Noncommutative Torus from Fibonacci Chains via Foliation

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Abstract

We classify the Fibonacci chains (F-chains) by their index sequences and construct an approximately finite dimensional (AF) C\textsuperscript{*}-algebra on the space of F-chains as Connes did on the space of Penrose tiling. The K-theory on this AF-algebra suggests a connection between the noncommutative torus and the space of F-chains. A noncommutative torus, which can be regarded as the C\textsuperscript{*}-algebra of a foliation on the torus, is explicitly embedded into the AF-algebra on the space of F-chains. As a counterpart of that, we obtain a relation between the space of F-chains and the leaf space of Kronecker foliation on the torus using the cut-procedure of constructing F-chains.

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1 Introduction

Recently, noncommutative geometry (NCG) has been one of the most active areas in mathematics with increasing interest and application to physics [1, 2, 3]. Not only it opens new areas in pure mathematics but its application to physics has now reached to the very frontier of fundamental physics, such as string and M theories [4, 5]. Many in the string/gravity circle, now consider NCG as a very possible candidate for the underlying mathematical framework of quantum theory of gravity [3, 4, 5]. However, applications of NCG to physical systems have not been confined to high energy physics. Bellissard already applied NCG to the quantum Hall effect and explained the Hall conductivity using the K-theory on the noncommutative algebra of functions on the Brillouin zone [6].

Quasicrystals seem to be another novel systems to which we may able to provide a “quantum-jump” progress when we adopt a NCG approach. Quasicrystals are new types of solids with ordered atomic arrangement but with a discrete point-group symmetry forbidden for periodic systems. Discovery of such materials in 1984 [10] has brought tremendous impact on condensed matter physics and material science. By then, the only known ordered solid state structures were crystals and solid states were considered either periodically ordered crystals or disordered amorphous materials. The structure of this new type of solids has been explained with Penrose tiling in which two types of prototiles arranged aperiodically [11]. One can show that the Penrose tiling lattice has the translational order and the rotational symmetry of observed quasicrystals by calculating their Fourier components [2]. Furthermore, Penrose tiling models provides clues to solving the puzzle of physical realization of such structure through atomic interaction, that is, the question of “why do atoms form complex Penrose tiling pattern rather than regularly repeating crystal arrangement?” [13, 14]. However, the study of its dynamical properties, the most important secret of quasicrystals, is in its infant stage yet. This may require a new tool to analyze since quasicrystals defy the standard classical classification of solids. We think that NCG can be a candidate for this. Connes already pointed out that the space of Penrose tiling can be analyzed nontrivially only with noncommutative algebra which has a quantum mechanical nature [2]. An indication of quantum nature may have already appeared in the fact that the symmetry of Penrose tiling is not intuitively observed from its real space lattice structure. As stressed by Rabson and Mermin [15], the symmetry of Penrose tiling is easily seen in Fourier transformed space through the phases of the wave-functions in a scattering process, which hardly play any role in the classical treatise.

Connes’ analysis of the Penrose tiling space is based on the scale invariance of the Penrose tilings. Using the inflation (see section 2) a Penrose tiling can be identified with a sequence consisted of 0’s and 1’s [4, 16]. Two different sequences correspond to the same tiling if their entries differ only in a finite number of terms. When this equivalence relation is taken into account, the space of tilings is given by the quotient
space obtained from the space of sequences mod out by the equivalence relation. As Connes pointed out in his book [2], one can hardly get any interesting information about this space if it is treated as an ordinary space with classical tools. For given any two Penrose tilings, one cannot distinguish one from the other with any finite portion of them since it appears in both tilings [16]. This tells us that the topology of the space of tilings is trivial, namely the space of tilings is equivalent to a single point. However, treating the space of tilings as a quantum space or noncommutative space, one can find its interesting topological invariant, the dimension group which is not trivial at all [2]. This is because a topologically trivial space cannot be described nontrivially by complex-valued functions. However, with operator-valued functions on this space one can explore the nontrivial structure of this seemingly trivial space.

The study of quasiperiodic structure along the noncommutative geometric approach was first done by Bellissard et al. [17] in a one-dimensional (1D) case. They investigated its spectral properties and tried to construct a quantal observable algebra which plays the role of the above mentioned operator-valued functions. However, their investigations fell short of geometric properties in the sense of Connes.

Recently, the study in the noncommutative geometry framework was done by Landi and companies [18] from the view point of noncommutative lattice which can be regarded as a finite topological approximation of a quantum physical model. They performed their investigation by studying the K-theory of the approximately finite dimensional (AF) $C^*$-algebra. For the Penrose tiling case, they retrieved the Connes’ result.

However, so far not much has been known about the underlying nature of the space of tilings. On the other hand, one can see a close resemblance of the K-theory result of Connes to that of the noncommutative torus.

In this paper, we investigate this aspect of the tiling space. We analyze the space of Fibonacci chains (F-chains) which is isomorphic to the space of Penrose tilings (see section 2). Using the “projection” method explained in section 3, both Fibonacci chains and Penrose tilings can be represented as points in the higher dimensional torus. However, we choose the space of F-chains (rather than the Penrose tiling space) since its geometrical interpretation is simpler in the torus representation [19]. By analyzing this torus parameterization from the perspective of foliation, which has become an important tool for the investigation of noncommutative geometric property, we show that the parameterized torus can be foliated to make a noncommutative torus and explain why the map from the F-chains to the leaves of foliation is surjective.

The organization of the paper is as follows. In section 2, we introduce the deflation method of obtaining the F-chains. We then construct the index sequences of the F-chains and explain the equivalence relation on them. With this, we construct an AF-algebraic structure in the manner that Connes formulated on the space of the Penrose tilings and calculate the K-theory in this scheme. In section 3, we “lift” the F-chains to a two-dimensional (2D) hyperspace. This procedure naturally leads to the
torus parameterization of the F-chains. We then show that the torus parameterization becomes the Kronecker foliation on the 2-torus when the equivalence relation of F-chains is applied. This mapping from the space of F-chains to the leaf space of the foliation is surjective. There is one “singular” leaf which corresponds to two different classes of F-chains. This “singularity” is explained in terms of both the projection method and the cut-procedure of obtaining the F-chains. In section 4, we extend the leaf space such that it can be isomorphic to the space of F-chains and embed the $C^*$-algebra of leaves of foliation into an AF-algebra on this extended space. We first obtain the equivalence relations on the extended leaf space using the equivalence relation of corresponding F-chains [2] in the finite steps. This equivalence relation partitions the space to the finite intervals. The AF-algebra is obtained as an inductive limit of the finite algebra on the space of the finite intervals. In our concluding remarks in section 5, we summarize our results and discuss the implication for future research in the properties of Penrose tiling.

2 Fibonacci chain and its K-theory

A typical example of a one-dimensional (1D) quasiperiodic structure is the so called Fibonacci chain (F-chain). An F-chain is a special infinite sequence of two segments, say, one short $S$-segment and one long $L$-segment with following properties;

1. Any finite part of the sequence appears infinite times but none of them are consequently repeated more than two times.

2. One type of segment (say $S$) cannot consequently repeated ($SS$ is not allowed).

One way to obtain this sequence of segments is the “deflation” method [16]. In this method, we start from a finite sub-chain of an F-chain. We then operate iterative substitution (deflation) rule, $S \rightarrow L$ and $L \rightarrow LS$ to build successive strings with increasing length. At any point in the chain, the type of segment ($L$ or $S$) is uniquely determined by the chosen starting sequence. Figure 1 shows such successive iterations when the starting sequence is just one segment $L$. An infinite number of iterative deflations produce an F-chain.

The inverse process of the “deflation” is the “inflation”. Now we begin with a F-chain $\mathcal{F}$ of two segments $L$ and $S$ and apply a composition (inflation), $LS \rightarrow L$ and $L \rightarrow S$. This produces another F-chain $\mathcal{F}_1$ of the two segments of $L_1$ and $S_1$, where $L_1 = LS$ and $S_1 = L$. Successive application of the compositions yields a series of F-chains $\mathcal{F}_n$ of two segments $L_n$ and $S_n$ where $L_n = L_{n-1}S_{n-1}$ and $S_n = L_{n-1}$ with $L_0 := L$ and $S_0 := S$ as shown in Fig. 2.

