Optimal time and space regularity for solutions of degenerate differential equations

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Abstract. We derive optimal regularity, in both time and space, for solutions of the Cauchy problem related to a degenerate differential equation in a Banach space $X$. Our results exhibit a sort of prevalence for space regularity, in the sense that the higher is the order of regularity with respect to space, the lower is the corresponding order of regularity with respect to time.

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1 Introduction

Let $X$ be a Banach space and let $M$ and $L$ be two closed linear operators from $X$ to itself, whose domains fulfill the relation $\mathcal{D}(L) \subseteq \mathcal{D}(M)$. Further, let $f$ be a continuous function from $[0,T]$ into $X$, $T > 0$, and let $u_0$ be a given element of $X$. The question of maximal regularity for the initial value problem

$$
\left\{ \begin{array}{l}
D_t(Mv(t)) = Lv(t) + f(t), \quad t \in (0,T], \\
Mv(0) = u_0,
\end{array} \right. \quad (D_t = \frac{d}{dt})
$$

(1.1)

concerns what kind of properties, in time and/or in space, the data need satisfy, in order that the solution $v$ to (1.1) exists and the derivative $D_tMv$ possesses similar regularity as the data.

Since the natural operator associated to (1.1) is $A = LM^{-1}$, we are led to consider the equivalent problem

$$
\left\{ \begin{array}{l}
D_tw(t) = Aw(t) + f(t), \quad t \in (0,T], \\
w(0) = u_0,
\end{array} \right. \quad (1.2)
$$

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where \( w = Mv \). Hence, the question of maximal regularity for (1.1) is strictly related to the regularity of the semigroup generated by \( A \). This yields to the analysis of the spectral equation \( \lambda u - Au = f, \lambda \in \mathbb{C}, f \in X \), in order to obtain an estimate of type

\[
\| (\lambda I - A)^{-1} f \|_X \leq C(\| \lambda \| + \lambda_0)^{-\beta} \| f \|_X, \quad \lambda \in \Sigma,
\]

(1.3)

\( I \) being the identity operator. Here, \( \beta \in (0, 1], \lambda_0 \geq 0 \) is large enough, and \( \Sigma \) is a complex region containing the half plane \( \Re \lambda \geq 0 \).

Of course (cf. [1, Theorem 3.17] and [2, Theorem 1]), if \( A \) satisfies assumption (1.3) with \( \beta = 1 \), then the results of maximal regularity are analogous to those exhibited in [7] for the non degenerate case, corresponding to \( M = I \) in (1.1), and for which, nowadays, a wide literature exists. In particular, in the case \( \beta = 1 \), \( D_t Mv \) has exactly the same regularity as the data. This extends to (1.1) the known results on the maximal regularity of solutions to (1.2) when the semigroup generated by \( A \) is analytic. On the contrary, according to [1] and [4], if \( \beta \in (0, 1) \), then, in general, the semigroup generated by \( A \) is no longer analytic, but only infinitely differentiable, and this implies that the time regularity of the solutions to (1.1) decreases. We refer to [1, Theorem 3.26], [2, Theorem 9] and [3, Theorem 7.2] for precise statements and amounts of the loss of regularity, but, briefly, the quoted theorems say that if \( f \in C^\nu([0, T]; X), \tau \in (1 - \beta, 1) \), and \( u_0 \) fulfills some natural consistency conditions, then \( D_t Mv \in C^\nu([0, T]; X) \), where \( \nu = \tau + \beta - 1 \).

Notice that, at present, one of the main deficiencies in the theory of degenerate equations is the absence of results of space regularity, in the case when \( \beta \in (0, 1) \) in (1.3). It is our aim, here, to give a contribution in this field providing an optimal “cross” regularity result, in which both time and space regularity for \( D_t Mv \) are established. As we shall see, the space regularity prevails, in the sense that the increase in space regularity reflects in a decrease of the order of time regularity.

The plan of the paper is the following. In Section 2, to a linear operator \( A \) from a Banach space \( X \) to itself, whose resolvent satisfies (1.3) in a region \( \Sigma \) depending on an additional parameter \( \alpha \in (\beta, 1] \), we associate the corresponding infinitely differentiable semigroup \( \{e^{tA}\}_{t \geq 0} \) on \( X \). Moreover, we recall the definition of the interpolation spaces \( (X, D(A))_{\gamma, p} \) between the domain \( D(A) \) of \( A \) and \( X \).

Section 3 is devoted to show that the uniform norm \( \| A^n e^{tA} \|_{\mathcal{L}(X; (X, D(A))_{\gamma, p})} \) blows up, as \( t \) goes to 0, as a suitable negative power of \( t \) depending on \( \alpha, \beta, \gamma \) and \( n \). Further, the blow-up rate is greater than the one observed in [6] for the non degenerate case. As a corollary, we show that for every \( \varepsilon \in (0, T] \) and \( \sigma \in (0, 1) \) the map \( t \to A^n e^{tA} \) belongs to \( C^\sigma([\varepsilon, T]; \mathcal{L}(X; (X, D(A))_{\gamma, p})) \).

Using the results of Section 3, in Section 4 we establish time and space regularity of some basic operator functions, which appear naturally when \( Mv \) and \( D_t Mv \) are represented in terms of the Volterra integral equation equivalent to (1.2). In particular, Lemmas 4.1, 4.3 highlight the above mentioned fact that the higher is the order \( \gamma \) of the interpolation space \( (X, D(A))_{\gamma, p} \), where we look for space regularity, the lower is the Hölder exponent \( \sigma \) of regularity in time.

Section 5 contains our main results. First, using Lemmas 4.1, 4.3 in Theorems 5.1 and 5.3 we show that if \( \gamma \) and \( \sigma \) are opportunely chosen, \( \sigma < \gamma \), then (1.1) has a unique strict solution \( u \) such that \( Mv \in C^\sigma([0, T]; (X, D(A))_{\gamma, p}) \). Then, combining Lemmas 4.3, 4.5 in Theorem 5.4 we prove that, if \( \alpha \) and \( \beta \) are large enough and the data pair \( (f, u_0) \) satisfies
some suitable space–time assumptions, the regularity $C^\alpha([0,T];(X,D(A))_{\gamma,p})$ holds for the derivative $D_tMv$, too.

2 Preliminary material and notations

Let $X$ be a Banach space endowed with norm $\| \cdot \|_X$ and let $A : D(A) \subset X \to X$ be a single valued linear operator. Recalling that the resolvent set $\rho(A)$ of $A$ is the set of values $\lambda \in \mathbb{C}$ such that $\lambda I - A$ has a bounded inverse $(\lambda I - A)^{-1}$ with domain dense in $X$, we assume that $A$ satisfies the following resolvent condition:

(H1) $\rho(A)$ contains the complex region $\Sigma = \{ \lambda \in \mathbb{C} : \Re \lambda \geq -c(|3m\lambda|+1)^\alpha \}$ and, for every $\lambda \in \Sigma$, the following estimate holds

$$\| (\lambda I - A)^{-1} \|_{\mathcal{L}(X)} \leq C(|\lambda|+1)^{-\beta},$$

for some exponents $0 < \beta < \alpha \leq 1$ and constants $c, C > 0$.

Here, as usual, $\mathcal{L}(X)$ denotes the Banach space $\mathcal{L}(X;X)$ of all bounded linear operators from $X$ to $X$, equipped with the uniform operator norm.

According to [1, Theorem 3.1], assumption (H1) implies that $A$ generates an infinitely differentiable semigroup on $X$. More precisely, introduce the family $\{e^{tA}\}_{t \geq 0} \subset \mathcal{L}(X)$ defined by the Dunford integral

$$e^{tA} = (2\pi i)^{-1} \int_{\Gamma} e^{\lambda t}(\lambda I - A)^{-1} d\lambda, \quad t > 0, \quad (2.1)$$

where $\Gamma \subset \Sigma$ is the contour parametrized by $\lambda = -c(|\eta|+1)^\alpha + i\eta, -\infty < \eta < \infty$. Define also $e^{0A} = 1$. Then $\{e^{tA}\}_{t \geq 0}$ is a semigroup on $X$, infinitely many times differentiable for $t > 0$ with $D_te^{tA} = Ae^{tA}$. In addition, $e^{tA}$ satisfy the estimates (see [1, Proposition 3.2])

$$\| A^k e^{tA} \|_{\mathcal{L}(X)} \leq \tilde{c}_k t^{(\beta-k-1)/\alpha}, \quad t > 0, \quad k \in \mathbb{N} \cup \{0\}, \quad (2.2)$$

where the $\tilde{c}_k$’s are positive constants depending on $k$. Of course, due to (2.2), if $\beta < 1$, then the function $t \to e^{tA}$ is not bounded as $t \to 0^+$. As a consequence, $e^{tA}$ is not necessarily strongly continuous in the norm of $X$ on the subspace $D(A)$.

