The tail process and tail measure of continuous time regularly varying stochastic processes

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Abstract

The goal of this paper is to investigate the tools of extreme value theory originally introduced for discrete time stationary stochastic processes (time series), namely the tail process and the tail measure, in the framework of continuous time stochastic processes with paths in the space $D$ of càdlàg functions indexed by $\mathbb{R}$, endowed with Skorohod’s $J_1$ topology. We prove that the essential properties of these objects are preserved, with some minor (though interesting) differences arising. We first obtain structural results which provide representation for homogeneous shift-invariant measures on $D$ and then study regular variation of random elements in $D$. We give practical conditions and study several examples, recovering and extending known results.

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1 Introduction

The goal of this paper is to study the extreme value theory of continuous time regularly varying processes stochastic processes in the light of the recent developments of this theory for regularly varying time series (that is stochastic processes indexed by $\mathbb{Z}$ with values in $\mathbb{R}^d$), which we recall now.

A time series $X = \{X_j, j \in \mathbb{Z}\}$ is said to be regularly varying with tail index $\alpha$ if all its finite dimensional distributions are regularly varying with the same index of regular variation $\alpha$ and under the same scaling. This means that the finite dimensional distributions are in the domain of attraction of a multivariate Fréchet distribution with the same tail index and under the same scaling. The extremal behaviour of such a stationary time series is now well understood and characterized by either one of two objects: the tail process, introduced by [BS09] and the tail measure introduced in the unpublished manuscript [SO12].

The tail process, which will be denoted $Y$ throughout this paper, describes the asymptotic behaviour of a stationary time series given an extreme value at time zero and provides convenient representations of the limiting quantities which arise in the statistics of extremes for time series. The tail process can be formally defined for a non stationary time series but will contain too little information to be useful.

Alternatively, a time series $X$ can be considered as a random element of the space $(\mathbb{R}^d)^\mathbb{Z}$ endowed with the product topology which makes it Polish. Regular variation in $(\mathbb{R}^d)^\mathbb{Z}$ can be defined using the theory of vague convergence of [Kal17] which will be described more precisely in Section 2 and appendix A. In that framework a time series is regularly varying if there exists a non zero measure $\nu$ on $(\mathbb{R}^d)^\mathbb{Z}$ scaling $a_n$ such that the measure $n\mathbb{P}(a_n^{-1}X \in A)$ converges to $\nu(A)$ for all Borel sets $A \subset (\mathbb{R}^d)^\mathbb{Z}$ which are contained in the complement of an open neighborhood of the null sequence $\mathbf{0}$, and are continuity sets of $\nu$. 

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In the terminology of extreme value theory for finite dimensional random vectors, \( \nu \) is the exponent measure of \( X \). If the time series is stationary, then its tail measure is shift-invariant. The finite dimensional projections of the tail measure are the exponent measures of the finite dimensional distributions of the time series \( X \), which therefore characterize the tail measure. Therefore, both definitions of regular variation of a time series are equivalent and since the tail process is defined by the finite dimensional distributions, the tail measure must entirely determine the tail process. Indeed, by a suitable choice of scaling, the tail measure can be made a probability measures on the set of sequences which exceed 1 (in norm) at time zero, and the distribution of the tail process is then the tail measure restricted to this set. The converse was proved (independently) by [PS18] and [Jan19]: the tail measure can be recovered from the tail process. This is essentially due to homogeneity and shift-invariance of the tail measure.

The tail process and tail measure of a stationary time series have been exhaustively studied and are extremely useful tool to understand the extremal behavior of a time series and describe the asymptotic distributions of the partial maximum process, the partial sum process (when the tail index is in \((0, 2)\)), and many statistics such as estimators of the tail index, the extremal index, and other extremal characteristics. A thorough treatment of the subject is given in [KS20].

There is a restriction to the validity of the previous statement: the extremal behaviour must not be the same as that of a sequence of i.i.d. random variables. In that case, the finite dimensional distributions and the time series are said to be extremally independent and the tail measure does not contain more information than the marginal distribution. Different tools are then needed and this case will not be considered in this paper.

The extreme value theory of continuous time stochastic processes is a very ancient and still active field of research. An important part of it is dedicated to Gaussian and related processes such as diffusion processes. See for instance the monographs [LLR83], [Ber92] and more recently [AW09]. There are some early references on extremes of continuous time regularly varying processes such as [Roo78] which deals with moving averages with stable innovations in discrete and continuous time, but the bulk of the literature seems to be more recent. See for instance among many other, [Sam04] (stable processes), [Fas05] (moving averages driven by a Lévy process), [FK07] (mixed moving averages), [WS10] (max-stable processes). There is one important difference between the Gaussian and related processes first mentioned and regularly varying processes. The former are typically extremally independent, that is the extremal behavior of their finite dimensional distributions is in first approximation the same as that of a vector with independent components, whereas the extremal behaviour of the latter typically inherits some form of serial dependence. Therefore, the tail process and tail measure are useless for the former class of processes (or rather for regularly varying transformations) but can be considered for the latter.

The main purpose of this work is to extend the theory of the tail process and the tail measure established for stationary time series to regularly varying stationary continuous time stochastic process. Stationarity is a restriction, but it is a usual assumption, especially
in view of statistical applications, and the tail process is only of interest in the context of stationarity.

The tail measure and the tail process of an $\mathbb{R}^d$-valued regularly varying stochastic process $X = \{X_t, t \in T\}$ indexed by an arbitrary index set $T$ can be defined exactly as in discrete time. If the finite dimensional distributions of the process are regularly varying, the admit an exponent measure and the family of these exponent measures satisfy a consistency property. These exponent measures are not finite but [SO12] proved that there exists a measure $\nu$ on $(\mathbb{R}^d)^T$ endowed with the product topology, whose finite dimensional projections are the exponent measures. As previously, the tail process can be defined as the weak limit of the finite dimensional distributions of $X$ given that $|X_0| > x$, as $x \to \infty$.

However, using this definition, no information is given on the paths of the tail process nor on the support of the tail measure. In full generality, the tail measure need not even be $\sigma$-finite, see [SO12, Proposition 2.4]. A nearly indispensable restriction is to consider only separable processes. For processes indexed by $\mathbb{R}$, a natural framework is to consider only processes with almost surely càdlàg paths, that is random element in the space $D(\mathbb{R}, \mathbb{R}^d)$ (hereafter simply written $D$) endowed with the $J_1$ topology, which is a Polish space. In this framework, it is then natural to define the regular variation of random element in $D$ as the convergence of the measure $TP(a^{-1}_T X \in \cdot)$ to a measure $\nu$ in the following sense: for all Borel sets $A$ which are continuity sets of $\nu$ and are separated from the null map $0$, that is sets which are contained in the complement of a neighborhood of $0$ \[ \lim_{n \to \infty} TP(X \in a_T A) = \nu(A). \]

This mode of convergence is simply called vague convergence in [Kal17]. This concept of regular variation in Polish spaces was originally developed in [HL06] and regular variation of càdlàg stochastic processes was first considered in [HL05], only for processes indexed by $[0,1]$, that is random elements in $D([0,1], \mathbb{R}^d)$.

The tail measure $\nu$ is then an exponent measure and as such must be homogeneous. In addition, if the process $X$ is stationary, the tail measure is shift-invariant. As already mentioned, its finite dimensional projections are the exponent measures of the finite dimensional distributions of $X$. It is also immediate that the distribution of the tail process is the tail measure restricted to the set of functions $f \in D$ such that $|f(0)| > 1$.

Taking these properties as definitions, the tail measure and tail process can be studied without reference to an underlying stochastic process. This is done in Section 2 whose main purpose is to extend the structural results obtained for tail measures on $(\mathbb{R}^d)^2$ by [PS18] to tail measures on $D$, defined as shift-invariant homogeneous measures, finite on sets separated from $0$.

The main result of Section 2 is Theorem 2.3 which states that, similarly to the discrete time case, the tail process determines the tail measure, and a tail measure always always has a spectral representation, that is a pseudo polar decomposition with respect to the semi-norm $f \mapsto |f(0)| \in D$. See (2.6) for a precise definition.

Then we obtain in Theorem 2.8 necessary and sufficient conditions for mixed-moving average representations of the tail measure. This result subsumes those originally obtained for max-
stable processes by [DK17] (in particular their Theorem 3), where no reference is made to the tail measure, although the link is implicit since the tail measure of a max-stable process determines its distribution. These results can also be expressed in the language of ergodic theory, in terms of dissipative and conservative flow representations. We will not pursue this direction in this paper to keep it at a reasonable length. See [WRS13] for similar results for sum-stable processes whose distribution is also determined by the tail measure.

Theorem 2.3 and Theorem 2.8 are mutatis mutandis the same as in discrete time. The main difference between discrete and continuous time is the role of certain maps, called anchoring maps by [BP18], one of which being the infargmax functional which finds the first time where the maximum of a sequence is achieved (see Example 2.11 for a precise definition). These maps play a crucial role in the study of the tail process in discrete time. In particular, and under certain conditions, the quantity \( P(\mathcal{I}(Y) = 0) \) is positive and the same for (almost) all anchoring maps \( \mathcal{I} \) and the tail measure can be expressed in terms of the tail process conditioned on \( \mathcal{I}(Y) = 0 \). See [PS18, Section 3.3]. This probability is denoted by \( \vartheta \) and called the candidate extremal index, since it is related to the classical extremal index which will be discussed hereafter.

In continuous time, the event \( \mathcal{I}(Y) = 0 \) has in general a zero probability, and conditioning is more difficult to handle. Therefore anchoring maps and conditioning had to be replaced. It turned out that the appropriate tool in continuous time is the exceedence functional \( \mathcal{E} \), defined for a measurable function \( f : \mathbb{R} \to \mathbb{R}^d \) by \( \mathcal{E}(f) = \int_{-\infty}^{\infty} \mathbb{1}\{|f(t)| > 1\} \, dt \). We will see in Sections 2.3 and 4 that anchoring maps of may behave very differently than in discrete time and that conditioning on different anchoring maps may produce different results.

Section 2 is concluded with certain identities for quantities which appear as limits of certain statistics of regularly varying processes. Depending on the method used to obtain these limits, they can be expressed in terms of the different objects related to the tail process. It is therefore convenient and important to know that these expressions are equivalent and that the summability or integrability conditions that guarantee their existence are equivalent. The usefulness of these identities will be illustrated in Section 3.4.

All the results of Section 2, in addition to be of intrinsic interest, are important to understand the extremal behaviour of regularly varying stochastic processes, and more particularly so for max-stable and sum-stable processes whose distribution is entirely determined by the tail measure, or equivalently the tail process. Thus they are also necessary preliminaries to the proper investigation of regularly varying stochastic processes, done in Section 3 which we describe now.

As already mentioned, in discrete time, the two definitions of regular variation of a time series, either by means of finite dimensional distributions or by considering the time series as a random element of the sequence space are equivalent. This is obviously not the case in continuous time. Thus our first task is to relate finite dimensional convergence and convergence in \( D \). This is done in Theorem 3.2. This result extends those of [HL05, HL06] which dealt only with \( D([0,1]) \). It states a necessary and sufficient condition for regular
variation in $\mathcal{D}$ in terms of convergence of finite dimensional distributions and a tightness criterion which extends the usual one in terms of the $J_1$-modulus of continuity. The proof of the direct implication is omitted since it is an immediate adaptation of the proof of the corresponding result in [HL05, Theorem 10]. However, for the converse, we take advantage of the results of Section 2 to obtain a more constructive proof.

Importantly, we also obtain in Theorem 3.2 that regular variation in $\mathcal{D}$ implies the weak convergence in $\mathcal{D}$ of the process $X$, conditioned on $|X_0| > x$ when $x \to \infty$, to its tail process $Y$ which is thus a random element in $\mathcal{D}$.

Note that the latter result entails an important difference with the discrete time case: we exclude extremal independence, that is the case where the extremal behaviour of the finite dimensional distributions are the same as those of i.i.d. vectors. This may happen for instance for a regularly varying marginal transformation of a Gaussian process. This is unavoidable, since the tail process of such a process would be identically zero, except at time zero. Thus it is not càdlàg and convergence to the tail process cannot hold in $\mathcal{D}$ endowed with the $J_1$ topology. However, this is not a considerable loss, since the tail process would be useless in that case.

From there on, we are able to easily extend to continuous time processes the main results of the extreme value theory discrete time series developed by means of the tail process in a series of papers ranging from [BS09] to [BPS18]. The most important object that we consider is related to the point process of exceedences which measures the time spent by the time series above a high threshold, introduced in discrete time as early as [Res86]. In continuous time, it was studied under the name excursion random measure by [HL98], which builds on the seminal paper [DH95] dealing with discrete time processes. Under the mixing condition (3.26) which is related to the well-known condition $D$ of [Lea74], and under condition (3.8) which yields the limit of excursions over a high level within a small portion of the path (first used in [DH95]), we obtain in Theorem 3.10 the weak convergence of a generalization of the excursion random measure to a Poisson point process on (a subspace of) $\mathcal{D}$.

The convergence of the excursion random measure has many application of which we cite only one, related to the convergence of the sample maxima. Recall that for an i.i.d. sequence with regularly varying marginal distribution, if $a_T$ is the quantile of order $1 - T^{-1}$, then $a_T \max(X_1, \ldots, X_T)$ ($T$ being restricted to integer values) converges weakly to a Fréchet distribution, say $F_\alpha$. For a stationary sequence, this convergence may still hold or may hold to $F_\alpha^\theta$, where $\theta \in [0, 1]$ is called the extremal index. Exact computation of the extremal index is not often easy or possible, but the tail process provides several convenient representations of the extremal index. In continuous time, it is still possible to define the extremal index as a real number $\theta$ such that $a_T^{-1} \sup_{0 \leq s \leq T} X_s$ converges weakly to $F_\alpha^\theta$. The essential difference is that in continuous time, the extremal index, if it exists is not confined to $[0, 1]$ but can take any value in $[0, \infty]$. The case $\theta = 1$ in discrete time or $\theta = \infty$ in continuous time corresponds to extremal independence; the case $\theta = 0$ is often called long range dependence in the extremes (different from other notions of long range dependence).
The conditions of Theorem 3.10 may be difficult to check. The only easy case is for \( m \)-dependent processes, that is processes such that past and future separated by \( m \) are independent. In Theorem 3.14, we show that if a process admits a suitable sequence of \( m \)-dependent approximations, then the conclusions of Theorem 3.10 hold, even if

As a consequence of Theorems 3.10 and 3.14, we prove the existence of the extremal index in \((0, \infty)\) and obtain representations in terms of the tail process. Some of these representations had been obtained in the context of max-stable processes by [DH19].

Here we must stress again that we only consider regulary varying processes which are not extremally independent, i.e. whose extremal behaviour has kept some form of temporal dependence. In particular, the assumptions of Theorem 3.10 exclude both extremal independence, i.e. \( \theta = \infty \) and long range dependence, i.e. \( \theta = 0 \). Convergence of the point process of clusters necessitates different normalization in both cases and much more sophisticated techniques. See for instance [Roy17, Section 8] for examples of (discrete time) stable processes with \( \theta = 0 \).

We conclude this paper with several illustrative but relatively simple examples in Section 4. We start with max-stable (Section 4.1) and sum-stable (Section 4.2) processes, for which we recover the know results of the literature and also prove the convergence of the point process of exceedences. Then we study a general class of functional moving averages in Section 4.3, the simplest example of which is the well-known shot noise process (Section 4.3.2).

**Notation**

We will use the usual letters \( f, g \), etc. to denote functions and also boldface letters such as \( x, y \), depending on the context. We use indifferently \( y_t \) of \( y(t) \) for the value of \( y \) at \( t \in \mathbb{R} \). We use capitalized boldface (\( X, Y \ldots \)) for stochastic processes indexed by \( \mathbb{R} \) (or any subset).

The space \( D(\mathbb{R}, \mathbb{R}^d) \) is the space of càdlàg functions defined on \( \mathbb{R} \) and when there is no risk of confusion, we simply write \( D \). The null function is denoted by \( 0 \). Given an arbitrary norm on \( \mathbb{R}^d \), we define \( D_0 = \{ y \in D : \lim_{|t| \to \infty} |y_t| = 0 \} \) and \( D_\alpha = \{ y \in D : \int_{-\infty}^{\infty} |y_t|^\alpha \, dt < \infty \} \).

We will also use the following notation. For a function \( y \) defined on \( \mathbb{R} \) and \( a < b \), we write \( y_{a,b} \) for the restriction of \( y \) to the interval \([a, b)\). With an abuse of notation, it will denote either a function defined on \([a, b)\) or the function defined on \( \mathbb{R} \) which is equal to \( y \) on \([a, b)\) and vanishes outside \([a, b)\). We further define \( y^* = \|y\|_\infty = \sup_{t \in \mathbb{R}} |y_t|, y^*_{a,b} = \sup_{a \leq t \leq b} |y_t| \). For a measurable \( y \) and \( p > 0 \), we set \( \|y\|^p_p = \int_{-\infty}^{\infty} |y(t)|^p \, dt \).

The backshift operator is defined by \( B^tf = f(\cdot - t) \) for all functions \( f \) defined on \( \mathbb{R} \).
2 Representations of tail measures on $\mathcal{D}$

Let the space $\mathcal{D}(\mathbb{R}, \mathbb{R}^d)$, hereafter simply $\mathcal{D}$, be endowed by the $J_1$ topology and the related Borel $\sigma$-field. See Appendix B for its definition and basic properties. We say that a subset $A$ of $\mathcal{D}$ is separated from 0 if it is included in the complement of an open neighborhood of the null function 0. This means that there exist real numbers $a \leq b$ such that

$$\inf_{y \in A} \sup_{a \leq t \leq b} |y_t| > 0 .$$

We will denote by $\mathcal{B}_0$ the class of sets separated from 0. A Borel measure $\mu$ on $\mathcal{D}$ will be said $\mathcal{B}_0$-boundedly finite if $\mu(\{0\}) = 0$ and $\mu(A) < \infty$ for all Borel sets $A$ in $\mathcal{B}_0$. The measurable sets in $\mathcal{B}_0$ are measure determining for $\mathcal{B}_0$-boundedly finite measures. Thus, a $\mathcal{B}_0$-boundedly finite measure is determined by the values $\nu(H)$ for all bounded or non-negative measurable maps $H$ with support in $\mathcal{B}_0$. See [Kal17, Theorem 4.11] or [BP19, Theorem 4.1].

**Definition 2.1.** A tail measure on $\mathcal{D}$ endowed with its Borel $\sigma$-field is a $\mathcal{B}_0$-boundedly finite Borel measure $\nu$ such that

(i) $\nu(\{0\}) = 0$;

(ii) $\nu(\{y \in \mathcal{D} : |y_0| > 1\}) = 1$;

(iii) there exists $\alpha > 0$ such that $\nu(tA) = t^{-\alpha} \nu(A)$ for all Borel subsets of $\mathcal{D}$.

For obvious reasons, the positive number $\alpha$ will be called the tail index of $\nu$. Since a $\nu$ is boundedly finite, it also holds that $\nu(\{y \in \mathcal{D} : |y_t| > 1\}) < \infty$ for all $t \in \mathbb{R}$, and more importantly $\nu$ is $\sigma$-finite.

By assumption, the measure $\nu$ restricted to $\{y \in \mathcal{D} : |y_0| > 1\}$ is a probability measure, so we can consider a $\mathcal{D}$-valued random element $Y$ defined on a probability space $(\Omega, \mathcal{F}, P)$ with distribution $\nu(\cdot \cap \{y \in \mathcal{D} : |y_0| > 1\})$, called a tail process. The homogeneity of $\nu$ implies that $|Y_0|$ has a Pareto distribution with tail index $\alpha$ and is independent of the process $\Theta$ defined by $\Theta = Y_0^{-1} Y$, called the spectral tail process. The spectral tail process can be viewed as a spectral decomposition of the tail measure with respect to the pseudo-norm $|y_0|$. That is

$$\int_{\mathcal{D}} H(y) 1\{|y_0| > 0\} \nu(dy) = \int_0^\infty \mathbb{E}[H(u\Theta)] \alpha u^{-\alpha - 1} du .$$

See [KS20, Section 5.2.1].

The shift-invariance of $\nu$ induces a very important property of the tail process.

**Lemma 2.2.** Let $\nu$ be a shift-invariant tail measure on $\mathcal{D}$ with tail index $\alpha$ and associated tail and spectral tail processes $Y$ and $\Theta$. For every non-negative measurable map $H$ on $\mathcal{D}$ and $x > 0$,

$$\mathbb{E}[H(Y) 1\{|Y_1| > x\}] = x^{-\alpha} \mathbb{E}[H(xB^t Y) 1\{|xY_{-1}| > 1\}] ,$$

$$\mathbb{E}[H(|\Theta_1|^{-1} \Theta) |\Theta_1|^\alpha] = \mathbb{E}[H(B^t \Theta) 1\{|\Theta_{-1}| \neq 0\}] .$$
These two properties are equivalent and the form \((2.3)\) was originally obtained in the context of discrete time stationary time series and called the time change formula by [BS09]. The version \((2.2)\) was obtained by [PS18]. The proof of the result in continuous time is exactly the same as in discrete time but we give the three-lines proof of \((2.2)\) for completeness.

**Proof.** By the definition of \(Y\), the shift-invariance and homogeneity of \(\nu\), we have
\[
E[H(Y)1\{|Y_t| > x\}] = \int_{\mathcal{D}} H(y)1\{|y_t| > x\}1\{|y_0| > 1\} \nu(dy)
\]
\[
= x^{-\alpha} \int_{\mathcal{D}} H(xB^t y)1\{|y_0| > 1\}1\{|xy_{-t}| > 1\} \nu(dy)
\]
\[
= x^{-\alpha} E[H(xB^t Y)1\{|xY_{-t}| > 1\}].
\]

\[\square\]

### 2.1 Spectral representation

The main result of this section is a representation theorem for shift-invariant tail measures. It extends or complements several results of the literature. It extends [DHS18, Theorem 2.4] to the continuous time case and provides a constructive proof in a restricted context of [EM18, Proposition 2.8] which deals with homogeneous measures in abstract cones.

**Theorem 2.3.** A Borel measure \(\nu\) on \(\mathcal{D}\) is a shift-invariant tail measure if and only if there exists a \(\mathcal{D}\)-valued process \(Z\), called a spectral process for \(\nu\) such that \(\mathbb{P}(Z = 0) = 0\),
\[
0 < E \left[ \sup_{a \leq s \leq b} |Z_s|^\alpha \right] < \infty , \tag{2.4}
\]
for all real numbers \(a \leq b\),
\[
E[|Z_t|^\alpha H(Z)] = E[|Z_0|^\alpha H(B^t Z)] \tag{2.5}
\]
for all \(t \in \mathbb{R}\) and bounded measurable \(0\)-homogeneous maps \(H\) on \(\mathcal{D}\), and
\[
\nu = \int_0^\infty E[\delta_u Z] \alpha u^{-\alpha - 1} du . \tag{2.6}
\]

Furthermore, a shift-invariant tail measure is entirely determined by its tail process \(Y\) whose distribution \(\mathbb{P}_Y\) is related to any spectral process \(Z\) by
\[
\mathbb{P}_Y = E \left[ |Z_0|^\alpha \delta_{\frac{Y}{Z_0}} \right] , \tag{2.7}
\]
where \(Y\) is a Pareto random variable with tail index \(\alpha\).
Proof. If \( \nu \) is defined by (2.6) with \( Z \) satisfying the stated properties, then it is a shift-invariant tail measure. Note that the standardization of the tail measure imposed in (ii) implies that \( \mathbb{E}[|Z_0|] = 1 \). We prove the converse. Let \( \nu \) be a shift-invariant tail measure and let \( f \) be a bounded positive continuous function on \( \mathbb{R} \) such that \( \int_{-\infty}^{\infty} f(t)dt = 1 \). For \( y \in \mathcal{D} \), define \( S_f(y) = \int_{-\infty}^{\infty} f(t)|y_t|^\alpha dt \). By the time change formula (2.3), we have \( \mathbb{E}[|\Theta_t|] = \mathbb{P}(\Theta \neq 0) \leq 1 \) for all \( t \in \mathbb{R} \), thus \( \int_{-\infty}^{\infty} \mathbb{E}[|\Theta_t-s|]f(s)ds < \infty \) for all \( t \) by assumption on \( f \).

Let \( T \) be a real-valued random variable, independent of \( Y \) with density \( f \). The previous property can be expressed as \( \mathbb{E}[S_f(B^T \Theta)] < \infty \). Thus \( \mathbb{P}(S_f(B^T Y) < \infty) = 1 \) and since \( Y \) is càdlàg, \( \mathbb{P}(S_f(Y) > 0) = 1 \). Therefore, we can define a process \( Z \) by

\[
Z = (S_f(B^T Y))^{-1/\alpha}B^T Y = (S_f(B^T \Theta))^{-1/\alpha}B^T \Theta.
\] (2.8)

Obviously, \( \mathbb{P}(Z = 0) = 0 \). Let the measure in the right-hand side of (2.6) be denoted by \( \nu_f \).

