Output Feedback $\mathcal{H}_2$ Model Matching for Decentralized Systems with Delays

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Abstract

This paper gives a new solution to the output feedback $\mathcal{H}_2$ model matching problem for a large class of delayed information sharing patterns. Existing methods for such problems typically reduce the decentralized problem to a centralized problem of higher state dimension. In contrast, the controller given in this paper is constructed from the solutions to the centralized control and estimation Riccati equations for the original system. The problem is solved by decomposing the controller into two components. One is centralized, but delayed, while the other is decentralized with finite impulse response (FIR). It is then shown that the optimal controller can be constructed through a combination of centralized spectral factorization and quadratic programming.

1 Introduction

Decentralized control problems arise when inputs to a dynamic system are chosen by multiple controllers with access to different information. In decentralized control with delays, local measurements are passed to the various controllers over a communication network with delays. As a result of the delays, some controllers will have access to measurements before others. This paper provides a new solution to the $\mathcal{H}_2$ model matching problem, subject to communication delays, based on spectral factorization.

1.1 Related Work

A large number of dynamic programming methods have been developed for decentralized optimal control problems. For the special case known as the one-step delay information sharing pattern, the output feedback $\mathcal{H}_2$ problem was solved in the 1970s by dynamic programming [1 2 3]. For more complex delay patterns, dynamic programming has extensions to decentralized state feedback [4 5 6], but output feedback is difficult because the separation principle fails [7 8 9]. Recently, methods based on POMDPs have been developed for output feedback control of nonlinear systems with general delay patterns [10 11].
In the past few years, spectral factorization has been employed to derive explicit solutions to the $H_2$ problem with sparsity constraints, but not delays. First, decentralized state feedback was addressed [12, 13], followed by restricted types of output feedback [14, 15], and most recently full output feedback [16]. In these works, it was shown how to efficiently construct decentralized solutions using standard Riccati equations. This paper applies spectral factorization to delayed information sharing patterns. As in the sparsity constrained case, the resulting controllers are efficiently computable from solutions to centralized Riccati equations.

1.2 Existing Solutions

The output feedback $H_2$ problem with communication delays, as studied in this paper, has been previously solved using approaches based on vectorization [17], linear matrix inequalities (LMIs) [18, 19]. The problem can also be solved as a special case of the work in [11]. All of these solutions reduce the decentralized control problem to a centralized problem of higher state dimension.

1.3 Contributions

The main contribution of this paper is a novel efficient solution to a general class of decentralized $H_2$ output feedback model matching problems with communication delays. Unlike the existing approaches mentioned above, the method of this paper works directly with the original state matrices. In fact, the solution is constructed from the classical control and estimation Riccati equations for the original system.

A key assumption made in this paper is that each local measurement eventually reaches each controller. This assumption allows the controller to be decomposed into a centralized, but delayed, component and a decentralized finite impulse response (FIR) component. Similar decompositions have been exploited in [3, 5, 6, 10, 11, 20]. Given the decomposition, the optimal centralized component can be computed as a function of the FIR component by a relatively straightforward extension of centralized spectral factorization. It is then shown that the optimal FIR component can be found by quadratic programming.

In the case of a quadratically invariant delay pattern [17], optimal decentralized feedback controllers can be computed via the Youla parametrization. When the delay pattern is not quadratically invariant, the model matching procedure of this paper is still optimal, but the feedback controller recovered by a linear fractional transformation is not guaranteed to satisfy the delay constraint.

1.4 Overview

The paper is structured as follows. Section 2 defines the general problem studied in this paper. Section 3 reviews spectral factorization for centralized $H_2$ model matching in both undelayed and delayed cases. Extending the delayed centralized model matching technique, the decentralized problem is solved in
Section 4. Numerical results are given in Section 5 and finally conclusions are given in 6.

2 Problem

This section introduces the basic notation and the model matching problem of interest. Subsection 2.3 describes how common delayed information sharing patterns can be cast in the framework of this paper.

2.1 Preliminaries on $\mathcal{H}_2$

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc of complex numbers. A function $G : (\mathbb{C} \cup \{\infty\}) \setminus \mathbb{D} \to \mathbb{C}^{p \times q}$ is in $\mathcal{H}_2$ if it can be expanded as

$$G(z) = \sum_{i=0}^{\infty} \frac{1}{z^i} G_i,$$

where $G_i \in \mathbb{C}^{p \times q}$ and $\sum_{i=0}^{\infty} \text{Tr}(G_i G_i^T) < \infty$. Define the conjugate of $G$ by

$$G(z)^\sim = \sum_{i=0}^{\infty} z^i G_i^*.$$

For a real rational transfer matrix, $G = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, the conjugate is given by

$$(C(zI - A)^{-1} B + D)^\sim = B^T \left( \frac{1}{z} I - A^T \right)^{-1} C^T + D^T.$$

The space $\mathcal{H}_2$ is a Hilbert space with inner product defined by

$$\langle G, H \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{Tr} \left( G(e^{j\theta}) H(e^{j\theta})^\sim \right) d\theta = \sum_{i=0}^{\infty} \text{Tr} (G_i H_i^*),$$

where the second equality follows from Parseval’s identity.

