Efficient Alternating Least Squares Algorithms for Truncated HOSVD of Higher-Order Tensors

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Abstract The truncated Tucker decomposition, also known as the truncated higher-order singular value decomposition (HOSVD), has been extensively utilized as an efficient tool in many applications. Popular direct methods for truncated HOSVD often suffer from the notorious intermediate data explosion issue and are not easy to parallelize. In this paper, we propose a class of new truncated HOSVD algorithms based on alternating least squares (ALS). The proposed ALS-based approaches are able to eliminate the redundant computations of the singular vectors of intermediate matrices and are therefore free of data explosion. Also, the new methods are more flexible with adjustable convergence tolerance and are intrinsically parallelizable on high-performance computers. Theoretical analysis reveals that the ALS iteration in the proposed algorithms is q-linear convergent with a relatively wide convergence region. Numerical experiments with both synthetic and real-world tensor data demonstrate that ALS-based methods can substantially reduce the total cost of the original ones and are highly scalable for parallel computing.

Keywords Tucker decomposition · Truncated higher-order singular value decomposition · Best low multilinear rank approximation · Alternating least squares · Parallelization

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1 Introduction

As a natural extension of vectors (order one) and matrices (order two), higher-order tensors have been receiving increasingly more attention in various applications, such as signal processing [16,50].
For decades, tensor decompositions have been extensively utilized as an efficient tool for dimension reductions, latent variable analysis and other purposes in a wide range of scientific and engineering fields. There exist a number of tensor decomposition models, such as canonical polyadic (CP) decomposition, also known as CANDECOMP/PARAFAC decomposition, Tucker decomposition, tensor train (TT) model, and hierarchical Tucker (HT) model. Among them, the Tucker decomposition, also known as the higher-order singular value decomposition (HOSVD), is regarded as a generalization of the matrix SVD and has been applied with significant successes in many applications.

In both theory and practice, a commonly considered problem is the truncated Tucker decomposition, which satisfies

$$\min_{\mathbf{B}} \|\mathbf{A} - \mathbf{B}\|,$$

(1.1)

where $\mathbf{A} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$ is a given order $N$ tensor and $\mathbf{B} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$ is its low multilinear rank approximation. Existing approaches solving (1.1) can be roughly divided into two categories: direct and iterative methods. The most popular direct algorithms for the low multilinear rank approximation of higher-order tensors is the truncated HOSVD (t-HOSVD) and its improved version, the sequentially truncated HOSVD (st-HOSVD). Despite the fact that the results of t-HOSVD and st-HOSVD are usually suboptimal, they can serve as good initial solution for popular iterative methods such as higher-order orthogonal iteration (HOOI). Other than the HOOI method, some efforts are also made in developing second-order approaches, such as Newton-type and trust-region algorithms. Although these methods can achieve faster convergence under certain conditions, they are still in early study and are usually not suitable for large-scale tensors.

In this paper, we focus on study how to efficiently compute the truncated Tucker decomposition (1.1) of higher-order tensors by using the direct algorithms, i.e., t-HOSVD and st-HOSVD. As a major cost of the two algorithms, the computation of tensor-matrix multiplications has been extensively optimized in a number of high-performance tensor libraries. Another potential bottleneck of the t-HOSVD and st-HOSVD algorithms is the calculation of the singular vectors of the intermediate matrices, which can be done by the truncated matrix SVD or eigen-decomposition of Gram matrix. The truncated matrix SVD can be obtained by using Krylov subspace methods, whilst the eigen-decomposition of the symmetric nonnegative definite Gram matrix can be done with a Krylov-Schur algorithm. Due to the fact that these methods rely on the factorization of intermediate matrices, they suffer from the notorious data explosion issue. And even if the hardware storage allowed, they are still not scalable for parallel computing and the total computation cost could be unbearably high.

In order to improve the performance of the t-HOSVD and st-HOSVD algorithms, we propose a class of alternating least squares (ALS) based algorithms for solving the truncated HOSVD problem. The key observation is that in the original algorithms the computations of singular vectors of the intermediate matrices are indeed not necessary and can be replaced with low rank approximations, and the low rank approximations can be done by using an ALS method which does not explicitly require the intermediate matrices. The proposed ALS-based algorithms enjoy advantages such as low computational cost, adjustable tolerance control, easy parallelization, and total avoidance of the intermediate data explosion issue. We present theoretical analysis and show that the ALS iteration in the proposed algorithms is q-linear convergent with a relatively wide convergence region.
Several numerical experiments with both synthetic and real-world tensor data demonstrate that new algorithms can substantially reduce the total cost of the original ones and are highly parallelizable.

The organization of the paper is as follows. In Sec. 2, we introduce some basic notations of tensor and the corresponding algorithms. In Sec. 3, the $t$-HOSVD-ALS and $st$-HOSVD-ALS algorithms are proposed. Some theoretical analysis on the convergence behavior of the ALS methods can also be found in Sec. 3. After that, computational complexity and the approximation errors of proposed algorithms are analyzed in Sec. 4. Test results on several numerical experiments are reported in Sec. 5. And the paper is concluded in Sec. 6.

2 Notations and Nomenclatures

Symbols frequently used in this paper can be found in the following table.

| Symbols | Notations |
|---------|-----------|
| $a$ | scalar |
| $a$ | vector |
| $A$ | matrix |
| $\mathbf{A}$ | three or higher-order tensor |
| $\circ$ | vector outer product |
| $\times_n$ | mode-$n$ product of tensor and matrix |
| $I_n$ | identity matrix with size $n \times n$ |
| $I_{n;N}$ | $\prod_{i=1}^{N} I_i$ |
| $\mathcal{R}(\mathbf{A})$ | a subspace formed by the columns of matrix $\mathbf{A}$ |
| $\sigma(\mathbf{A})$ | a set that consists of singular values of matrix $\mathbf{A}$ |
| $\mathbf{A}^\dagger$ | pseudo-inverse of matrix $\mathbf{A}$ |

Given an order $N$ tensor $\mathbf{A} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$, we denote $\mathbf{A}_{i_1,i_2,\cdots,i_N}$ as its $(i_1, i_2, \cdots, i_N)$-th element. In particular, rank one tensor is denoted as

$$u_1 \circ u_2 \circ \cdots \circ u_N,$$

where $u_n \in \mathbb{R}^{I_n}$ is a vector.

The norm of tensor $\mathbf{A}$ is defined as

$$\|\mathbf{A}\|_F = \sqrt{\sum_{i_1,i_2,\cdots,i_N} \mathbf{A}_{i_1,i_2,\cdots,i_N}^2}.$$  

The matricization of a higher-order tensor is a process of reordering the elements of the tensor into a matrix. For example, the mode-$n$ matricization of tensor $\mathbf{A}$ is denoted as $\mathbf{A}_{(n)}$, which is a matrix belonging to $\mathbb{R}^{I_n \times (I_1 \cdots I_{n-1} I_{n+1} \cdots I_N)}$. Specifically, the $(i_1, i_2, \cdots, i_N)$-th element of tensor $\mathbf{A}$ is mapped to the $(i_n, j)$-th entry of matrix $\mathbf{A}_{(n)}$, where

$$j = 1 + \sum_{k=1,k \neq n}^{N} (i_k - 1) J_k \text{ with } J_k = \prod_{m=1,m \neq n}^{k-1} I_m.$$
The multilinear rank of a higher-order tensor $\mathbf{A}$ is an integer array $(R_1, R_2, \cdots, R_N)$, where $R_n$ is the rank of its mode-$n$ matricization $\mathbf{A}_{(n)}$.

