Quaternionic approach to dual Magneto-hydrodynamics of dyonic cold plasma

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Abstract
The dual magneto-hydrodynamics of dyonic plasma describes the study of electrodynamics equations along with the transport equations in the presence of electrons and magnetic monopoles. In this paper, we formulate the quaternionic dual fields equations, namely, the hydro-electric and hydro-magnetic fields equations which are an analogous to the generalized Lamb vector field and vorticity field equations of dyonic cold plasma fluid. Further, we derive the quaternionic Dirac-Maxwell equations for dual magneto-hydrodynamics of dyonic cold plasma. We also obtain the quaternionic dual continuity equations that describe the transport of dyonic fluid. Finally, we establish an analogy of Alfven wave equation which may generate from the flow of magnetic monopoles in the dyonic field of cold plasma. The present quaternionic formulation for dyonic cold plasma is well invariant under the duality, Lorentz and CPT transformations.

Keywords: quaternion, dyons, magneto-hydrodynamics, cold plasma, Alfven wave, Lorentz invariant.

1 Introduction
In the past few decades, astronomers predicted that the universe was composed almost entirely of the baryonic matter (ordinary matter). According to Bachynski [1], more than 99% of the matter in the universe is in plasma state. This type of matter may be consist of baryonic and non-baryonic matter. The first experimental evidence of the existence of plasma was given by American Physicists [2]. In plasma, consisting of charged and neutral particles, the inter-ionic force between particles show electromagnetic in nature. Therefore, due to the long range order of Coulomb force charged particles interact with all other charged particles resulting in a collective behavior of plasma. In 1942, Alfven [3] gave the theory of Magnetohydrodynamics (MHD) and suggested that electrically conducting fluid can support the propagation of shear waves called the Alfven waves. Basically, MHD describes the behavior of electrically conducting fluid in the presence of magnetic field [4]. It is macroscopic theory that assumes the electrons, ions and charged particles moves together and treated them as a single fluid component known as single-fluid theory. The plasma along with MHD is simply described by a single temperature, velocity and density. However, when the MHD wave propagates faster than plasma thermal speed then the effect of temperature can be neglected [5]. This is called a cold plasma approximation (i.e., in cold plasma approximation, temperature doesn’t take into account). In this approximation, there is no wave related to pressure fluctuation (e.g. sound waves). On the other hand, the hot and warm plasmas are another sates of plasma where the collision between electrons and gas molecules are so frequent that there is a thermal equilibriu between electron and the gas molecules.

Meyer-Vernet [6] discussed the role of magnetic monopole in conducting fluid (plasma). The magnetic monopole proposed by Dirac [7], it is a hypothetical elementary particle having only one magnetic pole. Dirac also pointed out that if there exist any monopole in the universe then all the electric

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charge in the universe will be quantized [8]. Schwinger [9, 10], an exception to the argument against
the existence of monopole, and formulated relativistically covariant quantum field theory of magnetic
monopoles which maintained complete symmetry between electric and magnetic fields. Therefore, the
name of particles that carrying simultaneously the electric and magnetic charges called Dyons. Further,
the theoretical approach of Schwinger [9, 10] and Zwanziger [11] describe the theory of dyonic particles.
Peres [12] pointed out the controversial nature [13] of the singular lines of magnetic monopoles and
established the charged quantization condition in purely group theoretical manner without using them.
In view of mathematical physics, the study of four dimensional particles (dyons) in distinguish med-
iums can be explain by division algebras. There are four types of divisions algebras [14], namely the
real, complex, quaternion and octonion algebras. The complex algebra is an extension of real numbers,
the quaternion is an extension of complex numbers while the octonion is an extension of quaternions.
Quaternionic algebra [15] can also express by the four-dimensional Euclidean spaces [16, 17], and it
has vast applications in the multiple branches of physics.

Further, Rajput [18] pointed out an effective unified theory for quaternionic generalized electro-
magnetic and gravitational fields of dyons by using the quaternion algebra. The quaternionic form
of classical and quantum electrodynamics have been already discussed [19, 20, 21, 22]. Many authors
[23, 24, 25, 26, 27, 28, 29] have studied the role of hyper-complex algebras in various branches of physics.
Recently, Chanyal [30, 31] independently proposed a novel approach on the quaternionic covariant
theory for relativistic quantum mechanics, and established the quantized Dirac-Maxwell equations for
dyons. Besides, in literature [32, 33, 34], the reformulation of incompressible plasma fluids and MHD
equations have been discussed in terms of hyper-complex numbers. Keeping in view the importance
of quaternionic algebras, we establish the MHD field equations for dyonic cold plasma. Starting with
the definitions of one-fluid and two-fluid theory of plasma, we identify the cold plasma approxima-
tion where the thermal effects (or pressure effects) of conducting fluid will be neglected. Further, we
introduce the dual MHD equations of dyonic plasma consisted with electrons, magnetic monopoles
and their counter partners viz. ions and magneto-ions. In this study, we clarify that the dominating
aspect for the dyonic cold plasma approximation is the dynamics of electrons along with magnetic
monopoles. As we know that the generalized Dirac-Maxwell like equations are primary equations to
explain the dynamics of dyonic cold plasma. Therefore, undertaking the quaternionic dual-velocity
and dual-enthalpy of dyonic cold plasma, we have made an attempt to formulate the quaternionic
hydro-electric and hydro-magnetic fields equations, which are an analogous to the generalized Lamb
vector field and vorticity field of conducting dyonic fluid. The Lorenz gauge conditions for dyonic
cold plasma fluid are also obtained. Further, we derive the generalized quaternionic Dirac-Maxwell
equations to the case of dual magneto-hydrodynamics of dyonic cold plasma. We have discussed that
these Dirac-Maxwell equations for dyonic cold plasma are well invariant under the duality, Lorentz
and CPT transformations. Finally, the Alfv\äen wave like equation is established which may propagate
from the flow of magnetic monopoles in the dyonic cold plasma.

2 The quaternions

Through the extension of the set of natural numbers to the integers, a complex number \( \mathbb{C} \) is defined
by the set of all real linear combinations of the unit elements \((1, i)\), such that

\[
\mathbb{C} \leftrightarrow \{ \alpha = \alpha_1 + \alpha_2 i \mid (\alpha_1, \alpha_2 \in \mathbb{R}) \}, \quad (2.1)
\]

where the real number \( \alpha_1 \) is called the real part and \( \alpha_2 \) is called the imaginary part of a complex
number. If the real part \( \text{Re}(\alpha) = 0 \), then we can say that \( \alpha \) is purely imaginary. As such, the
Euclidean scalar product as \( \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{R} \) is then defined by

\[
\langle \alpha, \beta \rangle = \text{Re}(\alpha \cdot \beta) = \alpha_1 \beta_1 + \alpha_2 \beta_2, \quad (2.2)
\]

where \( \alpha = \alpha_1 + \alpha_2 i \) and \( \beta = \beta_1 + i \beta_2 \) are two complex numbers. The modulus of any complex number
is also defined by \( |\alpha| = \sqrt{\alpha \cdot \bar{\alpha}} = \sqrt{\alpha_1^2 + \alpha_2^2} \).
Thus the allowed four-dimensional Hamilton vector space is defined by quaternion algebra $\mathbb{H}$ over the field of real numbers $\mathbb{R}$ as

$$
\mathbb{H} \mapsto \left\{ \alpha = \sum_{j=0}^{3} e_j a_j = e_0 a_0 + e_1 a_1 + e_2 a_2 + e_3 a_3 \mid \forall a_j \in \mathbb{R} \right\}, \quad (2.3)
$$

where the Hamilton vector space ($\mathbb{H}$) has the quaternionic elements $(e_0, e_1, e_2, e_3)$, are called quaternion basis elements while $a_0$, $a_1$, $a_2$, $a_3$ are the real quaterate of a quaternion. As such the addition of two quaternions $\alpha = e_0 a_0 + e_1 a_1 + e_2 a_2 + e_3 a_3$ and $\beta = e_0 b_0 + e_1 b_1 + e_2 b_2 + e_3 b_3$ is given by

$$
\alpha + \beta = e_0 (a_0 + b_0) + e_1 (a_1 + b_1) + e_2 (a_2 + b_2) + e_3 (a_3 + b_3), \quad \forall (\alpha, \beta) \in \mathbb{H}. \quad (2.4)
$$