If $\mathcal{M}$ is a subspace of $\mathcal{H}_2$, denote the orthogonal projection onto $\mathcal{M}$ by $\mathbb{P}_{\mathcal{M}}$.

2.2 Formulation

This subsection introduces the generic problem of interest. Let $P$ be a stable discrete-time plant given by

$$P = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix},$$
with inputs of dimension $p_1, p_2$ and outputs of dimension $q_1, q_2$. Attention will be restricted to stable plants for simplicity. Unstable plants can be handled by first applying a stabilizing feedback and optimizing the resulting system.

For the existence of solutions of the appropriate Riccati equations, assume that

- $D_{12}^T D_{12}$ is positive definite,
- $(A, B_1)$ is stabilizable,
- $D_{21} D_{21}^T$ is positive definite,
- $(C_1, A)$ is detectable.

(Note that stabilizability and detectability follow immediately from the stability assumption.)

For $N \geq 1$, define the space of strictly proper finite impulse response (FIR) transfer matrices by $X = \bigoplus_{i=1}^N \mathbb{C}^{p_2 \times q_2}$. Note that $\frac{1}{z} \mathcal{H}_2$ can be decomposed into orthogonal subspaces as

$$\frac{1}{z} \mathcal{H}_2 = X \oplus \frac{1}{z^{N+1}} \mathcal{H}_2.$$  

Let $R_p$ be the space of proper real rational transfer matrices. Let $S \subset \frac{1}{z} R_p$ be a subspace of the form

$$S = \mathcal{Y} \oplus \frac{1}{z^{N+1}} R_p,$$

where $\mathcal{Y} \subset \bigoplus_{i=1}^N \mathbb{C}^{p_2 \times q_2} \subset X$.

The decentralized $\mathcal{H}_2$ model matching problem considered in this paper is given by

$$\min_Q \left\| P_{11} + P_{12} Q P_{21} \right\|_{\mathcal{H}_2}$$

s.t. $Q \in S \cap \frac{1}{z} \mathcal{H}_2$.  

(2)
A feedback controller for the plant can be defined from $Q$ by $K = Q(I + P_{22}Q)^{-1}$. If the space $S$ is quadratically invariant, then $Q \in S$ if and only if $K \in S$. [17]. Furthermore, since $Q$ is strictly proper and stable, and $P$ is stable, $K$ must be strictly proper and the closed loop system $P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$ must be stable. Furthermore, if $Q$ solves the model matching problem, then $K$ must solve the decentralized optimal control problem:

$$
\min_K \| P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} \|_2 \quad \text{s.t.} \quad K \in S.
$$

(3)

Note that even if $S$ is not quadratically invariant, the model matching problem is still convex, and can be solved by the methods in this paper. In this case, however, it could happen that $Q(I + P_{22}Q)^{-1} \notin S$, and thus the solution to Problem (2) need not lead to a solution of Problem (3).

For technical simplicity, controllers in this paper are assumed to be strictly proper (that is, in $\mathbb{R}^p$). The results in this paper can be extended to non-strictly proper controllers but more complicated formulas would result.

### 2.3 Communication Delay Patterns

Equation (1) can be used to model many delayed information sharing patterns. For instance, an infinite-horizon, strictly proper version of the 1-step delayed information sharing pattern studied in [1, 2, 3] is captured by the case that $N = 1$ and $Y$ corresponds to block diagonal FIR matrices

$$
Y = \frac{1}{z} \begin{bmatrix} R_{p21} & 0 \\ 0 & R_{p22} \end{bmatrix}.
$$

Similarly, for $N > 1$, the $N$-step delay information sharing pattern studied in [7, 8, 9, 10] can be characterized by $Y$ of the form

$$
Y = \bigoplus_{i=1}^{N} \frac{1}{z^i} \begin{bmatrix} R_{p21} & 0 \\ 0 & R_{p22} \end{bmatrix}.
$$

More general team problems with communication delays, such those studied in [11, 12, 13, 18, 19, 20], can also be captured by Equation (1). For instance, a strictly proper version of the three-player chain problem discussed in [5] is described by $N = 2$ and

$$
Y_{Ch} = \frac{1}{z} \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix} \oplus \frac{1}{z^2} \begin{bmatrix} * & * & 0 \\ * & * & * \\ 0 & * & * \end{bmatrix},
$$

(4)

where, for compactness, * is used to denote a space of appropriately sized real matrices.

The space $S$ is quadratically invariant if $KP_{22}K \in S$ for all $K \in S$. 

