**Article**

**Stability Analysis of Charged Rotating Black Ring**

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Received: 15 June 2020; Accepted: 5 July 2020; Published: 13 July 2020

**Abstract:** We study the electromagnetic field equation along with the WKB approximation. The boson tunneling phenomenon from charged rotating black ring (CRBR) is analyzed. It is examined that reserve radiation consistent with CRBR can be computed in general by neglecting back reaction and self-gravitational of the radiated boson particle. The calculated temperature depends upon quantum gravity and CRBR geometry. We also examine the corrected tunneling rate/probability of boson particles by assuming charge as well as energy conservation laws and the quantum gravity. Furthermore, we study the graphical behavior of the temperature and check the stability and instability of CRBR.

**Keywords:** charged rotating black ring; quantum gravity; stability analysis

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1. **Introduction**

Hawking studied the emission rate of radiation of all particles (photons, neutrinos, gravitons, positrons, etc.) from black holes (BHs) and also computed the temperature of the BH horizon.

Tunneling is the semiclassical method in which boson particles have ability to cross the black ring and BH horizon. Recent analysis show growing interest in Hawking temperature as a process of tunneling from black ring. The key component for examining modified tunneling radiation in the classical behavior of imaginary portion that includes the boson particles from the black ring horizon.

The tunneling radiation of charged fermion have been observed in BHs with Newman-Unti-Tamburino (NUT) in [1]. The authors concluded that the tunneling radiation depends on the electric charge, magnetic charge, acceleration, mass and rotation parameters, as well as the NUT parameter of the BHs pair. The analysis of the tunneling radiation from the global monopole Reissner Nordstöm-de Sitter BH in [2] and concluded that the corrected temperature depend on the global monopole. Sharif and Javed analyzed [3] radiation spectrum through tunneling fermions in the BHs horizon family. They concluded that the tunneling radiation must be depend on the horizon of outgoing positive particles and cosmological horizon of incoming particles. Sharif and Javed [4] devoted to analyzing the thermodynamic properties of BH with NUT, rotation and acceleration parameters and observed that the quantity of thermodynamics, such as surface gravity and the area-entropy relationship. The method of quantum tunneling of boson particles tunnel through the more generalized BHs horizons by applying the Proca equation has been analyzed [5]...
and examined that the tunneling radiation is independent of the types of particles at which particles through the horizon.

The application of the Hamilton-Jacobi method and the significance of field equation with electromagnetic interactions has been studied [6]. The massive boson in the electric and magnetic tunneling field and the temperature of BHs surrounded by fluid has been measured. The Hawking radiation from Myers Perry 5D BHs and black ring as a semiclassical tunneling have been investigated [7]. It has been analyzed that the Lagrangian charged field equation for charged boson and the WKB approximation. Javed et al analyzed [8] the radiation for kinds of Banados-Teitelboim-Zanelli-like BHs for higher dimensional spaces and studied that the rotation parameter effect on the tunneling radiation.

The generalized uncertainty principle (GUP) plays very significant role to study the gravity effects (GUP corrections). To assume the GUP effect on Hawking radiation, Dirac, Klein-Gordan and Lagrangian equations will be corrected by assuming gravity effects. Many authors have analyzed tunneling approach for different type particles with various spin such as fermions, scalar and bosons through the horizons of different black rings, wormholes and BHs, they also studied their corresponding modified Hawking temperature. The BHs are the main experimental scalar and vector fields for studying the effects of gravity and there is much literature on the thermodynamical properties of BHs to study the gravity effects under GUP. The BH thermodynamics have been analyzed within the GUP [9–16]. The field equation in the WKB approximation and tunneling of bosons from the BHs in different theory have been analyzed [17,18]. The GUP effect on BHs radiation and the instable and stable have been analyzed.

This paper is organized as follows. Section 2 contains the CRBR metric information, analyzes the imaginary part of boson particle action along the classical forbidden trajectory and finally compute the modified temperature. In Section 3, we analyzed the graphical behavior of temperature. In last section, final results and summery are made.

