On the Structure of Octonion Regular Functions

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Abstract. In this paper, we study octonion regular functions and the structural differences between regular functions in octonion, quaternion, and Clifford analyses.

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1. Introduction

In our recent papers [5,6], we started to study octonion algebraic methods in analysis. This work is a continuation of our studies in this fascinating field. Over the years, many results of octonion analysis have been published and studied since the fundamental paper of Dentoni and Sce [2] was published. One thing which has remained unclear to us is what is octonion analysis all about? A consensus has been that octonion, quaternion and Clifford analyses are similar from a theoretical point of view and, maybe for this reason, octonion analysis has received less attention. Our aim is to prove that octonion analysis and Clifford analysis are different theories from the perspective of regular functions. Thus, octonion analysis is a completely independent research topic.

We start by recalling preliminaries of octonions and Clifford numbers and their connections via triality. We define our fundamental function classes, i.e., left-, right- and bi-regular functions. We give characterizations for function classes in biaxial quaternion analysis and in Clifford analysis. The classical Riesz system of Stein and Weiss is used as a familiar reference to clearly see the differences.

The topic of this paper is highly technical, but we have tried to write everything as simply as possible. Hopefully we have succeeded in this job.

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Many questions remain open and the reader may find a lot of open research problems between the lines. We expect to answer some of these questions when the saga continues.

2. Preliminaries: Octonion and Clifford Algebras

In this algebraic part of the paper, we first recall briefly the basic definitions and notations related to octonion and Clifford algebras. In the second place, we study their connections in detail. In the whole paper, our principle is to consider the standard orthonormal basis \{e_0, e_1, \ldots, e_7\} for \(\mathbb{R}^8\) equipped with multiplications which lead to nonisomorphic algebras. We will denote the octonion product by \(e_i \circ e_j\), and the Clifford product by \(e_i e_j\).

2.1. Octonions

The algebra of octonions \(\mathbb{O}\) is the non-commutative and non-associative 8-dimensional algebra with the basis \{1, e_1, \ldots, e_7\}, where \(e_0\) is denoted by 1 (the identity) and omitted whenever clear from the context, and multiplication given by the following table.

\[
\begin{array}{cccccccc}
\circ & 1 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
1 & 1 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\
e_1 & e_1 & -1 & e_3 & -e_2 & e_5 & -e_4 & -e_7 & e_6 \\
e_2 & e_2 & -e_3 & -1 & e_1 & e_6 & e_7 & -e_4 & -e_5 \\
e_3 & e_3 & e_2 & -e_1 & -1 & e_7 & -e_6 & e_5 & -e_4 \\
e_4 & e_4 & -e_5 & -e_6 & -e_7 & -1 & e_1 & e_2 & e_3 \\
e_5 & e_5 & e_4 & -e_7 & e_6 & -e_1 & -1 & -e_3 & e_2 \\
e_6 & e_6 & e_7 & e_4 & -e_5 & -e_2 & e_3 & -1 & -e_1 \\
e_7 & e_7 & -e_6 & e_5 & e_4 & -e_3 & -e_2 & e_1 & 1 \\
\end{array}
\]

Let us point out that there are several ways to define an octonion product such that \(e_0 = 1\). Our choice is historically made, justified by tradition, and, for this reason, we may call it the canonical one. However, for instance, Lounesto uses a different multiplication table in his famous book [8].

For \(1 \leq i, j \leq 7\) we have

\[e_i \circ e_i = e_i^2 = -1, \quad \text{and} \quad e_i \circ e_j = -e_j \circ e_i \quad \text{if} \ i \neq j.\]

An element \(x \in \mathbb{O}\) may be represented in the forms

\[
x = x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4 + x_5 e_5 + x_6 e_6 + x_7 e_7 = x_0 + \underline{x}
\]

\[
= (x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3) + (x_4 + x_5 e_1 + x_6 e_2 + x_7 e_3) \circ e_4
\]

\[
= u + v \circ e_4
\]

\[
= (u_0 + u) + (v_0 + v) \circ e_4.
\]

Here, \(x_0, \ldots, x_7 \in \mathbb{R}\), \(x_0\) is the real part, \(\underline{x}\) is the vector part, and \(u\) and \(v \in \mathbb{H}\) are quaternions. The last form is called the quaternion form of an octonion.
The conjugate of \( x \) is denoted and defined by \( \bar{x} = x_0 - \sum_i x_i e_i \). Furthermore, the product of two octonions can be written as

\[
x \circ y = (x_0 + \bar{x}) \circ (y_0 + \bar{y}) = \sum_{i,j=0}^{7} x_i y_j e_i \circ e_j = \sum_{i=0}^{7} x_i y_i e_i^2 + \sum_{i,j=0}^{7} i \neq j x_i y_j e_i \circ e_j
\]

\[
= x_0 y_0 - \sum_{i=1}^{7} x_i y_i + x_0 \sum_{i=1}^{7} y_i e_i + y_0 \sum_{i=1}^{7} x_i e_i + \sum_{i,j=1}^{7} i \neq j x_i y_j e_i \circ e_j
\]

\[
= x_0 y_0 - x \cdot y + x_0 y + y_0 x + x \times y,
\]

where \( x \cdot y \) is the dot product and \( x \times y \) the cross product of vectors \( x \) and \( y \) in \( \mathbb{R}^7 \).

Denote the quaternion forms of the octonions \( x \) and \( y \) by

\[
x = (u_0 + u) + (v_0 + v) \circ e_4,
\]

\[
y = (a_0 + a) + (b_0 + b) \circ e_4.
\]

In Lemma 2.3 we will write the cross product \( x \times y \) of octonion vector parts \( x \) and \( y \) using the classical 3-dimensional cross products of the vector parts \( u, v, a, \) and \( b \) of quaternions (see, e.g., \([3,8,10]\)).

