CONSTRUCTING SEPARABLE STATES IN INFINITE-DIMENSIONAL SYSTEMS BY OPERATOR MATRICES

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Abstract. We introduce a class of states so-called semi-SSPPT (semi super strong positive partial transposition) states in infinite-dimensional bipartite systems by the Cholesky decomposition in terms of operator matrices and show that every semi-SSPPT state is separable. This gives a method of constructing separable states and generalizes the corresponding results in [Phys. Rev. A 77, 022113(2008); J. Phys. A: Math. Theor. 45 505303 (2012)]. This criterion is specially convenient to be applied when one of the subsystem is a qubit system.

1. Introduction and main result

Entanglement is an important resource in quantum information processing and quantum computation [1]. However, the detection of entanglement is one of the most difficult task in this area and much more effort had been paid to on this research field [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. Recall that, mathematically, a quantum state \( \rho \) (positive operator of trace 1) acting on a separable complex Hilbert space \( H = H_A \otimes H_B \) is called separable if it can be written as the form

\[
\rho = \sum_i p_i \rho_i^A \otimes \rho_i^B, \quad \sum_i p_i = 1, \quad p_i \geq 0
\]

or can be approximated in the trace-norm by the states of the above form, where \( \rho_i^A \) and \( \rho_i^B \) are respectively quantum states of subsystem A and B [13, 14]. Otherwise, \( \rho \) is called inseparable or entangled. A separable state with the form as in Eq. (1) is called countably separable [15, 16]. If \( \dim H_A \otimes H_B < +\infty \), it is known that all separable states \( \rho \) acting on \( H_A \otimes H_B \) are countably separable [14]. But, in the infinite-dimensional case, there exists separable states which are not countably separable [15].

One of the most famous and convenient criteria for detecting entanglement is the positive partial transpose (PPT) criterion proposed by Peres and Horodecki [17, 18] which asserts that if a quantum state \( \rho \) acting on \( H_A \otimes H_B \) is separable, then its partial transposes are positive.

Keywords: infinite-dimensional bipartite quantum systems, separable states, PPT states, Cholesky decomposition.

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operators, that is, $\rho^{TA/B} \geq 0$. There are entangled PPT states except for those in $2 \otimes 2$ and $2 \otimes 3$ systems [18]. So, it is important to know which PPT states are separable.

In this paper we present a method to construct a class of PPT states on infinite-dimensional bipartite systems by an operator-matrix trick and show that such states are separable.

In a bipartite system A+B described by $H_A \otimes H_B$ with $\dim H_A \otimes H_B = +\infty$, let $\{|i_a\}\}$ and $\{|j_b\}\}$ be any orthonormal bases of $H_A$ and $H_B$, respectively. Denote by $E^{a}_{kl} = |k_a\rangle\langle l_a|$ and $E^{b}_{kl} = |k_b\rangle\langle l_b|$. Then any state $\rho$ acting on $H_A \otimes H_B$ can be represented by

$$\rho = \sum_{k,l}^{\dim H_A} E^{a}_{kl} \otimes B_{kl} = \sum_{k,l}^{\dim H_B} A_{kl} \otimes E^{b}_{kl},$$

where $B_{kl}$s are trace class operators on $H_B$ and the series converges in trace-norm [22], that is,

$$\rho = \begin{pmatrix}
B_{11} & B_{12} & B_{13} & \cdots & \cdots \\
B_{21} & B_{22} & B_{23} & \cdots & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
B_{m1} & B_{m2} & B_{m3} & \cdots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots
\end{pmatrix} = \begin{pmatrix}
A_{11} & A_{12} & A_{13} & \cdots & \cdots \\
A_{21} & A_{22} & A_{23} & \cdots & \cdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
A_{m1} & A_{m2} & A_{m3} & \cdots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots
\end{pmatrix}$$

under the given bases. Take operator sequences $\{X_{ij}\}_{i,j=1}^{\dim H_A}$ and $\{S_{ij} : 1 \leq i < j \leq \dim H_A\}$ on $H_B$ so that the operator matrix (infinite if $\dim H_A = \infty$) of the form

$$X = \begin{pmatrix}
X_1 & S_{12}X_1 & S_{13}X_1 & \cdots & S_{1m}X_1 & \cdots \\
0 & X_2 & S_{23}X_2 & \cdots & S_{2m}X_2 & \cdots \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & X_{m-1} & S_{m-1,m}X_{m-1} & \cdots \\
0 & 0 & 0 & 0 & X_m & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots
\end{pmatrix}$$

is a Hilbert-Schmidt operator on $H_A \otimes H_B$, that is,

$$\text{Tr}(X^\dagger X) = \sum_i \text{Tr}(X_i^\dagger X_i) + \sum_{i<j} \text{Tr}(X_i^\dagger S_{ij} S_{ij} X_i) < \infty,$$

then

$$\rho_X^B = \frac{1}{\text{Tr}(X^\dagger X)} X^\dagger X$$

is a bipartite state in $H_A \otimes H_B$. One can construct $\rho_X^A$ in the same way. For convenience, we call $\{X_i\}_{i=1}^{\dim H_A}$ and $\{S_{ij} : 1 \leq i < j \leq \dim H_A\}$ Cholesky operators, and say Eq.(5) is a
Cholesky decomposition $\rho^X_A$. These terminologies come from the fact that every block matrix has a Cholesky decomposition of the form in Eq.(5)

