FANO THREEFOLDS WITH CANONICAL GORENSTEIN SINGULARITIES AND BIG DEGREE

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ABSTRACT. We provide a complete classification of Fano threefolds \( X \) with canonical Gorenstein singularities such that \((-K_X)^3 \geq 64\).

1. Introduction

Let \( X \) be a Fano threefold with canonical Gorenstein singularities. Then for the anticanonical degree \((-K_X)^3\) of \( X \) the following result holds:

**Theorem 1.1** (see [32, Theorem 1.5]). \((-K_X)^3 \leq 72\) with equality only for \( X = \mathbb{P}(3,1,1,1) \) or \( \mathbb{P}(6,4,1,1) \).

On the other hand, we have

**Theorem 1.2** (see [19, Theorem 1.5]). If \( 64 < (-K_X)^3 < 72 \), then \( X \) is one of the following:

- \( X_{70} \): the image of the anticanonically embedded threefold \( \mathbb{P}(6,4,1,1) \subset \mathbb{P}^{38} \) under birational projection from a singular cDV point on \( \mathbb{P}(6,4,1,1) \). In this case \((-K_X)^3 = 70\) and the singularities of \( X \) are non-cDV;

- \( X_{66} \): the anticanonical image of the \( \mathbb{P}^2 \)-bundle \( \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(5) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}) \). In this case \((-K_X)^3 = 66\) and the singularities of \( X \) are non-cDV.

The aim of the present paper is to classify those Fano threefolds \( X \) with \((-K_X)^3 = 64\). It follows from [14] and [15] that for such non-singular \( X \) the only possibility is \( X = \mathbb{P}^3 \). There are more examples in the singular case:

**Example 1.3.** Consider the weighted projective space \( X := \mathbb{P}(4,2,1,1) \). The singular locus of \( X \) consists of the curve \( L \approx \mathbb{P}^1 \) with the point \( P \in L \) such that \((P \in X)\) is the singularity of type \( \frac{1}{2}(2,1,1) \) and for every point \( O \in L \setminus \{P\} \) singularity \((O \in X)\) is analytically isomorphic to \((0,o) \in \mathbb{C} \times U\), where \((o \in U)\) is the singularity of type \( \frac{1}{2}(1,1) \) (see [12, 5.15]). Then Theorem 3.1 and Remark 3.2 in [35] imply that the singularities of \( X \) are canonical and Gorenstein. On the other hand, we have \( \mathcal{O}_X(-K_X) \cong \mathcal{O}_X(8) \) (see [5, Theorem 3.3.4]), which implies that the divisor \(-K_X\) is ample. Thus, \( X \) is a Fano threefold with canonical Gorenstein singularities. Furthermore, we have \((-K_X)^3 = 64\) (see [31]). Note that \( \mathbb{P}(4,2,1,1) \) is the unique weighted projective space which is a singular Fano threefold with canonical Gorenstein singularities and of the anticanonical degree 64 (see [31]).

**Example 1.4.** Let \( X \subset \mathbb{P}^9 \) be the cone over the anticanonically embedded surface \( S := \mathbb{P}^1 \times \mathbb{P}^1 \). Let \( f : Y \to X \) be the blow up of the vertex on \( X \). Then \( Y = \mathbb{P}(\mathcal{O}_S \oplus \mathcal{O}_S(-K_S)) \) and \( f \) is the birational contraction of the negative section of the \( \mathbb{P}^1 \)-bundle \( Y \). From the relative Euler exact sequence we obtain that \(-K_Y \sim 2M\), where \( \mathcal{O}_Y(M) \cong \mathcal{O}_Y(1) \) is the tautological sheaf on \( Y \) (see for example [28, Proposition 4.26]). On the other hand, \( f \) is given by the linear system \(|M|\), which implies that \( Y \) is a weak Fano threefold such that \( K_Y = f^*(K_X) \). In particular, \( X \) is a Fano threefold with canonical
Gorenstein singularities (see [20]). Moreover, from the Hirsch formula (see for example [8]) we get that \((-K_X)^3 = (-K_Y)^3 = 64\). Note also that \(X\) is toric.

**Example 1.5.** Let \(X \subset \mathbb{P}^9\) be the cone over the anticanonically embedded surface \(S := F_1\) (note that the divisor \(-K_S\) is ample on \(S\) by the Kleiman’s criterion for ampleness (see for example [21, Theorem 0-1-2]), and since \(S\) is a non-singular toric surface, this divisor is very ample (see for example [6])). Let \(f : Y \rightarrow X\) be the blow up of the vertex on \(X\). Then \(Y = \mathbb{P}(O_S \oplus O_S(-K_S))\) and \(f\) is the birational contraction of the negative section of the \(\mathbb{P}^1\)-bundle \(Y\). As in Example 1.4 we obtain that \(Y\) is a weak Fano threefold such that \(K_Y = f^*(K_X)\) and \((-K_Y)^3 = 64\), which implies that \(X\) is a Fano threefold with canonical Gorenstein singularities such that \((-K_X)^3 = 64\). Note also that \(X\) is toric.

**Remark 1.6.** It is not difficult to see that the singularities of the Fano threefolds from Examples 1.3–1.5 are non-terminal. On the other hand, Theorem 3.16 in Section 3 below implies that \(\mathbb{P}^3\) is the unique Fano threefold having terminal Gorenstein singularities and anticanonical degree 64.

Let us now state the main result of the present paper:

**Theorem 1.7.** If \((-K_X)^3 = 64\), then \(X\) is one of the following:

- \(\mathbb{P}^3\);
- the cone from Example 1.4;
- the cone from Example 1.5;
- the image of the anticanonically embedded threefold \(\mathbb{P}(3,1,1,1) \subset \mathbb{P}^{35}\) under birational projection from the tangent space at a smooth point on \(\mathbb{P}(3,1,1,1)\);
- the image of the anticanonically embedded threefold \(\mathbb{P}(6,4,1,1) \subset \mathbb{P}^{35}\) under birational projection from the tangent space at a smooth point on \(\mathbb{P}(6,4,1,1)\);
- the image of the anticanonically embedded threefold \(X_{66} \subset \mathbb{P}^{35}\) under birational projection from a singular cDV point on \(X_{66}\).

Moreover, in all cases the singularities of \(X\) are non-cDV, with the only exception for \(X = \mathbb{P}^3\).

Thus, Theorems 1.1, 1.2 and 1.7 exhaust all the Fano threefolds with canonical Gorenstein singularities having anticanonical degree \(\geq 64\).

**Remark 1.8.** Except for \(\mathbb{P}^3\) and the cones from Examples 1.4, 1.5, some of the threefolds from Theorem 1.7 may be isomorphic, since there are exactly five toric Fano threefolds having canonical Gorenstein singularities and anticanonical degree 64 (see [25]). It is interesting to investigate which of the threefolds from Theorem 1.7 are isomorphic to \(\mathbb{P}(4,2,1,1)\).

The proof of Theorem 1.7 relies on the methods developed in [32] and [19] to prove Theorems 1.1 and 1.2 respectively. In the present paper, for our convenience, we reproduce the proof of Theorem 1.2 adding minor corrections and technical improvements (see Sections 4–7). We prove Theorem 1.7 in Section 8.

In Sections 2 and 3 we fix the notation which is used in the paper, and provide some preliminary results and conventions.

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2. **Notation and conventions**

We use standard notions and facts from the theory of minimal models and singularities of pairs (see for example [24, 21] and [23]). We also use standard notions and facts from the theory of algebraic varieties and schemes (see for example [10]). All algebraic varieties are assumed to be projective and defined over \(\mathbb{C}\). Morphisms between algebraic varieties are assumed to be projective. A point on algebraic variety means a closed point.
Let us fix some notation and notions which we will use throughout the paper (see [10], [23], [24]):

- the linear equivalence of two Weil divisors $D_1$, $D_2$ on normal algebraic variety $V$ is denoted by $D_1 \sim D_2$. The Picard group of $V$ is denoted by $\text{Pic}(V)$;
- $\text{Sing}(V)$ denotes the singular locus of algebraic variety $V$. The analytic germ of point $O$ on $V$ is denoted by $(O \in V)$;
- for a $\mathbb{Q}$-Cartier divisor $L$ (respectively, linear system $\mathcal{L}$) and an algebraic cycle $Z$ on normal algebraic variety $V$, the restriction of $L$ (respectively, of $\mathcal{L}$) to $Z$ is denoted by $L|_Z$ (respectively, $\mathcal{L}|_Z$). For algebraic cycles $Z_1, \ldots, Z_k$ on $V$, the intersection of $Z_i$ in the Chow group of $V$ is denoted by $(Z_1 \cdot \ldots \cdot Z_k)_V, k \in \mathbb{N}$;
- the numerical equivalence of two $\mathbb{Q}$-Cartier divisors $L_1$, $L_2$ (respectively, 1-cycles $Z_1$, $Z_2$) on normal algebraic variety $V$ is denoted by $L_1 \equiv L_2$ (respectively, $Z_1 \equiv Z_2$). The Picard number of $V$ is denoted by $\rho(V)$;
- a normal algebraic variety $V$ is called a Fano variety (respectively, a weak Fano variety) if it has at most canonical Gorenstein singularities and the anticanonical divisor $-K_V$ is ample (respectively, nef and big). For $V$ we put $(-K_V)^3 := -(K_V^3)_V$ to be the anticanonical degree of $V$;
- for a Weil divisor $D$ on normal algebraic variety $V$, the corresponding divisorial sheaf is denoted by $\mathcal{O}_V(D)$;
- for a vector bundle $\mathcal{E}$ on non-singular algebraic variety $V$, the associated projective bundle is denoted by $\mathbb{P}(\mathcal{E})$ and the $i$-th Chern class of $\mathcal{E}$ is denoted by $c_i(\mathcal{E})$;
- for a coherent sheaf $\mathcal{F}$ on algebraic variety $V$, the $i$-th cohomology group of $\mathcal{F}$ is denoted by $H^i(V, \mathcal{F})$ and the Euler characteristic of $\mathcal{F}$ is denoted by $\chi(V, \mathcal{F})$;
- for a Cartier divisor $L$ on normal algebraic variety $V$, the corresponding complete linear system is denoted by $[L]$. For a linear system $\mathcal{L}$ on $V$, the base locus of $\mathcal{L}$ is denoted by $\text{Bs}(\mathcal{L})$. If $\mathcal{L}$ does not have base components, then the corresponding rational map is denoted by $\Phi_{\mathcal{L}}$;
- for a birational map $\chi : V' \dashrightarrow V$ between normal algebraic varieties and an algebraic cycle $Z$ (respectively, linear system $\mathcal{L}$) on $V$, the proper transform of $Z$ (respectively, of $\mathcal{L}$) on $V'$ is denoted by $\chi^*(Z)$ (respectively, by $\chi^*(\mathcal{L})$);
- for a rational map $\chi$ from algebraic variety $V$ and a subvariety $Z \subset V$, the restriction of $\chi$ to $Z$ is denoted by $\chi|_Z$;
- $\kappa(V)$ denotes the Kodaira dimension of a normal algebraic variety $V$;
- for a toric variety $V$, the torus $\mathbb{C}^\dim V$ with canonical action on $V$ is denoted by $T$;
- $\mathbb{F}_n$ denotes the Hirzebruch surface with a fibre $l$ and the minimal section $h$ such that $(h^2)_{\mathbb{F}_n} = -n, n \in \mathbb{Z}_{\geq 0}$.

3. Preliminaries

Let $X$ be a Fano threefold. From the Riemann–Roch Formula (see [37]) and Kawamata–Viehweg Vanishing Theorem (see for example [21]) we obtain

$$\dim | -K_X | = \frac{1}{2}K_X^3 + 2.$$ 

The number $g := -\frac{1}{2}K_X^3 + 1$ is integer and is called the genus of $X$. We have $\dim | -K_X | = g + 1$ and $(-K_X)^3 = 2g - 2$. In what follows we consider only such $X$ that $(-K_X)^3 \geq 64$. In particular, we have

$$(-K_X)^3 \in \{64, 66, 70, 72\}, \quad g \in \{33, 34, 36, 37\}$$

(see Theorems [1.1] and [1.2]). Further, the following results take place:
Theorem 3.2 (see [17]). If $|N| - K_X| \neq \emptyset$, then $(-K_X)^3 \leq 22$.

Thus, the anticanonical linear system $| - K_X|$ is base point free on $X$. Consider the morphism $\Phi_{|K_X|} : X \rightarrow \mathbb{P}^{g+1}$.

Theorem 3.3 (see [34], Theorem 1.5]). If $\Phi_{|K_X|}$ is not an embedding, then $(-K_X)^3 \leq 40$.

Theorem 3.4 (see [34], Theorem 1.6). If $\Phi_{|K_X|}$ is an embedding and $\Phi_{|K_X|}^{-1}(X)$ is not an intersection of quadrics, then $(-K_X)^3 \leq 54$.

From Theorems 3.2, 3.3 we get

Corollary 3.5. The linear system $| - K_X|$ gives an embedding of $X$ in $\mathbb{P}^{g+1}$ such that the image $X_{g-2} := \Phi_{|K_X|}(X)$ is an intersection of quadrics.

In what follows, we assume that $X = X_{g-2} \subset \mathbb{P}^{g+1}$ is anticanonically embedded.

Proposition 3.6 (see for example [24]). Let $V$ be a normal algebraic threefold with canonical singularities. Then there exists a normal algebraic threefold $V'$ with terminal $\mathbb{Q}$-factorial singularities and birational morphism $\tau : V' \rightarrow V$ such that $K_{V'} \equiv \tau^*(K_V)$.

Threefold $V'$ (or morphism $\tau$) from Proposition 3.6 is called a terminal $\mathbb{Q}$-factorial modification of $V$.

Now, let $f : Y \rightarrow X$ be the terminal $\mathbb{Q}$-factorial modification of $X$. Here $Y$ is a weak Fano threefold with terminal Gorenstein $\mathbb{Q}$-factorial singularities such that $(-K_X)^3 = (-K_Y)^3$. Moreover, according to [20], Lemma 5.1, $Y$ is factorial.

Remark 3.7. Conversely, for every weak Fano threefold $Y$ with terminal factorial singularities its image $X := f(Y)$ under the morphism $f := \Phi_{|nK_Y|}$ for some $n \in \mathbb{N}$ is a Fano threefold such that $K_Y \equiv f^*(K_X)$. In particular, $X$ has only canonical Gorenstein singularities (see [20]).

Remark 3.8. It follows from [16], Proposition 2.1.2 that $K_Y = f^*(K_X)$. This implies, since $O_X(-K_X) \simeq O_X(1)$, that the anticanonical linear system $| - K_Y|$ is base point free on $Y$.

Remark 3.9. Let $Y, Y'$ be two terminal $\mathbb{Q}$-factorial modifications of $X$. Then, since $Y$ and $Y'$ are relative minimal models over $X$ (see the proof of Proposition 3.6 in [24]), by [22], Theorem 4.3 birational map $Y \dasharrow Y'$ is either an isomorphism or a composition of $K_Y$-flops over $X$. In particular, it follows from [22], Theorem 2.4 that if $Y$ is non-singular, then $Y'$ is also non-singular.

Proposition 3.10 (see [32], Lemmas 4.2 and 4.3). In the above notations, the Mori cone $\overline{NE}(Y)$ is polyhedral and is generated by contractible extremal rays $R_i$ (i.e., there exists a morphism $f_{R_i} : Y \rightarrow Y'$ from a normal algebraic threefold $Y'$ such that the curve $Z \subset Y$ is mapped by $f_{R_i}$ to a point $f(Z) \in R_i$ for the numerical class of $Z$).

Remark 3.11. If $X \neq Y$, then Proposition 3.10 implies that $-K_Y$ determines a facet of the Mori cone $\overline{NE}(Y)$ and $f : Y \rightarrow X$ is its extremal contraction. In particular, $\rho(Y) > \rho(X)$ and $X$ is uniquely determined by $Y$.

Birational contractions $f : Y \rightarrow X$ from Examples 1.3 and 1.5 are precisely the terminal $\mathbb{Q}$-factorial modifications. Let us consider some other examples:

Example 3.12. Let $X \subset \mathbb{P}^4$ be the cone over the anticanonically embedded surface $\mathbb{P}^2$. Let $f : Y \rightarrow X$ be the blow up of the vertex on $X$. Then $Y = \mathbb{P}(O_{\mathbb{P}^2} \oplus O_{\mathbb{P}^2}(3))$ and $f$ is the birational contraction of the negative section of the $\mathbb{P}^1$-bundle $Y$. From the relative Euler exact sequence we obtain $-K_Y \sim 2M$, where $O_Y(M) \simeq O_Y(1)$ is the tautological sheaf on $Y$ (see for example [28], Proposition 4.26). On the other hand, $f$ is given by the linear system $|M|$, which implies that $Y$ is a weak Fano threefold such that $K_Y = f^*(K_X)$. In particular, $X$ is a Fano threefold with canonical Gorenstein singularities and $f : Y \rightarrow X$ is its terminal $\mathbb{Q}$-factorial modification (see Remark 3.7). Furthermore, it follows from the proof of Theorem 1.1 in [32] that $X \simeq \mathbb{P}(3,1,1,1)$.
Remark 3.13. In the notation from Example 3.12, the morphism $f$ is an extremal birational contraction with exceptional locus isomorphic to $\mathbb{P}^2$ (see also Remark 3.11). This implies that there are no small $K_Y$-trivial extremal contractions on $Y$. Then it follows from Remark 3.9 that every terminal $\mathbb{Q}$-factorial modification of the threefold $\mathbb{P}(3,1,1,1)$ is isomorphic to $Y$.

Example 3.14. Consider the weighted projective space $X := \mathbb{P}(6,4,1,1)$. The singular locus of $X$ is a curve $L \simeq \mathbb{P}^1$ such that for two points $P$ and $Q$ on $L$ singularities $(P \in X)$ and $(Q \in X)$ are of type $\frac{1}{6}(4,1,1)$ and $\frac{1}{4}(2,1,1)$, respectively, and for every point $O \in L \setminus \{P\}$ singularity $(O \in X)$ is analytically isomorphic to $((0,o) \in \mathbb{C} \times U)$, where $(o \in U)$ is the singularity of type $\frac{1}{2}(1,1)$ (see [12, 5.15]). Then Theorem 3.1 and Remark 3.2 in [35] imply that the singularities of $X$ are canonical and Gorenstein. On the other hand, we have $O_X(−K_X) \simeq O_X(12)$ (see [5, Theorem 3.3.4]), which implies that the divisor $−K_X$ is ample. Thus, $X$ is a Fano threefold with canonical Gorenstein singularities (see Theorem 1.1).

Let $f_1 : Y_1 \to X := \mathbb{P}_0$ be the weighted blow up of the point $P$ with weights $\frac{1}{6}(4,1,1)$. Then the singular locus of the threefold $Y_1$ is a curve $L_1$ such that for two points $P_1$ and $Q_1$ on $L_1$ singularities $(P_1 \in Y_1)$ and $(Q_1 \in Y_1)$ are of type $\frac{1}{4}(2,1,1)$ and for every point $O \in L_1 \setminus \{P_1, Q_1\}$ singularity $(O \in Y_1)$ is analytically isomorphic to $((0,o) \in \mathbb{C} \times U)$, where $(o \in U)$ is the singularity of type $\frac{1}{2}(1,1)$.

Let $f_2 : Y_2 \to Y_1$ be the weighted blow up of the points $P_1$ and $Q_1$ with weights $\frac{1}{4}(2,1,1)$. Then the singular locus of the threefold $Y_2$ is a curve $L_2 \simeq \mathbb{P}^1$ such that for every point $O$ on $L_2$ singularity $(O \in Y_2)$ is analytically isomorphic to $((0,o) \in \mathbb{C} \times U)$, where $(o \in U)$ is the singularity of type $\frac{1}{2}(1,1)$.

Finally, the blow up $f_3 : Y_3 \to Y_2$ of the curve $L_2$ on $Y_2$ leads to a non-singular threefold $Y := Y_3$ and a birational morphism $f : Y \to X$. By construction we have $K_{Y_3} = f_3^*(K_{Y_3})$ and $f$ is the composition of $f_i$, $1 \leq i \leq 3$. This implies that $K_Y = f^*(K_X)$ and $Y$ is a terminal $\mathbb{Q}$-factorial modification of $X$.

Remark 3.15. In the notation from Example 3.14, the morphism $f$ is a composition of extremal birational contractions (see Remark 3.11) and the exceptional locus of $f$ has pure codimension 1 on $Y$. This implies that there are no small $K_Y$-trivial extremal contractions on $Y$. Then it follows from Remark 3.9 that every terminal $\mathbb{Q}$-factorial modification of the threefold $\mathbb{P}(6,4,1,1)$ is isomorphic to $Y$.

Now, let $\text{ext} : Y \to Y'$ be a $K_Y$-negative extremal contraction (see Proposition 3.10). Then the following results take place:

Theorem 3.16. Let $V$ be a Fano threefold with terminal Gorenstein singularities. Then there exists a flat deformation of $V$ into non-singular Fano threefold, which implies that $(-K_V)^3 \leq 64$. Moreover, if $(-K_V)^3 = 64$, then $V = \mathbb{P}^3$.

