BOUNDS ON SOME EDGE FOLKMAN NUMBERS *

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Abstract

The edge Folkman numbers are defined by the equality

\[ F_e(a_1, \ldots, a_r; q) = \min \{|V(G)| : G \not\rightarrow (a_1, \ldots, a_r; q) \text{ and } cl(G) < q \}. \]

Lin proved that if \( R(a_1, a_2) = R(a_1-1, a_2) + R(a_1, a_2-1) \) then \( F_e(a_1, a_2; R(a_1, a_2)) = R(a_1, a_2) + 2 \), where \( R(a_1, a_2) \) is the Ramsey number showing that \( K_{R(a_1, a_2) - 3 + C_5} \rightarrow (a_1, a_2) \) where \( C_5 \) is the simple cycle on 5 vertices. We prove some upper bounds on edge Folkman numbers for which \( R(a_1, a_2) < R(a_1-1, a_2) + R(a_1, a_2-1) \) and we cite some lower bounds.

1 1. Introduction

Only finite non-oriented graphs without multiple edges and loops are considered. We call a \( p \)-clique of the graph \( G \) a set of \( p \) vertices each two of which are adjacent. The largest positive integer \( p \) such that \( G \) contains a \( p \)-clique is denoted by \( cl(G) \). A set of vertices of the graph \( G \) none two of which are adjacent is called an independent set. The largest positive integer \( p \) such that \( G \) contains an independent set on \( p \) vertices is called the independence number of the graph \( G \) and is denoted by \( \alpha(G) \). In this paper we shall also use the following notations:

- \( V(G) \) is the vertex set of the graph \( G \);
- \( E(G) \) is the edge set of the graph \( G \);
- \( N(v), v \in V(G) \) is the set of all vertices of \( G \) adjacent to \( v \);
- \( G[V], V \subseteq V(G) \) is the subgraph of \( G \) induced by \( V \);
- \( K_n \) is the complete graph on \( n \) vertices;
- \( \overline{G} \) is the complementary graph of \( G \).

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Let $G_1$ and $G_2$ be two graphs without common vertices. We denote by $G_1 + G_2$ the graph $G$ for which $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup E'$ where $E' = \{ xy : x \in V(G_1), y \in V(G_2) \}$. It is clear that
\[
cl(G_1 + G_2) = cl(G_1) + cl(G_2). \tag{1}
\]

**Definition 1.** Let $a_1, \ldots, a_r$ be positive integers. The symbol $G \not \rightarrow (a_1, \ldots, a_r)$ means that in every $r$-coloring of $V(G)$ there is a monochromatic $a_i$-clique in the $i$-th color for some $i \in \{1, \ldots, r\}$.

**Definition 2.** Let $a_1, \ldots, a_r$ be positive integers. We say that an $r$-coloring of $E(G)$ is $(a_1, \ldots, a_r)$-free if for each $i = 1, \ldots, r$ there is no monochromatic $a_i$-clique in the $i$-th color. The symbol $G \not \rightarrow (a_1, \ldots, a_r)$ means that there is no $(a_1, \ldots, a_r)$-free coloring of $E(G)$.

The smallest positive integer $n$ for which $K_n \not \rightarrow (a_1, \ldots, a_r)$ is called a Ramsey number and is denoted by $R(a_1, a_2)$. Let us denote $R(1, a_2)$ as $R(a_2)$. Note that the Ramsey number $R(a_1, a_2)$ can be interpreted as the smallest positive integer $n$ such that for every $n$-vertex graph $G$ either $cl(G) \geq a_1$ or $\alpha(G) \geq a_2$. The existence of such numbers was proved by Ramsey in [18]. We shall use only the values $R(3, 3) = 6$ and $R(3, 4) = 9$, [3].

The edge Folkman numbers are defined by the equality
\[
F_e(a_1, \ldots, a_r; q) = \min \{|V(G)| : G \not \rightarrow (a_1, \ldots, a_r) \text{ and } cl(G) < q\}.
\]
It is clear that $G \not \rightarrow (a_1, \ldots, a_r)$ implies $cl(G) \geq \max\{a_1, \ldots, a_r\}$. There exists a graph $G$ such that $G \not \rightarrow (a_1, \ldots, a_r)$ and $cl(G) = \max\{a_1, \ldots, a_r\}$. In the case $r = 2$ this was proved in [1] and in the general case in [16]. Therefore
\[
F_e(a_1, \ldots, a_r; q) \text{ exists if and only if } q > \max\{a_1, \ldots, a_r\}.
\]
It follows from the definition of $R(a_1, \ldots, a_r)$ that
\[
F_e(a_1, \ldots, a_r; q) = R(a_1, \ldots, a_r) \text{ if } q > R(a_1, \ldots, a_r).
\]

The smaller the value of $q$ in comparison to $R(a_1, \ldots, a_r)$ the more difficult the problem of computing the number $F_e(a_1, \ldots, a_r; q)$. Only ten edge Folkman numbers that are not Ramsey numbers are known. For the results see the papers: [2], [4], [6], [7], [9], [10], [11], [13].

