SCHOENFLIES SOLUTIONS WITH CONFORMAL BOUNDARY VALUES MAY FAIL TO BE SOBOLEV

YI RU-YA ZHANG

Abstract. We show that there exist planar Jordan domains \( \Omega_1 \) and \( \Omega_2 \) with boundaries of Hausdorff dimension 1 such that any conformal maps \( \varphi_1 : \mathbb{D} \rightarrow \Omega_1 \) and \( \varphi_2 : \Omega_2 \rightarrow \mathbb{D} \) cannot be extended as global homeomorphisms between the Riemann spheres of \( W^{1,1} \) class (or even not in \( BV \)).

Contents

1. Introduction 1
2. Proof of Theorem 1.2 3
3. Proof of Theorem 1.3 6
References 8

1. Introduction

Let \( \Gamma \subset \hat{\mathbb{C}} \) be a Jordan curve, namely there exists a homeomorphism \( \phi : S^1 \rightarrow \Gamma \), where \( \hat{\mathbb{C}} \) is the Riemann sphere and \( S^1 \) denotes the boundary of the unit disk \( \mathbb{D} \). According to Jordan curve theorem, the curve \( \Gamma \) divides the Riemann sphere into two components, and each component is a Jordan domain.

Jordan-Schoenflies theorem states that any homeomorphism between two Jordan curves on the Riemann spheres can be extended to a homeomorphism between the whole Riemann spheres; see [12 Corollary 2.9]. To be more specific, given two Jordan domains \( \Omega_1 \) and \( \Omega_2 \) together with the boundary value from \( \Gamma_1 = \partial \Omega_1 \) to \( \Gamma_2 = \partial \Omega_2 \) via a homeomorphism \( \varphi : \Gamma_1 \rightarrow \Gamma_2 \), there exists a homeomorphism \( \Phi \), which we call a Schoenflies solution with the boundary value \( \varphi \), from \( \hat{\mathbb{C}} \) to \( \hat{\mathbb{C}} \) such that the restriction of \( \Phi \) to \( \Gamma_1 \) coincides with \( \varphi \). Then a natural question rises:

**Question 1.1.** Given two Jordan domains \( \Omega_1, \Omega_2 \subset \hat{\mathbb{C}} \) together with a homeomorphism \( \varphi : \partial \Omega_1 \rightarrow \partial \Omega_2 \), what is the best regularity of Schoenflies solutions with the boundary value \( \varphi \)?

Date: August 31, 2018.

2010 Mathematics Subject Classification. 30C70.

Key words and phrases. Sobolev homeomorphism, Jordan-Schoenflies theorem.

This work was supported by the Academy of Finland via the Centre of Excellence in Analysis and Dynamics Research (Grant no. 271983) and the Hausdorff Center for Mathematics.
Certainly the answer to this question depends on the given boundary value. Recall Carathéodory’s theorem states that, given any Jordan domain $\Omega$, every conformal map $\varphi : \Omega_1 \to \Omega_2$ can be continuously extended to the boundary as a homeomorphism. In this paper we investigate Question 1.1 with the boundary value given by Carathéodory’s theorem, namely a conformal boundary value, and for simplicity we assume that $\Omega_1 = \mathbb{D}$.

Let us recall some known results. If $\Omega_2$ is bounded by a smooth Jordan curve, then by the techniques from differential topology for each conformal map we can find a smooth Schoenflies solution [2]. Moreover, if the complementary domain $\tilde{\Omega}$ is convex, then by the theory of harmonic maps, e.g. [3], there exists a harmonic Schoenflies solution falling into $W^{1, p}(\tilde{\mathbb{D}})$ for any $1 \leq p < 2$, namely the distributional derivative of the solution is in $L^p(\tilde{\mathbb{D}})$; see [13] and [7] for further information.

Actually we have even better regularity. Recall that convex domains are uniform, i.e. there exists a positive constant $\epsilon_0$ such that for any two distinct points $x, y \in \Omega$, there exists a rectifiable curve $\gamma \subset \Omega$ joining $x$ and $y$ which satisfies

$$\ell(\gamma) \leq \frac{1}{\epsilon_0}|x - y| \quad \text{and} \quad \text{dist} (z, \partial\Omega) \geq \epsilon_0 \min \{\ell(\gamma[x, z]), \ell(\gamma[z, y])\} \text{ for all } z \in \gamma,$$

where $\ell(\gamma[x, z])$ is the length of a subarc of $\gamma$ joining from $x$ to $z$, and $\ell(\gamma[z, y])$ corresponds to $z$ and $y$. It is known that the complementary domain of a Jordan uniform domain is also uniform.

