Uncertainty of Velocity in $\kappa$-Minkowski Spacetime

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Abstract

A velocity of a point particle in the $\kappa$-Minkowski spacetime is investigated. Characteristic points of the spacetime are that the Poincare group becomes a quantum group with $\kappa$, which is a mass dimension parameter, and is a kind of non-commutative geometry. We consider a particle in a coordinate space instead of it in a momentum space which is discussed in many articles. We see that the particle’s velocity has an uncertainty which depends on a length of particle’s propagation.

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1 Introduction

In cosmic ray physics, the following problems are not solved yet; anomalous detection of extremely high energy cosmic rays [1] above the GZK cutoff \( (5 \times 10^{19} \text{ eV}) \) [2], and TeV-\( \gamma \) rays [3] from Mrk 501 and Mrk 421. It was suggested that the absence of the GZK cutoff is closely related to violation of the Lorentz invariance [4]. If the Lorentz invariance is violated or deformed, there is the possibility to explain that there is not the GZK cutoff at \( E = 5 \times 10^{19} \text{ eV} \) [5, 6]. Therefore we have to consider a mechanism for the violation or deformation of the symmetry.

One of the characteristic features with respect to quantum gravity is the Plack length. If we would like to deal with this as an observer independent scale, the special relativity becomes a deformed one which has two scales; a velocity \( c \) and a mass \( \kappa \). This is called the doubly special relativity [7] which is explicitly realized by the \( \kappa \)-Poincare algebra [8].

The \( \kappa \)-Poincare algebra is defined by replacing the Poincare algebra with a quantum group (Hopf algebra) with a parameter \( \kappa \) [9]. In the limit \( \kappa \rightarrow \infty \), the Hopf algebra is reduced to the ordinary Lie algebra. Majid and Ruegg [10] have pointed out that the \( \kappa \)-deformed algebra is the symmetry in the \( \kappa \)-Minkowski spacetime in which time and spatial coordinates are non-commutative.

Some authors have discussed particle’s velocity formulae [11, 6] in the \( \kappa \)-Minkowski spacetime. In the arguments, it was assumed that a velocity is defined by \( v = \frac{\partial E}{\partial p} \) which is true in the commutative Minkowski spacetime. In [11], by applying the formula to a massless particle, it was predicted that the speed of light is dependent on its energy \( E \) and the parameter \( \kappa \). In [6], by changing the rule of differentiation with respect to momenta, it was mentioned that the speed of light is undeformed. However validity of using the formula \( v = \frac{\partial E}{\partial p} \) in the \( \kappa \)-Minkowski spacetime has not been discussed sufficiently yet.

It is known that there exists some bases of the \( \kappa \)-Poincare algebra [11, 12], which are related with each other by redefinition of the translation operators [13]. In other words, the \( \kappa \)-deformation of the Poincare algebra is not unique. Since the dispersion relation is given as the Casimir operator of the algebra, the particle’s velocity depends which bases we use, if the formula \( v = \frac{\partial E}{\partial p} \) is used. In this paper we would like to consider a velocity by avoiding using the formula, and would like to compare results.

The present paper is organized as follows. In the next section, we review the \( \kappa \)-Minkowski spacetime and the \( \kappa \)-Poincare algebra. Some velocity formulae are given explicitly by using the bicrossproduct basis. In the section 3, we will see that the action of a free particle does not depend on \( \kappa \), and \( c \) is interpreted as the expectation value of photon’s velocity. There is the deviation of the particle’s velocity from the value \( c \), because of the spacetime non-commutativity. The section 4 is devoted to the summary and the remarks.

2 Review of \( \kappa \)-Minkowski spacetime

In this section, we briefly review the \( \kappa \)-deformed Poincare algebra, and present discussions about velocities of particles in the \( \kappa \)-Minkowski spacetime.

Drinfeld [14] and Jimbo [15] have proposed a prescription how to define a quantum
enveloping algebra, which is \( q \)-deformation of an universal enveloping algebra of a simple Lie algebra, in terms of a Hopf algebra. Since the Poincare algebra is not a simple Lie algebra, it is not straightforward to apply the Drinfeld-Jimbo prescription to the Poincare algebra. It is well known that the Poincare algebra can be obtained by contraction of the \( 3 + 2 \) dimensional AdS algebra \( o(3,2) \) which is a simple Lie algebra. In the limit \( R \to \infty \) which is the AdS radius, the AdS algebra is reduced to the Poincare algebra. The contraction \([16]\) \( R \to \infty \) with \( iR \log q \to \kappa^{-1} \) of the \( q \)-deformed AdS algebra gives a \( \kappa \)-deformed Poincare algebra \([9]\). Here \( \kappa \) is a real constant, and \( q \) is an imaginary parameter\(^2\) with \( |q| = 1 \). The \( \kappa \)-deformation of the Poincare algebra is not unique \([11, 12]\), however which are related by redefinition of translation generators \([13]\). For example, the \( \kappa \)-Poincare algebra in the bicrossproduct basis \([10]\) is given by

