Super-Weyl anomalies in $\mathcal{N} = 2$ supergravity and (non)local effective actions

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Abstract

Using the formulation for $\mathcal{N} = 2$ conformal supergravity in SU(2) superspace, we define super-Weyl (or superconformal) anomalies and construct two types of nonlocal effective actions that generate these anomalies. We also present the local Wess-Zumino action for spontaneously broken $\mathcal{N} = 2$ superconformal symmetry, with the Goldstone supermultiplet identified with a reduced chiral superfield containing the dilaton and the axion among its components.
1 Introduction

Within the superconformal approach to $\mathcal{N} = 2$ locally supersymmetric theories in four dimensions (see, e.g., [1] for a pedagogical review), Poincaré or anti-de Sitter supergravity is realized as conformal supergravity coupled to two compensators, one of which is invariably a vector multiplet. If one makes use of the superspace formulation for $\mathcal{N} = 2$ conformal supergravity developed in [2], any supergravity-matter action is required to be invariant under arbitrary super-Weyl transformations generated by a covariantly chiral scalar parameter $\sigma$. Given a superconformal field theory coupled to background $\mathcal{N} = 2$ supergravity, its classical action must be independent of the compensators. At the quantum level, however, integrating out the matter fields in such a theory results in the breakdown of the Weyl and local $U(1)_R$ symmetries. There are two equivalent ways to describe this anomaly in terms of a nonlocal effective action that corresponds to the theory under consideration. Firstly, the effective action $\Gamma$ can be chosen in such a way that it does not depend explicitly on the compensating vector multiplet, but is not super-Weyl invariant, $\delta_\sigma \Gamma \neq 0$. The second option is that the effective action $\tilde{\Gamma}$ is super-Weyl invariant, $\delta_\sigma \tilde{\Gamma} = 0$, but it depends explicitly on the compensating vector multiplet. It turns out that $\Gamma$ is obtained from $\tilde{\Gamma}$ by choosing an appropriate super-Weyl gauge.

Recently, new $\mathcal{N} = 2$ locally supersymmetric higher-derivative invariants have been constructed that include the $\mathcal{N} = 2$ supersymmetric extension of the Gauss-Bonnet topological invariant [7]. This work has actually provided the missing tool one needs in order to give a simple and efficient superspace definition of the $\mathcal{N} = 2$ super-Weyl anomaly as well as to construct a nonlocal effective action that generates the anomaly. Moreover, it makes it possible to derive an $\mathcal{N} = 2$ supersymmetric extension of the local dilaton effective action introduced in [8].

This paper is organized as follows. In section 2 we define consistent $\mathcal{N} = 2$ super-Weyl anomalies and construct two types of nonlocal effective actions that generate these anomalies. The local dilaton effective action is introduced in section 3. Several implications of the results obtained are discussed in section 4. Appendix A contains a brief summary of the formulation for $\mathcal{N} = 2$ conformal supergravity [2] in $SU(2)$ superspace. Appendix B contains the expressions for the vector and tensor supergravity compensators in terms of prepotentials.

1 Although there exist two alternative superspace formulations for $\mathcal{N} = 2$ conformal supergravity [3, 4], the approach described in [2] is intimately related to the structure of superconformal transformations in $\mathcal{N} = 2$ Minkowski superspace as described in [5] (see also [6]).
2 Nonlocal effective action

Consider a superconformal field theory coupled to $\mathcal{N} = 2$ Poincaré or anti-de Sitter supergravity. The classical action of such a theory is invariant under the super-Weyl transformations, and it is independent of the supergravity compensators. In other words, the superconformal field theory couples to the Weyl multiplet. For concreteness, we use the supergravity formulation in which the second compensator is an improved tensor multiplet [9].

In the quantum theory, integrating out the matter fields leads to an effective action that is no longer a functional of the Weyl multiplet only. As mentioned in section 1, there are two equivalent ways to describe such a functional with broken super-Weyl invariance. Let us first consider the realization in which the effective action $\Gamma$ does not depend explicitly on the vector compensator, but is not invariant under the super-Weyl transformations.