2. Rotating Black Ring

The Einstein field equations in 5D have been expressed [19] a stationary flat regular solution and have an horizon outside of topology \( S^1 \times S^2 \). The BH solution is proposed to have a naked singularity, and the black ring solution also approach the similar naked singularity. A supergravity solution depicts a \( S^2 \times S^1 \) CRBR horizon and its properties in a 5D asymptotically flat metric have been studied [20]. For the CRBR, a curious relationship between the charge and the mass has been concluded. The 5D black ring horizon [21] has topology \( S^2 \times S^1 \) and self-gravitational and its tension are exactly balanced by the rotation of the black ring. The charge parameter increases the stability of black rings. The asymptotically flat stationary black ring [22] is a boundary value solution to the 5D Einstein vacuum equations. The asymptotically flat 5D black ring solutions for the Einstein vacuum equations has been tested taking three commuting Killing vector fields. The stationary axisymmetric nonlinear \( \sigma \)-model with self-gravitating in 5D metric taking three commutating Killing vector fields has been studied [23]. The regular rotating horizon of black ring have been generalized by two angular momenta and mass. A dipole charged black ring solution contains conical naked singularity and turns out to be generically unbalanced has been analyzed [24]. A dipole black ring can be used as an independent parameter to uniquely specify a black lens minimal supergravity solution.

The CRBR radiation has been investigated for the significance field and consider in physics of gravitational as well as in advanced theoretical frameworks for example brane/string dynamics. The CRBR radiation pulling attention of physicists which shows that CRBR is extremely excited states of quantum and so can be required to be fully realized in conditions of quantum gravity. Otherwise, the CRBR radiation is describe to establish the thermodynamics law in spacetimes effecting CRBR consistent. Generally approaches for the CRBR radiation, which takes spectrum of thermal only.

In this section, we analyzed the problem of Hawking temperature from CRBR result of the Einstein–Maxwell-dilation gravity 5-dimensional space theory. The first law of thermodynamics
exists for BRs and the emission probability is associated for the existence of Bekenstein Hawking BRs entropy during the radiation can be analyzed in Ref. [25]. This solution is agree with the results of Hawking temperature in conditions of tunneling for other case of BHs and it appears to be general axiom for the tunneling method. So, the BRs are new class of spacetime. It is analyzed on their different axioms and particular solution of quantum state using quantum tunneling method (the CRBR quantum state has been observed, to especially leading order is already observed in previous paper). The stability analyzes both the form of thermodynamical and dynamical views, and also the BRs are concerning vector field of further work.

The Einstein–Maxwell-dilaton 5D BR theory for Hawking radiation are analyzed by using the quantum tunneling method. To study the correct temperature, we assume and developed the quantum tunneling approach to calculate the effects of quantum gravity and charge. The effect of back reaction is not taken in this phenomenon. The neutral and dipole cases line element of a single electric charge black ring is written as [20]

$$
\begin{align*}
\text{ds}^2 &= - \frac{G(y)}{G(x)K^2(x,y)} [dt - C(v, \lambda) R \frac{1 + y}{G(y)} \cosh^2 \alpha d\phi^2 + \frac{R^2}{(x - y)^2} \\
G(x)[(F(y)G(y)] d\phi^2 - \frac{1}{F(y)} dy^2 + \frac{1}{F(x)} dx^2 + \frac{F(x)}{G(x)} \psi^2],
\end{align*}
$$

(1)

where $C(v, \lambda) = \sqrt{\frac{1 + y}{1 - \lambda(\lambda - v)}}$, $K(x, y) = \lambda(x - y) \sinh^2 \alpha F(x)$, $G(\xi) = 1 + \lambda \xi^2$, $F(\xi) = (1 + \xi^2)(1 + v \xi)$, parameters $v$ and $\lambda$ are dimensionless and take the values in range $1 > \lambda \geq v > 0$, we do not take the conical singularity at $x = 1$, $\lambda$ and $v$ are related to each other, say $\lambda = \frac{2v}{1 + v^2}$ and $a$ is a parameter representing as the charge of electric. The coordinate $\phi$ and $\psi$ are two cycles of BR and $x$ and $y$ the values range $1 \geq x \geq -1$ and $1 \geq y \geq -\infty$ and $R$ has the length of dimensional. The explicit computation of the electric charged is $Q = \frac{2M \sinh a}{3[1 + \frac{1}{4} \sinh^2 a]}$. The mass of BR is $M = \frac{3\pi R^2 \lambda}{4[1 - v]}$ in [26] and its angular momentum takes the form $J = \frac{\pi R^3 \sqrt{\lambda(\lambda - v)(1 + \lambda)}}{2[1 - v]^2}$.

