Relations Between Low-lying Quantum Wave Functions and
Solutions of the Hamilton-Jacobi Equation*

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Abstract

We discuss a new relation between the low lying Schroedinger wave function of a particle in a one-dimentional potential $V$ and the solution of the corresponding Hamilton-Jacobi equation with $-V$ as its potential. The function $V$ is $\geq 0$, and can have several minima ($V = 0$). We assume the problem to be characterized by a small anhamornicity parameter $g^{-1}$ and a much smaller quantum tunneling parameter $\epsilon$ between these different minima. Expanding either the wave function or its energy as a formal double power series in $g^{-1}$ and $\epsilon$, we show how the coefficients of $g^{-m}\epsilon^n$ in such an expansion can be expressed in terms of definite integrals, with leading order term determined by the classical solution of the Hamilton-Jacobi equation. A detailed analysis is given for the particular example of quartic potential $V = \frac{1}{2}g^2(x^2 - a^2)^2$.

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1. INTRODUCTION

We discuss a hitherto unexplored link between the low-lying Schroedinger wave function of a particle in a potential $+V$ and the solution of the corresponding Hamilton-Jacobi equation with $-V$ as its potential. In this paper, we restrict our study to the one dimensional problem. Extension to multi-dimensional space will be given in a separate publication.

Let $\Phi(x)$ be the solution of

$$H\Phi(x) = E\Phi(x), \quad (1.1)$$

where

$$H = -\frac{1}{2}\frac{d^2}{dx^2} + V(x) \quad (1.2)$$

describes the Hamiltonian of a non-relativistic particle of unit mass. Assume

$$V(x) \geq 0; \quad (1.3)$$

however, its minimum $V(x) = 0$ may occur at more than one point.

Introduce a scale factor $g^2$ by writing

$$V(x) = g^2v(x), \quad (1.4)$$

and consider the case of large $g$. We express

$$\Phi(x) = e^{-S(x)} \quad (1.5)$$

in terms of a formal expansion:

$$S(x) = gS_0(x) + S_1(x) + g^{-1}S_2(x) + g^{-2}S_3(x) + \cdots \quad (1.6)$$

and likewise its eigenvalue

$$E = gE_0 + E_1 + g^{-1}E_2 + g^{-2}E_3 + \cdots. \quad (1.7)$$
Substituting (1.5) - (1.7) into the Schroedinger equation (1.1) and equating the coefficients of $g^{-n}$ on both sides, we obtain

\begin{align*}
S_0'^2 &= 2v, \\
S_0'S_1' &= \frac{1}{2}S''_0 - E_0, \\
S_0'S_2' &= \frac{1}{2}(S''_1 - S''_1) - E_1
\end{align*}

(1.8)

eq 2v,

etc., where $S'_n = dS_n/dx$ and $S''_n = d^2S_n/dx^2$. For each $n \geq 1$, the $(n + 1)$th line of (1.8) determines $S'_n$ uniquely in terms of the $S_m$ for $m < n$. But the first line determines $S'_0$ only up to an ambiguity in sign. By continuity, the ambiguity makes itself felt only at a global maximum of $-v(x)$, that is where $v(x)$ vanishes quadratically and we must decide whether $S'_0$ should behave analytically, changing sign as it passes through the vanishing point, or whether $S'_0$ should have a kink and take the same sign on both sides of the vanishing point.

If $v(x)$ vanishes only once, say at $x = a$, our choice is clear. $S'_0$ should be positive for $x > a$ and negative for $x < a$, so that $S_0$ itself will increase on both sides as we go away from $x = a$, and the wavefunction in (1.5) will vanish at infinity in both directions. Thus we should take the "analytic" square root in the first line of (1.8).

If there are $M > 1$ values of $x$ where $v(x)$ vanishes, let us call one of them ($x = a$), where we want the wavefunction to have its major peak, the primary maximum of $-v$, and the others remote maxima. At the primary maximum $S'_0$ should behave analytically as before. But if we allow $S'_0$ to change sign also at a remote maximum, we shall find $S_0$ decreasing as we move farther on the other side, so that the wavefunction may grow to another peak competing with the primary one, or may blow up at infinity.

Thus, to have a well-behaved wavefunction with no remote peaks, it appears that $S'_0$ should change sign only at the primary maximum, and should have a kink at the remote maxima.

Further support for this choice arises from the generalization of our problem to many dimensions. The first line of (1.8) will then read $(\nabla S_0)^2 = 2v$, and at the primary maximum
it will be natural to make $\nabla S_0$ a vanishing vector with positive divergence. Elsewhere $S_0$ will be found by integrating this (Hamilton-Jacobi) equation along a family of trajectories radiating out from the primary maximum. In general (excluding a set of measure zero) these trajectories will not encounter any remote maximum. If we follow a trajectory that comes very near to a remote maximum, we shall see $-v$ rising almost to zero but not quite. Thus, $S'_0$ (differentiation along the trajectory) will not change sign, but will not quite have a kink. Passing to the limit, we see that in order to make $S_0$ continuous in all dimensions we should allow $S'_0$ to develop a kink on the singular trajectory that passes through the remote maximum.

In the present paper we shall be concerned more with the quantum mechanical wavefunction than with the Hamilton-Jacobi equation, as we wish to develop corrections to all orders in $g^{-1}$ and in $\epsilon$, the tunneling exponential. We shall find it convenient to introduce two types of wavefunction denoted respectively by $\phi$ and $\psi$. Generically, $\Phi$ refers to either $\phi$ or $\psi$.

The groundstate energy will have an $M$-fold quasi-degeneracy with a spread of $O(\epsilon)$. Correspondingly, there are $M$ eigenstates (called quasi-groundstates) of $H$. A $\phi$-function will be an exact solution of the Schroedinger equation with an energy $E$ that is not a true eigenvalue but lies within the groundstate fine structure. A $\phi$-function cannot behave well at both $+\infty$ and $-\infty$. On the other hand, a $\psi$-function is a linear combination of the true quasi-groundstate eigenfunctions. A $\psi$-function behaves well at both infinities, but is not an exact solution of the Schroedinger equation for any energy (except when $\psi$ happens to be just one of the $M$ quasi-groundstate wave functions).

In this and the following section we shall study only the leading order in $\epsilon$, and for this purpose we can restrict ourselves to the region excluding any remote maximum, i.e., to $x$ such that there is no second vanishing of $v$ between $x$ and $a$. In this region $\phi$- and $\psi$-functions are almost indistinguishable, and all the assertions in these sections will apply indifferently to both, except for the exact equations (2.9), (2.10) and (2.14) involving the Wronskian.

In section 3 ff., we shall have to go beyond the remote maxima, and for $M = 2$ at least we
shall find that for a $\phi$-function the energy $E$ can be tuned so that the Hamilton-Jacobi solution with analytic $S'_0$ remains a good approximation, whereas for a $\psi$-function the coefficients of the eigenfunctions can be tuned so that the Hamilton-Jacobi solution with a kink in $S'_0$ at the remote maximum is good.

It is convenient to write the first line of (1.8) as

$$\frac{1}{2}(dS_0/dx)^2 - v(x) = 0+,$$

which denotes the corresponding Hamilton-Jacobi equation with $0+$ as the total energy and $-v$ the potential. For its solution, we follow Hamilton’s action-integral by starting at the primary maximum of $-v(x)$, say $x = a$; i.e.,

$$-v(a) = 0.$$  \hspace{1cm} (1.10)

At this starting point, the magnitude of $S'_0$ is infinitesimal because the total energy is $0+$. However, we will allow $S'_0$ to change sign at $x = a$ in accordance with the discussion following (1.8). Thus, $S'_0$ is positive for $x > a$, and negative for $x < a$. In the limit $0+ \to 0$, $S'_0 = 0$ at $x = a$ and also at any other (degenerate) maximum of $-v(x)$, say $x = b$ and $b \neq a$. The limiting $S'_0$ becomes analytic at $x = a$, but has a kink at $x = b$. The resulting $\Phi = e^{-S}$ describes then a $\psi$-function.

Impose the "normalization" condition

$$\Phi(a) = 1$$

and, correspondingly,

$$S_0(a) = S_1(a) = S_2(a) = \cdots = 0.$$  \hspace{1cm} (1.12)

The solution of the Hamilton-Jacobi equation with the sign choices described above is

$$S_0(x) = \pm \int_a^x [2v(y) + (0+)]^{1/2} dy$$

(1.13)
with $[\cdots]^\frac{1}{2}$ denoting the principal value, the "+" sign for $x > a$ and the "−" sign for $x < a$, so that

\[ S_0(x) \geq 0 \quad \text{everywhere.} \]

As we shall see, the requirement that $S_1(x), S_2(x), \cdots$ be regular at $x = a$ determines $E_0, E_1, E_2, \cdots$.

As a first illustration, consider the simple harmonic oscillator with

\[ V(x) = \frac{g^2}{2}x^2. \tag{1.14} \]

There is only one minimum of $V(x)$ at

\[ x = a = 0. \]

Thus, (1.8) and (1.12) - (1.13) lead to

\[ S_1(x) = S_2(x) = \cdots = 0, \]
\[ E_1 = E_2 = \cdots = 0, \]

\[ S(x) = gS_0 = \frac{1}{2}gx^2 \tag{1.15} \]

and

\[ E = gE_0 = \frac{1}{2}g. \tag{1.16} \]

The solution of the Hamilton-Jacobi equation

\[ \frac{1}{2} \left( \frac{dS}{dx} \right)^2 - \frac{g^2}{2}x^2 = 0^+, \tag{1.17} \]

gives the exact groundstate Schrödinger wave function

\[ \Phi(x) = e^{-S(x)} = e^{-\frac{1}{2}gx^2}. \tag{1.18} \]
While the above approach resembles the familiar WKB method, it is different. The latter is good for excited levels, but not for the groundstate: the WKB wave function has an unnecessary "classical turning point" at \( x = \pm g^{-\frac{1}{2}} \) (even though it does give the correct eigenvalue). Note that WKB calls for introducing the energy \( E \) on the right hand side of the first line of (1.8), whereas we defer it to the second.

