A Family of Elliptic Curves With Rank \( \geq 5 \)

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Abstract In this paper, we construct a family of elliptic curves with rank \( \geq 5 \). To do this, we use the Heron formula for a triple \((A^2, B^2, C^2)\) which are not necessarily the three sides of a triangle. It turns out that as parameters of a family of elliptic curves, these three positive integers \(A, B,\) and \(C\), along with the extra parameter \(D\) satisfy the quartic Diophantine equation \(A^4 + D^4 = 2(B^4 + D^4)\).

Keywords Diophantine equation · elliptic curve · Heron formula

1 Introduction

As is well-known, the affine part of an elliptic curve \(E\) over a field \(K\) can be explicitly expressed by the generalized Weierstrass equation of the form

\[
E : y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6,
\]  

where \(a_1, a_2, a_3, a_4, a_6 \in K\). In this paper we are interested in the case of \(K = \mathbb{Q}\).

By the Mordell-Weil theorem \cite{23}, every elliptic curve over \(\mathbb{Q}\) has a commutative group \(E(\mathbb{Q})\) which is finitely generated, i.e., \(E(\mathbb{Q}) \cong \mathbb{Z}^r \times E(\mathbb{Q})_{\text{tors}}\), where \(r\) is a nonnegative integer called the rank of \(E(\mathbb{Q})\) and \(E(\mathbb{Q})_{\text{tors}}\) is the subgroup of elements of finite order called the torsion subgroup of \(E(\mathbb{Q})\).
By the Mazur theorem [21], the torsion subgroup $E(\mathbb{Q})_{\text{tors}}$ is one of the following 15 types: $\mathbb{Z}/n\mathbb{Z}$ with $1 \leq n \leq 10$ or $n = 12$, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2m\mathbb{Z}$ with $1 \leq m \leq 4$. Besides, it is not known which values of rank $r$ are possible. The folklore conjecture is that a rank can be arbitrarily large, but it seems to be very difficult to find examples with large ranks. The current record is an example of elliptic curve over $\mathbb{Q}$ with rank $\geq 28$, found by Elkies in May 2006 (see [5]). Having classified the torsion part, one interested in seeing whether or not the rank is unbounded among all the elliptic curves. There is no known guaranteed algorithm to determine the rank and it is not known which numbers can occur as the ranks.

2 Previous Works

Let $D$ be a non-zero integer. The curve

$$E_D : y^2 = x^3 + Dx$$

has been considered by many mathematicians. Note that the congruent number elliptic curve belongs to this category by $D = -n^2$. By taking $D = p$, where $p \equiv 5 \pmod{8}$ and less than 1000, Bremner and Cassels [1] show that the rank is always 1 in accordance with the conjecture of Selmer and Mordell. Kudo and Motose [11] studied the case $D = -p$, where $p$ is a Fermat or Mersenne prime and found the ranks 0, 1 and 2. By taking $D = pq$, where $p$ and $q$ are distinct odd primes, Yoshida [24] showed that the rank is at most 5. In [7], the authors take $D = -n$, where $n = u^4 + v^4 = r^4 + s^4$ and prove the rank is at least 3. Moreover, they show that if $n$ is odd and the parity conjecture is true, then the rank is at least 4. Maenishi [13] studied the case $D = -pq$, where $p$ and $q$ are distinct odd prime numbers and by imposing an extra condition found a rank 4 family.

In the case of $D = -n^2$ (congruent number elliptic curve), many authors have attempted to find curves with high ranks. Rogers [17], using an idea of Rubin and Silverberg [19], found two curves with ranks 5 and 6. Later, he found a curve with rank 7. In [3], by using Mestre-Nagao's sum [4,15,16], the authors wish to find congruent number elliptic curves with high ranks. They succeed to find new congruent elliptic curves with rank 6. Johnstone [10], in her Master thesis provides an in depth background on congruent numbers and elliptic curves and then, presents a family of congruent number elliptic curves with rank at least three.

In this work we consider the elliptic curve

$$E : y^2 = x^3 - 4S^2x,$$

over the $K3$ surface

$$T = V_{1,1,-2,-2} : A^4 + D^4 = 2(B^4 + C^4),$$
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where, $S = S(A, B, C, D)$ is a rational function of $A, B, C$ and $D$. We prove that the group of rational maps $P : T \to E$, that commute with the projection $E \to T$, has rank at least 5. We do this by exhibiting five explicit sections $P_1, P_2, P_3, P_4,$ and $P_5,$ and later showing that they are independent. We use the fact that $T(\mathbb{Q})$ is infinite to deduce that infinitely many specializations of $E$ have the rank at least 5 over $\mathbb{Q}$. This is done by using some elliptic curves of positive rank lying on $T$ that we found in [39].