**Definition 1.** A state $\rho \in S(H_A \otimes H_B)$ is called a semi-SSPPT state up to part B if it has a Cholesky decomposition as in Eq.(5) and the associated Cholesky operators $\{S_{ij} : 1 \leq i < j \leq \dim H_A\}$ satisfying the condition

$$[S_{ki}, S_{kj}^\dagger] = 0, \quad k < i \leq j.$$  

(6)

The semi-SSPPT states up to part A are defined similarly. A state is called a semi-SSPPT state if it is a semi-SSPPT states up to part B or a semi-SSPPT states up to part A.

It is easily checked that every semi-SSPPT state is PPT. The following is our may result.

**Theorem 1.** Let $\rho \in S(H_A \otimes H_B)$ be a state with $\dim H_A \otimes H_B \leq \infty$. If $\rho$ is semi-SSPPT, then $\rho$ is separable.

The terminology SSPPT (super strong positive partial transpose) comes from [19] for finite-dimensional systems and [21] for infinite-dimensional systems, where the additional assumption “every $S_{ij}$ is diagonalizable” is required. The main result in [19, 21] shows that the SSPPT states are countably separable. However, though the condition Eq.(6) ensures that every $S_{ij}$ is a normal operator, we know that there are many normal operators on infinite-dimensional Hilbert spaces that are not diagonalizable. Thus the above theorem 1 generalizes the result in [19, 21] greatly.

The proof of theorem 1 will be presented in Appendix B. We point out, our proof of theorem 1 needs new mathematical tools including introducing a concept of SOT-separability for bounded positive operators and establishing a Radon-Nikodym type theorem for spectral measure, which we present in Appendix A.

2. Corollaries and examples

Theorem 1 provides an easier way of constructing separable states in infinite-dimensional bipartite systems.

**Example 1.** Assume $\dim H_A = n \leq \infty$ and $\dim H_B = m \leq \infty$. Let $\{X_i\}_{i=1}^n$ and $\{S_i\}_{i=1}^n$ be two sequences of operators on $H_B$ such that $\sum_{i=1}^n \Tr(X_i^\dagger X_i) = \frac{1}{2}$, $S_i$ normal and $\Tr(X_i^\dagger S_i^\dagger S_i X_i) \sum_{i<j} \frac{1}{2} = \Tr(X_i^\dagger X_i)$ for each $i \in \{1, 2, \ldots, n\}$. Let $S_{ij} = \frac{1}{2} S_i$ for $j > i$ and let $X$ be the operator matrix as in Eq.(4). Then,

$$\Tr(X^\dagger X) = \sum_{i=1}^n \Tr(X_i^\dagger X_i) + \sum_{i=1}^n \Tr(X_i^\dagger S_i^\dagger S_i X_i) \sum_{i<j} \frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1.$$
Thus, \( \rho = X^\dagger X \) is a state in \( H_A \otimes H_B \) and is separable by theorem 1. In the case \( \dim H_A = n = \infty \), as \( \sum_{i<j}^{\infty} \frac{1}{2^{2i}} \), one may choose normal operator \( S_i \) so that \( \text{Tr}(X_i^\dagger S_i^\dagger S_i X_i) = \frac{9}{2} \) for each \( i \) to ensure that \( \sum_{i=1}^{\infty} \text{Tr}(X_i^\dagger S_i^\dagger S_i X_i) \sum_{i<j}^{\infty} \frac{1}{2^{2i}} = \frac{1}{2} \).

The following are some corollaries of theorem 1 which generalize the corresponding results in [21] from finite-dimensional systems to infinite-dimensional systems, and also illustrates the use of theorem 1 to detect the separability of a state in the case when \( \min\{\dim H_A, \dim H_B\} = 2 \). Note that every trace-class operator acting on an infinite-dimensional Hilbert space can not be invertible; so the corollary in [21] is not applicable to infinite-dimensional case.

Assume that \( \dim H_A = 2 \) (or \( \dim H_B = 2 \)) and \( \rho \in S(H_A \otimes H_B) \). Then \( \rho \) can be written in

\[
(7) \quad \rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \quad \text{or} \quad \rho = \begin{pmatrix} \tilde{\rho}_{11} & \tilde{\rho}_{12} \\ \tilde{\rho}_{21} & \tilde{\rho}_{22} \end{pmatrix}
\]

up to part B (or, up to part A).

