Interior properties of five-dimensional Schwarzschild black hole

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We investigate inner structure of Schwarzschild black hole on a five-dimensional spacetime $S^3 \times \mathbb{R}^2$. To do this, we exploit a fivebein scheme. In particular, we construct an equation of state of hydrostatic equilibrium for the five-dimensional Schwarzschild black hole, which is a five-dimensional version of the Tolman-Oppenheimer-Volkoff equation on four-dimensional manifold. We also investigate uniform density interior configuration of the five-dimensional black hole which consists of incompressible fluid of density, to find a general relativistic expression for pressure.

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I. INTRODUCTION

Since the five-dimensional Schwarzschild black hole metric has been proposed [1, 2], tremendous progresses have been made in higher-dimensional spacetime physics. A higher-dimensional Kerr-Schild manifold possesses non-trivial parameters such as mass, angular momenta [3], while a Kerr-AdS black hole is parameterized by mass, angular momenta and cosmological constant. Here, one notes that in $D$-dimensional manifold, the number of angular momenta is equal to the rank of rotation group $SO(D-1)$ [4]. For positive curvature Kähler-Einstein manifold in dimension $2n$, it has been shown for a countably infinite class of associated Sasaki-Einstein manifold to exist in dimension $2n+3$ [5]. Moreover, a countably infinite number of explicit co-homogeneity one Sasaki-Einstein metrics on $S^2 \times S^3$ have been presented in both the quasi-regular and irregular classes [5]. It is interesting to see that, in the Kerr-AdS metric, by twisting the Killing vector field of the black holes, the compact Sasaki-Einstein manifolds constructed previously [5, 6] have been reproduced [4].

The four-dimensional Plebanski metric is characterized by a double Kerr-Schild form [7], where both the mass and the NUT charge are involved in the metric linearly. The most general higher-dimensional AdS-Kerr-NUT solutions have been found [4]. This solutions can be considered as higher-dimensional generalizations of the Plebanski metric and they are parameterized by the mass, multiple NUT charges and arbitrary orthogonal rotations. The metric has been shown to possess $U(1)^n$ isometries with $n = [(D + 1)/2]$. The general AdS-Kerr-NUT solutions in $D$-dimensions with $(|D/2|, (|D+1|)/2)$ signature have been shown to admit $|D/2|$ linearly independent, mutually orthogonal and affinely-parameterized null geodesic congruences [8].

Recently, the five-dimensional Schwarzschild black hole metric has been exploited [1, 2], to construct the global embedding structure, the five-acceleration and the thermodynamic physical quantity such as the Hawking temperature and entropy [9]. Next, the hydrodynamic properties of the five-dimensional Schwarzschild black hole have been investigated by using the massive particles and photons which are moving around the black hole, to obtain the radial component equation for the the steady state axisymmetric accretion of the massive particles and photons. The radial component of the Einstein equation associated with the entropies of the massive particles and photons has been also evaluated. On the other hand, the stringy cosmology has been investigated in higher-dimensions [10, 11]. This cosmology is the string theory version of the standard Hawking-Penrose expansion theory and it treats the twist and the shear as well as the expansion of the universe.

In this paper, for the five-dimensional Schwarzschild black hole residing on the total manifold $S^3 \times \mathbb{R}^2$, we investigate the five-metric in terms of the fünfbein, to construct the Ricci tensors and the scalar curvature. Next, we consider the equation of state related to the inner structure of the five-dimensional Schwarzschild black hole. Here we exploit the perfect fluid stress-energy tensor to find the static, spherically symmetric solutions of Einstein equation. Finally, we investigate the Tolman-Oppenheimer-Volkoff type equation for the five-dimensional Schwarzschild black hole. This paper is organized as follows. In Section II, by exploiting the fünfbein, we set up the Ricci tensors and scalar curvature

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for five-dimensional Schwarzschild black hole metric. In Section III we investigate the interior solutions of the Einstein equation. Section IV includes the summaries and discussions.

