PRIME GRAPHS AND EXPONENTIAL COMPOSITION OF SPECIES

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ABSTRACT. In this paper, we enumerate prime graphs with respect to the Cartesian multiplication of graphs. We use the unique factorization of a connected graph into the product of prime graphs given by Sabidussi to find explicit formulas for labeled and unlabeled prime graphs. In the case of species, we construct the exponential composition of species based on the arithmetic product of species of Maia and Méndez and the quotient species, and express the species of connected graphs as the exponential composition of the species of prime graphs.

1. Introduction

Under the well-known concept of Cartesian product of graphs (Definition 2.1), prime graphs (Definition 2.2) are non-trivial connected graphs that are indecomposable with respect to the Cartesian multiplication. Hence any connected graph can be decomposed into a product of prime graphs, and this decomposition is shown by Sabidussi [14] to be unique.

To count labeled prime graphs, we express the Dirichlet exponential generating series (Definition 2.4) of connected graphs as the exponential of the Dirichlet exponential generating series of prime graphs. To count unlabeled prime graphs, we see that the set of unlabeled connected graphs has a free commutative monoid structure with its prime set being the set of unlabeled prime graphs. This free commutative monoid structure enables us to count unlabeled prime graphs in terms of unlabeled connected graphs.

Eventually, we aim at finding the cycle index of the species of prime graphs. To be more precise, we want to find the relation between the species of connected graphs and the species of prime graphs. To start with, we observe that the species associated to a graph is isomorphic to the molecular species corresponding to the automorphism group of this graph. This observation leads to a relation (Proposition 3.6) between the arithmetic product of species (Definition 3.2), studied by Maia and Méndez [9], and the Cartesian product of graphs. Moreover, a theorem (whose simplified but equivalent version is given by Proposition 2.3) of Sabidussi about the automorphism groups of connected graphs in terms of the automorphism groups of their prime factors plays an important role. We define a new operation, the exponential composition of species (Definition 3.10), which corresponds to the exponentiation group (Definition 1.3) in the case of molecular species and is related to the arithmetic product of species as the composition of species is related to the multiplication of species. We get a formula (Theorem 3.19) expressing the species of connected graphs as the exponential composition of the species of prime graphs. The enumeration of the species of prime graphs is therefore completed by applying the enumeration theorem (Theorem 3.15).

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for the exponential composition of species, which is a generalization of an enumeration theorem by Palmer and Robinson [11] on the cycle index polynomial of the exponentiation group.

An explicit formula for the inverse of the exponential composition would be nice to find, but that problem remains open.

1.1. Introduction to Species and Group Actions. The combinatorial theory of species was initiated by Joyal [6, 7]. For detailed definitions and descriptions about species, readers are referred to [2].

In short, species are classes of “labeled structures”. More formally, a species (of structures) is a functor from the category of finite sets with bijections $\mathbb{B}$ to itself. A species $F$ generates for each finite set $U$ a finite set $F[U]$, which is called the set of $F$-structures on $U$, and for each bijection $\sigma: U \to V$ a bijection $F[\sigma]: F[U] \to F[V]$, which is called the transport of $F$-structures along $\sigma$. The symmetric group $\mathfrak{S}_n$ acts on the set $F[n] = F[\{1, 2, \ldots, n\}]$ by transport of structures. The $\mathfrak{S}_n$-orbits under this action are called unlabeled $F$-structures of order $n$.

Each species $F$ is associated with three generating series, the exponential generating series $F(x) = \sum_{n \geq 0} |F[n]| x^n/n!$, the type generating series $\tilde{F}(x) = \sum_{n \geq 0} f_n x^n$, where $f_n$ is the number of unlabeled $F$-structures of order $n$, and the cycle index

$Z_F = Z_F(p_1, p_2, \ldots) = \sum_{n \geq 0} \left( \sum_{\lambda \vdash n} \text{fix } F[\lambda] \frac{p_{\lambda}}{z_{\lambda}} \right)$,

where $F[\lambda]$ denotes the number of $F$-structures on $[n]$ fixed by $F[\sigma]$, $\sigma$ is a permutation of $[n]$ with cycle type $\lambda$, $p_{\lambda}$ is the power sum symmetric function (see Stanley [15, p. 297]) indexed by the partitions $\lambda$ of $n$, and $z_{\lambda}$ is the number of permutations in $\mathfrak{S}_n$ that commute with a permutation of cycle type $\lambda$.

The following identities (see Bergeron, Labelle, and Leroux [2, p. 18]) illustrate the importance of the cycle index in the theory of species.

$F(x) = Z_F(x, 0, 0, \ldots)$,

$\tilde{F}(x) = Z_F(x, x^2, x^3, \ldots)$.

For example, let $\mathcal{G}$ be the species of graphs. Note that by graphs we mean simple graphs, that is, graphs without loops or multiple edges. The cycle index of $\mathcal{G}$ was given in [2, p. 76]:

$Z_{\mathcal{G}} = \sum_{n \geq 0} \left( \sum_{\lambda \vdash n} \text{fix } \mathcal{G}[\lambda] \frac{p_{\lambda}}{z_{\lambda}} \right)$,

where

$\text{fix } \mathcal{G}[\lambda] = 2^{\frac{1}{2}} \sum_{i,j \geq 1} \gcd(i, j) c_i(\lambda)c_j(\lambda) - \frac{1}{2} \sum_{k \geq 1} (k \mod 2) c_k(\lambda)$,

in which $c_i(\lambda)$ denotes the number of parts of length $i$ in $\lambda$. Let $\mathcal{G}^c$ be the species of connected graphs, and $\mathcal{E}$ the species of sets. The observation that every graph is a set of connected graphs gives rise to the following species identity:

$\mathcal{G} = \mathcal{E}(\mathcal{G}^c)$,
which can be read as “a graph is a set of connected graphs”, and gives rise to the identities
\[
\mathcal{G}^c(x) = \log(\mathcal{G}(x)),
\]
\[
\tilde{\mathcal{G}}^c(x) = \sum_{k \geq 1} \frac{\mu(k)}{k} \log(\tilde{\mathcal{G}}(x^k)),
\]
(1.1)
\[
Z_{\mathcal{G}^c} = \sum_{k \geq 1} \frac{\mu(k)}{k} \log(Z\mathcal{G} \circ p_k),
\]
where the operator $\circ$ on the right-hand side of (1.1) denotes the operation of plethysm on symmetric functions (see Stanley [15, p. 447]). For example, we can compute the first several terms of the cycle index of the species of connected graphs $\mathcal{G}^c$ using Maple:
\[
Z_{\mathcal{G}^c} = p_1 + \left( \frac{1}{2} p_1^2 + \frac{1}{2} p_2 \right) + \left( \frac{1}{3} p_3 + \frac{2}{3} p_1^3 + p_1 p_2 \right)
\]
\[
+ \left( \frac{19}{12} p_1^4 + 2 p_1^2 p_2 + \frac{5}{4} p_2^2 + \frac{2}{3} p_1 p_3 + \frac{1}{2} p_4 \right)
\]
\[
+ \left( 193 p_1^3 p_2 + \frac{2}{3} p_2 p_3 + \frac{91}{15} p_1^5 + 5 p_1 p_2^2 + \frac{4}{3} p_1^2 p_3 + \frac{3}{5} p_5 + p_1 p_4 \right)
\]
\[
+ \left( \frac{1669}{45} p_1^6 + \frac{91}{3} p_1^4 p_2 + \frac{38}{9} p_1^3 p_3 + \frac{43}{2} p_1^2 p_2^2 + 2 p_1^2 p_4 + \frac{8}{3} p_1 p_2 p_3 \right)
\]
\[
+ \left( \frac{4}{5} p_1 p_5 + \frac{26}{3} p_2^2 + \frac{5}{2} p_2 p_4 + \frac{25}{18} p_3^2 + \frac{5}{6} p_6 \right) + \cdots.
\]

For operations of species, readers are referred to [2, pp. 1–58] for more detailed definitions of the sum $F_1 + F_2$, the product $F_1 F_2 = F_1 \cdot F_2$, and the composition $F_1(F_2) = F_1 \circ F_2$ of arbitrary species $F_1$ and $F_2$.

The *quotient species* (see [2, p. 159]) is defined based on group actions. It appeared in [3] and [4] as an important tool in combinatorial enumeration. Suppose that a group $A$ acts naturally (see [2, p. 393]) on a species $F$. The quotient species of $F$ by $A$, denoted $F/A$, is defined to be such that for each finite set $U$, $F/A$-structures on $U$ is the set of $A$-orbits of $F$-structures on $U$, and for each bijection $\sigma : U \rightarrow V$, the transport of structures $(F/A)[\sigma] : F[U]/A \rightarrow F[V]/A$ is induced from the bijection $F[\sigma]$ that sends each $A$-orbit of the set $F[U]$ to an $A$-orbit of the set $F[V]$.

The notion of *molecular species* plays an important role in the analysis of species. Roughly speaking, a molecular species is one that is indecomposable under addition. More precisely, a species $M$ is molecular [16, 17] if there is only one isomorphism class of $M$-structures, i.e., if any two arbitrary $M$-structures are isomorphic.

If $M$ is molecular, then $M$ is concentrated on $n$ for some positive integer $n$, i.e., $M[U] \neq \emptyset$ if and only if $|U| = n$. If this is the case, then there is a subgroup $A$ of $\mathfrak{S}_n$ such that $M$ is isomorphic to the quotient species of $X^n$, the species of linear orders on an $n$-element set, by $A$, i.e., $M = X^n/A$. Furthermore, for $A$ and $B$ two subgroups of $\mathfrak{S}_n$ for some $n$, the molecular species $X^n/A$ is isomorphic to the molecular species $X^n/B$ if and only if $A$ and $B$ are conjugate subgroups of $\mathfrak{S}_n$. In other words, for each positive integer $n$, we get a
bijection \( \delta_n \) from the set of conjugate classes of subgroups of the symmetric group of order \( n \) to the set of molecular species concentrated on the cardinality \( n \). A formal construction for the molecular species \( X^n/A \) for a given subgroup \( A \) of \( S_n \) is given by Bergeron, Labelle, and Leroux [2, p. 144].

Polya’s cycle index polynomial [12, pp. 64–65] of a subgroup \( A \) of \( S_n \) is defined to be

\[
Z(A) = Z(A; p_1, p_2, \ldots, p_n) = \frac{1}{|A|} \sum_{\sigma \in A} \prod_{k=1}^{n} p_k^{c_k(\sigma)},
\]

where \( c_k(\sigma) \) denotes the number of \( k \)-cycles in the permutation \( \sigma \).

An application of Cauchy-Frobenius Theorem [13] (Lemma 3.14) gives that the cycle index polynomial of \( A \) is the same as the cycle index of the molecular species \( X^n/A \) (see [8, p. 117] Example 7.4):

\[
Z(A) = Z_{X^n/A}.
\]

This formula illustrates that the cycle index series of species is a generalization of Polya’s cycle index polynomial.