**Lemma 2.1.** ([5, Lemma 2.10]) Let \( u, v \in \mathbb{H} \). Then

\[
e_4 \circ u = \bar{u} \circ e_4,
\]

\[
e_4 \circ (u \circ e_4) = -\bar{u},
\]

\[
(u \circ e_4) \circ e_4 = -u,
\]

\[
u \circ (v \circ e_4) = (v \circ u) \circ e_4,
\]

\[
(u \circ e_4) \circ v = (u \circ \bar{v}) \circ e_4,
\]

\[
(u \circ e_4) \circ (v \circ e_4) = -\bar{v} \circ u.
\]

**Lemma 2.2.** If \( x, y \in \mathbb{O} \) are written as in \((2)\), then

\[
\bar{u} \times e_4 = u \circ e_4,
\]

\[
e_4 \times a = -a \circ e_4,
\]

\[
u \times (b \circ e_4) = -(u \times b) \circ e_4 - (u \cdot b)e_4,
\]

\[
e_4 \times (b \circ e_4) = b,
\]

\[
(v \circ e_4) \times a = -(v \times a) \circ e_4 + (a \cdot v)e_4,
\]

\[
(v \circ e_4) \times e_4 = -\bar{v},
\]

\[
(v \circ e_4) \times (b \circ e_4) = -v \times b.
\]

**Proof.** On the one hand, the first two equalities are direct consequences of \((1)\). On the other hand, Lemma 2.1 implies

\[
e_i \circ (e_j \circ e_4) = (e_j \circ e_i) \circ e_4,
\]

\[
e_4 \circ (e_j \circ e_4) = e_j,
\]

\[
(e_i \circ e_4) \circ e_j = -(e_i \circ e_j) \circ e_4,
\]
for \( 1 \leq i, j \leq 3 \). Then

\[
\mathbf{u} \times (\mathbf{b} \circ e_4) = \sum_{i,j=1}^{3} u_i b_j e_i \circ (e_j \circ e_4) = \left( \sum_{i,j=1}^{3} u_i b_j e_i \circ e_i \right) \circ e_4
\]

\[
= \left( - \sum_{i,j=1}^{3} u_i b_j e_i \circ e_j - \sum_{i=1}^{3} u_i b_i \right) \circ e_4 = -(\mathbf{u} \times \mathbf{b}) \circ e_4 - (\mathbf{u} \cdot \mathbf{b}) e_4,
\]

\[
e_4 \times (\mathbf{b} \circ e_4) = \sum_{j=1}^{3} b_j e_4 \circ (e_j \circ e_4) = \sum_{j=1}^{3} b_j e_j = \mathbf{b},
\]

\[
(\mathbf{v} \circ e_4) \times \mathbf{a} = \sum_{i,j=1}^{3} v_i a_j (e_i \circ e_4) \circ e_j = \left( - \sum_{i,j=1}^{3} v_i a_j e_i \circ e_j \right) \circ e_4
\]

\[
= \left( - \sum_{i,j=1}^{3} v_i a_j e_i \circ e_j + \sum_{i=1}^{3} v_i a_i \right) \circ e_4 = -(\mathbf{v} \times \mathbf{a}) \circ e_4 + (\mathbf{a} \cdot \mathbf{v}) e_4,
\]

\[
(\mathbf{v} \circ e_4) \times e_4 = \sum_{i=1}^{3} v_i (e_i \circ e_4) \circ e_4 = - \sum_{i=1}^{3} v_i e_i = -\mathbf{v},
\]

and

\[
(\mathbf{v} \circ e_4) \times (\mathbf{b} \circ e_4) = \sum_{i,j=1}^{3} v_i b_j (e_i \circ e_4) \circ (e_j \circ e_4) = \sum_{i,j=1}^{3} v_i b_j e_j \circ e_i
\]

\[
= - \sum_{i,j=1}^{3} v_i b_j e_i \circ e_j
\]

\[
= -\mathbf{v} \times \mathbf{b}.
\]

**Lemma 2.3.** Consider the quaternion representations of the vectors \( \mathbf{x} \) and \( \mathbf{y} \in \mathbb{O} \) as in (2). Then the cross product in quaternion form is

\[
\mathbf{x} \times \mathbf{y} = v_0 b - v_0 b + u \times a - \mathbf{v} \times \mathbf{b} \quad \in \text{span}\{e_1, e_2, e_3\}
\]

\[
+ (\mathbf{v} \cdot \mathbf{a} - \mathbf{u} \cdot \mathbf{b}) e_4 \quad \in \text{span}\{e_4\}
\]

\[
+ (w b_0 - v_0 a - u \times b - \mathbf{v} \times \mathbf{a}) \circ e_4 \quad \in \text{span}\{e_5, e_6, e_7\}
\]

**Proof.** By Lemma 2.2, we obtain

\[
\mathbf{x} \times \mathbf{y} = \mathbf{u} \times \mathbf{a} + \mathbf{u} \times (b_0 e_4) + \mathbf{u} \times (b \circ e_4)
\]

\[
+ (v_0 e_4) \times \mathbf{a} + (v_0 e_4) \times (b_0 e_4) + (v_0 e_4) \times (b \circ e_4)
\]

\[
+ (\mathbf{v} \circ e_4) \times \mathbf{a} + (\mathbf{v} \circ e_4) \times (b_0 e_4) + (\mathbf{v} \circ e_4) \times (b \circ e_4)
\]
\[ u \times a + ub_0 \circ e_4 - (u \times b) \circ e_4 \]
\[ - v_0 a \circ e_4 + 0 + v_0 b \]
\[ - (v \times a) \circ e_4 - vb_0 - v \times b \]
\[ = u \times a - v \times b + v_0 b - vb_0 \]
\[ + (v \cdot a - u \cdot b)e_4 \]
\[ + (ub_0 - v_0 a - u \times b - v \times a) \circ e_4. \]

\[ \square \]

**Corollary 2.4.** If \( x, y \in \mathbb{O} \) are written as in (2), then
\[ x \circ y = u_0a_0 - vb_0 + u \cdot a - v \cdot b \in \mathbb{R} \]
\[ + u_0a_0 + a_0u + vb_0 - vb_0 + u \times a - v \times b \in \text{span}\{e_1, e_2, e_3\} \]
\[ + (u_0b_0 + a_0v_0 + v \cdot a - u \cdot b)e_4 \in \text{span}\{e_4\} \]
\[ + (u_0a + a_0v - ub_0 - va_0 - u \times b - v \times a) \circ e_4 \in \text{span}\{e_5, e_6, e_7\} \]

2.2. The Clifford Algebra \( \mathcal{C}\ell_{0,7} \) and Triality

Since the dimension of the underlying vector space of the octonions and Clifford paravectors is 8, they behave similarly as vector spaces. Moreover, we may ask if there is a connection between the octonion product and the Clifford product? The answer is given by Pertti Lounesto in his book [8]. We will recall his ideas here in detail. Let us recall the basic definitions and properties of Clifford algebras.

We continue working with the basis \( \{e_0, e_1, \ldots, e_7\} \) for \( \mathbb{R}^8 \). The Clifford product is defined by
\[ e_i e_j + e_j e_i = -2\delta_{ij}, \quad i, j = 1, \ldots, 7, \]
where \( \delta_{ij} \) is the Kronecker delta symbol. Here, \( e_0 = 1 \). Then, similarly to the case of octonions, \( e_0^2 = 1 \), and \( e_j^2 = -1 \) for all \( j = 1, \ldots, 7 \). The Clifford product \( e_i e_j \) is not necessarily a vector or a scalar. This product generates an associative algebra, called the Clifford algebra, denoted by \( \mathcal{C}\ell_{0,7} \). The dimension of this Clifford algebra is \( 2^7 \), and an element \( a \in \mathcal{C}\ell_{0,7} \) may be represented as a sum
\[ a = \sum_{j=0}^{7} [a]_j \]
of a scalar part \( [a]_0 \), generated by 1, a 1-vector part \( [a]_1 \), generated by \( e_j \)'s, a 2-vector part \( [a]_2 \), generated by the products \( e_i e_j \), where \( 1 \leq i < j \leq 7 \), etc. Clifford numbers of the form \( [a]_1 \) are called vectors and those of the form \( [a]_{0,1} = [a]_0 + [a]_1 \) are called paravectors. The set of paravectors may be identified with \( \mathbb{R}^8 \).