\[
[p_{\mu}, p_{\nu}] = 0, \quad [p_0, M_i] = 0, \quad [p_i, M_j] = i \epsilon_{ijk} p_k \tag{2.1}
\]
\[
[M_i, M_j] = i \epsilon_{ijk} M_k, \quad [M_i, N_j] = i \epsilon_{ijk} N_k, \quad [N_i, N_j] = i \epsilon_{ijk} M_k \tag{2.2}
\]
\[
[N_i, p_0] = ip_0, \quad [N_i, p_j] = i \delta_{ij} \left( \frac{\kappa}{2}(1 - e^{-2p_0/\kappa}) + \frac{1}{2\kappa} p^2 \right) - \frac{i}{\kappa} p_i p_j \tag{2.3}
\]

with translation \( p_{\mu} \), rotation \( M_i \) and boost \( N_i \) generators. The \( o(3) \) subalgebra is undeformed. The coproducts are given by

\[
\Delta(M_i) = M_i \otimes 1 + 1 \otimes M_i \tag{2.4}
\]
\[
\Delta(N_i) = N_i \otimes 1 + \exp \left( -\frac{p_0}{\kappa} \right) \otimes N_i + \frac{1}{\kappa} \epsilon_{ijk} p_j \otimes M_k \tag{2.5}
\]
\[
\Delta(p_i) = p_i \otimes 1 + \exp \left( -\frac{p_0}{\kappa} \right) \otimes p_i \tag{2.6}
\]
\[
\Delta(p_0) = p_0 \otimes 1 + 1 \otimes p_0 \tag{2.7}
\]

When the coproducts are given, counits and antipodes are obtained uniquely. But we do not write the detailed form since we do not use them in the present paper. In the limit \( \kappa \to \infty \), the Hopf algebra are reduced to the Poincare algebra. The mass shell condition is obtained as the Casimir operator;

\[
m^2 = \left( 2\kappa \sinh \left( \frac{P_0}{2\kappa} \right) \right)^2 - P^2 e^\frac{P_0}{\kappa}. \tag{2.8}
\]

The mass \( m \) is not a physical one. A relation with a physical mass is suggested in \([13]\). We find that the expression gives the upper bound of the momentum; \( p_i p^i \leq \kappa^2 \).

It was noticed that quantum groups can be realized as symmetries in non-commutative spaces\(^3\) \([20]\). The \( \kappa \)-Poincare algebra is the symmetry in the \( \kappa \)-Minkowski spacetime \([10]\) which is non-commutative;

\[
[X^i, X^0] = \frac{1}{\kappa} X^i, \quad [X^i, X^j] = 0. \tag{2.9}
\]

Although there are some bases of the algebra \([11, 12]\), all of them correspond to the commutator \([13]\).

\(^2\)It is possible to use a real parameter \([17, 18]\) instead of the imaginary \( q \). In this case, contraction is \( R \log q \to \kappa^{-1} \).

\(^3\)As a review article, see \([19]\).
In the commutative Minkowski spacetime, a particle's velocity is written as

\[ v_i = \frac{\partial E}{\partial p^i} \]  \hspace{1cm} (2.10)

with \( E = \sqrt{p^2 + m^2} \). Many authors have assumed that this relation is true also in the \( \kappa \)-Minkowski spacetime [11]. For a massless particle, by solving (2.8) and using the formula (2.10) we have

\[ v^2 = c^2 \exp \left( \frac{2E}{\kappa} \right). \]

So the velocity of a massless particle is deformed from the constant \( c \), and has energy dependence. Because of the effect, the \( \kappa \) has the constraint \( |\kappa|^{-1} < 10^{-33} \text{m} \) from the astronomical observation of \( \gamma \) rays [21]. On the other hand, in [6], it was mentioned that the rule for the differentiation with the momentum is changed since the momentum sum rule is \( \kappa \)-deformed. From the coproduct (2.6), we notice that the sum of momenta \( p_1 \) and \( p_2 \) of two particles is deformed as

\[ p_1 + \exp \left( -\frac{E_1}{\kappa} \right) p_2 \]

which is non-Abelian. By applying this to the velocity formula, we have two kinds of velocities;

\[ V^i_l = \exp \left( -\frac{E}{\kappa} \right) V^i, \quad V^i_r = \left( 1 + \frac{1}{\kappa p \cdot V} \right)^{-1} V^i. \]

We find that these two velocities satisfies \( (V^i_l)^2 = (V^i_r)^2 = c^2 \) for massless particles.

