The singular value decomposition of the dynamic ray transforms operators acting on 2-tensor fields in $\mathbb{R}^2$

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Abstract. We consider the problem of the dynamic two-dimensional 2-tensor tomography. An object motion is a combination of rotation and shifting. Properties of the dynamic longitudinal, mixed and transverse ray transform operators are investigated. The singular value decompositions of the operators with usage of the classic orthogonal polynomials are constructed.

1. Introduction
Presently, the main trends in tomography, focused on studying vector and tensor characteristics of media, are intensively developing [1]–[3]. This is because of the fact that the areas of application of tomographic methods to studying nonscalar properties of objects are wide. There are conventional applications of the problem to photoelasticity and fiber optics [4], plasma diagnostics [5]. In recent years, there arise absolutely new directions of development such as the polarization tomography of quantum radiation [6], the tensor fields of residual stress [7], magneto-photoelasticity [8], the diffractive tomography of strains [9]. The new directions in the tensor tomography have applications in medicine and biology (see, for example, [10], [11]).

The inversion formulas [1], [12]–[14] are one of the main mathematical tools on which numerical methods and algorithms for solving the vector and the 2-tensor tomography problems are based. Moreover, of particular interest are three general methods: the least squares method, the method of approximate inverse and the method of singular value decomposition. In the numerical solution of the tomography problems in $\mathbb{R}^2$, the least squares method was used with approximating the sequences constructed using polynomials [15], [16] and $B$-splines [17]–[19]. The method of approximate inverse was successfully applied to solve the vector tomography problems in $\mathbb{R}^3$ [20]–[22] and the vector [23], [24], 2-tensor [25] and $m$-tensor tomography problems [26] in $\mathbb{R}^2$. Let us note the published works in which singular value decompositions of the ray transforms operators of a two-dimensional vector [27], [28] and 2-tensor fields were obtained [29], [30] and numerical studies of algorithms based on the truncated singular value decomposition method were performed [31], [32].

All the above results were obtained under the assumption that an object of the study is stationary. But in practice, this assumption in many cases is false. Such problems are called the dynamic tomography problems. There is a small number of papers, which are devoted to solving the dynamic tomography problems to restore the scalar [33]–[35] and the vector characteristics [36] of objects. However the 2-tensor case was not previously considered.
In this paper, we give definitions of the dynamic longitudinal, mixed and transverse ray tomography transforms, which act on symmetric 2-tensor fields. We consider the dynamic 2-tensor tomography problem in the following formulation. It is necessary to reconstruct a symmetric 2-tensor field (or a part of the field) by its known values of the dynamic longitudinal, and/or mixed, and/or transverse ray transforms. We assume that the movement of the object under study is known and is a combination of rotation and shifting. The properties of the dynamic ray transform operators are investigated, and their singular value decompositions are constructed.

2. Definitions and statement of the static tomography problems

Let \( x = (x_1, x_2) \), \( B = \{ x \in \mathbb{R}^2 \mid |x| = \sqrt{x_1^2 + x_2^2} < 1 \} \) be a unit disk with the boundary \( \partial B = \{ x \in \mathbb{R}^2 \mid |x| = 1 \} \) and \( Z = \{ (s, \xi) \mid \xi \in \mathbb{R}^2, |\xi| = 1, s \in \mathbb{R} \} \) be a cylinder. The functional space \( L_2(B) \) consists of functions, which are square integrable in \( B \). A set of symmetric \( m \)-tensor fields \( v(x) = (v_{i_1 \ldots i_m}(x)) \), where \( i_1, \ldots, i_m = 1, 2 \), defined in \( B \) is denoted by \( S^m(B) \). In this paper, we deal only with \( m = 1, 2 \). The inner product in \( S^m(B) \) is introduced by the formula

\[
\langle u(x), v(x) \rangle = \sum_{i_1, \ldots, i_m=1}^2 u_{i_1 \ldots i_m}(x)v_{i_1 \ldots i_m}(x).
\]