In order to study the tunneling probability for boson particles through the CRBR horizon $y_+$, we will assume the Lagrangian equation with gravity and electromagnetic effects. The charged motion with spin-1 fields is described out by the given Lagrangian gravity equation with vector field $\chi_\mu$ [15]

$$
\begin{align*}
\frac{1}{\sqrt{g}} \partial_\mu (\sqrt{-g} g^{\mu \nu}) + \frac{m^2}{\hbar^2} \chi^\nu + \frac{i}{\hbar} e A_\mu \chi^\nu + \frac{\beta h^2}{\hbar} \partial_\mu \partial_\nu (g^{00} \chi^\nu) \\
- \frac{\beta h^2}{\hbar} \partial_\mu \partial_\nu (g^{\mu \nu} \chi^\nu) = 0,
\end{align*}
$$

(2)

where $g$ is determinant of coefficients matrix, $m$ is particles mass and $\chi^{\mu \nu}$ is anti-symmetric tensor, i.e.,

$$
\chi_{\nu \mu} = (1 - \frac{\beta h^2}{\hbar^2} \partial^2_\nu) \partial_\nu \chi_\mu - (1 - \frac{\beta h^2}{\hbar^2} \partial^2_\mu) \partial_\mu \chi_\nu + (1 - \frac{\beta h^2}{\hbar^2} \partial^2_\nu) \frac{i}{\hbar} e A_\nu \chi_\mu \\
- (1 - \frac{\beta h^2}{\hbar^2} \partial^2_\mu) \frac{i}{\hbar} e A_\mu \chi_\nu.
$$

Here, $A_\mu$ is denoted as the electromagnetic vector potential of the CRBR and $e$ denotes the charge of the boson particle. Here $\beta$ is quantum gravity parameter. The $\chi$ components are calculated as...
\[
\begin{align*}
\chi^0 &= \frac{V}{f} \chi_0 - \frac{Z}{f} \chi_1, \\
\chi^1 &= \frac{1}{W} \chi_2, \\
\chi^2 &= \frac{1}{W} \chi_3, \\
\chi^3 &= \frac{1}{X} \chi_0 - \frac{U}{f} \chi_1, \\
\chi^4 &= \frac{1}{X} \chi_3, \\
\chi^{01} &= \frac{Z^2 \chi_{10} + UV \chi_{01}}{f^2}, \\
\chi^{02} &= \frac{V \chi_{02} - Z \chi_{12}}{W f}, \\
\chi^{03} &= \frac{-Z \chi_{03} + U \chi_{13}}{X f}, \\
\chi^{12} &= \frac{-Z \chi_{02} + U \chi_{12}}{W f}, \\
\chi^{13} &= \frac{-Z \chi_{03} + U \chi_{13}}{X f}, \\
\chi^{14} &= \frac{-Z \chi_{04} + U \chi_{14}}{Y f}, \\
\chi^{23} &= \frac{1}{W X} \chi_{23}, \\
\chi^{24} &= \frac{1}{W Y} \chi_{24}, \\
\chi^{34} &= \frac{1}{X Y} \chi_{34}
\end{align*}
\]

where \( f = UV - Z^2 \). The WKB approximation in [27] is given by

\[
\chi_v = c_v \exp \left[ \frac{i}{\hbar} \Theta_0(t, \phi, y, x, \psi) + \Sigma h^n \Theta_n(t, \phi, y, x, \psi) \right].
\]

Here \( \Theta_0, \Theta_n \) and \( c_v \) are arbitrary functions and constant. By neglecting the higher order terms for \( n = 1, 2, 3, \ldots \) and applying Equation (2), we obtain the set of field equations. The electromagnetic vector potential of black ring is given by

\[
A_\mu = A_1 dt + A_\phi d\phi,
\]

where

\[
A_1 = \frac{\lambda (x - y) \sin h \alpha \cos h \alpha}{G(x) K(x, y)}, \quad A_\phi = \frac{C(v, \lambda) R(1 + y) \sin h \alpha \cos h \alpha}{G(x) K(x, y)}.
\]

