Analytical and qualitative investigation of COVID-19 mathematical model under fractional differential operator

Kamal Shah1 | Muhammad Sher1 | Hussam Rabai’ah2,3 | Ali Ahmadian4,5 | Soheil Salahshour6 | Bruno A. Pansera7

1Department of Mathematics, University of Malakand, Chakdara, Pakistan
2College of Engineering, Al Ain University, Al Ain, UAE
3Mathematics Department, Tafila Technical University, Tafila, Jordan
4Institute of IR 4.0, The National University of Malaysia, Bangi, Malaysia
5Department of Mathematics, Near East University, Nicosia, Turkey
6Faculty of Engineering and Natural Sciences, Bahcesehir University, Istanbul, Turkey
7Department of Law, Economics and Human Sciences, Mediterranea University of Reggio Calabria, Reggio Calabria, Italy

Correspondence
Ali Ahmadian, Institute of IR 4.0, The National University of Malaysia, Bangi, 43600 UKM, Selangor, Malaysia and Department of Mathematics, Near East University, Nicosia, Turkey.
Email: ali.ahmadian@ukm.edu.my

Communicated by: M. Efendiev

1 INTRODUCTION

In the last of 2019, a dangerous outbreak was confirmed in the big city of China called Wuhan which was due to coronavirus. There are many theories behind the virus outbreak in present-day literature. Some researchers have studied that it originated from bats to humans owing to the complete unlawful transmission of animals on a Wuhan seafood market. The virus involved has been identified both in dogs and in pangolin. Hence, it has been considered that several infected cases claimed that they had been working in a local fish and wild animal market in Wuhan from where they got the infection of COVID-19. Afterward, the researchers have confirmed that the disease’s widespread nature is caused by direct contact between people. The mentioned disease was then named COVID-19. Nearly 100,000 people were infected in China up to March 2020. Then, the virus was transmitted from person to person very rapidly and spread up to the end of March 2020 in the whole globe. This pandemic is ongoing, and more than 7.38 million people almost in every country of the globe have been infected. Nearly 0.41 million people died from this pandemic. WHO announced that it is a global pandemic where scientists and researchers have not found success in controlling or curing this disease.1,2

Here, we remark that one way to open up this disease is due to immigration of the infected population from one area to another which affects the healthy population and spreads this disease very rapidly. In fact, it is like the flu which
has killed millions people in past. Therefore, globally, all the countries of the entire world have reduced their activities to some restricted domain. Every country of the world restricts the movement of the people from one place to other. Currently, researchers work on this outbreak. Various models in mathematical form have been developed for these diseases. For some study, we refer previous works and work cited therein. All the diseases are dangerous, but infectious diseases greatly effect the health as well as economy of a state. In the past for many diseases, controlling procedure and its future predictions are made by using mathematical modeling. Considering the threat of current pandemic, researchers have studied the COVID-19 disease from different aspects. Mathematical models in this regard can play a vital role to restrict the disease spreading. For this need very recently, some models have been studied for COVID-19; see detail in other works.

It is often desirable to describe the behavior of some real-life system or phenomenon, whether physical, sociological, or even economic, in mathematical terms. The mathematical description of a system of phenomenon is called a mathematical model and is constructed with certain goals in mind. Put it simply, mathematical modeling should become part of the toolbox of public health research and decision making. One way to study problem dynamics in the real world is to use mathematical modeling. Mathematical modeling is the ability to convert real-world problems into mathematical formulations whose theoretical and numerical analysis can provide useful insights and guidance for native applications. In the field of mathematics, many mathematical models have been built according to the assumptions of researchers. The purpose of mathematical modeling is to represent different types of real world situation in the language of mathematics. To find out the different dynamics of a disease and therefore to overcome it at an early stage, mathematical modeling plays an important role there. Thus simply, we can say that mathematical modeling has important role to study a disease and control of the disease specially when vaccination is absent. A lot of mathematical models exist in the literature on stability, existence theory, and optimization of biological models; for example, see (other works). Currently, for COVID-19, some models have been constructed to study its different aspects (see Lai et al. and Fanelli and Piazza). In this way, Lin and his coauthors in their study model COVID-19 under integer-order derivative as

\[
\begin{align*}
    S'(t) &= -\frac{b_0SG}{N} - \frac{b(t)SI}{N} - uS, \\
    E'(t) &= \frac{b_0SG}{N} + \frac{b(t)SI}{N} - (d + u)E, \\
    I'(t) &= dE - (l + u)I, \\
    R'(t) &= lI - uR, \\
    N'(t) &= -uN, \\
    D'(t) &= vlI - zD, \\
    C'(t) &= dE,
\end{align*}
\]

where \( b(t) = b_0(1 - a)\left(1 - \frac{D}{N}\right)^k \), \( S(t) \) is the susceptible populations, \( E(t) \) is exposed populations, \( I(t) \) is infectious populations, \( R(t) \) is removed population (recovered and dead), \( N(t) \) is total populations, \( D(t) \) is mimicking the public perception of risk regarding the number of severe and critical cases and deaths, and \( C(t) \) is the cumulative cases (reported and not reported). The authors establish this model on Wuhan city for the month of March 2020.

Detail of various parameters of model (1) is given as follows:

- \( G \) No. of zoonotic cases
- \( b_0 \) Rate of transmission
- \( a \) Action strength of government
- \( k \) Responds to intensity
- \( u \) Rate of emigration
- \( d^{-1} \) Latent mean period
- \( l^{-1} \) Infectious mean period
- \( v \) Severe cases proportion
- \( z^{-1} \) Public reaction mean duration

As we know that most of biological models are based on the classical approach to get a system of nonlinear autonomous first-order differential equations. Therefore, there is still room to develop such a mathematical models by the tools of advanced fractional calculus. Several novel studies, including numerous biological models, have shown to be extremely
helpful and more accurate than their counterparts. For example, Ullah et al.\(^3^0\) proposed a tuberculosis (TB) infection mathematical model for the Khyber Pakhtunkhwa province of Pakistan involving Caputo fractional-order derivative and tested it by the real data of the mentioned province and shows the advantages of fractional-order model. Khan et al.\(^3^1\) introduced pine wilt disease model involving fractional-order derivative of Caputo–Fabrizio type and confirmed its productiveness by establishing its unique solution. Qureshi and Atangana\(^3^2\) analyzed ordinary and fractional-order models of dengue outbreak in Cape Verde islands during 2009 and showed that fractional-order operator in the sense of Caputo–Fabrizio has the smallest squared sum of errors. Muhammad and Atangana\(^3^3\) analyzed the mathematical model of infectious disease called Ebola via Caputo, Caputo–Fabrizio, and Atanagana–Baleanu operators. The advanced studies show that the fractal–fractional-order operators are the best tools to analyze the mathematical models for real-world data; for detail, see previous studies.\(^3^4\)–\(^3^6\) The beauty of fractional calculus is that fractional derivatives (and integrals) are not a local (or point) property (or quantity). Thereby, this considers the history and nonlocal distributed effects. In other words, perhaps this subject translates the reality of nature better. Therefore, to make this subject available as popular subject to science and engineering community, it adds another dimension to understand or describe basic nature in a better way.