By the first part of the proof, \( \nu_f \) is a tail measure on \( \mathcal{D} \), hence is \( \sigma \)-finite. Let \( H \) be a non-negative measurable map, \( \epsilon > 0 \) and \( t \in \mathbb{R} \). Applying the time change formula (2.2) and the homogeneity of the functional \( S_f \), we obtain

\[
\int_{\mathcal{D}} H(y) 1\{|y_t| > \epsilon\} \nu_f(dy) = \int_0^\infty \mathbb{E}[H(uZ) 1\{|u| > \epsilon\}] \alpha u^{-\alpha-1} du
\]

\[
= \int_0^\infty \int_{-\infty}^{\infty} \mathbb{E}\left[H\left(\frac{uB^s Y}{S_f^{1/\alpha}(B^s Y)}\right) 1\left\{|u| > \epsilon\right\}\right] \alpha u^{-\alpha-1} f(s)ds du
\]

\[
= \epsilon^{-\alpha} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbb{E}\left[H\left(\epsilon u B^{s-t} Y\right) 1\{|u| > \epsilon\}\right] \alpha u^{-\alpha-1} f(s)ds du
\]

\[
= \epsilon^{-\alpha} \int_{-\infty}^{\infty} \mathbb{E}\left[H\left(\epsilon B^s Y\right) 1\{|y_{s-t}| > u\}\right] \alpha u^{-\alpha-1} f(s)ds
\]

\[
= \epsilon^{-\alpha} \int_{-\infty}^{\infty} H(\epsilon B^s Y) 1\{|y_0| > 1\} \nu(dy) = \int_{\mathcal{D}} H(\epsilon B^s Y) 1\{|y_t| > \epsilon\} \nu(dy).
\]

Let \( \epsilon > 0 \) and \( H \) be a non-negative measurable map on \( \mathcal{D} \) such that \( H(y) = 0 \) if \( y^* \leq \epsilon \). Then the previous identity yields

\[
\nu_f(H) = \int_{\mathcal{D}} \frac{H(y)}{\mathcal{E}_f(\epsilon^{-1} y)} 1\{|y_t| > \epsilon\} \nu_f(dy) = \int_0^\infty \int_{\mathcal{D}} \frac{H(y)}{\mathcal{E}_f(\epsilon^{-1} y)} 1\{|y_t| > \epsilon\} \nu_f(dy) dt
\]

\[
= \int_{-\infty}^{\infty} \int_{\mathcal{D}} \frac{H(y)}{\mathcal{E}_f(\epsilon^{-1} y)} 1\{|y_t| > \epsilon\} \nu(dy) dt = \int_{\mathcal{D}} H(y) \frac{\mathcal{E}_f(\epsilon^{-1} y)}{\mathcal{E}_f(\epsilon^{-1} y)} \nu(dy) = \nu(H)
\]

As already noted, the class of such maps \( H \) is measure determining, thus we have proved (2.6). Since \( Z \) depends only on the tail process \( Y \), this shows that the tail measure is completely determined by its tail process.
We now prove the other stated properties of $Z$. For all $a < b$,

\[
\mathbb{E}\left[\sup_{a \leq s \leq b} |Z_s|^\alpha\right] = \int_0^\infty \mathbb{P}(uZ_{a,b}^* > 1) \alpha u^{-\alpha - 1} du
\]

\[
= \nu\left(\{y \in \mathcal{D} : y_{a,b}^* > 1\}\right) < \infty,
\]

since the set $\{y \in \mathcal{D} : y_{a,b}^* > 1\}$ is separated from 0, hence has finite $\nu$-measure. This proves (2.4).

Finally, for a bounded measurable 0-homogeneous map $H$ on $\mathcal{D}$ and $t \in \mathbb{R}$, we have by (2.6),

\[
\mathbb{E}[|Z_t|^\alpha H(Z)] = \int_0^\infty \mathbb{E}[H(Z) \mathbb{1}\{|Z_t| > 1\}] \alpha r^{-\alpha - 1} dr
\]

\[
= \int_{\mathcal{D}} H(y) \mathbb{1}\{|y_t| > 1\} \nu(dy)
\]

\[
= \int_{\mathcal{D}} H(B^t y) \mathbb{1}\{|y_0| > 1\} \nu(dy) = \mathbb{E}[H(B^t Y)] .
\]

In the last line, we used the shift-invariance of $\nu$ and the definition of the tail process. For $t = 0$, this yields (2.7). Replacing $H$ by $H \circ B^t$ yields (2.5).

The main difference between the tail process and spectral processes related to a tail measure is that the former is unique in distribution. Note that the terminology is a bit confusing since in general, a spectral tail process $\Theta$ is not a spectral process, except if $\mathbb{P}(\Theta_t = 0) = 0$ for all $t$ since in that case the time change formula (2.3) is equivalent to (2.5), in view of the fact that $\mathbb{P}(|\Theta_0| = 1) = 1$.

A tail process $Y$ satisfies $\mathbb{P}(|Y_0| > 1) = 1$ and the time change formula (2.2). As a consequence of Theorem 2.3, we show that there is a one-to-one correspondence between $\mathcal{D}$-valued processes which satisfy these two properties and an additional boundedness condition and shift-invariant tail measures on $\mathcal{D}$.

**Corollary 2.4.** Let $Y$ be a random element in $\mathcal{D}$ such that

- $\mathbb{P}(|Y_0| > 1) = 1$;
- the time change formula (2.2) holds; 
- for all $a < b$,

\[
\int_a^b \mathbb{E}\left[\frac{1}{\int_a^b 1_{Y_{t-s} > 1} dt}\right] ds < \infty .
\]

Then there exists a unique shift-invariant tail measure $\nu$ such that the distribution of $Y$ is $\nu$ restricted to the set $\{y \in \mathcal{D} : |y_0| > 1\}$. 

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Proof. The properties of $Y$ used to define the process $Z$ in (2.8) are those assumed here so we can define $Z$ and a measure $\nu$ on $D$ by (2.6). This measure is homogeneous and shift-invariant by construction and we must prove that $Y$ is the tail process associated to $\nu$ and that $\nu$ is boundedly finite on $D \setminus \{0\}$. By definition and by the time change formula, we have

$$
\nu(H\{|y_0| > 1\}) = \int_0^\infty \mathbb{E}\left[ \frac{H(rB^T Y)1\{r|Y_t| > 1\}}{S_f(B^T Y)} \right] \alpha r^{-a-1} dr \\
= \int_0^\infty \mathbb{E}\left[ \frac{H(Y)1\{|Y_T| > r\}}{S_f(Y)} \right] \alpha r^{-a-1} dr \\
= \mathbb{E}\left[ \frac{H(Y)|Y_T|^a}{S_f(Y)} \right] = \int_{-\infty}^\infty \mathbb{E}\left[ \frac{H(Y)|Y_t|^a}{S_f(Y)} \right] f(t) dt = \mathbb{E}[H(Y)].
$$

We now prove that $\nu$ is boundedly finite. As already noted, this is equivalent to proving that $\mathbb{E}[(Z_{a,b}^*)^a] < \infty$ for all $a < b$. Define the map $\mathcal{E}_{a,b}$ on $D$ of exceedences between $a$ and $b$ by $\mathcal{E}_{a,b}(y) = \int_a^b \{\{|y_s| > 1\}\} ds$. By definition of $Z$, we have

$$
\mathbb{E}[(Z_{a,b}^*)^a] = \mathbb{E}\left[ \frac{(B^T Y_{a,b})^a}{S_f(B^T Y)} \right] = \int_0^\infty \mathbb{E}\left[ \frac{1\{rB^T Y_{a,b} > 1\}}{S_f(B^T Y)} \right] \alpha r^{-a-1} dr \\
= \int_a^b \int_0^\infty \mathbb{E}\left[ \frac{1\{rY_{s-t} > 1\}}{S_f(B^T Y)\mathcal{E}_{a,b}(rB^T Y)} \right] \alpha r^{-a-1} dr ds .
$$

Applying now the time change formula (2.2) and the definition of $T$ yields

$$
\mathbb{E}[(Z_{a,b}^*)^a] = \int_a^b \int_0^\infty \mathbb{E}\left[ \frac{1\{|Y_{s-t} > r\}}{S_f(B^T Y)\mathcal{E}_{a,b}(B^T Y)} \right] \alpha r^{-a-1} dr ds = \int_a^b \mathbb{E}\left[ \frac{1}{\mathcal{E}_{a,b}(B^T Y)} \right] ds .
$$

The last term is finite by assumption, thus $\nu$ is boundedly finite on $D_0$. 

Before turning to other representations of tail measures, we state and prove a simple lemma which will nevertheless be very important. Define the exceedence functional (or occupation time of $(1, \infty)$) $\mathcal{E}$ on $D$ by

$$
\mathcal{E}(y) = \int_{-\infty}^\infty \{\{|y_t| > 1\}\} dt.
$$

This map is well defined and takes value in $[0, \infty]$. If $Y$ is a tail process in $D$, then $\mathbb{P}(\mathcal{E}(Y) > 0) = 1$. In discrete time, the exceedence functional $\mathcal{E}_d$ is defined by replacing the integral is replaced by a series, and since a tail process $Y$ satisfies $|Y_0| > 1$ almost surely, it holds that $\mathcal{E}_d(Y) \geq 1$ almost surely, hence $\mathbb{E}[\mathcal{E}_d^{-1}(Y)] \leq 1$. In continuous time no such trivial bound holds. However, it still holds that the expectation of the inverse of the exceedence functional it is finite.
Lemma 2.5. Let $\nu$ be a shift-invariant tail measure on $\mathcal{D}$ and $Y$ be its tail process. Then

$$\mathbb{E}\left[\frac{1}{\mathcal{E}(Y)}\right] = \lim_{T \to \infty} \frac{1}{T} \nu\left(\left\{ y \in \mathcal{D} : \sup_{0 \leq t \leq T} |y_t| > 1 \right\}\right) < \infty.$$  \hspace{1cm} (2.10)

Proof. By Fatou’s lemma, we have

$$\mathbb{E}\left[\frac{1}{\mathcal{E}(Y)}\right] \leq \liminf_{T \to \infty} \int_0^1 \mathbb{E}\left[\frac{1}{\int_{-T}^{T(t-s)} 1\{|Y_t| > 1\} dt}\right] ds.$$  

By definition of the tail process and the shift-invariance of $\nu$ and by Fubini theorem, we have

$$\int_0^1 \mathbb{E}\left[\frac{1}{\int_{-T}^{T(t-s)} 1\{|Y_t| > 1\} dt}\right] ds = \frac{1}{T} \int_0^T \int_{-D} \int_{-T}^{T(s)} 1\{|y_s| > 1\} ds \nu(dy) = \frac{1}{T} \int_0^T \int_{-D} \int_{-T}^{T(s)} 1\{|y_s| > 1\} ds \nu(dy) = \frac{1}{T} \nu\left(\left\{ y \in \mathcal{D} : \sup_{0 \leq t \leq T} |y_t| > 1 \right\}\right) < \infty.$$  

The last term is finite since $\nu$ is boundedly finite and the last equality holds since for a càdlàg function $y$, $\sup_{0 \leq t \leq T} |y_t| > 1$ if and only if $\int_0^T 1\{|y_t| > 1\} dt > 0$, and we use the convention $0/0 = 0$. The shift-invariance of $\nu$ implies the subadditivity of the function $T \mapsto \nu\left(\left\{ y \in \mathcal{D} : \sup_{0 \leq t \leq T} |y_t| > 1 \right\}\right)$. Thus by Fekete’s lemma,

$$\lim_{T \to \infty} \frac{1}{T} \nu\left(\left\{ y \in \mathcal{D} : \sup_{0 \leq t \leq T} |y_t| > 1 \right\}\right) \in [0, \infty).$$

We conclude that $\mathbb{E}[\mathcal{E}^{-1}(Y)] < \infty$ and equality holds in (2.10) by the dominated convergence theorem. \hfill \square

2.2 Mixed moving average representations

We say that a tail measure $\nu$ has a mixed moving average representation if there exists $\vartheta \in (0, \infty)$ and a process $Q \in \mathcal{D}$ such that $P(Q^*) = 1$ and

$$\nu = \vartheta \int_{-\infty}^{\infty} \int_0^{\infty} \mathbb{E}[\delta_{u B(t)}] u^{-\alpha-1} du dt.$$  \hspace{1cm} (2.11)
Since \( \nu(\{ y \in \mathcal{D} : |y_0| > 1 \}) = 1 \), this representation entails
\[
\vartheta \int_{-\infty}^{\infty} E[|Q_t|^\alpha]dt = 1. \tag{2.12}
\]
The positive number \( \vartheta \) will be called the candidate extremal index. Taking \( H_e = \mathcal{E}^{-1}(y)1\{|y_0| > 1\} \), (2.11) and the property \( \mathbb{P}(Q^* = 1) = 1 \) yield
\[
\mathbb{E}[\mathcal{E}^{-1}(Y)] = \nu(H_e) = \vartheta \int_{-\infty}^{\infty} \int_{0}^{\infty} E \left[ \frac{1\{|Q_{-t}| > 1\}}{\mathcal{E}(rQ)} \right] \alpha r^{-\alpha-1}drdt = \vartheta.
\]
If \( Q \) is a random element in \( \mathcal{D} \) such that \( \mathbb{P}(Q^* = 1) = 1 \) and for all \( a \leq b \),
\[
0 < \int_{-\infty}^{\infty} \mathbb{E}[|Q_{a-t,b-t}^*|^\alpha]dt < \infty, \tag{2.13}
\]
then (2.11) with \( \vartheta = \left( \int_{-\infty}^{\infty} \mathbb{E}[|Q_t|^\alpha]dt \right)^{-1} \) defines a shift-invariant tail measure which is supported on \( \mathcal{D}_0 \). The purpose of this section is to show that a tail measure supported on \( \mathcal{D}_0 \) admits the representation (2.11) with \( Q \) satisfy (2.13) in addition to \( \mathbb{P}(Q^* = 1) = 1 \). We will need two preliminary lemmas.

**Lemma 2.6.** Assume that \( \mathbb{P}(\mathcal{E}(Y) = \infty) = 0 \). Let \( H \) be a non-negative shift-invariant and 0-homogeneous measurable map on \( \mathcal{D}(\mathbb{R}) \). Then, for all \( x > 1 \),
\[
\mathbb{E}\left[ \frac{H(Y)1\{Y^* > x\}}{\mathcal{E}(Y)} \right] = x^{-\alpha} \mathbb{E}\left[ \frac{H(Y)}{\mathcal{E}(Y)} \right]. \tag{2.14}
\]

**Proof.** For a càdlàg function, \( y^* > x \) is equivalent to \( \mathcal{E}(x^{-1}y) > 0 \) and \( \mathcal{E}(y) < \infty \) implies \( \mathcal{E}(x^{-1}y) < \infty \) for all \( x > 1 \). Thus, by Fubini theorem and the time change formula (2.2), we obtain
\[
\mathbb{E}\left[ \frac{H(Y)1\{Y^* > x\}}{\mathcal{E}(Y)} \right] = \mathbb{E}\left[ \frac{H(Y)\mathcal{E}(x^{-1}Y)}{\mathcal{E}(Y)\mathcal{E}(x^{-1}Y)} \right] = \int_{-\infty}^{\infty} \mathbb{E}\left[ \frac{H(Y)1\{|Y_t| > x\}}{\mathcal{E}(Y)\mathcal{E}(x^{-1}Y)} \right] dt
= x^{-\alpha} \int_{-\infty}^{\infty} \mathbb{E}\left[ \frac{H(Y)1\{|Y_{-t}| > 1\}}{\mathcal{E}(Y)\mathcal{E}(xY)} \right] dt = x^{-\alpha} \mathbb{E}\left[ \frac{H(Y)}{\mathcal{E}(Y)} \right].
\]

A set \( A \) is said to be homogeneous if \( tA = A \) for all \( t > 0 \).

**Lemma 2.7.** Let \( \nu \) be a shift-invariant tail measure which admits the representation (2.6). Let \( A \) be a homogeneous set. Then \( \mathbb{P}(Z \in A) = 0 \iff \mathbb{P}(Y \in A) = \mathbb{P}(\Theta \in A) = 0 \). Let \( A \) be a shift-invariant homogeneous set. Then \( \nu(A) = 0 \iff \mathbb{P}(Z \in A) = 0 \). If (2.11) holds, then \( \nu(A) = 0 \iff \mathbb{P}(Q \in A) = 0 \).
Proof. If $A$ is homogeneous, then
\[
\mathbb{P}(Y \in A) = \mathbb{P}(\Theta \in A) = \mathbb{E}[|Z_0|^\alpha 1\{Z \in A\}].
\]
Since $\mathbb{E}[|Z_0|^\alpha] = 1$, this shows that $\mathbb{P}(Y \in A) = 0 \iff \mathbb{P}(Z \in A) = 0$.

If $A$ is homogeneous and shift-invariant, then
\[
\nu(A) = \int_0^\infty \mathbb{P}(uZ \in A) \alpha u^{-\alpha-1}du = \infty \times \mathbb{P}(Z \in A).
\]
This proves the second claim.

If (2.11) holds, then
\[
\nu(A) = \int_{-\infty}^\infty \int_0^\infty \mathbb{P}(uB^tQ \in A) \alpha u^{-\alpha-1}du \, dt = \infty \times \mathbb{P}(Q \in A).
\]
This proves the last claim. □

We now prove the mentioned result. It is a complement of [DK17, Theorem 3] which is only concerned with max-stable processes (and fields) with a more constructive proof. [DHS18, Theorem 3] is expressed in terms of spectral processes. We state our result in terms of tail processes and tail measures. We also refer to [DK17] for the link between these representations and the dissipative/conservative decomposition of non-singular flows.

Theorem 2.8. Let $\nu$ be a shift-invariant tail measure with tail process $Y$. The following statements are equivalent.

(i) There exists a random element $Q$ in $\mathcal{D}$ such that $\mathbb{P}(Q^* = 1) = 1$ and (2.11) holds;
(ii) $\mathbb{P}(Y \in \mathcal{D}_0) = 1$;
(iii) $\mathbb{P}(Y \in \mathcal{D}_\alpha) = 1$;
(iv) $\mathbb{P}(\mathcal{E}(Y) < \infty) = 1$;
(v) $\nu$ is supported on $\mathcal{D}_0$;
(vi) $\nu$ is supported on $\mathcal{D}_\alpha$.

If these conditions hold, then $\vartheta > 0$ and the distribution of $Q$ is given by
\[
\mathbb{P}_Q = \vartheta^{-1} \mathbb{E}\left[\frac{\delta_Y \nu}{\mathcal{E}(Y)}\right].
\]
Proof. Since $D_0$ and $D_\alpha$ are shift-invariant and homogeneous, Lemma 2.7 implies that $(v) \iff (ii)$ and $(vi) \iff (iii)$. Thus we will only prove the equivalence of (i), (ii), (iii) and (iv).

If (2.11) holds, then for all $a \leq b$,

$$
\nu(\{y \in D : y_{a,b}^* > 1\}) = \dot{\nu} \int_{-\infty}^\infty \int_0^\infty P(uQ_{a-t,b-t}^* > 1) \alpha u^{-\alpha-1} du dt = \dot{\nu} \int_{-\infty}^\infty E[(Q_{a-t,b-t}^*)^\alpha] dt .
$$

The finiteness of this integral implies that $P(Q \in D_0 \cap D_\alpha) = 1$. Thus (i) implies both (ii) and (iii) (by the same argument as above). Obviously, (ii) implies (iv) and we next prove that (iv) implies (i).

Proof of (iv) $\implies$ (i). Since $P(E(Y) < \infty) = 1$ and $Y$ is càdlàg, $\dot{\nu} = E[E^{-1}(Y)] > 0$ and $\dot{\nu} < \infty$ by Lemma 2.5. Let $Q$ be a $D$-valued random process whose distribution is given by (2.15). Let $H$ be a non-negative measurable map on $D$ with support separated from $0$. Then, there exists $\epsilon > 0$ such that $H(y) = 0$ when $y^* \leq \epsilon$. Applying Fubini’s theorem and the shift-invariance and homogeneity of $\nu$, we obtain

$$
\nu(H) = \epsilon^{-\alpha} \int_D H(\epsilon y) \nu(dy) = \epsilon^{-\alpha} \int_D H(\epsilon y) \mathbb{1}\{E(y) > 0\} \nu(dy) = \epsilon^{-\alpha} \int_{-\infty}^\infty \int_D H(\epsilon y) \mathbb{1}\{|y_1| > 1\} \nu(dy) dt
$$

$$
= \epsilon^{-\alpha} \int_{-\infty}^\infty \int_D H(\epsilon y) \frac{1}{E(y)} \mathbb{1}\{|y_0| > 1\} \nu(dy) dt = \epsilon^{-\alpha} \int_{-\infty}^\infty E\left[\frac{H(\epsilon B^t Y)}{E(Y)}\right] dt .
$$

Applying now (2.14) yields

$$
\nu(H) = \epsilon^{-\alpha} \dot{\nu} \int_{-\infty}^\infty E[H(\epsilon Y^* B^t Q)] dt = \epsilon^{-\alpha} \dot{\nu} \int_{-\infty}^\infty \int_1^\infty E[H(\epsilon R B^t Q)] \alpha r^{-\alpha-1} dr dt
$$

$$
= \dot{\nu} \int_{-\infty}^\infty \int_\epsilon^{\infty} E[H(r B^t Q)] \alpha r^{-\alpha-1} dr dt = \dot{\nu} \int_{-\infty}^\infty \int_0^\infty E[H(r B^t Q)] \alpha r^{-\alpha-1} dr dt .
$$

In the last line, the lower bound of the integral is set equal to zero because $Q^* = 1$ by definition. This proves (i).

We have proved that (i), (ii) and (iv) are equivalent. Since (ii) implies (i) and (i) implies (iii), we have also proved that (ii) implies (iii). In discrete time, the converse is obvious but needs a proof in the present context. We will actually prove that (iii) implies (i).
Proof of \((iii) \implies (i)\). Assume that \((iii)\) holds. Using the homogeneity and shift-invariance of \(\nu\) and the fact that \(\nu(\{0\}) = 0\), we have

\[
\nu(H) = \int_{-\infty}^{\infty} H(y) \frac{\|y\|_\alpha^\alpha}{\|y\|_\alpha^\alpha} \nu(\mathrm{d}y) = \int_{0}^{\infty} \int_{-\infty}^{\infty} H(y) \frac{1}{\|y\|_\alpha^\alpha} \alpha u^{-1} \mathrm{d}u \right. \nu(\mathrm{d}y)
\]

\[
= \int_{0}^{\infty} \int_{-\infty}^{\infty} \frac{H(\theta_{y})}{S_\alpha(y)} \alpha u^{-1} \mathrm{d}u \nu(\mathrm{d}y)
\]

\[
= \int_{0}^{\infty} \int_{-\infty}^{\infty} \left[ \frac{(Y)^\alpha H \left( \frac{uB^\alpha Y}{Y^\alpha} \right)}{\|Y\|_\alpha^\alpha} \right] \alpha u^{-1} \mathrm{d}u \right. \nu(\mathrm{d}y)
\]

Recall that \(\vartheta = \mathbb{E}[\mathcal{E}^{-1}(Y)] < \infty\). Thus the previous identity yields

\[
\vartheta = \int_{-\infty}^{\infty} \int_{0}^{\infty} \mathbb{E} \left[ \frac{(Y)^\alpha \delta_{uY} Y}{\|Y\|_\alpha^\alpha} \mathcal{E}(u(Y^\alpha)^{-1}Y) \right] \alpha u^{-1} \mathrm{d}u \right. \mathbb{E} \left[ (Y)^\alpha \right].
\]

Thus (2.11) holds with \(Q\) whose distribution is given by

\[
\mathbb{P}_{Q} = \vartheta^{-1} \mathbb{E} \left[ \frac{(Y)^\alpha \delta_{Y} Y}{\|Y\|_\alpha^\alpha} \right].
\]

By construction, \(\mathbb{P}(Q^* = 1) = 1\). Thus we have proved that \((iii)\) implies \((i)\). 

In the course of the proof, we have obtained a representation of the tail measure in terms of the spectral tail process.

**Corollary 2.9.** Let the equivalent conditions of Theorem 2.8 hold. Then

\[
\vartheta = \mathbb{E} \left[ \frac{(\Theta)^\alpha}{\|\Theta\|_\alpha^\alpha} \right].
\] (2.16)

For all \(\alpha\)-homogeneous non-negative shift-invariant measurable maps \(H\) on \(\mathcal{D}\),

\[
\mathbb{E} \left[ \frac{H(\Theta)}{\|\Theta\|_\alpha^\alpha} \right] = \vartheta \mathbb{E}[H(Q)].
\] (2.17)

and

\[
\nu = \int_{-\infty}^{\infty} \int_{0}^{\infty} \mathbb{E} \left[ \frac{\delta_{uB^\alpha \Theta}}{\|\Theta\|_\alpha^\alpha} \right] \alpha u^{-1} \mathrm{d}u \mathrm{d}t.
\] (2.18)

Similarly to Corollary 2.4, we obtain a one-to-one equivalence between shift-invariant tail measures supported on \(\mathcal{D}_0\) and \(\mathcal{D}\)-valued random elements \(Q\) which satisfy (2.13).
Corollary 2.10. Let $Q$ be a $D$-valued random element such that $\mathbb{P}(Q^* = 1) = 1$ and (2.13) holds for all real numbers $a \leq b$. Then there exists a unique shift-invariant tail measure $\nu$ such that (2.11) holds. The tail measure is supported on $D_0$ and the tail process associated to it is expressed in terms of $Q$ by

$$Y = Y \frac{B^T Q}{|Q - T|}, \quad (2.19)$$

where $T$ is a real valued random variable whose distribution conditional to $Q$ admits the density $\vartheta Q^{-t}_{-t}$. $Y$ is a Pareto random variable with tail index $\alpha$, independent of $Q$ and $T$, and $\vartheta = (\int_{-\infty}^{\infty} \mathbb{E}[|Q|^{\alpha}] dt)^{-1}$. A spectral process $Z$ for $\nu$ is given by $Z = f(S)^{-1/\alpha} B^S Q$, with $f$ any strictly positive density on $\mathbb{R}$ and $S$ a real-valued random variable with density $f$, independent of $Q$.