Here, the quaternionic addition is clearly associative and commutative. The additive identity element is defined by the zero element, i.e.,

$$
0 = e_0 0 + e_1 0 + e_2 0 + e_3 0, \quad (2.5)
$$

and the additive inverse of $\alpha \in \mathbb{H}$ is given by

$$
-\alpha = e_0 (-a_0) + e_1 (-a_1) + e_2 (-a_2) + e_3 (-a_3). \quad (2.6)
$$

Correspondingly, the product of two quaternions, i.e. $(\alpha \circ \beta) \in \mathbb{H}$ can be expressed by

$$
\alpha \circ \beta = e_0 (a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3) + e_1 (a_0 b_1 + a_1 b_0 + a_2 b_3 - a_3 b_2) + e_2 (a_0 b_2 - a_1 b_3 + a_2 b_0 + a_3 b_1) + e_3 (a_0 b_3 + a_1 b_2 - a_2 b_1 + a_3 b_0). \quad (2.7)
$$

We may notice that this quaternionic product is associative, but not commutative. The quaternionic unit elements $(e_0, e_1, e_2, e_3)$ are followed the given relations,

$$
e_0^2 = 1, \quad e_2^2 = -1, \quad e_0 e_A = e_A e_0 = e_A, \quad e_A e_B = -\delta_{AB} e_0 + f_{ABC} e_C, \quad (\forall A, B, C = 1, 2, 3) \quad (2.8)
$$

where $\delta_{AB}$ is the delta symbol and $f_{ABC}$ is the Levi Civita three-index symbol having value $f_{ABC} = +1$ for cyclic permutation, $f_{ABC} = -1$ for anti-cyclic permutation and $f_{ABC} = 0$ for any two repeated indices. Further, we also may write the following relations to quaternion basis elements

$$
[e_A, e_B] = 2 f_{ABC} e_C, \quad \{e_A, e_B\} = -2 \delta_{AB} e_0, \quad e_A (e_B e_C) = (e_A e_B) e_C, \quad (2.9)
$$

where the brackets $[,]$ and $\{ , \}$ are used respectively for commutation and the anti-commutation relations. Thus the above multiplication rules governed the ordinary dot and cross product, i.e.,

$$
\alpha \circ \beta = (a_0 b_0 - \alpha \cdot \beta, \quad a_0 b_1 + b_0 \alpha + (\alpha \times \beta)), \quad (2.10)
$$
where we take $\alpha \times \beta \neq 0$ for non-commutative product of quaternion. The quaternionic product with the scalar quantity $\xi$ is given by

$$\xi \circ \alpha = e_0 (\xi \alpha_0) + e_1 (\xi \alpha_1) + e_2 (\xi \alpha_2) + e_3 (\xi \alpha_3).$$

(2.11)

As such, the multiplication identity element can expressed by the unit elements,

$$1 = e_0 1 + e_1 0 + e_2 0 + e_3 0.$$

(2.12)

Moreover, a quaternion can also be decomposed in terms of scalar $(S(\alpha))$ and vector $(V(\alpha))$ parts as

$$S(\alpha) = \frac{1}{2} (\alpha + \bar{\alpha}),$$

(2.13)

$$V(\alpha) = \frac{1}{2} (\alpha - \bar{\alpha}),$$

(2.14)

where the quaternionic conjugate $\bar{\alpha}$ is expressed by

$$\bar{\alpha} = e_0 \alpha_0 - (e_1 \alpha_1 + e_2 \alpha_2 + e_3 \alpha_3).$$

(2.15)

The real and imaginary parts of $\alpha$ can be written as

$$Re(\mathbb{H}) = \alpha_0,$$

(2.16)

$$Im(\mathbb{H}) = \{e_1 \alpha_1 + e_2 \alpha_2 + e_3 \alpha_3 \mid \forall \alpha_j = 1, 2, 3 \in \mathbb{R}\} \subseteq \mathbb{H}.$$  

(2.17)

If $Re(\mathbb{H}) = 0$ and $\alpha \neq 0$, then $\alpha$ is said to be purely imaginary quaternions. Therefore, all quaternions with zero real is simplified as imaginary space of $\mathbb{H}$, where the imaginary space $Im(\mathbb{H}) \in \mathbb{R}^3$ is a three dimensional real vector space,

$$Im(\alpha) = (\alpha_1, \alpha_2, \alpha_3) \implies Im(\alpha)^\dagger = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}.$$  

(2.18)

Interestingly, we may write the following form of quaternion as

$$\alpha = Re(\alpha) + \sum_{j=1}^{3} e_j Im(\alpha_j).$$

(2.19)

The quaternionic Euclidean scalar product $\mathbb{H} \times \mathbb{H} \rightarrow \mathbb{R}$ can also be expressed as

$$\langle \alpha, \beta \rangle = Re (\alpha \circ \bar{\beta}) = \alpha_0 \beta_0 + \alpha_1 \beta_1 + \alpha_2 \beta_2 + \alpha_3 \beta_3.$$  

(2.20)

Like complex numbers, the modulus of quaternion $\alpha$ is then defined as

$$|\alpha| = \sqrt{\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2}.$$  

(2.21)

Since, there exists the norm $N(\alpha) = \alpha \circ \bar{\alpha}$ of a quaternion, we have a division i.e., every $\alpha$ has an inverse of a quaternion and is expressed as

$$\alpha^{-1} = \frac{\bar{\alpha}}{|\alpha|}.$$  

(2.22)

While the quaternion conjugation satisfies the following property

$$\alpha_1 \circ \alpha_2 = \overline{\alpha_1 \circ \alpha_2}.$$  

(2.23)

The norm of the quaternion is positive definite and obey the composition law

$$N (\alpha_1 \circ \alpha_2) = N (\alpha_1) \circ N (\alpha_2).$$

(2.24)

The quaternion elements are non-Abelian in nature and thus represent a non-commutative division ring. Quaternion is an important fundamental mathematical tool that appropriate for four-dimensional world.
3 Magneto-hydrodynamics of cold plasma

Let us start with the basic parameters of the plasma. As we know that the plasma exists in many more forms in nature which has a wide spread use in the science and technology. The theory of plasma is divided into three categories, namely, the microscopic theory, kinetic theory and the fluid theory. In briefly, the microscopic theory is based on the motion of all the individual particles (e.g. electrons, ions, atoms, molecules, radicals, etc). According to Klimontovich, the time evolution of the particle density \( \rho_s \rightarrow \rho_s(r, v, t) \) is expressed by

\[
\frac{\partial \rho_s}{\partial t} + v \cdot \nabla \rho_s + \frac{q_s}{m_s} (E + v \times B) \cdot \nabla \rho_s = 0 , \tag{3.1}
\]

where \( v \) is the velocity of particles, \((q_s, m_s)\) are the effective charge and mass of the \( s \)-species particles and \((E, B)\) are the electric and magnetic field produced by the microscopic particles. Besides, the collisionless kinetic theory of plasma proposed by Vlasov, which has included the Boltzmann distribution function \( f_s \backsimeq \langle \rho_s \rangle \) as,

\[
\frac{\partial f_s}{\partial t} + v \cdot \nabla f_s + \frac{q_s}{m_s} (E + v \times B) \cdot \nabla f_s = 0 . \tag{3.2}
\]

