and orientations

For $k \in \mathbb{Z}_{\geq 0}$, let $[k] = \{1, \ldots, k\}$. If in addition $k \geq 3$, we denote by $\overline{M}_{0,k}$ the Deligne-Mumford moduli space of stable rational curves with $k$ marked points. For $k, l \in \mathbb{Z}_{\geq 0}$ with $k + 2l \geq 3$, we denote by $\overline{M}_{k,l}$ the Deligne-Mumford moduli space of stable real genus 0 curves

$$\mathcal{C} \equiv \{(\Sigma, (x_i)_{i \in [k]}, (z_i^+, z_i^-)_{i \in [l]}, \sigma)\}$$

with $k$ real marked points, $l$ conjugate pairs of marked points, and an anti-holomorphic involution $\sigma$ with separating fixed locus. This space is a smooth manifold of dimension $k + 2l - 3$, without boundary if $k \geq 1$ and with boundary if $k = 0$. The boundary of $\overline{M}_{0,l}$ parametrizes the curves with no irreducible component fixed by the involution; the fixed locus of the involution on a curve in $\partial \overline{M}_{0,l}$ is a single node. The strata of $\overline{M}_{0,l}$ parametrizing curves with two invariant irreducible components sharing a real node are of codimension 1, but are not part of $\partial \overline{M}_{0,l}$. The moduli space $\overline{M}_{k,l}$ is orientable if and only if $k = 0$ or $k + 2l \leq 4$; see [7] Prop. 1.5.

The main stratum $\mathcal{M}_{k,l}$ of $\overline{M}_{k,l}$ is the quotient of

$$\{(x_i)_{i \in [k]}, (z_i^+, z_i^-)_{i \in [l]} : x_i \in S^1, z_i^+ \in \mathbb{P}^1 - S^1, z_i^- = \tau(z_i^-), x_i \neq x_j, z_i^+ \neq z_j^+ \land i \neq j\}$$

by the natural action of the subgroup $\text{PSL}_2^\mathbb{C} \subset \text{PSL}_2^\mathbb{C}$ of automorphisms of $\mathbb{P}^1$ commuting with $\tau$. The topological components of $\mathcal{M}_{k,l}$ are indexed by the possible distributions of the points $z_i^+$ between the interiors of the two disks cut out by the fixed locus $S^1$ of the standard involution $\tau$ on $\mathbb{P}^1$ and by the orderings of the real marked points $x_i$ on $S^1$.

If $k + 2l \geq 4$ and $i \in [k]$, let

$$\iota^R_{k,l;i} : \overline{M}_{k,l} \longrightarrow \overline{M}_{k-1,l}$$

be the forgetful morphism dropping the $i$-th real marked point. The restriction of $\iota^R_{k,l;i}$ to the preimage of $\mathcal{M}_{k-1,l}$ is an $S^1$-fiber bundle. The associated short exact sequence (3.2) induces an isomorphism

$$\lambda(\mathcal{M}_{k,l}) \cong \iota^{R*}_{k,l;i} \lambda(\mathcal{M}_{k-1,l})|_{\mathcal{M}_{k,l}} \otimes (\ker d\iota^{R}_{k,l;i})|_{\mathcal{M}_{k,l}}.$$  

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If \( k + 2l \geq 5 \) and \( i \in [l] \), we similarly denote by
\[
f_{k,l;i} : \overline{\mathcal{M}}_{k,l} \to \overline{\mathcal{M}}_{k,l-1}
\] (4.4)
the forgetful morphism dropping the \( i \)-th conjugate pair of marked points. The restriction of \( f_{k,l;i} \) to the preimage of \( \mathcal{M}^r_{k,l-1} \) is a \( \mathbb{P}^1 \)-fiber bundle and thus induces an isomorphism
\[
\lambda(\mathcal{M}_{k,l}) \cong f_{k,l;i}^* \lambda(\mathcal{M}_{k,l-1})|_{\mathcal{M}_{k,l}} \otimes \lambda(\ker d_{f_{k,l;i}})|_{\mathcal{M}_{k,l}} .
\] (4.5)

For each \( C \in \mathcal{M}_{k,l} \) as in (4.1),
\[
\ker d_{f_{k,l;i}} \cong T_{z^+_i} \mathbb{P}^1
\]
is canonically oriented by the complex orientation of the fiber \( \mathbb{P}^1 \) at \( z_i^+ \). We denote the resulting orientation of the last factor in (1.3) by \( \sigma^+_i \).

Suppose \( l \in \mathbb{Z}^+ \) and \( C \in \mathcal{M}^r_{k,l} \) is as in (4.1) with \( \Sigma = \mathbb{P}^1 \). Let \( \mathbb{D}^2_+ \subset \mathbb{C} \subset \mathbb{P}^1 \) be the disk cut out by the fixed locus \( S^1 \) of \( \tau \) which contains \( z_i^+ \). We orient \( S^1 \subset \mathbb{D}^2_+ \subset \mathbb{C} \) in the standard way (this is the opposite of the boundary orientation of \( \mathbb{D}^2_+ \) as defined in Section 3.1). If \( 2k \geq 4 \) and \( i \in [k] \), this determines an orientation \( \sigma^+_i \) of the fiber
\[
\ker d_{f_{k,l;i}} \cong T_{x_i} S^1
\]
of the last factor in (1.3) over \( f_{k,l;i}^*(C) \). This orientation extends over the subspace
\[
\overline{\mathcal{M}}^r_{k,l;i} \subset \overline{\mathcal{M}}^r_{k,l}
\]
consisting of curves \( C \) as in (4.1) such that the real marked point \( x_i \) of \( C \) lies on the same irreducible component of \( \Sigma \) as the marked point \( z_i^+ \).

Let \( (x_1, x_{j_2}(C), \ldots, x_{j_k}(C)) \) be the ordering of the real marked points of \( C \) starting with \( x_1 \) and going in the direction of the standard orientation of \( S^1 \). We denote by \( \delta_\mathbb{R}(C) \in \mathbb{Z}_2 \) the sign of the permutation sending
\[
\varpi_C : \{2, \ldots, k\} \to \{2, \ldots, k\}, \quad \varpi_C(i) = j_i(C).
\]
If \( k = 0 \), we take \( \delta_\mathbb{R}(C) = 0 \). For \( l^* \in [l] \), let
\[
\delta_{l^*}(C) = \left| \left\{ i \in [l] - [l^*] : z_i^+ \notin \mathbb{D}^2_+ \right\} \right| + 2Z \in \mathbb{Z}_2.
\]
In particular, \( \delta_\mathbb{R}(C) = 0 \) if \( k \leq 2 \) and \( \delta_{l^*}(C) = 0 \). The functions \( \delta_\mathbb{R} \) and \( \delta_{l^*} \) are locally constant on \( \mathcal{M}^r_{k,l} \).

The space \( \mathcal{M}^r_{1,1} = \overline{\mathcal{M}}^r_{1,1} \) is a single point; we take \( \sigma_{1,1} = +1 \) to be its orientation as a plus point. We identify the one-dimensional space \( \overline{\mathcal{M}}^r_{0,2} \) with \([0, \infty]\) via the cross ratio
\[
\varphi_{0,2} : \overline{\mathcal{M}}^r_{0,2} \to [0, \infty], \quad \varphi([\{z_1^+, z_1^{-}\}, \{z_2^+, z_2^{-}\}]) = \frac{z_2^+ - z_1^-}{z_2^- - z_1^+} : \frac{z_2^+ - z_1^-}{z_2^- - z_1^+} = \left| \frac{1 - z_1^+/z_2^-}{z_1^+ - z_2^-} \right|^2 ;
\] (4.6)
see Figure 5. This identification, which is the opposite of [8 (3.1)] and [10 (1.12)], determines an orientation \( \sigma_{0,2} \) on \( \overline{\mathcal{M}}^r_{0,2} \).

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We now define an orientation $\sigma_{k,l}$ on $\mathcal{M}_{k,l}^r$ for $l \in \mathbb{Z}^+$ and $k+l \geq 3$ inductively. If $k \geq 1$, we take $\sigma_{k,l}$ to be so that the $i = k$ case of the isomorphism (4.3) is compatible with the orientations $\sigma_{k,l}$, $\sigma_{k-1,l}$, and $\sigma_{k}^R$ on the three line bundles involved. If $l \geq 2$, we take $\sigma_{k,l}$ to be so that the $i = l$ case of the isomorphism (4.5) is compatible with the orientations $\sigma_{k,l}$, $\sigma_{k,l-1}$, and $\sigma_{l}^R$. By a direct check, the orientations on $\mathcal{M}_{1,2}^r$ induced from $\mathcal{M}_{0,2}^r$ via (4.3) and $\mathcal{M}_{1,1}^r$ via (4.5) are the same. Since the fibers of $f_{k,l}|\mathcal{M}_{k,l}^r$ are even-dimensional, it follows that the orientation $\sigma_{k,l}$ on $\mathcal{M}_{k,l}^r$ is well-defined for all $l \in \mathbb{Z}^+$ and $k \in \mathbb{Z}_{\geq 0}$ with $k+2l \geq 3$. This orientation is as above [7, Lemma 5.4].

For $l^* \in [l]$, we denote by $\sigma_{k,l,l^*}$ the orientation on $\mathcal{M}_{k,l}^r$ which equals $\sigma_{k,l}$ at $C$ if and only if $\delta_{k,l}(C) = \delta_{k,l^*}(C)$. The next statement is straightforward.

**Lemma 4.1** The orientations $\sigma_{k,l,l^*}$ on $\mathcal{M}_{k,l}^r$ with $k, l \in \mathbb{Z}_{\geq 0}$ and $l^* \in [l]$ such that $k+2l \geq 3$ and satisfy the following properties:

- (M1) if $C \in \mathcal{M}_{k+1,l}^r$, the isomorphism (4.3) with $(k, i)$ replaced by $(k+1, j_{k+1}(C))$ respects the orientations $\sigma_{k+1,l,i^*}$, $\sigma_{k,l,i^*}$, and $\sigma_{k}^R_{k+1}$ at $C$;
- (M2) the isomorphism (4.5) with $(l, i)$ replaced by $(l+1, l^*+1)$ respects the orientations $\sigma_{k,l+1,i^*}^*$, $\sigma_{k,l,i^*}^*$, and $\sigma_{l}^R_{k,l+1}$;
- (M3) $\sigma_{k,l,i^*}$ is preserved by the interchange of two real points $x_i$ and $x_j$ with $2 \leq i, j \leq k$;
- (M4) if $2 \leq i \leq k$ and $C \in \mathcal{M}_{k,l}^r$, $\sigma_{k,l,i^*}$ is preserved at $C$ by the interchange of the real points $x_1$ and $x_{j_{k}(C)}$ with $2 \leq i \leq k$ if and only if $(k-1)(i-1) \in 2\mathbb{Z}$;
- (M5) $\sigma_{k,l,i^*}$ is preserved by the interchange of the points in a conjugate pair $(z_i^+, z_i^-)$ with $l^* < i \leq l$;
- (M6) $\sigma_{k,l,i^*}$ is preserved by the interchange of the points in a conjugate pair $(z_i^+, z_i^-)$ with $1 \leq i \leq l^*$;
- (M7) if $C \in \mathcal{M}_{k,l}^r$, $\sigma_{k,l,i^*}$ is preserved at $C$ by the interchange of the points in the conjugate pair $(z_i^+, z_i^-)$ if and only if

$$k \neq 0 \quad \text{and} \quad l-l^* \equiv \left(\frac{k}{2}\right) \mod 2 \quad \text{or} \quad k = 0 \quad \text{and} \quad l-l^* \equiv 1 \mod 2.$$ 

**4.2 Codimension 1 strata and degrees**

The (open) codimension 1 strata of $\mathcal{M}_{k,l}^r - \partial \mathcal{M}_{k,l}^r$ correspond to the sets $\{(K_1, L_1), (K_2, L_2)\}$ such that

$$[k] = K_1 \cup K_2, \quad [l] = L_1 \cup L_2, \quad |K_1| + 2|L_1| \geq 2, \quad |K_2| + 2|L_2| \geq 2.$$
The stratum $S$ corresponding to such a set parametrizes marked curves $C$ as in (4.1) so that the underlying surface $\Sigma$ consists of two real irreducible components with one of them carrying the real marked points $x_i$ with $i \in K_1$ and the conjugate pairs of marked points $(z_i^+, z_i^-)$ with $i \in L_1$ and the other component carrying the other marked points. A closed codimension 1 stratum $\bar{S}$ is the closure of such an open stratum $S$. Thus,

$$S \simeq \mathcal{M}^r_{|K_1|+1,|L_1|} \times \mathcal{M}^r_{|K_2|+1,|L_2|}, \quad \bar{S} \simeq \mathcal{M}^r_{|K_1|+1,|L_1|} \times \mathcal{M}^r_{|K_2|+1,|L_2|}. \quad (4.7)$$

Let $l \in \mathbb{Z}^+$. If $S$ is a codimension 1 stratum of $\mathcal{M}^r_{k,l} - \partial \mathcal{M}^r_{k,l}$ and $C \in S$, we denote by $\mathbb{P}^1_k$ the irreducible component of $C$ containing the marked points $z_i^\pm$, by $\mathbb{P}^1_2$ the other irreducible component, and by $S_{p}^1 \subset \mathbb{P}^1_1$ and $S_{q}^2 \subset \mathbb{P}^1_2$ the fixed loci of the involutions on these components. For $r = 1, 2$, we then take $K_r(S)$ and $L_r(S)$ to be the set of real marked points and the set of conjugate pairs of marked points, respectively, carried by $\mathbb{P}^1_r$ and define

$$k_r(S) = |K_r(S)| \quad \text{and} \quad l_r(S) = |L_r(S)|.$$

For $i \in [l]$, we denote by

$$S_i \subset \mathcal{M}^r_{k,l} \quad \text{and} \quad \bar{S}_i \subset \mathcal{M}^r_{k,l}$$

the open codimension 1 stratum parametrizing marked curves consisting of two real spheres with the marked points $z_i^\pm$ on one of them and all other marked points on the other sphere and its closure, respectively.

If $\bar{S} \subset \mathcal{M}^r_{k,l} - \partial \mathcal{M}^r_{k,l}$ is a closed codimension 1 stratum different from $\bar{S}_1$, let

$$\mathfrak{j}_{S,1}: \bar{S} \longrightarrow \mathcal{M}^r_{k_1(S),l_1(S)} \times \mathcal{M}^r_{k_2(S) + 1, l_2(S)} \quad (4.8)$$

denote the composition of the second identification in (4.7) with the forgetful morphism

$$\mathfrak{j}^\mathfrak{nd}_{\mathfrak{r}}: \mathcal{M}^r_{k_1(S) + 1, l_1(S)} \longrightarrow \mathcal{M}^r_{k_1(S), l_1(S)}$$

as in (4.2) dropping the marked point nd corresponding to the node. The vertical tangent bundle of $\mathfrak{j}_{S,1}|_{\bar{S}}$ is a pullback of the vertical tangent bundle of $\mathfrak{j}^\mathfrak{nd}_{\mathfrak{r}}|_{\mathcal{M}^r_{k_1(S) + 1, l_1(S)}}$ and thus inherits an orientation from the orientation $\mathfrak{o}^\mathfrak{nd}_{\mathfrak{r}}$ of the latter specified in Section 4.1; we denote the induced orientation also by $\mathfrak{o}^\mathfrak{nd}_{\mathfrak{r}}$. It extends over the subspace

$$S^\bullet \subset \bar{S} \subset \mathcal{M}^r_{k,l}$$

of curves $C$ so that the marked point nd of the first component of the image of $C$ under (4.7) lies on the same irreducible component of the domain as the marked point corresponding to $z_1^+$. Let $\mathcal{Y} \subset \mathcal{M}^r_{k,l}$ be a bordered hypersurface. If $k + 2l \geq 4$ and $i \in [k]$, we call $\mathcal{Y}$ regular with respect to $\mathfrak{j}^\mathfrak{r}_{k,l,i}$ if $\mathcal{Y} \subset \mathcal{M}^r_{k,l,i}$, $\mathfrak{j}^\mathfrak{r}_{k,l,i}(\mathcal{Y} - \mathcal{Y})$ is contained in the strata of codimension at least 2, i.e. the subspace of $\mathcal{M}^r_{k-1,l}$ parametrizing curves with at least two nodes, and $\mathfrak{j}^\mathfrak{r}_{k,l,i}(\partial \mathcal{Y})$ is contained in the union of $\partial \mathcal{M}^r_{k-1,l}$ and the strata of codimension at least 2. By the last two assumptions, $\mathfrak{j}^\mathfrak{r}_{k,l,i} \mathcal{Y}$ is a $\mathbb{Z}_2$-pseudocycle of codimension 0; see Section 33.3. By the first assumption, the orientation $\mathfrak{o}^\mathfrak{r}_{\mathfrak{i}}$ of
the last factor in (4.3) and a co-orientation \( \sigma_\Gamma^c \) on \( \Upsilon \) induce a relative orientation \( \sigma_\Gamma^c \sigma_i^R \) of \( f_{k,l;i}^{\tau} \mid \Upsilon \); see the paragraph above Lemma 3.1. Let
\[
\deg^R_i (\Upsilon, \sigma_\Gamma^c) \equiv \deg (f_{k,l;i}^{\tau} \mid \Upsilon, \sigma_\Gamma^c \sigma_i^R)
\]
be the degree of the Steenrod pseudocycle \( (f_{k,l;i}^{\tau} \mid \Upsilon, \sigma_\Gamma^c \sigma_i^R) \); see (3.15).

Suppose in addition that \( S \subset \overline{M}_{k,l}^\tau \) is a codimension 1 stratum. We call \( \Upsilon \) regular with respect to \( S \) if \( \Upsilon \) and \( \partial \Upsilon \) are transverse to \( \overline{S} \) in \( \overline{M}_{k,l}^\tau \).

\[
\Upsilon \cap \overline{S} \approx \Upsilon_1 \times \overline{M}_{k_2(S)+1,l_2(S)}^\tau
\]
under the second identification in (4.7) for some \( \Upsilon_1 \subset \overline{M}_{k_1(S)+1,l_1(S),nd}^\tau \); \( f_{S;1}((\Upsilon-\Upsilon) \cap \overline{S}) \) is contained in the strata of codimension at least 2 of the target of \( f_{S;1} \), and \( f_{S;1}((\partial \Upsilon \cap \overline{S}) \) is contained in the union of the boundary and the strata of codimension at least 2 of the target of \( f_{S;1} \). By the first and the last two assumptions, \( f_{S;1} | \Upsilon \cap \overline{S} \) is a \( \mathbb{Z}_2 \)-pseudocycle of codimension 0. By the first assumption, a co-orientation \( \sigma_\Gamma^c \) on \( \Upsilon \) in \( \overline{M}_{k,l}^\tau \) determines a co-orientation
\[
\sigma_\Upsilon^c \mid \Upsilon \cap \overline{S} = \sigma_\Upsilon^c \mid \Upsilon \cap \overline{S}
\]
on \( \Upsilon \cap \overline{S} \) in \( \overline{S} \). By the second assumption, \( \Upsilon \cap \overline{S} \subset S^* \). Along with the first assumption, this implies that \( S \neq S_1 \) if \( \Upsilon \cap \overline{S} \neq \emptyset \) and that \( \sigma_\Upsilon^c \) and the orientation \( \sigma_i^R \) of the fibers of the restriction of (4.8) to \( S \) specified above induce a relative orientation \( \sigma_\Upsilon^c \mid \Upsilon \cap \overline{S} \sigma_i^R \) of \( f_{S;1} | \Upsilon \cap \overline{S} \). Let
\[
\deg_S (\Upsilon, \sigma_\Upsilon^c) \equiv \deg (f_{S;1} | \Upsilon \cap \overline{S}, \sigma_\Upsilon^c \sigma_i^R) \equiv \deg (f_{S;1} | \Upsilon \cap \overline{S}, \sigma_\Upsilon^c \sigma_i^R).
\]

We call a bordered hypersurface \( \Upsilon \subset \overline{M}_{k,l}^\tau \) regular if \( \overline{\Upsilon-\Upsilon} \) is contained in the strata of codimension at least 2 and \( \Upsilon \) is regular with respect to the forgetful morphism \( f_{k,l;i}^{\tau} \) for every \( i \in [k] \) and with respect to every codimension 1 stratum \( S \subset \overline{M}_{k,l}^\tau \). For such a hypersurface, \( \Upsilon \cap S_1 = \emptyset \).

### 4.3 Strata orientations

Suppose \( l \geq 2 \) and \( k+2l \geq 5 \). The moduli space \( \overline{M}_{k,l}^\tau \) contains codimension 2 strata \( \Gamma \) that parametrize marked curves \( C \) as in (4.3) so that the underlying surface \( \Sigma \) consists of one real component \( \mathbb{P}_0^1 \) and one pair \( \mathbb{P}_+^1 \) of conjugate components; see Figure 1. We do not distinguish these strata based on the ordering of the marked points on the fixed locus \( S_1 \subset \mathbb{P}_0^1 \) of the involution. For such a stratum \( \Gamma \), let \( l_0(\Gamma), l_\Sigma(\Gamma) \in \mathbb{Z}_{\geq 0} \) be the number of conjugate pairs of marked points carried by \( \mathbb{P}_0^1 \) and \( \mathbb{P}_+^1 \cup \mathbb{P}_-^1 \), respectively. In particular,
\[
l_\Sigma(\Gamma) \geq 2 \quad \text{and} \quad l_0(\Gamma) + l_\Sigma(\Gamma) = l.
\]
The closure \( \overline{\Gamma} \) of \( \Gamma \) decomposes as
\[
\overline{\Gamma} \approx \overline{M}_{k,l_0(\Gamma)+1}^\tau \times \overline{M}_{0,l_\Sigma(\Gamma)+1}^\tau.
\]
(4.9)

We call a codimension 2 stratum as above primary if the marked point \( z_i^L \) of the curves \( C \in \Gamma \) is carried by \( \mathbb{P}_+^1 \cup \mathbb{P}_-^1 \).

For a primary codimension 2 stratum \( \Gamma \) and \( C \in \Gamma \), we denote by \( \mathbb{P}_+^1 \) the irreducible component of \( C \) carrying the marked point \( z_i^+ \). In this case, we choose the identification (4.9) so that
Suppose now that \( \lambda \) of \( o \) and an orientation \( B \) be the projections to the two factors. Denote by \( x \) thus, the real marked point \( l \) for codimension 2 stratum. The orientation \( o \) codimension 2 stratum. The orientation \( \lambda \) be as in Section 4.2. Define \( (\lambda) \) the remaining conjugate pairs of points and the real points in the first factor on the right-hand side are numbered in the same order as on the left-hand side.

If in addition \( l^* \in [l] \), let \( l_0^\pm(\Gamma) \) (resp. \( l^\pm(\Gamma) \)) be the number of marked points \( z \) with \( i \in [l^*] \) carried by \( \mathbb{P}^1_0\) (resp. \( \mathbb{P}^1_+ \)). The second factor in (4.9) is canonically oriented (being a complex manifold). We denote by \( s \) the orientation on \( \Gamma \) obtained via the identification (4.9) from the orientation \( o_{k,l} \) on \( \Gamma \) agrees with the complex orientation of \( \lambda \).

With the identification as above, let

\[
\pi_1, \pi_2 : \Gamma \rightarrow \mathcal{M}_{k,0}(\Gamma)+1, \mathcal{M}_{0,1}(\Gamma)+1
\]

be the projections to the two factors. Denote by

\[
\mathcal{L}_{\Gamma}^\mathbb{R} \rightarrow \mathcal{M}_{k,0}(\Gamma)+1 \quad \text{and} \quad \mathcal{L}_{\Gamma}^\mathbb{C} \rightarrow \mathcal{M}_{0,1}(\Gamma)+1
\]

the universal tangent line bundles at the first point of the first conjugate pair of marked points and at the first marked point, respectively. The normal bundle \( \mathcal{N} \) consists of conjugate smoothings of the two nodes of the curves in \( \Gamma \). Thus, it is canonically isomorphic to the complex line bundle

\[
\mathcal{L}_{\Gamma} = \pi_1^* \mathcal{L}_{\Gamma}^\mathbb{R} \otimes \pi_2 \mathcal{L}_{\Gamma}^\mathbb{C} \rightarrow \Gamma.
\]

The next observation is straightforward.

**Lemma 4.2** Suppose \( k, l \in \mathbb{Z}^{>0} \) and \( l^* \in [l] \) are such that \( k + 2l \geq 3 \). Let \( \Gamma \subset \mathcal{M}_{k,l}^\tau \) be a primary codimension 2 stratum. The orientation \( \sigma_\Gamma \) on \( \mathcal{N} \) induced by the orientations \( o_{k,l; l^*} \) on \( \mathcal{M}_{k,l}^\tau \) and \( o_{\Gamma; l^*} \) on \( \Gamma \) agrees with the complex orientation of \( \mathcal{L}_\Gamma \).

Suppose now that \( l \in \mathbb{Z}^+ \) and \( S \) is a codimension 1 stratum of \( \mathcal{M}_{k,l}^\tau \) \(-\partial \mathcal{M}_{k,l}^\tau \). For \( r = 1, 2 \), let

\[
K_r(S) \subset [k], \quad L_r(S) \subset [l], \quad \text{and} \quad k_r(S), \ell_r(S) \in \mathbb{Z}^{>0}
\]

be as in Section 4.2. Define

\[
r(S) = \begin{cases} 1, & \text{if } k = 0 \text{ or } 1 \in K_1(S); \\
2, & \text{if } 1 \in K_2(S); 
\end{cases}
\]

thus, the real marked point \( x \) lies on \( S_{r(S)}^{\tau} \) if \( k \geq 1 \).

For \( l^* \in [l] \) and \( r = 1, 2 \), define

\[
L_{r^*}(S) = [L_r(S) \cap [l^*]], \quad L_{r^*}(S) = [L_{r^*}(S)].
\]

An orientation \( \sigma_{S,C} \) of the normal bundle \( \mathcal{N}_C S \) of \( S \) in \( \mathcal{M}_{k,l}^\tau \) at \( C \in S \) determines a direction of degeneration of elements of \( \mathcal{M}_{k,l}^\tau \) to \( C \). The orientation \( o_{k,l, l^*} \) on \( \mathcal{M}_{k,l}^\tau \) limits to an orientation \( o_{k,l, l^*}, C \) of \( \lambda_C(\mathcal{M}_{k,l}^\tau) \) obtained by approaching \( C \) from this direction. Along with \( \sigma_{S,C} \), \( o_{k,l, l^*}, C \) determines an orientation \( \partial_{\sigma_{S,C}} o_{k,l, l^*}, C \) of \( \lambda_C(S) \) via the first isomorphism in (3.1). If in addition \( l^* \geq 1 \),
Lemma 4.3

Let \( i^* \in L^*_2(S) \) be the smallest element. The two directions of degeneration of elements of \( \mathcal{M}_{k,l}^r \) to \( C \) are then distinguished by whether the marked points \( z_i^+ \) and \( z_i^- \) of the degenerating elements lie on the same disk \( \mathbb{D}_r^2 \) or not. We denote by \( \sigma_{S,C}^{2\pm} \) the orientation of \( \mathcal{N}_C S \) which corresponds to the direction of degeneration for which \( z_i^+ \in \mathbb{D}_r^2 \) and by \( \sigma_{S,C}^{2\mp} \) the opposite orientation. Let \( \sigma_{S,k,l,i^*,C}^{2\pm} \) and \( \sigma_{S,l^*,C}^{2\pm} \) be the orientations of \( \lambda_C(\mathcal{M}_{k,l}) \) and \( \lambda_C(S) \), respectively, induced by \( \sigma_{S,C}^{2\pm} \) as in the previous paragraph.

