Constructing numbers in quantum gravity: infinions

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Abstract. Based on the Cayley-Dickson process, a sequence of multidimensional structured natural numbers (infinions) creates a path from quantum information to quantum gravity. Octonionic structure, exceptional Jordan algebra, and $E_8$ Lie algebra are encoded on a graph with $E_9$ connectivity, decorated by integral matrices. With the magic star, a toy model for a quantum gravity is presented with its naturally emergent quasicrystalline projective compactification.

1. Introduction

This paper is dedicated to the two centuries of the eight squares theorem, published in latin as “Adumbratio demonstrationis theorematis arithmici maximale universalis” by Carl Ferdinand Degen, (October 7th, 1818)[1] and the one century of the Cayley-Dickson process, “On quaternions and their generalization and the history of the eight square theorem” published by Leonard Eugene Dickson in March, 1919 [2]. Inspired by the Nag Hammadi codex VI (from 3rd century AD) ”The eighth reveals the ninth”, we take the ninth-dimensional vision of the lattice of octonion integrals, the $E_9$ affine Lie algebra, and get a new quasicrystalline view on quantum gravity.

1.1. Two hundred year old sedenion multiplication table

Figure 1. First sedenion multiplication table, named ”16 serierum”, published in 1818 by C. F. Degen. This table omits the $K_4$ coefficients, ”factoribus facile supplendis (et etiam tamdiu signis) omisis”– the sign factor is omitted, because it is easy to compute. He named $p, p^1, ... p^{15}$ the units $e_0, e_1, ... e_{15}$ and implemented the same xor table with $1 = e_0, a = e_1, b = e_2, c = e_4, d = e_8$. 
The eight squares theorem, upon which octonions were discovered 28 years later, claimed a possible generalization to any power of two, and gave the table of sedenions (see Figure 1), but omitted the computation of the sign coefficients. There is no possible combination of coefficients (i,j) realizing an 16 squares theorem, therefore the claim is wrong, but the intuition for the Infinions is there, several decades before Hamilton, Graves, Cayley and Dickson. In the next section we present the generalization to an infinite dimension, that we name Infinion, or simply \( I \), of the “16 sererium” from Degen, with a similar product rule \( e_i e_j = k_{ij} e_{i \oplus j} \) and we give an algorithm to compute \( k_{ij} \).

1.2. 9D coordinates

The roots of the affine Lie algebra \( E_9 \) constitute the even unimodular lattice of \( E_8 \), which is also the weight lattice of the \( E_9 \) Lie algebra. It is natural to represent the coordinates of the \( E_9 \) roots in 9 dimensions. We give them explicitly hereafter, and show how this is also convenient to represent the \( E_8 \) sublattice in the same basis. Section 3 will describe a natural action on the \( E_8 \) algebra obtained from the Tits magic square, using Jordan algebra and the 9D coordinates. Section 4 will show a quasicrystalline compactification of \( E_9 \) to a 4-dimensional quasicrystal, with an isomorphism between the 9D and the 4D coordinates.

- \( A_n \) Simplex lattices are naturally expressed in \( n + 1 \) dimension coordinates satisfying \( \sum_{k=1}^{n+1} x_k = 0 \).
- \( E_8 \) lattice is the superposition of three \( A_8 \) lattices: \( E_8 = \bigcup_{i=0}^{7} A_8^i \).
- 72 of its roots are permutations of \( \{3^1, -3^1, 0^7\} \), \( \cong 0[3] \), \( \in 3A_8^0 \).
- 84 of its roots are \( \mathcal{P}(-2^3, 1^6) \), \( \cong 1[3] \), \( \in 3A_8^1 \).
- 84 of its roots are \( \mathcal{P}(2^3, -1^6) \), \( \cong 2[3] \), \( \in 3A_8^2 \).
- \( E_6 \) lattice is the superposition of three \( A_8 \) lattices satisfying \( \sum_{k=1}^{3} x_k = \sum_{k=4}^{6} x_k = \sum_{k=7}^{9} x_k = 0 \).
- 18 of its roots are \( \mathcal{P}(3^1, -3^1, 0^7) \), \( \cong 0[3] \), \( \in 3A_8^0 \).
- 27 of its roots are \( \mathcal{P}(-2^3, 1^6) \), \( \cong 1[3] \), \( \in 3A_8^1 \).
- 27 of its roots are \( \mathcal{P}(2^3, -1^6) \), \( \cong 2[3] \), \( \in 3A_8^2 \).
- \( E_8 \) lattice is the superposition of two \( D_8 \) lattices: \( E_8 = \bigcup_{j=0}^{1} D_8^j \).
- 112 of its roots are \( \mathcal{P}(\pm 2^2, 0^6) \), \( \cong 0[2] \), \( \in 2D_8^0 \).
- 128 of its roots are \( \mathcal{P}(\pm 1^8) \), \( \cong 1[2] \), \( \in 2D_8^1 \), \( \sum_{k=1}^{8} x_k \cong 0[4] \).

