Hypergraph dynamics: assortativity and the expansion eigenvalue

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The largest eigenvalue of the matrix describing a network’s contact structure is often important in predicting the behavior of dynamical processes. We extend this notion to hypergraphs and motivate the importance of an analogous eigenvalue, the expansion eigenvalue, for hypergraph dynamical processes. Using a mean-field approach, we derive an approximation to the expansion eigenvalue and its associated eigenvector in terms of the degree sequence for uncorrelated hypergraphs. We introduce a generative model for hypergraphs that includes degree assortativity, and use a perturbation approach to derive an approximation to the expansion eigenvalue and its corresponding eigenvector for assortative hypergraphs. We validate our results with both synthetic and empirical datasets. We define the dynamical assortativity, a dynamically sensible definition of assortativity for uniform hypergraphs, and describe how reducing the dynamical assortativity of hypergraphs through preferential rewiring can extinguish epidemics.

Complex social systems often exhibit assortative mixing \[1\] \[2\], where individuals with similar characteristics connect with each other more often than it would be expected if they were connected at random. Assortativity has been extensively studied in network science \[3\] and found to have significant effects on synchronization \[4\], epidemic dynamics \[5\] \[6\], stability \[7\], game theory \[8\], and general diffusion processes \[9\].

Recently, there has been much work on using hypergraphs to describe systems with interactions involving multiple agents \[10\] \[11\]. Hypergraphs are useful to describe multi-way interactions in biology \[12\], social contagion \[13\] \[15\], synchronization \[16\], opinion models, infectious disease spread \[17\], and real data \[15\]. It has been shown that contagion on a hypergraph cannot be projected onto a pairwise network if the higher-order contagion function is non-linear \[19\]. Recently the pairwise notion of assortativity has been extended to hypergraphs for categorical node labels \[20\] \[22\] and continuous attributes \[18\]. Assortativity on hypergraphs can provide different insights on the structure of the interactions than assortativity on the pairwise network projection \[18\] and, as we will show, affect the outcome of hypergraph dynamical processes.

A fundamental problem when studying dynamics on networks is to determine how structural characteristics of the network affect the dynamical behavior. Many dynamical properties such as the onset of epidemic spreading \[23\], synchronization \[24\], and percolation \[25\] are determined by the largest eigenvalue of the network’s adjacency matrix. In turn, this eigenvalue is affected by the network’s degree distribution and assortative mixing properties \[26\] as well as other structural characteristics. In this paper we show how the expansion eigenvalue, a suitably generalized eigenvalue for hypergraphs, is similarly modified by the assortative properties of the hypergraph. This eigenvalue has been shown to determine the extinction threshold for the SIS model on hypergraphs \[27\], and we believe it will also prove useful in relating hypergraph assortative mixing patterns to other dynamical processes.

Our approach is as follows: first, we define and motivate the importance of the expansion eigenvalue on dynamical processes; second, we derive a mean-field approximation of this eigenvalue and its corresponding eigenvector for hypergraphs without assortativity; third, we present a generative model for assortative hypergraphs; fourth, we employ a perturbation approach to derive the effect of degree-degree mixing on these quantities and define the dynamical assortativity; and lastly, we show how our results can be used to modify hypergraph dynamics through preferential rewiring of hyperedges.

We start by defining terminology. A hypergraph is a mathematical object that describes group interactions among a set of nodes. We represent it as \(H = (V,E)\), where \(V\) is the set of nodes and \(E\) is the set of hyperedges, which are subsets of \(V\) and represent unordered interactions of arbitrary size. We call a hyperedge with cardinality \(m\) an \(m\)-hyperedge and a hypergraph with only \(m\)-hyperedges an \(m\)-uniform hypergraph. It is useful to consider weighted hypergraphs, where each edge \(e\) has an associated positive weight \(\beta_e\). We define the hyperdegree sequence as in Ref. \[14\], where the \(n\)th order hyperdegree of node \(i\), \(k^{(m)}_i\), is the number of \(m\)-hyperedges to which it belongs.

We now define the expansion eigenvalue and discuss its relevance to dynamical processes on hypergraphs. For a weighted hypergraph, the expansion eigenvalue \(\lambda\) and associated eigenvector \(u\) are defined by the eigenvalue equation

\[
\lambda u = \sum_{e=\{i_1,i_2,...,i_{m-1}\}\in E} \beta_e (u_{i_1} + \cdots + u_{i_{m-1}}),
\]

(1)

where \(\lambda\) and \(u\) are the Perron-Frobenius eigenvalue and eigenvector of the associated nonnegative matrix.

In the unweighted case (i.e., \(\beta_e = 1\) for every hyperedge \(e\)), the eigenvector \(u\) corresponds to the Clique motif Eigenvector Centrality \[28\]. Just as the largest eigenvalue
of a network’s adjacency matrix is determinant for network dynamics, the expansion eigenvalue plays an important role in dynamical processes on hypergraphs. For example, consider an SIS process on a hypergraph, where a healthy node can get infected via a hyperedge $e$ to which it belongs at rate $\beta_e$ if at least one other node in $e$ is infected (the case referred to as individual contagion in Ref. [14]) and heals spontaneously at rate $\gamma$. Then one can show (see Ref. [27] and Supplemental Material) that a sufficient condition for epidemic extinction is given by $\gamma > \lambda$. In addition, we can determine the relative importance of a node $i$ with respect to this contagion model (in terms of its probability of infection at the onset of the epidemic) from the $i$th entry of the associated eigenvector. More generally, the importance of the expansion eigenvalue in spreading processes can be understood from the fact that in the unweighted case the number of nodes reachable via hyperedges from a given starting node in $\ell$ steps grows asymptotically as $\lambda^\ell$ [28].

