Committees providing EJR can be computed efficiently

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Abstract

We identify a whole family of approval-based multi-winner voting rules that satisfy PJR. Moreover, we identify a subfamily of voting rules within this family that satisfy EJR. All these voting rules can be computed in polynomial time as long as the subalgorithms that characterize each rule within the family are polynomial. One of the voting rules that satisfy EJR can be computed in $O(nmk)$.

1 Introduction

The use of axioms in social choice theory dates back to the works of Arrow [11]. When the goal of the election is to select two or more candidates (i.e., a committee), one important type of axioms are those that try to identify which requirements must comply a representative set of winners.

In the context of approval-based multi-winner elections this topic has been addressed by Aziz et al. in [1] (the journal paper version is [2]). They proposed two axioms to capture the idea of representation: justified representation (JR) and extended justified representation (EJR). JR establishes requirements on when a set of voters deserves a representative while EJR establishes requirements on when a set of voters deserves several representatives. Aziz et al. analysed with these axioms several well-known multi-winner voting rules. The only voting rule that they found that satisfies EJR is the Proportional Approval Voting (PAV). Unfortunately, Aziz et al. [3] and Skowron et al. [10] have proved that computing PAV is NP-complete.
No voting rule computable in polynomial time that satisfies EJR has been found so far.

Sánchez-Fernández et al. proposed in [8] a relaxation of EJR that they called proportional justified representation (PJR). They show that the Greedy Monroe rule (which can be computed in polynomial time) satisfies PJR if the target committee size divides the total number of votes. Very soon, Brill et al. [4] and Sánchez-Fernández et al. [9] identified two voting rules that satisfy PJR in all cases and can be computed in polynomial time.

In this paper we identify a whole family of approval-based multi-winner voting rules that satisfy PJR. Moreover, we identify a subfamily of voting rules within this family that satisfy EJR. All these voting rules can be computed in polynomial time as long as the subalgorithms that characterize each rule within the family are polynomial.

2 Notation and preliminaries

We consider an approval-based multi-winner election with a set of voters \( N = \{1, \ldots, n\} \) and a set of candidates \( C = \{c_1, \ldots, c_m\} \). Each voter \( i \in N \) submits an approval ballot \( A_i \subseteq C \), which represents the subset of candidates that she approves of. We refer to the list \( \mathcal{A} = (A_1, \ldots, A_n) \) of approval ballots as the ballot profile. We will consider approval-based multi-winner voting rules that take as input \( (N, C, \mathcal{A}, k) \), where \( k \) is a positive integer that satisfies \( k \leq |C| \), and return a subset \( W \subseteq C \) of size \( k \), which we call the winning set. We omit \( N \) and \( C \) from the notation when they are clear from the context. We say that the exact quota \( q \) is equal to \( n/k \).

For each candidate \( c \in C \) we refer to \( N_c \) as the set of voters that approve \( c \) (\( N_c = \{i \in N : c \in A_i\} \)) and to \( n_c \) as the number of voters that approve \( c \) (\( n_c = |N_c| \)).

The maximum of a finite and non-empty set \( A \) of real numbers (represented as \( \max A \)) is the element of \( A \) strictly greater than all other elements of \( A \) (we assume that sets cannot contain duplicates). By convention we say that \( \max \emptyset = 0 \).

**Definition 1. Extended/proportional justified representation** Consider a ballot profile \( \mathcal{A} = (A_1, \ldots, A_n) \) over a candidate set \( C \) and a target committee size \( k, k \leq |C| \). Given a positive integer \( \ell \in \{1, \ldots, k\} \), we say that a set of voters \( N^* \subseteq N \) is \( \ell \)-cohesive if \( |N^*| \geq \ell \frac{n}{k} \) and \( \bigcap_{i \in N^*} A_i \geq \ell \). We say that a set of candidates \( W, |W| = k \), provides extended justified representation (EJR) (respectively, proportional justified representation (PJR)) for \( (\mathcal{A}, k) \) if for every \( \ell \in \{1, \ldots, k\} \) and every \( \ell \)-cohesive set of voters \( N^* \subseteq N \) it holds that exists a voter \( i \) in \( N^* \) such that \( |A_i \cap W| \geq \ell \) (respectively,
$|W \cap (\bigcup_{i \in N^*} A_i)| \geq \ell$. We say that an approval-based voting rule satisfies extended justified representation (EJR) (respectively, proportional justified representation (PJR)) if for every ballot profile $A$ and every target committee size $k$ it outputs a committee that provides EJR (respectively, PJR) for $(A, k)$.

3 Intuition

Aziz et al. [2] proposed a rule, that they called HareAV, that can be seen as an extension of the largest remainders apportionment method (with Hare quota) to approval-based multi-winner elections. HareAV is an iterative method in which at each step the most approved candidate that has not yet been elected is added to the set of winners, and at most $\lceil \frac{n}{k} \rceil$ of the votes that approve the elected candidate are also removed from the election (when the number of votes that approve the candidate is less than or equal to $\lceil \frac{n}{k} \rceil$, all such votes are removed from the election). Which particular votes are removed is left open.

Aziz et al. discuss in [2] whether HareAV satisfies EJR or not. They show an example in which HareAV fails EJR for some way of breaking intermediate ties. However, they were unable to construct an example where HareAV fails EJR for all ways of breaking intermediate ties. Based on this they say: “we now conjecture that it is always possible to break intermediate ties in HareAV so as to satisfy EJR”. Unfortunately, there are elections for which HareAV fails EJR regardless of the tie-breaking rule, as the following example, taken from Sánchez-Fernández et al. [8] shows:

Example 1. Let $n = 10$, $k = 7$, $C = \{c_1, \ldots, c_8\}$. Suppose that $A_i = \{c_i\}$ for $i = 1, \ldots, 4$ and $A_i = \{c_5, c_6, c_7, c_8\}$ for $i = 5, \ldots, 10$. Let $\ell = 4$. Then $\ell \cdot \frac{n}{k} = \frac{40}{7} < 6$, so the set of voters $\{5, 6, 7, 8, 9, 10\}$ form a 4-cohesive group of voters. However, under HareAV only three candidates from $\{c_5, c_6, c_7, c_8\}$ will be selected, regardless of the tie-breaking rule used.

Despite this negative result, the underlying idea of removing part of the votes in successive iterations offers great flexibility: many different ways of choosing which votes are removed at each iteration can be conceived. Based on this, we decided to explore the possibility of tweaking HareAV in pursuit of a rule that satisfies EJR and can be computed in polynomial time.

3.1 First ideas

The set of winners selected by HareAV in the above example fails to provide EJR because at each iteration too many votes are removed. In this example
$\left\lfloor \frac{n}{k} \right\rfloor = \left\lfloor \frac{120}{18} \right\rfloor = 2$. Therefore, after 3 iterations, there are $6 - 3 \times 2 = 0$ votes left that approve \{c_5, c_6, c_7, c_8\}. If instead we remove exactly $\frac{10}{7}$ votes at each iteration, after 3 iterations there would be $6 - 3 \times \frac{10}{7} = 12/7$ votes left that approve \{c_5, c_6, c_7, c_8\}. This is more than the number of votes received by any of the other candidates. Therefore, if we remove $\frac{10}{7}$ at each iteration, all the candidates in \{c_5, c_6, c_7, c_8\} would be elected. The problem, of course, is that $\frac{10}{7}$ is not an integer. To solve this we will allow to remove a fraction of one vote. For each voter $i$ we define $f_j^i$ as the fraction of the vote of voter $i$ that has not been removed from the election after $j$ candidates have been added to the set of winners. For each voter $i$ we define $f_0^i = 1$. At all time, it must be $0 \leq f_j^i \leq 1$.

One advantage of removing fractions of votes instead of whole votes is that it allows to treat all the votes that approve the elected candidate evenly: we do not need to ask ourselves which votes must be removed and which ones must not. We can, for instance, scale down all the fractions of the votes that approve the elected candidate by the same factor, so the remaining number of votes is reduced by $q = \frac{n}{k}$ units. In fact, this rule is not new. According to Svante Janson [6], it was proposed by the Swedish mathematicians Lars Edvard Phragmén and/or Gustaf Eneström in the 19th century (it seems that the original authorship is not absolutely clear). We will refer to this rule as Phragmén-STV. Phragmén-STV is formally defined in algorithm 1.

Unfortunately, this rule does not satisfy EJR. Here is one example in which this rule fails EJR:

**Example 2.** Let $n/k = 120$, $k = 18$, $C = \{c_1, \ldots, c_{21}\}$. Voters cast the following votes:

- 120 voters approve \{c_1, c_2, c_3\}
- 120 voters approve \{c_1, c_2, c_6\}
- 122 voters approve \{c_5, c_7\}
- 70 voters approve \{c_3, c_4, c_6\}
- 50 voters approve \{c_3, c_4\}
- 120 voters approve \{c_3, c_4, c_8\}
- 121 voters approve \{c_8, c_9\}
- 52 voters approve \{c_7\}
- 65 voters approve \{c_9\}
Algorithm 1 phragmén-STV

Input: an approval-based multi-winner election \((N, C, A, k)\)

Output: the set of winners \(W\)

1: \(q \leftarrow \frac{n}{k}\)
2: \(W \leftarrow \emptyset\)
3: for \(j = 1\) to \(k\) do
4:   \(\text{foreach } c \in C \setminus W \) do
5:      \(s_c^j \leftarrow \sum_{i \in A_i} f_i^{j-1}\)
6:   end foreach
7:   \(w \leftarrow \operatorname{argmax}_{c \in C \setminus W} s_c^j\)
8:   \(W \leftarrow W \cup \{ w \}\)
9: for \(i \in N \setminus N_w\) do
10:   \(f_i^j \leftarrow f_i^{j-1}\)
11: end foreach
12: for \(i \in N_w\) do
13:   if \(s_w^j \leq q\) then
14:      \(f_i^j \leftarrow 0\)
15:   else
16:      \(f_i^j \leftarrow f_i^{j-1} \frac{s_i^j - q}{s_w^j}\)
17: end if
18: end foreach
19: end for
20: return \(W\)

- For \(i = 1, \ldots, 12\), 110 voters approve \(\{c_9, i\}\).