Existence theory is one of the interesting areas of differential equations. This area has been studied very well in the last couple of decades. In the present literature, various approaches were utilized to show existence and uniqueness of solution of differential and integral equations. Krassnoselskii’s fixed point theorem, Leray–Schauder fixed point theorem, Schaefer’s fixed point theorem, and degree theory are commonly used to investigate the solutions for fractional-order differential equations. See previous studies.\(^3^7\)–\(^3^9\)–\(^4^0\)–\(^4^1\)

Because most of nonlinear problems cannot be solved for exact solution, we need powerful numerical or some analytical techniques. For good numerical results, one needs stable algorithms and methods. For such need, stability theory was founded. In the literature, there are different types of stability. But the most important type is Ulam–Hyers type stability which was introduced by Ulam\(^4^3\) in 1940 and further studied by Hyers\(^4^4\) in 1941. This stability answers the question when is it true that a function which approximately satisfies a functional equation must be close to an exact solution of this functional equation? If the problem accepts a solution, we say that the functional equation is stable. The mentioned stabilities have been studied for different physical problem; for example, see previous works.\(^4^2\)–\(^4^5\)–\(^4^6\)

Inspired of the work of Lin et al.\(^1^5\) and realistic nature of fractional calculus, we consider model (1) for existence, Ulam–Hyers stability, and for semianalytical solution under Caputo fractional-order derivative as

\[
\begin{align*}
{^c}D^\alpha_0 S(t) &= -\frac{b_0 SG}{N} - \frac{b(t)SI}{N} - uS, \\
{^c}D^\alpha_0 E(t) &= \frac{b_0 SG}{N} - \frac{b(t)SI}{N} - (d + u)E, \\
{^c}D^\alpha_0 I(t) &= dE - (l + u)I, \\
{^c}D^\alpha_0 R(t) &= lI - uR, \\
{^c}D^\alpha_0 N(t) &= -uN, \\
{^c}D^\alpha_0 D(t) &= vlI - zD, \\
{^c}D^\alpha_0 C(t) &= dE.
\end{align*}
\]

with \(0 < \alpha \leq 1\), \(b(t) = b_0(1 - a)\left(1 - \frac{p}{7}\right)^k\)

Initial conditions assume as follows:

\[
\begin{align*}
S(0) &= S_0, \\
E(0) &= E_0, \\
I(0) &= I_0, \\
R(0) &= R_0, \\
N(0) &= N_0, \\
D(0) &= D_0, \\
C(0) &= C_0.
\end{align*}
\]

First, we will develop qualitative theory for the above model, then we will discuss Ulam–Hyers stability and, finally, an approximate solution of the considered model (2) to be discussed. In our analysis, we used “Banach and Schaefer’s fixed point theorem” to study the existence and stability of the mentioned model. At last, a general scheme is established to
obtain series type solution. Also, we testify the approximate results for the model (2) on the real data of Pakistan for the last 60 days taken from a source. At the end, graphical results are given with detail discussion.

2 | FUNDAMENTAL RESULTS

Some basic results are recall as follows.

Definition 2.1 Kilbas et al.27 The fractional integral of order \( \alpha \in \mathbb{R}^+ \), for a function \( W(t) \in L^1([a, b], \mathbb{R}) \) is defined as

\[
I_0^\alpha W(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} W(s) \, ds,
\]

(4)

where \( 0 < \alpha \leq 1 \) and integral on the right is assumed to converge.

Definition 2.2 Kilbas et al.27 The Caputo fractional-order derivative of a function \( W(t) \in C^n[a, b] \) is defined as

\[
cD_0^\alpha W(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} W^{(n)}(s) \, ds;
\]

(5)

here, \( 0 < \alpha \leq 1 \), \( n = [\alpha] + 1 \), and \( [\alpha] \) shows integer part of \( \alpha \).

Lemma 2.3. Kilbas et al.27 Suppose \( U(t) \in C([0, \tau]) \), then the solution of fractional-order differential equations of the form

\[
\begin{cases}
cD_0^\alpha W(t) = U(t), \quad t \in [0, \tau], \quad n - 1 < \alpha < n, \\
W(0) = U_0
\end{cases}
\]

(6)

is given by

\[
W(t) = \sum_{i=0}^{n-1} c_i t^i,
\]

(7)

for \( c_i \in \mathbb{R}, \, i=0,1,2,\ldots,n-1 \).

Definition 2.4 Podlubny.48 The Laplace transform of Caputo fractional-order derivative is defined as follows:

\[
\mathcal{L}[cD_0^\alpha W(t)] = s^\alpha W(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} W^{(k)}(0), \quad n - 1 < \alpha < n.
\]

For a Banach’s space \( X = C([0, \tau]) \) with norm

\[
\|W\| = \max_{t \in [0, \tau]} \{ |W| \}, \text{ for all } W \in X,
\]

we have the following results.

Theorem 2.5. Granas and Dugundji49 Let \( X \) be a Banach space and \( T : X \to X \) compact and continuous. If the set

\[
E = \{ W \in X : W = \lambda TW, \lambda \in (0, 1) \},
\]

(8)

is bounded, then \( T \) has a fixed point.
First of all, we will consider model (2) for qualitative theory. For this, we put the right hand side of model (2) as follows:

\[
\begin{align*}
W_1(t, S, E, I, R, N, D, C) &= -\frac{b_0 S G}{N} - \frac{b(t)SI}{N} - uS, \\
W_2(t, S, E, I, R, N, D, C) &= \frac{b_0 SG}{N} + \frac{b(t)SI}{N} - (d + u)E, \\
W_3(t, S, E, I, R, N, D, C) &= dE - (l + u)I, \\
W_4(t, S, E, I, R, N, D, C) &= lI - uR, \\
W_5(t, S, E, I, R, N, D, C) &= -uN, \\
W_6(t, S, E, I, R, N, D, C) &= vI - zD, \\
W_7(t, S, E, I, R, N, D, C) &= dE.
\end{align*}
\] (9)

Using Equation (9), model (2) can be expressed as follows:

\[
\begin{align*}
\mathcal{D}_\alpha^\gamma W(t) &= Q(t, W(t)), \quad t \in [0, r], \quad 0 < \alpha \leq 1, \\
W(0) &= W_0.
\end{align*}
\] (10)

where

\[
W(t) = \begin{cases} 
S(t), & \quad W_0(t) = S_0, \\
E(t), & \quad W_0(t) = E_0, \\
I(t), & \quad W_0(t) = I_0, \\
R(t), & \quad W_0(t) = R_0, \\
N(t), & \quad W_0(t) = N_0, \\
D(t), & \quad W_0(t) = D_0, \\
C(t). & \quad W_0(t) = C_0,
\end{cases}
\]

(11)

On Lemma 2.3, Equation (10) can be converted to an equivalent integral form as follows:

\[
W(t) = W_0(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} Q(s, W(s))ds.
\] (12)

The following assumptions are key to our analysis:

**H1** If there exist constants $K_Q, M_Q$, and $q \in [0, 1)$, such that

\[
|Q(t, W(t))| \leq K_Q|W|^q + M_Q.
\]

**H2** If there exist constants $L_Q > 0$, and for each $W, \tilde{W} \in X$, such that

\[
|Q(t, W) - Q(t, \tilde{W})| \leq L_Q||W - \tilde{W}||.
\]

Define the map $T : X \rightarrow X$ as follows:

\[
TW(t) = W_0(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} Q(s, W(s))ds.
\] (13)

**Theorem 3.1.** If assumptions $H_1$ and $H_2$ hold, then problem (10) possesses at least one solution. Consequently, our considered model (2) has at least one solution.
Proof. With the help of Schaefer's fixed point theorem, the proof of the results will be present in four steps.

