Dust collapse in 4D Einstein-Gauss-Bonnet gravity

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We consider gravitational collapse in the recently proposed 4D limit of Einstein-Gauss-Bonnet gravity. We show that for collapse of a sphere made of homogeneous dust the process is qualitatively similar to the case of pure Einstein’s gravity. The singularity forms as the endstate of collapse and it is trapped behind the horizon at all times. However, and differently from Einstein’s theory, as a consequence of the Gauss-Bonnet term, the collapsing cloud reaches the singularity with zero velocity, and the time of formation of the singularity is delayed with respect to the pure Einstein case.

I. INTRODUCTION

Gauss-Bonnet (GB) gravity [1] is quadratic order Lovelock theory while Einstein is the linear order and it has been studied for decades as one of the most interesting higher order theories of gravity in D > 4 dimension. It also arises as low energy limit of heterotic superstring theory as the higher curvature correction to General Relativity (GR) [2]. Note that Einstein is linear order while GB is quadratic Lovelock theory, and Lovelock theorem states that any given order N theory is non-vacuous only in D > 2N; i.e., the GB term in the equations of motion gives contributions only in D > 4. However, it was recently proposed that a non trivial 4-dimensional limit of GB theory might exist, thus avoiding Lovelock’s theorem, if one considers a rescaling of the GB coupling constant α to cancel the vanishing term (D − 4) arising from the variation of the Gauss-Bonnet action [2].

The validity of this 4-dimensional Einstein-Gauss-Bonnet (4D-GB) theory is at present under debate and doubts have been raised in view of the fact that the rescaling proposed substitutes a vanishing factor with an undetermined one. Therefore it is not clear if the field equations provide valid equations of motions for systems without any special symmetry (see for example [3] and [4]). However, it is certain that 4-dimensional solutions with high symmetries, such as spherical ones, can be constructed from the 4-dimensional action of the 4D-GB obtained with the proposed redefinition of α. Furthermore it is possible to give such solutions a physical interpretation and investigate their properties, even in the eventual absence of a general theory, much in the same way as people investigate features of toy models in quantum-gravity even without a full quantum theory of the gravitational field. Thus it could be considered as an effective theory with a specific prescription. Comparison of the properties of such solutions with the corresponding ones in Einstein’s gravity can then help shed some light on the validity of the underlying theory.

For example, the properties of accretion disks around a static black hole were considered in [5], rotating black hole solutions were considered in [6], gravitational lensing and shadow was considered in [7] and [8], quasinormal modes were investigated in [9] and [10].

In the present work we consider another spherical solution of great physical interest, namely the gravitational collapse of a spherical cloud of non interacting particles (i.e. dust). Gravitational collapse has been a cornerstone of black hole physics for decades as it provides the mechanism through which black holes and space-time singularities can form via a dynamical process starting from an initially regular configuration (see for example [11] and [12]). Collapse in GB gravity was considered by several authors (see for example [13], [14], [15], [16] and [17]) and the final fate shows a non trivial dependence on the dimensionality of the space-time.

In this paper we specialize the framework for collapse to the 4D limit of the theory proposed in [1]. We show that the qualitative behaviour of homogeneous dust collapse is similar to the one in Einstein’s theory. The effect of the presence of the GB term can be seen in a delay in the co-moving time of formation of the singularity.

The paper is organized as follows: In section II we review the properties of spherical collapse in GB gravity. In section III we derive the equation of motion and the structure of singularity curve and apparent horizon for the 4-dimensional limit of the theory and compare with the corresponding results in pure Einstein’s gravity. Finally in section IV we briefly outline the implications of our result for the theory.

Throughout the paper we make use of units where \( G = c = 1 \) and absorb the term \( \kappa = 8\pi \) in the field equations into the definition of the energy-momentum.

