MOEBIUS TRANSFORMATIONS
PRESEVING FIXED
ANHARMONIC RATIO

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Abstract. O. Kobayashi [6] proved that $C^1$-mappings preserving anharmonic ratio are Moebius transformations. We strengthen his result and prove, that the requirement of differentiability and even of injectivity can be omitted.

1. Introduction

A concept of Apollonian tetrad in the complex plane $\mathbb{C}$ was introduced in the paper ([1], Def. 1, p.15): any ordered quadruple of distinct points $\{z_1, z_2, z_3, z_4\} \subset \mathbb{C}$ is called an Apollonian tetrad, if

$$|z_2 - z_3| \cdot |z_1 - z_4| = |z_3 - z_1| \cdot |z_2 - z_4| = |z_1 - z_2| \cdot |z_3 - z_4| \quad (0.1)$$

It was proved in ([1], Main Theorem, p. 19), that any univalent analytic function in the domain $D \subset \mathbb{C}$ is linear-fractional iff the image of any Apollonian tetrad in $D$ is also Apollonian tetrad.

Since that time more than ten articles appeared, in which different generalizations of the mentioned property of tetrad to finite set of points were introduced: Haruki H. and Rassias T.M. in [2] considered Apollonian triangles and hexagons, Bulut S. and Yilmaz Özgür N. in [3] considered Apollonian set consisting of $2n$ pairwise different points and proved, that the analytic univalent function is linear-fractional iff the image of any Apollonian set in $D$ is also an Apollonian set.

The constructions in these works have considerable interest from the point of view of the theory of quasimoebius transformations in $\mathbb{R}^n$ (see [4]) and in Ptolemaic spaces (see [5]).
O. Kobayashi [6] noticed, that relation (0.1) is equivalent to the equality \[ z_1 : z_2 : z_3 : z_4 = (1 \pm i\sqrt{3})/2, \] where \([z_1 : z_2 : z_3 : z_4]\) is the anharmonic ratio of the quadruple \(\{z_1, z_2, z_3, z_4\}\), and he obtained the following result:

**Theorem 1.** ( [6], Theorem 2.1, p. 118)

Let \(\lambda \in \mathbb{C} \setminus \mathbb{R}\) and \(U \subset \mathbb{C}\) be a domain. Suppose \(f : U \to \mathbb{C}\) is an injective differentiable mapping.

If for any quadruple of pairwise distinct points \(\{z_1, z_2, z_3, z_4\} \in U\) with anharmonic ratio \([z_1 : z_2 : z_3 : z_4] = \lambda\) the equality \([f(z_1) : f(z_2) : f(z_3) : f(z_4)] = \lambda\) holds, then \(f\) is a Moebius transformation.

In this paper we show, that the theorem is valid for any \(\lambda \in \mathbb{C} \setminus \{0, 1, \infty\}\) and for any continuous non-constant mapping \(f : U \to \mathbb{C}\) of the domain \(U \subset \mathbb{C}\) without the requirement of injectivity and differentiability of \(f\).

2. The tables of symbols, terminology and the main result

Let \(T\) be set of all ordered quadruples (tetrads) \(T = \{z_1, z_2, z_3, z_4\}\) in the extended complex plane \(\overline{\mathbb{C}}\), that do not contain any three coincident elements.

A tetrad with four pairwise different elements is called *nonsingular*.

For any tetrad we define its anharmonic ratio (see [7], §44) \(A(T) = [z_1 : z_2 : z_3 : z_4]\).

If all points in a tetrad \(T\) are finite and pairwise different, then \(A(T)\) is

\[
A(T) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_3 - z_2)(z_4 - z_1)}. \tag{1.1.1}
\]

Under these conditions \(A(T)\) is different from 0, 1 and \(\infty\). For tetrads \(s_{12}(T)\), \(s_{13}(T)\), \(s_{14}(T)\), obtained from \(T\) by permutation of first and second, of first and third, of first and fourth elements respectively the following equalities hold:

\[
A(s_{12}(T)) = \frac{1}{A(T)}; \quad A(s_{13}(T)) = \frac{A(T)}{A(T) - 1}; \quad A(s_{14}(T)) = 1 - A(T) \tag{1.1.2}
\]
If in the tetrad \( T = \{z_1, z_2, z_3, z_4\} \) neither three elements coincide, there exists a finite or infinite limit

\[
A(T) = \lim_{w_1 \to z_1, w_2 \to z_2, w_3 \to z_3, w_4 \to z_4} \frac{(w_1 - w_3)(w_2 - w_4)}{(w_3 - w_2)(w_4 - w_1)},
\]

where the limit is taken on a set of all nonsingular tetrads \( \{w_1, w_2, w_3, w_4\} \).