**Step 1:** First, we have to show \( T \) is continuous. Assume \( W_i \) is continuous for \( i = 1, 2, 3, \ldots, 7 \). Thus, \( Q(t, W(t)) \) is continuous. Let \( W_j, W \in X \) such that \( W_j \to W \), and we must have \( TW_j \to TW \).

Consider

\[
||TW_j - TW|| = \max_{t \in [0, r]} \left| \frac{1}{\Gamma(a)} \int_0^t (t-s)^{a-1}Q(s, W_j(s))ds - \frac{1}{\Gamma(a)} \int_0^t (t-s)^{a-1}Q(s, W(s))ds \right|
\]

\[
\leq \max_{t \in [0, r]} \left| \frac{1}{\Gamma(a)} \int_0^t (t-s)^{a-1} \left| Q(s, W_j(s)) - Q(s, W(s)) \right| ds \right|
\]

\[
\leq \frac{r^aLQ}{\Gamma(a+1)}||W_j - W|| \to 0 \text{ as } j \to \infty.
\]

Since \( Q \) is continuous, thus \( TW_j \to TW \); as a result, \( T \) is continuous.

**Step 2:** The operator \( T \) is bounded. For any \( W \in X \), the function \( T \) enjoys the growth condition as follows:

\[
||TW|| = \max_{t \in [0, r]} \left| W_0(t) + \frac{1}{\Gamma(a)} \int_0^t (t-s)^{a-1}Q(s, W(s))ds \right|
\]

\[
\leq |W_0| + \max_{t \in [0, r]} \frac{1}{\Gamma(a)} \int_0^t |(t-s)^{a-1}| |Q(s, W(s))| ds
\]

\[
\leq |W_0| + \frac{r^a}{\Gamma(a+1)} \left[ KQ||W||^q + M_Q \right].
\]

Let \( S \) be any bounded subset of \( X \). We need to show \( T(S) \) is bounded. For any \( W \in S \), since \( S \) is bounded, therefore, there exists \( K \geq 0 \), such that

\[
||W|| \leq K, \ \forall \ W \in S. \quad (14)
\]

Hence, for any \( W \in S \) using above growth conditions, one has

\[
||TW|| \leq |W_0| + \frac{r^a}{\Gamma(a+1)} \left[ KQ||W||^q + M_Q \right] \leq |W_0| + \frac{r^a}{\Gamma(a+1)} \left[ KQK^q + M_Q \right].
\]

Hence, \( T(S) \) is bounded.

**Step 3:** For equi-continuity, let \( t_1, t_2 \in [0, r] \), such that \( t_1 \geq t_2 \), then

\[
||TW(t_1) - TW(t_2)|| = \left| \frac{1}{\Gamma(a)} \int_0^{t_1} (t_1-s)^{a-1}Q(s, W(s))ds - \frac{1}{\Gamma(a)} \int_0^{t_2} (t_2-s)^{a-1}Q(s, W(s))ds \right|
\]

\[
\leq \left| \frac{1}{\Gamma(a)} \int_0^{t_1} (t_1-s)^{a-1} - \frac{1}{\Gamma(a)} \int_0^{t_2} (t_2-s)^{a-1} \right| |Q(s, W(s))| ds
\]

\[
\leq \frac{r^a}{\Gamma(a+1)} \left[ KQ||W||^q + M_Q \right] \to 0 \text{ as } t_1, t_2 \to t_2.
\]

Hence, by Arzelá–Ascoli theorem, \( T(S) \) is relatively compact.

**Step 4:** Finally, to show that the set

\[
E = \{ W \in X : W = \lambda TW, \ \lambda \in (0, 1) \}, \quad (15)
\]

is bounded, let \( W \in E \), then for each \( t \in [0, r] \), we have

\[
||W|| = \lambda ||TW|| \leq \lambda \left[ |W_0| + \frac{r^a}{\Gamma(a+1)} \left[ KQ||W||^q + M_Q \right] \right]
\]

which shows \( E \) is bounded. Hence, by Schaefer’s theorem, \( T \) possesses a fixed point; consequently, our considered problem (10) has at least one solution.
Remark 3.2. If the assumption \((H_1)\) is constructed for \(q = 1\), then Theorem 3.1 still holds if \(\frac{r^aK_0}{\Gamma(a+1)} < 1\).

**Theorem 3.3.** Our proposed model (10) has unique solution if \(\frac{r^aL_0}{\Gamma(a+1)} < 1\) hold.

**Proof.** Thanks to “Banach’s contraction theorem,” let \(W, \bar{W} \in X\), then
\[
\|TW - T\bar{W}\| \leq \max_{t \in [0, \tau]} \frac{1}{\Gamma(a)} \int_0^t (t - s)^{a-1} \|Q(s, W(s)) - Q(s, \bar{W}(s))\| ds
\]
\[
\leq \frac{r^aL_0}{\Gamma(a+1)} \|W - \bar{W}\|.
\]

Hence, \(T\) has unique fixed point. Therefore, our problem (10) has unique solution. \(\square\)

## 4 ULAM–HYERS STABILITY

Now in this part of the manuscript, we will discuss the Ulam–Hyers type stability which was introduced by Ulam, in 1940 and studied further by Hyers. The stated stability generalized further by Rassias, to Ulam–Hyer–Rassiass stability. For the last two decades this stability was studied very well. First, we will present different definition of Ulam–Hyers type stability which can be found in Ali et al.

Let \(H : X \to X\) be an operator satisfying
\[
HW = W, \text{ for } W \in X. \tag{16}
\]

**Definition 4.1.** Problem (16) is Ulam–Hyers type stable for \(\epsilon > 0\) and assume \(W \in X\) represents any solution of
\[
\|W - HW\| \leq \epsilon, \text{ for } t \in [0, \tau]. \tag{17}
\]

There exists unique solution \(\bar{W}\) of problem (16) such that \(C_q > 0\) satisfying
\[
\|\bar{W} - W\| \leq C_q \epsilon, \text{ } t \in [0, \tau]. \tag{18}
\]

**Definition 4.2.** For \(Y \in C([0, \tau]; \mathbb{R})\) with \(Y(0) = 0\), for any solution \(W\) of (17), and let \(\bar{W}\) be at most one solution of (16) such that
\[
\|W - \bar{W}\| \leq Y(\epsilon). \tag{19}
\]

then problem (16) is generalized Ulam–Hyers type stable.