II. DUST COLLAPSE IN GAUSS-BONNET GRAVITY

A model for collapse in D-dimensional GB theory is constructed from the action

\[
S = \int d^Dx \sqrt{-g} (R + \alpha L_{GB}) + S_{\text{matter}}. \tag{1}
\]

were \( R \) is the Ricci scalar which provides the general relativistic part of the action, \( g \) is the determinant of the metric and where the GB part of the Lagrangian \( L_{GB} \) is given by

\[
L_{GB} = R^2 + R_{\lambda\mu\nu\xi} R_{\lambda\mu\nu\xi} - 4R_{\mu\nu} R^{\mu\nu}. \tag{2}
\]
Here we consider the line element for a spherical n-
dimensional dynamical matter cloud in co-moving coordinates
\[ ds^2_\ominus = -e^{2\Phi}dt^2 + e^{2\Psi}dr^2 + R^2d\Omega_{n-2}^2, \]  
with \( \Phi(t, r) \) and \( R(t, r) \) to be determined from the field equations
\[ G_{\mu\nu} = G_{\mu\nu}^{(E)} + G_{\mu\nu}^{(GB)} = T_{\mu\nu}, \]  
where \( G_{\mu\nu}^{(E)} \) is the Einstein’s tensor, \( T_{\mu\nu} \) is the energy momentum tensor and \( G_{\mu\nu}^{(GB)} \) is the GB term given by
\[ G_{\mu\nu}^{(GB)} = \frac{\alpha}{2}g_{\mu\nu}L_{GB} + 2\alpha \left( R_{\mu\nu} - 2\Gamma_{\mu\nu}^\lambda R^\lambda + 2\Gamma_{\mu\lambda\nu}^\xi R^{\lambda\xi} - \Gamma_{\mu\xi\sigma} R^{\xi\sigma} \right). \]

For simplicity we will restrict to the case of dust collapse and therefore we take \( T_{\mu\nu}^{\text{dust}} \)\( = \text{diag}(\rho, 0, 0, 0) \). From the Bianchi identities we get \( \Phi' = 0 \), where primed quantities refer to derivatives with respect to \( r \). With an opportune rescaling of the co-moving time \( t \) we can then set \( \Phi = 0 \). The off diagonal component of the field equations is
\[ G_{t}^{\prime} = (D - 2) \frac{(\Psi R' - R')}{e^{2\Psi} R} + 2(2 - D) \frac{(\Psi R' - R')e^{2\Psi} R^2 e^{2\Psi} R^2}{e^{2\Psi} R^3} = 0, \]
where dotted quantities refer to derivatives with respect to \( r \) and we have introduced \( \alpha = (4 - D)(D - 3)\bar{\alpha} \). From the above equation we get two branches for collapse determined by two different forms for \( e^{2\Psi} \). One corresponds to taking
\[ 2\alpha(e^{2\Psi} + e^{2\Psi} R^2 - R^2) + e^{2\Psi} R^2 = 0, \]
and leads to
\[ e^{2\Phi} = \frac{2\alpha R^2}{R^2 + 2\alpha (R^2 + 1)}. \]
The other corresponds to taking
\[ \Psi R' - R' = 0, \]
and leads to
\[ e^{2\Psi} = \frac{R^2}{E}, \]
where \( E(r) \) is a free function of the radial coordinate, which is related to the initial velocity profile of the collapsing cloud. In the following we will focus on collapse satisfying equation (10) due to its similarities with the corresponding equation in Einstein’s gravity. The remaining non trivial field equations can be written as
\[ -G_{t}^{\prime} = \rho = \frac{(D - 2) F'}{2R^{D - 2} R'}, \]
\[ G_{r}^{\prime} = p_r = \frac{(D - 2) F}{2R^{D - 2} R} = 0, \]
where we have defined the mass function of the system \( F \), which is equivalent to the Misner-Sharp mass in the purely relativistic case, as
\[ F(r) = \alpha R^{D - 5} (\dot{R}^2 + 1 - E)^2 + R^{D - 3} (\dot{R}^2 + 1 - E). \]

Once equations (11) and (12) are satisfied, the remaining field equations, namely \( G_{\mu\nu}^{(GB)} \) \( (i = D - 2) \) are automatically satisfied. Equation (13) can be written in the form of the equation of motion for the system as
\[ \frac{F(r)}{R^{D - 5}} = \alpha \dot{R}^3 + [2\alpha (1 - E) + \Omega^2] \dot{R}^2 + \alpha (1 - E)^2 + R^2 (1 - E). \]

Notice that for \( \alpha = 0 \) the equation reduces to the usual equation of motion for dust collapse in GR. The above equation is a partial differential equation of the fourth power in \( \dot{R} \) and therefore it will in general admit four separate branches of solutions. In fact equation (13) is quadratic in \( \dot{R}^2 \) which for collapse ensuing from infinity with zero velocity \( (i.e. \dot{E} = 1) \) has one positive and the other negative root. Obviously the positive root is the only physically meaningful one, which will give two solutions corresponding to collapsing and expanding dust cloud. Also, the collapsing positive solution can be matched to a vacuum geometry at the boundary, and thus this is the branch on which we will focus in next section.

