Chapter 2

Boundary Triplets and Weyl Functions

The basic properties of boundary triplets for closed symmetric operators or relations in Hilbert spaces are presented. These triplets give rise to a parametrization of the intermediate extensions of symmetric relations, in particular of the self-adjoint extensions. Closely related is the Krešn formula which describes the resolvent operators of such intermediate extensions. The introduction of boundary triplets and a discussion of corresponding boundary value problems can be found in Section 2.1 and Section 2.2. Associated with a boundary triplet are the $\gamma$-field and the Weyl function, and these analytic objects are treated in Section 2.3. The existence and construction of boundary triplets is discussed in Section 2.4; their transformations are the contents of Section 2.5. Section 2.6 on Krešn’s resolvent formula for canonical extensions and a description of their spectra is central in this chapter. Furthermore, a discussion of self-adjoint exit space extensions, Straus families, and the Krešn–Našmark formula can be found Section 2.7. Some related perturbation problems are treated in Section 2.8.

2.1 Boundary triplets

The following definition introduces a boundary triplet, one of the key objects in this text. It is based on the well-known Green or Lagrange formula together with an additional maximality condition.

**Definition 2.1.1.** Let $S$ be a closed symmetric relation in a Hilbert space $\mathcal{G}$. Then \{\mathcal{G}, \Gamma_0, \Gamma_1\} is a boundary triplet for $S^*$ if $\mathcal{G}$ is a Hilbert space and $\Gamma_0, \Gamma_1 : S^* \to \mathcal{G}$ are linear mappings such that the mapping $\Gamma : S^* \to \mathcal{G} \times \mathcal{G}$ defined by

$$\Gamma \hat{f} = \{\Gamma_0 \hat{f}, \Gamma_1 \hat{f}\}, \quad \hat{f} = \{f, f'\} \in S^*,$$

is surjective and the identity

$$(f', g)_{\mathcal{G}} - (f, g')_{\mathcal{G}} = (\Gamma_1 \hat{f}, \Gamma_0 \hat{g})_{\mathcal{G}} - (\Gamma_0 \hat{f}, \Gamma_1 \hat{g})_{\mathcal{G}} \quad (2.1.1)$$

holds for all $\hat{f} = \{f, f'\}, \hat{g} = \{g, g'\} \in S^*$. 

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Note that a symmetric relation $S$ is densely defined if and only if $S^*$ is an operator. In this case the boundary mappings $\Gamma_0$ and $\Gamma_1$ can be defined on $\text{dom} \ S^*$ instead of (the graph of) $S^*$. More precisely, if $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a boundary triplet for $S^*$, then one defines boundary mappings $\Gamma_0$ and $\Gamma_1$ on $\text{dom} \ S^*$ by the following identifications

$$\Gamma_0 f = \Gamma_0 \hat{f}, \quad \Gamma_1 f = \Gamma_1 \hat{f}, \quad \hat{f} = \{f, f'\} \in S^*.$$ 

In the following treatment whenever $S$ is a densely defined operator, boundary mappings defined on $S^*$ and on $\text{dom} \ S^*$ will be identified in this sense. After this identification (2.1.1) turns into

$$\langle S^* f, g \rangle_{\mathcal{H}} - \langle f, S^* g \rangle_{\mathcal{H}} = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{G}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{G}}, \quad (2.1.2)$$

where $f, g \in \text{dom} \ S^*$. This formalism will be used in Chapter 6 and Chapter 8 in the treatment of ordinary and partial differential operators.

The identity (2.1.1) or the identity (2.1.2) is sometimes called the abstract Green identity or the abstract Lagrange identity; in this text mostly the terminology abstract Green identity will be used. This identity has a geometric interpretation which is best expressed in terms of the indefinite inner products

$$\left[ \cdot, \cdot \right]_{\mathcal{G}} := (\mathcal{J}_\mathcal{G} \cdot, \cdot)_{\mathcal{G}}, \quad \mathcal{J}_\mathcal{G} = \left( \begin{array}{cc} 0 & -iI \mathcal{G} \\ iI \mathcal{G} & 0 \end{array} \right),$$

where $\mathcal{J}_\mathcal{H} = \mathcal{J}_\mathcal{G}^* = \mathcal{J}_\mathcal{G}^{-1} \in \mathcal{B}(\mathcal{H}^2)$ and $\mathcal{J}_\mathcal{G} = \mathcal{J}_\mathcal{G}^* = \mathcal{J}_\mathcal{G}^{-1} \in \mathcal{B}(\mathcal{G}^2)$; cf. Section 1.8. By means of these inner products, the identity (2.1.1) can be rewritten as

$$\left[ \hat{f}, \hat{g} \right]_{\mathcal{G}} = \left[ \Gamma \hat{f}, \Gamma \hat{g} \right]_{\mathcal{G}}, \quad (2.1.4)$$

for $\hat{f} = \{f, f'\}, \hat{g} = \{g, g'\} \in S^*$. Later the scalar products in (2.1.1), (2.1.2), and (2.1.4) will be used without indices $\mathcal{H}$ and $\mathcal{G}$, respectively, when there is no danger of confusion. Recall that the adjoint $A^*$ of a relation $A$ in $\mathcal{H}$ can be written as an orthogonal complement with respect to the inner product $[\cdot, \cdot]$, that is $A^* = A^{[\perp]}$; cf. Section 1.8.

Some elementary but important properties of the boundary mappings are collected in the following proposition.

**Proposition 2.1.2.** Let $S$ be a closed symmetric relation in $\mathcal{H}$ and assume that $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a boundary triplet for $S^*$. Then the following statements hold:

(i) the mappings $\Gamma : S^* \rightarrow \mathcal{G} \times \mathcal{G}$ and $\Gamma_0, \Gamma_1 : S^* \rightarrow \mathcal{G}$ are surjective and continuous;

(ii) $\ker \Gamma = S$. 
Proof. (i) The continuity of $\Gamma : S^* \to \mathcal{S} \times \mathcal{S}$ is essentially a consequence of the fact that $\Gamma$ is isometric in the sense of (2.1.4). More precisely, by definition the mapping $\Gamma$ is surjective, and since dom $\Gamma = S^*$ is closed it follows from Lemma 1.8.1 that $\Gamma$ is continuous. Clearly, the mappings $\Gamma_0$ and $\Gamma_1$ are also surjective and continuous.

(ii) In order to show that ker $\Gamma \subset S$, let $\hat{f} \in \ker \Gamma$. Then it follows from (2.1.4) that $[\hat{f}, \hat{g}] = [\hat{f}, \hat{g}]_{g^2} = 0$ for all $\hat{g} \in S^*$, which implies $\hat{f} \in (S^*)_{[1]} = S^{**} = S$, since $S$ is closed. Hence, ker $\Gamma \subset S$ has been shown. To show that $S \subset \ker \Gamma$, let $\hat{f} \in S$. Since $\Gamma$ is surjective, for arbitrary fixed $\hat{\varphi} \in G \times G$ one can choose $\hat{g} \in S^*$ such that $\Gamma \hat{g} = J_G \hat{\varphi}$, with $J_G$ as in (2.1.3). Since $\hat{f} \in S$ and $\hat{g} \in S^*$ it follows from (2.1.4) that

$$ (\Gamma \hat{f}, \hat{\varphi})_{g^2} = (\Gamma \hat{f}, J_G^{-1} \Gamma \hat{g})_{g^2} = [\Gamma \hat{f}, \Gamma \hat{g}]_{g^2} = [\hat{f}, \hat{g}]_{g^2} = 0 $$

for all $\hat{\varphi} \in G^2$, which leads to $\Gamma \hat{f} = 0$. Thus, $S \subset \ker \Gamma$ has been shown. □

By means of a boundary triplet $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for $S^*$ the intermediate extensions of $S$ defined in Section 1.7 can be described via relations in the space $\mathcal{G}$. In particular, the one-to-one correspondence in the next theorem preserves adjoints, which is a consequence of the abstract Green identity (2.1.4).

**Theorem 2.1.3.** Let $S$ be a closed symmetric relation in $H$ and let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for $S^*$. Then the following statements hold:

(i) there is a bijective correspondence between the set of intermediate extensions $A_\Theta$ of $S$ and the set of relations $\Theta$ in $\mathcal{G}$, via

$$ A_\Theta := \{\hat{f} \in S^* : \Gamma \hat{f} \in \Theta\}; \quad (2.1.5) $$

(ii) $\overline{A_\Theta} = A_\overline{\Theta}$ and, in particular, the relation $A_\Theta$ is closed if and only if the relation $\Theta$ is closed;

(iii) $A_\Theta = \ker (\Gamma_1 - \Theta \Gamma_0)$;

(iv) $(A_\Theta^*)^* = A_{\overline{\Theta}^*}$ for every relation $\Theta$ in $\mathcal{G}$;

(v) $A_\Theta \subset A_{\Theta'}$ if and only if $\Theta \subset \Theta'$, when $\Theta$ and $\Theta'$ are relations in $\mathcal{G}$;

(vi) $A_\Theta$ is an operator if and only if $S$ is an operator and

$$ \Theta \cap \Gamma \{0\} \times \text{mul} S^* = \{0, 0\}. \quad (2.1.6) $$

**Proof.** (i) & (ii) The relation $S^* \subset \mathcal{G}^2$ is equipped with the Hilbert space inner product of $\mathcal{G}^2$. Now let $\mathcal{M} \subset \mathcal{G}^2$ be the orthogonal complement of $S$ in $S^*$, so that $S \oplus \mathcal{M} = S^*$. Since ker $\Gamma = S$, the restriction $\Gamma'$ of $\Gamma$ to $\mathcal{M}$ is an isomorphism between $\mathcal{M}$ and $\mathcal{G} \times \mathcal{G}$. Hence $\Gamma'$ gives a one-to-one correspondence between the subspaces $H'$ of $\mathcal{M}$ and the subspaces $\Theta$ of $\mathcal{G} \times \mathcal{G}$ via

$$ \Theta = \Gamma' H' \quad \text{or, equivalently,} \quad (\Gamma')^{-1} \Theta = H'. \quad (2.1.7) $$

By means of a boundary triplet $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ for $S^*$ the intermediate extensions of $S$ defined in Section 1.7 can be described via relations in the space $\mathcal{G}$. In particular, the one-to-one correspondence in the next theorem preserves adjoints, which is a consequence of the abstract Green identity (2.1.4).

**Theorem 2.1.3.** Let $S$ be a closed symmetric relation in $\mathcal{G}$ and let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for $S^*$. Then the following statements hold:

(i) there is a bijective correspondence between the set of intermediate extensions $A_\Theta$ of $S$ and the set of relations $\Theta$ in $\mathcal{G}$, via

$$ A_\Theta := \{\hat{f} \in S^* : \Gamma \hat{f} \in \Theta\}; \quad (2.1.5) $$

(ii) $\overline{A_\Theta} = A_\overline{\Theta}$ and, in particular, the relation $A_\Theta$ is closed if and only if the relation $\Theta$ is closed;

(iii) $A_\Theta = \ker (\Gamma_1 - \Theta \Gamma_0)$;

(iv) $(A_\Theta^*)^* = A_{\overline{\Theta}^*}$ for every relation $\Theta$ in $\mathcal{G}$;

(v) $A_\Theta \subset A_{\Theta'}$ if and only if $\Theta \subset \Theta'$, when $\Theta$ and $\Theta'$ are relations in $\mathcal{G}$;

(vi) $A_\Theta$ is an operator if and only if $S$ is an operator and

$$ \Theta \cap \Gamma \{0\} \times \text{mul} S^* = \{0, 0\}. \quad (2.1.6) $$

**Proof.** (i) & (ii) The relation $S^* \subset \mathcal{G}^2$ is equipped with the Hilbert space inner product of $\mathcal{G}^2$. Now let $\mathcal{M} \subset \mathcal{G}^2$ be the orthogonal complement of $S$ in $S^*$, so that $S \oplus \mathcal{M} = S^*$. Since ker $\Gamma = S$, the restriction $\Gamma'$ of $\Gamma$ to $\mathcal{M}$ is an isomorphism between $\mathcal{M}$ and $\mathcal{G} \times \mathcal{G}$. Hence $\Gamma'$ gives a one-to-one correspondence between the subspaces $H'$ of $\mathcal{M}$ and the subspaces $\Theta$ of $\mathcal{G} \times \mathcal{G}$ via

$$ \Theta = \Gamma' H' \quad \text{or, equivalently,} \quad (\Gamma')^{-1} \Theta = H'. \quad (2.1.7) $$
Clearly, this gives rise to a one-to-one correspondence between all intermediate extensions $H$ of $S$ and all subspaces $H'$ of $\mathfrak{M}$ via $H = S \oplus H'$, which is expressed in (2.1.5). Moreover, since $\Gamma'$ is an isomorphism it also follows from (2.1.7) that $\Theta = \Gamma' H' = \Gamma H'$ and hence the closure $\overline{H}$ of $H$ corresponds to the closure $\overline{\Theta}$ of $\Theta$. This implies via (2.1.5) that $A_{\overline{\Theta}} = A_{\Theta}$.

(iii) Let $A_{\Theta}$ be defined by (2.1.5). It will be verified that

$$A_{\Theta} = \ker(\Gamma_1 - \Theta\Gamma_0)$$

(2.1.8)

holds for any relation $\Theta$ in $\mathfrak{S}$. Note that (2.1.8) is clear in the special case that $\Theta$ is an operator, since $\Gamma \hat{f} = \{\Gamma_0 \hat{f}, \Gamma_1 \hat{f}\} \in \Theta$ means that $\Theta \Gamma_0 \hat{f} = \Gamma_1 \hat{f}$. Now assume that $\Theta$ is a relation.

First the inclusion ($\subset$) in (2.1.8) will be shown. For this consider $\hat{f} \in A_{\Theta}$. Hence, $\hat{f} \in S^*$ and $\{\Gamma_0 \hat{f}, \Gamma_1 \hat{f}\} \in \Theta$. Then $\{\hat{f}, \Gamma_0 \hat{f}\} \in \Gamma_0$ gives $\{\hat{f}, \Gamma_1 \hat{f}\} \in \Theta \Gamma_0$. Since $\{\hat{f}, \Gamma_1 \hat{f}\} \in \Gamma_1$ one finds $\{\hat{f}, 0\} \in \Gamma_1 - \Theta \Gamma_0$. In other words $\hat{f} \in \ker(\Gamma_1 - \Theta \Gamma_0)$.

For the inclusion ($\supset$) in (2.1.8) consider $\hat{f} \in \ker(\Gamma_1 - \Theta \Gamma_0)$. Then one has $\{\hat{f}, 0\} \in \Gamma_1 - \Theta \Gamma_0$ and hence there exists an element $\hat{\psi}$ such that $\{\hat{f}, \hat{\psi}\} \in \Gamma_1$ and $\{\hat{f}, \hat{\psi}\} \in \Theta \Gamma_0$. Thus $\{\hat{f}, \hat{\varphi}\} \in \Gamma_0$ and $\{\hat{\varphi}, \hat{\psi}\} \in \Theta$ for some $\hat{\varphi}$. Since both $\Gamma_0$ and $\Gamma_1$ are operators, one has $\hat{\psi} = \Gamma_1 \hat{f}$ and $\hat{\varphi} = \Gamma_0 \hat{f}$, and therefore $\{\Gamma_0 \hat{f}, \Gamma_1 \hat{f}\} \in \Theta$, that is, $\hat{f} \in A_{\Theta}$.

(iv) To show that $(A_{\Theta})^* \subset A_{\Theta^*}$, let $\hat{g} \in (A_{\Theta})^*$. Let $\hat{\varphi} \in \Theta$ and choose $\hat{f} \in A_{\Theta}$ such that $\Gamma \hat{f} = \hat{\varphi}$. Then one has

$$[\hat{\varphi}, \Gamma \hat{g}] = \Gamma \hat{f} = \hat{\varphi},$$

which implies $\Gamma \hat{g} \in \Theta^*$, that is, $\hat{g} \in A_{\Theta^*}$. One concludes $(A_{\Theta})^* \subset A_{\Theta^*}$. Since $A_{\Theta^*}$ is closed by (ii), the inclusion $A_{\Theta^*} \subset (A_{\Theta})^*$ follows together with (ii) from

$$A_{\Theta^*} = (A_{\Theta^*})^{**} \subset (A_{\Theta^{**}})^* = (A_{\Theta^*})^* = (A_{\Theta})^*.$$

(v) This assertion is obvious from the correspondence in (2.1.5).

(vi) Let $A_{\Theta}$ be an operator. Then clearly $S$ is an operator. Assume that $\Gamma \hat{f} \in \Theta$ for some element $\hat{f} = \{0, f'\} \in S^*$. Then $\hat{f} \in A_{\Theta}$ and hence $f' = 0$, so that (2.1.6) holds.

Conversely, assume now that $S$ is an operator and that (2.1.6) holds. If $\hat{f} = \{0, f'\} \in A_{\Theta}$, then $f' \in \text{mul} S^*$ and $\Gamma \{0, f'\} \in \Theta$. Hence, $\{0, f'\} = \{0, 0\}$ and as $S = \ker \Gamma$, it follows that $\{0, f'\} \in S$. This implies $f' = 0$. Therefore, $A_{\Theta}$ is an operator.

Due to the abstract Green identity (2.1.1), (2.1.2), or (2.1.4) some properties of intermediate extensions are preserved in the corresponding relations in $\mathfrak{S}$. 

\[ \square \]
Corollary 2.1.4. Let $S$ be a closed symmetric relation in $\mathcal{H}$, let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for $S^*$, and let $A_\Theta$ be the extension of $S$ in $\mathcal{G}$ corresponding to the relation $\Theta$ in $\mathcal{G}$ via (2.1.5). Then the following statements hold:

(i) $A_\Theta$ is dissipative (accumulative) if and only if $\Theta$ is dissipative (accumulative);
(ii) $A_\Theta$ is maximal dissipative (maximal accumulative) if and only if $\Theta$ is maximal dissipative (maximal accumulative);
(iii) $A_\Theta$ is symmetric if and only if $\Theta$ is symmetric;
(iv) $A_\Theta$ is maximal symmetric if and only if $\Theta$ is maximal symmetric;
(v) $A_\Theta$ is self-adjoint if and only if $\Theta$ is self-adjoint.

Proof. (i) This assertion follows immediately from the identity (see (1.8.4))

\[ \text{Im}(f', f) = \frac{1}{2} \left[ \hat{f}, \hat{f} \right]_{\mathcal{G}^2} \]

where $\hat{f} = \{f, f'\} \in S^*$, and (2.1.5).

(ii) According to Theorem 2.1.3 (v), for any two extensions $A_\Theta$ and $A_{\Theta'}$ of $S$ one has $A_\Theta \subset A_{\Theta'}$ if and only if $\Theta \subset \Theta'$ holds. Therefore, if $A_\Theta$ is maximal dissipative, then $\Theta$ is dissipative because of (i) and if $\Theta'$ is a dissipative extension of $\Theta$ in $\mathcal{G}$, then $A_{\Theta'}$ is a dissipative extension of $A_\Theta$, so that $A_\Theta = A_{\Theta'}$. Hence, $\Theta = \Theta'$ and $\Theta$ is maximal dissipative. The converse direction is proved in exactly the same way. The statement for maximal accumulative extensions follows analogously.

(iii)–(v) These assertions follow from the previous items, and the fact that a relation is symmetric (self-adjoint) if and only if it is (maximal) dissipative and (maximal) accumulative.

\[ S^* = \text{clos}(H \hat{\oplus} H'); \]

and in this case $H$ and $H'$ are transversal if and only if $H \hat{\oplus} H'$ is closed; cf. Lemma 1.7.7. In a similar way the closed relations $\Theta$ and $\Theta'$ in $\mathcal{G}$, as intermediate extensions of the trivial symmetric relation $\{0, 0\}$, are disjoint if $\Theta \cap \Theta' = \{0, 0\}$ and transversal if they are disjoint and $\Theta \hat{\oplus} \Theta' = \mathcal{G}^2$.

Lemma 2.1.5. Let $S$ be a closed symmetric relation in $\mathcal{H}$ and let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for $S^*$. Let $\Theta$ and $\Theta'$ be relations in $\mathcal{G}$. Then

\[ A_\Theta \cap A_{\Theta'} = A_{\Theta \cap \Theta'} \quad (2.1.9) \]

and

\[ A_\Theta \hat{\oplus} A_{\Theta'} = A_{\Theta \hat{\oplus} \Theta'} \quad (2.1.10) \]
In particular, if \( A_\Theta \) and \( A_{\Theta'} \) are closed or, equivalently, \( \Theta \) and \( \Theta' \) are closed, then the following statements hold:

(i) \( A_\Theta \) and \( A_{\Theta'} \) are disjoint if and only if \( \Theta \) and \( \Theta' \) are disjoint;
(ii) \( A_\Theta \) and \( A_{\Theta'} \) are transversal if and only if \( \Theta \) and \( \Theta' \) are transversal.

**Proof.** The identity (2.1.9) follows from

\[
A_\Theta \cap A_{\Theta'} = \{ \hat{f} \in S^* : \Gamma \hat{f} \in \Theta \} \cap \{ \hat{f} \in S^* : \Gamma \hat{f} \in \Theta' \} = \{ \hat{f} \in S^* : \Gamma \hat{f} \in \Theta \cap \Theta' \} = A_{\Theta \cap \Theta'},
\]

while the identity (2.1.10) follows from

\[
\Gamma(A_\Theta \hat{\oplus} A_{\Theta'}) = \Gamma(A_\Theta) \hat{\oplus} \Gamma(A_{\Theta'}) = \Theta \hat{\oplus} \Theta'.
\]

In particular, (2.1.9) together with \( S = \ker \Gamma \) shows that \( A_\Theta \cap A_{\Theta'} = S \) if and only if \( \Theta \cap \Theta' = \{0,0\} \), while (2.1.9) and (2.1.10) show that \( A_\Theta \cap A_{\Theta'} = S \) and \( A_\Theta \hat{\oplus} A_{\Theta'} = S^* \) if and only if \( \Theta \cap \Theta' = \{0,0\} \) and \( \Theta \hat{\oplus} \Theta' = \mathcal{G}^2 \). This completes the proof. \( \square \)

**Corollary 2.1.6.** Let \( S \) be a closed symmetric relation in \( \mathcal{H} \), let \( \{ \mathcal{G}, \Gamma_0, \Gamma_1 \} \) be a boundary triplet for \( S^* \) and assume that \( \dim \mathcal{G} < \infty \). If \( A_\Theta \) and \( A_{\Theta'} \) are self-adjoint extensions of \( S \) which are disjoint, then they are transversal.

Let \( S \) be a closed symmetric relation in \( \mathcal{H} \) and let \( \{ \mathcal{G}, \Gamma_0, \Gamma_1 \} \) be a boundary triplet for \( S^* \). There are two special extensions of \( S \) which will be frequently used in the following; they are defined by

\[
A_0 := \ker \Gamma_0 \quad \text{and} \quad A_1 := \ker \Gamma_1.
\]  

(2.1.11)

It is clear that \( A_0 \) and \( A_1 \) are self-adjoint extensions of \( S \), since they correspond to the self-adjoint parameters \( \Theta \) in \( \mathcal{G} \) in (2.1.5) given by

\[
\Theta = \{0\} \times \mathcal{G} \quad \text{and} \quad \Theta = \mathcal{G} \times \{0\},
\]  

(2.1.12)

respectively. Furthermore, the representations in (2.1.12) show that the self-adjoint extensions \( A_0 \) and \( A_1 \) are transversal; cf. Lemma 2.1.5. Note also in this context that a boundary triplet \( \{ \mathcal{G}, \Gamma_0, \Gamma_1 \} \) for \( S^* \) can only exist if the defect numbers of the closed symmetric relation \( S \) coincide (since it admits the self-adjoint extensions \( A_0 \) and \( A_1 \) in (2.1.11)); a more detailed discussion on the existence and uniqueness of boundary triplets will be provided in Section 2.4 and Section 2.5.

As \( S = \ker \Gamma \), it follows from von Neumann’s decomposition Theorem 1.7.11 that \( \Gamma \) is an isomorphism from \( \mathcal{H}_1(S^*) \hat{\oplus} \mathcal{H}_\lambda(S^*) \), \( \lambda \in \mathbb{C} \setminus \mathbb{R} \), onto \( \mathcal{G}^2 \). Due to the definitions in (2.1.11) a similar observation can be made for the components \( \Gamma_0 \) and \( \Gamma_1 \).
Lemma 2.1.7. Let \( S \) be a closed symmetric relation in \( \mathcal{H} \), let \( \{ \mathcal{G}, \Gamma_0, \Gamma_1 \} \) be a boundary triplet for \( S^* \), and let \( A_0 = \ker \Gamma_0 \) and \( A_1 = \ker \Gamma_1 \). Then the adjoint \( S^* \) admits the direct sum decompositions
\[
S^* = A_0 \overset{\sim}{\oplus} \hat{\mathcal{R}}_\lambda(S^*), \quad \lambda \in \rho(A_0),
\]
\[
S^* = A_1 \overset{\sim}{\oplus} \hat{\mathcal{R}}_\lambda(S^*), \quad \lambda \in \rho(A_1).
\] (2.1.13)

In particular, the restrictions \( \Gamma_0|\hat{\mathcal{R}}_\lambda(S^*) \) and \( \Gamma_1|\hat{\mathcal{R}}_\lambda(S^*) \) are isomorphisms from \( \hat{\mathcal{R}}_\lambda(S^*) \) onto \( \mathcal{G} \) for \( \lambda \in \rho(A_0) \) and \( \lambda \in \rho(A_1) \), respectively.

Proof. As \( A_0 \) and \( A_1 \) are self-adjoint, the direct sum decompositions (2.1.13) hold by Corollary 1.7.5. Since \( \Gamma_0 \) and \( \Gamma_1 \) map \( S^* \) onto \( \mathcal{G} \), and \( A_0 \) and \( A_1 \) are their respective kernels, it is clear that the restrictions \( \Gamma_0|\hat{\mathcal{R}}_\lambda(S^*) \) and \( \Gamma_1|\hat{\mathcal{R}}_\lambda(S^*) \) are isomorphisms from \( \hat{\mathcal{R}}_\lambda(S^*) \) onto \( \mathcal{G} \). \( \Box \)

In the rest of the text the self-adjoint extension \( A_0 = \ker \Gamma_0 \) will often serve as a point of reference due to the corresponding representation \( \{ 0 \} \times \mathcal{G} \) in the parameter space \( \mathcal{G} \). In the next proposition it is shown that \( A_0 \) and a given closed extension \( A_\Theta \) are disjoint (transversal) if and only if the parameter \( \Theta \) is a (bounded everywhere defined) operator.

Proposition 2.1.8. Let \( S \) be a closed symmetric relation in \( \mathcal{H} \), let \( \{ \mathcal{G}, \Gamma_0, \Gamma_1 \} \) be a boundary triplet for \( S^* \), let \( A_0 = \ker \Gamma_0 \), and let \( A_\Theta \) be the closed intermediate extension of \( S \) in \( \mathcal{H} \) corresponding to the closed relation \( \Theta \) in \( \mathcal{G} \) via (2.1.5). Then the following statements hold:

(i) \( A_\Theta \cap A_0 = S \) if and only if \( \Theta \) is a closed operator in \( \mathcal{G} \);

(ii) \( A_\Theta \cap A_0 = S \) and \( A_\Theta \overset{\sim}{\oplus} A_0 = S^* \) if and only if \( \Theta \in \mathcal{B}(\mathcal{G}) \).

Proof. Apply Lemma 2.1.5 to the self-adjoint extension \( A_0 = \ker \Gamma_0 \) that corresponds to \( \{ 0 \} \times \mathcal{G} \).

(i) \( A_\Theta \) and \( A_0 \) are disjoint if and only if \( \Theta \cap (\{ 0 \} \times \mathcal{G}) = \{ 0, 0 \} \), which is the same as saying that \( \text{mul} \Theta = \{ 0 \} \).

(ii) \( A_\Theta \) and \( A_0 \) are transversal if and only if
\[
\Theta \cap (\{ 0 \} \times \mathcal{G}) = \{ 0, 0 \} \quad \text{and} \quad \Theta \overset{\sim}{\oplus} (\{ 0 \} \times \mathcal{G}) = \mathcal{G} \times \mathcal{G},
\]
which is the same as saying that \( \text{mul} \Theta = \{ 0 \} \) and \( \text{dom} \Theta = \mathcal{G} \). By the closed graph theorem, the last two conditions are equivalent to \( \Theta \in \mathcal{B}(\mathcal{G}) \). \( \Box \)

Let \( S \) be a closed symmetric relation in \( \mathcal{H} \) with equal defect numbers and let \( H \) be a self-adjoint extension of \( S \). Later it will be shown that there exists a boundary triplet \( \{ \mathcal{G}, \Gamma_0, \Gamma_1 \} \) for \( S^* \) such that \( \ker \Gamma_0 \) coincides with \( H \); cf. Theorem 2.4.1. Furthermore, it will be shown that for a pair of self-adjoint extensions of \( S \) which are transversal, there exists a boundary triplet \( \{ \mathcal{G}, \Gamma_0, \Gamma_1 \} \) for \( S^* \) such that \( \ker \Gamma_0 \)
and \( \ker \Gamma_1 \) coincide with this pair; cf. Theorem 2.5.9. The notion of boundary triplet is not unique; in fact, a parametrization of all possible boundary triplets will be provided in Section 2.5.

The following theorem is of a different nature. It can be used to prove that a given relation \( T \) is the adjoint of a symmetric relation \( S \).

**Theorem 2.1.9.** Let \( T \) be a relation in \( \mathcal{H} \), let \( \mathcal{G} \) be a Hilbert space, and assume that

\[
\Gamma = \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix} : T \to \mathcal{G} \times \mathcal{G}
\]

is a linear mapping such that the following conditions are satisfied:

(i) \( \ker \Gamma_0 \) contains a self-adjoint relation \( A_0 \);
(ii) \( \text{ran} \Gamma = \mathcal{G} \times \mathcal{G} \);
(iii) for all \( \hat{f}, \hat{g} \in T \),

\[
(f', g)_{\mathcal{G}} - (f, g')_{\mathcal{G}} = (\Gamma_1 \hat{f}, \Gamma_0 \hat{g})_{\mathcal{G}} - (\Gamma_0 \hat{f}, \Gamma_1 \hat{g})_{\mathcal{G}}.
\]

Then \( S := \ker \Gamma \) is a closed symmetric relation in \( \mathcal{H} \) such that \( S^* = T \) and \( \{ \mathcal{G}, \Gamma_0, \Gamma_1 \} \) is a boundary triplet for \( S^* \) with \( A_0 = \ker \Gamma_0 \).

**Proof.** First note that condition (iii) implies that \( \ker \Gamma_0 \) is symmetric. To see this, let \( \hat{f}, \hat{g} \in \ker \Gamma_0 \). Then, by condition (iii),

\[
[\hat{f}, \hat{g}] = [\Gamma \hat{f}, \Gamma \hat{g}] = 0,
\]

and hence \( \ker \Gamma_0 \) is a symmetric relation in \( \mathcal{H} \). Then (i) gives \( A_0 = A_0^* \subset \ker \Gamma_0 \), which implies

\[
\ker \Gamma_0 \subset (\ker \Gamma_0)^* \subset A_0^* = A_0 \subset \ker \Gamma_0.
\]

Therefore, \( \ker \Gamma_0 = A_0 \) is self-adjoint in \( \mathcal{H} \). Moreover, \( S := \ker \Gamma \subset \ker \Gamma_0 \) is a symmetric relation in \( \mathcal{H} \).

