SPHERICALLY AVERAGED MAXIMAL FUNCTION AND SCATTERING FOR THE 2D CUBIC DERIVATIVE SCHRÖDINGER EQUATION

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Abstract. We prove scattering for the 2D cubic derivative Schrödinger equation with small data in the critical Besov space with some additional angular regularity. The main new ingredient is that we prove a spherically averaged maximal function estimate for the 2D Schrödinger equation. We also prove a global well-posedness result for the 2D Schrödinger map.

1. Introduction

In this paper, we study the Cauchy problem for the cubic derivative Schrödinger equation

\[
\begin{aligned}
    i\partial_t u + \Delta u &= |u|^2 a \cdot \nabla u + u^2 b \cdot \nabla \bar{u}, \\
    u(x, 0) &= u_0(x)
\end{aligned}
\]  

(1.1)

where \( u(x, t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{C}, \, a, b \in \mathbb{C}^2 \). The equation (1.1) arises from the strongly interacting many-body systems near criticality as recently described in terms of nonlinear dynamics \[8\]. The Schrödinger equation with derivative in the nonlinearity of the form

\[
i\partial_t u + \Delta u = F(u, \bar{u}, \nabla u, \nabla \bar{u}),
\]

(1.2)

has been studied extensively, e.g. see the introduction of \[15, 21\] for the history of the study. Besides equation (1.1) and the well-known one-dimension derivative Schrödinger equation, (1.2) contains another important model known as the Schrödinger maps

\[
\partial_t s = s \times \Delta_x s, \quad s(0) = s_0,
\]

(1.3)

where \( s : \mathbb{R}^n \times \mathbb{R} \to \mathbb{S}^2 \hookrightarrow \mathbb{R}^3 \). It was known that under stereographic projection (see Section 5 below) (1.3) is equivalent to

\[
i\partial_t u + \Delta u = \frac{2\bar{u}}{1 + |u|^2} \sum_{i=1}^{n} (\partial_{x_i} u)^2,
\]

(1.4)

and we see that the cubic term \( \bar{u}(\partial_{x_i} u)^2 \) serves as the first term in the Taylor expansion of the above nonlinear term.

Note that the equation (1.1) is invariant under the following scaling transform: for \( \lambda > 0 \)

\[
u(x, t) \to \lambda^{1/2} u(\lambda x, \lambda^2 t), \quad u_0(x) \to \lambda^{1/2} u_0(\lambda x),
\]

where \( u \) solves (1.1) and \( u_0 \) is an initial data.
then we see that the critical space for (1.1) is $H^{n-1}$ in the sense of scaling. Because of the loss of the derivative, the usual Strichartz analysis as for the power type non-linearity doesn’t work for (1.1). One needs some estimates with stronger smoothing effect. Kenig, Ponce, and Vega [15] introduced for the first time a method to obtain local well-posedness for general derivative Schrödinger equations. This method combines “local smoothing estimates”, “inhomogeneous local smoothing estimates”, which give the crucial gain of one derivative, and “maximal function estimates”. In the study of Schrödinger map, Ionescu and Kenig [12, 13] introduced the anisotropic local smoothing and maximal function estimates for Schrödinger equation. It was proved in [13] that the following local smoothing estimates hold

$$
\| e^{it\Delta} P_{k,e_1} f \|_{L^\infty_t L^2_x} \lesssim 2^{-k/2} \| f \|_2,
$$

where $P_{k,e_1} f = \mathcal{F}^{-1} \chi_{|\xi| \sim 2^k} \hat{f}$ (roughly, see Section 2 for the definition). In order to apply these estimates to deal with the cubic nonlinear terms, the following maximal function appears naturally

$$
\| e^{it\Delta} P_k f \|_{L^2_x L^\infty_t} \lesssim 2^{-(n-2)k/2} \| f \|_2.
$$

It was proved in [13] that (1.5) holds if $n \geq 3$. These estimates played key roles in the consequent study of Schrödinger map, e.g. [4]. For (1.1), in three dimensions and higher, one could gather these estimates to obtain global well-posedness and scattering in the critical Besov space, see [23] which also generalized the estimates and results to the non-elliptic case. In [21, 22] these estimates were generalized to the modulation space and sharp global well-posedness in modulation spaces for (1.1) (also in the non-elliptic) with $n \geq 3$ were obtained.

However, if $n = 2$, (1.5) fails. Thus the cubic nonlinear terms in two dimensions is more difficult. To the author’s knowledge, there are two approaches to deal with this difficulty. The first one was developed in [4] which uses the Galilean invariance of the Schrödinger propagator. The space $L^2_{x_1} L_x^\infty$ is replaced with a sum of Galilean transforms of it. The idea of using such sums of spaces as substitutes is due to Tataru [19]. The space is defined for any finite time interval $[-T, T]$, but the estimates in [4] are independent of $T$. The second one was developed in [21] which proves the following estimate via the Gabor frame representation of linear Schrödinger solution: for $1 \leq r < 2$

$$
\| e^{it\Delta} P_0 f \|_{L^r_t L^2_x} \lesssim \| f \|_r.
$$

Then with this well-posedness and scattering for (1.1) with $n = 2$ were proved for data in some modulation space.

In this paper, we take an another approach. Our ideas are inspired by [20] and the recent work [10]. First, since (1.5) only fails “logarithmically” for $n = 2$, we find that the spherically averaged maximal function estimate holds. This is like the spherically averaged endpoint Strichartz estimate for the 2D Schrödinger equation that was studied in [20]. More precisely, we show
Theorem 1.1. There exists $C > 0$ such that for $k \in \mathbb{Z}$, $u_0 \in L^2(\mathbb{R}^2)$, one has
\[ \|e^{it\Delta} P_k u_0\|_{L^2_t L^\infty_x L^2_\theta} \leq C 2^{k/2} \|u_0\|_2. \] (1.6)

See Section 2 for the definition of the space $L^2_t L^\infty_x L^2_\theta$ and $P_k$. Then we use an argument of [23] (which is in the spirit of [1]) to derive the corresponding inhomogeneous estimate. To use this norm to the equation (1.1), as in [10] we assume sufficient regularity on the sphere variable such that the space on the sphere is an algebra. Not like the Strichartz space, the local smoothing function space is anisotropic in $x$ which makes it not very compatible with the spherical average. For example, we do not have compare between $L^1_t L^\infty_x L^2_\theta$ and $L^1_t L^\infty_x L^2_t$. Fortunately, we can still close the iteration arguments in these spaces. We show

Theorem 1.2. Assume $n = 2$, $u_0 \in \dot{B}^{1/2,1}_{2,1,\theta}$ with $\|u_0\|_{\dot{B}^{1/2,1}_{2,1,\theta}} = \varepsilon_0 \ll 1$. Then there exists a unique global solution $u$ to (1.1) such that $\|u\|_{F^{1/2} \lesssim \varepsilon_0}$. Moreover, the map $u_0 \to u$ is Lipschitz from $\dot{B}^{1/2,1}_{2,1,\theta}$ to $C(\mathbb{R}; \dot{B}^{1/2,1}_{2,1,\theta})$, and scattering holds in this space.

Remark 1. In Theorem 1.2 $u_0 \in \dot{B}^{1/2,1}_{2,1,\theta}$ means that $u_0 \in \dot{B}^{1/2}_{2,1}$ and its spherical derivative $\partial_\theta u_0 \in \dot{B}^{1/2}_{2,1}$, and $\|u_0\|_{\dot{B}^{1/2,1}_{2,1,\theta}} = \|u_0\|_{\dot{B}^{1/2}_{2,1}} + \|\partial_\theta u_0\|_{\dot{B}^{1/2}_{2,1}}$. See Section 4 for the definition of $F^s$. Note that if $u_0 \in \dot{B}^{1/2}_{2,1}$ is radial, then $u_0 \in \dot{B}^{1/2,1}_{2,1,\theta}$. This is a bit surprising that for radial data the problem is relatively simpler even though the radial symmetry is not preserved under the flow of (1.1).