The quantity \( F(r) \) acts as a quasi-local mass for the collapsing dust cloud and can be understood as representing the amount of matter contained within the co-moving radius \( r \) at any time \( t \). The fact that \( F \) does not depend on \( t \) is a consequence of the choice of non interacting particles for the matter content, i.e. a consequence of equation (13) which implies \( \dot{F} = 0 \). This implies that there is no inflow or outflow of matter through any shell \( r \) and in particular through the boundary of the cloud and therefore the collapsing interior can be matched to a vacuum exterior. Setting the co-moving boundary of the collapsing cloud at \( r = r_b \) we can then interpret \( F(r_b) \) as related to the total mass of the system, and therefore it should be related to the mass parameter \( M \) in the external vacuum solution. This is indeed the case if the matching is performed with the vacuum solution discovered by Boulware and Deser (14) that has line element
\[ ds^2_\ominus = -H(S) dt^2 + \frac{dS^2}{H(S)} + S^2 d\Omega_{n-2}, \]
with
\[ H(S) = 1 + \frac{S^2}{2\alpha} \left( 1 \pm \sqrt{1 + \frac{4\alpha M}{(D - 2)^2 S^{D - 1}}} \right). \]

Notice that there are two branches of solutions depending on the sign in front of the square root, one referring to attractive and the other repulsive mass points and hence we choose the branch with minus sign. The two branches have the same behaviour for small \( S \), namely they approach a De Sitter-like solution (one attractive while the other repulsive), while they exhibit different behaviour asymptotically, with the branch with minus sign having Schwarzschild-like behaviour.
for large $S$ $[22, 24]$. The boundary radius $r = r_b$ in the interior corresponds to the collapsing boundary of the cloud $R_b(t) = R(r_b, t)$, while the same boundary as seen from the exterior is given by $S = S_b(T)$, with $T = T(t)$ on the boundary determined by the matching condition for continuity of $g_{00}$. The continuity of the metric on the unit $(D - 2)$-sphere implies that

$$R_b(t) = S_b(T(t)).$$

The total $D$-dimensional ADM mass $M$ is then related to $F(r_b)$ through the continuity condition for the equation of motion at the boundary (see $[36]$). In the simple case of marginally bound collapse, given by $E = 1$, this gives

$$R_b^2 = \frac{R_b^2}{2\alpha} \left(1 \pm \sqrt{1 + \frac{4\alpha F(r_b)}{R_b^{D-4}}} \right),$$

in the interior and

$$S_b(T(t)) = \frac{S_b^2}{2\alpha} \left(1 \pm \sqrt{1 + \frac{8\alpha M}{(D - 2)S_b^{D-4}}} \right),$$

in the exterior. The above equations, together with the boundary condition in equation $[17]$, can be satisfied only if

$$2M = (D - 2)F(r_b).$$

The above setup can be easily applied to the theory proposed in $[36]$ to investigate the final outcome of homogeneous collapse.

### III. DUST COLLAPSE IN 4D GAUSS-BONNET

We will now restrict the attention to the special case of collapse in $D = 4$ dimensions with the rescaling of the GB coupling given by $\tilde{\alpha} \rightarrow \tilde{\alpha}/(D - 4)$, which in our case, setting $D = 4$, corresponds to $\alpha = (D - 3)\tilde{\alpha} = \tilde{\alpha}$. It is important to point out that due to the spherical symmetry of the system, the reparameterization of $\alpha$ proposed in $[36]$ gives valid field equations for this model if the lower dimensional limit is taken suppressing the extra dimensions of the $(D - 2)$-sphere part of the metric. The line element $[23]$ in this case reads

$$ds^2 = -dt^2 + \frac{R^2}{E}dr^2 + R^2d\Omega_2^2,$$

with $d\Omega_2^2$ the usual line-element on the unit 2-sphere and the derivation of the field equations proceeds exactly as described in the previous section.

