The Domino Problem of the Hyperbolic Plane Is Undecidable: New Proof

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The present paper revisits the proof given in a paper of the author published in 2008 proving that the general tiling problem of the hyperbolic plane is algorithmically unsolvable by proving a slightly stronger version using only a regular polygon as the basic shape of the tiles. The problem was raised by a paper of Raphael Robinson in 1971, in his famous simplified proof that the general tiling problem is algorithmically unsolvable for the Euclidean plane, initially proved by Robert Berger in 1966. The present construction improves that of the 2008 paper. It also very strongly reduces the number of prototiles.

Keywords: hyperbolic plane; tilings; tiling problem; algorithmic unsolvability

1. Introduction

Throughout the paper, we say undecidable for algorithmically unsolvable.

Whether it is possible to tile the plane with copies of a fixed set of tiles was a question raised by Hao Wang [1] in the late 1950s. Wang solved the origin-constrained problem, which consists in fixing an initial tile in the above finite set of tiles. Indeed, fixing one tile is enough to entail the undecidability of the problem. Also called the general tiling problem later in this paper, the general case, free of any condition, in particular with no fixed initial tile, was proved undecidable by Robert Berger in 1966 [2]. Both Wang’s and Berger’s proofs deal with the problem in the Euclidean plane. In 1971, Raphael Robinson found an alternative, simpler proof of the undecidability of the general problem in the Euclidean plane [3]. In that paper, Robinson raised the question of the general problem for the hyperbolic plane. Seven years later, in 1978, he proved that in the hyperbolic plane the origin-constrained problem is undecidable [4]. Since then, the problem had remained open.

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In 2007, I proved the undecidability of the tiling problem in the hyperbolic plane, published in 2008; see [5]. In a recent arXiv paper [6], I presented a new proof of what was established in [5]. I follow the same general idea but the tiling itself is changed. As a result, a very significant reduction of the number of prototiles is obtained.

In order for this paper to be self-contained, I repeat the frame of the previous paper as well as the strategy used to address the tiling problem.

In the present introduction, we remind the reader of the general strategy to attack the tiling problem, as already established in the famous proofs dealing with the Euclidean case. We assume that the reader is familiar with the tiling \( \{7, 3\} \) of the hyperbolic plane. That tiling is the frame in which our solution of the problem lies. The reader familiar with hyperbolic geometry can skip that part of the paper. We also refer the reader to [7] and [8] for a more detailed introduction and for other bibliographical references. Also, in order that the paper can be self-contained, we briefly remind the reader about the notion of a space-time diagram of a Turing machine.

Following [6], I append a new section devoted to the construction of an aperiodic tiling. Wang mentioned in [1] that if any tiling of the hyperbolic plane were necessarily periodic, then the tiling problem would be decidable. Accordingly, the undecidability of the problem entails the existence of an aperiodic tiling of the hyperbolic plane. Section 2 deals with that point.

In Section 3, I reuse the construction of Section 2 to establish the properties of the particular tiling that we consider within the tiling \( \{7, 3\} \) and that are later used for the proof of Theorem 1.

In Section 3, we proceed to the proof itself, leaning on the definition of the needed tiles. In Section 3.6, we proceed to the counting of the needed prototiles. That allows us to prove:

**Theorem 1.** The domino problem of the hyperbolic plane is undecidable.

Reproducing similar sections from [5, 6], Section 4 gives several corollaries of Theorem 1. We conclude with the difference between the present paper and [5].

From Theorem 1, we immediately conclude that the general tiling problem is undecidable in the hyperbolic plane.

### 2. An Aperiodic Tiling of the Hyperbolic Plane

Section 2.1 briefly mentions the frame of our constructions. Then, in Section 2.4, we proceed to the construction of an aperiodic tiling of the hyperbolic plane. In Section 2.6 we define the prototiles that implement that construction.
2.1 The Tiling \{7, 3\}

We assume the reader to be familiar with hyperbolic geometry. See [7] for an introduction.

Regular tessellations are a particular case of tilings. They are generated from a regular polygon by reflection in its sides and, recursively, in the sides of the images. In the Euclidean case, there are, up to isomorphism and up to similarities, three tessellations, respectively based on the square, the equilateral triangle and the regular hexagon. Later on we say tessellation, for short.

In the hyperbolic plane, there are infinitely many tessellations that are based on the regular polygons with \( p \) sides and with \( 2\pi / q \) as vertex angle, and they are denoted by \( \{p, q\} \). This is a consequence of a famous theorem by Poincaré.

Among those tilings, we choose the tiling \( \{7, 3\} \), which we called the ternary heptagrid in [9]. It is illustrated by Figure 1. From now on, we call it the heptagrid.

![Figure 1. The heptagrid in Poincaré’s disc model.](https://doi.org/10.25088/ComplexSystems.32.1.19)

In [7, 9], many properties of the heptagrid are described. An important tool to establish them is the splitting method, prefigured in [10] and for which we refer to [7]. Here, we just suggest the use of this method, which allows us to exhibit a tree, spanning the tiling: the Fibonacci tree. Below, the left-hand side of Figure 2 illustrates the splitting of \( \mathbb{H}^2 \) into a central tile \( T \) and seven sectors dispatched around \( T \). Each sector is spanned by a Fibonacci tree. The right-hand side of Figure 2 illustrates how the sector can be split into subregions. Now, we notice that two of these regions are copies of the same sector and that the third region \( S \) can be split into a tile and then a copy of a sector and a copy of \( S \). Such a process gives rise to a tree whose nodes are in
bijection with the tiles of the sector. The tree structure will be used in the following, and other illustrations will allow the reader to better understand the process.

**Figure 2.** (a) The standard Fibonacci trees that span the heptagrid. (b) The splitting of a sector, spanned by a Fibonacci tree.

Another important tool to study the tiling \( \{7, 3\} \) is given by the *midpoint* lines, which are illustrated by Figure 3. The name comes from the fact that the lines join the midpoints of contiguous sides of tiles. Let \( s_0 \) be a side of a tile. Let \( M \) be the midpoint of \( s_0 \) and let \( A \) be one of the vertices defined by \( s_0 \). Two sides \( s_1 \) and \( s_2 \) of tiles of the heptagrid also share \( A \). They define a tile \( \mu \). Let \( u, v \) be the rays issued
from $A$ that cross the midpoint of $s_1, s_2$, respectively; see Figure 3. There, we can see how such rays allow us to delimit a sector, a property that is proved in [7, 9]. Later on, such a sector will be called a sector of the heptagrid. We say that $\mu$ is its root or that the sector is rooted at $\mu$.

### 2.2 Generating the Heptagrid with Tiles

The tiling that we have described in general terms in the previous section can effectively be generated from a set of four tiles that we call the prototiles. The basic colors we consider are green, yellow, blue and orange: we denote them by $G$, $Y$, $B$ and $O$, respectively. We say that the symbol attached to a tile is the status of the tile.

#### 2.2.1 Trees of the Heptagrid

Using the tiles defined previously, we define a tiling by applying the rules ($R_0$). Figure 4 illustrates the possible neighborhoods for each tile $G$, $Y$, $B$ and $O$. The tiles used for tiling the plane are copies of the prototiles:

$$G \rightarrow YBG, \ B \rightarrow BO, \ Y \rightarrow YBG, \ O \rightarrow YBO \quad (R_0)$$

![Figure 4](https://doi.org/10.25088/ComplexSystems.32.1.19)

Figure 4. The neighborhoods around each tile $B$, $Y$, $O$ or $G$. The neighborhoods around a tile of the same color correspond to the different occurrences of that color in the right-hand-side part of the rules ($R_0$).

Infinitely many tilings of the heptagrid can be constructed by applying the rules ($R_0$). Figure 4 illustrates all cases for a neighborhood of a tile.
We introduce a numbering of the sides of each tile of the heptagrid. Assume that the side that is given number 1 is fixed. We then number the other sides increasingly while turning counterclockwise around the tile. In a tile, side 1 is the side shared with its father. Since the central tile has no father, its side 1 is arbitrarily fixed. The neighbors of a tile are numbered after the side it shares with the tile we are considering. Accordingly, a side receives different numbers in the tiles that share it. Table (N) indicates the correspondence between those numbers. The index $g$, $y$, $b$ or $o$ refers to the status of the neighbor $\nu$ of a tile $\tau$. When the number in $\nu$ is 1, it means that $\nu$ is a son of $\tau$.

|       | 1  | 2  | 3  | 4  | 5  | 6  | 7  |
|-------|----|----|----|----|----|----|----|
| G-tile| $6_y$, $7_g$ | $7_b$ | $1_y$ | $1_b$ | $1_g$ | $2_b$ | $3_b$ |
| Y-tile| $4_y$, $3_g$, $6_o$, $7_o$ | $1_y$ | $1_b$ | $1_g$ | $2_b$ |
| B-tile| $5_y$, $4_g$, $7_y$, $6_g$, $7_g$ | $1_b$ | $1_o$ | $2_y$ | $2_g$, $o$ |
| O-tile| $5_b$, $o$, $7_b$ | $1_y$ | $1_b$ | $1_o$ | $2_y$ | $3_y$ |

Let us start with a few definitions.

Let $\tau_0$ and $\tau_1$ be two tiles of the heptagrid. A path between $\tau_0$ and $\tau_1$ is a finite sequence $\{\mu_i\}_{i=0}^n$ such that $\mu_i$ and $\mu_{i+1}$ when $0 \leq i < n$, share a side; say in that case the tiles are adjacent, and such that $\mu_0 = \tau_0$ and $\mu_n = \tau_1$. In that case, we say that $n + 1$ is the length of the path and we also say that the path joins $\tau_0$ to $\tau_1$. The distance between two tiles $\tau_0$ and $\tau_1$ is the smallest length reached among those of the paths joining $\tau_0$ to $\tau_1$.

