SUPER-RIGIDITY AND FINITENESS OF EMBEDDED J-HOLOMORPHIC CURVES ON CALABI-YAU THREEFOLDS

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Abstract. In [OZ], Zhu and the present author proved that for a generic choice of tame almost complex structures $J$ on a symplectic manifold $(M^{2n}, \omega)$ with its first Chern class $c_1 = 0$, any somewhere injective $J$-holomorphic maps $((\Sigma, j), u)$ in $M$ are Fredholm regular and embedded for any closed surface $\Sigma$ of any genus.

In this paper we prove that when $n = 3$ and $c_1(M) = 0$ all such $J$-holomorphic embedded curves are super-rigid for a generic choice of $J$: the normal linearizations of them and of all their multiple covers have trivial kernel. We also prove that there are only finitely many embedded $J$-holomorphic curves for each given homology class and given genus of the domain. In particular, we give the definition of integer counting invariants of embedded curves and derive a formula for their multiple cover contribution for such a choice of $J$. As a consequence, we prove a conjecture by Cox and Katz in all genus, which was originally stated in Conjecture 7.4.5 [CK] only for the genus zero case. A priori, this invariant may depend on the choice of $J$ and exhibit a wall-crossing phenomenon as suggested in [GV], [KS], and is believed to be the same as Gopakumar-Vafa’s BPS counting invariants [Kon].

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Date: June 15, 2010 (revised).

Key words and phrases. Calabi-Yau threefolds, embedded $J$-holomorphic curves, normal $\bar{\partial}$-operator, super-rigidity, Gopakumar-Vafa BPS count.

The author is partially supported by the NSF grant #DMS 0904197.
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References

1. Introduction

Let $(M^{2n}, \omega)$ be a symplectic manifold of dimension $2n$. We denote by $J$ an almost complex structure compatible with $\omega$ and by $\mathcal{J}_\omega = \mathcal{J}_\omega(M)$ the set of compatible almost complex structures on $M$. Let $\Sigma = \Sigma_g$ be a closed compact surface of genus $g$. We denote by $j$ an almost complex structure on $\Sigma$ and denote the set of almost complex structures on $\Sigma$ by $\mathcal{J}(\Sigma) = \{ j \in \text{End}_R(T\Sigma) \mid j^2 = -\text{id} \}$ and by $\mathcal{M}_g = \mathcal{M}(\Sigma_g)$ its quotient by $\text{Diff}(\Sigma_g)$ i.e., the moduli space of complex structures on $\Sigma$. We call a pair $((\Sigma, j), u)$ a $J$-holomorphic map if $u$ is $(j, J)$-holomorphic, i.e., if it satisfies

$$J \circ du = du \circ j.$$ 

We define the non-linear Cauchy-Riemann operator $\mathcal{J}_J$ by the assignment

$$\mathcal{J}_J : (j, u) \mapsto \mathcal{J}_{(j,J)}u$$

where $\mathcal{J}_{(j,J)}$ is defined by

$$\mathcal{J}_{(j,J)}u = \frac{du + J \circ du \circ j}{2}.$$ 

Recent development in the Gromov-Witten theory in relation to the integrality conjecture of the Gopakumar-Vafa BPS count [GV] unravels necessity of the count of embedded curves and of finer structure theorem on the image of $J$-holomorphic curves [P], [Kon]. In this regard, the following results are proved by Zhu and the present author in [OZ].

**Theorem 1.1** (Embeddedness, [OZ]). Suppose $c_1(M, \omega) = 0$. Let $(\Sigma, j)$ be a closed Riemann surface of genus $g$. Then there exists a dense subset $\mathcal{J}^{\text{nodal}}_\omega \subset \mathcal{J}_\omega$ such that for $J \in \mathcal{J}^{\text{nodal}}_\omega$, the following holds:

1. all somewhere injective $J$-holomorphic map $u : (\Sigma, j) \to (M, J)$ are Fredholm regular and embedded for any compact Riemann surface $(\Sigma, j)$.
2. when $n \geq 3$ no two somewhere injective $J$-holomorphic curves intersect unless the two curves have the same images.

In fact, the proof in [OZ] implies that the same is true for all $\mathcal{M}_g(J, \beta)$ as long as $c_1(M)(\beta) = 0$ even when $c_1(M) \neq 0$.

We denote by $\mathcal{M}^{\text{inj}}_g(M, J; \beta)$ the moduli space of somewhere injective $J$-holomorphic maps.

Combining this theorem with standard dimension counting argument, they also prove
Theorem 1.2 (Structure of stable maps, [OZ]). Suppose $n = 3$ in addition. There exists a subset $J \in J^\text{nodal} \subset J^\text{emb}$ which is dense in $J_\omega$ such that for any $J \in J^\text{nodal}$ the following hold:

1. it is either smooth and embedded or
2. it has smooth domain and factors through the composition
   
   $u = u' \circ \phi : \Sigma \to C \hookrightarrow M$
   
   for some embedding $u' : C \to M$ and ramified covering $\phi : \Sigma \to C$ or
3. it is a stable map of the type such that all of its irreducible components have the same locus of images $C \subset M$ and are ramified over $C$, except those of constant components.

The latter two cases above can occur only when $\beta$ is of the form $\beta = d\gamma$ with $d \geq 2$ for some other $\gamma \in H_2(M)$.

Remark 1.3. (1) In general these subsets are not open in $J_\omega$ and hence does not carry manifold structure themselves. However $J^\text{nodal}$ or $J^\text{emb}$ constructed in [OZ] is a countable intersection of dense open subsets of $J_\omega$ and we can always first work on those open subsets and then take the intersection. To be consistent with the later enumeration of ordered sets of $(G, K) \in \mathbb{N}^2$ controlling the genus $0 \leq g \leq G$ of the domain and the homology classes $\beta \in H_2(M)$ satisfying $\omega(\beta) \leq K$, we write

$$J_\omega = \bigcap_{(G, K) \in \mathbb{N}^2} J^\text{nodal}_{(G, K)}$$

where $J^\text{nodal}_{(G, K)} \subset J_\omega$ is open and dense. Similar discussion applies to $J^\text{emb}$ and write

$$J_\omega = \bigcap_{(G, K) \in \mathbb{N}^2} J^\text{emb}_{(G, K)}$$

so that $J^\text{nodal}_{(G, K)} \subset J^\text{emb}_{(G, K)}$. These are dense but still may not be open in $J_\omega$ due to the possible degeneration of domain complex structures.

(2) Considering $J^\text{nodal}_{(G, K)}$ is inevitable also to avoid simultaneous dealing with infinite number of bubbling degenerations and multicovering degenerations. Working with the moduli spaces with $0 \leq g \leq G$ and $\omega(\beta) \leq K$ enables us to always deal with a finite number of cases for each given $(G, K)$. It turns out that we need to introduce another parameter $L$ for each given $(G, K)$ to avoid degeneration of domain complex structure or of branched covering maps $\phi : \Sigma \to C$ to make our scheme work. See sections 11, 11 for the explanation of this choice $L$.

Theorem 1.1 motivated us to exploit embeddedness of $J$-holomorphic curves in the study of Gromov-Witten theory on Calabi-Yau threefolds and its relation to Gopakumar-Vafa formula [GV] (See [BP1, BP2] for a nice formulation of a precise conjecture and background materials.)

In this regard, the first main theorem is to prove finiteness of such embedded curves for each given genus and homology class of the curves. With the above two structure theorems in our hand, there are two major degenerations of embedded curves to study; one is the bubbling degeneration and the other multi-covering degeneration. The parameters $(G, K)$ partially control these two degenerations. On the other hand there is another possible degeneration of branched coverings
\( \phi : \Sigma \to C \) which is degeneration of complex structures of the domain \((\Sigma, j)\). The above mentioned parameter \( L \) is related to this degeneration.

One construction that exploits embeddedness (or immersedness) in the study of pseudo-holomorphic curves is construction of normal \( \overline{\partial} \)-operator introduced and studied by Gromov \cite{G}. This construction uses embeddedness of pseudo-holomorphic curves in an essential way. It considers the normal part of \( \overline{\partial} \)-operator of such a curve, denoted by \( \overline{\partial}_\nu \), and explains how the space of \textit{un-parameterized} holomorphic curves near given \( C \) can be identified locally with the zero set of \( \overline{\partial}_\nu \). This construction was further explained in \cite{HLS}. In this paper, we use the normal \( \overline{\partial} \)-operator and its generalization to \textit{smooth} maps into the image loci of \( J \)-holomorphic embedded curves in a crucial way.

For any given pair \((J, C)\) where \( C \) is an embedded \( J \)-holomorphic curve, we associate a Fredholm operator

\[
L(J, C) = D\overline{\partial}_\nu : \Omega^0(N_C) \to \Omega^{0,1}(N_C)
\]

by linearizing \( \overline{\partial}_\nu \) where \( N_C = TM|_C/TC \) is the normal bundle of \( C \) with complex structure \( i = i_{(J,C)} \) induced by \( J|_N_C \). We note that the pair \((J, C)\) with \( C \) a \( J \)-holomorphic curve induces a holomorphic structure on the normal bundle \( N_C \). This is because \( C \) is a real two dimensional surface on which obstruction to integrability of a complex structure on the complex vector bundle \( N_C \to C \) vanishes. We denote the associated holomorphic structure by \( i = i_{(J,C)} \).

We also want to study contributions by multiple coverings to the Gromov-Witten invariants. For this study, we need to study the super-rigidity in the sense of \cite{BFT, BP2}. For this purpose, we extend the above mentioned normal operator for any smooth map \( \phi : \Sigma \to C \subset M \) with smooth domain for each embedded \( J \)-holomorphic curve \( C \). It turns out that this study is also crucial in our proof of finiteness of embedded curves in a given class and for given genus.

For any given such a quadruple \((j, \phi; J, C)\) where \( C \) is embedded \( J \)-holomorphic curve and \( \phi : \Sigma \to C \) a smooth map, we associate a Fredholm operator

\[
\Psi(j, \phi; J, C) = D_\phi \overline{\partial}_\nu : \Omega^0(\phi^*N) \to \Omega^{0,1}(\phi^*N)
\]

where \( N = N_C \) is the normal bundle of \( C \) with complex structure \( i = i_{(J,C)} \) and \( \phi^*N \) equipped with the pull-back complex structure \( \phi^*i \).

**Definition 1.4.** Let \( C \subset M \) be a connected unparameterized embedded \( J \)-holomorphic curve of genus \( g \). We say that \( C \) is \((d,h)\)-rigid with respect to \( J \) if \( L = \Psi(j, \phi; J, C) \) has no kernel for any holomorphic \( \phi : \Sigma \to C \) of genus \( g_\Sigma = g + h \) and degree \( d \). We say \( C \) is \textit{super-rigid} with respect to \( J \) if it is \((d,h)\)-rigid for any \((d,h)\).

The following local perturbation theorem is a crucial ingredient in the proofs of global super-rigidity and finiteness results in this paper.

**Theorem 1.5** (Local super-rigidity). \( (M, \omega) \) be any symplectic manifold and \( J \) be a compatible almost complex structure. Suppose that \( C = \text{Im} u_0 \) is an isolated embedded \( J \)-holomorphic curve \( u_0 : \Sigma_0 \to M \) with \( c_1(M)([C]) = 0 \) which is Fredholm-regular with respect to \( J \). Then for any given neighborhood \( \mathcal{V}(J) \) of \( J \) in \( \mathcal{J}_\omega \), there exists \( J' \in \mathcal{V}(J) \) satisfying \( J' TC = TC \) with respect to which \( C \) becomes super-rigid.

The proof of this local perturbation result relies on deformations of the normal bundle of \( C \) and its complex structures via the deformation of ambient almost
complex structure $J$. In particular, a precise formula for the second derivative of the normal $\bar{\partial}$ operator along $C$ and its unexpected relationship to the curvature property of the normal bundle. (See Corollary 8.4.) We also obtain an explicit rigidity criterion in terms of the curvature property of the normal bundle $\phi^*N_C$. It roughly states that the curvature, or more precisely its $(0,1)$-component, of the normal bundle is ‘fat’ at an infinite number of points in $\Sigma$, then $(j, \phi; J, C)$ is rigid. This seems to have some independent interest in relation to the stability criterion. (See Theorem 13.4.)

From now on, we assume $n = 3$ and $c_1 = 0$ unless otherwise said explicitly.

If $J \in J^\text{emb}_\omega$, it is an immediate consequence of Theorem 1.1 that the unparameterized curve $C$ associated to any embedded $J$-holomorphic curve $u$ with any genus $g$ is $(1,0)$-rigid. This is because $\mathcal{M}^\text{emb}(M, J; \beta)$ has virtual dimension zero for all $g$ and so $\dim \text{Ker} \, D\bar{\partial}_J(u) = \dim \text{Coker} \, D\bar{\partial}_J(u)$ and hence Fredholm regularity of $u$ also implies the rigidity.

The following rigidity theorem is one of the key ingredients of our finiteness theorem of embedded counting. We recall that a Baire set in a complete metric space is any subset of second category, i.e., a countable intersection of open dense subsets.

**Theorem 1.6 (Super-rigidity).** There exists a subset $J^\text{rigid}_\omega \subset J^\text{emb}_\omega \subset J_\omega$ which is Baire in $J_\omega$ and such that for $J \in J^\text{rigid}_\omega$, all somewhere injective (and so embedded) $J$-holomorphic curves are super-rigid.

This kind of (perturbation) rigidity would be a delicate question to study in the algebraic category. For example, Bryan and Pandharipande constructed an embedded holomorphic curve of genus 3 that is not $(h,4)$-rigid for a suitable $h$ and hence not super-rigid [BP2]. But super-rigidity has been expected to hold in the more flexible category of almost complex manifolds (see [BP1] for such an expectation.)

Combining these theorems with a re-scaling argument similar to the one used by Ionel and Parker [IP1], we also prove the following finiteness result

**Theorem 1.7 (Finiteness).** For any $J \in J^\text{rigid}_\omega$ we have only a finite number of somewhere injective (and so embedded) $J$-holomorphic curves of genus $g$ in class $\beta$ for any given $\beta$ and $g$ (modulo reparametrization for the case of $g = 0, 1$).

As we mentioned above, the $(1,0)$-rigidity (for all genus $g$) is a consequence of the Fredholm regularity of the corresponding embedded curves because of the condition $c_1 = 0$ and $n = 3$. However for the multiple covers, Fredholm regularity and rigidity are quite independent of each other and the standard proof in [F], [M] for the generic transversality is irrelevant to the rigidity question for multiple covers. Recall that Fredholm regularity for the negative multiple cover cannot be achieved in general by a perturbation of $J$ alone, which is precisely reason why the Kuranishi structure and virtual fundamental technique has been developed.

The proof of Theorem 1.6 uses the idea of studying 1-jet transversality of $J$-holomorphic curves which appears in [IP1]. However our setting is very different from the type used in [IP1], because the above rigidity question concerns property of multiple covered genuine $J$-holomorphic maps, not that of solutions of inhomogeneous $J$-holomorphic equations as used in [IP1], [RT]. Our proof uses some carefully formulated off-shell Fredholm set-up, involves a key geometric calculations leading to a precise formula on the second derivatives of $\bar{\partial}_\nu$. 
And then transversality analysis of this second derivative involves another 2-jet transversality result along $\delta J$ for the above mentioned 1-jet transversality along $\delta \phi$ in which we use an analysis of the curvature of the Hermitian connection of the normal bundle $N_C$, (see Proposition 7.2 and Proposition 9.1).

The operator $\Psi(j, \phi; J, C)$ has the form $\overline{\partial} + a$ where $\overline{\partial} = \overline{\partial}_{\Phi(J, \phi; J, C)}$ is the Dolbeault operator of the holomorphic vector bundle $(\phi^* N_C, \phi^* i)$ with $i = i_{(J, C)}$ the complex structure on $N_C$ induced by $J$ and $a$ is a real zero-order operator contained in $\Omega^{(0,1)}(\phi^* \text{End}_g(N_C))$. Considering the spectral flow along a path from $L(J, C)$ to $\overline{\partial}$, we can assign a sign $\varepsilon(J, C) \in \{+1, -1\}$ to each such embedded curve and so we define

$$\tilde{n}_\beta^g(J) = \sum_{C \in \mathcal{M}^{\text{emb}}(M, J, \beta)} \varepsilon(J, C) \quad \text{(1.1)}$$

where $\mathcal{M}^{\text{emb}}(M, J; \beta)$ is the set of embedded $J$-holomorphic curves in class $\beta$ with genus $g$ for $J \in \mathcal{J}_\omega^{\text{rigid}}$.

**Question 1.8.** Are the integers $\tilde{n}_\beta^g$ invariant under the change of $J$? More precisely do we have

$$\tilde{n}_\beta^g(J_1) = \tilde{n}_\beta^g(J_2)$$

for $J_1, J_2 \in \mathcal{J}_\omega^{\text{rigid}}$ or is there any wall-crossing phenomenon?

We denote by $M_{g+h}(C; d[C])$ the set of pairs $((\Sigma, j), \phi)$ with $\phi : (\Sigma, j) \rightarrow (C, j_{(J, C)})$ being holomorphic and $\overline{M}_{g+h}(C; d[C])$ its stable map compactification. Theorem 1.9 implies $\ker \Psi(j, \phi; J, C) = 0$ for all $\phi \in M_{g+h}(C; d[C])$ and for all $g, h, d$ and hence the virtual bundle

$$\text{Coker } \Psi(\cdot, J, C) \rightarrow \overline{M}_{g+h}(C; d[C])$$

is the obstruction bundle of a Kuranishi neighborhood $\overline{M}_{g+h}(C; d[C])$ in $\mathcal{M}_{g+h}(M, J; \beta)$ [FO] at the multiple covering $\phi : \Sigma \rightarrow C \subset M$.

We denote the Euler class of the bundle by $E(\Psi(j, \phi; J, C))$ and its evaluation over the virtual fundamental cycle $[\overline{M}_{g+h}(C; d[C])]^{\text{virt}}$ by the rational number

$$e_g(h, d; (J, C)) := \int_{[\overline{M}_{g+h}(C; d[C])]^{\text{virt}}} E(\Psi(\cdot, J, C)). \quad \text{(1.2)}$$

Then we have the following structure theorem of Gromov-Witten invariant $N^g_\beta(M)$ defined by Ruan and Tian [RT]

**Theorem 1.9.** Let $J \in \mathcal{J}_\omega^{\text{rigid}}$. For any $\beta \in H_2(M, \mathbb{Z})$ and $g \in \mathbb{Z}$, we have

$$N^g_\beta(M) = \sum_{h=0}^g \sum_{\beta = d\gamma, d \geq 1} \left( \sum_{C \in \mathcal{M}^{\text{emb}}_{g-h}(J, M; \gamma)} e_{g-h}(h, d; (J, C)) \right). \quad \text{(1.3)}$$

In particular, Theorem 1.9 [RT] and 1.9 specialized to the genus zero case imply Conjecture 7.4.5 [CK].
The above definition (1.2) of \( e_g(h, d; (J, C)) \) is the real analog to the local Gromov-Witten invariants defined by Pandharipande [P] and Bryan-Pandharipande [BP1]

\[
C_g(h, d; C \subset M) = \int_{[\mathcal{M}_{g+h+k}(C; d[C])]^{vir}} c(R^1\pi_*\phi^*(N_C)).
\]

**Question 1.10.** Suppose \( J \in J^{rigid}_\omega \). Is the number \( e_g(h, d; (J, C)) \) independent of \( (J, C) \) but does it depend only on \( g, h \) and \( d \)?