The proof is omitted. Note that there is no issue with division by zero in (2.19) since

$$\mathbb{P}(|Q - T| = 0) = \vartheta \int_{-\infty}^{\infty} \mathbb{E}[\mathbb{1}\{|Q - t| = 0\} |Q - t|^\alpha] = 0.$$

Before turning to the next issues, we note that the representations (2.6) and (2.11) can be interpreted in terms of Poisson point processes (PPP) on $D$. A measure $\nu$ with the representation (2.6) is the mean measure of a PPP $N$ which can be expressed as

$$N = \sum_{i=1}^{\infty} \delta_{P_i, Z^{(i)}},$$

with $\sum_{i=1}^{\infty} \delta_{P_i, Z^{(i)}}$ a Poisson point process on $(0, \infty) \times D$ with mean measure $\nu_\alpha \otimes \mathbb{P}_Z$, $\nu_\alpha(du) = \alpha u^{-\alpha-1} du$ and $\mathbb{P}_Z$ is the distribution of the process $Z$. If $\nu$ admits the representation (2.11), then $N$ can be expressed as

$$N = \sum_{i=1}^{\infty} \delta_{P_i B^T_i, Q^{(i)}},$$

where $\sum_{i=1}^{\infty} \delta_{P_i, T_i, Q^{(i)}}$ is a PPP on $\mathbb{R} \times (0, \infty) \times D$ with mean measure $\vartheta \text{Leb} \otimes \nu_\alpha \otimes \mathbb{P}_Q$ (and $\mathbb{P}_Q$ denotes the distribution of $Q$).

2.3 Anchoring maps

Let $I : D \to [-\infty, \infty]$ be a measurable map with the following properties:

- $I(B^t y) = I(y) + t$ for all $t \in \mathbb{R}$;
- $|y_{I(y)}| > 0$ if $I(y) \in \mathbb{R}$. 

Such maps were introduced by [BP18] in the context of regularly varying random fields indexed by a lattice and called anchoring maps. We recall their properties in the case of time series indexed by \( \mathbb{Z} \). Two examples are the infargmax \( \mathcal{I}_0 \) and the first exceedence over 1 functionals \( \mathcal{I}_1 \), defined on \((\mathbb{R}^d)^\mathbb{Z}\) with values in \( \mathbb{Z} \cup \{-\infty, +\infty\} \) by

\[
\mathcal{I}_0(y) = \inf\{j \in \mathbb{Z} : |y_j| = \sup_{i \in \mathbb{Z}} |y_i|\}, \\
\mathcal{I}_1(y) = \inf\{j \in \mathbb{Z} : |y_j| > 1\},
\]

with the convention that \( \inf \emptyset = +\infty \). For a discrete time tail process \( Y \), that is a random element in \((\mathbb{R}^d)^\mathbb{Z}\) (endowed with the product topology) such that \( \mathbb{P}(|Y_0| > 1) = 1 \) and which satisfies the time change formula (2.2), if \( \mathbb{P}(\lim_{|j| \to \infty} |Y_j| = 0) \), then for any anchoring map \( \mathcal{I} \),

\[
\vartheta = \mathbb{E}[\mathcal{E}^{-1}(Y)] = \mathbb{P}(\mathcal{I}(Y) = 0) = \frac{1}{\mathbb{E}[\mathcal{E}(Y) | \mathcal{I}(Y) = 0]}.
\]  

(2.20)

See [BP18], [PS18] and [KS20, Chapter 5]. Also, the tail measure can be expressed as in (2.11) (with the integral over \( \mathbb{R} \) replaced by a sum indexed by \( \mathbb{Z} \)) in terms of a sequence \( Q \) whose distribution is that of \((Y^*)^{-1}Y\) conditionally on \( \mathcal{I}(Y) = 0 \). The goal of this section is to investigate the role of anchoring maps in continuous time. It turns out that their are subtle differences.

We assume throughout this section that the equivalent conditions of Theorem 2.8 hold. Assume that there exists an anchoring map \( \mathcal{I} \) such that \( \mathbb{P}(\mathcal{I}(Y) \in \mathbb{R}) = 1 \). Then, for all non-negative measurable map \( H \) defined on \( \mathbb{R} \times \mathcal{D} \), (2.11) and the fact that \( Q^* = 1 \) almost surely yield

\[
\mathbb{E}[H(\mathcal{I}(Y), Y)] = \vartheta \int_{-\infty}^{\infty} \int_{0}^{\infty} \mathbb{E}[H(t, uB^tQ)I\{u|Q_{-t}| > 1\}] \alpha u^{-\alpha-1} du dt
\]

\[
= \vartheta \int_{-\infty}^{\infty} \mathbb{E}[H(t, uB^tQ)I\{u|Q_{I(uQ)}| > 1\}] \alpha u^{-\alpha-1} du dt
\]

\[
= \vartheta \int_{-\infty}^{\infty} \mathbb{E}[H(t, YB^tQ)I\{Y|Q_{I(YQ)}| > 1\}] dt,
\]

with \( Y \) a random variable with a Pareto distribution with index \( \alpha \), independent of \( Q \). This shows that the distribution of \( \mathcal{I}(Y) \) is absolutely continuous with respect to Lebesgue’s measure with density \( f_\mathcal{I} \) given by

\[
f_\mathcal{I}(t) = \vartheta \mathbb{P}(Y|Q_{I(YQ)}| > 1),
\]  

(2.21)

and a regular version of the conditional distribution of \( Y \) given \( \mathcal{I}(Y) \) is

\[
\mathbb{E}[H(Y) | \mathcal{I}(Y) = t] = \frac{\mathbb{E}[H(YB^tQ)I\{Y|Q_{I(YQ)}| > 1\}]}{\mathbb{P}(Y|Q_{I(YQ)}| > 1)},
\]  

(2.22)
with the usual convention $\frac{0}{0} = 0$.

Assume furthermore that

$$\mathbb{P}(Y|Q_{\mathcal{I}(V)} > 1) = 1. \tag{2.23}$$

Then $\vartheta = f_{\mathcal{I}}(0)$ and (2.22) yields, for $t = 0$ and a shift-invariant measurable map $H$ on $\mathcal{D}$,

$$\mathbb{E}[H(Y)|\mathcal{I}(Y) = 0] = \mathbb{E}[H(Y)] \tag{2.22}$$

Since $Q^* = 1$, this proves that $Y^*$ has a Pareto distribution given $\mathcal{I}(Y) = 0$ and for any 0-homogeneous shift-invariant measurable map $g$, $Y^*$ and $g(Y)$ are conditionally independent given $\mathcal{I}(Y) = 0$.

Since for any non-negative measurable map $H$, the map $\int_{-\infty}^{\infty} H \circ B^t dt$ is shift-invariant, the previous relation yields a representation of the tail measure in terms of the conditional distribution of $Y$ given $\mathcal{I}(Y) = 0$ for anchoring maps which satisfy (2.23):

$$\int_{-\infty}^{\infty} \int_{0}^{\infty} \mathbb{E}[H(r(Y^*)^{-1} B^t Y)|\mathcal{I}(Y) = 0] \alpha r^{-\alpha - 1} dr = \int_{-\infty}^{\infty} \int_{0}^{\infty} \mathbb{E}[H(rQ)] \alpha r^{-\alpha - 1} dr = \nu(H). \tag{2.24}$$

This representation is always true in discrete time (with the integral over $\mathbb{R}$ replaced by a series). Taking $H = \mathcal{E}$, this shows that the expression (2.20) holds for anchoring maps satisfying (2.23):

$$\vartheta = \frac{1}{\mathbb{E}[\mathcal{E}(Y)|\mathcal{I}(Y) = 0]}. \tag{2.25}$$

**Example 2.11.** Let $\mathcal{I}_0$ be the inf arg max functional, that is

$$\mathcal{I}_0(y) = \inf\{t \in \mathbb{R}: y \in \{|y_t|, |y_{t-}|\}\}.$$  

If $\mathbb{P}(Y \in \mathcal{D}_0) = 1$, then $\nu(\{y \in \mathcal{D}: \mathcal{I}_0(y) \notin \mathbb{R}\}) = 0$ by Lemma 2.7. The map $\mathcal{I}_0$ is an anchoring map and is 0-homogeneous. The density $f_{\mathcal{I}_0}$ of $\mathcal{I}_0(Y)$ is then given by

$$f_{\mathcal{I}_0}(t) = \vartheta \mathbb{P}(Y|Q_{\mathcal{I}_0(Q) - t} > 1) = \vartheta \mathbb{E}[Q_{\mathcal{I}_0(Q) - t}^\alpha].$$

Since for each $t \in \mathbb{R}$ the map $y \mapsto |y_{\mathcal{I}_0(y) - t}^\alpha$ is shift-invariant and $\alpha$-homogeneous, (2.17) yields

$$f_{\mathcal{I}_0}(t) = \mathbb{E} \left[ \frac{\Theta_{\mathcal{I}_0(\Theta) - t}^\alpha}{\int_{-\infty}^{\Theta_{\mathcal{I}_0(\Theta) - t}} |\Theta_s|^\alpha ds} \right]. \tag{2.26}$$

Condition (2.23) holds if $Y$ is almost surely continuous or if $|Y|$ reaches its maximum by an upward jump. Examples of tail processes with these properties will be given in Example 4.3 and Section 4.3.2.

**Example 2.12.** Consider now the time of the first exceedance over 1, $\mathcal{I}_1(y) = \inf\{t \in \mathbb{R}: |y_t| > 1\}$. If $\mathbb{P}(Y \in \mathcal{D}_0) = 1$, then $\mathbb{P}(\mathcal{I}_1(Y) \in \mathbb{R}) = 1$. For this map, (2.23) holds when $|Y|$ exceeds 1 by an upward jump, but may not hold for continuous processes. See again Example 4.3 and Section 4.3.2.
2.4 Identities

In discrete time, the expectation of certain functionals of \( Q \) can be expressed in terms of the forward spectral tail process only. This is important since the forward spectral tail process is always easier to compute than the backward spectral tail process and much easier than the sequence \( Q \) which is obtained by a change of measure or by conditioning. For \( \alpha \)-homogeneous measurable functions satisfying certain additional conditions, it may be proved that

\[
\vartheta \mathbb{E}[H(Q)] = \mathbb{E}[H(\{\Theta_j, j \geq 0\})] - H(\{\Theta_j, j \geq 1\}).
\]

(2.27)

See [PS18, Section 3.3] for precise conditions. In particular, it always holds that

\[
\vartheta = \mathbb{E} \left[ \sup_{j \geq 0} |\Theta_j|^\alpha - \sup_{j \geq 1} |\Theta_j|^\alpha \right],
\]

(2.28)

\[
\vartheta \mathbb{E} \left[ \left( \sum_{j \in \mathbb{Z}} |Q_j| \right)^\alpha \right] = \mathbb{E} \left[ \left( \sum_{j=0}^\infty |\Theta_j| \right)^\alpha - \left( \sum_{j=1}^\infty |\Theta_j| \right)^\alpha \right].
\]

(2.29)

Both terms in (2.29) are finite when \( \alpha \leq 1 \) and are simultaneously finite or infinite if \( \alpha > 1 \) (the difference inside the expectation in the right-hand side being set equal to \( \infty \) if the series is not summable in the latter case). Such identities are important since the two different expressions appear as limits of the same statistics studied by different tools. In discrete time, these identities are obtained by a method of telescopic sums; in continuous time this technique is not available and must be replaced by some form of differentiation. Thus our only result is related to the identity (2.29). Its usefulness will be illustrated in Section 3.4. It would be of interest to obtain a formula extending (2.28), that is an expression of the candidate extremal index in terms of the forward spectral tail process only.

Lemma 2.13. Let \( \Theta \) be a \( d \)-dimensional spectral tail process such that \( \mathbb{P}(\Theta \in D_0) = 1 \) and let \( Q \) be the sequence defined by (2.15) or (2.17). Then

\[
\vartheta \mathbb{E} \left[ \left( \int_{-\infty}^{\infty} |Q_s| ds \right)^\alpha \right] = \alpha \mathbb{E} \left[ \left( \int_{0}^{\infty} |\Theta_s| ds \right)^{\alpha-1} \right].
\]

(2.30)

If \( \alpha \leq 1 \), both terms in (2.30) are finite, and if \( \alpha > 1 \) they are simultaneously finite or infinite. In the former case, for \( d = 1 \),

\[
\vartheta \mathbb{E} \left[ \left( \int_{-\infty}^{\infty} Q_s ds \right)^\alpha \right] = \alpha \mathbb{E} \left[ \Theta_0 \left( \int_{0}^{\infty} \Theta_s ds \right)^{\alpha-1} \right].
\]

(2.31)

In the case \( 0 < \alpha \leq 1 \), we use the convention \( x^{\alpha-1}_+ = 0 \) if \( x \leq 0 \).

Proof. To prove (2.30), we can assume that \( \Theta \) is a sequence of non-negative random variables.
Starting from (2.17), and applying the monotone convergence theorem, we obtain
\[
\vartheta E \left[ \left( \int_{-\infty}^{\infty} Q_s ds \right)^\alpha \right] = \mathbb{E} \left[ \frac{\left( \int_{-\infty}^{\infty} \Theta_t ds \right)^\alpha}{\|\Theta\|_\alpha} \right] = \lim_{t \to \infty} \mathbb{E} \left[ \frac{\left( \int_{-\infty}^{\infty} \Theta_t ds \right)^\alpha - \left( \int_{-t}^{\infty} \Theta_t ds \right)^\alpha}{\|\Theta\|_\alpha} \right]
\]
\[
= \alpha \lim_{t \to \infty} \int_{-t}^{t} \mathbb{E} \left[ \left( \int_{-t}^{\infty} \Theta_u du \right)^{\alpha - 1} \Theta_s \right] ds.
\]
Applying the time change formula (2.3) yields (2.30) and that both terms are simultaneously finite or infinite. If \( \alpha < 1 \), then (2.12) yields
\[
\vartheta E \left[ \left( \int_{-\infty}^{\infty} Q_s ds \right)^\alpha \right] \leq \vartheta E \left[ \int_{-\infty}^{\infty} Q_s^\alpha ds \right] = 1.
\]
Thus both terms are finite. If now \( \Theta \) is not non-negative, assuming that both terms in (2.27) are finite, we obtain (2.31) by applying the dominated convergence theorem instead of monotone convergence.

3  Regular variation in \( \mathcal{D}(\mathbb{R}) \)

Let \( X \) be a stationary process indexed by \( \mathbb{R} \), with values in \( \mathbb{R}^d \). We say that \( X \) is finite dimensional regularly varying if there exists a sequence \( a_n \) and for all \( k \geq 1, s_1, \ldots, s_k \in \mathbb{R} \) there exists a non-zero measure \( \nu_{s_1, \ldots, s_k} \) on \( \mathbb{R}^{dk} \setminus \{0\} \) such that
\[
\lim_{n \to \infty} n \mathbb{P} \left( \left( \frac{X_{s_1}}{a_n}, \ldots, \frac{X_{s_k}}{a_n} \right) \in A \right) \overset{v}{\longrightarrow} \nu_{s_1, \ldots, s_k}, \tag{3.1}
\]
as \( n \to \infty \). As in the infinite dimensional setting of Section 2, this means that
\[
\lim_{n \to \infty} n \mathbb{P} \left( \left( \frac{X_{s_1}}{a_n}, \ldots, \frac{X_{s_k}}{a_n} \right) \in A \right) = \nu_{s_1, \ldots, s_k} (A) < \infty,
\]
for all Borel sets \( A \) separated from \( 0 \) in \( \mathbb{R}^{dk} \) (i.e. included in the complement of a neighborhood (for the usual topology) of \( 0 \)) which are continuity sets of \( \nu_{s_1, \ldots, s_k} \). The measure \( \nu_{s_1, \ldots, s_k} \) is called the exponent measure of \((X_{s_1}, \ldots, X_{s_k})\) and there exists \( \alpha \) such that \( \nu_{s_1, \ldots, s_k} \) is \( \alpha \)-homogeneous, i.e. \( \nu_{s_1, \ldots, s_k}(tA) = t^{-\alpha} \nu_{s_1, \ldots, s_k}(A) \) for all \( t > 0, k \geq 1 \) and \( s_1, \ldots, s_k \).

Although these measures are not finite and Kolmogorov extension theorem cannot be applied, [SO12] proved that there exists an \( \alpha \)-homogeneous, shift-invariant measure \( \nu \) on \( \mathbb{R}^{dk} \) endowed with the product topology, called the tail measure, such that for all \( k \geq 1, s_1, \ldots, s_k \in \mathbb{R}, \nu_{s_1, \ldots, s_k} \) is the projection of \( \nu \). The result of [SO12] says nothing about the support of \( \nu \).
Let \(|\cdot|\) denote an arbitrary norm on \(\mathbb{R}^d\). We can and will henceforth assume that the norming sequence \(\{a_n\}\) is chosen such that
\[
\lim_{n \to \infty} n\mathbb{P}(|X_0| > a_n) = 1.
\]
With this choice, \(\nu(\{y \in \mathbb{R}^d : |y_0| > 1\}) = 1\).

According to [BS09, Theorem 2.1], finite dimensional regular variation of the process \(X\) is equivalent to the following conditions.

(i) \(|X_0|\) is regularly varying with tail index \(\alpha > 0\);

(ii) there exists a process \(Y\) such that for all \(s \leq t \in \mathbb{R}\),
\[
\lim_{x \to \infty} \mathcal{L}\left(\frac{X_s}{x}, \ldots, \frac{X_t}{x} | |X_0| > x\right) = \mathcal{L}(Y_s, \ldots, Y_t).
\]

The process \(Y\) is called the tail process. As shown in [SO12], the distribution of the tail process (seen as a random element in \(\mathbb{R}^d\)) is the tail measure restricted to the set \(\{y \in \mathbb{R}^d : |y_0| > 1\}\), which is a probability measure with the choice of norming constant \(a_n\) defined above. It follows from this definition that \(|Y|\) is a Pareto random variable with tail index \(\alpha\) and \(\Theta = |Y_0|^{-1} Y\) is independent of \(Y_0\). These properties were proved in [BS09] in the case of processes indexed by \(\mathbb{Z}\), but so far as finite dimensional distributions only are considered, they remain valid for processes indexed by \(\mathbb{R}\). However, this definition says nothing about the path properties of the process \(Y\).

In order to obtain more information on the support of \(\nu\) or the path properties of \(Y\), and make the link with the corresponding objects introduced in Section 2, the mode of convergence must be strengthened.

Recall from Section 2 the definition of the boundedness \(\mathcal{B}_0\), the class of subsets \(A\) separated from \(0\) in \(\mathcal{D}\) for the \(J_1\) topology, i.e. sets for which there exist \(a \leq b\) and \(\epsilon > 0\) such that \(\inf_{y \in A} y_{a,b}^* > \epsilon\). Recall also that a Borel measure \(\mu\) on \(\mathcal{D}\) is \(\mathcal{B}_0\)-boundedly finite if \(\mu(A) < \infty\) for all measurable sets \(A \in \mathcal{B}_0\). A sequence \(\{\mu_n, n \geq 1\}\) of \(\mathcal{B}_0\)-boundedly finite Borel measures on \(\mathcal{D}\) is said to converge \(\mathcal{B}_0\)-vaguely to a \(\mathcal{B}_0\)-boundedly finite Borel measure \(\mu\), denoted \(\mu_n \overset{v}{\to} \mu\), if \(\lim_{n \to \infty} \mu_n(A) = \mu(A)\) for all \(\mu\)-continuity measurable sets \(A \in \mathcal{B}_0\).

[HL05] introduced the notion of regular variation of stochastic processes indexed by \([0, 1]\), i.e. random elements in \(\mathcal{D}([0, 1], \mathbb{R}^d)\). Since \(\mathcal{D}(\mathbb{R}, \mathbb{R}^d)\) endowed with the \(J_1\) topology is a Polish space, the following extension is natural.

**Definition 3.1.** A \(\mathcal{D}\)-valued stationary stochastic process \(X\) is said to be regularly varying in \(\mathcal{D}\) if there exists a non-zero \(\mathcal{B}_0\)-boundedly finite measure \(\nu\) such that
\[
\frac{\mathbb{P}(X \in x \cdot)}{\mathbb{P}(|X_0| > x)} \overset{v}{\to} \nu.
\]
This definition entails that the limiting measure \( \nu \) is necessarily \( \alpha \)-homogeneous, i.e. \( \nu(t\cdot A) = t^{-\alpha} \nu(A) \) for all Borel sets \( A \subset D \). See [HL05, Remark 3].

Our first result is a necessary and sufficient condition for regular variation in \( D \) which adapts [HL05, Theorem 10] to \( D(\mathbb{R}) \). See Appendix B for the definition of the moduli of continuity \( w' \) and \( w'' \).

**Theorem 3.2.** Let \( X \) be a stationary, stochastically continuous \( D \)-valued process. The following statements are equivalent.

(i) \( X \) is regularly varying in \( D \).

(ii) (3.1) holds for all \( k \geq 1 \) and \( (s_1, \ldots, s_k) \in \mathbb{R}^k \) and for all \( a < b \),

\[
\lim_{\delta \to 0} \limsup_{x \to \infty} \frac{\mathbb{P}(w'(X, a, b, \delta) > x\epsilon)}{\mathbb{P}(|X_0| > x)} = 0 .
\]  

(3.3)

When these conditions hold, the tail measure of \( X \) is supported on \( D \), its tail process has almost surely càdlàg paths and conditionally on \( |X_0| > x, x^{-1}X \) converges weakly to \( Y \), as \( x \to \infty \), on \( D \) endowed with the \( J_1 \) topology.

In view of Theorem B.1, (3.3) can be equivalently replaced by

\[
\lim_{\delta \to 0} \limsup_{x \to \infty} \frac{\mathbb{P}(w''(X, a, b, \delta) > x\epsilon)}{\mathbb{P}(|X_0| > x)} = 0 .
\]  

(3.4)

**Proof of (i) \( \implies \) (ii).** The proof is essentially the same as the proof of the implication (ii) \( \implies \) (i) of [HL05, Theorem 10], replacing \( M_0 \)-convergence and \( D([0,1]) \) used therein by vague convergence and \( D(\mathbb{R}) \).

**Proof of (ii) \( \implies \) (i).** Define the measure \( \mu_x \) on \( D \) by

\[
\mu_x = \frac{\mathbb{E}[\delta_{x^{-1}X} \mathbf{1}\{X \neq 0\}]}{\mathbb{P}(|X_0| > x)} .
\]

For each \( n, k \in \mathbb{N}^* \), let \( D_{n,k} \) be the space of functions \( f \in D \) such that \( \sup_{-n \leq t < n} |f(t)| > k^{-1} \).

We first prove that \( \limsup_{x \to \infty} \mu_x(D_{n,k}) < \infty \) for each fixed \( n, k \in \mathbb{N}^* \). Indeed, for \( \delta > 0 \),

\[
\mu_x(D_{n,k}) = \frac{\mathbb{P}(kX^*_{-n,n} > x)}{\mathbb{P}(|X_0| > x)} \leq \frac{\mathbb{P}(2kw''(X, -n, n, \delta) > x)}{\mathbb{P}(|X_0| > x)} + \frac{\mathbb{P}(kX^*_{-n,n} > x, 2kw''(X, -n, n, \delta) \leq x)}{\mathbb{P}(|X_0| > x)} .
\]

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If $kX_{-n,n} > x$ and $2kw''(x,-n,n,\delta) \leq x$, then for every sequence $(t_0, \ldots, t_k)$ such that $a = t_0 < \cdots < t_k = b$ and $t_i - t_{i-1} \leq \delta$ for $i = 1, \ldots, k$, it necessarily holds that $2k \max_{0 \leq i \leq k} |X_{t_i}| > x$. Thus

$$
\mu_x(\mathcal{D}_{n,k}) \leq \frac{\mathbb{P}(2kw''(x,-n,n,\delta) > x)}{\mathbb{P}(|X_0| > x)} + \frac{\mathbb{P}(2k \max_{0 \leq i \leq k} |X_{t_i}| > x)}{\mathbb{P}(|X_0| > x)}.
$$

Under the assumptions of the theorem, both terms tend to zero by letting $x \rightarrow \infty$, then $\delta \rightarrow 0$. This proves our first claim, from which it ensues that the measure $\mu_x$ is $\mathcal{B}_0$-boundedly finite on $\mathcal{D}$ and we can define the finite measure $\mu_{n,k,x}$ on $\mathcal{D}_{n,k}$ by

$$
\mu_{n,k,x} = \frac{\mathbb{E}[\delta_x^{-1} \mathbb{1}\{X_{-n,n} > k^{-1}x\}]}{\mathbb{P}(|X_0| > x)}.
$$

Since $\mathbb{P}(w'(x,a,b,\delta) > x\epsilon; |X_0| > x) \leq \mathbb{P}(w'(x,a,b,\delta) > x\epsilon)$, the assumption (3.3) implies

$$
\lim_{\delta \rightarrow 0} \limsup_{x \rightarrow \infty} \mathbb{P}(w'(x,a,b,\delta) > x\epsilon; |X_0| > x) = 0. 
$$  \tag{3.5}

By Theorem B.1, the finite dimensional weak convergence of $x^{-1}X$ conditionally on $|X_0| > x$ (at all but countably many points of $\mathbb{R}$) and (3.5) yield the weak convergence in $\mathcal{D}$ of $x^{-1}X$ to $Y$, conditionally on $|X_0| > x$.