A frequently encountered operation in tensor computation is the tensor-matrix multiplication. In particular, the mode-$n$ tensor-matrix multiplication refers to the contraction of the tensor with a matrix along the $n$-th index. For example, suppose that $\mathbf{U} \in \mathbb{R}^{J_1 \times I_n}$ is a matrix, the mode-$n$ product of $\mathbf{A}$ and $\mathbf{U}$ is denoted as $\mathbf{A} \times_n \mathbf{U} \in \mathbb{R}^{I_1 \times \cdots \times I_{n-1} \times J \times I_{n+1} \times \cdots \times I_N}$. Elementwisely, one has

$$
\mathbf{B}_{i_1, \ldots, i_n} = (\mathbf{A} \times_n \mathbf{U})_{i_1, \ldots, i_n} = \sum_{i_n=1}^{I_N} \mathbf{A}_{i_1, \ldots, i_{n-1}, i_n} \mathbf{U}_{i_n}. 
$$

The Tucker decomposition [55,37], also known as the higher-order singular value decomposition (HOSVD) [14], is formally defined as

$$
\mathbf{A} = \mathbf{G} \times_1 \mathbf{U}^{(1)} \times_2 \mathbf{U}^{(2)} \cdots \times_N \mathbf{U}^{(N)},
$$

where $\mathbf{G} \in \mathbb{R}^{R_1 \times R_2 \times \cdots \times R_N}$ is referred to as the core tensor and $\mathbf{U}^{(n)} \in \mathbb{R}^{I_n \times R_n}$ are column orthogonal with each other for all $n \in \{1,2,\cdots,N\}$. We remark here that the size of the core tensor is often smaller than that of the original tensor, though it is hard to know how small it can be a priori [18,34]. In many applications, the Tucker decomposition is usually applied in its truncated form, which reads

$$
\mathbf{G}^{*} = \mathbf{A} \times_1 (\mathbf{U}^{(1)})^T \times_2 (\mathbf{U}^{(2)})^T \cdots \times_N (\mathbf{U}^{(N)})^T,
$$

which means

$$
\mathbf{A} \approx \mathbf{G} \times_1 \mathbf{U}^{(1)} \times_2 \mathbf{U}^{(2)} \cdots \times_N \mathbf{U}^{(N)}
$$

is the best low multilinear rank approximation of $\mathbf{A}$.

To compute the best low multilinear rank approximation of a higher-order tensor in the truncated Tucker decomposition, a popular approach is the truncated HOSVD ($t$-HOSVD, [55]) originally presented by Tucker himself [55]. Nowadays, it is better known with the effort of Lathauwer et al. [15], who analyzed the structure of core tensor and proposed to employ truncated SVD of the intermediate matrices in truncated HOSVD. The computing procedure of $t$-HOSVD is given in Algorithm 1.

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**Algorithm 1** $t$-HOSVD [55,15]

**Input:** Tensor $\mathbf{A} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$, truncation $(R_1, R_2, \cdots, R_N)$

**Output:** Low multilinear rank approximation $\mathbf{A} \approx \mathbf{G} \times_1 \mathbf{U}^{(1)} \times_2 \mathbf{U}^{(2)} \cdots \times_N \mathbf{U}^{(N)}$

1: for all $n \in \{1,2,\cdots,N\}$ do
2: \hspace{1cm} Compute $\mathbf{Q} \in \mathbb{R}^{I_n \times R_n}$ which is comprised of the $R_n$ leading left singular vectors of $\mathbf{A}_{(n)}$
3: \hspace{1cm} $\mathbf{U}^{(n)} = \mathbf{Q}$
4: end for
5: $\mathbf{G} = \mathbf{A} \times_1 (\mathbf{U}^{(1)})^T \times_2 (\mathbf{U}^{(2)})^T \cdots \times_N (\mathbf{U}^{(N)})^T$
Similarly, the computational cost when we select (\(N, \ldots, 2, 1\)) as the order of Algorithm 2 is

\[
O(\sum_{n=1}^{N} R_{n;N} I_{1:n}) \approx O(\sum_{n=1}^{N} R_{n;N} I_{N}^{n}). \tag{2.4}
\]
Clearly, (2.4) is smaller than (2.3). □

For the best low multilinear rank approximation (2.1), it is easy to see that $U^*_{(n)}$ is a column orthogonal factor matrix, therefore $(U^*)_{(n)} U^*(n)^T$ represents the orthogonal projection of subspace $R(U^*)_{(n)}$. Consequently, subspace represented by the optimal factor matrices are critical. Truncated SVD and eigen-decomposition are the commonly applied approaches to determine this subspace in the original t-HOSVD and st-HOSVD procedures, both of which have advantages and disadvantages. For instance, the truncated SVD has less computational complexity and can avoid the intermediate data explosion when an implicit algorithm is used, but is hard to parallelize. Eigen-decomposition, on the other hand, is more parallelization friendly than truncated SVD, but suffers from the intermediate data explosion issue and induces more computational cost.

In addition to truncated SVD or eigen-decomposition, tensor matricization and tensor-matrix multiplication are also important in the original t-HOSVD and st-HOSVD algorithms. Recently, some efforts on high-performance optimizations of basic tensor operations are made. For example, Li et al. proposed a shared-memory parallel implementation of dense tensor-matrix multiplication [38], and Smith et al. considered sparse tensor-matrix multiplications [51]. Nevertheless, the calculation of truncated SVD or eigen-decomposition is still the major challenge in the t-HOSVD and st-HOSVD algorithms.

3 Alternating Least Squares Algorithms for t-HOSVD and st-THOSVD

In this paper we tackle the challenges of the original t-HOSVD and st-HOSVD algorithms from an alternating least squares (ALS) perspective. Instead of utilizing truncated SVD or eigen-decomposition on the intermediate matrices, we propose to compute the dominant subspace with an ALS algorithm to solve a closely related matrix low rank approximation problem. The new method is referred to as t-HOSVD-ALS and st-HOSVD-ALS, respectively. Compared with the original t-HOSVD and st-HOSVD, the proposed algorithms enjoy advantages such as low computational cost, easy to parallelize, and free of the intermediate data explosion issue.

The classical ALS method for solving matrix low rank approximation problems was originally proposed by Leeuw et al., [17] and further applied in principal component analysis [63]. Algorithm 3 shows the detailed procedure of the ALS method.

Algorithm 3 $[L^*, R^*] = ALS(A, r)$

Input: Matrix $A \in \mathbb{R}^{m \times n}$, truncation $r < \min\{m, n\}$
Initial guesses $L_0 \in \mathbb{R}^{m \times r}$ or $R_0 \in \mathbb{R}^{n \times r}$
Output: Low rank approximation $\hat{A} = L^* R^* T$

1: $k = 0$
2: while not convergent do
3:     Solving multi-side least squares problem $\min_{R} \| L_k R^T - A \|^2_F$
4:     $R_k = (A^T L_k) (L_k^T L_k)^{-1}$
5:     Solving multi-side least squares problem $\min_{L} \| R_k L^T - A^T \|^2_F$
6:     $L_{k+1} = (AR_k) (R_k^T R_k)^{-1}$
7:     $k = k + 1$
8: end while
As an iterative method, the number of iterations for the ALS method has a dependency on the initial guess and the convergence criterion [54]. In what follows we will establish a rigorous convergence theory of the ALS method and derive an evaluation of the convergence region, which can help understand how the initial guess could affect the speed of convergence.