We stress that, even though here we are following the approach in [1], resolvent conditions of type (H1) were already introduced in [4]. In particular, in [4] Remark 383 it was showed that if $U$ is a closed linear operator with dense domain and such that

$$\| (\lambda I - U)^{-1} \|_{\mathcal{L}(X)} \leq C(\Re \lambda + |3m\lambda|^{\beta})^{-1}, \quad \Re \lambda > 0, \quad \beta \in (0,1),$$

then $U$ generates a semigroup $e^{tU}$ which is infinitely differentiable for $t > 0$.

For our purposes, we recall now the definitions of two classes of real interpolation spaces between $D(A)$ and $X$. First of all, we specify a topology on $D(A)$ equipping it with the norm $\| g \|_{D(A)} = \| y \|_X + \| Ay \|_X$ which makes $D(A)$ a Banach space. Now, if $Z$ is a Banach space, for an $Z$-valued strongly measurable function $g(\xi), \xi \in (0,\infty)$, we set

$$\| g \|_{L^\gamma(Z)} = \left( \int_0^\infty \| g(\xi) \|_Z^p \frac{d\xi}{\xi} \right)^{1/p}, \quad p \in [1, \infty),$$

$$\| g \|_{L^\infty(Z)} = \sup_{\xi \in (0,\infty)} \| g(\xi) \|_Z, \quad p = \infty.$$
Then, according to [1, pag. 26], for every $\gamma \in (0, 1)$ and $p \in [1, \infty]$ we introduce the (intermediate) spaces

$$X_A^{\gamma,p} = \{x \in X : \|\xi^\gamma A(\xi I - A)^{-1}x\|_{L^p_t(X)} < \infty\},$$

which becomes Banach spaces when endowed with the norm

$$\|x\|_{X_A^{\gamma,p}} := \|x\|_X + \|\xi^\gamma A(\xi I - A)^{-1}x\|_{L^p_t(X)}.$$

Also, for $\gamma \in (0, 1)$ and $p \in [1, \infty]$ we denote with $V(p, \gamma, D(A), X)$ the space of all $X$-valued functions $v(\xi), \xi \in (0, \infty)$, having the property that the maps $\xi \mapsto \xi^\gamma v(\xi)$ and $\xi \mapsto \xi^\gamma v'(\xi)$ belong, respectively, to $L^p_t(D(A))$ and $L^p_t(X)$. As it is well-known (cf. [5, Lemma 1.8.1]), the spaces $V(p, \gamma, D(A), X)$ are Banach spaces with the norm

$$\|v\|_{V(p,\gamma,D(A),X)} = \|\xi^\gamma v\|_{L^p_t(D(A))} + \|\xi^\gamma v'\|_{L^p_t(X)}$$

and any function $v \in V(p, \gamma, D(A), X)$ has a $X$-valued continuous extension at $t = 0$. This lead to define the trace spaces (cf. [5, Theorem 1.8.2])

$$(X, D(A))_{\gamma,p} = \{x \in X : x = v(0), \ v \in V(p, 1 - \gamma, D(A), X)\},$$

which turn out to be real interpolation spaces between $D(A)$ and $X$. They are Banach spaces endowed with the norm

$$\|x\|_{(X,D(A))_{\gamma,p}} = \inf\{\|v\|_{V(p,1-\gamma,D(A),X)} : x = v(0), \ v \in V(p,1 - \gamma, D(A), X)\}.$$

Further, for $\gamma \in (0, 1)$ and $1 \leq p_1 < p_2 \leq \infty$ we have

$$D(A) \hookrightarrow (X, D(A))_{\gamma,p_1} \hookrightarrow (X, D(A))_{\gamma,p_2} \hookrightarrow \overline{D(A)},$$

whereas, for $0 < \gamma_1 < \gamma_2 < 1$, we have

$$(X, D(A))_{\gamma_2,\infty} \hookrightarrow (X, D(A))_{\gamma_1,1}.$$
Here we show two preliminary results concerning the behaviour of \( X^t_A \), with respect the interpolation spaces \((X, \mathcal{D}(A))_{\gamma,p}\) say that, when \( t \) goes to zero, the norm \( \| A^t e^{tA} \|_{\mathcal{L}(X; (X, \mathcal{D}(A))_{\gamma,p})} \) may go to infinity, but not faster than a precise negative power of \( t \) depending on \( n, \gamma \) and the exponents \( \alpha, \beta \) appearing in (H1). A similar result is shown in [6, Proposition 2.3.9] for the non degenerate case, and in [1, Proposition 3.2] for the degenerate one. However, in [1], only the case \( n = 0 \) is treated and the role of the spaces \((X, \mathcal{D}(A))_{\gamma,p}\) is played there by the spaces \( X_{\gamma}^{\infty} \).

**Proposition 3.1.** Let \( \alpha, \beta \in (0,1), \beta < \alpha, \gamma \in (0,1), p \in [1, \infty] \) and \( n \in \mathbb{N} \cup \{0\} \). Then there exist positive constants \( C = C(\gamma, p, n) \) and \( C' = C'(\alpha, \beta, \gamma, p, n) \) such that the following estimates hold

\[
\begin{align*}
(i) \quad &\| A^n e^{tA} \|_{\mathcal{L}(X; (X, \mathcal{D}(A))_{\gamma,p})} \leq C t^{(\beta-n-1-\gamma)/\alpha}, \quad t \in (0,1], \\
(ii) \quad &\| A^n e^{tA} \|_{\mathcal{L}(X; (X, \mathcal{D}(A))_{\gamma,p})} \leq C' t^{(\beta-n-1)/\alpha}, \quad t \geq 1.
\end{align*}
\]

(3.1)

In particular, setting \( c_1(T) = C + C'T^{\gamma/\alpha} \), for every \( T > 0 \) we obtain

\[
\| A^n e^{tA} \|_{\mathcal{L}(X; (X, \mathcal{D}(A))_{\gamma,p})} \leq c_1(T) t^{(\beta-n-1-\gamma)/\alpha}, \quad \forall t \in (0, T].
\]

(3.2)

**Proof.** First, for \( t \in (0,1) \) and \( x \in X \), using the interpolation inequality \( \| y \|_{(X, \mathcal{D}(A))_{\gamma,p}} \leq c(\gamma, p) \| y \|^\gamma_X \| y \|^\gamma_{\mathcal{D}(A)}, \ y \in \mathcal{D}(A) \), and the estimate \( \| A^k e^{tA} \|_{\mathcal{L}(X)} \leq \widetilde{c}_k t^{(\beta-k-1)/\alpha}, \ t > 0, \)

We conclude the section introducing some notations we will largely use in the sequel. Given a Banach space \( Z \), \( C([0,T]; Z) \) and \( C^\delta([0,T]; Z), \delta \in (0,1) \), denote, respectively, the spaces of all continuous and \( \delta \)-Hölder continuous functions from \([0,T] \) into \( Z \). The shortenings \( \| \cdot \|_{0,T;Z} \) and \( \| \cdot \|_{\delta,T;Z} \) stand, respectively, for the usual sup-norm of \( C([0,T]; Z) \) and the norm \( \| \cdot \|_{C^\delta([0,T];Z)} \) of \( C^\delta([0,T]; Z) \), i.e.

\[
\| f \|_{0,T;Z} = \sup_{t \in [0,T]} \| f(t) \|_Z,
\]

\[
\| f \|_{\delta,T;Z} = \| f \|_{0,T;Z} + \| f \|_{\delta,T;Z}, \quad \| f \|_{\delta,T;Z} := \sup_{0 \leq s < t \leq T} \frac{\| f(t) - f(s) \|_Z}{(t-s)^\delta}.
\]

Moreover, \( B([0,T]; Z) \) and \( C^1((0,T]; Z) \) denote, respectively, the space of all bounded functions from \([0,T] \) into \( Z \) with the sup-norm, and the space of all strongly differentiable functions on \((0,T) \) whose derivatives are continuous from \((0,T] \) into \( Z \). Finally, if \( Z_1 \) and \( Z_2 \) are two different Banach space, \( \mathcal{L}(Z_1, Z_2) \) is the Banach space of bounded linear operators from \( Z_1 \) into \( Z_2 \) with the usual uniform operator norm.

