ALMOST NORMAL HEegaard SURFACES

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Abstract. We present a new and shorter proof of Stocking’s result that any strongly irreducible Heegaard surface of a closed orientable triangulated 3–manifold is isotopic to an almost normal surface. We also re-prove a result of Jaco and Rubinstein on normal spheres. Both proofs are based on the “reduction” technique introduced by the author.

1. Introduction

Any closed orientable triangulated 3–manifold $M$ has a Heegaard surface, which is an embedded surface that decomposes $M$ into two handlebodies. By adding trivial handles, one can construct Heegaard surfaces of $M$ of arbitrarily large genus. An important problem is to construct a Heegaard surface of $M$ of minimal genus. Though this problem has an algorithmic solution for closed orientable atoroidal Haken manifolds by work of Johannson [5], it is still open for non-Haken manifolds. One approach to the non-Haken case is based on a result of Casson and Gordon [1]. It states that any minimal genus Heegaard surface $H$ of a closed orientable irreducible non-Haken manifold $M$ is strongly irreducible, i.e., any two simple closed essential curves on $H$ bounding embedded discs in different connected components of $M \setminus H$ intersect each other in at least two points. In fact any compact orientable 3–manifold can be decomposed along incompressible surfaces so that the pieces have strongly irreducible Heegaard surfaces with boundary [13].

Stocking [14] has shown that any strongly irreducible Heegaard surface is isotopic to a so-called almost normal surface, a mild generalization of normal surfaces that goes back to Rubinstein [11]. This is amazing, since normal surfaces, introduced by Kneser [9] in his study of connected sums of 3–manifolds, have been designed to deal with incompressible surfaces, whereas Heegaard surfaces bound two handlebodies and are thus completely compressible on both sides. The aim of this paper is to present a new and shorter proof of Stocking’s result.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure1.png}
\caption{A triangle, a square and an octagon}
\end{figure}
We follow in this paper the terminology of Matveev \[10\]. We define it in detail in Section 3 and here give just an outline. Let $M$ be a closed orientable 3–manifold with a triangulation $\mathcal{T}$. A closed embedded surface in $M$ is 2–normal if its intersection with any tetrahedron is formed by copies of the pieces shown in Figure 1, which are called normal triangles, squares and octagons. The surface is 1–normal if it is formed by normal triangles and squares only.

Let $\gamma$ be an unknotted arc for some 1–normal surface $F \subset M$, i.e., $\gamma$ is contained in a tetrahedron $t$; connects two different components of $F \cap t$; and there is an embedded disc in $t$ whose boundary consists of $\gamma$, one arc in $\partial t$ and two arcs in $F \cap t$. From $F$ we cut out small discs around the endpoints of $\gamma$, glue in an annulus (a “tube”) along $\gamma$ and denote the result by $F^\gamma$ — see Section 2 for a more general definition.

**Theorem 1.** Let $M$ be a closed orientable irreducible triangulated 3–manifold, and let $H \subset M$ be a strongly irreducible Heegaard surface. Either there is a 1–normal surface $F \subset M$ and an unknotted arc $\gamma$ such that $F^\gamma \simeq H$, or $H$ is isotopic to a 2–normal surface with exactly one octagon.

We briefly outline our proof. It is easy to show that there are 1–normal surfaces $F_-, F_+$ that can be “completed” to form disjoint surfaces $S_-, S_+ \simeq H$, by adding some tubes disjoint from the edges of $\mathcal{T}$ (possibly multiple, knotted, nested tubes). Here $S_-$ is a so-called upper trivial and $S_+$ a lower trivial surface. By application of a classical finiteness result of Kneser for 1–normal surfaces, we choose $F_-$ and $F_+$ so that any 1–normal surface with an upper (resp. lower) trivial completion isotopic to $H$ separating $S_-$ from $S_+$ is “essentially the same” as $F_-$ (resp. $F_+$).

If $S_-$ (or similarly $S_+$) satisfies a certain technical condition then we apply the reduction technique that we have developed in \[6\] and \[7\]. If the first alternative of Theorem 1 does not hold then the result of the reduction is isotopic to $H$ and is an upper trivial completion of some 1–normal surface different from $F_-$ (in contradiction to the choice of $F_-, F_+$).

If, instead, $S_-$ and $S_+$ do not satisfy the mentioned technical condition, we apply a thin position argument. If the first alternative of Theorem 1 does not hold, then it yields a surface $S \simeq H$ that is an upper or lower trivial completion of a 2–normal surface with exactly one octagon. Again, we apply the reduction technique to $S$, and derive a contradiction to the choice of $F_-, F_+$, unless the second alternative of Theorem 1 holds. Thus in either case, one alternative is true.

The main reason why our proof is shorter than Stocking’s is the application of the completion and reduction techniques. This makes thin position a more efficient tool. Actually we only need to know one basic fact about thin position: A level surface in a thin position embedding has no pair of nested or independent compressing discs. The reduction technique was used in \[6\] to obtain bounds for the bridge number of links formed by edges of a triangulation of $S^3$. It also allows to simplify the correctness proof of the Rubinstein–Thompson algorithm (\[15\], \[10\]), which is implicit in \[6\].

We took a few technical results from the literature. To make this paper essentially self-contained, we include proofs or at least proof sketches for most cited results. Specifically, we re-prove Scharlemann’s “no nesting” lemma \[12\], that is at the base of our completion technique. It is convenient for us, though not strictly necessary, to work in a submanifold $N \subset M$ whose complement is a 3–ball containing all vertices of $\mathcal{T}$, and any 1–normal sphere in $N$ is a copy of $\partial N$. It is known \[4\]
that such a submanifold exists for any triangulated irreducible 3–manifold different from $S^3$. We include here a proof that is another application of the reduction technique. An algorithm that can be used to construct the submanifold appears in 3, with a bound for the complexity of $\partial N$.

The paper is organized as follows. In Sections 2–4 we expose some basic constructions and the notions of almost $k$–normal and impermeable surfaces. In Section 5 we recall the reduction technique and prove the existence of the above mentioned submanifold $N$. In Section 6 we introduce upper and lower completions of surfaces and re-prove the “no nesting” lemma of Scharlemann. Theorem 4 is proved in Section 7.

2. Notations and basic notions

In this paper, $M$ denotes a closed orientable 3–manifold with a triangulation $T$. We denote the $i$–skeleton of $T$ by $T^i$, for $i = 0, \ldots, 3$. An ambient isotopy that fixes each simplex of $T^i$ setwise is an isotopy mod $T^i$. We denote the number of connected components of a topological space $X$ by $\#(X)$. The notation $X \subset M$ stands for a tame embedding of $X$ into $M$, and $U(X)$ denotes an open regular neighborhood of $X$ in $M$.