This limit defines the anharmonic ratio of the tetrad \( T \) in general case.

Particularly, for nonsingular tetrad \( T = \{z_1, z_2, z_3, \infty\} \) we have \( A(T) = -(z_1 - z_3)/(z_3 - z_2) \). We notice, that the tetrad \( T \in T \) is nonsingular iff \( A(T) \) different from 0, 1 and \( \infty \).

A Moebius transformation \( \mu : \mathbb{C} \to \mathbb{C} \) is defined as a superposition of finite number of reflections (or inversions) with respect to generalized circles in \( \mathbb{C} \) (see [7], Def 3.1.1, P. 25) and it realized either by the linear-fractional function or by its conjugate.

If \( \mu \) is a linear-fractional mapping, then for any tetrad \( T \in T \) the equality \( A(\mu(T)) = A(T) \) (the invariance of anharmonic ratio by the linear-fractional mappings) holds. The opposite is also true: a bijective mapping \( \mu : \mathbb{C} \to \mathbb{C} \), which preserves an anharmonic ratio of all nonsingular tetrads, is realized by linear-fractional function (see [7], §4.4).

For given complex number \( \alpha \notin \{0, 1, \infty\} \), we denote by \( T(\alpha) \) the set of all tetrads \( T \in T \) with \( A(T) = \alpha \) (all that tetrads are nonsingular) and by \( T(\alpha, f) \) the set of all tetrads of \( T(\alpha) \), for which \( f(T) \in T \).

**Definition 2.** We say that a mapping \( f : U \to \mathbb{C} \), of a domain \( U \subset \mathbb{C} \) satisfies the condition \( \varphi(K, \alpha) \), if for any tetrad \( T \in T(\alpha, f) \) the equality \( A(f(T)) = \alpha = A(T) \) holds.

The main result of the paper is the following

**Theorem 3.** If \( \alpha \notin \{0, 1, \infty\} \), then any continuous mapping \( f : U \to \mathbb{C} \) of a domain \( U \subset \mathbb{C} \), which satisfies the condition \( \varphi(K, \alpha) \), is either a constant or the Moebius transformation; and if \( \alpha \) is not a real number, then a function \( f(z) \) is either a constant or a linear-fractional function.
3. The injectivity lemma

**Proposition 4.** If a continuous mapping $f : U \to \mathbb{C}$ satisfies the condition $\varphi(K, \alpha)$, $\alpha \notin \{0, 1, \infty\}$, then it satisfies the condition $\varphi(K, \beta)$, where $\beta$ is taken from the set

$$P_\alpha = \left\{ \frac{1}{\alpha}, 1 - \alpha, \frac{1}{1 - \alpha}, \frac{\alpha - 1}{\alpha} \right\}$$

(2.1.1)

**Proof.** The proof follows immediately from relations (1.1.2) and from the fact, that the equality $f(s(T)) = s(f(T))$ is preserved by the permutations $s$ of the elements of the tetrad, and therefore the relations $f(T) \in T$ and $f(s(T)) \in T$ are equivalent.

**Lemma 5.** Suppose $\alpha \notin \{0, 1, \infty\}$ and let $f : U \to \mathbb{C}$ be a continuous mapping of a domain $U \subset \mathbb{C}$ satisfying the condition $\varphi(K, \alpha)$. Then $f(z)$ is either injective in $U$ or $f$ is a constant map.

