CONGRUENCES ON BICYCLIC EXTENSIONS OF A LINEARLY ORDERED GROUP

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Abstract. In the paper we study inverse semigroups \( B(G) \), \( B^+(G) \), \( B(G) \) and \( B^+(G) \) which are generated by partial monotone injective translations of a positive cone of a linearly ordered group \( G \). We describe Green’s relations on the semigroups \( B(G) \), \( B^+(G) \), \( B(G) \) and \( B^+(G) \), their bands and show that they are simple, and moreover the semigroups \( B(G) \) and \( B^+(G) \) are bisimple. We show that for a commutative linearly ordered group \( G \) all non-trivial congruences on the semigroup \( B(G) \) (and \( B^+(G) \)) are group congruences if and only if the group \( G \) is archimedean. Also we describe the structure of group congruences on the semigroups \( B(G) \), \( B^+(G) \), \( B(G) \) and \( B^+(G) \).

1. Introduction and main definitions

In this article we shall follow the terminology of [7, 8, 14, 16, 20].

A semigroup is a non-empty set with a binary associative operation. A semigroup \( S \) is called inverse if for any \( x \in S \) there exists a unique \( y \in S \) such that \( x \cdot y \cdot x = x \) and \( y \cdot x \cdot y = y \). Such an element \( y \) in \( S \) is called the inverse of \( x \) and denoted by \( x^{-1} \). The map defined on an inverse semigroup \( S \) which maps every element \( x \) of \( S \) to its inverse \( x^{-1} \) is called the inversion.

If \( S \) is a semigroup, then we shall denote the subset of idempotents in \( S \) by \( E(S) \). If \( S \) is an inverse semigroup, then \( E(S) \) is closed under multiplication and we shall refer to \( E(S) \) as the band of \( S \). If the band \( E(S) \) is a non-empty subset of \( S \), then the semigroup operation on \( S \) determines the following partial order \( \preceq \) on \( E(S) \): \( e \preceq f \) if and only if \( ef = fe = e \). This order is called the natural partial order on \( E(S) \). A semilattice is a commutative semigroup of idempotents. A semilattice \( E \) is called linearly ordered or a chain if its natural order is a linear order.

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If $C$ is an arbitrary congruence on a semigroup $S$, then we denote by $\Phi_C : S \to S/C$ the natural homomorphisms from $S$ onto the quotient semigroup $S/C$. Also we denote by $\Omega_S$ and $\Delta_S$ the universal and the identity congruences, respectively, on the semigroup $S$, i.e., $\Omega(S) = S \times S$ and $\Delta(S) = \{(s, s) \mid s \in S\}$. A congruence $C$ on a semigroup $S$ is called non-trivial if $C$ is distinct from the universal and the identity congruence on $S$, and a group congruence if the quotient semigroup $S/C$ is a group. Every inverse semigroup $S$ admits a least group congruence $C_{mg}$: $aC_{mg}b$ if and only if there exists $e \in E(S)$ such that $ae = be$ (see [20, Lemma III.5.2]).

A map $h : S \to T$ from a semigroup $S$ to a semigroup $T$ is said to be an antihomomorphism if $(a \cdot b)h = (b)h \cdot (a)h$. A bijective antihomomorphism is called an antiisomorphism.

If $S$ is a semigroup, then we shall denote by $R$, $L$, $J$, $D$ and $H$ the Green’s relations on $S$ (see [8]):

- $aRb$ if and only if $aS^1 = bS^1$;
- $aLb$ if and only if $S^1a = S^1b$;
- $aJb$ if and only if $S^1aS^1 = S^1bS^1$;
- $D = L \circ R = R \circ L$;
- $H = L \cap R$.

Let $I_X$ denote the set of all partial one-to-one transformations of an infinite set $X$ together with the following semigroup operation: $x(\alpha \beta) = (x\alpha)\beta$ if $x \in \text{dom}(\alpha \beta) = \{y \in \text{dom} \alpha \mid y\alpha \in \text{dom} \beta\}$, for $\alpha, \beta \in I_X$. The semigroup $I_X$ is called the symmetric inverse semigroup over the set $X$ (see [8]). The symmetric inverse semigroup was introduced by Wagner [21] and it plays a major role in the theory of semigroups.

The bicyclic semigroup $C(p, q)$ is the semigroup with the identity 1 generated by two elements $p$ and $q$ subjected only to the condition $pq = 1$. The distinct elements of $C(p, q)$ are exhibited in the following useful array:

\[
\begin{array}{cccccc}
1 & p & p^2 & p^3 & \cdots \\
q & qp & qp^2 & qp^3 & \cdots \\
q^2 & q^2p & q^2p^2 & q^2p^3 & \cdots \\
q^3 & q^3p & q^3p^2 & q^3p^3 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
\end{array}
\]

and the semigroup operation on $C(p, q)$ is determined as follows:

\[q^k p^l \cdot q^m p^n = q^{k+\min(l,m)} p^{l+n-\min(l,m)}.\]

The bicyclic semigroup plays an important role in the algebraic theory of semigroups and in the theory of topological semigroups. For example the
well-known O. Andersen’s result [1] states that a (0–) simple semigroup is completely (0–) simple if and only if it does not contain the bicyclic semigroup. The bicyclic semigroup does not embed into stable semigroups [15].

**Remark 1.1.** We observe that the bicyclic semigroup is isomorphic to the semigroup $C_N(\alpha, \beta)$ which is generated by injective partial transformations $\alpha$ and $\beta$ of the set of positive integers $\mathbb{N}$, defined as follows:

$$(n)\alpha = n + 1 \quad \text{if } n \geq 1;$$

$$(n)\beta = n - 1 \quad \text{if } n > 1$$

(see Exercise IV.1.11(ii) in [20]).

Recall from [11] that a partially-ordered group is a group $(G, \cdot)$ equipped with a partial order $\leq$ that is translation-invariant; in other words, $\leq$ has the property that, for all $a, b, g \in G$, if $a \leq b$ then $a \cdot g \leq b \cdot g$ and $g \cdot a \leq g \cdot b$.

Later by $e$ we denote the identity of a group $G$. The set $G^+ = \{x \in G \mid e \leq x\}$ in a partially ordered group $G$ is called the positive cone, or the integral part, of $G$ and satisfies the properties:

1) $G^+ \cdot G^+ \subseteq G^+$;
2) $G^+ \cap (G^+)^{-1} = \{e\}$; and
3) $x^{-1} \cdot G^+ \cdot x \subseteq G^+$ for all $x \in G$.

Any subset $P$ of a group $G$ that satisfies the conditions 1)–3) induces a partial order on $G$ ($x \leq y$ if and only if $x^{-1} \cdot y \in P$) for which $P$ is the positive cone.

A linearly ordered or totally ordered group is an ordered group $G$ such that the order relation “$\leq$” is total [7].

In the remainder we shall assume that $G$ is a linearly ordered group.

For every $g \in G$ we denote

$$G^+(g) = \{x \in G \mid g \leq x\}.$$ 

The set $G^+(g)$ is called a positive cone on element $g$ in $G$.

For arbitrary elements $g, h \in G$ we consider a partial map $\alpha_{h}^{g} : G \rightarrow G$ defined by the formula

$$(x)\alpha_{h}^{g} = x \cdot g^{-1} \cdot h, \quad \text{for } x \in G^+(g).$$

We observe that Lemma XIII.1 from [9] implies that for such partial map $\alpha_{h}^{g} : G \rightarrow G$ the restriction $\alpha_{h}^{g} : G^+(g) \rightarrow G^+(h)$ is a bijective map.

We denote

$$\mathcal{B}(G) = \{\alpha_{h}^{g} : G \rightarrow G \mid g, h \in G\} \quad \text{and} \quad \mathcal{B}^+(G) = \{\alpha_{h}^{g} : G \rightarrow G \mid g, h \in G^+\},$$

and consider on the sets $\mathcal{B}(G)$ and $\mathcal{B}^+(G)$ the operation of the composition of partial maps. Simple verifications show that

$$\alpha_{h}^{g} \cdot \alpha_{l}^{k} = \alpha_{l}^{k}, \quad \text{where} \quad a = (h \lor k) \cdot h^{-1} \cdot g \quad \text{and} \quad b = (h \lor k) \cdot k^{-1} \cdot l, \quad (1)$$
for \( g, h, k, l \in G \). Therefore, property 1) of the positive cone and condition \([1]\) imply that \( \mathcal{B}(G) \) and \( \mathcal{B}^+(G) \) are subsemigroups of \( \mathcal{I}_G \).