**Corollary 1.** Let \( \rho \) be a state as in Eq.(7). If there is a Cholesky decomposition \( \rho = X^\dagger X \) up to part B/A with \( X = \begin{pmatrix} X_1 & S_{12}X_1 \\ 0 & X_2 \end{pmatrix} \) such that \( X_1 \) has dense range and \( \rho^{T_A/B} = Y^\dagger Y \) with \( Y = \begin{pmatrix} X_1 & S_{12}^\dagger X_1 \\ 0 & X_2 \end{pmatrix} \), then \( \rho \) is separable.

**Proof.** Since \( (Y^\dagger Y)^{T_A/B} = (\rho^{T_A/B})^{T_A/B} = \rho = X^\dagger X \), one gets \( X_1^\dagger S_{12} S_{12}^\dagger X_1 = X_1^\dagger S_{12}^\dagger S_{12} X_1 \), which entails that \( S_{12} S_{12}^\dagger = S_{12}^\dagger S_{12} \) as the range of \( X_1 \) is dense. So, \( \rho \) is semi-SSPPT and thus, by Theorem 1, is separable.

**Corollary 2.** Let \( \rho \) be a state as in Eq.(7). Then any one of the following conditions implies that \( \rho \) is separable.

1. \( \rho_{11} \geq \rho_{22} \) (or \( \tilde{\rho}_{11} \geq \tilde{\rho}_{22} \)).
2. \( \rho_{22} \geq \rho_{11} \) (or \( \tilde{\rho}_{22} \geq \tilde{\rho}_{11} \)).

**Proof.** Assume that \( \rho \) satisfies the condition \( \rho_{11} \geq \rho_{22} \). In this case, \( \dim H_A = 2 \) and \( \rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \) with \( \rho_{ij} \in T(H_B) \) and \( \rho_{22} \leq \rho_{11} \). We shall show that \( \rho \) is semi-SSPPT and hence is separable by Theorem 1.

Since \( \rho \geq 0 \) and \( \rho_{22} \leq \rho_{11} \), there are contractive operators \( T, S \) on \( H_B \) with \( \ker T \cap \ker S \cap \ker S^\dagger \supseteq \ker \rho_{11} \) such that \( \rho_{12} = \sqrt{\rho_{11}} T \sqrt{\rho_{22}} \) and \( \sqrt{\rho_{22}} = \sqrt{\rho_{11}} S = S^\dagger \sqrt{\rho_{11}} \) [23, Theorem 1.1], here, \( \ker L \) denotes the null space of the operator \( L \).

Let \( S_{12} = T S^\dagger \). Then we have \( \rho_{12} = \sqrt{\rho_{11}} S_{12} \sqrt{\rho_{11}} \). Note that

\[
\sqrt{\rho_{11}} S_{12}^\dagger S_{12} \sqrt{\rho_{11}} = \sqrt{\rho_{22}} T^\dagger T \sqrt{\rho_{22}} \leq \rho_{22}.
\]
Let
\[ X_2 = [\rho_{22} - \sqrt{\rho_{11}} S_{12}^{\dagger}] \sqrt{\rho_{11}}, \]
\[ X = \begin{pmatrix} \sqrt{\rho_{11}} & S_{12} \sqrt{\rho_{11}} \\ 0 & X_2 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} \sqrt{\rho_{11}} & S_{12}^{\dagger} \sqrt{\rho_{11}} \\ 0 & X_2 \end{pmatrix}. \]

Then
\[ \rho = X^\dagger X = \begin{pmatrix} \rho_{11} & \sqrt{\rho_{11}} S_{12} \sqrt{\rho_{11}} \\ \sqrt{\rho_{11}} S_{12}^{\dagger} \sqrt{\rho_{11}} & \sqrt{\rho_{11}} S_{12}^{\dagger} S_{12} \sqrt{\rho_{11}} + X_2^\dagger X_2 \end{pmatrix} \]
and
\[ \rho^{T_A} = Y^\dagger Y = \begin{pmatrix} \rho_{11} & \sqrt{\rho_{11}} S_{12}^{\dagger} \sqrt{\rho_{11}} \\ \sqrt{\rho_{11}} S_{12} \sqrt{\rho_{11}} & \sqrt{\rho_{11}} S_{12}^{\dagger} S_{12} \sqrt{\rho_{11}} + X_2^\dagger X_2 \end{pmatrix}. \]

Since \([\rho^{T_A}]^{T_A} = \rho\), we get
\[ \sqrt{\rho_{11}} S_{12} S_{12}^{\dagger} \sqrt{\rho_{11}} = \sqrt{\rho_{11}} S_{12}^{\dagger} S_{12} \sqrt{\rho_{11}}. \]

On the other hand, \(\ker T \cap \ker S \cap \ker S^{\dagger} \supseteq \ker \rho_{11}\) ensures that \(\ker S_{12} \supseteq \ker \sqrt{\rho_{11}}\). This entails that \(S_{12} S_{12}^{\dagger} = S_{12}^{\dagger} S_{12}\), that is, \(\rho\) is semi-SSPPT.