Having motivated the importance of the expansion eigenvalue $\lambda$, we now focus on how it is affected by assortative mixing by degrees in the hypergraph. In this paper we will consider random hypergraphs that are constructed from a prescribed hyperdegree sequence $\{k_1, \ldots, k_N\}$, where $N$ is the number of nodes, $k_i = [k_i^{(2)}, \ldots, k_i^{(M)}]$ is the target hyperdegree of node $i$, and $M$ is the maximum hyperedge size. The hypergraph is then constructed by creating a hyperedge $\{i_1, \ldots, i_m\}$ with probability $f_m(k_{i_1}, \ldots, k_{i_m})$, where the functions $f_m$ specify the assortative mixing properties of the hypergraph. The function $f_m(k_{i_1}, \ldots, k_{i_m}) = f_m^{(0)}(k_{i_1}^{(m)}, \ldots, k_{i_m}^{(m)}) = (m-1)! k_{i_1}^{(m)} \ldots k_{i_m}^{(m)} / (N(k))^{m-1}$, where we define $(x) = \sum_{i=1}^N x_i / N$, corresponds to the case where nodes are connected with hyperedges completely at random if the hyperdegree of node $i$ is $k_i$ (as in the hypergraph configuration model [14,18,29,30]). We start by deriving a mean-field approximation for the expansion eigenvalue $\lambda$ in this case, which we call the uncorrelated case, before considering hypergraphs with degree assortativity. For simplicity, from now on we will consider an unweighted $m$-uniform hypergraph, and will denote $k_i^{(m)}$ by $k_i$ and refer to it as the degree of node $i$. Now we assume that all nodes with the same degree are statistically equivalent and that the eigenvector entry of node $i$ depends only on its degree, i.e., $u_i \rightarrow u_k$. Defining $P(k)$ to be the number of nodes with degree $k$, the equation defining the expansion eigenvalue can be written as

$$
\lambda u_k = \frac{1}{(m-1)!} \sum_{k_1,\ldots,k_{m-1}} P(k_1) \ldots P(k_{m-1}) \times
f_m^{(0)}(k,k_1,\ldots,k_{m-1}) (u_{k_1} + \cdots + u_{k_{m-1}}). 
$$

(2)

By symmetry of the function $f_m^{(0)}$, we get

$$
\lambda u_k = \left[ (m-1) \sum_{k_1} P(k_1) k_1 u_{k_1} / N(k) \right] k,
$$

(3)

and multiplying both sides by $k P(k) / (N(k))$ and summing over $k$, we obtain for the uncorrelated case

$$
\lambda \approx (m-1) \langle k^2 \rangle / \langle k \rangle,
$$

(4)

and $u_k \propto k$ from Eq. (3). For non-uniform hypergraphs, assuming that $\beta_e = \beta_{[e]}$, the expansion eigenvalue is the largest eigenvalue of $K$, the degree-size correlation matrix, with entries $K_{ij} = \beta_i (i-1) (k_i^{(1)} k_j^{(1)}) / \langle k^{(1)} \rangle$ (see the Supplemental Material for the derivation).

In contrast with the uncorrelated case, we now assume that nodes are connected with an arbitrary function $f_m$ determining the connection probability. We define

$$
f_m(k_1, \ldots, k_m) = f_m^{(0)}(k_1, \ldots, k_m) \left[ 1 + \epsilon g_m(k_1, \ldots, k_m) \right],
$$

(5)

where $\epsilon$ is a parameter and $g_m$ an assortativity function for $m$-uniform hypergraphs. The assortativity function $g_m(k_1, \ldots, k_m)$ determines how likely it is that nodes with degrees $k_1, \ldots, k_m$ are joined by a $m$-hyperedge; if $\epsilon g_m > 0$ (or $\epsilon g_m < 0$) it is more (less) likely than it would be expected if they were connected at random. In order to preserve the degree sequence, $g_m$ must satisfy

$$
\sum_{k_1,\ldots,k_m} f_m^{(0)}(k_1,\ldots,k_m) g_m(k_1,\ldots,k_m) = 0.
$$

We present two examples of the assortativity function $g_m$: the aligned degrees and large degrees assortativity functions. The aligned degrees function measures the average pairwise correlation on a hyperedge. It is defined as

$$
g_m(k_1, \ldots, k_m) = \frac{1}{\langle k \rangle^m} \sum_{i<j} (k_i - \langle k \rangle) (k_j - \langle k \rangle) - \left[ \langle k^2 \rangle - \langle k \rangle^2 \right]^2 / \langle k^2 \rangle.
$$

(6)

This function is similar to the expected value of the uniform function described in Ref. [15]. Likewise, the large degrees function is defined as

$$
g_m(k_1, \ldots, k_m) = \prod_{i=1}^m \frac{k_i}{\langle k \rangle} - \left( \frac{\langle k^2 \rangle}{\langle k \rangle^2} \right)^m,
$$

(7)

and measures whether the product of degrees on a hyperedge is larger than expected for a configuration null model. The value of $\epsilon$ can be inferred from a realization of an $m$-uniform hypergraph if one assumes that the generative model is of the form presented in Eq. (5) (see Supplemental Material). Other assortativity functions can be constructed by considering, for example, only the
two largest degrees or the largest and smallest degrees in a hyperedge as in Ref. 15.

We now assume that the parameter \( \epsilon \) is small and develop perturbative approximations to the eigenvalue \( \lambda \) and its eigenvector \( u_k \). To first order these approximations are

\[
\lambda = \lambda^{(0)} + \epsilon \lambda^{(1)},
\]

\[
u_k = u_k^{(0)} + \epsilon u_k^{(1)},
\]

where \( \lambda^{(0)} = (m - 1)(k^2)/\langle k \rangle \) and \( u_k^{(0)} = \alpha k \), where \( \alpha \) is an arbitrary constant.

