Recurrence relation for the $6j$-symbol of $\text{su}_q(2)$ as a symmetric eigenvalue problem

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Abstract. A well known recurrence relation for the $6j$-symbol of the quantum group $\text{su}_q(2)$ is realized as a tridiagonal, symmetric eigenvalue problem. This formulation can be used to implement an efficient numerical evaluation algorithm, taking advantage of existing specialized numerical packages. For convenience, all formulas relevant for such an implementation are collected in the appendix. This realization is a biproduct of an alternative proof of the recurrence relation, which generalizes a classical ($q = 1$) result of Schulten and Gordon and uses the diagrammatic spin network formalism of Temperley-Lieb recoupling theory to simplify intermediate calculations.

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1. Introduction

Quantum groups first appeared in the study of quantum integrable systems. Since then, they have proven useful in many applications, including among others conformal field theory, statistical mechanics, representation theory and the theory of hypergeometric functions, along with exhibiting a rich internal structure. Quantum groups have appeared seminally in the mathematical physics literature in connection with topological quantum field theory [1]. More recently, the quantum group $\text{su}_q(2)$ was used to construct “loop” [2] and “spin foam” [3] models of quantum gravity with a positive cosmological constant.

The $6j$-symbol (or Racah-Wigner coefficients and henceforth the $6j$ for the classical and $q$-deformed versions respectively) has been studied extensively. It first appeared in work on $q$-hypergeometric functions [4]. Later, it was found to play a central role in the representation theory of $\text{su}_q(2)$ [5]. It is known to satisfy some recurrence relations [4, 6], including a particular linear, three term, single argument one [5, 7, 8]. This recurrence has been used to analyze the asymptotics of the classical [9, 10] and quantum [11] $6j$.

However, this $q$-$6j$ and the need to know its numerical values for a large number of arguments also appear in other segments of the literature [3, 12], less aware of these properties. An explicit formula for the $q$-$6j$ is well known and involves a number of arithmetic operations that is linear in its arguments (see Appendix). In applications where a large number of $6j$-symbols is needed at once, e.g., for all values of one argument with others fixed as is the case in [3], the total number of operations becomes quadratic in the arguments. The above mentioned recurrence relation can be used to greatly increase the efficiency of the calculation by reducing the total operation count to be linear in the arguments. The main goal of this paper is to show that, moreover, this recurrence can be realized as a tridiagonal, symmetric eigenvalue problem, a property not shared by most recurrence relations, especially since the equivalence is established using only rational operations.

A significant advantage of the eigenvalue formulation is the ability to make use of readily available, robust linear algebra packages, such as LAPACK [13], which automatically take care of the important issues of numerical accuracy and stability. When $q = 1$ or when $q$ is a primitive root of unity, the relevant inner product becomes either positive- or negative-definite and standard, specialized numerical methods can be exploited to increase the efficiency of the calculation even further. A secondary goal of this paper is to concisely collect all the relevant information needed to readily implement such an efficient $q$-$6j$ numerical evaluation algorithm without intimate familiarity with the literature on quantum groups or $q$-hypergeometric functions.

Sections 2 and 3, which can be skipped by those familiar with the mathematical literature on $\text{su}_q(2)$ recoupling theory, introduce the basic notions of the spin network formalism [14, 15, 16], define the Kauffman-Lins convention for the $6j$-symbol and summarize basic diagrammatic identities needed for Section 4, where the recurrence relation is realized as an eigenvalue problem. This is accomplished as a byproduct of an alternative proof of the recurrence itself that generalizes the classical argument from the Appendix of [10]. The diagrammatic spin network formalism makes all intermediate calculations easy to check and reproduce. The Appendix conveniently summarizes all formulas needed for a direct computer implementation of the recurrence-based evaluation of the $q$-$6j$, including the connection between the Kauffman-Lins and Racah-Wigner notational convention, which is traditionally used
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in physics.

