PROPAGATION OF SINGULARITIES FOR GRAVITY-CAPILLARY WATER WAVES

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Abstract. We generalize the wavefront set of Hörmander and the homogeneous wavefront set of Nakamura to the quasi-homogeneous wavefront set, which enables us to obtain the propagation of singularities for gravity-capillary water waves of finite depth. Consequences of this study include firstly the existence of solutions to the gravity-capillary water wave equation in weighted Sobolev spaces, secondly a microlocal smoothing effect for gravity-capillary water waves, and thirdly the propagation of singularities and microlocal smoothing effects for linear models such as fractional Schrödinger equations, the fourth order Schrödinger equation, etc. Our proof is based on the paradifferential calculus on weighted Sobolev spaces and the semiclassical paradifferential calculus.

1. Introduction

We are interested in the propagation of singularities for the gravity-capillary water wave equation, which is a quasilinear dispersive equation to be defined later. We shall first revisit some classical results about propagation of singularities for simpler linear dispersive equations, and see how they lead to a more generalized definition of singularities that is adaptive for gravity-capillary water waves.

1.1. Half Wave Equation. By the classical definition, \( x_0 \in \mathbb{R}^d \) is called a singularity of \( u \in \mathcal{D}'(\mathbb{R}^d) \), if \( u \) is not \( C^\infty \) in any neighborhood of \( x_0 \); the singular support of \( u \), denoted by \( \text{sing supp } u \), is the set of all singularities of \( u \). To study the propagation of \( \text{sing supp } u \) when \( u \) solves some PDEs, the information given by \( \text{sing supp } u \) alone is usually insufficient, as the direction of propagation for a singularity is not determined by its position, but rather by its “direction of oscillation”.

More precisely, in [15], Hörmander introduced the wavefront set \( \text{WF}(u) \), lifting \( \text{sing supp } u \) to the phase space \( \mathbb{R}^d \times (\mathbb{R}^d \setminus 0) \). By definition, \((x_0, \xi_0) \notin \text{WF}(u)\) if for some \( \varphi \in C^\infty_c(\mathbb{R}^d) \) with \( \varphi(x_0) \neq 0 \), \( \hat{\varphi}u \) decays rapidly within some conical neighborhood of \( \xi_0 \). If \((x_0, \xi_0) \in \text{WF}(u)\), then it is called a microlocal singularity of \( u \); we have \( x_0 \in \text{sing supp } u \) and \( \xi_0 \) is called the frequency of this microlocal singularity. By Sjöstrand-Zworski [30], \((x_0, \xi_0) \notin \text{WF}(u)\) if and only if for some \( a \in C^\infty_c(\mathbb{R}^{2d}) \) with \( a(x_0, \xi_0) \neq 0 \), \( a(x, hD_x)u = O(h^\infty)_{L^2}, 0 < h < 1 \). Therefore, \( \text{WF}(u) \) describes the mass accumulation of \( u \) in the phase space.

A classical result says that for solutions to the half wave equation

\[
\partial_t u + i\sqrt{-\Delta}u = 0,
\]

singularities travel at finite speeds along geodesics.

Theorem 1.1 (Lax [22], Courant-Lax [12], Hörmander [15, 16], etc.). Let \( u \) solve the half wave equation with initial data \( u_0 \in \mathcal{D}'(\mathbb{R}^d) \), and \( t_0 \in \mathbb{R} \), then \((x_0, \xi_0) \in \text{WF}(u_0)\) if and only if \((x_0 + t_0\xi_0/|\xi_0|, \xi_0) \in \text{WF}(u(t_0))\), i.e., \( \text{WF}(u) \) is propagated by the Hamiltonian flow of \(|\xi|\).

More generally, if \( u \) solves \( \partial_t u + iPu = 0 \) where \( P \) is a first order pseudodifferential operator admitting a principal symbol \( p \), then \( \text{WF}(u) \) is propagated by the Hamiltonian flow of \( p \).

As for semilinear wave equations, we refer to Bony [9], Lebeau [23], and the references therein, for classical results of propagation of singularities.

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1.2. Schrödinger Equation. For dispersive equations with an infinite speed of propagation, singularities can disappear and emerge. For example, if \( u \) solves the linear Schrödinger equation,

\[
\partial_t u - i \Delta u = 0,
\]

with \( u(0) \in \mathcal{D}'(\mathbb{R}^d) \) (the space of compactly supported distributions), then \( u(t) \in C^\infty(\mathbb{R}^d) \) for \( t \neq 0 \). Conversely, if \( u(0, x) = e^{-i|x|^2} \) with a quadratic oscillation at infinity, then \( u \) develops a delta-function singularity when \( t = 1 \).

Early works, including Lascar [21] and Boutet-de-Monvel [10], proved that singularities travel at an infinite speed along geodesics, but did not give time-dependent information in terms of initial data. Wunsch [34] however, revealed the time-dependent propagation of singularities by tracking the transformation between singularities and quadratic oscillations, using the quadratic-scattering wavefront set, \( \text{WF}_{\text{qsc}}(u) \). Similar results were later obtained, independently, by Nakamura [27], using the homogeneous wavefront set, \( \text{HWF}(u) \). These two wavefront sets, \( \text{WF}_{\text{qsc}}(u) \) and \( \text{HWF}(u) \), were proven to be essentially equivalent by Ito [18]. We shall recall the definition of \( \text{HWF}(u) \), because the similarity it shares with \( \text{WF}(u) \) inspires us to define the quasi-homogeneous wavefront set that is adaptive to the gravity-capillary water wave equation.

By definition, \( (x_0, \xi_0) \notin \text{HWF}(u) \) if for some \( a \in C_c^\infty(\mathbb{R}^d) \) with \( a(x_0, \xi_0) \neq 0 \), \( a(hx, hD_x)u = \mathcal{O}(h^{-\eta}) \), \( 0 < h < 1 \). Please allow us to call \( (x_0, \xi_0) \) a homogeneous singularity of \( u \) if it belongs to \( \text{HWF}(u) \). Therefore, \( \text{HWF}(u) \) describes the phase space distribution of \( u \) at the spatial infinity. Observe that the dispersion relation of the Schrödinger equation is \( \omega = |\xi|^2 \), so a wave packet near the frequency \( \xi \sim h^{-1} \) travels at the group velocity \( v = \frac{\partial \omega}{\partial \xi} = 2\xi \sim h^{-1} \). That explains Nakamura’s choice of the homogeneously scaled quantization \( a \mapsto a(hx, hD_x) \).

**Theorem 1.2** (Nakamura [27]). Let \( u \) solve the Schrödinger equation with initial data \( u_0 \in L^2(\mathbb{R}^d) \), and let \( t_0 \in \mathbb{R} \), then

\[
\begin{align*}
(S.1) & \quad (x_0, \xi_0) \in \text{HWF}(u_0) \text{ if and only if } (x_0 + 2t_0\xi_0, \xi_0) \in \text{HWF}(u(t_0)), \text{i.e., } \text{HWF}(u) \text{ is propagated by the Hamiltonian flow of } |\xi|^2. \\
(S.2) & \quad \text{if } (x_0, \xi_0) \in \text{HWF}(u_0) \text{ with } \xi_0 \neq 0, \text{ and } t_0 \neq 0, \text{ then } (2t_0\xi_0, \xi_0) \in \text{HWF}(u(t_0)).
\end{align*}
\]

In [27], Nakamura gave the credit of (S.1) to S. Doi, and extended Theorem 1.2 to asymptotically Euclidean manifolds. Theorem 1.2 states that, homogeneous singularities of \( u \) travel at a finite speed along geodesics. \( (x_0, \xi_0) \in \text{HWF}(u_0) \) may develop singularities, but only when \( x_0 + 2t_0\xi_0 = 0 \). By [27], if \( (0, \xi_0) \notin \text{HWF}(u) \), then \( (x_0, \xi_0) \notin \text{WF}(u), \forall x_0 \in \mathbb{R}^d \). Therefore, although we do not know where the newly developed singularities are, their frequencies must be \( \xi_0 \). Moreover, \( (x_0, \xi_0) \in \text{WF}(u_0) \) of \( u_0 \) instantaneously creates \( (2t_0\xi_0, \xi_0) \in \text{HWF}(u(t_0)) \) whenever \( t_0 \neq 0 \), due to the infinite speed of propagation. Conversely, if \( (2t_0\xi_0, \xi_0) \notin \text{HWF}(u(t_0)) \), then \( \text{WF}(u_0) \cap \{ \xi = \xi_0 \} = \emptyset \). Particularly, if \( \text{HWF}(u(t_0)) = \emptyset \), then \( \text{WF}(u_0) = \emptyset \). These microlocal smoothing effects, when extended to asymptotic Euclidean manifolds, generalize the pioneer work of Craig-Kappeler-Strauss [13]. We also refer to Robbiano-Zuily [29] for a microlocal analytic smoothing effect of linear Schrödinger equations, and Szeftel [31] for microlocal smoothing effects of semilinear Schrödinger equations.

1.3. Gravity-Capillary Water Wave Equation. The gravity-capillary water wave equation describes the evolution of inviscid, incompressible, and irrotational fluid with a free surface, in the presence of a gravitational field and the surface tension.

1.3.1. Eulerian Formulation. Let \( \eta = \eta(t, x), (t, x) \in \mathbb{R} \times \mathbb{R}^d, \) be real valued, and let \( 0 < b < \infty \). Define

\[
\Omega = \{ -b < y < \eta(t, x) \},
\]

which is a time-dependent domain in \( \mathbb{R}^{d+1} = \mathbb{R}^d \times \mathbb{R} \). Then \( \partial \Omega \) consists of a free surface

\[
\Sigma = \{ y = \eta(t, x) \}
\]

and a flat bottom

\[
\Gamma = \{ y = -b \}.
\]
Let $v : \Omega \to \mathbb{R}^d$ be the Eulerian vector field, $P : \Omega \to \mathbb{R}$ be the internal pressure, $g \in \mathbb{R}$ be the gravitational acceleration, and $e_y = (0, \ldots, 0, 1)$. Then
\begin{equation}
\nabla_{xy} \cdot v = 0, \quad \nabla_{xy} \times v = 0, \end{equation}
where $\nabla_{xy} = (\nabla, \partial_y)$, $\nabla = (\partial_{x_1}, \ldots, \partial_{x_d})$; $(v, P)$ satisfies the Euler equation,
\begin{equation}
\partial_t v + v \cdot \nabla_{xy} v + \nabla_{xy} P = -ge_y. \end{equation}
Let $n : \partial \Omega \to S^d$ be the exterior unit normal vector field of $\partial \Omega$, then
\begin{equation}
\partial_t \eta = \sqrt{1 + |\nabla \eta|^2} \cdot v |\Sigma \cdot n,
\end{equation}
implying that fluid particles that were initially on $\Sigma$ will stay on $\Sigma$; and
\begin{equation}
v |\Gamma \cdot n = 0,
\end{equation}
meaning that $\Gamma$ is impenetrable. Finally, let $\kappa > 0$ be the constant of surface tension, and let
\begin{equation}
H(\eta) = \nabla \cdot \left( \frac{\nabla \eta}{\sqrt{1 + |\nabla \eta|^2}} \right)
\end{equation}
be the mean curvature of the free surface, then $\kappa H(\eta)$ is the surface tension on $\Sigma$, which should be balanced out by $P|\Sigma$, i.e.,
\begin{equation}
-P|\Sigma = \kappa H(\eta).
\end{equation}
The equations (1.1), (1.2), (1.3), (1.4), (1.5) and (1.6) give the Eulerian formulation of the system of gravity-capillary water waves.

1.3.2. Zakharov [37] / Craig-Sulem [14] Formulation. By (1.1) and (1.4), there exists a real valued $\phi : \Omega \to \mathbb{R}$ such that
\begin{equation}
\nabla_{xy} \phi = v, \quad \Delta_{xy} \phi = 0, \quad \partial_y \phi |\Gamma = 0.
\end{equation}
Denote $\psi(t, x, \eta(t, x)) = \phi(t, x, \eta(t, x))$, then
\begin{equation}
\sqrt{1 + |\nabla \eta|^2} \cdot v |\Sigma \cdot n = G(\eta) \psi,
\end{equation}
where $G(\eta)$, called the Dirichlet-Neumann operator, is defined by
\begin{equation}
G(\eta) \psi(t, x) = \partial_y \phi(t, x, \eta(t, x)) - \nabla \eta(t, x) \cdot \nabla \phi(t, x, \eta(t, x)).
\end{equation}
Letting $\kappa = 1$, the gravity-capillarity water wave equation can be formulated in terms of $(\eta, \psi)$, which is called the Zakharov / Craig-Sulem formulation,
\begin{equation}
\begin{cases}
\partial_t \eta - G(\eta) \psi = 0, \\
\partial_t \psi + g\eta - H(\eta) = \frac{1}{2} |\nabla \psi|^2 - \frac{1}{2} \frac{(\nabla \eta \cdot \nabla \psi + G(\eta) \psi)^2}{1 + |\nabla \eta|^2} = 0.
\end{cases}
\end{equation}

The purpose of this paper is to present the propagation of singularities for solutions to (1.7), including a microlocal smoothing effect. To the best of our knowledge, these results are the first of this type for quasilinear dispersive equations.

1.3.3. Quasi-Homogeneous Wavefront Set and Model Equation. Recall that the linearization of (1.7) at the stationary state $(\eta, \psi) = (0, 0)$ is
\begin{equation}
\begin{cases}
\partial_t \eta - |D_x| \tanh(b|D_x|) \psi = 0, \\
\partial_t \psi + g\eta - \Delta \eta = 0.
\end{cases}
\end{equation}
Let us oversimplify (1.8) by setting $g = 0$, $b = \infty$, then $u := \psi - i \sqrt{|D_x|} \eta$ satisfies the fractional Schrödinger equation
\begin{equation}
\partial_t u + i |D_x|^{3/2} u = 0,
\end{equation}
for which neither $WF(u)$ nor $HWF(u)$ seems to suitably define singularities. Let us consider the more general model equation
\begin{equation}
\partial_t u + i |D_x|^{\gamma} u = 0, \quad \gamma \geq 1,
\end{equation}
which includes the half wave equation ($\gamma = 1$), the fractional Schrödinger equations ($1 < \gamma < 2$), the Schrödinger equation ($\gamma = 2$), the fourth order Schrödinger equation ($\gamma = 4$), etc., as special
cases. A wave packet of (1.9) near the frequency \( \xi \sim h^{-1} \) travels at the group velocity \( v = \frac{d\xi}{dt} = \gamma|\xi|^{-2} \xi \sim h^{-(\gamma-1)} \), so the quasi-homogeneously scaled quantization \( a \rightarrow a(h^{\gamma-1}x, hD_x) \) seems to be one of the rightful choice. By a more general scaling, we define the quasi-homogeneous wavefront set.

**Definition 1.3.** Let \( u \in \mathcal{S}' \), \( \delta \geq 0 \), \( \rho \geq 0 \) with \( \delta + \rho > 0 \), and \( \mu \in \mathbb{R} \cup \{\infty\} \). The quasi-homogeneous wavefront set \( \text{WF}_{\delta,\rho}^\mu(u) \) is a subset of \( \mathbb{R}^{2d} \) such that \( (x_0, \xi_0) \notin \text{WF}_{\delta,\rho}^\mu(u) \) if and only if for some \( a \in C_c^\infty(\mathbb{R}^d) \) with \( a(x_0, \xi_0) \neq 0 \), \( a(h^\delta x, h^\rho D_x)u = O(h^\mu) \), \( 0 < h < 1 \). Here,

\[
a(h^\delta x, h^\rho D_x)u(x) = (2\pi)^{-\frac{d}{2}} \int e^{i(x-y)\cdot \xi} a(h^\delta x, h^\rho \xi) u(y) \, dy \, d\xi.
\]

If \( (x_0, \xi_0) \in \text{WF}_{\delta,\rho}^\mu(u) \), we shall call it a quasi-homogeneous singularity of \( u \), or more precisely, a \((\delta, \rho)\)-singularity of \( u \) (of order \( \mu \)).

Clearly the quasi-homogeneous wavefront set generalizes \( \text{WF}(u) \) and \( \text{HWF}(u) \). It allows us to extend Theorem 1.1 and Theorem 1.2 to the model equation (1.9).

**Theorem 1.4.** Let \( u \) solve (1.9) with initial data \( u_0 \in H^{-\infty}(\mathbb{R}^d) \), \( \mu \in \mathbb{R} \cup \{\infty\} \) and \( t_0 \in \mathbb{R} \).

(M.1) If \( \rho \gamma = \delta + \rho \), \( (x_0, \xi_0) \in \text{WF}_{\delta,\rho}^\mu(u_0) \setminus \{\xi = 0\} \), then

\[
(x_0 + t_0 \gamma \xi_0)^{\gamma-2} \xi_0, \xi_0) \in \text{WF}_{\delta,\rho}^\mu(u(t_0)) \setminus \{\xi = 0\}.
\]

(M.2) If \( \gamma > 1 \), \( \rho \gamma > \delta + \rho \), \( t_0 \neq 0 \), \( (x_0, \xi_0) \in \text{WF}_{\delta,\rho}^\mu(u_0) \setminus \{\xi = 0\} \), then

\[
(t_0 \gamma \xi_0)^{\gamma-2} \xi_0, \xi_0) \in \text{WF}_{\rho(\gamma-1),\rho}^\mu(u(t_0)) \setminus \{\xi = 0\}.
\]

Indeed, letting \((\delta, \rho, \gamma, \mu) = (0, 1, 1, \infty)\) or \((1, 1, 2, \infty)\) in (M.1), we recover respectively Theorem 1.1 and (S.1) of Theorem 1.2; letting \((\delta, \rho, \gamma, \mu) = (0, 1, 2, \infty)\) in (M.2), we recover (S.2) of Theorem 1.2.

1.3.4. **Existence in Weighted Sobolev Spaces.** Instead of the linearization at \((\eta, \psi) = (0, 0)\), if we paralinearize and symmetrize (1.7) as Alazard-Burq-Zuily [1], we obtain a fractional Schrödinger equation (with lower order terms) on \( \Sigma \). The geometry of \( \Sigma \) is time dependent and is given by the solution itself, as (1.7) is quasilinear. We need this geometry to be asymptotically Euclidean to avoid the mess caused by the infinite speed of propagation, but the existence of such geometry is not cheap. We shall prove it by establishing the existence of asymptotically “flat” gravity-capillary water waves.

**Definition 1.5.** For \((\nu, k) \in \mathbb{R}^2 \), \( H^\mu_k \) consists of those \( u \in \mathcal{S}' \) with

\[
\|u\|_{H^\mu_k} := \|\langle \cdot \rangle^k (D_x)^\nu u\|_{L^2} < \infty.
\]

Here and throughout this paper, \( \langle \cdot \rangle = \sqrt{1 + |\cdot|^2} \). Moreover, given \((\mu, m) \in \mathbb{R} \times \mathbb{N} \), denote

\[
\mathcal{H}^\mu_m = \bigcap_{j=0}^m H^\mu_j - j/2.
\]

**Theorem 1.6.** Let \( d \geq 1 \), \( \mu > 3 + d/2 \), \( m \leq 2\mu - 6 - d \), and \((\eta_0, \psi_0) \in \mathcal{H}_m^{\mu+1/2} \times \mathcal{H}^\mu_m \), then for some \( T > 0 \), there exists a unique solution

\[
(\eta, \psi) \in C([-T, T], \mathcal{H}_m^{\mu+1/2} \times \mathcal{H}^\mu_m)
\]

to the Cauchy problem of (1.7) with initial data \((\eta_0, \psi_0)\).

For the Cauchy problem of water waves, we refer to the initial works of Kano-Nishida [19] and Yoshikazu [35, 36], the breakthroughs of Wu [32, 33] and Beyer-Günther [7] for the local well-posedness in Sobolev spaces with general initial data. Using the paradifferential calculus, Alazard-Burq-Zuily [1, 2, 3] proved the local well-posedness with low Sobolev regularity. We
shall prove Theorem 1.6 by combining the analysis of [1] and a spatial dyadic decomposition. More precisely, we define the dyadic parabolized operators
\[ P_a = \sum_{j \in \mathbb{N}} \psi_j T_{\psi_j a} \psi_j, \]
where (i) \( \psi_j, \psi_j \in C_c^\infty(\mathbb{R}^d) \) are supported in the dyadic annulus \( C^{-1}2^j \leq |x| \leq C2^j \), (ii) \( \psi_j \psi_j = \psi_j \), (iii) \( \{ \psi_j \}_{j \in \mathbb{N}} \) is a partition of unity of \( \mathbb{R}^d \), and (iv) \( T_{\psi_j a} \) is the usual parabola differential operator of Bony [8], which we shall review in §4.1. We show that the dyadic parabolized calculus naturally extends Bony’s parabola linearmization to weighted Sobolev spaces.

We do not attempt to lower \( \mu \) to \( > 2 + d/2 \), as it was the case in [1]. The range of \( m \) is so chosen that \( \mu - m/2 > 3 + d/2 \), enabling us to paralinearize (1.7) in \( \mathcal{H}_m^\mu \). We should mention that the well-posedness of gravity water waves, i.e., without surface tension, in uniformly local weighted Sobolev spaces was obtain by Nguyen [28] using a periodic spatial decomposition from [3].

### 1.3.5. Propagation at Spatial Infinity

Our first main result concerns about the propagation of (1/2, 1)-singularities at the spatial infinity, corresponding to (M.1) of Theorem 1.4.

**Theorem 1.7.** Let \( d \geq 1 \), \( \mu > 3 + d/2 \), \( 3 \leq m \leq 2\mu - 6 - d \). Suppose that for some \( T > 0 \), \( (\eta, \psi) \in C([-T, T], \mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^\mu) \) solves (1.7) with initial data \( (\eta_0, \psi_0) \). Let
\[ (x_0, \xi_0) \in WF_{1/2,1}^{\mu+1/2+\sigma}(\eta_0) \cup WF_{1/2,1}^{\mu+\sigma}(\psi_0), \quad \xi_0 \neq 0, \]
where \( 0 \leq \sigma \leq m/2 - 3/2 \). Let \( t_0 \in [-T, T] \) such that
\[ x_0 + \frac{3}{2} t_0 |\xi_0|^{-1} |\xi_0| \neq 0 \]
for all \( t \in [0, t_0] \) if \( t_0 \geq 0 \), or respectively for all \( t \in [t_0, 0] \) if \( t_0 \leq 0 \). Then
\[ (x_0 + \frac{3}{2} t_0 |\xi_0|^{-1} |\xi_0|, \xi_0) \in WF_{1/2,1}^{\mu+1/2+\sigma}(\eta(t_0)) \cup WF_{1/2,1}^{\mu+\sigma}(\psi(t_0)). \]

We will see that, by Lemma 2.16, as \( (\eta, \psi) \in \mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^\mu \),
\[ WF_{1/2,1}^{\mu+1/2}(\eta) \cup WF_{1/2,1}^{\mu}(\psi) \subset \{ \eta = 0 \} \cup \{ \xi = 0 \}. \]
By Alazard-Métivier [5], we expect \( \sigma \) to be at most \( \mu - \alpha - d/2 \) for some \( \alpha > 0 \), corresponding to the extra gain of regularity by the remainder of the paralinearization procedure. Although Theorem 1.7 does not give the optimal upper bound for \( \sigma \), as it is not our priority, but when \( m = 2\mu - 6 - d \), \( \sigma \) can still be as big as \( \mu - 9/2 - d/2 \).

### 1.3.6. Microlocal Smoothing Effect

Our second main result shows that a (0, 1)-singularity creates instantaneously a (1/2, 1)-singularity at the spatial infinity. Corresponding to (M.2) of Theorem 1.4. To state the result, observe that, given \( \eta_0 \in C^3(\mathbb{R}^d) \), the initial free surface \( \Sigma_0 = \Sigma_{t=0} \) is isometric to \( (\mathbb{R}^d, \varrho_0) \), with
\[ \varrho_0 = \left( \text{Id} + \frac{i}{\varrho_0} \text{grad} \eta_0 \cdot 1 \right)^{-1}. \]
We identify the co-geodesic flow \( G_t \) on \( T^* \Sigma_0 \) with the Hamiltonian flow on \( \mathbb{R}^{2d} \) of the symbol
\[ G(x, \xi) = (\varrho_0)^{1-1}(\xi, \xi) = |\xi|^{2} - \frac{(\text{grad} \eta_0 \cdot \xi)^2}{1 + |\text{grad} \eta_0|^2}. \]
That is,
\[ \partial_t G_t = X_G(G_t), \quad G_0 = \text{Id}_{\mathbb{R}^{2d}}, \]
where \( X_G = (\partial_\xi G, -\partial_x G) \).

**Theorem 1.8.** Let \( d \geq 1 \), \( \mu > 3 + d/2 \), \( 3 \leq m \leq 2\mu - 3 - d/2 \). Suppose that for some \( T > 0 \), \( (\eta, \psi) \in C([-T, T], \mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^\mu) \) solves (1.7) with initial data \( (\eta_0, \psi_0) \). Let
\[ (x_0, \xi_0) \in WF_{0,1}^{\mu+1/2+\sigma}(\eta_0) \cup WF_{0,1}^{\mu+\sigma}(\psi_0), \quad \xi_0 \neq 0, \]
where $0 \leq \sigma \leq \min\{\mu/2 - 3 - d/4, 3m/2\}$, and suppose that the co-geodesic $\{(x_t, \xi_t) := G_t(x_0, \xi_0)\}_{t \in \mathbb{R}}$ is forwardly resp. backwardly non-trapping, i.e., for any compact set $K \subset \mathbb{R}^d$, $x_t \notin K$, resp. $x_{-t} \notin K$ whenever $t > 0$ is sufficiently large. Then there exists $\xi_{+\infty} \in \mathbb{R}^{d\setminus\{0\}}$, resp. $\xi_{-\infty} \in \mathbb{R}^{d\setminus\{0\}}$ such that,

$$\lim_{t \to \infty} \xi_t = \xi_{+\infty}, \text{ resp. } \lim_{t \to -\infty} \xi_{-t} = \xi_{-\infty},$$

and for all $0 < t_0 \leq T$, resp. $-T \leq t_0 < 0$,

$$\left(\frac{3}{2} t_0^{a} |\xi_{+\infty} - \xi_{+\infty}|^{-1/2} \langle \xi_{+\infty}, \xi_{+\infty} \rangle \right) \in \mathcal{WF}_{1/2,1}^{\mu+1/2+\sigma} (\eta(t_0)) \cup \mathcal{WF}_{1/2,1}^{\mu+\sigma} (\psi(t_0)), \text{ resp. } \left(\frac{3}{2} t_0^{a} |\xi_{-\infty} - \xi_{-\infty}|^{-1/2} \langle \xi_{-\infty}, \xi_{-\infty} \rangle \right) \in \mathcal{WF}_{1/2,1}^{\mu+1/2+\sigma} (\eta(t_0)) \cup \mathcal{WF}_{1/2,1}^{\mu+\sigma} (\psi(t_0)).$$

We remark that the asymptotic directions, $\xi_{\pm\infty}$, are determined solely by the initial geometry given by $\eta_0$, due to the infinite speed of propagation.

One may wonder whether the non-trapping assumption in Theorem 1.8 is necessary. We are tempted to believe that the co-geodesic flow on $\Sigma_0$ is everywhere non-trapping, both forwardly and backwardly, because $\Sigma_0$ is the graph of a function from $\mathbb{R}^d$ to $\mathbb{R}$. However, there are only two cases that are known to us be true: either when $d = 1$, or when $\nabla \eta_0 \in L^\infty$ and $\| (x) \nabla^2 \eta_0 \|_{L^\infty}$ is sufficiently small, see §6.4. In both cases we obtain the following local smoothing effect.

**Corollary 1.9.** Under the hypothesis of Theorem 1.8, suppose that the following two conditions are satisfied,

1. either $d = 1$ or $\| (x) \nabla^2 \eta_0 \|_{L^\infty}$ is sufficiently small;
2. $\mathcal{WF}_{1/2,1}^{\mu+1/2+\sigma} (\eta_0) \cup \mathcal{WF}_{1/2,1}^{\mu+\sigma} (\psi_0) \subset \{x = 0\} \cup \{\xi = 0\}$,

then $\forall t_0 \in [-T, T]\setminus\{0\}$ and $\forall \varepsilon > 0$,

$$(\eta(t_0), \psi(t_0)) \in H_{\text{loc}}^{\mu+1/2+\sigma-\varepsilon} \times H_{\text{loc}}^{\mu+\sigma-\varepsilon}.$$ 

We remark that the second condition is satisfied if $(\eta_0, \psi_0) \in H_{2k}^{\mu+1/2+\sigma-k} \times H_{2k'}^{\mu+\sigma-k'}$ for some $(k, k') \in \mathbb{R}^2$, which is particularly the case if $(\eta_0, \psi_0) \in \mathcal{E}' \times \mathcal{E}'$.