Let $H$ be a bounded continuous map on $\mathcal{D}$ (with respect to the $J_1$ topology) with support separated from zero, i.e., there exists $a < b$ such that $H(y) = 0$ if $y_{a,b} \leq 2\epsilon$, or equivalently, $H = H[\mathcal{E}(2\epsilon^{-1}) > 0]$. Then, for $\eta > 0$,

$$
\mathbb{E}[H(x^{-1}X)\mathbb{1}\{\mathcal{E}_{a,b}(\epsilon^{-1}x^{-1}X) > x\eta\}]
= \int_a^b \mathbb{E}\left[\frac{H(x^{-1}X)\mathbb{1}\{|X_t| > x\epsilon\}\mathbb{1}\{\mathcal{E}_{a,b}(\epsilon^{-1}x^{-1}X) > x\eta\}}{\mathcal{E}_{a,b}(\epsilon^{-1}x^{-1}X)}\right] dt
= \int_a^b \mathbb{E}\left[\frac{H(x^{-1}B'X)\mathbb{1}\{|X_0| > x\epsilon\}\mathbb{1}\{\mathcal{E}_{a,b}(B'\epsilon^{-1}x^{-1}X) > x\eta\}}{\mathcal{E}_{a,b}(\epsilon^{-1}x^{-1}B'X)}\right] dt.
$$

The last line was obtained by stationarity of $X$. By the regular variation of $X_0$, the weak convergence in $\mathcal{D}$ stated above and dominated convergence, we now obtain

$$
\lim_{x \rightarrow \infty} \frac{\mathbb{E}[H(x^{-1}X)\mathbb{1}\{\mathcal{E}_{a,b}(\epsilon^{-1}x^{-1}X) > x\eta\}]}{\mathbb{P}(|X_0| > x)} = \epsilon^{-\alpha} \int_a^b \mathbb{E}\left[\frac{H(\epsilon B'Y)\mathbb{1}\{\mathcal{E}_{a,b}(\epsilon B'Y) > \eta\}}{\mathcal{E}_{a,b}(B'Y)}\right] dt.
$$

As $\eta \rightarrow 0$, since $H = H[\mathcal{E}(\epsilon^{-1}) > 0]$, we have by monotone convergence

$$
\lim_{\eta \rightarrow 0} \epsilon^{-\alpha} \int_a^b \mathbb{E}\left[\frac{H(\epsilon B'Y)\mathbb{1}\{\mathcal{E}_{a,b}(\epsilon B'Y) > \eta\}}{\mathcal{E}_{a,b}(B'Y)}\right] dt = \epsilon^{-\alpha} \int_a^b \mathbb{E}\left[\frac{H(\epsilon B'Y)}{\mathcal{E}_{a,b}(B'Y)}\right] dt. \tag{3.6}
$$

The latter quantity is finite by the first part of the proof and we denote it by $\mu(H)$. 


If \( f \in \mathcal{D} \) is such that \( w'(f, a, b, \eta) \leq \epsilon/2 \) and \( \mathcal{E}_{a,b}((2\epsilon)^{-1}f) > 0 \), then there exists \( t \in [a, b] \) such that \( f(t) > 2\epsilon \) and an interval \([u, v]\) such that \( t \in [u, v) \), \( v - u \geq \eta \) and \( \sup_{t' \in [u, v]}|f(t) - f(t')| \leq \epsilon \). Consequently, \( f(s) > \epsilon \) for all \( s \in [u, v] \) and \( \mathcal{E}_{a,b}(\epsilon^{-1}f) \geq \eta \). This yields

\[
\begin{align*}
\mathbb{E}[H(x^{-1}X) \mathbbm{1}\{\mathcal{E}_{a,b}(\epsilon^{-1}x^{-1}X) \leq \eta\}] &\leq \text{cst} \ P(X^*_{a,b} > 2\epsilon x; \mathcal{E}(\epsilon^{-1}x^{-1}X) \leq \eta) \\
&\leq \text{cst} \ P(w'(X, a, b, \eta) > \epsilon x) .
\end{align*}
\]

Here and throughout, cst denotes a numerical constant which depends on none of the variable parameters around it. Thus (3.3) implies that

\[
\lim_{\eta \to 0} \lim_{x \to \infty} \mathbb{E}[H(x^{-1}X) \mathbbm{1}\{\mathcal{E}_{a,b}(\epsilon^{-1}x^{-1}X) \leq \eta\}] = 0 .
\]

By the triangular argument [Bil99, Theorem 3.2], (3.6) and (3.7), we obtain

\[
\lim_{x \to \infty} \frac{\mathbb{E}[H(x^{-1}X)]}{\mathbb{P}(|X_0| > x)} = \mu(H) .
\]

This also proves that \( \mu(H) \) does not depend on the particular choice of \( a, b, \epsilon \). Thus, taking \( \epsilon = k^{-1}, a = -n \) and \( b = n \), we define a finite measure \( \mu_{n,k} \) on \( \mathcal{D}_{n,k} \) by

\[
\mu_{n,k}(H) = \mu(H) = k^n \int_{-n}^{n} \mathbb{E} \left[ \frac{H(k^{-1}B^tY)}{\mathcal{E}_{-n,n}(B^tY)} \right] dt .
\]

We have proved that \( \mu_{n,k,x} \overset{w}{\to} \mu_{n,k} \) on \( \mathcal{D}_{n,k} \) for all \( n, k \geq 1 \). This implies that \( \mu_x \overset{w}{\to} \mu \) on \( \mathcal{D} \setminus \{0\} \) by [Kal17, Lemma 4.6]. \( \square \)

### 3.1 The anticlustering condition

Let \( \mathcal{D}_0(\mathbb{R}, \mathbb{R}^d) \), hereafter abbreviated to \( \mathcal{D}_0 \), be the space of \( \mathbb{R}^d \)-valued càdlàg functions which tend to zero at \( \pm \infty \). Let \( \mathcal{H} \) be the set of one-to-one strictly increasing continuous maps on \( \mathbb{R} \) and \( d_{\infty} \) be the distance defined on \( \mathcal{D}_0 \) by

\[
d_{\infty}(f, g) = \inf_{u \in \mathcal{H}} \|f \circ u - g\|_{\infty} \vee \|u - \text{Id}\|_{\infty} .
\]

We denote the topology induced by the metric \( d_{\infty} \) by \( J^0_1 \). Obviously, \( d_{J_1}(f, g) \leq d_{\infty}(f, g) \leq \|f - g\|_{\infty} \), thus a sequence converging with respect to \( d_{\infty} \) converges with respect to \( d_{J_1} \) and an open set for \( d_{\infty} \) is also open for \( d_{J_1} \). The topology \( J^0_1 \) induced by \( d_{\infty} \) on \( \mathcal{D}_0 \) is Polish and the associated Borel \( \sigma \)-field is the product \( \sigma \)-field.

We now introduce an assumption which ensures that the tail process tends to zero at \( \infty \). This assumption is related to condition (2.8) of [DH95], see also [BS09, Condition 4.1]. To avoid repetitions, we define a scaling function as a non-decreasing unbounded function defined on \([0, \infty)\).
Assumption 3.3. There exist scaling functions \( a \) and \( r : \mathbb{R}_+ \to \mathbb{R}_+ \) such that

\[
\lim_{t \to \infty} \limsup_{T \to \infty} \mathbb{P} \left( \sup_{t \leq |s| \leq r_T} |X_s| > a_T x \mid |X_0| > a_T \right) = 0. \quad (3.8)
\]

For \( m > 0 \), we say that a stochastic process \( X \) is \( m \)-dependent if for all \( t \in \mathbb{R} \), \( \{X_s, s \geq t+m\} \) is independent of \( \{X_s, s \leq t\} \).

Lemma 3.4. Let \( X \) be an \( m \)-dependent \( \mathcal{D} \)-valued stationary stochastic process, regularly varying in \( \mathcal{D} \). Then (3.8) holds for all scaling functions \( a \) and \( r \) such that \( \lim_{T \to \infty} r_T \mathbb{P}(|X_0| > a_T) = 0 \).

Proof. For \( t > m \), we have by \( m \)-dependence

\[
\mathbb{P} \left( \sup_{t \leq |s| \leq r_T} |X_s| > a_T x \mid |X_0| > a_T \right) = \mathbb{P} \left( \sup_{t \leq |s| \leq r_T} |X_s| > a_T x \right)
\leq r_T \mathbb{P}(|X_0| > a_T) \frac{\mathbb{P} \left( \sup_{0 \leq |s| \leq 1} |X_s| > a_T x \right)}{\mathbb{P}(|X_0| > a_T)}.
\]

By regular variation in \( \mathcal{D} \), the fraction in the right-hand side converges to \( x^{-\alpha} \nu(\{y \in \mathcal{D} : y_{0,1} > 1\}) \) as \( T \) tends to \( \infty \). Thus (3.8) holds for every sequence \( r_T \) such that \( \lim_{T \to \infty} r_T \mathbb{P}(|X_0| > a_T) = 0 \) as claimed.

Lemma 3.5. Let \( X \) be a \( \mathcal{D} \)-valued stationary process, regularly varying in the sense of Definition 3.1. If Assumption 3.3 holds, then \( \mathbb{P}(Y \in \mathcal{D}_0) = 1 \) and conditionally on \( |X_0| > a_T x \), \( (xa_T)^{-1}X I_{[-r_T,r_T]} \overset{w}{\longrightarrow} Y \) in \( \mathcal{D}_0 \) endowed with the \( J^0 \) topology.

Proof. We first prove that \( \mathbb{P}(Y \in \mathcal{D}_0) = 1 \). By Theorem 3.2 and Assumption 3.3, we have, for \( \epsilon > 0 \) and large enough \( t \leq t' \),

\[
\mathbb{P} \left( \sup_{t \leq |s| \leq t'} |Y_t| > \epsilon \right) = \lim_{T \to \infty} \mathbb{P} \left( \sup_{t \leq |s| \leq t'} |X_t| > \epsilon a_T \mid |X_0| > a_T \right)
\leq \limsup_{T \to \infty} \mathbb{P} \left( \sup_{t \leq |s| \leq r_T} |X_t| > \epsilon a_T \mid |X_0| > a_T \right) \leq \epsilon.
\]

This proves the first claim.

To prove the stated weak convergence, we apply [Dud02, Theorem 11.3.3]. Let \( H \) be a bounded Lipschitz function with respect to the metric \( d_\infty \) on \( \mathcal{D}_0 \). Since the metric \( d_\infty \) is dominated by the uniform norm, we have

\[
\mathbb{E}[|H(a_T^{-1}X \mathbb{1}_{[-t,t]}) - H(a_T^{-1}X \mathbb{1}_{[-r_T,r_T]})| \mid |X_0| > xa_T]
\leq \text{cst } \epsilon + \mathbb{P} \left( \sup_{t \leq |s| \leq r_T} |X_s| > a_T \mid |X_0| > xa_T \right).
\]

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Thus, Assumption 3.3 yields
\[
\lim_{t \to \infty} \limsup_{T \to \infty} \mathbb{E} \left[ |H(a_T^{-1} X \mathbb{1}_{[-t,t]}) - H(a_T^{-1} X \mathbb{1}_{[-rT,rT]})| \mid |X_0| > xa_T \right] = 0. 
\] (3.9)

By the convergence (3.2), we have, for each \( t > 0 \),
\[
\lim_{T \to \infty} \mathbb{E} \left[ H(a_T^{-1} X \mathbb{1}_{[-t,t]}) \mid |X_0| > xa_T \right] = \mathbb{E} \left[ H(Y \mathbb{1}_{[-t,t]}) \right]. 
\] (3.10)

Since \( Y \) tends to zero at \( \infty \) and \( H \) is Lipschitz, we have
\[
\lim_{t \to \infty} \mathbb{E} \left[ H(Y \mathbb{1}_{[-t,t]}) \right] = \mathbb{E} \left[ H(Y) \right]. 
\] (3.11)

The convergences (3.9), (3.10) and (3.11) conclude the proof.

\[ \square \]

### 3.2 The cluster measure

Recall that we have defined \( \vartheta = \mathbb{E}[\mathcal{E}^{-1}(Y)] < \infty \) by Lemma 2.5. If \( \mathbb{P}(Y \in D_0) = 1 \), then \( \vartheta > 0 \) by Theorem 2.8. Let \( Q \) be be a random element on \( D_0 \) whose distribution is given by (2.15) and define the boundedly finite measure \( \nu^* \) on \( D_0 \) by
\[
\nu^* = \vartheta \int_{-\infty}^{\infty} \mathbb{E} \left[ \delta_{\partial Q} \right] \alpha r^{-\alpha - 1} dr.
\] (3.12)

This and (2.11) yield
\[
\nu = \int_{-\infty}^{\infty} \nu^* \circ B' dt.
\] (3.13)

**The space \( \tilde{D}_0 \)** In order to state our result on the convergence of the point process of exceedence, we need to introduce the space \( \tilde{D}_0 \) which is the quotient of the space \( D_0 \) by the relation of shift-equivalence. We say that two functions \( g, f \) defined on \( \mathbb{R} \) are shift-equivalent if there exists \( t \in \mathbb{R} \) such that \( f = B^t g \), i.e. \( f(x) = g(x - t) \) for all \( x \in \mathbb{R} \). This is an equivalence relation and the space \( \tilde{D}_0 \) is the set of equivalence classes for this relation. We endow it with the quotient topology which is metrizable with the metric \( \tilde{d}_\infty \) defined by
\[
\tilde{d}_\infty(\tilde{f}, \tilde{g}) = \inf_{f \in \tilde{f}, g \in \tilde{g}} d_\infty(f, g).
\]

The set \( \tilde{D}_0 \) endowed with this topology inherits the Polish property. A map \( \tilde{H} \) on \( \tilde{D}_0 \) is uniquely associated to a shift-invariant map \( H \) on \( D_0 \) by the relation \( H(f) = \tilde{H}(\tilde{f}) \) for all \( \tilde{f} \in \tilde{D}_0 \) and \( f \in \tilde{f} \).

We now define vague convergence of boundedly finite measures on \( \tilde{D}_0 \). Let \( \tilde{B}_0 \) be the class of subsets \( \tilde{A} \) of \( \tilde{D}_0 \) such that \( \tilde{y} \in \tilde{A} \) implies that there exist \( y \in \tilde{y} \) with \( y^* > \epsilon \). We
say that $\nu$ is $\tilde{\mathcal{B}}_0$-boundedly finite on $\tilde{\mathcal{D}}_0$ if $\nu(A) < \infty$ for all measurable sets $A \in \tilde{\mathcal{B}}_0$ and $\nu(\{0\}) = 0$. Vague convergence on $\tilde{\mathcal{D}}_0$ is defined with respect to $\tilde{\mathcal{B}}_0$: we say that a sequence of $\tilde{\mathcal{B}}_0$-boundedly finite measures $\nu_n$ converges vaguely to $\nu$ in $(\tilde{\mathcal{D}}_0, \tilde{\mathcal{B}}_0)$, denoted $\nu_n \xrightarrow{v} \nu$ if $\lim_{n \to \infty} \nu_n(A) = \nu(A)$ for all continuity sets for $\nu A \in \mathcal{B}_0$.

Any shift-invariant map $f$ on $\mathcal{D}_0$ can be seen as a map on $\tilde{\mathcal{D}}_0$ and conversely any map on $\tilde{\mathcal{D}}_0$ can be seen a shift-invariant map $f$ on $\mathcal{D}_0$, and it is Lipschitz with respect to $d_\infty$ if and only if it is Lipschitz with respect to the metric $d_\infty$. Thus a necessary and sufficient condition for vague convergence of a sequence of boundedly finite measures $\{\nu_n, n \geq 1\}$ on $\tilde{\mathcal{D}}_0$ to a measure $\nu$ is that $\lim_{n \to \infty} \nu_n(f) = \nu(f)$, for all shift-invariant maps $f$ in $\mathcal{D}_0$, with support in $\mathcal{B}_0$ and Lipschitz with respect to the metric $d_\infty$.

For functions $a_T$ and $r_T$, we define the measure $\nu_{T,rT}^*$ on $\mathcal{D}_0$ by

$$\nu_{T,rT}^* = \frac{\mathbb{E} \left[ \delta_{a_T^{-1}X_{0,rT}} \right]}{r_T \mathbb{P}(|X_0| > a_T)}.$$

(3.14)

**Lemma 3.6.** Let $X$ be a stationary $\mathcal{D}$-valued stochastic process, regularly varying in $\mathcal{D}$. Let $a_T$, $r_T$ be scaling functions such that $(3.8)$ holds. Then $\nu_{T,rT}^* \xrightarrow{v} \nu^*$ in $(\tilde{\mathcal{D}}_0, \tilde{\mathcal{B}}_0)$.

**Proof.** We must prove that for all $\epsilon > 0$ and shift-invariant bounded maps $H$ defined on $\mathcal{D}_0$, continuous with respect to the $J_0^0$ topology an such that $H(y) = 0$ if $y^* \leq 2\epsilon$, it holds that

$$\lim_{T \to \infty} \nu_{T,rT}^*(H) = \nu^*(H).$$

(3.15)

Write

$$\nu_{T}^*(H) = \frac{\mathbb{E}[H(a_T^{-1}X_{0,rT}) \mathbb{1}\{\mathcal{E}(\epsilon^{-1}a_TX_{0,rT}) > \eta\}]}{r_T \mathbb{P}(|X_0| > a_T)} + \frac{\mathbb{E}[H(a_T^{-1}X_{0,rT}) \mathbb{1}\{\mathcal{E}(\epsilon^{-1}a_TX_{0,rT}) \leq \eta\}]}{r_T \mathbb{P}(|X_0| > a_T)}.$$

As argued in the proof of Theorem 3.2, if $f \in \mathcal{D}$ is such that $w'(f, a, b, \eta) \leq \epsilon/2$ and $\mathcal{E}_{a,b}((2\epsilon)^{-1}f) > 0$, then $\mathcal{E}_{a,b}(\epsilon^{-1}f) \geq \eta$. Thus

$$\mathbb{E}[H(a_T^{-1}X_{0,rT}) \mathbb{1}\{\mathcal{E}(\epsilon^{-1}a_TX_{0,rT}) \leq \eta\}] \leq \text{cst} \mathbb{P}((X_{0,rT})^* > 2\epsilon a_T; \mathcal{E}(\epsilon^{-1}a_TX_{0,rT}) \leq \eta) \leq \text{cst} \mathbb{P}(w'(X, 0, r_T, \eta) > \epsilon a_T) \leq \text{cst} r_T \mathbb{P}(w'(X, 0, 1, \eta) > \epsilon a_T).$$

This yields, by [Bil99, Theorem 16.13],

$$\lim_{\eta \to 0} \lim_{T \to \infty} \frac{\mathbb{E}[H(a_T^{-1}X_{0,rT}) \mathbb{1}\{\mathcal{E}(\epsilon^{-1}a_TX_{0,rT}) \leq \eta\}]}{r_T \mathbb{P}(|X_0| > a_T)} \leq \lim_{\eta \to 0} \lim_{T \to \infty} \text{cst} \frac{\mathbb{P}(w'(X, 0, 1, \eta) > \epsilon a_T)}{r_T \mathbb{P}(|X_0| > a_T)} = 0.$$

(3.16)
The other term is dealt with by dominated convergence arguments and the convergence in $\mathcal{D}$ of $x^{-1}X$ conditionally on $|X_0| > x$. We first write
\[
\mathbb{E}[H(a_T^{-1}X_{0,r_T}) \mathbbm{1}\{\mathcal{E}(e^{-1}a_TX_{0,r_T}) > \eta}\] 
\[
r_T\mathbb{P}(|X_0| > a_T) 
\[
= \frac{1}{r_T\mathbb{P}(|X_0| > a_T)} \int_0^{rt} \mathbb{E}\left[\frac{H(a_T^{-1}X_{0,r_T}) \mathbbm{1}\{|X_s| > ea_T \mathbbm{1}\{\mathcal{E}(e^{-1}a_TX_{0,r_T}) > \eta}\}}{\mathcal{E}(e^{-1}a_T^{-1}X_{0,r_T,t})}\right] dt 
\[
= \frac{\mathbb{P}(|X_0| > a_T)}{\mathbb{P}(|X_0| > a_T)} \int_0^1 g_T(s) ds ,
\]
with $g_T$ defined by
\[
g_T(s) = \mathbb{E}\left[\frac{H(a_T^{-1}X \mathbbm{1}\{[-r_Ts, (1-s)r_T]\}) \mathbbm{1}\{\mathcal{E}(e^{-1}a_TX \mathbbm{1}\{[r_Ts, (1-s)r_T]\}) > \eta}\}}{\mathcal{E}(e^{-1}a_T^{-1}X \mathbbm{1}\{[-r_Ts, (1-s)r_T]\})}\right] | |X_0| > ea_T).
\]
Since $H$ is bounded and $g_T = O(\eta^{-1})$, we have by Lemma 3.5 and dominated convergence, for each $s \in [0, 1]$,
\[
\lim_{T \to \infty} g_T(s) \to e^{-a} \mathbb{E}\left[\frac{H(eY) \mathbbm{1}\{\mathcal{E}(Y) > \eta\}}{\mathcal{E}(Y)}\right] = \nu^*(H \mathbbm{1}\{\mathcal{E} > \eta\}).
\]
By dominated convergence again, we obtain
\[
\lim_{T \to \infty} \mathbb{E}[H(a_T^{-1}X_{0,r_T}) \mathbbm{1}\{\mathcal{E}(e^{-1}a_TX_{0,r_T}) > \eta\}] = \nu^*(H \mathbbm{1}\{\mathcal{E} > \eta\}).
\]
Furthermore, $\lim_{\eta \to 0} \nu^*(H \mathbbm{1}\{\mathcal{E} > \eta\}) = \nu^*(H)$. This convergence and (3.16) yield (3.15). □

The previous result states that if (3.8) holds with some functions $a_T$ and $r_T$, then $\nu_{T \cdot r_T}^* \overset{v}{\to} \nu^*$. We also know that (3.8) implies that the tail process converges almost surely to zero. There is no converse of this result. However, if we know that the tail process converges to zero, then for any given function $a_T$, we can prove the existence of a function $r_T$ such that $\nu_{T \cdot r_T}^* \overset{v}{\to} \nu^*$.

**Lemma 3.7.** Assume that $X$ is regularly varying in $\mathcal{D}$ with tail process $Y$ such that $\mathbb{P}(Y \in \mathcal{D}_0) = 1$. Then for each scaling function $a_T$, there exists a scaling function $r_T$ such that
\[
\lim_{T \to \infty} r_T\mathbb{P}(|X_0| > a_T) = 0 ,
\]
and $\nu_{T \cdot r_T}^* \overset{v}{\to} \nu^*$ in $(\mathcal{D}_0, \mathcal{B}_0)$.

**Proof.** Fix $m \geq 1$. For a map $H$ on $\mathcal{D}$, let $H_m$ be the map defined on $\mathcal{D}$ by $H_m(y) = H(y_{0,m})$. Define a measure $\tilde{\nu}_m^*$ on $\mathcal{D}_0$ by
\[
\tilde{\nu}_m^*(H) = \frac{1}{m} \nu(H_m).
\]

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Let $\epsilon > 0$ and $H$ be a bounded Lipschitz continuous map on $D_0$ such that $h(y) = 0$ if $y^* \leq 2\epsilon$. Regular variation in $D$ implies that

$$\lim_{T \to \infty} \frac{\mathbb{E}[H(a_T^{-1} X_{0,m})]}{m \mathbb{P}(|X_0| > a_T)} = \frac{1}{m} \nu(H_m) = \tilde{\nu}_m^*(H).$$

The latter quantity is finite since the support of $H_m(y) \leq \text{cst} \mathbb{1}\{y_{0,m}^* > 1\}$. Applying (2.11), we obtain

$$\tilde{\nu}_m^*(H) = \frac{1}{m} \nu(H_m) = \frac{1}{m} \int_0^\infty \int_0^\infty \mathbb{E}[H(rQ_{t,m-t})] \alpha r^{-\alpha-1} dr \, dt$$

$$= \int_0^1 \int_0^\infty \mathbb{E}[H(rQ_{mt,m(1-t)})] \alpha r^{-\alpha-1} dr \, dt + R_1 + R_2,$$

with $R_1$ and $R_2$ the integrals over $(-\infty, 0)$ and $(T, \infty)$, respectively. Since $H$ is Lipschitz continuous and bounded and $Q \in D_0$, we have by the dominated convergence theorem,

$$\lim_{m \to \infty} \int_0^1 \int_0^\infty \mathbb{E}[H(rQ_{mt,m(1-t)})] \alpha r^{-\alpha-1} dr \, dt = \int_0^\infty \mathbb{E}[H(rQ)] \alpha r^{-\alpha-1} dr = \nu^*(H).$$

We next prove that $R_1$ and $R_2$ tend to zero. Note that

$$\int_0^\infty |H(ry)| \alpha r^{-\alpha-1} dr \leq \text{cst} \int_0^\infty \mathbb{1}\{uy_{0,m}^* > \epsilon\} \alpha u^{-\alpha-1} du = \text{cst} \, \epsilon^{-\alpha} (y_{0,m}^*)^\alpha.$$

This yields, by subadditivity of the maximum,

$$R_1 \leq \frac{1}{m} \int_{-\infty}^0 \mathbb{E}[H_{a_s}(Q_{s,m,s})] \, ds \leq \frac{\text{cst}}{m} \int_0^\infty \mathbb{E}[(Q_{s,m+s}^*)^\alpha] \, ds$$

$$\leq \frac{\text{cst}}{m} \sum_{i=1}^m \int_0^\infty \mathbb{E}[(Q_{i-1+s,i+s}^*)^\alpha] \, ds = \frac{\text{cst}}{m} \sum_{i=1}^m \int_{i-1}^\infty \mathbb{E}[(Q_{i,s+1}^*)^\alpha] \, ds.$$

Note that $\int_{-\infty}^\infty \mathbb{E}[(Q_{s,s+1}^*)^\alpha] \, ds = \nu(\{y \in D : y_{0,1}^* > 1\}) < \infty$, thus

$$\lim_{i \to \infty} \int_i^\infty \mathbb{E}[(Q_{s,s+1}^*)^\alpha] \, ds = 0.$$  

By Cesaro’s theorem, this yields $\lim_{m \to \infty} R_1 = 0$. The proof for $R_2$ is along the same lines.

