ON THE GROTHENDIECK RING OF VARIETIES IN POSITIVE CHARACTERISTIC

KIRTI JOSHI

Abstract. Let \( k \) be an algebraically closed field of characteristic \( p > 0 \). I prove that the ring of smooth, complete \( k \)-varieties and Bittner relations contains zero divisors if \( p > 13 \) or \( p = 11 \). In particular it follows, under the same hypothesis, that the isomorphism class of any supersingular elliptic curve is a zero divisor in this ring.

1. Introduction

Let \( k \) be an algebraically closed field of characteristic \( p > 0 \). Let \( K_0(\mathcal{V}_k) \) be the Grothendieck ring of varieties over \( k \). This is the following: for an irreducible variety \( X/k \), write \([X]\) for its \( k \)-isomorphism class and consider the free abelian group generated by isomorphism classes \([X]\) of \( k \)-varieties with multiplication given product of varieties: \([X_1][X_2] = [X_1 \times X_2]\) of \( k \)-varieties \( X_1, X_2 \). This construction provides a ring in which \([\emptyset] = 0\) and \([\text{Spec}(k)] = 1\) is the multiplicative identity element. Let \( K_0(\mathcal{V}_k) \) be the quotient by the ideal generated by elements of the form \([X - Y] - [X] - [Y]\) where \( X \supseteq Y \) is a closed \( k \)-subvariety of \( X \). This makes \( K_0(\mathcal{V}_k) \) into a commutative ring with a unit (with \([\text{Spec}(k)] = 1\)) see [8, 3] for the basic properties of this ring.

While the simplicity of definition is one of the most attractive features of \( K_0(\mathcal{V}_k) \), it should be noted that this ring remembers far less algebraic geometry than what one learns in a first course on algebraic geometry. So one can ask what parts of algebraic geometry over \( k \) does \( K_0(\mathcal{V}_k) \) remember? The answer seems to be far from being simple.

Let us also introduce another related ring, denoted \( K_0^\text{bl}(\mathcal{C}_k^{sm}) \), which is generated by isomorphism classes of smooth complete \( k \)-varieties with product as multiplication and with relations \([\emptyset] = 0\) and \([\text{Bl}_Y(X)] - [E] = [X] - [Y]\) where \( \text{Bl}_Y(X) \) is the blowup of a smooth complete \( k \)-variety \( X \) along smooth, complete \( k \)-variety \( Y \subset X \) and \( E \) is the exceptional divisor (after [3] these relations are referred to as Bittner relations).

In [3] it was shown that:

**Theorem 1.1.** Let \( k \) be an algebraically closed field of characteristic zero. Then there is a natural isomorphism

\[
\tilde{K}_0^\text{bl}(\mathcal{C}_k^{sm}) \cong K_0(\mathcal{V}_k)
\]

given by sending \( [X] \mapsto [X] \).

The proof given in [3] also works if \( \text{char}(k) = p > 0 \) provided one knows that the weak factorization theorem of [1] holds for \( k \) and in particular proof of that theorem requires that embedded resolution of singularities (see [6, 13]) holds for \( k \) in all dimensions.

In characteristic zero it was shown in [11] that \( K_0(\mathcal{V}_k) \) is not a domain. The purpose of this note is to prove that \( K_0^\text{bl}(\mathcal{C}_k^{sm}) \) is not a domain (see Theorem 2.1.1) if \( p > 13 \) or \( p = 11 \). My approach differs from [11] in the following way: a key result of [8] which is needed in [11] is not available in characteristic \( p > 0 \); this difficulty is circumvented by working with
ring $K^0_{bl}(CV_{k}^{sm})$ which is conjecturally isomorphic to $K^0(Y_k)$ (under availability of embedded resolution of singularities in all dimensions); for $K^0_{bl}(CV_{k}^{sm})$ it is simpler to construct motivic measures (such as the Albanese measure); the other innovation is that I use a theorem of Pierre Deligne for constructing relations in $K^0_{bl}(CV_{k}^{sm})$.