**Proposition 1.2.** Let \( G \) be a linearly ordered group. Then the following assertions hold:

(i) elements \( \alpha^g_h \) and \( \alpha^h_g \) are inverses of each other in \( \mathcal{B}(G) \) for all \( g, h \in G \) (resp., \( \mathcal{B}^+(G) \) for all \( g, h \in G^+ \));

(ii) an element \( \alpha^g_h \) of the semigroup \( \mathcal{B}(G) \) (resp., \( \mathcal{B}^+(G) \)) is an idempotent if and only if \( g = h \);

(iii) \( \mathcal{B}(G) \) and \( \mathcal{B}^+(G) \) are inverse subsemigroups of \( \mathcal{I}_G \);

(iv) the semigroup \( \mathcal{B}(G) \) (resp., \( \mathcal{B}^+(G) \)) is isomorphic to \( S_G = G \times G \) (resp., \( S^+_G = G^+ \times G^+ \)) with the semigroup operation:

\[
(a, b) \cdot (c, d) = \begin{cases} 
(c \cdot b^{-1} \cdot a, d) , & \text{if } b < c; \\
(a, d), & \text{if } b = c; \\
(a, b \cdot c^{-1} \cdot d), & \text{if } b > c,
\end{cases}
\]

where \( a, b, c, d \in G \) (resp., \( a, b, c, d \in G^+ \)).

**Proof.** (i) Condition \([1]\) implies that

\[
\alpha^g_h \cdot \alpha^h_g \cdot \alpha^g_h = \alpha^g_h \quad \text{and} \quad \alpha^h_g \cdot \alpha^g_h \cdot \alpha^h_g = \alpha^h_g,
\]

and hence \( \alpha^g_h \) and \( \alpha^h_g \) are inverse elements in \( \mathcal{B}(G) \) (resp., \( \mathcal{B}^+(G) \)).

Statement (ii) follows from the property of the semigroup \( \mathcal{I}_G \) that \( \alpha \in \mathcal{I}_G \) is an idempotent if and only if \( \alpha : \text{dom} \alpha \rightarrow \text{ran} \alpha \) is an identity map.

Statements (i), (ii) and Theorem 1.17 from \([8]\) imply statement (iii). Statement (iv) is a corollary of condition \([1]\). \(\square\)

**Remark 1.3.** We observe that Proposition \([2]\) implies that:

(1) if \( G \) is the additive group of integers \((\mathbb{Z}, +)\) with usual linear order \( \leq \) then the semigroup \( \mathcal{B}^+(G) \) is isomorphic to the bicyclic semigroup \( \mathcal{C}(p, q) \);

(2) if \( G \) is the additive group of real numbers \((\mathbb{R}, +)\) with usual linear order \( \leq \) then the semigroup \( \mathcal{B}(G) \) is isomorphic to \( B_2^{(-\infty, \infty)} \) (see \([17, 18]\)) and the semigroup \( \mathcal{B}^+(G) \) is isomorphic to \( B_1^{(0, \infty)} \) (see \([2, 3, 4, 5, 6]\)) and

(3) the semigroup \( \mathcal{B}^+(G) \) is isomorphic to the semigroup \( S(G) \) which is defined in \([9, 10]\).

We shall say that a linearly ordered group \( G \) is a \( d \)-group if for every element \( g \in G^+ \setminus \{e\} \) there exists \( x \in G^+ \setminus \{e\} \) such that \( x < g \). We observe that a linearly ordered group \( G \) is a \( d \)-group if and only if the set \( G^+ \setminus \{e\} \) does not contain a minimal element.
Definition 1.4. Suppose that $G$ is a linearly ordered $d$-group. For every $g \in G$ we denote
\[ \hat{G}^+(g) = \{ x \in G \mid g < x \}. \]
The set $\hat{G}^+(g)$ is called a $\circ$-positive cone on element $g$ in $G$.

For arbitrary elements $g, h \in G$ we consider a partial map $\hat{a}_h^g : G \to G$ defined by the formula
\[ (x)\hat{a}_h^g = x \cdot g^{-1} \cdot h, \quad \text{for} \quad x \in \hat{G}^+(g). \]
We observe that Lemma XIII.1 from [7] implies that for such partial map $\hat{a}_h^g : G \to G$ the restriction $\hat{a}_h^g : \hat{G}^+(g) \to \hat{G}^+(h)$ is a bijective map.

We denote
\[ \hat{B}(G) = \{ \hat{a}_h^g : G \to G \mid g, h \in G \} \quad \text{and} \quad \hat{B}^+(G) = \{ \hat{a}_h^g : G \to G \mid g, h \in G^+ \}, \]
and consider on the sets $\hat{B}(G)$ and $\hat{B}^+(G)$ the operation of the composition of partial maps. Simple verifications show that
\[ \hat{a}_h^g \cdot \hat{a}_l^k = \hat{a}_h^k, \quad \text{where} \quad a = (h \vee k) \cdot h^{-1} \cdot g \quad \text{and} \quad b = (h \vee k) \cdot k^{-1} \cdot l, \quad (2) \]
for $g, h, k, l \in G$. Therefore, property 1) of the positive cone and condition (2) imply that $\hat{B}(G)$ and $\hat{B}^+(G)$ are subsemigroups of the symmetric inverse semigroup $\hat{I}_G$.

Proposition 1.5. If $G$ is a linearly ordered $d$-group then the semigroups $\hat{B}(G)$ and $\hat{B}^+(G)$ are isomorphic to $B(G)$ and $B^+(G)$, respectively.

Proof. A map $\hat{h} : B(G) \to \hat{B}(G)$ (resp., $\hat{h} : B^+(G) \to \hat{B}^+(G)$) we define by the formula:
\[ (a^B_h)\hat{h} = \hat{a}_h^g, \quad \text{for} \quad g, h \in G \quad \text{(resp.,} \quad g, h \in G^+ \text{)).} \]
Simple verifications show that such map $\hat{h}$ is an isomorphism of the semigroups $\hat{B}(G)$ and $B(G)$ (resp., $\hat{B}^+(G)$ and $B^+(G)$). \( \square \)

Suppose that $G$ is a linearly ordered $d$-group. Then obviously $\hat{B}(G) \cap B(G) = \emptyset$ and $\hat{B}^+(G) \cap B^+(G) = \emptyset$. We define
\[ \overline{B}(G) = \hat{B}(G) \cup B(G) \quad \text{and} \quad \overline{B}^+(G) = \hat{B}^+(G) \cup B^+(G). \]

Proposition 1.6. If $G$ is a linearly ordered $d$-group then $\overline{B}(G)$ and $\overline{B}^+(G)$ are inverse semigroups.

Proof. Since $\hat{B}(G)$, $B(G)$, $\hat{B}^+(G)$ and $B^+(G)$ are inverse subsemigroups of the symmetric inverse semigroup $\hat{I}_G$ over the group $G$ we conclude that it is sufficient to show that $\overline{B}(G)$ and $\overline{B}^+(G)$ are subsemigroups of $\hat{I}_G$. 
We fix arbitrary elements $g, h, k, l \in G$. Since $\alpha_g^h, \alpha_k^h, \alpha_g^k$ and $\alpha_l^k$ are partial injective maps from $G$ into $G$ we have that

$$\alpha_g^h \cdot \alpha_l^k = \begin{cases} \alpha_{h-k,l}^{g-h}, & \text{if } h < k; \\ \alpha_l^k, & \text{if } h = k; \\ \alpha_g^k, & \text{if } h > k. \end{cases}$$

Hence $\overline{B}(G)$ is a subsemigroup of $\mathcal{I}_G$.

Similar arguments and property 1) of the positive cone imply that $\overline{B}^+(G)$ is a subsemigroup of $\mathcal{I}_G$. This completes the proof of our proposition.  

In our paper we study semigroups $\overline{B}(G)$ and $\overline{B}^+(G)$ for a linearly ordered group $G$, and semigroups $\overline{B}(G)$ and $\overline{B}^+(G)$ for a linearly ordered $d$-group $G$. We describe Green’s relations on the semigroups $\overline{B}(G)$, $\overline{B}^+(G)$, $\overline{B}(G)$ and $\overline{B}^+(G)$, their bands and show that they are simple, and moreover the semigroups $\overline{B}(G)$ and $\overline{B}^+(G)$ are bisimple. We show that for a commutative linearly ordered group $G$ all non-trivial congruences on the semigroup $\overline{B}(G)$ (and $\overline{B}^+(G)$) are group congruences if and only if the group $G$ is archimedean. Also, we describe the structure of group congruences on the semigroups $\overline{B}(G)$, $\overline{B}^+(G)$, $\overline{B}(G)$ and $\overline{B}^+(G)$.