In general, the super-Weyl variation of the effective action $\Gamma$ has the form

$$\delta_\sigma \Gamma = (c - a) \int d^4x d^4\theta \mathcal{E} W^{\alpha \beta} W_{\alpha \beta} + a \int d^4x d^4\theta \mathcal{E} \sigma \Xi + \text{c.c.} , \quad (2.1)$$

for some anomaly coefficients $a$ and $c$. Here $\mathcal{E}$ denotes the chiral density (see, e.g., [10] for more details), $W_{\alpha \beta} = W_{\beta \alpha}$ is the covariantly chiral super-Weyl tensor, and $\Xi$ denotes the following composite scalar [7]:

$$\Xi := \frac{1}{6} \bar{D}_{ ij} S_{ij} + \bar{S}^{ij} S_{ij} + Y_{\dot{\alpha} \dot{\beta}} Y^{\dot{\alpha} \dot{\beta}} , \quad (2.2)$$

The torsion superfields $S_{ij}$, $W_{\alpha \beta}$ and $Y_{\alpha \beta}$ and their conjugates $\bar{S}^{ij}$, $\bar{W}_{\dot{\alpha} \dot{\beta}}$ and $\bar{Y}^{\dot{\alpha} \dot{\beta}}$ are defined in Appendix A. The fundamental properties of $\Xi$ are as follows [7]:

(i) $\Xi$ is covariantly chiral,

$$\bar{D}_{ i} \bar{\Xi} = 0 ; \quad (2.3a)$$

(ii) the super-Weyl transformation of $\Xi$ is (see (2.4) for the definition of $\bar{\Delta}$)

$$\delta_\sigma \Xi = 2\sigma \Xi - 2\bar{\Delta} \bar{\sigma} ; \quad (2.3b)$$

(iii) the functional

$$- \int d^4x d^4\theta \mathcal{E} \left\{ W^{\alpha \beta} W_{\alpha \beta} - \Xi \right\} \quad (2.3c)$$

is a topological invariant, which is related to the difference of the Gauss-Bonnet and Pontryagin invariants. The chirality of $\Xi$ is quite a nontrivial property which follows from
The super-Weyl transformation law (2.3b) may be derived with the aid of the equations (A.6b), (A.8a) and (A.8b).

In the transformation law (2.3b), $\bar{\Delta}$ denotes the chiral projection operator \[ \bar{\Delta} = \frac{1}{96} \left( \bar{D}^{ab} + 16 \bar{S}^{ab} \right) \bar{D}_{ab} - \bar{D}^{\dot{a}\dot{b}} \bar{D}_{\dot{a}\dot{b}} (\bar{D}^{\dot{a}\dot{b}} - 16 \bar{Y}^{\dot{a}\dot{b}}) \right), \] with $\bar{D}^{\dot{a}\dot{b}} := \bar{D}_k^{(\dot{a})} \bar{D}^{(\dot{b})k}$. Its main properties are the following: for any super-Weyl inert scalar $U$

$$\bar{D}^{\dot{a}\dot{b}} \bar{\Delta} U = 0, \quad (2.5a)$$
$$\delta_{\sigma} U = 0 \implies \delta_{\sigma} \bar{\Delta} U = 2\sigma \bar{\Delta} U, \quad (2.5b)$$
$$\int d^4x d^4\theta d^4\bar{\theta} E U = \int d^4x d^4\theta d^4\bar{\theta} E \bar{\Delta} U. \quad (2.5c)$$

The super-Weyl invariance of the second term in (2.3c) follows from the relations (2.3b) and (2.5c) in conjunction with the identity

$$\bar{D}_i^{\dot{a}} \sigma = 0 \implies \int d^4x d^4\theta d^4\bar{\theta} E \sigma = 0, \quad (2.6)$$

for any covariantly chiral scalar $\sigma$.

One can check that the super-Weyl variation (2.1) obeys the Wess-Zumino consistency condition \[ (\delta_{\sigma_1} \delta_{\sigma_2} - \delta_{\sigma_2} \delta_{\sigma_1}) \Gamma = 0. \] (2.7) This property guarantees the existence of $\Gamma$.

To construct $\Gamma$ explicitly, we introduce two scalar Green's functions $G_{+-}(z, z')$ and $G_{-+}(z, z')$ that are related to each other by the rule

$$G_{+-}(z, z') = G_{-+}(z', z) \quad (2.8)$$

and obey the following conditions:

(i) the two-point function $G_{-+}(z, z')$ is covariantly antichiral in $z$ and chiral in $z'$,

$$\bar{D}^{\dot{a}}_i G_{-+}(z, z') = 0, \quad \bar{D}^{\dot{a}} \bar{D}^{\dot{b}} G_{-+}(z, z') = 0; \quad (2.9)$$

(ii) the two-point function $G_{-+}(z, z')$ satisfies the differential equation

$$\bar{\Delta} G_{-+}(z, z') = \delta_+(z, z'). \quad (2.10)$$
Here we have used the chiral delta-function
\[ \delta_+(z, z') := \bar{\Delta} \left\{ E^{-1} \delta^4(x-x')\delta^4(\theta-\theta') \delta^4(\bar{\theta}-\bar{\theta}') \right\} = \delta_+(z', z), \] (2.11)
which is covariantly chiral with respect to each of its arguments,
\[ \bar{D}^\alpha_\dot{\alpha}_i \delta_+(z, z') = 0, \quad \bar{D}^\dot{\alpha}_\alpha_i \delta_+(z, z') = 0. \] (2.12)