As a second example, we consider the quartic potential

\[
V(x) = \frac{1}{2} g^2 (x^2 - a^2)^2. \tag{1.19}
\]

The minimum \( V(x) = 0 \) is realized at two separate points

\[
x = \pm a. \tag{1.20}
\]

In this problem, in addition to the small anharmonicity parameter

\[
(ga^3)^{-1} << 1, \tag{1.21}
\]

there is a much smaller associated parameter

\[
\epsilon = e^{-\frac{4}{3}g a^3} \tag{1.22}
\]

that characterizes the tunneling between the two minima \( x = a \) and \( -a \). We shall start with the two solutions \( S_0(\pm) \) of the Hamilton-Jacobi equation (1.9),

\[
\frac{1}{2} \left[ \frac{d}{dx} S_0(\pm) \right]^2 - \frac{1}{2} (x^2 - a^2)^2 = 0 + \tag{1.23}
\]

by requiring

\[
S_0(+) = 0 \quad \text{at} \quad x = a \tag{1.24}
\]

and

\[
S_0(-) = 0 \quad \text{at} \quad x = -a.
\]
Hence,

\[ S_0(+) = \pm \int_a^x [(y^2 - a^2)^2 + (0+)]^{1/2} dy \]

and

\[ S_0(-) = \pm \int_{-a}^x [(y^2 - a^2)^2 + (0+)]^{1/2} dy \]

in accordance with (1.13) and the sign choices described below (1.10), so that \( S_0(+) \) and \( S_0(-) \) are positive everywhere. In addition, they are related to each other by

\[ x \to -x, \quad S_0(\pm) \to S_0(\mp). \] (1.26)

Let \( H \) be the Hamiltonian associated with the quartic potential (1.19):

\[ H = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} g^2 (a^2 - a^2)^2, \] (1.27)

and \( \psi_{even}, \psi_{odd} \) its even and odd ground-states with eigenvalues \( E_{even} \) and \( E_{odd} \). Write

\[ \mathcal{E} = \frac{1}{2} (E_{odd} + E_{even}), \]

\[ \Delta = \frac{1}{2} (E_{odd} - E_{even}) \] (1.28)

and

\[ \psi_{\pm}(x) = \frac{1}{2} [\psi_{even}(x) \pm \psi_{odd}(x)]. \] (1.29)

Thus,

\[ \psi_{+}(x) = \psi_{-}(-x) \] (1.30)

\[ (H - \mathcal{E})\psi_{\pm}(x) = -\Delta \psi_{\mp}(x) \] (1.31)

and

\[ (\psi_+^\prime \psi_- - \psi_-^\prime \psi_+)^\prime = 2\Delta \cdot (\psi_-^2 - \psi_+^2), \] (1.32)
where, as well as throughout the paper, the prime denotes $\frac{d}{dx}$.

We may expand $\mathcal{E}$ and $\Delta$ both formally as the following double series:

$$\mathcal{E} = ga \sum_{m,n} C_{mn} (ga^3)^{-m} e^{-\frac{4}{3} nga^3}$$  \hspace{1cm} (1.33)

and

$$\Delta = 4\left(\frac{2}{\pi} ga^3 a^5 \right)^{1/4} e^{-\frac{4}{3} ga^3} \sum_{m,n} c_{mn} (ga^3)^{-m} e^{-\frac{4}{3} nga^3}$$  \hspace{1cm} (1.34)

where $C_{mn}$ and $c_{mn}$ are numerical coefficients. The main body of this paper is to show how by using the solution (1.25) of the Hamilton-Jacobi Equation as the zeroth approximation to the Schroedinger wave function (1.5), we can obtain explicit expressions for these coefficients.

In Section 2, we derive

$$C_{00} = c_{00} = 1,$$

$$C_{10} = -\frac{1}{4}, \quad C_{20} = -\frac{9}{2^6}, \quad (1.35)$$

$$C_{30} = -\frac{89}{2^9}, \quad \text{etc.}$$

The general formulas for $C_{mn}$ and $c_{mn}$ are obtained in Section 3 and 4. Discussions for other examples of $V(x)$ are given in Section 5. In the Appendix, we analyze further the difference between a $\psi$-function and a $\phi$-function.
2. ZEROTH AND FIRST ORDERS IN $\epsilon$

We continue our analysis of the quartic potential case:

$$V(x) = \frac{1}{2}g^2(x^2 - a^2)^2.$$ 

To the zeroth order in

$$\epsilon = e^{-\frac{4}{3}ga^3},$$

$$E_{\text{even}} \cong E_{\text{odd}} \cong E.$$  \quad (2.1)

We write in the notation of (1.33),

$$E = ga \sum_{m=0}^{\infty} C_m (ga^3)^{-m}. \quad (2.2)$$

(This definition is a bit fuzzy since the series is asymptotic. In the next section we shall define $E$ precisely. For the purpose of this section, it suffices that $E \cong \frac{1}{2}(E_{\text{even}} + E_{\text{odd}})$ correct to $O(\epsilon)$ and that (2.2) follows as a consequence. We define $\phi_{\pm}$ as the exact solution of (1.1) with $\phi_+(\infty) = \phi_-(\infty) = 0$. But since $E$ is not an eigenvalue of $H$, $\phi_+(\infty)$ and $\phi_-(\infty)$ are both divergent.)

For clarity, we reserve $\psi_{\pm}$ for the exact wave functions given by (1.29), and use

$$\psi_+(x) \cong \phi_+(x) \quad \text{for} \quad x > -a,$$

$$\psi_-(x) \cong \phi_-(x) \quad \text{for} \quad x < a \quad (2.3)$$

in the approximation of neglecting $\epsilon$.

Following (1.5) and setting the generic $\Phi$ as $\phi_{\pm}$ we write

$$\phi_{\pm}(x) = e^{-S(\pm)} \quad (2.4)$$

with
\[ \phi_+(x) = \phi_-(\omega) \]
\[ \phi_+(\infty) = \phi_-(\infty) = 0 \]  \hspace{1cm} (2.5)

and
\[ \phi_+(a) = \phi_-(\omega a) = 1. \]

In accordance with (1.6) - (1.7) and writing \( S \) as \( S(\pm) \), we expand

\[ S(\pm) = g \sum_m g^{-m} S_m(\pm) \]

and

\[ E = g \sum_m g^{-m} E_m \]

where

\[ S_m(+) = 0 \text{ at } x = a \]
\[ S_m(-) = 0 \text{ at } x = -a, \]  \hspace{1cm} (2.7)

and the corresponding \( S'_m(+) \) and \( S'_m(-) \) are finite at \( x = a \) and \( x = -a \), respectively. As mentioned before (after (2.2)), we have

\[ (H - E)\phi_\pm = 0, \]  \hspace{1cm} (2.8)

which leads to

\[ (\phi'_+ \phi_- - \phi'_- \phi_+)' = 0 \]  \hspace{1cm} (2.9)

and therefore

\[ \phi'_+ \phi_- - \phi'_- \phi_+ = \lambda = \text{const}. \]  \hspace{1cm} (2.10)

By using the Hamiltonian (1.27) and the set of equations (1.8) but replacing \( S_m \) by \( S_m(\pm) \), we derive
This expression of $E$ gives the coefficients $C_{10} - C_{30}$ quoted in (1.35).

We note that for $x < -2a$, (2.11) gives a negative $S_0(\cdot)$, and as expected, when $x \to -\infty$, $S_0(\cdot) \to -\infty$ and therefore $\phi_+(\infty)$ is divergent; likewise, as $x \to +\infty$, $S_0(\cdot) \to -\infty$ and $\phi_-(-\infty)$ is divergent.

In order to evaluate $\Delta$, we return to the exact wave functions $\psi_\pm$, given by (1.29), and integrate (1.32) from $-\infty$ to $x$. Since $\psi_{even}(x)$ and $\psi_{odd}(x)$ are eigenfunctions of $H$, both are well behaved at $x = \pm \infty$, so are $\psi_\pm(x)$ of (1.29). Hence, (1.32) gives
\[
\psi_+^\prime(x)\psi_-(x) - \psi_-^\prime(x)\psi_+(x) = 2\Delta \int_{-\infty}^x [\psi_+^2(y) - \psi_-^2(y)] dy. \tag{2.14}
\]

According to (2.3) - (2.4) and (2.6), the zeroth order of the exponents in \(\psi_{\pm} = e^{-S_{\pm}}\) are determined by (2.11) provided \(x > -a + O\left(\frac{1}{\sqrt{ga}}\right)\) for \(\psi_+(x)\) and \(x < a - O\left(\frac{1}{\sqrt{ga}}\right)\) for \(\psi_-(x)\).

Take

\[-a < x < 0 \tag{2.15}\]

and also

\[x + a >> \left(\frac{1}{ga}\right)^{\frac{3}{2}}.\]

Hence, the left hand side of (2.14) is given by (2.13), while its right hand side is \(2\Delta\) times

\[
= \int_{-\infty}^x e^{-2S_{\pm}} dy + O(\epsilon) = \int_{-\infty}^x e^{-2ga(y+a)^2} dy [1 + O\left(\frac{1}{ga^3}\right) + O(\epsilon)]
\]

\[
= \left(\frac{\pi}{2ga}\right)^{\frac{3}{2}} [1 + O\left(\frac{1}{ga^3}\right) + O(\epsilon)], \tag{2.16}
\]

which leads to

\[2\Delta \cong \lambda \left(\frac{2ga}{\pi}\right)^{\frac{1}{2}},\]

and i.e.,

\[\Delta = 4ga^2\left(\frac{2ga}{\pi}\right)^{\frac{3}{2}} e^{-\frac{4}{3}ga^3} [1 + O\left(\frac{1}{ga^3}\right) + O(\epsilon)]. \tag{2.17}\]

Thus, \(c_{00} = 1\) in (1.34). By using the first line of (2.16) and expanding \(exp[-S_1(-) - g^{-1}S_2(-) - g^{-2}S_3(-) \cdots]\) as a formal series in \(g^{-1}\), we can evaluate \(c_{10}, c_{20}, \cdots\) in terms of moments of Gaussian integrals.

In the next two sections, we develop a systematic method which enables us to evaluate all other \(c_{mn}\) and \(C_{mn}\) in (1.33) - (1.34) in terms of definite integrals.
3. DEFINITIONS AND PRELIMINARIES

In this section we introduce several definitions and establish some useful formulas for the quartic potential case.