Let $S \subset M \setminus \partial^0$ be a closed surface, which is allowed to be empty or non-connected. The weight of $S$ is $\|S\| = \#(S \cap T^1)$. The surface $S$ is splitting if any connected component of $S$ decomposes $M$ into two parts. We fix a vertex $x_0 \in T^0$ and define $B^+(S)$ (resp. $B^-(S)$) as the closure of the union of components of $M \setminus S$ that are connectible with $x_0$ by a path transversely intersecting $S$ in an odd (resp. even) number of points. In particular, $B^-(\emptyset) = M$. We do not include $x_0$ in the notation “$B^+(S)$”, since in our applications the choice of $x_0$ plays no role.

Let $\Gamma \subset B^+(S)$ be an embedded graph, not necessarily transversal to $T^2$. We define $S^\Gamma = \partial(B^-(S) \cup U(\Gamma))$. Similarly, if $\Gamma \subset B^-(S)$ then we define $S^\Gamma = \partial(B^+(S) \cup U(\Gamma))$.

If $S$ is in general position to $T$ then $G(S)$ denotes the union of those connected components of $S \cap T^2$ that intersect $T^1$. For any tetrahedron $t$ of $T$, $G(S) \cap \partial t$ is a disjoint union of circles. Hence by inserting disjoint discs in $M \setminus T^2$, one obtains from $G(S)$ a closed embedded surface $S^x$ (pronounced “$S$ cut”), so that $S^x \cap T^2 = G(S) \subset S \cap T^2$ and $S^x \setminus T^2$ is a disjoint union of discs. One obtains $S^x$ from $S$ by cut-and-paste operations along discs in $M \setminus T^2$, and omission of connected components that are disjoint from $T^1$.

3. Almost $k$–normal surfaces

An embedded arc $\gamma$ in a closed 2–simplex $\sigma$ of $T$ disjoint from the vertices of $\sigma$ with $\gamma \cap \partial \sigma = \partial \gamma$ is a normal arc, if $\gamma$ connects different edges of $\sigma$. Otherwise, $\gamma$ is a return. A closed surface $S \subset M$ in general position to $T^2$ is almost $k$–normal, if $S \cap T^2$ is a union of normal arcs and of circles in $T^2 \setminus T^1$, and for any tetrahedron $t$ of $T$, any edge $e$ of $t$ and any connected component $c$ of $S \cap \partial t$ holds $\#(c \cap e) \leq k$.

Almost $k$–normal surfaces are difficult to deal with. For instance, if an almost $k$–normal surface meets a tetrahedron in an annulus then this annulus can be arbitrarily knotted. These problems do not occur if one makes the additional assumption that all intersections with tetrahedra are discs, as in the following definition.
Definition 1. Let \( F \subset M \) be an almost \( k \)-normal surface. If \( F \setminus T^2 \) is a disjoint union of discs and \( F \cap T^2 \) is a union of normal arcs in the 2–simplices of \( T \) then \( F \) is a \( k \)-normal surface.

Note that we do not assume \( F \neq \emptyset \). Actually it is convenient in our proofs to consider the empty set as a 1–normal surface. If \( S \subset M \) is an almost \( k \)-normal surface, then \( S^\times \) is \( k \)-normal. It is easy to show that \( S^\times \) is determined by \( S \cap T^1 \) up to isotopy mod \( T^2 \). We have \( S^\times = \emptyset \) if and only if \( \|S\| = 0 \).

In this paper, we mainly consider 1– and 2–normal surfaces. A 2–normal surface intersects a tetrahedron \( t \) of \( T \) in a disjoint union of triangles, squares and octagons, as in Figure 1. Up to isotopy mod \( T^2 \) there are four types of triangles (one for each vertex of \( t \)), three types of squares (one for each pair of opposite edges of \( t \)), and three types of octagons in \( t \). A 1–normal surface is formed by triangles and squares.

Theorem 2 (Lemma 4 in [3]). Let \( n \) be the number of tetrahedra of \( T \). Let \( F \subset M \) be a 1–normal surface with more than \( 20n \) connected components. Then two connected components of \( F \) are isotopic mod \( T^2 \). \( \square \)

This finiteness result goes back to Kneser [9]. There are better bounds for the number of components, but the proofs are more complicated than the proof of Lemma 4 in [3].

4. Compressing and essential discs

Here we study some properties of embedded discs whose interior is disjoint from \( T^1 \). In the next section, these discs will be used to construct isotopies of \( S \).

Definition 2. Let \( S \subset M \) be a closed surface in general position to \( T^1 \). Let \( \alpha \subset T^1 \setminus T^0 \) and \( \beta \subset S \) be embedded arcs with \( \partial \alpha = \partial \beta \). A compact embedded disc \( D \subset M \) in general position to \( S \) is a compressing disc for \( S \) with string \( \alpha \) and base \( \beta \), if \( \partial D = \alpha \cup \beta \) and \( D \cap T^1 = \alpha \).

Definition 3. Let \( S \subset M \) be a closed surface in general position to \( T^1 \). A compact embedded disc \( D \subset M \setminus T^2 \) is essential for \( S \), if \( \partial D \subset S \) is not null-homotopic in \( S \setminus T^2 \) and \( \#(D \cap S) < \#(D' \cap S) \) for any compact embedded disc \( D' \subset M \setminus T^2 \) bounded by \( \partial D \).

Let \( D \) be a compressing or essential disc for a splitting surface \( S \subset M \). If \( \partial D \cap S \) has a collar in \( D \cap B^\pm(S) \) (resp. \( D \cap B^-(S) \)) then \( D \) is an upper (resp. lower) compressing or essential disc. If \( D \cap S \subset \partial D \) then \( D \) is strict.

Obviously, \( k \)-normal surfaces have no essential discs. In fact, as stated in the following lemma, all discs in the complement of \( T^1 \) with boundary in a 1–normal surface are trivial.

Lemma 1 (Lemma 10 and 11 in [3]). A 1–normal surface \( F \subset M \) has no compressing discs. Let \( D \subset M \setminus T^1 \) be a closed embedded disc in general position to \( F \) with \( \partial D \subset F \). Then \( \partial D \) bounds a disc in \( F \setminus T^1 \).

Proof. We choose the disc \( D \) up to isotopy of the pair \( (D, \partial D) \) in \( (M \setminus T^1, F \setminus T^1) \) so that \( \#(\partial D \cap T^2) \) is minimal. By cut-and-paste arguments, there are no circles in \( D \cap T \). If \( \partial D \) is not contained in a single tetrahedron then there is a component \( D' \) of \( D \setminus T^2 \) such that \( \partial D' \cap F \) is a single arc, contained in some tetrahedron \( t \). Since \( F \) is 1–normal, \( \partial D' \cap F \) splits off a disc \( D'' \subset t \) from some component of \( F \cap t \) so that \( \partial D'' \cap T^1 = \emptyset \). By an isotopy of \( D \) with support in \( U(D'') \), one can remove \( \partial D \)
from $D''$, decreasing #($\partial D \cap \mathcal{T}^2$) — in contradiction to our choice. Therefore $\partial D$ is contained in a single tetrahedron, thus bounding a disc in $F \setminus \mathcal{T}^2$. The arguments are similar when $D$ is a compressing disc (see [3] for details).