II. SETUP OF EINSTEIN EQUATION VIA FÜNFBEIN

The five-dimensional Schwarzschild black hole defined on the total manifold $S^3 \times \mathbb{R}^2$ is described in terms of the scalar functions $f(r)$ and $g(r)$ as follows

$$ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2 [d\alpha^2 + \sin^2 \alpha(d\theta^2 + \sin^2 \theta d\phi^2)].$$

Here we have three angles of the three-sphere whose ranges are defined by $0 \leq \alpha \leq \pi$, $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. In the five-dimensional Schwarzschild black hole, the fünfbein is given by

$$(e_0)_a = f^{1/2}(r)(dt)_a, \quad (e_1)_a = h^{1/2}(r)(dr)_a,$$
$$(e_2)_a = r(d\alpha)_a, \quad (e_3)_a = r \sin \alpha (d\theta)_a,$$
$$(e_4)_a = r \sin \alpha \sin \theta (d\phi)_a,$$

and the inverse fünfbein is also described as follows

$$(e_0)^a = -f^{-1/2}(r)(\partial_t)^a, \quad (e_1)^a = h^{-1/2}(r)(\partial_r)^a,$$
$$(e_2)^a = r^{-1}(\partial_\alpha)^a, \quad (e_3)^a = (r \sin \alpha)^{-1}(\partial_\theta)^a,$$
$$(e_4)^a = (r \sin \alpha \sin \theta)^{-1}(\partial_\phi)^a.$$

The fünfbein structure will be also exploited in the next section to investigate the equation of state of hydrostatic equilibrium for the five-dimensional Schwarzschild black hole, namely the five-dimensional Tolman-Oppenheimer-Volkoff type equation. Moreover, we have the identities among the fünfbein and inverse fünfbein

$$(e_\mu)^a (e_\nu)_a = \eta_{\mu\nu}, \quad (e_\mu)^a (e_\nu)^b \eta^{\mu\nu} = g^{ab}, \quad (e_\mu)^a (e_\nu)_b \eta^{\mu\nu} = \delta^a_b.$$  

Exploiting the above metric (2.1), after some algebra we obtain the Ricci tensors $R_{ab}$ $(a,b = 0, 1, 2, 3, 4)$ as follows

$$R_{00} = \frac{f''}{2h} - \frac{(f')^2}{4fh} - \frac{f'h'}{4h^2} + \frac{3f'}{2rh},$$
$$R_{11} = -\frac{f''}{2f} + \frac{(f')^2}{4f^2} + \frac{f'h'}{4fh} + \frac{3h'}{2rh},$$
$$R_{22} = -\frac{rf'}{2fh} + \frac{r'h'}{2h^2} + 2 - \frac{2}{h},$$
$$R_{33} = \left(-\frac{rf'}{2fh} + \frac{r'h'}{2h^2} + 2 - \frac{2}{h}\right) \sin^2 \alpha,$$
$$R_{44} = \left(-\frac{rf'}{2fh} + \frac{r'h'}{2h^2} + 2 - \frac{2}{h}\right) \sin^2 \alpha \sin^2 \theta.$$  

(2.5)

Here the primes denote the time derivatives. With the Ricci tensors in mind, we find the Einstein equation

$$G_{ab} = R_{ab} - \frac{1}{2}R = 8\pi T_{ab},$$  

(2.6)

where the scalar curvature $R$ is given by

$$R = \frac{f''}{fh} + \frac{(f')^2}{2f^2h} + \frac{f'h'}{2fh^2} - \frac{3f'}{rfh} + \frac{3h'}{rh^2} + \frac{6}{r^2} - \frac{6}{r^2h}.$$  

(2.7)

and $T_{ab}$ is the fluid stress-energy tensor which will be described in the next section in detail.