We remark that if the answer to this question is affirmative, (1.3) would imply the identity

\[
N^g_\beta(M) = \sum_{h=0}^{g} \sum_{d'=d, \beta} n^g_{\beta} \cdot e_g - h(h, d; (J, C))
\]

which is close to the form suggested in [GV]. In this regard, it is expected that the integer invariant \( n^g_{\beta} \) is closely related to the BPS count introduced therein.

We refer readers to section 14 for further discussion on the structure of Gromov-Witten invariants.

Many of the argument used in [OZ] and in this paper immediately generalize to the embedded count for the case of Fano with constraints at marked points. We will come back to this study and others in a sequel to the present paper.

Now organization of the contents of the paper is in order. In section 2, we give a review of normal \( \bar{\partial} \)-operator introduced by Gromov. We partially follow and refine the exposition presented in [HLS]. In section 3, we study multi-sections of the normal bundle of embedded \( J \)-holomorphic curves and introduce a decomposition of \( \bar{\partial} J \) into the horizontal and vertical parts. In section 4, we provide the functional analytic setting of study of the linearization of normal \( \bar{\partial} \)-operator and derive a precise formula of the linearization operator denoted by \( \Psi(j, \phi; J, C) \). In section 5 and 6, we prove a theorem which states that whenever a degeneration of embedded curves either into a multiple curve or a singular stable map, there exists an embedded curve \( C \) among the irreducible components of the limit map whose normal linearization has non-trivial kernel. In section 7 and 8, we derive a precise formula of the derivative \( (D_\phi \Psi)(j, \phi; J, C) \). This formula is one of the essential ingredients in our proof for the 1-jet transversality along \( \delta \phi \) which in turn depends on partial 2-jet transversality along \( \delta J \). In section 9, we study this 2-jet transversality by studying curvature variation of the normal bundle under the deformation of almost complex structure \( J \) satisfying \( JTC = TC \) (with another additional condition \( \delta_J \gamma_J = 0 \)). In section 10 - 13, we wrap-up our proof of the super-rigidity and finiteness of embedded \( J \)-holomorphic curves. We first prove that the rigidity and the finiteness hold at each finite stage of the area and the genus of the embedded curves and the degrees of their multiple covers for an open and dense subset \( J^{rigid}_\omega \subset J_\omega \) and then take their intersection to define the required \( J^{rigid}_\omega \). In section 14, we give the precise definitions of counting of embedded curves and their multiple cover contributions and prove the main structure theorem Theorem 1.9 of Gromov-Witten invariant.

Finally we would like to comment on the improvements achieved in this version when compared to the original version that had appeared in the arXiv [O2] and subsequently withdrawn. The main ideas used in the version [O2], considering the normal linearized operator combined with an explicit calculation of its linearization.

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and studying the curvature of the associated normal bundle to the embedded $J$-holomorphic curves still turns out to be a correct scheme of the proof. In hindsight, although the present author was not successful in his attempt at the time of writing $[O2]$, the main thrust of the ideas underlying in $[O2]$ was attempting to prove the kind of transversality statement presented here in Proposition 13.2.

Besides the precise transversality statement mentioned above, there are two major improvements in this paper: one is our derivation of the correct explicit formula for the linearization $(D_{\phi} \Psi)(j, \phi; J, C)$ (Proposition 9.1) of the normal linearized operator. In $[O2]$, there are many inaccuracies in our computations largely due to our confusing notations. In the present version, we made our notations straight and carried out this calculations in a much more transparent and precise way. The second improvement is the replacement of our complicated induction argument used in $[O2]$ by the step-by-step induction according to the order of $(G, K; L)$ given in section 10.1. This replacement is partially motivated by a proof of generic nondegeneracy of multiple Reeb orbits in contact geometry presented by Albers, Bramham and Wendl in Appendix of $[ABW]$. The structure of their step-by-step induction is very similar to the one presented in this paper although they apply the induction scheme in a much simpler context of ODE. More specifically, inclusion of the condition (4) in Definition 10.2 is motivated by a similar condition imposed in their proof.

Finally we would like to point out that the analogs to the results from $[OZ]$ and the present paper are expected to hold in the open string context with (Calabi-Yau) Lagrangian $D$-branes. One nontrivial issue to be resolved before heading such results is to understand the structure of non-simple (pseudo)-holomorphic maps for the higher genus case. The relevant structure theorem for the genus 0 case was obtained in $[KwO]$. Even for the genus 0 case, the structure theorem of non-simple bordered holomorphic curves with Lagrangian (or more generally totally real) boundary condition is much more complicated than that of closed holomorphic curves. One also needs to understand how such a structure theorem can be utilized to prove the relevant rigidity and finiteness results for the open string case. We hope to investigate this in a near future.

We work mostly with $C^\infty$ function spaces. As usual one should complete various spaces of smooth sections appearing in this paper by a suitable $W^{k,p}$-norms. Since we work directly with $C^\infty$ function spaces, it would be most straightforward to use the norm Floer used in $[Fl]$ for the space of almost complex structures. Because we mostly deal with embedded curves, one could also work with $C^k$-norms first and then derive the generic transversality result for the $C^\infty$ case by carrying out an induction over $(G, K) \in \mathbb{N}^2$ that is similar to the kind of argument used in the present paper. See also Appendix of $[ABW]$ for the relevant argument used in the proof for the transversality result for the closed Reeb orbits for a generic choice of $C^\infty$ contact forms. Since this process is more or less straightforward, we will not mention them from now on unless it is absolutely necessary. We refer readers to $[Fl], [M], [OZ], [O3]$ for some detailed discussions on the related issues.

We thank Rahul Pandharipande for his interest on this work and some useful enlightening discussion we had 2 years ago on super-rigidity in the algebraic context and on the counter examples in the reference $[BP2]$ in the beginning stage of the current work. We also thank Eleny Ionel, Tom Parker and Junho Lee for their meticulous reading of and pointing out some errors in the previous arXiv version.
of the present paper. We also thank Chris Wendl for a useful e-mail correspondence in which he explained to the author their proof of generic nondegeneracy of multiple Reeb orbits in [ABW].

2. Review of normal $\overline{\partial}$-operator $\overline{\partial}_\nu$

Let $(M, \omega, J)$ be a given almost Kähler manifold.

When a $J$-holomorphic curve is embedded, denote by $C \subset M$ its image. Gromov [G] considers the normal part of $\partial$-operator of such a curve, denoted by $\partial_\nu$, explains how the space of un-parameterized holomorphic curves near $C$ can be identified locally with the zero set of $\overline{\partial}_\nu$. This construction was further expounded in [HLS] which we will follow with some changes and clarifications in our exposition of this section.

Let $C$ be an embedded un-parameterized $J$-holomorphic curve. We consider the normal bundle $N_C = T_C M/TC$, $\pi_C : N_C \to C$.

This normal bundle carries the structure of holomorphic vector bundle: each fiber $N_x$, $x \in C$ is isomorphic to $T_x M/T_x C$ and $T_x C \subset T_x M$ is a complex subspace. Obviously this complex multiplication on $N_x$ changes smoothly as $x$ varies over $C$. Then since $C$ is a two dimensional surface, this complex structure $N \to C$ is integrable which makes $N$ to be a holomorphic vector bundle of rank $n - 1$. We denote by $i = i_{(J,C)}$ the induced complex structure on $N$ and by $\overline{\partial}$ the associated Dolbeault operator acting on the space $\Omega^0(N_C) = \Gamma(N_C)$ of sections of $N_C$.

By the tubular neighborhood theorem, we can identify a neighborhood $U \subset M$ of $C$ with that of the zero section in $N$ (as a differentiable manifold). Using the almost Kähler metric $g = \omega(\cdot, J \cdot)$, we fix an identification once and for all denote it by $\exp : V \subset N_C \to U \subset M$

and the projection $N \to C$ by $\pi$. By re-scaling the metric, we may assume

$$D_1(N_C) \subset V$$

where $D_1(N_C)$ is the unit disc bundle of $N_C$. We will often identify $V$ and $U$ and regard $U$ as a subset of $N_C$ equipped with the Hermitian metric of $i_{(J,C)}$.

Every real surface which is $C^1$-close to $C$ can be seen as the graph of a $C^1$-small section $\varphi \in \Omega^0(N_C)$. From now on, restricting ourselves to the tubular neighborhood, we identify $M$ with $N$.

We fix a Hermitian metric, denoted by $h$, induced from $g$ and its associated canonical (or Chern) connection $\nabla$ on the (holomorphic) vector bundle $\pi : N \to C$. We recall that we have $\overline{\partial} = \nabla^{(0,1)}$ for such a connection. We consider the induced splitting

$$T_n N = HT_n N \oplus VT_n N \cong T_{\varphi(n)} C \oplus N_{\varphi(n)}.$$ (2.2)

We can define the associated complex structure on the total space $N$ as a manifold, which by definition can be written as

$$J_0(n) = \begin{pmatrix} j(\pi(n)) & 0 \\ 0 & i \end{pmatrix}$$

with respect this splitting. We note

$$J = J_0 \quad \text{on } TM|_C \cong TN|_C.$$ (2.3)
And on a small neighborhood $U$ of $C$, we can write
\[ J(n) = \Phi(n)^{-1} J_0(n) \Phi(n) \]
where $\Phi(n)$ is an automorphism of $T_nN$. With respect to the above decomposition, we can write
\[ \Phi(n) = \begin{pmatrix} \alpha(n) & \beta(n) \\ \gamma(n) & \delta(n) \end{pmatrix} \] (2.4)
where
\[ \alpha(n) \in \text{End}_\mathbb{R}(T_{\pi(n)}C), \quad \beta(n) \in \text{Hom}_\mathbb{R}(N_{\pi(n)}, T_{\pi(n)}C) \]
\[ \gamma(n) \in \text{Hom}_\mathbb{R}(T_{\pi(n)}C, N_{\pi(n)}), \quad \delta(n) \in \text{End}_\mathbb{R}(N_{\pi(n)}). \]

We may also choose the map $n \mapsto \Phi(n)$ to be smooth, and satisfy
\[ \Phi|_C \equiv \text{id} \] (2.5)
by (2.3).

In particular when a section $\varphi \in \Gamma(N_C) = \Omega^0(N_C)$ is given $\alpha(\varphi)$, $\beta(\varphi)$, $\gamma(\varphi)$ and $\delta(\varphi)$ define sections of the corresponding vector bundles: for example, $\gamma(\varphi)$ defines an element $\Gamma(\text{Hom}(TC, N_C)) = \Omega^1(N_C)$ by
\[ \gamma(\varphi)(z) := \gamma(\varphi(z)) \]
for $z \in C$. If we denote by $C = C_\varphi$ the graph of $\varphi$, then we have $TC = \text{graph}(\nabla \varphi)$ and
\[ \Phi(TC) = \{ (\alpha(\varphi) + \beta(\varphi)\nabla \varphi) \cdot v, (\gamma(\varphi) + \delta(\varphi)\nabla \varphi) \cdot v ) \mid v \in TC \}. \]

The following lemma is proved in [HLS].

**Lemma 2.1.** The followings are equivalent:

1. $C$ is a $J$-holomorphic curve.
2. $\Phi(TC)$ is $J_0$-invariant.
3. $P(\varphi)$ is $i$-linear.

We note that on the zero section $o_{N_C} \cong C$ of $N$ we have $\alpha = \text{id}$ and $\beta = 0$. Therefore if $\varphi \in \Omega^0(N_C)$ is $C^1$-small, the matrix
\[ \alpha(\varphi)(z) + \beta(\varphi)(z)\nabla \varphi(z) : T_zC \to T_zC \]
is invertible and hence $\Phi(TC) = \text{graph}(P(\varphi))$ where $P(\varphi) : TC \to N$ is defined by
\[ P(\varphi) = (\gamma(\varphi) + \delta(\varphi)\nabla \varphi)(\alpha(\varphi) + \beta(\varphi)\nabla \varphi)^{-1}. \] (2.6)

Now we give the definition of $\overline{\partial}_\nu$. It goes from an open subset $U \subset \Omega^0(N_C)$ containing the zero section to $\Omega^{(0,1)}(C; N)$, and is given by
\[ \overline{\partial}_\nu(\varphi) = (P(\varphi) + i \circ P(\varphi) \circ j)/2; \quad U \to \Omega^{(0,1)}(N_C). \] (2.7)

Then holomorphic curves which are $C^1$-close to $C$ are in one-to-one correspondence with $\overline{\partial}_\nu^{-1}(0)$ [G], [HLS].
3. Multi-sections of normal bundle

In this section, we will modify construction of the normal $\overline{\partial}$-operator in the previous section for the multi-sections of the normal bundle in relation to the ramified holomorphic covering of the embedded curve $C \subset M$

$$\phi : \Sigma \to C \subset M$$

where $\Sigma$ is a compact Riemann surface, not necessarily connected and $\phi$ is not constant on any connected component. By the connectedness of $C$, $\phi$ on each connected component will be a ramified covering of $C$. Once this said, we will assume that $\Sigma$ is also connected.

We consider the set of smooth maps $\phi : \Sigma \to C$ with degree $d \geq 0$ and consider the pull-back $$(\phi^* N, \phi^* i_{(J, C)})$$

over the smooth map $\phi : \Sigma \to C$. We denote the canonical map $\tilde{\phi} : E = \phi^* N \to N$ defined by $\tilde{\phi}(\sigma, n) = (\phi(\sigma), n)$ which defines a smooth map. Considering the pull-back Hermitian metric on $E = \phi^* N$ from $N$ via $\phi : \Sigma \to C$ and the associated Hermitian connection, we obtain the induced splitting

$$T_e E = HT_e E \oplus VT_e E \cong T_{\sigma} \Sigma \oplus N_n$$

where $e = (\sigma, n)$ with $\phi(\sigma) = \pi(n)$.

We denote by $\nabla_{(J, C)}^{(0,1)}$ the $(0, 1)$-part of the connection $\nabla$. We note that if $\phi$ is holomorphic, $\nabla_{(J, C)}^{(0,1)} = \overline{\partial}$ where $\overline{\partial}$ is the standard Dolbeault operator.

**Definition 3.1.** Let $\phi : \Sigma \to C$ be a smooth map and consider the pull-back $\phi^* N_C$.

We define a $\phi$-section of $\pi : N \to C$ by a map $s : \Sigma \to N$ such that

$$\pi \circ s = \phi$$

i.e., a section of the pull-back bundle $\phi^* N$. If $\phi$ is holomorphic and $s$ is holomorphic, then we say that $s$ is a holomorphic $\phi$-section of $\pi : N \to C$ along $\phi$, or just a holomorphic multi-section.

Since it will be important to consider the graph of a multi-section $s$ as a subset of $N$, not that of $\phi^* N$, we sometimes denote the set of multi-sections by

$$\Omega_\phi^0(C, N) := \Omega^0(\phi^* N).$$

Similarly we denote

$$\Omega_\phi^{0,1}(C, N) := \Omega^{(0,1)}(\phi^* N).$$

For a multi-section $s : \Sigma \to N$, we define the covariant derivative of the multi-section, denoted by $\nabla s$, by

$$\nabla s(\sigma) = \Pi_{n(\sigma)}^n \circ ds(\sigma)$$

where $\Pi_{n}^n : T_n N \to VT_n N \cong N_n$ is the vertical projection.

Replacing the section and $\nabla s$ in the previous section by the multi-section and $\nabla s$ respectively, we can construct the maps $\alpha(s)$, $\beta(s)$, $\gamma(s)$ and $\delta(s)$ as before: for example, $\gamma(s)$ defines an element

$$\Gamma(\text{Hom}_\mathbb{R}(T\Sigma, \phi^* N)) = \Omega^1(\phi^* N).$$
If we denote by \( S \subset N \) the graph of a multi-section \( s \), then we have \( TS = \text{graph} (\nabla s) \) and
\[
\Phi(TS) = \{(\alpha(s) + \beta(s)\nabla s) \cdot w, (\gamma(s) + \delta(s)\nabla s) \cdot w) \mid w \in T\Sigma\}.
\]
We note that on the ‘zero section’, \( \phi : \Sigma \to C \) we have \( \alpha \equiv \text{id} \) and \( \beta \equiv 0 \). Therefore if \( s \in \Omega^0_\phi(\Sigma; N) \) is \( C^1 \)-small, the matrix
\[
\alpha(s) + \beta(s)\nabla s
\]
is invertible and hence \( \Phi(TS) = \text{graph} (P(s)) \) where \( P(s) : T\Sigma \to N \) is defined by
\[
P(s) = (\gamma(s) + \delta(s)\nabla s)(\alpha(s) + \beta(s)\nabla s)^{-1}.
\]
(3.4)

Now we give the definition of the normal operator for the multi-sections which we again denote by \( \overline{\delta}_\nu \). We first recall that \( \Phi \) depends on \( J \) and so do \( \alpha(s) \), \( \beta(s) \), \( \gamma(s) \) and \( \delta(s) \). Hence \( \overline{\delta}_\nu \) is given by
\[
\overline{\delta}_\nu(s) = \overline{\delta}_{\nu; (j, C)}(s) = (P(J, s) + i \circ P(J, s) \circ j)/2; \quad J \times U \to \Omega^0(\phi^* N) \quad (3.5)
\]
where \( i = i_{(j, C)} \) is the complex structure on \( N \) induced from \( J \), and \( U \) is the open neighborhood given by
\[
U = \{s \in \Omega^0(\phi^* N) \mid \text{graph} s \subset U \subset N_C\}.
\]
We consider the pairs \( (J, C) \) where \( C \subset M \) is an embedded \( J \)-holomorphic curve. We denote by \( \mathcal{M}^{hol}(M, \omega; \beta) \) the set of such pairs \( (J, C) \) with genus \( g \) in class \( [C] = \gamma \). Since this set coincides with the set of un-parameterized somewhere injective \( J \)-holomorphic curves, it carries a \( C^k \) Banach manifold structure for any given positive integer \( k \geq 1 \). It is a fiber bundle over \( J_\omega \).

Now we decompose the standard \( \overline{\delta}_J \) into the horizontal and vertical parts thereof for the set of maps \( u : \Sigma \to M \) whose image is \( C^0 \)-close to \( C \) or to the zero section in \( N_C \). Any such map can be decomposed into
\[
u = (\pi \circ u, s_u), \quad s_u \in (\pi \circ u)^* N
\]
and vice versa. Denote
\[
\mathcal{F}_{g+h}(C; d[\gamma]) = \{(\phi, s) | \phi \in C^\infty(\Sigma, C), s \in \Omega^0(\Sigma, \phi^* N), \alpha(s) + \beta(s)\nabla s \text{ is invertible}\}
\]
\[
\mathcal{F}_{g+h}(g, d; \beta) = \bigcup_{\gamma; \beta = d\gamma} \mathcal{F}_{g+h}(C; d[\gamma])
\]
and define a map
\[
\overline{\delta} : J(\Sigma) \times \mathcal{F}_{g+h}(g, d; \beta) \to \mathcal{H}''
\]
by \( \overline{\delta}(j, u; J, C) = \overline{\delta}_{C}(j, u, J) \) with
\[
\overline{\delta}_{C}(j, u, J) = (\overline{\delta}_{h, C}(\pi \circ u), \overline{\delta}_{v, C}s_u) \quad (3.6)
\]
where \( \mathcal{H}'' := \mathcal{H}_h'' \times \mathcal{H}_v'' \) and
\[
\mathcal{H}_h'' = \bigcup_{C \in \mathcal{M}^{hol}(M, J; \beta)} \mathcal{H}_{h, C}
\]
\[
\mathcal{H}_v'' = \bigcup_{C \in \mathcal{M}^{hol}(M, J; \beta)} \mathcal{H}_{v, C}
\]
and
\[ \mathcal{H}^\nu_{h,C} = \bigcup_{(j,\phi,J)} \mathcal{H}^\nu_{h,C;(j,\phi,J)}; \quad \mathcal{H}^\nu_{v,C} = \bigcup_{(j,\phi,J)} \mathcal{H}^\nu_{v,C;(j,\phi,J)}; \]
\[ \mathcal{H}^\nu_{v,C} = \bigcup_{(j,\phi,J)} \mathcal{H}^\nu_{v,C;(j,\phi,J)}; \quad \mathcal{H}^\nu_{v,C;\phi,J} = \Omega^{(0,1)}_{\phi}\nu(C^*N_C) \]

and \( \overline{\partial}_{h,C} : C^\infty(\Sigma, C) \to \mathcal{H}^\nu_{h,C} \) is the standard Gromov’s \( \overline{\partial}_{(j,\phi,J)} \) and \( \overline{\nu}_{v,C} = \overline{\nu} \) the normal \( \overline{\partial} \)-operator defined before.