To establish the convergence theory of the ALS method, we first require the following lemma, which was proved in [60].

**Lemma 1** Let $A, B \in \mathbb{R}^{n \times n}$ be symmetric positive definite matrices and satisfy $B \leq A$, then the following inequalities hold

$$\|A^{-1}B\|_2 \leq 1 \text{ and } \|BA^{-1}\|_2 \leq 1,$$

where $B \leq A$ represents $A - B$ is symmetric semi-positive matrix.

The convergence theorem of Algorithm 3 is summarized in the theorem below.

**Theorem 1** Let $A \in \mathbb{R}^{m \times n}$ be a matrix, and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{\min(m,n)}$ be the singular values. Suppose that the following conditions hold:

1. $\sigma(L_k), \sigma(R_k)$ are uniformly bounded.
2. $R(L)_0$ is in a neighborhood of the exact solution.

Then Algorithm 3 is local $q$-linear convergent, and the convergence ratio is approximately $\sigma^2_{r+1}/\sigma^2_r$, where $\sigma_{r+1} < \sigma_r$.

This theorem illustrates the convergence of the ALS method in a viewpoint of subspace, and the convergence ratio depends on the gap of $\sigma_r$ and $\sigma_{r+1}$. The detailed proof can be found in Appendix A.

**Remark 1** If condition 1° in Theorem 1 is not satisfied, then either $L_k$ or $R_k$ is close to singular. This implies that the truncation $r$ is inappropriately chosen, i.e., greater than the numerical rank of $A$.

An evaluation of the convergence region of the ALS method can be found in the following theorem.

**Theorem 2** Under the assumption of Theorem 1, provided that the initial guess $L_0$ satisfies

$$\|L_0^{(2)}L_0^{(1)}L_0^{(1)}L_0^{(2)}\|_2 \leq \sqrt{\frac{\sigma^2_r - (\sigma_r - \varepsilon)^2}{\sigma^2_r - \sigma^2_{\min}}},$$

then the ALS method converges to the exact solution. Here

$$U^T L_0 = \begin{pmatrix} U_1^T L_0^{(1)} \\ U_2^T L_0^{(2)} \end{pmatrix} = \begin{pmatrix} L_0^{(1)} \\ L_0^{(2)} \end{pmatrix},$$

$A = U \Sigma V^T$ is the full SVD of $A$, $U = [U_1, U_2]$ is the block form of $U$, and $\varepsilon$ is an arbitrary positive number such that $\sigma_r - \varepsilon > \sigma_{r+1}$. 
The proof of Theorem 2 can be found in Appendix B. We remark that it can be seen from the theorem that, within the convergence region, better initial guess is guaranteed to lead to faster convergence. It is also worth noting that (3.1) indicates that the convergence region depends on $\varepsilon$. A smaller $\varepsilon$ means higher requirement for the initial guess, but less number of iterations.

With the help of Algorithm 3, we are able to solve the rank-$R_n$ approximation problem to obtain the dominant subspace of $A_{(n)}$ in $t$-HOSVD. Based on it, we derive the ALS accelerated versions of the $t$-HOSVD algorithm, namely $t$-HOSVD-ALS, presented in Algorithm 4.

**Algorithm 4** $t$-HOSVD-ALS

**Input:** Tensor $\mathbf{A} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$, truncation $(R_1, R_2, \cdots, R_N)$

**Output:** Low multilinear rank approximation $\mathbf{A} \approx \mathbf{G} \times_1 U^{(1)} \times_2 U^{(2)} \cdots \times_N U^{(N)}$

1: for all $n \in \{1, 2, \cdots, N\}$ do
2: $[L, \sim] = \text{ALS}(\mathbf{A}_{(n)}, R_n)$
3: Reduced QR decomposition $L = \hat{Q}\hat{R}$
4: $U^{(n)} = \hat{Q}$
5: end for
6: $\mathbf{G} = \mathbf{A} \times_1 U^{(1)T} \times_2 U^{(2)T} \cdots \times_N U^{(N)T}$

The ALS improved $t$-HOSVD algorithm, referred to as $st$-HOSVD-ALS can be analogously derived, as presented in Algorithm 5.

**Algorithm 5** $st$-HOSVD-ALS

**Input:** Tensor $\mathbf{A} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$, truncation $(R_1, R_2, \cdots, R_N)$

**Output:** Low multilinear rank approximation $\mathbf{A} \approx \mathbf{G} \times_1 U^{(1)} \times_2 U^{(2)} \cdots \times_N U^{(N)}$

1: Select an order of $\{1, 2, \cdots, N\}$, i.e., $\{i_1, i_2, \cdots, i_N\}$
2: Let $\mathbf{B} = \mathbf{A}$
3: for all $n \in \{i_1, i_2, \cdots, i_N\}$ do
4: Mode-$n$ matricization $\mathbf{B}_{(n)} \leftarrow \mathbf{B}$
5: $[L, R] = \text{ALS}(\mathbf{B}_{(n)}, R_n)$
6: Reduced QR decomposition $L = \hat{Q}\hat{R}$
7: $U^{(n)} = \hat{Q}$
8: Update $\mathbf{B}_{(n)} = RR^T$
9: $\mathbf{B} \leftarrow \mathbf{B}_{(n)}$ in tensor format
10: end for
11: $\mathbf{G}_{(i_N)} = \mathbf{B}_{(i_N)}$
   $\mathbf{G} \leftarrow \mathbf{G}_{(i_N)}$ in tensor format

The difference between Algorithm 4 and 5 is whether or not to store $R$ and $\hat{R}$, the right factor matrices of the ALS method and reduced QR decomposition, respectively. Storing them will help reduce the overall computational cost when updating tensor $\mathbf{B}$, and core tensor $\mathbf{G}$ can be calculated with the last factor matrix simultaneously in Algorithm 5. Apart from the computational cost of
ALS in the $t$-HOSVD-ASL algorithm, calculating the core tensor $G$ is also critical, especially for higher-order tensors. An extra advantage of the $st$-HOSVD-ALS is that the computational cost of core tensor is avoided as much as possible.

Compared with $t$-HOSVD and $st$-HOSVD, the proposed algorithms exhibit several advantages. First, the redundant computations of the singular vectors are totally avoided, thus the overall cost of the algorithm can be substantially reduced. Second, the convergence of the ALS procedure is controllable by adjusting the convergence tolerance. This is helpful considering the fact that $t$-HOSVD and $st$-HOSVD are quasi-optimal, and are often used as the initial guess for other iterative algorithms such as HOOI. Third, the algorithms are free of intermediate data explosion since the least square problems can be solved without explicitly computing $A_{(n)}$ or $B_{(n)}$.

An added benefit of the proposed $t$-HOSVD-ALS and $st$-HOSVD-ALS algorithms is that the solution of the multi-side least squares problems is intrinsically parallelizable. By using the ALS method, each row of the factor matrix $L$ or $R$ can be independently updated. Therefore, one can distribute the computation of the rows over multiple computing units. Since the workload for each row is almost identical, a simple static load distribution strategy suffices. All other operations in the algorithms, such as the matrix-matrix multiplication, the QR reduction and the matrix inversion, can also be easily parallelized by calling vendor-supplied highly optimized linear algebra libraries.

