Abstract

We obtain sharp estimates for the localized distribution function of $\mathcal{M}\phi$, when $\phi$ belongs to $L^{p,\infty}$ where $\mathcal{M}$ is the dyadic maximal operator. We obtain these estimates given the $L^1$ and $L^q$ norm, $q < p$ and certain weak $L^p$-conditions.

Keywords: Dyadic, Maximal

1. Introduction

The dyadic maximal operator on $\mathbb{R}^n$ is a useful tool in analysis and is defined by:

$$\mathcal{M}_d\phi(x) = \sup \left\{ \frac{1}{|Q|} \int_Q |\phi(u)| \, du : x \in Q, \, Q \subseteq \mathbb{R}^n \, \text{dyadic cube} \right\} \quad (1.1)$$

for every $\phi \in L^1_{loc}(\mathbb{R}^n)$ where $| \cdot |$ is the Lebesgue measure on $\mathbb{R}^n$ and the dyadic cubes are those formed by the grids $2^{-N}\mathbb{Z}^n$ for $N = 1, 2, \ldots$.

As it is well known it satisfies the following weak type $(1,1)$ inequality

$$|\{x \in \mathbb{R}^n : \mathcal{M}_d\phi(x) \geq \lambda\}| \leq \frac{1}{\lambda} \int_{\{\mathcal{M}_d\phi \geq \lambda\}} |\phi(u)| \, du \quad (1.2)$$
for every $\phi \in L^1(\mathbb{R}^n)$ and every $\lambda > 0$ from which it is easy to get the following $L^p$ inequality:

$$\|\mathcal{M}_d \phi\|_p \leq \frac{p}{p-1}\|\phi\|_p. \quad (1.3)$$

For every $p > 1$ and $\phi \in L^p(\mathbb{R}^n)$ it is easy to see that the weak type inequality $(1.2)$ is best possible and it is proved in [9] that $(1.3)$ is also best possible (for general martingales see [2] and [3]).

In studying the dyadic maximal operator it would be convenient to work with functions supported in the unit cube $[0,1]^n$ and more generally defined on a non-atomic probability measure space $(X, \mu)$ where the dyadic sets are given in a family $\mathcal{T}$ of measurable subsets of $X$ that has a tree-like structure similar to the one in the dyadic case. Then we replace $\mathcal{M}_d$ by

$$\mathcal{M}_T \phi(x) = \sup \left\{ \frac{1}{\mu(I)} \int_I |\phi|d\mu : x \in I \subseteq X, \ I \in \mathcal{T} \right\} \quad (1.4)$$

and $(1.2)$ and $(1.3)$ remain true and sharp is this setting.

Actually, in this general setting $(1.3)$ has been improved even more by inserting the $L^1$-norm of $\phi$ as a variable giving the so called Bellman functions of the dyadic maximal operator. In fact in [5] the following function of variables $f,F$ has been explicitly computed

$$B(f,F) = \sup \left\{ \int_X (\mathcal{M}_T \phi)^pd\mu : \phi \geq 0, \ \int_X \phi d\mu = f, \ \int_X \phi^p d\mu = F \right\} \quad (1.5)$$

where $0 < f^p \leq F$.

The related Bellman functions for the case $p < 1$ have been also computed in [6].

It is interesting now to search what happens in case we replace the $L^p$-norm with the quasi norm $\| \cdot \|_{p,\infty}$ defined in $L^{p,\infty}$, where

$$\|\phi\|_{p,\infty} = \sup \{ \lambda \mu(\{\phi \geq \lambda\})^{1/p} : \lambda > 0 \} \quad (1.6)$$
for every $\phi$ such that this supremum is finite.

It is known that $L^{p,\infty} \supseteq L^p$ and $\mathcal{M}$ can be defined on $L^{p,\infty}$ with values on $L^{p,\infty}$. As a matter of fact it is not difficult to see that $\mathcal{M}_T$ satisfies the following

$$
\|\mathcal{M}_T \phi\|_{p,\infty} \leq \frac{p}{p-1} \|\phi\|_{p,\infty}
$$

(1.7)

for every $\phi \in L^{p,\infty}$.

In [8] it is proved that (1.7) is best possible.

Actually, a stronger fact is proved there, namely that

$$
\sup \left\{ \|\mathcal{M}_T \phi\|_{p,\infty} : \phi \geq 0, \int_X \phi d\mu = f, \|\phi\|_{p,\infty} = F \right\} = \frac{p}{p-1} F
$$

(1.8)

for every $(f,F)$ such that $0 < f \leq \frac{p}{p-1} F$. That is (1.7) is sharp allowing every value for the $L^1$-norm of $\phi$.

In the present paper we compute

$$
\sup \left\{ \mu((\{\mathcal{M}_T \phi \geq \lambda\}) : \phi \geq 0, \int_X \phi d\mu = f, \int_X \phi^q d\mu = A, \|\phi\|_{p,\infty} = F \right\}.
$$

(1.9)

for a fixed $q$ such that $1 < q < p$, and for all allowable values of $(f,A,F)$.

Actually doing this we improve (1.2) even more by inserting as variables the $L^q$-norm and the $L^{p,\infty}$-quasi norm of $\phi$. From this we have as a consequence that

$$
\sup \left\{ \|\mathcal{M}_T \phi\|_{p,\infty} : \phi \geq 0, \int_X \phi d\mu = f, \int_X \phi^q d\mu = A, \|\phi\|_{p,\infty} = F \right\}
$$

$$
= \frac{p}{p-1} F,
$$

(1.10)

that is (1.7) is best possible allowing every possible value of the $L^1$ and $L^q$-norm.

At last we mention that all the above calculations are independent of the measure space and the associated tree. We begin now with:
2. Preliminaries

Let \((X, \mu)\) be a non-atomic probability space.

The following holds:

**Lemma 2.1** Let \(\phi : X \to \mathbb{R}^+\) be measurable and \(I \subseteq X\) be measurable with \(\mu(I) > 0\). Suppose that \(\frac{1}{\mu(I)} \int_I \phi d\mu = s\). Then for every \(t\) such that \(0 < t \leq \mu(I)\) there exists a measurable set \(E_t \subseteq I\) with \(\mu(E_t) = t\) and \(\frac{1}{\mu(E_t)} \int_{E_t} \phi d\mu = s\).

**Proof.** Consider the measure space \((I, \mu/I)\) and let \(\psi : I \to \mathbb{R}^+\) be the restriction of \(\phi\) on \(I\) that is \(\psi = \phi/I\). Then if \(\psi^* : [0, \mu(I)] \to \mathbb{R}^+\) is the decreasing rearrangement of \(\psi\), we have that

\[
\frac{1}{t} \int_0^t \psi^*(u) du \geq \frac{1}{\mu(I)} \int_0^{\mu(I)} \psi^*(u) du = s \geq \frac{1}{t} \int_{\mu(I)-t}^{\mu(I)} \psi^*(u) du. \tag{2.1}
\]

Since \(\psi^*\) is decreasing we get the inequalities in (2.1), while the equality is obvious since

\[
\int_0^{\mu(I)} \psi^*(u) du = \int_I \phi d\mu.
\]

From (2.1) it is easily seen that there exists \(r \geq 0\) such that \(t + r \leq \mu(I)\) with

\[
\frac{1}{t} \int_r^{t+r} \psi^*(u) du = s. \tag{2.2}
\]

It is also easily seen that there exists \(E_t\) measurable subset of \(I\) such that

\[
\mu(E_t) = t \quad \text{and} \quad \int_{E_t} \phi d\mu = \int_r^{t+r} \psi^*(u) du \tag{2.3}
\]

since \((X, \mu)\) is non-atomic.