We need the space of the square integrable symmetric \( m \)-tensor fields \( L_2(S^m(B)) \). The inner product in \( L_2(S^m(B)) \) is defined as

\[
(u, v)_{L_2(S^m(B))} = \int_B \langle u(x), v(x) \rangle dx.
\]

The Sobolev spaces are denoted by \( H^k(S^m(B)) \) and \( H^k_0(S^m(B)) \). We also use functions in the weight space \( L_2(Z, \rho) \), \( \rho > 0 \). The inner product in the space \( L_2(Z, \rho) \) is defined as

\[
(f, g)_{L_2(Z, \rho)} = \int_Z f(s, \xi)g(s, \xi)\rho(s, \xi) \, ds \, d\xi.
\]

The operators of inner derivation \( d \) and inner \( \perp \) derivation \( d^\perp \) are the compositions of operators of a covariant derivation and symmetrization \( d, d^\perp : H^k(S^m(B)) \rightarrow H^{k-1}(S^{m+1}(B)) \) and act on the function \( f \) and the vector field \( v \) according to the formulas

\[
(dv)_i = \frac{\partial f}{\partial x_i}, \quad (dv)_{ij} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right),
\]

\[
(d^\perp f)_i = (-1)^{i} \frac{\partial f}{\partial x_{3-i}}, \quad (dv)_{ij} = \frac{1}{2} \left( (-1)^{i} \frac{\partial v_i}{\partial x_{3-j}} + (-1)^{j} \frac{\partial v_j}{\partial x_{3-i}} \right).
\]

The divergence operator \( \text{div} : H^k(S^m(B)) \rightarrow H^{k-1}(S^{m-1}(B)) \) acts on the vector field \( v \) and on the symmetric 2-tensor field \( w \) according to the formulas

\[
\text{div} v = \sum_{i=1}^2 \frac{\partial v_i}{\partial x_i}, \quad \text{div} w = \sum_{i=1}^2 \frac{\partial w_{ji}}{\partial x_i}.
\]

Recall that a symmetric \( m \)-tensor field \( w \in H^k(S^m(B)) \) is called potential if there is a tensor field \( v \in H^{k+1}(S^{m-1}(B)) \) such that \( w = \text{div} v \). A tensor field \( w \in H^k(S^m(B)) \) is called solenoidal
if $\text{div} \, w = 0 \in H^{k-1}(S^{m-1}(B))$. In other words, there is a function $\psi$ such that $w = (d^\perp)^2 \psi$ (see, for example, [12]). It is well known [1] that there is a unique decomposition of an arbitrary symmetric $m$-tensor field $w$ to a sum of potential and solenoidal parts

$$w = s w + d w, \quad \text{div} \, s w = 0, \quad v \in H^1_0(S^{m-1}(B)).$$

There exists a more detailed decomposition [12], [14] of a symmetric 2-tensor field $w$ to a sum of the three terms

$$w = d^2 \varphi + dd^\perp \phi + (d^\perp)^2 \psi,$$

where

$$\varphi \in H^2_0(B), \quad \phi \in H^2(B), \quad d^\perp \phi \in H^1_0(S^1(B)), \quad \psi \in H^2(B).$$

The Radon transform $\mathcal{R} f : L_2(B) \to L_2(\mathbb{Z})$ of the function $f$ is defined by the formula

$$[\mathcal{R} f](s, \xi) = \int_B f(x) \delta(\langle \xi, x \rangle - s) \, dx,$$

where $\delta$ denotes the delta distribution, the unit vector $\eta = \xi^\perp = (-\sin \alpha, \cos \alpha)$ specifies the direction of integration, $\xi = (\cos \alpha, \sin \alpha)$ is the normal vector.