We refer to Christianson-Hur-Staffilani [11] and Alazard-Burq-Zuily [1] for local smoothing effects of 2D capillary-gravity water waves, and Alazard-Ifrim-Tataru [4] for a Morawetz inequality of 2D gravity water waves.

### 1.4. Outline of Paper.

In §2, we present basic properties of weighted Sobolev spaces and the quasi-homogeneous wavefront set. In §3, we prove Theorem 1.4 by extending the idea of Nakamura [27]. In §4, we review the paradifferential calculus of Bony, and extend it to weighted Sobolev spaces by a spatial dyadic decomposition. We also develop a quasi-homogeneous semiclassical paradifferential calculus, and study its relations with the quasi-homogeneous wavefront set. In §5, we study the Dirichlet-Neumann operator in weighted Sobolev spaces and prove the existence of asymptotically flat gravity-capillary water waves, i.e., Theorem 1.6. In §6, we prove our main results, Theorem 1.7, Theorem 1.8 and Corollary 1.9, by extending the proof of Theorem 1.4 to the quasilinear equation using the paradifferential calculus.

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2. **QUASI-HOMOGENEOUS MICROLOCAL ANALYSIS**

#### 2.1. Quasi-Homogeneous Semiclassical Calculus.

**Definition 2.1.** For $(\mu, k) \in \mathbb{R}^2$, set $\mathcal{S}_{\mu}^k(x, \xi) = \langle x \rangle^k \langle \xi \rangle^\mu$. Let $a_h \in C^\infty (\mathbb{R}_x^d \times \mathbb{R}_\xi^d)$, we say that $a_h \in S_{\mu}^k$ if $\forall \alpha, \beta \in \mathbb{N}^d$, $\exists C_{\alpha\beta} > 0$, such that $\forall (x, \xi) \in \mathbb{R}_x^d \times \mathbb{R}_\xi^d$,

$$\sup_{0 < h < 1} |\partial_x^\alpha \partial_\xi^\beta a_h(x, \xi)| \leq C_{\alpha\beta} m_{k-|\beta|}(x, \xi).$$
We say that \( a_h \in S^\mu_k \) is \((\mu,k)\)-elliptic if \( \exists R > 0, C > 0 \) such that for \( |x| + |\xi| \geq R \),

\[
\inf_{0 < h < 1} \inf_{x, \xi} |a_h(x, \xi)| \geq Cm^\mu_k(x, \xi).
\]

Denote

\[
S^\infty_\infty = \bigcup_{(\mu,k) \in \mathbb{R}} S^\mu_k, \quad S^-\infty = \bigcap_{(\mu,k) \in \mathbb{R}} S^\mu_k.
\]

We say that \( a_h \in S^-\infty \) is elliptic at \((x_0, \xi_0)\) if for some neighborhood \( \Omega \) of \((x_0, \xi_0)\),

\[
\inf_{0 < h < 1} \inf_{(x, \xi) \in \Omega} |a_h(x, \xi)| > 0.
\]

**Definition 2.2.** Given \( \delta \geq 0, \rho \geq 0 \) with \( \delta + \rho \geq 0 \), for \( h > 0 \), define

\[
\vartheta^\delta_\rho_h : \mathbb{R}^d_x \times \mathbb{R}^d_\xi \to \mathbb{R}^d_x \times \mathbb{R}^d_\xi \quad (x, \xi) \mapsto (h^\delta x, h^\rho \xi),
\]

which induces a pullback \( \vartheta^\delta_\rho_h \) on \( S^\infty_\infty \), \( \vartheta^\delta_\rho_h a_h = a_h \circ \vartheta^\delta_\rho_h \). Then set

\[
\Op_h^\delta_\rho(a_h) = \Op(\vartheta^\delta_\rho_h a_h),
\]

where

\[
\Op(a)u(x) := (2\pi)^{-d} \iint e^{i(x-y)\cdot \xi} a(x, \xi) u(y) \, dy \, d\xi.
\]

By the formula \((\vartheta^\delta_\rho_h)^{-1} \Op_h^\delta_\rho(a) \vartheta^\delta_\rho_h = \Op_h^{0, \delta + \rho}(a)\), where \( \vartheta^\delta_\rho_h u(x) := h^{d/2}\vartheta^\delta_\rho_h u(h^\delta x) \) is an isometry of \( L^2(\mathbb{R}^d) \), we induce the following results from the usual semiclassical calculus for which we refer to the book [40] of Zworski.

**Proposition 2.3.** There exists \( K > 0 \) such that, if \( a \in C^\infty(\mathbb{R}^d_x \times \mathbb{R}^d_\xi) \) with

\[
M = \sum_{|\alpha| + |\beta| \leq d} \| \partial_x^\alpha \partial_\xi^\beta a \|_{L^\infty} < \infty,
\]

then \( \| \Op_h^\delta_\rho(a) \|_{L^2 \to L^2} \leq KM \).

**Proposition 2.4** (Sharp Gårding Inequality). If \( \delta + \rho > 0 \) and \( a_h \in S^-\infty \) such that \( \text{Re} a_h \geq 0 \), then for some \( C > 0 \), all \( u \in L^2(\mathbb{R}^d) \), and \( 0 < h < 1 \),

\[
\text{Re}(\Op_h^\delta_\rho(a_h)u, u)_{L^2} \geq -C h^{\delta + \rho}\|u\|_{L^2}^2.
\]

**Proposition 2.5.** There exists a bilinear operator

\[
\vartheta^\delta_\rho_h : S^\infty_\infty \times S^\infty_\infty \to S^\infty_\infty,
\]

such that

\[
\Op_h^\delta_\rho(a_h) \Op_h^\delta_\rho(b_h) = \Op_h^\delta_\rho(a_h \vartheta^\delta_\rho_h b_h).
\]

If \( a_h \in S^\mu_h \), \( b_h \in S^\nu_h \), then \( a_h \vartheta^\delta_\rho_h b_h \in S^{\mu + \nu}_{h+k} \). We denote through this article that \( D_x = \frac{1}{i} \nabla_x \). For \( \gamma > 0 \), define

\[
a_h \vartheta^\delta_\rho_h \gamma b_h = \sum_{|\alpha| < \gamma} \frac{h^{\alpha(\delta + \rho)}}{\alpha!} \partial_\xi^\alpha a_h \partial_x D^\rho_x b_h,
\]

then \( a_h \vartheta^\delta_\rho_h \gamma b_h - a_h \vartheta^\delta_\rho_h b_h = \mathcal{O}(h^{\gamma(\delta + \rho)} S_{h^{k+\ell-\gamma}}^{\nu + \nu - \gamma}) \). If \( \delta + \rho > 0 \) and either \( a_h \in S^-\infty \) or \( b_h \in S^-\infty \), then \( a_h \vartheta^\delta_\rho_h \gamma b_h - a_h \vartheta^\delta_\rho_h b_h = \mathcal{O}(h^{\gamma(\delta + \rho)} S^-\infty) \).

**Proposition 2.6.** Then there exists a linear operator

\[
\vartheta^\delta_\rho_h : S^\infty_\infty \to S^\infty_\infty
\]

such that

\[
\Op_h^\delta_\rho(a_h)^* = \Op_h^\delta_\rho(\vartheta^\delta_\rho_h a_h).
\]
If $a_h \in S^\mu_k$, then $\zeta_h^{\delta,\rho} a_h \in S^\mu_k$. For $\gamma > 0$, define

$$\zeta_h^{\delta,\rho} a_h = \sum_{|\alpha| \leq \gamma} \frac{h^{|\alpha| (\delta + \rho)}}{\alpha!} \partial^\alpha \xi^\rho D^\alpha z a_h,$$

then $\zeta_h^{\delta,\rho} a_h - \zeta_h^{\delta,\rho} a_h = O(h^{\gamma (\delta + \rho)})_{S^\mu_k - \gamma}$. If $\delta + \rho > 0$ and $a_h \in S^{-\infty}$, then $\zeta_h^{\delta,\rho} a_h - \zeta_h^{\delta,\rho} a_h = O(h^{\gamma (\delta + \rho)})_{S^{-\infty}}$.

2.2. Weighted Sobolev Spaces. Let us recall that $\mathcal{S}(\mathbb{R}^d)$ is the Schwartz space, and $\mathcal{S}'(\mathbb{R}^d)$ is the space of tempered distributions.

**Definition 2.7.** We say that a linear operator $A : \mathcal{S} \to \mathcal{S}'$ is of order $(\nu, \ell) \in \mathbb{R}^2$, and denote $A \in \mathcal{O}_\nu^{\ell}$ if for all $(\mu, k) \in \mathbb{R}^2$, there exists $C > 0$ such that $\forall u \in \mathcal{S}$, $\|Au\|_{H^{\mu-k,\ell}_k} \leq C\|u\|_{H^{\mu}_k}$. Therefore, $A$ extends to a bounded linear operator from $H^\mu_k$ to $H^{\mu-k,\ell}_k$. We denote $A \in \mathcal{O}_{-\infty}^{\ell}$ if $A \in \mathcal{O}_\nu^{\ell}$, $\forall (\nu, \ell) \in \mathbb{R}^2$. Let $A_\alpha : \mathcal{S} \to \mathcal{S}'$ and $C_\alpha > 0$ be indexed by $\alpha \in \mathcal{A}$. We say that $A_\alpha = O(C_\alpha \sigma_\tau)$, if $\forall (\mu, k) \in \mathbb{R}^2$, $\exists K > 0$, such that $\forall \alpha \in \mathcal{A}$, $\|A_\alpha\|_{H^{\mu-k,\ell}_k} \leq KC_\alpha$.

By Proposition 2.3 and Proposition 2.5, we obtain

**Proposition 2.8.** If $a_h \in S^\mu_k$ with $(\nu, \ell) \in \mathbb{R}^2$, then $\text{Op}(a_h) \in \mathcal{O}_\nu^{\ell}$.

**Proposition 2.9.** $u \in H^\mu_k$ if and only if for some $a_h \in S^\mu_k$ which is $(\mu, k)$-elliptic, $\text{Op}(a_h) u \in L^2$.

**Proposition 2.10.** $\mathcal{S} = \cap_{(\mu, k) \in \mathbb{R}^2} H^\mu_k$, $\mathcal{S}' = \cup_{(\mu, k) \in \mathbb{R}^2} H^\mu_k$.

**Proof.** Clearly $\mathcal{S} \subset \cap_{(\mu, k) \in \mathbb{R}^2} H^\mu_k$. The converse follows by Sobolev injection. As for the second statement, clearly $\cup_{(\mu, k) \in \mathbb{R}^2} H^\mu_k \subset \mathcal{S}'$. Conversely, let $u \in \mathcal{S}'$, then $\exists N > 0$, such that $\forall \varphi \in \mathcal{S}$,

$$(u, \varphi)_{\mathcal{S}'_\alpha} \lesssim \sum_{|\alpha| + |\beta| \leq N} \|\alpha^\alpha \partial^\beta \varphi\|_{L^\infty} \lesssim \|\text{Op}(m_M^\alpha)\varphi\|_{L^2}$$

with $M$ sufficiently large, implying that $u \in H^{-M}_{-\infty}$.

**Lemma 2.11.** Let $u \in \mathcal{S}'$, then $u = h^{-M} \text{Op}_h^{\delta,\rho}(m_M^{-\alpha}) \text{Op}(1)_{L^2}$ for some $M > 0$. Therefore, if $\delta + \rho > 0$, and $a_h \in \mathcal{O}(h^\infty)_{S^{-\infty}}$, then $\text{Op}_h^{\delta,\rho}(a_h) u_h = \mathcal{O}(h^\infty)_{\mathcal{S}}$.

**Proof.** By the proof of Proposition 2.10, $\exists M, N > 0$, such that $\forall \varphi \in \mathcal{S}$,

$$(u, \varphi)_{\mathcal{S}'_\alpha} \lesssim \sum_{|\alpha| + |\beta| \leq N} \|\alpha^\alpha \partial^\beta \varphi\|_{L^\infty} \lesssim h^{-M} \|\text{Op}_h^{\delta,\rho}(m_M^\alpha)\varphi\|_{L^2}$$

Next, we characterize weighted Sobolev spaces by a dyadic decomposition.

**Definition 2.12.** Let $\psi : \mathbb{N} \to C_0^\infty(\mathbb{R}^d)$, $j \mapsto \psi_j$, we denote $\psi \in \mathcal{P}$ if

(i) $\text{supp} \psi_j \subset \{ C^{-1}2^j \leq |x| \leq C2^j \}$ for some $C > 1$ and all $j \geq 1$;
(ii) $\psi_j \geq 0$ for all $j \in \mathbb{N}$ and $C^{-1} \leq \sum_{j \in \mathbb{N}} \psi_j \leq C$ for some $C > 1$;
(iii) $\|\partial^\alpha \psi_j\|_{L^\infty} \leq C_\alpha 2^{-j|\alpha|}$ for all $\alpha \in \mathbb{N}$, $j \in \mathbb{N}$, and some $C_\alpha > 0$ depending solely on $\alpha$.

Given $\psi \in \mathcal{P}$, we denote $\psi \in \mathcal{P}_s$ if

(i) $\sum_{j \in \mathbb{N}} \psi_j = 1$;
(ii) $\text{supp} \psi_j \cap \text{supp} \psi_k = \emptyset$ whenever $|j - k| > 2$.

**Proposition 2.13.** Let $(\mu, k) \in \mathbb{R}^2$, $\psi \in \mathcal{P}$, and $u \in \mathcal{S}'$, then $u \in H^\mu_k$ if and only if $\sum_{j \in \mathbb{N}} 2^{2j} \|\psi_j u\|_{H^\mu}^2 < \infty$. Moreover, $\exists C > 1$, such that $\forall u \in H^\mu_k$,

$$C^{-1} \|u\|_{H^\mu}^2 \leq \sum_{j \in \mathbb{N}} 2^{2j} \|\psi_j u\|_{H^\mu}^2 \leq C \|u\|_{H^\mu}^2.$$
Proof. We may assume that $\psi \in \mathcal{P}_4$, because $\forall \phi^1, \phi^2 \in \mathcal{P},$

$$\sum_{j \in \mathbb{N}} 2^{2jk} \| \psi_j \|^2_{H^k} \lesssim \sum_{j \in \mathbb{N}} 2^{2jk} \| \phi_j^2 \|^2_{H^k},$$

Define $\tilde{\psi} \in \mathcal{P}$ by setting $\tilde{\psi}_j = \sum_{|k-j| \leq 2} \psi_k$, then $\tilde{\psi}_j \psi_j = \psi_j$, $\forall j$. For $u \in H^\mu_k$, $2^{2j} \| \psi_j u \|^2_{H^\mu} \lesssim \| \tilde{\psi}_j (D_x)^\mu \psi_j (x)^k u \|^2_{L^2} + \| (1 - \tilde{\psi}_j) (D_x)^\mu \psi_j (x)^k u \|^2_{L^2}$.

Apply Proposition 2.5 with $(\delta, \rho) = (1, 0)$, we have

$$(1 - \tilde{\psi}_j) (D_x)^\mu \psi_j (D_x)^{-\mu} = \mathcal{O}(2^{-jN})_{L^2 \to L^2}, \quad \forall N > 0.$$  

Therefore,

$$\sum_{j \in \mathbb{N}} \| (1 - \tilde{\psi}_j) (D_x)^\mu \psi_j (x)^k u \|^2_{L^2} \lesssim \sum_{j \in \mathbb{N}} 2^{-2jN} \| u \|^2_{H^\mu_k} \lesssim \| u \|_{H^\mu_k}^2.$$  

For $r = 0, 1, \ldots, 9$, set

$$a_r = \sum_{j \in 10N+r} \tilde{\psi}_j \langle \xi \rangle^\mu \psi_j (x)^k u \in S^\mu_k.$$  

Observe that for $0 \neq j - j' \in 10N$, supp $\tilde{\psi}_j \cap$ supp $\tilde{\psi}_{j'} = \emptyset$, therefore,

$$\sum_{j \in \mathbb{N}} \| \tilde{\psi}_j (D_x)^\mu \psi_j (x)^k u \|^2_{L^2} = \sum_{r=0}^9 \sum_{j \in 10N+r} \| \tilde{\psi}_j (D_x)^\mu \psi_j (x)^k u \|^2_{L^2} = \sum_{r=0}^9 \| \text{Op}(a_r) u \|^2_{L^2} \lesssim \| u \|^2_{H^\mu_k}.$$  

And we prove that

$$\sum_{j \in \mathbb{N}} 2^{2j} \| \psi_j u \|^2_{H^\mu} \lesssim \| u \|^2_{H^\mu_k}.$$  

Conversely, observe that $a := \sum_{r=0}^9 a_r$ is $(\mu, k)$-elliptic. So for some $r \in S^{-\infty}_0$,

$$\| u \|^2_{H^\mu_k} \lesssim \| \text{Op}(a) u \|^2_{L^2} + \| \text{Op}(r) u \|^2_{L^2}.$$  

Similarly as above, we have

$$\| \text{Op}(a) u \|^2_{L^2} \lesssim \| \text{Op}(a_r) u \|^2_{L^2} \lesssim \sum_{r=0}^9 \sum_{j \in 10N+r} \| \tilde{\psi}_j (D_x)^\mu \psi_j (x)^k u \|^2_{L^2} \lesssim \sum_{j \in \mathbb{N}} 2^{2j} \| (D_x)^\mu \psi_j u \|^2_{L^2};$$

while for the second term, we denote $\sharp = \sharp_1^{0,0}$ for simplicity and write

$$\| \text{Op}(r) u \|^2_{L^2} = (u, \text{Op}(r^\sharp r) u)_{L^2} = \sum_{j \in \mathbb{N}} (u, \text{Op}(r^\sharp r) \psi_j u)_{L^2}.$$  

For each term in the summation, we have for all $N > 0$ and $\varepsilon > 0$,

$$(u, \text{Op}(r^\sharp r) \psi_j u)_{L^2} = (\text{Op}(m^\mu_k) u, \text{Op}(m^\mu_k r^\sharp r^\sharp m^\mu_{-N+k}) (D_x)^\mu (x)^{-N+k} \psi_j u)_{L^2} \lesssim \| u \|_{H^\mu_k} \| (D_x)^\mu (x)^{-N+k} \psi_j u \|_{L^2} \lesssim 2^{-jN} \| u \|_{H^\mu_k} 2^{2j} \| (D_x)^\mu \psi_j u \|_{L^2} \lesssim 2^{-jN} (\varepsilon \| u \|_{H^\mu_k}^2 + \varepsilon^{-1} 2^{2j} \| (D_x)^\mu \psi_j u \|_{L^2}^2).$$

Summing up in $j$,

$$\| \text{Op}(r) u \|^2_{L^2} \lesssim \varepsilon \| u \|_{H^\mu_k}^2 + \varepsilon^{-1} \sum_{j \in \mathbb{N}} 2^{2j} \| \psi_j u \|_{H^\mu_k}^2.$$  

We conclude by choosing $\varepsilon$ sufficiently small. \qed
2.3. Quasi-Homogeneous Wavefront Sets. The following characterization is easy to prove by a routine construction of parametrices.

**Proposition 2.14.** Let \( u \in \mathcal{S}' \), then \((x_0, \xi_0) \notin \WF^\mu_{\delta,\rho}(u)\) if and only if for some \( a_h \in S_{-\infty}^-\) which is elliptic at \((x_0, \xi_0)\), \( \Op^\delta_{\rho}(a_h)u = \mathcal{O}(h^\mu)_{L^2} \).

**Lemma 2.15.** Let \( u \in \mathcal{S}' \) and \( a_h \in S_{-\infty}^- \) such that

\[
\supp a_h \subset K \subset \mathbb{R}^d_x \times \mathbb{R}^d_\xi \setminus \WF^\mu_{\delta,\rho}(u)
\]

for some compact set \( K \) and all \( 0 < h < 1 \), then

\[
\langle u, \Op^\delta_{\rho}(a_h)u \rangle_{\mathcal{S}', \mathcal{S}} = \mathcal{O}(h^{2\mu}).
\]

Consequently, \( \Op^\delta_{\rho}(a_h)u = \mathcal{O}(h^\mu)_{L^2} \).

**Proof.** Let \( \{ \Omega_i \}_{i \in I} \) be an open cover of \( K \). Let \( b^i_h \in S_{-\infty}^- \) be elliptic everywhere in \( \Omega_i \), such that \( \Op^\delta_{\rho}(b^i_h)u = \mathcal{O}(h^\mu)_{L^2} \). By a partition of unity, we may assume that \( K \subset \Omega := \Omega_{i_0} \) for some \( i_0 \in I \), and set \( b_h = b_{i_0}^0 \). By the ellipticity of \( b_h \), we can find \( c_h \in S_{-\infty}^- \) and \( r_h = O(h^\infty)_{S_{-\infty}^-} \) such that \( a_h = (\zeta^\delta_h b_h) h^\delta \gamma r_h + c_h \). Therefore, by Lemma 2.11,

\[
\langle u, \Op^\delta_{\rho}(a_h)u \rangle_{\mathcal{S}', \mathcal{S}} = \langle u, \Op^\delta_{\rho}(b_h)u \rangle_{\mathcal{S}', \mathcal{S}} + \langle u, \Op^\delta_{\rho}(c_h) \Op^\delta_{\rho}(b_h)u \rangle_{L^2} + \langle u, \Op^\delta_{\rho}(r_h)u \rangle_{\mathcal{S}', \mathcal{S}} = \mathcal{O}(h^{2\mu}) + \mathcal{O}(h^\infty) = \mathcal{O}(h^{2\mu}).
\]

**Lemma 2.16.** Let \( u \in \mathcal{S}' \), then

1. \( \WF^\mu_{\delta,\rho}(u) \) is a closed \((\delta, \rho)\)-cone, i.e., \( \theta_\lambda^\delta \rho \WF^\mu_{\delta,\rho}(u) = \WF^\mu_{\delta,\rho}(u), \forall \lambda > 0 \);
2. \( \WF^\mu_{\delta,\rho}(u) = \WF^\mu_{\delta,\rho}(u), \forall \gamma > 0 \);
3. \((x_0, \xi_0) \in \WF^\mu_{\delta,\rho}(u) \iff (\xi_0, -x_0) \in \WF^\mu_{\rho,\delta}(\hat{u})\);
4. \((x_0, \xi_0) \in \WF^\mu_{\delta,\rho}(u) \iff (x_0, -\xi_0) \in \WF^\mu_{\rho,\delta}(\hat{u})\);
5. Denote \( \WF^\mu_{\delta,\rho}(u)^\circ = \WF^\mu_{\delta,\rho}(u) \setminus \mathcal{N}_{\delta,\rho} \), where

\[
\mathcal{N}_{\delta,\rho} = \begin{cases}
\{ x = 0 \} \times \mathbb{R}^d_\xi, & \delta > 0, \rho = 0; \\
\mathbb{R}^d_x \times \{ \xi = 0 \}, & \delta = 0, \rho > 0; \\
\{ x = 0 \} \times \mathbb{R}^d_\xi \cup \mathbb{R}^d_x \times \{ \xi = 0 \}, & \delta > 0, \rho > 0.
\end{cases}
\]

If \( u \in H^\mu_k \) with \((\mu, k) \in \mathbb{R}^2 \), and \( a_h \in S_{-\infty}^- \) such that

\[
\mathcal{N}_{\delta,\rho} \cap \bigcup_{0 < h < 1} \supp a_h = \emptyset,
\]

then \( \Op^\delta_{\rho}(a_h)u = \mathcal{O}(h^{\delta \mu + \rho k})_{L^2} \). In particular, \( \WF^\mu_{\delta,\rho}(u)^\circ = \emptyset \).

**Proof.** (1) and (2) are easy. To prove (3), we use \( \mathcal{F}^{-1} \Op^\delta_{\rho}(a_h) \mathcal{F} = \Op^\delta_{\rho}(\zeta^\delta_{\rho} b_h) \mathcal{F} \) where \( \mathcal{F} \) is the Fourier transform and \( b_h(x, \xi) = a_h(\xi, -x) \). To prove (4), we use \( \Op(a_h)u_h = \Op(b_h)u_{\#} \), where \( b_h(x, \xi) = a_h(x, -\xi) \). To prove the case where \( \delta > 0, \rho > 0 \) of (5), we simply observe that,

\[
(\hat{\theta}^\delta_{\rho} a_h)(\xi) = \mathcal{O}(h^{\delta \mu + \rho k})_{S_0^0}.
\]

The other two cases are similar. \( \square \)

3. Model Equation

We prove Theorem 1.4 by combining the ideas of Nakamura [27] and simple methods of scaling. There is no harm in assuming that \( u_0 \in L^2(\mathbb{R}^d) \).
3.1. **Proof of (M.1).** For \( a \in W^{1,\infty}_{\text{loc}}(\mathbb{R} \times \mathbb{R}^d) \) and \( \mathcal{A} \in W^{1,\infty}_{\text{loc}}(\mathbb{R}, L^2 \rightarrow L^2) \), define
\[
\mathcal{L}_a a = \partial_a a + \{\xi|\gamma, a\}, \quad \mathcal{L}_a \mathcal{A} = \partial_a \mathcal{A} + i[|D_\xi|\gamma, \mathcal{A}].
\]
Here \( \{\cdot, \cdot\} \) is the Poisson bracket, i.e. \( \{f, g\} = \nabla_\xi f \cdot \nabla_x g - \nabla_x f \cdot \nabla_\xi g \).

**Lemma 3.1.** For \( a_h \in W^{1,\infty}_{\text{loc}}(\mathbb{R}, S^-_{\infty}) \) with
\[
\bigcup_{0 < h < 1} \text{supp } a_h \cap \{\xi = 0\} = \emptyset,
\]
there exists \( b_h \in L^\infty_{\text{loc}}(\mathbb{R}, S^-_{\infty}) \) with \( \text{supp } b_h \subset \text{supp } a_h \), such that
\[
\mathcal{L}_t \text{Op}^{\delta, \rho}_h(a_h) = \text{Op}^{\delta, \rho}_h(\mathcal{L}_t a_h) + h^{\delta + \rho} \text{Op}^{\delta, \rho}_h(b_h) + O(h^\infty)_{L^\infty_{\text{loc}}(\mathbb{R}, L^2 \rightarrow L^2)}.
\]

**Proof.** \( \forall T > 0, \exists \varepsilon > 0 \) such that
\[
\bigcup_{t \in [-T,T]} \bigcup_{0 < h < 1} \text{supp } a_h(t, \cdot) \cap \{[\xi] \leq \varepsilon\} = \emptyset.
\]
Let \( \pi \in C^\infty(\mathbb{R}^d) \) such that \( 0 \leq \pi \leq 1 \), \( \pi(\xi) = 0 \) for \( |\xi| \leq \varepsilon/3 \), \( \pi(\xi) = 1 \) for \( |\xi| \geq 2\varepsilon/3 \). Then
\[
i[|D_\xi|\gamma, \text{Op}^{\delta, \rho}_h(a_h)] = ih^{\delta - \rho} h^{\delta} h^{\delta + \rho} \text{Op}^{\delta, \rho}_h(b_h) + O(h^\infty)_{L^\infty_{\text{loc}}(\mathbb{R}, L^2 \rightarrow L^2)}.
\]
Now that \( |\xi| \gamma(\xi) \in S^0_0 \), we conclude by Proposition 2.5 and the hypothesis \( \rho \gamma = \delta + \rho \).

Assume that \( \mu = \infty \), as the proof is similar for \( \mu < \infty \). Let \( (x_0, \xi_0) \notin W^{1,\infty}_{\text{loc}}(u_0) \) with \( \xi_0 \neq 0 \).

By Lemma 2.15, \( \exists \phi \in C^\infty_c(\mathbb{R}^d \times (\mathbb{R}^d \setminus 0)) \) with \( \phi(x_0, \xi_0) \neq 0 \), such that \( \text{Op}^{\delta, \rho}_h(\phi) u = O(h^\infty)_{L^2} \).