**Proof.** We assume that $f$ is not injective. Then we can find different points $\zeta_0, \zeta_\infty \in U$, for which $f(\zeta_0) = f(\zeta_\infty) = a \in \mathbb{C}$. Take such linear-fractional mappings $\mu$ and $\nu$ that $\mu(\zeta_\infty) = \nu(a) = \infty$. Evidently $\infty \in \mu(U) = U'$. A continuous mapping $g = \nu \circ f \circ \mu^{-1} : U' \to \mathbb{C}$ satisfies the condition $\varphi(K, \gamma)$ for any constant $\gamma \in P_\alpha$; whereas $g(\infty) = g(\mu(\zeta_\infty)) = \infty$.

Consider the set $M = g^{-1}(\{\infty\}) \cap C = \{z \in C : g(z) = \infty\}$. Since $g$ is continuous, the set $M$ is closed in $U' \setminus \{\infty\}$. We show, that $M$ is an open set.

Let $z_0$ be an any point in $M$.

Build an open disc $B = B(z_0, r) \subset U'$ with radius

$$r = \frac{\max\{1, \text{dist}(z_0, \partial U')\}}{2 + |\alpha|}$$

(2.2.1)

If $g(z) \equiv \infty$ in $B$, then $B \subset M$, and $z_0$ is an interior point of the set $M$. Consider the point $z_1 \in B$, in which $g(z_1) \neq \infty$. The point $z_2 = z_0 + \alpha(z_1 - z_0)$ lies in $U'$, whereas

$$|z_2 - z_0| = |\alpha| \cdot |z_1 - z_0| \leq \text{dist}(z_0, \partial U') \frac{|\alpha|}{2 + |\alpha|} < \text{dist}(z_0, \partial U').$$

For $T = \{z_2, z_1, z_0, \infty\} \subset U'$ we have $A(T) = -(z_2 - z_0)/(z_0 - z_1) = \alpha$. The assumption, that $f(T) = \{g(z_2), g(z_1), \infty, \infty\} \in T$, contradicts
the condition $\varphi(K, \alpha)$, because in this case $A(f(T)) \in \{0, 1, \infty\}$, that is $f(A) \neq \alpha$.

Therefore the tetrad $f(T)$ has three equal elements. But $g(z_1) \neq \infty$ implies that $g(z_2) = \infty$, that is $z_2 \in M$.

Since $g(z_1) \neq \infty$ and $f$ is continuous in the point $z_1$, take such $\varepsilon > 0$ that the disc $B(z_1, \varepsilon) \subset B$ and a function $g(z) \neq \infty$ in $B(z_1, \varepsilon)$. From the inequality $\varepsilon < r$ and from (2.2.1) it follows that

$$B(z_0, \frac{|\alpha|}{1 + |\alpha|}) \subset B \quad (2.2.2)$$

Putting $\delta = \varepsilon \frac{|\alpha|}{1 + |\alpha|}$, consider any point $z \in B(z_0, \delta)$. For $w = z + \alpha^{-1}(z_2 - z)$ the inequality holds:

$$|w - z_1| = |z - z_1 + z_1 - z_0 + \alpha^{-1}(z_0 - z)| < |z - z_0| \left|1 - \frac{1}{\alpha}\right| < \varepsilon,$$

which means, that $w \in B(z_1, \varepsilon)$ and therefore $g(w) \neq \infty$. For $T' = \{z_2, w, z, \infty\} \subset U'$ we have $A(T') = -(z_2 - z)/(z - w) = \alpha$. The assumption that $g(T') = \{g(z_2) = \infty, g(w), g(z), \infty\} \in T$ contradicts the condition $\varphi(K, \alpha)$, because $A(g(T')) \in \{0, 1, \infty\}$ and therefore $A(g(T')) \neq \alpha$. Then in the tetrad $g(T')$ there are three equal elements. But $g(w) \neq \infty$ and therefore $g(z) = \infty$. Thus we see, that $g(z) \equiv \infty$ in the disc $B(z_0, \delta)$. So $B(z_0, \delta) \subset M$ and $z_0$ is an interior point of the set $M$. Since any point of the set $M$ it is an interior point, the set $M$ is open.

Since the set $U'\backslash\{\infty\}$ is connected, the open-and-closed set $M$ is an empty set or $U'\backslash\{\infty\}$. But $\mu(\zeta_0) \in M$, the $M \neq \emptyset$. Therefore $M = U'\backslash\{\infty\}$, and it means that $g(z) \equiv \infty$ in $U'$. Thus $f(\zeta) \equiv a$ in $U$.