This naturally introduces the index sequences of the chains [16]. For a given segment $\alpha$ in the original F-chain $\mathcal{F}$, the index sequence $i(\mathcal{F}, \alpha)$ is defined as an infinite sequence.
of integers \((a_0, a_1, a_2, \ldots)\) where \(a_n = 1\) or \(0\) according as whether \(\alpha\) belongs to an \(S_n\) or \(L_n\) segment in \(F_n\) for \(n = 0, 1, \ldots\). From the inflation rule \((LS \rightarrow L, L \rightarrow S)\), it is clear that an \(S\) segment in \(F_n\) must belong to an \(L\) segment in \(F_{n+1}\), that is, \(a_n = 1\) implies \(a_{n+1} = 0\) for an index sequence \((a_n)\). In fact, one can show that the set of index sequences of F-chain \(i(F, \alpha)\) is isomorphic to the set \(Z\) of sequences \((a_n)\), with \(a_n = 1\) or \(0\) such that \(a_n = 1 \implies a_{n+1} = 0\) \([16, 20]\).

Figure 2 illustrates the way of constructing the index sequence using inflation. For the segment denoted by the triangle in \(F\), the index sequence \((a_n)\) is given by \((a_n) = (0, 0, 1, 0, 0, 0, \ldots)\) since this segment belongs to \(L, L, S, L, L, L\) segments in the \(F, F_1, F_2, F_3, F_4\) and \(F_5\) chains respectively. Similarly, the index sequence \((b_n)\) for the segment denoted by the circle is given by \((b_n) = (1, 0, 1, 0, 1, 0, 0, \ldots)\). Note that the indices in both sequences are the same for \(n \geq 5\) since both the triangle and circle segments in \(F\) belongs to the same segments for \(n \geq 5\) chains. In fact, the inflation will make any two segments in \(F\) separated by a finite distance belong to the same segment in \(F_n\) for sufficiently large \(n\). Therefore, for the index sequences, \((a_n) = i(F, \alpha)\) and \((a'_n) = i(F, \alpha')\) from two given segments in the same chain, there must be an integer \(M\) such that \(a'_n = a_n\) for all \(n > M\). This naturally leads to the following definition of the equivalence relation \(\mathcal{R}\) on \(Z\):

\[
(a_n) \sim (a'_n) \quad \text{iff} \quad \text{there is an integer } M > 0 \text{ such that } a_n = a'_n \text{ for all } n > M.
\]  

(1)
Figure 2: Successive inflations of the F-chain $F$ produce a sequence of F-chains $F_1$, $F_2$, and so on. For the segment denoted by the triangle in $F$, the index sequence is given by $(0, 0, 1, 0, 0, 0, \ldots)$ whereas that for the segment denoted by the circle is given by $(1, 0, 1, 0, 1, 0, \ldots)$.

With this equivalence relation, it is trivial that any two index sequences from the same F-chain are in the same equivalence class.

Conversely, one can also show that any two different sequences in $Z$ with $a_n = a'_n$ for all $n > M$ can be constructed as index sequences from two different segments in the same F-chain [10]. Therefore, the space of F-chain is given by the quotient space $X = Z/R$. In fact this identification allows us to see the space of F-chains as a noncommutative space, as was noted in [3]. In other words one can define the $C^*$-algebra associated to the quotient space $X$.

In what follows we review the construction of the $C^*$-algebra and the computation of its $K$-theory following the lines of [3]. Consider the set

$$Z_n = \{(a_0, \cdots, a_n) \mid a_j \in \{0, 1\} \text{ and } a_j = 1 \implies a_{j+1} = 0\}.$$ 

These sets form an inverse system of sets:

$$\cdots \rightarrow Z_{n+1} \rightarrow Z_n \rightarrow \cdots \rightarrow Z_1$$

under the projection maps $Z_{n+1} \rightarrow Z_n$ given by $(a_0, \cdots, a_n, a_{n+1}) \mapsto (a_0, \cdots, a_n)$. Note that the inverse limit $\lim Z_n = Z$ is simply the set of all F-chains. On each $Z_n$, there is an equivalence relation $R_n$ given by

$$(a_0, \cdots, a_n) \sim (a'_0, \cdots, a'_n) \iff a_n = a'_n.$$  

Let $X_n = Z_n/R_n$ be the set of all equivalence classes. Since each entries of sequences in $Z_n$ are either 0 or 1, there are only two elements in $X_n$. Those two elements correspond
to 0 or 1 in the final entry. Thus the space $X_n$ cannot be described non-trivially by means of functions with values in complex numbers, $\mathbb{C}$. Instead, if we take operator-valued functions on $X_n$, there exists a very rich class of such functions. For this, each $[x] \in X_n$, one can associate a finite dimensional Hilbert space $l^2_x$ having elements of $[x]$ for an orthonormal basis and the algebra is given by the set of all functions on $X_n$ with values in operators on $l^2_x$. Note that if the dimension of $l^2_x$ is $k$, then the algebra of operators on $l^2_x$ is the algebra of all $k \times k$ matrices. More explicitly, if $x_0 (x_1 \text{ resp.})$ represents the class in $Z_n$ with 0 (1 resp.) in the final entry, then the dimension of $l^2_{x_0}$ ($l^2_{x_1}$ resp.) is the number of distinct elements in $Z_n$ that end with 0 (1 resp.). Let $k_n$ and $k'_n$ be the dimension of $l^2_{x_0}$ and $l^2_{x_1}$, respectively. Now the algebra of functions on $[x_0]$ ($[x_1]$ resp.) with values in $M_{k_n}(\mathbb{C})$ ($M_{k'_n}(\mathbb{C})$ resp.) is simply $M_{k_n}(\mathbb{C}) \oplus M_{k'_n}(\mathbb{C})$ and thus the $C^*$-algebra $A_n$ of operator-valued functions on $X_n$ is identified with $M_{k_n}(\mathbb{C}) \oplus M_{k'_n}(\mathbb{C})$. Also we have an inclusion map $A_n \rightarrow A_{n+1}$ and it is uniquely determined by the equalities

$$
\begin{pmatrix}
k_{n+1} \\
k'_{n+1}
\end{pmatrix} =
\begin{pmatrix}
1 & 1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
k_n \\
k'_n
\end{pmatrix}.
$$

(3)

It allows embedding $A_n$ as block matrices in $A_{n+1}$ i.e.,

$$
\begin{pmatrix}
M & 0 \\
0 & N
\end{pmatrix} \mapsto
\begin{pmatrix}
M & 0 & 0 \\
0 & N & 0 \\
0 & 0 & M
\end{pmatrix}
$$

where $M \in M_{k_n}(\mathbb{C})$ and $N \in M_{k'_n}(\mathbb{C})$. Now we have an inductive system of $C^*$-algebras:

$$
A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_n \rightarrow \cdots
$$

(4)