Proof. By the main result in [30] there exists a flat morphism $\nu : V \to \Delta$, where $\Delta \ni 0$ is the unit disk in $\mathbb{C}$, such that $\nu^{-1}(0) \simeq V$ and $V_t := \nu^{-1}(t)$ is a non-singular Fano threefold for all $0 \neq t \in \Delta$. In particular, the results of [14] and [15] imply that $(-K_V)^3 \leq 64$.

Further, it follows from the proof of Theorem 1.4 in [18] that for every $0 \neq t \in \Delta$ there exists an isomorphism $\varphi : \text{Pic}(V) \simeq \text{Pic}(V_t)$ such that $\varphi(K_V) = K_{V_t}$. Now, suppose that $(-K_V)^3 = 64$. Then $(-K_{V_t})^3 = 64$, and the results of [14] and [15] imply that $V_t \simeq \mathbb{P}^3$ for all $0 \neq t \in \Delta$. Moreover, the above isomorphism $\varphi$ implies equalities $r(V) = r(V_t) = 4$ for the Fano indexes of $V$ and $V_t$ for all $t \in \Delta$, since $\text{Pic}(V) \simeq \text{Pic}(V_t) \simeq \mathbb{Z}$. Then by [16, Theorem 3.1.14] we have $V = \mathbb{P}^3$. \hfill $\Box$

Thus, if $\dim Y' = 0$, then $X = Y$, and Theorem 3.16 implies that $(-K_X)^3 \leq 64$ with equality only for $X = \mathbb{P}^3$.

Theorem 3.17 (see [32, Proposition 4.11]). If $\dim Y' = 1$, i.e., ext : $Y \to Y'$ is a del Pezzo fibration, then $(-K_X)^3 \leq 54$.\hfill 5
Theorem 3.18 (see [32, Proposition 5.2]). If $\dim Y' = 2$, i.e., $\text{ext} : Y \to Y'$ is a conic bundle, then we have:

- $Y'$ is a non-singular surface such that the divisor $-K_{Y'}$ is nef and big;
- if $\text{ext} : Y \to Y'$ is not a $\mathbb{P}^1$-bundle, then $(-K_X)^3 \leq 64$;
- if $\text{ext} : Y \to Y'$ is a $\mathbb{P}^1$-bundle, then either $(-K_X)^3 \leq 64$ or $(-K_X)^3 = 72$ with $X \cong \mathbb{P}(3, 1, 1, 1)$.

Theorem 3.19. If $\text{ext}$ is the birational morphism, then we have:

- $\text{ext}$ is divisorial with exceptional divisor $E$;
- if $O := \text{ext}(E)$ is a point, then $\text{ext}$ is the blow up of $Y'$ at $O$. Furthermore, the threefold $Y'$ is a weak Fano threefold with terminal singularities such that $(-K_{Y'})^3 > (-K_Y)^3$. Moreover, $Y'$ is factorial except for the case when $E \cong \mathbb{P}^2$, $\mathcal{O}_E(E) \cong \mathcal{O}_{\mathbb{P}^2}(2)$ and $(O \in Y')$ is the singularity of type $\frac{1}{2}(1, 1, 1)$;
- if $C := \text{ext}(E)$ is a curve, then $Y'$ is non-singular near $C$, the curve $C$ is reduced and irreducible and $\text{ext}$ is the blow up of $Y'$ at $C$.

Proof. The statement follows from [3], [27] and the equality

$$K_Y = \text{ext}^*(K_{Y'}) + \alpha E$$

on $Y$ for some positive $\alpha \in \mathbb{Q}$. \hfill \Box

Corollary 3.20. In the assumptions of Theorem 3.19 if $(\text{ext}(E) \in Y')$ is the singularity of type $\frac{1}{2}(1, 1, 1)$, then $f(E)$ is a plane on $X$ (i.e., the surface $\Pi \cong \mathbb{P}^2$ on $X$ such that $(K_X^3 \cdot \Pi)_X = 1$).

Proof. Since $E \cong \mathbb{P}^2$ and $\mathcal{O}_E(E) \cong \mathcal{O}_{\mathbb{P}^2}(2)$, we have

$$(K^2_Y \cdot E)_Y = ((\text{ext}^*(K_{Y'})) + \frac{1}{2}E)^2 \cdot E)_Y = \frac{1}{4}(E^3)_Y = 1,$$

which implies that $f(E)$ is the surface $\Pi \cong \mathbb{P}^2$ on $X$ such that $(K_X^3 \cdot \Pi)_X = (K_Y^3 \cdot E)_Y = 1$. \hfill \Box

Proposition 3.21 (see [32, Proposition 4.5]). If $C := \text{ext}(E)$ is a curve, then $Y'$ has terminal factorial singularities and one of the following holds:

- $Y'$ is a weak Fano threefold with $(-K_{Y'})^3 \geq (-K_Y)^3$;
- for $C$ we have $(K_{Y'} \cdot C)_{Y'} > 0$ and it is the only curve on $Y'$ which intersects $K_{Y'}$ positively. Furthermore, in this case $C \cong \mathbb{P}^1$ and $E \cong \mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_1$.

Proof. Let us use the arguments from the proof of Proposition 4.5 in [32]. By Theorem 3.19 the curve $C$ is reduced and irreducible, the threefold $Y'$ is non-singular near $C$ and $\text{ext}$ is the blow up of $Y'$ at $C$. In particular, $Y'$ has terminal factorial singularities and $K_Y = \text{ext}^*(K_{Y'}) + E$ on $Y$. Further, we have

$$(K^3_Y)_Y = (K^3_{Y'})_{Y'} + 3(\text{ext}^*(K_{Y'}) \cdot E^2)_Y + (E^3)_Y,$$

and if $Y'$ is a weak Fano threefold, then we obtain

$$0 \leq ((-K_Y)^2 \cdot E)_Y = 2(\text{ext}^*(K_{Y'}) \cdot E^2)_Y + (E^3)_Y$$

and

$$0 \leq ((-K_Y) \cdot (-\text{ext}^*(K_{Y'})) \cdot E)_Y = (\text{ext}^*(K_{Y'}) \cdot E^2)_Y.$$

This together with (3.22) gives inequality $(-K_Y)^3 \geq (-K_Y)^3$.

Now, if $Y'$ is not a weak Fano threefold, then for some irreducible curve $Z$ on $Y'$ we get $(K_{Y'} \cdot Z)_{Y'} > 0$. It is easy to see that in fact $Z = C$. Then Proposition 3.10 and [27, Corollary 1.3] imply that $C \cong \mathbb{P}^1$. 

In particular, $E \cong \mathbb{F}_n$ for some $n \geq 0$. Further, since $-K_Y|_E$ is a section of the $\mathbb{P}^1$-bundle $E \to C$, we have

$$-K_Y|_E \sim h + (n + a)l$$

on $E$ for some $a \in \mathbb{Z}$. Moreover, we have $a \geq 0$ because the divisor $-K_Y$ is nef on $Y$. Then we obtain

$$0 > \left( -K_Y \cdot C \right)_Y = (K_Y^2 \cdot E)_Y - 2 + 2p_a(C) = \left( ( -K_Y|_E )^2 \right)_E - 2 = n + 2a - 2,$$

which implies that $a = 0$ and $n < 1$. Proposition 3.21 is completely proved. \qed

**Corollary 3.23.** In the assumptions of Proposition 3.21,

- if $E \cong \mathbb{P}^1 \times \mathbb{P}^1$, then $f(E)$ is a line on $X$ (i.e., a curve $\Gamma \cong \mathbb{P}^1$ on $X$ such that $(-K_X \cdot \Gamma)_X = 1$) and $X$ is singular along $f(E)$;
- if $E \cong \mathbb{F}_1$, then $f(E)$ is a plane on $X$ (i.e., a surface $\Pi \cong \mathbb{P}^2$ on $X$ such that $(K^2_X \cdot \Pi)_X = 1$).

**Proof.** In the notation from the proof of Proposition 3.21, if $E \cong \mathbb{P}^1 \times \mathbb{P}^1$, then $(-K_Y \cdot h)_Y = 0$ and $(-K_Y \cdot l)_Y = 1$. This implies that $f(E)$ is the curve $\Gamma \cong \mathbb{P}^1$ on $X$ such that $(-K_X \cdot \Gamma)_X = 1$.

If $E \cong \mathbb{F}_1$, then $(-K_Y \cdot h)_Y = 0$ and $h$ is the only curve on $E$ which intersects $K_Y$ by zero. This implies that $f(E)$ is the surface $\Pi \cong \mathbb{P}^2$ on $X$ such that $(K^2_X \cdot \Pi)_X = (K^2_X \cdot E)_Y = 1$. \qed

In the conclusion of the present Section let us prove some statements about projections of Fano threefolds. For this we will need the following result:

**Theorem 3.24** (see [37]). Let $V$ be a Fano threefold. Then the general surface in $| - K_V |$ has only Du Val singularities.

**Lemma 3.25.** Let $\pi : X \to X'$ be projection from a linear space such that $\dim X' = 3$. Then $\pi$ is birational.

**Proof.** This is obvious because $X$ is an intersection of quadrics (see Corollary 3.23). \qed

In the notation from Lemma 3.25 let $\Omega$ be the center of the projection $\pi : X \to X'$.

**Lemma 3.26.** If $X'$ is an anticanonically embedded Fano threefold, then the set $\Omega \cap \operatorname{Sing}(X)$ consists only of cDV points on $X$.

**Proof.** Suppose that $P \in \Omega \cap \operatorname{Sing}(X)$ is a non-cDV point on $X$. Then the general surface $S \in |-K_X|$ through $P$ is a K3 surface whose singularities are worse than Du Val. In particular, we have $\kappa(S) < 0$. On the other hand, $\pi$ gives a birational map $S \dashrightarrow S'$ on the general surface $S' \in |-K_X|$, which implies that $\kappa(S) = \kappa(S') < 0$. But by Theorem 3.24 $S'$ is a K3 with only Du Val singularities, which implies that $\kappa(S) = 0$, a contradiction. \qed

There is the following inversion of Lemma 3.26.

**Lemma 3.27.** Let $\pi : X \to X'$ be projection from the singular $cA_1$ point $O \in X$. Then $X'$ is an anticanonically embedded Fano threefold such that $(-K_X')^3 = (-K_X)^3 - 2$.

**Proof.** Consider the blow up $\sigma : W \to X$ of the threefold $X$ at the point $O$ and the commutative diagram

\[
\begin{array}{ccc}
 W & \xrightarrow{\sigma} & X \\
 \downarrow{\tau} & & \downarrow{\pi} \\
 & X' & \\
\end{array}
\]
Projection $\pi$ is given by the linear system $\mathcal{H} \subset -K_X$ of all hyperplane sections of $X$ passing through $O$. Since $O \in X$ is a singular cA$_1$ point, $W$ has at most canonical Gorenstein singularities$^1$ and for the general surface $H \in \mathcal{H}$ we have

$$\sigma^{-1}(H) = \sigma^*(H) - E_\sigma,$$

where $E_\sigma$ is the $\sigma$-exceptional divisor. On the other hand, from the adjunction formula we obtain equality

$$K_W = \sigma^*(K_X) + E_\sigma$$
on $W$. Thus, the morphism $\tau : W \to X'$ is given by the linear system $\sigma^{-1}(\mathcal{H}) \subseteq -K_W$. In particular, $W$ is a weak Fano threefold because $\sigma^{-1}(\mathcal{H})$ is base point free on $W$ and $(-K_W)^3 = (-K_X)^3 - 2 > 0$. This implies that $\dim X' = 3$ and the projection $\pi$ is birational (see Lemma 3.25). Then, since $\tau$ is a crepant morphism, threefold $X'$ has only canonical Gorenstein singularities (see [20]). Moreover, we have $O_{X'}(-K_{X'}) \cong O_{X'}(1)$, which implies that $X'$ is an anticanonically embedded Fano threefold such that $(-K_{X'})^3 = (-K_X)^3 - 2$.

$\square$

4. Constructions of special type

We use notation and conventions from Section 3. In the present Section we consider the case when $\text{ext} : Y \to Y'$ is the contraction on a weak Fano threefold $Y'$ with terminal factorial singularities. We also assume here that $64 < (-K_X)^3 < 72$.

According to Remark 3.7 threefold $Y'$ is a terminal $\mathbb{Q}$-factorial modification of some Fano threefold $X'$. Denote by $f' : Y' \to X'$ the corresponding crepant morphism (see Proposition 3.6) and let $E_{f'}$ be the $f'$-exceptional locus. By Theorem 3.19 and Proposition 3.21 we have $(-K_{Y'})^3 \geq (-K_Y)^3$. Then Corollary 3.5 implies that the anticanonical linear system $|-K_{X'}|$ gives an embedding of $X'$ in $\mathbb{P}^{g'+1}$ such that the image $X_{2g'-2} := \Phi_{-K_{X'}}(X')$ is an intersection of quadrics (here $g'$ is the genus of $X'$). In what follows, we assume that $X' = X_{2g'-2} \subset \mathbb{P}^{g'+1}$ is anticanonically embedded.

Consider the commutative diagram

$$(4.1)$$

$$\begin{array}{ccc}
Y' & \to & Y' \\
\downarrow f' & & \downarrow f' \\
X & \leftarrow & X' \\
\downarrow p & & \\
\end{array}$$

with the induced birational map $p$.

**Lemma 4.2.** If $C := \text{ext}(E)$ is a curve, then $p$ is the birational projection with center which cuts out $f'(C)$ on $X'$.

**Proof.** By Theorem 3.19 the curve $C$ is reduced and irreducible, the threefold $Y'$ is non-singular near $C$ and $\text{ext}$ is the blow up of $Y'$ at $C$. In particular, on $Y$ we have equality

$$K_Y = \text{ext}^*(K_{Y'}) + E.$$ (4.3)

Thus, the morphism $f$ is given by the linear system $|-K_Y| = |\text{ext}^*(-K_{Y'}) - E|$ (see Remark 3.8). This implies that the map $f \circ \text{ext}^{-1}$ is given by the linear system $|-K_{Y'} - C|$. On the other hand, one of the following holds:

- $E_{f'} \cap C = \emptyset$;
- $f'(C)$ is a point and $C$ belongs to that component of the divisor $E_{f'}$, which is mapped by $f'$ onto a curve;
- $f'$ is a small contraction of the curve $C$.

$^1$This easily follows from the Morse Lemma (see for example [11]).
Indeed, otherwise one can find a curve \( Z \subset Y \) such that \((K_{Y'} \cdot \text{ext}_s(Z))_{Y'} = 0 \) and \((E \cdot Z)_Y > 0 \). Then from equality (4.3) we obtain that \((K_Y \cdot Z)_Y > 0 \), which is impossible because the divisor \(-K_Y\) is nef on \(Y\).

Since \( K_{Y'} = f'^*(K_{X'}) \) on \(Y'\), we get that the map \( p \) is given by the linear system \(|-K_{X'} - f'(C)|\). □

**Lemma 4.4.** Inequality \((-K_{Y'})^3 > (−K_{Y})^3 \) holds.

**Proof.** By Theorem 3.19 if \( \text{ext}(E) \) is a point, then \((-K_{Y'})^3 > (−K_{Y})^3 \).

Now, let \( \text{ext}(E) \) be a curve. Then from Lemma 4.2 we get that \( g < g' \), since the dimension of the projective space decreases under projection. The inequality \((-K_{Y'})^3 > (−K_{Y})^3 \) now follows from the definition of genus (see Section 3). □

**Proposition 4.5.** If \( C := \text{ext}(E) \) is a curve and \((-K_{Y'})^3 = 72 \), then \( X' = \mathbb{P}(6,4,1,1) \), \((-K_{X'})^3 = 70 \) and \( X \) is the image of the threefold \( X' \) under birational projection from a singular cDV point on \(X'\).

**Proof.** By Theorem 1.1 we have \( X' = \mathbb{P}(3,1,1,1) \) or \( \mathbb{P}(6,4,1,1) \). Then it follows from Remarks 3.13 and 3.15 that \( X' \) is isomorphic to one of the threefolds constructed in Examples 3.12 and 3.14.

**Lemma 4.6.** \( X' \neq \mathbb{P}(3,1,1,1) \).

**Proof.** Suppose that \( X' = \mathbb{P}(3,1,1,1) \). Then from Example 3.12 we get that \( E_{P'} \) is an irreducible divisor which is contracted to the unique singular point on \(X'\). This implies that \( E_{P'} \cap C = \emptyset \) (see the proof of Lemma 4.2).

Further, by Lemma 4.2 birational map \( p : X' \rightarrow X \) (see 4.1) is the projection from some linear space \( V \subset \mathbb{P}^3 \) which cuts out the curve \( f'(C) \) on \(X'\). Moreover, it follows from 3.1 that \( \dim V \leq 2 \), which implies that \((-K_{X'} \cdot f'(C))_{X'} \leq 2 \). But for \( X' = \mathbb{P}(3,1,1,1) \) we have \( O_{X'}(-K_{X'}) \simeq O_{X'}(6) \) (see 5. Theorem 3.3.4)). Then for the divisor \( M \) on \( X' \) with \( O_{X'}(M) \simeq O_{X'}(1) \) we obtain

\[
0 < (M \cdot f'(C))_{X'} \leq \frac{1}{3},
\]

which implies that the curve \( f'(C) \) passes through the singular point on \(X'\). This contradicts \( E_{P'} \cap C = \emptyset \).

It follows from Lemma 4.6 that \( X' = \mathbb{P}(6,4,1,1) \). In the notation of Example 3.14 let \( P \) and \( Q \) be two singular points on \(X'\).

**Lemma 4.7.** \( f'(C) \) is a point on \(X'\), other than \( P \) and \( Q \).

**Proof.** By Remark 3.15 locus \( E_{P'} \) is of pure codimension 1 on \(Y'\). Suppose that \( f'(C) \) is a curve. Then we get \( E_{P'} \cap C = \emptyset \) (see the proof of Lemma 4.2). Now, as in the proof of Lemma 4.6 we obtain that \((-K_{X'} \cdot f'(C))_{X'} \leq 2 \). But for \( X' = \mathbb{P}(6,4,1,1) \) we have \( O_{X'}(-K_{X'}) \simeq O_{X'}(12) \) (see 5. Theorem 3.3.4)). Then for the divisor \( M \) on \( X' \) with \( O_{X'}(M) \simeq O_{X'}(1) \) we obtain

\[
0 < (M \cdot f'(C))_{X'} \leq \frac{1}{6},
\]

which implies that the curve \( f'(C) \) passes through singular points on \(X'\). In particular, since \( Y' \) is non-singular by Remark 3.15 we get \( E_{P'} \cap C \neq \emptyset \), a contradiction.

Thus, \( f'(C) \) is a point. Then by Lemma 4.2 birational map \( p \) (see 4.1) is the projection with center which cuts out the point \( f'(C) \) on \(X'\). Suppose that \( f'(C) = P \) or \( Q \). Then it follows from the arguments in Example 3.14 that \( f' \) contracts a component of \( E_{P'} \) into \( f'(C) \). Thus, there is an exceptional divisor over \(X'\) with the zero discrepancy and the center at \( f'(C) \), which implies that \( f'(C) \) is non-cDV (see for example 24. Theorem 5.34)). This contradicts Lemma 3.26. □

Set \( O := f'(C) \). Lemma 4.7 and Example 3.14 imply that \( (O \in X') \) is the cA\(_1\) singularity. Then Lemmas 4.2 and 3.27 complete the proof of Proposition 4.5. □
Proposition 4.8. In the assumptions of Proposition 4.3, a threefold \(X\) has the unique singular point, and this point is non-cDV.

Proof. In the notation of Example 3.14 let \(L\) be the singular locus of the threefold \(X' = \mathbb{P}(6, 4, 1, 1)\).

Lemma 4.9. The curve \(L\) is a line on \(X'\) (i.e., \(L \cong \mathbb{P}^1\) and \((-K_{X'} \cdot L)_{X'} = 1\)).

Proof. The curve \(L\) is given by equations \(x_2 = x_3 = 0\) on \(X'\), where \(x_0, x_1, x_2, x_3\) are weighted projective coordinates on \(X'\) of weights \(6, 4, 1, 1\), respectively (see [12, 5.15]). This implies that \(L \cong \mathbb{P}^1\). It remains to show that \((-K_{X'} \cdot L)_{X'} = 1\).

Let \(S\) be the surface on \(X'\) with equation \(x_3 = 0\). Then \(L \subset S\) and

\[
(-K_{X'} \cdot L)_{X'} = (-K_{X'}|_S \cdot L)_S = 1,
\]

where the last intersection is taken on \(S = \mathbb{P}(6, 4, 1) \cong \mathbb{P}(3, 2, 1)\) (see [12, 5.7]).