The longitudinal $\mathcal{P}$, transverse $\mathcal{P}^\perp$ and mixed $\mathcal{P}^*$ ray transforms

$$\mathcal{P}, \mathcal{P}^\perp, \mathcal{P}^* : L_2(S^2(B)) \to L_2(\mathbb{Z})$$

of a symmetric 2-tensor field $w = (w_{11}, w_{12}, w_{22})$ are defined by the formulas

$$[\mathcal{P} w](s, \xi) = \int_B \langle w(x), \eta^2 \rangle \delta(\langle \xi, x \rangle - s) \, dx = \int_B \left( \sum_{i,j=1}^2 w_{ij}(x) \eta_i \eta_j \right) \delta(\langle \xi, x \rangle - s) \, dx,$$

$$[\mathcal{P}^\perp w](s, \xi) = \int_B \langle w(x), \xi^2 \rangle \delta(\langle \xi, x \rangle - s) \, dx = \int_B \left( \sum_{i,j=1}^2 w_{ij}(x) \xi_i \xi_j \right) \delta(\langle \xi, x \rangle - s) \, dx,$$

$$[\mathcal{P}^* w](s, \xi) = \int_B \langle w(x), \xi \eta \rangle \delta(\langle \xi, x \rangle - s) \, dx = \int_B \left( \sum_{i,j=1}^2 w_{ij}(x) \xi_i \eta_j \right) \delta(\langle \xi, x \rangle - s) \, dx.$$

The operators of longitudinal, transverse and mixed ray transforms have nonzero kernels [12]. Namely, for each $\varphi \in H^2_0(B)$, we have

$$[\mathcal{P} (d^\perp)^2 \varphi](s, \xi) = [\mathcal{P} \, dd^\perp \varphi](s, \xi) = 0,$$

$$[\mathcal{P}^\perp d^2 \varphi](s, \xi) = [\mathcal{P}^\perp \, dd^\perp \varphi](s, \xi) = 0,$$

$$[\mathcal{P}^* d^2 \varphi](s, \xi) = [\mathcal{P}^* (d^\perp)^2 \varphi](s, \xi) = 0.$$

Moreover, there are connections between the ray transforms and the Radon transform of the same potential $\varphi \in H^2_0(B)$ [12]:

$$[\mathcal{P} (d^\perp)^2 \varphi](s, \xi) = [\mathcal{P}^\perp d^2 \varphi](s, \xi) = 2[\mathcal{P}^* \, dd^\perp \varphi](s, \xi) = \frac{\partial^2 (\mathcal{R} \varphi)}{\partial s^2}(s, \xi).$$

In other words, in the general case, for the complete recovery of a symmetric 2-tensor field, it is necessary to know the values of all the three ray transforms.

We can formulate the 2-tensor tomography problem in the stationary case. Let the longitudinal ray transform $\mathcal{P} w$, and/or the transverse ray transform $\mathcal{P}^\perp w$, and/or the mixed ray transform $\mathcal{P}^* w$ of a symmetric 2-tensor field $w$ be known for all $(s, \xi) \in \mathbb{Z}$. It is required to determine the unknown tensor field $w(x)$, $x \in B$ (or a part of $w$) from these data.
3. The singular value decomposition of the static ray transform operators

It is necessary to recall about a singular value decomposition of the ray transforms operators in the stationary case [30], [32]. Taking in consideration the connection between the ray transform operators, we can consider only the transverse ray transform operator, acting on a potential symmetric 2-tensor field \( P_\perp d^2 \varphi, \varphi \in H_0^2(B) \). The singular value decompositions for the longitudinal and mixed ray transform operators can be analogously obtained.

**Lemma.** The singular value decomposition of the transverse ray transform operator

\[ P_\perp : L^2(S^2(B)) \to L^2(Z, (1 - s^2)^{-1/2}) \]

is given by

\[ \{ (\sigma_{kn}, u_{kni}, v_{kni}) | k, n = 0, 1, \ldots, i = 1, 2 \}, \]

where

\[ \sigma_{kn} = \sqrt{\frac{4\pi}{k + 2n + 3}}, \]

\[ u_{kni}(x) = \begin{cases} d^2 \left( \frac{k + 2n + 3}{8\pi} \frac{C_{n+k}^k}{(n+1)(n+2)} (1 - |x|^2)^2 H_k^i(x) P_{n}^{(k+3,k+1)}(|x|^2) \right), & k > 0, \\ d^2 \left( \frac{2n + 3}{16\pi} \frac{1}{(n+1)(n+2)} (1 - |x|^2)^2 P_{n}^{(3,1)}(|x|^2) \right), & k = 0, \end{cases} \]

\[ v_{kni}(s, \alpha) = (-1)^n \frac{\sqrt{2}}{\pi} \sqrt{1 - s^2} C_{k+2n+2}^{(1)}(s) Y_k^i(\alpha). \]