For this example the set of winners selected by phragmén-STV is one of \(c_3\) or \(c_4\) plus \(\{c_5, \ldots, c_{21}\}\). However, the 240 voters that approve \(\{c_1, c_2, c_5\}\) or \(\{c_1, c_2, c_6\}\) form a 2-cohesive group of voters but none of these voters approves 2 of the winners.

In the next section we will examine this example in detail to understand how to tweak further HareAV to make it satisfy EJR. Observe, however, that it is very easy to prove that phragmén-STV satisfies PJR. Suppose that \(N^*\) is an \(\ell\)-cohesive group of voters (that is, \(|N^*| \geq \ell \frac{n}{k}\) and \(|\cap_{i \in N^*} A_i| \geq \ell\)). At each iteration in which a candidate that is approved by some voters in \(N^*\) is elected at most \(q = \frac{n}{k}\) votes from \(N^*\) are removed. So, after \(\ell - 1\) candidates that are approved by some voters in \(N^*\) have been selected, at least \(\ell \frac{n}{k} - (\ell - 1) \frac{n}{k} = \frac{n}{k}\) votes from \(N^*\) remain in the election, and there exists at least one candidate that is approved by all such votes. But it is impossible
that after all the $k$ candidates have been selected a group of $\frac{n}{k}$ votes that approve a common candidate still remains in the election, so we know that at least one more candidate approved by some voter in $N^*$ must be elected.

In fact, as long as we remove at most $q = \frac{n}{k}$ votes at each iteration (and when the number of remaining votes that approve the selected candidate is less than or equal to $q$ all such votes are removed from the election), and we select always a candidate that is approved by at least $q$ of the votes remaining when the candidate is selected whenever such candidate exists, the rule will satisfy PJR. This motivates the following definition.

**Definition 2.** We say that a voting rule belongs to the PJR-Exact family (this family of voting rules is characterized by the use of the exact quota as the (maximum) amount of votes removed from the election at each iteration) when the following conditions hold:

1. The voting rule consists of an iterative algorithm in which at each iteration one candidate is added to the set of winners $W$.

2. At each iteration fractions of the votes are removed from the election. For each voter $i$ we define $f^j_i$ as the fraction of the vote of voter $i$ that has not been removed from the election after $j$ candidates have been added to the set of winners. For each voter $i$ we define $f^0_i = 1$. At all time, it must be $0 \leq f^j_i \leq 1$, for all $i, j$. For each voter $i$ and for each $j = 0, \ldots, k - 1$, it must be $f^{j+1}_i \leq f^j_i$.

   If at certain iteration $j$ the number of remaining (fractions of the) votes that approve the selected candidate $c$ is greater than $q = \frac{n}{k}$, then only $q$ of such votes are removed from the election (that is, $\sum_{c \in A_i}(f^j_i - f^{j-1}_i) = q$). If at certain iteration $j$ the number of remaining (fractions of the) votes that approve the selected candidate $c$ is less than or equal to $q$, then all such votes are removed from the election (that is, for all $i$ such that $c \in A_i$, it is $f^j_i = 0$).

   For each voter $i$ that does not approve the candidate $c$ selected at iteration $j$, it is $f^j_i = f^{j-1}_i$.

3. At each iteration $j$, if exists at least one not yet elected candidate $c$ such that it is approved by at least $q$ (fractions of the) remaining votes, no candidate that is approved by less than $q$ votes can be selected.

**Theorem 1.** All the voting rules that belong to the PJR-Exact family satisfy PJR.

Then, the next question is: can we find a subfamily of voting rules in PJR-Exact that satisfy EJR? We address this question in the next section.
3.2 Next ideas

First of all, let’s analyze in detail what happens in the first three iterations of Phragmén-STV for example 2.

**Iteration 1**  The first candidate selected is the most approved. Such candidate is $c_5$, that is approved by 242 voters. For each voter that approves $c_5$ (that is, the voters that approve $\{c_1, c_2, c_5\}$ and the voters that approve $\{c_5, c_7\}$) it is $f_i^1 = f_i^0 \frac{242 - 120}{242} = \frac{122}{242}$. For all the other voters it is $f_i^1 = 1$.

To analyze what has happened in this iteration, let’s introduce first some ideas. First of all, for each not yet elected candidate $c_i$, we can establish an upper bound on the possible values of $\ell$, for which an $\ell$-cohesive group of voters can exist such that all the members in the group approve $c_i$. Let’s call such bound the *dissatisfaction level* of such candidate. One possible way (although a not accurate enough) of defining such bound is $\lceil \frac{kn}{c_n} \rceil$.

For instance, we can establish the following bounds: $2 = \lceil \frac{120 + 120}{120} \rceil$ for $c_1$ and $c_2$, $2 = \lceil \frac{120 + 122}{120} \rceil$ for $c_5$ and $1 = \lceil \frac{122 + 52}{120} \rceil$ for $c_7$.

Then, for each voter $i$ we can also establish an upper bound on the possible values of $\ell$, for which voter $i$ can be a member of a $\ell$-cohesive group of voters. Let’s call this again the *dissatisfaction level* of the voters. One possible way (although again a not accurate enough) of defining such bound is the maximum value of the dissatisfaction levels of the candidates in $A_i$. To reach that bound would require that the maximum dissatisfaction level $\ell$ of a candidate in $A_i$ is reached by $\ell$ of such candidates and that all that $\ell$ candidates are (mostly) approved by the same voters, so this is a rather pessimistic bound.

Taking into account these ideas, at iteration 1 things seem to be going well. Observe that for all the voters that approve $\{c_1, c_2, c_5\}$ and all the voters that approve $\{c_5, c_7\}$, the dissatisfaction level is clearly less than or equal to 2 (because all of $c_1$, $c_2$, $c_5$ and $c_7$ have a dissatisfaction level smaller than or equal to 2). The EJR axiom requires that certain voter in an $\ell$-cohesive group approves $\ell$ winners. If a voter $i$ has a dissatisfaction level of $x$, we can think that it would be desirable that for each candidate approved by $i$ that is added to the set of winners no more than $\frac{1}{x}$ units of the vote of $i$ are removed from the election. This would allow that $x$ candidates approved by $i$ could be added to the set of winners before the entire vote of $i$ is exhausted. This is the case of iteration 1, because we remove $\frac{120}{242}$ from each voter that approves $c_5$, which is less that $\frac{1}{2}$, and the dissatisfaction level of the voters that approve $c_5$ is at most 2.

Observe, however, that we can be more accurate in the computation of the dissatisfaction levels. First of all, we observe that the dissatisfaction level
of a candidate can decrease during the execution of a rule. Suppose that the
dissatisfaction level of certain candidate $c$ is initially $\ell$. Suppose that after
certain number of iterations some voters that approve $c$ have already at least $\ell$
of their approved candidates in the set of winners. In that situation we cannot
expect that such voters contribute with $\frac{1}{\ell}$ units of vote when candidate $c$ is
elected, so we must take this into account when computing the dissatisfaction
levels.

An example of this is the situation of candidate $c_7$ after iteration 1. Since
$122 + 52 = 174 > 120$ voters approve $c_7$ we must expect that the initial
dissatisfaction level of candidate $c_7$ must be 1. However, after candidate $c_5$
is elected at iteration 1 the 122 voters that approve $\{c_5, c_7\}$ already approve
one candidate in the set of winners. For such voters $f_i^1 = \frac{122}{242} < \frac{1}{\ell}$ (here $\frac{1}{\ell}$ is
the inverse of the initial dissatisfaction level of candidate $c_7$). So we can not
count any more on these voters to compute the dissatisfaction level of $c_7$ (and
therefore the dissatisfaction level of $c_7$ after iteration 1 must be $\left\lfloor \frac{52}{120} \right\rfloor = 0$).
We observe also that after iteration 1 the remaining (fractions of) votes that
approve $c_7$ are $122 \frac{122}{242} + 52 \approx 113.5 < 120$, so it seems difficult to guarantee
that $c_7$ will be elected at a future iteration.

Observe also that the decrease of the dissatisfaction level of $c_7$ does not
look dangerous with respect to our goal of satisfying EJR. In fact, the voters
that approve $c_7$ constitute a $1$-cohesive group of voters. For such voters the
requirements imposed by EJR are satisfied with the election of candidate $c_5$
because the number of voters that do not approve at least one candidate
(that is, voters that approve only $c_7$) is 52 which is less that 120.