5
3 Centralized Spectral Factorization

This section gives spectral factorization solutions to centralized model matching problem in both delayed and undelayed cases. While the solutions are classical, they will be presented in detail, as the decentralized model matching problem relies heavily on the terms and ideas in the centralized solutions.

3.1 Undelayed Case

The undelayed case corresponds to

$$\min_Q \| P_{11} + P_{12}QP_{21} \|_{\mathcal{H}_2}$$

s.t. $Q \in \frac{1}{2}\mathcal{H}_2$. \hfill (5)

A necessary condition for optimality is given by

$$P_{12}P_{11} P_{21} + P_{12}P_{21} Q P_{21} \in \left( \frac{1}{2}\mathcal{H}_2 \right)^\perp.$$ \hfill (6)

A simple argument shows that $\left( \frac{1}{2}\mathcal{H}_2 \right)^\perp = \frac{1}{2}\mathcal{H}_2^\perp$, a fact that will be used several times.

To derive the optimality condition, let $\delta \in \frac{1}{2}\mathcal{H}_2$ be a small perturbation. The perturbed norm can be expanded as

$$\| P_{11} + P_{12}(Q + \delta)P_{21} \|_{\mathcal{H}_2}^2 = \langle P_{11} + P_{12}(Q + \delta)P_{21}, P_{11} + P_{12}(Q + \delta)P_{21} \rangle + 2\text{Re}(P_{12}(P_{11} + P_{12}QP_{21})P_{21}, \delta) + O(\|\delta\|_{\mathcal{H}_2}^2).$$

In particular, if $Q$ is optimal, then the second term must vanish for any $\delta$, and it follows that Equation (6) must hold.

The following classical lemmas show how to factorize $P_{12}P_{12}$ and $P_{21}P_{21}$ as products of causally invertible transfer matrices.

**Lemma 1.** Let $X$ be the stabilizing solution of the Riccati equation

$$X = C_1^T C_1 + A^T X A - (A^T X B_2 + C_1^T D_{12}) \Omega^{-1}(B_2^T X A + D_{12}^T C_1),$$

where $\Omega = D_{12}^T D_{12} + B_2^T X B_2$. Define the (linear quadratic regulator) gain by

$$K = -\Omega^{-1}(B_2^T X A + D_{12}^T C_1).$$

The transfer matrix $P_{12}P_{12}$ has a left spectral factorization $P_{12}P_{12} = W_L^{-1} W_L^{-1}$, where $W_L$ is given by

$$W_L = \begin{bmatrix} A + B_2 K & B_2 \\ K & I \end{bmatrix} \Omega^{-1/2},$$

$$W_L^{-1} = \Omega^{1/2} \begin{bmatrix} A & -B_2 \\ K & I \end{bmatrix}.$$
Lemma 2. Let \( Y \) be the stabilizing solution of the Riccati equation

\[
Y = B_1 B_1^T + AYA^T - (AYC_2^T + B_1 D_2^T) \Psi^{-1} (C_2 Y A^T + D_2 B_1^T),
\]

where \( \Psi = D_2 D_2^T + C_2 Y C_2^T \). Define the (Kalman filter) gain by

\[
L = -(AYC_2^T + B_1 D_2^T) \Psi^{-1}.
\]

The transfer matrix \( P_{21} P_{21}^\sim \) has a right spectral factorization

\[
P_{21} P_{21}^\sim = W_{R}^{-1} W_{R}^\sim
\]

where \( W_{R} \) is given by

\[
W_{R} = \Psi^{-1/2} \begin{bmatrix} A + L C_2 & L \\ C_2 & I \end{bmatrix},
\]

\[
W_{R}^{-1} = \begin{bmatrix} A & L \\ -C_2 & I \end{bmatrix} \Psi^{1/2}
\]

The following standard theorem gives the spectral factorization solution to the model matching problem. The presentation is slightly non-standard, in that the optimal matrix \( Q_0 \) is defined in terms of an auxiliary matrix \( T \), which is used in the delayed and decentralized solutions.

Theorem 1. Define \( T \) by

\[
T = \Omega^{1/2} \begin{bmatrix} A \\ K \end{bmatrix} \Psi^{1/2}.
\]

The optimal solution to the model matching problem of Equation (5) is given by

\[
Q_0 = -W_L TW_R.
\]

Proof. Assume that Equation (6) holds. Plugging in the spectral factorizations shows that

\[
P_{21}^\sim P_{21} P_{21}^\sim + W_{L}^\sim W_{L}^{-1} Q W_{R}^{-1} W_{R}^\sim \in \frac{1}{z} \mathcal{H}_2.
\]

Anticausality of \( W_{L}^\sim \) and \( W_{R}^\sim \) implies that

\[
W_{L}^\sim P_{21} P_{21}^\sim W_{R}^\sim + W_{L}^{-1} Q W_{R}^{-1} \in \frac{1}{z} \mathcal{H}_2.
\]