A common property with respect to them is that the velocities are determined uniquely when their energies are given. In other words, the particle velocities have no uncertainty. However, there have to exist such an uncertainty, because time and spatial coordinates of particles cannot be observed simultaneously by the non-commutativity (2.9). We would like to investigate the point in the next section.

### 3 Uncertainty of velocity

Here we would like to see how velocities of point particles are deformed in the \( \kappa \)-Minkowski spacetime by using a method by which we can avoid arguments which depend on the bases of the \( \kappa \)-Poincare algebra. In order to do this, we consider a particle in coordinate space \( X^\mu \) instead of the momentum space.

When a space is a non-commutative one, there is the possibility that a symmetry on the space becomes a quantum group [20]. The quantum group makes a commutator of coordinates covariant. In this formalism, a parameter of the non-commutativity is an invariant scale. By virtue of this, the doubly special relativity can be realized as a quantum group [8]. We apply this to the \( \kappa \)-Minkowski spacetime (2.9).

All of known bases of the \( \kappa \)-Poincare algebra correspond to the commutator (2.9) [13]. So we can perform an analysis which is independent on a choice of the basis if we
start from the commutator (2.9). The Poincare quantum group transformation of the
coordinates is defined by

\[ X'_{\mu} = \Lambda_{\nu}^{\mu} X^\nu + a^\mu \]

with

\[ \eta_{\mu\nu}\Lambda_{\rho}^{\mu}\Lambda_{\sigma}^{\nu} = \eta_{\rho\sigma}. \] (3.1)

Here \( \eta^{\mu\nu} = (-, +, +, +) \) is the flat metric tensor. We demand that the transformed
coordinates \( X'^\mu \) also satisfy the commutator (2.9). The Poincare quantum group does
not determined only by the covariance of the commutator. Furthermore we demand that
differentiation with respect to \( X^\mu \) is not deformed;

\[ dX^\mu \wedge dX^\nu = -dX^\nu \wedge dX^\mu. \]

In order for the condition (3.1) to be compatible with the Poincare quantum group, (3.1)
have to be commutative with all of the elements of the quantum group. By using the
conditions, the Poincare quantum group [22] is determined uniquely;

\[ [\Lambda^\mu_{\rho}, \Lambda^\nu_{\sigma}] = 0, \quad [a^\mu, a^\nu] = \frac{1}{\kappa} (\delta^\mu_{\xi} \delta^\nu_0 - \delta^\nu_{\xi} \delta^\mu_0) a^i \] (3.2)

\[ [a^\mu, \Lambda^\nu_k] = \frac{1}{\kappa} [(\Lambda^\nu_0 - \delta^\nu_0) \Lambda^\mu_{\rho} + (\Lambda^0_{\rho} - \delta^0_{\rho}) \eta^{\mu\nu}] \] (3.3)

which is the dual Hopf algebra [23] of (2.1)-(2.3).

Next we would like to see kinematics of a particle in the \( \kappa \)-Minkowski spacetime. The
action of a free particle which is invariant under the Poincare quantum group (3.2) and
(3.3) have the usual form;

\[ S = -\frac{1}{2} \int d\tau \left[ \frac{1}{e} (\dot{X}^\mu)^2 - e m^2 \right] \] (3.4)

with \( \dot{X}^\mu = \frac{dx^\mu}{d\tau} \). On the other hand, when we consider the \( \kappa \)-deformed Poincare algebra
(2.1)-(2.3) as a bicrossproduct Hopf algebra [10], the invariant metric is

\[ (X_0)^2 - (X^i)^2 + \frac{3}{\kappa} X_0 \]

which is different from ours. It is not clear why they are different. The canonical Hamiltonian
is also \( \kappa \)-independent

\[ H = \frac{1}{2e} [P^\mu P^\mu - m^2]. \]

The action of the free particle does not depend on the non-commutativity parameter
\( \kappa \) in the \( \kappa \)-Minkowski spacetime. It is known that, in the non-commutative spacetime
with \( [X^\mu, X^\nu] = i \theta^{\mu\nu} \), \( \theta^{\mu\nu} \)-dependent parts of free field theories’ action vanish. This
is the characteristic point in common. Although the action is \( \kappa \)-independent, the non-
commutativity effects a change of physical quantities. In the rest of the present section,
we see this explicitly.
Let’s consider a velocity of a particle which is described by the action (3.4). Because of the spacetime non-commutativity, we cannot observe strict values of $X^i$ and $X^0$ simultaneously, and there exists an ambiguity of the particle’s velocity. At first, we see the expectation value of the velocity. It is natural to make the expectation values of the coordinates $x^\mu := \langle X^\mu \rangle$ be a solution of the equations of motion which are derived from the action (3.4) with $X^\mu \to x^\mu$. The expectation value of the particle velocity is

$$v^i = \frac{dx^i}{dt} = \frac{\dot{x}^i}{t}.$$  

By using the equation of motion, we have $v^2 = c^2$ for massless particles. The expectation value of the velocity satisfies the ordinary velocity formula. The invariant scale $c$ can be interpreted as the expectation value of a velocity of a massless particle.

