The fluctuating gravitational field in inhomogeneous and clustered self-gravitating systems.

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Abstract. In this paper I extend the results of Ahmad & Cohen (1973), regarding the study of the probability distribution of the stochastic force in homogeneous gravitational systems, to inhomogeneous gravitational ones. To this aim, I study the stochastic force distribution using N-body realizations of Plummer’s spherically symmetric models. I find that the stochastic force distribution obtained for the evolved system is in good agreement with Kandrup’s (1980) theory of stochastic force in inhomogeneous systems. Correlation effects that arise during the evolution of the system of particles are well described by Antonuccio-Delogu & Atrio-Barandela’s (1992) theory.

Key words: stars: statistics

1. Introduction

The analysis of the dynamics of stellar systems such as globular clusters or clusters of galaxies has shown that the gravitational stochastic force plays a fundamental role in their evolution. In these systems the stochastic force, arising from statistical fluctuations

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in the number of neighbours of a test star, perturbs the stars orbits from the orbits they would have if the density distribution in the system were perfectly smooth. The first consequence produced by the stochastic force is the existence of a frictional force that implies a preferential deceleration of a particle in the direction of motion (Chandrasekhar 1943, Ahmad & Cohen 1974). The existence of the stochastic force is due to the discreteness of gravitational systems, i.e. to the fact that the mass is concentrated into discrete objects like stars. The gravitational field inside such systems can be in principle computed by summing up the potentials of all stars. Obviously, this is not practicable when the number of stars constituting an object (e.g. a globular cluster or a galaxy) is large and so it is generally assumed that the potential can be expressed as an uncorrelated sum of two fields: a mean field obtained from the smoothed mass distribution and a rapidly varying random fluctuation, representing stochastic deviations from the mean field conditions. The mean field can be determined through the Poisson's equation when the density of the system is given, while the determination of the fluctuating field is a more complex task.

The analysis of fluctuations has been formulated through different approaches. One of them is the two body approximation (see Kandrup 1980). In this approach, fluctuations are approximated by a sum of binary interactions of the test particle with the remaining particles. As a consequence, fluctuations are required to have a timescale, $T$, small in comparison with any other relevant timescale, (the dynamical timescale and the relaxation timescale). The main assumptions of this approach are that the probability distribution verifies a Fokker-Planck equation and that the fluctuations in velocity of a test star are due to the sum of binary encounters with neighbouring stars. Finally, there is the inherent assumption that the stochastic force on a test star, at a given time, is equal to the sum of the effects due to each field star. As a result, in the limit of a small angle of scattering there is a logarithmic divergence in the velocity fluctuation $\frac{d\delta v}{dt}$. Physically, this arises because the sum of the forces acting on a test star in an infinite system diverges. The divergence can be solved introducing a cut-off at a distance of the order of the mean particle distance, $r_{med}$ (Kandrup 1980).

To solve these problems it is necessary to use a full stochastic treatment which takes correctly into account the effects of the field stars whose distance from a test star is greater than the interparticle separation.

The stochastic treatment of gravitational fluctuations is connected with the evaluation of a density probability, $W(F)$, i.e. the probability that a test star is subject at a given time to a stochastic force, $F$. In the simplest case (homogeneous, unclustered, infinite
system) the distribution $W(F)$ was calculated by Chandrasekhar & von Neumann (1942) (hereafter CN), while the theory of stochastic force in inhomogeneous, unclustered, infinite systems was developed by Kandrup (1980). The results of Kandrup (1980) showed that the CN theory is in reality independent of the condition of uniform density. Also in inhomogeneous systems, like in homogeneous, the cancellation of contributions to the force from distant field stars is shown to yield essentially negligible contribution to the total magnitude of the fluctuating force (Kandrup 1980) (while the field stars have a nontrivial role in the determination of the distribution of stochastic force in the weak force limit) and so the basic results given by CN are not dependent on the density profile and can be easily extended to inhomogeneous systems.