Replacing \( f_m^{(0)} \) on the right-hand side of Eq. (2) with the \( f_m \) in Eq. (5), using Eq. (8), assuming symmetry of \( f_m \), multiplying by \( k P(k)/\langle N(k) \rangle \), summing over \( k \), and canceling the zero-order terms, we obtain to first order

\[
\lambda^{(1)} = (m - 1) \langle k \rangle \langle kk_1 \rangle_E \frac{2 k k_1 \ldots k_{m-1}}{\langle k \rangle^m} g_m(k, k_1, \ldots, k_{m-1}).
\]

Removing the reference to \( g_m \) using the relation in Eq. (5) we find

\[
\epsilon \lambda^{(1)} = (m - 1) \langle k \rangle \langle kk_1 \rangle_E \frac{2 k k_1 \ldots k_{m-1}}{\langle k \rangle^m} - \lambda^{(0)},
\]

where \( \langle kk_1 \rangle_E \) is the mean pairwise product of degrees over all possible 2-node combinations in each hyperedge in the hypergraph, \( \langle kk_1 \rangle_E = \sum_{e \in E} \sum_{(i,j) \subseteq e} k_i k_j / (|E|_2^{(m)}) \).

Therefore, the expansion eigenvalue can be written, to first order, as

\[
\lambda = \lambda^{(0)} + \epsilon \lambda^{(1)} = (m - 1) \langle k \rangle \langle kk_1 \rangle_E \frac{2 k k_1 \ldots k_{m-1}}{\langle k \rangle^m} - \lambda^{(0)} (1 + \rho),
\]

where we defined

\[
\rho = \frac{\langle k \rangle^2 \langle kk_1 \rangle_E}{\langle k_1 \rangle^2} - 1.
\]

We refer to \( \rho \) as the dynamical assortativity for its relation to hypergraph dynamics. One can verify that the expected value of \( \rho \) for an uncorrelated hypergraph is 0. Interestingly, to first order the expansion eigenvalue does not depend on the particular assortativity function \( g_m \) used, but only on the average of pairwise products of the degrees belonging to the same hyperedge. A schematic of disassortative \( (\rho < 0) \) and assortative \( (\rho > 0) \) hypergraphs is shown in Fig. 1.

We validate our results with numerical simulations on both synthetic and empirical hypergraphs. For both types of data, we modify the dynamical assortativity of the datasets by performing preferential double edge swaps on the hypergraphs. For each dataset hypergraph \( H \), we focus on an \( m \)-uniform partition \( H_m \) (i.e., we only consider its hyperedges of size \( m \)). We set a target dynamical assortativity \( \rho \) and swap edges as follows. We choose two hyperedges \( e_1 = \{i_1, i_2, \ldots, i_m\} \) and \( e_2 = \{j_1, j_2, \ldots, j_m\} \) and a node from each uniformly at random, say \( i_1 \) and \( j_1 \). Then we consider the rewired hypergraph \( H_m' \) obtained by replacing \( e_1 \) and \( e_2 \) with \( e_1' = \{i_1, i_2, \ldots, i_m\} \) and \( e_2' = \{j_1, j_2, \ldots, j_m\} \) respectively. If the assortativity of \( H_m' \) with this edge swap, \( \rho' \), reduces the difference between the current assortativity, \( \rho \), and the desired assortativity, \( \hat{\rho} \), the swap is accepted and we set \( H_m = H_m' \). To ensure that the algorithm explores the space of possible hypergraphs, we accept edge swaps which increase the difference between the desired assortativity and the current assortativity with probability \( e^{-|\hat{\rho} - \rho|} e^{-|\rho - \rho'|}/T \) (we set \( T = 10^{-5} \)). We terminate the algorithm when \( |\hat{\rho} - \rho| \) is smaller than a prescribed tolerance or when a maximum number of edge swaps have been performed (we used a tolerance of \( 10^{-2} \) and \( 10^6 \) maximum edge swaps).
FIG. 2. A comparison of the actual expansion eigenvalue $\lambda$ (connected triangles) to the first-order approximation of the eigenvalue $\lambda^{(0)} + \varepsilon \lambda^{(1)}$ (connected circles) for (a) the configuration model, (b) the tags-ask-ubuntu dataset, (c) the congress-bills dataset, and (d) the Eu-Emails dataset. The square marker denotes the original $(\rho, \lambda)$ value of the dataset.

For the synthetic hypergraph, we constructed a 3-uniform configuration model (CM) hypergraph of size $N = 10^5$ according to the algorithm described in Ref. 14 with a degree sequence drawn from a truncated power-law distribution, $P(k) \propto k^{-3}$ on $[10, 100]$. We also used the Eu-Emails (EE), congress-bills (CB), and tags-ask-ubuntu (TAU) hypergraph datasets from Refs. 31–33, filtered to only include hyperedges of size 3.

In Fig. 2, the expansion eigenvalue $\lambda$ calculated numerically via the power method from Eq. (1) (connected triangles) and the first-order approximation $\lambda^{(0)} + \varepsilon \lambda^{(1)}$ (connected circles) are plotted as a function of $\rho$ for the four datasets mentioned above. For each dataset, the starting point [i.e., the point $(\rho, \lambda)$ for the original hypergraph] is shown with a square marker. For the synthetic hypergraph (a), as expected, the first order approximation works well for small values of dynamical assortativity. For the TAU dataset (b), surprisingly, the agreement is even better than for the synthetic dataset for larger values of $\rho$. Interestingly, for the CB (c) and EE (d) datasets, and to a much lesser extent for the TAU dataset, the value of $\lambda$ changes sharply when first increasing (CB dataset and EE datasets), or both increasing and decreasing (TAU dataset) the assortativity. We hypothesize that initial edge swaps might be destroying other structure (such as community structure, clustering, or assortative mixing by unaccounted attributes), causing $\lambda$ to change abruptly as this structure is destroyed, and then to change slowly as the effects of changing the assortativity dominate. We note that there appear to be limitations to the extent to which $\rho$ can be modified. This is similar to the limitations to the values of assortativity that networks and hypergraphs can achieve [34–37].