Now we turn to the study of the Schrödinger map. It has also been studied extensively (also in the case in which the sphere $\mathbb{S}^2$ is replaced by more general targets). Based on variants of the energy method, the local existence of the sufficiently smooth solutions were obtained, even for large data, see, for example, [18] [6] [9] [14] and the references therein. Similarly as (1.1), by the scaling we see the critical space for (1.3) is $\dot{H}^{d/2}$. Local well-posedness were obtained [12] for small data in $H^s_0(\mathbb{R}^n : \mathbb{S}^2)$, $s > (n + 1)/2$. This was improved to $s > n/2$ by Bejenaru [2]. Bejenaru observed for the first time in the setting of Schrödinger maps, that the gradient part of the nonlinearity in (1.4) has a certain null structure. Global well-posedness for small data in the critical Besov space in dimensions $n \geq 3$ were obtained in [13], and independently in [3]. Recently, global well-posedness for small data in the critical Sobolev space were proved in [3] first for $n \geq 4$, and in [4] for $n \geq 2$ where some state of art techniques were built. We revisit the case $n = 2$ using the new maximal function estimate. We prove

Theorem 1.3. Assume $n = 2$, the Schrödinger map initial value problem (1.3) is globally well-posed for small data $s_0 \in \dot{B}^{1,1}_Q \cap \dot{B}^{3,1,2}_Q(\mathbb{R}^2; \mathbb{S}^2)$, $Q \in \mathbb{S}^2$.

Remark 2. The space $\dot{B}^{s,1}_Q$ is defined by
\[ \dot{B}^{s,1}_Q = \{ f : \mathbb{R}^2 \to \mathbb{R}^3 ; f - Q \in \dot{B}^{s,1}_{2,1,\theta} \mid |f(x)| \equiv 1 \text{ a.e. in } \mathbb{R}^2 \}. \]

In the proof of Theorem 1.3 we don’t use $X^{s,b}$ type space and the null structure observed in [2], to make our argument as simple as possible. It is very likely that using $X^{s,b}$ type space and null structure as in [13], one can improve Theorem 1.3 to the critical space $s_0 \in \dot{B}^{1,1}_Q$, $Q \in \mathbb{S}^2$. However, we do not pursue it in this paper.
2. Definitions and Notations

For $x, y \in \mathbb{R}$, $x \lesssim y$ means that there exists a constant $C$ such that $x \leq C y$, and $x \sim y$ means that $x \lesssim y$ and $y \lesssim x$. We use $\mathcal{F}(f)$, $\hat{f}$ to denote the space-time Fourier transform of $f$, and $\mathcal{F}_{x_i, t} f$ to denote the Fourier transform with respect to $x_i, t$.

Let $\eta : \mathbb{R} \to [0, 1]$ be an even, non-negative, radially decreasing smooth function such that: a) $\eta$ is compactly supported in $\{ \xi : |\xi| \leq 8/5 \}$; b) $\eta \equiv 1$ for $|\xi| \leq 5/4$. For $k \in \mathbb{Z}$ let $\chi_k(\xi) = \eta(|\xi|/2^k) - \eta(|\xi|/2^{k-1})$ and $\chi_{\leq k}(\xi) = \eta(|\xi|/2^k)$, $\tilde{\chi}_k(\xi) = \sum_{l=-9}^9 \chi_{k+l}(\xi)$ and then define the Littlewood-Paley projector $P_k, P_{\leq k}$ on $L^2(\mathbb{R}^2)$ by

$$
P_k u(\xi) = \chi_k(|\xi|) \hat{u}(\xi), \quad P_{\leq k} u(\xi) = \chi_{\leq k}(|\xi|) \hat{u}(\xi).
$$

Let $S^1$ be the unit circle in $\mathbb{R}^2$. For $e \in S^1$, define $P_{k,e} u(\xi) = \tilde{\chi}_k(|\xi|) \chi_k(|\xi|) \hat{u}(\xi)$. Since for $|\xi| \sim 2^k$ we have $\sum_{l=-5}^5 \chi_{k+l}(\xi_1) + \sum_{l=-5}^5 \chi_{k+l}(\xi_2) \sim 1$, then let

$$
\beta_j^2(\xi) = \frac{\sum_{l=-5}^5 \chi_{k+l}(\xi_j)}{\sum_{l=-5}^5 \chi_{k+l}(\xi_1) + \sum_{l=-5}^5 \chi_{k+l}(\xi_2)} \cdot \frac{1}{\sum_{l=-1}^9 \chi_{k+l}(|\xi|)}, \quad j = 1, 2.
$$

Define the operator $Q_j^1$ on $L^2(\mathbb{R}^2)$ by $Q_j^1 f(\xi) = \beta_j^2(\xi) \hat{f}(\xi)$, $j = 1, 2$. Let $e_1 = (1, 0)$, $e_2 = (0, 1)$. Then we have

$$
P_k = P_{k,e_1} Q_1^1 + P_{k,e_2} Q_2^1.
$$

(2.1)

For any $e \in S^1$, we can decompose $\mathbb{R}^2 = \lambda e \oplus H_e$, where $H_e$ is the line with normal vector $e$, endowed with the induced measure. For $1 \leq p, q < \infty$, we define $L_{e}^{p,q}$ the anisotropic Lebesgue space by

$$
\| f \|_{L_{e}^{p,q}} = \left( \int_{\mathbb{R}} \left( \int_{H_e \times \mathbb{R}} |f(\lambda e + y, t)|^q d\lambda d\mu_t \right)^{p/q} d\lambda \right)^{1/p}
$$

with the usual definition if $p = \infty$ or $q = \infty$. We write $L_{e_1}^{p,q} = L_{x_1}^{p,q}, L_{e_2}^{p,q} = L_{x_2}^{p,q}$. $L_{x_1,t}^{p,q}$.

For any spherical variable $\theta \in S^1$, $\partial_\theta$ be the spherical derivative and $\Delta_\theta = \sqrt{1 - \Delta_\theta}$. We identify $S^1 = \mathbb{R}/(2\pi \mathbb{Z}) := \mathbb{T}$. Denote by $H^{s,p}_\theta = \Lambda^{s,p}_\theta L^p$ the standard $L^p$ Sobolev space on $\mathbb{T}$. We define $L_{e}^{p,q} H^{s,r}_\theta$ by the norm

$$
\| f \|_{L_{e}^{p,q} H^{s,r}_\theta} = \| \| f( |x| \cos \theta, |x| \sin \theta, t) \|_{H^{s,r}_\theta} \|_{L_{e}^{p,q}}.
$$

By the $SO(2)$ integration, we will also use the following form

$$
\| f \|_{L_{e}^{p,q} H^{s,r}_\theta} = \left( \int_0^{2\pi} |\Lambda^s_\theta[f(A_\theta \cdot x, t)]|^r d\theta \right)^{1/r}_{L_{e}^{p,q}},
$$

where $A_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$. For any function $f$, we denote the action $A_\theta f(x) = f(A_\theta \cdot x)$. It’s easy to see that $\partial_\theta[A_\theta f] = A_\beta(\partial_\theta f)$. We use $\hat{B}_p^{s,q}$ to denote the
homogeneous Besov spaces on $\mathbb{R}^2$ which is the completion of the Schwartz function under the norm

$$\|f\|_{\dot{B}^s_{p,q}} = \left(\sum_{k \in \mathbb{Z}} 2^{sqk}\|P_k f\|_{L^p}^q\right)^{1/q}.$$ 

We define $\dot{B}^{s,\alpha}_{p,q,\theta}$ to be the space with the norm $\|f\|_{\dot{B}^{s,\alpha}_{p,q,\theta}} = \|A^\alpha_\theta f\|_{\dot{B}^s_{p,q}}$. Then it’s easy to see that $\|f\|_{\dot{B}^{s,1}_{p,q,\theta}} \sim \|f\|_{\dot{B}^s_{p,q}} + \|\partial_\theta f\|_{\dot{B}^s_{p,q}}$. For any space-time norm $X$, we define $X_{H}^{s,p,\theta}$ by the norm $\|f\|_{X_{H}^{s,p,\theta}} = \|A^\theta f\|_{X_{H}^{s,p}}$.

We conclude this section by a convolution property of the spherical average space which implies that $P_k, Q^j_k$ are bounded operators in $XL^q_\theta$ if $X$ is space translation invariant.

