A note on axial symmetries

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Abstract
This note describes how to characterize and normalize an axial Killing field on a general Riemannian geometry or four-dimensional Lorentzian geometry. No global assumptions are necessary, such as that the orbits of the Killing field all have period $2\pi$. Rather, any Killing field that vanishes at at least one point necessarily has the expected global properties.

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Axial Killing fields play an important role in classical general relativity when defining the angular momentum of a black hole or a similar compact, gravitating body. The simplest example is the Komar angular momentum, which applies to regions of spacetime that admit a global axial Killing field $\phi^a$. The formula is

$$J_{\text{Komar}} := \frac{1}{16\pi} \oint_S V_a \phi_b \, dS^{ab} = \frac{1}{8\pi} \oint_S K_{ab} \, s^a \phi^b \, \epsilon,$$

(1)

where $S$ is a spacelike two-sphere, $dS^{ab}$ is the area bivector normal to $S$, $K_{ab}$ is the extrinsic curvature of a Cauchy surface $\Sigma$ containing $S$, $s^a$ is the spacelike normal to $S$ within $\Sigma$, and $\epsilon$ is the intrinsic area element on $S$. Similar integrals—such as the quasi-local formulae due to Brown and York [1] and for dynamical horizons [2, 3]—apply more generally, when only the intrinsic, two-dimensional metric on $S$ is symmetric. But the definitions do rely on the existence of such an intrinsic symmetry.

To interpret an integral like (1) as a physical angular momentum, the Killing field $\phi^a$ should have some global characteristics of an axial Killing field. For a general Riemannian geometry ($\mathcal{M}, g_{ab}$), these include:

(a) $\phi^a$ vanishes at at least one fixed point $p_0 \in \mathcal{M}$,
(b) every orbit of the flow $\Phi(t)$ generated by $\phi^a$ is either a fixed point or a circle, and
(c) every circular orbit closes only at integer multiples of a common period $t = t_0$. 

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One can scale such a Killing field $\phi^a$ by a constant so that $t_0 = 2\pi$, thereby giving integrals like (1) a definite value and effectively completing the definition of the physical angular momentum.

Conditions (b) and (c) above are of course global, and would be difficult to check explicitly in practice, for example, in numerical relativity. Fortunately, well-known results in Riemannian geometry (see, e.g., chapter 6 of [4]) provide a much simpler, local way to check them. Specifically, condition (a) implies that the orbits of $\Phi(t)$ have exactly the same global characteristics as those of the associated flow

$$\Phi'(t) := D_{\phi_0} \Phi(t) : T_{p_0} \mathcal{M} \rightarrow T_{p_0} \mathcal{M}$$

in the tangent space at the fixed point. The fundamental reason for this is that the symmetries $\Phi(t)$ map geodesics of $(\mathcal{M}, g_{ab})$ to other geodesics. In particular, recall the exponential map, which maps a tangent vector $v^a \in T_p \mathcal{M}$ to the point $p' \in \mathcal{M}$ at unit affine parameter along the unique geodesic starting at $p$ with initial velocity $v^a$:

$$\gamma(0) = p, \quad \gamma^a(0) = v^a \quad \text{and} \quad \gamma^a(s) \nabla_a \gamma^b(s) = 0 \quad \Rightarrow \quad \exp v := \gamma(1) \in \mathcal{M}. \quad (3)$$

Taking $p = p_0$ to be the fixed point, $\Phi(t)$ maps $\gamma(s)$ to another geodesic having the same initial position, but a rotated initial velocity. In mathematical terms,

$$\Phi(t) \circ \exp(v) = \exp \circ \Phi'(t)(v) \quad \text{for all} \quad v^a \in T_{p_0} \mathcal{M}. \quad (4)$$