The Clifford product of two paravectors \( x \) and \( y \) can be written as
\[ xy = (x_0 + x)(y_0 + y) = \sum_{i,j=0}^{7} x_i y_j e_i e_j \]
\[ = \sum_{i=0}^{7} x_i y_i e_i^2 + \sum_{i,j=0}^{7} x_i y_j e_i e_j \]
\[ = x_0y_0 - \sum_{i=1}^{7} x_iy_i + x_0 \sum_{i=1}^{7} y_ie_i + y_0 \sum_{i=1}^{7} x_ie_i + \sum_{i,j=1}^{7} x_iy_je_ie_j \]
\[ = x_0y_0 - \bar{x} \cdot y + x_0y_0 \bar{x} + \bar{x} \wedge y, \quad (3) \]

where \( \bar{x} \wedge y \) is the wedge product of vectors \( \bar{x} \) and \( y \). In particular, \( \bar{x}y = \bar{x} \wedge y - \bar{x} \cdot y \).

The reader can see that formally the octonion and the Clifford products are similar, and a reasonable question is “how they are connected?”. We would like to construct the octonion product using the Clifford algebra \( \mathbb{C} \ell_{0,7} \).

Let us consider the octonion product of the basis elements \( e_i \) and \( e_j \), where \( 1 \leq i, j \leq 7, i \neq j \):

\[ e_i \circ e_j = e_k. \]

Then \( 1 \leq k \leq 7, \) and \( i \neq k \neq j \). The corresponding Clifford product \( e_i e_j \) may be mapped to \( e_k \) by multiplying it by the trivector \( e_j e_i e_k \), i.e.,

\[ (e_i e_j)(e_j e_i e_k) = e_i^2 e_j^2 e_k = e_k. \]

Using the same trivector, \( e_j e_i \) is mapped to \(-e_k\). If \( \bar{a} \) and \( \bar{b} \) are vectors, then

\[ \bar{a} \bar{b}(e_j e_i e_k) = (a_i b_j - a_j b_i)e_k + [\bar{a} \bar{b}(e_j e_i e_k)]_3 + [\bar{a} \bar{b}(e_j e_i e_k)]_5. \]

Picking the 1-vector part

\[ [\bar{a} \bar{b}(e_j e_i e_k)]_1 = (a_i b_j - a_j b_i)e_k, \]

we get a part of the \( k \)th component of the octonion product \( \bar{a} \circ \bar{b} \). Using this idea, we may express the octonion product \( \bar{a} \circ \bar{b} \) as the paravector part of the Clifford product \( \bar{a} \bar{b}(1 - W) \), where \( W \) is a suitable 3-vector.

**Lemma 2.5.** ([8, Sec 23.3], [12, Lemma 4.1]) Define

\[ W = e_{123} + e_{145} + e_{176} + e_{246} + e_{257} + e_{347} + e_{365}. \]

Let \( a = a_0 + \bar{a} \) and \( b = b_0 + \bar{b} \) be paravectors. Then

\[ a \circ b = [\bar{a} \bar{b}(1 - W)]_{0,1} \]

and in particular, \( a \times b = -[(a \wedge \bar{b})W]_{1,1} \).

Lounesto states Lemmas 2.5 and 2.7 without proofs at pages 303–304 in [8]. He uses a different multiplication table of octonions, and therefore the seven basis vectors \( e_{ijk} \) have different indices \( i, j, \) and \( k \) in his trivector \( W \). Venäläinen gives a proof for Lemma 2.5 in her licentiate thesis [12]. For the convenience of the reader, we give a proof of Lemma 2.5 here.

**Proof.** To begin with, we have

\[ [\bar{a} \bar{b}(1 - W)]_{0,1} = [\bar{a} \bar{b}]_{0,1} - [\bar{a} \bar{b}W]_{1} = a_0 b_0 - a \cdot \bar{b} + a_0 \bar{b} + b_0 a - [\bar{a} \bar{b}W]_1. \]

By (1) and (3), it is enough to show that \( \bar{a} \times \bar{b} = -[(a \wedge \bar{b})W]_{1,1} \). Consider the triplets

\[ \nu = 123, 145, 176, 246, 257, 347, 365. \]
The product $e_i e_j e_\nu$ is a vector only if the pair of indices $ij$ belongs to the triplet $\nu$. Since the cross and the wedge products

$$a \times b = \sum_{i,j=1 \atop i<j}^7 (a_i b_j - a_j b_i)e_i \circ e_j \quad \text{and} \quad a \wedge b = \sum_{i,j=1 \atop i<j}^7 (a_i b_j - a_j b_i)e_i e_j$$

have the same coefficients, and each pair $ij$, $1 \leq i < j \leq 7$, is contained in exactly one of the triplets $\nu$, say $\nu_0$, it is enough to check that $e_i e_j e_{\nu_0} = -e_i \circ e_j$ for all such pairs $ij$.

A straightforward computation shows:

**Lemma 2.6.** The trivector $W = e_{123} + e_{145} + e_{176} + e_{246} + e_{257} + e_{347} + e_{365}$ is invertible with

$$W^{-1} = \frac{1}{7}(W - 6e_{12...7}).$$

In the above, we identified octonions $\mathbb{O}$ with the 8-dimensional paravectors. The dimension 8 plays a special role in the theory of spin groups, since $\text{Spin}(8)$ has the so-called *exceptional automorphisms*. This feature is called *triality*, and the first time it was noticed was in the book of Study [11]. For modern references, see [4, 8, 10]. The triality means that in addition to paravectors, we may identify octonions with the spinor spaces $S^\pm$. The spinor spaces may be realized by

$$S^\pm = C \ell_{0,7} I^\pm,$$

where $I^\pm$ is a primitive idempotent,

$$I^\pm = \frac{1}{16}(1 + We_{12...7})(1 \pm e_{12...7}), \quad (4)$$

see [1, 8]. A straightforward computation shows that the octonion product in spinor spaces may be determined as follows.