We aim to find \( a_h \in W^{1,\infty}_{\text{loc}}(\mathbb{R}, S^-_{\infty}) \) of the asymptotic expansion
\[
a_h \sim \sum_{j \in \mathbb{N}} h^{j(\delta + \rho)} a_j^h,
\]
where \( a_j^h \in W^{1,\infty}_{\text{loc}}(\mathbb{R}, S^-_{\infty}) \), such that \( \forall t \in \mathbb{R}, a_h(t, \cdot) \) is elliptic at \( (x_0 + t\gamma|\xi_0|^{\gamma - 2}\xi_0, \xi_0) \), and
\[
\mathcal{L}_t \text{Op}^{\delta, \rho}_h(a_h) = O(h^\infty)_{L^\infty_{\text{loc}}(\mathbb{R}, L^2 \rightarrow L^2)}.
\]

If such \( a_h \) is found, we set \( \mathcal{A}_h = \text{Op}^{\delta, \rho}_h(a_h) \), then
\[
\frac{d}{dt} \|A_h u\|_{L^2}^2 = 2\text{Re}(\mathcal{L}_t A_h u, A_h u)_{L^2} + 2\text{Re}(i|D_\xi|\gamma A_h u, A_h u)_{L^2} \leq O(h^\infty) \|A_h u\|_{L^2} + 0.
\]
We deduce by Gronwall’s inequality that \( \mathcal{A}_h u = O(h^\infty)_{L^\infty_{\text{loc}}(\mathbb{R}, L^2)} \) and conclude.

To construct \( a_h \), we define \( a_j^h \in W^{1,\infty}_{\text{loc}}(\mathbb{R}, S^-_{\infty}) \) by solving iteratively the transportation equations
\[
\begin{aligned}
\mathcal{L}_t a_j^0 = 0, \\
\partial_{t} a_j^0 + b_{j-1}^h = 0, \\
\partial_{t} a_j^0|_{t=0} = \phi, \\
a_j^0|_{t=0} = 0, \quad j \geq 1,
\end{aligned}
\]
where \( b_j^h \in W^{1,\infty}_{\text{loc}}(\mathbb{R}, S^-_{\infty}) \) satisfies, by Lemma 3.1,
\[
\mathcal{L}_t \text{Op}^{\delta, \rho}_h(b_j^h) = \text{Op}^{\delta, \rho}_h(\mathcal{L}_t a_j^h) + h^{\delta + \rho} \text{Op}^{\delta, \rho}_h(b_j^h) + O(h^\infty)_{L^\infty_{\text{loc}}(\mathbb{R}, L^2 \rightarrow L^2)}.
\]

\( \Box \)

3.2. **Proof of (M.2).** Let \( \beta = \rho \gamma - (\delta + \rho) > 0 \), introduce the semiclassical time variable \( s = h^{-\beta} t \), and rewrite (1.9) as
\[
(3.1) \quad \partial_s u + ih^{\beta} |D_\xi|\gamma u = 0.
\]

For \( a \in W^{1,\infty}_{\text{loc}}(\mathbb{R} \times \mathbb{R}^d) \) and \( \mathcal{A} \in W^{1,\infty}_{\text{loc}}(\mathbb{R}, L^2 \rightarrow L^2) \), define
\[
\mathcal{L}_a a = \partial_a a + \{\xi|\gamma, a\}, \quad \mathcal{L}_a \mathcal{A} = \partial_a \mathcal{A} + ih^{\beta} [|D_\xi|\gamma, \mathcal{A}].
\]
Lemma 3.2. Let $\phi \in C_\infty^0(\mathbb{R}^d)$ such that $\phi \geq 0$, $\phi(0) > 0$, and $x \cdot \nabla \phi(x) \leq 0$, $\forall x \in \mathbb{R}^d$. Let $\epsilon > 0$, $(x_0, \xi_0) \in \mathbb{R}^d_+ \times (\mathbb{R}^d_+ \setminus 0)$. For $s \geq 0$ and $(x, \xi) \in \mathbb{R}^d_+ \times \mathbb{R}^d_+$, set
\[
\chi(s, x, \xi) = \phi\left(\frac{x - s\gamma|x^2| - x_0}{1 + s}\right)\phi\left(\frac{\xi - \xi_0}{\epsilon}\right).
\]
Then $\chi \in W^{\infty, \infty}(\mathbb{R}_0^+; \mathbb{S}^{-1}_0)$, $\mathcal{L}_s\chi \in W^{\infty, \infty}(\mathbb{R}_0^+; \mathbb{S}^{-\infty}_1)$, and
\[
(3.2)
\]
$\mathcal{L}_s\chi \geq 0$.
Let $t_0 > 0$, and set $(\tau u)(s, x, \xi) = u(s, \frac{x}{t_0}, x, \xi)$, then $\tau \chi \in W^{\infty, \infty}(\mathbb{R}_0^+; \mathbb{S}^{-\infty}_1)$. Let $\epsilon$ be sufficiently small, then $(\tau \chi)(s, \cdot)$ is elliptic at $(t_0\gamma|\xi_0|^{-2}\xi_0, \xi_0)$ for $s$ sufficiently large.

Proof. Each time we differentiate $\chi$ with respect to $x$, we get a multiplicative factor $(1 + s)^{-1}$, which is of size $(x)^{-1}$ on supp $\chi$, as supp $\chi \subset \{C^{-1}s \leq |x| \leq Cs\}$ for some $C > 0$. Therefore $\chi \in W^{\infty, \infty}(\mathbb{R}_0^+; \mathbb{S}^{-\infty}_0)$. It is easy to see that $\tau \chi(s, \cdot)$ is bounded in $C_\epsilon^0(\mathbb{R}^{2d})$. We write
\[
(3.3)
(\tau \chi)(s, x, \xi) = \phi\left(\frac{x - t_0\gamma|\xi|^{-2}\xi_0}{1 + s}\right) - \gamma|\xi|^{-2}\xi_0 - \gamma|\xi_0|^{-2}\xi_0 + \frac{x_0}{1 + s}\phi\left(\frac{\xi - \xi_0}{\epsilon}\right),
\]
where $|\xi|^{-2}\xi_0 - |\xi_0|^{-2}\xi_0 = o(1)$ as $\epsilon \to 0$, whence $\tau \chi(s, \cdot)$ elliptic at $(t_0\gamma|\xi_0|^{-2}\xi_0, \xi_0)$ for sufficiently large $s$. To estimate $\mathcal{L}_s\chi$, we perform an explicit computation,
\[
\partial_s \chi = -(\nabla \phi)\left(\frac{x - s\gamma|x^2| - x_0}{1 + s}\right)\phi\left(\frac{\xi - \xi_0}{\epsilon}\right) \times \left(\frac{x - s\gamma|x^2| - x_0 + (1 + s)\gamma|\xi|^{-2}\xi}{(1 + s)^2}\phi\left(\frac{\xi - \xi_0}{\epsilon}\right) \times \frac{1}{1 + s}\right).
\]
Therefore,
\[
\mathcal{L}_s\chi = -(\nabla \phi)\left(\frac{x - s\gamma|x^2| - x_0}{1 + s}\right)\phi\left(\frac{\xi - \xi_0}{\epsilon}\right) \times \left(\frac{x - s\gamma|x^2| - x_0}{(1 + s)^2}\phi\left(\frac{\xi - \xi_0}{\epsilon}\right) \times \frac{1}{1 + s}\right) \geq 0.
\]
Similarly as above, we prove that $\mathcal{L}_s\chi \in W^{\infty, \infty}(\mathbb{R}_0^+, \mathbb{S}^{-\infty}_1)$.

We assume $t_0 > 0$ and $\mu = \infty$ as the other cases are similar. Let $\epsilon > 0$ be sufficiently small and let $\{\lambda_j\}_{j \in \mathbb{N}} \subset [1, 1 + \epsilon]$ be strictly increasing. Let $\phi$ be as in Lemma 3.2, and set $\forall j \in \mathbb{N}$,
\[
\chi_j(s, x, \xi) = \phi\left(\frac{x - s\gamma|x^2| - x_0}{\lambda_j(1 + s)}\right)\phi\left(\frac{\xi - \xi_0}{\lambda_j\epsilon}\right).
\]
Then supp $\chi_j \subset \{\chi_{j+1} > 0\}$. We aim to construct $a_h \in W^{\infty, \infty}(\mathbb{R}_0^+, \mathbb{S}^{-\infty}_0)$, such that
(i) supp $a_h \subset \bigcup_{j \in \mathbb{N}}$ supp $\chi_j$;
(ii) $a_h(0, \cdot) - (\chi_j^0, \rho)_{s_{1/0}}^0 \rho \chi_0(0, \cdot) = O(h^\infty)_{S^{-\infty}_0}$;
(iii) $\tau a_h \in W^{\infty, \infty}(\mathbb{R}_0^+, \mathbb{S}^{-\infty}_0)$;
(iv) $\tau a_h(s, \cdot)$ is elliptic at $(t_0\gamma|\xi_0|^{-2}\xi_0, \xi_0)$ for sufficiently large $s$;
(v) $\mathcal{L}_s^0 \rho_{h_0}^{\rho}(a_h) \geq O(h^\infty)_{L^\infty(\mathbb{R}_0^+, \mathbb{S}^{-\infty}_0)}$.
If such $a_h$ is found, and assume that
\[
(t_0\gamma|\xi_0|^{-2}\xi_0, \xi_0) \notin WF_{\rho(\gamma, -1), \rho}(u(t = t_0)).
\]
By (i) and (3.3), if we replace $\phi$ with $\phi(\lambda \cdot)$ for some $\lambda > 1$ sufficiently large, then for some compact set $K$, and sufficiently small $h > 0$,
\[
\text{supp} \theta_{1/0, h}^{\beta_0} a_h|_{s = h^{-\beta} t_0} \subset K \subset \mathbb{R}^{d_x} \times \mathbb{R}_t \times \text{WF}_{\rho(\gamma, -1), \rho}(u(t = t_0)).
\]
By (iv), $\theta_{1/0, h}^{\beta_0} a_h|_{s = h^{-\beta} t_0} \in S^{-\infty}_0$ is elliptic at $(t_0\gamma|\xi_0|^{-2}\xi_0, \xi_0)$. Therefore, by Lemma 2.15,
\[
(u, \text{Op}_{h}^{\rho}(a_h) u_{L^2})|_{s = h^{-\beta} t_0} = (u, \text{Op}_{h}^{\rho(\gamma, -1), \rho}(\theta_{1/0, h}^{\beta_0} a_h) u_{L^2})|_{s = h^{-\beta} t_0} = O(h^\infty).
\]
By (3.1),
\[
\frac{d}{ds} (u, \text{Op}_{h}^{\rho}(a_h) u)_{L^2} = (u, \mathcal{L}_s^0 \text{Op}_{h}^{\rho}(a_h) u)_{L^2},
\]
which implies by (v) that
\[
(u, \mathcal{O}_h^{\delta, \rho}(a_h)u)_{L^2} |_{s=0} = (u, \mathcal{O}_h^{\delta, \rho}(a_h)u)_{L^2} |_{s=\delta t_0} - \int_0^{\delta t_0} (u, \mathcal{L}_s^{h} \mathcal{O}_h^{\delta, \rho}(a_h)u)_{L^2} \, ds
\]
\[
\leq \mathcal{O}(h^\infty) + \mathcal{O}(h^{-\beta} \times h^\infty) = \mathcal{O}(h^\infty).
\]
Therefore, by (ii),
\[
\|\mathcal{O}_h^{\delta, \rho}(\chi_0)u\|_{L^2}^2 = (u, \mathcal{O}_h^{\delta, \rho}(a_h)u)_{L^2} |_{s=0} + \mathcal{O}(h^\infty) = \mathcal{O}(h^\infty).
\]
And we conclude that \((x_0, \xi_0) \not\in W^{\delta, \rho}(u_0)\).

We shall construct \(a_h\) in the following form of asymptotic expansion
\[
a_h(s, x, \xi) \sim \sum_{j \in \mathbb{N}} h^{j(\delta + \rho)} \varphi^j(s) a^j_h(s, x, \xi),
\]
where \(a^j_h \in W^{\infty, \infty}(\mathbb{R}_0, S_{0}^{-\infty})\), with \(\text{supp} a^j_h \subset \text{supp} \chi_j\), and \(\varphi^j \in P_j\), with
\[
(3.4) P_j = \left\{ f(\ln(1 + s)) : f(X) = \sum_{k=0}^j c_k X^k; c_k \geq 0, \forall k \right\}.
\]
All functions belonging to \(P_j\) is smooth and non-negative for \(s \geq 0\). Moreover, if \(\psi \in P_j\), then
\[
((1 + s) \partial_s)^{-1} \psi(s) := \int_0^s (1 + \sigma)^{-1} \psi(\sigma) \, d\sigma \in P_{j+1}.
\]
Indeed,
\[
((1 + s) \partial_s)^{-1}(\ln(1 + \cdot))^n = (n + 1)^{-1}(\ln(1 + \cdot))^{n+1}, \quad \forall n \in \mathbb{N}.
\]
The above asymptotic expansion is in the weak sense that, for some \(\epsilon' > 0\), and all \(N \in \mathbb{N}, \nabla a_h - \sum_{j < N} h^{j(\delta + \rho)} \varphi^j a^j_h \in \mathcal{O}(h^{N(\delta + \rho - \epsilon')} W^{\infty, \infty}([0, h^{-\beta} T], S_{0}^{-\infty})).
\]
We begin by setting \(\varphi^0 \equiv 1\) and choosing \(a^0_h\) satisfying
\[
a^0_h - \left( \delta, \rho \right) \partial_h \chi_h a^0_h = \mathcal{O}(h^\infty) W^{\infty, \infty}(\mathbb{R}_0, S_{0}^{-\infty}),
\]
\[
a^0_h(0, \cdot) - \left( \delta, \rho \right) \partial_h \chi_h a^0_h(0, \cdot) = \mathcal{O}(h^\infty) S_{-\infty}.
\]
By the definition of \(\beta\) and Proposition 2.5, Proposition 2.6,
\[
\mathcal{L}_s^{h} \mathcal{O}_h^{\delta, \rho}(a^0_h) = 2 \mathcal{O}_h^{\delta, \rho}(\rho_0 \mathcal{L}_s \chi_0) + h^{\delta + \rho} \mathcal{O}_h^{\delta, \rho}(r_h^0) + \mathcal{O}(h^\infty) L^{\infty}\infty(\mathbb{R}_0, L^2),
\]
where \(r_h^0 \in L^\infty(\mathbb{R}_0, S_{-\infty}^{-\infty})\), with \(\text{supp} r_h^0 \subset \text{supp} \chi_0\). So \(\langle s \rangle r_h^0 \in L^\infty(\mathbb{R}_0, S_{0}^{-\infty})\). Similarly, \(\langle s \rangle \chi_0 \mathcal{L}_s \chi_0 \in L^\infty(\mathbb{R}_0, S_{0}^{-\infty})\). By Lemma 3.2, \(\chi_0 \mathcal{L}_s \chi_0 \geq 0\). So by the sharp Gårding inequality, \(\exists b_h^0 \in L^\infty(\mathbb{R}_0, S_{-\infty}^{-\infty})\), with \(\text{supp} b_h^0 \subset \{ \chi_1 > 0 \}\), such that
\[
\mathcal{L}_s^{h} \mathcal{O}_h^{\delta, \rho}(b_h^0) \geq -\langle s \rangle^{-1} h^{\delta + \rho} \mathcal{O}_h^{\delta, \rho}(b_h^0) + \mathcal{O}(h^\infty) L^{\infty}\infty(\mathbb{R}_0, L^2).
\]
Suppose that we have found \(\varphi^j \in P_j\), \(a^j_h\) for \(j = 0, \ldots, \ell - 1\) and \(\psi^{\ell-1} \in P_{\ell-1}\), \(b_h^{\ell-1} \in L^\infty(\mathbb{R}_0, S_{-\infty}^{-\infty})\), with \(\text{supp} b_h^{\ell-1} \subset \{ \chi_\ell > 0 \}\), such that
\[
(3.5) \mathcal{L}_s^{h} \mathcal{O}_h^{\delta, \rho} \left( \sum_{j=0}^{\ell-1} h^{j(\delta + \rho)} \varphi^j a^j_h \right) \geq -\langle s \rangle^{-1} \psi^{\ell-1} h^{\ell(\delta + \rho)} \mathcal{O}_h^{\delta, \rho}(b_h^{\ell-1}) + \mathcal{O}(h^\infty) L^{\infty}\infty(\mathbb{R}_0, L^2).
\]
Then we set
\[
\varphi^\ell = ((1 + s) \partial_s)^{-1} \psi^{\ell-1}, \quad a^\ell_h(s, x, \xi) = B \chi(\ell)(s, x, \xi),
\]
for some constant \(B > 0\) sufficiently large, such that
\[
\mathcal{L}_s(\varphi^\ell a^\ell_h) = B(1 + s)^{-1} \psi^{\ell-1} \chi_\ell + B \varphi^\ell \mathcal{L}_s \chi_\ell \geq B(1 + s)^{-1} \psi^{\ell-1} \chi_\ell \geq \langle s \rangle^{-1} \psi^{\ell-1} b_h^{\ell-1}.
\]
Observe that
\[
\mathcal{L}_s(\varphi^\ell a^\ell_h) = \mathcal{O}(\langle s \rangle^{-1}(\psi^{\ell-1} + \varphi^\ell)) S_{-\infty}, \quad \langle s \rangle^{-1} \psi^{\ell-1} b_h^{\ell-1} = \mathcal{O}(\langle s \rangle^{-1} \psi^{\ell-1} S_{-\infty})\text{.}
\]
By the sharp Gårding inequality, \( \exists b^\ell_h \in L^\infty(\mathbb{R}_{\geq 0}, S^0_{-\infty}) \), with \( \text{supp} b^\ell_h \subseteq \{ \chi_{\ell+1} > 0 \} \), such that
\[
\mathcal{L}^k_h \text{Op}^k_h(\varphi^a_h) - \langle s \rangle^{-1} \psi^{e-1} \text{Op}^k_h(b^\ell_h - 1) \geq -\langle s \rangle^{-1} \psi^{e} \delta^{+\rho} \text{Op}^k_h(b^\ell_h) + O(h^\infty)_{L^2 \rightarrow L^2},
\]
with \( \psi^\ell = \psi^{e-1} + \varphi^e \in P_k \). Summing up (3.5) and \( h^{k(\delta^{+\rho})} \times (3.6) \), we close the induction procedure.

We prove the asymptotic expansion by observing that, for all \( \ell' > 0 \),
\[
\| \psi^\ell \|_{L^\infty([0, h^{-\beta} T])} = O(\| \log h^\ell \|) = O(h^{-\ell'}). \tag{4.1}
\]

4. PARADIFFERENTIAL CALCULUS

We develop a paradifferential calculus on weighted Sobolev spaces, and a semiclassical paradifferential calculus.

4.1. Classical Paradifferential Calculus. We recall some classical results of the paradifferential calculus. We refer to the original work of Bony [8], and the books [17, 25, 6]. The proofs below are mainly based on [25], so we shall only sketch them. In the meanwhile, we shall also make some refinements regarding the estimates of the remainder terms, for the sake of the semiclassical paradifferential calculus that will be developed later.

**Definition 4.1.** For \( m \in \mathbb{R}, r \geq 0 \), \( \Gamma^{m,r} \) is the space of locally bounded functions \( a(x, \xi) \) on \( \mathbb{R}^d_x \times (\mathbb{R}^d_\xi \setminus \{0\}) \) which are \( C^\infty \) with respect to \( \xi \), and \( \forall \alpha \in \mathbb{N}^d, \exists C_\alpha > 0 \) such that
\[
\| \partial_\xi^\alpha a(\cdot, \xi) \|_{W^{r,\infty}} \leq C_\alpha (\xi)^{m-|\alpha|}, \quad \forall |\xi| \geq 1/2.
\]
Moreover we denote
\[
M^{m,r}(a) = \sup_{|\alpha| \leq 2(d+2)+r} \sup_{|\xi| \geq 1/2} \langle \xi \rangle^{m-|\alpha|} \| \partial_\xi^\alpha a(\cdot, \xi) \|_{W^{r,\infty}}.
\]

**Definition 4.2.** \( (\chi, \pi) \in C^\infty(\mathbb{R}^d_\eta) \times C^\infty(\mathbb{R}^d_\eta) \) is called admissible if
\begin{enumerate}
\item \( \pi(\eta) = 1 \) for \( |\eta| \geq 1 \), \( \pi(\eta) = 0 \) for \( |\eta| \leq 1/2 \);
\item \( \chi \) is an even function, homogeneous of degree 0, and for some \( 0 < \epsilon_1 < \epsilon_2 < 1 \),
\end{enumerate}

\[
\left\{ \begin{array}{l}
\chi(\theta, \eta) = 1, \quad |\theta| \leq \epsilon_1 |\eta|,
\chi(\theta, \eta) = 0, \quad |\theta| \geq \epsilon_2 |\eta|.
\end{array} \right.
\]

**Definition 4.3.** For \( a \in \Gamma^{m,0} \), \( m \in \mathbb{R} \), the paradifferential operator \( T_a \) is defined by
\[
\bar{T}_a \hat{u}(\xi) = (2\pi)^{-d} \int_{\mathbb{R}^d} \chi(\xi - \eta, \eta) \pi(\eta) \hat{a}(\xi - \eta, \eta) \hat{u}(\eta) \, d\eta,
\]
where \( (\chi, \pi) \) is admissible and
\[
\hat{a}(\theta, \xi) = \int e^{-ix.\theta} a(x, \xi) \, dx.
\]

In other words, \( T_a = \text{Op}(\sigma_a) \) with
\[
\sigma_a(\cdot, \xi) = \pi(\xi) \chi(D_x, \xi) a(\cdot, \xi).
\]

**Proposition 4.4.** Let \( a \in \Gamma^{m,0} \), \( m \in \mathbb{R} \), then \( T_a = O(M^{m,0}(a))_{S^0_0} \).

**Lemma 4.5.** Let \( a \in \Gamma^{m,r} \) with \( m \in \mathbb{R} \) and \( r \geq 0 \), then \( M^{m,r}(\sigma_a) \lesssim M^{m,r}(a) \). If \( r \in \mathbb{N} \), then \( \forall \beta \in \mathbb{N}^d \) with \( |\beta| \leq r \),
\[
M^{m-r+|\beta|,0}(\partial_\beta^\beta (\sigma_a - a\pi)) \lesssim M^{m,0}(\nabla^r a).
\]

**Proof.** The first statement is proven in [25]. We only prove the second statement. There is no harm in assuming that \( \beta = 0 \). By [25], for \( \xi \neq 0 \),
\[
(\sigma_a - a\pi)(x, \xi) = \pi(\xi) \int \rho(x, y, \xi) \Phi(y, \xi) \, dy,
\]
Proof. Let \( a \in \Gamma^{r,m}_0 \) with \( m \in \mathbb{R} \) and \( r \in \mathbb{N} \). Let \( (\chi, \pi) \) and \( (\chi', \pi') \) be admissible. Denote by \( T_a \) and \( T'_a \) the paradifferential operators respectively defined by these two admissible pairs. Then
\[
T_a - T'_a = O(\langle M^{m,0}(\nabla^r a) \rangle)_{\mathcal{E}_0} + O(\langle M^{m,r}(a) \rangle)_{\mathcal{E}_0}.
\]

By Corollary 4.6, let \( \sigma \) be admissible such that \( \sigma(\cdot, \xi) \) satisfies for \(|\xi| \geq 1/2\) and \(|\alpha| \leq 2(d + 2) + r\) the estimates
\[
\|\partial^\alpha_x \rho(\cdot, \xi)\|_{L^1} \lesssim \|\partial^\alpha_a \rho(\cdot, \xi)\|_{L^\infty} \lesssim \|\rho(\cdot, \xi)\|_{L^\infty} \lesssim \|\partial^\alpha_x \rho(\cdot, \xi)\|_{L^\infty},
\]
and \( \Phi(a, \xi) = F^{-1}(\sigma(a, \xi)) \). We conclude with simple integral inequalities as in [25].

**Corollary 4.7.** Let \( a \in \mathcal{W}^\infty,\infty(\mathbb{R}^d_+) \). If \( T_a \psi - \psi \in \mathcal{E}_{0 \infty} \).

**Proposition 4.8.** Let \( a \in \Gamma^{r,m}_0 \), \( b \in \Gamma^{r,m'}_0 \) with \( r \in \mathbb{N} \), \( m \in \mathbb{R} \), \( m' \in \mathbb{R} \), then
\[
T_aT_b - T_{ab} = O\left(\langle M^{m,r}(a)M^{m',r}(b) \rangle\right)_{\mathcal{E}_{0}} + O\left(\langle M^{m,r}(a)M^{m',r}(b) \rangle\right)_{\mathcal{E}_{0}}.
\]

where \( a\sharp b = \sum_{\alpha \in \mathbb{N}^d} \frac{1}{\alpha!} \partial^\alpha_x aD^r_x b \).