We have thus proved that $\nu_{T,m}^* \xrightarrow{v} \tilde{\nu}_m^*$ and $\tilde{\nu}_m^* \xrightarrow{v} \nu^*$. Since vague convergence is metrizable, this implies that there exists a sequence $r_T$ such that $\nu_{T,r_T}^* \xrightarrow{v} \nu^*$. (See for instance [Dud02, p.395, comment after Proposition 11.3.2].)

Assume that the function $r_T$ does not satisfy (3.17). Then, along a subsequence, we would have $r_T \mathbb{P}(|X_0| > a_T) \to c > 0$ and thus

$$\lim_{T \to \infty} \mathbb{P}(X_{0,T}^* > a_T x) = c \nu^*(\{y^* > w\}) = c \nu x^{-\alpha}.$$

This is a contradiction, since the left-hand side must be less than 1, and the right-hand side can be arbitrarily large. Thus the scaling function $r_T$ must satisfy (3.17).
The previous results allow to prove convergence of the measure $\nu^*_{T,r}$ when the process $X$ admits a suitable sequence of approximations.

**Lemma 3.8.** Let $X$ be a stationary process, regularly varying in $D$ with tail process $Y$ such that $\mathbb{P}(Y \in D_0) = 1$. Assume that there exists a sequence of stationary $m$-dependent processes $X^{(m)}$, regularly varying in $D$, such that $(X, X^{(m)})$ is stationary and

$$
\lim_{m \to \infty} \limsup_{x \to \infty} \frac{\mathbb{P}\left(\sup_{0 \leq s \leq 1} \left| X_s - X_s^{(m)} \right| > x \right)}{\mathbb{P}(|X_0| > x)} = 0. \tag{3.19}
$$

Let $\nu^*$ and $\nu^*_m$ be the cluster measures of $X$ and $X^{(m)}$, respectively. Then $\nu^*_m \xrightarrow{v} \nu^*$ and $\nu^*_{T,rT} \xrightarrow{v} \nu^*$ in $(\tilde{D}_0, \tilde{B}_0)$ for all scaling functions $a_T, r_T$ such that (3.17) holds.

Note that by Lemma 3.7, we already know that there exists at least one sequence $r_T$ such that $\nu^*_{T,rT} \xrightarrow{v} \nu^*$. The goal of this lemma is two fold: to prove that $\nu^*_m \xrightarrow{v} \nu^*$ and that the former convergence holds for all scaling functions $r_T$ such that $\lim_{T \to \infty} r_T \mathbb{P}(|X_0| > a_T) = 0$ for any scaling function $a_T$.

**Proof of Lemma 3.8.** Note first that (3.19) implies that

$$
\lim_{m \to \infty} \limsup_{T \to \infty} \left| \frac{\mathbb{P}(|X^{(m)}_0| > T x)}{\mathbb{P}(|X_0| > T)} - x^{-\alpha} \right| = 0. \tag{3.20}
$$

See [KS20, Proposition 5.2.5]. This implies that there exists $m_0 \geq 1$ such that for all $m \geq m_0$,

$$
\frac{1}{2} \leq \limsup_{T \to \infty} \frac{\mathbb{P}(|X^{(m)}_0| > T)}{\mathbb{P}(|X_0| > T)} \leq 2. \tag{3.21}
$$

As a consequence, if $r_T$ is a scaling function such that

$$
\lim_{T \to \infty} r_T \mathbb{P}(|X_0| > a_T) = 0, \tag{3.22}
$$

then it also holds that

$$
\lim_{T \to \infty} r_T \mathbb{P}(|X^{(m)}_0| > a_T) = 0, \tag{3.23}
$$

for all $m \geq m_0$. Define now the measure $\nu^*_{T,rT,m}$ by

$$
\nu^*_{T,rT,m} = \frac{\mathbb{E}\left[\delta_{a_T^{-1}X^{(m)}_{0,rT}}\right]}{r_T \mathbb{P}(|X^{(m)}_0| > a_T)}. \tag{3.24}
$$

By Lemma 3.4 and the previous considerations, the process $X^{(m)}$ being $m$-dependent, it satisfies condition (3.8) for all sequences $r_T$ such that (3.22) for $m \geq m_0$. Thus $\nu^*_{T,rT,m} \xrightarrow{v} \nu^*_m$ for such sequences.
Let $H$ be a shift-invariant map on $D_0$, Lipschitz with respect to the metric $d_\infty$ and such that $H(y) = 0$ if $y^* \leq 2\epsilon$ for $\epsilon > 0$ depending on $H$. Fix $\eta < \epsilon$. Then $|x - y| \leq \eta$ and $|x| \wedge |y| \leq \epsilon$ imply $H(x) = H(y) = 0$. Thus,

$$
|\nu_{T,r}^*(H) - \nu_{T,r,m}^*(H)| \leq \frac{1}{r_T} \mathbb{E}[|H(X_0/\nu_T) - H(X_0^{(m)})|] + \text{cst} \left( \frac{\mathbb{P}(|X_0| > a_T)}{\mathbb{P}(|X_0^{(m)}| > a_T)} - 1 \right)
$$

$$
\leq \text{cst} \eta \nu_{T,r}^*(\{y^* > \epsilon\}) + \frac{\mathbb{P}\left(\sup_{0 \leq s \leq r_T} |X_s - X_s^{(m)}| > a_T \eta\right)}{r_T \mathbb{P}(|X_0| > a_T)}
$$

$$
+ \text{cst} \left| \frac{\mathbb{P}(|X_0| > a_T)}{\mathbb{P}(|X_0^{(m)}| > a_T)} - 1 \right| \eta \nu_{T,r}^*(\{y^* > \epsilon\}) + \frac{\mathbb{P}(\sup_{0 \leq s \leq 1} |X_s - X_s^{(m)}| > a_T \eta)}{\mathbb{P}(|X_0| > a_T)}
$$

$$
\leq \text{cst} \eta \nu_{T,r}^*(\{y^* > \epsilon\}) + \text{cst} \left| \frac{\mathbb{P}(|X_0| > a_T)}{\mathbb{P}(|X_0^{(m)}| > a_T)} - 1 \right| . \quad (3.24)
$$

By Lemma 3.7, there exists a sequence $r_T^0$ such that (3.22) holds and $\nu_{T,r_T^0}^* \xrightarrow{v} \nu^*$. Thus, for this sequence applying (3.19) and (3.20), we obtain

$$
\lim_{m \to \infty} \limsup_{T \to \infty} \left| \nu_{T,r_T^0}^*(H) - \nu_{T,r_T^0,m}^*(H) \right| \leq \text{cst} \eta \nu^*(\{y^* > \epsilon\}) .
$$

Since $\eta$ is arbitrary, the right-hand side is actually 0. By (3.20), we can also choose $r_T^0$ satisfying (3.23) for $m$ large enough, thus $\nu_{T,r_T^0,m}^* \xrightarrow{v} \nu_{m}^*$. By Lemma A.3, this proves that $\nu_m^* \xrightarrow{v} \nu^*$.

By inverting the roles of $X$ and $X^{(m)}$ in the derivations that lead to (3.24), we obtain

$$
\left| \nu_{T,r_T}^*(H) - \nu_{T,r_T,m}^*(H) \right|
$$

$$
\leq \text{cst} \eta \nu_{T,r_T,m}^*(\{y^* > \epsilon\}) + \frac{\mathbb{P}\left(\sup_{0 \leq s \leq 1} |X_s - X_s^{(m)}| > a_T \eta\right)}{\mathbb{P}(|X_0^{(m)}| > a_T)}
$$

$$
+ \text{cst} \left| \frac{\mathbb{P}(|X_0^{(m)}| > a_T)}{\mathbb{P}(|X_0| > a_T)} - 1 \right| \eta \nu_{T,r_T}^*(\{y^* > \epsilon\}) + \frac{\mathbb{P}(\sup_{0 \leq s \leq 1} |X_s - X_s^{(m)}| > a_T \eta)}{\mathbb{P}(|X_0| > a_T)}
$$

$$
\leq \text{cst} \eta \nu_{T,r_T}^*(\{y^* > \epsilon\}) + \text{cst} \left| \frac{\mathbb{P}(|X_0| > a_T)}{\mathbb{P}(|X_0^{(m)}| > a_T)} - 1 \right| .
$$

Since $\nu_{T,r_T,m}^* \xrightarrow{v} \nu_m^*$ for all sequences $r_T$ satisfying (3.22) and $m$ large enough, this yields

$$
\lim_{m \to \infty} \limsup_{T \to \infty} \left| \nu_{T,r_T}^*(H) - \nu_{T,r_T,m}^*(H) \right| \leq \text{cst} \eta \nu_m^*(\{y^* > \epsilon\}) .
$$

Since $\eta$ is arbitrary, the right-hand side is actually 0. By Lemma A.3 again, this proves that $\nu_{T,r_T}^* \xrightarrow{v} \nu^*$ in $(\mathcal{D}_0, \mathcal{B}_0)$. \qed
3.3 The point process of clusters

We now define the functional point process of clusters on \([0, \infty) \times \tilde{D}_0 \setminus \{0\}\). For \(i \in \mathbb{N}^\ast\), set 
\[ X_{T,i} = a_{T}^{-1}X_{[(i-1)r_T,ir_T)}, \]
identified with its equivalence class in \(\tilde{D}_0\). Set also 
\[ m_T = Tr_T^{-1} \]
and
\[ N_T = \sum_{i=1}^{\infty} \delta_{\frac{X_{T,i}}{m_T}}. \tag{3.25} \]

This point process is related to the excursion random measure \(\zeta_T\) of [HL98] defined by
\[ \zeta_T = \int_0^T \delta_{\frac{X_{T,i}}{m_T}} \, dt. \]

Indeed, for a function \(f\) defined on \([0, \infty) \times \mathbb{R}\), let the map \(H_f\) be (formally) defined on \([0, \infty) \times \tilde{D}_0\) by
\[ H_f(t, y) = \int_{-\infty}^{\infty} f(t, y_s) \, ds. \]

This yields
\[ N_T(H_f) = \sum_{i=1}^{\infty} \int_{(i-1)r_T}^{ir_T} f \left( \frac{i}{m_T}, \frac{X_{T,i}}{a_T} \right) \, ds. \]

If \(f\) vanishes for large \(t\) and for \(|x| > \epsilon\) and is Lipschitz continuous with respect to \((t, x)\), then \(N_T(H_f) - \zeta_T(f) \xrightarrow{P} 0\). Thus the convergence of \(N_T\) implies that of \(\zeta_T\). The random measures contains more information than \(\zeta_T\). See [BPS18] for the corresponding discussion in discrete time.

The convergence of \(N_T\) will be established under the following unprimitive assumption which validates a blocking method. It is a classical assumption in extreme value theory for stochastic processes. See e.g. Condition \(\mathcal{A}(a_n)\) in [DH95] for time series. It is implied by condition \(\Delta(u_T)\) of [HL98] in continuous time.

**Assumption 3.9.** There exist scaling functions \(a\) and \(r: \mathbb{R}_+ \to \mathbb{R}_+\) such that for all bounded continuous maps \(f: \mathbb{R}_+ \times \tilde{D}_0 \setminus \{0\} \to \mathbb{R}_+\) such that \(f(t, y) = 0\) if \(|t| > A\) or \(y^\ast \leq \epsilon\) for some \(A\) and \(\epsilon > 0\) (depending on \(f\)),
\[ \lim_{T \to \infty} \sup \left| \mathbb{E} \left[ e^{-N_T(f)} \right] - \prod_{i=1}^{\infty} \mathbb{E} \left[ e^{-f(i r_T, X_{T,i})} \right] \right| = 0. \tag{3.26} \]

Under this assumption, we extend [BPS18, Theorem 3.6] to the continuous time framework and [HL98, Theorem 4.1].

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Theorem 3.10. Let $X$ be a stationary $\mathcal{D}$-valued stochastic process, regularly varying in $\mathcal{D}$, with cluster measure $\nu^*$. Assume that there exist scaling functions $a_T$ and $r_T$ such that $\lim_{T \to \infty} TP(|X_0| > a_T) = 1$ and (3.8) holds. Let $N_T$ be the point process of cluster defined in (3.25) with the same functions $a$ and $r$. The following statements are equivalent:

- Assumption 3.9 holds with the same functions $a$ and $r$;
- $N_T \xrightarrow{w} N$ with $N$ a PPP on $\mathbb{R}_+ \times \tilde{\mathcal{D}}_0 \setminus \{0\}$ with mean measure $\text{Leb} \otimes \nu^*$.

A PPP $N$ with mean measure $\text{Leb} \otimes \nu^*$ has the representation

$$N = \sum_{i=1}^{\infty} \delta_{T,i} P_i Q^{(i)}, \quad (3.27)$$

where $\sum_{i=1}^{\infty} \delta_{T,i} P_i$ is a PPP on $\mathbb{R}_+ \times (0, \infty)$ with mean measure $\text{Leb} \otimes \nu_\alpha$ and $Q^{(i)}$ are i.i.d. copies of a process $Q$ whose distribution is given by (2.15).

Proof of Theorem 3.10. The proof is essentially the same as the proof of [BPS18, Theorem 3.6]. See also [KS20, Theorem 7.3.1]. We briefly sketch it. Let $X^\dagger_{T,i}, i \geq 1$, be independent random elements in $\mathcal{D}_0$ with the same distribution as $X_{T,1}$. Define $\xi_{T,i} = \delta_{T,i} \odot X_{T,i}, i \geq 1$. Then $\{\xi_{T,i}, i \geq 1\}$ is a null array of point processes in the sense of [Kal17, Section 4.3]. By stationarity, the mean measure of $\xi_{T,i}$ is $\delta_{\frac{i}{mT}} \otimes \nu^*_T$. Denote $\nu_T = \sum_{i=1}^{\infty} \delta_{\frac{i}{mT}} \otimes \nu^*_T$. By [KS20, Theorem 7.1.6], the stated convergence is equivalent to the convergence $\nu_T \xrightarrow{v} \text{Leb} \otimes \nu^*$. This boils down to proving that $\sum_{i=1}^{\infty} \delta_{\frac{i}{mT}} \xrightarrow{v} \text{Leb} \otimes \nu^*$ which is trivial.

A real-valued stationary process $X$ is said to have extremal index $\theta$ if for every $\tau > 0$ and sequence $u_T$ such that $\lim_{T \to \infty} TP(X_0 > u_T) = \tau$, it holds that $\lim_{T \to \infty} P(\sup_{0 \leq s \leq T} X_s \leq u_T) = e^{-\theta \tau}$. For a stochastic process indexed by $Z$, the extremal index, if it exists, must be in $[0, 1]$. For a continuous time process it can be in $[0, \infty]$. Under Definition 3.1 and Assumption 3.9, we show that it exists in $(0, \infty)$.

Corollary 3.11. Let $X$ be a stationary $\mathcal{D}$-valued process, regularly varying in $\mathcal{D}$ with tail index $\alpha > 0$. If Assumptions 3.3 and 3.9 hold with a scaling function $a_T$ such that $\lim_{T \to \infty} TP(|X_0| > a_T) = 1$,

$$\lim_{T \to \infty} TP\left(\sup_{0 \leq t \leq T} |X_s| \leq a_T x\right) = e^{-\theta x^{-\alpha}}.$$

Proof. The point process convergence of Theorem 3.10 yields:

$$P\left(\sup_{0 \leq t \leq T} |X_s| \leq a_T x\right) = P(N_T([0, 1] \times \{y^* > x\}) = 0)$$

$$\rightarrow P(N([0, 1] \times \{y^* > x\}) = 0)$$

$$= e^{-\nu^*\{y^* > x\}} = e^{-\theta x^{-\alpha}}.$$

\[\square\]
We now provide conditions which ensure Assumption 3.3 or Assumption 3.9. For the definition of $\beta$-mixing, see [Bra05].

**Lemma 3.12.** Let $X$ be $\beta$-mixing with rate $\beta_t$ and assume that there exist sequences $\{r_T\}$ and $\{\ell_T\}$ such that

$$\lim_{T \to \infty} \frac{r_T}{T} = \lim_{T \to \infty} \frac{\ell_T}{T} = \lim_{T \to \infty} \frac{T\beta_{\ell_T}}{r_T} = 0.$$ 

Then Assumption 3.9 holds with $a_T$ such that $T\mathbb{P}(|X_0| > a_T) \to 1$.

**Proof.** The proof is similar to the proof in the case of discrete time processes, see [BPS18, Lemma 6.2].

**Lemma 3.13.** If $X$ is an $m$-dependent regularly varying stationary $\mathcal{D}$-valued stochastic process, then Assumptions 3.3 and 3.9 hold for all scaling functions $a_T$ and $r_T$ such that (3.17) holds.

**Proof.** We already know that Assumption 3.3 holds by Lemma 3.4. Since $m$-dependent sequences are $\beta$-mixing with arbitrarily fast rates, Assumption Assumption 3.9 holds by Lemma 3.12.

We now consider processes which admit a sequence of tail equivalent approximations.

**Theorem 3.14.** Let $X$ be a stationary process, regularly varying in $\mathcal{D}$ with tail process $Y$ such that $\mathbb{P}(Y \in \mathcal{D}_0) = 1$ and cluster measure $\nu^*$. Assume that there exists a sequence of $m$-dependent stationary processes $X^{(m)}$, regularly varying in $\mathcal{D}$, such that $(X, X^{(m)})$ is stationary for every $m$ and (3.19) holds. Then the point process of clusters $N_T$ converges weakly to a Poisson point process $N$ on $[0, \infty) \times \tilde{\mathcal{D}} \setminus \{0\}$ with mean measure $\text{Leb} \otimes \nu^*$.

**Proof.** Let $N_T^{(m)}$ be the point process of clusters of the process $X^{(m)}$. Since $X^{(m)}$ is regularly varying in $\mathcal{D}$ and $m$-dependent, $N_T^{(m)} \xrightarrow{w} N^{(m)}$ with $N^{(m)}$ a Poisson point process with mean measure $\text{Leb} \otimes \nu^*_m$. Since $\nu^*_m \xrightarrow{v^*} \nu^*$, it also holds that $N^{(m)} \xrightarrow{w} N$. If we moreover prove that for all $\epsilon > 0$ and Lipschitz (with respect to the $d_{J_1}$ distance) continuous maps $H$ such that $H(y) = 0$ if $y^* \leq \epsilon$ and for all $\eta > 0$,

$$\lim_{m \to \infty} \limsup_{T \to \infty} \mathbb{P}(|N_T(H) - N_T^{(m)}(H)| > \eta) = 0,$$

(3.28)

then by Proposition A.5, we will have proved that $N_T \xrightarrow{w} N$.

We now prove (3.28). For simplicity, we drop the time component which brings no difficulty but only additional notational complexity. This means that we replace $N$ by $N([0,1] \times \cdot)$. Let $\epsilon > 0$ and $H$ be a shift-invariant map on $\mathcal{D}_0$, Lipschitz continuous with respect to the
metric $d_{J_1}$ and such that $H(y) = 0$ if $y^* \leq 2\varepsilon$. Let $B^c_\varepsilon$ denote the complement of the ball centered at 0 with radius $\varepsilon$ with respect to the metric $d_{J_1}$. Then, for $\delta < \varepsilon$,

$$
\mathbb{P}(|N_T(H) - N_{T,m}(H)| > \eta) \leq \mathbb{P} \left( \sum_{i=1}^{[T/r_T]} |H(X_{T,i}) - H(X^{(m)}_{T,i})| > \eta \right)
$$

$$
\leq \frac{T}{r_T} \mathbb{P}(d_{J_1}(X_{T,1}, X^{(m)}_{T,1}) > \delta) + \mathbb{P} \left( K \varepsilon N^{(m)}_T(B^c_\varepsilon) > \eta \right)
$$

$$
\leq \frac{T}{r_T} \mathbb{P} \left( \sup_{0 \leq s \leq r_T} |X_s - X^{(m)}_s| > a_T \delta \right) + \mathbb{P} \left( K \delta N^{(m)}_T(B^c_\varepsilon) > \eta \right)
$$

By (3.19), the first term in the last equation vanishes when $n$, then $m$ tend to $\infty$. The weak convergence of $N^{(m)}_T$ and Markov inequality yield

$$
\lim_{T \to \infty} \mathbb{P} \left( K \delta N^{(m)}_T(B^c_\varepsilon) > \eta \right) = \mathbb{P} \left( K \delta N^{(m)}_T(B^c_\varepsilon) > \eta \right) \leq K \delta \eta^{-1} \nu^*_m(B^c_\varepsilon).
$$

Since $\nu^*_m \xrightarrow{w} \nu^*$, we obtain

$$
\lim_{m \to \infty} \lim_{T \to \infty} \mathbb{P} \left( K \delta N^{(m)}_T(B^c_\varepsilon) > \eta \right) \leq \text{cst} \, \delta.
$$

Since $\delta$ is arbitrary, this proves (3.28) and concludes the proof of Theorem 3.14.

\[\square\]

### 3.4 Illustrations

We now describe informally the type of results that can be obtained with the tools of Sections 2 and 3, which highlight the practical usefulness of the identities of Section 2.4 and the measure $\nu^*$ introduced in Section 3.2. A rigorous investigation of these problems is beyond the scope of this paper. See [KS20, Chapters 8-10] for a review of the corresponding results in discrete time.

**Convergence to $\alpha$-stable processes**

The convergence of the point process of clusters can be used to prove limit theorems such as convergence of the partial sum process. For $\alpha \in (1, 2)$, under Assumptions 3.3 and 3.9, it can be proved by a truncation and continuous mapping argument applied to the point process of clusters that

$$
a_T^{-1} \int_0^{T_t} X_s ds - c_T \xrightarrow{f.i.d.} \Lambda(t),
$$
with $c_T$ a suitable centering, $\Lambda$ a Lévy stable process such that,

$$\log \mathbb{E}[e^{iz\Lambda(1)}] = -\sigma^\alpha |z|^\alpha \{1 - i \beta \text{sign}(z) \tan(\pi \alpha/2)\},$$

with

$$\sigma^\alpha = \alpha \Gamma(1 - \alpha) \cos(\pi \alpha/2) \mathbb{E}\left[\int_0^\infty \Theta_s ds \right]^{-1},$$

$$\beta = \frac{\mathbb{E}\left[\Theta_0 \left(\int_0^\infty \Theta_s ds \right)^{<\alpha>-1}\right]}{\mathbb{E}\left[\int_0^\infty \Theta_s ds \right]^{-1}}$$

with $x^{<a>} = \text{sign}(x)|x|^a$ for all $x \neq 0$ and $a \neq 0$, and assuming that $\sigma \neq 0$. If $\alpha < 1$, we have seen in Section 2.4 that the above quantities are always finite. If $1 < \alpha < 2$, this convergence can also be proved to hold under an extra assumption which guarantees that $\sigma$ and $\beta$ are still well defined and a negligibility assumption is needed to handle the small jumps of the Lévy process. For $\alpha = 1$, an additional centering term appears in the limiting stable law. The convergence can be proved in the $J_1$ or $M_1$ topology under some additional conditions. Since this convergence is used using the point process of clusters, the expressions of $\sigma^\alpha$ and $\beta$ are obtained first in terms of the sequence $Q$ and then translated in terms of the forward spectral tail process using Lemma 2.13. See [BPS18] for exhaustive results in discrete time.

**Estimation of cluster functionals**

Many extreme value statistical problems involved so-called cluster functionals, introduced in discrete time by [DR10] without the formalism of the tail process. See [KS20, Chapter 10] for a complete presentation of such results in discrete time.

In the language of Section 3.3, quantities of interest are often of the form $\nu^*(H)$ with $H$ a functional on $D_0$. For instance, the extreme value index $\gamma = \alpha^{-1}$ can be expressed as $\gamma = \nu^*(H_{\log^+})$ with $\log^+(x) = \log(x \wedge 1)$ and $H_{\log}(y) = \int_{-\infty}^\infty \log^+(|y_s|)ds$. For $y \in D_0$, the integral is over a finite interval hence well defined and finite. By (3.12) and (2.12), we have

$$\nu^*(H_{\log^+}) = \vartheta \int_{-\infty}^\infty \int_0^\infty \mathbb{E}[\log^+(r|Q_s|)]\alpha r^{-\alpha-1}drds$$

$$= \vartheta \int_{-\infty}^\infty \mathbb{E}[|Q_s|]ds \int_0^\infty \log^+(r)\alpha r^{-\alpha-1}dr = \gamma .$$

Another example is given by the relation $\vartheta = \nu^*(H_\theta)$ which is a direct consequence of (3.12) with $H_\theta(y) = 1\{y^* > 1\}$. The assumptions which are needed to obtain results for estimators of $\vartheta$ also imply that $\vartheta = \theta$, the true extremal index.

