Electric-circuit realization of non-Hermitian higher-order topological systems

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Topological phases are characterized by the bulk topological number, and signaled by the emergence of zero-energy boundary modes. LC electric circuits appropriately designed are known to have topological phases together with topological phase transitions. We show that LCR circuits realize them in non-Hermitian topological systems. In particular, higher-order topological phases are obtained in the anisotropic honeycomb and diamond circuits. Topological phase transitions, being induced by tuning variable capacitors, are clearly observable by measuring the impedance resonance due to zero-admittance corner modes. Remarkably, only the topological resonance peaks remain prominent in non-Hermitian systems, since all other resonances are suppressed by resistors.

I. INTRODUCTION

Topological physics is one of the most important concepts in contemporary physics, among which topological insulators and its generalization to higher-order topological insulators are fascinating. They are characterized by the bulk topological numbers, where the bulk-boundary correspondence and its generalization play a key role. In particular, topological zero-energy corner modes emerge for the second-order topological insulators in two dimensions and for the third-order topological insulators in three dimensions. They are robust against impurities. They have so far been studied mainly in fermionic systems in materials. However, since topological features are based solely on the homotopy of the linear algebra inherent to the system, various systems such as photonic, phononic, and microwave systems share the same topological properties. LC electric circuits have also topological phases. As great merits, electric circuits are easily designed to yield topological phases, and furthermore, topological phase transitions are induced simply by controlling variable capacitors. Topological zero-energy boundary modes are observed as zero-admittance boundary modes.

Recently, non-Hermitian topological systems attract increasing attentions. Although quantum mechanics should be Hermitian, the dissipation effects can be simulated by introducing non-Hermitian terms into the system. They are realized in photonic systems, microwave resonators, wave guides, quantum walks, and cavity systems. In the parity-time-reversal (PT) symmetric non-Hermitian systems, the bulk energy remains to be real. On the other hand, in the chiral-symmetric non-Hermitian systems, the bulk energy becomes complex in general.

The non-Hermitian Su-Schrieffer-Heeger (SSH) model is the most studied example. The Hermitian SSH model has been generalized to higher dimensions. Especially, they are realized in the anisotropic honeycomb and diamond lattice models. These models realize higher-order topological phases, which are characterized by the emergence of topological zero-energy corner modes. A natural question is whether there is a non-Hermitian counterpart part of higher-order topological phases and how to realize them in electric circuits.

In this work, we show that LCR electric circuits present a concrete playground to investigate non-Hermitian topological physics, where resistors naturally lead to non-Hermitian terms by way of the Joule heating energy dissipation. We focus on the chiral symmetric topological electric circuits. We find that zero-admittance modes in the absence of resistors remain as they are even when non-Hermitian terms are introduced by resistors although the bulk admittance becomes complex. Furthermore, in the presence of resistors, all impedance resonances are drastically suppressed except for the topological zero-admittance modes. It will be easy to perform experimental observations of these predictions electrically since we have variable resistors and capacitors. We explicitly investigate topological properties in the SSH, anisotropic honeycomb and diamond LCR circuits as non-Hermitian generalizations, where the ordinary, second-order and third-order topological systems are realized.
II. RESULTS

A. Kirchhoff’s law and non-Hermitian systems

We investigate a class of electric circuits, where each node \( a \) is connected to the ground via the inductance \( L \). See examples in Fig. 1. Let \( I_a \) be the current between node \( a \) and the ground via the inductance, \( V_a \) be the voltage at node \( a \), \( C_{ab} \) and \( R_{ab} \) be the capacitance and the resistance between nodes \( a \) and \( b \), respectively. When we apply an AC voltage \( V(t) = V(0) e^{i \omega t} \), the Kirchhoff’s current law reads

\[
I_a(\omega) = \sum_b J_{ab}(\omega) V_b(\omega),
\]

where the sum is taken over all adjacent nodes \( b \), and \( J_{ab}(\omega) \) is the circuit Laplacian,

\[
J_{ab}(\omega) = i \omega \delta_{ab} \left( -\frac{1}{\omega^2 L} + \sum_{c \neq a} \frac{1}{1/C_{ac} + i \omega R_{ac}} \right) - i \omega H_{ab}(\omega),
\]

where

\[
H_{ab}(\omega) = \frac{1}{1/C_{ab} + i \omega R_{ab}}
\]

for \( a \neq b \) and \( H_{aa}(\omega) = 0 \). An important observation is that we are able to regard \( H_{ab}(\omega) \) as the tight-binding Hamiltonian of a lattice system in condensed-matter physics, where the transfer integral between adjacent sites \( a \) and \( b \) is given by

\[
t_{ab} = C_{ab}/(1 + i \omega C_{ab} R_{ab}).
\]