**Remark 4.9.** If \( a = a \frac{\pi}{b} \) and \( b = b \pi \), then by the proof below,
\[
T_aT_b - T_{ab} = O\left(\langle M^{m,r}(a)M^{m',r}(b) \rangle\right)_{\mathcal{E}_{0}} + O\left(\langle M^{m,r}(a)M^{m',r}(b) \rangle\right)_{\mathcal{E}_{0}}.
\]

**Proof.** By Corollary 4.6, we may fix \( \epsilon_2 < 1/4 \). We decompose
\[
T_aT_b - T_{ab} = \| (I) + (II),
\]
where
\[
(I) = Op(\sigma_a)Op(\sigma_b) - Op(\sigma_a \sharp \sigma_b), \quad (II) = Op(\sigma_a \sharp \sigma_b) - Op(\sigma_{ab}).
\]
Let \( \theta \) be admissible such that \( \theta(\chi) = \chi \). Then by [25], \( Op(\sigma_a)Op(\sigma_b) = Op(\sigma) \),
\[
\sigma(x, \xi) = \frac{1}{(2\pi)^d} \int e^{i(x-y) \cdot \theta} \sigma_a(x, \xi) \sigma_b(y, \xi) dy d\eta = \sigma_a \sharp \sigma_b(x, \xi) + \sum_{|\alpha| = r} q_\alpha(x, \xi),
\]
where
\[
q_\alpha(x, \xi) = \int R_\alpha(x, \xi) (D^r_x \sigma_b)(y, \xi) dy,
\]
with \( R_\alpha \) satisfying
\[
\|\partial^\beta_x R_\alpha(x, \cdot, \xi)\|_{L^1} \lesssim M^{m,r}(a)\langle \xi \rangle^{m-|\alpha|-|\beta|},
\]
and consequently, as in [25],
\[
\| (I) \|_{H^s \to H^{m-m'-\rho}} \lesssim \sum_{|\alpha| = r} M^{m+m'-|\alpha|,0}(q_\alpha) \lesssim M^{m,r}(a)M^{m',0}(\nabla^r b).
\]

To Estimate (II), for each \(|\alpha| < r\), we decompose
\[
\partial^\alpha_x \sigma_a D^r_x \sigma_b - \sigma \partial^\alpha_x aD^r_x b = (i) + (ii) + (iii) + (iv),
\]
where
\[
(i) = \partial^\alpha_x (\sigma_a - a\pi)D^r_x \sigma_b,
\]
\[
(ii) = \partial^\alpha_x aD^r_x (\sigma_b - b\pi),
\]
\[
(iii) = \partial^\alpha_x (a\pi)D^r_x (b\pi) - \sigma \partial^\alpha_x (a\pi)D^r_x (b\pi),
\]
\[
(iv) = \sigma \partial^\alpha_x (a\pi)D^r_x (b\pi) - \sigma \partial^\alpha_x aD^r_x b.
\]
By Lemma 4.5,
\begin{align*}
M^{m+m'-r,0}(i) & \lesssim M^{m-r,0}(\sigma_a - a\pi)M^{m',0}(D^2_x\sigma_b) \lesssim M^{m,0}(\nabla r)aM^{m',r}(b), \\
M^{m+m'-r,0}(ii) & \lesssim M^{m-\alpha,0}(\partial^\alpha\nabla^r a)M^{m-r+\alpha,0}(D^{\alpha}_x(\sigma_b - b\pi)) \lesssim M^{m,0}(a)M^{m',0}(\nabla r)b,
\end{align*}

By Lemma 4.5, Leibniz’s rule and interpolation,
\begin{align*}
M^{m+m'-r,0}(iii) & \lesssim M^{m+m'-\alpha,0}(\nabla^\alpha(\partial^\alpha\nabla^r a)(D^2_x\sigma_b)) \\
& \lesssim M^{m-\alpha,0}(\nabla^r a)M^{m,0}(b) + M^{m-\alpha,0}(\partial^\alpha\nabla^r a)M^{m',0}(\nabla r)b
\end{align*}

And we easily verify that,
\begin{align*}
M^{-N,0}(iv) & \lesssim M^{m,r}(a)M^{m',r}(b), \quad \forall N > 0.
\end{align*}

These estimates implies that,
\begin{align*}
(II) = \mathcal{O}(M^{m,r}(a)M^{m,0}(\nabla r) + M^{m,0}(\nabla r)aM^{m',r}(b))_{\mathcal{E}^{m+m'-r}_0} + \mathcal{O}(M^{m,r}(a)M^{m',r}(b))_{\mathcal{E}^{m'-r}_0}.
\end{align*}

\begin{proposition}
Let $a \in \Gamma^{m,r}$ with $r \in \mathbb{N}$ and $m \in \mathbb{R}$, then
\begin{align*}
T^*_a - T_a = \mathcal{O}(M^{m,0}(\nabla r)a)_{\mathcal{E}^{m'-r}_0},
\end{align*}
where $a^* = \sum_{\alpha \in \mathbb{N}^d} \frac{\partial^\alpha a}{\alpha!} D_x^\alpha a$.
\end{proposition}
\begin{proof}
Let $(\theta, \pi)$ be admissible such that $\theta \chi = \chi$, then $T^*_a = \text{Op}(\sigma^*_a)$ with
\begin{align*}
\sigma^*_a(x, \xi) = (2\pi)^{-d} \int e^{-i\xi \cdot \sigma_a(x+y, \xi + \eta)} d\eta dy = a^*(x, \xi) + \sum_{|\alpha| = r} r\alpha(x, \xi),
\end{align*}
where by [25],
\begin{align*}
r\alpha(x, \xi) = \frac{2\pi}{\alpha!} \int_{\mathbb{R}^{2d}\times[0,1]} r(1-t)^{r-1} e^{-i\xi \cdot \sigma_a(x+y, \xi + \eta)} \theta(\eta, \xi) dt d\eta dy,
\end{align*}
Similarly, observing the term $D^\alpha_x \partial^\alpha \nabla^r a(x, \xi + \eta)$ in the integral, the analysis in [25] implies that
\begin{align*}
M^{m-r,0}(\sigma^*_a - \sigma_a) \lesssim \sum_{|\alpha| = r} M^{m-r,0}(r\alpha) + M^{m-r,0}(a^* - \sigma_a) \lesssim M^{m,0}(\nabla r)a.
\end{align*}
\end{proof}

\begin{proposition}
Let $a \in H^\alpha$ and $b \in H^\beta$ with $\alpha > d/2$, and $\beta > d/2$, then
\begin{align*}
\|ab - T^*_a b - T_a a\|_{H^{\alpha+\beta-d/2}} \lesssim \|a\|_{H^\alpha}\|b\|_{H^\beta}.
\end{align*}
\end{proposition}

\begin{proposition}
Let $F \in C^\infty(\mathbb{R})$ such that $F(0) = 0$, then $\forall \mu > d/2$, there exists a monotonically increasing function $C : \mathbb{R}_\geq 0 \rightarrow \mathbb{R}_\geq 0$, such that $\forall u \in H^\mu$,
\begin{align*}
\|F(u)\|_{H^\mu} + \|F(u) - T_F(\mu)u\|_{H^{2\mu-d/2}} \leq C(\|u\|_{H^\mu}^\mu)\|u\|_{H^\mu}.
\end{align*}
\end{proposition}

\subsection{4.2. Dyadic Paradifferential Calculus}
We develop the theory of paradifferential calculus on weighted Sobolev spaces.

\begin{definition}
Let $r \in \mathbb{N}$, $k \in \mathbb{R}$, and $0 \leq \delta \leq 1$. Given $u \in \mathcal{S}'$, we say that $u \in W^{r,\infty}_{k,\delta}$ if for all $\alpha \in \mathbb{N}$ with $|\alpha| \leq r$, $(x)^{k+\delta|\alpha|} \partial^\alpha_x u \in L^\infty(\mathbb{R}^d)$.
\end{definition}

\begin{definition}
Let $m \in \mathbb{N}$, $k \in \mathbb{R}$, $r \in \mathbb{N}$, $0 \leq \delta \leq 1$, $\Gamma^{m,r}_{k,\delta}$ is the space of locally bounded functions $a(\xi) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus 0)$ which are $C^\infty$ respect to $\xi$, and $\forall \alpha \in \mathbb{N}^d$, $\exists C_\alpha > 0$, such that
\begin{align*}
\|\partial^\alpha_x a(\cdot, \xi)\|_{W^{r,\infty}_{k,\delta}} \leq C_\alpha \langle \xi \rangle^{m-|\alpha|}, \quad \forall \langle \xi \rangle \geq 1/2.
\end{align*}
Moreover, we denote
\begin{align*}
M^{m,r}_{k,\delta}(a) = \sup_{|\alpha| \leq 2(d+2)+r} \sup_{|\xi| \geq 1/2} \langle \xi \rangle^{-m} \|\partial^\alpha_x a(\cdot, \xi)\|_{W^{r,\infty}_{k,\delta}}.
\end{align*}
\end{definition}
Let
\[ \Gamma^{-\infty,r}_{k,\delta} = \bigcap_{m \in \mathbb{R}} \Gamma^{m,r}_{k,\delta}, \quad \Gamma^{m,r}_{-\infty,\delta} = \bigcap_{k \in \mathbb{R}} \Gamma^{m,r}_{k,\delta}. \]
Then for \((m,k) \in (\mathbb{R} \cup \{-\infty\})^2\) and \(h > 0\), define
\[ h\Sigma_{k,\delta}^{m,r} = \sum_{0 < j < r} h^j \Gamma^{m-j,r-j}_{k-\delta,\delta}. \]
Let \(f(h) > 0\) be a positive function of \(0 < h < 1\), we denote \(a_h \in f(h)\Sigma_{k,\delta}^{m,r}\) if \(a_h = \mathcal{O}(1)_{f(h)\Sigma_{k,\delta}^{m,r}}\) for \(0 < h < 1\).

**Remark 4.15.** Observe that \(W^{r,\infty}_{k,0} = W^{r,\infty}_k, \Gamma^{0,\infty}_{0,0} = \Gamma^{m,r}.\) We shall denote for simplicity
\[ W^{r,\infty}_k = W^{r,\infty}_{k,0} \quad \Sigma_{k,\delta}^{m,r} = \sum_{k,\delta}^{m,r}, \quad h\Sigma_{k,\delta}^{m,r} = h\Sigma_{k,\delta}^{m,r}. \]

**Lemma 4.16.** Let \(A : \mathcal{S} \to \mathcal{S}, \psi, \phi, \phi_0 \in \mathcal{S}; (m, k) \in \mathbb{R}^2\). If \(\forall (u, \ell) \in \mathbb{R}^2, \exists C > 0\), such that \(\forall u \in \mathcal{S}, \|\psi_j A u\|_{H^{\ell-m}} \leq C \) \(2^j \|\phi_j u\|_{H^{\ell}},\) then \(A \in \mathcal{O}_{M}^n\). Particularly, let \(\{A_j\}_{j \in \mathbb{N}} \in \mathcal{L}^\infty(\mathcal{O}_{M}^n),\) then
\[ A := \sum_{j \in \mathbb{N}} 2^{jk} \psi_j A_j \phi_j \in \mathcal{O}_{M}^n. \]

**Definition 4.17.** Fix \(\psi \in \mathcal{S}_k\), and define \(\tilde{\psi} \in \mathcal{S}_{k'}\) by setting \(\tilde{\psi}_j = \sum_{|j-k| \leq 10} \psi_k\). Let \(a \in \Gamma^{m,r}_{k,\delta}\), the dyadic paradifferential operator \(\mathcal{P}_a\) is defined by
\[ \mathcal{P}_a = \sum_{j \in \mathbb{N}} \psi_j T_{\psi_j a} \tilde{\psi}_j. \]

**Proposition 4.18.** Let \(a \in \Gamma^{m,r}_{k,\delta}\), then \(\mathcal{P}_a = \mathcal{O}(M^{m,0}_{k,0}(a))_{\mathcal{O}_{M}^n}\).

**Proof.** Observe that \(\|T_{\psi_j a}\|_{H^{\ell-m}} \leq M^{m,0}_{0,0}(\psi_j a) \leq 2j^k M^{m,0}_{k,0}(a).\)

**Proposition 4.19.** Let \(a \in \Gamma^{m,r}_{k,\delta}, b \in \Gamma^{m',r}_{k',\delta}, r \in \mathbb{N}, (m, k), (m', k') \in \mathbb{R}^2, 0 \leq \delta \leq 1,\)
\[ \mathcal{P}_a \mathcal{P}_b - \mathcal{P}_{ab} = \mathcal{O}(M^{m,0}_{k,0}(a) M^{m',r}_{k',\delta}(b))_{\mathcal{O}_{M}^{m+m'-r}}. \]

where \(a \in \mathbb{R}^{m+m'-r} \sum_{k+k',\delta}.\)

**Proof.** Let \(\tilde{\psi}_j : \mathbb{N} \to C^\infty, \tilde{\psi}_j = \sum_{|j-k| \leq 10} \psi_j, \) so \(\psi_j \tilde{\psi}_j = \tilde{\psi}_j.\) If \(|j - j'| \leq 20\), then write,
\[ \mathcal{P}_a \mathcal{P}_b = \sum_{(j,j') \in \mathbb{N}^2} \tilde{\psi}_j T_{\psi_j a} \tilde{\psi}_{j'} T_{\psi_{j'} b} - \sum_{(j,j') \not\in \mathbb{N}^2} \tilde{\psi}_j T_{\psi_j a} \tilde{\psi}_{j'} T_{\psi_{j'} b} + \tilde{\psi}_j R_{j,j'} \tilde{\psi}_j, \]
the remainder being
\begin{align*}
R_{j,j'} &= \psi_j T_{\psi_j a} \tilde{\psi}_{j'} T_{\psi_{j'} b} - \psi_j T_{\psi_j a} \tilde{\psi}_{j'} T_{\psi_{j'} b} \\
&= \mathcal{O}(2^{(k+k'-\delta)} M^{m,r}_{k,\delta}(a) M^{m',r}_{k',\delta}(b))_{\mathcal{O}_{M}^{m+m'-r}} + \mathcal{O}(2^{(k+k')} M^{m,r}_{k,\delta}(a) M^{m',r}_{k',\delta}(b))_{\mathcal{O}_{M}^{m+m'-r}}.
\end{align*}
by Proposition 4.14, Proposition 4.8 and Corollary 4.7. More precisely, when composing \(T_{\psi_j a}\) and \(T_{\tilde{\psi}_j}\), we use \(\psi_j \tilde{\psi}_j = \psi_j\) and have
\[ T_{\psi_j a} T_{\tilde{\psi}_j} = T_{\psi_j a} + \mathcal{O}(M^{m,r}(\psi_j a) M^{0,0}(\psi_j a))_{\mathcal{O}_{M}^{m-r}} + \mathcal{O}(M^{m,0}(\tilde{\psi}_j a) M^{0,r}(\tilde{\psi}_j a))_{\mathcal{O}_{M}^{m-r}} + \mathcal{O}(M^{m,0}(\psi_j a) M^{0,r}(\tilde{\psi}_j a))_{\mathcal{O}_{M}^{m-r}}.\]
Next observe that
\[ \sum_{j' : |j - j'| \leq 20} (\psi_{j'}a) \bar{z}_{j'}b = (\psi_ja) \bar{z}b. \]
Hence
\[ \sum_{j' : |j - j'| \leq 20} T_{\psi_ja} T_{\psi_jb} = \psi_j T_{\psi_j(a \bar{b})} \bar{\psi}_j + R_j, \]
where the remainder can be estimated similarly as above,
\[ R_j = O(2^{(k^2 - 5r)} M_{k, \delta}^{m,r} (a) M_{k', \delta}^{m', r} (b)) \epsilon_0^{m - m' - 5r} + O(2^{(k + k')} M_{k, \delta}^{m,r} (a) M_{k', \delta}^{m', r} (b)) \epsilon_0^{-\infty}. \]
We conclude by Lemma 4.16. \hfill \Box

**Proposition 4.20.** Let \( a \in \Gamma_{k, \delta}^{m,r} \) with \( (m, k) \in \mathbb{R}^2 \), and \( r \in \mathbb{N}, 0 \leq \delta \leq 1 \), then
\[ \mathcal{P}_a^* - \mathcal{P}_a = O(M_{k, \delta}^{m,r} (a)) \epsilon_k^{m, r} + \epsilon_{-\infty}, \]
where \( a^* = \sum_{\alpha \in \mathbb{N}^d} a_{\alpha} \partial_{\alpha} \bar{a} \in \sum_{k, \delta}^{m,r} \).

**Proof.** Observe that for any real valued \( \psi \in \mathcal{C}_c^\infty (\mathbb{R}^d) \),
\[ (\psi a)^* = a^* \psi. \]
More precisely, this means that,
\[
(\psi a)^* = \sum_{|\gamma| < r} \frac{1}{\gamma!} \partial_\gamma^\psi \bar{D}_x^\gamma (\psi \bar{a}) = \sum_{|\gamma| < r} \frac{1}{\gamma!} \sum_{\alpha + \beta = \gamma} \frac{\gamma!}{\alpha! \beta!} D_\alpha^\psi \bar{D}_x^\alpha \bar{D}_x^\beta \partial_\xi^\psi \bar{a}
= \sum_{|\alpha| + |\beta| < r} \frac{1}{\alpha!} \partial_{\alpha}^\psi \bar{D}_x^\alpha \bar{D}_x^\beta \bar{a}
\]
\[ = a^* \psi. \]
Then write
\[ \mathcal{P}_a^* - \mathcal{P}_a = \sum_{j \in \mathbb{N}} \psi_j (R_j^1 + R_j^2) \psi_j, \]
where by (4.5),
\[ R_j^1 = T_{\psi_j a}^* - T_{(\psi_j a)^*}, \quad R_j^2 = T_{(\psi_j a)^*} - T_{\psi_j a}^* = T_{a^* \psi_j - \psi_j a^*}. \]
For \( R_j^1 \) we use Proposition 4.10,
\[ R_j^1 = O(M_{k, \delta}^{m,0} (\nabla_x \psi_j a)) \epsilon_0^{m, r} = O(2^{(k - 5r)} M_{k, \delta}^{m,r} (a)) \epsilon_0^{m, r}. \]
By Lemma 4.16,
\[ \sum_{j \in \mathbb{N}} \psi_j R_j^1 \psi_j = O(M_{k, \delta}^{m,r} (a)) \epsilon_{k - 5r}^{m, r} + \epsilon_{-\infty}. \]
Using \( \sum_{j \in \mathbb{N}} \psi_j = 1 \), we induce that \( \sum_{j \in \mathbb{N}} \partial_{\alpha}^\psi \psi_j \equiv 0, \forall \alpha \in \mathbb{N}^d \setminus 0 \). Therefore,
\[ \sum_{j \in \mathbb{N}} a^* \psi_j - \psi_j a^* = 0. \]
Then we write
\[ a^* \psi_j - \psi_j a^* = \sum_{|\alpha| + |\beta| < r} D_\alpha^\psi \psi_j \cdot w_{\alpha \beta}, \quad w_{\alpha \beta} \in \Gamma_{k - |\beta|, \delta}^{m - |\alpha|, r - |\beta|}, \]
where the symbols \( w_{\alpha \beta} \) are independent of \( j \). Set
\[ R_{\alpha \beta} = \sum_{j \in \mathbb{N}} \psi_j T_{D_\alpha^\psi \psi_j w_{\alpha \beta} \psi_j}, \]
then
\[ \sum_{j \in \mathbb{N}} \psi_j R_j^2 \psi_j = \sum_{\alpha, \beta} R_{\alpha \beta}. \]
By (4.6), we prove similarly as in Proposition 4.19 that
\[ \psi_j R_{\alpha \beta} = \psi_j \sum_{|j-j'| \leq 20} \psi_j T_{dj}^2 \psi_j \cdot w_{\alpha \beta} \psi_j = O(2^{j(|\alpha|+k-|\beta|+r-|\beta|)} M_{k-|\beta|,|\delta|}^{|\beta|,|\beta|,|\beta|}(w_{\alpha \beta})) e_m^{r-|\alpha|} \]
\[ + O(2^{j(|\alpha|+k-|\beta|+r-|\beta|)} M_{k-|\beta|,|\delta|}^{|\beta|,|\beta|,|\beta|}(w_{\alpha \beta})) e_0^{-\infty} \]
\[ = O(2^{j(k-\delta r)} M_{k,d}^{m,r}(a)) e_m^{r-\infty} + O(2^{j k} M_{k,d}^{m,r}(a)) e_0^{-\infty}. \]

Setting \( \psi_j' = \sum_{j'|j| \leq 100} \psi_j' \). We again conclude by Lemma 4.16, and the identity
\[ R_{\alpha \beta} = \sum_{j \in \mathbb{N}} \psi_j R_{\alpha \beta} \psi_j', \]
that \( R_{\alpha \beta} = O(M_{k,d}^{m,r}(a)) e_m^{r-\delta r} + e_0^{-\infty} \).

**Proposition 4.21.** Let \( a \in H^\alpha_k, b \in H^{\beta}_\ell \), with \( \alpha > d/2, \beta > d/2, k \in \mathbb{R}, \ell \in \mathbb{R} \), then
\[ \|a b - P_a b - P_b a\|_{H^{\alpha+\beta-d/2}_{k+\ell}} \lesssim \|a\|_{H^\alpha_k} ||b||_{H^{\beta}_\ell}. \]

**Proof.** Decompose the product \( ab \) as follows,
\[ ab = \sum_{j \in \mathbb{N}} \psi_j(\psi_j a)(\psi_j b) = P_a b + P_b a + R_1 + R_2, \]
where
\[ R_1 = \psi_j(\psi_j a \cdot \psi_j b - \psi_j b \psi_j a), \]
\[ R_2 = \psi_j(\psi_j T_{\psi_j a \psi_j} - \psi_j T_{\psi_j b}) \psi_j b + \psi_j(\psi_j T_{\psi_j b} \psi_j) - \psi_j T_{\psi_j a} \psi_j a. \]
By Proposition 4.11,
\[ \|R_1\|_{H^{\alpha+\beta-d/2}} \lesssim \|\psi_j a\|_{H^\alpha_k} \|\psi_j b\|_{H^{\beta}_\ell} \lesssim 2^{-j(k+\ell)} \|a\|_{H^\alpha_k} \|b\|_{H^{\beta}_\ell}. \]
By Proposition 4.8 and Corollary 4.7,
\[ \psi_j T_{\psi_j a \psi_j} - \psi_j T_{\psi_j a} = 2^{-j k} O(||a||_{H^\alpha_k}) e_0^{m-r}, \]
\[ \psi_j T_{\psi_j b} \psi_j - \psi_j T_{\psi_j b} = 2^{-j \ell} O(||b||_{H^{\beta}_\ell}) e_0^{m-r}. \]
We conclude by Proposition 2.13. \( \square \)

**Proposition 4.22.** Suppose \( F \in C^\infty(\mathbb{R}) \) with \( F(0) = 0 \), then for \( \mu > d/2 \), there exists some monotonically increasing function \( C : \mathbb{R}_+ \to \mathbb{R}_+ \), such that for \( u \in H^\mu_k \), with \( k \geq 0 \),
\[ \|F(u)\|_{H^\alpha_k} + \|F(u) - F'(u) u\|_{H^{2\mu-d/2}_k} \leq C(||u||_{H^\mu_k}) ||u||_{H^\mu_k}. \]

**Proof.** Decompose
\[ F(u) = \sum_{j \geq 0} \psi_j F(\psi_j u). \]
By Proposition 4.12,
\[ \|F(\psi_j u)\|_{H^\mu} \leq C(||\psi_j u||_{H^\mu}) \|\psi_j u||_{H^\mu} \leq C(||u||_{H^\mu}) \|\psi_j u||_{H^\mu}. \]
Then write
\[ \psi_j F(\psi_j u) = \psi_j T_{\psi_j F'(u) \psi_j} u + \psi_j R_j, \]
with
\[ R_j = \psi_j (T_{F'(\psi_j u) \psi_j} - F(\psi_j u)) + \psi_j (T_{F'(\psi_j u) - \psi_j T_{F'(\psi_j u)}} \psi_j u). \]
By Proposition 4.12, Proposition 4.11 and Corollary 4.7,
\[ \|R_j\|_{H^{2\mu-d/2}} \leq C(||u||_{H^\mu}) \|\psi_j u||_{H^\mu}. \]
We conclude with Proposition 2.13. \( \square \)
4.3. Semiclassical Paradifferential Calculus. We develop a semiclassical dyadic paradifferential calculus, and a quasi-homogeneous semiclassical paradifferential calculus, using scaling arguments inspired by M´etivier-Zumbrun [26].

Definition 4.23. Define the scaling operator \( \tau_h : u(\cdot) \mapsto h^d u(h \cdot) \). For \( b \in \Gamma^{m,r}, h > 0 \), we define the semiclassical paradifferential operator

\[
T^h_b = \tau_h^{-1} T^{h,0}_{h \ast} \tau_h.
\]

For \( a \in \Gamma^{m,r}, h > 0 \), we define the semiclassical dyadic paradifferential operator

\[
\mathcal{P}^h_a = \sum_{j \in \mathbb{N}} \psi_j T^h_{\psi_j,0} \psi_j.
\]

For \( \epsilon \geq 0 \), we define the quasi-homogeneous semiclassical paradifferential operator

\[
\mathcal{P}^{h,\epsilon}_a = \mathcal{P}^h_{0,0} a.
\]

Proposition 4.24. If \( m \leq 0 \), and \( k \leq 0 \), then \( \sup_{0<h<1} ||\mathcal{P}^{h,\epsilon}_a||_{L^2 \rightarrow L^2} < \infty \).

Proof. Observe that \( \theta^{1+\epsilon}_{h,0} a = O(1)_{1,0} \). We conclude with Lemma 4.16. \( \square \)

Definition 4.25. For \( \epsilon \geq 0, a_h \in \mathcal{D}'(\mathbb{R}^d_x \times \mathbb{R}^d_\xi) \), we say \( a_h \in \mathcal{A}_\epsilon \) if

\[
\bigcup_{0<h<1} \text{supp } a_h \bigcap \mathcal{N}_{r,1} = \emptyset,
\]

recalling the definition (2.1) of \( \mathcal{A}_{\delta,\rho} \).

Proposition 4.26. Let \( (m, k), (m', k') \in (\mathbb{R} \cup \{-\infty\})^2 \), \( r \in \mathbb{N} \), with \( r \geq m + m' \), \( \delta r \geq k + k' \). Let \( a_h \in \Gamma^{m,r}_{k,\delta} \cap \sigma_0 \), \( b_h \in \Gamma^{m',r}_{k',\delta} \cap \sigma_0 \), such that for some \( R_h \geq 0 \) depending on \( h \),

\[
(4.7) \quad \text{supp } a_h \bigcap \text{supp } b_h \subset \{ |x| \geq R_h \} \times \mathbb{R}^d_\xi,
\]

then for \( h > 0 \) sufficiently small,

\[
\mathcal{P}^h_{a_h} \mathcal{P}^h_{b_h} - \mathcal{P}^h_{a_h b_h} = O(h^r (1 + R_h)^{k+k'-\delta r})_{L^2 \rightarrow L^2},
\]

where \( a_h b_h = \sum_{|\alpha|<r} h^{\alpha} \partial^\alpha a_h D_x^\alpha b_h \in \mathcal{H}^{m+m',r}_{k+k',\delta} \).

Proof. By (4.7), \( \psi_j a_h \neq 0 \) and \( \psi_j b_h \neq 0 \) implies that \( j \gtrsim \log_2 (1 + R_h) \). We claim that

\[
\mathcal{P}_{a_h}^h \mathcal{P}_{b_h}^h = \sum_{j \geq \log_2 (1 + R_h)} \psi_j T_{\psi_j a_h} \psi_j T_{\psi_j b_h} \psi_j.
\]

And then we can conclude by the identity

\[
\sum_{j' : |j'-j| \leq 20} (\psi_j a_h) (\psi_j b_h) = \psi_j (a_h b_h).
\]

Indeed, we use \( b_h \in \sigma_0 \) and (4.1) to induce that \( \mathcal{F}(T^{h,0}_{h \ast}(\psi_j b_h) u) \) vanishes in a neighborhood of \( \xi = 0 \). By (4.3), for some \( \pi \in C^\infty(\mathbb{R}^d) \) which vanishes near \( \xi = 0 \) and equals to 1 outside a neighborhood of \( \xi = 0 \), and for all \( m + m' \leq N \in \mathbb{N} \),

\[
\tau_h T_{\psi_j a_h} \psi_j T_{\psi_j b_h} \psi_j T_{\psi_j b_h} \psi_j T_{\psi_j b_h} \psi_j T_{\psi_j b_h} \psi_j T_{\psi_j b_h} \psi_j T_{\psi_j b_h} = O(M^{m,0}(\psi_j a_h)) a^{m} O(2^{-jN} h^N) \psi_j T_{\psi_j b_h} + O(M^{m',0}(\psi_j b_h)) \psi_j T_{\psi_j b_h}.
\]
Then we use Proposition 4.8 and Remark 4.9,
\[
T_{\theta_{a h}} T_{\theta_{a h}} T_{\theta_{a h}} = T_{\theta_{a h}} + O(M_{m, r}(\nabla^{1/0}(\theta_{a h} a_h)) M_{m, r}(\theta_{a h} a_h)) \rho_0^{m + m'}
\]

\[
+ O(M_{m, r}(\theta_{a h} a_h))^2 M_{m, r}(\theta_{a h} a_h)) \rho_0^{m + m'}
\]

\[
+ O(M_{m, r}(\theta_{a h} a_h))^2 M_{m, r}(\theta_{a h} a_h)) \rho_0^{m + m'}
\]

\[
+ O(M_{m, r}(\theta_{a h} a_h))^2 M_{m, r}(\theta_{a h} a_h)) \rho_0^{m + m'}
\]

\[
\]

\[
= T_{\theta_{a h}} + O(h^r(1 + R_h)_{k, k'} \delta r)_{L^2 \to L^2}.
\]

To estimate the remainders, we see that for each \(a \in \mathbb{N}^d\) with \(|a| = r\),
\[
\frac{\partial^2 \theta_{a h}}{\partial x^a} = \sum_{\alpha_1 + \alpha_2 = a} \frac{\alpha_1}{\alpha_1 ! \alpha_1 !} \theta_{a h} \nabla \theta_{a h} \nabla \theta_{a h} a_h
\]

\[
= \sum_{\alpha_1 + \alpha_2 = a} \frac{\alpha_1}{\alpha_1 ! \alpha_1 !} O(h^{1 - j} \alpha_1 \times H^{2j(k - \delta r)})_{L^2 \to L^2},
\]

where we use \(0 \leq \delta \leq 1\). Therefore, the first term in the remainder is
\[
O(h^r(1 + R_h)_{k, k'} \delta r)_{L^2 \to L^2} = O(h^r(1 + R_h)_{k, k'} \delta r)_{L^2 \to L^2}.
\]

Similar methods apply to the other two terms.

Combining the analysis of Proposition 4.26, Proposition 4.20, using Proposition 4.8, we obtain a similar result for the adjoint, whose proof we shall omit, as it is similar as above.

**Proposition 4.27.** Let \((m, k) \in (\mathbb{R} \cup \{-\infty\})^2\), \(r \in \mathbb{N}\), with \(r \geq m\), \(\delta r \geq k\). Let \(a_h \in \Gamma_{k, \delta} \cap \Sigma_0\), such that for some \(R_h \geq 0\) depending on \(h\),
\[
\text{supp } a_h \subset \{|x| \geq R_h\} \times \mathbb{R}^d,
\]

then for \(h > 0\) sufficiently small,
\[
\left(\mathcal{P}_{a_h}^h\right)^* - \mathcal{P}_{a_h}^h = O(h^r(1 + R_h)_{k, k'} \delta r)_{L^2 \to L^2},
\]

where \(a_h = \sum_{|a| < r} \frac{\alpha}{\alpha_1 ! \alpha_1 !} D_x^a a_h \in \mathcal{H}^{k, \delta} \).