## 2 Upper bounds on the edge Folkman numbers

We obtained the following result:

**Theorem 1** Let $a$ and $\alpha$ be nonnegative integers. Let us denote $R = R(3, a)$. Let $R = R(3, a - 1) + a - \alpha$ and $R - 3a + \alpha + 5 \geq R(3, a - 2)$, $a \geq 4$. If there exists a graph $U$ with the properties
\[
cl(U) = a - 1
\]
\[
U \not \rightarrow (a - 1, a - 2)
\]
\[
U \not \rightarrow (a - 3, \ldots, a - 3, 3) \text{ times}
\]
\[
1 \text{ times}
\]
Then

\[ F_\epsilon(3, a; R - a + \alpha + 4) \leq R - 2a + \alpha + 4 + |V(U)|. \]

Proof Consider the graph \( G = K_{R - 2a + \alpha + 4} + U \) where \( U \) is the graph from the statement of the theorem. We will prove that \( G \sim (3, a) \) and thus the theorem will be proved (here we use (1) to compute that \( cl(G) = R - 2a + \alpha + 4 + cl(U) = R - a + \alpha + 3 \).

Assume the opposite: that there exists a \((3, a)\)-free 2-coloring of \( E(G) \)

\[ E(G) = E_1 \cup E_2, \quad E_1 \cap E_2 = \emptyset. \]  

(2)

We shall call the edges in \( E_1 \) blue and the edges in \( E_2 \) red.

We define for an arbitrary vertex \( v \in V(G) \) and index \( i = 1, 2 \):

\[ N_i(v) = \{ x \in N(v) \mid [v, x] \in E_i \}, \]

\[ G_i(v) = G[N_i(v)] \]

\[ A_i(v) = N_i(v) \cap V(U) \]

Let \( H \) be a subgraph of \( G \). We say that \( H \) is a monochromatic subgraph in the blue-red coloring (2) if \( E(H) \subseteq E_1 \) or \( E(H) \subseteq E_2 \). If \( E(H) \subseteq E_1 \) we say that \( H \) is a blue subgraph and if \( E(H) \subseteq E_2 \) we say that \( H \) is a red subgraph.

It follows from the assumption that the coloring (2) is \((3, a)\)-free that

\[ cl(G_1(v)) \leq a - 1 \text{ and } cl(G_2(v)) \leq R(3, a - 1) - 1 \text{ for each } v \in V(G) \]  

(3)

Indeed, assume that \( cl(G_1(v)) \geq a \). Then there must be no blue edge connecting any two of the vertices in \( G_1(v) \) because otherwise this blue edge together with the vertex \( v \) would give a blue triangle. As we assumed \( cl(G_1(v)) \geq a \) then we have a red \( a \)-clique. Analogously assume \( cl(G_2(v)) \geq R(3, a - 1) \). Then we have either a blue \( 3 \)-clique or a red \((a - 1)\)-clique in \( G_2(v) \). If we have a blue \( 3 \)-clique in \( G_2(v) \) then we are through. If we have a red \((a - 1)\)-clique then this \((a - 1)\)-clique together with the vertex \( v \) gives a \( a \)-clique. Thus (3) is proved.

We shall prove that

\[ cl(G[A_1(v)]) + cl(G[A_2(v)]) \leq 2a - 5 \text{ for each } v \in V(K_{R - 2a + \alpha + 4}) \]  

(4)

Assume that (4) is not true, i.e. that there exists a vertex \( v \in V(K_{R - 2a + \alpha + 4}) \) such that

\[ cl(G[A_1(v)]) + cl(G[A_2(v)]) \geq 2a - 4. \]

Then as there are \( R - 2a + \alpha + 3 \) more vertices in \( V(K_{R - 2a + \alpha + 4}) \) with the exception of \( v \), it follows that

\[ cl(G_1(v)) + cl(G_2(v)) \geq R - 2a + \alpha + 3 + 2a - 4 = R + \alpha - 1 = R(3, a - 1) + a - 1 \]

(here we use the statement of the theorem that \( R(3, a) = R(3, a - 1) + a - \alpha \).)

