Cohomological rigidity and the Anosov-Katok construction

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Abstract

We provide a general argument for the failure of Anosov-Katok-like constructions (as in [AFK15] and [Kar14]) to produce Cohomologically Rigid diffeomorphisms in manifolds other than tori. A \( C^\infty \) smooth diffeomorphism \( f \) of a compact manifold \( M \) is Cohomologically Rigid iff the equation, known as Linear Cohomological one,

\[
\psi \circ f - \psi = \varphi
\]

admits a \( C^\infty \) smooth solution \( \psi \) for every \( \varphi \) in a codimension 1 closed subspace of \( C^\infty (M, \mathbb{C}) \). As an application, we show that no Cohomologically Rigid diffeomorphisms exist in the Almost Reducibility regime for quasiperiodic cocycles in homogeneous spaces of compact type, even though the Linear Cohomological equation over a generic such system admits a solution for a dense subset of functions \( \varphi \). We thus confirm a conjecture by M. Herman and A. Katok in that context and provide some insight in the mechanism obstructing the construction of counterexamples.

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1 Introduction

1.1 Generalities and statement of the results

Let $M$ be a compact, $C^\infty$-smooth oriented $d$-dimensional manifold without boundary, furnished with the volume form $\mu$. Let us also consider $C^\infty_\mu(M, \mathbb{C})$, the space of smooth functions having 0 mean with respect to $\mu$. We study the solvability of the linear cohomological equation over any given volume preserving diffeomorphism $f \in \text{Diff}^\infty_\mu(M)$, i.e., of the equation

$$\psi \circ f - \psi = \varphi\quad (1.1)$$

where the function $\varphi \in C^\infty_\mu(M)$ is known and the unknown is $\psi \in C^\infty_\mu(M)$. This equation is central in the study of dynamical systems, as it arises naturally (cf. [KH96], §2.9), in the construction of smooth volume forms (cf. op. cit. §5.1), in the construction of conjugations as in K.A.M. theory (cf. op. cit. §15.1), and in the study of ergodic sums (cf. [FF03]).

Let us establish some terminology concerning eq. 1.1. A function $\varphi \in C^\infty_\mu(M)$ is called a coboundary over $f$ if eq. 1.1 admits a solution $\psi \in C^\infty_\mu(M)$. The space of coboundaries over $f$ is denoted by $\text{Cob}^\infty_\mu(f) \subset C^\infty_\mu(M)$. A diffeomorphism $f \in \text{Diff}^\infty_\mu(M)$ is called Cohomologically Rigid ($CR$), iff eq. 1.1 admits a solution for every $\varphi \in C^\infty_\mu(M)$, i.e.

$$f \in CR(M) \iff \text{Cob}^\infty_\mu(f) \equiv C^\infty_\mu(M)$$

The diffeomorphism $f$ is called Distributionally Uniquely Ergodic ($DUE$), iff coboundaries are dense in $C^\infty_\mu(M)$, i.e.

$$f \in DUE(M) \iff \overline{\text{Cob}^\infty_\mu(f)}^\text{cl} = C^\infty_\mu(M)$$

A celebrated example of $DUE$ diffeomorphisms are minimal rotations in tori. Up to date, the only known examples of $CR$ diffeomorphisms are Diophantine...
rotations in tori, i.e. rotations that are badly approximated by periodic ones (cf. §2.3 for the precise definition). M. Herman (cf. [Her80]1) and then A. Katok (cf. [Hur85]) have posed the following conjecture.

**Conjecture 1.** The only examples, up to smooth diffeomorphism and conjugacy, of $CR(M)$ diffeomorphisms are Diophantine translations in tori.

Anticipating the statement of cor. B, we mention that the goal of the present article is to show that the Anosov-Katok construction is not an appropriate tool for constructing counter-examples to this conjecture. This result is, in fact, a positive one in guise of a negative one, as it allows us to verify the conjecture in certain cases, cf. thm C.

The importance of this conjecture comes from K.A.M. theory, named after Kolmogorov, Arnol’d and Moser. A classical theorem of Arnol’d (cf. eg. [KH96] §15.1)2 states that a Diophantine rotation $R_\alpha : x \to x + \alpha$ on the circle $T^1 = \mathbb{R}/\mathbb{Z}$, if perturbed to a real analytic diffeomorphism whose rotation number is $\alpha$, will be analytically conjugate to the unpertrubed rotation. The proof is carried out by constructing successive conjugations that make the perturbation ever smaller and showing that the product of conjugations converges. The very construction of conjugations consists in efficiently solving the cohomological equation over the rotation $R_\alpha$, and the argument works precisely because Diophantine rotations are $CR(T^1)$. This theorem of local linearizability is known to be false for non-Diophantine (i.e. Liouville) rotations, see [Yoc95]. K.A.M. theory studies perturbations of Diophantine rotations in different contexts, and the general conclusion is that they tend to be rigid, i.e. persist under perturbations. Application of the K.A.M. machinery, however, depends crucially and in a general way on the Cohomological Rigidity of the unperturbed model. The conjecture, put informally, states that this very powerful tool case’s application is restricted to the local study of Diophantine rotations.

On the other side of the spectrum, the Anosov-Katok method of approximation by conjugation (cf. §1.2) works for Liouville-type rotations, i.e. rotations that are very well approximated by periodic ones. There, K.A.M. theory fails to apply and establish rigidity, and the Anosov-Katok method actually shows that there is no rigidity by constructing, for example, diffeomorphisms of the disc $\{ x \in \mathbb{R}^2, |x| \leq 1 \}$ that are weakly mixing and arbitrarily close to given Liouville rotations around $0 \in \mathbb{R}^2$ (cf. the original paper [AK70]). In the same context but for perturbations of Diophantine rotations, K.A.M. theory concludes the persistence of invariant circles (cf. [R02]). Weak mixing is an equidistribution
property for the orbits of a dynamical system, and it is stronger than ergodicity. Consequently, the existence of an invariant curve is an obstruction to weak mixing.

The efficiency of the Anosov-Katok construction in producing realizations of exotic dynamics makes it a good candidate for producing counter-examples to the conjecture. To our best knowledge, there have been two recent such attempts, both of them in spaces of quasi-periodic skew-product diffeomorphism spaces, [AFK15] and [Kar14]. Both attempts fail for slightly different reasons, but in the present article we will establish the reason why such attempts should not be expected to produce counter-examples to the conjecture. The reason is that the respective constructions share a key ingredient, the Anosov-Katok method.

Before coming to the two articles cited here above, let us quickly establish some notation, which we will also use in §5. The notation concerns the space of skew-product diffeomorphisms
\[ \text{SW}^\infty(T^d, P), \]
where \( T = \mathbb{R}^d/\mathbb{Z}^d \) and either \( P = G \) is a compact Lie group or a homogeneous space \( P = G/H \), where \( H \) is a closed subgroup of \( G \). If we let \( \alpha \in T^d \) be a translation and \( A(\cdot) : T^d \to G \) be \( C^\infty \)-smooth, then the element \((\alpha, A(\cdot)) \in \text{SW}^\infty(T^d, P)\) acts on \( T^d \times P \) by
\[
(\alpha, A(\cdot))(x, s.H) \mapsto (x + \alpha, A(x).s.H), \forall (x, s) \in T^d \times G
\]
Such an action is called a cocycle over \( \alpha \), or simply a cocycle. The space of cocycles over a fixed rotation \( \alpha \) is denoted by \( \text{SW}^\infty_{\alpha}(T^d, P) \) and \( \alpha \) is called the frequency of the cocycle. We obviously have \( \text{SW}^\infty(T^d, P) \subset \text{Diff}^\infty(T^d \times P) \) where \( \mu \) is the product of the Lebesgue-Haar measures in \( T^d \) and \( P \).

Conjugation within the class of cocycles is given by fibered conjugation. Two cocycles over the same rotation, \((\alpha, A_i(\cdot)) \) for \( i = 1, 2 \), are conjugate iff there exists a \( C^\infty \)-smooth mapping \( B(\cdot) : T^d \to G \) such that
\[
(\alpha, A_1(\cdot)) = (\alpha, B(\cdot + \alpha).A_2(\cdot).B^{-1}(\cdot))
\]
This corresponds to a change of variables in the phase space \( T^d \times P \) following
\[
(x, s.H) \mapsto (x, B(x).s.H)
\]
In [AFK15], the authors worked in the space of skew-product diffeomorphisms of \( M = T \times P \), where \( T = \mathbb{R}/\mathbb{Z} \) is the one-dimensional torus and \( P \) is either a compact nil-manifold or a homogeneous space of compact type.

They established genericity of \( \text{DUE}(M) \) in \( \text{AK}^\infty(C) \), the closure of the conjugacy class of cocycles that are periodic diffeomorphisms of \( T \times P \):
\[
C = \{(p/q, A(\cdot), p, q \in \mathbb{Z}^+, A(\cdot) \in C^\infty(T, P), A(\cdot + (q - 1)p/q) \cdots A(\cdot) \equiv \text{Id}\}
\]
Note that if \((p/q, A(\cdot)) \) is as above, then \((p/q, A(\cdot))^q \equiv \text{Id} \in \text{Diff}^\infty_T(T^d \times P)\), i.e.
\[(p/q, A(\cdot)) \) is a periodic diffeomorphism of period \( q \).

The space \( \text{AK}^\infty(C) \) was named after the Anosov-Katok construction for the exact reason that the proof was based on periodic approximation for the
frequency (i.e. the translation acting on $\mathbb{T}$), and on approximation by periodic diffeomorphisms in the fibres, i.e. $P$. For these reasons, the authors established genericity of $DUE$ in $\mathcal{AK}^\infty(\mathcal{C})$, but were not able to address Cohomological Rigidity: Liouville numbers are generic in $\mathbb{T}$, and they are those expected to be produced by a periodic approximation argument.

In [Kar14], we studied a non-generic slice of the space of quasi-periodic skew-product diffeomorphisms of $\mathbb{T} \times SU(2)$, where the frequency was fixed and satisfied a condition called Recurrent Diophantine, slightly stricter than a classical Diophantine one. The result obtained was that $DUE$ is generic even within this non-generic slice. Additionally, the techniques involved in the proof were precise enough so that the author was able to establish the non-existence of $CR$ diffeomorphisms in that space. The technique of the proof, even though more precise than that of [AFK15], shares a basic ingredient with the latter: the Anosov-Katok argument in the fibers.

The goal of the present article is to show that the construction of counter-examples to the conjecture, if any such counter-examples exist, is beyond present understanding of the Anosov-Katok construction. An informal statement of the main result of the paper, to be made precise by thm 3.1, is the following.

**Theorem A.** Let $\mathcal{C}$ be a space of diffeomorphisms for which there exists $\sigma \in \mathbb{R}$ such that for every $f \in \mathcal{C}$,

$$\overline{\text{Cob}^\infty(f)}^{cl} \subseteq C^\sigma_\mu(M)$$

Then, a generic diffeomorphism in $\mathcal{AK}^\infty(\mathcal{C}) \subset \text{Diff}^\infty_\mu(M)$ is not Cohomologically Rigid.

The space $\mathcal{AK}^\infty(\mathcal{C})$ is defined in §1.2 as the closure of the conjugacy class of a class of diffeomorphisms $\mathcal{C}$. Theorem 3.7 provides a similar but less precise statement on the meagreness of $CR$ in a space where $DUE \setminus CR$ is dense (plus a technical but important assumption). By $cl_\sigma$, we denote the closure of a space in the $C^\sigma$ topology.

From the proof of thm. A and its proof, we also obtain the following corollary. It is a statement on the Anosov-Katok method, which is an object that does not admit an unambiguous definition. In §4, and more precisely in propositions 4.4 and 4.5, we quantify what we mean by "Anosov-Katok-like construction" in the statement of the corollary, thus making it a meaningful mathematical proposition. We would like to point out that obtaining a property by a construction for us means that the proof that the object satisfies a certain property uses only information coming from its construction (c.f def. 4.1). An informal statement of the corollary is the following.

**Corollary B.** Counter-examples to Katok’s conjecture, if they exist, cannot be obtained by an Anosov-Katok-like construction.

The reason why such constructions fail to produce $CR$ objects is that fast approximation is, to a certain extent, incompatible with Cohomological Rigidity. The precise statements provided in the proof cover with some margin the known
constructions, and the proof is structured so that all assumptions are explicitly stated and introduced when they become relevant in the argument. We hope that treating cases where the estimates differ slightly from our assumptions will be facilitated this way.

Corollary B and its proof seem to indicate that the conjecture 1 is true. Even though it is not known, as L. Flaminio pointed out to us, whether Cohomological Rigidity implies the vanishing of all Lyapunov Exponents, the failure of the most powerful method in elliptic dynamics to produce counter-examples (unless a new arsenal of examples, allowing considerably more efficient Anosov-Katok constructions, is discovered) suggests quite strongly that the conjecture be true.

In particular, cor. B and its proof allow us to verify the conjecture in the following setting.

**Theorem C.** Given $P$ a homogeneous space of compact type and $\alpha$ Diophantine rotation, there exists an open set of cocycles in $SW_\alpha^\infty(\mathbb{T}^d, P)$ where $DUE \setminus CR$ is generic but no Cohomologically Rigid cocycles exist.

Genericity of $DUE$ is of course provided by [AFK15], but our theorem is more precise, since it proves inexistence of $CR$ cocycles. The theorem is made more precise in §5 by thm 5.2.

Combining the above theorem with the so-called renormalization scheme, [Kri01] and [Kar16b], we obtain the following corollary, valid for cocycles in $SW_\alpha^\infty(\mathbb{T}, P)$ ($d = 1$) and whose rotation satisfies a Recurrent Diophantine Condition.

**Corollary D.** Given $P$ a homogeneous space of compact type and $\alpha$ Recurrent Diophantine rotation, $DUE \setminus CR$ is generic in $SW_\alpha^\infty(\mathbb{T}, P)$, but no Cohomologically Rigid cocycles exist.

Recent advances in non-standard K.A.M. techniques (cf. [AFK11]) suggest that the arithmetic condition can be relaxed to a classical Diophantine one. The corollary would then hold true in

$$SW^\infty(\mathbb{T}, P) = \bigcup_{\alpha \in \mathbb{T}} SW_\alpha^\infty(\mathbb{T}, P)$$

### 1.2 The Anosov-Katok method

A general description of the Anosov-Katok method (see [AK70], [FK04]) for constructing realizations of wild dynamical behaviours is the following. One defines a class of diffeomorphisms $\mathcal{C}$, each preserving a rich structure (invariant manifolds, measures, distributions) or even the class of periodic diffeomorphisms, and whose dynamics are quite explicit. One then considers the conjugacy class $\mathcal{T}$ of such diffeomorphisms (where conjugacy is in the right regularity class, usually $C^\infty$ or $C^\omega$, and of the correct type, i.e. volume preserving, fibered, etc.). Subsequently, one looks for realizations of the sought after behaviour in $\mathcal{AK}^\sigma(\mathcal{C}) = \mathcal{T}^\sigma$, where $\sigma = \infty$ for smooth realizations and $\sigma = \omega$ for real analytic ones. To this end, a sequence $f_n \in \mathcal{C}$ and a sequence of conjugations $H_n$
are constructed so that the representatives \( f_n = H_n \circ \tilde{f}_n \circ H_n^{-1} \in T \) satisfy

\[
f_n \to f \in AK^\infty(C) \setminus T
\]

The conjugations \( H_n \) are constructed iteratively,

\[
H_n = H_{n-1} \circ h_n \quad \text{and} \quad H_0 = \text{Id}
\]

and \( h_n \) is chosen so that

\[
h_n \circ \tilde{f}_{n-1} = \tilde{f}_{n-1} \circ h_n \tag{1.2}
\]

The representative at the next step of the construction is then defined by

\[
f_n = H_n \circ \tilde{f}_n \circ H_n^{-1}
\]

where \( \tilde{f}_n \) has to be very close to \( \tilde{f}_{n-1} \) so that the \( f_n \) converge despite the divergence of \( H_n \).