2. Algebraic properties of the semigroups $\overline{B}(G)$ and $\overline{B}^+(G)$

**Proposition 2.1.** Let $G$ be a linearly ordered group. Then the following assertions hold:

(i) if $\alpha_g^h, \alpha_h^g \in E(\overline{B}(G))$ (resp., $\alpha_g^h, \alpha_h^g \in E(\overline{B}^+(G))$), then $\alpha_g^h \preceq \alpha_h^g$ if and only if $g \geq h$ in $G$ (resp., in $G^+$);

(ii) the semilattice $E(\overline{B}(G))$ (resp., $E(\overline{B}^+(G))$) is isomorphic to $G$ (resp., $G^+$), considered as a $\vee$-semilattice under the mapping $(\alpha_h^g)\mathcal{H} = g$;

(iii) $\alpha_h^g \mathcal{B} \alpha_l^k$ in $\overline{B}(G)$ (resp., in $\overline{B}^+(G)$) if and only if $g = k$ in $G$ (resp., in $G^+$);

(iv) $\alpha_h^g \mathcal{L} \alpha_l^k$ in $\overline{B}(G)$ (resp., in $\overline{B}^+(G)$) if and only if $h = k$ in $G$ (resp., in $G^+$);

(v) $\alpha_h^g \mathcal{R} \alpha_l^k$ in $\overline{B}(G)$ (resp., in $\overline{B}^+(G)$) if and only if $g = k$ and $h = l$ in $G$ (resp., in $G^+$), and hence every $\mathcal{H}$-class in $\overline{B}(G)$ (resp., in $\overline{B}^+(G)$) is a singleton set;

(vi) $\alpha_h^g \mathcal{D} \alpha_l^k$ in $\overline{B}(G)$ (resp., in $\overline{B}^+(G)$) for all $g, h, k, l \in G$ and hence $\overline{B}(G)$ (resp., $\overline{B}^+(G)$) is a bisimple semigroup;

(vii) $\overline{B}(G)$ (resp., $\overline{B}^+(G)$) is a simple semigroup.

**Proof.** Statements (i) and (ii) are trivial and follow from the definition of the semigroup $\overline{B}(G)$.

(iii) Let $\alpha_h^g, \alpha_l^k \in \overline{B}(G)$ be such that $\alpha_h^g \mathcal{B} \alpha_l^k$. Since $\alpha_h^g \overline{B}(G) = \alpha_l^k \overline{B}(G)$ and $\overline{B}(G)$ is an inverse semigroup, Theorem 1.17 from \cite{S} implies that
Given two partially ordered sets \((A, \leq_A)\) and \((B, \leq_B)\), the \textit{lexicographical order} \(\leq_{\text{lex}}\) on the Cartesian product \(A \times B\) is defined as follows:

\((a, b) \leq_{\text{lex}} (a', b')\) if and only if \(a <_A a'\) or \((a = a'\) and \(b \leq_B b')\).

In this case we shall say that the partially ordered set \((A \times B, \leq_{\text{lex}})\) is the \textit{lexicographic product} of partially ordered sets \((A, \leq_A)\) and \((B, \leq_B)\) and it is denoted by \(A \times_{\text{lex}} B\). We observe that the lexicographic product of two linearly ordered sets is a linearly ordered set.

**Proposition 2.2.** Let \(G\) be a linearly ordered \(d\)-group. Then the following assertions hold:

1. \(E(\mathcal{B}(G)) = E(\mathcal{B}(G)) \cup E(\hat{\mathcal{B}}(G))\) and \(E(\mathcal{B}^+(G)) = E(\mathcal{B}^+(G)) \cup E(\hat{\mathcal{B}}^+(G))\).
2. If \(\alpha_g^\circ, \alpha_h^\circ, \alpha_g^\circ, \alpha_h^\circ \in E(\mathcal{B}(G))\) (resp., \(\alpha_g^\circ, \alpha_h^\circ, \alpha_h^\circ, \alpha_h^\circ \in E(\mathcal{B}^+(G))\)) then:
   - (a) \(\alpha_g^\circ \leq \alpha_h^\circ\) if and only if \(g \geq h\) in \(G\) (resp., in \(G^+\));
   - (b) \(\hat{\alpha}_g^\circ \leq \hat{\alpha}_h^\circ\) if and only if \(g \geq h\) in \(G\) (resp., in \(G^+\));
   - (c) \(\alpha_g^\circ \leq \hat{\alpha}_h^\circ\) if and only if \(g > h\) in \(G\) (resp., in \(G^+\));
   - (d) \(\hat{\alpha}_g^\circ \leq \hat{\alpha}_h^\circ\) if and only if \(g > h\) in \(G\) (resp., in \(G^+\)).
3. The semilattice \(E(\mathcal{B}(G))\) (resp., \(E(\mathcal{B}^+(G))\)) is isomorphic to the lexicographic product \(G \times_{\text{lex}} \{0, 1\}\) (resp., \(G^+ \times_{\text{lex}} \{0, 1\}\)) of semilattices \((G, \lor)\) (resp., \((G^+, \lor)\)) and \(\{0, 1\}\) under the mapping \((\alpha_g^\circ i) = (g, 1)\) and \((\alpha_g^\circ i) = (g, 0)\), and hence \(E(\mathcal{B}(G))\) (resp., \(E(\mathcal{B}^+(G))\)) is a linearly ordered semilattice.
4. The elements \(\alpha\) and \(\beta\) of the semigroup \(\mathcal{B}(G)\) (resp., \(\mathcal{B}^+(G)\)) are equivalent in \(\mathcal{B}(G)\) (resp., in \(\mathcal{B}^+(G)\)) provides either \(\alpha, \beta \in \mathcal{B}(G)\)
(resp., \(\alpha, \beta \in \mathbb{B}^+(G)\)) or \(\alpha, \beta \in \mathbb{B}(G)\) \(\alpha, \beta \in \mathbb{B}(G)\) \(\alpha, \beta \in \mathbb{B}(G)\); and moreover, we have that
(a) \(\alpha_h^k \mathbb{B} \alpha_i^k\) in \(\mathbb{B}(G)\) \(\alpha, \beta \in \mathbb{B}(G)\) if and only if \(g = k\); and
(b) \(\alpha_h^k \mathbb{B} \alpha_i^k\) in \(\mathbb{B}(G)\) \(\alpha, \beta \in \mathbb{B}(G)\) if and only if \(g = k\).

(v) The elements \(\alpha\) and \(\beta\) of the semigroup \(\mathbb{B}(G)\) \(\alpha, \beta \in \mathbb{B}(G)\) \(\alpha, \beta \in \mathbb{B}(G)\) \(\alpha, \beta \in \mathbb{B}(G)\); and moreover, we have that
(a) \(\alpha_h^k \mathbb{L} \alpha_i^k\) in \(\mathbb{B}(G)\) \(\alpha, \beta \in \mathbb{B}(G)\) \(\alpha, \beta \in \mathbb{B}(G)\) \(\alpha, \beta \in \mathbb{B}(G)\) \(\alpha, \beta \in \mathbb{B}(G)\); and moreover, we have that
(b) \(\alpha_h^k \mathbb{L} \alpha_i^k\) in \(\mathbb{B}(G)\) \(\alpha, \beta \in \mathbb{B}(G)\) \(\alpha, \beta \in \mathbb{B}(G)\) if and only if \(h = l\).

(vi) The elements \(\alpha\) and \(\beta\) of the semigroup \(\mathbb{B}(G)\) \(\alpha, \beta \in \mathbb{B}(G)\) \(\alpha, \beta \in \mathbb{B}(G)\) \(\alpha, \beta \in \mathbb{B}(G)\); and moreover, we have that
(a) \(\alpha_h^k \mathbb{H} \alpha_i^k\) in \(\mathbb{B}(G)\) \(\alpha, \beta \in \mathbb{B}(G)\) \(\alpha, \beta \in \mathbb{B}(G)\) \(\alpha, \beta \in \mathbb{B}(G)\); and moreover, we have that
(b) \(\alpha_h^k \mathbb{H} \alpha_i^k\) in \(\mathbb{B}(G)\) \(\alpha, \beta \in \mathbb{B}(G)\) \(\alpha, \beta \in \mathbb{B}(G)\) if and only if \(g = k\) and \(h = l\); and
(c) every \(\mathbb{H}\)-class in \(\mathbb{B}(G)\) \(\alpha, \beta \in \mathbb{B}(G)\) \(\alpha, \beta \in \mathbb{B}(G)\) is a singleton set.

(vii) \(\mathbb{B}(G)\) \(\alpha, \beta \in \mathbb{B}(G)\) \(\alpha, \beta \in \mathbb{B}(G)\) is a simple semigroup.

(viii) The semigroup \(\mathbb{B}(G)\) \(\alpha, \beta \in \mathbb{B}(G)\) \(\alpha, \beta \in \mathbb{B}(G)\) has only two distinct \(\mathbb{D}\)-classes which are inverse subsemigroups \(\mathbb{B}(G)\) \(\alpha, \beta \in \mathbb{B}(G)\) \(\alpha, \beta \in \mathbb{B}(G)\) \(\alpha, \beta \in \mathbb{B}(G)\).

**Proof.** Statements (i), (ii) and (iii) follow from the definition of the semigroup \(\mathbb{B}(G)\) and Proposition 1.4.5.

Proofs of statements (iv), (v) and (vi) follow from Proposition 1.16 and Theorem 1.17 [8] and are similar to statements (ii), (iv) and (v) of Proposition 2.1.