Let $A = \varinjlim A_n$ be the inductive limit of the system. Then $A$ is an AF-algebra and is considered as the $C^*$-algebra of $X$. In general, approximately finite $C^*$-algebra or AF-algebra is defined by an inductive limit of a sequence of finite-dimensional $C^*$-algebras and such algebras can be completely classified by its $K$-theory \[24\]. By applying basic properties of $K$-theory \[1, 21, 22\] to the system (4), one can see that the $K$-theory of $A$ is also determined by the equations (3). Note that for each $n$,

$$
K_i(A_n) = K_i(M_{k_n}(\mathbb{C}) \oplus M_{k'_n}(\mathbb{C})) = \begin{cases} 
\mathbb{Z} \oplus \mathbb{Z} & \text{if } i = 0 \\
0 & \text{if } i = 1,
\end{cases}
$$

and the positive cone is given by

$$
K_0^+(A_n) = \mathbb{Z}^+ \oplus \mathbb{Z}^+.
$$

The map $K_0(A_n) \rightarrow K_0(A_{n+1})$ is uniquely determined by the equation (3) and is represented by $\begin{pmatrix}
1 & 1 \\
1 & 0
\end{pmatrix}$. Since $\begin{pmatrix}
1 & 1 \\
1 & 0
\end{pmatrix}$ is invertible in $\mathbb{Z} \oplus \mathbb{Z}$, it is an isomorphism on
\[ K_0(A_n) \longrightarrow K_0(A_{n+1}) \text{ for all } n \geq 0 \text{ and we have} \]
\[ K_0(A) = \lim_{n \to \infty} K_0(A_n) \cong \mathbb{Z} \oplus \mathbb{Z}. \]

On the other hand, \( \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \) is not invertible in \( \mathbb{Z}^+ \oplus \mathbb{Z}^+ \). To compute \( K_0^+(A) \), let \( (a, b) \in \mathbb{Z} \oplus \mathbb{Z} \) and then
\[ K_0^+(A_1) = \{(a, b) \in \mathbb{Z} \oplus \mathbb{Z} \mid a + b \geq 0 \text{ and } b \geq 0\}, \]
and if we let \( \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} f_{11}^n & f_{12}^n \\ f_{21}^n & f_{22}^n \end{pmatrix} \), then
\[ K_0^+(A_n) = \{(a, b) \in \mathbb{Z} \oplus \mathbb{Z} \mid f_{11}^na + f_{12}^nb \geq 0 \text{ and } f_{21}^na \geq 0\}. \]

From the computation of \( \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} f_{11}^n & f_{12}^n \\ f_{21}^n & f_{22}^n \end{pmatrix} \), we see that
\[ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix} \]
with the defining relation:
\[ f_{n+1} = f_n + f_{n-1}, \text{ and } f_1 = f_2 = 1. \]

Thus
\[ K_0^+(A_n) = \{(a, b) \in \mathbb{Z} \oplus \mathbb{Z} \mid f_na + f_{n+1}b \geq 0 \text{ and } f_{n-1}a + f_nb \geq 0\} \]
\[ = \{(a, b) \in \mathbb{Z} \oplus \mathbb{Z} \mid a + \frac{f_{n+1}}{f_n}b \geq 0\}. \]

From this
\[ K_0^+(A) = \lim_{n \to \infty} K_0^+(A_n) = \{(a, b) \in \mathbb{Z} \oplus \mathbb{Z} \mid a + \tau b \geq 0\}, \]
where \( \lim_{n \to \infty} \frac{f_{n+1}}{f_n} = \tau \) is the Golden mean. Now the space of F-chains is completely characterized by the ordered group
\[ (K_0(A), K_0^+(A)) = (\mathbb{Z}^2, \{(a, b) \in \mathbb{Z} \oplus \mathbb{Z} \mid a + \tau b \geq 0\}). \]

Recall that the noncommutative torus is the \( C^* \)-algebra generated by two operators \( u, v \) subject only to
\[ uv = e^{2\pi i \Theta} vu \]
where \( \Theta \) is a real number. It is well-known that the \( K \)-theory of the noncommutative 2-torus \( A_\Theta \) is given by \( K_i(A_\Theta) \cong \mathbb{Z}^2 \), where \( i = 0, 1 \). In particular, \( K_0(A_\Theta) \) is isomorphic
to \( \mathbb{Z} \oplus \Theta \mathbb{Z} \) as ordered groups by a theorem of Pimsner and Voiculescu [25]. Furthermore, the noncommutative torus can be embedded into a certain type of AF-algebra as discussed in Landi, Lizzi, and Szabo recently [26]. In the above we have shown that the \( C^* \)-algebra of the space of F-chains is an AF-algebra and its \( K \)-groups were computed. Furthermore, the Bratteli diagram of the AF-algebra satisfies the condition required by Landi \emph{et al.}'s work with \( c_n = 1 \) in their notation [18, 1]. Thus one might expect that the torus \( A_\tau \) can be embedded into the \( C^* \)-algebra of the space of F-chains. As a dual picture, if we can realize the noncommutative torus as a geometric object, then we may characterize the space of F-chains from the space associated to the algebra \( A_\tau \). In the next section we will show that the space of F-chains can be determined by submanifolds of ordinary torus \( T^2 \).

3 Torus representation and foliation

In this section we establish a precise relation between the space of F-chains and the leaf space of the Kronecker foliation on the torus. This will be done in the torus representation of F-chains [19]. We first review the definition of the Kronecker foliation and study the correspondence between the noncommutative torus and the Kronecker foliation. We then show how they are related to the space of F-chains.

In general, a foliation of codimension \( q \) on an \( n \)-dimensional manifold is a partition of the manifold into \( p \)-dimensional connected submanifolds, where \( n = p + q \). Such submanifolds are called the leaves of the foliation. Locally the leaves look like a set of parallel planes of codimension \( q \) in Euclidean space. The space of leaves can be understood as families of solutions of systems of differential equations and the study of foliation is the study of the global behavior of such solutions. For example, a first order differential equation is a vector field. For a vector field without zeros, the orbits of the flow generated by the vector field form a 1D foliation. See [23] for details for the theory of foliations.

It is well-known that the 2-torus \( T^2 \) is the only oriented compact 2-dimensional manifold which admits a non-singular codimension 1 foliation. Up to topological equivalence one can classify smooth foliations of \( T^2 \) [23]. In particular, there is a foliation which contains no closed leaves and this foliation is equivalent to the Kronecker foliation with irrational slope. Let \( T^2 = S^1 \times S^1 = \mathbb{R}^2 / \mathbb{Z}^2 \) with natural coordinates \((x, y) \in \mathbb{R}^2\). For non-zero constants \( a_1 \) and \( a_2 \), a smooth one-form \( \omega = a_1 \, dx + a_2 \, dy \) on the torus defines a foliation on \( T^2 \). The leaves of this foliation are the solutions of the differential equation

\[
dy = -\frac{a_1}{a_2} \, dx.
\]

If \( \frac{a_1}{a_2} \) is rational, then each leaf is closed and hence a circle on the torus. If \( \frac{a_1}{a_2} \) is irrational, then all the leaves are diffeomorphic to \( \mathbb{R} \) and each leaf is dense in \( T^2 \). This foliation is called the Kronecker foliation associated to a real number \( -\frac{a_1}{a_2} \). From now
on, we will restrict ourselves to the case when \( \frac{a_1}{a_2} = \frac{1}{\tau} \), where \( \tau \) is the Golden mean. Each leaf in this case can be seen as a straight line in \( \mathbb{R}^2 \) with the fixed slope, \( y = \frac{1}{\tau}x + b \).

Since a straight line \( y = \frac{1}{\tau}x + b \) is determined by its \( y \)-cut, we see that the space of leaves of the foliation is parameterized by the \( y \)-cuts. On the torus, two lines \( \frac{1}{\tau}x + b \) and \( \frac{1}{\tau}x + b' \) represent the same leaf if \( b - b' = \frac{1}{\tau}n \), for some integer \( n \). This defines an equivalence relation on the \( y \)-cuts and the leaf space \( X_F \) of the Kronecker foliation can be identified with the set of equivalence classes. The topology on the space of leaves is the same as the quotient topology of \( S^1 = \mathbb{R}/\mathbb{Z} \) divided by the partition into orbits of the rotation given by \( z \mapsto z + \frac{1}{\tau} \), where \( z \in S^1 \), and hence there are no open sets in \( X_F \) except \( \emptyset \) and \( X_F \). Therefore, the leaf space has the trivial topology as in the case of the space of \( F \)-chains.