Informally, the diffeomorphism \( \tilde{f}_n \circ \tilde{f}_{n-1} \) is constructed in a scale finer than the one where \( \tilde{f}_{n-1} \circ \tilde{f}_{n-2} \) was constructed, and the condition in eq. 1.2 assures that the constructions in the respective different scales are independent.

Since omitting a finite number of steps of the construction does not change the asymptotic properties of the limit object \( f \), we immediately get the following consequence. If realizations of a behaviour can be constructed in \( AK^\infty(C) \), then such realizations exist arbitrarily close to the class \( C \) in the \( C^\infty \) topology. This is related to the concept of Almost Reducibility, cf. §5.

Theorem 3.1 imposes a rate of convergence of the approximant diffeomorphisms to the limit object in order to exclude Cohomological Rigidity. This type of fast rate of convergence is what makes in general the above construction work, and inasmuch as such a condition has to be built into the construction, the latter should be expected not to produce CR diffeomorphisms.

1.3 The proofs in a nutshell

The proof of thm. A says that, given a class \( C \) of diffeomorphisms whose coboundary space has codimension at least 1 in \( C^\infty_\mu(M) \), the elements \( f \in AK^\infty(C) \) for which the approximation \( T \ni f_n \to f \) is fast will not be CR. The speed is measured with respect to the failure to solving the cohomological equation over \( f_n \) for functions that oscillate slowly (the low modes of a given Laplacian on \( M \)). The strength of these obstructions is measured by comparing their speed of oscillation with their distance from \( \text{Cob}^\infty(f_n) \). The fast approximation condition is proved to be generic.

The rate of approximation required so that thm. A be true is fast, i.e. exponential with respect to the strength of the obstructions, which makes it Liouville-like. The proof of cor. B focuses on the \( f \in AK^\infty(C) \) for which this rate fails, and becomes Diophantine-like. Then, under reasonable assumptions on the class \( C \), or even more generously on \( T \), the diffeomorphisms that are approximated at a polynomial rate will still not be CR.
The proof of thm. C is based on the fact that we can identify a class C for which the open set of cocycles of the statement is contained in AK∞(C). This class is that of resonant cocycles (cf. §5.2 for the definition), the important fact being that their analysis is very efficient and that their coboundary space is of large codimension. In [Kar17], we established that a sharply polynomial rate of approximation (i.e. polynomial and not exponential) implies the existence of a smooth invariant foliation into tori, an obvious obstruction to DUE. Therefore, no CR examples exist in the corresponding AK∞ space.

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2 Definitions, notation and preliminaries

2.1 General notation and calculus

By M we will denote a C∞ compact oriented manifold without boundary, and by C∞(M) the space of smooth (complex-valued) functions ϕ on M such that ∫_M ϕdµ = 0, where µ will denote a fixed smooth probability measure equivalent to Lebesgue, i.e. a volume form.

We will denote by ∥·∥_{C^s} the standard C^s norms of mappings M → E, E a normed vector space,

\[ \|f\|_{C^s} = \max_{0 \leq \sigma \leq s} \|\partial^\sigma f\|_{L^\infty} \]

We use the same notation for the countable family of semi-norms or semi-metrics defining the topology in C^∞(M) and Diff^∞(M). These are

\[ d_s(ψ, ψ') = \max_{0 \leq \sigma \leq s} \|\partial^\sigma ψ - \partial^\sigma ψ'\|_{L^\infty} \]

for functions, and

\[ d_s(f_1, f_2) = \max\{d_s(f_1 \circ f_2^{-1}, \text{Id}), d_s(f_2 \circ f_1^{-1}, \text{Id})\} \]

for diffeomorphisms.

We will use the inequalities concerning the composition of functions with mappings (see [Kar16b] or [Kri99]). Here, ψ ∈ C^∞(E) and f, f_1 and f_2 are smooth mappings M → E, where E is a normed vector space.

\[ \|ψ \circ f\|_{C^s} \leq C_s\|ψ\|_{C^{s}} |||f|||_s \] (2.1)

\[ \|ψ \circ f_2 - ψ \circ f_1\|_{C^s} \leq C_s\|ψ\|_{C^{s+1}} |||f_2\|_{C^s}, |||f_1|||_s, \|f_2 \circ f_1^{-1}\|_{C^s} \] (2.2)

where

\[ |||f|||_s = C_s(1 + \|f\|_{C^0})^s(1 + \|f\|_{C^{s}}) \]
When $f_1 \equiv \text{Id}$ the second inequality reads simply
\[
\|\psi \circ f_2 - \psi\|_{C^s} \leq C_s \|\psi\|_{C^{s+1}} \|f_2\|_{C^s}
\]
For mappings $M \to M$ and functions $\psi : M \to \mathbb{C}$, this inequality stays true as long as we admit an apriori bound on $\|f_2\|_{C^0}$ (proof by fixing a system of charts such that the ball of a fixed radius $\delta > 0$ around each point is contained in a chart). The constants would then depend on the a priori bound.

If $g$ is a fixed Riemannian metric on $M$, inducing the measure $\mu$, then we have a natural basis for the space $C_\infty^\mu(M)$. The eigenfunctions of $\Delta_g$, the Laplace-Beltrami operator associated to $g$, $\{\phi_i\}_{i=0}^\infty$ are the functions satisfying
\[
\Delta_g \phi_i = -\lambda_i^2 \phi_i \\
(\phi_j(\cdot), \phi_i(\cdot))_{L^2(\mu)} = \delta_{ij} \\
0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_i \leq \cdots < \infty
\]
Moreover, the sum
\[
\psi(\cdot) = \sum_{i=1}^{\infty} \hat{\psi}_i \phi_i(\cdot)
\]
defines a $C^\infty$-smooth function if, and only if,
\[
\hat{\psi}_i = O(\lambda_i^{-\infty})
\]
and every $C^\infty$ function admits such a representation which is unique, with
\[
\hat{\psi}_i = (\psi(\cdot), \phi_i(\cdot))_{L^2(\mu)}
\]
The functions $\phi_i$ satisfy the following estimate on the growth of derivatives
\[
\|\phi_i\|_{C^s} \leq C_s \lambda_i^{s+d/2}
\]
see [PAifASP65] or [Kuk00]. We also define the Sobolev spaces $H^s \equiv H^s_2$ for $s \in \mathbb{R}$, where we drop the reference to the fixed metric $g$, by
\[
\{\psi \in L^2(\mu), \sum_{i \in \mathbb{N}} (1 + \lambda_i)^{2s} |\hat{\psi}_i|^2 < \infty\}
\]
and, as usual, define the Sobolev norm in $H^s$ as the square root of the sum in the definition,
\[
\|\psi\|_{H^s} = \|\psi\|_s = \sum_{i \in \mathbb{N}} (1 + \lambda_i)^{2s} |\hat{\psi}_i|^2
\]
and the inner product giving rise to the norm
\[
\langle \psi, \psi' \rangle_{H^s} = \sum_{i \in \mathbb{N}} (1 + \lambda_i)^{2s} \hat{\psi}_i \overline{\hat{\psi}'_i}
\]

The space $H^{-s}$ is the dual of $H^s$, but the only self-dual space in the classical chain of inclusions

$$C^\infty \equiv H^\infty \subset \cdots \subset H^{s_1} \subset \cdots \subset H^{s_2} \subset \cdots \subset H^{-\infty} \equiv D'$$

where $-\infty < s_1 < s_2 < \infty$, is $L^2 \equiv H^0$. In fact, if we fix $g$ and $\{\phi_i\}$, the duality between $H^{-s}$ and $H^s$ is given by

$$\langle u, \psi \rangle_{H^{-s},H^s} = \sum_{i \in \mathbb{N}} (1 + \lambda_i)^{-s} \hat{u}_i \hat{\phi}_i(\cdot), \sum_{i \in \mathbb{N}} (1 + \lambda_i)^s \hat{\psi}_i \hat{\phi}_i(\cdot) \rangle_{L^2}$$

$$= \sum_{i \in \mathbb{N}} \hat{u}_i \hat{\psi}_i$$

where $u = \sum_{i \in \mathbb{N}} \hat{u}_i \phi_i(\cdot)$ and $\psi = \sum_{i \in \mathbb{N}} \hat{\psi}_i \phi_i(\cdot)$.

For $s > 0$, we will need the regularisation operators $T_N$, $\hat{T}_N$ and $R_N$ defined by

$$T_N \psi = \sum_{i \in \mathbb{N}} \hat{\psi}_i \phi_i(\cdot)$$

$$\hat{T}_N \psi = \sum_{0 < i \in \mathbb{N}} \hat{\psi}_i \phi_i(\cdot)$$

$$R_N \psi = \sum_{i > N} \hat{\psi}_i \phi_i(\cdot)$$

The operators $T_N$ and $\hat{T}_N$ coincide when restricted to $C^\infty\mu(M)$ or $H^s_\mu = \{\psi \in H^s, \int \psi d\mu = 0\}$. These operators satisfy the estimates

$$\|T_N \psi(\cdot)\|_{C^s} \leq C_s \lambda_N^{s+d/2} \|\psi(\cdot)\|_{C^0}$$

$$\|R_N \psi(\cdot)\|_{C^s} \leq C_{s,s'} \lambda_N^{s-s'+d} \|\psi(\cdot)\|_{C^{s'}}$$

Since we consider a fixed volume form on $M$, namely $\mu$, we will also need the homogeneous Sobolev spaces

$$\dot{H}^s = \{u \in H^s, \hat{u}_0 = 0\}, s \in \mathbb{R}$$

We will conserve the notation $\dot{H}^s$ for distributions, i.e. for $s < 0$, and the notation $H^s_\mu$ for functions, i.e. for $s \geq 0$, even though there is an overlap.

The introduction of the metric $g$ serves only for providing a basis of $L^2$ consisting of smooth functions and whose growth of $C^s$ norms satisfies the above useful properties. Alternatively, it can be interpreted as a ruler for measuring the characteristic scale at the successive steps of the Anosov-Katok construction by comparing them with $\lambda_k^{-1}$.

### 2.2 Diffeomorphisms, cocycles and cohomology

For this section, see [Koc09]. By $\text{Diff}^\infty(M)$ we will denote the space of $C^\infty$ diffeomorphisms on $M$, and by $\text{Diff}^\infty_\mu(M)$ those that preserve the measure $\mu$. If
\( \mathcal{U} \) is a subspace of \( C^\infty_\mu(M) \), and \( \sigma \in \mathbb{N} \), we will denote by
\[
\mathcal{U}^{cl}_\sigma \subset H^\sigma_\mu(M)
\]
the closure of the space \( \mathcal{U} \) in the \( H^\sigma \) topology.

The group \( \text{Diff}^\infty_\mu(M) \) acts on \( C^\infty_\mu \) by composition: for \( f \in \text{Diff}^\infty_\mu(M) \) and \( \varphi \in C^\infty_\mu(M) \),
\[
f^* \varphi = \varphi \circ f
\]
For every \( \varphi \in C^\infty(M) \) and \( f \in \text{Diff}^\infty(M) \), we can define the real \( \mathbb{Z} \) cocycle in \( M \) over \( f \) by
\[
\Phi_{\varphi,f}(x,n) \mapsto \sum_{i=0}^{n-1} \varphi \circ f^i
\]
A cocycle \( \Phi_{\varphi,f} \) is \( C^s \)-cohomologous to \( \Phi_{\varphi',f} \) iff there exists \( \psi \in C^s(M) \) such that
\[
\Phi_{\varphi,f}(x,n) = \psi \circ f^n - \psi + \Phi_{\varphi',f}(x,n)
\]
This is equivalent to \( \psi \circ f - \psi = \varphi - \varphi' \).

We shall say that \( \Phi_{\varphi,f} \) is an \( H^s \)-coboundary iff it is \( H^s \)-cohomologous to the null cocycle, \( \Phi_{0,f} \), which amounts to \( \psi \in H^s \) satisfying eq. 1.1. For \( f \in \text{Diff}^\infty_\mu(M) \) and \( 0 \leq s \leq \infty \) we will denote by \( \text{Cob}^s(f) \subset C^\infty_\mu \) the space of smooth functions which are \( H^s \)-coboundaries:
\[
\text{Cob}^s(f) = \{ \varphi \in C^\infty_\mu(M), \exists \psi \in H^s_\mu, \psi \circ f - \psi = \varphi \}
\]
This space is obtained by seeing the coboundary operator as an operator \( H^s_\mu \to H^s_\mu \) and intersecting its image with \( C^\infty_\mu \).

A first obstruction to a function \( \varphi \) being a coboundary over \( f \) is related to distributions preserved by \( f \) (see, e.g. [Kat01]). These are the distributions satisfying
\[
(f^* u, \psi) = (u, \psi \circ f) = (u, \psi), \forall \psi \in C^\infty(M)
\]
Definition 2.1. For every \( f \in \text{Diff}^\infty(M) \), we will denote the distributions in \( H^{-s} \setminus \{0\} \) (resp. \( H^{-s} \setminus \{0\} \)) that are preserved by \( f \) by \( H^{-s}(f) \) (resp. \( \dot{H}^{-s}(f) \)). By \( \mathcal{D}'(f) \) we denote the distributions in \( \mathcal{D}' \equiv H^{-\infty} \) preserved by \( f \). Since \( M \) is compact,
\[
\mathcal{D}'(f) = \{0\} \cup \bigcup_{s \in \mathbb{R}} H^s(f)
\]
Clearly, \( \mathbb{R}_\mu \subset \mathcal{D}'(f) \), for all \( f \in \text{Diff}^\infty_\mu(M) \). Whenever we write \( u \in \dot{H}^{-s}(f) \), we implicitly assume that \( \|u\|_{H^{-s}} = 1 \).

We also denote by \( \mathcal{H}^{-s}(M) \) the diffeomorphisms in \( \text{Diff}^\infty_\mu(M) \) that preserve a distribution in \( \dot{H}^{-s}(M) \):
\[
\mathcal{H}^{-s}(M) = \{ f \in \text{Diff}^\infty_\mu(M), \dot{H}^{-s}(f) \neq \emptyset \}
\]
It follows immediately from the definition that any $C^\infty$-coboundary $\varphi \in \text{Cob}^\infty(f)$ must satisfy
\[ \langle u, \varphi \rangle = 0, \forall u \in \mathcal{D}'(f) \] (2.7)
Under this condition on $\varphi$, the Hahn-Banach theorem shows that it is actually an approximate coboundary, i.e. that for every $\varepsilon > 0$ and $s_0 \in \mathbb{N}$, there exist $\psi, \epsilon \in C^\infty_{\mu}(M)$ satisfying
\[ \psi \circ f - \psi = \varphi + \epsilon \]
\[ \|\epsilon\|_{C^{s_0}} < \varepsilon \]
However, the condition of eq. 2.7 is not sufficient for a function to be a coboundary, and the application of the Hahn-Banach theorem gives an optimal answer in full generality. A celebrated example is that of Liouvillean rotations, for which we refer the reader to the next section, and especially to prop. 2.1.