In the remainder of this section, we recall the notion of impermeable surfaces and its relationship with almost 2–normal surfaces. Let $S \subset M$ be a splitting surface in general position to $\mathcal{T}$ and let $D_1, D_2$ be upper and lower compressing discs for $S$ with strings $\alpha_1, \alpha_2$. If $D_1 \subset D_2$ or $D_2 \subset D_1$ then $D_1$ and $D_2$ are nested. If $D_1 \cap D_2 \subset \partial \alpha_1 \cap \partial \alpha_2$ then $D_1$ and $D_2$ are independent from each other. If $S$ has both strict upper and strict lower compressing discs and has no pair of nested or independent upper and lower compressing discs, then $S$ is impermeable. Note that this property does not change under an isotopy of $S$ mod $\mathcal{T}^1$. Impermeable surfaces are closely related to (almost) 2–normal surfaces, by the following lemma that is implicit in the literature [10], [4]. For completeness, we resume the proof.

**Lemma 2.** Any impermeable surface $S \subset M$ with strict upper and lower compressing discs $D_1$ and $D_2$ is related to an almost 2–normal surface with exactly one octagon by an isotopy mod $\mathcal{T}^1$ with support in $U(D_1 \cup D_2)$.

**Proof.** By isotopy of $S$ mod $\mathcal{T}^1$ with support in $U(D_1)$, we pull the base of the strict upper compressing disc $D_1$ into a single tetrahedron, close to the string of $D_1$, changing $S$ into a surface $S_1$. Since $S_1$ is impermeable, no lower compressing disc for $S_1$ is contained in a 2–simplex of $\mathcal{T}$. Hence any return of $S_1$ gives rise to an upper compressing disc. Similarly, we pull the base of $D_2$ into a single tetrahedron by isotopy mod $\mathcal{T}^1$ with support in $U(D_2)$, changing $S$ into a surface $S_2$ so that any return of $S_2$ gives rise to a lower compressing disc.

We transform $S_1$ into $S_2$ by isotopy mod $\mathcal{T}^1$ with support in $U(D_1 \cup D_2)$. No surface occurring in the transformation has both upper and lower compressing discs in $\mathcal{T}^2$, since this would give rise to a pair of nested or independent compressing discs. Thus, in the course of the isotopy, we obtain a surface $\tilde{S}$ in general position to $\mathcal{T}$ that has neither upper nor lower compressing discs in $\mathcal{T}^2$. Hence $\tilde{S}$ is almost $k$–normal for some natural number $k$. It is easy to verify (see [10] for details) that an almost $k$–normal surface has a pair of nested or independent compressing discs, if it is not almost 2–normal or has more than one octagon.

There remains to show that $\tilde{S}$ is not almost 1–normal — for details see Lemma 21 in [3]. Assume that $\tilde{S}$ is almost 1–normal. Let $D_u$ be a strict upper compressing disc for $\tilde{S}$. There is a subdisc $D \subset D_u$ such that $\partial D \cap \tilde{S}$ is a single arc, and the closure of $\partial D \setminus \tilde{S}$ is an arc $\gamma \subset \mathcal{T}^2$ that connects two different normal arcs of $\tilde{S}$ and is the only arc in $D \cap \mathcal{T}^2$ with that property. By pulling $\tilde{S} \cap \partial D$ along $D$ into a single tetrahedron, we obtain a surface $\tilde{F} = \partial(B^+(\tilde{S}) \cup U(D \setminus U(\gamma)))$. By the choice of $D$, $\tilde{F}$ has no returns, hence it is almost 1–normal and $\tilde{F}^\times$ is isotopic mod $\mathcal{T}^2$ to $\tilde{S}^\times$. The union of $D \cap U(\gamma)$ with a stripe in $\mathcal{T}^2$ yields a not necessarily strict upper compressing disc $\tilde{D}_u$ for $\tilde{F}$ contained in a single tetrahedron $t$ whose string $\alpha$ is contained in $B^+(\tilde{F})$. We can assume that $\tilde{D}_u$ is contained in a single connected component $K$ of $t \cap B^+(\tilde{F}^\times)$ and meets $\partial K$ only in its string $\alpha$. Since $\tilde{F}$ is almost 1–normal, the two points of $\partial \alpha$ belong to different components of $\partial t \cap \tilde{F}$.

Since $\tilde{F}$ is impermeable, it has a strict lower compressing disc $D_l$. Let $\delta$ be a component of $D_l \cap \partial K$. Since any component of $\tilde{F} \cap \partial t$ contains at most one point of $\tilde{D}_u$, there is a subdisc $C' \subset (\partial K \cap \tilde{F}) \setminus \mathcal{T}^1$ with $\partial C' \subset \tilde{F} \cup \delta$ and $\delta \subset \partial C'$, such that $\partial C'$ is disjoint from $D_u$. Hence, by cut-and-paste operations and since no
lower compressing disc is completely contained in $K$, we can assume that $D_1$ and $D_u$ are independent, $D_1 \cap D_u \subset \alpha$. This is impossible, as $F$ is impermeable. This finally proves that $\tilde{S}$ is not almost $1$–normal and has thus exactly one octagon. 

5. Upper and lower reductions

We recall here the reduction technique that we have introduced in [6] and refined in [7]. If a surface $S \subset M$ has a strict compressing disc then one can pull $S$ along it, as in the next definition, in order to decrease the weight $\|S\| = \#(S \cap T^1)$. Under certain conditions on $S$, as stated below, one eventually comes to an almost $1$–normal surface by repeating this process. This fact is the basis for our main applications of the reduction technique.

Definition 4. Let $S \subset M$ be a splitting surface that is in general position to $T^2$. Let $D \subset M$ be a strict upper (resp. lower) compressing disc for $S$. The surface $\partial(B^-(S) \cup U(D))$ (resp. $\partial(B^+(S) \cup U(D))$) obtained by pulling $S$ along $D$ is an elementary reduction of $S$ along $D$. An upper (resp. lower) reduction of $S$ is a surface $S' \subset M$ obtained from $S$ by successive elementary reductions along strict upper (resp. strict lower) compressing discs. If these discs are contained in $T^2$ then $S'$ is an upper (resp. a lower) $T^2$–reduction of $S$.