Other cases can be dealt with similarly. \(\square\)

We give an example to illustrate how to apply Corollary 2.

**Example 2.** Assume \(\dim H_A = 2\) and \(\dim H_B \leq \infty\). For any operators \(\rho_{11}, D\) and \(T\) acting on \(H_B\) with \(\rho_{11}\) a positive trace-class operator, \(\|D\| \leq 1\) and \(\|T\| \leq 1\). Obviously, by the corollary 2 the state \(\rho \in S(H_A \otimes H_B)\) constructed by
\[ \rho = \frac{1}{\text{Tr}(\rho_{11} + \sqrt{\rho_{11}} D D^{\dagger} \sqrt{\rho_{11}})} \begin{pmatrix} \rho_{11} & \sqrt{\rho_{11}} T [\sqrt{\rho_{11}} D D^{\dagger} \sqrt{\rho_{11}}]^{\dagger} \\ [\sqrt{\rho_{11}} D D^{\dagger} \sqrt{\rho_{11}}]^{\dagger} T^{\dagger} \sqrt{\rho_{11}} & \sqrt{\rho_{11}} D D^{\dagger} \sqrt{\rho_{11}} \end{pmatrix} \]
is separable since
\[ \rho_{11} \geq \sqrt{\rho_{11}} D D^{\dagger} \sqrt{\rho_{11}} = \rho_{22}. \]

3. **Conclusions**

In terms of the Cholesky decomposition and local commutativity, we introduce a notion of semi strongly super positive partial transpose (semi-SSPPT) states and establish a criterion of separability: if a quantum state in an infinite-dimensional bipartite system is semi-SSPPT, then it is separable. This criterion generalizes the corresponding results in [19, 21] and gives
a way of constructing separable states by operator matrices. This criterion is specially convenient to be applied when one of the subsystem is a qubit system. To prove this criterion, we establish a Radon-Nikodym type theorem for the spectral measure. However, our Radon-Nikodym type theorem is stated in term of unbounded operators and the strong operator topology (SOT) convergence. This forces us to introduce a notion of SOT-separability for positive operators acting on tensor product of two Hilbert spaces, and show that a state is separable if and only if it is SOT-separable. These results together enable us to give a proof of the main criterion. Our discussion also reveals that introduce and study separability for positive operators acting on tensor product of Hilbert spaces are helpful for solving some problems raised in quantum information theory.

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Appendix A:

SOT-separability for positive operators and Radon-Nikodym type theorem for spectral measure

To prove theorem 1, we need generalize the concept of separability from states to bounded positive operators and Radon-Nikodym theorem from vector measures in Hilbert space to the spectral measures.

Denote by $\mathcal{B}(H)$ and $\mathcal{B}_+(H)$ the set of all bounded linear operators and the set of all positive bounded operators acting on a complex Hilbert space $H$, respectively.

**Definition A.1.** A positive operator $T \in \mathcal{B}_+(H_A \otimes H_B)$ is called SOT-separable if there exist positive operators $\{A_k\} \subset \mathcal{B}_+(H_A)$ and $\{B_k\} \subset \mathcal{B}_+(H_B)$ such that

$$T = \sum_k A_k \otimes B_k$$

or if $T$ is the limit of the operators of the form as in Eq.(8) under the strong operator topology (briefly, SOT). Otherwise, $T$ is said to be SOT-inseparable.

We remark that, if the sum in Eq.(8) is a series, we mean that the series is convergent under SOT. It is obvious that 0 is SOT-separable and, if $\dim H_A \otimes H_B < \infty$, then a nonzero positive operator $T$ is SOT-separable if and only if $\frac{1}{\text{Tr}(T)} T$ is a separable quantum state.

Denote by $\mathcal{S}_{\text{SOT}}(H_A \otimes H_B)$ the set of all SOT-separable operators, which is a SOT-closed convex cone in $\mathcal{B}(H_A \otimes H_B)$.

The following is a SOT-separability criterion, which is similar to the entanglement witness criterion for states.

**Proposition A.1.** A positive operator $T$ is SOT-inseparable if and only if there is a self-adjoint operator $W \in \mathcal{B}(H_A \otimes H_B)$ of finite rank such that

1. $\text{Tr}(W(A \otimes B)) \geq 0$ holds for any $A \in \mathcal{B}_+(H_A)$ and $B \in \mathcal{B}_+(H_B)$;
2. $\text{Tr}(WT) < 0$.