In (2.3), the $\psi_\pm(x)$ on its left hand side are defined in terms of the eigenfunction $\psi_{even}$ and $\psi_{odd}$ of the quartic-potential Hamiltonian $H$, in accordance with (1.27) - (1.29), whereas the $\phi_\pm(x)$ on its right hand side are introduced as solutions of

\[(H - E)\phi_\pm(x) = 0, \quad (3.1)\]

with $E$ between the two lowest eigenvalues $E_{even}$ and $E_{odd}$,

\[E_{odd} > E > E_{even}, \quad (3.2)\]

and $\phi_\pm$ satisfying the additional conditions

\[\phi_+(x) = \phi_-(x), \quad (3.3)\]

and \[\phi_+(\infty) = \phi_-(\infty) = 0.\]

It follows from the Sturm-Liouville Theorem that $\phi_+(x)$ and $\phi_-(x)$ must have one (and only one) zero each at a finite $x$. Let

\[\phi_+(-\alpha) = \phi_-(\alpha) = 0. \quad (3.4)\]

When $E$ decreases from $E_{odd}$ to $E_{even}$, $\alpha$ moves from $x = 0$ to $\infty$. It is convenient to choose $\alpha$ to be near $a$, with

\[|\alpha - a| \leq O\left(\frac{1}{\sqrt{ga}}\right). \quad (3.5)\]

As we shall see in Section 4, (3.5) is equivalent to constraint
\[ |E - \frac{1}{2}(E_{\text{even}} + E_{\text{odd}})| \leq O(\epsilon^2) \]  

(3.6)

where \( \epsilon \) is given by (2.1). Because of (3.1), \( (\phi'_+ \phi_+ - \phi'_- \phi_-)' = 0 \) is exact, valid to all orders of \( \epsilon = e^{-\frac{4}{3}a^3} \), and so is (2.10):

\[ \phi'_+ \phi_- - \phi'_- \phi_+ = \lambda = \text{const.} \]  

(3.7)

Likewise, the asymptotic expansion (2.13)

\[ \lambda = 8ga^2e^{-\frac{4}{2}a^3}[1 - \frac{3}{8ga^3} - \frac{53}{256(ga^3)^2} + \cdots] \]

holds at all \( x \). From (3.3) - (3.4) and (3.7), we have

\[ \phi_+(x) = \lambda \phi_-(x) \cdot \begin{cases} \int_{-\alpha}^{x} \frac{dy}{\phi_-(y)} & \text{for } x < \alpha \\ \int_{x}^{\alpha} \frac{dy}{\phi_-(y)} & \text{for } x > \alpha \end{cases} \]  

(3.8)

and

\[ \phi_-(x) = \lambda \phi_+(x) \cdot \begin{cases} \int_{x}^{\alpha} \frac{dy}{\phi_+(y)} & \text{for } x > -\alpha \\ \int_{-\infty}^{x} \frac{dy}{\phi_+(y)} & \text{for } x < -\alpha \end{cases} \]  

(3.9)

For \(|x + a| > O(\frac{1}{\sqrt{ga}})\), the exponent \( S(+) \) of

\[ \phi_+(x) = e^{-S(+)} \]  

(3.10)

has an asymptotic expansion given by (2.8) and (2.11). Through

\[ \phi_+(-x) = \phi_-(x) = e^{-S(-)} \]  

(3.11)

we have the corresponding asymptotic expansion of \( S(-) \) for \(|x - a| > O(\frac{1}{\sqrt{ga}})\). These expressions and the Schroedinger equation (3.1) lead to the previously derived asymptotic expansion (2.12) for \( E \), independent of \( \alpha \). To the zeroth order in \( \epsilon \), we note that for \( \psi_+(x) \) of (1.29), the asymptotic series is valid only for \( x > -a + O(\frac{1}{\sqrt{ga}}) \); however, as will be discussed
later, for $\phi_+(x)$ the same asymptotic series also holds for $x < -a - O\left(\frac{1}{\sqrt{gA}}\right)$ if $\alpha$ satisfies (3.5). A similar difference exists between $\psi_-(x)$ and $\phi_-(x)$ through $x \to -x$.

Different choices of $\pm \alpha$ give different $\phi_{\pm}(x)$, and therefore also different $E$. The asymptotic expansion for $E$ implies that these different $E$ become the same to the zeroth order in $\epsilon$. (It will be shown in Section 4 that actually the constraint (3.5) determines $E$ to the first order in $\epsilon$.)

Similarly, for $x > -a + O\left(\frac{1}{\sqrt{gA}}\right)$, the $\alpha$-independent asymptotic expansion (2.11) for $S(\pm)$ implies that these different $\phi_+(x)$ are also the same if we neglect the $O(\epsilon)$ correction. For

$$x < -a + O\left(\frac{1}{\sqrt{gA}}\right),$$

(3.12)

$\phi_+(x)$ is most conveniently given by (3.8). We note that for $x < a - O\left(\frac{1}{\sqrt{gA}}\right)$, which includes the above region (3.12), the asymptotic expansion of (2.11) for $S(-)$ determines $\phi_-(x)$ to the zeroth order in $\epsilon$. Thus, in the region (3.12), under $\alpha \to \alpha + \delta \alpha$, we have from (3.8)

$$\delta \phi_+(x) = \lambda [\phi_-(x)/\phi_-^2(-\alpha)] \delta \alpha,$$

(3.13)

in which $\lambda \delta \phi_- = O(\epsilon^2)$ is neglected. Since $[1/\phi_-^2(y)]$ is minimum (for $y < \alpha$) at $y = -a$, the restriction (3.5) limits the corresponding $\delta \phi_+$ to $O(\epsilon)$ or smaller. Likewise, through $x \to -x$, the corresponding $\delta \phi_-$ is also $O(\epsilon)$ or smaller. Furthermore, for $\alpha$ satisfying the restriction (3.5), in the region $x$ is less than $-a$ by $O\left(\frac{1}{\sqrt{gA}}\right)$

$$x < -a - O\left(\frac{1}{\sqrt{gA}}\right),$$

through the upper expression of (3.8) one can use the asymptotic expansion of $\phi_- = e^{-S(-)}$ to derive the asymptotic expansion of $\phi_+ = e^{-S(\pm)}$, in accordance with (2.11). A similar statement holds in which $x \to -x$ and the roles of $\phi_-$ and $\phi_+$ are reversed.

In Section 4 we shall relate the eigenfunctions $\psi_{even}$ and $\psi_{odd}$ to these $\phi_{\pm}$. For this purpose it is useful to introduce two Green’s functions (“right” and “left”):
\[
< x | G_R | y > = \frac{2}{\lambda} \begin{cases} 
-\phi_+(x)\phi_-(y) + \phi_-(x)\phi_+(y), & \text{for } y > x \\
0 & \text{for } x > y 
\end{cases}
\] (3.14)

\[
< x | G_L | y > = \frac{2}{\lambda} \begin{cases} 
0 & \text{for } y > x \\
\phi_+(x)\phi_-(y) - \phi_-(x)\phi_+(y), & \text{for } x > y 
\end{cases}
\] (3.15)

and two corresponding step functions:

\[
S_R(x - y) = \begin{cases} 
-1 & \text{if } y > x \\
0 & \text{if } x > y 
\end{cases}
\] (3.16)

and

\[
S_L(x - y) = \begin{cases} 
0 & \text{if } y > x \\
1 & \text{if } x > y 
\end{cases}
\] (3.17)

Hence,

\[
< x | G_{RL} | y > = \frac{2}{\lambda} [\phi_+(x)\phi_-(y) - \phi_-(x)\phi_+(y)] \cdot S_R(x - y).
\] (3.18)

Both \( S_R \) and \( S_L \) satisfy

\[
\frac{\partial}{\partial x} S_{RL}(x - y) = \delta(x - y).
\] (3.19)

Since

\[
[\phi_+(x)\phi_-(y) - \phi_-(x)\phi_+(y)]\delta(x - y) = 0
\]

we find

\[
\frac{\partial}{\partial x} < x | G_{RL} | y > = \frac{2}{\lambda} [\phi'_+(x)\phi_-(y) - \phi'_-(x)\phi_+(y)] S_{RL}(x - y),
\] (3.20)

\[
-\frac{1}{2} \frac{\partial^2}{\partial x^2} < x | G_{RL} | y > = \frac{2}{\lambda} \left( -\frac{1}{2} [\phi''_+(x)\phi_-(y) - \phi''_-(x)\phi_+(y)] S_{RL}(x - y) + \right)
\]

\[
+ \frac{2}{\lambda} \left( \phi'_+(x)\phi_-(y) - \phi'_-(x)\phi_+(y) \right) \delta(x - y)
\] (3.21)
and

$$(H - E) < x|G_{R(L)}|y > = -\frac{1}{\lambda} \lfloor \phi'_+(x)\phi_-(y) - \phi'_-(x)\phi_+(y) \rfloor \cdot \delta(x-y) = -\delta(x-y). \quad (3.22)$$

Note that at a fixed $x$, as $y \to \infty$

$$< x|G_R|y > \to -\frac{2}{\lambda} \phi_+(x)\phi_-(y) \to \infty \quad (3.23)$$

and as $y \to -\infty$,

$$< x|G_L|y > \to -\frac{2}{\lambda} \phi_-(x)\phi_+(y) \to \infty \quad (3.24)$$

Because of (3.7) and (3.10) - (3.11), at all $x$ we have

$$\phi_+(x)\phi_-(x) = \frac{\lambda}{S(-)' - S(+)}; \quad (3.25)$$

Furthermore, for

$$|x^2 - a^2| >> O(\sqrt{a/g}), \quad (3.26)$$

we find, by using (2.11),

$$S(-)' - S(+) = 2g(a^2 - x^2) - \frac{2a}{a^2 - x^2} - \frac{3a^4 + 6a^2x^2 - x^4}{2ga^2(a^2 - x^2)^3} + O(1/g^2). \quad (3.27)$$

Therefore, integrals such as

$$\int_{-\infty}^{\infty} < x|G_R|y > \phi_+(y)dy \quad (3.28)$$

and

$$\int_{-\infty}^{\infty} < x|G_L|y > \phi_-(y)dy \quad (3.29)$$

are convergent.