From (2.2) and (2.3) we get the conclusion of the lemma. \(\square\)

We now call two measurable subsets of \(X\) almost disjoint if \(\mu(A \cap B) = 0\).

We give now the following
Definition 2.1 A set $T$ of measurable subsets of $X$ will be called a tree if the following conditions are satisfied.

(i) $X \in T$ and for every $I \in T$ we have that $\mu(I) > 0$.

(ii) For every $I \in T$ there corresponds a finite or countable subset $C(I) \subseteq T$ containing at least two elements such that:

(a) the elements of $C(I)$ are pairwise almost disjoint subsets of $I$.

(b) $I = \bigcup C(I)$.

(iii) $T = \bigcup_{m \geq 0} T_m$ where $T_0 = \{X\}$ and

$$ T_{m+1} = \bigcup_{I \in T_m} C(I). $$

(iv) $\lim_{m \to +\infty} \sup_{I \in T_m} \mu(I) = 0$. □

From [5] we have the following

Lemma 2.2 For every $I \in T$ and every $\alpha$ such that $0 < \alpha < 1$ there exists subfamily $F(I) \subseteq Y$ consisting of pairwise almost disjoint subsets of $I$ such that

$$ \mu \left( \bigcup_{J \in F(I)} J \right) = \sum_{J \in F(I)} \mu(J) = (1 - \alpha) \mu(I). \quad \square $$

Let now $(X, \mu)$ be a non-atomic probability measure space and $T$ a tree as in Definition 1.1. We define the associated maximal operator to the tree $T$ as follows: For every $\phi \in L^1(X, \mu)$ and $x \in X$, then

$$ M\phi(x) = M_T\phi(x) = \sup \left\{ \frac{1}{\mu(I)} \int_I |\phi| d\mu : x \in I \in T \right\}. $$
3. Domain of the extremal problem

Our aim is to find for every $\lambda > 0$ the following

$$B(f, A, \lambda) = \sup \left\{ \mu(\{M \phi \geq \lambda\}) : \phi \geq 0, \int_X \phi d\mu = f, \int_X \phi^q d\mu = A, \|\phi\|_{p, \infty} = \frac{p - 1}{p} \right\}. \quad (3.1)$$

For this reason we define

$$B_1(f, A, \lambda) = \sup \left\{ \mu(\{M \phi \geq \lambda\}) : \phi \geq 0, \int_X \phi^q d\mu = A, \|\phi\|_{p, \infty} \leq \frac{p - 1}{p} \right\}. \quad (3.2)$$

In order to find (3.1) and (3.2) it is necessary to find the allowable values of $f$ and $A$. That is the values for which there exists $\phi : (X, \mu) \to \mathbb{R}^+$ such that

$$\int_X \phi d\mu = f, \int_X \phi^q d\mu = A \quad \text{and} \quad \|\phi\|_{p, \infty} = \frac{p - 1}{p} \quad \text{or} \quad \|\phi\|_{p, \infty} \leq \frac{p - 1}{p}. \quad (3.3)$$

For the beginning let $f, A$ and $\phi$ such that

$$\int_X \phi d\mu = f, \int_X \phi^q d\mu = A, \quad \|\phi\|_{p, \infty} \leq \frac{p - 1}{p}. \quad (3.3)$$

Consider the decreasing rearrangement of $\phi$, $g = \phi^* : [0, 1] \to \mathbb{R}^+$. Then for every $\lambda > 0|\{g \geq \lambda\}| = \mu(\{\phi \geq \lambda\})$ where $|\cdot|$ the Lebesgue measure on $[0, 1]$. As a consequence

$$\int_0^1 g = f, \quad \int_0^1 g^q = A \quad (3.3)$$

and

$$\sup \{\lambda|\{g \geq \lambda\}|^{1/p} : \lambda > 0\} \leq \frac{p - 1}{p} \quad (3.4)$$
(3.4) now gives for every $\lambda > 0$ that

$$|\{g \geq \lambda\}| \leq \left[\frac{p - 1/p}{\lambda}\right]^p.$$  \hspace{1cm} (3.5)

But if $\psi : (0, 1] \to \mathbb{R}^+$ defined by $\psi(t) = (1 - p)t^{-1/p}$, then (3.5) means that

$$g(t) \leq \psi(t) \quad \text{for every} \quad t \in (0, 1],$$  \hspace{1cm} (3.6)

since $g$ is decreasing. Now from (3.6) we easily get $0 < f \leq 1$. Fix such a $f$. Obviously from Holder’s inequality $f^q \leq A$. We search now for the minimum and maximum values of $A$ for which there exist $g : [0, 1] \to \mathbb{R}^+$ decreasing such that (3.3) and (3.5) hold.

We have the following simple

**Lemma 3.1** If $g$ satisfies (3.3) and (3.6) then $A \leq \Gamma f^{p-1/q-p-1}$ where $\Gamma = \left(\frac{p-1/p}{p}\right)^q \frac{p}{p-q}$.

**Proof.** It is easy to see that the function which gives the maximum value of $A$ for which there exists $g$ such that (3.3) and (3.6) hold (for a fixed $f$) is that with the largest possible values.

As a matter of fact if $g$ does not have the largest possible values we can arrange things in such a way to produce a function $g_1$ with the same integral and bigger $L^q$-norm. This is done by increasing $g$ to $g_2$ in suitable sets such that $g_2 \leq \psi$ and decreasing $g$ analogously again in suitable sets.

Then since $1 < q$ we easily get that the $L^q$-norm of $g_2$ is bigger than that of $g_1$.

So we set $g_1 : (0, 1] \to \mathbb{R}^+$ such that

$$g_1(t) = \psi(t) = \frac{p - 1}{p} t^{-1/p}, \quad t \in (0, c] \quad \text{for} \quad c < 1$$

suitable such that $\int_0^c g_1(t)dt = f$ which is equivalent to

$$\int_0^c \psi(t)dt = f \Leftrightarrow c^{1-\frac{1}{p}} = f \Leftrightarrow c = f^{p/p-1}.$$  \hspace{1cm} (3.7)
Then
\[
\int_0^1 g_1^q(t)dt = \left( \frac{p-1}{p} \right)^q \int_0^c t^{-q/p}dt
\]
\[
= \left( \frac{p-1}{p} \right)^q \frac{1}{1-\frac{q}{p}} c^{1-\frac{q}{p}} = \Gamma f^{p-q/p-1}.
\]

After the comments and the calculations we get the proof of the Lemma. □

In a similar way for a fixed $0 < f \leq 1$ we need to find the smallest value of $A$ for which there exist $g$ such that (3.3) and (3.6) hold.

This is done in the following steps

**Lemma 3.2** If $0 < f \leq 1$ and $g : [0, 1] \to \mathbb{R}^+$ such that $g$ is decreasing, $g \leq \psi$, $\int_0^1 g(t)dt = f$ and $\int_0^1 g^q(t)dt = A$ then $A \geq A_f$ where

\[
A_f = \begin{cases} 
  f^q, & \text{if } 0 < f \leq \frac{p-1}{p} \\
  \left( \frac{p-1}{p} \right)^q \frac{1}{p-q}\left\{ p - q[p(1 - f)]^{p-q/p-1} \right\}, & \text{if } \frac{p-1}{p} < f \leq 1.
\end{cases}
\]

**Proof.** Indeed since $f^q \leq A$ for every $f \leq 1$ we need only to check the case $\frac{p-1}{p} < f \leq 1$. (Notice that for $A_1 = f^q$ and $g$ such that $g(t) = f$, for every $t \in [0, 1]$ we have that $g \leq \psi$, $\int_0^1 g(t)dt = f$ and $\int_0^1 g^q(t)dt = A_1$ in case where $0 < f \leq \frac{p-1}{p}$). As before we need to find that $g : [0, 1] \to \mathbb{R}^+$ with the smallest values such that $\int_0^1 g(t)dt = f$ and $g \leq \psi$. Arguing as before, we consider the function $g_2 : [0, 1] \to \mathbb{R}^+$ defined by:

\[
g_2(t) = \frac{p-1}{p} c^{-1/p}, \quad t \in (0, c] \]
\[
= \frac{p-1}{p} t^{-1/p}, \quad t \in [c, 1]
\]
where $c$ is such that
\[ \int_0^1 g_2(t) dt = f. \] (3.8)