Using separation of variables technique [7], we can choose

\[
\Theta_0 = -E_0 t + J \phi + I(x, y) + L \psi,
\]

where \( E_0 = (E - \sum_{i=1}^t j_i \Omega_i) \), \( E \) is the energy of particle, \( J \) and \( L \) are represent the particles angular momentums corresponding to the angles \( \phi \) and \( \psi \) respectively. From set of field equations, we can obtain a \( 5 \times 5 \) matrix equation \( G(c_0, c_1, c_2, c_3, c_4)^T = 0 \), the components of the expected matrix have the following form

\[
G(c_0, c_1, c_2, c_3, c_4)^T = 0,
\]

which implies a \( 5 \times 5 \) matrix denoted as “\( G \)”, whose entries are devoted as follows:

\[
\begin{align*}
G_{00} &= \frac{Z^2}{f} [E_0^2 + \beta E_0^4] - \frac{UV}{f} [J^2 + \beta J^4] - \frac{V}{W f} [I_1^2 + \beta I_1^4] \\
&\quad - \frac{V}{2} [L^2 + \beta L^4] - m^2 V, \\
G_{01} &= \frac{Z^2}{f} \bar{A} J - \frac{UV}{f} \bar{A} J + \frac{Z}{W} \bar{I}_1 + \frac{Z}{2} [L^2 + \beta L^4] - m^2 Z, \\
G_{02} &= -\frac{V}{W} \bar{A} I_2 - \frac{Z}{W f} [J + \beta J^3], \\
G_{03} &= -\frac{V}{X} \bar{A} I_3, \\
G_{04} &= -\frac{V}{Y} \bar{A} L, \\
G_{10} &= \frac{Z^2}{f} [E_0 L + \beta E_0 J^3] - \frac{UV}{f} [E_0 J + \beta E_0 J^3] + \frac{Z}{W f} [I_1^2 + \beta I_1^4] + \\
&\quad \frac{Z}{X} [E_0 + \beta E_0] I_2 + \frac{e A_0 Z^2}{f} [E_0 + \beta E_0^3] + e A_0 UV [J + \beta J^3] + m^2 Z,
\end{align*}
\]
\[ G_{11} = \frac{Z^2}{f} E_0 \tilde{A} - \frac{U}{f} E_0 \tilde{A} - \frac{U}{W} [I_1^2 + \beta I_4^2] - \frac{U}{X} [I_1^2 + \beta I_4^2] I_2 - \frac{U}{Y} [L^2 + \beta L^4] \\
- m^2 Z X - eA_0 Z^2 \tilde{A} + eA_0 UV \tilde{A}, \]
\[ G_{12} = -\frac{Z}{W} \tilde{A} I_1 + \frac{U}{WJ} [J + \beta J^3] I_1 + \frac{U}{X} [J + \beta J^3] I_2, \]
\[ G_{13} = \frac{Z}{X} \tilde{A} I_2, \quad G_{14} = \frac{Z}{Y} \tilde{A} L + \frac{U}{Y J} [J + \beta J^3] L, \]
\[ G_{20} = -V [E_0 I_1 + \beta E_0 I_1^3] + Z \int [I_1 + \beta I_1^3] + \frac{eA_0 V}{Wf} [I_1 + \beta I_1^3], \]
\[ G_{21} = \frac{Z}{f} [E_0 I_1 + \beta E_0 I_1^3] + \frac{Z}{f} [I_1 + \beta I_1^3] J + \frac{eA_0 V}{Wf} [I_1 + \beta I_1^3], \]
\[ G_{22} = -\frac{V}{f^2} E_0 \tilde{A} - \frac{Z}{f^2} E_0 \tilde{A} - \frac{Z}{f} [E_0 I + \beta E_0 J^3] - \frac{Z}{f} \tilde{A} J - \frac{U}{f} [J + \beta J^3] \\
- m^2 - \frac{1}{X} [I_1^2 + \beta I_1^4] - \frac{1}{Y} [L^2 + \beta L^4] - \frac{eA_0 V}{f^2} \tilde{A} + \frac{eA_0 V}{f^2} [J_1 + \beta J_1^3], \]
\[ G_{23} = \frac{1}{X} [I_1 + \beta I_1^3] I_2, \quad G_{24} = \frac{1}{Y} [I_1 + \beta I_1^3] L, \]
\[ G_{30} = -\frac{V}{f} [I_2 + \beta I_2^3] E_0 - \frac{Z}{f} [I_2 + \beta I_2^3] J + \frac{eA_0 V}{f} [I_2 + \beta I_2^3], \]
\[ G_{31} = \frac{Z}{f} [I_2 + \beta I_2^3] E_0 - \frac{U}{f} [I_2 + \beta I_2^3] J - \frac{ZeA_0}{f} [I_2 + \beta I_2^3], \]
\[ G_{32} = \frac{1}{W} [I_2 + \beta I_2^3] I_1, \]
\[ G_{33} = -\frac{V}{f} E_0 \tilde{A} - \frac{Z}{f} \tilde{A} J - \frac{U}{f} [J^2 + \beta J^4] - \frac{1}{W} [I_1^2 + \beta I_1^4] - \frac{1}{Y} [L^2 + \beta L^4] \\
- m^2 + \frac{eA_0 V}{Wf} \tilde{A} + \frac{ZeA_0}{f^2} [J + \beta J^3], \]
\[ G_{34} = \frac{1}{Y} [I_2 + \beta I_2^3] L, \]
\[ G_{40} = -\frac{V}{f} [L + \beta L^3] E_0 - \frac{V}{f} [L + \beta L^3] J + \frac{eA_0 V}{f} [L + \beta L^3], \]
\[ G_{41} = \frac{Z}{f} [L + \beta L^3] E_0 + \frac{U}{f} [L + \beta L^3] J - \frac{eA_0 V}{f} [L + \beta L^3], \]
\[ G_{42} = \frac{1}{W} [L + \beta L^3] I_1, \quad G_{43} = \frac{1}{X} [L + \beta L^3] I_2, \]
\[ G_{44} = -\frac{V}{f} E_0 \tilde{A} - \frac{Z}{f} [J + \beta J^3] E_0 - \frac{Z}{f} \tilde{A} J - \frac{U}{f} [J^2 + \beta J^4] - \frac{1}{W} [I_1^2 + \beta I_1^4] \\
- \frac{1}{X} [I_1^2 + \beta I_1^4] - m^2 - \frac{eA_0 V}{f} \tilde{A} + \frac{ZeA_0}{f^2} [J + \beta J^3], \]
\]
where \( \tilde{A} = E_0 + \beta E_0 - eA_0 - \beta eA_0 E_0^2, \) \( J = \partial_\phi \Phi_0, \) \( I_1 = \partial_\psi \Phi_0, \) \( I_2 = \partial_\psi \Phi_0 \) and \( L = \partial_\psi \Phi_0. \) For the non-trivial solution, the determinant \( G \) is equal to zero. So, we get
\[ Im I^\pm = \int \sqrt{\frac{E_0^2 + X_1 [1 + \beta X_2]}{-W}} dy \]
\[ = \pm i \pi \frac{E_0 + [1 + \beta]}{2X(y_+)}, \]