In all cases, we see that rewiring the hypergraph to increase the average value of $\langle kk_1 \rangle_E$ (or, equivalently, $\rho$) has a dramatic effect on the expansion eigenvalue. For example, for the EE dataset $\lambda$ can be reduced threefold by the rewiring process. Thus, hypergraph rewiring might be a useful theoretical tool to control dynamical processes that depend on the expansion eigenvalue.

We can also use this perturbation approach to derive a first-order approximation to the eigenvector. Expanding Eq. (3) to first order we obtain a nonhomogeneous linear equation for the perturbation $u_k^{(1)}$ with solution

$$u_k^{(1)} = \alpha k \frac{\langle k \rangle}{\langle k^2 \rangle} \sum_{k_1, \ldots, k_{m-1}} P(k_1) \cdots P(k_{m-1}) \times$$

$$\frac{k_1^2 k_2 \cdots k_{m-1}}{(N \langle k \rangle)^{m-1}} g_m(k, k_1, \ldots, k_{m-1}) + \beta k,$$

where the constants $\alpha$ and $\beta$ are determined by the normalization of $u_k^{(0)}$ and $u_k^{(0)} + \varepsilon u_k^{(1)}$.

Lastly, we show how modifying the dynamical assortativity by rewiring hypergraphs can extinguish an epi-
As an example, consider a hypergraph SIS contagion spreading amongst groups of size $m$ at a fixed rate $\beta_m$. In Ref. [27], the authors derive a sufficient condition for epidemic extinction for such models. For $m$-uniform hypergraphs and $\beta_e = \beta_m$, the extinction threshold for the individual contagion model is $\beta_m < \beta^*_m = \gamma / \lambda$. By decreasing $\lambda$ through hyperedge swaps and thus increasing $\beta^*_m$ so that $\beta^*_m > \beta_m$, the epidemic can be extinguished. (Note, however, that this is a sufficient condition; $\beta_m > \beta^*_m$ may not lead to an epidemic.)

We present an example based on the CB dataset, and additional cases in the Supplemental Material. In this case, we consider $m = 3$, $\gamma = 1$, and $\beta_3 = 7.9 \times 10^{-3}$. In Fig. 3(a), we plot the chosen value of $\beta_3$ as a fraction of the extinction threshold, $\beta_3 / \beta^*_3$ (solid line with markers), which decreases as $\beta^*_3$ is increased by hyperedge swaps, and the threshold for extinction (dashed line) $\beta_3 / \beta^*_3 = 1$. Below the dashed line, epidemics are impossible. Above the dashed line, they may be possible. In Fig. 3(b), we plot the percentage of the population infected as a function of $\rho$ (averaged over 10 realizations of the epidemic). For all values of $\rho$ such that $\beta_3 / \beta^*_3 < 1$, no epidemics occur. For large enough values of $\rho$, however, we see that epidemics occur.

We caution, however, that decreasing $\lambda$ via edge swaps might not necessarily suppress epidemics if $\beta_3 / \beta^*_3$ is not reduced below 1. In principle, epidemics will occur for values of $\beta_3$ larger than a threshold $\beta^*_3 \geq \beta^*_3$ which depends on the hypergraph structure. If the edge swaps modify this threshold in such a way that $\beta^*_3 < \beta^*_3 < \beta_3$, when originally $\beta^*_3 < \beta^*_3 < \beta_3$, epidemics can actually be promoted by the rewiring process (we show an example in the Supplemental Material). Therefore, reduction of $\beta_3 / \beta^*_3$ by preferential edge swaps should be attempted only when one can guarantee that $\beta_3 / \beta^*_3$ can be reduced below 1 or when there is already an epidemic.

In this paper, we have presented a novel definition of assortativity for hypergraphs, related it to the expansion eigenvalue, and motivated its use in relating assortative structure in hypergraphs to the epidemic behavior. A key limitation of our theoretical approach is that it assumes that connections between nodes are made probabilistically based solely on nodal attributes, while real networks are not necessarily constructed from such generative models. Despite these limitations, our results provide a way to connect various measures of hypergraph structure with dynamical processes in a systematic way and for a large class of tunable null models. We believe that exploring the role of the expansion eigenvalue in other dynamical processes on hypergraphs will be a fruitful research direction.

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SUPPLEMENTAL MATERIAL FOR HYPERGRAPH DYNAMICS: ASSORTATIVITY AND THE EXPANSION EIGENVALUE

In this Supplemental Material we present calculations not included in the main text, including the derivation of the extinction threshold, the mean-field approximation for the expansion eigenvalue for non-uniform hypergraphs, the perturbed eigenvalue and associated eigenvector, and the null values of the assortativity functions on an uncorrelated hypergraph. We also present additional properties of the dynamical assortativity and derive a relationship between the assortativity functions $g_m$, the dynamical assortativity $\rho$, and the parameter $\epsilon$ using our generative model.