Consider a sector $S$ as defined earlier, rooted at a tile $\mu$; see Figure 3. We know that the set of tiles whose center is contained in $S$ is spanned by a tree rooted at $\mu$. In [7, 10], it is proved that such a tree is spanned by the rules of $(R_0)$. Call such a tree a tree of the heptagrid when its root is not a B-tile. We can say either that $A$ (see the figure) is the origin of the tree or of the sector and that $A$ points at $\mu$. We say that the origin of the tree points at the root of the tree. We denote that tree by $T(\mu)$. A tree of the heptagrid is a set of tiles; it is not the set of points contained in those tiles. We call the left-, right-hand-side border of $T(\mu)$ the set of tiles of the tree that are crossed by the left-, right-hand-side ray, respectively, that delimit the sector defining $T(\mu)$.

In a tree of the heptagrid $T$, the level $m$ is the set of tiles of $T$ that are at the distance $m$ from the root. By induction, it is easy to prove that the sons of a tile on the level $m$ belong to the level $m + 1$.

A tiling of the heptagrid can be defined by the following process:

**Construction 1.**

- Time 0: fix a tile $\tau$ as a root of a tree $T(\tau_0)$ of the heptagrid; that root is the level 0 of $T(\tau_0)$ and chooses its status among Y, G or O.

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– Time \(m + 1, m \in \mathbb{N}\): construct \(\tau_{m+1}\) as a father of \(\tau_m\), taking \(\tau_{m+1}\) with a status that is compatible with that of \(\tau_m\); construct the level 1 of \(T(\tau_{m+1})\) and for \(i \in \{0..m\}\), if \(h_i\) is the level of \(T(\tau_i)\) constructed at time \(m\), construct all the levels of \(T(\tau_{i+1})\) up to \(h_i + 2\).

We have the following property proved in [6]:

**Proposition 1.** Let \(T_0, T_1\) be two trees of the heptagrid with \(T_1 \subset T_0\). Then a level of \(T_1\) is contained in a level of \(T_0\). More precisely, let \(\rho_1\) be the root of \(T_1\). Let \(b\) be the level of \(\rho_1\) in \(T_0\). Let \(\tau\) be a tile of \(T_1\). If the level of \(\tau\) in \(T_1\) is \(k\), its level in \(T_0\) is \(b + k\).

Figure 5 illustrates two different applications of Construction 1. The left-hand-side image represents an implementation where at time 0 the initial root is a \(G\)-tile and, at each time, the father of \(\tau_{m+1}\) is also a \(G\)-tile for several consecutive values of \(m\), and then it is a \(Y\)-tile. In the central and right-hand-side images, we have two views of the same implementation: \(G\)-tiles in an infinite sequence are crossed by a line \(\ell\) so that an infinite sequence of consecutive \(B\)-tiles is also crossed.

![Figure 5](https://example.com/fig5.jpg)

**Figure 5.** Two examples of an implementation of the rules \((R_0)\) to tile the heptagrid.

In Figure 5, we can notice levels for which Proposition 1 is, of course, true. But the figures lead us to introduce a definition. From Construction 1 and from Proposition 1, we can see that a level of \(T(\tau_i)\) is continued in \(T(\tau_{i+1})\) and, a fortiori, in all \(T(\tau_{i+k})\) for all positive integers \(k\).

**Proposition 2.** For any tree of the heptagrid \(T(\tau)\), if \(\mu\) is not a \(B\)-tile belonging to that tree, then \(T(\mu) \subset T(\tau)\).

Proof in [6].

We remind the reader that in \(T(\tau)\), whatever the status of \(\tau\) that is assumed to be not \(B\), the number of tiles belonging to the level \(m\) of that tree is \(f_{2m+1}\), where \(\{f_n\}_{n \in \mathbb{N}}\) is the Fibonacci sequence satisfying \(f_0 = f_1 = 1\); see [7, 10]. If the tiles \(\mu\) and \(\nu\) belong to the same level of
$T(\tau)$, we call *apartness* between $\mu$ and $\nu$ the number $n$ of tiles $\omega_i$ with $i \in \{0 \ldots n\}$ such that those $\omega_i$ belong to the same level and such that $\omega_0 = \mu$, $\omega_n = \nu$ and that for $i$ with $0 \leq i < n$, we have that $\omega_i$ and $\omega_{i+1}$ are adjacent. Accordingly, denoting the apartness between $\mu$ and $\nu$ by $\text{apart}(\mu, \nu)$, we get that if those tiles belong to the level $m$, then $\text{apart}(\mu, \nu) \leq f_{2m+1}$. From Proposition 1, it is plain that the definition of the apartness of two tiles does not depend on the tree of the heptagrid to which they belong, provided that then, they belong to the same tree.

Let us look more closely at the two implementations of Construction 1 illustrated by Figure 5. Fix a $G$-tile $\tau$ of the heptagrid. Fix a midpoint $A$ of a side of another tile sharing a vertex $V$ only with $\tau$. From $A$, draw two rays issued from $A$: one of them $u$ passes through the midpoint of one of the sides of $\tau$ sharing $V$, while $v$ passes through the midpoint of the other side of $\tau$ sharing $V$. The rays $u$ and $v$ allow us to define a sector of the heptagrid pointed by $A$. Applying the rule $(R_0)$ to $\tau$, to its sons and, recursively to the sons of its sons, we define a tree of the heptagrid. Let $v$ be the ray issued from $A$ that also crosses the $G$-son of $\tau$. From the rules $(R_0)$ and from Figure 4, it is not difficult to establish that in $T(\tau)$, $v$ crosses only $G$-tiles. We can notice that time $m+1$ of Construction 1 gives us only two possibilities to define the father of a root: indeed, such a father cannot be either a $B$-tile or an $O$-tile, since $G$-tiles are never sons of either a $B$-tile or an $O$-tile.

![Figure 6](image_url). A tree $T(\tau)$ of the heptagrid with two subtrees $T(\rho)$ and $T(\sigma)$, where $\rho$ is the $G$-son of the $Y$-son of $\tau$ and $\sigma$ is the $G$-son of $\tau$. In each tile where it occurs, side 1 is defined according to Table (N).
Let \( \kappa \) be a \( Y \)- or a \( G \)-tile. Let \( \varphi \) be such a tile that \( \kappa \) becomes a son of \( \varphi \). The status of \( \varphi \) is \( Y \) or \( G \). Accordingly, it may be the same as \( \kappa \) or different from \( \kappa \). Consider \( \psi \) such that \( \kappa \) is a son of \( \psi \). Again, the status of \( \psi \) is either \( Y \) or \( G \). If the statuses of \( \kappa \), \( \varphi \) and \( \psi \) are \( YGY \) or \( GYG \) in easy notations, we say that we have a \( YGY \)-, a \( GYG \)-alternation, respectively. In the case of an alternation, it is proved in [6] that \( T(\kappa) \subset T(\psi) \). We generalize the definition of alternation by considering the cases when we have \( YG'Y \) or \( GY^mG \), with \( \ell, m \geq 1 \). We again have \( T(\kappa) \subset T(\psi) \), where \( \kappa \) and \( \psi \) are the first and the last tiles in the patterns, respectively, the inclusion being proper.

From those observations, we conclude that there are two basic situations: either, starting from a time \( k \), the father appended at time \( k + i \) always has the same status as the root at time \( k \) or there are infinitely many times \( i_j \), with \( i_j > k \), such that the situation at times \( k + i_j, k + i_j + h, h \in \{ 1 \ldots \} \) and \( k + i_j + \ell + 1 \) is an alternation.

Consider the first case. There are two subcases: We define a process starting from time 0 with \( \kappa \) such that, at each time \( k + 1 \), a father is defined for the tile defined at time \( k \). If starting from a time \( k \) the appended father is always a \( G \)-, \( Y \)-tile, we say that we have a \( G \), \( Y \)-ultimate configuration, respectively.

It is proved in [6] that:

**Lemma 1.** Let \( \{ \tau_i \}_{i \in \mathbb{Z}} \) be a sequence of tiles in a tiling constructed with the rules \((R_0)\), such that for all \( i \) in \( \mathbb{Z} \) \( T(\tau_i) \subset T(\tau_{i+1}) \). If the sequence is in an alternating configuration, then for any tile \( \mu \) there is an index \( i \) such that \( \mu \) falls within \( T(\tau_i) \). If it is not the case:

If the sequence is in a \( Y \)-ultimate configuration, let \( k \) be the smallest integer such that \( \tau_i \) is a \( Y \)-tile for all \( i \geq k \). Then, let \( \omega_i \) be the tile sharing a side with \( \tau_{i+1} \) and another one with \( \tau_i \). Then for any tile \( \mu \) there is either an index \( i \) such that \( \mu \) falls within \( T(\tau_i) \) or there is an index \( j, j \geq k \) such that \( \mu \) falls within \( T(\omega_j) \).

If the sequence is in a \( G \)-ultimate configuration, let \( k \) be the smallest integer such that \( \tau_i \) is a \( G \)-tile for all \( i \geq k \). Let \( \beta_i \) be the \( B \)-son of \( \tau_i \) for \( i \geq k \) and let \( \omega_i \) be the \( O \)-son of \( \beta_i \) for \( i \geq k \) too. Then for any tile \( \mu \) that is not a \( \beta_i \) with \( i \geq k \), there is an index \( i \) such that \( \mu \) falls within \( T(\tau_i) \) or there is an index \( j, j \geq k \) such that \( \mu \) falls within \( T(\omega_j) \).