1.3. Group Theoretic View

\[
E_8 = SU(9) + \mathbf{84} + \overline{\mathbf{84}}. \tag{1}
\]

- The relationship between the \( E_8 \) lattice and the Simplex lattice, \( E_8 = 3A_8 \), is illustrated and will be extended to exceptional periodicity algebras [3, 4],
- exceptionally \( \mathbf{84} = \Lambda^3 C^0 \) 3-form and \( \overline{\mathbf{84}} = \Lambda^6 C^0 \) 6-form in \( SU(9) \) [5],
- or generally \( \mathbf{84} = 28 + 56 = \Lambda^2 C^8 \oplus \Lambda^3 C^8 \) 2-form and 3-form, and \( \overline{\mathbf{84}} = 56 + 28 = \Lambda^6 C^8 \oplus \Lambda^5 C^8 \) 6-form and 5-form in \( Cl(8) \).

2. Infinion

Infinion\(^1\) is the infinite limit of the algebra of dimension \( 2^n \) built by the Cayley-Dickson process.

\(^1\) Some results presented in this section were first presented in the talk ”Non associative quantum gravity” at the conference on non-associative algebra in Coimbra in 2011. No proceedings, but the book of abstracts were available [6].
The Cayley-Dickson process is based on the two recursion rules,
\[
(a, b) = (\pi, -\overline{b}),
\]
\[
(a, b)(c, d) = (ac - \overline{db}, \overline{bc} + da).
\] (2)

Binary \textit{xor} operation arises from the iteration of equation 2. The organization of numbers on a cubic dice illustrates the binary \textit{xor} operation, where any two opposite faces sum to seven, but also \textit{xor} to seven: up \textit{xor} bottom = 7 = 1, \textit{xor} 6 = 2, \textit{xor} 5 = 3, \textit{xor} 4 = 001, \textit{xor} 110 = 010, \textit{xor} 101 = 011, \textit{xor} 111 = 011 = 3, 4 \textit{xor} 5 = 1... a \textit{xor} b \textit{xor} a = b, a \textit{xor} a = 0, a \textit{xor} 0 = a, \textit{xor} is commutative and associative. The unique canonical basis is ordering the units such that the index of a product is the \textit{xor} of the indices,
\[
e_i e_j = s_i j e_i \textit{xor} j, s_i j \in S^0 = \{-1, 1\},
\]
\[
e_0 = 1, s_0 j = s_i 0 = 1, s_i i = -1 + 2\delta_{i0},
\]
\[
s_i j = -s_j i(1 - 2\delta_{i0} + \delta_{i1} + \delta_{i2}) + 4(\delta_{i1}\delta_{i2}).
\] (3)

**Theorem 1.** Giving \(K_n = (k_{i j} = 2s_i j - 1)|0 \leq i < 2^n \& 0 \leq j < 2^n\) and, similarly \(C_n = (c_{i j}), D_n = (d_{i j}), R_n = (r_{i j})\), the Cayley-Dickson process (equation 2) is equivalent to the recursion relation \(K_{n+1} = \left( \begin{array}{cc} K_n & R_n \\ C_n & D_n \end{array} \right) \& K_0 = 1\), where
\[
c_{i j} = \delta_{i j} - (1 - \delta_{i j})k_{i j},
\]
\[
d_{i j} = -\delta_{i j} + (1 - \delta_{i j})(\delta_{i0} + (1 - \delta_{i0})k_{i j}),
\]
\[
r_{i j} = \delta_{i j} - (1 - \delta_{i j})d_{i j}.
\] (4)

