SEMIRING IDENTITIES OF POWER GROUPS

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ABSTRACT. For every semigroup $S$, the set $\mathcal{P}(S)$ of its subsets forms a semiring under set-theoretical union $\cup$ and element-wise multiplication $\cdot$. We show that if $S$ is a finite non-abelian solvable group, the semiring $(\mathcal{P}(S), \cup, \cdot)$ admits no finite identity basis. We also solve the finite basis problem for the semiring of all Hall relations over any finite set.

1. INTRODUCTION

We assume the reader’s acquaintance with the concepts of an algebra of type $\tau$ and an identity of type $\tau$; see, e.g., [3]. We use expressions like $A := B$ to emphasize that $A$ is defined to be $B$.

An additively idempotent semiring (ai-semiring, for short) is an algebra $\mathcal{A} = (S, +, \cdot)$ of type $(2, 2)$ whose operations are associative and fulfill the distributive laws

$$x(y + z) = xy + xz \quad \text{and} \quad (y + z)x = yx + zx,$$

and besides that, addition is commutative and fulfills the idempotency law $x + x = x$. We refer to the semigroup $(S, \cdot)$ as to the multiplicative reduct of $\mathcal{A}$.

Examples of ai-semirings are plentiful and include many objects of importance for computer science, idempotent analysis, tropical geometry, and algebra such as, e.g., semirings of binary relations [14], syntactic semirings of languages [22, 23], tropical semirings [21], endomorphism semirings of semilattices [13]. A natural family of ai-semirings comes from the powerset construction applied to an arbitrary semigroup. Namely, for any semigroup $(S, \cdot)$, one can multiply its subsets element-wise: for any $A, B \subseteq S$, put

$$A \cdot B := \{ ab \mid a \in A, b \in B \}.$$

It is known and easy to verify that this multiplication distributes over the set-theoretical union of subsets of $S$. Thus, denoting the powerset of $S$ by $\mathcal{P}(S)$, we see that $(\mathcal{P}(S), \cup, \cdot)$ is an ai-semiring, called the power semiring of $(S, \cdot)$. In semigroup theory, there exists a vast literature on power semigroups, i.e., the multiplicative reducts of power semirings. The interest in power semigroups is mainly motivated by their applications in the algebraic theory of regular languages; see [11 Chapter 11] and references therein. Power semirings have attracted less attention so far, but we think they deserve study no less than power semigroups as $(\mathcal{P}(S), \cup, \cdot)$ carries more information than its multiplicative reduct $(\mathcal{P}(S), \cdot)$.
(For instance, it is known that in general, the power semigroup \((\mathcal{P}(S), \cdot)\) does not determine its ‘parent’ semigroup \((S, \cdot)\) up to isomorphism \[16, 17\] while it is easy to see that the power semiring \((\mathbb{P}(S), \cup, \cdot)\) does.)

The power semigroups of finite groups have quite a specific structure discussed in detail in \[1\ Section 11.1\]. In particular, they satisfy the following implications:

\begin{align}
\text{(1.1)} \\
& ef = e^2 = e \& fe = f^2 = f \to e = f, \\
\text{(1.2)} \\
& ef = f^2 = f \& fe = e^2 = e \to e = f.
\end{align}

Semigroups satisfying (1.1) and (1.2) are called block-groups. We refer the reader to Pin’s survey \[20\] for an explanation of the name ‘block-group’ and an overview of the remarkable role played by finite block-groups in the theory of regular languages. In this paper, we deal with power semirings of finite groups, which we call power groups for brevity, and more broadly, with ai-semirings whose multiplicative reducts are block-groups. We address the finite axiomatizability question (aka the Finite Basis Problem) for semiring identities satisfied by such ai-semirings. For power groups, we prove the following.

**Theorem 1.1.** For any finite non-abelian solvable group \((G, \cdot)\), the identities of the power group \((\mathbb{P}(G), \cup, \cdot)\) admit no finite basis.

In 2009, Dolinka \[5\ Problem 6.5\] asked for a description of finite groups whose power groups have no finite identity basis (provided such finite groups exist, which was unknown at that time). Theorem 1.1 gives a partial answer to Dolinka’s question.

Theorem 1.1 is a consequence of quite a general condition ensuring the absence of a finite identity basis for ai-semirings whose multiplicative reducts are block-groups with solvable subgroups (see Theorem 4.2 below). Yet another application of this condition, combined with a result from \[11\], gives a complete solution to the Finite Basis Problem for a natural family of semirings of binary relations. Recall that a Hall relation on a finite set \(X\) is a binary relation that contains all pairs \((x, x\pi), x \in X\), for a permutation \(\pi : X \to X\). Clearly, the set \(H(X)\) of all Hall relations on \(X\) is closed under unions and products of relations, and hence \((H(X), \cup, \cdot)\) is a subsemiring of the \((\cup, \cdot)\)-semiring of all binary relations on \(X\).

**Theorem 1.2.** The identities of the ai-semiring of all Hall relations on a finite set \(X\) admit a finite basis if and only if \(|X| = 1\).

The paper is structured as follows. Sect. 2 collects preliminaries, including the necessary properties of block-groups. The technical core of the paper is Sect. 3 where we show that every finite block-group with solvable subgroups satisfies certain semigroup identities from a family of identities introduced in our earlier paper \[7\]. This allows us in Sect. 4 to quickly deduce the main results from those in \[7\].

2. Preliminaries

The present paper uses a few basic notions of semigroup theory such as idempotents, ideals, and Rees quotients. They all can be found in the early chapters of any general semigroup theory text, e.g., \[4,10\]. We also assume the reader’s acquaintance with rudiments of group theory, including the concept of a solvable group, see, e.g., \[9\].

We recall the definition of a Brandt semigroup as we will need some calculations in such semigroups. Let \(I\) be a non-empty set, \(G = (G, \cdot)\) a group, and \(0 \notin G\) a fresh symbol. The Brandt semigroup \(B_{G,I}\) over \(G\) has the set \(B_{G,I} := I \times G \times I \cup \{0\}\) as its carrier, and the multiplication in \(B_{G,I}\) is defined by
Letting $e_b = 0$, holds. By symmetry, the implication (1.2) holds as well. Let $I$ be a non-empty set, let $G = (G, \cdot)$ be a group, and for any $\ell, r \in I$, let $G_{\ell r} := \{(\ell, g, r) \mid g \in G\}$. Then $(G_{\ell r}, \cdot)$ is a maximal subgroup of the Brandt semigroup $B_{G, I}$ if $\ell = r$; otherwise the product of any two elements from $G_{\ell r}$ is 0.

An element $a$ of a semigroup $(S, \cdot)$ is said to be regular if there exists a $b \in S$ satisfying $aba = a$ and $bab = b$; any such $b$ is called an inverse of $a$. A semigroup is called regular [resp., inverse] if every its element has an inverse [resp., a unique inverse]. Every group $G = (G, \cdot)$ is an inverse semigroup, and the unique inverse of an element $g \in G$ is nothing but its group inverse $g^{-1}$. The Brandt semigroup $B_{G, I}$ over $G$ is inverse as well: for $\ell, r \in I$ and $g \in G$, the unique inverse of $(\ell, g, r)$ is $(r, g^{-1}, \ell)$ and the unique inverse of 0 is 0.

Now we present a few properties of block-groups needed for Sect. 3. They all are easy direct proofs for the reader’s convenience.