We observe also that when computing the dissatisfaction level of the
voters we do not need to take care of the dissatisfaction levels of candidates
that have already being added to the set of winners. Suppose that for voter $i$
after $j$ candidates have been added to the set of winners $W$, the maximum
dissatisfaction level of candidates in $A_i \cap W$ is $\ell$. If $|A_i \cap W| \geq \ell$, then
this voter satisfies the requirements imposed by EJR if she belongs to a $\ell$-
cohesive group of voters. If $|A_i \cap W| < \ell$ and the dissatisfaction levels of
all candidates in $A_i \setminus W$ are strictly smaller than $\ell$, then this voter cannot
belong to a $\ell$-cohesive group of voters.\footnote{It may happen that even if all the candidates in $A_i \setminus W$ have a dissatisfaction level
strictly smaller than $\ell$ some of them had initially a dissatisfaction level greater than or
equal to $\ell$. It is then possible that voter $i$ belongs to a $\ell$-cohesive group of voters, but
in such case, other voters in the cohesive group must have at least $\ell$ of their approved
candidates in the set of winners, and thus the requirements imposed by EJR would be
also satisfied.} In summary, in the computation of the dissatisfaction level of each voter $i$ we only need to take into account the candidates in $A_i \setminus W$ and the number of candidates
approved by voter $i$ that have already being added to the set of winners (that is, $|A_i \cap W|$).

The above discussion motivates the following definitions.

**Definition 3.** Consider a ballot profile $A = (A_1, \ldots, A_n)$ over a candidate set $C$ and a target committee size $k$, $k \leq |C|$. Given a set of candidates $W$, $|W| \leq k$, we will represent the dissatisfaction level of a candidate $c \in C \setminus W$ with respect to the set of candidates $W$ as $\ell(c, W)$ and its value is the highest non-negative integer such that $\ell(c, W) = \left\lfloor \frac{k}{n} \left| \{i : c \in A_i \land |A_i \cap W| < \ell(c, W)\} \right| \right\rfloor$.

When certain rule is used to obtain the set of winners for a given election, and the rule consists of an iterative algorithm in which at each iteration one candidate is added to the set of winners, we will use the notation $\ell_j(c)$ to refer to dissatisfaction level of a candidate $c$ with respect to the first $j$ ($j = 0, \ldots, k$) candidates selected by the rule. That is, for each $j = 0, \ldots, k$, if $W_j$ is the set of the first $j$ candidates added to the set of winners by the given rule ($W_0 = \emptyset$), and $c \in C \setminus W_j$, then $\ell_j(c) = \ell(c, W_j)$.

**Definition 4.** Consider a ballot profile $A = (A_1, \ldots, A_n)$ over a candidate set $C$ and a target committee size $k$, $k \leq |C|$. The set of winners $W$ is computed with certain rule that belongs to the PJR-Exact family. For $j = 0, \ldots, k - 1$, let $W_j$ be the set of the first $j$ candidates added by the rule to the set of winners ($W_0 = \emptyset$). For $j = 0, \ldots, k - 1$, for each candidate $c \in C \setminus W_j$ and for each voter $i$ such that $c \in A_i$, the dissatisfaction level $\ell_j(i, c)$ of voter $i$ if candidate $c$ is added to the set of winners in the next iteration is 

$$\ell_j(i, c) = \max_{c' \in A_i \setminus (W_j \cup \{c\})} \ell_j(c'),$$

and the minimum fraction of $i$'s vote that should be kept for other candidates if candidate $c$ is added to the set of winners in the next iteration is $g^j_i(c)$, where

$$g^j_i(c) = \begin{cases} 0 & \text{if } \ell_j(i, c) \leq |A_i \cap W_j| \\
\frac{\ell_j(i, c) - |A_i \cap W_j| - 1}{\ell_j(i, c)} & \text{if } \ell_j(i, c) > |A_i \cap W_j| \end{cases}$$

For $j = 0, \ldots, k - 1$, we say that a candidate $c \in C \setminus W_j$ is in a normal state after $j$ candidates have been added to the set of winners if the following conditions hold:

1. For each voter $i$ such that $c \in A_i$, if $\ell_j(i, c) > |A_i \cap W_j|$ then it has to be $f^j_i \geq \frac{\ell_j(i, c) - |A_i \cap W_j|}{\ell_j(i, c)}$, and

2. \[ \sum_{i : c \in A_i} (f^j_i - g^j_i(c)) \geq q. \]
Candidates in normal state are those candidates that can be safely added to the set of winners. First of all, for each voter \( i \) such that \( i \) approves the considered candidate \( c \) the dissatisfaction level of voter \( i \) is \( \ell_j(i, c) \) and this means that we wish that voter \( i \) will approve at least \( \ell_j(i, c) \) winners at the end of the election. We note that in the computation of \( \ell_j(i, c) \) we excluded candidate \( c \) because if \( c \) is added to the set of winners we would not need to consider the value of \( \ell_j(c) \) in the computation of the dissatisfaction level of voter \( i \). Then, the first condition states that for each voter \( i \) that approves \( c \) either such voter is already satisfied (this means that the number of already elected winners that the voter approves is at least \( \ell_j(i, c) \)) or the voter has enough fraction of vote left so as to be able to assign \( \frac{1}{\ell_j(i, c)} \) to each of the additional winners that such voter expects to approve in future. This is the total number of winners that the voter wish to approve \( (\ell_j(i, c)) \) minus the number of already elected winners that the voter approves \( (|A_i \cap W_j|) \).

The second condition establishes that if candidate \( c \) is added to the set of winners it will be possible to remove \( q \) votes from the voters that approve \( c \) in such a way that each voter that needs to approve additional winners could still assign \( \frac{1}{\ell_j(i, c)} \) to each of such winners.

We observe that in example 2 candidate \( c_5 \) is in normal state before iteration 1. For the voters that approve \( \{c_1, c_2, c_5\} \) we have \( \ell_0(i, c_5) = \ell_0(c_1) = \ell_0(c_2) = 2 \) and for the voters that approve \( \{c_5, c_7\} \) we have \( \ell_0(i, c_5) = \ell_0(c_7) = 1 \). For the first condition, and for all voters that approve \( c_5 \) we have \( f^0_i = 1 \geq \frac{\ell_0(i, c_5) - |A_i \cap W_5|}{\ell_0(i, c_5)} = \frac{\ell_0(i, c_5) - 0}{\ell_0(i, c_5)} = 1 \). For the second condition we have \( \sum_{i \in A_i} (f_i^0 - g_i^0(c_5)) = 120(1 - \frac{1}{2}) + 122(1 - 0) \geq 120 \). However, candidate \( c_7 \) is in normal state before iteration 1 because for candidate \( c_7 \) we have \( \sum_{i \in A_i} (f_i^0 - g_i^0(c_7)) = 122(1 - \frac{1}{2}) + 52(1 - 0) = 113 \leq 120 \).

The following lemma relates the concept of dissatisfaction level with EJR.

**Lemma 1.** Consider a ballot profile \( A = (A_1, \ldots, A_n) \) over a candidate set \( C \) and a target committee size \( k, k \leq |C| \). Suppose that for certain candidate set \( W, |W| = k \) it holds that the dissatisfaction level of each candidate \( c \in C \setminus W \) is 0. Then \( W \) provides EJR.

**Proof.** For the sake of contradiction suppose that for certain ballot profile \( A = (A_1, \ldots, A_n) \) over a candidate set \( C \) and certain target committee size \( k, k \leq |C| \), for certain candidate set \( W, |W| = k \) it holds that for each candidate \( c \in C \setminus W \) it is \( \ell(c, W) = 0 \), but \( W \) does not provide EJR.

Then there exists a set of voters \( N^* \subseteq N \) and a positive integer \( \ell \) such that \( |N^*| \geq \ell |C| \) and \( \bigcap_{i \in N^*} A_i \geq \ell \) but \( |A_i \cap W| < \ell \) for each \( i \in N^* \). Moreover, a candidate must exist such that it is approved by all the voters in \( N^* \) but she is not in \( W \). Let \( c^* \) be such candidate. Clearly, \( N^* \subseteq \{ i : c^* \in \)
\[ A_i \land |A_i \cap W| < \ell \}, \text{ and therefore, } \left| \frac{1}{n} \{ i : c^* \in A_i \land |A_i \cap W| < \ell \} \right| \geq \ell. \text{ This implies that } \ell(c^*, W) \geq \ell > 0, \text{ a contradiction.} \]

\[ \Box \]

Let’s see now what happens in the next two iterations.

**Iteration 2** The second candidate selected is \( c_8 \), that is approved by 241 of the remaining votes. For each voter that approves \( c_8 \) (that is, the voters that approve \( \{ c_3, c_4, c_8 \} \) and the voters that approve \( \{ c_8, c_9 \} \)) it is \( f_i^2 = \frac{241 - 120}{241} f_i^1 = \frac{121}{241} \).