2. Spin networks

In a variety of physical and mathematical applications, one often encounters tensor contraction expressions of the form

$$T_{i_1 \ldots i_n}^{j_1 \ldots j_m} = A_{i_1 \ldots i_n}^{k_1 \ldots k_p} B_{j_1 \ldots j_m}^{l_1 \ldots l_q} Z_{\ldots},$$

(1)

where $T$, $A$, $B$, $Z$ are invariant tensors, with each index transforming under a representation of a group or an algebra. The application at hand usually calls for evaluating $T$, or at least simplifying it. An extensive literature on this subject exists for the classical group $SU(2)$ or its Lie algebra, a subject known as angular momentum recoupling \[17, 18\]. It is well known that such tensor contractions can be very efficiently expressed, manipulated, and simplified using diagrams known as spin networks \[14\]. Extensions of these techniques \[15, 16\] are also known for the quantum (or $q$-deformed, since they depend on an arbitrary complex number $q \neq 0$) analogs, the quantum group $su_q(2)$ or $U_q(su(2))$. The basics of this diagrammatic formalism, as needed for the derivation of the recurrence relation, are given in this and next sections. All relevant formulas, including explicit spin network evaluations in terms of quantum integers are listed in the Appendix.

Single spin networks are edge-labeled graphs \[1\], where each vertex has valence either 1 or 3. General spin networks are formal linear combinations of single spin networks. Edges attached to univalent vertices are called free. Spin networks without free edges are called closed. Conventionally, the labels are either integers (spins) of half-integers (twice-spins), which correspond to irreducible representations of $su_q(2)$. Reference \[15\] labels all spin networks with twice-spins. Unless otherwise indicated, all conventions in this paper follow \[15\]. Two spin networks may be equal even if not represented with identical labeled graphs. A complete description of these identities are given in \[15\] and \[16\]; their study constitutes spin network recoupling and is what allows us to equate spin networks with $su_q(2)$-invariant tensors and their contractions.

In this correspondence, each index of a tensor, transforming under an irreducible representation, corresponds to a spin network edge, labeled by the same representation (free indices correspond to free edges). In particular, a closed spin network corresponds to a complex number. Spin networks form a graded algebra over $\mathbb{C}$ (as do tensors). The grading is given by the number of free edges (free indices) and the product is diagrammatic juxtaposition (tensor product).

3. Diagrammatic identities

The spin networks with $n$ free edges with fixed labels ($n$-valent spin networks) form a linear space with a natural bilinear form (or inner product). Suppose that the free edges are ordered in some canonical way, then, given two spin networks, we can reflect one of them in a mirror and connect the free edges in order. The value of the resulting closed spin network defines the bilinear form, which is symmetric and

\[\downarrow\] Spin networks are actually ribbon graphs, but since all diagrams in this paper are planar, the ribbon structure can be added through blackboard framing.

\[\uparrow\] A down-to-earth guide to this correspondence, for the classical $q = 1$ case, can be found in Appendix A of \[19\]. Complete details with proofs can be found in \[16\].
non-degenerate [15]. We use the bra-ket notation for this inner product \( \langle s'|s \rangle \), where \( s \) and \( s' \) are two spin networks. We also let \( |s \rangle \) stand for \( s \) and \( \langle s'| \) for the reflection of \( s' \). The existence of an inner product allows the following identities, whose proofs can be found in [15]. For each identity, the corresponding well known fact of \( SU(2) \) representation theory is given.

The space of 2-valent spin networks, with ends labeled \( a \) and \( b \), is 1-dimensional if \( a = b \), and 0-dimensional otherwise. For non-trivial dimension, the single edge gives a complete basis and therefore the bubble identity:

\[
a \quad b = \delta_{ab} a \quad a . \tag{2}
\]

This identity the diagrammatic analog of Schur's lemma for intertwiners between irreducible representations.

The space of 3-valent spin networks, with ends labeled \( a, b \) and \( c \), is also 1-dimensional if the triangle inequalities (A.6) and parity constraints (A.7) are satisfied, and 0-dimensional otherwise, if \( q \) is generic. When \( q \) is a primitive root of unity, the dimension also vanishes whenever the further \( r \)-boundedness constraint (A.8) is violated. In the case of nontrivial dimension, the canonical trivalent vertex gives a complete basis and therefore the vertex collapse identity:

\[
\begin{align*}
\begin{array}{c}
\quad a \\
\quad b \\
\quad c
\end{array}
\begin{array}{c}
\quad a \\
\quad b \\
\quad c
\end{array}
= \theta(a, b, c)
\begin{array}{c}
\quad a \\
\quad b \\
\quad c
\end{array}
, \tag{3}
\end{align*}
\]

\[
\theta(a, b, c) = \begin{array}{c}
\quad a \\
\quad b \\
\quad c
\end{array} . \tag{4}
\]

The normalization of the vertex, the value of the \( \theta \)-network, is evaluated in Equation (A.5). This identity is the diagrammatic analog of the uniqueness (up to normalization) of the Clebsch-Gordan intertwiner.