A natural estimator of $\nu^*(H)$ given the observation of a path $X_s, s \in [0, T]$ is obtained by
\[ \hat{\nu}^*_T(H) \] with

\[ \hat{\nu}_T^* = \frac{1}{T \mathbb{P}(|X_0| > u_T)} \sum_{i=1}^{K_T} \delta_{X_{T,i}}, \]

with \( X_{T,i} = (X_s)_{(i-1)r_T \leq s \leq ir_T}, r_T \) and \( u_T \) are increasing functions such that \( T/r_T \to 0 \), \( r_T \mathbb{P}(|X_0| > u_T) \to 0 \), \( T \mathbb{P}(|X_0| > u_T) \to \infty \) and \( K_T = \lceil T/r_T \rceil \). Such estimators are called block estimators. The random measure \( \hat{\nu}^*_T \) can be called an empirical cluster measure, since it is expected to converge weakly to \( \nu^* \).

Actually, \( \hat{\nu}^*_T \) is not a feasible estimator, since the factor in the denominator depends on the marginal distribution. There are different ways to deal with this issue which we will not discuss here. Under mixing conditions such as \( \beta \)-mixing and under condition which guarantee the existence of the limiting variance, we expect to prove a central limit theorem of the form

\[ \sqrt{T \mathbb{P}(|X_0| > u_T)} \{ \hat{\nu}^*_T - \nu^*(H) \} \overset{d}{\to} N(0, \nu^*(H^2)). \]

We can compute \( \nu^*(H^2) \) for the examples cited above. This is straightforward for the estimator of the extremal index since \( H^2_v = H_v \), thus \( \nu^*(H^2_v) = \vartheta \). The computations are harder for the estimator of the extreme value index. For \( y \in \mathcal{D}_0 \), define \( \ell_s(y) = \log_y(y_s) \).

Under assumptions which guarantee that the variance is finite, we have, using (3.13) and the shift-invariance of \( \nu \),

\[ \nu^*(H^2_{\log}) = \int_{-\infty}^\infty \int_{-\infty}^\infty \nu^*(\ell_s(y)\ell_t(y))dsdt = \int_{-\infty}^\infty \nu(\ell_0(y)\ell_t(y))dt. \]

Since \( \ell_0(y) = 0 \) if \( |y_0| \leq 1 \), we obtain, using the definition of the tail process in terms of the tail measure,

\[ \nu^*(H^2_{\log}) = \int_{-\infty}^\infty \mathbb{E}[\log_+(|Y_0|)\log_+(|Y_t|)]dt = \gamma \int_{-\infty}^\infty \mathbb{E}[|Y_t| \wedge 1]dt. \]

The last identity is obtained by a repeated application of the time change formula (2.2) and the identity \( \log_+(y) = \int_1^\infty 1\{y > u\}u^{-1}du \). Conditions are needed to ensure that the integral is finite.

### 4 Examples

In this section we will study several classes of stochastic processes. For each, we will prove regular variation in \( \mathcal{D} \), check the conditions of Theorem 3.14 and thus obtain their extremal
The novelty of our results is the regular variation in $D$ and the convergence of the point process of clusters. We start with max- and sum-stable processes whose distributions are entirely determined by the tail measure. Then we consider functionally weighted sums of i.i.d. regularly varying random variables.

The list is limited to keep the paper at reasonable length. Other interesting examples include Lévy driven (mixed) moving average processes: [HL05, Section 4], [Fas06, Section 4], [FK07] for which we expect that regular variation in $D$ holds and Theorem 3.14 can be applied. An important class of models already studied in discrete time consists of regularly functions of Markov processes. It is proved in [KSW19] that Assumption 3.3 holds for functions of geometrically ergodic Markov chains whose kernel satisfies a suitable drift condition. Since these chains are also $\beta$-mixing with geometric rate, they fulfill the conditions of Theorem 3.10. Extending these results for continuous time Markov processes would provide a wealth of useful examples.

### 4.1 Max-stable processes

Let $\eta$ be a stationary max-stable process with unit scale $\alpha$-Fréchet marginal distributions, i.e. for all $k \geq 1$, $t_1, \ldots, t_k \in \mathbb{R}$ and $x_1, \ldots, x_k > 0$,

$$
\mathbb{P}\left( \bigvee_{i=1}^{k} \frac{\eta(t_i)}{x_i} \leq 1 \right) = e^{-\nu_1 x_i - \alpha}
$$

The finite dimensional distributions of $\eta$ are regularly varying and are completely determined by their exponent measures. Hence the tail measure $\nu$ in the sense of [SO12] entirely determines the distribution of $\zeta$.

If $\text{tailmeasure}$ is a tail measure in the sense of Definition 2.1 with spectral process $Z$, then we can define a max-stable process $\tilde{\eta}$ by

$$
\tilde{\eta}_t = \bigvee_{i=1}^{\infty} P_i Z_t^{(i)}, \quad t \in \mathbb{R}.
$$

with $\sum_{i=1}^{\infty} \delta_{P_i}$ a Poisson point process on $(0, \infty)$ with mean measure $\nu_\alpha$ and $Z_t^{(i)}, i \geq 1$ i.i.d. copies of $Z$. Then obviously $\tilde{\eta}$ has the same distribution as $\eta$. We now prove the converse.

**Theorem 4.1.** Let $\alpha > 0$, $\nu$ be a tail measure on $D$ with tail index $\alpha$ and $Z$ be a spectral process for $\nu$. Let $\eta$ be a $D$-valued process. The following statements are equivalent:

(i) $\eta$ is a max-stable process with $\alpha$-Fréchet marginal distributions, regularly varying in $D(\mathbb{R})$, with tail measure $\nu$;

(ii) $\eta$ admits the representation (4.1).
Proof. Since $\eta_0$ has an $\alpha$-Fréchet distribution, we must prove that for all $\epsilon > 0$, $a > 0$ and bounded Lipschitz continuous maps $H$ on $D$ such that $H(y) = 0$ if $y^{*-a, a} \leq \epsilon$,

$$\lim_{T \to \infty} T^\alpha \mathbb{E}[H(T^{-1}\eta)] = \nu(H). \quad (4.2)$$

Let $W$ be a process whose distribution is given by

$$\mathbb{E}[H(W)] = \frac{\mathbb{E}[H((Z_{a,b}^*)^{-1}Z)(Z_{a,b}^*)^\alpha]}{\mathbb{E}[(Z_{a,b}^*)^\alpha]}.$$ 

Let $W^{(i)}$ be i.i.d. copies of $W$. Then, $\eta$ has the same distribution as

$$c_{\alpha}^{1/\alpha} \sum_{i=1}^{\infty} P_i W^{(i)},$$

with $c_{\alpha} = \mathbb{E}[\sup_{a \leq s \leq b} |Z(s)|^\alpha]$. Since $P_1$ has a Fréchet distribution and $\sup_{-a \leq t \leq a} |W_t| = 1$, we have

$$\lim_{T \to \infty} T^\alpha \mathbb{E}[H(c_{\alpha}^{1/\alpha} T^{-1} P_1 W)] = c_{\alpha} \int_{0}^{\infty} \mathbb{E}[H(uW)] u^{-\alpha-1} du$$

$$= \int_{0}^{\infty} \mathbb{E}[H(uZ)] u^{-\alpha-1} du = \nu(H).$$

By definition of the metric $d_{J_1}$, for every $f, g \in D(\mathbb{R}, \mathbb{R}_+)$ and $a > 0$, we have

$$d_{J_1}(f, f \lor g) \leq \sup_{-a \leq s \leq a} |f(s) - f \lor g(s)| + e^{-a} \leq \sup_{-a \leq s \leq a} g(s) + e^{-a}.$$ 

Since $H$ is bounded, Lipschitz continuous with respect to the $J_1$ metric and has support separated from 0, and $W_{a,b}^{*} = 1$, we have, for all $\beta < \epsilon/2$,

$$T^\alpha \mathbb{E}[H(T^{-1} P_1 W^{(1)})] - \mathbb{E}[H(T^{-1}\eta)] \leq \text{cst} \ T^\alpha \mathbb{P}(d_{J_1}(P_1 W^{(1)}, \vee_{i=1}^{\infty} P_i W^{(i)}) > T\beta) + \text{cst} \ T^\alpha \mathbb{P}(P_1 > T\epsilon/2)$$

$$\leq \text{cst} \ T^\alpha \mathbb{P}(P_2 + e^{-a} > T\beta) + \beta T^\alpha \mathbb{P}(P_1 > T\epsilon/2).$$

This yields, for all $\beta < \epsilon/2$,

$$\limsup_{T \to \infty} T^\alpha \mathbb{E}[H(T^{-1} P_1 W^{(1)})] - \mathbb{E}[H(T^{-1}\eta)] \leq \text{cst} T^\alpha \mathbb{E}[H(T^{-1} P_1 W^{(1)})] - \mathbb{E}[H(T^{-1}\eta)] \leq \text{cst} \beta.$$ 

Since $\beta$ is arbitrary, the lim sup is actually zero. Altogether, we have proved $(4.2)$. \qed

Applying the results of Section 2, we obtain an explicit mixed moving average representation of max-stable processes generated by a dissipative flow. We also recover the result of [DH19, Theorem 2.1 and 2.3] and give a new expression for the extremal index.
Corollary 4.2. Let $\nu$ be a shift-invariant tail measure on $\mathcal{D}$ with tail process $Y$. Let $\eta$ be the associated max-stable process defined by (4.1). Then $\eta$ admits an extremal index $\theta$ equal to $\vartheta$, i.e.

$$\theta = \vartheta = \mathbb{E} \left[ \frac{1}{\mathcal{E}(Y)} \right] < \infty.$$ 

The extremal index is positive if and only if $\nu(\mathcal{D}_0) > 0$. The process $\eta$ admits a mixed moving average representation if and only if $\nu(\mathcal{D}_c^0) = 0$. In the latter case, $\vartheta > 0$ and

$$\eta_t = \bigvee_{i=1}^{\infty} P_i Q^i_{t-T_i}$$

with $\{T_i, P_i, Q_i\}$ the points of a Poisson point process on $\mathbb{R} \times (0, \infty) \times \mathcal{D}$ with mean measure $\text{Leb} \otimes \nu' \otimes \mathbb{P}_Q$ and with $Q$ a random element in $\mathcal{D}$ whose distribution is given in (2.15).

Proof. We only prove the statement on the extremal index for completeness, slightly simplifying the argument in the proof of [DH19, Theorem 2.1]. Without loss of generality, we can assume that $\nu_C = 0$ since $\eta = \eta_D \vee \eta_C$ and $\eta_C$ has zero extremal index. Then,

$$- \log \mathbb{P} \left( \sup_{0 \leq s \leq T} \eta_s \leq T^{1/\alpha} x \right) = \nu \left( \{ y \in \mathcal{D} : y_{0,T}^* > T^{1/\alpha} x \} \right)$$

$$= \vartheta x^{-\alpha} T^{-1} \int_{-\infty}^{\infty} \mathbb{E} \left[ \sup_{-s \leq u \leq T-s} Q_u^\alpha \right] ds = x^{-\alpha} \tilde{\nu}_T^*(H),$$

with $\tilde{\nu}_T^*$ defined in (3.18) and $H(y) = 1 \{ y^* > 1 \}$. Since we have obtained in the proof of Lemma 3.7 that $\tilde{\nu}_T^* \overset{\nu}{\rightarrow} \nu^*$ and $\nu^*(H) = \vartheta$, we obtain

$$\lim_{T \to \infty} - \log \mathbb{P} \left( \sup_{0 \leq s \leq T} \eta_s \leq T^{1/\alpha} x \right) = \vartheta x^{-\alpha}.$$ 

This proves that $\vartheta$ is the true extremal index of the process $\eta$. \qed

Example 4.3. Let $W$ be a Gaussian process with continuous paths, stationary increments and such that $\mathbb{P}(W_0 = 0) = 1$. Let $\sigma_t^2 = \text{var}(W_t)$. For $\alpha > 0$ define the process $Z$ by $Z_t = e^{W_t - \alpha \sigma_t^2/2}$, $t \in \mathbb{R}$. Let $\nu$ be the tail measure on $\mathcal{D}$ defined by $\nu = \int_{0}^{\infty} \mathbb{E}[\delta_u Z] \alpha u^{-\alpha-1} du$ and $\eta$ the associated max-stable process. Then $\eta$ is stationary and has almost surely continuous paths. Cf. [KSdH09]. Assume that

$$\mathbb{P} \left( \lim_{|t| \to \infty} W_t - \alpha \sigma_t^2/2 = -\infty \right) = 1.$$ 

Then the extremal index $\theta$ of $\eta$ is positive and given by

$$\theta = \vartheta = \mathbb{E} \left[ \sup_{t \in \mathbb{R}} e^{\alpha W_t - \alpha^2 \sigma_t^2/2} \right] = \mathbb{E} \left[ \frac{1}{\int_{-\infty}^{\infty} 1 \{ \alpha W_t - \alpha^2 \sigma_t^2/2 > -E \} dt} \right],$$

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with $E$ a random variable with a standard exponential distribution, independent of $W$. The first expression for $\theta$ was obtained by [DH19].

Since $W$ has continuous paths, we can also apply the results of Section 2.3 and Example 2.11. Let $\hat{W}$ be the process defined by $\hat{W}_t = W_t - \alpha \sigma^2 t / 2$ and $I_0$ be the infargmax functional. By homogeneity of $I_0$ and the monotonicity of the exponential, $I_0(\hat{W})$ admits a continuous density $q_0$ with respect to Lebesgue's measure on $\mathbb{R}$ given by

$$q_0(t) = \mathbb{E}\left[ \frac{e^{\alpha \hat{W}_{I_0(\hat{W}) - t}}}{\int_{-\infty}^{\infty} e^{\alpha \hat{W}_s} ds} \right], \quad t \in \mathbb{R}.$$ 

In this case, condition (2.23) holds by continuity and the representations (2.24) and (2.25) hold.

Consider now the first exceedence map $I_1$. Then the continuity of the sample paths imply that $\mathbb{P}(Y|Q_{I_1(YQ)}>1) = 0$, thus (2.23) does not hold.

### 4.2 Sum-stable processes

Let $\zeta$ be a separable $\alpha$-stable process $(0 < \alpha < 2)$. This means that its finite dimensional distributions are $\alpha$-stable, hence regularly varying with tail index $\alpha$ and characterized by their exponent measures. Consequently, the distribution of $\zeta$ is characterized by its tail measure on $\mathbb{R}$ endowed with the product topology.

If we assume further that $\zeta$ is regularly varying on $\mathcal{D}$ then its tail measure is a tail measure on $\mathcal{D}(\mathbb{R}, \mathbb{R})$ in the sense of Definition 2.1. By Theorem 2.3, it admits a spectral process $Z$ which satisfies $\mathbb{E}[\sup_{a \leq s \leq b} |Z(s)|^\alpha] < \infty$ for all $a \leq b$.

Let $\sum_{i=1}^\infty \delta_{P_iZ(i)}$ is a PPP on $\mathcal{D}$ with mean measure $\nu$. If $\alpha < 1$, the series $\sum_{i=1}^\infty P_i$ is summable, thus the series

$$\sum_{i=1}^\infty P_iZ(i), \quad (4.4)$$

is almost surely locally uniformly convergent. If $\alpha \in [1, 2)$, we will assume furthermore that $\nu$, or equivalently $Z$, is symmetric (i.e. $\nu(A) = \nu(-A)$ for all Borel measurable subset $A$ of $\mathcal{D}$). This allows to avoid the issue of centering and to use the maximal inequality recalled in Appendix C. Then the series in (4.4) is pointwise almost surely convergent, and defines a symmetric $\alpha$-stable process. In both cases, the series in (4.4) defines an $\alpha$-stable process with the same finite-dimensional distributions as $\zeta$. Cf. [ST94, Theorem 1.4.2].

**Theorem 4.4.** Let $\alpha \in (0, 2)$, $\nu$ be a tail measure on $\mathcal{D}$ with tail index $\alpha$, $Y$ be the associated tail process and $Z$ be a spectral process for $\nu$. If $\alpha \in [1, 2)$, assume furthermore that $\nu$ (or equivalently $Y$ or $Z$) is symmetric. Let $\zeta$ be a $\mathcal{D}$-valued process. The following statements are equivalent:
(i) \( \zeta \) is an \( \alpha \)-stable process, regularly varying in \( D(\mathbb{R}) \), with tail measure \( \nu \);

(ii) \( \zeta \) admits the representation (4.4) with \( Z^{(i)} \), \( i \geq 1 \), i.i.d. copies of \( Z \).

**Proof.** We have already proved the implication \((i) \implies (ii)\). We prove the converse. We must prove that for all \( \epsilon > 0 \), \( a > 0 \) and bounded Lipschitz continuous maps \( H \) on \( D \) such that \( H(y) = 0 \) if \( y^*_{a,a} \leq \epsilon \),

\[
\lim_{x \to \infty} \frac{\mathbb{E}[H(x^{-1} \zeta)]}{\mathbb{P}(|\zeta| > x)} = \nu(H). \tag{4.5}
\]

As in the proof of Theorem 4.1, let \( W \) be a process whose distribution is given by

\[
\mathbb{E}[H(W)] = \frac{\mathbb{E}[H((Z_{a,b}^*)^{-1} Z)(Z_{a,b}^*)^\alpha]}{\mathbb{E}[(Z_{a,b}^*)^\alpha]}.
\]

Let \( W^{(i)} \) be i.i.d. copies of \( W \). Then \( \zeta \) has the same distribution as

\[
c^1_\alpha \sum_{i=1}^\infty P_i W^{(i)},
\]

with \( c_\alpha = \mathbb{E}[\sup_{a \leq s \leq b} |Z(s)|^\alpha] \). Since \( P_1 \) has a Fréchet distribution and \( \sup_{-a \leq t \leq a} |W_t| = 1 \), we have

\[
\lim_{x \to \infty} x^{\alpha} \mathbb{E}[H(c_\alpha^{1/\alpha} x^{-1} P_1 W)] = c_\alpha \int_0^\infty \mathbb{E}[H(u W)] \alpha u^{-\alpha-1} du = \int_0^\infty \mathbb{E}[H(u Z)] \alpha u^{-\alpha-1} du = \nu(H).
\]

Write \( \zeta^{(1)} = c_\alpha^{1/\alpha} P_1 W^{(1)} \). Since \( H \) is Lipshitz-continuous with respect to \( d_{J_1} \) with support separated from \( 0 \), and \( W^{*}_{-a,a} = 1 \) almost surely, we have, for every \( \eta \in (0, \epsilon/2) \),

\[
\left| \mathbb{E}[H(x^{-1} \zeta)] - \mathbb{E}[H(x^{-1} \zeta^{(1)})] \right| \leq \text{cst } \eta \mathbb{P}(P_1 > a\epsilon/2) + \text{cst } \mathbb{P}(d_{J_1}(\zeta, \zeta^{(1)}) > x\eta).
\]

Since the \( J_1 \) metric is bounded by the uniform metric on any compact interval, we obtain, for any two functions \( f, g \in D \),

\[
d_{J_1}(f, g) \leq \sup_{-a \leq s \leq a} |f(s) - g(s)| + e^{-a}.
\]

Thus, applying the bound \((C.5)\) in the proof of Lemma \( C.1 \), we obtain

\[
\limsup_{x \to \infty} x^\alpha \mathbb{P}(d_{J_1}(\zeta, \zeta^{(1)}) > x\eta) \leq \limsup_{x \to \infty} x^\alpha \mathbb{P}\left( \sup_{a \leq s \leq b} \left| \sum_{j=2}^\infty P_j W^{(j)}(s) \right| + e^{-a} > x\eta \right) = 0.
\]

This proves that

\[
\limsup_{x \to \infty} x^\alpha \left| \mathbb{E}[H(x^{-1} \zeta)] - \mathbb{E}[H(x^{-1} \zeta^{(1)})] \right| \leq \text{cst } \eta.
\]

Since \( \eta \) is arbitrary, we have proved that \((ii) \implies (i)\). \( \square \)
If $\nu(D^c_0) = 0$, then $\vartheta > 0$ and $\zeta$ can be expressed as

$$\zeta_t = \sum_{i=1}^{\infty} P_i Q^{(i)}(t - T_i), \quad (4.6)$$

where $\sum_{i=1}^{\infty} \delta_{T_i, P_i Q^{(i)}}$ is a PPP on $\mathbb{R} \times (0, \infty) \times D_0$ with mean measure $\vartheta \text{Leb} \otimes \nu \otimes \mathbb{P} Q$ and the process $Q$ satisfies (2.13). Applying Theorem 3.14, we obtain the convergence of the point process of clusters $N_T$ defined in (3.25).

**Theorem 4.5.** Let $\zeta$ be $\mathcal{D}$ valued $\alpha$-stable process, regularly varying in $\mathcal{D}$ with tail measure $\nu$ such that $\nu(D^c_0) = 0$. Then $\zeta$ admits the representation (4.6) with $Q$ satisfying (2.13) and the point process of clusters $N_T$ converges to a Poisson point process with mean measure $\text{Leb} \otimes \nu^*$ for all sequences $r_T$ such that $r_T/T \to 0$.

**Proof.** Define

$$Q^{(m)}_t = Q_t \mathbb{1}\{|t| \leq m\}.$$

Let $Q^{(m,i)}$, $i \geq 1$, be i.i.d. copies of $Q^{(m)}$ and define $X^{(m)}$ by

$$X^{(m)}_t = \sum_{i=1}^{\infty} P_i Q^{(m,i)}_{t - T_i}.$$

Then $(X, X^{(m)})$ is stationary, $X^{(m)}$ is $m$-dependent and regularly varying in $\mathcal{D}$. We prove that the condition (3.19) of Theorem 3.14 holds. Applying (C.2) of Lemma C.1 (with $Z$ expressed in terms of $Q$ as in Corollary 2.10), we obtain

$$\lim_{m \to \infty} \limsup_{T \to \infty} T \mathbb{P} \left( \sup_{0 \leq s \leq 1} |X_s - X^{(m)}_s| > a_T \epsilon \right) \leq \text{cst} \lim_{m \to \infty} \limsup_{T \to \infty} T^{-1} \int_{-\infty}^{\infty} \mathbb{E} \left[ \sup_{0 \leq s \leq T} |Q(t + s)|^\alpha \mathbb{1}\{|t + s| > m\} \right] dt \leq \text{cst} \lim_{m \to \infty} \int_{-\infty}^{\infty} \mathbb{E} \left[ \sup_{0 \leq s \leq 1} |Q(t + s)|^\alpha \mathbb{1}\{|t + s| > m\} \right] dt = 0.$$

Thus (3.19) holds.

Thus we recover [Sam04, Theorem 2.2] which obtained the extremal index of a general stable process and [Roo78, Corollary 5.3] which obtained the extremal index of moving average with respect to a Lévy stable process.