4 Computational Cost and Error Analysis

In the proposed $t$-HOSVD-ALS and $st$-HOSVD-ALS algorithms, the performance of the ALS iteration depends on several factors, such as the initial guess and the convergence criterion. Based on the convergence property and the convergence condition of the ALS method, we suggest to set the initial guess $L_0$ as follows.

1. Generate a random matrix $S$, whose entries are uniform distributions on interval $[0,1]$.
2. Compute the reduced QR decomposition $A_{(n)}S = QR$.
3. Let $Q$ be the initial guess, i.e., $L_0 = Q$.

In this way, it is assured that $\mathcal{R}(L_0)$ is a subspace of $\mathcal{R}(A_{(n)})$, which is closer to the left dominant subspace of $A_{(n)}$ than a random initial guess. Also, step 3 makes sure that the initial guess is properly normalized.

The stopping condition of the ALS iteration can be set to

$$\|A_{(n)} - L_k R_k^T \|_F - \|A_{(n)} - U_1 U_1^T A_{(n)} \|_F \leq \eta \|A\|_F,$$

where $\mathcal{R}(U_1)$ is the left dominant subspace of $A_{(n)}$, and $\eta$ is an accuracy tolerance parameter. In practice, however, $U_1$ is often not available. We therefore advise to replace (4.1) by

$$\|A_{(n)} - L_k R_k^T \|_F - \|A_{(n)} - L_{k+1} R_{k+1}^T \|_F \leq \eta \|A\|_F$$

as the stop criterion.

Next, we will discuss truncation $R_n$ and how to select the tolerance parameter $\eta$ by error analysis. To analyze the approximation error of ALS-based algorithms, we first recall a useful lemma.

**Lemma 2** [56] Let $U^{(n)} \in \mathbb{R}^{I_n \times R_n}$, $n \in \{1, 2, \cdots, N\}$ be a sequence of column orthogonal matrices, calculated via the $t$-HOSVD or $st$-HOSVD algorithm, and suppose that $\hat{A} = A \times_1 (U^{(1)} U^{(1)T}) \times_2$
\((U^{(2)}U^{(2)T}) \cdots \times_N (U^{(N)}U^{(N)T})\) is an approximation of \(A \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}\). Then
\[
\| \hat{A} - A \|_F^2 \leq \sum_{n=1}^N \gamma_n \leq N\| A - A_{opt} \|_F^2, \tag{4.3}
\]
where \(\gamma_n = \sum_{r=R_n+1}^{I_n} (\sigma_r^{(n)})^2\), and \(A_{opt}\) is the optimal solution of problem (1.1).

It is worth noting that although estimation (4.3) ignores the computation error, it is still useful in practice. By Lemma 2, the error analysis of our algorithms is described in Theorem 3, with proof given in Appendix C.

**Theorem 3** If the stop criterion of ALS is set to (4.1), then the approximation errors of Algorithm 4 and 5 are bounded by
\[
\frac{\| \hat{A} - A \|_F}{\| A \|_F} \leq \sqrt{\sum_{n=1}^N (\eta_n^2 + \frac{\gamma_n}{\| A \|_F^2})} \leq \sqrt{N(\eta + \frac{\| A - A_{opt} \|_F}{\| A \|_F})}, \tag{4.4}
\]
where \(\eta = \max_{n \in \{1, 2, \cdots, N\}} \eta_n\).

We remark that although in practice (4.1) is replaced by (4.2), numerical tests indicate that the main result (4.4) still holds. From (4.4), we advice to choose the tolerance parameter \(\eta_n\) such that the dominant term in the right hand side of (4.4) is \(\gamma_n/\| A \|_F^2\) or \(\| A - A_{opt} \|_F/\| A \|_F\). Furthermore, if truncation \(R_n\) is selected appropriately, both \(\gamma_n\) and \(\eta_n\) will be small, and the ALS will converge very fast since \(\sigma_{R_n+1}/\sigma_{R_n} \ll 1\). On the other hand, less suitable truncation \(R_n\) represents larger \(\gamma_n\) and therefore larger \(\eta_n\), which in turn reduces the required number of ALS iterations.

Also of interest to us is the overall costs of the proposed algorithms. We analyze cases related to both general higher-order tensor \(A \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}\) with truncation \((R_1, R_2, \cdots, R_N)\) and cubic tensor \(\mathcal{A} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}\) with truncation \((R, R, \cdots, R)\). The computational complexities of the \(t\)-HOSVD-ALS and \(st\)-HOSVD-ALS algorithms are listed in Table 1, where \(\text{iter}_n\) is the number of ALS iterations for mode \(n\).

| Algorithm   | \(A \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}\) | \(A \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}\) |
|-------------|-------------------------------------------------|-------------------------------------------------|
| \(t\)-HOSVD-ALS | \(O(\sum_{n=1}^{N} (R_nI_1;N)\text{iter}_n)\) | \(O(\sum_{n=1}^{N} (RI_{N})\text{iter}_n)\) |
| \(st\)-HOSVD-ALS | \(O(\sum_{n=1}^{N} (R_nI_{n;N})\text{iter}_n)\) | \(O(\sum_{n=1}^{N} (RI_{N-n+1})\text{iter}_n)\) |

From the table we can see that the computational costs rely greatly on \(\text{iter}_n\), which in turn depends on the initial guess, the truncation \(R_n\) and the accuracy requirement. Our numerical results reveal that \(\text{iter}_n\) is usually far smaller than \(R_n\), indicating the low computational costs of the proposed algorithms.
5 Numerical Experiments

In this section, we will compare the proposed ALS-based algorithms with the original \(t\)-HOSVD and \(st\)-HOSVD algorithms by several numerical experiments related to both synthetic and real-world tensors. The implementation of the original algorithms includes \(t(st)\)-HOSVD-svds which uses matrix SVD to calculate the leading left singular vectors of matrix and \(t(st)\)-HOSVD-eigs which uses eigen-decomposition of the Gram matrix instead. To examine the numerical behaviors of these algorithms, we carry out most the experiments in MATLAB v2017b on a laptop computer equipped with an Intel Core i5-8250U CPU of 1.60 GHz. And to study the parallel performance of the proposed algorithms, we implement the algorithms in C and run them on a workstation equipped with an 32-core Intel Xeon E5-2620 CPU of 2.10 GHz. Unless mentioned otherwise, the tolerance parameter is set to \(\eta = 10^{-4}\), and the maximum numer of ALS iterations is limited to 1,000 in all tests.

5.1 Reconstruction of random low-rank tensors with noise

In the first set of experiments we examine the performance of the original truncated HOSVD algorithms and the proposed ALS-based ones for the reconstruction of random low-rank tensors with noise. The tests are designed following the work of Zhang and Golub [64]. Specifically, the input tensor is randomly generated as

\[
\hat{A} = A + \delta E,
\]

where the elements of \(E\) follow the standard Gaussian distribution, and the noisy level is controlled by \(\delta = 10^{-2}\). The base tensor \(A \in \mathbb{R}^{I \times I \times I}\) has a low multilinear rank structure, which is constructed by

\[
A = \lambda_1 \cdot a_1 \circ b_1 \circ c_1 + \lambda_2 \cdot a_2 \circ b_2 \circ c_2 + \cdots + \lambda_R \cdot a_R \circ b_R \circ c_R,
\]

where \(a_r, b_r, c_r \in \mathbb{R}^I\) are randomly generated normalized vectors, and coefficients \(\lambda_r \in [5, 10]\) for all \(r \in \{1, 2, \cdots, R\}\).