In equations (3.1) and (3.2), we may consider that the two dominating particles (i.e. electrons and ions both) constitute the dynamics of plasma, called the two-fluid theory of plasma. For the two-fluid theory of plasma, at a given position \( x \) the mass and charge densities become

\[
\rho_M(x) = m_e n_e(x) + m_i n_i(x) , \tag{3.3}
\]

\[
\rho_c(x) = q_e n_e(x) + q_i n_i(x) , \tag{3.4}
\]

and the current density becomes

\[
J = q_e n_e v_e + q_i n_i v_i . \tag{3.6}
\]

The continuity equations can be written as

\[
\frac{\partial \rho_M}{\partial t} + \nabla \cdot (\rho_M v) = 0 , \quad \text{(mass conservation law)} \tag{3.7}
\]

\[
\frac{\partial \rho_c}{\partial t} + \nabla \cdot J = 0 , \quad \text{(charge conservation law)} \tag{3.8}
\]

As such, the momentum equation for plasma fluid is expressed as [35],

\[
\rho_M \left( \frac{\partial}{\partial t} + v \cdot \nabla \right) v = (J \times B) + \rho_c E - \nabla p , \tag{3.9}
\]

where \( \nabla p \) is the pressure force introduced due to the inhomogeneity of the plasma and \((J \times B)\) is a Lorentz force per unit volume element. Now, we introduce an acceleration to the conducting fluid,

\[
\frac{\partial v}{\partial t} \rightarrow \left( \frac{\partial}{\partial t} + v \cdot \nabla \right) v , \tag{3.10}
\]
where the term \((v \cdot \nabla) v\) is used for the convective acceleration of fluid. Furthermore, the generalized Ohm’s law becomes [35]

\[
\frac{m_e m_i}{\rho M e^2} \frac{\partial \mathbf{J}}{\partial t} = \frac{m_i}{2 \rho_M e} \nabla p + E + (v \times \mathbf{B}) - \frac{m_i}{\rho_M e} (\mathbf{J} \times \mathbf{B}) - \frac{\mathbf{J}}{\sigma},
\]

(3.11)

where \(\sigma\) denotes the conductivity of fluid. One can define the Maxwell’s equations with natural unit \((\hbar = c = 1)\) as,

\[
\nabla \cdot \mathbf{E} = \rho c,
\]

(3.12)

\[
\nabla \cdot \mathbf{B} = 0,
\]

(3.13)

\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},
\]

(3.14)

\[
\nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}.
\]

(3.15)

Interestingly, if we combine together the conducting fluidic field and electromagnetic field then the relevant theory comes out called MHD. The MHD of cold plasma is an approximation theory of fluid dynamics where we neglect temperature effect and combine the electron equation with ionic equation to form a one-fluid model [39]. For the cold plasma model, many researchers [40, 41] suggested that at a given position, all particle-species (mostly ions and electrons) have comparable temperatures \((T)\), energies \((E)\) (equivalent to masses) and velocities \((v)\). It follows that the fluid velocity is identical for particle velocity. Now, we may summarize the following conditions for the cold plasma approximation, i.e.,

\[
T_e \sim T_i \text{ (neglected)}
\]

\[
\delta_e \sim \delta_i
\]

\[
v_e \sim v_i
\]

\[
\rho_e \sim \rho_i
\]

\[
\nabla p \sim 0.
\]

(3.16)

We consider that the effected behavior of electrons are comparable to the ions, while their temperatures and pressure-gradients are taken negligible in case of homogeneous cold plasmas. Thus, using approximation (3.16), the average mass and charge densities to cold plasma are expressed as

\[
\rho \mapsto \rho_M(x) \simeq m_e n_e(x) \equiv m_i n_i(x),
\]

(3.17)

\[
\rho \mapsto \rho_e(x) \simeq q_e n_e(x) \equiv q_i n_i(x).
\]

(3.18)

As such, the Navier-Stokes and Ohm’s equations become

\[
\rho \left(\frac{\partial}{\partial t} + v \cdot \nabla\right) v = \rho \mathbf{E},
\]

(3.19)

\[
\mathbf{J} = \sigma (\mathbf{E} + v \times \mathbf{B}),
\]

(3.20)

where \((\mathbf{J} \times \mathbf{B}) \sim 0\) to the case if the current is small compared to \((v \times \mathbf{B})\). The ideal MHD equations \((\rho \sim 0)\) for cold plasma may then be expressed as

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0,
\]

(3.21)

\[
\rho \left(\frac{\partial}{\partial t} + v \cdot \nabla\right) v = 0,
\]

(3.22)

\[
\nabla \times (v \times \mathbf{B}) = \frac{\partial \mathbf{B}}{\partial t},
\]

(3.23)

\[
\nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}.
\]

(3.24)
To considering wave behavior of cold particles, the cold plasma wave has temperature independent dispersion relation. If \( v_A \) is Alfven velocity, then the dispersion relation for cold plasma waves become \[ \omega^2 = \frac{\kappa^2 v_A^2}{1 + v_A^2}. \] Interestingly, the cold plasma waves propagate like as Alfven waves which are independent on temperature.

## 4 Dual MHD equations for dyonic cold plasma

The dual MHD field consists not only electrons and ions but also having the magnetic monopole and their ionic partners magneto-ions. Generally, the composition of an electron and a magnetic monopole referred a dyon. In this study, we may neglect the magneto-ionic contribution like ions to continue the dyonic cold plasma approximations. Dirac proposed the symmetrized field equations by postulating the existence of magnetic monopoles, i.e.,

\[
\begin{align*}
\nabla \cdot E &= \rho^e, \\
\nabla \cdot B &= \rho^m, \\
\nabla \times E &= -\frac{\partial B}{\partial t} - J^m, \\
\nabla \times B &= \frac{\partial E}{\partial t} + J^e.
\end{align*}
\]

In the above generalized Dirac Maxwell’s equations, \( \rho^e \) and \( \rho^m \) are the electric and magnetic charge densities while \( J^e \) and \( J^m \) are the corresponding current densities. To study the dyonic cold plasma field, there are a couple of masses and charges species in presence of dyons. Thus, the generalized dual densities (mass and charge densities) may be expressed for one-fluid theory of dyonic cold plasma as

\[
\begin{align*}
\varrho^D (\varrho^e, \varrho^m) &\rightarrow (m^e n^e + m^m n^m), \\
\rho^D (\rho^e, \rho^m) &\rightarrow (q^e n^e + q^m n^m),
\end{align*}
\]

where \( m^e, n^e, \) and \( q^m \) are defined the mass, total number and charge of magnetic monopoles, respectively. As such, we can express the center of mass velocity of dyonic fluid in cold plasma as

\[
v^D \approx \frac{1}{\varrho^D} \left( v^e m^e n^e(x) + v^m m^m n^m(x) \right),
\]

whereupon the dual current densities (electric and magnetic) are defined by

\[
J^e = q^e n^e v^e, \quad \text{and} \quad J^m = q^m n^m v^m.
\]

The conservation laws for the dynamics of dyonic cold plasma can be written as

\[
\begin{align*}
\frac{\partial \varrho^D}{\partial t} + \nabla \cdot (\varrho^D v^D) &= 0, \quad \text{(dyons mass conservation law)} \\
\frac{\partial \rho^e}{\partial t} + \nabla \cdot J^e &= 0, \quad \text{(electric charge conservation law)} \\
\frac{\partial \rho^m}{\partial t} + \nabla \cdot J^m &= 0, \quad \text{(magnetic charge conservation law)}.
\end{align*}
\]