Next, we examine the equation
\[ H \phi_+ = -\frac{1}{2} \phi_+'' + V \phi_+ = E \phi_+, \]  
(3.30)

in which \( E \) does not have to be an eigenvalue of \( H \). Write, as before, \( \phi_+ = e^{-S(+)}/n \) but introduce a new function \( \theta \), defined to be

\[ \theta \equiv S(+) - \sqrt{2V}. \]  
(3.31)

Thus, \( \theta \) satisfies

\[ \frac{1}{2}(\theta' - \theta^2) - \sqrt{2V} \theta + \frac{1}{2} \frac{V'}{\sqrt{2V}} = E \]  
(3.32)

which, for the quartic potential

\[ V(x) = \frac{1}{2} g^2 (x^2 - a^2)^2, \]

becomes

\[ \frac{1}{2}(\theta' - \theta^2) - g(x^2 - a^2)\theta + gx = E. \]  
(3.33)

The boundary condition \( \phi_+(\infty) = 0 \) gives, for large \( x \),

\[ \theta = \frac{1}{x} - \frac{E}{gx^2} + \frac{a^2}{x^3} + O\left(\frac{1}{gx^4}\right). \]  
(3.34)

The asymptotic behavior of \( \phi_+(x) \) as \( x \to \infty \) is therefore insensitive to \( E \); the same holds for \( \phi_-(x) \) as \( x \to -\infty \). Consequently, the convergences of (3.28) - (3.29) also imply the convergences of

\[ \int_{-\infty}^{\infty} <x|G_{R(L)}|y> \psi_{even}(y) dy \]  
(3.35)

and

\[ \int_{-\infty}^{\infty} <x|G_{R(L)}|y> \psi_{odd}(y) dy \]  
(3.36)
According to (3.14) - (3.15), each of these Green’s functions \( G_R \) and \( G_L \) consists of two terms, \(-\phi_+(x)\phi_-(y)\) and \(\phi_-(x)\phi_+(y)\). It is useful to examine their relative magnitudes. Since \(\phi_+(-\alpha) = \phi_-(\alpha) = 0\), by using (2.11) we see that \(\phi_+(x) > 0\) for \(x > -\alpha\) and \(\phi_-(x) > 0\) for \(x < \alpha\). Because

\[
\frac{[\phi_-(x)]'}{\phi_+(x)} = -\frac{\lambda}{\phi_+^2(x)} < 0, \tag{3.37}
\]

for any \(x < y\) we have

\[
\frac{\phi_-(x)}{\phi_+(x)} > \frac{\phi_-(y)}{\phi_+(y)}. \tag{3.38}
\]

Hence for \(-\alpha < x < y\)

\[
\phi_-(x)\phi_+(y) > \phi_-(y)\phi_+(x). \tag{3.39}
\]

For

\[-\alpha < x < y < \alpha, \tag{3.40}\]

both sides of (3.39) are positive, which gives

\[
0 < \phi_-(x)\phi_+(y) - \phi_-(y)\phi_+(x) < \phi_-(x)\phi_+(y). \tag{3.41}
\]

For

\[\alpha < x < y, \tag{3.42}\]

both sides of (3.39) are negative. We have

\[
0 < -\phi_-(x)\phi_+(y) < -\phi_+(x)\phi_-(y) \tag{3.43}
\]

and therefore

\[
0 < \phi_-(x)\phi_+(y) - \phi_+(x)\phi_-(y) < -\phi_+(x)\phi_-(y). \tag{3.44}
\]
These inequalities help us to study the bounds on the integrals (3.28) - (3.29) and (3.35) - (3.36). For example, for $x$ large $>> \alpha$,

$$
\int_{-\infty}^{\infty} <x|G_{R}|y> \phi_{+}(y)dy < -2 \frac{\phi_{+}(x)}{x} \int_{x}^{\infty} \phi_{-}(y)\phi_{+}(y)dy = -2\phi_{+}(x) \int_{x}^{\infty} \frac{1}{S(-)' - S(+)' dy}. \quad (3.45)
$$

From (3.27) we find, as $x \to \infty$

$$
-2 \int_{x}^{\infty} \frac{1}{S(-)' - S(+)' dy} \to \frac{1}{gx}; \quad (3.46)
$$

i.e.,

$$
\int_{-\infty}^{\infty} <x|G_{R}|y> \phi_{+}(y)dy \to \frac{1}{gx} \phi_{+}(x). \quad (3.47)
$$

By using (3.31) and (3.34), we see that as $x \to \infty$, apart from a relative constant normalization factor between $\psi_{even(odd)}$ and $\phi_{+}$,

$$
\int_{-\infty}^{\infty} <x|G_{R}|y> \psi_{even(y)}dy \to \phi_{+}(x)/gx \quad (3.48)
$$

and also

$$
\int_{-\infty}^{\infty} <x|G_{R}|y> \psi_{odd(y)}dy \to \phi_{+}(x)/gx. \quad (3.49)
$$
4. ITERATIVE PROCEDURE

In this section, we present a systematic method to evaluate the eigenfunctions $\psi_{\text{even}}, \psi_{\text{odd}}$ and the eigenvalues $E_{\text{even}}$ and $E_{\text{odd}}$ for the quartic potential case (1.27). Write

$$E_{\text{even}} = E + \Delta_e$$

and

$$E_{\text{odd}} = E + \Delta_{od},$$

where $E$ satisfies (3.2) (and can be further tuned so that (3.3) holds). By definitions, $\psi_{\text{even}}$ and $\psi_{\text{odd}}$ satisfy

$$(H - E)\psi_{\text{even}} = \Delta_e \psi_{\text{even}}$$

and

$$(H - E)\psi_{\text{odd}} = \Delta_{od} \psi_{\text{odd}}.$$

From (3.22), we have

$$(H - E) \Delta_j \int_{-\infty}^{\infty} <x|G_R|y> \psi_j(y)dy = -\Delta_j \psi_j(x),$$

(4.3)

where $j = \text{even} (e)$ or $\text{odd} (od)$. Introduce

$$\chi_j \equiv \psi_j + \Delta_j \int_{-\infty}^{\infty} <x|G_R|y> \psi_j(y)dy,$$

(4.4)

which, on account of (4.2) - (4.3), satisfies

$$(H - E)\chi_j(x) = 0.$$  

(4.5)

This is the same as equation (3.1) for $\phi_+(x)$. Since both $\chi_j(x)$ and $\phi_+(x) \to 0$, as $x \to \infty$, we have

$$\chi_j(x) \propto \phi_+(x).$$  

(4.6)

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From (3.48) - (3.49), the proportionality constant must not be zero, and can always be taken as unity by adjusting the normalization of $\psi_j$. Therefore, we can write

$$\psi_{\text{even}}(x) = \phi_+(x) - \Delta_e \int_{-\infty}^{\infty} <x|G_R|y > \psi_{\text{even}}(y) dy$$

(4.7)

and

$$\psi_{\text{odd}}(x) = \phi_+(x) - \Delta_{od} \int_{-\infty}^{\infty} <x|G_R|y > \psi_{\text{odd}}(y) dy$$

(4.8)

Impose the boundary conditions, at $x = 0$

$$\psi'_{\text{even}}(0) = 0 \quad \text{and} \quad \psi_{\text{odd}}(0) = 0$$

(4.9)

From (4.7) - (4.8) and regarding $G_R$ as a square matrix, $\psi_{\text{even}}, \psi_{\text{odd}}, \phi_+$ as column matrices, we derive

$$\psi_{\text{even(odd)}} = (1 + \Delta_{e(od)} G_R)^{-1} \phi_+;$$

(4.10)

i.e.,

$$\psi_{\text{even}} = (1 - \Delta_e G_R + \Delta_e^2 G_R^2 - \cdots) \phi_+$$

(4.11)

$$\psi_{\text{odd}} = (1 - \Delta_{od} G_R + \Delta_{od}^2 G_R^2 - \cdots) \phi_+$$

(4.12)

Combining (4.3) with (4.11) and using $\phi'_+(0) = -\phi'_-(0)$, we find

$$1 + \frac{2\Delta_e}{\lambda} \int_{0}^{\infty} [\phi_+^2(y) + \phi_+(y)\phi_-(y)]dy$$

$$-\left(\frac{2\Delta_e}{\lambda}\right)^2 \int_{0}^{\infty} [\phi_+(y_1) + \phi_-(y_1)] dy_1 \int_{y_1}^{\infty} [\phi_-(y_1)\phi_+(y_2) - \phi_+(y_1)\phi_-(y_2)] \phi_+(y_2) dy_2$$

$$+ \frac{2\Delta_e}{\lambda} \int_{0}^{\infty} [\phi_+(y_1) + \phi_-(y_1)] dy_1 \int_{-\infty}^{\infty} dy_2 \int_{-\infty}^{\infty} dy_3 <y_1|G_R|y_2 > <y_2|G_R|y_3 > \phi_+(y_3)$$

$$+ \cdots = 0. \quad (4.13)$$
Likewise, combining (4.9) with (4.12), we derive

\[ 1 - \frac{2 \Delta_{od}}{\lambda} \int_0^\infty [\phi_+(y) - \phi_-(y)] \phi_+(y) dy \]

\[ + \left( \frac{2 \Delta_{od}}{\lambda} \right)^2 \int_0^\infty [\phi_+(y_1) - \phi_-(y_1)] dy_1 \int_y^{y_1} [\phi_-(y_1) \phi_+(y_2) - \phi_+(y_1) \phi_-(y_2)] \phi_+(y_2) dy_2 \]

\[ - \frac{2}{\lambda} \Delta_{od}^3 \int_0^\infty [\phi_+(y_1) - \phi_-(y_1)] dy_1 \int_{-\infty}^y dy_2 \int_{-\infty}^{y_2} dy_3 <y_1|G_R|y_2 >> <y_2|G_R|y_3 > \phi_+(y_3) \]

\[ + \cdots = 0 \quad (4.14) \]

From (4.13) - (4.14), \( \Delta_e \) and \( \Delta_{od} \) can be evaluated to successive orders of \( \epsilon^n \).