It will be shown that

\[
S = T^*, \tag{2.1.14}
\]

so that, in particular, \( S \) is closed. To see \((\subset)\) in (2.1.14), let \( \hat{f} \in S = \ker \Gamma \). For any \( \hat{g} \in T \) one has \([\hat{f}, \hat{g}] = [\Gamma \hat{f}, \Gamma \hat{g}] = 0\), so that \( \hat{f} \in T^* \). To see \((\supset)\) in (2.1.14), let \( \hat{f} \in T^* \). Since \( A_0 \) is self-adjoint and \( A_0 \subset T \), it follows that \( T^* \subset A_0 = \ker \Gamma_0 \), so that \( \Gamma_0 \hat{f} = 0 \). For arbitrary \( \hat{g} \in T \) it therefore follows that

\[
0 = [\hat{f}, \hat{g}] = [\Gamma \hat{f}, \Gamma \hat{g}] = -i(\Gamma_1 \hat{f}, \Gamma_0 \hat{g}).
\]

From condition (ii) one concludes \( \text{ran} \Gamma_0 = \mathcal{G} \) and this leads to \( \Gamma_1 \hat{f} = 0 \). Hence, \( \hat{f} \in \ker \Gamma_0 \cap \ker \Gamma_1 = S \). Therefore, (2.1.14) is proved.
It follows from \( S = T^* \) that \( S^* = T^{**} = T \). Hence, it remains to show that \( T \) is closed. Let \( (\hat{f}_n) \) be a sequence in \( T \) converging to \( \hat{f} \). It suffices to show that \( \hat{f} \in T \). Let \( \hat{\psi} \in \mathcal{G}^2 \) and let \( \hat{g} \in T \) be such that \( \hat{\psi} = \mathcal{G}^{-1}\hat{g} \) (here condition (ii) is being used). Using the continuity of the indefinite inner product \([.,.]\) (see Section 1.8) one obtains
\[
(\Gamma \hat{f}_n, \hat{\psi}) = (\Gamma \hat{f}_n, \mathcal{G}^{-1}\hat{g}) = [\Gamma \hat{f}_n, \hat{g}] = [\hat{f}_n, \hat{g}] \to [\hat{f}, \hat{g}].
\] (2.1.15)
This shows that \( \Gamma \hat{f}_n \) is a weak Cauchy sequence in \( \mathfrak{N} \), hence weakly bounded and thus bounded. It follows that there exists a subsequence, again denoted by \( \Gamma \hat{f}_n \), which converges weakly to some \( \hat{\varphi} \in \mathfrak{N} \times \mathfrak{N} \). Now let \( \hat{h} \in T \) be such that \( \Gamma \hat{h} = \hat{\varphi} \) (again condition (ii) is being used). Choose \( \hat{g} \in T \) and let, as above, \( \hat{\psi} = \mathcal{G}^{-1}\Gamma \hat{g} \), so that (2.1.15) remains valid. Then (2.1.15) implies
\[
[\hat{f}, \hat{g}] = \lim_{n \to \infty} (\Gamma \hat{f}_n, \hat{\psi}) = (\hat{\varphi}, \hat{\psi}) = (\Gamma \hat{h}, \mathcal{G}^{-1}\Gamma \hat{g}) = [\Gamma \hat{h}, \hat{g}] = [\hat{h}, \hat{g}],
\]
and therefore \([\hat{f} - \hat{h}, \hat{g}] = 0 \). Since \( \hat{g} \in T \), one concludes that \( \hat{f} - \hat{h} \in T^* = S \subset T \). Now \( \hat{h} \in T \) implies that \( \hat{f} \in T \). Therefore, \( T \) is closed and it follows that \( S^* = T \).

By conditions (ii) and (iii) \( \{\mathfrak{N}, \Gamma_0, \Gamma_1\} \) is a boundary triplet for \( S^* \). Above it was also shown that \( A_0 = \ker \Gamma_0 \). □

## 2.2 Boundary value problems

Let \( S \) be a closed symmetric relation in \( \mathfrak{N} \) and let \( \{\mathfrak{N}, \Gamma_0, \Gamma_1\} \) be a boundary triplet for \( S^* \). Due to Theorem 2.1.3 one may think of the intermediate extensions of \( S \) being parametrized by the relations in the space \( \mathfrak{N} \); for this reason the space \( \mathfrak{N} \) will often be called the boundary space or parameter space associated with the boundary triplet. Let \( \Theta \) be a closed relation in \( \mathfrak{N} \) and let \( A_\Theta \) be the corresponding closed extension of \( S \) in \( \mathfrak{N} \) via (2.1.5):

\[
A_\Theta = \{ \hat{f} \in S^* : \Gamma \hat{f} \in \Theta \} = \ker (\Gamma_1 - \Theta \Gamma_0). \tag{2.2.1}
\]

Recall from Section 1.10 that any closed relation \( \Theta \) in \( \mathfrak{N} \) has a parametric representation of the form \( \Theta = \{ \mathcal{A} e, \mathcal{B} e : e \in \mathcal{E} \} \), i.e.,
\[
\Theta = \{ \{ \mathcal{A} e, \mathcal{B} e \} : e \in \mathcal{E} \} \tag{2.2.2}
\]
with some operators \( \mathcal{A}, \mathcal{B} \in \mathcal{B}(\mathcal{E}, \mathfrak{N}) \) and a Hilbert space \( \mathcal{E} \). Likewise, since \( \Theta^* \) is closed, it has a representation of the form
\[
\Theta^* = \{ \{ \mathcal{C} e', \mathcal{D} e' \} : e' \in \mathcal{E}' \} \tag{2.2.3}
\]
with some operators \( \mathcal{C}, \mathcal{D} \in \mathcal{B}(\mathcal{E}', \mathfrak{N}) \) and a Hilbert space \( \mathcal{E}' \). Thus, (2.2.3) gives
\[
\Theta = \{ \{ \varphi, \varphi' \} \in \mathfrak{N} \times \mathfrak{N} : \mathcal{D}^* \varphi = \mathcal{C}^* \varphi' \}. \tag{2.2.4}
\]
Therefore, it follows that $A_\Theta$ in (2.2.1) can be written as

$$A_\Theta = \{ \hat{f} \in S^* : \mathcal{D}_\Gamma \hat{f} = \mathcal{E}_\Gamma \hat{f} \}. \quad (2.2.5)$$

In the following it will be shown how the pair $\{ \mathcal{E}, \mathcal{D} \}$ in (2.2.3) and (2.2.5) can be expressed in terms of the original pair $\{ \mathcal{A}, \mathcal{B} \}$ in (2.2.2). The main result is contained in the next proposition.

Recall that the condition that $\Theta = \{ \mathcal{A}, \mathcal{B} \}$ is closed with some $\mathcal{A}, \mathcal{B} \in \mathcal{B}(\mathcal{E}, \mathcal{G})$ is equivalent to the condition that $\text{ran}(\mathcal{A}^* \mathcal{A} + \mathcal{B}^* \mathcal{B})$ is closed in $\mathcal{E}$; cf. Proposition 1.10.3. In fact, in the case where $\Theta$ is closed one may assume that the representing pair $\{ \mathcal{A}, \mathcal{B} \}$ satisfies the normalization condition $\mathcal{A}^* \mathcal{A} + \mathcal{B}^* \mathcal{B} = I$; cf. Proposition 1.10.3.

**Proposition 2.2.1.** Let $S$ be a closed symmetric relation in $\mathcal{H}$, let $\{ \mathcal{G}, \Gamma_0, \Gamma_1 \}$ be a boundary triplet for $S^*$, and let $A_\Theta$ be the closed extension of $S$ in $\mathcal{H}$ corresponding to the closed relation $\Theta$ in $\mathcal{G}$ via (2.1.5). Assume that $\Theta$ has the representation $\Theta = \{ \mathcal{A}, \mathcal{B} \}$ with $\mathcal{A}, \mathcal{B} \in \mathcal{B}(\mathcal{E}, \mathcal{G})$ such that $\mathcal{A}^* \mathcal{A} + \mathcal{B}^* \mathcal{B} = I$. \quad (2.2.6)

Then the intermediate extension $A_\Theta$ in (2.2.1) can be described as

$$A_\Theta = \{ \hat{f} \in S^* : \left( \frac{\mathcal{B} \mathcal{A}^*}{I - \mathcal{A} \mathcal{A}^*} \right) \Gamma_0 \hat{f} = \left( I - \frac{\mathcal{B} \mathcal{B}^*}{\mathcal{A} \mathcal{B}^*} \right) \Gamma_1 \hat{f} \}. \quad (2.2.7)$$

**Proof.** By Proposition 1.10.10, condition (2.2.6) implies that the relation $\Theta$ is given by

$$\Theta = \left\{ \{ \varphi, \varphi' \} \in \mathcal{G}^2 : \left( \frac{\mathcal{B} \mathcal{A}^*}{I - \mathcal{A} \mathcal{A}^*} \right) \varphi = \left( I - \frac{\mathcal{B} \mathcal{B}^*}{\mathcal{A} \mathcal{B}^*} \right) \varphi' \right\}. \quad (2.2.8)$$

Then (2.2.7) follows from (2.1.5). \qed

In the next proposition it will be assumed, in addition, that $\rho(\Theta) \neq \emptyset$. The following result is a reformulation of Theorem 1.10.5 and formula (2.1.5).

**Proposition 2.2.2.** Let $S$ be a closed symmetric relation in $\mathcal{H}$, let $\{ \mathcal{G}, \Gamma_0, \Gamma_1 \}$ be a boundary triplet for $S^*$, and let $A_\Theta$ be the closed extension of $S$ in $\mathcal{H}$ corresponding to the closed relation $\Theta$ in $\mathcal{G}$ via (2.1.5). Then $\mu \in \rho(\Theta)$ if and only if $\Theta$ has the representation $\Theta = \{ \mathcal{A}, \mathcal{B} \}$ with $\mathcal{A}, \mathcal{B} \in \mathcal{B}(\mathcal{G})$ such that $(\mathcal{B} - \mu \mathcal{A})^{-1} \in \mathcal{B}(\mathcal{G})$. Moreover, the pair $\{ \mathcal{A}, \mathcal{B} \}$ may be chosen such that $\Theta^* = \{ \mathcal{A}^*, \mathcal{B}^* \}$. In this case the intermediate extension $A_\Theta$ in (2.2.1) can be described as

$$A_\Theta = \{ \hat{f} \in S^* : \mathcal{B} \Gamma_0 \hat{f} = \mathcal{A} \Gamma_1 \hat{f} \}. \quad (2.2.8)$$

For $\mu \in \rho(\Theta)$ it follows from (1.10.9) that in Proposition 2.2.2 one can choose

$$\mathcal{A} = (\Theta - \mu)^{-1} \quad \text{and} \quad \mathcal{B} = I + \mu(\Theta - \mu)^{-1}. \quad (2.2.9)$$
In the case that $\mu \in \mathbb{C} \setminus \mathbb{R}$ one may also choose
\[ A = I - C_\mu[\Theta] \quad \text{and} \quad B = \mu - \mu C_\mu[\Theta] \] (2.2.10)
by (1.10.10), where $C_\mu$ denotes the Cayley transform.

The next corollary is a translation of Corollary 2.1.4 and Corollary 1.10.8. In each of the cases in this corollary one may apply Proposition 2.2.2 by choosing the pair $\{A, B\}$ as in (2.2.9) or (2.2.10) with $\mu \in \mathbb{C} \setminus \mathbb{R}$, $\mu \in \mathbb{C}^+$, or $\mu \in \mathbb{C}^-$, respectively.

**Corollary 2.2.3.** Let $S$ be a closed symmetric relation in $\mathcal{H}$, let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for $S^*$, and let $A_\Theta$ be the closed extension of $S$ in $\mathcal{G}$ corresponding to the closed relation $\Theta$ in $\mathcal{G}$ via (2.1.5). Assume that $\Theta = \{A, B\}$. Then the following statements hold:

(i) $A_\Theta$ is self-adjoint if and only if
\[ \text{Im}(A^*B) = 0 \quad \text{and} \quad (B - \mu A)^{-1} \in \mathcal{B}(\mathcal{G}) \]
for some, and hence for all $\mu \in \mathbb{C}^+$ and for some, and hence for all $\mu \in \mathbb{C}^-$;

(ii) $A_\Theta$ is maximal dissipative if and only if
\[ \text{Im}(A^*B) \geq 0 \quad \text{and} \quad (B - \mu A)^{-1} \in \mathcal{B}(\mathcal{G}) \]
for some, and hence for all $\mu \in \mathbb{C}^-$;

(iii) $A_\Theta$ is maximal accumulative if and only if
\[ \text{Im}(A^*B) \leq 0 \quad \text{and} \quad (B - \mu A)^{-1} \in \mathcal{B}(\mathcal{G}) \]
for some, and hence for all $\mu \in \mathbb{C}^+$.

In the case that the representation $\Theta = \{A, B\}$ is chosen so that $\Theta^* = \{A^*, B^*\}$, the extension $A_\Theta$ is given by (2.2.8).

Now the converse question will be addressed. Let $A$ be a closed extension of $S$ given in terms of boundary conditions. The problem is to determine a corresponding parameter $\Theta$ in $\mathcal{G}$ such that $A = A_\Theta$.

**Proposition 2.2.4.** Let $S$ be a closed symmetric relation in $\mathcal{H}$, and let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for $S^*$. Assume that $\mathfrak{F}$ is a Hilbert space, $\mathcal{M}, \mathcal{N} \in \mathcal{B}(\mathcal{G}, \mathfrak{F})$, and that, without loss of generality, the space $\mathfrak{F}$ is minimal:
\[ \mathfrak{F} = \text{span}\{\text{ran } \mathcal{M}, \text{ran } \mathcal{N}\}. \]

Furthermore, assume that $\mathcal{M} - \mu \mathcal{N} \in \mathcal{B}(\mathcal{G}, \mathfrak{F})$ is bijective for some $\mu \in \mathbb{C}$ and let $A$ be an intermediate extension of $S$ of the form
\[ A = \{\hat{f} \in S^* : \mathcal{M}\Gamma_0\hat{f} = \mathcal{N}\Gamma_1\hat{f}\}. \] (2.2.11)

Then $A$ is closed and $A = A_\Theta$, where the parameter $\Theta = \{A, B\}$ is given by
\[ \{A, B\} = \{(M - \mu N)^{-1} N, (M - \mu N)^{-1} M\}. \]
Proof. First observe that the intermediate extension $A$ in (2.2.11) is closed since $M, N \in \mathcal{B}(\mathcal{G}, \mathcal{F})$. Moreover, $A$ corresponds to the closed relation $\Theta$ in $\mathcal{G}$ given by

$$\Theta = \{\{\varphi, \varphi'\} \in \mathcal{G} \times \mathcal{G} : M\varphi = N\varphi'\}.$$ 

Now the assertion follows from Proposition 1.10.7. $\square$

Let again $\Theta$ be a closed relation in $\mathcal{G}$ and let $A_{\Theta}$ be the corresponding closed extension in $H$ via (2.1.5). Assume, in addition, that $\Theta$ admits an orthogonal decomposition $\Theta = \Theta_{op} \oplus \Theta_{mul}$, $\mathcal{G} = \mathcal{G}_{op} \oplus \mathcal{G}_{mul}$, into a (not necessarily densely defined) operator part $\Theta_{op}$ acting in the Hilbert space $\mathcal{G}_{op} = \text{dom} \Theta^* = (\text{mul} \Theta)^\perp$ and a multivalued part $\Theta_{mul} = \{0\} \times \text{mul} \Theta$ in the Hilbert space $\mathcal{G}_{mul} = \text{mul} \Theta$; cf. Theorem 1.3.16 and the discussion following it.

Recall from Theorem 1.4.11, Theorem 1.5.1, and Theorem 1.6.12 that any closed symmetric, self-adjoint, (maximal) dissipative, or (maximal) accumulative relation $\Theta$ in $\mathcal{G}$ gives rise to such a decomposition. If $P_{op}$ denotes the orthogonal projection in $\mathcal{G}$ onto $\mathcal{G}_{op}$, then the closed extension $A_{\Theta}$ in (2.1.5) has the form

$$A_{\Theta} = \{\widehat{f} \in S^* : \Theta_{op} P_{op} \Gamma_0 \widehat{f} = P_{op} \Gamma_1 \widehat{f}, (I_{\mathcal{G}} - P_{op}) \Gamma_0 \widehat{f} = 0\}. \quad (2.2.12)$$

Note that this abstract boundary condition also requires $P_{op} \Gamma_0 \widehat{f} \in \text{dom} \Theta_{op}$.

2.3 Associated $\gamma$-fields and Weyl functions

Let $S$ be a closed symmetric relation in the Hilbert space $\mathcal{H}$ and let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for $S^*$. Recall from Lemma 2.1.7 that $\Gamma_0$ maps $\widehat{\mathcal{H}}_\lambda(S^*)$ bijectively onto $\mathcal{G}$ when $\lambda \in \rho(A_0)$. Hence, the inverse mapping

$$\widehat{\gamma}(\lambda) := (\Gamma_0 | \widehat{\mathcal{H}}_\lambda(S^*))^{-1}, \quad \lambda \in \rho(A_0),$$

maps $\mathcal{G}$ bijectively onto $\widehat{\mathcal{H}}_\lambda(S^*)$. Let $\pi_1$ be the orthogonal projection from $\mathcal{H} \times \mathcal{H}$ onto $\mathcal{H} \times \{0\}$. Then $\pi_1$ maps $\widehat{\mathcal{H}}_\lambda(S^*)$ bijectively onto $\mathcal{H}_\lambda(S^*)$.

**Definition 2.3.1.** Let $S$ be a closed symmetric relation in $\mathcal{H}$, let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for $S^*$, and let $A_0 = \ker \Gamma_0$. Then

$$\rho(A_0) \ni \lambda \mapsto \gamma(\lambda) = \{\Gamma_0 \widehat{f}_\lambda, f_\lambda \} : \widehat{f}_\lambda \in \widehat{\mathcal{H}}_\lambda(S^*)\}$$ \quad (2.3.1)

or, equivalently,

$$\rho(A_0) \ni \lambda \mapsto \gamma(\lambda) = \pi_1 \widehat{\gamma}(\lambda) = \pi_1 (\Gamma_0 | \widehat{\mathcal{H}}_\lambda(S^*))^{-1},$$

is called the $\gamma$-field associated with the boundary triplet $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$. 
The main properties of the $\gamma$-field will now be discussed.

**Proposition 2.3.2.** Let $S$ be a closed symmetric relation in $\mathfrak{S}$, let $\{\mathfrak{S}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for $S^*$, and let $A_0 = \ker \Gamma_0$. Then the following statements hold for the corresponding $\gamma$-field $\gamma$:

(i) $\gamma(\lambda) \in B(\mathfrak{S}, \hat{\mathfrak{S}})$ for all $\lambda \in \rho(A_0)$ and, in fact, $\gamma(\lambda)$ maps $\mathfrak{S}$ isomorphically onto $\mathfrak{N}_\lambda(S^*) \subset \hat{\mathfrak{S}}$;

(ii) for all $\lambda, \mu \in \rho(A_0)$ the operators $\gamma(\lambda)$ and $\gamma(\mu)$ are connected via

$$\gamma(\lambda) = (I + (\lambda - \mu)(A_0 - \lambda)^{-1})\gamma(\mu);$$

(iii) the operator function $\gamma : \rho(A_0) \to B(\mathfrak{S}, \hat{\mathfrak{S}})$, $\lambda \mapsto \gamma(\lambda)$, is holomorphic, i.e., the limit

$$\frac{d}{d\mu} \gamma(\mu) = \lim_{\lambda \to \mu} \frac{\gamma(\lambda) - \gamma(\mu)}{\lambda - \mu}$$

exists for all $\mu \in \rho(A_0)$ in $B(\mathfrak{S}, \hat{\mathfrak{S}})$;

(iv) for all $\lambda \in \rho(A_0)$ the operator $\gamma(\lambda)^* \in B(\hat{\mathfrak{S}}, \mathfrak{S})$ is given by

$$\gamma(\lambda)^* h = \Gamma_1 \{(A_0 - \bar{\lambda})^{-1} h, (I + \bar{\lambda}(A_0 - \bar{\lambda})^{-1}) h\}, \quad h \in \mathfrak{S}, \quad (2.3.2)$$

and $\ker \gamma(\lambda)^* = (\mathfrak{N}_\lambda(S^*))^\perp = \text{ran}(S - \bar{\lambda})$ holds. Moreover, one has

$$\Gamma\{(A_0 - \bar{\lambda})^{-1} h, (I + \bar{\lambda}(A_0 - \bar{\lambda})^{-1}) h\} = \{0, \gamma(\lambda)^* h\}, \quad h \in \mathfrak{S}. \quad (2.3.3)$$

**Proof.** (i) Let $\lambda \in \rho(A_0)$. Since the restriction of $\Gamma_0$ to $\hat{\mathfrak{N}}_\lambda(S^*)$ is an isomorphism from $\hat{\mathfrak{N}}_\lambda(S^*)$ onto $\mathfrak{S}$ (see Lemma 2.1.7), while $\pi_1$ is an isomorphism from $\hat{\mathfrak{N}}_\lambda(S^*)$ onto $\mathfrak{N}_\lambda(S^*)$, it follows from Definition 2.3.1 that the mapping $\gamma(\lambda)$ is an isomorphism from $\mathfrak{S}$ onto $\mathfrak{N}_\lambda(S^*)$. From this it is also clear that $\gamma(\lambda) \in B(\mathfrak{S}, \hat{\mathfrak{S}})$.

(ii) Let $\lambda, \mu \in \rho(A_0)$ and let $\varphi \in \mathfrak{S}$. Then there exists $\hat{f}_\mu = \{f_\mu, \mu f_\mu\} \in \hat{\mathfrak{N}}_\mu(S^*)$ such that $\varphi = \Gamma_0 \hat{f}_\mu$ and hence $f_\mu = \gamma(\mu) \varphi$. Due to $S^* = A_0 \tilde{+} \hat{\mathfrak{N}}_\lambda(S^*)$ there exist $\tilde{h} \in A_0$ and $\tilde{f}_\lambda = \{f_\lambda, \lambda f_\lambda\} \in \hat{\mathfrak{N}}_\lambda(S^*)$ such that

$$\hat{f}_\mu = \tilde{h} + \tilde{f}_\lambda.$$

Observe that $\tilde{f}_\lambda - \tilde{f}_\mu = -\tilde{h} \in A_0$, which gives $\Gamma_0 \tilde{f}_\lambda = \Gamma_0 \tilde{f}_\mu$, so that $\Gamma_0 \tilde{f}_\lambda = \varphi$ and $f_\lambda = \gamma(\lambda) \varphi$. Moreover, this observation also shows that for some $g \in \mathfrak{S}$

$$\{f_\lambda, \lambda f_\lambda\} = \{f_\mu, \mu f_\mu\} + \{(A_0 - \lambda)^{-1} g, (I + \lambda(A_0 - \lambda)^{-1}) g\}.$$

Hence, $\{f_\lambda, 0\} = \{f_\mu, (\mu - \lambda) f_\mu\} + \{(A_0 - \lambda)^{-1} g, g\}$, so that $g = (\lambda - \mu) f_\mu$. Therefore, $f_\lambda = f_\mu + (\lambda - \mu)(A_0 - \lambda)^{-1} f_\mu$, which implies

$$\gamma(\lambda) \varphi = (I + (\lambda - \mu)(A_0 - \lambda)^{-1}) \gamma(\mu) \varphi.$$
(iii) Fix some $\mu \in \rho(A_0)$. Then it follows from (ii) and the fact that the mapping $\lambda \mapsto (A_0 - \lambda)^{-1}$ is a holomorphic operator function with values in $\mathcal{B}(\mathfrak{g})$ that $\lambda \mapsto \gamma(\lambda)$ is a holomorphic operator function on $\rho(A_0)$ with values in $\mathcal{B}(\mathfrak{g}, \mathfrak{g})$.

(iv) Fix $\lambda \in \rho(A_0)$ and let $h \in \mathfrak{g}$. Then there exists $\hat{k} = \{k, k'\} \in A_0$ with $h = k' - \bar{\lambda}k$. Let $\varphi \in \mathfrak{g}$; then $\gamma(\lambda)\varphi = f_{\lambda}$ for some $f_{\lambda} \in \mathcal{H}(S^*)$. Hence, with the abstract Green identity for $\hat{k} = \{k, k'\}$ and $\tilde{f}_{\lambda} = \{f_{\lambda}, \lambda f_{\lambda}\}$ it follows from $\Gamma_0\hat{k} = 0$ that

$$(\varphi, \gamma(\lambda)^*h) = (\gamma(\lambda)\varphi, k' - \bar{\lambda}k)$$

$$= (f_{\lambda}, k' - \bar{\lambda}k)$$

$$= -(f_{\lambda}, k' - (f_{\lambda}, k'))$$

$$= -(\Gamma_1\tilde{f}_{\lambda}, \Gamma_0\hat{k}) - (\Gamma_0\tilde{f}_{\lambda}, \Gamma_1\hat{k})$$

$$= (\tilde{f}_{\lambda}, \Gamma_0\hat{k})$$

$$= (\varphi; \Gamma_1\hat{k}),$$

which implies

$$\gamma(\lambda)^*h = \Gamma_1\hat{k} = \Gamma_1\{(A_0 - \bar{\lambda})^{-1}h, (I + \bar{\lambda}(A_0 - \bar{\lambda})^{-1})h\}.$$  

The identity $\ker \gamma(\lambda)^* = (\mathcal{H}_\lambda(S^*))^\perp$ follows from $\text{ran} \gamma(\lambda) = \mathcal{H}_\lambda(S^*)$. Furthermore, the identity $(\mathcal{H}_\lambda(S^*))^\perp = \text{ran} (S - \bar{\lambda})$ is clear and $(2.3.3)$ follows from $(2.3.2)$ and

$$\{(A_0 - \bar{\lambda})^{-1}h, (I + \bar{\lambda}(A_0 - \bar{\lambda})^{-1})h\} \in A_0 = \ker \Gamma_0.$$

This completes the proof.

In the case where the symmetric relation $S$ is a densely defined symmetric operator and $\{\mathfrak{g}, \Gamma_0, \Gamma_1\}$ is a boundary triplet for $S^*$ with boundary mappings $\Gamma_0$ and $\Gamma_1$ defined on $\text{dom} S^*$ (see the text below Definition 2.1.1 and (2.1.2)) the formula for the adjoint $\gamma(\lambda)^*$ of the corresponding $\gamma$-field in Proposition 2.3.2 (iv) has the simpler form

$$\gamma(\lambda)^*h = \Gamma_1(A_0 - \bar{\lambda})^{-1}h, \quad \lambda \in \rho(A_0), \ h \in \mathfrak{g}.$$  

According to Proposition 2.3.2 (iv), the action of $\Gamma_1$ on a general element of $A_0$ is expressed in terms of the operator $\gamma(\lambda)^*$. The form of this action is particularly simple on eigenelements of $A_0$.

**Corollary 2.3.3.** Let $\lambda \in \rho(A_0)$ and assume that $\{h, xh\} \in A_0$ with $x \in \mathbb{R}$. Then

$$\Gamma_1\{h, xh\} = (x - \bar{\lambda})\gamma(\lambda)^*h.$$  

If $\{0, h\} \in A_0$, then

$$\Gamma_1\{0, h\} = \gamma(\lambda)^*h.$$
2.3. Associated $\gamma$-fields and Weyl functions

Proof. Let $\lambda \in \rho(A_0)$ and let $\{h, xh\} \in A_0$ with $x \in \mathbb{R}$. Then

$$h = (A_0 - \lambda)^{-1}(x - \lambda)h,$$

which together with Proposition 2.3.2 (iv) leads to

$$(x - \lambda)\gamma(\lambda)^*h = (x - \lambda)\Gamma_1\{(A_0 - \lambda)^{-1}h, (I + \lambda(A_0 - \lambda)^{-1})h\}$$

$$= \Gamma_1\{h, xh\}.$$ 

If $\{0, h\} \in A_0$, then $h \in \ker(A_0 - \lambda)^{-1}$ and the expression for $\Gamma_1\{0, h\}$ follows directly from Proposition 2.3.2 (iv). \qed

The definition and properties of the $\gamma$-field now give rise to the notion of Weyl function. It is defined, as in the case of the $\gamma$-field, for a closed symmetric relation $S$ in terms of the boundary triplet for $S^*$ and the eigenspaces of $S^*$.

**Definition 2.3.4.** Let $S$ be a closed symmetric relation in $\mathcal{H}$, let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for $S^*$, and let $A_0 = \ker \Gamma_0$. Then

$$\rho(A_0) \ni \lambda \mapsto M(\lambda) = \{(\Gamma_0 \hat{f}_\lambda, \Gamma_1 \hat{f}_\lambda) : \hat{f}_\lambda \in \mathcal{H}_\lambda(S^*)\}$$

(2.3.4)

or, equivalently,

$$\rho(A_0) \ni \lambda \mapsto M(\lambda) = \Gamma_1 \hat{\gamma}(\lambda) = \Gamma_1(\Gamma_0 | \mathcal{H}_\lambda(S^*))^{-1},$$

is called the Weyl function associated with the boundary triplet $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$.

Here is a simple example of a Weyl function for a trivial symmetric relation $S$ in $\mathcal{H}$. Note that in this example one has $\mathcal{G} = \mathcal{H}$, i.e., the corresponding boundary triplet maps onto $\mathcal{H} \times \mathcal{H}$; this situation is not typical in standard applications; cf. Chapters 6, 7, and 8.

**Example 2.3.5.** Let $S = \{0, 0\}$ be the trivial symmetric relation in $\mathcal{H}$. It is clear that $S^* = \mathcal{H} \times \mathcal{H}$ and $\mathcal{H}_\lambda(S^*) = \mathcal{H}$ for $\lambda \in \mathbb{C}$. Now define

$$\Gamma_0 \hat{f} = f' \quad \text{and} \quad \Gamma_1 \hat{f} = -f,$$

so that $\Gamma : S^* \to \mathcal{H} \times \mathcal{H}$ is surjective and (2.1.1) is satisfied. Hence, $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ is a boundary triplet for $S^*$. Note that

$$A_0 = \ker \Gamma_0 = \mathcal{H} \times \{0\}$$

is a self-adjoint extension of $S$ with $\rho(A_0) = \mathbb{C} \setminus \{0\}$, $\sigma(A_0) = \{0\}$, and $\mathcal{N}_0(A_0) = \mathcal{H}$. It follows from Definition 2.3.1 and Definition 2.3.4 that the $\gamma$-field and the Weyl function are given by $\gamma(\lambda) = (1/\lambda)I$ and $M(\lambda) = -(1/\lambda)I$, respectively.