Its main property is
\[ \Psi(z) = \int d^4x' d^4\theta' \delta^+_+(z, z') \Psi(z'), \quad \bar{D}^\alpha_i \Psi = 0, \] (2.13)
for any covariantly chiral scalar \( \Psi \). Under a finite super-Weyl transformation, the full superspace delta-function \( \delta(z, z') = E^{-1} \delta^4(x-x')\delta^4(\theta-\theta')\delta^4(\bar{\theta}-\bar{\theta}') \) is inert, which implies that the chiral delta-function \( \delta_+(z, z') \) changes as follows:
\[ \delta_+(z, z') \to e^{2\sigma} \delta_+(z, z'). \] (2.14)

From the relations (2.5b), (2.10) and (2.14) it follows that the Green’s functions \( G_{++}(z, z') \) and \( G_{+-}(z, z') \) are super-Weyl inert.

Using the above relations allows us to determine a nonlocal effective action generating the super-Weyl anomaly (2.1). It is
\[ \Gamma = -\frac{1}{2} (c - a) \int d^4x d^4\theta \mathcal{E} \int d^4x' d^4\theta' \mathcal{E}' W^\alpha\beta(z) W_{\alpha\beta}(z) G_{+-}(z, z') \Xi(z') + \text{c.c.} \]
\[ -\frac{1}{2} a \int d^4x d^4\theta \mathcal{E} \int d^4x' d^4\theta' \mathcal{E}' \Xi(z) G_{+-}(z, z') \Xi(z'). \] (2.15)

This nonlocal effective action may be compared with the analogous results in \( \mathcal{N} = 1 \) supergravity [13, 14].

In the superconformal approach to \( \mathcal{N} = 2 \) locally supersymmetric theories, any supergravity-matter system is described by a super-Weyl invariant action functional that in general depends on two compensators, one of which is always a vector multiplet. To construct a super-Weyl invariant extension, \( \bar{\Gamma} \), of the effective action (2.15), we only need to take into account the compensating vector multiplet described by a covariantly chiral field strength \( W \) and its conjugate \( \bar{W} \),
\[ \bar{D}^\dot{\alpha}_i W = 0, \] (2.16)
subject to the constraint [2]
\[ \left( \bar{D}^\dot{\alpha}_i + 4 S^\dot{\alpha}_i \right) W = \left( \bar{D}^\dot{\alpha}_i + 4 S^\dot{\alpha}_i \right) \bar{W}, \] (2.17)
which reduces to that given in [15] in the rigid supersymmetric limit. In order to comply with the compensator interpretation, the field strength has to be nowhere vanishing, \( \mathcal{W} \neq 0 \). The constraints (2.16) and (2.17) define a reduced chiral superfield. Under the super-Weyl transformation, \( \mathcal{W} \) varies as

\[
\delta_\sigma \mathcal{W} = \sigma \mathcal{W} .
\]  

(2.18)

The super-Weyl invariant extension of \( \Gamma \) is

\[
\tilde{\Gamma} = \Gamma - \int d^4x d^4\theta E \left\{ (c - a) \mathcal{W}^\alpha \mathcal{W}_\alpha + a \Xi \right\} \ln \mathcal{W} + \text{c.c.}
- 2a \int d^4x d^4\theta d^4\bar{\theta} E \ln \mathcal{W} \ln \bar{\mathcal{W}} .
\]  

(2.19)

It is easy to see that

\[
\delta_\sigma \tilde{\Gamma} = 0 .
\]  

(2.20)

One may always choose a super-Weyl gauge \( \mathcal{W} = 1 \) in which the effective action \( \tilde{\Gamma} \) reduces to the expression (2.15). Since the vector multiplet described by \( \mathcal{W} \) and \( \bar{\mathcal{W}} \) is just a compensator, the super-Weyl symmetry is not anomalous in this approach, in accordance with the general analysis given in [16].

3 Dilaton effective action

We now present a local Wess-Zumino action, \( S_D \), that reproduces the \( \mathcal{N} = 2 \) super-Weyl anomaly (2.1). It may be obtained by following the Wess-Zumino construction of the local action via integration of the anomaly [12]. In complete analogy with the \( \mathcal{N} = 1 \) case studied by Schwimmer and Theisen [8], our approach makes use of an off-shell dilaton supermultiplet that contains the dilaton and the axion among its component fields. We choose it to be the \( \mathcal{N} = 2 \) vector multiplet described by a covariantly chiral field strength \( \mathcal{Z} \) and its conjugate \( \bar{\mathcal{Z}} \). The field strength \( \mathcal{Z} \) is a reduced chiral superfield obeying the constraints (2.16) and (2.17), with \( \mathcal{W} \to \mathcal{Z} \). In addition, \( \mathcal{Z} \) is required to be nowhere vanishing, \( \mathcal{Z} \neq 0 \). The super-Weyl transformation of \( \mathcal{Z} \) written in the form

\[
\ln \mathcal{Z} \to \ln \mathcal{Z}' = \ln \mathcal{Z} + \sigma
\]  

(3.1)

tells us that \( \ln \mathcal{Z} \) is a Goldstone multiplet of spontaneously broken super-Weyl symmetry.