### 3 Regularity of \( e^{tA} \) with respect the spaces \((X, \mathcal{D}(A))_{\gamma,p}\)

Here we show two preliminary results concerning the behaviour of \( A^n e^{tA}, n \in \mathbb{N} \cup \{0\}, \) with respect the interpolation spaces \((X, \mathcal{D}(A))_{\gamma,p}\). Essentially, the following Proposition 3.1 says that, when \( t \) goes to zero, the norm \( \| A^n e^{tA} \|_{\mathcal{L}(X; (X, \mathcal{D}(A))_{\gamma,p})} \) may go to infinity, but not faster than a precise negative power of \( t \) depending on \( n, \gamma \) and the exponents \( \alpha, \beta \) appearing in (H1). A similar result is shown in [6, Proposition 2.3.9] for the non degenerate case, and in [1, Proposition 3.2] for the degenerate one. However, in [1], only the case \( n = 0 \) is treated and the role of the spaces \((X, \mathcal{D}(A))_{\gamma,p}\) is played there by the spaces \( X_{\gamma}^{\infty} \).

**Proposition 3.1.** Let \( \alpha, \beta \in (0,1), \beta < \alpha, \gamma \in (0,1), p \in [1, \infty] \) and \( n \in \mathbb{N} \cup \{0\} \). Then there exist positive constants \( C = C(\gamma, p, n) \) and \( C' = C'(\alpha, \beta, \gamma, p, n) \) such that the following estimates hold

\[
\begin{align*}
(i) \quad &\| A^n e^{tA} \|_{\mathcal{L}(X; (X, \mathcal{D}(A))_{\gamma,p})} \leq C t^{(\beta-n-1-\gamma)/\alpha}, \quad t \in (0,1], \\
(ii) \quad &\| A^n e^{tA} \|_{\mathcal{L}(X; (X, \mathcal{D}(A))_{\gamma,p})} \leq C' t^{(\beta-n-1)/\alpha}, \quad t \geq 1.
\end{align*}
\]

(3.1)

In particular, setting \( c_1(T) = C + C'T^{\gamma/\alpha} \), for every \( T > 0 \) we obtain

\[
\| A^n e^{tA} \|_{\mathcal{L}(X; (X, \mathcal{D}(A))_{\gamma,p})} \leq c_1(T) t^{(\beta-n-1-\gamma)/\alpha}, \quad \forall t \in (0, T].
\]

(3.2)

**Proof.** First, for \( t \in (0,1) \) and \( x \in X \), using the interpolation inequality \( \| y \|_{(X, \mathcal{D}(A))_{\gamma,p}} \leq c(\gamma, p) \| y \|^\gamma_X \| y \|^\gamma_{\mathcal{D}(A)}, \ y \in \mathcal{D}(A) \), and the estimate \( \| A^k e^{tA} \|_{\mathcal{L}(X)} \leq \widetilde{c}_k t^{(\beta-k-1)/\alpha}, \ t > 0, \)
$k \in \mathbb{N} \cup \{0\}$, we get
\[
\|t^n A^ne^{tA}x\|_{(X,D(A))_{\gamma,p}} \leq c(\gamma, p)\|t^n A^ne^{tA}x\|_{X}^{1-\gamma}\|t^n A^ne^{tA}x\|_{\gamma}^\gamma
\]
\[
\leq c(\gamma, p)c_n^{-1}\gamma t^{(1-\gamma)((\beta+(\alpha-1)n-1)/\alpha)}\|x\|_{X}^{1-\gamma}\|t^n A^ne^{tA}x\|_{X} + \|t^n A^{n+1}e^{tA}x\|_{X}^\gamma
\]
\[
\leq c(\gamma, p)c_n^{-1}\gamma t^{(1-\gamma)((\beta+(\alpha-1)n-1)/\alpha)}\|\tilde{c}_n(t\gamma^{(\beta+(\alpha-1)n-1)/\alpha} + \tilde{c}_{n+1}t^{(\beta+(\alpha-1)n-2)/\alpha})\|_{X}
\]
\[
\leq c(\gamma, p)c_n^{-1}\gamma(\tilde{c}_n + \tilde{c}_{n+1})\gamma t^{(\beta+(\alpha-1)n-1-\gamma)/\alpha}\|x\|_{X}
\]
This proves (3.1)(i) with $C = c(\gamma, p)c_n^{-1}\gamma(\tilde{c}_n + \tilde{c}_{n+1})\gamma$. Concerning (3.1)(ii), instead, for $t \geq 1$ and $x \in X$, using (3.1)(i) with $t = 1/2$ and $n = 0$, we easily derive:
\[
\|t^n A^ne^{tA}x\|_{(X,D(A))_{\gamma,p}} = \|(t-t/2)^n A^ne^{(t-t/2)A}e^{\frac{t}{2}A}x\|_{(X,D(A))_{\gamma,p}}
\]
\[
\leq \left(\frac{t}{t-1/2}\right)^n \overline{c}_n(t-1/2)^{\beta+(\alpha-1)n-1/\alpha}\|e^{\frac{t}{2}A}\|\|\mathcal{L}(X;X,D(A))_{\gamma,p}\|x\|_{X}
\]
\[
\leq 2^n\overline{c}_n[1-(2t)]^{\beta+(\alpha-1)n-1/\alpha}\|e^{\frac{t}{2}A}\|\|\mathcal{L}(X;X,D(A))_{\gamma,p}\|x\|_{X}
\]
\[
\leq 2^{n+1-\beta/\alpha}\overline{c}_n^{(\beta+(\alpha-1)n-1)/\alpha}\|e^{\frac{t}{2}A}\|\|\mathcal{L}(X;X,D(A))_{\gamma,p}\|x\|_{X}
\]
\[
\leq 2^{2(n+1-\beta/\alpha)}\overline{c}_n^{(\beta+(\alpha-1)n-1)/\alpha}\|\mathcal{L}(X;X,D(A))_{\gamma,p}\|x\|_{X}.
\]
This proves (3.1)(ii) with $C' = 2^{2(n+1-\beta/\alpha)}\overline{c}_n C$. Hence, if $1 \leq T$, from (3.1)(ii) we get
\[
\|A^ne^{tA}\|_{\mathcal{L}(X;X,D(A))_{\gamma,p}} \leq C't^{\gamma\alpha}(\beta-n-1)/\alpha, 1 \leq T \leq T.\]
Combining this latter inequality with (3.1)(i) we obtain (3.2).

\textbf{Remark 3.2.} Observe that, due to the continuous embedding (2.3), if $(n, p) = (0, \infty)$ then (3.2) agrees with estimate $\|e^{tA}\|_{\mathcal{L}(X;X)} \leq C't^{\gamma\alpha}(\beta-n-1)/\alpha$ in [11 Proposition 3.2]. However, since $(X, D(A))_{\gamma,p} \hookrightarrow (X, D(A))_{\gamma,\infty}$ for $p \in [1, \infty)$, our estimate really refines that in [11], even in the case $n = 0$.

Proposition 3.1 easily implies that, when $t$ is bounded away from zero, then, for every $\sigma \in (0, 1)$, the operator function $t \mapsto A^ne^{tA}$ is $\sigma$-Hölder continuous in time with values in $\mathcal{L}(X;X,D(A))_{\gamma,p})$. Indeed, the following corollary holds.