Accordingly we may examine various topological concepts by using electric circuits. We are particularly interested in the topological boundary modes associated with the bulk-boundary correspondence in non-Hermitian topological systems. Note that the Hamiltonian \( H_{ab}(\omega) \) is naturally non-Hermitian due to the Joule heating loss by resistors.

We analyze an electric circuit forming a lattice structure. A lattice is constructed by translating a unit cell repeatedly, which allows us to define the crystal momentum together with the Brillouin zone even in electric circuits.

Bipartite circuits: To make the idea as concrete as possible, we focus on a bipartite electric circuit, where a unit cell contains only two nodes. As we illustrate in Fig. 1, a unit electric circuit consists of two nodes \( A \) and \( B \), two capacitors \( C_A \) and \( C_B \), and two resistors \( R_A \) and \( R_B \). In addition, we require the chiral symmetry \( \sigma_\tau \) satisfying \( \{H(k), \sigma_z\} = 0 \), which assures the symmetric spectrum \( E \leftrightarrow -E \).

Topological numbers: The generic non-Hermitian bipartite Hamiltonian with chiral symmetry is given by

\[
H(k) = \begin{pmatrix} 0 & h_1(k) \\ h_2(k) & 0 \end{pmatrix},
\]

It allows us to define two winding numbers \( W_\alpha \)

\[
W_\alpha = \int_{-\pi}^{\pi} \frac{dk}{2\pi i} \partial_k \log h_\alpha.
\]

Hence the system is characterized by \( \mathbb{Z} \oplus \mathbb{Z} \). They define the bulk topological numbers. In our cases, we find \( W_2 = -W_1 \) and the topological number is given by one winding number \( W = (W_2 - W_1)/2 = W_2 \).

Admittance spectrum: The admittance spectrum consists of the eigenvalues of the circuit Laplacian. It corresponds to the band structure in condensed-matter physics: See an instance in Fig. 2. The emergence of the zero-admittance modes is the signal of the bulk topology owing to the bulk-boundary correspondence.

Impedance: A measurable quantity of electric circuits is the two-point impedance, which is given by

\[
Z_{ab} = \frac{V_a - V_b}{I_{ab}} = G_{aa} + G_{bb} - G_{ab} - G_{ba},
\]

where \( G \) is the green function defined by the inverse of the Laplacian \( G = J^{-1} \). It diverges at the frequency satisfying \( J = 0 \). Especially, it is possible to detect the topological zero-admittance modes by measuring the divergence of the impedance.
is in the unit of Ω, but all of them are washed away except for the topological impedance peaks for $R \neq 0$. 

After the diagonalization, the circuit Laplacian has the form

$$J_n(\omega) = i\omega \left( -\frac{1}{\omega^2 L} + \sum_{\alpha=A,B} \frac{n_\alpha C_\alpha}{1 + i\omega C_\alpha R_\alpha} \right) - i\omega \varepsilon_n(\omega),$$

where $n_\alpha$ is the number of the nodes adjacent to node $\alpha$, and $\varepsilon_n$ is the eigen-modes of the circuit Laplacian. When $R_\alpha = 0$, the impedance diverges at the resonance frequencies

$$\omega_R(\varepsilon_n) = \sqrt{-\varepsilon_n + \sum_{\alpha} n_\alpha C_\alpha},$$

which is the solution of $J_\alpha(\omega) = 0$. Especially, we find a resonance at the zero-admittance mode ($\varepsilon_0 = 0$) which is a topological impedance resonance. Indeed, the emergence of the zero-admittance modes is a requisite of the topological phase. On the other hand, all other resonances are trivial in the sense that they have no topological origin. However, it is nontrivial to differentiate these resonances unless we know the value of the resonance frequency. We shall soon see that only the topological resonances survive in the presence of resistors ($R_\alpha \neq 0$), where the imaginary part of $\varepsilon_n$ introduced by resistors suppresses trivial resonances.