A comparison of the two-body approximation (with cut off at radius $r_{med}$) with the full stochastic theory shows that for large forces the two treatments are equivalent while, in the limit of weak forces, they disagree because distant field stars play a nontrivial role in the origin of the stochastic force.

A self-consistent mean field theory, like CN and Kandrup’s theory, can describe correctly the stochastic force in a system only if there are no correlations among the positions of particles so that the probability of finding one particle at a position $x$ and another at $y$ is given by:

$$n_2(x, y) = n_1(x)n_2(y)$$

where $n_1(x)$ is the probability density of finding a particle at $x$. This assumption requires that the potential experienced by a test particle is on average larger than the binary potential $\frac{Gm}{R}$ ($m$ is the particle mass and $R$ the radius of the system) connected with binary encounters. Only with the simplifying hypothesis of weakly clustered systems it is possible to extrapolate the results of Kandrup (1980) to clustered systems (Antonuccio-Delogu & Atrio-Barandela 1992, hereafter AB92).

Numerical experiments on the CN distribution were done by Ahmad & Cohen (1973). As previously stressed, this distribution is a theoretical description of the stochastic force in the homogeneous systems only. In this particular case they found a good agreement of numerical experiments with the theoretical distribution.

The only test of the theoretical distribution of stochastic force in inhomogeneous systems (Kandrup 1980) and clustered systems (Antonuccio-Delogu & Atrio-Barandela 1992) is that performed using unevolved systems by A. Del Popolo (1995), who found a good agreement of the theoretical distribution with that derived from numerical experiments. In the quoted paper the experimental stochastic force was calculated in unevolved inho-
mogeneous and clustered systems of particles and directly compared with Kandrup’s and AB92 distributions. Something similar was performed by Hunger et al (1965). In their paper numerical experiments were performed to verify the theoretical stochastic distribution of electric field magnitude into a homogeneous system of particles (Holtsmark distribution) without evolving the N-body system.

This procedure can be justified as follows:

the force per unit mass acting on a test star is given by:

\[ \mathbf{F} = -G \sum_{i=1}^{3} \frac{m_i}{|\mathbf{r}_i - \mathbf{r}|^3} (\mathbf{r}_i - \mathbf{r}) \] (2)

where the sum is extended to the N stars of the system, \( m_i \) is the mass of the i-th field star and \( r_i \) is its distance relative to the origin. Owing to the motions of the stars, the force acting on a point P will change with time. If \( \psi(\mathbf{F}_0; \mathbf{F}, t) \) denotes the probability that a force of intensity \( \mathbf{F} \) acts at P after a time \( t \) then:

\[ \psi(\mathbf{F}_0; \mathbf{F}, t) \rightarrow W(\mathbf{F}) \quad t >> T \] (3)

(Chandrasekhar & von Neumann 1942), where \( \mathbf{F}_0 \) is the force at \( t = 0 \),

\[ T \simeq \left( \frac{2\pi Gm}{3 \sqrt{<v^2>}} \right)^{1/2} \frac{|\mathbf{F}|}{[Q_H^{3/2} + |\mathbf{F}|^{3/2}]} \] (4)

is the mean life of the state \( \mathbf{F} \), \( Q_H = 2.6031Gmn^{2/3} \), \( n \) is the number density and \( \sqrt{<v^2>} \) is the root mean squared velocity of the field particles. By a state \( \mathbf{F} \) I mean (according to Chandrasekhar & von Neumann 1942) that at a fixed point P a force per unit mass of intensity \( \mathbf{F} \) is acting at an instant \( t \). After a sufficient length of time, the force acting on P will be uncorrelated with that acting at time \( t = 0 \). If the process that generates the change in time of the force is a Markoff process the correlations between the force \( \mathbf{F}(t_1) \) and \( \mathbf{F}(t_2) \) acting on the same point, but at two different instants, decrease exponentially as \( \exp \left( -\frac{t}{T} \right) \). In this way the notion of the mean life of the state \( \mathbf{F} \) is made clear. Then, for a time greater than the mean life of the state \( \mathbf{F} \) the stochastic force distribution reduces to a stationary distribution (Chandrasekhar & von Neumann 1942) \( W(\mathbf{F}) \) that is a function of the density \( n \) of the system.