A topological component \( S_* \) of \( S \) is characterized by the distribution of the points \( z_i^\pm \) with \( i \in L_r(S) \) between the interiors of the two disks cut out by the fixed locus \( S_1 \) in each component \( \mathbb{P}_r^1 \) of the domain of the curves in \( S \) and by the orderings of the real marked points \( x_i \) with \( i \in K_r(S) \) on \( S_r^1 \).

Thus,

\[
S_* \approx M_1 \times M_2 \subset M_{k_1(S)+1,l_1(S)}^r \times M_{k_2(S)+1,l_2(S)}^r
\]

for some topological components \( M_1 \) and \( M_2 \) of the moduli spaces on the right-hand side above. We choose this identification so that

(\( \sigma_{S1} \)) the orderings of the conjugate pairs of marked points on the two sides are consistent,

(\( \sigma_{S2} \)) the nodal point on each of the irreducible components on the left-hand side corresponds to the first real marked point in the associated factor on the right-hand side.

If in addition \( l^*_2(S) \geq 1 \) and \( i^* \in L^*_2(S) \) is the smallest element as before, we denote by \( \sigma_{S,l^*} \) the orientation on \( S \) obtained via the identification (4.10) from the orientations \( \sigma_{k_1(S)+1,l_1(S);l^*_2(S)} \) on \( M_{k_1(S)+1,l_1(S)}^r \) and \( \sigma_{k_2(S)+1,l_2(S);l^*_2(S)} \) on \( M_{k_2(S)+1,l_2(S)}^r \). The orientation \( \sigma_{S,l^*} \) does not depend on the orderings of the real points on \( S_1 \) and \( S_2 \). In this case, both fixed loci \( S_1^0 \subset \mathbb{P}_1^1 \) are canonically oriented. For a topological component \( S_* \) of \( S \), let \( j'_1(S_*) \in \mathbb{Z}^{\geq 0} \) be the number of real marked points that lie on the oriented arc of \( S_r^1 \) between the nodal point of \( \mathbb{P}_r^1 \) and the real marked point \( x_1 \) of any \( C \in S_* \); if \( k=0 \), we take \( j'_1(S_*)=0 \). Define

\[
\delta^+_{C,l^*}(S) = 1, \quad \delta^+_{R}(S_*) = (k-1)j'_1(S_*) + (r(S)-1)k_1(S)k_2(S),
\]

\[
\delta^-_{C,l^*}(S) = l^*_2(S)-l^*_2(S), \quad \delta^-_{R}(S_*) = (k-1)j'_1(S_*) + \left(\frac{k_2(S)+1}{2}\right) + (r(S)-1)(k-1).
\]

Lemma 4.3 Suppose \( k, l \in \mathbb{Z}^{\geq 0} \) and \( l^* \in [l] \) are such that \( k+2l \geq 3 \). Let \( S_* \subset \mathcal{M}_{k,l}^r - \partial \mathcal{M}_{k,l}^r \) be a topological component of codimension 1 stratum such that \( l^*_2(S) \geq 1 \). The orientations \( \sigma_{S,l^*}^\pm \) and \( \sigma_{S,l^*} \) on \( \lambda(S)|S_* \) are the same if and only if \( \delta^\pm_{C,l^*}(S) \equiv k+\delta^\pm_{R}(S_*) \mod 2 \).

Proof. For \( r=1,2 \), let

\[
l_r = l_r(S), \quad l^*_r = l^*_r(S), \quad k_r = k_r(S), \quad j'_1 = j'_1(S_*)).
\]

If \( l^*_r = l = 2 \) and \( k=0, S = S_1 = S_2 \) is a point and \( \sigma_{S,l^*} = +1 \). The claim in this case thus holds by the definition of the orientations \( \sigma_{0,2;2} = \sigma_{0,2} \) on \( M_{0,2}^r \) and \( \sigma_{S,C}^{2\pm} \) on \( \mathcal{N}S \). Since the orientation \( \sigma_{0,1;l} \equiv \sigma_{0,l} \) with \( l \geq 2 \) is obtained from the orientations \( \sigma_{0,l-1;1,l-1} \) (resp. \( \sigma_{1,l-1;1,l-1} \)) and \( \sigma_{1;l}^+ \), it follows that the claim holds whenever \( l^*_r = 1 \) and \( k=0 \).

Let \( C \in S_* \) be as in (4.11). Suppose \( l^*_r < l \) and \( k = 0 \). Let \( l_1^r \) and \( l_2^r \) be the numbers of the marked points \( z_i^- \) of \( C \) with \( i \in [l]-[l^*_r] \) on the same disk as \( z_i^+ \) and on the same disk as \( z_i^\pm \), respectively.
By definition,
\[ \sigma_{1,t_1;\tau} = (-1)^{t_1} \sigma_{1,t_1;\tau} \mid_{M_1}, \quad \sigma_{t_1;\tau}^+ = (-1)^{t_1} \sigma_{t_1;\tau}^+ \mid_{M_1}, \quad \sigma_{t_1;\tau}^- = (-1)^{t_1} \sigma_{t_1;\tau}^- \mid_{M_1}. \]

Thus, the claim in this case follows from the \( t^* = t \) case above.

Suppose \( k > 0 \), \( S' \subset M_{0,1}^r \) is the image of \( S \) under the forgetful morphism \( \tilde{f} \) dropping all real marked points, \( C' = \tilde{f}(C) \), and
\[(C'_1, C'_2) \in M'_1 \times M'_2\]
is the corresponding pair of marked irreducible components (with 1 real marked point each). Let \((x_{i_1}, \ldots, x_{i_{k_1}})\) be the ordering of the real marked points on \( S'_1 \) along its canonical direction starting from the first point after the node and \((x_{j_1}, \ldots, x_{j_{k_2}})\) be the analogous ordering of the real marked points on \( S'_2 \). The orientation \( \sigma_{t_1;\tau} \mid_{TCS} \) on \( TCS \) is obtained via isomorphisms
\[
(TCS, \sigma_{t_1;\tau}) \approx (T_{C'_1} M'_1, \sigma_{1,t_1;\tau}) \oplus \bigoplus_{m=1}^{k_1} T_{x_{i_{m}}} S'_1 \oplus (T_{C'_2} M'_2, \sigma_{t_1;\tau}^+) \oplus \bigoplus_{m=1}^{k_2} T_{x_{j_{m}}} S'_2
\]
(4.11)
\[
\approx (T_{C'_1} M'_1, \sigma_{1,t_1;\tau}) \oplus (T_{C'_2} M'_2, \sigma_{t_1;\tau}^+) \oplus \bigoplus_{m=1}^{k_1} T_{x_{i_{m}}} S'_1 \oplus \bigoplus_{m=1}^{k_2} T_{x_{j_{m}}} S'_2
\]

from the standard orientations on \( S'_1 \) and \( S'_2 \) determined by the marked points \( z^+_1 \) and \( z^+_2 \). The second isomorphism above is orientation-preserving because the dimension of \( T_{C'_2} M'_2 \) is even.

Let \( \tilde{C} \in M_{k,l}^r \) be a smooth marked curve close to \( C \) from the direction of degeneration determined by \( o_{S'}^{t_1} \). Let \((x_1, x_{i_2}, \ldots, x_{i_{k_1}})\) be the ordering of the real marked points of \( \tilde{C} \) along the standard direction of \( S' \) determined by \( z^+_1(\tilde{C}) \). The orientation \( \sigma_{S'_1;\tau}^\pm \) at \( C \) is obtained via isomorphisms
\[
(TCS, \sigma_{S'_1;\tau}^\pm) \oplus (NC S, o_{S'}^{t_1}) \approx (T_{C} M_{k,l}^r, \sigma_{0,\tau}^\pm) \approx (T_{C'} M_{k,l}^r, \sigma_{0,\tau}^\pm) \oplus \bigoplus_{m=1}^{k} T_{x_{i_{m}}} S'_{1}
\]
(4.12)
\[
\approx (T_{C'} S', \sigma_{S'_1;\tau}^\pm) \oplus (NC S', o_{S'}^{t_1}) \oplus \bigoplus_{m=1}^{k} T_{x_{i_{m}}} S'_{1}
\]
\[
\approx (-1)^k (T_{C'} S', \sigma_{S'_1;\tau}^\pm) \oplus \bigoplus_{m=1}^{k} T_{x_{i_{m}}} S'_{1} \oplus (NC S, o_{S'}^{t_1}).
\]

By (4.11), (4.12), and the \( k = 0 \) case above, the claim in the general case holds if \( o_{S'}^{t_1} \) has the same parity as the parity of the permutation
\[
(i_1, \ldots, i_{k_1}, j_1, \ldots, j_{k_2}) \longrightarrow (i_{1}^+, \ldots, i_{k_2}^+)
\]
(4.13)
plus the parity of \( k_2 \) in the minus case, since the tangent spaces \( T_{x_{j_{m}}} S'_2 \) then enter with the reversed orientations.
Suppose \( r(S) = 1 \). The plus case of (4.13) then moves the indices \((i_1, \ldots, i_{j_1'})\) to the end preserving their order. The parity of this permutation is

\[
\delta^+_{\mathcal{R}}(S_s) = j'_1(k-j'_1) \equiv (k-1)j'_1 \mod 2.
\]

The minus case of (4.13) is the composition of the permutation

\[
(j_1, \ldots, j_{k_2}) \rightarrow (j_{k_2}, \ldots, j_1)
\]

with the transposition in the plus case. This adds an extra \( k_2(k_2-1)/2 \) to the parity.

Suppose \( r(S) = 2 \). The plus case of (4.13) then moves \((j_{j'_1+1}, \ldots, j_{j_1'})\) to the front preserving their order. The parity of this permutation is

\[
\delta^+_{\mathcal{R}}(S_s) = (k_2-j'_1)(k_1+j'_1) \equiv (k-1)j'_1 + k_1 k_2 \mod 2.
\]

The minus case of (4.13) consists of the permutation (4.14) followed by moving \((j_{j'_1+1}, \ldots, j_1)\) to the front of the entire \( k \) tuple. The parity of this permutation plus \( k_2 \) is

\[
\delta^-_{\mathcal{R}}(S_s) = \left(\frac{k_2+1}{2}\right) + (j'_1+1)(k-1-j'_1) \equiv (k-1)j'_1 + k-1 + \left(\frac{k_2+1}{2}\right).
\]

This establishes the claim. \( \square \)

### 4.4 Bordisms in \( \overline{M}_{1,2} \) and \( \overline{M}_{0,3} \)

The two relations of Theorem 1.1 are proved by applying (2.5) with the hypersurfaces \( \Upsilon \subset \overline{M}_{1,2} \) and \( \Upsilon \subset \overline{M}_{0,3} \) of Lemmas 4.4 and 4.5 below. These hypersurfaces are regular, in the sense defined at the end of Section 4.2, and in particular are disjoint from the codimension 1 stratum \( S_1 \) of the moduli space. We determine the degrees of these hypersurfaces with respect to the other non-boundary codimension 1 strata and with respect to the forgetful morphism \( f_{1,2,1}^{\mathbb{R}} \) in the first case. These degrees are essential for computing the right-hand side of (2.5); see Proposition 5.7.

Orientations are interpreted below as relative orientations of maps to a point; see Section 4.1. All notation for the codimension 1 strata and the degrees is as is in Section 4.2. For a primary codimension 2 stratum \( \Gamma \) of \( \overline{M}_{k,l} \), we denote by \( \sigma^\Gamma_{\mathbb{R}} \) the canonical orientation on \( \mathcal{NT} \) as in Lemma 4.2 and by \( \sigma^{\Gamma,l} \) the orientation on \( \Gamma \) as in the first half of Section 4.3. Since \( \sigma_{k,l} = \sigma_{k,l}^{\Gamma,l} \) for \( k = 0, 1 \),

\[
\sigma^{\Gamma,l} = \sigma^\Gamma_{\mathbb{R}} \sigma_{k,l}
\]

in the cases of Lemmas 4.4 and 4.5. Let \( P^\pm \in \overline{M}_{1,2}^\Gamma \) be the three-component curve so that \( z^+_1 \) and \( z^+_2 \) lie on the same irreducible component.

**Lemma 4.4** There exists an embedded closed path \( \Upsilon \subset \overline{M}_{1,2}^\Gamma \) with a co-orientation \( \sigma^\Upsilon_{\mathbb{R}} \) so that \( \Upsilon \) is a regular hypersurface and

\[
\partial(\Upsilon, \sigma^\Upsilon_{\mathbb{R}}) = (P^+, \sigma^P_{+}) \cup (P^-, \sigma^P_-), \quad \deg^S_{\mathbb{R}}(\Upsilon, \sigma^\Upsilon_{\mathbb{R}}) = 1, \quad \deg_{S_2}(\Upsilon, \sigma^\Upsilon_{\mathbb{R}}) = -1.
\]

(4.16)
Proof. Since \((P^+, o_{P^+,2})\) is a ±-point, (4.15) gives
\[
o^c_{P^+,0}o_{1,2} = ±1.
\]

Let \(\widetilde{M}_{1,2}’ \cong S^2\) be the space obtained by contracting \(S_1\) to a point \(P_0\). By [7, Lemma 5.4], the orientation \(o_{1,2}\) on \(M_{1,2}\) extends over \(\widetilde{M}_{1,2}\); this can also be readily seen from the definitions. The morphisms \(f^R_{1,2,1}\) and \(f^R_{1,2,2}\) descend to smooth maps
\[
f^R_{1,2,1} : \widetilde{M}_{1,2}’ \longrightarrow \overline{M}_{0,2}’ \quad \text{and} \quad f^R_{1,2,2} : \widetilde{M}_{1,2}’ \longrightarrow \overline{M}_{1,1}’.
\]

We can identify \(\overline{M}_{1,2}’\) with \(S^2 \subset \mathbb{R}^3\) and \(\overline{M}_{0,2}’\) with \([-1,1]\) so that \(P^z = (±1,0,0)\) and \(f^R_{1,2,1}\) is the height function. The fibers of \(f^R_{1,2,1}\) over \(M_{0,2}’\) are then the circles of constant latitude. The orientation \(o^R_1\) of the fibers of \(f^R_{1,2,1}|_{\overline{M}_{1,2}’}\) specified in Section 4.1 extends over the equator \(\overline{S}_2 \cong S^1\). By Lemma [4.14(8)],
\[
o_{1,2}|_{\overline{M}_{1,2}’} = (o^R_1o_{0,2})|_{\overline{M}_{1,2}’}.
\]

Let \(Y' \subset \widetilde{M}_{1,2}\) be a meridian running from \(P^-\) to \(P^+\) disjoint from \(P_0\) and \(o_{Y'}\) be its canonical orientation. Thus, the restriction
\[
f^R_{1,2,1} : (Y', o_{Y'}) \longrightarrow (\overline{M}_{0,2}’, o_{0,2})
\]
is an orientation-preserving diffeomorphism. We take \(o^r_{Y'}\), to be the orientation of \(N\overline{Y}'\) so that the projection
\[
(\ker d f^R_{1,2,1}, o^R_1) \longrightarrow (N\overline{Y}', o^r_{Y'})
\]
is an orientation-preserving isomorphism. By (4.18) and Lemma [3.1(1)]
\[
o_{Y'} = o^r_{Y'}o_{1,2} \quad \text{and} \quad \deg^R(Y', o^r_{Y'}) = \deg(f^R_{1,2,1}|_{Y'}, o^r_{Y'}o^R_1) = 1.
\]

By Lemma [4.14(8)], the orientation \(o_{1,2}\) corresponds to the natural orientation of the complex coordinate \(z^+_2\) with \(z^+_1 = 0\) and \(x_1 = 1\) fixed. Thus, \(o^r_{Y'}\) is the negative rotation in the \(z^+_2\)-coordinate. Along \(\overline{S}_2\), it corresponds to the negative rotation of the node. Thus,
\[
\deg_{S_2}(Y', o^r_{Y'}) = \deg(f_{0,2,2}|_{Y', \overline{S}_2}, o^r_{Y'}o_{\text{nd}}^R) = -1.
\]

Since the outer normal co-orientation \(o^c_{Y'}\) of \(\partial Y'\) agrees with the restriction of \(\pm o_{Y'}\) at \(P^\pm\), i.e.
\[
(o^c_{\partial Y'}, o_{Y'})|_{P^\pm} = ±1,
\]
the first statement in (4.19) gives
\[
(o^c_{\partial Y'}, o^r_{Y'})|_{P^\pm}o_{1,2}|_{P^\pm} = o^c_{\partial Y'}|_{P^\pm}o^r_{Y'}|_{P^\pm} = ±1.
\]

Comparing with (4.18), we conclude that
\[
(o^c_{\partial Y'}, o_{Y'})|_{P^\pm} = o^c_{P^\pm},
\]
i.e. the first equality in (4.16) with \(Y\) replaced by \(Y'\) holds as well.

We take \(Y \subset \overline{M}_{1,2}’\) to be the preimage of \(Y'\) under the blowdown map (which is a diffeomorphism on a neighborhood of \(Y\)) and \(o^c_{Y}\) to be the pullback of \(o^c_{Y'}\). \(\square\)
The moduli space $\overline{\mathcal{M}}_{0,3}$ is a 3-manifold with the boundary

$$\partial \overline{\mathcal{M}}_{0,3} = S_{23}^{++} \sqcup S_{23}^{+-} \sqcup S_{23}^{-+} \sqcup S_{23}^{--},$$

where

$$S_{ij}^{\pm} = \overline{\mathcal{M}}_{0,4} \approx S^2$$

is the closure of the open codimension 1 stratum $S_{ij}^{\pm}$ of curves consisting of a pair of conjugate spheres with the marked points $z_i^\pm$ and $z_j^\pm$ on the same sphere as $z_i^\mp$; see [8, Fig. 4] and the first diagram in Figure 6. There are four primary codimension 2 strata $\Gamma_i^\pm$, with $i = 2, 3$, in $\overline{\mathcal{M}}_{0,3}$. The closed interval $\Gamma_i^+$ (resp. $\Gamma_i^-$) is the closure of the open codimension 2 stratum $\Gamma_i^+$ (resp. $\Gamma_i^-$) of curves consisting of one real sphere and a conjugate pair of spheres so that the real sphere carries the marked points $z_i^\pm$ and the decorations $\pm$ of the marked points on each of the conjugate spheres are the same (resp. different); see the last pair of diagrams in Figure 6. Let

$$\hat{\Gamma}_i^+ = \Gamma_i^+ \cup (\Gamma_i^+ \cap S_i) \subset \Gamma_i^+$$

be the complement of the endpoints.

**Lemma 4.5** There exist a bordered surface $\Upsilon \subset \overline{\mathcal{M}}_{0,3}$ with a co-orientation $\sigma_\Upsilon^\gamma$ and a one-dimensional manifold $\gamma' \subset \overline{\mathcal{M}}_{0,3}$ with a co-orientation $\sigma_{\gamma'}$, so that $\Upsilon$ is transverse to all open strata of $\overline{\mathcal{M}}_{0,3}$ not contained in any $\hat{\Gamma}_i^\pm$ with $i = 2, 3$, $\Upsilon$ is a regular hypersurface, and

$$\partial(\Upsilon, \sigma_\Upsilon^\gamma) = (\hat{\Gamma}_2^+, \sigma_{\Gamma_2^+}) \cup (\hat{\Gamma}_3^+, -\sigma_{\Gamma_3^+}) \cup (\hat{\Gamma}_2^-, \sigma_{\Gamma_2^-}) \cup (\hat{\Gamma}_3^-, -\sigma_{\Gamma_3^-}) \cup (\gamma', \sigma_{\gamma'}),$$

$$\deg_S(\Upsilon, \sigma_\Upsilon^\gamma) = 1, \quad \deg_S(\Upsilon, \sigma_{\gamma'}) = -1. \quad (4.20)$$

Proof. For $i = 2, 3$, $z_i^\pm$ moves in $(\Gamma_i^+, \sigma_{\Gamma_i^+})$ (resp. $(\Gamma_i^-, \sigma_{\Gamma_i^-})$) from the node separating the sphere carrying $z_i^-$ (resp. $z_i^+$) to the other node. Each closed interval $\hat{\Gamma}_i^\pm$ intersects $S_i$ transversally at one point $P_i^\pm$ and does not intersect $S_j$ for $j = 1, 2, 3$ with $j \neq i$. It intersects $\partial \overline{\mathcal{M}}_{0,3}$ at its endpoints; we denote the starting point by $P_i^{+-}$ and the ending point by $P_i^{\pm}$ by [8 Section 3], the orientation $\sigma_{0,3} = \sigma_{0,3,3}$ on $\mathcal{M}_{0,3}$ extends over $\overline{\mathcal{M}}_{0,3}$. 29

![Diagrams](image-url)
By [8, Remark 3.5], $\overline{M}_{0,3}'$ is the blowup of a bordered manifold $\overline{M}_{0,3}'$ at a point $P_0$ with the exceptional divisor $S_1$. Denote by
\[ p : \overline{M}_{0,3}' \rightarrow \overline{M}_{0,3}' \]
the blowdown map. The morphisms $f_{0,3,2}$ and $f_{0,3,3}$ descend to smooth maps
\[ f_{0,3,2} : \overline{M}_{0,3}' \rightarrow \overline{M}_{0,2}' \quad \text{and} \quad f_{0,3,3} : \overline{M}_{0,3}' \rightarrow \overline{M}_{0,2}'. \tag{4.21} \]
Since $S_1$ is disjoint from the four spheres of $\partial \overline{M}_{0,3}'$ and the four intervals $P_i^\pm$ with $i = 2, 3$, $p$ is a diffeomorphism on neighborhoods of these spaces. We denote the images of these intervals and the twelve points $P_i^\pm, P_i^\pm$ on them under $p$ in the same way. The spaces
\[ \tilde{S}_2 = p(S_2) \approx \overline{M}_{1,2}' \approx S^2 \quad \text{and} \quad \tilde{S}_3 = p(S_3) \approx \overline{M}_{1,2}' \] are the fibers of $f_{0,3,3}$ and $f_{0,3,2}$, respectively, over the curve consisting of two real components, which corresponds to $1 \in [0, \infty]$ under the identification $\varphi_{0,2}$ in (4.20).

Setting $(z_1^+, z_1^-) = (0, \infty)$, we obtain a natural identification
\[ \overline{M}_{0,3}' - \partial \overline{M}_{0,3}' \approx \{(z_2^+, z_2^-), (z_3^+, z_3^-) \in (\mathbb{P}^1)^4 : \tau(z_1^+) = \tau(z_1^-), (z_1^+, z_1^-) \neq (0, 0)\}/\sim, \]
\[ \{(z_2^+, z_2^-), (z_3^+, z_3^-) \in (\mathbb{P}^1)^4, (z_1^+, z_1^-) = (0, 0)\}/\sim. \]
The condition $z_1^+ = \tau(z_1^-)$ implies that the points $z_1^+$ and $z_1^-$ lie on a great arc through the poles $z_1^+ = 0$ and $z_1^- = \infty$ (or lie at $z_1^+$). The blowup point $P_0$ in this identification corresponds to the point $[(1, 1), (1, 1)]$. The projections (4.21) in this identification are given by
\[ f_{0,3,2}([(z_1^+, z_2^-), (z_3^+, z_3^-)]) = [(0, \infty), (z_5^+, z_5^-)]. \]
In particular, the fiber of $f_{0,3,2}$ over a point of $[(0, \infty), (z_5^+, z_5^-)]$ of $\overline{M}_{0,2}' - \partial \overline{M}_{0,2}'$ can be identified via $z_1^+$ with $\mathbb{P}^1$ by choosing $z_5^+ \in \mathbb{R}^+$.

The space $\overline{M}_{0,3}' - \partial \overline{M}_{0,3}'$ is covered by two charts
\[ \mathbb{R}^+ \times \mathbb{P}^1 \rightarrow \overline{M}_{0,3}' - \partial \overline{M}_{0,3}', \quad (r_2, z_3) \rightarrow [(r_2, 1/r_2), (z_3, 1/\overline{z}_3)], \]
\[ \mathbb{P}^1 \times \mathbb{R}^+ \rightarrow \overline{M}_{0,3}' - \partial \overline{M}_{0,3}', \quad (z_2, r_3) \rightarrow [(z_2, 1/\overline{z}_2), (r_3, 1/r_3)]. \tag{4.22} \]
In these charts,
\[ \varphi_{0,2}(f_{0,3,3}(r_2, z_3)) = 1/r_2^2, \quad \tilde{S}_2 = \{(r_2, z_3) \in \mathbb{R}^+ \times \mathbb{P}^1 : r_2 = 1\}, \]
\[ \varphi_{0,2}(f_{0,3,2}(z_2, r_3)) = 1/r_3^2, \quad \tilde{S}_3 = \{(z_2, r_3) \in \mathbb{P}^1 \times \mathbb{R}^+ : r_3 = 1\}. \tag{4.23} \]
The overlap map between the two charts,
\[ (r_2, r_3 e^{i\theta}) \rightarrow (r_2 e^{-i\theta}, r_3), \]
is orientation-preserving with respect to the standard orientations $\sigma_{\mathbb{R}^+}$ on $\mathbb{R}^+$ and $\sigma_{\mathbb{P}^1}$ on $\mathbb{P}^1$. We take $\tilde{\sigma}_{0,3}$ to be the orientation on $\overline{M}_{0,3}'$ opposite to the orientation determined by $\sigma_{\mathbb{R}^+}$ and $\sigma_{\mathbb{P}^1}$ via the two charts in (4.22). Since the map
\[ (r_2, \sigma_{\mathbb{R}^+}) \rightarrow (r_2, -\sigma_{\mathbb{R}^+}), \quad r_2 \rightarrow \varphi_{0,2}(f_{0,3,3}(r_2, z_3)), \tag{4.24} \]
is orientation-preserving for each $z_i^\pm \in \mathbb{P}^1$ fixed, Lemma 4.1([4.1] and (4.15) give
\[
\mathcal{M}_{0,3}(\mathbb{P}^1, \mathcal{M}_{0,3}^r) = P^q_{0,3}(\mathbb{P}^1, \mathcal{M}_{0,3}^r), \quad \mathcal{M}_{0,3}^r = \mathcal{M}_{0,3}^r. \tag{4.25}
\]

The moduli space $\mathcal{M}_{0,3}$ is a submanifold of $\mathcal{M}_{0,6}$. By [19] Appendix D.4.5, the four cross-ratios
\[
\text{CR}_{\pm} : \mathcal{M}_{0,3}^r \to \mathbb{P}^1, \quad \text{CR}_{\pm} \left( \left[ (z_i^+ , z_i^-)_{i=1}^3 \right] \right) = \frac{z_2^+ - z_1^-}{z_3^- - z_1^+} : \frac{z_2^+ - z_1^+}{z_3^- - z_1^-},
\]
extend over $\mathcal{M}_{0,3}^r$ and descend to smooth maps from $\tilde{\mathcal{M}}_{0,3}^r$. The subspace
\[
\tilde{\mathcal{Y}}' \subset \tilde{\mathcal{M}}_{0,3}^r - \{ q_{1,2}^\pm , q_{2,3}^\pm \}
\]
where all four cross-ratios take values in
\[
\mathbb{R}^\mathbb{P}^1 = \left[ -\infty, \infty \right] / -\infty \sim \infty \tag{4.26}
\]
is an orientable surface, as explained in the next paragraph. The boundary of $\tilde{\mathcal{Y}}'$ consists of the complement of two points in a circle on each boundary sphere of $\partial \tilde{\mathcal{M}}_{0,3}^r$.