**Definition 1.1.** An example of molecular species is the species associated to a graph. For each graph \( G \) we assign a species \( \mathcal{O}_G \) to it such that for any finite set \( U \), the set \( \mathcal{O}_G[U] \) is the set of graphs isomorphic to \( G \) with vertex set \( U \). The species \( \mathcal{O}_G \) is the molecular species corresponding to the automorphism group of \( G \) as a subgroup of the symmetric group on the vertex set of \( G \). We write \( Z(G) \) for the cycle index of the species associated to the graph \( G \), which is the same as the cycle index polynomial of the automorphism group of \( G \). In other words,

\[
Z(G) = Z_{\mathcal{O}_G} = Z(\text{aut}(G)).
\]

The fact that molecular species are indecomposable under addition leads to a molecular decomposition of any species [2, p. 141]. That is, every species of structures \( F \) is the sum of its molecular subspecies:

\[
F = \sum_{M \subseteq F, M \text{ molecular}} M.
\]

Let \( A \) be a subgroup of \( S_m \), and let \( B \) be a subgroup of \( S_n \). We can construct new groups based on \( A \) and \( B \).

**Definition 1.2.** The product group whose elements are of the form \((a, b)\), where \( a \in A \) and \( b \in B \), and whose group operation is given by \((a_1, b_1) \cdot (a_2, b_2) = (a_1a_2, b_1b_2)\), where \( a_1 \) and \( a_2 \) are elements of \( A \), and \( b_1 \) and \( b_2 \) are elements of \( B \), has two group representations, denoted by \( A \ast B \) and \( A \times B \), where the group \( A \ast B \) acts on the set \([m + n]\) by

\[(1.3) \quad (a, b)(i) = \begin{cases} 
  a(i), & \text{if } i \in \{1, 2, \ldots, m\}, \\
  b(i - m) + m, & \text{if } i \in \{m + 1, m + 2, \ldots, m + n\},
\end{cases}
\]

and the group \( A \times B \) acts on the set \([m] \times [n]\) by \((a, b)(i, j) = (a(i), b(j))\), for all \( i \in [m] \) and \( j \in [n] \).

Therefore, we can identify the group \( A \ast B \) with a subgroup of \( S_{m+n} \), and the group \( A \times B \) with a subgroup of \( S_{mn} \).
Definition 1.3. The wreath product of $A$ and $B$ has group elements of the form $(\alpha, \tau)$, where $\alpha$ is a permutation in $A$ and $\tau$ is a function from $[m]$ to $B$. The composition of two elements $(\alpha, \tau)$ and $(\beta, \eta)$ of $B \wr A$ is given by

$$(\alpha, \tau)(\beta, \eta) = (\alpha \beta, (\tau \circ \beta) \eta),$$

where $\beta \in A$ is viewed as a function from $[m]$ to $[m]$, and $(\tau \circ \beta) \eta$ denotes the point-wise multiplication of $\tau \circ \beta$ and $\eta$, both functions from $[m]$ to $B$.

We introduce two group representations of the wreath product of $A$ and $B$, denoted $B \wr A$ and $B^A$, which were studied in full detail by Palmer and Robinson [11].

First, the group $B \wr A$ acts on the set $[m] \times [n]$ by letting $(\alpha, \tau)(i, j) = (\alpha i, \tau(i) j)$, for all $i \in [m]$ and $j \in [n]$. Hence the group $B \wr A$ can be identified with a subgroup of $\mathfrak{S}_{mn}$.

Second, the group $B^A$ acts on the set of functions from $[m]$ to $[n]$ by letting $(\alpha, \tau)(f) = g$ for $f : [m] \to [n]$, where $g : [m] \to [n]$ is defined by

$$((\alpha, \tau)f)(i) = g(i) = \tau(i)(f(\alpha^{-1} i)),$$

for any $i \in [m]$. We observe that the group $B^A$ can be identified with a subgroup of $\mathfrak{S}_{nm}$.

Yeh [16, 17] proved the following species identities:

$$\frac{X^m X^n}{A \ B} = \frac{X^{m+n}}{A \ast B},$$

$$\frac{X^m}{A} \frac{X^n}{B} = \frac{X^{mn}}{B \wr A}.$$  

Note that these results agree with Pólya’s Theorems [12] for the cycle index polynomials of $A \ast B$ and $B \wr A$. In this paper, we will study the molecular species $X^{mn}/(A \times B)$ (Section 3.1) and the molecular species $X^{nm}/B^A$ (Section 3.2).

2. Labeled and Unlabeled Prime Graphs

2.1. Cartesian Product of Graphs. For any graph $G$, we let $V(G)$ be the vertex set of $G$, $E(G)$ the edge set of $G$, and $l(G) = |V(G)|$ the number of vertices in $G$. Two graphs $G$ and $H$ with the same number of vertices are said to be isomorphic, denoted $G \cong H$, if there exists a bijection from $V(G)$ to $V(H)$ that preserves adjacency. Such a bijection is called an isomorphism from $G$ to $H$. In the case when $G$ and $H$ are identical, this bijection is called an automorphism of $G$. The collection of all automorphisms of $G$, denoted $\text{aut}(G)$, constitutes a group called the automorphism group of $G$. We set $L(G)$ to be the number of graphs isomorphic to $G$ with vertex set $V(G)$. It is easy to see that $L(G) = l(G)!/|\text{aut}(G)|$.

We use the notation $\sum_{i=1}^{n} G_i = G_1 + G_2 + \cdots + G_n$ to mean the disjoint union of a set of graphs $\{G_i\}_{i=1, \ldots, n}$.

Definition 2.1. The Cartesian product of graphs $G_1$ and $G_2$, denoted $G_1 \odot G_2$, as defined by Sabidussi [14] under the name the weak Cartesian product, is the graph whose vertex set is $V(G_1 \odot G_2) = V(G_1) \times V(G_2) = \{(u, v) : u \in V(G_1), v \in V(G_2)\}$, in which $(u, v)$ is adjacent to $(w, z)$ if either $u = w$ and $\{v, z\} \in E(G_2)$ or $v = z$ and $\{u, w\} \in E(G_1)$. 


An example of the Cartesian product of two graphs is given in Figure 1.

Figure 1. The Cartesian product of a graph with vertex set \{1, 2, 3, 4\} and a graph with vertex set \{a, b, c\} is a graph with vertex set \{(i, j)\}, where \(i \in \{1, 2, 3, 4\}\) and \(j \in \{a, b, c\}\).

For simplicity and without ambiguity, we call \(G_1 \odot G_2\) the product of \(G_1\) and \(G_2\).

It can be verified straightforwardly that the Cartesian multiplication is commutative and associative up to isomorphism. We denote by \(G^n\) the Cartesian product of \(n\) copies of \(G\).

**Definition 2.2.** A graph \(G\) is prime with respect to Cartesian multiplication if \(G\) is a connected graph with more than one vertex such that \(G \cong H_1 \odot H_2\) implies that either \(H_1\) or \(H_2\) is a singleton vertex.

Two graphs \(G\) and \(H\) are called relatively prime with respect to Cartesian multiplication, if and only if \(G \cong G_1 \odot J\) and \(H \cong H_1 \odot J\) imply that \(J\) is a singleton vertex.

We denote by \(\mathcal{P}\) the species of prime graphs. We see from Definition 2.2 that any non-trivial connected graph can be decomposed into a product of prime graphs. Sabidussi [14] proved that such a prime decomposition is unique up to isomorphism.

The automorphism groups of the Cartesian product of a set of graphs was studied by Sabidussi [14] and Palmer [10]. For example, Sabidussi proved that the automorphism group of the disjoint union of a set of graphs is isomorphic to the automorphism group of the Cartesian product of these graphs. Sabidussi also showed that the automorphism group of the Cartesian product of the disjoint union of two relatively prime graphs is the product of the automorphism group of these two graphs.
2.2. Labeled Prime Graphs. In this section all graphs considered are connected.

Sabidussi gave an important formula about the automorphism group of a connected graph using its prime factorization: If \( G \) is a connected graph with prime factorization

\[
G \cong P_{s_1}^{r_1} \odot P_{s_2}^{r_2} \odot \cdots \odot P_{s_k}^{r_k},
\]

where for \( r = 1, 2, \ldots, k \), all \( P_r \) are distinct prime graphs, and all \( s_r \) are positive integers, then

\[
\text{aut}(G) \cong \prod_{r=1}^{k} \text{aut}(P_{s_r}^{r}) \cong \prod_{r=1}^{k} \text{aut}(P_r)^{s_r}.
\]

Note that the \( P_{s_r}^{r} \), for \( r = 1, 2, \ldots, k \), are pairwise relatively prime. Since the automorphism group of the Cartesian product of the disjoint union of two relatively prime graphs is the product of the automorphism groups of the graphs, we see that Sabidussi’s formula reduces equivalently to the following proposition:

**Proposition 2.3.** (Sabidussi [14]) Let \( P \) be a prime graph, and let \( k \) be a nonnegative integer. Then the automorphism group of \( P^k \) is the exponentiation group \( \text{aut}(P)^{\otimes k} \), i.e.,

\[
\text{aut}(P^k) = \text{aut}(P)^{\otimes k}.
\]

In particular,

\[
|\text{aut}(P^k)| = |\text{aut}(P)^{\otimes k}| = k! \cdot |\text{aut}(P)|^k.
\]

**Definition 2.4.** The Dirichlet exponential generating series for a sequence of numbers \( \{a_n\}_{n \in \mathbb{N}} \) is defined by \( \sum_{n \geq 1} a_n/(n!n^s) \).

Multiplication of Dirichlet exponential generating series is given by

\[
\left( \sum_{n \geq 1} \frac{a_n}{n!n^s} \right) \left( \sum_{n \geq 1} \frac{b_n}{n!n^s} \right) = \sum_{n \geq 1} \frac{c_n}{n!n^s},
\]

where

\[
c_n = \sum_{k|n} \binom{n}{k} a_k b_{n/k} = \sum_{k|n} \frac{n!}{k!(n/k)!} a_k b_{n/k}.
\]

The Dirichlet exponential generating function for a species \( F \) with the restriction \( F[\emptyset] = \emptyset \) is defined by

\[
\mathfrak{D}(F) = \sum_{n \geq 1} \frac{|F[n]|}{n!n^s}.
\]

The Dirichlet exponential generating function for a graph \( G \) is defined by

\[
\mathfrak{D}(G) = \frac{L(G)}{l(G)! \cdot l(G)^s},
\]

where \( L(G) \) is the number of graphs isomorphic to \( G \) with vertex set \( V(G) \), and \( l(G) \) is the number of vertices of \( G \). In other words,

\[
\mathfrak{D}(G) = \mathfrak{D}(\Theta_G),
\]
where \( \mathcal{G} \) is the species associated to a graph defined by Definition 1.1. Recall that
\[
L(G) = \frac{l(G)!}{|\text{aut}(G)|}
\]
Therefore,
\[
\mathcal{D}(G) = \frac{1}{|\text{aut}(G)| \cdot l(G)^s}.
\]

**Example 2.5.** Let \( \mathcal{P} \) be the species of prime graphs, let \( \mathcal{G}^c \) be the species of connected graphs, let \( \mathcal{C} \) be the set of unlabeled connected graphs, and let \( \mathcal{P} \) be the set of unlabeled prime graphs. Then \( \mathcal{D}(\mathcal{G}^c) \) and \( \mathcal{D}(\mathcal{P}) \) are the Dirichlet exponential generating functions for these two species, respectively:
\[
\mathcal{D}(\mathcal{G}^c) = \sum_{n \geq 1} \frac{[\mathcal{G}^c[n]]}{n! n^s} = \sum_{G \in \mathcal{C}} \mathcal{D}(G), \quad \mathcal{D}(\mathcal{P}) = \sum_{n \geq 1} \frac{[\mathcal{P}[n]]}{n! n^s} = \sum_{P \in \mathcal{P}} \mathcal{D}(P).
\]

Propositions 3.4 and 3.6 lead straightforwardly to the following lemma.