(3.8) now give $\frac{p-1}{p} c^{1-rac{1}{p}} + \frac{1}{c} \int_0^1 \psi(t) dt = f \Rightarrow \frac{p-1}{p} c^{1-rac{1}{p}} + \left(1 - c^{1-rac{1}{p}}\right) = f \Rightarrow c^{1-rac{1}{p}} = p(1-f) \Rightarrow c = \left[p(1-f)\right]^{p/p-1}. \] So we can easily see that
\[ \int_0^1 g_1^q(t) dt = \left(\frac{p-1}{p}\right)^q \frac{1}{p-q} \{p - q[p(1-f)]^{p-q/p-1}\} \] so the lemma is proved. □

So we proved that for every $f, A$ such that there exists a $g : [0,1] \to \mathbb{R}^+$ with $\int_0^1 g = f$, $\int_0^1 g^q = A$, $g \leq \psi$ we have that $f \leq 1$ and $A_f \leq A \leq \Gamma f^{p-q/p-1}$.

In fact we additionally proved that for every $f$ there exist functions $g_1, g_2 \leq \psi$ such that $\int_0^1 g_i = f$ and $\int_0^1 g_1^q(t) dt = \Gamma f^{p-q/p-1}$, $\int_0^1 g_2^q(t) dt = A_f$.

We use this to prove the following

**Lemma 3.3** If $0 < f \leq 1$ and $A_f \leq A \leq \Gamma f^{p-q/p-1}$ then there exists $g : [0,1] \to \mathbb{R}^+$ such that
\[ g \leq \psi, \quad \int_0^1 g(t) dt = f \quad \text{and} \quad \int_0^1 g^q(t) dt = A. \]

**Proof.** Let $0 < f \leq 1$ and $g_1, g_2$ as before. For every $\ell \in [0,1]$ we define $h_\ell := \ell g_1 + (1 - \ell) g_2$. Then $h_\ell \leq \psi$, $\int_0^1 h_\ell = f$ and $h_1 = g_1$, $h_0 = g_2$.

Then we consider the function $T : [0,1] \to \mathbb{R}^+$ defined by $T(\ell) = \int_0^1 h_\ell^q$. It is obvious that $T$ is continuous on $[0,1]$ and that $T(0) = A_f \leq A \leq T(1) = \Gamma f^{p-q/p-1}$.

As a consequence we have that there exists a $\ell \in [0,1]$ such that $T(\ell) = \int_0^1 h_\ell^q(t) dt = A$. By setting $g = h_\ell$ the lemma is proved. □
Remark 3.1 Suppose that $f, A$ are such that there exists $g : [0, 1] \to \mathbb{R}^+$ decreasing with $\int_0^1 g = f$, $\int_0^1 g^q = A$, $g \leq \psi$. Then since $(X, \mu)$ is non-atomic there exists $\phi : (X, \mu) \to \mathbb{R}^+$ such that $\phi^* = g$. Then obviously

$$\int_X \phi dt = f, \quad \int_X \phi^q dt = A, \quad \|\phi\|_{p, \infty} \leq \frac{p-1}{p}.$$ 

We collect all the above in the following

Corollary 3.1 For $f$ and $A$ positive constants the following are equivalent

(i) There exists $\phi : X \to \mathbb{R}^+$ measurable such that

$$\int_X \phi d\mu = f, \quad \int_X \phi^q d\mu = A, \quad \|\phi\|_{p, \infty} \leq \frac{p-1}{p}.$$ 

(ii) $0 < f \leq 1$ and $A_f \leq A \leq \Gamma f^{p/q - p - 1}$.

We say then that $(f, A) \in D$.

Actually using the above arguments it is easy to see that the following is true.

Corollary 3.2 For $f$ and $A$ positive constants with $A \neq f^q$ the following are equivalent

(i) There exists $\phi : (X, \mu) \to \mathbb{R}^+$ measurable such that

$$\int_X \phi d\mu = f, \quad \int_X \phi^q d\mu = A, \quad \|\phi\|_{p, \infty} = \frac{p-1}{p}.$$ 

(ii) $0 < f \leq 1$ and $A_f \leq A \leq \Gamma f^{p/q - p - 1}$. 


4. The extremal problem

Suppose now that \((f, A) \in D\) and \(\lambda > 0\).

We remind that

\[
B(f, A, \lambda) = \sup \left\{ \mu(\{M\phi \geq \lambda\}) : \phi \geq 0, \int_X \phi d\mu = f, \int_X \phi^q d\mu = A, \|\phi\|_{p,\infty} = \frac{p-1}{p} \right\}
\] (4.1)

and

\[
B_1(f, A, \lambda) = \sup \left\{ \mu(\{M\phi \geq \lambda\}) : \phi \geq 0, \int_X \phi d\mu = f, \int_X \phi^q d\mu = A, \|\phi\|_{p,\infty} \leq \frac{p-1}{p} \right\}
\] (4.2)

Our aim is to find the above functions.

First observe that 
\(B(f, A, \lambda) = B_1(f, A, \lambda) = 1\), for \(\lambda < f\) so we can suppose that \(\lambda \geq f\).

Obviously

\[
B_1(f, A, \lambda) \geq B(f, A, \lambda).
\] (4.3)

As we shall see later we have equality in (4.3). We work out (4.2). Let \(\phi\) be as in there and \(E = \{M\phi \geq \lambda\}\). Then \(E\) is the almost disjoint union of elements of \(T\), \(I_j, j = 1, 2, \ldots\). Indeed, we just need to consider those \(I \in T\) maximal under the condition \(\frac{1}{\mu(I)} \int_I \phi d\mu \geq \lambda\).

For every \(j\) we have that

\[\int_{I_j} \phi d\mu \geq \lambda \mu(I_j).\]

Summing (4.4) up to \(j\) we get

\[\int_E \phi d\mu \geq \lambda \mu(E).\] (4.4)
We again consider the decreasing rearrangement of \( \phi \), let \( \phi^* : [0, 1] \rightarrow \mathbb{R}^+ \).

In this point we need a fact which is true on every non-atomic finite measure space and can be seen in [1].

Namely that for every \( \delta \in [0, 1] \)

\[
\int_0^\delta \phi^*(t) dt = \sup \left\{ \int_\mathcal{K} \phi d\mu : \mathcal{K} \text{ measurable subset of } X \text{ such that } \mu(\mathcal{K}) = \delta \right\}
\]

where the supremum is actually attained.