In the tests, we gradually increase the tensor size \(I\) from 20 to 280 with step 20 and set the truncation \(R = 0.2I\). We carry out the tests for 20 times and draw the averaged reconstruction errors and running time in Fig. 1. From the figure, it is observed that there is almost no difference in reconstruction error among all tested algorithms, indicating that the proposed ALS-based methods can maintain the accuracy of the original ones. In terms of the running time, \(t\)-HOSVD-ALS is 2.4\( \times \sim 32\times\) faster than \(t\)-HOSVD-svds, although not significantly faster than \(t\)-HOSVD-eigs. For \(st\)-HOSVD the performance improvement from the ALS-based method is much higher. In particular, by using \(st\)-HOSVD-ALS, a speedup of 58.1\( \times \sim 212\times\) and 8.4\( \times \sim 17.6\times\) is achieved as compared to \(st\)-HOSVD-svds and \(st\)-HOSVD-eigs, respectively.

5.2 Approximation of sparse random tensors

The second set of experiments is designed for testing the capability of the original truncated HOSVD algorithms and the proposed ALS-based ones on approximating randomly generated sparse...
tensors. Inspired by [39], we set the input fourth-order sparse tensor $\mathbf{A} \in \mathbb{R}^{100 \times 100 \times 100 \times 100}$ to be

$$
\mathbf{A} = \sum_{i=1}^{10} \gamma_i \cdot \mathbf{a}_i \circ \mathbf{b}_i \circ \mathbf{c}_i \circ \mathbf{d}_i + \sum_{i=11}^{100} \frac{1}{i^2} \cdot \mathbf{a}_i \circ \mathbf{b}_i \circ \mathbf{c}_i \circ \mathbf{d}_i,
$$

where $\gamma = 10$ is an adjustable parameter, $\{\mathbf{a}_i, \mathbf{b}_i, \mathbf{c}_i, \mathbf{d}_i\}$ are sparse unit vectors for all $i \in \{1, 2, \cdots, 100\}$, randomly generated with 5% nonzeros.

Considering that the input tensor is sparse, it would be of interest to investigate how different algorithms work with small truncation. We therefore carry out the experiments with the truncation $(R, R, R, R)$ set to $R = 5, 10, 15, 20$, respectively. We run the test for 20 times and record the averaged relative residuals and running time of the tested algorithms in Table 2. We can see from the table that relative residuals of all tested algorithms are almost indistinguishable, which again validates the accuracy of the proposed methods. Table 2 also shows that the original svds-based algorithms are the slowest across all tests. For $t$-HOSVD, the ALS-based method does not show much advantage in terms of running time as compared with the eigs-based one, especially when the truncation is very small. But for $st$-HOSVD, the ALS-based approach is substantially faster, cutting the running time by 41.3$\times$~933$\times$ as compared with the original eigs-based algorithm.

To further investigate the performance of the proposed ALS-based algorithms, we show the averaged numbers of ALS iterations in Fig. 2, in which $\text{iter}_n$ represents the number of ALS iterations for mode $n$. Overall it can be seen that the required numbers of ALS iterations are quite low. Especially for all cases with $R \geq 10$, only around 4 iterations are needed for the ALS iteration to converge. The reason behind this interesting observation is that by construction of the tensor $\mathbf{A}$, the gap of singular values $\sigma_R$ and $\sigma_{R+1}$ is large when $R \geq 10$ but small otherwise. This is consistent with our previous theoretical analysis on the convergence behavior of the ALS method.

### 5.3 Compression tensors arising from fluid dynamics simulations

The purpose of this set of experiments is to examine the performance of different truncated HOSVD algorithms for compressing tensors generated from the simulation results of a lid-driven
Table 2: Relative residuals and running time of various truncated HOSVD algorithms for approximating sparse random tensors with different truncations.

| Algorithm | $t$-HOSVD | $st$-HOSVD |
|-----------|-----------|------------|
| R = 5     | svds 2.569 | eigs 14.74 | ALS 2.595 | svds 2.424 | eigs 0.84 | ALS 2.425 |
|           | running time (s) 6.88 | 3.51 | 0.009 |
| R = 10    | relative residual ($\times 10^{-1}$) 1.3592 | eigs 26.56 | ALS 1.3592 | svds 7.08 | eigs 0.87 | ALS 1.3592 |
|           | running time (s) 4.12 | 3.03 | 0.001 |
| R = 15    | relative residual ($\times 10^{-3}$) 7.8033 | eigs 42.69 | ALS 8.5311 | svds 11.91 | eigs 1.00 | ALS 8.0237 |
|           | running time (s) 4.81 | 4.14 | 0.01 |
| R = 20    | relative residual ($\times 10^{-4}$) 5.4019 | eigs 66.17 | ALS 5.6548 | svds 18.45 | eigs 1.24 | ALS 5.3724 |
|           | running time (s) 6.71 | 5.32 | 0.03 |

cavity flow, which is a standard benchmark for incompressible fluid dynamics [10]. The simulation is done in a square domain of length 1 m with the speed of the top plate setting to 1 m/s and all other boundaries with no slip. The kinematic viscosity is $\nu = 1.0 \times 10^{-4}$ m$^2$/s, and the fluid properties is assumed to be laminar. We use the OpenFOAM software package [2] to conduct the simulation on a uniform grid with 512 grid cells in each direction. The simulation is run with time step $\Delta t = 2.0 \times 10^{-4}$ s and terminated at $t = 1.0$ s. We record the magnitude of velocity at $t = 0.01$ s, 0.02 s, $\cdots$, 1.0 s. The simulation results of the lid-driven cavity flow are stored in a third-order tensor of size $512 \times 512 \times 100$. To test the tensor approximation algorithms, we fix the truncation to $(64, 64, 20)$, corresponding to a compression ratio of 320 : 1.
Table 3: Relative residuals and running time of various truncated HOSVD algorithms for compressing tensors arising from fluid dynamics simulations with different tolerance parameters.

| Algorithm       | $t$-HOSVD | st-HOSVD |
|-----------------|-----------|----------|
|                 | svds | eigs | ALS | svds | eigs | ALS |
| $\eta = 10^{-2}$ | 1.124 | 1.124 | 1.160 | 1.124 | 1.124 | 1.187 |
| relative residual ($\times 10^{-3}$) | 24.10 | 1.24 | 0.73 | 11.30 | 0.45 | 0.006 |
| running time (s) | 24.10 | 1.24 | 0.73 | 11.30 | 0.45 | 0.006 |
| $\eta = 10^{-4}$ | 1.124 | 1.124 | 1.139 | 1.124 | 1.124 | 1.145 |
| relative residual ($\times 10^{-3}$) | 25.02 | 1.62 | 1.30 | 11.83 | 0.58 | 0.01 |
| running time (s) | 25.02 | 1.62 | 1.30 | 11.83 | 0.58 | 0.01 |
| $\eta = 10^{-6}$ | 1.124 | 1.124 | 1.124 | 1.124 | 1.124 | 1.124 |
| relative residual ($\times 10^{-3}$) | 25.19 | 1.44 | 2.19 | 11.54 | 0.56 | 0.03 |
| running time (s) | 25.19 | 1.44 | 2.19 | 11.54 | 0.56 | 0.03 |

First, we study the efficiency of tested algorithms under different accuracy requirements. We run the test for 20 times with tolerance parameter $\eta$ adjusted to different values and record the averaged relative residual and running time for each value of $\eta$; the test results are listed in Table 3. From the table we have the following observations.