Next some elementary properties of the Weyl function are discussed. Recall that the real part and imaginary part of a bounded operator $T \in \mathcal{B}(\mathcal{G})$ are defined as $\text{Re} T = \frac{1}{2}(T + T^*)$ and $\text{Im} T = \frac{1}{2i}(T - T^*)$, respectively.

**Proposition 2.3.6.** Let $S$ be a closed symmetric relation in $\mathcal{H}$, let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for $S^*$, and let $A_0 = \ker \Gamma_0$. Then the following statements hold.
for the corresponding $\gamma$-field $\gamma$ and Weyl function $M$:

(i) $M(\lambda) \in \mathcal{B}(\mathcal{G})$ for all $\lambda \in \rho(A_0)$;

(ii) for all $\lambda \in \rho(A_0)$ one has $M(\lambda)\Gamma_0\hat{f}_\lambda = \Gamma_1\hat{f}_\lambda$ for every $\hat{f}_\lambda \in \hat{\mathcal{H}}_\lambda(S^*)$ or, equivalently, one has

$$\Gamma\hat{\gamma}(\lambda)\varphi = \{\Gamma_0\hat{\gamma}(\lambda)\varphi, \Gamma_1\hat{\gamma}(\lambda)\varphi\} = \{\varphi, M(\lambda)\varphi\}$$

for every $\varphi \in \mathcal{G}$;

(iii) for all $\lambda, \mu \in \rho(A_0)$ the identity

$$M(\lambda) - M(\mu)^* = (\lambda - \bar{\mu})\gamma(\mu)^*\gamma(\lambda)$$

holds, and, in particular, the symmetry condition $M(\lambda)^* = M(\bar{\lambda})$ holds for all $\lambda \in \rho(A_0)$;

(iv) $\text{Im} M(\lambda) \in \mathcal{B}(\mathcal{G})$ is a nonnegative (nonpositive) self-adjoint operator for all $\lambda \in \mathbb{C}^+ (\lambda \in \mathbb{C}^-)$ and $0 \in \rho(\text{Im} M(\lambda))$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$;

(v) for any fixed $\lambda_0 \in \rho(A_0)$ and all $\lambda \in \rho(A_0)$ the resolvent of $A_0$ and the function $M$ are connected via

$$M(\lambda) = \text{Re} M(\lambda_0) + \gamma(\lambda_0)^* \left[ \lambda - \text{Re} \lambda_0 + (\lambda - \lambda_0)(\lambda - \bar{\lambda}_0)(A_0 - \lambda)^{-1} \right] \gamma(\lambda_0);$$

(vi) the identity

$$\gamma(\bar{\mu})^*(A_0 - \lambda)^{-1}\gamma(\nu) = \frac{M(\lambda)}{(\lambda - \nu)(\lambda - \mu)} + \frac{M(\mu)}{(\mu - \nu)(\mu - \lambda)} + \frac{M(\nu)}{(\nu - \lambda)(\nu - \mu)}$$

holds for $\lambda, \mu, \nu \in \rho(A_0)$ such that $\lambda \neq \nu$, $\lambda \neq \mu$, and $\nu \neq \mu$.

Proof. (i) Let $\lambda \in \rho(A_0)$. By Lemma 2.1.7, the restriction of $\Gamma_0$ to $\hat{\mathcal{H}}_\lambda(S^*)$ is an isomorphism between $\hat{\mathcal{H}}_\lambda(S^*)$ and $\mathcal{G}$. Hence, the inverse $\hat{\gamma}(\lambda)$ is an isomorphism between $\mathcal{G}$ and $\hat{\mathcal{H}}_\lambda(S^*)$, and since the operator $\Gamma_1 : S^* \to \mathcal{G}$ is continuous by Proposition 2.1.2 (i), it follows from Definition 2.3.4 that $M(\lambda) = \Gamma_1\hat{\gamma}(\lambda) \in \mathcal{B}(\mathcal{G})$.

(ii) It is clear from (i) and the definition of $M(\lambda)$ that $M(\lambda)\Gamma_0\hat{f}_\lambda = \Gamma_1\hat{f}_\lambda$ for every $\hat{f}_\lambda \in \hat{\mathcal{H}}_\lambda(S^*)$. Now $\hat{\gamma}(\lambda)\varphi$ belongs to $\hat{\mathcal{H}}_\lambda(S^*)$ for $\varphi \in \mathcal{G}$, so that

$$\{\Gamma_0\hat{\gamma}(\lambda)\varphi, \Gamma_1\hat{\gamma}(\lambda)\varphi\} = \{\Gamma_0\hat{\gamma}(\lambda)\varphi, M(\lambda)\Gamma_0\gamma(\lambda)\varphi\} = \{\varphi, M(\lambda)\varphi\}.$$ 

Conversely, assume that $\{\Gamma_0\hat{\gamma}(\lambda)\varphi, \Gamma_1\hat{\gamma}(\lambda)\varphi\} = \{\varphi, M(\lambda)\varphi\}$ for all $\varphi \in \mathcal{G}$ and let $\hat{f}_\lambda \in \hat{\mathcal{H}}_\lambda(S^*)$. Then $\hat{f}_\lambda = \hat{\gamma}(\lambda)\varphi$ for some $\varphi \in \mathcal{G}$ and hence

$$\{\Gamma_0\hat{f}_\lambda, \Gamma_1\hat{f}_\lambda\} = \{\Gamma_0\hat{\gamma}(\lambda)\varphi, \Gamma_1\hat{\gamma}(\lambda)\varphi\} = \{\varphi, M(\lambda)\varphi\}$$

yields $M(\lambda)\Gamma_0\hat{f}_\lambda = \Gamma_1\hat{f}_\lambda$.

(iii) Let $\lambda, \mu \in \rho(A_0)$. For given $\varphi, \psi \in \mathcal{G}$ one can choose

$$\hat{h}_\lambda = \{h_\lambda, \lambda h_\lambda\} \in \hat{\mathcal{H}}_\lambda(S^*) \quad \text{and} \quad \hat{k}_\mu = \{k_\mu, \mu k_\mu\} \in \hat{\mathcal{H}}_\mu(S^*),$$

for all $\lambda \in \rho(A_0)$;
such that \( \varphi = \Gamma_0 \hat{h}_\lambda \) and \( \psi = \Gamma_0 \hat{k}_\mu \). Clearly, \( \gamma(\lambda) \varphi = h_\lambda \), \( \gamma(\mu) \psi = k_\mu \), and the abstract Green identity applied to \( \hat{h}_\lambda \) and \( \hat{k}_\mu \) shows that

\[
(M(\lambda) - M(\mu)^*) \varphi, \psi = (M(\lambda) \varphi, \psi) - (\varphi, M(\mu) \psi)
\]

\[
= (M(\lambda) \Gamma_0 \hat{b}_\lambda, \Gamma_0 \hat{k}_\mu) - (\Gamma_0 \hat{b}_\lambda, M(\mu) \Gamma_0 \hat{k}_\mu)
\]

\[
= (\Gamma_1 \hat{h}_\lambda, \Gamma_0 \hat{k}_\mu) - (\Gamma_0 \hat{h}_\lambda, \Gamma_1 \hat{k}_\mu)
\]

\[
= (\lambda h_\lambda, k_\mu) - (h_\lambda, \mu k_\mu)
\]

\[
= (\lambda - \mu)(h_\lambda, k_\mu)
\]

\[
= ((\lambda - \mu)\gamma(\lambda) \varphi, \gamma(\mu) \psi).
\]

Thus, one has the identity \( M(\lambda) - M(\mu)^* = (\lambda - \mu)\gamma(\mu)^* \gamma(\lambda) \). Setting \( \mu = \overline{\lambda} \) it follows that \( M(\lambda) = M(\overline{\lambda})^* \) and therefore \( M(\lambda)^* = M(\overline{\lambda}), \lambda \in \rho(A_0) \).

(iv) The assertion (iii) gives for \( \lambda \in \mathbb{C} \setminus \mathbb{R} \)

\[
\frac{\text{Im} M(\lambda) \varphi, \varphi}{\text{Im} \lambda} = (\gamma(\lambda)^* \gamma(\lambda) \varphi, \varphi) = \|\gamma(\lambda) \varphi\|^2, \quad \varphi \in S.
\]

Hence, for \( \lambda \in \mathbb{C}^+ \) or \( \lambda \in \mathbb{C}^- \) the operator \( \text{Im} M(\lambda) \) is nonnegative or nonpositive, respectively. As \( \gamma(\lambda) \) is an isomorphism from \( S \) onto \( \mathfrak{H}_\lambda(S^*) \) it follows that for all \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) the operator \( \text{Im} M(\lambda) \) is boundedly invertible.

(v) Let \( \lambda_0 \in \rho(A_0) \) be fixed. Then assertion (iii) implies

\[
\text{Im} M(\lambda_0) = (\text{Im} \lambda_0) \gamma(\lambda_0)^* \gamma(\lambda_0),
\]

while \( \gamma(\lambda) = (I + (\lambda - \lambda_0)(A_0 - \lambda)^{-1}) \gamma(\lambda_0), \lambda \in \rho(A_0) \), by Proposition 2.3.2. Using (iii) this leads to

\[
M(\lambda) = M(\lambda_0)^* + (\lambda - \lambda_0) \gamma(\lambda_0)^* \gamma(\lambda)
\]

\[
= \text{Re} M(\lambda_0) - i \text{Im} M(\lambda_0) + (\lambda - \lambda_0) \gamma(\lambda_0)^* [I + (\lambda - \lambda_0)(A_0 - \lambda)^{-1}] \gamma(\lambda_0)
\]

\[
= \text{Re} M(\lambda_0) + \gamma(\lambda_0)^* [(\lambda - \text{Re} \lambda_0) + (\lambda - \lambda_0)(\lambda - \lambda_0)(A_0 - \lambda)^{-1}] \gamma(\lambda_0)
\]

for all \( \lambda \in \rho(A_0) \).

(vi) It follows from item (iii) and \( \gamma(\lambda) = (I + (\lambda - \nu)(A_0 - \lambda)^{-1}) \gamma(\nu) \) in Proposition 2.3.2 (ii) that

\[
\gamma(\overline{\mu})^*(A_0 - \lambda)^{-1} \gamma(\nu) = \gamma(\mu)^* \frac{\gamma(\lambda) - \gamma(\nu)}{\lambda - \nu}
\]

\[
= \frac{1}{\lambda - \nu} \left( \gamma(\overline{\mu})^* \gamma(\lambda) - \gamma(\mu)^* \gamma(\nu) \right)
\]

\[
= \frac{1}{\lambda - \nu} \left( \frac{M(\lambda) - M(\overline{\mu})^*}{\lambda - \mu} - \frac{M(\nu) - M(\overline{\nu})^*}{\nu - \mu} \right),
\]

and a simple calculation using \( M(\overline{\mu})^* = M(\mu) \) then yields the assertion. \( \square \)
In the next corollary it turns out that the Weyl function $M$ is a uniformly strict Nevanlinna function; cf. Definition A.4.1 and Definition A.4.7.

**Corollary 2.3.7.** The Weyl function $M$ in Definition 2.3.4 is a uniformly strict $\mathcal{B}(\mathcal{G})$-valued Nevanlinna function. Its values $M(\lambda)$ are maximal dissipative (maximal accumulative) operators for $\lambda \in \mathbb{C}^+$ ($\lambda \in \mathbb{C}^-$), and $-\lambda \in \rho(M(\lambda))$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

**Proof.** According to Proposition 2.3.2 (iii), the function $\lambda \mapsto \gamma(\lambda)$ is holomorphic on $\rho(A_0)$. Hence, it follows from Proposition 2.3.6 (iii) with fixed $\mu \in \rho(A_0)$ that the function $\lambda \mapsto M(\lambda)$ is holomorphic on $\rho(A_0)$ and hence, in particular, on the possibly smaller subset $\mathbb{C} \setminus \mathbb{R}$. Clearly, according to Proposition 2.3.6 (iii) and (iv) one has $M(\lambda)^* = M(\overline{\lambda})$ and $(\text{Im } \lambda)(\text{Im } M(\lambda)) \geq 0$ for $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and hence $M$ is a $\mathcal{B}(\mathcal{G})$-valued Nevanlinna function. It follows from Proposition 2.3.6 (iv) that $M$ is uniformly strict. □

**Corollary 2.3.8.** Let $M$ be the Weyl function in Definition 2.3.4. Then the following statements hold:

(i) if $x \in \rho(A_0) \cap \mathbb{R}$, then $M(x) \in \mathcal{B}(\mathcal{G})$ is self-adjoint;

(ii) if $x \in \rho(A_0) \cap \mathbb{R}$, then the derivative $M'(x) \in \mathcal{B}(\mathcal{G})$ is a nonnegative self-adjoint operator and $0 \in \rho(M'(x));$

(iii) if $(a, b) \subset \mathbb{R}$ belongs to $\rho(A_0)$, then for all $\varphi \in \mathcal{G}$ the function

$$x \mapsto (M(x)\varphi, \varphi)$$

is nondecreasing on $(a, b)$;

(iv) if $(a, b) \subset \mathbb{R}$ belongs to $\rho(A_0)$, then there exist self-adjoint relations $M(a)$ and $M(b)$ in $\mathcal{G}$ such that

$$M(b) = \lim_{x \uparrow b} M(x) \quad \text{and} \quad M(a) = \lim_{x \downarrow a} M(x)$$

in the strong graph sense or, equivalently, in the strong resolvent sense on $\mathbb{C} \setminus \mathbb{R}$.

**Proof.** (i) It follows from $M(\lambda)^* = M(\overline{\lambda})$, $\lambda \in \rho(A_0)$, that for $x \in \rho(A_0) \cap \mathbb{R}$ one has that $M(x)^* = M(x)$, i.e., $M(x) \in \mathcal{B}(\mathcal{G})$ is self-adjoint.

(ii) Since $M$ is holomorphic on $\rho(A_0)$, it is clear that the derivative $M'(x) \in \mathcal{B}(\mathcal{G})$ exists. Moreover, for all $\varphi \in \mathcal{G}$ and $y \neq x$ Proposition 2.3.6 (iii) shows that

$$M'(x)\varphi, \varphi = \lim_{y \to x} \frac{(M(x)\varphi, \varphi) - (M(y)\varphi, \varphi)}{x - y} = \lim_{y \to x} (\gamma(x)\varphi, \gamma(y)\varphi) = \|\gamma(x)\varphi\|^2$$

and hence $M'(x) \geq 0$ is self-adjoint. Since $\gamma(x)$ maps $\mathcal{G}$ isomorphically onto $\mathcal{Q}_x(S^*)$ it also follows that $0 \in \rho(M'(x))$. 

(iii) If \((a, b) \subset \mathbb{R}\) belongs to \(\rho(A_0)\), then for all \(\varphi \in \mathcal{G}\) the mapping \(x \mapsto (M(x)\varphi, \varphi)\) is differentiable and (ii) implies that \(x \mapsto (M(x)\varphi, \varphi)\) is nondecreasing on \((a, b)\).

(iv) For \(a < y < x < b\) it follows from (iii) that \((M(y)\varphi, \varphi) \leq (M(x)\varphi, \varphi)\) for all \(\varphi \in \mathcal{G}\). If \(\gamma_y\) is a lower bound for \(M(y)\), then Corollary 1.9.10 implies that there exists a semibounded self-adjoint relation \(M(b)\) such that \(M(x)\) converges in the strong resolvent sense to \(M(b)\) on \(\mathbb{C} \setminus [\gamma_y, \infty)\) when \(x\) tends to \(b\). According to Corollary 1.9.6 (i), this is equivalent to strong graph convergence of \(M(x)\) to \(M(b)\).

The same considerations as above show that \((-M(y)\varphi, \varphi) \leq (-M(x)\varphi, \varphi)\) for \(a < x < y < b\) and \(\varphi \in \mathcal{G}\), and hence \(-M(x)\) converges in the strong resolvent sense to a semibounded self-adjoint relation \(-M(a)\) on \(\mathbb{C} \setminus [\gamma_y, \infty)\) when \(x\) tends to \(a\); here \(\gamma_y\) is a lower bound for \(-M(y)\). This implies that \(M(x)\) tends to \(M(a)\) in the strong graph sense and in the strong resolvent sense. □

It is known that every isolated spectral point of a self-adjoint operator or relation \(A_0\) is an eigenvalue and a pole of first order of the resolvent \(\lambda \mapsto (A_0 - \lambda)^{-1}\). As a consequence of Proposition 2.3.6 (v), the isolated singularities of the Weyl function \(M\) are poles of first order. This is formulated in the next corollary, which can also be regarded as a simple example of the connection between the properties of the Weyl function \(M\) and the spectrum of \(A_0\). The full connection between these objects is studied in detail in Section 3.5 and Section 3.6.

**Corollary 2.3.9.** If \(x \in \mathbb{R}\) is an isolated singularity of \(M\) and \(B_x\) is a disc centered at \(x\) such that \(M\) is holomorphic in \(\overline{B_x} \setminus \{x\}\), then \(M\) admits a norm convergent Laurent series expansion of the form

\[
M(\lambda) = \frac{M_{-1}}{\lambda - x} + \sum_{k=0}^{\infty} M_k(\lambda - x)^k, \quad M_{-1}, M_0, M_1, \ldots \in \mathcal{B}(\mathcal{G}).
\]

In particular,

\[
\lim_{\lambda \to x} (\lambda - x)M(\lambda) = M_{-1} = \frac{1}{2\pi i} \int_{C} M(\lambda) \, d\lambda,
\]

where \(C\) denotes the boundary of \(B_x\).

In the next remark it is explained that a self-adjoint part of a symmetric relation has, roughly speaking, no influence on the corresponding boundary triplet, \(\gamma\)-field, and Weyl function.

**Remark 2.3.10.** Let \(S\) be a closed symmetric relation in \(\mathcal{H}\), let \(\{\mathcal{G}, \Gamma_0, \Gamma_1\}\) be a boundary triplet for \(S^*\), and let \(A_0 = \ker \Gamma_0\). Assume that \(\mathcal{H}\) admits an orthogonal decomposition \(\mathcal{H} = \mathcal{H}' \oplus \mathcal{H}''\) and that \(S\) has the orthogonal decomposition

\[
S = S' \oplus A,
\]

(2.3.5)
where $S'$ is a closed symmetric relation in $\mathcal{H}'$ and $A$ is a self-adjoint relation in $\mathcal{H}''$. Then it follows from (2.3.5) and Proposition 1.3.13 that

$$S^* = (S')^* \oplus A,$$  \hspace{1cm} (2.3.6)

where $(S')^*$ stands for the adjoint of $S'$ in the space $\mathcal{H}'$. Observe that according to (2.3.6) every element $\{f, f'\} \in S^*$ has the decomposition

$$\{f, f'\} = \{h, h'\} + \{k, k'\}, \quad \{h, h'\} \in (S')^*, \quad \{k, k'\} \in A.$$  \hspace{1cm} (2.3.7)

Since $A \subset S$, (2.3.7) shows that $\Gamma_0 \{f, f'\} = \Gamma_0 \{h, h'\}$ and $\Gamma_1 \{f, f'\} = \Gamma_1 \{h, h'\}$. Hence, if $\Gamma'_0$ and $\Gamma'_1$ denote the restrictions of $\Gamma_0$ and $\Gamma_1$ to $(S')^*$, then it is easily seen that $\{S, \Gamma'_0, \Gamma'_1\}$ is a boundary triplet for $(S')^*$ such that

$$A'_0 = \ker \Gamma'_0, \quad A_0 = \ker \Gamma_0 = A'_0 \oplus A.$$  

Moreover, note that (2.3.6) shows

$$\tilde{\mathcal{R}}_\lambda(S^*) = \tilde{\mathcal{R}}_\lambda((S')^*), \quad \lambda \in \rho(A_0) = \rho(A'_0) \cap \rho(A),$$

which implies that the Weyl function $M'$ and the $\gamma$-field $\gamma'$ satisfy

$$M'(\lambda) = M(\lambda), \quad \gamma'(\lambda) = \gamma(\lambda), \quad \lambda \in \rho(A_0).$$

For completeness observe that if $H$ is a closed intermediate extension of $S$ and $H' = H \cap (S')^*$, then $S' \subset H' \subset (S')^*$ and $H = H' \oplus A$. Hence, one may discard the self-adjoint part $A$ in $\mathcal{H}''$ without disturbing the boundary triplet structure.

### 2.4 Existence and construction of boundary triplets

Here the existence and possible constructions of boundary triplets based on decompositions in Section 1.7 are addressed. Recall first from Corollary 1.7.13 that a closed symmetric relation $S$ in a Hilbert space $\mathcal{H}$ admits self-adjoint extensions in $\mathcal{H}$ if and only if the defect numbers

$$\dim \tilde{\mathcal{R}}_\lambda(S^*) \quad \text{and} \quad \dim \tilde{\mathcal{R}}_\mu(S^*)$$  \hspace{1cm} (2.4.1)

of $S$ are equal for some, and hence for all $\lambda \in \mathbb{C}^+$ and some, and hence for all $\mu \in \mathbb{C}^-$. Since any boundary triplet for $S^*$ induces two self-adjoint extensions $A_0$ and $A_1$ of $S$ it is clear that a boundary triplet can only exist if the defect numbers are equal. It turns out that this condition is also sufficient.

The following main result makes explicit how to construct a boundary triplet in terms of a given self-adjoint extension of a closed symmetric relation $S$ (which exists if and only if the defect numbers in (2.4.1) coincide). The following notation will be used. For $\mu \in \mathbb{C}$ the natural embedding of $\mathcal{R}_\mu(S^*)$ into $\mathcal{H}$ is denoted by $\iota_{\mathcal{R}_\mu(S^*)}$ and its adjoint is the orthogonal projection $P_{\mathcal{R}_\mu(S^*)}$ from $\mathcal{H}$ onto $\mathcal{R}_\mu(S^*)$. 

**Theorem 2.4.1.** Let $S$ be a closed symmetric relation in $\mathcal{H}$ and assume that $H$ is a self-adjoint extension of $S$ in $\mathcal{H}$. Fix $\mu \in \rho(H)$ and decompose $S^*$ as

$$S^* = H + \hat{\mathcal{N}}_\mu(S^*)$$

(2.4.2)

Let $\tilde{f} = \{f, f'\} \in S^*$ have the corresponding decomposition

$$\tilde{f} = \tilde{f}_0 + \tilde{f}_\mu,$$

(2.4.3)

with $\tilde{f}_0 = \{f_0, f'_0\} \in H$ and $\tilde{f}_\mu = \{f_\mu, \mu f_\mu\} \in \hat{\mathcal{N}}_\mu(S^*)$. Then

$$\Gamma_0 \tilde{f} := f_\mu \text{ and } \Gamma_1 \tilde{f} := P_{\mathcal{N}_\mu(S^*)}(f'_0 - \bar{\mu} f_0 + \mu f_\mu)$$

(2.4.4)

define a boundary triplet $\{\mathcal{N}_\mu(S^*), \Gamma_0, \Gamma_1\}$ for $S^*$ such that $H = \ker \Gamma_0$. Moreover, for $\lambda \in \rho(H)$ the corresponding $\gamma$-field $\gamma$ is given by

$$\gamma(\lambda) = (I + (\lambda - \mu)(H - \lambda)^{-1})\mathcal{N}_\mu(S^*)$$

(2.4.5)

and the corresponding Weyl function $M$ is given by

$$M(\lambda) = \lambda + (\lambda - \mu)(\lambda - \bar{\mu})P_{\mathcal{N}_\mu(S^*)}(H - \lambda)^{-1}\mathcal{N}_\mu(S^*)$$

(2.4.6)

**Proof.** Since $H$ is a self-adjoint extension of $S$, the direct sum decomposition (2.4.2) with $\mu \in \rho(H)$ follows from Corollary 1.7.5. Hence, for every $\tilde{f} \in S^*$ there is a unique decomposition as in (2.4.3). Let $\tilde{g} \in S^*$ have a corresponding decomposition

$$\tilde{g} = \tilde{g}_0 + \tilde{g}_\mu,$$

(2.4.7)

where $\tilde{g}_0 = \{g_0, g'_0\} \in H$ and $\tilde{g}_\mu = \{g_\mu, \mu g_\mu\} \in \hat{\mathcal{N}}_\mu(S^*)$. Then it follows directly from the decompositions (2.4.3), (2.4.7), and $(f'_0, g_0) = (f_0, g'_0)$ that

$$(f', g) - (f, g') = (f'_0 + \mu f_\mu, g_0 + g_\mu) - (f_0 + \mu f_\mu, g'_0 + g_\mu)$$

$$= (f'_0 + \mu f_\mu, g_\mu) + (\mu f_\mu, g_0) - (f_\mu, g'_0 + g_\mu) - (f_0, g_\mu)$$

(2.4.8)

Moreover, it follows from the definition (2.4.4) applied to $\tilde{f}$ and $\tilde{g}$ that

$$(\Gamma_1 \tilde{f}, \Gamma_0 \tilde{g})_{\mathcal{N}_\mu(S^*)} - (\Gamma_0 \tilde{f}, \Gamma_1 \tilde{g})_{\mathcal{N}_\mu(S^*)}$$

$$= (P_{\mathcal{N}_\mu(S^*)}(f'_0 - \bar{\mu} f_0 + \mu f_\mu), g_\mu)_{\mathcal{N}_\mu(S^*)}$$

$$- (f_\mu, P_{\mathcal{N}_\mu(S^*)}(g'_0 - \bar{\mu} g_0 + \mu g_\mu))_{\mathcal{N}_\mu(S^*)}$$

(2.4.9)

$$= (f'_0 - \bar{\mu} f_0 + \mu f_\mu, g_\mu) - (f_\mu, g'_0 - \bar{\mu} g_0 + \mu g_\mu).$$

A combination of (2.4.8) and (2.4.9) shows that the abstract Green identity (2.1.1) holds.
It will now be verified that $\Gamma = (\Gamma_0, \Gamma_1)^T : S^* \to \mathfrak{N}_\mu(S^*) \times \mathfrak{N}_\mu(S^*)$ is surjective. To see this, let $\varphi, \varphi' \in \mathfrak{N}_\mu(S^*)$. Since $\mu \in \rho(H)$, one can choose $\{f_0, f_0'\} \in H$ such that
\[ f_0' - \bar{\mu}f_0 = \varphi' - \mu\varphi. \tag{2.4.10} \]
It is clear from (2.4.2) that
\[ \hat{f} := \{f_0, f_0'\} + \{\varphi, \mu\varphi\} \in S^*, \]
and therefore (2.4.4) shows that
\[ \Gamma_0 \hat{f} = \varphi, \quad \Gamma_1 \hat{f} = P_{\mathfrak{N}_\mu(S^*)}(f_0' - \bar{\mu}f_0 + \mu\varphi) = \varphi', \]
where (2.4.10) was used in the last equality. Hence, $\text{ran} \Gamma = \mathfrak{N}_\mu(S^*) \times \mathfrak{N}_\mu(S^*)$ and thus $\{\mathfrak{N}_\mu(S^*), \Gamma_0, \Gamma_1\}$ is a boundary triplet for $S^*$. It follows from the definition of $\Gamma_0$ and the decomposition (2.4.2) that $H = \ker \Gamma_0$.

Now (2.4.5) and (2.4.6) will be verified. Let $\hat{f}_\mu = \{f_\mu, \mu f_\mu\} \in \hat{\mathfrak{N}}_\mu(S^*)$. Then (2.4.4) gives
\[ \Gamma_0 \hat{f}_\mu = f_\mu \quad \text{and} \quad \Gamma_1 \hat{f}_\mu = \mu f_\mu. \]
Therefore, Definition 2.3.1 leads to
\[ \gamma(\mu) = \{\{\Gamma_0 \hat{f}_\mu, f_\mu\} : \hat{f}_\mu \in \hat{\mathfrak{N}}_\mu(S^*)\} = \{\{f_\mu, f_\mu\} : \hat{f}_\mu \in \hat{\mathfrak{N}}_\mu(S^*)\} \]
or, equivalently, $\gamma(\mu) : \mathfrak{N}_\mu(S^*) \to \mathfrak{H}$ acts as $f_\mu \mapsto f_\mu$. Thus, $\gamma(\mu)$ is the canonical embedding of $\mathfrak{N}_\mu(S^*)$ into $\mathfrak{H}$,
\[ \gamma(\mu) = \iota_{\mathfrak{N}_\mu(S^*)}, \tag{2.4.11} \]
and $\gamma(\mu)^* : \mathfrak{H} \to \mathfrak{N}_\mu(S^*)$ is the orthogonal projection onto $\mathfrak{N}_\mu(S^*)$, that is, $\gamma(\mu)^* = P_{\mathfrak{N}_\mu(S^*)}$. Proposition 2.3.2 (ii) and (2.4.11) imply that the $\gamma$-field of $\{\mathfrak{N}_\mu(S^*), \Gamma_0, \Gamma_1\}$ is of the required form. Furthermore, Definition 2.3.4 leads to
\[ M(\mu) = \{\{\Gamma_0 \hat{f}_\mu, \Gamma_1 \hat{f}_\mu\} : \hat{f}_\mu \in \hat{\mathfrak{N}}_\mu(S^*)\} = \{\{f_\mu, \mu f_\mu\} : \hat{f}_\mu \in \hat{\mathfrak{N}}_\mu(S^*)\} \]
or, equivalently,
\[ M(\mu) = \mu. \]
Hence, Proposition 2.3.6 (v) with $\lambda_0 = \mu$ gives the desired result. \hfill $\square$

It is interesting to see what Theorem 2.4.1 means in the simple case when the underlying symmetric relation is trivial; cf. Example 2.3.5, which is opposite in the sense that there $\ker \Gamma_0 = \mathfrak{H} \times \{0\}$ and $M(\lambda) = -(1/\lambda)I$. Again this example is not typical, since in standard applications $\mathfrak{G} \neq \mathfrak{H}$; cf. Chapters 6, 7, and 8.
Example 2.4.2. Let $S = \{0, 0\}$ be the trivial symmetric relation in $\mathcal{H}$. Note that
\[ H = \{0\} \times \mathcal{H} \]
is a self-adjoint extension of $S$ with $0 \in \rho(H)$. It is clear that $S^* = \mathcal{H} \times \mathcal{H}$ and that
$\widehat{\mathcal{H}}_0(S^*) = \mathcal{H} \times \{0\}$. Therefore, one has the direct sum decomposition
\[ S^* = H \oplus \widehat{\mathcal{H}}_0(S^*) \]
and any $\hat{f} \in S^*$ has the corresponding decomposition
\[ \hat{f} = \{f, f'\} = \{0, f'\} + \{f, 0\}, \quad \{0, f'\} \in H, \quad \{f, 0\} \in \widehat{\mathcal{H}}_0(S^*). \]
According to (2.4.12), one sees that
\[ \Gamma_0 \hat{f} = f \quad \text{and} \quad \Gamma_1 \hat{f} = f', \quad \hat{f} = \{f, f'\} \in S^*, \]
defines a boundary triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for $S^*$ with $\ker \Gamma_0 = H$. Note that $\rho(H) = \mathbb{C}$
and that for every $\lambda \in \mathbb{C}$ the resolvent $(H - \lambda)^{-1}$ is the zero operator. Hence, the
$\gamma$-field is given by $\gamma(\lambda) = I$ and the Weyl function is given by $M(\lambda) = \lambda I$.