Note that \( W_{L}^{-1} Q W_{R}^{-1} \in \frac{1}{z} \mathcal{H}_2 \). It follows that Equation (8) can be set to zero by applying the projection operator:

\[
P_{\frac{1}{z} \mathcal{H}_2} (W_{L}^\sim P_{12} P_{21} P_{21}^\sim W_{R}^\sim + W_{L}^{-1} Q W_{R}^{-1})
\]

\[
= P_{\frac{1}{z} \mathcal{H}_2} (W_{L}^\sim P_{12} P_{21} P_{21}^\sim W_{R}^\sim) + W_{L}^{-1} Q W_{R}^{-1}
\]

\[
= 0.
\]

Let \( T = P_{\frac{1}{z} \mathcal{H}_2} (W_{L}^\sim P_{12} P_{21} P_{21}^\sim W_{R}^\sim) \), Equation (9) shows that \( Q = -W_L TW_R \). Furthermore, standard state space manipulations show that \( T \) has the form in Equation (7), and the proof is complete.
3.2 Delayed Case

The delayed case corresponds to the following model matching problem:

\[
\begin{align*}
\min_Q & \|P_{11} + P_{12}QP_{21}\|_{\mathcal{H}_2} \\
\text{s.t.} & \quad Q \in \frac{1}{z}N^{+1}\mathcal{H}_2.
\end{align*}
\]

An argument analogous to the derivation of Equation (6) shows that a necessary condition for optimality in the delayed case is

\[
P_{12}^-P_{11}P_{21}^- + P_{12}^-P_{12}QP_{21}^- \in \left(\frac{1}{z^{N+1}}\mathcal{H}_2\right)^{\perp}.
\]

(11)

As in the undelayed case, a simple argument shows that \(\frac{1}{z}N^{+1}\mathcal{H}_2\) is orthogonal to \(\frac{1}{z}N^{+1}\mathcal{H}_2\). Theorem 2. The optimal solution to the delayed model matching problem is given by

\[
Q_N = -W_L \mathbb{P}_{\frac{1}{z^{N+1}}\mathcal{H}_2}(T)W_R.
\]

(12)

Proof. Assume that \(Q \in \frac{1}{z}\mathcal{H}_2\) satisfies Equation (11) and thus

\[
\begin{align*}
P_{12}^-P_{11}P_{21}^- + W_L^-W_L^{-1}QW_R^{-1}W_R^- & \in \frac{1}{z^{N+1}}\mathcal{H}_2, \\
W_L^-P_{12}^-P_{11}P_{21}^-W_R^- + W_L^{-1}QW_R^{-1} & \in \frac{1}{z^{N+1}}\mathcal{H}_2,
\end{align*}
\]

(13)

where the second line follows from anticausality of \(W_L^-\) and \(W_R^-\). As in the proof of the case with no delays, \(W_L^{-1}QW_R^{-1} \in \frac{1}{z^{N+1}}\mathcal{H}_2\) and the left side of Equation (13) can be set to zero by projection:

\[
\begin{align*}
\mathbb{P}_{\frac{1}{z^{N+1}}\mathcal{H}_2} (W_L^-P_{12}^-P_{11}P_{21}^-W_R^- + W_L^{-1}QW_R^{-1}) \\
&= \mathbb{P}_{\frac{1}{z^{N+1}}\mathcal{H}_2} (W_L^-P_{12}^-P_{11}P_{21}^-W_R^-) + W_L^{-1}QW_R^{-1} \\
&= 0.
\end{align*}
\]

Furthermore, since \(\frac{1}{z^{N+1}}\mathcal{H}_2 \subset \frac{1}{z}\mathcal{H}_2\), it follows that \(\mathbb{P}_{\frac{1}{z^{N+1}}\mathcal{H}_2} = \mathbb{P}_{\frac{1}{z^{N+1}}\mathcal{H}_2} \mathbb{P}_{\frac{1}{z}\mathcal{H}_2}\). Thus, the projection can be computed in terms of \(T\) as

\[
W_L^{-1}QW_R^{-1} = -\mathbb{P}_{\frac{1}{z^{N+1}}\mathcal{H}_2} \left(\mathbb{P}_{\frac{1}{z}\mathcal{H}_2} (W_L^-P_{12}^-P_{11}P_{21}^-W_R^-)\right) \\
= -\mathbb{P}_{\frac{1}{z^{N+1}}\mathcal{H}_2}(T).
\]

Multiplying on the left and right by \(W_L\) and \(W_R\), respectively, completes the proof. \(\square\)
4 Decentralized Model Matching

This section presents the main results of the paper. Recall that in centralized model matching, from Equation (2), that \( Q \) is constrained to be in the space \( \frac{1}{\gamma} \mathcal{H}_2 \oplus \mathcal{Y} \). It follows that without loss of generality, \( Q \) can be decomposed as
\[
Q = U + V
\]
with \( U \in \frac{1}{\gamma} \mathcal{H}_2 \) and \( V \in \mathcal{Y} \).