The definition of a block-group can be restated in terms of inverse elements, and it is this version of the definition that is often used in the literature to introduce block-groups.

**Lemma 2.2.** A semigroup $(B, \cdot)$ is a block-group if and only if every element in $B$ has at most one inverse.

**Proof.** The ‘if’ part. Take any $e, f \in B$ that satisfy the antecedent of the implication (1.1), that is, $ef = e^2 = e$ and $fe = f^2 = f$. Multiplying $ef = e$ by $e$ on the right and using $e^3 = e$, we get $efe = e$, and similarly, multiplying $fe = f$ by $f$ on the right and using $f^2 = f$, we get $f ef = f$. Hence $f$ is an inverse of $e$. On the other hand, we have $e = e^2 = e^3$ so that $e$ is an inverse of itself. Since $e$ has at most one inverse, we conclude that $e = f$ so that (1.1) holds. By symmetry, the implication (1.2) holds as well.

The ‘only if’ part. Take any $a \in B$ and suppose that both $b$ and $c$ are inverses of $a$. Letting $e := ba$ and $f := ca$, we have $ef = baca = ba = e$, $e^2 = bab = e$, and similarly, $fe = caba = ca = f$, $f^2 = caca = f$. Since $(B, \cdot)$ satisfies the implication (1.1), we get $e = f$, that is, $ba = ca$. By symmetry, the implication (1.2) ensures $ab = ac$. Now we have $b = bab = cab = caca = c$. □

Lemma 2.2 shows that regular block-groups are inverse, so that one can think of block-groups as a sort of non-regular analogs of inverse semigroups.

For any semigroup $(S, \cdot)$, we denote the set of all its idempotents by $E(S)$. A semigroup $(S, \cdot)$ is called $\lambda$-trivial if every principal ideal of $(S, \cdot)$ has a unique generator and periodic if every one-generated subsemigroup of $(S, \cdot)$ is finite. In [15], Proposition 2.3] it is proved that a finite monoid $(S, \cdot)$ is a block-group if and only if the subsemigroup $(E(S), \cdot)$ generated by $E(S)$ is $\lambda$-trivial. In fact, the proof in [15] uses periodicity of $(S, \cdot)$ rather than its finiteness and does not use the identity element of $(S, \cdot)$. Thus, we have the following characterization of periodic block-groups.

**Proposition 2.3.** A periodic semigroup is a block-group if and only if the subsemigroup generated by its idempotents is $\lambda$-trivial.

As a consequence, we have the following observation.
Lemma 2.4. If \((S, \cdot)\) is a periodic block-group, then every regular element of the subsemi-group \((\langle E(S), \cdot \rangle)\) lies in \(E(S)\).

Proof. Let \(a \in \langle E(S) \rangle\) be a regular element and \(b\) its inverse. The FitzGerald Trick [8, Lemma 1] shows that \(b \in \langle E(S) \rangle\) as well. Since \(a = aba\) and \(b = bab\) generate the same ideal in \((\langle E(S), \cdot \rangle)\), Proposition 2.3 implies \(a = b\). Thus, \(a = a^2\), whence \(a\) and \(a^2\) also generate the same ideal in \((\langle E(S), \cdot \rangle)\). Now Proposition 2.3 implies \(a = a^2 \in E(S)\). \(\square\)

Lemma 2.5. If a Brandt semigroup \((B, \cdot)\) occurs as an ideal in a periodic block-group \((S, \cdot)\), then for all \(b \in B\), \(f \in E(S)\), either \(fb = b\) or \(fb = 0\).

Proof. If the semigroup \((B, \cdot)\) is represented as \(\mathcal{B}_{G,I}\) for some group \((G, \cdot)\) and some non-empty set \(I\), then each non-zero idempotent of is of the form \((i, e, i)\) where \(i \in I\) and \(e\) is the identity element of the group \(G\). Take an arbitrary element \(b = (\ell, g, r) \in I \times G \times I\). Then \(b = (\ell, e, \ell)b\) whence \(fb = f(\ell, e, \ell)b\) for each \(f \in E(S)\). The product \(f(\ell, e, \ell)\) lies in \(B\) and so it is regular. By Lemma 2.4 \(f(\ell, e, \ell)\) is an idempotent. If \(f(\ell, e, \ell) \neq 0\), then \(f(\ell, e, \ell) = (j, e, j)\) for some \(j \in I\). Multiplying the equality through by \((j, e, j)\) on the right yields \(f(\ell, e, \ell)(j, e, j) = (j, e, j)\) whence \(j = \ell\). We conclude that either \(f(\ell, e, \ell) = (\ell, e, \ell)\) or \(f(\ell, e, \ell) = 0\), whence either \(fb = b\) or \(fb = 0\). \(\square\)

A principal series of a semigroup \((S, \cdot)\) is a chain

\[
S_0 \subset S_1 \subset \cdots \subset S_h = S
\]

of ideals \(S_j\) of \((S, \cdot)\) such that \(S_0\) is the least ideal of \((S, \cdot)\) and there is no ideal of \((S, \cdot)\) strictly between \(S_{j-1}\) and \(S_j\) for \(j = 1, \ldots, h\). The factors of the series (2.1) are the Rees quotients \((S_j/S_{j-1}, \cdot)\), \(j = 1, \ldots, h\). The number \(h\) is called the height of the series (2.1).

Recall that a semigroup \((Z, \cdot)\) with a zero element 0 is called a zero semigroup if \(ab = 0\) for all \(a, b \in Z\). A semigroup \((T, \cdot)\) with a zero element 0 is 0-simple if it is not a zero semigroup and \(\{0\}\) and \(T\) are the only ideals of \((T, \cdot)\). A semigroup \((T, \cdot)\) is simple if \(T\) is its only ideal. (Observe that a simple semigroup may consist of a single element while any 0-simple semigroup has at least two elements.) By [4, Lemma 2.39], if (2.1) is a principal series, then \((S_0, \cdot)\) is a simple semigroup and every factor \((S_j/S_{j-1}, \cdot)\), \(j = 1, \ldots, h\), is either a 0-simple semigroup or a zero semigroup with at least two elements.

Restricting a classical result of semigroup theory (see [4, Theorem 3.5] or [10, Theorem 3.2.3]) to the case of periodic block-groups yields the following.

Lemma 2.6. If a periodic block-group is simple, it is a group, and if it is 0-simple, it is a Brandt semigroup.

Corollary 2.7. If a periodic block-group possesses a principal series, then its least ideal is a group and the non-zero factors of the series are Brandt semigroups.

We call (2.1) a Brandt series if \((S_0, \cdot)\) is a group and every factor \((S_j/S_{j-1}, \cdot)\) with \(j = 1, \ldots, h\), which is not a zero semigroup, is a Brandt semigroup. Thus, Corollary 2.7 can be restated as saying that a principal series of a periodic block-group is a Brandt series (provided the series exists). The converse is also true, even without periodicity.

Lemma 2.8. If a semigroup possesses a Brandt series, it is a block-group.