where

\[
X_1 = -\frac{2XZE_0}{f} - \frac{XZeA_0E_0^3}{f} - \frac{XU}{f}j^2 - \frac{X}{Y}L^2 - m^2X, \\
X_2 = -\frac{XE_0^4}{f^2} - \frac{XZeA_0E_0^3}{f} - \frac{XZeA_0E_0^2}{f} + \frac{UX}{f}j^4 - \frac{I_2}{2}XL^0Y.
\]

The boson tunneling probability is given as

\[
\Gamma = \frac{\Gamma_{\text{emission}}}{\Gamma_{\text{absorption}}} = \exp\left[-4\pi \left(E - \sum_{i=1}^{2} j_i\Omega_i - eA_0\right) / 2\kappa (y+)\right] [1 + \Xi \beta].
\] (7)

We obtained the Hawking temperature at outer CRBR horizon, which is similar to the massless particle. This method can also be applied to the all other kinds of BRs. The Hawking temperature CRBR is given as

\[
T_H = \frac{\sqrt{(x - y)^2(1 - x^2)(\nu x + 1)\ell}}{4\pi \sqrt{(1 + y\nu)(1 - y^2)}} [1 + \Xi \beta]
\] (8)

where \(\ell = \nu y^3 + \nu y + \nu x - 2xy - 3xy^2x + 2\). The CRBR temperature depends upon parameter \(\nu, \beta, x\) and \(y\). The modified temperature (if \(\beta = 0\)) at which boson particle tunnel through the CRBR horizon is similar to the temperature of particles [7].