**Proof.** Formula 5 shows how to build \(K_{N+1}\) from \(K_N\) with the help of three binary matrices of dimension \(2^N\), the row \(\rho_{i j} = \delta_{i j}\), the column \(\Gamma_{i j} = \delta_{i j}\), the diagonal matrix \(\Delta_{i j} = \delta_{i j}\). The following recursion relation, also illustrated graphically in Figure 2,
\[
K_{n+1} = \left( \begin{array}{cc} K_n & R_n \\ C_n & D_n \end{array} \right) = \left( \begin{array}{cc} K_n & (K_n + \rho_n)\Delta_n + \Gamma_n \\ \tilde{K}_n + \Gamma_n & (K_n + \rho_n)\Delta_n \end{array} \right),
\] (5)
is equivalent to the relations defining \(c_{i j}, d_{i j}, r_{i j}\) in (4), themselves implying realization of the recursive equation (2) while in the canonical basis (3).

Implementation code and examples are given in the Appendix.

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**Figure 2.**
Left: \(K_2, K_3\) and \(K_4\),
White= 1 (+), black= 0 (-).
Right: \(K_3\) is built from \(K_2\),
top: \(K_2, K_2, \Gamma_2, C_2\),
middle: \(\tilde{K}_2, 0\), \(K_2 + 0\), \(\Delta_2, D_2\),
bottom: \(D_2, \Gamma_2, K_2\).

Applying our theorem, we can compute the multiplication table of Infinion, restricted to any power of two. We represent \(e_i e_j = s_i j e_i \textit{xor} j\) by a 4 by 4 bitmap, where the central 2 by 2 square is either white if \(s_i j\) is positive, or black otherwise. The 12 bits around encodes in binary the value \(i \textit{xor} j\). This is clearly visible on the sedenion table given in figure 3. Figures 4 and 5 show self-similarity and a fractal behavior structured from the \textit{xor} product.
3. Magic star

The Gosset polytope of the \( E_8 \) roots is projected to 13 vertices drawing an hexagram. This Magic star [8] in figure 6 (projected using [9]) is slightly rotated to show the central \( E_6 \) and the three Jordan pairs around.

3.1. Jordan algebra

![Figure 6. Magic Star in \( e_6 \) Coxeter plane at the left, rotated to \( g_2 \) Coxeter plane on the right. Projective forms \( H \) and \( V \) are given in table 1, columns \( e_6 \) and \( g_2 \), such that \( x = H \cdot R \) and \( y = V \cdot R \) for each root \( R \).](image)

Jordan Matrix:

- Each \( E_8 \) vertex holds an exceptional Jordan [10, 11] matrix \( J \in \mathcal{M}_8^3 \),
- 10D Minkowski Spacetime with a transversal octonion \( o \) as

\[
J_2 = \begin{pmatrix} t - x_8 \\ o = x^0 e_0 + \sum_{k=1}^7 x^k e_k \\ \bar{x} = x^0 e_0 - \sum_{k=1}^7 x^k e_k \end{pmatrix} \in \mathcal{M}_8^2 = SL_2(\mathbb{O}),
\]
Table 1. $H$ and $V$ forms.

|    | $e_6$ |    | $g_2$ |    | $e_6'$ |    | $h_4$ |    |
|----|-------|----|-------|----|--------|----|-------|----|
| $H$ | $V$   | $H$ | $V$   | $H$ | $V$    | $H$ | $V$   |
| 0.00 | 0.00  | -0.06 | -0.25 | 0.00 | 0.00  | -0.30 | 0.05  |
| 0.00 | 0.00  | 0.11  | 0.07  | 0.00 | 0.00  | -0.11 | 0.19  |
| 0.00 | 0.00  | -0.06 | 0.12  | 0.00 | 0.00  | 0.08  | 0.05  |
| 0.00 | 0.00  | 0.24  | -0.07 | 0.00 | 0.00  | 0.11  | 0.29  |
| 0.00 | 0.00  | 0.00  | 0.00  | 0.12 | 0.00  | -0.46 | 0.80  |
| -0.70 | -0.41 | -0.68 | -0.39 | -0.70 | -0.47 | 0.58  | 0.33  |
| 0.70 | -0.41 | 0.68  | -0.39 | 0.70 | -0.47 | 0.46  | 0.13  |
| 0.00 | 0.82  | 0.00  | 0.78  | 0.00 | 0.74  | 0.34  | 0.33  |