Since \(\mathcal{O}_{X'}(-K_{X'}) \cong \mathcal{O}_{X'}(12)\) (see [5, Theorem 3.3.4]), the general element \(D \in | -K_{X'}|\) is given by equation

\[
a_0x_0^2 + a_0x_2x_3) + a_4(x_2, x_3)x_1 + a_8(x_2, x_3)x_1 + a_12(x_2, x_3) = 0
\]

on \(X'\), where \(a_i(x_2, x_3)\) are homogeneous polynomials of degree \(i\) in the variables \(x_2, x_3\).

If \(y_0, y_1, y_2\) are weighted projective coordinates on the surface \(S\) with weights \(3, 2, 1\), respectively, then \(D|_S\) is given by equation

\[
b_0y_0^3 + b_1y_0y_1y_2 + b_3y_1^3 + b_4y_1^2y_2^2 + b_5y_1y_2^4 + b_6y_2^6 = 0
\]

on \(S\), where \(b_i \in \mathbb{C}\) (see [12, 5.10]). This implies that \(\mathcal{O}_S(D|_S) \cong \mathcal{O}_S(6)\). On the other hand, we have \(\mathcal{O}_S(L) \cong \mathcal{O}_S(1)\). Thus, we obtain

\[
(-K_{X'} \cdot L)_{X'} = (-K_{X'}|_S \cdot L)_S = 1,
\]

where the last intersection is taken on \(S = \mathbb{P}(3, 2, 1)\) (see [5]). \(\square\)

Lemma 4.10. \(L\) is the unique line on \(X'\).

Proof. Let \(L_0 \neq L\) be another line on \(X'\). Denote by \(L'_0\) the proper transform of \(L_0\) on \(Y'\). Since \((-K_{Y'} \cdot L'_0)_{Y'} = 1\) and \(\rho(X') = 1\), it follows from the Cone Theorem (see for example [21, Theorem 4.2-1]) that \(R := \mathbb{R}_{\geq 0}[L'_0]\) is an extremal ray. Then it follows from the Contraction Theorem (see for example [21, Theorem 3.2-1]), Theorems 3.16, 3.17, 3.18 and 3.19 that there exists a \(K_{Y'}\)-negative extremal birational contraction \(f_R : Y' \rightarrow Y'_R\) on \(R\) with exceptional divisor \(E_R\).

If \(Y'_R\) is a weak Fano threefold with terminal factorial singularities, then from Theorem 3.19 and Proposition 3.21 we obtain that \((-K_{Y'_R})^3 \geq -K_{Y'_R})^3 = 72\). Hence by Theorem 1.1 and Remark 3.7 threefold \(Y'_R\) is a terminal Q-factorial modification of either \(\mathbb{P}(3, 1, 1, 1)\) or of \(\mathbb{P}(6, 4, 1, 1)\). Then it follows from Remarks 3.13 and 3.15 that either \(Y'_R \cong Y'\) or \(Y'_R\) is isomorphic to the weak Fano threefold from Example 3.12. This implies that \(4 = \rho(Y') - 1 = \rho(Y'_R) = 5\) or \(2\), a contradiction.

If \(Y'_R\) has non-factorial singularities, then it follows from Theorem 3.19 and Proposition 3.21 that \(f_R(E_R)\) is a point. In this case, for two non-cDV points \(P\) and \(Q\) on \(X'\) from Example 3.14 we have \(f^{-1}(P) \cap E_R = f'^{-1}(Q) \cap E_R = \emptyset\), and since \(L_0 \cap L \neq \emptyset\) (see the proof of Lemma 4.7), this is possible only if the curve \(Z := f'^{-1}(L) \cap E_R\) is contained in the fibres of the morphism \(f'\) (see Example 3.14). But this implies that \(0 = (K_{Y'} \cdot Z)_{Y'} < 0\), a contradiction.

Finally, suppose that the divisor \(-K_{Y'_R}\) is not nef. Then it follows from Theorem 3.19 and Proposition 3.21 that \(f_R(E_R)\) is a curve and \(E_R \cong \mathbb{F}_1\) or \(\mathbb{P}^1 \times \mathbb{P}^1\). But, if \(E_R \cong \mathbb{F}_1\), then \(f'(E_R)\) is a plane on \(X'\) such that \(L \not\subset f'(E_R)\) (see the proof of Corollary 3.23). This implies that there exists a line on \(X'\) not intersecting \(L\), which is impossible (see the proof of Lemma 4.7). Further, in case when \(E_R \cong \mathbb{P}^1 \times \mathbb{P}^1\) we have \(L_0 \subset E_R \subset E'_R\) (see the proof of Corollary 3.23), which implies that \(L_0 = L\) (see Example 3.14), a contradiction. Lemma 4.10 is completely proved. \(\square\)
It follows from Lemmas 4.9, 4.10 and the construction of threefold $X$ that $X$ has the unique singular point $o$. If $(o \in X)$ is a cDV singularity, then $X$ has terminal Gorenstein singularities (see for example [24, Theorem 5.34]). But this contradicts Theorem 3.16 since $(-K_X)^3 = 70$.

Proposition 4.8 is completely proved. □

Remark 4.11. It follows easily from the previous arguments that, in the assumptions of Proposition 4.5 for Fano threefold $X$ the equality $\rho(X) = 1$ holds. Let us denote this $X$ by $X_70$. By construction Fano threefold $X_70$ is unique up to isomorphism. It was found by I. A. Cheltsov.

Lemma 4.12. Let $f_70 : Y_70 \rightarrow X_70$ be a terminal $\mathbb{Q}$-factorial modification of the threefold $X_70$. Then the threefold $Y_70$ is non-singular and the exceptional locus of the morphism $f_70$ has pure codimension 1 on $Y_70$.

Proof. It follows from Example 3.14, Remark 3.15 and the construction of the threefold $X_70$ (see the proof of Proposition 4.5) that there exists a non-singular terminal $\mathbb{Q}$-factorial modification $f'_70 : Y'_70 \rightarrow X_70$ such that the exceptional locus of the morphism $f'_70$ has pure codimension 1 on $Y_70$. Now the statement follows from Remark 3.9.

Let us now return to the beginning of the present Section and prove the following results:

Lemma 4.13. If $\text{ext}(E)$ is a point, then

- $(-K_{Y'})^3 \neq 72$;
- $X' \neq X_70$.

Proof. Suppose that $(-K_{Y'})^3 = 72$. Then by Theorem 1.1 threefold $Y'$ is a terminal $\mathbb{Q}$-factorial modification either of $\mathbb{P}(3,1,1,1)$ or of $\mathbb{P}(6,4,1,1)$. In particular, by Remarks 3.13 and 3.15 $Y'$ is non-singular. On the other hand, by Theorem 3.19 morphism $E$ is the blow up of $Y'$ at the point $\text{ext}(E)$, and since $Y'$ is non-singular, this implies that $K_{Y'} = \text{ext}^*(K_{Y'}) + 2E$ on $Y'$ and $(E^3)' = 1$. Then we get

$$(-K_{Y'})^3 = 72 - 8(E^3)' = 64,$$

a contradiction, since $64 < (-K_X)^3 < 72$ by our assumption.

In case when $X' = X_70$, from Lemma 4.12 and the above arguments we obtain that $(-K_{Y'})^3 = 62$, which again is impossible. □

Lemma 4.14. If $C := \text{ext}(E)$ is a curve, then $X' \neq X_70$.

Proof. Suppose that $X' = X_70$. Then, as in the proof of Lemma 4.10 $f'(C)$ is either a point or a line, and the threefold $X$ is the image of $X'$ under birational projection from $f'(C)$. Moreover, if $f'(C)$ is a point, then it is non-cDV (see Proposition 4.8), which contradicts Lemma 3.26.

Thus, $f'(C)$ is a line on $X'$. Since $f' : Y' \rightarrow X'$ contracts exceptional locus of pure codimension 1 on $Y'$ to point (see Lemma 4.12 and Proposition 4.18), it follows from the proof of Lemma 4.12 that $E' \cap C = \emptyset$. Further, since $\rho(X') = 1$ (see Remark 4.11), as in the proof of Lemma 4.10 there exists a $K_{Y'}$-negative extremal birational contraction $f_C : Y' \rightarrow Y'_C$ of the curve $C$ with exceptional divisor $E_C$.

If $Y'_C$ is a weak Fano threefold with terminal factorial singularities, then from Theorem 1.1 Remark 3.7 and Lemmas 4.4, 4.13, 4.7 we get that $E' \cap C \neq \emptyset$, a contradiction.

If $Y'_C$ has non-factorial singularities, then, since $Y'$ is non-singular (see Lemma 4.12), it follows from $E' \cap C = \emptyset$, Theorem 3.19, Propositions 3.21, 4.8 and Corollary 3.20 that $f_C(E_C)$ is a point and $X'$ contains a plane $\Pi$ not passing through the singular point on $X'$. Further, from Remark 4.11 we obtain that $\Pi \sim -\frac{1}{70}K_{X'}$ on $X'$, since $(-K_{X'})^3 = 70$, $\mathcal{O}_{X'}(-K_{X'}) \simeq \mathcal{O}_{X'}(1)$ and $(K_{X'}^2, \Pi)_{X'} = 1$. But then for every line $Z \subset \Pi$ we have $(\Pi \cdot Z)_{X'} = \frac{1}{70}$, which implies that $Z$ passes through the singular point on $X'$, a contradiction.
Finally, suppose that the divisor \(-K_Y\) is not nef. Then it follows from Theorem 3.19 and Proposition 3.21 that \(f_C(E_C)\) is a curve and \(E_C = \mathbb{P}^1 \times \mathbb{P}^1\) or \(\mathbb{F}_1\). But, if \(E_C = \mathbb{P}^1 \times \mathbb{P}^1\), then \(X'\) is singular along a line (see the proof of Corollary 3.23), which is impossible by Proposition 4.8. On the other hand, in case when \(E_C = \mathbb{F}_1\) we have \(E' \cap C \neq \emptyset\) (see the proof of Corollary 3.23), a contradiction. Lemma 4.14 is completely proved.

Remark 4.15. It follows from the proof of Lemma 4.14 that every line on the threefold \(X_{70}\) passes through the singular point on \(X_{70}\). Moreover, the same arguments imply that there are no reduced and irreducible conics on \(X_{70}\).

Remark 4.16. It follows from Theorems 1.1, 3.19, 3.17 and 3.18, Lemmas 4.4, 4.13, 4.14 and Propositions 3.21, 4.5, 4.8 that to complete the proof of Theorem 1.2 it remains to consider the case when for \(K_Y\)-negative extremal birational contraction \(\text{ext} : Y \rightarrow Y' Y'\) is not a weak Fano threefold with terminal factorial singularities. Then by Corollaries 3.20 and 3.23 the corresponding Fano threefold \(X\) is either singular along a line or contains a plane. We also distinguish the case when \(X\) has a non-cDV point.

5. General case: reduction to the log Mori fibration

In the present Section we follow §6 in [32]. We use notation and conventions from Section 3. We also assume that \(64 < (-K_X)^3 < 72\). Recall that by Remark 4.16 to complete the proof of Theorem 1.2 it is enough to consider the case when the threefold \(X\) satisfies one of the following conditions:

- \(X\) is singular along a line \(\Gamma\) (case A);
- \(X\) contains a plane \(\Pi\) (case B);
- \(X\) contains a non-cDV point \(O\) (case C).

Set \(\mathcal{L} := | - K_X | \) and consider the following linear systems:

- \(\mathcal{H} := \{H \in \mathcal{L} | H \supset \Gamma\} \) in case A;
- \(\mathcal{H} := \{H | H + \Pi \in \mathcal{L}\} \) in case B;
- \(\mathcal{H} := \{H \in \mathcal{L} | H \ni O\} \) in case C.

Take a terminal \(\mathbb{Q}\)-factorial modification \(f : Y \rightarrow X\) of the threefold \(X\). Put \(\mathcal{L}_Y := f_*^{-1}(\mathcal{L})\) and \(\mathcal{H}_Y := f_*^{-1}(\mathcal{H})\). Then for the general element \(H \in \mathcal{H}\) and \(H_Y := f_*^{-1}(H) \in \mathcal{H}_Y\) we have

\[
K_Y + H_Y + D_Y = f^*(K_X + H) \sim 0 \quad \text{in cases A and C}
\]

and

\[
K_Y + H_Y + D_Y = f^*(K_X + H + \Pi) \sim 0 \quad \text{in case B}
\]
on \(Y\), where \(D_Y\) is an effective integral non-zero \(f\)-exceptional divisor in cases A and C, and a sum of \(f_*^{-1}(\Pi)\) and effective integral \(f\)-exceptional divisor in case B.

Remark 5.3. By construction and [35 Corollary 2.14] every irreducible component of the divisor \(D_Y\) is a surface of negative Kodaira dimension.

Further, for general element \(L \in \mathcal{L}\) and \(L_Y := f_*^{-1}(L) \in \mathcal{L}_Y\) we have

\[
K_Y + L_Y = f^*(K_X + L) \sim 0
\]
on \(Y\).

Lemma 5.5 (see [32 Lemma 6.5]). The image of the threefold \(X\) under the map \(\Phi_{\mathcal{H}}\) is three-dimensional.

It follows from Theorem 3.24 and the Inversion of Adjunction (see for example [23, Theorem 7.5]) that the pair \((X, L)\) is canonical. Then from (5.4) and [32 Lemma 3.1] we get the following
Lemma 5.6. The pair $(Y, L_Y)$ is canonical.

Moreover, we have the following

Lemma 5.7 (see [32] Lemma 6.8). Threefold $Y$ can be chosen in such a way that
- the pair $(Y, H_Y)$ is canonical;
- the linear system $H_Y$ consists of nef Cartier divisors.

In what follows, we assume that $Y$ satisfies the conditions of Lemma 5.7. Now, let us apply the log minimal model program to the pair $(Y, H_Y)$. Then on each step the equality $K + H \equiv -D$ for some divisor $D > 0$ is preserved (see (5.1) and (5.2)). This implies that at the end of the program we obtain a pair $(W, H_W)$ with a $(K_W + H_W)$-negative extremal contraction $\text{ext}_W : W \to W'$ to a lower-dimensional variety $W'$.

Remark 5.8. Denote by $H_W$ the proper transform on $W$ of the linear system $H_Y$. Then $H_W \in H_W$, which implies by Lemma 5.7 and [32] Lemma 3.4 that the pair $(W, H_W)$ is canonical and $H_W$ consists of nef Cartier divisors. In particular, $\text{ext}_W$ is a $K_W$-negative extremal contraction and hence $W$ has only terminal Q-factorial singularities. Furthermore, denote by $L_W$ the proper transform on $W$ of the linear system $L_Y$. Then for the general element $L_W \in L_W$ we have $K_W + L_W \equiv 0$ (see (5.4)), which implies by [32], Lemma 3.1 that the pair $(W, L_W)$ is canonical. In particular, since $L_W \subseteq |−K_W|$, the linear system $|−K_W|$ does not have fixed components. Finally, $(W, L_W)$ is a generating 0-pair (see [32], Definition 4.1 and [33]).

Remark 5.9. By construction the initial Fano threefold $X$ is the image of the threefold $W$ under birational map $\Phi_{L_W}$ (see Remark 3.5). Furthermore, it follows from Lemma 5.5 that the divisor $H_W$ is ample over $W'$ and the linear system $H_W$ is movable.

Further, we have

\begin{equation}
K_W + H_W + D_W \equiv 0
\end{equation}

on $W$ for some $D_W > 0$ with irreducible components of negative Kodaira dimension (see Remark 5.8).

Remark 5.11. We have $\dim |−K_W| \geq \dim L_W = \dim |−K_X|$ (see Remarks 5.8 and 5.9). Moreover, by construction of the linear system $H$ inequality $\dim H \geq \dim |−K_W| − 3$ holds. Then it follows from $64 < (−K_X)^3 < 72$ that $\dim |−K_W| \geq 35$ and $\dim |H_W| \geq 32$ (see (3.1)).

Since $\dim W' < \dim W$, $W'$ is either a point, a curve or a surface. But if $W'$ is a point, then it follows from [32], Proposition 7.2 that $\dim |−K_W| \leq 34$, which contradicts the estimate in Remark 5.11. Two other cases will be treated in the next two Sections.

6. Contraction to curve

We use notation and conventions from Section 5. In the present Section we consider the case when $\dim W' = 1$. We note that the curve $W'$ is smooth. Then it follows from the relative Kawamata–Viehweg Vanishing Theorem (see for example [21]) and the Leray spectral sequence that $H^1(W', O_{W'}) = H^1(W, O_W) = 0$. Thus, we have $W' \simeq \mathbb{P}^1$.

Further, the general fibre $W_\eta$ of the morphism $\text{ext}_W : W \to W'$ is a smooth del Pezzo surface. On the other hand, by construction for $H_W \in H_W$ the divisor $−(K_{W_\eta} + H_W|_{W_\eta})$ is ample on $W_\eta$. Moreover, by Remark 5.9 the divisor $H_W|_{W_\eta}$ is also ample on $W_\eta$. This implies that $W_\eta \simeq \mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$, and in the first case $H_W|_{W_\eta} \simeq O_{\mathbb{P}^2}(1)$ or $O_{\mathbb{P}^2}(2)$.

Lemma 6.1. If $W_\eta \simeq \mathbb{P}^2$ and $H_W|_{W_\eta} \simeq O_{\mathbb{P}^2}(1)$, then $W$ is a $\mathbb{P}^2$-bundle such that the morphism $\text{ext}_W$ is its natural projection to $\mathbb{P}^1$. Moreover, $W = \mathbb{P}(O_{\mathbb{P}^2}(5) \oplus O_{\mathbb{P}^1}(2) \oplus O_{\mathbb{P}^1})$ or $\mathbb{P}(O_{\mathbb{P}^1}(6) \oplus O_{\mathbb{P}^1}(2) \oplus O_{\mathbb{P}^1})$. 

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Proof. According to [32, Lemma 8.1], we have \( W = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(d_1) \oplus \mathcal{O}_{\mathbb{P}^1}(d_2) \oplus \mathcal{O}_{\mathbb{P}^1}) \) with \( d_1 \geq d_2 \geq 0 \) so that \( \text{ext}_W \) is the natural projection of \( W \) to \( \mathbb{P}^1 \). Moreover, it follows from the proof of Proposition 8.2 in [32] and Remark 5.8 that only the following values for \( (d_1, d_2) \) are possible:

\[
(1, 1), \quad (2, 1), \quad (2, 2), \quad (3, 1), \quad (3, 2), \quad (4, 1), \quad (4, 2), \quad (5, 2), \quad (6, 2).
\]

It is easy to compute (see for example [36, Ch. 2]) that in all cases except for the last two \( \dim | -K_W | \leq 33 \). Then from Remark 5.11 we obtain that \( (d_1, d_2) = (5, 2) \) or \( (6, 2) \).

**Proposition 6.2.** In the assumptions of Lemma 6.1, if \( W = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(5) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}) \), then \( (-K_X)^3 = 66 \) and \( X \) is the anticanonical image of \( W \).

**Proof.** For \( W = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(5) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}) \) we have \( \dim | -K_W | = 35 \) (see for example [36, Ch. 2]). Then from Remark 5.8 we get equality \( | -K_W | = L_W \). In particular, for the initial Fano threefold \( X \) we have \( (-K_X)^3 = 66 \). Let us show that this case really occurs.

On \( W \) we have \( -K_W \simeq 3M - 5F \), where \( M \) is the tautological divisor and \( F \) is the fibre of the projection \( \text{ext}_W \) (see for example [36, Ch. 2]). Then the linear system \( | -K_W | \) is generated by the following polynomials:

\[
(6.3) \quad g_1x_3^2, \quad x_1x_2^2, \quad g_2x_1x_2x_3, \quad g_4x_1x_2^2, \quad g_5x_2^2x_3, \quad g_7x_2^2, \quad g_{10}x_3^3,
\]

where \( x_1, x_2, x_3 \) are projective coordinates on \( F \simeq \mathbb{P}^2 \), \( g_i \) is a homogeneous polynomial of degree \( i \) in projective coordinates \( t_0, t_1 \) on the base \( W' \simeq \mathbb{P}^1 \) (see for example [36, Ch. 2]). In particular, the base locus of the linear system \( L_W = | -K_W | \) is the curve \( C_0 \simeq \mathbb{P}^1 \) given by equations \( x_1 = x_2 = 0 \) on \( W \).

To resolve the indeterminacy locus of the rational map \( \Phi_{C_0} : W \dashrightarrow X := \Phi_{L_W}(W) \), one may assume that in (6.3) the equalities \( t_2 = x_3 = 1 \) hold. We put \( x_1 =: x, x_2 =: y \) and \( t_1 =: z \).

Let \( \sigma_1 : W_1 \longrightarrow W_0 =: W \) be the blow up of the curve \( C_0 \), and \( E_1 \) be the \( \sigma_1 \)-exceptional divisor. Threefold \( W_1 \) is covered by two affine charts \( U_1^{(1)} \), each of which is isomorphic to \( \mathbb{C}^3 \). Again we denote by \( x, y, z \) the coordinates on each of \( U_1^{(1)} \).