Proof. Let (2.1) be a Brandt series in a semigroup \((S, \cdot)\). Take any regular element \(a \in S\) and let \(j \in \{0, 1, \ldots, h\}\) be the least number with \(a \in S_j\). Recall that each inverse of \(a\) generates the same ideal of \((S, \cdot)\) as does \(a\). Therefore, if \(j = 0\), all inverses of \(a\) lie in the least ideal of \((S, \cdot)\) which is a group. Then all inverses of \(a\) are equal since they coincide with the group.
inverse of \(a\) in the group \((S_0, \cdot)\). If \(j > 0\), we have \(a \in S_j \setminus S_{j-1}\), and all inverses of \(a\) also lie in \(S_j \setminus S_{j-1}\). Their images in the Rees quotient \((S_j/S_{j-1}, \cdot)\) are inverses of the image of \(a\), and hence, they coincide as the quotient must be an inverse semigroup. Since non-zero elements of \(S_j/S_{j-1}\) are in a 1-1 correspondence with the elements of \(S_j \setminus S_{j-1}\), all inverses of \(a\) are equal also in this case. Now Lemma \(2.2\) shows that \((S, \cdot)\) is a block-group. \(\square\)

We also need the following property of \(\beta\)-trivial semigroups.

**Lemma 2.9** (Lemma 8.2.2(iii)). If a \(\beta\)-trivial semigroup \((S, \cdot)\) satisfies the identity \(x^p = x^{p+1}\) and \(w\) is an arbitrary semigroup word, then \((S, \cdot)\) satisfies the identity \(w^p = w^p\) where \(u\) is the product, in any order, of the variables that occur in \(w\).

### 3. Identities Holding in Block-Groups with Solvable Subgroups

Here we aim to show that every finite block-group with solvable subgroups satisfies certain identities constructed in \([7]\). For the reader’s convenience, we reproduce the construction in detail so that no acquaintance with \([7]\) is necessary for understanding the results and the proofs of this section.

For any \(i, h \geq 1\), consider the set \(X^{(h)}_i := \{x_{i_1i_2\ldots i_h} \mid i_1, i_2, \ldots, i_h \in \{1, 2, \ldots, i\}\}\). We fix arbitrary \(n, m \geq 1\) and introduce a family of words \(v_{n,m}\) over \(X^{(h)}_{2n}\), \(h = 1, 2, \ldots\), by induction. For \(h = 1\), let

\[
v^{(1)}_{n,m} := x_1 x_2 \cdots x_{2n} \left( x_n x_{n+1} \cdots x_{n+m-1} \right)^2.
\]

Assuming that \(h > 1\) and the word \(v^{(h-1)}_{n,m}\) over \(X^{(h-1)}_{2n}\) has already been defined, we create \(2n\) copies of this word over the alphabet \(X^{(h)}_{2n}\) as follows. We start by taking for every \(j \in \{1, 2, \ldots, 2n\}\), the substitution \(\sigma^{(h)}_{2n,j} : X^{(h-1)}_{2n} \to X^{(h)}_{2n}\) that appends \(j\) to the indices of its arguments, that is,

\[
\sigma^{(h)}_{2n,j}(x_{i_1i_2\ldots i_{h-1}}) := x_{i_1i_2\ldots i_{h-1}j} \quad \text{for all } i_1, i_2, \ldots, i_{h-1} \in \{1, 2, \ldots, 2n\}.
\]

Then we let \(v^{(h-1)}_{n,m,j} := \sigma^{(h)}_{2n,j}(v^{(h-1)}_{n,m})\) and define

\[
v^{(h)}_{n,m} := \sigma^{(h)}_{2n,1}(v^{(h-1)}_{n,m,1}) \cdots \sigma^{(h)}_{2n,2n}(v^{(h-1)}_{n,m,2n}) \left( v^{(h-1)}_{n,m,n+1} \cdots v^{(h-1)}_{n,m,n+m-1} \right)^2.
\]

Comparing the definitions (3.1) and (3.3), one readily sees that the word \(v^{(h)}_{n,m}\) is nothing but the image of \(v^{(1)}_{n,m}\) under the substitution \(x_j \mapsto v^{(h-1)}_{n,m,j}\), \(j \in \{1, 2, \ldots, 2n\}\).

**Remark 1.** It is easy to see that the word \(v^{(h+r)}_{n,m}\) is the image of the word \(v^{(h)}_{n,m}\) under a substitution \(\zeta\) that sends every variable from \(X^{(h)}_{2n}\) to a word obtained from \(v^{(r)}_{n,m}\) by renaming its variables. Indeed, in terms of the substitutions \(\sigma^{(h)}_{2n,j}\) from (3.2), one can express \(\zeta\) as follows:

\[
\zeta(x_{i_1i_2\ldots i_{h}}) := \sigma^{(h+r)}_{2n,i_1i_2} \cdots (\sigma^{(h+r)}_{2n,i_2}(v^{(r)}_{n,m})) \cdots \quad \text{for all } i_1, i_2, \ldots, i_{h} \in \{1, 2, \ldots, 2n\}.
\]

Simply put, \(\zeta(x_{i_1i_2\ldots i_{h}})\) appends \(i_1i_2\ldots i_{h}\) to the indices of all variables of \(v^{(r)}_{n,m}\).

For any \(n, k \geq 0\) with \(n + k > 0\) and any \(m \geq 1\), we define the following word over \(X^{(1)}_{n+k}\):

\[
u_{n,k,m} := x_1 x_2 \cdots x_{n+k} \left( x_n x_{n+1} \cdots x_{n+m+k} \right)^2.
\]

Notice that \(u_{n,n,m} = v^{(1)}_{n,m}\). 

Lemma 3.1. Let $I$ be a non-empty set, $G = (G, \cdot)$ a group, and $BG_{G}$ the Brandt semigroup over $G$. For any $m \geq 1$ and $n, k \geq 0$ with $n + k > 0$ and any substitution $\tau : X^{(1)}_{n+k} \to BG_{G}$, the element $\tau(u_{n,k,m})$ lies in a subgroup of $BG_{G}$.

Proof. Lemma 2.1 implies that for any $a \in BG_{G}$, the power $a^{2n}$ lies in a subgroup of $BG_{G}$. Since $u_{0,k,m} = (x_{1}x_{2} \cdots x_{k})^{2n}$ and $u_{1,k,m} = (x_{1}x_{2} \cdots x_{k+1})^{2n}$, the claim holds for $n \in \{0, 1\}$.

For the rest of the proof, assume $n \geq 2$. If $\tau(u_{n,k,m}) = 0$, there is nothing to prove. Let $\tau(u_{n,k,m}) \neq 0$. Then all elements $\tau(x_{i}), i = 1, 2, \ldots, n + k$, must lie in the set $I \times G \times I$ of non-zero elements of $BG_{G}$. Hence $\tau(x_{i}) = (\ell_{i}, g_{i}, r_{i})$ for some $\ell_{i}, r_{i} \in I$ and $g_{i} \in G$.