The following nomenclature concerning the properties of a diffeomorphism relative to the space of its coboundaries is more or less standard.

**Definition 2.2.** A diffeomorphism $f \in \text{Diff}^\infty_{\mu}(M)$ is called DUE, standing for Distributionally Uniquely Ergodic, iff $\dim \mathcal{D}'(f) = 1$, in which case $\mathcal{D}'(f)$ is the vector space generated by the unique invariant probability measure $\mu$:
\[ f \in \text{DUE}(M) \iff \mathcal{D}'(f) = \mathbb{R}\mu \]

**Definition 2.3.** The diffeomorphism $f$ is called CS, Cohomologically Stable, iff the space of coboundaries $\text{Cob}^\infty(f)$ is closed in the $C^\infty$ topology.
\[ f \in \text{CS}(M) \iff \text{Cob}^\infty(f) = \overline{\text{Cob}^\infty(f)} \]

**Definition 2.4.** A diffeomorphism $f \in \text{Diff}^\infty_{\mu}(M)$ is called CR, Cohomology Rigid, iff it is both DUE and CS, i.e. iff $\text{Cob}^\infty(f)$ is closed and of codimension 1 in $C^\infty(M)$. We thus have
\[ f \in \text{CR}(M) \iff \text{Cob}^\infty(f) = C^\infty_{\mu}(M) \]

### 2.3 Rotations in tori and arithmetics

A vector $\alpha \in \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ will be called irrational iff
\[ \langle q, \alpha \rangle \in \mathbb{Z} \text{ and } q \in \mathbb{Z}^d \Rightarrow q = 0 \]
The vector $\alpha \in \mathbb{T}^d$ induces a minimal rotation $x \to x + \alpha$ on the torus $\mathbb{T}^d$ iff it is irrational. We can distinguish between two types of irrational vectors through the following definitions. The justification is given just below, in prop. 2.1.

**Definition 2.5.** A vector $\alpha \in \mathbb{T}^d$ is called Diophantine iff it satisfies a Diophantine Condition of type $\tilde{\gamma}, \tilde{\tau}$, for some $\tilde{\gamma} > 0$ and $\tilde{\tau} > d$. Such a condition, denoted by $DC(\tilde{\gamma}, \tilde{\tau})$, is defined by
\[ \alpha \in DC(\tilde{\gamma}, \tilde{\tau}) \iff |\langle q, \alpha \rangle|_\mathbb{Z} \geq \frac{\tilde{\gamma}^{-1}}{|q|^\tilde{\tau}}, \forall q \in \mathbb{Z}^d \setminus \{0\} \]
The distance from $Z$, $|\omega|_Z$, for $\omega \in \mathbb{R}$ is defined by

$$|\omega|_Z = \min_{p \in Z} |\omega - p|$$

and the norm on $\mathbb{Z}^d$ is the $\ell^1$ norm. It is a classical fact that Diophantine vectors form a full Haar measure meagre set in $\mathbb{T}^d$.

**Definition 2.6.** Liouville vectors in $\mathbb{T}^d$ are denoted by $L$ and are the irrational vectors which do not satisfy any Diophantine Condition.

It is an equally well established result that Liouville vectors form a residual set of 0 measure in $\mathbb{T}^d$. We now recall the proof of the fact that an irrational rotation in $\alpha \in \mathbb{T}^d$ is $CR$ iff $\alpha \in DC$.

**Proposition 2.1.** Let a rotation $R_\alpha : \mathbb{T}^d \to \mathbb{T}^d$, $R_\alpha : x \mapsto x + \alpha \mod \mathbb{Z}^d$. Then, $R_\alpha \in CR(\mathbb{T}^d)$ iff $\alpha \in DC$.

**Proof.** If $\alpha \in DC(\tilde{\gamma}, \tilde{\tau})$ and $\varphi \in C^\infty_\mu$, then there is a unique $\psi \in C^\infty_\mu$ solving

$$\psi(\cdot + \alpha) - \psi(\cdot) = \varphi(\cdot)$$

and satisfying the estimate $\|\psi\|_s \leq C_s \gamma \|\varphi\|_{s+\tau}$. The first step of the proof is application of the Fourier transform in order to obtain the equation

$$\hat{\psi}(k) = \frac{1}{e^{2i\pi \langle k, \alpha \rangle} - 1} \hat{\varphi}(k)$$

We then estimate the norm using the definition of the Diophantine condition. The factor $(e^{2i\pi \langle k, \alpha \rangle} - 1)^{-1}$ is known as a small denominator.

If $\alpha \in L$, let $q_n \in \mathbb{Z}^d$ be such that $|\langle q_n, \alpha \rangle|_Z \leq |q_n|^{-n}$. Then, the function

$$\varphi(\cdot) = \sum_n (e^{2i\pi \langle q_n, \alpha \rangle} - 1)^{1/2} e^{2i\pi \langle q_n, \cdot \rangle}$$

is in $C^\infty_\mu(\mathbb{T}^d)$, but the solution is not defined in any function or distribution space, since the modulus of its $q_n$-th Fourier coefficient grows faster than any power of $q_n$.

A straightforward application of the proposition above and of the definition of Cohomological Rigidity shows that the only $CR(\mathbb{T}^d)$ diffeomorphisms homotopic to the Id are, up to smooth conjugation, Diophantine translations (see [Koc09] for the details).

Let us also define the Recurrent Diophantine condition. We call $G : \mathbb{T} \setminus \{0\} \to \mathbb{T}$ the Gauss map $x \mapsto \{x^{-1}\}$, where $\{\cdot\}$ denotes the fractional part of a real number. For $\alpha \in \mathbb{T} \setminus \mathbb{Q}$, call $\alpha_n = G^n(\alpha)$.

**Definition 2.7.** A rotation $\alpha \in \mathbb{T}$ satisfies a Recurrent Diophantine condition of type $\tilde{\gamma}, \tilde{\tau}$ iff $\alpha_n \in DC(\tilde{\gamma}, \tilde{\tau})$ for infinitely many $n \in \mathbb{N}$.

It is a full Haar measure condition for every $\tilde{\gamma} > 0$ and $\tilde{\tau} > 1$, and, put informally, states that when we apply the continued fractions algorithm on $\alpha$, the remainders of the Euclidean division satisfy a fixed Diophantine condition infinitely often.
3 Proof of theorem A

We can now state a precise version of thm A.

**Theorem 3.1.** Suppose that $S \subset \text{Diff}_\mu^\infty(M)$ is a closed subspace such that $\mathcal{H}^{-\sigma}(M) \cap S$ is dense in $S$ for some $\sigma \geq 0$. Then, $CR(M) \cap S$ is meagre (or empty).

Before providing the proof for this theorem, we remark that if it also happens (as in [AFK15] and [Kar14]) that $DUE(M) \cap S$ is dense, then $DUE$ is actually generic in $S$, since for general reasons $DUE$ is a $G_\delta$ property:

$$DUE(M) = \bigcap_{m,n,k} \{ f \in \text{Diff}_\mu^\infty(M), \exists \psi \in C^\infty_{\mu}, \| \psi \circ f - \psi - \phi_k \|_{C^n} < m^{-1} \} \quad (3.1)$$

where $\{\phi_k\}$ is the basis of eigenfunctions of the Laplacian on $M$. However, there is no apriori topological reason why $CR(M) \cap S$ should not be empty. The initial goal of this paper was in fact to prove that $CR$ is an $F_\sigma$ property (just as $DC$, thus establishing the difficulty of the conjecture in full generality. We still do not know whether this is true.

Theorem 3.1 explains why the techniques of [AFK15] fail to conclude about the existence of a counterexample to the conjecture, since they only provide information on generic diffeomorphisms in the space $AK^\infty$ as it is defined in the reference, where $\mathcal{H}^{-\sigma}(T \times P)$ is dense for every $\sigma \geq 0$: a generic diffeomorphism in that space has to be $DUE$ and not $CR$. It also explains why the hands-on approach of [Kar14] is needed in order to exclude the existence of $CR$ in the respective space of dynamical systems.

3.1 Preparation of the proof

We now prepare the proof of thm A, by stating and proving two lemmas. They can be seen as abstractions of what happens when we perturb a rational, resp. a Liouvillean, rotation and look for a solution to the cohomological equation for a rhs function supported in the modes where the denominator is 0, resp. Liouville-small.

This first lemma, basic ingredient of the proof of thm. 3.1, provides an estimate which quantifies the following fact. Given a diffeomorphism $f' \in \text{Diff}_\mu^\infty(M)$ which preserves a distribution $u \in \mathcal{H}^{-\sigma}(M)$ and a function $\varphi \in H^s_{\mu}$, $\varphi \not\in \ker u$, if we perturb $f'$ to $f$ in the $C^s$ topology and assume that $\varphi \in \text{Cob}^\infty(f)$, then the estimates on the norms of the solution (or an approximate one) should be expected to be bad. The following lemma provides a precise statement, and its proof is to be compared with the small denominator estimate for irrational rotations.

**Lemma 3.2.** Let $f \in \text{Diff}_\mu^\infty(M)$ and suppose that there exists $f' \in \text{Diff}_\mu^\infty(M)$ such that $\| f \circ (f')^{-1} \|_{C^s} = \delta_\sigma > 0$ is small and such that there exists some
Suppose, now, that there exists an approximate solution \( \psi \in C^{\sigma+1} \) to the cohomological equation, i.e.

\[
\psi \circ f - \psi = \varphi + \epsilon
\]

with \( \varphi \in C^\infty \setminus \ker u \) and \( \|\epsilon\|_{H^\sigma} = \varepsilon, \) small enough (in fact \( \langle u, \epsilon \rangle \leq \frac{1}{2} \langle u, \varphi \rangle \)), will suffice. Then, \( \psi \) satisfies the following estimate

\[
\|\psi\|_{C^{\sigma+1}} \geq C_{\sigma}|||f|||_{\sigma} \delta \langle u, \varphi \rangle
\]

**Proof.** The proof uses the estimates for composition of mappings and the invariance of the objects. Eq. 3.2 and the fact that \( (f')^* u = u \) imply that

\[
(u, \psi \circ (f \circ (f')^{-1}) - \psi) = (u, \varphi + \epsilon)
\]

Estimation by duality, the assumed smallness of \( \langle u, \epsilon \rangle \) and the triangle inequality imply directly that

\[
\|u\|_{H^{-\sigma}} \|\psi \circ (f \circ (f')^{-1}) - \psi\|_{H^\sigma} \geq \frac{1}{2} \langle u, \varphi \rangle
\]

The estimate announced in the statement of the lemma follows from the inequality on the composition of functions with mappings. \( \square \)

This second lemma is more qualitative in its nature. It is used in the proof of thm. 3.7, which is consequently less precise than thm. 3.1.

**Lemma 3.3.** Let \( f \in \text{Diff}^\infty_{\mu}(M) \), and suppose that it is not cohomologically stable, i.e. that

\[
\text{Cob}^\infty_{\mu}(f) \subsetneq \text{Cob}^\infty(f)^{\text{cl}}
\]

and fix \( \varphi \in \text{Cob}^\infty(f)^{\text{cl}} \setminus \text{Cob}^\infty(f) \), an approximate but not exact coboundary over \( f \).

Fix some \( \delta > 0 \) and a \( s_0 \in \mathbb{N} \) big enough. Then, for every \( M > 0 \), there exists \( \varepsilon > 0 \) such that, for every \( s_1 \geq s_0 + 1 \), if we call \( \delta_{s_1} = d_{s_1}(f, f') < \delta \), then

\[
\psi \circ f' - \psi = \varphi
\]

implies that

\[
\begin{cases}
\|\psi\|_{s_0} > M, \text{ or } \\
\|\psi\|_{s_1} > C_{s_1}^{-1} \varepsilon \|f\|_{s_1}^{-1} \delta_{s_1}^{-1}
\end{cases}
\]

In particular,

\[
\|\psi\|_{s_0+1} > \max\{M, C_{s_0}^{-1} \varepsilon \|f\|_{s_0}^{-1}\delta_{s_0}^{-1}\}
\]

**Proof.** Let \( f \) and \( \varphi \) be as in the statement of the lemma. Then, by the Ascoli-Arzelà theorem, there exists \( s_0 \) such that if \( s_1 \geq s_0 + 1 \), then for every \( M > 0 \) there exists \( \varepsilon > 0 \) such that if \( \psi \in C^\infty_{\mu} \) and

\[
\|\psi \circ f - \psi - \varphi\|_{s_1} < \varepsilon \Rightarrow \|\psi\|_{s_0} > M
\]
Let us fix such $M$ and $\varepsilon$, and suppose that $d_s(f, f') = \delta$ is small enough for some $s \geq s_1$. If, now, $\psi$ is such that $\psi \circ f' - \psi = \phi$, then

$$
\psi \circ f - \psi = \phi + \psi \circ f - \psi \circ f'
$$

Now, either

$$
\|\psi\|_{s_0+1} \geq C_{s_0}^{-1} \|f\|^{\lambda}_{s_0} \delta^{s_0-1}
$$

where $C_{s_0}$ is the constant appearing in eq. 2.2, and

$$
\|\psi\|_s \to \infty \text{ as } \delta_{s_0} \to 0, \forall s \geq s_0 + 1
$$

or

$$
\|\psi \circ f - \psi \circ f'\|_{s_0} \leq C_{s_0} \|\psi\|_{s_0+1} \|f\|^{\lambda}_{s_0} \delta_{s_0} < \varepsilon
$$

In the second case, by the Cohomological Instability of $f$ we obtain that

$$
\|\psi\|_{s_1} > M
$$

and therefore $\|\psi\|_{s_1} > \max\{M, C_{s_1}^{-1} \|f\|^{\lambda}_{s_1} \delta_{s_1}^{-1}\}$.

### 3.2 A proposition on approximation

We now prove the following proposition concerning the instability of a diffeomorphism $f$ that is well approximated by diffeomorphisms preserving distributions. The proposition shows that, under quite mild conditions, $f$ will not be $CR$.

The proof consists in producing a function $\phi \in C^\infty\mu$, $\phi \neq 0$, which is not a coboundary over $f$.

**Proposition 3.4.** Let $f \in \text{Diff}_\mu^\infty(M)$ and suppose that there exists a sequence of diffeomorphisms $f_n \in \text{Diff}_\mu^\infty(M)$, $f_n \to f$ in $C^\infty$, satisfying the following properties:

1. There exists $\sigma \geq 0$ such that, for every $n \in \mathbb{N}$, $f_n \in \hat{H}^{-\sigma}(M)$.

2. If we let $\delta_n = \delta_{\sigma,n} = d_\sigma(f, f_n) \searrow 0$, then there exist a sequence $u_n \in \hat{H}^{-\sigma}(f_n)$ with $\|u_n\|_{H^{-\sigma}} = 1$, a sequence $N_n \in \mathbb{N}$, $K > 0$ and $s_0 \geq 0$, such that

$$
\|T_{N_n} u_n\|_{H^{-\sigma}} \geq K \lambda_{N_n}^{-s_0} \text{ and } \delta_n = O(\lambda_{N_n}^{-s_0})
$$

Then, there exists $\omega \in C^\infty_{\mu}(M)$ which is not an exact coboundary over $f$: $\omega \notin \text{Cob}^\infty(f)$.