- Central cross encoding scalar $\phi$ and $Spin(9,1)$ spinor $\Psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$,

$$J = \begin{pmatrix} t-x_8 & \psi_+ = \sum_{k=0}^7 \psi_k e_k \\ \phi - 2t & \psi_- = \sum_{k=0}^7 \psi_k e_k \\ o & t + x_8 \end{pmatrix} \in \mathfrak{m}_8^3 = SL_3(\mathbb{O}), \quad (7)$$

- Jordan product: $J_1 \cdot J_2 = \frac{1}{2}(J_1 J_2 + J_2 J_1)$ [10],

- Freudenthal product: $J_1 \times J_2 = \frac{1}{2}(2J_1 J_2 - Tr(J_1)J_2 - Tr(J_2)J_1 + I(Tr(J_1)Tr(J_2) - Tr(J_1 \cdot J_2)))$ [12],

- Associator: $[J_1, J_2, J_3] = (J_1 \cdot J_2) \cdot J_3 - J_1 \cdot (J_2 \cdot J_3)$ [13],

- Left quasi multiplication: $L_x : L_x(y) = x \cdot y$,

- Quadratic map: $U_x = 2L_x^2 - L_x^2$ [14],

- Linearized map: $V_{x,y} : V_{x,y}(z) = (U_{x+z} - U_x - U_z)(y)$ [15],

- Trilinear map: $\{x, y, z\} = V_{x,y}(z) = 2(L_{x,y} + [L_x, L_y])(z)$,

- Axioms: $A1 : U_x V_{y,z} = V_{x+y} U_z, A2 : U_{x,y} = U_z U_y U_x$,

- Jordan pair: $x, y A1 \& A2 \& V_{x,y,z} = V_{x+y, z}$.

Discrete Jordan Matrix:

Each octonion $o$ in $J$ (see figure 7), induced by lattice coordinates, can be restricted to integer [16] and can be encoded by its $9D$ coordinates in a $3 \times 3$ matrix, by applying the rotation (11).

![Figure 7](image-url)

**Figure 7.** An $A_2$ plane cuts several shells of the $E_8$ lattice, each shell being associated to a color for the vertex. The Jordan matrix can become integral, and represented as a $9 \times 9$ matrix in $\mathbb{Z}$

$$J' = \begin{pmatrix} t-x_8 & \psi_+ & \psi_- \\ \phi - 2t & \psi_- & t + x_8 \end{pmatrix}.$$
3.2. $F_4$ action, $E_6$ action

$F_4$ action: $F_4$ action is a derivation [17] on $\mathfrak{m}_8^3$:

- An element of 52D algebra $F_4$ is represented by two traceless $H_+$ and $H_-$,
- Its action [18] on $J = H + \Phi$ is
  \[ F_4(H_+, H_-)(J) = \delta J = [H_+, J, H_-], \]  
  (8)
- Invariants are $I_1 = Tr(J)$, $I_2 = Tr(J^2)$, $I_3 = Det(J) = \frac{1}{2}Tr(J \cdot J \times J)$.

$E_6$ action: $E_6$ action is a derivation [17] on $\mathfrak{m}_8^3 \otimes \mathbb{C}$:

- An element of 78D algebra $E_6$ is represented by $H_1$, $H_+ H_- \in Tr_0(\mathfrak{m}_8^3)$,
- Its action [18] on $J$ is
  \[ E_6(H_1, H_+, H_-)(J) = \delta J = [H_+, J, H_-] + \varepsilon_1 H_1 \cdot J, \]  
  (9)
- Invariants are $I_2 = Tr(J^2)$, $I_3 + \imath I_4^3 = 3Det(J) = Tr((J \times J)^*)$, $I_4 = Tr((J \times J \cdot (J^* \times J^*)^*)$.

$E_6(-26)$ action: An action on the reduced structure group is proposed in [19],
- $J = \Xi + \Psi + \Phi$,
- $S = \frac{1}{8\pi} Tr \int d\sigma d\tau (\delta_{\alpha} \Xi^\alpha \Xi + \delta_{\alpha} \Psi^\alpha \Psi + \delta_{\alpha} \Phi^\alpha \Phi)$.

3.3. $E_7$ action, $E_8$ action

$E_7$ action: An action of $E_7$ by a Freudenthal triple system on $E_8$ was proposed in [20] :

- 56D representation of $E_7$ as $\mathfrak{m}_{27}^2(\mathfrak{m}_8^3)$.