The Kronecker foliation can also be obtained from the suspension of diffeomorphisms \( [23] \). Let \( \psi_\tau : S^1 \to S^1 \) be the diffeomorphism which is the rotation through angle \( \frac{2\pi}{\tau} \), i.e., \( \psi_\tau(z) = e^{\frac{2\pi}{\tau}} \cdot z \), \( z \in S^1 \). The product manifold \( S^1 \times \mathbb{R} \) is foliated by the leaves of the form \( \{z\} \times \mathbb{R} \). This foliation on \( S^1 \times \mathbb{R} \) is invariant under the \( \mathbb{Z} \)-action on \( S^1 \times \mathbb{R} \): for \( z \in S^1 \), \( b \in \mathbb{R} \),

\[
(z, b)^n = (\psi_\tau^n(z), b + n), \quad n \in \mathbb{Z}.
\]

This means that the quotient \( S^1 \times_{\mathbb{Z}} \mathbb{R} \cong \mathbb{T}^2 \) carries a 1D foliation whose leaves are the image of \( \{z\} \times \mathbb{R} \) under the quotient map \( S^1 \times \mathbb{R} \to \mathbb{T}^2 \). Equivalently, the leaves are transverse to the fibers of \( S^1 \times_{\mathbb{Z}} \mathbb{R} \to \mathbb{R}/\mathbb{Z} \cong S^1 \). Thus the space of leaves of the foliation on \( \mathbb{T}^2 \) are parameterized by \( \mathbb{R} \) together with the \( \mathbb{Z} \)-action associated to the action of Eq. (5). This is exactly the same relation as the one in the Kronecker foliation and hence the foliation obtained via the diffeomorphism \( \psi_\tau \) is equivalent to the Kronecker foliation on \( \mathbb{T}^2 \) with the irrational slope \( \frac{1}{\tau} \). This is in fact followed by the Denjoy’s theorem which asserts that if a foliation of \( \mathbb{T}^2 \) does not have compact leaves, then it is topologically equivalent to a foliation obtained by a suspension of an irrational rotation of the circle. Also, Denjoy constructed examples of foliations on \( \mathbb{T}^2 \) with exceptional minimal sets and this motivated the study of minimal sets of foliations of codimension one on compact manifolds of dimension \( \geq 3 \). Here we briefly review Denjoy’s example which is obtained by the suspending the diffeomorphism \( \psi_\tau : S^1 \to S^1 \) with an exceptional minimal set \([24]\). In section 4, we will identify the set of all \( F \)-chains with the exceptional minimal set. A subset \( E \) of \( S^1 \) is said to be minimal if it is closed, nonempty and invariant under \( \psi_\tau \) and also if \( E' \subset E \) is closed, invariant subset then either \( E' = \emptyset \) or \( E' = E \). A minimal set is called exceptional if it is homeomorphic to a subset of the Cantor set on \( S^1 \). The exceptional minimal set for \( \psi_\tau \) is constructed in the following manner. First, cut the circle \( S^1 \) at all the points of an orbit \( \{\theta_n \mid n \in \mathbb{Z}\} \) of the given irrational rotation. At the \( n \) cutting point, insert a segment \( J_n \) of length \( l_n \) with \( \sum l_n < \infty \). Then we get a new circle and the set \( S^1 - \bigcup_{n \in \mathbb{Z}} J_n = E \) is homeomorphic to the Cantor set and this is the desired exceptional minimal set.
From the above construction of the Kronecker foliation, one can relate the $C^*$-algebra of the foliation to the noncommutative torus. Let $C(S^1)$ be the $C^*$-algebra of continuous functions on $S^1$. Then the rotation $\psi_\tau : S^1 \to S^1$ induces the automorphism $\psi_\tau^* : C(S^1) \to C(S^1)$ given by $\psi_\tau^*(f) = f \circ \psi_\tau$, where $f \in C(S^1)$, or
\[
(\psi_\tau^* f)(z) = (f \circ \psi_\tau)(z) = f(e^{\frac{2\pi i}{\tau}} \cdot z), \quad z \in S^1.
\]
Let us denote the group of all automorphisms of $C(S^1)$ by $\text{Aut}(C(S^1))$. Then the action of Eq. (3) can be given as the group homomorphism $\alpha : \mathbb{Z} \to \text{Aut}(C(S^1))$ which is given by
\[
(\alpha(n)f)(z) := (\alpha_n f)(z) = (f \circ \psi_{\tau n}^*)(z) = f(e^{\frac{2\pi i n}{\tau}} \cdot z).
\]

Now the $C^*$-algebra of this action is so called the cross product $C^*$-algebra $C(S^1) \rtimes_{\alpha} \mathbb{Z}$ [21]. As we have seen above, this algebra is generated by the rotation and the $\mathbb{Z}$-action. More explicitly, the $C^*$-algebra is represented on $L^2(S^1)$ with generators $U$ and $V$ according to the rotation and $\mathbb{Z}$-action:
\[
(Uf)(z) = zf(z) \quad \text{and} \quad (Vf)(z) = f(e^{\frac{2\pi i}{\tau}} \cdot z), \quad f \in L^2(S^1), \quad z \in S^1.
\]
It is easy to verify that the operators $U$ and $V$ satisfy the relation
\[
UV = e^{\frac{2\pi i}{\tau}} VU.
\]

Hence the $C^*$-algebra $C(S^1) \rtimes_{\alpha} \mathbb{Z}$ is identified with the noncommutative torus $A_{\frac{1}{\tau}}$ and also it can be regarded as the $C^*$-algebra on the leaf space of the Kronecker foliation with the slope $\frac{1}{\tau}$. Also this is Morita equivalent to the noncommutative torus $A_r$ [2] and hence its $K$-theory is given by $K_i(A_r) \cong \mathbb{Z}^2$, where $i = 0, 1$. In particular, $K_0(A_r)$ is isomorphic to $\mathbb{Z} \oplus \tau \mathbb{Z}$ as ordered groups as discussed in section 2. In below we will establish a relation between the space of the F-chains and the leaf space of the Kronecker foliation on the torus $\mathbb{T}^2$ appearing in the torus representation [19].

An F-chain can be represented as a point in a 2-torus $\mathbb{T}^2$ [19]. Here we will show that all F-chains in an equivalent class can be represented as a leaf of the foliation on the torus. First we construct a Fibonacci lattice (F-lattice) from an F-chain. An F-lattice is a 1D tiling consists of two prototiles $L$ and $S$ whose arrangements form an F-chain. The ratio of the lengths of the two tiles, $m(L)$ and $m(S)$, is given by
\[
\frac{m(L)}{m(S)} = \tau.
\]
Figure 3 shows an F-lattice (upper part of (a)) and a way of lifting it into a 2D hyper-space which is a direct product of two 1D spaces; the “parallel space” $\mathbb{R}_{\parallel}$ and the “perp-space” $\mathbb{R}_{\perp}$. The parallel space $\mathbb{R}_{\parallel}$ is a straight line parallel to the F-lattice. The perp-space $\mathbb{R}_{\perp}$ is the 1D space perpendicular to $\mathbb{R}_{\parallel}$.

The coordinate of a vertex (F-lattice point; the boundary between two given consequent tiles) relative to any reference vertex can be expressed in the form $n_L m(L) + n_S m(S)$, where $(n_L, n_S) \in \mathbb{Z}^2$, $m(S) = \sin \theta$ and $m(L) = \cos \theta$ with $\theta = \arctan(1/\tau)$. 

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Figure 3: The lifting of a Fibonacci chain (F-chain) into a 2D hyper-space. For a given F-chain, we can construct a 1D tiling consists of two prototiles $L$ and $S$ which can be lifted into a 2D square lattice whose $x$-axis has the slope $-1/\tau$ with respect to the 1D tiling on $\mathbb{R}_\parallel$. The embedded step (solid thick line) in the 2D lattice can be covered by a strip parallel to $\mathbb{R}_\parallel$ with width $\Delta = \cos \theta + \sin \theta$ when the position of the strip is well chosen. The perp coordinate $t_\perp$ is given by the $\mathbb{R}_\perp$ coordinate of the strip bottom. This value is uniquely determined for a given infinite F-chain. In the upper part of (a), only a finite part of the F-chain is shown. Therefore, the position of the strip which covers the finite embedded step is not uniquely determined. All three strips in (a), (b) and (c) cover the shown finite part but correspond to different (infinite) F-chains. The highest and the lowest strips, shown in (b) and (c) respectively, corresponds to the singular F-chains. The parallel coordinate $t_\parallel$ of the F-chain depends on the choice of the vertex in the tiling.