Actually, a resonance does not always occur at all resonance frequencies given by $(9)$. For instance, although both solid and dotted curves are solutions of $(9)$ in Fig.3(a), actual resonances appear only on solid curves in Fig.3(b). It is due to the exact cancellation between the contributions from $G_{aa} + G_{bb}$ and $G_{ab} + G_{ba}$ in $(7)$.

### B. Non-Hermitian SSH circuits with chiral symmetry

The simplest model of the non-Hermitian topological system is the SSH model\cite{PT,PTSSH}. The $PT$ symmetric SSH model is especially well studied, where the gain and loss are balanced\cite{PTSSH}. However, it is nontrivial to construct this model in passive electric circuits consisting only capacitors, inductors and resistors due to the absence of gains. On the other hand, the chiral symmetric SSH model is adequate for electric circuits.

We consider a circuit shown in Fig.1(a), where capacitors and resistors are connected directly, and all nodes are grounded by inductors. The circuit Laplacian is written in the form

$$J_{ab}(\omega) = i\omega \delta_{ab} \left( -\frac{1}{\omega^2 L} + \sum_{\alpha=A,B} \frac{C_\alpha}{1 + i\omega C_\alpha R_\alpha} \right) - i\omega H_{ab}(\omega),$$

with the Hamiltonian

$$H = \begin{pmatrix}
0 & t_A + t_B e^{ik} \\
t_A + t_B e^{-ik} & 0
\end{pmatrix}. \quad \text{(11)}$$

Here,

$$t_A = \frac{C_A}{1 + i\omega C_A R_A}, \quad t_B = \frac{C_B}{1 + i\omega C_B R_B} \quad \text{(12)}$$

are complex numbers. It is reduced to the normal SSH model when $t_A$ and $t_B$ are real, i.e., $R_A = R_B = 0$. Otherwise, the Hamiltonian is non-Hermitian. By evaluating the winding number\cite{PT}, we find that the system is topological ($W = 1$) for $|t_A/t_B| < 1$ and trivial ($W = 0$) for $|t_A/t_B| > 1$. 

![Impedance spectrum in the chiral non-Hermitian SSH model](image)
We show the admittance spectrum as a function of $t_{AB}/t_B$ in Fig.2. First we calculate the spectrum without resistors, where the Hamiltonian is Hermitian. There appear zero-admittance modes for $|t_{AB}/t_B| < 1$, indicating that the system is in the topological phase. When resistors are included, the admittance of the bulk becomes complex but the zero-admittance modes remain as they are.

Next we study a finite chain containing $L$ unit cell made of the SSH circuit. We show the two-point impedance between the two outermost edge nodes in Fig.3. First, we note that the frequency formula (9) explains very well the numerical results of resonances: See Fig.3(a) and (b). Next, there appear many resonant peaks without resistors $(R_A = R_B = 0)$ both in the trivial and topological phases, as in Fig.3(c) and (e). Especially, there is a topological resonance at $\omega_0 = 1/\sqrt{L(C_A + C_B)}$. Nevertheless, it is actually difficult to distinguish these two phases by the emergence of large resonant peaks. However, once resistors are included, all resonance peaks are washed away except for the resonance peak in the topological phase. When resistors are included, the admittance spectrum by choosing $\delta = 0.2$. We find the zero-admittance modes survive even in the presence of the randomness whether resistors exist or not: See Fig.3(d) and (f).

Topological stability against randomness: We study the effects of the randomness of the capacitors and resistors. For this purpose, we make substitution $C_\alpha \rightarrow C_\alpha (1 + \eta_\alpha)$ and $R_\alpha \rightarrow R_\alpha (1 + \xi_\alpha)$, where $\eta_\alpha$ and $\xi_\alpha$ are uniformly distributed random variables ranging from $-\delta$ to $\delta$. We have calculated the admittance spectrum by choosing $\delta = 0.2$. We find the zero-admittance modes survive even in the presence of the randomness whether resistors exist or not: See Fig.3(g) and (h). This is because there is always zero-admittance solution for a finite chain due to its edges even in the presence of the randomness. See Methods B.