$E_8$ action: $E_8$ proposed action is a derivation on $\mathfrak{m}_8^3 \otimes \mathbb{C}$:

- The action is extrapolated from Tits-Rosenfeld-Freudenthal magic square [21] expressed by Vinberg [22] as:
  \[ \mathbb{L}(\mathbb{A}, J^3(\mathbb{B})) = \text{Der}(\mathbb{A}) \bigoplus Im(\mathbb{A}) \bigotimes Tr_0(J^3(\mathbb{B})) \bigoplus \text{Der}(J^3(\mathbb{B})). \]  
  (10)

4. Quantum gravity

Our model is based on the $E_8$ Lattice decorated with a Jordan matrix at each vertex. We compute the 9D coordinates of each vertex by the following rotation [23]:

\[ \mathcal{R} = \frac{1}{6} \begin{pmatrix} 5 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & 5 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & 5 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 5 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & 5 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & 5 & -1 & -1 & -1 \\ 2 & 2 & 2 & 2 & 2 & 2 & -2 & -2 & -2 \end{pmatrix}. \]  
(11)

4.1. Induced Fano plane

As illustrated by figure 8, each vertex, like the vertex $\{1, 1, 1, 1, -2, -2, 1, -2\}$ of the blue $A_8$, $3A_8^3$ (having 8D coordinates $\{0, 0, 0, 0, -2, -2, 0\}$ in $2D_8^0$), is the center of a magic star. It intertwinings the down-pointing red triangle of $3A_8^3$ and the up-pointing green triangle of $3A_8^0$. An enclosed green hexagon gives the 6 $g2$ elements, and the green and red tips of the regular tetrahedrons completing the triangles give the two other matrices to complete the set of 9 Jordan matrices (the central one being traceless) and 6 scalars needed to define an element of the $e8$ algebra.

The correspondence between the Fano plane shown in figure 9, and the seven vertices in the magic star of figure 8 is obvious, and gives the key to the use of $Im(\mathbb{A})$ in equation (10).
4.2. \( E_8 \) Quasi-lattice compactification
A golden selective projection operates the \( E_8 \) to \( H_4 \) folding. The rings in figure 10 will project to 4D in the Elser-Sloane quasicrystal[24] as the green points represented in figure 11. Around the central 600-cell, there are 120 other 600-cells, whose centers are the vertices of a larger 600-cell, thanks to the self-similarity of the quasicrystal. In figure 11 we represent each of the 121 600-cells by only one 30-ring, made of 30 tetrahedrons (while they are made of 20 similar rings), to keep our diagram readable.

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{figure10}
\caption{Rotation of \( E_8 \) from the \( E_6 \) Coxeter plane close to the magic star, to the \( H_4 \) Coxeter plane where 4 of the 8 rings of 30 roots will match a 600-cell. Projective forms \( H \) and \( V \) are given in the table 1, columns \( e'_6 \) and \( h_4 \), such that \( x = H \cdot R \) and \( y = V \cdot R \) for each root \( R \).}
\end{figure}

4.3. Quasi-lattice action
A, the observer, chooses a tetrahedron, selects a vertex in it, selects an operation:

- Operation \( F_4 \) involves two \( E_8 \) vertices and updates one \( E_8 \) vertex,
- Operation \( E_6 \) involves three vertices and updates two,
- Operation \( E_7 \) needs choosing a top in the tetrahedron, involves seven vertices,
- Operation \( E_8 \) involves the full magic star.
Figure 11. Elser-Sloane quasicrystal triacontagonally projected.

\( B \), the observed Jordan matrices affected to lattice vertices are initially blank.  
\( P \) the observation, is operated as follows:

- Once the observer and its operation are chosen, selected vertices, if blank, are initialized,
- The operation is performed and vertices are updated.

Figure 12 shows one 30-ring from a perspective where the facing tetrahedron, having big blue spheres as vertices, belongs to only one \( A_8^1 \) (the blue one), while its opposite belongs to the green \( A_8^2 \). The magic star can now be identified in the quasicrystal in a similar way that it is in the crystal in figure 8.

Figure 12. 30-ring: a ring of 30 of the 600 tetrahedrons from the 600-cell, projected from 4D to 3D. Colors indicate to which \( A_8 \) lattice each vertex belongs before projection from 8D to 4D.