We do not allow mixing of elementary reductions along upper and lower compressing discs in the transition from $S$ to $S'$. We have $\|S'\| \leq \|S\|$ with equality if and only if $S = S'$. The following lemma is the key tool in applications of the reduction technique.

Lemma 3 (Corollary 1 in [7], p. 57). Let $N \subset M$ be a sub–3–manifold such that $\partial N$ is $k$–normal for some $k \in \mathbb{N}$. Let $S \subset N$ be a closed connected splitting surface in general position to $T$. Assume that $S$ has only strict upper essential discs, and has no lower compressing discs contained in a single tetrahedron. Then $S$ has an almost $1$–normal upper $T^2$–reduction in $N$ that has only strict upper essential discs. 

The idea of the proof of the preceding lemma is as follows. Since $\partial N$ is $k$–normal, it has no returns and no circles in $T^2 \setminus T^1$. Hence if a surface in $N$ has a strict compressing disc in $T^2$ then this disc is contained in $N$. Thus any upper $T^2$–reduction $S'$ of $S$ is contained in $N$. One proves by induction on the number of elementary reductions that $S'$ has only strict upper essential discs and has no lower compressing discs contained in a tetrahedron. Hence any return of $S'$ gives rise to an upper compressing disc $D \subset T^2$ for $S'$ with string in $B^+(S')$. If there is a circle $\gamma$ in $D \cap S'$ then $\gamma$ bounds discs in $B^+(S')$ (e.g., a strict upper essential disc), in both adjacent tetrahedra. It follows that $\gamma$ and $\partial D \cap S'$ belong to distinct components of $S'$. This contradicts the hypothesis that $S$ is connected. Hence $D$ is strict and gives rise to another elementary $T^2$–reduction of $S'$. Thus we can assume that $S'$ has no returns at all, i.e., $S'$ is almost $k$–normal for some $k \in \{1, 2, \ldots\}$. Since $S'$ has no lower compressing discs contained in a tetrahedron, $S'$ is almost $1$–normal.

Theorem 3 (Implicit in [4], Proposition 3.3). Let $M \neq S^3$ be a closed orientable irreducible 3–manifold with a triangulation $T$. There is a $1$–normal sphere $F_0 \subset M$ such that $B^-(F_0)$ is a 3–ball and any 1–normal sphere in $B^+(F_0)$ is isotopic mod $T^2$ to $F_0$. 

Proof. By Theorem 2 there is a disjoint union \( \Sigma \subset M \) of finitely many 1–normal spheres \( S_1, \ldots, S_m \) such that any 1–normal sphere in \( M \setminus \Sigma \) is isotopic mod \( \mathcal{T}^2 \) to a connected component of \( \Sigma \). The link \( \partial U(x) \) of a vertex \( x \) of \( \mathcal{T} \) is a 1–normal sphere that can be removed from any 1–normal surface by isotopy mod \( \mathcal{T}^2 \). Hence, \( \Sigma \) contains a copy of the link of any vertex.

Since \( M \) is irreducible and not homeomorphic to \( S^3 \), there is a unique open 3–ball \( B_i \subset M \) with \( \partial B_i = S_i \), for \( i = 1, \ldots, m \). Let \( N = M \setminus \bigcup_{i=1}^m B_i \). For \( i = 1, \ldots, m \), if \( S_i \) is the link \( \partial U(x) \) of a vertex \( x \) then \( x \in B_i \). Hence by definition, \( N \) contains no vertex of \( \mathcal{T} \). Therefore \( B^+(\partial N) \) does not depend on the choice of the vertex \( x_0 \) in the definition of \( B^+(\cdot) \), and \( N = B^+(\partial N) \).

We now show that \( \partial N \) is connected, which proves the theorem with \( F_0 = \partial N \). Let us assume that \( \partial N \) is not connected, say \( \partial N = S_1 \cup \cdots \cup S_n \) with \( n > 1 \). Since \( N \cap \mathcal{T}^2 \) is connected, there is a system \( \Gamma \subset (N \cap \mathcal{T}^2) \setminus \mathcal{T}^1 \) of \( n - 1 \) disjoint simple arcs with \( \partial \Gamma \subset \partial N \), such that \( S = (\partial N)^\Gamma \) is a sphere. Note that \( S \) is not almost 1–normal, since \( \Gamma \) is not in general position to \( \mathcal{T} \). Since \( \Gamma \subset \mathcal{T}^2 \cap B^+(\partial N) \), \( S \) has only strict upper essential discs, and \( S \) has no lower compressing discs contained in a single tetrahedron. Hence by Lemma 4 \( S \) has an almost 1–normal upper \( \mathcal{T}^2 \)–reduction \( S' \subset N \) that has only strict upper essential discs. Any connected component of \( (S')^\times \) is a 1–normal sphere in \( N \). Hence, by the choice of \( \Sigma \), \( (S')^\times \) is formed by copies of boundary components of \( N \). Since \( S' \) is connected and has only strict upper essential discs, \( (S')^\times \) contains at most one copy of each connected component of \( \partial N \). Since \( S' \) does not separate two components of \( \partial N \), we have either \( (S')^\times = \partial N \) or \( (S')^\times = \emptyset \). But since \( S \) is not almost 1–normal, it follows \( S \neq S' \). Therefore \( \|S'\| < \|S\| \), thus \( (S')^\times = \emptyset \) and \( \|S'\| = 0 \). By consequence, \( S' \) bounds a ball \( B' \subset M \setminus \mathcal{T}^1 \), hence \( B^+(S') = B' \). But \( B^-(S') \simeq B^-(S) \) is a ball as well, since it is a \( \partial \)–connected sum of the 3–balls forming \( M \setminus N \). This contradicts the hypothesis \( M \neq S^3 \), which finally proves that \( \partial N \) is in fact connected. The theorem follows with \( F_0 = \partial N \).

We remark that, since \( M \neq S^3 \), there is no 2–normal sphere in \( B^+(F_0) \) with exactly one octagon (see [15, 10]). A proof using the reduction technique is implicit in [7], Chapter 4. A construction algorithm for the system \( \Sigma \) occurring in the preceding proof and an estimate for \( \|\Sigma\| \) can be found in [6]. It was applied to a study of bridge numbers of links in [6] and to convex 4–polytopes in [8].

6. Completions of surfaces

In this section, we introduce a bunch of notions that naturally occur when considering strongly irreducible Heegaard surfaces. Let \( S \subset M \) be a splitting surface. A simple closed curve on \( S \) contained in the boundary of a tetrahedron is a **short upper** (resp. lower) **meridian** of \( S \) if it does not bound a disc in \( S \) and does bound an embedded disc in \( B^+(S) \) (resp. in \( B^-(S) \)).