Next we would like to see a deviation from the expectation value. The expectation value of a deviation from the expectation value $x^\mu$ satisfy the inequality [24]

$$\langle (\Delta X^i)^2 \rangle \langle (\Delta T)^2 \rangle \geq \frac{1}{4c^2\kappa^2} |\langle X^i \rangle|^2$$  

with $\Delta X^\mu := X^\mu - x^\mu$. Here the index $i$ is not summed. The inequality means that time and spatial coordinates of a particle cannot be detected precisely at once except for at the origin in a coordinate system. When $|\langle X^i \rangle|$ is large, so is the uncertainty. This depends which coordinate systems we use.

Let’s consider a case in which a free particle starts from the origin $\langle X^\mu \rangle = 0$ of the spacetime. At a later time, the particle is at $\langle X^\mu \rangle = x^\mu$ with $x^0 \neq 0$. Since the $o(3)$ subalgebra of the Poincare algebra is not $\kappa$-deformed, without loss of generality, we can choose a coordinate system with

$$L := x^1 \neq 0, \quad x^2 = x^3 = 0.$$  

In this case $\langle X^i \rangle$ in the right hand side of (3.5) is an expectation value of a propagation length of the particle during the time $x^0$. Hence when the length $L$ is very large so is the spacetime uncertainty.

Here we would like to see a deviation from the expectation value $v := \langle V \rangle$. We denote $\delta X^i = \sqrt{\langle (\Delta X^i)^2 \rangle}$ and $\delta T = \sqrt{\langle (\Delta T)^2 \rangle}$. Although the inequality (3.5) does not forbid both of $\delta X^i$ and $\delta T$ to be large, by analogy with quantum mechanics, we assume that they are small with the equality (3.5) satisfied. Under the condition with the configuration (3.6), we find that the uncertainties along the directions $x^2$ and $x^3$ vanish; $\delta X^2 = \delta X^3 = 0$, and $\delta T$ and $\delta X^1$ have non-zero values. The spacetime uncertainty brings about the following velocity’s uncertainty;

$$\delta V^2 = \pm 2\frac{L^2}{t^3} \delta T \pm 2\frac{L}{t^2} \delta X^1 - 4\frac{L}{t^3} \delta X^1 \delta T \pm \frac{3L^2}{t^4} (\delta T)^2 + \frac{1}{t^2} (\delta X^1)^2$$  

up to second order in $\delta X^i$ and $\delta T$. We assume that $\delta X^i$ and $\delta T$ are so small that the first two terms in (3.7) are the dominant part. If we choose a configuration in which
the first order terms are canceled with each other, this gives the minimum value of $\delta V^2$ approximately. So the minimum value of $\delta V^2$ is

$$\delta V^2\big|_{\text{min}} \approx -\frac{3v^3}{|\kappa L|c}$$  \hspace{1cm} (3.8)

Here we have used $v := \frac{L}{t}$.

4 Summary and remarks

In this paper, we have investigated free particle’s motion in the $\kappa$-Minkowski spacetime in which the Poincare group becomes the quantum group. The particle’s action is not $\kappa$-deformed, since the invariant metric of the $\kappa$-Minkowski spacetime does not have $\kappa$-dependence in our formalism. When we define particle’s velocity, there exists uncertainty of the velocity because the spacetime coordinates are non-commutative operators with the commutator (2.9). The point is different from the discussions in momentum spaces [11, 6]. We saw that the particle’s velocity can depend on $\kappa$, even if the action is $\kappa$-independent.

In ordinary relativistic classical theories, massless particles propagate only with the velocity $c$ which is defined by $c = 299792458\text{m/s}$. However in the $\kappa$-Minkowski spacetime, the massless particle’s velocity have the uncertainty

$$\delta V^2\big|_{\text{min}} \approx -\frac{3c^2}{|\kappa L|}$$

which does not depend on the energy of the particle but on the propagation length $L$ of the particle. When $L$ is small, so is the spacetime uncertainty (3.5). However for a small $L$, the velocity’s uncertainty is large. The constant $c$ can be interpreted as the expectation value of the photon’s velocity.

In order to compatible with the observed value of $c$, we have the bound $|L\kappa| > 10^{20}$. The typical length of light paths in experiments of velocity of light is of order $10^{2}-10^{3}\text{m}$ [26]. So $\kappa$ is $|\kappa| > 10^{17}\text{m}$ which is weaker restriction than the one which is derived by using the formula (2.10).

In a non-relativistic limit $\frac{v}{c} \to 0$, the uncertainty (3.8) of the velocity can be negligible. The result is compatible with the fact that the only Lorentz boost part of the Poincare algebra is $\kappa$-deformed.

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