Let $\mathbb{L} = [A^1] \in K^0(Y_k)$ be the isomorphism class of the affine line in $K^0(Y_k)$. In [4] it was shown (assuming $k = \mathbb{C}$) that $\mathbb{L}$ is a zero divisor in $K^0(Y_k)$. In Section 2.2 I observe that there is an infinite collection of natural candidate relations in $K^0(Y_k)$ which show that $\mathbb{L}$ is a zero divisor (for any field $k$ with $\text{char}(k) = p > 0$). But I do not know how to prove this at the moment (see Section 2.2 for more precise statements).

I was introduced to the circle of ideas surrounding the Grothendieck ring $K^0(Y_k)$ through Ravi Vakil’s lecture course at the Arizona Winter School (AWS 2015). It is a pleasure to thank Ravi Vakil for a stimulating series of lectures. It is also a pleasure to thank Research Institute of Mathematical Sciences (RIMS, Kyoto) for its excellent hospitality and Shinichi Mochizuki for answering questions related to his construction of Albanese varieties and also for hosting my visit to RIMS in Spring 2018.

2. Main Theorem

2.1. The main theorem of this note is the following:

**Theorem 2.1.1.** Let $k$ be an algebraically closed field of characteristic $p > 0$. If $p > 13$ or $p = 11$ then the isomorphism class of any supersingular elliptic curve is a zero divisor in $K^0_{bl}(CV_{k}^{sm})$. In particular $K^0_{bl}(CV_{k}^{sm})$ contains zero-divisors.

**Proof.** Let $AV_k$ denote the multiplicative monoid of isomorphism classes of abelian varieties over $k$ and let $\mathbb{Z}[AV_k]$ denote the monoid ring. To prove the theorem I construct a natural homomorphism (the Albanese motivic measure)

$$\text{Alb} : K^0_{bl}(CV_{k}^{sm}) \to \mathbb{Z}[AV_k]$$

given by $[X] \to [\text{Alb}(X)]$. This construction is the content of Proposition 2.1.6 below. Now let me prove Theorem 2.1.1 assuming the construction of this homomorphism (see Proposition 2.1.6).

Recall from [14, Chapter V, Theorem 4.1(c)] that there are exactly

$$(2.1.2) \quad \delta_p = \left[\frac{p-1}{12}\right] + \varepsilon_p$$

isomorphism classes of supersingular elliptic curve over an algebraically closed field of characteristic $p > 0$, see [14, Chapter 5] for the definition of $\varepsilon_p$ and at any rate note that $\delta_p \geq 2$ if and only if $p > 13$ or $p = 11$.

Suppose that $\delta_p \geq 2$. Let $E_1/k$ be any supersingular elliptic curve and let $E_2/k$ be any supersingular elliptic curve not isomorphic to $E_1$. Then by a beautiful theorem of Pierre Deligne (see [7]) there exists an isomorphism $E_1 \times E_1 \simeq E_1 \times E_2$. Thus one has

$$(2.1.3) \quad [E_1 \times E_1] = [E_1] \cdot [E_1] = [E_1] \cdot [E_2] = [E_1 \times E_2].$$

in $K^0_{bl}(CV_{k}^{sm})$. In particular we have

$$(2.1.4) \quad [E_1] \cdot ([E_1] - [E_2]) = 0.$$ 

Observe that as the images of $[E_1], [E_1] - [E_2]$ are non-zero in $\mathbb{Z}[AV_k]$, and hence one sees that $[E_1], [E_1] - [E_2] \neq 0 \in K^0_{bl}(CV_{k}^{sm})$ so both are zero divisors in $K^0_{bl}(CV_{k}^{sm})$. In particular $[E_1]$ is a zero-divisor in $K^0_{bl}(CV_{k}^{sm})$ as claimed. This proves Theorem 2.1.1.
Remark 2.1.5. More generally Deligne’s Theorem (see [7]) asserts that if \( n \geq 2 \) and \( E_1, \ldots, E_{2n} \) are any supersingular elliptic curves then \( E_1 \times \cdots \times E_n \cong E_{n+1} \times \cdots \times E_{2n} \). So this theorem provides many more relations in \( K^0_{bl}(\mathcal{C}V^m_k) \).