**Theorem 3.** The optimal solution to the decentralized model matching problem (Equation (2)) is given by
\[
Q^* = U^* + V^*
\]
where \( V^* \) is the unique minimizer of
\[
\| P_X (W_L^{-1}VW_R^{-1}) \|_{\mathcal{H}_2}^2 + 2 \langle P_X (W_L^{-1}VW_R^{-1}), T \rangle
\]
and
\[
U^* = Q_N - W_L P \frac{1}{\gamma} \mathcal{H}_2 (W_L^{-1}V^*W_R^{-1}) W_R.
\]
Here \( Q_N \) is the optimal centralized delayed controller from Theorem 2.

The theorem can be proved by combining the following two lemmas:

**Lemma 3.** For any \( V \in \mathcal{Y} \), the optimal solution to
\[
\min_U \| P_{11} + P_{12}VP_{21} + P_{12}UP_{21} \|_{\mathcal{H}_2}
\]
\[\text{s.t. } U \in \frac{1}{\gamma} \mathcal{H}_2 \]
is given by
\[
U(V) = Q_N - W_L P \frac{1}{\gamma} \mathcal{H}_2 (W_L^{-1}VW_R^{-1}) W_R.
\]
with optimal cost given by
\[
\| P_{11} + P_{12}VP_{21} + P_{12}U(V)P_{21} \|_{\mathcal{H}_2}^2 = \| P_{11} + P_{12}Q_NP_{21} \|_{\mathcal{H}_2}^2 + \| P_X (W_L^{-1}VW_R^{-1}) \|_{\mathcal{H}_2}^2 + 2 \langle P_X (W_L^{-1}VW_R^{-1}), T \rangle
\]

**Lemma 4.** The expression in Equation (14) has a unique minimum \( V^* \) which can be efficiently computed by quadratic programming.

**Remark 1.** Note that Equation (17) implies that the optimal \( U \) is always the sum of the optimal delayed controller, \( Q_N \), and a correction term that depends linearly on \( V \). Furthermore, Equation (18) shows that optimal decentralized cost is the cost of the delayed controller minus benefits gained from choosing \( V \). In particular if \( V = 0 \), then the delayed cost is recovered.
To see how the lemmas prove Theorem 3, assume that $U^*$ and $V^*$ are optimal. By optimality, $U^*$ must solve Problem (16) with $V = V^*$. Thus Equation (15) holds. Furthermore, optimality of $V^*$ implies that it must minimize the right side of Equation (18), which is equivalent to minimizing Equation (14).

To complete the proof of Theorem 3, the lemmas will now be proved.

Lemma 3. First Equation (17) will be derived, and then the form will be used to derive Equation (18). If $U$ solves Problem (16), then, as in the proof of the centralized delayed case (Theorem 2), a necessary condition for optimality is given by

$$P_{12}P_{11}P_{21}^\sim + P_{12}^2P_{12}V^*P_{21}^*P_{21}^\sim \in \frac{1}{z^{N+1}}\mathcal{H}_2^\perp.$$  

Plugging in the spectral factorizations shows that

$$P_{12}^*P_{11}^*P_{21}^* + W_L^*W_L^{-1}VW_R^{-1}W_R^* \in \frac{1}{z^{N+1}}\mathcal{H}_2^\perp.$$  

By anticausality, multiplying on the left and right by $W_L^*$ and $W_R^*$, respectively, gives,

$$W_L^*P_{12}^*P_{11}^*P_{21}^*W_R^* + W_L^{-1}VW_R^{-1}W_R^* \in \frac{1}{z^{N+1}}\mathcal{H}_2^\perp.$$  

As in the centralized delayed case, $W_L^{-1}UW_R^{-1} \in \frac{1}{z^{N+1}}\mathcal{H}_2$ and the left side can be set to zero by projection:

$$\begin{align*}
\mathcal{P}_{z^{N+1}} \mathcal{H}_2 \left( W_L^*P_{12}^*P_{11}^*P_{21}^*W_R^* \right) \\
\mathcal{P}_{z^{N+1}} \mathcal{H}_2 \left( W_L^{-1}VW_R^{-1} + W_L^{-1}UW_R^{-1} \right) \\
= \mathcal{P}_{z^{N+1}} \mathcal{H}_2 (T) \\
+ \mathcal{P}_{z^{N+1}} \mathcal{H}_2 \left( W_L^{-1}VW_R^{-1} \right) + W_L^{-1}UW_R^{-1} \\
= 0.
\end{align*}$$