There is an addendum to Theorem 2.4.1 when the decomposition (2.4.2) is
replaced by a decomposition involving
\[ \widehat{\mathcal{H}}_\infty(S^*) = \{\{0, f'\} : f' \in \mul S^*\}. \]
In fact, the following result may be seen as a limit result obtained from (2.4.2)
with $\mu \to \infty$. The embedding of $\mathcal{H}_\infty(S^*) = \mul S^*$ into $\mathcal{H}$ is denoted by $\iota_{\mathcal{H}_\infty(S^*)}$
and its adjoint is the orthogonal projection $P_{\mathcal{H}_\infty(S^*)}$ from $\mathcal{H}$ onto $\mathcal{H}_\infty(S^*)$. The
proof of Proposition 2.4.3 is straightforward. Observe that Example 2.3.5 is an
illustration of the following proposition.

Proposition 2.4.3. Let $S$ be a closed symmetric operator in $\mathcal{H}$ and assume that $H$
is a self-adjoint extension of $S$ which belongs to $\mathcal{B}(\mathcal{H})$. Then $S^*$ can be decomposed as
\[ S^* = H \oplus \widehat{\mathcal{H}}_\infty(S^*), \quad \text{direct sum.} \tag{2.4.12} \]
Let $\hat{f} = \{f, f'\} \in S^*$ have the corresponding decomposition
\[ \hat{f} = \hat{f}_0 + \hat{f}_\infty, \]
with $\hat{f}_0 = \{f_0, H f_0\} \in H$ and $\hat{f}_\infty = \{0, f_\infty\} \in \widehat{\mathcal{H}}_\infty(S^*)$. Then
\[ \Gamma_0 \hat{f} := f_\infty \quad \text{and} \quad \Gamma_1 \hat{f} := -P_{\mathcal{H}_\infty(S^*)} f_0 \tag{2.4.13} \]
define a boundary triplet $\{\mathcal{H}_\infty(S^*), \Gamma_0, \Gamma_1\}$ for $S^*$ such that $H = \ker \Gamma_0$. Moreover,
for $\lambda \in \rho(H)$ the corresponding $\gamma$-field $\gamma$ is given by
\[ \gamma(\lambda) = -(H - \lambda)^{-1} \iota_{\mathcal{H}_\infty(S^*)}, \]
and the corresponding Weyl function $M$ is given by
\[ M(\lambda) = P_{\mathcal{H}_\infty(S^*)} (H - \lambda)^{-1} \iota_{\mathcal{H}_\infty(S^*)}. \]
Remark 2.4.4. Let $S$ be a closed symmetric operator in $\mathcal{H}$ and let $H \in \mathcal{B}(\mathcal{H})$ be a self-adjoint extension of $S$ as in Proposition 2.4.3. Then $S$ is bounded and hence $\text{dom } S$ is closed. Decompose $\mathcal{H} = \text{dom } S \oplus \mathcal{N}_\infty(S^*)$ and note that

$$S = \begin{pmatrix} S_{11} \\ S_{21} \end{pmatrix} : \text{dom } S \to \begin{pmatrix} \text{dom } S \\ \mathcal{N}_\infty(S^*) \end{pmatrix},$$

and in a similar way

$$H = \begin{pmatrix} H_{11} & H_{21}^* \\ H_{21} & H_{22} \end{pmatrix} : \begin{pmatrix} \text{dom } S \\ \mathcal{N}_\infty(S^*) \end{pmatrix} \to \begin{pmatrix} \text{dom } S \\ \mathcal{N}_\infty(S^*) \end{pmatrix}.$$}

It follows that $H_{11} = S_{11}$, $H_{21} = S_{21}$, and $H_{21}^* = S_{21}^*$. Relative to the decomposition (2.4.12) of $S^*$, the boundary triplet in (2.4.13) can be written as

$$\Gamma_0 \hat{f} = f_\infty \quad \text{and} \quad \Gamma_1 \hat{f} = -f_2, \quad \text{where} \quad \hat{f} = \left\{ \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} f_1' \\ f_2' \end{pmatrix} \right\} \in S^*.$$}

Let $\Theta$ be a closed relation in $\mathcal{G} = \mathcal{N}_\infty(S^*)$. Then the corresponding extension $A_\Theta$ of $S$ is given by

$$A_\Theta = \left\{ \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} S_{11}f_1 + S_{21}^*f_2 \\ S_{21}f_1 + H_{22}f_2 + f_\infty \end{pmatrix} : \{f_\infty, -f_2\} \in \Theta \right\},$$

which can formally be written as

$$A_\Theta = \begin{pmatrix} S_{11} & S_{21}^* \\ S_{21} & H_{22} - \Theta^{-1} \end{pmatrix}.$$}

Therefore, the extensions of $S$ may be interpreted as solutions of the completion problem posed by the incomplete $2 \times 2$ operator matrix

$$\begin{pmatrix} S_{11} & S_{21}^* \\ S_{21} & * \end{pmatrix}.$$}

Theorem 2.4.1 has some variations when the self-adjoint extension $H$ in (2.4.2) is further decomposed; cf. Section 1.7. The most straightforward results are presented in the following corollaries. In the next result the direct sum decomposition from Corollary 1.7.10 (with $\mu = \bar{\lambda}$) is used.

Corollary 2.4.5. Let $S$ be a closed symmetric relation in $\mathcal{H}$, assume that $H$ is a self-adjoint extension of $S$ in $\mathcal{H}$, and fix $\mu \in \mathbb{C} \setminus \mathbb{R}$. Then

$$S^* = S \tilde{+} \{(H - \bar{\mu})^{-1}k, (I + \bar{\mu}(H - \bar{\mu})^{-1})k : k \in \mathcal{N}_\mu(S^*)\} \tilde{+} \tilde{\mathcal{N}}_\mu(S^*),$$

where the sums are direct. Let $\hat{f} = \{f, f'\} \in S^*$ have the corresponding decomposition

$$\{f, f'\} = \{h, h'\} + \{(H - \bar{\mu})^{-1}k, (I + \bar{\mu}(H - \bar{\mu})^{-1})k\} + \{f_\mu, \mu f_\mu\}, \quad (2.4.14)$$

(2.4.14)
where \( \hat{h} = \{ h, h' \} \in S \), \( k \in \mathfrak{N}_\mu(S^*) \), and \( f_\mu \in \mathfrak{N}_\mu(S^*) \). Then
\[
\Gamma_0 \hat{f} := f_\mu \quad \text{and} \quad \Gamma_1 \hat{f} := k + \mu f_\mu
\]
define a boundary triplet \( \{ \mathfrak{N}_\mu(S^*), \Gamma_0, \Gamma_1 \} \) for \( S^* \) such that \( H = \ker \Gamma_0 \). The corresponding \( \gamma \)-field and Weyl function are given by (2.4.5) and (2.4.6).

**Proof.** It follows from Corollary 1.7.10 that every \( \hat{f} = \{ f, f' \} \in S^* \) can be written as
\[
\{ f, f' \} = \{ f_0, f_0' \} + \{ f_\mu, \mu f_\mu \},
\]
where \( \{ f_0, f_0' \} \in H \), \( \{ f_\mu, \mu f_\mu \} \in \hat{\mathfrak{N}}_\mu(S^*) \), and
\[
\{ f_0, f_0' \} = \{ h, h' \} + \{ (H - \bar{\mu})(I + \bar{\mu}(H - \bar{\mu})^{-1}) k \},
\]
with \( \{ h, h' \} \in S \) and \( k \in \mathfrak{N}_\mu(S^*) \). The boundary mappings in Theorem 2.4.1 then have the form \( \Gamma_0 \hat{f} = f_\mu \) and
\[
\Gamma_1 \hat{f} = P_{\mathfrak{N}_\mu(S^*)}(f_0' - \bar{\mu} f_0 + \mu f_\mu)
= P_{\mathfrak{N}_\mu(S^*)}(h' - \bar{\mu} h + k + \mu f_\mu)
= k + \mu f_\mu,
\]
where \( h' - \bar{\mu} h \in \text{ran}(S - \bar{\mu}) = \mathfrak{N}_\mu(S^*) \) and \( k + \mu f_\mu \in \mathfrak{N}_\mu(S^*) \) was used in the last step. This shows that the mappings in (2.4.15) form a boundary triplet with the same \( \gamma \)-field and Weyl function as in Theorem 2.4.1. \( \square \)

In Theorem 2.4.1, Proposition 2.4.3, and Corollary 2.4.5 the boundary triplets were based on decompositions of \( S^* \) in Section 1.7. The following result gives a boundary triplet for a decomposition of \( S^* \) which is a mixture of the above decompositions.

**Corollary 2.4.6.** Let \( S \) be a closed symmetric relation in \( \mathfrak{N}_\mu \), assume that \( H \) is a self-adjoint extension of \( S \) in \( \mathfrak{N}_\mu \), and fix \( \mu \in \mathbb{C} \setminus \mathbb{R} \). Every \( \hat{f} = \{ f, f' \} \in S^* \) has the unique decomposition
\[
\{ f, f' \} = \{ h, h' \} + \{ (I + \bar{\mu}(H - \bar{\mu})^{-1}) \psi, (\mu + \bar{\mu} + \bar{\mu}^2(H - \bar{\mu})^{-1}) \psi \}
+ \{ (H - \bar{\mu})^{-1} \varphi, (I + \bar{\mu}(H - \bar{\mu})^{-1}) \varphi \},
\]
with \( \hat{h} = \{ h, h' \} \in S \) and \( \psi, \varphi \in \mathfrak{N}_\mu(S^*) \). Then
\[
\Gamma_0 \hat{f} = \psi \quad \text{and} \quad \Gamma_1 \hat{f} = \varphi + 2(\text{Re} \, \mu) \psi
\]
define a boundary triplet \( \{ \mathfrak{N}_\mu(S^*), \Gamma_0, \Gamma_1 \} \) for \( S^* \) such that \( H = \ker \Gamma_0 \). The corresponding \( \gamma \)-field and Weyl function are given by (2.4.5) and (2.4.6).
Proof. Let \( \hat{f} = \{f, f'\} \in S^* \). Then according to (2.4.14) there is the decomposition

\[
\{f, f'\} = \{h, h'\} + \{(H - \overline{\mu})^{-1}k, (I + \overline{\mu}(H - \overline{\mu})^{-1})k\} + \{\psi, \mu \psi\},
\]

where \( \hat{h} = \{h, h'\} \in S, k \in \mathfrak{M}_\mu(S^*), \) and \( \psi \in \mathfrak{M}_\mu(S^*) \) are uniquely determined. Define the element \( \varphi \) by \( k = \overline{\mu} \psi + \varphi \), so that \( \varphi \in \mathfrak{M}_\mu(S^*) \) and the right-hand side of the above decomposition can be rewritten as

\[
\{h, h'\} + \{(H - \overline{\mu})^{-1}(\overline{\mu} \psi + \varphi), (I + \overline{\mu}(H - \overline{\mu})^{-1})(\overline{\mu} \psi + \varphi)\} + \{\psi, \mu \psi\}.
\]

This yields the decomposition for \( \{f, f'\} \) in the statement. The boundary triplet in (2.4.15) now reads as (2.4.16); the corresponding \( \gamma \)-field and Weyl function are given by (2.4.5) and (2.4.6). \( \square \)

By von Neumann’s second formula (see Theorem 1.7.12) one can describe the self-adjoint extension \( H \) in Theorem 2.4.1 by means of an isometric operator from \( \mathfrak{M}_\pi(S^*) \) onto \( \mathfrak{M}_\mu(S^*) \). This observation also gives rise to the construction of a boundary triplet, where the parameter space is given by \( \mathfrak{M}_\mu(S^*) \).

**Theorem 2.4.7.** Let \( S \) be a closed symmetric relation in \( \mathfrak{H} \), let \( H \) be a self-adjoint extension of \( S \), and fix some \( \mu \in \mathbb{C} \setminus \mathbb{R} \). Let \( W \) be the isometric mapping from \( \mathfrak{M}_\pi(S^*) \) onto \( \mathfrak{M}_\mu(S^*) \) such that

\[
H = S + (I - \overline{W})\mathfrak{M}_\pi(S^*) = S + \{f_\pi - Wf_\pi, \overline{\mu}f_\pi - \mu Wf_\pi\}
\]

(2.4.17)

and decompose \( \hat{f} = \{f, f'\} \in S^* \) according to von Neumann’s first formula:

\[
\hat{f} = \{h, h'\} + \{f_\mu, \mu f_\mu\} + \{f_\pi, \overline{\mu}f_\pi\},
\]

(2.4.18)

where \( \hat{h} = \{h, h'\} \in S, \hat{f}_\mu = \{f_\mu, \mu f_\mu\} \in \mathfrak{M}_\mu(S^*), \) and \( \hat{f}_\pi = \{f_\pi, \overline{\mu}f_\pi\} \in \mathfrak{M}_\pi(S^*) \).

Then

\[
\Gamma_0 \hat{f} = f_\mu + Wf_\pi \quad \text{and} \quad \Gamma_1 \hat{f} = \mu f_\mu + \overline{\mu} Wf_\pi
\]

(2.4.19)

define a boundary triplet \( \{\mathfrak{M}_\mu(S^*), \Gamma_0, \Gamma_1\} \) for \( S^* \) such that \( H = \ker \Gamma_0 \). The corresponding \( \gamma \)-field and the Weyl function are given by (2.4.5) and (2.4.6).

**Proof.** Let \( \hat{f} = \{f, f'\} \in S^* \) be decomposed as in (2.4.18) and let \( \hat{g} = \{g, g'\} \in S^* \) be decomposed in the analogous form

\[
\hat{g} = \{k, k'\} + \{g_\mu, \mu g_\mu\} + \{g_\pi, \overline{\mu} g_\pi\},
\]

where \( \hat{k} = \{k, k'\} \in S, \hat{g}_\mu = \{g_\mu, \mu g_\mu\} \in \mathfrak{M}_\mu(S^*), \) and \( \hat{g}_\pi = \{g_\pi, \overline{\mu} g_\pi\} \in \mathfrak{M}_\pi(S^*) \).

Since \( \{h, h'\}, \{k, k'\} \in S \), one has \( (h', k) = (h, k') \),

\[
(h', g_\mu + g_\pi) - (h, \mu g_\mu + \overline{\mu} g_\pi) = 0, \\
(\mu f_\mu + \overline{\mu} f_\pi, k) - (f_\mu + f_\pi, k') = 0,
\]
and therefore
\[
(f', g) - (f, g') = (h' + \mu f_\mu + \overline{\mu} f_{\overline{\mu}}, k + g_\mu + g_{\overline{\mu}}) - (h + f_\mu + f_{\overline{\mu}}, k' + \mu g_\mu + \overline{\mu} g_{\overline{\mu}})
\]
\[
= (\mu f_\mu + \overline{\mu} f_{\overline{\mu}}, g_\mu + g_{\overline{\mu}}) - (f_\mu + f_{\overline{\mu}}, \mu g_\mu + \overline{\mu} g_{\overline{\mu}})
\]
\[
= (\mu - \overline{\mu})(f_\mu, g_\mu) + (\overline{\mu} - \mu)(f_{\overline{\mu}}, g_{\overline{\mu}})
\]
\[
= (\mu - \overline{\mu})(f_\mu, g_\mu)_{\mathfrak{H}_\mu(S^*)} + (\overline{\mu} - \mu)(W f_\overline{\mu}, W g_{\overline{\mu}})_{\mathfrak{H}_\mu(S^*)}.
\]

On the other hand it follows from (2.4.19) that
\[
(\Gamma_1 \hat{f}, \Gamma_0 \hat{g})_{\mathfrak{H}_\mu(S^*)} - (\Gamma_0 \hat{f}, \Gamma_1 \hat{g})_{\mathfrak{H}_\mu(S^*)}
\]
\[
= (\mu f_\mu + \overline{\mu} W f_{\overline{\mu}}, g_\mu + W g_{\overline{\mu}})_{\mathfrak{H}_\mu(S^*)} - (f_\mu + W f_\overline{\mu}, \mu g_\mu + \overline{\mu} W g_{\overline{\mu}})_{\mathfrak{H}_\mu(S^*)}
\]
\[
= (\mu - \overline{\mu})(f_\mu, g_\mu)_{\mathfrak{H}_\mu(S^*)} + (\overline{\mu} - \mu)(W f_\overline{\mu}, W g_{\overline{\mu}})_{\mathfrak{H}_\mu(S^*)},
\]
i.e., the abstract Green identity (2.1.1) holds.

In order to see that \( \Gamma = (\Gamma_0, \Gamma_1)^\top : S^* \to \mathfrak{H}_\mu(S^*) \times \mathfrak{H}_\mu(S^*) \) is surjective consider \( \varphi, \psi \in \mathfrak{H}_\mu(S^*) \) and define \( \tilde{f} = \{f_\mu, \mu f_\mu\} + \{f_\overline{\mu}, \overline{\mu} f_{\overline{\mu}}\} \in S^* \) by
\[
\tilde{f} = \frac{1}{\mu - \overline{\mu}} \left\{ \mu \varphi - \psi, \mu (\overline{\mu} \varphi - \psi) \right\} + \frac{1}{\mu - \overline{\mu}} \left\{ W^* (\mu \varphi - \psi), \overline{\mu} W^* (\mu \varphi - \psi) \right\}.
\]
Then
\[
\Gamma_0 \tilde{f} = f_\mu + W f_{\overline{\mu}} = \varphi \quad \text{and} \quad \Gamma_1 \tilde{f} = \mu f_\mu + \overline{\mu} W f_{\overline{\mu}} = \psi,
\]
and therefore \( \Gamma = (\Gamma_0, \Gamma_1)^\top \) maps onto \( \mathfrak{H}_\mu(S^*) \times \mathfrak{H}_\mu(S^*) \). This implies that \( \{\mathfrak{H}_\mu(S^*), \Gamma_0, \Gamma_1\} \) is a boundary triplet for \( S^* \). Note that \( \tilde{f} \) in (2.4.18) is in \( \ker \Gamma_0 \) if and only if \( f_\mu = -W f_{\overline{\mu}} \) and from (2.4.17) it then follows that \( H = \ker \Gamma_0 \).

Finally, to describe the \( \gamma \)-field and the Weyl function, consider the decomposition
\[
S^* = H \oplus \widehat{\mathfrak{H}_\mu(S^*)} = \ker \Gamma_0 \oplus \widehat{\mathfrak{H}_\mu(S^*)}
\]
and note that if \( \tilde{f} \) in (2.4.18) belongs to \( \widehat{\mathfrak{H}_\mu(S^*)} \), then \( \hat{f} = f_\mu = \{f_\mu, \mu f_\mu\} \). Hence, (2.4.19) gives
\[
\Gamma_0 \hat{f}_\mu = f_\mu \quad \text{and} \quad \Gamma_1 \hat{f}_\mu = \mu f_\mu.
\]
In the same way as in the proof of Theorem 2.4.1 one concludes that \( \gamma(\mu) = \iota_{\mathfrak{H}_\mu(S^*)} \) and \( M(\mu) = \mu \). Now Proposition 2.3.2 (ii) yields (2.4.5) and Proposition 2.3.6 (v) implies (2.4.6). \( \square \)

Note that the strategy in the proof of Theorem 2.4.7 is different from the strategy in the two previous results. The connection will be sketched now. In Theorem 2.4.7 the isometric mapping \( W \) from \( \mathfrak{H}_\mu(S^*) \) onto \( \mathfrak{H}_\mu(S^*) \) determines the boundary triplet \( \{\mathfrak{H}_\mu(S^*), \Gamma_0, \Gamma_1\} \) for \( S^* \) in (2.4.19). The self-adjoint extension \( H \) of \( S \) determined by \( W \) in (2.4.17) then satisfies \( H = \ker \Gamma_0 \). Now apply
Theorem 2.4.1 with this particular self-adjoint extension. Hence, \( \hat{f} \in S^* \) in Theorem 2.4.1 is decomposed in the form

\[
\hat{f} = \{f_0, f'_0\} + \{\varphi_\mu, \mu \varphi_\mu\},
\]

(2.4.20)

where \( \{f_0, f'_0\} \in H \) and \( \{\varphi_\mu, \mu \varphi_\mu\} \in \mathcal{H}_\mu(S^*) \). Making use of the decomposition (2.4.17) of \( H \) it follows that

\[
\{f_0, f'_0\} = \{h, h'\} + \{-W \psi_\mu, -\mu W \psi_\mu\} + \{\psi_\mu, \mu \psi_\mu\}
\]

(2.4.21)

with \( \{h, h'\} \in S \) and \( \psi_\mu \in \mathcal{H}_\mu(S^*) \). Therefore, \( \hat{f} \) in (2.4.20) is given by

\[
\hat{f} = \{h, h'\} + \{f_\mu, \mu f_\mu\} + \{f_\mu, \mu f_\mu\},
\]

where \( \{f_\mu, \mu f_\mu\} = \{\varphi_\mu - W \psi_\mu, \mu \varphi_\mu - \mu W \psi_\mu\} \) and \( \{f_\mu, \mu f_\mu\} = \{\psi_\mu, \mu \psi_\mu\} \). Now the identity

\[
\varphi_\mu = f_\mu + W \psi_\mu = f_\mu + W f_\mu
\]

shows that the boundary maps \( \Gamma_0 \) in Theorem 2.4.1 and Theorem 2.4.7 coincide. Moreover, as \( P_{\mathcal{H}_\mu(S^*)} (h' - \mu h) = 0 \), the identity

\[
P_{\mathcal{H}_\mu(S^*)} (f'_0 - \mu f_0) = -\mu W \psi_\mu + \mu W f_\mu + \mu \varphi_\mu = \mu f_\mu + \mu W f_\mu
\]

follows from (2.4.21), and shows that the boundary maps \( \Gamma_1 \) in Theorem 2.4.1 and Theorem 2.4.7 are the same.

### 2.5 Transformations of boundary triplets

Let \( S \) be a closed symmetric relation in \( \mathcal{H} \) with equal defect numbers. Then \( S \) admits self-adjoint extensions in \( \mathcal{H} \) and each self-adjoint extension gives rise to a boundary triplet as in Theorem 2.4.1. Hence, boundary triplets for \( S^* \) are not uniquely determined, with the exception of the trivial case \( S = S^* \). A complete description of all boundary triplets for \( S^* \) will be given with the help of block operator matrices that are unitary with respect to the indefinite inner products in Section 1.8; cf. (2.1.3). The transformation properties of the corresponding boundary parameters, \( \gamma \)-fields, and Weyl functions are discussed afterwards.

The main result on the description of all boundary triplets for \( S^* \) is the following theorem. It describes the transformation of boundary triplets.

**Theorem 2.5.1.** Let \( S \) be a closed symmetric relation in \( \mathcal{H} \), assume that \( \{\mathcal{G}, \Gamma_0, \Gamma_1\} \) is a boundary triplet for \( S^* \), and let \( \mathcal{G}' \) be a Hilbert space. Then the following statements hold:

(i) Let \( W \in \mathcal{B}(\mathcal{G} \times \mathcal{G}, \mathcal{G}' \times \mathcal{G}') \) satisfy

\[
W^* \mathcal{G}' W = \mathcal{G} \quad \text{and} \quad W \mathcal{G} W^* = \mathcal{G}',
\]

(2.5.1)
and define
\[
\left( \Gamma'_{0} \right) = W \left( \begin{pmatrix} \Gamma_{0} \\ \Gamma_{1} \end{pmatrix} \right) = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \left( \begin{pmatrix} \Gamma_{0} \\ \Gamma_{1} \end{pmatrix} \right).
\]  

(2.5.2)

Then \( \{ \mathcal{S}', \Gamma'_{0}, \Gamma'_{1} \} \) is a boundary triplet for \( S^* \).

(ii) Let \( \{ \mathcal{S}', \Gamma'_{0}, \Gamma'_{1} \} \) be a boundary triplet for \( S^* \). Then there exists a unique operator \( W \in \mathcal{B}(\mathcal{S} \times \mathcal{S}, \mathcal{S}' \times \mathcal{S}') \) satisfying (2.5.1) such that (2.5.2) holds.

Proof. (i) Note that the operator \( W \in \mathcal{B}(\mathcal{S} \times \mathcal{S}, \mathcal{S}' \times \mathcal{S}') \) is unitary from \( (\mathcal{S}^2, [\cdot, \cdot]_{\mathcal{S}^2}) \) to \( (\mathcal{S}'^2, [\cdot, \cdot]_{\mathcal{S}'^2}) \); cf. Proposition 1.8.2. Hence, for \( \hat{f}, \hat{g} \in S^* \) one has
\[
[\Gamma' \hat{f}, \Gamma' \hat{g}]_{\mathcal{S}'^2} = [W \Gamma \hat{f}, W \Gamma \hat{g}]_{\mathcal{S}^2} = [\Gamma' \hat{f}, \Gamma' \hat{g}]_{\mathcal{S}'^2} = [\hat{f}, \hat{g}]_{\mathcal{S}'^2}.
\]

Therefore, \( \Gamma' \) and \( \Gamma'' \) satisfy the abstract Green identity (2.1.1); cf. (2.1.4). Since \( W \) is surjective by Proposition 1.8.2, \( \Gamma' = W \Gamma \) is also surjective thanks to the surjectivity of \( \Gamma \). It follows that \( \{ \mathcal{S}', \Gamma'_{0}, \Gamma'_{1} \} \) is a boundary triplet for \( S^* \).

(ii) Assume that \( \{ \mathcal{S}', \Gamma'_{0}, \Gamma'_{1} \} \) is a boundary triplet for \( S^* \) and define a linear relation \( W \subset \mathcal{S}^2 \times \mathcal{S}^2 \) by
\[
W := \{ \Gamma' \hat{f}, \Gamma' \hat{g} : \hat{f} \in S^* \}.
\]

(2.5.3)

It follows from Proposition 2.1.2 (ii) that \( W \) is an operator. Indeed, if \( \Gamma' \hat{f} = 0 \), then \( \hat{f} \in \mathcal{S} \) and thus \( \Gamma' \hat{f} = 0 \). For the operator \( W \) one has dom \( W = \mathcal{S} \times \mathcal{S} \) and ran \( W = \mathcal{S}' \times \mathcal{S}' \) since ran \( \Gamma = \mathcal{S} \times \mathcal{S} \) and ran \( \Gamma' = \mathcal{S}' \times \mathcal{S}' \).

Define the inner product \( [\cdot, \cdot]_{\mathcal{S}^2} \) on \( \mathcal{S}^2 \) as in (2.1.3). Let \( \hat{\varphi}, \hat{\psi} \in \mathcal{S} \times \mathcal{S} \) and let \( \hat{f}, \hat{g} \in S^* \) be such that \( \Gamma' \hat{f} = \hat{\varphi} \) and \( \Gamma' \hat{g} = \hat{\psi} \). Then one has
\[
[\mathcal{W} \hat{\varphi}, \mathcal{W} \hat{\psi}]_{\mathcal{S}'^2} = [W \Gamma \hat{f}, W \Gamma \hat{g}]_{\mathcal{S}^2} = [\Gamma' \hat{f}, \Gamma' \hat{g}]_{\mathcal{S}'^2} = [\hat{f}, \hat{g}]_{\mathcal{S}'^2},
\]

and hence \( \mathcal{W} \) is an isometric operator from \( (\mathcal{S}^2, [\cdot, \cdot]_{\mathcal{S}^2}) \) to \( (\mathcal{S}'^2, [\cdot, \cdot]_{\mathcal{S}'^2}) \). This implies that the first identity in (2.5.1) is satisfied and from Lemma 1.8.1 it follows that \( \mathcal{W} \in \mathcal{B}(\mathcal{S} \times \mathcal{S}, \mathcal{S}' \times \mathcal{S}') \). Furthermore, as \( \mathcal{W} \) is surjective, Proposition 1.8.2 implies that also the second identity in (2.5.1) holds.

By the definition (2.5.3) of the operator \( \mathcal{W} \), the boundary triplets \( \{ \mathcal{S}, \Gamma_{0}, \Gamma_{1} \} \) and \( \{ \mathcal{S}', \Gamma'_{0}, \Gamma'_{1} \} \) are connected via (2.5.2). Moreover, the operator \( \mathcal{W} \) is unique. Indeed, if \( \Gamma' = W \Gamma \) and \( \Gamma' = W \Gamma \), then \( (W - W) \Gamma \hat{f} = 0 \) for all \( \hat{f} \in S^* \) and as ran \( \Gamma = \mathcal{S} \times \mathcal{S} \) it follows that \( \mathcal{W} = \mathcal{W} \). \( \square \)

The transformation of a boundary triplet \( \{ \mathcal{S}, \Gamma_{0}, \Gamma_{1} \} \) in Theorem 2.5.1 induces a transformation of the closed relations in the parameter space. Assume that \( \mathcal{W} \in \mathcal{B}(\mathcal{S} \times \mathcal{S}, \mathcal{S}' \times \mathcal{S}') \) satisfies the identities in (2.5.1) and let \( \{ \mathcal{S}', \Gamma'_{0}, \Gamma'_{1} \} \) be the corresponding transformed boundary triplet in (2.5.2). Let \( \Theta \) be a relation in \( \mathcal{S} \) and define \( \Theta' \) in \( \mathcal{S}' \) as a Möbius transform of \( \Theta \) by
\[
\Theta' = \mathcal{W}[\Theta] = \{ W_{11} \varphi + W_{12} \varphi', W_{21} \varphi + W_{22} \varphi' : \{ \varphi, \varphi' \} \in \Theta \};
\]

(2.5.4)
Proposition 2.5.2. Let $S$ be a closed symmetric relation in $\mathcal{H}$ and assume that \{${\mathcal{G}}, {\Gamma}_0, {\Gamma}_1$\} and \{${\mathcal{G}}', {\Gamma}_0', {\Gamma}_1'$\} are boundary triplets for $S^*$ connected via $\Gamma' = \mathcal{W}\Gamma$ in Theorem 2.5.1. Let $\Theta$ be a closed relation in $\mathcal{G}$ and let $\Theta'$ be defined by (2.5.4). Then the closed intermediate extensions

$$A_{\Theta} = \ker (\Gamma_1 - \Theta \Gamma_0) \quad \text{and} \quad A'_{\Theta'} = \ker (\Gamma_1' - \Theta' \Gamma_0')$$

coincide, that is, for $\hat{f} \in S^*$ one has

$$\Gamma' \hat{f} \in \Theta' \quad \iff \quad \Gamma \hat{f} \in \Theta.$$

Proof. Let $\hat{f} \in S^*$. Then the transformation formulas (2.5.2), (2.5.4), and the fact that $\mathcal{W}$ is bijective imply

$$\Gamma' \hat{f} \in \Theta' \quad \iff \quad \mathcal{W} \Gamma \hat{f} \in \mathcal{W}[\Theta] \quad \iff \quad \Gamma \hat{f} \in \Theta.$$

Hence, $\ker (\Gamma_1 - \Theta \Gamma_0)$ and $\ker (\Gamma_1' - \Theta' \Gamma_0')$ coincide; cf. Theorem 2.1.3 (iii). □

Likewise, the transformation of the boundary triplet leads to a transformation of the $\gamma$-field and the Weyl function.