**Corollary 4.6.** Let $\zeta$ be $\mathcal{D}$ valued $\alpha$-stable process, regularly varying in $\mathcal{D}$ with tail measure $\nu$. Then $\zeta$ admits an extremal index $\theta = \vartheta$, i.e.

$$\lim_{T \to \infty} T \mathbb{P} \left( \max_{0 \leq s \leq T} |\zeta_s| \leq a_T x \right) = \lim_{T \to \infty} T \mathbb{P} \left( \max_{0 \leq s \leq T} \eta_s \leq a_T x \right) = e^{-\vartheta x^{-\alpha}}.$$
Proof. Let $Z$ be a spectral process for $\nu$. By the bound (C.1) in Lemma C.1,
\[ P \left( \sup_{0 \leq s \leq T} |\zeta(s)| > T^{1/\alpha} x \right) \leq \text{cst} \ E^{-1} \left[ \sup_{0 \leq s \leq T} |Z(s)|^\alpha \right]. \]
Lemma 2.5 implies that $\lim_{T \to \infty} T^{-1} E \left[ \sup_{0 \leq s \leq T} |Z(s)|^\alpha \right] = \vartheta$. Thus if $\vartheta = 0$, we obtain that
\[ \lim_{T \to \infty} P \left( \sup_{0 \leq s \leq T} |\zeta(s)| > T^{1/\alpha} x \right) = 0, \]
which means that $\vartheta = 0$ and also that
\[ T^{-1/\alpha} \sup_{0 \leq s \leq T} |\zeta(s)| \xrightarrow{P} 0, \text{ as } T \to \infty. \]
Assume now that $\vartheta > 0$, which is equivalent to $\nu(D_0) > 0$. If $\nu(D_0) = 0$, then we can apply Theorem 4.5 to conclude. Otherwise, define $p = E[(Z(0))^\alpha 1\{Z \in D_0\}]$, $Z_0 = p^{-1/\alpha} Z_1 \{Z \in D_0\}$ and $Z_1 = (1-p)^{-1/\alpha} Z_1 \{Z \notin D_0\}$. Let $Z_0^{(i)}$, $Z_1^{(i)}$, $i \geq 1$ be i.i.d. copies of $Z_0$ and $Z_1$. Define $\zeta_0$ and $\zeta_1$ by (4.4) with $Z_0$ and $Z_1$ respectively. By the previous part of the proof we have $T^{-1/\alpha} \sup_{0 \leq s \leq T} |\zeta_1(s)| \xrightarrow{P} 0$ and by Theorem 4.5, we have
\[ P \left( \sup_{0 \leq s \leq T} |\zeta(s)| \leq T^{1/\alpha} x \right) \sim P \left( \sup_{0 \leq s \leq T} |\zeta_0(s)| > T^{1/\alpha} x \right) \to e^{-p^{-1}\vartheta_0 x^{-\alpha}}, \]
with $\vartheta_0 = \lim_{T \to \infty} E[\sup_{0 \leq s \leq T} |Z_0(s)|^\alpha] = p \lim_{T \to \infty} E[\sup_{0 \leq s \leq T} |Z(s)|^\alpha] = p \vartheta. \]

Integral representations

Stable processes can also be defined by their integral representations. For completeness, we briefly rewrite (without proof) our results using these representations. Let $M$ be a $\alpha$-stable random measure with independent increments and control measure $m$ on a measurable space $(E, \mathcal{E})$. For simplicity, following [Sam04], we assume that $M$ (hence $m$) is symmetric. This means that for every measurable function $f$ such that $\int_E |h(x)|^\alpha m(dx) < \infty$,
\[ \log E[e^{izM(h)}] = -C_{\alpha} |z|^{\alpha} \int_E |h(x)|^\alpha m(dx). \]
with $C_{\alpha} = \int_0^\infty \sin(x)x^{-\alpha}dx$. Let $f : E \times \mathbb{R} \to \mathbb{R}$ be a measurable function, such that $t \mapsto f(x, t)$ is càdlàg for all $x \in E$ and for all $a \leq b$,
\[ \int_{E_a \leq s \leq b} |f(x, s)|^\alpha m(dx) < \infty. \]
By [ST94, Theorem 10.2.3], this is a necessary condition for local boundedness of the stable process $\zeta$ defined by
\[ \zeta_t = \int_E f(x, t)M(dx), \ t \in \mathbb{R}. \]
The tail measure $\nu$ is given by

$$
\nu = \frac{1}{\int_E |f(x,0)|^\alpha m(dx)} \int_E \int_0^\infty \delta_{u f(x,\cdot)}|u|^{-\alpha-1}du \, m(dx) . \tag{4.7}
$$

See [SO12, Section 3]. If $\nu(D^\alpha_0) = 0$, then $X$ admits a mixed moving average representation. This means that there exist a measured space $(F, \mathcal{F}, \mu)$ and a measurable function $g : F \times \mathbb{R} \to \mathbb{R}$ such that

$$
\int_F \int_{-\infty}^\infty \sup_{a \leq s \leq b} |g(w, t + s)|^\alpha \mu(dw)dt < \infty , \tag{4.8}
$$

for all $a \leq b$. The latter condition implies that $\lim_{|t| \to \infty} g(w, t) = 0$ for $\mu$-almost all $w$. Then $\zeta$ can be defined as

$$
\zeta_t = \int_F \int_{-\infty}^\infty g(w, t - s) \Lambda(dwds) .
$$

where $\Lambda$ is an $\alpha$-stable random measure with independent increments and control measure $\mu \otimes \text{Leb}$ on $F \times \mathbb{R}$.

Define $g^* = \sup_{t \in \mathbb{R}} |g(\cdot, t)|$. Then, the extremal index is given by

$$
\vartheta = \frac{\int_E (g^*(x))^\alpha \mu(dx)}{\int_{-\infty}^\infty |g(x, t)|^\alpha \mu(dx) dt} .
$$

See [Sam04, Theorem 22(i)]. A mixed-moving average representation of the tail measure is given by

$$
\nu = \vartheta \int_{-\infty}^\infty \int_0^\infty \mathbb{E}\left[ \delta_{uh(W, -t)} \right] \alpha u^{-\alpha-1}du \, dt ,
$$

with $W$ a random variable whose distribution admits a density proportional to $(g^*)^\alpha$ with respect to Lebesgue’s measure and $h(w, t) = (g^*)^{-1}(w)g(w, t)$.

### 4.3 Functional weighted sums

Let $\{f_k \in \mathbb{Z}\}$ be a sequence of random element in $D(\mathbb{R}, \mathbb{R})$ and let $\{Z_k, k \in \mathbb{Z}\}$ be a sequence of i.i.d. random variables, regularly varying with index $\alpha$ and extremal skewness $p_Z$, independent of the previous sequence. We (formally) define the process $X$ by

$$
X_t = \sum_{k \in \mathbb{Z}} f_k(t)Z_k , \quad t \in \mathbb{R} . \tag{4.9}
$$
If this process is well defined, it is stationary if \( \{B^k f_k, k \in \mathbb{Z}\} \overset{d}{=} \{f_k, k \in \mathbb{Z}\} \) for all \( t \in \mathbb{R} \). If \( \alpha \leq 1 \) or \( \alpha > 1 \) and \( \mathbb{E}[Z_0] = 0 \) and if there exists \( \beta \in (0, \min(\alpha, 1)) \) such that for every \( t \in \mathbb{R} \),

\[
\mathbb{P} \left( \sum_{k \in \mathbb{Z}} |f_k(t)|^\beta < \infty \right) = 1 ,
\]

then the series \( \sum_{k \in \mathbb{Z}} f_k(t)Z_k \) is almost surely convergent; see [HS08, Section 3] and [MS00, Lemma A.3]. If furthermore

\[
\sum_{k \in \mathbb{Z}} \mathbb{E}[|f_k(t)|^\beta] < \infty ,
\]

then \( \mathbf{X} \) is finite dimensional regularly varying. Before stating and proving rigorously our results, we give some heuristics. The main argument to obtain the extremal behavior of the process \( \mathbf{X} \) is the so-called “single large jump principle”. The series \( X_t \) is large if and only if there is a single jump \( Z_k \) which is extremely large and it is chosen “at random” among the sequence \( \{Z_j, j \in \mathbb{Z}\} \) according to the distribution of a random variable \( N \) such that

\[
\mathbb{P}(N = k) = \frac{\mathbb{E}[|f_k(0)|^\alpha]}{\sum_{j \in \mathbb{Z}} \mathbb{E}[|f_j(0)|^\alpha]} , \quad k \in \mathbb{Z} ,
\]

and the law of \( \{f_j, j \in \mathbb{Z}\} \) given \( N = k \) is given (for suitable maps \( H \)) by

\[
\mathbb{E}[H(f_j, j \in \mathbb{Z}) | N = k] = \frac{\mathbb{E}[H(f_j, j \in \mathbb{Z})|f_k(0)|^\alpha]}{\mathbb{E}[|f_k(0)|^\alpha]} .
\]

There is no issue with division by zero since \( \mathbb{P}(f_N(0) = 0) = 0 \) by definition of \( N \). For finite dimensional distribution, the tail process \( \mathbf{Y} \) of a series of the form (4.9) was obtained in [MS10, Section 8]:

\[
Y_t = \frac{f_N(t)}{|f_N(0)|} Y \epsilon_0 , \quad t \in \mathbb{R} ,
\]

with \( Y \) and \( \epsilon_0 \) independent of \( N \) and \( \{f_k, k \in \mathbb{Z}\} \), \( Y \) with a Pareto distribution with tail index \( \alpha > 0 \) and \( \mathbb{P}(\epsilon_0 = 1) = 1 - \mathbb{P}(\epsilon_0 = -1) = p_Z \).

In the functional framework, we expect that applying the single big jump heuristics yields, for a continuous map \( H \) on \( \mathcal{D} \) endowed with the \( J_1 \) topology with support separated from \( \mathbf{0} \),

\[
\frac{\mathbb{E}[H(x^{-1}X)]}{\mathbb{P}(|Z_0| > x)} \sim \sum_{k \in \mathbb{Z}} \frac{\mathbb{E}[H(x^{-1}f_k Z_k)]}{\mathbb{P}(|Z_0| > x)} \rightarrow \sum_{k \in \mathbb{Z}} \int_0^\infty \mathbb{E}[H(u \epsilon_0 f_k)] \alpha u^{-\alpha} \, du .
\]
Since the tail measure is normalized by the condition $\nu(\{ y \in D : |y_0| > 1 \}) = 1$, we obtain

$$
\nu(H) = \frac{\sum_{k \in \mathbb{Z}} \int_{0}^{\infty} \mathbb{E}[H(u \epsilon_0 f_k)] \alpha u^{-\alpha-1} du}{\sum_{k \in \mathbb{Z}} \mathbb{E}[|f_k(0)|^{\alpha}]} 
= \frac{\sum_{k \in \mathbb{Z}} \int_{0}^{\infty} \mathbb{E}[|f_k(0)|^{\alpha} H(u |f_k(0)|^{-1} \epsilon_0 f_k)] \alpha u^{-\alpha-1} du}{\sum_{k \in \mathbb{Z}} \mathbb{E}[|f_k(0)|^{\alpha}]} 
= \int_{0}^{\infty} \mathbb{E}[H(u |f_N(0)|^{-1} \epsilon_0 f_N)] \alpha u^{-\alpha-1} du.
$$

Thus the tail measure is given by

$$
\nu = \int_{0}^{\infty} \mathbb{E}[\delta_{u |f_N(0)|^{-1} \epsilon_0 f_N}] \alpha u^{-\alpha-1} du. \quad (4.13)
$$

Replacing the function $H$ by $H \mathbb{1}\{|y_0| > 1\}$, we recover (4.11). In order to prove rigorously (4.13) we will need a truncation argument. We now state and prove a result with ad-hoc assumptions.

**Proposition 4.7.** Let $\{Z_k, k \in \mathbb{Z}\}$ be a sequence of i.i.d. random variables, regularly varying with index $\alpha > 0$ and extremal skewness $p_Z$ and such that $\mathbb{E}[Z_0] = 0$ if $\alpha > 1$. Assume that $\{f_k, k \in \mathbb{Z}\}$ is a sequence of random functions, independent of $\{Z_k, k \in \mathbb{Z}\}$, such that $\mathbb{P}(f_k \in \mathcal{D}) = 1$, $f_k$ is stochastically continuous for all $k \in \mathbb{Z}$ and there exists $\beta \in (0, \min(\alpha, 1))$ such that (4.10) holds. Assume that the process $X$ defined in (4.9) is stochastically continuous and that for all $a < b$ and $x > 0$,

$$
\sum_{k \in \mathbb{Z}} \mathbb{E} \left[ \sup_{a \leq s \leq b} |f_k(s)|^{\alpha} \right] < \infty, \quad (4.14)
$$

$$
\lim_{m \to \infty} \limsup_{T \to \infty} \left( \sup_{a \leq s \leq b} \left| \sum_{|k| > m} f_k(s) Z_k \right| > a_T x \right) = 0, \quad (4.15)
$$

with $a_T$ such that $\lim_{T \to \infty} T \mathbb{P}(|X_0| > a_T) = 1$. Then $\mathbb{P}(X \in \mathcal{D}) = 1$, $X$ is regularly varying in $\mathcal{D}$ and (4.13) holds.

Assume moreover $\mathbb{P}(f_k \in \mathcal{D}_0) = 1$ for all $k \in \mathbb{Z}$ and let $N_T$ be the point process of clusters of $X$ as defined in (3.25). Then $N_T \xrightarrow{w} N$, where $N$ is a Poisson point process on $\mathbb{R} \times \mathcal{D}_0$ with mean measure $\text{Leb} \otimes \nu^*$ and $\nu^*$ defined by

$$
\nu^* = \int_{0}^{\infty} \mathbb{E} \left[ \delta_{u |f_N|^{-1} \epsilon_0 f_N} \right] \alpha u^{-\alpha-1} du.
$$

The extremal index $\vartheta$ of $X$ exists and is given by

$$
\vartheta = \mathbb{E} \left[ \sup_{t \in \mathbb{R}} \left| f_N(t) \right|^{\alpha} \right] \left( \int_{-\infty}^{\infty} \left| f_N(t) \right|^{\alpha} dt \right)^{-1}.
$$
Proof. For a positive integer $m$, define 
\[ X_t^{(m)} = \sum_{|k| \leq m} f_k(t)Z_k \quad \hat{X}_t^{(m)} = \sum_{|k| > m} f_k(t)Z_k, \quad t \in \mathbb{R}. \] (4.16)

Then $X = X^{(m)} + \hat{X}^{(m)}$. The process $X^{(m)}$ is almost surely càdlàg as a finite sum of almost surely càdlàg functions and stochastically continuous. Condition (4.15) implies that $X^{(m)}$ converges in probability (hence almost surely along a subsequence) locally uniformly to $X$, thus $X$ is also almost surely càdlàg. Define $c_m$ as
\[ c_m = \lim_{T \to \infty} T \mathbb{P}(|X_0^{(m)}| > a_T) = \frac{\sum_{|k| \leq m} \mathbb{E}[|f_k(0)|^\alpha]}{\sum_{k \in \mathbb{Z}} \mathbb{E}[|f_k(0)|^\alpha]} \cdot \]

We will prove that for all $a < b \in \mathbb{R}$, $\epsilon > 0$ and Lipschitz continuous maps $H$ with respect to $d_{J_1}$ (defined in (B.1)) such that $H(y) = 0$ if $y^*_a \leq \epsilon$, it holds that
\[ \lim_{T \to \infty} T \mathbb{E}[H(a_T^{-1} X^{(m)})] = c_m \sum_{|k| \leq m} \int_0^\infty \mathbb{E}[H(u\epsilon_0 f_k)]\alpha u^{-\alpha-1}du, \] (4.17)
\[ \lim_{m \to \infty} \limsup_{T \to \infty} T \mathbb{E}[|H(a_T^{-1} X) - H(a_T^{-1} X^{(m)})|] = 0. \] (4.18)

In view of (4.14), the series $\sum_{k \in \mathbb{Z}} \int_0^\infty \mathbb{E}[H(u\epsilon_0 f_k)]\alpha u^{-\alpha-1}du$ is summable and
\[ \lim_{m \to \infty} \sum_{|k| \leq m} \int_0^\infty \mathbb{E}[H(u\epsilon_0 f_k)]\alpha u^{-\alpha-1}du = \sum_{k \in \mathbb{Z}} \int_0^\infty \mathbb{E}[H(u\epsilon_0 f_k)]\alpha u^{-\alpha-1}du. \]

Thus (4.17) and (4.18) imply the convergence (4.12). Because of the Lipschitz property of $H$, we have, for every $\eta \in (0, \epsilon/2)$,
\[ \mathbb{E}[|H(a_T^{-1} X) - H(a_T^{-1} X^{(m)})|] \leq \text{cst} \eta \mathbb{P}(X^*_a > a_T \epsilon/2) + \text{cst} \mathbb{P}(d_{J_1}(X, X^{(m)}) > a_T \eta). \]

By definition of the metric $d_{J_1}$ and since the $J_1$ metric on an interval is bounded by the uniform metric, we have, for all functions $f, g \in \mathcal{D}$ and $t > 0$,
\[ d_{J_1}(f, g) \leq \sup_{-t \leq s \leq t} |f(s) - g(s)| + e^{-t}. \]

Therefore,
\[ \limsup_{T \to \infty} T \mathbb{E}[|H(a_T^{-1} X) - H(a_T^{-1} X^{(m)})|] \]
\[ \leq \text{cst} \eta + \text{cst} \limsup_{T \to \infty} T \mathbb{P}\left( \sup_{-t \leq s \leq t} |X_s - X_s^{(m)}| + e^{-t} > a_T \eta \right). \]

Thus (4.15) implies that (4.18) holds and there only remains to prove (4.17).
• The regular variation of \( Z_0 \) implies that for all \( \eta > 0 \) and bounded continuous functions \( g \) on \( \mathbb{R} \) such that \( g(x) = 0 \) if \( |x| \leq \eta \),

\[
\lim_{x \to \infty} \frac{\mathbb{E}[g(x^{-1}Z_0)]}{\mathbb{P}(|Z_0| > x)} = \int_0^\infty \mathbb{E}[g(u\epsilon_0)]\alpha u^{-\alpha-1}du ,
\]

with \( \epsilon_0 \) as above.

• For every \( f \in \mathcal{D} \), the map \( u \mapsto uf \) is continuous on \( \mathbb{R} \). Thus, for every map \( H \) on \( \mathcal{D} \), continuous with respect to the \( J_1 \) topology, the map \( g : u \mapsto H(uf) \) is continuous on \( \mathbb{R} \).

• If \( H \) is moreover bounded and there exist \( a < b \) and \( \eta > 0 \) such that \( H(y) = 0 \) if \( y_{a,b} \leq \eta \), then for every function \( f \in \mathcal{D} \) (which is necessarily locally bounded), the map \( u \mapsto H(uf) \) is bounded, continuous with support separated from zero: if \( |u| \leq \epsilon (f_{a,b})^{-1} \), then \( H(uf) = 0 \). Consequently,

\[
\lim_{x \to \infty} \frac{\mathbb{E}[H(x^{-1}fZ)]}{\mathbb{P}(|Z_0| > x)} = \int_0^\infty \mathbb{E}[H(u\epsilon_0) | f]\alpha u^{-\alpha-1}du ,
\]

• If \( f \) is a \( \mathcal{D} \)-valued random map, independent of \( Z \), then, conditionally on \( f \), we have almost surely,

\[
\lim_{x \to \infty} \frac{\mathbb{E}[H(x^{-1}fZ) | f]}{\mathbb{P}(|Z_0| > x)} = \int_0^\infty \mathbb{E}[H(u\epsilon_0) | f]\alpha u^{-\alpha-1}du .
\]

Since \( H(x^{-1}fZ) = 0 \) if \( f_{a,b}^*Z \leq x\epsilon \), By Potter’s bound (cf. [KS20, Proposition 1.4.2]), we have, for \( x \geq 1 \),

\[
\frac{\mathbb{E}[H(x^{-1}fZ) | f]}{\mathbb{P}(|Z_0| > x)} \leq \frac{\mathbb{P}(f_{a,b}^*|Z_0| > x\epsilon)}{\mathbb{P}(|Z_0| > x)} \leq \text{cst}(f_{a,b}^* \lor 1)^{\alpha+\epsilon} .
\]

Thus, by the dominated convergence theorem, if \( \mathbb{E}[(f_{a,b}^*)^{\alpha+\epsilon}] < \infty \), we obtain

\[
\lim_{x \to \infty} \frac{\mathbb{E}[H(x^{-1}fZ)]}{\mathbb{P}(|Z_0| > x)} = \int_0^\infty \mathbb{E}[H(u\epsilon_0)]\alpha u^{-\alpha-1}du .
\]

• Consider now i.i.d. random variables \( Z_1, \ldots, Z_k \). If \( g : \mathbb{R}^k \to \mathbb{R} \) is continuous and bounded with support separated from zero, regular variation yields (cf. [KS20, Proposition 2.1.1])

\[
\lim_{x \to \infty} \frac{\mathbb{E}[g(x^{-1}(Z_1, \ldots, Z_k))]}{\mathbb{P}(|Z_0| > x)} = \sum_{i=1}^k \int_0^\infty \mathbb{E}[g_i(u\epsilon_0)]\alpha u^{-\alpha-1}du ,
\]

with \( g_i(u) = g(0, \ldots, u, \ldots, 0) \) with the only nonzero component in the \( i \)-th position. Each function \( g_i \) is bounded, continuous with support separated from zero so each integral in (4.19) is well defined and finite.
• Since the functions \( f_i \) have no common discontinuities, the map \( (u_1, \ldots, u_k) \mapsto \sum_{i=1}^k u_i f_i \) is continuous with respect to the \( J_1 \) topology. Thus, defining \( X = \sum_{i=1}^k f_i Z_i \), we have, for a bounded continuous (with respect to the \( J_1 \) topology) map \( H \) and \( a < b, \epsilon > 0 \) such that \( H(y) = 0 \) if \( y^*_{a,b} \leq \epsilon \), applying (4.19) with \( g(u_1, \ldots, u_k) = H(u_1 f_1 + \cdots + u_k f_k) \), we obtain by the same arguments as in the case \( k = 1 \),

\[
\lim_{x \to \infty} \frac{\mathbb{E}[H(x^{-1}(f_1 Z_1 + \cdots + f_k Z_k))]}{\mathbb{P}(|Z_0| > x)} = \sum_{i=1}^k \int_0^\infty \mathbb{E}[H(s \epsilon_0)] \alpha s^{-\alpha-1} ds .
\]

Thus (4.17) holds and this proves that \( X \) is regularly varying in \( \mathcal{D} \) with tail measure given by (4.13).

The convergence of the point process of clusters and the expression of the extremal index follow from Theorem 3.14.

Obtaining the bound (4.15) may be a hard task. We will pursue the investigation on two examples.

4.3.1 Functional moving average

We now consider the case

\[
f_k(t) = f(t - T_k), \quad k \in \mathbb{Z}, \quad t \in \mathbb{R}
\]

where \( \{T_k, k \in \mathbb{Z}\} \) are the points of a unit rate homogeneous Poisson point process on \( \mathbb{R} \) and \( f \in \mathcal{D}_0 \) is a deterministic function such that

\[
\int_{-\infty}^\infty |f(t)|^\beta dt < \infty .
\]

with \( \beta \in (0, \min(\alpha, 1)) \). Since \( f \) is bounded, this implies that \( \int_{-\infty}^\infty |f(t)|^q dt < \infty \) for all \( q \geq \beta \). The tail process \( Y \) is given by

\[
Y_t = \frac{f(t - T)}{|f(T)|} Y_{\epsilon_0}
\]

with \( T \) a random variable with density \( \|f\|^{-\alpha}_\alpha |\tilde{f}|^{\alpha} \) with respect to Lebesgue’s measure on \( \mathbb{R} \), with \( \tilde{f}(t) = f(-t) \). The condition (4.15) becomes

\[
\lim_{m \to \infty} \limsup_{T \to \infty} T \mathbb{P} \left( \sup_{a \leq s \leq b} \left| \sum_{|k| > m} f(s - T_k) Z_k \right| > a_T x \right) = 0 , \quad (4.20)
\]

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for all \( x > 0 \). Instead of (4.20), a different truncation may be used. If for all \( x > 0 \),
\[
\lim_{m \to \infty} \limsup_{T \to \infty} T \mathbb{P} \left( \sup_{a \leq s \leq b} \left| \sum_{k \in \mathbb{Z}} f(s - T_k) \mathbb{1} \{ |s - T_k| > m \} Z_k \right| \right) = 0 ,
\]
then the process \( X \) can be approximated by the sequences of processes \( \tilde{X}^{(m)} \) defined by
\[
\tilde{X}^{(m)}_t = \sum_{|k| \in \mathbb{Z}} f(s - T_k) \mathbb{1} \{ |s - T_k| \leq m \} Z_k .
\]
The process \( \tilde{X}^{(m)} \) is \( m \)-dependent by the independent increment property of the Poisson process. If either (4.20) or (4.21) holds, then \( X \) is regularly varying in \( \mathcal{D} \) and its tail measure is given by
\[
\nu(H) = \| f \|_\alpha^{-\alpha} \int_{-\infty}^{\infty} \int_{0}^{\infty} \mathbb{E} [ H(\epsilon_0 u B^{-\alpha} f) ] du \alpha u^{-\alpha-1} du .
\]
By Corollary 3.11, the extremal index is given by
\[
\vartheta = \mathbb{E} \left[ \frac{\sup_{t \in \mathbb{R}} |f(t-T)|^\alpha}{\int_{-\infty}^{\infty} f(t-T)^\alpha dt} \right] = \sup_{t \in \mathbb{R}} \frac{|f(t)|^\alpha}{\int_{-\infty}^{\infty} |f(t)|^\alpha dt} .
\]
Proving (4.21) is easy in the case \( \alpha < 1 \). Indeed, by [HS08, Theorem 3.1], we obtain
\[
T \mathbb{P} \left( \sup_{a \leq s \leq b} \left| \sum_{k \in \mathbb{Z}} f(s - T_k) Z_k \right| > a_T x \right)
\leq T \mathbb{P} \left( \sum_{k \in \mathbb{Z}} \sup_{a \leq s \leq b} |f(s - T_k)||Z_k| > a_T x \right)
\to x^{-\alpha} \sum_{k \in \mathbb{Z}} \mathbb{E} \left[ \sup_{a \leq s \leq b} |f(s - T_k)|^\alpha \right] = x^{-\alpha} \int_{-\infty}^{\infty} \sup_{-a+t \leq s \leq b+t} |f(s)|^\alpha dt .
\]
Thus both methods of truncation are suitable. We leave the case \( \alpha \geq 1 \) for future work.