\footnote{In case that \( \mathcal{W} = \text{const} \), the second line of (2.19) vanishes, due to (2.6), for the choice \( \sigma = \text{const} \).}
The local effective action is

\[
S_D[Z, \bar{Z}] = \frac{1}{4} f^2 \int d^4x \, d^4\theta \, \mathcal{E} \, Z^2 + \int d^4x \, d^4\theta \, \mathcal{E} \, \left\{ (c - a) W^\alpha{}^\beta W_{\alpha\beta} + a \bar{Z} \right\} \ln Z + \text{c.c.}
+ 2a \int d^4x \, d^4\theta \, d^4\bar{\theta} \, E \, \ln Z \, \ln \bar{Z}.
\]  

(3.2)

Its super-Weyl variation is

\[
\delta_\sigma S_D[Z, \bar{Z}] = (c - a) \int d^4x \, d^4\theta \, \mathcal{E} \, \sigma W^\alpha{}^\beta W_{\alpha\beta} + a \int d^4x \, d^4\theta \, \mathcal{E} \, \sigma \bar{Z} + \text{c.c.}
\]  

(3.3)

This variation coincides with (2.1) and is independent of the dilaton supermultiplet. The kinetic term in the first line of (3.2) is super-Weyl invariant. This term is added by hand.

In the flat-superspace limit, the functional in the second line of (3.2) reduces to the unique superconformal \(F^4\) term discussed in [17, 18, 19].

The super-Weyl invariant extension of \(S_D\) is

\[
\tilde{S}_D[Z, \bar{Z}] = \frac{1}{4} f^2 \int d^4x \, d^4\theta \, \mathcal{E} \, Z^2 + \int d^4x \, d^4\theta \, \mathcal{E} \, \left\{ (c - a) W^\alpha{}^\beta W_{\alpha\beta} + a \bar{Z} \right\} \ln \frac{Z}{W} + \text{c.c.}
+ 2a \int d^4x \, d^4\theta \, d^4\bar{\theta} \, E \left\{ \ln Z \ln \bar{Z} - \ln W \ln \bar{W} \right\}.
\]  

(3.4)

This action reduces to (3.2) in the super-Weyl gauge \(W = 1\).

In the quantum theory, integrating out the dilaton supermultiplet leads to a nonlocal effective action depending only on the supergravity fields. By construction, this action reproduces the super-Weyl anomaly. At tree level we can approximate \(Z\) by a chiral superfield \(\mathcal{Z}\) obeying the equation

\[
(D_{ij} + 4S_{ij}) \mathcal{Z} = 0,
\]  

(3.5)

which is the equation of motion corresponding to the kinetic term in (3.2). This equation has a simple geometric interpretation if we recall the finite super-Weyl transformation of the torsion tensor \(S_{ij}\) [20],

\[
S_{ij} \rightarrow S'_{ij} = \frac{1}{4} e^{\sigma} e^{\bar{\sigma}} (D_{ij} + 4S_{ij}) e^{-\sigma}.
\]  

(3.6)

Thus the chiral parameter \(\Sigma\) defined by \(\mathcal{Z} = e^{-\Sigma}\) generates a super-Weyl transformation such that \(S'_{ij} = 0\). This transformation is unique under natural boundary conditions, which implies that \(\mathcal{Z}\) is a well-defined functional of the supergravity fields. Using the interpretation of \(\mathcal{Z}\) described, one may see that its super-Weyl transformation law is

\[
\mathcal{Z} \rightarrow \mathcal{Z}' = e^{\sigma} \mathcal{Z}.
\]  

(3.7)
For the transformation law (3.1) and for the effective action (3.2), it is not essential that $Z$ obeys the Bianchi identity (2.17). In principle, the constraint (2.17) may be replaced with a more general off-shell condition
\[(D^{ij} + 4S^{ij})Z - (\bar{D}^{ij} + 4\bar{S}^{ij})\bar{Z} = i\mathcal{H}^{ij}, \quad D^{(ij}\mathcal{H}^{jk)\alpha} = \bar{D}_{(ij}^{\alpha}\mathcal{H}^{jk)} = 0,\]
with $\mathcal{H}^{ij}$ a real SU(2) triplet.