The intersections of $\tilde{\mathcal{Y}}'$ with the charts (4.22) are given by
\[
\begin{align*}
\mathbb{R}^+ \times \mathbb{R}^1 & \to \tilde{\mathcal{Y}}' - f_{0,3,2}(\mathcal{M}_{0,2}^r), \quad (r_2, r_3) \mapsto \left[ (r_2, 1/r_2), (r_3, 1/r_3) \right], \\
\mathbb{R}^1 \times \mathbb{R}^+ & \to \tilde{\mathcal{Y}}' - f_{0,3,3}(\mathcal{M}_{0,2}^r), \quad (r_2, r_3) \mapsto \left[ (r_2, 1/r_2), (r_3, 1/r_3) \right]. \tag{4.27}
\end{align*}
\]

An element $\left[ (z_2^+, z_2^-), (z_3^+, z_3^-) \right]$ of $\tilde{\mathcal{M}}_{0,3}^r - \partial \tilde{\mathcal{M}}_{0,3}^r$ belongs to $\tilde{\mathcal{Y}}'$ if and only if all four points $z_i^\pm \in \mathbb{P}^1$ with $i = 2, 3$ lie on a great circle through $z_i^-$ and $z_i^+$. The structure of $\tilde{\mathcal{Y}}'$ along $\partial \tilde{\mathcal{M}}_{0,3}^r$ is described by the local coordinates of [8] Remark 3.5 with $z \in \mathbb{R}$. The overlap map between the charts (4.27),
\[
\begin{align*}
\mathbb{R}^+ \times \mathbb{R}^* & \to \mathbb{R}^* \times \mathbb{R}^+, \quad (r_2, r_3) \mapsto \begin{cases} (r_2, r_3), & \text{if } r_3 \in \mathbb{R}^+; \\
(-r_2, -r_3), & \text{if } r_3 \in \mathbb{R}^-;
\end{cases}
\end{align*}
\]
is orientation-preserving with respect to the orientation $\mathcal{O}_{\mathbb{R}^+}$ on $\mathbb{R}^+$ and the orientation $\mathcal{O}_{\mathbb{R}^1}$ on $\mathbb{R}^1$ induced by the standard orientation of $[-\infty, \infty]$ via (4.26). We take $\mathcal{O}_{\mathbb{R}^+}$ to be the orientation on $\tilde{\mathcal{Y}}'$ determined by $\mathcal{O}_{\mathbb{R}^+}$ and $\mathcal{O}_{\mathbb{R}^1}$ via the two charts in (4.27).

The surface $\tilde{\mathcal{Y}}'$ contains the four open intervals
\[
\tilde{\Gamma}_{i}^\pm = \mathcal{T}_{i}^\pm - \{ P_{i}^\pm, P_{i}^{\mp} \}
\]
with $i = 2, 3$; the closures of these intervals connect the components of the closure of $\partial \tilde{\mathcal{Y}}'$. In the two charts (4.22),
\[
\begin{align*}
\tilde{\Gamma}_{2}^+ & = \{ (r_2, z_3) \in \mathbb{R}^+ \times \mathbb{P}^1 : z_3 = 0 \}, \quad \tilde{\Gamma}_{2}^- = \{ (r_2, z_3) \in \mathbb{R}^+ \times \mathbb{P}^1 : z_3 = \infty \}, \\
\tilde{\Gamma}_{3}^+ & = \{ (z_2, r_3) \in \mathbb{P}^1 \times \mathbb{R}^+ : z_2 = 0 \}, \quad \tilde{\Gamma}_{3}^- = \{ (z_2, r_3) \in \mathbb{P}^1 \times \mathbb{R}^+ : z_2 = \infty \}.
\end{align*}
\]
The cut $\tilde{\gamma}'$ of $\tilde{\gamma}'$ along the four open intervals has two components, $\tilde{\gamma}^+$ and $\tilde{\gamma}^-$. They are distinguished by whether $CR^+ (C)$ lies in $\mathbb{R}^+$ or $\mathbb{R}^-$ for the elements $C$ of $\tilde{\gamma}' - \partial \tilde{\gamma}'$, i.e. whether the points $z_2^+$ lie on the same great arc through $z_1^-$ and $z_1^+$ as the points $z_3^+$ or on the opposite arc.

Let $\gamma' = \tilde{\gamma}^+$ and $\gamma' = \gamma' \cap \partial \tilde{\gamma}'$. The former is a surface with boundary
$$\partial \gamma' = \tilde{\gamma}_2^+ \cup \tilde{\gamma}_3^+ \cup \tilde{\gamma}_2^- \cup \tilde{\gamma}_3^- \cup \gamma'.$$

By (4.23) and (4.27), this surface intersects $\tilde{S}_2$ and $\tilde{S}_3$ transversely along the closed line segments given by
$$\tilde{\gamma}_2 \equiv \gamma' \cap \tilde{S}_2 = \{ (r_2, z_3) \in \mathbb{R}^+ \times \mathbb{P}^1 : r_2 = 1, z_3 \in [0, \infty) \},$$
$$\tilde{\gamma}_3 \equiv \gamma' \cap \tilde{S}_3 = \{ (z_2, r_3) \in \mathbb{P}^1 \times \mathbb{R}^+ : r_3 = 1, z_2 \in [0, \infty) \}$$
in the charts (4.22). For $i = 2, 3$, let $\sigma_{\tilde{\gamma}_i}$ be the \textit{opposite} of the orientation on $\tilde{\gamma}_i$ given by the $r_{5-i} = |z_{5-i}|$ coordinate. The restriction
$$f_{0,3;3} : (\tilde{\gamma}_2, \sigma_{\tilde{\gamma}_2}) \longrightarrow (\mathcal{M}_{0,2}, \sigma_{0,2}) \quad (4.28)$$
is then an orientation-preserving diffeomorphism (because (4.24) is orientation-preserving).

We denote by $\sigma_{\gamma'}$ and $\sigma^c_{\gamma'}$ the restrictions of the orientation $\sigma_{\tilde{\gamma}}$ and the co-orientation $\sigma^c_{\tilde{\gamma}}$ to $\gamma'$, by $\sigma_{\gamma'}$ the boundary orientation on $\gamma'$ induced by $\sigma_{\gamma'}$, and by $\sigma^c_{\gamma'}$ the orientation on $\mathcal{N} \gamma'$ determined
by $\partial_{0,3}$ and $o_\gamma$. Thus,

$$o_{\gamma'} = o_{\gamma'}^c \partial_{0,3}, \quad o_\gamma = (o_{\gamma'}^c o_{\gamma'})|_{\gamma'} = o_\gamma^c \partial_{0,3}. \quad (4.29)$$

At the point $P_2^+ \in \Gamma_2^+ \cup \tilde{\Gamma}_2$, the orientation $o_{\Gamma_2^+:3}$ on $\Gamma_2^+$ is the opposite of the orientation given by the $x_2$-coordinate (because $z_2^+$ moves from $\tilde{z}_1^- = \infty$ to $z_1^+ = 0$); see Figure 7. Since the natural isomorphisms

$$(T_{P_2} \Gamma_2^+, o_{\Gamma_2^+}) \longrightarrow \left(N_{\gamma'} \Gamma_2^+:3, o_{\gamma'}^c \partial_{\gamma'} \right) \quad \text{and} \quad (T_{P_2} \tilde{\Gamma}_2^+, o_{\Gamma_2^+:3}) \otimes (T_{P_2} \tilde{\Gamma}_2, o_{\Gamma_2}) \longrightarrow (T_{P_2} \Gamma_2^+, o_{\gamma'})$$

are orientation-preserving,

$$\partial o_{\gamma'}|_{P_2^+} \equiv (o_{\gamma'}^c o_{\gamma'})|_{P_2^+} = o_{\Gamma_2^+:3}|_{P_2^+}.$$

Since the right-hand side of

$$\partial \gamma' \gamma' = (\tilde{\Gamma}_2^+, o_{\Gamma_2^+:3}) \cup (\tilde{\Gamma}_3^+, -o_{\Gamma_3^+:3}) \cup (\tilde{\Gamma}_2^-, o_{\Gamma_2^-:3}) \cup (\tilde{\Gamma}_3^-, -o_{\Gamma_3^-:3}) \cup (\gamma', o_\gamma)$$

is an oriented loop and the equality above respects the orientations at $P_2^+$, it follows that this equality respects the orientations everywhere. Combining it with the second equality in (4.25) and the first and last equalities in (4.29), we obtain (4.20) with $\gamma$ replaced by $\gamma'$.

We now compute the degree

$$\deg \{f_{0,3}^c|_{\tilde{\gamma}_1}, o_{\gamma'}^c|_{\tilde{\gamma}_1} o_{\gamma'}^c \} \in \mathbb{Z} \quad (4.30)$$

of $f_{0,3}^c|_{\tilde{\gamma}_1}$ with respect to the co-orientation $o_{\gamma'}^c|_{\tilde{\gamma}_1}$ on $\tilde{\gamma}_1$ and the natural orientation $o_{\gamma'}^c$ of the fibers of

$$\tilde{f}_{0,3}^c|_{\tilde{\gamma}_1} \approx \tilde{\gamma}_1^c \tilde{\gamma}_1 \approx \overline{\mathcal{M}}_{0,2}^\gamma$$

over $\mathcal{M}_{0,2}^\gamma$ as in the proof of Lemma 4.4. By Lemma 4.1 (3.2), the orientation $o_{\gamma}^c$ on $\tilde{\gamma}_1 \cap f_{0,3}^c(\mathcal{M}_{0,2}^\gamma)$ is given by the $z_{5-x}$-coordinate under the corresponding identification in (4.23). Since the diffeomorphism (4.22) is orientation-preserving, it follows that the vertical orientation $o_{\gamma}^c$ on $\tilde{\gamma}_1$ is given by the negative rotation in the $z_{5-x}$-coordinate. Since the charts (4.22) are orientation-reversing with respect to $\partial_{0,3}$ and the charts (4.27) are orientation-preserving with respect to $o_\gamma$, the orientation $o_{\gamma'}^c$ on $\gamma'$ is given by the negative rotation in the $z_3$-coordinate in the first chart in (4.22) and the positive rotation in the $z_3$-coordinate in the second chart in (4.22). Thus, the projection

$$(\ker d \{f_{0,3}^c|_{\tilde{\gamma}_1}, o_{\gamma'}^c|_{\tilde{\gamma}_1} \} \otimes (\gamma', o_{\gamma'}), (-1)^i o_{\gamma'}^c) \longrightarrow (\gamma', o_{\gamma'}^c)|_{\tilde{\gamma}_1}$$

is an orientation-preserving isomorphism and the number in (4.30) is $(-1)^i$; see Lemma 3.1 (1).

The surface $\gamma'$ is transverse to $\tilde{\gamma}_1 \cup \tilde{\gamma}_2$, but passes through $P_0$. Let $(\gamma'', o_{\gamma''})$ be a co-oriented surface in $\tilde{\mathcal{M}}_{0,3}^\gamma$ obtained from $(\gamma', o_{\gamma'})$ by a small deformation around $P_0$ so that $\gamma''$ is still transverse to $\tilde{\gamma}_1 \cup \tilde{\gamma}_2$ and $P_0 \notin \gamma''$. We take $\gamma \subset \mathcal{M}_{0,3}$ to be the preimage of $\gamma''$ under the blowdown map $p$ (which is a diffeomorphism on a neighborhood of $\gamma$) and $o_{\gamma'} = p^* o_{\gamma''}$.

**Remark 4.6.** We could have taken $\gamma' = \tilde{\gamma}^-$ in the proof of Lemma 4.5 to avoid $P_0$, but with $o_{\gamma'} = -o_{\gamma'}^c|_{\gamma'}$. 

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5 Real GW-invariants

5.1 Moduli spaces of stable maps

Let $(X, \omega, \phi)$ be a real symplectic manifold and $k, l \in \mathbb{Z}^{\geq 0}$ with $k + 2l \geq 3$. We denote by $\mathcal{H}^\omega_{k,l}$ the space of pairs $(J, \nu)$ consisting of $J \in \mathcal{J}_u^\omega$ and a real perturbation $\nu$ of the $\mathcal{C}_J$-equation as in [7, Section 2]. For $(J, \nu) \in \mathcal{H}^\omega_{k,l}$, a real genus 0 $(J, \nu)$-map with $k$ real marked points and $l$ conjugate pairs of marked points is a tuple

$$u = (u: \Sigma \rightarrow X, (x_i)_{i \in [k]}, (z_i^+, z_i^-)_{i \in [l]}, \sigma) \quad (5.1)$$

such that

$$\mathcal{C} = (\Sigma, (x_i)_{i \in [k]}, (z_i^+, z_i^-)_{i \in [l]}, \sigma) \quad (5.2)$$

is a real genus 0 nodal curve with complex structure $j$, $k$ real marked points, and $l$ conjugate pairs of marked points and $u$ is a smooth map satisfying

$$u \circ \sigma = \phi \circ u, \quad \mathcal{C}_J u|_z = \frac{1}{2} (d_z u + J \circ d_z u \circ j) = \nu(z, u(z)) \quad \forall z \in \Sigma.$$

Such a map is called simple if the restriction of $u$ to each unstable irreducible component of the domain is simple (i.e. not multiply covered) and no two such restrictions have the same image.

For $B \in H_2(X)$ and $(J, \nu) \in \mathcal{H}^\omega_{k,l}$, we denote by $\overline{\mathcal{M}}_{k,l}(B; J, \nu)$ the moduli space of the equivalence classes of stable real genus 0 degree $B$ $(J, \nu)$-maps with $k$ real marked points and $l$ conjugate pairs of marked points modulo the reparametrizations. Let

$$\overline{\mathcal{M}}_{k,l}^{\nu}(B; J, \nu) \subset \overline{\mathcal{M}}_{k,l}(B; J, \nu) \quad \text{and} \quad \mathcal{M}_{k,l}(B; J, \nu) \subset \overline{\mathcal{M}}_{k,l}^{\nu}(B; J, \nu)$$

be the subspace of simple maps and the (virtually) main stratum, i.e. the subspace consisting of maps as in (5.1) from smooth domains $\Sigma$, respectively.

The forgetful morphisms

$$\tilde{f}_{k+1,i; i}^{\nu} : \overline{\mathcal{M}}_{k+1,i} \rightarrow \overline{\mathcal{M}}_{k,l}^{\nu}, \quad i \in [k+1], \quad \text{and} \quad \tilde{f}_{k,l+1; i}^{\nu} : \overline{\mathcal{M}}_{k,l+1} \rightarrow \overline{\mathcal{M}}_{k,l}^{\nu}, \quad i \in [l+1],$$

induce maps

$$\tilde{f}_{k+1,i; i}^{\nu \ast} : \mathcal{H}^\omega_{k+1,l} \rightarrow \mathcal{H}^\omega_{k,l}^{\nu \ast} \quad \text{and} \quad \tilde{f}_{k,l+1; i}^{\nu \ast} : \mathcal{H}^\omega_{k,l} \rightarrow \mathcal{H}^\omega_{k,l+1}^{\nu \ast},$$

respectively. For each $\nu \in \mathcal{H}^\omega_{k,l}$, we also denote by

$$\tilde{f}_{k+1,i; i}^{\nu} : \overline{\mathcal{M}}_{k+1,l}(B; J, \nu) \rightarrow \overline{\mathcal{M}}_{k,l}(B; J, \nu),$$

$$\tilde{f}_{k,l+1; i}^{\nu} : \overline{\mathcal{M}}_{k,l+1}(B; J, \nu) \rightarrow \overline{\mathcal{M}}_{k,l}(B; J, \nu) \quad (5.3)$$

the forgetful morphisms dropping the $i$-th real marked point and the $i$-th conjugate pair of marked points, respectively. The restriction of the second morphism in (5.3) to the preimage of $\mathcal{M}_{k,l}(B; J, \nu)$ is a $\mathbb{P}^1$-fiber bundle. We denote by $\sigma_i^+$ the relative orientation of this restriction induced by the position of the marked point $z_i^+$.

For $c \in \mathbb{Z}^{\geq 1}$, a (virtually) codimension $c$ stratum $S$ of $\overline{\mathcal{M}}_{k,l}(B; J, \nu)$ is a subspace of maps from domains $\Sigma$ with precisely $c$ nodes and thus with $c+1$ irreducible components isomorphic to $\mathbb{P}^1$. It is characterized by the distributions of

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In particular, if in addition $u$ is in a codimension 1 stratum distinguished by whether the fixed locus $\Sigma^\sigma$ of $(\Sigma, \sigma)$ consists of a single point or a wedge of two circles. These two types are known as sphere bubbling and disk bubbling, respectively. If $k$ and $B$ satisfy (2.1), as is the case if (1.3) holds, then the fixed locus $\Sigma^\sigma$ of the domain $(\Sigma, \sigma)$ of every element $u$ of $\mathcal{M}_{k,l}(B; J, \nu)$ is a circle or a tree of two or more circles. In this case, sphere bubbling does not occur.

Suppose $l \in \mathbb{Z}^+$ and $S$ is a codimension 1 disk bubbling stratum of $\mathcal{M}_{k,l}(B; J, \nu)$. We define

$$K_1(S), K_2(S) \subset [k], \quad l_1(S), l_2(S) \subset [l], \quad k_1(S), k_2(S), l_1(S), l_2(S) \in \mathbb{Z}_{\geq 0}$$

analogously to $K_r(S), L_r(S), k_r(S), l_r(S)$ in Section 1.2. We denote by $B_1(S) \in H_2(X)$ the degree of the restriction of the map components $u$ of the elements $u$ of $S$ to the irreducible component $\mathbb{P}_1$ of the domain carrying the marked points $z^\pm_i$ and by $B_2(S) \in H_2(X)$ the degree of the restriction of $u$ to the other irreducible component $\mathbb{P}_2$ of the domain. Let

$$\mathcal{S} \subset \mathcal{M}_{k,l}(B; J, \nu)$$

be the virtual closure of $S$, i.e. the subspace of maps $u$ as in (5.1) so that the domain $\Sigma$ can be split at a node into two connected (possibly reducible) surfaces, $\Sigma_1$ and $\Sigma_2$, so that the degree of the restriction of the map component $u$ to $\Sigma_1$ is $B_1(S)$, the real marked points $x_i$ with $i \in K_1(S)$ lie on $\Sigma_1$, and so do the conjugate pairs of marked points $z^\pm_i$ with $i \in L_1(S)$.

If in addition $l^* \in [l]$, let

$$L^*_1(S) = L_1(S) \cap [l^*], \quad L^*_2(S) = L_2(S) \cap [l^*], \quad l^*_1(S) = |L^*_1(S)|, \quad l^*_2(S) = |L^*_2(S)|,$$

$$\varepsilon_{l^*}(S) = \ell_{\omega}(B_2(S)) + 1 - (k_2(S) + 2(l_2(S) - l^*_2(S))).$$

In particular,

$$\ell_{\omega}(B_1(S)) + \ell_{\omega}(B_2(S)) = \ell_{\omega}(B) - 1, \quad 1 \leq l^*_1(S) \leq l_1(S), \quad l^*_2(S) \leq l_2(S), \quad k_1(S) + k_2(S) = k, \quad l_1(S) + l_2(S) = l, \quad l^*_1(S) + l^*_2(S) = l^*.$$

We denote by

$$\mathcal{M}_{k,l,l^*}^*(B; J, \nu) \subset \mathcal{M}_{k,l}^*(B; J, \nu)$$

the subspace of simple maps that have no nodes, or lie in a codimension 1 stratum $S$ with $\varepsilon_{l^*}(S) \equiv 0, 1 \mod 4$, or have only one conjugate pair of nodes. Let $\mathcal{M}_{k,l,l^*}(B; J, \nu)$ be the space obtained by cutting $\mathcal{M}_{k,l}(B; J, \nu)$ along the closures $\overline{S}$ of the codimension 1 strata $S$ with $\varepsilon_{l^*}(S) \equiv 2, 3 \mod 4$. Thus, $\mathcal{M}_{k,l,l^*}(B; J, \nu)$ contains a double cover of $\overline{S}$ for each codimension 1 stratum $S$ of $\mathcal{M}_{k,l}(B; J, \nu)$ with $\varepsilon_{l^*}(S) \equiv 2, 3 \mod 4$; the union of these covers forms the (virtual) boundary of $\mathcal{M}_{k,l,l^*}(B; J, \nu)$. Let

$$q: \mathcal{M}_{k,l,l^*}(B; J, \nu) \longrightarrow \mathcal{M}_{k,l}^*(B; J, \nu)$$

be the quotient map. We denote by

$$\mathcal{M}_{k,l,l^*}^*(B; J, \nu) \subset \mathcal{M}_{k,l,l^*}(B; J, \nu)$$

the subspace of simple maps that
- have no nodes, or
- have only one real node, or
- have only one conjugate pair of nodes.

The boundary $\partial\overline{\mathcal{M}}_{k,l;\ast}(B; J, \nu)$ of this subspace consists of double covers $\tilde{S}^\ast$ of the subspaces $S^\ast$ of simple maps of the codimension 1 strata $S$ of $\overline{\mathcal{M}}_{k,l}(B; J, \nu)$ with $\varepsilon\ast(S) \cong 2, 3 \mod 4.$

For each $i \in [k]$, let
\[
ev_i^R: \overline{\mathcal{M}}_{k,l}(B; J, \nu) \longrightarrow X^\phi, \quad \ev_i^R([u, (x_i)_{i \in [k]}, (z_i^+, z_i^-)_{i \in [l]}, \sigma]) = u(x_i),
\]
be the evaluation morphism for the $i$-th real marked point. For each $i \in [l]$, let
\[
ev_i^+: \overline{\mathcal{M}}_{k,l}(B; J, \nu) \longrightarrow X, \quad \ev_i^+([u, (x_i)_{i \in [k]}, (z_i^+, z_i^-)_{i \in [l]}, \sigma]) = u(z_i^+),
\]
be the evaluation morphism for the positive point of the $i$-th conjugate pair of marked points. Let
\[
\ev = \prod_{i=1}^k \ev_i^R \times \prod_{i=1}^l \ev_i^+: \overline{\mathcal{M}}_{k,l}(B; J, \nu) \longrightarrow X_{k,l} \equiv (X^\phi)^k \times X^l \quad (5.6)
\]
be the total evaluation map. We also denote by
\[
\ev_i^R: \overline{\mathcal{M}}_{k,l;\ast}(B; J, \nu) \longrightarrow X^\phi, \quad \ev_i^+: \overline{\mathcal{M}}_{k,l;\ast}(B; J, \nu) \longrightarrow X,
\]
\[
\ev: \overline{\mathcal{M}}_{k,l;\ast}(B; J, \nu) \longrightarrow X_{k,l} \quad (5.7)
\]
the compositions of the evaluation maps above with the quotient map $q$ in (5.4). We will use the same notation for the compositions of the first three evaluation maps with all obvious maps to $\overline{\mathcal{M}}_{k,l}(B; J, \nu)$.

For $l^* \in [l]$ and a tuple
\[
h = (h_i: H_i \longrightarrow X)_{i \in [l^*]} \quad (5.8)
\]
of maps, define
\[
\mathcal{Z}_{k,l;\ast}(B; J, \nu) = \{(u, (y_i)_{i \in [l^*]}): (\overline{\mathcal{M}}_{k,l;\ast}(B; J, \nu) \times \prod_{i=1}^{l^*} H_i : \ev_i^+(u) = h_i(y_i) \forall i \in [l^*]) \}.
\]

We denote by
\[
\ev_{k,l;h}: \mathcal{Z}_{k,l;\ast}(B; J, \nu) \longrightarrow X_{k,l;\ast-l^*} \quad (5.9)
\]
the map induced by (5.6). Orientations on $H_i$ determine an orientation $\alpha_h$ on
\[
M_h = \prod_{i=1}^{l^*} H_i. \quad (5.10)
\]
Along with the symplectic orientation $\alpha_{\omega}$ of $X$ and a relative orientation $\alpha_{\ev}$ of
\[
\ev: \overline{\mathcal{M}}_{k,l;\ast}(B; J, \nu) \longrightarrow X_{k,l}, \quad (5.11)
\]
the orientation $\alpha_h$ determines a relative orientation $\alpha_{\ev\alpha_h}$ of (5.9).
A dimension $n$ pseudocycle $h: H \rightarrow X$ in the usual sense determines an element $[h]$ of $H_n(X; \mathbb{Z})$; see [30]. If in addition $B$ is a homology class in $X$ in the complementary dimension, let

$$h \cdot X B = \langle \text{PD}_X([h]), B \rangle \in \mathbb{Z}$$

denote the homology intersection product of $[h]$ with $B$. If $h$ and $B$ are not of complementary dimensions, we set $h \cdot X B = 0$. The next two statements follow readily from [22]; see Section 6.2.

**Lemma 5.1** Suppose $(X, \omega, \phi)$ is a real symplectic fourfold, $k, l \in \mathbb{Z}^+ \cap [l]$, $B \in H_2(X)$, and $(J, \nu) \in \mathcal{H}_{k,l}^{\omega,\phi}$ is generic. If $k$ and $B$ satisfy (2.1), then a Pin$^-$-structure $p$ on $X^\phi$ determines relative orientations $\alpha_{p,l^*}$ and $\alpha_{p,l^*}$ of the maps

$$ev: \mathcal{M}_{k,l;1}(B; J, \nu) \rightarrow X_{k,l} \quad \text{and} \quad ev: \widehat{\mathcal{M}}_{k,l;1}(B; J, \nu) \rightarrow X_{k,l},$$

respectively, with the following properties:

- $(\alpha_{p1})$ the restrictions of $\alpha_{p,l^*}$ and $\alpha_{p,l^*}$ to $\mathcal{M}_{k,l}(B; J, \nu)$ are the same;
- $(\alpha_{p2})$ the restrictions of $\alpha_{p,l^*+1}\omega$ and $\alpha_{p,l^*+1}^{\phi_{B+\nu}}$ to $\mathcal{M}_{k,l+1}(B; J, \nu_{k,l+1;l^*+1})$ are the same;
- $(\alpha_{p3})$ the interchange of two real points $x_i$ and $x_j$ with $1 \leq i, j \leq k$ preserves $\alpha_{p,l^*}$;
- $(\alpha_{p4})$ the interchange of the points in a conjugate pair $(z_i^+, z_i^-)$ with $l^* < i \leq l$ preserves $\alpha_{p,l^*}$;
- $(\alpha_{p5})$ the interchange of the points in a conjugate pair $(z_i^+, z_i^-)$ with $1 \leq i < l^*$ reverses $\alpha_{p,l^*}$;
- $(\alpha_{p6})$ if $u \in \mathcal{M}_{k,l}^\phi(B; J)$, $\alpha_{p,l^*}$ is reversed at $u$ by the interchange of the points in the conjugate pair $(z_i^+, z_i^-)$ if and only if

$$l_\omega(B) \equiv k + 2(l - l^*) \mod 4;$$

- $(\alpha_{p7})$ if $k, l, l^* = 1$ and $B = 0$, $(ev^\mathbb{R} \times id_X, \alpha_{p,l^*} \omega)$ is a Steenrod pseudocycle of degree 1.