For a systematic analysis, we expand

\[ \frac{2 \Delta_e}{\lambda} = \delta_0(e) + \epsilon \delta_1(e) + \epsilon^2 \delta_2(e) + \cdots + \epsilon^n \delta_n(e) + \cdots \]

\[ \frac{2 \Delta_{od}}{\lambda} = \delta_0(od) + \epsilon \delta_1(od) + \epsilon^2 \delta_2(od) + \cdots + \epsilon^n \delta_n(od) + \cdots \quad (4.15) \]

Define

\[ < x|f_R|y > \equiv \frac{\lambda}{2} < x|G_R|y > \]

\[ = \begin{cases} 
\phi_-(x) \phi_+(y) - \phi_+(x) \phi_-(y), & x < y \\
0, & x > y.
\end{cases} \quad (4.16) \]

Cast (4.13) and (4.14) into the forms

\[ 1 + a_0 \left( \frac{2 \Delta_e}{\lambda} \right) + \epsilon a_1 \left( \frac{2 \Delta_e}{\lambda} \right)^2 + \epsilon^2 a_2 \left( \frac{2 \Delta_e}{\lambda} \right)^3 + \cdots + \epsilon^n a_n \left( \frac{2 \Delta_e}{\lambda} \right)^{n+1} + \cdots = 0 \quad (4.17) \]

and

\[ 1 + b_0 \left( \frac{2 \Delta_{od}}{\lambda} \right) + \epsilon b_1 \left( \frac{2 \Delta_{od}}{\lambda} \right)^2 + \epsilon^2 b_2 \left( \frac{2 \Delta_{od}}{\lambda} \right)^3 + \cdots + \epsilon^n b_n \left( \frac{2 \Delta_{od}}{\lambda} \right)^{n+1} + \cdots = 0 \quad (4.18) \]

where

\[ a_0 = \int_0^\infty [\phi_+^2(y) + \phi_-(y) \phi_+(y)] dy, \quad (4.19) \]
\[ b_0 = -\int_0^\infty [\phi^2_+(y) - \phi_-(y)\phi_+(y)]dy, \quad (4.20) \]

\[ e_{a_1} = \int_0^\infty [\phi_+(y_1) + \phi_-(y_1)]dy_1 \int_{-\infty}^{< y_1 |} < f_R | y_2 > \phi_+(y_2)dy_2, \quad (4.21) \]

\[ e_{b_1} = -\int_0^\infty [\phi_+(y_1) - \phi_-(y_1)]dy_1 \int_{-\infty}^{< y_1 |} < f_R | y_2 > \phi_+(y_2)dy_2, \quad (4.22) \]

\[ e^n a_n = \int_0^\infty [\phi_+(y_1) + \phi_-(y_1)]dy_1 \int_{-\infty}^{< y_1 |} < f_R | y_2 > \phi_+(y_2)dy_2, \quad (4.23) \]

\[ e^n b_n = -\int_0^\infty [\phi_+(y_1) - \phi_-(y_1)]dy_1 \int_{-\infty}^{< y_1 |} < f_R | y_2 > \phi_+(y_2)dy_2. \quad (4.24) \]

**Theorem**

\[ a_n = O(1) \quad \text{and} \quad b_n = O(1). \quad (4.25) \]

**Proof.** Write

\[ e^n a_n = (-)^n J(+) \quad \text{and} \quad e^n b_n = -(-)^n J(-), \quad (4.26) \]

where, in accordance with (4.23) - (4.24),

\[ J(\pm) \equiv \int_0^\infty dy_1 \int_{y_1}^\infty dy_2 \cdots \int_{y_{n-1}}^\infty dy_n \int_{y_n}^\infty I(\pm) \quad (4.27) \]

and

\[ I(\pm) = [\phi_+(y_1) \pm \phi_-(y_1)] \cdot [\phi_-(y_1)\phi_+(y_2) - \phi_+(y_1)\phi_-(y_2)] \cdots \]

\[ \cdot [\phi_-(y_m)\phi_+(y_{m+1}) - \phi_+(y_m)\phi_-(y_{m+1})] \cdots \]

\[ \cdot [\phi_-(y_n)\phi_+(y_{n+1}) - \phi_+(y_n)\phi_-(y_{n+1})] \cdot \phi_+(y_{n+1}). \quad (4.28) \]
Decompose the integration range of (4.27) into the sum

\[(0) + (1) + (2) + \cdots + (m) + \cdots (n) + (n + 1),\]

where

\[(0) : \quad \alpha < y_1 < y_2 < \cdots < y_n < y_{n+1} \quad (4.29)\]

\[(m) : \quad 0 < y_1 < y_2 < \cdots < y_m < \alpha, \quad \alpha < y_{m+1} < \cdots < y_n < y_{n+1} \quad (4.30)\]

with \(m = 1, 2, \cdots, n,\) and

\[(n + 1) : \quad 0 < y_1 < y_2 < \cdots < y_n < y_{n+1} < \alpha . \quad (4.31)\]

The integral (4.27) can be written as

\[J(\pm) = \sum_{m=0}^{n+1} J_m(\pm), \quad (4.32)\]

where

\[J_m(\pm) = \int_{(m)} \prod_{1}^{n+1} dy_i I(\pm) = \int_0^\alpha \cdots \int_{y_{m-1}}^\alpha dy_m \int_{y_{m+1}}^\infty \cdots \int_{y_{n}}^\infty I(\pm). \quad (4.33)\]

Consider first \(J_0.\) From (3.44) and (4.28), the magnitude of \(I(\pm)\) satisfies the inequality

\[|I(\pm)|_{in(0)} < (-)^n |\phi_+(y_1) \pm \phi_-(y_1)|[\phi_+(y_1)\phi_-(y_2)]|\phi_+(y_2)\phi_-(y_3)| \cdots [\phi_+(y_n)\phi_-(y_{n+1})]\phi_+(y_{n+1}). \quad (4.34)\]

Since at any \(x,\) on account of (5.23),

\[\phi_+(x)\phi_-(x) = O(\epsilon) \quad (4.35)\]

and for \(x > -\alpha,\)
\[ \phi_+^2(x) \leq O(1), \quad (4.36) \]

we have

\[ |\phi_+(y_1) \pm \phi_-(y_1)|\phi_+(y_1) \leq O(1) \quad (4.37) \]

and therefore

\[ J_{m=0}(\pm) = O(\epsilon^n). \quad (4.38) \]

Next, for any \( m > 0 \), in the region \((m)\) given by \((4.30)\), we have, on account of \((3.41)\) and \((3.44)\),

\[
|I(\pm)|_{in(m)} < (-)^{n-m}[\phi_+(y_1) \pm \phi_-(y_1)] \cdot [\phi_-(y_1)\phi_+(y_2)] \cdot [\phi_-(y_2)\phi_+(y_3)] \cdots \\
\quad [\phi_-(y_{m-1})\phi_+(y_m)] \cdot [\phi_-(y_m)\phi_+(y_{m+1}) - \phi_+(y_m)\phi_-(y_{m+1})] \cdot \\
\quad [\phi_+(y_{m+1})\phi_-(y_{m+2})] \cdots [\phi_+(y_n)\phi_-(y_{n+1})]\phi_+(y_{n+1}). \quad (4.39) 
\]

For \( 0 < y_1 < \alpha \), we have \( \phi_+(y_1) > \phi_-(y_1) > 0 \),

\[
\phi_+(y_1)\phi_-(y_1) = O(\epsilon) \\
\phi_-^2(y_1) = O(\epsilon), 
\]

and therefore

\[ [\phi_+(y_1) \pm \phi_-(y_1)]\phi_-(y_1) = O(\epsilon). \]

Along the sequence on the right hand side of \((4.39)\), we find

\[
\phi_+(y_2)\phi_-(y_2) = O(\epsilon), \cdots \\
\phi_+(y_{m-1})\phi_-(y_{m-1}) = O(\epsilon), 
\]

and since both \( \phi_+(y_m)\phi_-(y_m) \) and \( \phi_+(y_{m-1})\phi_-(y_{m-1}) \) are \( O(\epsilon) \),
\[
\phi_+(y_m)\left[\phi_-(y_m)\phi_+(y_{m+1}) - \phi_+(y_m)\phi_-(y_{m+1})\right]\phi_+(y_{m+1})
\]
\[
= \phi_+(y_m)\phi_-(y_m)\phi_+^2(y_{m+1}) - \phi_+^2(y_m)\phi_-(y_{m+1})\phi_+(y_{m+1})
\]
\[
= O(\epsilon).
\]
Likewise,
\[
\phi_-(y_{m+2})\phi_+(y_{m+2}) = O(\epsilon), \quad \ldots
\]
\[
\phi_-(y_{n+1})\phi_+(y_{n+1}) = O(\epsilon).
\]

Thus, in the region \((m)\), \(I(\pm) = O(\epsilon^n)\), so is \(J_m(\pm)\); i.e.,
\[
J_m(\pm) = O(\epsilon^n).
\]
(4.40)

Likewise, we can show \(J_{n+1}(\pm) = O(\epsilon^n)\) and that leads to (4.25). QED

Substituting (4.15) into (4.17), and setting the coefficients of different powers of \(\epsilon\) separately to be zero, we have

\[
1 + a_0 \delta_0(\epsilon) = 0, \quad (4.41)
\]
\[
a_0 \delta_1(\epsilon) + a_1 \delta_0(\epsilon)^2 = 0, \quad (4.42)
\]
\[
a_0 \delta_2(\epsilon) + 2a_1 \delta_0(\epsilon)\delta_1(\epsilon) + a_2 \delta_0^3(\epsilon) = 0, \quad etc.. \quad (4.43)
\]

Likewise,
\[
1 + b_0 \delta_0(\od) = 0, \quad (4.44)
\]
\[
b_0 \delta_1(\od) + b_1 \delta_0(\od)^2 = 0, \quad (4.45)
\]
\[
b_0 \delta_2(\od) + 2b_1 \delta_0(\od)\delta_1(\od) + a_2 \delta_0^3(\od) = 0, \quad etc.. \quad (4.46)
\]
(We recognize that, in view of (4.19) - (4.24), $a_n$ and $b_n$ also contain $\epsilon$-dependence., although each is $O(1)$ to the leading order. We likewise permit the $\delta_n$ to have implicit $\epsilon$-dependences so that they may not be fixed uniquely by (4.15). We have removed the ambiguity by requiring that when (4.15) is substituted into (4.17) - (4.18), the resulting equation should be true to each explicit power of $\epsilon$, disregarding the implicit entrance of $\epsilon$ through the $a_n$, $b_n$ and $\delta_n$. The important point is that if $\Delta_e$, $\Delta_{od}$ are given by (4.15) with (4.41) - (4.46), then (4.17) - (4.18) will follow.)

To $O(\epsilon)$,

\[ 2\Delta_e = \lambda \delta_0 = -\lambda/a_0 \]  
\[ 2\Delta_{od} = \lambda \delta_0 = -\lambda/b_0. \]  

Since

\[ \frac{\int_0^\infty \phi_-(y)\phi_+(y)dy}{\int_0^\infty \phi_+^2(y)dy} = O(\epsilon) \]  
\[ (4.48) \]

We find, from (4.19) - (4.20),

\[ a_0 = \int_0^\infty \phi_+^2(y)dy + O(\epsilon) \]
\[ b_0 = -\int_0^\infty \phi_+^2(y)dy + O(\epsilon) \]  
\[ (4.49) \]

Thus

\[ \Delta_e = -\left[ \frac{1}{2} / \int_0^\infty \phi_+^2(y)dy \right] + O(\epsilon^2) \]
\[ \Delta_{od} = +\left[ \frac{1}{2} / \int_0^\infty \phi_+^2(y)dy \right] + O(\epsilon^2) \]  
\[ (4.50) \]

which lead to, neglecting $O(\epsilon^2)$,

\[ \frac{1}{2}(E_{even} + E_{odd}) = ga - \frac{1}{4a^2} - \frac{9}{2g(2a)^5} - \frac{89}{2g^2(2a)^8} + O\left(\frac{1}{g^3a^{11}}\right), \]
\[ -\Delta_e = \Delta_{od} = \lambda/2/ \int_0^\infty \phi_+^2 dx, \]
\[ \lambda = 8ga^2e^{-\frac{4}{3}ga^3}\left[1 - \frac{3}{8ga^3} - \frac{53}{256(ga^3)^2} + O\left(\frac{1}{(ga^3)^3}\right)\right], \]

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and \( \phi_+ = e^{-S(+)} \) given by (2.6) and (2.11).