From (4.5) we now get in view of the previous comment for \( a = \mu(E) \) that \( \int_0^a \phi^*(t) dt \geq \lambda a \), so if we define by

\[
T(f, A, \lambda) = \sup \left\{ \alpha \in (0, 1] : \exists g : [0, 1] \rightarrow \mathbb{R}^+ \text{ decreasing such that } \int_0^1 g = f, \int_0^1 g^q = A, g \leq \psi \text{ and } \int_0^\alpha g \geq \alpha \lambda \right\}
\]

we have that

\[
B_1(f, A, \lambda) \leq T(f, A, \lambda). \quad (4.5)
\]

In fact in relation (4.5) the converse inequality is also true. We state is as a

**Lemma 4.1** If \( a \in (0, 1] \) and \( g : [0, 1] \rightarrow \mathbb{R}^+ \) such that \( g \) is decreasing and

\[
\int_0^a g(t) dt \geq a\lambda, \int_0^1 g(t) dt = f, \int_0^1 g^q(t) dt = A, g \leq \psi,
\]

then there exists \( \phi : (X, \mu) \rightarrow \mathbb{R}^+ \) measurable with

\[
\int_X \phi d\mu = f, \int_X \phi^q d\mu = A, \|\phi\|_{p, \infty} \leq \frac{p - 1}{p}
\]

with the additional property:

\[
\mu(\{M\phi \geq \lambda\}) \geq a.
\]
Proof. Indeed from Lemma 2.2 setting \( I = X \) we guarantee the existence of a sequence \((I_j)\) of pairwise almost disjoint elements of \( T \) in such a way that
\[
\mu(\bigcup_j I_j) = \sum_j \mu(I_j) = a. \tag{4.6}
\]
We consider the measure space \(([0,a], | \cdot |)\) where \( | \cdot | \) is Lebesgue measure. Because of
\[
\int_0^a g(t)dt \geq a\lambda,
\]
applying Lemma 2.1 repeatedly we have as a consequence the existence of a partition \( S = \{A_j, j = 1,2,\ldots\} \) of \([0,a]\), which consists of Lebesgue measurable subsets of \([0,a]\) such that
\[
|A_j| = \mu(I_j) \text{ and } \int_{A_j} g(t)dt \geq \lambda|A_j|. \tag{4.7}
\]
For every \( j = 1,2,\ldots \) let now \( g_j = \left(\frac{g}{A_j}\right)^* \) defined on \([0,|A_j|]\). Since \((X,\mu)\) is non-atomic and \( \mu(I_j) = |A_j| \) we easily see that for every \( j \) there exists \( \phi_j : I_j \to \mathbb{R}^+ \) measurable such that \( \phi_j^* = g_j \). Additionally suppose that \( g' = \left(\frac{g}{(a,1]}\right)^* \) and set \( Y = X \setminus \bigcup I_j \). Since \( \mu(Y) = 1 - a \) for the same reasons we get a \( \phi' : Y \to \mathbb{R}^+ \) such that \( \phi'^* = g' \). Then since \( I_j \) are pairwise almost disjoint there exists a measurable function \( \phi : X \to \mathbb{R}^+ \) such that \( \phi|_{I_j} = \phi_j \) almost everywhere for every \( j \) and \( \phi|_Y = \phi' \). Then it is easy to see that
\[
\phi^* = g \leq \psi, \quad \int_{I_j} \phi d\mu = \int_{A_j} g d\mu \geq \lambda|A_j| = \lambda \mu(I_j)
\]
that is
\[
\frac{1}{\mu(I_j)} \int_{I_j} \phi d\mu \geq \lambda \text{ for every } j = 1,2,\ldots.
\]
So, \( \{\mathcal{M} \phi \geq \lambda\} \supseteq \bigcup I_j \). As a consequence we get \( \mu(\{\mathcal{M} \phi \geq \lambda\}) \geq a \) and the lemma is proved. \( \square \)

Using now Lemma 4.1 we see that
\[
B(f,A) = T(f,A) \lambda. \tag{4.8}
\]
In fact we have equality in (4.8) even if we replace the inequality:

\[ \int_0^a g(t) dt \geq a\lambda \]

given in the definition of \( T(f, A, \lambda) \) by equality, thus getting the function \( S(f, A, \lambda) \), we state it as

**Remark 4.1** \( B_1(f, A, \lambda) = S(f, A, \lambda) \) where

\[
S(f, A, \lambda) = \sup \left\{ a \in (0, 1] : \begin{array}{c}
\exists g : [0, 1] \to \mathbb{R}^+ \text{ decreasing such that} \\
\int_0^a g = f, \int_0^1 g^q = A, g \leq \psi \\
\text{and } \int_0^a g(t) dt = a\lambda.
\end{array} \right\}
\]

This is true because if \( g \) is as in the definition of \( T(f, A, \lambda) \), then of course \( a \int_0^a g(t) dt \geq a\lambda \). But then there exists \( \beta \geq a \) such that \( a \int_0^\beta g(t) dt = \beta\lambda \) since \( \theta(t) \) is a decreasing function of \( t \), where \( \theta \) is defined by \( \theta : (0, 1] \to \mathbb{R}^+ \) with \( \theta(t) = \frac{1}{t} \int_0^t g(u) du \) (\( g \) is decreasing). But then the remark follows by applying Lemma 4.1 with \( \beta \) in place of \( a \). □

We collect all the above as

**Corollary 4.1** For \((f, A) \in D_1\), \( B_1(f, A, \lambda) \) equals the supremum of all \( a \in (0, 1] \) for which there exists \( g : [0, 1] \to \mathbb{R}^+ \) decreasing and \( A_1, A_2 \geq 0 \) such that

\[
\int_0^a g = f_1, \int_0^a g^q = A_1, \int_a^1 g = f_2, \int_a^1 g^q = A_2
\]

and \( g \leq \psi \) where \( A_1 + A_2 = A, f_1 = \lambda a, f_2 = f - \lambda a \). □

**Remark 4.2** Notice that we can ignore the demand that \( g \) is decreasing in Corollary 4.1 since we can repeat the proof of Lemma 4.1 without this hypothesis. □
Using now the techniques of Section 3 we can prove the following generalizations of Corollary 3.1.

**Proposition 4.1** Let $\alpha \in (0, 1]$ $f_1, A_1$ positive numbers, were $f_1 \leq \alpha^{1-\frac{1}{p}}$.

Then the following are equivalent

(i) $\exists g : [0, \alpha] \rightarrow \mathbb{R}^+$ Lebesgue measurable such that $g \leq \psi$ on $[0, a]$ and

\[ \int_0^\alpha \phi = f_1, \int_0^\alpha \phi_1^q = A_1. \]

(ii) a) If $0 < f_1 \leq \frac{p-1}{p} \alpha^{1-\frac{1}{p}}$ then $\frac{f_1^q}{\alpha^{q-1}} \leq A_1 \leq \Gamma f_1^{p-q/p-1}$

b) If $\frac{p-1}{p} \alpha^{1-\frac{1}{p}} \leq f_1 \leq \alpha^{1-\frac{1}{p}}$ then

\[ \Delta f(\alpha) \leq A_1 \leq \Gamma f_1^{p-q/p-1} \]

where

\[ \Delta f(\alpha) = \left( \frac{p-1}{p} \right)^p \frac{1}{p-q} \left\{ p\alpha^{1-\frac{2}{p}} - q \left[ p\left( \alpha^{1-\frac{1}{p}} - f_1 \right) \right]^{p-q/p-1} \right\}. \]

\[ \square \]

**Proposition 4.2** For $a \in (0, 1]$ and $f_2, A_2$ such that $f_2 \leq 1 - \alpha^{1-\frac{1}{p}}$ the following are equivalent

(i) $\exists g : [a, 1] \rightarrow \mathbb{R}^+$ Lebesgue measurable such that

\[ g \leq \psi \text{ on } [\alpha, 1] \text{ and } \int_\alpha^1 g = f_2, \int_\alpha^1 g^q = A_2. \]