**Proposition 5.2** Suppose $(X, \omega, \phi)$ is a real symplectic fourfold, $p$ is a Pin$^-$-structure on $X^\phi$, $l \in \mathbb{Z}^+ \cap [l]$, and $B \in H_2(X)$ are such that

$$k \equiv \ell_\omega(B) - 2(l - l^*) \geq \max(0, 3 - 2l).$$

Let $h = (h_i)_{i \in [l^*]}$ be a tuple of pseudocycles of codimension 2 in general position so that $\phi_{\ast}[h_1] = -[h_1]$. For a generic choice of $(J, \nu) \in \mathcal{H}_{k,l}^{\omega,\phi}$, the map (5.1) with the relative orientation $\alpha_{p,l^*} \cdot h$ is a codimension 0 Steenrod pseudocycle and

$$\deg(ev_{k,l;1}, \alpha_{p,l^*} \cdot h) = \left(\prod_{i=1}^{l^*} h_i \cdot X B\right) N_{B, l-l^*}^{\phi, p}.$$

### 5.2 Decomposition Formulas

Let $(X, \omega, \phi)$ be a real symplectic fourfold, $p$ be a Pin$^-$-structure on $X^\phi$, $k, k', l \in \mathbb{Z}^+ \cap [l]$, and $B \in H_2(X)$, and $(J, \nu) \in \mathcal{H}_{k,l}^{\omega,\phi}$. If $k$ and $B$ satisfy (2.1), there is a well-defined forgetful morphism

$$f_{k', \nu}: \overline{\mathcal{M}}_{k,l}(B; J, \nu) \rightarrow \overline{\mathcal{M}}_{k', \nu}$$

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which drops the last $k-k'$ real marked points and the last $l-l'$ conjugate pairs from the nodal marked curve $\Gamma$ associated with each tuple $u$ as in (5.2) and contracts the unstable irreducible components of the resulting curve. Let $h$ as in (5.8) be a tuple of smooth maps from oriented manifolds and

$$p = ((p^{\mathbb{R}}_i)_{i \in [k]}, (p^{\mathbb{T}}_i)_{i \in [l]-[l^*]}) \in X_{k,l-l^*} = (X^\phi)^k \times (X - X^\phi)^{l-l^*}. \tag{5.17}$$

Let $\Gamma \subset \overline{\mathcal{M}}_{k',l'}$ be a primary codimension 2 stratum and $\sigma^c_\Gamma$ be its canonical co-orientation as in Lemma 4.2. We denote by

$$\mathfrak{M}_{\Gamma;k,l}(B; J, \nu) \subset \mathfrak{M}_{k,l}(B; J, \nu)$$

the subspace consisting of maps from three-component domains. The domain of every element $u$ of $\mathfrak{M}_{\Gamma;k,l}(B; J, \nu)$ is stable and thus $u$ is automatically a simple map. Define

$$\mathcal{Z}_{\Gamma;k,l,h}^*(B; J, \nu) = \{ (u, (y_i)_{i \in [l^*]}) \in \mathcal{Z}_{k,l,h}^*(B; J, \nu): u \in \mathfrak{M}_{\Gamma;k,l}(B; J, \nu) \}. \tag{5.18}$$

For generic choices of $(J, \nu)$ and $h$,

$$\mathcal{Z}_{\Gamma;k,l,h}^*(B; J, \nu) \subset \mathcal{Z}_{k,l,h}^*(B; J, \nu)$$

is a smooth submanifold of a smooth manifold with the normal bundle canonically isomorphic to $\mathfrak{f}^*_{k',l'}\sigma_\Gamma^c$. We denote by

$$\sigma_{\Gamma; p; h} = (\mathfrak{f}^*_{k',l'}\sigma_\Gamma^c) \circ p_{l^*} \circ h$$

the relative orientation of the restriction

$$\text{ev}_{\Gamma; h}: \mathcal{Z}_{\Gamma;k,l,h}^*(B; J, \nu) \longrightarrow X_{k,l-l^*\nu}. \tag{5.19}$$

With $l_0(\Gamma), l_C(\Gamma), l_0^*(\Gamma) \in \mathbb{Z}^{>0}$ as at the beginning of Section 4.3 and $B \in H_2(X)$, define

$$\langle l^* \rangle_{\Gamma} = l' - l^* - (l_0(\Gamma) - l_0^*(\Gamma)) \in \mathbb{Z}^{>0}; \quad \langle B; k \rangle_{\Gamma} = \begin{cases} 1, & \text{if } k' = k = 1, l_0(\Gamma) = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Let $L_0^*(\Gamma) \subset \llbracket l^* \rrbracket$ be the subset indexing the conjugate pairs of marked points $(z_i^+, z_i^-)$ with $i \in \llbracket l^* \rrbracket$ carried by the real component of the curves in $\Gamma$. Define

$$L_C^*(\Gamma) = \llbracket l^* \rrbracket - L_0^*(\Gamma), \quad \langle h \rangle_{l^*: l'} = \begin{cases} h_i \cdot X^h_j, & \text{if } \llbracket L_C^*(\Gamma) \rrbracket = l_C(\Gamma), L_C^*(\Gamma) = \{ i, j \}; \\ 0, & \text{otherwise.} \end{cases}$$

**Proposition 5.3** Suppose $(X, \omega, \phi)$ is a real symplectic fourfold, $p$ is a Pin$^-$-structure on $X^\phi$, $k', l \in \mathbb{Z}^{>0}$, $l' \in [l]$, $l^* \in [l']$, and $B \in H_2(X)$ are such that

$$k \equiv k_{\omega}(B) - 2(l-l^*) - 2 \geq \max(0, 3-2l) \tag{5.19}$$
and (5.15) holds. Let $\Gamma \subset \overline{M}_{k', \ell'}$ be a primary codimension 2 stratum, $h$ as in (5.8) be a tuple of pseudocycles of codimension 2 with $\phi_*(h_i) = -[h_i]$ for every $i \in \{\ast\}$, and $p$ be as in (5.17). If the elements of $h$ and $p$ are in general position and $(J, \nu) \in H_{k, \ell}$ is generic, then $p$ is a regular value of (5.18) and the set $ev_{\Gamma; h}^{-1}(p)$ is finite. Furthermore,

$$
|ev_{\Gamma; h}^{-1}(p)|_{\delta_{\Gamma; p, h}}^{\pm} = 2\omega(B/2) - 1 - \langle \ast \rangle \Gamma - \langle \ast \rangle \Gamma \left( \prod_{i \in [\ast]} h_i \cdot X \right)_B \sum_{B' \in H_2(X)} N_{B'}^{X} \left( \prod_{i \in \{\ast\}} h_i \cdot X \right)_B \sum_{B' \in H_2(X)} N_{B'}^{X} \left( \prod_{i \in \{\ast\}} h_i \cdot X \right)_B + \langle h \rangle_{\ell; \Gamma} \left( \prod_{i \in \ell_0^* \Gamma} h_i \cdot X \right)_B \sum_{B' \in H_2(X)} N_{B'}^{X} \left( \prod_{i \in \ell_0^* \Gamma} h_i \cdot X \right)_B \sum_{B' \in H_2(X)} N_{B'}^{X} \left( \prod_{i \in \ell_0^* \Gamma} h_i \cdot X \right)_B + \langle h \rangle_{\ell; \Gamma} \left( \prod_{i \in \ell_0^* \Gamma} h_i \cdot X \right)_B \sum_{B'' \in H_2(X)} N_{B''}^{X} \left( \prod_{i \in \ell_0^* \Gamma} h_i \cdot X \right)_B \sum_{B'' \in H_2(X)} N_{B''}^{X} \left( \prod_{i \in \ell_0^* \Gamma} h_i \cdot X \right)_B \left( \prod_{i \in \ell_0^* \Gamma} h_i \cdot X \right)_B \sum_{B'' \in H_2(X)} N_{B''}^{X} \left( \prod_{i \in \ell_0^* \Gamma} h_i \cdot X \right)_B \sum_{B'' \in H_2(X)} N_{B''}^{X} \left( \prod_{i \in \ell_0^* \Gamma} h_i \cdot X \right)_B \left( \prod_{i \in \ell_0^* \Gamma} h_i \cdot X \right)_B .
$$

(5.20)

Remark 5.4. The domain of (5.18) can be completed to a Steenrod pseudocycle by adding in the codimension 1 strata of its closure. This implies that the set $ev_{\Gamma; h}^{-1}(p)$ is finite for a generic choice of $p$ and $|ev_{\Gamma; h}^{-1}(p)|_{\delta_{\Gamma; p, h}}^{\pm}$ is the degree of this pseudocycle. The proof of Proposition 5.3 in Section 6.4 instead identifies $ev_{\Gamma; h}^{-1}(p)$ with a finite subset of the cross product of two moduli spaces with the signed cardinality given by the right-hand side of (5.20).

Suppose $S$ is an open codimension 1 stratum of $\overline{M}_{k, \ell}(B; J, \nu)$. Let

$$
K_r(S) \subset [k], \quad L_r(S) \subset \{\ast\}, \quad \ell^* r(S) \subset \{\ast\}, \quad \ell_r(S), \ell^* r(S), \epsilon_r(S) \in \mathbb{Z}_{\geq 0}, \quad B_r(S) \in H_2(X)
$$

be as in Section 5.1 and $S^* \subset S$ be the subspace of simple maps. With $M_h$ given by (5.10), define

$$
S^*_{h} = \left\{ (u, (y_i)_{i \in \{\ast\}}) \in S^* \times M_h : ev^+_i(u) = h_i(y_i) \forall i \in \{\ast\} \right\}.
$$

The (virtual) normal bundles $N_S$ of $S$ in $\overline{M}_{k, \ell}(B; J, \nu)$ and $N_S^*$ of $S^*_h$ in

$$
\overline{M}_{k, \ell; h}(B; J, \nu) \equiv \left\{ (u, (y_i)_{i \in \{\ast\}}) \in \overline{M}_{k, \ell}(B; J, \nu) \times M_h : ev^+_i(u) = h_i(y_i) \forall i \in \{\ast\} \right\}
$$

are canonically isomorphic.

If $u \in S^*_h$, an orientation $\mathcal{o}_{S; u}^c$ of $N_u S$ determines a direction of degeneration of elements of $Z^*_{k, \ell; h}(B; J, \nu)$ to $u$. The relative orientation $\mathcal{o}_{S; u}^c$ of (5.3) limits to a relative orientation $\mathcal{o}_{p; h; u}$ of $ev_{k, \ell; h}: \overline{M}_{k, \ell; h}(B; J, \nu) \to X_{k, \ell; \ast}$

at $u$ obtained by approaching $u$ from this direction. Along with $\mathcal{o}_{S; u}^c$, $\mathcal{o}_{p; h; u}$ determines a relative orientation $\mathcal{o}_{S; u}^c \mathcal{o}_{p; h; u}$ of the restriction

$$
ev_{S; h}: S^* \to X_{k, \ell; \ast}
$$

(5.22)

of ev$_{k, \ell; h}$ via the first isomorphism in (5.1).
Lemma 5.5 Suppose \((X, \omega, \phi), p, k, l, l^*, B,\) and \((J, \nu)\) are as in Lemma 5.1 and \(h\) as in (5.8) is a generic tuple of smooth maps from oriented manifolds. If \(k\) and \(B\) satisfy \((2.1)\), \(S\) is an open codimension 1 stratum of \(\overline{M}_{k,l}(B; J, \nu)\), and \(u \in S^*_h\), then the relative orientation \(\partial_{\delta_S, u} o_{p; h; u}\) of (5.22) at \(u\) does not depend on the choice of \(o_{\delta_S, u}\) if and only if \(\epsilon_{l^*}(S) \equiv 2, 3 \mod 4\).

The relative orientation \(o_{p; l^*} o_h\) of the restriction of (5.21) to

\[
\mathcal{M}_{k,l,h}(B; J, \nu) = \{((u', (y_i)) \in [l]) \in \mathcal{M}_{k,l}(B; J, \nu) \times M_h : \text{ev}^+_i(u') = h_i(y_i) \forall i \in [l^*]\}
\]

extends across \(S^*_h\) if and only if \(\partial_{\delta_S, u} o_{p; h; u}\) depends on the choice of \(o_{\delta_S, u}\) for every \(u \in S^*_h\). In particular, the first statement of Lemma 5.1 is an immediate consequence of Lemma 5.5. If \(\mathcal{M}_{k,l}(B; J, \nu)\) is cut along \(\mathcal{S}\) and \(\mathcal{S}\) is the double cover of \(S^*\) in the cut, then \(\partial_{\delta_S, u} o_{p; h; u}\) is the boundary relative orientation induced by \(o_{p; l^*} o_h\) at one of the copies \(\mathcal{S}\) of \(u\) in

\[
\mathcal{S}^*_h = \{(\mathcal{S}, (y_i)) \in \mathcal{S} \times M_h : \text{ev}^+_i(u') = h_i(y_i) \forall i \in [l^*]\};
\]

we then denote it by \(\partial_{\delta_{\mathcal{S}}, u} o_{p; h; u}\). If \(\epsilon_{l^*}(S) \equiv 2, 3 \mod 4\), we abbreviate \(\partial_{\delta_S, u} o_{p; h; u}\) as \(\partial_{\delta_{\mathcal{S}}, u} o_{p; h; u}\).

Remark 5.6. While Lemma 5.5 follows readily from [22 Prop. 5.3], it is also immediately implied by our Lemmas 4.3 and 6.2 (which are also needed to establish Proposition 5.7 below). The terms \(u_2(\psi(d'))\) in [22 (17), (18)] appear to be extra (they are omitted in the key invariance on [22, p53]). The first equation in [22] with

\[
\mu(d') = \ell_\omega(B_2(S)), \quad k'' = k_2(S), \quad l'' = l_2(S) - l_2(S)
\]

compares the two possibilities for \(\partial_{\delta_{\mathcal{S}}, u} o_{p; h; u}\) when \(k = 0\) or the real marked point \(x_1\) lies on the same component of \(u\) as the marked points \(z_i^+\); the second equation treats the remaining case. The right-hand side of the latter reduces to the right-hand side of the former if \((2.1)\) holds. The right-hand side of [22 (17)], without the \(u_2(\psi(d'))\) term, in turn reduces to

\[
\frac{(\mu(d') - k'' - 2l'')(\mu(d') - k'' - 2l'' - 1)}{2} + 1 \equiv \frac{\epsilon_{l^*}(S)(\epsilon_{l^*}(S) - 1)}{2} + 1 \mod 2;
\]

the last expression vanishes (i.e. the two orientations are the same) if and only if \(\epsilon_{l^*}(S) \equiv 2, 3, 3\).

The stratum \(S\) satisfies exactly one of the following conditions:

(S0) \(K_2(S) \cap [k'] = \emptyset\) and \(L_2(S) \cap [l'] = \emptyset\);

(S1) \([K_2(S) \cap [k'] = 1\) and \(L_2(S) \cap [l'] = \emptyset\);

(S2) there exists a codimension 1 stratum \(S \subset \overline{M}_{k', l'}\) such that \(f_{k', l', l}(S) \subset S\).

We call a pair \((S, \Upsilon)\) consisting of \(S\) and a (possibly bordered) hypersurface \(\Upsilon \subset \overline{M}_{k,l}\) admissible if one of the following conditions holds:

(S1) \(K_2(S) \cap [k'] = \emptyset\), \(L_2(S) \cap [l'] = \emptyset\), and \(\Upsilon\) is regular with respect to \(f_{k', l', i}^\Upsilon\);

(S2) there exists a codimension 1 stratum \(S \subset \overline{M}_{k', l'}\) such that \(f_{k', l', l}(S) \subset S\) and and \(\Upsilon\) is regular with respect to \(S\).

The notions of \(\Upsilon\) being regular with respect to \(f_{k', l', i}^\Upsilon\) and \(S\) are defined in Section 4.2.
For $\Upsilon \subset \overline{M}_{k^*}\nu^*$, define

$$f_{p;\Upsilon} : \Upsilon \longrightarrow X_{k, l-1^*} \times \overline{M}_{k^*}\nu^*, \quad f_{p;\Upsilon}(P) = (p, P),$$

(5.24)

$$S_{h; p, \Upsilon}^* = \{((u, y), P) \in S_h^* \times \Upsilon : f_{k, l}^*(u) = P\}.$$  

(5.25)

If $(S, \Upsilon)$ is an admissible pair and $\sigma_\Upsilon$ is a co-orientation on $\Upsilon$, we denote by $\text{deg}(S, \sigma_\Upsilon) \in \mathbb{Z}$ the corresponding degree $\text{deg}_{\overline{M}_k}(\Upsilon, \sigma_\Upsilon)$ or $\text{deg}_{S, \Upsilon}(\sigma_\Upsilon)$ defined in Section 1.2.

**Proposition 5.7** Suppose $(X, \omega, \phi), \ p, k, k^*, l, l^*, B, p, h$ are in Proposition 5.3

$$S \subset \overline{M}_{k^*}\nu^*$$

and $\Upsilon \subset \overline{M}_{k^*}\nu^*$

form an admissible pair, and $\sigma_\Upsilon$ is a co-orientation on $\Upsilon$. If $\varepsilon_{\ast\ast}(S) = 2$, the elements of $h$ and $p$ are in general position, and $(J, \nu) \in \mathcal{H}_{k^*}\nu^*$ is generic, then

$$(\text{ev}_{S; h, l} : S_h^* \longrightarrow X_{k, l-1^*} \times \overline{M}_{k^*}\nu^*)$$

and $f_{p; \Upsilon} : \Upsilon \longrightarrow X_{k, l-1^*} \times \overline{M}_{k^*}\nu^*$ are transverse maps from manifolds of complementary dimensions and the set $S_{h; p, \Upsilon}^*$ is finite. Furthermore,

$$\left| S_{h; p, \Upsilon}^* \right| \frac{\pm}{\pm} \text{deg}(S, \sigma_\Upsilon) \left( \prod_{i \in L_1^S} h_i \cdot X B_1(S) \right) \left( \prod_{i \in L_2^S} h_i \cdot X B_2(S) \right)$$

$$\times N_{\overline{M}_{k^*}\nu^*} \left( B_1(S); l_1(S), l_2(S) - l_1^*(S) \right) \left( B_2(S); l_2(S) - l_1^*(S) \right).$$

(5.26)

Due to the condition $\varepsilon_{\ast\ast}(S) = 2$,

$$k_1(S) = \ell_\omega(B_1(S)) - 2(l_1(S) - l_1^*(S)) \quad \text{and} \quad k_2(S) + 1 = \ell_\omega(B_2(S)) - 2(l_2(S) - l_2^*(S)),$$

i.e. the second irreducible component of the maps in $S$ passes through an extra real point.

**Remark 5.8.** The crucial property of $(S, \Upsilon)$ used in the proof is that the condition $f_{k, l}^*(u) \in \Upsilon$ in (5.25) factors through

$$S \longrightarrow \overline{M}_{k^*\nu^*}(1) \left( B_1(S); J, \nu_1 \right) \times \overline{M}_{k^*\nu^*}(2) \left( B_2(S); J, \nu_2 \right)$$

$$\longrightarrow \overline{M}_{k^*\nu^*}(3) \left( B_1(S); J, \nu_1 \right)$$

for a good choice of $\nu$. Thus, Lemma 3.3 applies.

### 5.3 Proof of Theorem 1.1

We now deduce the two relations of Theorem 1.1 with $N_{B_{k^*}}(X^\phi)$ replaced by $N_{B_{k^*}}$, from Propositions 5.3 and 5.7 and several lemmas stated so far. We fix $(X, \omega, \phi), \ p, B$ as in Theorem 1.1 take $k$ as in (1.3) and $l^* \in \{2, 3\}$, and choose generic tuples

$$h = (h_i : H_i \longrightarrow X)_{i \in [k^*]} \quad \text{and} \quad p = (p_i^\pm_{i \in [k^*]} \left( p_i^\pm_{i \in [l^* - 1] - [l^*]} \right) \in (X^\phi)^{k^*} \times (X - X^\phi)^{l^* - 1}$$

so that each $h_i$ is a codimension 2 pseudocycle,

$$\ell_\omega(B) = 2l + k, \quad \text{and} \quad h_1^* h_3 = 0.$$  

(5.27)
We establish the three relations of Theorem 1.1 with the left-hand sides multiplied by $h_1 \cdot h_2$.

Denote by

$$f_{k', l'} : \widetilde{M}_{k, l + t^* - 1, 1}(B; J, \nu) \to \overline{M}_{k', l'}$$

the composition of (5.16) and the quotient map $q$ in (5.4) with $l$ replaced by $l + l^* - 1$. For a stratum $S$ of $\overline{M}_{k, l + t^* - 1}(B; J, \nu)$, let

$$\hat{S}^* = q^{-1}(S^*) \subset \widetilde{M}_{k, l + t^* - 1, 1}(B; J, \nu).$$

With the notation as in (5.5), let

$$M_h = H_1 \times \ldots \times H_{l^*},$$

$$\hat{Z}_{k, l + t^* - 1, h}^* (B; J, \nu) = \{(u, y_1, \ldots, y_{l^*}) \in \widetilde{M}_{k, l + t^* - 1, h}^* (B; J, \nu) \times M_h : \text{ev}_i^+(u) = h_i(y_i) \; \forall \; i \in [l^*]\}.$$

For $(J, \nu) \in H_{k, l + l^* - 1}^{\omega, \phi}$ generic, the relative orientation $\hat{\sigma}_{p, h}^*$ of Lemma 5.1 and the orientation $\sigma_h$ of $M_h$ determine a relative orientation $\hat{\sigma}_{p, h}$ of the map

$$\text{ev}_{k, l + l^* - 1, h} : \hat{Z}_{k, l + l^* - 1, h}^* (B; J, \nu) \to X_{k, l - 1}$$

induced by (5.6) with $l$ replaced by $l + l^* - 1$.

Below we take $\Upsilon \subset \overline{M}_{k', l'}$ to be the bordered compact hypersurfaces of Lemmas 4.4 and 4.5 with their co-orientations $\sigma_T^c$. For a stratum $S$ of $\overline{M}_{k, l + l^* - 1}(B; J, \nu)$, let

$$S_h^* \subset S^* \times M_h \times \Upsilon \quad \text{and} \quad \hat{S}_h^* \subset \hat{S}^* \times M_h$$

be as in (5.25) and (5.23), respectively, and

$$\hat{S}_{h, p, \Upsilon}^* = \{((u, y), P) \in \hat{S}_h^* \times \Upsilon : f_{k', l'}(u) = P\}.$$

We establish the next two statements at the end of this section.

**Lemma 5.9** For a generic choice of $(J, \nu) \in H_{k, l + l^* - 1}^{\omega, \phi}$, the map

$$\text{ev}_{k, l + l^* - 1, h} \times f_{k', l'} : \hat{Z}_{k, l + l^* - 1, h}^* (B; J, \nu) \to X_{k, l - 1} \times \overline{M}_{k', l'}$$

is a bordered $\mathbb{Z}_2$-pseudocycle of dimension $4l + 2k - 2$ transverse to (5.24).

**Corollary 5.10** For a generic choice of $(J, \nu) \in H_{k, l + l^* - 1}^{\omega, \phi}$,

$$M_{(\text{ev}_{k, l + l^* - 1, h} \times f_{k', l'}) : p, \Upsilon} \subset \hat{Z}_{k, l + l^* - 1, h}^* (B; J, \nu) \times \Upsilon$$

is a compact one-dimensional manifold with boundary (2.4) and

$$\left(\hat{Z}_{k, l + l^* - 1, h}^* (B; J, \nu) \times f_{k', l'} : p, \Upsilon\right)_{\epsilon \neq 0 (S)} = \bigsqcup_{\epsilon \neq 0 (S)} \hat{S}_{h, p, \Upsilon}^*.$$

with the union taken over the codimension 1 strata $S$ of $\overline{M}_{k, l + l^* - 1}(B; J, \nu)$ that satisfy either [S1] or [S2] above Proposition 5.7.
In our case, $\Upsilon \cap \overline{S}_1 = \emptyset$. If $S^*_{h,p;\Upsilon} \neq \emptyset$ and $S$ satisfies (S2) then $S \neq S_1$. Combined with Corollary 5.10 and the assumption that $(k',l')$ is either $(1,2)$ or $(0,3)$, this implies that the pair $(S, \Upsilon)$ is admissible in the sense defined above Proposition 5.7 whenever $S^*_{h,p;\Upsilon} \neq \emptyset$.

The identity (2.5) follows from Lemma 3.5. We use Corollary 5.10 and Proposition 5.7 to express the right-hand side of this identity, i.e. the signed cardinality of (5.29), in terms of the real invariants $N^\phi_{B',l'}$. We use Lemma 3.3(1) and Proposition 5.3 to express the left-hand side of (2.5) in terms of the real invariants $N^\phi_{B',l}$ and the complex invariants $N^\chi_{B'}$. Setting the two expressions equal and dividing by 2, we obtain the two identities of Theorem 1.1.

**Proof of (2.5).** We take $l^* = 2$. Since $k, l \geq 1$ in this case, the morphism

$$f_{1,2} : \hat{Z}^*_{k,l+1;h}(B; J, \nu) \rightarrow \mathcal{M}^\Upsilon_{1,2}$$

is well-defined. Let $P_{m} \in \mathcal{M}^\Upsilon_{1,2} \subset \mathcal{M}^\Upsilon_{1,2}$, $\sigma_{P_{m}}^\Upsilon$, and $\sigma_{P_{m}}^\Upsilon$ be as in the statement of Lemma 3.4. Let $A^R_1$ (resp. $A_2$) be the collection of the codimension 1 strata of $\mathcal{M}^\Upsilon_{k,l+1}(B; J, \nu)$ with $\epsilon_2(S) = 2$ such that the irreducible component $P^1_{m}$ carrying $(x^+_1, x^-_1)$ (resp. the other component $P^2_{n}$) of the maps in $S$ carries the conjugate pair $(z^+_2, z^-_2)$, but not the real marked point $x_1$. Each such stratum is doubly covered by a stratum $\hat{S}$ of $\partial \mathcal{M}^*_{k,l+1;h}(B; J, \nu)$.