First, consider the case $k = 0$. Since $u_{0,0,m} = x_{1}x_{2} \cdots x_{n} \left(x_{n}x_{n-1} \cdots x_{1}\right)^{2m-1}$, we have $\tau(u_{0,0,m}) = (\ell_{1}, g, r_{1})$ for some $g \in G$. By Lemma 2.1, we have to show that $\ell_{1} = r_{1}$. We verify that $\ell_{j} = r_{j}$ for all $j = n, n - 1, \ldots, 1$ by backwards induction. Since $x_{n}^{2}$ occurs as a factor in the word $u_{0,0,m}$, we must have $\tau(x_{n}^{2}) \neq 0$, whence $\ell_{n} = r_{n}$. If $j > 1$, both $x_{j-1}x_{j}$ and $x_{j}x_{j-1}$ occur as factors in $u_{0,0,m}$. We then have

$$r_{j-1} = \ell_{j} \quad \text{since } \tau(x_{j-1}x_{j}) \neq 0,$$

$$= r_{j} \quad \text{by the induction assumption},$$

$$= \ell_{j-1} \quad \text{since } \tau(x_{j}x_{j-1}) \neq 0.$$

Now, let $k > 0$. Since $u_{n,k,m} = x_{1}x_{2} \cdots x_{n+k} \left(x_{n}x_{n-1} \cdots x_{1} \cdot x_{n+k} \cdots x_{n+2} \cdots x_{n+k}\right)^{2m-1}$, we have $\tau(u_{n,k,m}) = (\ell_{1}, g, r_{n+k})$ for some $g \in G$. Here we have to show that $\ell_{1} = r_{n+k}$. Since $\tau(x_{1}x_{2} \cdots x_{n+k}) \neq 0$ and $\tau(x_{n+k}x_{n+1}) \neq 0$, we have

$$r_{1} = \ell_{2}, r_{2} = \ell_{3}, \ldots, r_{n-1} = \ell_{n}, r_{n} = \ell_{n+1} \quad \text{and } r_{n+k} = \ell_{n}.$$

Since $\tau(x_{n}x_{n-1} \cdots x_{1}x_{n+1}) \neq 0$, we also have

$$r_{n} = \ell_{n-1}, r_{n-1} = \ell_{n-2}, \ldots, r_{2} = \ell_{1}, r_{1} = \ell_{n+1}.$$

Therefore, we obtain

$$r_{n+k} = \ell_{n}, r_{n} = \ell_{n-1}, r_{n-1} = \ell_{n-2}, \ldots, r_{2} = \ell_{1}, r_{1} = \ell_{n+1}.$$  

In addition, if $n$ is even, then

$$r_{n+k} = \ell_{n}, r_{n} = \ell_{n-1}, r_{n-1} = \ell_{n-2}, \ldots, r_{2} = \ell_{1}.$$  

We see that $r_{n+k} = \ell_{1}$ in either case.

Several statements in this section deal with semigroups that possess Brandt series and have subgroups subject to certain restrictions. To keep the premises of these statements compact, we introduce a few short names. In the sequel:

- groups of exponent dividing $m$ are called $m$-groups;
- solvable $m$-groups of derived length at most $k$ are called $[m,k]$-groups;
- a semigroup with a Brandt series of height $h$ all of whose subgroups are $m$-groups (resp., $[m,k]$-groups) is called an $(h,m)$-semigroup (resp., an $(h,m,k)$-semigroup).

Lemma 3.2. Any $(h,m)$-semigroup satisfies the identity $x^{20m} = x^{2h+1m}$. 

\[ \square \]
Lemma 3.3. Let \((S, \cdot)\) be an \((h, m)\)-semigroup and \((2.1)\) its Brandt series. We induct on \(h\). If \(h = 0\), then \((S, \cdot)\) is an \(m\)-group, and any \(m\)-group satisfies \(x^m = x^{2m}\).

Let \(h > 0\) and let \(a\) be an arbitrary element in \(S\); we have to verify that

\[ a^{2h} = a^{2h+1}m. \]

If \(a\) lies in a subgroup of \((S, \cdot)\), then \(a^m = a^{2m}\) since all subgroups of \((S, \cdot)\) are \(m\)-groups. Squaring this \(h\) times gives (3.6). If \(a\) lies in no subgroup of \((S, \cdot)\), then \(a^2 \in S_{h-1}\) — this is clear if \(a \in S_{h-1}\) or the factor \((S_h/S_{h-1}, \cdot)\) is a zero semigroup and follows from Lemma 2.1 if \((S_h/S_{h-1}, \cdot)\) is a Brandt semigroup. By the induction assumption, \((a^2)^{2h-1} = (a^2)^{2h},\) that is, (3.6) holds again.

Recall that if \((S, \cdot)\) is a semigroup, \((< E(S) >, \cdot)\) stands for its subsemigroup generated by the set \(E(S)\) of all idempotents of \((S, \cdot)\).

Lemma 3.3. Let \((S, \cdot)\) be an \((h, m)\)-semigroup. For any \(k \geq 1\), if a substitution \(\tau: X_2n^{(k)} \rightarrow S\) is such that \(\tau(x_{i_1, i_2, \ldots, i_k}) \in \langle E(S) \rangle\) for all \(i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, 2n\}\), then \(\tau(v_{n,2^b)m}) \in E(S)\).

Proof. By Lemma 3.2, the semigroup \((S, \cdot)\) satisfies \(x^{2h} = x^{2h+1}m\) whence for each \(p \geq 2h\) and each \(a \in \langle E(S) \rangle\), the elements \(a^p\) and \(a^{p+1}\) generate the same ideal in the subsemigroup \(\langle E(S) >, \cdot \rangle\). The periodic semigroup \((S, \cdot)\) is a block-group by Lemma 2.8 whence the subsemigroup \(\langle E(S) >, \cdot \rangle\) is \(\beta\)-trivial by Proposition 2.3. Therefore, we have \(a^p = a^{p+1}\), i.e., \(\langle E(S) >, \cdot \rangle\) satisfies the identity \(x^p = x^{p+1}\).

Letting \(a_j := \begin{cases} \tau(x_j), & \text{if } k = 1, \\ \tau(v_{n,m, k}), & \text{if } k > 1, \end{cases}\) for \(j = 1, 2, \ldots, 2n\), we have from (3.1) and (3.3)

\[
\tau(v_{n,2^b)m}) = a_1a_2 \cdots a_{2n} = (a_n a_{n-1} \cdots a_1 \cdot a_n+1 a_{n+2} \cdots a_{2n})^{2h+1}m-1
\]

By Lemma 2.9

\[
= a_1a_2 \cdots a_{2n} = (a_1 \cdots a_{n-1} a_n a_{n+1} a_{n+2} \cdots a_{2n})^{2h+1}m-1
\]

\[
= (a_1 \cdots a_{n-1} a_n a_{n+1} a_{n+2} \cdots a_{2n})^{2h+1}m - (a_1 \cdots a_{n-1} a_n a_{n+1} a_{n+2} \cdots a_{2n})^{2h+1}m
\]

\[
= (a_1 \cdots a_{n-1} a_n a_{n+1} a_{n+2} \cdots a_{2n})^{2h+1}m
\]

Hence \(\tau(v_{n,2^b)m}) \in E(S)\) as required.

Lemma 3.4. Let \((S, \cdot)\) be an \((h, m)\)-semigroup with Brandt series \((2.1)\). If \((S, \cdot)\) is a Brandt semigroup and \(S = \langle E(S) >, \cdot \rangle \cup S_1\), then for an arbitrary substitution \(\tau: X_2n^{(1)} \rightarrow S\), the element \(\tau(v_{n,2^b)m})\) lies in a subgroup of \(S\).

Proof. If \(\tau(x_k) \in \langle E(S) >\) for all \(k \in \{1, 2, \ldots, 2n\}\), then \(\tau(v_{n,2^b)m}) \in E(S)\) by Lemma 3.3.