\textbf{Corollary 3.3.} Let $\alpha, \beta \in (0, 1), \beta < \alpha, \gamma \in (0, 1), p \in [1, \infty]$ and $n \in \mathbb{N} \cup \{0\}$. Then, for every $\sigma \in (0, 1)$ and $0 < s < t \leq T$ we have
\[
\|A^n e^{tA} - A^n e^{sA}\|_{\mathcal{L}(X;X,D(A))_{\gamma,p}} \leq \sigma^{-1}c_1(T)s^{(\alpha+\beta-n-2-\gamma-a)/\alpha}(t-s)^\gamma.
\]
\textbf{Proof.} For every $x \in X$ and $0 < s < t \leq T$, using the identity $[A^n e^{tA} - A^n e^{sA}]x = \int_s^t A^{n+1}e^{rA}x\,dr$, inequality (3.3) with $n$ replaced by $n + 1$ and the well-known inequality $t^{\gamma} - s^{\gamma} \leq (t-s)^\gamma, \gamma \in (0, 1)$, we easily obtain
\[
\|\|A^n e^{tA} - A^n e^{sA}\|_{X,D(A))_{\gamma,p}} \leq c_1(T)\|x\|_X \int_s^t \xi^{(\beta-n-2-\gamma)/\alpha}\,d\xi
\]
\[
\leq c_1(T)\|x\|_X s^{(\alpha+\beta-n-2-\gamma-a)/\alpha} \int_s^t \xi^{\alpha-1}\,d\xi
\]
\[
\leq \sigma^{-1}c_1(T)\|x\|_X s^{(\alpha+\beta-n-2-\gamma-a)/\alpha}(t-s)^\gamma.
\]

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This completes the proof. □

4 Time and space regularity of the basic operator functions

Proposition 3.1 and Corollary 3.3 enable us to prove some Hölder-in-time regularity with respect the spaces \( (X, D(A))_{\gamma,p} \) for those operator functions involving the semigroup \( e^{tA} \) which we will encounter later. Through the rest of the paper, \( c_j(T), j = 2, 3, \ldots, \) shall denote positive nondecreasing functions of \( T \) depending also on \( \alpha, \beta, \gamma, p \) and \( \sigma \in (0, 1) \).

**Lemma 4.1.** Let \( \alpha, \beta \in (0, 1) \) such that \( \beta < \alpha \) and \( 2\alpha + \beta > 2 \). Then, for every \( \gamma \in (0,2\alpha + \beta - 2) \) and \( \sigma \in (\gamma, (2\alpha + \beta - 2 - \gamma)/\alpha) \) the linear operator

\[
[Q_1 g](t) := \int_0^t e^{(t-\xi)A} g(\xi) \, d\xi \tag{4.1}
\]

maps \( C([0,T];X) \) into \( C^\gamma([0,T];(X, D(A))_{\gamma,p}) \), \( p \in [1, \infty] \), and satisfies the estimate:

\[
\|Q_1 g\|_{\sigma,T; (X, D(A))_{\gamma,p}} \leq T^{(2\alpha+\beta-2-\gamma-\alpha\sigma)/\alpha} \gamma c_2(T) \|g\|_{0,T;X}. \tag{4.2}
\]

**Proof.** First, for every \( t \in [0,T] \), inequality (3.2) with \( n = 0 \) implies

\[
\|Q_1 g\|(t) \|_{(X, D(A))_{\gamma,p}} \leq \frac{\sigma c_1(T)}{\alpha + \beta - 1 - \gamma} \|g\|_{0,T;X} t^{(\alpha+\beta-1-\gamma)/\alpha}, \tag{4.3}
\]

where the exponent \( (\alpha + \beta - 1 - \gamma)/\alpha \) is positive since \( 0 < \gamma < 2\alpha + \beta - 2 \leq \alpha + \beta - 1 \). Moreover, when \( \sigma \in (0,(2\alpha + \beta - 2 - \gamma)/\alpha) \) and \( 0 < s < t \leq T \), from both (3.2) and (3.3) with \( n = 0 \) we obtain

\[
\|Q_1 g\|(t) - [Q_1 g](s) \|_{(X, D(A))_{\gamma,p}} \\
\leq c_1(T) \|g\|_{0,T;X} \int_0^s (s - \xi)^{(\alpha+\beta-2-\gamma-\alpha\sigma)/\alpha} \, d\xi \\
+ \int_s^t (t - \xi)^{(\beta-1-\gamma)/\alpha} \, d\xi \\
\leq c_1(T) \|g\|_{0,T;X} \left[ \frac{\sigma}{2\alpha + \beta - 2 - \gamma - \alpha\sigma} (t - s)^\sigma + \frac{(t - s)^{(\alpha+\beta-1-\gamma)/\alpha}}{\alpha + \beta - 1 - \gamma} \right] \\
\leq c_1(T) \|g\|_{0,T;X} \left[ \frac{\sigma}{2\alpha + \beta - 2 - \gamma - \alpha\sigma} + \frac{1}{\alpha + \beta - 1 - \gamma} \right] (t - s)^\sigma. \tag{4.4}
\]

Finally, summing up (4.3) and (4.4), we derive (4.2) with

\[
c_2(T) = \alpha c_1(T) \left[ \frac{T^{(1-\alpha)/\alpha}}{\alpha + \beta - 1 - \gamma} + \frac{\sigma}{2\alpha + \beta - 2 - \gamma - \alpha\sigma} \right]. \tag{4.5}
\]

This completes the proof. □

\[1\] Since \( \sigma \in (0,(2\alpha + \beta - 2 - \gamma)/\alpha) \), the case \( s = 0 \) follows from inequality (4.3) once we observe that

\[
(\alpha + \beta - 1 - \gamma)/\alpha - \sigma > (\alpha + \beta - 1 - \gamma - (2\alpha + \beta - 2 - \gamma))/\alpha = (1-\alpha)/\alpha \geq 0
\]
If \( g \) is not only merely continuos from \([0, T]\) to \( X \), but \( \sigma \)-Hölder continuous, then the thesis of Lemma 4.1 follows by a weaker assumption on \( \alpha \) and \( \beta \) and for larger values of \( \gamma \) and \( \sigma \). Indeed, the proof can be modified in order to avoid Corollary 3.3 as it is shown in the following lemma.

**Lemma 4.2.** Let \( \alpha, \beta \in (0,1] \) such that \( \beta < \alpha \) and \( \alpha + \beta > 1 \). Then, for every \( \gamma \in (0, \alpha + \beta - 1) \) and \( \sigma \in (0, (\alpha + \beta - 1 - \gamma)/\alpha) \) the linear operator \( Q_1 \) defined by (4.4) maps \( C^\sigma([0,T];X) \) into \( C^\sigma([0,T];(X,D(A))_{\gamma,p}) \), \( p \in [1,\infty] \), and satisfies the estimate:

\[
\|Q_1g\|_{\sigma,T,(X,D(A))_{\gamma,p}} \leq T^{(\alpha+\beta-1-\gamma-\alpha\sigma)/\alpha}c_3(T)\|g\|_{\sigma,T,X}. \tag{4.6}
\]

If \( g \) is a constant function then (4.6) can be improved until the value \( \sigma = (\alpha+\beta-1-\gamma)/\alpha \).