The condition of item 2 of the theorem compares the rate of convergence of the $f_n$ to $f$ with the spectrum of the $u_n$, and imposes a compatibility condition between the two (to be compared with the approximation of a Liouville number by its best rational approximations). We point out for later use that, up to considering a subsequence depending on a fixed $M > 0$, we can impose that

$$
\delta_n^{-1} \sum_{k>n} \delta_k < M
$$
Under the assumption of the existence a sequence $f_n \in \text{Diff}^\infty_\mu(M)$ such that $f_n \to f$ fast enough and such that $H^{-\sigma}(f_n) \neq \emptyset$ for some $\sigma > 0$ and for every $n \in \mathbb{N}$, then lemma 3.2 becomes relevant. If we let $\delta_{\sigma,n} = d_\sigma(f,f_n) \perp 0$, and we suppose that $\omega \in C^\infty_\mu(M)$ and $u_n \in H^{-\sigma}(f_n)$ is such that

$$\delta_{\sigma,n}^{-1} \langle u_n, \omega \rangle \to \infty \quad (3.3)$$

by lemma 3.2, this would force a solution $\psi$ of the cohomological equation

$$\psi \circ f - \psi = \omega$$

to satisfy $\|\psi\|_{\sigma+1} = \infty$, so that no such smooth function can exist. We stress that we do not claim that $\omega \in C^\infty_\mu(f)$. We now construct such a function $\omega$.

**Proof.** Let us choose $u_n \in H^{-\sigma}(f_n)$, $\|u_n\|_{H^{-\sigma}} = 1$, and $\omega_n \in H^{-\sigma}_\mu(M)$, $\|\omega_n\|_{H^{-\sigma}} = 1$ with $^3$

$$\omega_n \perp H^{-\sigma} \ker u_n$$

Since our goal is to construct a $C^\infty_\mu$ function, we truncate the functions $\omega_n$ in order to obtain

$$\omega^{(n)}(\cdot) = T_N \omega_n(\cdot) \in C^\infty_\mu(M)$$

and sum them following

$$\omega(\cdot) = \sum_{k=1}^{\infty} c_k \delta_k \omega^{(k)}(\cdot)$$

where $\delta_n = \delta_{\sigma,n}$. The function thus defined will be smooth provided that $\delta_n = O(\lambda_n^{-s_1})$, and $|c_n| = O(\lambda_n^{s_1})$ for some $s_1 < \infty$.

Under these conditions, let us calculate and estimate the lhs of the limit in eq. 3.3:

$$\delta_{\sigma,n}^{-1} \langle u_n, \omega \rangle = \delta_{\sigma,n}^{-1} \sum_{k<n} c_k \delta_k \langle u_n, \omega^{(k)} \rangle + c_n \langle u_n, \omega^{(n)} \rangle + \delta_{\sigma,n}^{-1} \sum_{k>n} c_k \delta_k \langle u_n, \omega^{(k)} \rangle \quad (3.4)$$

No reasonable assumption seems to exist that imposes restrictions on the first sum. For example, $\langle u_n, \omega^{(k)} \rangle \ll \delta_n, 0 < k < n$, seems to be needlessly restrictive. Fortunately, such an assumption appears to be unnecessary: we need only consider the sign of the sum,

$$\begin{cases} +1, & \text{if } \sum_{k<n} c_k \delta_k \langle u_n, \omega^{(k)} \rangle \geq 0 \\ -1, & \text{if } \sum_{k<n} c_k \delta_k \langle u_n, \omega^{(k)} \rangle < 0 \end{cases}$$

and chose the sign of $c_n = \pm \lambda_n^{s_1}$ accordingly. Then, the first two terms in eq. 3.4 are, in absolute value, $\geq |c_n \langle u_n, \omega^{(n)} \rangle|$ so that, under our assumptions,

$$|\delta_{\sigma,n}^{-1} \sum_{k<n} c_k \delta_k \langle u_n, \omega^{(k)} \rangle + c_n \langle u_n, \omega^{(n)} \rangle| \geq |c_n \langle u_n, \omega^{(n)} \rangle| \geq K \lambda_n^{s_1-s_0} \to \infty$$

---

$^3$Here we implicitly identify $H^\sigma$ with its dual. Since we keep $\sigma$ fixed, there are no complications. The argument would become more complicated if one wishes to allow $\sigma = \sigma_n \to \infty$. 

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so long as \( s_1 > s_0 \).

In order to establish the divergence of the limit in eq. 3.3, we have to be able to conclude that the last sum in eq. 3.4 is \( \omega (\lambda_{N_n}^{s_1} - s_0) \). To this end, it actually suffices to estimate brutally \(|c_k(u_n, \omega^{(k)})| \leq |c_k| = \lambda_{N_k}^{s_1}\), which implies that

\[
|\delta_n^{-1} \sum_{k>n} c_k \delta_k(u_n, \omega^{(k)})| \leq \delta_n^{-1} \sum_{k>n} |c_k| \delta_k
\]

Then, up to considering a subsequence, we can bound the rhs of the inequality by an absolute constant, and this concludes the construction of \( \omega(\cdot) \).

We remark that the proposition shows that, under quite mild conditions, \( Cob(f) \) may not even contain the space

\[
\{ \varphi \in C^\infty_{\mu}, \langle u_n, \varphi \rangle \to 0 \}
\]

We also remark that we do not need to assume that \( N_n \to \infty \), even though this is expected to occur in general. In particular, if the limit diffeomorphism \( f \) is \( DUE \), the sequence \( N_n \) has to diverge, as implies the following lemma and its corollary.

**Lemma 3.5.** Let \( f \in Diff^\infty_{\mu}(M) \), and suppose that there exists a sequence \( \{f_n\} \in Diff^\infty_{\mu}(M) \), \( f_n \to f \) and satisfying the following condition:

1. \( f_n \in \dot{H}^{-\sigma}(M) \)
2. there exists a pre-compact sequence \( \{u_n\} \in \dot{H}^{-\sigma}(M) \) with \( u_n \in \dot{H}^{-\sigma}(f_n) \) for every \( n \in \mathbb{N} \).

Then, \( f \notin DUE(M) \).

**Proof.** Let \( u \in \overline{\{u_n\}}_{\mu^{-\sigma}} \subset \dot{H}^{-\sigma}(M) \), and let \( \varphi \in C^\infty_{\mu}(M) \) be such that \( \langle u, \varphi \rangle = 1 \). If \( \varphi \) were an approximate coboundary over \( f \), then there would exist \( \epsilon(\cdot) \) and \( \psi(\cdot) \) such that

\[
\psi \circ f(\cdot) - \psi(\cdot) = \varphi(\cdot) + \epsilon(\cdot)
\]

with \( \epsilon \in C^\infty_{\mu} \) arbitrarily small in the \( C^\infty \) topology, and \( \psi \in C^\infty_{\mu} \) depending on \( \epsilon(\cdot) \). Given such \( \epsilon(\cdot) \) and \( \psi(\cdot) \), for \( n \) big enough

\[
\psi \circ f_n(\cdot) - \psi(\cdot) = \varphi(\cdot) + \epsilon(\cdot) + \delta_n(\cdot)
\]

with \( \delta_n(\cdot) = \psi \circ f_n(\cdot) - \psi \circ f(\cdot) \) arbitrarily small in the \( C^\infty \) topology as \( n \to \infty \). For \( n \) big enough and in a subsequence, the rhs tests \( > 1/2 \) against \( u_n \), while the lhs test 0, a contradiction.

In fact, \( u \in \dot{H}^{-\sigma}(f) \). The proof grants the following corollary.

**Corollary 3.6.** Under the hypotheses of prop. 3.4, if additionally \( f \in DUE(M) \), then \( N_n \to \infty \).
Proof. If $N_n$ can be chosen to be bounded by $M \in \mathbb{N}^{*}$, then there exists $\varphi \in C_{\mu}^{\infty}(M)$, spectrally supported in the first $M$ modes,

$$T_M \varphi(\cdot) = \varphi(\cdot)$$

and such that

$$\lim \sup |\langle u_n, \varphi \rangle| > 0$$

The proof of lemma 3.5 implies that $\varphi$ cannot be an approximate coboundary over $f$.\qed

Remark 3.1. We point out that we have shown that, if there exists $\varphi \in C_{\mu}^{\infty}(M)$, a sequence $f_n \to f$ and $u_n \in H^{\sigma}(f_n)$, $\|u\|_{-\sigma} = 1$, such that

$$\lim \sup |\langle u_n, \varphi \rangle| > 0$$

then $f \notin DUE$.

Lemma 3.5 and its corollary imply the intuitively obvious fact that if a diffeomorphism $f$, built by approximation by the sequence $f_n$, is to be $DUE$, then the obstructions of the $f_n$ have to recede to infinity, precisely as in [AFK15] and [Kar14]. The important point in the proof is of course the compactness of functions with bounded spectrum.

3.3 Proof of thm A

We now provide the proof of the precise version of thm A.

Proof of thm 3.1. Let $\{f_n\}_{\mathbb{N}}$ be a dense set in $\mathcal{H}^{-s}(M) \cap \mathcal{S}$. Let, now, $i, k \in \mathbb{N}^{*}$, and for $n \in \mathbb{N}$ choose $u_n \in H^{-\sigma}(f_n)$ of norm 1 and $N_n = N_n(i, k)$ such that

$$\|T_{N_n} u_n\|_{H^{-\sigma}} \geq k^{-1} \lambda_{N_n}^{-i}$$

For each $n$, choose $u_n$ so that $N_n$ is minimal. Then, $N_n$ is non-increasing as $i, k$ increase.

Now, for $j, l \in \mathbb{N}^{*}$ define

$$V_{j,l} = V_{j,l}(i, k) = \{f \in \mathcal{S}, \exists n \in \mathbb{N}, d_{\sigma}(f, f_n) < l^{-1} \lambda_{N_n}^{-j} \}$$

By definition, $V_{j,l}$ is open and non-empty, since it contains $\{f_n\}$.

Lemma 3.2 shows that $\mathcal{S} \setminus CR(M)$ contains the $G_{\delta}$ set

$$\bigcup_{i, k} \bigcap_{j, l} V_{j,l}(i, k)$$

and thus $\mathcal{S} \setminus CR(M)$ is of the second category.\qed
Lemma 3.3 becomes relevant in a space where Cohomological Instability is a priori known to be dense. The following theorem shows that it is sufficient for \( CR \) to be meagre, independently of the presence of distribution-preserving diffeomorphisms.

Before stating the theorem, we need to establish some notation. Let us call for \(-\infty \leq \sigma \leq \infty\)

\[
Q(\sigma) = \{ f \in \text{Diff}_\mu^\infty(M), \text{Cob}^\sigma(f) \subseteq \text{Cob}^{\sigma+1}_\mu(M) \}
\]

We have clearly \( Q(\sigma') \subset Q(\sigma) \) if \( \sigma < \sigma' \). Moreover, as soon as the the coboundary operator is not open, its image is meagre in \( \mathcal{C}_\mu^\infty(M) \) (this is true for linear operators in Banach spaces). A Diophantine rotation is not in any \( Q(\sigma) \), while a Liouvilean one is in \( Q(\sigma) \) for all \(-\infty \leq \sigma \leq \infty\). A non-irrational rotation in \( \mathbb{T}^d \) inducing a minimal rotation in a torus of dimension \( 0 < d' < d \) will be in all, resp. no, \( Q(\sigma) \), if the induced rotation is Liouville, resp. Diophantine.

**Theorem 3.7.** Let \( S \) be a closed subset of \( \text{Diff}_\mu^\infty(M) \) and suppose that there exists \( \sigma > -\infty \) such that \( Q(\sigma) \) is dense in \( S \). Then \( CR(M) \cap S \) is meager in \( S \) (or empty).

Naturally, the theorem becomes relevant when \( Q(\sigma) \cap \text{DUE} \) is a priori known to be dense in \( S \), for otherwise this theorem is a weaker version of thm. 3.1. This is a good moment to remark that we do not know whether

\[
Q(\sigma) \subset \mathcal{H}^{-\sigma}(M)^{\sigma+1}
\]

is true, i.e. whether Cohomological Instability is caused by approximation of distribution-preserving diffeomorphisms. We now come to the proof of thm. 3.7

**Proof.** Let \( \{f_n\}_{n \in \mathbb{N}} \subset Q(\sigma) \) be dense in \( S \). Then, there exists a dense subset \( \{\varphi_n\}_{n \in \mathbb{N}} \subset C^\infty_\mu(M) \), such that

\[
\varphi_n \notin \text{Cob}^\sigma(f_m), \forall n, m \in \mathbb{N}
\]

This follows from the fact that the image of a linear operator (in our case of the coboundary operator \( \psi \mapsto \psi \circ f - \psi \), for any fixed \( f \in \text{Diff}_\mu^\infty(M) \)) is either the full space or meagre. By the hypothesis, we can chose \( \sigma \) uniformly in \( n \) so that \( \text{Cob}^\sigma(f_n) \) be meagre in \( C^\infty_\mu(M) \) for all \( n \in \mathbb{N} \).

Lemma 3.3 then implies that the set

\[
\mathcal{V}_{M,n} = \{ f \in S, \psi \circ f - \psi = \varphi_n \Rightarrow \|\psi\|_{C^{\sigma+1}} > M \}
\]

is open for \( M, n \in \mathbb{N} \). We formally define \( \|\psi\|_{C^{\sigma+1}} = \infty \) if no such solution exists. Clearly, if we call \( \mathcal{R} \) the complement of \( CR(M) \) in \( S \),

\[
\bigcap_{M,n} \mathcal{V}_{M,n} \subset \mathcal{R}
\]

Thus, the set \( \mathcal{R} \) contains a \( G_\delta \) set and is assumed dense. It is consequently generic. \( \Box \)
The condition that $Q(\sigma)$ be dense in $S$ for some $\sigma > -\infty$ is satisfied by all known $DUE \setminus CR$ examples. It does not seem to be implied by mere density of $DUE$, and this is indicated by the following lemma, or rather by its proof. We remark that the hypothesis demands that $f$ not be $CS$, which is weaker than it not being $CR$.

**Lemma 3.8.** Let $f \in \text{Diff}_\mu(M) \setminus CS(M)$. Then, there exists $-\infty \leq s_0 < \infty$ such that $f \in Q(s)$ for all $-\infty \leq s < -s_0$.