A natural question is that whether the set of rational points $T(\mathbb{Q})$ is Zariski dense in the $K3$ surface $T$. The following theorem answers the question affirmatively.

**Theorem 1** Let $a,b,c,d \in \mathbb{Q}^*$ be nonzero rational numbers with $abcd$ square. Let $P = [x_0 : y_0 : z_0 : w_0]$ be a rational point on $V_{a,b,c,d}: ax^4 + by^4 + cz^4 + dw^4 = 0$, and suppose that $x_0y_0z_0w_0 \neq 0$ and let $P$ is not contained in one of the 48 lines of the surface. Then the set of rational points of $V$ is Zariski dense in $V$ as well as dense in the real analytic topology on $V(\mathbb{R})$.

**Proof** [12], Theorem 1.1.

The point $(21, 19, 20, 7)$ is on $V_{1,1,-2,-2}$ and satisfies the hypotheses of above theorem, so it shows that the rational points on the surface $V_{1,1,-2,-2}$ are indeed dense in Zariski topology, and also in the real analytic topology.

3 Definition of rational function $S$

In this work we deal with a family of elliptic curves which are related to the positive integer solutions of the diophantine equation

$$A^4 + D^4 = 2(B^4 + C^4)$$

as follows.

Heron formula states that for a triangle with sides $a, b$ and $c$, one can get the area of the triangle by the formulae:

$$S = \sqrt{p(p-a)(p-b)(p-c)},$$

where, $p = \frac{a+b+c}{2}$.

Take $(A^2, B^2, C^2)$, where $(A, B, C)$ are as in equation (2). Since the triple $(A, B, C)$ is arising from the Diophantine equation (2), there is not guarantee to have a real triangle. Now by taking $a = A^2$, $b = B^2$, and $c = C^2$, we find that

$$S = \sqrt{(A^2 + B^2 + C^2)(A^2 + B^2 - C^2)(A^2 + C^2 - B^2)(B^2 + C^2 - A^2) / 16}.$$  

Expanding (4), one gets

$$S^2 = -\left(\frac{A^8 + B^8 + C^8 - 2A^4B^4 - 2A^4C^4 - 2B^4C^4}{16}\right),$$
equivalently,

\[16S^2 = 2A^4B^4 + 2A^4C^4 + 2B^4C^4 - A^8 - B^8 - C^8,\]

or

\[\left(\frac{A^4 + B^4 - C^4}{2}\right)^2 + 4S^2 = A^4B^4.\]

Multiplying both sides by \(A^2B^2\) yields

\[A^2B^2\left(\frac{A^4 + B^4 - C^4}{2}\right)^2 + 4A^2B^2S^2 = A^6B^6.\]

Taking \(y = AB\left(\frac{A^4 + B^4 - C^4}{2}\right)\) and \(x = A^2B^2\), we get the following family of elliptic curves:

\[E : y^2 = x^3 - 4S^2x.\] (5)

Since the roles of \(A\), \(B\), and \(C\) are symmetric in the Heron formula, we have the following points on the family (5) also.

\[P_1 = (A^2B^2, \frac{AB(A^4 + B^4 - C^4)}{2}),\]
\[P_2 = (A^2C^2, \frac{AC(A^4 + C^4 - B^4)}{2}),\]
\[P_3 = (B^2C^2, \frac{BC(B^4 + C^4 - A^4)}{2}).\]

Next we wish to impose two more points on the curve (5) with \(x\)-coordinates as \(B^2D^2\) and \(C^2D^2\). Substituting \(x = B^2D^2\) in (6) yields

\[y^2 = B^2D^2\left(\frac{4B^4D^4 + A^8 + B^8 + C^8 - 2A^4B^4 - 2A^4C^4 - 2B^4C^4}{4}\right),\]

or

\[y^2 = B^2D^2\left(\frac{A^4(A^4 - 2B^4 - 2C^4) + B^8 + C^8 - 2B^4C^4 + 4B^4D^4}{4}\right).\] (6)

Let

\[A^4 - 2B^4 - 2C^4 = -D^4,\] (7)

then by substituting \(A^4 = 2B^4 + 2C^4 - D^4\) in (6), we get

\[y^2 = B^2D^2\left(\frac{B^2 + D^2 - C^2}{2}\right)^2.\]

Thus, the point \(P_4 = (B^2D^2, \frac{BD(B^4 + D^4 - C^4)}{2})\) is a new point on (5). Similarly, one can easily check that the point \(P_5 = (C^2D^2, \frac{CD(C^4 + D^4 - B^4)}{2})\) lies
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It is clear that the existence of these extra points on the family depends exactly on the existence of the solutions of the Diophantine equation $A^4 + D^4 = 2(B^4 + C^4)$.