By Corollary 5.10 and Proposition 5.7, half of the right-hand side of (2.5), not including the sign in front, equals

$$\sum_{S \in A^R_1} |S^*_{h,p;\Upsilon}|^\pm |\hat{S}_{k,l+1;h}(B; J, \nu)_{\sigma_{P_{m}}, \sigma_{P_{n}}}| + \sum_{S \in A_2} |S^*_{h,p;\Upsilon}|^\pm |\hat{S}_{k,l+1;h}(B; J, \nu)_{\sigma_{P_{m}}, \sigma_{P_{n}}}| = \sum_{S \in A^R_1} (h_1 \cdot X B_1(S))(h_2 \cdot X B_1(S)) N^\phi_{B_1(S), l_1(S), 1}^{\phi_{B_1(S)}} - 2 N^\phi_{B_2(S), l_2(S)} - \sum_{S \in A_2} (h_1 \cdot X B_1(S))(h_2 \cdot X B_2(S)) N^\phi_{B_1(S), l_1(S), 1}^{\phi_{B_1(S)}} - N^\phi_{B_2(S), l_2(S), 1}^{\phi_{B_2(S)}}.

Summing over all splittings of $B \in H_2(X)$ into $B_1$ and $B_2$, of $l-1$ conjugate pairs of points into sets of cardinalities $l_1$ and $l_2$, and of $k-1$ real points into sets of cardinalities $\ell_\omega(B_1) - 2l_1$, we obtain

$$\frac{1}{2} \left( \hat{Z}^*_{k,l+1;h}(B; J) |_{\sigma_{P_{m}}, \sigma_{P_{n}}} \right)_{f_{1,2}} = \sum_{B_1, B_2 \in H_2(X), l_1 + l_2 = l} \left( h_1 \cdot X B_1 \right) \left( h_2 \cdot X B_2 \right) \left( \frac{l-1}{l} \right) \left( \ell_\omega(B) - 2l_1 + 1 \right) \left( \ell_\omega(B_1) - 2l_1 \right) N^\phi_{B_1, l_1} N^\phi_{B_2, l_2}$$

(5.30)

We note that $l_1 \equiv l_1(S) - 2$ in the $A^R_1$ sum above and $l_1 \equiv l_1(S) - 1$ in the $A_2$ sum, because the subtractions from $l_1(S)$ correspond to the insertions of the divisors $H_1, H_2$; the meaning of $l_2$ is analogous.
By Lemma 3.3(1) and Proposition 5.3, half of the left-hand side of (2.5) equals

\[ |\text{ev}_{\leftarrow, h}^{-1}(p)|_{\partial_{p+h}}^\pm = (h_1 \cdot x h_2) N_{B, l}^{\phi \mu} + 2^{\ell(B/2) - 3} \langle h_1 \cdot x B \rangle (h_2 \cdot x B) \sum_{B' \in H_2(X)} N_{B'}^{X} \]

\[ + \sum_{B_0, B' \in H_2(X) \backslash \{0\}} 2^{\ell(B')} \langle B_0 \cdot x B' \rangle (h_1 \cdot x B') (h_2 \cdot x B') \left( \ell(B') \right) N_{B'}^{X} N_{B_0, l-1-\ell(B')}^{\phi \mu}. \]

Equating this expression with the negative of (5.30), as dictated by (2.5), we obtain the first identity in Theorem 1.1 with the left-hand side multiplied by \(h_1 \cdot x h_2\).

**Proof of (5.31).** We again take \(l^* = 2\). Since \(l \geq 2\) in this case, the morphism

\[ f_{0,3} : \hat{Z}_{k,l+1,h}(B; J, \nu) \rightarrow \overline{M}_{0,3} \]

is well-defined. Let \(\Gamma^+_1, \gamma^+ \in \mathcal{M}_{0,3}^1, \sigma^+_1, \sigma^+_2\), and \(\sigma^-_1, \sigma^-_2\) be as in the statement of Lemma 3.5. Let \(A_2\) (resp. \(A_3\)) be the collection of the codimension 1 strata \(S\) of \(\overline{M}_{k,l+1}(B; J, \nu)\) with \(\epsilon_2(S) = 2\) such that \(\mathbb{P}^1_1\) (resp. \(\mathbb{P}^1_2\)) carries the conjugate pair \((z_3^+, z_3^-)\), but not \((z_2^+, z_2^-)\).

By Corollary 5.10 and Proposition 5.7, half of the right-hand side of (2.5) equals

\[ \sum_{S \in A_2} |S^*_{h, p; Y}|_{\partial_{p+h}}^{\pm} \sigma^+_S \] \[ + \sum_{S \in A_3} |S^*_{h, p; Y}|_{\partial_{p+h}}^{\pm} \sigma^-_S = \sum_{S \in A_3} \left( h_1 \cdot x B_1(S) (h_2 \cdot x B_1(S)) N_{B_1, l_1(S)-1}^{\phi \mu} N_{B_2, l_2(S)-1}^{\phi \mu} \right) \]

\[ - \sum_{S \in A_2} \left( h_1 \cdot x B_1(S) (h_2 \cdot x B_2(S)) N_{B_1, l_1(S)-1}^{\phi \mu} N_{B_2, l_2(S)-1}^{\phi \mu} \right). \]

Summing over all splittings of \(B \in H_2(X)\) into \(B_1\) and \(B_2\), of \(l-2\) conjugate pairs of points into sets of cardinalities \(l_1\) and \(l_2\), and of \(k\) real points into sets of the appropriate cardinalities, we obtain

\[
\frac{1}{2} \left( \hat{Z}_{k,l+1,h}(B; J) \right)_{\partial_{h,l+1,h,l_1,l_2}}^{\pm} \times f_{0,3} \mathcal{X}^*_{h, p; h, \sigma^+_S}
\]

\[
= \sum_{B_1, B_2 \in H_2(X) \backslash \{0\}} \left( h_1 \cdot x B_1 \right) (h_2 \cdot x B_1) \left( \ell(B) - 2 \right) \left( \ell(B_1) - 2 \right) N_{B_1, l_1}^{\phi \mu} N_{B_2, l_2+1}^{\phi \mu}
\]

\[
- \sum_{B_1, B_2 \in H_2(X) \backslash \{0\}} \left( h_1 \cdot x B_1 \right) (h_2 \cdot x B_2) \left( \ell(B) - 2 \right) \left( \ell(B_1) - 2 \right) N_{B_1, l_1+1}^{\phi \mu} N_{B_2, l_2}^{\phi \mu}.
\]

We note that \(l_1 = l_1(S) - 2\) in both sums above, because \(l_1(S) - 1\) includes the conjugate pair \((z_3^+, z_3^-)\) in the \(A_2\) sum; this pair is included into \(l_2(S)\) in the \(A_3\) sum.
By Lemma 3.3(1) and Proposition 5.3, half of the negative of the left-hand side of (2.5) equals

$$
\left|\text{ev}_{r+2;h}^{-1}(p)\right|_{r+2;h}^{+} - \left|\text{ev}_{r+1;h}^{-1}(p)\right|_{r+1;h}^{+} = (h_1 \cdot X \cdot h_2)N_{B, l}
$$

$$
+ \sum_{B_0, B' \in \mathcal{H}_2(X) - \{0\}} 2^\omega(B') (B_0^* \cdot X \cdot B') (h_1 \cdot X \cdot B') (h_2 \cdot X \cdot B') \left( \frac{l - 2}{\ell_\omega(B')} \right) N_{B_0, l}^{-1} N_{B_0, l}^{-1} - \ell_\omega(B')
$$

$$
- \sum_{B_0, B' \in \mathcal{H}_2(X) - \{0\}} 2^\omega(B') (B_0^* \cdot X \cdot B') (h_1 \cdot X \cdot B') (h_2 \cdot X \cdot B_0) \left( \frac{l - 2}{\ell_\omega(B')} \right) N_{B_0, l}^{-1} N_{B_0, l}^{-1} - \ell_\omega(B').
$$

Equating this expression with the negative of (5.31), we obtain the second identity in Theorem 1.1 with the left-hand side multiplied by $h_1 \cdot X \cdot h_2$.

**Proof of [**WDV3**]**. We now take $l^* = 3$. Since $l \geq 1$ in this case, the morphism

$$
\hat{f}_{0,3} : \hat{\mathcal{M}}_{k, l+2}^* (B; J, \nu) \longrightarrow \hat{\mathcal{M}}_{0,3}^*
$$

is well-defined. Let $\Gamma_{l+2}^+, \gamma, \gamma' \subset \hat{\mathcal{M}}_{0,3}^*$, $\sigma_{\alpha_{l+2}}$, $\sigma_{\alpha_{l+1}}$, and $\sigma_{\alpha_{l+0}}$ be as in the statement of Lemma 4.5. Let $\mathcal{A}_2$ (resp. $\mathcal{A}_3$) be the collection of the codimension 1 strata $S$ of $\hat{\mathcal{M}}_{k, l+2}^*(B; J, \nu)$ with $\epsilon_3(S) = 2$ such that $\mathbb{P}_1^1$ (resp. $\mathbb{P}_2^1$) carries the conjugate pair $(z_3^+, z_3^-)$, but not $(z_2^+, z_2^-)$.

By Corollary 5.10 and Proposition 5.7, half of the right-hand side of (2.5) equals

$$
\sum_{S \in \mathcal{A}_2} |S_{h, p; T}|_{\partial \partial_{h, p; T}}^{+} + \sum_{S \in \mathcal{A}_3} |S_{h, p; T}|_{\partial \partial_{h, p; T}}^{+}
$$

$$
= \sum_{S \in \mathcal{A}_3} (h_1 \cdot X \cdot B_1(S) (h_2 \cdot X \cdot B_1(S)) (h_3 \cdot X \cdot B_2(S)) N_{B_1(S), l_1(S) - 1} N_{B_2(S), l_2(S) - 1}^{-1}
$$

$$
- \sum_{S \in \mathcal{A}_2} (h_1 \cdot X \cdot B_1(S) (h_3 \cdot X \cdot B_1(S)) (h_2 \cdot X \cdot B_2(S)) N_{B_1(S), l_1(S) - 1} N_{B_2(S), l_2(S) - 1}^{-1}.
$$

Summing over all splittings of $B \in \mathcal{H}_2(X)$ into $B_1$ and $B_2$, of $l - 1$ conjugate pairs of points into sets of cardinalities $l_1$ and $l_2$, and of $k$ real points into sets of the appropriate cardinalities, we obtain

$$
\frac{1}{2} \left| \hat{\mathcal{M}}_{k, l+1}^* (B; J) \right|_{\text{ev}_{l+1;h}^{+}, \text{ev}_{l+1;h}^{-}} \hat{f}_{l+1;h}^{+} T|_{\partial \partial_{h, p; T}}^{+}
$$

$$
= \sum_{B_1, B_2 \in \mathcal{H}_2(X) - \{0\}} (h_1 \cdot X \cdot B_1) (h_2 \cdot X \cdot B_1) (h_3 \cdot X \cdot B_2) \left( \frac{l - 2}{\ell_\omega(B)} \right) N_{B_1, l_1}^{-1} N_{B_2, l_2}^{-1} - \ell_\omega(B)
$$

$$
- \sum_{B_1, B_2 \in \mathcal{H}_2(X) - \{0\}} (h_1 \cdot X \cdot B_1) (h_3 \cdot X \cdot B_1) (h_2 \cdot X \cdot B_2) \left( \frac{l - 2}{\ell_\omega(B)} \right) N_{B_1, l_1}^{-1} N_{B_2, l_2}^{-1}.
$$

We note that $l_1 \equiv l_1(S) - 2$ and $l_2 \equiv l_2(S) - 1$ in both sums above, because the subtractions from $l_1(S)$ and $l_2(S)$ correspond to the insertions of the divisors $H_1, H_2, H_3$. 

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By Lemma 3.3(1), Proposition 5.3 and the last equation in (5.27), half of the negative of the left-hand side of (2.5) equals

\[
\left| \text{ev}^{-1}_{\gamma, h}(p) \right|_{j, p, h} - \left| \text{ev}^{-1}_{\gamma, h}(p) \right|_{0, j, p, h} = (h_1 \cdot X, h_2) (h_3 \cdot X, B) N_{B, l}^{\phi, p}
\]

\[
+ \sum_{B_0, B' \in H_2(X) - \{0\}} \mathcal{O}(B')(B_0 \cdot X, B') (h_1 \cdot X, B'') (h_2 \cdot X, B_0) \left( \frac{l - 1}{l \omega(B')} \right) N_{B, l}^{\phi, p} N_{B_0, l - 1}^{\phi, p} N_{B_0, l - 1}^{\phi, p}.
\]

Equating this expression with the negative of (5.31), we obtain the last identity in Theorem 5.1 with the left-hand side multiplied by $h_1 \cdot X, h_2$.

**Proof of Lemma 5.9** It is sufficient to show that

\[\text{ev} \times f_{k', J'} : \widehat{M}_{k, l + l^* - 1, j}^* (B; J') \longrightarrow X_{k, l + l^* - 1} \times \overline{M}_{k', l'} \]

(5.33)
is a bordered $\mathbb{Z}_2$-pseudocycle of dimension $4l + 2k + 2(l^* - 1)$ transverse to

\[f_{h; k, l^* - 1, j} : X_{k, l + l^* - 1} \equiv X_{l^*} \times X_{k, l - 1}, \]

(5.34)
\[f_{h; k, l^* - 1, j} : (y_1, \ldots, y_{l^*}, P) = (h_1(y_1), \ldots, h_{l^*}(y_{l^*}), p, P).\]

Since $\dim X = 4$, $\langle c_1(X, \omega), B \rangle \geq 1$ for every $B' \in H_2(X) - \{0\}$ which can be represented by a $J$-holomorphic map $u : \mathbb{P}^1 \longrightarrow X$ for a generic $J \in \mathcal{J}_\omega$.

For a stratum $S$ of $\widehat{M}_{k, l + l^* - 1, j}^* (B; J')$, we denote by $S^* \subset S$ the subspace of simple maps and by $c(S^*)$ the number of nodes of maps in $S$. The image of $S$ under $f_{k', J'}$ is contained in a stratum $S'$ of $\overline{M}_{k', l'}$ with $c(S') \equiv c(S)$. For a generic choice of $(J', \nu)$, $S^* \subset S$ is a smooth manifold of dimension

\[\dim S^* = \ell_\omega(B) + 2(l + l^* - 1) + k - c(S) = 4l + 2k + 2(l^* - 1) - c(S). \]

(5.35)
The image of $S - S^*$ under

\[\text{ev} \times f_{k', J'} : \widehat{M}_{k, l + l^* - 1, j}^* (B; J') \longrightarrow X_{k, l + 1} \times \overline{M}_{k', l'} \]

(5.36)
is covered by smooth maps from manifolds $S'$ with

\[\dim S' \leq \ell_\omega(B) + 2(l + l^* - 1) + k - 2 - c(S') = 4l + 2k + 2(l^* - 1) - c(S'); \]

(5.37)
see [21 Section 3] and [31 Section 3.4].

The space $\widehat{M}_{k, l + l^* - 1, j}^* (B; J')$ consists of the main stratum $\mathcal{M}_{k, l + l^* - 1, j}^* (B; J')$ and the subspaces $S^*$ of the strata $S$ with either one real node only or one conjugate pair of nodes only. Each stratum has a disjoint open neighborhood in $\widehat{M}_{k, l + l^* - 1, j}^* (B; J')$. Thus, the gluing maps in (13) for these strata can be chosen so that their images do not overlap. Along with the smooth structure of $\mathcal{M}_{k, l + l^* - 1, j}^* (B; J')$, these maps then determine a smooth structure on $\widehat{M}_{k, l + l^* - 1, j}^* (B; J')$ with respect to which the map (5.33) is smooth.
Since the space \( \wM_{k,l+l^*-1,l^*}(B; \nu) \) is compact,
\[
\text{Bd}(\text{ev} \times f_{k',l'}|_{\wM_{k,l+l^*-1,l^*}(B; \nu)}) \subset \text{ev} \times f_{k',l'}(\wM_{k,l+l^*-1,l^*}(B; \nu) - \wM_{k,l+l^*-1,l^*}(B; \nu)).
\] (5.38)
The complement on the right-hand side above consists of the subspaces \( S^* \) of the strata \( S \) with \( c(S) \geq 2 \) and of the subspaces \( S - S^* \) with \( c(S) \geq 1 \). Combining this with (5.33) and (5.37), we conclude that the left-hand side of (5.38) is covered by smooth maps from manifolds of dimension at most \( 4l+2k+2(l^*-2) \). Thus, (5.33) is a bordered \( \mathbb{Z}_2 \)-pseudocycle of dimension \( 4l+2k+2(l^*-1) \).

For a generic \((J, \nu)\), the restriction
\[
\text{ev} \times f_{k',l'}: S^* \longrightarrow X_{k,l+l^*-1} \times S^\vee
\] (5.39)
of (5.36) to \( S^* \) is transverse (in the target above) to \( f_{h,p;Y'} \) for every given submanifold \( Y' \subset S^\vee \).
Along with the smoothings of the nodes, this implies that (5.33) is transverse to \( f_{h,p;Y'} \). Since
\[
f_{k',l'}(\partial \wM_{k,l+l^*-1,l^*}(B; \nu)) \cap \partial Y = \emptyset
\]
with our choices of \( \nu \), we conclude that the restriction of (5.33) to \( \partial \wM_{k,l+l^*-1,l^*}(B; \nu) \) is transverse to (5.34) and to the restriction of (5.34) to the boundary \( M_h \times \partial Y \) of its domain.

Since the image of \( f_{k',l'} \) is disjoint from \( \partial \wM_{k',l} \), it is also disjoint from \( \text{Bd} \ Y \) (which is empty in the case of Lemma 4.1) and consists of 8 points in \( \partial \wM_{0,3} \) in the case of Lemma 4.5. It remains to show that smooth maps from manifolds of dimensions at most \( 4l+2k+2(l^*-2) \) covering the right-hand side of (5.38) can be chosen so that they are transverse to (5.34) and to the restriction of (5.34) to \( M_h \times \partial Y \). If \( c(S) \geq 2 \) and \( S \) is not a stratum of \( \wM_{k,l+l^*-1,l^*}(B; \nu) \), i.e. the maps in \( S \) do not just contain a conjugate pair of nodes, then the transversality of \( \partial Y \) to every stratum of \( \wM_{k',l'} - \partial Y \), the transversality of (5.39) to \( f_{h,p;Y'} \) for every given submanifold \( Y' \subset S^\vee \), and (5.35) imply that
\[
\text{ev} \times f_{k',l'}(S^*) \cap f_{h,p;Y}(M_h \times Y) = \emptyset.
\] (5.40)
For any stratum \( S \) of \( \wM_{k,l+l^*-1,l^*}(B; \nu) \), the image of \( S - S^* \) under (5.33) is covered by smooth maps
\[
h_S: S' \longrightarrow X_{k,l+l^*-1} \times S^\vee
\]
satisfying (5.37); these maps are transverse (in the target above) to \( f_{h,p;Y'} \) for every given submanifold \( Y' \subset S^\vee \) for a generic \((J, \nu)\). This implies that (5.40) holds with \( S^* \) replaced by \( S - S^* \). Thus, the bordered \( \mathbb{Z}_2 \)-pseudocycle (5.33) is transverse to (5.34).

**Proof of Corollary 5.10.** For a codimension 1 stratum \( S \) of \( \wM_{k,l+l^*-1}(B; \nu) \) and \( i = 1, 2 \), let
\[
k_i = k_i(S), \quad l_i = l_i(S), \quad l_i^* = l_i^*(S), \quad B_i = B_i(S)
\] (5.41)
be as in Section 5.1. Suppose that \( \hat{S}^* \) is a stratum of \( \wM_{k,l+l^*-1,l^*}(B; \nu) \), i.e. \( \epsilon(S) \equiv 2, 3 \) mod 4.

For good choices of \( \nu \) (still sufficiently generic), the restriction to \( \hat{S}^* \) of (5.7) with \( l \) replaced by \( l+l^*-1 \) factors as
\[
\hat{S}^* \longrightarrow \mathcal{M}_{k_1,l_1}^1(B_1; \nu_1) \times \mathcal{M}_{k_2,l_2}^1(B_1; \nu_2)
\longrightarrow \mathcal{M}_{k_1,l_1}^1(B_1; \nu_1') \times \mathcal{M}_{k_2,l_2}^1(B_2; \nu_2') \longrightarrow X_{k_1,l_1} \times X_{k_2,l_2} \longrightarrow X_{k,l+l^*-1}.
\]
Thus, $\tilde{S}^*_{h,p;\Upsilon} = \emptyset$ for a generic choice of $(J, \nu)$ unless
\[ \ell_\omega(B_i) + 2(l_i - l^s_i) + k_i \geq 4(l_i - l^s_i) + 2k_i \quad \forall i = 1, 2. \]
Since
\[ \ell_\omega(B_1) + \ell_\omega(B_2) = \ell_\omega(B) - 1 = 2l + k - 1, \quad k_1 + k_2 = k, \quad l_1 + l_2 = l + 1, \quad l^s_1 + l^s_2 = l^s, \]
and $\epsilon_t(S) \equiv 2, 3 \mod 4$, it follows that $\epsilon_t(S) = 2$ and
\[ \ell_\omega(B_1) = 2(l_1 - l^s_1) + k_1. \tag{5.42} \]
If $S$ satisfies (S(0)) above Proposition 5.7 the restriction to $\tilde{S}^*$ of the composition of $\tilde{S}^*$ with the projection to the product $X_{k_1,l_1} \times M^\nu_{k',l'}$ factors as
\[ \tilde{S}^* \rightarrow \mathfrak{m}_{k_1+1,l_1}(B_1; J, \nu_1) \times \mathfrak{m}_{k_2+1,l_2}(B_2; J, \nu_2) \rightarrow \mathfrak{m}_{k_1,l_1}(B_1; J, \nu_1) \rightarrow X_{k_1,l_1} \times M^\nu_{k',l'}. \]
Since the restriction of (5.33) to $\tilde{S}^*$ is transverse to (5.34) and $\Upsilon$ is a real hypersurface, (5.42) then implies that $\tilde{S}^*_{h,p;\Upsilon} = \emptyset$.

6 Proofs of structural statements

6.1 Orienting the linearized $\overline{\partial}$-operator

For $u$ as in (5.1), let
\[ D^\phi_{J,\nu;u} : \Gamma(u) \equiv \{ \xi \in \Gamma(\Sigma; u^* TX) : \xi \circ \sigma = d\phi \circ \xi \} \rightarrow \Gamma^{0,1}(u) \equiv \{ \zeta \in \Gamma(\Sigma; (T^* \Sigma, j)^{0,1} \otimes \mathbb{C} u^*(TX, J)) : \zeta \circ \sigma = d\phi \circ \zeta \} \]
be the linearization of the $\{ \overline{\partial}_J - \nu \}$-operator on the space of real maps from $(\Sigma, \sigma)$ with its complex structure $j$. We define
\[ \lambda^c_u(X) = \bigotimes_{i=1}^l \lambda(T_{u(x_i)}^*) X_i, \quad \lambda^\beta_u(X) = \lambda \left( \bigoplus_{i=1}^k T_{u(x_i)} X_i^\phi \right) = \bigotimes_{i=1}^k \lambda(T_{u(x_i)} X_i^\phi), \]
\[ \lambda_u(D^\phi_{J,\nu}) = \det D^\phi_{J,\nu;u}, \quad \tilde{\lambda}_u(D^\phi_{J,\nu}, X) = \lambda^\beta_u(X)^* \otimes \lambda^c_u(X)^* \otimes \lambda_u(D^\phi_{J,\nu}); \]
the summands and the factors in the definition of $\lambda^\beta_u(X)$ are not ordered. By [9 Appendix], the projection
\[ \tilde{\lambda}(D^\phi_{J,\nu}, X) = \bigcup_{u \in \mathfrak{m}_{k,l}(B; J, \nu)} \{ u \} \times \tilde{\lambda}_u(D^\phi_{J,\nu}, X) \rightarrow \mathfrak{m}_{k,l}(B; J, \nu) \tag{6.1} \]
is a line orbi-bundle with respect to a natural topology on its domain.

For $i \in [k]$ and $u \in \mathfrak{m}_{k,l}(B; J, \nu)$ with the associated marked curve $C$ as in (5.2), let
\[ j_i(u) = j_i(C) \in [k] \]
be as in Section 1.1. The next statement is a consequence of the orienting construction of [22, Prop. 3.1], a more systematic perspective of which appears in the proof of [5] Thm. 7.1. 48
Lemma 6.1 Suppose \((X,\omega,\phi)\) is a real symplectic fourfold, \(l \in \mathbb{Z}^+\), \(k \in \mathbb{Z}^{\geq 0}\), \(B \in H_2(X)\), and \((J,\nu) \in \mathcal{H}_{k,l}^{\omega,\phi}\). If \(k\) and \(B\) satisfy (2.1), then a Pin\(^-\)structure \(p\) on \(X^\phi\) determines an orientation \(\sigma_p^D\) on the restriction of (6.1) to \(\mathcal{M}_{k,l}(B; J, \nu)\) with the following properties:

\((\sigma_p^D1)\) \(\sigma_p^D\) is preserved by the interchange of two real points \(x_i\) and \(x_j\) with \(2 \leq i, j \leq k\);

\((\sigma_p^D2)\) if \(2 \leq i \leq k\) and \(u \in \mathcal{M}_{k,l}(B; J, \nu)\), \(\sigma_p^D\) is preserved at \(u\) by the interchange of the real points \(x_1\) and \(x_{j_i(u)}\) if and only if \((k-1)(i-1) \in 2\mathbb{Z}\);

\((\sigma_p^D3)\) \(\sigma_p^D\) is preserved by the interchange of the points in a conjugate pair \((z_i^+, z_i^-)\) with \(1 < i \leq l\); and

\((\sigma_p^D4)\) if \(u \in \mathcal{M}_{k,l}(B; J, \nu)\), \(\sigma_p^D\) is preserved at \(u\) by the interchange of the points in the conjugate pair \((z_i^+, z_i^-)\) if and only if

\[ k \neq 0 \quad \text{and} \quad \ell_\omega(B) \cong 2, 3 \mod 4 \] or

\[ k = 0 \quad \text{and} \quad \ell_\omega(B) \cong 0 \mod 4; \]

\((\sigma_p^D5)\) if \(k, l, l = 1, B = 0, \) and \(\nu\) is small, then \(\sigma_p^D\) is the orientation induced by the evaluation at \(x_1\).

Proof. Let \(u\) be as in (5.1). For the purposes of applying [5, Thm. 7.1], we take the distinguished half-surface \(\mathbb{D}^2 \subset \mathbb{P}^1\) to be the disk so that \(\mathbb{D}^2\) is the fixed locus \(S^1\) of \(\tau\) and \(z_i^+ \in \mathbb{D}^2\). A Pin\(^-\)structure \(p\) on \(X^\phi\) determines an orientation \(\sigma_{p,\mathbb{R}}^D\) on the line \(\lambda_u^R(X)^* \otimes \lambda_u(D^\phi_{J,\nu})\) varying continuously with \(u\). By the CROrient 1p property in [5, Section 7.2], \(\sigma_{p,\mathbb{R}}^D\) satisfies the first four properties of this lemma. By the CROrient 3 and 6a properties in [5, Section 7.2], it also satisfies the last property of the lemma. Along with the symplectic orientations of \(T_{u(z_i^+)} X\), \(\sigma_{p,\mathbb{R}}^D\) determines an orientation \(\sigma_{p,\mathbb{R}}^D\) on \(\lambda_u(D^\phi_{J,\nu}, X)\) varying continuously with \(u\). Since the complex dimension of \(X\) is even, \(\sigma_p^D\) also satisfies all five properties.