EPIDEMIC THRESHOLD

Consider the individual contagion process for the hypergraph susceptible-infected-susceptible (SIS) model defined in Ref. [14] on an undirected hypergraph of size $N$. We denote $\gamma$ as the healing rate and $\beta_e$ as the infection rate of a hyperedge $e$. As discussed in Ref. [27] in Theorem 9.1 ($e_1 = e_2 = 1$ in our case), the extinction threshold for the exact stochastic process can be bounded above by that for the mean-field dynamics. The mean-field equation for $x_i$, the probability that node $i$ is infected, is given by

$$\frac{dx_i}{dt} = -\gamma x_i + (1 - x_i) \sum_{e = \{i, i_1, \ldots, i_{m-1}\} \in E} \beta_e [1 - (1 - x_{i_1}) \cdots (1 - x_{i_{m-1}})].$$  \hfill (14)

By inspection, $x_i = 0$ is always a fixed point of this equation. We write an ODE for linear perturbations around this equilibrium to derive conditions for the system’s stability. To first order, the equation for the perturbations, $\delta x_i$, are

$$\frac{d(\delta x_i)}{dt} = -\gamma(\delta x_i) + \sum_{\{i, i_1, \ldots, i_{m-1}\} \in E} \beta_e (\delta x_{i_1} + \cdots + \delta x_{i_{m-1}}),$$  \hfill (15)

If we assume $\delta x_i = u_i e^{rt}$, then

$$(r + \gamma) u_i = \lambda u_i = \sum_{\{i, i_1, \ldots, i_{m-1}\} \in E} \beta_e (u_{i_1} + \cdots + u_{i_{m-1}}),$$  \hfill (16)

where $\lambda$ is the expansion eigenvalue and so, $r = \lambda - \gamma$. Therefore, a sufficient condition for epidemic extinction is $\gamma > \lambda$ [27]. For an $m$-uniform hypergraph with $\beta_e = \beta_m$, the extinction threshold is $\beta_m/\gamma < 1/\lambda$, where $\lambda$ is the expansion eigenvalue of the unweighted hypergraph.

If we rewrite the last term of Eq. (15) as a sum over uniform hypergraphs, then to first order

$$\frac{d(\delta x_i)}{dt} = -\gamma(\delta x_i) + \sum_{m=2}^M (m-1) \left(W^{(m)}\delta x\right)_i,$$  \hfill (17)

where $W^{(m)}$ is the weighted version of the clique motif matrix defined in Ref. [28] and $\delta x = [\delta x_1, \ldots, \delta x_N]$. We can define $W = \sum_{m=2}^M (m-1)W^{(m)}$ as a linear operator with eigenvalue $\lambda$ and as before, the extinction threshold is $\gamma > \lambda$.

DERIVATION OF THE NON-UNIFORM UNCORRELATED EXPANSION EIGENVALUE

We consider an uncorrelated hypergraph with edges of sizes $m = 2, \ldots, M$ and edge weights of the form $\beta_e = \beta_{|e|}$. The expansion eigenvalue equation can be written

$$\lambda u_i = \sum_{m=2}^M \beta_m \sum_{\{i, i_1, \ldots, i_{m-1}\} \in E} (u_{i_1} + \cdots + u_{i_{m-1}}).$$  \hfill (18)

The degree-based mean-field eigenvalue equation, where we assume $u_i = u_{k_i}$, can be written as

$$\lambda u_k = \sum_{m=2}^M \beta_m \frac{1}{(m-1)!} \sum_{k_1, \ldots, k_{m-1}} P(k_1) \ldots P(k_{m-1}) f_m(k,k_1,\ldots,k_{m-1})(u_{k_1} + \cdots + u_{k_{m-1}}).$$  \hfill (19)
Focusing on the uncorrelated case, we assume that
\[ f_m(k, k_1, \ldots, k_{m-1}) = f_m^{(0)}(k^{(m)}_1, k^{(m)}_2, \ldots, k^{(m)}_{m-1}) = \frac{(m-1)!k^{(m)}_1 \ldots k^{(m)}_{m-1}}{(N(k^{(m)}))^{m-1}} \]
so
\[ \lambda u_k = \sum_{m=2}^{M} \beta_m \sum_{k_1, \ldots, k_{m-1}} P(k_1) \ldots P(k_{m-1}) \frac{k^{(m)}_1 \ldots k^{(m)}_{m-1}}{(N(k^{(m)}))^{m-1}} (u_{k_1} + \cdots + u_{k_{m-1}}), \]
and from symmetry,
\[ \lambda u_k = \sum_{m=2}^{M} k^{(m)} \beta_m (m-1) \sum_{k_i} \frac{P(k_1) k^{(m)}_1}{N(k^{(m)})} u_{k_1}. \]
From Eq. (20), we can see that \( u_k \) must be a linear combination of \( k^{(m)} \). We assume an ansatz of the form
\[ u_k = \sum_{m=2}^{M} \alpha_m k^{(m)} = k^T \alpha, \]
where \( \alpha = (\alpha_1, \ldots, \alpha_M) \) is an unknown vector of nonnegative weights. Renaming the summation indices and evaluating this ansatz in the eigenvalue equation,
\[ \lambda \sum_{j=2}^{M} \alpha_j k^{(j)} = \sum_{i=2}^{M} k^{(i)} \beta_i (i-1) \sum_{k_i} \frac{P(k^{(i)}_1) k^{(i)}_1 \sum_{j=2}^{M} \alpha_j k^{(j)}_1}{N(k^{(i)})}. \]
Changing the order of summation,
\[ \lambda k^T \alpha = \sum_{i=2}^{M} \sum_{j=2}^{M} k^{(i)} \beta_i (i-1) \frac{k^{(j)}_1}{N} \sum_{k_1} \frac{P(k^{(i)}_1) k^{(i)}_1 k^{(j)}_1}{N}, \]
\[ = \sum_{i=2}^{M} \sum_{j=2}^{M} k^{(i)} \beta_i (i-1) \frac{k^{(j)}_1}{N} \alpha_j, \]
\[ = k^T K \alpha, \]
We call \( K \) the \textit{degree-size correlation matrix}, with entries \( K_{ij} = \beta_i (i-1) \langle k^{(i)} k^{(j)} \rangle / \langle k^{(i)} \rangle \) which we call the \textit{inter-size correlations}. Generically (when \( k \) is not orthogonal to the range of \( K - \lambda I \)), this equation has a solution if and only if \( \lambda \) and \( \alpha \) solve the eigenvalue equation \( \lambda \alpha = K \alpha \). Notice that in the \( m \)-uniform case, we recover the expression we previously derived. Since \( K \) is a nonnegative matrix, we can use the Perron-Frobenius theorem and choose the non-negative eigenvalue with a corresponding non-negative eigenvector. If \( K \) is primitive, then \( \lambda \) is the eigenvalue of \( K \) with largest magnitude.