4 Discussion

It is of interest to discuss the effective action (2.19) in the context of the $\mathcal{N} = 2$ supercurrent multiplet introduced by Sohnius [21] in the super-Poincaré case (see also [5]). Within the superconformal setting for $\mathcal{N} = 2$ supergravity, the supercurrent multiplet was defined in [22], and here we recall the key relations. To start with, we have to recall the prepotential structure of $\mathcal{N} = 2$ supergravity realized as the Weyl multiplet coupled to vector and tensor compensators. The Weyl multiplet can be described in terms of a real unconstrained prepotential $\mathcal{H}$ (introduced at the linearized level in [23, 24]) which naturally originates (see [5] for a detailed derivation) within the harmonic-superspace approach to $\mathcal{N} = 2$ supergravity [25]. A prepotential for the compensating vector multiplet can be chosen to be an unconstrained real SU(2) triplet $V^{ij} = V^{ji}$, $\bar{V}^{ij} = \varepsilon_{ik}\varepsilon_{jl}V^{kl}$. The explicit expression for the chiral field strength $W$ in terms of the prepotential is given by eq. (B.1). Prepotentials for the compensating tensor multiplet can be chosen to be a covariantly chiral scalar $\Psi$ and its conjugate $\bar{\Psi}$. The explicit expression for the field strength, $G^{ij}$, of the tensor multiplet in terms of the prepotentials is given by eq. (B.5).

The supercurrent conservation equation [22] is
\[\frac{1}{4}(D_{ij} + 4\bar{S}_{ij})J = W\mathcal{T}_{ij} - G_{ij}Y,\]
where the supercurrent $J$ and the trace multiplets $\mathcal{T}_{ij}$ and $Y$ are defined as
\[J = \frac{\delta S}{\delta \mathcal{H}}, \quad \mathcal{T}_{ij} = \frac{\delta S}{\delta \mathcal{V}^{ij}}, \quad Y = \frac{\delta S}{\delta \Psi},\]
with $\delta S/\delta \mathcal{H}$ a covariantized variational derivative. The superfields $J$ and $\mathcal{T}_{ij}$ are real, while $Y$ is covariantly chiral. In addition, $Y$ and $\mathcal{T}_{ij}$ must obey the constraints
\[D_{\alpha}^{(k}\mathcal{T}_{ij)}^{\alpha} = \bar{D}_{\alpha}^{(k}\mathcal{T}_{ij)}^{\alpha} = 0, \quad (D^{ij} + 4S^{ij})Y = (\bar{D}^{ij} + 4\bar{S}^{ij})\bar{Y}.\]
These constraints are due to the property that the prepotential $V^{ij}$ and $J^i$ are defined modulo the gauge transformations (B.2) and (B.6) respectively. In the case of a superconformal field theory, both trace multiplets vanish,

$$T^{ij} = 0, \quad J^i = 0. \quad (4.4)$$

For the effective action (2.19), we easily read off the trace multiplets

$$\langle T_{ij} \rangle = -\frac{1}{4}(D_{ij} + 4S_{ij}) \left\{ (c - a) \frac{W^{\alpha\beta} W_{\alpha\beta}}{W} + a \frac{\Xi + 2 \Delta \ln \hat{W}}{W} \right\} + \text{c.c.}, \quad \langle J \rangle = 0. \quad (4.5)$$

The effective action is independent of the tensor compensator, which implies $\langle J \rangle = 0$.

It is instructive to compare the anomalous supertrace (4.5) with that describing the super-Weyl anomalies in $\mathcal{N} = 1$ supergravity [26, 27] (see [28] for a review)

$$\langle T \rangle = (c - a) W^{\alpha\beta\gamma} W_{\alpha\beta\gamma} + \frac{a}{4}(\bar{D}^2 - 4R)(G^a G_a + 2R \bar{R}), \quad (4.6)$$

with $a$ and $c$ numerical parameters (containing a factor of $1/\pi^2$). Here $W_{\alpha\beta\gamma}$ is the covariantly chiral super-Weyl tensor, and $G_a$, $R$ and $\bar{R}$ are the other torsion superfields of the Wess-Zumino superspace geometry [29, 30]. The $a$-terms in (4.6) constitute the chiral density of the $\mathcal{N} = 1$ topological invariant [31]

$$\int d^4x \, d^2\theta \, E W^{\alpha\beta\gamma} W_{\alpha\beta\gamma} + \int d^4x \, d^2\theta \, d^2\bar{\theta} \, E (G^a G_a + 2R \bar{R}). \quad (4.7)$$