If, for example, the particles are distributed with a probability density in the configuration space given by:

\[ \tau(r) = \frac{a}{r^p} \] (5)

where \( a \) and \( p \) are two constants, the theoretical distribution of the stochastic force is:
\[ W(F) = \frac{2F}{\pi} \int_0^\infty t dt \sin(tF) \exp \left[ -\frac{\alpha}{2}(Gm t)^{(3-p)/2} \int_0^\infty \frac{dz}{z^{(7-p)/2}} \right] \] (6)

where \( \alpha \) is connected to the number density \( n \) (\( \alpha = 4\pi n \) for \( p = 0 \)). As shown in Sect. 2 of this paper, in the case of a Plummer model \( W(F) \) is again function of the density (Eq. 10). If the system is generated using a distribution function that satisfies Vlasov equation (as in the case of Plummer’s model) it remains time-stationary on time interval \( t_{\text{cross}} < \Delta t < t_{\text{relax}} \), where \( t_{\text{cross}} \) is the crossing time and \( t_{\text{relax}} \) the relaxation time. In the case of a Plummer model this is shown in a paper by L. Hernquist (1987).

This paper shows how the density profile remains unchanged during time evolution over 50 crossing times. Then the shape of \( W(F) \) remains unchanged in the quoted interval until correlations effects are developed. This justifies the procedure by A. Del Popolo’s (1995) paper.

The aim of A. Del Popolo’s (1995) paper was to show that, given an inhomogeneous density profile, the stochastic force can be described by Kandrup’s or the AB92 theory. Different is the purpose of this paper. Here I study the stochastic force in an evolving inhomogeneous system to find if and when Kandrup’s and AB92 theories describe correctly the stochastic force in the system. This paper deals with the stochastic force in real systems. It is then necessary to distribute the system particles in an inhomogeneous initial configuration and to evolve it. The evolution of the system is necessary at least for two reasons:

firstly, the structure of real systems like globular clusters is fundamentally due to their dynamical evolution. To compare the theoretical results to real inhomogeneous systems it is then necessary to evolve an inhomogeneous initial configuration and to calculate the stochastic force during the evolution;

secondly, the evolution of a system produces correlations in the configuration space. Self-gravitating systems are unstable with respect to the development of macroscopic correlations. Correlations produce an enhancement in the probability that a test star experiences large forces (AB92) and this produces a change in the stochastic force distribution \( W(F) \).

The aim of this paper is to study the stochastic force in real, evolved systems.

In this paper it is shown that the stochastic force in inhomogeneous evolved systems is well described by Kandrup’s theory as long as correlations are developed. When correlations are present a better description of the stochastic force can be obtained using AB92 theory. This was done by numerically evolving an isotropic Plummer model containing 8000 particles for 150 dynamical times. The stochastic force distribution was calculated
during the evolution on a test star set at the system centre and it was compared with
Kandrup’s theoretical distribution. The force was also calculated using 1000 points dis-
tributed into a sphere of radius 0.01R (R is the maximum radius) centred on the system
centre. The disagreement observed on occasion between Kandrup’s distribution and the
experimental one was removed using Antonuccio-Delogu & Atrio-Barandela theory.
The paper has the following structure:
in Sect. 2 I describe how the system was built and how the stochastic force was calcu-
lated. In Sect. 3 I show the results of the numerical experiments when correlations are not
present. In Sect. 4 the disagreement between Kandrup’s distribution and the experimen-
tal one, observed in the presence of correlations, is removed using the AB92 theoretical
distribution.