**Proof.** Since $\mathcal{S}_{\text{SOT}}(H_A \otimes H_B)$ is a SOT-closed convex subset of $\mathcal{B}(H_A \otimes H_B)$, by Hahn-Banach theorem, $T \notin \mathcal{S}_{\text{SOT}}(H_A \otimes H_B)$ if and only if there exists a SOT-continuous linear functional $\phi$ on $\mathcal{B}(H_A \otimes H_B)$ and a real number $c$ such that $\text{Re}(\phi(S)) \geq c$ for all $S \in \mathcal{S}_{\text{SOT}}(H_A \otimes H_B)$ but $\text{Re}(\phi(WT)) < c$. As $\phi$ is SOT-continuous, there are vectors $x_1, \ldots, x_r; y_1, \ldots, y_r \in H_A \otimes H_B$ with $r < \infty$ such that $\phi(X) = \sum_{i=1}^r \langle y_i | X | x_i \rangle$ holds for all $X \in \mathcal{B}(H_A \otimes H_B)$. Let $E = \sum_{i=1}^r | x_i \rangle \langle y_i |$. Then $E \in \mathcal{B}(H_A \otimes H_B)$ is a finite-rank operator which satisfies
\(\phi(X) = \text{Tr}(EX)\) for all \(X\) (Ref., for example, [24]). If \(X\) is self-adjoint, that is, if \(X^\dagger = X\), then \(\phi(X)^* = \text{Tr}(EX)^* = \text{Tr}((EX)^\dagger) = \text{Tr}(E^\dagger X)\) and hence \(\text{Re}(\phi(X)) = \text{Tr}(WX)\), where \(W = \text{Re}(E) = \frac{1}{2}(E + E^\dagger)\). It follows that \(\text{Tr}(WS) = \text{Re}(\phi(S)) \geq c\) for all \(S \in \mathcal{S}_{\text{SOT}}(H_A \otimes H_B)\) and \(\text{Tr}(WT) = \text{Re}(\phi(T)) < c\). Note that \(0 \in \mathcal{S}_{\text{SOT}}(H_A \otimes H_B)\). So we must have \(c \leq 0\). We assert that \(\text{Tr}(WS) \geq 0\) holds for any \(S \in \mathcal{S}_{\text{SOT}}(H_A \otimes H_B)\). If, on the contrary, there is \(S \in \mathcal{S}_{\text{SOT}}(H_A \otimes H_B)\) so that \(\text{Tr}(WS) = a < 0\). As \(\mathcal{S}_{\text{SOT}}(H_A \otimes H_B)\) is a convex cone, \(tS \in \mathcal{S}_{\text{SOT}}(H_A \otimes H_B)\) for any \(t > 0\). Thus we have \(ta = \text{Tr}(W(tS)) \geq c\) for all \(t > 0\), which is a contradiction since \(ta \to -\infty\) when \(t \to \infty\). Therefore, we have found a self-adjoint operator \(W\) of finite rank such that (1) and (2) hold.

Conversely, if there is some self-adjoint operator \(W\) of finite rank such that (1) and (2) hold, then \(\text{Tr}(WS) \geq 0\) holds for all \(S \in \mathcal{S}_{\text{SOT}}(H_A \otimes H_B)\) since \(\phi : X \mapsto \text{Tr}(WX)\) is a SOT-continuous linear functional. Thus \(\phi\) separates strictly \(T\) and \(\mathcal{S}_{\text{SOT}}(H_A \otimes H_B)\). So \(T \not\in \mathcal{S}_{\text{SOT}}(H_A \otimes H_B)\). \(\square\)

Next we discuss the relationship between separability and SOT-separability for a quantum state. Notice that, generally speaking, though a sequence \(\{T_n\}\) of positive trace-class operators converges to a state \(\rho\) under SOT, one can not assert that \(\{T_n\}\) converges to \(\rho\) under the trace-norm. For instance, let \(\{a_n\}_{n=1}^\infty\) be a sequence of positive numbers so that \(\sum_{n=1}^\infty a_n = a < \infty\). For any \(n\), let \(T_n = \text{diag}(t_1, t_2, \ldots, t_k \ldots)\) with \(t_k = 0\) if \(k \leq n\) and \(t_{n+m} = a_m\). Then \(\{T_n\}_{n=1}^\infty\) is a sequence of positive trace-class operators and \(T_n \to 0\) in SOT. However, \(\text{Tr}(T_n) = a\) which does not converge to 0. In addition, for a state \(\rho\), let \(T_n' = \rho + T_n\). Then \(T_n' \to \rho\) under SOT but \(\text{Tr}(T_n') = 1 + a\) does not converge to \(\text{Tr}(\rho) = 1\). Hence, \(\{T_n'\}\) does not converge to \(\rho\) in trace-norm. This suggests that a SOT-separable state may not be separable. However, the following result reveals surprisingly that this is not the case.

**Proposition A.2.** Let \(\rho \in \mathcal{S}(H_A \otimes H_B)\) be a state. Then \(\rho\) is separable if and only if \(\rho\) is SOT-separable.

**Proof.** Clearly, \(\rho\) is separable implies that \(\rho\) is SOT-separable by the definitions as the convergence in trace-class norm implies the convergence in SOT.