Suppose now that \(l \in \mathbb{Z}^+\) and \(S\) is an open codimension 1 stratum of \(\overline{\mathcal{M}}_{k,l}(B; J, \nu)\). Define

\[ r(S) = \begin{cases} 1, & \text{if } k = 0 \text{ or } 1 \in K_1(S); \\ 2, & \text{if } 1 \in K_2(S). \end{cases} \]

An orientation \(\sigma_{S,u}^\circ\) of \(\mathcal{N}_u S\) determines a direction of degeneration of elements of \(\mathcal{M}_{k,l}(B; J, \nu)\) to \(u\). The orientation \(\sigma_p^D\) on (6.1) limits to an orientation \(\sigma_{p,u}^D\) of \(\tilde{\lambda}_u(D^\phi_{J,\nu}, X)\) by approaching \(u\) from this direction. The orientation \(\sigma_{p,u}^D\) is called the limiting orientation induced by \(p\) and \(\sigma_{S,u}^\circ\) in [5, Section 7.3]. If in addition \(l_2(S) \geq 1\), the possible orientations \(\sigma_{S,u}^\circ\) of \(\mathcal{N}_u S\) are distinguished as above Lemma 4.3. We denote by \(\sigma_{p,u}^{D,\pm}\) the limiting orientation of \(\tilde{\lambda}_u(D^\phi_{J,\nu}, X)\) induced by \(p\) and \(\sigma_{S,u}^\circ\).

The domain of each element \(u \in S\) consists of an irreducible component \(\mathbb{P}^1_1\) carrying the marked points \(z_i^+\) with fixed locus \(S^1_i\) and another irreducible component \(\mathbb{P}^1_2\) with fixed locus \(S^2_1\). The fixed locus \(S^1_i\) splits \(\mathbb{P}^1_1\) into two disks. Let \(S_u \subset S\) be the subspace of all maps with fixed distributions of the marked points \(z_i^+\) with \(i \in [l]\) between the two disks and with fixed orderings of the marked points \(x_i\) with \(i \in [k]\) and the nodal points on the two fixed loci. We call such a subspace a substratum of \(S\). If \(k_2(S) + 2l_2(S) \geq 2\), i.e. the marked domain (5.2) of every element \(u \in S\) is stable, then the image of \(S\) under the forgetful morphism

\[ f_{k,l}: \overline{\mathcal{M}}_{k,l}(B; J, \nu) \to \overline{\mathcal{M}}_{k,l}^0 \]

is proper.
is contained in a codimension 1 stratum $\mathcal{S}^\vee$. In such a case, a substratum $\mathcal{S}_*$ of $\mathcal{S}$ is given by

$$\mathcal{S}_* = \mathcal{S} \cap f_{k,l}^{-1}(\mathcal{S}^\vee)$$

for some topological component $\mathcal{S}_*^\vee$ of $\mathcal{S}^\vee$.

For good choices of $\nu$, there are a natural embedding

$$\mathcal{S}_* \hookrightarrow \mathcal{M}_1 \times \mathcal{M}_2 \subset \mathcal{M}_{k_1(S)+1,l_1(S)}(B_1(S); J, \nu_1) \times \mathcal{M}_{k_2(S)+1,l_2(S)}(B_2(S); J, \nu_2) \quad (6.2)$$

for some unions $\mathcal{M}_1$ and $\mathcal{M}_2$ of topological components of the moduli spaces on the right-hand side above and forgetful morphisms

$$f_{\text{nd}}: \mathcal{M}_{k_1(S)+1,l_1(S)}(B_1(S); J, \nu_1) \to \mathcal{M}_{k_1(S),l_1(S)}(B_1(S); J, \nu'_1),$$

$$f_{\text{nd}}: \mathcal{M}_{k_2(S)+1,l_2(S)}(B_2(S); J, \nu_2) \to \mathcal{M}_{k_2(S),l_2(S)}(B_2(S); J, \nu'_2) \quad (6.3)$$

dropping the real marked points corresponding to the nodal points nd on the two components. We choose the embedding in (6.2) so that it satisfies (0.81) and (0.82) in Section 4.3. For an element $u \in \mathcal{S}$, we denote by

$$u_1 \in \mathcal{M}_{k_1(S)+1,l_1(S)}(B_1(S); J, \nu_1) \quad \text{and} \quad u_2 \in \mathcal{M}_{k_2(S)+1,l_2(S)}(B_2(S); J, \nu_2)$$

the pair of maps corresponding to $u$ via (6.2). Let

$$u'_1 \in \mathcal{M}_{k_1(S),l_1(S)}(B_1(S); J, \nu'_1) \quad \text{and} \quad u'_2 \in \mathcal{M}_{k_2(S),l_2(S)}(B_2(S); J, \nu'_2)$$

be the images of $u_1$ and $u_2$ under the forgetful morphisms in (6.3).

Suppose $k$ and $B$ satisfy (2.1), $l_2(S) \geq 1$, and $i^* \in L_2(\mathcal{S}_*)$ is as above Lemma 4.3. For each $u \in \mathcal{S}_*$, the exact sequences

$$0 \to D_{J,\nu,u}^\phi \to D_{J,\nu,u'}^\phi \oplus D_{J,\nu,u''}^\phi \to T_{u(\text{nd})}X^\phi \to 0, \quad (\xi_1, \xi_2) \to \xi_2(\text{nd}) - \xi_1(\text{nd}), \quad (6.4)$$

$$0 \to D_{J,\nu,u}^\phi \to D_{J,\nu,u'}^\phi \oplus D_{J,\nu,u''}^\phi \to T_{u(\text{nd})}X^\phi \to 0, \quad (\xi_1, \xi_2) \to \xi_2(\text{nd}) - \xi_1(\text{nd}),$$

of Fredholm operators determine isomorphisms

$$\lambda_u(D_{J,\nu,u}^\phi) \otimes \lambda(T_{u(\text{nd})}X^\phi) \approx \lambda_{u'_1}(D_{J,\nu,u'_1}^\phi) \otimes \lambda_{u'_2}(D_{J,\nu,u'_2}^\phi), \quad (6.5)$$

$$\lambda_u(D_{J,\nu,u}^\phi) \otimes \lambda(T_{u(\text{nd})}X^\phi) \approx \lambda_{u'_1}(D_{J,\nu,u'_1}^\phi) \otimes \lambda_{u'_2}(D_{J,\nu,u'_2}^\phi).$$

If $\epsilon_l(S) \in 2\mathbb{Z}$ (for any $l^* \in [l]$), a Pin$^-$-structure $\mathfrak{p}$ on $X^\phi$ determines homotopy classes of isomorphisms

$$\lambda_{u'_1}(D_{J,\nu,u'_1}^\phi) \to \lambda_{u'_1}(X) \otimes \lambda_{u'_1}(X) \quad \text{and} \quad \lambda_{u'_2}(D_{J,\nu,u'_2}^\phi) \to \lambda_{u'_2}(X) \otimes \lambda_{u'_2}(X);$$

see Lemma 6.1. Combining these isomorphisms with the first isomorphism in (6.5), we obtain a homotopy class of isomorphisms

$$\lambda_u(D_{J,\nu,u}^\phi) \otimes \lambda(T_{u(\text{nd})}X^\phi) \approx \lambda_{u'_1}(X) \otimes \lambda_{u'_1}(X) \otimes \lambda_{u'_2}(X) \otimes \lambda_{u'_2}(X) \approx \lambda_{u'_1}(X) \otimes \lambda_{u'_2}(X) \otimes \lambda_{u'_2}(X) \otimes \lambda(T_{u(\text{nd})}X^\phi), \quad (6.6)$$

$$\approx \lambda_{u'_1}(X) \otimes \lambda_{u'_2}(X) \otimes \lambda(T_{u(\text{nd})}X^\phi).$$
If $\epsilon_1(S) \not\equiv 2\mathbb{Z}$, a Pin$^-$-structure $\mathfrak{p}$ on $X^\phi$ similarly determines a homotopy class of isomorphisms
\[
\lambda_u(D_{j,l}^\phi) \otimes \lambda(T_{u(nd)}X^\phi) \approx \lambda_{u_1}^R(X) \otimes \lambda_{u_1}^C(X) \otimes \lambda_{u_2}^R(X) \otimes \lambda_{u_2}^C(X) \\
\approx \lambda(T_{u(nd)}X^\phi) \otimes \lambda_{u_1}^R(X) \otimes \lambda_{u_1}^C(X) \otimes \lambda_{u_2}^R(X) \otimes \lambda_{u_2}^C(X) \\
\approx \lambda_u^R(X) \otimes \lambda_u^C(X) \otimes \lambda(T_{u(nd)}X^\phi).
\]

In either case, we denote the associated orientation on $\widetilde{\lambda}_u(D_{j,l}^\phi)$ by $\partial_p^D$.

If $l_2(S) \geq 1$, we choose the embedding $(6.2)$ so that the real marked points of the tuples of $u_1$ and $u_2$ corresponding to $u \in S_\ast$ are ordered by their position on $S_1^1 \subset \mathbb{P}_1$ and $S_2^1 \subset \mathbb{P}_2$, respectively, starting from the node in the counterclockwise direction with respect to $z_1^+ \in \mathbb{P}_1$ and $z_2^+ \in \mathbb{P}_2$. We define $\delta^+_D(S_\ast) \in \mathbb{Z}$ as above Lemma 4.3 and set
\[
\delta^+_D(S) = k_2(S) \langle w_2(X), B_1(S) \rangle, \quad \delta^+_D(S) = \frac{(\ell_\omega(B_2(S)) - k_2(S))(\ell_\omega(B_2(S)) - k_2(S) + 1)}{2}.
\]

**Lemma 6.2** Suppose $(X, \omega, \phi)$, $k, l, B$, and $(J, \nu) \in \mathcal{H}_{k,l}^\phi$ are as in Lemma 6.1 the pair $(k, B)$ satisfies $(2.1)$, and $S_\ast$ is a stratum of a codimension 1 stratum $S$ of $\mathfrak{M}_{k,l}(B; J, \nu)$ with $l_2(S) \geq 1$. The orientations $\partial_p^{D_+}$ and $\partial_p^{D_-}$ on $\hat{\lambda}(D_{j,l}^\phi)$ are the same if and only if $\delta_S^+(S) \equiv \delta^+_S(S_\ast)$ mod 2.

**Proof.** Let $u \in S_\ast$. We define $r_\varepsilon(S)$ to be 1 if $\epsilon_1(S) \equiv 2\mathbb{Z}$ and 2 if $\epsilon_1(S) \not\equiv 2\mathbb{Z}$. Let $j'_\varepsilon(u) \in \mathbb{Z}$ and the number of real marked points that lie on the oriented arc of $S_1^1$ between the node and the real marked point $z_1^+ \in \mathbb{P}_1$ is the smallest value of $i$; if $k_{r_\varepsilon(S)}(S) = 0$, we take $j'_\varepsilon(u) = 0$. The marked points $z_1^+ \in \mathbb{P}_1$ and $z_2^+ \in \mathbb{P}_2$ determine the distinguished disks as in the proof of Lemma 6.1. By Lemma 6.1, the orientation $\partial_p^D$ at $u$ agrees with the split orientation of $\mathfrak{M}_{k,l}(B; J, \nu)$ if and only if $(k_{r_\varepsilon(S)} - 1)j'_\varepsilon(u) = 0$. Thus, $[5]$ Cor. 7.5 implies the claim for $\lambda^R_u(X)^* \otimes \lambda_u(D_{j,l}^\phi)$. Since the conjugate pairs of marked points have the same effect on $\partial_p^{D_+}$ and $\partial_p^{D_-}$, the claim follows.

**6.2 Proofs of Lemmas 5.1 and 5.5 and Proposition 5.2**

Suppose $(X, \omega, \phi)$, $p, k, l$, $B$, and $(J, \nu)$ are as in Lemma 5.1 the pair $(k, B)$ satisfies $(2.1)$, and $(J, \nu)$ is generic. The exact sequences
\[
0 \to \ker D_{j,l}^\phi \to T_u \mathfrak{M}_{k,l}^*(B; J, \nu) \to T_{j,l} \mathcal{M}_{k,l}^r \to 0
\]
with $u \in \mathfrak{M}_{k,l}^*(B; J, \nu)$ induced by the forgetful morphism $f_{k,l}$ determine an isomorphism
\[
\lambda(\text{ev}^* \mathfrak{M}_{k,l}^*(B; J, \nu)) \equiv \text{ev}^* \lambda^R(X)^* \otimes \text{ev}^* \lambda^C(X)^* \otimes \lambda(\mathfrak{M}_{k,l}^*(B; J, \nu)) \\
\approx \hat{\lambda}(D_{j,l}^\phi, X) \otimes f_{k,l}^* \lambda(\mathcal{M}_{k,l}^r)
\]
(6.7)
of line bundles over $\mathfrak{M}_{k,l}^*(B; J, \nu)$. By Lemma 6.1 the Pin$^-$-structure $\mathfrak{p}$ on $X^\phi$ induces an orientation $\partial_p^D$ on the first factor on the right-hand side above. Along with the orientation $\partial_{k,l}^*$ on the second factor defined in Section 5.1 it determines a relative orientation $\partial_{p,j,l}^*$ on the restrictions of (5.12) to $\mathfrak{M}_{k,l}^*(B; J, \nu)$ via (6.7).
Proofs of Lemma 5.1 and 5.5. By Lemmas 4.1 and 6.1, the relative orientation $o_{p,l*}$ above satisfies all properties listed in Lemma 5.1 wherever it is defined. Every (continuous) extension of $o_{p,l*}$ to subspaces of the domains of the maps in (5.12) satisfies the same properties. The relative orientation $o_{p,l*}$ automatically extends over all strata of codimension 2 and higher. By Lemma 5.5, it extends over the codimension 1 strata of the two domains as well. Lemma 5.5 in turn follows immediately from Lemmas 4.3 and 6.2.

The next observation, which is used in the proof of Proposition 5.2, is straightforward.

**Lemma 6.3** Suppose $A_{ij}$ with $i,j \in [3]$ are oriented finite-dimensional vector spaces, the rows and columns in the diagram in Figure 8 are exact sequences of vector-space homomorphisms, and this diagram commutes. The total number of rows and columns in this diagram which (do not) respect the orientations is congruent to $\dim A_{13} - \dim A_{31} \mod 2$.

**Proof of Proposition 5.2**. We continue with the notation in the proof of Lemma 5.9, but apply it to the strata $S$ of $\mathcal{M}_{k,l}(B;J,\nu)$. Let

$$ev_{l*} = \prod_{i=1}^{l*} ev_i^+: \mathcal{M}_{k,l}(B;J,\nu) \to X^{l*}$$

(6.8)

and $h: Z \to X_{l*}$ be a smooth map from a manifold of dimension $2l* - 2$ that covers $\text{Bd} \ f_h$. Let

$$ev_{k,l;h}: Z_{k,l;h}(B;J,\nu) \equiv \{(u,y) \in \mathcal{M}_{k,l}(B;J,\nu) \times M_h : ev_{l*}(u) = f_h(y)\} \to X_{k,l-l*},$$

$$ev_{k,l;h}: Z_{k,l;h}(B;J,\nu) \equiv \{(u,z) \in \mathcal{M}_{k,l}(B;J,\nu) \times Z : ev_{l*}(u) = h(z)\} \to X_{k,l-l*}$$

(6.9)

be the maps induced by (5.6).

For each stratum $S$ of $\mathcal{M}_{k,l}(B;J,\nu)$, define

$$S_h = Z_{k,l;h}(B;J,\nu) \cap (S \times M_h), \quad S_h^* = Z_{k,l;h}(B;J,\nu) \cap (S^* \times M_h).$$

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For a generic \((J, \nu)\), the subspace \(S^*\) of simple maps in \(S\) is a smooth manifold of dimension
\[
\dim S^* = \ell_w(B) + 2l + k - c(S) = 4l - 2l^* + 2k - c(S) \tag{6.10}
\]
and the restriction of \([5.9]\) to \(S^*\) is transverse to \(f_h\) and to \(h\). Along with \([6.10]\), the first transversality property implies that \(S^*_h\) is a smooth manifold of dimension
\[
\dim S^*_h = \dim S^* - 2l^* = 4(l - l^*) + 2k - c(S). \tag{6.11}
\]

For every stratum \(S\) of \(\mathfrak{M}_{k,l}(B; J, \nu)\), there is a smooth manifold \(S'\) and smooth maps
\[
ev_{[s^*]}: S' \rightarrow X^{l^*} \quad \text{and} \quad ev_{k,l;*} : S' \rightarrow X_{k,l-l^*}
\]
such that \(ev_{[s^*]}\) is transverse to \(f_h\) and to \(h\),
\[
ev_{[s^*]}(S - S^*) \subset ev_{[s^*]}(S'), \quad \text{and} \quad \dim S' \leq \ell_w(B) + 2l + k - 2 = 4l - 2l^* + 2k - 2; \tag{6.12}
\]
see [21] Section 3] and [31] Section 3.4]. In particular, the map
\[
ev_{k,l;h} : S_h^* \equiv \{(u, y) \in S' \times M_h; ev_{[s^*]}(u) = f_h(y)\} \rightarrow X_{k,l-l^*}
\]
induced by \(ev_{k,l;*}\) is smooth,
\[
ev_{k,l;h}(S_h - S_h^*) \subset ev_{[s^*]}(S'_h), \quad \text{and} \quad \dim S'_h \leq 4(l - l^*) + 2k - 2. \tag{6.13}
\]

By the reasoning in the proof of Lemma [5.9] applied to \(\mathfrak{M}_{k,l;*}^*(B; J, \nu)\) instead of \(\mathfrak{M}_{k,l;*}^{*+1:2}(B; J, \nu)\), \(\mathfrak{M}_{k,l;*}^*(B; J, \nu)\) is a smooth manifold. Along with the first transversality property after \([6.10]\), this implies that \([5.9]\) is a smooth map between smooth manifolds of the same dimension. The relative orientation \(o_{p;*}\) of the first map in \([5.12]\) and the orientation \(o_h\) determine a relative orientation \(o_{p;*} o_h\) of \([5.9]\).

Since the space \(\mathfrak{M}_{k,l}(B; J, \nu)\) is compact,
\[
\text{Bd}(ev_{k,l;h} | Z_{k,l;h}^*(B; J, \nu)) \subset ev_{k,l;h}(Z_{k,l;h}(B; J, \nu) - Z_{k,l;h}^*(B; J, \nu)) \cup ev_{k,l;h}(Z_{k,l;h}(B; J, \nu)). \tag{6.14}
\]

By \([6.10]\), \([6.12]\), and the transversality of the maps \(ev_{[s^*]}\) on \(S^*\) and \(S'\) to \(h\), the last set above is covered by smooth maps from finitely manifolds of dimension at most
\[
\dim Z_{k,l;h}^*(B; J, \nu) - 2 = 4(l - l^*) + 2k - 2. \tag{6.15}
\]

The set
\[
Z_{k,l;h}(B; J, \nu) - Z_{k,l;h}^*(B; J, \nu) \subset \mathfrak{M}_{k,l}(B; J, \nu) \times M_h
\]
consists of the subspaces \(S^*_h\) corresponding to the strata \(S\) of \(\mathfrak{M}_{k,l}(B; J, \nu)\) with either \(c(S) \geq 2\) nodes or \(c_l (S) \equiv 2, 3 \mod 2\) and of the subspaces \(S - S^*\) with \(c(S) \geq 1\). By \([6.11]\) and \([6.13]\), a smooth map from manifold of dimension \([6.15]\) covers \(ev_{k,l;h}(S^*)\) if \(c(S) \geq 2\) and \(ev_{k,l;h}(S - S^*)\) for any stratum \(S\) of \(\mathfrak{M}_{k,l}(B; J, \nu)\).
Suppose $S$ is a stratum of $\overline{M}_{\kappa,l}(B; J, \nu)$ with $\zeta(S) = 1$ and $k_i, l_i, l_i^*$, $B_i$ are as in (5.41). Let
\[ f_{h_1}: M_{h_1} \to X^{l_i^*} \] and \[ f_{h_2}: M_{h_2} \to X^{l_i^*} \] be the pseudocycles determined by the maps $h_1, \ldots, h_i$ corresponding to the conjugate pairs of marked points indexed by $i \in [l^*]$ that are carried by the first and second components of the maps in $S$, respectively. For good choices of $\nu$ (still sufficiently generic), the restriction of (6.9) to $S_h^{*}$ factors as
\[ S_h^{*} \xrightarrow{\text{ev}_{k_1,l_1;h_1}} Z_{k_1,l_1;h_1}(B_1; J, \nu_1) \times Z_{k_2,l_2;h_2}(B_2; J, \nu_2) \]
\[ \xrightarrow{\text{ev}_{k_2,l_2;h_2}} X_{k_1,l_1-l_i^*} \times X_{k_2,l_2-l_i^*} \longrightarrow X_{k,l-l^*}. \]

Thus, $\text{ev}_{k,l,h}(S_h^{*})$ is covered by a smooth map from a manifold of dimension
\[ \dim Z_{k_i,l_i;h_1}(B_i; J, \nu_i) + \dim X_{k_{3-i},l_{3-i}-l_i^*} = \ell_\omega(B_i) + 2(l_i-l_i^*) + k_i + 4(l_{3-i} - l_{3-i}^*) + 2k_{3-i} = 4(l - l^*) + 2k \]
for $i = 1, 2$. Since
\[ (\ell_\omega(B_1) - 2(l_1 - l_1^*) - k_1) + (\ell_\omega(B_2) - 2(l_2 - l_2^*) - k_2) = \ell_\omega(B) - 2(l - l^*) - k = -1, \]
it follows that $\text{ev}_{k,l,h}(S_h^{*})$ is covered by a smooth map from a manifold of dimension (6.15) unless
\[ \varepsilon_l(S) \equiv (\ell_\omega(B_2) - 2(l_2 - l_2^*) - k_2) + 1 \]
is either 0 or 1. Combining this with the preceding paragraph, we conclude that the left-hand side of (6.14) is covered by a smooth map from a manifold of dimension $4(l - l^*) + 2k - 2$. This establishes the first claim of the proposition.

It remains to establish (5.14). We can assume that $B \neq 0$ and can be represented by a $J$-holomorphic map; thus, $\langle \omega, B \rangle \neq 0$. Let $H \in H^2(X; \mathbb{Z})$ be such $\phi^*H = -H$ and $\langle H, B \rangle \neq 0$; such a class $H$ can be obtained by slightly deforming $\omega$ so that it represents a rational class, taking a multiple of the deformed class that represents an integral class, and then taking the anti-invariant part of the multiple. Let $h_1$ and $h_2$ be two pseudocycles as in the statement of the proposition representing the Poincaré dual of $H$. By definition,
\[ N_{B,l-l^*}^{\phi,p} = \frac{1}{\langle H, B \rangle^2} \deg(\text{ev}_{k,l-l^*+2;(h_1,h_2), o_{p;2}^0(h_1,h_2)}) \].

An implicit implication of a similar definition in [22, Section 4] is that $N_{B,l-l^*}^{\phi,p}$ does not depend on the choices of $H$, $h_1$, and $h_2$. This follows from (6.16) below, which also implies (5.14).

Let $k, l, l^*, B, h$ be as in the statement of the proposition and $h': H' \to X$ be another codimension 2 pseudocycle in general position. We denote by $h'h'$ the tuple $(h_1, \ldots, h_{l^*}, h')$ and show below that
\[ \deg(\text{ev}_{k,l+1;h'h'}, o_{p;l^*} + 1 o_{h'h'}) = (h' \cdot B) \deg(\text{ev}_{k,l;h}, o_{p;l^*} o_h), \] (6.16)
with $\text{ev}_{k,l;h}$ as in (5.9) and

$$\text{ev}_{k,l+1;hh'}: Z^*_{k,l+1;hh'}(B; J, f^*_{k,l+1;ll^*+1} \nu) \rightarrow X_{k,l(l+1)-(l^*+1)} = X_{k,l-l^*}.$$ 

The second forgetful morphism in (5.3) with $(l, i)$ replaced by $(l+1, l^*+1)$ induces a morphism

$$f: Z^*_{k,l+1;hh'}(B; J, f^*_{k,l+1;ll^*+1} \nu) \rightarrow Z^*_{k,l;h}(B; J, \nu)$$

so that $\text{ev}_{k,l+1;hh'} = \text{ev}_{k,l;h} \circ f$. The relative orientations $o_{ll^*+1} o_{hh'}$ of $\text{ev}_{k,l+1;hh'}$ and $o_{ll^*} o_h$ of $\text{ev}_{k,l;h}$ determine a relative orientation $o_p$ of $f$. The number of the preimages

$$\tilde{u} \equiv (u, (z^+_{ll^*+1}, z^-_{ll^*+1}), y, y')$$

of a generic point

$$(u, y) \in Z^*_{k,l;h}(B; J, \nu) \cap (\mathcal{M}_{k,l}(B; J, \nu) \times M_h)$$

under $f$ is finite. For such a preimage $\tilde{u}$, $d_{u} f$ is an isomorphism. With $u$ as in (5.1), the homomorphism

$$T_{z^+_{ll^*+1}} \mathbb{P}^1 \oplus T_y H' \rightarrow T_{u(z^+_{ll^*+1})} X = T_{h'(y')} X, \quad (v, w) \mapsto d^+_{z^+_{ll^*+1}} u(v) + d_y h'(w), \quad (6.17)$$

is an isomorphism. Its domain and target are oriented by the complex orientation of $\mathbb{P}^1$ (i.e. the vertical orientation $o^*_{ll^*+1}$ in the notation of Lemma 5.1[a]), the given orientation $o_{hh'}$ of $H'$, and the symplectic orientation $o_w$ of $X$. We set $s_{\tilde{u}}$ to be $+1$ if this isomorphism is orientation-preserving and to be $-1$ if it is orientation-reversing. We show below that $s_{\tilde{u}}(o_p) = s_{\tilde{u}}$. Since

$$\sum_{\tilde{u} \in \tilde{f}^{-1}(u, y)} s_{\tilde{u}} = h'.X B,$$ (6.18)

the desired identity (6.16) then follows from (4.7).
Let \( \Delta \subset X^2 \) and \( \Delta^i \subset (X^i)^2 \) denote the diagonals. The orientation \( \sigma_\omega \) of \( X \) induces an orientation \( \sigma_\Delta \) on the normal bundle \( N\Delta \) of \( \Delta \) and an orientation \( \sigma_{\Delta^i} \) on the normal bundle \( N\Delta^i \) of \( \Delta^i \). Define

\[
Z_h = Z_{k,l,h}^*(B; J, \nu), \\
Z_{h^*} = Z_{k,l+1,h^*}^*(B; J, f_{k,l+1,i}^* \nu), \\
M_l = M_{k,l}(B; J, \nu), \\
M_{l+1} = M_{k,l+1}(B; J, f_{k,l+1,i}^* \nu).
\]

Let \( (u, y) \) and \( \tilde{u} \) be as above. Fix an orientation \( \sigma \) on \( T_{ev}(u) X_{k,l} \). The differentials of the obvious maps induce a commutative square in Figure 9 with exact rows and columns. Since the dimensions of \( X \) and \( H' \) are even, the sign \( s_u \) of (6.17) is the sign of the isomorphism in the left column with respect to the orientations \( \sigma_{\Delta^i} \), \( \sigma_h \), and \( \sigma_\Delta \). Along with the relative orientation \( \sigma_{p,i} \) and \( \sigma_{h,i} \), \( \sigma \) induces an orientation \( \sigma_{(u,y)} \) on \( T_{(u,y)} (M_l \times M_h) \) (resp. \( \sigma_{(u,y)} \)) on \( T_{(u,y)} M_{h^*} \). Since the dimension of \( H' \) is even, Lemma 5.3 implies that the middle row respects the orientations. Along with the orientation \( \sigma_{\Delta^i} \) on \( N\Delta^i \) (resp. \( \sigma_{\Delta^i} \) on \( N\Delta^i \)), \( \sigma_{(u,y)} \) (resp. \( \sigma_{(u,y)} \)) induces an orientation \( \sigma'_{(u,y)} \) on \( T_{(u,y)} Z_h \) (resp. \( \sigma'_{(u,y)} \) on \( T_{(u,y)} Z_{h^*} \)), \( \sigma'_{(u,y)} \) if and only if the isomorphism in the left column is. The latter is the case if and only if \( s_u = -1 \). These two statements imply that \( s_u = s_u \).