Then by construction the following is obvious.

**Proposition 3.2.** Suppose \( u = (\phi, s) : \Sigma \to N \) with its image contained in the tubular neighborhood of \( U \subset N \) and \( (\alpha(s) + \beta(s)\nabla s) \) is invertible. Then \( u \) is \((j,J)\)-holomorphic if and only if \( \overline{\partial}C(j,u) = 0 \).

4. Off-shell setting of linearization of \( \overline{\partial}_v \)

We now consider the set of maps \( u = (\phi, 0) \) with \( \phi : \Sigma \to C \subset M \) where \( C \) is an unparameterized embedded \( J \)-holomorphic curve and consider the normal linearization of the map i.e., the linearization of \( \overline{\partial}_v \) at each given \((j,\phi; J,C)\).

Each \((j,\phi)\) with \( j \in \mathcal{J}(\Sigma) \) and \( \phi \in C^\infty(\Sigma, C) \) having degree \( d \) give rise to the linearization operator of \( \overline{\partial}_v \) at \((j,\phi; J,C)\) which we denote by

\[ D_v \overline{\partial}_v(j,\phi; J,C) : \Omega^0(\phi^*N) \to \Omega^{(0,1)}(\phi^*N). \]

We now interpret the map

\[ (j,\phi; J,C) \mapsto D_v \overline{\partial}_v(j,\phi; J,C) \]

as a section of some bundle over a certain space which we now describe precisely.

Denote by \( \mathcal{J}(\Sigma) \) the set of almost complex structures on \( \Sigma \), i.e., the set of \( j \in \operatorname{End}_\Sigma(T\Sigma) \) satisfying \( j^2 = -\operatorname{id} \). We then denote by \( \mathcal{J}(\Sigma) \times \mathcal{F}_{g,h}(g,d;\beta) \) the set of quadruples \((j,\phi; J,C)\) such that

1. \( C \) is an embedded \( J \)-holomorphic curve of genus \( g \), \( d|C| = \beta \) in \( H_2(M) \).
2. \( \Sigma \) is a compact surface with genus \( g_{\Sigma} = g + h \) with \( h \geq 0 \).
3. \( \phi : \Sigma \to C \) is a smooth map from \( \Sigma \) to \( C \) with degree \( d \geq 1 \).

It is an infinite dimensional smooth manifold fibered over \( \mathcal{M}_{g}^{\text{emb}}(M; J; \beta) \). We denote

\[ \mathcal{H} \text{ol}_{g+h}(g,d;\beta) = \{(j,\phi; J,C) \mid C \text{ embedded } J\text{-holomorphic}, \quad j_{(j,C)} \circ d\phi = d\phi \circ j \} \]

and

\[ \mathcal{M}_{g+h}(g,d;\beta) = \mathcal{H} \text{ol}_{g+h}(g,d;\beta)/\sim \]

where \( \sim \) is the gauge equivalence i.e.,

\[ (j,\phi) \sim (\psi^*j, \phi \circ \psi) \]

for \( \psi \in \text{Diff}(\Sigma) \). Here \( \mathcal{M}_{g+h}(g,d;\beta) \) is nothing but the standard moduli space of \( J \)-holomorphic curves \( \phi : \Sigma \to M \) in \( M \) which is a covering of an embedded \( J \)-holomorphic curves \( C \) in genus \( g \) with \( \beta = d|C| \).

We will consider another action of \( \text{Diff}(\Sigma) \) on the set \( \mathcal{J}(\Sigma) \times \mathcal{F}_{g+h}(g,d;\beta) \) given by

\[ (j,\phi) \mapsto (j,\phi \circ \psi), \quad \psi \in \text{Diff}(\Sigma). \]
This preserves the given embedded curve $C$ but changes the branched holomorphic covering $\phi : (\Sigma, j) \to (C, i)$ to a non-holomorphic map $(j, \phi \circ \psi)$. We would like to emphasize that the action is different from the gauge action of $Diff(\Sigma)$

$$\psi : (j, \phi) \mapsto (\psi^* j, \phi \circ \psi).$$

For example, the action $\Psi$ does not preserve the set $Hol_{g+h}(g, d; \beta)$, while the gauge action does. Not to confuse readers on these two actions of $Diff(\Sigma)$, we denote the universal moduli space $M_{g+h}(g, d; \beta)$ as $\mathcal{M}_{g+h}(g, d; \beta)$ instead of $Hol_{g+h}(g, d; \beta)/Diff(\Sigma)$.

We denote the set of Fredholm operators

$$D : \Omega^0(\phi^* N) \to \Omega^{(0,1)}(\phi^* N)$$

with $\dim \ker L = k$ by $Fred_k(j, \phi; J, C)$ and their union by $Fred_k$ i.e.,

$$Fred_k = \bigcup_{(j, \phi; J, C) \in \mathcal{J}(\Sigma) \times \mathcal{F}_{g+h}(g, d; \beta)} Fred_k(j, \phi; J, C).$$

Finally we denote

$$Fred = \bigcup_{k \in \mathbb{Z}_+} Fred_k.$$

The following is by now well-known (see [Kos] for example).

**Lemma 4.1.** $Fred_k$ is a submanifold of Fred with real codimension $k(k - i)$ and the fiber of the normal bundle of $Fred_k$ in Fred at an operator $D \in Fred_k$ is $Hom_{\mathbb{R}}(\ker D, \text{Coker } D)$.

Each $Fred_k$ defines a fibration over $\mathcal{F}_{g+h}(g, d; \beta)$. Then we obtain a smooth map

$$\Psi : \mathcal{J}(\Sigma) \times \mathcal{F}_{g+h}(g, d; \beta) \to Fred$$

defined by the linearization map

$$\Psi(j, \phi; J, C) := D, J_{\psi}(j, \phi; J, C)$$

(4.5) defines a smooth section of the bundle

$$Fred \to \mathcal{J}(\Sigma) \times \mathcal{F}_{g+h}(g, d; \beta) = \bigcup_{(J, C; \beta = d[C])} \mathcal{J}(\Sigma) \times C^\infty(\Sigma, C; d).$$

In fact, this is nothing but the projection of the standard linearization to the normal bundle $N_C$ with respect to the splitting $TM|_C = TC \oplus N$.

The space $Hol_{g+h}(g, d; \beta)$ is the union of $Hol_{g+h}(C; d[C])$ over $(J, C)$ with $C$ being $J$-holomorphic embedded curve where $Hol_{g+h}(C; d[C])$ is the set of $(j, \phi)$ with $\phi : (\Sigma, j) \to (C, i_{J, C})$ being holomorphic. And $M_{g+h}(g, d; \beta)$ is the union of

$$M_{g+h}(C; d[C]) = Hol_{g+h}(C; d[C])/\sim.$$  

$M_{g+h}(C; d[C])$ is a complex manifold (or an orbifold) of complex dimension given by

$$r = 2(g(\Sigma) - 1) - 2d(g(C) - 1) = 2(g + h - 1) - 2d(g - 1).$$

$r$ is also the same as the degree of branching divisor $br(\phi)$ for $\phi \in M_{g+h}(C; d[C])$ by the standard Riemann-Hurwitz formula (see [FP] for example). In particular we have $r \geq 0$.

We denote by

$$p : Hol_{g+h}(g, d; \beta) \to M_{g+h}(g, d; \beta)$$

(4.6)
the canonical projection. We note that $\mathcal{J}(\Sigma) \times \mathcal{F}_{g+h}(g, d; \beta)$ has the structure of smooth bundle over

$$\bigcup_{\gamma : d\gamma = \beta} \mathcal{M}^\text{emb}_g(M, \omega; \gamma)$$

whose fiber at $(J, C)$ given by $\mathcal{J}(\Sigma) \times C^\infty(\Sigma, C; d[C])$. This ends our discussion on the off-shell setting of the map $\mathcal{G}$.

**Remark 4.2.** We would like to note that although $\text{Hol}_{g+h}(g, d; \beta)$ or $\mathcal{M}_{g+h}(g, d; \beta)$ is a smooth manifold (or more precisely orbifold for the latter case), its smooth structure does not come from the transversality of the $\overline{\mathcal{G}}$ against $\phi_{\mathcal{H}_{\gamma}} \subset \mathcal{H}_{\gamma}^*$: In general the linearization of $\overline{\mathcal{G}}_h$ has non-trivial cokernel. However it has the structure of fibration

$$\mathcal{M}_{g+h}(g, d; \beta) \to \bigcup_{\gamma : d\beta = \gamma} \mathcal{M}^\text{emb}_g(M; \gamma)$$

for which the base is a smooth manifold and a fiber is isomorphic to $M_{g+h}(C; d[C])$ which has the smooth structure.

The following simple observation is one of the crucial ingredients used in our proof of the main rigidity result.

**Proposition 4.3.** The action $\Gamma(\Sigma)$ of $\text{Diff}(\Sigma)$ on $C^\infty(\Sigma, C)$ is transverse to $\text{Hol}_{g+h}(C; d[C]) \subset C^\infty(\Sigma, C)$.

**Proof.** Let $\phi \in \text{Hol}_{g+h}(C; d[C]) \subset C^\infty(\Sigma, C)$. The normal space

$$T_{\phi}C^\infty(\Sigma, C)/T_{\phi}\text{Hol}_{g+h}(C; d[C])$$

is nothing but $\Gamma(\phi^*TC)/H^0(\phi^*TC)$ where $H^0(\phi^*TC)$ is the space of holomorphic sections of the holomorphic line bundle $\phi^*TC$.

On the other hand the tangent space of the orbit $\text{Diff}(\Sigma) \cdot \{\phi\} \subset C^\infty(\Sigma, C)$ consists of the variations of the type $\phi \circ \psi_t$. We denote

$$W = \frac{d}{dt} \bigg|_{t=0} \psi_t.$$ We need prove

$$T_{\phi}(\text{Diff}(\Sigma) \cdot \{\phi\}) + H^0(\phi^*TC) = \Gamma(\phi^*TC).$$

To prove this, it will be enough to prove the following lemma

**Lemma 4.4.** Let $X \in \Gamma(\phi^*TC)$. If $\int_\Sigma \langle X, Y \rangle = 0$ for all $Y \in d\phi(\Gamma(T\Sigma)) + H^0(\phi^*TC)$, then $X = 0$. Here $\Gamma(T\Sigma)$ denotes the space of vector fields on $\Sigma$.

**Proof.** In fact, we will prove a slightly stronger statement with the phrase “for all $Y \in d\phi(\Gamma(T\Sigma)) + H^0(\phi^*TC)$” replaced by “for all $Y \in d\phi(\Gamma(T\Sigma))$”.

Suppose $X$ satisfies

$$\int_\Sigma \langle X, Y \rangle = 0$$

for all $Y \in d\phi(\Gamma(T\Sigma))$. Let $Y = d\phi(W)$ for $W \in \Gamma(T\Sigma)$. If $X$ is not a zero vector field, there exists $\sigma_0 \in \Sigma$ with $X(\sigma_0) \neq 0$. Since there are only finite number of branch points of $\phi$, we may assume $\phi(\sigma_0) \in C \setminus \text{br}(\phi)$: if $X(\sigma_0) = 0$ for all $\Sigma \setminus \phi^{-1}(\text{br}(\phi))$, the by continuity $X$ must be zero. We recall $X(\sigma_0) \in T_{\phi(\sigma_0)}C$. 
Since $\phi: \Sigma \to C$ is a covering map away from $br(\phi)$, we can find a local vector field $W_0$ defined on a sufficiently small neighborhood $U \subset \Sigma \setminus br(\phi)$ of $\sigma_0$ such that $d\phi(W_0(\sigma)) = X(\sigma)$.

Multiplying a suitable cut-off function $\chi: \Sigma \to [0, 1]$ supported in $U$ and $\chi \equiv 1$ on another neighborhood $V$ with $\sigma_0 \in V \subset \nabla \subset U$, we will have
$$\int_{\Sigma} \langle X, d\phi(\tilde{W}) \rangle > 0, \quad \tilde{W} = \chi W$$

a contradiction to (4.7). This finishes the proof. □

This in turn finishes the proof of Proposition 4.3. □

We recall the following standard index formula

**Proposition 4.5.** The index of $D_v \partial_\nu(j, \phi; J, C)$ is given by
$$\iota := \text{Index } D_v \partial_\nu(j, \phi; J, C) = 2(c_1(\phi^*N)[\Sigma] + (n - 1)\chi(\Sigma)) = 2(c_1(\phi^*N)[\Sigma] + (n - 1)(1 - g_{\Sigma})) \quad (4.8)$$
in general dimension. In particular, when $n = 3$, $c_1(M, \omega) = 0$ and $C$ is embedded, we have
$$\iota = 2(2d(g - 1) + 2(1 - g - h)) = -r \leq 0 \quad (4.9)$$

**Proof.** Riemann-Roch formula immediately implies (4.8). On the other hand, when $n = 3$, $c_1(M, \omega) = 0$ and $C$ is embedded, we have $c_1(\phi^*N) = c_1(\omega_{\Sigma})$ where $\omega_{\Sigma}$ is the canonical line bundle of $\Sigma$. Therefore (4.9) follows. □

Next we compute an explicit formula for $D_v \partial_\nu(j, \phi; J, C)$. Let $\gamma \in \text{Hom}_R(TC, N_C)$ be the map given in (2.4). We define the vertical partial derivative $D^v_\nu \gamma(J, \phi; s)$ at $s = 0$ by
$$D^v_\nu \gamma(J, \phi; s) = \frac{d}{dt} \big|_{t=0} \gamma(J, \exp_\phi(t\xi)) \quad (4.10)$$
where $\exp_\phi(t\xi)$ with $-\varepsilon < t < \varepsilon$ is a family of sections along $\phi$ i.e., in $\Omega^0(\phi^*N)$ such that $J_0 = J$ and $s_0 = \phi$ and $\partial s_t/\partial t|_{t=0} = \xi$. Similarly we define the derivative $D^v_\nu P(J, \phi, s)$ of $P(J, \phi, s)$.

The following result is a variation of a result from [HLS].

**Proposition 4.6.** Let $(j, \phi; J, C)$ be as above. Then we have
$$D_v \partial_\nu(j, \phi; J, C) = \phi^* \left( (\nabla + \mu)_{(0,1)}^{(0,1)}(N_C, i_{(0,1)}(J, C)) \right) \quad (4.11)$$
where $\nabla$ is the Hermitian connection of the Hermitian holomorphic vector bundle $(N_C, i_{(0,1)}(J, C), h_{(0,1)}(J, C)), \mu = D_v^c \gamma_{J}$ and $(\cdot)_{(0,1)}^{(0,1)}(J, C)$ is $(0, 1)$-part. In particular when $\phi$ is holomorphic, $L$ has the form of
$$L = \partial + \phi^* \mu_{(0,1)}^{(0,1)}(J, C)$$
where $\partial$ is the standard Dolbeault operator of the holomorphic vector bundle $(\phi^*N, \phi^*i)$. 

Proof. In the course of calculations, we denote $j_C = j_{(J,C)}$ and $i = i_{(J,C)}$.

For $\xi \in \Omega^0_\phi(C; N)$, let $s_t$ with $-\varepsilon < t < \varepsilon$ be a family of multi-sections in $\Omega^0_\phi(C; N)$ such that

$$s_0 \equiv 0, \quad \frac{d}{dt}\bigg|_{t=0} s_t(\sigma) = \xi(\sigma)$$

for $\sigma \in \Sigma$. Using (3.4), (3.5) and recalling that on the zero section we have $\alpha = id$, $\beta = 0 = \gamma$, we get

$$D_v \bar{\partial}_\nu(\xi) = \frac{1}{2} \left( D_v P(\phi)(\xi) + \phi^* i \circ D_v s P(\phi)(\xi) j_C d\phi \right)$$

$$D_v P(\phi)(\xi) = \nabla^\phi \xi + D_v^\phi \gamma(\phi)(\xi) d\phi.$$ 

Therefore we have

$$D_v \bar{\partial}_\nu(\xi) = \frac{1}{2} \left( \nabla^\phi \xi + \phi^* i \circ \nabla^\phi \xi j_C d\phi \right) + \frac{1}{2} \left( D_v^\phi \gamma(\phi)(\xi) + \phi^* i \circ D_v^\phi \gamma(\phi)(d\phi(\xi)) j_C d\phi \right).$$

From its expression, we have

$$\frac{1}{2} \left( D_v^\phi \gamma(\phi)(d\phi(\xi)) + \phi^* i \circ D_v^\phi \gamma(\phi)(d\phi(\xi)) j_C d\phi \right) = \phi^* \left( \mu_J \right)^{(0,1)}_{(J,C), i_{(J,C)}}$$

with the bundle map $\mu_J : N_C \to \Lambda^1(N_C)$ which is defined by

$$\mu_J = D_v^\phi(\gamma).$$

This finishes the proof of (4.11).

The second statement follows from the well-known fact that the $(0,1)$-part of the Hermitian connection on the Hermitian holomorphic vector bundle is the same as $\bar{\partial}$ since the pull-back connection via the holomorphic map $\phi$ is again a Hermitian connection on the holomorphic vector bundle $\phi^* N_C$.

Remark 4.7. We note that dependence of the operator $D_v \bar{\partial}_\nu(j, J, C; \phi, 0)$ on $J$ is only on its 1-jet of $J$ along $C$, i.e., on the restriction of $J$ on $TM|_C \cong TN|_C$.

5. Degeneration to multiple covering with smooth domain

The material of this section will be used to carry out the first step of induction scheme carried out in sections (10).

Let $J \in J^\text{nodal}_\omega \subset J^\text{emb}_\omega$. In this section, we will prove the following theorem. An analogous theorem was mentioned in [IP1], [E].

**Theorem 5.1.** Suppose that there exists a sequence of maps $u_i : (\Sigma_i, j_i) \to M$ with distinct images such that it converges to a smooth map $u_\infty : (\Sigma, j) \to M$ which factors through $u_\infty = u_0 \circ \phi$ where $u_0 : \Sigma' \to M$ is a $J$-holomorphic embedding and $\phi : \Sigma \to \Sigma'$ is a ramified covering of degree $\geq 1$. Denote $C = \text{Im} u_0$. Then the linearization $\Psi(j, \phi, J, C) = D_v \bar{\partial}_\nu$ has non-trivial kernel.