Let \( c \subset S \) be any simple closed curve contained in the boundary of a tetrahedron. The curve bounds an embedded disc in the tetrahedron. By Scharlemann’s “no nesting” lemma (Lemma 2.2 in [12]), if \( S \) is a strongly irreducible Heegaard surface then \( c \) bounds an embedded disc in \( B^+(S) \) or in \( B^-(S) \). Hence, if \( c \) does not bound an embedded disc in \( B^-(S) \), then \( c \) is a short upper meridian of \( S \). If \( S \) is in general position to \( \mathcal{T} \) then any pair of short meridians can be made disjoint, by pushing one of them into the interior of a tetrahedron. Thus, a strongly irreducible Heegaard
surface in general position to $\mathcal{T}$ either has no short upper meridians or no short lower meridians.

This gives rise to the following definition. Let $S \subset M$ be a splitting surface. If any simple closed curve on $S$ contained in the boundary of a tetrahedron bounds an embedded disc in $B^-(S)$ (resp. in $B^+(S)$) then $S$ is upper trivial (resp. lower trivial). If $S^\times$ is defined then we call $S$ an upper completion (resp. lower completion) of $S^\times$. Evidently, if $S$ is upper trivial and $\Gamma \subset B^+(S)$ is a graph in general position to $\mathcal{T}^2$ then $S^\Gamma$ is upper trivial as well. By the preceding paragraph, a strongly irreducible Heegaard surface in general position to $\mathcal{T}$ is upper (resp. lower) trivial if and only if it has no short upper (resp. lower) meridian.

The following two lemmas are formulated for an upper trivial surface. Of course, these lemmas remain true if we exchange “upper” with “lower” and $B^+$ with $B^-$. 

**Lemma 4.** Let $M$ be irreducible and let $S \subset M$ be a connected upper trivial surface. Then either $S$ has a short lower meridian or $S$ is isotopic to a connected component of $S^\times$ or $S$ is contained in a 3–ball.

**Proof.** Let us assume that $S$ has no short lower meridian. Since $S$ is upper trivial, any simple closed curve $c \subset S$ contained in the boundary of a tetrahedron $t$ bounds a disc in $B^-(S)$. By assumption, $c$ actually bounds a disc $D \subset S$, and it also bounds a disc $D' \subset t$. By taking subdiscs, we assume that $D \cap D' = \partial D = \partial D'$. Since $M$ is irreducible, the sphere $D \cup D'$ bounds a 3–ball $B$. We change $S$ by an isotopy with support in $U(B)$, replacing $D$ with $D'$. An iteration of this process yields a surface $\tilde{S} \subset M$ so that $G(\tilde{S}) \subset G(S)$, and any simple closed curve on $\tilde{S}$ contained in the boundary of a tetrahedron $t$ bounds a disc in $\tilde{S} \cap t$.

If $G(\tilde{S}) \neq \emptyset$ then $\tilde{S}$ is isotopic mod $\mathcal{T}^2$ to a connected component of $S^\times$. Otherwise, $\tilde{S}$ is contained in a single tetrahedron ($\tilde{S} \cap \mathcal{T}^2 = \emptyset$) or in the union of two tetrahedra (in this case, $\tilde{S}$ is a sphere). Hence, $S \cong \tilde{S}$ is contained in a 3–ball. $\Box$

**Lemma 5.** Let $M$ be irreducible. Let $S \subset M$ be an almost 1–normal upper trivial strongly irreducible Heegaard surface with a strict upper compressing disc $\hat{D}$. Assume that $S$ is not isotopic to a connected component of $S^\times$ and $S$ is not contained in a 3–ball. Then there is an arc $\gamma \subset (B^+(S^\times) \cap \mathcal{T}^2) \setminus \mathcal{T}^1$ such that $S$ is related to an upper completion of $\langle S^\times \rangle^\gamma$ by isotopy mod $\mathcal{T}^1$ with support in $U(\hat{D})$.

**Proof.** By taking a parallel copy of a subdisc of $\hat{D}$, there is a compact embedded disc $D \subset U(\hat{D}) \setminus \mathcal{T}^1$ in general position to $S$ such that the closure of $\partial D \setminus S$ is an arc $\gamma \subset B^+(S) \cap \mathcal{T}^2$ connecting two different normal arcs of $S \cap \mathcal{T}^2$, and no other arc in $(D \cap \mathcal{T}^2) \setminus \gamma$ connects two different normal arcs of $S$. When we pull $S$ across $D$ by an isotopy mod $\mathcal{T}^1$ with support in $U(\hat{D})$, we obtain the surface $S_D = \partial(B^-(S) \cup U(D))$. We will show that $S_D$ is an upper completion of $\langle S^\times \rangle^\gamma$.

Consider a subdisc $D' \subset D$ such that $\partial D'$ is formed by an arc $\alpha \subset \partial t_1$ and an arc $\beta \subset S \cap t_1$, for some tetrahedron $t_1$. Such a subdisc exists by an innermost arc argument. Let $t_2$ be the other tetrahedron whose boundary contains $\alpha$. When we pull $S$ across $D'$ (as part of the transition from $S$ to $S_D$), we obtain $S' = \partial(B^-(S) \cup U(D'))$. If $\alpha \neq \gamma$ then $\alpha$ does not connect different normal arcs of $S$, thus $(S')^\times$ is isotopic mod $\mathcal{T}^2$ to $S^\times$. If $\alpha = \gamma$ then $S' = S_D$.

To prove that $S'$ is upper trivial, we have to consider the parts of $S' \cap \mathcal{T}^2$ that do not belong to $S \cap \mathcal{T}^2$. If $D' \cap \mathcal{T}^2$ contains simple closed curves (in the interior
of $D'$ then these give rise to simple closed curves of $S' \cap \mathcal{T}^2$ bounding discs in $S'$. There remains to consider curves adjacent to $\alpha$, in the following two cases.

1. We assume that $\partial \alpha$ is contained in a single component of $(S \cap \mathcal{T}^2) \setminus \mathcal{T}^1$. Let $\alpha' \subset (S \cap \mathcal{T}^2) \setminus \mathcal{T}^1$ be the arc that connects the two endpoints of $\alpha$, and let $c_1, c_2$ be the components of $S \cap \partial t_1, S \cap \partial t_2$ that contain $\partial \alpha$. By hypothesis and by Lemma \[1\] $S$ has a short lower meridian. The curves $\alpha' \cup \beta$, $(c_1 \setminus \alpha') \cup \beta$ and $c_2$ bound discs in $M$. They are contained in a single tetrahedron, hence are disjoint from the short lower meridian. Thus, since $S$ is strongly irreducible and by the “no nesting” lemma, they bound discs in $B^-(S)$. Transforming the union of discs in $B^-(S)$ bounded by $c_2$ and $\alpha' \cup \beta$, we see that $(c_2 \setminus \alpha') \cup \beta$ bounds a disc in $B^-(S)$ as well. Transforming $S$ into $S'$, $\alpha' \cup \beta$ becomes a new connected component of $(S' \cap \mathcal{T}^2) \setminus \mathcal{T}^1$, $(c_1 \setminus \alpha') \cup \beta$ replaces $c_1$, and $(c_2 \setminus \alpha') \cup \beta$ replaces $c_2$ — compare the left part of Figure 2. Since all these curves bound discs in $B^-(S')$, $S'$ is upper trivial.