By using the asymptotic expansion of \( S(+) \) in \( \phi_+ = e^{-S(+)}, \)

\[
\phi_-(x) = \lambda \phi_+(x) \int_{-\alpha}^{x} \frac{dy}{\phi_+(y)}, \quad x > -\alpha
\]

and (4.41) - (4.46), we can evaluate the coefficients of \( (ga^3)^{-m}e^{-\frac{1}{2}nga^3} \) in the double series expansion (1.33) - (1.34) for \( E_{even} \) and \( E_{odd} \) in terms of definite integrals. In particular, the corresponding asymptotic expansion (2.12) of \( E \) is automatically agreeing with \( \frac{1}{2}(E_{even}+E_{odd}) \) to \( O(\epsilon) \). The underlying reason is: from (3.13), a variation of \( \alpha \rightarrow \alpha + \delta\alpha \) gives \( \delta\phi_/\delta\alpha \leq O(\epsilon) \), which leads to \( \leq O(\epsilon^2) \) to \( \Delta_e \) and \( \Delta_{od} \).

As remarked in Section 3, different choices of \( \pm\alpha \) within the range (3.5) give different \( E \) and \( \phi_\pm \) within \( O(\epsilon^2) \); however, they lead to the same \( \psi_{even(od)} \) and \( E_{even(od)} \).
5. REMARKS AND OTHER EXAMPLES

(1) Ansatz of trial wave functions.

In the notation of (1.1) - (1.4), let all the minima of the positive potential

\[ V(x) = g^2 v(x) \]  \hspace{1cm} (5.1)

be at points

\[ l_1, l_2, \ldots, l_N; \]  \hspace{1cm} (5.2)

i.e.,

\[ V(l_j) = 0, \]  \hspace{1cm} (5.3)

with \( j = 1, \ldots, N \). For each \( j \), there exists a solution of the Hamilton-Jacobi equation (1.9):

\[ \frac{1}{2} \left( \frac{dS_0}{dx} \right)^2 - v(x) = 0 +. \]  \hspace{1cm} (5.4)

According to (1.13), this solution can be written as

\[ S_0(j) = \pm \int_{l_j}^{x} \left[ 2v(y) + (0+) \right]^\frac{1}{2} dy, \]  \hspace{1cm} (5.5)

with the \( \pm \) sign chosen to make

\[ S_0(j) \geq 0 \hspace{1cm} \text{everywhere.} \]  \hspace{1cm} (5.6)

These \( N \) functions are all linearly independent and can be used as a set of trial functions. In the neighborhood of \( x = l_j \), the corresponding \( \phi(j) \) describes the correct harmonic oscillator ground state behavior. The value of \( \phi(j) \) at a different minimum \( x = l_i \) \((i \neq j)\) gives the
barrier penetration amplitude from \( j \) to \( i \), similar to the instanton description\(^{[1,2]} \). Therefore, the set \((5.7)\) forms a convenient ansatz of trial wave functions for the \( N \) low-lying eigenstates of the Hamiltonian \((1.2)\). In the case when \( V(x) \) is periodic, from this ansatz we can construct the lowest energy band structure of the problem.

(2) The systematic procedure developed in Section 4 can be extended to a much larger class of potentials with degenerate minima. The limitation depends on the applicability of equations similar to \((4.7) - (4.8)\). Here we give two examples that illustrate this limitation.

Consider first the special potential

\[
V(x) = u[-\delta(x-l) - \delta(x+l) + q\delta(x)],
\]

where \( l, u \) and \( q \) are positive constants with

\[
q \leq 1,
\]

so that

\[
\int_{-\infty}^{\infty} V(x)dx \leq -u.
\]

The ground state of \( H = -\frac{1}{2} \left( \frac{d^2}{dx^2} \right) + V(x) \) is

\[
\psi_{even}(x) = \psi_{even}(-x) = \begin{cases} 
  e^{-\kappa(x-l)}, & x > l \\
  (1 - \frac{u}{\kappa})e^{-\kappa(x-l)} + \frac{u}{\kappa}e^{\kappa(x-l)}, & 0 < x < l
\end{cases}
\]

with its eigenvalue

\[
E_{ev} = -\frac{1}{2} \kappa^2.
\]

It can be readily shown that

\[
\kappa = u[1 + \frac{\kappa - qu}{\kappa + qu} e^{-2\kappa l}].
\]
Thus, for $q \leq 1$, $\kappa$ is $> u$, consistent with (5.10). If we neglect the barrier penetration factor $e^{-2\kappa l}$ then $\kappa \cong u$, as would be the case for a single $-u\delta(x - l)$ potential.

Introduce $\phi_+(x)$ for this problem as the solution of

$$(H - E)\phi_+(x) = 0, \quad (5.14)$$

with $\phi_+(\infty) = 0$, $\phi_+(l) = 1$ and

$$E = -\frac{1}{2}u^2. \quad (5.15)$$

We have

$$\phi_+(x) = \begin{cases} 
  e^{-u(x-l)}, & x > l \\
  e^{u(x-l)}, & 0 < x < l \\
  qe^{-u(x+l)} + (1 - q)e^{u(x-l)}, & -l < x < 0 \\
  (q + 2(1 - q)e^{-2ul})e^{u(x+l)} + (-1 + q)e^{-2ul-u(x+l)}, & x < -l. 
\end{cases} \quad (5.16)$$

Construct the Green’s function $G_R$ in terms of $\phi_+(x)$ and $\phi_-(x) = \phi_+(-x)$:

$$<x|G_R|y> = \frac{2}{\lambda} \begin{cases} 
  -\phi_+(x)\phi_-(y) + \phi_-(x)\phi_+(y), & \text{for } x < y \\
  0, & \text{for } x > y 
\end{cases} \quad (5.17)$$

where, similar to (3.7)

$$\lambda = \phi'_+(x)\phi_-(x) - \phi'_-(x)\phi_+(x) = 2u(1 - q)e^{-2ul}. \quad (5.18)$$

Consequently, as in (3.22)

$$(H - E) <x|G_R|y> = -\delta(x - y). \quad (5.19)$$

Because $E_{ev} < E$, the integral

$$\int_{-\infty}^{\infty} <x|G_R|y > \psi_{even}(y)dy \quad (5.20)$$
is convergent. Therefore, as in (4.5),
\[(H - E)[\psi_{even} + \Delta_e \int_{-\infty}^{\infty} <x|G_R|y > \psi_{even}(y)dy] = 0,\]  
(5.21)
where \(\Delta_e = E_{ev} - E\). However, unlike (4.7) - (4.8), we find that
\[\chi_{even} \equiv \psi_{even} + \Delta_e \int_{-\infty}^{\infty} <x|G_R|y > \psi_{even}(y)dy\]  
(5.22)
satisfies
\[\chi_{even} = 0.\]  
(5.23)

For the quartic potential, instead of (5.23), we have (4.6), which can be written as
\[\psi_{even}(x) = \phi_+ - \Delta_e \int_{-\infty}^{\infty} <x|G_R|y > \psi_{even}(y)dy\]  
(5.24)
in accordance with (4.7); furthermore
\[\psi_{even}(x) \to \phi_+ \quad \text{as} \quad x \to \infty.\]  
(5.25)

Recall that equations (3.30) - (3.32) are applicable for any \(V(x)\). Assume
\[V(x) \to \frac{g^2}{2}x^n, \quad \text{as} \quad x \to \infty;\]  
(5.26)
then instead of (3.34), we have for large \(x\)
\[\theta = \frac{n}{4} \frac{1}{x} - \frac{E}{gx^{n/2}} + O\left(\frac{1}{x^2} \text{or} \frac{1}{x^{2+n/2}}\right).\]  
(5.27)

For \(n > 2\), \(\theta\) is insensitive to \(E\). The marginal case is \(n = 2\); in that case (5.24) remains valid, but both terms on its right hand side are important as \(x \to \infty\). For the potential (5.8) of the above example, we have \(n = 0\); therefore (5.24) is not valid, being replaced by (5.22) - (5.23).

(3) We will now examine the marginal case \(n = 2\). Assume
\[
V(x) = g^2 v(x) = \begin{cases} 
\frac{1}{2} g^2 (x - l)^2 & \text{for } x > 0 \\
\Lambda \delta(x) & \\
\frac{1}{2} g^2 (x + l)^2 & \text{for } x < 0.
\end{cases}
\]  

(5.28)

Let \( \phi_+ \) be the solution of

\[
H \phi_+ = -\frac{1}{2} \phi_+''(x) + V(x) \phi_+(x) = E \phi_+(x).
\]  

(5.29)

Write

\[
\phi_+(x) = e^{-S(\cdot)}
\]  

(5.30)

with \( \phi_+(\infty) = 0 \), and expand, as in (1.6) - (1.7),

\[
S(\cdot) = g S_0(\cdot) + S_1(\cdot) + g^{-1} S_2(\cdot) + \cdots
\]  

(5.31)

\[
E = g E_0 + E_1 + g^{-1} E_2 + \cdots
\]  

(5.32)

From (5.29), we have

\[
- S'(\cdot)^2 + S''(\cdot) + 2 g^2 v(x) = 2E.
\]  

(5.33)

Substituting (5.31) and (5.32) into (5.33), and setting the coefficients of different powers of \( g \) to be zero, we derive

\[
S_0(\cdot)' = (x - l),
\]

which, together with the “normalization” condition \( \phi_+(l) = 1 \), gives

\[
S_0(\cdot) = \frac{1}{2} (x - l)^2;
\]

\[
E_0 = \frac{1}{2}, \quad E_1 = E_2 = \cdots = 0
\]

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and

\[ S_1(+) = S_2(+) = \cdots = 0. \]