Lemma 1 and Proposition 1 allow us to call *isocline* the bi-infinite extension of any level of a tree \( T(\tau_m) \), for any value of \( m \) in \( \mathbb{N} \). Note that Figure 4 allows us to define isoclines in a \( G \)-ultimate configuration by completing the levels for the exceptional \( B \)-tiles indicated in the lemma and by joining the levels of the \( T(\omega_i) \) with the corresponding \( T(\tau_j) \).

Note that the images of Figure 5 also represent the tilings with their isoclines.

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Let $T$ be a tree of the heptagrid, let $\rho$ be its root and let $\tau$ be a tile in $T$. Say that a path $\pi = \{\pi_i\}_{i \in \{0..n\}}$ joining $\rho$ to $\tau$ is a tree path if and only if for each non-negative integer $i$, with $i < n$, $\pi_{i+1}$ is a son of $\pi_i$. It is not difficult to see that, in that case, $n$ is the level of $\tau$, which can easily be proved by induction on $n$.

An infinite sequence $\beta = \{\beta_i\}_{i \in \mathbb{N}}$ such that $\beta_0 = \rho$ and for each $i$ in $\mathbb{N}$, $\beta_{i+1}$ is a son of $\beta_i$ is called a branch in a tree $T = T(\rho)$ of the heptagrid. Accordingly, a tree path in $T$ is a path from the root of $T$ to a tile $\tau$ in $T$ that is contained in the branch of $T$ that passes through $\tau$.

We can state:

**Proposition 3.** Let $T$ be a tree of the heptagrid. Let $\mu$ and $\nu$ be tiles in $T$. Let $\pi_{\mu}$, $\pi_{\nu}$ be the tree path from the root $\rho$ of $T$ to $\mu$, $\nu$, respectively. Then, $T(\mu) \subset T(\nu)$ if and only if $\pi_{\mu} \subset \pi_{\nu}$ and $T(\mu) \cap T(\nu) = \emptyset$ if and only if we have both $\pi_{\mu} \not\subset \pi_{\nu}$ and $\pi_{\nu} \not\subset \pi_{\mu}$.

Proof in [6].

We have an important property:

**Lemma 2.** Two distinct trees of the heptagrid are either disjoint or one of them contains the other.

The lemma is an immediate corollary of Proposition 3. As proved in [6], we also derive the following result:

**Proposition 4.** Let $T(\tau)$ be a tree of the heptagrid. Let $T(\mu)$ be another tree of the heptagrid with $\mu$ within $T(\tau)$. Let $\pi_{\mu}$ be a tree path from the root of $T(\tau)$ to $\mu$. Then $\pi_{\mu}$ contains at least one tile $\nu$ that is not a $B$-tile. Moreover, for any tile $\omega$ that is not a $B$-tile in $\pi_{\mu}$, we have $T(\omega) \subset T(\tau)$.

Note that in Figure 4, the curves representing the isoclines are constituted by two kinds of segments defined as follows. Those segments join the midpoints of two different sides of a tile: one kind, denoted by $w$, is defined by joining side 2 and side 7; the other kind, denoted by $b$, is defined by joining side 2 and side 6 or joining side 3 and side 7. Call these marks on a tile its level marks. The distribution of the level marks obeys the following rules:

$w \rightarrow \text{bww} \quad \text{b} \rightarrow \text{bw}$, \hspace{1cm} (S)

**Lemma 3.** When tiling the heptagrid with the prototiles $Y$, $G$, $B$ and $O$ by applying the rules $(R_0)$, the rules of $(S)$ also apply if we put $w$ marks on $B$- and $O$-tiles only and $b$ marks on $Y$- and $G$-tiles only.

Proof in [6].

Note that in $w$-tiles, sides 2 and 7 are joined by the marks, while in $Y$-tiles it is the case for sides 3 and 7, and in $G$-tiles it is the case for sides 2 and 6.

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Construction 1 allows us to tile the whole hyperbolic plane in infinitely many ways, even uncountably many, since at each time we have a choice between two possibilities and the number of steps is infinite.

It can be argued that any construction of a tiling that, by definition, starts with any tile, is in some sense described by Construction 1. Indeed, whatever the starting tile, we find at some point a G-tile since in a tiling, there is a G-tile at a distance at most 3 from any tile μ. That distance can be observed for a B-tile: its O-son has a Y-son, which in its turn has a G-son.

2.2.2 The Trees of the Tiling

We now introduce two new colors for the tiles, mauve and red, which we denote by M and R, respectively. We decide that M-tiles duplicate the B-tiles when they are sons of a G-tile and only in that case, and that an R-tile duplicates the O-son of an M-tile, so that the rules \((R_0)\) are replaced by the following:

\[
\begin{align*}
G & \rightarrow YMG, \\
B & \rightarrow BO, \\
Y & \rightarrow YBG, \\
O & \rightarrow YBO, \\
R & \rightarrow YBO, \\
M & \rightarrow BR,
\end{align*}
\]

\((R_1)\)

As previously, the status of a tile is, by definition, its color.

Figure 7 illustrates the application of the rules \((R_1)\) by giving what they induce for the neighbors of a tile given its status and the status of its father. Like what is done in Figure 4, we also define levels by using the rules \((S)\). As far as a B-tile or an O is w, M- and R-tiles are also w. As in Figure 4, the number of central tiles associated to the same status is the number of occurrences of the status in the right-hand-side parts of the rules \((R_1)\).

We call tree of the tiling any \(T(ν)\) where ν is an R-tile. We repeat that a tree of the tiling is a set of tiles, not the set of points contained in those tiles. We also indicate here that, as far as M-, R-tiles behave like B-, O-tiles, respectively, we later refer to B-, O-tiles only unless the specificity of M-, R-tiles is required.

From Lemma 2 we can state:

Lemma 4. Let \(T_1\) and \(T_2\) be two trees of the tiling. Either those trees are disjoint or one of them contains the other. Moreover, a ray that delimits one of those trees does not intersect any of the rays delimiting the other tree. The same also applies to the rightmost and the leftmost branches of those trees.

Proof in [6].

Figure 8 illustrates Lemma 4. Note that the figure does not mention all trees of the tiling that can be drawn within the limits of that figure.
Figure 7. The prototiles generating the tiling: we describe all cases for the neighborhood of a tile, whatever it is: B, Y, O, G, M or R. The neighborhoods around a tile of the same color correspond to the different occurrences of that color in the right-hand-side part of rules \( R_1 \).

Figure 8. A tree \( T \) of the tiling with three subtrees of the tiling contained in \( T \). One of them is contained in another one, while two of those trees of the tiling inside \( T \) are disjoint.

Let us go back to the process described by Construction 1. The process leads us to introduce the following notion:

**Definition 1.** A thread is a set \( F \) of trees of the tiling such that:

(i) If \( A_1, A_2 \in F \), then either \( A_1 \subset A_2 \) or \( A_2 \subset A_1 \).
If $A \in \mathcal{F}$, then there is $B \in \mathcal{F}$ with $B \subset A$, the inclusion being proper.

(iii) If $A_1, A_2 \in \mathcal{F}$ with $A_1 \subset A_2$ and if $A$ is a tree of the tiling with $A_1 \subset A$, and $A \subset A_2$, then $A \in \mathcal{F}$.

Said in words, a thread is a set of trees of the tiling on which the inclusion defines a linear order that has no smaller element and that contains all trees of the tiling that belong to a segment of the set, according to the order defined by inclusion.

**Definition 2.** A thread $\mathcal{F}$ of the tiling is called an ultra-thread if it possesses the following additional property:

(iv) There is no $A \in \mathcal{F}$ such that for all $B \in \mathcal{F}$, $B \subset A$.

**Lemma 5.** A set $\mathcal{F}$ of trees of the tiling is an ultra-thread if and only if it possesses properties (i), (ii) and (iii) of Definition 1 together with the following:

(v) For any $A \in \mathcal{F}$ and for any tree $B$ of the tiling, if $A \subset B$, then $B \in \mathcal{F}$.

The proof of the lemma requires a kind of converse of Proposition 4.

**Proposition 5.** Let $A$ and $B$ be two trees of the tiling with $A \subset B$. Let $\rho$ be the root of $B$ and let $\pi$ be the tree path from $\rho$ to the root $\tau$ of $A$. Let $C$ be a tree of the tiling such that $A \subset C \subset B$. Then there is a tile $\nu$ of $\pi$ such that $C = T(\nu)$.

Proof of the lemma and of the proposition in [6].

Accordingly, an ultra-thread is a maximal thread with respect to the inclusion. A thread $\mathcal{F}$ that is not an ultra-thread is called *bounded* and there is a tree $A$ in $\mathcal{F}$ such that for each $B$ in $\mathcal{F}$, we get $B \subset A$. In that case, $A$ is called the *bound* of $\mathcal{F}$.

Consider the following construction:

**Construction 2.**

- Time 0: fix an R-son $\rho$ of an M-tile that is itself the son of a G-tile; let $F_0$ be $T(\rho)$.

- Time 1: at the level 3 of $F_0$, and on its left-hand-side border, there is another R-tile $\rho_{-1}$; let $F_{-1}$ be $T(\rho_{-1})$; clearly, $F_{-1} \subset F_0$. Repeating that process by induction, we produce a sequence $\{F_i\}_{i \leq 0}$ of trees of the tiling such that $F_i$ is contained in $F_{i-1}$ for all negative $i$; denote by $\rho_i$ the root of $F_i$. If $\tau_{i+1}$ is the son of a tile $\tau_i$, we say that $T(\tau_i)$ completes $T(\tau_{i+1})$.