Now, consider the space of 4-valent networks with free edges labeled \( a, b, c \) and \( d \). There are two natural bases, the vertical \( \langle l | \) and the horizontal \( | j \rangle \):

\[
\begin{align*}
\begin{array}{c}
\quad a \\
\quad b \\
\quad c \\
\quad d
\end{array}
\begin{array}{c}
\quad a \\
\quad b \\
\quad c \\
\quad d
\end{array}
\quad j . \tag{5}
\end{align*}
\]

The admissible ranges for \( j \) and \( l \), the dimension \( n \) of this space, and the conditions on \( (a, b, c, d) \) under which \( n > 0 \) are given by Equations (A.10) through (A.27). The transition matrix between the two bases is given by the so-called Tet-network:

\[
\begin{align*}
\text{Tet}(a, b, c, d; j, l) = \begin{array}{c}
\quad a \\
\quad b \\
\quad c \\
\quad d
\end{array}
\begin{array}{c}
\quad l \\
\quad j \\
\quad l
\end{array}
= \langle j | l \rangle . \tag{6}
\end{align*}
\]
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The coefficients expressing the vertical basis in terms of the horizontal one define the 6j-symbol, which can be expressed in terms of the Tet-network:

$$|l⟩ = \sum_j \begin{vmatrix} a & b & j \\ c & d & l \end{vmatrix}_{KL} |j⟩,$$

(7)

$$\begin{vmatrix} a & b & j \\ c & d & l \end{vmatrix}_{KL} = \begin{array}{c} a \\ c \\ d \\ j \\ l \end{array} \begin{array}{c} b \\ c \\ d \\ j \\ a \end{array}.$$  (8)

Note the subscript KL for Kauffman-Lins, since this 6j-symbol is defined with respect to the conventions of [15]. The relation to the classical Racah-Wigner 6j-symbol used in the physics literature is given explicitly in Equation A.28.

4. Recurrence relation for the Tet-network

The identities given in the previous section allow an alternative, elementary derivation of the three-term recurrence relation for the Tet-network, distinct from the standard one. The standard derivation is given in [7] and another is possible using the general theory of recurrences for q-hypergeometric functions [6], but neither directly yields the symmetric eigenvalue problem form.

It is easy to check, using the bubble identity, that both the vertical and horizontal bases are orthogonal and that they are normalized as

$$⟨\bar{j}|\bar{j}⟩ = \begin{array}{c} a \\ b \\ c \\ d \\ j \end{array} \begin{array}{c} b \\ a \\ c \\ d \\ j \end{array} = \begin{array}{c} a \\ b \\ c \\ d \end{array}$$  (9)

$$⟨l|l⟩ = \begin{array}{c} a \\ b \\ c \\ d \end{array} \begin{array}{c} b \\ a \\ c \\ d \end{array}.$$  (10)

Curiously, when these normalizations are fully expanded using formulas from the Appendix, they take the form $$(-)^\sigma P/Q$$, where P and Q are products of positive quantum integers. In both cases, $$\sigma = (a + b + c + d)/2$$, is an integer independent of j or l. When q = 1 or when q is a primitive root of unity, positive quantum integers are positive real numbers. Hence the above inner product is real and either positive- or negative-definite. On the other hand, for arbitrary complex q, the normalizations (9) and (10) can be essentially arbitrary complex numbers.

If we can find a linear operator L that is diagonal in one basis, but not in the other, then we can obtain $$⟨\bar{j}|l⟩$$ as matrix elements of the diagonalizing transformation. Furthermore, if the non-diagonal form of L is tridiagonal, then the linear equations defining $$⟨\bar{j}|l⟩$$ reduce to a three-term recurrence relation.
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We can construct such an operator by generalizing the argument for the classical case, found in the Appendix of [10]. For brevity of notation, we introduce a special modified version of the trivalent vertex:

\[
\begin{array}{c}
\quad a \\
\quad a
\end{array}
\begin{array}{c}
\quad 2 \\
\quad a
\end{array}
= \begin{array}{c}
\quad a \\
\quad a
\end{array}
\]
\[ (11) \]

The unlabeled edge implicitly carries twice-spin 2 and the bold dot indicates the multiplicative factor of \([a]\). Using it, we can define a symmetric operator \(L\). Its diagrammatic representation and its matrix elements are given below.