**Remark 4.3.** For \(\chi(t) \in C([0, \tau]; \mathbb{R})\), then \(W \in X\) satisfies (17) if
(i) \(|\chi(t)| \leq \epsilon, \forall t \in [0, \tau] \]
(ii) \(HW(t) = W + \chi(t), \forall t \in [0, \tau]. \]

The following relation is needed in future work. The perturb problem corresponding to Equation (10) is given by
\[
\begin{cases}
\displaystyle s^aD_{+0}^a W(t) = Q(t, W(t)) + \chi(t), \\
W(0) = W_0.
\end{cases} \tag{20}
\]

**Lemma 4.4.** Perturb problem (20) satisfies the relation given by
\[
|W(t) - TW(t)| \leq a \epsilon, \text{ where } a = \frac{r^a}{\Gamma(a+1)}. \tag{21}
\]

**Proof.** By Remark 4.3 and Lemma 2.3, one may obtain the required result. \(\square\)
Theorem 4.5. Solution to problem (10) is Ulam–Hyers type and generalized-Ulam–Hyers type stable on Lemma 4.4 if \( \frac{r^\alpha L_Q}{\Gamma(\alpha+1)} < 1 \).

Proof. For any solution \( W \in X \) of problem (20) and \( \tilde{W} \in X \) represents unique solution to problem (10), then

\[
|W(t) - \tilde{W}(t)| = |W(t) - T\tilde{W}(t)| \leq |W(t) - TW(t)| + |TW(t) - T\tilde{W}(t)|
\]

\[
\leq ae + \frac{r^\alpha L_Q}{\Gamma(\alpha+1)} |W(t) - \tilde{W}(t)|
\]

\[
\leq \frac{ae}{1 - \frac{r^\alpha L_Q}{\Gamma(\alpha+1)}}.
\]

Thus, problem (10) is Ulam–Hyers and generalized Ulam–Hyers type stable. 

Definition 4.6. Problem (16) is Ulam–Hyers–Rassias type stable with \( \Omega \in C[[0, \tau], R] \), if for \( \epsilon > 0 \) and for any solution \( W \in X \) of

\[
||W - HW|| \leq \Omega(t)\epsilon, \text{ for } t \in [0, \tau].
\]

there exists unique solution \( \tilde{W} \) to problem (16) with \( C_q > 0 \) satisfying

\[
||\tilde{W} - W|| \leq C_q\Omega(t)\epsilon, \forall t \in [0, \tau].
\]

Definition 4.7. For \( \Omega \in C[[0, \tau], R] \) if there exist \( C_q, \Omega \) and for \( \epsilon > 0 \), let \( W \) represent any solution to problem (23) and \( \tilde{W} \) represent unique solution to problem (16) such that

\[
||W - \tilde{W}|| \leq C_q\Omega(t)\epsilon, \forall t \in [0, \tau],
\]

then problem (16) is generalized Ulam–Hyers–Rassias type stable.

Remark 4.8. If there exist \( \chi(t) \in C[[0, \tau]; R] \), then \( \tilde{W} \in X \) satisfies (17) if

(i) \( |\chi(t)| \leq \epsilon\Omega(t), \forall t \in [0, \tau] \)
(ii) \( H\tilde{W}(t) = \tilde{W} + \chi(t), \forall t \in [0, \tau] \).

Lemma 4.9. For perturb problem (20), the following hold.

\[
|W(t) - TW(t)| \leq a\Omega(t)\epsilon, \text{ where } a = \frac{r^\alpha}{\Gamma(\alpha+1)}.
\]

Proof. Using Remark (4.8) and Lemma 2.3, one may get the required result. 

Theorem 4.10. Solution to problem (10) is Ulam–Hyers–Rassias and generalized-Ulam–Hyers–Rassias type stable on Lemma 4.9 if \( \frac{r^\alpha L_Q}{\Gamma(\alpha+1)} < 1 \).

Proof. Let \( W \in X \) be any solution and \( \tilde{W} \in X \) represent unique solution to problem (10), then

\[
|W(t) - \tilde{W}(t)| = |W(t) - T\tilde{W}(t)| \leq |W(t) - TW(t)| + |TW(t) - T\tilde{W}(t)|
\]

\[
\leq ae + \frac{r^\alpha L_Q}{\Gamma(\alpha+1)} |W(t) - \tilde{W}(t)|
\]

\[
\leq \frac{ae}{1 - \frac{r^\alpha L_Q}{\Gamma(\alpha+1)}}.
\]

Thus, problem (10) is Ulam–Hyers–Rassias stable and generalized Ulam–Hyers–Rassias type stable. 

TABLE 1 Interpretation of the numerical values and parameters

| G    | No. of zoonotic cases | 0.100 |
|------|-----------------------|-------|
| $b_0$ | Rate of transmission | 0.5944, 0.000003345/day |
| $a$  | Action strength of the Government | 0.4239, 0.00004239 |
| $k$  | Responds intensity | 1117.3 |
| $\mu$ | Rate of emigration | 0.0205/day, 0.000002205/day |
| $d^{-1}$ | Latent period mean | 3 |
| $l^{-1}$ | Infectious period mean | 5 |
| $z^{-1}$ | Public reaction mean duration | 11.2/day |
| $v$ | Severe cases proportion | 0.2 |
| $N_0$ | Population size initially | 220 Millions |
| $S_0$ | Susceptible population initially | 10.25N_0 |
| $E_0$ | Exposed population initially | 12 Millions |
| $I_0$ | Infection population initially | 0.01194 Millions |
| $R_0$ | Recovered population initially | 0.003008 Millions |
| $D_0$ | Mimicking people initially | 7 Millions |
| $C_0$ | Cumulative cases initially (both reported and not reported) | 0.054 Millions |

FIGURE 1 Dynamics of susceptible people $S$ at various fractional order [Colour figure can be viewed at wileyonlinelibrary.com]

FIGURE 2 Dynamics of exposed population $E$ at various fractional order [Colour figure can be viewed at wileyonlinelibrary.com]
In this part of the article, a general algorithm for series solution to model (2) is developed. Using Laplace transform, we have

\[
\begin{align*}
\mathcal{L}[S(t)] &= \frac{S_0}{s} + \frac{1}{s^\alpha} \mathcal{L} \left[ -\frac{b_0SG}{N} - \frac{b(t)SI}{N} - uS \right], \\
\mathcal{L}[E(t)] &= \frac{E_0}{s} + \frac{1}{s^\alpha} \mathcal{L} \left[ \frac{b_0SG}{N} + \frac{b(t)SI}{N} - (d + u)E \right], \\
\mathcal{L}[I(t)] &= \frac{I_0}{s} + \frac{1}{s^\alpha} \mathcal{L} \left[ dE - (l + u)I \right], \\
\mathcal{L}[R(t)] &= \frac{R_0}{s} + \frac{1}{s^\alpha} \mathcal{L} \left[ II - uR \right], \\
\mathcal{L}[N(t)] &= \frac{N_0}{s} + \frac{1}{s^\alpha} \mathcal{L} \left[ -uN \right], \\
\mathcal{L}[D(t)] &= \frac{D_0}{s} + \frac{1}{s^\alpha} \mathcal{L} \left[ vlI - zD \right], \\
\mathcal{L}[C(t)] &= \frac{C_0}{s} + \frac{1}{s^\alpha} \mathcal{L} \left[ dE \right].
\end{align*}
\]