This contradicts (3). Thus (4) is proved.

Now we shall prove that

\[ cl(G[A_1(v)]) = a - 1 \text{ or } cl(G[A_2(v)]) = a - 1 \text{ for each } v \in V(K_{R - 2a + \alpha + 4}) \]  

(5)

Assume that (5) is not true. Then we obtain from \( cl(U) = a - 1 \) that

\[ cl(G[A_1(v)]) \leq a - 2 \text{ and } cl(G[A_2(v)]) \leq a - 2 \text{ for some } v \in V(K_{R - 2a + \alpha + 4}). \]  

(6)
It follows from the statement of the theorem that $U \rightarrow (a-1, a-2)$ that in every 2-coloring of $V(U)$, in which there are no $(a-1)$-cliques in none of the two colors then there are $(a-2)$-cliques in the both colors. Hence $A_1(v)$ and $A_2(v)$ contain $(a-2)$-clique. Therefore the inequalities in (6) are in fact equalities, which contradicts (4). Thus (5) is proved. Let us note that it follows from (5) and (4) that for each $v \in V(K_{R−2a+α+4})$

$$\text{cl}(G[A_2(v)]) = a−1 \text{ then } \text{cl}(G[A_1(v)]) \leq a−4.$$  

(7)

If we assume that there are $a$ vertices $v$ in $V(K_{R−2a+α+4})$ with the property $\text{cl}(G[A_1(v)]) = a−1$ then it follows from (3) that there are only red edges between these $a$ vertices and hence this is a red $a$-clique. Therefore having in mind the statement of the theorem there are at least $R−2a + α + 4 − (a−1) \geq R(3, a−2)$ vertices $v$ in $V(K_{R−2a+α+4})$ with the property $\text{cl}(G[A_1(v)]) \leq a−2$. Let us denote the set of these vertices by $S$. It follows by (5) that $\text{cl}(G[A_2(v)]) = a−1$ for each $v \in S$. Then we obtain from (7) that

$$\text{cl}(G[A_1(v)]) \leq a−4 \text{ for each } v \in S.$$  

(8)

As there are no blue 3-cliques among the vertices in $S$ there is a red $(a−2)$-clique among them. Let us denote the vertices of this $(a−2)$-clique by $w_1, w_2, \ldots, w_{a−2}$. Let us partition the vertices of $U$ in $a−1$ colors in the following way:

$$V_1 = A_1(w_1)$$

and for $j ∈ 2, \ldots, a−2$

$$V_j = A_1(w_j) \setminus V_1 \cup \ldots \cup V_{j−1}$$

$$V_{a−1} = V(U) \setminus V_1 \cup \ldots \cup V_{a−2}.$$

According to (8) it is impossible $V_j$, $j ∈ 1, \ldots, a−2$ to contain a $(a−3)$-clique because according to its definition $V_j$ is a subset of $A_1(w_j)$. But we know from the statement of the theorem that $U \rightarrow (a−3, \ldots, a−3, 3)$. Hence $V_{a−1}$ contains a $3$-

$a-2$ times

clique. Then this $3$-clique must contain a red edge (otherwise it is a blue $3$-clique and we are through). But it follows from the definition of $V_{a−1}$ that all the edges between the vertices in $V_{a−1}$ and the vertices $w_1, \ldots, w_{a−2}$ are red. Then the red edge in $V_{a−1}$ and the red $(a−2)$-clique $w_1, \ldots, w_{a−2}$ form a red $a$-clique. Thus the theorem is proved.

If we put $α = 0$, $a = 5$, $U = Q$, where $Q$ denotes the graph whose complementary is the graph given here we obtain the following Kolev’s result

$$F_ε(3; 5; 13) \leq 21, [5].$$

What is novel in this result is that for the first time the graph $Q$ is used in the theory of edge Folkman numbers.

We also prove the following result which unfortunately is not a particular case of

**Theorem 1:**

**Theorem 2** $F_ε(4; 4; 17) \leq 25$.

*Proof:* Consider the graph $G = K_{12} + Q$ and we shall show that $G \rightarrow (4, 4)$. Assume the opposite: that there exists a $(4, 4)$-free 2-coloring of $E(G)$

$$E(G) = E_1 \cup E_2, \quad E_1 \cap E_2 = \emptyset.$$  

(9)
We shall call the edges in $E_1$ blue and the edges in $E_2$ red.