**Lemma 2.6.** Let \( G_1 \) and \( G_2 \) be relatively prime graphs. Then
\[
\mathcal{D}(G_1 \circ G_2) = \mathcal{D}(G_1) \mathcal{D}(G_2)
\]

**Lemma 2.7.** Let \( P \) be any prime graph. Let \( T \) be the set of all nonnegative integer powers of \( P \), i.e., \( T = \bigcup_{k \geq 0} P^k \). Then the Dirichlet exponential generating functions for \( T \) and \( P \) are related by
\[
\mathcal{D}(T) = \exp(\mathcal{D}(P)).
\]

**Proof.** We start with
\[
\mathcal{D}(P) = \frac{L(P)}{l(P)! \cdot l(P)^s} = \frac{1}{|\text{aut}(P)| \cdot l(P)^s}.
\]
It follows from Proposition 2.3 that
\[
L(P^k) = \frac{l(P^k)!}{|\text{aut}(P^k)|} = \frac{l(P^k)!}{k! \cdot |\text{aut}(P)|^k},
\]
and that
\[
\mathcal{D}(P^k) = \frac{L(P^k)}{l(P^k)! \cdot l(P^k)^s} = \frac{1}{k! \cdot |\text{aut}(P)|^k \cdot l(P)^s} = \frac{\mathcal{D}(P)^k}{k!}.
\]
Summing up on \( k \), we get
\[
\mathcal{D}(T) = \sum_{k \geq 0} \frac{\mathcal{D}(P)^k}{k!} = \exp(\mathcal{D}(P)).
\]

**Theorem 2.8.** For \( \mathcal{D}(\mathcal{G}^c) \) and \( \mathcal{D}(\mathcal{P}) \), we have
\[
\mathcal{D}(\mathcal{G}^c) = \exp(\mathcal{D}(\mathcal{P})).
\]
Proof. Lemma 2.6 gives that the Dirichlet exponential generating function of a product of two relatively prime graphs is the product of the Dirichlet exponential generating functions of the two graphs. Since the operation of Cartesian product on graphs is associative up to isomorphism, it follows that if we have a set of pairwise relatively prime graphs \( \{G_i\}_{i=1,2,\ldots,r} \), and let \( G = \bigcirc_{i=1}^r G_i \), then

\[
D(G) = \prod_{i=1}^r D(G_i).
\]

(2.3)

Now according to the definition of the Dirichlet exponential generating function for graphs, we get

\[
D(G^c) = \sum_{G \in \mathcal{G}^c} D(G) = \prod_{P \in \mathcal{P}} D\left(\sum_{k \geq 0} P^k\right) = \prod_{P \in \mathcal{P}} \exp(D(P)) = \exp\left(\sum_{P \in \mathcal{P}} D(P)\right) = \exp(D(\mathcal{P})).
\]

It is well-known that the exponential generating series of the species of connected graphs \( \mathcal{G}^c \) is

\[
\mathcal{G}^c(x) = \sum_{n \geq 1} [\mathcal{G}^c[n]] \frac{x^n}{n!} = \log \left( \sum_{n \geq 1} \frac{2^\binom{n}{2}}{n!} \right)
\]

\[
= \frac{x}{1!} + \frac{x^2}{2!} + \frac{4}{3!} \frac{x^3}{3!} + 38 \frac{x^4}{4!} + 728 \frac{x^5}{5!} + 26704 \frac{x^6}{6!} + 1866256 \frac{x^7}{7!} + \ldots.
\]

We obtain \( D(\mathcal{G}^c) \) by replacing \( x^n \) with \( n^{-s} \) for each \( n \) in the above expression:

\[
D(\mathcal{G}^c) = \sum_{n \geq 1} [\mathcal{G}^c[n]] \frac{1}{n! n^s}
\]

\[
= \frac{1}{1! 1^s} + \frac{1}{2! 2^s} + \frac{4}{3! 3^s} + 38 \frac{1}{4! 4^s} + 728 \frac{1}{5! 5^s} + 26704 \frac{1}{6! 6^s} + 1866256 \frac{1}{7! 7^s} + \ldots.
\]

Theorem 2.8 gives a way of counting labeled prime graphs by writing

\[
D(\mathcal{P}) = \log D(\mathcal{G}^c).
\]

For example, we write down the first terms of \( D(\mathcal{P}) \) as follows:

\[
D(\mathcal{P}) = \frac{1}{2! 2^s} + \frac{1}{3! 3^s} + 35 \frac{1}{4! 4^s} + 728 \frac{1}{5! 5^s} + 26464 \frac{1}{6! 6^s} + 1866256 \frac{1}{7! 7^s} + \ldots.
\]
2.3. Unlabeled Prime Graphs. In this section all graphs considered are unlabeled and connected.

**Definition 2.9.** The *(formal)* Dirichlet series of a sequence \( \{a_n\}_{n=1,2,\ldots,\infty} \) is defined to be \( \sum_{n=1}^{\infty} \frac{a_n}{n^s} \).

The multiplication of Dirichlet series is given by
\[
\sum_{n \geq 1} a_n \cdot \sum_{m \geq 1} \frac{b_m}{m^s} = \sum_{n \geq 1} \left( \sum_{k|n} a_k b_{n/k} \right) \frac{1}{n^s}.
\]

**Definition 2.10.** A *monoid* is a semigroup with a unit. A *free commutative monoid* is a commutative monoid \( M \) with a set of primes \( P \subseteq M \) such that each element \( m \in M \) can be uniquely decomposed into a product of elements in \( P \) up to rearrangement. Let \( M \) be a free commutative monoid. We get a monoid algebra \( \mathbb{C}M \), in which the elements are all formal sums \( \sum_{m \in M} c_m m \), where \( c_m \in \mathbb{C} \), with addition and multiplication defined naturally. For each \( m \in M \), we associate a length \( l(m) \) that is compatible with the multiplication in \( M \). That is, for any \( m_1, m_2 \in M \), we have \( l(m_1) l(m_2) = l(m_1 m_2) \).

Let \( M \) be a free commutative monoid with prime set \( P \). The following identity holds in the monoid algebra \( \mathbb{C}M \):
\[
\sum_{m \in M} m = \prod_{p \in P} \frac{1}{1 - p}. 
\]

Furthermore, we can define a homomorphism from \( M \) to the ring of Dirichlet series under which each \( m \in M \) is sent to \( 1/l(m)^s \), where \( l \) is a length function of \( M \). Therefore,
\[
\sum_{m \in M} \frac{1}{l(m)^s} = \prod_{p \in P} \frac{1}{1 - l(p)^{-s}}.
\]

Recall that \( \mathbb{C} \) is the set of unlabeled connected graphs under the operation of Cartesian product. The unique factorization theorem of Sabidussi gives \( \mathbb{C} \) the structure of a commutative free monoid with a set of primes \( \mathbb{P} \), where \( \mathbb{P} \) is the set of unlabeled prime graphs. This is saying that every element of \( \mathbb{C} \) has a unique factorization of the form \( b_1^{e_1} b_2^{e_2} \cdots b_k^{e_k} \), where the \( b_i \) are distinct primes in \( \mathbb{P} \). Let \( l(G) \), the number of vertices in \( G \), be a length function for \( \mathbb{C} \). We have the following proposition.

**Proposition 2.11.** For \( \mathbb{C} \) and \( \mathbb{P} \), we have
\[
\sum_{G \in \mathbb{C}} \frac{1}{l(G)^s} = \prod_{p \in \mathbb{P}} \frac{1}{1 - l(P)^{-s}}.
\]

The enumeration of prime graphs was studied by Raphaël Bellèc [11]. We use Dirichlet series to count unlabeled connected prime graphs.
Theorem 2.12. Let \( \tilde{c}_n \) be the number of unlabeled connected graphs on \( n \) vertices, and let \( b_m \) be the number of unlabeled prime graphs on \( m \) vertices. Then we have

\[
\sum_{n \geq 1} \frac{\tilde{c}_n}{n^s} = \prod_{m \geq 2} \frac{1}{(1 - m^{-s})^{b_m}}.
\]

Furthermore, if we define numbers \( d_n \) for positive integers \( n \) by

\[
\sum_{n \geq 1} \frac{d_n}{n^s} = \log \sum_{n \geq 1} \tilde{c}_n n^s,
\]

then

\[
d_n = \sum_{m' = n} \frac{b_m}{l},
\]

where the sum is over all pairs \((m, l)\) of positive integers with \( ml = n\).

The proof of Theorem 2.12 follows Remark 2.13 and Proposition 2.14 below.

Remark 2.13. In what follows, we introduce an interesting recursive formula for computing \( d_n \). To start with, we differentiate both sides of Equation (2.5) with respect to \( s \) and simplify. We get that

\[
\sum_{n \geq 2} \log n \frac{\tilde{c}_n}{n^s} = (\sum_{n \geq 1} \frac{\tilde{c}_n}{n^s}) \left( \sum_{n \geq 2} \log n \frac{d_n}{n^s} \right),
\]

which gives

\[
\tilde{c}_n \log n = \sum_{ml = n} \tilde{c}_m d_l \log l.
\]

Since \( \tilde{c}_1 \) is the number of connected graphs on 1 vertex, \( \tilde{c}_1 = 1 \). It follows easily from Equation (2.7) that \( d_p = \tilde{c}_p \) when \( p \) is a prime number. Therefore, if \( p \) is a prime number, \( b_p = d_p = c_p \). This fact can be seen directly, since a connected graph with a prime number of vertices is a prime graph.

Raphaël Bellec used Equation (2.7) to find formulae for \( d_n \) where \( n \) is a product of two different primes or a product of three different primes:

If \( n = pq \) where \( p \neq q \),

\[
d_n = \tilde{c}_n - \tilde{c}_p \tilde{c}_q;
\]

If \( n = pqr \) where \( p, q \) and \( r \) are distinct primes,

\[
d_n = \tilde{c}_n + 2\tilde{c}_p \tilde{c}_q \tilde{c}_r - \tilde{c}_p \tilde{c}_q \tilde{c}_r - \tilde{c}_q \tilde{c}_p \tilde{c}_r - \tilde{c}_r \tilde{c}_q \tilde{c}_p.
\]

In fact, Equations (2.8) and (2.9) are special cases of the following proposition.