Proposition 2.5.3. Let $S$ be a closed symmetric relation in $\mathcal{H}$ and assume that \{${\mathcal{G}}, {\Gamma}_0, {\Gamma}_1$\} and \{${\mathcal{G}}', {\Gamma}_0', {\Gamma}_1'$\} are boundary triplets for $S^*$ connected via $\Gamma' = \mathcal{W}\Gamma$ in Theorem 2.5.1. Let $A_0 = \ker \Gamma_0$, $A'_0 = \ker \Gamma'_0$, and let $\gamma, \gamma'$ and $M, M'$ be the $\gamma$-fields and Weyl functions corresponding to \{${\mathcal{G}}, {\Gamma}_0, {\Gamma}_1$\} and \{${\mathcal{G}}', {\Gamma}_0', {\Gamma}_1'$\}, respectively. Then for all $\lambda \in \rho(A_0) \cap \rho(A'_0)$ the operator

$$W_{11} + W_{12} M(\lambda)$$

is an isomorphism from $\mathcal{G}$ onto $\mathcal{G}'$, and the identities

$$\gamma'(\lambda) = \gamma(\lambda) (W_{11} + W_{12} M(\lambda))^{-1} \quad (2.5.5)$$

and

$$M'(\lambda) = (W_{21} + W_{22} M(\lambda)) (W_{11} + W_{12} M(\lambda))^{-1} \quad (2.5.6)$$

hold.

Proof. For $\lambda \in \rho(A_0)$ and $\hat{f}_\lambda \in \widehat{\mathcal{R}}(S^*)$ one has

$$\Gamma'_0 \hat{f}_\lambda = W_{11} \Gamma_0 \hat{f}_\lambda + W_{12} \Gamma_1 \hat{f}_\lambda = (W_{11} + W_{12} M(\lambda)) \Gamma_0 \hat{f}_\lambda,$$

which leads to

$$\Gamma'_0 | \widehat{\mathcal{R}}(S^*) = (W_{11} + W_{12} M(\lambda)) (\Gamma_0 | \widehat{\mathcal{R}}(S^*)). \quad (2.5.7)$$
If, in addition, \( \lambda \in \rho(A) \cap \rho(A') \), then, by Lemma 2.1.7, \( \Gamma_0 \) and \( \Gamma_0' \) are isomorphisms from \( \hat{\mathfrak{H}}_\lambda(S^*) \) onto \( \mathcal{G} \) and \( \mathcal{G}' \), respectively. Hence, it follows from (2.5.7) that \( W_{11} + W_{12}M(\lambda) \) is an isomorphism from \( \mathcal{G} \) onto \( \mathcal{G}' \), and therefore

\[
(\Gamma_0' \mid \hat{\mathfrak{H}}_\lambda(S^*))^{-1} = (\Gamma_0 \mid \hat{\mathfrak{H}}_\lambda(S^*))^{-1}(W_{11} + W_{12}M(\lambda))^{-1}.
\]

If one applies the orthogonal projection \( \pi_1 \) from \( H \times H \) onto \( H \times \{0\} \) to both sides, then (2.5.5) follows. Similarly, for \( \lambda \in \rho(A) \cap \rho(A') \), one finds that

\[
(W_{21} + W_{22}M(\lambda))(W_{11} + W_{12}M(\lambda))^{-1}\Gamma_0'f_\lambda = (W_{21} + W_{22}M(\lambda))\Gamma_0f_\lambda
\]

\[
= W_{21}\Gamma_0f_\lambda + W_{22}\Gamma_1f_\lambda
\]

\[
= \Gamma_1'f_\lambda,
\]

which yields (2.5.6).

The second corollary treats the situation in which a bijective operator \( D \) dilates the Weyl function \( M \) and a self-adjoint operator \( P \) produces a shift of the dilated Weyl function \( D^*MD \).

**Corollary 2.5.4.** Let \( S \) be a closed symmetric relation in \( \mathcal{S} \) and assume that \( \{\mathcal{G}, \Gamma_0, \Gamma_1\} \) is a boundary triplet for \( S^* \) with \( \gamma \)-field \( \gamma \) and Weyl function \( M \), and define

\[
\Gamma_0' = \Gamma_1 \quad \text{and} \quad \Gamma_1' = -\Gamma_0.
\]

Then \( \{\mathcal{G}, \Gamma_0', \Gamma_1'\} \) is a boundary triplet for \( S^* \) and \( \ker \Gamma_1' = \ker \Gamma_1 \). Moreover, for \( \lambda \in \rho(A) \cap \rho(A') \) the corresponding \( \gamma \)-field \( \gamma' \) and the Weyl function \( M' \) are given by

\[
\gamma'(\lambda) = \gamma(\lambda)M(\lambda)^{-1} \quad \text{and} \quad M'(\lambda) = -M(\lambda)^{-1},
\]

respectively.

**Proof.** The operator

\[
W = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \in \mathcal{B}(\mathcal{G} \times \mathcal{G})
\]

satisfies both identities in (2.5.1). Now the assertions follow from Theorem 2.5.1 and Proposition 2.5.3.

The second corollary treats the situation in which a bijective operator \( D \) dilates the Weyl function \( M \) and a self-adjoint operator \( P \) produces a shift of the dilated Weyl function \( D^*MD \).

**Corollary 2.5.5.** Let \( S \) be a closed symmetric relation in \( \mathcal{S} \) and assume that \( \{\mathcal{G}, \Gamma_0, \Gamma_1\} \) is a boundary triplet for \( S^* \) with \( \gamma \)-field \( \gamma \) and Weyl function \( M \). Let \( \mathcal{G}' \) be a Hilbert space, let \( D \in \mathcal{B}(\mathcal{G}', \mathcal{G}) \) be boundedly invertible, let \( P \in \mathcal{B}(\mathcal{G}') \) be self-adjoint, and define

\[
\Gamma_0' = D^{-1}\Gamma_0 \quad \text{and} \quad \Gamma_1' = D^*\Gamma_1 + PD^{-1}\Gamma_0.
\]
Then \( \{ S', \Gamma_0', \Gamma_1' \} \) is a boundary triplet for \( S^* \) and \( \ker \Gamma_0' = \ker \Gamma_0 \). Moreover, for \( \lambda \in \rho(A_0) \) the corresponding \( \gamma \)-field \( \gamma' \) and the Weyl function \( M' \) are given by

\[
\gamma'(\lambda) = \gamma(\lambda)D \quad \text{and} \quad M'(\lambda) = D^* M(\lambda) D + P,
\]

respectively.

**Proof.** It is not difficult to check that the operator

\[
W = \begin{pmatrix} D^{-1} & 0 \\ PD^{-1} & D^* \end{pmatrix} \in \mathcal{B}(\mathcal{G} \times \mathcal{G}, \mathcal{G}' \times \mathcal{G}')
\]

satisfies both identities in (2.5.1). Now the assertions follow from Theorem 2.5.1 and Proposition 2.5.3. \( \square \)

The next corollary complements Corollary 2.5.5.

**Corollary 2.5.6.** Let \( S \) be a closed symmetric relation in \( \mathcal{H} \) and assume that \( \{ \mathcal{G}, \Gamma_0, \Gamma_1 \} \) and \( \{ \mathcal{G}', \Gamma_0', \Gamma_1' \} \) are boundary triplets for \( S^* \) such that \( \ker \Gamma_0 = \ker \Gamma_0' \).

Then there exist a boundedly invertible operator \( D \in \mathcal{B}(\mathcal{G}', \mathcal{G}) \) and a self-adjoint operator \( P \in \mathcal{B}(\mathcal{G}') \) such that

\[
\Gamma_0' = D^{-1} \Gamma_0 \quad \text{and} \quad \Gamma_1' = D^* \Gamma_1 + PD^{-1} \Gamma_0.
\]

In particular, the \( \gamma \)-fields and Weyl functions corresponding to the boundary triplets \( \{ \mathcal{G}, \Gamma_0, \Gamma_1 \} \) and \( \{ \mathcal{G}, \Gamma_0', \Gamma_1' \} \) satisfy (2.5.8).

**Proof.** It follows from Theorem 2.5.1 that there exists \( W \in \mathcal{B}(\mathcal{G} \times \mathcal{G}, \mathcal{G}' \times \mathcal{G}') \) with the properties (2.5.1) such that

\[
\begin{pmatrix} \Gamma_0' \\ \Gamma_1' \end{pmatrix} = W \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix} = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix}.
\]

The assumption \( \ker \Gamma_0 = \ker \Gamma_0' \) implies \( W_{12} = 0 \). In fact, if \( \tilde{f} \in \ker \Gamma_0 = \ker \Gamma_0' \), then \( W_{12} \tilde{f} = 0 \) by (2.5.10) and hence Proposition 2.1.2 (i) implies \( W_{12} = 0 \). Therefore, the first identity \( W^* \mathcal{G} W = \mathcal{G} \) in (2.5.1) means that

\[
W_{11} W_{22} = I_{\mathcal{G}}, \quad W_{22} W_{11} = I_{\mathcal{G}}, \quad W_{11}^* W_{21} = W_{21}^* W_{11}.
\]

Likewise, the second equality \( W \mathcal{G} W^* = \mathcal{G} \) in (2.5.1) means that

\[
W_{11} W_{22}^* = I_{\mathcal{G}'}, \quad W_{22} W_{11}^* = I_{\mathcal{G}'}, \quad W_{21} W_{22}^* = W_{22} W_{21}^*.
\]

It follows that \( D := W_{22}^* \in \mathcal{B}(\mathcal{G}', \mathcal{G}) \) is boundedly invertible with \( D^{-1} = W_{11} \), the operator \( P := W_{21} D \in \mathcal{B}(\mathcal{G}') \) is self-adjoint and (2.5.9) is satisfied. \( \square \)
A combination of Corollary 2.5.5 with \( G = G, D = I, P = -\Theta \), and Corollary 2.5.4 leads to the following statement.

**Corollary 2.5.7.** Let \( S \) be a closed symmetric relation in \( \mathcal{H} \) and assume that \( \{ G, \Gamma_0, \Gamma_1 \} \) is a boundary triplet for \( S^* \) with \( \gamma \)-field \( \gamma \) and Weyl function \( M \). Let \( \Theta \in \mathcal{B}(\mathcal{G}) \) be self-adjoint, and define

\[
\Gamma'_0 = \Gamma_1 - \Theta \Gamma_0 \quad \text{and} \quad \Gamma'_1 = -\Gamma_0.
\]

Then \( \{ G, \Gamma'_0, \Gamma'_1 \} \) is a boundary triplet for \( S^* \) and \( \ker \Gamma'_0 = \ker (\Gamma_1 - \Theta \Gamma_0) = A_\Theta \) holds. Moreover, for \( \lambda \in \rho(A_0) \cap \rho(A_\Theta) \) the corresponding \( \gamma \)-field \( \gamma' \) and the Weyl function \( M' \) are given by

\[
\gamma'(\lambda) = -\gamma(\lambda)(\Theta - M(\lambda))^{-1} \quad \text{and} \quad M'(\lambda) = (\Theta - M(\lambda))^{-1},
\]

respectively.

The following statement is also a direct consequence of Corollary 2.5.5.

**Corollary 2.5.8.** Let \( S \) be a closed symmetric relation in \( \mathcal{H} \) and assume that \( \{ G, \Gamma_0, \Gamma_1 \} \) is a boundary triplet for \( S^* \) with \( \gamma \)-field \( \gamma \) and Weyl function \( M \). Let \( Q(\lambda) \in \mathcal{B}(\mathcal{G}), \lambda \in \mathbb{C} \setminus \mathbb{R}, \) be a family of operators which satisfy

\[
\frac{Q(\lambda) - Q(\mu)}{\lambda - \mu} = \gamma(\mu)^* \gamma(\lambda), \quad \lambda, \mu \in \mathbb{C} \setminus \mathbb{R}. \tag{2.5.11}
\]

Let \( \lambda_0 \in \mathbb{C} \setminus \mathbb{R} \) and define the self-adjoint operator \( P \in \mathcal{B}(\mathcal{G}) \) by

\[
P = \text{Re} Q(\lambda_0) - \text{Re} M(\lambda_0).
\]

Then \( Q \) is the Weyl function corresponding to the boundary triplet \( \{ \mathcal{G}, \Gamma'_0, \Gamma'_1 \} \), where

\[
\Gamma'_0 = \Gamma_0 \quad \text{and} \quad \Gamma'_1 = \Gamma_1 + P \Gamma_0,
\]

and \( Q(\lambda) = M(\lambda) + P \) holds for all \( \lambda \in \rho(A_0) \).

**Proof.** Due to Proposition 2.3.6 (iii) it follows from the identity (2.5.11) that

\[
Q(\lambda) - Q(\lambda_0)^* = M(\lambda) - M(\lambda_0)^*, \quad \lambda, \lambda_0 \in \mathbb{C} \setminus \mathbb{R},
\]

and, in particular, \( \text{Im} Q(\lambda_0) = \text{Im} M(\lambda_0) \). Hence, one obtains

\[
Q(\lambda) - M(\lambda) = Q(\lambda_0)^* - M(\lambda_0)^* = \text{Re} Q(\lambda_0) - \text{Re} M(\lambda_0) = P.
\]

With the choice \( D = I \) and \( P \) as above, the result follows from Corollary 2.5.5. \( \square \)

Now it will be shown that a pair of transversal self-adjoint extensions induces a boundary triplet \( \{ \mathcal{G}, \Gamma_0, \Gamma_1 \} \) which determines these extensions via the boundary conditions \( \ker \Gamma_0 \) and \( \ker \Gamma_1 \). The following theorem is a consequence of Theorem 2.4.1 and Corollary 2.5.5.
Chapter 2. Boundary Triplets and Weyl Functions

Theorem 2.5.9. Let $S$ be a closed symmetric relation in $\mathcal{H}$ and assume that $H$ and $H'$ are transversal self-adjoint extensions of $S$ in $\mathcal{H}$, that is,

$$S^* = H \oplus H'$$

holds; cf. Lemma 1.7.7. Then there exists a boundary triplet $\{G, \Gamma_0, \Gamma_1\}$ for $S^*$ such that

$$H = \ker \Gamma_0 \quad \text{and} \quad H' = \ker \Gamma_1.$$

(2.5.12)

Proof. As $H$ is a self-adjoint extension of $S$, there is a boundary triplet $\{G, \Upsilon_0, \Upsilon_1\}$ for $S^*$ such that $H = \ker \Upsilon_0$; cf. Theorem 2.4.1. Since $H'$ is a self-adjoint extension of $S$ it follows from Corollary 2.1.4 (v) that there exists a self-adjoint relation $\Theta$ in $\mathcal{G}$ such that

$$H' = \ker (\Upsilon_1 - \Theta \Upsilon_0).$$

Furthermore, since $H'$ and $H$ are transversal, it follows from Proposition 2.1.8 (ii) that $\Theta \in B(\mathcal{G})$. Now define

$$\begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix} = \begin{pmatrix} I & 0 \\ -\Theta & I \end{pmatrix} \begin{pmatrix} \Upsilon_0 \\ \Upsilon_1 \end{pmatrix},$$

so that $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a boundary triplet for $S^*$ by Corollary 2.5.5. By construction, the boundary triplet $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ has the properties (2.5.12). □

Corollary 2.5.10. Let $S$ be a closed symmetric relation in $\mathcal{H}$ and assume that $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a boundary triplet for $S^*$. Let $A_\Theta = \ker (\Gamma_1 - \Theta \Gamma_0)$ be a self-adjoint extension of $S$ corresponding to the self-adjoint relation $\Theta$ in $\mathcal{G}$ via (2.1.5). Then there exists a boundary triplet $\{\mathcal{G}, \Gamma_0', \Gamma_1'\}$ such that

$$\ker \Gamma_0' = \ker (\Gamma_1 - \Theta \Gamma_0) \quad \text{and} \quad \ker \Gamma_1' = \ker (\Gamma_1 + \Theta^{-1} \Gamma_0),$$

(2.5.13)

that is, $A_0' = A_\Theta$ and $A_1' = A_{-\Theta^{-1}}$.

Proof. Recall that $\Theta^* = (J\Theta)^\perp$, where $J$ denotes the flip-flop operator in $\mathcal{G}^2$ from (1.3.1). Since $\Theta$ is self-adjoint it follows that $\Theta = (J\Theta)^\perp$ or, in other words,

$$\mathcal{G}^2 = \Theta \oplus J\Theta.$$

In particular, one sees that $\Theta$ and $J\Theta = -\Theta^{-1}$ are transversal in $\mathcal{G}$. Therefore, the self-adjoint extensions $\ker (\Gamma_1 - \Theta \Gamma_0)$ and $\ker (\Gamma_1 + \Theta^{-1} \Gamma_0)$ are transversal; cf. Lemma 2.1.5 (ii). Now the assertion follows from Theorem 2.5.9. □

Corollary 2.5.10 is concerned with the existence of the boundary triplet $\{\mathcal{G}, \Gamma_0', \Gamma_1'\}$ with the properties (2.5.13). In fact, it is possible to explicitly construct such a boundary triplet via the choice of an appropriate operator $W$ such that $\Gamma' = WG$; cf. Corollary 2.5.7 for the special case $\Theta \in B(\mathcal{G})$. 
Corollary 2.5.11. Let $S$ be a closed symmetric relation in $\mathcal{H}$ and assume that $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ is a boundary triplet for $S^*$. Let $\Theta$ be a self-adjoint relation in $\mathcal{G}$ and choose $A, B \in B(\mathcal{G})$ such that

$$\Theta = \{\{A\varphi, B\varphi\} : \varphi \in \mathcal{G}\}$$

and the identities

$$A^*B = B^*A, \quad AB^* = BA^*, \quad A^*A + B^*B = I = AA^* + BB^*, \quad (2.5.14)$$

hold; cf. Corollary 1.10.9. Then $\{\mathcal{G}, \Gamma'_0, \Gamma'_1\}$, where

$$\Gamma'_0 = B^*\Gamma_0 - A^*\Gamma_1 \quad \text{and} \quad \Gamma'_1 = A^*\Gamma_0 + B^*\Gamma_1, \quad (2.5.15)$$

is a boundary triplet for $S^*$ such that both identities in (2.5.13) hold, that is, $A'_0 = A\Theta$ and $A'_1 = A_{-\Theta^{-1}}$.

Proof. It is not difficult to check that

$$W = \begin{pmatrix} B^* & -A^* \\ A^* & B^* \end{pmatrix} \in B(\mathcal{G} \times \mathcal{G}) \quad (2.5.16)$$

satisfies (2.5.1) and it is clear from (2.5.15) that $\{\mathcal{G}, \Gamma'_0, \Gamma'_1\}$ and $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ are connected via $W$ as in Theorem 2.5.1 (i). Hence, $\{\mathcal{G}, \Gamma'_0, \Gamma'_1\}$ is a boundary triplet for $S^*$. It follows from (2.5.14) that

$$W[\Theta] = \{\{B\varphi - A\varphi, A\varphi + B\varphi\} : \{A\varphi, B\varphi\} \in \Theta\}$$

$$= \{0\} \times \mathcal{G},$$

and since $-\Theta^{-1} = \{\{B\varphi, -A\varphi\} : \varphi \in \mathcal{G}\}$, it follows in the same way that

$$W[-\Theta^{-1}] = \{\{B\varphi + A\varphi, A\varphi + B\varphi\} : \{B\varphi, -A\varphi\} \in -\Theta^{-1}\}$$

$$= \mathcal{G} \times \{0\}.$$

Recall from Proposition 2.5.2 that

$$\Gamma'\hat{f} \in W[\Xi] \iff \Gamma\hat{f} \in \Xi, \quad \hat{f} \in S^*,$$

holds for any closed relation $\Xi$ in $\mathcal{G}$. With $\Xi = \Theta$ and $\Xi = -\Theta^{-1}$ one then has

$$\Gamma'\hat{f} \in \{0\} \times \mathcal{G} \iff \Gamma\hat{f} \in \Theta, \quad \hat{f} \in S^*,$$

and

$$\Gamma'\hat{f} \in \mathcal{G} \times \{0\} \iff \Gamma\hat{f} \in -\Theta^{-1}, \quad \hat{f} \in S^*,$$

respectively. Now (2.1.11)–(2.1.12) imply

$$A'_0 = \ker\Gamma'_0 = \ker (\Gamma_1 - \Theta\Gamma_0) = A\Theta$$

and

$$A'_1 = \ker\Gamma'_1 = \ker (\Gamma_1 + \Theta^{-1}\Gamma_0) = A_{-\Theta^{-1}}. \quad \Box$$
Assume that the boundary triplets \( \{ G, \Gamma_0, \Gamma_1 \} \) and \( \{ G, \Gamma_0', \Gamma_1' \} \) are as in Corollary 2.5.11 and let \( \gamma \) and \( M \), and \( \gamma' \) and \( M' \), be the corresponding \( \gamma \)-fields and Weyl functions, respectively. Then it follows from Proposition 2.5.3 that for all \( \lambda \in \rho(A_0) \cap \rho(A_0') \) one has

\[
\gamma'(\lambda) = \gamma(\lambda) \left( B^* - A^* M(\lambda) \right)^{-1} \tag{2.5.17}
\]

and

\[
M'(\lambda) = (A^* + B^* M(\lambda)) \left( B^* - A^* M(\lambda) \right)^{-1}. \tag{2.5.18}
\]

In the special case where the defect numbers of \( S \) are \((1,1)\) one may choose \( A = \frac{1}{\sqrt{s^2 + 1}} \) and \( B = \frac{s}{\sqrt{s^2 + 1}}, \ s \in \mathbb{R} \cup \{\infty\}, \)

where \( A = 0 \) and \( B = 1 \) if \( s = \infty \); this interpretation will be used also in the following. With this choice of \( A \) and \( B \) the operator in (2.5.16) reduces to the \( 2 \times 2 \)-matrix

\[
W = \frac{1}{\sqrt{s^2 + 1}} \begin{pmatrix} s & -1 \\ 1 & s \end{pmatrix}, \ s \in \mathbb{R} \cup \{\infty\}.
\]

In this case

\[
\Gamma_0' = \frac{1}{\sqrt{s^2 + 1}} (s\Gamma_0 - \Gamma_1), \quad \Gamma_1' = \frac{1}{\sqrt{s^2 + 1}} (\Gamma_0 + s\Gamma_1), \quad s \in \mathbb{R} \cup \{\infty\}, \tag{2.5.19}
\]

and for \( \lambda \in \rho(A_0) \cap \rho(A_0') \) the corresponding \( \gamma \)-field and Weyl function are given by

\[
\gamma'(\lambda) = \frac{\sqrt{s^2 + 1}}{s - M(\lambda)} \gamma(\lambda) \quad \text{and} \quad M'(\lambda) = \frac{1 + s M(\lambda)}{s - M(\lambda)}, \ s \in \mathbb{R} \cup \{\infty\}. \tag{2.5.20}
\]

Now let \( S \) be a closed symmetric relation in \( \mathcal{H} \) and assume that \( \{ G, \Gamma_0, \Gamma_1 \} \) is a boundary triplet for \( S^* \). Consider a closed symmetric extension \( S' \) of \( S \) with the property \( S' \subset A_0 = \ker \Gamma_0 \). Then the boundary triplet \( \{ G, \Gamma_0, \Gamma_1 \} \) can be restricted to \( (S')^* \subset S^* \) and \( A_0 \) coincides with the kernel of the restriction of \( \Gamma_0 \). The Weyl function corresponding to this restricted boundary triplet is a compression of the original Weyl function onto a subspace of \( G \). In the following proposition this is made precise from the point of view of an orthogonal decomposition of \( G \).

**Proposition 2.5.12.** Let \( S \) be a closed symmetric relation in \( \mathcal{H} \) and assume that \( \{ G, \Gamma_0, \Gamma_1 \} \) is a boundary triplet for \( S^* \) with \( A_0 = \ker \Gamma_0 \), and corresponding \( \gamma \)-field \( \gamma \) and Weyl function \( M \). Assume that \( G \) has the orthogonal decomposition

\[
G = G' \oplus G'' \tag{2.5.21}
\]

with corresponding orthogonal projections \( P' \) and \( P'' \) and canonical embedding \( \iota' \). Then the following statements hold:
2.5. Transformations of boundary triplets

(i) The relation

\[ S' = \{ \hat{f} \in S^* : \Gamma_0 \hat{f} = 0, \ P' \Gamma_1 \hat{f} = 0 \} \quad (2.5.22) \]

is closed and symmetric with \( S \subset S' \subset A_0 \).

(ii) The adjoint \((S')^*\) of \( S' \) is given by

\[ (S')^* = \{ \hat{f} \in S^* : P'' \Gamma_0 \hat{f} = 0 \}. \]

(iii) The triplet \( \{ \mathcal{G}', \Gamma_0', \Gamma_1' \} \), where

\[ \Gamma_0' = \Gamma_0 \upharpoonright (S')^* \quad \text{and} \quad \Gamma_1' = P' \Gamma_1 \upharpoonright (S')^*, \]

is a boundary triplet for \((S')^*\) such that \( A_0 = \ker \Gamma_0' \).

(iv) The \( \gamma \)-field \( \gamma' \) and Weyl function \( M' \) corresponding to the boundary triplet \( \{ \mathcal{G}', \Gamma_0', \Gamma_1' \} \) are given by

\[ \gamma'(\lambda) = \gamma(\lambda) \iota' \quad \text{and} \quad M'(\lambda) = P'M(\lambda) \iota', \quad \lambda \in \rho(A_0). \]

Moreover, for every closed symmetric extension \( S' \) with \( S \subset S' \subset A_0 \) there exists an orthogonal decomposition \((2.5.21)\) of \( \mathcal{G} \) such that \((2.5.22)\) holds.

Proof. (i) & (ii) It is clear from the definition that \( S \subset S' \subset A_0 \) and that \( S' \) can be written as

\[ S' = \{ \hat{f} \in S^* : \Gamma \hat{f} \in \{ 0 \} \times \mathcal{G}'' \}. \]

Hence, \( S' = A_0 \Theta \) when \( \Theta = \{ 0 \} \times \mathcal{G}'' \). It follows that \( S' \) is closed, and by \((1.3.4)\) one has \( \Theta^* = \mathcal{G}' \times \mathcal{G} \), so that Theorem 2.1.3 (iv) shows

\[ (S')^* = \{ \hat{f} \in S^* : \Gamma \hat{f} \in \mathcal{G}' \times \mathcal{G} \} = \{ \hat{f} \in S^* : P'' \Gamma_0 \hat{f} = 0 \}. \]

(iii) With the choice \( \hat{f}, \hat{g} \in (S')^* \) one has \( \Gamma_0 \hat{f} = P' \Gamma_0 \hat{f} \) and \( \Gamma_0 \hat{g} = P' \Gamma_0 \hat{g} \). Then \((2.1.1)\) yields

\[ (f', g)_\mathcal{G} - (f, g')_\mathcal{G} = (\Gamma_1 \hat{f}, \Gamma_0 \hat{g})_\mathcal{G} - (\Gamma_0 \hat{f}, \Gamma_1 \hat{g})_\mathcal{G} \\
= (\Gamma_1 \hat{f}, P' \Gamma_0 \hat{g})_{\mathcal{G}'} - (P' \Gamma_0 \hat{f}, \Gamma_1 \hat{g})_{\mathcal{G}'} \\
= (\Gamma_1' \hat{f}, \Gamma_0 \hat{g})_{\mathcal{G}'} - (\Gamma_0' \hat{f}, \Gamma_1' \hat{g})_{\mathcal{G}'} \quad (2.5.23) \]

It follows from the surjectivity of \( \Gamma \) and the identity \( \Gamma_0 \hat{f} = P' \Gamma_0 \hat{f} \) when \( \hat{f} \in (S')^* \), that

\[ \Gamma' = \left( \begin{array}{c} \Gamma_0 \upharpoonright (S')^* \\ P' \Gamma_1 \upharpoonright (S')^* \end{array} \right) = \left( \begin{array}{c} P' \Gamma_0 \upharpoonright (S')^* \\ P' \Gamma_1 \upharpoonright (S')^* \end{array} \right) : (S')^* \to (\mathcal{G}') \]

maps \((S')^*\) onto \( \mathcal{G}' \times \mathcal{G}' \). Together with \((2.5.23)\) this shows that \( \{ \mathcal{G}', \Gamma_0', \Gamma_1' \} \) is a boundary triplet for \((S')^*\). It is clear that \( A_0 = \ker \Gamma_0' \) holds.
(iv) Since $\Gamma_0$ maps $\hat{\mathcal{H}}_\lambda(S^*)$, $\lambda \in \rho(A_0)$, one-to-one onto $\mathcal{G}$, the restriction $\Gamma'_0$ maps $\hat{\mathcal{H}}_\lambda((S')^*)$, $\lambda \in \rho(A_0)$, one-to-one onto $\mathcal{G}'$. Hence, (2.3.1) implies that

$$\rho(A_0) \ni \lambda \mapsto \gamma'_{\lambda} = \{ \{ \Gamma'_0 \hat{f}_\lambda, \Gamma'_1 \hat{f}_\lambda \} : \hat{f}_\lambda \in \hat{\mathcal{H}}_\lambda((S')^*) \}$$

and therefore $\gamma'_{\lambda} = \gamma(\lambda)_{\nu'}$. It follows from (2.3.4) that

$$\rho(A_0) \ni \lambda \mapsto \gamma_\lambda = \{ \{ \Gamma_0 \hat{f}_\lambda, \Gamma_1 \hat{f}_\lambda \} : \hat{f}_\lambda \in \hat{\mathcal{H}}_\lambda((S')^*) \},$$

which shows that $M'_{\lambda} = P'M_{\lambda})_{\nu'}$.

Finally, if $S'$ is a closed symmetric extension of $S$ with the property $S' \subset A_0$, then $S' = A_0$ for some closed symmetric relation $\Theta$ in $\mathcal{G}$ such that $\Theta \subset \{0\} \times \mathcal{G}$ by Theorem 2.1.3 (v). As $\Theta$ is closed, there exists a closed subspace $\mathcal{G}'' \subset \mathcal{G}$ such that $\Theta = \{0\} \times \mathcal{G}''$. With $\mathcal{G}' = (\mathcal{G}'')^\perp$ it is clear that the orthogonal decomposition (2.5.21) of $\mathcal{G}$ holds and $S'$ is of the form (2.5.22).

In the situation of Proposition 2.5.12 the intermediate extensions of $S'$ can also be interpreted as intermediate extensions of $S$. In the next corollary the connection between these extensions relative to the appropriate boundary triplets is explained.

**Corollary 2.5.13.** Assume that the parameter space $\mathcal{G}$ has the orthogonal decomposition (2.5.21) and let $S'$ be as in Proposition 2.5.12. Let $\Theta'$ be a closed relation in $\mathcal{G}'$ and let $\Theta$ be the closed linear relation in $\mathcal{G}$ defined by

$$\Theta = \Theta' \oplus (\{0\} \times \mathcal{G}'').$$

For the intermediate extensions induced by $\Theta$ and $\Theta'$ one has

$$\{ \hat{f} \in S^* : \Gamma \hat{f} \in \Theta \} = \{ \hat{f} \in (S')^* : \Gamma' \hat{f} \in \Theta' \}$$

or, equivalently, $\ker (\Gamma_1 - \Theta \Gamma_0) = \ker (\Gamma'_1 - \Theta' \Gamma'_0)$.