**Corollary 4.28.** Let \(\varepsilon \geq 0\), \((m, k)(m', k') \in (\mathbb{R} \cup \{-\infty\})^2\), \(r \in \mathbb{N}\) with \(r \geq \max\{m + m', k + k'\}\), \(k \leq 0\). Let \(a_h \in \Gamma_{k, \delta} \cap \Sigma_0\), \(b_h \in \Gamma_{k, \delta} \cap \Sigma_0\). Then
\[
\mathcal{P}_{a_h}^h - \mathcal{P}_{b_h}^h = O(h^{1 + \varepsilon} - \varepsilon(k + k'))_{L^2 \to L^2},
\]

\[
\mathcal{P}_{b_h}^h - \mathcal{P}_{a_h}^h = O(h^{1 + \varepsilon} - \varepsilon(k + k'))_{L^2 \to L^2}.
\]

**Proof.** It suffices to observe that, if \(\varepsilon > 0\) then \(\text{supp } \theta_{h, k}^r a_h \subset \{|x| \geq h^{-\varepsilon}\} \), \(\theta_{h, k}^r a_h = O(h^{-\varepsilon k}_{m, r})_{1, 0}\).

We conclude by Proposition 4.26.

**Corollary 4.29.** Let \(\varepsilon \geq 0\), \((m, k)(m', k') \in (\mathbb{R} \cup \{-\infty\})^2\), \(r \in \mathbb{N}\) with \(r \geq m + m', k \leq 0\), \(k' \leq 0\). Let \(a_h \in \Gamma_{k, \delta} \cap \Sigma_0\), \(b_h \in \Gamma_{k, \delta} \cap \Sigma_0\). Then for \(h > 0\) sufficiently small,
\[
\mathcal{P}_{a_h}^h - \mathcal{P}_{b_h}^h = O(h^{1 + \varepsilon} - \varepsilon(k + k'))_{L^2 \to L^2},
\]

where \(a_h b_h = \sum_{|a| < r} \frac{\alpha}{\alpha_1 ! \alpha_1 !} D_x^a a_h D_x^b b_h \in \mathcal{H}^{k, \delta} \).

**Proof.** It suffices to use the identity \((\theta_{h, k}^r a_h)(\theta_{h, k}^r b_h) = \theta_{h, k}^r(a h b_h)\).

The results above only concerned about the high frequency regime. The next lemma studies the interaction of high frequencies and low frequencies.
Lemma 4.30. Let $m \in \mathbb{R}$, $a_h \in \Gamma^{0,0}$, $b_h \in \Gamma^{0,0}$, such that for some $R > 0$, 
$$\text{supp } a_h \in \{ |\xi| \geq R \}, \quad \text{supp } b_h \in \{ |\xi| \leq h^{-1}R/4 \}.$$ 
Then $\mathcal{P}_{a_h} \mathcal{P}_{b_h} = \mathcal{O}(h^\infty)_{L^2 \to L^2}.$

Proof. By definition

$$T_{\psi_j b_h} u(\xi) = (2\pi)^{-d} \int \chi(\xi - \eta, \eta) \pi(\eta) \psi_j b_h(\xi - \eta, \eta) \hat{u}(\eta) d\eta.$$ 

The admissibility of $\chi$ implies that

$$\text{supp } T_{\psi_j b_h} u \subset \{ |\xi| \leq h^{-1}R/3 \}.$$ 

Therefore, for any $|j' - j| \leq 20$, 

$$\sum_j \psi_j T^h_{\psi_j a_h} \psi_j \sum_j \psi_j T^h_{\psi_j b_h} \psi_j$$ 

$$= \sum_j \psi_j T^h_{\psi_j a_h} \pi(hD_x / R) \sum_j \psi_j \pi(hD_x / R) T^h_{\psi_j b_h} \psi_j$$ 

$$= \sum_j \psi_j \mathcal{O}(h^\infty)_{L^2 \to L^2} \psi_j.$$ 

We conclude by Lemma 4.16. \qed

Corollary 4.31. Let $a \in \Gamma^{m,0}$ which is homogeneous of degree $m$ with respect to $\xi$, then 

$$h^m \mathcal{P}_a = \mathcal{P}^h_a + R_h.$$ 

where for $b \in \Gamma^{0,0} \cap \sigma_0$ and $h > 0$ sufficiently small, $\mathcal{P}_b R_h = \mathcal{O}(h^\infty)_{L^2 \to L^2}.$

Proof. By a direct verification using (4.2), the homogeneity of $a$ and the admissible function $\chi$, we see that $h^m \mathcal{P}_a - \mathcal{P}^h_a = \mathcal{P}^h_{a_h}$ where $a_h \in \Gamma^{0,0}$ satisfies

$$\text{supp } a_h \subset \mathbb{R}_x^d \times \text{supp } \xi(1 - \pi(h\cdot)) \subset \mathbb{R}_x^d \times \{ |\xi| \leq 2h^{-1} \}.$$ 

We conclude by Lemma 4.30. \qed

Lemma 4.32. Let $a_h \in \Gamma^{m,r} \cap \sigma_0$ with $r \geq \max\{m,0\} + 1 + d/2$, then for $h > 0$ sufficiently small,

$$T^h_{a_h} - \text{Op}_h(a_h) = \mathcal{O}(h^r)_{L^2 \to L^2}.$$ 

Proof. By (4.3) and Calderón-Vaillancourt theorem, we have

$$T^h_{a_h} - \text{Op}_h(a_h) = \mathcal{O}(M^{m,0}(\nabla_x (\partial_{\Gamma}^{1,0} a_h)))_{L^2 \to L^2} = \mathcal{O}(h^r M^{m,0}(a_h))_{L^2 \to L^2}.$$ 

\qed

Lemma 4.33. Let $a_h \in \Gamma^{m,\infty} \cap \sigma_0$ with $m \in \mathbb{R} \cup \{-\infty\}$, then for $h > 0$ sufficiently small,

$$\mathcal{P}^h_{a_h} - \text{Op}_h(a_h) = \mathcal{O}(h^\infty)_{L^2 \to L^2}.$$ 

Proof. By Lemma 4.32 and Lemma 4.16,

$$\mathcal{P}^h_{a_h} - \sum_{j \in \mathbb{N}} \psi_j \text{Op}_h(\psi_j a_h) \psi_j = \mathcal{O}(h^\infty)_{L^2 \to L^2}.$$ 

By the usual semiclassical symbolic calculus,

$$\sum_{j \in \mathbb{N}} \psi_j (\psi_j a_h) \psi_j = \psi_j a_h + \psi_j \mathcal{O}(h^\infty)_{\Gamma^{-\infty,\infty}},$$ 

which is uniform with respect to $j \in \mathbb{N}$. Therefore,

$$\sum_{j \in \mathbb{N}} \psi_j (\psi_j a_h) \psi_j = a_h + \mathcal{O}(h^\infty)_{\Gamma^{-\infty,\infty}}.$$ 

\qed
Lemma 4.34 (Paradifferential Gårding Inequality). Let \( a_h \in M_{n \times n}(\Gamma^{0,r}) \cap \sigma_0 \) with \( n \geq 0 \) and \( r \geq 1 + d/2 \). Suppose that \( \text{Re} a_h \geq 0 \), then \( \exists C > 0 \) such that \( \forall u \in L^2 \),

\[
\text{Re}(T_{a_h}^h u, u)_{L^2} \geq -C h \| u \|_{L^2}^2.
\]

Consequently, for some \( C' > 0 \),

\[
\text{Re}(\mathcal{P}_{a_h}^h u, u)_{L^2} \geq -C' h \| u \|_{L^2}^2.
\]

Proof. (4.8) results from Lemma 4.32 and Proposition 2.4. Therefore,

\[
\text{Re}(\mathcal{P}_{a_h}^h u, u)_{L^2} = \sum_{j \in \mathbb{N}} \text{Re}(\psi_j T_{a_h}^h \psi_j^* u, u)_{L^2} = \sum_{j \in \mathbb{N}} \text{Re}(T_{a_h}^h \psi_j^* u, \psi_j^* u)_{L^2} \geq -C h \sum_{j \in \mathbb{N}} \| \psi_j^* u \|_{L^2}^2 \geq -C' h \| u \|_{L^2}^2.
\]

\( \square \)

4.4. Link with Quasi-Homogeneous Wavefront Sets.

Lemma 4.35. Let \( a_h \in N_{-\infty}^{(r)} \cap \sigma_0 \), \( r > 1 + d/2 \), be elliptic at \((x_0, \xi_0) \in \mathbb{R}_x^d \times (\mathbb{R}_\xi^d \setminus 0)\), in the sense that, for some neighborhood \( \Omega \) of \((x_0, \xi_0)\),

\[
\inf_{0 < h < 1} \inf_{(x, \xi) \in \Omega} |a_h(x, \xi)| > 0.
\]

Let \( u \in L^2 \), and suppose that \( T_{a_h}^h u = \mathcal{O}(h^\sigma)_{L^2} \) with \( 0 \leq \sigma \leq r \), then \((x_0, \xi_0) \not\in \mathcal{WF}^\sigma_{0,1}(u)\).

Proof. Assume that \( \Omega \subset \mathbb{R}_x^d \times (\mathbb{R}_\xi^d \setminus 0) \). Let \( b_h \in S_{-\infty}^\infty \) with \( \text{supp} b_h \subset \Omega \). Then for some \( c_h \in N_{-\infty}^{(r)} \), \( T_{b_h} = T_{a_h}^h T_{b_h} + \mathcal{O}(h^r)_{L^2 - L^2} \). Therefore, \( T_{b_h} u = \mathcal{O}(h^\sigma)_{L^2} \), and by Lemma 4.32, \( \mathcal{O}_h(b_h) u = \mathcal{O}(h^\sigma)_{L^2} \).

\( \square \)

Lemma 4.36. Let \( \epsilon \geq 0 \), \( \epsilon \in \Gamma^{m,r}_{0,0} \) (if \( \epsilon = 0 \)) resp. \( \Gamma^{m,r}_{0,1} \) (if \( \epsilon > 0 \)), and suppose that \( e \) is homogeneous of degree \( m \) with respect to \( \xi \). Then for \( f \in H^s \) and \( 0 \leq \sigma \leq (1 + \epsilon)r \),

\[
\mathcal{WF}_{\epsilon,1}^{\delta - \sigma, -m}(\mathcal{P}_\epsilon f)^0 \subset \mathcal{WF}_{\epsilon,1}^{\delta + \sigma}(f)^0.
\]

If in addition \( e \) is elliptic, i.e., for some \( C > 0 \) and \( |\xi| \) sufficiently large, \( |e(x, \xi)| \geq C|\xi|^m \), then

\[
\mathcal{WF}_{\epsilon,1}^{\delta - \sigma, -m}(\mathcal{P}_\epsilon f)^0 = \mathcal{WF}_{\epsilon,1}^{\delta + \sigma}(f)^0.
\]

Recall the definition of \( \mathcal{WF}_{\delta,\rho}^\mu(u)^0 \) in §2.

Proof. For \( \mu \in \mathbb{R} \), denote \( Z^\mu = \mathcal{P}_{|\xi|^{\mu}} \). Then \( Z^{-\mu} Z^\mu - \text{Id} \in \mathcal{O}_-^\infty \). Therefore,

\[
f - Z^{-\mu} Z^\mu f \in H^\infty_{\epsilon,1}, \quad \mathcal{P}_\epsilon f - \mathcal{P}_{\epsilon,1}(|\xi|^{-\sigma} Z^\mu f) \in H^\infty_{\epsilon,1},
\]

where \( \delta = 0 \) if \( \epsilon = 0 \), while \( \delta = 1 \) if \( \epsilon > 0 \). By Lemma 2.16

\[
\mathcal{WF}_{\epsilon,1}^{\delta + \sigma}(f)^0 = \mathcal{WF}_{\epsilon,1}^{\delta, \sigma}(Z^\mu f)^0, \quad \mathcal{WF}_{\epsilon,1}^{\delta - \sigma, -m}(\mathcal{P}_\epsilon f)^0 = \mathcal{WF}_{\epsilon,1}^{\delta - (m-s)}(\mathcal{P}_{\epsilon,1}(|\xi|^{-\sigma} Z^\mu f)^0).
\]

So we may assume that \( s = 0 \). Let \( a, b \in S_{-\infty} \cap \sigma_\epsilon \), such that

\[
\text{supp} b \subset \{ a \not= 0 \} \subset \text{supp} a \subset \mathbb{R}_x^d \times \mathbb{R}_\xi^d \setminus \mathcal{WF}_{\epsilon,1}^\mu(f),
\]

then by Lemma 2.15, \( \mathcal{O}_{b}^{c,e}(a) f = \mathcal{O}(h^\sigma)_{L^2} \). By Corollary 4.31, Lemma 4.33, Proposition 4.26, and Corollary 4.28,

\[
h^m \mathcal{O}_{b}^{c,e}(a) \mathcal{P}_\epsilon f = \mathcal{O}_{b}^{c,e}(a) \mathcal{P}_\epsilon f + \mathcal{O}(h^\infty)_{L^2} = \mathcal{O}_{b}^{c,e}(a) \mathcal{P}_\epsilon f + \mathcal{O}_{b}^{c,e}(a) f + \mathcal{O}_{b}^{c,e}(a) f + \mathcal{O}(h^\infty)_{L^2} = \mathcal{O}(1)_{L^2 - L^2} \mathcal{O}_{b}^{c,e}(a) f + \mathcal{O}(h^r(1+\epsilon))_{L^2} = \mathcal{O}(h^\sigma)_{L^2},
\]

proving the first statement. The second statement follows by a construction of parametrix. \( \square \)
5. Asymptotically Flat Water Waves

In this section we prove Theorem 1.6. The idea is to combine the analysis in [1] with the dyadic paradifferential calculus on weighted Sobolev spaces. We shall use the following notations for simplicity. Let \( w \in L^\infty(\mathbb{R}^d) \) which is nowhere vanishing, then for \( A : \mathcal{S}' \to \mathcal{S}' \) and \( f \in \mathcal{S}' \), we denote \( A^{(w)} = w A^{-1}, f^{(w)} = w f \). Particularly, \( (A f)^{(w)} = A^{(w)} f^{(w)} \). For \( k \in \mathbb{R} \), we also denote, by an abuse of notation, \( A^{(k)} = A^{(\langle x \rangle^k)}, f^{(k)} = f^{(\langle x \rangle^k)} \), when there is no ambiguity. Observe that \( L_k^2 = H_k^0 \) is an Hilbert space with the inner product

\[
(f, g)_{L_k^2} = (f^{(k)}, g^{(k)})_{L^2}.
\]

5.1. Dirichlet-Neumann Operator. We study the Dirichlet-Neumann operator on weighted Sobolev spaces and its paralinearization. The time variable will be temporarily omitted for simplicity.

5.1.1. Boundary Flattening. Let \( \eta \in W^{1,\infty}(\mathbb{R}^d) \), such that

\[
\delta := b + \inf_{x \in \mathbb{R}^d} \eta(x) > 0,
\]

and define the Lipschitzian diffeomorphism

\[
\tau : \mathbb{R}^d_+ \times \mathbb{R}_+ \to \mathbb{R}^d_+ \times \mathbb{R}_+, \quad (x, z) \mapsto (x, z + \eta(x)).
\]

Set \( \hat{\Omega} = \tau^{-1}(\Omega), \hat{\Sigma} = \tau^{-1}(\Sigma), \hat{\Gamma} = \tau^{-1}(\Gamma) \), then

\[
\hat{\Omega} = \{ -b - \eta(x) < z < 0 \}, \quad \hat{\Sigma} = \{ z = 0 \}, \quad \hat{\Gamma} = \{ z = b - \eta(x) \}.
\]

Let \( \tau_\ast \) be the pullback induced by \( \tau \), then

\[
\tau_\ast(\text{d}x^2 + \text{d}y^2) = (\text{d}x \, \text{d}z) \varrho \left( \frac{\text{d}x}{\text{d}z} \right)
\]

with

\[
\varrho = \begin{pmatrix}
\text{Id} + (\nabla \eta)^\top \\
\nabla \eta \\
\nabla \eta^\top \\
1
\end{pmatrix}, \quad \varrho^{-1} = \begin{pmatrix}
\text{Id} \\
-\nabla \eta \\
-\nabla \eta^\top \\
1
\end{pmatrix}
\]

Let \( \nabla_{xz} = (\nabla, \partial_z) \), then the divergence, gradient and Laplacian operators with respect to the metric \( \varrho \) are

\[
div_{\varrho} u = (\nabla, \partial_z) \cdot u,
\]

\[
\nabla_{\varrho} u = (\nabla u - \nabla \eta \partial_z u, -\nabla \eta \cdot \nabla u + (1 + |\nabla \eta|^2) \partial_z u),
\]

\[
\Delta_{\varrho} u = \partial_z^2 u + (\nabla - \nabla \eta \partial_z)^2 u.
\]

And the exterior unit normal to \( \partial \hat{\Omega} = \hat{\Sigma} \cup \hat{\Gamma} \) is

\[
\textbf{n}_\varrho = \left( (D\tau)^{-1} \right)_{T \partial \hat{\Omega}, \varrho}, \textbf{n} = \begin{cases}
\frac{t(-\nabla \eta)^\top + (2|\nabla \eta|^2)}{\sqrt{1 + |\nabla \eta|^2}}, & \text{if } \hat{\Sigma}, \\
t(0, 1), & \text{if } \hat{\Gamma}.
\end{cases}
\]

Let \( \psi \in H^{1/2} \), and suppose that \( \phi \) satisfies the equation

\[
\Delta_{xy} \phi = 0, \quad \phi|_{\Sigma} = \psi, \quad \partial_n \phi|_{\Gamma} = 0,
\]

then \( v = (\tau|_{\hat{\Omega}})_\ast \phi \) satisfies

\[
\Delta_{\varrho} v = 0, \quad v|_{\hat{\Sigma}} = \psi, \quad \partial_{n_\varrho} v|_{\hat{\Gamma}} = 0.
\]

And

\[
\sqrt{1 + |\nabla \eta|^2} G(\eta) \psi = \partial_{n_\varrho} v|_{\hat{\Sigma}} = \textbf{n}_\varrho \cdot (\nabla_{xz} v)|_{z=0}.
\]
5.1.2. Elliptic Estimate. Let $\chi_0 \in C^\infty(\mathbb{R}_{z})$ with $\chi(z) = 0$ for $z \leq -\delta/2$ and $\chi(z) = 1$ for $z \geq 0$. Decompose $v = \tilde{v} + \psi$, where

$$\psi(x, z) = \chi_0(z)e^{z(D_x)}\psi(x).$$

**Lemma 5.1.** Let $n \geq 0$, $m \in \mathbb{R}$, $\mu \in \mathbb{R}$, $k \in \mathbb{R}$, $a \in S^m_0$, then

$$\|\partial^\mu_z \text{Op}(a)\psi\|_{L^2(\mathbb{R}_{z} \leq 0, H^\mu_k - \mu + n + 1/2)} \lesssim \|\psi\|_{H^\mu_k}.$$

**Proof.** We only prove the case with $n = 0$. The general case follows with similar arguments and the identity

$$\partial^\mu_z \psi(x, z) = \sum_{j=0}^{n} \binom{n}{j} \chi_0^{(n-j)}(z)(\partial^j z)\partial_z \psi(x).$$

Let $b(x, \xi) = a(x, \xi)(\xi)^{\mu - m} \in S^\mu_0$, $\lambda(z, \xi) = \chi_0(z)e^{z(\xi)}(\xi)^{1/2} \in L_{z \leq 0}S^1_0$, then $\forall N \geq 0$,

$$\|\text{Op}(a)\psi\|_{L^2(\mathbb{R}_{z} \leq 0, H^\mu_k - \mu + n + 1/2)} \lesssim \|\text{Op}(\lambda)\text{Op}(b)\psi\|_{L^2(\mathbb{R}_{z} \leq 0, L^2_k)} + \|\psi\|_{H^\mu_k}.$$

Observe that,

$$\text{Op}(\lambda)^{(k)} - (\text{Op}(\lambda)^{(k)})^* \in L_{z \leq 0}^{2} \Theta^{-1/2}_0, \hspace{1em} (\text{Op}(\lambda)^{(k)})^2 - \text{Op}(\lambda)^{(k)} \in L_{z \leq 0}^{\infty} \Theta^{0}_0,$$

and

$$\sigma(\xi) := \int_{-\infty}^{0} \lambda^2(z, \xi) \, dz = \langle \xi \rangle \int_{-\infty}^{0} \chi_0^2(z)e^{z(\xi)z} \, dz \in \Theta_0,$$

therefore,

$$\|\text{Op}(\lambda)\text{Op}(b)\psi\|_{L^2(\mathbb{R}_{z} \leq 0, L^2_k)}^2 = (\text{Op}(\lambda^2)\text{Op}(b)\psi, \text{Op}(b)\psi)_{L^2(\mathbb{R}_{z} \leq 0, L^2_k)} + \mathcal{O}(\|\psi\|_{H^\mu_k}^2)$$

$$= (\text{Op}(\sigma)\text{Op}(b)\psi, \text{Op}(b)\psi)_{L^2_k} + \mathcal{O}(\|\psi\|_{H^\mu_k}^2)$$

$$= \mathcal{O}(\|\psi\|_{H^\mu_k}^2).$$

**Lemma 5.2.** $\forall k \in \mathbb{R}$, $\|\tilde{v}\|_{H^k_1} \leq C(||\eta||_{W^{1, \infty}})\|\psi\|_{H^k_1}$.

**Proof.** Let $H^1_{0, 0}$ be the completion of the space

$$\{f \in C^\infty(\Omega) : f \text{ vanishes in a neighborhood of } \tilde{\Sigma}\}$$

with respect to the norm

$$\|u\|_{H^1_{0, 0}} := \|\nabla \varphi u\|_{L^2_0} = (\nabla \varphi u, \nabla \varphi u)_{L^2_0},$$

where $(X, Y)_{L^2_0} := \int_{\Omega} \varphi(X, Y) \, dx \, dz$. As $b < \infty$, by Poincaré inequality,

$$\|u\|_{L^2} \leq C(\|\eta\|_{L^\infty})\|\partial_z u\|_{L^2} \leq C(\|\eta\|_{W^{1, \infty}})\|u\|_{H^1_{0, 0}}, \hspace{1em} \forall u \in H^1_{0, 0}.$$

Let $0 < \zeta \in C^\infty(\mathbb{R})$ be such that $\zeta(z) = 1$ for $|z| \leq 1$, and $\zeta(z) = 0$ for $|z| \geq 2$. For some $R > 0$ sufficiently large to be determined later, set $w(x) = R \zeta(|x|^2/R)$. Then $(x)^{k} \lesssim w(x) \lesssim R(x)^{k}$, supp $\nabla w \subset \{x \geq R^{(k-1)/k}\}$, and $|\nabla w(x)| \lesssim R^{(k-1)/k}$.

Because $\tilde{v}$ satisfies the equation $\Delta_{\varphi} \tilde{v} = -\Delta_{\varphi} \psi$, we consider $\tilde{v}(w)$ as the variational solution to the equation

$$B(\tilde{v}(w), \cdot) = -L(\cdot),$$

where for $u, \varphi \in H^1_{0, 0}$,

$$B(u, \varphi) = \left(\nabla \varphi u, \nabla \varphi w\right)_{L^2_0}, \hspace{1em} L(\varphi) = \left(\nabla \varphi \psi(w), \nabla \varphi w\right)_{L^2_0}.$$

Observe that $\nabla_{\varphi}(w^{k+1}) = \nabla_{\varphi} + b_w$, where $b_w = (w^{-1}\nabla w, -\nabla \eta \cdot w^{-1}\nabla w) \in L^\infty$ satisfies

$$\|b_w\| \leq C(\|\eta\|_{W^{1, \infty}})R^{-1/k}.$$
We easily verify that $L$ and $B$ are continuous linear and bilinear forms on $H^{1,0}_δ$. Moreover $B$ is coercive when $R$ is sufficiently large, indeed,

$$B(\varphi, \varphi) = \|\nabla_δ \varphi\|_{L^2_\delta}^2 - \|b_\omega \varphi\|_{L^2_\delta}^2 \geq (1 - C(\|\eta\|_{W^{1,\infty}})R^{-2/k})\|\nabla_\delta \varphi\|_{L^2_\delta}^2.$$ 

Therefore, by Lax-Milgram’s Theorem and Lemma 5.1,

$$\|\overline{\varphi}\|_{H^1_k} \lesssim \|\overline{\varphi}^{(\omega)}\|_{H^{1,0}_k} \lesssim \|L\|_{(H^{1,0}_k)^*} \lesssim \|\psi\|_{H^1} \lesssim \|\psi\|_{H^{1,0}_k}.$$ 

\qed

**Theorem 5.3.** Suppose that $(\eta, \psi) \in W^{1, \infty} \times H^{1/2}_k$ with $k \in \mathbb{R}$, then

$$\|G(\eta)\psi\|_{H^{-1/2}_k} \leq C(\|\eta\|_{W^{1, \infty}})\|\psi\|_{H^{1/2}_k}.$$ 

**Proof.** By Lemma 5.1 and Lemma 5.2,

$$v \in L^2([-1, 0], H^1_k) \cap H^1([-1, 0], L^2_\delta).$$

By interpolation,

$$v \in C^0([-1, 0], H^{1/2}_k) \cap C^1([-1, 0], H^{-1/2}_k).$$ 

\qed

5.1.3. **Higher Regularity.**

**Proposition 5.4.** Suppose that $(\eta, \psi) \in H^{\mu+1/2} \times H^{\mu}_k$ with $\mu > 1/2 + d/2$, $1/2 \leq \sigma \leq \mu$ and $k \in \mathbb{R}$, then

$$\|G(\eta)\psi\|_{H^{\mu-1}_k} \leq C(\|\eta\|_{H^{\mu+1/2}})\|\psi\|_{H^{\mu}_k}.$$ 

Consequently, if $(\eta, \psi) \in H^{\mu+1/2} \times H^{\sigma}_m$ with $\sigma - m/2 \geq 1/2$, then

$$\|G(\eta)\psi\|_{H^{\sigma-1}_m} \leq C(\|\eta\|_{H^{\mu+1/2}})\|\psi\|_{H^{\sigma}_m}.$$ 

**Proof.** Let $\Lambda = \langle D_x \rangle^{\sigma-1/2}$, and let $\chi, \check{\chi} \in C^\infty([-\delta, \delta])$, where $\delta$ is defined by (5.1), such that $\chi(0) \neq 0$ and $\check{\chi}(z) = 1$ for $z \in \text{supp } \chi$. Then we have the following equation for $\overline{\varphi}$,

$$-\Delta_\delta (\chi \Lambda \overline{\varphi}) + K(\check{\chi} \Lambda \overline{\varphi}) = \Delta_\delta (\chi \Lambda \varphi) - K(\check{\chi} \Lambda \varphi).$$

where $K = [\Delta_\delta, \chi \Lambda]^{-1}$. Observe that $\Delta_\delta = P \cdot P$ with $P = \nabla \varphi - \nabla \eta \partial_\sigma$, we expand

$$K = P \cdot Q + Q \cdot P + Q, \quad Q = [P, \chi \Lambda]^{-1}.$$ 

By Kato-Ponce commutator estimates, we verify that

$$\|Q\|_{H^1 L^2_k \to L^2 L^2_k} \leq C(\|\varphi\|_{H^{\mu+1/2}}).$$ 

Then observe that $\langle x \rangle^k P(\langle x \rangle^{-k}) - P = \alpha \cdot \nabla \varphi$ with $\alpha \in L^\infty$, we obtain the estimate

$$\| (Kf, g)_{L^2 L^2_k} \| \lesssim \|\nabla \varphi \|_{L^2 L^2_k} \|g\|_{H^1 L^2_k} + \|f\|_{H^1 L^2_k} \|\nabla \varphi \|_{L^2 L^2_k}.$$ 

Suppose that we have already proven that

$$\overline{\varphi} \in L^2([-\delta, 0], H^{\sigma-1/2}_k) \cap H^1([-\delta, 0], H^{\sigma-3/2}_k),$$

with the norm denoted by $N_\sigma$. Then,

$$\| (K(\chi \Lambda \overline{\varphi}), \chi \Lambda \varphi)_{L^2 L^2_k} \| \lesssim \|\chi \Lambda \overline{\varphi}\|_{H^1 L^2_k} \|\chi \Lambda \varphi\|_{H^1 L^2_k} + \|\chi \Lambda \overline{\varphi}\|_{H^1 L^2_k} \|\chi \Lambda \varphi\|_{H^1 L^2_k} \lesssim N_\sigma \|\chi \Lambda \overline{\varphi}\|_{H^1 L^2_k}.$$ 

$$\| (K(\check{\chi} \Lambda \overline{\varphi}), \chi \Lambda \varphi)_{L^2 L^2_k} \| \lesssim \|\check{\chi} \Lambda \overline{\varphi}\|_{H^1 L^2_k} \|\chi \Lambda \varphi\|_{H^1 L^2_k} + \|\check{\chi} \Lambda \overline{\varphi}\|_{H^1 L^2_k} \|\chi \Lambda \varphi\|_{H^1 L^2_k} \lesssim \|\check{\chi} \Lambda \varphi\|_{H^1 L^2_k} \|\chi \Lambda \varphi\|_{H^1 L^2_k}.$$ 

$$\| (\Delta_\delta (\chi \Lambda \overline{\varphi}), \chi \Lambda \varphi)_{L^2 L^2_k} \| \lesssim \|P(\chi \Lambda \overline{\varphi})\|_{L^2 L^2_k} \|P(\chi \Lambda \varphi)\|_{L^2 L^2_k} + \|\chi \Lambda \overline{\varphi}\|_{L^2 L^2_k} \|\chi \Lambda \varphi\|_{L^2 L^2_k} \|\chi \Lambda \varphi\|_{L^2 L^2_k} \|\nabla \varphi\|_{L^2 L^2_k} \lesssim \|\chi \Lambda \overline{\varphi}\|_{H^1 L^2_k} \|\chi \Lambda \varphi\|_{L^2 L^2_k} \|\chi \Lambda \varphi\|_{L^2 L^2_k} \|\nabla \varphi\|_{L^2 L^2_k} \lesssim N_\sigma \|\chi \Lambda \overline{\varphi}\|_{H^1 L^2_k} \|\chi \Lambda \varphi\|_{L^2 L^2_k}.$$ 

$$\| (\Delta_\delta (\chi \Lambda \overline{\varphi}), \chi \Lambda \varphi)_{L^2 L^2_k} \| \lesssim \|P(\chi \Lambda \overline{\varphi})\|_{L^2 L^2_k} \|P(\chi \Lambda \varphi)\|_{L^2 L^2_k} + \|\chi \Lambda \overline{\varphi}\|_{L^2 L^2_k} \|\chi \Lambda \varphi\|_{L^2 L^2_k} \|\chi \Lambda \varphi\|_{L^2 L^2_k} \|\nabla \varphi\|_{L^2 L^2_k} \lesssim N_\sigma \|\chi \Lambda \overline{\varphi}\|_{H^1 L^2_k} \|\chi \Lambda \varphi\|_{L^2 L^2_k}.$$
Combing the estimates above, by Lemma 5.1, we have \( \forall \varepsilon > 0 \)
\[
\| \tilde{\chi} \Lambda \tilde{v} \|_{H^1 H^1_k}^2 \lesssim \varepsilon \| \tilde{\chi} \Lambda \tilde{v} \|_{H^1 H^1_k}^2 + \varepsilon^{-1} N^2_{\delta} + \varepsilon^{-1} \| \psi \|_{H^2_k}^2.
\]
By choosing \( \varepsilon \) sufficiently small, this \textit{a priori} estimate implies that
\[
\tilde{v} \in L^2([-\delta, 0], H^{\sigma+1/2}_k) \cap H^1([-\delta, 0], H^{\sigma-1/2}_k).
\]
Therefore
\[
v = \tilde{v} + \psi \in L^2([-\delta, 0], H^{\sigma+1/2}_k) \cap H^1([-\delta, 0], H^{\sigma-1/2}_k).
\]
By interpolation,
\[
v \in C^0([\delta, 0], H^2_{\sigma}) \cap C^1([-\delta, 0], H^{\sigma-1}).
\]
\[\square\]

5.2. Paralinearization. Now we paralinearize the system of water waves. The following results can be proven directly by combining the analysis in [1] and our dyadic paradiﬀerential calculus, mainly Proposition 4.19, Proposition 4.21, and Proposition 4.22. We shall work in the Sobolev spaces \( \mathcal{H}_m^\sigma \), recalling Deﬁnition 1.5.