**Proof.** In this case, by composing with a diffeomorphism, we may assume that $\Sigma_i = \Sigma$. We use the argument similar to those used in the proofs of Theorem 7.5 (i) [O1] or of Lemma 5.1 [E] in the following proof.
Using the normal exponential map \( \exp : \mathcal{V} \subset N_C \to \mathcal{U} \subset M \), we can define a function \((j', u') \mapsto (\zeta', \xi')\) for a unique choice of \((\zeta', \xi') \in H^1(\Sigma, T\Sigma) \times \Omega^0(\Sigma; \phi^*N_C)\) so that

\[
(j', u') = (\exp_j \zeta', \Pi \exp_\phi(\xi'))
\]
as long as \((j', u')\) is \(C^1\)-close to \((j, u_\infty)\). Here \(\Pi : N_{\pi(u(\sigma))} \to N_{\phi(\sigma)}\) is the parallel transport along the curve

\[
t \mapsto \exp_{(j, \phi)}(t(\zeta', \xi'))
\]
with respect to the Hermitian connection of \(i\) of \(N\). We denote \(C' = \text{Im } u'\). Then we have \(\xi' \neq 0\) if \(C' \neq C\).

We define a map \(\mathcal{F} : \mathcal{J}(\Sigma) \times \mathcal{U}(u_\infty) \to H^1(\Sigma, T\Sigma) \times \Omega^{0,1}(\Sigma; \phi^*N_C)\) by

\[
\mathcal{F}(j', u') = (\zeta', \mathcal{G}_\nu(\xi'))
\]
where \(\mathcal{U}(u_\infty) \subset C^\infty(\Sigma, M)\) is a \(C^1\)-neighborhood of \(u_\infty\) and \(\mathcal{G}_\nu(\xi') = \mathcal{G}_{\nu(j', J, C)}(\xi')\) defined in section 2. We will apply this to \((j', u') = (j_i, u_i)\) for sufficiently large \(i\) later.

Denote \(L = \Psi(j, \phi; J, C)\). Considering a right inverse \(Q : \text{Im } L \to \text{Im } L^\dagger \subset \Omega^0(\Sigma; \phi^*N_C)\), we have the decomposition

\[
\Omega^0(\Sigma; \phi^*N_C) = \text{Ker } L \oplus \text{Im } L^\dagger
\]
where \(L^\dagger\) is the \(L^2\)-adjoint of \(L\). For the simplicity of notation, we denote \(\theta = (\zeta', \xi')\) and decompose

\[
\theta = \tilde{\theta} + \hat{\theta} \in \text{Ker } L \oplus \text{Im } L^\dagger.
\]

By definition of \(Q\), we have \(\hat{\theta} = Q(\gamma)\) for a unique \(\gamma \in \text{Im } L\).

Suppose now \(u' : (\Sigma_i, j_i') \to (M, J)\) is \((j', J)-\text{holomorphic}\) and so \(\mathcal{F}(j', u') = 0\). We denote the norm of \(||Q|| = C\). Then we derive

\[
||\hat{\theta}|| = ||Q\gamma|| \leq c_0 ||\gamma|| = c_0 ||L(Q(\gamma))|| = c_0 ||d\mathcal{F}(j, \phi)\hat{\theta}||
\]

\[
= ||\mathcal{F}(j', u') - \mathcal{F}(j, \phi) - d\mathcal{F}(j, \phi)(\hat{\theta})||
\]

\[
\leq CC_1(||\hat{\theta}\|_{\infty}) ||\hat{\theta}||
\]

by the standard quadratic estimate, where \(C_1(r)\) is a function such that \(C_1(r) \to 0\) as \(r \to 0\). Therefore if the \(C^1\)-distance of \((j', u')\) to \((j, \phi)\) so small so that

\[
CC_1(||\hat{\theta}\|_{\infty}) < 1
\]

we must have \(\hat{\theta} = Q(\gamma) = 0\) and hence \(\theta = \tilde{\theta} \in \text{Ker } L\).

Now suppose there exists a sequence \((j_i, u_i)\) with \(C_i \neq C\) whenever \(i \neq j\) but \((j_i, u_i) \to (j, u_\infty)\) in \(C^1\)-topology. Since \(C_i \neq C\), \(\xi_i \neq 0\). For the corresponding \(\theta_i = (\zeta_i, \xi_i)\). But the above conclusion implies that \(\xi_i\) must be contained in \(\text{Ker } L\) and so \(\text{Ker } L \neq \{0\}\). This finishes the proof.

6. Degeneration to Multiple Covering with Singular Domain

In this section, we study the case of degeneration to a stable map whose image locus is an embedded \(J\)-holomorphic curve \(C\) but whose domain becomes singular.

If a stable map \(u : \Sigma \to M\) with singular domain occurs as a limit of \(u_i : (\Sigma_i, j_i) \to (M, J)\) with \([u_i] = \beta\), then \(\beta = d\gamma\) for some \(d \geq 2\) where \([C] = \gamma\) by the structure theorem on the image of stable maps stated in Theorem 1.2.

We prove the following theorem by a re-scaling argument in a tubular neighborhood of \(C \subset N_C\) similarly as in [IP]. One difference of the current case therefrom
is that our case is a re-scaling along a 2-dimensional surface which is of higher codimension than 2 while [IP1] deals with the situation of codimension 2.

We first recall that the normal bundle $N_C$ has a canonical holomorphic bundle structure $i = i(J,C)$ induced by $(J,C)$.

**Theorem 6.1.** Let $J \in \mathcal{J}_\omega$ be any tame almost complex structure of $(M,\omega)$ and $C$ be a $J$-holomorphic embedding. Suppose that $u : \Sigma \to C \subset M$ is a stable map with singular domain that occurs as the stable limit of a sequence of embedded $J$-holomorphic curves as above. Then there exists a ramified holomorphic covering $\phi_1 : \Sigma_1 \to C$ such that

$$\text{Ker} \Psi(j_1,\phi_1; J,C) \neq \{0\}.$$  

**Proof.** We symplectically identify neighborhoods of $C \subset M$ and $o_C \subset N_C$ using the symplectic neighborhood theorem and so we may assume $M = N_C$ and $C = o_C$ in $N = N_C$ in this proof. More precisely we consider a symplectic blow-up $(\tilde{M}_\varepsilon,\tilde{\omega}_k)$ of $(M,\omega)$ along $C \subset M$. We identify an $\varepsilon$-neighborhood of $C$ in $M$ with a corresponding neighborhood of the zero section $o_C$ of $N_C \subset \tilde{M}_\varepsilon$.

We further consider the projectivization of $N$

$$N \hookrightarrow \mathbb{P}(O \oplus N)$$

with $N = \mathbb{P}(1 \oplus N)$. Denote by $R_t : N \to N$ the scalar multiplication $R_t(n) = n/t$ for $t \in \mathbb{R}^*$ which also induces scaling map on $\mathbb{P}(O \oplus N)$ and we also denote by $R_t$. Then $R_t \circ u_i$ is a $(R_t)^*J$-holomorphic map for any $t \in \mathbb{R}^*$. We note that the almost complex structure

$$J_t := (R_t)^*J \to J_0$$

in $C^\infty$ as $t \to 0$ on compact sets in $N_C \subset \mathbb{P}(O \oplus N_C)$.

Now we can imitate the proof of Proposition 6.6 [IP1] to find a sequence $t_i \to \infty$ such that there exists an irreducible component $(\Sigma_1,\phi_1)$ of $(\Sigma,\phi)$ on which the re-scaled map $R_{t_i} \circ u_i$ produces a non-constant $R_0^*J = J_0$-holomorphic section of $\phi^\infty_N$.  

**Remark 6.2.** This case, where the image completely sinks into the symplectic hypersurface $V$ with singular domain, was not explicitly singled out in the proof of Proposition 6.6 [IP1]. However the argument of Step 3 in the proof of Proposition 6.6 [IP1] still applies to this case with some minor changes in their condition (6.13) as explained to the author by Ionel and Parker [IP2]. Because of this and because the current case deals with higher codimension case, not the case of symplectic hypersurface as in [IP1], we give the full details of this re-scaling argument in our case for the readers’ convenience. In addition, we use the $C^0$-distance of the image of $u_i$ from the locus $C$ as the re-scaling parameter instead of the energy parameter used in [IP1]. Our choice of scaling parameter seems to give simpler re-scaling argument for the purpose of just producing a kernel element for the linearization in this particular case.

To carry out the re-scaling argument, we need to make precise the description of symplectic form near $C$. We will follow the approach taken by Li and Ruan [LR]. We first choose a symplectic form on $N_C$ as follows. Take a Hermitian metric on the vector bundle $\pi : N_C \to C$ and denote by $| \cdot |$ the associated norm. Then the function $|n|^2$ defines a smooth function on $N_C$ and $i\partial \bar{\partial}|n|^2$ is a closed 2-form fiberwise nondegenerate. Let $\omega_0$ be an area form on $o_C$. Then

$$\omega_\varepsilon := \pi^* \omega_0 + \varepsilon i\partial \bar{\partial}|n|^2$$  

(6.2)
is a symplectic form on the total space of $N_C$ in a neighborhood of the zero section if $\varepsilon$ is sufficiently small. Considering the function

$$H_\varepsilon(z, u) = |u|^2 - \varepsilon$$

and performing the symplectic cutting (see [Le] for the precise definition) along the hypersurface $H_\varepsilon^{-1}(0)$, we obtain a symplectic manifold

$$(\mathbb{P}(\mathcal{O} \oplus N_C), \omega^-_\varepsilon).$$  

(6.3)

By the symplectic neighborhood theorem, the symplectic structure of a neighborhood of $C$ is uniquely determined by the area of $C$ and the almost complex structure of the symplectic normal bundle. Identifying the $\varepsilon$-neighborhood of $C$ with that of $N_C$, we consider the family of hypersurface

$$(\mathbb{P}(\mathcal{O} \oplus N_C), \omega^-_\varepsilon).$$

$N_\varepsilon = \{|u|^2 = \varepsilon\}.$

Applying the symplectic cut, we obtain two symplectic manifolds $M^+_\varepsilon$ and $M^-_\varepsilon$. Here $M^-_\varepsilon$ is nothing but the above projectivization

$$M^-_\varepsilon = (\mathbb{P}(\mathcal{O} \oplus N_C), \omega^-_\varepsilon).$$

We fix one such $\varepsilon > 0$.

Now we can identify the symplectic blow up $\widetilde{M}_\varepsilon$ with the symplectic sum

$$\widetilde{M}_\varepsilon = M^-_\varepsilon \cup_{N_\varepsilon} M^+_\varepsilon, \quad \omega = \omega_\varepsilon \# \omega.$$  

(6.4)

By denoting $\mathbb{P}_\infty = \mathbb{P}(\mathcal{O} \oplus N_C) \setminus \mathbb{P}(1 \oplus N_C)$ and $S(N_C)$ the unit sphere bundle of $N_C$, $\mathbb{P}(\mathcal{O} \oplus N_C) \setminus \mathbb{P}(1 \oplus N_C) \cong M^-_\varepsilon \setminus N_\varepsilon$ is a symplectic manifold with positive end with its asymptotic boundary given by the contact manifold $N_\varepsilon \cong S(N_C)$ and $M^+_\varepsilon \cong M \setminus C$ is one with negative end.

Focusing on the $\varepsilon$-neighborhood of $C$, we pull back the pair $(J, g)$ from $M$ to the corresponding neighborhood in $\mathbb{P}(\mathcal{O} \oplus N_C) \setminus \mathbb{P}_\infty$. We also consider the pair $(J_0, g_\varepsilon)$ where $J_0$ is the almost complex structure

$$J_0(n) = j(\pi(n)) \oplus i_{(J,C)}$$

and $g_\varepsilon$ is the metric induced by a given Hermitian metric on the bundle $N_C$ and the associated metric thereon (as a manifold) in the fixed $\varepsilon$-neighborhood and extended suitably outside. We fix a cut-off function $\chi$ supported on the neighborhood of $C$ with $\chi \equiv 1$ on the $\varepsilon/2$-neighborhood. For small $t$, we set $\chi_t = \chi \circ R_\sqrt{t}$.

Starting with the background metric $g' = \chi_t g + (1 - \chi_t) g_\varepsilon$, and a fixed symplectic form $\omega_\varepsilon$, we are given a tame triple $(\omega_\varepsilon, J_t, g_t)$ with $J_t = (R_t)^* J$, $g_t = (R_t)^* g'$ on $\mathbb{P}(\mathcal{O} \oplus N_C)$. As $t \to 0$, we have $J_t \to J_0$ in $C^0$ compact sets in $N_C = \mathbb{P}(\mathcal{O} \oplus N_C) \setminus \mathbb{P}_\infty$ and $(J_t, g_t) \to (J, g)$ in $C^\infty$ on compact sets of $M \setminus C$.

Since $u_i \to \phi$ in stable map topology, the image of $u_i$ is contained in the $\varepsilon'$-neighborhood of $C$ for all sufficiently large $i$ with $\varepsilon' \leq \varepsilon/4$ and so we can work on $N_C$ inside $\mathbb{P}(\mathcal{O} \oplus N_C)$ with respect to the $(\omega', J_t, g_t)$.

We need to recall the precise definition of stable map topology for the following arguments. We follow the exposition of Fukaya and Ono [FO] for this definition of stable map topology and refer readers thereto for complete details of the definition.

If we decompose the stable map $\phi$ into its irreducible components

$$(\Sigma, \phi) = \cup_{\nu}(\Sigma_{\nu}, \phi_{\nu}),$$
after stabilizing the domains by adding suitable number of auxiliary marked points to \(\Sigma_\infty\) and \(\Sigma_t\), we can express \(u_i\) as a resolution of \(\phi\)

\[
u_i = \text{Res}_{\tilde{\alpha}_i}(\phi) : \]

Here

\[
\tilde{\alpha}_i = \{\alpha_{x,i} \mid x \in \text{Sing}(\Sigma'_i), \alpha_{x,i} \in T_{x,i,\nu} \Sigma'_{i,\nu} \times T_{x,i,\nu} \Sigma'_{i,\nu}\}
\]

are the deformation parameters of nodes of \(\Sigma'_i\) where \(\Sigma'_i\) has the same intersection pattern as \(\Sigma_\infty\) and \(\Sigma'_i \to \Sigma_\infty\) in the Deligne-Mumford space and \(\tilde{\alpha}_i \to \tilde{0}\) as \(i \to \infty\) (see \([\text{FO}]\)).

Consider any non-constant component \(\phi_{\nu} : \Sigma_{\nu} \to C \subset N_C\) and the restrictions of \(u_i\) to the corresponding component \(\Sigma'_{i,\nu} \setminus W_i(\delta)\) where \(W_i(\delta)\) is the union of neck regions of \(\Sigma_i\) of the size \(\delta > 0\). We recall from section 10 \([\text{FO}]\) that we have the canonical embedding of \(\Sigma'_{i,\nu} \subset \Sigma_i\) so that we can identify

\[
\Sigma_i \cong \bigcup_{\nu} (\Sigma'_{i,\nu} \setminus \cup_{x,\nu} D_{x,\nu})
\]

where \(D_{x,\nu}\) is the disc neighborhood of \(x \in \Sigma_\nu\) of size \(|\alpha_{x,i}|^{3/4}\). By the definition of stable map topology, for any given \(\delta > 0\), \(u_i \to \phi\) on \(\Sigma'_{i,\nu} \setminus W_i(\delta)\) on any given irreducible component \(\Sigma_\nu\). Since \(\Sigma'_{i,\nu} \to \Sigma_\nu\) as \(i \to \infty\), this convergence statement makes sense. We consider a non-constant component \((\phi_{\nu}, \Sigma_\nu)\) and the corresponding restriction of \(u_i\) to \(\Sigma'_{i,\nu} \setminus W_i(\delta)\).

Now we apply the above stable map convergence to the re-scaled sequence \(R_{c_t} \circ u_i : \Sigma \to N_{C} \subset \mathbb{P}(O \oplus N_C)\) with

\[
c_i = \max_{z \in \Sigma_i}|u_i|(z).
\]

Then \(|R_{c_t} \circ u_i(z)| \leq 1\) in \(U \cong \mathbb{V} \subset N_{C}\). (See (2.4) and the paragraph right after it.)

By the choice of the symplectic form (6.3) on \(\mathbb{P}(O \oplus N_C)\), there exists a constant \(K > 0\) independent of \(i\) such that

\[
\int_{\Sigma_i} \omega_\nu < K
\]

and so we can apply the stable map convergence to the sequence since \(J_{c_i} \to J_0\) in \(C_0\) on compact sets in \(N_C\). Since \(|R_{c_t} \circ u_i(z)| \leq 1\) and hence the images of all the irreducible components of the limit of \(R_{c_i} \circ u_i\) are contained in the \(D(N_C)\) and are \(J_0\)-holomorphic where \(D(N_C)\) is the unit disc-bundle of \(N_C\). Then we can write \(u_i\) as

\[
u_i(z) = \exp_{\tau \circ u_i(z)}(c_i \xi_i(z))
\]

with \(\xi_i(z) \in N_{\tau \circ u_i(z)}\). Then we have

\[
\max_{z \in \Sigma_i}|\xi_i(z)| = 1.
\]

Next we remark

\[
\pi \circ R_{c_i} \circ u_i \equiv \pi \circ u_i.
\]

Because of the (local) convergence of \(R_{c_i} \circ u_i\) and \(\pi \circ R_{c_i} \circ u_i \equiv \pi \circ u_i\), the limit of \(R_{c_i} \circ u_i\) produces a component \((\Sigma_1, \tilde{u}_1)\) and \(\xi_1 \in \Omega^0(\Sigma_1, \phi_1^*\nu)\) with \(\phi_1 \equiv \pi \circ \tilde{u}_1\) such that

\[
\max_{z \in \Sigma_1}|\tilde{u}_1(z)| = 1.
\]

There are two possibilities: one is where \(\tilde{u}_1\) is a fiber component and the other where \(\pi \circ \tilde{u}_1\) is not a point.
In the first case, we can write
\[ \bar{u}_1(z) = \exp_{x_1} \xi_1(z) \]
where \( x_1 \) is a point in \( C \) and \( \xi_1 \) satisfies \( \partial \xi_1 = 0 \) and \( \max_z |\xi_1(z)| = 1 \). But this is impossible since \( \xi_1 \) is mapped into \( (N_{x_1}, i) \cong \mathbb{C}^{n-1} \).

In the latter case we note that \( \pi \circ \bar{u}_1 \) is a \( J \)-holomorphic curve whose image is contained in \( C \). Since \( C \) is connected and embedded \( J \)-holomorphic curve, \( \pi \circ \bar{u}_1 \) must be onto \( C \) and so \( \phi_1 \) is a ramified covering of \( C \) followed by the embedding of \( C \) into \( M \). In this case, we can write
\[ \bar{u}_1(z) = \exp_{\phi_1(z)}(\xi_1(z)) \]
for a non-constant section \( \xi_1 \in \phi_1^*N \) with
\[ \max_z |\xi_1(z)| = 1 \] (6.5)

Now we will show that \( \xi_1 \) is contained in \( \text{Ker} \, \Psi(j_1, \phi_1; J, C) \). We denote \( \phi_n = \pi \circ R_{c_n} \circ u_n = \pi \circ u_n \) where \( \pi : N_C \to C \) is the projection and write
\[ R_{c_n} \circ u_n = \exp_{\phi_n}(\xi_n) \]
for some \( \xi_n \in \phi_n^*N \). Take the sequence of the component \( \Sigma_{n,\nu} \setminus W_n(\delta_n) \) of \( \Sigma_n \setminus W_n(\delta_n) \) with \( \delta_n \to 0 \) on which the 'coning-off' of \( R_{c_n} \circ u_n \) converges to \( \phi_1 \). We define a map \( \bar{u}_{n,\nu} : \Sigma_{n,\nu} \to M \) defined by the coning-off of the restriction of \( R_{c_n} \circ u_n \) to \( \Sigma_{n,\nu} \setminus W_n(\delta_n) \) along the union of discs of the size \( \delta_n \) removed from \( \Sigma_{n,\nu} \). Then we still have
\[ \pi \circ \bar{u}_{n,\nu} \to \phi_1. \]

We write
\[ R_{c_n} \circ u_{n,\nu} = \exp_{\phi_{n,\nu}} \xi_{n,\nu} \]
for some \( u_{n,\nu} \) and \( R_{c_n} \circ u_{n,\nu} \to \bar{u}_1 \).