Therefore, the vertices can be lifted into a square lattice of $\mathbb{Z}^2$ as shown in the lower part of Fig. 3(a). All pairs of adjacent vertices in the 1D tiling separated by a tile $L$ or $S$ are mapped onto neighboring vertices of a 2D square lattice in the $x$ or the $y$
directions respectively where the $x$-axis has the slope $-1/\tau$. The embedded step (thick solid line) in the 2D lattice by this lifting can be covered by a strip parallel to the $\mathbb{R}_\parallel$ with width $\Delta = \cos \theta + \sin \theta$ if the position of the strip is well chosen.

For a given (infinite) F-chain, the “perpendicular space” $\mathbb{R}_\perp$ coordinate $t_\perp$ of the strip (defined as the $\mathbb{R}_\perp$-coordinate of the bottom boundary of the strip), which covers the embedded step completely, is uniquely determined. Therefore, we can assign a $t_\perp$ value for a given F-chain. The $\mathbb{R}_\parallel$ coordinate $t_\parallel$ of the chain is not uniquely determined but depends on the choice of the vertex in the 1D tiling. Figure 3(a) shows two different values; $t_{1,\parallel}$ for the triangle vertex and $t_{2,\parallel}$ for the circle vertex. In section 2, we mentioned that two sequences in $\mathbb{Z}$ which are equivalent by $\mathcal{R}$ of Eq. (1) can be constructed from two different segments in the same F-chain. In other words, two F-chains in an equivalence class can be considered as a finite translation of each other. Since the translation in the $\mathbb{R}_\parallel$ direction corresponds to the movement along the leaf in the torus representation, all F-chains in an equivalent class can be represented as the points on the same leaf on the torus no matter what vertices we choose for $t_\parallel$.

Conversely, an F-lattice (hence an F-chain) can be obtained from a 2D square lattice by the projection methods. The lattice sites of the 2D square structure can be projected onto the 1D parallel space, $\mathbb{R}_\parallel$ at the slope $\tan \theta = 1/\tau$ with respect to the horizontal rows of the square lattices. Since the slope of the line is irrational, the projection of all 2D lattice points to $\mathbb{R}_\parallel$ form a dense set of points. If we restrict projections on $\mathbb{R}_\parallel$ to the points confined within a strip which is parallel to $\mathbb{R}_\parallel$ and has a cross section $\Delta$ in $\mathbb{R}_\perp$ equals to the perp-space projection of a square unit cell ($\Delta = \cos \theta + \sin \theta$), then the projection to $\mathbb{R}_\parallel$ gives an F-lattice [27]. The movement of the strip along the perp-space $\mathbb{R}_\perp$ gives rise to rearrangement of tiles from one perfect F-lattice to another and the strip is called the “window” of the corresponding F-lattice. In general, the windows should include one and only one boundary to produce a perfect F-lattice. Figure 3 can be also used to illustrate the “projection methods”. Now we first choose a window and select the 2D lattice points which are in the window. Then the projection of those lattice points into the $\mathbb{R}_\parallel$ space gives the vertices of an F-lattice. The boundaries are irrelevant to the projected structure unless they pass a 2D lattice point since all vertices of the F-lattice are produced from the 2D lattice points inside the window (Fig. 3(a)). When a boundary of the window intersects with a 2D lattice point, so does the other boundary as shown in Fig. 3(b) since the width of the window equals to the perp-space projection of a square unit cell. If the window included both boundaries, the projection would produce an extra vertex from the 2D unit cell denoted by hatching. On the other hands, if it excluded both, the projected lattice would miss

---

1 In the upper part of Fig. 3(a), only a finite part of the F-chain is shown. Therefore, the position of the strip which covers the finite embedded step is not uniquely determined. All three strips in the figure cover the finite embedded step. The highest strip shown in (b), whose lower boundary passes the lattice point, corresponds to the infinite F-chain whose index sequence is (101010...). The lowest strip in (c) corresponds to (01010...). Details will be discussed later.
a vertex. Therefore, the “proper” windows must include one and only one boundary. If we include the lower boundary as shown in Fig. 3(b), then we get the ‘tile arrangement ’LS’ from the hatched unit cell, while we get the ‘SL’ arrangement for the other case. In other words, there are two different F-lattices (and hence two different F-chains) corresponding two different proper windows in spite of the $R_{\perp}$ position of the window is the same.

This “singularity” for the windows whose boundaries pass the lattice points can be more clearly understood in the cut-procedure. A leaf on the torus can produce an F-chains (but not an F-lattice) directly (instead of going through a strip or a window) in the cut-procedure. Figure 4 illustrates a way to get an F-chain by this method from a square lattice in a 2D “hyper-space”. We consider a 2D square lattice and the lines with the slope of $1/\tau$. (A straight line in a square lattice can be considered as a representation of a leaf on the torus in the “extended” scheme.) We can produce an F-chain associated to the line in the following way. If the line intersects the $y$-axis, we give the segment “L” while we assign the segment “S” when the line intersects with the $x$-axis. For example, the line $l_r$ in Fig. 4 intersects $\cdots$, $y$, $x$, $y$, $x$, $y$, $x$, $y$, $x$, $y$, $\cdots$ axes and hence produces an F-chain $F_r = \cdots LSLLSLLSLLSLL \cdots$. The correspondence between a straight line and an F-chain is one-to-one except the “singular” line $l_s$ which passes the origin. For the singular case, the three lines, the parallel line $l_s$, the $x$-axis and the $y$-axis meet at a point (at the origin). Since $l_s$ meets both the $x$ and the $y$ axes at the same time, a pair of segments (one $S$ and one $L$ segments) should be assigned at the origin. However, assigning two segments at the same point is impossible. This “singularity” can be resolved by moving the line $l_s$ infinitesimally. If we move $l_s$ slightly upward, it first meets the $x$-axis and then the $y$-axis and ‘SL’ is assigned at the origin. Therefore, the parallel line $l_s$ produces the F-chain $F_{s1}$ in Fig. 4 when it is moved upward infinitesimally. In contrast, if we move $l_s$ slightly downward, it meets the $x$-axis first and then meets the $y$-axis. Therefore, ‘LS’ is assigned at the origin and we have $F_{s2}$ in this case.

Now, the space of F-chains can be parameterized by the $y$-axis in $\mathbb{R}$ since the line $y = \frac{1}{\tau}x + b$ is determined by the $y$-intercepts. As in the leaf space of the Kronecker foliation, two F-chains that correspond to two lines $\frac{1}{\tau}x + b$ and $\frac{1}{\tau}x + b'$ are in the same equivalence class if $b - b' = n/\tau$. This is because the arrangements of intersections from the two lines, hence the two corresponding F-chains, are the same up to the finite translation (by $n\sqrt{\tau^2 + 1}/\tau$) when $b - b' = n/\tau$. If two F-chains $F$ and $F'$ are the same up to a finite translation, they are in the same class by the equivalence relation $\mathcal{R}$ of Eq. (2). Let $(a_n)$ and $(a'_n)$ be index sequences of the two F-chains with $(a_n) = i(F, \alpha)$ and $(a'_n) = i(F', \alpha')$ where $\alpha$ and $\alpha'$ are the segments at the origins of $F$ and $F'$ respectively. Then, there is a segment $\alpha'' \in F$ within a finite distance from the origin such that $(a''_n) = i(F, \alpha'')$ be the identically same as $(a'_n)$ since $F'$ is a finite translation of $F$. Now, we have an integer $M$ such that $a_n = a'_n$ for all $n > M$ since the inflation will make two segments $\alpha$ and $\alpha''$ belong to the same segment in $F_n$ for sufficiently
Figure 4: An F-chain can be obtained from the sequence of intersections between a line with slope $1/\tau$ and the $x$ and $y$ axes. A regular line $l_r$ intersects \ldots, $y, x, y, x, y, x, y, x, y, \ldots$ axes and corresponds to a unique F-chain, \ldots $LSLLSLLSLL$ \ldots$. The singular line $l_s$, which passes through the origin, corresponds to the two different F-chains, $F_{s_1}$ and $F_{s_2}$.

large $n$.