**Proof.** If $f \in \text{Diff}_\mu(M) \setminus CS(M)$, then there exists a sequence $n_k \nearrow +\infty$ such that, if we call

$$U = \{ \sum_k \hat{\psi}_{n_k} \phi_{n_k}(\cdot), \hat{\psi}_{n_k} = O(\lambda_{n_k}^{-\infty}) \} \subset C_\mu^\infty(M)$$

and

$$V = \{ \sum_{n \notin \{n_k\}} \hat{\psi}_n \phi_n(\cdot), \hat{\psi}_n = O(\lambda_n^{-\infty}) \} \subset C_\mu^\infty(M)$$

then

1. the space of coboundaries over functions in $U$,

$$U(f) = \{ \psi \circ f - \psi, \psi \in U \} \subset C_\mu^\infty(M)$$

2. with the same notational convention, codim $\left( \overline{V(f)}^{C_\mu^\infty(M)} \right) = \infty$ in $\{Cob^\infty(f)\}^{C_\mu^\infty}$

3. There exists $s_1 < \infty$ such that for every $s \geq 0$,

$$\| \phi_{n_k} \circ f(\cdot) - \phi_{n_k}(\cdot) \|_s = O(\lambda_{n_k}^{s_1})$$

The third item is due to the fact that, if the polynomial rate of growth of each $C^s$ norm is not bounded uniformly\(^4\) for $s$, i.e. if there does not exist such an $s_1$, then

$$\sum_k \hat{\psi}_{n_k} (\phi_{n_k} \circ f(\cdot) - \phi_{n_k}(\cdot))$$

converges to a $C^\infty$ function iff $\hat{\psi}_{n_k} = O(\lambda_{n_k}^{-\infty})$, i.e iff the function

$$\sum_k \hat{\psi}_{n_k} \phi_{n_k}(\cdot)$$

is itself $C^\infty$ which implies that $U(f)$ is closed.

The conclusion of the lemma now follows as in the case of Liouvillian rotations. The space $Cob^s(f)$ is meager in $Cob^{s_1}(f)$ whenever $s < s_1$, since

$$\{ \hat{\psi}_{n_k} = O(\lambda_{n_k}^{s_1}) \}$$

is meager in

$$\{ \hat{\psi}_{n_k} = O(\lambda_{n_k}^{s_1}) \}$$

\(^4\)Uniformity is in $s$, not in the constant involved.
The reason why the condition of thm. 3.7 does not seem to be implied by density of DUE in \( \mathcal{S} \) is the following. Cohomological instability is caused by an at most polynomial growth of the \( C^s \) norms of the "elementary coboundaries"

\[
\phi^f_{n_k}(\cdot) = \phi_{n_k} \circ f(\cdot) - \phi_{n_k}(\cdot)
\]

along a subsequence \( \{n_k\} \) depending on \( f \). A generic function has full spectrum, and, given two DUE \( \setminus CR \) diffeomorphisms \( f \) and \( f' \), arbitrarily close in \( C^\infty \), a generic function in \( C^\infty_\mu \) will not be an exact coboundary over either of the two. However, the norms of the elementary coboundaries

\[
\phi^{f'}_{n_k}(\cdot) = \phi_{n_k} \circ f'(\cdot) - \phi_{n_k}(\cdot)
\]

over \( f' \) might grow fast, so that instability of \( f' \) may be caused by the slow growth along a different subsequence \( \{n'_k\} \). In that case, trying to control \( \varphi^{f'}_{n'_k}(\cdot) \) using information on \( \varphi^f_{n_k}(\cdot) \) seems hopeless in general.

In the known DUE \( \setminus CR \) examples of Liouvillean rotations and of DUE cocycles in \( \mathbb{T}^d \times SU(2) \), the rate of growth (actually decay) is constant and equal to \( \sigma = -\infty \), but instability can be caused by the fast decay of the norms of elementary coboundaries along different subsequences.

Given the above, the following question seems interesting and beyond the scope of present technology.

**Question 1.** Does there exist a DUE diffeomorphism \( f \in \text{Diff}^\infty_\mu(M) \) of a compact manifold \( M \) for which there exists \( \sigma \in [-\infty, \infty) \) such that

\[
\text{Cob}^\sigma(f) = C^\infty_\mu(M)
\]

### 3.4 Some comments on the proof

Proposition 3.4 shows that fast approximation of a diffeomorphism by diffeomorphisms that preserve distributions creates an obstruction to certain functions being exact coboundaries. An Anosov-Katok type argument can make sure that the obstructions of \( f_n \) recede to infinity as \( n \to \infty \) and \( f_n \to f \), which would make every function in \( C^\infty_\mu(M) \) an approximate coboundary. On the other hand, theorem 3.1 shows that the limiting procedure leaves a trace, and, for a generic diffeomorphism \( f \) obtained in this way, a generic function will not be an exact coboundary.

Since we are working in the measure preserving category, we also have a canonical way of identifying functions with distributions, via the \( L^2 \) self duality. If, now, one wishes to obtain a weak solution of eq. 1.1 in some space of distributions \( H^{-s} \), then the existence of a non-zero function \( \omega \in H^s_\mu \), invariant under \( f \), poses an obstruction, as the rhs function \( \varphi \) has to satisfy

\[
\langle \omega, \varphi \rangle_{L^2} = 0
\]

for it to be in the range of the coboundary operator \( \psi \mapsto \psi \circ f - \psi \) on \( \dot{H}^{-\sigma} \to \dot{H}^{-\sigma} \). For the same reason, fast approximation of \( f \) by diffeomorphisms \( f_n \),
preserving functions in any $\dot{H}^s$, would imply that functions $\varphi \in C^\infty_\mu$ that are coboundaries in the sense of distributions,

$$Cob^\sigma(f) = \{ \varphi \in C^\infty_\mu, \exists u \in \dot{H}^{-\sigma}, f^* u - u = \varphi \text{ in } H^{-\sigma} \}$$

will still be meagre in $C^\infty_\mu$, for a generic such $f$ and for every $\sigma \in [0, \infty)$.

### 4 Proof of corollary B

In this section we provide justification for our claim in corollary B that the Anosov-Katok construction cannot provide counter-examples to the conjecture. Since the Anosov-Katok construction is not a well-defined object, we produce a statement which imitates the fast convergence scheme of approximation by conjugation. Let $C$ be a space of cohomologically stable diffeomorphisms, whose coboundary space is of infinite codimension in $C^\infty_\mu(M)$. We also assume some control on the norms of the solution of the cohomological equation. We then define the corresponding $\mathcal{AK}^\infty = \mathcal{AK}^\infty(C)$ space as the closure of $T$, the conjugacy class of $C$, and show the incompatibility of fast approximation under the restrictions imposed by lem. 3.5 and cor. 3.6. These restrictions are necessary for $DUE$ to be generic in $\mathcal{AK}^\infty$.

In order to prove cor. B, we impose some fast approximation conditions, and make assumptions that favour $CR(M) \cap \mathcal{AK}^\infty(C) \neq \emptyset$. We produce two example propositions, prop 4.4 and 4.5, establishing that even under such favourable conditions, the diffeomorphisms produced by the construction will not be $CR$. Consequently, even if $CR(M) \cap \mathcal{AK}^\infty(C) \neq \emptyset$, the construction will not be able to establish that.

Consider, for example, the case of the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. If $C$ is the class of periodic rotations, then

$$\left( \bigcup_{\tilde{\gamma} > 0, \tilde{\tau} > 1} DC(\tilde{\gamma}, \tilde{\tau}) \right) \subset \mathcal{AK}^\infty(C)$$

but periodic approximation should be expected to converge to rotations in $C$ and to diffeomorphisms of the circle that have Liouvillean rotation number.

The Anosov-Katok construction consists in inductively constructing a sequence $f_n \in T$, converging "fast" to $f \in \mathcal{AK}^\infty(C) \setminus T$ with some prescribed dynamical properties. These properties are algorithmically verifiable in the context of the construction in the sense that the dynamical system $f$ can be analyzed only through the limit procedure that constructs and defines it. In our context, we wish to construct $CR$ diffeomorphisms using the class of diffeomorphisms preserving a large space of distributions as $C$, and this forces $CR$ to be meagre in $\mathcal{AK}^\infty(C)$. Consistently with the constructive approach described above, we give the following definition.
Definition 4.1. Suppose that a class \(C \subset \text{Diff}_\mu^\infty(M)\) is given, with a certain set of properties \(\Pi\) satisfied by the elements of \(C\), and consider the corresponding Anosov-Katok space \(\text{AK}^\infty(C)\).

We will say that \(f \in \text{AK}^\infty(C)\) satisfies a certain property \(\varpi\) in the constructive sense in the context of the corresponding Anosov-Katok construction (and abridge to “\(f\) has this certain property constructively”), if the Anosov-Katok method constructs a sequence \(f_n \in T, f_n \to f\) such that

\[
\forall n \in \mathbb{N} \exists \Pi(n) \subset \Pi, f_n \in \Pi_n \wedge f_n \overset{\text{C}^\infty}{\to} f \Rightarrow f \in \varpi
\]

More informally, \(f\) satisfies the property \(\varpi\) in the constructive sense iff said property for \(f\) can be established through a limit procedure using only the known properties of the class \(C\) and the corresponding Anosov-Katok construction. An important remark is that we do not demand that \(\varpi \in \Pi\). For example, periodic rotations in the circle are not uniquely ergodic, but irrational rotations are.

This definition is essentially an acknowledgement of the fact that the limit objects \(f\) obtained by the construction are analyzable only as limit objects, and therefore all the properties established within the context of the construction are obtained by analyzing objects in \(T\), thus analyzing objects in \(C\), and passing to the limit. Consequently, the statement the ”\(f\) does not satisfy constructively a certain property” means only that the known properties of the class \(C\) do not allow us to conclude whether \(f\) has the property or not, at least in the context of the construction.

Let us explain the definition through the example properties in which we are interested in the present article. In the context of the Anosov-Katok construction, a diffeomorphism \(f \in \text{AK}^\infty(C)\) is constructively DUE if the construction provides an approximating sequence \(\{f_n\} \subset T\) satisfying the following property.

For every function \(\varphi \in \text{C}^\infty_\mu\), for every \(s \in \mathbb{N}\) and for every \(\varepsilon > 0\), there exists \(n \in \mathbb{N}\) and a function \(\varphi_n \in \text{C}^\infty_\mu\) such that

1. \(\|\varphi - \varphi_n\|_s < \varepsilon\)
2. There exists \(\psi_n \in \text{C}^\infty_\mu\) such that \(\psi_n \circ f_n - \psi_n = \varphi_n\) (i.e. \(\varphi_n \in \text{Cob}^\infty(f_n)\))
3. The solution \(\psi_n\) satisfies the following estimate (cf eq. 2.2)

\[
C_s\|\psi_n\|_{s+1}\|f_n\|_s d_s(f, f_n) < \varepsilon
\]

This set of properties clearly implies that \(f\) is DUE, since

\[
\psi_n \circ f - \psi_n = \varphi + (\varphi_n - \varphi) + (\psi_n \circ f - \psi_n \circ f_n)
\]

and the terms in the parentheses sum up to \(< 2\varepsilon\) in the \(H^s\) norm by assumption. This is the constructive interpretation of the approach adopted in [AFK15]. Since the class \(C\) and the approximant sequence \(f_n\) are constructed in some sense, the solution \(\psi_n\) should also be expected to be obtainable in a constructive way. More precisely, we should expect the equation

\[
\tilde{\psi} \circ \tilde{f}_n - \tilde{\psi} = \tilde{\varphi}
\]
to be solvable with a good control for a sufficiently large space of functions $\tilde{\varphi}$, and that this space should grow with $n$ and cover $C^\infty_\mu(M)$ in the limit $n \to \infty$. Then, the conjugation $H_n$ (through which $f_n = H_n \circ \tilde{f}_n \circ H_n^{-1}$ is defined), acting on the space of such functions $\tilde{\varphi}$ and on the solutions $\tilde{\psi}$ by pullback should allow the three properties listed here above to be established.

For this reason, we introduce the first standing assumption on the construction.

**Assumption 4.1.** We suppose that the class of mappings $\mathcal{C}$ is Cohomologically Stable, and that mappings in $\mathcal{C}$ preserve infinitely many linearly independent distributions of bounded regularity $\sigma$:

$$\exists \sigma \geq 0 \text{ such that } h \in \mathcal{C} \Rightarrow \dim H^{-\sigma}(h) = \infty$$

Clearly, the same holds for $\mathcal{T}$.

Accordingly, we will say that $f \in \mathcal{AK}^\infty(\mathcal{C})$ is constructively $CR$ if we can algorithmically solve the cohomological equation over $f$, by successively solving cohomological equations over the $f_n$ and summing the solutions up in the following sense. We suppose that to every $h \in \mathcal{T}$ we can associate operators

$$P = P_h : C^\infty_\mu \to C^\infty_\mu$$
$$R = R_h = \text{Id} - P_h : C^\infty_\mu \to C^\infty_\mu$$

who satisfy the following property: if $V_h = P_h(C^\infty_\mu) \cap \text{Cob}^\infty(h)$, then $V_h \to C^\infty_\mu$. We also suppose that we can construct $\text{Ob}_h : P_h(C^\infty_\mu) \to V_h$, $\text{Ob}_h \not= 0$, satisfying

$$\left\{ \begin{array}{l}
(\text{Id} - \text{Ob}_h)(P_h(C^\infty_\mu)) \subset V_n \\
\text{Ob}_h(P_h(C^\infty_\mu)) \cap \text{Cob}^\infty(h) = \{0\}
\end{array} \right.$$ 

It is an operator taking a function $\varphi \in P_n(C^\infty_\mu)$, the space where we suppose that we can solve the cohomological equation over $h$ with good estimates, and keeps the part of $\varphi$ which is not a coboundary.

We shall also make the assumption that both $P$ and $\text{Ob}$ are continuous operators $H^s \to H^s$, for all $s \geq 0$, and that they are uniformly bounded:

**Assumption 4.2.** There exist constants $K_s$, $s \geq 0$, such that for each $s \geq 0$ and for every $h \in \mathcal{T}$

$$\|P_h\|_{s,s} \leq K_s$$
$$\|\text{Ob}_h\|_{s,s} \leq K_s$$

The constants $K_s$ depend only on the specifics of each Anosov-Katok construction. Realistically, such a property can be established for mappings in $\mathcal{C}$ and not in $\mathcal{T}$. The corresponding bounds for mappings in $\mathcal{T}$ would then deteriorate by the norms of the conjugations, see eq. 2.1. In general, a loss of derivatives could also be expected.

Let, now, $\varphi \in C^\infty_\mu$ and apply the following algorithm.
1. Let \( \varphi_1 = \varphi \in C_{\mu}^\infty \).

2. Suppose that \( n-1 \) steps of the algorithm have been executed, thus defining \( \varphi_n \).

3. Calling \( P_{f_n} = P_n \), same for \( \text{Ob}_{f_n} = \text{Ob}_n \), solve the equation

\[
\psi_n \circ f_n - \psi_n = P_n \varphi_n - \text{Ob}_n \varphi_n
\]

4. Define

\[
\varphi_{n+1} = \varphi_n - (\psi_n \circ f - \psi_n) = R_n \varphi_n + \text{Ob}_n \varphi_n + (\psi_n \circ f - \psi_n \circ f)
\]

and iterate the algorithm.

The diffeomorphism \( f \) will be constructively CR iff

1. it is DUE, for which amounts to \( \varphi_n \to 0 \) in \( C_{\mu}^\infty \), for every \( \varphi \in C_{\mu}^\infty \). For this to hold for every \( \varphi \in C_{\mu}^\infty \), and since we have no control over the linearization error term \( \psi_n \circ f_n - \psi_n \circ f \) except for its size, we need to be able to conclude that

\[
\begin{align*}
\text{For every } \varphi_n \in \mathcal{V}_n, \\
\psi_n \circ f_n - \psi_n \circ f &\to 0 \text{ in } C^\infty \\
\text{Ob}_n \varphi &\to 0 \text{ in } C^\infty
\end{align*}
\]

2. The solutions form a convergent series:

\[
\sum_{k=1}^{n} \psi_k \to \psi \in C_{\mu}^\infty
\]

If both conditions are verified, we clearly have

\[
\psi \circ f - \psi = \varphi
\]

The construction of the \( f_n, P_n \) and \( \text{Ob}_n \), the topology in which \( P_n \to \text{Id} \) and the meaning of \( \mathcal{V}_n \to C_{\mu}^\infty \) are to be determined in each specific application. This was the strategy followed by the author in [Kar14].