**Proof.** Since \( g \in C^\sigma([0,T];X) \), \( \sigma \in (0, (\alpha + \beta - 1 - \gamma)/\alpha) \), when \( 0 < s < t \leq T \) from (5.2) with \( n = 0 \) it follows

\[
\|[Q_1g](t) - [Q_1g](s)\|_{(X,D(A))_{\gamma,p}} \\
\leq \int_s^t \|e^{\xi A}[g(t-\xi) - g(s-\xi)]\|_{(X,D(A))_{\gamma,p}} \, d\xi + \int_s^t \|e^{\xi A}g(t-\xi)\|_{(X,D(A))_{\gamma,p}} \, d\xi \\
\leq c_1(T)\|g\|_{\sigma,T,X}[(t-s)^{\sigma}\int_s^t \xi^{(\beta-1-\gamma)/\alpha} \, d\xi + \int_s^t \xi^{(\beta-1-\gamma)/\alpha} \, d\xi] \\
\leq \frac{\alpha c_1(T)}{\alpha + \beta - 1 - \gamma}\|g\|_{\sigma,T,X}[(t-s)^{\sigma}s^{(\alpha+\beta-1-\gamma)/\alpha} + (t-s)^{(\alpha+\beta-1-\gamma)/\alpha}] \\
\leq \frac{\alpha c_1(T)}{\alpha + \beta - 1 - \gamma}\|g\|_{\sigma,T,X}[s^{(\alpha+\beta-1-\gamma)/\alpha} + (t-s)^{(\alpha+\beta-1-\gamma-\sigma)/\alpha}](t-s)^{\sigma}. \tag{4.7}
\]

Summing up (4.3) and (4.7), we derive (4.6) with

\[
c_3(T) = \alpha(\alpha + \beta - 1 - \gamma)^{-1}c_1(T)[2T^\sigma + 1]. \tag{4.8}
\]

Last assertion trivially follows simply observing that \( g(t-\xi) - g(s-\xi) = 0 \) in the previous computations. \( \square \)

**Lemma 4.3.** Let \( \alpha, \beta \in (0,1] \) such that \( \beta < \alpha \) and \( \alpha + \beta > 1 \). Moreover, let \( x \in D(A) \), \( \gamma \in (0, \alpha + \beta - 1) \) and \( \sigma \in (0, (\alpha + \beta - 1 - \gamma)/\alpha) \). Then \( e^{tA}x \in C^\sigma([0,T];(X,D(A))_{\gamma,p}) \), \( p \in [1,\infty] \), and satisfies the estimate:

\[
\|e^{tA}x\|_{\sigma,T,(X,D(A))_{\gamma,p}} \leq c_4(T)\|x\|_{D(A)}. \tag{4.9}
\]

**Proof.** First, from Lemma 2.3 in [5], with the triplet \((L^p(\Omega),A(t),f)\) being replaced by \((X,A,x)\), we have \( \int_0^t e^{\xi A}x \, d\xi \in D(A) \) for every \( t > 0 \) and \( A\int_0^t e^{\xi A}x \, d\xi = e^{tA}x - x \). Now, since \( x \in D(A) \) from the equality \( A(\lambda - A)^{-1}x = (\lambda - A)^{-1}Ax \) and the definition of \( \{e^{\xi A}\}_{t > 0} \) by Dunford integrals, we have \( Ae^{tA}x = e^{tA}Ax \) for every \( t > 0 \). Hence

\[
\int_0^t \|Ae^{tA}x\|_X \, d\xi = \int_0^t \|e^{tA}Ax\|_X \, d\xi \leq c_0 \int_0^t \xi^{(\beta-1)/\alpha}\|Ax\|_X \, d\xi \leq c_0\|x\|_{D(A)}t^{(\alpha+\beta-1)/\alpha}. \]

\(^2\)Since \( \sigma \in (0, (\alpha + \beta - 1 - \gamma)/\alpha) \), the case \( s = 0 \) follows from inequality (4.3).
It follows that the map $\xi \rightarrow \|A e^{\xi A}x\|_X$ belongs to $L^1((0, t); X)$ for all $t \in (0, T]$ and $e^{tA}x - x = A \int_0^t e^{\xi A}x \, d\xi = \int_0^t e^{\xi A}Ax \, d\xi$. Thus, (3.2) with $n = 0$ yields to
\[
\|e^{tA}x - x\|_{(X, D(A))_{\gamma, p}} \leq c_1(T) \int_0^t \xi^{(\gamma - 1)/\alpha} \|Ax\|_X \, d\xi \leq c_1(T)\|x\|_{D(A)} t^{(\alpha + \beta - 1)/\alpha}. \tag{4.10}
\]

Consequently, for every $t \in [0, T]$, we get
\[
\|e^{tA}x\|_{(X, D(A))_{\gamma, p}} \leq \|e^{tA}x - x\|_{(X, D(A))_{\gamma, p}} + \|x\|_{(X, D(A))_{\gamma, p}}
\]
\[
\leq [c_1(T) t^{(\alpha + \beta - 1)/\alpha} + c(\gamma, p)] \|x\|_{D(A)}. \tag{4.11}
\]

Now, let $\sigma \in (0, (\alpha + \beta - 1 - \gamma)/\alpha)$ and $0 < s < t \leq T_3$. Then, reasoning as in the derivation of (4.10), we obtain
\[
\|e^{tA}x - e^{sA}x\|_{(X, D(A))_{\gamma, p}} = \| \int_s^t A e^{\xi A}x \, d\xi\|_{(X, D(A))_{\gamma, p}}
\]
\[
\leq c_1(T)\|x\|_{D(A)} (t - s)^{(\alpha + \beta - \gamma)/\alpha}
\]
\[
\leq c_1(T) T^{(\alpha + \beta - 1 - \gamma - \sigma)/\alpha} \|x\|_{D(A)} (t - s)^{\sigma}. \tag{4.12}
\]

From (4.11) and (4.12) we deduce (4.9) with
\[
c_4(T) = c(\gamma, p) + c_1(T) T^{(\alpha + \beta - 1 - \gamma - \sigma)/\alpha} (T^{\sigma} + 1) \tag{4.13}
\]
and the proof is complete.

In the next section, in order to obtain optimal regularity for solutions of degenerate parabolic equations, we will need to estimate the maps $t \rightarrow e^{\xi A} [f(t) - f(0)]$ and $t \rightarrow \int_0^t A e^{(t-\xi)A} [f(\xi) - f(t)] \, d\xi$, where $f$ is Hölder continuous. The next two lemmas give us the desired results in this direction.

**Lemma 4.4.** Let $\alpha, \beta \in (0, 1)$ such that $\beta < \alpha$ and $2\alpha + \beta > 2$. Then, for every $\mu \in ((2 - \alpha - \beta)/\alpha, 1)$, $\gamma \in (0, \alpha \mu + \alpha + \beta - 2)$ and $\sigma \in (0, (\alpha \mu + \alpha + \beta - 2 - \gamma)/\alpha)$ the linear operator
\[
[Q_2f](t) := e^{tA} [f(t) - f(0)] \tag{4.14}
\]
maps $C^\mu([0, T]; X)$ into $C^\sigma([0, T]; (X, D(A))_{\gamma, p})$, $p \in [1, \infty]$, and satisfies the estimate:
\[
\|Q_2f\|_{(X, D(A))_{\gamma, p}} \leq T^{(\alpha \mu + \alpha + \beta - 2 - \gamma - \sigma)/\alpha} c_5(T) \|f\|_{\mu, T, X}. \tag{4.15}
\]

**Proof.** When $f \in C^\mu([0, T]; X)$, $\mu \in ((2 - \alpha - \beta)/\alpha, 1)$, from (3.2) with $n = 0$ it follows
\[
\|[Q_2f](t)\|_{(X, D(A))_{\gamma, p}} \leq c_1(T) \|f\|_{\mu, T, X} t^{(\alpha \mu + \alpha + \beta - 1 - \gamma)/\alpha}, \quad \forall t \in [0, T], \tag{4.16}
\]
where $(\alpha \mu + \beta - 1 - \gamma)/\alpha > 0$, due to $0 < \gamma < \alpha \mu + \alpha + \beta - 2 \leq \alpha \mu + \beta - 1$. Now, let $\sigma \in (0, (\alpha \mu + \alpha + \beta - 2 - \gamma)/\alpha)$ and $0 < s < t \leq T_3$. We have
\[
\|[Q_2f](t) - [Q_2f](s)\|_{(X, D(A))_{\gamma, p}} \leq \sum_{k=1}^2 I_k(s, t), \tag{4.17}
\]
---

3. Since $\sigma \in (0, (\alpha + \beta - 1 - \gamma)/\alpha)$ and $e^{\xi A}$ is defined to be 1, the case $s = 0$ follows from (4.10).