Thus,

\[ \phi_+(x) = e^{-\frac{1}{2}g(x-l)^2} \quad \text{for} \quad x > 0. \]  

(5.34)

At \( x = 0^+ \),

\[ \phi_+(0+) = e^{-\frac{1}{2}gl^2} \quad \text{and} \quad \phi'_+(0+) = gle^{-\frac{1}{2}gl^2}. \]  

(5.35)

On account of the \( \Lambda \delta(x) \) term in \( V(x) \), at \( x = 0^- \) we have

\[ \phi_+(0-) = e^{-\frac{1}{2}gl^2} \quad \text{and} \quad \phi'_+(0-) = -2\Lambda \phi_+(0) + \phi'_+(0+) = (-2\Lambda + gl)e^{-\frac{1}{2}gl^2}. \]  

(5.36)

For \( x < 0 \), \( \phi_+(x) \) satisfies the second order differential equation (5.29)

\[ -\frac{1}{2} \phi''_+ + \frac{1}{2} g^2(x+l)^2 \phi_+ = \frac{1}{2} g \phi_+, \]  

(5.37)

whose general solution is

\[ \phi_+(x) = a\chi(x) + be^{-\frac{1}{2}g(x+l)^2} \]  

(5.38)

where \( a \) and \( b \) are constants and

\[ \chi(x) = e^{-\frac{1}{2}(x+l)^2} \int_0^x e^{g(y+l)^2} dy. \]  

(5.39)

Because of the boundary conditions (5.36),

\[ a = 2[gl - \Lambda]e^{-g^2} \quad \text{and} \quad b = 1. \]  

(5.40)

Hence, for \( x < 0 \)
\[ \phi_+(x) = e^{-\frac{1}{2}g(x+l)^2} [1 + a \int_0^x e^{g(y+l)^2} dy]. \] (5.41)

Introduce, as before,

\[ \phi_-(x) = \phi_+(-x). \] (5.42)

we have, for \( x > 0 \),

\[ \phi_-(x) = e^{-\frac{1}{2}g(x-l)^2} [1 - a \int_0^x e^{g(y-l)^2} dy]. \] (5.43)

When \( \Lambda = gl \), we have \( a = 0 \) and the exact groundstate wave function given by

\[
\begin{cases}
  e^{-\frac{1}{2}g(x-l)^2}, & x > 0 \\
  e^{-\frac{1}{2}g(x+l)^2}, & x < 0
\end{cases}
\] (5.44)

the corresponding groundstate energy is

\[ \frac{1}{2}g. \] (5.45)

For \( \Lambda \neq gl \), the Hamiltonian \( H \) can be written as

\[ H = H_0 + h, \] (5.46)

where

\[ H_0 = -\frac{1}{2} \frac{d^2}{dx^2} + \begin{cases}
  \frac{1}{2}g^2(x-l)^2, & x > 0 \\
  gl\delta(x) \\
  \frac{1}{2}g^2(x+l)^2, & x < 0
\end{cases} \] (5.47)

and

\[ h = (\Lambda - gl)\delta(x). \] (5.48)

Regarding \( h \) as the perturbation, to the first order in \( h \), the change of energy is given by the expectation value of \( h \), with (5.44) as the unperturbed wave function; i.e.,
\[ E_{\text{even}} \cong \frac{1}{2} g + \frac{(\Lambda - gl)e^{-g^2}}{2 \int_0^\infty e^{-g(x-l)^2} dx}. \quad (5.49) \]

We now turn to an alternative derivation of \( E_{\text{even}} \) by using (4.7) and (4.9). Let \( \phi_+(x) \) and \( \phi_-(x) \) be the same functions that satisfy (5.29)-(5.43). From (5.34) and (5.43), it follows that for \( x \) and \( y \) both positive definite,

\[ \phi_+(x)\phi_-(y) - \phi_-(x)\phi_+(y) = a\phi_+(x)\phi_+(y)[f(x) - f(y)], \quad (5.50) \]

where

\[ f(x) = \int_0^x e^{g(z-l)^2} \, dz, \quad (5.51) \]

and

\[ \phi'_+(x)\phi_-(y) - \phi'_-(x)\phi_+(y) = -g(x-l)[\phi_+(x)\phi_-(y) - \phi_-(x)\phi_+(y)] + \frac{\phi_+(y)}{\phi_+(x)} \phi'_+(x) \phi'(y) \quad (5.52) \]

which, at \( x = y \), becomes

\[ \phi'_+(x)\phi_-(x) - \phi'_-(x)\phi_+(x) = a. \quad (5.53) \]

As in (5.24), we write

\[ \psi_{\text{even}}(x) = \phi_+(x) - \Delta e \int_{-\infty}^\infty < x | G_R | y > \psi_{\text{even}}(y) \, dy, \quad (5.54) \]

where

\[ E_{\text{even}} = \frac{g}{2} + \Delta e \quad (5.55) \]

and

\[ < x | G_R | y > = \begin{cases} \frac{2a}{g}[\phi_-(x)\phi_+(y) - \phi_+(x)\phi_-(y)] & , \quad x < y \\ 0 & , \quad x > y \end{cases} \quad (5.56) \]
For \( y > x \), by using (5.52) - (5.53) we see that as \( x \to 0^+ \)

\[
\frac{\partial}{\partial x} < x | G_R | y > \to gl < 0 | G_R | y > -2[\phi_+(y)/\phi_+(0+)].
\]  

(5.57)

The function \( \psi_{\text{even}}(x) \) and \( \phi_+(x) \) are continuous at \( x = 0 \), but not their derivatives. We have

\[
\phi_+(0) = e^{-\frac{1}{2}gL^2}
\]  

(5.58)

\[
\psi_{\text{even}}(0) = e^{-\frac{1}{2}gL^2} - \Delta e \int_{-\infty}^{\infty} < 0 | G_R | y > \psi_{\text{even}}(y) dy
\]  

(5.59)

\[
\phi^\prime_+(0+) = g e^{-\frac{1}{2}gL^2}
\]  

(5.60)

and

\[
\psi^\prime_{\text{even}}(0+) = g l \psi_{\text{even}}(0) + 2\Delta e e^{\frac{1}{2}gL^2} \int_{0}^{\infty} \phi_+(x) \psi_{\text{even}}(x) dx.
\]  

(5.61)

From the Schroedinger equation with \( V(x) \) given by (5.28), the groundstate wave function must satisfy

\[
\frac{\psi^\prime_{\text{even}}(0+)}{\psi_{\text{even}}(0)} = -\frac{\psi^\prime_{\text{even}}(0-)}{\psi_{\text{even}}(0)} = \Lambda.
\]  

(5.62)

Substituting (5.61) into (5.62), we derive

\[
\Lambda - gl = \frac{2\Delta e}{\psi_{\text{even}}(0)} e^{\frac{1}{2}gL^2} \int_{0}^{\infty} \phi_+(x) \psi_{\text{even}}(x) dx.
\]  

(5.63)

To derive \( \Delta e \) to \( O(e^{-gL^2}) \), we need only the zeroth order approximation in the wave function

\[
\psi_{\text{even}} \simeq \phi_+
\]  

(5.64)

which gives

\[
\Delta e = \frac{(\Lambda - gl)e^{-gL^2}}{2 \int_{0}^{\infty} \phi^2_+(x) dx} + O(e^{-2gL^2})
\]  

(5.65)

in agreement with (5.49).
To derive the next order correction, we replace (5.64) by

$$\psi_{\text{even}}(x) \cong \phi_{+}(x) - \Delta_e \int_{-\infty}^{\infty} <x|G_R|y > \phi_{+}(y)dy. \tag{5.66}$$

From (5.50), (5.53) and (5.56), we have for $y > x > 0$,

$$<x|G_R|y > = 2\phi_{+}(x)\phi_{+}(y)\int_{x}^{y} \phi_{+}^{-2}(z)dz. \tag{5.67}$$

Hence, the approximation (5.66) leads to

$$\psi_{\text{even}}(0) \cong \phi_{+}(0)[1 - 2\Delta_e \int_{0}^{\infty} \phi_{+}^{2}(y)dy \int_{0}^{\infty} \phi_{+}^{-2}(z)dz], \tag{5.68}$$

$$\psi_{\text{even}}^{-1}(0) \cong \phi_{+}^{-1}(0)[1 + 2\Delta_e \int_{0}^{\infty} \phi_{+}^{2}(y)dy \int_{0}^{\infty} \phi_{+}^{-2}(z)dz] \tag{5.69}$$

and

$$\int_{0}^{\infty} \phi_{+}(x)\psi_{\text{even}}(x)dx \cong \int_{0}^{\infty} dx\phi_{+}(x)[\phi_{+}(x) - \Delta_e \int_{x}^{\infty} <x|G_R|y > \phi_{+}(y)dy]$$

$$= \int_{0}^{\infty} dy\phi_{+}^{2}(y)[1 - 2\Delta_e \int_{0}^{\infty} \phi_{+}^{2}(x)F(x)dx \int_{0}^{\infty} \phi_{+}^{-2}(z)dz], \tag{5.70}$$

where

$$F(x) = \int_{x}^{\infty} \phi_{+}^{2}(y)dy \int_{x}^{\infty} \phi_{+}^{-2}(z)dz. \tag{5.71}$$

Thus, (5.69) can also be written as

$$\psi_{\text{even}}^{-1}(0) \cong \phi_{+}^{-1}(0)[1 + 2\Delta_e F(0)] \tag{5.72}$$

which, together with (5.70), gives

$$\frac{1}{\psi_{\text{even}}(0)} \int_{0}^{\infty} \phi_{+}(x)\psi_{\text{even}}(x)dx \cong \frac{1}{\phi_{+}(0)} \int_{0}^{\infty} \phi_{+}^{2}(y)dy$$

$$\cdot \{1 - \frac{2\Delta_e}{\int_{0}^{\infty} \phi_{+}^{2}(z)dz} \int_{0}^{\infty} \phi_{+}^{2}(x)[F(x) - F(0)]dx\}. \tag{5.73}$$
Because the potential $V(x) \to \frac{1}{2} g^2 x^2$ as $x \to \infty$, which is the marginal case $n = 2$ in accordance with (5.27), the corresponding integral $F(x)$ given by (5.74) has a logarithmic divergence as the range of the $y$-integration $\to \infty$. However, as we shall see, the difference $F(x) - F(0)$ and, therefore, $\Delta_e$ are well-defined.