6.3 Proof of Proposition 5.7

For \( k', l' \in \mathbb{Z}^+ \) with \( k' + 2l' \leq 2 \), we denote by \( H_{k',l}' \) the set of pairs \((J,0)\) with \( J \in \mathcal{J}_{0}^\circ \). We continue with the notation in the statement of this proposition and just above. Let \( k_1, l_1, l_i, B_i \) be as in (5.41) and

\[
M^* = M_{k,l}^*(B; J, \nu).
\]

Since \( (S, \Upsilon) \) is admissible, \( 2l_1 + k_1 \geq 3 \), and \( k_2 = 0 \). We assume that there exist \( \nu_1' \in H_{k_1,l_1}^{\omega,\phi} \) and \( \nu_2 \in H_{k_2+1,l_2}^{\omega,\phi} \) so that every stratum \( S_{i} \subset S \) admits a decomposition as in (6.2) with \( \nu_1 = f_{i}^* \nu_1' \) subject to the conditions specified below (6.3) and above Lemma 5.2.

We first assume that \( l_2 \neq 0 \) and take \( i^* \in L^*_S(S) \) to be the smallest element of this set. By this assumption, the image of \( S \) under the forgetful morphism \( f_{k,l} \) is contained in a codimension 1 stratum \( \mathcal{M}_{k,l}^\phi \) of \( \mathcal{M}_{k,l} \). By Lemma 5.5, we can assume that the orientation \( \sigma_S \) of \( NS \) used to define the relative orientation \( \partial \sigma_{p,i} = \partial \sigma_S \sigma_{p,i} \) of (5.22) is \( \sigma_S^+ \) in the notation of Lemma 6.2.

For \( u \in S \), let

\[
\begin{align*}
\text{if } u_1 = M_1 & = M_{k_1+1,l_1}(B_1; J, \nu_1), \\
\text{if } u_2 = M_2 & = M_{k_2+1,l_2}(B_2; J, \nu_2), \\
D_{\phi}^u & = D_{j,v,\phi}^u, \\
D_{\phi}^u & = D_{j,\nu_1,\phi}^u = D_{j,\nu_2,\phi}^u, \\
D_{\phi}^u & = D_{j,\nu_1,\nu_2}^u,
\end{align*}
\]

be as above Lemma 6.2 and in Section 4.2. We denote by

\[
C = f_{k,l}(u) \in S^\vee \subset \mathcal{M} \subset \mathcal{M}_{k,l}^\phi, \\
C_1 = f_{k+1,l_1}(u_1) \in M_1 = M_{k+1,l_1}^\phi, \\
C_1' = f_{k+1,l_1}(u_1') \in M_1' = M_{k+1,l_1}^\phi, \\
C_2 = f_{k+2,l_2}(u_2) \in M_2 = M_{k+2,l_2}^\phi.
\]
The exact sequence
\[ 0 \rightarrow T_u S \rightarrow T_{u_1} M_1 \otimes T_{u_2} M_2 \rightarrow T_{u_{(nd)}} X^\phi \rightarrow 0, \quad (\xi_1, \xi_2) \rightarrow \xi_2 - \xi_1, \]
(6.19)
of vector spaces determines an isomorphism
\[ \lambda_u(S) \otimes \lambda(T_{u_{(nd)}} X^\phi) \cong \lambda_{u_1}(M_1) \otimes \lambda_{u_2}(M_2). \]
(6.20)
Since \( \epsilon_{l^*}(S) \in 2\mathbb{Z} \), the Pin\(^{-}\)-structure \( p \) on \( X^\phi \) determines homotopy classes \( a_{p,l^*} \) and \( a_{p,l^*} \) of isomorphisms
\[ \lambda_{u_1}'(M_1') \rightarrow \lambda_{u_1}'(X) \otimes \lambda_{u_1}^C(X) \quad \text{and} \quad \lambda_{u_2}(M_2) \rightarrow \lambda_{u_2}(X) \otimes \lambda_{u_2}^C(X), \]
(6.21)
respectively; see Lemma 5.1. Combining the first homotopy class of isomorphisms above with the first \( S^1 \)-fibration in (6.3) and the orientation \( a_{\text{nd}} \) on its vertical tangent bundle \( T_{u_1} M_1 = T_{\text{nd}} S^1 \), we obtain a homotopy class \( \tilde{a}_{p,l^*} = a_{\text{nd}} \circ a_{p,l^*} \) of isomorphisms
\[ \lambda_{u_1}(M_1) \cong \lambda_{u_1}'(M_1') \otimes T_{\text{nd}} S^1 \cong \lambda_{u_1}'(X) \otimes \lambda_{u_1}^C(X). \]
(6.22)
Along with (6.20) and the second homotopy class of isomorphisms in (6.21), it determines a homotopy class of isomorphisms
\[ \lambda_u(S) \rightarrow \lambda^R_u(X) \otimes \lambda^C_u(X) \]
determined by (6.23). The next lemma is deduced from Lemmas 4.3 and 6.2 at the end of this section.

**Lemma 6.4** The orientations \( \tilde{a}_{p,l^*} \) and \( a_{p,l^*} \) of \( \lambda(\text{ev}|_S) \) are opposite.

We take \( h_1 \) and \( h_2 \) to be the components of \( h \) as in the proof of Proposition 5.2 and
\[ p_1 \in X_{k_1,l_1 - l_1^*} \quad \text{and} \quad p_2 \in X_{k_2,l_2 - l_2^*} \]
to be the components of \( p \in X_{k,l - l^*} \) defined analogously. Let
\[ Z_1 = Z_{k_1+1,l_1,h_1}^*(B_1; J, \nu_1) \cap (M_1 \times M_{h_1}), \quad Z_1' = Z_{k_1+1,l_1,h_1}^*(B_1'; J, \nu_1') \cap (M_1' \times M_{h_1}), \]
\[ Z_2 = Z_{k_2+1,l_2,h_2}^*(B_2; J, \nu_2) \cap (M_2 \times M_{h_2}). \]
The first forgetful morphism in (6.3) induces a fibration \( f_{Z_1} \) so that the diagram
\[ \begin{array}{ccc}
Z_1 & \xrightarrow{f_{Z_1}} & Z_1' \\
\pi_{Z_1} & & \pi_{Z_1'} \\
M_1 & \xrightarrow{f_{\text{nd}}} & M_1'
\end{array} \]
commutes. Since \( \pi_Z \) induces an isomorphism between the vertical tangent bundles \( TZ_1^v \) of \( f_Z \) and \( T\mathcal{M}_1^v \) of \( f_{nd} \), it pulls back \( \sigma_{nd}^R \) to an orientation \( \sigma_{Z_1}^v \) on the fibers of \( f_Z \). The relative orientations \( \sigma_{nd}^S \cdot \sigma_{P,l_2}^v \), and \( \sigma_{P,l_2}^v \) on

\[
ev : \mathcal{M}_1 \to X_{k_1,l_1}, \quad \ev' : \mathcal{M}_1' \to X_{k_1,l_1}, \quad \text{and} \quad \ev : \mathcal{M}_2 \to X_{k_2+1,l_2},
\]

respectively, the orientations \( \sigma_{h_1} \) of \( H_1 \), and the symplectic orientation \( \omega_\cdot \) on \( X \) determine relative orientations \( \tilde{\sigma}_{p_1,h_1} \), \( \sigma_{p_1,h_1} \), and \( \sigma_{p_2,h_2} \) of

\[
ev_{h_1} : Z_1 \to X_{k_1,l_1-t_1^*}, \quad \ev'_{h_1} : Z_1' \to X_{k_1,l_1-t_1^*}, \quad \text{and} \quad \ev_{h_2} : Z_2 \to X_{k_2+1,l_2-t_2^*},
\]

respectively. Since the dimensions of \( X \) and \( H_1 \) are even,

\[
\tilde{\sigma}_{p_1,h_1} = \omega_{Z_1} \cdot \sigma_{p_1,h_1} = (\pi_Z \sigma_{nd}^R) \sigma_{p_1,h_1}.
\]

(6.24)

For \( \tilde{u} \in \mathcal{S}_h^s \), we denote by

\[
\tilde{u}_1 \in Z_1, \quad \tilde{u}_1' \in Z_1', \quad \tilde{u}_2 \in Z_2
\]

the images of \( \tilde{u} \) under the projections induced by the embedding (6.2), the first forgetful morphism in (6.3), and the decomposition

\[
M_h \cong M_{h_1} \times M_{h_2}.
\]

The exact sequence

\[
0 \to T\tilde{u} \mathcal{S}_h^s \to T\tilde{u}_1 Z_1 \oplus T\tilde{u}_2 Z_2 \to T(u_{nd}) X^\phi \to 0, \quad (\xi_1, \xi_2) \to (\xi_2 - \xi_1),
\]

of vector spaces determines an isomorphism

\[
\lambda_{\tilde{u}} (\mathcal{S}_h^s) \otimes \lambda (T(u_{nd}) X^\phi) \cong \lambda_{\tilde{u}_1} (Z_1) \otimes \lambda_{\tilde{u}_2} (Z_2).
\]

Along with the relative orientations \( \tilde{\sigma}_{p_1,h_1} \) and \( \sigma_{p_2,h_2} \) above, this isomorphism determines a homotopy class of isomorphisms

\[
\lambda_{\tilde{u}} (\mathcal{S}_h^s) \otimes \lambda (T(u_{nd}) X^\phi) \approx \lambda_{\tilde{u}_1} (Z_1) \otimes \lambda_{\tilde{u}_2} (Z_2).
\]

We denote the associated relative orientation of (6.22) by

\[
\sigma_{S,h} = (\tilde{\sigma}_{p_1,h_1})_{nd} \cdot \sigma_{p_2,h_2}.
\]

(6.25)

Since the dimensions of \( X \) and \( H_1 \) are even, Lemma (6.4) implies that

\[
|\mathcal{S}_h^s|_{\sigma_{S,h}, \sigma_T} = -|\mathcal{S}_h^s|_{\sigma_{p_1,h}, \sigma_T}.
\]

(6.26)

If \( S \) and \( \Upsilon \) satisfy \( \text{[STY]} \) above Proposition (5.7) with \( i \in [k] \) as in \( \text{[STY]} \) let

\[
k' = k' - 1, \quad l' = l', \quad k'' = 1, \quad l'' = 0, \quad \Upsilon_1 = \Upsilon, \quad nd = i.
\]

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If $S$ and $\Upsilon$ satisfy $[S2\Upsilon]$ and $S \subset M_{k', l'}$ in $[S2\Upsilon]$ let

$$k'_1 = k_1(S), \quad l'_1 = l_1(S), \quad k'_2 = k_2(S), \quad l'_2 = l_2(S)$$

and denote by

$$\pi_1: S \approx M_{k'_1 + 1, l'_1} \times M_{k'_2 + 1, l'_2} \longrightarrow M_{k'_1 + 1, l'_1}$$

the projection to the first component in the second identification in $[17]$. In this case, $\Upsilon \cap S \approx \Upsilon_1 \times M_{k'_1 + 1, l'_1} \subset M_{k'_1 + 1, l'_1} \times M_{k'_1 + 1, l'_1}$ for some $\Upsilon_1 \subset M_{k'_1 + 1, l'_1}$:nd. The co-orientation $\sigma^\ast_{\Upsilon_1 \cap S}$ on $\Upsilon \cap S$ in $S$ induced by $\sigma^\ast_{\Upsilon}$ is the pullback by $\pi_1$ of a co-orientation $\sigma^\ast_{\Upsilon_1}$ on $\Upsilon_1$ in $M_{k'_1 + 1, l'_1}$. Let

$$f_{k'_1 + 1, l'_1} = \pi_1 \circ f_{k', l'}: S_h \longrightarrow S \longrightarrow M_{k'_1 + 1, l'_1}.$$ 

In both cases,

$$\dim \Upsilon_1 = \dim \Upsilon + 1 - k'_2 - 2l'_2$$

(6.27)

and the forgetful morphism $f_{k'_1 + 1, l'_1}$ factors as

$$S_h \longrightarrow Z_1 \times Z_2 \longrightarrow Z_1 \longrightarrow M_{k'_1 + 1, l'_1}.$$ 

We define

$$f_M = f_{k'_1 + 1, l'_1}: M_{k'_1 + 1, l'_1} \longrightarrow M_{k'_1, l'_1}.$$ 

If $S$ and $\Upsilon$ satisfy $[S2\Upsilon] [2]$ and $[3]$ in Lemma 3.3 give

$$|S^\ast_{h, p, \Upsilon}|_{\sigma_{S, h}, \sigma^\ast_{\Upsilon}} = -|M_{(ev_{S, h}, f_{k', l'})}, f_{p, \Upsilon}|_{\sigma_{S, h}, \pi^\ast \sigma^\ast_{\Upsilon}} = -(-1)^{k'_2} M_{(ev_{S, h}, f_{k'_1 + 1, l'_1}), f_{p, \Upsilon}}_{\sigma_{S, h}, \sigma^\ast_{\Upsilon}};$$

(6.28)

the signed fiber products in the second and third expressions above are taken with respect to $X_{k, l} \times S$ and $X_{k, l} \times \overline{M}_{k'_1, l'_1}$, respectively. The first and last expressions in (6.28) are the same if $S$ and $\Upsilon$ satisfy $[S1\Upsilon]$. By (6.28) and Lemma 3.4

$$|M_{(ev_{S, h}, f_{k'_1 + 1, l'_1}), f_{p, \Upsilon}}|_{\sigma_{S, h}, \sigma^\ast_{\Upsilon}} = |M_{(ev_{h_1}, f_{k'_1 + 1, l'_1}), f_{p, \Upsilon}}|_{\sigma_{h_1}, \sigma^\ast_{\Upsilon}} \deg(ev_{h_2}, \sigma_{p, h});$$

(6.29)

By Lemma 3.3 (1),

$$|M_{(ev_{h_1}, f_{k'_1 + 1, l'_1}), f_{p, \Upsilon}}|_{\sigma_{h_1}, \sigma^\ast_{\Upsilon}} = (-1)^{\dim \Upsilon} \deg(ev_{h_1}, f_{k'_1 + 1, l'_1}^{\ast \Upsilon}(\Upsilon_1); f_{k'_1 + 1, l'_1}^{\ast \Upsilon}(\Upsilon_1), \sigma_{h_1}^{\ast \Upsilon}) .$$

By the first identity in (6.24) and 3.7,

$$\deg(ev_{h_1} f_{k'_1 + 1, l'_1}^{\ast \Upsilon}(\Upsilon_1), (f_{k'_1 + 1, l'_1}^{\ast \Upsilon}(\Upsilon_1), \sigma_{h_1}^{\ast \Upsilon}) = \deg(f_{p, \Upsilon})_{\sigma_{h_1}, \sigma^\ast_{\Upsilon}}(\Upsilon_1), (f_{k'_1 + 1, l'_1}^{\ast \Upsilon}(\Upsilon_1), \sigma_{h_1}^{\ast \Upsilon}) \deg(ev'_{h_1}, \sigma_{p, h}).$$

By the second identity in (6.24) and Lemma 3.2,

$$s_{\Upsilon}(f_{k'_1 + 1, l'_1}^{\ast \Upsilon}(\Upsilon_1), \sigma_{\Upsilon}^{\ast \Upsilon}) = s_{\Upsilon}(f_{k'_1 + 1, l'_1}^{\ast \Upsilon}(\Upsilon_1), \pi^\ast \sigma_{\Upsilon}^{\ast \Upsilon}, \sigma_{\Upsilon}^{\ast \Upsilon}) s_{\Upsilon}(f_{k'_1 + 1, l'_1}^{\ast \Upsilon}(\Upsilon_1), \sigma_{\Upsilon}^{\ast \Upsilon}) s_{\Upsilon}(f_{k'_1 + 1, l'_1}^{\ast \Upsilon}(\Upsilon_1), \sigma_{\Upsilon}^{\ast \Upsilon}).$$
for a generic $\tilde{u} \in \mathcal{f}_{k'_{1}+1,l'_1}^{-1}(\mathcal{Y}_1)$.

Combining the last three equations with (6.27), we obtain

$$|M_{(ev_{h_1}; l_{1}' = k'_1)}|_{\mathcal{T}_1} \mathcal{F}_{\mathcal{O}_{h_1}; \mathcal{O}_{l_1}} = -(1)^{\dim \mathcal{V} + k'_1} \deg \mathcal{F}_{\mathcal{O}_{h_1}; \mathcal{O}_{l_1}} \deg (ev_{h_1}'_1, \mathcal{O}_{h_1})$$

Along with (6.28) and (6.29), this gives

$$|\mathcal{S}_{h_1; l_1}^*|_{\mathcal{T}_1} \mathcal{F}_{\mathcal{O}_{h_1}; \mathcal{O}_{l_1}} = -(1)^{\dim \mathcal{V} + k'_1} \deg \mathcal{F}_{\mathcal{O}_{h_1}; \mathcal{O}_{l_1}} \deg (ev_{h_1}'_1, \mathcal{O}_{h_1}) \deg (ev_{h_2}, \mathcal{O}_{h_2}).$$

Suppose $l'_2 = 0$. Choose another codimension 2 pseudocycle $h': H' \to X$ in general position with $h' \cdot B_2 \neq 0$. Let

$$\mathcal{S}_{h'}^{*} \subset Z_{\mathcal{S}_{h', l'_1}^*}(B; j_{k', l'_1 + 1})$$

be the codimension 1 stratum so that $j_{k', l'_1 + 1}(\mathcal{S}_{h'}^{*}) = \mathcal{S}_{h}^{*}$ and the irreducible component $\mathbb{P}^1_2$ of the maps in this stratum carries the marked points $z_{i_2+1}$. The vertical tangent bundle of the projection

$$j_{k', l'_1 + 1}: Z_{\mathcal{S}_{h', l'_1}^*}(B; j_{k', l'_1 + 1}) \to Z_{\mathcal{S}_{h}^{*}}(B; j)$$

is oriented by the position of $z_{i_2+1}$; we denote this orientation by $\mathcal{O}_{\mathcal{S}_{h}}^{+}$. By the $l'_2 \neq 0$ case above,

$$|\mathcal{S}_{h'}^{*} \cdot \mathcal{T}_1 \mathcal{F}_{\mathcal{O}_{h'}} = -(1)^{\dim \mathcal{V}} \deg \mathcal{F}_{\mathcal{O}_{h'}} \deg (ev_{h_1}'_1, \mathcal{O}_{h_1}) \deg (h' \cdot X B_2) N_{B_2, l_2}.$$

Since the dimensions of $X$ and $H'$ are even,

$$\partial \mathcal{O}_{h'} = \partial (\mathcal{O}_{h'} \mathcal{O}_{h'}) = \mathcal{O}_{h'} \partial \mathcal{O}_{h'}.$$

By the reasoning in the proof of (6.14), this implies that

$$|\mathcal{S}_{h'}^{*} \cdot \mathcal{T}_1 \mathcal{F}_{\mathcal{O}_{h'}} = (h' \cdot X B_2) N_{B_2, l_2}.$$

Combining (6.30) and (6.31) with (6.14), we again obtain (5.26).

**Proof of Lemma 6.4.** Fix orientations of $T_{u(\text{ind})}X^\phi$ and $T_{u(x_i)}X^\phi$ for all $i \in [k]$. We can then view all relevant relative orientations as orientations in the usual sense. Let $\mathcal{S}_{\mathcal{S}} \subset \mathcal{S}$ be the substratum containing $\mathcal{S}$. The two spaces in the bottom row are oriented by $\mathcal{O}_{\mathcal{S}}^{+}$ with the isomorphism between them being orientation-preserving. This orientation and the orientations $\mathcal{O}_{p}^D$, $\mathcal{O}_{p,l}$, and $\mathcal{O}_{p,l,i}$ determine the limiting orientations $\mathcal{O}_{p}^{D, +}$ on ker $D^l_p$, $\mathcal{O}_{p,l,i}$ on $T_{u(\mathcal{M})}$, and $\mathcal{O}_{p,l,i}$ on $T_{C\mathcal{M}}$; by (6.7), the middle row respects these orientations. The middle (resp. right) column respects the orientations $\partial \mathcal{O}_{p,l}$ on $T_{u(\mathcal{S})}$, $\mathcal{O}_{p,l,i}$
Figure 10: Commutative squares of vector spaces with exact rows and columns for the proof of Lemma 6.4
$T_u \mathcal{M}^\star$, and $\phi^+_{k,l;i}^\star$ on $N_u S$ (resp. $\partial o_{k,l;i}^\star$ on $T_u S^\vee$, $\phi^+_{k,l;i}^\star$ on $T_u M$, and $\phi^+_{k,l;i}^\star$ on $N_u S^\vee$). Lemma 6.3 then implies that the top row in the first exact square of Figure 10 respects the orientations $\phi^+_{p;i}^\star$ on $\ker D_{u}^\phi$, $\partial o_{p;i}^\star$ on $T_u S$, and $\partial o_{k,l;i}^\star$ on $T_u S^\vee$.

The differentials of forgetful morphisms induce the second exact square of Figure 10. The two spaces in the top row are oriented by $\phi^\star_{nd}$ as in Section 4.2 with the isomorphism between them being orientation-preserving. The first real marked point of $u_1$ is the node, while the second one (if $k_1 \neq 0$) is the next marked point on the fixed locus $S^\perp_1 \subset \mathbb{P}^1_1$ in the counterclockwise direction with respect to $z_1^\perp$. By (6.11) and (6.14) in Lemma 4.1, the right column thus does not respect the orientations $\phi^\star_{nd}$ on $T_{nd} S^\perp_1$, $\phi_{k_1+1,l_1;i}^\star$ on $T_{C_1} M_1$, and $\phi_{k_1,l;i}^\star$ on $T_{C_1} M_1'$ because

$$k_1 + \dim M_1' = 2k_1 + 2l_1 - 3 \neq 2 \mathbb{Z}.$$ 

By (6.7), the bottom row respects the orientations $\phi^D_{p;i}^\star$ on $\ker D_{u_1}^\phi$, $\phi_{p;i}^\star$ on $T_{u_1} M_1'$, and $\phi_{k_1,l;i}^\star$ on $T_{C_1} M_1$. By (6.22), the middle column respects the orientations $\phi^\star_{nd}$ on $T_{nd} S^\perp_1$, $\partial o_{p;i}^\star$ on $T_u S_1$, and $\phi_{k_1+1,l_1;i}^\star$ on $T_{C_1} M_1$ if and only if

$$1 + \dim \ker D_{u_1}^\phi = 1 + 2 + \langle c_1 (X, \omega), B_1 \rangle$$

is even. Since $\epsilon_i(S) = 2$, we conclude that the middle row in the second exact square of Figure 10 respects the orientations $\phi^D_{p;i}^\star$ on $\ker D_{u_2}^\phi$, $\phi_{p;i}^\star$ on $T_{u_2} S_1$, and $\phi_{k_1+1,l_1;i}^\star$ on $T_{C_1} M_1$ if and only if $k_1$ is even.