Rearranging and multiplying on the left and right by $W_L$ and $W_R$, gives the form of $U$:

$$U = -W_L\mathcal{P}_{z^{N+1}} \mathcal{H}_2 (T)W_R \\
- W_L\mathcal{P}_{z^{N+1}} \mathcal{H}_2 (W_L^{-1}VW_R^{-1})W_R \\
= Q_N - W_L\mathcal{P}_{z^{N+1}} \mathcal{H}_2 (W_L^{-1}VW_R^{-1})W_R,$$

where $Q_N$ is the solution from Theorem 2. Thus Equation (17) has been proved. Now Equation (18) must be proved. The full controller, $Q$, is given by

$$Q = Q_N - W_L\mathcal{P}_{z^{N+1}} \mathcal{H}_2 (W_L^{-1}VW_R^{-1})W_R + V.$$  

(19)
The second and third terms can be expressed as
\[-W_L P_{X \rightarrow \mathcal{H}_2} (W_L^{-1} V W!_R^{-1}) W_R + V\]  \hspace{1cm} (20)
\[= W_L \left( W_L^{-1} V W!_R^{-1} \right) \left( P_{X \rightarrow \mathcal{H}_2} (W_L^{-1} V W!_R^{-1}) \right) W_R \]
\[= W_L \left( \left( P_{X \rightarrow \mathcal{H}_2} - P_{X \rightarrow \mathcal{H}_2} \right) \left( W_L^{-1} V W!_R^{-1} \right) \right) W_R \]
\[= W_L \mathbb{P}_X \left( W_L^{-1} V W!_R^{-1} \right) W_R\]

Note that the third equality follows since \( P_{X} = P_{X \rightarrow \mathcal{H}_2} - P_{X \rightarrow \mathcal{H}_2} \). Defining \( G \) by
\[G = \mathbb{P}_X \left( W_L^{-1} V W!_R^{-1} \right),\]  \hspace{1cm} (21)
the controller \( Q \) can now be written as
\[Q = Q_N + W_R G W_L.\]  \hspace{1cm} (22)

Plugging Equation (22) into \( \|P_{11} + P_{12} Q P_{21}\|_{\mathcal{H}_2}^2 \) gives a quadratic function of \( G \):
\[\|P_{11} + P_{12} Q P_{21}\|_{\mathcal{H}_2}^2\]
\[= \|P_{11} + P_{12} (Q_N + W_L G W_R) P_{21}\|_{\mathcal{H}_2}^2\]
\[= \|P_{11} + P_{12} Q N P_{21}\|_{\mathcal{H}_2}^2 + \|P_{12} W_L G W_R P_{21}\|_{\mathcal{H}_2}^2\]
\[+ 2 \langle P_{11} + P_{12} Q N P_{21}, P_{12} W_L G W_R P_{21} \rangle,\]  \hspace{1cm} (23)

where the third term is real because \( Q_N \) and \( G \) must have real coefficients.

The second term of Equation (23) can be simplified as
\[\langle P_{12} W_L G W_R P_{21}, P_{12} W_L G W_R P_{21} \rangle\]
\[= \langle W_L^* P_{12} W_L G W_R P_{21} W_R^*, G \rangle\]
\[= \langle W_L^* W_L W_L^{-1} W_L G W_R W_R^{-1} W_R^{-1} G \rangle\]
\[= \langle G, G \rangle.\]  \hspace{1cm} (24)

Similarly, the third term of Equation (23) can be simplified as
\[\langle P_{11} + P_{12} Q_d P_{21}, P_{12} W_L G W_R P_{21} \rangle\]
\[= \langle W_L^* P_{11} P_{12} W_L^* W_R^{-1}, G \rangle\]
\[+ \langle W_L^* P_{12} Q N P_{21} P_{21} W_R^*, G \rangle\]
\[= \langle T, G \rangle + \langle W_L^* W_L^{-1} Q_N W_R^{-1} W_R^{-1} W_R^{-1} G \rangle\]
\[= \langle T, G \rangle + \langle W_L^{-1} Q_N W_R^{-1}, G \rangle\]
\[= \langle T, G \rangle.\]  \hspace{1cm} (25)

The fourth equality follows because \( G \in \mathcal{X}^* \) and \( W^{-1}_L Q_N W^{-1}_R \in \frac{1}{\mathcal{X}} \mathcal{H}_2 \), which are orthogonal spaces. Combining Equation (24) and (25) with Equation (23) proves that the cost can be decomposed as
\[\|P_{11} + P_{12} Q P_{21}\|_{\mathcal{H}_2}^2\]
\[= \|P_{11} + P_{12} Q N P_{21}\|_{\mathcal{H}_2}^2 + \|G\|_{\mathcal{H}_2}^2 + 2 \langle G, T \rangle.\]
Substituting the definition of $G$ (Equation (21)), proves Equation (18) and the proof of the lemma is complete.