By definition of linearization of \( D\bar{\varphi}_{J} \) at \( (j_{n,\nu}, \phi_{n,\nu}) \), we obtain the mapping component
\[ \Pi_n \overline{\varphi}_{(j_{n,\nu}, J)}u_{n,\nu} = \overline{\varphi}_{(j_{n,\nu}, J)}\phi_{n,\nu} = D_{\phi_{n,\nu}} \overline{\varphi}_{j_{n,\nu}}(c_n \xi_{n,\nu}) + O(|c_n \xi_{n,\nu}|^2) \]
where \( \Pi_n \) is the parallel transport along the short geodesic from \( \pi \circ u_{n,\nu} \) to \( \phi_{n,\nu} \). Dividing the equation by \( c_n \), we obtain
\[ \frac{1}{c_n} \Pi_n \overline{\varphi}_{(j_{n,\nu}, J)}u_{n,\nu} - \frac{1}{c_n} \overline{\varphi}_{(j_{n,\nu}, J)}\phi_{n,\nu} = D_{\phi_{n,\nu}} \overline{\varphi}_{j_{n,\nu}}(\xi_{n,\nu}) + O(|c_n| \xi_{n,\nu}^2) \]
(6.6)
Since the image of \( \phi_{n,\nu} \) is contained in \( C \) and \( C \) is \( J \)-holomorphic, we have
\[ ( \overline{\varphi}_{(j_{n,\nu}, J)}(\phi_{n,\nu}))^\perp = 0 : \]
(6.7)
In fact, we have
\[ \overline{\varphi}_{(j_{n,\nu}, J)}(\phi_{n,\nu}) = \frac{d\phi_{n,\nu} + J \circ d\phi_{n,\nu} \circ j_{n,\nu}}{2} \]
Since the image of \( \phi_{n,\nu} \) is contained in \( C \), the image of \( d\phi_{n,\nu} \) is tangent to \( C \). Furthermore since \( C \) is \( J \)-holomorphic, the image of \( J \circ d\phi_{n,\nu} \circ j_{n,\nu} \) is also tangent to \( C \) and hence \( ( \overline{\varphi}_{(j_{n,\nu}, J)}(\phi_{n,\nu}))^\perp = 0 \) regarding \( \phi_{n,\nu} \) as a map to \( u_{n,\nu} ; \Sigma_{n,\nu} \to C \leftarrow M \).

On the other hand, we have
\[ \frac{1}{c_n} \Pi_n \overline{\varphi}_{(j_{n,\nu}, J)}u_{n,\nu} = \Pi_n \overline{\varphi}_{(j_{n,\nu}, J)}(R_{c_n} \circ u_{n,\nu}) \to 0 \]
as \( n \to \infty \) since \( R_{c_n} \circ u_{n,\nu} \to \overline{u}_1 \) and \( R^c_{c_n} J = J_{c_n} \to J_0 \) by (6.1).

Therefore by taking the limit of the normal component of (6.6) and noting 
\[ |c_n||\xi_{n,\nu}|^2 = c_n^{-1}|c_n\xi_{n,\nu}|^2 \leq c_n, \]
we obtain
\[ 0 = (D_{u_n} \overline{\mathcal{G}}_{(j, J)}(\xi_1))^\perp. \]

Now we prove the following proposition.

**Proposition 6.3.** Regard \( \phi \) as the \((j, J)\)-holomorphic map \( u \) which is \( \phi \) followed by the inclusion \( C \to M \) and regard \( \xi \in \Gamma(\phi^*N_C) \) as an element of \( \Gamma(u^*TM) \) via the splitting (6.1). Then we have
\[ (D_{\overline{\mathcal{G}}_{(j, J)}(u)}(\xi))^\perp = \Psi(j, \phi; J, C)(\xi). \]  

\[(6.8)\]

for all \( \xi \in \Gamma(\phi^*N_C) \).

**Proof.** Let \( \xi \in \Gamma(u^*TM) \). Identifying a neighborhood of \( C \subset M \) with that of the zero section of \( N = N_C \), we will consider the variation
\[ u_t = (\phi, t\xi) \]
where \( \xi \) is a section of \( \phi^*N \to \Sigma \). With respect to the canonical connection of \( i = i(J, C) \) on \( N \), we decompose
\[ T_nN \cong T_{\pi(n)}C \oplus N_{\pi(n)}. \]

Then we have
\[ \Pi(du_t) = (d\phi, t\nabla^\phi \xi) \]
where \( \nabla^\phi \) is the pull-back connection on \( \phi^*N \) of the canonical connection of \( N \), and \( \Pi \) is the parallel translation from \( T_{u(z)}N \) to \( T_{\pi(u(z))}N \).

We recall the almost complex structure \( J \) is represented by \( J(n) = \Phi(n)^{-1}J_0(n)\Phi(n) \) where
\[ J_0(n) = \begin{pmatrix} j(\pi_1(n)) & 0 \\ 0 & i \end{pmatrix} \]
and
\[ \Phi(n) = \begin{pmatrix} \alpha(n) & \beta(n) \\ \gamma(n) & \delta(n) \end{pmatrix}. \]

We compute
\[ \Pi \overline{\mathcal{G}}_{(j, J)}(u_t) = \Pi \left( \frac{du_t + J \circ du_t \circ j}{2} \right) = \frac{1}{2} \Pi(du_t) + \frac{1}{2} \Pi(Jdu_t \circ j) \]
\[ = \frac{1}{2} \left( \frac{d\phi}{t\nabla^\phi \xi} \right) + \frac{1}{2} \Pi(\Phi(u_t)^{-1}J_0(u_t)\Phi(u_t) \left( \frac{d\phi \circ j}{t\nabla^\phi \xi \circ j} \right)) \]  

\[(6.9)\]

Using the fact \( \alpha(u_t) = id + o(|t|), \beta(u_t) = o(|t|), \delta(u_t) = id + o(|t|) \), we compute, modulo \( o(|t|) \),
\[ \Phi(u_t)^{-1}J_0(u_t)\Phi(u_t) \left( \frac{d\phi \circ j}{t\nabla^\phi \xi \circ j} \right) \]
\[ = \begin{pmatrix} j(J, C) \circ \phi & 0 \\ -\gamma(\exp_\phi(t\xi)j(J, C) \circ \phi + i(J, C) \circ \phi \gamma(\exp_\phi(t\xi)J(J, C)) & 0 \end{pmatrix} \left( \frac{d\phi \circ j}{t\nabla^\phi \xi \circ j} \right) \]
\[ = \frac{1}{2} (d\phi + \phi^*j(J, C) d\phi) \]
\[ + \frac{1}{2} (\nabla^\phi \xi + \phi^*i\nabla^\phi \xi \circ j) + \frac{1}{2} (\gamma(\exp_\phi(t\xi)) + \phi^*i\gamma(\exp_\phi(t\xi)) \circ j). \]
Therefore substituting this into \ref{dual} and taking the derivative, we obtain
\[
\nabla_{\nu} j_{(j,J)}(u_t) = \left( \frac{1}{2}(\nabla^s j + \phi^s j_{(j,J)} \nabla^s j \circ j) + \frac{1}{2}(D^s_\gamma \phi + \phi^s i D^s_\gamma \phi \circ j) \right)
\]
where \( j_{(j,J)} = J_{|TC} \) and \( i = i_{(j,J)} = J_{|NC} \) as before. Therefore we derive the identity
\[
(D\overline{\phi}_{(j,J)}(u)(\xi))^\perp = (\nabla^\phi)^{(0,1)} j_{(j,\phi^s i)} \xi + (D^\phi_{\gamma}(\phi))^{(0,1)} \xi = \Psi(j, \phi; J, C)(\xi).
\]
This finishes the proof. \( \Box \)

Proposition \ref{prop:1} then implies
\[
(D_{u_t} \overline{\phi}_{(j,J)}(\xi_1))^\perp = \Psi(j_1, \phi_1; J, C)(\xi_1)
\]
and hence \( \xi_1 \in \text{Ker} \Psi(j_1, \phi_1; J, C) \). Furthermore we have
\[
\max_{z \in \Sigma_1} |\xi_1(z)| = 1
\]
from \ref{dual} and so \( \xi_1 \) is not zero. Therefore we have produced a ramified covering \( \phi_1 : \Sigma_1 \to C \) such that \( \text{Ker} \Psi(j_1, \phi_1; J, C) \neq 0 \). This finishes the proof. \( \Box \)

7. Calculation of \( \phi \)-variations

The main goal of this section and the next six is to prove rigidity of embedded \( J \)-holomorphic curves and their multiple covers \((j, \phi)\), i.e., to prove \( \text{Ker} \Psi(j, \phi; J, C) = \{0\} \) for all such \((j, \phi)\).

Our proof of rigidity will rely on the study of 1-jet transversality, i.e., transversality of the section
\[
\phi \to \Psi(j, \phi; J', C)
\]
of the bundle \( \text{Fred}_C \) against \( \text{Fred}_{C,k} \) in \( \text{Fred}_C \) for a suitable choice of \( J' \in \mathcal{J}_{(J,C)} \).

Here
\[
\text{Fred}_C \to C^\infty(\Sigma, C) \times \mathcal{J}_{(J,C)}
\]
is the bundle of Fredholm operators whose fiber at \((\phi; J')\)
\[
\text{Fred}_{C, (\phi; J')} = \text{Fred} \left( W^{\ell,p}(T\Sigma), W^{\ell-1,p}((\phi^{(0,1)}(\phi^* T C))) \right)
\]
and \( \text{Fred}_{C,k} \) is the subbundle consisting of those whose kernel has dimension \( k \). Here \( \ell \geq 2 \) is an integer. (Strictly speaking, we should also include a fixed \( j \) in our notation as \( \text{Fred}_{k;C} \) instead of \( \text{Fred}_C \). For the simplicity of notation, we suppress it.) We first recall that the normal space to \( \text{Fred}_k \) in \( \text{Fred} \) at an operator \( D \in \text{Fred}_k \) is isomorphic
\[
\text{Hom}_{\mathbb{R}}(\text{Ker} D, \text{Coker} C).
\]
We recall
\[
\Psi(j, \phi; J, C) = \phi^* \left( (\nabla + \mu_J)^{(0,1)}_{(j,J,C)} \right)
\]
for any \((j, \phi; J, C)\). The latter coincides with
\[
(\phi^* (\nabla + \mu_J))^{(0,1)}_{j, \phi^s i}
\]
whenever \((j, \phi)\) is a pair such that \( \phi \) is \((j, j_{(J,C)})\)-holomorphic.

We fix the Poincaré metric on \((\Sigma, J)\). Using the Hermitian connection on \( N \) and the associated \( L^2 \)-metric on the space of sections, we identify \( \text{Coker} \Psi \) with \( \text{Ker} \Psi^* \) where \( \Psi^* (j, \phi; J, C) \) is the \( L^2 \)-adjoint of \( \Psi(j, \phi; J, C) \).
We will be more interested in the pull-back of \((7.1)\) under the map
\[
\text{Diff} (\Sigma) \times \text{Hol}_{g+h}(C; d[C]) \to C^\infty (\Sigma, C); (\psi, \phi) \mapsto \phi \circ \psi.
\]
We will also denote the pull-back bundle by Fred. The following is an obvious lemma but will play a crucial role in our argument.

**Lemma 7.1.** Let \(\psi \in \text{Diff} (\Sigma)\) and \(\phi \in C^\infty (\Sigma, C)\). Then we have
\[
\dim \text{Ker} \, \Psi (j, \phi; J, C) = \dim \text{Ker} \, \Psi (j, \phi \circ \psi; J, C).
\]
The same identity holds also for the cokernel. In particular, the action of \(\text{Diff} (\Sigma)\) preserves each stratum \(\text{Fred}_{C, k}\) of \(\text{Fred}_{C}\) for all \(k \geq 0\).

**Proof.** First we recall
\[
\Psi (j, \phi \circ \psi; J, C) = (\phi \circ \psi)^* \left( (\nabla + \mu)^{\langle 0, 1 \rangle}_{(j, i)} \right) = \psi^* \left( (\nabla + \mu)^{\langle 0, 1 \rangle}_{(j, i)} \right)
\]
from \((4.1)\). Since \(\psi\) is a diffeomorphism, we have \(\Psi (j, \phi; J, C)(\xi) = 0\) for \(\xi \in \Omega^1 (\phi^* N_C)\) if and only if \(\Psi (j, \phi \circ \psi; J, C)(\xi^\psi) = 0\) for \(\xi^\psi \in \Omega^1 ((\phi \circ \psi)^* N_C)\) is the section defined by
\[
\xi^\psi (\sigma) = \xi (\psi^{-1} (\sigma)).
\]
Since \(\xi \mapsto \xi^\psi\) is one-one correspondence, the lemma follows. \(\square\)

Now we carry out somewhat subtle calculations to obtain a precise formula for the variation \(D_\phi \Psi (\kappa)\) for a holomorphic map \(\phi : \Sigma \to C\) under the above mentioned action \(\text{Diff} (\Sigma)\). This will play a crucial role in our scheme of the proof of super-rigidity later. (One could extend our calculations to derive the variation \(\delta (j, \phi)\). Since we do not need this general variation for our purpose and want to make it clear that the \(\phi\)-variation is what we need, we intentionally do not include this general variational formula.)

First we consider a path \(t \mapsto \phi_t\) with \(\phi_0 = \phi\) and \(\phi_t = \phi \circ \psi_t\) where \(\psi_t : \Sigma \to \Sigma\) an isotopy of diffeomorphisms of \(\Sigma\). We define the vector field \(W\) on \(\Sigma\) by
\[
W = \frac{d}{dt} \bigg|_{t=0} \psi_t.
\]
We compute
\[
\frac{d}{dt} \bigg|_{t=0} \left\langle c, (\phi_t^* (\nabla + \mu))^{\langle 0, 1 \rangle}_{(j, i); \phi_t^* \phi_i^* K} \right\rangle
\]
for the variation \(\phi_t = \phi \circ \psi_t\).

We will use \(\langle \cdot, \cdot \rangle_2\) for the \(L^2\)-inner product and \(\langle \cdot, \cdot \rangle\) the pointwise inner product in the following calculations. We denote \(\nabla^\phi\) the pull-back connection on \(\phi^* N_C\) of the connection \(\nabla\) on \(N_C\).

We denote by \(R_C\) the curvature of the Hermitian connection of the holomorphic vector bundle \((N_C, i_{(J, C)}, h_{(J, C)})\) where the Hermitian metric \(h_{(J, C)}\) is defined by
\[
h_{(J, C)} = g_J |_{N_C}
\]
where \(g_J = \omega (\cdot, J \cdot)\) is the metric associated to \(J\) compatible to \(\omega\). We recall by definition \(R_C\) we have
\[
R_C (X, Y) \xi = \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X, Y]} \xi
\]
for any vector fields \(X, Y\) on \(C\) and \(\xi\) a section of \(N_C\).
We define a map
\[
R_{(j,\phi;J,C)}^{(0,1)} : \Gamma(\phi^*TC) \to \Omega_{(j,\phi;J,C)}^{(0,1)}(\text{End}(\phi^*N))
\]
such that for \( W \in \Gamma(\phi^*TC) \) its value \( R_{(j,\phi;J,C)}^{(0,1)}(W) \) is defined by
\[
R_{(j,\phi;J,C)}^{(0,1)}(W) = \frac{1}{2} \left( R_C(d\phi(\cdot), d\phi(W)) + i_{(J,C)}R_C(J,\phi\cdot, d\phi(\cdot), d\phi(W)) \right)
\]
as an element \( \Omega_{(j,\phi;J,C)}^{(0,1)}(\phi^*N) \).

We now derive the following general variation formula of \( \Psi(j,\phi;J,C) \) along \((j,\phi) \in M_{g+h}(C;d[C])\) towards the non-holomorphic direction of \( C^\infty(\Sigma,C) \) with \( j \) fixed.

**Proposition 7.2.** Let \((j,\phi;J,C) \in M_{g+h}(g,d;\beta)\) and denote \( i = i_{(J,C)} \) the complex structure on \( N \). Then we have
\[
(D_\phi \Psi)(j,\phi;J,C)(W) = \left( \nabla^\phi + \mu^\phi_{(j,\phi;J,C)} \right) \nabla W + \nabla_W^\phi \left( \left( \nabla^\phi + \mu^\phi_{(j,\phi;J,C)} \right) \nabla W \right) \tag{7.5}
\]
as a one-form on \( \Sigma \). Here \( \nabla W \) is the covariant derivative of the vector field \( W \) with respect to the Levi-Civita connection associated to a Kähler metric of \( j \) on \( \Sigma \).

**Proof.** For the simplicity of notation, we will just denote \( \phi^*i = \phi^*i_{(J,C)} \) in this proof.

Let \( \psi_t \in \text{Diff}(\Sigma) \) for \(-\varepsilon \leq t \leq \varepsilon\) be one-parameter families of diffeomorphisms of \( \Sigma \) and of (almost) complex structures of \( \Sigma \) respectively with
\[
W = \left. \frac{d}{dt} \right|_{t=0} \psi_t.
\]
We need to compute
\[
\nabla_t^\phi \left|_{t=0} \Psi(j,\phi \circ \psi_t;J,C) \right.
\]
which is a first-order differential operator acting on \( \Omega^0(\phi^*N_C) \) with values in \( \Omega_{(j,\phi;J,C)}^{(0,1)}(\phi^*N_C) \). Therefore for given \( \kappa \in \Omega^0(\phi^*N_C) \) we compute
\[
\left( \nabla_t^\phi \right|_{t=0} \Psi(j,\phi \circ \psi_t;J,C) (\kappa) = \left( \nabla_t^\phi \right|_{t=0} \psi_t^* \left( \left( \nabla^\phi + \mu^\phi_{(j,\phi;J,C)} \right) \nabla W \right) \right) (\kappa) \tag{7.6}
\]
which is a one-form on \( \Sigma \). By definition, we have
\[
\left( \nabla_t^\phi \right|_{t=0} \psi_t^* \left( \left( \nabla^\phi + \mu^\phi_{(j,\phi;J,C)} \right) \nabla W \right) \right) (\kappa) = \left. \nabla_t^\phi \right|_{t=0} \left( \psi_t^* \left( \left( \nabla^\phi + \mu^\phi_{(j,\phi;J,C)} \right) \nabla W \right) \right) (\kappa) \tag{7.7}
\]
The second term becomes
\[
\left( \nabla^\phi + \mu^\phi_{(j,\phi;J,C)} \right) \left. \nabla_t^\phi \right|_{t=0} \kappa(\psi_t) = \left( \nabla^\phi + \mu^\phi_{(j,\phi;J,C)} \right) \left. \nabla_W^\phi \kappa \right|_{t=0} (\kappa(\psi_t)).
\]
For the first term, we compute
\[
\left. \nabla_t^\phi \right|_{t=0} \left( \psi_t^* \left( \nabla^\phi \kappa \right) \right) = \left. \nabla_t^\phi \right|_{t=0} \left( \nabla_{\psi_t^* \kappa} \psi_t \right) = \left. \nabla_{\psi_t^* \kappa} \psi_t \right|_{t=0} = \nabla_{\psi_t^* \kappa} \psi_t (\kappa(\psi_t)).
\]

Next we compute
\[
\nabla^\phi_i |_{t=0} \psi^*_i (\phi^* i \nabla^\phi \kappa \circ j) = \phi^* i \left( \nabla^\phi_j \nabla^\phi_{W(j)} \kappa + \nabla^\phi_j \nabla^\phi_W \kappa \right).
\]
(7.9)

Here we use the identities
\[

\nabla^\phi_W i = 0, \quad \nabla_W j = 0
\]
where the second vanishing holds because \(\nabla_W\) is the covariant derivative on \(\Sigma\) with respect to the Kähler metric on \(\Sigma\) and the first vanishing occurs since \(\nabla^\phi\) is the pull-back of a Hermitian connection on \(N_C\) associated to the complex structure \(i = i_{(J,C)}\).