4. Since $0 < \sigma < (\alpha \mu + \alpha + \beta - 2 - \gamma)/\alpha \leq (\alpha \mu + \beta - 1 - \gamma)/\alpha$ the case $s = 0$ follows from (4.16).
where
\[ I_1(s, t) := \|e^{tA}[f(t) - f(s)]\|_{(X, \mathcal{D}(A))_{\gamma, p}}, \]
\[ I_2(s, t) := \|(e^{tA} - e^{sA})[f(s) - f(0)]\|_{(X, \mathcal{D}(A))_{\gamma, p}}. \]

Concerning \( I_1(s, t) \), the same reasoning made to derive (4.16) lead to
\[ I_1(s, t) \leq c_1(T) t^{(\alpha \mu + \beta - 1 - \gamma - \alpha \sigma)/\alpha} |f|_{\mu, T; X} (t - s)^\sigma, \tag{4.18} \]
the exponent \((\alpha \mu + \beta - 1 - \gamma - \alpha \sigma)/\alpha \) being positive, since \( 0 < \sigma < (\alpha \mu + \alpha + \beta - 2 - \gamma)/\alpha \leq (\alpha \mu + \beta - 1 - \gamma)/\alpha \). Instead, using (3.3) with \( n = 0 \) we obtain
\[ I_2(s, t) \leq \sigma^{-1} g^{(\alpha \mu + \alpha + \beta - 2 - \gamma - \alpha \sigma)/\alpha} c_1(T) |f|_{\mu, T; X} (t - s)^\sigma \tag{4.19} \]
Therefore, (4.16)–(4.19) yield to (4.15) with
\[ c_5(T) = c_1(T) |T^{(1 - \alpha)/\alpha} (T^\sigma + 1) + \sigma^{-1}|. \tag{4.20} \]
The proof is now complete. \( \square \)

**Lemma 4.5.** Let \( \alpha, \beta \in (0, 1) \) such that \( \beta < \alpha \) and \( 3\alpha + \beta > 3 \). Then, for every \( \mu \in ((3 - 2\alpha - \beta)/\alpha, 1) \), \( \gamma \in (0, \alpha \mu + 2\alpha + \beta - 3) \) and \( \sigma \in (0, (\alpha \mu + 2\alpha + \beta - 3 - \gamma)/\alpha) \) the linear operator
\[ [Q_3 f](t) := \int_0^t A e^{(t - \xi)A} [f(\xi) - f(t)] \, d\xi \tag{4.21} \]
maps \( C^\mu([0, T]; X) \) into \( C^\sigma([0, T]; \gamma, p) \), \( p \in [1, \infty] \), and satisfies the estimate:
\[ \|Q_3 f\|_{\sigma; T; (X, \mathcal{D}(A))_{\gamma, p}} \leq T^{(\alpha \mu + 2\alpha + \beta - 3 - \gamma - \alpha \sigma)/\alpha} c_6(T) |f|_{\mu, T; X}. \tag{4.22} \]

**Proof.** Let \( f \in C^\mu([0, T]; X) \), \( \mu \in ((3 - 2\alpha - \beta)/\alpha, 1) \). Then, for every \( t \in [0, T] \), using (3.2) with \( n = 1 \) we find
\[ \|Q_3 f(t)\|_{(X, \mathcal{D}(A))_{\gamma, p}} \leq c_1(T) |f|_{\mu, T; X} \int_0^t (t - \xi)^{(\alpha \mu + \beta - 2 - \gamma)/\alpha} \, d\xi \]
\[ \leq \frac{\alpha c_1(T)}{\alpha \mu + \alpha + \beta - 2 - \gamma} |f|_{\mu, T; X} t^{(\alpha \mu + \alpha + \beta - 2 - \gamma)/\alpha}. \tag{4.23} \]
Notice that the choice \( \gamma \in (0, \alpha \mu + 2\alpha + \beta - 3) \) implies \( (\alpha \mu + \alpha + \beta - 2 - \gamma)/\alpha > 0 \) in the latter inequality. Now, let \( \sigma \in (0, (\alpha \mu + 2\alpha + \beta - 3 - \gamma)/\alpha) \) and \( 0 < s < t \leq T \)
\[ \text{We have} \]
\[ \|Q_3 f(t) - Q_3 f(s)\|_{(X, \mathcal{D}(A))_{\gamma, p}} \leq \sum_{k=1}^3 J_k(s, t), \tag{4.24} \]
\[ \text{Since } \sigma \in (0, (\alpha \mu + 2\alpha + \beta - 3 - \gamma)/\alpha), \text{ the case } s = 0 \text{ follows from (4.23) once we observe that} \]
\[ (\alpha \mu + \alpha + \beta - 2 - \gamma)/\alpha - \sigma > (\alpha \mu + \alpha + \beta - 2 - \gamma - (\alpha \mu + 2\alpha + \beta - 3 - \gamma))/\alpha = (1 - \alpha)/\alpha \geq 0. \]
where

\[ J_1(s, t) := \| \int_0^s [A e^{(t-\xi)A} - A e^{(s-\xi)A}] [f(\xi) - f(s)] \, d\xi \|_{(X, D(A))_{\gamma, p}}, \]

\[ J_2(s, t) := \| \int_0^s A e^{(t-\xi)A} [f(s) - f(t)] \, d\xi \|_{(X, D(A))_{\gamma, p}}, \]

\[ J_3(s, t) := \| \int_s^t A e^{(t-\xi)A} [f(\xi) - f(t)] \, d\xi \|_{(X, D(A))_{\gamma, p}}. \]

We examine first \( J_1(s, t) \). To this purpose, from inequality (3.3) with \( n = 1 \) we deduce

\[ J_1(s, t) \leq \sigma^{-1} c_1(T) \| f \|_{\mu, T; X} \left[ \int_0^s (s - \xi)^{(\alpha\mu + \alpha + \beta - 3 - \gamma - \alpha\sigma)/\alpha} \, d\xi \right] (t - s)^\sigma \]

\[ \leq \frac{\sigma^{-1} \alpha c_1(T)}{\alpha\mu + 2\alpha + \beta - 3 - \gamma - \alpha\sigma} \| f \|_{\mu, T; X} s^{(\alpha\mu + 2\alpha + \beta - 3 - \gamma - \alpha\sigma)/\alpha} (t - s)^\sigma. \]  

(4.25)

Let us turn to \( J_2(s, t) \). Since \( \alpha + \beta - 2 - \gamma < 0 \), inequality (3.2) with \( n = 1 \) yields to

\[ J_2(s, t) \leq c_1(T) \| f \|_{\mu, T; X} \left[ \int_0^s (t - \xi)^{(\beta - 2 - \gamma)/\alpha} \, d\xi \right] (t - s)^\mu \]

\[ \leq \frac{\alpha c_1(T)}{2 + \gamma - \alpha - \beta} \| f \|_{\mu, T; X} [(t - s)^{(\alpha + \beta - 2 - \gamma)/\alpha} - t^{(\alpha + \beta - 2 - \gamma)/\alpha}] (t - s)^\mu \]

\[ \leq \frac{\alpha c_1(T)}{2 + \gamma - \alpha - \beta} \| f \|_{\mu, T; X} (t - s)^{(\alpha + \alpha + \beta - 2 - \gamma)/\alpha} \]

\[ \leq \frac{\alpha c_1(T)}{2 + \gamma - \alpha - \beta} \| f \|_{\mu, T; X} t^{(\alpha + \alpha + \beta - 2 - \gamma - \alpha\sigma)/\alpha} (t - s)^\sigma. \]  

(4.26)

Finally, concerning \( J_3(s, t) \), still from (3.2) with \( n = 1 \) we get

\[ J_3(s, t) \leq c_1(T) \| f \|_{\mu, T; X} \int_s^t (t - \xi)^{(\alpha\mu + \beta - 2 - \gamma)/\alpha} \, d\xi \]

\[ \leq \frac{\alpha c_1(T)}{\alpha\mu + \alpha + \beta - 2 - \gamma} \| f \|_{\mu, T; X} (t - s)^{(\alpha\mu + \alpha + \beta - 2 - \gamma)/\alpha} \]

\[ \leq \frac{\alpha c_1(T)}{\alpha\mu + \alpha + \beta - 2 - \gamma} \| f \|_{\mu, T; X} t^{(\alpha\mu + \alpha + \beta - 2 - \gamma - \alpha\sigma)/\alpha} (t - s)^\sigma. \]  