We define for an arbitrary vertex $v \in V(G)$ and index $i = 1, 2$:

$$N_i(v) = \{ x \in N(v) \mid [v, x] \in E_i \},$$

$$G_i(v) = G[N_i(v)],$$

$$A_i(v) = N_i(v) \cap V(Q).$$

We say that $H$ is a monochromatic subgraph in the blue-red coloring (9) if $E(H) \subseteq E_1$ or $E(H) \subseteq E_2$. If $E(H) \subseteq E_1$ we say that $H$ is a blue subgraph and if $E(H) \subseteq E_2$ we say that $H$ is a red subgraph.

It follows from the assumption that the coloring (9) is $(4,4)$-free and $R(3,4) = 9$ that

$$\text{cl}(G_1(v)) \leq 8 \text{ and } \text{cl}(G_2(v)) \leq 8 \text{ for each } v \in V(G) \quad (10)$$

Assume $\text{cl}(G_2(v)) \geq 9$. Since $R(3,4) = 9$, then we have either a blue 3-clique or a red 4-clique in $G_1(v)$. If we have a blue 3-clique in $G_1(v)$ then this blue 3-clique together with the vertex $v$ forms a blue 4-clique. If we have a red 4-clique then we are through. Analogously we disprove $\text{cl}(G_1(v)) \geq 9$. Thus (10) is proved.

We shall prove that

$$\text{cl}(G[A_1(v)]) + \text{cl}(G[A_2(v)]) \leq 5 \text{ for each } v \in V(K_{12}) \quad (11)$$

Assume that (11) is not true, i.e. that there exists a vertex $v \in V(K_{12})$ such that

$$\text{cl}(G[A_1(v)]) + \text{cl}(G[A_2(v)]) \geq 6.$$

Then as there are eleven more vertices in $V(K_{12})$ with the exception of $v$, it follows that

$$\text{cl}(G_1(v)) + \text{cl}(G_2(v)) \geq 17.$$

It follows from the pigeonhole principle that either $\text{cl}(G_1(v)) \geq 9$ or $\text{cl}(G_2(v)) \geq 9$, which contradicts (10). Thus (11) is proved.
It is clear from the definition of $V$.

It follows from (11) and (12) that

if $cl(G[A_1(v)]) = 4$ then $cl(G[A_2(v)]) \leq 1$ for each $v \in V(K_{12})$. 

Assume that there are two vertices $a, b \in V(K_{12})$ such that the edge $ab$ is blue and $cl(G[A_1(a)]) = cl(G[A_1(b)]) = 4$. Then it follows from (13) that $cl(G[A_2(a)]) \leq 1$ and $cl(G[A_2(b)]) \leq 1$. Let us consider the following 3-coloring of $V(Q)$:

$$V_1 = A_2(a)$$
$$V_2 = A_2(b) \setminus A_1(a)$$
$$V_3 = V(Q) \setminus (V_1 \cup V_2).$$

It is clear from the definition of $V_1, V_2, V_3$ that $V_3$ consists of such vertices that are connected with both $a$ and $b$ with blue edges. It is clear from $cl(G[A_2(a)]) \leq 1$ and $cl(G[A_2(b)]) \leq 1$ that $V_1$ and $V_2$ contain no 2-cliques. But we know from [8] that $Q \not\rightarrow (2, 2, 4)$ and hence $V_3$ contains a 4-clique. If this 4-clique has a blue edge then this blue edge together with the edge $ab$ is a blue 4-clique. If this 4-clique does not have a blue edge then it is a red 4-clique- a contradiction. Thus we obtained that

if $cl(G[A_1(a)]) = cl(G[A_1(b)]) = 4$ then the edge $ab$ is not blue, $a, b \in K_{12}$. 

(14)

Analogously we obtain

if $cl(G[A_2(a)]) = cl(G[A_2(b)]) = 4$ then the edge $ab$ is not red, $a, b \in K_{12}$. 