Here \( P_n^{(\rho,\eta)} \) are the Jakobi polynomials, \( C_n^{(\mu)} \) are the Gegenbauer polynomials. The harmonic polynomials \( H_k^i(x) \) are defined in the polar coordinate system \( (x = r \xi) \) by the formulas

\[ H_k^1(r, \alpha) = r^k Y_k^1(\alpha) = r^k \cos k\alpha, \quad k \geq 0, \]

\[ H_k^2(r, \alpha) = r^k Y_k^2(\alpha) = r^k \sin k\alpha, \quad k \geq 1. \]

4. Statement of the dynamic 2-tensor tomography problems

Due to the rotation of the x-ray source, the data collected in computer tomography takes a definite amount of time. Via \( \xi = (\cos(\phi t), \sin(\phi t))^T \) we denote the position of a radiation source. Here \( t \) is the time instant, \( \phi \) is the source rotation angle. Therefore, we have \( \eta = (-\sin(\phi t), \cos(\phi t))^T \).

We assume that the motion of the investigated object is described by

\[ \Gamma_{\theta} x = A_{\theta} x + b_{\theta}, \quad (1) \]

where the matrix

\[ A_{\theta} = \begin{pmatrix} \cos(\theta t) & -\sin(\theta t) \\ \sin(\theta t) & \cos(\theta t) \end{pmatrix} \]

sets the rotation by the angle \( \theta t \) at the time instant \( t \) from the beginning of data collection, the vector \( b_{\theta} \in \mathbb{R}^2 \) specifying the shifting.

In the frame of the setting, the dynamic Radon transform

\[ \mathcal{R}_{\Gamma} : L^2(B) \to L^2(Z), \]

is given by

\[ (\mathcal{R}_{\Gamma} f)(s, \xi) = \int_B f(\Gamma_{\theta} x) \delta(|\xi, x| - s) \, dx. \]
Then the dynamic longitudinal, transverse and mixed ray transforms

\[ \mathcal{P}_\Gamma, \mathcal{P}_\Gamma^\perp, \mathcal{P}_\Gamma^* : L_2(S^2(B)) \to L_2(Z) \]

can be defined by

\[
\begin{align*}
(\mathcal{P}_\Gamma w)(s, \xi) &= \int_B \langle w(\Gamma_\theta x), (A_\theta \eta)^2 \rangle \delta(\langle \xi, x \rangle - s) \, dx, \\
(\mathcal{P}_\Gamma^\perp w)(s, \xi) &= \int_B \langle w(\Gamma_\theta x), (A_\theta \xi)^2 \rangle \delta(\langle \xi, x \rangle - s) \, dx, \\
(\mathcal{P}_\Gamma^* w)(s, \xi) &= \int_B \langle w(\Gamma_\theta x), A_\theta \xi A_\theta \eta \rangle \delta(\langle \xi, x \rangle - s) \, dx.
\end{align*}
\]

The statement of the 2-tensor tomography problem in the dynamic case: let the longitudinal ray transform \( \mathcal{P}_\Gamma w \), and/or the transverse ray transform \( \mathcal{P}_\Gamma^\perp w \), and/or the mixed ray transform \( \mathcal{P}_\Gamma^* w \) of a symmetric 2-tensor field \( w \) be known for all \( (s, \xi) \in Z \). It is required to determine the unknown 2-tensor field \( w(x), x \in B \) (or a part of \( w \)) from these data at the time moment \( t = 0 \).

5. The singular value decomposition of the dynamic ray transform operators

To solve the dynamic 2-tensor tomography problem, we construct singular value decompositions of the dynamic ray transform operators.