(ii) a) If $f_2 \leq (1 - \alpha)^{p-1} \frac{1}{p}$ then $\frac{f_2^q}{(1 - \alpha)^{q-1}} \leq A_2 \leq E_{f_2}(\alpha)$ where

\[ E_{f_2}(\alpha) = \Gamma \left[ \left( f_2 + \alpha^{1-\frac{1}{p}} \right)^{p-q/p-1} - \alpha^{1-\frac{2}{p}} \right]. \]
b) If \((1 - \alpha)\frac{p - 1}{p} \leq f_2 \leq 1 - \alpha^{1 - \frac{q}{p}}\) then

\[ \Gamma_{f_2}(\alpha) \leq A_2 \leq E_{f_2}(\alpha) \]

where

\[ \Gamma_{f_2}(\alpha) = \left( \frac{p - 1}{p} \right)^q c^{-q/p}(c - \alpha) + \Gamma(1 - c^{1 - \frac{q}{p}}) \]

where \(c\) satisfies

\[ \frac{1}{p} c^{1 - \frac{q}{p}} + \left(1 - \frac{1}{p}\right)ac^{-1/p} = 1 - f_2. \] \hspace{1cm} (4.9)

**Remark 4.3** Notice that since \((1 - \alpha)\frac{p - 1}{p} \leq f_2 \leq 1 - \alpha^{1 - \frac{1}{p}}\) then there exists unique \(c\) satisfying (4.9). \(\Box\)

In light now of Corollary 4.1 and Proposition 4.1 for a fixed \(\lambda > f\) we define the following functions \(T_\lambda, S_\lambda : [0, 1/\lambda^p] \to \mathbb{R}^+\) by

\[
T_\lambda(\alpha) = \begin{cases} 
\lambda^\alpha, & \text{for } \alpha \leq \left[ \frac{p - 1}{p}/\lambda \right]^p \\
\Delta_f(\alpha), & \text{for } \left[ \frac{p - 1}{p}/\lambda \right]^p < \alpha \leq \frac{1}{\lambda^p}.
\end{cases}
\]

where \(f_1 = \lambda\alpha\) and \(S_\lambda(\alpha) = \Gamma(\lambda\alpha)^{p-q/p-1}\).

In light of Proposition 4.2 and Corollary 4.1 we also define \(F_\lambda, G_\lambda : [0, f/\lambda] \to \mathbb{R}^+\) for \(\alpha\) such that \(f - \lambda\alpha \leq 1 - \alpha^{1 - \frac{q}{p}}\).

a) If \(0 < f \leq \frac{p - 1}{p}\)

\[ F_\lambda(\alpha) = \frac{(f - \lambda\alpha)^q}{(1 - \alpha)^{q-1}} \text{ and } G_\lambda(\alpha) = E_{f_2}(\alpha). \]

b) While if \(\frac{p - 1}{p} < f \leq 1\)

\[
F_\lambda(\alpha) = \begin{cases} 
\frac{(f - \lambda\alpha)^q}{(1 - \alpha)^{q-1}}, & \text{for } \frac{f - \lambda\alpha}{\lambda^{1-p}/p} \leq \alpha \leq \frac{f}{\lambda} \\
\Gamma_{f_2}(\alpha), & \text{for } \alpha \leq \frac{f - \lambda\alpha}{\lambda^{1-p}/p}
\end{cases}
\]

and \(G_\lambda(\alpha) = E_{f_2}(\alpha)\) where \(f_2 = f - \lambda\alpha\).
After giving the definitions of $T_\lambda, S_\lambda, F_\lambda, G_\lambda$ we can rewrite Corollary 4.1 as

**Corollary 4.2** For $(f, A) \in D$ and $\lambda > f$, $B_1(f, A, \lambda)$ equals to supremum of all $\alpha \in (0, 1]$ such that $\alpha \leq \min \{f/\lambda, 1/\lambda^p\}$ and $f - \lambda \alpha \leq 1 - \alpha^{1 - \frac{1}{p}}$ for which there exist $A_1, A_2 \geq 0$ with

$$
\begin{align*}
T_\lambda(\alpha) &\leq A_1 \leq S_\lambda(\alpha) \\
F_\lambda(\alpha) &\leq A_2 \leq G_\lambda(\alpha)
\end{align*}
$$
and $A = A_1 + A_2$.

**Remark 4.4**

i) After stating Corollary 4.2 it is easy to see, in view of Propositions 4.1 and 4.2, that the supremum in (4.2) is actually maximum.

ii) If $\alpha = B_1(f, A, \lambda)$ then obviously $\alpha \leq f/\lambda$, while $\alpha \leq 1/\lambda^p$. This is true because by i) there exists $g : [0, 1] \to \mathbb{R}^+$ such that $\int_0^\alpha g = \alpha \lambda$, $g \leq \psi$, $\int_0^1 g = f$, $\int_0^1 g^q = A$. But then, by the first two relations we get easily $\alpha \leq 1/\lambda^p$.

iii) Notice also that because of Propositions 4.1 and 4.2, $F_\lambda(\beta), G_\lambda(\beta), T_\lambda(\beta), S_\lambda(\beta)$ have geometric interpretations as $L^q$-norms of essentially unique functions on the respective intervals when $\beta \leq \min \left\{\frac{f}{\lambda}, \frac{1}{\lambda^p}\right\}$ and $f - \lambda \beta \leq 1 - \beta^{1 - \frac{1}{p}}$.

We state now the following

**Lemma 4.2** For $(f, A) \in D$ such that $A_f \not\leq A$ and $\alpha(\lambda) = B_1(f, A, \lambda)$, there exists $\lambda_1 \geq \left(\frac{1}{f}\right)^{1/p-1}$ such that $\alpha(\lambda) = \frac{1}{\lambda^p}$, for every $\lambda \geq \lambda_1$.

**Proof.** If $\lambda \geq \left(\frac{1}{f}\right)^{1/p-1}$ then $\frac{1}{\lambda^p} \leq \frac{1}{f}$.

We consider the equation

$$F_\lambda(1/\lambda^p) = A - \frac{\Gamma}{\lambda^{p-q}}.$$ 

We easily see that

$$\lim_{\lambda \to +\infty} F_\lambda(1/\lambda^p) = A_f \not\leq A = \lim_{\lambda \to +\infty} \left( A - \frac{\Gamma}{\lambda^{p-q}} \right).$$
For $\lambda = \lambda_0 = \left(\frac{1}{f}\right)^{1/p-1}$ we have that

$$F_{\lambda_0}(1/\lambda_0^p) = F_{\lambda_0}\left(\frac{f}{\lambda_0}\right) = 0 \geq A - \Gamma f^{p-q/p-1} = A - \frac{\Gamma}{\lambda_0^{p-q}}.$$  

So, there exists $\lambda \geq \lambda_0$ such that

$$F_{\lambda}(1/\lambda^p) = A - \frac{\Gamma}{\lambda^{p-q}}.$$  

Let

$$\lambda_1 = \inf\left\{\lambda \geq \lambda_0 : F_{\lambda}(1/\lambda^p) = A - \frac{\Gamma}{\lambda^{p-q}}\right\}$$

which is obviously a minimum. Then

$$F_{\lambda_1}\left(\frac{1}{\lambda_1^p}\right) = A - \frac{\Gamma}{\lambda_1^{p-q}}. \quad (4.10)$$

Consider the following function defined on $[0, 1/\lambda_1^p]$: $g_1(t) = \psi(t)$, $0 \leq t \leq 1/\lambda_1^p$. Applying Proposition 4.2 for $\alpha = \frac{1}{\lambda_1^p}$, $f_2 = f - \frac{1}{\lambda_1^{p-1}}$ we obtain that there exists $g_2 : [\frac{1}{\lambda_1^p}, 1] \to \mathbb{R}^+$ such that

$$g_2 \leq \psi / \left[\frac{1}{\lambda_1^p}, 1\right], \quad \int_{1/\lambda_1^p}^1 g_2 = f - \frac{1}{\lambda_1^{p-1}}, \quad \int_{1/\lambda_1^p}^1 g_2^q = F_{\lambda_1}\left(\frac{1}{\lambda_1^p}\right).$$