The operator \(L\) is diagonal in the \(|l\rangle\)-basis and its matrix elements \(L_{ll'} = \langle l | L | l' \rangle\) are

\[
L_{ll'} = \begin{array}{c}
\quad a \\
\quad c \\
\quad d
\end{array}
\begin{array}{c}
\quad b \\
\quad e
\end{array}
\quad l
= \frac{[a][b]}{[2]} \delta_{ll'}
\]
\[ (12) \]

with

\[
\lambda(a, b, l) = \frac{[a - b + l + 1]}{2} \delta_{ll'} - \frac{[a + b - l + 1]}{2} [a + b + l + 2]_{[2]},
\]
\[ (13) \]

where we have evaluated Tet\((a, a, b, b; l, 2)\) as

\[
\begin{array}{c}
\quad a \\
\quad b \\
\quad c \\
\quad d
\end{array}
\begin{array}{c}
\quad a \\
\quad b \\
\quad c \\
\quad d
\end{array}
\quad l
\]
\[ (14) \]

This result may be obtained directly from Equation (A.9), where the sum reduces to two terms, or from more fundamental considerations [20]. In the limit, \(q \to 1\), the eigenvalues simplify to \(\lambda(a, b, l) = \frac{1}{4} [l(l + 2) - a(a + 2) - b(b + 2)]\), which shows that the operator \(L\) is closely related to the “square of angular momentum” in quantum mechanics, which was used to obtain the classical version of this recurrence relation [10].

On the other hand, in the \(|\bar{j}\rangle\) basis, the operator \(L\) is not diagonal and the matrix elements \(\bar{L}_{jj'} = \langle \bar{j} | L | \bar{j}' \rangle\), making use of the vertex collapse identity, are

\[
\begin{array}{c}
\quad a \\
\quad b \\
\quad c \\
\quad d
\end{array}
\begin{array}{c}
\quad a \\
\quad b \\
\quad c \\
\quad d
\end{array}
\quad j
\]
\[ (15) \]

with the special case \(\bar{L}_{00} = 0\). Fortunately, though \(\bar{L}_{jj'}\) is not diagonal, it is tridiagonal. This property is a consequence of the conditions enforced at the central vertex in both Tet-networks above: the triangle inequality, \(|j - j'| \leq 2\), and the parity constraint, which forces admissible values of \(j\) to change by 2. If these conditions are violated, the matrix element \(\bar{L}_{jj'}\) vanishes.
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The diagonal elements $\bar{L}_{jj}$ can be evaluated using (15). For the off-diagonal elements $\bar{L}_{j,j+2} = \bar{L}_{j,j+2}$ we also need

\begin{equation}
\begin{pmatrix} a & 2 \\ j & d \end{pmatrix} = \frac{1}{[a]} \begin{pmatrix} a + d - j \\ 2 \end{pmatrix} \begin{pmatrix} j + 2 \\ a \end{pmatrix}
\end{equation}

which can be obtained in the same way as (15). Finally, we need the identities

\begin{equation}
\begin{pmatrix} 2 \\ j \end{pmatrix} = -\frac{[j + 2]}{[2]} \bigcirc j \quad \text{and} \quad \begin{pmatrix} 2 \\ j \end{pmatrix} = \bigcirc j + 2.
\end{equation}

The $|j\rangle$-basis matrix elements can now be expressed as (again, recall the special case $\bar{L}_{00} = 0$)

\begin{align}
\bar{L}_{jj} &= -\langle j|j\rangle \frac{[2] \lambda(a, j, d) \lambda(b, j, c)}{[j][j + 2]}, \\
\bar{L}_{j,j+2} &= \langle j+2|j+2\rangle \left[ \frac{a + d - j}{2} \right] \left[ \frac{b + c - j}{2} \right].
\end{align}

The transition matrix elements $\langle j\,|l\rangle$ can now be obtained by solving an eigenvalue problem in the $|j\rangle$-basis:

\begin{align}
\langle j\,|l\rangle - \lambda_l|l\rangle &= \sum_{j'} \frac{\langle j|L - \lambda_l|j'\rangle}{\langle j'|j'\rangle} \langle j'|l\rangle, \\
0 &= \sum_{j'} \left( \frac{\bar{L}_{j,j'}}{\langle j'|j'\rangle} - \lambda_l \delta_{jj'} \right) \langle j'|l\rangle, \\
0 &= \sum_{j'} \left( \bar{L}_{j,j'} - \lambda_l \langle j'|j'\rangle \delta_{jj'} \right) \frac{\langle j'|l\rangle}{\langle j'|j'\rangle}.
\end{align}