Now let \(v \in T_z \Sigma\) and \(W'\) be a locally defined vector field near \(z\) with \(W'(z) = v\).

We can choose the extension \(W'\) so that
\[
[W, W'](z) = 0
\]
(7.10)
at a given point \(z \in \Sigma\). We evaluate \(\nabla^\phi \nabla^\phi_W \kappa\) against \(v\) at \(z\) and obtain
\[
(\nabla^\phi_W \nabla^\phi_W \kappa)(z) = (\nabla^\phi_W \nabla^\phi_W \kappa)(z) + R_C(d\phi(v), d\phi(W(z)))\kappa(z)
\]
\[
(\nabla^\phi_j \nabla^\phi_W \kappa)(z) = (\nabla^\phi_j \nabla^\phi_W \kappa)(z) + R_C(d\phi(j(v)), d\phi(W(z)))\kappa(z)
\]
by the choice of \(W'\) satisfying (7.11) and substitute it into (7.8), (7.9) and get

\[
\nabla^\phi_i |_{t=0} \psi^*_i \left( (\nabla^\phi)(^{(0,1)}_{(\phi^* \iota)}) \kappa \right) = (\nabla^\phi)(^{(0,1)}_{(\phi^* \iota)}) \circ \nabla^\phi + \nabla^\phi_W \left( (\nabla^\phi)(^{(0,1)}_{(\phi^* \iota)}) \kappa \right) + R^{(0,1)}_{(J,\phi; J,C)}(W)\kappa.
\]
(7.11)

Next we compute
\[
\nabla^\phi_i |_{t=0} \left( \psi^*_i \left( (\mu)(^{(0,1)}_{(\phi^* \iota)}) \kappa \right) = \mu^{(0,1)}_{(\phi^* \iota)}(d\phi(\nabla W))\kappa + \nabla^\phi_W \left( \mu^{(0,1)}_{(\phi^* \iota)}(d\phi)(\kappa) \right).
\]
(7.12)

Adding (7.11) and (7.12) and subtracting (7.7), we obtain
\[
\nabla^\phi_i |_{t=0} \psi^*_i \left( (\nabla^\phi + \mu^{(0,1)}_{j,\phi \iota}) \kappa \right) = (\nabla^\phi + \mu^{(0,1)}_{j,\phi \iota}) \circ \nabla^\phi + \nabla^\phi_W \left( (\nabla^\phi + \mu^{(0,1)}_{j,\phi \iota}) \kappa \right) + R^{(0,1)}_{(J,\phi; J,C)}(W)\kappa - (\nabla^\phi + \mu^{(0,1)}_{j,\phi \iota}) \nabla^\phi_W \kappa.
\]
(7.13)

This finishes the proof. \(\square\)

**Remark 7.3.**

1. We would like to point out that the variation
\[
\nabla^\phi_i |_{t=0} \psi^*_i \left( (\nabla^\phi + \mu^{(0,1)}_{j,\phi \iota}) \kappa \right)
\]
is a tensor, more precisely a section of \(\Omega^{(0,1)}(\phi^* N_C)\). Therefore its evaluation at a given point \(\sigma_0\) against a vector field \(W'\) on \(\Sigma\) depends only on the value \(W'(\sigma_0)\), not on the vector field \(W'\) itself. This enables us to choose \(W'\) suitably without changing its value at \(\sigma_0\) when we evaluate the variation at a given point.

2. We note that \(D^\gamma_j = \mu_j\) and \(\mu_j\) depends on the 1-jet of \(J\) along \(C\), while the term \(R^{(0,1)}_{(J,\phi; J,C)}(W)\kappa\) depends on the 2-jet of \(J\) along \(C\).

Now we specialize Proposition (7.2) to the element for \(\kappa\) lying in Ker(\(\nabla^\phi + \mu^{(0,1)}\)).
Let \( \text{Proposition 8.2.} \) defined as before in section 7.

\( \partial \) is holomorphic. For given \( k \)

\[ \sum_{c} \text{as a one-form on } \mathcal{J}. \]

\( \text{Suppose } \text{Lemma 8.3.} \) We start with the following lemma

\[ \Delta \phi \]

\( \text{where } \langle \cdot, \cdot \rangle \] orthonormal basis in the induced complex structure on \( \mathcal{J}. \)

\( \text{We denote by } \) and non-zero \( v \) carry a Banach manifold itself. Similarly \( \text{Lemma 8.1.} \)

\( \text{We then consider the set of pairs } \)

\( \text{We also consider the set of pairs } \) and non-zero \( \) and non-zero \( \) in \( \text{Fred} \) which is \( C^{k} \) with \( k \geq 1. \)

\( \text{The following lemma is easy to check.} \)

\( \text{Lemma 8.1. } \text{For } k \geq 1, \mathcal{J}^{k}_{(J,C)} \) is a closed submanifold of the Banach manifold \( \mathcal{J}^{k}_{\ast} \) and so carries a Banach manifold itself. Similarly \( \) is a closed submanifold of the Frechet manifold \( \mathcal{J}. \)

\( \text{We then consider the set of pairs } \)

\( \text{We also consider the set of pairs } \) for all pairs \( (c, \kappa) \) of non-zero \( \) and non-zero \( c \in \text{Coker } \Psi(j, \phi; J, C) \) and non-zero \( c \in \text{Coker } \Psi(j, \phi; J, C). \)

\( \text{Proposition 8.2. } \text{Let } j \in \mathcal{J} \) be given and assume that \( \phi : (\Sigma, j) \to (C, j_{(J,C)}) \) is holomorphic. For given \( k \in \mathbb{N}, \) we consider the subset \( \text{Fred}_{C,k} \subset \text{Fred} \) and \( \text{Fred} \) at \( (\phi; J) \) if

\[ \left\langle c, R^{(0,1)}_{(J,\phi;J,C)}(\cdot)\kappa \right\rangle \not= 0 \quad \text{pointwise} \]

\( \text{as a one-form on } \Sigma, \) for all pairs \( (c, \kappa) \) of non-zero \( \kappa \in \text{Ker } \Psi(j, \phi; J, C) \) and non-zero \( c \in \text{Coker } \Psi(j, \phi; J, C). \)

\( \text{Proof. } \text{We start with the following lemma} \)

\( \text{Lemma 8.3. } \text{Suppose } \kappa \in \text{Ker } \Psi(j, \phi; J, C) \) and \( c \in \text{Coker } \Psi(j, \phi; J, C). \) Consider the variation \( \delta \phi = d \phi(W). \) Then we have

\[ \left\langle c, (D_{\phi} \Psi)(j, \phi; J, C)(W)\kappa \right\rangle = \left\langle c, R^{(0,1)}_{(J,\phi;J,C)}(W)\kappa \right\rangle \]

where \( \langle \cdot, \cdot \rangle \) denotes the \( L^{2} \)-inner product.
Proof. We recall from Proposition 7.2
\[(D_{\phi}\Psi)(j, \phi; J, C)(W)\kappa = R^{(0,1)}_{(j, \phi; J, C)}(W)\kappa - (\nabla^{\phi} + \mu^{\phi})^{(0,1)}_{(j, \phi^{*} i)}(\nabla^{\phi}_{W}\kappa)\].

Therefore we have
\[\langle c, D_{\phi}\Psi(j, \phi; J, C)(W)\kappa \rangle_{2} = \langle c, R^{(0,1)}_{(j, \phi; J, C)}(\cdot)\kappa \rangle_{2} - \langle c, (\nabla^{\phi} + \mu^{\phi})^{(0,1)}_{(j, \phi^{*} i)}(\nabla^{\phi}_{W}\kappa) \rangle_{2}\].

But since \(c\) lies in \(L^{2}\)-cokernel of the operator \((\nabla^{\phi} + \mu^{\phi})^{(0,1)}_{(j, \phi^{*} i)}\), the second summand in the right hand side vanishes and hence the proof. \(\square\)

Going back to the proof of the proposition, let \((c, \kappa) \in \text{Ker}\Psi(j, \phi; J, C) \times \text{Coker}\Psi(j, \phi; J, C)\). By the hypothesis, we have
\[\langle c, R^{(0,1)}_{(j, \phi; J, C)}(\cdot)\kappa \rangle \neq 0\]
as a one form on \(\Sigma\). Then there exists \(\sigma_{0} \in \Sigma \setminus \text{br}(\phi)\) and a vector field \(w_{0} \in T_{\sigma_{0}}\Sigma\) such that
\[\langle c(\sigma_{0}), R^{(0,1)}_{(j, \phi; J, C)}(w_{0}\kappa(\sigma_{0})) \rangle \neq 0\].

Then we can first choose a vector field \(W\) with \(W(\sigma_{0}) = w_{0}\) and supported in a small neighborhood of \(\sigma_{0}\) and then multiply a suitable cut-off function \(\chi\) to \(W\) supported in a sufficiently small neighborhood and obtain
\[\langle c, R^{(0,1)}_{(j, \phi; J, C)}(\tilde{W})\kappa \rangle_{2} = \int_{\Sigma} \langle c, R^{(0,1)}_{(j, \phi; J, C)}(\tilde{W})\kappa \rangle \neq 0\]

for the globally defined vector field \(\tilde{W} = \chi W\). Therefore by Lemma 8.3 this implies
\[\langle c, (D_{\phi}\Psi)(j, \phi; J, C)(\tilde{W})\kappa \rangle_{2} \neq 0\].

Combining the above lemma and 8.3, we have proven that the section \(T_{(j, C)}\) is transversal to \(\text{Fred}_{C, k} \subset \text{Fred}_{C}\) which finishes the proof of transversality of the map \(\phi \mapsto \Psi(j, \phi; J, C)\) against \(\text{Fred}_{C, k} \subset \text{Fred}_{C}\). \(\square\)

Combining Lemma 7.4 and Proposition 8.2 we immediately obtain the following corollary.

**Corollary 8.4.** Let \(\Psi(j, \phi; J, C) \in \text{Fred}_{C}\). Suppose \(k = \dim \text{Ker}\Psi(j, \phi; J, C) \geq 1\). Then there exists a non-zero pair
\[(c, \kappa) \in \text{Ker}\Psi(j, \phi; J, C) \times \text{Coker}\Psi(j, \phi; J, C)\]
such that \(\langle c, R^{(0,1)}_{(j, \phi; J, C)}(\cdot)\kappa \rangle = 0\) as a one-form on \(\Sigma\).

**Proof.** Suppose to the contrary. Under this hypothesis, Proposition 8.2 says that the partial derivative \(D_{\phi}\Psi(j, \phi; J, C)\) is transversal to \(\text{Fred}_{C, k} \subset \text{Fred}_{C}\). On the other hand, Lemma 7.4 says that the action of \(\text{Diff}(\Sigma)\) preserves strata \(\text{Fred}_{C, k}\). These two can happen simultaneously only when the stratum \(\text{Fred}_{C, k}\) is an open stratum, i.e., when \(k = 0\). This contradicts to the standing hypothesis of the corollary. This finishes the proof. \(\square\)

In section 13 we will prove the hypothesis of this corollary will hold for a dense set of \(J\)'s.
Remark 8.5. The geometric meaning of (8.3) is that the graph of the section pair \((c, \kappa)\) over \(\Sigma\) is contained in the ‘flat locus’ of the curvature \(R^{(0,1)}_{(j,\phi;J,C)}\). In the next section, we will show that this phenomenon cannot happen for a generic choice of \(J\) from \(\mathcal{J}_{(J,C)}\) (9.1). (See Proposition 9.1.) This is because we can make the curvature ‘so bumpy as’ we want so that whenever there exists a non-zero pair \((c, \kappa)\) their graph cannot be completely contained in the flat locus of the curvature \(R^{(0,1)}_{(j,\phi;J,C)}\). As a result, there cannot be a non-zero such pair \((c, \kappa)\) generically.

This argument applies to general situations, not restricted to the Calabi-Yau case. We will come back to this question again elsewhere. See Theorem 13.4.

9. Analysis of the curvature of \(N_C\); 2-jet transversality

We would like to point out that unlike the case of somewhere injective curves in which we vary \(J\) in \(M\), we will not be able to prove rigidity for \(d \geq 2\) by varying \(J\) due to the possible presence of automorphism group. It turns out that for given pair \((j, \phi)\) with \(\phi\) being holomorphic with respect to \(j\), we need to vary \(\phi\) in a way that \(\phi\) is not holomorphic in terms of \(j\). One crucial difference between \(\phi\)-variations and \(J\)-variations is that the former can be cut-off arbitrarily so that it becomes supported in a small neighborhood at any given point of \(\Sigma\) while \(J\) cannot be so but should satisfy some symmetry caused by the multiple covering.

To be able to exploit \(\phi\)-variations, we had to search for a right Fredholm formulation and the off-shell setting for this rigidity proof and carry out deliberate calculation of the variation of \(\phi\) and obtain a precise formula. And we will also need to carry out some careful analysis of the curvature operator of the normal bundle \(N_C\) of the associated canonical connection as carried out in the previous section. This kind of calculation is common in geometric analysis but not has been so common in relation to the study of pseudo-holomorphic curves. Beside the freedom of localizing the \(\phi\)-variations, it is also worthwhile to mention that the curvature \(R\) involves 2-jet data of \(J\) along \(C\) while the definition of linearization involves only 1-jet data.

Motivated by the formula (7.14) and Lemma 8.3, we carry out a detailed study of curvature operator \(R^{(0,1)}_{(j,\phi;J,C)}\) in this section.

Let \((j, \phi; J, C)\) be given and recall from section 2 the description of \(J\) satisfying \(TC = JTC\),

\[
J(n) = \Phi_J(n)^{-1}J_0(n)\Phi_J(n)
\]

where \(\Phi_J\) is an automorphism of \(TN\) (not of \(N\)!) satisfying \(\Phi_J|_C \equiv id\).

\[
\Phi_J(n) = \begin{pmatrix}
\alpha_J(n) & \beta_J(n) \\
\gamma_J(n) & \delta_J(n)
\end{pmatrix}
\]

as in section 2.

With respect to the above description of \(J\), we then consider the subset

\[
\mathcal{J}_{(J,C)} := \{ J' \in \mathcal{J}_\omega \mid TC = J'TC, \gamma_{J'} = \gamma_J \text{ on } C \} \quad (9.1)
\]

By the condition \(TC = J'TC\), \(C\) is also an embedded unparameterized \(J'\)-holomorphic curve and so if \(\phi : \Sigma \to C\) is holomorphic with respect to \(i_{(J,C)}\), then it is also so with respect to \(i_{(J',C)}\). Therefore we can still consider the operator \(\Psi(j, \phi; J', C)\) for the same \(C\), \((j, \phi)\). Recall the definition (4.1), (4.6) of \(\Psi\).
Proposition 9.1. Let \((j, \phi; J, C)\) be given. Consider the curvature operator
\[
R^{(0,1)}_{(j, \phi; J, C)} : T\Sigma \otimes \phi^* TC \to \text{Hom}''(\phi^* N C)
\]
as a function of \((j, \phi; J') \in \text{Hol}_{g+h}(C; d[C]) \times \mathcal{J}(J, C)\). Let \(\kappa_0 \in N_{\phi(\sigma_0)}\) and \(c_0 \in \Lambda^{(0,1)}(N_{\sigma_0})\) be non-zero vectors and \(\sigma_0 \notin \phi^{-1}(br(\phi))\), and define a subset of \(T^*_{\phi(\sigma_0)} C\)
by
\[
T^*_{\sigma_0} := \left\{ \alpha \in T^*_{\phi(\sigma_0)} C \mid \alpha = \left\langle c_0, \delta J R^{(0,1)}_{(j, \phi; J, C)}(\sigma_0) \kappa_0 \right\rangle, \delta J \gamma = 0, \delta J \in T \mathcal{J}(J, C) \right\}
\]
(9.2)
Then \(T^*_{\sigma_0} = T^*_{\phi(\sigma_0)} C\).

This proposition seems to have some interest of its own. We need some digression on the curvature of Hermitian connections on holomorphic vector bundles in general before giving the proof of this proposition.

We start with a general curvature formula for the canonical connection on a Hermitian holomorphic vector bundle on a Riemann surface \(C\) expressed in terms of holomorphic frame \(e = \{e_1, \cdots, e_n\}\) and its associated dual frame by \(f = \{f^1, \cdots, f^n\}\). Denote the associated matrix of the Hermitian metric by \(h = h(f)\) which is a complex positive definite Hermitian matrix. We denote the associated coordinates by \(h = (h_{\alpha \beta})\), i.e.,
\[
h = \sum_{\alpha, \beta} h_{\alpha \beta} f_\alpha \otimes \bar{f}_\beta.
\]
We denote a variation of \(h\) by \(\delta h = \dot{h}\) for the simplicity of notation. Denote the component of curvature by \(K_{\alpha \beta \bar{\gamma} \bar{\varepsilon}}\) in general given by
\[
K_{\alpha \beta \bar{\gamma} \bar{\varepsilon}} = h \left( R \left( \frac{\partial}{\partial \bar{\gamma}}, \frac{\partial}{\partial \bar{\varepsilon}} \right) \bar{e}_\beta, e_\alpha \right).
\]
With respect to this notation, it is well-known in complex geometry (see [KN] for example)
\[
K_{\alpha \beta \bar{\gamma} \bar{\varepsilon}} = \frac{\partial^2 h_{\alpha \beta}}{\partial \bar{\gamma} \partial \bar{\varepsilon}} - \sum_{\gamma, \varepsilon} h_{\gamma \varepsilon} \frac{\partial h_{\alpha \gamma}}{\partial \bar{\gamma}} \frac{\partial h_{\varepsilon \beta}}{\partial \bar{\varepsilon}}.
\]
(9.3)
Therefore we have the variational formula for \(K\) at \(z_0 \in C\)
\[
\delta K_{\alpha \beta \bar{\gamma} \bar{\varepsilon}} = \frac{\partial^2 \dot{h}_{\alpha \beta}}{\partial \bar{\gamma} \partial \bar{\varepsilon}} - \delta \left( \sum_{\gamma, \varepsilon} h_{\gamma \varepsilon} \frac{\partial h_{\alpha \gamma}}{\partial \bar{\gamma}} \frac{\partial h_{\varepsilon \beta}}{\partial \bar{\varepsilon}} \right)
\]
(9.4)
inside the deformation space of Hermitian holomorphic vector bundles of \((N; i, h) \to C\). For example, this deformation formula can be applied to the case of our interest
\[
(N_{C}, i_{(j, C)}, h_0) \to C
\]
(9.5)
induced by a Kähler deformation at \(z_0\) obtained by deforming \(J'\) to the direction tangent to \(\mathcal{J}(J, C)\). We note that the terms inside the parenthesis in (9.4) involves at most the first derivatives of \(\dot{h}_{\alpha \beta}\) while the first terms purely involves the second derivatives for \(\dot{h}_{\alpha \beta}\).