**Proof.** It is clear that the relation $\Theta$ defined in (2.5.24) is a closed relation in $\mathcal{G}$. The identity in (2.5.25) now follows from

$$\{ \hat{f} \in S^* : \Gamma \hat{f} \in \Theta \} = \{ \hat{f} \in S^* : \Gamma \hat{f} \in \Theta' \oplus (\{0\} \times \mathcal{G}'') \}
\quad = \{ \hat{f} \in S^* : \{(P' \Gamma_0 \hat{f}, P' \Gamma_1 \hat{f}) \in \Theta', P'' \Gamma_0 \hat{f} = 0 \} \}
\quad = \{ \hat{f} \in (S')^* : \{\Gamma_0 \hat{f}, \Gamma_1 \hat{f} \} \in \Theta' \}
\quad = \{ \hat{f} \in (S')^* : \Gamma' \hat{f} \in \Theta' \},$$

where (2.5.24) has been used in conjunction with the boundary triplet in Proposition 2.5.12.
Let $S$ and $S'$ be closed symmetric relations in $\mathcal{H}$ and $\mathcal{H}'$ which are unitarily equivalent, and let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ and $\{\mathcal{G}', \Gamma'_0, \Gamma'_1\}$ be boundary triplets for $S^*$ and $(S')^*$, respectively. The notion of unitary equivalence for these boundary triplets will now be introduced which leads to unitary equivalence of the corresponding extensions, $\gamma$-fields, and Weyl functions.

Let $\mathcal{H}$ and $\mathcal{H}'$ be Hilbert spaces and let $U \in \mathcal{B}(\mathcal{H}, \mathcal{H}')$ be a unitary operator from $\mathcal{H}$ onto $\mathcal{H}'$. Let $S$ and $S'$ be closed symmetric relations in $\mathcal{H}$ and $\mathcal{H}'$, respectively, such that they are unitarily equivalent by means of $U$, that is,

$$S' = \{\{Uf,Uf'\} : \{f,f'\} \in S\} \quad (2.5.26)$$

in the sense of Definition 1.3.7. It follows from (1.3.7) that this assumption is equivalent to $S^*$ and $(S')^*$ being equivalent under $U$,

$$(S')^* = \{\{Uf,Uf'\} : \{f,f'\} \in S^*\}.$$ 

Then $U$ maps $\mathcal{H}_\lambda(S^*)$ unitarily onto $\mathcal{H}_\lambda((S')^*)$, and hence

$$\mathcal{H}_\lambda((S')^*) = \{\{Uf, \lambda Uf_\lambda\} : \{f, \lambda f_\lambda\} \in S^*\}.$$ 

Furthermore, let $V \in \mathcal{B}(\mathcal{G}, \mathcal{G}')$ be a unitary mapping from $\mathcal{G}$ onto $\mathcal{G}'$. Then the closed relations $\Theta$ in $\mathcal{G}$ and $\Theta'$ in $\mathcal{G}'$ are unitarily equivalent if

$$\Theta' = \{\{Vf,Vf'\} : \{f,f'\} \in \Theta\}.$$ 

(2.5.27)

The notion of unitary equivalence of two boundary triplets involves not only the unitary equivalence between $\mathcal{H}$ and $\mathcal{H}'$, but also the unitary equivalence between $\mathcal{G}$ and $\mathcal{G}'$.

**Definition 2.5.14.** Let $S$ and $S'$ be closed symmetric relations in $\mathcal{H}$ and $\mathcal{H}'$, and let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ and $\{\mathcal{G}', \Gamma'_0, \Gamma'_1\}$ be boundary triplets for $S^*$ and $(S')^*$, respectively. Then $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ and $\{\mathcal{G}', \Gamma'_0, \Gamma'_1\}$ are said to be unitarily equivalent if there exist a unitary operator $U \in \mathcal{B}(\mathcal{H}, \mathcal{H}')$ and a unitary operator $V \in \mathcal{B}(\mathcal{G}, \mathcal{G}')$ such that

(i) $S$ and $S'$ are unitarily equivalent by means of $U$ as in (2.5.26);

(ii) $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ and $\{\mathcal{G}', \Gamma'_0, \Gamma'_1\}$ are connected via

$$\Gamma'_0\{Uf,Uf'\} = V\Gamma_0\{f,f'\} \quad \text{and} \quad \Gamma'_1\{Uf,Uf'\} = V\Gamma_1\{f,f'\} \quad (2.5.28)$$

for all $\{f,f'\} \in S^*$.

In the next proposition it will be shown that for unitarily equivalent boundary triplets the corresponding closed extensions, $\gamma$-fields, and Weyl functions are unitarily equivalent.

**Proposition 2.5.15.** Let $S$ and $S'$ be closed symmetric relations in $\mathcal{H}$ and $\mathcal{H}'$, and let $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ and $\{\mathcal{G}', \Gamma'_0, \Gamma'_1\}$ be boundary triplets for $S^*$ and $(S')^*$, respectively, which are unitarily equivalent by means of the unitary operators $U \in \mathcal{B}(\mathcal{H}, \mathcal{H}')$ and $V \in \mathcal{B}(\mathcal{G}, \mathcal{G}')$. Then the following statements hold:
(i) For all closed relations $\Theta$ in $\mathcal{G}$ and $\Theta'$ in $\mathcal{G}'$ connected via \eqref{2.5.27} the closed extensions

$$A_\Theta = \{f \in S^* : \Gamma f \in \Theta\} \quad \text{and} \quad A'_{\Theta'} = \{\hat{h} \in (S')^* : \Gamma' \hat{h} \in \Theta'\}$$

are unitarily equivalent by means of $U \in \mathcal{B}(\mathcal{G}, \mathcal{G}')$, that is,

$$A'_{\Theta'} = \{\{U f, U f'\} : \{f, f'\} \in A_\Theta\}.$$

(ii) The $\gamma$-fields $\gamma$ and $\gamma'$ corresponding to $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ and $\{\mathcal{G}', \Gamma'_0, \Gamma'_1\}$ are related by

$$\gamma'(\lambda) = U \gamma(\lambda)V^{-1}, \quad \lambda \in \rho(A_0) = \rho(A_0') \Rightarrow \lambda \in \rho(A_0) = \rho(A'_0).$$

(iii) The Weyl functions $M$ and $M'$ corresponding to $\{\mathcal{G}, \Gamma_0, \Gamma_1\}$ and $\{\mathcal{G}', \Gamma'_0, \Gamma'_1\}$ are related by

$$M'(\lambda) = VM(\lambda)V^{-1}, \quad \lambda \in \rho(A_0) = \rho(A'_0).$$

Proof. (i) It follows from the definition that

$$A_\Theta = \{\{f, f'\} \in S^* : \Gamma \{f, f'\} \in \Theta\}$$

and

$$A'_{\Theta'} = \{\{g, g'\} \in (S')^* : \Gamma' \{g, g'\} \in \Theta'\}.$$ 

Since $S$ and $S'$ are unitarily equivalent, so are $S^*$ and $(S')^*$, and hence one has $\{g, g'\} \in (S')^*$ if and only if $\{f, f'\} \in S^*$ for some $\{f, f'\} \in S^*$. Thus, by the unitary equivalence of the boundary triplets one obtains

$$A'_{\Theta'} = \{\{U f, U f'\} : \{f, f'\} \in S^*, \{\Gamma'_0 \{U f, U f'\}, \Gamma'_1 \{U f, U f'\}\} \in \Theta'\}$$

$$= \{\{U f, U f'\} : \{f, f'\} \in S^*, \{VT_0 \{f, f'\}, VT_1 \{f, f'\}\} \in \Theta'\}$$

$$= \{\{U f, U f'\} : \{f, f'\} \in S^*, \{\Gamma_0 \{f, f'\}, \Gamma_1 \{f, f'\}\} \in \Theta\}$$

$$= \{\{U f, U f'\} : \{f, f'\} \in A_\Theta\}.$$

(ii) By item (i), the self-adjoint relations $A_0 = \ker \Gamma_0$ and $A'_0 = \ker \Gamma'_0$ are unitarily equivalent by means of $U$, which implies that $\rho(A_0) = \rho(A_0')$, and hence the $\gamma$-fields $\gamma$ and $\gamma'$ are defined on the same subset of $\mathbb{C}$. For $\lambda \in \rho(A_0)$ one computes

$$\gamma'(\lambda) = \{\{\Gamma'_0 \{g_{\lambda}, \lambda g_{\lambda}\}, g_{\lambda}\} \in \hat{\mathcal{H}}(\lambda)((S')^*)\}$$

$$= \{\{\Gamma'_0 \{U f_{\lambda}, \lambda U f_{\lambda}\}, U f_{\lambda}\} \in \hat{\mathcal{H}}(S^*)\}$$

$$= \{\{VT_0 \{f_{\lambda}, \lambda f_{\lambda}\}, U f_{\lambda}\} : \{f_{\lambda}, \lambda f_{\lambda}\} \in \hat{\mathcal{H}}(S^*)\}$$

$$= U \gamma(\lambda)V^{-1}.$$
(iii) Since $\rho(A_0) = \rho(A_0')$, the Weyl functions $M$ and $M'$ are defined on the same subset of $\mathbb{C}$. Fix $\lambda \in \rho(A_0)$, let $\psi \in \mathcal{S}'$ and choose $\{f_\lambda, \lambda f_\lambda\} \in \hat{\mathcal{R}}_\lambda(S^*)$ such that $\Gamma_0'\{Uf_\lambda, \lambda U f_\lambda\} = \psi$. Since $\{Uf_\lambda, \lambda U f_\lambda\} \in \hat{\mathcal{R}}_\lambda((S')^*)$, it follows from the definition of the Weyl function and (2.5.28) that

$$M'(\lambda)\psi = M'(\lambda)\Gamma_0'\{Uf_\lambda, \lambda U f_\lambda\}$$

$$= \Gamma_1'\{Uf_\lambda, \lambda U f_\lambda\}$$

$$= VM(\lambda)\Gamma_0\{f_\lambda, \lambda f_\lambda\}$$

$$= VM(\lambda)V^{-1}\Gamma_0'\{Uf_\lambda, \lambda U f_\lambda\}$$

$$= VM(\lambda)V^{-1}\psi.$$ 

This yields $M'(\lambda) = VM(\lambda)V^{-1}$ for all $\lambda \in \rho(A_0)$. \qed

Let $S$ and $S'$ be closed symmetric relations in $\mathcal{H}$ and $\mathcal{H}'$ with boundary triplets $\{\mathcal{S}, \Gamma_0, \Gamma_1\}$ and $\{\mathcal{S}', \Gamma_0', \Gamma_1'\}$ that are unitarily equivalent by means of a unitary operator $U \in \mathcal{B}(\mathcal{H}, \mathcal{H}')$ and a unitary operator $V \in \mathcal{B}(\mathcal{S}, \mathcal{S}')$ as in Proposition 2.5.15. Then according to Proposition 2.5.15 one has that

$$A_0' = \{\{Uf, U f'\} : \{f, f'\} \in A_0\};$$

cf. (2.5.28). In particular, this implies that the multivalued parts of $A_0$ and $A_0'$ are connected by

$$\text{mul} A_0' = U(\text{mul} A_0),$$

and since $U$ is unitary it also follows that

$$\overline{\text{dom} A_0'} = U(\overline{\text{dom} A_0}).$$

The following corollary is an immediate consequence of Proposition 2.5.15 (ii) and Proposition 2.3.2 (ii).

**Corollary 2.5.16.** Let $P$ and $P'$ be the orthogonal projections in $\mathcal{S}$ and $\mathcal{S}'$ onto $(\text{mul} A_0)^\perp$ and $(\text{mul} A_0')^\perp$, respectively. Then

$$\left( \begin{array}{c} P'\gamma'(\lambda) \\ (I - P')\gamma'(\lambda) \end{array} \right) = U \left( \begin{array}{c} P\gamma(\lambda) \\ (I - P)\gamma(\lambda) \end{array} \right) V^{-1}, \quad \lambda \in \rho(A_0) = \rho(A_0'),$$

where $(I - P)\gamma(\lambda) = (I - P)\gamma(\lambda_0)$ and $(I - P')\gamma'(\lambda) = (I - P')\gamma'(\lambda_0)$ are parts that do not depend on $\lambda$.

In Theorem 4.2.6 it will be shown that if $S$ and $S'$ are simple (see Section 3.4) and their Weyl functions are unitarily equivalent, then in fact the corresponding boundary triplets are unitarily equivalent.
2.6 Kreĭn’s formula for intermediate extensions

Let \( S \) be a closed symmetric relation in a Hilbert space \( \mathcal{H} \) and assume that \( \{\mathcal{G}, \Gamma_0, \Gamma_1\} \) is a boundary triplet for \( S^\ast \). According to Theorem 2.1.3, the mapping \( \Gamma = (\Gamma_0, \Gamma_1)^\top \) induces a bijective correspondence between the set of (closed) intermediate extensions \( A_\Theta \) of \( S \) and the set of (closed) relations \( \Theta \) in \( \mathcal{G} \), via

\[
\Theta \mapsto A_\Theta = \{\hat{f} \in S^\ast : \Gamma\hat{f} \in \Theta\} = \ker(\Gamma_1 - \Theta\Gamma_0);
\]

and \( A_0 = \ker\Gamma_0 \) corresponds to \( \Theta = \{0\} \times \mathcal{G} \). For \( \lambda \in \rho(A_0) \) the relation \( (A_\Theta - \lambda)^{-1} \) will be regarded as a perturbation of the resolvent of the self-adjoint extension \( A_0 \) of \( S \). This fact is expressed by the formula provided in Theorem 2.6.1 and some variants under the additional assumption \( \lambda \in \rho(A_\Theta) \) are discussed afterwards. In the special case \( \lambda \in \rho(A_0) \) one has \( (A_0 - \lambda)^{-1} \in B(\mathcal{H}) \) and the resolvent difference, and hence also the perturbation term, are bounded operators. Moreover, it is shown later how the different types of spectral points \( \lambda \in \sigma(A_\Theta) \) which are contained in \( \rho(A_0) \) are related to the Weyl function and the parameter \( \Theta \). A more in-depth treatment of the connection of the spectrum and the Weyl function can be found in Chapter 3.

In the next theorem the difference of \( (A_\Theta - \lambda)^{-1} \) and \( (A_0 - \lambda)^{-1} \), \( \lambda \in \rho(A_0) \), is expressed in a perturbation term which involves the Weyl function \( M \) and the parameter \( \Theta \). This results in a general version of Kreĭn’s formula for intermediate extensions.

**Theorem 2.6.1.** Let \( S \) be a closed symmetric relation in \( \mathcal{H} \), let \( \{\mathcal{G}, \Gamma_0, \Gamma_1\} \) be a boundary triplet for \( S^\ast \), \( A_0 = \ker\Gamma_0 \), and let \( \gamma \) and \( M \) be the corresponding \( \gamma \)-field and Weyl function, respectively. Moreover, let \( \Theta \) be a closed relation in \( \mathcal{G} \) and let

\[
A_\Theta = \{\hat{f} \in S^\ast : \Gamma\hat{f} \in \Theta\} \quad (2.6.1)
\]

be the corresponding extension via (2.1.5). Then for all \( \lambda \in \rho(A_0) \) one has the equality

\[
(A_\Theta - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda)\Theta - (M(\lambda))^{-1} \gamma(\bar{\lambda})^*, \quad (2.6.2)
\]

where the inverses in the first and the last term are taken in the sense of relations. Moreover, if \( \lambda \in \rho(A_0) \cap \rho(A_\Theta) \), then \( (\Theta - M(\lambda))^{-1} \in B(\mathcal{G}) \) and (2.6.2) holds in the sense of bounded linear operators.

**Proof.** Assume that \( \lambda \in \rho(A_0) \). In order to establish the identity (2.6.2) it must be shown that the relations on the left-hand side and right-hand side coincide.

First the inclusion \((\subset)\) in (2.6.2) will be shown. For this purpose, consider \( \{g, g'\} \in (A_\Theta - \lambda)^{-1} \) so that, equivalently, \( \hat{g}_\Theta = \{g', g + \lambda g'\} \in A_\Theta \). Moreover, denote

\[
\hat{g}_0 = \{(A_0 - \lambda)^{-1}g, (I + \lambda(A_0 - \lambda)^{-1})g\} \in A_0.
\]
Then
\[ \hat{g}_\Theta - \hat{g}_0 = \{ g' - (A_0 - \lambda)^{-1}g, \lambda(g' - (A_0 - \lambda)^{-1}g) \}, \]
and hence \( \hat{g}_\Theta - \hat{g}_0 \in \hat{\mathcal{R}}_\lambda(S^*) \). Since \( \hat{\gamma}(\lambda) \) maps \( \mathcal{S} \) onto \( \hat{\mathcal{R}}_\lambda(S^*) \) there exists an element \( \varphi \in \mathcal{S} \) such that
\[ \hat{g}_\Theta = \hat{g}_0 + \hat{\gamma}(\lambda)\varphi. \tag{2.6.3} \]
By Proposition 2.3.6 (ii) one has \( \Gamma\hat{\gamma}(\lambda)\varphi = \{ \varphi, M(\lambda)\varphi \} \) and, moreover, Proposition 2.3.2 (iv) shows that \( \Gamma\hat{g}_0 = \{ 0, \gamma(\lambda)^*g \} \). Since \( \hat{g}_\Theta \in A_\Theta \), an application of \( \Gamma \) to (2.6.3) yields
\[ \{ 0, \gamma(\lambda)^*g \} + \{ \varphi, M(\lambda)\varphi \} = \Gamma\hat{g}_0 + \Gamma\hat{\gamma}(\lambda)\varphi = \Gamma\hat{g}_\Theta \in \Theta, \]
see (2.6.1). Thus, \( \{ \varphi, \gamma(\lambda)^*g + M(\lambda)\varphi \} \in \Theta \) and \( \{ \varphi, \gamma(\lambda)^*g \} \in \Theta - M(\lambda) \) or, equivalently, \( \{ g, \varphi \} \in (\Theta - M(\lambda))^{-1}\gamma(\lambda)^* \), which implies that
\[ \{ g, \gamma(\lambda)\varphi \} \in \gamma(\lambda)(\Theta - M(\lambda))^{-1}\gamma(\lambda)^*. \tag{2.6.4} \]
Now consider the first component \( g' = (A_0 - \lambda)^{-1}g + \gamma(\lambda)\varphi \) in the identity (2.6.3). Then one has
\[ \{ g, g' \} = \{ g, (A_0 - \lambda)^{-1}g + \gamma(\lambda)\varphi \}, \tag{2.6.5} \]
and due to \( \{ g, (A_0 - \lambda)^{-1}g \} \in (A_0 - \lambda)^{-1} \) and (2.6.4) it follows from (2.6.5) that
\[ \{ g, g' \} \in (A_0 - \lambda)^{-1} + \gamma(\lambda)(\Theta - M(\lambda))^{-1}\gamma(\lambda)^*, \]
and hence the inclusion \( (\subseteq) \) in (2.6.2) holds.

Next the inclusion \( (\supseteq) \) in (2.6.2) will be shown. For this purpose, let
\[ \{ g, g' \} \in (A_0 - \lambda)^{-1} + \gamma(\lambda)(\Theta - M(\lambda))^{-1}\gamma(\lambda)^*. \]
By the definition of the sum of relations, this means that
\[ g' = (A_0 - \lambda)^{-1}g + h, \quad \text{where} \quad \{ g, h \} \in \gamma(\lambda)(\Theta - M(\lambda))^{-1}\gamma(\lambda)^*. \]
Recall from Proposition 2.3.2 (i) that \( \gamma(\lambda) \in \mathcal{B}(\mathcal{S}, \mathcal{S}) \), since \( \lambda \in \rho(A_0) \). Hence, \( h = \gamma(\lambda)\varphi \), where \( \{ \gamma(\lambda)^*g, \varphi \} \in (\Theta - M(\lambda))^{-1} \). Consequently, it is clear that \( \{ \varphi, \gamma(\lambda)^*g + M(\lambda)\varphi \} \in \Theta \). Next observe that
\[ \{ g', g + \lambda g' \} = \{ (A_0 - \lambda)^{-1}g, (I + \lambda(A_0 - \lambda)^{-1})g \} + \{ \gamma(\lambda)\varphi, \lambda\gamma(\lambda)\varphi \}, \]
which implies that
\[ \Gamma\{ g', g + \lambda g' \} = \{ 0, \gamma(\lambda)^*g \} + \{ \varphi, M(\lambda)\varphi \} \in \Theta \]
or \( \{ g', g + \lambda g' \} \in A_\Theta \). Thus, \( \{ g, g' \} \in (A_\Theta - \lambda)^{-1} \) and therefore
\[ (A_0 - \lambda)^{-1} + \gamma(\lambda)(\Theta - M(\lambda))^{-1}\gamma(\lambda)^* \subset (A_\Theta - \lambda)^{-1}. \]
Hence, the inclusion \( (\supseteq) \) in (2.6.2) has been shown.
To prove the last assertion in the theorem, assume that \( \lambda \in \rho(A_0) \cap \rho(A_\Theta) \). It is first shown that \( \ker(\Theta - M(\lambda)) = \{0\} \). To see this, let \( \varphi \in \ker(\Theta - M(\lambda)) \). Then clearly \( \{\varphi, M(\lambda)\varphi\} \in \Theta \). Define \( \hat{f}_\lambda = \gamma(\lambda)\varphi \), so that \( \hat{f}_\lambda = \{f_\lambda, \lambda f_\lambda\} \in \hat{\mathfrak{F}}_\lambda(S^*) \) and
\[
\Gamma \hat{f}_\lambda = \{\Gamma_0 \hat{f}_\lambda, \Gamma_1 \hat{f}_\lambda\} = \{\Gamma_0 \hat{f}_\lambda, M(\lambda) \Gamma_0 \hat{f}_\lambda\} = \{\varphi, M(\lambda)\varphi\} \in \Theta.
\]
Thus, \( \hat{f}_\lambda \in A_\Theta \) and \( f_\lambda = \gamma(\lambda)\varphi \in \ker(A_\Theta - \lambda) \). Since \( \lambda \in \rho(A_\Theta) \), one concludes that \( \gamma(\lambda)\varphi = 0 \) and \( \varphi = 0 \). Hence, \( \ker(\Theta - M(\lambda)) = \{0\} \).

Next it is shown that \( (\Theta - M(\lambda))^{-1} \in \mathcal{B}(\mathcal{G}) \). Since \( \lambda \in \rho(A_0) \), the identity (2.6.2) holds and as it is assumed that \( \lambda \in \rho(A_\Theta) \), one has \( \text{dom}(A_\Theta - \lambda)^{-1} = \mathfrak{G} \). Therefore,
\[
\text{dom} \left[ (\Theta - M(\lambda))^{-1} \gamma(\lambda)^* \right] = \text{dom} \left[ \gamma(\lambda)(\Theta - M(\lambda))^{-1} \gamma(\lambda)^* \right] = \mathfrak{G}, \tag{2.6.6}
\]
where the first identity is clear since \( \gamma(\lambda) \in \mathcal{B}(\mathcal{G}, \mathfrak{G}) \). As \( \text{ran} \gamma(\lambda)^* = \mathfrak{G} \), one concludes from (2.6.6) that \( \text{dom}(\Theta - M(\lambda))^{-1} = \mathfrak{G} \). By assumption, \( \Theta \) is closed and then \( M(\lambda) \in \mathcal{B}(\mathfrak{G}) \) implies that \( \Theta - M(\lambda) \) is closed. Then \( (\Theta - M(\lambda))^{-1} \) is a closed operator and by the closed graph theorem \( (\Theta - M(\lambda))^{-1} \in \mathcal{B}(\mathcal{G}) \). \( \square \)

Assume that \( \lambda \in \rho(A_0) \). In Theorem 2.6.1 it is shown that then \( \lambda \in \rho(A_\Theta) \) leads to \( (\Theta - M(\lambda))^{-1} \in \mathcal{B}(\mathcal{G}) \). In fact, there is a one-to-one correspondence between the part of the spectrum of \( A_\Theta \) contained in \( \rho(A_0) \) and the spectrum of \( \Theta - M(\lambda) \) contained in \( \rho(A_0) \). The following theorem and its corollary are direct consequences of the Kreîn formula (2.6.2). A complete description of the spectrum of self-adjoint extensions \( A_\Theta \) in terms of the singularities of the function \( \lambda \mapsto (\Theta - M(\lambda))^{-1} \) is given in Section 3.8.

**Theorem 2.6.2.** Let \( S \) be a closed symmetric relation in \( \mathfrak{G} \), let \( \{\mathcal{G}, \Gamma_0, \Gamma_1\} \) be a boundary triplet for \( S^* \), \( A_0 = \ker \Gamma_0 \), and let \( \gamma \) and \( M \) be the corresponding \( \gamma \)-field and Weyl function, respectively. Moreover, let \( \Theta \) be a closed relation in \( \mathcal{G} \) and let
\[
A_\Theta = \{f \in S^* : \Gamma f \in \Theta\}
\]
be the corresponding extension via (2.1.5). Then the following statements hold for all \( \lambda \in \rho(A_0) \):

(i) \( \lambda \in \sigma_p(A_\Theta) \Leftrightarrow 0 \in \sigma_p(\Theta - M(\lambda)) \), and in this case
\[
\ker(A_\Theta - \lambda) = \gamma(\lambda) \ker(\Theta - M(\lambda)); \tag{2.6.7}
\]

(ii) \( \lambda \in \sigma_r(A_\Theta) \Leftrightarrow 0 \in \sigma_r(\Theta - M(\lambda)) \);

(iii) \( \lambda \in \sigma_c(A_\Theta) \Leftrightarrow 0 \in \sigma_c(\Theta - M(\lambda)) \);

(iv) \( \lambda \in \rho(A_\Theta) \Leftrightarrow 0 \in \rho(\Theta - M(\lambda)) \).
Proof. Assume that $\lambda \in \rho(A_0)$ and consider the right-hand side of (2.6.2) as the sum of the operator $(A_0 - \lambda)^{-1} \in B(\mathcal{H})$ and the relation

$$\gamma(\lambda)(\mathcal{H} - M(\lambda))^{-1}\gamma(\lambda)^*$$

in $\mathcal{H}$. Hence, the domain of the right-hand side of (2.6.2) is given by

$$\text{dom} \left[ \gamma(\lambda)(\mathcal{H} - M(\lambda))^{-1}\gamma(\lambda)^* \right] = \text{dom} \left[ (\mathcal{H} - M(\lambda))^{-1}\gamma(\lambda)^* \right],$$

where it was used that $\gamma(\lambda) \in B(\mathcal{G}, \mathcal{H})$. Thus, it follows from (2.6.2) that

$$\text{dom}(A_\mathcal{H} - \lambda)^{-1} = \text{dom} \left( \mathcal{H} - M(\lambda) \right)^{-1}\gamma(\lambda)^*.$$

Due to the definition of the sum of relations and $\text{mul}(A_0 - \lambda)^{-1} = \{0\}$ the multivalued part of the right-hand side of (2.6.2) is given by

$$\text{mul} \left[ \gamma(\lambda)(\mathcal{H} - M(\lambda))^{-1}\gamma(\lambda)^* \right] = \text{mul} \left[ (\mathcal{H} - M(\lambda))^{-1}\gamma(\lambda)^* \right] = \gamma(\lambda) \text{mul} \left( \mathcal{H} - M(\lambda) \right)^{-1}.$$

Thus, it follows from (2.6.2) that

$$\text{mul}(A_\mathcal{H} - \lambda)^{-1} = \gamma(\lambda) \text{mul} \left( \mathcal{H} - M(\lambda) \right)^{-1}. \quad (2.6.8)$$

The proof of the theorem is based on the identities (2.6.8) and (2.6.9).

For the interpretation of (2.6.8) recall that $\gamma(\lambda)$, $\lambda \in \rho(A_0)$, maps $\mathcal{G}$ isomorphically onto $\mathcal{H}_\lambda(\mathcal{S}^*)$; see Proposition 2.3.2 (i). This implies that the restriction

$$\gamma(\lambda)^*: \mathcal{H}_\lambda(\mathcal{S}^*) \rightarrow \mathcal{G}$$

is an isomorphism.

In particular, $\gamma(\lambda)^*$ is a bijection between closed or dense subspaces in $\mathcal{H}_\lambda(\mathcal{S}^*)$ and closed or dense subspaces in $\mathcal{G}$, respectively. Now assume that $V$ is a closed relation in $\mathcal{G}$. Since $\ker \gamma(\lambda)^* = (\mathcal{H}_\lambda(\mathcal{S}^*))^1$, it follows that

$$\text{dom} \gamma(\lambda)^* \text{ is closed in } \mathcal{H} \iff \text{dom} \gamma(\lambda)^* \text{ is closed in } \mathcal{G},$$

and

$$\text{dom} \gamma(\lambda)^* \text{ is dense in } \mathcal{H} \iff \text{dom} \gamma(\lambda)^* \text{ is dense in } \mathcal{G}. \quad (2.6.10)$$

(i) The identity (2.6.7) follows from (2.6.9). It is clear that (2.6.7) implies the equivalence $\lambda \in \sigma_p(A_0) \iff 0 \in \sigma_p(\mathcal{H} - M(\lambda))$.

(iii) It follows from (2.6.8) that $\text{ran}(A_\mathcal{H} - \lambda)$ is a dense nonclosed subspace of $\mathcal{H}$ if and only if $\text{dom}(\mathcal{H} - M(\lambda))^{-1}\gamma(\lambda)^*$ is a dense nonclosed subspace of $\mathcal{H}$. By (2.6.10) and (2.6.11) with $V = (\mathcal{H} - M(\lambda))^{-1}$, this is equivalent to $\text{ran}(\mathcal{H} - M(\lambda))$ being a dense nonclosed subspace of $\mathcal{G}$. In addition, it follows from (i) that $A_\mathcal{H} - \lambda$ is injective if and only if $\mathcal{H} - M(\lambda)$ is injective. This proves the assertion.
(iv) The implication \((\Rightarrow)\) holds by Theorem 2.6.1. The implication \((\Leftarrow)\) is easy to see. Indeed, assume that \(0 \in \rho(\Theta - M(\lambda))\). Then \((\Theta - M(\lambda))^{-1} \in \mathcal{B}(\mathcal{G})\) and since \(\gamma(\lambda) \in \mathcal{B}(\mathcal{G}, \mathcal{G})\) and \(\gamma(\lambda)^* \in \mathcal{B}(\mathcal{G}, \mathcal{G})\) for \(\lambda \in \rho(A_0)\), one concludes from (2.6.2) that \((A_\Theta - \lambda)^{-1} \in \mathcal{B}(\mathcal{G})\), i.e., \(\lambda \in \rho(A_\Theta)\).

(ii) This assertion is a consequence of (i), (iii), and (iv). \(\square\)

The Kreĭn formula (2.6.2) was formulated above in terms of the closed relation \(\Theta\) in the Hilbert space \(\mathcal{G}\). Now the form of the Kreĭn formula will be given when a tight parametric representation of \(\Theta\) is chosen; cf. Section 1.10.