4. PROOF OF THE MAIN THEOREM

We prove the main theorem in several steps arranging them as independent propositions.

Proposition 6. Suppose the domain $U \subset \overline{C}$ contains $\infty$, and a continuous injective mapping $f : U \to \overline{C}$, $f(\infty) = \infty$ satisfies the condition
\( \varphi(K, \alpha), \alpha \notin \{0, 1, \infty\}. \) Then for any \( z_0 \in U \) and \( a_0 = z_0 + w_0, \ b_0 = z_0 - w_0, \) where \( |w_0| < \text{dist}(z_0, \partial U), \) the equality holds:

\[
f(z_0) = \frac{f(a_0) + f(b_0)}{2} \tag{3.1.1}
\]

**Proof.** Let \( \alpha \in \{2, \ 1/2, \ -1\} \). Then by Proposition 4 the mapping \( f \) satisfies the condition \( \varphi(K, \beta) \). For \( T = \{a_0, \ b_0, \ z_0, \ \infty\} \) we obtain

\[
2 = A(T) = [f(a_0) : f(z_0) : f(b_0) : \infty] = -\frac{f(a_0) - f(b_0)}{f(b_0) - f(z_0)},
\]

from which the desired equation (3.1.1) immediately follows.

Suppose now \( \alpha \notin \{0, 1, \infty, \ 2, \ 1/2, \ -1\} \).

If \( |\alpha| > 1 \), then we take \( \beta = \alpha \); in otherwise we take \( \beta = 1/\alpha \). Then \( f \) satisfies the condition \( \varphi(K, \beta) \), \( \beta \notin \{0, 1, \infty, \ 1/2\} \) (see proposition 4).

For \( \beta' = (1 - \beta)/(1 - 2\beta) \) we have the equation \((1 - 2\beta)/(1 - 2\beta') = -1\).

As \( \beta \notin \overline{B}(1/2, \ 1/2) \), then \( |\beta - 1/2| > 1/2 \), that is \( |1 - 2\beta'| > 1. \)

Therefore \(|1 - 2\beta'| = q < 1 \).

Put \( R = \text{dist}(z_0, \partial U) \) and \( w_k = (2\beta' - 1)^k w_0, \ k = 1, 2, \ldots \)

As \(|w_k| = q^k |w_0| < R \), then all points \( a_k = z_0 + w_k \) and \( b_k = z_0 - w_k \)

lies in the disc \( B(z_0, \ R) \subset U \). We show that for any \( k = 0, 1, 2, \ldots \)

the follow equality is valid:

\[
f(a_k) + f(b_k) = f(a_0) + f(b_0) \tag{3.1.2}
\]

For \( k = 0 \) (3.1.2) is trivial. We suggest, that it holds for some \( k \) and show, that \( f(a_{k+1}) + f(b_{k+1}) = f(a_0) + f(b_0) \).

For the tetrad \( T_1 = \{b_k, \ a_{k+1}, \ b_{k+1}, \ \infty\} \) we have

\[
A(T_1) = -\frac{b_k - b_{k+1}}{b_{k+1} - a_{k+1}} = -\frac{w_{k+1} - w_k}{-2w_{k+1}} = \frac{\beta' - 1}{2\beta' - 1} = \beta.
\]

The equality

\[
A(T_1) = -\frac{f(b_k) - f(b_{k+1})}{f(b_{k+1}) - f(a_{k+1})} = \beta
\]

follows from the condition \( \varphi(K, \beta) \). For \( T_2 = \{a_k, \ b_{k+1}, \ a_{k+1}, \ \infty\} \) we have

\[
A(T_2) = -\frac{a_k - a_{k+1}}{a_{k+1} - b_k} = -\frac{w_k - w_{k+1}}{2w_{k+1}} = \beta.
\]

Therefore

\[
A(T_2) = -\frac{f(a_k) - f(a_{k+1})}{f(a_{k+1}) - f(b_{k+1})} = \beta.
\]
Thus we come to the equality
\[
\frac{f(b_{k+1}) - f(b_k)}{f(b_{k+1}) - f(a_{k+1})} = \frac{f(a_k) - f(a_{k+1})}{f(b_{k+1}) - f(a_{k+1})},
\]
from which it follows that \(f(b_{k+1}) - f(b_k) = f(a_k) - f(a_{k+1})\).