- The relative residuals and running time of the original HOSVD algorithms are insensitive to the change of the tolerance parameter $\eta$. This is due to the usage of Krylov subspace method for computing matrix truncated SVD or eigen-decomposition.
- For the ALS-based methods, the relative residuals and the running time both depend on $\eta$. With $\eta$ decreased, the relative residuals are reduced to a similar level that the original HOSVD can attain but more running time is required.
- The svds-based algorithms are the slowest in all tests. $t$-HOSVD-ALS is slightly faster than $t$-HOSVD-eigs when $\eta = 10^{-2}$ and $10^{-4}$ but the advantage is lost when $\eta = 10^{-6}$. The st-HOSVD-ALS method, on the other hand, is substantially faster than st-HOSVD-eigs, with a speedup of around $18.7 \times \sim 384.7 \times$.

It seems from the tests that despite the excellent performance of the proposed st-HOSVD-ALS method, the proposed $t$-HOSVD-ALS algorithm is not quite advantageous, especially when the tolerance parameter $\eta$ is small. In practice an HOSVD algorithm is often used as the initial guess of a supposedly more accurate iterative method such as HOOI. In this case it is not necessary to use a tight tolerance parameter. To examine whether $\eta = 10^{-4}$ is a suitable choice for the ALS-based algorithms in the same tests, we draw in Fig. 3 the contours of the original and compressed velocity data at $t = 0.51$ s. It clearly shows that when $\eta = 10^{-4}$, the compressed results are consistent with each other with no distinguishable difference. In fact, the measured maximum difference between the compressed data obtained by the original $t$-HOSVD and that by $t$-HOSVD-ALS is around $1.25 \times 10^{-4}$ which is nearly one order of magnitude smaller than the measured compression error, which is around $1.12 \times 10^{-3}$ in this case.
To further investigate the applicability of the compressed results, we use the computed truncated HOSVD with tolerance parameter $\eta = 10^{-4}$ as the initial guess of the HOOI method with stopping criterion $10^{-12}$. The HOOI method is taken from the MATLAB Tensor Toolbox v3.1 [5]. The test results are presented in Table 4, in which we list the relative residuals with the HOSVD provided initial guesses, the final relative residuals of HOOI, and the number of HOOI iterations, all averaged on 20 independent runs. From the table we can see that although the initial residual provided by the ALS-based algorithms are slightly larger than those provided by the original truncated HOSVD methods, same final residuals can be achieved after HOOI iterations nevertheless. And more importantly, the required numbers of HOOI iterations are insensitive to which specific HOSVD algorithms, original or not, are used as shown in the tests. In other words, the proposed ALS-based methods are able to deliver similar results as the original ones when applying in HOOI, even when the tolerance parameter is relatively loose.

| Algorithm          | Relative residual | Number of HOOI iterations |
|--------------------|-------------------|---------------------------|
|                    | Initial          | Final                     |                           |
| $t$-HOSVD-svds     | $1.1240 \times 10^{-3}$ | $1.1209 \times 10^{-3}$ | 5.0                       |
| $t$-HOSVD-eigs     | $1.1240 \times 10^{-3}$ | $1.1209 \times 10^{-3}$ | 5.0                       |
| $t$-HOSVD-ALS      | $1.1396 \times 10^{-3}$ | $1.1209 \times 10^{-3}$ | 5.5                       |
| $st$-HOSVD-svds    | $1.1240 \times 10^{-3}$ | $1.1209 \times 10^{-3}$ | 6.0                       |
| $st$-HOSVD-eigs    | $1.1240 \times 10^{-3}$ | $1.1209 \times 10^{-3}$ | 6.0                       |
| $st$-HOSVD-ALS     | $1.1454 \times 10^{-3}$ | $1.1209 \times 10^{-3}$ | 6.0                       |

Table 4: A comparison of HOOI results for compressing tensors arising from fluid dynamics simulations with initial solutions provided by various truncated HOSVD algorithms.
5.4 Parallel performance

An advantage of the proposed ALS-based methods is that they are easy to parallelize. In this experiment, we implement the $t$-HOSVD-ALS and $st$-HOSVD-ALS algorithms in C++ with OpenMP multi-threading parallelization [3]. The involved linear algebra operations are available with parallelization from the Intel MKL [1,59] and the open-source ARMA DILLO [45,46] libraries. In the test we choose the input tensor to be the randomly generated ones in the first set of experiments and set the size of the tensor to be $400 \times 400 \times 400$ and the truncation to be $R = 80$. We break down the running time into different portions, including the ALS iterations along the three dimensions ($ALS-i, i \in \{1, 2, 3\}$) and the calculation of the core tensor. The test results are drawn in Fig. 4. Also shown in the figure is the parallel scalability of the proposed algorithms.

From Fig. 4(a), one can observe that $t$-HOSVD-ALS is highly scalable. A speedup of $22.7 \times$ can be achieved when the number of processor cores is increased from 1 to 32, corresponding to a parallel efficiency of 70.9%. In particular, major operations of the $t$-HOSVD-ALS, including ALS iterations and core tensor calculations, are all accelerated efficiently with the increased number of processor cores. On the other hand, it can be seen from Fig. 4(b) that $st$-HOSVD-ALS can also achieve satisfactory parallel scalability of 9.6× when the number of processor cores is increased from 1 to 16, corresponding to a parallel efficiency of 60.0%. There is a drop of parallel efficiency when using 32 processor cores, this is expected considering the fact that the $st$-HOSVD-ALS algorithm is nearly an order of magnitude faster than $t$-HOSVD-ALS as tested, therefore the running time with 32 processor cores is too short, not able to compensate the parallelization overhead. Another interesting observation one can make is that in $st$-HOSVD-ALS the calculation of core tensor no long plays a role. Instead, the ALS along different dimensions are the major cost. Overall, the test results clearly show that the proposed ALS-based algorithms can be easily scalable with high parallel efficiency.
6 Conclusions

In this paper, we proposed a class of ALS-based algorithms for solving the truncated HOSVD problem. Compared with the original t-HOSVD and st-HOSVD algorithms, the proposed algorithms are superior in several ways. First, by eliminating the redundant computations of the singular vectors, the overall costs of the algorithms are substantially reduced. Second, the proposed algorithms are more flexible with adjustable convergence tolerance, which is especially useful when the algorithms are used to generate initial solutions for iterative methods such as HOOI. Third, the proposed algorithms are free of the notorious data explosion issue due to the fact that the ALS procedure does not explicitly require the intermediate matrices. And fourth, the ALS-based approaches are parallelization friendly on high-performance computers. Theoretical analysis shows that the ALS iteration in the proposed algorithms is q-linear convergent with a relatively wide convergence region. Numerical experiments with both synthetic and real-world tensor data demonstrate that proposed ALS-based algorithms can substantially reduce the total cost of truncated HOSVD and are highly parallelizable.

Possible future works could include applying of the proposed ALS-based algorithms to more applications, among which we are especially interest in large-scale scientific computing. It would also be of interest to study randomization techniques to further improve the performance of the proposed algorithms, considering the fact that solving multiple least squares problems with different right-hand sides is the major cost. Some of the ideas presented in this work, such as the utilization of ALS for solving the intermediate low rank approximation problem, might be possible to extend to other tensor decomposition models such as tensor-train (TT) and hierarchical Tucker (HT) decompositions.