**Corollary 2.6.3.** Let \(S\) be a closed symmetric relation in \(\mathcal{H}\), let \(\{\mathcal{G}, \Gamma_0, \Gamma_1\}\) be a boundary triplet for \(S^*\), \(A_0 = \ker \Gamma_0\), and let \(\gamma\) and \(M\) be the corresponding \(\gamma\)-field and Weyl function, respectively. Let the closed relation \(\Theta\) have the parametric representation
\[
\Theta = \{\{Ae, Be\} : e \in \mathcal{E}\},
\]
(2.6.12)
where \(\mathcal{E}\) is a Hilbert space and \(A, B \in \mathcal{B}(\mathcal{E}, \mathcal{H})\), and assume that this representation of \(\Theta\) is tight, i.e., \(\ker A \cap \ker B = \{0\}\) holds. Then for all \(\lambda \in \rho(A_0)\) one has
\[
\lambda \in \rho(A_\Theta) \iff (B - M(\lambda)A)^{-1} \in \mathcal{B}(\mathcal{G}, \mathcal{E}),
\]
and in this case
\[
(A_\Theta - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda)A(B - M(\lambda)A)^{-1} \gamma(\lambda)^*.
\]
(2.6.13)

**Proof.** According to Theorem 2.6.2, for \(\lambda \in \rho(A_0)\) one has
\[
\lambda \in \rho(A_\Theta) \iff 0 \in \rho(\Theta - M(\lambda)),
\]
that is, \(\lambda \in \rho(A_\Theta)\) if and only if \((\Theta - M(\lambda))^{-1} \in \mathcal{B}(\mathcal{G})\). Due to the tightness of the representation (2.6.12), Lemma 1.11.6 shows that
\[
(\Theta - M(\lambda))^{-1} \in \mathcal{B}(\mathcal{G}) \iff (B - M(\lambda)A)^{-1} \in \mathcal{B}(\mathcal{G}, \mathcal{E}),
\]
as for all \(\lambda \in \rho(A_0)\) one has that \(M(\lambda) \in \mathcal{B}(\mathcal{G})\). In this case it follows that \((\Theta - M(\lambda))^{-1} = A(B - M(\lambda)A)^{-1}\). Furthermore, the resolvent formula (2.6.13) follows from (2.6.2). \(\square\)

Let again \(\Theta\) be a closed relation in \(\mathcal{G}\) and assume, in the same way as at the end of Section 2.2, that \(\Theta\) admits an orthogonal decomposition
\[
\Theta = \Theta_{\text{op}} \oplus \Theta_{\text{mul}}, \quad \mathcal{G} = \mathcal{G}_{\text{op}} \oplus \mathcal{G}_{\text{mul}},
\]
(2.6.14)
into a (not necessarily densely defined) operator part \(\Theta_{\text{op}}\) acting in the Hilbert space \(\mathcal{G}_{\text{op}} = \text{dom} \Theta^* = (\text{mul} \Theta)^\perp\) and a multivalued part \(\Theta_{\text{mul}} = \{0\} \times \text{mul} \Theta\) in the Hilbert space \(\mathcal{G}_{\text{mul}} = \text{mul} \Theta\); cf. Section 1.3. Recall that, in particular, closed symmetric, self-adjoint, (maximal) dissipative, and (maximal) accumulative relations \(\Theta\) in \(\mathcal{G}\) admit such a decomposition.
Corollary 2.6.4. Assume that the closed relation $\Theta$ in Theorem 2.6.1 has the orthogonal decomposition (2.6.14), let $P_{op}$ be the orthogonal projection onto $G_{op}$, and denote the canonical embedding of $G_{op}$ into $G$ by $\iota_{op}$. Let

$$A_{\Theta} = \{ \hat{f} \in S^* : \Gamma \hat{f} \in \Theta \}$$

be the intermediate extension of $S$ via (2.2.12). Then for all $\lambda \in \rho(A_0) \cap \rho(A_{\Theta})$ one has $(\Theta_{op} - P_{op}M(\lambda)\iota_{op})^{-1} \in \mathcal{B}(G_{op})$ and

$$(A_{\Theta} - \lambda)^{-1} = (A_{0} - \lambda)^{-1} + \gamma(\lambda)\iota_{op}(\Theta_{op} - P_{op}M(\lambda)\iota_{op})^{-1}P_{op}\gamma(\lambda)^*.$$  

Proof. In view of (2.6.14), one sees that for $\lambda \in \rho(A_0) \cap \rho(A_{\Theta})$

$$\Theta - M(\lambda) = \left\{ \left\{ \left( \varphi, 0 \right), \left( \Theta_{op}\varphi - P_{op}M(\lambda)\iota_{op}\varphi \right) \right\} : \varphi \in \text{dom} \Theta_{op}, \psi \in G_{mul} \right\},$$

and hence

$$(\Theta - M(\lambda))^{-1} = \left\{ \left\{ \left( \Theta_{op}\varphi - P_{op}M(\lambda)\iota_{op}\varphi \right), \left( \varphi, 0 \right) \right\} : \varphi \in \text{dom} \Theta_{op}, \chi \in G_{mul} \right\}.$$  

Since $(\Theta - M(\lambda))^{-1} \in \mathcal{B}(G)$, one has

$$\ker (\Theta_{op} - P_{op}M(\lambda)\iota_{op}) = \{0\} \quad \text{and} \quad \text{ran} (\Theta_{op} - P_{op}M(\lambda)\iota_{op}) = G_{op}.$$  

This shows that $\Theta_{op} - P_{op}M(\lambda)\iota_{op}$ is a bijective closed operator in $G_{op}$. Hence, $(\Theta_{op} - P_{op}M(\lambda)\iota_{op})^{-1} \in \mathcal{B}(G_{op})$ and so

$$(\Theta - M(\lambda))^{-1} = \begin{pmatrix} (\Theta_{op} - P_{op}M(\lambda)\iota_{op})^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

with respect to the decomposition (2.6.14). Now the identity

$$(\Theta - M(\lambda))^{-1} = \iota_{op}(\Theta_{op} - P_{op}M(\lambda)\iota_{op})^{-1}P_{op}$$

is an immediate consequence. This together with Theorem 2.6.1 implies the statement. \qed

If the closed relation $\Theta$ in $G$ admits a decomposition of the form

$$\Theta = \Theta' \oplus (\{0\} \times G'')$$  

as in Corollary 2.5.13, where $\Theta'$ is a closed relation in the Hilbert space $G'$ and $G = G' \oplus G''$, then Kreĭn’s formula can also be interpreted in the context of the intermediate symmetric extension $S'$ of $S$ in Proposition 2.5.12 and the corresponding restriction of the boundary triplet $\{G, \Gamma_0, \Gamma_1\}$. More precisely, if $\Theta$ is of the form
(2.6.15) and $S'$ and the boundary triplet \{\mathcal{G}', \Gamma_0', \Gamma_1'\} are as in Proposition 2.5.12 with corresponding $\gamma$-field $\gamma'$ and Weyl function $M'$, and

$$A_{\Theta'} = \{ \hat{f} \in (S')^*: \Gamma' \hat{f} \in \Theta' \} = \ker \left( \Gamma_1' - \Theta' \Gamma_0' \right),$$

then $A_{\Theta'} = \ker \left( \Gamma_1 - \Theta \Gamma_0 \right) = A_{\Theta}$ and $A_0' = \ker \Gamma_0' = \ker \Gamma_0 = A_0$ hold by Corollary 2.5.13 and Proposition 2.5.12, respectively. Moreover, by Theorem 2.6.1 one has $(\Theta' - M'(\lambda))^{-1} \in \mathcal{B}(S')$ for all $\lambda \in \rho(A_{\Theta'}) \cap \rho(A_0')$ and

$$(A_{\Theta'} - \lambda)^{-1} = (A_0' - \lambda)^{-1} + \gamma'(\lambda) \left( \Theta' - M'(\lambda) \right)^{-1} \gamma'(\bar{\lambda})^*. $$

In the special case where $\Theta' = \Theta_{\text{op}}$ and $\{0\} \times \mathcal{G}' = \Theta_{\text{mut}}$ as in (2.6.14) one has

$$M'(\lambda) = P_{\text{op}} M(\lambda)_{\text{t}_{\text{op}}}$$

and $\gamma'(\lambda) = \gamma(\lambda)_{\text{t}_{\text{op}}}$, so that Kreĭn’s formula in Corollary 2.6.4 can be rewritten in the form

$$(A_{\Theta_{\text{op}}} - \lambda)^{-1} = (A_0' - \lambda)^{-1} + \gamma'(\lambda) \left( \Theta_{\text{op}} - M'(\lambda) \right)^{-1} \gamma'(\bar{\lambda})^*. $$

The behavior of Kreĭn’s formula under transformations of boundary triplets will be discussed next. To this end suppose that $S$ is a closed symmetric relation in \mathcal{H}, let \{\mathcal{G}, \Gamma_0, \Gamma_1\} be a boundary triplet for $S^*$, $A_0 = \ker \Gamma_0$, and let $\gamma$ and $M$ be the corresponding $\gamma$-field and Weyl function, respectively. Consider a closed extension

$$A_{\Theta} = \{ \hat{f} \in S^*: \Gamma \hat{f} \in \Theta \} = \ker \left( \Gamma_1 - \Theta \Gamma_0 \right)$$

corresponding to a closed relation $\Theta$ in \mathcal{G}. Then for all $\lambda \in \rho(A_{\Theta}) \cap \rho(A_0)$ one has

$$(A_{\Theta} - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda) \left( \Theta - M(\lambda) \right)^{-1} \gamma(\bar{\lambda})^*$$

according to Theorem 2.6.1. Let $\mathcal{G}'$ be a further Hilbert space and assume that $\mathcal{W} \in \mathcal{B}(\mathcal{G} \times \mathcal{G}')$ satisfies the identities in (2.5.1). Let \{\mathcal{G}', \Gamma_0', \Gamma_1'\} be the corresponding transformed boundary triplet in (2.5.2) with $\gamma$-field $\gamma'$ and Weyl function $M'$ specified in Proposition 2.5.3. Let $A_0' = \ker \Gamma_0'$ and define the closed relation $\Theta'$ in $\mathcal{G}'$ by $\Theta' = \mathcal{W}[\Theta]$; cf. (2.5.4). By Proposition 2.5.2, one has

$$A_{\Theta} = \ker \left( \Gamma_1 - \Theta \Gamma_0 \right) = \ker \left( \Gamma_1' - \Theta' \Gamma_0' \right) = A_{\Theta'},$$

and hence for all $\lambda \in \rho(A_{\Theta}) \cap \rho(A_0')$ Kreĭn’s formula in Theorem 2.6.1 has the form

$$(A_{\Theta} - \lambda)^{-1} = (A_0' - \lambda)^{-1} + \gamma'(\lambda) \left( \Theta' - M'(\lambda) \right)^{-1} \gamma'(\bar{\lambda})^* = (A_{\Theta'} - \lambda)^{-1}. $$

In this sense Kreĭn’s formula is invariant under transformations of boundary triplets.

Next, Theorem 2.6.2 will be complemented for the case where the extensions are self-adjoint. Recall from Section 1.5 that for a self-adjoint relation $\hat{H}$ a spectral point $\lambda \in \mathbb{R}$ belongs to the discrete spectrum $\sigma_d(\hat{H})$ if $\lambda$ is an eigenvalue with finite multiplicity which is an isolated point in $\sigma(\hat{H})$. It will be used that $\lambda \in \sigma_d(\hat{H})$ if
and only if
\[ \dim \ker (H - \lambda) < \infty \quad \text{and} \quad \ran (H - \lambda) = \ran (H - \lambda). \] \hspace{1cm} (2.6.16)

The complement of the discrete spectrum of \( H \) in \( \sigma(H) \) is the essential spectrum, denoted by \( \sigma_{\text{ess}}(H) \).

**Theorem 2.6.5.** Let \( S \) be a closed symmetric relation, let \( \{ S, \Gamma_0, \Gamma_1 \} \) be a boundary triplet for \( S^* \), \( A_0 = \ker \Gamma_0 \), and let \( M \) be the corresponding Weyl function. Let \( \Theta \) be a self-adjoint relation in \( S \) and let
\[ A_\Theta = \{ \hat{f} \in S^* : \Gamma \hat{f} \in \Theta \} \]
be the corresponding self-adjoint extension via (2.1.5). Then the following statements hold for all \( \lambda \in \rho(A_0) \):

(i) \( \lambda \in \sigma_d(A_\Theta) \iff 0 \in \sigma_d(\Theta - M(\lambda)) \);
(ii) \( \lambda \in \sigma_{\text{ess}}(A_\Theta) \iff 0 \in \sigma_{\text{ess}}(\Theta - M(\lambda)) \).

*Proof.* Here one relies on the observations made in the proof of Theorem 2.6.2. Assume that \( \lambda \in \rho(A_0) \).

(i) It follows from Theorem 2.6.2 (i) that
\[ 0 < \dim \ker (A_\Theta - \lambda) < \infty \iff 0 < \dim \ker (\Theta - M(\lambda)) < \infty, \]
and it follows from (2.6.8) and (2.6.10) with \( V = (\Theta - M(\lambda))^{-1} \) that
\[ \ran (A_\Theta - \lambda) \text{ closed} \iff \ran (\Theta - M(\lambda)) \text{ closed}. \]

Now assertion (i) is a consequence of the above equivalences and the characterization (2.6.16) of discrete eigenvalues of self-adjoint relations.

(ii) Note that \( \lambda \in \sigma(A_\Theta) \) if and only if \( 0 \in \sigma(\Theta - M(\lambda)) \) by Theorem 2.6.2. Hence, this assertion is a consequence of item (i), \( \sigma_{\text{ess}}(A_\Theta) = \sigma(A_\Theta) \setminus \sigma_d(A_\Theta) \), and \( \sigma_{\text{ess}}(\Theta - M(\lambda)) = \sigma(\Theta - M(\lambda)) \setminus \sigma_d(\Theta - M(\lambda)) \). \( \square \)

### 2.7 Krein’s formula for exit space extensions

The Krein formula in Theorem 2.6.1 holds for intermediate extensions of a symmetric relation \( S \) in a Hilbert space \( \mathcal{H} \). In particular, these intermediate extensions contain maximal dissipative, maximal accumulative, and self-adjoint extensions. Now consider larger Hilbert spaces \( \mathcal{K} \) which contain \( \mathcal{H} \) as a closed subspace and self-adjoint relations \( \hat{A} \) in \( \mathcal{K} \) which extend \( S \) as studied by Krein and Naimark. It will be shown that such self-adjoint extensions induce families of relations in \( \mathcal{H} \) which also extend \( S \). For these families of relations there is a version of the Krein formula, which will also be called Krein–Naimark formula in this text.
The following notions are useful. Let $\mathcal{H}$ and $\mathcal{H}'$ be Hilbert spaces and let $\tilde{A}$ be a self-adjoint relation in the Hilbert space $\mathcal{H} \oplus \mathcal{H}'$. The Štraus family $T(\lambda), \lambda \in \mathbb{C}$, in $\mathcal{H}$ corresponding to the self-adjoint relation $\tilde{A}$ in $\mathcal{H} \oplus \mathcal{H}'$ is defined by

$$T(\lambda) = \left\{ (f, f') \in \mathcal{H} \times \mathcal{H} : \left\{ \left( \begin{array}{c} f \\ h \\ \end{array} \right), \left( \begin{array}{c} f' \\ h' \\ \end{array} \right) \right\} \in \tilde{A}, h' = \lambda h \right\}. \quad (2.7.1)$$

Here a vector notation is used for the elements

$$\left( \begin{array}{c} f \\ h \\ \end{array} \right) \in \text{dom} \tilde{A} \subset \mathcal{H} \oplus \mathcal{H}' \quad \text{and} \quad \left( \begin{array}{c} f' \\ h' \\ \end{array} \right) \in \text{ran} \tilde{A} \subset \mathcal{H} \oplus \mathcal{H}' ,$$

where $f, f' \in \mathcal{H}$ and $h, h' \in \mathcal{H}'$. This notation will be frequently used in the rest of this section. Closely associated with the Štraus family $T(\lambda)$ is the compressed resolvent $R(\lambda) \in B(\mathcal{H})$ of the self-adjoint relation $\tilde{A}$ in $\mathcal{H} \oplus \mathcal{H}'$, defined by

$$R(\lambda) = P_{\mathcal{H}}(\tilde{A} - \lambda)^{-1}i_{\mathcal{H}}, \quad \lambda \in \rho(\tilde{A}); \quad (2.7.2)$$

here $P_{\mathcal{H}} : \mathcal{H} \oplus \mathcal{H}' \to \mathcal{H}$ denotes the orthogonal projection from $\mathcal{H} \oplus \mathcal{H}'$ onto $\mathcal{H}$ and $i_{\mathcal{H}} : \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}'$ is the canonical embedding of $\mathcal{H}$ into $\mathcal{H} \oplus \mathcal{H}'$.

**Lemma 2.7.1.** Let $\tilde{A}$ be a self-adjoint relation in $\mathcal{H} \oplus \mathcal{H}'$. Then the following statements hold for the Štraus family $T(\lambda)$ in (2.7.1):

1. $T(\lambda)$ is maximal accumulative (maximal dissipative) for $\lambda \in \mathbb{C}^+$ ($\lambda \in \mathbb{C}^-$);
2. $T(\lambda) = T(\overline{\lambda})^*, \lambda \in \mathbb{C} \setminus \mathbb{R}$;
3. $\lambda \mapsto (T(\lambda) - \lambda)^{-1}$ is holomorphic on $\mathbb{C} \setminus \mathbb{R}$ with values in $B(\mathcal{H})$.

Moreover, the following statements hold for the compressed resolvent $R(\lambda) \in B(\mathcal{H})$ in (2.7.2):

1. $\lambda \mapsto R(\lambda)$ is holomorphic on $\mathbb{C} \setminus \mathbb{R}$ with values in $B(\mathcal{H})$;
2. $R(\lambda) = R(\overline{\lambda})^*, \lambda \in \mathbb{C} \setminus \mathbb{R}$;
3. for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$,

$$\frac{\text{Im} R(\lambda)}{\text{Im} \lambda} - R(\lambda)R(\lambda)^* \geq 0. \quad (2.7.3)$$

Furthermore, the Štraus family $T(\lambda)$ in (2.7.1) and the compressed resolvent in (2.7.2) are related via

$$R(\lambda) = (T(\lambda) - \lambda)^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \quad (2.7.4)$$

**Proof.** It is clear from (2.7.1) that for $\lambda \in \mathbb{C}$ the Štraus family $T(\lambda)$ satisfies

$$(T(\lambda) - \lambda)^{-1} = \left\{ \{f' - \lambda f, f\} \in \mathcal{H} \times \mathcal{H} : \left\{ \left( \begin{array}{c} f \\ h \\ \end{array} \right), \left( \begin{array}{c} f' \\ h' \\ \end{array} \right) \right\} \in \tilde{A}, h' = \lambda h \right\} .$$
On the other hand, it is clear that
\[(\widetilde{A} - \lambda)^{-1} = \left\{ \left\{ \left( f' - \lambda f, \frac{f}{h} \right), \left( \frac{f}{h} - \lambda \frac{f'}{h'} \right) \right\} : \left( f', \frac{f}{h} \right) \in \widetilde{A} \right\},\]
so that the compressed resolvent of \(\widetilde{A}\) in (2.7.2) is given for \(\lambda \in \rho(\widetilde{A})\) by
\[R(\lambda) = P_{\mathcal{H}}(\widetilde{A} - \lambda)^{-1} \iota_{\mathcal{H}}\]
\[= \left\{ \left\{ f' - \lambda f, f \right\} \in \mathcal{H} \times \mathcal{H} : \left( \left( f, \frac{f}{h} \right), \left( f, \frac{f}{h} \right) \right) \in \widetilde{A}, h' = \lambda h \right\} .\]
Comparison of the above identities shows that (2.7.4) holds.
(iv) & (v) follow immediately from (2.7.2).
(i) Let \(\{ f, f' \} \in T(\lambda)\). Then there exists a pair \(\{ h, h' \} \) such that
\[\left\{ \left( f, \frac{f}{h} \right), \left( f, \frac{f}{h} \right) \right\} \in \widetilde{A} \text{ and } h' = \lambda h .\]
Since \(\widetilde{A}\) is self-adjoint, it follows that
\[0 = \text{Im} \left( \left( f', f \right) + \left( h', h \right) \right) = \text{Im} \left( f', f \right) + \text{Im} \lambda \left( h, h \right)\]
and this implies that \(T(\lambda)\) is accumulative (dissipative) for \(\lambda \in \mathbb{C}^+ \) (\(\lambda \in \mathbb{C}^-\)). The maximality follows from (2.7.4) since \(\text{ran} \left( T(\lambda) - \lambda \right) = \mathcal{H} \); cf. Theorem 1.6.4.
(ii) It follows from (v) and (2.7.4) that
\[(T(\lambda) - \lambda)^{-1} = (T(\lambda) - \lambda)^{-*},\]
and this implies \(T(\lambda) = T(\lambda)^*\).
(iii) This is clear from (iv) and (2.7.4).
(vi) Let \(\varphi \in \mathcal{H} \) and \(\varphi' = R(\lambda)\varphi\). Then (2.7.4) implies \(\{ \varphi', \varphi + \lambda \varphi' \} \in T(\lambda)\). For \(\lambda \in \mathbb{C}^+ \) the relation \(T(\lambda)\) is maximal accumulative and hence (v) implies
\[0 \geq \text{Im} \left( \varphi + \lambda \varphi', \varphi' \right) = \text{Im} \left( \varphi, R(\lambda)\varphi \right) + \text{Im} \lambda \| R(\lambda)\varphi \|^2\]
\[= \text{Im} \left( R(\lambda')\varphi, \varphi \right) - (\text{Im} \lambda') \| R(\lambda')\varphi \|^2 .\]
Thus, it follows for \(\lambda \in \mathbb{C}^+ \) and \(\varphi \in \mathcal{H} \) that
\[\frac{\text{Im} \left( R(\lambda')\varphi, \varphi \right)}{\text{Im} \lambda} - (R(\lambda')R(\lambda')^* \varphi, \varphi) \geq 0 ,\]
which implies (2.7.3) on \(\mathbb{C}^-\). A similar reasoning leads to (2.7.3) on \(\mathbb{C}^+\). □

In Chapter 4 it will be shown how the properties of the Štraus family and the compressed resolvent in Lemma 2.7.1 determine the space \(\mathcal{H}'\) and the self-adjoint relation \(\widetilde{A}\) in \(\mathcal{H} \oplus \mathcal{H}'\).
In the present context the Štraus family and the compressed resolvent appear when one considers self-adjoint extensions of a closed symmetric relation \( S \) in a Hilbert spaces \( \mathcal{H} \). Let the Hilbert space \( \mathcal{H} \oplus \mathcal{H}' \) be an extension of \( \mathcal{H} \), where the Hilbert space \( \mathcal{H}' \) is an exit space. Assume that the self-adjoint relation \( \bar{A} \) in \( \mathcal{H} \oplus \mathcal{H}' \) is an extension of the symmetric relation \( S \) in \( \mathcal{H} \), i.e., \( S \subset A \). The Štraus family and the compressed resolvent of \( \bar{A} \) consist of relations in the closed subspace \( \mathcal{H} \) that extend \( S \) in the following sense.

**Proposition 2.7.2.** Let \( S \) be a closed symmetric relation in \( \mathcal{H} \) and let \( \bar{A} \) be a self-adjoint extension of \( S \) in \( \mathcal{H} \oplus \mathcal{H}' \). Then the Štraus family \( T(\lambda) \) in (2.7.1) satisfies

\[
S \subset T(\lambda) \subset S^*, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},
\]

and the compressed resolvent \( R(\lambda) \) in (2.7.2) satisfies

\[
R(\lambda)\upharpoonright_{\text{ran}(S - \lambda)} = (S - \lambda)^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\]

**Proof.** In order to prove (2.7.5), let \( \{f, f'\} \in S \). Then one has

\[
\left\{ \begin{pmatrix} f \\ 0 \end{pmatrix}, \begin{pmatrix} f' \\ 0 \end{pmatrix} \right\} \in \bar{A},
\]

and hence \( \{f, f'\} \in T(\lambda) \) for all \( \lambda \in \mathbb{C} \setminus \mathbb{R} \). This shows \( S \subset T(\lambda) \) for all \( \lambda \in \mathbb{C} \setminus \mathbb{R} \) and making use of Lemma 2.7.1 (ii) one concludes that \( T(\lambda) = (T(\lambda))^* \subset S^* \). The identity (2.7.6) follows from the inclusion \( S \subset T(\lambda) \) and (2.7.4). \( \square \)

Assume in the context of Proposition 2.7.2, that \( \{\mathcal{G}, \Gamma_0, \Gamma_1\} \) is a boundary triplet for \( S^* \). Then each relation \( T(\lambda), \lambda \in \mathbb{C} \setminus \mathbb{R}, \) in (2.7.5), being an intermediate extension of \( S \), can be described by the relation \( \Gamma(T(\lambda)) \) in the parameter space \( \mathcal{G} \). It follows from Lemma 2.7.1 and Proposition 1.12.6 that the family \( -T(\lambda) \), \( \lambda \in \mathbb{C} \setminus \mathbb{R} \), is a Nevanlinna family in \( \mathcal{G} \) in the sense of Definition 1.12.1. Note that for the holomorphy condition in Definition 1.12.1 it is necessary to apply Proposition 1.12.6. A similar reasoning will also be used in the proof of the next theorem, which relates a Nevanlinna family in \( \mathcal{G} \) to the Štraus family \( T(\lambda) \).

**Theorem 2.7.3.** Let \( S \) be a closed symmetric relation in \( \mathcal{H} \) and let \( \{\mathcal{G}, \Gamma_0, \Gamma_1\} \) be a boundary triplet for \( S^* \). Let \( \bar{A} \) be a self-adjoint extension of \( S \) in \( \mathcal{H} \oplus \mathcal{H}' \) and let \( T(\lambda), \lambda \in \mathbb{C} \setminus \mathbb{R}, \) be the corresponding Štraus family in (2.7.1). Then

\[
T(\lambda) = \{ \hat{f} \in S^* : \Gamma \hat{f} \in -\tau(\lambda) \} = \ker (\Gamma_1 + \tau(\lambda)\Gamma_0),
\]

where \( \tau(\lambda), \lambda \in \mathbb{C} \setminus \mathbb{R}, \) is a Nevanlinna family in \( \mathcal{G} \).

**Proof.** It follows from Lemma 2.7.1 and Proposition 2.7.2 that \( T(\lambda) \) is a closed extension of \( S \) for each \( \lambda \in \mathbb{C} \setminus \mathbb{R} \). According to Theorem 2.1.3, the extension \( T(\lambda) \) of \( S \) can be written in the form (2.7.7), where \( \tau(\lambda) = -\Gamma(T(\lambda)) \).

It will be shown that \( \tau(\lambda), \lambda \in \mathbb{C} \setminus \mathbb{R}, \) is a Nevanlinna family in \( \mathcal{G} \). Since \( T(\lambda) \) is maximal accumulative (maximal dissipative) for \( \lambda \in \mathbb{C}^+ \) (\( \lambda \in \mathbb{C}^- \)) it follows
from Corollary 2.1.4 (ii) that $\tau(\lambda)$ is maximal dissipative (maximal accumulative) for $\lambda \in \mathbb{C}^+ (\lambda \in \mathbb{C}^-)$. The property $T(\lambda) = T(\lambda)^*$ and Theorem 2.1.3 imply $\tau(\lambda) = \bar{\tau(\lambda)}^*$. Denote the $\gamma$-field and the Weyl function corresponding to the boundary triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ by $\gamma$ and $M$. Then for $\lambda \in \mathbb{C}^+$ and $\mu \in \mathbb{C}^\pm$ it follows from Lemma 1.11.5 that $(-\tau(\lambda) - M(\mu))^{-1} \in \mathcal{B}(\mathcal{H})$ and

$$(T(\lambda) - \mu)^{-1} = (A_0 - \mu)^{-1} - \gamma(\mu)(\tau(\lambda)^{+} + M(\mu))^{-1}\gamma(\mu)^*$$

holds by Theorem 2.6.1. According to Lemma 2.7.1, the mapping $\lambda \mapsto (T(\lambda) - \mu)^{-1}$ is holomorphic and hence by Proposition 1.12.6 it follows that also the mapping $\lambda \mapsto (T(\lambda) - \mu)^{-1}$ is holomorphic. Now (2.7.8) shows that $\lambda \mapsto (\tau(\lambda)^{+} + \mu)^{-1}$ is holomorphic. Therefore, $\tau(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, is a Nevanlinna family.

The Kreïn–Naïmark formula in the following theorem is now an immediate consequence.

**Theorem 2.7.4.** Let $S$ be a closed symmetric relation in $\mathcal{H}$, let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for $S^*$, $A_0 = \ker \Gamma_0$, and let $\gamma$ and $M$ be the corresponding $\gamma$-field and Weyl function, respectively. Let $\tilde{A}$ be a self-adjoint extension of $S$ in $\mathcal{H} \oplus \mathcal{H}'$. Then with the Nevanlinna family $\tau$ in $\mathcal{H}$ from Theorem 2.7.3 the compressed resolvent $R(\lambda)$ in (2.7.2) of $\tilde{A}$ is given by the Kreïn–Naïmark formula

$$R(\lambda) = (A_0 - \lambda)^{-1} - \gamma(\lambda)(M(\lambda) + \tau(\lambda))^{-1}\gamma(\lambda)^*,$$

$\lambda \in \mathbb{C} \setminus \mathbb{R}$.  \hfill (2.7.9)

*Proof.* As in the proof of Theorem 2.7.3, it follows from Theorem 2.6.1 that for $\lambda \in \mathbb{C} \setminus \mathbb{R}$ one has

$$(T(\lambda) - \lambda)^{-1} = (A_0 - \lambda)^{-1} - \gamma(\lambda)(\tau(\lambda)^{+} + M(\lambda))^{-1}\gamma(\lambda)^*.$$  \hfill (2.7.4)

Hence, the formula (2.7.9) follows from (2.7.4). \hfill \Box

In Chapter 4 the converse of Theorem 2.7.4 will be proved: for every Nevanlinna family in $\mathcal{H}$ there exists a self-adjoint exit space extension $\tilde{A}$ of $S$ such that (2.7.9) holds for the compressed resolvent of $A$.

Just as in the case of Corollary 2.6.3, there is now a formulation of the Kreïn formula for exit space extensions in terms of a parametric representation of the Nevanlinna family $\tau$. Assume that the Nevanlinna pair $\{A, B\}$ is a tight representation of the Nevanlinna family $\tau$; cf. Section 1.12. Then the next corollary can be shown in the same way as Corollary 2.6.3 by applying Proposition 1.12.6.

**Corollary 2.7.5.** Let $S$ be a closed symmetric relation in $\mathcal{H}$, let $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for $S^*$, $A_0 = \ker \Gamma_0$, and let $\gamma$ and $M$ be the corresponding $\gamma$-field and Weyl function, respectively. Let the Nevanlinna family $\tau$ in Theorem 2.7.4
have the tight representation \( \tau = \{A, B\} \) with the Nevanlinna pair \( \{A, B\} \). Then the compressed resolvent \( R(\lambda) \) of \( \tilde{A} \) has the form

\[
R(\lambda) = (A_0 - \lambda)^{-1} - \frac{\gamma(\lambda)A(\lambda)(B(\lambda) + M(\lambda)A(\lambda))^{-1}}{\lambda} \gamma(\tilde{\lambda})^*, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\]

The Štraus family in Theorem 2.7.3 can also be described in terms of a representing Nevanlinna pair \( \{A, B\} \).