But then if $g : [0, 1] \to \mathbb{R}^+$ with $g / \left[0, \frac{1}{\lambda_1^p}\right] = \psi$ and $g / \left[\frac{1}{\lambda_1^p}, 1\right] = g_2$ we have because of (4.9) that

$$\int_0^1 g_1 = f, \quad \int_0^1 g^q = A, \quad g \leq \psi, \quad \int_{1/\lambda_1^p}^{1/\lambda_1^p} g = \frac{1}{\lambda_1^{p-1}} = \frac{1}{\lambda_1^p} \cdot \lambda_1$$

and according to Lemma 4.1 we have that $\alpha_1 = B_1(f, A, \lambda_1) \geq \frac{1}{\lambda_1^p}$. But of course $\alpha_1 \leq \frac{1}{\lambda_1^p}$, so that $\alpha_1 = \frac{1}{\lambda_1^p}$. But then we easily see that $\alpha(\lambda) = \frac{1}{\lambda^p}$ for every $\lambda \geq \lambda_1$. This is true because $g : [0, 1] \to \mathbb{R}^+$ as mentioned before satisfies:

$$g \leq \psi, \quad \int_0^{1/\lambda_1^p} g = \frac{1}{\lambda_1^{p-1}} = \frac{1}{\lambda_1^p} \cdot \lambda \quad (4.11)$$
Applying now Lemma 4.1 we obtain the result, that is \( \alpha(\lambda) = \frac{1}{\lambda^p} \) \( \forall \lambda \geq \lambda_1 \).

Let now \( \lambda_2 = \min \left\{ \lambda : \alpha(\lambda) = \frac{1}{\lambda^p} \right\} \) and \( \lambda \) such that \( \lambda : \alpha(\lambda) = \frac{1}{\lambda^p} \). Then \( \frac{1}{\lambda^p} \leq \frac{1}{\lambda} \Rightarrow \lambda \geq \left( \frac{1}{\lambda} \right)^{1/p-1} = \lambda_0 \), so that \( \lambda_2 = \min \left\{ \lambda \geq \lambda_0 : \alpha(\lambda) = \frac{1}{\lambda^p} \right\} \).

Let \( \lambda_1 \) as defined in Lemma 4.2. Obviously \( \lambda_1 \geq \lambda_2 \). We state now the following

**Lemma 4.3** If \( (f, A) \in D \) and \( \lambda > f \) such that \( \alpha = \alpha(\lambda) = B_1(f, A, \lambda) \not\leq \frac{1}{\lambda^p} \)

then there exists \( g : [0, 1] \to \mathbb{R}^+ \) such that \( g \leq \psi, \int_0^\alpha g = \alpha \lambda, \int_0^1 g = f, \int_0^1 g^q = A_2 \) where \( A_2 = F_\lambda(\alpha) \).

**Proof.** The existence of such \( \alpha \) is guaranteed but with \( A_2 \) such that \( F_\lambda(\alpha) \leq A_2 \leq G_\lambda(\alpha) \).

Suppose that \( A_2 > F_\lambda(\alpha) \). Then for a suitable \( \varepsilon > 0 \) which will be chosen later there is a \( g_1 : [0, 1] \to \mathbb{R}^2 \) such that \( g_1 \leq \psi, \int_0^1 g_1 = f, \int_0^1 g_1^q = A_2 - \varepsilon \geq F_\lambda(\alpha) \) and \( g_1 = g \) on \([0, \alpha]\). This is true because of Proposition 4.2. Since now \( \alpha(\lambda) < \frac{1}{\lambda^p} \) then \( g_1 \not\leq \psi \) on a subset of \([0, \frac{1}{\lambda^p}]\) with positive measure, that is there is space between \( g \) and \( \psi \) on \([0, \frac{1}{\lambda^p}]\). Indeed if \( g_1 = \psi \) on \([0, \frac{1}{\lambda^p}]\) then we would have that \( \int_0^{1/\lambda^p} g_1 = \int_0^{1/\lambda^p} \psi = \frac{1}{\lambda^p} \cdot \lambda \) and so Lemma 4.1 would give \( \alpha(\lambda) = \frac{1}{\lambda^p} \) which is contradiction by assumption. So that

\[
\left| \{g_1 \not\leq \psi \} \cap \left[0, \frac{1}{\lambda^p}\right]\right| > 0.
\]

So, since \( q > 1 \) we can increase \( g_1 \) to \( g_2 \) on \([0, 1/\lambda^p]\) and decrease \( g_1 \) to \( g_2 \) on \([1/\lambda^p, 1]\) in a way that \( \int_0^1 g_2 = f, \int_0^1 g_2^q > \int_0^1 g_1^q = A - \varepsilon \) such that there exists \( \beta > \alpha \) so that \( \int_0^\beta g_2 \geq \beta \lambda \). Actually, if \( \varepsilon > 0 \) is small enough we can
arrange everything so that \( \int_0^1 g_2^q = A \). This gives \( B(f, A, \lambda) > \alpha \) which is a contradiction. So the lemma is proved. □

Now, let \( \lambda_2 \) be as before. Since \( \lambda_2 \) is the minimum positive \( \lambda \) such that \( \alpha(\lambda) = 1/\lambda^p \), we have from Lemma 4.3 and by continuity reasons that there exists \( g : [0, 1] \to \mathbb{R}^+ \) such that

\[
\int_0^{1/\lambda_2^p} g = \frac{1}{\lambda_2^{p-1}} = \frac{1}{\lambda_2^p} \cdot \lambda_2, \quad \int_0^1 g = f, \quad \int_0^1 g^q = A, \quad g \leq \psi
\]

and such that

\[
A_2 = \int_{1/\lambda_2^p}^1 g^q = F_{\lambda_2}(\alpha)
\]

where \( \alpha = \alpha(\lambda_2) = \frac{1}{\lambda_2^p} \). But then

\[
T_{\lambda_2}(\alpha(\lambda_2)) = S_{\lambda_2}(\alpha(\lambda_2)) = \frac{\Gamma}{\lambda_2^{p-q}}.
\]

Since

\[
T_{\lambda_2}(\alpha) \leq A_1 = \int_0^{1/\lambda_2^p} g \leq S_{\lambda_2}(\alpha)
\]

we have that

\[
F_{\lambda_2} \left( \frac{1}{\lambda_2^p} \right) + \frac{\Gamma}{\lambda_2^{p-q}} = A
\]

that is \( \lambda_1 = \lambda_2 \).

Let now \( \lambda = \lambda_1 = \lambda_2 \).

Then

\[
F_{\lambda} \left( \frac{1}{\lambda^p} \right) + \frac{\Gamma}{\lambda^{p-q}} = A,
\]

and \( \lambda \geq \left( \frac{1}{7} \right)^{1/p-1} \). For \( \mu > \lambda \) and \( \beta = \frac{1}{\mu^p} \) we have that

\[
\frac{1}{\mu^p} \leq \frac{f}{\mu} \quad \text{and} \quad f - \mu \beta \leq 1 - \beta^{1-\frac{1}{p}}.
\]
Then $F_\mu(\beta) = F_\mu(1/\mu^p)$ describes the minimum $L^q$-norm value of functions $g$ defined on

$$\left[\frac{1}{\mu^p}, 1\right] = [\beta, 1]$$

for which $\int_\beta^1 g = f - \mu \beta$.

So

$$F_\mu\left(\frac{1}{\mu^p}\right) + \frac{\Gamma}{\mu^{p-q}} = \int_0^1 g_\mu^q$$

where $g_\mu$ is defined such that

$$g_\mu := \psi, [0, \beta], \quad \int_\beta^1 g_\mu^q = F_\mu\left(\frac{1}{\mu^p}\right) \quad \text{and} \quad g_\mu \leq \psi.$$ 

But then it is easy to see because of the form of $g_\mu$ that $\int_0^1 g_\mu^q$ decreases when $\mu$ increases.