The anomalous supertrace (4.6) corresponds to the conservation equation [32]

$$\bar{D}^\dot{a} T_{a\dot{a}} + \frac{2}{3} \mathcal{D}_a T = 0, \quad \bar{D}_a T = 0, \quad (4.8)$$

with $T_a = \bar{T}_a$ the supercurrent and $T$ the trace multiplet. Unlike the $\mathcal{N} = 2$ supercurrent equation (4.11), the conservation law (4.8) does not involve any compensator. The point is that (4.8) originates within the ordinary Wess-Zumino formulation for $\mathcal{N} = 1$ supergravity. One can develop a superconformal extension of this formulation that makes use of a covariantly chiral scalar compensator $\Phi$, $\bar{D}_a \Phi = 0$, and any supergravity-matter action is invariant under super-Weyl transformations. From the point of view of such a formulation, eq. (4.8) corresponds to a super-Weyl gauge $\Phi = 1$. Without imposing this gauge condition, the supercurrent equation (4.8) has to be replaced with a more general equation

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3Direct calculations of the anomalous supertrace in concrete models [27] produce an additional contribution to $\langle T \rangle$ which is proportional to $(D^2 - 4R)D^2R$. It can be removed by adding a local finite counterterm $\int d^4x \, d^2\theta \, d^2\bar{\theta} \, E R \bar{R}$ to the effective action.
given in [33]. Then, the anomalous supertrace (4.6) gets modified to include an explicit dependence on Φ.

The \( \mathcal{N} = 1 \) anomalous supertrace (4.6) was originally computed in [27] for the scalar and vector multiplets coupled to \( \mathcal{N} = 1 \) supergravity. It is an interesting open problem to compute the \( \mathcal{N} = 2 \) anomalous supertrace (4.5) for vector and hyper multiples in a manifestly supersymmetric setting.

The analysis given in section 2 implies the existence of the following super-Weyl inert covariantly chiral scalar:

\[
\Psi^2 := \frac{1}{\mathcal{W}^2} \left( \Delta \ln \mathcal{W} + \frac{1}{2} \Xi \right), \quad \bar{D}^\dot{i} \Psi^2 = 0, \quad \delta_{\sigma} \Psi^2 = 0. \tag{4.9}
\]

This is a curved superspace generalization of the conformal primary weight-zero chiral superfield introduced in [19]. The superfield \( \mathcal{W}^2 \Psi^2 \) naturally originates within the approach developed in [7]. The chiral scalar \( \Psi^2 \) and its conjugate \( \bar{\Psi}^2 \) can be used to construct higher derivative super-Weyl invariants

\[
\int d^4 x d^4 \theta \mathcal{E} \mathcal{W}^2 \mathcal{F}(\Psi^2), \tag{4.10a}
\]

\[
\int d^4 x d^4 \theta d^4 \bar{\theta} \mathcal{E} \Upsilon(\Psi^2, \bar{\Psi}^2), \tag{4.10b}
\]

which generalize those introduced in [19]. At the component level, such functionals generate \( F^{2n} \) contributions, with \( n = 3, 4, \ldots \), and \( F \) the U(1) field strength.

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**A Conformal supergravity**

This appendix contains a summary of the formulation for \( \mathcal{N} = 2 \) conformal supergravity [2] in SU(2) superspace [34]. A curved \( \mathcal{N} = 2 \) superspace parametrized by local coordinates \( z^M = (x^m, \theta^\mu, \bar{\theta}^\dot{\mu} = (\theta^\mu)_\ast) \), where \( m = 0, 1, \ldots, 3, \mu = 1, 2, \dot{\mu} = 1, 2 \) and
\( \iota = \frac{1}{2} \). The structure group is chosen to be \( \text{SL}(2, \mathbb{C}) \times \text{SU}(2) \), and the covariant derivatives \( D_A = (D_a, D^i, D^i_\hat{a}) \) read
\[
D_A = E_A + \Phi_A^{kl} J_{kl} + \frac{1}{2} \Omega_A^{bc} M_{bc} \\
= E_A + \Phi_A^{kl} J_{kl} + \Omega_A^{\hat{b}\gamma} M_{\hat{b}\gamma} + \bar{\Omega}_A^{\hat{b}\gamma} \bar{M}_{\hat{b}\gamma} .
\]

Here \( M_{cd} \) and \( J_{kl} \) are the generators of the Lorentz and \( \text{SU}(2) \) groups respectively, and \( \Omega_A^{bc} \) and \( \Phi_A^{kl} \) the corresponding connections. The action of the generators on the covariant derivatives are defined as:
\[
[M_{\alpha\beta}, D^i] = \varepsilon_{\gamma(\alpha} D^i_{\beta)} , \quad [\bar{M}_{\hat{a}\hat{b}}, \bar{D}^i_\hat{a}] = \varepsilon_{\gamma(\hat{a}} \bar{D}^i_{\hat{b})} , \quad (A.2a) \\
[J_{kl}, D^i_\alpha] = -\delta^i_{(k} D_{\alpha l)} , \quad [J_{kl}, \bar{D}^i_\hat{a}] = -\varepsilon_{i(k} \bar{D}^i_{\hat{a})} . \quad (A.2b)
\]