(vii) We shall show that \(\mathbb{B}(G) \cdot \alpha \cdot \mathbb{B}(G) = \mathbb{B}(G)\) for every \(\alpha \in \mathbb{B}(G)\). We fix arbitrary \(\alpha, \beta \in \mathbb{B}(G)\) and show that there exist \(\gamma, \delta \in \mathbb{B}(G)\) such that \(\gamma \cdot \alpha \cdot \delta = \beta\).

We consider the following cases:

1. \(\alpha = \alpha_h^g \in \mathbb{B}(G)\) and \(\beta = \alpha_i^k \in \mathbb{B}(G)\);
2. \(\alpha = \alpha_h^g \in \mathbb{B}(G)\) and \(\beta = \alpha_i^k \in \mathbb{B}(G)\);
3. \(\alpha = \alpha_h^g \in \mathbb{B}(G)\) and \(\beta = \alpha_i^k \in \mathbb{B}(G)\);
4. \(\alpha = \alpha_h^g \in \mathbb{B}(G)\) and \(\beta = \alpha_i^k \in \mathbb{B}(G)\),

where \(g, h, k, l \in G\).

We put:
\[ \gamma = \alpha_g^k \text{ and } \delta = \alpha_h^l \text{ in case (1)}; \]
\[ \gamma = \hat{\delta}_g^k \text{ and } \delta = \hat{\alpha}_l^h \text{ in case (2)}; \]
\[ \gamma = \alpha_a^g \text{ and } \delta = \alpha_{a^{-1}}^g \text{, where } a \in G^+(g) \setminus \{g\}, \text{ in case (3)}; \]
\[ \gamma = \hat{\delta}_g^k \text{ and } \delta = \hat{\alpha}_l^h \text{ in case (4)}.. \]

Elementary verifications show that \( \gamma \cdot \alpha \cdot \delta = \beta \), and this completes the proof of assertion (\( \text{vii} \)).

Statement (\( \text{viii} \)) follows from statements (\( \text{iv} \)) and (\( \text{v} \)).

The proof of the statements of the proposition for the semigroup \( \mathcal{B}^+(G) \) is similar.

\begin{proof}
Since the semigroup \( \mathcal{C} \) which is generated by elements \( \alpha_g^q \) and \( \alpha_h^r \) is isomorphic to the semigroup \( \mathcal{C}_N(\alpha, \beta) \) (this isomorphism \( i : \mathcal{C} \to \mathcal{C}_N(\alpha, \beta) \)) we can determine on generating elements of \( \mathcal{C} \) by the formulae \( (\alpha_g^q)i = \alpha \) and \( (\alpha_h^r)i = \beta \) we conclude that the first part of the proposition follows from Remark 1.1. Obviously, the element \( \alpha_g^q \) is a unity of the semigroup \( \mathcal{C} \). \( \square \)

3. CONGRUENCES ON THE SEMIGROUPS \( \mathcal{B}(G) \) AND \( \mathcal{B}^+(G) \)

The following lemma follows from the definition of a congruence on a semilattice:

\begin{lemma}
Let \( \mathcal{C} \) be an arbitrary congruence on a semilattice \( S \) and let \( \preceq \) be the natural partial order on \( S \). Let \( a \) and \( b \) be idempotents of the semigroup \( S \) such that \( a \mathcal{C} b \). Then the relation \( a \preceq b \) implies that \( a \mathcal{C} c \) for all idempotents \( c \in S \) such that \( a \preceq c \preceq b \).
\end{lemma}

A linearly ordered group \( G \) is called archimedean if for each \( a, b \in G^+ \setminus \{e\} \) there exist positive integers \( m \) and \( n \) such that \( b \leq a^m \) and \( a \leq b^n \) [7].

Linearly ordered archimedean groups may be described as follows (Hölder’s Theorem): A linearly ordered group is Archimedean if and only if it is isomorphic to some subgroup of the additive group of real numbers with the natural order [13].

\begin{theorem}
Let \( G \) be an archimedean linearly ordered group. Then every non-trivial congruence on \( \mathcal{B}^+(G) \) is a group congruence.
\end{theorem}

\begin{proof}
Suppose that \( \mathcal{C} \) is a non-trivial congruence on the semigroup \( \mathcal{B}^+(G) \). Then there exist distinct elements \( \alpha_a^g \) and \( \alpha_a^h \) of the semigroup \( \mathcal{B}^+(G) \) such
that $\alpha_b^a \alpha_d^c$. Since by Proposition 2.1(iv) every $\mathcal{H}$-class of the semigroup $\mathcal{B}^+(G)$ is a singleton set we conclude that either $\alpha_b^a \cdot (\alpha_b^a)^{-1} \neq \alpha_d^c \cdot (\alpha_d^c)^{-1}$ or $(\alpha_b^a)^{-1} \cdot \alpha_b^a \neq (\alpha_d^c)^{-1} \cdot \alpha_d^c$. We shall consider case $\alpha_a^a = \alpha_b^a \cdot (\alpha_b^a)^{-1} \neq \alpha_d^c \cdot (\alpha_d^c)^{-1} = \alpha_c^c$. In the other case the proof is similar. Since by Proposition 2.1(ii) the band $E(\mathcal{B}^+(G))$ is a linearly ordered semilattice without loss of generality we can assume that $\alpha_c^c \leq \alpha_a^a$. Then by Proposition 2.1(i) we have that $a < c$ in $G$. Since $\alpha_b^a \alpha_d^c$ and $\mathcal{B}^+(G)$ is an inverse semigroup Lemma III.1.1 from [20] implies that $(\alpha_b^a \cdot (\alpha_b^a)^{-1}) \mathcal{C} (\alpha_d^c \cdot (\alpha_d^c)^{-1})$, i.e., $\alpha_b^a \alpha_d^c$. Then we have that

\[
\begin{align*}
\alpha_a^c \cdot \alpha_a^a \cdot \alpha_c^a &= \alpha_c^c, \\
\alpha_c^c \cdot \alpha_a^a \cdot \alpha_c^a &= \alpha_c^{c-1} \cdot \alpha_c^a; \\
\alpha_c^c \cdot \alpha_c^{a-1} \cdot \alpha_c^a &= \alpha_c^{(a-1)^2}; \\
\ldots \\
\alpha_c^c \cdot \alpha_c^{(a-1)^n-1} \cdot \alpha_c^a &= \alpha_c^{(a-1)^n},
\end{align*}
\]

and hence $\alpha_b^a \alpha_d^c \alpha_c^{(a-1)^n}$ for every non-negative integer $n$. Since $a < c$ in $G$ we get that $a^1 \cdot c$ is a positive element of the linearly ordered group $G$. Since the linearly ordered group $G$ is archimedean we conclude that for every $g \in G$ with $g > a$ there exists a positive integer $n$ such that $a^1 \cdot g < (a^1 \cdot c)^n$ and hence $g < c \cdot (a^1 \cdot c)^{n-1}$. Therefore Lemma 3.1 and Proposition 2.1(i) imply that $\alpha_b^a \alpha_d^c$ for every $g \in G$ such that $a < g$.

If $a = e$ then we have that all idempotents of the semigroup $\mathcal{B}^+(G)$ are $\mathcal{C}$-equivalent. Since the semigroup $\mathcal{B}^+(G)$ is inverse we conclude that the quotient semigroup $\mathcal{B}^+(G)/\mathcal{C}$ contains only one idempotent and by Lemma II.1.10 from [20] the semigroup $\mathcal{B}^+(G)/\mathcal{C}$ is a group.

Suppose that $e < a$. Then by Proposition 2.2 we have that the semigroup $\mathcal{C}^*$ which is generated by elements $\alpha_e^c$ and $\alpha_e^g$ is isomorphic to the bicyclic semigroup for every element $g$ in $G^+$ such that $e < a \leq g$. Hence we have that the following conditions hold:

\[
\begin{align*}
\ldots \ll \alpha_g^g \ll \alpha_g^{g^i} \ll \ldots \ll \alpha_g^{g_{i-1}} \ll \ldots \ll \alpha_g^{g_j} \ll \alpha_a^a \quad \text{and} \\
\alpha_g^{g_i} \neq \alpha_g^{g_j} 
\end{align*}
\]

for distinct positive integers $i$ and $j$, in $E(\mathcal{B}^+(G))$. Since the linearly ordered group $G$ is archimedean we conclude that $\alpha_b^a \alpha_d^c \alpha_g^{g_i}$ for every positive integer $i$. Since the semigroup $\mathcal{C}^*$ is isomorphic to the bicyclic semigroup we have that Corollary 1.32 of [8] and Lemma 3.1 imply that all idempotents of the semigroup $\mathcal{B}^+(G)$ are $\mathcal{C}$-equivalent. Since the semigroup $\mathcal{B}^+(G)$ is inverse we conclude that the quotient semigroup $\mathcal{B}^+(G)/\mathcal{C}$ contains only one idempotent and by Lemma II.1.10 from [20] the semigroup $\mathcal{B}^+(G)/\mathcal{C}$ is a group. \qed
**Theorem 3.3.** Let $G$ be an archimedean linearly ordered group. Then every non-trivial congruence on $\mathcal{B}(G)$ is a group congruence.