We have shown that the space of F-chains is the same as the leaf space of Kronecker foliation except that the singular leaf which corresponds to the two F-chains. One may think that the two F-chains corresponding to the singular leaf are the same class. The only difference between two chains are at the origin which can be removed by a local surgery; we can obtain one chain from the other by flipping one pair of segment at the origin. Furthermore, one is a mirror image of the other and related by a 180 degree rotation. However, they are not in the same class by the equivalence relation $R$ of Eq. (1). If we construct the index sequences of the two F-chains from the segment at the origin, one chain corresponds to the index sequence $(a_n) = (0101010101\ldots)$ and the other corresponds to $(a_n') = (1010101010\ldots)$ (see section 4). In other words, $a_n = \delta_{n,2k}$ for one chain while $a_n' = \delta_{n,2k+1}$ for the other chain. Clearly $a_n$ and $a_n'$ are different for all $n$ and they are in different classes.
Since we have two different F-chains on the singular leaf (which is only one leaf on the Kronecker foliation) on $\mathbb{T}^2$, we have a surjective map from the space of F-chains to the space of leaves. Both spaces have trivial topology and the map is open and continuous. Now, the surjectiveness corresponds to that the map from the C*-algebra of leaves of foliation (noncommutative torus) to the C*-algebra of F-chains (we already showed in section 2 that it is an AF-algebra) is injective. In this sense our discussion above can be seen as a dual picture of the embedding of noncommutative torus into a certain type of AF-algebra.

In the following section, we construct such an AF-algebra by introducing an extended space of leaves which is isomorphic to the space of F-chains.

4 An extended space of leaves

Since the map from the space of F-chains to the leaf space of the Kronecker foliation is surjective, we cannot retrieve all F-chains from the leaves on $\mathbb{T}^2$. However, there is only one leaf which corresponds to more than one class of F-chains. Furthermore, this singular leaf corresponds to only two classes of F-chains. Therefore, if we assign one class of F-chain to every leaf (including the singular leaf), all F-chains except only one class of F-chains are obtained from the leaf space. For example, if we assign $F_{s_2}$ in Fig. 4 to the singular leaf, then $F_{s_1}$-class is “missing” but all other F-chains are in the leaf space on $\mathbb{T}^2$. In this section, we show that an extended leaf space, which is isomorphic to the space of F-chains, is naturally obtained if we construct the equivalence relations on the leaf space using the equivalence relation $R_n$ of the finite subsequences of the index sequences given by Eq. (2). In the limit of the length of the subsequences goes to infinity, we get the extended leaf space which is the sum of two spaces; the leaf space on $\mathbb{T}^2$ and the space consists of one leaf corresponding the “missing” class.

In section 3, the index subsequences of the F-chains are constructed using the inflation and the AF-algebra is introduced as an inductive limit of the finite algebra on the space of the finite subsequences of the index sequences. There were only two sets of equivalence classes on the space of the finite subsequences since the last entries of sequences $a_n$ in $\mathbb{Z}_n$ can have only two values, either 0 or 1. Here, we consider the space of straight lines in the cut-procedure of Fig. 4. Except for the singular lines which passes the lattice points, each line produces one and only one F-chain. The equivalence relation between the lines is constructed according to the equivalence relation $R_n$ for their F-chains (Eq. (3)). As before, the straight lines are parameterized by the $y$-intercepts. To define the equivalence relation between the lines, we consider the index sequences of their F-chains $i(F(b), \alpha)$ where $\alpha$ is the segment at $x = 0$ (at the $y$-axis) and $F(b)$ is the F-chain corresponds to the line $y = \frac{1}{\tau}x + b$. The index $a_0$ of the original (uninflated) F-chain $F$ is 0 for every $b \in W_0 := (0, 1)$ since the type of the segment at the origin is always $L$ by the definition of the cut-procedure (at $x = 0$, the leaf $y = \frac{1}{\tau}x + b$ always cut the $y$-axis). Figure 3(a) illustrates this; any
Figure 5: A sequence of partitions of the transversal (1,0) using the cut-procedure and the inflations of the $F$-chain. $F_{n+1}$ is the inflation of $F_n$ as in Fig. 2.
$b \in W_0$ corresponds to the $L$ segment in $\mathcal{F}$ at the origin. However, the type of the segments at the origin in the inflated F-chains $\mathcal{F}_1$ can be both $L$ and $S$ depending on the value of $b$. As shown in Fig. 3(b), the segment arrangement in $\mathcal{F}$ at the origin is $LS$ for $b \in A = (1/\tau^2, 1)$, and $LL$ for $b \in B = (0, 1/\tau^2)$. Since $LS$ becomes $L$ and $L$ becomes $S$ by inflation, the segment type in $\mathcal{F}_1$ at the origin is $L$ for $b \in A$ and $S$ for $b \in B$. Therefore, $W_0$ is partitioned by two open intervals, $A$ and $B$ and a boundary point $b_1 = 1/\tau^2$ for $n = 1$. Let us denote the union of the two open intervals by $W_1$; $W_1 = (0, 1/\tau^2) \cup (1/\tau^2, 1) = W_0 - b_1$. Note that the boundary point $b_1$ is given by the intersection between $W_0$ and the line which passes the 2D lattice point $(1, 1)$. For this boundary line $y = 1/\tau x + (1 - 1/\tau)$, the segment type at the origin of $\mathcal{F}_1$ is not well defined, that is, $b_1$ is the singular point for $\mathcal{F}_1$. Similarly, we can partition $W_1$ by considering the doubly inflated F-chains $\mathcal{F}_2$. Since an $S$-segment in $\mathcal{F}_1$ becomes an $L$-segment in $\mathcal{F}_2$, the interval $B$ in Fig. 3(b) is not divided for $n = 2$ (interval $C$ in Fig. 3(c)). The interval $A$ in Fig. 3(b) which represents the class of $a_1 = 0$ is divided by two intervals by the line which passes the 2D lattice point $(2, 2)$. Therefore, we get the three intervals, $W_2 = (0, 1/\tau^2) \cup (1/\tau^2, 2/\tau^2) \cup (2/\tau^2, 1)$ for $n = 2$ as shown in Fig. 3(c). In general, an interval corresponding to $a_n = 1$ becomes an interval corresponding to $a_{n+1} = 0$ in the next step while an interval corresponding to $a_n = 0$ will be divided as two neighboring intervals, one for $a_{n+1} = 0$ and the other for $a_{n+1} = 1$. Therefore, the partitioned interval for the $n$th inflated chain, which will be denoted by $W_n$, is given by the union of $f_{n+2}$ intervals with $f_{n+1}$ $L$-intervals and $f_n$ $S$-intervals;

$$W_n = W_{n,L} + W_{n,S},$$

where

$$W_{n,L} = \sum_{k=1}^{f_{n+1}} I_{n,L_k}, \quad W_{n,S} = \sum_{k=1}^{f_n} I_{n,S_k}.$$

Here, $I_{n,L_k}$ ($I_{n,S_k}$ resp.) is the $k$th interval of $L$-type ($S$-type resp.) in $W_n$ and $f_k$ is the Fibonacci number introduced in section 2. The lengths of the $L$ and the $S$-intervals in $W_n$ are $1/\tau^n$ and $1/\tau^{n+1}$ respectively.