Conversely, assume that \(\rho\) is inseparable (i.e., entangled). We have to show that \(\rho\) is also SOT-inseparable. Let \(\{|i_a\rangle\}\) and \(\{|j_b\rangle\}\) be arbitrarily given orthonormal bases for \(H_A\) and \(H_B\), respectively. Let \(P_k\) and \(Q_k\) be finite-rank projections on \(H_A^{(k)}\) and \(H_B^{(k)}\), the span of \(\{|i_a\rangle\}_{i=1}^k\) and the span of \(\{|j_b\rangle\}_{j=1}^k\), respectively. If \(k \geq \dim H_A\) (or \(k \geq \dim H_B\)), let \(P_k = I_A\) (or \(Q_k = I_B\)) with \(I_A\) the identity operator on \(H_A\). Let

\[
\rho_k = \frac{1}{\text{Tr}((P_k \otimes Q_k)\rho(P_k \otimes Q_k))}(P_k \otimes Q_k)\rho(P_k \otimes Q_k).
\]
Obviously, \( \rho = \| : \|_{\text{Tr}} \lim_{k \to \infty} \rho_k \). As \( \rho \) is inseparable, there exists infinitely many \( k \) so that \( \rho_k \) is inseparable (otherwise, \( \rho \) should be separable). Take such a \( k \). Then \( \rho_k \) can be regarded as an inseparable state in the finite-dimensional system \( H_A(k) \otimes H_B(k) \). Thus, by the entanglement witness criterion, there is a self-adjoint operator \( W_k \) on \( H_A(k) \otimes H_B(k) \) such that \( \text{Tr}(W_k \rho_k) < 0 \). Let \( W = (P_k \otimes Q_k)W_k(P_k \otimes Q_k) \). \( W \) is a self-adjoint operator of rank \( \leq k^2 < \infty \) and is an entanglement witness for \( \rho \) because for any pure states \( P_A \in \mathcal{S}(H_A) \) and \( Q_B \in \mathcal{S}(H_B) \),

\[
\text{Tr}(W(P_A \otimes Q_B)) = \text{Tr}(W_k((P_kP_A P_k) \otimes (Q_kQ_BQ_k))) \geq 0
\]

and

\[
\text{Tr}(W \rho) = \text{Tr}((P_k \otimes Q_k)W_k(P_k \otimes Q_k) \rho) = \text{Tr}(W_k \rho_k) < 0.
\]

For any \( A \otimes B \in \mathcal{B}_+(H_A \otimes H_B) \), \( (P_k \otimes Q_k)(A \otimes B)(P_k \otimes Q_k) = (P_kA P_k) \otimes (Q_kBQ_k) \) is either zero or a positive multiple of a finite rank separable state. So, we still have

\[
\text{Tr}(W(A \otimes B)) \geq 0.
\]

This implies by Proposition A.1 that \( \rho \) is SOT-inseparable, as desired.

Now, let us turn to the question of establishing Radon-Nikodym type theorem for spectral measure. Let \( \Omega \) be a nonempty set, \( \mathcal{B} \) be a \( \sigma \)-algebra of subsets of \( \Omega \), \( H \) be a Hilbert space. Recall that a spectral measure for \( (\Omega, \mathcal{B}, H) \) is an operator-valued function \( E : \mathcal{B} \to \mathcal{B}(H) \) such that

(i) for each \( \Delta \) in \( \mathcal{B} \), \( E(\Delta) \) is a projection;

(ii) \( E(\emptyset) = 0 \) and \( E(\Omega) = I \);

(iii) for \( \Delta_1, \Delta_2 \in \mathcal{B} \), \( E(\Delta_1 \cap \Delta_2) = E(\Delta_1)E(\Delta_2) \).

(iv) if \( \{\Delta_i\} \subset \mathcal{B} \) are pairwise disjoint sets, then \( E(\bigcup_i \Delta_i) = \sum_i E(\Delta_i) \), here the sum converges in SOT (Ref. [24]).

\textbf{Proposition A.3.} (The Radon-Nikodym type theorem for spectral measure) Let \( H \) be a complex Hilbert space, \( \Omega \) be a nonempty set, \( \mathcal{B} \) be a \( \sigma \)-algebra of subsets of \( \Omega \). Assume that \( E \) is a spectral measure for \( (\Omega, \mathcal{B}, H) \) and \( \mu \) is a positive measure on \( (\Omega, \mathcal{B}) \). If \( E \ll \mu \), that is, if \( \mu(\Delta) = 0 \Rightarrow E(\Delta) = 0 \), then there is an operator-valued function \( D : \Omega \to \mathcal{B}(H) \) such that \( \langle x|D(\omega)|x \rangle \geq 0 \) a.e. \( \mu \) and

\[
E(\Delta)x = (B) \int_\Delta D(\omega)xd\mu_\omega
\]

holds for every \( x \in H \) and \( \Delta \in \mathcal{B} \), where \((B) \int_\Delta \) means the Bochner integral.

We remark that \( D(\omega) \) may take an unbounded operator.