Then by induction we obtain
\[
f(a_{k+1}) + f(b_{k+1}) = f(a_k) + f(b_k) = f(a_0) + f(b_0).
\]
Thus we have proved (3.1.2) for all \(k = 0, 1, 2, \ldots\)

Since \(w_k \to 0\), when \(k \to \infty\), then \(a_k \to z_0\) and \(b_k \to z_0\).

Next from (3.1.2) we obtain the desired relation (3.1.1).

**Proposition 7.** Suppose a domain \(U \subset \mathbb{C}\) contains \(\infty\), and a continuous mapping \(f : U \to \mathbb{C}\), \(f(\infty) = \infty\) satisfies the condition \(\varphi(K, \alpha)\), \(\alpha \notin \{0, 1, \infty\}\). Then the mapping \(f\) moves any linear segment \(L \subset U \setminus \{\infty\}\) to a linear segment; any ray \(P \subset U \setminus \{\infty\}\) to some ray; any line \(Q \subset U \setminus \{\infty\}\) to some line.

**Proof.** We show that for any point \(c \in U \setminus \{\infty\}\) the mapping \(f\) is linear on any linear segment \(L \subset B(c, r)\), where \(r = \text{dist}(c, \partial U)/3\).

Let \(a, b\) be the endpoints of the segment \(L\). For any point \(z \in L\),
\[
\max\{|z - a|, |z - b|\} \leq 2r < \text{dist}(z, \partial U).
\]
That is, we may apply the Proposition 6 to any point \(z \in L\) and any \(w\) such that \(|w| < \max\{|z - a|, |z - b|\}\). Therefore for any pair of points \(t_1, t_2 \in L\) we have
\[
f((t_1 + t_2)/2) = (f(t_1) + f(t_2))/2.
\]
We conclude that the function \(f\) is linear on a dense subset of the segment \(L\) and so, by continuity of \(f\), it is linear on \(L\).

Thus the mapping \(f\) is locally linear on any connected subset \(S \subset U \setminus \{\infty\}\), which lies on a straight line. Therefore by connectedness of \(S\), \(f\) is linear on all of \(S\). Particularly, an image of any segment is some segment, the image of any ray is some ray, the image of any line is some line.

Next we use a criterion of Moebiusness for mappings of \(n\)-dimensional domains, proposed by Zelinsky Y.B. for the mapping of (we take this theorem for case \(n = 2\)).

**Theorem 8.** ([8], Th. 8, P. 35) Suppose a continuous mapping \(f : D \to \mathbb{C}\) of a domain \(D \subset \mathbb{C}\) moves any set \(P \subset D\), which lies on
a generalized circle to a set, which lies on a generalized circle. If \( f(D) \) does not lie on a generalized circle, then \( f \) is a Moebius transformation.

**Lemma 9.** Any continuous injective mapping \( f : U \to \mathbb{C} \) of a domain \( U \subset \overline{\mathbb{C}} \), which satisfies the condition \( \varphi(K, \alpha), \alpha \notin \{0, 1, \infty\} \), is a Moebius transformation.

**Proof.** Take any open disc \( D \subset U \) such that \( \overline{D} \subset U \) and a set \( P \subset D \) lies on a generalized circle \( S \subset \mathbb{C} \). Then \( S \cap D \) is a connected subset of a generalized circle \( S \).

The situation 1.

Suppose \( S \cap \partial D \neq \emptyset \) and \( a \in S \cap \partial D \). We build linear-fractional mappings \( \mu : \mathbb{C} \to \mathbb{C} \) and \( \eta : \mathbb{C} \to \mathbb{C} \) such that \( \mu(a) = \infty, \eta(f(a)) = \infty \) and consider the mapping \( g = \eta \circ f \circ \mu^{-1} : \mu(U) \to \mathbb{C} \). This mapping satisfies the condition \( \varphi(K, \alpha) \) and \( \infty \in \mu(U), \ g(\infty) = \infty \). The set \( L' = \mu((S \cap \overline{D})\{a\}) \subset \mu(U)\{\infty\} \) is either a ray or a line. By Proposition 7, the set \( g(L') \) is also either some ray or a line. Therefore \( f = \eta^{-1} \circ g \circ \mu \) maps a set \( S \cap D \) (and any subset \( P \) also) to a subset of a circle.