5. Concluding remarks

We have given the canonical explicit construction of the algebra we name infinion, properly generalizing the division algebras. The table used is based on \( \text{xor} \) and is different from the octonion table often used in physics. Infinion opens the way to new Jordan matrices \( \mathfrak{M}_3^n \) to \( \mathfrak{M}_\infty \) which could be useful to describe exceptional periodicity [4].

We have proposed a lattice model based on the \( E_8 \) lattice, where each vertex belongs to one of three copies of the \( A_8 \) simplex lattice. The explicit transformation is given, and also the procedure to find a magic star attached to any vertex and build an exceptional action from the geometric neighborhood where each vertex has a Jordan matrix. Projecting it to a quasi-lattice
model on the Elser-Sloane quasicrystal, we have realized a new type of compactification from an 8D integral point set to a dense 4D quasicrystal.

We expect to use these new mathematical tools to build a theory of Quantum Gravity sharing some properties (E8, compactification) with String Theory and some (discreteness, topology) with Loop Quantum Gravity.

Appendix

Listing 1 gives the code to compute signs for any n, as a bitmap. Listing 2 shows how to use the bitmap to get the signs. Listing 3 implements multiplication of Infinions expressed as a list. The algorithm uses the basic operators on a square bitmap:

- **ColorNegate**: reverse each pixel between white and black
- **ImageMultiply**: apply a logic and, or a binary multiplication (where white is 1, black is 0). The resulting pixel will be white only if it is white in both operated images.
- **ImageAdd**: apply a logic or, or a binary addition (where white is 1, black is 0). The resulting pixel will be black only if it is black in both operated images.

**Listing 1.** Recursive algorithm computing the matrix of signs $K_n = (k_{ij})$ as an image.

\[
\begin{align*}
\text{ImDoub}[n_-, i_-, l_-, c_] & := \text{ImageAssemble}\{\{a, \\
\text{ImageAdd}[\text{ImageMultiply}[\text{ImageAdd}[\text{ColorNegate}@a, l], \text{ColorNegate}@i], c]\}, \\
\{\text{ImageAdd}[\text{ColorNegate}@a, c], \text{ImageMultiply}[\text{ImageAdd}[\text{ColorNegate}@a, l], \text{ColorNegate}@i]\}\}
\end{align*}
\]

\[
\begin{align*}
\text{ImDoub}[a_-] & := \text{ImDoub}[a, \text{Image@IdentityMatrix}[\text{First@ImageDimensions}@a], \\
\text{Image@SparseArray}[\{(1, i_-) \to 1\}, \text{ImageDimensions}@a], \\
\text{Image@SparseArray}[\{(i_-, l) \to 1\}, \text{ImageDimensions}@a]]
\end{align*}
\]

\[
K\text{Image}[n_+] := K\text{Image}[n] = \begin{cases} 
\text{If}[n == 0, \text{Image}[\{\{1\}\}], \text{ImDoub}[\text{KImage}[n - 1]]] 
\end{cases}
\]

**Listing 2.** Code example computing the matrix $K_4$ from KImage.

\[
(k = \text{Round}/@(2 \text{ ImageData}@K\text{Image}[4] - 1)) //\text{MatrixForm}
\]

**Listing 3.** Multiplication of two infinions represented as lists.

\[
\text{pad}[z_] := \text{PadRight}[(\text{If}[\text{Length}@z > 0, z, \{z\}]), \\
2 \text{ Ceiling}[(-10^-50 + \text{Log}[\text{Max}[1, \text{Length}@z]])/\text{Log}[2]]] 
\]

\[
\text{Mul1}[a_, b_] := \text{Sort}@\text{Flatten}[\text{Table}[[\text{BitXor}[i - 1, j - 1], a[[i]] b[[j]] k[[i, j]]], \\
\{i, 1, \text{Length}@a\}, \{j, 1, \text{Length}@b\}]] 
\]

\[
\text{Mul2}[a_, b_] := (\# [[2]] &)/@\text{Partition}[	ext{Mul1}[a, b], \text{Length}@a]] 
\]

\[
\text{Mul}[a_, b_] := \text{Mul2}[\text{PadRight}[\text{pad}@a, \text{Max}[\text{Length}@\text{pad}@a, \text{Length}@\text{pad}@b]], \\
\text{PadRight}[\text{pad}@b, \text{Max}[\text{Length}@\text{pad}@a, \text{Length}@\text{pad}@b]] 
\]