**Proposition 5.5.** Let \( (\eta, \psi) \in \mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^\sigma \) with \( \mu > 3 + d/2, m < 2\mu - 6 - d \). Set \( \lambda = \lambda^{(1)} + \lambda^{(0)} \), where
\[
\lambda^{(1)}(x, \xi) = \sqrt{(1 + |\nabla \eta|^2)|\xi|^2 - (\nabla \eta \cdot \xi)^2},
\]
\[
\lambda^{(0)}(x, \xi) = \frac{1 + |\nabla \eta|^2}{2\lambda^{(1)}} \left\{ \nabla \cdot \left( \alpha^{(1)}(1) \nabla \eta \right) + i \partial_\xi \lambda^{(1)} \cdot \nabla \alpha^{(1)} \right\},
\]
with \( \alpha^{(1)}(x, \xi) = \frac{\lambda^{(1)} + i \nabla \eta \cdot \xi}{1 + |\nabla \eta|^2} \). Then
\[
G(\eta) \psi = \mathcal{P}_\lambda(\psi - \mathcal{P}_B \eta) - \mathcal{P}_V \cdot \nabla \eta + R(\eta, \psi),
\]
where
\[
B = \frac{\nabla \eta \cdot \nabla \psi + G(\eta) \psi}{1 + |\nabla \eta|^2}, \quad V = \nabla \psi - B \nabla \eta.
\]
and \( R(\eta, \psi) \in \mathcal{H}_m^\mu+1/2 \).

We shall denote \( \omega = \psi - \mathcal{P}_B \eta \), which is called the good unknown of Alinhac.

**Proof.** We only sketch the proof, for the key ingredients are already given in [1]. Let \( v \) be deﬁned as in §5.1, and set \( u = v - \mathcal{P} \partial_n \eta \). By (5.2), Proposition 4.21, and [1],
\[
\mathcal{P}_\alpha \partial^2_n u + \Delta u + \mathcal{P}_\beta \cdot \nabla \partial_n u - \mathcal{P}_\gamma \partial_\xi u \in \mathcal{H}_m^\mu,
\]
where \( \alpha = 1 + |\nabla \eta|^2, \beta = -2 \nabla \eta, \gamma = \Delta \eta \). Then we ﬁnd symbols \( a_{\pm} = a_{\pm}^{(1)} + a_{\pm}^{(0)} \), whose explicit expressions are given later in Proposition 6.2, such that
\[
(\partial_\xi - \mathcal{P}_{a_-})(\partial_\xi - \mathcal{P}_{a_+})u \in \mathcal{H}_m^\mu.
\]
The exact same proof as Proposition 3.19 in [1] implies that
\[
(\partial_z u - \mathcal{P}_{a_+ z})|_{z = 0} \in \mathcal{H}_m^{\mu+1/2}.
\]
We conclude by setting \( \lambda = (1 + |\nabla \eta|^2)a_+ - i \nabla \eta \cdot \xi \).

The proofs of following results are in the same spirit and simpler, and we shall omit them.

**Proposition 5.6.** Let \( \mathcal{H}_m^{\mu+1/2} \) with \( \mu > 3 + d/2 \) and \( m < 2\mu - 6 - d/2 \), then
\[
H(\eta) = -\mathcal{P}_\ell \eta + f(\eta),
\]
where \( \ell = \ell^{(2)} + \ell^{(1)} \) is deﬁned by
\[
\ell^{(2)} = \frac{(1 + |\nabla \eta|^2)|\xi|^2 - (\nabla \eta \cdot \xi)^2}{(1 + |\nabla \eta|^2)^{3/2}}, \quad \ell^{(1)} = \frac{1}{2} \partial_\xi \cdot D_\xi \ell^{(2)},
\]
and \( f(\eta) \in \mathcal{H}_m^{2\mu-2-d/2} \).
Lemma 5.7. Let \((\eta, \psi) \in \mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^\mu\), with \(\mu > 3 + d/2\) and \(m < 2\mu - 6 - d/2\), then
\[
\frac{1}{2} |\nabla \psi|^2 - \frac{1}{2} \frac{(\nabla \psi \cdot \nabla \psi + G(\eta)\psi)^2}{1 + |\nabla \eta|^2} = \mathcal{P}_\gamma \cdot \nabla \psi - \mathcal{P}_B \mathcal{P}_V \cdot \nabla \eta - \mathcal{P}_B G(\eta)\psi + f(\eta, \psi),
\]
where \(f(\eta, \psi) \in \mathcal{H}_m^{2\mu-2-d/2}\).

Proposition 5.8. Let \((\eta, \psi) \in \mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^\mu\), with \(\mu > 3 + d/2\) and \(m < 2\mu - 6 - d/2\), then \((\eta, \psi)\) solves the water wave equation if and only if
\[
(\partial_t + \mathcal{P}_V \cdot \nabla + \mathcal{L}) \left( \begin{array}{c} \eta \\ \psi \end{array} \right) = f(\eta, \psi)
\]
where
\[
\mathcal{L} = Q^{-1} \left( \begin{array}{cc} 0 & -\mathcal{P}_\gamma \\ \mathcal{P}_\gamma & 0 \end{array} \right) Q, \quad \text{with} \quad Q = \left( \begin{array}{cc} \text{id} & 0 \\ -\mathcal{P}_B & \text{id} \end{array} \right),
\]
and \(f(\eta, \psi) = Q^{-1}(f_1, f_2) \in \mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^\mu\) is defined by
\[
\begin{align*}
f_1 &= G(\eta)\psi - \{\mathcal{P}_\lambda(\psi - \mathcal{P}_B \eta) - \mathcal{P}_V \cdot \nabla \eta\}, \\
f_2 &= -\frac{1}{2} |\nabla \psi|^2 + \frac{1}{2} \frac{(\nabla \psi \cdot \nabla \psi + G(\eta)\psi)^2}{1 + |\nabla \eta|^2} + H(\eta) \\
&\quad + \mathcal{P}_V \cdot \nabla \psi - \mathcal{P}_B \mathcal{P}_V \cdot \nabla \eta - \mathcal{P}_B G(\eta)\psi + \mathcal{P}_B \eta - \gamma n.
\end{align*}
\]

5.3. Symmetrization.

Definition 5.9. For \(T > 0\), \(\gamma \in \mathbb{R}\) and two operators \(\mathcal{A}, \mathcal{B} \in L^\infty([0, T], \mathcal{G}_0^{3/2})\), we say that \(\mathcal{A} \sim \mathcal{B}\), or simply \(\mathcal{A} \sim \mathcal{B}\) when there is no ambiguity of the choice of \(\gamma\), if
\[
\mathcal{A} - \mathcal{B} \in L^\infty([0, T], \mathcal{G}_0^{\gamma-3/2}).
\]

According to [1], there exists symbols which depend solely on \(\eta\),
\[
\gamma = \gamma^{(3/2)} + \gamma^{(1/2)}, \quad p = p^{(1/2)} + p^{(-1/2)}, \quad q = q^{(0)},
\]
whose principal symbols being explicitly
\[
\gamma^{(3/2)} = \sqrt{\ell(2)\lambda(1)}, \quad p^{(1/2)} = (1 + |\nabla \eta|^2)^{-1/2} \sqrt{\lambda(1)}, \quad q^{(0)} = (1 + |\nabla \eta|^2)^{1/4},
\]
such that
\[
\mathcal{P}_p \mathcal{P}_\lambda \sim \frac{1}{2} \mathcal{P}_\gamma \mathcal{P}_q, \quad \mathcal{P}_q \mathcal{P}_\ell \sim \frac{1}{2} \mathcal{P}_\gamma \mathcal{P}_p, \quad \mathcal{P}_\gamma \sim \frac{1}{2} (\mathcal{P}_\gamma)^*.
\]

Define the symmetrizer
\[
S = \left( \begin{array}{cc} \mathcal{P}_p & 0 \\ 0 & \mathcal{P}_q \end{array} \right) Q,
\]
then
\[
(5.3) \quad \mathcal{S} \mathcal{L} \sim \left( \begin{array}{cc} 0 & -\mathcal{P}_\gamma \\ \mathcal{P}_\gamma & 0 \end{array} \right) S,
\]
where the equivalence relation is applied separately to each component of the matrices.

5.4. Approximate System. Set the mollifier \(J_\varepsilon = \mathcal{P}_{j_\varepsilon}\) where \(j_\varepsilon = j_\varepsilon^{(0)} + j_\varepsilon^{(-1)}\),
\[
j_\varepsilon^{(0)} = \exp(-\varepsilon \gamma^{(3/2)}), \quad j_\varepsilon^{(-1)} = \frac{1}{2} \partial_\xi \cdot D_x j_\varepsilon^{(0)}.
\]

Then we have, uniformly for \(\varepsilon > 0\),
\[
J_\varepsilon \mathcal{P}_\gamma \sim \frac{1}{2} \mathcal{P}_\gamma J_\varepsilon, \quad J_\varepsilon^* \sim_0 J_\varepsilon.
\]

Let \(\bar{p} = \bar{p}^{(-1/2)} + \bar{p}^{(-3/2)}\) with
\[
\bar{p}^{(-1/2)} = 1/p^{(1/2)}, \quad \bar{p}^{(-3/2)} = -(\bar{p}^{(-1/2)} p^{(-1/2)} + \frac{1}{i} \partial_\xi \bar{p}^{(-1/2)} \cdot \partial_x p^{(1/2)}) / p^{(1/2)},
\]
then
\[
\mathcal{P}_p \bar{p} \sim_0 \text{id}, \quad \mathcal{P}_q \mathcal{P}_1 q \sim_0 \text{id}.
\]
Finally we define
\[ \mathcal{L}_\varepsilon = \mathcal{L}Q^{-1} \left( \begin{array}{cc} P_p J_\varepsilon P_p & 0 \\ 0 & P_{1/q} J_\varepsilon P_q \end{array} \right) Q, \]
and the approximate system
\[ (\partial_t + P_V \cdot \nabla J_\varepsilon + \mathcal{L}_\varepsilon) \frac{\eta}{\psi} = f(J_\varepsilon \eta, J_\varepsilon \psi). \] (5.4)
A key identity of the operator \( \mathcal{L}_\varepsilon \) is that
\[ S \mathcal{L}_\varepsilon \sim \begin{pmatrix} 0 & -\mathcal{P}_\gamma \\ \mathcal{P}_\gamma & 0 \end{pmatrix} J_\varepsilon S. \]

5.5. A Priori Estimate.

Proposition 5.10. Let \((\eta, \psi) \in C^1([0, T], \mathcal{H}_{2m}^{\mu+1/2} \times \mathcal{H}_{2m}^{\mu})\) with \(\mu > 3 + d/2\) and \(m < 2\mu - 6 - d/2\) solve the approximate system (5.4). Define
\[ M_T = \sup_{0 \leq t \leq T} \| (\eta, \psi)(t) \|_{\mathcal{H}_{2m}^{\mu+1/2} \times \mathcal{H}_{2m}^{\mu}}, \quad M_0 = \| (\eta, \psi)(0) \|_{\mathcal{H}_{2m}^{\mu+1/2} \times \mathcal{H}_{2m}^{\mu}}. \]
Then
\[ M_T \leq C(M_0) + TC(M_T) \]
for some non-decaying function \(C : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}\).

Proof. For \(0 \leq k \leq m\), set
\[ M_k^0 = \sup_{0 \leq t \leq T} \| (\eta, \psi)(t) \|_{\mathcal{H}_k^{\mu+1/2-k} \times \mathcal{H}_k^{\mu-k/2}}, \quad M_k^0 = \| (\eta, \psi)(0) \|_{\mathcal{H}_k^{\mu+1/2-k} \times \mathcal{H}_k^{\mu-k/2}}. \]
By [1],
\[ M_k^0 \leq C(M_0^0) + TC(M_T^0). \]
It remains to prove that for \(1 \leq k \leq m\),
\[ M_k^T \leq C(M_0^T) + TC(M_T). \]
To do this, let \(\Lambda_k^\mu \equiv \mathcal{P}_{m_k^{\mu-k/2}}\), and set \(\Phi = \Lambda_k^\mu S(\eta/\psi)\). Then
\[ (\partial_t + P_V \cdot \nabla J_\varepsilon) \Phi = \begin{pmatrix} 0 & -\mathcal{P}_\gamma \\ \mathcal{P}_\gamma & 0 \end{pmatrix} J_\varepsilon \Phi = F_\varepsilon \]
where \(F_\varepsilon = F_\varepsilon^1 + F_\varepsilon^2 + F_\varepsilon^3\), with
\[ F_\varepsilon^1 = \Lambda_k^\mu S f(J_\varepsilon \eta, J_\varepsilon \psi), \]
\[ F_\varepsilon^2 = [\partial_t + P_V \cdot \nabla J_\varepsilon, \Lambda_k^\mu S] \left( \frac{\eta}{\psi} \right), \]
\[ F_\varepsilon^3 = \begin{pmatrix} 0 & -\mathcal{P}_\gamma \\ \mathcal{P}_\gamma & 0 \end{pmatrix} J_\varepsilon \Lambda_k^\mu S \left( \frac{\eta}{\psi} \right) - \Lambda_k^\mu S \mathcal{L}_\varepsilon \left( \frac{\eta}{\psi} \right). \]
By Proposition 5.8, Proposition 5.5, Proposition 5.6, and Lemma 5.7,
\[ \| f(J_\varepsilon \eta, J_\varepsilon \psi) \|_{\mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^{\mu}} \leq C(\| (J_\varepsilon \eta, J_\varepsilon \psi) \|_{\mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^{\mu}} \leq C(\| (\eta, \psi) \|_{\mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^{\mu}}), \]
therefore,
\[ \| F_\varepsilon^1 \|_{L^\infty([0, T], L^2)} \leq C(M_T). \]
Then we estimate
\[ \| [\partial_t + P_V \cdot \nabla J_\varepsilon, \Lambda_k^\mu S] \|_{L^\infty([0, T], H_k^{\mu+1/2-k/2} \times H_k^{\mu-k/2} \to L^2 \times L^2)} \leq C(M_T), \]
which implies
\[ \| F_\varepsilon^2 \|_{L^\infty([0, T], L^2)} \leq C(M_T). \]
Unfortunately, the operator
\[ R := \begin{pmatrix} 0 & -\mathcal{P}_\gamma \\ \mathcal{P}_\gamma & 0 \end{pmatrix} J_\varepsilon \Lambda_k^\mu S - \Lambda_k^\mu S \mathcal{L}_\varepsilon \]

\[ \begin{pmatrix} 0 & -P_\gamma \\ P_\gamma & 0 \end{pmatrix} J_x[A_\mu^+, S] + [S L_\varepsilon, A_\mu^+] + \left( \begin{pmatrix} 0 & -P_\gamma \\ P_\gamma & 0 \end{pmatrix} J_x S - S L_\varepsilon \right) A_\mu^+ \]

do not send \( H_\mu^{k+1/2-k/2} \times H_\mu^{k-1/2} \) to \( L^2 \) because of the sub-principal symbols do not vanish in the symbolic calculus, due to the existence of the commutators with \( \Lambda_\mu^+ \). However, by Proposition 4.19,
\[
\left\| R(\eta, \psi) \right\|_{L^2 \times L^2} \lesssim \left\| (\eta, \psi) \right\|_{H_\mu^{k+1/2} \times H_\mu^{k+1/2-k/2} + \left\| (\eta, \psi) \right\|_{H_\mu^{k+1/2-k/2} \times H_\mu^{k-1/2}}.
\]
So when \( k \geq 1 \) we still have
\[
\left\| F_k^3 \right\|_{L^\infty([0, T], L^2)} \leq C(M_T).
\]
Finally by an exact same energy estimate as in [1], we deduce that
\[
M_T^k \lesssim \left\| \Phi \right\|_{L^\infty([0, T], L^2)} \leq C(M_0^n) + TC(M_T).
\]

5.6. Existence.

**Lemma 5.11.** For all \((\eta_0, \psi_0) \in \mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^{\mu} \) and all \( \varepsilon > 0 \), the Cauchy problem of the approximate system (5.4) has a unique maximal solution
\[
(\eta, \psi) \in C([0, T], \mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^{\mu}).
\]
Moreover, \( \exists T_\varepsilon > 0 \) such that
\[
\inf_{\varepsilon \in [0, 1]} T_\varepsilon \geq T_0.
\]

**Proof.** Following [1], the existence follows from the existence theory of ODEs by writing (5.4) in the compact form
\[
\partial_t X = \mathcal{F}_\varepsilon(X),
\]
where \( \mathcal{F}_\varepsilon \) is a Lipschitz map on \( \mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^{\mu} \). The only nontrivial term is the Dirichlet-Neumann operator, whose regularity follows by combining Proposition 5.4 and the shape derivative formula (which goes back to Zakharov [38]),
\[
\langle dG(\eta, \psi), \varphi \rangle := \lim_{h \to 0} \frac{1}{h} (G(\eta + h \varphi) - G(\eta)) \psi = -G(\eta)(B \varphi) - \nabla_x \cdot (V \varphi).
\]
A standard abstract argument then shows that \( T_\varepsilon \) has a strictly positive lower bound, we refer to [1] for more details. \( \square \)

**Proof of Theorem 1.6.** By Lemma 5.11, we obtain a sequence \( \{(\eta, \psi)\} \) which satisfies (5.4) and is uniformly bounded in \( L^\infty([0, T], \mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^{\mu}) \) for some \( T > 0 \). By (5.4), \( \{(\partial_t \eta, \partial_t \psi)\} \) is uniformly bounded in \( L^\infty([0, T], \mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^{\mu-1/2}) \). By [1], there exists
\[
(\eta, \psi) \in C([0, T], H_\mu^{\mu+1/2} \times H_\mu^{\mu})
\]
which solves (1.7), such that as \( \varepsilon \to 0 \), \( (\eta, \psi) \to (\eta, \psi) \) weakly in \( L^2([0, T], \mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^{\mu}) \), and strongly in \( C([0, T], \mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^{\mu-1/2}) \). We then prove that for \( 1 \leq k \leq m \),
\[
\Phi = \Phi(\eta, \psi) := \Lambda_\mu^+ S(\eta, \psi) \left( \begin{pmatrix} \eta \\ \psi \end{pmatrix} \right)
\]
lies in \( C([0, T], L^2) \), where \( \Lambda_\mu^+ \) is defined in Proposition 5.10, and \( S = S(\eta, \psi) \) is the symmetrizer. Up to an extract of a subsequence, we may assume by weak convergence that
\[
(\eta, \psi) \in L^\infty([0, T], \mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^{\mu}),
(\partial_t \eta, \partial_t \psi) \in L^\infty([0, T], \mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^{\mu-1/2}),
\]
with
\[
\left\| (\eta, \psi) \right\|_{L^\infty([0, T], \mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^{\mu}) \cap W^{1, \infty}([0, T], \mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^{\mu-1/2})} \leq C\left( \left\| (\eta_0, \psi_0) \right\|_{\mathcal{H}_m^{\mu+1/2} \times \mathcal{H}_m^{\mu}} \right).
\]
This already implies that \((\eta, \psi)\) is weakly continuous in \(H^{\mu+1/2}_m \times H^\mu_m\). By the analysis in the previous section,
\[
(\partial_t + \mathcal{P}_V \cdot \nabla)\Phi + \begin{pmatrix} 0 & -\mathcal{P}_\gamma \\ \mathcal{P}_\gamma & 0 \end{pmatrix} \Phi = F,
\]
with
\[
\|F\|_{L^\infty([0,T],L^2)} \leq C(\|\eta_0, \psi_0\|_{H^{\mu+1/2}_m \times H^\mu_m}).
\]
Let \(J_h = \text{Op}_h(e^{-h^2|x|^2-|\xi|^2})\). Now that \(e^{-h^2|x|^2-|\xi|^2} \in S^0\), we have the commutator estimate
\[
[J_h, \mathcal{P}_V \cdot \nabla] = O(1)_{\mathcal{O}^{-1}}, \quad [J_h, \mathcal{P}_\gamma] = O(1)_{\mathcal{O}^{1/2}}.
\]
Because \(k \geq 1\), by the same spirit of estimating \(R\) in Proposition 5.10, we obtain the following energy estimate
\[
\frac{d}{dt} \|J_h \Phi(t)\|_{L^2}^2 \leq C(\|\eta_0, \psi_0\|_{H^{\mu+1/2}_m \times H^\mu_m}).
\]
Therefore, \(t \mapsto \|J_h \Phi(t)\|_{L^2}^2\) are uniformly Lipschitzian. Consequently, by Arzelà-Ascoli theorem, \(t \mapsto \|\Phi(t)\|_{L^2}^2\) is continuous, because \(J_h \Phi \to \Phi\) as \(h \to 0\). Combining the weak continuity, we induce by functional analysis that \(\Phi \in C([0,T],L^2)\). By (5.5), the paradifferential calculus, and the definition of \(\Phi\), we easily induce that
\[
(\eta, \psi) \in C([0,T],H^{\mu+1/2}_m \times H^\mu_m).
\]
Thus we finish the proof of Theorem 1.6.

\[
\therefore
\]

6. Propagation of Singularities for Water Waves

6.1. Finer Paralinearization and Symmetrization. To study the propagation of singularities, we need much finer results of paralinearization and symmetrization than Proposition 5.5 and Proposition 5.8 so as to gain regularities in the remainder terms.

**Proposition 6.1.** Let \((\eta, \psi) \in H^{\mu+1/2} \times H^\mu\), with \(\mu > 3 + d/2\). Then \(\exists \lambda \in \Sigma^{1,\mu-1/2-d/2}\), such that
\[
G(\eta)\psi = \mathcal{P}_\lambda(\psi - \mathcal{P}_B \eta) - \mathcal{P}_V \cdot \nabla \eta + R(\eta, \psi),
\]
where \(R(\eta, \psi) \in H^{2\mu-3-d/2}\).

**Proof.** This theorem follows by replacing the usual paradifferential calculus with the dyadic paradifferential calculus in the analysis of [5]. In [5], the explicit expression for \(\lambda\) is given. We write it down for the sake of later applications.
\[
\lambda = (1 + |\nabla \eta|^2)a_+ - i\nabla \eta \cdot \xi,
\]
where \(a_\pm = \sum_{j \leq 1} a^{(j)}_\pm\) is defined as follows. Setting \(c = \frac{1}{1+|\nabla \eta|^2}\), then
\[
a_\pm = \begin{pmatrix} 1 \\ 0 \\ \frac{\partial_k \xi a_\pm^{(1)} - \partial_k a_\pm^{(1)} - c \Delta \eta a_\pm^{(1)}}{a_\pm^{(1)}} \end{pmatrix},
\]
\[
\lambda = \begin{pmatrix} ic \nabla \eta \cdot \xi - \sqrt{c} |\xi|^2 - (c \nabla \eta \cdot \xi)^2 \\ ic \nabla \eta \cdot \xi + \sqrt{c} |\xi|^2 - (c \nabla \eta \cdot \xi)^2 \\ ic \nabla \eta \cdot \xi \end{pmatrix},
\]
\[
\mathcal{P}_\lambda(\psi - \mathcal{P}_B \eta) \in H^{2\mu-3-d/2}.
\]
Suppose that \(a^{(j)}_\pm\) are defined for \(m \leq j \leq 1\), then we define
\[
a_\pm^{(m-1)} = \begin{pmatrix} 1 \\ -a_\pm^{(1)} \end{pmatrix},
\]
\[
a_\pm^{(0)} = \begin{pmatrix} 1 \\ -a_\pm^{(1)} \end{pmatrix} - \frac{1}{a_\pm^{(1)}} \sum_{m \leq k \leq 1} \sum_{m \leq \ell \leq 1} \sum_{|a|=k+\ell-m} \frac{1}{a_\pm^{(1)}} \partial_\xi a_\pm^{(k)} D_x a_\pm^{(\ell)}.
\]
The principal and sub-principal symbols of \(\lambda\) clearly coincide with the ones given by Proposition 5.5.

\[
\therefore
\]
Proposition 6.2. Set \( w = \Lambda^\mu US(\psi/\psi) \), where \( \Lambda^\mu = \mathcal{P}_{(\gamma^{(3/2)})^{\mu/3}} \) and
\[
U = \begin{pmatrix}
-\iota & 1 \\
\iota & 1
\end{pmatrix}.
\]
Then, for some \( Q \in \mathcal{M}_{2 \times 2}(\Sigma^{0,\mu-5/2}), \) \( \zeta \in \Sigma^{-1/2,\mu-5/2}, \)
\[
(6.1) \quad (\partial_t + \mathcal{P}_V \cdot \nabla + \mathcal{P}_Q)w + i\mathcal{P}_\gamma \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} w + \frac{ig}{2} \mathcal{P}_{\zeta} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \in H^{\mu-4-d/2}.
\]

Remark 6.3. Because \( \chi \) in the definition of paradifferential operators is an even function, we easily verify that \( \Lambda^\mu, \) \( \mathcal{P}_p, \) \( \mathcal{P}_q, \) \( \mathcal{P}_B \) all map real-valued functions to real-valued functions. Therefore, \( w = \binom{\eta}{\psi} \) with
\[
(6.2) \quad u = \Lambda^\mu(-\iota,1)S\binom{\eta}{\psi} = \Lambda^\mu\mathcal{P}_p\eta - i\Lambda^\mu\mathcal{P}_q\omega,
\]
recalling that \( \omega = \psi - \mathcal{P}_B\eta \) is the good unknown of Alinhac.