Suppose that $\Omega_2$ is a uniform domain. Then by classical results on quasiconformal mappings, there exists a Schoenflies solution $\Phi$ such that both $\Phi$ and $\Phi^{-1}$ are in $W^{1,2}$ (or even $W^{1,p}$ for some $p > 2$) if the boundary value $S^1 \to \partial \Omega_2$ is a quasisymmetry; see [11] and [5]. Recently P. Koskela, P. Pankka and the author have been working on a version of this result on domains satisfying Gehring–Martio conditions [9], where the boundary value is given by a certain conformal map.

The main results of this paper are the following.

**Theorem 1.2.** There exists a Jordan domain $\Omega \subset \hat{\mathbb{C}}$ with boundary of Hausdorff dimension 1 such that, for every conformal map $\varphi : \mathbb{D} \to \Omega$, any Schoenflies solution with the conformal boundary value $\varphi$ on $S^1$ is not in $W^{1,1}$ (or even not in BV).

**Theorem 1.3.** There exists a Jordan domain $\Omega \subset \hat{\mathbb{C}}$ with rectifiable boundary such that, for every conformal map $\varphi : \mathbb{D} \to \Omega$, any Schoenflies solution with the conformal boundary value $\varphi^{-1}$ on $\partial \Omega$ is not in $W^{1,1}$ (or even not in BV).

These two results indicate that, in general, one cannot expect the regularity of Schoenflies solutions to a given boundary value to be better than homeomorphism; even if the boundary value is given by a (extended) conformal map (which is a quite natural choice). Thus, geometric assumptions on the Jordan domain in question and (energy) controls on the boundary value are necessary. One can see e.g. [11, 13, 7, 10] for recent results in this direction.

The notation in the paper is quite standard. The Euclidean distance between two sets $A, B \subset \mathbb{R}^2$ is denoted by $\text{dist} (A, B)$. We denote by $\ell(\gamma)$ the length of a curve $\gamma$. For a set $A \subset \mathbb{R}^2$, we write its boundary as $\partial A$, and its closure as $\overline{A}$, respectively, with respect to the Euclidean topology. We use the notation $\mathcal{H}^1$ for 1-dimensional Hausdorff measure.
2. Proof of Theorem 1.2

Define the inner distance with respect to $\Omega$ between $x, y \in \Omega$ by
\[ \text{dist}_{\Omega}(x, y) = \inf_{\gamma \subset \Omega} \ell(\gamma), \]
where the infimum runs over all curves joining $x$ and $y$ in $\Omega$.

The idea of the proof is that, we construct a Jordan domain $\Omega \subset \hat{\mathbb{C}}$ satisfying that there exists a set $E \subset \partial \Omega$ such that,
(i) for any conformal $\varphi : \mathbb{D} \to \Omega$, i.e. for any conformal boundary value, we have
\[ \mathcal{H}^1(\varphi^{-1}(E)) > 0; \]
(ii) for any point $\tilde{x}$ in the complementary domain $\tilde{\Omega}$,
\[ \text{dist}_{\tilde{\Omega}}(x, E) = \infty. \]

If such a Jordan domain exists, then by (i) and (ii), any Schoenflies solution with the conformal boundary value $\varphi$ fails to be in $W^{1,1}$ (even not in $BV$) by Fubini’s theorem; indeed, such a solution maps a family of radial segments in the exterior of the unit disk (with finite length) into curves of infinite length in $\tilde{\Omega}$, and by calculating in the polar coordinate we know that such a map cannot be in $W^{1,1}$ (even not in $BV$). Hence Theorem 1.2 follows.

We first construct a Jordan curve $\Gamma$ in the plane. Towards this, let us recall the construction of a fat Cantor set $E \subset [0,1]$ on the real axis. Let $C_0 = I_{0,1} = [0,1]$ and $C_i$ with $i \geq 1$ recursively as follows: When $I_{i,j} = [a,b]$ has been defined, let
\[ I_{i+1,2j-1} = \left[ a, \frac{a + b - 4^{-i}}{2} \right] \quad \text{and} \quad I_{i+1,2j} = \left[ \frac{a + b + 4^{-i}}{2}, b \right]; \]
i.e. we remove an open interval of length $4^{-i}$ from the middle of the interval $I_{i,j}$. Then we set
\[ C_i = \bigcup_{j=1}^{2^i} I_{i,j}. \]
The set $E$ is finally given by
\[ E = \bigcap_{i=1}^{\infty} C_i. \]
A simple calculation shows that, for every $i \in \mathbb{N}$ and $1 \leq j \leq 2^i$, each interval $I_{i,j}$ has length
\[ \frac{2^i + 1}{2^{2i+1}} \in (2^{-i-1}, 2^{-i}). \]
(2.1)