**MORE DETAILED DERIVATION OF THE PERTURBED EIGENVALUE**

We start with the expansion of Eq. (2) in the main text to first order (we recall that we are considering an \( m \)-uniform hypergraph), which is
\[ \alpha \lambda^{(0)} k + \epsilon \lambda^{(0)} u^{(1)} + \alpha \epsilon \lambda^{(1)} k \]
\[ = \alpha \sum_{k_1, \ldots, k_{m-1}} P(k_1) \ldots P(k_{m-1}) \frac{k^{(0)} k^{(1)} \ldots k^{(m-1)}_{m-1}}{(N(k^{(m)}))^{m-1}} (k_1 + \cdots + k_{m-1}) \]
\[ + \epsilon \sum_{k_1, \ldots, k_{m-1}} P(k_1) \ldots P(k_{m-1}) \frac{k^{(0)} k^{(1)} \ldots k^{(m-1)}_{m-1}}{(N(k^{(m)}))^{m-1}} (u^{(1)}_{k_1} + \cdots + u^{(1)}_{k_{m-1}}) \]
\[ + \alpha \epsilon \sum_{k_1, \ldots, k_{m-1}} P(k_1) \ldots P(k_{m-1}) \frac{k^{(0)} k^{(1)} \ldots k^{(m-1)}_{m-1}}{(N(k^{(m)}))^{m-1}} g_m(k, k_1, \ldots, k_{m-1}) (k_1 + \cdots + k_{m-1}). \]
From the 0th order approximation, the first terms on both sides of the equation are equal and we can cancel them. Secondly, assuming symmetry of \( f_m \) and \( g_m \), we can simplify the right-hand side as

\[
\epsilon \lambda^{(0)} u_k^{(1)} + \alpha \epsilon \lambda^{(1)} k \\
= \epsilon (m-1) k \sum_{k_1} \frac{k_1 P(k_1) u_{k_1}^{(1)}}{N(k)} \\
+ \alpha \epsilon (m-1) \sum_{k_1, \ldots, k_{m-1}} P(k_1) \cdots P(k_{m-1}) \frac{k_1^2 k_2 \cdots k_{m-1}}{(N(k))^{m-1}} g_m(k, k_1, \ldots, k_{m-1}).
\]

We multiply both sides by \( k P(k)/(N(k)) \) and sum over \( k \) which yields

\[
\epsilon \lambda^{(0)} \sum_{k} \frac{k P(k) u_{k}^{(1)}}{N(k)} + \alpha \epsilon \lambda^{(1)} \sum_{k} \frac{k^2 P(k)}{N(k)} \\
= \epsilon (m-1) \sum_{k} \frac{k^2 P(k)}{N(k)} \sum_{k_1} \frac{k_1 P(k_1) u_{k_1}^{(1)}}{N(k)} \\
+ \alpha \epsilon (m-1) \sum_{k, k_1, \ldots, k_{m-1}} P(k) P(k_1) \cdots P(k_{m-1}) \frac{k_1^2 k_2 \cdots k_{m-1}}{(N(k))^{m}} g_m(k, k_1, \ldots, k_{m-1}).
\]

Because \( \lambda^{(0)} = (m-1)/k^2/k \), the first terms on both sides are equal and we cancel them, yielding

\[
\epsilon \lambda^{(1)} = \epsilon (m-1) \frac{k}{(k^2)} \sum_{k, k_1, \ldots, k_{m-1}} P(k) P(k_1) \cdots P(k_{m-1}) \frac{k_1^2 k_2 \cdots k_{m-1}}{(N(k))^{m}} g_m(k, k_1, \ldots, k_{m-1}).
\tag{23}
\]

We can use the relation that \( f_m(k_1, \ldots, k_m) = (m-1)!k_1 \cdots k_m/(N(k))^{m-1} [1 + \epsilon g_m(k_1, \ldots, k_m)] \) to remove the reference to \( g_m \), obtaining

\[
\epsilon \lambda^{(1)} = \frac{(m-1)}{(m-1)!} \frac{k}{(k^2)} \sum_{k, k_1, \ldots, k_{m-1}} P(k) P(k_1) \cdots P(k_{m-1}) \frac{k_1^2}{(N(k))^{2}} f_m(k, k_1, \ldots, k_{m-1})
\]

\[
- \frac{(m-1)}{(m-1)!} \frac{k}{(k^2)} \sum_{k, k_1} P(k) P(k_1) \frac{k^2}{(N(k))^2}.
\]