**Lemma 2.7.** [8, Sec 23.3] For paravectors $a$ and $b$, we have

$$a \circ b = 16[abI^-]_{0,1}. \quad (5)$$

### 3. Octonion Analysis

In this section, we recall basic facts of octonion analysis, i.e., the theory of Cauchy–Riemann operators in the octonionic setting. After that, we carefully study the general structure of the null solutions of these operators and define four different classes of regular functions: left-, right-, B-, and R-regular functions. R-regular functions are just solutions of the classical Riesz system. We use the Riesz system here as a familiar reference to better understand the structure of octonion regular functions. In Clifford analysis the corresponding function classes are equal. This structural difference is a fundamental difference between octonion and Clifford analyses.
3.1. Cauchy–Riemann Operators

A function \( f : \mathbb{R}^8 \to \mathbb{O} \) is of the form \( f = f_0 + f_1 e_1 + \cdots + f_7 e_7 = f_0 + f \), where \( f_j : \mathbb{R}^8 \to \mathbb{R} \). We define the Cauchy–Riemann operator

\[
D_x = \partial x_0 + e_1 \partial x_1 + \cdots + e_7 \partial x_7.
\]

Its vector part,

\[
D_x = e_1 \partial x_1 + \cdots + e_7 \partial x_7
\]

is called the Dirac operator. If the coordinate functions of \( f \) have partial derivatives, then \( D_x \) operates on \( f \) from the left and from the right as

\[
D_x f = \sum_{i,j=0}^7 e_i \partial x_j f_j \quad \text{and} \quad f D_x = \sum_{i,j=0}^7 e_j \partial x_i f_j.
\]

Decomposition (1) gives (see [7])

\[
D_x f = \partial x_0 f_0 - D_x \cdot f + \partial x_0 f + D_x f_0 + D_x \times f \quad \text{and} \quad f D_x = \partial x_0 f_0 - D_x \cdot f + \partial x_0 f + D_x f_0 - D_x \times f,
\]

where \( \partial x_0 f_0 - D_x \cdot f \) is the divergence of \( f \) and \( D_x \times f \) is the rotor of \( f \).

If \( D_x f = 0 \) (resp., \( f D_x = 0 \)), then \( f \) is called left (resp., right) regular.

In Clifford analysis one studies functions \( f : \mathbb{R}^8 \to \mathbb{C} \ell_{0,7} \). We define the Cauchy–Riemann operator similarly as in octonion analysis:

\[
\partial x = \partial x_0 + e_1 \partial x_1 + \cdots + e_7 \partial x_7 = \partial x_0 + \partial x.
\]

Functions satisfying \( \partial x f = 0 \) (resp., \( f \partial x = 0 \)) on \( \mathbb{R}^8 \) are called left (resp., right) monogenic. In this paper we only need to consider paravector valued functions

\[
f = f_0 + f_1 e_1 + \cdots + f_7 e_7.
\]

3.2. Left-, Right-, B- and R-Regular Functions

Comparing the real and vector parts in (6) and (7) yields the following well known results.

**Proposition 3.1.** A function \( f : \mathbb{R}^8 \to \mathbb{O} \) is left regular if and only if it satisfies the Moisil–Teodorescu type system

\[
\partial x_0 f_0 - D_x \cdot f = 0, \\
\partial x_0 f + D_x f_0 + D_x \times f = 0,
\]

(8)
whose componentwise form is
\[
\begin{align*}
\partial_{x_0}f_0 - \partial_1f_1 - \ldots - \partial_7f_7 &= 0, \\
\partial_{x_0}f_1 + \partial_1f_0 + \partial_2f_3 - \partial_3f_2 + \partial_4f_5 - \partial_5f_4 - \partial_6f_7 + \partial_7f_6 &= 0, \\
\partial_{x_0}f_2 + \partial_2f_0 - \partial_1f_3 + \partial_3f_1 + \partial_4f_6 - \partial_6f_4 + \partial_5f_7 - \partial_7f_5 &= 0, \\
\partial_{x_0}f_3 + \partial_3f_0 + \partial_1f_2 - \partial_2f_1 + \partial_4f_7 - \partial_7f_4 - \partial_5f_6 + \partial_6f_5 &= 0, \\
\partial_{x_0}f_4 + \partial_4f_0 - \partial_1f_5 + \partial_5f_1 - \partial_2f_6 + \partial_6f_2 - \partial_3f_7 + \partial_7f_3 &= 0, \\
\partial_{x_0}f_5 + \partial_5f_0 + \partial_1f_4 - \partial_4f_1 - \partial_2f_7 + \partial_7f_2 + \partial_3f_6 - \partial_6f_3 &= 0, \\
\partial_{x_0}f_6 + \partial_6f_0 + \partial_1f_7 - \partial_7f_1 + \partial_2f_4 - \partial_4f_2 - \partial_3f_5 + \partial_5f_3 &= 0, \\
\partial_{x_0}f_7 + \partial_7f_0 - \partial_1f_6 + \partial_6f_1 + \partial_2f_5 - \partial_5f_2 + \partial_3f_4 - \partial_4f_3 &= 0.
\end{align*}
\]

We will denote the space of left regular functions by $\mathcal{M}^{(l)}$, and similarly, right regular functions by $\mathcal{M}^{(r)}$.

**Proposition 3.2.** A function $f: \mathbb{R}^8 \to \mathbb{O}$ is both left and right regular if and only if it satisfies the system
\[
\begin{align*}
\partial_{x_0}f_0 - D_{\underline{x}} \cdot f &= 0, \\
\partial_{x_0}f + D_{\underline{x}}f_0 &= 0, \\
D_{\underline{x}} \times f &= 0,
\end{align*}
\]
whose componentwise form is
\[
\begin{align*}
\partial_{x_0}f_0 - \partial_1f_1 - \ldots - \partial_7f_7 &= 0, \\
\partial_{x_0}f_i + \partial_1f_0 &= 0, \quad i = 1, \ldots, 7, \\
\partial_{x_2}f_3 - \partial_{x_3}f_2 + \partial_{x_4}f_5 - \partial_{x_5}f_4 - \partial_{x_6}f_7 + \partial_{x_7}f_6 &= 0, \\
- \partial_{x_1}f_3 + \partial_{x_3}f_1 + \partial_{x_4}f_6 - \partial_{x_6}f_4 + \partial_{x_5}f_7 - \partial_{x_7}f_5 &= 0, \\
\partial_{x_1}f_2 - \partial_{x_2}f_1 + \partial_{x_4}f_7 - \partial_{x_7}f_4 - \partial_{x_5}f_6 + \partial_{x_6}f_5 &= 0, \\
- \partial_{x_1}f_5 + \partial_{x_5}f_1 - \partial_{x_2}f_6 + \partial_{x_6}f_2 - \partial_{x_3}f_7 + \partial_{x_7}f_3 &= 0, \\
\partial_{x_1}f_4 - \partial_{x_4}f_1 - \partial_{x_2}f_7 + \partial_{x_7}f_2 + \partial_{x_3}f_6 - \partial_{x_6}f_3 &= 0, \\
\partial_{x_1}f_7 - \partial_{x_7}f_1 + \partial_{x_2}f_4 - \partial_{x_4}f_2 - \partial_{x_3}f_5 + \partial_{x_5}f_3 &= 0, \\
- \partial_{x_1}f_6 + \partial_{x_6}f_1 + \partial_{x_2}f_5 - \partial_{x_5}f_2 + \partial_{x_3}f_4 - \partial_{x_4}f_3 &= 0.
\end{align*}
\]