Candidate \( c_8 \) is in normal state before iteration 2. For the voters that approve \( \{ c_3, c_4, c_8 \} \) we have \( \ell_1(i, c_8) = \ell_1(c_3) = \ell_1(c_4) = 2 \) and for the voters that approve \( \{ c_8, c_9 \} \) we have \( \ell_1(i, c_8) = \ell_1(c_8) = 1 \). For the first condition, and for all voters that approve \( c_8 \) we have \( f_i^1 = 1 \geq \frac{\ell_1(i, c_8) - |A_i \cap W|}{\ell_1(i, c_8)} = \frac{\ell_1(i, c_8) - 0}{\ell_1(i, c_8)} = 1 \). For the second condition we have \( \sum_{i : c_8 \in A_i} (f_i^1 - g_i^1(c_8)) = 120(1 - \frac{1}{2}) + 121(1 - 0) \geq 120. \)

Things continue to look good.

**Iteration 3** The third candidate selected is \( c_6 \), that is approved by 190 of the remaining votes. For each voter that approves \( c_6 \) (that is, the voters that approve \( \{ c_1, c_2, c_6 \} \) and the voters that approve \( \{ c_3, c_4, c_6 \} \)) it is \( f_i^3 = \frac{190 - 120}{190} f_i^2 = \frac{70}{190} \).

Candidate \( c_6 \) is not in normal state before iteration 3. For the voters that approve \( \{ c_1, c_2, c_6 \} \) we have \( \ell_2(i, c_6) = \ell_2(c_1) = \ell_2(c_2) = 2 \) and for the voters that approve \( \{ c_3, c_4, c_6 \} \) we have \( \ell_2(i, c_6) = \ell_2(c_3) = \ell_2(c_4) = 2 \). Candidate \( c_6 \) is not in normal state because \( \sum_{i : c_6 \in A_i} (f_i^2 - g_i^2(c_6)) = 120(1 - \frac{1}{2}) + 70(1 - \frac{1}{2}) = 95 < 120. \) Moreover, we observe that the election of candidate \( c_6 \) causes a big harm to candidates \( c_1 \) and \( c_2 \). Voters that approve \( c_1 \) and \( c_2 \) form a 2-cohesive group of voters. After iteration 3, all such voters approve only one of the candidates in the set of winners. This means that to satisfy EJR we need that an additional candidate approved by such voters would have to be added to the set of winners (this has to be one of \( c_1 \) or \( c_2 \)). However, after iteration 3 the remaining votes that approve \( c_1 \) and \( c_2 \) are \( 120 \frac{122}{242} + 120 \frac{70}{190} \approx 104,7 < 120, \) and therefore it seems difficult to guarantee that one of \( c_1 \) or \( c_2 \) will be added to the set of winners (in fact they will never be added to the set of winners). We observe that if the fraction of vote that was removed from the voters that approve \( \{ c_1, c_2, c_6 \} \) were less than or equal to \( \frac{1}{2} \) more than 120 voters that approve \( c_1 \) and \( c_2 \) would remain in the election, and therefore we would be in the safe side.

The situation that has happened in iteration 3 is captured by the following definitions.
Definition 5. Consider a ballot profile $\mathcal{A} = (A_1, \ldots, A_n)$ over a candidate set $C$ and a target committee size $k$, $k \leq |C|$. The set of winners $W$ is computed with certain rule that belongs to the PJR-Exact family. For $j = 0, \ldots, k$, let $W_j$ be the set of the first $j$ candidates added by the rule to the set of winners ($W_0 = \emptyset$). For $j = 0, \ldots, k - 1$, for each candidate $c \in C \setminus W_j$ and for each voter $i$ such that $c \in A_i$ let $\ell_j(i, c) = \max_{c' \in A_i \setminus (W_j \cup \{c\})} \ell_j(c')$.

For $j = 0, \ldots, k - 1$, we say that a candidate $c \in C \setminus W_j$ is in a starving state after $j$ candidates have been added to the set of winners if for some voter $i$ such that $c \in A_i$ it is $\ell_j(i, c) > |A_i \cap W_j|$ but $f_j^i < \frac{\ell_j(i, c) - |A_i \cap W_j|}{\ell_j(i, c)}$.

For $j = 0, \ldots, k - 1$, we say that a candidate $c \in C \setminus W_j$ is in an insufficiently supported state after the execution of iteration 3 and we say that iteration $j$ is neither a normal iteration nor an insufficiently supported iteration.

Finally, for $j = 0, \ldots, k - 1$, we say that a candidate $c \in C \setminus W_j$ is in an insufficiently supported state after $j$ candidates have been added to the set of winners if it is not in a starving state and $\sum_{i \in A_i} f_i^j \geq q$ but $\sum_{i \in A_i} (f_i^j - g_i^j(c)) < q$, where

$$g_i^j(c) = \begin{cases} 0 & \text{if } \ell_j(i, c) \leq |A_i \cap W_j| \\ \frac{\ell_j(i, c) - |A_i \cap W_j| - 1}{\ell_j(i, c)} & \text{if } \ell_j(i, c) > |A_i \cap W_j| \end{cases}.$$

Definition 6. Consider a ballot profile $\mathcal{A} = (A_1, \ldots, A_n)$ over a candidate set $C$ and a target committee size $k$, $k \leq |C|$. The set of winners $W$ is computed with certain rule that belongs to the PJR-Exact family. For $j = 0, \ldots, k$, let $W_j$ be the set of the first $j$ candidates added by the rule to the set of winners ($W_0 = \emptyset$). For $j = 0, \ldots, k - 1$, for each candidate $c \in C \setminus W_j$ and for each voter $i$ such that $c \in A_i$ let $\ell_j(i, c) = \max_{c' \in A_i \setminus (W_j \cup \{c\})} \ell_j(c')$.

For $j = 0, \ldots, k$, we say that iteration $j$ is normal if the following conditions hold:

1. The candidate $c$ that is added to the set of winners is in a normal state.
2. For each voter $i$ such that $c \in A_i$ and $\ell_{j-1}(i, c) > |A_i \cap W_{j-1}|$, it must be $f_i^j \geq \frac{\ell_{j-1}(i, c) - |A_i \cap W_j|}{\ell_{j-1}(i, c)}$.

According to these definitions, we observe that for example candidate $c_6$ was in an eager state before the execution of iteration 3 and that candidates $c_1$ and $c_2$ are in a starving state after the execution of iteration 3.
We observe that candidates that after certain iteration are in an eager state do not need to last in such state forever. For the election in example 2 suppose that in the first iteration we select candidate $c_5$ but that we remove all the 120 votes from the voters that approve $\{c_5, c_7\}$; then, in the second iteration we add candidate $c_1$ to the set of winners and we remove all the 120 votes that approve $\{c_1, c_2, c_5\}$. Now candidate $c_6$ is in a normal state.

As we have already discussed, the selection of a candidate that is in an eager state at iteration 3 is closely related to the fact that the set of winners outputted by phragmén-STV for example 2 fails to provide EJR. To avoid this situation we are going to follow a very drastic but simple approach: we are going to avoid selecting candidates in an eager state as much as possible (and we are going to see soon that we can do much to avoid selecting candidates in an eager state).

Normal iterations have a number of nice properties that we are going to discuss now.

**Lemma 2.** Consider a ballot profile $A = (A_1, \ldots, A_n)$ over a candidate set $C$ and a target committee size $k, k \leq |C|$. The set of winners $W$ is computed with certain rule that belongs to the PJR-Exact family. For $j = 0, \ldots, k$, let $W_j$ be the set of the first $j$ candidates added by the rule to the set of winners ($W_0 = \emptyset$). For $j = 0, \ldots, k$, and for each voter $i$, let $\ell_j(i) = \max_{c \in A_i \setminus W_j} \ell_j(c)$.

Fix a value $j$ such that $1 \leq j \leq k$ and suppose that the first $j$ iterations of the rule are all normal. Then for each voter $i$ it holds that either $|A_i \cap W_j| \geq \ell_j(i)$ or $f^j_i \geq \ell_j(i) - |A_i \cap W_j| / \ell_j(i)$.

**Proof.** For $h = 0, \ldots, k-1$, for each candidate $c \in C \setminus W_h$ and for each voter $i$ such that $c \in A_i$ let $\ell_h(i, c) = \max_{c' \in A_i \setminus (W_h \cup \{c\})} \ell_h(c')$. Fix a voter $i \in N$. If no candidate approved by voter $i$ is added to the set of winners in the first $j$ iterations then the lemma trivially holds. Suppose that some candidates approved by voter $i$ have been added to the set of winners during the first $j$ iterations but $|A_i \cap W_j| < \ell_j(i)$. Let $r$ be the last of the first $j$ iterations in which a candidate approved by voter $i$ is added to the set of winners and let $c_r$ be such candidate. Then, $f^j_i = f^r_i$. We have already seen that the dissatisfaction levels of the voters are monotonically non-increasing, and therefore, it is $\ell_j(i) \leq \ell_{j-1}(i, c_r)$. Moreover, since iteration $r$ is normal we have that $f^r_i \geq \ell_{r-1}(i, c_r) - |A_i \cap W_{r-1}| / \ell_{r-1}(i, c_r)$. We have

\[ f^j_i \geq \frac{\ell_{r-1}(i, c_r) - |A_i \cap W_{r-1}|}{\ell_{r-1}(i, c_r)}. \]

\[ \text{And maybe antidemocratic.} \]
For each voter \( C \) rule no candidate in \( j \) then iterations of the rule are all normal. Then, for each candidate Clearly, for each candidate \( c \) \( \ell \) such that \( \ell_c \) and for each voter \( i \) such that \( \ell_i \) be the set of the first \( j \) candidates added by the rule to the set of winners \( W_0 = \emptyset \). Fix a value \( j \) such that \( 1 \leq j \leq k - 1 \) and suppose that the first \( j \) iterations of the rule are all normal. Then, after the first \( j \) iterations of the rule no candidate in \( C \setminus W_j \) can be in starving state.