The generalized Navier-Stokes force equation can also be exhibited in presence of magnetic monopole, i.e.,

\[
\varrho^D \left( \frac{\partial}{\partial t} + v^D \cdot \nabla \right) v^D = (J^e \times B) - (J^m \times E) + \rho^e E + \rho^m B - (\nabla p)^D,
\]
where the duality invariant Lorentz force equation for dyons is

\[ \mathbf{F}^D = \rho^e \mathbf{E} + (\mathbf{J}^e \times \mathbf{B}) + \rho^m \mathbf{B} - (\mathbf{J}^m \times \mathbf{E}) \]  

(4.13)

and the dyonic pressure gradient term \((\nabla p)^D\) takes negligible to the case of cold plasma approximation. Conditionally, if the influence of dyonic current is small then the force equation can be written as

\[ \rho^e \mathbf{E} + \rho^m \mathbf{B} \]  

(4.14)

In the same way, the Ohm’s law for the dyonic cold plasma is expressed as

\[ \mathbf{J}^e = \sigma^e (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \]  

(4.15)

\[ \mathbf{J}^m = \sigma^m (\mathbf{B} - \mathbf{v} \times \mathbf{E}) \]  

(4.16)

where \(\sigma^m\) is the magnetic conductivity. Therefore, from equations (4.15)-(4.16), we can conclude that for infinite conductivity of dyons \((\sigma^e, m \to \infty)\) the electric and magnetic field vectors constitute from the rotation of each other, i.e., \(\mathbf{E} = - (\mathbf{v} \times \mathbf{B})\), and \(\mathbf{B} = (\mathbf{v} \times \mathbf{E})\). The above classical field equations given by (4.1) to (4.16) of dyons are referred to dual MHD field equations of dyonic cold plasma.

5 Quaternionic formulation to dual fields of dyonic cold plasma

In order to write the dual MHD field equations for dyonic cold plasma, we may start with quaternionic two-velocity \((\mathbf{u}, \mathbf{v})\) and two-enthalpy \((h, k)\) of dyons for plasma fluid dynamics as

\[ \mathbb{U} (e_1, e_2, e_3, e_0) = \{ u_x, u_y, u_z, -i \frac{h}{a_0} \} \],  

(5.1)

\[ \mathbb{V} (e_1, e_2, e_3, e_0) = \{ v_x, v_y, v_z, -ia_0k \} \],  

(5.2)

where \((\mathbb{U}, \mathbb{V})\) are quaternionic variables associated with two four-velocities of electrons and magnetic monopoles of dyons and, \(a_0\) denoted the speed of particles (dyons) moving in conducting cold plasma. Here, we have taken the two-enthalpy of dyons i.e. the internal energy of dyons associated with electrons and magnetic monopoles. Like many physicists \([32, 43, 44]\), there is an analogy between the electromagnetic and hydrodynamic. Thus, we may write the analogy of two four-potentials \((A, B)\) of dyons as

\[ A \left( A, -\frac{i}{c} \phi^e \right) \quad \longrightarrow \quad \mathbb{U} \left( \mathbf{u}, -\frac{i}{a_0}h \right) \],  

(5.3)

\[ B \left( B, -ic\phi^m \right) \quad \longrightarrow \quad \mathbb{V} \left( \mathbf{v}, -ia_0k \right) \],  

(5.4)

where the vector components \(\mathbf{u} \rightarrow (u_x, u_y, u_z)\), \(\mathbf{v} \rightarrow (v_x, v_y, v_z)\) are analogous to electric and magnetic vector potentials of dyons while the scalar components \((h, k)\) are analogous to their scalar potentials. It should be notice that the role of quaternionic two-velocities of dyonic-fluid in generalized hydrodynamics of cold plasma is similar as the quaternionic two-four-potentials of dyons in generalized electrodynamics. Now, we may summarize the dyonic potentials corresponding to its fluid behavior in table-1.

| Electrodyamics case | Hydrodynamics case |
|---------------------|--------------------|
| \(A\) (electric vector potential) \(\longrightarrow\) \(\mathbf{u}\) (electric velocity of the fluid) |
| \(B\) (magnetic vector potential) \(\longrightarrow\) \(\mathbf{v}\) (magnetic velocity of the fluid) |
| \(\phi^e\) (electric scalar potential) \(\longrightarrow\) \(h\) (electric enthalpy of the fluid) |
| \(\phi^m\) (magnetic scalar potential) \(\longrightarrow\) \(k\) (magnetic enthalpy of the fluid) |

Tab. 1: Analogies between electrodynamics and hydrodynamics in presence of dyons
The unified structure of quaternionic two four-velocities \((\mathbb{W} \in \mathbb{H})\) for the generalized fields of dyonic cold plasma can be written as

\[
\mathbb{W} = \left( \mathbb{U} - \frac{i}{a_0} \mathbb{V} \right)
\]

\[
= e_1 \left( u_x - \frac{i}{a_0} u_z \right) + e_2 \left( u_y - \frac{i}{a_0} u_y \right) + e_3 \left( u_z - \frac{i}{a_0} u_z \right) - \frac{i}{a_0} e_0 (h - ia_0 k),
\]

it reduces to

\[
\mathbb{W} = \sum_{j=1}^{3} e_j w_j - \frac{i}{a_0} e_0 \Omega_0
\]

\[
= \sum_{j=1}^{3} e_j \left( u_j - \frac{i}{a_0} v_j \right) - \frac{i}{a_0} e_0 (h - ia_0 k),
\]

where \(w \to \left( u - \frac{i}{a_0} v \right)\) and \(\Omega_0 \to (h - ia_0 k)\) are dyonic fluid-velocity and dyonic enthalpy in cold plasma, respectively. Here, the scalar component \((\Omega_0)\) represents the amount of dyonic internal energy required to move one kilogram of the fluid element. Now, to formulate the quaternionic dual MHD field equations for dyonic cold plasma, it is necessary to define quaternionic space-time differential operator as

\[
\mathbb{D} = \left( \nabla, - \frac{i}{a_0} \frac{\partial}{\partial t} \right) \simeq e_1 \frac{\partial}{\partial x} + e_2 \frac{\partial}{\partial y} + e_3 \frac{\partial}{\partial z} - \frac{i}{a_0} \frac{\partial}{\partial t},
\]

its quaternionic conjugate is

\[
\mathbb{D} = \left( - \nabla, - \frac{i}{a_0} \frac{\partial}{\partial t} \right) \simeq - e_1 \frac{\partial}{\partial x} - e_2 \frac{\partial}{\partial y} - e_3 \frac{\partial}{\partial z} - \frac{i}{a_0} \frac{\partial}{\partial t}.
\]

The quaternionic product of \(\mathbb{D} \circ \mathbb{D}\) will be

\[
\mathbb{D} \circ \mathbb{D} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{a_0^2} \frac{\partial^2}{\partial t^2}
\]

\[
= \nabla^2 - \frac{1}{a_0^2} \frac{\partial^2}{\partial t^2} = \mathbb{D} \circ \mathbb{D},
\]

where \(\mathbb{D} \circ \mathbb{D}\) or \(\mathbb{D} \circ \mathbb{D}\) is defined by the D’Alembert operator \(\Box\). In order to emphasize the variation of quaternionic space-time to two four-velocities of dyonic fluid plasma, we may operate the quaternionic differential operator \(\mathbb{D}\) on generalized two four-velocities \(\mathbb{W}\) as