The term

\[
\frac{1}{2!(m-2)!} \sum_{k, k_1, \ldots, k_{m-1}} P(k) P(k_1) \cdots P(k_{m-1}) k k_1 f_m(k, k_1, \ldots, k_{m-1})
\]

represents the expected sum of all products of degrees for pairs of nodes belonging to the same hyperedge (where the factors 2! and \((m-2)!\) correct for overcounting permutations of \( k, k_1 \) and \( k_2, k_3, \ldots, k_{m-1} \) respectively). Since the number of possible pairwise products in a hypergraph is given by

\[
\sum_{k, k' \in E, k \neq k'} 1 = \left( \frac{m}{2} \right) |E| = \left( \frac{N(k)}{m} \right) \left( \frac{m(m-1)}{2} \right) = \frac{(m-1) N \langle k \rangle}{2},
\tag{24}
\]

letting \(|E|\) be the number of edges, we can express \( \lambda^{(1)} \) in terms of

\[
\langle kk_1 \rangle_E = \frac{1}{(m-1)!} \sum_{k_1, \ldots, k_{m-1}} P(k) P(k_1) \cdots P(k_{m-1}) \frac{k_1}{N(k)} f_m(k, k_1, \ldots, k_{m-1}),
\]

the average of pairwise degree products over pairs of connected nodes, as

\[
\epsilon \lambda^{(1)} = (m-1) \frac{\langle k \rangle \langle kk_1 \rangle_E}{\langle k^2 \rangle} - \lambda^{(0)}.
\]
Therefore,

$$\lambda = \lambda^{(0)} + \epsilon \lambda^{(1)} = (m - 1) \frac{\langle k \rangle \langle kk_1 \rangle E}{\langle k^2 \rangle} = \lambda^{(0)} (1 + \rho), \quad (25)$$

where

$$\rho = \frac{\langle k^2 \rangle \langle kk_1 \rangle E}{\langle k^2 \rangle^2} - 1. \quad (26)$$

**DERIVING THE PERTURBED EIGENVECTOR**

With the 0th order terms canceled, Eq. (22) becomes

$$\lambda^{(0)} u_k^{(1)} + \alpha \lambda^{(1)} k\langle k \rangle \langle k_2 \rangle \sum_{k_1, k_2, \ldots, k_{m-1}} P(k_1) \ldots P(k_{m-1}) \frac{k_2 \ldots k_{m-1}}{(N \langle k \rangle)^{m-1}} g_m(k, k_1, \ldots, k_{m-1}),$$

and we can substitute the expressions for $\lambda^{(0)}$ and $\lambda^{(1)}$ derived above, yielding the nonhomogeneous linear equation for $u_k^{(1)}$

$$\begin{align*}
(m - 1) \frac{\langle k^2 \rangle}{\langle k \rangle} u_k^{(1)} + (m - 1) \frac{\langle k \rangle}{\langle k^2 \rangle} k \sum_{k_1, \ldots, k_{m-1}} P(k_1) \ldots P(k_{m-1}) \frac{k_2 \ldots k_{m-1}}{(N \langle k \rangle)^{m-1}} g_m(k, k_1, \ldots, k_{m-1}) \\
= (m - 1) k \sum_{k_1} \frac{k_1 P(k_1) u_k^{(1)}}{N \langle k \rangle} \\
+ \alpha(m - 1) k \sum_{k_1, \ldots, k_{m-1}} P(k_1) \ldots P(k_{m-1}) \frac{k_2 \ldots k_{m-1}}{(N \langle k \rangle)^{m-1}} g_m(k, k_1, \ldots, k_{m-1}). \quad (27)
\end{align*}$$

The following expression, where the first term is a particular solution of (27) and the second a solution of the corresponding homogeneous equation,

$$u_k^{(1)} = \alpha k \frac{\langle k \rangle}{\langle k^2 \rangle} \sum_{k_1, \ldots, k_{m-1}} P(k_1) \ldots P(k_{m-1}) \frac{k_1^2 \ldots k_{m-1}}{(N \langle k \rangle)^{m-1}} g_m(k, k_1, \ldots, k_{m-1}) + \beta k, \quad (28)$$

is the general solution of Eq. (27), where $\alpha$ and $\beta$ are determined by the chosen normalization.

**RELATING THE VALUE OF EPSILON TO OTHER QUANTITIES**

When we write $f_m(k_1, \ldots, k_m) = (m - 1)! k_1 \ldots k_m / (N \langle k \rangle)^{m-1} [1 + \epsilon g_m(k_1, \ldots, k_m)]$, we can use the mean-field approximation to derive a relation that allows us to infer the value of $\epsilon$ from the empirically observed average of the assortativity function over hyperedges, $\langle g_m \rangle$. We derive this relation for the large degrees assortativity function,
although we can use the same approach to derive a similar relation for the aligned degrees assortativity function.

\[
\langle g_m \rangle = \frac{1}{m!} \sum_{k_1, \ldots, k_m} P(k_1) \ldots P(k_m) f_m(k_1, \ldots, k_m) g_m(k_1, \ldots, k_m) / (N(k)/m),
\]

\[
= \frac{1}{N(k)(m-1)!} \sum_{k_1, \ldots, k_m} P(k_1) \ldots P(k_m) \frac{(m-1)!k_1 \ldots k_m}{(N(k))^m} [1 + \epsilon g_m(k_1, \ldots, k_m)] g_m(k_1, \ldots, k_1),
\]

\[
= \epsilon \sum_{k_1, \ldots, k_m} P(k_1) \ldots P(k_m) \frac{k_1 \ldots k_m}{(N(k))^m} g_m(k_1, \ldots, k_m)^2,
\]

\[
= \epsilon \sum_{k_1, \ldots, k_m} P(k_1) \ldots P(k_m) \frac{k_1 \ldots k_m}{(N(k))^m} \left[ \frac{k_1 \ldots k_m}{\langle k \rangle^m} - \left( \frac{\langle k^2 \rangle}{\langle k \rangle^2} \right)^m \right]^2,
\]

\[
= \epsilon \left[ \frac{\langle \langle k^3 \rangle \rangle_m}{\langle k^3 \rangle_m^2} - \left( \frac{\langle k^2 \rangle}{\langle k \rangle^2} \right)^{2m} \right],
\]

and so,

\[
\epsilon = \frac{\langle g_m \rangle}{\left( \frac{\langle k^3 \rangle}{\langle k^3 \rangle_m^2} \right)^m - \left( \frac{\langle k^2 \rangle}{\langle k \rangle^2} \right)^{2m}}.
\]