So that for every $\mu > \lambda$ $F_\mu\left(\frac{1}{\mu^p}\right) + \frac{\Gamma}{\mu^{p-q}} < A$.

Summarizing all the above we obtain the following

**Theorem 4.1** If $\alpha = B_1(f, A, \lambda)$ where $(f, A) \in D$ with $A_f \notin A$, then

(i) $\alpha(\lambda) = \frac{1}{\lambda^p}$ for every $\lambda \geq \lambda_1$, where $\lambda_1$ is the unique root of the equation

$$F_\lambda\left(\frac{1}{\lambda^p}\right) + \frac{\Gamma}{\lambda^{p-q}} = A \quad \text{on the interval} \quad \left(\left(\frac{1}{f}\right)^{1/p-1}, +\infty\right).$$

(ii) For every $f < \lambda < \lambda_1$ $\alpha$ equals the supremum of all $\beta$ such that $\beta \leq \frac{\min\left(\frac{f}{\lambda}, \frac{1}{\lambda^p}\right)}{\lambda}$ and $f - \lambda \beta \leq 1 - \beta^{1-1/p}$ for which

$$T_\lambda(\beta) \leq A - F_\lambda(\beta) \leq S_\lambda(\beta).$$

We now analyze part (ii) of Theorem 4.1.

Let $f < \lambda < \lambda_1$, so that $\alpha = \alpha(\lambda) = B_1(f, \alpha, \lambda) < \frac{1}{\lambda^p}$. Of course, we must also have that $\alpha \leq \frac{f}{\lambda}$. We search now for those $\beta \in \left[0, \frac{1}{\lambda^p}\right]$ such that $f - \lambda \beta \leq 1 - \beta^{1-1/p}$.
Consider $K$ defined on $[0, \frac{1}{\lambda^p}]$ by $K(\beta) = f - 1 + \beta^{1-q/p} - \lambda \beta$. Since $K'(\beta) = \frac{p-1}{p} \beta^{-1/p} - \lambda$, $K$ increasing on $[0, \beta_0]$, decreasing on $[\beta_0, \frac{1}{\lambda^p}]$ with maximum value at the point $\beta_0$ where $\beta_0 = \left[ \frac{p-1/p}{\lambda} \right]^p$. Then

$$K(\beta_0) = f - 1 + \left[ \frac{p-1/p}{\lambda} \right]^{p-1} \cdot \frac{1}{p}$$

which may be positive as well as negative. We first work in case that $K(\beta_0) > 0$ and $\frac{p-1}{p} < f \leq 1$. From the above we have that there exist $\beta_1, \beta_2 \leq \frac{1}{\lambda^p}$ with $\beta_1 < \beta_2$ so that $f - \lambda \beta_i = 1 - \beta_i^{1-\frac{1}{p}}$ for $i = 1, 2$ and for $\beta \leq \frac{1}{\lambda^p}$ we have that $f - \lambda \beta \leq 1 - \beta^{1-\frac{1}{p}}$ if and only if $\beta \in [0, \beta_1] \cup [\beta_2, \frac{1}{\lambda^p}]$. With the above hypothesis we prove the following

**Lemma 4.4** For $(f, A)$ such that $A > A_f$, $f < \lambda < \lambda_1$ we have that

$$\alpha = B_1(f, A, \lambda) \in \left[ \beta_2, \min \left\{ \frac{f}{\lambda}, \frac{1}{\lambda^p} \right\} \right].$$

**Proof.** Obviously, for $\gamma = \frac{f}{\lambda}$, $f - \gamma \lambda \leq 1 - \gamma^{1-\frac{1}{p}}$ and $\lambda \beta_2 \leq \beta_2^{1-\frac{1}{p}}$ so by means of Proposition 4.1 there exists $\phi : [0, \beta_2] \to \mathbb{R}^+$ such that

$$\int_0^{\beta_2} \phi = \lambda \beta_2, \phi \leq \psi, \int_0^{\beta_2} \phi^q = T_\lambda(\beta_2).$$

Now since $\beta_2 > \beta_0 = \left[ \frac{p-1/p}{\lambda} \right]^p$, $T_\lambda(\beta_2) = \Delta_f(\beta_2)$. We extend now $\phi$ on $[0, 1]$ by defining $\phi = \psi$ on $[\beta_2, 1]$. Then since $f - \lambda \beta_2 = 1 - \beta_2^{1-\frac{1}{p}}$ we have that $
abla 0^1 \phi = f$.

By definition now of $\phi$ and $\Delta_f(\beta_2)$ the form of $\phi$ must be such that

$$\int_0^1 \phi^q = A_f \quad \text{(since } \int_0^1 \phi = f).$$

But we remind that $\int_0^1 \phi = \lambda \beta_2$. Then, since

$$\int_0^1 \phi^q = A_f < A$$

it is easy to construct a function $g : [0, 1] \to \mathbb{R}^+$ such that
\[ g \leq \psi, \int_0^1 g = f, \int_0^1 g^q = A \] which for a \( \gamma \) with \( \gamma > \beta_2 \) satisfies \( \int_0^\gamma g = \gamma \lambda \). But then, applying Lemma 4.1 \( B_1(f, A, \lambda) \geq \gamma > \beta_2 \) or \( \alpha \in [\beta_2; \min \{ f, \frac{1}{\lambda p} \}] \) that is, what we needed to prove. \( \square \)

Consider now the following function defined on \( R_{\lambda} : \Delta = [\beta_2, \min \{ f, \frac{1}{\lambda p} \}] \to \mathbb{R}^+ \)

with \( R_{\lambda}(\beta) = F_{\lambda}(\beta) + S_{\lambda}(\beta), \beta \in \Delta \).

Because of the definitions of \( F_{\lambda} \) and \( S_{\lambda} \) and having in mind the geometric interpretations of them, we easily see that \( R_{\lambda} \) is increasing on \( \Delta \). In fact \( S_{\lambda}(\alpha) \) represents the \( L^q \)-norm of a function \( g_1 \) defined on \([0, \alpha]\) such that \( \int_0^\alpha g_1 = \alpha \lambda \) and \( g_1 = \begin{cases} \psi, & \text{on } [0, c] \\ 0, & \text{on } (c, \alpha] \end{cases} \) for suitable \( c \). But \( F_{\lambda}(\alpha) \) represents the minimum \( L^q \)-norm of all functions \( \phi \) defined on \([\alpha, 1]\) such that \( \int_\alpha^1 \phi = f - \lambda \alpha \). Then there exists essentially unique \( g_2 : [\alpha, 1] \to \mathbb{R}^+ \) such that \( \int_\alpha^1 g_2 = f - \lambda \alpha \) and \( F_{\lambda}(\alpha) = \int_0^1 g_2^q \). We set \( g = \begin{cases} g_1, & \text{on } [0, \alpha] \\ g_2, & \text{on } (\alpha, 1] \end{cases} \).

Increasing now \( \alpha \), it is obvious that we increase \( \int_0^1 g^q \), that is \( R_{\lambda}(\alpha) \) increases.

Let now \( \alpha = B_1(f, A, \lambda) \) then as we mentioned before, \( \alpha \in \Delta \), and of course \( T_{\lambda}(\alpha) \leq A - F_{\lambda}(\alpha) \leq S_{\lambda}(\alpha) \) because of Lemma 4.3 If \( \alpha < \min \{ f, \frac{1}{\lambda p} \} \) and \( F_{\lambda}(\alpha) + T_{\lambda}(\alpha) < A \) then since \( \lambda - \lambda \alpha < 1 - \alpha \lambda \) there exists \( \gamma \) such that \( \alpha < \gamma < \min \{ f, \frac{1}{\lambda p} \} \) such that \( F_{\lambda}(\gamma) + T_{\lambda}(\gamma) < A \) and of course

\[ R_{\lambda}(\gamma) = F_{\lambda}(\gamma) + S_{\lambda}(\gamma) \geq A \] (\( R_{\lambda} \) is increasing).