### 4.3.2 Shot noise process

Let \( \{ T_k, Z_k, \eta_k, k \in \mathbb{Z} \} \) are the points of a Poisson point process on \( \mathbb{R} \times \mathbb{R} \times (0, \infty) \) with mean measure \( \text{Leb} \otimes \mathbb{P}_Z \otimes \mathbb{P}_\eta \) where \( \mathbb{P}_Z \) and \( \mathbb{P}_\eta \) denote the distribution of \( Z_0 \) and \( \eta_0 \), respectively. Consider the model defined in (4.9) with
\[
f_k = \mathbb{1}_{[T_k, T_k + \eta_k)} , \quad k \in \mathbb{Z} ,
\]

that is

\[ X_t = \sum_{j \in \mathbb{Z}} Z_j 1\{T_j \leq t < T_j + \eta_j\} . \]

This process is also known as the infinite source Poisson process; see [RSS12]. The number of non-zero terms in the sum is almost surely finite with a Poisson distribution with mean \( \mathbb{E}[\eta_0] \). Thus the assumption \( \mathbb{E}[Z_0] = 0 \) is not needed in the case \( \alpha \geq 1 \), the sample paths are piecewise constant and càdlàg and the process is stationary. Furthermore, by standard results on random sums of regularly varying random variables (see e.g. [HS08, Corollary 3.2]), we have

\[ P(|X_0| > x) \sim \mathbb{E}[\eta_0] P(|Z_0| > x) . \] (4.22)

For \( m > 0 \), define

\[ X_t^{(m)} = \sum_{j \in \mathbb{Z}} Z_j 1\{T_j \leq t < T_j + \eta_j \land m\} . \]

Then

\[ X_t - X_t^{(m)} = \sum_{j \in \mathbb{Z}} Z_j 1\{T_j + \eta_j \land m \leq t < T_j + \eta_j\} . \]

Since \( \{(T_j, \eta_j), j \in \mathbb{Z}\} \) are the point of a marked Poisson point process with independent i.i.d. marks, the process \( \mathbf{X} - \mathbf{X}^{(m)} \) has the same distribution as the process \( \tilde{X}^{(m)} \) defined by

\[ \tilde{X}_t^{(m)} = \sum_{j \in \mathbb{Z}} Z_j 1\{T_j \leq t < T_j + (\eta_j - m)_+\} . \]

Thus, for \( a < b \),

\[ \lim_{T \to \infty} T P\left( \sup_{a \leq s \leq b} |X_s - X_t^{(m)}| > a_T x \right) = \lim_{T \to \infty} T P\left( \sup_{a \leq s \leq b} |\tilde{X}_s^{(m)}| > a_T x \right) . \]

Forgetting the truncation for the moment, we have

\[ TP\left( \sup_{a \leq s \leq b} |X_s| > a_T x \right) \leq TP\left( \sum_{k \in \mathbb{Z}} |Z_k| \sup_{a \leq s \leq b} \mathbb{1}\{T_k \leq s < T_k + \eta_k\} > a_T x \right) \]

\[ \leq TP\left( \sum_{k \in \mathbb{Z}} |Z_k| \mathbb{1}\{(T_k, \eta_k) \in A\} > a_T x \right) , \]

with \( A = \{(t, u) \in \mathbb{R} \times (0, \infty) : a - u < t \leq b\} \). We have

\[ \text{Leb} \otimes \mathbb{P}_\eta(A) = \mathbb{E} \left[ \int_{-\infty}^{b} 1\{t + \eta_0 > a\} dt \right] = b - a + \mathbb{E}[\eta_0] . \]

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Let \( N \) be a Poisson random variable, independent of \( \{Z_k\} \) with mean \( b - a + \mathbb{E}[\eta_0] \). Then, applying (4.22) and the same result for the tail of a Poisson sum of regularly varying summands, we have

\[
T \mathbb{P} \left( \sum_{k \in \mathbb{Z}} |Z_k| 1\{ (T_k, \eta_k) \in A \} > a_T x \right) = T \mathbb{P} \left( \sum_{k=1}^{N} |Z_k| > a_T x \right) 
\sim x^{-\alpha} \mathbb{E}[\eta_0] \mathbb{E}[N] = x^{-\alpha} \mathbb{E}[\eta_0](b - a + \eta_0)
\]

Returning to the truncated sum, we obtain

\[
\limsup_{T \to \infty} T \mathbb{P} \left( \sup_{0 \leq s \leq 1} |\bar{X}_s^{(m)}| > a_T x \right) \leq x^{-\alpha}(b - a + \mathbb{E}[(\eta_0 - m)_+] \mathbb{E}[(\eta_0 - m)_+]) .
\]

The term in the right-hand side tends to zero as \( m \) tends to \( \infty \). Thus we can apply the truncation argument and this proves that \( X \) is regularly varying in \( \mathcal{D} \) and its tail measure is given by

\[
\nu(H) = \mu^{-1} \sum_{k \in \mathbb{Z}} \int_0^\infty \mathbb{E}[\{T_k \leq 0 < T_k + \eta_0\} H(u \epsilon_0 1_{[\eta_0, T_k + \eta_0]})] \alpha u^{-\alpha - 1} du
\]

\[
= \mu^{-1} \int_0^\infty \int_0^\infty \mathbb{E}[\{t \leq 0 < t + \eta_0\} H(u \epsilon_0 1_{[t, t + \eta_0]})] \alpha u^{-\alpha - 1} du
\]

\[
= \mu^{-1} \int_0^\infty \mathbb{E} \left[ \int_0^{\eta_0} H(u \epsilon_0 1_{[-t, t + \eta_0 - t]}) dt \right] \alpha u^{-\alpha - 1} du
\]

\[
= \mu^{-1} \int_0^\infty \mathbb{E} \left[ H(u \epsilon_0 1_{[-\zeta', \zeta]} \right] \alpha u^{-\alpha - 1} du
\]

with \( \zeta, \zeta' \) such that

\[
\mathbb{P}(\zeta' > s, \zeta > t) = \mu^{-1} \mathbb{E}[(\eta_0 - s - t)_+].
\]

**Remark 4.8.** The law of \((-\zeta', \zeta)\) is the law of the points \((T_0, T_1)\) of a stationary renewal process with interarrival times distributed as \( \eta_0 \) under the Palm measure. Cf. [BB03, Section 1.4.1].

The tail process is given by

\[
Y = Y \epsilon_0 1_{[-\zeta', \zeta]}.
\]

By Corollary 3.11, the extremal index of \( X \) is given by

\[
\vartheta = \mathbb{E} \left[ \frac{1}{\int_{-\infty}^{\infty} 1\{Y_s > 1\} ds} \right] = \mathbb{E}[(\zeta + \zeta')^{-1}] = \frac{1}{\mu}.
\]

For this simple process, we can also confirm the findings of Section 2.3. Let \( \mathcal{I}_0 \) and \( \mathcal{I}_1 \) be the infargmax functional and the time of the first exceedence over 1 as in Examples 2.11 and 2.12. Here, \( \mathcal{I}_0(Y) = \mathcal{I}_1(Y) \) and in view of Remark 4.8, the law of \( Y \) given \( \mathcal{I}_0(Y) = \mathcal{I}_1(Y) = 0 \) is \( Y \epsilon_0 1_{[0, \eta_0]} \). Thus

\[
\mathbb{E}[\mathcal{E}(Y) | \mathcal{I}_1(Y) = 0] = \mathbb{E}[\eta] = \mu = \vartheta^{-1}.
\]

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Appendix A  Vague convergence

Let $E$ be a non-empty set. A boundedness on $E$ is a subset $B$ of $\mathcal{P}(E)$ with the following properties:

- a finite union of elements of $B$ is in $B$;
- a subset of an element of $B$ is in $B$.

The elements of $B$ are called bounded sets. In a metric space, the class of metrically bounded sets is a boundedness. Let now $E$ be a Polish space, endowed with its Borel $\sigma$-field. We will also need the following class of sets.

- A sequence $\{U_n, n \in \mathbb{N}\}$ of open sets if called a localizing sequence if for all $n \geq 0$, $U_n \in B$, $\overline{U}_n \subset U_{n+1}$, $\cup_{n \geq 0} U_n = E$ and every bounded set is included in one of the $U_n$.

Such a sequence $\{U_n\}$ is called a localizing sequence for $E$.

A Borel measure $\mu$ is said to be $B$-boundedly finite if $\mu(B) < \infty$ for all Borel sets $B \in B$. A sequence of $B$-boundedly finite measures $\{\mu_n, n \in \mathbb{N}\}$ is said to converge vaguely to a boundedly finite measure $\mu$, denoted $\mu_n \xrightarrow{v} \mu$, if $\lim_{n \to \infty} \mu_n(A) = \mu(A)$ for all bound Borel sets $B$ such that $\mu(\partial B) = 0$. A version of the Portmanteau theorem is available, [BP19, Theorem 2.7]. Let $\mathcal{M}_B$ be the set of boundedly finite Borel measures on $E$. The topology of vague convergence is the smallest topology on $\mathcal{M}_B$ which makes the maps $\mu \mapsto \mu(f)$ continuous for all continuous functions $f$ with bounded support. Endowed with this topology, the space $\mathcal{M}_B$ is Polish, [BP19, Theorem 3.1].

If there exists a localizing sequence, then vague convergence can be related to weak convergence. This is a consequence of [Kal17, Lemma 4.6].

**Proposition A.1.** Let $\{\nu_n, n \in \mathbb{N}\}$ be a sequence of boundedly finite Borel measures on $E$. Let $\{U_n, n \in \mathbb{N}\}$ be a localizing sequence and $\nu$ a boundedly finite measure such that $U_k$ is a continuity set of $\nu$ for all $k \geq 1$. Then $\nu_n \xrightarrow{v} \nu$ if and only if, for each $k \in \mathbb{N}$, the restrictions $\mu_n$ to $U_k$ converge weakly to the restriction of $\mu$ to $U_k$.

Another characterization of vague convergence is by means of Lipschitz functions. Let $E$ be a Polish space endowed with a boundedness $B$. We say that a metric $d$ on a Polish space is compatible with $B$ if $d$ induces the topology of $E$ and for every $B \in B$ there exists $\epsilon > 0$ such that the $\epsilon$-enlargement of $B$ with respect to $d$ is still bounded. (The $\epsilon$-enlargement with respect to $d$ of a subset $B$ is the set $\{x \in E : \exists y \ni B, d(x, y) \leq \epsilon\}$.) A real-valued function $f$ on $E$ is said to be $d$-Lipschitz if there exists a constant $K$ such that $|f(x) - f(y)| \leq Kd(x, y)$ for all $x, y \in E$. The following result is essentially [Kal17, Lemma 4.1]. See also [KS20, Theorem B.1.17].
**Proposition A.2.** Let $E$ be a Polish space endowed with a boundedness $B$ and $d$ be a compatible metric. Let $\{\nu_n, n \geq 1\}$ be $B$-boundedly finite Borel measures. Then $\nu_n \xrightarrow{v} \nu$ if and only if $\lim_{n \to \infty} \nu_n(f) = \nu(f)$ for all bounded $d$-Lipschitz functions $f$ with support in $B$.

As a consequence of metrizability and the characterization of vague convergence by Lipschitz functions, we obtain the following triangular argument.

**Lemma A.3.** Let $\{\nu_n, \nu_{m,n}, m \geq 1, n \geq 1\}$ be boundedly finite Borel measures on a Polish space $E$ endowed with a boundedness $B$. Assume that for each $m \geq 1$, $\nu_{n,m} \xrightarrow{v} \nu^{(m)}$ as $n \to \infty$, $\nu^{(m)} \xrightarrow{v} \nu$ as $m \to \infty$ and for every non-negative bounded measurable map $f$ with bounded support and Lipschitz with respect to an arbitrary compatible metric,

$$\lim_{m \to \infty} \limsup_{n \to \infty} |\nu_n(f) - \nu_{m,n}(f)| = 0.$$ 

Then $\nu_n \xrightarrow{v} \nu$ as $n \to \infty$.

A random measure is a random element of $M_B$ endowed with the topology of vague convergence. A sequence of random measures $N_n, n \in \mathbb{N}$ is said to converge weakly to a random measure $N$, denoted $N_n \xrightarrow{w} N$, if $N_n(f) \xrightarrow{d} N(f)$ for all bounded continuous functions $f$ with bounded support, and $\xrightarrow{d}$ denotes weak convergence of real valued random variables.

The following result, [BP19, Proposition 4.6], provides a useful characterization of vague convergence of weak convergence of random measures.

**Theorem A.4.** Let $E$ be a Polish space endowed with a boundedness $B$ and let $d$ be a compatible metric. Let $\{N_n, n \in \mathbb{N}\}$ be random measures in $M_B$. Then the following statements are equivalent:

(i) $N_n \xrightarrow{w} N$ as $n \to \infty$;

(ii) $N_n(f) \xrightarrow{d} N(f)$ as $n \to \infty$ for all bounded, non-negative $d$-Lipschitz continuous functions with bounded support;

(iii) $\lim_{n \to \infty} \mathbb{E}[e^{-N_n(f)}] = \mathbb{E}[e^{-N(f)}]$ for all bounded, non-negative $d$-Lipschitz continuous functions with bounded support.

Similarly to Lemma A.3, we obtain a triangular argument for weak convergence of random measures.

**Proposition A.5.** Let $\{N_n, N_{m,n}, n \geq 1, m \geq 1\}$ be random measures in $M_B$.

Assume that $N_{m,n} \xrightarrow{w} N^{(m)}$ as $n \to \infty$, $N^{(m)} \xrightarrow{w} N$ as $m \to \infty$ and for all $\eta > 0$,

$$\lim_{m \to \infty} \limsup_{n \to \infty} \mathbb{P}(|N_{n,m}(f) - N_n(f)| > \eta) = 0.$$ 

Then $N_n \xrightarrow{w} N$. 

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The fundamental example which covers all the situations of this paper is investigated in [HL06].

Example A.6. Let E be a Polish space and let 0 be an element of E. Let $E_0 = E \setminus \{0\}$. The boundedness $B_0$ on $E_0$ is the class of sets separated from 0: $B \in B_0$ if and only if there exists an open set $U$ of $E$ such that $B \subset U$ (the complement of $U$). If $d$ is any metric which induces the topology of $E$, then $B \in B_0$ if and only if there exists $\epsilon > 0$ such that $x \in B$ implies $d(x, 0) > \epsilon$. The sequence $U_n = \{x \in E : d(x, 0) > n^{-1}\}$, $n \geq 1$ is a localizing sequence. Also, every bounded set has an $\epsilon$-enlargement with respect to $d$ which is still bounded. Thus, vague convergence on $E_0$ is characterized by non-negative bounded Lipschitz functions with respect to any metric which induces the topology of $E$.

Appendix B  The $J_1$ topology

For an $I \subset \mathbb{R}$, we define the $J_1$ metric on $I$, denoted $d_I$, as follows. Let $B_I$ be the set of one-to-one strictly increasing continuous maps on $I$. Then, for $f, g \in D(I)$,

$$d_I(f, g) = \inf_{u \in B_I} \|f \circ u - g\|_I \lor \|u - \text{Id}\|_I.$$ 

Oviously, $d_I(f, g) \leq \|f - g\|_I$. For $I = \mathbb{R}$, we write $d_\infty$ and $\|\cdot\|_\infty$.

For fixed $f, g \in D$, the map $t \mapsto d_{[-t, t]}(f, g)$ is càdlàg and continuous at every $t$ such that $t$ and $-t$ are continuity points of both $f$ and $g$.

The $J_1$ topology on $D(\mathbb{R})$ is the topology of $J_1$ convergence on compact subsets of $\mathbb{R}$, induced by the metric

$$d_{J_1}(f, g) = \int_0^\infty \{d_{[-t, t]}(f, g) \wedge 1\} e^{-t} dt. \quad (B.1)$$

The space $D$ endowed with the $J_1$ topology is Polish and the Borel $\sigma$-field associated to the $J_1$ topology on $D$ is the product $\sigma$-field. See [Whi80, Section 2]. Note that for all $f, g \in D$,

$$d_{J_1}(f, g) \leq \|f - g\|_\infty.$$ 

Furthermore, equality holds if $g = 0$, i.e. $d_\infty(0, f) = \|f\|$ for all $f \in D$.

For $a < b$ and $\eta > 0$, $\eta < b - a$, let $\mathcal{P}(a, b, \eta)$ be the set of finite increasing sequences $(t_0, \ldots, t_k)$ with $k \geq 1$, $t_0 = a$, $t_k = b$ and $\inf_{1 \leq i \leq k}(t_i - t_{i-1}) \geq \eta$. Define for a function $f \in D$,

$$w'(f, a, b, \eta) = \inf_{(t_0, \ldots, t_k) \in \mathcal{P}(a, b, \eta)} \sup_{1 \leq k \leq t_{i-1} \leq s < t_i} |f(s) - f(t)|,$$

$$w''(f, a, b, \delta) = \sup_{a \leq s \leq t \leq u \leq b} |f(t) - f(s)| \wedge |f(u) - f(t)|.$$ 

It generally holds that $w''(f, a, b, \delta) \leq w'(f, a, b, \delta)$ ([Bil99, Eq. (12.28)]) but both quantities can be used to characterize relative compactness in $D$. 

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Theorem B.1. Let \( \{X, X_n, n \in \mathbb{N}\} \) be a sequence of \( D(R) \)-valued stochastic processes. Then \( X_n \xrightarrow{w} X \) in \( D(\mathbb{R}) \) endowed with the \( J_1 \) topology if and only if for all \( a < b \) such that \( P(X \text{ discontinuous at } a) = P(X \text{ discontinuous at } b) = 0, X_n \xrightarrow{f.d.} X \) (in a dense subset of \([a, b]\)) and for all \( \epsilon > 0 \), either (both) of the following conditions hold:

\[
\begin{align*}
\lim_{\delta \to 0} \limsup_{n \to \infty} P(w'(X_n, a, b, \delta) > \epsilon) &= 0, \\ \lim_{\delta \to 0} \limsup_{n \to \infty} P(w''(X_n, a, b, \delta) > \epsilon) &= 0.
\end{align*}
\] (B.2)

Proof. For \( a \leq b \) and \( f \in D \), let \( R_{a,b} f \) be the restriction of \( f \) to \([a, b]\). By [Whi80, Theorem 2.8], a sequence of probability measures \( P_n \) on \( D(\mathbb{R}) \) converges weakly to \( P \) if and only if \( P_n \circ R^{-1}_{a,b} \xrightarrow{w} P \circ R^{-1}_{a,b} \) for all \( k \in \mathbb{N} \) and a sequence \( \{(a_k, b_k), k \in \mathbb{N}\} \) such that \( \bigcup_{k \geq 0}[a_k, b_k] = \mathbb{R} \). The sequence \( a_k \) can be chosen non-increasing, the sequence \( b_k \) can be chosen non-decreasing and the points \( a_k, b_k \) can be chosen as continuity points of \( P \), i.e. \( P(y \text{ not continuous at } t) = 0 \) for all \( t \in \{a_k, b_k, k \in \mathbb{N}\} \).

Thus it suffices to prove that \( R_{a,b} X_n \xrightarrow{w} R_{a,b} X \) for all continuity points \( a, b \) of \( X \). By [Bil99, Theorem 13.2 and Theorem 13.3] this follows from the stated finite dimensional weak convergence and (B.3) or (B.2). \qed

Appendix C  A maximal inequality for stable processes

We summarize here and give a self-contained proof of certain arguments used in [Sam04] that are needed in Section 4.2.

Lemma C.1. Let \( \alpha \in (0, 2) \) and \( \{P_i, i \geq 1\} \) be the points of a Poisson point process on \((0, \infty)\) with mean measure \( \nu_\alpha \). Let \( \{Z, Z_j, j \geq 1\} \) be i.i.d. càdlàg stochastic processes, independent of \( \{P_i, i \geq 1\} \), such that \( 0 < \mathbb{E}[\sup_{a \leq s \leq b} |Z(t)|^\alpha] < \infty \). If \( \alpha \in [1, 2) \), assume furthermore that the distribution of \( Z \) is symmetric. Then it is possible to define an \( \alpha \)-stable process \( X = \sum_{j=1}^{\infty} P_j Z_j \) and there exists a constant such that for all \( a \leq b \) and \( x > 0 \),

\[
\mathbb{P}\left( \sup_{a \leq s \leq b} |X(s)| > x \right) \leq \text{cst} \ x^{-\alpha} \mathbb{E}\left[ \sup_{a \leq s \leq b} |Z(s)|^\alpha \right].
\] (C.1)

Furthermore for all \( x > 0 \),

\[
\lim_{x \to \infty} x^\alpha \mathbb{P}\left( \sup_{a \leq s \leq b} |X(s)| > x \right) = \mathbb{E}\left[ \sup_{0 \leq s \leq 1} |Z(s)|^\alpha \right].
\] (C.2)

Proof. The proof in the case \( \alpha \in (0, 1) \) is straightforward since the sum \( \sum_{j=1}^{\infty} P_j \) is almost surely convergent. We only prove the case \( 1 \leq \alpha < 2 \).
Recall that \( Z_{a,b}^* = \sup_{a \leq s \leq b} |Z(s)| \) and write \( c_\alpha(a, b) = \mathbb{E}[(Z_{a,b}^*)^\alpha] \). Let \( W \) be a stochastic process whose distribution is given by

\[
\mathbb{P}_W = \frac{\mathbb{E}[(Z_{a,b}^*)^\alpha \delta(Z_{a,b}^*)^{-1} Z]}{\mathbb{E}[(Z_{a,b}^*)^\alpha]}
\]

Let \( W^{(i)}, i \geq 1 \) be i.i.d. copies of \( W \). Then \( X \overset{d}{=} c_\alpha^{1/\alpha} \sum_{i=1}^{\infty} P_i W_i \). See [ST94, Section 3.10].

The interest of replacing the process \( Z \) by \( W \) is that the latter satisfies \( \mathbb{P}(\sup_{a \leq s \leq b} |W(s)| = 1) = 1 \).

Let the points \( P_i, i \geq 1 \) be numbered in decreasing order. Then \( P_i = \Gamma_i^{1/\alpha} \) with \( \Gamma_i \) the points of a unit rate Poisson point process on \([0, \infty)\). Since \( \Gamma_i \) has a \( \Gamma(j, 1) \) distribution, we have, for every \( k \geq 1 \),

\[
\mathbb{P}
\left(
\sup_{a \leq s \leq b} \left| \sum_{j=k+1}^{\infty} P_i W_j(t) \right| > x
\right)
\] = \( \frac{1}{k!} \int_{0}^{\infty} \mathbb{P}(G(y) > x)y^k e^{-y} dy \),

with \( G(y) = \sup_{a \leq s \leq b} \left| \sum_{j=1}^{\infty} (y + \Gamma_j)^{-1/\alpha} Z_j(s) \right| \). Since \( \mathbb{P}(\sup_{a \leq s \leq b} |W(s)| = 1) = 1 \), as shown in the proof of [Sam04, Theorem 2.2, bottom of p.814] (which uses the symmetry assumption and takes its argument from the proof of [RS93, Lemma 2.2]), there exists \( r > 0 \) such that for all \( y > 0 \),

\[
\mathbb{E}
\left[
\exp\left\{ \frac{-\log 2}{r + 2y^{-1/\alpha}} G(y) \right\}
\right]
\] \( \leq 4 \).

Thus

\[
\mathbb{P}
\left(
\sup_{a \leq s \leq b} \left| \sum_{j=k+1}^{\infty} \epsilon_i P_i W_j(t) \right| > x
\right)
\] = \( \frac{1}{k!} \int_{0}^{\infty} \mathbb{P}(G(y) > x)y^k e^{-y} dy \)

\( \leq \frac{4}{k!} \int_{0}^{\infty} e^{-\frac{x \log 2}{r + 2y^{-1/\alpha}}} y^k e^{-y} dy \)

\( = \frac{4}{k!} x^{-2\alpha} \int_{0}^{\infty} e^{-\frac{x \log 2}{r + 2y^{-1/\alpha}}} y^{k e^{-y x^{-\alpha}}} dy \).

Note that

\[
e^{-\frac{x \log 2}{r + 2y^{-1/\alpha}}} e^{-y x^{-\alpha}} \leq \begin{cases} e^{-\sqrt{y}} & \text{if } y > x^{2\alpha}, \\ e^{-\frac{y^{1/\alpha} \log 2}{r y^{1/2 \alpha} + 2}} & \text{if } y \leq x^{2\alpha}. \end{cases}
\]

Therefore, the integral above is bounded with respect to \( x \) and

\[
\mathbb{P}
\left(
\sup_{a \leq s \leq b} \left| \sum_{j=k+1}^{\infty} \epsilon_i P_i W_j(t) \right| > x
\right)
\] \( \leq \text{cst} \ x^{-k\alpha}. \) (C.3)
To prove (C.1), we write
\[
P \left( \sup_{a \leq s \leq b} |X(s)| > x \right)
\leq P \left( c_1^{1/\alpha} P_1 > (1 - \epsilon)x \right) + P \left( c_1^{1/\alpha} \sup_{a \leq s \leq b} \left| \sum_{j=2}^{\infty} P_j W_j(t) \right| > \epsilon x/2 \right),
\] (C.4)
and apply (C.3) with \(k = 1\). To prove (C.2), note that
\[
P \left( c_1^{1/\alpha} \sup_{a \leq s \leq b} \left| \sum_{j=2}^{\infty} P_j W_j(t) \right| > \epsilon x/2 \right)
\leq P \left( c_1^{1/\alpha} P_2 > \epsilon x/2 \right) + P \left( c_1^{1/\alpha} \sup_{a \leq s \leq b} \left| \sum_{j=3}^{\infty} P_j W_j(t) \right| > \epsilon x/2 \right).
\]
Since \(P_2\) is regularly varying with tail index 2\(\alpha\), applying (C.3) with \(k = 2\) yields
\[
P \left( \sup_{a \leq s \leq b} \left| \sum_{j=2}^{\infty} \epsilon_i P_i W_j(t) \right| > x \right) \leq \text{cst} \ x^{-2\alpha}.
\] (C.5)
Plugging this bound into (C.4) yields
\[
\limsup_{x \to \infty} x^{\alpha} P \left( \sup_{a \leq s \leq b} |X(s)| > x \right) \leq (1 - \epsilon)^{-\alpha} c_\alpha.
\]
Since \(\epsilon\) is arbitrary, the lim sup is actually equal to 1. A lower bound for the lim inf is obtained similarly. This proves (C.2).

Acknowledgements

This work has been greatly influenced by many discussions with Clément Dombry and Enkelejd Hashorva. The author is also grateful to Olivier Wintenberger for organizing a seminar on regular variation for continuous time stochastic processes in 2018-2019.

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