**Proof.** For each voter \( i \), let \( \ell_j(i) = \max_{c \in A_i \setminus W_j} \ell_j(c) \). For each candidate \( c \in C \setminus W_j \) and for each voter \( i \) such that \( c \in A_i \), let \( \ell_j(i, c) = \max_{c' \in A_i \setminus W_j \cup \lbrace c \rbrace} \ell_j(c') \).

Clearly, for each candidate \( c \in C \setminus W_j \) and for each voter \( i \) such that \( c \in A_i \), it is \( \ell_j(i) \geq \ell_j(i, c) \). Moreover, by lemma 2 it is either \( |A_i \cap W_j| = \ell_j(i) \) or \( f_i^j \geq \frac{\ell_j(i) - |A_i \cap W_j|}{\ell_j(i)} \). Therefore, if for certain voter \( i \) it is \( \ell_j(i, c) > |A_i \cap W_j| \), then \( f_i^j \geq \frac{\ell_j(i) - |A_i \cap W_j|}{\ell_j(i)} \).

**Corollary 1.** Consider a ballot profile \( A = (A_1, \ldots, A_n) \) over a candidate set \( C \) and a target committee size \( k \), \( k \leq |C| \). The set of winners \( W \) is computed with certain rule that belongs to the PJR-Exact family. For \( j = 0, \ldots, k \), let \( W_j \) be the set of the first \( j \) candidates added by the rule to the set of winners \( W_0 = \emptyset \). Fix a value \( j \) such that \( 1 \leq j \leq k - 1 \) and suppose that the first \( j \) iterations of the rule are all normal. Then, after the first \( j \) iterations of the rule no candidate in \( C \setminus W_j \) can be in starving state.

**Lemma 3.** Consider a ballot profile \( A = (A_1, \ldots, A_n) \) over a candidate set \( C \) and a target committee size \( k \), \( k \leq |C| \). The set of winners \( W \) is computed with certain rule that belongs to the PJR-Exact family. For \( j = 0, \ldots, k \), let \( W_j \) be the set of the first \( j \) candidates added by the rule to the set of winners \( W_0 = \emptyset \). Fix a value \( j \) such that \( 1 \leq j \leq k \) and suppose that the first \( j \) iterations of the rule are all normal. Then, for each candidate \( c \in C \setminus W_j \) such that \( \ell_j(c) \geq 1 \), it holds that \( \sum_{i \in A_i \cap A_i \setminus W_j} \ell_j \left( \frac{f_i^j - \ell_j(c) - |A_i \cap W_j| - 1}{\ell_j(c)} \right) \geq q \).

**Proof.** For each voter \( i \), let \( \ell_j(i) = \max_{c \in A_i \setminus W_j} \ell_j(c) \). Clearly, for each voter \( i \) such that \( c \in A_i \) it is \( \ell_j(i) \geq \ell_j(c) \). By lemma 2 for each voter \( i \) such that \( |A_i \cap W_j| < \ell_j(i) \) it is \( f_i^j \geq \frac{\ell_j(i) - |A_i \cap W_j|}{\ell_j(i)} \). Definition implies that \( \left\{ i : c \in A_i \setminus |A_i \cap W_j| < \ell_j(c) \right\} \geq \ell_j(c) \frac{q}{\ell_j(i)} \). We have

\[
f_i^j = f_i^r \geq \frac{\ell_{r-1}(i, c_r) - |A_i \cap W_r|}{\ell_{r-1}(i, c_r)} = \frac{\ell_{r-1}(i, c_r) - |A_i \cap W_j|}{\ell_{r-1}(i, c_r)} \geq \frac{\ell_j(i) - |A_i \cap W_j|}{\ell_j(i)} \]

\[\square\]
Theorem 2. Consider a ballot profile \( \mathcal{A} = (A_1, \ldots, A_n) \) over a candidate set \( C \) and a target committee size \( k \), \( k \leq |C| \). The set of winners \( W \) is computed with certain rule that belongs to the PJR-Exact family. For \( j = 0, \ldots, k \), let \( W_j \) be the set of the first \( j \) candidates added by the rule to the set of winners \( W_0 = \emptyset \). Fix a value \( j \) such that \( 0 \leq j \leq k - 1 \) and suppose that the first \( j \) iterations of the rule are all normal \( (j = 0 \text{ means that no iteration has been executed yet}) \). Suppose that after the first \( j \) iterations certain candidate \( c_1 \in C \setminus W_j \) is in an eager state. Then, there exists other candidate \( c_2 \in C \setminus W_j \) that is in a normal state.

Proof. For each candidate \( c \in C \setminus W_j \) and for each voter \( i \) such that \( c \in A_i \) let \( \ell_j(i, c) = \max_{c' \in A_i \setminus (W_j \cup \{c\})} \ell_j(c') \). Also, for each candidate \( c \in C \setminus W_j \) and each voter \( i \) such that \( c \in A_i \) let \( g_i^j(c) \) be

\[
g_i^j(c) = \begin{cases} 0 & \text{if } \ell_j(i, c) \leq |A_i \cap W_j| \\ \frac{\ell_j(i, c) - |A_i \cap W_j| - 1}{\ell_j(i,c)} & \text{if } \ell_j(i, c) > |A_i \cap W_j| \end{cases}
\]

First of all, we observe that by lemma 2 all the candidates \( c \in C \setminus W_j \) satisfy the first condition required to be in normal state, that is, for each voter \( i \) such that \( c \in A_i \) if \( \ell_j(i, c) > |A_i \cap W_j| \) it has to be \( f_i^j \geq \frac{\ell_j(i,c) - |A_i \cap W_j|}{\ell_j(i,c)} \).

Now, if candidate \( c_1 \) is in an eager state, this means that \( \sum_{i : c_1 \in A_i} f_i^j \geq q \) but \( \sum_{i : c_1 \in A_i} (f_i^j - g_i^j(c_1)) < q \). Suppose first that \( \ell_j(c_1) = 0 \). Since for each \( i \) such that \( c_1 \in A_i \) and \( \ell_j(i, c_1) = 0 \) it is \( g_i^j(c_1) = 0 \), there should be certain voter \( i_1 \) such that \( c_1 \in A_{i_1} \) and \( \ell_j(i_1, c_1) > 0 \) (otherwise \( \sum_{i : c_1 \in A_i} (f_i^j - g_i^j(c_1)) = \)).
\[ \sum_{i \in A_i} f_i^j \geq q, \text{ and } c_1 \text{ would be in a normal state}. \] If \( \ell_j(i_1, c_1) > 0 \) this means that there must exist a candidate \( c_3 \) in \( C \setminus W_j \) such that \( c_3 \in A_{i_1} \) and \( \ell_j(c_3) > 0 \).

If \( c_3 \) is in a normal state then the theorem holds. By lemma 3 it has to be \( \sum_{i \in A_i : A_i \cap W_j \subset \ell_j(c_3)} (f_i - \frac{\ell_j(c_3) - |A_i \cap W_j| - 1}{\ell_j(c_3)}) \geq q \) and therefore \( \sum_{i \in A_i} f_i^j \geq q \). Thus, \( c_3 \) cannot be in an insufficiently supported state. By corollary 1 \( c_3 \) cannot be in a starving state. Therefore, if \( c_3 \) is not in a normal state then \( c_3 \) has to be in an eager state. If \( c_3 \) is in an eager state then it is \( \sum_{i \in A_i} (f_i^j - g_i^j(c_3)) < q \). Combining this with lemma 3 we have \( \sum_{i \in A_i} (f_i^j - g_i^j(c_3)) - \sum_{i : A_i \cap W_j < \ell_j(c_3)} (f_i^j - \frac{\ell_j(c_3) - |A_i \cap W_j| - 1}{\ell_j(c_3)}) < 0 \). But

\[
\sum_{i \in A_i} (f_i^j - g_i^j(c_3)) - \sum_{i : A_i \cap W_j < \ell_j(c_3)} (f_i^j - \frac{\ell_j(c_3) - |A_i \cap W_j| - 1}{\ell_j(c_3)}) \\
\geq \sum_{i : A_i \cap W_j < \ell_j(c_3)} (f_i^j - g_i^j(c_3)) - (f_i^j - \frac{\ell_j(c_3) - |A_i \cap W_j| - 1}{\ell_j(c_3)}) \\
= \sum_{i : A_i \cap W_j < \ell_j(c_3)} (\frac{\ell_j(c_3) - |A_i \cap W_j| - 1}{\ell_j(c_3)} - g_i^j(c_3))
\]

This implies that for some voter \( i_3 \) such that \( c_3 \in A_{i_3} \) and \( |A_{i_3} \cap W_j| < \ell_j(c_3) \) it has to be \( \frac{\ell_j(c_3) - |A_{i_3} \cap W_j| - 1}{\ell_j(c_3)} - g_i^j(c_3) < 0 \). Since \( |A_{i_3} \cap W_j| < \ell_j(c_3) \), it is \( \frac{\ell_j(c_3) - |A_{i_3} \cap W_j| - 1}{\ell_j(c_3)} \geq 0 \). Therefore, \( g_i^j(c_3) > 0 \). It follows that \( \ell_j(i_3, c_3) > |A_{i_3} \cap W_j| \) and that \( g_i^j(c_3) = \frac{\ell_j(i_3, c_3) - |A_{i_3} \cap W_j| - 1}{\ell_j(i_3, c_3)} \). But \( \ell_j(i_3, c_3) = \frac{\ell_j(c_3) - |A_{i_3} \cap W_j| - 1}{\ell_j(c_3)} - \frac{\ell_j(i_3, c_3) - |A_{i_3} \cap W_j| - 1}{\ell_j(i_3, c_3)} < 0 \) implies that \( \ell_j(i_3, c_3) > \ell_j(c_3) \). Then, a candidate \( c_4 \in C \setminus W_j \) must exist such that \( c_4 \in A_{i_3} \) and \( \ell_j(c_4) > \ell_j(c_3) \).