Proposition 2.14. Let \( d_n, \tilde{c}_n \) be defined as above. Then we have

\[
d_n = \tilde{c}_n - \frac{1}{2} \sum_{n_1 n_2 = n} \tilde{c}_{n_1} \tilde{c}_{n_2} + \frac{1}{3} \sum_{n_1 n_2 n_3 = n} \tilde{c}_{n_1} \tilde{c}_{n_2} \tilde{c}_{n_3} - \ldots .
\]
Proof. We can use the identity
\[ \log(1 + x) = x - \frac{1}{2} x^2 + \frac{1}{3} x^3 - \frac{1}{4} x^4 + \ldots \]
to compute from Equation (2.5) that
\[
\sum_{n \geq 1} \frac{d_n}{n^s} = \log \left( 1 + \sum_{n \geq 2} \frac{\tilde{c}_n}{n^s} \right)
= \sum_{n \geq 2} \frac{\tilde{c}_n}{n^s} - \frac{1}{2} \left( \sum_{n \geq 2} \frac{\tilde{c}_n}{n^s} \right)^2 + \frac{1}{3} \left( \sum_{n \geq 2} \frac{\tilde{c}_n}{n^s} \right)^3 - \ldots .
\]
Equating coefficients of \(n^{-s}\) on both sides, we get equation (2.10) as a result. \(\square\)

Proof of Theorem 2.12. We start with
\[
(2.11) \quad \sum_{m} \frac{1}{l(m)^s} = \prod_{p} \frac{1}{1 - l(p)^{-s}},
\]
where the left-hand side is multiplied over all connected graphs, and the right-hand side is summed over all prime graphs. Regrouping the summands on the left-hand side with respect to the number of vertices in \(m\), we get the left-hand side of Equation (2.4). Regrouping the factors on the right-hand side with respect to the number of vertices in \(p\), we get the right-hand side of Equation (2.4).

Taking the logarithm of both sides of Equation (2.4), we get
\[
\log \sum_{n \geq 1} \frac{\tilde{c}_n}{n^s} = \log \prod_{m \geq 2} \left( \frac{1}{1 - l(m)^{-s}} \right)^{b_m} = \sum_{m \geq 2} b_m \log \frac{1}{1 - m^{-s}}
= \sum_{m \geq 2} \left( b_m \sum_{l \geq 1} \frac{m^{-sl}}{l} \right) = \sum_{m \geq 2, l \geq 1} \frac{b_m}{l \cdot m^s},
\]
and Equation (2.10) follows immediately. \(\square\)

Next, we will compute the numbers \(b_n\) in terms of the numbers \(d_n\) using the following lemma.

Lemma 2.15. Let \(\{D_i\}_{i=1,\ldots}\) and \(\{J_i\}_{i=1,\ldots}\) be sequences of numbers satisfying
\[
(2.12) \quad D_k = \sum_{l \mid k} \frac{J_{k/l}}{l},
\]
and let \(\mu\) be the Möbius function. Then we have
\[
J_k = \frac{1}{k} \sum_{l \mid k} \mu \left( \frac{k}{l} \right) l D_l.
\]
Proof. Multiplying by $k$ on both sides of Equation (2.12), we get

$$kD_k = \sum_{l \mid k} \frac{k}{l} J_{k/l} = \sum_{l \mid k} lJ_l.$$

Applying the Möbius inversion formula, we get

$$kJ_k = \sum_{l \mid k} \mu\left(\frac{k}{l}\right) lD_l.$$

Therefore,

$$J_k = \frac{1}{k} \sum_{l \mid k} \mu\left(\frac{k}{l}\right) lD_l.$$

□

Given any natural number $n$, let $e$ be the largest number such that $n = r^e$ for some $r$. Note that $r$ is not a power of a smaller integer. We let $D_k = d_{r^k}$, $J_k = b_{r^k}$. It follows that Equation (2.6) is equivalent to Equation (2.12).

**Theorem 2.16.** For any natural number $n$, let $e, r$ be as described in above. Then we have

$$b_n = \frac{1}{e} \sum_{l \mid e} \mu\left(\frac{e}{l}\right) ld_{r^e}.$$

**Proof.** The result follows straightforwardly from Lemma 2.15 □

Table 1 in the Appendix gives the numbers of labeled and unlabeled prime graphs with no more than 16 vertices.

3. **Exponential Composition of Species**

3.1. **Arithmetic Product of Species.** The arithmetic product was studied by Maia and Méndez [9]. The arithmetic product of two molecular species $X^m/A$ and $X^n/B$, where $A$ is a subgroup of $\mathfrak{S}_m$ and $B$ is a subgroup of $\mathfrak{S}_n$, can be defined to be the molecular species $X^{mn}/(A \times B)$, where $A \times B$ is the group representation of the product group of $A$ and $B$ acting on the set $[m] \times [n]$ (Definition 1.2).

In order to define the arithmetic product of general species, Maia and Méndez developed a decomposition of a set, called a rectangle.

**Definition 3.1.** Let $U$ be a finite set. A rectangle on $U$ of height $a$ is a pair $(\pi_1, \pi_2)$ such that $\pi_1$ is a partition of $U$ with $a$ blocks, each of size $b$, where $\#U = ab$, and $\pi_2$ is a partition of $U$ with $b$ blocks, each of size $a$, and if $B$ is a block of $\pi_1$ and $B'$ is a block of $\pi_2$ then $|B \cap B'| = 1$.

A $k$-rectangle on $U$ is a $k$-tuple of partitions $(\pi_1, \pi_2, \ldots, \pi_k)$ such that

i) for each $i \in [k]$, $\pi_i$ has $a_i$ blocks, each of size $|U|/a_i$, where $|U| = \prod_{i=1}^k a_i$. 

ii) for any $k$-tuple $(B_1, B_2, \ldots, B_k)$, where $B_i$ is a block of $\pi_i$ for each $i \in [k]$, we have $|B_1 \cap B_2 \cap \cdots \cap B_k| = 1$. See Figure 2 for a 3-rectangle $(\pi_1, \pi_2, \pi_3)$ represented by a 3-partite graph.

![Figure 2](image)

$\pi_1 = \{A_1, A_2, A_3, A_4\}$
$\pi_2 = \{B_1, B_2, B_3\}$
$\pi_3 = \{C_1, C_2\}$

**Figure 2.** A 3-rectangle $(\pi_1, \pi_2, \pi_3)$, represented by a 3-partite graph, and labeled on the triangles.

We denote by $\mathcal{N}$ the species of rectangles, and by $\mathcal{N}^{(k)}$ the species of $k$-rectangles. Let $n = \prod_{i=1}^{k} a_i$, and let $\Delta$ be the set of bijections of the form

$$\delta : [a_1] \times [a_2] \times \cdots \times [a_k] \to [n].$$

Note that the cardinality of the set $\Delta$ is $n!$. The group

$$\prod_{i=1}^{k} \mathcal{S}_{a_i} = \{\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_k) : \sigma_i \in \mathcal{S}_{a_i}\}$$

acts on the set $\Delta$ by setting

$$(\sigma \cdot \delta)(i_1, i_2, \ldots, i_k) = \delta(\sigma_1(i_1), \sigma_2(i_2), \ldots, \sigma_k(i_k)),$$

for each $(i_1, i_2, \ldots, i_k) \in [a_1] \times [a_2] \times \cdots \times [a_k]$. We observe that this group action result in a set of $\prod_{i=1}^{k} \mathcal{S}_{a_i}$-orbits, and that each orbit consists of exactly $a_1!a_2!\cdots a_k!$ elements of $\Delta$. Observe further that there is a one-to-one correspondence between the set of $\prod_{i=1}^{k} \mathcal{S}_{a_i}$-orbits on the set $\Delta$ and the set of $k$-rectangles of the form $(\pi_1, \ldots, \pi_k)$, where each $\pi_i$ has $a_i$ blocks. Therefore, the number of such $k$-rectangles is

$$\left\{ \frac{n}{a_1, a_2, \ldots, a_k} \right\} := \frac{n!}{a_1!a_2!\cdots a_k!}.$$

**Definition 3.2.** Let $F_1$ and $F_2$ be species of structures with $F_1[\emptyset] = F_2[\emptyset] = \emptyset$. The **arithmetic product** of $F_1$ and $F_2$, denoted $F_1 \boxplus F_2$, is defined by setting for each finite set $U$,

$$(F_1 \boxplus F_2)[U] = \sum_{(\pi_1, \pi_2) \in \mathcal{N}^{(k)}[U]} F_1[\pi_1] \times F_2[\pi_2],$$

where the sum represents the disjoint union (See Figure 3).
In other words, an $F_1 \Box F_2$-structure on a finite set $U$ is a tuple of the form $((\pi_1, f_1), (\pi_2, f_2))$, where $(\pi_1, \pi_2)$ is a rectangle on $U$ and $f_i$ is an $F_i$-structure on the blocks of $\pi_i$ for each $i$. A bijection $\sigma : U \to V$ sends a partition $\pi$ of $U$ to a partition $\pi'$ of $V$, namely, $\sigma(\pi) = \pi' = \{\sigma(B) : B$ is a block of $\pi\}$. Thus $\sigma$ induces a bijection $\sigma_{\pi} : \pi \to \pi'$, sending each block of $\pi$ to a block of $\pi'$. The transport of structures for any bijection $\sigma : U \to V$ is defined by

$$(F_1 \Box F_2)[\sigma]((\pi_1, f_1), (\pi_2, f_2)) = ((\pi'_1, F_1[\sigma_{\pi_1}](f_1)), (\pi'_2, F_2[\sigma_{\pi_2}](f_2))).$$

Maia and Méndez showed that the arithmetic product of species is commutative, associative, distributive, and with a unit $X$, the species of singleton sets:

$$F_1 \Box X = X \Box F_1 = F_1.$$

**Definition 3.3.** The arithmetic product of species $F_1, F_2, \ldots, F_k$ with $F_i(\emptyset) = \emptyset$ for all $i$ is defined by setting $\bigboxtimes_{i=1}^k F_i = F_1 \Box F_2 \Box \cdots \Box F_k$, which sends each finite set $U$ to the set

$$\bigboxtimes_{i=1}^k F_i[U] = \sum F_1[\pi_1] \times F_2[\pi_2] \times \cdots \times F_k[\pi_k],$$

where the sum is taken over all $k$-rectangles $($\pi_1, \pi_2, \ldots, \pi_k$) of $U$, and represents the disjoint union. We denote by $F^{\Box k}$ the arithmetic product of $k$ copies of $F$.