The short exact sequences (6.4) and (6.19) and the differential of the forgetful morphism $f_{k,l}$ induce the third exact square of Figure 10. By (6.7), the short exact sequence of the middle summands in the middle row respects the orientations $\phi^D_{p;i}^\star$ on $\ker D_{u_2}^\phi$, $\phi_{p;i}^\star$ on $T_{u_2} M_2$, and $\phi_{k_2,l;i}^\star$ on $T_{C_2} M_2$. Along with the conclusion of the previous paragraph and Lemma 6.3 this implies that the middle row respects the orientations $\phi^D_{p;i}^\star \oplus \phi^D_{p;i}^\star$, $\phi_{p;i}^\star \oplus \phi_{p;i}^\star$, and $\phi_{k_1+1,l_1;i}^\star \oplus \phi_{k_2+1,l_2;i}^\star$ if and only if

$$k_1 + \dim \ker D_{u_2}^\phi (\dim M_1) = k_1 + (2 + \langle c_1 (X, \omega), B_2 \rangle)(k_1 + 2l_1 - 2)$$

is even. Since $\epsilon_i(S) = 2$, this is the case if and only if $k+1+k_2 \in 2 \mathbb{Z}$. By Lemma 6.2, the left column respects the orientations $\phi^D_{p;i}^\star$, $\phi^D_{p;i}^\star \oplus \phi^D_{p;i}^\star$, and the chosen orientation $\phi^\star_{nd}$ on $T_{u(nd)} X^\phi$ if and only if $\delta^\phi_{p;i} (S) \cong \delta^+ (S_2) \mod 2$. Since $\epsilon_i(S) = 2$, this is the case if and only if the number $k_2 + k_1 k_2 + \delta^+ (S_2)$ is even. By Lemma 4.3, the non-trivial isomorphism in the right column respects the orientations $\phi_{k_1+1,l_1;i}^\star \oplus \phi_{k_2+1,l_2;i}^\star$ if and only if $\delta^\phi_{p;i} (S) \cong k_1 + 2l_1 - 2 \mod 2$. By (6.23), the middle column respects the orientations $\phi^D_{p;i}^\star$, $\phi_{p;i}^\star \oplus \phi_{p;i}^\star$, and $\phi^\star_{nd}$. Combining these statements with Lemma 6.3 we conclude that the top row does not respect the orientations $\phi^D_{p;i}^\star$, $\phi^S_{p;i}^\star$, and $\partial o_{k,l;i}^\star$ because

$$(k_1 + k_2)(k_1 + k_1 k_2 + \delta^+ (S_2)) + (k_1 + \delta^+ (S_2) + \dim S^\vee)(\dim X^\phi) = 1 \mod 2.$$ 

Comparing this conclusion with the conclusion concerning the top row in the first exact square of Figure 10 above, we obtain the claim. □
6.4 Proof of Proposition 5.3

We denote by \( \mathcal{H}_{0,m}^{\omega} \) the space of pairs \((J, \nu')\) consisting of \( J \in \mathcal{J}_\omega \) and a Ruan-Tian perturbation \( \nu' \) of the \( \overline{\partial}_J \)-equation if \( m \geq 3 \) and take \( \mathcal{H}_{0,2}^{\omega} \) to be the set of pairs \((J, 0)\) with \( J \in \mathcal{J}_\omega \). For \( B' \in H_2(X) \) and \( \nu' \in \mathcal{H}_{0,m}^{\omega} \), we denote by \( \mathcal{M}_{m}^{\nu}(B'; J, \nu') \) the moduli space of (complex) genus 0 degree \( B' \) \((J, \nu')\)-holomorphic maps from smooth domains with \( m \) marked points and by

\[
ev_i : \mathcal{M}_{m}^{\nu}(B'; J, \nu') \to X, \quad i \in [m],
\]

the evaluation maps at the marked points. For \( I \subset [m] \), let \( \sigma_I : \nu' \) be the orientation of \( \mathcal{M}_{m}^{\nu}(B'; J, \nu') \) obtained by twisting the standard complex orientation by \((-1)^{|I|}\). Define

\[
\Theta_i^J : X \to X, \quad \Theta_i^J = \begin{cases} id_X, & \text{if } i \notin I; \\ \phi, & \text{if } i \in I; \end{cases}
\]

\[
ev^J : \mathcal{M}_{m}^{\nu}(B; J, \nu) \to X^m, \quad \ev^J(u) = ((\Theta_i^J(\ev_i(u)))_{i \in [m]}).
\]

We continue with the notation in the statement of Proposition 5.3 and just above. The co-orientation \( \sigma^c_\Gamma \) of \( \Gamma \) in \( \mathcal{M}_{k,l}^c(B; J, \nu) \) and the relative orientation \( \sigma_{p;l}^* \) of Lemma 5.1 induce a relative orientation \( (\sigma^c_{k,l}(\sigma^c_\Gamma)\sigma_{p;l}^*)^\Gamma \) of the restriction

\[
ev_\Gamma : \mathcal{M}_{k,l}^c(B; J, \nu) \to X_{k,l}
\]

of \((6.11)\). Fix a stratum \( S \subset \mathcal{M}_{k,l}^c(B; J, \nu) \). Let \( B_\mathbb{R} \) be the degree of the restriction of the maps in \( S \) to the real component \( \mathbb{P}_1^0 \) of the domain and \( B_\mathbb{C} \) be the degree of their restrictions to the component \( \mathbb{P}_1^+ \) of the domain carrying the marked point \( z^+_i \). Denote by \( L_0, L_\mathbb{C} \subset [l] \) the subsets indexing the conjugate pairs of marked points carried by \( \mathbb{P}_1^0 \) and \( \mathbb{P}_1^+ \), respectively. Define

\[
L_0^c = L_0^c(\Gamma), \quad L_\mathbb{C}^c = L_\mathbb{C}^c(\Gamma), \quad l_0 = |L_0|, \quad l_\mathbb{C} = |L_\mathbb{C}|, \quad l_0^c = l_0^c(\Gamma), \quad l_\mathbb{C}^c = l_\mathbb{C} - l_0^c.
\]

By \((6.19)\)

\[
(l_0 - l_0^c) + (l_\mathbb{C} - l_\mathbb{C}^c) = l - l^*;
\]

\[
6(l_\omega(B_\mathbb{R}) - (k + 2(l_0 - l_0^c))) + 2(l_\omega(B_\mathbb{C}) - (l_\mathbb{C} - l_\mathbb{C}^c)) = 0. \quad (6.32)
\]

Let \( L_\mathbb{C}^c \subset L_\mathbb{C}^c \) be the subset indexing the conjugate pairs of marked points \( (z^+_i, z^-_i) \) of curves in \( \Gamma \) so that \( z^-_i \) lies on \( \mathbb{P}_1^+ \).

For a good choice of \( \nu_\mathbb{R} \), there exist \( \nu_{\mathbb{C}} \in \mathcal{H}_{k,l_0+1}^{\omega,\phi} \), \( \nu_{\mathbb{C}} \in \mathcal{H}_{l_\mathbb{C}+1}^{\omega} \), and a natural embedding

\[
\iota_\Delta : S \hookrightarrow \mathcal{M}_{k,l_0+1} \times \mathcal{M}_{l_\mathbb{C}} \equiv \mathcal{M}_{k,l_0+1}(B; J, \nu_{\mathbb{R}}) \times \mathcal{M}_{l_\mathbb{C}+1}(B; J, \nu_{\mathbb{C}}) \quad (6.33)
\]

satisfying \((\sigma_\Gamma 1) - (\sigma_\Gamma 3)\) in Section 4.3. If \( B_\mathbb{R} \neq 0 \), we also assume that there exists \( \nu'_{\mathbb{R}} \in \mathcal{H}_{k,l_0}^{\omega,\phi} \) so that the forgetful morphism

\[
f_{\mathbb{R}d} : \mathcal{M}_{k,l_0}^{\nu_{\mathbb{R}}} \to \mathcal{M}_{k,l_0}^{\nu_{\mathbb{R}}}(B; J, \nu'_{\mathbb{R}}) \quad (6.34)
\]

dropping the conjugate pair corresponding to the node \( nd \) (i.e. the first one) is defined. If \( B_\mathbb{C} \neq 0 \), we similarly assume that there exists \( \nu'_{\mathbb{C}} \in \mathcal{H}_{l_\mathbb{C}}^{\omega} \) so that the analogous forgetful morphism

\[
f_{\mathbb{C}d} : \mathcal{M}_{l_\mathbb{C}}^{\nu_{\mathbb{C}}} \to \mathcal{M}_{l_\mathbb{C}}^{\nu_{\mathbb{C}}}(B; J, \nu'_{\mathbb{C}}) \quad (6.35)
\]
is defined. Denote by $I \subset \{l_{C} + 1\}$ (resp. $I' \subset \{l_{C}\}$) the subset indexing the marked points of a map in $\mathcal{M}_{C}$ (resp. $\mathcal{M}_{C}'$) corresponding to the marked points on the left-hand side of (6.33) indexed by $I'$ under (6.33) (resp. (6.33) and (6.35)).

For an element $u \in S$, we denote by $u_{0} \in \mathcal{M}_{R}$ and $u_{+} \in \mathcal{M}_{C}$ the pair of maps corresponding to $u$ via (6.33). Let $u_{0}' \in \mathcal{M}_{R}'$ and $u_{+}' \in \mathcal{M}_{C}'$ be the image of $u_{0}$ under (6.34) if $B_{R} \neq 0$ and the image of $u_{+}$ under (6.35) if $B_{C} \neq 0$, respectively. The exact sequence

$$0 \rightarrow T_{u}S \rightarrow T_{u_{0}}\mathcal{M}_{R} \oplus T_{u_{+}}\mathcal{M}_{C} \rightarrow T_{u_{(nd)}}X \rightarrow 0,$$

of vector spaces determines an isomorphism

$$\lambda_{u}(S) \otimes \lambda(T_{u_{(nd)}}X) \approx \lambda_{u_{0}}(\mathcal{M}_{R}) \otimes \lambda_{u_{+}}(\mathcal{M}_{C}).$$

The Pin$^{-}$-structure $p$ on $X^{\phi}$ determines a homotopy class $\sigma_{p,t_{0}^{*}}$ of isomorphisms

$$\lambda_{u_{0}}(\mathcal{M}_{R}) \rightarrow \lambda_{u_{0}}^{R}(X) \otimes \lambda_{u_{0}}^{C}(X);$$

see Lemma 5.1. Combining the above two homotopy classes of isomorphisms with the complex orientations of $\lambda(T_{u_{(nd)}}X)$ and $\lambda_{u_{+}}^{C}(X)$ and the orientation $\sigma_{C;I}$ of $\lambda_{u_{+}}(\mathcal{M}_{C})$, we obtain a homotopy class $\sigma_{p,t_{0}^{*};u}^{\Gamma}$ of isomorphisms

$$\lambda_{u}(S) \rightarrow \lambda_{u}^{R}(X) \otimes \lambda_{u}^{C}(X).$$

Lemma 6.5 The relative orientations $(t_{k,l}^{p}, t_{\Gamma}^{C})\sigma_{p,t_{0}^{*}}$ and $\sigma_{p,t_{0}^{*};u}^{\Gamma}$ of $\lambda(\text{ev}_{\Gamma})$ are the same.

Proof. The proof of Lemma 6.3 readily adapts using Lemma 4.2. The relevant analogue of Lemma 6.2 follows readily from [5] Cor. 7.3. In light of Lemma 4.2, the $(k',l') = (0,3)$ case of Lemma 6.5 also follows from Lemma 5.2 and Remark 5.3 in [8]; the proof in [8] extends to arbitrary $(k',l')$. □

We denote by

$$h_{R}: M_{h_{R}} \rightarrow X_{t_{0}^{k}}, \quad h_{C}: M_{h_{C}} \rightarrow X_{t_{C}}, \quad p_{R} \in X_{k,l_{0}-t_{0}^{*}}, \quad \text{and} \quad p_{C} \in X_{lc-l_{C}^{*}}$$

the components of $h$ and $p$ corresponding to the marked points on $\mathbb{P}_{0}$ and $\mathbb{P}_{1}$ for the maps $u \in S$. Let

$$S_{h} = S_{\text{ev} \times h} M_{h}, \quad Z_{R} = (\mathcal{M}_{R})_{\text{ev} \times h_{R}} M_{h_{R}}, \quad Z_{C} = (\mathcal{M}_{C})_{\text{ev} \times h_{C}} M_{h_{C}}$$

be the corresponding spaces cut out by $h$ and

$$\text{ev}_{C}: Z_{C} \rightarrow X_{lc-l_{C}^{*}}$$

be the evaluation map induced by $\text{ev}$.

The relative orientations $\sigma_{p,t_{0}^{*}+1}$ and $\sigma_{C;I}$ of

$$\text{ev}: \mathcal{M}_{R} \rightarrow X_{k,l_{0}+1} \quad \text{and} \quad \text{ev}^{I}: \mathcal{M}_{C} \rightarrow X_{lc+1},$$

respectively, the orientations $\sigma_{h_{l}}$ of $H_{l}$, and the symplectic orientation $\sigma_{w}$ on $X$ determine relative orientations $\sigma_{p,h_{l}}$ and $\sigma_{h_{l};I}$ of the induced maps

$$\text{ev}_{R}: Z_{R} \rightarrow X_{k,l_{0}-t_{0}^{*}} \quad \text{and} \quad \text{ev}_{C}: Z_{C} \rightarrow X_{lc-l_{C}^{*}}$$

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induced by ev. If \( B_\mathbb{C} \neq 0 \) (resp. \( B_\mathbb{R} \neq 0 \), we also define \( \bar{\mathbf{u}}_0' \in \mathbb{Z}'_\mathbb{R} \) (resp. \( \bar{\mathbf{u}}_+ \in \mathbb{Z}'_\mathbb{C} \)) to be the image of \( \bar{\mathbf{u}}_0 \) (resp. \( \bar{\mathbf{u}}_+ \)) under the forgetful morphism \( \bar{f}_\mathbb{R} : \mathbb{Z}'_\mathbb{R} \rightarrow \mathbb{Z}'_\mathbb{R} \) (resp. \( \bar{f}_\mathbb{C} : \mathbb{Z}'_\mathbb{C} \rightarrow \mathbb{Z}'_\mathbb{C} \))

dropping the conjugate pair corresponding to the nodes. The exact sequence

\[
0 \rightarrow T\bar{\mathbf{u}}\mathcal{S}_\mathbb{R} \rightarrow T\bar{\mathbf{u}}_0 \mathbb{Z} \mathbb{R} \oplus T\bar{\mathbf{u}}_+ \mathbb{Z} \mathbb{C} \rightarrow T_{u(\text{nd})}X \rightarrow 0
\]

(6.36)

determines an isomorphism

\[
\lambda_{\bar{\mathbf{u}}}(\mathcal{S}_\mathbb{R}) \otimes \lambda_{T_{u(\text{nd})}X} \approx \lambda_{\bar{\mathbf{u}}_0}(\mathbb{Z}_\mathbb{R}) \otimes \lambda_{\bar{\mathbf{u}}_+}(\mathbb{Z}_\mathbb{C}).
\]

Along with \( o_{p;h_\mathbb{R}} \) and \( o_{h_\mathbb{C};I} \), these isomorphisms determine a relative orientation

\[
o_{p;h}^\Gamma = (o_{p;h_\mathbb{R}})_{\text{nd}} \otimes o_{h_\mathbb{C};I}
\]

(6.37)

of the restriction of \( \mathcal{S} \) to \( S \). Since the dimensions of \( H_i \) and \( X \) are even, Lemma [6.35] implies that

\[
|\text{ev}_{\Gamma, h_\mathbb{R}}(p) \cap \mathcal{S}_h|_{\text{nd}} = |\text{ev}_{\Gamma, h_\mathbb{C}}(p) \cap \mathcal{S}_h|_{\text{nd}}^\pm
\]

(6.38)

We denote by \( o^\Gamma_\mathcal{S} \) the co-orientation of \( S \) in \( \mathbb{Z}_\mathbb{R} \times \mathbb{Z}_\mathbb{C} \) determined by the symplectic orientation \( o_\omega \) of \( X \) via (6.36).

If \( B_\mathbb{R} \neq 0 \) (resp. \( B_\mathbb{C} \neq 0 \), we also define

\[
\mathbb{Z}'_\mathbb{R} = (\mathbb{M}'_\mathbb{R})_{\text{ev}} \times h_\mathbb{R} M_{h_\mathbb{R}} \quad (\text{resp. } \mathbb{Z}'_\mathbb{C} = (\mathbb{M}'_\mathbb{C})_{\text{ev}} \times h_\mathbb{C} M_{h_\mathbb{C}}).
\]

If \( B_\mathbb{R} \neq 0 \) and \( l_0^* \neq 0 \), \( o_{p;I_0^*} \) determines a relative orientation \( o_{p;h_\mathbb{R}}' \) of the evaluation map

\[
\text{ev}'_\mathbb{R} : \mathbb{Z}'_\mathbb{R} \rightarrow X_{k, l_0 - l_0^*} \quad (6.39)
\]

induced by ev. If \( B_\mathbb{C} \neq 0 \), \( o_{h;I'} \) determines a relative orientation \( o_{h_\mathbb{C};I'} \) of the evaluation map

\[
\text{ev}'_\mathbb{C} : \mathbb{Z}'_\mathbb{C} \rightarrow X_{l_\mathbb{C} - l_\mathbb{C}} \quad (6.40)
\]

induced by \( \text{ev}'_C \). Since the diagrams

\[
\begin{array}{ccc}
\mathbb{Z}'_\mathbb{R} & \xrightarrow{f_\mathbb{R}} & \mathbb{Z}'_\mathbb{R} \\
\pi'_\mathbb{R} \downarrow & & \downarrow \pi'_\mathbb{R} \\
\mathbb{M}'_\mathbb{R} & \xrightarrow{f_{\text{nd}}} & \mathbb{M}'_\mathbb{R}
\end{array}
\quad \quad \quad
\begin{array}{ccc}
\mathbb{Z}'_\mathbb{C} & \xrightarrow{f_\mathbb{C}} & \mathbb{Z}'_\mathbb{C} \\
\pi'_\mathbb{C} \downarrow & & \downarrow \pi'_\mathbb{C} \\
\mathbb{M}'_\mathbb{C} & \xrightarrow{f_{\text{nd}}} & \mathbb{M}'_\mathbb{C}
\end{array}
\]

commute, the projections \( \pi'_{\mathbb{R}} \) and \( \pi'_{\mathbb{C}} \) pull back the relative orientations \( o_{\text{nd}}^\pm \) of (6.34) and (6.35) to relative orientations \( o_{\text{nd}}^\pm \) of (6.39) and (6.40), respectively. Since the dimensions of \( X \) and \( H_i \) are even,

\[
o_{p;h_\mathbb{R}} = o_{p;h_\mathbb{R}}^\pm o_{h_\mathbb{R}}' \quad \text{and} \quad o_{h_\mathbb{C};I} = o_{h_\mathbb{C};I'} (\pi'_{\mathbb{C}} o_{\text{nd}}^\pm o_{h_\mathbb{C};I'}),
\]

(6.41)

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Figure 11: Computation of (6.38) in the \(B_R, B_C \neq 0\) case

whenever \(B_R \neq 0\) and \(B_C \neq 0\), respectively.

Suppose \(B_R, B_C \neq 0\). By (6.32), we can assume that

\[
\ell_\omega(B_R) = k + 2(l_0 - l_0^*) \quad \text{and} \quad \ell_\omega(B_C) = l_C - l_C^*; \quad (6.42)
\]

otherwise, either \(Z_R = \emptyset\) or \(Z_C = \emptyset\) for a generic \(p \in X_{k, l - l^*}\). By the first statement above and Lemma 5.1(1), the interchange of the marked points of the elements in \(Z_R\) reverses the orientation \(\sigma_{p, hR}\). This interchange also reverses the vertical orientation \(\sigma_+^R\) of the fibration \(f_R\). Thus, the orientation \(\sigma_{p, hR}^+\) on \(Z_R^+\) can be defined by the first equation in (6.41) if \(l_0^* = 0\). By the second equation in (6.42) and the assumption that \(\sigma_+(h_i) = -[h_i]\) for every \(i \in [l^*]\),

\[
|\text{ev}^{-1}_C(p)\rangle_{\text{co}}^\pm = |(M_{C})_{\text{ev}, t \times \text{co}, p} M_{h_C}\rangle^\pm_{\text{co}, t, h_C} = \prod_{i \in [l_C]} (h_i \cdot B_C) N_{BC}^X; \quad (6.43)
\]

see [8] Prop. 4.3.

Since the real dimension of \(X\) is 4, the homomorphism

\[
d_{\text{nd}}u_+ - d_{\text{nd}}u_0 : T_{\text{nd}} \mathbb{P}^1_0 \oplus T_{\text{nd}} \mathbb{P}^1_+ \longrightarrow T_{u(\text{nd})} X = N_u S_h
\]

in the commutative diagram of Figure 11 is an isomorphism for a generic element \(u \in S_h\). So is the bottom row in this diagram. The number of the preimages \((\tilde{u}_0, \tilde{u}_+)^*\) of a generic point \((u_0, u_+)\) of \(\{Z_R^+ \times Z_C^+\}_{\text{co}, t, s}^+\) under \(f_R \times f_C\) is finite. Since the dimensions of \(T_{\text{nd}} \mathbb{P}^1_0\) and \(T_{\text{nd}} \mathbb{P}^1_+\) are even,

\[
\sigma_R^\pm = \sigma_R^+ \otimes \sigma_C^+
\]

is the vertical orientation of the fibration \(f_R \times f_C\). By Lemma 3.1(1) and the reasoning in the proof of (5.14) at the end of Section 6.2,

\[
\mathcal{S}(\tilde{u}_0, \tilde{u}_+) (\sigma_S^\pm \sigma_R^\pm) \in \{ \pm 1 \}
\]
Combining this statement with (6.38), Lemma 5.1

By the proof of (5.14), in Figure 12, applied to the right diagram in Figure 12,

\[ Z \]

dimensions of

Thus, by (6.32) and (5.19). By (6.37) and Lemma 3.4 with \( \Upsilon \)

Suppose of (5.20).

Combining this statement with (6.38) and (6.44), we obtain the second term on the right-hand side

for generic

Summing this over all possibilities for \( S \) with \( B_R, B_C \neq 0 \) that satisfy (6.42), we obtain the \((B_0, B')\) sum in (5.20).

Suppose \( B_C = 0 \) and thus \( B_R = B \). We can assume that \( l^*_k = l_C = 2 \); otherwise, \( Z_C = \emptyset \) for generic \( h \) and \( p \). Thus, \( Z_C \) is a finite collection of points, while the dimensions of \( Z_R \) and \( X \times X_{k,l-l^*} \) are the same by (6.32) and (5.19). By (6.37) and Lemma 3.4 with \( \Upsilon, M = pt \) applied to the left diagram in Figure 12

\[ |ev^{-1}_{\Gamma,h}(p) \cap S_h|_{h^\prime, p, h}^\pm = |Z_C|_{h^\prime, i}^\pm |\{ ev_{nd} \times ev_{\Gamma} \}^{-1}(pt, p)|_{p, h^\prime, h, \omega}^\pm \].

By the proof of (5.14),

\[ |\{ ev_{nd} \times ev_{\Gamma} \}^{-1}(pt, p)|_{p, h^\prime, h, \omega}^\pm = \left( \prod_{i \in I^*_0} h_i \cdot X \right) N^X_{B_0} \]

Combining this statement with (6.38) and (6.44), we obtain the second term on the right-hand side of (5.20).

Suppose \( B_R = 0 \) and thus \( \delta(B_C) = B \). We can assume that \( k^*, k = 1 \) and \( l_0 = 0 \); otherwise, \( Z_R = \emptyset \) for generic \( h \) and \( p \). Thus, \( \langle B; k \rangle = 1 \), the dimensions of \( Z_R \) and \( X^\phi \) are the same, and so are the dimensions of \( Z_C \) and \( X \times X_{l-l^*} \) by (6.32) and (5.19). By (6.37) and Lemma 3.4 with \( \Upsilon, M = pt \) applied to the right diagram in Figure 12

\[ |ev^{-1}_{\Gamma,h}(p) \cap S_h|_{h^\prime, p, h}^\pm = |ev^{-1}_{\Gamma,h}(p)|_{h^\prime, i}^\pm |\{ ev_{nd} \times ev_{\Gamma} \}^{-1}(pt, p)|_{p, h^\prime, h, \omega}^\pm \].

Combining this statement with (6.38), Lemma 5.3 (if \( \Phi \)) and (6.43), we conclude that

\[ |ev^{-1}_{\Gamma,h}(p) \cap S_h|_{h^\prime, p, h}^\pm = \left( \prod_{i \in I^*_0} h_i \cdot X \right) N^X_{B_C} = 2^{-l^*} \left( \prod_{i \in I^*_0} h_i \cdot X \right) N^X_{B_C} \].
Summing this over all possibilities for $S$ with $B_2 = 0$, we obtain the first term on the right-hand side of (5.20).

References

[1] A. Arroyo, E. Brugallé, and L. López de Medrano, Recursive formulas for Welschinger invariants of the projective plane, Int. Math. Res. Not. 2011, no. 5, 1107-1134

[2] E. Brugallé, Floor diagrams relative to a conic, and GW-W invariants of del Pezzo surfaces, Adv. Math. 279 (2015), 438-500

[3] E. Brugallé and N. Puignau, Behavior of Welschinger invariants under Morse simplifications, Rend. Semin. Mat. Univ. Padova 130 (2013), 147-153

[4] X. Chen, Solomon’s relations for Welschinger’s invariants: examples, preprint

[5] X. Chen and A. Zinger, Spin/Pin Structures and Real Enumerative Geometry, in preparation

[6] X. Chen and A. Zinger, WDVV type relations for real Gromov-Witten invariants, work in progress

[7] P. Georgieva, Open Gromov-Witten invariants in the presence of an anti-symplectic involution, Adv. Math. 301 (2016), 116-160

[8] P. Georgieva and A. Zinger, Enumeration of real curves in $\mathbb{CP}^{2n-1}$ and a WDVV relation for real Gromov-Witten invariants, Duke Math. J. 166 (2017), no. 17, 3291-3347

[9] P. Georgieva and A. Zinger, Real Gromov-Witten theory in all genera and real enumerative geometry: construction, math/1504.06617v3

[10] P. Georgieva and A. Zinger, Real Gromov-Witten theory in all genera and real enumerative geometry: properties, math/1507.06633v2

[11] A. Givental, Semisimple Frobenius structures at higher genus, IMNR 2001, no. 23, 1265-1286

[12] A. Horev and J. Solomon, The open Gromov-Witten-Welschinger theory of blowups of the projective plane, math/1210.4034

[13] I. Itenberg, V. Kharlamov, and E. Shustin, A Caporaso-Harris type formula for Welschinger invariants of real toric del Pezzo surfaces, Comment. Math. Helv. 84 (2009), no. 1, 87-126

[14] I. Itenberg, V. Kharlamov, and E. Shustin, Welschinger invariants of small non-toric Del Pezzo surfaces, JEMS 15 (2013), no. 2, 539-594

[15] I. Itenberg, V. Kharlamov, and E. Shustin, Welschinger invariants of real Del Pezzo surfaces of degree 3, Math. Ann. 355 (2013), no. 3, 849-878
[16] I. Itenberg, V. Kharlamov, and E. Shustin, Welschinger invariants of real del Pezzo surfaces of degree $\geq 2$, Internat. J. Math. 26 (2015), no. 8, 1550060, 63pp

[17] M. Kontsevich and Y. Manin, Gromov-Witten classes, quantum cohomology, and enumerative geometry, Comm. Math. Phys. 164 (1994), no. 3, 525–562

[18] J. Li and G. Tian, Virtual moduli cycles and Gromov-Witten invariants of general symplectic manifolds, Topics in Symplectic 4-Manifolds, 47-83, First Int. Press Lect. Ser., I, Internat. Press, 1998

[19] D. McDuff and D. Salamon, J-holomorphic Curves and Symplectic Topology, Colloquium Publications 52, AMS, 2012

[20] Y. Ruan and G. Tian, A mathematical theory of quantum cohomology, J. Differential Geom. 42 (1995), no. 2, 259–367

[21] Y. Ruan and G. Tian, Higher genus symplectic invariants and sigma models coupled with gravity, Invent. Math. 130 (1997), no. 3, 455–516

[22] J. Solomon, Intersection theory on the moduli space of holomorphic curves with Lagrangian boundary conditions. [math/0606429]

[23] J. Solomon, A differential equation for the open Gromov-Witten potential, pre-print 2007

[24] N. Steenrod, Homology with local coefficients, Ann. of Math. 44 (1943), no. 4, 610-627

[25] C. Teleman, The structure of 2D semi-simple field theories, Invent. Math. 188 (2012), no. 3, 525-588

[26] F. Warner, Foundations of Differentiable Manifolds and Lie Groups, GTM 94, Springer-Verlag, 1983

[27] J.-Y. Welschinger, Invariants of real symplectic 4-manifolds and lower bounds in real enumerative geometry, Invent. Math. 162 (2005), no. 1, 195-234

[28] J.-Y. Welschinger, Spinor states of real rational curves in real algebraic convex 3-manifolds and enumerative invariants, Duke Math. J. 127 (2005), no. 1, 89-121

[29] J.-Y. Welschinger, Optimalité, congruences et calculs d’invariants des variétés symplectiques réelles de dimension quatre, math/0707.4317

[30] A. Zinger, Pseudocycles and integral homology, Trans. Amer. Math. Soc. 360 (2008), no. 5, 2741–2765

[31] A. Zinger, Real Ruan-Tian perturbations, math/1701.01420