**Theorem 1.** In the case of motion (1), there is a connection between the dynamic Radon transform and the dynamic ray transforms

\[
[\mathcal{P}_\Gamma \,(d^I)^2 \varphi](s, \xi) = [\mathcal{P}_\Gamma^\perp \, d^2 \varphi](s, \xi) = 2[\mathcal{P}_\Gamma^* \, d^I \varphi](s, \xi) = \frac{\partial^2 [\mathcal{R}_\Gamma \varphi]}{\partial s^2}(s, \xi), \quad \varphi \in H^2_0(B).
\]

**Proof.** Introduce the notation \( \Gamma_\theta x := \tilde{x} = (\tilde{x}_1, \tilde{x}_2) \), then the coordinates of points of the integrating line \( P_{s,\xi} \) are defined by the following formulas

\[
\begin{align*}
\tilde{x}_1 &= s \cos(\phi + \theta)t - p \sin(\phi + \theta)t + (b_\theta)_1, \\
\tilde{x}_2 &= s \sin(\phi + \theta)t + p \cos(\phi + \theta)t + (b_\theta)_2,
\end{align*}
\]

and we have

\[
\begin{align*}
\frac{\partial^2 [\mathcal{R}_\Gamma \varphi]}{\partial s^2}(s, \xi) &= \frac{\partial^2}{\partial s^2} \int_B \varphi(\Gamma_\theta x) \delta(\langle \xi, x \rangle - s) \, dx = \frac{\partial^2}{\partial s^2} \int_{B \cap P_{s,\xi}} \varphi(\tilde{x}_1(s, p), \tilde{x}_2(s, p)) \, dp \\
&= \int_{B \cap P_{s,\xi}} \left( \frac{\partial^2 \varphi}{\partial x_1^2} \left( \frac{\partial \tilde{x}_1}{\partial s} \right)^2 + 2 \frac{\partial^2 \varphi}{\partial x_1 \partial x_2} \frac{\partial \tilde{x}_1}{\partial s} \frac{\partial \tilde{x}_2}{\partial s} + \frac{\partial^2 \varphi}{\partial x_2^2} \left( \frac{\partial \tilde{x}_2}{\partial s} \right)^2 \right) \, dp \\
&= \int_B \langle d^2 \varphi(\Gamma_\theta x), (A_\theta \xi)^2 \rangle \delta(\langle \xi, x \rangle - s) \, dx = [\mathcal{P}_\Gamma^\perp \, d^2 \varphi](s, \xi).
\end{align*}
\]
On the other hand, there are equalities

\[ \frac{\partial^2 [R \varphi]}{\partial s^2} (s, \xi) = \int_{B \cap P_s \xi} \left( \frac{\partial^2 \varphi}{\partial x_1^2} \left( \frac{\partial \bar{x}_1}{\partial s} \right)^2 + 2 \left( - \frac{\partial^2 \varphi}{\partial x_1 \partial \bar{x}_2} \right) \cdot \frac{\partial \bar{x}_1}{\partial s} \left( - \frac{\partial \bar{x}_2}{\partial s} \right) + 2 \frac{\partial^2 \varphi}{\partial x_2^2} \cdot \left( - \frac{\partial \bar{x}_2}{\partial s} \right)^2 \right) dp \]

\[ = \int_{B} \langle (d^+ \varphi (\Gamma \theta \mathbf{x}), (A_0 \eta)^2) \delta (\xi, \mathbf{x}) - s \rangle d\mathbf{x} = [P \varphi] (s, \xi). \]

Now we consider \([P^* \varphi \langle (\Gamma \theta \mathbf{x}), (A_0 \eta)^2) \delta (\xi, \mathbf{x}) - s \rangle d\mathbf{x} = [P \varphi] (s, \xi).\]

Therefore, we have

\[ [\varphi] (d^+ \varphi) (s, \xi) - [P \varphi \langle (\Gamma \theta \mathbf{x}), (A_0 \eta)^2) \delta (\xi, \mathbf{x}) - s \rangle d\mathbf{x} = \frac{1}{2} \int_{B \cap P_s \xi} \left( \frac{\partial^2 \varphi}{\partial x_1^2} \left( \frac{\partial \bar{x}_1}{\partial s} \right)^2 + \frac{\partial^2 \varphi}{\partial x_2^2} \left( \frac{\partial \bar{x}_2}{\partial s} \right)^2 \right) dp \]

\[ = \frac{1}{2} \int_{B \cap P_s \xi} \left( \frac{\partial^2 \varphi}{\partial x_1^2} + \frac{\partial^2 \varphi}{\partial x_2^2} \right) dp = \frac{1}{2} [R \varphi] (s, \xi). \]