Thus $C_i$, and hence $E$ is well-defined. Moreover, $E$ has positive $\mathcal{H}^1$-measure; note that at step $i$, $i \geq 1$ there are $2^i$ intervals removed with total length $2^{-i-1}$.

We now construct a sequence of simple curves $\gamma_i$ based on the construction of $E$. Again we proceed inductively according to the index $i$. For $i \in \mathbb{N}$ and $1 \leq j \leq 2^i$, denote by $I'_{i,j} \subset I_{i,j}$ the interval removed from $I_{i,j}$ in the iteration step in the construction of $E$. Let
\( \gamma_0 \) be the interval \([0, 1]\). When \( \gamma_{i-1} \), \( i \geq 1 \) has been defined, we replace every open interval \( I'_{i,j} \), \( 1 \leq j \leq 2^i \), contained in \( \gamma_{i-1} \), by a curve
\[
\gamma_{i,j} = \partial(I'_{i,j} \times [0, 2^{-i}]) \setminus (I'_{i,j} \times \{0\}),
\]
consisting of three line segments, where \( \times \) means the Cartesian product. Via the modification on \( \gamma_{i-1} \), we obtain a new curve \( \gamma_i \). See Figure 1. Since \( \{\gamma_i\} \) (under suitable parameterizations) is a Cauchy sequence of curves in the plane with respect to the supremum distance, the limit \( \gamma \) exits and is a curve. Moreover, observe that \( \gamma \) is simple.

For fixed \( n \in \mathbb{N} \), there are \( 2^{n+1} - 1 \) curves \( \gamma_{i,j} \) intersecting \( \mathbb{R} \times (2^{-n-1}, 2^{-n}] \). Indeed, if \( \gamma_{i,j} \cap (\mathbb{R} \times (2^{-n-1}, 2^{-n}]) \neq \emptyset \), then \( i \leq n \). The distance between any two of these curves is strictly larger than \( 2^{-n-1} \) by (2.1).

We next construct a sequence of new curves \( \Gamma_n \) according to the index \( n \). First of all define \( \Gamma_0 = \gamma \). When \( \Gamma_{n-1} \), \( n \geq 1 \) has been defined, we modify the segments in
\[
\gamma_{i,j} \cap (\mathbb{R} \times (2^{-n-1}, 2^{-n}]) \setminus \emptyset \leq i \leq n
\]
to obtain \( \Gamma_n \). Recall that \( \gamma_{i,j} \) replaced the interval \( I'_{i,j} \) in the construction of \( \gamma_i \). Denote by \( a_{i,j} \) and \( b_{i,j} \) the end points of \( I'_{i,j} \) with \( a_{i,j} < b_{i,j} \). Then for every \( 1 \leq i \leq n \),
\[
\gamma_{i,j} \cap (\mathbb{R} \times (2^{-n-1}, 2^{-n})] = ([a_{i,j}] \times (2^{-n-1}, 2^{-n}] \cup ([b_{i,j}] \times (2^{-n-1}, 2^{-n}])
\]
and each \( 0 \leq k \leq 2^n \), we replace each segment
\[
\{a_{i,j}\} \times [2^{-n-1} + (4k)2^{-2n-3}, 2^{-n-1} + (4k + 1)2^{-2n-3}]
\]
by
\[
\partial \left( [a_{i,j} - 2^{-n-1}, a_{i,j}] \times [2^{-n-1} + (4k)2^{-2n-3}, 2^{-n-1} + (4k + 1)2^{-2n-3}] \right)
\]
\[
\setminus \{a_{i,j}\} \times [2^{-n-1} + (4k)2^{-2n-3}, 2^{-n-1} + (4k + 1)2^{-2n-3}],
\]
and
\[
\{b_{i,j}\} \times [2^{-n-1} + (4k + 2)2^{-2n-3}, 2^{-n-1} + (4k + 3)2^{-2n-3}]
\]