We will call functions satisfying (11) \emph{B-regular}, and denote the space of such functions by $\mathcal{M}_B$. Naturally
\[
\mathcal{M}_B = \mathcal{M}^{(l)} \cap \mathcal{M}^{(r)}.
\]

The fundamental difference between octonion and Clifford analyses is that in Clifford analysis the paravector valued null solutions to the Cauchy–Riemann operator satisfy the Riesz system and are at the same time left and right monogenic, which is not true in octonion analysis. The following well-known proposition follows from the definitions, similarly as in octonion analysis, by comparing the scalar parts, 1-vector parts, and 2-vector parts.
Proposition 3.3. Suppose $f: \mathbb{R}^8 \to \mathbb{C}\ell_{0,7}$ is a paravector valued function. Then $\partial_x f = 0$ if and only if $f \partial_x = 0$, and this is equivalent to $f$ satisfying the Riesz-system

$$\begin{align*}
\partial_{x_0} f_0 - \partial_x \cdot f &= 0, \\
\partial_{x_0} f + \partial_x f_0 &= 0, \\
\partial_x \wedge f &= 0,
\end{align*}$$

(12)

whose componentwise form is

$$\begin{align*}
\partial_{x_0} f_0 - \partial_x f_1 - \ldots - \partial_x f_7 &= 0, \\
\partial_{x_0} f_i + \partial_x f_0 &= 0, \quad i = 1, \ldots, 7, \\
\partial_x f_j - \partial_x f_i &= 0, \quad i, j = 1, \ldots, 7, \quad i \neq j,
\end{align*}$$

(13)

Functions satisfying (13) are called $R$-regular, and the space of such functions is denoted by $\mathcal{M}_R$.

To convince the reader about the existence of these function classes, we recall the following classical method from Clifford analysis.

Remark 3.4. (Cauchy–Kovalevskaya extension) If $f: \Omega \to \mathbb{R}$ is a real analytic function defined on an open $\Omega \subset \mathbb{R}^7 \cong \mathbb{O} \cap \{x_0 = 0\}$ we may construct its Cauchy–Kovalevskaya extension analogously to Clifford analysis (see [1]) by defining

$$CK[f](x) = e^{-x_0 D_x f}(x).$$

It is easy to see that since $f$ is real valued, $D_x CK[f] = CK[f] D_x = 0$, i.e., $CK[f] \in \mathcal{M}_B$. Since $\mathbb{O}$ is an alternative division algebra, that is $x(xy) = x^2 y$ for all $x, y \in \mathbb{O}$, the Cauchy–Kovalevskaya extension may be extended to octonion valued real analytic functions. A necessary condition for $CK[f] \in \mathcal{M}^{(\ell)}$ is that $f$ is an octonion valued real analytic function with $\neq 0$. This condition is not sufficient since, e.g.,

$$CK[x](x) = 7x_0 + x$$

belongs to $\mathcal{M}_B$.

We may conclude that although Clifford and octonion analyses have formally very similar definitions, the corresponding function spaces are different.

Proposition 3.5. $\mathcal{M}_R \subsetneq \mathcal{M}_B \subsetneq \mathcal{M}^{(\ell)}$

Proof. Two inclusions follow from Propositions 3.1–3.3. The examples showing that the inclusions are strict, respectively, are: if $f = x_2 e_1 - x_7 e_4$, then $D_x f = f D_x = 0$, but $\partial_{x_2} f_1 - \partial_{x_1} f_2 = 1 \neq 0$, and if $f = x_1 - x_2 e_3$, then $D_x f = 0$, but $f D_x = 2e_1 \neq 0$. }

This result is crucial in understanding the fundamental character of octonion analysis and the structural differences between octonion, quaternion, and Clifford analyses.
Remark 3.6. (Quaternion analysis) If we make the corresponding definitions for quaternion regular function classes by considering the Cauchy–Riemann operator $D_x = \partial x_0 + e_1 \partial x_1 + e_2 \partial x_2 + e_3 \partial x_3$ acting on quaternion valued functions $f = f_1 + f_1 e_1 + f_2 e_2 + f_3 e_3$, then, by comparing (11) and (13), we observe immediately that

$$\mathcal{M}_R = \mathcal{M}_B \subset \mathcal{M}(\ell).$$

Remark 3.7. (Clifford analysis) If we make the corresponding definitions for paravector valued monogenic functions, then, by Proposition 3.3,

$$\mathcal{M}_R = \mathcal{M}_B = \mathcal{M}(\ell).$$

4. Function Classes in Biaxial Quaternion Analysis

In the preceding section we gave characterizations for left-, B-, and R-regular functions using componentwise and vector forms. In this section we write the three systems of Sect. 3.2 in quaternion forms. The use of the quaternion forms of the function and the Cauchy–Riemann operator is called the biaxial quaternion analysis.

Consider the Cauchy–Riemann operator $D_x$ and the function $f: \mathbb{R}^8 \to \mathbb{O}$ in the quaternion forms:

$$D_x = \partial u_0 + \partial u + (\partial v_0 + \partial v) \circ e_4,$$

$$f = g_0 + g + (h_0 + h) \circ e_4.$$  

According to Corollary 2.4, we can write

$$(D_x f) = \partial u_0 g_0 + \partial v_0 h_0 + \partial u g - \partial u \cdot h = 0,$$

$$(\partial u_0 h_0 + \partial v_0 g_0 + \partial u \cdot g - \partial u \cdot h) e_4 = 0,$$

$$(\partial u_0 h + \partial v_0 g + \partial u h_0 - \partial v_0 g - \partial u \times h - \partial u \times g) \circ e_4 = 0.$$  

This implies the quaternion forms (14) and (15) of, respectively, the Moisil–Teodorescu type system (8) and the system (10).