where $\lambda_l = \lambda(a, b, l)$. Since $\bar{L}_{jj'}$ is tridiagonal, we obtain a three-term recurrence relation for the $\langle j|l\rangle$ transition coefficients. Expanding the expression for $\bar{L}_{jj'}$, we find the following general form of the recurrence relation:

\begin{equation}
\frac{\bar{L}_{jj-2}}{[j-2][j-2]} \langle j-2|l\rangle + \left( \frac{\bar{L}_{jj}}{[j]|j\rangle} - \lambda_l \right) \langle j|l\rangle + \frac{\bar{L}_{jj+2}}{[j+2][j+2]} \langle j+2|l\rangle = 0,
\end{equation}

with the provision that $\bar{L}_{jj'}$ vanishes whenever either of the indices fall outside the admissible range or $j = j' = 0$. Finally, the transition coefficients are uniquely determined (up to sign) by requiring the normalization condition

\begin{equation}
\sum_j \frac{|l|\langle j|l\rangle}{\langle j|j\rangle} = |l|l\rangle.
\end{equation}

Practically, it is more convenient to recover the correct normalization for all $j$ and fixed $l$, or vice versa, by requiring $\langle j|l\rangle$ to agree with (A.9) for $j = j'$, cf. (A.16), where the sum reduces to a single term.
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Once the Tet-network has been evaluated recursively, the 6j-symbol can be obtained from Equation (7). Alternatively, a linear, three-term recurrence relation directly for the 6j-symbol follows from (24) and the linear, two-term recurrence relations for the bubble and \( \theta \)-networks, obvious from (A.4) and (A.5). However, because of the additional normalization factors in Equation (7), this direct recurrence relation cannot be cast in the form of a symmetric eigenvalue problem like (23) using rational operations alone.

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Appendix A. Formulas

For a complex number \( q \neq 0 \) and an integer \( n \) the corresponding quantum integer is defined as

\[
[n] = \frac{q^n - q^{-n}}{q - q^{-1}}.
\]

(A.1)

In the limit \( q \to 1 \), we recover the regular integers, \( [n] \to n \). When \( q = \exp(i\pi/r) \), for some integer \( r > 1 \), it is a primitive root of unity and the definition reduces to

\[
[n] = \frac{\sin(n\pi/r)}{\sin(\pi/r)},
\]

(A.2)

This expression is clearly real and positive in the range 0 < \( n < r \). Quantum factorials are direct analogs of classical factorials:

\[
[0]! = 1, \quad [n]! = [1][2] \cdots [n].
\]

(A.3)

Next, we give the evaluations of some spin networks needed in the paper. They are reproduced from Ch. 9 of [15]. The bubble diagram evaluates to

\[
\bullet j = (-)^{j}[j + 1]
\]

(A.4)

whenever it is non-vanishing. For generic \( q \), it vanishes if \( j < 0 \) and if \( q \) is a primitive root of unity then it also vanishes when \( j > r - 2 \). The \( \theta \)-network evaluates to

\[
\theta(a, b, c) = \frac{(-)^s(s + 1)!|s - a|!||s - b|||s - c||}{[a]![b]![c]!},
\]

(A.5)

with \( s = (a + b + c)/2 \), whenever the twice-spins \( (a, b, c) \) are admissible and vanishes otherwise. Admissibility consists of the following criteria (besides the obvious \( a, b, c \geq 0 \)):

\[
\begin{align*}
\text{triangle inequalities} & \quad \begin{cases} 
a \leq b + c \\
b \leq c + a \\
c \leq a + b
\end{cases}, \\
\text{parity} & \quad a + b + c \equiv 0 \pmod{2}.
\end{align*}
\]

(A.6)
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When \( q \) is a primitive root of unity, further constraints needs to be satisfied:

\[
\begin{aligned}
r\text{-boundedness} & \quad \left\{ \begin{array}{l}
a, b, c \leq r - 2 \\
a + b + c \leq 2r - 4
\end{array} \right. \\
\end{aligned}
\]

(A.8)

The tetrahedral- or Tet-network evaluates to

\[
\text{Tet}(a, b, c, d; j, l) = \frac{T!}{E!} \sum_{S} \frac{(-)^S[S + 1]!}{S! \prod_{j} [S - a_j]!} \prod_{j} |b_j - S|!
\]