For each bijection $\sigma : U \to V$, the transport of structures of $\bigboxtimes_{i=1}^k F_i$ along $\sigma$ sends an $\bigboxtimes_{i=1}^k F_i$-structure on $U$ of the form

$$( (\pi_1, f_1), (\pi_2, f_2), \ldots, (\pi_k, f_k) )$$

to an $\bigboxtimes_{i=1}^k F_i$-structure on $V$ of the form

$$( (\pi'_1, F_1[\sigma_{\pi_1}]f_1), (\pi'_2, F_2[\sigma_{\pi_2}]f_2), \ldots, (\pi'_k, F_k[\sigma_{\pi_k}]f_k) ),$$

where $\sigma_{\pi_i}$ is the bijection induced by $\sigma$ sending blocks of $\pi_i$ to blocks of $\pi'_i$.

Maia and Méndez proved the following proposition which illustrates that the Dirichlet exponential generating functions are useful for enumeration involving the arithmetic product of species.
Proposition 3.4. (Maia and Ménendez) Let $F_1$ and $F_2$ be species with $F_i[\emptyset] = \emptyset$ for $i = 1, 2$. Then
\begin{equation}
D(F_1 \boxtimes F_2) = D(F_1)D(F_2).
\end{equation}

Theorem 3.5. (Maia and Ménendez) Let species $F_1$ and $F_2$ satisfy $F_1[\emptyset] = F_2[\emptyset] = \emptyset$. Then we have
\begin{equation}
Z_{F_1 \boxtimes F_2} = Z_{F_1} \boxtimes Z_{F_2},
\end{equation}
where the operation $\boxtimes$ on the right-hand side of the equation is a bilinear operation on symmetric functions defined by setting
\[ p_{\nu} := p_{\lambda} \boxtimes p_{\mu}, \]
where
\[ c_k(\nu) = \sum_{\text{lcm}(i,j) = k} \gcd(i, j)c_i(\lambda)c_j(\mu), \]
in which $\text{lcm}(i, j)$ denotes the least common multiple of $i$ and $j$, and $\gcd(i, j)$ denotes the greatest common divisor of $i$ and $j$.

Furthermore, the arithmetic product of molecular species and the Cartesian product of graphs are closely related, as shown in the following proposition.

Proposition 3.6. Let $G_1$ and $G_2$ be two graphs that are relatively prime to each other. Then the species associated to the Cartesian product of $G_1$ and $G_2$ is equivalent to the arithmetic product of the species associated to $G_1$ and the species associated to $G_2$. That is,
\begin{equation}
\mathcal{O}_{G_1 \otimes G_2} = \mathcal{O}_{G_1} \boxtimes \mathcal{O}_{G_2}.
\end{equation}

Proof. Let $l(G_1) = m$ and $l(G_2) = n$. Then $l(G_1 \otimes G_2) = mn$.

Since $G_1$ and $G_2$ are relatively prime, we get
\[ \text{aut}(G_1 \otimes G_2) = \text{aut}(G_1) \times \text{aut}(G_2). \]
Therefore,
\[ \mathcal{O}_{G_1 \otimes G_2} = \frac{X^{l(G_1 \otimes G_2)}}{\text{aut}(G_1 \otimes G_2)} = \frac{X^{mn}}{\text{aut}(G_1) \times \text{aut}(G_2)} = \frac{X^m}{\text{aut}(G_1)} \boxtimes \frac{X^n}{\text{aut}(G_2)} = \mathcal{O}_{G_1} \boxtimes \mathcal{O}_{G_2}. \]

Note that if $G_1$ and $G_2$ are not relatively prime to each other, then the species associated to the Cartesian product of $G_1$ and $G_2$ is generally different from the arithmetic product of $\mathcal{O}_{G_1}$ and $\mathcal{O}_{G_2}$. This is because the automorphism group of the product of the graphs is no longer the product of the automorphism groups of the graphs.
3.2. Exponential Composition of Species. Let $A$ be a subgroup of $\mathfrak{S}_m$, and let $B$ be a subgroup of $\mathfrak{S}_n$. The group $B^A$ defined by Definition 1.3 acts on the set of functions from $[m]$ to $[n]$, and hence can be identified with a subgroup of $\mathfrak{S}_{nm}$. This gives rise to a molecular species $X^{nm}/B^A$, which is defined to be the exponential composition of species. A more general definition is given in the following.

Let $F$ be a species of structures with $F[\emptyset] = \emptyset$, let $k$ be a positive integer, and let $A$ be a subgroup of $\mathfrak{S}_k$. Recall that an $F \boxtimes k$-structure on a finite set $U$ is a tuple of the form

\[
((\pi_1, f_1), (\pi_2, f_2), \ldots, (\pi_k, f_k)),
\]

where $(\pi_1, \pi_2, \ldots, \pi_k)$ is a $k$-rectangle on $U$, and each $f_i$ is an $F$-structure on the blocks of $\pi_i$. The group $A$ acts on the set of $F \boxtimes k$-structures by permuting the subscripts of $\pi_i$ and $f_i$, i.e.,

\[
\alpha((\pi_1, f_1), \ldots, (\pi_k, f_k)) = ((\pi_{\alpha(1)}, f_{\alpha(1)}), \ldots, (\pi_{\alpha(k)}, f_{\alpha(k)})),
\]

where $\alpha$ is an element of $A$, $(\pi_{\alpha_1}, \pi_{\alpha_2}, \ldots, \pi_{\alpha_k})$ is a $k$-rectangle on $U$, and each $f_{\alpha_i}$ is an $F$-structure on the blocks of $\pi_{\alpha_i}$. It is easy to check that this action of $A$ on $F \boxtimes k$-structures is natural, that is, it commutes with any bijection $\sigma : U \rightarrow V$. Hence we get a quotient species under this group action.

**Definition 3.7.** (Exponential Composition with a Molecular Species) Let $F$ be a species with $F[\emptyset] = \emptyset$. We define the exponential composition of $F$ with the molecular species $X^{k}/A$ to be the quotient species, denoted $(X^{k}/A)\langle F \rangle$, under the group action described in above. That is,

\[
(X^{k}/A)\langle F \rangle := F \boxtimes k/A.
\]

**Theorem 3.8.** Let $A$ and $B$ be subgroups of $\mathfrak{S}_m$ and $\mathfrak{S}_n$, respectively, and let $B^A$ be the exponentiation group of $A$ with $B$. Then we have

\[
\frac{X^m}{A} \langle \frac{X^n}{B} \rangle = \frac{X^{nm}}{B^A}.
\]

As a consequence, we have

\[
Z_{(X^{m}/A)(X^{n}/B)} = Z(B^A).
\]

**Proof.** Since the arithmetic product is associative, we have

\[
\left(\frac{X^n}{B}\right)^{\boxtimes m} = \frac{X^N}{B^m},
\]

where $B^m$ is the the product of $m$ copies of $B$, acting on the set

\[
[n] \times [n] \times \cdots \times [n]
\]

piecewisely, and hence viewed as a subgroup of $\mathfrak{S}_{nm}$. Therefore, the set of $(X^n/B)^{\boxtimes m}$-structures on $[N]$ can be identified with the set of $B^m$-orbits of linear orders on $[N]$. 

The group $A$ acts on these $B^m$-orbits of linear orders by permuting the subscripts. This action results in the quotient species

$$\frac{X^m}{A} \langle \frac{X^n}{B} \rangle = \left( \frac{X^n}{B} \right)^m / A = \left( \frac{X^N}{B^m} \right) / A.$$ 

We observe that an $A$-orbit of $B^m$-orbits of linear orders on $[N]$ admits an automorphism group isomorphic to the exponentiation group $B^A$, hence the quotient species $(X^N/B^m)/A$ is the same as the molecular species $X^N/B^A$. Figure 4 illustrates a group action of $A$ on a set of $(X^n/B)^m$-structures. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.png}
\caption{($(X^n/B)^m)/A = X^m/(B^A)$.}
\end{figure}

**Definition 3.9.** Let $k$ be a positive integer, and $F$ a species with $F[\emptyset] = \emptyset$. We define the exponential composition of $F$ of order $k$ to be the species

$$\mathcal{E}_k \langle F \rangle = F^{\square k} / \mathcal{S}_k.$$ 

We set $\mathcal{E}_0 \langle F \rangle = X$.

**Definition 3.10.** (Exponential Composition of Species) Let $F$ be a species with $F[\emptyset] = F[1] = \emptyset$. We define the exponential composition of $F$, denoted $\mathcal{E} \langle F \rangle$, to be the sum of
\( \mathcal{E}_k(F) \) on all nonnegative integers \( k \), i.e.,

\[
\mathcal{E}(F) = \sum_{k \geq 0} \mathcal{E}_k(F).
\]

The exponential composition of species has properties listed in the following theorems. Theorem 3.11 gives a connection between the exponential composition and the Dirichlet exponential generating function of species. Theorem 3.12 lists further properties of the exponential composition of the sum of two species.

**Theorem 3.11.** Let \( F \) be a species with \( F[\emptyset] = F[1] = \emptyset \). Then

\[
\mathcal{D}(\mathcal{E}(F)) = \exp(\mathcal{D}(F)).
\]

**Proof.** Each \( F \bigotimes_k / \mathfrak{S}_k \)-structure on a finite set \( U \) is an \( \mathfrak{S}_k \)-orbit of \( F \bigotimes_k \)-structures on \( U \), where the action is taken by permuting the subscripts of the \( F \bigotimes_k \)-structures. We observe that there are \( k! \) \( F \bigotimes_k \)-structures in each of the \( \mathfrak{S}_k \)-orbits. Therefore,

\[
\left| \frac{F \bigotimes_k}{\mathfrak{S}_k} [n] \right| = \frac{|F \bigotimes_k[n]|}{k!},
\]

and

\[
\mathcal{D}(\mathcal{E}_k(F)) = \mathcal{D}(F \bigotimes_k / \mathfrak{S}_k) = \frac{\mathcal{D}(F \bigotimes_k)}{k!} = \frac{\mathcal{D}(F)^k}{k!}.
\]

It follows that

\[
(3.4) \quad \mathcal{D}(\mathcal{E}(F)) = \mathcal{D} \left( \sum_{k \geq 0} \mathcal{E}_k(F) \right) = \sum_{k \geq 0} \mathcal{D}(\mathcal{E}_k(F)) = \sum_{k \geq 0} \frac{\mathcal{D}(F)^k}{k!} = \exp(\mathcal{D}(F)).
\]

\( \Box \)

**Theorem 3.12.** (Properties of the Exponential Composition) Let \( F_1 \) and \( F_2 \) be species with \( F_1[\emptyset] = F_2[\emptyset] = F_1[1] = F_2[1] = \emptyset \), and let \( k \) be any nonnegative integer. Then

\[
\mathcal{E}_k(F_1 + F_2) = \sum_{i=0}^{k} \mathcal{E}_i(F_1) \bigotimes \mathcal{E}_{k-i}(F_2),
\]

(3.5)

\[
\mathcal{E}(F_1 + F_2) = \mathcal{E}(F_1) \bigotimes \mathcal{E}(F_2).
\]