Otherwise let \(\{k_1, k_2, \ldots, k_{p+q}\}\) with

\[
1 \leq k_1 < k_2 < \cdots < k_p \leq n < k_{p+1} < k_{p+2} < \cdots < k_{p+q} \leq 2n
\]

be the set of all indices \(k\) such that \(\tau(x_k) \notin \langle E(S) >\). (Here \(p = 0\) or \(q = 0\) is possible but \(p+q > 0\).) Since \(S = \langle E(S) >, \cdot \rangle \cup S_1\), we have \(\tau(x_{k_1}), \tau(x_{k_2}), \ldots, \tau(x_{k_{p+q}}) \in S_1\). By Lemma 2.5
and its dual, either $\tau(v^{(1)}_{n,2^m}) = 0$ or removing all $\tau(x_i)$ such that $\tau(x_i) \in \langle E(S) \rangle$ does not change the value of $\tau(v^{(1)}_{n,2^m})$. If $\tau(v^{(1)}_{n,2^m}) = 0$, the claim holds. Otherwise, consider the substitution $\tau': X^{(1)}_{p+q} \to S_1$ given by $\tau'(x_i) := \tau(x_{i_s})$ for all $s \in \{1, 2, \ldots, p + q\}$. Then $\tau(v^{(1)}_{n,2^m}) = \tau'(u_{p,q,2^m})$. By Lemma 3.1, the element $\tau'(u_{p,q,2^m})$ lies in a subgroup of the Brandt semigroup $(S_1, \cdot)$. Therefore, $\tau(v^{(1)}_{n,2^m})$ belongs to a subgroup of $(S, \cdot)$. 

Lemma 3.5. Any $[m,k]$-group satisfies the identity $v^{(k)}_{m,n} = 1$.

Proof. Let $G = (G, \cdot)$ be an $[m,k]$-group and $a \in G$. Since $a^m$ is the identity element of $G$, we have $a^{2^m-1} = a^{-1}$. Consider any substitution $\tau: X^{(1)}_{2n} \to G$ and let $a_i := \tau(x_i)$. Then

$$\tau(v^{(1)}_{m,n}) = a_1 a_2 \cdots a_{2^n} (a_n a_{n-1} \cdots a_1 \cdot a_{n+1} a_{n+2} \cdots a_{2^n})^{-1} = a_1 a_2 \cdots a_n \cdot a_1^{-1} a_2^{-1} \cdots a_n^{-1}.$$ 

Recall that the commutator of elements $a$ and $b$ of a group is defined as $[a, b] := a^{-1} b^{-1} a b$. It is easy to verify that the right-hand side of (3.7) can be written as a product of commutators, namely,

$$(a_1 a_2 \cdots a_n \cdot a_1^{-1} a_2^{-1} \cdots a_n^{-1}) = [a_1^{-1}, a_2^{-1}] [a_2 a_1^{-1}, a_3^{-1}] \cdots [a_{n-1} a_{n-2} \cdots a_n^{-1}, a_n^{-1}].$$

Therefore, $\tau(v^{(1)}_{m,n})$ belongs to the derived subgroup $(G^{(1)}(\cdot))$ of $G$. We use this as the induction basis and show by induction on $h$ that any substitution $\theta: X^{(h)}_{2n} \to G$ sends the word $v^{(h)}_{m,n}$ to the $h$-th derived subgroup $(G^{(h)}(\cdot))$ of $G$.

Indeed, let $h > 1$. Since for each $j \in \{1, 2, \ldots, 2^n\}$, the word $v^{(h-1)}_{m,n,j}$ is obtained from the word $v^{(h-1)}_{m,n-1}$ by renaming its variables, $\theta(v^{(h-1)}_{m,n,j}) \in G^{(h-1)}$ by the induction assumption. Combining (3.3) and (3.7), we see that

$$\theta(v^{(h)}_{m,n}) = \theta(v^{(h-1)}_{m,n,1}) \theta(v^{(h-1)}_{m,n,2}) \cdots \theta(v^{(h-1)}_{m,n,2^n}) \cdot \theta(v^{(h-1)}_{m,n,1})^{-1} \theta(v^{(h-1)}_{m,n,2})^{-1} \cdots \theta(v^{(h-1)}_{m,n,2^n})^{-1}. $$

Now rewriting the last expression as in (3.8), we see that $\theta(v^{(h)}_{m,n})$ is a product of commutators of elements from $G^{(h-1)}$. Hence $\theta(v^{(h)}_{m,n}) \in G^{(h)}$. Since the group $G$ is solvable of derived length at most $k$, its $k$-th derived subgroup is trivial, whence the word $v^{(k)}_{m,n}$ is sent to the identity element of $G$ by any substitution $X^{(k)}_{2n} \to G$. This means that $G$ satisfies the identity $v^{(k)}_{m,n} = 1$.

Lemma 3.6. Let $(S, \cdot)$ be an $(h,m,k)$-semigroup and $\langle E(S) \rangle$ its Brandt series. If $S_0 = \{0\}$ and $S = \langle E(S) \rangle \cup S_1$, then $(S, \cdot)$ satisfies the identity $v^{(k+1)}_{n,2^m} = (v^{(k+1)}_{n,2^m})^2$.

Proof. We have to verify that $\tau(v^{(k+1)}_{n,2^m}) \in E(S)$ for every substitution $\tau: X^{(k+1)}_{2n} \to S$. Recall from Remark 1 that the word $v^{(k+1)}_{n,2^m}$ is the image of the word $v^{(k)}_{n,2^m}$ under the substitution that sends every variable $x_{i_1 \ldots i_k}$ to $X^{(1)}_{n,2^m,i_1 \ldots i_k}$ obtained from $v^{(1)}_{n,2^m}$ by renaming its variables. Hence if all elements of the form $\tau(v^{(1)}_{n,2^m,i_1 \ldots i_k})$ lie in $\langle E(S) \rangle$, then $\tau(v^{(k+1)}_{n,2^m}) \in E(S)$ by Lemma 3.2. So, we may assume that at least one element of the form $\tau(v^{(1)}_{n,2^m,i_1 \ldots i_k})$ lies in $S_1 \setminus \langle E(S) \rangle$. 


By the definition of an \((h, m, k)\)-semigroup, \((S_1, \cdot)\) is either a zero semigroup or a Brandt semigroup over an \([m, k]\)-group. If \((S_1, \cdot)\) is a zero semigroup, we have \(\tau(v_{n,2^km}^{(k+1)}) = 0\) since \(S_1\) is an ideal of \((S, \cdot)\) and each factor of the form \(v_{n,2^km}^{(k+1)}\) occurs in the word \(v_{n,2^km}^{(k+1)}\) at least twice. If \((S_1, \cdot)\) is a Brandt semigroup, Lemma \(\ref{lem:brandt-series}\) and its dual imply that either \(\tau(v_{n,2^km}^{(k+1)}) = 0\) or removing all factors \(\tau(v_{n,2^km}^{(1)})\) that lie in \(\langle E(S) \rangle\) does not change the value of \(\tau(v_{n,2^km}^{(k+1)})\). In the former case, the claim holds, and in the latter case, the remaining elements belong to subgroups of \((S_1, \cdot)\) by Lemma \(\ref{lem:subgroups}\). If some of these elements belong to different maximal subgroups of \((S_1, \cdot)\), then \(\tau(v_{n,2^km}^{(k+1)}) = 0\) by the definition of multiplication in Brandt semigroups. Thus, it remains to consider only the situation when, for all \(i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, 2n\}\), all elements \(\tau(v_{n,2^km}^{(1)}) \in S_1 \setminus \langle E(S) \rangle\) belong to the same maximal subgroup \((G, \cdot)\) of \((S_1, \cdot)\).