**Lemma 2.1.** Let $X$ be a space-time function space on $\mathbb{R}^2 \times \mathbb{R}$ that is space translation invariant. Then for $1 \leq q \leq \infty$

$$\|f * g\|_{XL^q_\theta} \leq C\|f\|_{L^1_1 L^\infty_\theta}\|g\|_{XL^q_\theta}.$$ 

**Proof.** We have

$$\|f * g\|_{XL^q_\theta} \sim \left\|\int f(A^\theta_\beta x - y)g(y)dy\right\|_{XL^q_\theta} \sim \left\|\int f(A^\theta_\beta x - A^\theta_\beta y)g(A^\theta_\beta y)dy\right\|_{XL^q_\theta} \leq \|f\|_{L^1_1 L^\infty_\theta}\|g\|_{XL^q_\theta},$$

where in the last inequality we used that $X$ is space translation invariant. \qed

### 3. Spherically averaged maximal function estimates

In this section, we prove the spherically averaged maximal function estimate. First, we consider the homogeneous case.

**Lemma 3.1.** Assume $k \in \mathbb{Z}$. Then

$$\|e^{it\Delta}P_k f\|_{L^\infty_{\theta}L^2_\theta} \lesssim 2^{k/2}\|f\|_2$$

**Proof.** By the scaling invariance, we may assume $k = 0$. Moreover, since $e^{it\Delta}$ commutes with rotation, then by a rotation transform we may assume $e = (1, 0)$. It reduces to prove

$$\|e^{it\Delta}P_0 f\|_{L^\infty_{\theta}L^2_\theta} \lesssim \|f\|_2$$

Using the Hölder inequality and Bernstein’s inequality, we easily get

$$\|e^{it\Delta}P_0 f\|_{L^\infty_{\theta}L^2_\theta} \lesssim \|e^{it\Delta}P_0 f\|_{L^\infty_{\theta}L^2_\theta} \leq C\|f\|_2.$$ 

Then it remains to show

$$\|e^{it\Delta}P_0 f\|_{L^2_{|x|\geq 0}L^\infty_{\theta}L^2_\theta} \leq C\|f\|_2.$$ 

We will prove (3.3) by two steps.

**Step 1.** radial case
We assume $f$ is radial. It is well known that if $G(x) = g(|x|)$ is radial and $G \in L^2(\mathbb{R}^n)$, then the Fourier transform of $G$ is also radial (cf. [16]), and
\begin{equation}
\hat{G}(\xi) = 2\pi \int_0^\infty g(s)s^{n-1}(s|\xi|)^{-\frac{n+2}{2}} J_{\frac{n-2}{2}}(s|\xi|)ds,
\end{equation}
where $J_m(r)$ is the Bessel function
\begin{equation}
J_m(r) = \frac{(r/2)^m}{\Gamma(m + 1/2)\pi^{1/2}} \int_{-1}^1 e^{ir(1 - t^2)^{m-1/2}}dt, \quad m > -1/2.
\end{equation}
Since $f$ is radial and denote $\hat{f}(\xi) = h(|\xi|)$, then by the formula (3.4) we get $e^{it\Delta}P_0f(x_1,x_2) = F_0(t,\sqrt{x_1^2 + x_2^2})$, where for $\rho \geq 0$
\begin{equation}
F_0(t,\rho) = 2\pi \int_0^\infty e^{-its^2}\eta_0(s)h(s)sJ_0(s\rho)ds.
\end{equation}
Therefore, to show (3.3) it is equivalent to show
\begin{equation}
\int_1^\infty \sup_{\rho \geq x,t \in \mathbb{R}} |F_0(t,\rho)|^2 dx \leq C\|h\|_2^2.
\end{equation}
To prove (3.6), we will use the decay properties at the infinity of the Bessel function. More precisely, for $n \geq 2$
\begin{equation}
J_{\frac{n-2}{2}}(r) = c_n e^{ir} - c_n^* e^{-ir} r^{\frac{n-2}{2}} E_+(r) - c_n^* e^{-ir} E_-(r),
\end{equation}
where $E_{\pm}(r) \lesssim r^{-(n+1)/2}$ if $r \geq 1$, see [17]. Inserting (3.7) into (3.5), we then divide $F_0(t,|x|)$ into two parts: the main term and the error term, namely
\begin{equation}
F_0(t,\rho) = M(t,\rho) + E(t,\rho)
\end{equation}
with
\begin{align*}
cM(t,\rho) &= \rho^{-\frac{1}{2}} \int_\mathbb{R} \eta_0(s)h(s)\frac{1}{2}e^{i(\rho s - ts^2)}ds + \rho^{-\frac{1}{2}} \int_\mathbb{R} \eta_0(s)h(s)\frac{1}{2}e^{-i(\rho s + ts^2)}ds, \\
cE(t,\rho) &= \int_\mathbb{R} \eta_0(s)h(s)se^{-its^2 - i\rho s}E_+(\rho s)ds - \int_\mathbb{R} \eta_0(s)h(s)se^{-its^2 + i\rho s}E_-(\rho s)ds.
\end{align*}
For the error term, since $|E(t,\rho)| \lesssim \rho^{-3/2}\|h\|_2$, then one get that
\begin{equation}
\int_1^\infty \sup_{\rho \geq x,t \in \mathbb{R}} |E(t,\rho)|^2 dx \lesssim \int_1^\infty x^{-3}\|h\|_2^2 dx \lesssim \|h\|_2^2.
\end{equation}
It remains to bound the main term. From symmetry, it suffices to show that
\begin{equation}
\int_1^\infty \sup_{\rho \geq x,t \in \mathbb{R}} \frac{1}{\rho} \int_0^\infty |\eta_0(s)h(s)e^{i(\rho s - ts^2)}| ds \leq \|h\|_2^2.
\end{equation}
Obviously,
\begin{align*}
\int_1^\infty \sup_{\rho \geq x,t \in \mathbb{R}} \left| \int_0^\infty \eta_0(s)h(s)e^{i(\rho s - ts^2)}ds \right|^2 dx \\
\lesssim \sum_{k=0}^\infty 2^{-k} \int_1^\infty \sup_{2^k \leq \rho \leq 2^{k+1},x,t \in \mathbb{R}} \left| \int_0^\infty \eta_0(s)h(s)e^{i(\rho s - ts^2)}ds \right|^2 dx.
\end{align*}
Define the operator $T$ acting on $h \in L^2([1, 3])$ as follows
\[
T(h)(x) = \frac{1}{x^{1/2}} \int_0^\infty \eta_0(s)h(s)e^{i(xps-\xi s^2)}ds.
\]
Thus it suffices to show
\[
\|Th\|_{L^2_x[1, \infty]} \leq C\|h\|_2, \forall k \in \mathbb{Z}_+.
\]
By $TT^*$ argument, it suffices to show
\[
\|TT^* f\|_{L^2_x[1, \infty]} \leq C\|f\|_{L^2_x[1, \infty]} L^1_{\rho-2^k, t \in \mathbb{R}}.
\]
Indeed, we have
\[
TT^* f = \frac{1}{x^{1/2}} \int \left( \int_0^\infty \eta_0^2(s)e^{i((xp-x')s-(t-t')s^2)} ds \right) \frac{1}{x^{1/2}} f(x', \rho', t') dx' d\rho' dt'.
\]
By the stationary phase method, we have
\[
\left| \int_0^\infty \eta_0^2(s)e^{i((xp-x')s-(t-t')s^2)} ds \right| \lesssim (1 + |x \rho - x' \rho'|)^{-1/2}.
\]
Thus, we get
\[
|TT^* f| \lesssim \frac{1}{x^{1/2}} \int (1 + |x \rho - x' \rho'|)^{-1/2} \frac{1}{x^{1/2}} |f(x', \rho', t')| dx' d\rho' dt'
\]
\[
\lesssim \int_{|x| \sim |x'|} \frac{|f(x', \rho', t')|}{x^{1/2}} dx' d\rho' dt' + \int_{|x| \gg |x'|} \frac{|f(x', \rho', t')|}{2k/2x^{1/2}} dx' d\rho' dt'
\]
\[
+ \int_{|x| \ll |x'|} \frac{|f(x', \rho', t')|}{2k/2x^{1/2}} dx' d\rho' dt'
\]
\[
:= I + II + III.
\]
Now we show (3.14). For the contribution of the term $I$, we have
\[
I \lesssim M(\|f(\cdot, \rho, t)\|_{L^1_{\rho, t}})(x)
\]
where $M$ is the Hardy-Littlewood maximal operator. Then from the $L^2$ boundedness of $M$, we see the estimate of $I$ is fine. The estimate of $II, III$ simply follows from the Hölder inequality.