\[
\mathbb{D} \circ \mathbb{W} = e_1 \left\{ \left( \frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z} \right) \frac{1}{a_0^2} \frac{\partial}{\partial t} + \frac{i}{a_0} \left( \frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} \right) + \frac{\partial h}{\partial x} \right\} + e_2 \left\{ \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \frac{1}{a_0^2} \frac{\partial}{\partial t} + \frac{i}{a_0} \left( \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right) + \frac{\partial h}{\partial y} \right\} + e_3 \left\{ \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} \frac{1}{a_0^2} \frac{\partial}{\partial t} + \frac{i}{a_0} \left( \frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \right) + \frac{\partial h}{\partial z} \right\} - e_0 \left\{ \frac{\partial u_t}{\partial x} + \frac{\partial u_x}{\partial t} + \frac{1}{a_0^2} \frac{\partial}{\partial t} + \frac{i}{a_0} \left( \frac{\partial v_t}{\partial x} + \frac{\partial v_x}{\partial t} \right) + \frac{\partial h}{\partial y} \right\} \right\}.
\]
Equation (5.10) governed the following quaternionic hydrodynamics field equation for dyonic cold plasma, i.e.,
\[ \mathbb{D} \circ \mathbb{W} = \Psi \simeq e_1 \psi_1 + e_2 \psi_2 + e_3 \psi_3 + e_0 \chi, \]  
(5.11)

where \( \psi \rightarrow (\psi_1, \psi_2, \psi_3) \) and \( \chi \) are the vector and scalar fields connected to the hydrodynamics of dyonic cold plasma, respectively. Further, the unified structure of quaternionic hydrodynamics field components can be expressed as
\[ \psi_1 = \left( \nabla \times \mathbf{u} \right)_x - \frac{1}{a_0^2} \frac{\partial \mathbf{v}_x}{\partial t} - \frac{\partial k}{\partial x} + \frac{i}{a_0} \left\{ - \left( \nabla \times \mathbf{v} \right)_x - \frac{\partial \mathbf{u}_x}{\partial t} - \frac{\partial h}{\partial x} \right\}, \]  
(5.12)

\[ \psi_2 = \left( \nabla \times \mathbf{u} \right)_y - \frac{1}{a_0^2} \frac{\partial \mathbf{v}_y}{\partial t} - \frac{\partial k}{\partial y} + \frac{i}{a_0} \left\{ - \left( \nabla \times \mathbf{v} \right)_y - \frac{\partial \mathbf{u}_y}{\partial t} - \frac{\partial h}{\partial y} \right\}, \]  
(5.13)

\[ \psi_3 = \left( \nabla \times \mathbf{u} \right)_z - \frac{1}{a_0^2} \frac{\partial \mathbf{v}_z}{\partial t} - \frac{\partial k}{\partial z} + \frac{i}{a_0} \left\{ - \left( \nabla \times \mathbf{v} \right)_z - \frac{\partial \mathbf{u}_z}{\partial t} - \frac{\partial h}{\partial z} \right\}, \]  
(5.14)

\[ \chi = - \left\{ \left( \nabla \cdot \mathbf{u} + \frac{1}{a_0^2} \frac{\partial h}{\partial t} \right) - \frac{i}{a_0} \left( \nabla \cdot \mathbf{v} + \frac{\partial k}{\partial t} \right) \right\}. \]  
(5.15)

We may consider the generalized dual hydrodynamics fields namely the hydro-electric and hydro-magnetic fields of dyonic-fluid associated with the dynamics of electrons and magnetic monopoles in dyonic cold plasma. Thus, the unified fields can be rewrite as
\[ \psi_1 \leftrightarrow \left( B_x + \frac{i}{a_0} E_x \right), \]  
(5.16)

\[ \psi_2 \leftrightarrow \left( B_y + \frac{i}{a_0} E_y \right), \]  
(5.17)

\[ \psi_3 \leftrightarrow \left( B_y + \frac{i}{a_0} E_y \right), \]  
(5.18)

\[ \chi \leftrightarrow - \left( \mathcal{L} - \frac{i}{a_0} \tilde{\mathcal{L}} \right). \]  
(5.19)

The hydro-electric field vector \( \mathbf{E} \) plays as the generalized Lamb vector field and the hydro-magnetic field vector \( \mathbf{B} \) plays as the generalized vorticity field \([45, 46, 47]\) to the case of dual MHD. The generalized Lamb vector field may be used to accelerate the dyonic fluid flow while the vorticity field is its counterpart. Thus, the generalized dual fields \( \mathbf{E}, \mathbf{B} \) for dyonic fluid become,
\[ \mathbf{E} = - \nabla \times \mathbf{v} - \frac{\partial \mathbf{u}}{\partial t} - \nabla h, \]  
(5.20)

\[ \mathbf{B} = \nabla \times \mathbf{u} - \frac{1}{a_0^2} \frac{\partial \mathbf{v}}{\partial t} - \nabla k, \]  
(5.21)

and the dual Lorenz gauge conditions \( \mathcal{L}, \tilde{\mathcal{L}} \) for the continuous flow of incompressible dyonic fluid plasma are
\[ \mathcal{L} :\rightarrow \nabla \cdot \mathbf{u} + \frac{1}{a_0^2} \frac{\partial h}{\partial t} = 0, \]  
(5.22)

\[ \tilde{\mathcal{L}} :\rightarrow \nabla \cdot \mathbf{v} + \frac{\partial k}{\partial t} = 0. \]  
(5.23)

The unified quaternionic Lamb-vorticity field vector \( \Psi \) (or generalized hydro-electromagnetic field vector) for dyons can be expressed as
\[ \Psi = e_1 \left( B_x + \frac{i}{a_0} E_x \right) + e_2 \left( B_y + \frac{i}{a_0} E_y \right) + e_3 \left( B_z + \frac{i}{a_0} E_z \right). \]  
(5.24)
Now, applying the quaternionic conjugate of differential operator $\bar{D}$ to equation (5.24), we obtain

$$
\bar{D} \circ \Psi = -e_1 \left\{ (\nabla \times B)_x - \frac{1}{a_0^2} \frac{\partial E_x}{\partial t} \right\} + \frac{i}{a_0} \left\{ (\nabla \times E)_x + \frac{\partial B_x}{\partial t} \right\} \\
- e_2 \left\{ (\nabla \times B)_y - \frac{1}{a_0^2} \frac{\partial E_y}{\partial t} \right\} + \frac{i}{a_0} \left\{ (\nabla \times E)_y + \frac{\partial B_y}{\partial t} \right\} \\
- e_3 \left\{ (\nabla \times B)_z - \frac{1}{a_0^2} \frac{\partial E_z}{\partial t} \right\} + \frac{i}{a_0} \left\{ (\nabla \times E)_z + \frac{\partial B_z}{\partial t} \right\} \\
+ e_0 \left[ \nabla \cdot B + \frac{i}{a_0} \nabla \cdot E \right].
$$

Equation (5.25) shows the quaternionic space-time evaluation of generalized Lamb-vorticity fields in the incompressible fluid of dyonic cold plasma. The dynamics of dyonic cold plasma fluid can be expressed by following equation

$$
\bar{D} \circ \Psi = -S (S, \varphi) \simeq - (e_1 S_1 + e_2 S_2 + e_3 S_3 + e_0 \varphi),
$$

where $S$ is the quaternionic source for the dyonic cold plasma. Moreover, the quaternionic vector and scalar components of dyonic sources, i.e., $(S, \varphi)$ can be written as

$$
S_1 \leftrightarrow \left( \frac{\mu J^e_x}{a_0} - \frac{i e_0}{a_0} \frac{\rho^m}{\epsilon} \right),
$$

$$
S_2 \leftrightarrow \left( \frac{\mu J^e_y}{a_0} - \frac{i e_0}{a_0} \frac{\rho^m}{\epsilon} \right),
$$

$$
S_3 \leftrightarrow \left( \frac{\mu J^e_z}{a_0} - \frac{i e_0}{a_0} \frac{\rho^m}{\epsilon} \right),
$$

$$
\varphi \leftrightarrow \left( \frac{\mu \rho^m}{a_0} - \frac{i e_0}{a_0} \frac{\rho^e}{\epsilon} \right),
$$