2. Stochastic force in inhomogeneous systems

To calculate the stochastic force in an inhomogeneous system I used an initial configu-
ration in which particles were distributed according to a Plummer model with density
profile:
\[
\rho(r) = \frac{3M}{4\pi r_0^3} \frac{1}{\left[1 + \left(\frac{r}{r_0}\right)^2\right]^{5/2}}
\]
where \(M\) is the total mass of the system and \(r_0\) is the scale length. The isotropic
distribution that reproduces the mass distribution given by Eq. 7 is:
\[
f(E) = \frac{\sqrt{2}}{378\pi^3 Gr_0^2 \sigma_0^2} \left(-\frac{E}{\sigma_0^2}\right)^{7/2} \quad -6\sigma_0^2 \leq E \leq 0
\]
(Edington 1916)

where \(\sigma_0\) is the central velocity dispersion and \(E\) is the total energy. This distribution
function was chosen for three reasons:
a) being it a function of an integral of motion according to Jeans theorem it must be a
steady state solution to Vlasov equation and in absence of numerical errors and dynam-
ical instabilities (Merrit & Aguilar 1985) it remains stationary;
b) according to Doremus-Feix-Baumann theorem (1971) and to Antonov’s second law the
Plummer model is stable.

On the other hand Antonov (1973) concluded that any spherical system composed en-
tirely of stars on radial orbits is unstable. This conclusion can be extended to anisotropic
spherical systems in which, as shown by Henon’s (1973 b) and Barnes’ (1986) numerical
simulations, there exists a so called radial-orbit-instability that leads to a triaxial or bar-
like final configuration. In the case of Plummer’s isotropic model (Eq. 8) this problem is
not present in my simulations, while an anisotropic Plummer model (Merrit 1985) shows the radial orbit instability;
c) the density distribution in a Plummer model is similar to the density distribution of many real systems. Plummer (1911) showed that the density distribution given by Eq. 7 provides a good fit to the density profile of globular clusters.

The initial conditions were generated from the distribution function given in Eq. 8 assuming a cut-off radius $R = 1$, the mass of the system $M = 1$, $r_0 = 0.15$ and $G = 1$. All the particles had equal mass. To have a system whose total mass is contained into a unitary sphere, Eq. 7 was renormalized and consequently also the potential of the system which is obtained from Eq. 7 through Poisson’s equation. The system of 8000 particles was evolved over 150 dynamical times using a N-body code (L.Hernquist 1987). During the evolution of the system the total force acting on a test point at the centre of the system was sampled every $\frac{1}{20}$ of a dynamical time. The stochastic force, $F_{\text{stoch}}$, was calculated observing that at the centre of a spherical system we have:

$$F_{\text{tot}} = F_{\text{stoch}} + F_{\text{med}} = F_{\text{stoch}} \quad (9)$$

because the mean field force, $F_{\text{med}}$, is equal to zero. The force was calculated on a point at the centre of the system because theoretically Kandrup’s distribution gives the probability distribution of the stochastic force for a particle at the centre only. When points displaced away from the centre are used the stochastic force distribution must be calculated as follows:

**Fig. 1.** Comparison of the theoretical stochastic force distribution (solid line) with the experimental distribution (histogram) for an inhomogeneous system of 8000 particles evolved over 25 dynamical times. The force is calculated using a test particle set at the centre of the system.

**Fig. 2.** Same as Fig. 1 but now the system is evolved over 50 dynamical times.
a) the stochastic force should be calculated by subtracting the mean field force from the total force:

\[ F_{\text{stoch}} = F_{\text{tot}} - F_{\text{med}} \]  \hspace{1cm} (10)

b) the theoretical distribution must be numerically simulated as done by Ahmad & Cohen (1973).

The stochastic force was also calculated using a series of snapshots of the evolved system.

\[ \text{Fig. 3.} \] Comparison of the theoretical stochastic force (solid line) with the experimental distribution (histogram) for an inhomogeneous system of 8000 particles evolved for 20 dynamical times. The force is calculated as described on the text using 1000 test particles set into a sphere of radius 0.01R centred in the centre of the system.