For the sake of completeness and ensuing uses in the next section, we first consider the vacuum solutions of the Einstein equation with $T_{ab} = 0$,

$$R_{ab} = 0.$$  

(2.8)
outside the five-dimensional Schwarzschild black hole. By adding the Ricci tensors $(1/f)R_{00}$ and $(1/h)R_{11}$ we obtain
\[
\frac{df}{f} + \frac{dh}{h} = 0,
\]
to yield with a constant $C_1$
\[
f = \frac{C_1}{h} = \frac{1}{h}.
\]
In the second equality in (2.10), we have used the fact that by rescaling the time coordinate, $t \rightarrow C_1^{1/2}t$, we may set $C_1 = 1$. Next, $R_{22} = 0$ in (2.5) produces
\[
\frac{d}{dr}(r^2 f) - 2r = 0,
\]
which yields
\[
f = 1 + \frac{C_2}{r^2}.
\]
We next consider the behavior of a test body in the five-dimensional Newtonian gravitational field of mass $M$, to fix the coefficient $C_2$ as follows: $C_2 = -\beta M$ with a new constant $\beta$
\[
\beta = \frac{8}{3\pi}.
\]
Substituting $f$ and $h$ in (2.10), (2.12) and (2.13) into (2.1), we obtain the vacuum solution outside the five-dimensional Schwarzschild black hole, whose metric is of the form
\[
ds_5^2 = -\left(1 - \frac{\beta M}{r^2}\right) dt^2 + \left(1 - \frac{\beta M}{r^2}\right)^{-1} dr^2 + r^2 [d\alpha^2 + \sin^2 \alpha (d\theta^2 + \sin^2 \theta d\phi^2)].
\]
Here one notes that at $r = 0$ we have a physical singularity while at $r = r_S$ we have a coordinate singularity where $r_S$ is given by
\[
r_S = (\beta M)^{1/2}.
\]

III. INTERIOR SOLUTIONS OF EINSTEIN EQUATION

In this section, we will investigate the equation of state associated with the inner structure of the Schwarzschild black hole in the five-dimensional total manifold. We observe here that the details of the ingredients such as the Ricci tensors and Einstein tensors in the five-dimensional spacetime are quite nontrivially different from those in the four-dimensional one, as shown in (2.5) and (3.4)-(3.6) below.

We now find the static and spherically symmetric solutions of Einstein equation. To do this, we assume the perfect fluid stress-energy tensor of the form
\[
T_{ab} = \rho u_a u_b + P(g_{ab} + u_a u_b),
\]
where $\rho$ and $P$ are the mass-energy density and pressure of the fluid. Here one notes that the fluid four-velocity $u^a$ should point in the same direction as the static Killing vector field, to yield
\[
u^a = -(e_0)^a = f^{1/2}(\partial_t)^a.
\]
The perfect fluid stress-energy tensor in (3.1) with flat indices ($A, B = 0, 1, 2, 3, 4$) then becomes
\[
T_{AB} = \text{diag} (\rho, P, P, P, P).
\]
Substituting (3.3) into (2.6), we next obtain the three independent equations
\[
8\pi \rho = G_{00} f^{-1} = \frac{3h'}{2rh^2} + \frac{3}{r^2} - \frac{3}{r^2 h},
\]
\[
8\pi P = G_{11} h^{-1} = \frac{3f'}{2rfh} - \frac{3}{r^2} + \frac{3}{r^2 h},
\]
\[
8\pi P = G_{22} r^{-2} = \frac{f'}{rfh} - \frac{h'}{2fh} + \frac{f''}{4fh} - \frac{(f')^2}{4fh^2} - \frac{f' h'}{4fh^2} - \frac{1}{r^2} + \frac{1}{r^2 h}.
\]
Here one notes that $G_{33}$ and $G_{44}$ reproduce the result in (3.6), since we have

$$G_{33} = G_{22} \sin^2 \alpha, \quad G_{33} = G_{22} \sin^2 \alpha \sin^2 \theta.$$  

(3.7)