**Step 2.** general case

We assume $f$ is nonradial. First, we make some reductions using the spherical harmonics on $S^1$. For any function $f \in L^2(\mathbb{R}^2)$, we can write
\[
f_n(r) = \sum_{n \in \mathbb{Z}} f_n(r)e^{in\theta}.
\]
Hence by the property of Fourier transform (see [10])
\[
e^{it\Delta} P_0 f(re^{i\theta}) = \sum_{n \in \mathbb{Z}} 2\pi i^{-n} T_n(f_n)(t, r)e^{in\theta},
\]
where
\[
T_n(f)(t, r) = \int e^{-itr^2} J_n(r \rho) \rho \chi_0(\rho) f(\rho) d\rho.
\]
Thus (3.3) becomes
\[ \|T_n(f_n)(t, |x|)\|_{L^2_{L^2_{r_1}, L^2_{r_2}, t_n}^r} \lesssim \|f_n(|x|)\|_{L^2_{t_n}}. \] (3.11)

To prove (3.11), it is equivalent to show
\[ \|T_n(f)(t, r)\|_{L^2_{L^2_{r_{|x|}, t}}^r} \lesssim \|f\|_{L^2}, \] (3.12)
with a bound independent of \( n \geq 0 \).

To prove (3.12), we need to use the uniform property of \( J_n \) with respect to \( n \). We have
\[ \|T_n(f)(t, r)\|_{L^2_{L^2_{r_{|x|}, t}}^r} \lesssim \|T_n(f)(t, r)\|_{L^2_{|x| n, r_{|x|}, t}} + \|T_n(f)(t, r)\|_{L^2_{|x| n, r_{|x|}, t}} := A + B. \]

First, we estimate the term \( A \). By the Cauchy-Schwartz inequality, we have
\[ A \leq n^{1/2} \|T_n(f)(t, r)\|_{L^2_{r_{|x|}, t}}. \]
Thus it suffices to show \( |T_n(f)(t, r)| \lesssim n^{-1/2} \). If \( r \gg n \) or \( r \ll n \), this follows easily from the fact that \( |J_n(r)| \lesssim n^{-1/2} \). It remains to show
\[ \left\| \int e^{-it\rho^2} J_n(r\rho)\chi_0(\rho) f(\rho) d\rho \right\|_{L^2_{r_{|x|}, t}} \lesssim n^{-1/2} \|f\|_2. \]
By \( TT^* \) argument, it suffices to show
\[ \left\| \int \left( \int e^{-i(t-t')\rho^2} J_n(r\rho) J_n(r'\rho) \chi_0(\rho) d\rho \right) g(t', r') dt' dr' \right\|_{L^2_{r_{|x|}, t}} \lesssim n^{-1} \|g\|_{L^1_{r_{|x|}, t}}. \]
By the uniform decay of Bessel function (e.g. see Lemma 2.2 in [11]),
\[ |J_n(r)| \lesssim (1 + |r^2 - n^2|)^{-1/4}, \]
it suffices to show
\[ \sup_{r, r' \sim 1} \left| \int (1 + |r^2 - n^2|)^{-1/4} (1 + |r'^2 - n^2|)^{-1/4} \chi_0(\rho) d\rho \right| \lesssim n^{-1}, \]
which follows from the Cauchy-Schwartz inequality.

Now we estimate the term \( B \). Since \( r \geq |x| \gg n \), we have (given in [1], for the proof see Lemma 2.5 in [11])
\[ J_n(r) = \frac{1}{\sqrt{2\pi}} \frac{e^{i\theta(r)} + e^{-i\theta(r)}}{(r^2 - n^2)^{1/4}} + h(n, r) := J_n^1(r) + J_n^2(r) + J_n^3(r), \]
where
\[ \theta(r) = (r^2 - n^2)^{1/2} - \nu \arccos \frac{n}{r} - \frac{\pi}{4} \]
and
\[ |h(n, r)| \lesssim r^{-1}. \]
Thus, we get
\[ B \leq \sum_{j=1}^3 \|T_n^j(f)(t, r)\|_{L^2_{|x| n, r_{|x|}, t}} := \sum_{j=1}^3 B_j \]
where
\[ T^j_n(f)(t, r) = \int e^{-itp^2} J^j_n(r\rho)\rho\chi_0(\rho)f(\rho)d\rho, \quad j = 1, 2, 3. \]

For \( B_3 \), we use the decay of \( h \) and get
\[ \sup_{r \geq |x|} |T^3_n(f)(t, r)| \leq \sup_{r \geq |x|} r^{-1/2} \| f \|_2 \leq |x|^{-1} \| f \|_2 \]
which suffices to give the bound as desired. It remains to control \( B_1 \) since the estimate for \( B_2 \) follows in the same way.

It suffices to show that
\[
\int_{10n}^{\infty} \sup_{\rho \leq x, t \in \mathbb{R}} \left| \int_0^\infty \frac{1}{(\rho^2 s^2 - n^2)^{1/4}} \chi_0(s)e^{i(\theta(\rho s) - ts^2)} h(s)ds \right|^2 dx \lesssim \| h \|_2^2.
\]

Since \( \rho \geq |x| \gg n \) and \( s \sim 1 \), then \(|(\rho^2 s^2 - n^2)^{-1/4} - (\rho s)^{-1/2}| \lesssim |x|^{-5/2} n^2 \), and thus we get
\[
\int_{10n}^{\infty} \sup_{\rho \leq x, t \in \mathbb{R}} \left| \int_0^\infty \frac{1}{(\rho^2 s^2 - n^2)^{1/4}} - \frac{1}{(\sqrt{\rho} s)} \chi_0(s)e^{i(\theta(\rho s) - ts^2)} h(s)ds \right|^2 dx \lesssim \int x^{-5} n^4 dx \cdot \| h \|_2^2 \lesssim \| h \|_2^2.
\]

Therefore, it remains to show
\[
\int_{10n}^{\infty} \sup_{\rho \leq x, t \in \mathbb{R}} \frac{1}{\rho} \left| \int_0^\infty \chi_0(s)e^{i(\theta(\rho s) - ts^2)} h(s)ds \right|^2 dx \lesssim \| h \|_2^2. \tag{3.13}
\]

We proceed as in Step 1. Obviously,
\[
\int_{10n}^{\infty} \sup_{\rho \leq x, t \in \mathbb{R}} \frac{1}{\rho} \left| \int_0^\infty \chi_0(s)e^{i(\theta(\rho s) - ts^2)} h(s)ds \right|^2 dx \lesssim \sum_{k=0}^{\infty} 2^{-k} \int_{10n}^{\infty} \sup_{2^k \rho \leq x, t \in \mathbb{R}} \frac{1}{x} \left| \int_0^\infty \chi_0(s)e^{i(\theta(\rho s) - ts^2)} h(s)ds \right|^2 dx.
\]

Define the operator \( L \) acting on \( h \in L^2([1, 3]) \) as follows
\[ L(h)(x) = \frac{1}{x^{1/2}} \int_0^\infty \eta_0(s)h(s)e^{i(\theta(x\rho s) - ts^2)} ds. \]

Thus it suffices to show
\[ \| Lh \|_{L^2_{[1,\infty]}} \lesssim C \| h \|_2, \quad \forall k \in \mathbb{Z}_+. \]

By \( TT^* \) argument, it suffices to show
\[
\| LL^*f \|_{L^2_{[1,\infty]}} \lesssim C \| f \|_{L^1_{[1,\infty]}} \| L^1_{[1,\infty]} \|_{L^1_{[2k,\infty]}}. \tag{3.14}
\]

Indeed, we have
\[
LL^*f = \frac{1}{x^{1/2}} \int \left( \int_0^\infty \chi_0^2(s)e^{i(\theta(x\rho s) - \theta(x\rho' s) - (t-t')s^2)} ds \right) \frac{1}{x^{1/2}} f(x', \rho', t') dx' d\rho' dt'.
\]
Direct computation shows that for $r \gg n$

$$\theta'(r) = (r^2 - n^2)^{1/2}r^{-1} \sim 1,$$

$$\theta''(r) = (r^2 - n^2)^{-1/2} - (r^2 - n^2)^{1/2}r^{-2} = (r^2 - n^2)^{-1/2}n^2r^{-2} \lesssim r^{-1}.$$ 

Thus by the stationary phase method, we have

$$\left| \int_0^\infty \chi_0^2(s)e^{i(\theta(x_0s) - \theta(x'_0s) - (t-t')s^2)}ds \right| \lesssim \begin{cases} 1, & |x| \sim |x'| \\ 2^{-k/2} \max(|x|, |x'|)^{-1/2}, & |x| \sim |x'|. \end{cases}$$

With this the rest proof is the same as in step 1. We complete the proof. \hfill \Box

Next, we derive the inhomogeneous estimate. Here we use an direct argument of Lemma 7.5 in [4].