![Figure 1](image-url)
SCHÖNFLES SOLUTIONS WITH CONFORMAL BOUNDARY VALUES MAY FAIL TO BE SOBOLEV

Figure 2. The curve $\Gamma_2$ is shown in the figure, with the replacement of certain segments contained in $\Gamma_1$ by parts of boundaries of some rectangles, respectively.

by

$$\partial \left( [b_{i,j}, b_{i,j} + 2^{-n-1}] \times [2^{-n-1} + (4k + 2)2^{-2n-3}, 2^{-n-1} + (4k + 3)2^{-2n-3}] \right) \setminus \{b_{i,j}\} \times [2^{-n-1} + (4k + 2)2^{-2n-3}, 2^{-n-1} + (4k + 3)2^{-2n-3}] .$$

This gives us the new curve $\Gamma_n$. See Figure 2.

Again since $\{\Gamma_n\}$ (under suitable parameterizations) is a Cauchy sequence of curves in the plane with respect to the supremum distance, we conclude that $\Gamma_n$ converges uniformly to some curve $\Gamma_\infty$ as $n \to \infty$, and according to our construction $\Gamma_\infty$ is simple. Define

$$\Gamma = \Gamma_\infty \cup (\partial([0, 1] \times [-1, 0]) \setminus [0, 1] \times \{0\}) .$$

Since $\Gamma_\infty$ is simple, then also $\Gamma$ is simple, and hence Jordan as it is closed. Via the standard stereographic projection we have a Jordan curve on $\mathbb{C}$, and denote by $\Omega$ the component containing the preimage of the square $[0, 1] \times [-1, 0]$ under the stereographic projection.

Recall the Cantor set $E$. Now let us check that the Jordan domain $\Omega$ satisfies the two properties (i) and (ii). We remark that, in our construction, for any point $x$ in $\Omega$, we have

$$\text{dist}_\Omega(x, E) < \infty .$$

Before showing (i), we note that, property (i) is stated with respect to all conformal maps. However, since two such Riemann maps differ from each other by a Möbius transform on the unit disk, we may assume that $\varphi(0)$ is the preimage of the center of the square $[0, 1] \times [-1, 0]$ under the stereographic projection.

Recall that the harmonic measure in the unit disk is defined via the Poisson kernel, and then in any Jordan domain via the (extended) Riemann mapping. For $E \subset \partial \Omega$, we use $\omega(x_0, E, \Omega)$ to designate the harmonic measure of $E$ at $x_0$ in $\Omega$. It is known that $\omega(x_0, E, \Omega) = u(x_0)$ where $u$ is the (unique) harmonic function in $\Omega$ whose boundary value is the characteristic function of $E$ on $\partial \Omega$. We refer to [4] for more details.

Lemma 2.1. The Jordan domain $\Omega$ constructed above satisfies properties (i) and (ii).
Proof. Towards (i), we first observe that
\[ \omega(\varphi(0), E, \Omega) \geq \omega(\varphi(0), E, Q) > 0, \] (2.2)
where \( Q \) is the preimage of the square \([0, 1] \times [-1, 0]\) under the stereographic projection. This estimate comes from the comparison principle of harmonic functions. Indeed, let \( u_1 \) and \( u_2 \) be the harmonic functions on \( \Omega \) and \( Q \subset \Omega \), respectively, with the characteristic function of \( E \) on the corresponding boundaries as the boundary values. By zero-extension to \( \Omega \) we have that \( u_2 \) is a subharmonic function on \( \Omega \) with the same boundary value as \( u_1 \). Thus
\[ u_1(\varphi(0)) \geq u_2(\varphi(0)) > 0, \]
where the second inequality follows from the strong minimum principle. and hence we obtain (2.2). By the conformal invariance of harmonic measure we have
\[ \omega(0, \varphi^{-1}(E), \mathbb{D}) > 0. \]
According to the definition of harmonic measures in the unit disk, we conclude (i).

To show (ii), note that in our construction, any curve in the unbounded component of \( \mathbb{R}^2 \setminus \Gamma \) towards \( E \) has length at least 1 in the region \( \mathbb{R} \times (2^{-n-1}, 2^{-n}] \) for \( n \) large enough; the curve has to oscillate \( 2^n \) times and each time it goes at least \( 2^{-n} \). This implies that any curve in the unbounded component of \( \mathbb{R}^2 \setminus \Gamma \) towards \( E \) has infinite length. Since the stereographic projection is biLipschitz away from the north pole, we conclude that (ii) is satisfied. □

3. Proof of Theorem 1.3

First let us recall the definition of John domains.