- Time $2n+1$, $n \geq 0$: complete $T(\rho_{2n})$ by $T(\rho_{2n+1})$, where $\rho_{2n+1}$ is an M-tile that is the son of a G-tile $\omega_{2n+1}$, which we take as the G-son of a Y-tile $\xi_{2n+1}$.
– Time $2n+2$: complete $T(\xi_{2n+1})$ by $T(\rho_{2n+2})$, where $\rho_{2n+2}$ is an R-tile whose Y-son is $\xi_{2n+1}$; let $F_{n+1}$ be $T(\rho_{2n+2})$ that contains $F_n$.

**Proposition 6.** The sequence constituted by the $F_n$, $n \in \mathbb{Z}$ of Construction 2 is an ultra-thread.

Proof in [6].

### 2.3 Isoclines

We go back to the rules $(S)$ defined in Section 2.2. We proved there that it is possible to tile the plane with the rules $(R_0)$ so that the rules $(S)$ also apply, provided that w-marks are put on B- and O-tiles exactly and that b-marks are put on Y- and G-tiles exactly. In fact, we can prove more:

**Lemma 6.** Consider a tiling of the heptagrid with Y, G, B and O as prototiles obtained by applying the rules $(R_0)$. Then, defining w-marks on B- and O-tiles exactly and b-marks on Y- and G-tiles exactly, the b- and w-marks obey the rules of $(S)$.

Proof in [6].

Figure 9 illustrates the property that the levels of a tree of the tiling coincide with those of its subtrees that are also trees of the tiling. We already noticed that property for the trees of the heptagrid; see Proposition 1. That allows us to continue the levels to infinity on both sides of a tree of the tiling. We call isoclines the curves obtained by the marks contained in each level of $T$.

![Figure 9](image-url) **Figure 9.** Illustration of the levels in the tiling. Seven of them are indicated in the figure. Four trees of the tiling are shown with the rays defining the corresponding tree of the tiling.
In the following, it will be important to mark the path of some isoclines on each tile of the tiling. The isoclines are unchanged if some B- and O-tiles are replaced by M- and R-tiles, respectively, provided that rules \((R_1)\) are applied. Figure 7 shows us that the levels are also defined in the same way.

### 2.4 Constructing an Aperiodic Tiling
We remind the reader that in the heptagrid, a tiling is periodic if there is a shift \(\tau\) such the tiling is globally invariant under the application of \(\tau\). The goal of the present subsection is to prove:

**Theorem 2.** There is a tiling of the heptagrid that is not periodic. It can be constructed with 157 prototiles.

Goodman-Strauss, see [11], got a better result, proving Theorem 2 with a set of 85 prototiles.

We presently turn to the construction that will be reused to prove Theorem 1.

The construction is performed as follows.

We define two families of trilaterals, a red one and a blue one. In each family, we have triangles and phantoms, so that we have blue and red triangles and also blue and red phantoms. We call them the interwoven triangles since the blue trilaterals generate the red ones, which in turn generate the blue trilaterals. Blue and red are the colors of the trilateral. Blue and red are called opposite colors. Triangle or phantom is the attribute of a trilateral. Triangle and phantom are terms applied to opposite attributes.

The first steps of the construction are represented by Figure 10. Although the figure is drawn in the Euclidean plane, it can be implemented in the heptagrid.

![Figure 10](https://doi.org/10.25088/ComplexSystems.32.1.19)
the borders of such a tree do not intersect those of another one. The legs of a triangle or those of a phantom will follow the borders of a tree \( T(\tau) \) of the tiling. The basis of the triangle or of the phantom will follow a level of \( T(\tau) \).

For properties shared by both triangles and phantoms whichever the color, we will speak of trilaterals. For the set of all trilaterals, we will speak of the *interwoven triangles*.

For the construction, we consider a sequence of \( \mathbb{R} \)-tiles \( \{\rho_i\}_{i \in \mathbb{N}} \) such that for each \( i \) in \( \mathbb{N} \), \( T(\rho_{i+1}) \subset T(\rho_i) \), and such that \( \rho_{i+1} \) is the \( \mathbb{R} \)-son of an \( \mathbb{M} \)-tile that is the \( \mathbb{M} \)-son of a \( \mathbb{G} \)-tile that is the \( \mathbb{Y} \)-son of \( \rho_i \). We say that the pattern YGMR joins \( \rho_i \) to \( \rho_{i+1} \). Now, we require that \( \rho_0 \) belong to an isocline, chosen at random and that we call *isocline 0*. We number the isoclines with numbers in \( \mathbb{Z} \). Each isocline \( 8n, n \in \mathbb{N} \) is called *green* and each isocline \( n \) with \( n \equiv 4 \pmod{8} \) is called *orange*. Under that condition, the sequence of the \( \rho_i \) is called a *wire*. For any \( \rho_i \), we say that \( i \) is its *abscissa*. We say that \( \rho_{2i+1} \) is the *midpoint* between \( \rho_{2i} \) and \( \rho_{2i+2} \). Note that, by construction, the midpoint lies on an orange isocline and each \( \rho_{2i} \) lies on a green isocline.

The role of the green isoclines is to construct generation 0 of the trilaterals whose color is blue. From now on, each \( \mathbb{R} \)-tile on a green or an orange isocline is called an *active seed*. It triggers a trilateral, moreover: for each \( i \) in \( \mathbb{N} \), the trilaterals raised at \( \rho_{2i} \) and \( \rho_{2(i+1)} \) have the same color and opposite attributes. The \( \mathbb{R} \)-tiles on an orange isocline raise a blue trilateral or a red one.

**Construction 3.** Along each wire \( \{\rho_i\}_{i \in \mathbb{N}} \) of the tiling:

- Step 0: define the trilaterals of generation 0 that are blue; \( \rho_{2i} \) emit legs of a trilateral \( T_0 \) that are stopped by the isocline passing through \( \rho_{2i+2} \); \( \rho_{2i+2} \) emit legs of a trilateral \( T_1 \) that has the same color as \( T_0 \) but the opposite attribute with respect to \( T_0 \); the \( \rho_{2i+1} \) that lies inside a triangle of generation 0 emits a red trilateral; let \( T_1 \) and \( T_2 \) be the trilaterals raised at \( \rho_{2i+1} \) and \( \rho_{2i+5} \), respectively, for the same \( i \); \( T_1 \) and \( T_2 \) are both red and they have opposite attributes; accordingly, the basis of \( T_1 \) is raised at \( \rho_{2i+5} \); the seeds at \( \rho_{2i+1} \) also emit a mauve signal along their orange isocline from side to side.

- Step \( n + 1, n \in \mathbb{N} \): for each trilateral \( T \) of the generation \( n \), let \( \rho_i \) be its vertex and let \( \rho_{j} \) emit its basis; then \( \rho_k \) is its midpoint where \( k \) satisfies \( 2k = i + j \); also, \( j - i \) is the *height* of \( T \); the isocline passing through \( \rho_k \) is called the *midline* of \( T \); then for each triangle \( T_0 \) of the generation \( n \), its midpoint emits the vertex of a trilateral \( T_1 \) and the basis of a trilateral \( T_2 \); \( T_1 \) and \( T_2 \) have opposite attributes and both have the opposite color with respect to \( T_0 \); when the mauve signal \( \mu \) emitted at step 0 is accompanied by the basis of a phantom, it is stopped by the legs of the first triangle \( T \) that it meets and the isocline of \( \mu \) is the midline of \( T \); when the mauve signal is accompanied by the
basis $\beta$ of a triangle $T$, it is stopped by the first legs of the same color as $\beta$, which completes the construction of $T$; the trilaterals of the generation $n+1$ are the trilaterals whose vertex is raised at the midpoint of a triangle of the generation $n$.

The construction is illustrated by Figure 11.

**Figure 11.** Illustrating the construction of the interwoven triangles. We can see how to construct a triangle of the generation $n+1$ from triangles of the generation $n$.

**Proposition 7.** The trilaterals of the odd generations are red; the even generations are blue. If $h_n$ is the height of a trilateral of the generation $n$, we have $h_n = 2^{n+1}$. The abscissa $\xi_{n,m}$ of the vertex of the $m$th trilateral of the generation $n$, $m \in \mathbb{N}$, is given by $\xi_{n,m} = 2^n - 1 + m \cdot 2^{n+1}$, assuming that $\xi_{0,0} = 0$.

Proof in [6].

Denote by $\mu_{n,m}$ the midpoint of the $m$th trilateral of the generation $n$. From the proof of the proposition, we note that

$$\mu_{n,m} = (m + 1) \cdot 2^{n+1} - 1 = (2m + 2) \cdot 2^n - 1,$$

which means $\mu_{n,m}$ is also the midpoint of a trilateral of the previous generation. In fact, each second midpoint of trilaterals of the previous generations is still the midpoint of a trilateral of the generation $n$. The other midpoints are midpoints of triangles, so that they emit vertices of trilaterals of the generation $n$. That proves the construction too. Note that the proof is illustrated by Figure 11. The reader is referred to the Appendix of [6] for other graphics illustrating the first five steps of the construction.

Together with Proposition 7, we have additional properties:

**Lemma 7.** A trilateral $T$ of the generation $n+1$ contains a single phantom $P$ of the generation $n$ and there are two triangles $T_0$, $T_1$ of the generation $n$ such that $T_0$, $T_1$ contains the vertex, the basis of $T$,
respectively, in both cases on their midline. Moreover, $T$ and $P$ have the same midline. A trilateral $T$ of the generation $n + 2$ contains three trilaterals that are of the same color as the generation $n$ when $n \geq 1$, two of them being triangles and, in between them, a phantom $P$, the third one. Also, $T$ and $P$ have the same midpoint.