Proposition 4.1. $f: \mathbb{R}^8 \to \mathbb{O}$ is left regular if and only if it satisfies the system

$$\partial u_0 h_0 + \partial v_0 g_0 + \partial u \cdot g - \partial u \cdot h = 0,$$

$$\partial u_0 g_0 - \partial v_0 h_0 - \partial u \cdot g - \partial u \cdot h = 0,$$

$$\partial u_0 g + \partial u_0 h_0 + \partial v_0 h - \partial u \cdot g - \partial u \cdot h = 0,$$

$$\partial u_0 h + \partial v_0 g + \partial u_0 h_0 - \partial v_0 g - \partial u \times h - \partial u \times g = 0.$$  

Proposition 4.2. $f: \mathbb{R}^8 \to \mathbb{O}$ is left and right regular if and only if it satisfies the system

$$\partial u_0 h_0 + \partial v_0 g_0 + \partial u \cdot g - \partial u \cdot h = 0,$$

$$\partial u_0 g_0 - \partial v_0 h_0 - \partial u \cdot g - \partial u \cdot h = 0,$$

$$\partial u_0 g + \partial u_0 h_0 + \partial v_0 h - \partial u \cdot g - \partial u \cdot h = 0,$$

$$\partial u_0 h + \partial v_0 g + \partial u_0 h_0 - \partial v_0 g - \partial u \times h - \partial u \times g = 0.$$  

\[
\begin{align*}
\partial_{u_0} h_0 + \partial_{v_0} g_0 + \partial_u \cdot g - \partial_u \cdot h &= 0, \\
\partial_{u_0} g_0 - \partial_{v_0} h_0 - \partial_u \cdot g - \partial_u \cdot h &= 0, \\
\partial_{u_0} g + \partial_{v_0} h + \partial_u \cdot g - \partial_u \cdot h &= 0, \\
\partial_{u_0} h + \partial_{v_0} g + \partial_u h_0 - \partial_v g_0 &= 0, \\
\partial_u \times g - \partial_u \times h &= 0, \\
\partial_u \times h + \partial_v \times g &= 0.
\end{align*}
\]

(15)

One example of the use of biaxial quaternion analysis is the proof of the following vector calculus identity in the octonionic case.

**Lemma 4.3.** Let the coordinates of \( f: \mathbb{R}^8 \to \mathbb{O} \) and \( g: \mathbb{R}^8 \to \mathbb{O} \) have partial derivatives. Then

\[
D_x \cdot (f \times g) = (D_x \times f) \cdot g - f \cdot (D_x \times g).
\]

**Proof.** We use quaternion decompositions

\[
D_x = \partial_u + \partial_{v_0} e_4 + \partial_u \circ e_4,
\]

\[
f = f_1 + F_0 e_4 + F_1 \circ e_4,
\]

\[
g = g_1 + G_0 e_4 + G_1 \circ e_4.
\]

On the left-hand side we apply Lemma 2.3 to the cross product \( f \times g \), and use the classical vector calculus identity

\[
\nabla : (u \times v) = (\nabla \times u) \cdot v - u \cdot (\nabla \times v)
\]

for \( u, v: \mathbb{R}^3 \to \mathbb{R}^3 \):

\[
D_x \cdot (f \times g) = (\partial_u + \partial_{v_0} e_4 + \partial_u \circ e_4) \cdot (F_0 G_1 - F_1 G_0 + f_1 \times g_1 - F_1 \times G_1)
\]

\[
+ (F_1 \circ g_1 - f_1 \circ G_1) e_4
\]

\[
+ (f_1 G_0 - F_0 g_1 - f_1 \times G_1 - F_1 \times g_1) \circ e_4
\]

\[
= \partial_u \cdot (F_0 G_1) - \partial_u \cdot (F_1 G_0) + \partial_u \cdot (f_1 \times g_1) - \partial_u \cdot (F_1 \times G_1)
\]

\[
+ \partial_{v_0} (F_1 \circ g_1) - \partial_{v_0} (f_1 \circ G_1)
\]

\[
+ \partial_{u_0} (f_1 G_0) - \partial_{u_0} (F_0 g_1) - \partial_{u_0} (f_1 \times G_1) - \partial_{u_0} (F_1 \times g_1)
\]

\[
= (\partial_u F_0) \cdot G_1 + F_0 (\partial_u \cdot G_1) - (\partial_u \cdot F_1) G_0 - F_1 \cdot (\partial_u G_0)
\]

\[
+ (\partial_{v_0} f_1) \cdot g_1 - f_1 \cdot (\partial_{u_0} \times g_1) - (\partial_u \times F_1) \cdot G_1 + F_1 \cdot (\partial_u \times G_1)
\]

\[
+ (\partial_{u_0} F_1) \cdot g_1 + F_1 \cdot (\partial_{v_0} g_1) - (\partial_u f_1) \cdot G_1 - f_1 \cdot (\partial_u G_1)
\]

\[
+ (\partial_{u_0} F_1) G_0 + f_1 \cdot (\partial_u G_0) - (\partial_u F_0) \cdot g_1 - F_0 (\partial_u \times g_1)
\]

\[
- (\partial_{u_0} f_1) \cdot G_1 + f_1 \cdot (\partial_u \times G_1) - (\partial_u \times F_1) \cdot g_1 + F_1 + (\partial_u \times g_1).
\]

On the right-hand side we apply Lemma 2.3 to the rotors \( D_x \times f \) and \( D_x \times g \):

\[
(D_x \times f) \cdot g = (\partial_{v_0} F_1 - \partial_u F_0 + \partial_u \times f_1 - \partial_u \times F_1)
\]

\[
+ (\partial_{u_0} F_0) - \partial_{u_0} F_1 \circ e_4
\]

\[
+ (\partial_{u_0} F_0 - \partial_{v_0} f_1) \circ e_4)
\]

\[
(g_1 + G_0 e_4 + G_1 \circ e_4)
\]

\[
= (\partial_{v_0} F_1) \cdot g_1 - (\partial_u F_0) \cdot g_1 + (\partial_u \times f_1) \cdot g_1 - (\partial_u \times F_1) \cdot g_1
\]
\[ + (\partial_u \cdot f_1)G_0 - (\partial_u \cdot F_1)G_0 \]
\[ + (\partial_u F_0) \cdot G_1 - (\partial_v \cdot f_1) \cdot G_1 - (\partial_u \times f_1) \cdot G_1, \]

and
\[ \ddot{f} \cdot (D_x \times g) = (f_1 + F_0 e_4 + F_1 \circ e_4) \cdot \]
\[ (\partial_{v_0} G_1 - \partial_u G_0 + \partial_u \times g_1 - \partial_u \times G_1) \]
\[ + (\partial_u \cdot g_1 - \partial_u \cdot G_1) e_4 \]
\[ + (\partial_u G_0 - \partial_{v_0} g_1 - \partial_u \times G_1 - \partial_u \times g_1) \circ e_4 \]
\[ = f_1 \cdot (\partial_{v_0} G_1) - f_1 \cdot (\partial_u G_0) + f_1 \cdot (\partial_u \times g_1) - f_1 \cdot (\partial_u \times G_1) \]
\[ + F_0(\partial_u \cdot g_1) = 0 \]
\[ + F_1 \cdot (\partial_u G_0) - F_1 \cdot (\partial_{v_0} g_1) - F_1 \cdot (\partial_u \times G_1) - F_1 \cdot (\partial_u \times g_1) . \]

\[ \square \]