This implies (at least when $\mathcal{M}$ is complete and path-connected) that every orbit of $\Phi(t)$ is the image under the exponential of at least one orbit of $\Phi'(t)$ in the Euclidean vector space $\left( T_{p_0} \mathcal{M}, g_{ab}(p_0) \right)$. This mapping of orbits might not be faithful in the sense that multiple orbits of $\Phi'(t)$ map to the same orbit of $\Phi(t)$, that non-trivial orbits of $\Phi'(t)$ map to fixed-point orbits of $\Phi(t)$, or that an orbit of $\Phi'(t)$ winds multiple times around the corresponding orbit of $\Phi(t)$. Regardless of these subtleties, however, if all orbits of $\Phi'(t)$ close, then all orbits of $\Phi(t)$ will as well. Moreover, a generic orbit of $\Phi(t)$ has the same period as (any one of the corresponding orbit(s) of $\Phi'(t)$.

It is comparatively straightforward to analyze the flow of $\Phi'(t)$ in the tangent space at the fixed point. Since each transformation fixes the origin of $T_{p_0} \mathcal{M}$, the $\Phi'(t)$ form a one-dimensional subgroup of the isometry group $\text{SO}(n)$ for $g_{ab}(p_0)$. In fact, this subgroup is generated by the covariant derivative of the Killing field at the fixed point:

$$\Phi'(t) = e^{tF} \quad \text{with} \quad F^b_a := \nabla_c \phi^b \big|_{p_0}. \quad (5)$$

The exponential here is in the usual sense of matrices. Now, by suitable choice of orthonormal basis, any anti-symmetric two-tensor on a Euclidean vector space can be put in canonical, block diagonal form

$$F^b_a \sim \begin{pmatrix} 0 & -f_1 & \cdots & \cdots & \cdots & -f_p \\ f_1 & 0 & \cdots & \cdots & \cdots & f_p \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ f_p & -f_1 & \cdots & \cdots & \cdots & 0 \end{pmatrix} \quad (6)$$
with $0 < f_k \leq \cdots \leq f_1$. The $\Phi^i(t)$ from (5) therefore describe simultaneous SO(2) rotations in a set of orthogonal two-planes in $T_{p_0}N, M$, each occurring at a potentially distinct frequency $f_k$ in the parameter $t$. It follows that all orbits of $\Phi^i(t)$ close if and only if all of the $f_k$ are integer multiples of a single frequency $f_0$, and that all orbits of $\Phi^i(t)$ have the same period if and only if all of the $f_k$ are equal. One can write the condition that all the $f_k$ are equal in a basis-independent way as

$$\left(\text{tr} \ F^2\right) F^3 = \left(\text{tr} \ F^4\right) F.$$  

(7)

Thus, any Killing field $\varphi^a$ on a Riemannian manifold $(M, g_{ab})$ that has a fixed point $p_0$, and whose covariant derivative at that fixed point satisfies (7), necessarily has the global properties from conditions (b) and (c) above. That is, these global properties of an axial Killing field are evident from its local properties near a fixed point.

We conclude with several comments.

First, as mentioned above, the preferred normalization for an axial Killing field $\varphi^a$ is such that the orbits of $\Phi^i(t)$ close at integer multiples of $t_0 = 2\pi$. Equivalently, it is such that $f_k = 1$ for all $k$ in (6), or simply

$$F^3 = -F.$$  

(8)

One can use the freedom to scale a Killing field by a constant factor to satisfy this normalization condition for any Killing field that has a fixed point where (7) holds.

Second, (7) holds identically for any anti-symmetric matrix in two or three dimensions. Indeed, (6) can only have one non-zero block along its diagonal in such cases, and thus only one non-zero $f_k$. It follows that every Killing field $\varphi^a$ with a fixed point on a two- or three-dimensional Riemannian manifold $(M, g_{ab})$ necessarily has the global characteristics from conditions (b) and (c) above.