We also note the variation \(\dot{h}_{\alpha \beta}\) is a Hermitian matrix and we can solve
\[
\frac{\partial^2 \dot{h}_{\alpha \beta}}{\partial \bar{\gamma} \partial \bar{\varepsilon}} = A_{\alpha \beta}
\]
(9.6)
for $h_{\alpha\beta}$ for any given Hermitian matrix $A_{\alpha\beta}$ at a given point $z_0$ as long as we are free to vary $h$ inside the set of complex matrices up to the second order of a given point without restriction on the choice of $h$.

**Proof of Proposition 4.4** We specialize the above discussion on the curvature to our situation. We first reformulate Proposition 9.1 in terms of complex notation. We complexify $N_C = N \otimes \mathbb{C}$ and regard $N$ as its $(1,0)$-part of $N_C$. We denote by $i = i_{(J,C)}$ the holomorphic structure on $N_C$ associated to $(J,C)$ and by $h_{(J,C)}$ the Hermitian metric on $N_C$ induced by the almost Kähler metric

$$g = \omega(\cdot, J\cdot)$$

for the compatible $J$. We denote $\dot{h} = \delta h$. Using this complex notation, the proposition will follow from the following complex version.

**Lemma 9.2.** Let $z_0 \in C$ and $A_{\alpha\beta}$ be any given $(n-1) \times (n-1)$ Hermitian matrix.

1. There exists a complex coordinates $z = x + \sqrt{-1}y$ and a holomorphic frame \{$e_1, e_2, \ldots, e_{n-1}$\} of $N_C$ on a neighborhood of $U \subset C$ such that

$$\frac{\partial h_{\alpha\beta}}{\partial z}(z_0) = 0 = \frac{\partial h_{\alpha\beta}}{\partial \bar{z}}(z_0), \quad h_{\alpha\beta}(z_0) = id. \quad (9.7)$$

2. Then we can find some variation $\dot{h} = \delta h$ such that

$$\frac{\partial^2 h_{\alpha\beta}}{\partial z \partial \bar{z}}(z_0) = A_{\alpha\beta}, \quad \frac{\partial h_{\alpha\beta}}{\partial z}(z_0) = 0, \quad \dot{h}_{\alpha\beta}(z_0) = 0 \quad (9.8)$$

for all $1 \leq \alpha, \beta \leq n-1$.

**Proof.** We start with the variation formula of $\delta J$ in terms of $\delta \Phi$. We note that as we vary almost complex structures inside $J_{(J,C)}$, the given embedded $J$-holomorphic curve $C$ will remain fixed and so the normal bundle $N \to C$ also deforms as a rank $(n-1)$ holomorphic vector bundle together with the complex structure on the same base. We recall that our base $C$ is complex 1-dimensional and so the (almost) complex vector bundle $(N,J)$ is integrable and so holomorphic.

Let $(J^t, C)$ with $-\varepsilon < t < \varepsilon$ be such a one-parameter family with $(J^0, C) = (J, C)$. We write

$$J^t = (\Phi^t)^{-1} J_0 \Phi^t$$

or equivalently $\Phi^t J^t = J_0 \Phi^t$ with $\Phi^0 = \Phi$ and $J^0 = J = \Phi^{-1} J_0 \Phi$. We recall $J_0$ is the distinguished almost complex structure which we do not vary and that the splitting of $TN \cong \pi^* TC \oplus N$ is also fixed. Therefore we have the identity

$$\Phi \delta J + \delta \Phi J = J_0 \delta \Phi.$$

We recall from (2.3) that $J = J_0$ on $TM|_C$ and $\Phi|_C \equiv id$.

Therefore we obtain on $C$

$$\delta J = J_0 \delta \Phi - \delta \Phi J_0 \quad (9.9)$$

where $A''$ is the anti-complex linear component of $A$ with respect to the corresponding complex structures $j = j_{(J,C)}$ on the domain and target $i = i_{(J,C)}$ respectively. Again for the simplicity of notations, we denote by $\delta \alpha$ by $\dot{\alpha}$ and so on.
We recall that we are varying \((J, C)\) so that \(TC = J(TC)\) and \(\dot{\gamma} = 0\). Except these requirements, we are completely free to choose \(\dot{\alpha}, \dot{\beta}, \dot{\delta}\), especially for the choice of \(\dot{\delta}\).

Therefore (9.10) becomes
\[
\dot{j} = \begin{pmatrix} 0 & 2j\dot{\beta}'' \\ 0 & 2i\dot{\delta}'' \end{pmatrix}
\]
which can be chosen arbitrarily. In particular \(\delta''\) covers all \(i\)-anti-invariant matrices.

Now we express the variation \(\dot{h}\) of Hermitian metric \(h_J\) on \(N\). We note that the variation of the compatible metric \(h\) with \(\omega\) fixed over \(J\) is given by
\[
\dot{h} = \omega(\cdot, \dot{J}\cdot).
\]

At the given point \((z_0, 0) \in N \cong M\), by the symplectic neighborhood theorem we may assume that the symplectic form has the standard form
\[
dx \wedge dy + (dq_1 \wedge dp_1 + dq_2 \wedge dp_2)
\]
on some neighborhood \(U\) of \(z_0\) with complex coordinates \(z = x + \sqrt{-1}y\) and \(J(z_0, 0) = \begin{pmatrix} j & 0 \\ 0 & i \end{pmatrix}\).

Furthermore can choose the above mentioned holomorphic frame \(\{e_1, e_2, \ldots, e_{n-1}\}\) near \(z_0\) so that it is Hermitian orthogonal i.e.,
\[
e_i(z_0) = \frac{1}{2} \left( \frac{\partial}{\partial q_i} + \sqrt{-1} \frac{\partial}{\partial p_i} \right)(z_0), \quad i = 1, \ldots, n-1
\]
and \(\nabla e_i(z_0) = 0\). In particular the Hermitian metric \(h_J\) has the form
\[
dh_{\alpha\beta}(z_0) = 0.
\]
This is precisely statement (1) of the lemma which will also imply
\[
d\dot{h}_{\alpha\beta}(z_0) = 0.
\]

Now we vary \(J\) by \(\dot{J}\) so that \(\dot{\delta''}\) remains to be \(i\)-anti-invariant. Then (9.10) is equivalent to the complex expression
\[
\dot{j} = \begin{pmatrix} 0 & 2\sqrt{-1}\dot{\beta}_{z\bar{z}} \\ 0 & 2\sqrt{-1}\dot{\delta}_{\alpha\bar{\beta}} \end{pmatrix}
\]
where \(\dot{\delta}_{\alpha\bar{\beta}}\) is \((n-1) \times (n-1)\) Hermitian matrix and \(\dot{\beta}_{z\bar{z}}\) is a \(1 \times 2\) complex valued matrix which are arbitrary.

Combining all these, we can vary \(J\) so that \(h_{\alpha\beta}\) satisfies (9.8) and that
\[
\frac{\partial^2 h_{\alpha\bar{\beta}}}{\partial z\partial \bar{z}}(z_0) = \frac{\partial^2 \delta_{\alpha\bar{\beta}}}{\partial z\partial \bar{z}}(z_0)
\]
can become any given \(2 \times 2\) complex matrix. This verifies that there is no restriction on the choice of variation \(\dot{h}\) that is both complex and of the form of \(\delta_J h_j\) at the given point \(z_0\). This finishes the proof of Lemma 9.2 by the remark around (9.6). \(\square\)

Finally it is obvious that the set of \(2(n-1) \times 2(n-1)\) matrices acts transitively on \(\mathbb{R}^{2(n-1)}\). Therefore Lemma 9.2 enables us to choose \(\delta J\) so that
\[
\delta_J R^{(0,1)}_{(\xi, \phi, J, C)} \left( \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right) \kappa(\sigma_0) = c(\sigma_0) \left( \frac{\partial}{\partial z} \right)
\]
and hence \( \left< c(\sigma_0), \delta_J R_{(j,\phi;J,C)}^{(0,1)} \kappa(\sigma_0) \right> \neq 0 \), provided \( \kappa(\sigma_0) \neq 0 \neq c(\sigma_0) \). Recalling the definition (7.3) of \( R_{(j,\phi;J,C)}^{(0,1)} \), we can immediately translate this transitivity into the statement of Proposition 9.1. This finishes the proof of Proposition 9.1. □

10. INITIAL STEP OF THE INDUCTION

In this section and the next few, we will carry out a double induction and prove both super-rigidity and finiteness simultaneously. From now on, we will assume the basic hypotheses

\[ c_1(M,\omega) = 0, \quad n = 3 \]  

(10.1)

unless otherwise said explicitly. The important consequence of this restriction is that the moduli space \( M_{\text{emb}}(J;\beta) \) has virtual dimension 0 and the index of \( \Psi(j,\phi;J,C) \) satisfies

\[ \iota = 2(2d(g - 1) + 2(1 - g - h)) = -r \leq 0 \]

where \( r \) is the number of branch points as discussed in Proposition 4.5.

We would like to recall that neither of the subsets \( J_{\omega} \) or \( J_{\text{nodal}} \) is in general open and so one cannot use the Fredholm framework using these subsets of \( J_{\omega} \). For this reason, we will use an inductive scheme over the genus of the domain \( \Sigma \) and the energy of the map \( \phi \).

We recall the map \( \Psi \) given in (4.5)

\[ \Psi : F_{g,h}(g,d;\beta) \rightarrow \text{Fred}. \]

Denote by \( \pi : F_{g+h}(g,d;\beta) \rightarrow J_{\omega} \) the obvious projection.

Let \( (j,\phi;J,C) \) be given with \( \phi : (\Sigma,j) \rightarrow (C,j(J,C)) \) being holomorphic. We recall the formula

\[ \Psi(j,\phi;J,C) = \phi^*(\nabla + \mu_J)^{(0,1)}_{(j,\phi;J,C)} \]  

(10.2)

where \( \nabla \) is the Hermitian connection of \( (N_C,i(J,C)) \). We use the following notion of rigidity of embedded \( J \)-holomorphic curves.

**Definition 10.1.** Let \( C \subset M \) be a connected unparameterized embedded \( J \)-holomorphic curve of genus \( g \). We say that \( C \) is \((d,h)\)-rigid if \( L = \Psi(j,\phi;J,C) \) has no kernel for any holomorphic map \( \phi : \Sigma \rightarrow C \) of genus \( g \Sigma = g + h \) and degree \( d \). We say \( C \) is super-rigid if it is \((d,h)\)-rigid for any \((d,h)\).

By the structure theorem, Theorem 1.2 mentioned in section 1, any element \( (j,u) \in M_g(M,J;\beta) \) has decomposition

\[ u' = u \circ \phi : \]

Here \( u \) is an embedded Fredholm-regular \( J \)-holomorphic curve and \( \phi : (\Sigma,j) \rightarrow (C,j(J,C)) \) is a holomorphic map of degree \( d' \geq 1 \) where \( C \) is the image locus of \( u \).

We denote by \( M^{inj} \) the universal moduli space of somewhere injective pseudo-holomorphic curves, i.e., the set of \((j,u,J)\) such that \( u \) is somewhere injective \((j,J)\)-holomorphic curve, and \( M^{emb} \) that of embedded curves.

The following is the key definition which enables us to carry out the step-by-step proof at each finite stage of \((G,K)\).

**Definition 10.2.** For each given pair \((G,K) \in \mathbb{N}^2\), we consider the subset

\[ M^{emb}_{(G,K)} \subset M^{inj} \]

consisting of the triple \((j,u,J)\) that satisfies
(1) \( u : \Sigma \to M \) is \((j, J)\)-holomorphic and embedded
(2) \( \omega([u]) \leq K \)
(3) \( g_\Sigma \leq G \).
(4) There exists an automorphism \( \psi \in \text{Aut}(\Sigma, j) \) such that for any \( \sigma \neq \sigma' \)
\[
\frac{1}{K} \leq \frac{\text{dist}(u \circ \psi(\sigma), u \circ \psi(\sigma'))}{\text{dist}(\sigma, \sigma')} \leq K
\] (10.3)

He would like to particularly highlight condition (4) in the definition of \( \mathcal{M}_{(G, K)}^{emb} \), which is partially motivated by the corresponding condition in the generic transversality of multiple Reeb orbits used in Appendix of [ABW]. This condition prevents embedded curves from giving rise either to bubbling degeneration or to multiple covering degeneration. In [O2], the author attempted to use a very complicated induction procedure instead precisely to handle these degenerations. We would like to emphasize that imposing the lower bound in condition (4) makes sense only for embedded curves.

The following is an immediate consequence of condition (4).

**Lemma 10.3.** Suppose \( \tilde{u} = u \circ \psi \) satisfies the condition (4) in Definition 10.2. Then we have
\[
\frac{1}{K} \leq |d\tilde{u}(\sigma)| \leq K
\] (10.4)

for all \( \sigma \in \Sigma \).

**Proof.** Let \( \sigma \in \Sigma \) and \( v \in T_\sigma \Sigma \) with \( |v| = 1 \). Set \( \gamma : (-\varepsilon, \varepsilon) \to M \) to be the path \( \gamma(t) = \exp_{u(\sigma)}(sv) \). Then we have
\[
|du(\sigma(v))| = \lim_{s \to 0} \frac{\text{dist}(\tilde{u}(\gamma(s)), \tilde{u}(\sigma))}{\text{dist}(\gamma(s), \sigma)}.
\]
Therefore we obtain \( \frac{1}{K} \leq |d\tilde{u}(\sigma(v))| \leq K \) for all \( \sigma \) and \( v \) with \( |v| = 1 \). This finishes the proof. \( \square \)

The following lemma is the basis of considering the above definition.

**Lemma 10.4.** We have
\[
\mathcal{M}^{emb} = \bigcup_{(G, K) \in \mathbb{N}^2} \mathcal{M}_{(G, K)}^{emb}.
\]

**Proof.** Let \((\Sigma, j, u, J) \in \mathcal{M}^{emb} \). The first 3 conditions are trivial to check. We let \( K_1 = \omega([u]) \) and \( G = g_\Sigma \).

It remains to check condition (4) for some \( K \). We will show (10.3) indeed holds for \( \psi = id \) for some \( K > 0 \). Since \( u \) is immersed, we have
\[
\min_{\sigma \in \Sigma} |du(\sigma)| \geq C > 0
\] (10.5)
for some \( C > 0 \).

We first prove that there exists some \( K_2 > 0 \) such that
\[
\inf_{\sigma' \in \Sigma \setminus \{\sigma\}} \frac{\text{dist}(u(\sigma), u(\sigma'))}{\text{dist}(\sigma, \sigma')} \geq \frac{1}{K_2}.
\]
Suppose to the contrary that
\[
\inf_{\sigma' \in \Sigma \setminus \{\sigma\}} \frac{\text{dist}(u(\sigma), u(\sigma'))}{\text{dist}(\sigma, \sigma')} = 0.
\]
Let \( \sigma_i \neq \sigma'_i \) be a pair of sequences such that

\[
\frac{\text{dist}(u(\sigma_i), u(\sigma'_i))}{\text{dist}(\sigma_i, \sigma'_i)} \to 0
\]
as \( i \to \infty \). By the compactness of \( \Sigma \), we may assume, by choosing a subsequence if necessary, that \( \sigma_i \to \sigma_0 \). Choose complex coordinates \((U, z)\) at \( \sigma_0 \) and regard \( u \) as a holomorphic map on an open subset \( U \subset \mathbb{C} \) to \( \mathbb{C} \). Since \( u \) is an embedding, we have

\[
\frac{\text{dist}(u(\sigma_0), u(\sigma'))}{\text{dist}(\sigma_0, \sigma')} \geq \delta_{\sigma_0} > 0
\]
for all \( \sigma' \notin U \) for some \( \delta_{\sigma_0} > 0 \). Therefore \( \sigma'_i \in U \) for all sufficiently large \( i \)'s. Since this is true for any given small neighborhood \( U \) of \( \sigma_0 \), we conclude

\[
|\sigma'_i - \sigma_0| \to 0
\]
as \( i \to \infty \). Then we have

\[
u(\sigma'_i) - u(\sigma_0) = du(\sigma_0)(\sigma'_i - \sigma_0) + o(|\sigma'_i - \sigma_0|) \tag{10.6}
\]
since \( u \) is differentiable at \( \sigma_0 \). Dividing this by \( (\sigma'_i - \sigma_0) \) and taking the absolute value, we obtain

\[
\frac{|u(\sigma'_i) - u(\sigma_0)|}{|\sigma'_i - \sigma_0|} \geq |du(\sigma_0)| - \frac{o(|\sigma'_i - \sigma_0|)}{|\sigma'_i - \sigma_0|}
\]
i.e.,

\[
|du(\sigma_0)| \leq \frac{|u(\sigma'_i) - u(\sigma_0)|}{|\sigma'_i - \sigma_0|} + \frac{o(|\sigma'_i - \sigma_0|)}{|\sigma'_i - \sigma_0|}. \tag{10.7}
\]
Since \( |\sigma'_i - \sigma_0| \to 0 \) and \( \frac{o(|\sigma'_i - \sigma_0|)}{|\sigma'_i - \sigma_0|} \to 0 \) as \( i \to \infty \) and by the hypothesis, there exists some \( N \) such that for \( i \geq N \), we have

\[
\frac{|u(\sigma'_i) - u(\sigma_0)|}{|\sigma'_i - \sigma_0|}, \quad \frac{o(|\sigma'_i - \sigma_0|)}{|\sigma'_i - \sigma_0|} \leq \frac{C}{3}
\]
Substituting these and (10.6) into (10.7), we obtain

\[
C \leq \frac{C}{3} + \frac{C}{3} = \frac{2C}{3}
\]
a contradiction. This proves

\[
\inf_{\sigma' \in \Sigma \setminus \{\sigma\}} \frac{\text{dist}(u(\sigma), u(\sigma'))}{\text{dist}(\sigma, \sigma')} > 0
\]
and so there exists some \( K_2 > 0 \) such that

\[
\inf_{\sigma' \in \Sigma \setminus \{\sigma\}} \frac{\text{dist}(u(\sigma), u(\sigma'))}{\text{dist}(\sigma, \sigma')} \geq \frac{1}{K_2},
\]

Next we prove there exists some \( K_3 > 0 \) such that

\[
\sup_{\sigma' \in \Sigma \setminus \{\sigma\}} \frac{\text{dist}(u(\sigma), u(\sigma'))}{\text{dist}(\sigma, \sigma')} \leq K_3.
\]
Let \( K' = \|du\|_{\infty} \). From (10.6), we obtain

\[
\frac{|u(\sigma'_i) - u(\sigma_0)|}{|\sigma'_i - \sigma_0|} \leq |du(\sigma_0)| + \frac{o(|\sigma'_i - \sigma_0|)}{|\sigma'_i - \sigma_0|}
\]
Here the compactness of $\Sigma$ and the derivative bound implies that there exists some $\delta = \delta(K') > 0$ such that

$$\frac{o(|\sigma'_i - \sigma_0|)}{|\sigma'_i - \sigma_0|} \leq C$$

for some $C = C(\delta) > 0$ independent of all $\sigma, \sigma'$ with $\text{dist}(\sigma, \sigma') \leq \delta$. Therefore Lemma 10.5 implies

$$\frac{|u(\sigma'_i) - u(\sigma_0)|}{|\sigma'_i - \sigma_0|} \leq K' + C$$

for all $\sigma, \sigma'$ with $\text{dist}(\sigma, \sigma') \leq \delta$. On the other hand if $\text{dist}(\sigma, \sigma') \geq \delta$, we have

$$\frac{|u(\sigma'_i) - u(\sigma_0)|}{|\sigma'_i - \sigma_0|} \leq \frac{\text{diam}(M)}{\delta}.$$ (10.8)

We set $K_3 = \max\{K' + C, \frac{\text{diam}(M)}{\delta}\}$. Finally we choose $K = \max\{K_1, K_2, K_3\}$.