Morphism \( \sigma_1|_{U_1^{(1)}} : U_1^{(1)} \longrightarrow W_0 \) is given by the formula

\[
(x, y, z) \mapsto (x, xy, z).
\]

Then the linear system \( L_{W_1} := \sigma_1^{-1}(L_W) \), when restricted to \( U_1^{(1)} \), is generated by the following polynomials:

\[
g_1y^2, \quad x, \quad g_2xy, \quad g_4x^2y^2, \quad g_5x, \quad g_7x^2y, \quad g_{10}x^2,
\]

which implies that \( L_{W_1} \) is base point free on \( U_1^{(1)} \).

Morphism \( \sigma_1|_{U_1^{(2)}} : U_1^{(2)} \longrightarrow W_0 \) is given by the formula

\[
(x, y, z) \mapsto (xy, y, z).
\]

Then the linear system \( L_{W_1} \), when restricted to \( U_1^{(2)} \), is generated by the following polynomials:

\[
(6.4) \quad g_1y^2, \quad x, \quad g_2xy, \quad g_4x^2y^2, \quad g_5x^2y, \quad g_7x^2y^2, \quad g_{10}x^3y^2,
\]

which implies that the base locus of \( L_{W_1} \) is an irreducible rational curve \( C_1 \subset E_1 \) given by equations \( x = y = 0 \) on \( W_1 \).

To resolve the indeterminacy locus of the rational map \( \Phi_{L_{W_1}} : W_1 \longrightarrow X \) one may assume that \( W_1 = U_2^{(1)} \) with coordinates \( x, y, z \). Then (6.4) is a basis of the linear system \( L_{W_1} \).

Let \( \sigma_2 : W_2 \longrightarrow W_1 \) be the blow up of the curve \( C_1 \), and \( E_2 \) be the \( \sigma_2 \)-exceptional divisor. Threefold \( W_2 \) is covered by two affine charts \( U_i^{(2)} \), each of which is isomorphic to \( \mathbb{C}^3 \). Again we denote by \( x, y, z \) the coordinates on each of \( U_i^{(2)} \).
Morphism $\sigma_{2,1}^{(2)} : U_1^{(2)} \to W_1$ is given by the formula
\[(x, y, z) \mapsto (x, xy, z),\]
and again the linear system $L_{W_2} := \sigma_{2}^{-1}(L_{W_1})$ is base point free on $U_1^{(2)}$.

Morphism $\sigma_{3,1}^{(3)} : U_1^{(3)} \to W_2$ is given by the formula
\[(x, y, z) \mapsto (x, y, z),\]
and again the linear system $L_{W_2}$, when restricted to $U_2^{(2)}$, is generated by the following polynomials:
\[
g_1y, \ x, \ g_2xy, \ g_3xy^2, \ g_4x^2y, \ g_5x^2y^2, \ g_6x^2y^3, \ g_7x^2y^4, \ g_8x^2y^5, \]
which implies that the base locus of $L_{W_2}$ is an irreducible rational curve $C_2 \subset E_2$ given by equations $x = y = 0$ on $W_2$.

Finally, to resolve the indeterminacy locus of the rational map $\Phi_{L_{W_2}} : W_2 \dashrightarrow X$ one may assume that $W_2 = U_2^{(2)}$ with coordinates $x, y, z$. Then (6.5) is a basis of the linear system $L_{W_2}$.

Let $\sigma_3 : W_3 \to W_2$ be the blow up of the curve $C_2$, and $E_3$ be the $\sigma_3$-exceptional divisor. Threefold $W_3$ is covered by two affine charts $U_3^{(3)}$, each of which is isomorphic to $\mathbb{C}^3$. Again we denote by $x, y, z$ the coordinates on each of $U_3^{(3)}$.

Morphism $\sigma_{3,1}^{(3)} : U_3^{(3)} \to W_2$ is given by the formula
\[(x, y, z) \mapsto (x, y, z),\]
and again the linear system $L_{W_3} := \sigma_{3}^{-1}(L_{W_2})$ is base point free on $U_1^{(3)}$.

Morphism $\sigma_{3,1}^{(3)} : U_3^{(3)} \to W_2$ is given by the formula
\[(x, y, z) \mapsto (x, y, z),\]
and again the linear system $L_{W_3}$, when restricted to $U_2^{(3)}$, is generated by the following polynomials:
\[
g_1, \ x, \ g_2xy, \ g_3xy^2, \ g_4x^2y, \ g_5x^2y^2, \ g_6x^2y^3, \ g_7x^2y^4, \ g_8x^2y^5, \]
which implies that $L_{W_3}$ is base point free on $U_2^{(3)}$ and hence on $W_3$.

**Lemma 6.6.** $W_3$ is a non-singular weak Fano threefold such that its image $X = \Phi_{L_{W_3}}(W_3)$ is a Fano threefold with $(-K_X)^3 = 66$.

**Proof.** In the above notation, we have
\[K_{W_i} = \sigma_i^*(K_{W_{i-1}}) + E_i\]
on $W_i$, and the general element in $L_{W_{i-1}}$ is non-singular along the base curve $C_{i-1}$, $1 \leq i \leq 3$ (see (6.3) and (6.5)). This implies, since $L_{W_0} = |-K_W|$, that $L_{W_i} \subseteq |-K_W|$ for all $1 \leq i \leq 3$. In particular, the divisor $-K_{W_3}$ is nef on $W_3$ because the linear system $L_{W_3}$ is base point free on $W_3$.

Further, since $(K_W \cdot C_0)_w = 5$, $(K_W^3)_w = -54$ (see for example [36 A. 4]) and
\[
(-K_{W_i})|_{E_i} = C_i
\]
for $i = 1, 2$ (see the above arguments), it is easy to compute that
\[
(K_{W_i})|_{W_i} = (K_{W_{i-1}})|_{W_{i-1}} - 2(5 - 2(i - 1)) + 2 = (K_{W_{i-1}})|_{W_{i-1}} - 8 + 4(i - 1)
\]
for $1 \leq i \leq 3$ (see the proof of Proposition [3.21]). In particular, we have $(K_{W_3})|_{W_3} = 66$, which implies that $W_3$ is a non-singular weak Fano threefold. Then $X = \Phi_{L_{W_3}}(W_3)$ is a Fano threefold with canonical Gorenstein singularities (see [20]). Moreover, by construction we have $O_X(-K_X) \sim O_X(1)$ and $\dim |-K_W| = 35$. Thus, we obtain that $(-K_X)^3 = 66$ (see (5.1)).
Proposition 6.2 is completely proved. □

Remark 6.7. By construction Fano threefold $X$ from Proposition 6.2 is toric. Let us denote it by $X_{66}$. It follows from the explicit description of the fan of $X_{66}$ in [25] that $X_{66}$ is not $\mathbb{Q}$-factorial and $\rho(X) = 1$ (see also the proof of Lemma 8.7 in Section 8).

**Proposition 6.8.** If $X = X_{66}$, then the singularities of $X$ are non-cDV.

**Proof.** We use notation from the proof of Proposition 6.2. It follows from the proof of Lemma 6.6 that $\nu W_3 = | - K_{W_3} |$ and $\Phi : W_3 \rightarrow X$ is a terminal $\mathbb{Q}$-factorial modification of $X$.

Let $L_t \subset E_2$ be the fibre $\sigma_2^{-1}(t)$, $t \in C_1 \simeq \mathbb{P}^1$, and $L'_t$ its proper transform on $W_3$.

**Lemma 6.9.** We have $(K_{W_3} \cdot L'_t)_{w_3} = 0$.

**Proof.** Set $E^*_1 := (\sigma_2 \circ \sigma_3)^*(E_1)$ and $E^*_2 := \sigma_3^*(E_2)$. Then we obtain equality

$$K_{W_3} = (\sigma_1 \circ \sigma_2 \circ \sigma_3)^*(K_W) + E^*_1 + E^*_2 + E_3$$
on $W_3$. Furthermore, we have

$$(E_3 \cdot L'_t)_{w_3} = 1, \quad (E_2 \cdot L'_t)_{w_3} = (E_2 \cdot L_t)_{w_2} = -1, \quad (E^*_1 \cdot L'_t)_{w_3} = (\sigma_2^*(E_1) \cdot L_t)_{w_2} = 0,$$

which implies that $(K_{W_3} \cdot L'_t)_{w_3} = 0$.

Let $E'_2$ be the proper transform of the surface $E_2$ on $W_3$.

**Lemma 6.10.** We have $(K_{W_3} \cdot E'_2 \cdot E_3)_{w_3} = 0$.

**Proof.** In the notation from the proof of Proposition 6.2, let $\sigma_3 : W_3 \rightarrow W_2$ be the blow up of the curve $C_2$, and $E_3$ be the $\sigma_3$-exceptional divisor. Without loss of generality one may assume that $W_2 = U_2^{(2)}$ with coordinates $x, y, z$. Then the threefold $W_3$ is covered by two affine charts $U_i^{(3)}$, each of which is isomorphic to $\mathbb{C}^3$. Again we denote by $x, y, z$ the coordinates on each of $U_i^{(3)}$.

Morphism $\sigma_3|_{U_i^{(3)}} : U_1^{(3)} \rightarrow W_2$ is given by the formula

$$(x, y, z) \mapsto (x, xy, z).$$

Then the general surface $S \in L_{W_3} = | - K_{W_3} | = \sigma_{3*}^{-1}(L_{W_2})$, when restricted to $U_1^{(3)}$, is given by equation

$$g_1 y + g_0 + g_2 xy + g_3 x^2 y^2 + g_5 x^3 y^2 + g_7 x^4 y^3 + g_{10} x^6 y^4 = 0,$$

where $g_0 \in \mathbb{C}^*$ (see (6.5)). On the other hand, the surface $E_3$ is given by equation $x = 0$ on $U_1^{(3)}$ and the surface $E_2$ is given by equation $y = 0$ on $W_2$. This implies that

$$S \cap E'_2 \cap E_3 = \emptyset$$
on $U_1^{(3)}$.

Further, morphism $\sigma_3|_{U_2^{(3)}} : U_2^{(3)} \rightarrow W_2$ is given by the formula

$$(x, y, z) \mapsto (xy, y, z).$$

Then we have $E'_2 \cap U_2^{(3)} \cap E_3 = \emptyset$ because the surface $E_3$ is given by equation $y = 0$ on $U_2^{(3)}$ and the surface $E_2$ is given by equation $y = 0$ on $W_2$. This implies that for the general surface $S \in L_{W_3} = | - K_{W_3} |$ we have

$$S \cap E'_2 \cap E_3 = \emptyset$$
on $U_2^{(3)}$ and hence $(K_{W_3} \cdot E'_2 \cdot E_3)_{w_3} = 0$. □

It follows from Lemmas 6.9 and 6.10 that $o := \Phi|_{W_3}(E'_2)$ is a point on $X$ and the divisor $E'_2$ over $X$ has zero discrepancy. This implies that $(o \in X)$ is a non-cDV singularity (see for example [21, Theorem 5.34]). Proposition 6.8 is completely proved. □
Now we turn to the second case in Lemma 6.1 and prove the following

**Proposition 6.11.** In the assumptions of Lemma 6.1, if \( W = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(6) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}) \), then \( X = X_{70} \) is the Fano threefold constructed in Proposition 4.5.

**Proof.** For \( W = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(6) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}) \), its image \( \Phi_{-K_W}(W) \subset \mathbb{P}^{38} \) is the anticanonically embedded Fano threefold \( \mathbb{P}(6,4,1,1) \) (see [13] Ch. 4, Remark 4.2). Then it follows from Remarks 5.9, 5.8 and the assumption \( 64 < (-K_X)^3 < 72 \) that \( \dim \mathcal{L}_W \in \{35,36,37\} \) (see (3.1)), and hence the initial Fano threefold \( X \) is the image of \( \mathbb{P} := \mathbb{P}(6,4,1,1) \subset \mathbb{P}^{38} \) under birational projection from a point, a line or a plane. Let \( \pi : \mathbb{P} \rightarrow X \) be this projection. Note that by Theorem 3.3, threefold \( \mathbb{P} \subset \mathbb{P}^{38} \) is an intersection of quadrics.

**Lemma 6.12.** If \( \pi \) is the projection from a point, then \( X = X_{70} \) is the Fano threefold constructed in Proposition 4.5.

**Proof.** Let \( O \) be the center of the projection \( \pi \). Then \( O \) belongs to \( \mathbb{P} \), since otherwise \( \pi \) is an isomorphism, which is impossible because \( 64 < (-K_X)^3 < 72 \). Moreover, if \( O \) is a smooth point on \( \mathbb{P} \), then \( (-K_X)^3 = 71 \), which is also impossible (see (3.1)). Thus, we have \( O \in \text{Sing}(\mathbb{P}) \).

In the notation of Example 6.14, if \( O \) is different from the points \( P \) and \( Q \), then \( (O \in \mathbb{P}) \) is the cA_1 singularity, and by definition we obtain that \( X = X_{70} \). Now, let \( O = P \) or \( Q \). Then \( O \) is a non-cDV point on \( \mathbb{P} \) (see the proof of Lemma 4.7), which contradicts Lemma 3.26.

By Lemma 6.12 it remains to consider the cases when \( \pi \) is the projection from a line and a plane. Let us consider the case of the projection from a line first. Let \( \gamma \) be the center of the projection \( \pi \). Since \( \mathbb{P} \subset \mathbb{P}^{38} \) is an intersection of quadrics, either \( \mathbb{P} \) contains the line \( \gamma \) or intersects with \( \gamma \) at \( \leq 2 \) points (note that \( \gamma \cap \mathbb{P} \neq \emptyset \), since otherwise \( \pi \) is an isomorphism, which is impossible because \( 64 < (-K_X)^3 < 72 \)).

**Lemma 6.13.** The line \( \gamma \) is not contained in \( \mathbb{P} \).

**Proof.** Suppose that \( \gamma \subset \mathbb{P} \). Then by Lemma 4.10 \( \gamma \) coincides with the singular locus of \( \mathbb{P} \). Then, as in the proof of Lemma 6.12 we get a contradiction with Lemma 3.26.

Thus, \( \gamma \) intersects \( \mathbb{P} \) by \( \leq 2 \) points.

**Lemma 6.14.** We have \( \gamma \cap \mathbb{P} \cap \text{Sing}(\mathbb{P}) = \emptyset \).

**Proof.** In the notation of Example 6.14, if \( \gamma \) contains a singular point on \( \mathbb{P} \), other than \( P \) and \( Q \), then \( X \) is the image of the anticanonically embedded Fano threefold \( X_{70} \) under birational projection from some point \( O \). We have \( O \in X_{70} \), since otherwise \( \pi \) is an isomorphism, which is impossible because \( (-K_X)^3 < (-K_{X_{70}})^3 \) (see the proof of Lemma 4.4). Since \( X_{70} \) has the unique singular point, and this point is non-cDV (see Proposition 4.8), \( O \) must be a smooth point (see Lemma 3.26). But then we have \( (-K_X)^3 = 69 \), which is impossible (see (3.1)).

The obtained contradiction implies that the set \( \gamma \cap \text{Sing}(\mathbb{P}) \) can contain only \( P \) and \( Q \). Then, as in the proof of Lemma 6.12 we get a contradiction with Lemma 3.26.

Thus, \( \gamma \) intersects \( \mathbb{P} \) by \( \leq 2 \) smooth points. In these assumptions, we have

**Lemma 6.15.** The equality \( (-K_X)^3 = 72 \) holds.

**Proof.** Let \( S \) be the general hyperplane section of \( \mathbb{P} \) passing through \( \gamma \). Then \( S \) is a non-singular K3 surface near \( \gamma \cap \mathbb{P} \) because \( \gamma \cap \mathbb{P} \) consists of \( \leq 2 \) smooth points. Furthermore, projection \( \pi \) gives a birational map \( \chi := \pi|_S : S \rightarrow S' \) on the general surface \( S' \in |(-K_X)| \) with Du Val singularities (see Theorem 3.24). The map \( \chi \) is undefined exactly at the points from \( \gamma \cap \mathbb{P} \), which implies that \( S \) is a partial minimal resolution of \( S' \) and \( \chi \) is a morphism.
Further, projection $π$ is given by the linear system $L \subset |−K_\mathbb{P}|$ of all hyperplane sections of $\mathbb{P}$ passing through $γ$. Set $L_S := L|_S$ and $L_{S'} := |−K_X|_{S'}$. Then we have $L_S = χ^*(L_{S'})$, which implies equality $(L^2)_S = (L^2)_{S'}$, for $L ∈ L_S$ and $L' ∈ L_{S'}$. On the other hand, we have $(L^2)_{S'} = (−K_X)^3$ and $(L^2)_S = (−K_\mathbb{P})^3$. Thus, we obtain

$$(−K_X)^3 = (L^2)_{S'} = (L^2)_S = 72.$$  

From Lemma 6.15 we get a contradiction with $64 < (−K_X)^3 < 72$. Now, let us turn to the last case when $π$ is the projection from a plane. Let $Ω$ be the center of projection. Since $\mathbb{P} ⊂ \mathbb{P}^{38}$ is an intersection of quadrics, it follows from Lemma 4.10 that $\mathbb{P}$ intersects with $Ω$ either at $≤ 4$ points or by an irreducible reduced conic (note again that $Ω ∩ \mathbb{P} ≠ ∅$, since otherwise $π$ is an isomorphism, which is impossible because $64 < (−K_X)^3 < 72$).

**Lemma 6.16.** If $Ω ∩ \mathbb{P}$ is a finite set, then it consists only of smooth points on $\mathbb{P}$.

**Proof.** In the notation of Example 3.14, if $Ω$ contains a singular point on $\mathbb{P}$, other than $P$ and $Q$, then $X$ the image of the anticanonically embedded Fano threefold $X_{70}$ under birational projection from some line $Γ$. We have $X_{70} ∩ Γ ≠ ∅$, since otherwise $π$ is an isomorphism, which is impossible because $64 < (−K_X)^3 < 72$ (see the proof of Lemma 4.4). Furthermore, the line $Γ$ is not contained in $X_{70}$, since otherwise $Γ$ passes through the non-cDV point on $X_{70}$ (see Remark 4.15 and Proposition 4.8), which, as in the proof of Lemma 6.12, gives a contradiction. These arguments imply that $Γ ∩ X_{70}$ consists of $≤ 2$ smooth points of $X_{70}$ (see Theorem 3.4). Then, as in the proof of Lemma 6.15 we obtain that $(−K_X)^3 = (−K_X)^3$. On the other hand, we have $−K_X)^3 < (−K_X)^3$, a contradiction.

Thus, the set $Ω ∩ Sing(\mathbb{P})$ can contain only $P$ and $Q$. Then, as in the proof of Lemma 6.12 we get a contradiction with Lemma 3.26.

If $Ω$ intersects $\mathbb{P}$ at $≤ 4$ points, then it follows from Lemma 6.16 and the proof of Lemma 6.15 that $−K_X)^3 = 72$, which is impossible because $64 < (−K_X)^3 < 72$. Thus, $Ω ∩ \mathbb{P}$ is an irreducible reduced conic $C$.

**Lemma 6.17.** The set $C ∩ Sing(\mathbb{P})$ is non-empty and consists only of $cA_1$ points on $\mathbb{P}$.

**Proof.** Since $(−K_\mathbb{P} · C)_\mathbb{P} = 2$ and $O_\mathbb{P}(−K_\mathbb{P}) ≃ O_\mathbb{P}(12)$ (see [5, Theorem 3.3.4]), for the divisor $M$ on $\mathbb{P}$ with $O_\mathbb{P}(M) ≃ O_\mathbb{P}(1)$ we obtain

$$(M · C)_\mathbb{P} = \frac{1}{6},$$

which implies that the curve $C$ passes through singular points on $\mathbb{P}$. Further, in the notation of Example 3.14, if $C$ contains either $P$ or $Q$, then, as in the proof of Lemma 6.12 we get a contradiction with Lemma 3.26. Now the statement follows from the arguments in Example 3.14.

It follows from Lemma 6.17 that $X$ is the image of the anticanonically embedded Fano threefold $X_{70}$ under birational projection from some line on $X_{70}$. Then, as in the proof of Lemma 6.16 we get a contradiction. Proposition 6.11 is completely proved.

Let us now return to the beginning of the present Section. In the case when $W_η ≃ \mathbb{P}^2$ and $H_W|_{W_η} ≃ O_{\mathbb{P}^2}(2)$ by Proposition 8.7 there exists a birational map $W → W_0$ on a $\mathbb{P}^2$-bundle $W_0$ over $W'$. Furthermore, the proper transforms $L_{W_0}$ and $H_{W_0}$ on $W_0$ of linear systems $L_W$ and $H_W$, respectively, and the threefold $W_0$ posses the same properties as $L_W$, $H_W$ and $W$ (see Proposition 8.10, Remarks 5.8, 5.9 and 5.11). Then again by Lemma 6.1, $W_0 = \mathbb{P}(O_{\mathbb{P}^1}(5) ⊕ O_{\mathbb{P}^1}(2) ⊕ O_{\mathbb{P}^1})$ or $\mathbb{P}(O_{\mathbb{P}^1}(6) ⊕ O_{\mathbb{P}^1}(2) ⊕ O_{\mathbb{P}^1})$, and as in the proof of Propositions 6.2, 6.11 we obtain that the initial Fano threefold $X$ is isomorphic to $X_{70}$ or $X_{66}$.