(28)
Assuming solution in the form of series as

\[
\begin{align*}
S(t) &= \sum_{j=0}^{\infty} S_j(t), \\
E(t) &= \sum_{j=0}^{\infty} E_j(t), \\
I(t) &= \sum_{j=0}^{\infty} I_j(t), \\
R(t) &= \sum_{j=0}^{\infty} R_j(t), \\
N(t) &= \sum_{j=0}^{\infty} N_j(t), \\
D(t) &= \sum_{j=0}^{\infty} D_j(t), \\
C(t) &= \sum_{j=0}^{\infty} C_j(t).
\end{align*}
\]

Decomposing the nonlinear terms \(S(t)I(t)\) by Adomian polynomials as

\[
S(t)I(t) = \sum_{j=0}^{\infty} A_j(t), \quad \text{where} \quad A_j(t) = \frac{1}{j!} \frac{d^j}{dz^j} \left[ \sum_{k=0}^{j} z^k S_k(t) \sum_{k=0}^{j} z^k I_k(t) \right] \bigg|_{z=0}.
\]

**FIGURE 5** Dynamics of total population \(N\) at various fractional order [Colour figure can be viewed at wileyonlinelibrary.com]

**FIGURE 6** Dynamics of mimicking population \(D\) at various fractional order [Colour figure can be viewed at wileyonlinelibrary.com]
By system (29) and (30), system (28) gives

\[
\begin{align*}
\mathcal{L}\left[\sum_{j=0}^{\infty} S_j(t)\right] &= \frac{S_0}{s} + \frac{1}{s^2} \mathcal{L}\left[-b_0 \sum_{j=0}^{\infty} S_j(t)G \frac{b(t) \sum_{j=0}^{\infty} A_j(t)}{\sum_{j=0}^{\infty} N_j(t)} - \sum_{j=0}^{\infty} N_j(t) - u \sum_{j=0}^{\infty} S_j(t) + \sum_{j=0}^{\infty} N_j(t) - (d + u) \sum_{j=0}^{\infty} E_j(t)\right], \\
\mathcal{L}\left[\sum_{j=0}^{\infty} S_j(t)\right] &= \frac{E_0}{s} + \frac{1}{s^2} \mathcal{L}\left[\frac{b_0 \sum_{j=0}^{\infty} S_j(t)G}{\sum_{j=0}^{\infty} N_j(t)} + \frac{b(t)S \sum_{j=0}^{\infty} A_j(t)}{\sum_{j=0}^{\infty} N_j(t)} - \sum_{j=0}^{\infty} N_j(t)\right], \\
\mathcal{L}\left[\sum_{j=0}^{\infty} I_j(t)\right] &= \frac{I_0}{s} + \frac{1}{s^2} \mathcal{L}\left[\frac{d \sum_{j=0}^{\infty} E_j(t) - (l + u) \sum_{j=0}^{\infty} I_j(t)}{\sum_{j=0}^{\infty} N_j(t)}\right], \\
\mathcal{L}\left[\sum_{j=0}^{\infty} R_j(t)\right] &= \frac{R_0}{s} + \frac{1}{s^2} \mathcal{L}\left[\frac{l \sum_{j=0}^{\infty} I_j(t) - u \sum_{j=0}^{\infty} R_j(t)}{\sum_{j=0}^{\infty} N_j(t)}\right], \\
\mathcal{L}\left[\sum_{j=0}^{\infty} N_j(t)\right] &= \frac{N_0}{s} + \frac{1}{s^2} \mathcal{L}\left[-u \sum_{j=0}^{\infty} N_j(t)\right], \\
\mathcal{L}\left[\sum_{j=0}^{\infty} D_j(t)\right] &= \frac{D_0}{s} + \frac{1}{s^2} \mathcal{L}\left[\frac{v \sum_{j=0}^{\infty} I_j(t) - z \sum_{j=0}^{\infty} D_j(t)}{\sum_{j=0}^{\infty} N_j(t)}\right], \\
\mathcal{L}\left[\sum_{j=0}^{\infty} C_j(t)\right] &= \frac{C_0}{s} + \frac{1}{s^2} \mathcal{L}\left[d \sum_{j=0}^{\infty} E_j(t)\right].
\end{align*}
\]
Compare like terms on each sides of (31), we get

\[
\begin{align*}
\mathcal{L}[S_0(t)] &= \frac{S_0}{s}, \\
\mathcal{L}[E_0(t)] &= \frac{E_0}{s}, \\
\mathcal{L}[I_0(t)] &= \frac{I_0}{s}, \\
\mathcal{L}[R_0(t)] &= \frac{R_0}{s}, \\
\mathcal{L}[N_0(t)] &= \frac{N_0}{s}, \\
\mathcal{L}[D_0(t)] &= \frac{D_0}{s}, \\
\mathcal{L}[C_0(t)] &= \frac{C_0}{s},
\end{align*}
\]\n
(32)

\[
\begin{align*}
\mathcal{L}[S_1(t)] &= \frac{1}{s^2} \mathcal{L} \left[ -b_0 S_0(t) G \frac{N_j(t)}{N_0(t)} - b(t) A_0(t) - u S_0(t) \right], \\
\mathcal{L}[E_1(t)] &= \frac{1}{s^2} \mathcal{L} \left[ b_0 S_0(t) G \frac{N_j(t)}{N_0(t)} + b(t) S A_0(t) - (d + u) E_0(t) \right], \\
\mathcal{L}[I_1(t)] &= \frac{1}{s^2} \mathcal{L} \left[ d E_0(t) - (l + u) I_0(t) \right], \\
\mathcal{L}[R_1(t)] &= \frac{1}{s^2} \mathcal{L} \left[ I I_0(t) - u R_0(t) \right], \\
\mathcal{L}[N_1(t)] &= \frac{1}{s^2} \mathcal{L} \left[ -u N_0(t) \right], \\
\mathcal{L}[D_1(t)] &= \frac{1}{s^2} \mathcal{L} \left[ v I I_0(t) - z D_0(t) \right], \\
\mathcal{L}[C_1(t)] &= \frac{1}{s^2} \mathcal{L} \left[ d E_0(t) \right].
\end{align*}
\]\n
(33)

\[
\begin{align*}
\mathcal{L}[S_2(t)] &= \frac{1}{s^3} \mathcal{L} \left[ -b_0 S_1(t) G \frac{N_j(t)}{N_1(t)} - b(t) A_1(t) - u S_1(t) \right], \\
\mathcal{L}[E_2(t)] &= \frac{1}{s^3} \mathcal{L} \left[ b_0 S_1(t) G \frac{N_j(t)}{N_0(t)} + b(t) S A_1(t) - (d + u) E_1(t) \right], \\
\mathcal{L}[I_2(t)] &= \frac{1}{s^3} \mathcal{L} \left[ d E_1(t) - (l + u) I_1(t) \right], \\
\mathcal{L}[R_2(t)] &= \frac{1}{s^3} \mathcal{L} \left[ I I_1(t) - u R_1(t) \right], \\
\mathcal{L}[N_2(t)] &= \frac{1}{s^3} \mathcal{L} \left[ -u N_1(t) \right], \\
\mathcal{L}[D_2(t)] &= \frac{1}{s^3} \mathcal{L} \left[ v I I_1(t) - z D_1(t) \right], \\
\mathcal{L}[C_2(t)] &= \frac{1}{s^3} \mathcal{L} \left[ d E_1(t) \right].
\end{align*}
\]\n
(34)
For $j \geq 0$, we generalized the terms as