Since the error term \( \psi_n \circ f_n - \psi_n \circ f \) is unavoidable, and since it cannot be analyzed further than estimating its size (we are admitting that \( f \) is analyzable only as a limit object), we are forced make the following assumption. When \( \varphi_n \) runs through \( \mathcal{V}_n \), the corresponding error terms run through \( C_{\mu}^\infty (M) \): for every \( n \in \mathbb{N} \),

\[
\{\psi_n \circ f_n - \psi_n \circ f, \varphi_n \in \mathcal{V}_n\}^3 = C_{\mu}^\infty (M)
\]

Thus, in general, \( P_n(\varphi_{n+1}) \neq 0 \), and therefore, at the \( n+1 \)-th step we will have to solve again for functions in \( \mathcal{V}_n \). This will be so for a function of smaller norm (provided that our construction works well) and (inevitably) with worse estimates, since at the \( n+1 \)-th step we are tuning the dynamics in a finer scale. As a consequence, we need the following assumption.
Assumption 4.3. The vector spaces $V_n$ form a filtration of $C^\infty_\mu(M)$:

$$
\forall n \in \mathbb{N}, V_{n-1} \subsetneq V_n \text{ and } \bigcup_k V_k = C^\infty_\mu(M)
$$

Moreover, we assume that the construction proceeds step-by-step, tuning the dynamics in ever finer scales. At each step, we try to clear the greatest possible part of the obstructions coming from the previous steps, while unavoidably imposing new ones. Since the construction is inductive, the following assumption is natural.

Assumption 4.4. The obstructions at each step cannot be solved for at a previous step: for $m < n$

$$
P_m(\text{Ob}_n(V_n)) = \{0\}
$$

To our best knowledge, in the implementations of the Anosov-Katok method, the following situation appears to occur. With the notation used in the introduction (cf. par. 1.2), at the $n$-th step, the diffeomorphism $\tilde{f}_n$ is constructed, so that for some $N_n$ the perturbation with respect to the previous step, $\tilde{f}_n \circ \tilde{f}^{-1}_{n-1}$ acts in a prescribed way in a scale $\sim \lambda_{N_n}^{-1}$. The construction comes with the existence of obstructions to the solution of the cohomological equation for functions spectrally supported in the first $\sim N_n$ modes, i.e. in the image of the operator $T_{cN_n}$ for some constant $c \sim 1$.

The dimension of these obstructions is small with respect to $N_n$, and the estimates for the solution of the cohomological equation in the complementary space are good, but naturally depend on $\lambda_{N_n}$. Then, the conjugation $\eta_n$ is constructed in a way that keeps the scales $\sim \lambda_{N_{n-1}}^{-1}$ and $\sim \lambda_{N_n}^{-1}$, $N_{n-1} \ll N_n$, independent (cf. eq. 1.2), and the construction is iterated.

The space $P_n(C^\infty_\mu(M))$ is, informally, a space of functions that oscillate considerably slower than $\lambda_{N_n}^{-1}$ and are therefore analyzable in the scale of the $n$-th step of the construction. The space $V_n$ is the space of coboundaries that do not oscillate too fast, and solution of the cohomological equation for a function in $V_n$ over $f_n$ can be expected to give good information on the solution for the same function over $f$.

In all known implementations, the assumption of the existence of constants $K, \sigma$ and $s_0$ and of sequences $N_n$ and $u_n \in H^{-\sigma}(f_n)$ such that

$$
\|T_{N_n}u_n\|_{H^{-\sigma}} \geq K\lambda_{N_n}^{-s_0}
$$

as in prop. 3.4 is actually an understatement of the strength of the obstructions, since the parameters can assume their limit values $K = 1, \sigma = s_0 = 0$. In what follows, we consider a fixed set of such parameters $K, \sigma, s_0$ and $N_n$.

The notion of fast convergence of $f_n$ to $f$ is specified in this context in the following way, where we introduce the notation $\eta_{s,n} = d_s(f_{n-1}, f_n)$ for the size of the perturbation at the $n$-th step of the construction.

\[^5\text{In } [\text{Kar14}] \text{ the constant } c \text{ can actually be replaced by a fixed power of } N_n.\]
Definition 4.2. We will say that \( f_n \to f \) fast if for every \( s, \eta_{s,n} = O(\eta_{s,n-1}^\infty), \) i.e.
\[
\eta_{s,n-1}^{-l} \eta_{s,n} \to 0, \forall l \in \mathbb{N}
\]

Assumption 4.5. We assume that the Anosov-Katok construction that we are studying satisfies this fast convergence assumption.

Prop. 3.4 implies that, for \( f \) to be CR, approximation should not be fast with respect to \( \lambda_{N_n} \). We thus introduce the following assumption, which implies that our estimates work with rates of convergence that do not satisfy the hypotheses of the proposition, thus not precluding that the limit object \( f \) be CR.

Assumption 4.6. We suppose that, for the construction to converge, the sequence \( f_n \) has to satisfy the following. For any choice of the sequence \( \{N_n\} \) as in prop. 3.4, there exists \( \tau > 0 \) such that for every \( s \in \mathbb{N} \) as in def. 4.2, there exists \( \gamma = \gamma_s > 0 \) such that
\[
\eta_{s,n} \leq \gamma \lambda_{N_{n-1}}^{-\tau}
\]

This polynomial decay of \( \eta_{s,n} \), coupled with the assumed fast convergence of \( f_k \to f \), implies directly the fast growth of the \( N_n \).

Lemma 4.1. If \( f_n \to f \) fast, then, under assumption 4.6, we have fast growth of the sequence \( N_n \):
\[
\lambda_{N_{n-1}}^{-l} \lambda_{N_n} \to \infty, \forall l \in \mathbb{N}
\]

The following corollary is immediate. The conclusion is weaker than the one implying Cohomological Instability by prop. 3.4. We remind the notation \( \delta_{s,n} = d_s(f_n, f) \).

Corollary 4.2. Under the conditions of lem. 4.1,
\[
\delta_{s,n} = O(\lambda_{N_n}^{-\tau}), \eta_{s,n} = O(\lambda_{N_{n-2}}^{-\infty}) \text{ and } \delta_{s,n} = O(\lambda_{N_{n-1}}^{-\infty})
\]

We can also show there exists some competition between \( f \in DUE \), the estimates for the solution \( \psi_n \) and the rate of decay of \( \eta_{s,n} \) as in assumption 4.6.

The following lemma implies that the slower \( \eta_{s,n} \) decays with respect to \( \lambda_{N_n} \), the better the control has to be over the solution of the cohomological equation at the \( n \)-th step of the algorithm.

Lemma 4.3. Under assumption 4.6, if \( f \) is constructively DUE, then
\[
\|\psi_n\|_{\sigma+1} = o(\lambda_{N_n}^\infty)
\]

Proof. For \( f \) to be DUE, the error term
\[
\psi_n \circ f - \psi_n \circ f_n
\]
has to tend to 0 in the \( C^\infty \) topology. The supposed slow decay of \( \delta_{s,n} \) implies, together with eq. 4.1, the conclusion of the lemma. \( \square \)
For \( f \) to be \( DUE \), we need additionally that for every \( \varphi \in C^\infty_\mu \) and every \( n \) there has to exist \( m > n \) such that

\[
\tilde{\text{Ob}}_{m,n-1}\varphi_{n-1} = \text{Ob}_m (\text{Ob}_{m-1} \cdots (\text{Ob}_{n-1}\varphi_{n-1})) \approx 0
\]

By lem. 3.2, and the fast convergence of \( f_k \rightarrow f \), we get a lower bound for the solution to the obstructions. This bound is a function of the size of the obstructions carried over from the preceding step \( m - 1 \) and of \( \eta_{\sigma,m-1} \):

\[
\|\psi_m\|_{\sigma+1} \geq C_\sigma \|f_m\|_\sigma \|\tilde{\text{Ob}}_{m,n-1}\varphi_{n-1}\|_{\sigma,\mu}^{-1}_{\eta_{\sigma,m-1}}
\]  

(4.3)

where for \( \omega \in C^\infty_\mu(M) \) and \( f \in \text{Diff}_\mu(M) \) we call

\[
|\omega|_s = \sup_{u \in H^{-s}(f)} |\langle u, \omega \rangle|
\]  

(4.4)

For the limit object \( f \) to be constructively \( CR \), this lower bound needs to be summable, thus improving significantly the conclusion of lem 4.3.

We consider two cases, the distinguishing factor being the number of steps \( m - (n - 1) > 0 \) that the construction needs for the obstructions to become coboundaries. The break is between \( m - (n - 1) = 1 \), which we call \textit{rotation vector case} and treat first, and \( m - (n - 1) > 1 \) (the exact value being irrelevant) which we call \textit{compact group case}.

The \textit{rotation vector case} Suppose that the most favourable scenario occurs, where the obstructions related to the approximant at the \( n - 1 \)-th step can be solved for at the \( n \)-th step. It is indeed most favourable, since the denominator \( \eta_{\sigma,k} \) appearing in eq. 4.3 only worsens the estimates as \( k \) increases (see the \textit{compact group case} for more details). It occurs, for example, when the \( \tilde{f}_k \) are rigid periodic translations in tori and the \( f_k \) are conjugate to them. The relevant assumption for this case is the following one.

**Assumption 4.7.** In the rotation vector case, we assume that all the obstructions arising at the \( n - 1 \)-th step are coboundaries in the \( n \)-th step:

\[
P_n\text{Ob}_{n-1}\varphi_{n-1} = \text{Ob}_{n-1}\varphi_{n-1}
\]

In this case, assumptions concerning only rates of convergence are not sufficient for deciding in a general way whether \( f \in CR \) or not, since the lower bound of eq. 4.3 does not produce a diverging series.

The problem here lies in the nature of the construction. The diffeomorphism \( \tilde{f}_n \circ \tilde{f}_{n-1} \), of the size \( \eta_{s,n} \), appearing in the calculation of the estimate in eq. 4.3 (see the proof of lem. 3.2) is constructed in the scale \( \sim \lambda_{N_n}^{-1} \) and the conjugation producing \( f_n \) leaves the constructions in the scales \( \sim \lambda_{N_n}^{-1} \) and \( \sim \lambda_{N_n}^{-1} \) independent. Consequently, the information on the solution of the equation

\[
\psi_n \circ f_n - \psi_n = P_n\text{Ob}_{n-1}\varphi_{n-1} \approx \text{Ob}_{n-1}\varphi_{n-1}
\]
and, in particular, the norm of the solution, will by construction be related to $\lambda_{N_n}$. By lem. 4.1, we have

$$\lambda^{-l}_{N_{n-1}} \lambda_{N_n} \to \infty, \forall l \in \mathbb{N}$$

Consequently, even assuming that $\text{Ob}_{n-1} \varphi_{n-1} = O(\lambda_{N_{n-1}}^{-\infty})$, the estimates will not be sufficient for establishing the summability of $\psi_n$, unless the norm of the mapping $\varphi_k \mapsto \psi_k$ grows very slowly with $\lambda_{N_n}$. If, as in the known implementations, the norm of the inverse of the coboundary operator grows like some power of $\lambda_{N_n}$ and such estimates are essentially optimal, then, using similar arguments as in the proof of thm 3.1, one can actually show that the limit object $f$ is actually not $CR$. This is summed up in the following example proposition.

**Proposition 4.4** (Example proposition for the rotation vector case). Let an Anosov-Katok construction satisfy all standing assumptions.

Suppose, in particular, that for any sequence $f_n \in \mathcal{T}$ obtained via the Anosov-Katok construction and converging fast to $f \in \mathcal{AK}^{\infty}(C) \setminus \mathcal{T}$, and for every choice of the parameters $\sigma$, so, $\{N_n\}$, $\gamma$ and $\tau$, assumption 4.6 holds. We also suppose that the construction satisfies the assumption 4.7, i.e. that for all $n \in \mathbb{N}$,

$$\mathcal{P}_n \mid_{\text{Ob}_{n-1}(\mathcal{V}_{n-1})} = \text{Id}$$

Using the notation established, suppose additionally that

1. For every $n \in \mathbb{N}$, the solution $\psi$ of the cohomological equation over $f_n$ for a function $\varphi \in \mathcal{V}_n$ is estimated by

$$\|\psi\|_s \leq C_s \lambda_{N_n}^{\tau'} \|\varphi\|_s$$

for some $0 < \tau' < \tau$.

2. The norm of functions outside $\mathcal{V}_n$ decays fast: for every $\varphi \in C^{\infty}_{\mu}$,

$$\|(R_n \circ R_{n-1} \circ \cdots \circ R_1)\varphi\|_s = O(\lambda_{N_n}^{-\infty})$$

Then, the limit objects $f$ are not constructively $CR$.

If, moreover, there exists $0 < \tau'' \leq \tau'$ such that for every $n$,

$$\omega_n \in \mathcal{P}_n \circ \text{Ob}_{n-1}(\mathcal{V}_{n-1})$$

implies any function $\chi_n$ such that $\chi_n \circ f_n - \chi_n = \omega_n$ satisfies

$$\|\chi_n\|_{\sigma+1} \geq C \lambda_{N_n}^{\tau''} \|\omega_n\|_{\sigma}, \forall \omega_n \in \mathcal{P}_n \circ \text{Ob}_{n-1}(\mathcal{V}_{n-1})$$

(4.5)

then the limit objects obtained by the construction are not $CR$.

Informally, we are supposing that we are able to treat all obstructions arising in the scale $\lambda_{N_{n-1}}^{-1}$, by tuning the dynamics at the next scale $\lambda_{N_n}^{-1}$. If solution of the cohomological equation costs a positive power of $\lambda_{N_n}$, then $f$ is not constructively $CR$. If the cost of treating the obstructions of the previous step costs indeed a power of $\lambda_{N_n}$, then the limit diffeomorphism cannot be $CR$. 

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Proof. Suppose that $f_k \to f$, and that the construction satisfies the hypotheses of the proposition. Fix $\varphi \in C^\infty_0(M)$ and apply the algorithm for the solution of the equation  

$$\psi \circ f - \psi = \varphi$$

Then, at the 1-st step, the unknown function $\varphi_2$ satisfies  

$$\varphi_2(\cdot) = \text{Ob}_1 \varphi_1(\cdot) + (\psi_1 \circ f_1 - \psi_1 \circ f) + O(C^\infty(\lambda_{N_1}^{-\infty}))$$

where the $O(\lambda_{N_1}^{-\infty})$ is due to the estimation on the rest $R_1 \varphi_1$. Estimation of the error term in the parentheses gives  

$$\|\psi_1 \circ f_1 - \psi_1 \circ f\|_s \leq C\lambda_{N_1}^\gamma \|P_1 \varphi_1\|_{s+1} \delta_s, 1 \leq C\gamma \lambda_{N_1}^\gamma \|P_1 \varphi_1\|_{s+1}$$

If this term is to converge to 0 we need $\tau' < \tau$, which is a hypothesis.