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Let us now consider the case when $W_\eta \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Then by [32, Proposition 9.2] there exists an embedding $W \hookrightarrow \mathbb{F}$ over $W'$, where

$$F := \mathbb{P}(\mathcal{E}), \quad \mathcal{E} := \bigoplus_{i=1}^{4} \mathcal{O}_{\mathbb{P}^1}(d_i),$$

$d_1 \geq d_2 \geq d_3 \geq d_4 = 0$, such that $W_\eta \subset F \simeq \mathbb{P}^3$ is the non-singular quadric. Let $M$ and $F$ be the tautological divisor and the fibre of the $\mathbb{P}^3$-bundle $F \to \mathbb{P}^1$, respectively. Then we have $W \sim 2M + rF$ for some $r \in \mathbb{Z}$. Furthermore, from [32, Lemma 9.5] and the Adjunction Formula we obtain

$$-K_W = 2G + (2 - d - r)N,$$

where $G := M|_W$, $N := F|_W$.

**Lemma 6.19** (see [32, Lemma 9.7]). The inequality $d + r \geq 3$ holds.

**Proof.** If $d + r < 2$, then the divisor $-K_W$ is ample on the threefold $W$, and since $W$ has terminal singularities (see Remark 5.8), it follows from Theorem 3.16 that $\dim |-K_W| \leq 34$. This contradicts the estimate in Remark 5.11.

Now, let $d + r = 2$. Then we have $-K_W = 2G$ (see (6.18)). In particular, $-K_W$ is a nef and big Cartier divisor on $W$. Note that $(-K_W)^3 \neq 72$. Indeed, since $W$ has Gorenstein terminal $\mathbb{Q}$-factorial singularities (see Remark 5.8), it follows from [20, Lemma 5.1] that $W$ is factorial. Then by Theorem 1.1 and Remark 5.7 $W$ is a terminal $\mathbb{Q}$-factorial modification either of $\mathbb{P}(3,1,1,1)$ or of $\mathbb{P}(6,4,1,1)$, and hence $W$ is isomorphic to one of the threefolds constructed in Examples 3.12 and 3.14 (see Remarks 3.13 and 3.15). On the other hand, we have $\rho(W) = 2$, which implies that $W$ is a terminal $\mathbb{Q}$-factorial modification of $\mathbb{P}(3,1,1,1)$. Then $W$ contains a surface swapped by the curves having zero intersection with $K_W$ (see Example 3.12). But this is impossible, since the curves in the fibres of the morphism $\text{ext}_W$ are numerically proportional and have negative intersection with $K_W$ (see Remark 5.8).

Further, since $W$ is a weak Fano threefold, from the Riemann–Roch Formula (see [37]) and Kawamata–Viehweg Vanishing Theorem (see for example [21]) we obtain

$$\dim |-K_W| = -\frac{1}{2}K_W^3 + 2.$$

Then from Remark 5.11 we get

$$(-K_W)^3 \in \{66, 68, 70\}.$$

But the general element in the linear system $[G]$ has Du Val singularities (see for example [38]), and from the Adjunction Formula we obtain that $(K_G^2)_G = \frac{1}{8}(-K_X)^3$. Then it follows from (6.21) that $(K_G^2)_G \not\in \mathbb{Z}$, which is impossible. \hfill $\square$

Repeating word by word the proof of Proposition 9.4 in [32], from Lemma 6.19 we deduce that there exists a birational map $W \to W_0$ on a $\mathbb{P}^2$-bundle $W_0$ over $W'$. Furthermore, the proper transforms $L_{W_0}$ and $H_{W_0}$ on $W_0$ of linear systems $L_W$ and $H_W$, respectively, and the threefold $W_0$ posses the same properties as $L_W$, $H_W$ and $W$ (see (5.10), Remarks 5.8, 5.9 and 5.11). As above, this is enough for the proof of Theorem 1.2 in the case when $W'$ is a curve.

7. Contraction to Surface

We use notation and conventions from Section 5. In the present Section we consider the case when $\dim W' = 2$.

By [32, Lemma 10.1] the surface $W'$ is non-singular and $\text{ext}_W : W \to W'$ is a $\mathbb{P}^1$-bundle. Furthermore, for the fibre $F$ of the morphism $\text{ext}_W$ we have

$$( -K_W \cdot F )_{W'} = 2.$$
This and (6.10) imply that $H_W \in \mathcal{H}_W$ and $D_W$ are sections of $\text{ext}_W$ because $H_W$ is ample over $W'$ (see Remark 5.9 and $\text{ext}_W$ is a $(K_W + H_W)$-negative extremal contraction. In particular, $W'$ is covered by irreducible component of the divisor $D_W$ which has negative Kodaira dimension (see Section 5). On the other hand, it follows from the relative Kawamata–Viehweg Vanishing Theorem (see for example [21]) and the Leray spectral sequence that $H^1(W', \mathcal{O}_{W'}) = H^1(W, \mathcal{O}_W) = 0$. Thus, we obtain the following

**Lemma 7.1** (see [32]). $W'$ is a rational surface.

Further, by Grothendieck Theorem on triviality of the Brauer group of the smooth rational surface (see [9]) we have $W \cong \mathbb{P}(\mathcal{E})$ for the rank 2 vector bundle

$$\mathcal{E} := \text{ext}_{W'}(\mathcal{O}_W(H_W))$$

over $W'$. Then, since the divisor $H_W$ is nef on $W$, from Remark 5.8 and [32 Lemma 4.4] we obtain

$$\dim |H_W| + 1 = h^0(W', \mathcal{E}) = \chi(W', \mathcal{E}).$$

**Proposition 7.3.** $\dim |H_W| \in \{32, 33, 34, 35\}$.

**Proof.** According to Remark 5.11, we have $\dim |H_W| \geq 32$.

Suppose that $\dim |H_W| \geq 36$. Then it follows from the proof of Proposition 10.3 in [32] that $W = \mathbb{P}(\mathcal{O}_\mathbb{P}(3) \oplus \mathcal{O}_\mathbb{P}(6))$ and the image $\Phi_{-K_W}(W)$ is the anticanonically embedded Fano threefold $\mathbb{P}(3,1,1,1) \subset \mathbb{P}^{38}$. From Remarks 5.9, 5.8 and the assumption $64 < (-K_X)^3 < 72$ we get that $\dim \mathcal{L}_W \in \{35,36,37\}$ (see (3.1)), and hence the initial Fano threefold $X$ is the image of $\mathbb{P} := \mathbb{P}(3,1,1,1) \subset \mathbb{P}^{38}$ under birational projection from a point, a line or a plane. Let $\pi : \mathbb{P} \dashrightarrow X$ be this projection. Note that by Theorem 3.4 threefold $\mathbb{P} \subset \mathbb{P}^{38}$ is an intersection of quadrics.

**Lemma 7.4.** If $\pi$ is the projection from a point, then $(-K_X)^3 \notin \{66, 68, 70\}$.

**Proof.** Let $O$ be the center of the projection $\pi$. Then $O$ belongs to $\mathbb{P}$, since otherwise $\pi$ is an isomorphism, which is impossible because $64 < (-K_X)^3 < 72$. Moreover, if $O$ is a smooth point on $\mathbb{P}$, then $(-K_X)^3 = 71$, which is also impossible (see (3.1)).

Thus, we have $O \in \text{Sing}(\mathbb{P})$. Then it follows from the arguments in Example 3.12 that there is an exceptional divisor over $\mathbb{P}$ with the zero discrepancy and the center at $O$, which implies that $O$ is non-cDV (see for example [21 Theorem 5.34]). This contradicts Lemma 3.26.

**Lemma 7.5.** If $\pi$ is the projection from a line, then $(-K_X)^3 \notin \{66, 68, 70\}$.

**Proof.** Let $\gamma$ be the center of the projection $\pi$. Then $\gamma$ intersects $\mathbb{P}$, since otherwise $\pi$ is an isomorphism, which is impossible because $64 < (-K_X)^3 < 72$. If $\gamma$ contains the singular point on $\mathbb{P}$, then, as in the proof of Lemma 7.4, we get a contradiction with Lemma 3.26.

Now, suppose that the set $\gamma \cap \mathbb{P}$ consists only of smooth points on $\mathbb{P}$. Note that the line $\gamma$ is not contained in $\mathbb{P}$, since the divisor $K_\mathbb{P}$ is divisible in $\text{Pic}(\mathbb{P})$ (see Example 3.12), and hence intersects $\mathbb{P}$ at $\leq 2$ points, since $\mathbb{P} \subset \mathbb{P}^{38}$ is an intersection of quadrics. Then the general hyperplane section $S$ of $\mathbb{P}$ through $\gamma$ is a non-singular $K3$ surface. Furthermore, projection $\pi$ gives a birational map

$$\chi := \pi \big|_S : S \dashrightarrow S'$$

on the general surface $S' \in | - K_X |$ with Du Val singularities (see Theorem 3.24). This implies that $S$ is the minimal resolution of $S'$ and $\chi$ is a morphism.

Projection $\pi$ is given by the linear system $\mathcal{L} \subset | - K_\mathbb{P} |$ of all hyperplane sections of $\mathbb{P}$ passing through $\gamma$. Set $\mathcal{L}_S := \mathcal{L} \big|_S$ and $\mathcal{L}_{S'} := | - K_X | \big|_{S'}$. Then we have $\mathcal{L}_S = \chi^*(\mathcal{L}_{S'})$, which in particular implies the equality $(L^2)_S = (L^2)_{S'}$ for $L \in \mathcal{L}_S$ and $L' \in \mathcal{L}_{S'}$. On the other hand, we have $(L^2)_{S'} = (-K_X)^3$ and $(L^2)_S = (-K_\mathbb{P})^3$. Thus, we obtain

$$(-K_X)^3 = (L^2)_{S'} = (L^2)_S = 72.$$
Lemma 7.6. If \( \pi \) is the projection from a plane, then \((-K_X)^3 \not\in \{66, 68, 70\} \).

Proof. Let \( \Omega \) be the center of the projection \( \pi \). Then \( \Omega \) intersects \( \mathbb{P} \), since otherwise \( \pi \) is an isomorphism, which is impossible because \( 64 < (-K_X)^3 < 72 \). Moreover, the plane \( \Omega \) is not contained in \( \mathbb{P} \), since the divisor \( K_F \) is divisible in \( \text{Pic}(\mathbb{P}) \) (see Example 3.12), and hence intersects \( \mathbb{P} \) either at \( \leq 4 \) points or by a conic, since \( \mathbb{P} \subset \mathbb{P}^3 \) is an intersection of quadrics.

If \( \Omega \) intersects \( \mathbb{P} \) at \( \leq 4 \) points, then, as in the proof of Lemmas 7.4 and 7.5, we obtain a contradiction with inequality \((-K_X)^3 = 72 \).

Now, let \( B := \Omega \cap \mathbb{P} \) be a conic. Note that the image of the threefold \( \mathbb{P} \) under the isomorphism \( \Phi \) is the cone over del Pezzo surface of degree 9 (see Example 3.12). This implies that \( \Phi^{-1} \mathbb{P}(B) \) is a generatrix of this cone. In particular, \( B \) contains the singular point on \( \mathbb{P} \). Then, as in the proof of Lemma 7.1, we get a contradiction with Lemma 3.26.

From Lemmas 7.4, 7.6 we get a contradiction with \( |H_W| \geq 36 \), the assumption \( 64 < (-K_X)^3 < 72 \) and (3.1). Proposition 7.3 is completely proved.

Our next goal is to get a contradiction with (3.1), the assumption \( 64 < (-K_X)^3 < 72 \) and the estimates in Proposition 7.3.

Remark 7.7. It follows from [32, Lemma 10.4] and Lemma 7.1 that there exists a birational map \( W \rightarrow W_0 \) on a \( \mathbb{P}^1 \)-bundle \( W_0 \) over a non-singular rational surface \( W'_0 \) without \((-1\text{-})\)-curves. Furthermore, the proper transforms \( L_{W_0} \) and \( H_{W_0} \) on \( W_0 \) of linear systems \( L_W \) and \( H_W \), respectively, and the threefold \( W_0 \) possess the same properties as \( L_W \), \( H_W \) and \( W \) (see (5.10), Remarks 5.8, 5.9 and 5.11). Moreover, for the general elements \( H_W \in H_W \) and \( H_{W_0} \in H_{W_0} \) inequality \( \dim |H_W| \leq \dim |H_{W_0}| \) holds. Thus, without loss of generality one may assume that \( W' \simeq \mathbb{P}^2 \) or \( F_n \) for some \( n \neq 1 \). Moreover, by [32, Lemma 10.12] in the last case we have \( n \leq 4 \).

For \( i \in \{1, 2\} \) set \( c_i := c_i(\mathcal{E}) \). From the relative exact Euler sequence (see for example [28, Proposition 4.26]) we obtain

(7.8) \[-K_W \sim 2H_W + \text{ext}_W(-K_{W'} - c_1)\]

on \( W \). From this and the Hirsch formula (see for example [8]), since \( O_W(H_W) \) is the tautological line bundle of \( W \), we obtain

(7.9) \[(H^2_W)_W \equiv (H_W \cdot \text{ext}_W(c_1) - \text{ext}_W(c_2))_W \quad \text{ and } \quad (H^3_W)_W = c_1^2 - c_2,

which implies the following formula

(7.10) \[(-K_W)^3 = 6(K^2_{W'})_W + 2c_1^2 - 8c_2.

Further, by the Riemann–Roch Formula for rank 2 vector bundles over a rational surface we have

(7.11) \[\chi(\mathcal{E}) = \frac{1}{2}((c_1^2)_{W'} - 2c_2 - (K_{W'} \cdot c_1))_{W'} + 2\]

(see for example [11]).

Lemma 7.12 (see [32]). For every nef divisor \( B \) on \( W' \), inequality

(7.12) \[(c_1 \cdot B)_{W'} \leq 3(-K_{W'} \cdot B)_{W'},\]

holds.

Proof. Since

(7.13) \[D_W \sim H_W + \text{ext}_W(-K_{W'} - c_1)\]
on $W$ and the linear system $|-K_W|$ does not have fixed components (see Remark 5.8), for the nef divisor $B$ on $W'$ we have (see (7.9))

$$0 \leq (-K_W \cdot D_W \cdot \text{ext}_W^*(B))_W = (2H_W^2 \cdot \text{ext}_W^*(B))_W + 3(( -K_W - c_1) \cdot B)_W = 3((-K_W \cdot B)_W - (c_1 \cdot B)_W).$$

Lemma 7.13 (see [32]). Let $B > 0$ be a nef divisor on $W'$ such that $K_W \cdot D_W \cdot \text{ext}_W^*(B) = 0$. Then the divisor $-K_W$ is nef on $W$.

Proof. Suppose that $Z$ is a horizontal curve on $W$ such that $(-K_W \cdot Z)_W < 0$. Then we have

$$(D_W \cdot Z)_W = ((-K_W - H_W) \cdot Z)_W < 0.$$ 

Thus, for the general element $L \in |-K_W|$ we get $Z \subseteq L \cap D_W$. On the other hand, by the assumption the intersection $L \cap D_W$ consists of the fibres of the morphism $\text{ext}_W$, a contradiction.

Lemma 7.14. The cycle $-K_W + c_1$ is nef on $W'$.

Proof. Since the linear system $|-K_W|$ does not have fixed components and the divisor $H_W$ is nef on $W$ (see Remark 5.8), for every irreducible curve $B$ on $W'$ we have (see (7.9))

$$0 \leq (-K_W \cdot H_W \cdot \text{ext}_W^*(B))_W = (2H_W^2 \cdot \text{ext}_W^*(B))_W + \left((-K_W - c_1) \cdot B\right)_W = 2(c_1 \cdot B)_W + (\left((-K_W - c_1) \cdot B\right)_W = 3((-K_W \cdot B)_W - (c_1 \cdot B)_W).$$

Lemma 7.15 (see [32, Lemma 10.6]). Let $B \subset W'$ be an irreducible rational curve such that $\dim |B| > 0$ and $E|_B \simeq \mathcal{O}_{P^1}(d_1) \oplus \mathcal{O}_{P^1}(d_2)$ for some $d_1, d_2 \in \mathbb{Z}$. Then $|d_1 - d_2| \leq 2 + (B^2)_{W'}$.

Proof. Set $m := |d_1 - d_2|$ and $G := g^{-1}(B)$. Then $G \simeq \mathbb{F}_m$ and for the minimal section $h$ on the surface $G$ we have

$$(7.16) \quad -2 + m = -2 - (h^2)_G = (K_G \cdot h)_G = (K_W \cdot h)_W + (G \cdot h)_W = (K_W \cdot h)_W + (B^2)_{W'}.$$

Since the linear system $|-K_W|$ does not have fixed components (see Remark 5.8), we have $(K_W \cdot h)_W \leq 0$, which implies that $m \leq 2 + (B^2)_{W'}$.

Proposition 7.17. We have $W' \neq \mathbb{P}^2$.

Proof. Suppose that $W' = \mathbb{P}^2$. Then we get $H^{2i}(\mathbb{P}^2, \mathbb{Z}) \simeq \mathbb{Z}$, $i = 1, 2$. Thus, one may assume that $c_1$ and $c_2$ are the integers. Then by Lemma 7.12 we have $0 \leq c_1 \leq 9$.

Lemma 7.18. Vector bundle $E$ is indecomposable.

Proof. Suppose that $E$ is decomposable. Then $E \simeq \mathcal{O}_{P^2}(a) \oplus \mathcal{O}_{P^2}(a + b)$ for some $b \geq 0$. Moreover, since the divisor $H_W$ is nef, we also have $a \geq 0$.

Further, it follows from the properties of the Chern classes (see for example [7]) that $c_1 = 2a + b$ and $c_2 = a^2 + ab$. Then from (7.11) we obtain

$$\chi(W', E) = \frac{1}{2} \left(2a^2 + 2ab + b^2 + 6a + 3b\right) + 2.$$ 

According to (7.2) and Proposition 7.3 we have $\chi(W', E) \in \{33, 34, 35, 36\}$. Then it follows easily from the estimates $a, b \geq 0$ and $0 \leq c_1 \leq 9$ that either $(a, b) = (2, 4)$ or $(4, 1)$.
Suppose that \((a, b) = (2, 4)\). Then \(W \simeq \mathbb{P}(\mathcal{E}')\), where
\[
\mathcal{E}' = \mathcal{O}_{W'} \oplus \mathcal{O}_{W'}(4),
\]
which implies that \(W\) is the blow up of the vertex of the cone \(S \subset \mathbb{P}^{15}\) over \(\mathbb{P}^2\) embedded in \(\mathbb{P}^{14}\) by the linear system \(|\mathcal{O}_{\mathbb{P}^2}(4)|\). Let \(N\) be the negative section of \(W\) which is contracted to the vertex of the cone \(S\). Since \(c_1(\mathcal{E}') = 4 > -K_{W'} = 3\), it follows from the formula similar to (7.8) that
\[
(-K_W \cdot Z)_W < 0
\]
for every curve \(Z \subset N\). In particular, the surface \(N\) is a base component of the linear system \(|-K_W|\), which is impossible by Remark 5.8.

Now, suppose that \((a, b) = (4, 1)\). In this case we have equality \(c_1 = -3K_{W'}\). Then it follows from the proof of Lemma 7.12 that
\[
(-K_W \cdot D_W \cdot \text{ext}_W^*(B))_W = 0
\]
for every curve \(B\) on the surface \(W'\). Then by Lemma 7.13 the divisor \(-K_W\) is nef on \(W\), and hence \(W\) is a weak Fano threefold because \(-K_W\) is also big by Remark 5.9. On the other hand, for \((a, b) = (4, 1)\) we have \(c_2 = 20\), and from (7.10) we obtain that \((-K_W)^3 = 56\). Then form the Riemann–Roch Formula (see [37]) and Kawamata–Viehweg Vanishing Theorem (see for example [21]) we get
\[
\dim |-K_W| = -\frac{1}{2}K_W^3 + 2 = 30,
\]
which contradicts the estimate in Remark 5.11. Lemma 7.18 is completely proved.

Remark 7.19. The arguments from the proof of Proposition 10.3 in [32] imply that if \(c_1 = 9\), then the vector bundle \(\mathcal{E}\) is decomposable. Thus, by Lemma 7.18 we have \(0 \leq c_1 \leq 8\).

Lemma 7.20. The number \(c_1\) is even.