That is

\[ T_{\lambda}(\gamma) \leq A - F_{\lambda}(\gamma) \leq S_{\lambda}(\gamma). \]
Then Corollary 4.2 gives \( B_1(f, A, \lambda) \geq \gamma > \alpha \), a contradiction that is if \( \alpha = B(f, A, \lambda) < \min \left\{ \frac{f}{\lambda}, \frac{1}{\lambda^p} \right\} \) we must have that \( F_\lambda(\alpha) + T_\lambda(\alpha) = A \).

Consider now \( \lambda_0 = \left( \frac{1}{f} \right)^{1/p-1} \) and the function \( h : E = \left[ f, \left( \frac{1}{\lambda} \right)^{1/p-1} \right] \to \mathbb{R}^+ \) defined by \( h(\lambda) = T_\lambda(f/\lambda) \). Notice that for \( f \leq \lambda \leq \left( \frac{1}{f} \right)^{1/p-1} \) we have that \( \frac{1}{\lambda} \leq \frac{1}{\lambda^p} \), so this definition makes sense.

Then

\[
h(f) = T_f(1) = A_f \not\leq A \quad \text{and} \quad h(\lambda_0) = \Gamma \frac{1}{\lambda_0^{p-q}} = \Gamma f^{p-q/p-1} \geq A.
\]

Again, having in mind the geometric interpretation of \( T_\lambda(\alpha) \) it is easy to see that \( h \) is strictly increasing on \( E \). So there exists unique \( \lambda_3 \in E \) such that \( T_{\lambda_3}(f/\lambda_3) = A \). Now for \( f < \lambda \leq \lambda_3 \)

\[
T_\lambda(f/\lambda) \leq A = A - F_\lambda(f/\lambda) \leq \Gamma f^{p-q/p-1} = S_\lambda(f/\lambda)
\]

that is in view of Corollary 4.2 \( B_1(f, A, \lambda) = \frac{f}{\lambda} \).

For \( \lambda_3 < \lambda < \lambda_1 \) we obviously have that

\[
B(f, A, \lambda) = \max \{ \alpha \in \Delta : F_\lambda(\alpha) + T_\lambda(\alpha) = A \}.
\]

So we found \( B_1(f, A, \lambda) \) in case that \( f < \lambda < \lambda_1 \) and \( K(\beta_0) > 0, \frac{p-1}{p} < f \leq 1 \).

The case \( K(\beta_0) = 0 \) is worked out in the same way where we replace \( \beta_2 \) by \( \beta_0 \), while the case \( K(\beta_0) < 0 \) is worked out for

\[
\Delta = \left[ 0, \min \left\{ \frac{f}{\lambda}, \frac{1}{\lambda^p} \right\} \right].
\]

Analogous results are obtained when \( 0 < f \leq \frac{p-1}{p} \) where

\[
\Delta = \left[ 0, \min \left\{ \frac{f}{\lambda}, \frac{1}{\lambda^p} \right\} \right],
\]

since then

\[
f - \lambda \beta \leq \frac{p-1}{p} (1 - \beta) \leq 1 - \beta^{1-\frac{1}{p}} \quad \text{for every} \quad \beta \leq \frac{1}{\lambda^p}.
\]

We state all the above results in the following

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Theorem 4.2 If \((f, A) \in D, A \not\leq A_f\) then \(B_1(f, A, \lambda)\) is given by

\[
B_1(f, A, \lambda) = \begin{cases} 
1, & 0 < \lambda \leq f \\
\frac{f}{\lambda}, & f < \lambda \leq \lambda_3 \\
\delta, & \lambda_3 < \lambda \leq \lambda_1 \\
\frac{1}{\lambda_1}, & \lambda_1 \leq \lambda
\end{cases}
\]

where

\[
d = \max\{\gamma \in \Delta : F_\lambda(\gamma) + T_\lambda(\gamma) = A\}. \quad \Box
\]

That is we found sharp inequalities concerning the localized distribution function of \(\mathcal{M}\phi\) given the \(L^1\) and \(L^q\)-norms and the usual quasi-norm \(\| \cdot \|_{p, \infty}\) of \(\phi\) for \(1 < q < p\) as variables.

Remark 4.5 i) The case where \(A = A_f\) can be worked out separately because there exists essentially unique function \(g : [0, 1] \rightarrow \mathbb{R}^+\) such that \(\int_0^1 g = f\), \(\int_0^1 g^q = A\), \(\int_0^1 g \leq \psi\).

ii) We have that \(B(f, A, \lambda) = B_1(f, A, \lambda)\), for \(A \not= f^q\) as mentioned in the beginning of this section. This is true of course for \(\lambda \geq \lambda_1\), that is for \(\lambda\) such that \(\alpha(\lambda) = B_1(f, A, \lambda) = \frac{1}{\lambda^p}\).

Now for \(\lambda < \lambda_1\) let \(\alpha = B_1(f, A, \lambda)\). Then there exists \(g : [0, 1] \rightarrow \mathbb{R}^+\) such that \(\int_0^1 g = f\), \(\int_0^1 g^q = A\), \(\int_0^1 g = \alpha \lambda\), \(g \leq \psi\). Then it is easy to see that for every \(\varepsilon > 0\) small enough we can change \(g\) to \(g_\varepsilon\) in a way that

\[
\int_0^{\alpha - \varepsilon} g_\varepsilon \geq (\alpha - \varepsilon)\lambda, \quad \int_0^1 g_\varepsilon = f, \quad \int_0^1 g_\varepsilon^q = A + \delta_\varepsilon, \quad \|g_\varepsilon\|_{p, \infty} = \frac{p - 1}{p}
\]

and \(\delta_\varepsilon \to 0\) as \(\varepsilon \to 0^+\). This using continuity arguments gives \(B(f, A, \lambda) = \alpha\).

iii) In the statement of Theorem 4.2 it is not difficult to see (by doing some tedious calculations in any case according to the way that \(F_\lambda, T_\lambda\) are defined) that for the range \(\lambda_3 < \lambda < \lambda_1\) there is in fact unique \(\gamma \in \Delta\) such that \(F_\lambda(\gamma) + T_\lambda(\gamma) = A\), because \(F_\lambda + T_\lambda\) is increasing on \(\Delta\).
iv) Notice the continuity of the function as calculated on Theorem 4.2 at the point \( \lambda = \lambda_1 \). As a matter of fact \( \delta \) is such that 
\[
F_{\lambda_1}(\delta) + T_{\lambda_1}(\delta) = A.
\]
But \( \lambda_1 \) is such that
\[
F_{\lambda_1}(\delta) + T_{\lambda_1}(\delta) = F_{\lambda_1}\left(\frac{1}{\lambda_1^p}\right) + T_{\lambda_1}\left(\frac{1}{\lambda_1^p}\right) = A,
\]
which in view of Remark iii) above, gives \( \delta = \frac{1}{\lambda_1^p} \).

We state now the following immediate corollary of Theorem 4.2 as

**Theorem 4.3** For \((f, A) \in D, \lambda > f, F = \frac{p-1}{p} \) and \( A \geq A_f \) the following holds:

\[
\sup\left\{ \|\mathcal{M}\phi\|_{p,\infty} : \phi \geq 0, \int_X \phi d\mu = f, \int_X \phi^q d\mu = A, \|\phi\|_{p,\infty} = F \right\} = \frac{p}{p-1} F
\]

with the supremum attained.

That is (1.7) is best possible allowing every value of the \( L^1 \) and \( L^q \)-norm.

\[\square\]

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