(A.9)

where the summation is over the range \( m \leq S \leq M \) and

\[
\begin{aligned}
T! = \prod_{i,j} [b_j - a_i]!, & \quad E! = [a]![b]![c]![d]!/[j]!/[l]!,
\end{aligned}
\]

(A.10)

\[
\begin{aligned}
a_1 = (a + d + j)/2, & \quad b_1 = (b + d + j + l)/2, \\
a_2 = (b + c + j)/2, & \quad b_2 = (a + c + j + l)/2, \\
a_3 = (a + b + l)/2, & \quad b_3 = (a + b + c + d)/2, \\
a_4 = (c + d + l)/2, & \quad m = \max\{a_i\}, \\
& \quad M = \min\{b_j\}.
\end{aligned}
\]

(A.11)

(A.12)

(A.13)

(A.14)

(A.15)

The indices \( i \) and \( j \) fully span the defined ranges. Each of the triples of twice-spins \( (a, b, c, d) \) are fixed, the admissibility conditions for generic \( q \) enforce the ranges of \( j \leq j \leq J \) and \( l \leq l \leq L \) to

\[
\begin{aligned}
j = \max\{|a - d|, |b - c|\}, & \quad \overline{j} = \min\{a + d, b + c\}, \\
\underline{l} = \max\{|a - b|, |c - d|\}, & \quad \overline{l} = \min\{a + b, c + d\},
\end{aligned}
\]

(A.16)

(A.17)

with

\[
\begin{aligned}
j \equiv a + b \equiv c + d \pmod{2}, & \quad (A.18) \\
l \equiv a + d \equiv b + c \pmod{2}.
\end{aligned}
\]

(A.19)

The number of admissible values is the same for \( j \) and \( l \) and is equal to \( n = \max\{0, \bar{n}\} \), where

\[
\begin{aligned}
\bar{n} = \min\{m, s - M\} + 1, & \quad m = \min\{a, b, c, d\}, \\
s = (a + b + c + d)/2, & \quad M = \max\{a, b, c, d\}.
\end{aligned}
\]

(A.20)

(A.21)

This number \( n \) is also the dimension of the space of 4-valent spin networks with fixed twice-spins \( (a, b, c, d) \) labeling the free edges. This dimension is non-vanishing, \( n > 0 \), precisely when the twice-spins satisfy the conditions

\[
\begin{aligned}
a + b + c + d \leq 2 \max\{a, b, c, d\}, & \quad (A.22) \\
a + b + c + d \equiv 0 \pmod{2}.
\end{aligned}
\]

(A.23)

When \( q \) is a primitive root of unity, the admissible ranges shrink to \( \underline{l} \leq j \leq \overline{j} \), and \( \underline{l} \leq l \leq \overline{l} \), where

\[
\begin{aligned}
\overline{j} = \min\{j, r - 2, 2r - 4 - \max\{a + b + c\}\}, & \quad (A.24) \\
\overline{l} = \min\{l, r - 2, 2r - 4 - \max\{a + b + c + d\}\}. & \quad (A.25)
\end{aligned}
\]
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The number of admissible values in each range is thus restricted to $n = \max\{0, \bar{n}_r\}$, where

$$\bar{n}_r = \min\{\bar{n}, r - 1 - \max\{M, s - m\}\}.$$  \hfill (A.26)

The condition $n > 0$ requires (A.22), (A.23) and

$$a + b + c + d \leq 2 \min\{a, b, c, d\} + 2r - 4.$$  \hfill (A.27)

The above admissibility criteria are well known. However, the consequent explicit expressions for the constraints on $(a, b, c, d)$, the bounds on $j$ and $l$, and the dimension $n$ are not easily found in the literature.

In the classical $q = 1$ case, the Kauffman-Lins version of the $6j$-symbol (7) differs from the Racah-Wigner convention used in the physics literature, which preserves the symmetries of the underlying Tet-network. The two $6j$-symbols are related through the formula

$$\left\{\frac{j_1}{2} \quad \frac{j_2}{2} \quad \frac{j_3}{2}\right\}_{RW} = \frac{\text{Tet}(J_1, J_2, j_1, j_2; J_3, j_3)}{\sqrt{\theta(J_1, J_2, j_3)\theta(j_1, j_2, j_3)\theta(J_1, J_2, J_3)\theta(j_1, j_2, J_3)}}.$$  \hfill (A.28)

which can be obtained by comparing the explicit expressions (A.9) and (6.3.7) of [17]. Note that the argument of the absolute value under the square root has sign $(-)^{j_3-J_3}$.

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