2. We assume that $\alpha$ connects two different components of $(S \cap \mathcal{T}^2) \setminus \mathcal{T}^1$. Since $S$ is almost 1–normal, $\alpha$ connects two different components $c_i, c'_i$ of $S \cap \partial t_i$, for $i = 1, 2$ — see the right part of Figure 2. Since $S$ is upper trivial, $c_i, c'_i$ bound discs in $B^-(S)$. Transforming $S$ into $S'$, these discs become connected along a stripe in $U(\alpha) \subset B^-(S')$, and therefore any connected component of $S' \cap \partial t_i$ bounds a disc in $B^-(S')$. Thus $S'$ is upper trivial.

We replace $S$ with $S'$ and repeat until $\alpha = \gamma$. In the final step of this iteration, we have $S' = S_D$, hence $S_D$ is upper trivial. Furthermore, $G(S_D)$ is isotopic mod $\mathcal{T}^2$ to $G((S^\times)\gamma)$. Since $S^\times$ is 1–normal, $(S^\times)\gamma \setminus \mathcal{T}^2$ is a disjoint union of discs. Hence $S_D^\times$ is isotopic mod $\mathcal{T}^2$ to $(S^\times)\gamma$. \[ \square \]

Scharlemann’s “no nesting” lemma is fundamental for our applications of upper and lower completions. Scharlemann’s short proof (Lemma 2.2 in \[12\]) refers to a lemma stated in \[1\], but the proof there is also based on citation. To avoid nested citations, we include here an elementary proof of the “no nesting” lemma.

**Lemma 6** (Lemma 2.2 in \[12\]). Let $H \subset M$ be a strongly irreducible Heegaard surface, and let $\alpha \subset H$ be a simple closed curve bounding an embedded disc in $M$. Then $\alpha$ bounds an embedded disc in $B^+(H)$ or $B^-(H)$.

**Proof.** Let $D \subset M$ be an embedded disc with $\partial D = \alpha$ such that $D$ is in general position to $H$ and $\#((D \cap H))$ is minimal. We can assume that $D \cap H \neq \partial D$, since otherwise there is nothing to prove. In particular, $\alpha$ is essential. We can assume...
by induction that if a simple closed curve \( \alpha \subset H \) bounds a disc \( \tilde{D} \subset M \) with \( #(\tilde{D} \cap H) < #(D \cap H) \) then \( \alpha \) bounds a disc in \( B^+(H) \) or \( B^-(H) \).

Let \( \delta \) be a connected component of \( (D \cap H) \setminus \partial D \). It bounds a subdisc \( \tilde{D} \subset D \) with \( #(\tilde{D} \cap H) < #(D \cap H) \). Hence by induction, \( \delta \) bounds a disc in one handlebody of the decomposition, say, in \( B^-(H) \). Cut-and-paste arguments and minimality of \( #(D \cap H) \) imply that \( \delta \) is essential on \( H \) and bounds a component of \( D \cap B^-(H) \). Since \( H \) is strongly irreducible, the other components of \( (D \cap H) \setminus \partial D \) are not meridians of \( B^+(H) \). Hence when we apply the same argument, it follows that \( D \cap B^-(H) \) is a non-empty disjoint union of meridional discs, and \( P = D \cap B^+(H) \) is a non-empty planar surface since \( D \cap H \neq \partial D \) by assumption.

Let \( C \subset B^+(H) \) be a meridional disc transversely intersecting \( P \) such that \( #(C \cap P) \) is minimal. Since \( H \) is strongly irreducible and \( \partial P \) contains a meridian of \( B^-(H) \), we have \( C \cap P \neq \emptyset \). Let \( \gamma \subset C \cap P \) be an innermost arc, cutting off a disc \( C' \subset C \) with \( \partial C' \cap \partial P = \partial \gamma \). By cut-and-paste arguments, there is no circle in \( C' \cap P \), hence \( C' \cap P = \gamma \). Let \( \beta = \partial C' \cap H \) and \( \Delta = \partial P \setminus \partial D \). We consider three cases.

1. If \( \gamma \) connects two different components of \( \Delta \) then one can decrease \( #(D \cap H) \) by isotopy of \( D \) along \( C' \), in contradiction to the choice of \( D \).

2. Assume that \( \partial \gamma \) is contained in a single component \( \delta \subset \partial P \). If \( \gamma \) cuts off a disc \( D' \) in \( P \) then a copy of \( C' \cup D' \) is a disc in \( B^+(H) \) that is disjoint from \( \partial P \). Since \( \partial P \) contains a meridian of \( B^-(H) \) and \( H \) is strongly irreducible, \( \partial(C' \cup D') \) bounds a disc in \( H \). Thus we can decrease \( #(C \cap P) \), in contradiction to the choice of \( C \). Hence, \( \gamma \) does not cut off a disc in \( P \).

Let \( \delta_1, \delta_2 \) be the two connected components of \( \delta \setminus \partial \gamma \). If \( \delta_1 \cup \gamma \) (resp. \( \delta_2 \cup \gamma \)) bounds a disc \( D'' \subset D \) then \( D'' \not\subset P \) by the preceding paragraph. Therefore \( #(C' \cup D'') \cap H < #(D \cap H) \), and hence \( \delta_1 \cup \beta \) (resp. \( \delta_2 \cup \beta \)) bounds an embedded disc in \( B^+(H) \) or \( B^-(H) \) by induction. It actually bounds a disc in \( B^-(H) \), since \( \partial P \) contains a meridian of \( B^-(H) \), since a copy of \( \delta_1 \cup \gamma \) is disjoint from \( \partial P \), and since \( H \) is strongly irreducible.

If \( \delta = \partial D \) then both \( \delta_1 \cup \gamma \) and \( \delta_2 \cup \gamma \) bound discs in \( D \). Hence by the preceding paragraph, \( \partial D \) bounds a disc in \( B^-(H) \), in contradiction to our assumption. If \( \delta \subset \Delta \) then either \( \delta_1 \cup \gamma \) or \( \delta_2 \cup \gamma \) bounds a disc \( D' \subset D \). Hence it bounds a disc \( D'' \subset B^-(H) \). Replacing \( D' \) with \( D'' \cup C' \), we can decrease \( #(D \cap H) \), since \( D'' \not\subset P \). We obtain a contradiction to the minimality of \( #(D \cap H) \).