We can then repeat for candidate \( c_4 \) and successively the reasoning we did for \( c_3 \). At each step either the candidate that we find is in normal state or there exists another candidate in \( C \setminus W_j \) with a strictly higher dissatisfaction level. We note, however, that dissatisfaction levels are bounded by \( k \), and therefore, at some point we have to find a candidate \( c_2 \) that is in a normal state.

The proof when \( \ell_j(c_1) > 0 \) is similar but we can start reasoning directly as we did with candidate \( c_3 \).

We have now identified our strategy for the \textit{EJR-Exact} family of voting rules: we will run normal iterations as much as possible. If at certain point no candidate is available that is in a normal state, this means that all the unelected candidates have to be in an insufficiently supported state. We
then simply run enough insufficiently supported iterations to complete the required number of candidates.

4 The EJR-Exact family of voting rules

The EJR-Exact family of voting rules is defined in algorithms 2 and 3.

Algorithm 2 The EJR-Exact family (part 1)
Given the algorithms Alg1, Alg2 and Alg3 that characterize each rule

| Input | an approval-based multi-winner election \((N, C, A, k)\) |
|-------|-----------------------------------------------|
| Output | the set of winners \(W\) |

1: \(W \leftarrow \emptyset\)
2: foreach \(c \in C\) do
3: \(\ell_0(c) \leftarrow \left\lfloor \frac{kn_c}{n} \right\rfloor\)
4: endforeach
5: for \(j = 1\) to \(k\) do
6: \(C^{st1} \leftarrow \emptyset\)
7: foreach \(c \in C \setminus W\) do
8: \(x_c \leftarrow 0\)
9: foreach \(i : c \in A_i\) do
10: \(\ell_{j-1}(i, c) \leftarrow \max_{c' \in A_i \setminus (W \cup \{c\})} \ell_{j-1}(c')\)
11: if \(\ell_{j-1}(i, c) > |A_i \cap W| + 1\) then
12: \(x_c \leftarrow x_c + f_{i}^{j-1} - \frac{\ell_{j-1}(i, c) - (|A_i \cap W| + 1)}{\ell_{j-1}(i, c)}\)
13: else
14: \(x_c \leftarrow x_c + f_{i}^{j-1}\)
15: end if
16: end foreach
17: if \(x_c \geq q\) then
18: \(C^{st1} \leftarrow C^{st1} \cup \{c\}\)
19: end if
20: end foreach

Each instance of EJR-Exact family is characterized by 3 algorithms: Alg1, Alg2 and Alg3. At each normal iteration first Alg1 is executed to select which of the candidates in a normal state will be added to the set of winners. Then Alg2 is executed to remove \(q\) votes from the voters that approve the selected candidate in such a way that the rules that must be followed in a normal
iteration are respected. Finally Alg\textsc{3} is executed to select the candidates that will be added to the set of winners in insufficiently supported iterations.

\begin{algorithm}[H]
\begin{algorithmic}[1]
\State if $C^{st1} \neq \emptyset$ then
\Comment{Stage 1: normal iteration}
\State Execute Alg\textsc{1} to select a candidate $w$ from $C^{st1}$
\State \hspace{1em} $f^j_i \leftarrow f^{j-1}_i$
\State \hspace{1em} end foreach
\State \hspace{1em} end while
\State $W \leftarrow W \cup \{w\}$
\State \hspace{1em} end foreach
\State \hspace{1em} end if
\State \hspace{1em} break
\State \hspace{1em} end if
\State \hspace{1em} end for
\State if $|W| < k$ then
\Comment{Stage 2: insufficiently supported iterations}
\State $j \leftarrow |W|$
\State Execute Alg\textsc{3} to add $k-j$ candidates from $C \setminus W$ to $W$
\State end if
\State return $W$
\end{algorithmic}
\end{algorithm}

The execution of a rule in the EJR-Exact family is as follows. First, for each candidate $c \in C$, its initial dissatisfaction level $\ell_0(c)$ is computed in lines 2–4 of algorithm \textsc{2}. Then, a loop is executed (at most) $k$ times to add candidates to the set of winners. At each iteration $j$ of the loop, first for each candidate $c \in C \setminus W$ we check if such candidate is in a normal state. The candidates that are in a normal state are stored in $C^{st1}$. This is done in lines 6–20 of algorithm \textsc{2}.

If $C^{st1}$ is not empty then we run a normal iteration (lines 22–34 of algorithm \textsc{3}). Algorithm Alg\textsc{1} is executed to select one candidate $w$ from $C^{st1}$ that will be added to the set of winners and then algorithm Alg\textsc{2} is executed.
to remove \( q \) votes from the voters that approve candidate \( w \). This has to be done in such a way that the rules established for normal iterations are respected (see definition 3). Finally, we update the dissatisfaction levels of the candidates.

If \( C_{st1} \) is empty then we exit the loop and, if necessary, run enough insufficiently supported iterations to complete \( W \) (lines 40–42 of algorithm 3).

Theorem 3. All the voting rules in the EJR-Exact family satisfy EJR.

Proof. It is very easy to prove that all the rules in the EJR-Exact family satisfy EJR. For the sake of contradiction suppose that for certain ballot profile \( A = (A_1, \ldots, A_n) \) over a candidate set \( C \) and certain target committee size \( k \), \( k \leq |C| \), the set of winners \( W \) that certain rule that belongs to the EJR-Exact family outputs does not provide EJR. Then there exists a set of voters \( N* \subseteq N \) and a positive integer \( \ell \) such that \( |N*| \geq \ell \) and \( \bigcap_{i \in N*} A_i \geq \ell \) but \( |A_i \cap W| < \ell \) for each \( i \in N* \). Moreover, a candidate must exist such that candidate is approved by all the voters in \( N* \) but she is not in the set of winners. Let \( c \) be such candidate. Observe that according to definition 3 for each iteration \( j \) it is \( \ell_j(c) \geq \ell \).

Suppose first that for that election and rule all the iterations are normal. Then, since at each iteration we remove \( q = \frac{n}{k} \) votes at the end of the election we have removed all the votes from the election. However, lemma 2 says that for each voter \( i \) in \( N* \) it must be \( f_i^k \geq \frac{\ell_i(c) - |A_i \cap W|}{\ell_j(c)} > 0 \), a contradiction.

Suppose then that for that election and rule we have some insufficiently supported iterations and let \( j \) be the first insufficiently supported iteration and \( W_{j-1} \) be the set of the first \( j - 1 \) candidates that have been added to the set of winners. We have seen that when we ran the first insufficiently supported iteration all the candidates are in an insufficiently supported state, that is, for each \( c' \in C \setminus W_{j-1} \) it is \( \sum_{i \in A_c} f_i^{j-1} < q \). However, lemma 3 says that for candidate \( c \) it is \( \sum_{i \in A_c} f_i^{j-1} \geq q \) which is again a contradiction. \( \Box \)

We establish now bounds in computational complexity of the instances of the EJR-Exact family in terms of the number of arithmetic operations. We note that to take into account the size of the operands implies to establish bounds on the number of bits required to represent each \( f_i^j \). While this cannot be done in general for all the instances of the EJR-Exact family, it can be easily done for particular instances.

Theorem 4. Suppose that for certain instance of EJR-Exact the number of arithmetic operations required to execute Alg1, Alg2 and Alg3 is bounded, respectively, by \( O(o_1) \), \( O(o_2) \) and \( O(o_3) \) (depending on the particular instance,
such bounds can depend on \( n, m, k, \) and other factors). Then, the number of arithmetic operations required to execute such instance in the worst case is bounded by \( O(nm^2k + k(o_1 + o_2) + o_3) \).

Proof. It is enough to review the cost of each part of the algorithm. We assume that the profile is stored in a table with a row per voter and a column per candidate. Each cell of the table is a bit that is set to 1 if the corresponding voter approves the corresponding candidate. We also assume that the value of \(|A_i \cap W|\) for each voter \( i \) is stored in an array of \( n \) counters (one for each voter). Initially, the value of all the counters is set to zero. Each time that a candidate \( w \) is added to the set of winners the counters corresponding to the voters that approve \( w \) are incremented. This can be done in \( O(n) \).

The initial dissatisfaction levels computed in lines 2–4 of algorithm 2 requires to compute \( n_c \) (this can be done in \( O(n) \)) for each candidate, so the total cost of computing the initial dissatisfaction levels is \( O(nm) \).