By hypothesis,  

$$\text{Ob}_1 \varphi_1(\cdot) \in V_2 \subset C^\infty_0(f_2)$$

and any function $\psi_2^{\text{Ob}}$ solving the cohomological equation with $\text{Ob}_1 \varphi_1(\cdot)$ as rhs satisfies  

$$\|\psi_2^{\text{Ob}}\|_{\sigma+1} \leq C\lambda_{N_2}^\gamma \|\text{Ob}_1 \varphi_1\|_{\sigma} \leq C\lambda_{N_2}^\gamma \|P_1 \varphi_1\|_{\sigma}$$

The solution for the error term will satisfy the estimate  

$$\|\psi_2^{\text{error}}\|_{\sigma+1} \leq C^2 \gamma \lambda_{N_2}^\gamma \lambda_{N_1}^{\gamma - \tau} \|P_1 \varphi_1\|_{\sigma} \leq K_\sigma C^2 \gamma \lambda_{N_2}^\gamma \lambda_{N_1}^{\gamma - \tau} \|\varphi_1\|_{\sigma}$$

Iteration of the procedure results in the norms of the solutions $\psi_n$ being estimated by quantities that blow up, so that $CR$ cannot be concluded, since terms of the type  

$$O(\lambda_{N_n}^{\gamma} \lambda_{N_{n-1}}^{\gamma - \tau}) \to \infty$$

will appear in the rhs of the estimate for the solution $\psi_n$. Such terms diverge independently of the sign of $\tau' - \tau$, as log as $\tau' > 0$.

Let us now prove that $f \notin CR(M)$ as soon as $\tau''$ as in eq. 4.5 exists. Suppose that $f \in CR(M)$, let $u_{n-1} \in H^{-\sigma}(f_{n-1})$, and define $\omega_{n-1} \in C^\infty_0$ as in the proof of prop. 3.4. Let us assume, for the moment, that  

$$\omega_{n-1}(\cdot) \in V_n \quad (4.6)$$

Then, the function $\chi_n$ solving the cohomological equation for $\omega_{n-1}$ over $f_n$ satisfies  

$$\|\chi_n\|_{s+1} \geq C\lambda_{N_n}^{\gamma''} \|\omega_{n-1}\|_s$$

Incidentally, this shows that assuming only polynomial decay of $u_{n,s}$ makes $DUE$ a difficult property to establish. Compare with lem. 4.3. Inspection of the proof shows that $\tau''$ may be allowed to depend on $s$, as long as $0 < \tau''(s) < \tau'$. We can even have $\tau''(s) \to 0$ as $s \to \infty$.  

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On the other hand the functions $\chi_n$ satisfy

$$\chi_n \circ f - \chi_n = \omega_{n-1} + (\chi_n \circ f - \chi_n \circ f_n)$$

The error term in the parentheses is

$$\|\chi_n \circ f - \chi_n \circ f_n\|_s = O(\lambda^{\tau'_n - \tau} N_n)$$

in the $C^\infty$ topology.

We remark that, since the functions $\omega_n$ have finite spectrum, supposing that

$$\mathcal{O}(\lambda^{\tau'_n - \tau} N_n) \leq \|\omega_n\|_s \leq 1$$

as we have, for any given $s \geq 0$ we can have $\nu_n \omega_n \to 0$ in $H^s$ as soon as

$$\nu_n = \nu_n(s) = \lambda^{-\xi}_n$$

with $\xi$ big enough (depending on $s$).

Since for any such fixed $s$,

$$\int \nu_{n-1}(s)\omega_{n-1} + \nu_{n-1}(s)(\chi_n \circ f - \chi_n \circ f_n) \overset{H^s}{\Rightarrow} 0$$

we have a contradiction with Cohomological Stability for $f$.

If eq. 4.6 is not true, then call $r_{\sigma,n} = \|R_n \omega_{n-1}\|_s > 0$. If $R_n \omega_{n-1} \in \mathcal{V}_{n+1}$, then the solution $\chi_{n+1}$ is estimated by below by

$$\|\chi_{n+1}\|_s \gtrsim \lambda^{\tau''}_{N_{n+1}} r_{\sigma,n}$$

Therefore, if $r_{\sigma,n}$ is not of the order of $\lambda^{\tau''}_{N_{n+1}}$, the estimates become worse. If $r_{\sigma,n}$ is indeed of the order of $\lambda^{\tau''}_{N_{n+1}}$, the estimates assuming eq. 4.6 work just as well, since $\lambda_N \ll \lambda_{N_{n+1}}$. This concludes the proof.

The compact group case  If the algorithm fails to solve for all obstructions arising at the step $n-1$ in the $n$-th step, but needs to wait for arbitrarily many steps of the algorithm, the situation is clearer, and the corollary can be proved without any assumptions on the norm of the solution, since the orders of magnitude allow us to conclude. If $\tilde{H}^{-\sigma}(f_{n-1}) \cap \tilde{H}^{-\sigma}(f_n) \neq \emptyset$, then an obstruction is carried on to the $n+1$-th step, and the solution is bounded below by a factor with $\eta_{\sigma,n}$ in the denominator instead of $\eta_{\sigma,n-1}$. The relevant assumption is now the following, which allows us to show that carrying an obstruction even for one step precludes Cohomological Rigidity.

Assumption 4.8. In the compact group case, we assume that, for every $n$, there exists an obstruction arising at the $n-1$-th step, that is an obstruction in the $n$-th step. More explicitly, assume that

$$\tilde{H}^{-\sigma}(f_{n-1}) \cap \tilde{H}^{-\sigma}(f_n) \neq \emptyset$$

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In such a case, when solving the equation
\[
\psi_{n+1} \circ f_{n+1} - \psi_{n+1} = \mathcal{P}_{n+1} \tilde{\text{Ob}}_{n,n-1} \varphi_{n-1} - \text{Ob}_{n-1} \varphi_{n-1}
\]
where the rhs is precisely the obstructions carried over from step \( n-1 \), one should expect that
\[
\| \text{Ob}_{n-1} \varphi_{n-1} \|_\sigma \approx \| \text{Ob}_{n} \text{Ob}_{n-1} \varphi_{n-1} \|_\sigma \approx \| \mathcal{P}_{n+1} \text{Ob}_{n} \text{Ob}_{n-1} \varphi_{n-1} \|_\sigma \to 0
\]
Then, the rhs of the bound in eq. 4.3, together with assumption 4.6, becomes
\[
||| f_n |||_\sigma |\tilde{\text{Ob}}_{n,n-1} \varphi_{n-1} |_{\sigma}^{-1} > \gamma^{-1} ||| f_n |||_\sigma |\text{Ob}_{n,n-1} \varphi_{n-1} |_{\sigma}^{\lambda_{N_n}}
\]
and the rhs goes to infinity as soon as
\[
|\tilde{\text{Ob}}_{n,n-1} \varphi_{n-1} |_{\sigma}^{f_n} \geq C \lambda_{N_n}^{-\tau'}
\]
for some \( \tau' < \tau \). This is a generous condition since \( \text{Ob}_{n-1} \) is related with the scale \( \lambda_{N_{n-1}}^{-1} \).

This sums up to the following example proposition.

**Proposition 4.5** (Example proposition for the compact group case). Let an Anosov-Katok construction satisfy all standing assumptions save for assumption 4.7.

Suppose, in particular, that for any sequence \( f_n \in \mathcal{T} \) obtained via the Anosov-Katok construction and converging fast to \( f \in \mathcal{T}^{\mathcal{I}_\infty} \setminus \mathcal{T} \), and for every choice of the parameters \( \sigma, s_0, \{N_n\}, \gamma \) and \( \tau \), assumption 4.6 holds. We also suppose that the construction satisfies the assumption 4.8, i.e. that for all \( n \in \mathbb{N} \),
\[
\hat{H}^{-\sigma}(f_{n-1}) \cap \hat{H}^{-\sigma}(f_n) \neq \emptyset
\]
so that it corresponds to the compact group case.

Then, the limit objects \( f \) obtained by the construction are not CR.

The proof goes exactly as that of prop. 3.4.

**Proof.** Suppose that \( f \in CR(M) \), let \( u_{n-1} \in \hat{H}^{-\sigma}(f_{n-1}) \cap \hat{H}^{-\sigma}(f_n) \), and define \( \omega_{n-1} \in C^\infty_\mu \) as in the proof of prop. 3.4. The function \( \chi_{n-1} \) solving the cohomological equation for \( \omega_{n-1} \) over \( f \) satisfies
\[
\| \chi_{n-1} \|_{\sigma} \| | f_n \| |_{\sigma} | \omega_{n-1} |_{\sigma}^{\delta_{\sigma,n}} \geq C^{-1} \gamma \| | f_n \| |_{\sigma} \lambda_{N_{n-1}}^{-\delta_{\sigma,n}} \lambda_{N_n}^\gamma
\]
The conclusion is as in the end of the proof of prop. 4.4.

This concludes the proof of cor. B.
A remark on the proof The proof of cor. B can be summarized as follows. An Anosov-Katok-like construction proceeds either by pushing all obstructions further to infinity at each step ("rotation vector case"), or by pushing only a part of them ("compact group case"). In the second case, the orders of magnitude are sufficient in order to conclude that the $f \in \mathcal{AK}^\infty(\mathcal{C}) \setminus \mathcal{T}$ for which the construction converges are not CR. In the first case, we need an assumption on the solution of the cohomological equation for diffeomorphisms in $\mathcal{T}$ in order to conclude, but the assumptions are natural and cover all known implementations.

We also make the following remark.

Remark 4.1. The assumptions at the $n$-th step that are stated as equalities have a tolerance of $O(\lambda^{-\infty}_n)$, where $m$ is the correct scale, i.e. $m = n$ or $m = n \pm 1$.

Even though we refrained from stating such sharp conditions in order to keep the argument transparent, we invite the reader to verify for themselves this fact.

For example, even the assumption that the $V_n$ form a filtration can be weakened to the following, a lot messier and more technical: For every $n$, there exists a space $V_{n-1}' \subset V_n$, isomorphic to $V_{n-1}$ and such that the isomorphism is close to the Id (meaning that the spaces $V_{n-1}$ and $V_{n-1}'$ form a small angle. The condition in assumption 4.8 can accordingly be relaxed to the existence of $u_{n-1} \in H^{-\sigma}(f_{n-1})$ and $u_n \in H^{-\sigma}(f_n)$ forming a small enough angle in $H^{-\sigma}(M)$.

Such considerations are in line with the fact that usually the properties satisfied by diffeomorphisms constructed by the Anosov-Katok construction are $G_\delta$ (e.g. DUE in the case of the present article).

We also make the almost obvious but important remark that the fast approximation assumption, ass. 4.5, need only be verified by a subsequence of the approximants for cor. B to be true (consider, e.g. Liouvillean rotations and their best rational approximations).

5 Relation with Almost Reducibility

5.1 Introduction

The two cases treated in propositions 4.4 and 4.5 cover the Almost Reducibility regime (see [Eli01], [Kri99],[Kar16b]) whenever it is obtained via a K.A.M. constructive procedure. As a consequence, in spaces of Dynamical Systems where Almost Reducibility can be established in a non highly exotic way, the Herman-Katok conjecture should be expected to be true.

Let us quickly explain the concept of Almost Reducibility, using the notation of §1.1: we consider the space of skew-product diffeomorphisms $SW^\infty_\alpha(T^d, P)$ over a fixed rotation $\alpha \in T^d$.

A cocycle is called constant iff $A(\cdot) \equiv A \in G$ is a constant mapping, and reducible iff it is smoothly conjugate to a constant one. Obviously, reducible
coclones cannot be CR unless \( P \approx \mathbb{T}^{d'} \) for some \( d' \in \mathbb{N}^* \). A celebrated theorem by H. Eliasson, [Eli01], establishes that even in the favourable case where \( \alpha \in DC \) and \( A(\cdot) \) is a perturbation of a constant mapping in \( SO(3) \), the cocycle \((\alpha, A(\cdot))\) might not be (and in fact will generically not be) reducible. The author showed in [Kar14] that generically it will actually be DUE \( \setminus \) CR and never CR. Such a cocycle is nonetheless almost reducible, i.e. there exists \( G_n(\cdot), A_n(\cdot) \) a sequence of mappings \( \mathbb{T}^{d'} \rightarrow G \) and \( A_n \) a sequence of constants in \( G \) such that

\[
A_n^{-1}(G_n(\cdot + \alpha).A(\cdot).G_n^{-1}(\cdot)) \rightarrow \text{Id} \quad \text{in} \quad C^\infty(\mathbb{T}^{d'}, G)
\]

The sequence \( B_n(\cdot) \) generically diverges, and generically it diverges precisely because the cocycle \((\alpha, A(\cdot))\) is not reducible (cf. [Kar17], [Kar14]).

The reason for the divergence of the sequence \( B_n(\cdot) \) is the phenomenon of resonances. Before defining the notion of resonances, let us briefly recall some facts from the theory of compact Lie groups (see [Die75] or [Kar16b]).

### 5.2 Facts from the theory of compact Lie groups

For each \( A \in G \) there exists at least one torus \( T \approx \mathbb{T}^{d'} \), \( T \hookrightarrow G \), of maximal dimension \( d' \) (depending only on \( G \)) such that \( A \in T \). Obviously, \( \{A^k\}_{k \in \mathbb{Z}} \subseteq T \) for every such torus \( T \). Given such a torus \( T \), called a maximal torus, we can decompose the adjoint action of \( G \) on \( g = T_{Id}G \), the Lie algebra of \( G \),

\[
\text{Ad}_A : g \ni s \mapsto \frac{d}{dt}A.e^{ts}.A^{-1} \big|_{t=0} \in g
\]

for every \( A \in G \). By \( \exp \) we denote the exponential mapping \( g \rightarrow G \) with respect to the natural metric on \( G \) given by the Cartan-Killing form

\[
\langle a, b \rangle = -\text{tr} (s \mapsto [a, [b, s]])
\]

and \([\cdot, \cdot]\) is the Lie bracket \( g \times g \rightarrow g \). The commutator \([a, b]\) is given by the derivative of the mapping \( \mathbb{R} \rightarrow G, t \mapsto e^{a}.e^{tb}.e^{-a} \) at \( t = 0 \).

The decomposition of the adjoint action into eigenspaces, known as root-space decomposition, reads as follows.

Firstly, let us denote by \( t \subset g \) the Lie algebra of \( T \), i.e. \( t = T_{Id}T \). It is a maximal abelian algebra, i.e. a maximal subspace of \( g \) where the Lie bracket vanishes identically. There exist pairwise orthogonal spaces \( \mathbb{C}j_\rho \approx \mathbb{C} \approx \mathbb{R}^2 \), \( \mathbb{C}j_\rho \hookrightarrow g \), which are orthogonal to \( t \) and indexed by a finite set \( \Delta \subset t^* \setminus \{0\} \) (called the roots of \( g \) with respect to \( t \)), satisfying the following properties:

- for every \( \rho \in \Delta \), the vectors \( j_\rho \) and \( i.j_\rho \) ( \( i \in \mathbb{C} \) is the imaginary unit) are orthogonal.
- for every \( \rho \in \Delta \), there exists a vector \( h_\rho \in t \setminus \{0\} \), orthogonal to \( \mathbb{C}j_\rho \) and such that

\[
[h_\rho, j_\rho] = i.j_\rho
\]

\[\text{The cyclicity of the definitions is only apparent.}\]
plus cyclic permutations. This can be summarized by saying that the vector space generated by \( \{ h_\rho, j_\rho, i_\rho \} \) defines an embedding of \( su(2) \), the Lie algebra of \( SU(2) \) which is isomorphic to \( \mathbb{R}^3 \) equipped with its scalar and vector product, into \( g \). This embedding is denoted by \( (su(2))_\rho \).