**Proof.** Suppose that $\mathcal{C}$ is a non-trivial congruence on the semigroup $\mathcal{B}(G)$. Similar arguments as in the proof of Theorem 3.2 imply that there exist distinct idempotents $\alpha_a$ and $\alpha_b$ in the semigroup $\mathcal{B}(G)$ such that $\alpha_a \mathcal{C} \alpha_b$ and $\alpha_b \not\leq \alpha_a$, for $a, b \in G$ with $a \not\leq b$ in $G$. Then we have that

$$\alpha_a \cdot \alpha_a \cdot \alpha_e = \alpha_e \quad \text{and} \quad \alpha_a \cdot \alpha_b \cdot \alpha_e = \alpha_b^{-a} \cdot \alpha_e = \alpha_b^{a-1},$$

and hence $\alpha_a \mathcal{C} \alpha_b^{a-1}$. Since $a \leq b$ in $G$ we conclude that $e \leq b \cdot a^{-1}$ in $G$ and hence Theorem 3.2 implies that $\alpha_c \mathcal{C} \alpha_d$ for all $c, d \in G^+$. We fix an arbitrary element $g \in G \setminus G^+$. Then we have that $g^{-1} \in G^+ \setminus \{e\}$ and hence $\alpha_e \mathcal{C} \alpha_g^{g^{-1}}$. Since

$$\alpha^g \cdot \alpha^g \cdot \alpha^g = \alpha^g \quad \text{and} \quad \alpha^g \cdot \alpha_g^{-1} \cdot \alpha^g = \alpha_g^{-1} \cdot \alpha^g = \alpha_g^{-1} \cdot c \cdot \alpha^g = \alpha^g,$$

we conclude that $\alpha_e \mathcal{C} \alpha_g$. Therefore all idempotents of the semigroup $\mathcal{B}(G)$ are $\mathcal{C}$-equivalent. Since the semigroup $\mathcal{B}(G)$ is inverse we conclude that the quotient semigroup $\mathcal{B}(G)/\mathcal{C}$ contains only one idempotent and by Lemma II.1.10 from [20] the semigroup $\mathcal{B}(G)/\mathcal{C}$ is a group. \hfill $\square$

**Remark 3.4.** We observe that Proposition 1.5 implies that if $G$ is a linearly ordered $d$-group then the statements similar to Propositions 2.1 and 2.3 and Theorems 3.2 and 3.3 hold for the semigroups $\mathcal{B}(G)$ and $\mathcal{B}^+(G)$.

**Theorem 3.5.** If $G$ is the lexicographic product $A \times_{\text{lex}} H$ of non-singleton linearly ordered groups $A$ and $H$, then the semigroups $\mathcal{B}(G)$ and $\mathcal{B}^+(G)$ have non-trivial non-group congruences.

**Proof.** We define a relation $\sim_\zeta$ on the semigroup $\mathcal{B}(G)$ as follows:

$$\alpha_{(a_1,b_1)} \sim_\zeta \alpha_{(a_2,b_2)} \quad \text{if and only if} \quad a_1 = a_2, \ c_1 = c_2 \quad \text{and} \quad d_1^{-1}b_1 = d_2^{-1}b_2.$$

Simple verifications show that $\sim_\zeta$ is an equivalence relation on the semigroup $\mathcal{B}(G)$.
Next we shall prove that \( \sim_{\epsilon} \) is a congruence on \( \mathcal{B}(G) \). Suppose that \( \alpha_{(a_1,b_1)}^{(a_2,b_2)} \) for some \( \alpha_{(a_1,b_1)}^{(a_2,b_2)} \in \mathcal{B}(G) \). Let \( \alpha_{(u,v)} \) be an arbitrary element of \( \mathcal{B}(G) \). Then we have that

\[
\alpha_{(k_1,l_1)}^{(m_1,n_1)} = \alpha_{(a_1,b_1)}^{(c_1,d_1)} \cdot \alpha_{(u,v)}^{(x,y)} = \begin{cases} 
\alpha_{(a_1,b_1)}^{(c_1,d_1)} \cdot \alpha_{(u,v)}^{(x,y)} & \text{if } (c_1, d_1) \leq (u, v); \\
\alpha_{(a_1,b_1)}^{(c_1,d_1)} \cdot \alpha_{(u,v)}^{(x,y)} & \text{if } (u, v) \leq (c_1, d_1) \\
\end{cases}
\]

and

\[
\alpha_{(k_2,l_2)}^{(m_2,n_2)} = \alpha_{(a_2,b_2)}^{(c_2,d_2)} \cdot \alpha_{(u,v)}^{(x,y)} = \begin{cases} 
\alpha_{(a_2,b_2)}^{(c_2,d_2)} \cdot \alpha_{(u,v)}^{(x,y)} & \text{if } (c_2, d_2) \leq (u, v); \\
\alpha_{(a_2,b_2)}^{(c_2,d_2)} \cdot \alpha_{(u,v)}^{(x,y)} & \text{if } (u, v) \leq (c_2, d_2) \\
\end{cases}
\]

and since \( a_1 = a_2, c_1 = c_2 \) and \( d_1^{-1}b_1 = d_2^{-1}b_2 \) we conclude that the following conditions hold:

1. If \( c_1 = c_2 < u \), then \( k_1 = u c_1^{-1}a_1 = u c_2^{-1}a_2 = k_2, m_1 = x = m_2 \) and
   \( n_1^{-1}l_1 = y^{-1}vd_1^{-1}b_1 = y^{-1}vd_2^{-1}b_2 = n_2^{-1}l_2; \)
2. If \( c_1 = c_2 = u \) and \( d_1 \leq v \), then \( k_1 = a_1 = a_2 = k_2, m_1 = x = m_2 \) and
   \( n_1^{-1}l_1 = y^{-1}vd_1^{-1}b_1 = y^{-1}vd_2^{-1}b_2 = n_2^{-1}l_2; \)
3. If \( u < c_1 = c_2 \), then \( k_1 = a_1 = a_2 = k_2, m_1 = c_1u^{-1}x = c_2u^{-1}x = m_2 \) and
   \( n_1^{-1}l_1 = y^{-1}vd_1^{-1}b_1 = y^{-1}vd_2^{-1}b_2 = n_2^{-1}l_2; \)
(4) if $u = c_1 = c_2$ and $v \leq d_1$, then $k_1 = a_1 = a_2 = k_2$, $m_1 = x = m_2$ and
\[ n_1^{-1}l_1 = y^{-1}vd_1^{-1}b_1 = y^{-1}vd_2^{-1}b_2 = n_2^{-1}l_2. \]
Hence we get that $\alpha_{(m_1,n_1)}^{(k_1,l_1)} \sim \alpha_{(m_2,n_2)}^{(k_2,l_2)}$. Similarly we have that
\[
\alpha_{(r_1,s_1)}^{(p_1,q_1)} = \alpha_{(x,y)}^{(u,v)} \cdot \alpha_{(c_1,d_1)}^{(a_1,b_1)} \begin{cases}
\alpha_{(c_1,d_1)}^{(a_1,b_1)} - (x,y)^{-1} \cdot (u,v), & \text{if } (x,y) \leq (a_1,b_1); \\
\alpha_{(x,y)}^{(u,v)} \cdot \alpha_{(c_1,d_1)}^{(a_1,b_1)} - (c_1,d_1), & \text{if } (a_1,b_1) \leq (x,y),
\end{cases}
\]
\[
= \begin{cases}
\alpha_{(c_1,d_1)}^{(a_1,b_1)} - (c_1,d_1), & \text{if } (x,y) \leq (a_1,b_1); \\
\alpha_{(x,y)}^{(u,v)} \cdot \alpha_{(c_1,d_1)}^{(a_1,b_1)} - (c_1,d_1), & \text{if } (a_1,b_1) \leq (x,y),
\end{cases}
\]
\[
\alpha_{(r_2,s_2)}^{(p_2,q_2)} = \alpha_{(x,y)}^{(u,v)} \cdot \alpha_{(c_2,d_2)}^{(a_2,b_2)} \begin{cases}
\alpha_{(c_2,d_2)}^{(a_2,b_2)} - (x,y)^{-1} \cdot (u,v), & \text{if } (x,y) \leq (a_2,b_2); \\
\alpha_{(x,y)}^{(u,v)} \cdot \alpha_{(c_2,d_2)}^{(a_2,b_2)} - (c_2,d_2), & \text{if } (a_2,b_2) \leq (x,y),
\end{cases}
\]
\[
= \begin{cases}
\alpha_{(c_2,d_2)}^{(a_2,b_2)} - (c_2,d_2), & \text{if } (x,y) \leq (a_2,b_2); \\
\alpha_{(x,y)}^{(u,v)} \cdot \alpha_{(c_2,d_2)}^{(a_2,b_2)} - (c_2,d_2), & \text{if } (a_2,b_2) \leq (x,y),
\end{cases}
\]
and since $a_1 = a_2$, $c_1 = c_2$ and $d_2^{-1}b_1 = d_2^{-1}b_2$ we conclude that the following conditions hold:

(1) if $x < a_1 = a_2$, then $p_1 = a_1x^{-1}u = a_2x^{-1}u = p_2$, $r_1 = c_1 = c_2 = r_2$ and
\[ s_1^{-1}q_1 = d_1^{-1}b_1y^{-1}v = d_2^{-1}b_2y^{-1}v = s_2^{-1}q_2; \]

(2) if $x = a_1 = a_2$ and $y \leq b_1$, then $p_1 = u = p_2$, $r_1 = c_1 = c_2 = r_2$ and
\[ s_1^{-1}q_1 = d_1^{-1}b_1y^{-1}v = d_2^{-1}b_2y^{-1}v = s_2^{-1}q_2; \]

(3) if $a_1 = a_2 < x$, then $p_1 = u = p_2$, $r_1 = xa_1^{-1}c_1 = xa_2^{-1}c_2 = r_2$ and
\[ s_1^{-1}q_1 = d_1^{-1}b_1y^{-1}v = d_2^{-1}b_2y^{-1}v = s_2^{-1}q_2; \]
(4) if $a_1 = a_2 = x$ and $b_1 \leq y$, then $p_1 = u = p_2$, $r_1 = c_1 = v = r_2$ and

$$s_1^{-1}q_1 = d_1^{-1}b_1y^{-1}v = d_2^{-1}b_2y^{-1}v = s_2^{-1}q_2.$$ 

Hence we get that $\alpha_{(p_1,q_1)}^{(r_1,s_1)} \sim_c \alpha_{(p_2,q_2)}^{(r_2,s_2)}$.

We fix any $a_1,a_2,b_1,b_2 \in G$. If $a_1 \neq a_2$ then we have that the elements $\alpha_{(a_1,b_1)}^{(a_1,b_1)}$ and $\alpha_{(a_2,b_2)}^{(a_2,b_2)}$ are idempotents of the semigroup $\mathcal{B}(G)$, and moreover the elements $\alpha_{(a_1,b_1)}^{(a_1,b_1)}$ and $\alpha_{(a_2,b_2)}^{(a_2,b_2)}$ are not $\sim_c$-equivalent. Since a homomorphic image of an idempotent is an idempotent too, we conclude that $\pi_c: \mathcal{B}(G) \rightarrow \mathcal{B}(G)/\sim_c$ is the natural homomorphism which is generated by the congruence $\sim_c$ on the semigroup $\mathcal{B}(G)$. This implies that the quotient semigroup $\mathcal{B}(G)/\sim_c$ is not a group, and hence $\sim_c$ is not a group congruence on the semigroup $\mathcal{B}(G)$.

The proof of the statement that the semigroup $\mathcal{B}^+(G)$ has a non-trivial non-group congruence is similar.

**Theorem 3.6.** Let $G$ be a commutative linearly ordered group. Then the following conditions are equivalent:

(i) $G$ is archimedean;

(ii) every non-trivial congruence on $\mathcal{B}(G)$ is a group congruence; and

(iii) every non-trivial congruence on $\mathcal{B}^+(G)$ is a group congruence.

**Proof.** Implications (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) follow from Theorems 3.3 and 3.5, respectively.

(ii) $\Rightarrow$ (i) Suppose the contrary that there exists a non-archimedean commutative linearly ordered group $G$ such that every non-trivial congruence on $\mathcal{B}(G)$ is a group congruence. Then by Hahn Theorem (see [12] or [16, Section VII.3, Theorem 1]) $G$ is isomorphic to a lexicographic product $\prod_{\alpha \in \mathcal{J}} H_\alpha$ of some family of non-singleton subgroups $\{H_\alpha \mid \alpha \in \mathcal{J}\}$ of the additive group of real numbers with a non-singleton linearly ordered index set $\mathcal{J}$. We fix a non-maximal element $\alpha_0 \in \mathcal{J}$, and put

$$A = \prod_{\alpha \leq \alpha_0} H_\alpha \quad \text{and} \quad H = \prod_{\alpha_0 < \alpha} H_\alpha.$$ 

Then we have that $G$ is isomorphic to a lexicographic product $A \times_{\text{lex}} H$ of non-singleton linearly ordered groups $A$ and $H$, and hence by Theorem 3.5 the semigroup $\mathcal{B}(G)$ has a non-trivial non-group congruence. The obtained contradiction implies that the group $G$ is archimedean.

The proof of implication (iii) $\Rightarrow$ (i) is similar to (ii) $\Rightarrow$ (i). 

On the semigroup $\overline{\mathcal{B}}(G)$ (resp., $\overline{\mathcal{B}}^+(G)$) we determine a relation $\sim_{\overline{\mathcal{I}}}^\emptyset$ in the following way. We define a map $\overline{\mathcal{I}}: \overline{\mathcal{B}}(G)$ (resp., $\overline{\mathcal{I}}: \overline{\mathcal{B}}^+(G)$
for $\alpha, \beta, \gamma, \delta \in B(G)$ (resp., $\alpha, \beta, \gamma, \delta \in \mathcal{F}(G)$). Simple verifications show that $\sim_{10}$ is an equivalence relation on the semigroup $\mathcal{F}(G)$ (resp., $\mathcal{F}^+(G)$).

**Proposition 3.7.** If $G$ is a linearly ordered $d$-group then $\sim_{10}$ is a congruence on semigroups $\mathcal{F}(G)$ and $\mathcal{F}^+(G)$. Moreover, quotient semigroups $\mathcal{F}(G)/\sim_{10}$ and $\mathcal{F}^+(G)/\sim_{10}$ are isomorphic to semigroups $\mathcal{F}(G)$ and $\mathcal{F}^+(G)$, respectively.

**Proof.** It is sufficient to show that if $\alpha \sim_{10} \beta$ and $\gamma \sim_{10} \delta$ then $(\alpha \cdot \gamma) \sim_{10} (\beta \cdot \delta)$ for $\alpha, \beta, \gamma, \delta \in \mathcal{F}(G)$ (resp., $\alpha, \beta, \gamma, \delta \in \mathcal{F}^+(G)$). Since the case $\alpha = \beta$ and $\gamma = \delta$ is trivial we consider the following cases:

(i) $\alpha = a^g_b$, $\beta = a^g_b$ and $\gamma = \delta = a^c_d$;
(ii) $\alpha = a^g_b$, $\beta = a^g_b$ and $\gamma = \delta = a^c_d$;
(iii) $\alpha = a^g_b$, $\beta = a^g_b$ and $\gamma = \delta = a^c_d$;
(iv) $\alpha = a^g_b$, $\beta = a^g_b$ and $\gamma = \delta = a^c_d$;
(v) $\alpha = a^g_b$, $\beta = a^g_b$, $\gamma = \delta = a^c_d$;
(vi) $\alpha = a^g_b$, $\beta = a^g_b$, $\gamma = \delta = a^c_d$;
(vii) $\alpha = a^g_b$, $\beta = a^g_b$, $\gamma = \delta = a^c_d$;
(viii) $\alpha = a^g_b$, $\beta = a^g_b$, $\gamma = \delta = a^c_d$;
(ix) $\alpha = a^g_b$, $\beta = a^g_b$, $\gamma = \delta = a^c_d$;
(x) $\alpha = a^g_b$, $\beta = a^g_b$, $\gamma = \delta = a^c_d$;
(xi) $\alpha = a^g_b$, $\beta = a^g_b$, $\gamma = \delta = a^c_d$;
(xii) $\alpha = a^g_b$, $\beta = a^g_b$, $\gamma = \delta = a^c_d$;

where $a, b, c, d \in G$ (resp., $a, b, c, d \in G^+$).

In case (i) we have that

$$\alpha \cdot \gamma = a^g_b \cdot a^c_d = \begin{cases} a^{c-b-1}_a, & \text{if } b < c; \\ a^c_d, & \text{if } b = c; \\ a^b_{c-1}d, & \text{if } b > c, \end{cases} \quad \text{and } \beta \cdot \delta = a^g_b \cdot a^c_d = \begin{cases} a^{c-b-1}_a, & \text{if } b < c; \\ a^c_d, & \text{if } b = c; \\ a^b_{c-1}d, & \text{if } b > c, \end{cases}$$

and hence $(\alpha \cdot \gamma) \sim_{10} (\beta \cdot \delta)$ in $\mathcal{F}(G)$ (resp., $\mathcal{F}^+(G)$). In other cases verifications are similar.