Noticing that the equation in (3.4) contains only $h$, we shuffle this $\rho$-equation as follows

$$8\pi \rho = \frac{3}{2r^3} \frac{d}{dr} \left[ r^2 \left( 1 - \frac{1}{h} \right) \right],$$  

(3.8)

which yields

$$h(r) = \left[ 1 - \frac{\beta m(r)}{r^2} \right]^{-1}.$$  

(3.9)

Here $\beta$ is the constant in (2.13) and $m(r)$ is a mass in the five-dimensional spacetime defined by

$$m(r) = 2\pi^2 \int_0^r \rho(r') r' r^3 dr'.$$  

(3.10)

Moreover, we observe that a necessary condition for staticity is $h \geq 0$ to yield $r \geq [\beta m(r)]^{1/2}$. Assuming that $\rho = 0$ for $r > R$, we notes that the interior solution $h$ in (3.9) joins on the vacuum solution (2.14) with the total mass $M$ as follows

$$M = m(R) = 2\pi^2 \int_0^R \rho(r) r^3 dr,$$  

(3.11)

which is identical to the form for the total mass in Newtonian gravity in the five-dimensional spacetime. The total proper mass $M_p$ should be now defined on the proper volume element

$$[(^{(4)}g)]^{1/2} d^4x = h^{1/2} r^3 \sin^2 \alpha \sin \theta \ dr \ d\alpha \ d\theta \ d\phi,$$  

(3.12)

to yield

$$M_p = 2\pi^2 \int_0^R \rho(r) \left[ 1 - \frac{\beta m(r)}{r^2} \right]^{-1/2} r^3 dr.$$  

(3.13)

The difference $E_B = M_p - M$ can be interpreted as the gravitational binding energy of the configuration.

Now we assume that $f$ is given by the following form

$$f = e^{\beta \phi}$$  

(3.14)

then, from (3.5), we obtain

$$\frac{d\phi}{dr} \approx \frac{2[m(r) + \pi^2 r^4 P]}{r[r^2 - \beta m(r)]}.$$  

(3.15)

In the Newtonian limit where $r^4 P \ll m(r)$ and $m(r) \ll r^2$, (3.15) reduces to

$$\frac{d\phi}{dr} \approx \frac{2m(r)}{r^3}.$$  

(3.16)

Next, we need to find the equation of state inside the given black hole by exploiting the remnant equation (3.6). However it is not so trivial to manipulate (3.6) in order to arrive at the desired equation of state. Instead, we use the connection one-form $\omega_{\alpha\mu\nu}$ defined in terms of the fünfbein and inverse fünfbein in (2.2) and (2.3), respectively, as follows

$$\omega_{\alpha\mu\nu} = (e_{\mu})^b \nabla_a (e_{\nu})_b = (e_{\mu})^b \omega_{ab\nu}.$$  

(3.17)

Exploiting the Einstein equation in (2.6), we readily obtain

$$(e_{\mu})_b 8\pi \nabla_a T^{ab} = \omega_{ab\mu}(G_{ab} - 8\pi T_{ab}) - \nabla_a [(e_{\mu})_b (G^{ab} - 8\pi T^{ab})],$$  

(3.18)
which implies that the Einstein equation (2.6) is equal to
\[ \nabla_a T^{ab} = 0. \] (3.19)

The identity in (3.19) produces the following two equations
\[ (\rho + P) u^a \nabla_a u_b + (g_{ab} + u_a u_b) \nabla^a P = 0, \] (3.20)
\[ u^a \nabla_a \rho + (\rho + P) \nabla^a u_a = 0, \] (3.21)
which hold regardless of the dimensionality of the given target manifold. Exploiting (3.20), we arrive at an equation of state of hydrostatic equilibrium for the five-dimensional Schwarzschild black hole as follows
\[ \frac{dP}{dr} = -\beta (\rho + P) \frac{m(r) + 2\pi^2 r^4 P}{r[r^2 - 2m(r)]}. \] (3.22)
where we have used a Killing vector condition
\[ \nabla_a u_b + \nabla_b u_a = 0, \] (3.23)
and the following relations
\[ (e_1)_b u^a \nabla_a u_b = \frac{1}{2} \beta h^{-1/2} g^r, \]
\[ (e_1)_b g^{ab} \nabla_a P = h^{-1/2} \partial_r P. \] (3.24)