Proof. The lemma is an easy consequence of Proposition 7, whose proof is in [6]. □

Proposition 8. The legs of a trilateral do not intersect the legs of another one, whichever its color, whichever its attribute. Moreover, two triangles of the same color are either disjoint or one of them is embedded in the other one.

Proof. Immediate corollary of Lemma 4 and of Proposition 7. □

The color, the attribute and the generation of a trilateral constitute its characteristics.

The trilaterals we defined in Section 2 can be embedded in the tiling of the hyperbolic plane illustrated by Figure 12. Later we see how to implement the property that each fourth isocline is either green or orange, with green and orange isoclines alternating.

Figure 12. Representation of seeds and isoclines in two tilings of the heptagrid. (a) Two trees of the tiling are illustrated, both rooted at an R-tile. They belong to an ultra-thread. (b) The tiling has no ultra-threads, only bounded threads.

Proposition 9. In each triangle $T$ of the generation $n$, $n \geq 1$, its midline $\mu$ crosses $n$ phantoms $P_m$ of the generation $m$ with $0 \leq m < n$. Moreover, $\mu$ is also a midline for each $P_m$, where $0 \leq m < n$.

Proof in [6].
2.5 Application to Isoclines and to Threads

Going back to isoclines, we already noticed that they allow us to define levels in the whole hyperbolic plane. As shown by Figures 4 and 7, isoclines do not intersect and above an isocline there is always an isocline, so that isoclines constitute a partition of the tiling. We need more information than Construction 2. In that construction, we defined a wire, denoted by $Q$, as the sequence of tiles joining all $\rho_i$, $i \in \mathbb{N}$, in the tree. The sequence of $T(\rho_i)$, $i \in \mathbb{N}$ defines a thread.

Let us remember that Construction 2 defines an ultra-thread $\mathcal{F}_{i \in \mathbb{Z}}$, where each $\mathcal{F}$ is $T(\rho_i)$ where the tiles joining $\rho_i$ to $\rho_{i+1}$ have the statuses R, Y, G, M and R in that order. Since $i$ runs over $\mathbb{Z}$, we may, in that case, define $Q$ as a sequence of tiles indexed in $\mathbb{Z}$ with the property that $Q_{4i}$ is exactly $\rho_i$. We again call that new sequence the quasi-axis of that ultra-thread. Then, it is possible to prove:

Lemma 8. Let $\{T(\rho_i)\}_{i \in \mathbb{Z}}$ be the sequence of trees of the tiling defined by Construction 2. Then, for each tile $\tau$ of the heptagrid, there is $i \in \mathbb{Z}$ such that $\tau \in T(\rho_i)$. Accordingly, for any tile $\tau$ of the heptagrid that is not a B-tile, there is $i \in \mathbb{Z}$ such that $T(\tau) \subset T(\rho_i)$.

Proof in [6].

As a corollary of Lemma 8, we can deduce the following property of the ultra-thread obtained from Construction 2:

Lemma 9. Let $\mathcal{F}$ be the ultra-thread given by Construction 2 and let $\mathcal{G}$ be another ultra-thread. Then, for each tree of the tiling $G$ in $\mathcal{G}$, there is a tree of the tiling $F$ belonging to $\mathcal{F}$ such that $G \subset F$.

Proof. Immediate. □

Lemma 10. Let $\mathcal{F} = \bigcup_{n \in \mathbb{Z}} F_n$ be an ultra-thread and let $\tau$ be a tile. Then there is $m$ in $\mathbb{Z}$ such that $\tau \in F_m$.

Proof in [6].

Lemma 11. There are tilings of the heptagrid with the tiles Y, G, B, O, M and R and the application of the rules ($R_1$) such that all its threads are bounded.

Proof in [6].

Accordingly, some realizations of the tiling contain ultra-threads and some realizations of it contain none, as illustrated by Figure 12, whose right-hand-side part shows us a tiling where all threads are bounded.

2.6 The Prototiles for an Aperiodic Tiling

Remember that each eighth isocline is green and that in between two consecutive green isoclines, the fourth one from the first green isocline...
is orange. That defines the directions \textit{up} and \textit{down} in the hyperbolic plane. The isoclines also allow us to define the directions \textit{to left} and \textit{to right}, since such directions can be defined on each isocline and it is not difficult to do that in such a way that to left is the same direction on consecutive isoclines.

Remember that the active seeds sitting on a green isocline are the vertices of a trilateral of the generation 0, so that they trigger the construction of that trilateral. A seed that sits on an orange isocline is the vertex of a trilateral of the generation \( n \) with \( n \geq 1 \). An isocline that is neither green nor orange is said to be \textit{blue}.

In this subsection, we implement Construction 3 as a tiling. To that purpose, we define a set of \textit{prototiles}: the tiles of the tiling are copies of prototiles. By copy we mean an isometric image that places a tile from an isocline onto another one such that left, right, up and down of the former place coincide with those directions on the new isocline. Figure 13 defines the tiles required for the implementation of the isoclines.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure13.png}
\caption{Implementation of the rules \((R_1)\) and of the isoclines by 29 prototiles. Note the convention for representing heptagonal tiles by squares.}
\end{figure}
To force the succession of green and orange isoclines, we use the B-tiles since they occur in most rules in \((R_1)\). The first row in Figure 13 gives the tiles needed by the B-tiles to convey the signals organizing the alternation between green and orange isoclines with the separating blue isoclines. The second and third rows convey the marks of the isoclines for the other tiles, G-, Y-, O-, M- and R-tiles, those latter ones with blue isoclines only as far as the active seeds sit on green or orange isoclines only. The last row illustrates the tiles needed to start marking the B-tiles: it is the case for the B-son of a Y-, an O-, an M- or an R-tile. Depending on the isocline of the father of such a B-tile and the surrounding isoclines, we have the appropriate tile to be synchronized with B-tiles on the same isocline.

Consider a tree of the heptagrid \(\mathcal{T} = T(\tau)\). Define \(\{\beta_i\}_{i \in \mathbb{N}}\) to be the branch of \(\mathcal{T}\) as follows. If \(\tau\) is not a G-tile, then \(\beta_0 = \tau\). Otherwise, \(\beta_0\) is the B-son of the M-son of \(\tau\). Then, for any non-negative integer \(n\), \(\beta_{n+1}\) is the B-son of \(\beta_n\). Call that branch the \(\beta\)-branch from \(\tau\). We say that the branch consists of \(\tau\) and of its recursive B-offspring. The interest of that definition is that the \(\beta\)-branch of \(\mathcal{T}\) does not intersect any tree of the tiling contained in \(\mathcal{T}\).

It can easily be seen from Figure 7 that the prototiles of the figure can tile \(T(\tau)\), the active seeds being excepted. The first three rows of the figure indicate the convention we use to represent the heptagonal prototiles by square prototiles. Since we have mainly a top-down direction and a left-right direction given by the isoclines, the square format is convenient. The top number indicates 1, the side to the father. At the bottom side of the square, we have the numbers of the sides to the sons of the tile. On the left- and right-hand-side edges, we have the number of the sides crossed by the isocline on which the tile sits.

The tiles with a blue isocline can build the images of Figure 7 and only them. Since the tiles with a green isocline of Figure 13 look like those with a blue isocline, we obtain the images of Figure 7 and only them with the tiles of Figure 13. We also clearly obtain that green, orange and blue isoclines do not mix and do not cross each other. The first row of Figure 13 allows us to build a \(\beta\)-branch in any tree of the tiling. But the first row alone generates a \(\beta\)-branch whose root is rejected at infinity. For a true \(\beta\)-branch rooted at a tree of the heptagrid, we need the tiles of the last row of Figure 13: the father of a B-tile is either a Y-, an O- or an R-tile. In each case, the father may be on a green, an orange or a blue isocline, while the B-tile may be on a blue, an orange or a green isocline. Of course, if the father of the B-tile is on a green tile, then that tile’s father and the B-tile are on a blue isocline.

Note that the green, orange isoclines defined by a first tile 1, 5, respectively, impose the position of all other green, orange isoclines.
by the fact that the tiles bearing a green, orange isocline can only abut on the same level tiles also bearing a green, orange isocline, respectively.

In order to define the prototiles to construct the trilaterals, we need another property, which can be deduced from Proposition 7 and Lemma 7:

**Proposition 10.** The legs of a trilateral $T$ of the generation $n + 1$ are cut once by the basis $B$ of the triangle $T_0$ of the generation $n$ whose midpoint is the vertex $V$ of $T$. The isocline $\beta$ that contains $B$ is issued from $\rho_j$ where $\rho_j$ is the midpoint between $V$ and the midpoint of $T$. In between $V$ and $\beta$, the legs of $T$ are cut by bases of phantoms only. In between $\beta$ and the basis of $T$, the legs of $T$ are not cut by any trilateral of whichever generation.

Proof in [6].

Figure 14 gives the prototiles for constructing the trilaterals that have to be appended to those of Figure 13. Note that the prototiles 1 to 4 of Figure 14 complete the prototiles of Figure 13 for what are the seeds on a green or an orange isocline.

The first row of the second part of Figure 14 illustrates prototiles to trigger the construction of the legs of a trilateral. Note that both left- and right-hand-side legs are represented. Moreover, as indicated by the figure itself, we use a few gray colors to be replaced by various hues of blue and red. It is the reason why the 128 prototiles are illustrated by only 40 samples. In fact, as indicated in the propositions and lemmas devoted to the trilaterals, the legs can be uniformly dealt with. Note the fact that for a leg of either side, we use two hues of the same color: a light version for the first half of the leg starting from the vertex and a darker version for the second half. The rightmost part of the first row in the second part of the figure illustrates the conventions we use for the hues that represent two or three colors.