**Lemma 3.2.** Let $k \in \Z$. Assume $u, F$ solves the equation

$$iu_t + \Delta u = F(x, t), \quad u(x, 0) = 0.$$ 

Then for any $e \in S^1$ we have

$$\|P_k u\|_{L^2_{2, t}L^2_{x} \leq \sup_{e \in S^1} \|F\|_{L^1_{2, t}}. \quad (3.15)$$

**Proof.** By the scaling and rotational invariance, we may assume $k = 0$ and $e = (1, 0)$. $P_0u = U + V$ such that $\mathcal{F}_t U$ is supported in $\{\xi_1 \sim 1 : |\xi_2| \sim 1\} \times \mathbb{R}$ and $\mathcal{F}_t V$ is supported in $\{\xi_1 \sim 1 : |\xi_2| \sim 1\} \times \mathbb{R}$. Thus it suffices to show

$$\|U\|_{L^2_{2, t}L^\infty_{x}, L^2_{\theta \leq \|F\|_{L^1_{2, t}} \leq \|F\|_{L^1_{2, t}, t}} \cdot \|V\|_{L^2_{2, t}L^\infty_{x}, L^2_{\theta \leq \|F\|_{L^1_{2, t}} \leq \|F\|_{L^1_{2, t}, t}} \cdot (3.16)$$

We only need to estimate $F$, since the estimate for $V$ is identical. We will write $u = U$. We assume $\mathcal{F}_t F$ is supported in $\{\xi_1 \sim 1 : |\xi_2| \sim 1\} \times \mathbb{R}$. We have

$$u(t, x) = \int_{\mathbb{R}^3} \frac{e^{it\tau e^{ik\xi}}}{\tau - |\xi|^2} \mathcal{F}(\xi, \tau) d\xi d\tau$$

$$= \int_{\mathbb{R}^3} \frac{e^{it\tau e^{ik\xi}}}{\tau - |\xi|^2} \mathcal{F}(\xi, \tau)(1_{\mathbb{R}^2 \setminus (\tau, \xi_2) : \tau - \xi_2^2 \sim 1} + 1_{\tau - \xi_2^2} d\xi d\tau$$

$$= u_1 + u_2.$$

For $u_1$, we simply use the Plancherel equality and get

$$\|\Delta u_1\|_{L^2_x} + \|\partial_t u_1\|_2 \leq \|F\|_2,$$

and thus by Sobolev embedding and Bernstein’s inequality we obtain the desired estimate. Now we estimate $u_2$. Let $G(x_1, \xi_2, \tau) = 1_{\tau - \xi_2^2 \sim 1} F(x_2, \tau)$. Then

$$u_2 = \int_{\mathbb{R}} \int_{\mathbb{R}^3} \frac{e^{it\tau e^{ik\xi}}}{\tau - |\xi|^2} [e^{-iy_1 \xi_1} G(y_1, \xi_2, \tau)] d\xi d\tau dy_1$$

$$= \int_{\mathbb{R}} T_{y_1}(G(y_1, \cdot))(t, x) dy_1$$

where

$$T_{y_1}(f)(t, x) = \int_{\mathbb{R}^3} \frac{e^{it\tau e^{ik\xi}}}{\tau - |\xi|^2} [e^{-iy_1 \xi_1} 1_{\xi_2 \leq 1} f(\xi_2, \tau)] d\xi d\tau.$$
Thus it suffices to prove
\[
\|T_{y_1}(f)\|_{L^2_{x_1}L^\infty_{z_2},L^2_\theta} \lesssim \|f\|_{L^2}, \quad \forall y_1 \in \mathbb{R}. \tag{3.17}
\]

Define \(s = s(\tau, \xi_2) = \tau - \xi_2^2\), we have
\[
T_{y_1}(f)(t, x) = \int_{\mathbb{R}^2} 1_{|\xi_2| \leq 1, \tau - \xi_2^2 > 0} \left( \int \frac{e^{i(x_1-y_1)\xi_1}}{\tau - \xi_2^2 - \xi_1^2} d\xi_1 \right) e^{i\tau} e^{ixz_2^2} f(\xi_2, \tau) d\xi_2 d\tau
\]
\[
= \int_{\mathbb{R}^2} \left( \frac{1}{2\sqrt{s}} \left( \frac{1}{\sqrt{s} + \xi_1} + \frac{1}{\sqrt{s} - \xi_1} \right) \right) e^{i\tau} e^{ixz_2^2} 1_{|\xi_2| \leq 1, \tau - \xi_2^2 > 0} f(\xi_2, \tau) d\xi_2 d\tau
\]
\[
:= I_1(f) + I_2(f).
\]

We only estimate \(I_1\), since \(I_2\) follows in the same way. By the property of Hilbert transform, we get
\[
I_1(f)(t, x) = \int_{\mathbb{R}^2} \frac{e^{-i(x_1-y_1)\sqrt{s}}}{2\sqrt{s}} \text{isgn}(x_1 - y_1) \cdot e^{i\tau} e^{ixz_2^2} 1_{|\xi_2| \leq 1, \tau - \xi_2^2 > 0} f(\xi_2, \tau) d\xi_2 d\tau.
\]

Making a change of variable \(\eta_1 = -\frac{1}{2}(\sqrt{s} - \xi_2^2), d\tau = 2\eta_1 d\eta_1\), we obtain
\[
I_1(f)(t, x) = \text{isgn}(x_1 - y_1) \int_{\mathbb{R}^2} e^{i\eta_1^2} e^{i(x_1+2\eta_1)\eta_1} e^{i(x_1-y_1)\eta_1} \cdot e^{-iy_1\eta_1} 1_{|\xi_2| \leq 1, \eta_1 - 1} f(\xi_2, \eta_1^2 + \xi_2^2) d\xi_2 d\eta_1.
\]

Thus, by Lemma 3.1, we get
\[
\|I_1(f)\|_{L^2_{x_1}L^\infty_{x_2},L^2_\theta} \lesssim \|1_{|\xi_2| \leq 1, \eta_1 - 1} f(\xi_2, \eta_1^2 + \xi_2^2)\|_{L^2} \lesssim \|f\|_{L^2}.
\]

We complete the proof of the lemma. \(\square\)