The algebra of covariant derivatives is \([2]\)
\[
\{D^i_\alpha, D^j_\beta\} = 4 S^{ij} M_{\alpha\beta} + 2 \varepsilon^{ij} \varepsilon_{\alpha\beta} Y^{\gamma\delta} M_{\gamma\delta} + 2 \varepsilon^{ij} \varepsilon_{\alpha\beta} \bar{W}^{\gamma\delta} \bar{M}_{\gamma\delta} \\
+ 2 \varepsilon_{\alpha\beta} \varepsilon^{ij} S_{kl} J_{kl} + 4 Y_{\alpha\beta} J^{ij} , \quad (A.3a) \\
\{D^i_\hat{a}, D^j_\hat{b}\} = -4 S^{ij} M^{\hat{a}\hat{b}} - 2 \varepsilon^{ij} \varepsilon^{\hat{a}\hat{b}} \bar{Y}^{\gamma\delta} \bar{M}_{\gamma\delta} - 2 \varepsilon^{ij} \varepsilon^{\hat{a}\hat{b}} W^{\gamma\delta} M_{\gamma\delta} \\
- 2 \varepsilon_{ij} \varepsilon^{\hat{a}\hat{b}} \bar{S}_{kl} J_{kl} + 4 \bar{Y}^{\hat{a}\hat{b}} J_{ij} , \quad (A.3b) \\
\{D^i_\alpha, \bar{D}^j_\hat{a}\} = -2 i \delta^i_j (\sigma^c)_{\alpha}^{\hat{a}} D_c + 4 \delta^i_j G^{a\hat{b}} M_{a\hat{b}} + 4 \delta^i_j G_{a\hat{b}} \bar{M}^{\hat{a}\hat{b}} + 8 G^{a\hat{b}} J_{ij} . \quad (A.3c)
\]

see \([2]\) for the explicit expressions for commutators \([D_a, D^j_\beta]\) and \([D_\alpha, D^j_\hat{a}]\). Here the real four-vector \( G_{a\hat{a}} \), the complex symmetric tensors \( S^{ij} = S^{ji}, W_{\alpha\beta} = W_{\beta\alpha}, Y_{\alpha\beta} = Y_{\beta\alpha} \) and their complex conjugates \( S_{ij} := \overline{S^{ij}}, W_{\hat{a}\hat{b}} := \overline{W_{a\hat{b}}}, Y_{\hat{a}\hat{b}} := \overline{Y_{a\hat{b}}} \) are constrained by the Bianchi identities \([31][2]\). The latter comprise the dimension-3/2 identities
\[
D^j_a S^{ik} = \bar{D}^j_\alpha S^{ik} = 0 , \quad (A.4a) \\
\bar{D}^i_\hat{a} W_{\beta\gamma} = 0 , \quad (A.4b) \\
D^i_\alpha Y_{\beta\gamma} = 0 , \quad (A.4c) \\
D^i_\alpha S_{ij} + D^j_\beta Y_{\beta\alpha} = 0 , \quad (A.4d) \\
D^i_\alpha G_{\beta\hat{b}} = -\frac{1}{4} \bar{D}^i_\beta Y_{\alpha\beta} + \frac{1}{12} \varepsilon_{\alpha\beta} \bar{D}^i_\beta S_{ij} - \frac{1}{4} \varepsilon_{\alpha\beta} \bar{D}^i_\beta \bar{W}_{\beta\gamma} , \quad (A.4e)
\]

as well as the dimension-2 relation
\[
(D^i_\alpha D^j_\beta - 4 Y_{\alpha\beta}) W^{\alpha\beta} = (D^j_\alpha D^i_\hat{a} - 4 \bar{Y}^{\hat{a}\hat{b}})(\bar{W}_{a\hat{b}}) . \quad (A.5)
\]
The algebra of covariant derivatives (A.3) is invariant under the super-Weyl transformations [2]

$$\delta_\sigma D^i_\alpha = \frac{1}{2} \sigma D^i_\alpha + (D^{\gamma i}_\sigma \sigma) M_{\gamma \alpha} - (D_{\alpha k} \sigma) J^{ki} , \quad (A.6a)$$

$$\delta_\sigma \bar{D}_{\dot{a}i} = \frac{1}{2} \sigma \bar{D}_{\dot{a}i} + (\bar{D}^{\dot{a} i}_\dot{\sigma} \dot{\sigma}) \bar{M}_{\dot{a} \dot{\alpha}} + (\bar{D}^k_{\dot{a} \dot{\sigma}} \dot{\sigma}) J_{ki} , \quad (A.6b)$$