C. Non-Hermitian honeycomb circuits

Next we generalize our investigation to electric circuits corresponding to two-dimensional lattices. A typical example is the anisotropic honeycomb circuit consisting of two types of the capacitors and resistors, as in Fig.4(b). The anisotropic honeycomb lattice model is known to become a second-order topological insulator, where topological corner modes emerge.

The circuit Laplacian is written in the form

$$J_{ab}(\omega) = i\omega h_{ab} \left( -\frac{1}{\omega^2 L} + \frac{2C_A}{1 + i\omega C_A R_A} + \frac{C_B}{1 + i\omega C_B R_B} \right)$$

$$- i\omega H_{ab}(\omega),$$

with the Hamiltonian

$$H = \begin{pmatrix} 0 & h_1 \\ h_2 & 0 \end{pmatrix},$$

and

$$h_1 = 2t_A \cos \frac{\sqrt{3}k_x}{2} + t_B e^{-i\frac{\sqrt{3}k_y}{2}};$$

$$h_2 = 2t_A \cos \frac{\sqrt{3}k_x}{2} + t_B e^{i\frac{\sqrt{3}k_y}{2}}.$$
has a third-order topological insulator phase for ₂ and is ( \(p = 0\)).

When we fix one terminal at  (A) in the vicinity of the rhombus center, a huge ball is found at the corner  (B). Non-Hermitian polarization is half quantized, \(p = 0/2\), which we may as the topological numbers as in the case of Hermitian higher-order topological insulators. The Wannier center is \((0, 1/2)\) for the topological phase and \((0, 0)\) for the trivial phase.

D. Non-Hermitian diamond circuits

Finally, we study the anisotropic diamond lattice, which has a third-order topological insulator phase for \(|t_A/t_B| < 1/3\). For the anisotropic diamond lattice, we take

\[
\begin{align*}
\alpha_1 &= t_A \left( e^{i k_x X_2} + e^{i k_y X_3} + e^{i k_x X_1} \right) + t_B e^{i k_y X_1}, \\
\alpha_2 &= t_A \left( e^{-i k_x X_2} + e^{-i k_y X_3} + e^{-i k_x X_1} \right) + t_B e^{-i k_y X_1}
\end{align*}
\]

with the four lattice vectors pointing the tetrahedron directions \(X_1 = (1, 1, 1), X_2 = (1, -1, -1), X_3 = (-1, 1, -1)\) and \(X_4 = (-1, -1, 1)\). We show the corner impedance in Fig. 5. When we fix one terminal at  (A) in the vicinity of the rhombus center, an enhanced topological resonance emerges at the corner  (B).

III. METHODS

A. Topological numbers

The chiral symmetric \(2 \times 2\) non-Hermitian Hamiltonian systems is

\[
H = \begin{pmatrix}
0 & h_1 \\
h_2 & 0
\end{pmatrix}.
\]

There are several way of defining topological numbers. Here we prove their equivalence.

i) We have adopted the definition \(W_\alpha = \int_{-\pi}^\pi dk \partial_{k_\alpha} \log h_\alpha\), or

\[
W_{\alpha, \mu} = \frac{\int_{-\pi}^\pi dk \partial_{k_\mu} \log h_\alpha}{2\pi i}.
\]

For the non-Hermitian SSH model, we obtain

\[
\begin{align*}
W_{1, x} &= \begin{cases}
-1 & \text{for } |t_A/t_B| < 1 \\
0 & \text{for } |t_A/t_B| > 1
\end{cases}, \\
W_{2, x} &= \begin{cases}
1 & \text{for } |t_A/t_B| < 1 \\
0 & \text{for } |t_A/t_B| > 1
\end{cases}.
\end{align*}
\]

We find the relation \(W_2 = -W_1\), and hence there is only one winding number, \(W/W_2 = W_2 - W_1\), in the present system.