### 4. Cubic Derivative NLS

In this section we prove Theorem 1.2. The ideas is from [4]. First, we define the main dyadic function space \(F_k\) and \(N_k\) for \(k \in \mathbb{Z}\). If \(f(x, t) \in L^2(\mathbb{R}^2 \times \mathbb{R})\) has spatial frequency localized in \(\{\|\xi_2\| \sim 2^k\}\), define
\[
\|f\|_{F_k} = \|f\|_{L^8_z L^\infty_x} + \|f\|_{L^4_z L^4_x} + 2^{k/6} \sup_{|j-k| \leq 20} \sup_{e \in S^1} \|P_{j,e} f\|_{L^6_z L^3_x}
\]
\[
+ 2^{-k/2} \sup_{e \in S^1} \|f\|_{L^4_z L_\theta} + 2^{k/2} \sup_{e \in S^1} \|P_{j,e}(A_\beta f)\|_{L^\infty_z L^2_{\beta'}}.
\]
\[
\|f\|_{G_k} = \|f\|_{L^8_z L^\infty_x} + \|f\|_{L^4_z L^4_x} + 2^{-k/2} \sup_{e \in S^1} \|f\|_{L^2_z L_\theta}.
\]
\[
\|f\|_{N_k} = \inf_{f = f_1 + f_2 + f_3 + f_4} (\|f_1\|_{L^{4/3}_{t,x}} + 2^{k/6} \|f_2\|_{L^{3/2,6/5}_{t,x}}
\]
\[
+ 2^{k/6} \|f_3\|_{L^{3/2,6/5}_{t,x}} + 2^{-k/2} \sup_{e \in S^1} \|f_4\|_{L^4_z}).
\]
Then we define the space $F^s, N^s$ with the following norm
\[
\|u\|_{F^s} = \sum_{k \in \mathbb{Z}} 2^{ks} (\|P_k u\|_{F_k} + \|P_k \partial_\theta u\|_{F_k} ) := \sum_{k \in \mathbb{Z}} 2^{ks} \|P_k u\|_{\tilde{F}_k},
\]
\[
\|u\|_{G^s} = \sum_{k \in \mathbb{Z}} 2^{ks} (\|P_k u\|_{G_k} + \|P_k \partial_\theta u\|_{G_k} ) := \sum_{k \in \mathbb{Z}} 2^{ks} \|P_k u\|_{\tilde{G}_k},
\]
\[
\|u\|_{N^s} = \sum_{k \in \mathbb{Z}} 2^{ks} (\|P_k u\|_{N_k} + \|P_k \partial_\theta u\|_{N_k} ) := \sum_{k \in \mathbb{Z}} 2^{ks} \|P_k u\|_{\tilde{N}_k}.
\]
Note that to use the spherically averaged maximal function estimate, we need the spherically averaged local smoothing estimate.

**Lemma 4.1** (Linear estimates). Assume $u, F, u_0$ solves the following equation
\[
i \partial_t u + \Delta u = F, \quad u(0, x) = u_0.
\]
Then for any $s \in \mathbb{R}$, we have
\[
\|u\|_{F^s} = \|u_0\|_{H^{s,1, \rho}} + \|F\|_{N^s}.
\]
**Proof.** By the definition, it suffices to show
\[
\|P_k u\|_{F_k} \lesssim \|P_k u_0\|_2 + \|P_k F\|_{N_k}.
\]
(4.1)
Since $\Delta$ commutes with rotation and the local smoothing estimate (see [12]), we have
\[
\|P_k e^{A_\theta u}\|_{L^\infty_t L^2_\rho} \lesssim \|P_k e^{A_\theta u}\|_{L^2_t L^\infty_\rho} \lesssim 2^{-k/2} \|A_\theta u_0\|_{L^2_t L^2_\rho} + 2^{-k} \sup_{e \in \mathbb{S}^1} \|A_\theta F\|_{L^1_t L^2_\rho} \lesssim 2^{-k/2} \|u_0\|_2 + 2^{-k} \sup_{e \in \mathbb{S}^1} \|F\|_{L^1_t L^2_\rho}.
\]
Similarly, in the above inequality we can replace $\sup_{e \in \mathbb{S}^1} \|F\|_{L^1_t L^2_\rho}$ by $\sup_{e \in \mathbb{S}^1} \|F\|_{L^2_t L^2,6/5}$ and $\|F\|_{L^1_t L^2,3/4}$. The other components except for the maximal function follow from the known linear estimate. For the maximal function component, we use Lemma [3.1][3.2] and the Christ-Kiselev lemma [17] (or Lemma 7.3 in [4]).

To prove Theorem [1.2] by the standard iteration method, it suffices to show the trilinear estimates. We need the following lemma.

**Lemma 4.2.** Assume $k_1, k_2 \in \mathbb{Z}$. Then
\[
\|P_{k_1} u P_{k_2} \bar{v}\|_{L^2_{t,x}} + \|P_{k_1} \partial_\theta u P_{k_2} \bar{v}\|_{L^2_{t,x}} + \|P_{k_1} u P_{k_2} \partial_\theta \bar{v}\|_{L^2_{t,x}} \lesssim 2^{k_1/2} 2^{-k_2/2} \|P_{k_1} u\|_{\tilde{G}_{k_1}} \|P_{k_2} \bar{v}\|_{\tilde{F}_{k_2}}.
\]
**Proof.** Since $A_\beta$ commute with $P_k$, and by Lemma [2.1] we have
\[
\|P_{k_1} u P_{k_2} \bar{v}\|_{L^2_{t,x}} = \|P_{k_1} A_\beta u P_{k_2} A_\beta \bar{v}\|_{L^2_{t,x} L^\infty_\rho} \lesssim \|P_{k_1} A_\beta u P_{k_2, e_1} A_\beta \bar{v}\|_{L^2_{t,x} L^\infty_\rho} + \|P_{k_1} A_\beta u P_{k_2, e_2} Q_{k_2}^2 A_\beta \bar{v}\|_{L^2_{t,x} L^\infty_\rho}
\]
\[
\lesssim \|P_{k_1} A_\beta u\|_{L^\infty_t L^\infty_\rho} \|P_{k_2, e_1} A_\beta \bar{v}\|_{L^\infty_t L^\infty_\rho} + \|P_{k_1} A_\beta u\|_{L^\infty_t L^\infty_\rho} \|P_{k_2, e_2} A_\beta \bar{v}\|_{L^\infty_t L^\infty_\rho}
\]
\[
\lesssim 2^{k_1/2} 2^{-k_2/2} \|P_{k_1} u\|_{\tilde{G}_{k_1}} \|P_{k_2} \bar{v}\|_{\tilde{F}_{k_2}}.
\]
For the other component, we have
\[
\|P_k \partial_\theta u P_k \tilde{v}\|_{L^{2}_{t,x}} + \|P_k u P_k \partial_\theta \tilde{v}\|_{L^{2}_{t,x}} \\
= \|P_k \partial_\beta (A_\beta u) \cdot P_k \bar{A}_\beta \tilde{v}\|_{L^{2}_{t,x}} + \|P_k (A_\beta u) \cdot P_k \partial_\beta (A_\beta \tilde{v})\|_{L^{2}_{t,x}} \leq I + II.
\]

For \(I\), by the Sobolev embedding \(H^1_\theta(\mathbb{T}^1) \hookrightarrow L^\infty_\theta(\mathbb{T}^1)\) we have
\[
I \lesssim \|P_k \partial_\beta (A_\beta u) \cdot P_k, e_1, A_\beta \tilde{v}\|_{L^2_{t,x}L^2_\beta} + \|P_k \partial_\beta (A_\beta u) \cdot P_k, e_2, A_\beta \tilde{v}\|_{L^2_{t,x}L^2_\beta} \\
\lesssim \|P_k \partial_\beta (A_\beta u)\|_{L^2_{\infty t}L^1_\beta} \|P_k, e_1, A_\beta \tilde{v}\|_{L^2_{\infty t}L^1_\beta} \\
+ \|P_k \partial_\beta (A_\beta u)\|_{L^2_{\infty t}L^1_\beta} \|P_k, e_2, A_\beta \tilde{v}\|_{L^2_{\infty t}L^1_\beta} \\
\lesssim 2^{k_1/2 - k_2} \|P_k, e_1, \tilde{v}\|_{\tilde{F}_{k_2}} \|P_k, e_2, \tilde{v}\|_{\tilde{F}_{k_2}}.
\]