We observe that an \( (F_1 + F_2) \bigotimes_k \)-structure on a finite set \( U \) is a rectangle on \( U \) with each partition in the rectangle enriched with either an \( F_1 \) or an \( F_2 \)-structure. Taking the \( \mathfrak{S}_k \)-orbits of these \( (F_1 + F_2) \bigotimes_k \)-structures on \( U \) means basically making every partition of the rectangle “indistinguishable”. Hence in each \( \mathfrak{S}_k \)-orbit, all partitions enriched with an \( F_1 \)-structure are grouped together to give an \( \mathfrak{S}_{k_1} \)-orbit of \( F_1 \bigotimes_{k_1} \)-structures, and the remaining partitions are grouped together to give an \( \mathfrak{S}_{k_2} \)-orbit of \( F_2 \bigotimes_{k_2} \)-structures, where \( k_1 \) and \( k_2 \) are nonnegative integers whose sum is equal to \( k \).
Proof of Theorem 3.12. First, we prove that for any nonnegative integer $k,$
\[ \mathcal{E}_k(F_1 + F_2) = \sum_{i=0}^{k} \mathcal{E}_i(F_1) \boxtimes \mathcal{E}_{k-i}(F_2). \]

The case when $k = 0$ is trivial. Let us consider $k$ to be a positive integer. Let $s$ and $t$ be nonnegative integers whose sum equals $k$. Let $U$ be a finite set. To get an $\mathcal{E}_s(F_1) \boxtimes \mathcal{E}_t(F_2)$-structure on $U,$ we first take a rectangle $(\rho, \tau)$ on $U,$ and then take an ordered pair $(a, b),$ where $a$ is an $\mathcal{E}_s(F_1)$-structure on the blocks of $\rho,$ and $b$ is an $\mathcal{E}_t(F_2)$-structure on the blocks of $\tau.$ That is,
\[ a = \{(\rho_1, f_1), \ldots, (\rho_s, f_s)\}, \quad b = \{(\tau_1, g_1), \ldots, (\tau_t, g_t)\}, \]
where $(\rho_1, \ldots, \rho_s)$ is a rectangle on the blocks of $\rho,$ $(\tau_1, \ldots, \tau_t)$ is a rectangle on the blocks of $\tau,$ $f_i$ is an $F_1$-structure on the blocks of $\rho_i,$ and $g_j$ is an $F_2$-structure on the blocks of $\tau_j.$

As pointed out by Maia and Méndez [9], for any nonnegative integers $i, j,$ the species of $(i + j)$-rectangles is isomorphic to the arithmetic product of the species of $i$-rectangles and the species of $j$-rectangles:
\[ \mathcal{N}(i+j) = \mathcal{N}(i) \boxtimes \mathcal{N}(j). \]

It follows that $(\rho_1, \ldots, \rho_s, \tau_1, \ldots, \tau_t)$ is a rectangle on $U.$

On the other hand, let $x$ be an $\mathcal{E}_k(F_1 + F_2)$-structure on $U$. We can write $x$ as a set of the form
\[ x = \{(\pi_1, f_1), \ldots, (\pi_r, f_r), (\pi_{r+1}, g_{r+1}), \ldots, (\pi_k, g_k)\}, \]
where $(\pi_1, \pi_2, \ldots, \pi_k)$ is a $k$-rectangle on $U,$ $r$ is a nonnegative integer between 0 and $k,$ each $f_i$ is an $F_1$-structure on $\pi_i$ for $i = 1, \ldots, r,$ and each $g_j$ is an $F_2$-structure on $\pi_j$ for $j = r + 1, \ldots, k.$

We then write $x = (x_1, x_2),$ where
\[ x_1 = \{(\pi_1, f_1), \ldots, (\pi_r, f_r)\}, \quad x_2 = \{(\pi_{r+1}, g_{r+1}), \ldots, (\pi_k, g_k)\}. \]

Hence running through values of $s$ and $t,$ we get that the set of $\mathcal{E}_s(F_1) \boxtimes \mathcal{E}_t(F_2)$-structures on $U$, written in the form of the pairs $(a, b)$ whose construction we described in above, corresponds naturally to the set of $\mathcal{E}_k(F_1 + F_2)$-structures on $U.$

The proof of
\[ \mathcal{E}(F_1 + F_2) = \mathcal{E}(F_1) \boxtimes \mathcal{E}(F_2). \]
is straightforward using the properties of the arithmetic product, namely, the commutativity, associativity and distributivity:
\[ \mathcal{E}(F_1 + F_2) = \sum_{k \geq 0} \mathcal{E}_k(F_1 + F_2) = \sum_{k \geq 0} \sum_{i+j=k} \mathcal{E}_i(F_1) \boxtimes \mathcal{E}_j(F_2) \]
\[ = \left( \sum_{i \geq 0} \mathcal{E}_i(F_1) \right) \boxtimes \left( \sum_{j \geq 0} \mathcal{E}_j(F_2) \right) = \mathcal{E}(F_1) \boxtimes \mathcal{E}(F_2). \]
\[ \square \]
Note that identity (3.5) is analogous to the identity about the composition of a sum of species with the species of sets $\mathcal{E}$:
\[
\mathcal{E}(F_1 + F_2) = \mathcal{E}(F_1) \mathcal{E}(F_2).
\]
What is more, (3.5) illustrates a kind of distributivity of the exponential composition. In fact, if a species of structures $F$ has its molecular decomposition written in the form
\[
F = \sum_{M \in \text{molecular}} M,
\]
then the exponential composition of $F$ can be written as
\[
\mathcal{E}(F) = \bigoplus_{M \in \text{molecular}} \mathcal{E}(M).
\]

3.3. Cycle Index of Exponential Composition. The cycle index polynomial of the exponentiation group was given by Palmer and Robinson [11]. They defined the following operators $I_k$ for positive integers $k$.

Let $\mathcal{R} = \mathbb{Q}[p_1, p_2, \ldots]$ be the ring of polynomials with the operation $\boxtimes$ as defined in Theorem 3.5. Palmer and Robinson defined for positive integers $k$ the $\mathbb{Q}$-linear operators $I_k$ on $\mathcal{R}$ as follows:

Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition of $n$. The action of $I_k$ on the monomial $p_\lambda$ is given by
\[
I_k(p_\lambda) = p_\gamma,
\]
where $\gamma = (\gamma_1, \gamma_2, \ldots)$ is the partition of $n^k$ with
\[
c_j(\gamma) = \frac{1}{j} \sum_{l | j} \mu \left( \left( \sum_{i | l / \gcd(k, l)} ic_i(\lambda) \right)^{\gcd(k, l)} \right).
\]

Furthermore, $\{I_k\}$ generates a $\mathbb{Q}$-algebra $\Omega$ of $\mathbb{Q}$-linear operators on $\mathcal{R}$. For any elements $I, J \in \Omega$, any $r \in \mathcal{R}$ and $a \in \mathbb{Q}$, we set
\[
(aI)(r) = a(I(r)), \quad (I + J)(r) = I(r) + J(r), \quad (IJ)(r) = I(r) \boxtimes J(r).
\]

As discussed in Palmer and Robinson’s paper [11], if $I_m(p_\mu) = p_\nu$, then $\nu$ is the cycle type of an element $(\alpha, \tau)$ of the exponentiation group $B^A$ acting on $[n]^m$, where $\alpha$ is a permutation in $A$ with a single $m$-cycle, and $\tau \in B^m$ is such that $\mu$ is the cycle type of the permutation $\tau(m)\tau(m - 1)\cdots\tau(2)\tau(1)$.

**Definition 3.13.** Let $f_1$ and $f_2$ be elements of the ring $\mathcal{R} = \mathbb{Q}[p_1, p_2, \ldots]$. We define the exponential composition of $f_1$ and $f_2$, denoted $f_1 \ast f_2$, to be the image of $f_2$ under the operator obtained by substituting the operator $I_r$ for the variables $p_r$ in $f_1$.

Note that the operation $\ast$ is linear in the left parameters, but not on the right parameters. We call this the partial linearity of the operation $\ast$. 

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Let $A$ be a subgroup of $S_m$, and let $B$ be a subgroup of $S_n$. Palmer and Robinson \[11, pp. 128–131\] proved that the cycle index polynomial of $B^A$ is the exponential composition of $Z(A)$ with $Z(B)$. That is,

$$Z(B^A) = Z(A) \ast Z(B).$$

As a consequence of Theorem 3.8 we get the cycle index of the species $(X^m/A)/(X^n/B)$:

$$Z((X^m/A)/(X^n/B)) = Z(A) \ast Z(B).$$

Next we generalize Palmer and Robinson’s result to get the formula for the cycle index of the exponential composition of an arbitrary species. First, we introduce a lemma that is a generalization of the Cauchy-Frobenius Theorem, alias Burnside’s Lemma. For the proof of a more general result, with applications and further references, see Robinson \[13\]. Another application is given in \[4\].

**Lemma 3.14.** (Cauchy-Frobenius) Suppose that a finite group $M \times N$ acts on a set $S$. The groups $M$ and $N$, considered as subgroups of $M \times N$, also act on $S$. The group $N$ acts on the set of $M$-orbits. Then for any $g \in N$, the number of $M$-orbits fixed by $g$ is given by

$$\frac{1}{|M|} \sum_{f \in M} \text{fix}(f, g),$$

where $\text{fix}(f, g)$ denotes the number of elements in $S$ that are fixed by $(f, g) \in M \times N$.

**Theorem 3.15.** (Cycle Index of the Exponential Composition) Let $A$ be a subgroup of $S_k$, and let $F$ be a species of structures concentrated on the cardinality $n$. Then the cycle index of the species $(X^k/A)(F)$ is given by

$$Z((X^k/A)(F)) = Z(A) \ast Z_F,$$

where the expression $Z(A) \ast Z_F$ denotes the image of $Z_F$ under the operator obtained by substituting the operator $I_r$ for the variables $p_r$ in $Z(A)$.

**Remark 3.16** (Notation and Set-up). We denote by Par$_n$ the set of partitions of $n$, and by Par$_n^k$ the set of $k$-sequences of partitions of $n$.

For fixed integers $n$, $k$, and $N = n^k$, we denote by $\mathcal{N}_N$ the species of $k$-dimensional cubes, or $k$-cubes, on $[N]$, defined by

$$\mathcal{N}_N = \mathcal{E}_n^\square_k[N].$$

We also call the elements of the set $(X^n)^\square_k[N]$ $k$-dimensional ordered cubes on $[N]$.