Figure 3 shows the arrangement of $I_{n,L_k}$ and $I_{n,S_k}$ in $W_n$. From Fig. 3, we see that the intervals are divided by the lines which pass the 2D lattice point $(r, s)$ such that $0 < s - r/\tau < 1$. Let us arrange such lattice points according to the “parallel” distance $d(r, s) = \tau s (r + s/\tau)$ and denote them as $P_k = (r_k, s_k)$ where $d(r_{k'}, s_{k'}) < d(r_k, s_k)$ for $k' < k$. Now, let $l_k$ be the line which passes the lattice point $P_k$. Then the $f_{n+1}$ lines, $l_{f_{n+1}}, \ldots, l_{f_{n+2}}$, divide the $f_{n+1}$ $L$-intervals in $W_n$. For example, $l_1$ which passes $P_1 = (1, 1)$ divides the $L$-interval in $W_0$, $l_2$ which passes $P_2 = (2, 2)$ divides the $L$-interval in $W_1$, and $l_3$ and $l_4$ which pass $P_3 = (3, 2)$ and $P_4 = (4, 3)$ divide the $L$-intervals in $W_2$. Note that $l_1$ divides the $L$-interval of $W_0$ such that the lower part of it becomes an $S$-interval in $W_1$ while $l_2$ divides the $L$-interval in $W_1$ such that the
up to a lattice translation. In other words, relative arrangements of the lattice points from the “boundary” lattice points are the same for all L-intervals in $W_n$ and hence divided in the same pattern by the boundary lines in the following inflations. The statement (2) can be shown with an inductive way. Here we outline the proof for even $n$. The odd $n$ case can be proven similarly. For $n = 2$, the bottommost interval is an L-interval and the upper boundary is given by the line which passes the lattice point $(1, 1)$. Let the bottommost interval of $n = 2k$ case be an L-interval. Then the size of the interval is $\tau^{-n}$ and the upper boundary line passes the lattice point $(f_n, f_{n-1})$. This interval will be divided by the lines which pass the lattice points $(r, s)$ such that $0 < r - s/\tau < \tau^{-n}$. Since $f_n/f_{n-1}$ is the “best” rational approximant of $\tau$ [28], the lattice point $(f_{n+2}, f_{n+1})$ has the smallest parallel distance and hence the line which passes it divides the interval first. The “perpendicular” distance of this lattice point is $f_{n+1} - f_{n+2}/\tau = \tau^{-(n+2)}$ and we see that the lower partition of the interval becomes an S-interval in $W_{n+1}$. Furthermore, this implies the bottommost interval for $n + 2$ is the L-type and the above argument can be applied inductively for all even $n$. 

\[ \begin{array}{cccccc}
W_0 & W_1 & W_2 & W_3 & W_4 & W_5 \\
\text{L} & \text{L} & \text{S} & \text{L} & \text{S} & \text{L} \\
\text{S} & \text{L} & \text{L} & \text{S} & \text{L} & \text{L} \\
\text{S} & \text{L} & \text{L} & \text{S} & \text{L} & \text{L} \\
\text{S} & \text{L} & \text{L} & \text{S} & \text{L} & \text{L} \\
\text{S} & \text{L} & \text{L} & \text{S} & \text{L} & \text{L} \\
\text{S} & \text{L} & \text{L} & \text{S} & \text{L} & \text{L} \\
\text{S} & \text{L} & \text{L} & \text{S} & \text{L} & \text{L} \\
\end{array} \] 

Figure 6: A sequence of partitions of the transversal for the construction of a sequence of finite dimensional algebras. An L-interval in $W_n$ is divided by an L- and an S-interval in $W_{n+1}$ but an S-interval becomes an L-interval without partition in the next step. For even $n$, the lower part of an L-interval becomes an S-interval in the $(n+1)$th inflation while the upper part becomes an S-interval for odd $n$.\footnote{This can be shown by two steps; (1) All L-intervals in $W_n$ are divided in the same pattern. (2) A particular L-interval in $W_n$ is divided as the way mentioned above. To prove (1), let $l_k$ and $l_{k'}$ be the two boundary lines of an L-interval in $W_n$ and pass the lattice points $p_k = (r_k, s_k)$ and $p_{k'} = (r_{k'}, s_{k'})$ respectively. The irrationality of the slope guarantees the same $|\Delta r| = |r_k - r_{k'}|$ and $|\Delta s| = |s_k - s_{k'}|$ for all L-intervals in $W_n$; all of them are given by $f_n$ and $|\Delta s| = f_{n-1}$ due to the relations $f_{n-1} - f_n/\tau = (-1)^n \frac{1}{\tau^k}$. Now, the L-intervals in $W_n$ can be mapped to pairs of 2D lattice points which are identical up to a lattice translation. In other words, relative arrangements of the lattice points from the “boundary” lattice points are the same for all L-intervals in $W_n$ and hence divided in the same pattern by the boundary lines in the following inflations. The statement (2) can be shown with the bottommost interval for even $n$ and the topmost interval for odd $n$ with an inductive way. Here we outline the proof for even $n$. The odd $n$ case can be proven similarly. For $n = 2$, the bottommost interval is an L-interval and the upper boundary is given by the line which passes the lattice point $(1, 1)$. Let the bottommost interval of $n = 2k$ case is an L-interval. Then the size of the interval is $\tau^{-n}$ and the upper boundary line passes the lattice point $(f_n, f_{n-1})$. This interval will be divided by the lines which pass the lattice points $(r, s)$ such that $0 < r - s/\tau < \tau^{-n}$. Since $f_n/f_{n-1}$ is the “best” rational approximant of $\tau$ [28], the lattice point $(f_{n+2}, f_{n+1})$ has the smallest parallel distance and hence the line which passes it divides the interval first. The “perpendicular” distance of this lattice point is $f_{n+1} - f_{n+2}/\tau = \tau^{-(n+2)}$ and we see that the lower partition of the interval becomes an S-interval in $W_{n+1}$. Furthermore, this implies the bottommost interval for $n + 2$ is the L-type and the above argument can be applied inductively for all even $n.}$
(S resp.) in $W_1$ if $a_1 = 0$ ($a_1 = 1$ resp.). Then choose the interval $L$ ($S$ resp.) in $W_2$, which is a subinterval of the previously chosen one if $a_2 = 0$ ($a_2 = 1$ resp.). For $a_n = 0$ ($a_n = 1$ resp.), choose the interval $L$ ($S$ resp.) which is a subinterval of the chosen interval in $W_{n-1}$. Then, there is always an interval in $W_n$ for a given sequence in $Z_n$ since an interval $S$ in $W_k$ becomes an interval $L$ in $W_{k+1}$. Conversely, an interval in $W_n$ can be indexed by a sequence in $Z_n$ by recording the types of the intervals in $W_k$ (for $k = 1, \ldots, n$) which the chosen interval belongs to. Now we can identify $W := \lim_{n \to \infty} W_n$ with the set $Z = \lim_{n \to \infty} Z_n$ and hence $W$ is the set of all F-chains. Note that the boundary points excluded from the $n$th partitioned interval, $W_0 - W_n$, are the first $f_{n+1}$ orbit points of the irrational rotation $-1/\tau$ from $1 - 1/\tau = 1/\tau^2$. In other words, the lines through lattice points corresponds to the orbit of the rotation defined by the diffeomorphism $\psi_{\tau}$ introduced in section 3. In fact the construction of $W$ is exactly the same as that of the exceptional minimal set for the suspension of diffeomorphism $\psi_{\tau}$. Thus the set $W$ or the set of all F-chains is the exceptional minimal set and also the set $Z$ is homeomorphic to the Cantor set as asserted in [2, 29].