For \(II\), we have
\[
II \lesssim \|P_k (A_\beta u) \cdot P_k, e_1, \partial_\beta (A_\beta \tilde{v})\|_{L^2_{t,x}L^2_\beta} + \|P_k (A_\beta u) \cdot P_k, e_2, \partial_\beta (A_\beta \tilde{v})\|_{L^2_{t,x}L^2_\beta} \\
\lesssim \|P_k (A_\beta u)\|_{L^2_{\infty t}L^1_\beta} \|P_k, e_1, \partial_\beta (A_\beta \tilde{v})\|_{L^2_{\infty t}L^1_\beta} \\
+ \|P_k (A_\beta u)\|_{L^2_{\infty t}L^1_\beta} \|P_k, e_2, \partial_\beta (A_\beta \tilde{v})\|_{L^2_{\infty t}L^1_\beta} \\
\lesssim 2^{k_1/2 - k_2} \|P_k, e_1, \tilde{v}\|_{\tilde{F}_{k_2}} \|P_k, e_2, \tilde{v}\|_{\tilde{F}_{k_2}}.
\]

We complete the proof of the lemma. \(\square\)

**Lemma 4.3** (Nonlinear estimates). Assume \(i = 1, 2, s \geq 1/2\). Then
\[
\|u \tilde{v} \partial_\xi \tilde{w}\|_{N^s} \lesssim \|u\|_{F^s} \|v\|_{F^{1/2}} \|w\|_{F^{1/2}} + \|u\|_{F^{1/2}} \|v\|_{F^s} \|w\|_{F^{1/2}} + \|u\|_{F^s} \|v\|_{F^{1/2}} \|w\|_{F^s}.
\]

**Proof.** We only prove the case \(s = 1/2\), since the other case are similar. By the definition, we have
\[
\|u \tilde{v} \partial_\xi \tilde{w}\|_{N^{1/2}} \\
= \sum_{k_4} 2^{k_4/2} (\|P_{k_4} [u \tilde{v} \partial_\xi \tilde{w}]\|_{N_{k_4}} + \|\tilde{v} P_{k_4} [u \tilde{v} \partial_\xi \tilde{w}]\|_{N_{k_4}}) \\
\leq \sum_{k_1, k_2, k_3, k_4} 2^{k_4/2} (\|P_{k_4} [P_{k_1} u P_{k_2} \bar{v} \partial_\xi , P_{k_3} w]\|_{N_{k_4}} + \|\tilde{v} P_{k_4} [P_{k_1} u P_{k_2} \bar{v} \partial_\xi , P_{k_3} w]\|_{N_{k_4}}) \\
:= I + II.
\]

We will estimate the sum above case by case, according to the type of frequency interactions. By symmetry, we may assume \(k_1 \leq k_2\). We also assume that \(k_2 \leq k_3\), namely the derivative falls on the largest frequency, since the other case \(k_2 > k_3\) can be handled similarly.

**Case 1:** \(k_4 \leq k_1 + 200\).
For this case, we use the Strichartz norm $L^{4/3}$ for $N_{k,\alpha}$. By the properties of Fourier support of input functions, we may assume $k_3 \leq k_2 + 300$. Thus we have

$$I \lesssim \sum_{k_1, k_4 \leq \min(k_1, k_2, k_3)+5} 2^{k_4/2} \| P_{k_4} [P_{k_1} u P_{k_2} \tilde{v} \partial_x P_{k_3} w] \|_{L^{4/3}}$$

$$\leq \sum_{k_1, k_4 \leq k_2 + 5} 2^{k_4/2} \| P_{k_1} u \|_{L^3_{t,x}} 2^{k_2/2} \| P_{k_2} v \|_{L^3_{t,x}} 2^{k_3/2} \| P_{k_3} w \|_{L^3_{t,x}}$$

$$\lesssim \| u \|_{F^1/2} \| v \|_{F^1/2} \| w \|_{F^1/2}.$$

The estimate for $II$ is the same as $I$, since $\partial_\theta$ commutes with $P_k$.

Case 2: $k_1 + 200 < k_4 \leq k_2 + 100$.

In this case we have $k_1 < k_2 - 100$ and $k_3 \leq k_2 + 200$. Then we get

$$I \lesssim \sum_{k_4} 2^{k_4/2} 2^{k_4/6} \| P_{k_1} u P_{k_2} \tilde{v} \|_{L^3_{t,x}} \| P_{k_3, e_1} w \|_{L^{6/5}_{t,x}}$$

$$+ \| P_{k_4} [P_{k_1} u P_{k_2} \tilde{v} \partial_x P_{k_3, e_2} w] \|_{L^{3/2, 6/5}_{t,x}} := I_1 + I_2.$$

By symmetry we only estimate $I_1$. By Lemma 4.2 we get

$$I_1 \lesssim \sum_{k_4} 2^{k_4/2} 2^{k_4/6} 2^{k_3} \| P_{k_1} u P_{k_2} \tilde{v} \|_{L^3_{t,x}} \| P_{k_3, e_1} w \|_{L^{6/5}_{t,x}}$$

$$\lesssim \sum_{k_1} 2^{k_1/2} 2^{k_2/2} 2^{k_3/2} \| P_{k_1} u \|_{L^3_{t,x}} \| P_{k_2, v} \|_{L^6_{t,x}} \| P_{k_3, w} \|_{L^{6/5}_{t,x}} \lesssim \| u \|_{F^1/2} \| v \|_{F^1/2} \| w \|_{F^1/2}.$$

For the term $II$, we have

$$II \lesssim \sum_{k_1} \| P_{k_4} [\partial_\theta (P_{k_1} u P_{k_2} \tilde{v}) \partial_x P_{k_3} w] \|_{N_{k_4}} + \| P_{k_4} [P_{k_1} u P_{k_2} \tilde{v} \partial_x P_{k_3, e_2} w] \|_{N_{k_4}}$$

$$:= II_1 + II_2.$$

For the term $II_1$, as for the term $I$ we get

$$II_1 \lesssim \sum_{k_4} 2^{k_4/2} 2^{k_3} \| \partial_\theta (P_{k_1} u P_{k_2} \tilde{v}) \|_{L^3_{t,x}} \| P_{k_3} w \|_{L^{6/5}_{t,x}} \lesssim \| u \|_{F^1/2} \| v \|_{F^1/2} \| w \|_{F^1/2}.$$

It remains to estimate the term $II_2$. Note that $[\partial_\theta, \partial_x] = \partial_x$, so we get

$$II_2 \lesssim \sum_{k_1} \| P_{k_4} [P_{k_1} u P_{k_2} \tilde{v} \partial_x P_{k_3} w] \|_{N_{k_4}} + \| P_{k_4} [P_{k_1} u P_{k_2} \tilde{v} \partial_x P_{k_3} \partial_\theta w] \|_{N_{k_4}}.$$

The first term on the righthand side above is just $I$, while the second term can be handled exactly as for $I$.

Case 3: $k_4 > \max(k_1 + 200, k_2 + 100)$.

In this case we have $|k_4 - k_3| \leq 5$. For the term $I$ by Lemma 4.2 and noting that $\| F \|_{L^{2 \infty}_{t,x}} \lesssim \| f \|_{L^{2 \infty}_{t,x}}$, we get

$$I \lesssim \sum_{k_1} 2^{k_4/2} 2^{k_4/2} \| P_{k_4} [P_{k_1} u P_{k_2} \tilde{v} \partial_x P_{k_3} w] \|_{L^{2}_{t,x}}$$

$$\lesssim \sum_{k_1} \| P_{k_1} u P_{k_2} \partial_x w \|_{L^{2}_{t,x}} \| P_{k_2} v \|_{L^{2 \infty}_{t,x}} \lesssim \| u \|_{F^1/2} \| v \|_{F^1/2} \| w \|_{F^1/2}.$$
For the term $II$, we have

$$II \lesssim \sum_{k_i} \|P_{k_i}[P_{k_i}(\partial_\theta u)P_{k_2}\bar{v}\partial_{x_i}P_{k_3}w)]\|_{L^2_v} + \sum_{k_i} \|P_{k_i}[P_{k_i}u(P_{k_2}\bar{v})\partial_{x_i}P_{k_3}w]\|_{L^2_v}$$

$$+ \sum_{k_i} \|P_{k_i}[P_{k_i}uP_{k_2}\bar{v}\partial_\theta\partial_{x_i}P_{k_3}w]\|_{L^2_v} := II_1 + II_2 + II_3.$$ 

For the term $II_1$ we have

$$II_1 \lesssim \sum_{k_i} \|P_{k_i}(\partial_\theta u)P_{k_3}\partial_{x_i}w\|_{L^2_v} \|P_{k_2}v\|_{L^2_{\infty}} \lesssim \|u\|_{F^{1/2,1}} \|v\|_{F^{1/2,1}} \|w\|_{F^{1/2,1}}.$$ 

Similarly, we can bound the term $II_2$. For the term $II_3$, we use the commutator as in Case 2 and then bound as $II_1$. Thus we finish the proof.