We modify (5.74) by introducing a regulator $(1 + \lambda y)^{-1}$ in its integrand. Let

$$F_\lambda(x) \equiv \int_x^\infty (1 + \lambda y)^{-1} \phi_+^2(y) dy \int_x^y \phi_-^2(z) dz$$

(5.74)

and define

$$F(x) - F(0) = \lim_{\lambda \to 0^+} [F_\lambda(x) - F_\lambda(0)].$$

(5.75)

In (5.74), the $z$-integration is

$$\int_x^y \phi_-^2(z) dz = \int_x^y e^{g(z-l)^2} dz$$

$$= \frac{e^{g(y-l)^2}}{2g(y-l)} - \frac{e^{g(x-l)^2}}{2g(x-l)} + \int_x^y \frac{e^{g(z-l)^2}}{2g(z-l)^2} dz.$$ (5.76)

The subsequent $y$-integration for the first term on the right-hand side of (5.76) leads to:

$$\int_x^\infty (1 + \lambda y)^{-1} \phi_+^2(y) \frac{e^{g(y-l)^2}}{2g(y-l)} dy$$

$$= -\frac{1}{2g(1 + \lambda l)} \left[ \ln \lambda + \ln \frac{|l - x|}{1 + \lambda x} \right].$$ (5.77)

In the difference $F_\lambda(x) - F_\lambda(0)$, the above $\ln \lambda$ term cancels. The limit $\lambda \to 0^+$ gives a finite $F(x) - F(0)$, therefore, a finite ratio

$$\frac{1}{\psi_{even}(0)} \int_0^\infty \phi_+(x) \psi_{even}(x) dx$$

(5.78)

in accordance with (5.73). The substitution of (5.73) into (5.63) leads to a well-defined quadratic equation for $\Delta_e$; its solution determines $\Delta_e$ to the $O(e^{-2gl^2})$. 

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APPENDIX

In this Appendix, we discuss further the difference between a $\psi$-function and a $\phi$-function, introduced in Section 1 (between (1.8) and (1.9). For clarity, we return to the quartic potential case in which Hamiltonian $H$ is given by (1.27) and its even and odd groundstate wave functions are $\psi_{even}$ and $\psi_{odd}$. As in (1.29),

$$\psi_{\pm}(x) = \frac{1}{2}[\psi_{even}(x) \pm \psi_{odd}(x)]. \quad (A1)$$

Let $\phi_{\pm}(x)$ and $E$ satisfy (3.1) - (3.6). At $x = \pm \infty$, $\psi_{\pm}(x)$ are well-behaved, with

$$\psi_{\pm}(\infty) = \psi_{\pm}(-\infty) = 0. \quad (A2)$$

However, since

$$(H - E)\phi_{\pm}(x) = 0, \quad (A3)$$

with $E \neq$ an eigenvalue of $H$, only

$$\phi_{+}(\infty) = \phi_{-}(-\infty) = 0, \quad (A4)$$

but $\phi_{+}(-\infty)$ and $\phi_{-}(\infty)$ are divergent.

From (4.1) - (4.5), we express (4.6), $\chi_{j}(x) \propto \phi_{\pm}(x)$, as

$$\chi_{j}(x) = c_{j}\phi_{\pm}(x), \quad (A5)$$

where $c_{j}$ are constants and, as before, $j = even(e)$ or odd(od). Thus, (4.7) and (4.8) now become

$$\psi_{even}(x) = c_{e}\phi_{+}(x) - \Delta_{e} \int_{-\infty}^{\infty} <x|G_{R}|y> \psi_{even}(y)dy \quad (A6)$$

and
\[ \psi_{\text{odd}}(x) = c_{\text{od}} \phi_{+}(x) - \Delta_{\text{od}} \int_{-\infty}^{\infty} < x | G_{R} | y > \psi_{\text{odd}}(y) dy. \quad (A7) \]

Through \( x \to -x \), it follows that

\[ \psi_{\text{even}}(x) = c_{e} \phi_{-}(x) - \Delta_{e} \int_{-\infty}^{\infty} < x | G_{L} | y > \psi_{\text{even}}(y) dy \quad (A8) \]

and

\[ \psi_{\text{odd}}(x) = -c_{\text{od}} \phi_{-}(x) - \Delta_{\text{od}} \int_{-\infty}^{\infty} < x | G_{L} | y > \psi_{\text{odd}}(y) dy. \quad (A9) \]

Combining (A1) with (A6) - (A7) and using the matrix notation of (4.10), we derive

\[ \psi_{+} = c_{+} \phi_{+} - G_{R} [\Delta_{+} \psi_{+} + \Delta_{-} \psi_{-}] \quad (A10) \]

and

\[ \psi_{-} = c_{-} \phi_{+} - G_{R} [\Delta_{-} \psi_{+} + \Delta_{+} \psi_{-}] \quad (A11) \]

where

\[ c_{\pm} = \frac{1}{2} (c_{e} \pm c_{\text{od}}) \]

and

\[ \Delta_{\pm} = \frac{1}{2} (\Delta_{e} \pm \Delta_{\text{od}}). \quad (A12) \]

Likewise, from (A8) - (A9)

\[ \psi_{+} = c_{-} \phi_{-} - G_{L} [\Delta_{+} \psi_{+} + \Delta_{-} \psi_{-}] \quad (A13) \]

and

\[ \psi_{-} = c_{+} \phi_{-} - G_{L} [\Delta_{-} \psi_{+} + \Delta_{+} \psi_{-}] \quad (A14) \]

On account of (3.48) - (3.49) and the corresponding formulas, replacing \( G_{R} \) by \( G_{L} \), we find
\[ \begin{align*}
\text{as } x \to \infty, \quad & \psi_+(x) \to c_+ \phi_+(x) \to 0 \quad (A15) \\
& \psi_-(x) \to c_- \phi_-(x) \to 0 \quad (A16)
\end{align*} \]

and

\[ \begin{align*}
\text{as } x \to -\infty, \quad & \psi_+(x) \to c_- \phi_-(x) \to 0 \quad (A17) \\
& \psi_-(x) \to c_+ \phi_+(x) \to 0 \quad (A18)
\end{align*} \]

in agreement with (A2). As we shall see, it is convenient to choose

\[ c_+ = 1 \quad \text{and} \quad c_- = \epsilon = e^{-\frac{4}{3}g a^3}. \quad (A19) \]

Thus,

\[ \psi_+(x) \to \begin{cases} 
\phi_+(x) & \text{as } x \to \infty \\
\epsilon \phi_-(x) & \text{as } x \to -\infty 
\end{cases} \quad (A20) \]

and

\[ \psi_-(x) \to \begin{cases} 
\epsilon \phi_+(x) & \text{as } x \to \infty \\
\phi_-(x) & \text{as } x \to -\infty.
\end{cases} \quad (A21) \]

We now examine the expressions

\[ \begin{align*}
\psi_\pm &= e^{-S(\pm)}, \\
\phi_\pm &= e^{-S(\pm)}
\end{align*} \quad (A22) \]

and the asymptotic expansions (neglecting \(O(\epsilon)\) corrections)

\[ \begin{align*}
S(x) &= g S_0(x) + S_1(x) + g^{-1} S_2(x) + \cdots \\
\hat{S}(x) &= g S_0(x) + S_1(x) + g^{-1} S_2(x) + \cdots \quad (A23)
\end{align*} \]

From (A20) and (A21), we see that
\[
S(+) \to \begin{cases} 
S(+) & \text{as } x \to \infty \\
\frac{4}{3}ga^3 + S(-) & \text{as } x \to -\infty.
\end{cases} \tag{A24}
\]

and

\[
S(-) \to \begin{cases} 
\frac{4}{3}ga^3 + S(+) & \text{as } x \to \infty \\
S(-) & \text{as } x \to -\infty.
\end{cases} \tag{A25}
\]

Since, neglecting \(O(\epsilon)\), \(\{S_n(\pm)\}\) and \(\{S_n(\pm)\}\) satisfy the same set of differential equations we have

\[
S_n(+) = S_n(+) \quad \text{for } x > -a + O\left(\frac{1}{\sqrt{m}}\right), \tag{A26}
\]

and

\[
S_0(+) = \frac{4}{3}a^3 + S_0(-) \quad \text{for } x < -a \tag{A27}
\]

\[
S_m(+) = S_m(-) \quad \text{for } x < -a \text{ and } m \neq 0. \tag{A28}
\]

Likewise,

\[
S_n(-) = S_n(-) \quad \text{for } x < a - O\left(\frac{1}{\sqrt{m}}\right), \tag{A29}
\]

and

\[
S_0(-) = \frac{4}{3}a^3 + S_0(+) \quad \text{for } x > a \tag{A30}
\]

\[
S_m(-) = S_m(+) \quad \text{for } x > a \text{ and } m \neq 0. \tag{A31}
\]

We note that \(S_0(\pm)\) and \(S_0(\pm)\) are both solutions of

\[
[S_0'(\pm)]^2 = 2v = (x^2 - a^2)^2 \tag{A32}
\]

and

\[
[S_0'(\pm)]^2 = 2v = (x^2 - a^2)^2. \tag{A33}
\]
In accordance with (2.11), at all $x$

$$S_0(+) = \frac{1}{3}(x - a)^2(x + 2a)$$

and

$$S_0(-) = \frac{1}{3}(x + a)^2(-x + 2a).$$

(A34)

However, from (A26) - (A27) and (A29) - (A30),

$$S_0(+) = \begin{cases} S_0(+) = \frac{1}{3}(x - a)^2(x + 2a) & \text{for } x > -a \\ \frac{4}{3}a^3 + S_0(-) = \frac{4}{3}a^3 + \frac{1}{3}(x + a)^2(-x + 2a) & \text{for } x < -a. \end{cases}$$

(A35)

and

$$S_0(-) = \begin{cases} \frac{4}{3}a^3 + S_0(+) = \frac{4}{3}a^3 + \frac{1}{3}(x - a)^2(x + 2a) & \text{for } x > a \\ S_0(-) = \frac{1}{3}(x + a)^2(-x + 2a) & \text{for } x < a, \end{cases}$$

(A36)

which are solutions of the Hamilton-Jacobi equation

$$\frac{1}{2} \left( \frac{dS_0(\pm)}{dx} \right)^2 - v(x) = 0 +$$

(A37)

with $0+$ as the total energy. By requiring $S_0(\pm) = \infty$ at $x = \pm\infty$ and $S_0(\pm) = 0$ at $x = \pm a$ respectively, we derive (A35) - (A36) in the limit $0+ \to 0$.

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