Since the restriction $\Phi_{10} : \mathcal{F}(G) \to \mathcal{F}(G)$ of the natural homomorphism $\Phi_{10} : \mathcal{F}(G) \to \mathcal{F}(G)$ is a bijective map we conclude that the semigroup $(\mathcal{F}(G))\Phi_{10}$ is isomorphic to the semigroup $\mathcal{F}(G)$. Similar arguments show that the semigroup $\mathcal{F}^+(G)/\sim_{10}$ is isomorphic to $\mathcal{F}^+(G)$. \hfill $\square$

**Theorem 3.8.** Let $G$ be an archimedean linearly ordered $d$-group. If $\mathcal{C}$ is a non-trivial congruence on $\mathcal{F}(G)$ (resp., $\mathcal{F}^+(G)$) then the quotient semigroup
\( \mathcal{B}(G)/\mathcal{E} \) (resp., \( \mathcal{B}^+(G)/\mathcal{E} \)) is either a group or \( \mathcal{B}(G)/\mathcal{E} \) (resp., \( \mathcal{B}^+(G)/\mathcal{E} \)) is isomorphic to the semigroup \( \mathcal{B}(G) \) (resp., \( \mathcal{B}^+(G) \)).

Proof. Since the subsemigroup of idempotents of the semigroup \( \mathcal{B}(G) \) is linearly ordered we have that similar arguments as in the proof of Theorem 3.2 imply that there exist distinct idempotents \( \varepsilon \) and \( \iota \) of \( \mathcal{B}(G) \) such that \( \varepsilon \mathcal{E} \iota = \emptyset \). If the set \( (\varepsilon, \iota) = \{ v \in E(\mathcal{B}(G)) \mid \varepsilon < v < \iota \} \) is non-empty then Lemma 3.4 and Theorem 3.5 imply that the quotient semigroup \( \mathcal{B}(G)/\mathcal{E} \) is inverse and has only one idempotent, and hence by Lemma II.1.10 from 20 it is a group. Otherwise there exists \( g \in G \) such that \( \iota = \varepsilon g \) and \( \varepsilon = \varepsilon g \). Since \( \alpha^k_l = \alpha^k_q \cdot \alpha^q_g \cdot \alpha^g_l \) and \( \tilde{\alpha}^k_l = \alpha^k_q \cdot \tilde{\alpha}^q_g \cdot \alpha^g_l \) for every \( k, l \in G \) we conclude that the congruence \( \mathcal{E} \) coincides with the congruence \( \sim_{13} \) on \( \mathcal{B}(G) \), and hence by Proposition 3.7 the quotient semigroup \( \mathcal{B}(G)/\mathcal{E} \) is isomorphic to the semigroup \( \mathcal{B}(G) \).

In the case of the semigroup \( \mathcal{B}^+(G) \) the proof is similar. \( \square \)

**Theorem 3.9.** Let \( G \) be a commutative linearly ordered \( d \)-group. Then the following conditions are equivalent:

1. \( G \) is archimedean;
2. every non-trivial congruence on \( \mathcal{B}(G) \) is a group congruence; and
3. every non-trivial congruence on \( \mathcal{B}^+(G) \) is a group congruence;
4. the semigroup \( \mathcal{B}(G) \) has a unique non-trivial non-group congruence;
5. the semigroup \( \mathcal{B}^+(G) \) has a unique non-trivial non-group congruence.

Proof. The equivalence of statements (i), (ii) and (iii) follows from Proposition 3.5 and Theorem 3.6. Also Theorem 3.8 implies that implications (i) \( \Rightarrow \) (iv) and (i) \( \Rightarrow \) (v) hold.

Next we shall show that implication (iv) \( \Rightarrow \) (i) holds. Suppose the contrary: there exists a commutative linearly ordered non-archimedean \( d \)-group \( G \) such that the semigroup \( \mathcal{B}(G) \) has a unique non-trivial non-group congruence. Then by Proposition 3.1 we have that \( \sim_{13} \) is a unique non-trivial non-group congruence on the semigroup \( \mathcal{B}(G) \). Therefore, similarly as in the proof of Theorem 3.6 we get that \( G \) is isomorphic to the lexicographic product \( A \times \text{lex} H \) of non-singleton linearly ordered groups \( A \) and \( H \), and hence by Theorem 3.6 we have that the semigroup \( \mathcal{B}(G) \) has a non-trivial non-group congruence \( \sim \). We define a relation \( \equiv \) on the semigroup \( \mathcal{B}(G) \) as follows:

1. \( \left( \alpha^{(a,b)}_{(c,d)}, \alpha^{(p,q)}_{(r,s)} \right) \in \equiv \) if and only if \( \left( \alpha^{(a,b)}_{(c,d)}, \alpha^{(p,q)}_{(r,s)} \right) \in \sim \), for \( \alpha^{(a,b)}_{(c,d)}, \alpha^{(p,q)}_{(r,s)} \in \mathcal{B}(G) \subseteq \mathcal{B}(G) \);
2. \( \left( \alpha^{(p,q)}_{(r,s)}, \alpha^{(p,q)}_{(r,s)} \right), \left( \alpha^{(p,q)}_{(r,s)}, \alpha^{(p,q)}_{(r,s)} \right), \left( \alpha^{(p,q)}_{(r,s)}, \alpha^{(p,q)}_{(r,s)} \right) \in \equiv \), for all \( p, r \in A \) and \( q, s \in H \);
\((iii)\) \(\left(\tilde{\alpha}_{(a,b)} \circ \tilde{\alpha}_{(p,q)}\right) \in \sim\) if and only if \(\left(\alpha_{(a,b)}^{(c,d)}, \alpha_{(r,s)}^{(p,q)}\right) \in \sim\), for \(\alpha_{(a,b)}^{(c,d)}, \alpha_{(r,s)}^{(p,q)} \in \mathcal{B}(G) \subseteq \mathcal{B}(G)\) and \(\tilde{\alpha}_{(a,b)} \circ \tilde{\alpha}_{(r,s)} \in \mathcal{B}(G) \subseteq \mathcal{B}(G)\);

\((iv)\) \(\left(\tilde{\alpha}_{(a,b)} \circ \tilde{\alpha}_{(r,s)}\right) \in \sim\) if and only if \(\left(\alpha_{(a,b)}^{(c,d)}, \alpha_{(r,s)}^{(p,q)}\right) \in \sim\), for \(\alpha_{(a,b)}^{(c,d)}, \alpha_{(r,s)}^{(p,q)} \in \mathcal{B}(G) \subseteq \mathcal{B}(G)\) and \(\tilde{\alpha}_{(a,b)} \circ \tilde{\alpha}_{(r,s)} \in \mathcal{B}(G) \subseteq \mathcal{B}(G)\);

\((v)\) \(\left(\alpha_{(a,b)}^{(c,d)}, \alpha_{(r,s)}^{(p,q)}\right) \in \sim\) if and only if \(\left(\alpha_{(a,b)}^{(c,d)}, \alpha_{(r,s)}^{(p,q)}\right) \in \sim\), for \(\alpha_{(a,b)}^{(c,d)}, \alpha_{(r,s)}^{(p,q)} \in \mathcal{B}(G) \subseteq \mathcal{B}(G)\) and \(\tilde{\alpha}_{(a,b)} \circ \tilde{\alpha}_{(r,s)} \in \mathcal{B}(G) \subseteq \mathcal{B}(G)\).

Then simple verifications show that \(\sim\) is a congruence on the semigroup \(\mathcal{B}(G)\), and moreover the quotient semigroup \(\mathcal{B}(G)/\sim\) is isomorphic to the quotient semigroup \(\mathcal{B}(G)/\sim_{\#}\). This implies that the congruence \(\sim\) is different from \(\sim_{\#}\). This contradicts that \(\sim_{\#}\) is a unique non-trivial non-group congruence on the semigroup \(\mathcal{B}(G)\). The obtained contradiction implies implication \((iv) \Rightarrow (i)\).

The proof of implication \((v) \Rightarrow (i)\) is similar to implication \((iv) \Rightarrow (i)\). \(\square\)

**Theorem 3.10.** Let \(G\) be a linearly ordered group and \(\mathcal{E}_{mg}\) be the least group congruence on the semigroup \(\mathcal{B}(G)\) (resp., \(\mathcal{B}^+(G)\)). Then the quotient semigroup \(\mathcal{B}(G)/\mathcal{E}_{mg}\) (resp., \(\mathcal{B}^+(G)/\mathcal{E}_{mg}\)) is antiisomorphic to the group \(G\).