Denote by \(e\) the identity element of \((G, \cdot)\) and define the substitution \(\tau: X_{2^km}^{(k)} \to G\) by

\[
\tau(x_{i_1, i_2, \ldots, i_k}) := \begin{cases} 
\tau(v_{n,2^km}^{(1)}) & \text{if } \tau(v_{n,2^km}^{(1)}) \text{ belongs to } G, \\
e & \text{otherwise,}
\end{cases}
\]

for each \(i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, 2n\}\). As we know that \(\tau(v_{n,2^km}^{(1)}) \in \langle E(S) \rangle \cup G\) for all \(i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, 2n\}\) and \(fg = gf = g\) for all \(g \in G, f \in E(S)\), we conclude that \(\tau(v_{n,2^km}^{(k+1)}) = \tau(v_{n,2^km}^{(k)})\). Since \((G, \cdot)\) is an \([m, k]\)-group and so a \([2^m, k]\)-group, it satisfies the identity \(v_{n,2^km}^{(k)} = 1\) by Lemma \(\ref{lem:brandt-series}\). Hence \(\tau(v_{n,2^km}^{(k+1)}) = \tau(v_{n,2^km}^{(k)}) = e\). Since the substitution \(\tau\) is arbitrary, \((S, \cdot)\) satisfies the identity \(v_{n,2^km}^{(k+1)} = (v_{n,2^km}^{(k)})^2\).

**Proposition 3.7.** Let \((S, \cdot)\) be an \((h, m, k)\)-semigroup with Brandt series \((\ref{equ:brandt-series})\) and \(S_0 = \{0\}\). For any \(n \geq 2\), \((S, \cdot)\) satisfies the identity

\begin{equation}
\tag{3.9}
v_{n,2^km}^{(kh+h)} = (v_{n,2^km}^{(h)})^2.
\end{equation}

**Proof.** We induct on \(h\). If \(h = 1\), then the claim follows from Lemma \(\ref{lem:zero-identity}\). Let \(h > 1\). The Rees quotient \((S/S_1, \cdot)\) is an \((h-1, m, k)\)-semigroup whose Brandt series starts with the zero term. By the induction assumption, \((S/S_1, \cdot)\) satisfies the identity \(v_{n,2^km}^{(kh-k+h-1)} = (v_{n,2^km}^{(h)})^2\) for any \(n \geq 2\). By Lemma \(\ref{lem:brandt-series}\) \((S/S_1, \cdot)\) satisfies the identity \(v_{n,2^km}^{(kh-k+h-1)} = (v_{n,2^km}^{(kh-k+h-1)})^2\) as well. This readily implies that any substitution \(X_{2^km}^{(kh-k+h-1)} \to S\) sends the word \(v_{n,2^km}^{(kh-k+h-1)}\) to either an idempotent in \(S \setminus S_1\) or an element in \(S_1\). Recall from Remark \(\ref{rem:idempotent}\) that the word \(v_{n,2^km}^{(kh-h)}\) is the image of the word \(v_{n,2^km}^{(k+1)}\) under the substitution that sends every variable \(x_{i_1, i_2, \ldots, i_k+1} \in X_{2^km}^{(k+1)}\) to the word \(v_{n,2^km}^{(kh-k+h-1)}\) obtained from \(v_{n,2^km}^{(kh-k+h-1)}\) by renaming its variables. Therefore, for every substitution \(\tau: X_{2^km}^{(k+1)} \to S\), all elements of the form \(\tau(v_{n,2^km}^{(kh-k+h-1)})\) lie in either \(E(S)\) or \(S_1\).

Now consider the substitution \(\tau: X_{2^km}^{(k+1)} \to E(S) \cup S_1\) defined by

\[
\tau(x_{i_1, i_2, \ldots, i_k+1}) := \tau(v_{n,2^km}^{(kh-k+h-1)}), \quad \text{for each } i_1, i_2, \ldots, i_k+1 \in \{1, 2, \ldots, 2n\}.
\]
Clearly, $\tau(v_{n,2^m}^{kh+h}) = \tau(v_{n,2^m}^{(k+1)})$. However, $(E(S) \cup S_1, \cdot)$ is an $(h, m, k)$-semigroup satisfying the assumption of Lemma 3.6. By this lemma, the semigroup $(E(S) \cup S_1, \cdot)$ satisfies the identity $v_{n,2^m}^{(k+1)} = (v_{n,2^m}^{(k+1)})^2$. This ensures that the element $\tau(v_{n,2^m}^{kh+h}) = \tau(v_{n,2^m}^{(k+1)})$ is an idempotent. Since the substitution $\tau$ is arbitrary, $(S, \cdot)$ satisfies the identity (3.9). □

Finally, we remove the restriction $S_0 = \{0\}$.

**Proposition 3.8.** The identity $v_{n,2^m}^{(kh+h+k)} = (v_{n,2^m}^{kh+h+k})^2$ with $n \geq 2$ holds in each $(h, m, k)$-semigroup.

**Proof.** Let $(S, \cdot)$ be an $(h, m, k)$-semigroup with Brandt series (2.1). Consider the Rees quotient $(S/S_0, \cdot)$. This is an $(h, m, k)$-semigroup whose Brandt series starts with the zero term. By Proposition 3.7, the semigroup $(S/S_0, \cdot)$ satisfies (3.9). Recall from Remark 1 that the word $v_{n,2^m}^{kh+h}$ is the image of the word $v_{n,2^m}^{(k+1)}$ under the substitution that sends every variable $x_{i_1,\ldots,i_k} \in X_{2^n}^{(k)}$ to the word $v_{n,2^m}^{kh+h}$ obtained from $v_{n,2^m}^{(k+1)}$ by renaming its variables. Then, for every substitution $\tau: X_{2^n}^{(k)} \to S$, the element $\tau(v_{n,m,i_1,\ldots,i_k}^{kh+h})$ is an idempotent of $(S/S_0, \cdot)$. Therefore, $\tau(v_{n,m,i_1,\ldots,i_k}^{kh+h})$ lies in either $E(S) \setminus S_0$ or $S_0$. Recall that by the definition of an $(h, m, k)$-semigroup, $(S_0, \cdot)$ is an $[m, k]$-group. By e the identity element of this group. For each $f \in E(S)$, the product $fe$ lies in $S_0$ because $S_0$ is an ideal in $S$. Then $fe$ is an idempotent in $S_0$ by Lemma 2.4 which applies since $(S, \cdot)$ is a periodic block-group by Lemmas 3.2 and 2.8. Hence $fe = e$ since a group has no idempotents except its identity element. Consequently, for every $g \in S_0$, we have $fg = feg = eg = g$. Dually, $gf = g$ for all $g \in S_0$, $f \in E(S)$. Now consider the substitution $\tau: X_{2^n}^{(k)} \to S_0$ defined by

$$\tau(x_{i_1,\ldots,i_k}) := \begin{cases} \tau(v_{n,m,i_1,\ldots,i_k}^{kh+h}) & \text{if } \tau(v_{n,m,i_1,\ldots,i_k}^{kh+h}) \text{ belongs to } S_0, \\ e & \text{otherwise}, \end{cases}$$

for each $i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, 2^n\}$. As we know that $\tau(v_{n,m,i_1,\ldots,i_k}^{kh+h}) \in E(S) \cup S_0$ for all $i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, 2^n\}$ and $f \neq g = g$ for all $g \in S_0$, $f \in E(S)$, we conclude that $\tau(v_{n,2^m}^{kh+h+k}) = \tau(v_{n,2^m}^{kh+h+k})$. Since $(S_0, \cdot)$ is an $[m, k]$-group (and so a $[2^m, k]$-group), it satisfies the identity $v_{n,2^m}^{kh+h+k} = 1$ by Lemma 3.5. Hence $\tau(v_{n,2^m}^{kh+h+k}) = \tau(v_{n,2^m}^{kh+h+k}) = e$. Since the substitution $\tau$ is arbitrary, $(S, \cdot)$ satisfies the identity $v_{n,2^m}^{kh+h+k} = (v_{n,2^m}^{kh+h+k})^2$. □

Now we are ready to complete the mission of this section.