**Definition 3.1 (John domain).** A domain \( \Omega \subset \mathbb{R}^2 \) is called \( J \)-John with \( J \geq 0 \) if there exist a distinguished point \( x_0 \in \Omega \) such that, for every \( x \in \Omega \), there is a curve \( \gamma : [0, \ell(\gamma)] \to \Omega \) parameterized by the arclength satisfying \( \gamma(0) = x, \gamma(\ell(\gamma)) = x_0 \) and for all \( t \in [0, \ell(\gamma)] \)
\[ \text{dist} (\gamma(t), \mathbb{R}^2 \setminus \Omega) \geq Jt. \]
The curve \( \gamma \) is called a John curve, and \( J \) is called a John constant.

We say that a homeomorphism \( \varphi : \mathbb{D} \to \Omega \) is quasisymmetric with respect to the inner distance if there is a homeomorphism \( \eta : [0, \infty) \to [0, \infty) \) so that
\[ |z - x| \leq t|y - x| \text{ implies } \text{dist}_\Omega(\varphi(z), \varphi(x)) \leq \eta(t) \text{dist}_\Omega(\varphi(y), \varphi(x)) \]
for each triple \( z, x, y \) of points in \( \mathbb{D} \). It is clear from the definition that the inverse of a quasisymmetric map is also quasisymmetric. By iteration, one can show that, there exist constant \( C \) and \( 0 < s \leq 1 \) depending only on \( \eta(1) \) such that
\[ \eta(t) \leq C \max\{t^s, t^{1/s}\}; \] (3.1)
see [Page 25].

**Lemma 3.2 ([6], Theorem 3.1).** Let \( \Omega \subset \mathbb{R}^2 \) be a simply connected domain, and \( \varphi : \mathbb{D} \to \Omega \) be a conformal map. Then \( \Omega \) is John if and only if \( \varphi \) is quasisymmetric with respect to the inner distance. This statement is quantitative in the sense that the John constant and the function \( \eta \) in quasisymmetry depend only on each other and \( \text{diam}(\Omega)/\text{dist}(\varphi(0), \partial \Omega) \).
Figure 3. The construction of the John domain $\Omega$ is shown in the figure.

The rough idea to show Theorem 1.3 is that we construct a John Jordan domain $\Omega$ such that there exist infinitely many pairs of subarcs on $\partial \Omega$ satisfying that, the inner distance with respect to $\Omega$ between the elements in each pair is much larger than their inner distance with respect to $\bar{\Omega} = \hat{\mathbb{C}} \setminus \Omega$. However, Lemma 3.2 tells that under $\varphi: \mathbb{D} \rightarrow \Omega$, the preimages of the elements in each pair have large distance in the interior and exterior of the unit circle, respectively. This implies that any Schoenflies solution with conformal boundary value $\varphi^{-1}$ should have huge energy.

To be more specific, let us first construct a planar Jordan curve in the following way. First define

$$\xi(t) = \min \{ t^2, Ct^{1+1/s} \}$$

for some constant $C$ and $0 \leq s < 1$ to be determined. Notice that $\xi(t)$ is a homeomorphism from $[0, \infty)$ to $[0, \infty)$, and for $0 \leq t \leq 1$, we have $0 \leq \xi(t) \leq t$.

Let $\Gamma_0$ be the boundary of the square $[0, 1] \times [0, 1]$. When the curve $\Gamma_{i-1}, i \geq 1$ has been defined, we set $\Gamma_i$ via modifying $\Gamma_{i-1}$ as follows. We replace the line segment

$$\{1\} \times [2^{-i}, 2^{-i} + \xi(2^{-i})],$$

contained in $\Gamma_{i-1}$ by the following curve

$$[1 - 2^{-i}, 1] \times \{2^{-i}\} \cup \{(t, \xi(t - 1 + 2^{-i}) + 2^{-i}) : t \in [1 - 2^{-i}, 1]\}.$$

Observe that, since $\xi(t) \leq t$ when $0 \leq t \leq 1$, $\Gamma_i$ is well-defined and also a Jordan curve. See Figure 3.

Moreover, $\{\Gamma_i\}$ (under suitable parameterizations) is a Cauchy sequence of curves with respect to the supremum distance, and hence converges to some curve $\Gamma$ as $i \rightarrow \infty$. Additionally, by our construction $\Gamma$ is Jordan and the Jordan domain $\Omega$, the bounded component
of $\mathbb{R}^2 \setminus \Gamma$, is uniformly $J$-John with $J$ independent of the choice of $C$ and $s$; indeed $J$ can be chosen to be absolute.