Consider a triangle $T$ whose vertex is denoted by $V$ and its midpoint by $\omega$. From Proposition 10, the first basis of a triangle $T_0$ cutting a leg of $T$ cuts the path from $V$ to the basis of $T$ at the midpoint $\nu$ between $V$ and $\omega$. Such a basis met by a leg when running over it from $V$ occurs at the one-quarter point of the leg. From Lemma 7, the other bases cutting the leg of $T$ in between $V$ and $\nu$ are bases of trilaterals of lower generations. Clearly, the mauve signals running on the isoclines of those bases meet triangles of a generation lower than that of $T_0$, so that when those bases cut the leg of $T$ there is no mauve signal with them. So that the first time a leg of $T$ meets a mauve signal, it is on the isocline passing through $\omega$. Accordingly, the change of color for the leg of $T$ occurs at that moment. Later, there is no meeting of a basis of a trilateral, the basis of $T$ being excepted. When it is the case, the basis does not contain a mauve signal at that meeting.
Figure 14. The 128 prototiles for constructing the trilaterals. Among those prototiles, 28 of them represent a red or a blue trilateral. Note the conventions of colors in order to restrict the number of images to 40.

From Proposition 7, we know that the height of a triangle of the generation $n$ is $2^{n+1}$, so that the triangles are bigger and bigger, making it impossible for the tiling to be periodic. Accordingly, the proof of Theorem 2 is completed. The number of prototiles needed is the sum of the numbers indicated in Figures 13 and 14.

Figure 10 illustrates the proof of Theorem 2.

3. Completing the Proof of Theorem 1

Presently, we shall see how to obtain the prototiles we need in order to prove Theorem 1, whose proof is completed only once we have produced the set of prototiles.

Since we have bigger and bigger triangles, taking triangles as the frame of the simulation of a Turing machine is a possible solution. It
is enough that the set of prototiles is adapted to a given Turing machine $M$ in order to perform its computation in any triangle. If $M$ does not halt, the computation is stopped when the computing signal meets the basis of the triangle and it will be the case in all triangles. If $M$ halts, the halting will be observed in some triangle. It is easy to implement the halting state by a prototile, one side of which cannot abut any prototile.

That condition can be fulfilled if we can perform the computation in a triangle. The scenario is the following. The initial configuration is displayed along the right-hand leg $\ell_r$ of the triangle $T$. That leg consists of O-tiles, the vertex of $T$ being excepted: it is the tile 14.3, an R-tile sitting on an orange isocline. From the O-tiles, we consider the path in $T$ that goes from a tile $\tau$ of $\ell_r$ to a tile of the basis of $T$ by following the Y-son of $\tau$ and, recursively, the Y-sons of those Y-sons. Call such a path the Y-path from $\tau$ and say that $\tau$ is its source. From Lemma 2, we know that a Y-path from a tile $\ell_r$ does not meet the legs of a triangle contained in $T$. The role of a Y-path from $\tau$ is to convey the content of the square of the tape of $M$ that lies in $\tau$. The Y-path updates that content as soon as the appropriate state is seen, so that the Y-path records the history of the computation on the square represented by its source. A computing signal $\xi$ starts from the root $\rho$ of $T$ and it visits the Y-paths according to the program of $M$. In order to go from one Y-path to the next one, the $\xi$ travels on a level of $T(\rho)$. That signal conveys the current state $\eta$ of $M$. When $\xi$ meets a Y-path conveying the current content $\sigma$ of the square of the tape that is the source of that Y-path, $\xi$ performs the instruction associated to $\eta$ and $\sigma$ in the program of $M$. The Y-path conveys the new letter contained by the square at the source of the Y-path. It also conveys the new state of $M$ as well as the direction $\delta$ toward the Y-path whose source is a neighbor of the source from which the previous Y-path originated. To that goal, $\xi$ goes to the next level along the Y-path it met and, on that level, goes to the new Y-path in the direction given by $\delta$.

Since $T$ may contain other triangles of the same color in which the same computation of $M$ is performed, those computations should not interfere with each other. We already know that the Y-paths generated in $T$ do not meet those of a triangle inside $T$. It is also necessary that the levels on which $\xi$ travels in $T$ are not those on which a similar signal travels in a triangle contained in $T$. Accordingly, we have to deal with that point.

Call free row of a trilateral $T$, the intersection with $T$ of an orange or a green isocline that does not meet the legs of a red triangle contained in $T$. Note that the notion might be applied to phantoms as well, but we reserve it for triangles. We deal with that problem in Section 3.1.
We also notice from Figure 14 that active seeds trigger both the construction of legs of a trilateral $T$ and the construction of the basis of a trilateral whose status is opposite to that of $T$. However, as indicated by Figure 12, it may happen that the basis triggered by an active seed will not meet legs of an appropriate triangle. It is the case if the tree of the tiling raised by the active seed is the bound of a thread. That raises another problem dealt with in Section 3.2.

## 3.1 Free Rows in Red Triangles
Before considering how to detect the free rows in a triangle, it is important to know whether there are enough of them for the computation purpose.

**Lemma 12.** In a red trilateral of the generation $2n+1$ there are $2^{n+1} + 1$ free rows.

Proof in [6].

It is worth noticing that if we choose the blue triangles instead of the red ones in order to simulate the computation of the Turing machine, using a similar definition for free rows with the help of blue signals instead of red ones, we would obtain that in each blue triangle there is a single free row, the midline of the triangle; see [5] for the proof. The reason is that generation 0 consists of blue trilaterals in which there is a single free row, while in a red triangle of generation 1 there are three free rows.

Accordingly, it is worth dealing with the detection of the free rows in red triangles. Note that a red triangle of the generation $2n+1$ is crossed by $4^{2n+2} = 8n+1$ isoclines. Accordingly, if the number of free rows of a red triangle of the generation $2n+1$ is very small with respect to its height, it still tends to infinity as $n$ tends to infinity.

To detect the free rows of a red triangle $T$, we decide that the legs of a red triangle send a red signal outside and inside the leg along any green or orange isocline crossed by the leg, the basis being included. It goes on if it meets the legs of a blue triangle or the legs of a phantom, whichever the color. Accordingly, the signal also runs inside a red triangle. However, we forbid the meeting of the red signal with a leg of the same color and of the opposite side. When that is the case, the leg of $T$ sends a yellow signal inside $T$ that may meet the opposite leg only. Accordingly, the yellow signal runs only on free rows. Note that the yellow signal may freely cross legs of blue triangles and of phantoms whatever their color. Accordingly, each green or orange isocline inside $T$ conveys a signal: a red one if on that isocline the signal meets the leg of a red triangle inside $T$, and a yellow one if on that isocline the signal does not meet a red triangle inside $T$. In Section 3.6, we show how the problem is solved.
3.2 Synchronization

We already noted the problem of possible active seeds that are the origin of a bound for some bounded thread.

Another problem arises: on the same green or orange isocline there might be several active seeds. It is important that the red signals raised in a triangle do not meet on the same isocline a yellow signal running inside another triangle. Call *latitude* a finite set of consecutive green and orange isoclines. The *latitude of a trilateral* is the set of green and orange isoclines from the isocline of its vertex to that of its basis.

Note that the lateral red signals give rise to signals that may travel along an isocline far away from the legs of any triangle. Those signals of opposite laterality may meet in between two red triangles and outside them: in that case, a left-hand-side signal coming from the right meets a right-hand-side signal coming from the left. It is important that the latitude of a trilateral coincide with that of trilaterals of the same characteristics belonging to different threads. The red signals used for detecting the free rows are not enough for that property.

To better see what is involved, we need the following notion. Consider a trilateral \( T \) of generation \( n + 1 \). If the vertex \( V \) of \( T \) is inside a triangle \( T_1 \), we say that \( T_1 \) is the *father* of \( T \). Note that a trilateral may have no father: this is the case in a wire defined by a bounded thread. If \( T \) has a father of \( T_1 \), we may define the father of \( T_1 \) if it exists later. Accordingly, for any trilateral \( T \), we construct a sequence \( \{T_k\}_{k \in [0..h]} \) such that \( T_0 = T \) and for each \( k \) in \([0..h-1]\), \( T_{k+1} \) is the father of \( T_k \) and \( T_h \) has no father. Each \( T_k \) with \( k \) in \([0..h]\) is called an *ancestor* of \( T \), \( k \) is its *rank* and \( T_h \) is called the *remotest ancestor* of \( T \). Note that the generation of the remotest ancestor of a trilateral \( T \) depends on the wire to which \( T \) belongs.

We append two kinds of signals. We consider a special signal for blue trilaterals: the vertex of a blue trilateral as well as the ends of its basis trigger a *blue* signal, the same one whichever the laterality of the end emitting it, whichever the attribute of the trilateral. Such a signal is important due to the fact that a thread may not cross the latitude of a given trilateral. Also, to distinguish latitudes of red trilaterals, we need to mark the isoclines passing through the vertices of red triangles. We call that latter mark the *silver* signal. It is raised by the vertex \( V \) of a red triangle and it travels on the orange isocline that passes through \( V \).

The silver and the blue signals allow us to prove the following property:

**Lemma 13.** Let \( T \) be a trilateral belonging to a wire \( \mathcal{W} \). Let \( S \) be a trilateral whose characteristics are those of \( T \), \( S \) belonging to a wire \( \mathcal{V} \), with \( \mathcal{V} \neq \mathcal{W} \). Then \( T \) and \( S \) have the same latitude if and only if \( T, S \)
have an ancestor $X$, $Y$, respectively, of the same rank, such that the vertex of $X$ and that of $Y$ lie on the same isocline.