Now, let us give an equivalence relation $\tilde{R}_n$ on $W_n$ as $R_n$ of Eq. (2) on $Z_n$. Then the set of equivalence classes, $\tilde{X}_n = W_n/\tilde{R}_n$ has only two elements, $W_{n,L}$ and $W_{n,S}$ which have $f_{n+1}$ and $f_n$ intervals respectively. By taking these $f_{n+1}$ and $f_n$ intervals as the bases of the $W_{n,L}$ and $W_{n,S}$ classes respectively, we recover the sequence of finite algebras described in section 3. In the limit of $n$ goes to infinity, we obtain an AF-algebra which is the same as in section 2.

We now show that the space $\tilde{X} = \lim_{n \to \infty} \tilde{X}_n$, which is isomorphic to the space of F-chains $X$, is given by the quotient space obtained from $W$ mod out by the “leaf equivalence relation”; $b \sim b'$ iff $b - b' = n/\tau$ for some integer $n$, and call $\tilde{X}$ as “extended leaf space”. An important consequence of the partition sequence of Fig. 3 is that all $L$-intervals in $W_n$ are divided in the same pattern in $W_{n+1}$ (see footnote (1) also). Furthermore, all $S$-intervals in $W_n$ become the $L$-intervals in $W_{n+1}$. Therefore, all intervals of the same type in $W_n$ are divided in the same pattern in $W_m$ for all $m > n$. This observation provides the relation between two points, $b_z$ and $b_{z'}$, in an equivalence class which can be indexed by two sequences $z = (a_k)$ and $z' = (a'_k)$ with $a_m = a'_m$ for all $m \geq n$. If $I_n$ ($I'_n$ resp.) is the interval in $W_n$, which $b_z$ ($b'_z$ resp.) belongs to, the relative distance from a reference point (say, the center) of $I_n$ to $b_z$ is the same as that from the center of $I'_n$ to $b_z$ because $a_m = a'_m$ for all $m \geq n$. Since the lengths of $L$-intervals and $S$-intervals in $W_n$ are $\tau^{-n}$ and $\tau^{-(n+1)}$ respectively, the distance between $b_z$ and $b_{z'}$ is given by

$$b_z - b_{z'} = k_L \tau^{-n} + k_s \tau^{-(n+1)} = (-1)^n \left[ (k_S f_{n+1} - k_L f_n) \frac{1}{\tau} + (k_L f_{n-1} - k_S f_n) \right] = \frac{k_1}{\tau} - k_2$$  \hspace{1cm} (6)

with integers $k_1 = (-1)^n (k_S f_{n+1} - k_L f_n)$ and $k_2 = (-1)^n (k_L f_{n-1} - k_S f_n)$. Here $k_L$ and
$k_S$ are the number of $L$ and $S$-intervals between the two chosen intervals in $W_n$ and we used the relation $\tau^{-k} = (-1)^k (f_{k-1} - f_k/\tau)$. These are the exactly the same condition for the same leaf on the torus in section 3.

We should note that the space obtained by the limit of the above procedure is not the space of leaves of the Kronecker foliation on $T^2$. If we follow the very bottom intervals of $W$ in Fig. 4, we get the F-chain whose index sequence is given by \((01010101\ldots)\) while we get \((001010101\ldots)\) when we follow the very top intervals. Therefore, the limit points of these two sequences, 0 and 1 represent different classes. In the foliation on the torus, above two leaves had to be identified since 0 and 1 are identical point in $S_1$. This was the reason that the singular leaf on the torus corresponds two distinct F-chains.

5 Concluding remarks

In this paper, we studied the space of F-chains from the perspective of noncommutative geometry. We defined the equivalence relation of the F-chains based on their index sequences and the space of F-chains $X$ is given as the quotient set $Z/R$ of all F-chains $Z$ divided by the equivalence relation $R$ of Eq. (1). This space is exactly the same as the space of Penrose tiling considered by Connes [1]. From the calculation of its $K$-theory, we know that the $K_0$-group of Penrose tiling (and hence F-chains) is isomorphic to that of the noncommutative torus. Furthermore, an F-chain can be parameterized as a point on the torus $T^2$ [13]. These facts suggest a strong connection between the noncommutative torus and the space of F-chains. However, a $C^*$-algebra on the space of F-chains cannot be a noncommutative torus. From the Connes’ work [2], we know that the construction of a nontrivial algebra on the space of F-chains gives rise to an AF-algebra whose $K_1$ vanishes while the $K_1$ of the noncommutative torus does not.

Here, we studied the exact relationship between the noncommutative torus and the AF-algebra on the space of F-chains. Using the torus representation and the cut-procedure, we found a surjective map from the space of F-chains to the space of leaves on Kronecker foliation. The surjectiveness of the map and the embedding of the $C^*$-algebra of noncommutative torus to an AF-algebra was explicitly shown by considering a sequence of finite-algebra constructed on the finite partitions $I_{n,L_k}$ and $I_{n,S_k}$, the quotient space $W_n/R_n$. In the limit of $n$ goes to infinity, the quotient space is identified with the space of $F$-chains. This is a dual approach to the way of embedding of the $C^*$-algebra of the space of leaves on Kronecker Foliation into the $C^*$-algebra of the space of F-chains.

We think the current method of finding the relationship between the space of F-chains and the space of leaves of Kronecker Foliation can be applied to the space of Penrose tilings. As an F-chain can be represented as a leaf on $T^2$, which is a line parallel to the 1D space spanned by a 2D vector $(\cos \theta, \sin \theta)$ with $\tan \theta = 1/\tau$, a Penrose tiling can be represented as a plane on $T^5$ which are parallel to the 2D space.
spanned by two 5D vectors, \((1, c_1, c_2, c_3, c_4)\) and \((1, s_1, s_2, s_3, s_4)\) where \(c_k = \cos\left(\frac{2\pi}{5} k\right)\) and \(s_k = \sin\left(\frac{2\pi}{5} k\right)\). Recall that a leaf on \(T^2\), \(y = \frac{1}{7} x + b\) can be parameterized by the \(y\)-cut \(b\) by identifying the position of a leaf as the point of \(x = 0\) on the leaf. Similarly, a Penrose tiling can be parameterized by the position of the origin of the plane. By introducing the equivalence relation between the positions of the planes according to the equivalence relationship of their Penrose tiling, we can construct a space of wrapping 2D planes (“2D leaf”) in the 5D space. From the identity between the space of F-chains and that of Penrose tiling, one may expect that this space should be very similar to the space of leaves of Kronecker foliation. However, a preliminary study shows that this may not be the case. The properties of the singular plane in this space, which passes through the origin of the 5D space, may behave quite differently from the singular leaf of F-chains. The singular plane corresponds to 5 different Penrose tilings but their index sequences are the same and given by \((0, 0, 0, \ldots)\). Therefore, all of them are in the same equivalence class unlike the F-chains from the singular leaf. Further work on this issue may have practical application for the study of quasicrystal structures. The decapod defects, which can be a seed for rapid quasicrystal growth \([30]\), are known to be related to the singularity of the plane which passes the origin of 5D space \([31]\). If future studies establish the role of the higher dimensional singular “leaf” in the hyper lattice space for the space of general quasiperiodic tilings, they may provide a new clue to solve the old puzzle of the topological character of the decapod defects \([27]\).

We hope the current study on the space of F-chains provides a motive for the proliferation of noncommutative geometrical approaches for the properties of Penrose tiling and quasicrystals. At the moment, the progress on the dynamical properties of the quasicrystalline structure seems to be slow. There have been a great deal of studies on the dynamical properties on the 1D quasiperiodic systems but a very little innocuous extensions to the higher dimensional quasicrystalline structure have been successfully made. We speculate that the study on the dynamics of 1D quasiperiodic system from the noncommutative geometrical aspect may provide a new tool for the investigation to the dynamics of quasicrystals. As shown in this paper, both the space of F-chains (1D quasiperiodic systems) and the space of Penrose tiling (2D quasicrystalline lattice) show the same non trivial structure only with a noncommutative geometrical approach.

In summary, we show that the noncommutative torus can be obtained from the space of Fibonacci chains via foliation. We hope that this understanding will help to enhance the understanding of the dynamics of the quasicrystalline structure in future.

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