Let $\eta$ be the homeomorphism given by Lemma 3.2 and choose $C$ and $s$ as in (3.1). By Lemma 3.2 the conformal map $\varphi : \mathbb{D} \to \Omega$ is $\eta$-quasisymmetric. Thus we have

$$
\eta(\text{dist}(\varphi^{-1}(A_i), \varphi^{-1}(B_i))) \geq 2^{-i-3}
$$

where

$$
A_i = [1 - 2^{-i-1}, 1] \times \{2^{-i}\},
$$

$$
B_i = \{(t, \xi(t - 1 + 2^{-i}) + 2^{-i}) : t \in [1 - 2^{-i-1}, 1]\}.
$$

According to (3.1), we have

$$
2^{-i+3} \leq c \text{dist}(\varphi^{-1}(A_i), \varphi^{-1}(B_i))
$$

with the constant $c$ depending only on the John constant $J$. Thus, for any homeomorphism from $\overline{\Omega}$ to $\overline{\mathbb{D}}$ with the boundary value $\varphi^{-1}$, the image (under such a homeomorphism) of every vertical line segment joining $A_i$ and $B_i$ in $\overline{\Omega}$ has length at least $c^{-1}2^{-i+3}$. Hence via Fubini’s theorem and the stereographic projection, for any Schoenflies solution $\Phi$ with the conformal boundary value $\varphi^{-1}$, we have

$$
\int_\mathbb{C} |D\Phi| \, dx \geq c' \sum_{i=1}^{\infty} 2^{-i-1} \frac{2^{-i+3}}{\xi(2^{-i})} \geq c' \sum_{i=1}^{\infty} 1 = \infty,
$$

where the constant $c'$ depending only on the John constant $J$. Thus $\Phi$ cannot be in $W^{1,1}$, and a similar argument also shows that $\Phi$ cannot be $BV$. Therefore we conclude Theorem 1.3.

References

[1] K. Astala, T. Iwaniec, G. J. Martin, and J. Onninen, *Extremal mappings of finite distortion*, Proc. London Math. Soc. 91 (2005), no. 3, 655-702.

[2] S. R. Bell, S. G. Krantz, *Smoothness to the boundary of conformal maps*. Rocky Mountain J. Math. 17 (1987), no. 1, 23–40.

[3] P. Duren, *Harmonic mappings in the plane*. Cambridge Tracts in Mathematics, 156. Cambridge University Press, Cambridge, 2004.

[4] J. B. Garnett, D. E. Marshall, *Harmonic measure*. New Mathematical Monographs, 2. Cambridge University Press, Cambridge, 2005.

[5] F. W. Gehring, *The $L^p$-integrability of the partial derivatives of a quasiconformal mapping*. Acta Math. 130 (1973), 265–277.

[6] J. Heinonen, *Quasiconformal mappings onto John domains*. Rev. Mat. Iberoamericana 5 (1989), 3-4, 97–123.

[7] T. Iwaniec, G. Martin, C. Sbordone, *$L^p$-integrability & weak type $L^2$-estimates for the gradient of harmonic mappings of $D$. Discrete Contin. Dyn. Syst. Ser. B 11 (2009), no. 1, 145–152.

[8] P. Koskela, *Lectures on quasiconformal and quasisymmetric mappings*, University of Jyväskylä.

[9] P. Koskela, P. Pankka, Y. R.-Y, Zhang, *Generalized uniform domains*, In progress.

[10] P. Koskela, Z. Wang, H. Xu, *Controlled diffeomorphic extension of homeomorphisms*, arXiv:1805.02906

[11] O. Martio, J. Sarvas, *Injectivity theorems in plane and space*. Ann. Acad. Sci. Fenn. Ser. A I Math. 4 (1979), no. 2, 383–401.

[12] Ch. Pommerenke, *Boundary behaviour of conformal maps*. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 299. Springer-Verlag, Berlin, 1992.
G. C. Verchota. *Harmonic homeomorphisms of the closed disc to itself need be in $W^{1,p}$, $p < 2$, but not $W^{1,2}$. Proc. Amer. Math. Soc. 135* (2007), no. 3, 891–894.

Hausdorff Center for Mathematics, Endenicher Allee 60, D-53115 Bonn, Germany.

E-mail address: yizhang@math.uni-bonn.de