Third, if $M \sim S^2$ is a topological two-sphere, then every vector field, and in particular every Killing field, vanishes at at least one point $p_0$. Thus, every Killing field on a topological two-sphere is axial, and can be normalized locally by imposing (8) at a fixed point. One can also prove this using the classical uniformization theorem because every Killing field $\varphi^a$ of $(S^2, g_{ab})$ is also a Killing field of the conformally-related round geometry $(S^2, \tilde{g}_{ab})$, which are all axial. The proof described here, however, is more direct, and in fact has been used to give an independent proof of the uniformization theorem based on Ricci flow [5, 6].

Fourth, we have focussed above on strictly Riemannian geometries, but the results extend more or less intact to the pseudo-Riemannian case and, in particular, the physically most important case of a Lorentzian four-manifold $(M, g_{ab})$. As before, if $g_{ab}$ admits a Killing field $\varphi^a$ with a fixed point $p_0 \in M$, then the exponential maps orbits of $\Phi^i(t)$ in the tangent space at the fixed point $p_0$ onto orbits of $\Phi^i(t)$ in $M$. The only difference from the Riemannian case is that $F^b := \nabla_a \varphi^b |_{p_0}$ may generate several different types of flow in the tangent space. These can be classified [7] based on the linear algebraic characteristics of $F^b$. The classification is straightforward in four dimensions using the spinor approach, where

$$F_{ab} = 2 \ i_{(A \ o_B)} \ e_{A'B'} + 2 \ i_{(A' \ o_B')} \ e_{AB} \quad \text{with} \quad t^A \ o_A = \lambda$$  

(9)

for some complex eigenvalue $\lambda$. If $\lambda \neq 0$, then $F^a_{ab}$ has a basis of eigenvectors consisting of the real null vectors $t^a := o^A \ o_A$ and $n^a := t^A \ o_A$, which have real eigenvalues $\pm 2 \ Re \ \lambda$, and the complex null vectors $\tilde{m}^a := t^A \ o_A$ and its complex conjugate $\tilde{\tilde{m}}^a$, which have imaginary eigenvalues $\pm 2i \ Im \ \lambda$. The corresponding flow $\Phi^i(t)$ describes boosts in the time-like plane spanned by $\tilde{m}^a$ and $\tilde{m}^a$, and simultaneous rotations in the space-like plane spanned by $\tilde{m}^a$ and $\tilde{\tilde{m}}^a$. If $\lambda = 0$, then $t^A \ o_A^A$ and $F_{ab} = 2 \ i_{[a \ \chi_b]}$ for some real, space-like vector $\chi^a$ orthogonal to
ell. The flow $\Phi'(t)$ generated by $F^a_{\ b}$ in this case corresponds to a ‘null rotation’ in the Newman–Penrose formalism [8–10]. In any event, selecting the purely rotational isometries in the tangent space amounts to restricting $\lambda \neq 0$ to be pure imaginary, which one can do by demanding that

$$\text{tr} \ F^2 < 0 \quad \text{and} \quad 2 \ \text{tr} \ F^4 = \left( \text{tr} \ F^2 \right)^2. \quad (10)$$

Any Killing field $\varphi^a$ on a Lorentzian four-manifold $(\mathcal{M}, g_{ab})$ that has a fixed point $p_0$, and whose derivative $F^b_{\ a} := \nabla_a \varphi^b |_{p_0}$ satisfies these conditions, is necessarily axial throughout $\mathcal{M}$. As in the Riemannian case, it can be normalized using (8).

Finally, one important feature of these results is that they suggest a natural and convenient way to normalize an approximate Killing field on a Riemannian geometry $(\mathcal{M}, g_{ab})$ that has no actual symmetries [11–13]. Namely, suppose that a candidate approximate Killing field $u^a$ can be found up to constant scaling on $\mathcal{M}$ and vanishes at at least one point, which again is guaranteed if $\mathcal{M} \sim S^2$. Then, one may define $F_{ab} := 2 V_{[a} \ u_{b]}$, and use (8) to normalize $u^a$ throughout $\mathcal{M}$ without having to compute the orbits of $u^a$ and ensure that they close [14].

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