Let $\sigma$ be a permutation on $[k]$ with cycle type

$$\text{ct}(\sigma) = (r_1, r_2, \ldots, r_d).$$

Then $\sigma$ acts on the $F^\square_k$-structures by permuting the subscripts. Let $\nu$ be a partition of $N$. Let $\delta$ be a permutation of $[N]$ with cycle type $\nu$. Then $\delta$ acts on the $F^\square_k$-structures by transport of structures. We also introduce the notation

$$I(\text{ct}(\sigma); \lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(d)}) = I_{r_1}(p_{\lambda^{(1)}}) \boxtimes I_{r_2}(p_{\lambda^{(2)}}) \boxtimes \cdots \boxtimes I_{r_d}(p_{\lambda^{(d)}}).$$
We denote by $\text{Rec}_F(\sigma, \nu)$ a function on the pair $(\sigma, \nu)$ defined by

\begin{equation}
\text{Rec}_F(\sigma, \nu) := \frac{d}{\prod_{i=1}^{d} \text{fix} \ F[\lambda^{(i)}]} \cdot \prod_{i=1}^{d} z_{\lambda^{(i)}} \cdot z_{\lambda^{(d)}},
\end{equation}

where the summation is over all sequences $(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(d)})$ in $\text{Par}_n^d$ with

$I(\text{ct}(\sigma); \lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(d)}) = p_\nu$. \hfill (3.9)

We denote by $\text{fix}_F(\sigma, \delta)$ the number of $F\boxtimes_k$-structures on the set $[N]$ fixed by the joint action of the pair $(\sigma, \delta)$.

**Proof of Theorem 3.15.** Let $\nu$ be a partition of $N$. It suffices to prove that the coefficients of $p_\nu$ on both sides of Equation (3.8) are equal.

The right-hand side of Equation (3.8) is

\begin{equation}
Z(A) * Z_F = \left( \frac{1}{|A|} \sum_{\sigma \in A} p_\text{ct}(\sigma) \right) \ast \left( \sum_{\lambda \vdash n} \text{fix} \ F[\lambda] \frac{P_\lambda}{z_\lambda} \right) = \frac{1}{|A|} \sum_{\sigma \in A} I_{\text{ct}(\sigma)} \left( \sum_{\lambda \vdash n} \text{fix} \ F[\lambda] \frac{P_\lambda}{z_\lambda} \right).
\end{equation}

For $\sigma \in A$ with $\text{ct}(\sigma) = (r_1, r_2, \ldots, r_d)$, we have

$I_{\text{ct}(\sigma)} = I_{r_1} \cdots I_{r_d}$,

and

\begin{equation}
I_{\text{ct}(\sigma)} \left( \sum_{\lambda \vdash n} \text{fix} \ F[\lambda] \frac{P_\lambda}{z_\lambda} \right) = I_{r_1} \left( \sum_{\lambda \vdash n} \text{fix} \ F[\lambda] \frac{P_\lambda}{z_\lambda} \right) \boxtimes I_{r_2} \left( \sum_{\lambda \vdash n} \text{fix} \ F[\lambda] \frac{P_\lambda}{z_\lambda} \right) \boxtimes \cdots \boxtimes I_{r_d} \left( \sum_{\lambda \vdash n} \text{fix} \ F[\lambda] \frac{P_\lambda}{z_\lambda} \right).
\end{equation}

Therefore, the coefficient of $p_\nu$ in the expression $Z(A) * Z_F$ is

\begin{equation}
\frac{1}{|A|} \sum_{\sigma \in A} \prod_{i=1}^{d} \text{fix} \ F[\lambda^{(i)}] = \frac{1}{|A|} \sum_{\sigma \in A} \text{Rec}_F(\sigma, \nu),
\end{equation}

where the summation on the left-hand side is taken over all sequences $(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(d)})$ in $\text{Par}_n^d$ for some $d \geq 1$ and all $\sigma \in A$ with $\text{ct}(\sigma) = (r_1, r_2, \ldots, r_d)$ such that

$I(\text{ct}(\sigma); \lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(d)}) = p_\nu$,

and $\text{Rec}_F(\sigma, \nu)$ on the right-hand side is as defined by (3.9) in Remark 3.16.

The left-hand side of Equation (3.8) is

\begin{equation}
Z_F\boxtimes_k/A = \sum_{\nu \vdash N} \frac{P_{\nu}}{z_\nu} \left( \frac{F_{\boxtimes_k}}{A} \right) \left[ \nu \right] \frac{P_\nu}{z_\nu}.
\end{equation}
Therefore, the coefficient of $p_\nu$ in the expression $Z_{F^\boxtimes k/A}$ is
\[
\frac{1}{z_\nu} \text{fix } \frac{F^\boxtimes k}{A} [\nu].
\]
We then apply Theorem 3.14 to get that the number of $A$-orbits of $F^\boxtimes k$-structures on $[N]$ fixed by a permutation $\delta \in \mathfrak{S}_N$ of cycle type $\nu$ is
\[
(3.11) \quad \text{fix } \frac{F^\boxtimes k}{A} [\nu] = \text{fix } \frac{F^\boxtimes k}{A} [\delta] = \frac{1}{|A|} \sum_{\sigma \in A} \text{fix}_F(\sigma, \delta),
\]
where $\text{fix}_F(\sigma, \delta)$ is as defined in Remark 3.16.

Therefore, combining (3.11) and (3.10), the proof of Equation (3.8) is reduced to showing that
\[
(3.12) \quad \text{fix}_F(\sigma, \delta) = z_\nu \text{Rec}_F(\sigma, \nu),
\]
for any $\delta, \nu$ and $\sigma$.

To prove (3.12), we start with observing that in order for an $F^\boxtimes k$-structure on $[N]$ of the form
\[
((\pi_1, f_1), (\pi_2, f_2), \ldots, (\pi_k, f_k))
\]
to be fixed by the pair $(\sigma, \delta)$, it is necessary that $(\sigma, \delta)$ fixes the $k$-cube of the form $(\pi_1, \pi_2, \ldots, \pi_k) \in \mathcal{M}_N$. This is equivalent to saying that
\[
(3.13) \quad \mathcal{M}_N[\delta](\pi_1, \pi_2, \ldots, \pi_k) = (\pi_{\sigma(1)}, \pi_{\sigma(2)}, \ldots, \pi_{\sigma(k)}).
\]

Suppose (3.13) holds for some $k$-cube $(\pi_1, \pi_2, \ldots, \pi_k) \in \mathcal{M}_N$. We let $\beta_i \in \mathfrak{S}_n$ be the induced action of $\delta$ on the blocks of $\pi_i$, for $i = 1, 2, \ldots, k$. That is,
\[
\mathcal{M}_N[\delta](\pi_i) = \beta_i(\pi_{\sigma(i)})
\]
for all $i \in [k]$.

Now we consider the simpler case when $\sigma$ is a $k$-cycle, say, $\sigma = (1, 2, \ldots, k)$. Then the action of $\delta$ sends $(\pi_1, \pi_2, \ldots, \pi_k)$ to $(\pi_2, \pi_3, \ldots, \pi_1)$. Let $\beta = \beta_1 \beta_2 \cdots \beta_k$. The above discussion is saying that
\[
I_k(p_{\text{ct}(\beta)}) = p_\nu.
\]
On the other hand, given a partition $\lambda$ of $n$ satisfying $I_k(p_\lambda) = p_\nu$, there are $n!/z_\lambda$ permutations in $\mathfrak{S}_n$ with cycle type $\lambda$. Let $\beta$ be one of such. Then the number of sequences $(\beta_1, \beta_2, \ldots, \beta_k)$ whose product equals $\beta$ is $(n!)^{k-1}$, since we can choose $\beta_1$ up to
ordered cubes, we count the number of further condition that the product of the induced permutations on the $\beta$ we group the $k$ 

$\nu$ and only one of them is the $\delta$ that we started with. Considering that the $k$-cubes are just $S_k$-orbits of the $k$-dimensional ordered cubes, we count the number of $k$-cubes that are fixed by the pair $(\sigma, \delta)$ with the further condition that the product of the induced permutations on the $\pi_i$ by $\delta$ has cycle type $\lambda$:

$$\frac{\# \left\{ (\beta_1, \beta_2, \ldots, \beta_k) \in S_k \text{ with } ct(\beta_1 \cdots \beta_k) = \lambda \right\} \cdot \# \{ k\text{-dimensional ordered cubes} \}}{\# \{ \text{permutations on } [N] \text{ with cycle type } \nu \right\} \cdot \# \{ \text{k-dimensional ordered cubes in each equivalence class} \}} = \left[ \begin{array}{c} n! \\ k \end{array} \right] \cdot n! / z_\lambda = \frac{z_\nu}{z_\lambda}.$$

Now we try to compute how many $F_k$-structures of the form $((\pi_1, f_1), (\pi_2, f_2), \ldots, (\pi_k, f_k))$, based on a given rectangle $(\pi_1, \pi_2, \ldots, \pi_k)$ that is fixed by the $\beta_i$ with $ct \left( \prod_i \beta_i \right) = \lambda$, are fixed by the pair $(\sigma, \delta)$. We observe that the action of $(\sigma, \delta)$ determines that $f_k = F[\beta_1] f_1$ and $f_i = F[\beta_{i-1}] f_{i-1}$ for $i = 2, 3, \ldots, k$, and hence

$$f_k = F[\beta_1] F[\beta_2] \cdots F[\beta_k] f_k = F[\beta] f_k = F[\lambda] f_k.$$

In other words,

$$f_k \in \text{Fix } F[\lambda].$$

Hence as long as we choose an $f_k$ from $\text{Fix } F[\lambda]$, then all the other $f_i$ for $i < k$ are determined by our choice of $f_k$. There are $\text{fix } F[\lambda]$ such choices for $f_k$.

Therefore, in the case when $\sigma$ is a $k$-cycle, we get that the number of $F_k$-structures on the set $[N]$ fixed by the pair $(\sigma, \delta)$ is

$$\text{fix}_F(\sigma, \delta) = \sum_{I_k(\nu) = \nu} \text{fix } F[\lambda] \frac{z_\nu}{z_\lambda} = z_\nu \text{Rec}_F(\sigma, \nu).$$

Now let us consider the general case when $\sigma$ contains $d$ cycles of lengths $r_1, r_2, \ldots, r_d$. Let $(\pi_1, \pi_2, \ldots, \pi_k)$ be a $k$-cube fixed by the pair $(\sigma, \delta)$. Again we have (3.13), and we get an induced $\beta_i$ on the blocks of $\pi_{\sigma^{-1} i}$ for each $i$.

We observe that the action of $\sigma$ on the subscripts of the $k$-cube partitions the list $\pi_1, \pi_2, \ldots, \pi_k$ into $d$ parts, of lengths $r_1, r_2, \ldots, r_d$, within each of which we get a $r_i$-cycle. We group the $\beta_i$ on each of the $d$ parts and get $d$ permutations in the group $S_n$, whose cycle types are denoted by $\lambda(1), \lambda(2), \ldots, \lambda(d)$. This construction gives that such a sequence of partitions $(\lambda(1), \lambda(2), \ldots, \lambda(d))$ will be those that satisfy

$$I(\text{ct}(\sigma); \lambda(1), \lambda(2), \ldots, \lambda(d)) = p_\nu.$$

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Therefore, the number of $k$-cubes fixed by $(\sigma, \delta)$ corresponding to such a sequence of partitions $(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(d)})$ is

$$\frac{(n!)^{r_1-1} \cdot n!}{N! / z_\nu} \cdot \frac{N!}{(n!)^k} = \frac{(n!)^{r_1+\cdots+r_d} \cdot z_\nu}{z_{\lambda^{(1)}} \cdots z_{\lambda^{(d)}}} = \frac{z_\nu}{z_{\lambda^{(1)}} \cdots z_{\lambda^{(d)}}}.$$ 

The number of $F$-structures that are assigned to this $k$-cube $(\pi_1, \pi_2, \ldots, \pi_k)$ that will be fixed under the action of the pair $(\sigma, \delta)$ corresponding to the sequence of partitions $(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(d)})$ is hence

$$\text{fix } F[\lambda^{(1)}] \cdots \text{fix } F[\lambda^{(d)}],$$

since, similarly to our previous discussion, within each of the $d$ parts, we only need to pick an $F$-structure that is fixed by a permutation of cycle type $\lambda^{(i)}$, and all other $F$-structures are left determined.