In the next stage, we will talk about the torsion subgroup of the (5).

4 The Torsion subgroup of (5)

Before starting the argument on the torsion subgroup of (5), let us recall the following theorem which reveals the structure of torsion subgroup for elliptic curves of the form $y^2 = x^3 + Dx$, where $D \in \mathbb{Z}$ and is fourth-power-free integer.

Theorem 2 Let $D \in \mathbb{Z}$ be a fourth-power-free integer, and $E_D$ be the elliptic curve $E_D : y^2 = x^3 + Dx$.

Then,

$$E_{D,tors}(\mathbb{Q}) \simeq \begin{cases} \mathbb{Z}/4\mathbb{Z}, & \text{if } D = 4, \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, & \text{if } -D \text{ is a perfect square,} \\ \mathbb{Z}/2\mathbb{Z}, & \text{otherwise.} \end{cases}$$

Proof See [22, Proposition 6.1, page 346]

Remark 1 From the appearance of (5) and also Theorem 2, one may guess that the family defined in (5) is a congruent number elliptic curve family and so, the torsion group is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. In the following we show this is not the case. First of all, one should be aware of the probable values of $S$ in (4). As $(A^2, B^2, C^2)$ does not necessarily construct a triangular, there is no guarantee for $S$ to be a real or an imaginary number in equation (4). For example, for the 4-tuple $(A, D, B, C) = (21, 19, 20, 7)$, we get $S = 180\sqrt{979}$, while $(A, D, B, C) = (1661081, 988521, 336280, 1437599)$ leads us to $S = 840\sqrt{962357334498800500956761065836542898196489}i$. In the former case $S$ is real, while in the later one $S$ is an imaginary number. In the case of real valued $S$, we see that the coefficient of $x$ in (5) is negative. But if $S$ is an imaginary number, the coefficient of $x$ would be positive in (5). Moreover, $S = 180\sqrt{979}$ shows that it is not necessary for $4S^2$ to be a perfect square. In this case, whether or not the value of $4S^2$ be a perfect square, the torsion subgroup of (5) is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$. If $S$ is an imaginary number, then the coefficient of $x$ in (5) is positive. Therefore, the torsion subgroup of (5) is $\mathbb{Z}/2\mathbb{Z}$. 
Finding the solutions of the equation $A^4 + D^4 = 2(B^4 + C^4)$

In [8] Izadi and Nabardi found infinitely many integer solutions of this equation. Their method is based on the points of the elliptic curve $y^2 = x^3 - 36x$ with a generator $(-3, 9)$ explained as follows:

Let $P_n = (x_n, y_n)$, where $P_n = n \cdot (-3, 9)$ ($n \in \mathbb{N}$) is a point on the elliptic curve $y^2 = x^3 - 36x$, one gets

\[ A_n = \phi_n^4 + 1296\psi_n^8 + 864\phi_n\psi_n^6 + 72\phi_n^2\psi_n^4 + 144\omega_n\psi_n^5 - 24\phi_n^3\psi_n^2 + 4\phi_n^2\omega_n\psi_n, \]
\[ D_n = -864\phi_n\psi_n^6 - \phi_n^4 - 1296\psi_n^8 - 72\phi_n^2\psi_n^4 + 144\omega_n\psi_n^5 + 24\phi_n^3\psi_n^2 + 4\phi_n^2\omega_n\psi_n, \]
\[ B_n = 4(\phi_n^2 + 36\psi_n^4)\omega_n\psi_n, \]
\[ C_n = (\phi_n^2 - 36\psi_n^4 - 12\phi_n\psi_n^2)(\phi_n^2 - 36\psi_n^4 + 12\phi_n\psi_n^2), \]

such that $A_n^4 + D_n^4 = 2(B_n^4 + C_n^4)$, where $\phi_n$ and $\psi_n$ are $n$-th division polynomials (see [23, pp.80-83]). Therefore, we define a family of elliptic curves by

\[ E_n : y^2 = x^3 + \left( \frac{A_n^8 + B_n^6 + C_n^6 - 2A_n^4B_n^4 - 2A_n^4C_n^4 - 2B_n^4C_n^4}{4} \right)x. \]  