Proof. Suppose that \(c_1\) is odd. Then from Remark 7.19 we get \(c_1 = 2m - 3\) for some \(2 \leq m \leq 5\). Further, since by (7.2) and Proposition 7.3 we have \(\chi(W', \mathcal{E}) \in \{33, 34, 35, 36\}\), from (7.11) we obtain that \(2m^2 - 3m - c_2 \geq 31\). Then it follows from the properties of the Chern classes (see for example [7]) that
\[
c_1(\mathcal{E}(-m)) = -3 \quad \text{and} \quad c_2(\mathcal{E}(-m)) = c_2 - m^2 + 3m \leq m^2 - 31 < 0.
\]

On the other hand, from the Riemann–Roch Formula (see (7.11)) and Serre Duality we obtain
\[
h^0(W', \mathcal{E}(-m)) + h^0(W', \mathcal{E}(-m) \otimes \det \mathcal{E}(-m)^* \otimes \mathcal{O}_{W'}(-3)) \geq \chi(W', \mathcal{E}(-m)) \geq 1,
\]
which implies that \(H^0(W', \mathcal{E}(-m)) \neq 0\) because \(\det \mathcal{E}(-m)^* \simeq \mathcal{O}_{W'}(3)\).

Let \(s \in H^0(W', \mathcal{E}(-m))\) be a non-zero section and \(Z \subset W'\) the zero locus of \(s\). Since \(c_2(\mathcal{E}(-m)) < 0\), we have \(\dim Z = 1\) (see [8]). Take the general line \(\Gamma \subset W' = \mathbb{P}^2\) and set \(k := (\Gamma \cdot Z)_{W'}\). Since \(c_1(\mathcal{E}(-m)) = -3\), it follows from the properties of the Chern classes (see for example [7]) that
\[
\mathcal{E}(-m)|_\Gamma = \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(-k - 3).
\]

On the other hand, by Lemma 7.15 we have \(2k + 3 \leq 3\), which implies that \(k = 0\) and \(Z = \emptyset\), a contradiction.

It follows from Lemma 7.20 and Remark 7.19 that \(c_1 = 2m - 2\) for some \(1 \leq m \leq 5\). Further, since by (7.2) and Proposition 7.3 we have \(\chi(W', \mathcal{E}) \in \{33, 34, 35, 36\}\), from (7.11) we obtain that \(2m^2 - m - c_2 \geq 32\). Then it follows from the properties of the Chern classes (see for example [7]) that
\[
c_1(\mathcal{E}(-m)) = -2 \quad \text{and} \quad c_2(\mathcal{E}(-m)) = c_2 - m^2 + 2m \leq m^2 + m - 32 < 0.
\]
Now, as in the proof of Lemma 7.20, there exists a section \( s \in H^0(W', \mathcal{E}(-m)) \) with the one-dimensional zero locus \( Z \subset W' \). Take the general line \( \Gamma \subset W' = \mathbb{P}^2 \) and set \( k := (\Gamma \cdot Z)_{W'} \). Since \( c_1(\mathcal{E}(-m)) = -2 \), it follows from the properties of the Chern classes (see for example [7]) that
\[
\mathcal{E}(-m)|_{\Gamma} = \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(-k - 2).
\]

On the other hand, by Lemma 7.15 we have \( 2k + 2 \leq 3 \), which implies that \( k = 0 \) and \( Z = \emptyset \), a contradiction. Proposition 7.17 is completely proved.

**Proposition 7.21.** For \( n \in \{0, 2, 3, 4\} \), we have \( W' \neq \mathbb{F}_n \).

**Proof.** Suppose that \( W' = \mathbb{F}_n \), where \( n \in \{0, 2, 3, 4\} \). Then we get \( H^4(\mathbb{F}_n, \mathbb{Z}) \simeq \mathbb{Z} \) and \( H^2(\mathbb{F}_n, \mathbb{Z}) \simeq \mathbb{Z} \cdot h \oplus \mathbb{Z} \cdot l \). Set \( c_1 := ah + bl, c_2 := c, \) where \( a, b, c \in \mathbb{Z} \). Then by Lemma 7.12 we have
\[
(7.22) \quad a \leq 3(-K_{W'} \cdot l)_{W'} = 6 \quad \text{and} \quad b \leq 3(-K_{W'} \cdot (h + nl))_{W'} = 3(2 + n).
\]
Moreover, \( a \geq 0 \) and \( b \geq na \), since the divisor \( H_W \) is nef on \( W' \).

Let \( p, q \) be the integers such that \( a = 2p + a' \) and \( b = 2q + b' \) for some \( a', b' \in \mathbb{Z} \) with \( -2 \leq a', b' \leq -1 \). Consider the twisted vector bundle \( \mathcal{E}' := \mathcal{E} \oplus \mathcal{O}_{\mathbb{P}^2}(-ph - ql) \) and set \( c_i' := c_i(\mathcal{E}'), i = 1, 2 \). Then it follows from the properties of the Chern classes (see for example [7]) that
\[
(7.23) \quad c_1' = a'h + b'l \quad \text{and} \quad c_2' = c + nap - aq - bp - npq + 2pq.
\]

Further, from (7.11) we obtain
\[
(7.24) \quad \chi(W', \mathcal{E}) = -\frac{1}{2} na(a + 1) + ab + a + b - c + 2
\]
and
\[
(7.25) \quad \chi(W', \mathcal{E}') = (b' - \frac{1}{2} na') (a' + 1) + a' - c_2' + 2.
\]

**Lemma 7.26.** If \( n = 0 \), then \( c_2' < 0 \) and \( \chi(W', \mathcal{E}') > 0 \).

**Proof.** By (7.22) we have \( 0 \leq a, b \leq 6 \). Further, from (7.23) and (7.24) we get
\[
c_2' = c - aq - bp + 2pq = \frac{1}{2} ab + a + b - \chi(W', \mathcal{E}) + 2 + \frac{1}{2} a'b'.
\]
According to (7.2) and Proposition 7.3 we have \( \chi(W', \mathcal{E}) \in \{33, 34, 35, 36\} \). Then it is easy to see that \( c_2' < 0 \) when either \( a \) or \( b \) is less than 6. Moreover, for all \( 0 \leq a, b \leq 6 \) and \( \chi(W', \mathcal{E}) \in \{35, 36\} \) we also have \( c_2' < 0 \).

Now, let \( a = b = 6 \) and \( \chi(W', \mathcal{E}) \in \{33, 34\} \). Then \( c_1 = 6h + 6l = -3K_{W'} \), and it follows from the proof of Lemma 7.12 that
\[
(-K_{W'} \cdot D_{W'} \cdot \text{ext}_W(B))_{W'} = 0
\]
for every curve \( B \) on the surface \( W' \). Then by Lemma 7.13 the divisor \(-K_{W'} \) is nef on \( W' \), and hence \( W' \) is a weak Fano threefold because \(-K_{W} \) is also big by Remark 5.9. On the other hand, from (7.24) we get
\[
\chi(W', \mathcal{E}) = ab + a + b - c + 2.
\]

Then for \( a = b = 6 \) and \( \chi(W', \mathcal{E}) \in \{33, 34\} \) we obtain that \( c \in \{16, 17\} \). This and (7.10) imply that \((-K_{W})^3 \in \{64, 56\} \). Now, form the Riemann–Roch Formula (see [37]) and Kawamata–Viehweg Vanishing Theorem (see for example [21]) we get
\[
\dim | -K_W | = \frac{1}{2} K_W^3 + 2 \leq 34,
\]
which contradicts the estimate in Remark 5.11.

Thus, \( c_2' < 0 \). Further, from (7.25) we obtain
\[
\chi(W', \mathcal{E}') = b'(a' + 1) + a' - c_2' + 2.
\]
Since \( -2 \leq a', b' \leq -1 \) and \( c_2' < 0 \), we have \( \chi(W', \mathcal{E}') > 0 \). Lemma 7.26 is completely proved.
Lemma 7.27. If $n = 2$, then $c'_2 < 0$ and $\chi(W', E') > 0$.

Proof. By (7.22) we have $2a \leq b \leq 12$ and $0 \leq a \leq 6$. Further, from (7.23) and (7.24) we get

$$c'_2 = c + 2ap - aq - bq - 2p^2 + 2pq = -\frac{1}{2}a^2 + ab + 1 - \frac{1}{2}ab + b - \chi(W', E') + 2 - \frac{1}{2}a^2 + \frac{1}{2}ab'.$$

According to (7.2) and Proposition 7.3, we have $\chi(W', E') \in \{33, 34, 35, 36\}$. Then

$$c'_2 \leq -\frac{1}{2}a^2 + 6a + 14 - \chi(W', E') - \frac{1}{2}a^2 - a' \leq 32 - \chi(W', E') - \frac{1}{2}a^2 - a' \leq -1 - \frac{1}{2}a^2 - a' < 0.$$  

Further, from (7.25) we obtain

$$\chi(W', E') = (b' - a')(a' + 1) + a' - c'_2 + 2.$$  

Then $\chi(W', E') \leq 0$ only for $a' = -2$, $b' = -1$ and $c'_2 = -1$ because $-2 \leq a', b' \leq -1$ and $c'_2 < 0$. But in this case we have $b \leq 11$, which implies that

$$-1 = c'_2 \leq -\frac{1}{2}a^2 + \frac{11}{2}a + 12 - \chi(W', E') < -1,$$

a contradiction. Thus, $\chi(W', E') > 0$. Lemma 7.27 is completely proved.

Lemma 7.28. If $n = 3$, then $c'_2 < 0$ and $\chi(W', E') > 0$.

Proof. According to (7.22), we have $3a \leq b \leq 15$, which implies that $0 \leq a \leq 5$. Further, from (7.23) and (7.24) we get

$$c'_2 = c + 3ap - aq - bp - 3p^2 + 2pq = -\frac{3}{4}a^2 + \frac{13}{2}a + 16 - \chi(W', E') - \frac{3}{4}a^2 - a' \leq 29 - \chi(W', E') < 0,$$

and for $b = 15$ and $0 \leq a \leq 5$ we obtain

$$c'_2 = -\frac{3}{4}a^2 + 7a + 17 - \chi(W', E') - \frac{3}{4}a^2 - \frac{1}{2}a' \leq 33 - \chi(W', E') < 0.$$  

If $\chi(W', E') = 33$, then for $3a \leq b \leq 14$ and $0 \leq a \leq 5$ we get

$$c'_2 \leq -\frac{3}{4}a^2 + 13a - 17 - \frac{3}{4}a^2 + \frac{1}{2}a' \leq -\frac{3}{4}a^2 + \frac{13}{2}a - 15 - \frac{3}{4}a^2 < 0.$$  

Now, $\chi(W', E') = 33$ and let $b = 15$. Then it follows from the proof of Lemma 7.12 that

$$(-K_W \cdot D_W \cdot \text{ext}^n_w(B))_W = 0$$

for the curve $B \sim h + 3l$ on the surface $W' = F_3$. Then by Lemma 7.13 the divisor $-K_W$ is nef on $W$, and hence $W$ is a weak Fano threefold because $-K_W$ is also big by Remark 5.9. Then by Theorem 3.18 the divisor $-K_{W'}$ is nef on $W' = F_3$, a contradiction.

Thus, $c'_2 < 0$. Further, from (7.25) we get

$$\chi(W', E') = (b' - \frac{3}{2}a')(a' + 1) + a' - c'_2 + 2.$$  

Then $\chi(W', E') \leq 0$ only for $a' = -2$ because $-2 \leq a', b' \leq -1$ and $c'_2 < 0$. In particular, $a$ must be even.

If $a' = -2$ and $b' = -2$, then $\chi(W', E') = -1 - c'_2$ and

$$c'_2 = -\frac{3}{4}a^2 + \frac{1}{2}ab - \frac{1}{2}a + b - \chi(W', E') + 1.$$
On the other hand, since $a$ and $b$ are even, we have $a \leq 4$ and $3a \leq b \leq 14$, which implies that 
\[
c'_2 \leq -\frac{3}{4}a^2 + \frac{13}{2}a - 18 < -1.\]
Thus, for $a' = -2$ and $b' = -2$ we have $\chi(W', E') > 0$.

Now, let $a' = -2, b' = -1$. Then $\chi(W', E') = -2 - c'_2$ and 
\[
c'_2 = -\frac{3}{4}a^2 + \frac{1}{2}ab - \frac{1}{2}a + b - \chi(W', E).\]
On the other hand, since $a$ is even, we have $0 \leq a \leq 4$. Then for $3a \leq b \leq 13$ we get 
\[
c'_2 \leq -\frac{3}{4}a^2 + 6a + 13 - \chi(W', E) \leq -\frac{3}{4}a^2 + 6a - 20 < -2,\]
which implies that $\chi(W', E') > 0$. Furthermore, since $a$ is odd, for $b > 13$ we have $b = 15$. Then, as above, we obtain that $-K_{W'}$ is nef on $W' = \mathbb{P}_3$, a contradiction.

Thus, $\chi(W', E') > 0$. Lemma 7.28 is completely proved.

**Lemma 7.29.** If $n = 4$, then $c'_2 < 0$ and $\chi(W', E') > 0$.

**Proof.** According to (7.22), we have $4a \leq b \leq 18$, which implies that $0 \leq a \leq 4$. Further, from (7.23) and (7.24) we get 
\[
c'_2 = c + 4ap - aq - bp + 4p^2 + 2pq = -a^2 + \frac{1}{2}ab - a + b - \chi(W', E) + 2 - a^2 + \frac{1}{2}a'b'.\]
By (7.24) and Proposition 7.3 we have $\chi(W', E) \in \{33, 34, 35, 36\}$. If $b \leq 17$, then for $0 \leq a \leq 4$ we get 
\[
c'_2 \leq -a^2 + \frac{15}{2}a - 14 - a^2 + \frac{1}{2}a'b' \leq -a^2 + \frac{15}{2}a - 14 - a^2 - a' < 0.\]
Now, let $b = 18$. Then it follows from the proof of Lemma 7.12 that 
\[
( -K_W \cdot D_W \cdot \text{ext}_W (B))_W = 0\]
for the curve $B \sim h + 4l$ on the surface $W' = \mathbb{P}_4$. Then by Lemma 7.13 the divisor $-K_W$ is nef on $W$, and hence $W$ is a weak Fano threefold because $-K_W$ is also big by Remark 5.9. Then by Theorem 3.18 the divisor $-K_{W'}$ is nef on $W' = \mathbb{P}_4$, a contradiction.

Thus, $c'_2 < 0$. Further, from (7.25) we get 
\[
\chi(W', E') = (b' - 2a')(a' + 1) + a' - c'_2 + 2.\]
Then $\chi(V, E') \leq 0$ only for $a' = -2$ because $-2 \leq a', b' \leq -1$ and $c'_2 < 0$. In particular, $a$ must be even. If $a' = -2$ and $b' = -1$, then $\chi(W', E') = -3 - c'_2$ and 
\[
c'_2 = -a^2 + \frac{1}{2}ab - a + b - \chi(W', E) - 1.\]
Since $b$ is odd, we have $4a \leq b \leq 17$. Then for $0 \leq a \leq 3$ we get 
\[
c'_2 \leq -a^2 + \frac{15}{2}a - 17 < -3,\]
and for $a = 4$ and $4a \leq b \leq 16$ we get 
\[
c'_2 \leq -54 + 3b < -3,\]
which implies that $\chi(W', E') > 0$. Furthermore, for $a = 4$ and $b = 17$ the cycle $-K_{W'} + c_1 = 6h + 23l$ is not nef on $W' = \mathbb{P}_4$, which contradicts Lemma 7.14. Thus, for $a' = -2$ and $b' = -1$ we have $\chi(W', E') > 0$.

Now, let $a' = -2$ and $b' = -2$. Then $\chi(W', E') = -2 - c'_2$ and 
\[
c'_2 = -a^2 + \frac{1}{2}ab - a + b - \chi(W', E).\]
For $4a \leq b \leq 16$ and $0 \leq a \leq 4$ we have
\[ c_2' \leq -a^2 + 7a - 17 < -2, \]
which implies that $\chi(W', \mathcal{E}') > 0$. Furthermore, for $b > 16$ we have $b = 18$, since $b$ is even. Then, as above, we obtain that $-K_{W'}$ is nef on $W' = \mathbb{F}_4$, a contradiction.

Thus, $\chi(W', \mathcal{E}') > 0$. Lemma 7.29 is completely proved. \hfill \Box

**Lemma 7.30** (see [32, Claim 10.18]). Inequality $H^0(W', \mathcal{E}') \neq 0$ holds.

**Proof.** Suppose that $H^0(W', \mathcal{E}') = 0$. Then it follows from Lemmas 7.26–7.29 that $H^2(W', \mathcal{E}') \neq 0$. Furthermore, by Serre Duality we have
\[ H^2(W', \mathcal{E}')^* \cong H^0(W', \mathcal{E}'^* \otimes K_{W'}) \cong H^0(W', \mathcal{E}' \otimes \det \mathcal{E}'^* \otimes K_{W'}). \]
But
\[ (\det \mathcal{E}'^* \otimes K_{W'})^* = \mathcal{O}_{W'}((a' + 2)h + (b' + n + 2)l), \]
which implies that $H^0(W', (\det \mathcal{E}'^* \otimes K_{W'}))^*) \neq 0$, a contradiction. \hfill \Box

Let $s \in H^0(W', \mathcal{E}')$ be a non-zero section (see Lemma 7.30) and $Z \subset W'$ the zero locus of $s$. Since $c_2' < 0$ (see Lemmas 7.26–7.29), we have $\dim Z = 1$ (see [8]). Put $Z \sim q_1h + q_2l$. It follows from the properties of the Chern classes (see for example [7]) that
\[ \mathcal{E}'|_l = \mathcal{O}_{\mathbb{P}^1}(q_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a' - q_1) \]
for the general curve $l \in |l|$ on the surface $W'$. Then by Lemma 7.15 we have
\[ 2q_1 - a' \leq 2, \]
which implies that $q_1 = 0$ because $q_1 \geq 0$ and $a' \leq -1$. Thus, the zero locus of $s$ is contained in the fibres of the $\mathbb{P}^1$-bundle $W' = \mathbb{F}_n \to \mathbb{P}^1$. In particular, we have $q_2 > 0$. On the other hand, it follows from the properties of the Chern classes that
\[ (7.31) \quad \mathcal{E}'|_{l'} = \mathcal{O}_{\mathbb{P}^1}(q_2) \oplus \mathcal{O}_{\mathbb{P}^1}(b' - q_2) \]
for the general curve $l' \in |h + nl|$ on the surface $W'$. Fix a point $o \in l'$ and the local coordinate $t$ in a neighborhood $o \in U \subset W'$ such that the equation $t = t_0$ for fixed $t_0 \in \mathbb{C}$ determines the fibre of the natural morphism $U \subset W' \to \mathbb{P}^1$. Then, since $(c_1(\mathcal{E}'))_{|l'} = c_1(\mathcal{E}'|_{l'}) = b' < 0$ and hence $H^0(l', \mathcal{O}_{\mathbb{P}^1}(c_1(\mathcal{E}'|_{l'}))) = 0$, and $Z \sim q_2l$, the section $s|_{l'}$ of $\mathcal{E}'|_{l'}$ on $U \cap l'$ is determined as the set of pairs $(t^\alpha, t^\beta)$, for some $\alpha, \beta \in \mathbb{N}$, when $t$ runs through $U \cap l'$. This and (7.31) imply that
\[ 2q_2 - b' \leq \alpha q_2 + \beta(q_2 - b') = 0, \]
which is impossible because $q_2 > 0$ and $b' \leq -1$. Proposition 7.21 is completely proved. \hfill \Box

From Remark 7.7, Propositions 7.17 and 7.21 we obtain that $\text{ext}_W : W \to W'$ can not be a contraction onto a surface. Together with the results of the previous Sections, this completes the proof of Theorem 1.2.

**8. Fano threefolds of degree 64**

We use notation and conventions from Section 3. In the present Section we will prove Theorem 1.7. So, now $X$ is a Fano threefold with $(-K_X)^3 = 64$.

Let $f : Y \to X$ be a terminal $\mathbb{Q}$-factorial modification of $X$ and $\text{ext} : Y \to Y'$ a $K_Y$-negative extremal contraction. It follows from Theorem 3.16 that to prove Theorem 1.7 one may assume that $\dim Y' \neq 0$. Then by Remark 3.11 and Theorem 3.17 we have $X \neq Y$, $\rho(Y) > 1$ and $\dim Y' \geq 2$.

**Proposition 8.1.** If $\dim Y' = 2$, then $X$ is isomorphic to one of the cones from Examples 1.4, 1.5.
Proof. It follows from Theorem 3.18 that ext : Y → Y’ is a \( \mathbb{P}^1 \)-bundle such that the divisor \(-K_{Y’}\) is nef and big on the non-singular surface \( Y’ \). Then by Grothendieck Theorem on triviality of the Brauer group of the non-singular rational surface (see [3]) we have \( Y = \mathbb{P}(\mathcal{E}) \) for some rank 2 vector bundle \( \mathcal{E} \) over \( Y’ \).

Lemma 8.2. If \( Y’ \) does not contain \((-1)\)-curves, then \( X \) is isomorphic to the cone from Example 1.4.

Proof. Since \(-K_{Y’}\) is nef and big on \( Y’ \), we have \( Y’ = \mathbb{P}^2 \) or \( \mathbb{P}^1 \times \mathbb{P}^1 \), or \( \mathbb{F}_2 \).