**Remark 4.4.** (The set of Regular functions is not a module) In quaternion analysis \( \partial_u g = 0 \) implies \( \partial_u (g \circ a) = 0 \) for all \( a \in \mathbb{H} \) (see Lemma 4.5). The same does not hold in octonion analysis. For example, define \( g: \mathbb{H} \to \mathbb{H} \)
\[ g(x) = x_1 - x_2 e_3. \]
Then \( D_x g = e_1 - e_2 e_3 = 0 \), but \( D_x (g \circ e_4) = D_x (x_1 e_4 - x_2 e_7) = e_1 e_4 - e_2 e_7 = 2 e_5. \)

For quaternion functions we have the product rules (16) and (17) for the Cauchy–Riemann operator. Remark 4.4 suggests that we do not have any kind of a non-trivial product rule for octonion valued functions. In practice, one way to compute \( D_x (fg) \) for octonion valued functions is to use biaxial quaternion analysis, and then to apply (16)–(24).

**Lemma 4.5.** ([3, Thm 1.3.2]) Let the coordinates of \( f: \mathbb{H} \to \mathbb{H} \) and \( g: \mathbb{H} \to \mathbb{H} \) have partial derivatives. Then
\[ \partial_u (f \circ g) = (\partial_u f) \circ g + \bar{f} \circ (\partial_u g) - 2(f \cdot \partial_u) g \]

(16)

and
\[ (f \circ g) \partial_u = (f \partial_u) \circ \bar{g} + f \circ (g \partial_u) - 2(g \cdot \partial_u) f. \]

(17)

Here, \( (f \cdot \partial_u) g) = \sum_{i=1}^{3} f_i \partial_x_i g. \)

**Corollary 4.6.** Let the coordinates of \( f: \mathbb{H} \to \mathbb{H} \) and \( g: \mathbb{H} \to \mathbb{H} \) have partial derivatives. Then
\[ \partial_u ((f \circ e_4) \circ g) = [(f \partial_u) \circ g + f \circ (\bar{g} \partial_u) + 2(g \cdot \partial_u) f] \circ e_4 \]

(18)
\[ \partial_u ((f \circ g \circ e_4)) = [(g \partial_u) \circ \bar{f} + g \circ (f \partial_u) - 2(f \cdot \partial_u) g] \circ e_4 \]

(19)
\[ \partial_u ((f \circ e_4) \circ (g \circ e_4)) = -(\partial_u \bar{g}) \circ f - g \circ (\partial_u f) - 2(g \cdot \partial_u) f \]

(20)
\[ (\partial_u \circ e_4)(f \circ g) = [\partial_u \bar{g}) \circ \bar{f} + g \circ (\partial_u \bar{f}) + 2(g \cdot \partial_u) \bar{f}] \circ e_4 \]

(21)
\[ (\partial_u \circ e_4)((f \circ e_4) \circ g) = - (g \partial_u) \circ f - g(\bar{f} \partial_u) - 2(f \cdot \partial_u) g \]

(22)
\[ (\partial_u \circ e_4)(f \circ (g \circ e_4)) = -(\bar{f} \partial_u) \circ g - \bar{f} \circ (g \partial_u) - 2(g \cdot \partial_u) \bar{f} \]

(23)
\[ (\partial_u \circ e_4)((f \circ e_4) \circ (g \circ e_4)) = -(\partial_u \bar{f}) \circ g - f(\partial_u g) - 2(f \cdot \partial_u) g \circ e_4 \]

(24)

**Proof.** Apply Lemmas 2.1 and 4.5, and use the fact \( \bar{f} g = g \bar{f}. \)  \[ \square \]
5. Function Classes in Clifford Analysis

In this last section, we study the classes of left-, B-, and R-regular functions using Clifford analysis. We begin with the following algebraic lemma.

**Lemma 5.1.** Let $I$ be the primitive idempotent

$$I = \frac{1}{16}(1 + We_{12\ldots7})(1 - e_{12\ldots7}),$$

where $W = e_{123} + e_{145} + e_{176} + e_{246} + e_{257} + e_{347} + e_{365}$, and let $a = a_0 + a$ and $b = b_0 + b \in \mathcal{C}_{0,7}$ be paravectors. Then

$$16[abI]_0 = a_0b_0 - a \cdot b,$$

$$16[abI]_1 = a_0b + ab_0 - [(a \wedge b)W]_1,$$

$$16[abI]_2 = a \wedge b - [(a_0b + ab_0)W]_2 + [(a \wedge b)We_{1\ldots7}]_2,$$

$$16[abI]_3 = -[(a_0b + a \cdot b)W] + [(a_0b + ab_0)We_{1\ldots7}]_3 - [(a \wedge b)We_{1\ldots7}]_3,$$

$$16[abI]_4 = (a_0b_0 - a \cdot b)We_{1\ldots7} - [(a_0b + ab_0)W]_4 + [(a \wedge b)We_{1\ldots7}]_4,$$

$$16[abI]_5 = [(a_0b + ab_0)We_{1\ldots7}]_5 - [(a \wedge b)W]_5 - (a \wedge b)e_{1\ldots7},$$

$$16[abI]_6 = -(a_0b + ab_0)e_{1\ldots7} + [(a \wedge b)We_{1\ldots7}]_6,$$

$$16[abI]_7 = -[(a_0b + a \cdot b)e_{1\ldots7},$$

and

$$[abI]_k = 0 \iff [abI]_{7-k} = 0, \quad k = 0, 1, \ldots, 7. \quad (33)$$

If $[abI]_0 = 0$, then the conditions $[abI]_j = 0$, $j = 2, 3, 4, 5$, are pairwise equivalent. In particular, if $[abI]_{0,1,2} = 0$, then $abI = 0$.

**Proof.** Write the real part and the 1- and 2-vector parts of $ab$ using (3), and expand the definition (4) of $I$ using the fact $e_{12\ldots7}^2 = 1$:

$$ab = (a_0b_0 - a \cdot b) + (a_0b + ab_0) + a \wedge b,$$

$$16I = 1 - W + We_{12\ldots7} - e_{12\ldots7}.$$

Here, $W$ is a 3-vector and $We_{12\ldots7}$ is a 4-vector. Then, for example, $aW$ only contains 2- and 4-vector parts, and therefore $[aW]_3 = 0$. This kind of reasoning implies (25)–(32).