As we discussed above, there are 5 points on (9) which are given by:

\[ P_{n1} = \left( A_n^2B_n^2, \frac{A_nB_n(A_n^4 + B_n^4 - C_n^4)}{2} \right), \]
\[ P_{n2} = \left( A_n^2C_n^2, \frac{A_nC_n(A_n^4 + C_n^4 - B_n^4)}{2} \right), \]
\[ P_{n3} = \left( B_n^2C_n^2, \frac{B_nC_n(B_n^4 + C_n^4 - A_n^4)}{2} \right), \]
\[ P_{n4} = \left( B_n^2D_n^2, \frac{B_nD_n(B_n^4 + D_n^4 - C_n^4)}{2} \right), \]
\[ P_{n5} = \left( C_n^2D_n^2, \frac{C_nD_n(C_n^4 + D_n^4 - B_n^4)}{2} \right). \]

Let $n = 2$, then

\[ E_2 : y^2 = x^3 + 2716157340889414533900362432217058675869770553600x, \]
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and

\[ P_{21} = (110502951275524201934400, 549083548316905650689533416877852800), \]
\[ P_{22} = (2019516118036396685564241, -3704296107487960167032542005050395239), \]
\[ P_{23} = (23710164715943220558400, 804710464588380886496762950328163200), \]
\[ P_{24} = (312020909765749236942400, 936950008965894699667383086290460800), \]
\[ P_{25} = (5702393005462282638861361, 141744467546800549687092185696272575). \]

Using SAGE \cite{20} software, shows that the determinant of height pairing matrix of \([P_{21}, P_{22}, P_{23}, P_{24}, P_{25}]\) equals \(\sim 30739535349\) and so, these points are independent. Since specialization is an injective homomorphism \cite[pp.456-457]{22}, it follows that the equation \(9\) is a family of elliptic curves with rank $\geq 5$.

In a separate paper \cite{9}, the authors found another set of solutions for \(7\). Let us briefly recall this method. The smallest known solution for \(7\) is \((A_0, D_0, B_0, C_0) = (21, 19, 20, 7)\). Dividing the equation \(7\) by \(C^4\), we get

\[ x^4 + y^4 - 2u^4 - 2 = 0 \quad x, y, u \in \mathbb{Q}. \quad (10) \]

Now \((x_0, y_0, u_0) = (3, 19/7, 20/7)\) is a solution for \((10)\). In this stage we define

\[ x = at + x_0, \quad y = bt + y_0, \quad u = ct + u_0. \quad (11) \]

Putting these in \((10)\), yields

\[ Mt^4 + Nt^3 + Rt^2 + St = 0, \quad (12) \]

where

\[ M = a^4 + b^4 - 2c^4, \]
\[ N = 12a^3 + \frac{76}{7}b^3 - \frac{160}{7}c^3, \]
\[ R = 54a^2 + \frac{2166}{49}b^2 - \frac{4800}{49}c^2, \]
\[ S = 108a + \frac{27436}{443}b - \frac{6400}{443}c. \quad (13) \]

By letting \(S = 0\), one has

\[ c = \frac{9261}{16000}a + \frac{6850}{16000}b. \quad (14) \]

If we let \(R = 0\), then from \((14)\) we have

\[ b = \frac{12147}{10507}a, \quad \frac{93}{133}a. \quad (15) \]
Let $b = \frac{93}{133}a$, so we can take $c = \frac{123}{140}a$. From (12) we have $t = -\frac{N}{M}$ and so
\[
 t = -\frac{1732800}{389209a}
\]
Therefore the new rational solution for (13) is:
\[
 x_1 = -\frac{565173}{389209}, \quad y_1 = -\frac{1086621}{2724463}, \quad u_1 = -\frac{2872540}{2724463}
\]
Equivalently,
\[
 A_1 = -3956211, \quad D_1 = -1086629, \quad B_1 = -2872540, \quad C_1 = 2724463
\]
is a new solution for (7). Again by repeating the same argument, we can find another solution for (17). Corresponding to (607, 1999, 951, 1640), we have
\[
 y^2 = x^3 + 9749352988442901002400000x,
\]
using MWRANK [2], one can show the rank is 8.

Let (181, 2077, 1247, 1620), then we get
\[
 y^2 = x^3 + 4988940634912192616750400x,
\]
In this case, the rank is 6.

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