In each case the force was computed using 1000 randomly distributed test points in a sphere of radius 0.01R. The force on each test point was obtained by subtracting the mean field force from the total force. In both cases the stochastic force was expressed in terms of \( \frac{Gm}{r_0^2} \), where \( m \) is the mass of one particle. The theoretical distribution for the density profile given in Eq. (7) was calculated following the same technique followed in the Appendix A of A. Del Popolo (1995):

\[ W(F) = A \cdot F \int_0^\infty \rho d\rho \sin(\rho F) \exp \left[ -\frac{\alpha}{2} (Gm\rho)^{3/2} B(\rho) \right] \]  \hspace{1cm} (11)

where \( A \) is a constant of normalization, \( \alpha = \frac{3N}{R^3} \left[ 1 + (R/r_0)^2 \right]^{3/2} \), \( N \) is the number of particles and \( B(\rho) \) is given by:

\[ B(\rho) = \int_0^\infty \frac{z - \sin z}{z^{7/2}} \left( 1 + \frac{Gm\rho}{r_0^2} \right)^{3/2} \]  \hspace{1cm} (12)

This theoretical distribution is slightly different from that given by Kandrup (1980) for a power-law density profile. In particular the parameter \( \alpha \) and \( B(\rho) \) are different from
the previous ones. The $\alpha$ parameter is strictly connected to the density in the system and on it depends the magnitude of the stochastic force per unit mass, $<|F_{stoch}|>$. As shown by Kandrup, this has the magnitude which may be expected from the interaction of the test star with a few nearby field stars. In the case of the Plummer model this is an order of magnitude greater than the value obtained in the homogeneous case, for which is:

$$<|F_{stoch}|> = 8.879Gmn^{2/3} \approx 0.4$$

where $m$ is the mass of one particle, $n$ the mean density (the units previously introduced were used). The theoretical distribution of the stochastic force was finally compared with the histogram of force obtained from the evolved system.

3. Results of the numerical experiments

The results obtained from the numerical experiments performed are shown in Figs. 1-8. Fig. 1 and Fig. 2 show the distribution of the stochastic force calculated using a test point at the centre of the system. Fig. 1 is obtained from the system evolved over 25 dynamical times. It shows a good agreement of the theoretical distribution (solid line) with the experimental distribution of force obtained from the system (histogram). Fig. 2 is the same as the previous one but in this case the system is evolved over 50 dynamical times. During the evolution of the system I observed a disagreement between the theoretical and experimental distributions is due to the correlations in configuration space that arise during the evolution of the system.

Fig. 5. Comparison of the theoretical stochastic force (solid line) with the experimental distributions for an inhomogeneous system of 8000 particles evolved over 85 (solid line histogram), 100 (dashed line histogram), 150 (dotted line histogram) dynamical times. The force is calculated using a test particle set at the centre of the system. The disagreement between experimental and theoretical distributions is due to the correlations in configuration space that arise during the evolution of the system.

Fig. 3 and 4, are obtained from the system of particles evolved respectively over 20 and 80 dynamical times. Now the force is calculated using 1000 points randomly distributed into a sphere of radius $0.01R$. Also in these cases the theoretical distributions (solid line) are in good agreement with the experimental ones (histogram).

During the evolution of the system I observed a disagreement between the theoretical
Fig. 6. Distribution of cosines of the angle between pairs of particles as measured from centre of the system for a system evolved over 100 dynamical times.

and the experimental distributions, plotted in Fig. 5. This figure shows the comparison between the theoretical stochastic force (solid line) and the experimental distributions in a system evolved over 85 (solid line histogram), 100 (dashed line histogram), 150 (dotted line histogram) dynamical times and with the stochastic force calculated using a test point at the centre of the system. The disagreement is observed when the system is evolved over times larger than 80 dynamical times. Disagreements of this kind were found by Ahmad & Cohen (1973) in their simulations of homogeneous systems and were attributed to correlations in the configuration space. Then it is interesting to test this hypothesis also in the inhomogeneous system I evolved.