3. We are left with the case that \( \gamma \) connects \( \partial D \) with a component \( \delta \) of \( \Delta \). Let \( D_{\delta} \) be the disc in \( D \cap B^-(H) \) bounded by \( \delta \). Let \( \tau \subset H \) be a simple closed curve obtained by connecting \( \delta \setminus (U(\partial \beta)) \) and \( \partial D \setminus (U(\partial \beta)) \) with two copies of \( \beta \). By taking the union of the disc \( D \setminus (U(\gamma)) \) with \( \partial D \setminus (U(\gamma)) \) with two copies of \( C' \), \( \tau \) bounds an embedded disc \( \tilde{D} \subset M \) with \( #(\tilde{D} \cap H) < #(D \cap H) \). Hence, using the same arguments as in Case 2, \( \tau \) bounds a disc \( D' \) in \( B^-(H) \). Taking the union of \( D' \) with a stripe along \( \beta \) and with \( D_{\delta} \), we obtain a disc in \( B^-(H) \) bounded by \( \partial D \). This contradicts our assumption and proves the lemma. \( \square \)

7. Proof of Theorem 1

Let \( M \) be a closed orientable irreducible 3–manifold with a triangulation \( T \), and let \( H \subset M \) be a strongly irreducible Heegaard surface of \( M \). By Waldhausen’s theorem on Heegaard surfaces of \( S^3 \), \( H \) is an embedded 2–sphere if \( M \approx S^3 \). In this case, we pick a vertex \( x \in T^0 \), and connect two copies of the 1–normal sphere
\partial U(x) along an unknotted arc. We obtain an almost 1–normal sphere isotopic to \( H \) and satisfying Theorem 1. There remains to consider the case \( M \not\approx S^3 \). In particular, \( H \) is not contained in a 3–ball. Let \( F_0 \subset M \) be as in Theorem 3 and let \( N = B^+(F_0) \).

By isotopy of \( H \), we can assume that the ball \( M \setminus N \) is contained in a single connected component \( \mathcal{H} \) of \( M \setminus H \). Since \( \mathcal{H} \) is a handlebody, there is a graph \( \Gamma_- \subset \mathcal{H} \cap N \) in general position to \( T \) with \( \partial \Gamma_- \subset \partial N \), such that \( \mathcal{H} \) collapses onto \( (M \setminus N) \cup \Gamma_- \). Hence \( S_- = (\partial N)^{T_-} \) is an almost 1–normal upper trivial surface isotopic to \( H \). Similarly, there is a graph \( \Gamma_+ \subset M \setminus \mathcal{H} \) in general position to \( T \) such that \( M \setminus \mathcal{H} \) collapses onto \( \Gamma_+ \). Hence \( S_+ = \partial U(\Gamma_+) \) is an almost 1–normal lower trivial surface isotopic to \( H \). We have \( N \supset B^+(S_-) \supset B^+(S_+) = U(\Gamma_+) \).

Thus there are 1–normal surfaces \( F_-, F_+ \subset N \), an upper completion \( S_+ \simeq H \) of \( F_- \) and a lower completion \( S_- \simeq H \) of \( F_+ \) with \( B^+(S_-) \subset B^+(S_+) \). Let \( S \simeq H \) be a lower (resp. upper) trivial almost 1–normal surface that is nested between \( S_- \) and \( S_+ \), i.e., \( B^+(S_-) \subset B^+(S) \subset B^+(S_+) \). Assume that \( \| S \| < \| S_- \| \) (resp. \( \| S \| < \| S_+ \| \)) or there is a connected component of \( S \times I \) that can not be isotoped mod \( T^2 \) into \( B^+(F_-) \) (resp. of \( B^+(F_+) \)). Then we replace \( F_- \) by \( S \) (resp. \( F_+ \) by \( S \)), and iterate. By Kneser’s finiteness result (Theorem 2), the iteration stops after a finite number of steps. We choose \( F_-, F_+ \) so that a further iteration is impossible.

If some component \( F \) of \( F_- \) or \( F_+ \) is isotopic to \( H \) then we connect \( F \) with \( \partial \mathcal{N} \) along an unknotted arc, and obtain the first alternative of Theorem 1. By now we assume that no connected component of \( F_- \) or \( F_+ \) is isotopic to \( H \). Hence by Lemma 3, \( S_- \) has a short lower meridian and \( S_+ \) has a short upper meridian.

**Case 1.** We assume that \( S_- \) has a strict upper compressing disc in \( B^-(S_+) \), for some choice of \( S_- \), \( S_+ \). According to Lemma 4, \( S_- \) is isotopic mod \( T^1 \) to an upper completion of \( F_- \) contained in \( B^+(S_-) \cap B^-(S_+) \), for some arc \( \gamma \subset (B^+(F_-) \cap T^2) \setminus T^1 \). The surface \( F_- \) satisfies the hypothesis of Lemma 4, hence \( F_- \) has an almost 1–normal upper \( T^2 \)-reduction \( S' \subset B^+(F_-) \cap B^-(F_+) \). We use the freedom in the choice of \( S_- \), \( S_+ \) and isotope it so that \( S_- \cup S_+ \) has empty intersection with the strict compressing discs involved in the elementary reductions transforming \( F_- \) into \( S \). In particular, the upper \( T^2 \)-reduction \( S' \) of \( F_- \) gives rise to an almost 1–normal upper \( T^2 \)-reduction \( S'_- \subset B^+(S_-) \cap B^-(S_+) \) of \( S_+ \).

If \( F'_2 \simeq H \) then we get the first alternative of Theorem 1 by pushing \( \gamma \) into the interior of a tetrahedron. Otherwise, by Lemma 4, \( S_- \) has a small lower meridian. We take a copy \( c \subset S_- \setminus T^2 \) of this meridian. It is not affected by elementary reductions along compressing discs in \( T^2 \). Thus \( S'_- \) has no short upper meridian, since it would be disjoint from the meridian \( c \) of \( B^-(S'_-) \). Hence \( S'_- \simeq H \) is upper trivial. This is a contradiction to the choice of \( F_- \), \( F_+ \), as \( \| S'_- \| < \| S_- \| = \| F_- \| \).

In conclusion, if \( S_- \) has a strict upper compressing disc in \( B^+(S_+) \), or similarly if \( S_+ \) has a strict lower compressing disc in \( B^+(S_-) \), for some choice of \( S_- \), \( S_+ \), then the first alternative of Theorem 1 holds.