Here we note that, in the four-dimensional black hole, the corresponding equation of state is called Tolman-Oppenheimer-Volkoff equation and is given by [12–14]
\[ \frac{dP}{dr} = - (\rho + P) \frac{m(r) + 4\pi r^3 P}{r[r^2 - 2m(r)]}. \] (3.25)

In the Newtonian limit where \( r^4 P \ll m(r) \) and \( m(r) \ll r^2 \), (3.22) reduces to
\[ \frac{dP}{dr} \approx -\beta \rho \frac{m(r)}{r^3}. \] (3.26)

Here one notes that the spacetime metric inside the five-dimensional Schwarzschild black hole is given by
\[ ds^2 = -e^{\beta \phi} dt^2 + \left[ 1 - \frac{\beta m(r)}{r^2} \right]^{-1} dr^2 + r^2 [d\alpha^2 + \sin^2 \alpha (d\theta^2 + \sin^2 \theta d\phi^2)]. \] (3.27)

IV. CONCLUSIONS AND DISCUSSIONS

In summary, we have investigated the inner structure of the Schwarzschild black hole defined on the five-dimensional spacetime. More specifically, we have obtained the equation of state of hydrostatic equilibrium for the five-dimensional Schwarzschild black hole by exploiting the fünfbein scheme. We have formulated the Einstein equation for this five-dimensional Schwarzschild black hole. To do this, we have found the Ricci tensors and scalar curvature for the five-dimensional target manifold. Using the Einstein equation, we have constructed the interior solutions for the five-dimensional Schwarzschild black hole and the corresponding the spacetime metric inside the black hole.

We have several comments to address. First, we consider uniform density interior configuration of the five-dimensional black hole which consists of incompressible fluid of density \( \rho_0 \):
\[ \rho(r) = \begin{cases} \rho_0 & (r \leq R) \\ 0 & (r > R) \end{cases}, \] (4.1)
from which exploiting (3.10) we obtain
\[ m(r) = \frac{1}{2} \pi^2 \rho_0 r^4. \] (4.2)
In the Newtonian limit where $r^4 P \ll m(r)$ and $m(r) \ll r^2$, (3.26) yields
\[ P(r) \approx \frac{1}{4} \beta \pi^2 \rho_0^2 (R^2 - r^2), \]
(4.3)
which is consistent with the boundary condition $P(R) \equiv 0$. Moreover the pressure at the core is readily obtained to yield
\[ P_c \approx \frac{1}{4} \beta \pi^2 \rho_0^2 R^2, \]
(4.4)
which is also rewritten as follows
\[ P_c \approx \frac{1}{2^{3/2}} \beta \pi \rho_0^{3/2} M^{1/2}. \]
(4.5)
Here we have used the identity
\[ R = \left( \frac{2M}{\pi \rho_0} \right)^{1/4}. \]
(4.6)
After some algebra together with (4.2), the general relativistic expression for $P$ is then given by
\[ P(r) = \rho_0 \frac{(1 - \beta M/R^2)^{1/2} - (1 - \beta Mr^2/R^4)^{1/2}}{(1 - \beta Mr^2/R^4)^{1/2} - 2(1 - \beta M/R^2)^{1/2}}, \]
(4.7)
which yields the pressure at the core of the black hole defined in the four-dimensional spacetime
\[ P_c = \rho_0 \frac{(1 - \beta M/R^2)^{1/2} - 1}{1 - 2(1 - \beta M/R^2)^{1/2}}. \]
(4.8)
Here one can readily observe that, in the Newtonian limit $P_c$, the central pressure in (4.8) reduces to (4.4).

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