**Lemma 4** Recalling Equation (21), $G$ can be expanded as an FIR transfer matrix

$$ G = \sum_{i=1}^{N} \frac{1}{z^i} G_i $$

Now the coefficients of $G$ will be computed in terms of $V$, $W_L^{-1}$, and $W_R^{-1}$. For notional simplicity, let $H = W_L^{-1}$ and $J = W_R^{-1}$. The matrices $H$ and $J$ can be expanded as

$$ H = \sum_{i=0}^{\infty} \frac{1}{z^i} H_i = \Omega^{1/2} (I - \frac{1}{z} \sum_{i=0}^{\infty} \frac{1}{2} K A^i B_2) $$
$$ J = \sum_{i=0}^{\infty} \frac{1}{z^i} J_i = (I - \frac{1}{z} \sum_{i=0}^{\infty} \frac{1}{2} C_2 A^i L) \Psi^{1/2}. $$

Since $V \in Y \subset X$ it can be expanded as $V = \sum_{i=1}^{N} \frac{1}{z} V_i$. It follows that $G_i$ can be written as a linear function of $V$:

$$ G_i = \sum_{j,l \geq 0, k \geq 1, j+k+l=i} H_j V_k J_l. \quad (26) $$

Similar to $H$ and $J$, $T$ can be expanded as

$$ T = \sum_{i=0}^{\infty} \frac{1}{z^i} T_i = \frac{1}{z} \sum_{i=0}^{\infty} \frac{1}{z^i} \Omega^{1/2} K A^i L \Psi^{1/2}. $$

The expansions of $G$ and $T$ can now be used to express Equation (14) in a form suitable for numerical evaluation:

$$ \| P_X (W_L^{-1} V W_R^{-1}) \|_{H_2}^2 + 2 \langle P_X (W_L^{-1} V W_R^{-1}) , T \rangle \\
= \| G \|_{H_2}^2 + 2 \langle G , T \rangle \\
= \sum_{i=1}^{N} \text{Tr} (G_i G_i^T) + 2 \sum_{i=1}^{N} \text{Tr} (G_i T_i^T). \quad (27) $$

Note that Equations (26) and (27) can be used to define a convex quadratic program in $V$. If the quadratic form $\sum_{i=1}^{N} \text{Tr} (G_i G_i^T)$ is positive definite in $V$, then the right side of Equation (27) must have a unique minimum which is efficiently computable.

The proof can thus be completed by showing that $\| G \|_{H_2}^2 = 0$ implies that $V = 0$. Assume that $\| G \|_{H_2}^2 = 0$. By the positive definiteness of norms, it must be that $G = 0$. Equations (20) and (21) imply that

$$ W_L GW_R = V - W_L \frac{1}{z^{N+1}} H_2 \left( W_L^{-1} V W_R^{-1} \right) W_R, $$

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and thus, by projection,

\[ V = P_X (W_LGW_R). \]

Therefore, \( G = 0 \) implies that \( V = 0 \) and the proof is complete.

\[ \square \]

5 Numerical Examples

The results in this paper demonstrate that decentralized model matching with communication delays can be efficiently solved by in terms of the original state matrices. In particular, aside from centralized Riccati equations, the only numerical computation required is a quadratic program specified by Equations (26) and (27). This section demonstrates the method with a few examples.

5.1 The Chain Problem

The three-player chain structure, [5], is a delayed information sharing pattern specified by the graph in Figure 2. In the frequency domain, the information structure is represented by the constraint \( K \in \mathcal{S}_{Ch} = \mathcal{Y}_{Ch} \oplus \frac{1}{s} \mathcal{R}_p \), where \( \mathcal{Y}_{Ch} \) is given in Equation (4). Consider the plant specified by

\[
A = \begin{bmatrix}
0.5 & 0.2 & 0 \\
0.2 & 0.5 & 0.2 \\
0 & 0.2 & 0.5 \\
\end{bmatrix},
\]

\[
B = \begin{bmatrix}
I_{3 \times 3} & 0_{3 \times 3} & I_{3 \times 3} \\
\end{bmatrix},
\]

\[
C = \begin{bmatrix}
0_{3 \times 3} \\
I_{3 \times 3} \\
\end{bmatrix},
\]

\[
D = \begin{bmatrix}
0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} \\
0_{3 \times 3} & 0_{3 \times 3} & I_{3 \times 3} \\
0_{3 \times 3} & I_{3 \times 3} & 0_{3 \times 3} \\
\end{bmatrix}.
\]