It is well known [37] that

\[ [R \varphi] (s, \xi) = \frac{\partial^2 [R \varphi]}{\partial s^2} (s, \xi). \]

For the dynamic Radon transform \( R \varphi \), a similar equality is also true. Thus, we obtain

\[ [P^* \varphi \langle (\Gamma \theta \mathbf{x}), (A_0 \eta)^2) \delta (\xi, \mathbf{x}) - s \rangle d\mathbf{x} = \frac{1}{2} \frac{\partial^2 [R \varphi]}{\partial s^2} (s, \xi). \]

**Theorem 2.** In the case of motion (1), there are connections between the dynamic ray transforms and the static ray transforms:

\[ \langle P \varphi \rangle (A_0 \xi, b_\theta), A_0 \xi), \]

\[ \langle P^* \varphi \rangle (A_0 \xi, b_\theta), A_0 \xi). \]

**Proof.** We demonstrate this for the transverse ray transforms (static and dynamic). For the longitudinal and mixed ray transforms proof is analogously. We have

\[ [P^\perp \varphi] (s, \xi) = \int_{B} \langle w (A_0 \mathbf{x} + b_\theta), (A_0 \xi)^2 \rangle \delta (\xi, \mathbf{x}) - s \rangle d\mathbf{x} \]

\[ = \int_{B} \langle w (y), (A_0 \xi)^2 \rangle \delta (\xi, A_0^{-1} (y - b_\theta)) - s \rangle d\mathbf{y} \]

\[ = \int_{B} \langle w (y), (A_0 \xi)^2 \rangle \delta ((A_0^{-1})^T \xi, y - b_\theta) - s \rangle d\mathbf{y} \]

\[ = \int_{B} \langle w (y), (A_0 \xi)^2 \rangle \delta ((A_0 \xi, y) - (A_0 \xi, b_\theta) + s) \rangle d\mathbf{y} \]

\[ = [P^\perp w] (s + (A_0 \xi, b_\theta), A_0 \xi). \]
**Theorem 3.** In the case of motion (1), the singular value decomposition of the dynamic transverse ray transform operator

\[ \mathcal{P}_\perp^\Gamma : L_2(S^2(B)) \to L_2(Z, (1 - s^2)^{-1/2}) \]

is given by

\[ \{ (\sigma_{kn}, u_{kni}, v_{kni}) \mid k, n = 0, 1, ..., i = 1, 2 \}, \]

where

\[ \sigma_{kn} = \sqrt{\frac{4\pi}{k + 2n + 3}}, \]

\[ u_{kni}(x) = \begin{cases} 
  d^2 \left( \sqrt{\frac{k + 2n + 3}{8\pi}} \frac{C_n^{(k+k)}(1 - |x|^2)^2 H_k^{(1)}(x) P_n^{(k+3,k+1)}(|x|^2)}{(n+1)(n+2)} \right), & k > 0, \\
  d^2 \left( \sqrt{\frac{2n + 3}{16\pi}} \frac{1}{(n+1)(n+2)} (1 - |x|^2)^2 P_n^{(3,1)}(|x|^2) \right), & k = 0, 
\end{cases} \]

\[ v_{kni}(\tilde{s}, \tilde{\alpha}) = (-1)^n \sqrt{\frac{2}{\pi}} \sqrt{1 - s^2} C_{k+2n+2}(\tilde{s}) Y_k^{(1)}(\tilde{\alpha}). \]

with \( \tilde{s} = s + \langle A\theta \xi, b_\theta \rangle, \tilde{\alpha} = (\phi + \theta)t \).

**Proof.** This follows from the Lemma and Theorem 2. □

Recall that the singular value decompositions of the dynamic longitudinal and mixed ray transform operators can be obtained using Theorems 1 and 3.

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