where $(J^e, \rho^e)$ are the quaternionic electric source current and source density associated with the dynamics of hydro-electric field while $(J^m, \rho^m)$ are corresponding magnetic sources associated with the dynamics of hydro-magnetic field of dyonic fluid. Therefore, the quaternionic unified hydro-electromagnetic source for dyonic cold plasma can be expressed by

$$
S = \mu \left( e_1 J^e_x + e_2 J^e_y + e_3 J^e_z - e_0 \rho^m \right) - \frac{i}{a_0} \left( e_1 \frac{J^m_y}{\epsilon} + e_2 \frac{J^m_y}{\epsilon} + e_3 \frac{J^m_z}{\epsilon} + e_0 \frac{\rho^e}{\epsilon} \right)
$$

$$
= \left( J - \frac{i e_0}{a_0} \frac{\rho^e}{\epsilon} \right).
$$

Here, $\mathbb{J}(e_j, e_0) \rightarrow \left( \mu J^e_j - \frac{i e_0}{a_0} \frac{\rho^e_j}{\epsilon} \right)$, $\mathbb{K}(e_j, e_0) \rightarrow \left( \mu J^m_j, -\rho^m_j \right)$ are quaternionic two four-fluid sources of dyons and $(\epsilon, \mu)$ are considering the permittivity and permeability satisfy $a_0 = \frac{1}{\sqrt{\mu \epsilon}}$. Now,
equating quaternionic imaginary and real coefficients in equation (5.26), and obtain,

\[ \nabla \cdot E = \frac{\rho^e}{\epsilon}, \quad \text{Imaginary part of } e_0 \]  
(5.32)

\[ \nabla \cdot B = \mu \rho^m, \quad \text{Real part of } e_0 \]  
(5.33)

\[ (\nabla \times E)_x = -\frac{\partial B_y}{\partial t} - \frac{J^m_y}{\epsilon}, \quad \text{Imaginary part of } e_1 \]  
(5.34)

\[ (\nabla \times E)_y = -\frac{\partial B_z}{\partial t} - \frac{J^m_z}{\epsilon}, \quad \text{Imaginary part of } e_2 \]  
(5.35)

\[ (\nabla \times E)_z = -\frac{\partial B_x}{\partial t} - \frac{J^m_x}{\epsilon}, \quad \text{Imaginary part of } e_3 \]  
(5.36)

\[ (\nabla \times B)_x = \frac{1}{a_0^2} \frac{\partial E_y}{\partial t} + \mu J^e_x, \quad \text{Real part of } e_1 \]  
(5.37)

\[ (\nabla \times B)_y = \frac{1}{a_0^2} \frac{\partial E_z}{\partial t} + \mu J^e_y, \quad \text{Real part of } e_2 \]  
(5.38)

\[ (\nabla \times B)_z = \frac{1}{a_0^2} \frac{\partial E_x}{\partial t} + \mu J^e_z, \quad \text{Real part of } e_3 \]  
(5.39)

The above eight equations represent the quaternionic field equations for hydrodynamics of dyonic cold plasma. These obtained equations are primary equations for dual MHD of dyonic cold plasma, which are exactly same as the generalized Dirac-Maxwell equations given by (4.1)-(4.4). As such, we also may write the unified dual MHD field equations for dyonic cold plasma as

\[ \nabla \cdot \Psi = i \varphi, \]  
(5.40)

\[ \nabla \times \Psi = -i \frac{\partial \Psi}{a_0 \partial t} + S. \]  
(5.41)

The present quaternionic formulation describes the macroscopic cold plasma behavior. The solution of differential equations (5.40)-(5.41) provide the evolution of generalized lamb vector field and generalized vorticity field to the presence of dyonic cold plasma. Now, we may check the validity of dual MHD field equations for dyonic cold plasma in given subsections.

### 5.1 Duality invariant

Let us check the duality invariant symmetry for generalized hydro-electric and hydro-magnetic fields of dyonic cold plasma. The duality transformation defines the rotation of hydro-electric and hydro-magnetic field components in the quaternionic space such that the physics behind the quantity remains the same after the transformation is performed. Suppose, \( F^{\alpha \beta} \) and \( \mathcal{F}^{\alpha \beta} \) are the field and dual field tensor, then the duality transformation becomes [48]

\[
F^{\prime \alpha \beta} : \rightarrow F^{\alpha \beta} \cos \theta + \mathcal{F}^{\alpha \beta} \sin \theta, \\
\mathcal{F}^{\prime \alpha \beta} : \rightarrow -\mathcal{F}^{\alpha \beta} \sin \theta + F^{\alpha \beta} \cos \theta, \quad (0 \leq \theta \leq \frac{\pi}{2}).
\]  
(5.42)

Correspondingly, the quaternionic hydro-electric and hydro-magnetic fields can also transform as

\[
\begin{pmatrix}
E \\
B
\end{pmatrix} \rightarrow \mathcal{D}_{2 \times 2} \begin{pmatrix}
E \\
B
\end{pmatrix},
\]  
(5.43)
where \( \mathcal{D}_{2 \times 2} = \begin{pmatrix} \cos \theta & a_0 \sin \theta \\ -\frac{1}{a_0} \sin \theta & \cos \theta \end{pmatrix} \) is an unitary matrix called the duality transformation matrix (or simply D-matrix). For general case \( \theta = \frac{\pi}{2} \), the generalized dual fields will be transform as

\[
\begin{pmatrix} E \\ B \end{pmatrix} = \begin{pmatrix} 0 & a_0 \\ -\frac{1}{a_0} & 0 \end{pmatrix} \begin{pmatrix} E \\ B \end{pmatrix} : \implies \begin{cases} E \mapsto a_0 B \\ B \mapsto -\frac{1}{a_0} E \end{cases},
\]

(5.44)

Here, the D-matrix \( \mathcal{D}_{2 \times 2} = \begin{pmatrix} 0 & a_0 \\ -\frac{1}{a_0} & 0 \end{pmatrix} \). For quaternionic dual-velocity and dual-enthalpy of dyons fluid, the following duality transformation relations governed the streamline flow, i.e.,

\[
\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & a_0 \\ -\frac{1}{a_0} & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} : \implies \begin{cases} u \mapsto a_0 v \\ v \mapsto -\frac{1}{a_0} u \end{cases},
\]

(5.45)

\[
\begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} 0 & a_0 \\ -\frac{1}{a_0} & 0 \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} : \implies \begin{cases} h \mapsto a_0 k \\ k \mapsto -\frac{1}{a_0} h \end{cases}.
\]

(5.46)

Accordingly, the dual-current and dual-density of dyonic plasma will be transform as

\[
\begin{pmatrix} J^e \\ J^m \end{pmatrix} = \begin{pmatrix} 0 & a_0 \\ -\frac{1}{a_0} & 0 \end{pmatrix} \begin{pmatrix} J^e \\ J^m \end{pmatrix} : \implies \begin{cases} J^e \mapsto a_0 J^m \\ J^m \mapsto -\frac{1}{a_0} J^e \end{cases},
\]

(5.47)

\[
\begin{pmatrix} \rho^e \\ \rho^m \end{pmatrix} = \begin{pmatrix} 0 & a_0 \\ -\frac{1}{a_0} & 0 \end{pmatrix} \begin{pmatrix} \rho^e \\ \rho^m \end{pmatrix} : \implies \begin{cases} \rho^e \mapsto a_0 \rho^m \\ \rho^m \mapsto -\frac{1}{a_0} \rho^e \end{cases}.
\]

(5.48)

Interestingly, from relations (5.44) to (5.48), we can conclude that the generalized Dirac- Maxwell equations for dyonic fluid of cold plasma are invariant under the duality transformations and showing the highly symmetric nature in presence of dyonic fluid.