**Proposition 3.9.** For each finite block-group $(S, \cdot)$ with solvable subgroups, there exist positive integers $q$ and $r$ such that $(S, \cdot)$ satisfies the identities $v_{n,q}^{(r)} = (v_{n,q}^{(r)})^2$ for all $n \geq 2$.

**Proof.** Clearly, each finite semigroup possesses a principal series. By Corollary 2.7 every principal series of a block-group is a Brandt series. Let $h$ be the height of a Brandt series of $(S, \cdot)$, let $m$ be the least common multiple of the exponents of subgroups of $(S, \cdot)$, and let $k$ be the maximum of the derived lengths of these subgroups. Then $(S, \cdot)$ is an $(h, m, k)$-semigroup. Now Proposition 3.8 applies, showing that $(S, \cdot)$ satisfies all identities $v_{n,2^m}^{kh+h+k} = (v_{n,2^m}^{kh+h+k})^2$ with $n \geq 2$. Hence one can choose $q = 2^h m$ and $r = kh + k + h$. □

Proposition 3.9 gives a vast generalization of [7] Proposition 2.4 where a similar result was established for finite inverse semigroups with abelian subgroups.
4. PROOFS OF THE MAIN RESULTS

Recall that the 6-element Brandt monoid \( B_2^1 \cdot \) is formed by the following zero-one \( 2 \times 2 \)-matrices

\[
\begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix},
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix},
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 \\
1 & 0
\end{pmatrix},
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}
\]

under the usual matrix multiplication \( \cdot \). In terms of the construction for Brandt semigroups presented in Sect. 2 \((B^1_2, \cdot)\) is obtained by adjoining the identity element to the Brandt semigroup over the trivial group with the 2-element index set. It is known and easy to verify that the set \( B_2^1 \) admits a unique addition \(+\) such that \( B_2^1 := (B_2^1, +, \cdot) \) becomes an ai-semiring. The addition is nothing but the Hadamard (entry-wise) product of matrices:

\[(a_{ij}) + (b_{ij}) := (a_{ij}b_{ij}).\]

The identities of the semiring \( B_2^1 \) were studied by the second author \cite{P20} and, independently, by Jackson, Ren, and Zhao \cite{J11}; it was proved that these identities admit no finite basis. Moreover, in \cite{P9} we proved that this property of \( B_2^1 \) is very contagious: if all identities of an ai-semiring hold in \( B_2^1 \), then under some rather mild conditions, the identities have no finite basis. Here is the key result of \cite{P9} stated in the notation adopted here.

**Theorem 4.1** (\cite{P9} Theorem 4.2). Let \( I \) be an ai-semiring whose multiplicative reduct satisfies the identities \( v_{n,q}^{(r)} = (v_{n,q}^{(r)})^2 \) for all \( n \geq 2 \) and some \( q, r \geq 1 \). If all identities of \( I \) hold in the ai-semiring \( B_2^1 \), then the identities admit no finite basis.

In \cite{P9}, Theorem 4.1 was applied to finite ai-semirings whose multiplicative reducts were inverse semigroups with abelian subgroups. Due to Proposition 3.9 we are in a position to enlarge its application range as follows.

**Theorem 4.2.** Let \( I = (S, +, \cdot) \) be a finite ai-semiring whose multiplicative reduct is a block-group with solvable subgroups. If all identities of \( I \) hold in the ai-semiring \( B_2^1 \), then the identities admit no finite basis.

**Proof.** Proposition 3.9 implies that the semigroup \( (S, \cdot) \) satisfies the identities \( v_{n,q}^{(r)} = (v_{n,q}^{(r)})^2 \) for all \( n \geq 2 \) and some \( q, r \geq 1 \), and therefore, Theorem 4.1 applies. \( \square \)

We are ready to deduce Theorems 1.1 and 1.2.

**Proof of Theorem 1.** We aim to prove that for any finite non-abelian solvable group \( (G, \cdot) \), the identities of the power group \( (\mathcal{P}(G), \cup, \cdot) \) admit no finite basis. It is shown in \cite{P19} Proposition 2.4, see also \cite{P11} Proposition 11.1.2, that the semigroup \( (\mathcal{P}(G), \cdot) \) is a block-group and all maximal subgroups of \( (\mathcal{P}(G), \cdot) \) are of the form \( N_G(H)/H \), where \( (H, \cdot) \) is a subgroup of \( (G, \cdot) \) and \( N_G(H) = \{ g \in G \mid gH = Hg \} \) is the normalizer of \( H \) in \( (G, \cdot) \). Hence each subgroup of \( (\mathcal{P}(G), \cdot) \) embeds in a quotient of a subgroup of \( (G, \cdot) \) and so it is solvable. We see that the multiplicative reduct of \( (\mathcal{P}(G), \cup, \cdot) \) is a block-group with solvable subgroups.

First suppose that some subgroup \( (H, \cdot) \) is not normal in the group \( (G, \cdot) \). Then, of course, \( 1 < |H| < |G| \) and one can choose an element \( g \in G \) such that \( g^{-1}Hg \neq H \). Denote by \( e \) the identity element of \( (G, \cdot) \) and let \( E := \{ e \} \). Define the following subset of \( \mathcal{P}(G) \):

\[ B := \{ E, H, g^{-1}H, Hg, g^{-1}Hg \} \cup J, \]

where \( J := \{ A \subseteq G \mid |A| > |H| \} \). Observe that \( g^{-1}H \neq Hg \); indeed \( g^{-1}H \cdot Hg = g^{-1}Hg \) while \( Hg \cdot g^{-1}H = H \), so that the assumption \( g^{-1}H = Hg \) forces the equality \( g^{-1}Hg = H \).
that contradicts the choice of \( g \). It is easy to verify that \( B \) is closed under union and elementwise multiplication of subsets so that \( (B, \cup, \cdot) \) is a subsemiring in \((\mathcal{P}(G), \cup, \cdot)\). The set \( J \) is an ideal in both \((B, \cup)\) and \((B, \cdot)\) whence one can form the Rees quotient \((B/J, \cup, \cdot)\). This 6-element Rees quotient is isomorphic to the ai-semiring \( \mathcal{B}_2^1 \) via the following map:

\[
\begin{array}{ccccccc}
J & E & Hg & g^{-1}H & H & g^{-1}Hg \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
(0 \ 0) & (1 \ 0) & (0 \ 1) & (0 \ 0) & (1 \ 0) & (0 \ 0) & (0 \ 1) \\
\end{array}
\]

We omit the verification that is routine.