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A Proof of Theorem 1

Based on the assumption of Theorem 1, \( L_k \) is nonsingular and \( L_k^T L_k \) is positive definite. Thus, the iterative form of Algorithm 3 is

\[
R_k = A^T L_k (L_k^T L_k)^{-1},
\]

\[
L_{k+1} = A R_k (R_k^T R_k)^{-1},
\]

i.e.,

\[
L_{k+1} = AA^T L_k (L_k^T A A^T L_k)^{-1} (L_k^T L_k).
\]

Suppose that the full SVD of \( A \) is \( A = U \Sigma V^T \), where

\[
U = [U_1, U_2], \quad V = [V_1, V_2], \quad \Sigma = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix}.
\]

Then from (A.2), we have

\[
U^T L_{k+1} = U^T AVV^T A^T UU^T L_k (L_k^T U U^T A V V^T A^T U U^T L_k)^{-1} (L_k^T U U^T L_k),
\]

which can be rewritten into block form

\[
\begin{pmatrix} L_{k+1}^1 \\ L_{k+1}^2 \end{pmatrix} = \begin{pmatrix} \Sigma_1^2 & 0 \\ 0 & \Sigma_2 \end{pmatrix} \begin{pmatrix} L_k^1 \\ L_k^2 \end{pmatrix} (L_k^1)^T \Sigma_1^2 L_k^1 + L_k^2)^{-1} (L_k^1)^T L_k^1 + L_k^2)^{-1} (L_k^1)^T L_k^1 + L_k^2)^{-1} (L_k^1)^T L_k^1 + L_k^2).
It then follows that
\[
L_{k+1}^{(1)} = \Sigma_k^2 L_k^{(1)} (L_k^{(1)T} \Sigma_k^2 L_k^{(1)})^{-1} (L_k^{(1)T} \Sigma_k^2 L_k^{(1)})^{-1} (L_k^{(1)T} L_k^{(1)})^{-1} (L_k^{(1)T} L_k^{(1)})^{-1},
\]
\[
L_{k+1}^{(2)} = \Sigma_k^2 (L_k^{(2)T} \Sigma_k^2 L_k^{(2)})^{-1} (L_k^{(2)T} \Sigma_k^2 L_k^{(2)})^{-1} (L_k^{(2)T} L_k^{(2)})^{-1} (L_k^{(2)T} L_k^{(2)})^{-1}.
\]

Furthermore,
\[
(L_k^{(1)T} \Sigma_k^2 L_k^{(1)})^{-1} = (I + (L_k^{(1)T} \Sigma_k^2 L_k^{(1)})^{-1} (L_k^{(2)T} \Sigma_k^2 L_k^{(2)})^{-1} (L_k^{(1)T} \Sigma_k^2 L_k^{(1)})^{-1}.
\]

Here we suppose that the distance between \(R(L_k)\) and \(R(U_2)\) is small enough, therefore can be denoted as \(\delta_k\), which only depends on \(\|L_k^{(2)}\|_2\) (i.e., there exist two constants \(\alpha, \beta > 0\) such that \(\alpha \delta_k \leq \|L_k^{(2)}\|_2 \leq \beta \delta_k\)). We can then obtain the lower bound of the distance between \(R(L_k)\) and \(R(U_1)\), which is \(1 - \delta_k^2\), and
\[
\|L_k^{(1)}\|_2 \leq C_1, \quad \|L_k^{(1)}\|_2 \leq \frac{C_2}{1 - \delta_k^2},
\]
where \(C_1, C_2\) are constants independent on \(k\) and \(\delta_k\). From (A.5) and (A.6), there exists a constant \(C\) that is only dependent on \(C_1, C_2\) so that the following inequality holds.
\[
(L_k^{(1)T} \Sigma_k^2 L_k^{(1)})^{-1} \leq (L_k^{(1)T} \Sigma_k^2 L_k^{(1)})^{-1} + \frac{C \delta_k^2}{(1 - 2\delta_k^2)^2},
\]

Further, from (A.7) and (A.4), assume that \(\sigma_r > \sigma_{r+1}\), we have
\[
L_{k+1}^{(2)} \leq \Sigma_k^2 \frac{L_k^{(2)}}{\sigma_r^2} (L_k^{(1)T} \Sigma_k^2 L_k^{(1)})^{-1} (L_k^{(1)T} L_k^{(1)})^{-1} + \hat{C} \left( \frac{\delta_k^2}{1 - 2\delta_k^2} + \frac{\delta_k^2}{1 - \delta_k^2} + \frac{\delta_k^2}{1 - 2\delta_k^2} \right),
\]
where \(\hat{C}\) is a constant. Clearly,
\[
0 < L_k^{(1)T} L_k^{(1)} \leq L_k^{(1)T} \Sigma_k^2 L_k^{(1)},
\]
and by Lemma 1, we have
\[
\|L_k^{(1)T} \Sigma_k^2 L_k^{(1)})^{-1} (L_k^{(1)T} L_k^{(1)})\|_2 \leq 1.
\]
Since \(\delta_k\) is small enough, we obtain
\[
\|L_k^{(2)}\|_2 \leq \frac{\sigma_r^2}{\sigma_r^2} \|L_k^{(2)}\|_2 + \hat{C} \delta_k^2 \leq \frac{\sigma_r^2}{\sigma_r^2} \|L_k^{(2)}\|_2 + \frac{\hat{C}}{\sigma_r^2} \|L_k^{(2)}\|_2,
\]
where \(\alpha, \hat{C}\) do not depend on \(k\) and \(\|L_k^{(2)}\|_2\).

Denote
\[
q = \frac{\sigma_r^2}{\sigma_r^2} + \frac{\hat{C}}{\sigma_r^2} \|L_0^{(2)}\|_2.
\]
Since we assume that \(R(L_0)\) is close to \(R(U_1)\) enough, \(\|U_k^{(2)} L_0\|_2\) is sufficiently small, i.e., \(\|L_0^{(2)}\|_2 = o(1)\). In other words, we assume that \(q < 1\). From (A.8), we have
\[
\|L_k^{(2)}\|_2 \leq q \|L_k^{(2)}\|_2
\]
for all \(k\), which leads to
\[
\lim_{k \to +\infty} \|L_k^{(2)}\|_2 \to 0.
\]
Combining with the assumption of $L_k$, it is verified that $R(L_k)$ is orthogonal to $R(U_2)$ with $k \to +\infty$. Since the orthogonal complement space of $R(U_2)$ is unique, we have

$$R(L_k) = R(U_1), \quad k \to +\infty,$$

where $R(U_1)$ is the dominant subspace of $A$. In other words, we have

$$\lim_{k \to +\infty} \|L_k L_k^\dagger - U_1 U_1^T\|_2 = 0,$$

where $L_k^\dagger$ is the pseudo-inverse of $L_k$. Further, from the iterative form of the ALS method, we have

$$R_k = A^T(L_k^\dagger)^T,$$

thus

$$L_k R_k^T = L_k L_k^\dagger A \to U_1 U_1^T A, \quad k \to +\infty.$$  

And since the Frobenius norm $\| \cdot \|_F$ is continuous,

$$\lim_{k \to +\infty} \|A - L_k R_k^T\|_F = \|A - U_1 U_1^T A\|_F.$$  

Since $U_1 U_1^T A$ is the exact solution of low rank approximation of $A$, the convergence of the ALS method is proved.