### 5.2 Lorentz invariant

Let us start with the most usual transformation \([49, 50]\) that preserves the quaternionic intervals

\[
ds^2 = dx^2 + dy^2 + dz^2 - a_0^2 dt^2,
\]

i.e.,

\[
X^\xi = \Lambda_\xi^\eta X^\eta,
\]

(5.49)

where \( X \) is any four-vector and the Lorentz transformation matrix element \( \Lambda_\xi^\eta \) is

\[
\Lambda_\xi^\eta \mapsto \begin{pmatrix} \cosh \phi & 0 & 0 & -i \sinh \phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i \sinh \phi & 0 & 0 & \cosh \phi \end{pmatrix}.
\]

(5.50)

Here \( \phi \) is the boost parameter. Using the above Lorentz transformation matrix, we may obtain the following transformation equations for quaternionic four-velocity (\( \mathcal{W} \)) of dyonic cold plasma which are an analogous to quaternionic potentials of dyons, i.e.,

\[
w'_x = \gamma (w_x - a_0 \Omega_0), \quad w'_y = w_y, \quad w'_z = w_z,
\]

\[
\Omega'_0 = \gamma (\Omega_0 - a_0 w_x),
\]

(5.51)

where

\[
cosh \phi = \frac{1}{\sqrt{1 - \tanh^2 \phi}} = \frac{1}{\sqrt{1 - a_0^2}} = \gamma,
\]

\[
sinh \phi = a_0 \gamma.
\]

(5.52)
If we consider the massive dyonic particles [51], then the transformation relations (5.51) lead to the energy-momentum transformations for dyonic cold plasma,

\[
\begin{align*}
\mathcal{P}'_x &= \gamma (\mathcal{P}_x - a_0 \mathcal{E}), \\
\mathcal{P}'_y &= \mathcal{P}_y, \\
\mathcal{P}'_z &= \mathcal{P}_z,
\end{align*}
\]

where the quaternionic four-momentum is defined by \(\mathcal{P} = (e_1, e_2, e_3, e_0) = (\mathcal{P}_x, \mathcal{P}_y, \mathcal{P}_z, \mathcal{E})\). It should be noticed that the obtained relations (5.53) are similar to the usual relativistic Lorentz energy-momentum transformation relations [49, 50], where we assume that the speed of dyons \((a_0)\) is comparable to the speed of light \((c \sim 1)\). As such, we also may establish the following transformation relations for quaternionic source current and source density, i.e.,

\[
\begin{align*}
S'_x &= \gamma (S_x - a_0 \psi), \\
S'_y &= S_y, \\
S'_z &= S_z, \\
\psi' &= \gamma (\psi - a_0 S_x),
\end{align*}
\]

Correspondingly, we obtain the Lorentz transformation relations for unified hydro-electromagnetic field of dyonic cold plasma, so that,

\[
\begin{align*}
\psi'_x &= \psi_x, \\
\psi'_y &= \gamma (\psi_y - i a_0 \psi_z), \\
\psi'_z &= \gamma (\psi_z + i a_0 \psi_y),
\end{align*}
\]

along with

\[
\begin{align*}
\frac{\partial}{\partial x}' &= \gamma \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial t} \right), \\
\frac{\partial}{\partial t}' &= \gamma \left( \frac{\partial}{\partial t} + a_0 \frac{\partial}{\partial x} \right).
\end{align*}
\]

The beauty of the transformation relations (5.54)-(5.56) is that, the generalized Dirac-Maxwell equations for dyonic fluid of cold plasma are well invariant under these Lorentz transformation.

### 5.3 CPT Invariant

In order to check the CPT invariance [52] for the dual MHD field equations of dyonic cold plasma, we may write the charge conjugation matrix \((C)\) to the case of quaternionic dual-current sources and hydro-electromagnetic fields of dyonic fluid as \(C \rightarrow \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\), where the charge conjugation transformation plays as

\[
\begin{align*}
C : \begin{pmatrix} J'^e \\ J'^m \end{pmatrix} \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} J^e \\ J^m \end{pmatrix}, \\
C : \begin{pmatrix} E' \\ B' \end{pmatrix} \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} E \\ B \end{pmatrix}.
\end{align*}
\]

Correspondingly, the parity matrix \(P \rightarrow \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\) can govern the following transformations for the dyonic fluid,

\[
\begin{align*}
P : \begin{pmatrix} J'^e \\ J'^m \end{pmatrix} \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} J^e \\ J^m \end{pmatrix}, \\
P : \begin{pmatrix} E' \\ B' \end{pmatrix} \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} E \\ B \end{pmatrix}.
\end{align*}
\]
As such, we can write the time reversal matrix, i.e. $T \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and the transformation perform as

$$T : \begin{pmatrix} J^m \\ J^e \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} J^m \\ J^e \end{pmatrix}, \quad (5.61)$$

$$T : \begin{pmatrix} E' \\ B' \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} E \\ B \end{pmatrix}. \quad (5.62)$$

The forth component of quaternionic sources can also be transform for charge conjugation, parity and time reversal as the following ways

$$C : \begin{pmatrix} \rho^e' \\ \rho^m' \end{pmatrix} \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \rho^e \\ \rho^m \end{pmatrix}, \quad (5.63)$$

$$P : \begin{pmatrix} \rho^e' \\ \rho^m' \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \rho^e \\ \rho^m \end{pmatrix}, \quad (5.64)$$

$$T : \begin{pmatrix} \rho^e' \\ \rho^m' \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \rho^e \\ \rho^m \end{pmatrix}. \quad (5.65)$$

We can summarize the quaternionic physical quantities of dual MHD fields and their changes under charge conjugation, parity inversion and time reversal given by table-2 [53, 54].

| Physical quantities | Charge conjugation ($C$) | Parity inversion ($P$) | Time reversal ($T$) |
|--------------------|------------------------|-----------------------|-------------------|
| $\partial_t$       | $\partial_t$           | $-\partial_t$         | $-\partial_t$     |
| $\nabla$            | $\nabla$               | $-\nabla$             | $\nabla$          |
| $a_0$               | $a_0$                  | $-a_0$                | $-a_0$            |
| $J^e$               | $-J^e$                 | $-J^e$                | $-J^e$            |
| $J^m$               | $-J^m$                 | $J^m$                 | $J^m$             |
| $E$                 | $-E$                   | $-E$                  | $E$               |
| $B$                 | $-B$                   | $B$                   | $-B$              |
| $\rho^e$            | $-\rho^e$              | $\rho^e$              | $\rho^e$          |
| $\rho^m$            | $-\rho^m$              | $-\rho^m$             | $-\rho^m$         |

Tab. 2: Quaternionic physical quantities and their CPT transformations

Now, we may apply the CPT transformation relations on generalized Dirac-Maxwell equations for dyonic fluid of cold plasma as [54],

$$CPT (\nabla \cdot E) T^{-1} P^{-1} C^{-1} = CPT \left( \frac{\rho^e}{\epsilon} \right) T^{-1} P^{-1} C^{-1},$$

$$CPT (\nabla \cdot B) T^{-1} P^{-1} C^{-1} = CPT (\mu \rho^m) T^{-1} P^{-1} C^{-1},$$

$$CPT (\nabla \times B) T^{-1} P^{-1} C^{-1} = CPT \left( \frac{1}{\epsilon} \frac{\partial E}{\partial t} + \mu J^e \right) T^{-1} P^{-1} C^{-1} + CPT (\mu J^e) T^{-1} P^{-1} C^{-1},$$

$$CPT (\nabla \times E) T^{-1} P^{-1} C^{-1} = CPT \left( -\frac{\partial B}{\partial t} \right) T^{-1} P^{-1} C^{-1} + CPT \left( \frac{\epsilon}{\mu} J^m \right) T^{-1} P^{-1} C^{-1}. \quad (5.66)$$