Proof. Combining Proposition 6.1 and Proposition 5.8, moving the term \( g\eta \) to the left hand side,
\[
(\partial_t + \mathcal{P}_V \cdot \nabla + \mathcal{L})\binom{\eta}{\psi} = f(\eta,\psi),
\]
where \( f(\eta,\psi) = Q^{-1}(f_1)(\eta) \in H^{2\mu-7/2-d/2} \times H^{2\mu-4-d/2} \) is defined by
\[
f_1 = G(\eta)\psi - \{\mathcal{P}_\gamma(\psi - \mathcal{P}_B\eta) - \mathcal{P}_V \cdot \nabla\eta\},
\]
\[
f_2 = -\frac{1}{2} |\nabla\psi|^2 + \frac{1}{2} \frac{\nabla^2 \nabla \cdot \nabla\psi + G(\eta)\psi^2}{1 + |\nabla\eta|^2} + H(\eta)
\]
\[
+ \mathcal{P}_V \cdot \nabla\psi - \mathcal{P}_B \mathcal{P}_V \cdot \nabla\eta - \mathcal{P}_B G(\eta)\psi + \mathcal{P}_\ell.\eta.
\]

Given two time-dependent operators \( \mathcal{A}, \mathcal{B} : \mathcal{S} \to \mathcal{S}' \), we say that \( \mathcal{A} \sim \mathcal{B} \) if
\[
\mathcal{A} - \mathcal{B} \in L^\infty([0,T], C^{0,\mu+d/2+4}_0).
\]
By the ellipticity of \( \gamma^{(3/2)}, \) \( p^{(1/2)} \) and \( q^{(0)} \), we can find paradifferential operators \( \tilde{\Lambda}^\mu \) and \( \tilde{S} \) by a routine construction of parametrix such that
\[
\tilde{\Lambda}^\mu \Lambda^\mu \sim \text{Id}, \quad \tilde{S} S \sim \text{Id}.
\]
Observe that \( S \) is a lower triangular matrix, we can also choose \( \tilde{S} \) to be lower triangular. Therefore, we can find \( \zeta \in \Sigma^{-1/2,\mu-5/2} \) by the symbolic calculus, such that
\[
\begin{pmatrix} 0 & 0 \\ \mathcal{P}_\zeta & 0 \end{pmatrix} \Lambda^\mu S - \Lambda^\mu S \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \sim 0.
\]
Moreover, the principal symbols of \( S \) is
\[
\begin{pmatrix} p^{(1/2)} & 0 \\ B^{(1/2)} & q^{(0)} \end{pmatrix}.
\]
We verify that the principal symbol of \( \zeta \) is
\[
\zeta^{(-1/2)} = q^{(0)}/p^{(1/2)}.
\]
Then by (5.3) and the fact that the Poisson bracket between the symbol of \( \Lambda^\mu \) and \( \gamma \) vanishes, we find by the symbolic calculus two symbols \( A, B \in \mathcal{M}_{2 \times 2}(\Sigma^{0,\mu-5/2}) \) such that
\[
[\partial_t + \mathcal{P}_V \cdot \nabla, \Lambda^\mu S] \sim [\partial_t + \mathcal{P}_V \cdot \nabla, \Lambda^\mu S]\tilde{S}\Lambda^\mu S \sim \mathcal{P}_A \Lambda^\mu S,
\]
\[
\begin{pmatrix} 0 & -\mathcal{P}_\gamma \\ \mathcal{P}_\gamma & 0 \end{pmatrix} \Lambda^\mu S - \Lambda^\mu S \mathcal{L} \sim \begin{pmatrix} 0 & -\mathcal{P}_\gamma \\ \mathcal{P}_\gamma & 0 \end{pmatrix} - \Lambda^\mu S \mathcal{L} \tilde{\Lambda}^\mu \Lambda^\mu S \sim \mathcal{P}_B \Lambda^\mu S.
\]
Therefore, let \( \Phi = \Lambda^\mu S(\psi/\psi) \), and write
\[
g(0) = \begin{pmatrix} 0 & 0 \\ g & 0 \end{pmatrix} \begin{pmatrix} \eta \\ \psi \end{pmatrix},
\]
we obtain by the analysis above that
\[(\partial_t + \mathcal{P}_u \cdot \nabla)\Phi + \begin{pmatrix} 0 & -\mathcal{P}_\gamma \\ \mathcal{P}_\gamma & 0 \end{pmatrix} \Phi + \begin{pmatrix} 0 & 0 \\ g\mathcal{P}_\xi & 0 \end{pmatrix} \Phi = \mathcal{P}_A \Phi + \mathcal{P}_B \Phi + F,
\]
where
\[F = (A + B) \begin{pmatrix} \eta \\ \psi \end{pmatrix} - \mathcal{P}_A + B \Phi + \begin{pmatrix} 0 & 0 \\ g\mathcal{P}_\xi & 0 \end{pmatrix} \Phi - g\Lambda^\mu S \begin{pmatrix} 0 \\ \eta \end{pmatrix} + \Lambda^\mu f(\eta, \psi) \in H^{\mu-4-d/2}.
\]
Finally, observe that
\[U \begin{pmatrix} 0 & -\mathcal{P}_\gamma \\ \mathcal{P}_\gamma & 0 \end{pmatrix} U^{-1} = i \begin{pmatrix} \mathcal{P}_\gamma & 0 \\ 0 & -\mathcal{P}_\gamma \end{pmatrix}, \quad U \begin{pmatrix} 0 & 0 \\ \mathcal{P}_\xi & 0 \end{pmatrix} U^{-1} = \frac{i}{2} \begin{pmatrix} \mathcal{P}_\xi & -\mathcal{P}_\xi \\ -\mathcal{P}_\xi & \mathcal{P}_\xi \end{pmatrix},\]
We conclude by setting \( Q = -\frac{i}{4} U(A + B) U^{-1}. \)

**Remark 6.4.** If \( \eta \in H^{\mu+1/2}_m \) with \( m < 2\mu - 6 - d/2 \), then for \( 0 \leq j \leq \mu + 1/2 - d/2 \),
\[\nabla^j \eta \in H^{\mu+1/2-j}_m \subset W^{\rho_{j},0},\]
where
\[\rho_j = \min \{2(\mu + 1/2 - j - d/2), m \}.\]
By the construction of \( \lambda \) in the proof of Proposition 6.1, we can prove by induction that
\[\lambda^{(1-j)}(t,x,\xi) = f_j(\nabla \eta(t,x), \ldots, \nabla^{j+1} \eta(t,x), \xi),\]
where \( f_j(u_0, \ldots, u_j, \xi) \) is smooth with respect to \( u_0, \ldots, u_j \) and homogeneous of degree \( 1-j \) with respect to \( \xi \), and moreover
\[|f_j(u_0, \ldots, u_j, \xi)| \leq C(|u_0, \ldots, u_j|)|u_j||\xi|^{1-j}.
\]
Indeed, this is clearly true when \( j = 0 \), and when \( j \geq 1 \), \( f_j \) is linear with respect to \( u_j \). Therefore,
\[\lambda^{(1)} - |\xi| \in \Gamma^{1,0}_{m,0}, \quad \lambda^{(1-j)} \in \Gamma^{1-j,0}_{\rho_j+1,0}, \quad \forall j \geq 1.
\]
Similarly, we have
\[\gamma^{(3/2)} - |\xi|^{3/2} \in \Gamma^{3/2,0}_{m,0}, \quad \lambda^{(3/2-j)} \in \Gamma^{3/2-j,0}_{\rho_j+1,0}, \quad \forall j \geq 1.
\]
\[\zeta^{(-1/2)} - |\xi|^{-1/2} \in \Gamma_{m,0}, \quad \zeta^{(-1/2-j)} \in \Gamma^{1-j,0}_{\rho_j+1,0}, \quad \forall j \geq 1.
\]
\[Q^{(-j)} \in \Gamma^{j,0}_{\rho_j+2,0}, \quad \forall j \geq 0.
\]

**Lemma 6.5.** Let \( u \) be defined as (6.2). If \( (\eta, \psi) \in H^{\mu+1/2} \times H^{\mu} \), then for \( 0 \leq \sigma \leq r \) where \( r \) is the largest integer such that \( r < \mu - 1 - d/2 \),
\[WF_{0,1}^\sigma(u)^0 = WF_{0,1}^{\mu+1/2+\sigma}(\eta)^0 \cup WF_{0,1}^{\mu+\sigma}(\psi)^0.
\]
If \( (\eta, \psi) \in H^{\mu+1/2}_m \times H^{\mu}_m \), with \( m < \frac{2}{3}(\mu - 1 - d/2) \), then for \( 0 \leq \sigma \leq \frac{2}{3}m \),
\[WF_{1,2}^\sigma(u)^0 = WF_{1,2}^{\mu+1/2+\sigma}(\eta)^0 \cup WF_{1,2}^{\mu+\sigma}(\psi)^0.
\]
**Proof.** If \( \eta \in H^{\mu+1/2} \), then \( (\gamma^{(3/2)})^{2\mu/3} \in \Gamma^{\mu,r}, p^{(1/2)} \in \Gamma^{1/2,r}, q^{(0)} \in \Gamma^{0,r}, B \in \Gamma^{0,r} \). If \( \eta \in H^{\mu+1/2}_m \), then for \( 0 \leq j \leq m \),
\[\nabla^j \eta \in H^{\mu+1/2-j}_m \subset H^{\mu+1/2-3j/2}_j \subset W_{0,j}^{\rho_{j},0}.
\]
Therefore, \( (\gamma^{(3/2)})^{2\mu/3} \in \Gamma^{\mu,m}_0, p^{(1/2)} \in \Gamma^{1/2,m}_0, q^{(0)} \in \Gamma^{0,m}_0, B \in \Gamma^{0,m}_0 \). By Lemma 4.36 and (6.2), for each \( \epsilon = 0 \) or \( \epsilon = 1/2 \),
\[WF_{\epsilon,1}^\sigma(u)^0 = WF_{\epsilon,1}^\sigma(\Lambda^{\mu} P_\rho \eta)^0 \cup WF_{\epsilon,1}^\sigma(\Lambda^{\mu} P_\sigma (\psi - P_B \eta))^0
\]
\[= WF_{\epsilon,1}^{\mu+1/2+\sigma}(\eta)^0 \cup WF_{\epsilon,1}^{\mu+\sigma}(\psi - P_B \eta)^0
\]
\[\subset WF_{\epsilon,1}^{\mu+1/2+\sigma}(\eta)^0 \cup (WF_{\epsilon,1}^{\mu+\sigma}(\psi)^0 \cup WF_{\epsilon,1}^{\mu+\sigma}(P_B \eta)^0)
\]
\[\subset WF_{\epsilon,1}^{\mu+1/2+\sigma}(\eta)^0 \cup WF_{\epsilon,1}^{\mu+\sigma}(\psi)^0 \cup WF_{\epsilon,1}^{\mu+\sigma}(\eta)^0
\]
Conversely, as $WF_{\epsilon,1}^{\mu+1/2+\sigma}(P_B\eta)$ \(\subset WF_{\epsilon,1}^{\mu+1/2+\sigma}(\eta)\), we have
\[
WF_{\epsilon,1}^{\mu+1/2+\sigma}(\eta) \cup WF_{\epsilon,1}^{\mu+\sigma}(\psi) = WF_{\epsilon,1}^{\mu+1/2+\sigma}(\eta) \cup (WF_{\epsilon,1}^{\mu+\sigma}(\psi) \setminus WF_{\epsilon,1}^{\mu+1/2+\sigma}(\eta)) = WF_{\epsilon,1}^{\mu+1/2+\sigma}(\eta) \cup (WF_{\epsilon,1}^{\mu+\sigma}(\psi - P_B\eta) \setminus WF_{\epsilon,1}^{\mu+1/2+\sigma}(\eta)) = WF_{\epsilon,1}^{\mu+1/2+\sigma}(\eta) \cup WF_{\epsilon,1}^{\mu+\sigma}(\psi - P_B\eta) = WF_{\epsilon,1}^{\sigma}(u) .
\]

6.2. Proof of Theorem 1.7. By Lemma 6.5, it is equivalent to prove the following theorem.

**Theorem 6.6.** Under the hypothesis of Theorem 1.7, let $u$ be defined by (6.2), and let
\[
(x_0, \xi_0) \in WF_{1/2,1}^\sigma(u_0) \quad \text{with } 0 \leq \sigma \leq \max\{m/2 - 3/2, 0\} .
\]
Let $t_0 \in [0, T]$, and suppose that
\[
x_0 + \frac{3}{2}t_1 |\xi_0|^{-1/2}\xi_0 \neq 0 \quad \forall t \in [0, t_0] ,
\]
then
\[
(x_0 + \frac{3}{2}t_0 |\xi_0|^{-1/2}\xi_0, \xi_0) \in WF_{1/2,1}^\sigma(u(t_0)) .
\]

**Proof.** For $\nu \in \mathbb{R}$, denote $X^\nu = \sum_{k\in\mathbb{Z}} H_k^{m-k/2}$. By Lemma 2.16, if $f \in X^\nu$, then $WF_{1/2,1}^\nu(f) = \emptyset$. By Remark 6.4,
\[
\mathcal{P}_V \cdot \nabla w \in H^{-1}_m \subset X^{m/2-1},
\]
\[
\mathcal{P}_{\gamma^{(j)}} w - \mathcal{P}_{\xi^{(j)}} w \in H^{-j/2}_m \subset X^{m/2-j/2},
\]
\[
\mathcal{P}_{\gamma^{(j)}} w - \mathcal{P}_{\xi^{(j)}} w \in H^{-j/2}_m \subset X^{m/2-j/2},
\]
\[
\mathcal{P}_{\gamma^{(j)}} w - \mathcal{P}_{\xi^{(j)}} w \in H^{-j/2}_m \subset X^{m/2-j/2},
\]
\[
\mathcal{P}_{\gamma^{(j)}} w - \mathcal{P}_{\xi^{(j)}} w \in H^{-j/2}_m \subset X^{m/2-j/2},
\]
By the hypothesis on $m$, we thus obtain
\[
(6.3) \quad \partial_t u' + i |D_x|^{3/2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} w' + \frac{i g}{2} |D_x|^{-1/2} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} w' \in X^{m/2-3/2},
\]
where $w' = \pi(D_x)w$, and $\pi \in C^\infty(\mathbb{R}^d)$ which vanishes near the origin, and equals to 1 out side a neighborhood of the origin. Observe that the matrix
\[
M = |D_x|^{3/2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{g}{2} |D_x|^{-1/2} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}
\]
is symmetrizable. Indeed, let $\theta = \sqrt{g|D_x|^{-2} + 1}$, and set
\[
P = \begin{pmatrix} 1 & (1 - \theta)(1 + \theta)^{-1} \\ -1 & -(1 + \theta)(1 - \theta)^{-1} \end{pmatrix},
\]
then $P \in \mathcal{C}_0^\infty$, and
\[
PMP^{-1} = \sqrt{g|D_x| + |D_x|^3} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in X^{m/2-3/2}.
\]
Let $\tilde{w} = Pw'$, then
\[
\partial_t \tilde{w} + \sqrt{g|D_x| + |D_x|^3} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tilde{w} \in X^{m/2-3/2}.
\]
Let \( u' = \pi(D_x)u \), then
\[
\tilde{w} = P\left( \frac{u'}{\overline{w}} \right) = 2 \begin{pmatrix} (1 + \theta)^{-1} & 0 \\ 0 & -(1 + \theta)^{-1} \end{pmatrix} \begin{pmatrix} \Re u' + i\theta \Im u' \\ \Re u' - i\theta \Im u' \end{pmatrix}.
\]

Let \( v = \Re u' + i\theta \Im u' \), then \( \WF_{1/2,1}(u)^\circ = \WF_{1/2,1}(v)^\circ \), and
\[
\partial_t v + \sqrt{|g|} |D_x| + |D_x|^3 v \in X^{m/2-3/2}.
\]

We are left to prove that if \((x_0, \xi_0) \in \WF_{1/2,1}(v(0))^\circ \), then
\[
(x_0 + \frac{3}{2} \eta_0 |\xi_0|^{-1/2} \xi_0, \xi_0) \in \WF_{1/2,1}(v(t_0)).
\]

Because \( \sqrt{|g|} |\xi| + |\xi|^{3/2} \sim |\xi|^{3/2} \) in the high frequency regime, similar proof as (M.1) of Theorem 1.4 yields the conclusion. \( \square \)

### 6.3. Proof of Theorem 1.8.

#### 6.3.1. Hamiltonian Flow.

Let \( \Phi = \Phi_s : \mathbb{R}^d_x \times (\mathbb{R}^d_\xi \setminus 0) \to \mathbb{R}^d_x \times (\mathbb{R}^d_\xi \setminus 0) \) be the Hamiltonian flow of
\[
H(x, \xi) = \gamma^{(3/2)}(0, x, \xi) = \left( |\xi|^{2} - \frac{(\nabla \eta_0 \cdot \xi)^2}{1+|\nabla \eta_0|^2} \right)^{3/4}.
\]

That is
\[
\partial_t \Phi(x, \xi) = X_H(\Phi(x, \xi)), \quad \Phi|_{s=0} = \text{Id}_{\mathbb{R}^d_x \times (\mathbb{R}^d_\xi \setminus 0)},
\]

where \( X_H = (\partial_3 H, -\partial_2 H) \). We use \( s \) to denote the time variable in accordance to the semiclassical time variable in the following section. Observe that

**Lemma 6.7.** For \((x, \xi) \in \mathbb{R}^d_x \times (\mathbb{R}^d_\xi \setminus 0)\), we have
\[
\Phi_s(x, \xi) = \mathcal{G}_{\varphi_s(x, \xi)}(x, \xi),
\]
where \( \mathcal{G} \) is the geodesic defined in §1.3.6, and
\[
\varphi_s(x, \xi) = \frac{3}{4} \int_0^s G(\Phi_\sigma(x, \xi))^{-1/4} \, d\sigma.
\]

**Proof.** We have \( \mathcal{G}_{\varphi_0(x, \xi)}(x, \xi) = \mathcal{G}_0(x, \xi) = (x, \xi) = \Phi_0(x, \xi) \). Then observe that
\[
H(x, \xi) = G(x, \xi)^{3/4} = g_{2}^{-1}(\xi, \xi) \xi^{3/4}.
\]

Therefore,
\[
\frac{d}{ds} \mathcal{G}_{\varphi_s(x, \xi)}(x, \xi) = \frac{d}{ds} \varphi_s(x, \xi) \left( \frac{d}{ds} \mathcal{G}_{\varphi_s(x, \xi)}(x, \xi) \right) = \frac{3}{4} \mathcal{G}(\varphi_s(x, \xi))^{-1/4} X_G(\varphi_s(x, \xi)) = X_H(\mathcal{G}_{\varphi_s(x, \xi)}(x, \xi)).
\]

We conclude by the uniqueness of solutions to Hamiltonian ODEs. \( \square \)

**Lemma 6.8.** Suppose that for some \( \epsilon > 0 \), \( \nabla \eta_0 \in W^{0, \infty}_{1/2+\epsilon} \), \( \nabla^2 \eta_0 \in W^{0, \infty}_{1+\epsilon} \). Let \((x_0, \xi_0) \in \mathbb{R}^d_x \times (\mathbb{R}^d_\xi \setminus 0) \) such that the co-geodesic \( \{(x_s, \xi_s) = \Phi_s(x_0, \xi_0) \}_{s \in \mathbb{R}} \) is forwardly non-trapping. Set
\[
z_s = x_s - x_0 - \frac{3}{2} \int_0^s |\xi_\sigma|^{-1/2} \xi_\sigma \, d\sigma,
\]
then \( \exists (z_{+\infty}, \xi_{+\infty}) \in \mathbb{R}^d_x \times (\mathbb{R}^d_\xi \setminus 0) \) such that
\[
\lim_{s \to +\infty} (z_s, \xi_s) = (z_{+\infty}, \xi_{+\infty}).
\]

Consequently, by Lemma 6.7, let \((x'_s, \xi'_s) = \mathcal{G}_s(x_0, \xi_0) \), then \( \lim_{s \to +\infty} \xi'_s = \xi_{+\infty} \).
Proof. Because \{ (x_s, \xi_s) \}_{s \in \mathbb{R}} is forwardly non-trapping, and we only consider the limiting behavior when \( s \to +\infty \), we may assume that \( \varepsilon_0 := \| x \nabla^2 \eta_0 \|_{L^{\infty}} \) is sufficiently small. As \( \nabla \eta_0 \in L^\infty \), we have \( H(\cdot, \xi) \simeq |\xi|^2 \). Then

\[
\frac{d}{ds} (x_s \cdot \xi_s) = \partial_\xi H(x_s, \xi_s) \cdot \xi_s - x_s \cdot \partial_x H(x_s, \xi_s),
\]

where

\[
\partial_\xi H(x_s, \xi_s) \cdot \xi_s = \frac{3}{2} H(x_s, \xi_s) = \frac{3}{2} H(x_0, \xi_0) \simeq |\xi_0|^{3/2},
\]

and

\[
\partial_x H(x_s, \xi_s) = \frac{3}{4} H(x_s, \xi_s)^{-1/3} \partial_x G(x_s, \xi_s) = \frac{3}{4} H(x_0, \xi_0)^{-1/3} \left( \frac{2 \nabla \eta_0 \cdot \xi_s}{1 + |\nabla \eta_0|^2} \nabla^2 \eta_0 \xi_s - \frac{2 (\nabla \eta_0 \cdot \xi_s)^2}{1 + |\nabla \eta_0|^2} \nabla^2 \eta_0 \nabla \eta_0 \right) \rvert_{x=x_s}.
\]

Therefore

\[
x_s \cdot \partial_x H(x_s, \xi_s) = O(|\xi_s|^{3/2}) = O(|\xi_0|^{3/2}),
\]

and consequently,

\[
\frac{d}{ds} (x_s \cdot \xi_s) \gtrsim |\xi_0|^{3/2}. \tag{6.4}
\]

So for any bounded set \( B \subset \mathbb{R}^d \),

\[
\lambda(s \geq 0 : x_s \in B) \lesssim \sup \{ |x \cdot \xi| : (x, \xi) \in B \times \mathbb{R}^d, H(x, \xi) = H(x_0, \xi_0) \} \frac{1}{|\xi_0|^{3/2}} \lesssim \sup_{x \in B} |x| (|\xi_0|^{-1/2}, \tag{6.5}
\]

where \( \lambda \) is the Lebesgue measure on \( \mathbb{R} \). Let \( E(x, \xi) = H(x, \xi) - |\xi|^{3/2} \), then by the hypothesis of the decay of \( \eta_0, E \in H^{3/2,1} \). By the definition of \( z_s \), we have

\[
\frac{d}{ds} (z_s, \xi_s) = (\partial_\xi E, -\partial_x E)(x_s, \xi_s) = O(|x_s|^{-1-\epsilon}).
\]

By (6.4) and (6.5),

\[
\int_0^\infty (x_s)^{-1-\epsilon} ds = (1+\epsilon) \int_0^\infty t^\epsilon \lambda(s \geq 0 : (x_s)^{-1} > t) dt \lesssim \int_0^1 t^\epsilon \sqrt{t^2 - 1} dt < \infty.
\]

Therefore, for any \( 0 < s^- < s^+ \) with \( s^- \to \infty \),

\[
|(z_{s^+}, \xi_{s^+}) - (z_{s^-}, \xi_{s^-})| \lesssim \int_{s^-}^{s^+} (x_\sigma)^{-1-\epsilon} d\sigma \to 0,
\]

implying that \( (x_s, \xi_s) \) is a Cauchy sequence as \( s \to \infty \). \( \square \)

6.3.2. Construction of Symbol. For \( h \geq 0 \), and \( h^{1/2} s \leq T \), set

\[
H_h(s, x, \xi) = \gamma^{(3/2)}(h^{1/2} s, x, \xi),
\]

so in particular \( H(x, \xi) \equiv H_0(s, x, \xi) \). For \( h > 0 \), the semiclassical time variable \( s = h^{-1/2} t \) was inspired by Lebeau [24], see also Zhu [39] for an application in theory of water waves.

For \( a \in C^\infty([0, h^{-1/2} T] \times \mathbb{R}^d \times \mathbb{R}^d) \), set

\[
\mathcal{L}^\pm_{h, a} = \partial_\xi a \pm \{ H_h, a \}.
\]

Lemma 6.9. Suppose that for some \( \epsilon > 0 \), \( \nabla \eta_0 \in W^{1,\infty}(0, 1/2+\epsilon), \nabla^2 \eta_0 \in W^{1,\infty}, \nabla^3 \eta_0 \in W^{0,\infty}(1/2+\epsilon) \). Let \( (x_0, \xi_0) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus 0) \) such that the co-geodesic \( \{ (x_s, \xi_s) = \Phi_s(x_0, \xi_0) \}_{s \in \mathbb{R}} \) is forwardly non-trapping, then for some \( s_0 > 0 \), there exists

\[
\chi^\pm \in W^{1,\infty}(\mathbb{R}_{\geq s_0}, \mathbb{R}^d), \Gamma_{-\infty, \mu - 3 - d/2} \cap W^{1,\infty}(\mathbb{R}_{\geq s_0}, S_0^{-\infty})
\]

such that

1. \( \chi^\pm(0, x, \xi) \in S_{-\infty}^{\infty} \) is elliptic at \( (x_0, \pm \xi_0) \);
(2) \(\forall t_0 > 0, \chi^\pm(s, t_0 x, \xi) \in S^{-\infty}_\infty\) is elliptic at \((\frac{3}{2} t_0 |\xi_\infty|^{-1/2} \xi_\infty, \pm \xi_\infty)\) for sufficiently large \(s\).

Moreover, if \((\eta, \psi) \in H^{m+1/2}_0 \times H^m_0\) with \(m > 3 + d/2\) and \(m \geq 2\), then

\[
\mathcal{L}_{h,a}^\pm \chi^\pm \in L^\infty([0, h^{-1/2} T], \Gamma_{-1,0,0}^{-\infty,0-4-d/2}),
\]

and

\[(6.7)\]

\[
\mathcal{L}_{h,a}^\pm \chi^\pm \geq \mathcal{O}(h^{1/2}) L^\infty([0, h^{-1/2} T], \Gamma_{-1,0,0}^{-\infty,0-4-d/2}).
\]

**Proof.** Let \(\phi \in C^\infty_c(\mathbb{R}^d)\) such that

(i) \(\phi \geq 0\), \(\phi(x) = 1\) for \(|x| \leq 1/2\), \(\phi(x) = 0\) for \(|x| \geq 1\), \(\text{supp} \phi = \{ |x| \leq 1 \}\);

(ii) \(x \cdot \nabla \phi(x) \leq 0\) for all \(x \in \mathbb{R}^d\);

(iii) \(y \cdot \nabla \phi(x) = 0\) for all \(x, y \in \mathbb{R}^d\) with \(x \cdot y = 0\).