Therefore, we get that for any pair $(\sigma, \delta)$,

$$\text{fix}_F(\sigma, \delta) = z_\nu \text{Rec}_F(\sigma, \nu),$$

which concludes our proof. \qed

**Remark 3.17.** We can use the molecular decomposition to define the exponential composition of a species $F$ with a species $H$. That is, if the molecular decomposition of $H$ is given by

$$H = \sum_{M \subseteq H, M \text{ molecular}} M,$$

then we define $H\langle F \rangle$ by

$$H\langle F \rangle = \sum_{M \subseteq H, M \text{ molecular}} M\langle F \rangle.$$

The left-linearity of the operation $*$ gives that the cycle index of $H\langle F \rangle$ is

$$Z_{H\langle F \rangle} = Z_H * Z_F = \left( \sum_{M \subseteq H, M \text{ molecular}} Z_M \right) * Z_F = \sum_{M \subseteq H, M \text{ molecular}} Z_M * Z_F.$$

### 3.4. Cycle Index of the Species of Prime Graphs

Now we are ready to come back to the species of prime graphs.

**Lemma 3.18.** Let $P$ be any prime graph, and let $k$ be any nonnegative integer. Then the species associated to the $k$-th power of $P$ is the exponential composition of $\mathcal{O}_P$ of order $k$. That is,

$$\mathcal{O}_P^k = \mathcal{O}_k\langle \mathcal{O}_P \rangle.$$

**Proof.** We apply Theorem 3.8 and get

$$\mathcal{O}_k\langle \mathcal{O}_P \rangle = \mathcal{O}_P^{\boxtimes k} / \mathcal{S}_k = \left( \frac{X^n}{\text{aut}(P)} \right)^{\boxtimes k} / \mathcal{S}_k = \frac{X^{nk}}{\text{aut}(P)^{\boxtimes k}}.$$
It follows from Proposition 2.3 that
\[ E_k(\mathcal{O}_P) = X^\frac{n_k}{\text{aut}(P^k)} = \mathcal{O}_{P^k}. \]

\[ \square \]

We can verify Lemma 3.18 in an intuitive way. Note that the set of \( E_k(\mathcal{O}_P) \)-structures on a finite set \( U \) is the set of \( \mathcal{S}_k \)-orbits of \( \mathcal{O}_P^{\square k} \)-structures on \( U \), and an element of \( E_k(\mathcal{O}_P)[U] \) of the form \( \{ (\pi_1, f_1), \ldots, (\pi_k, f_k) \} \) is such that \( (\pi_1, \pi_2, \ldots, \pi_k) \) is a \( k \)-rectangle on \( U \), and each \( f_i \) is a graph isomorphic to \( P \) whose vertex set equal to the blocks of \( \pi_i \). Such a set \( \{ (\pi_1, f_1), \ldots, (\pi_k, f_k) \} \) corresponds to a graph \( G \) isomorphic to \( P^k \) with vertex set \( U \). More precisely, \( G \) is the Cartesian product of the \( f_i \) in which each vertex \( u \in U \) is of the form \( u = B_1 \cap B_2 \cap \cdots \cap B_k \), where each \( B_i \) is one of the blocks of \( \pi_i \). In this way, we get a one-to-one correspondence between the \( E_k(\mathcal{O}_P) \)-structures on \( U \) and the set of graphs isomorphic to \( P^k \) with vertex set \( U \).

**Theorem 3.19.** The species \( \mathcal{G}^c \) of connected graphs and \( \mathcal{P} \) of prime graphs satisfy
\[ \mathcal{G}^c = \mathcal{S}(\mathcal{P}). \]

**Proof.** In this proof, all graphs considered are unlabeled.

The molecular decomposition of the species of prime graphs is
\[ \mathcal{P} = \sum_{P \text{ prime}} \mathcal{O}_P, \]
where each \( \mathcal{O}_P \) is a molecular species which is isomorphic to \( X^{|P|}/\text{aut}(P) \).

Let \( \{ P_1, P_2, \ldots \} \) be the set of unlabeled prime graphs. We have
\[
\mathcal{S}(\mathcal{P}) = \mathcal{S}(\mathcal{O}_{P_1} + \mathcal{O}_{P_2} + \cdots)
= \mathcal{S}(\mathcal{O}_{P_1}) \boxdot \mathcal{S}(\mathcal{O}_{P_2}) \boxdot \cdots
= (X + \mathcal{O}_{P_1} + \mathcal{O}_{P_2} + \cdots) \boxdot (X + \mathcal{O}_{P_2} + \mathcal{O}_{P_2} + \cdots) \boxdot \cdots
= \sum_{i_1, i_2, \ldots \geq 0} \mathcal{O}_{P_1}^{i_1} \boxdot \mathcal{O}_{P_1}^{i_2} \boxdot \cdots
= \sum_{i_1, i_2, \ldots \geq 0} \mathcal{O}_{P_1}^{i_1} \boxdot P_2^{i_2} \boxdot \cdots
= \sum_{C \text{ connected}} \mathcal{O}_C
= \mathcal{G}^c.
\]

\[ \square \]

Note that Theorem 2.8 follows as a corollary of Theorems 3.11 and 3.19.
Remark 3.20. Recall that the exponential composition of a species \( F \) is the sum of \( \mathcal{E}_k(F) \) on all nonnegative integers \( k \):

\[
\mathcal{E}(F) = \mathcal{E}_0(F) + \mathcal{E}_1(F) + \mathcal{E}_2(F) + \cdots = X + F + \mathcal{E}_2(F) + \cdots.
\]

Theorem 3.19 gives that

\[
\mathcal{G}^c = X + \mathcal{P} + \text{higher terms},
\]

\[
\mathcal{P} = \mathcal{G}^c - X - \text{higher terms},
\]

\[
Z_{\mathcal{P}} = Z_{\mathcal{G}^c} - p_1 - \text{higher terms}.
\]

Now we can compute the cycle index of the species of prime graphs \( Z_{\mathcal{P}} \) from the cycle index of the species of connected graphs \( Z_{\mathcal{G}^c} \), given by formula (1.2), recursively using maple:

\[
Z_{\mathcal{P}} = \left( \frac{1}{2} p_1^2 + \frac{1}{2} p_2 \right) + \left( \frac{2}{3} p_1^3 + p_1 p_2 + \frac{1}{3} p_3 \right)
+ \left( \frac{35}{24} p_1^4 + \frac{7}{4} p_1^2 p_2 + \frac{2}{3} p_1 p_3 + \frac{7}{8} p_2^2 + \frac{1}{4} p_4 \right)
+ \left( \frac{91}{15} p_1^5 + \frac{19}{3} p_1^2 p_2 + \frac{4}{3} p_1 p_3 + \frac{5}{4} p_2 p_2 + p_1 p_4 + \frac{2}{3} p_2 p_3 + \frac{3}{5} p_5 \right)
+ \left( \frac{1654}{45} p_1^6 + \frac{91}{3} p_1^4 p_2 + \frac{38}{9} p_1^3 p_3 + \frac{21}{4} p_1^2 p_2^2 + 2 p_1 p_2 p_2 + \frac{8}{3} p_1 p_2 p_3
+ \frac{4}{5} p_1 p_5 + \frac{47}{6} p_2^3 + \frac{5}{2} p_2 p_4 + \frac{11}{9} p_3^2 + \frac{2}{3} p_6 \right) + \cdots.
\]

Figure 5 shows the unlabeled prime graphs on no more than 4 vertices.

![Unlabeled prime graphs with 4 vertices](image)

**Figure 5.** Unlabeled prime graphs with \( n \) vertices, \( n \leq 4 \).

Hence we write down the beginning terms of the molecular decomposition of the species \( \mathcal{P} \):

\[
\mathcal{P} = \mathcal{E}_2 + (X \cdot \mathcal{E}_2 + \mathcal{E}_3) + (\mathcal{E}_2 \circ X^2 + X \cdot \mathcal{E}_3 + X^2 \cdot \mathcal{E}_2 + \mathcal{E}_2 \cdot \mathcal{E}_2 + \mathcal{E}_4) + \cdots.
\]

Comparing Figure 5 with unlabeled connected graphs with no more than 4 vertices, we see that there is only one unlabeled connected graph with 4 vertices that is not prime. In
fact, if we compare the first several terms of $Z_{Gc}$, given by (1.2), and $Z_P$ of order no more than 6, we get that

$$Z_{Gc} - Z_P = p_1 + \frac{1}{8} \left( p_1^4 + 2p_1^2p_2 + 3p_2^2 + 2p_4 \right)$$

$$+ \frac{1}{4} \left( p_1^6 + p_1^2p_2^2 + 2p_2^3 \right) + \frac{1}{12} \left( p_1^6 + 3p_1^2p_2^2 + 4p_2^3 + 2p_3^2 + 2p_6 \right) \cdots ,$$

which is the cycle index of connected non-prime graphs on no more than 6 vertices, as shown in Figure 6, which consist of a single vertex, a graph with 4 vertices, and two graphs with 6 vertices.

Figure 6. Unlabeled non-prime graphs on $n$ vertices, $n \leq 6$.

Table 1. Numbers of labeled and unlabeled prime graphs on $n$ vertices, denoted $p^l_n$ and $p^u_n$, respectively, for $n \leq 16$.

| $n$ | $p^l_n$ | $p^u_n$ |
|-----|---------|---------|
| 1   | 0       | 0       |
| 2   | 1       | 1       |
| 3   | 4       | 2       |
| 4   | 35      | 5       |
| 5   | 728     | 21      |
| 6   | 26464   | 110     |
| 7   | 1866256 | 853     |
| 8   | 251518352 | 11111  |
| 9   | 66296210432 | 261077 |
| 10  | 34496477587456 | 11716550 |
| 11  | 35641657548953344 | 1006700565 |
| 12  | 73354596197458024448 | 164059830354 |
| 13  | 301272202649664088951808 | 50335907869219 |
| 14  | 2471648811030427594714599424 | 29003487462847208 |
| 15  | 40527680937730480229320939012096 | 31397381142761241918 |
| 16  | 1328578958335783200943054119287117312 | 6396956011322517616514 |

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