For the sake of clarity, we can now implement some scaling relying on the gauge freedom to set the initial radius as $R(0, r) = r$, the arbitrariness of the function $E(r)$ and some physical requirement for the mass function (see $[36]$ for details). We thus set

$$R(r, t) = ra(r, t),$$

$$E(r) = 1 - r^2 f(r),$$

$$F(r) = m(r)r^3,$$

which coincides with the equation of motion for the Oppenheimer-Snyder-Datt (OSD) collapse $[25, 26]$ when $\alpha = 0$. Notice that with the above rescaling Einstein’s equation for $\rho(t)$, i.e. equation $[19]$, becomes $\rho = 3m_0a^3$ which is finite at the onset of collapse and leads to divergent density for $a \rightarrow 0$. For illustrative purposes it is useful to consider the simplest case of marginally bound collapse, which corresponds to $k = 0$. Then equation $[25]$ reduces to

$$\alpha\dot{a}^2 + a^2 \dot{a}^2 = m_0a,$$

which gives valid field equations for this model if the lower dimensional limit is taken suppressing the extra dimensions of the $(D - 2)$-sphere part of the metric. The line element $[23]$ in this case reads

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$$\alpha\dot{a}^2 + a^2 \dot{a}^2 = m_0a,$$
FIG. 2. Comparison between the Kretschmann scalar for collapse in Einstein’s gravity (dashed line) and in the 4D-EGB theory (solid line). Here we assume that the singularity occurs at the same time $t_s$ for both scenarios (thus shifting the scale factor in the GR case). Then we can see that, as a consequence of the GB term, $K$ diverges faster in GR with respect to the 4D-EGB case.

where

$$z = \frac{1}{\sqrt{2}} \sqrt{1 + \frac{4\alpha m_0}{a^3} - 1},$$  \hspace{1cm} (29)$$

and $c$ is an integration constant that is found by setting the initial condition as $a(0) = 1$:

$$c = \frac{2}{3} \sqrt{\alpha} \left( \frac{1}{z_0} - \arctan(z_0) \right),$$  \hspace{1cm} (30)$$

with

$$z_0 = \sqrt{1 + \frac{4\alpha m_0}{2} - 1}.$$  \hspace{1cm} (31)$$

Then the scale factor $a(t)$ is simply given by the inverse of equation (28). Comparison between the behaviour of the solution of equation (27) and the OSD case is shown in figure 1.

The Kretschmann scalar for the above scenario is given by

$$K = 12 \left( \frac{a^2}{a^3} + \frac{\dot{a}^4}{a^4} \right),$$  \hspace{1cm} (32)$$

which diverges for $a \to 0$, thus showing that a curvature singularity forms as the endstate of collapse, similarly to the purely relativistic case. However, as a consequence of the GB term the singularity is “weaker” than in its relativistic counterpart. This can be seen from the fact that the Kretschmann scalar diverges faster in the GR case, as shown in figure 2. From the above considerations we see that the singularity forms at the time $t_s$ for which $a(t_s) = 0$. Due to the homogeneous nature of collapse, all shells fall into the singularity at the same time $t_s$ and the singularity is space-like. In the present case we have

$$t_s = \frac{2}{3} \sqrt{\alpha} \left( \frac{1}{z_0} - \arctan(z_0) \right) - \frac{4}{3} \sqrt{\alpha}.$$  \hspace{1cm} (33)$$

We shall now investigate the behaviour of trapped surfaces for the marginally bound and homogeneous dust collapse scenario. The apparent horizon in the interior is defined via the condition that the surface $R(t, r)$ becomes null, i.e. $g^{\mu\nu}(\partial_\mu R)(\partial_\nu R) = 0$. This corresponds to the requirement that

$$1 - \frac{F(r)}{R(t, r)} = 1 - \frac{r^2 m_0}{a(t)} = 0,$$  \hspace{1cm} (34)$$

which implicitly defines the apparent horizon curve $r_{ah}(t)$ from

$$r_{ah}(t) = \sqrt{\frac{a(t)}{m_0}}.$$  \hspace{1cm} (35)$$

In figure 3 we show the evolution of the apparent horizon radius in the 4D-EGB theory as well as in Einstein’s theory. Similarly to what we have discussed for the scale factor, the main effect of the GB term is to delay the formation of trapped surfaces.