- For \( a \in \mathfrak{t} \) and \( z \in \mathbb{C} \),
  \[ \text{Ad}_e^a(zj_\rho) = e^{2i\pi \rho(a)\cdot z}j_\rho \]

- There exists a subset \( \Delta_+ \subset \Delta = \Delta_+ \cup (-\Delta_+) \) of roots such that every root \( \rho \in \Delta \) can be written in the form
  \[ \rho = \sum_{\rho' \in \Delta_+} m(\rho, \rho') \rho' \]

  with \( m(\rho, \rho') \) integers of the same sign. The distinction with respect to the sign is essentially the same as that between elements below and above the diagonal of a unitary matrix.

Obviously, the eigenvalues of \( \text{Ad}_e^a \) are \( \{ e^{2i\pi \rho(a)} \}_{\rho \in \Delta \cup \{0\}} \). We remark that \( \rho : \mathfrak{t} \to \mathbb{R} \) take values in the real line, so that all eigenvalues are in the unit circle. Root-space decompositions with respect to different maximal tori are equivalent, since they are obtained by the adjoint action of the group onto itself.

In the context of the study of quasi-periodic cocycles, the following definition is very important.

**Definition 5.1.** Given \( \alpha \in \mathbb{T}^d \), an element \( A \) of \( G \) will be called resonant with respect to \( \alpha \) iff there exists \( \rho \in \Delta \) and \( k_\rho \in \mathbb{Z}^d \setminus \{0\} \) such that

\[ \rho(a) - k_\rho \cdot \alpha \in \mathbb{Z} \]

where \( a \in \mathfrak{t} \) is any preimage of \( A \) under the exponential mapping.

A constant cocycle \( (\alpha, A) \in SW_\alpha^\infty(\mathbb{T}^d, G) \) is resonant iff \( A \in G \) is resonant with respect to \( \alpha \), and the integer vector \( k_\rho \) (which is unique if \( \alpha \) is irrational) is called resonance.

### 5.3 The almost reducibility theorem

Given the above, we can state the almost reducibility theorem referred to in §5.1 as follows.

**Theorem 5.1** ([Eli01], [Kri99], [Kar16b]). Let \( G \) be a compact Lie group, and \( \alpha \in DC(\tilde{\gamma}, \tilde{\tau}) \). Then, there exist \( s_0 > 0 \) and \( \varepsilon > 0 \) such that if the cocycle \( (\alpha, A e^{F(\cdot)}) \), \( A \in G \) and \( F(\cdot) \in C^\infty(\mathbb{T}^d, g) \) satisfies

\[ ||F(\cdot)||_0 < \varepsilon \text{ and } ||F(\cdot)||_{s_0} < 1 \]

then it is Almost Reducible.

More precisely, the K.A.M. scheme that proves Almost Reducibility produces:
1. a fast increasing sequence \( N_n \in \mathbb{N}^* \), \( N_{n+1} = N_n^{1+\delta} \) for some \( 0 < \delta < 1 \)

2. a sequence of constants \( A_n \in G \)

3. a number \( \tau > \tilde{\tau} \), a subsequence \( n_k \) and a set of resonant constants \( \{ \Lambda_k \}_{k=1}^M \subset G \) whose resonance \( k_{n_k}^\rho \in \mathbb{Z}^* \) in each \( (su(2))_\rho \), if it is defined, satisfies \( |k_{n_k}^\rho| \leq N_{n_k} \), as well as the following properties. The constant \( \Lambda_k \) commutes with \( A_{n_k} \) and \( d(\Lambda_k, A_{n_k}) < N_{n_k}^{-\tau} \), if such an \( \Lambda_{n_k} \) exists. The number \( M \in \mathbb{N} \cup \{ \infty \} \) counts the number of resonant steps of the K.A.M. scheme, i.e. those for which \( \Lambda_{n_k} \) is defined

4. a sequence of conjugations \( Y_n(\cdot) \in C^\infty(\mathbb{T}^d, g) \) satisfying
\[
\|Y_n(\cdot)\|_s = O(N_n^{-\infty}), \forall s \geq 0
\]

5. a sequence of conjugations \( B_{n_k}(\cdot) \in C^\infty(\mathbb{T}^d, G) \) satisfying
\[
\|B_{n_k}(\cdot)\|_s \simeq C_s N_{n_k}^{s+\lambda}, \forall s \geq 0
\]

for some constant \( \lambda > 0 \). These conjugations commute with the respective \( A_{n_k} \) and \( \Lambda_{n_k} \), and the constant
\[
B_{n_k}(\cdot + \alpha).\Lambda_{n_k}.B_{n_k}^{-1}(\cdot)
\]
is \( N_{n_k}^{-\tau} \)-away from resonant constants. If \( A_n \) is \( N_n^{-\tau} \)-away from resonant constants (i.e. if \( n \) is not a resonant step) then \( B_{n_k}(\cdot) \) is not defined

6. a sequence of mappings \( F_n(\cdot) \in C^\infty(\mathbb{T}^d, g) \) satisfying
\[
\|F_n(\cdot)\|_s = O(N_n^{-\infty}), \forall s \geq 0
\]

and such that the conjugation constructed iteratively following \( G_0 = \text{Id} \) and
\[
G_n(\cdot) = \begin{cases} e^{Y_n(\cdot)}G_{n-1}(\cdot) & \text{if } n \notin \{n_k\} \\ B_n(\cdot)e^{Y_n(\cdot)}G_{n-1}(\cdot) & \text{if } n \in \{n_k\} \end{cases}
\]
satisfies
\[
G_n(\cdot + \alpha)Ae^{F(\cdot)}G_n^{-1}(\cdot) = A_n.e^{F_n(\cdot)}
\]

As we have already pointed out in [Kar14] and [Kar15], the close-to-the-\( \text{Id} \) conjugations \( Y_n \) are highly redundant. If we rearrange them with the \( B_{n_k} \) following
\[
B_{n_1}e^{Y_{n_1}} \cdots e^{Y_{n_1+1}}B_{n_2}e^{Y_{n_2}} \cdots e^{Y_{n_2+1}} = B_{n_2} \cdots B_{n_1} \cdot \tilde{e}^{Y_{n_1}} \cdots \tilde{e}^{Y_{n_1+1}}
\]
and similarly for the rest of the steps, the product formed by the \( e^{Y_n} \) converges thanks to items 4 and 5 of thm. 5.1. If we call the resulting product \( D(\cdot) \), then the cocycle
\[
\tilde{A}(\cdot) = D(\cdot + \alpha)Ae^{F(\cdot)}D^{-1}(\cdot)
\]
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is in what we called K.A.M. normal form in the references. This means that, up to a second order perturbation, the cocycle

\[(B_{n_k} \cdots B_{n_1})(\cdot + \alpha) \tilde{A}(\cdot)((B_{n_k} \cdots B_{n_1}))^{-1}(\cdot) = \tilde{A}_k e^{\tilde{F}_k(\cdot)}\]

is either constant or has the following particular structure. Consider a root-space with respect to a torus passing by \(\tilde{A}_k\). If the root \(\rho\) is not resonant for \(\Lambda_k\), then the restriction of \(\tilde{F}_k(\cdot)\) in \((su(2))\rho\) is a constant. The constant is in \(t\) if the corresponding eigenvalue of \(\Lambda_n\) is not equal to 1, and the constant is in \((su(2))\rho\) if the eigenvalue is equal to 1. If \(\rho\) is resonant, for the resonance \(k_\rho\), the restriction is a constant in \(t\) plus \(\tilde{F}(k_\rho)e^{2\pi i k_\rho \cdot j_\rho}\). In short, to the first order only the resonant modes are active, and this particular structure allows us to accurately estimate the commutativity (or lack thereof) of the constants \(\tilde{A}_{n_k}\) and \(\tilde{A}_{n_k+1}\). Informally, if the commutator is significantly away from the \(\text{Id}\) for an infinite number of resonant steps, then the dynamics will be weakly mixing in the fibers (c.f. [Kar15]). If the commutators visit a certain small set depending only on \(G\) infinitely often, the dynamics will be \(DUE\) (c.f. [Kar14] for the case \(G = SU(2)\)).

In what follows, we assume the cocycle in normal form and omit the tilde in the notation, while keeping the rest of the notation the same.

5.4 The study of the cohomological equation

The study of the invariant distributions of almost reducible cocycles was possible because of the relation between Almost Reducibility and the Anosov-Katok construction. The former, obtained by an application of K.A.M. theory proves "almost rigidity" for perturbations of constant cocycles (i.e. perturbations of constant cocycles are almost conjugate to constant ones) and gives very good control on the failure of rigidity. When rigidity fails, i.e. when a perturbation of a constant cocycle is not conjugate to a constant one, this control allows almost reducibility to be interpreted as a fast approximation by conjugation scheme for the given cocycle. The approximant cocycles are resonant (their coboundary space is smaller than that of generic constant ones), and the estimates furnished by the K.A.M. scheme make analysis extremely efficient.

In what follows, we adapt notation from the references to notation of the present work. In [Kar17], we proved, for cocycles in \(T^d \times SU(2)\), that when prop. 3.4 is not applicable because the rate of convergence is polynomial with respect to the corresponding sequence \(\lambda_{N_k}\), the cocycle is \(C^\infty\) reducible. In [Kar14] we showed that when the rate is exponential (and it is so for a generic Almost Reducible cocycle), the cocycle is generically \(DUE\). In the same work we showed that such dynamical systems are never \(CR\), and the proof of cor. B is an abstraction of that proof.

In fact, the following theorem is well in the reach of the techniques developed in [Kar17] and [Kar14], but without the tools developed in the present article, the proof would be unnecessarily involved.
Theorem 5.2. Let \( d \in \mathbb{N}^* \) and \( G \) be a compact group. Then, in the K.A.M. regime of \( SW^\infty(\mathbb{T}^d, P) \), there are no counter-examples to conjecture 1.

Proof. The K.A.M. regime is the set of cocycles \((\alpha, A(\cdot))\) to which thm. 5.1 applies. We assume the cocycle in K.A.M. normal form.

When the space \( \mathbb{T}^d \times P \) is equipped with its natural Riemannian structure, almost reducibility, interpreted as in [Kar14] or [Kar15], produces a sequence of cocycles \((\alpha, A_n(\cdot))\) converging to \((\alpha, A(\cdot))\) which, together with the sequence \( N_n \) satisfy the conditions of prop. 3.4 for any \( \sigma \geq 0 \), except possibly for the fast approximation condition \( \delta_{\sigma,n} = O(\lambda^{-\infty}_n) \).

Thus, two cases can occur.

1. Either the fast approximation condition is satisfied, something which translates to

\[
d(A_{n_k}, \Lambda_k) = O(N^{-\infty}_{n_k})
\]

and \((\alpha, A(\cdot)) \notin CR\) by prop 3.4.

2. Or there exist \( \gamma', \tau' > 0 \) such that

\[
d(A_{n_k}, \Lambda_k) \geq \gamma' N^{-\tau'}_{n_k}
\]

In this case, an adjustment of the parameters of the scheme, as in [Kar17] can show that the cocycle is actually reducible, which results in the phase space foliating in invariant tori.

The proof is complete.

The other important model of cocycles, apart from constant ones, is given by the periodic geodesics of the group \( G \) (see [Kri01] or [Kar16b], chapters 4 and 8). They do not constitute a good basis for an \( \mathcal{AC}^\infty \) space, though, for the following reason. They are modeled upon the parabolic map

\[
\mathbb{T} \times \mathbb{T} \ni (x, y) \mapsto (x + \alpha, y + rx) \in \mathbb{T} \times \mathbb{T}
\]

for some \( r \in \mathbb{N}^* \), instead of a the quasi-periodic mapping

\[
\mathbb{T} \times \mathbb{T} \ni (x, y) \mapsto (x + \alpha, y + \beta) \in \mathbb{T} \times \mathbb{T}
\]

modelling the constant ones. Invariant distributions for the parabolic map can be calculated by hand, or see [Kat01]. The calculation shows that, unless one allows \( r \to \infty \), the assumptions of lem. 3.5 are satisfied and no \( DUE \) example can be constructed in the corresponding space.

As long as Almost Reducible cocycles and cocycles that can be conjugated arbitrarily close to periodic geodesics of \( G \) fill \( SW^\infty(\mathbb{T}, G) \) for some \( \alpha \), then no counter-examples to conj. 1 exist in that space. Theorem 1.3 in [Kar16b] argues that this is the case when \( \alpha \in RDC \), which proves corollary D.
6 Conclusions and comments

The proof of corollary B shows that, if one wishes to construct a counter-example to the Herman-Katok conjecture, they have indeed a very difficult task to accomplish, since using the most powerful method for constructing realizations of non-standard dynamics in the elliptic case appears to be a bad strategy. As soon as approximation is fast, cf. def. 4.2, they should be able to treat all obstructions arising at each step in the immediately next one, and they should be able to do so with estimates that seem to be out of reach for the existing arsenal of examples. On the other hand, slow approximation (i.e. at a polynomial rate) seems to result in the persistence of some structure obstructing DUE.

The whole approach of the article comes from intuition built on elliptic dynamics, and especially quasi-periodic dynamics. This context is precisely the origin of the Katok-Herman conjecture, which informally states that K.A.M. theory is perturbation theory for rotations in tori, and that only they can serve as its linear model.

Our approach seems to be disjoint from those in the literature. For example, the proof of the conjecture for flows in dimension 3 ([For08], [Koc09], [Mat09], [RHRH06]), the first non-trivial case for the continuous-time version of the conjecture, is based on techniques and results from dynamical systems and differential topology, and some very heavy machinery from symplectic topology. This symplectic topology machinery, namely the Weinstein conjecture, is used in order to exclude the case where the vector field is the Reeb vector field of a contact form in a cohomological sphere, in which case CR fails quite dramatically, due to the existence of periodic orbits.

The study of the cohomological equation for circle diffeomorphisms in [AK11] uses the renormalization scheme and depends very heavily on the existence of an order in $T^1$ which has been systematically exploited throughout the development of the theory (cf. [Her79], [Yoc95]).

The study of the cohomological equation for homogeneous flows ([FFRH13], [FF07]), is naturally based on representation theory, as was the article by the author, [Kar14]. The works by L. Flaminio, G. Forni and F. Rodriguez-Hertz go further and deeper than the verification of conj. 1, but as far as the conjecture itself is concerned, a posteriori they seem to be more or less the end of the road. This is so, because such flows or diffeomorphisms are found to always have an infinite codimensional space of coboundaries, and, consequently, in those classes CR fails quite dramatically. Of course, the infinite codimensionality of the coboundary space is precisely the object of the proof and it is a priori not at all obvious.

On the other hand, in the space of cocycles in $T^d \times SU(2)$, the space studied by the author in [Kar14], DUE is a generic property, and the proof of the inexistence of CR examples can give some insight into the mechanism that both creates DUE and obstructs CR. This insight is precisely what led to the present article.

Despite the fact that harmonic analysis on manifolds is less efficient and less elegant than representation theory for homogeneous spaces, the present article
suggests that its use can lead to advances in the study of the conjecture.

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