\textbf{Proof.} Recall that a Banach space \( X \) is said to have the Radon-Nikodym Property (RNP) if for any finite positive measure space \( (\Omega, \mathcal{F}, \mu) \) and vector-valued measure \( F : \mathcal{F} \to X \), if \( F \ll \mu \), then there exists a Bochner integrable vector-valued function \( f : \Omega \to X \) such that
$F(\Delta) = (B) \int_{\Delta} f(\omega) d\mu_{\omega}$ holds for any $\Delta \in \mathcal{F}$. Not every Banach space has RNP. However it is well-known that every Hilbert space has RNP (ref. [25, 26]).

Now let $(\Omega, \mathcal{B}, \mu)$ be a finite positive measure space and $(\Omega, \mathcal{B}, H, E)$ be a spectral measure space so that $E \ll \mu$. We remark here that we can not use the result in [26] because the spectral measure is not $\sigma$-bounded. For any vector $x \in H$, it is clear that $F_x : \mathcal{B} \rightarrow H$ defined by $F_x(\Delta) = E(\Delta) x$ is a $H$-valued measure satisfying $F_x \ll \mu$ and $\sigma$-boundedness. As $H$ has RNP, there is a Bochner integrable vector-valued function $D_x : \Omega \rightarrow H$ such that $E(\Delta) x = F_x(\Delta) = (B) \int_{\Delta} D_x(\omega) d\mu_{\omega}$ holds for all $\Delta \in \mathcal{B}$. Note that, $D_{\alpha x+y}(\omega) = \alpha D_x(\omega) + D_y(\omega)$ a.e. $\mu$ for each $x \in H$. And then $(B) \int_{\Delta} D(\omega) x d\mu_{\omega} = (B) \int_{\Delta} D_x(\omega) d\mu_{\omega}$ for any $x \in H$. Since, $\int_{\Delta} \langle x | D(\omega) | x \rangle d\mu_{\omega} = \langle x | E(\Delta) | x \rangle \geq 0$ for any Borel set $\Delta$, one sees that $\langle x | D(\omega) | x \rangle \geq 0$ a.e. $\mu$ for each $x \in H$. So, almost all $D(\omega)$ are (may unbounded) operators with domain $H$ satisfying $\langle x | D(\omega) | x \rangle \geq 0$ and

$$E(\Delta) x = (B) \int_{\Delta} D(\omega) x d\mu_{\omega} \quad (A.2)$$

holds for all $x \in H$ and $\Delta \in \mathcal{B}$. \hfill \Box

Some times we denote the relation in Eq.(A.2) by

$$E(\Delta) = (\text{SOT}) \int_{\Delta} D(\omega) d\mu_{\omega} \quad (A.3)$$

holds for any $\Delta \in \mathcal{B}$.

**Appendix B: Proof of main result**

Now we are at a position to give a proof of the main result theorem 1.

**Proof of Theorem 1.** Assume that $\rho \in \mathcal{S}(H_A \otimes H_B)$ is a semi-SSPPT state. We have to show that $\rho$ is separable. We only need to check the case that $\rho$ is semi-SSPPT up to part B since the proof for the case of semi-SSPPT up to A is similar.

As $\rho$ is a semi-SSPPT state up to part B, we may write $\rho = X^\dagger X$, where $X$ upper triangular operator matrices of the form mentioned in Eq.(4) with respect to an orthonormal basis $\{|i_a\rangle\}$ of $H_A$. Let $C_k$ be the operator matrix with the same size as that of $X$, which is induced from
X by replacing all entries by zero except for the kth row of X, i.e.,

\[
C_k = \begin{pmatrix}
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\
0 & \cdots & 0 & X_k & S_{k,k+1}X_k & S_{k,k+2}X_k & \cdots & S_{km}X_k \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots \\
\vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\
\end{pmatrix},
\]

\[k = 1, 2, \ldots. \] Then \(C_k\) is a Hilbert-Schmidt operator and

\[\rho = \sum_k C_k^\dagger C_k, \tag{8}\]

Here the series converges in the trace-norm. If \(C_k \neq 0\), write \(C_k^\dagger C_k = p_k \rho_k\) where \(p_k = \text{Tr}(C_k^\dagger C_k)\). We will show that \(\rho_k\) is separable for any \(k\) whenever \(C_k \neq 0\), and then, \(\rho = \sum_k p_k \rho_k\) is separable, too.

Consider the case when \(k = 1\). We have

\[p_1 \rho_1 = (X_1^\dagger S_{1j}^\dagger S_{1j} X_1) = \sum_{i,j} |i_a\rangle \langle j_a| \otimes (X_1^\dagger S_{1j}^\dagger S_{1j} X_1) \tag{B.1}\]