5. Schrödinger map in two dimensions

In this section, we prove Theorem 1.3. Consider the Schrödinger maps

$$\partial_\theta s = s \times \Delta_x s, \quad s(0) = s_0, \quad (5.1)$$

where $s : \mathbb{R}^2 \times \mathbb{R} \to S^2 \hookrightarrow \mathbb{R}^3$. Using the stereographic projection

$$u = \frac{s_1 + is_2}{1 + s_3},$$

we see $u$ solves the equation

$$i\partial_t u + \Delta u = \frac{2\bar{u}}{1 + |u|^2} \sum_{i=1}^n (\partial_{x_i} u)^2. \quad (5.2)$$

Conversely, if $u$ solves (5.2), then

$$s = \left(\frac{2\Re u}{1 + |u|^2}, \frac{2\Im u}{1 + |u|^2}, 1 - |u|^2\right)$$

solves (5.1). To prove Theorem 1.3 it suffices to prove

**Theorem 5.1.** Assume $n = 2$, $u_0 \in \dot{B}^{1,1}_{2,1,\theta} \cap \dot{B}^{3/2,1}_{2,1,\theta}$ with $\|u_0\|_{\dot{B}^{1,1}_{2,1,\theta} \cap \dot{B}^{3/2,1}_{2,1,\theta}} = \varepsilon_0 \ll 1$. Then there exists a unique global solution $u$ to (5.2) such that $\|u\|_{F^{1/2,1} \cap F^{3/2,1}} \lesssim \varepsilon_0$. Moreover, the map $u_0 \to u$ is Lipschitz from $\dot{B}^{1,1}_{2,1,\theta} \cap \dot{B}^{3/2,1}_{2,1,\theta}$ to $C(\mathbb{R}; \dot{B}^{1,1}_{2,1,\theta} \cap \dot{B}^{3/2,1}_{2,1,\theta})$, and scattering holds in this space.

As the proof for Theorem 1.2 it reduces to prove the nonlinear estimates. We use Taylor’s expansion to rewrite the nonlinear term: if $\|u\|_\infty < 1$

$$\frac{2\bar{u}}{1 + |u|^2} \sum_{i=1}^n (\partial_{x_i} u)^2 = \sum_{k=0}^\infty 2\bar{u}(-|u|^2)^k \sum_{i=1}^n (\partial_{x_i} u)^2.$$ 

We prove

**Lemma 5.2.** The following estimates holds

$$\|\bar{u}(-|u|^2)^k \sum_{i=1}^n (\partial_{x_i} u)^2\|_{N^{1/2,1} \cap N^{3/2,2}} \lesssim C^{2k} \|u\|_{F^{1/2,1} \cap F^{3/2,1}}^{2k} \|u\|_{F^{1/2,1} \cap F^{3/2,1}} \|u\|_{F^{1/2,1} \cap F^{3/2,1}}.$$ 

The above lemma will follow from the following two lemmas.
Lemma 5.3. We have
\[ \|uv\|_{G^1 \cap G^{3/2}} \lesssim \|u\|_{G^1 \cap G^{3/2}} \|v\|_{G^{1/2} \cap G^{3/2}}. \]

Proof. By the definition we have \( \|u\|_{L^\infty_x B^{1/2}_2} \lesssim \|\partial_x u\|_{L^\infty_x B^{1/2}_2} \lesssim \|u\|_{F^1}. \) Thus by para-product decomposition we can prove that both \( G^1 \) and \( G^1 \cap G^{3/2} \) are algebras under multiplication.

Lemma 5.4. Assume \( i = 1, 2. \) Then
\[ \|u\partial_x v\partial_x w\|_{N^1 \cap N^{3/2}} \lesssim \|u\|_{G^1 \cap G^{3/2}} \|v\|_{F^1 \cap F^{3/2}} \|w\|_{F^{1/2} \cap F^{3/2}}. \]

Proof. By the definition, we have: \( s = 1, 3/2 \)
\[ \|u\partial_x v\partial_x w\|_{N^s} \]
\[ \leq \sum_{k_i} 2^{k_i s}(\|P_{k_4}[P_{k_1}uP_{k_2}\partial_x v\partial_x w]\|_{N_{k_4}} + \|\partial_x P_{k_4}[P_{k_1}uP_{k_2}\partial_x v\partial_x w]\|_{N_{k_4}}) \]
\[ := I + II. \]

We estimate the two terms case by case. By symmetry we may assume \( k_2 \leq k_3 \) in the summation.

Case 1: \( k_3 \leq k_1 + 9 \) or \( k_1 \leq \min(k_1, k_2) + 200. \)
In this case, \( k_4 \leq k_1 + 200. \) Thus we have
\[ I \lesssim \sum_{k_i} 2^{k_i s}(\|P_{k_4}[P_{k_1}uP_{k_2}\partial_x v\partial_x w]\|_{F^{4/3}} \]
\[ \lesssim \sum_{k_i} 2^{k_i s}\|P_{k_4}u\|_{L^4_{x,t}} 2^{2k_2}\|P_{k_2}v\|_{L^4_{x,t}} 2^{2k_3}\|P_{k_3}w\|_{L^4_{x,t}} \]
\[ \lesssim \|u\|_{G^1} \|v\|_{F^1} \|w\|_{F^1}. \]

The estimate for \( II \) is the same as \( I. \)

Case 2: \( k_1 \leq k_3 - 9, k_4 \leq k_2 + 100. \)
In this case we have \( k_1 < k_2 - 100 \) and \( k_3 \leq k_2 + 200. \) Then we get
\[ I \lesssim \sum_{k_i} 2^{k_i s}2^{k_i/6}(\|P_{k_4}[P_{k_1}uP_{k_2}\partial_x v\partial_x w]\|_{L^{3/2}_{x_1}} \]
\[ + \|P_{k_4}[P_{k_1}uP_{k_2}\partial_x v\partial_x w]\|_{L^{3/2}_{x_2}}) := I_1 + I_2. \]

By symmetry we only estimate \( I_1. \) By Lemma 4.2 we get
\[ I_1 \lesssim \sum_{k_i} 2^{k_i s}2^{k_i/6}2^{k_3}\|P_{k_1}uP_{k_2}\partial_x v\|_{L^2_{x,t}} \|P_{k_3}w\|_{L^5_{x_1}} \]
\[ \lesssim \sum_{k_i} 2^{k_i s}2^{k_3+2k_2/2}2^{k_1/2}\|P_{k_1}u\|_{\tilde{G}_{k_1}} \|P_{k_2}v\|_{\tilde{F}_{k_2}} \|P_{k_3}w\|_{\tilde{F}_{k_3}} \]
\[ \lesssim \|u\|_{G^{1/2} \cap G^{3/2}} \|v\|_{F^{1/2} \cap F^{3/2}} \|w\|_{F^{1/2} \cap F^{3/2}}. \]

For the term \( II, \) we can control it similarly.

Case 3: \( k_4 > \max(k_1 + 200, k_2 + 100). \)
In this case we have $|k_4 - k_3| \leq 5$. By Lemma 4.2 we get

$$I \lesssim \sum_{k_i} 2^{k_4} s^{2-k_4/2} \left\| P_{k_4} \left[ P_{k_1} u P_{k_2} \partial_{x_1} \bar{v} \partial_{x_2} P_{k_3} w \right] \right\|_{L^2_t L^2_x} \lesssim \sum_{k_i} 2^{k_4} s^{2-k_4/2} \left\| P_{k_1} u P_{k_3} \partial_{x_1} w \right\|_{L^2_t L^2_x} \lesssim \left\| u \right\|_{G^{1/2} \cap G^{3/2}} \left\| v \right\|_{F^{1/2} \cap F^{3/2}} \left\| w \right\|_{F^{1/2} \cap F^{3/2}}.$$

For the term $II$, we can control it similarly.

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