The following proposition, which constructs the Albanese motivic measure on \( K^0_{bl}(\mathcal{C}V^m_k) \) circumvents the deeper construction of motivic measures on \( K^0(\mathcal{V}_k) \) due to [8] which is crucially dependent on (embedded) resolution of singularities.

Proposition 2.1.6. Let \( AV_k \) be the multiplicative monoid of isomorphism classes of abelian \( k \)-varieties and let \( \mathbb{Z}[AV_k] \) be the monoid ring. Then one has a natural morphism

\[
\text{Alb} : K^0_{bl}(\mathcal{C}V^m_k) \to \mathbb{Z}[AV_k]
\]

which is given by \([X] \to [\text{Alb}(X)]\).

Proof. Recall that if \( X \) is a smooth, complete \( k \)-variety then \( \text{Alb}(X) \) is an abelian variety equal to the dual of the reduced Picard scheme \( \text{Pic}(X)_{\text{red}} \) of \( X \). For understanding properties of Albanese varieties of arbitrary varieties over arbitrary fields arbitrary characteristics readers may find [12], [9] useful.

It is clear that \([X] \to [\text{Alb}(X)] \in AV_k \) can be extended linearly (with \([0] \mapsto 0\)) to define a homomorphism of groups from the free abelian group generated by isomorphism classes of smooth, complete \( k \)-varieties. By Lemma 2.1.8, for complete \( k \)-varieties one has\([X] \times [Y] = [X \times_k Y] \mapsto [\text{Alb}(X \times_k Y)] = [\text{Alb}(X)] \times [\text{Alb}(Y)]\). So the mapping respects multiplication. By Lemma 2.1.7 \( \text{Alb}(X') = [\text{Alb}(X)] \) and \([\text{Alb}(Y')] = [\text{Alb}(Y)]\). Thus the element \([X'] - [X] - ([E] - [Y])\) which maps to \([\text{Alb}(X')] - [\text{Alb}(X)] - ([\text{Alb}(E)] - [\text{Alb}(Y)]) = 0\). Thus this homomorphism respects multiplication and Bittner relations and hence this homomorphism factors through the quotient ring \( K^0_{bl}(\mathcal{C}V^m_k) \). Hence one has the induced morphism \( K^0_{bl}(\mathcal{C}V^m_k) \to \mathbb{Z}[AV_k] \). □

Lemma 2.1.7. Suppose \( k \) is algebraically closed and \( Y \subset X \) be smooth, complete \( k \)-varieties. Let \( X' = \text{Bl}_{Y}(X) \) and \( Y' = E \subset X' \) be the exceptional divisor. Then one has natural isomorphisms

\[
\text{Alb}(X') \cong \text{Alb}(X)
\]

and

\[
\text{Alb}(Y') \cong \text{Alb}(Y).
\]

Proof. The proof follows from the well-known fact: if \( A \) is an abelian variety then there are no non-constant morphisms from a projective space \( \mathbb{P}^m \to A \) [10]. Suppose \( A' = \text{Alb}(X') \) and \( A = \text{Alb}(X) \). Since there are no non-constant morphisms from a projective space to an abelian variety, the tautological morphism \( X' \to A' \) factors as \( X' \to X \to A' \) and by the universal property of Albanese variety \( A \) of \( X \) this factors as \( X' \to X \to A \to A' \) and in particular it follows that any morphism from \( X' \) to an abelian variety factors through \( A \) so \( A \) is the Albanese variety of \( X' \) hence \( A = A' \) and similarly one sees that \( \text{Alb}(Y') = \text{Alb}(Y) \). □

Lemma 2.1.8. Let \( X_1, X_2 \) be two smooth, complete \( k \)-varieties. Then

\[
\text{Alb}(X_1 \times_k X_2) \cong \text{Alb}(X_1) \times \text{Alb}(X_2).
\]