Therefore, it may conclude that the generalized Dirac-Maxwell equations for dyonic cold plasma are invariant under CPT transformations.
6 Quaternionic hydro-electromagnetic wave propagation

To establish the dual hydrodynamics wave equations for dyonic cold plasma, we can start with the following quaternionic relation,

$$\mathbb{D} \circ (\mathbb{D} \circ \Psi) = - \mathbb{D} \circ S,$$  (6.1)

where the left hand part of equation (6.1) can be written as

$$\mathbb{D} \circ (\mathbb{D} \circ \Psi) = e_1 \left\{ \left( \frac{\partial^2 B_x}{\partial x^2} - \frac{1}{a_0^2} \frac{\partial^2 B_x}{\partial t^2} \right) + \frac{i}{a_0} \left( \frac{\partial^2 E_x}{\partial x^2} - \frac{1}{a_0^2} \frac{\partial^2 E_x}{\partial t^2} \right) \right\}$$

$$+ e_2 \left\{ \left( \frac{\partial^2 B_y}{\partial y^2} - \frac{1}{a_0^2} \frac{\partial^2 B_y}{\partial t^2} \right) + \frac{i}{a_0} \left( \frac{\partial^2 E_y}{\partial y^2} - \frac{1}{a_0^2} \frac{\partial^2 E_y}{\partial t^2} \right) \right\}$$

$$+ e_3 \left\{ \left( \frac{\partial^2 B_z}{\partial z^2} - \frac{1}{a_0^2} \frac{\partial^2 B_z}{\partial t^2} \right) + \frac{i}{a_0} \left( \frac{\partial^2 E_z}{\partial z^2} - \frac{1}{a_0^2} \frac{\partial^2 E_z}{\partial t^2} \right) \right\}.$$  (6.2)

Accordingly, the right hand part of equation (6.1) can be expressed as

$$\mathbb{D} \circ S = e_1 \left\{ \mu \left( \frac{\partial J_x^e}{\partial y} - \frac{\partial J_y^e}{\partial x} \right) - \frac{1}{a_0^2 \mu} \frac{\partial}{\partial t} \right\} + \frac{i}{a_0 \epsilon} \left( \frac{\partial J_y^m}{\partial y} - \frac{\partial J_y^m}{\partial x} \right)$$

$$+ e_2 \left\{ \mu \left( \frac{\partial J_x^e}{\partial z} - \frac{\partial J_z^e}{\partial x} \right) - \frac{1}{a_0^2 \mu} \frac{\partial}{\partial t} \right\} + \frac{i}{a_0 \epsilon} \left( \frac{\partial J_z^m}{\partial z} - \frac{\partial J_z^m}{\partial x} \right)$$

$$+ e_3 \left\{ \mu \left( \frac{\partial J_x^e}{\partial x} + \frac{\partial J_y^e}{\partial y} + \frac{1}{a_0^2 \mu} \frac{\partial}{\partial t} \right) + \frac{i}{a_0 \epsilon} \left( \frac{\partial J_y^m}{\partial x} + \frac{\partial J_x^m}{\partial y} + \frac{\partial J_z^m}{\partial z} + \frac{\partial J_z^m}{\partial y} + \frac{\partial J_x^m}{\partial z} + \frac{\partial J_y^m}{\partial x} \right) \right\}.$$  (6.3)

Now, equating the real and imaginary parts of quaternionic basis vectors in equation (6.1), and obtained the following relations

$$\nabla \cdot J^e + \frac{\partial \rho^e}{\partial t} = 0,$$  (6.4)

$$\nabla \cdot J^m + \frac{1}{a_0^2} \frac{\partial \rho^m}{\partial t} = 0,$$  (6.5)

$$\nabla^2 B - \frac{1}{a_0^2} \frac{\partial^2 B}{\partial t^2} - \mu (\nabla \rho^m) - \frac{1}{a_0^2 \epsilon} \frac{\partial J^e}{\partial t} = 0,$$  (6.6)

$$\nabla^2 E - \frac{1}{a_0^2} \frac{\partial^2 E}{\partial t^2} - \frac{1}{\epsilon} (\nabla \rho^e) - \mu (\nabla \times J^e) = 0.$$  (6.7)

Equations (6.4) and (6.5) are defined the well-known dual continuity equations while equations (6.6) and (6.7) are represented the generalized hydro-magnetic and hydro-electric wave equations for dyonic cold plasma in presence of electrons and magnetic monopoles. The beauty of equation (6.6) is that, it is an analogous to Alfven wave propagation [55, 56] associated with magnetic monopoles, and the same way equation (6.7) describes the counterpart of Alfven wave propagation associated with the electrons. Thus, the unified hydro-electromagnetic wave equations for dyonic fluid of cold plasma can also be expressed as

$$\nabla^2 \Psi - \frac{1}{a_0^2} \frac{\partial^2 \Psi}{\partial t^2} - i (\nabla \psi) - \frac{i}{a_0} \frac{\partial S}{\partial t} + (\nabla \times S) = 0.$$  (6.8)

Interestingly, the generalized wave equation (6.8) is invariant under the duality, Lorentz and CPT transformations.
7 Conclusion

The dyons are high energetic soliton particles existed in the cold plasma. The cold plasma model is the simplest model where we assume negligible plasma temperature, and the corresponding distribution function shows the Dirac delta function centered at the macroscopic flow of linearized velocity. Dyonic cold plasma model can be used in the study of small amplitude electromagnetic waves propagating in the conducting plasma. In this study, we have applied the four-dimensional space-time algebra (quaternionic algebra) to elaborate the dynamics of dyonic fluid in cold plasma field. In section-2, we have explained in detail the properties of quaternionic algebra. However, the quaternion is an important and appropriate fundamental mathematical tool to understand the four-dimension space-time world. In section-3 & 4, the fundamental equations for MHD field and their cold plasma approximation have been defined. The interesting part we have mentioned here that the dual MHD equations for massive dyons consisted with electrons and magnetic monopoles. The generalized equations involving the mass and charge densities are expressed in terms of one-fluid theory of dyonic cold plasma. Accordingly, we have discussed the dual current densities given by equation (4.8). The mass conservation law, dual-charge conservation law, Lorentz force equation and Ohm’s law for dyonic cold plasma have been defined. In section-5, we have described the quaternionic formulation for moving massive dyonic fluid of incompressible cold plasma. The advantage of the quaternionic formulation is that, it is better to explain two four-velocities, hydro-electric (Lamb vector) and hydro-magnetic (vorticity) fields and the dual Lorenz gauge conditions for dyonic cold plasma. It has been emphasized that the dual hydrodynamics field of dyons (i.e., hydro-electric and hydro-magnetic fields) deal with both electro-hydrodynamic and magnetic-hydrodynamics. In present study, the existence of magnetic monopoles has been visualized to MHD field. It has been shown that the two current sources are also associated with the quaternionic hydro-electric and hydro-magnetic fields of dyonic plasma fluid. We have established the eight primary equations of dual MHD field in presence of dyonic fluid. Interestingly, the unified macroscopic Dirac-Maxwell equations (5.40), (5.41) have been obtained in the case of dyonic dual MHD. It has been noticed that like electrodynamics, the Dirac-Maxwell fluid equations are mandatory to describe the dynamics of MHD plasma. The beauty of cold plasma field equations is that, these equations are well invariant under the duality, Lorentz and CPT transformations. In section-6, we have obtained the quaternionic dual continuity equations for incompressible dyonic fluid. The generalized hydro-electric and hydro-magnetic wave equations have been established for dyonic cold plasma in presence of electrons and magnetic monopoles. It has been emphasized that, the obtained Alfvén wave like equation associated with magnetic monopoles, while the counterpart of Alfvén wave equation plays as electric-plasma waves in presence of electrons.

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