Note that Lemma 13 mentions an ancestor within the same latitude and it says nothing of the remotest ancestors of $T$ and $S$ that may belong to different latitudes. As a consequence of the lemma, we can say that the latitudes of red triangles of the generation $2n + 1$ are the same whatever the wire giving rise to the interwoven triangles and for any $n \in \mathbb{N}$.

Proof of the Lemma in [6].

Later on, we refer to Lemma 13 when we say that the silver and the blue signals allow us to synchronize all wires of the tiling.

The problem raised by possible bounds of threads is dealt with as follows. The blue signal emitted by the basis of a trilateral of a wire may meet the basis emitted on the same isocline by a blue trilateral of another wire. Such a meeting is permitted: it solves the problem of possibly missing trilaterals in a bounded thread.

From our description of the signals emitted by the legs of a triangle in order to detect free rows inside them, we can see that such signals must cross legs of the same laterality. As illustrated by Figure 15, we can see that a red signal emitted by a red triangle $T_0$ included into another red triangle $T_1$ also cuts the legs of $T_1$. Those red signals are similar to the blue signals defined earlier for both trilaterals. The difference here is that they concern red triangles only and that they are not emitted only by the vertex and the basis: they are emitted on each green or orange isocline crossing the leg. We also decide that right-, left-hand-side red signals coming from the left, the right, respectively, may match with a red basis coming from the right, the left, respectively.

![Figure 15](https://doi.org/10.25088/ComplexSystems.32.1.19)

Figure 15. The free rows in the red triangles. They are in yellow in the figure. Note that the yellow signal is superposed with the mauve signal on the mid-line of red triangles.
Note that appending the silver signal means just changing a bit the tiles conveying an orange isocline, but it also requires to append five tiles, since there are tiles outside legs and bases of trilaterals that convey an orange isocline with no signal at all.

We remain with the condition meant by the general tiling problem. We borrow the next subsection from [5] with a few changes.

### 3.3 The General Tiling Problem

In the proofs of the general tiling problem in the Euclidean plane by Berger [2] and by Robinson [3], there is an assumption that is implicit and that was, most probably, considered as obvious at that time.

Consider a finite set $S$ of prototiles. We call solution of the tiling of the plane by $S$ a partition $\mathcal{P}$ such that the closure of any element of $\mathcal{P}$ is a copy of an element of $S$. We notice that the definition contains the traditional condition on matching signs in the case when the elements of $S$ possess signs. We also notice that a copy means an isometric image. In that problem, we assume that only shifts are allowed and we exclude rotations. Note that, in the Euclidean case, rotations are also ruled out. Here rotations have to be explicitly ruled out, since the shifts leaving the tiling globally invariant also generate the rotations that leave the tiling globally invariant. In fact, we accept isometries and only those such that a tile marked $w$ or $b$ on a given isocline is transformed into a tile marked $w$ or $b$, respectively, on an isocline, the same one or another one.

Note that the general tiling problem can be formalized as follows:

$$\forall S(\exists \mathcal{P} \text{sol}(\mathcal{P}, S) \lor \neg (\exists \mathcal{P} \text{sol}(\mathcal{P}, S))),$$

where $\text{sol}(\mathcal{P}, S)$ means that $\mathcal{P}$ is a solution of $S$ and where $\lor$ is interpreted in a constructive way: there is an algorithm that, applied to $S$, provides us with “yes” if there is a solution and “no” if there is none.

The origin-constrained problem can be formalized in a similar way by:

$$\forall (S, a) (\exists \mathcal{P} \text{sol}(\mathcal{P}, S, a) \lor \neg (\exists \mathcal{P} \text{sol}(\mathcal{P}, S, a))),$$

where $a \in S$, with the same algorithmic interpretation of $\lor$ and where the formula $\text{sol}(\mathcal{P}, S, a)$ means that $\mathcal{P}$ is a solution of $S$ that starts with $a$. Note that if $a$ is a blocking tile, that is, a tile that cannot abut any tile in $S$, then we may face a situation where we cannot tile the plane because $a$ was chosen at random, while it is possible to tile the plane. A solution is to exclude $a$ from the choice. Another solution is to allow the occurrence of contradictions because a wrong tile was chosen, while the appropriate tile would raise no contradiction. Of course, there must be a restriction: such a change should occur finitely many times at most for the same place.
Obviously, if we have a solution of the general tiling problem for the considered instance, we also have a solution of the origin-constrained problem, with the given advantage that we may choose the first tile. To prove that the general tiling problem, in the considered instance, has no solution, we have to prove that, whatever the initial tile, except the blocking one, the corresponding origin-constrained problem has no solution either.

The present construction aims at the same goal.

From Proposition 7, we know that the trilaterals are bigger and bigger once their generation is triggered along a wire. Consequently, what we suggested with the Y-paths and the free rows answers positively the possibility to simulate any Turing machine working on a semi-infinite tape, which, as is well known, does not alter the generality. We are left with the way to force such computations.

### 3.4 The Seeds

We establish that there are enough seeds for starting the computation of a Turing machine in the interwoven triangles.

We have the important property:

**Lemma 14.** Let the root of a tree of the tiling $T$ be on a green or an orange isocline. Then, there is a seed in the tiles of $T$ on the next orange or green isocline, respectively, downward. Starting from that last isocline, there are seeds, downward, on all the isoclines.

As shown in [6], it is not difficult to derive from Lemma 14 that:

**Lemma 15.** Assume that there is a seed on a green isocline. Then, there is at least a seed on the next green isocline and on each further isocline, whichever its color.

In each red triangle, we define a limited grid in which we simulate the execution of the same Turing machine starting from the same initial finite configuration. Of course, the whole initial configuration occurs in a big enough red triangle. If the configuration is not complete in a red triangle, the computation halts on the basis of the red triangle. Accordingly as the red triangles are bigger and bigger, if the machine does not stop, it is possible to tile the plane. If the machine halts, the halting produces a tile that prevents the tiling from being completed. As far as the halting problem of Turing machines starting from a finite initial configuration is undecidable, that reduction proves that the tiling problem of the hyperbolic plane is also undecidable.

Since we know that there are enough active seeds, we have to look at how the triangles constructed from them behave.

The construction performed in Section 2 required the realization of the interwoven triangles starting from at least one wire, since that
alone entails the construction of bigger and bigger triangles that are
disjoint from each other. The synchronization property of the silver
and blue signals guarantees that the same computation is performed
in the red triangles of the same latitude, which proves Theorem 1, pro-
vided that we produce the needed tiles.

3.5 The Implementation

As can immediately be seen, the important feature is not that we have
strictly parallel lines, and that squares are aligned along lines that are
perpendicular to the tapes. What is important is that we have a grid,
which may be a more or less distorted image of the representation just
described.

We can reinforce Lemma 15:

Lemma 16. For each tile $\tau$, there is an active seed whose distance from
$\tau$ is at most 12.

Proof in [6].

In Figure 12, we can see two active seeds and several seeds that are
not active. Accordingly, most seeds are not active, but the active ones
are also dense in the heptagrid. It means that if we start to tile the hep-
tagrid from an arbitrary tile, the blocking one being excepted, sooner
or later we fall upon an active seed. We go back to that topic a bit
later and will also discuss the exact set of prototiles.

As already mentioned, the legs issued from an active seed $\sigma$ follow
the borders of $T(\sigma)$. Note that the active seeds also send signals on
the green and the orange isoclines.

What is important is the thread structure and Lemma 7. Note that
the silver and the blue signals prevent the occurrence of two active
seeds on the same isocline giving rise to trilaterals of different charac-
teristics.

3.6 The Tiles

In this subsection, we describe as precisely as possible the tiles needed
for the constructions defined in the previous subsections. The descrip-
tion is split into two parts.

We revisit the prototiles defined in Section 2.6 with Figures 13 and
14. Indeed, we have to implement the detection of the free rows, the
construction of the red and blue signals and then the travel of the
computing signal $\xi$. That latter is tightly connected with the program
of the simulated Turing machine $M$, so that the related prototiles are
more accurately called metatiles, since they bear variable signs for
the content of a square of the tape of $M$, for the state of $M$ and for
the direction $\delta$ that has to be followed in order to meet the next $Y$-
path. The detection of the free rows and the construction of red and
blue signals are defined by Section 3.6.1. The management of the signal $\xi$ is performed in Section 3.6.2.

### 3.6.1 The Prototiles

With the silver signal, we fix the implementation of the triangles and of the phantom. The actual place of the generation $n + 1$ is fixed by the first choice of an active seed, which is in a free green isocline. If the active seed triggers a triangle, a phantom, the active seeds of the basis of the trilateral trigger a phantom, a triangle, respectively, whence the expected alternation that the whole construction is based upon.

The set of tiles we turn to now is called the set of prototiles. We subdivide the set into two parts: the construction of the isoclines and the construction of the trilaterals. A prototile is a pattern. Indeed, a tile is the indication of two data points: the location of a tile in the heptagrid and the indication of a copy of the prototile that is placed at that location. The mark of the isoclines indicates which shifts are allowed: from a tile on an isocline to another tile on another isocline, provided that the marks of the image match with those of the new isocline.

The set of prototiles forces the construction of the tiling with the isoclines. It also forces the activation of the seeds and the consecutive construction of the embedded triangles, including the detection of the free rows in the triangles. Figures 13 and 14 illustrate the prototiles. Each figure defines marks on the tiles for the construction of the tiling, of the isoclines, of the triangles and of the phantoms, respectively. We append to those figures two new prototiles in order to introduce the newly defined red and yellow signals that allow us to locate the free rows.