Proof. This is immediate from the universal property of Albanese varieties. □
2.2. A relation which may be used to prove that $L$ is a zero divisor in $K^0(\mathcal{V}_k)$. By \cite{2,5} there exist a remarkable collection of explicitly defined smooth affine algebras $A/k$, given explicitly by $A = k[x, y, z, w]/(x^m y + z^e + w + w^s p)$ where $e, m, s$ are positive integers such that $p^e \nmid sp, sp \nmid p^e$, and with the following remarkable properties:

\begin{align}
(2.2.1) \quad & \text{Spec}(A) \times_k A^1 \cong A^4 \\
(2.2.2) \quad & \text{Spec}(A) \not\cong A^3.
\end{align}

This example shows that in $K^0(\mathcal{V}_k)$ one has the relation

\[ [\text{Spec}(A)] \cdot L^4 = ([\text{Spec}(A)] - L^3) \cdot L = 0. \]

and in particular $L$ is a zero-divisor if

\[ [\text{Spec}(A)] - L^3 \neq 0. \]

But I do not know how to prove that $[\text{Spec}(A)] \neq L^3$ in $K^0(\mathcal{V}_k)$. In fact suppose $A, A'$ are two such algebras with different values of parameters $e, m, s$ chosen so that $\text{Spec}(A), \text{Spec}(A')$ are not isomorphic as schemes then it is enough to prove that $[\text{Spec}(A)] \neq [\text{Spec}(A')]$ in $K^0(\mathcal{V}_k)$ as one also has the relation

\[ [\text{Spec}(A)] \cdot L = L^4 = [\text{Spec}(A')] \cdot L. \]

But again I do not know how to prove that $[\text{Spec}(A)] \neq [\text{Spec}(A')]$ in $K^0(\mathcal{V}_k)$.

REFERENCES

[1] Dan Abramovich, Kalle Karu, Kenji Matsuki, and Jaroslaw Wlodarczyk. Torification and factorization of birational maps. \emph{J. Amer. Math. Soc.}, 15(3):531–572, 2002.

[2] T. Asanuma. Polynomial fibre rings of algebras over Noetherian rings. \emph{Inventiones Math.}, 1987.

[3] Franziska Bittner. The universal Euler characteristic for varieties of characteristic zero. volume(number):pages.

[4] Lev Borisov. The class of the affine line is a zero-divisor in the Grothendieck ring. \emph{Preprint: arXiv:1412.6194}, 2014.

[5] Neena Gupta. On the cancellation problem for the affine space $A^3$ in characteristic $p$. \emph{Invent. Math.}, 2013.

[6] Heisuke Hironaka. Resolution of singularities of an algebraic variety over a field of characteristic zero, i, ii. 79:109–203,205–326, 1964.

[7] Toshiyuki Katsura and Frans Oort. Families of supersingular abelian surfaces. \emph{Compositio Math.}, 1987.

[8] Michael Larsen and Valery Lunts. Motivic measures and stable birational geometry. 2001.

[9] Shinichi Mochizuki. Topics in Anabelian Geometry I: Generalities. \emph{Preprint: http://www.kurims.kyoto-u.ac.jp/~motizuki}.

[10] D. Mumford. \emph{Abelian varieties}. Oxford University Press, Bombay.

[11] Bjorn Poonen. The Grothendieck ring of varieties is not a domain. \emph{Math. Res. Lett.}, 2002.

[12] Niranjan Ramachandran. Duality of Albanese and Picard 1-Motives. \emph{K-Theory}, 2001.

[13] Shreeram Shankar Abhyankar. \emph{Resolution of singularities of embedded algebraic surfaces}. Pure and Applied Math. Academic Press, New York, 1966.

[14] Joseph Silverman. \emph{The arithmetic of elliptic curves}, volume 106 of \emph{Graduate Text in Mathematics}. Springer-Verlag, Berlin, 1985.

\begin{center}
\text{Math. department, University of Arizona, 617 N Santa Rita, Tucson 85721-0089, USA.}
\end{center}

\begin{center}
\text{E-mail address: kirti@math.arizona.edu}
\end{center}