Now, (33) follows from the facts that for any $c \in \mathcal{C}_{0,7}$,

$$c = 0 \iff ce_{12\ldots7} = 0, \quad \text{and}$$

$$[c]_k e_{12\ldots7} = [ce_{12\ldots7}]_{7-k}, \quad k = 0, 1, \ldots, 7.$$

To prove the last claim, it is now enough to show that in the case $[abI]_0 = 0$, $[abI]_2 = 0$ if and only if $[abI]_3 = 0$. This can be seen by computing

$$16[abI]_2 = (a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2 + a_4b_5 - a_5b_4 - a_6b_7 + a_7b_6)(e_{23} + e_{45} - e_{67}) + (a_0b_2 + a_1b_3 + a_2b_6 + a_3b_1 + a_4b_6 + a_5b_7 - a_6b_4 - a_7b_5)(-e_{13} + e_{46} + e_{57}) + (a_0b_3 + a_1b_2 - a_2b_1 + a_3b_0 + a_4b_7 - a_5b_6 + a_6b_5 - a_7b_4)(e_{12} + e_{47} - e_{56}).$$
\[ + (a_0 b_4 - a_1 b_5 - a_2 b_6 - a_3 b_7 \]
\[ + a_4 b_0 + a_5 b_1 + a_6 b_2 + a_7 b_3)(-e_{15} - e_{26} - e_{37}) \]
\[ + (a_0 b_5 + a_1 b_4 - a_2 b_7 + a_3 b_6 \]
\[ - a_4 b_1 + a_5 b_0 - a_6 b_3 + a_7 b_2)(e_{14} + e_{27}e_{36}) \]
\[ + (a_0 b_6 + a_1 b_7 + a_2 b_4 - a_3 b_5 \]
\[ - a_4 b_2 + a_5 b_3 + a_6 b_0 - a_7 b_1)(e_{17} + e_{24} - e_{35}) \]
\[ + (a_0 b_7 - a_1 b_5 + a_2 b_6 + a_3 b_4 - a_4 b_3 \]
\[ - a_5 b_2 + a_6 b_1 + a_7 b_0)(-e_{16} + e_{25} + e_{34}), \]

and in the case \([abI]_0 = 0,\]
\[ 16[abI]_3 = (a_0 b_1 + a_1 b_0 + a_2 b_3 - a_3 b_2 + a_4 b_5 - a_5 b_4 \]
\[ - a_6 b_7 + a_7 b_6)(e_{247} - e_{256} - e_{346} - e_{357}) \]
\[ + (a_0 b_2 - a_1 b_3 + a_2 b_0 + a_3 b_1 + a_4 b_6 \]
\[ + a_5 b_7 - a_6 b_4 - a_7 b_5)(e_{147} + e_{156} + e_{345} - e_{367}) \]
\[ + (a_0 b_3 + a_1 b_2 - a_2 b_1 + a_3 b_0 \]
\[ + a_4 b_7 - a_5 b_6 + a_6 b_5 - a_7 b_4)(e_{146} + e_{157} - e_{245} + e_{267}) \]
\[ + (a_0 b_4 - a_1 b_5 - a_2 b_6 - a_3 b_7 + a_4 b_0 \]
\[ + a_5 b_1 + a_6 b_2 + a_7 b_3)(e_{127} - e_{136} + e_{235} - e_{567}) \]
\[ + (a_0 b_5 + a_1 b_4 - a_2 b_7 + a_3 b_6 - a_4 b_1 \]
\[ + a_5 b_0 - a_6 b_3 + a_7 b_2)(-e_{126} - e_{137} - e_{234} + e_{467}) \]
\[ + (a_0 b_6 + a_1 b_7 + a_2 b_4 - a_3 b_5 - a_4 b_2 \]
\[ + a_5 b_3 + a_6 b_0 - a_7 b_1)(e_{125} + e_{134} - e_{237} - e_{457}) \]
\[ + (a_0 b_7 - a_1 b_6 + a_2 b_5 + a_3 b_4 - a_4 b_3 \]
\[ - a_5 b_2 + a_6 b_1 + a_7 b_0)(-e_{124} + e_{135} + e_{236} + e_{456}). \]

\[ \square \]

We infer that left-, B-, and R-regularity can be studied by considering paravector-spinor valued functions \(fI\).

**Theorem 5.2.** Suppose that \(f: \mathbb{R}^8 \to \mathbb{R}^8\) is a paravector valued function such that the coordinate functions have partial derivatives.

(a) \(f\) is left-regular if and only if
\[ [\partial_x fI]_j = 0 \text{ for } j = 0, 1. \tag{34} \]

(b) \(f\) is B-regular if and only if
\[ [\partial_x fI]_j = 0 \text{ for } j = 0, 1, \text{ and } [\partial_x fW]_1 = 0. \tag{35} \]

**Proof.** (a) follows using Lemma 2.7:
\[ D_x f = 16[\partial_x fI]_0 + 16[\partial_x fI]_1. \]
(b) From (3) and (26), we obtain

\[
\partial_x f W_1 = \left[ (\partial_x f_0 - \partial_x \cdot f) W \right]_1 + \left[ (\partial_x f_0 + \partial_x \cdot f) W \right]_1 + \left[ (\partial_x \wedge f) W \right]_1 = -16 [\partial_x f I]_1 + \partial_x f_0 + D_x f_0.
\]

Since \( D_x \times f = -[\partial_x \wedge f] W_1 \) (Lemma 2.5), the claim now follows from (a) and Propositions 3.1–3.2.

\[\square\]

**Remark 5.3.** If \( \partial_x f = 0 \), then (trivially) \( [\partial_x f I]_j = 0 \) for all \( j = 0, 1, \ldots, 7 \). The converse does not hold. This follows from the fact that the equation \( aI = 0 \) does not have a unique solution \( a = 0 \) in the Clifford algebra. Hence, paravector spinor valued solutions of the Cauchy–Riemann equations form a bigger function class, and the class of R-regular solutions is

\[\mathcal{M}_R \subsetneq \{ f : \partial_x f I = 0 \} = \{ f : [\partial_x f I]_j = 0, \ j = 0, 1, \ldots, 7 \} \]

\[= \{ f : [\partial_x f I]_j = 0, \ j = 0, 1, 2 \}.\]

The equality of the latter two function classes follows from Lemma 5.1. An example showing that the inclusion is strict: if \( f = x_2 e_1 - x_7 e_4 \), then \( \partial_x f = e_4 e_7 - e_1 e_2 \), but \( [\partial_x f I]_j = 0 \) for \( j = 0, 1, 2 \).

### 6. Conclusion

The key idea of this paper is to study differences between octonion and Clifford analyses. This leads us to observe the fundamental difference between octonion regular and Clifford monogenic functions. The structure of octonion regular functions is studied by comparing left-, right-, \( B \)-, and \( R \)-regular functions. The existence of these classes is a consequence of different algebraic properties of the algebras. In the heart of octonion analysis is the study of the properties of these function classes and their relations, which distinguishes it essentially from Clifford analysis.

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