The situation 2.

Take \( S \subset D \) and \( a \in S \). We build linear-fractional mappings \( \mu \) and \( \eta \) by analogy with the situation 1 and consider the mapping \( g = \eta \circ f \circ \mu^{-1} \). By Proposition 7, the mapping \( g \) moves the line \( L' = \mu((S \setminus \{a\}) \subset \mu(U)\{\infty\} \) to some line. Therefore \( g(L') \) is a generalized circle and \( f \) moves the generalized circle \( S = \mu^{-1}(L') \) to a generalized circle, and any subset \( P \subset S \) to a subset of a generalized circle. Thus \( f \) satisfies the conditions of the theorem 8 in the disc \( D \) and is injective. Therefore \( f \) is a Moebius transformation on the disc \( D \). Since our choice of the disc \( D \subset U \) is arbitrary, the mapping \( f \) is locally Moebius on a domain \( U \). Moebius transformations are explicitly defined by their values on some quadruple, that does not lie on a generalized circle. From local Moebiusness of \( f \) it follows, that \( f \) is Moebius on a domain \( U \). ■

**Lemma 10.** Any continuous injective mapping \( f : U \to \mathbb{C} \) of a domain \( U \subset \overline{\mathbb{C}} \), which satisfies the condition \( \varphi(K, \alpha), \alpha \in \mathbb{C}\setminus\mathbb{R} \) is a linear-fractional function.
Proof. By Lemma 9 the mapping $f$ is a Moebius transformation, so $f(z)$ is either a linear-fractional function or its conjugate. We show that if $\alpha = a + ib$ and $b \neq 0$, then the second is impossible. Let $f(z) = \mu(z)$, where $\mu(z)$ is a linear-fractional function. Take any tetrad $T = \{z_1, z_2, z_3, z_4\} \subset U$ with anharmonic ratio $A(T) = \alpha = a + ib$. Then $A(\mu(T)) = A(T) = a + ib$ and $A(f(T)) = A(\mu(T)) = a - ib \neq \alpha$. This contradicts the condition $\varphi(K, \alpha)$. Therefore $f(z)$ is a linear-fractional function. ■

The proof of the main Theorem follows immediately from Lemma 5, Lemma 9 and Lemma 10.

References

[1] Haruki H. and Rassias T.M. A new characteristic of Möbius transformations by use of Appolonius points of triangles. // J. Math. Anal. Appl—1996.—Vol. 197.—P. 14-22.
[2] Haruki H. and Rassias T.M. A new characteristic of Möbius transformations by use of Appolonius hexagons. // Proc. Amer. Math. Soc.—2000.—Vol. 128.—P. 2105-2109.
[3] Bulut S., Yılmaz Öğür N. A new characterization of Möbius transformations by use of Apollonius points of $(2n-1)$-gons. // Acta Math. Sinica. Engl. ser.—2005.—No 3.—P. 667-672.
[4] Väisälä J. Quasimöbius maps. // J. Analyse Math.—1984/85.—44.—P. 218-234.
[5] Aseev V.V., Sychev A.V. and Tetenov A.V. Moebius-invariant metrics and generalized angles in ptolemaic spaces. // Siberian Mathematical Journal.—Vol. 46.—No. 2.—pp. 189-204.—2005.
[6] Kobayashi O. Apollonius points and anharmonic ratios. // Tokyo Math. J.—2007.—Vol. 30.—No 1—P. 117-119.
[7] Beardon A.F. The geometry of discrete groups. // Springer-Verlag.—New York-Heidelberg-Berlin.—1983.
[8] Zelinsky Y.B. On maps invariant on subsets mappings. Approximation theory and allied questions of analysis and topology. Collection of a scientific works. // Kiev, Institute of mathematics of Academy of Sciences of Ukrainian SSR—1987.—P. 25-35.(Russian)
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