(4.27)

As a consequence, replacing (4.25)–(4.27) in (4.24), we obtain

\[ \| \| Q_3 f(t) - [Q_3 f](s) \|_{(X, D(A))_{\gamma, p}} \leq T^{(\alpha\mu + 2\alpha + \beta - 3 - \gamma - \alpha\sigma)/\alpha} c_7(T) \| f \|_{\mu, T; X} (t - s)^\sigma, \]  

(4.28)

where \( c_7(T) = \alpha c_1(T) c_8(T) \), \( c_8(T) \) being defined by

\[ c_8(T) = \left[ \frac{\sigma^{-1}}{\alpha\mu + 2\alpha + \beta - 3 - \gamma - \alpha\sigma} + \frac{\alpha\mu T^{(1-\alpha)/\alpha}}{(2 + \gamma - \alpha - \beta)(\alpha\mu + \alpha + \beta - 2 - \gamma)} \right]. \]

Summing up (4.28) and (4.28) we easily derive (4.22) with

\[ c_6(T) = \alpha c_1(T) \left[ (\alpha\mu + \alpha + \beta - 2 - \gamma)^{-1} T^{(1-\alpha + \alpha\sigma)/\alpha} + c_8(T) \right]. \]

(4.29)

This completes the proof.
5 Maximal regularity for degenerate equations

In this section we apply the preliminary lemmata of Section 4 for proving time and space regularity of solutions to the following degenerate first-order initial value problem

$$\begin{cases}
D_t(Mv(t)) = Lv(t) + f(t), & t \in (0, T], \\
Mv(0) = u_0,
\end{cases}$$

(5.1)
in a Banach space $X$. Here $M$ and $L$ are two closed linear operators from $X$ to itself having domains, respectively, $\mathcal{D}(M)$ and $\mathcal{D}(L)$ with $\mathcal{D}(L) \subset \mathcal{D}(M)$, $f \in C([0, T]; X)$ is a given function and $u_0 \in X$ is a given initial value.

We want to stress that, since the Cauchy problem (5.1) coincides with that of type (D-E.1) in [1, Section 3.3], the range of the possible applications of our results turns out to be very large. To this purpose, we refer the interested reader to [1], and to the references therein, for a list of boundary value problems related to degenerate parabolic equations which can be reduced to (5.1) via an abstract reformulation.

According to [1], we recall that the $M$-modified resolvent set of $L$ is the set $\rho_M(L) = \{ \lambda \in \mathbb{C} : \lambda M - L \text{ has "bounded inverse" } M(\lambda M - L)^{-1} \text{ on } X \}$. It is easy to prove that $\rho_M(L) \subset \rho(LM^{-1})$ and that $M(\lambda M - L)^{-1} = (\lambda - LM^{-1})^{-1} = \lambda \in \rho_M(L)$ (cf. [1, Theorem 1.14]). With the notion of $M$-modified resolvent set of $L$ at hand, we assume:

(H2) $\rho_M(L)$ contains the region $\Sigma = \{ \lambda \in \mathbb{C} : \Re \lambda \geq -c(|\Re \lambda| + 1)^\alpha \}$ and, for every $\lambda \in \Sigma$, the following estimate holds

$$\|M(\lambda M - L)^{-1}\|_{\mathcal{L}(X)} \leq C(|\lambda| + 1)^{-\beta},$$

for some exponents $0 < \beta < \alpha \leq 1$ and constants $c, C > 0$.

Of course, assumption (H2) implies that the operator $A = LM^{-1}$ with domain $\mathcal{D}(A) = M(\mathcal{D}(L))$ satisfies assumption (H1) and hence that it generates a semigroup $\{e^{tA}\}_{t \geq 0}$ defined by (2.1) and satisfying (2.2).

Notice that, due to the identity $L(\lambda M - L)^{-1} = \lambda M(\lambda M - L)^{-1} - I$, (H2) reads equivalently to

$$\|L(\lambda M - L)^{-1}\|_{\mathcal{L}(X)} \leq C|\lambda|(|\lambda| + 1)^{-\beta} + 1 \leq (C + 1)(|\lambda| + 1)^{1-\beta}. \tag{5.2}$$

However, until now, under assumption (5.2) only results of time regularity have been established. See, for instance, [2, Theorem 9] and [3, Theorem 7.2]. A result of space regularity has been obtained in [2], but with a stronger hypothesis of abstract potential type on the operator $T = ML^{-1} = A^{-1}$, precisely

$$\|L(\lambda M - L)^{-1}\|_{\mathcal{L}(X)} = \|L(\lambda I - T)^{-1}\|_{\mathcal{L}(X)} \leq C, \quad \lambda > 0. \tag{5.3}$$

In this case $T$, the part of $T$ in the closure $\overline{R(T)}$ of its range, has a densely defined inverse $T^{-1}$ (unbounded, in general) which generates an analytic semigroup in $\overline{R(T)}$. Then, denoted by $P$ the projection operator onto the null space $N(T)$ of $T$ and provided that some suitable assumptions are satisfied on $(I - P)f$ and $(I - P)Lu_0$, $v_0 = Mu_0$, in [2, Theorem 5] it is shown that $D_tMv$ belongs to $B([0, T], X^{\theta, \infty}_{T-1})$, $\theta \in (0, 1)$. This
is done by means of customary techniques of analytic semigroup theory. In particular, since assumption (5.3) implies the identity $X^{t,\infty}_T = (R(T), D(T^{-1}))_{\theta, \infty}$, the quoted result extends [7, Theorem 5.5] to degenerate equations.

The main problem in [2] lies in the characterization of projection $P$, which is crucial when space regularity is investigated. From this point of view, our aim is twofold. At first, to replace (5.3) with the more general assumption (H2) removing the analyticity of the semigroup $e^{tA}$. Then, to show both time and space regularity for $D_t M v$ without invoking $P$.

We begin proving two theorems concerning the regularity of the strict solution to (5.1). Recall that, according to [2] page 53, by a strict solution $v$ to (5.1) we mean a function $v \in C((0,T], \mathcal{D}(L))$ such that $M v \in C^1((0, T]; X)$ and (5.1) holds, where $M v(0) = u_0$ is understood in the sense that $\lim_{t \to 0} \|M L^{-1}(M v(t) - u_0)\|_X = 0$.

**Theorem 5.1.** Let assumption (H2) be fulfilled with $2\alpha + \beta > 2$ and let $u_0 \in M(\mathcal{D}(L))$ and $f \in C^\mu([0,T]; X)$, $\mu \in ((2 - \alpha - \beta)/\alpha, 1)$. Then, for every $\gamma \in (0, 2\alpha + \beta - 2)$, $\sigma \in (0, (2\alpha + \beta - 2 - \gamma)/\alpha)$ and $p \in [1, \infty]$, problem (5.1) has a unique strict solution $v$ such that

$$M v \in C^1((0, T]; X) \cap C((0, T]; X) \cap C^\sigma((0, T]; (X, \mathcal{D}(A))_{\gamma,p}).$$

Moreover, the following estimate holds true:

$$\|M v\|_{\sigma, T; (X, \mathcal{D}(A))_{\gamma,p}} \leq c_4(T)\|u_0\|_{\mathcal{D}(A)} + T^{\mu}c_2(T)\|f\|_{0, T; X},$$

where $\nu = (2\alpha + \beta - 2 - \gamma - \sigma)/\alpha$. Here $c_2(T)$ and $c_4(T)$ are the positive nondecreasing functions of $T$ defined, respectively, in (4.3) and (4.13).