ii) Another definition of the non-Hermitian winding number is

\[
W_\mu = \frac{1}{2\pi i} \int \langle \psi^R | \partial_{k_\mu} | \psi^L \rangle dk,
\]

where \(\langle \psi^L \rangle\) is the left eigen-function satisfying

\[
H |\psi^L\rangle = \varepsilon^L |\psi^L\rangle,
\]

and \(\langle \psi^R \rangle\) is the right eigen-function satisfying

\[
H^\dagger |\psi^R\rangle = \varepsilon^R |\psi^R\rangle.
\]

It is identical to the non-Hermitian Wannier center defined by \(\langle \psi^L | \partial_{k_\mu} | \psi^L \rangle\) when \(A_\mu\) depends only on \(k_\mu\), which is the case in gapped phases since the integration over the codimension of \(k_\mu\) is constant and cancels with \(V\).

In our model, the relation \(|h_1| = |h_2|\) holds and they are given by

\[
\begin{align*}
|\psi^L\rangle &= \frac{1}{\sqrt{2}} \left\{ \frac{1}{\sqrt{h_1}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}, \\
|\psi^R\rangle &= \frac{1}{\sqrt{2}} \left\{ \frac{1}{\sqrt{h_2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}
\end{align*}
\]

and hence

\[
\begin{align*}
W &= \frac{1}{2\pi i} \int \partial_{k_2} \log h_1 - \partial_{k_1} \log h_2 \, dk \\
&= \frac{W_1 - W_2}{4}.
\end{align*}
\]

We conclude that the non-Hermitian polarization is half quantized for the topological phase since there is a relation \(p = -W/2\).

iii) In the chiral symmetric model there exists the non-Hermitian chiral index,

\[
\gamma = \int_{-\pi}^\pi \frac{dk}{2\pi i} \text{Tr} \left( \sigma_2 H^{-1} \partial_k H \right) = W_2 - W_1.
\]

This index is equivalent to the topological number since \(\gamma = 2W\).
B. Zero-admittance edge modes of SSH circuits

We construct an analytic form of the eigen-functions at the zero-energy state in the SSH model \([11]\). We label the eigen-function at the outer most node as \(\psi_1\), and that of the node next to it as \(\psi_2\), and so on. The eigen-function is \(\psi = \{\psi_1, \psi_2, \ldots, \psi_N\}\) if there are \(N\) nodes across the chain. The Hamiltonian is explicitly written as

\[
H = \begin{pmatrix}
0 & t_A & 0 & 0 & \cdots \\
t_A & 0 & t_B & 0 & \cdots \\
0 & t_B & 0 & t_A & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots 
\end{pmatrix}.
\] (33)

The eigenvalue problem \(H\psi = 0\) is explicitly given by

\[
t_A \psi_1 + t_B \psi_2 = 0,
\]
\[
t_A \psi_2 + t_B \psi_3 = 0,
\]
\[
\vdots
\]
\[
t_A \psi_{2n} + t_B \psi_{2n+2} = 0,
\]
\[
t_A \psi_{2n+1} + t_B \psi_{2n+3} = 0.
\] (34)

By solving (34) recursively from the outer most cite, we obtain the relation \(\psi_{2n+1} = [t_A/t_B]^n \psi_1\) and the analytic form of the eigen mode for odd cite \(n\),

\[
\psi_{2n+1} = \sqrt{1 - \left|\frac{t_A}{t_B}\right|^2} \left(\frac{t_A}{t_B}\right)^n \psi_1.
\] (35)

On the other hand, the wave function is zero for even cite \(\psi_{2n} = 0\). In order for the edge modes to exist, the eigen-function must be normalizable, whose condition is

\[
\left|\frac{t_A}{t_B}\right| < 1.
\] (36)

It is consistent with the topological phase.

For the random case, the Hamiltonian is altered to be

\[
H = \begin{pmatrix}
0 & t_{A,1} & 0 & 0 & \cdots \\
t_{A,1} & 0 & t_{B,1} & 0 & \cdots \\
0 & t_{B,1} & 0 & t_{A,2} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots 
\end{pmatrix}.
\] (37)

Then the zero-admittance solution is given by

\[
\psi_{2n+1} = \frac{\prod t_{A,n}}{\sqrt{1 + \sum \prod t_{B,n}^2}}
\] (38)

and \(\psi_{2n} = 0\).

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