(15)

We have from (12) that $cl(G[A_1(v)]) = 4$ or $cl(G[A_2(v)]) = 4$ for each $v \in K_{12}$. For a fixed $v \in K_{12}$ we may assume without loss of generality that $cl(G[A_1(v)]) = 4$. Assume that for this $v$ there exist three distinct vertices $b_1, b_2, b_3 \in K_{12}$ such that the edges $vb_i, i \in \{1, 2, 3\}$ are blue. Now it follows from (13), (14), (15) and the fact that the edges $vb_i, i \in \{1, 2, 3\}$ are blue that

$$cl(G[A_2(b_i)]) = 4 \text{ and } cl(G[A_1(b_i)]) \leq 1$$

(16)

for $i \in \{1, 2, 3\}$. Now it follows from (15) that the edges $b_ib_j$ are blue for $i, j \in \{1, 2, 3\}$. Hence $vb_1b_2b_3$ is a blue 4-clique and the theorem is proved.

Thus we obtained that there are no more than two blue edges from the vertex $v$. Therefore there are at least 9 vertices connected to $v$ with red edges which contradicts (10). We proved that $G \not\rightarrow (4, 4)$. As $cl(G) = 4$ then we obtain that $F_v(4, 4; 17) \leq 25$, which we wanted to prove.

Remark So far it was known by [1] that $F_v(4, 4; 17) < \infty$.

The last known lower bound of the edge Foklman number $F_v(4, 4; 17)$ is $F_v(4, 4; 17) \geq 22$ - see [15].
3 References

[1] Folkman, J. Graphs with monochromatic complete subgraphs in every edge coloring. *SIAM J. Appl. Math.* 18, 1970, 19–24.
[2] Graham R.L. On edgewise 2-colored graphs with monochromatic triangles containing no complete hexagon. *J. Comb. theory* 4, 1968, 300.
[3] Greenwood, R., A. Gleason. Combinatorial relation and chromatic graphs. *Canad. J. Math.*, 7, 1955, 1–7.
[4] Lin S. On Ramsey numbers and $K_r$-coloring of graphs. *J. Comb. theory Ser B* 12, 1972, 82-92.
[5] Kolev N. New upper bound for the edge Folkman number $F_e(3, 5; 13)$. *Serdica math. J.*, 34, 2008, 783-790.
[6] Kolev, N., N. Nenov. An example of 16-vertex Folkman edge $(3,4)$-graph without 8-cliques. *Annuaire Univ. Sofia Fac. Math. Inform.*, 98, 2008 101-126, see http://arxiv.org/PS_cache/math/pdf/0602/0602249v1.pdf.
[7] Kolev N., N. Nenov. The Folkman number $F_e(3,4; 8)$ is equal to 16. *Compt. rend. Acad. bulg. Sci.*, 59, No 1, 2006, 25-30.
[8] Nenov N. On the vertex Folkman number $F(3,4)$. *Compt. rend. Acad. bulg. Sci.*, 54, 2, 2001, 23-26.
[9] Nenov N. On an assumption of Lin about Ramsey-Graham-Spencer numbers. (Russian) *Compt. rend. Acad. bulg. Sci.*, 33, 9, 1980, 1171-1174.
[10] Nenov N. Generalization of a certain theorem of Greenwood and Gleason on three-color colorings of the edges of a complete graph with 17 vertices. (Russian) *Compt. rend. Acad. bulg. Sci.*, 34, 1981, 1209-1212.
[11] Nenov N. Lower estimates for some constants related to Ramsey graphs. (Russian) *Annuaire Univ. Sofia Fac. Math. Inform.*, 75, 1984, 27-38.
[12] Nenov N. On the $(3,4)$-Ramsey graphs without 9-cliques. (Russian) *Univ. Sofia Fac. Math. Inform.*, 85, 1991, 71-81.
[13] Nenov N. An example of 15-vertex Ramsey $(3,3)$-graph with clique number 4. (Russian) *Compt. rend. Acad. bulg. Sci.*, 34, 1981, 1487-1489.
[14] Nenov N. On the Zykov numbers and some of its applications to Ramsey theory. (Russian) *Compt. rend. Acad. bulg Serdica Bulg. math. Publ.*, 9, 1983, 161-167.
[15] Nenov N. On the vertex Folkman numbers $F_v(2,\ldots, 2; q)$. *Serdica Math. J.*, 35, 2009 251-272.
[16] Nesetril J, Rodl V. The Ramsey property for graphs with forbidden complete subgraphs. *J. Comb Th.* B20, 1976, 243-249.
[17] Piwakowski K., Radziszowski S., Urbanski S. Computation of the Folkman number $F_e(3,3; 5)$. *J. Graph theory* 32, 1999, 41-49.
[18] Ramsey P. On a problem of formal logic. *Proc. London Math. Soc.*, 30, 1930, 264-286.

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