**Case 2.** We assume that \( S_- \) has no strict upper compressing disc in \( B^-(S_+) \) and \( S_+ \) has no strict lower compressing disc in \( B^+(S_-) \), for \( n \) choice of \( S_- \), \( S_+ \). Let \( \Sigma_g \) be the closed orientable surface whose genus \( g \) coincides with the genus of \( H \). By hypothesis on \( F_- \), \( F_+ \), there is an embedding \( J : \Sigma_g \times [0,1] \to M \) in general position to \( T \) such that \( J_\xi = J(\Sigma_g \times \{ \xi \}) \simeq H \) for all \( \xi \in [0,1] \). \( J_0 \) is an upper completion of \( F_- \), and \( J_1 \) is a lower completion of \( F_+ \).
For any \( \xi \in [0, 1] \), \( J_\xi \) contains no vertex (\( T^0 \) is in the complement of \( N \)), and it has at most one point of tangency with \( T^1 \setminus T^0 \) or \( T^2 \setminus T^1 \) by general position of \( J \). A parameter \( \xi \in [0, 1] \) is critical if \( J_\xi \) is tangent to \( T^1 \). There are only finitely many critical parameters. The complexity of \( J \) is

\[
\kappa(J) = \sum_{\xi \in \mathcal{I}_{\text{critical}}} \|J_\xi\|.
\]

We choose \( J \) so that \( \kappa(J) \) is minimal among all embeddings with the above properties. This is called thin position, a notion with many fruitful applications — for instance in \([2, 15, 10, 13, 6]\).

By assumption on \( F_-, F_+ \), there is a short lower meridian of \( J_0 \) and a short upper meridian of \( J_1 \). Hence there is a non-critical parameter \( \xi \in (0, 1] \) such that \( J_\xi \) has neither short upper nor short lower meridians, or has both short upper and short lower meridians. But the latter case can not occur since otherwise by small isotopy we obtain meridians of \( B^-(J_\xi) \) and \( B^+(J_\xi) \) that intersect in at most one point (namely in a point of tangency of \( J_\xi \) with \( T^2 \)), in contradiction to strong irreducibility of \( J_\xi \). Thus \( J_\xi \) is both upper and lower trivial, and by small isotopy we can assume that it is in general position to \( T \).

If \( J \) has no critical parameter then it easily follows that \( J_\xi \) has no returns, since \( J_0 \) and \( J_1 \) have no returns, \( G(J_0) \subset B^-(J_\xi), G(J_1) \subset B^+(J_\xi), \) and \( G(J_0) \) is isotopic mod \( T^2 \) to \( G(J_1) \). Thus \( J_\xi \) is almost \( k \)-normal for some \( k \in \mathbb{N} \). Since a \( k \)-normal surface is determined by its intersection with \( T^1 \), \( J_\xi \) is isotopic mod \( T^2 \) to \( F_- \).

Hence \( J_\xi \) is an upper completion of \( F_- \) isotopic to \( H \) without a short lower meridian, in contradiction to the assumption. Therefore, \( J \) has a critical parameter.

Since we are in Case 2, \( J_1 = S_+ \) has no strict lower compressing discs. Hence there is a smallest critical parameter \( \xi_0 \in [0, 1] \) with \( \|J_{\xi_0+\epsilon}\| < \|J_{\xi_0-\epsilon}\| \) for small \( \epsilon > 0 \). Since \( J_0 \) has no strict upper compressing disc and \( \xi_0 \) is minimal, we have \( \|J_0\| < \|J_{\xi_0-\epsilon}\| \). Thus \( J_{\xi_0-\epsilon} \) has both strict upper and lower compressing discs in \( B^+(J_0) \cap B^-(J_1) \).

Assume that \( J_{\xi_0-\epsilon} \) has a pair of nested or independent upper and lower compressing discs \( D_1, D_2 \subset M \). By Lemma \([11]\) and since \( F_-, F_+ \) are \( 1 \)-normal, we can choose \( D_1, D_2 \subset B^+(F_-) \cap B^-(F_+) \). We can assume that the intersection of \( D_i \) with \( S_{\pm} \) is disjoint from \( T^2 \). Thus \( D_1 \cup D_2 \) does not contain a meridian of \( B^+(S_-) \) or of \( B^-(S_+) \), by the existence of short meridians of \( S_{\pm} \) and strong irreducibility. Hence an isotopy of \( J \) along \( D_1 \cup D_2 \) yields an embedding \( J' : \Sigma_g \times I \to M \) so that \( J'_0 \) is an upper completion of \( F_- \) and \( J'_1 \) is a lower completion of \( F_+ \). Moreover, the number of critical parameters does not increase and the weight of some critical levels do decrease, which implies \( \kappa(J') < \kappa(J) \) — see \([10, 15]\) for details. This contradicts the choice of \( J \) and disproves the existence of \( D_1, D_2 \).

By the preceding paragraphs, \( J_{\xi_0-\epsilon} \subset B^+(J_0) \cap B^-(J_1) \) is impermeable. It has strict upper and lower compressing discs in \( B^+(J_0) \cap B^-(J_1) \) yield by critical points of \( J \). Thus by Lemma \([2]\) \( J_{\xi_0-\epsilon} \) is isotopic mod \( T^1 \) to an impermeable almost 2-normal surface \( S \subset B^+(J_0) \cap B^-(J_1) \) that has exactly one octagon.

We assume that \( S \) has a short lower meridian; the opposite case of a short upper meridian is of course symmetric. Let \( F \) be the component of \( S^x \) containing the octagon. The octagon yields a strict upper compressing disc for \( F \) contained in a tetrahedron. Let \( F' \) be the result of an elementary reduction of \( F \) along that strict upper compressing disc. \( F' \) satisfies the hypothesis of Lemma \([3]\). Thus \( F' \) has an almost \( 1 \)-normal upper \( T^2 \)-reduction \( F'' \subset B^+(S^x) \cap B^-(F_+) \). As in Case 1, by an
appropriate choice of $S_-$ and $S_+$, we obtain an almost 1–normal upper reduction $S' \subset B^+(S_-) \cap B^-(S_+)$ of $S$ that has a short lower meridian. Hence $S'$ is upper trivial. The 1–normal surface $(S')^\times$ is separated from $F_-$ by $F$, which has an octagon. Hence some connected component of $(S')^\times$ can not be isotoped mod $T^2$ into $B^-(F_-)$. This is a contradiction to the choice of $F_-$. 

Thus $S$ has neither short upper nor short lower meridians, i.e., $S$ is both upper and lower trivial. Therefore $H$ is isotopic to a connected component $F_1$ of $S^\times$, by Lemma [1] If $F_1$ contains the octagon then we get the second alternative of Theorem [1] Otherwise, we join $F_1$ with $\partial N$ along an unknotted arc and get the first alternative of Theorem [1] This finishes the proof of Theorem [1].

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