For comparison purposes, the optimal \( \mathcal{H}_2 \) norm was computed using model matching from this paper, the LMI method of [18, 19], and the vectorization method of [17]. In all three cases the norm was found to be 2.1082. In contrast, the centralized controller, \( Q_0 \), gives a norm of 2.0853, while the delayed controller, \( Q_2 \), gives a norm of 2.1780. This is to be expected, since the controller
Figure 3: This plot shows the closed-loop norm for $Q^N_{\text{Tri}}$, $Q^N_{\text{Di}}$, $Q^N_{\text{Low}}$, and $Q_N$ (the pure delay case). For a given $N$, the controllers with fewer sparsity constraints give rise to lower norms. As $N$ increases, all of the norms increase monotonically since the controllers have access to less information. The dotted lines correspond to the optimal norms for sparsity structures given in Equation 28. For pure delay, $Q_N \to 0$ as $N \to \infty$, and thus the norm approaches the open-loop value.

obeying the three-player chain structure is more constrained than $Q_0$, but less constrained than $Q_2$: $\frac{1}{2} \mathcal{H}_2 \subset (S_{\text{Ch}} \cap \frac{1}{2} \mathcal{H}_2) \subset \frac{1}{2} \mathcal{H}_2$.

5.2 Increasing Delays

Consider the plant with matrices given by

$$A = \begin{bmatrix} 1 & 0.2 & 0 & 0 \\ -0.2 & 0.8 & 0 & 0 \\ 0 & 0 & 1 & 0.2 \\ 0 & -0.2 & -0.2 & 0.8 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.2 & -0.2 & 0 & 0.2 \\ 0 & 0 & 0 & 0 \\ 0.2 & 0.2 & 0 & 0.2 \end{bmatrix}.$$
\begin{align*}
C &= \begin{bmatrix}
10 & 0 & -10 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\end{bmatrix}, \\
D &= \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
\end{bmatrix}.
\end{align*}

For \( N \geq 1 \), let \( Q^N_{Tri}, Q^N_{Di}, \) and \( Q^N_{Low} \) solve the decentralized model matching problem, Equation (2), with the form
\begin{align*}
Q^N_{Tri} &= U^N_{Tri} + V^N_{Tri}, \\
Q^N_{Di} &= U^N_{Di} + V^N_{Di}, \\
Q^N_{Low} &= U^N_{Low} + V^N_{Low}.
\end{align*}

Here \( U^N_{Tri}, U^N_{Di}, U^N_{Low} \in \mathbb{F}^{1 \times z \mathcal{H}_2} \) and \( V^N_{Tri}, V^N_{Di}, V^N_{Low} \) are FIR transfer matrices with sparsity structure given by
\begin{align*}
V^N_{Tri} &= \sum_{i=1}^{N} \frac{1}{z^i} \begin{bmatrix}
* & 0 \\
* & * \\
\end{bmatrix}, \\
V^N_{Di} &= \sum_{i=1}^{N} \frac{1}{z^i} \begin{bmatrix}
* & 0 \\
0 & * \\
\end{bmatrix}, \\
V^N_{Low} &= \sum_{i=1}^{N} \frac{1}{z^i} \begin{bmatrix}
0 & 0 \\
0 & * \\
\end{bmatrix}.
\end{align*}

The resulting norms are plotted in Figure (3).

As \( N \to \infty \), the resulting controllers appear to approach optimal sparse controllers
\begin{align*}
Q^\infty_{Tri} &\in \begin{bmatrix}
\frac{1}{z} \mathcal{H}_2 & 0 \\
\frac{1}{z} \mathcal{H}_2 & \frac{1}{z} \mathcal{H}_2 \\
\end{bmatrix}, \\
Q^\infty_{Di} &\in \begin{bmatrix}
\frac{1}{z} \mathcal{H}_2 & 0 \\
0 & \frac{1}{z} \mathcal{H}_2 \\
\end{bmatrix}, \\
Q^\infty_{Low} &\in \begin{bmatrix}
0 & 0 \\
0 & \frac{1}{z} \mathcal{H}_2 \\
\end{bmatrix},
\end{align*}
(28)
which can be computed by the vectorization technique from [17]. Evidence for the convergence is shown by the fact that the norms limit to the values computed for the sparse controllers (Figure 3).
6 Conclusion

This paper derives a novel solution for a class of output feedback $H_2$ model matching problems with communication delays. To find the optimal solution, the controller is decomposed into orthogonal components, both of which are easily computable. In particular, centralized delayed controllers that optimally correct for the FIR component are computed by spectral factorization. Then, the problem is then reduced to optimization over the FIR component.

The results of this paper indicate that the optimal control can be computed in terms of the centralized Riccati equations for the system. Existing time-domain methods, such as [11, 18, 19], work with state variables that have been augmented to include memory vectors required by the various controllers. The optimal controllers are then constructed based on centralized solutions to the augmented-state problems. It would be interesting to see if these alternative constructions can be mapped onto one another. In particular, the augmented-state solutions could lend insight into the computation of the FIR terms, while the method of this paper might be used to construct solutions to the augmented-state problems in terms of optimal controllers for the original centralized system.

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