\[
\mathcal{L} [S_{j+1}(t)] = \frac{1}{s^\alpha} \mathcal{L} \left[ \frac{-b_0 S_j(t)G}{N_j(t)} - \frac{b(t)A_j(t)}{N_j(t)} - uS_j(t) \right],
\]

\[
\mathcal{L} [E_{j+1}(t)] = \frac{1}{s^\alpha} \mathcal{L} \left[ \frac{b_0 S_j(t)G}{N} + \frac{b(t)SA_j(t)}{N_j(t)} - (d + u)E_j(t) \right],
\]

\[
\mathcal{L} [I_{j+1}(t)] = \frac{1}{s^\alpha} \mathcal{L} \left[ dE_j(t) - (l + u)I_j(t) \right],
\]

\[
\mathcal{L} [R_{j+1}(t)] = \frac{1}{s^\alpha} \mathcal{L} \left[ lI_j(t) - uR_j(t) \right],
\]

\[
\mathcal{L} [N_{j+1}(t)] = \frac{1}{s^\alpha} \mathcal{L} \left[ -uN_j(t) \right],
\]

\[
\mathcal{L} [D_{j+1}(t)] = \frac{1}{s^\alpha} \mathcal{L} \left[ vI_j(t) - zD_j(t) \right],
\]

\[
\mathcal{L} [C_{j+1}(t)] = \frac{1}{s^\alpha} \mathcal{L} \left[ dE_j(t) \right].
\]
By application of inverse Laplace transform to systems (32)–(35), we have

\[
\begin{align*}
S_0(t) &= S_0, \\
E_0(t) &= E_0, \\
I_0(t) &= I_0, \\
R_0(t) &= R_0, \\
N_0(t) &= N_0, \\
D_0(t) &= D_0, \\
C_0(t) &= C_0.
\end{align*}
\] (36)

\[
\begin{align*}
S_1(t) &= \mathcal{L}^{-1} \left[ \frac{1}{s^2} \mathcal{L} \left[ \frac{-b_0S_0(t)G}{N_1(t)} - \frac{b(t)A_0(t)}{N_0(t)} - uS_0(t) \right] \right], \\
E_1(t) &= \mathcal{L}^{-1} \left[ \frac{1}{s^2} \mathcal{L} \left[ \frac{b_0S_0(t)G}{N_0(t)} + \frac{b(t)SA_0(t)}{N_0(t)} - (d + u)E_0(t) \right] \right], \\
I_1(t) &= \mathcal{L}^{-1} \left[ \frac{1}{s^2} \mathcal{L} \left[ dE_0(t) - (l + u)I_0(t) \right] \right], \\
R_1(t) &= \mathcal{L}^{-1} \left[ \frac{1}{s^2} \mathcal{L} \left[ II_0(t) - uR_0(t) \right] \right], \\
N_1(t) &= \mathcal{L}^{-1} \left[ \frac{1}{s^2} \mathcal{L} \left[ -uN_0(t) \right] \right], \\
D_1(t) &= \mathcal{L}^{-1} \left[ \frac{1}{s^2} \mathcal{L} \left[ vI_0(t) - zD_0(t) \right] \right], \\
C_1(t) &= \mathcal{L}^{-1} \left[ \frac{1}{s^2} \mathcal{L} \left[ dE_0(t) \right] \right].
\end{align*}
\] (37)

\[
\begin{align*}
S_2(t) &= \mathcal{L}^{-1} \left[ \frac{1}{s^2} \mathcal{L} \left[ \frac{-b_0S_1(t)G}{N_1(t)} - \frac{b(t)A_1(t)}{N_1(t)} - uS_1(t) \right] \right], \\
E_2(t) &= \mathcal{L}^{-1} \left[ \frac{1}{s^2} \mathcal{L} \left[ \frac{b_0S_1(t)G}{N_0(t)} + \frac{b(t)A_1(t)}{N_1(t)} - (d + u)E_1(t) \right] \right], \\
I_2(t) &= \mathcal{L}^{-1} \left[ \frac{1}{s^2} \mathcal{L} \left[ dE_1(t) - (l + u)I_1(t) \right] \right], \\
R_2(t) &= \mathcal{L}^{-1} \left[ \frac{1}{s^2} \mathcal{L} \left[ II_1(t) - uR_1(t) \right] \right], \\
N_2(t) &= \mathcal{L}^{-1} \left[ \frac{1}{s^2} \mathcal{L} \left[ -uN_1(t) \right] \right], \\
D_2(t) &= \mathcal{L}^{-1} \left[ \frac{1}{s^2} \mathcal{L} \left[ vI_1(t) - zD_1(t) \right] \right], \\
C_2(t) &= \mathcal{L}^{-1} \left[ \frac{1}{s^2} \mathcal{L} \left[ dE_1(t) \right] \right].
\end{align*}
\] (38)
Thus, for $j \geq 0$, the general term is given by

$$
\begin{align*}
S_j(t) &= \mathcal{L}^{-1} \left[ \frac{1}{s^a} \mathcal{L} \left\{ \frac{-b_0 S(t) G}{N(t)} - \frac{b(t) A(t)}{N(t)} - u S(t) \right\} \right], \\
E_j(t) &= \mathcal{L}^{-1} \left[ \frac{1}{s^a} \mathcal{L} \left\{ \frac{b_0 S(t) G}{N} + \frac{b(t) S(t) A(t)}{N(t)} - (d + u) E(t) \right\} \right], \\
I_j(t) &= \mathcal{L}^{-1} \left[ \frac{1}{s^a} \mathcal{L} \left\{ d E(t) - (l + u) I(t) \right\} \right], \\
R_j(t) &= \mathcal{L}^{-1} \left[ \frac{1}{s^a} \mathcal{L} \left\{ d E(t) - (l + u) I(t) \right\} \right], \\
N_j(t) &= \mathcal{L}^{-1} \left[ \frac{1}{s^a} \mathcal{L} \left\{ -u N(t) \right\} \right], \\
D_j(t) &= \mathcal{L}^{-1} \left[ \frac{1}{s^a} \mathcal{L} \left\{ v I(t) - z D(t) \right\} \right], \\
C_j(t) &= \mathcal{L}^{-1} \left[ \frac{1}{s^a} \mathcal{L} \left\{ d E(t) \right\} \right].
\end{align*}
$$

Similarly, we obtain series solution as

$$
\begin{align*}
S(t) &= S_0(t) + S_1(t) + S_2(t) + S_3(t) + \ldots, \\
E(t) &= E_0(t) + E_1(t) + E_2(t) + E_3(t) + \ldots, \\
I(t) &= I_0(t) + I_1(t) + I_2(t) + I_3(t) + \ldots, \\
R(t) &= R_0(t) + R_1(t) + R_2(t) + R_3(t) + \ldots, \\
N(t) &= N_0(t) + N_1(t) + N_2(t) + N_3(t) + \ldots, \\
D(t) &= D_0(t) + D_1(t) + D_2(t) + D_3(t) + \ldots, \\
C(t) &= C_0(t) + C_1(t) + C_2(t) + C_3(t) + \ldots.
\end{align*}
$$

#### 6 | NUMERICAL RESULTS AND DISCUSSION

For justification of developed results, values have been taken about COVID-19 in Pakistan and also presented in Table 147:

**Case-I:** $G = 0, u = 0.02205/day, a = 0.4239/day, b_0 = 0.5944/day$

In Figures 1–7, we draw the graph of the solution for the given values of parameter defined as above. Using Matlab-16, the series solutions defined in Equation (40) have been plotted.