**Proof.** First, when $f \in C^\mu([0,T]; X)$, $\mu \in ((2 - \alpha - \beta)/\alpha, 1)$, and $u_0 \in M(\mathcal{D}(L)) = \mathcal{D}(A)$, [1, Theorem 3.9] ensures that problem (5.1) admits a unique strict solution $v$ such that $M v \in C^1((0,T]; X) \cap C([0,T]; X)$. In particular, the following representation holds:

$$(M v)(t) = e^{tA}u_0 + [Q_1 f](t), \quad t \in [0, T],$$

$Q_1$ being defined in (4.1). This is a consequence of [1, Theorem 3.7] and the Remark to it, changing the unknown function to $w = M v$ and rewriting (5.1) into the equivalent form

$$D_t w(t) = A w(t) + f(t), \quad t \in (0,T], \quad w(0) = u_0.$$  

Further, for every $\gamma \in (0, 2\alpha + \beta - 2)$ and $\sigma \in (0, (2\alpha + \beta - 2 - \gamma)/\alpha)$, Lemmas 4.1 and 4.3 imply that $Q_1 f$ and $e^{tA}u_0$ belong to $C^\sigma((0, T]; (X, \mathcal{D}(A))_{\gamma,p})$, $p \in [1, \infty]$, and the same assert is true for $M v$ by virtue of (5.6). Finally, estimate (5.5) follows from (4.2), (4.9) and (5.6). \hfill \Box

**Remark 5.2.** We stress that, even if in Theorem 5.1 $f$ is assumed $\mu$-Hölder continuous in time, during the proof we have used Lemma 4.1 which requires only the mere continuity of $f$. This is, for, $\mu$ being in $((2 - \alpha - \beta)/\alpha, 1)$, it is not guaranteed that $f \in C^\sigma([0, T]; X)$, $\sigma \in (0, (\alpha + \beta - 1 - \gamma)/\alpha)$, $\gamma \in (0, \alpha + \beta - 1)$, in order to apply Lemma 4.2. Indeed, provided $\alpha + \beta > 3/2$ and $\gamma \in (0, 2(\alpha + \beta) - 3) \subset (0, \alpha + \beta - 1)$, it may happen that $(2 - \alpha - \beta)/\alpha < \mu < \sigma < (\alpha + \beta - 1 - \gamma)/\alpha$, so that $f \notin C^\sigma([0, T]; X)$. Since $\alpha + \beta > 3/2$
implies $2\alpha + \beta > 2$, such a case may effectively take place if $\alpha$ and $\beta$ are large enough. Situation is not better if we try to apply Lemma 4.2 restricting Theorem 5.4. Moreover, the following estimate holds true:

$$
\|Mv\|_{\sigma,T;\langle X,D(A)\rangle_{\gamma,p}} \leq c_3(T)\|u_0\|_{D(A)} + T^\nu c_3(T) \max\{1,T^{\mu-\sigma}\}\|f\|_{\mu,T;X},
$$

(5.7)

where $\nu = (\alpha + \beta - 1 - \gamma - \alpha\sigma) / \alpha$. Here $c_3(T)$ is the positive nondecreasing function of $T$ defined in (4.3).

**Proof.** As before, since $\alpha + \beta > 3/2$ implies $2\alpha + \beta > 2$, the belonging of $Mv$ to $C^1([0,T];X) \cap C([0,T];X)$ follows from [1, Theorem 3.9]. Now, our assumptions on $u_0$, $\gamma$ and $\sigma$ enable us to use Lemma 4.3 to ensure that $e^{A}u_0$ belong to $C^\sigma([0,T];X,\mathcal{D}(A))_{\gamma,p}$, $p \in [1,\infty]$. In addition, the assumption $\gamma \geq 2(\alpha + \beta) - 3$ imply the following chain of inequalities

$$
0 < \sigma < (\alpha + \beta - 1 - \gamma) / \alpha \leq (2 - \alpha - \beta) / \alpha < \mu < 1.
$$

As a consequence, we have $f \in C^\sigma([0,T];X)$, $\sigma \in (0,(\alpha + \beta - 1 - \gamma) / \alpha)$, and we are in position to apply Lemma 4.2. Thus, $Q_1f$ is in $C^\sigma([0,T];X,\mathcal{D}(A))_{\gamma,p}$, $p \in [1,\infty]$, and satisfies (4.6), with $g$ being replaced by $f$. Hence, from (5.6) we deduce that $Mv \in C^\sigma([0,T];X,\mathcal{D}(A))_{\gamma,p}$, $p \in [1,\infty]$. Finally, estimate (5.7) follows from (4.6), (4.9), (5.6) and the inequality $\|f\|_{\sigma,T;X} \leq \max\{1,T^{\nu-\sigma}\}\|f\|_{\mu,T;X}$. \( \square \)

We now come to our main theorem, which provides regularity in both time and space for the derivative $D_t Mv$. The following statement improves [1, Theorem 3.26] and the results in [2] and [3] mentioned before.

**Theorem 5.4.** Let assumption (H2) be fulfilled with $3\alpha + \beta > 3$ and let $u_0 \in M(D(L))$ and $f \in C^\mu([0,T];X)$, $\mu \in ((3 - 2\alpha - \beta) / \alpha, 1)$. Further, let assume that

$$
L u_0 + f(0) =: g_0 \in M(D(L)), \quad u_0 = M v_0, \quad v_0 \in D(L).
$$

(5.8)

Then, for every $\gamma \in (0,\sigma(\mu + 2\alpha + \beta - 3)), \sigma \in (0,(\sigma(\mu + 2\alpha + \beta - 3) - \gamma) / \alpha)$ and $p \in [1,\infty]$, problem (5.1) has a unique strict solution $v$ such that

$$
D_t Mv \in C^\sigma([0,T];X,\mathcal{D}(A))_{\gamma,p}.
$$

Moreover, the following estimate holds true:

$$
\|D_t Mv\|_{\sigma,T;\langle X,D(A)\rangle_{\gamma,p}} \leq c_4(T)\|g_0\|_{D(A)} + T^\nu C(T)\|f\|_{\mu,T;X},
$$

(5.9)

where $\nu = (\sigma(\mu + 2\alpha + \beta - 3) - \gamma - \sigma\alpha) / \alpha$ and $C(T) = T(1 - \alpha) / \alpha c_5(T) + c_6(T)$. Here $c_5(T)$ and $c_6(T)$ are the positive nondecreasing functions of $T$ defined, respectively, in (4.20) and (4.29).
Proof. First, since \(3\alpha + \beta > 3\) implies \(2\alpha + \beta > 2\) and \(f \in C^\mu([0,T];X)\), where \(\mu \in ((3 - 2\alpha - \beta)/\alpha, 1) \subset ((2 - \alpha - \beta)/\alpha, 1)\), Theorem 5.1 applies and (5.6) holds. Hence, as shown in [1, Remark page 55], differentiating (5.6) with respect to \(t\) and using (5.8), we deduce

\[
D_t(M\nu(t)) = e^{tA}g_0 + [Q_2f](t) + [Q_3f](t), \quad t \in [0,T],
\]

the \(Q_j\)'s, \(j = 2, 3\), being defined, respectively, in (4.14) and (4.21). In particular, \(M\nu \in C^1([0,T];X)\) and the equation in (5.1) makes sense even at \(t = 0\). Now, notice that

\[
\gamma \in (0, \alpha\mu + 2\alpha + \beta - 3) \subset (0, \alpha\mu + \alpha + \beta - 2) \subset (0, \alpha + \beta - 1),
\]

\[
\sigma \in \left(0, \frac{\alpha\mu + 2\alpha + \beta - 3 - \gamma}{\alpha}\right) \subset \left(0, \frac{\alpha\mu + \alpha + \beta - 2 - \gamma}{\alpha}\right) \subset \left(0, \frac{\alpha + \beta - 1 - \gamma}{\alpha}\right),
\]

so that all the assumptions of Lemmas 4.3–4.5 are satisfied. Therefore, for every \(\gamma \in (0, \alpha\mu + 2\alpha + \beta - 3)\) and \(\sigma \in (0, (\alpha\mu + 2\alpha + \beta - 3 - \gamma)/\alpha)\), \(e^{tA}g_0\) and \(Q_jf\), \(j = 2, 3\), belong to \(C^\sigma([0,T];(X, D(A))_{\gamma,p})\), \(p \in [1, \infty]\). Of course, due to (5.10), the same belonging holds for the derivative \(D_tM\nu\). Finally, estimates (4.9), (4.14) and (4.22) yields to (5.9), and the proof is complete.

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