4. Correlations and stochastic force.

Correlation effects in gravitational system were studied in several papers (Prigogine & Severne 1966; Gilbert 1970; AB92). Already Gilbert (1970) predicted that correlations produce an enhancement of the stochastic force, this conclusion being confirmed in the study of AB92. Thus a cause of the disagreement observed in my simulations between theoretical and experimental stochastic force could be the correlations arising during the evolution. To test this hypothesis and verify the possible presence of correlations, the test of the distribution of cosines of the angle between pairs of particles was used (Miller 1971). If in a system correlations are not present, the distribution of cosines is uniform between -1 and 1 with a probability density of 0.5. A larger value for a large value of the cosines indicates the presence of clustering. The test for the system evolved over 85, 100, 150 dynamical times was performed. In all three cases there was evidence for clustering in the configuration space. The result of Miller’s test for the system evolved over 100 dynamical times is shown in Fig. 6. For $\cos \theta \approx 1$ the density distribution is larger than 0.5 indicating a correlation in the configuration space. To measure quantitatively the correlations in the system of particles it is necessary to measure the two points correlation function $\xi$. This is possible using the counts in cell analysis (Ripley, 1980; Mo,
Once known the two points correlation function the theory developed in AB92 was used to calculate the theoretical distribution of a correlated system. The equation used to calculate the distribution of force is:

$$W_N(F) = 4\pi^2|F|^2W_N(F) = \frac{2F}{\pi} \int_0^\infty t\sin(tF) \left[ \frac{\alpha}{N}(\text{Gm}t)^{(3-p)/2} \int_{\text{gmt}^2}^{\infty} \exp(-\frac{Gmt}{t_0^2 z}) \sin\frac{z}{z(7-p)/2} \right]^N$$

$$\cdot \left[ 1 + \frac{1}{2}(1 - \frac{1}{N}) \frac{\Sigma(t)}{A_2(t)} \right]$$

where $\Sigma(t)$ and $A_2(t)$ are given in the quoted paper (Eq. 34 and Eq. 32 respectively).

In Figs. 7 and 8 this theoretical distribution (dotted line) is compared with the experimental ones (histogram). The disagreement between the two distribution can be removed when correlation effects are taken into account and the theoretical distribution given by Kandrup is replaced by Antonuccio-Delogu & Atrio-Barandela distribution for clustered systems (dotted line).

Kandrup’s distribution and the experimental one is due, as previously written, to the correlations that arise during the evolution. Correlations in the configuration space produce an enhancement of the probability that a test particle is subject to a large force, as can be seen from Figs. 7 and 8. The disagreement between Kandrup’s distribution and the experimental one is resolved using a theoretical distribution for clustered systems.

**Conclusions.**

In this paper the theoretical stochastic force distribution in inhomogeneous systems has been studied (Kandrup 1980) using a system of 8000 particles distributed according to
an isotropic Plummer model. The theoretical distribution has been compared to that of stochastic force obtained from the system evolved over 150 dynamical times. The results of the comparison has shown that Kandrup’s theory describes correctly the stochastic force in inhomogeneous, evolved systems as long as correlations are not present. Disagreements between the theoretical distribution and the experimental one, observed during the evolution, are due to correlations arising during the evolution of the system. The presence of correlations has been shown using Miller’s (1971) test and the two point correlation function has been calculated using the count in cell analysis (Mo, 1991; Ripley 1980) and it has been used in the AB92 theory to calculate a theoretical distribution of the stochastic force taking into account the correlation effects. The final comparison has shown a good agreement between the last theoretical distribution and the experimental distribution. Then the stochastic force in inhomogeneous unclustered systems are well described by Kandrup’s theory while when correlations are present it must be substituted by the AB92 theory.

In conclusion I observe that the role of the stochastic force in the evolution of gravitational systems increases when correlations are present because in such systems there is a greater probability that stars are subject to large stochastic forces with respect to the uncorrelated one.

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