with \(S_{11} = I_B\). Since \(\rho\) is semi-SSPPT up to part B, \(\{S_{1i}\}\) is a commutative set of normal operators. Then there exists a normal operator \(N_1 \in \mathcal{B}(H_B)\) and bounded Borel functions \(\{f_{1i}\}\) such that \(S_{1i} = f_{1i}(N_1)\). Let \(N_1 = \int_{\sigma(N_1)} \omega dE_\omega\) be the spectral decomposition, where \((\sigma(N_1), \mathcal{B}, H, E)\) is the spectral measure of \(N_1\), \(\mathcal{B}\) is the \(\sigma\)-algebra of all Borel subsets of \(\sigma(N_1)\), the spectrum of \(N_1\). Thus, we have \(S_{1i} = \int_{\sigma(N_1)} f_{1i}(\omega)dE_\omega\). Because \(H_B\) is separable, there exists a probability measure, that is, the scalar spectral measure, \((\sigma(N_1), \mathcal{B}, \mu)\) so that, for any \(\Delta \in \mathcal{B}, E(\Delta) = 0\) if and only if \(\mu(\Delta) = 0\) (Ref. [24]). Then, by Proposition A.3, there exists an operator-valued function \(D\) such that \(\langle x|D(\omega)|x \rangle \geq 0\) a.e. \(\mu\) for each \(x \in H\) and

\[E(\Delta) = \text{(SOT)} \int_\Delta D(\omega) d\mu_\omega\]

holds for all \(\Delta \in \mathcal{B}\). By Eq.(B.1) one gets, for each product vector \(x_A \otimes x_B \in H_A \otimes H_B\),

\[
\rho_1(x_A \otimes x_B) = p_1^{-1} \sum_{i,j} |i_a\rangle \langle j_a| \otimes (X_1^\dagger \int_{\sigma(N_1)} f_{1i}(\omega)^* f_{1j}(\omega) dE_\omega X_1)(x_A \otimes x_B) = p_1^{-1} \sum_{i,j} |i_a\rangle \langle j_a| x_A \otimes ((B) \int_{\sigma(N_1)} f_{1i}(\omega)^* f_{1j}(\omega) X_1^\dagger D(\omega) X_1 x_B d\mu_\omega) = \sum_{i,j} (B) \int_{\sigma(N_1)} [p_1^{-1} f_{1i}(\omega)^* f_{1j}(\omega) |i_a\rangle \langle j_a| x_A \otimes [X_1^\dagger D(\omega) X_1 x_B d\mu_\omega].
\]
Take an orthonormal basis \( \{|j_a\}\) of \( H_B \). For any \( n \), let \( P_n \) be the \( n \)-rank projection onto \( \text{span}\{i_a\}_{i=1}^n \) and \( Q_n \) be the \( n \)-rank projection onto \( \text{span}\{|j_b\\}_{j=1}^n \). Then

\[
(P_n \otimes Q_n) \rho_1 (P_n \otimes Q_n) (x_A \otimes x_B)
\]

\[
= (B) \int_{\sigma(N_1)} [p_1^{-1} \sum_{i,j=1}^n f_{i1}(\omega)^* f_{1j}(\omega)|i_a\langle j_a|] \otimes [Q_n X_1^d(\omega) X_1 Q_n] d\mu(\omega) (x_A \otimes x_B)
\]

holds for any \( x_A \otimes x_B \in H_A \otimes H_B \), which entails that

\[
(P_n \otimes Q_n) \rho_1 (P_n \otimes Q_n)
\]

\[
= (B) \int_{\sigma(N_1)} [p_1^{-1} \sum_{i,j=1}^n f_{i1}(\omega)^* f_{1j}(\omega)|i_a\langle j_a|] \otimes [Q_n X_1^d(\omega) X_1 Q_n] d\mu(\omega).
\]

Let \( A_n(\omega) = \sum_{i,j=1}^n p_1^{-1} f_{i1}(\omega)^* f_{1j}(\omega)|i_a\langle j_a| \) and \( B_n(\omega) = Q_n X_1^d(\omega) X_1 Q_n \). It is easily seen that \( A_n(\omega) \) is a rank one positive operator on \( H_A \) for each \( \omega \in \sigma(N_1) \). As \( B_n(\omega) \) is bounded and satisfies \( \langle x_B| B_n(\omega)|x_B \rangle \geq 0 \) for any \( x_B \in H_n \), \( B_n(\omega) \) must be a positive operator. Therefore,

\[
\sigma_n = \frac{1}{\text{Tr}((P_n \otimes Q_n) \rho_1 (P_n \otimes Q_n))} (P_n \otimes Q_n) \rho_1 (P_n \otimes Q_n)
\]

\[
= \frac{1}{\text{Tr}((P_n \otimes Q_n) \rho_1 (P_n \otimes Q_n))} (B) \int_{\sigma(N_1)} A_n(\omega) \otimes B_n(\omega) d\mu(\omega)
\]

is a separable state. Since \( \rho_1 = \text{SOT-lim}_{n \to \infty} (P_n \otimes Q_n) \rho_1 (P_n \otimes Q_n) \), we see that \( \rho_1 \) is SOT-separable. Then the proposition A.2 ensures that \( \rho_1 \) is a separable state, as desired.

Similarly, one can check that \( \rho_k \) is separable for each \( k, k \geq 1 \). Hence, we see that \( \rho \) is a separable state, as desired.