Assuming the same generative model described above, we can relate the value of \( \epsilon \) to \( \rho \) given a particular choice of assortativity function. We derive this relation for the large degrees assortativity function, although we can use the same approach to derive a similar relation for the aligned degrees assortativity function. In the mean field approximation, the \( \langle kk_1 \rangle_E \) term is

\[
\langle kk_1 \rangle_E = \frac{\langle k^2 \rangle^2}{\langle k \rangle^2} - \epsilon \sum_{k, k_1, \ldots, k_{m-1}} P(k) P(k_1) \ldots P(k_{m-1}) \frac{k^2 k_1^2 \ldots k_{m-1}^2}{(N(k))^m} \left[ \frac{k_1 \ldots k_{m-1}}{\langle k \rangle^m} - \left( \frac{\langle k^2 \rangle}{\langle k \rangle^2} \right)^m \right],
\]

\[
= \frac{\langle k^2 \rangle^2}{\langle k \rangle^2} - \epsilon \left[ \frac{\langle k^2 \rangle^{m-2} \langle k^3 \rangle^2}{\langle k \rangle^{2m}} - \frac{\langle k^2 \rangle^2}{\langle k \rangle^2} \left( \frac{\langle k^2 \rangle}{\langle k \rangle^2} \right)^{m} \right],
\]

and

\[
\rho = \frac{\langle k^2 \rangle^2 \langle kk_1 \rangle_E}{\langle k^2 \rangle^2} \epsilon - 1 = \left( \frac{\langle k^2 \rangle}{\langle k \rangle^2} \right)^m \left[ \frac{\langle k^2 \rangle^2 \langle k^3 \rangle^2}{\langle k \rangle^{4}} - 1 \right] \epsilon.
\]

One can easily verify that the expected value of \( \rho \) on an uncorrelated hypergraph is 0 by setting \( \epsilon = 0 \).

**Calculating the Null Value of Assortativity Functions**

For each assortativity function, we derive the expected value for a null-model (the uncorrelated case) as

\[
\langle g_m \rangle = \frac{1}{m!} \sum_{k_1, \ldots, k_m} P(k_1) \ldots P(k_m) \frac{(m-1)!k_1 \ldots k_m}{(N(k))^m} g_m(k_1, \ldots, k_m)
\]

\[
\langle g_m \rangle = \frac{1}{\langle k \rangle^m} \sum_{k_1, \ldots, k_m} P(k_1) \ldots P(k_m) \frac{k_1 \ldots k_m}{\langle k \rangle^m} \prod_{i=1}^{m} \frac{k_i}{\langle k \rangle^i}.
\]

For the large degrees case,

\[
\langle g_m \rangle = \langle k^2 \rangle^m \frac{1}{\langle k \rangle^m} \sum_{k_1, \ldots, k_m} P(k_1) \ldots P(k_m) \frac{k_1^2 \ldots k_m^2}{\langle k \rangle^m}.
\]
For the aligned degrees case,

\[
\langle g_m \rangle = \sum_{k_1, \ldots, k_m} P(k_1) \cdots P(k_m) \frac{k_1 \cdots k_m}{(N\langle k \rangle)^m} \left( \begin{array}{c} m \\ 2 \end{array} \right) \sum_{i<j} \left( \frac{k_i - \langle k \rangle}{\langle k \rangle} \right) \left( \frac{k_j - \langle k \rangle}{\langle k \rangle} \right),
\]

WLOG \[
= \sum_{k_1, \ldots, k_m} P(k_1) \cdots P(k_m) \frac{k_1 \cdots k_m}{(N\langle k \rangle)^m} \left( \frac{k_1 - \langle k \rangle}{\langle k \rangle} \right) \left( \frac{k_2 - \langle k \rangle}{\langle k \rangle} \right),
\]

\[
= \left[ \frac{\langle k^2 \rangle - \langle k \rangle^2}{\langle k \rangle^2} \right]^2.
\]

SUPPRESSING EPIDEMICS THROUGH PREFERENTIAL REWIRING

In this Section, we include additional plots of the effect of disassortative rewiring on the epidemic extent. We consider the CM and EE datasets described in the main text. The following plots have the same structure as that in the main text so we omit the legend for simplicity.

In Fig. 4, we see the same behavior as that of the CB dataset. We comment that, as we expect, the epidemic threshold is fairly close to the predicted extinction threshold. In Figs. 5 and 7, we see behavior that differs from that of the CB dataset, but is consistent with our theoretical approach. In Fig. 5, we see that the epidemic extent is roughly less than 0.25% for all values of \( \rho \). This does not contradict the bounds we derived because there is no epidemic below the extinction threshold. The behavior in Fig. 7 indicates that additional structure is present in the original hypergraph that seems to be suppressing the epidemic as well and warrants further study.

As discussed in the text, it is possible that if edge swaps do not bring \( \beta_3/\beta_c^3 \) below 1 as in Fig. 7, the process results in an epidemic. While we only see this for the EE dataset, one should be cautious of rewiring the hypergraph unless one can guarantee that \( \beta_3/\beta_c^3 < 1 \) can be achieved.

FIG. 4. CM dataset, \( \beta_3 = 1.78 \times 10^{-2} \)
FIG. 5. EE dataset, $\beta_3 = 1.3 \times 10^{-3}$

FIG. 6. EE dataset, $\beta_3 = 2.1 \times 10^{-3}$

FIG. 7. EE dataset, $\beta_3 = 3.2 \times 10^{-3}$
DATA AVAILABILITY

All code used in our analysis can be found at https://github.com/nwlandry/hypergraph-assortativity [38].