Figure 16 illustrates the generation of the yellow and the red signals by the legs of a triangle. The figure makes use of metatiles to do that, meaning the light mauve color indicating the isocline can be freely replaced either by the color orange or green, depending on which isocline we consider: remember that the red, blue and yellow signals run on green or orange isoclines only. Accordingly, that color represents a variable for colors of the isoclines. A part of the tiles of Figure 16 is already present in Figure 14: the new red, blue or yellow signs are appended to those of Figure 14 for 20 of them.

Note that the tiles allowing red and blue signals to meet are attached to B-tiles: there are enough of them on each isocline. The distance between two consecutive B-tiles on an isocline is at most 5, as can be checked on Figure 12.

From that remark and summing the prototiles defined in Figures 13, 14 and 16, we get 232 prototiles.
Figure 16. Generation of the red and the yellow signals by the legs of the triangle. Note that there are two red signals: one for the left-hand-side legs and the other for the right-hand-side legs. Taking into account the possible colors of the isocline, we get 75 tiles to be appended for those from Figures 13 and 14.

Lemma 17. There is a set of 232 prototiles that allows us to construct a tiling of the heptagrid implementing the embedded triangles with their isoclines together with the detection of the free rows in each triangle, the latitudes of trilaterals with identical attributes being synchronized.

Note that the number of free rows in a trilateral is that of Lemma 12, since the basis of a triangle is not signalized as a free row.

In the Appendix of [6], several figures illustrate the construction of the tiling by focusing each one on the images belonging to Figure 7.

3.6.2 The Metatiles

Let us now examine the construction of prototiles for simulating a Turing machine. As already mentioned, that part of the prototiles depends upon the Turing machine \( M \) that is simulated. It also depends on the data \( D \) to which \( M \) is applied. Of course, it would be possible to consider Turing machines starting from an empty tape. The consequence would be a huge complexification of the machine, which would store the data into its states. It is simpler to consider that \( M \) applies to data distinguished from the states. It is the reason we call these tiles metatiles.

As already mentioned, the simulation of the computation of \( M \) is organized in the red triangles, starting from generation 0. The interest
of those infinitely many generations is the fact that since the number $n$ of the generation increases, the number of free rows in the corresponding triangles also increases, which gives the tiling more time in the simulation of $M$. Also note that the space also increases since the height of a red triangle of the generation $2n + 1$ is $8^{n+1}$ according to Lemma 6. As already mentioned, the initial configuration is displayed along the rightmost branch of a red triangle $T$ which, outside the head of $T$, consists of O-tiles. A tree of the heptagrid rooted at a tile on the rightmost branch of $T$ has its leftmost branch constituted of Y-tiles. Now, from Lemma 4, those borders do not meet the legs of a triangle inside $T$. Accordingly, the computation signal $\xi$ travels on Y-tiles only when it goes from a free row of $T$ to the next one and it crosses consecutive tiles when it travels on a free row of $T$.

The metatiles are illustrated by Figure 17.

![Figure 17. The metatiles for simulating the working of $M$.](image)

Metatile 17.1 sends a white signal to the right-hand-side leg of the triangle it generates. It triggers tile 17.3, which is on that leg, in order to represent the squares of the Turing tape of $M$. Metatiles 17.6 up to 17.11 illustrate the travel of the current state of $M$ on a free row. Note that if a seed occurs on that isocline, it must be active and it is either metatile 17.1 or 17.10, depending on whether the tile triggers a red triangle or a blue trilateral, respectively. Metatiles 17.12 up to 17.15 illustrate the execution of an instruction: the current state travels on the free row going to the left or to the right, metatiles 17.14, 17.15 or 17.12, 17.13, respectively. The difference is seen on the Y-son: if the new state goes to the left, to the right, the mark is put to the left, to the right, respectively, of the yellow mark of the Y-son. Metatiles 17.16 up to 17.19 allow the new signal following the Y-path to cross nonfree rows. When it reaches the free row, the new current state goes to the left, to the right, metatiles 17.20, 17.21, respectively, depending on the side from which the current state came along the Y-path. Metatile 17.22 is used when the current state
reaches the basis of the red triangle: the computation is stopped, since there are not enough free rows in that triangle for the computation of $M$. Metatiles 17.23 and 17.24 are used when the current state is the halting state: when it meets the free row, the halting of the computation of $M$ is implemented. Note that those latter tiles cannot abut any other tiles of those we defined and cannot abut each other or each one with itself. And so, we can see that the computation in a triangle always stops. Either it happens by the meeting of the computing signal along a $Y$-path with the basis of the triangle, or it happens by the halting of $M$ itself. Metatiles 17.22 up to 17.24 illustrate those situations.

The number of metatiles depends upon the number $s$ of states and the number $\ell$ of letters of $M$. From Figure 17, we can make the following counting:

1–5: 5 tiles;
6–11: $12 \times \ell$ tiles: two possible isoclines and $\ell$ possible states;
12–15: $2 \times I$ tiles; $I$: number of instructions of $M$;
16–19: $3 \times I$ tiles: the four tiles together, three possible isoclines and $I\ell$ possible instructions;
20–21: $2 \times I$ tiles; two possible isoclines, $I\ell$ possible instructions;
22–24: 3 tiles,

where $I$ is the number of instructions of the program of $M$, $\ell$ is the number of states and $s$ is the number of letters. Note that $I \leq \ell \times s$. Also $D$ is the length of the data written with letters of $M$.

Accordingly, the total number of metatiles is $7I + 12\ell + D + 8$.

Combining that result with the previous countings we get:

**Lemma 18.** For each Turing machine $M$ with $s$ states and $\ell$ letters whose program contains $I$ instructions exactly, and whose data requires $D$ letters, there is a set $\mathcal{P}_{M,D}$ of $240 + 7I + 12\ell + D$ prototiles, so that the tiling problem is undecidable for the set of all $\mathcal{P}_{M,D}$ applied to data written with letters of $M$.

The exact description of Figures 14 and 17 together with the counting of Lemma 18 constitutes an algorithmic reduction from the set of Turing machines as defined in Section 3.5 to the set of finite sets of prototiles we consider. That completes the proof of Theorem 1.

### 4. A Few Corollaries for Connected Tiling Problems

For the convenience of the reader, this section reproduces the similar section of [6, Section 7].
The construction leading to the proof of Theorem 1 allows us to get a few results in the same line of problems.

As indicated in [12, 13], there is a connection between the general tiling problem and the Heesch number of a tiling. That number is defined as the maximum number of rings around a disc that can be formed with the tiles of a given set of tiles; see [14] for more information. As indicated in [13], and as our construction fits in the case of domino tilings, we have the following corollary of Theorem 1.

**Theorem 3.** There is no computable function that bounds the Heesch number for the tilings of the hyperbolic plane.

The construction of [15] gives the following result; see [16, 17].

**Theorem 4.** The finite tiling problem is undecidable for the hyperbolic plane.

Indeed, when the Turing machine halts, the halting state triggers a signal that encloses the computation area and that compels the tiling to be completed by blank tiles only; see [17].

Combining the construction proving Theorem 4 and the partition theorem that is proved in [7, Chapter 4, Section 4.5.2] about the splitting of Fibonacci patchworks (also see [18]), the construction of this paper allows us to establish the following result [19].

**Theorem 5.** The periodic tiling problem is undecidable for the hyperbolic plane, also in its domino version.

Note that the analog of Theorem 5 for the Euclidean plane was proved by Gurevich and Koriakov; see [20].

In the statement of Theorem 5, periodic means that there is a shift that leaves the tiling globally invariant. The construction mimics that of Theorem 4 in the fact that if the simulated Turing machine halts, we also enclose the computing area. But this time, we enlarge the notion of computing area and of triangles so as to also permit black trees to support embedded triangles. In this way, we can define areas of the kind defined by Fibonacci patchworks and of the size dictated by the halting of the machine. We define colors for these surrounding signals in such a way that they entail a construction of a scaled Fibonacci tree; see [21]. Next, it is not difficult to construct a tiling of the hyperbolic plane in this way, periodically, applying the shifts already mentioned in [7, Chapter 4, Section 4.5.3; also see [18].

At last, in another direction, we may apply the arguments of Hanf and Myers [22, 23], and prove the following result.

**Theorem 6.** There is a finite set of tiles such that it generates only non-recursive tilings of the hyperbolic plane.

The proof makes use of the construction of two incomparable recursively enumerable sets $A$ and $B$ of integers. The set of tiles defines the computation of these sets by a Turing machine. Moreover,
the set of tiles tiles the plane if and only if there is a set to separate A from B. As such a set cannot be constructed by an algorithm, we obtain the result stated in Theorem 6.

5. Conclusion

It seems to me that the construction of Section 2 could be applied to prove undecidability problems on cellular automata. Of course, the halting problem for cellular automata is undecidable, but this is a simple consequence of the undecidability of the same problem for Turing machines.

In fact, it is interesting to notice that the construction of Section 2, which is based on Construction 3, can be performed by a cellular automaton. The working of the automaton is briefly sketched in [6].

As indicated in the Introduction, the construction is inspired by the construction in [5]. However, and this was the main goal of the present paper, the number of needed prototiles is significantly reduced.

From Lemma 18, simulating a Turing machine $M$ whose program contains $I$ instructions for $s$ states and $\ell$ and which letters applied to a data point $D$ of length $D$ with a tiling, $240 + 7.I + 12.\ell + D$ prototiles are needed in our simulation. In contrast, $18 \ 870 + 880.s + 1852.\ell + 192.I + D$ prototiles are required by [5]. If we apply those formulas to the universal Turing machine with six states and four letters from [24] we get 449 prototiles with the present paper, while 35782 of them are required for [5]. The present result is more than 79 times better than that of [5].

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