3. Gravitational Stability Analysis

The positive boson particles are tunneling outside of the CRBR horizon and there exists some fundamental interaction with them. The negative boson particles are tunneling inside from the CRBR horizon. The original spectrum of a boson will have the probability that inside the surrounding or nearby region of a CRBR and negative particle, it must be absorbed and the boson particle would be irradiated and thus many spectrums of tunneling particles would be almost totally disseminated. The condition of ionized (positive) boson particles is formed and these particles will be radiated.

This section gives the analysis of corrected Hawking temperature with rotation parameter for different values of gravity parameter. The calculated Hawking temperature \(T_H\) depends upon metric parameters \((\nu, x, y)\) as well as on the quantum gravity.

Figure 1, we concluded that \(T_H\) does not depend upon the mass of the CRBR but only depends upon the quantum gravity of the outgoing boson particles which is due to the influence of gravity. The \(T_H\) increases due to quantum gravity and remains constant when \(0 \leq y \leq 0.65\) and CRBR remains stable in this range. The \(T_H\) increases when \(0.65 < y < \infty\) and CRBR is unstable in this range. The results obtained from 2D in Figure 1 are similar to the results obtained from 3D Figure 2.

![Figure 1. Hawking temperature \(T_H\) versus \(y\) for \(x = 2, \nu = 10, \beta = 50\) to \(150\) and \(\Xi = 1\)](image)
4. Conclusions

In this work, we have applied the Hamilton–Jacobi method to compute tunneling probability of boson particles in the CRBR horizon. Law of energy and angular momentum effects are conserved. The back-reaction and self-interaction effect are neglected. For the 5-dimensional CRBR, we computed that the modified tunneling probability and modified temperature are not only associated on the horizon of the 5-dimensional CRBR but also the quantum gravity. When $\beta \geq 50$, then the positive modified temperature is received. In our analysis, we obtained that the quantum corrections accelerate the high gain values in modified temperature during the boson particles emission. Here, it is significant to discuss that the value of the modified temperature is larger than the original temperature and the CRBR stops emitting particles when its mass approaches to the smaller value. Equation (8) proved that CRBR radiation is a like BH radiation. It shows that the CRBR will radiate all types of particles as like BH radiation. Our result gives a correction to the Hawking temperature of CRBR. The result discovered out to the CRBR temperature depends on the quantum gravity.

The results of corrected tunneling radiation in Equation (7) for boson particle looks similar in (Equation (2.14) in [7]) if $\beta = 0$, but the mass, charge and angular momentum are same. Moreover, the CRBR with the quantum gravity have more temperature than CRBR without quantum gravity. We can observe that the larger quantum gravity supply gives more radiation. Moreover, this solution is even satisfied if background CRBR metric is more generalized. The results obtained from 3D graphs are similar to the results obtained from 2D graphs. From our CRBR graphical analysis, the temperature increase with the increasing CRBR horizon and CRBR reflects the stable condition for small values of quantum gravity parameter.

Author Contributions: Conceptualization, R.A.; methodology, R.A. and M.A.; formal analysis, R.A. and K.B.; investigation, R.A., K.B. M.A., M.F.M. and S.A.A.S.; software, R.A. and M.A.; writing-original draft preparation, R.A. and M.A.; writing-review and editing, R.A.; visualization, R.A.; supervision, R.A.; project administration, R.A. and K.B.; funding acquisition, K.B. All authors have read and agreed to the published version of the manuscript.

Funding: The work of KB was partially supported by the JSPS KAKENHI Grant Number JP 25800136 and Competitive Research Funds for Fukushima University Faculty (19RI017).

Conflicts of Interest: The authors declare no conflict of interest.

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