From Figure 1, we see that the susceptible population was decreasing for the initial population at given time in different orders. In same line as in Figure 2, exposed class was increasing with the decrease of susceptible class, and hence, the
number of infection is increasing as in Figure 3 because people are exposed to infection. This was due to the fact that people were not taking the matter seriously in Pakistan so the number of infections grew with proper speed. Initially, in first 15–20 days, the numbers of recovered class were increasing as people initially were getting ride easily due to the reason that cases were limited and treatment was proper as in Figure 4. After that, the recovered rate is becoming slow now as numbers of infection increased, and also there is no proper cure yet introduced. Also now the infection has attacked mostly on aged people whose recovery will take time. From Figure 5, we see that total population of healthy people is gradually decreasing with different rate due to fractional order derivative. Similarly, in Figure 6, the people initially not taking the matter seriously but now the people are taking it seriously and so the Mimicking class is decreasing with different rates because of fractional order. The class of cumulative cases is also increasing with different rate as in Figure 7. From Figures 1 to 7, we see that fractional calculus provides a global dynamics of the novel coronavirus infection model. The smaller the fractional order, the faster the concerned decay or growth and vice versa.

**Case-II:** \( G = 100, \ u = 0.000002205/day, \ a = 0.0004239/day, \ b_0 = 0.00003345/day.\)
In Figures 8–14, we draw the graph of the solution for the given values of parameter defined above. Using Matlab-16, the series solutions defined in Equation (40) have been plotted.

From Figure 8, we see that as susceptible population is decreasing in the presence of zoonotic cases and immigration for the given initial population at given time at different order, then in the same line as in Figure 9, exposed class was also decreasing with the decrease of susceptible class, and hence, the number of infection was increasing as in Figure 10, because more people were exposed to catch infection. This is due to the reason that people first were not taking the matter serious in Pakistan so the number of infection rate raised up with proper fast speed. Initially, in first 15–20 days, the numbers of recovered class were increasing as people initially were getting ride easily due to the reason that cases were limited and treatment was proper as in Figure 11. The recovery rate is also gradually increased due to the increase of people getting ride from infection or going to die. From Figure 12, we see that total population of healthy people is gradually decreasing with different rate due to fractional order derivative. Similarly, in Figure 13, the people initially not taking the matter seriously but now the people are taking it seriously and so the Mimicking class is decreasing with different rates because of fractional order. The class of cumulative cases is also increasing with different rates as in Figure 14. From Figures 8–14, we see that fractional calculus provides a global dynamics of the novel coronavirus-19 infection model. The smaller the fractional order, the faster the concerned decay or growth and vice versa.

7 | CONCLUSION

We have examined a novel model of coronavirus-19 under fractional order derivative. We first have established the existence theory of the model along with stability results of Ulam type via the use of nonlinear analysis. After that by a coupled method of Laplace transform and Adomian decomposition, we have established a general algorithm for series type solution to the considered model. By using Matlab, we have plotted the graphs against some real data of Pakistan, we observed that permitting immigration and not taking the matter seriously by the people in first sixty days the infection rate has been raised roughly in the country. But now the people are taking the matter seriously, but in the previous sixty days proper, great numbers of people have been coughing throughout the country. By fractional calculus approach, we have examined the global dynamics of the current novel disease. We see that fractional calculus approach globally describes the dynamics of all the compartments of the considered model more comprehensively.

ACKNOWLEDGEMENT

We are thankful to the reviewers for useful comments which have improved this work very well. No funding source is available.

CONFLICT OF INTEREST

There exists no competing interest regarding this research work.

AUTHOR CONTRIBUTIONS

Authors have equal contribution in this paper.
ORCID

Kamal Shah https://orcid.org/0000-0002-8851-4844
Hussam Rabai’ah https://orcid.org/0000-0003-4597-5787
Ali Ahmadian https://orcid.org/0000-0002-0106-7050

REFERENCES

1. World Health Organization. Coronavirus disease 2019 (COVID-19) Situation Report-62 https://www.who.int/docs/default-source/coronaviruse/situation-reports/20200322-sitrep-62-covid-19.pdf?sfvrsn=f7764c462, 2020.

2. WHO. Coronavirus disease 2019 (COVID-19), Situation Report - 72, 1 April 2020.

3. Lu H, Stratton CW, Tang YW. Outbreak of pneumonia of unknown etiology in Wuhan, China: the mystery and the miracle. J Med Viro. 2020;92(4):401-402.

4. Qasim M, Ahmad W, Zhang S, Yasir M, Azhar M. Data model to predict prevalence of COVID-19 in Pakistan. medRxiv 2020.

5. Qasim M, Ahmad W, Yoshida M, Gould M, Yasir M. Analysis of the worldwide corona virus (COVID-19) pandemic trend; a modelling study to predict its spread. medRxiv 2020.

6. Goyal M, Baskonus HM, Prakash A. An efficient technique for a time fractional model of lassa hemorrhagic fever spreading in pregnant women. Euro Phys J Plus. 2019;134(10):482.

7. Kumar D, Singh J, Al Quraishi M, Baleanu D. A new fractional SIRS-SI malaria disease model with application of vaccines, antimalarial drugs, and spraying. Adv Diff Equ. 2019;2019(1):1-19.

8. Shah K, Alqudah MA, Jarad F, Abdeljawad T. Semi-analytical study of Pine Wilt Disease model with convex rate under Caputo-Febrizio fractional order derivative. Chaos Solitons Fract. 2020;135:109754.

9. Nár LM. Mathematical Biology: An Introduction. Photosynthetica. 2002;40(3):414-414.

10. Sher M, Shah K, Khan ZA, Khan H, Khan A. Computational and theoretical modeling of the transmission dynamics of novel COVID-19 under Mittag-Leffler power law. Alex Eng J. 2020;59(5):3133-3147.

11. Tian X, Li C, Huang A, et al. Potent binding of 2019 novel coronavirus spike protein by a SARS coronavirus-specific human monoclonal antibody. Emerg Micro Infec. 2020;9(1):382-385.

12. Ahmed I, Baba IA, Yusuf A, Kumam P, Yusuf I. A mathematical model of Coronavirus Disease (COVID-19) containing asymptomatic and symptomatic classes. Inter J Infece Dise. 2020;93:211-216.