Since \( \mathcal{B}_1 \) is isomorphic to a Rees quotient of a subsemiring of \((\mathcal{P}(G), \cup, \cdot)\), all identities of \((\mathcal{P}(G), \cup, \cdot)\) hold in \( \mathcal{B}_1 \). Thus, Theorem 4.2 applies to \((\mathcal{P}(G), \cup, \cdot)\), showing that its identities admit no finite basis.

If all subgroups of \((G, \cdot)\) are normal, then since \((G, \cdot)\) is non-abelian, it has a subgroup isomorphic to the 8-element quaternion group; see [9, Theorem 12.5.4]. This subgroup is isomorphic to a subgroup in \((\mathcal{P}(G), \cdot)\), and by [11, Theorem 6.1] the identities of every ai-semiring whose multiplicative reduct possesses a non-abelian nilpotent subgroup admit no finite basis. We see that \((\mathcal{P}(G), \cup, \cdot)\) has no finite identity basis in this case as well. \( \square \)

**Remark 2.** Given a semigroup \((S, \cdot)\), one sometimes considers, along with the power semigroup \((\mathcal{P}(S), \cdot)\), its subsemigroup \((\mathcal{P}'(S), \cdot)\) where \(\mathcal{P}'(S)\) is the set of all *non-empty* subsets of \(S\); see, e.g., [1, Chapter 11]. Clearly, the empty set serves as a zero in power semigroups so that one can think of \((\mathcal{P}'(S), \cdot)\) as the result of adjoining a new zero to \((\mathcal{P}(S), \cdot)\). It is well-known that the property of a semigroup to have or not to have a finite identity basis does not change when a new zero is adjoined; see [28, Section 3]. Therefore \((\mathcal{P}'(S), \cdot)\) and \((\mathcal{P}'(S), \cdot)\) admit or do not admit a finite identity basis simultaneously.

Since the set \(\mathcal{P}'(S)\) is closed under union, one can consider the semiring \((\mathcal{P}'(S), \cup, \cdot)\). The power semiring \((\mathcal{P}(S), \cup, \cdot)\) can be thought of as the result of adjoining a new element, which is both an additive identity and multiplicative zero, to \((\mathcal{P}'(S), \cup, \cdot)\). However, it is not yet known whether adjoining such an element preserves the property of having/not having a finite identity basis. Nevertheless, the above proof of Theorem 6.1 applies without a hitch to ai-semirings of the form \((\mathcal{P}'(G), \cup, \cdot)\) where \((G, \cdot)\) is any finite non-abelian solvable group and shows that the identities of every such ai-semiring admit no finite basis.

**Remark 3.** It is quite natural to consider \((\mathcal{P}(S), \cup, \cdot, 0)\) as an algebra of type \((2, 2, 0);\) this was actually the setting adopted by Dolinka [5]. One can verify that the conclusion of Theorem 6.1 persists in this case as well. The main part of the proof of Theorem 6.1 invokes our results from [7] via Theorem 4.2. These results work fine for ai-semirings with multiplicative zero treated as algebras of type \((2, 2, 0);\) this is explicitly registered in [7, Remark 3]. In the part of the proof that deals with the case where all subgroups of the group \((G, \cdot)\) are normal, one should refer to Theorem 7.6 from [11] that transfers the result of [11, Theorem 6.1] to the signature \{+, \cdot, 0\}.

**Proof of Theorem 4.2** The theorem claims that the identities of the semiring \((H(X), \cup, \cdot)\) of all Hall relations on a finite set \(X\) admit a finite basis if and only if \(|X| = 1\). The ‘if’ part is obvious, since if \(|X| = 1\), then \(|H(X)| = 1\), and the identities of the trivial semiring have a basis consisting of the single identity \(x = y\).

For the ‘only if’ part, assume that \(X = X_n := \{1, 2, \ldots, n\}\) and \(n \geq 2\). The semigroup \((H(X_n), \cdot)\) is a block-group; see, e.g., [6, Proposition 2]. It can be easily verified that the ai-semiring \(\mathcal{B}_2^1\) embeds into \((H(X_2), \cup, \cdot)\) via the following map:
The matrices in the bottom row are matrices over the Boolean semiring $\mathbb{B} := (\{0,1\},+\cdot)$ with $0 \cdot 1 = 1 \cdot 0 = 0 + 0 = 0$, $1 \cdot 1 = 1 + 0 = 0 + 1 = 1 + 1 = 1$, and we mean the standard representation of binary relations as matrices over the Boolean semiring (in which Hall relations correspond to matrices with permanent 1). Obviously, the ai-semiring $\mathbb{A}_i$-semiring the standard representation of binary relations as matrices over the Boolean semiring (in identities of $(H(X_n),\cup,\cdot)$ for each $n \geq 2$, whence so does $\mathcal{B}_1^2$. Therefore all identities of $(H(X_n),\cup,\cdot)$ hold in $\mathcal{B}_1^2$.

Let $\text{Sym}_n$ stand for the group of all permutations of the set $X_n$. It is known that all subgroups of the semigroup of all binary relations on $X_n$ embed into $\text{Sym}_n$; see, e.g., [18, Theorem 3.5]. In particular, all subgroups of the semigroup $(H(X_n),\cdot)$ with $n \leq 4$ embed into $\text{Sym}_n$, and hence, are solvable. We see that Theorem 4.2 applies to the ai-semiring $(H(X_n),\cup,\cdot)$ with $n = 2, 3, 4$, whence the identities of the ai-semiring admit no finite basis.

To cover the case $n > 4$, observe that the semigroup $(H(X_n),\cdot)$ has $\text{Sym}_n$ as its group of units, and for $n > 4$ (even for $n \geq 4$), this group possesses non-abelian nilpotent subgroups, for instance, the 8-element dihedral group. By [11, Theorem 6.1] the ai-semiring $(H(X_n),\cup,\cdot)$ has no finite identity basis in this case as well. \hfill $\Box$

In the introduction, we mentioned Dolinka’s problem [5, Problem 6.5] as the main motivation of our study. Recall that the problem asks for a description of finite groups whose power groups have no finite identity basis. Combining Theorem 1.1 of the present paper and Theorem 6.1 of [11] reduces the problem to two special cases.

First, the problem remains open for non-solvable groups whose nilpotent subgroups are abelian. The structure of such groups is well understood, see [2]; an interesting example here is the sporadic simple group of order 175560 found by Janko [12]. While arguments in the proof of Theorem 1.1 do not cover this case, we conjecture that the conclusion of the theorem persists. In other words, we believe that the power group of any finite non-abelian group has no finite identity basis.

Second, the problem is open for a majority of abelian groups. This may seem surprising as power groups of abelian groups have commutative multiplication, but finite ai-semirings with commutative multiplication and without finite identity basis are known to exist, see [11, Corollary 4.11]. This does not exclude, of course, that all power groups of finite abelian groups admit a finite identity basis, but to the best of our knowledge, the only group for which such a basis is known is the 2-element group. Namely, Ren and Zhao [24] have shown that the identities $xy = yx$ and $x = x^3$ form a basis for identities of a 4-element ai-semiring that can be easily identified with the power group of the 2-element group.

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