**Listing 4.** Multiplication of two quaternions, two conjugate octonions and two sedenion zero-dividors ($e_4 + e_{10})(e_7 - e_9)$.

\[
\text{Mul}[\{0, 1\}, \{0, 0, 1\}] 
\]

\[
\text{Mul}[\{1, -1, 1, -1, -1, 1, -1, 1\}, \{1, 1, 1, 1, 1, 1, 1, 1\}] 
\]

\[
\text{Mul}[\{0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1\}, \{0, 0, 0, 0, 0, 0, 1, 0, -1\}] 
\]

\[
\{0, 0, 0, 1\} 
\]

\[
\{8, 0, 0, 0, 0, 0, 0\} 
\]

\[
\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\} 
\]
References

[1] Degen C F 1818 Adumbratio demonstrationis theorematis arithmetici maxime universalis. Memoires de l’Académie impériale des sciences de St. Petersbourg 5e série

[2] Dickson L E 1919 On quaternions and their generalization and the history of the eight square theorem The Annals of Mathematics 20, 3 155

[3] Marrani A and Truini P 2016 Exceptional lie algebras at the very foundations of space and time p-Adic Numbers, Ultrametric Analysis and Applications 8, 1 68

[4] Truini P, Rios M and Marrani A 2017 The magic star of exceptional periodicity Preprint 1711.07881

[5] Ferrara S, Marrani A and Trigiante M 2012 Super-ehlers in any dimension J. High Energy Phys. 11

[6] Albuquerque H 2011 Program and abstracts Conf. Non-Associative Algebras and Related Topics (Coimbra: University)

[7] Pozzi L 2011 T.O.E. Theory of everything C.A.B. Centre d’Art Bastille, Grenoble http://1995-2015.undo.net/it/mostra/125825

[8] Truini P 2012 Exceptional lie algebras, su(3) and jordan pairs Pacific J. Math. 260, 227 (Preprint math-ph/1112.1258 )

[9] Lisi G 2007 Elementary particle explorer http://deferentialgeometry.org/epe/

[10] Jordan P, von Neumann J and Wigner E 1934 On an algebraic generalization of the quantum mechanical formalism The Annals of Mathematics 35, 1 29

[11] Albert A A 1934 On a certain algebra of quantum mechanics The Annals of Mathematics 35, 1 65

[12] Freudenthal H 1954 Beziehungen der e7 und e8 zur oktavenbene, i, ii Indag. Math. 16 218

[13] Gürsey F and Tze C H 1996 On the Role of Division, Jordan and Related Algebras in Particle Physics (World Scientific)

[14] McCrimmon K 2004 A Taste of Jordan Algebras (Universitext. New York: Springer)

[15] Jacobson N 1971 Exceptional Lie Algebras (Lectures notes. New York: Dekker) 14

[16] Catto S, Yasemin G, Amish K and Levent K 2013 Root structures of infinite gauge groups and supersymmetric field theories Journal of Physics: Conference Series 474 012013

[17] Chevalley C and Schafer R D 1950 The exceptional simple lie algebras f4 and e6 Proc. of the National Academy of Sciences of the United States of America 36, 2 pp 13741

[18] Catto S, Yoon S C, and Levent K 2013 Invariance properties of the exceptional quantum mechanics (f4) and its generalization to complex jordan algebras (e6) in Lie Theory and Its Applications in Physics 46975 (Springer Proceedings in Mathematics & Statistics. Tokyo:Springer) pp 46975

[19] Foot R and Joshi G C 1989 Space-time symmetries of superstring and jordan algebras International Journal of Theoretical Physics 28, 12 pp 144962

[20] Faulkner J R 1971 A construction of lie algebras from a class of ternary algebras Transactions of the American Mathematical Society 155, 2 pp 397408

[21] Tits J 1962 Une classe d’algébres de lie en relation avec les algèbres de jordan Proceedings of the Koninklijke Nederlandse Academie van Wetenschappen. Series A. Mathematical sciences 65 pp 53035

[22] Vinberg E B 1966 A construction of exceptional simple Lie groups (Russian) Tr. Semin. Vektorn. Tensorn. Anal. 13 pp 7-9

[23] Coxeter H S M 1930 The polytopes with regular-prismatic vertex figures Phil. Trans. R. Soc. Lond. A 229, no. 670-680

[24] Moody R V and Patera J 1993 Quasicrystals and icosians Journal of Physics A: Mathematical and General 26, 12 p 2829