Then we ran at most \( k \) times a loop in which the following tasks are done:

1. First, for each candidate \( c \), we have to check if such candidate is in normal state (lines 7–20 of algorithm 2). To do that, for each voter \( i \) that approves the candidate under consideration we have to compute \( \ell_j(i, c) \). This depends on the number of candidates approved by voter \( i \), and therefore, it can be done in \( O(m) \) operations. In summary, lines 7–20 of algorithm 2 can be executed in \( O(nm^2) \).

2. In line 23 of algorithm 3 a candidate in normal state is selected in \( O(o_1) \).

3. In line 27 of algorithm 3 \( q \) votes are removed from the election in \( O(o_2) \).

4. In line 28 of algorithm 3 a candidate \( w \) is added to the set of winners. As we explained before, also the counters that store the value of \(|A_i \cap W|\) for the voters that approve \( w \) are incremented. This can be done in \( O(n) \).

5. In lines 29–34 of algorithm 3 the dissatisfaction levels are updated for each candidate. The while loop in lines 31–33 is executed at most \( k \) times, because the dissatisfaction levels are bounded by \( k \). Each time that we execute such while loop we need to compute \( \{|i : c \in A_i \land |A_i \cap W| < \ell_j(c)\}| \). This can be done in \( O(n) \) because we assume that the values of \(|A_i \cap W|\) and \( \ell_j(c) \) have already been computed and stored in memory. Therefore, the total computational cost of the execution of lines 29–34 is bounded by \( O(nmk) \).
Finally, the cost of executing line 42 of algorithm 3 is bounded by $O(o_3)$. Combining all these data we obtain a worst case bound of $O(nm^2k + nk^2 + k(o_1 + o_2) + o_3)$.

It is however possible to establish a better bound for the cost of the computation of the dissatisfaction levels. As we have already seen, the initial dissatisfaction level of each candidate is bounded by $k$. Based on this we have established that each time that the while loop in lines 31–33 of algorithm 3 is reached, such loop is executed at most $k$ times. This is because each time we enter such loop the dissatisfaction level of the candidate under consideration is decremented and the dissatisfaction levels cannot fall below 0.

However, a similar reasoning allows us to conclude that for each candidate the while loop in lines 31–33 of algorithm 3 can be entered at most $k$ times during the whole execution of an EJR-Exact rule. To see why we note that the dissatisfaction level of a candidate is never incremented, its initial value is at most $k$, its value cannot fall below 0 and each time that the while loop in lines 31–33 of algorithm 3 is entered the dissatisfaction level of the candidate under consideration is decremented by one unit.

We have to be careful here. The condition in line 31 of algorithm 3 is checked each time that the loop is reached. Therefore, for each candidate such condition is checked at most $2k$ times (once for each iteration of the outer loop from line 5 of algorithm 2 to line 38 of algorithm 3 plus at most $k$ additional times when the condition holds). The cost of the evaluation of the condition in line 31 of algorithm 3 is $O(n)$. The body of the while loop (line 32 of algorithm 3) is executed in constant time at most $k$ times for each candidate. Therefore, the total cost of updating the dissatisfaction levels of the candidates during an execution of an EJR-Exact rule is at most $O(2nmk + mk) = O(nmk)$ and the total cost of an execution of an EJR-Exact rule is at most $O(nm^2k + k(o_1 + o_2) + o_3)$.

A very natural instance of the EJR-Exact family is EJR-LR-Even, defined in algorithms 4, 5 and 6.

**Example 3.** The set of winners produced by EJR-LR-Even for the election shown in example 2 is (in this order) $c_5$, $c_8$, $c_1$ or $c_2$, $c_3$ or $c_4$, $c_{10}$, $\ldots$, $c_{21}$, $c_6$, and $c_9$.

We illustrate the operation of EJR-LR-Even with the first iteration for the election shown in example 2. The selected candidate is $c_5$. We have already seen that this candidate is initially in a normal state. $x_{c_5}$ stores the amount of vote that can be removed from the election while keeping $\ell_{j-1(i,c_5)}$ for additional candidates when necessary, and its value is $120(1 - \frac{1}{2}) + 122 = 182$. 21
Algorithm 4 EJR-LR-Even (part 1)

Input: an approval-based multi-winner election \((N, C, A, k)\)

Output: the set of winners \(W\)

1: \(W \leftarrow \emptyset\)
2: foreach \(c \in C\) do
3: \(\ell_0(c) \leftarrow \left\lfloor \frac{kn_c}{n} \right\rfloor\)
4: end foreach
5: for \(j = 1\) to \(k\) do
6: \(C^{\text{st1}} \leftarrow \emptyset\)
7: foreach \(c \in C \setminus W\) do
8: \(s_j^c \leftarrow \sum_{i : c \in A_i} f_i^{j-1}\)
9: \(x_c \leftarrow 0\)
10: foreach \(i : c \in A_i\) do
11: \(\ell_{j-1}(i, c) \leftarrow \max_{c' \in A_i \setminus (W \cup \{c\})} \ell_{j-1}(c')\)
12: if \(\ell_{j-1}(i, c) > |A_i \cap W| + 1\) then
13: \(x_c \leftarrow x_c + f_i^{j-1} - \frac{\ell_{j-1}(i, c) - (|A_i \cap W| + 1)}{\ell_{j-1}(i, c)}\)
14: else
15: \(x_c \leftarrow x_c + f_i^{j-1}\)
16: end if
17: end foreach
18: if \(x_c \geq q\) then
19: \(C^{\text{st1}} \leftarrow C^{\text{st1}} \cup \{c\}\)
20: end if
21: end foreach

An important difference between phargmén-STV and EJR-LR-Even is that in EJR-LR-Even we only scale down the fractions of votes that are contained in \(x_c\), and therefore the fractions of vote that remain in the election after iteration 1 are as follows: 
\[
f_i^1 = f_i^0 - \frac{q}{x_{c_5}} (f_i^0 - \frac{\ell_0(i, c_5) - (|A_i \cap W| + 1)}{\ell_0(i, c_5)}) = 1 - \frac{120}{182} (1 - \frac{2 - (0+1)}{2}) = \frac{122}{182}\]
for the voters that approve \(\{c_1, c_2, c_4\}\), and 
\[
f_i^1 = f_i^0 - \frac{2}{x_{c_5}} f_i^0 = 1 - \frac{120}{182} = \frac{62}{182}\]
for the voters that approve \(\{c_5, c_7\}\).

Lemma 4. The number of arithmetic operations required to compute EJR-LR-Even in the worst case is bounded by \(O(nm^2k)\).
Algorithm 5 EJR-LR-Even (part 2)

22: \textbf{if} $C^{st1} \neq \emptyset$ \textbf{then}
23:   \hspace{1em} $w \leftarrow \arg\max_{c \in C^{st1}} s^j_c$ \hspace{1em} $\triangleright$ \text{Stage 1: normal iteration}
24:   \hspace{1em} \textbf{foreach} $i \in N \setminus N_w$ \textbf{do}
25:     \hspace{2em} $f^j_i \leftarrow f^{j-1}_i$
26:   \hspace{1em} \textbf{end foreach}
27:   \hspace{1em} \textbf{foreach} $i \in N_w$ \textbf{do}
28:     \hspace{2em} \textbf{if} $\ell_{j-1}(i, c) > |A_i \cap W| + 1$ \textbf{then}
29:       \hspace{3em} $f^j_i \leftarrow f^{j-1}_i - \frac{q}{x_w} \left( f^{j-1}_i - \frac{\ell_{j-1}(i, c) - (|A_i \cap W| + 1)}{\ell_{j-1}(i, c)} \right)$
30:     \hspace{2em} \textbf{else}
31:       \hspace{3em} $f^j_i \leftarrow f^{j-1}_i - \frac{q}{x_w} f^{j-1}_i$
32:     \hspace{2em} \textbf{end if}
33:   \hspace{1em} \textbf{end foreach}
34:   \hspace{1em} $W \leftarrow W \cup \{w\}$
35:   \hspace{1em} \textbf{foreach} $c \in C \setminus W$ \textbf{do}
36:     \hspace{2em} $\ell_j(c) \leftarrow \ell_{j-1}(c)$
37:     \hspace{2em} \textbf{while} $\ell_j(c) > \left| \{i : c \in A_i \land |A_i \cap W| < \ell_j(c) \} \right|$ \textbf{do}
38:       \hspace{3em} $\ell_j(c) \leftarrow \ell_j(c) - 1$
39:     \hspace{2em} \textbf{end while}
40:   \hspace{1em} \textbf{end foreach}
41: \hspace{1em} \textbf{else}
42:   \hspace{2em} \textbf{break}
43: \hspace{1em} \textbf{end if}
44: \hspace{1em} \textbf{end for}

5 Some interesting instances of EJR-Exact

The results that we have obtained in the previous section are quite surprising because before only the PAV rule was known to satisfy EJR. In contrast, we can define as many rules as we want by choosing what we do in Alg1, Alg2 and Alg3. In this section we discuss some interesting alternatives. This section is organized in two parts. First we discuss possible alternatives for Alg1 and Alg2 and then we consider alternatives for Alg3.