Suppose that \( Y’ = \mathbb{P}^2 \). Then we get \( H^2(Y’, \mathbb{Z}) \simeq \mathbb{Z}, i = 1, 2 \). Thus, one may assume that \( c_1(\mathcal{E}) \) and \( c_2(\mathcal{E}) \) are the integers. Moreover, if \( c_1(\mathcal{E}) \) is even, then it follows from the properties of the Chern classes (see for example [7]) that one may assume that \( c_1(\mathcal{E}) = 0 \). In this case from (7.10) we obtain

\[ 64 = (-K_Y)^3 = 54 - 8c_2(\mathcal{E}), \]

which implies that \( c_2(\mathcal{E}) = -5/4 \not\in \mathbb{Z} \), a contradiction.

Now, let \( c_1(\mathcal{E}) \) be odd. Then it follows from the properties of the Chern classes that one may assume that \( c_1(\mathcal{E}) = -K_{Y’}, \) which implies the equality \(-K_Y = 2D \) on \( Y’ \), where \( D \) is the tautological section of the \( \mathbb{P}^1 \)-bundle ext : \( Y → Y’ \) (see (7.8)). From the Adjunction Formula we obtain

\[ -K_D \sim D|_D, \]

and since \((-K_Y)^3 = 64\), we have \( K_D^2 = 8 \). On the other hand, \( D \simeq \mathbb{P}^2 \), a contradiction.

Suppose that \( Y’ = \mathbb{P}^1 \times \mathbb{P}^1 \). Then we get \( H^1(Y’, \mathbb{Z}) \simeq \mathbb{Z} \) and \( H^2(Y’, \mathbb{Z}) \simeq \mathbb{Z} \cdot h \oplus \mathbb{Z} \cdot l \). This implies that \( c_1(\mathcal{E}) := ah + bl \) and \( c_2(\mathcal{E}) := c \), where \( a, b, c \in \mathbb{Z} \). Moreover, if \( a \) or \( b \) is odd, then exactly as in the proof of Proposition 5.2 in [32] we get a contradiction in this case.

Now, let \( a \) and \( b \) be both even. Then it follows from the properties of the Chern classes that one may assume that \( c_1(\mathcal{E}) = -K_{Y’}, \) which implies the equality \(-K_Y = 2D \) on \( Y’ \), where \( D \) is the tautological section of the \( \mathbb{P}^1 \)-bundle ext : \( Y → Y’ \) (see (7.8)). Note that the morphism \( f : Y → X \) contracts a divisor, since otherwise \( X \) has only terminal Gorenstein singularities and Theorem 3.16 implies that \( X = Y = \mathbb{P}^3 \), which contradicts our assumption. Furthermore, it follows from the equalities \( c_1(\mathcal{E}) = -K_{Y’}, -K_Y = 2D \) and (7.9) that the \( f \)-exceptional locus \( E_f \) is horizontal with respect to ext. Moreover, from the Adjunction Formula

\[ -K_D \sim D|_D \]

and an isomorphism \( D \simeq \mathbb{P}^1 \times \mathbb{P}^1 \) we obtain that either \( E_f \subset D \) or \( E_f \cap D = \emptyset \). Since by Kawamata–Viehweg Vanishing Theorem \( H^1(Y, \mathcal{O}_Y) = 0 \), the restriction \( H^0(Y, \mathcal{O}_Y(D)) \to H^0(D, \mathcal{O}_D(D)) \) is surjective. This implies that the linear system \( |D| \) gives a birational morphism with exceptional divisor \( E_f \) mapping \( Y \) onto the cone in \( \mathbb{P}^9 \) over anticanonically embedded \( \mathbb{P}^1 \times \mathbb{P}^1 \), and contracting \( E_f \) into the vertex. Thus, \( X \) is isomorphic to the cone from Example 1.4.

Finally, suppose that \( Y’ = \mathbb{F}_2 \). Then, as above, we have \( c_1(\mathcal{E}) := ah + bl \) and \( c_2(\mathcal{E}) := c \), where \( a, b, c \in \mathbb{Z} \). Let us consider the case when \( a \) and \( b \) are both even (cases of the other parities of \( a \) and \( b \) are treated in exactly the same way as in the proof of Proposition 5.2 in [32]). Then, as above, one may assume that \( c_1(\mathcal{E}) = -2h - 2l \). From (7.10) we obtain

\[ 64 = (-K_Y)^3 = 48 - 8c, \]

which implies that \( c = -2 \). Further, from (7.11) and Serre Duality we get

\[ h^0(Y’, \mathcal{E}) + h^0(Y’, \mathcal{E} \otimes \det \mathcal{E}^* \otimes \omega_{Y’}) \geq \frac{1}{2} \left( \left( (2h + 2l)^2 \right)_{Y’} - 2c + 2(\det \mathcal{E}^* \otimes \omega_{Y’} + K_{Y’} \cdot (h + l))_{Y’} \right) + 2 = 2, \]

and since \( \det \mathcal{E}^* \otimes \omega_{Y’} \simeq \mathcal{O}_{Y’}(-2l) \), this implies that \( H^0(Y’, \mathcal{E}) \neq 0 \).

Let \( s \in H^0(Y’, \mathcal{E}) \) be a non-zero section and \( Z \subset Y’ \) the zero locus of \( s \). Since \( c_2 = -2 \), we have \( \dim Z = 1 \) (see [8]). Put \( Z \simeq q_1h + q_2l \). It follows from the properties of the Chern classes (see for example [7]) that

\[ \mathcal{E}|_l = \mathcal{O}_{\mathbb{P}^1}(q_1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2 - q_1) \]
for the general curve $l \in |l|$ on the surface $Y'$. Since the linear system $|−K_Y|$ is base point free on $Y$ (see Remark 3.8), as in the proof of Lemma 7.15 we get

$$2q_1 + 2 \leq 2,$$

which implies that $q_1 = 0$ because $q_1 \geq 0$. Thus, the zero locus of $s$ is contained in the fibres of the $\mathbb{P}^1$-bundle $Y' = \mathbb{F}_1$ into $\mathbb{P}^1$. In particular, we have $q_2 > 0$. On the other hand, since $c_1(E|_{l'}) = −2$ for the general curve $l' \in |h + 2l|$, as in the proof of Proposition 7.21 we get $2q_2 + 2 \leq 0$, a contradiction. Lemma 8.2 is completely proved.

**Lemma 8.3.** If $Y'$ contains a $(-1)$-curve, then $X$ is isomorphic to the cone from Example 1.5.

**Proof.** It follows from [32, Lemma 5.16], Theorem 3.18 and Remark 3.7 that $Y' = \mathbb{F}_1$. Then we get $H^4(Y', \mathbb{Z}) \simeq \mathbb{Z}$ and $H^2(Y', \mathbb{Z}) \simeq \mathbb{Z} \cdot h + \mathbb{Z} \cdot l$. This implies that $c_1(E) := ah + bl$ and $c_2(E) := c$, where $a, b, c \in \mathbb{Z}$. Moreover, if $a$ is odd and $b$ is even, then, as in the proof of Lemma 8.2 one may assume that $c_1(E) = −h$. Then from (7.10) we obtain

$$64 = (−K_Y)^3 = 46 − 8c,$$

which implies that $c = −9/4 \notin \mathbb{Z}$, a contradiction.

Now, let $a$ be even and $b$ odd. Then, as in the proof of Lemma 8.2 one may assume that $c_1(E) = −K_{Y'}$. In this case, arguing exactly as in the proof of Lemma 8.2 and substituting $\mathbb{P}^1 \times \mathbb{P}^1$ with $\mathbb{F}_1$, we obtain that $X$ is isomorphic to the cone from Example 1.5.

Finally, suppose that $a$ and $b$ are both even. Then, as in the proof of Lemma 8.2 one may assume that $c_1(E) = −2h − 2l$. From (7.10) we obtain

$$64 = (−K_Y)^3 = 56 − 8c,$$

which implies that $c = −1$. Further, from (7.11) and Serre Duality we get

$$h^0(Y', E) + h^0(Y', E \otimes \det E^* \otimes \omega_{Y'}) ≥ \frac{1}{2}((−2h − 2l)^2)_{Y'} − 2c + 2(K_{Y'} \cdot (h + l))_{Y'} + 2 = 2,$$

and since $\det E^* \otimes \omega_{Y'} \simeq O_Y(−l)$, this implies that $H^0(Y', E) \neq 0$.

Let $s \in H^0(Y', E)$ be a non-zero section and $Z \subset Y'$ the zero locus of $s$. Since $c_2 = −1$, we have $\dim Z = 1$ (see [8]). Put $Z \sim q_1 h + q_2 l$. It follows from the properties of the Chern classes (see for example [7]) that

$$E|_l = O_{\mathbb{P}^1}(q_1) \oplus O_{\mathbb{P}^1}(−2 − q_1)$$

for the general curve $l \in |l|$ on the surface $Y'$. Since the linear system $|−K_Y|$ is base point free on $Y$ (see Remark 3.8), as in the proof of Lemma 7.15 we get

$$2q_1 + 2 \leq 2,$$

which implies that $q_1 = 0$ because $q_1 \geq 0$. Thus, the zero locus of $s$ is contained in the fibres of the $\mathbb{P}^1$-bundle $Y' = \mathbb{F}_1$ into $\mathbb{P}^1$. In particular, we have $q_2 > 0$. On the other hand, it follows from the properties of the Chern classes that

$$E|_{l'} = O_{\mathbb{P}^1}(q_2) \oplus O_{\mathbb{P}^1}(−2 − q_2)$$

for the general curve $l' \in |h + l|$ on the surface $Y'$. Then, as in the proof of Lemma 7.15 we get

$$2q_2 + 2 \leq 3,$$

which implies that $q_2 = 0$, a contradiction. Lemma 8.3 is completely proved.

Apart from Lemmas 8.2 and 8.3 prove Proposition 8.1.

According to Proposition 8.1 it remains to consider the case when the contraction $\text{ext} : Y \rightarrow Y'$ is birational. Let $E$ be the exceptional divisor of $\text{ext}$ (see Theorem 3.19).
Proposition 8.4. In the above assumptions, if $Y'$ is a weak Fano threefold with terminal factorial singularities, then $X$ is one of the following:

- the image of the anticanonically embedded threefold $\mathbb{P}(3,1,1,1) \subset \mathbb{P}^{38}$ under birational projection from the tangent space at a smooth point on $\mathbb{P}(3,1,1,1)$;
- the image of the anticanonically embedded threefold $\mathbb{P}(6,4,1,1) \subset \mathbb{P}^{38}$ under birational projection from the tangent space at a smooth point on $\mathbb{P}(6,4,1,1)$;
- the image of the anticanonically embedded threefold $X_{66} \subset \mathbb{P}^{35}$ under birational projection from a singular cDV point on $X_{66}$.

Proof. By Remark 3.7 threefold $Y'$ is a terminal $\mathbb{Q}$-factorial modification of some Fano threefold $X'$. Denote by $f' : Y' \to X'$ the corresponding crepant morphism and let $E_{f'}$ be the $f'$-exceptional locus. Further, the anticanonical linear system $| -K_{X'}|$ gives an embedding of $X'$ in $\mathbb{P}^{g'+1}$ such that the image $X_{2g'-2} := \Phi_{| -K_{X'}|}(X')$ is an intersection of quadrics (here $g'$ is the genus of $X'$). In what follows, we assume that $X' = X_{2g'-2} \subset \mathbb{P}^{g'+1}$ is anticanonically embedded (see Section 4).

Lemma 8.5. If $\text{ext}(E)$ is a point, then $X$ is one of the following:

- the image of the anticanonically embedded threefold $\mathbb{P}(3,1,1,1) \subset \mathbb{P}^{38}$ under birational projection from the tangent space at a smooth point on $\mathbb{P}(3,1,1,1)$;
- the image of the anticanonically embedded threefold $\mathbb{P}(6,4,1,1) \subset \mathbb{P}^{38}$ under birational projection from the tangent space at a smooth point on $\mathbb{P}(6,4,1,1)$.

Proof. Since $(-K_{Y'})^3 > (-K_Y)^3$ by Lemma 1.2 and terminal $\mathbb{Q}$-factorial modifications of $\mathbb{P}(3,1,1,1)$, $\mathbb{P}(6,4,1,1)$, $X_{70}$ and $X_{66}$ are non-singular (see Remarks 3.9, 3.13, 3.15 and Lemmas 4.12, 6.6), it follows from Theorems 1.1, 1.2 and 3.19 that $\text{ext}$ is the blow up of the smooth point $\text{ext}(E)$ on $Y'$. This implies that

$$K_Y = \text{ext}^*(K_{Y'}) + 2E$$
onumber

on $Y$, $(E^3)_Y = 1$ and hence $(-K_{Y'})^3 = 72$. Then by Theorem 1.1 and Remark 3.7 threefold $Y'$ is a terminal $\mathbb{Q}$-factorial modification either of $\mathbb{P}(3,1,1,1)$ or of $\mathbb{P}(6,4,1,1)$. Then, as in the proof of Lemma 1.2 we obtain that $\text{ext}(E) \cap E_{f'} = \emptyset$ and the map $p : X' \to X$ is given by the linear system $| -K_{X'} - 2f'(\text{ext}(E))|$ (see 1.1). This gives the projection from the tangent space at the smooth point $f'(\text{ext}(E))$ on $X'$. Since $X'$ is an intersection of quadrics and $-K_X)^3 = 64$, $p$ is birational.

Conversely, it is easy to see (see for example [29, Section 7]) that the projection from the tangent space at a smooth point on $X' = \mathbb{P}(3,1,1,1)$ or $\mathbb{P}(6,4,1,1)$ leads to a Fano threefold $X$ with $(-K_X)^3 = 64$. Lemma 8.5 is completely proved.

Now, let $C := \text{ext}(E)$ be a curve. Since $(-K_{Y'})^3 > (-K_Y)^3$ by Lemma 1.2 as in the proof of Lemma 1.2 we obtain that $X$ is the image of $X' = \mathbb{P}(3,1,1,1)$ or $\mathbb{P}(6,4,1,1)$, or $X_{70}$, or $X_{66}$ under birational projection from the linear space $V$ such that dim $V \leq 3$ and $V$ cuts out the scheme $f'(C)$ on $X'$.

Lemma 8.6. In the above assumptions, we have $X' \neq \mathbb{P}(3,1,1,1)$, $\mathbb{P}(6,4,1,1)$ and $X_{70}$.

Proof. Suppose that $X' = \mathbb{P}(3,1,1,1)$. Then dim $V = 3$, which implies that

$$(-K_{X'} \cdot f'(C))_{X'} \leq 4,$$

since $X'$ is an intersection of quadrics. As in the proof of Lemma 4.6 we obtain that $f'(C)$ is a curve which passes through the singular point on $X'$. On the other hand, as in the proof of Lemma 4.6 we have $E_{f'} \cap f'(C) = \emptyset$, a contradiction.

Now, suppose that $X' = \mathbb{P}(6,4,1,1)$. Again, since dim $V = 3$ and

$$(-K_{X'} \cdot f'(C))_{X'} \leq 4,$$

...
Proof. O
Thus, we are led to the case when \( X' = X_{70} \) and \( \dim V = 2 \) (see Proposition 4.5). It follows from Lemma 3.26 and Proposition 4.8 that the set \( f'(C) = V \cap X' \) consists only of smooth points on \( X' \). Moreover, since \( X' \) is an intersection of quadrics, the set \( V \cap X' \) consists either of \( \leq 4 \) points or of a conic. But in the last case from Remark 4.15 and Proposition 4.8 we get that \( V \cap X' \) contains the unique non-cDV point on \( X' \), which contradicts Lemma 3.26. Thus, the set \( V \cap X' \) consists of \( \leq 4 \) smooth points on \( X' \). Then, as in the proof of Lemma 6.15, we get \( 64 = (-K_X)^3 = (-K_{X'})^3 = 70 \), a contradiction. □

It follows from Lemma 8.6 that one must have \( X' = X_{66} \) and \( V \subset X' \) is a point. Moreover, \( V \) is a singular point on \( X' \), since otherwise \( (-K_X)^3 = 65 \), which is impossible.

**Lemma 8.7.** In the above assumptions, threefold \( X' \) contains a singular cA1 point.

**Proof.** Threefold \( X' \) as a toric variety (see Remark 6.7) is given by the fan \( \Sigma \subset N \otimes \mathbb{Z} \cong \mathbb{R}^3 \), generated by the vectors
\[
e_1 := (-1,0,0), \quad e_2 := (1,-1,0), \quad e_3 := (-1,-1,2), \quad e_4 := (-1,-1,-3), \quad e_5 := (-1,2,1)\]
in \( N \), where \( N := \mathbb{Z}^3 \) is the standard lattice in \( \mathbb{R}^3 \) (see [25]). Then the affine \( T \)-invariant cover of \( X' \) is determined by decomposition of \( \Sigma \) into the following cones (see for example [6]):
\[
\Sigma_1 := < e_1,e_2,e_3 >, \quad \Sigma_2 := < e_1,e_3,e_4,e_5 >, \quad \Sigma_3 := < e_2,e_3,e_4,e_5 >, \quad \Sigma_4 := < e_1,e_2,e_3 >.
\]
Calculating the volume of corresponding simplex, we obtain that \( \Sigma_1 \) determines a singular chart \( U_{\Sigma_1} \) isomorphic to \( \mathbb{C}^3 / \mu_2 \) (see for example [4, 2.6.2] and [6, 2.2]). Note that singularities of the threefold \( X' \) in \( U_{\Sigma_1} \) are non-isolated, since otherwise \( X' \) has unique non-gorenstein singularity in \( U_{\Sigma_1} \) of type \( \frac{1}{2}(1,1,1) \), which is impossible. Thus, we have
\[
U_{\Sigma_1} \cong \mathbb{C} \times (\mathbb{C}^2 / \mu_2),
\]
and hence threefold \( X' \) contains a singular cA1 point. □

Lemmas 3.26 and 8.4 imply that \( X \) is the image of \( X' \) under birational projection from the singular cDV point \( V \) on \( X' \). This and Lemma 8.5 prove Proposition 8.4. □

It follows from Proposition 8.4 and Corollaries 3.20 and 3.23 that to the threefold \( X \) one can apply the construction from Section 5. Thus, we obtain the pair \((W,H_W)\) with the \((K_W + H_W)\)-negative extremal contraction \( \text{ext}_W : W \rightarrow W' \) to a lower-dimensional variety \( W' \) (we use the notation from Section 5). Furthermore, for \( W \) all the conditions from Remarks 5.8 and 5.9 are satisfied, and, as in Remark 5.11 we get the estimates
\[
(8.8) \quad \dim | - K_W | \geq \dim L_W = \dim | - K_X | = 34 \quad \text{and} \quad \dim | H_W | \geq 31.
\]
If \( \dim W' = 0 \), then \( W \) is a Q-Fano threefold, and it follows from [32, Proposition 7.2] that \( \dim | - K_W | \leq 34 \). This and (8.8) imply that \( | - K_W | = L_W \). Thus, we have \( X = W = Y \), which contradicts our assumption.

**Proposition 8.9.** If \( \dim W' = 1 \), then \( X \) is one of the following:

- the image of the anticanonically embedded threefold \( \mathbb{P}(6,4,1,1) \subset \mathbb{P}^{38} \) under birational projection from the tangent space at a smooth point on \( \mathbb{P}(6,4,1,1) \);
- the image of the anticanonically embedded threefold \( X_{66} \subset \mathbb{P}^{35} \) under birational projection from a singular cDV point on \( X_{66} \).

**Proof.** Let us use the notation from Section 5. We consider the case when \( W_\eta \cong \mathbb{P}^2 \) and \( \mathcal{O}_{W_\eta}(H_W |_{W_\eta}) \cong \mathcal{O}_{\mathbb{P}^2}(1) \) first. It follows from (8.8) and the proof of Lemma 6.11 and Propositions 6.2 and 6.11 respectively, that \( X \) is one of the following:
• the image of the anticanonically embedded threefold $\mathbb{P}(6,4,1,1) \subset \mathbb{P}^{38}$ under birational projection;
• the image of the anticanonically embedded threefold $X_{66} \subset \mathbb{P}^{35}$ under birational projection.

Lemma 8.10. If $X$ is the image of the anticanonically embedded threefold $\mathbb{P}(6,4,1,1) \subset \mathbb{P}^{38}$ under birational projection, then $X$ is the threefold constructed in Lemma 8.3.

Proof. Set $\mathbb{P} := \mathbb{P}(6,4,1,1) \subset \mathbb{P}^{38}$ and let $\pi: \mathbb{P} \rightarrow X$ be the given projection. Let us denote also by $V$ the center of the projection $\pi$.

Since $\dim V = 3$ and $\mathbb{P}$ is an intersection of quadrics (see Theorem 3.3), it follows from Lemma 8.10 that $V$ intersects $\mathbb{P}$ either at $\leq 8$ points or by a curve of degree $\leq 4$ (note that $V \cap \mathbb{P} \neq \emptyset$, since otherwise $\pi$ is an isomorphism, which is impossible because $(-K_X)^3 = 64$). Moreover, if $V \cap \mathbb{P}$ is a curve, then, as in the proof of Lemma 8.10 we get a contradiction.