13. Ahmed I, Goufo EFD, Yusuf A, Kumam P, Chaipanya P, Nonlaopon K. An epidemic prediction from analysis of a combined HIV-COVID-19 co-infection model via ABC-fractional operator. Alex Eng J. 2021;60(3):2979-2995.

14. Ali A, Mashwani WK, Naem S, et al. COVID-19 Infected Lung Computed Tomography Segmentation and Supervised Classification Approach. CMC-COMPU Mate Conti. 2021;68(1):391-407.

15. Shoaib M, Raja MAZ, Sabir MT, et al. A stochastic numerical analysis based on hybrid NAR-RBFs networks nonlinear SITR model for novel COVID-19 dynamics. Compu Meth Prog Biomed. 2021;202:105973.

16. Shah K, Din RU, Deebani W, Kumam P, Shah Z. On nonlinear classical and fractional order dynamical system addressing COVID-19. Respir Physiol. 2021;2021:104069.

17. Arfan M, Alraabiah H, Rahman MU, et al. Investigation of fractal-fractional order model of COVID-19 in Pakistan under Atangana-Baleanu Caputo (ABC) derivative. Results Phys. 2021;24:104046.

18. Sahai P, Mukherjee D, Singh PK, Ahmadian A, Ferrara M, Sarkar R. GraphCovidNet: A graph neural network based model for detecting COVID-19 from CT scans and X-rays of chest. Sci Rep. 2021;11(1):1-16.

19. Gupta V, Jain N, Katariya P, et al. An emotion care model using multimodal textual analysis on COVID-19. Chaos Solitons Fract. 2021;144:110708.

20. Jain N, Jhunthra S, Garg H, et al. Prediction modelling of COVID using machine learning methods from B-cell dataset. Results Phys. 2021;21:103813.

21. Ghorui N, Ghosh A, Mondal SP, et al. Identification of dominant risk factor involved in spread of COVID-19 using hesitant fuzzy MCDM methodology. Results Phys. 2021;21:103811.

22. Toledo-Hernandez R, Rico-Ramirez V, Iglesias-Silva GA, Diwekar UM. A fractional calculus approach to the dynamic optimization of biological reactive systems. Part I: Fractional models for biological reactions. Chem Eng Sci. 2014;117:217-228.

23. Miller KS, Ross B. An introduction to the fractional calculus and fractional differential equations. Wiley; 1993.

24. Kilbas AA, Srivastava HM, Trujillo JJ. Theory and Applications of Fractional Differential Equations. Vol. 204 of North-Holland Mathematics Studies. Dordrecht: Elsevier; 2006.
28. Lai CC, Shih TP, Ko WC, Tang HJ, Hsueh PR. Severe acute respiratory syndrome coronavirus 2 (SARS-CoV-2) and coronavirus disease-2019 (COVID-19): The epidemic and the challenges. *Inter J Antimic Agents*. 2020;55(3):105924.
29. Fanelli D, Piazza F. Analysis and forecast of COVID-19 spreading in China, Italy and France. *Chaos Solitons Fract*. 2020;134:109761.
30. Ullah S, Khan MA, Farooq M. A fractional model for the dynamics of TB virus. *Chaos Solitons Fract*. 2018;116:63-71.
31. Khan MA, Ullah S, Okosun KO, Shah K. A fractional order pine wilt disease model with Caputo–Fabrizio derivative. *Adv Diff Equ*. 2018;2018(1):1-18.
32. Qureshi S, Atangana A. Mathematical analysis of dengue fever outbreak by novel fractional operators with field data. *Physica A: Stat Mech Appl*. 2019;526:121127.
33. Muhammad AK, Atangana A. Dynamics of Ebola disease in the framework of different fractional derivatives. *Entropy*. 2019;21(3):303.
34. El-Dessoky MM, Khan MA. Corrigendum “Modeling and analysis of the polluted lakes system with various fractional approaches”. *Chaos Solitons Fract*. 2020;135:109776.
35. Wang W, Khan MA. Analysis and numerical simulation of fractional model of bank data with fractal–fractional Atangana–Baleanu derivative. *J Comput Appl Math*. 2020;369:112646.
36. Li Z, Liu Z, Khan MA. Fractional investigation of bank data with fractal-fractional Caputo derivative. *Chaos Solit Fract*. 2020;131:109528.
37. Dhage BC. A nonlinear alternative in Banach algebras with applications to functional differential equations. *Nonlinear Funct Anal Appl*. 2004;8:563-575.
38. Dhage BC. Fixed point theorems in ordered Banach algebras and applications. *Pan Amer Math J*. 1999;9:83-102.
39. Dhage BC. A fixed point theorem in Banach algebras involving three operators with applications. *Kyungpook Math J*. 2004;44:145-155.
40. Sher M, Shah K, Feckan M, Khan RA. Qualitative analysis of multi-terms fractional order delay differential equations via the topological degree theory. *Math*. 2020;8(2):218.
41. Sher M, Shah K, Rassias J. On qualitative theory of fractional order delay evolution equation via the prior estimate method. *Math Meth Appl Sci*. 2020;1-12.
42. Sher M, Shah M, Abdeljawad T. Study of evolution problem under Mittag–Leffler type fractional order derivative. *Alex Eng J*. 2020;59(5):3945-3951.
43. Ullam SM. *Problems in Modern Mathematics (Chapter VI)*. New York: Wiley; 1940.
44. Hyers DH. On the stability of the linear functional equation. *Proc Natl Acad Sci U S A*. 1941;27(4):222-224.
45. Sher M, Shah K, Chu YM, Khan RA. Applicability of topological degree theory to evolution equation with proportional delay. *Fractals*. 2020;28(8):204028.
46. Sher M, Shah K, Khan ZA. Study of time fractional order problems with proportional delay and controllability term via fixed point approach. *AIMS Math*. 2021;6(5):5387-5396.
47. Official Updates Coronavirus COVID-19 in Pakistan: Ministry of National Health Services Regulations & Coordination, 25 April, 2020. *Www.COVID.Govt.Pk*.
48. Podlubny I. *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*. San Diego: Elsevier; 1998.
49. Granas A, Dugundji J. *Fixed Point Theory*. New York: Springer-Verlag; 2003.
50. Rassias TM. On the stability of the linear mapping in Banach spaces. *Proc Ameri Math Soc*. 1978;72(2):297-300.
51. Ali Z, Zada A, Shah K. On Ulam’s stability for a coupled systems of nonlinear implicit fractional differential equations. *Bull Malay Math Sci Soci*. 2019;42(5):2681-2699.
52. Ali Z, Zada A, Shah K. Ulam stability to a toppled systems of nonlinear implicit fractional order boundary value problem. *Bound Value Prob*. 2018;2018(1):1-16.

**How to cite this article:** Shah K, Sher M, Rabai’ah H, Ahmadian A, Salahshour S, Pansera BA. Analytical and qualitative investigation of COVID-19 mathematical model under fractional differential operator. *Math Meth Appl Sci*. 2023;46(7):8223-8242. [https://doi.org/10.1002/mma.7704](https://doi.org/10.1002/mma.7704)