5.1 Alternatives for Alg1 and Alg2

5.1.1 Simple EJR or SEJR

Suppose that we want to compute a committee that provides EJR for certain ballot profile, candidate set and target committee size. Suppose that we
run some normal iterations to add some candidates to the set of winners. After such normal iterations we find that certain candidate has the maximum dissatisfaction level and that such dissatisfaction level is greater than or equal to 1. Then, corollary \ref{cor:1} says that such candidate cannot be in a starving state, lemma \ref{lem:3} says that she cannot be in an insufficiently supported state and theorem \ref{thm:2} says that she cannot be in an eager state. Thus, she is in a normal state. Observe that we have not needed to look to the votes that remain in the election. That is, if we choose at each iteration the candidate with the highest dissatisfaction level (when ties happen we can choose any of the tied candidates) we know that there exists a way to run normal iterations. Thus, we do not need to compute the fractions of votes that remain in the election.

If at certain point we find that all the candidates that have not already been added to the set of winners have a dissatisfaction level equal to 0, it is possible that some of them are in a normal state and others are in an insufficiently supported state. However, lemma \ref{lem:4} ensures that in that situation the set of winners will provide EJR and therefore we do not need to worry about this.

Algorithm \ref{alg:7} describes the subfamily of voting rules that operate under these ideas. We refer to this subfamily as Simple EJR (SEJR) because this is the simplest way that we know to compute committees that provide EJR.

\textbf{Example 4.} We consider again the election described in example \ref{ex:2}.

Algorithm 7 The SEJR subfamily

Given the algorithm Alg3 that characterize each rule

Input: an approval-based multi-winner election \((N, C, A, k)\)
Output: the set of winners \(W\)

1: \(W \leftarrow \emptyset\)
2: \(j \leftarrow 0\)
3: foreach \(c \in C\) do
4: \(\ell_0(c) \leftarrow \left\lfloor \frac{k n_c}{n} \right\rfloor\)
5: end foreach
6: \(m_\ell \leftarrow \max_{c \in C \setminus W} \ell_0(c)\)
7: while \(j < k\) and \(m_\ell > 0\) do
8: \(w \leftarrow \arg\max_{c \in C \setminus W} \ell_j(c)\)
9: \(W \leftarrow W \cup \{w\}\)
10: \(j \leftarrow j + 1\)
11: foreach \(c \in C \setminus W\) do
12: \(\ell_j(c) \leftarrow \ell_{j-1}(c)\)
13: while \(\ell_j(c) > \left\lfloor \frac{k}{n}\{i : c \in A_i \land |A_i \cap W| < \ell_j(c)\}\right\rfloor\) do
14: \(\ell_j(c) \leftarrow \ell_j(c) - 1\)
15: end while
16: end foreach
17: \(m_\ell \leftarrow \max_{c \in C \setminus W} \ell_j(c)\)
18: end while
19: if \(|W| < k\) then
20: \(j \leftarrow |W|\)
21: Execute Alg3 to add \(k - j\) candidates from \(C \setminus W\) to \(W\)
22: end if
23: return \(W\)

pose that we compute the first winners using SEJR (that is, lines 1–18 of algorithm [7]). Suppose that we break ties first selecting the most approved candidates and in the second place by lexicographic order. The initial dissatisfaction levels of the candidates is: 2 for \(c_1, c_2, c_3, c_4, c_5,\) and \(c_8; 1\) for \(c_6, c_7,\) and \(c_9\) and 0 for \(c_{10}, \ldots, c_{21}\). Therefore, we have to choose first one of \(c_1, c_2, c_3, c_4, c_5,\) or \(c_8\). It suffices to use the first tie-breaking rule to select candidate \(c_5\). Then, the dissatisfaction level of candidate \(c_7\) falls to 0 and the dissatisfaction levels of the other candidates do not change. Therefore, in the second iteration we have to choose one of \(c_1, c_2, c_3, c_4,\) or \(c_8\). It suffices
again to use the first tie-breaking rule to select candidate $c_8$. The dissatisfaction level of candidate $c_9$ falls to 0 and the dissatisfaction levels of the other candidates do not change. Thus, in the third iteration we have to choose one of $c_1, c_2, c_3,$ or $c_4$. All these candidates are approved by 240 votes and therefore we need to make use of the second tie-breaking rule to select $c_1$. The dissatisfaction levels of $c_2$ and $c_6$ fall to 0. Finally, in the fourth iteration $c_3$ is selected, and the dissatisfaction level of $c_4$ falls to 0. All the candidates in $C \setminus W$ have now a dissatisfaction level of 0, and therefore we exit the SEJR loop. We may choose freely any 14 of the remaining candidates and the set of winners will always provide EJR.

**Lemma 5.** Suppose that for certain instance of SEJR the number of arithmetic operations required to execute Alg3 is bounded, by $O(o_3)$. Then, such instance can be computed in $O(nmk + o_3)$.

**Proof.** We have seen at the end of the proof of theorem 4 that the total cost of updating the dissatisfaction levels of the candidates during an execution of an EJR-Exact rule is at most $O(nmk)$. It follows immediately that the cost of the execution of an instance of SEJR is bounded by $O(nmk + o_3)$.

We note that since in SEJR we only make use of integer arithmetic it is reasonable to assume that arithmetic operations can be done in constant time.

**Corollary 2.** For any ballot profile, candidate set and target committee size, it is possible to compute a set of winners that provides EJR in $O(nmk)$.

### 5.1.2 Minimizing Wasted Votes

It seems reasonable to desire that Alg2 removes votes in such a way that it tries to minimize wasted votes. Suppose that for a given election after running $j$ normal iterations set of winners is $W_j$. The next iteration is also normal and candidate $w$ is selected to be added to the set of winners. Now, we have to remove $q$ of the votes that approve $w$ from the election. Under the idea of trying to minimize the number of wasted votes we should probably first remove the votes from voters that have all their approved candidates in the set of winners (that is, if for a voter $i$ that approves $w$ it is $A_i \subseteq W_j \cup \{w\}$, then the vote of this voter should be one of the first removed in the election.

In the second place, we believe that we should remove the votes of voters that are already satisfied (we say that a voter $i$ is satisfied if $|A_i \cap (W_j \cup \{w\})| \geq \ell_j(i, w)$), because for these voters there is no need to add any other of their approved candidates to get a set of winners that provides EJR.
5.2 Alternatives for Alg3: EJR-Exact rules as apportionment methods

Brill et al. [5] presents the following analogy between multi-winner elections and apportionment problems: “Any apportionment problem can be seen as a very simple approval voting instance: all voters approve all the candidates from their chosen party, and only those.”. Such analogy can be used as a way to classify multi-winner voting rules according to which party-list proportional representation system they reduce.

Basically, the idea is to map any party-list election to an approval-based multi-winner election. For each list in the original party-list election $k$ candidates ($k$ is the number of seats that must be allocated) are created. Then, if list $A$ received $n_A$ votes in the original party-list election, also $n_A$ voters approve only all the candidates created for list $A$ in the approval-based multi-winner election. For additional details we refer to [5].

The party-list proportional representation system to which a particular instance of EJR-Exact reduces mainly depends on the algorithm chosen for Alg3. In particular, EJR-LR-Even reduces to largest remainders (and hence the LR in its name). In fact, any rule in the EJR-Exact family that uses the same algorithm as EJR-LR-Even for Alg3 reduces to largest remainders.

Largest remainders assigns seats to each list in two steps: first, as many seats as its lower quota; secondly, it assigns the last seats to the lists with largest remainders after subtracting to each list total vote as many quotas as such list has received. In the equivalent multi-winner election the first step is equivalent to normal iterations for any instance of EJR-Exact. Then, EJR-LR-Even in the insufficiently supported iterations assigns seats to the candidates with higher remaining approval votes; this is equivalent to the second step of largest remainders.

Interestingly, there are also rules in EJR-Exact that reduce to D’Hondt. Since D’Hondt satisfies also lower quota, running normal iterations until there are no candidates left in normal state and then using any approval-based iterative rule such that their iterations are equivalent to D’Hondt iterations in the apportionment scenario, like ODH [9], seq-phragmén [6, 4] or RAV (surveyed by Kilgour in [7]) for Alg3 will produce an instance that reduce to D’Hondt. Due to its simplicity, we will use RAV to illustrate the idea.

Definition 7. Reweighted Approval Voting (RAV) RAV is a multi-round rule that in each round selects a candidate and then reweights the approvals for the subsequent rounds. Specifically, it starts by setting $W = \emptyset$. Then in round $j, j = 1, \ldots, k$, it computes the approval-weight of each candidate $c$ as:
\[
\sum_{i \in A_i} \frac{1}{1 + |W \cap A_i|},
\]

At each iteration, the candidate with largest approval weight is added to the set of winners.

**Example 5.** We can consider using SEJR for the initial iterations and then running RAV. We refer to this as SEJR-RAV. We use again example 4 to illustrate how SEJR-RAV works. We first run SEJR as described in example 4 and get \(W = \{c_1, c_3, c_5, c_8\}\). In the first RAV iteration the candidate with largest approval weight is \(c_9\). Its approval weight is

\[
121 \frac{1}{1 + |W \cap \{c_8, c_9\}|} + 65 \frac{1}{1 + |W \cap \{c_9\}|} = 125.5.
\]

The remaining candidates added to the set of winners are \(c_7\) and \(c_{10}, \ldots, c_{21}\).

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