Suppose that $V \cap \mathbb{P}$ is a finite set. Then, as in the proof of Lemma 6.16, $V \cap \mathbb{P}$ consists only of smooth points on $\mathbb{P}$. Furthermore, if the scheme $V \cap \mathbb{P}$ is reduced, then, as in the proof of Lemma 6.15 we get that $(-K_X)^3 = 72$, a contradiction. Now, if the scheme $V \cap \mathbb{P}$ is non-reduced, then, since $\mathbb{P}$ is an intersection of quadrics, $V$ coincides with the tangent space at a smooth point on $\mathbb{P}$. Lemma 8.10 is completely proved.

Now, as in the proof of Proposition 8.4 we obtain the following

Lemma 8.11. If $X$ is the image of the anticanonically embedded threefold $X_{66} \subset \mathbb{P}^{35}$ under birational projection, then $X$ is the threefold constructed in Proposition 8.4.

Lemmas 8.10 and 8.11 prove Proposition 8.9 in the case when $W_\eta \cong \mathbb{P}^2$ and $O_{W_\eta}(H_W|_{W_\eta}) \cong O_{\mathbb{P}^2}(1)$. Furthermore, the case when $W_\eta \cong \mathbb{P}^2$ and $O_{W_\eta}(H_W|_{W_\eta}) \cong O_{\mathbb{P}^2}(2)$ is treated in exactly the same way as in Section 6.

Finally, in the case when $W_\eta \cong \mathbb{P}^1 \times \mathbb{P}^1$, as in Section 6, it is enough to prove Lemma 6.19 under assumption that $(-K_X)^3 = 64$. It follows from the proof of Lemma 6.19 that one should only exclude the case when $W$ is a weak Fano threefold with $(-K_W)^3 = 64$ and $-K_W = 2G$ on $W$. But in this case, 6.20 and 8.8 imply that $L_W = |-K_W|$. On the other hand, since $\rho(W) = 2$, the negative section of the $\mathbb{P}^3$-bundle $F$ containing $W$ is a curve. Then the equality $-K_W = 2G$ implies that $\Phi_{L_W}: W \rightarrow X$ is a small contraction such that $K_W = \Phi_{L_W}^*(K_X)$ on $W$, and hence $X$ has only terminal Gorenstein singularities. Then from Theorem 3.16 we get that $X = Y = \mathbb{P}^3$, which contradicts our assumption. Proposition 8.9 is completely proved.

Proposition 8.12. If $\dim W' = 2$, then $X$ is one of the following:

• the image of the anticanonically embedded threefold $\mathbb{P}(3,1,1,1) \subset \mathbb{P}^{38}$ under birational projection from the tangent space at a smooth point on $\mathbb{P}(3,1,1,1)$;
• the cone from Example 7.4.

Proof. Let us use the notation from the Section 7.

Lemma 8.13. One of the following holds:

• $\dim |H_W| \in \{31,32,33,34,35\}$;
• $\dim |H_W| \geq 36$ and $X$ is the image of the anticanonically embedded threefold $\mathbb{P}(3,1,1,1) \subset \mathbb{P}^{38}$ under birational projection from the tangent space at a smooth point on $\mathbb{P}(3,1,1,1)$.

Proof. According to (8.8), we have $\dim |H_W| \geq 31$. Further, if $\dim |H_W| \geq 36$, then, as in the proof of Proposition 7.3, the initial Fano threefold $X$ is the image of the anticanonically embedded threefold $\mathbb{P} := \Phi_{-K_W}(W) = \mathbb{P}(3,1,1,1) \subset \mathbb{P}^{38}$ under birational projection from a linear subspace $V \subset \mathbb{P}^{38}$ with $\dim V = 3$. 32
If \( \dim V \cap \mathbb{P} > 1 \), then, since \( \mathbb{P} \) is an intersection of quadrics (see Theorem 3.31), \( \mathbb{P} \) contains a line. But this is impossible because the divisor \( K_\mathbb{P} \) is divisible in \( \text{Pic}(\mathbb{P}) \) (see Example 3.12).

If \( \dim V \cap \mathbb{P} \leq 1 \), then, as in the proof of Lemma 8.10 we obtain that \( V \) coincides with the tangent space at a smooth point on \( \mathbb{P} \). \( \square \)

Recall that by Remark 7.7 one may assume that \( W' = \mathbb{P}^2 \) or \( \mathbb{F}_n \) with \( 0 < n \neq 1 \leq 4 \).

**Lemma 8.14.** If \( \dim |H_W| \in \{31, 32, 33, 34, 35\} \), then \( W' \neq \mathbb{P}^2 \).

*Proof.* Suppose that \( W' = \mathbb{P}^2 \). Let us use the notation from the proof of Proposition 7.17.

Suppose that the vector bundle \( \mathcal{E} \) is decomposable. We have

\[
\chi(W', \mathcal{E}) \in \{32, 33, 34, 35, 36\},
\]

and if \( \chi(W', \mathcal{E}) > 32 \), then, as in the proof of Lemma 7.18 we get a contradiction. Now, let \( \chi(W', \mathcal{E}) = 32 \). Then we have

\[
2a^2 + 2ab + b^2 + 6a + 3b = 60,
\]

where \( a, b \geq 0 \) and \( 0 \leq 2a + b \leq 9 \). The direct substitution implies that under these conditions the above equality is never true. Thus, \( \mathcal{E} \) is indecomposable. Then, as in the proof of Proposition 7.17 in this case we obtain

\[
c_1(\mathcal{E}(-m)) = -2 \text{ or } -3 \quad \text{and} \quad c_2(\mathcal{E}(-m)) < 0
\]

for some \( m \in \mathbb{N} \). Now, as in the proof of Proposition 7.17 we get a contradiction. \( \square \)

**Lemma 8.15.** If \( \dim |H_W| \in \{31, 32, 33, 34, 35\} \) and \( W' = \mathbb{F}_6 \), then \( X \) is isomorphic to the cone from Example 1.4.

*Proof.* Let us use the notation from the proof of Lemma 7.26. We have

\[
\chi(W', \mathcal{E}) \in \{32, 33, 34, 35, 36\},
\]

and if \( \chi(W', \mathcal{E}) > 33 \), then from the proof of Lemma 7.26 and the arguments at the end of the Section 7 we get a contradiction.

Further, if \( a \) or \( b < 6 \), then for \( \chi(W', \mathcal{E}) \in \{32, 33\} \) we have

\[
c'_2 = \frac{1}{2}ab + a + b - \chi(W', \mathcal{E}) + 2 + \frac{1}{2}a'b' < 0.
\]

Moreover, in this case \( \chi(W', \mathcal{E}') > 0 \) (see the proof of Lemma 7.26). Then from the arguments at the end of the Section 7 we get a contradiction.

Now, let \( a = b = 6 \). Then, as in the proof of Lemma 7.26 we obtain that \( W \) is a weak Fano threefold with \( (-K_W)^3 \in \{64, 72\} \). But the case \( (-K_W)^3 = 72 \) is impossible because otherwise by Theorem 1.1 and Remarks 3.7 3.13 3.15 one must have \( \rho(W) = 2 \) or \( 5 \). On the other hand, we have \( \rho(W) = 3 \), a contradiction.

Finally, in the case when \( (-K_W)^3 = 64 \) from the Riemann–Roch Formula (see [37]), Kawamata–Viehweg Vanishing Theorem (see for example [21]) and (8.8) we obtain that \( | -K_W | = \mathcal{L}_W \), which implies that \( W \) is a terminal \( \mathbb{Q} \)-factorial modification of \( X \) (see Remarks 3.7 and 3.8). Then, as in the proof of Lemma 8.2 we obtain that \( X \) is isomorphic to the cone from Example 1.4.

Lemma 8.15 is completely proved. \( \square \)

**Lemma 8.16.** If \( \dim |H_W| \in \{31, 32, 33, 34, 35\} \), then \( W' \neq \mathbb{F}_2 \).

*Proof.* Let us use the notation from the proof of Lemma 7.27. According to the arguments at the end of the Section 7 it is enough to prove that \( \chi(W', \mathcal{E}') > 0 \) and \( c'_2 < 0 \).

We have

\[
\chi(W', \mathcal{E}) \in \{32, 33, 34, 35, 36\},
\]

and if \( \chi(W', \mathcal{E}) > 32 \), then, as in the proof of Lemma 7.27 we get what we need.
Now, let $\chi(W', \mathcal{E}) = 32$. Then we obtain
\[ c_2 = -\frac{1}{2}a^2 + \frac{1}{2}ab + b - 30 - \frac{1}{2}a'^2 + \frac{1}{2}a'b', \]
where $0 \leq a \leq 6$, $2a \leq b \leq 12$ and $-2 \leq a', b' \leq -1$ are the integers.

If $b \leq 11$, then we have
\[ c_2' \leq -\frac{1}{2}a^2 + \frac{11}{2}a - 19 - \frac{1}{2}a'^2 + \frac{1}{2}a'b' \leq -4. \]
Now, let $b = 12$. Then for $a \leq 4$ we have
\[ c_2' = -\frac{1}{2}a^2 + 6a - 18 - \frac{1}{2}a'^2 - a' < 0. \]
Further, for $a \in \{5, 6\}$ from (8.14) we get that $c_2(\mathcal{E}) \in \{17, 18\}$. Then from (7.10) we obtain
\[ (8.17) \quad (-K_W)^3 = 48 + 2((ah + 12b)^2)_{W'}, -8c_2(\mathcal{E}) = 48 + 4(-a'^2 + 12a) - 8c_2(\mathcal{E}) \leq 52. \]
On the other hand, it follows from the proof of Lemma 7.12 that
\[ (\neg W' \cdot D_W \cdot ext_{W}(B))_W = 0 \]
for the curve $B \sim h + 2l$ on the surface $W' = \mathbb{F}_2$. Then by Lemma 7.13 the divisor $-K_W$ is nef on $W$, and hence $W$ is a weak Fano threefold because $-K_W$ is also big by Remark 5.9. This and (8.17), as in the proof of Lemma 6.19 imply that $\dim | -K_W | \leq 28$ (see (6.20)), which contradicts the estimate in (8.8).

Thus, $c_2' < 0$. In particular, $\chi(W', \mathcal{E}') \leq 0$ only for $a' = -2$, $b' = -1$ and $c_2' = -1$ (see the proof of Lemma 7.27). But in this case we have $b \leq 11$, which implies that
\[ -1 = c_2' \leq -\frac{1}{2}a^2 + \frac{11}{2}a - 20 < -1 \]
a contradiction. Thus, $\chi(W', \mathcal{E}') > 0$. Lemma 8.16 is completely proved.

\[ \square \]

**Lemma 8.18.** If $\dim |H_W| \in \{31, 32, 33, 34, 35\}$, then $W' \neq \mathbb{F}_3$.

**Proof.** Let us use the notation from the proof of Lemma 7.28. According to the arguments at the end of the Section 7 it is enough to prove that $\chi(W', \mathcal{E}') > 0$ and $c_2' < 0$.

We have
\[ \chi(W', \mathcal{E}) \in \{32, 33, 34, 35, 36\}, \]
and if $\chi(W', \mathcal{E}) > 32$, then, as in the proof of Lemma 7.28 we get what we need.

Now, let $\chi(W', \mathcal{E}) = 32$. Then we obtain
\[ c_2 = -\frac{3}{4}a^2 + \frac{1}{2}ab - \frac{1}{2}a + b - 30 - \frac{3}{4}a'^2 + \frac{1}{2}a'b', \]
where $0 \leq a \leq 5$, $3a \leq b \leq 15$ and $-2 \leq a', b' \leq -1$ are the integers.

If $b = 15$, then, as in the proof of Lemma 7.28 we obtain that the divisor $-K_{W'}$ is nef on $W' = \mathbb{F}_3$, which is impossible. Further, for $b \leq 14$ we have
\[ c_2' \leq -\frac{3}{4}a^2 + \frac{13}{2}a - 16 - \frac{3}{4}a'^2 + \frac{1}{2}a'b' \leq -3. \]
Thus, $c_2' < 0$. In particular, $\chi(W', \mathcal{E}') \leq 0$ only for $a' = -2$ (see the proof of Lemma 7.28), which implies that $a$ is even.

Let $b' = -2$. Then for $a' = -2$ we get $\chi(W', \mathcal{E}') = -1 - c_2'$. On the other hand, we have $b \leq 14$, which implies that
\[ c_2' \leq -\frac{3}{4}a^2 + \frac{13}{2}a - 17 < -1. \]
for $0 \leq a \leq 4$, and hence $\chi(W', \mathcal{E}') > 0$ in this case.

Now, let $b' = -1$. Then for $a' = -2$ we get $\chi(W', \mathcal{E}') = -2 - c'_2$. On the other hand, by the above arguments we have $b \leq 13$, which implies that

$$c'_2 \leq -\frac{3}{4}a^2 + 6a - 19 < -2$$

for $0 \leq a \leq 4$, and hence $\chi(W', \mathcal{E}') > 0$ in this case.

Thus, $\chi(W', \mathcal{E}') > 0$. Lemma 8.18 is completely proved. □

**Lemma 8.19.** If $\dim |H_W| \in \{31, 32, 33, 34, 35\}$, then $W' \neq \mathbb{F}_4$.

**Proof.** Let us use the notation from the proof of Lemma 7.29. According to the arguments at the end of the Section 7, it is enough to prove that $\chi(W', \mathcal{E}') > 0$ and $c'_2 < 0$.

We have

$$\chi(W', \mathcal{E}) \in \{32, 33, 34, 35\},$$

and if $\chi(W', \mathcal{E}) > 32$, then, as in the proof of Lemma 7.29 we get what we need.

Now, let $\chi(W', \mathcal{E}) = 32$. Then we obtain

$$c'_2 = -a^2 + \frac{1}{2}ab - a + b - 30 - a'^2 + \frac{1}{2}a'b',$n

where $0 \leq a \leq 4$, $4a \leq b \leq 18$ and $-2 \leq a', b' \leq -1$ are the integers.

If $b = 18$, then, as in the proof of Lemma 7.29 we obtain that the divisor $-K_{W'}$ is nef on $W' = \mathbb{F}_4$, which is impossible. Further, for $b \leq 16$ we have

$$c'_2 \leq -a^2 + 7a - 14 - a'^2 - a' \leq -2.$$

Now, let $b = 17$. Then we have

$$c'_2 = -a^2 + \frac{15}{2}a - 13 - a'^2 - \frac{1}{2}a'.$$

The direct substitution implies that $c'_2 < 0$ for $a \neq 3$. If $a = 3$, then $c'_2 = 0$.

Thus, $c'_2 < 0$. In the following we will show that $c'_2 < 0$. But first let us prove the inequality $\chi(W', \mathcal{E}') > 0$. Since $c'_2 \leq 0$, $\chi(W', \mathcal{E}') \leq 0$ only for $a' = -2$ (see the proof of Lemma 7.29), which implies that $a$ is even.

Let $b' = -1$. Then for $a' = -2$ we get $\chi(W', \mathcal{E}') = -3 - c'_2$. On the other hand, by the above arguments we have $b \leq 17$, which implies that

$$c'_2 = -a^2 + \frac{1}{2}ab - a + b - 33 \leq -a^2 + 15 - 2a - 16 < -3$$

for even $0 \leq a \leq 3$. Further, for $b \leq 16$ and $a = 4$ we have

$$c'_2 = -53 + 3b < -3.$$

Finally, the case $a = 4$ and $b = 17$ is impossible (see the proof of Lemma 7.29). Thus, for $a' = -2$ and $b' = -1$ we have $\chi(W', \mathcal{E}') > 0$.

Now, let $b' = -2$. Then for $a' = -2$ we get $\chi(W', \mathcal{E}') = -2 - c'_2$. On the other hand, by the above arguments we have $b \leq 16$, which implies that

$$c'_2 = -a^2 + \frac{1}{2}ab - a + b - 32 \leq -a^2 + 7a - 16 < -3$$

for $0 \leq a \leq 4$, and hence $\chi(W', \mathcal{E}') > 0$ in this case.

Thus, $\chi(W', \mathcal{E}') > 0$. In particular, as in the proof of Lemma 7.30 we have $H^0(W', \mathcal{E}') \neq 0$. Now we are ready to exclude the above case when $a = 3$, $b = 17$ and $c'_2 = 0$. 

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Let $s \in H^0(W', \mathcal{E}')$ be a non-zero section and $Z \subset W'$ the zero locus of $s$. Since $c'_2 = 0$, either\dim Z = 1$ or $Z = \emptyset$ (see [8]). But in the first case from the arguments at the end of the Section 7, we get a contradiction. Thus, $s$ does not have zeroes on $W'$, and we get the following exact sequence

$$0 \to \mathcal{O}_{W'} \xrightarrow{s} \mathcal{E} \xrightarrow{r} \mathcal{O}_{W'}(3h + 17l) \to 0,$$

since $c_1(\mathcal{E}) = 3h + 17l$ and $c_2(\mathcal{E}) = 0$, where $\mathcal{O}_{W'} \xrightarrow{s} \mathcal{E}$ is the multiplication by $s$ (see [8]). This implies, since the divisor $h + 11l$ is ample on $W' = \mathbb{P}_4$ by the Kleiman’s criterion for ampleness (see for example [21, Theorem 0-1-2]) and hence

$$H^1(W', \mathcal{O}_{W'}(−3h − 17l)) \simeq H^1(W', \mathcal{O}_{W'}(h + 11l)) = 0,$$

that $r$ has a section. Thus, we have $\mathcal{E} \cong \mathcal{O}_{W'} \oplus \mathcal{O}_{W'}(3h + 17l)$ and $W \cong \mathbb{P}(\mathcal{O}_{W'} \oplus \mathcal{O}_{W'}(3h + 17l))$ is the blow up of the vertex of the cone $S$ over $\mathbb{P}_4$ embedded by the linear system $|3h + 17l|$ (note that the divisor $3h + 17l$ is ample on $W' = \mathbb{P}_4$ by the Kleiman’s criterion for ampleness, and since $W'$ is a non-singular toric surface, this divisor is very ample (see for example [6])).

Let $N$ be the negative section on $W$ which is contracted to the vertex of the cone $S$. Since

$$K_{W'} + c_1(\mathcal{E}) = h + 11l$$

is an ample divisor on $W' = \mathbb{P}_4$ by the Kleiman’s criterion for ampleness, it follows from the formula similar to (7.8) that

$$(- K_W \cdot Z)_W < 0$$

on $W$ for every curve $Z \subset N$. In particular, the surface $N$ is a base component of the linear system $|− K_W|$, which is impossible by Remark 5.8. Lemma 8.19 is completely proved. □

Lemmas 8.13, 8.14, 8.15, 8.16, 8.17 and 8.19 prove Proposition 8.12.

Theorem 3.16 and Propositions 8.1, 8.4, 8.9, 8.10 and 8.12 prove the first part of Theorem 1.7.

**Corollary 8.20.** Let $X$ be a Fano threefold from Theorem 1.7. If $X \neq \mathbb{P}^3$, then the singularities of $X$ are non-cDV.

**Proof.** Theorem 3.16 and [24, Corollary 5.38] imply that the cones from Examples 1.4 and 1.5 have non-cDV singularities. In the remaining cases one has the commutative diagram

$$\begin{array}{ccc}
X'' & \xrightarrow{\pi} & X' \\
\sigma \downarrow & & \uparrow \tau \\
X' \xrightarrow{\pi} & & X,
\end{array}$$

where $X' = \mathbb{P}(3, 1, 1, 1)$ or $\mathbb{P}(6, 4, 1, 1)$, or $X_{66}$, $\pi$ is a birational projection, $\sigma$ is the blow up of a cDV point on $X'$ and $\tau$ is a crepant morphism. In particular, threefolds $X'$ and $X''$ are isomorphic near their non-cDV points (see [19, Corollary 1.7]), which implies that singularities of $X$ are non-cDV (see for example [24, Theorem 5.34]). □

Theorem 1.7 is completely proved.

**References**

[1] Arnold V., Gusein-Zade S., Varchenko A. Singularities of differentiable maps // Progress in Mathematics. 1988.
[2] Borisov A. Boundedness of Fano threefolds with log-terminal singularities of given index // J. Math. Sci. Univ. Tokyo. 2001. V. 8. P. 329–342.
[3] Cutkosky S. Elementary contractions of Gorenstein threefolds // Math. Ann. 1988. V. 280. P. 521–525.
[4] Danilov V. I. The geometry of toric varieties // Uspekhi Mat. Nauk. 1978. V. 33(2). P. 85-134.
[5] Dolgachev I. V. Weighted projective varieties // Lecture Notes in Math. 1982. V. 956. P. 34–71.
[6] Fulton W. Introduction to toric varieties // Princeton University Press. 1993.
[7] Griffiths P., Harris J. Principles of Algebraic Geometry // New York: John Wiley. 1978.