$$\delta_\sigma D_a = \frac{1}{2} (\sigma + \bar{\sigma}) D_a + \frac{i}{4} (\sigma_a)^\alpha_\beta (D^{k} \sigma) \bar{D}^{\dot{a}}_{\dot{\alpha}} + \frac{i}{4} (\sigma_a)^\alpha_\beta (\bar{D}^{k} \bar{\sigma}) D^k_\alpha - \frac{1}{2} (D^b(\sigma + \bar{\sigma})) M_{ab} , \quad (A.6c)$$

with the parameter $\sigma$ being an arbitrary covariantly chiral superfield,

$$\bar{D}_{\dot{a}i} \sigma = 0 , \quad (A.7)$$

provided the dimension-1 components of the torsion transform as follows:

$$\delta_\sigma S^{ij} = \bar{\sigma} S^{ij} - \frac{1}{4} D^{\gamma(i} \bar{D}^{j)}_\gamma \sigma , \quad (A.8a)$$

$$\delta_\sigma Y_{\alpha\beta} = \bar{\sigma} Y_{\alpha\beta} - \frac{1}{4} D^k_{(\alpha} D_{\beta)} k \sigma , \quad (A.8b)$$

$$\delta_\sigma W_{\alpha\beta} = \sigma W_{\alpha\beta} , \quad (A.8c)$$

$$\delta_\sigma G_{\alpha\dot{\beta}} = \frac{1}{2} (\sigma + \bar{\sigma}) G_{\alpha\dot{\beta}} - \frac{i}{4} D_{\alpha\dot{\beta}} (\sigma - \bar{\sigma}) . \quad (A.8d)$$

As is seen from (A.8c), the covariantly chiral symmetric spinor $W_{\alpha\beta}$ transforms homogeneously, and therefore it is a superfield extension of the Weyl tensor.

### B Compensators as gauge-invariant field strengths

In this appendix we recall expressions for the two supergravity compensators in terms of prepotentials.

The first compensator is identified with the reduced chiral superfield $W$ and its conjugate $\bar{W}$. These superfields can be expressed in terms of the curved-superspace extension [35] of Mezincescu’s prepotential [36], $V^{ij} = V^{ji}$, which is an unconstrained real SU(2) triplet. The expression for $W$ in terms of $V^{ij}$ [35] is

$$W = \frac{1}{4} \Lambda^{\alpha} \left( D_{ij} + 4 S_{ij} \right) V^{ij} . \quad (B.1)$$

The prepotential is defined only up to gauge transformations [35]

$$\delta V^{ij} = D^\alpha_k \Lambda^{kij} + D_{\dot{a}k} \bar{\Lambda}^{\dot{a}kij} , \quad \Lambda^{kij} = \Lambda^{(kij)} , \quad \bar{\Lambda}^{\dot{a}kij} := (\Lambda^{kij})^* , \quad (B.2)$$

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with the gauge parameter $\Lambda_{\alpha}^{kij}$ being completely arbitrary modulo the algebraic condition given. The super-Weyl transformation of $V^{ij}$ is

$$\delta_\sigma V^{ij} = -(\sigma + \bar{\sigma})V^{ij} . \quad (B.3)$$

The second compensator is a tensor (or linear) multiplet. It is described by a real $\text{SU}(2)$ triplet $G^{ij}$, with the algebraic properties $G^{ij} = G^{ji}$ and $\bar{G}_{ij} := (G^{ij})^* = G_{ij}$, subject to the constraints

$$\mathcal{D}^{(i}G^{jk)} = \bar{\mathcal{D}}^{(i\bar{G}^{jk)} = 0 . \quad (B.4)$$

These constraints are solved in terms of a covariantly chiral scalar prepotential $\Psi$ and its conjugate $\bar{\Psi}$ as follows:

$$G^{ij} = \frac{1}{4}(\mathcal{D}^{ij} + 4S^{ij})\Psi + \frac{1}{4}(\bar{\mathcal{D}}^{ij} + 4\bar{S}^{ij})\bar{\Psi} , \quad \mathcal{D}^{i}\Psi = 0 . \quad (B.5)$$

The prepotential is defined up to gauge transformations of the form

$$\delta \Psi = i \Lambda , \quad \left(\mathcal{D}^{ij} + 4S^{ij}\right)\Lambda = \left(\bar{\mathcal{D}}^{ij} + 4\bar{S}^{ij}\right)\bar{\Lambda} , \quad (B.6)$$

with $\Lambda$ an arbitrary reduced chiral superfield. The super-Weyl transformation laws of $G^{ij}$ and $\Psi$ were given in [2] and [43] respectively:

$$\delta_\sigma G^{ij} = (\sigma + \bar{\sigma})G^{ij} , \quad \delta_\sigma \Psi = \sigma \Psi . \quad (B.7)$$

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