**Proof.** By Proposition 1.2 \((ii)\) and Lemma III.5.2 from [20] we have that elements \(\alpha_{b}^{g}\) and \(\alpha_{d}^{g}\) are \(\mathcal{E}_{mg}\)-equivalent in \(\mathcal{B}(G)\) (resp., in \(\mathcal{B}^+(G)\)) if and only if there exists \(x \in G\) such that \(\alpha_{b}^{g} \cdot \alpha_{d}^{x} = \alpha_{d}^{g} \cdot \alpha_{b}^{x}\). Then Proposition 2.4 \((i)\) implies that \(\alpha_{b}^{g} \cdot \alpha_{d}^{g} = \alpha_{d}^{g} \cdot \alpha_{b}^{g}\) for all \(g \in G\) such that \(g \geq x\) in \(G\). If \(g \geq b\) and \(g \geq d\) then the definition of the semigroup operation in \(\mathcal{B}(G)\) (resp., in \(\mathcal{B}^+(G)\)) implies that \(\alpha_{b}^{g} \cdot \alpha_{d}^{g} = \alpha_{b}^{g \cdot b^{-1} \cdot c}\) and \(\alpha_{d}^{g} \cdot \alpha_{b}^{g} = \alpha_{d}^{g \cdot d^{-1} \cdot c}\), and since \(G\) is a group we get that \(b^{-1} \cdot a = d^{-1} \cdot c\).

Conversely, suppose that \(\alpha_{b}^{g}\) and \(\alpha_{d}^{g}\) are elements of the semigroup \(\mathcal{B}(G)\) (resp., in \(\mathcal{B}^+(G)\)) such that \(b^{-1} \cdot a = d^{-1} \cdot c\). Then for any element \(g \in G\) such that \(g \geq b\) and \(g \geq d\) in \(G\) we have that \(\alpha_{b}^{g} \cdot \alpha_{d}^{g} = \alpha_{b}^{g \cdot b^{-1} \cdot c}\) and \(\alpha_{d}^{g} \cdot \alpha_{b}^{g} = \alpha_{d}^{g \cdot d^{-1} \cdot c}\), and hence since \(b^{-1} \cdot a = d^{-1} \cdot c\) we get that \(\alpha_{b}^{g} \mathcal{E}_{mg} \alpha_{d}^{g}\). Therefore, \(\alpha_{b}^{g} \mathcal{E}_{mg} \alpha_{d}^{g}\) in \(\mathcal{B}(G)\) (resp., in \(\mathcal{B}^+(G)\)) if and only if \(b^{-1} \cdot a = d^{-1} \cdot c\).

We determine a map \(f: \mathcal{B}(G) \to G\) (resp., \(f: \mathcal{B}^+(G) \to G\)) by the formula \((\alpha_{b}^{g})f = b^{-1} \cdot a\), for \(b, a \in G\). Then we have that

\[
(\alpha_{b}^{g} \cdot \alpha_{d}^{g})f = \begin{cases} 
(\alpha_{b}^{g \cdot b^{-1} \cdot c})f, & \text{if } b < c; \\
(\alpha_{b}^{g})f, & \text{if } b = c; \\
(\alpha_{b}^{g \cdot c^{-1} \cdot d})f, & \text{if } b > c,
\end{cases} = \begin{cases} 
(d^{-1} \cdot c \cdot b^{-1} \cdot a), & \text{if } b < c; \\
(d^{-1} \cdot a), & \text{if } b = c; \\
(d^{-1} \cdot c \cdot d^{-1} \cdot a), & \text{if } b > c,
\end{cases}
\]
for \( a, b, c, d \in G \). This completes the proof of the theorem. \( \Box \)

Hölder’s Theorem and Theorem 3.10 imply the following:

**Theorem 3.11.** Let \( G \) be an archimedean linearly ordered group and \( \mathcal{C}_{mg} \) be the least group congruence on the semigroup \( \mathcal{B}(G) \) (resp., \( \mathcal{B}^+(G) \)). Then the quotient semigroup \( \mathcal{B}(G)/\mathcal{C}_{mg} \) (resp., \( \mathcal{B}^+(G)/\mathcal{C}_{mg} \)) is isomorphic to the group \( G \).

Theorems 3.2, 3.3 and 3.11 imply the following:

**Corollary 3.12.** Let \( G \) be an archimedean linearly ordered group and \( \mathcal{C}_{mg} \) be the least group congruence on the semigroup \( \mathcal{B}(G) \) (resp., \( \mathcal{B}^+(G) \)). Then every non-isomorphic image of the semigroup \( \mathcal{B}(G) \) (resp., \( \mathcal{B}^+(G) \)) is isomorphic to some homomorphic image of the group \( G \).

**Theorem 3.13.** Let \( G \) be a linearly ordered \( d \)-group and \( \mathcal{C}_{mg} \) be the least group congruence on the semigroup \( \mathcal{B}(G) \) (resp., \( \mathcal{B}^+(G) \)). Then the quotient semigroup \( \mathcal{B}(G)/\mathcal{C}_{mg} \) (resp., \( \mathcal{B}^+(G)/\mathcal{C}_{mg} \)) is antiisomorphic to the group \( G \).

**Proof.** Similar arguments as in the proofs of Theorem 3.10 and Proposition 5.7 show that the following assertions are equivalent:

1. \( \alpha_b^a \mathcal{C}_{mg} \alpha_d^c \) in \( \mathcal{B}(G) \) (resp., in \( \mathcal{B}^+(G) \));
2. \( \alpha_b^a \mathcal{C}_{mg} \alpha_d^c \) in \( \mathcal{B}(G) \) (resp., in \( \mathcal{B}^+(G) \));
3. \( \hat{\alpha}_b^a \mathcal{C}_{mg} \hat{\alpha}_d^c \) in \( \mathcal{B}(G) \) (resp., in \( \mathcal{B}^+(G) \));
4. \( b^{-1} \cdot a = d^{-1} \cdot c \).

We determine a map \( f : \mathcal{B}(G) \to G \) (resp., \( f : \mathcal{B}^+(G) \to G \)) by the formulae \( (\alpha_b^a)f = b^{-1} \cdot a \) and \( (\hat{\alpha}_b^a)f = b^{-1} \cdot a \), for \( a, b \in G \). Then we have that

\[
(\alpha_b^a \cdot \alpha_d^c)f = \begin{cases}
(\hat{\alpha}_d^{c \cdot b^{-1} \cdot a})f, & \text{if } b < c; \\
(\hat{\alpha}_d^a)f, & \text{if } b = c; \\
(\hat{\alpha}_{b \cdot c^{-1} \cdot d})f, & \text{if } b > c,
\end{cases}
\]

\[
(\hat{\alpha}_b^a \cdot \hat{\alpha}_d^c)f = \begin{cases}
(\hat{\alpha}_d^{c \cdot b^{-1} \cdot a})f, & \text{if } b < c; \\
(\hat{\alpha}_d^a)f, & \text{if } b = c; \\
(\hat{\alpha}_{b \cdot c^{-1} \cdot d})f, & \text{if } b > c,
\end{cases}
\]
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\[
(\alpha^a_b \cdot \alpha^c_d)f = \begin{cases} 
(\alpha^a_b d^{-1} - b^{-1} a)_f, & \text{if } b < c; \\
(\alpha^a_b a)_f, & \text{if } b = c; \\
(\alpha^a_b c^{-1} d)_f, & \text{if } b > c,
\end{cases}
\]

\[
= \begin{cases} 
1^{-1} \cdot c \cdot b^{-1} \cdot a, & \text{if } b < c; \\
1^{-1} \cdot c \cdot b^{-1} \cdot a, & \text{if } b = c; \\
1^{-1} \cdot c \cdot b^{-1} \cdot a, & \text{if } b > c,
\end{cases}
\]

for \(a, b, c, d \in G\). This completes the proof of the theorem.

\[\Box\]

Hölder’s Theorem and Theorem 3.13 imply the following:

**Theorem 3.14.** Let \(G\) be an archimedean linearly ordered \(d\)-group and \(\mathcal{C}_{mg}\) be the least group congruence on the semigroup \(\mathcal{B}(G)\) (resp., \(\mathcal{B}^+(G)\)). Then the quotient semigroup \(\mathcal{B}(G)/\mathcal{C}_{mg}\) (resp., \(\mathcal{B}^+(G)/\mathcal{C}_{mg}\)) is isomorphic to the group \(G\).

Theorems 3.8 and 3.14 imply the following:

**Corollary 3.15.** Let \(G\) be an archimedean linearly ordered \(d\)-group, \(T\) be a semigroup and \(h: \mathcal{B}(G) \to T\) (resp., \(h: \mathcal{B}^+(G) \to T\)) be a homomorphism. Then only one of the following conditions holds:

(i) \(h\) is a monomorphism;

(ii) the image \((\mathcal{B}(G))h\) (resp., \((\mathcal{B}^+(G))h\)) is isomorphic to some homomorphic image of the group \(G\);

(iii) the image \((\mathcal{B}(G))h\) (resp., \((\mathcal{B}^+(G))h\)) is isomorphic to the semigroup \(\mathcal{B}(G)\) (resp., \(\mathcal{B}^+(G)\)).

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