**Corollary 2.7.6.** Let \( S \) be a closed symmetric relation in \( \mathcal{H} \), let \( \{\mathcal{G}, \Gamma_0, \Gamma_1\} \) be a boundary triplet for \( S^* \), and let \( T(\lambda) \), \( \lambda \in \mathbb{C} \setminus \mathbb{R} \), be the Štraus family in Theorem 2.7.3. Let the corresponding Nevanlinna family \( \tau \) have the tight representation \( \tau = \{A, B\} \) with the Nevanlinna pair \( \{A, B\} \). Then

\[
T(\lambda) = \{ \hat{f} \in S^* : B(\tilde{\lambda})^* \Gamma_0 \hat{f} = -A(\tilde{\lambda})^* \Gamma_1 \hat{f} \}. \tag{2.7.10}
\]

**Proof.** By assumption \( \tau(\lambda) \), \( \lambda \in \mathbb{C} \setminus \mathbb{R} \), is given as

\[
\tau(\lambda) = \{ \{A(\lambda)\varphi, B(\lambda)\varphi\} : \varphi \in \mathcal{G} \},
\]

with a Nevanlinna pair \( \{A, B\} \) and this representation is tight. The symmetry property \( \tau(\lambda)^* = \tau(\tilde{\lambda}) \) implies that

\[
\tau(\lambda)^* = \{ \{A(\tilde{\lambda})\varphi, B(\tilde{\lambda})\varphi\} : \varphi \in \mathcal{G} \},
\]

so that \( \tau(\lambda) \) can also be written as

\[
\tau(\lambda) = \{ \{\varphi, \varphi'\} \in \mathcal{G}^2 : B(\tilde{\lambda})^* \varphi = A(\tilde{\lambda})^* \varphi' \},
\]

and hence

\[
-\tau(\lambda) = \{ \{\varphi, \varphi'\} \in \mathcal{G}^2 : B(\tilde{\lambda})^* \varphi = -A(\tilde{\lambda})^* \varphi' \};
\]

cf. (2.2.3) and (2.2.4). Thus, (2.7.10) follows from (2.7.7). \qed

In the following a particular self-adjoint exit space extension of \( S \) will be studied. Here the exit space is the parameter space \( \mathcal{G} \).

**Proposition 2.7.7.** Let \( S \) be a closed symmetric relation in \( \mathcal{H} \) and let \( \{\mathcal{G}, \Gamma_0, \Gamma_1\} \) be a boundary triplet for \( S^* \). Then

\[
\tilde{A} = \left\{ \left\{ \begin{pmatrix} f \\ \Gamma_0 \hat{f} \end{pmatrix}, \begin{pmatrix} f' \\ -\Gamma_1 \hat{f} \end{pmatrix} \right\} : \hat{f} = \{f, f'\} \in S^* \right\} \tag{2.7.11}
\]

is a self-adjoint extension of \( S \) in \( \mathcal{H} \oplus \mathcal{G} \). The corresponding Štraus family \( T(\lambda) \), \( \lambda \in \mathbb{C} \setminus \mathbb{R} \), in \( \mathcal{H} \) has the form

\[
T(\lambda) = \{ \hat{f} \in S^* : -\Gamma_1 \hat{f} = \lambda \Gamma_0 \hat{f} \} = \ker (\Gamma_1 + \lambda \Gamma_0) \tag{2.7.12}
\]

and the compressed resolvent \( R(\lambda) \) onto \( \mathcal{H} \) is given by

\[
R(\lambda) = (A_0 - \lambda)^{-1} - \frac{\gamma(\lambda)(M(\lambda) + \lambda)^{-1}}{\lambda} \gamma(\tilde{\lambda})^*.
\]
Proof. Observe that $S \subset \tilde{A}$. Indeed, for $\hat{f} = \{f, f'\} \in S$ one has $\Gamma_0 \hat{f} = \Gamma_1 \hat{f} = 0$ by Proposition 2.1.2 (ii) and hence

\[ \left\{ \begin{pmatrix} f \\ 0 \\ \end{pmatrix}, \begin{pmatrix} f' \\ 0 \end{pmatrix} \right\} \in \tilde{A}. \]

It follows from the abstract Green identity (2.1.1) and the definition of $\tilde{A}$ in (2.7.11) that the relation $\tilde{A}$ is symmetric, that is, $\tilde{A} \subset (\tilde{A})^*$. Now let the element

\[ \left\{ \begin{pmatrix} g \\ \alpha \end{pmatrix}, \begin{pmatrix} g' \\ \alpha' \end{pmatrix} \right\}, \quad g, g' \in \mathcal{H}, \alpha, \alpha' \in \mathcal{G}, \quad (2.7.13) \]

belong to $(\tilde{A})^*$. Then for all $\hat{f} = \{f, f'\} \in S^*$ one has

\[ \left( \begin{pmatrix} f' \\ -\Gamma_1 \hat{f} \end{pmatrix}, \begin{pmatrix} g \\ \alpha \end{pmatrix} \right) = \left( \begin{pmatrix} f \\ \Gamma_0 \hat{f} \end{pmatrix}, \begin{pmatrix} g' \\ \alpha' \end{pmatrix} \right) \]

or, equivalently,

\[ (f', g) - (f, g') = (\Gamma_1 \hat{f}, \alpha) + (\Gamma_0 \hat{f}, \alpha'). \quad (2.7.14) \]

In particular, since $\ker \Gamma = S$, it follows from (2.7.14) that if $\hat{f} \in S$, then $\hat{g} \in S^*$. Therefore, the abstract Green identity (2.1.1) together with (2.7.14) imply

\[ (\Gamma_1 \hat{f}, \Gamma_0 \hat{g}) - (\Gamma_0 \hat{f}, \Gamma_1 \hat{g}) = (\Gamma_1 \hat{f}, \alpha) + (\Gamma_0 \hat{f}, \alpha') \]

for all $\hat{f} \in S^*$. By definition, the mapping $\Gamma : S^* \to \mathcal{G} \times \mathcal{G}$ is surjective, and consequently

\[ \alpha = \Gamma_0 \hat{g} \quad \text{and} \quad \alpha' = -\Gamma_1 \hat{g}. \]

Thus, the element in (2.7.13) belongs to $\tilde{A}$. Hence, $\tilde{A}$ is a self-adjoint extension of $S$ in $\mathcal{H} \oplus \mathcal{G}$.

The Štraus family $T(\lambda)$, $\lambda \in \mathbb{C}$, in $\mathcal{H}$ corresponding to the self-adjoint relation $\tilde{A}$ in (2.7.11) has the form

\[ T(\lambda) = \left\{ \hat{f} = \{f, f'\} \in S^* : \left\{ \begin{pmatrix} f \\ \Gamma_0 \hat{f} \end{pmatrix}, \begin{pmatrix} f' \\ -\Gamma_1 \hat{f} \end{pmatrix} \right\} \in \tilde{A}, \quad -\Gamma_1 \hat{f} = \lambda \Gamma_0 \hat{f} \right\}, \]

and hence is given by (2.7.12). The statement concerning the compressed resolvent of $\tilde{A}$ onto $\mathcal{H}$ follows from the Krein–Naǐmark formula in Theorem 2.7.4. □

Finally, the Štraus family and the compressed resolvent of the self-adjoint relation $A$ in $\mathcal{H} \oplus \mathcal{G}$ can be regarded from a slightly different point of view. Thus far, the Štraus family and the compressed resolvent were given as notions in the Hilbert space $\mathcal{H}$; more structure was added by considering a closed symmetric relation $S$ in $\mathcal{H}$ and assuming that $\tilde{A}$ is a self-adjoint extension of $S$ in $\mathcal{H} \oplus \mathcal{G}$. Now the role of the original space and the exit space will be interchanged and
a self-adjoint relation \( \tilde{A} \) in \( \mathcal{H} \oplus \mathcal{H}' \) will be viewed as a self-adjoint extension of the trivial symmetric relation \( S' \) in \( \mathcal{H}' \). The Štraus family \( T'(\lambda), \lambda \in \mathbb{C}, \) in \( \mathcal{H}' \) corresponding to \( \tilde{A} \) in \( \mathcal{H} \oplus \mathcal{H}' \) is defined by

\[
T'(\lambda) = \left\{ \{h, h'\} \in \mathcal{H}' \times \mathcal{H}' : \left( \begin{pmatrix} f \\ \bar{h} \end{pmatrix}, \begin{pmatrix} f' \\ \bar{h}' \end{pmatrix} \right) \in \tilde{A}, \ f' = \lambda f \right\}
\]

and the corresponding compressed resolvent \( R'(\lambda) \in \mathcal{B}(\mathcal{H}') \) is given by

\[
P_{\mathcal{H}'}(\tilde{A} - \lambda)^{-1} \iota_{\mathcal{H}'} = (T'(\lambda) - \lambda)^{-1}, \quad \lambda \in \rho(\tilde{A}); \tag{2.7.15}
\]

here \( P_{\mathcal{H}'} : \mathcal{H} \oplus \mathcal{H}' \rightarrow \mathcal{H}' \) denotes the orthogonal projection from \( \mathcal{H} \oplus \mathcal{H}' \) onto \( \mathcal{H}' \) and \( \iota_{\mathcal{H}'} : \mathcal{H}' \rightarrow \mathcal{H} \oplus \mathcal{H}' \) denotes the canonical embedding of \( \mathcal{H}' \) into \( \mathcal{H} \oplus \mathcal{H}' \). The adjoint of the trivial symmetric relation \( S' = \{0, 0\} \) in \( \mathcal{H}' \) is \( (S')^* = \mathcal{H}' \times \mathcal{H}' \) and

\[
\Gamma_0 \hat{h} = h \quad \text{and} \quad \Gamma_1 \hat{h} = h', \quad \hat{h} = \{h, h'\} \in (S')^*,
\]

defines a boundary triplet \( \{\mathcal{H}', \Gamma_0, \Gamma_1\} \) for \( (S')^* \). Then \( A'_0 = \{0\} \times \mathcal{H}' \), so that \( (A'_0 - \lambda)^{-1} = 0, \lambda \in \mathbb{C} \), and the \( \gamma \)-field and the Weyl function are given by

\[
\gamma'(\lambda) = I \quad \text{and} \quad M'(\lambda) = \lambda I;
\]

cf. Example 2.4.2. In this situation the Štraus family \( T'(\lambda), \lambda \in \mathbb{C} \setminus \mathbb{R}, \) in \( \mathcal{H}' \) induces a Nevanlinna family \( \tau(\lambda), \lambda \in \mathbb{C} \setminus \mathbb{R}, \) in the same space \( \mathcal{H}' \) as in (2.7.7) via

\[
T'(\lambda) = \ker (I_1^* + \tau(\lambda)I_0^*),
\]

so that \( \tau(\lambda) = -T'(\lambda) \). Then the compressed resolvent (2.7.15) takes the form

\[
P_{\mathcal{H}'}(\tilde{A} - \lambda)^{-1} \iota_{\mathcal{H}'} = -(\tau(\lambda) + \lambda)^{-1}, \tag{2.7.16}
\]

which can be viewed as the Kreĭn–Naĭmark formula in \( \mathcal{H}' \) for the extension \( \tilde{A} \) of \( S' \).

For the self-adjoint relation \( \tilde{A} \) in Proposition 2.7.7 it turns out in this new context that the corresponding Štraus family in \( \mathcal{H} \) is given by the function \(-M\).

**Proposition 2.7.8.** Let \( S \) be a closed symmetric relation in \( \mathcal{H} \), let \( \{\mathcal{G}, \Gamma_0, \Gamma_1\} \) be a boundary triplet for \( S^* \), and let \( M \) be the corresponding Weyl function. Consider the self-adjoint relation

\[
\tilde{A} = \left\{ \left\{ \begin{pmatrix} f \\ \Gamma_0 \hat{f} \end{pmatrix}, \begin{pmatrix} f' \\ -\Gamma_1 \hat{f} \end{pmatrix} \right\} : \hat{f} = \{f, f'\} \in S^* \right\}
\]

in \( \mathcal{H} \oplus \mathcal{G} \). Then the corresponding Štraus family in \( \mathcal{G} \) is given by

\[
\left\{ \{\Gamma_0 \hat{f}, -\Gamma_1 \hat{f}\} \in \mathcal{G} \times \mathcal{G} : \left( \begin{pmatrix} f \\ \Gamma_0 \hat{f} \end{pmatrix}, \begin{pmatrix} f' \\ -\Gamma_1 \hat{f} \end{pmatrix} \right) \in \tilde{A}, \hat{f} \in \mathcal{R}(S^*) \right\} \tag{2.7.17}
\]
2.8. Perturbation problems

and coincides with \(-M(\lambda), \lambda \in \mathbb{C} \setminus \mathbb{R}\). Furthermore, the compressed resolvent of 
\(\tilde{A}\) to \(\mathcal{G}\) is given by

\[
P_\mathcal{G}(\tilde{A} - \lambda)^{-1} \iota_\mathcal{G} = -(M(\lambda) + \lambda)^{-1}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R};
\]

(2.7.18)

here \(P_\mathcal{G} : \mathcal{H} \oplus \mathcal{G} \to \mathcal{G}\) denotes the orthogonal projection from \(\mathcal{H} \oplus \mathcal{G}\) onto \(\mathcal{G}\) and
\(\iota_\mathcal{G} : \mathcal{G} \to \mathcal{H} \oplus \mathcal{G}\) is the canonical embedding of \(\mathcal{G}\) into \(\mathcal{H} \oplus \mathcal{G}\).

Proof. It follows from the definition of the Štraus family in (2.7.1) that the Štraus family corresponding to
\(\tilde{A}\) in the Hilbert space \(\mathcal{G}\) has the form (2.7.17). Since \(\{\Gamma_0 \tilde{f}, -\Gamma_1 \tilde{f}\}\) belongs to (2.7.17) if and only if \(\tilde{f} \in \mathcal{H}_\lambda(S^*)\), it is also clear that
for all \(\lambda \in \mathbb{C} \setminus \mathbb{R}\) the Štraus family coincides with the values \(-M(\lambda)\) of the Weyl
function corresponding to the boundary triplet \(\{\mathcal{G}, \Gamma_0, \Gamma_1\}\). The formula (2.7.18)
follows from (2.7.16) in this special case. \(\square\)

2.8 Perturbation problems

Let \(A\) be a self-adjoint relation in the Hilbert space \(\mathcal{H}\), let \(V \in \mathcal{B}(\mathcal{H})\) be a bounded
self-adjoint operator in \(\mathcal{H}\), and consider the self-adjoint relation

\[
B = A + V.
\]

(2.8.1)

For \(\lambda \in \rho(A) \cap \rho(B)\) one can rewrite (2.8.1) in the form

\[
(B - \lambda)^{-1} - (A - \lambda)^{-1} = -(B - \lambda)^{-1}V(A - \lambda)^{-1};
\]

this follows from Lemma 1.11.2 with \(H = A\), \(R = \lambda - V\) and \(S = \lambda\). In particular,
if \(V\) in (2.8.1) belongs to some left-sided or right-sided operator ideal, then the
same is true for the difference of the resolvents of \(A\) and \(B\). From this point of
view perturbation problems in the resolvent sense are more general than additive
perturbations of the form (2.8.1). Such perturbation problems embed naturally in
the framework of the extension theory that has been discussed in this chapter.

In the next theorem the particularly simple case of finite-rank perturbations
is treated.

Theorem 2.8.1. Let \(A\) and \(B\) be self-adjoint relations in \(\mathcal{H}\) and assume that

\[
\dim \text{ran } ((B - \lambda_0)^{-1} - (A - \lambda_0)^{-1}) = n < \infty
\]

(2.8.2)

for some, and hence for all \(\lambda_0 \in \rho(A) \cap \rho(B)\). Then \(S = \text{ker} \Gamma_0 \cap \text{ker} \Gamma_1\) is a closed symmetric
relation in \(\mathcal{H}\) and there exists a boundary triplet \(\{\mathcal{H}^n, \Gamma_0, \Gamma_1\}\) for \(S^*\) such that

\[
A = \text{ker} \Gamma_0 \quad \text{and} \quad B = \text{ker} \Gamma_1.
\]

(2.8.3)
If $\gamma$ and $M$ are the $\gamma$-field and the Weyl function, respectively, corresponding to \({\mathbb{C}^n, \Gamma_0, \Gamma_1}\), then
\[
(B - \lambda)^{-1} - (A - \lambda)^{-1} = -\gamma(\lambda)M(\lambda)^{-1}\gamma(\bar{\lambda})^* 
\]
(2.8.4)
for all $\lambda \in \rho(A) \cap \rho(B)$. Moreover, if $\lambda \in \rho(A)$, then $\lambda \in \sigma_p(B)$ if and only if $0 \in \sigma_p(M(\lambda))$, and the multiplicities are at most $n$ and coincide.

Proof. Let $\lambda_0 \in \rho(A) \cap \rho(B)$ be such that (2.8.3) holds, and consider the closed symmetric relation $S = A \cap B$ in $\mathfrak{A}$. By construction, $A$ and $B$ are disjoint self-adjoint extensions of $S$, and hence
\[
\text{ran} (S - \bar{\lambda}_0) = \ker ((B - \bar{\lambda}_0)^{-1} - (A - \lambda_0)^{-1})
\]
by Theorem 1.7.8. This leads to
\[
\ker (S^* - \lambda_0) = (\text{ran} (S - \bar{\lambda}_0))^\perp = \text{ran} ((B - \lambda_0)^{-1} - (A - \lambda_0)^{-1}),
\]
where (2.8.2) was used in the last equality. Now Theorem 1.7.8 implies that $A$ and $B$ are transversal self-adjoint extensions of $S$. Theorem 2.5.9 shows that there exists a boundary triplet \({\mathbb{C}^n, \Gamma_0, \Gamma_1}\) such that (2.8.3) holds, and the formula (2.8.4) follows from Theorem 2.6.1. One also concludes from (2.8.4) and the fact that $M(\lambda)$ is bijective for $\lambda \in \rho(A) \cap \rho(B)$ (see Corollary 2.5.4) that the difference of the resolvents in (2.8.2) is of rank $n$ for all $\lambda \in \rho(A) \cap \rho(B)$. The last statement on the eigenvalues of $B$ follows from Theorem 2.6.2 (i). \hfill \Box

The following result is a generalization of Theorem 2.8.1 that applies to non-self-adjoint intermediate extensions $B$.

Theorem 2.8.2. Let $A$ be a self-adjoint relation in $\mathfrak{A}$ and let $B$ be a closed relation in $\mathfrak{A}$ such that $\rho(B) \neq \emptyset$. Then $S = A \cap B$ is a closed symmetric relation in $\mathfrak{A}$ and there exist a boundary triplet \({\mathfrak{A}, \Gamma_0, \Gamma_1}\) for $S^*$ and a closed operator $\Theta$ in $\mathfrak{A}$ such that
\[
A = \ker \Gamma_0 \quad \text{and} \quad B = \ker (\Gamma_1 - \Theta \Gamma_0).
\]
If $\gamma$ and $M$ are the $\gamma$-field and the Weyl function, respectively, corresponding to \({\mathfrak{A}, \Gamma_0, \Gamma_1}\), then
\[
(B - \lambda)^{-1} - (A - \lambda)^{-1} = \gamma(\lambda)(\Theta - M(\lambda))^{-1}\gamma(\bar{\lambda})^*
\]
for all $\lambda \in \rho(A) \cap \rho(B)$. Moreover, for all $\lambda \in \rho(A)$ one has $\lambda \in \sigma_i(B)$ if and only if $0 \in \sigma_i(\Theta - M(\lambda))$, $i = p, c, r$, and for $i = p$ the geometric multiplicities coincide.

Proof. It is clear that $S = A \cap B$ is a closed symmetric relation and hence there exists a boundary triplet \({\mathfrak{A}, \Gamma_0, \Gamma_1}\) for $S^*$ such that $A = \ker \Gamma_0$; cf. Theorem 2.4.1. Since $B$ is a closed extension of $S$, there exists a closed relation $\Theta$ in $\mathfrak{A}$ such that $B = \ker (\Gamma_1 - \Theta \Gamma_0)$. By construction, the relations $A$ and $B$ are disjoint and hence it follows from Proposition 2.1.8 (i) that $\Theta$ is a closed operator in $\mathfrak{A}$. The resolvent formula and the assertion on the spectrum of $B$ are immediate consequences of Theorem 2.6.1 and Theorem 2.6.2. \hfill \Box
Let $\mathfrak{K}$ and $\mathcal{L}$ be Hilbert spaces and let $T \in \mathcal{B}(\mathfrak{K}, \mathcal{L})$ be a compact operator. Recall that the singular values $s_k(T)$, $k \in \mathbb{N}$, of $T$ are defined as the eigenvalues of the nonnegative compact operator $(T^*T)^{1/2} \in \mathcal{B}(\mathfrak{K})$ (enumerated in nonincreasing order). The Schatten–von Neumann ideal $\mathfrak{S}_p(\mathfrak{K}, \mathcal{L})$, $1 \leq p < \infty$, consists of all compact operators $T \in \mathcal{B}(\mathfrak{K}, \mathcal{L})$ such that the singular values are $p$-summable, that is,

$$\sum_{k=1}^{\infty} (s_k(T))^p < \infty.$$ 

If $\mathfrak{K} = \mathcal{L}$ the notation $\mathfrak{S}_p(\mathfrak{K})$ is used instead of $\mathfrak{S}_p(\mathfrak{K}, \mathfrak{K})$. Observe that the nonzero singular values of $T$ coincide with the nonzero singular values of the restriction of $T$ to $(\ker T)^\perp$ as the corresponding restriction of $T^*T$ is a nonnegative compact operator in the Hilbert space $(\ker T)^\perp$. This fact will be used in the proof of the following theorem.

**Theorem 2.8.3.** Let $S$ be a closed symmetric relation in $\mathfrak{H}$, let $\{\mathfrak{G}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for $S^*$, let $A_{\Theta_1}$ and $A_{\Theta_2}$ be closed extensions of $S$ corresponding to closed relations $\Theta_1$ and $\Theta_2$ in $\mathfrak{G}$ via (2.1.5), and assume that $\rho(A_{\Theta_1}) \cap \rho(A_{\Theta_2}) \neq \emptyset$ and $\rho(\Theta_1) \cap \rho(\Theta_2) \neq \emptyset$. Then

$$(A_{\Theta_1} - \lambda)^{-1} - (A_{\Theta_2} - \lambda)^{-1} \in \mathfrak{S}_p(\mathfrak{G})$$

for some, and hence for all $\lambda \in \rho(A_{\Theta_1}) \cap \rho(A_{\Theta_2})$ if and only if

$$(\Theta_1 - \xi)^{-1} - (\Theta_2 - \xi)^{-1} \in \mathfrak{S}_p(\mathfrak{G})$$

for some, and hence for all $\xi \in \rho(\Theta_1) \cap \rho(\Theta_2)$.

**Proof.** Let $A_0 = \ker \Gamma_0$ and let $\gamma$ and $M$ be the $\gamma$-field and the Weyl function corresponding to the boundary triplet $\{\mathfrak{G}, \Gamma_0, \Gamma_1\}$. Then one has

$$(A_{\Theta_1} - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda)(\Theta_1 - M(\lambda))^{-1}\gamma^*(\lambda),$$

$$(A_{\Theta_2} - \lambda)^{-1} = (A_0 - \lambda)^{-1} + \gamma(\lambda)(\Theta_2 - M(\lambda))^{-1}\gamma^*(\lambda),$$

for all $\lambda \in \rho(A_{\Theta_1}) \cap \rho(A_{\Theta_2}) \cap \rho(A_0)$, and hence

$$(A_{\Theta_1} - \lambda)^{-1} - (A_{\Theta_2} - \lambda)^{-1} = \gamma(\lambda)\left[(\Theta_1 - M(\lambda))^{-1} - (\Theta_2 - M(\lambda))^{-1}\right]\gamma^*(\lambda).$$

It will be shown that (2.8.5) holds if and only if

$$(\Theta_1 - M(\lambda))^{-1} - (\Theta_2 - M(\lambda))^{-1} \in \mathfrak{S}_p(\mathfrak{G}).$$

In fact, it is clear that if (2.8.8) holds, then so does (2.8.5). Conversely, if (2.8.5) holds, then

$$\gamma(\lambda)\left[(\Theta_1 - M(\lambda))^{-1} - (\Theta_2 - M(\lambda))^{-1}\right]\gamma^*(\lambda) \in \mathfrak{S}_p(\mathfrak{G})$$
follows directly from \((2.8.7)\). Since \(\gamma(\lambda)\) is an isomorphism from \(\mathcal{G}\) onto \(\mathcal{M}_\lambda(S^*)\) and \(\ker \gamma(\lambda)^* = \mathcal{M}_\lambda(S^*)^\perp\), it follows that the restriction of \(\gamma(\lambda)^*\) to \(\mathcal{M}_\lambda(S^*)\) is an isomorphism onto \(\mathcal{G}\). Hence, the operator in \((2.8.8)\) may also be viewed as a bounded operator from \(\mathcal{M}_\lambda(S^*)\) to \(\mathcal{M}_\lambda(S^*)\) and thus belongs to the Schatten–von Neumann ideal \(\mathcal{S}_p(\mathcal{M}_\lambda(S^*), \mathcal{N}_\lambda(S^*))\). In this context \(\gamma(\lambda): \mathcal{G} \rightarrow \mathcal{M}_\lambda(S^*)\) and \(\gamma(\lambda)^*: \mathcal{M}_\lambda(S^*) \rightarrow \mathcal{G}\) are bounded invertible and hence it follows that \((2.8.8)\) holds. Therefore, if \(\lambda \in \rho(A_{\Theta_1}) \cap \rho(A_{\Theta_2}) \cap \rho(A_0)\), then \((2.8.5)\) is equivalent to \((2.8.8)\). Note that if \((2.8.5)\) holds for some \(\lambda \in \rho(A_{\Theta_1}) \cap \rho(A_{\Theta_2})\), then it holds for all \(\lambda \in \rho(A_{\Theta_1}) \cap \rho(A_{\Theta_2})\) by Lemma 1.11.4.

It remains to show that for all \(\lambda \in \rho(A_{\Theta_1}) \cap \rho(A_{\Theta_2}) \cap \rho(A_0)\) \((2.8.8)\) is equivalent to \((2.8.6)\). By Lemma 1.11.4,

\[
(\Theta_1 - M(\lambda))^{-1} - (\Theta_2 - M(\lambda))^{-1} = [I - (\Theta_1 - \xi)^{-1}(M(\lambda) - \xi)]^{-1} \\
\quad [(\Theta_1 - \xi)^{-1} - (\Theta_2 - \xi)^{-1}] [I - (M(\lambda) - \xi)(\Theta_2 - \xi)^{-1}]^{-1}
\]

and since the factors around \((\Theta_1 - \xi)^{-1} - (\Theta_2 - \xi)^{-1}\) on the right-hand side are bounded invertible by Lemma 1.11.3, this establishes the equivalence of \((2.8.8)\) and \((2.8.6)\).

If \(\Theta_1\) and \(\Theta_2\) in Theorem 2.8.3 are bounded operators in \(\mathcal{G}\) the condition \(\rho(\Theta_1) \cap \rho(\Theta_2) \neq \emptyset\) is automatically satisfied and the identity

\[
(\Theta_1 - \xi)^{-1} - (\Theta_2 - \xi)^{-1} = (\Theta_1 - \xi)^{-1}(\Theta_2 - \Theta_1)(\Theta_2 - \xi)^{-1}
\]

shows that \((\Theta_1 - \xi)^{-1} - (\Theta_2 - \xi)^{-1} \in \mathcal{S}_p(\mathcal{G})\) for \(\xi \in \rho(\Theta_1) \cap \rho(\Theta_2)\) if and only if \(\Theta_1 - \Theta_2 \in \mathcal{S}_p(\mathcal{G})\). This leads to the following corollary.

**Corollary 2.8.4.** Let \(S\) be a closed symmetric relation in \(\mathcal{H}\), let \(\{\mathcal{G}, \Gamma_0, \Gamma_1\}\) be a boundary triplet for \(S^*\), let \(A_{\Theta_1}\) and \(A_{\Theta_2}\) be closed extensions of \(S\) which correspond to bounded operators \(\Theta_1, \Theta_2 \in \mathcal{B}(\mathcal{G})\) via \((2.1.5)\), and assume that \(\rho(A_{\Theta_1}) \cap \rho(A_{\Theta_2}) \neq \emptyset\). Then

\[
(A_{\Theta_1} - \lambda)^{-1} - (A_{\Theta_2} - \lambda)^{-1} \in \mathcal{S}_p(\mathcal{G})
\]

for some, and hence for all \(\lambda \in \rho(A_{\Theta_1}) \cap \rho(A_{\Theta_2})\) if and only if \(\Theta_1 - \Theta_2 \in \mathcal{S}_p(\mathcal{G})\).

The following proposition is an addendum to Theorem 2.8.2 in the special case where \(B\) is an \(\mathcal{S}_p\)-perturbation of \(A\) in the resolvent sense.

**Proposition 2.8.5.** Let \(A\) be a self-adjoint relation in \(\mathcal{H}\), let \(B\) be a closed relation in \(\mathcal{H}\) with \(\rho(B) \neq \emptyset\), assume that

\[
(B - \lambda_0)^{-1} - (A - \lambda_0)^{-1} \in \mathcal{S}_p(\mathcal{G})
\]

(2.8.10)
for some \( \lambda_0 \in \rho(A) \cap \rho(B) \) and that (2.8.10) is not a finite-rank operator. Let \( S = A \cap B \) and \( \{G, \Gamma_0, \Gamma_1\} \) be a boundary triplet for \( S^* \) as in Theorem 2.8.2 such that

\[
A = \ker \Gamma_0 \quad \text{and} \quad B = \ker (\Gamma_1 - \Theta \Gamma_0)
\]

for some closed operator \( \Theta \) in \( G \). If \( \rho(\Theta) \neq \emptyset \), then \( \Theta \) is an unbounded closed operator and \( (\Theta - \xi)^{-1} \in \mathcal{S}_p(G) \) for all \( \xi \in \rho(\Theta) \).

**Proof.** Assume that (2.8.10) holds and that \( \rho(\Theta) \neq \emptyset \). As \( \Theta_0 = \{0\} \times G \) is the self-adjoint relation in \( G \) which corresponds to \( A = \ker \Gamma_0 \) and \( (\Theta_0 - \xi)^{-1} = 0 \) for \( \xi \in \mathbb{C} \), one concludes from Theorem 2.8.3 and (2.8.10) that

\[
(\Theta - \xi)^{-1} = (\Theta - \xi)^{-1} - (\Theta_0 - \xi)^{-1} \in \mathcal{S}_p(G), \quad \xi \in \rho(\Theta).
\]

Together with the assumption that (2.8.10) is of infinite rank, this implies that \( \Theta \) is an unbounded closed operator in \( G \). \( \square \)