From (A.8), we further confirm the $q$-linear convergence of the ALS method, with approximate convergence ratio $\sigma_{r+1}/\sigma_r^2$.

\[ \square \]

**B Proof of Theorem 2**

The assumption of Theorem 1 implies that $L_k$ is nonsingular at every iteration $k$. We assume that

$$L_k^{(1)T} \Sigma^2 \Sigma_k^{(2)T} L_k^{(1)} + L_k^{(2)T} \Sigma^2 \Sigma_k^{(2)T} L_k^{(2)}$$

is positive definite. Let $\varepsilon$ be a positive number such that $\sigma_r > \sigma_{r+1} - \varepsilon$. By (A.4), we know

$$L_{k+1}^{(1)} = \frac{\Sigma_1^2}{(\sigma_r - \varepsilon)^2} L_{k}^{(1)} + \frac{\Sigma_2^2}{(\sigma_r - \varepsilon)^2} L_{k}^{(2)} + L_{k}^{(1)T} \Sigma_2 \Sigma_k^{(2)T} L_{k}^{(2)} - L_{k}^{(2)T} \Sigma_2 \Sigma_k^{(2)T} L_{k}^{(1)},$$

and

$$L_{k+1}^{(2)} = \frac{\Sigma_1^2}{(\sigma_r - \varepsilon)^2} L_{k}^{(2)} + \frac{\Sigma_2^2}{(\sigma_r - \varepsilon)^2} L_{k}^{(1)} + L_{k}^{(2)T} \Sigma_2 \Sigma_k^{(2)T} L_{k}^{(2)} - L_{k}^{(1)T} \Sigma_2 \Sigma_k^{(2)T} L_{k}^{(1)},$$

which means

$$L_{k+1}^{(1)} = \frac{\Sigma_1^2}{(\sigma_r - \varepsilon)^2} L_{k}^{(1)} + \frac{\Sigma_2^2}{(\sigma_r - \varepsilon)^2} L_{k}^{(2)} + \frac{\Sigma_1^2}{(\sigma_r - \varepsilon)^2} L_{k}^{(1)} + \frac{\Sigma_2^2}{(\sigma_r - \varepsilon)^2} L_{k}^{(2)} - \frac{\Sigma_1^2}{(\sigma_r - \varepsilon)^2} L_{k}^{(1)} + \frac{\Sigma_2^2}{(\sigma_r - \varepsilon)^2} L_{k}^{(2)}.$$

Clearly it holds that

$$\|L_{k+1}^{(2)}\|_2 \leq \frac{\sigma_{r+1}^2}{(\sigma_r - \varepsilon)^2} \|L_k^{(2)}\|_2$$

under the condition that

$$L_k^{(1)T} L_k^{(1)} + L_k^{(2)T} L_k^{(2)} \leq \frac{\Sigma_1^2}{(\sigma_r - \varepsilon)^2} L_k^{(1)} + \frac{\Sigma_2^2}{(\sigma_r - \varepsilon)^2} L_k^{(2)} + \frac{\Sigma_1^2}{(\sigma_r - \varepsilon)^2} L_k^{(1)} + \frac{\Sigma_2^2}{(\sigma_r - \varepsilon)^2} L_k^{(2)}.$$

If $L_k^{(1)}$ is nonsingular, then (B.3) implies

$$\left(L_k^{(2)} L_k^{(1)^{-1}}\right)^T (I - \frac{\Sigma_2 \Sigma_k^{(2)T}}{(\sigma_r - \varepsilon)^2}) L_k^{(1)^{-1}} \leq \frac{\Sigma_2^2}{(\sigma_r - \varepsilon)^2} - I.$$  

(B.4)
It follows to see that
\[ \|L_k^{(2)}L_k^{(1)}\|_2 \leq \sqrt{\frac{\sigma^2 - (\sigma - \epsilon)^2}{(\sigma - \epsilon)^2 - \sigma^2}} \] (B.5)
is a sufficient condition of (B.4).

Next we will prove that if the initial guess $L_0$ satisfies condition (B.5), then $L_k^{(1)}$ is nonsingular and (B.2) is satisfied at every iteration $k$.

Provided that $L_0$ satisfies (B.5), we obtain
\[ \|L_1^{(2)}\|_2 \leq \frac{\sigma_0^2 + 1}{\sigma^2} \|L_0^{(2)}\|_2. \] (B.6)

And according to the proof of Theorem 1, we know $L_1^{(1)}$ is also nonsingular, which implies
\[
L_1^{(1)T} \Sigma_1^2 L_1^{(1)} + L_1^{(2)T} \Sigma_2^2 L_2^{(2)}
\] is positive definite. Then by (B.1), we have
\[
\|L_1^{(2)}L_1^{(1)T}\|_2 \leq \|\Sigma_2^2 L_2^{(2)}L_0^{(1)T} L_1^{(1)} - 1\|_2 \leq \frac{\sigma_0^2 + 1}{\sigma^2} \|L_0^{(2)}\|_2 \leq \sqrt{\frac{\sigma^2 - (\sigma - \epsilon)^2}{(\sigma - \epsilon)^2 - \sigma^2}}. \] (B.7)

Analogously, we can prove that for every iteration $k$, $L_k^{(1)}$ is nonsingular, i.e., $L_k^{(1)T} \Sigma_1^2 L_k^{(1)} + L_k^{(2)T} \Sigma_2^2 L_2^{(2)}$ is positive definite, and (B.5) is satisfied. Since (B.5) is a sufficient condition of (B.4) and (B.3), inequality (B.2) is true at every iteration $k$, which implies that
\[ \lim_{k \to \infty} \|L_k\|_2 \leq - U_k U_k^T = 0. \]

The rest part of the proof is analogous to the proof of Theorem 1, which is omitted for brevity. \(\square\)

### C Proof of Theorem 3

In Algorithm 4, $U^{(n)}$ is obtained from the rank-$R_n$ approximation of $A_{(n)}$, which is done in an iterative manner and allows a tolerance parameter $\eta_n$. Therefore, we have
\[ \|A_{(n)} - L^* R^T\|_F^2 \leq \eta_n^2 \|A\|_F^2 + \gamma_n, \] (C.1)
where $L^*$ and $R^*$ are the same as in Algorithm 4. Note that $R$ is updated by solving a multi-side least squares problem
\[ \min_{R} \|A_{(n)} - LR^T\|_F, \]
whose exact solution is $R = A_{(n)}^T(L^*)^T$. Thus
\[ \|A_{(n)} - L^* R^T\|_F = \|A_{(n)} - L^* L^* A_{(n)}\|_F, \] (C.2)
where $L^* L^*$ represents an orthogonal projection on subspace $R(L^*)$. Consequently, by (C.1), (C.2) and (4.3), we have
\[ \|A - A\|_F^2 \leq \sum_{n=1}^N (\eta_n^2 \|A_{(n)}\|_F^2 + \gamma_n), \]
which means
\[ \|A - A\|_F^2 \leq \sum_{n=1}^N (\eta_n^2 + \frac{\gamma_n}{\|A\|_F^2}). \]
Combining with (4.3), we obtain (4.4).

The error analysis of Algorithm 5 can be analogously done. \(\square\)