Such \(\phi\) can be constructed by setting \(\phi(x) = \varphi(|x|)\) where \(\varphi : \mathbb{R} \to \mathbb{R}\) satisfies \(0 \leq \varphi \leq 1\), \(\varphi(z) = 1\) if \(z \leq 1/2\), \(\varphi(z) = 0\) if \(z \geq 1\). For \(\delta > 0\), \(\lambda > 0\), \(\nu > 0\) and sufficiently large \(s > 0\), set

\[
\tilde{\chi}^\pm(s, x, \xi) = \phi\left(\frac{x - x_s}{\lambda \delta s}\right) \phi\left(\frac{\xi + \xi_s}{\delta - s^{-\nu}}\right).
\]

We verify that \(\mathcal{L}_{0,a}^\pm \tilde{\chi}^\pm(s, \cdot) \geq 0\) for \(s > 0\) sufficient large. Indeed,

\[
\mathcal{L}_{0,a}^\pm \tilde{\chi}^\pm(s, x, \xi) = \left(\pm \frac{\partial_t H(x, \xi) - \partial_\xi H(x_s, \xi_s)}{\lambda \delta s^2} - \frac{x - x_s}{\lambda \delta s^2}\right) \nabla \phi\left(\frac{x - x_s}{\lambda \delta s}\right) \phi\left(\frac{\xi + \xi_s}{\delta - s^{-\nu}}\right)
+
\left(\pm \frac{\partial_x H(x_s, \xi_s) - \partial_\xi H(x, \xi)}{(\delta - s^{-\nu})^2} - \nu \frac{\xi + \xi_s}{(\delta - s^{-\nu}) s^\nu + 1}\right) \nabla \phi\left(\frac{x - x_s}{\lambda \delta s}\right) \phi\left(\frac{\xi + \xi_s}{\delta - s^{-\nu}}\right).
\]

By (i),

\[
\text{supp} \phi\left(\frac{-x_s}{\lambda \delta s}\right) \subset \{ x \in \mathbb{R}^d : |x - x_s| \leq \lambda \delta s \},
\]

\[
\text{supp} \phi\left(\frac{\xi_s}{\delta - s^{-\nu}}\right) \subset \{ \xi \in \mathbb{R}^d : |\xi + \xi_s| \leq \delta - s^{-\nu} \},
\]

\[
\text{supp} \nabla \phi\left(\frac{-x_s}{\lambda \delta (1 + s)}\right) \subset \{ x \in \mathbb{R}^d : \frac{1}{2} \lambda \delta s \leq |x - x_s| \leq \lambda \delta s \},
\]

\[
\text{supp} \nabla \phi\left(\frac{\xi_s}{\delta - s^{-\nu}}\right) \subset \{ \xi \in \mathbb{R}^d : \frac{1}{2} (\delta - s^{-\nu}) \leq |\xi + \xi_s| \leq \delta - s^{-\nu} \}.
\]

By Lemma 6.8,

\[
x_s = x_0 + \frac{3}{2} \int_0^s |\xi_\sigma|^{-1/2} \xi_\sigma d\sigma + z_s = \frac{3}{2} s|\xi_\infty|^{-1/2} \xi_\infty + o(s).
\]

Therefore, by writing

\[
\tilde{\chi}^\pm(s, t_0 x, \xi) = \phi\left(\frac{x - x_0}{\lambda \delta t_0^{3/2} |\xi_\infty|^{-1/2} \xi_\infty + o(1)}\right) \phi\left(\frac{\xi + \xi_0 + o(1)}{\delta - s^{-\nu}}\right),
\]

we see that \(\tilde{\chi}^\pm(s, \cdot, \xi)\) is elliptic at \((\frac{3}{2} t_0 |\xi_\infty|^{-1/2} \xi_\infty, \pm \xi_\infty)\) for sufficiently large \(s\). Moreover, if \(\lambda \delta\) is sufficiently small and \(s\) is sufficiently large, then

\[
\text{supp} \phi\left(\frac{-x_s}{\lambda \delta s}\right) \subset \{ x \in \mathbb{R}^d : |x| \geq s \}.
\]

Therefore, by the hypothesis on \(t_0\), we have for \((x, \xi) \in \text{supp} \tilde{\chi}^\pm(s, \cdot, \cdot),

\[
\nabla^2_{xy} H(x, \xi) = \left(\begin{array}{cc}
\nabla^2_{xx} H & \nabla_x \nabla_\xi H \\
\nabla_\xi \nabla_x H & \nabla^2_{\xi \xi} H
\end{array}\right)(x, \xi) = \left(\begin{array}{cc}
\mathcal{O}(s^{-2-\epsilon}) & \mathcal{O}(s^{-3/2-\epsilon}) \\
\mathcal{O}(s^{-3/2-\epsilon}) & \mathcal{O}(1)
\end{array}\right),
\]

and consequently, by the finite increment formula,

\[
|\partial_\xi H(x_s, \xi_s) - \partial_\xi H(x, \xi)| \lesssim s^{-3/2-\epsilon} |x - x_s| + |\xi + \xi_s| \lesssim s^{-1/2-\epsilon} \lambda \delta + \delta;
\]

\[
|\partial_x H(x_s, \xi_s) - \partial_x H(x, \xi)| \lesssim s^{-2-\epsilon} |x - x_s| + s^{-3/2-\epsilon} |\xi + \xi_s| \lesssim \lambda \delta s^{-1-\epsilon} + \delta s^{-3/2-\epsilon}.
\]
By (iii),
\[
(\partial_t H(x, \xi) - \partial_t H(x_s, \xi_s)) \cdot \nabla \phi \left( \frac{x-x_s}{\lambda \delta s} \right) = \frac{x-x_s}{|x-x_s|^2} (x-x_s) \cdot \nabla \phi \left( \frac{x-x_s}{\lambda \delta s} \right) \]
\[
= O(s^{-3/2-\epsilon} + \lambda^{-1} s^{-1})(x-x_s) \cdot \nabla \phi \left( \frac{x-x_s}{\lambda \delta s} \right);
\]
\[
(\partial_x H(x_s, \xi_s) - \partial_x H(x, \xi)) \cdot \nabla \phi \left( \frac{\xi + \xi_s}{\delta - s^{-\nu}} \right) \]
\[
= (\partial_x H(x_s, \xi_s) - \partial_x H(x, \xi)) \cdot \nabla \phi \left( \frac{\xi + \xi_s}{\delta - s^{-\nu}} \right) \]
\[
= O(\lambda s^{-1-\epsilon} + s^{-3/2-\epsilon})(\xi \pm \xi_s) \cdot \nabla \phi \left( \frac{\xi \pm \xi_s}{\delta - s^{-\nu}} \right).
\]
Finally, we fix $0 < \nu < \epsilon$, $\delta > 0$. Then, when $\lambda$ is sufficiently large, and $s \geq s_0 - 1 > 0$ with $s_0$ being sufficiently large, by (ii),
\[
\mathcal{L}_{0,s}^{\pm} \chi^\pm = -1 + \frac{1 + O(s^{-1/2-\epsilon} + \lambda^{-1})}{\lambda \delta s^2}(x-x_s) \cdot \nabla \phi \left( \frac{x-x_s}{\lambda \delta s} \right) \phi \left( \frac{\xi \pm \xi_s}{\delta - s^{-\nu}} \right) \]
\[
- \frac{\nu(\delta - s^{-\nu}) - O(\lambda) s^{\nu-\epsilon}}{(\delta - s^{-\nu})^2 s^\nu + 1} (\xi \pm \xi_s) \cdot \nabla \phi \left( \frac{\xi \pm \xi_s}{\delta - s^{-\nu}} \right) \geq 0.
\]
We verify as in Lemma 3.2 that
\[
\chi^\pm \in W^{\infty, \infty}(\mathbb{R}_{\geq s_0}, S_0^-), \quad \mathcal{L}_{0,s}^{\pm} \chi^\pm \in W^{\infty, \infty}(\mathbb{R}_{\geq s_0}, \Gamma_{1,0}^-; \mu - d/2).
\]
Next, we set for $s \geq s_0$,
\[
\chi^\pm(s, x, \xi) = \tilde{\chi}^\pm(s, x, \xi).
\]
To define $\chi^\pm$ for $s \leq s_0$, we choose $\rho \in C^\infty(\mathbb{R})$ such that $0 \leq \rho \leq 1$, $\rho(s) = 1$ for $s \geq s_0$, and $\rho(s) = 0$ for $s \leq s_0 - \alpha$ for some small $\alpha > 0$ to be specified later, and solve the transport equation on $[0, s_0]$,
\[
\mathcal{L}_{0,s}^{\pm} \chi^\pm(s, x, \xi) = \rho(s) \mathcal{L}_{0,s}^{\pm} \tilde{\chi}^\pm(s, x, \xi), \quad \chi^\pm(s_0, x, \xi) = \tilde{\chi}^\pm(s_0, x, \xi).
\]
Clearly $\chi^\pm$ satisfies (6.6), and
\[
\mathcal{L}_{0,s}^{\pm} \chi^\pm \geq 0.
\]
Moreover, because
\[
\chi^\pm(s, x, \xi) = \tilde{\chi}^\pm(s_0, \Phi \pm(\rho - s)(x, \xi)) - \int_{s_0}^{s_0} \rho(\sigma) \mathcal{L}_{0,s}^{\pm} \tilde{\chi}^\pm(\sigma, \Phi \pm(\rho - s)(x, \xi)) \, d\sigma,
\]
if we choose $\alpha > 0$ sufficiently small, then
\[
\chi^\pm(0, x_0, \pm \xi_0) = \tilde{\chi}^\pm(s_0, x_0, \pm \xi_0) - \int_{s_0-\alpha}^{s_0} \rho(\sigma) \mathcal{L}_{0,s}^{\pm} \tilde{\chi}^\pm(\sigma, x_0, \mp \xi_0) \, d\sigma \geq 1 - \| \mathcal{L}_{0,s}^{\pm} \tilde{\chi}^\pm(x_0, \pm \xi_0) \|_{L^1([s_0 - \alpha, s_0])} > 0.
\]
Therefore, $\chi^\pm(0, \cdot, \cdot)$ is elliptic at $(x_0, \pm \xi_0)$.
To estimate $\mathcal{L}_{h,s}^{\pm} \chi^\pm$, we use
\[
H_h(s, x, \xi) - H_0(s, x, \xi) = H_h(s, x, \xi) - H_h(0, x, \xi) \]
\[
= \int_0^s (\partial_s H_h)(\sigma, x, \xi) \, d\sigma = h^{1/2} \int_0^s (\partial_\gamma (3/2)(h^{1/2} \sigma, x, \xi) \, d\sigma,
\]
and write
\[
\mathcal{L}_{h,s}^{\pm} \chi^\pm(s, \cdot) - \mathcal{L}_{0,s}^{\pm} \chi^\pm(s, \cdot) = \pm \{ H_h - H_0, \chi^\pm \}(s, \cdot) = \pm h^{1/2} \int_0^s \{ \partial_\gamma (3/2)(h^{1/2} \sigma, \cdot), \chi^\pm(\cdot, \cdot) \} \, d\sigma.
\]
Observe that
\[ \partial_t\gamma^{(3/2)} = -\frac{3}{2}\left(\frac{1}{1 + |\eta|^2}\right)^{-1/4} \left( \frac{\nabla \eta \cdot \xi}{1 + \eta^2} \nabla G(\eta) \psi \cdot \xi - \frac{(\nabla \eta \cdot \xi)^2}{(1 + |\eta|^2)^2} \nabla G(\eta) \psi \cdot \nabla \eta \right). \]

By hypothesis and Proposition 5.4, \( \nabla G(\eta) \psi \in H_{m-2}^2 \subset H_2^{\mu-3} \) as \( m \geq 2 \). Therefore,
\[ \partial_t\gamma^{(3/2)}(h^{1/2}, \cdot, \cdot) \in L^\infty([0, h^{-1/2}T], \Gamma_{-\gamma_{-1,0}}^{3/2, \mu-3-d/2}). \]

Using \( |x| \sim s \) on supp \( \chi^\pm(s, \cdot) \), we have, uniformly for all \( s \in [0, h^{-1/2}T] \),
\[ \langle s \rangle \langle \partial_t\gamma^{(3/2)}(h^{1/2}, \sigma, \cdot) \rangle \chi^\pm(s, \cdot) \in L^\infty_s([0, h^{-1/2}T], \Gamma_{-\gamma_{-1,0}}^{\mu-4-d/2}). \]

Therefore,
\[ \mathcal{L}_{h,s}^\pm \chi^\pm(s, \cdot) - \mathcal{L}_{0,s}^0 \chi^\pm(s, \cdot) = \pm h^{1/2}(s)^{-1} \int_0^s \mathcal{O}(1)_{L^\infty([0, h^{-1/2}T], \Gamma_{-\gamma_{-1,0}}^{\mu-4-d/2})} d\sigma \]
\[ = \pm h^{1/2}(s)^{-1} \mathcal{O}(s)_{\Gamma_{-\gamma_{-1,0}}^{\mu-4-d/2}} \]
\[ = \mathcal{O}(h^{1/2})_{\Gamma_{-\gamma_{-1,0}}^{\mu-4-d/2}}, \]

which, together with (6.9), proves (6.7).

6.3.3. Propagation. Now we prove Theorem 1.8. By Lemma 6.5 and Lemma 6.7, it suffices to prove the following theorem.

**Theorem 6.10.** Under the hypothesis of Theorem 1.8, and let \( u \) be defined as (6.2). Let
\[(x_0, \xi_0) \in \text{WF}_{0,1}^\sigma(u_0)^0, \]
with \( 0 \leq \sigma < \min\{\mu/2 - 3 - d/4, 3m/2\} \), such that the co-geodesic \( \{(x_s, \xi_s) = \Phi_s(x_0, \xi_0)\}_{s \in \mathbb{R}} \) is forwardly non-trapping. Set
\[ \xi_\infty = \lim_{s \to +\infty} \xi_s, \]
then \( \forall t_0 \in [0, T] \),
\[ \left( \frac{3}{2} t_0 |\xi_\infty|^{-1/2} \xi_\infty, \xi_\infty \right) \in \text{WF}_{1/2,1}^\sigma(u(t_0)). \]

Under the semiclassical time variable \( s = h^{-1/2}t, (6.1) \) becomes
\[(\partial_s + h^{1/2}\mathcal{P}_V \cdot \nabla + h^{1/2}\mathcal{P}_Q)w + ih^{1/2} \begin{pmatrix} \mathcal{P}_\gamma & 0 \\ 0 & -\mathcal{P}_s \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} w = \mathcal{F}_h = \mathcal{O}(h^{1/2})_{L^\infty_{1/2}}. \]

We define \( \mathcal{L}^h_s \) which applies to time dependent operators \( \mathcal{A} : \mathcal{S}' \to \mathcal{S}' \),
\[ \mathcal{L}_h^s \mathcal{A} = \partial_s \mathcal{A} + h^{1/2} \left[ \mathcal{P}_V \cdot \nabla + \mathcal{P}_Q + i\mathcal{P}_\gamma \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{ih}{2} \mathcal{P}_s \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \right] \mathcal{A}. \]

We also define \( \mathcal{L}_s^h \) which applies to symbols of the diagonal form \( A = \begin{pmatrix} A^+ & 0 \\ 0 & A^- \end{pmatrix} \),
\[ \mathcal{L}_s^h A = \begin{pmatrix} \mathcal{L}_{h,s}^+ A^+ & 0 \\ 0 & \mathcal{L}_{h,s}^- A^- \end{pmatrix}. \]

**Proof of Theorem 6.10.** We shall from now on denote \( \rho = \lfloor \mu - 4 - d/2 \rfloor \), \( I_h = [0, h^{-1/2}T] \) and
\[ Y_h^\rho = L^\infty(I_h, M_{2 \times 2}(h\Sigma^{-\infty, \rho})) \]
for simplicity. Choose a strictly increasing sequence \( \{\lambda_j\}_{j \geq 0} \subset [1, 1 + \varepsilon] \) with \( \varepsilon > 0 \) being sufficiently small. Define \( \chi^\pm_j \) as in Lemma 6.9 where we replace \( \phi \) with \( \phi(\cdot/\lambda_j) \). Then

\[ \text{supp} \chi^\pm_j \subset \{ \chi^\pm_{j+1} > 0 \}, \quad \forall j \in \mathbb{N}. \]

And we set
\[ \chi_j = \begin{pmatrix} \chi_j^+ & 0 \\ 0 & \chi_j^- \end{pmatrix}. \]
We shall construct an operator $\mathcal{A}_h \in L^\infty(I_h, L^2 \to L^2)$ satisfying the following properties:

(i) $\mathcal{A}_h$ is a paradofferential operator, more precisely, there exists

$$A^\pm_h \in W^{1,\infty}(\mathbb{R}_{\geq 0}, \Gamma^{-\infty, \rho+1}) \cap W^{1,\infty}(\mathbb{R}_{\geq s_0}, \Gamma^{-\infty})$$

for some $s_0 > 0$, such that

$$\mathcal{A}_h - P^h_{A_h} = \mathcal{O}(h^\rho)_{L^\infty(I_h, L^2 \to L^2)}, \quad A_h = \begin{pmatrix} A^+_h & 0 \\ 0 & A^-_h \end{pmatrix}.$$

Moreover,

$$\text{supp } A^\pm_h \subset \bigcup_{j \geq 0} \text{supp } \chi_j^\pm.$$

(ii) $A^+_h(0, x, \xi)$ is elliptic at $(x_0, \pm \xi_0)$;

(iii) $A^+_h(s, \xi) \in S^{-\infty}_{-\infty}$ is elliptic at $(\frac{3}{2}t_0|\xi_0|^{-1/2}\xi_\infty, \xi_\infty)$ for $s > 0$ sufficiently large;

(iv) $L_h^h \mathcal{A}_h \geq \mathcal{O}(h^\rho)_{L^\infty(I_h, L^2 \to L^2)}$.

We shall construct $\mathcal{A}_h$ of the form

$$\mathcal{A}_h = \sum_{j \geq 0} h^{j/2} \varphi^j A^j_h,$$

where $\varphi \in P_j$, recalling the definition (3.4), and $A^j_h \in L^\infty(I_h, L^2 \to L^2)$. We begin by setting

$$A^0_h = (P^h_{\chi_{\lambda_0}}^\ast P^h_{\chi_{\lambda_0}}), \quad \varphi^0 \equiv 1.$$

Therefore, by the symbolic calculus, Lemma 6.9 and Corollary 4.31,

$$\partial_x A^0_{h} + h^{1/2} \left[i P_{\gamma} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, A^0_{h} \right] = 2P^h_{\chi_0 \gamma} \chi_0 + hP^h_{b^0_h} + \mathcal{O}(h^\rho)_{L^\infty(I_h, L^2 \to L^2)},$$

for some $b^0_h \in L^\infty(I_h, \Sigma_{-\infty, \rho})$, with $\text{supp } b^0_h \subset \text{supp } \chi_0$. Therefore $\langle s \rangle b^0_h \in Y^\rho_h$. Similarly,

$$h^{1/2}[\mathcal{P}_{\chi_0} \cdot \nabla, A^0_{h}] = h^{1/2}P^h_{b^0_h} + \mathcal{O}(h^\rho)_{L^\infty(I_h, L^2 \to L^2)},$$

where $\langle s \rangle b^0_h \in Y^\rho_h$, with $\text{supp } b^0_h \subset \text{supp } \chi_0$. Be careful that, because $Q$ and $P_{\gamma} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ are not diagonal matrices, their commutators with $A^0_h$ do not gain an extra $h$, for the principal symbols do not cancel each other. So,

$$h^{1/2}[\mathcal{P}_{\gamma}, A^0_{h}] = h^{1/2}P^h_{b^0_h} + \mathcal{O}(h^\rho)_{L^\infty(I_h, L^2 \to L^2)};$$

$$h^{1/2}[\mathcal{P}_{\chi_0} \cdot \nabla, A^0_{h}] = h^{1/2}P^h_{b^0_h} + \mathcal{O}(h^\rho)_{L^\infty(I_h, L^2 \to L^2)},$$

where $\langle s \rangle b^0_h, \langle s \rangle b^0_h \in Y^\rho_h$, with $\text{supp } b^0_h \cup \text{supp } b^0_h \subset \text{supp } \chi_0$. By Lemma 6.9,

$$\chi_0 \gamma \chi_0 \geq h^{1/2} b^4_h,$$

where $\langle s \rangle b^4_h \in L^\infty(I_h, \Gamma^{-\infty, \rho}) \subset Y^\rho_h$, with $\text{supp } b^4_h \subset \text{supp } \chi_0$. Therefore, by the paradofferential Garding inequality, i.e., Lemma 4.34,

$$\mathcal{P}_{\chi_0} \cdot \nabla b^4_h - h^{1/2}P^h_{b^4_h} \geq \mathcal{O}(h^\rho)_{L^2 \to L^2},$$

for some $b^5_h \in Y^\rho_h$ with $\text{supp } b^5_h \subset \{ \chi_1 > 0 \}$. Set

$$\alpha^0_h = \langle s \rangle (b^0_h + b^2_h + b^4_h) \in Y^\rho_h, \quad \beta^0_h = \langle s \rangle (2b^0_h + b^2_h + b^4_h) \in Y^\rho_h.$$

Then

$$L^h_{s} \mathcal{A}^0_{h} \geq h^{1/2}(s)^{-1}P^h_{\alpha^0_h + h^{1/2} \beta^0_h} + \mathcal{O}(h^\rho)_{L^\infty(I_h, L^2 \to L^2)}.$$
Suppose that we have found $A_h^j \in L^\infty(I_h, L^2 \rightarrow L^2)$, $\varphi^j \in P_j$ for $j = 0, \ldots, \ell - 1$, and $\psi^{j-1} \in P_{\ell-1}$, $\alpha_h^{\ell-1}, \beta_h^{\ell-1} \in Y_h^\rho$ with $\text{supp} \alpha_h^{\ell-1} \cup \text{supp} \beta_h^{\ell-1} \subset \{ \chi_\ell > 0 \},$ such that

$$L_h^\ell \left( \sum_{j=0}^{\ell-1} h^{j/2} \varphi^j A_h^j \right) \geq h^{\ell/2} (s)^{-1} \psi^{j-1} P_{\alpha_h^{\ell-1} + h^{1/2} \beta_h^{\ell-1}} + O(h^\rho)_{L^\infty(I_h, L^2 \rightarrow L^2)}.$$  

Then as in the proof of (M.2) of Theorem 1.4, we set

$$\varphi^\ell(s) = \int_0^s (1 + \sigma)^{-1} \psi^{j-1}(\sigma) \, d\sigma, \quad A_h^\ell = C_\ell \varphi^\ell P_{\chi_\ell},$$

where the constant $C_\ell$ is sufficiently large, such that by Lemma 6.9, in the sense of positivity of matrices,

$$C_\ell \mathcal{L}_h^\rho (\varphi^\ell \chi_\ell) = C_\ell (1 + s)^{-1} \psi^{j-1} \chi_\ell + C_\ell \varphi^\ell \mathcal{L}_h^\rho \chi_\ell \geq (s)^{-1} \psi^{j-1} \alpha_h^{\ell-1} + \varphi^\ell h^{1/2} (s)^{-1} \beta_h^{\ell-1}.$$

for some $\beta_h^{\ell-1} \in Y_h^\rho$. By the paradiiferential Garding inequality, and a routine construction of parametrix, we find $\tilde{a}_h^\ell \in Y_h^\rho$, with $\text{supp} \tilde{a}_h^\ell \subset \{ \chi_{\ell+1} > 0 \}$, such that

$$P_{\epsilon^\rho} h^\rho \chi_\ell \leq \mathcal{L}_h^\rho (\varphi^\ell \chi_\ell) \leq \mathcal{L}_h^\rho (\varphi^\ell \chi_\ell) + \epsilon^\rho h^\rho \chi_\ell.$$ 

Similarly as in the estimate of $A_h^0$, by a symbolic calculus, we find $\tilde{a}_h^\ell, \beta_h^\ell \in Y_h^\rho$, with $\text{supp} \tilde{a}_h^\ell \cup \text{supp} \beta_h^\ell \subset \text{supp} \chi_\ell$, such that

$$L_h^\ell A_h^\ell = \mathcal{L}_h^\rho (\varphi^\ell \chi_\ell) + h^{1/2} (s)^{-1} \varphi^\ell P_{a_h^\ell + h^{1/2} \beta_h^\ell} + O(h^\rho)_{L^\infty(I_h, L^2 \rightarrow L^2)}.$$ 

Summing up the two inequalities above,

$$L_h^\ell A_h^\ell - (s)^{-1} \psi^{j-1} P_{\alpha_h^{\ell-1}} \geq h^{1/2} (s)^{-1} \psi^{j-1} P_{\alpha_h^{\ell-1} + h^{1/2} \beta_h^\ell} + O(h^\rho)_{L^\infty(I_h, L^2 \rightarrow L^2)}.$$ 

Therefore, combining (6.10) and (6.11),

$$L_h^\ell \left( \sum_{j=0}^{\ell} h^{j/2} \varphi^j A_h^j \right) \geq h^{(\ell+1)/2} (s)^{-1} \psi^{j-1} P_{\alpha_h^{\ell-1} + h^{1/2} \beta_h^\ell} + O(h^\rho)_{L^\infty(I_h, L^2 \rightarrow L^2)}.$$ 

with

$$\psi^\ell = 1 + \psi^{j-1} + \varphi^\ell, \quad \alpha_h^\ell = \psi^{j-1} \psi^{-1} \beta_h^{\ell-1} + \varphi^\ell (\alpha_h^\ell + \beta_h^\ell), \quad \beta_h^\ell = \psi^{j-1} + \varphi^\ell \alpha_h^\ell + \varphi^\ell \beta_h^\ell.$$ 

And we close the induction procedure.

To finish the proof, suppose that

$$\left( \frac{3}{2} t_0 \right)^{1/2} \xi_{\infty}^{-1/2} \xi_{\infty} \notin \text{WF}^\sigma_{1/2,1} \left( u(t_0) \right),$$

then by Lemma 2.16,

$$\left( \frac{3}{2} t_0 \right)^{1/2} \xi_{\infty}^{-1/2} \xi_{\infty} \notin \text{WF}^\sigma_{1/2,1} \left( u(t_0) \right).$$

By (i) and (iii), if we replace $\phi$ with $\phi(\lambda \cdot)$ for some sufficiently large $\lambda > 0$, then for sufficiently small $h > 0$,

$$\text{supp} \theta^{1/2,0}_h \chi_j |_{s=h^{-1/2}t_0} \subset \mathbb{R}^d \times \mathbb{R}_\xi^d \setminus \text{WF}^\sigma_{1/2,1} \left( u(t_0) \right),$$

$$\text{supp} \theta^{1/2,0}_h \chi_j |_{s=h^{-1/2}t_0} \subset \mathbb{R}^d \times \mathbb{R}_\xi^d \setminus \text{WF}^\sigma_{1/2,1} \left( u(t_0) \right).$$

So by Proposition 4.33 and Lemma 2.15,

$$\left( A_h w, w \right)_{L^2} \big|_{s=h^{-1/2}t_0} = O(h^{2\sigma}).$$

Because by our construction, $\varphi^\ell(0) = 0$, $\forall \ell \geq 1$,

$$A_h |_{s=0} = A_h^0 |_{s=0} = \left( \mathcal{P}_{\chi_0}^h \right)^* \mathcal{P}_{\chi_0}^h |_{s=0}.$$
Because $F_h = \mathcal{O}(h^{1/2})_{H^s}$, we have, by Lemma 2.16,
\[
\mathcal{A}_h F_h = \mathcal{O}(h^{\beta+1/2})_{L^2}.
\]
Therefore,
\[
\|T^h_{\chi_0} w|_{s=0}\|_{L^2}^2 = \frac{\text{Re}(\mathcal{A}_h w, w)_{L^2}}{\mathcal{A}_h F_h, w}_{L^2} ds - \int_0^{h^{-1/2}} t_0 \mathcal{A}_h F_h, w)_{L^2} ds 
\leq \mathcal{O}(h^{2\sigma}) + \mathcal{O}(h^{\beta-1/2}) = \mathcal{O}(h^{2\sigma}).
\]
Observe that $\chi_0|_{s=0}$ is of compact support with respect to $x$, we have
\[
T^h_{\chi_0|_{s=0}} = T^h_{\beta} + \mathcal{O}(h^\sigma)_{L^2 \to L^2},
\]
where
\[
\beta_h = \sum_{j \geq 0} \psi_j \chi_0|_{s=0} \psi_j \in h^{-\infty}.
\]
is a finite summation. We conclude by Lemma 4.35 that $(x_0, \xi_0) \notin WF_{0,1}(u_0)$. \qed

6.4. **Proof of Corollary 1.9.** The case when $d = 1$ is trivial. For the second case, we shall prove that on any co-geodesic $\{(x_t, \xi_t)\}_{t \in \mathbb{R}}$,
\[
\lim_{t \to \pm \infty} x_t \cdot \xi_t = \infty,
\]
so no geodesics can be trapped. The proof of (6.12) is almost finished by the proof of Lemma 6.8. Indeed, similar calculations imply that
\[
\frac{d}{dt} (x_t \cdot \xi_t) \gtrsim |\xi_0|^2.
\]
\qed

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