IV. CONCLUSIONS

We considered gravitational collapse of homogeneous dust in the newly proposed 4-dimensional limit of EGB gravity. We showed that the scenario obtained within the theory proposed in [4] does not depart significantly from the dynamical structure of dust collapse in Einstein’s gravity. An appealing feature of collapse in 4D-EGB theory is that the scale factor reaches the singularity with zero velocity thus making the singularity at the end of collapse “weaker” than in the corresponding relativistic case. It is known that the metric function (16) remains regular as $r \to 0$ as the approach to singularity occurs via de Sitter-like space [23]. It is perhaps because of
this reason that the singularity is weaker also in collapse and the collapsing cloud reaches the singularity with zero velocity. Also, the 4D-EGB theory allows for new branches of collapsing solutions that don’t have a relativistic limit for $\alpha = 0$ or at large distances. These solutions will be investigated in future work. On the other hand, for the branch of solution that reproduces to GR in the limit of $\alpha = 0$ with asymptotically flat vacuum exterior, the behaviour of the singularity and apparent horizon is qualitatively similar to the case of Einstein’s theory thus suggesting that black holes may form in 4D-EGB theory much in the same way as they do in GR.

It is well known that the introduction of inhomogeneities in dust collapse may alter dramatically the formation of trapped surfaces leading to the appearance of naked singularities (see for example [27]). Therefore, it will be interesting to see if such conclusions remain valid for inhomogeneous dust collapse in the 4D-EGB theory.

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[1] D. Lovelock, J. Math. Phys. 12, 498 (1971).
[2] D. J. Gross and J. H. Sloan, Nucl. Phys. B 291, 41 (1987).
[3] M. C. Bento and O. Bertoni, Phys. Lett. B 368, 198 (1996).
[4] D. Glavan and C. Lin, Phys. Rev. Lett. 124, 081301 (2020).
[5] W.-Y. Ai, arXiv:2004.02858 [gr-qc].
[6] M. Gurses, T. C. Sisman, and B. Tekin, arXiv:2004.03390 [gr-qc].
[7] C. Liu, T. Zhu, and Q. Wu, arXiv:2004.01662 [gr-qc].
[8] M. Guo and P.-C. Li, arXiv:2003.02523 [gr-qc].
[9] R. Kumar and S. G. Ghosh, arXiv:2003.08927 [gr-qc].
[10] S.-W. Wei and Y.-X. Liu, arXiv:2003.07769 [gr-qc].
[11] S. Ul Islam, R. Kumar, and S. G. Ghosh, arXiv:2004.01038 [gr-qc].
[12] R. A. Konoplya and A. F. Zinhailo, arXiv:2003.01188 [gr-qc].
[13] M. S. Churilova, arXiv:2004.00513 [gr-qc].
[14] P. S. Joshi, Gravitational collapse and spacetime singularities, Cambridge University Press, Cambridge, UK (2007).
[15] P. S. Joshi and D. Malafarina, Int. J. Mod. Phys. D 20, 2641 (2011).
[16] H. Maeda, Phys. Rev. D 73, 104004 (2006).
[17] S. Jhingan and S. G. Ghosh, Phys. Rev. D 81, 024010 (2010).
[18] S. G. Ghosh and S. Jhingan, Phys. Rev. D 82, 024017 (2010).
[19] K. Zhou, Z.-Y. Yang, D.-C. Zou, and R.H. Yue, Mod. Phys. Lett. A 26, 2135 (2011).
[20] N. Dadhich, S. G. Ghosh, and S. Jhingan, Phys. Rev. D 88, 084024 (2013).
[21] C. Misner and D. Sharp, Phys. Rev. 136, B571 (1964).
[22] D. G. Boulware and S. Deser, Phys. Rev. Lett. 55, 2656 (1985).
[23] N. Dadhich, Proceedings of 12th Regional Conference on Mathematical Physics, eds. M. J. Aslam, F. Hussain, A. Qadir, Riazuddin, H. Saleem, pp.331 (World Scientific, 2007), arxiv:hep-th/0509126.
[24] T. Torii and H. Maeda, Phys. Rev. D 71, 124002 (2005).
[25] J. R. Oppenheimer and H. Snyder, Phys. Rev. 56, 455 (1939).
[26] S. Datt, Z. Phys. 108, 314 (1938).
[27] P. S. Joshi and I. H. Dwivedi, Phys. Rev. D 47, 5357 (1993).