This finishes the proof.

Next we consider $(C_g, i_g)$ the abstract Riemann surface of genus $g$ with fixed complex structure $i_g$. We denote by $M_{g+h}(C_g; d[C_g])$ for $d \geq 1$ the moduli space of holomorphic maps $\phi : (\Sigma, j) \to (C_g, i_g)$ with degree $d$. We fix the increasing exhaustion by open subsets $U_{g+h,d}^f$ with compact closure $\overline{U}_{g+h,d}$:

$$M_{g+h}(C_g; d[C_g]) = \bigcup_{\ell = 1}^{\infty} \overline{U}_{g+h,d}^\ell.$$ (10.10)

In our situation, $i_g$ will be the one corresponds to $i_{(J,C)}$. The open sets $U_{g+h,d}^f$ consist of $((\Sigma, j), \phi)$ such that the corresponding equivalence class $[((\Sigma, j), \phi)] \in M_{g+h}(C_g; d[C_g])$ is a finite distance away from the singular strata of compactification $\overline{M}_{g+h}(C_g; d[C_g])$.

From now on, we will often denote by $((\Sigma, j), \phi)$ also the corresponding equivalence class as long as it does not cause much confusion.

**Definition 10.5.** We define

$$\mathcal{J}_{(G,K)}^{\text{nodal}} = \{ J \in \mathcal{J}_\omega \mid \text{The properties in Theorem 10.1, 10.2 hold for all } (j, \phi; J, C) \text{ with } g\Sigma \leq G, \omega(d[C]) \leq K \}.$$ (10.11)

Obviously we have $\mathcal{J}_{(G,K)}^{\text{nodal}} = \cap_{(G,K) \in \mathbb{N}} \mathcal{J}_{(G,K)}^{\text{nodal}}$. Since $\mathcal{J}_{(G,K)}^{\text{nodal}}$ is Baire in $\mathcal{J}_\omega$, so is each $\mathcal{J}_{(G,K)}^{\text{nodal}}$. In particular they are dense in $\mathcal{J}_\omega$. However a priori they may not be open in $\mathcal{J}_\omega$. Similarly we define $\mathcal{J}_{(G,K)}^{\text{emb}}$.

**Definition 10.6.** We say that an almost complex structure $J \in \mathcal{J}_\omega$ is $(G, K; L)$-rigid if the following holds:

1. $J \in \mathcal{J}_{(G,K)}^{\text{nodal}}$
2. For any element $u \in \mathcal{M}_{(G,K)}^{\text{emb}}(J) := \mathcal{M}_{(G,K)}^{\text{emb}} \cap \pi^{-1}(J)$ with $C = \text{Im} u \subset M$ of genus $g$ and its multiple covers $u \circ \phi : \Sigma \to M$ with branched covering $\phi : (\Sigma, j) \to (C, i_{(J,C)})$ such that

$$\omega([u \circ \phi]) \leq K, g\Sigma = g + h \leq G, \phi \in \overline{U}_{g+h,d}^L,$$ (10.12)

the normal linearization operator $\Psi(j, \phi; J, C)$ has trivial kernel.

We say $J$ is $(G, K)$-rigid, if it is $(G, K; L)$-rigid for all $L \in \mathbb{N}$.
We would like to remark that the choice of $L$ depends on the moduli space $M_{g+h}(C_g; d[C_g])$ and so depends on $g$, $h$, and $d$. But when $(G, K)$ is given, there are only finitely many such $g$, $h$ and $d$ to be considered and so we can make this choice of $L$ all at once for the finite number of possibilities. In this regard, we should first fix $(G, K)$ and then $L$ afterwards. Our choice of the notation $(G, K; L)$ instead of $(G, k, L)$ reflects this fact.

We denote by $\mathcal{J}^{\text{rigid}}_{(G, K; L)} \subset \mathcal{J}_\omega$ the set of $(G, K; L)$-rigid almost complex structures and by $\mathcal{J}^{\text{rigid}}_{(G, L)}$ the set of $(G, K)$-rigid ones. By definition, we have

$$\mathcal{J}^{\text{rigid}}_{(G, L)} = \bigcap_{L=1}^{\infty} \mathcal{J}^{\text{rigid}}_{(G, K; L)}.$$ 

A priori $\mathcal{J}^{\text{rigid}}_{(G, L)}$ may not be open. However we will prove the following theorem in the next 3 sections.

**Theorem 10.7.** For every $(G, K; L)$, $\mathcal{J}^{\text{rigid}}_{(G, K; L)}$ is open and dense in $\mathcal{J}_\omega$.

Once we have proved this, the intersection

$$\mathcal{J}^{\text{rigid}}_{\omega} = \bigcap_{(G, K) \in \mathbb{N}^2} \bigcap_{L \in \mathbb{N}} \mathcal{J}^{\text{rigid}}_{(G, K; L)}$$

will be the Baire subset of $\mathcal{J}_\omega$ such that for any $J \in \mathcal{J}^{\text{rigid}}_{\omega}$, the required super-rigidity and finiteness results hold. This will finish the proof of Theorem 1.6.

11. $\mathcal{J}^{\text{rigid}}_{(G, K; L)}$ is open

For each $(G, K; L) \in \mathbb{N}^3$, we define

$$\mathcal{M}_{(G, K; L)}(J) = \{(u, \phi) \mid u \in \mathcal{M}_{\text{emb}}^{(G, K)}(J), g_{\Sigma} = g + h \leq G, \omega([u \circ \phi]) \leq K, \phi \in \mathcal{C}_{g+h,d}\}.$$ 

We define

$$\mathcal{M}_{\text{emb}}^{(G, K; L)}(J) := \mathcal{M}_{\text{emb}}^{(G, K)}(J) \cap \mathcal{M}_{(G, K; L)}(J).$$

We note that for any element $u \circ \phi \in \mathcal{M}_{\text{emb}}^{(G, K; L)}(J)$, we have $\deg \phi = 1$ and $\phi$ cannot have a critical point. Therefore $\phi$ is a biholomorphism.

We start with the following lemma using Theorem 5.1 and the definition of $(G, K; L)$-rigidity.

**Lemma 11.1.** Suppose $J \in \mathcal{J}^{\text{nodal}}_{(G, K)}$. Consider sequences et $J_i \to J$ in $\mathcal{J}_\omega$ and $u_i : (\Sigma_i, j_i) \to (M, J_i)$ of $J_i$-holomorphic curves in $\mathcal{M}_{\text{emb}}^{(G, K; L)}(J_i)$. Then there exists a subsequence of $u_i$ that converges in $C^\infty$ to $u_\infty : \Sigma \to M$ that has smooth domain $(\Sigma, j)$ and is somewhere injective and also contained in $\mathcal{M}_{\text{emb}}^{(G, K; L)}(J)$. Therefore $\mathcal{M}_{\text{emb}}^{(G, K; L)}(J)$ is a compact zero dimensional manifold and in particular $\#(\mathcal{M}_{\text{emb}}^{(G, K; L)}(J)) < \infty$.

**Proof.** Let $u_i \circ \phi_i$ be a sequence that lies in $\mathcal{M}_{\text{emb}}^{(G, K; L)}(J_i)$ for each $i$.

By the energy bound $\omega([u_i \circ \phi_i]) \in K$ and $g_{\Sigma_i} \leq G$, there exists a subsequence, again denoted by $u_i \circ \phi_i$, we may assume, by taking a subsequence, $g_{\Sigma_i} = g$, $g_{\Sigma_i} = g + h$, $[u_i] = \beta$ and $\deg \phi_i = d \geq 1$ for all $i$. We know $\phi_i$ is a biholomorphism and so after applying an automorphism of $(\Sigma, j_i)$, we may assume that $\phi_i : \Sigma \to C$ is a fixed biholomorphism and so can be omitted for the discussion in this proof.
By Lemma 10.3, we have the derivative bound

\[ \frac{1}{K} \leq |du_i(\sigma)| \leq K \]

and so \( u_i \) converges to a smooth map \( u_\infty : \Sigma_\infty \to M \) after choosing a subsequence.

Due to the condition (10.3), \( u_\infty \) cannot be multiply covered and so must be somewhere injective. Therefore it must be embedded and Fredholm regular because \( J \in J_{\text{nodal}}^{(G,K)} \subset J_{\text{emb}}^{(G,K)} \). And \( u_\infty \) should satisfy Condition (4) of the definition of \( M_{(G,K)}^{\text{emb}}(J) \) since the condition is closed in \( C_\infty \). Therefore \( u_\infty \) lies in \( M_{(G,K)}^{\text{emb}}(J) \). Then Fredholm regularity of \( u_\infty \) of \( J \) implies

\[ \text{Coker}(D_\nu \overline{\phi}(J_\infty)) = \text{Coker}(\Psi(J_\infty, \phi, J, C_\infty)) = \{0\} \]

where \( \phi : \Sigma \to C \) is the above mentioned fixed biholomorphism. Then it follows \( \text{Ker}(\Psi(J_\infty, \phi, J, C_\infty)) = \{0\} \) by the fact that Index \( \Psi(J_\infty, \phi, J, C_\infty) = 0 \). (Recall the remark right after Definition 1.1)

This implies that the image of the sequence \( u_i \) eventually stabilizes, i.e., \( C_i = C_\infty \) for all \( i \geq i_0 \) for some \( i_0 \in \mathbb{N} \). For otherwise Theorem 5.1 would imply \( \text{Ker}(\Psi(J_\infty, \phi_\infty, J, C_\infty)) \neq \{0\} \) which would give rise to a contradiction.

Altogether the chosen subsequence of \( u_i \) must converge in \( C_\infty \) to a smooth embedded curve which lie in \( M_{(G,K)}^{\text{emb}}(J) \). This finishes the proof. \( \square \)

**Corollary 11.2.** Let \( J \in J_{\text{nodal}}^{(G,K)} \). Then there exists an open neighborhood \( \mathcal{V}(J) \) of \( J \) in \( J_\omega \) such that for any \( J' \in \mathcal{V}(J) \), \( M_{(G,K)}^{\text{emb}}(J') \) is a compact zero dimensional manifold (and so is a finite set) and there is a canonical one-one correspondence

\[ M_{(G,K)}^{\text{emb}}(J) \to M_{(G,K)}^{\text{emb}}(J') \]

and the image loci of the corresponding embedded curves remain disjoint.

**Proof.** Combining Theorem 1.1, 1.2 and Lemma 11.1 we derive that \( M_{(G,K)}^{\text{emb}}(J) \) consists of a finite number of embedded curves which are disjoint and Fredholm regular with index 0. Let \( \{C_1, \ldots, C_m\} \) be the set of embedded \( J \)-holomorphic curves in \( M_{(G,K)}^{\text{emb}}(J) \). Since embeddedness, disjointness and Fredholm regularity all are open conditions, the corollary is an immediate consequence of the implicit function theorem. \( \square \)

Now we are ready to prove the main theorem in this section.

**Theorem 11.3.** For any \( (G, K; L) \in \mathbb{N}^3 \), \( J_{(G,K,L)}^{\text{rigid}} \) is open in \( J_\omega \).

**Proof.** Suppose \( J \in J_{(G,K,L)}^{\text{rigid}} \). Consider an arbitrary sequence \( J_i \to J \). We will prove that \( J_i \in J_{(G,K,L)}^{\text{rigid}} \) for all sufficiently large \( i \)'s, which will then prove that \( J_{(G,K,L)}^{\text{rigid}} \) is open.

Suppose to the contrary that there exists a sequence of \( u_i \in M_{(G,K)}^{\text{emb}}(J_i) \) and \( (j_i, \phi_i) \in U_{g+h,d}^\psi \) such that \( \text{Ker}(\Psi(j_i, \phi_i; J_i, C_i) \neq \{0\} \) for all \( i \)'s. By Lemma 11.1, \( u_i \) has a convergent subsequence. Let \( u_\infty \) be the limit of \( u_i \).

We first consider the case where the subsequence consists of embedded curves, i.e., \( \text{deg} \phi_i = 1 \). By the proof of Lemma 11.1 any such sequence converging in stable map topology must converge to a somewhere injective (and so embedded) \( J \)-holomorphic map. Therefore the limit \( u_\infty \) is an embedded \( J \)-holomorphic curve with \( g_{2,\infty} \leq G \) and \( \omega([u_\infty]) \leq K \). Then \( (G, K; L) \)-rigidity of \( J \) implies its normal
linearization should have trivial kernel. Since the dimension of the kernel of the $(D_u\partial_{(j,J)}(u))$ is upper semi-continuous in $u$ and hence $\dim \text{Ker}(D_u\partial_{(j,J)}(u)) = 0$ for sufficiently large $i$. But we have $D_u\partial_{(j,J)}(u))^\perp = \Psi(j, \phi_i; J, C_i)$ by Proposition 6.3 and so $\dim \text{Ker} \Psi(j, \phi_i; J, C_i) = 0$ gives rise to a contradiction to the standing hypothesis. This proves that the normal linearization of $u_i$ must have trivial kernel for sufficiently large $i$.

Now we consider the case of multiple covers $u_i \circ \phi_i$ i.e., with $\deg \phi_i \geq 2$. Since there are only finitely many $\beta$’s with $\omega(\beta) \leq K$ and $g+h \leq G$, we may assume, by taking a subsequence, $g_{i_0} = g$, $g_{i_0} = g+h$, $[u_i] = \beta$ and $\deg \phi_i = d \geq 2$ for all $i$. By the above paragraph, some subsequence of $u_i$ converges in $C^\infty$ to an embedded curve $u$. Since $\phi_i \in \mathcal{U}_{g+h,d}$, it also converges to $\phi \in \overline{\mathcal{U}}_{g+h,d}$. This implies that $u_i \circ \phi_i$ converges in $C^\infty$ to a smooth $u \circ \phi \in \mathcal{M}_{(G,K;L)}(J)$.

By the $(G,K;L)$-rigidity, we have $\text{Ker} \Psi(j, \phi; J, C) = \{0\}$ where $C = \text{Im} u$. Again by the upper semi-continuity of the dimension of the kernel of the $\text{Ker}(D_u\partial_{(j,J)}(u))$, we must have $\text{Ker}(D_u\partial_{(j,J)}(u \circ \phi_i))^\perp = \Psi(j, \phi_i; J_i, C_i) = \{0\}$ if $i$ is sufficiently large. This proves that $J_i$ is $(G,K;L)$-rigid for sufficiently large $i$.

Since this is true for any given sequence $J_i \to J$, we have proven openness of $\mathcal{J}^{\text{rigid}}_{(G,K;L)}$ in $\mathcal{J}$.

\[\square\]

12. DEFORMING NORMAL BUNDLE VIA $\delta J$ TO MAKE IT FAT

In this section, let $J \in \mathcal{J}^{\text{emb}}_\omega$ and let $(J, C)$ and $(G, K)$ be given. We consider the subset

$$\mathcal{J}_{(J,C)} \cap \mathcal{V}(J) \subset \mathcal{J}_\omega.$$ 

Motivated by Proposition 5.2 we need to consider evaluation of the one-form

$$\sigma \to \left\langle c(\sigma), R^{(0,1)}_{(j,\phi,\sigma;J,C)}\kappa(\sigma) \right\rangle : \Sigma \to T^*\Sigma$$

at a point $\sigma$ with $\phi(\sigma) \in C \setminus \text{br}(\phi)$. We define

$$H\alpha^0_{g+h,1}(C; d[C]) = \{(j, \phi, \sigma) | (j, \phi) \in H\alpha_{g+h}(C; d[C]), \sigma \in \phi^{-1}(C \setminus \text{br}(\phi))\}.$$ 

Similarly we define

$$M^0_{g+h,1}(C; d[C]) := H\alpha^0_{g+h,1}(C; d[C])/\sim.$$ 

It follows that $H\alpha^0_{g+h,1}(C; d[C])$ (respectively $M^0_{g+h,1}(C; d[C])$) is an open subset of $H\alpha_{g+h,1}(C; d[C])$ (respectively $M_{g+h,1}(C; d[C])$).

For given $(J, C)$, we regard the assignment

$$(j, \phi, J') \mapsto R^{(0,1)}_{(j,\phi,J';C)}$$

as a section of the bundle of finite rank

$$E \to H\alpha^0_{g+h,1}(C; d[C]) \times \mathcal{J}_{(J,C)}$$

whose fiber at $(j, \phi, J')$ is given by

$$E_{(j,\phi,\sigma;J')} := T^*\Sigma \otimes T^*_\phi(\sigma)C \otimes \Lambda_{(j,\phi,J';C)}^{(0,1)}(N_C|_{\sigma}).$$

We consider contraction of $R^{(0,1)}_{(j,\phi,\sigma;J',C)}$ against the pair $(\sigma, \kappa_{\sigma})$ at a marked point $\sigma$, which we denote by the bundle map over $\phi$

$$R_{(j,\phi,\sigma,J';C)}(\sigma, \kappa_{\sigma}) = \left\langle c_{\sigma}, R^{(0,1)}_{(j,\phi,J',C)}(\sigma)\kappa_{\sigma} \right\rangle,$$

(12.1)
\[
R_{(j,φ,j′,C)} : S(TΣ) ⊕ S(Λ^{(0,1)}(φ^*TC)) \longrightarrow T^*C
\]

Here we denote by \( S(·) \) the corresponding unit sphere bundle of \( · \).

**Remark 12.1.** Introduction of this kind of bundle of finite rank over the infinite dimensional base is somewhat reminiscent of similar bundles considered in [OZ], [O3] in the study of higher jet transversality. We would like to emphasize the map \( R_{(j,φ,j′,C)}(1) \) can be studied in the level of the bundle \( S^1(TΣ) \oplus S^1(Λ^{(0,1)}(φ^*TC)) \) in terms of the fiber element \((c_σ,κ_σ)\) at each \( σ \), while the pairing

\[
⟨c, (DφΨ)(j, φ; J,C)κ⟩
\]

involves the global section \( c, κ \). This local reduction is a crucial ingredient in the argument used in our proof of Proposition 12.2.

**Proposition 12.2.** For each given \((j, φ, σ)\) and non-zero pair 

\[(c_σ,κ_σ) ∈ Λ^{(0,1)}_{(J,i,(j,φ),C)}(Tφ(σ)C) × TσΣ,\]

the map

\[J′ ↦ R_{(j,φ,σ,j′,C)}(c_σ,κ_σ); J′(J,C) → T^*_σC\]

is transversal to \( φ_{T^*_C} \) as long as \( σ ∈ Σ \setminus br(φ) \).

**Proof.** We compute

\[δ_{J′} \left( (R_{(j,φ,σ,j′,C)})(c_σ,κ_σ) \right)\]

which is nothing but

\[⟨c_σ,δ_{J′}R_{(j,φ,σ,j′,C)}(σ)κ_σ⟩.\]

Then Proposition 9.1 is immediately translated into the transversality statement of the map \( R_{(j,φ,j′,C)} \).

\[\square\]

The following theorem will play an important role in our proof of density of \( J_{rigid}^{(G,K,L)} \).

**Theorem 12.3.** Let \((J, C)\) and \((j, φ)\) be given as above. And let \( V(J) ⊂ J_ω \) be a neighborhood of \( J \) and let a compact subset \( B ⊂ Σ \setminus br(φ) \) be given. Then there exists \( J′ ∈ V(J) \cap J(J,C) \) such that

\[⟨c_σ,R_{(j,φ,j′,C)}(σ)κ_σ⟩ \neq 0\]

for any non-zero pair \((κ_σ,c_σ) ∈ NC|σ × Λ^{(0,1)}_{(j,φ^*c)}(φ^*NC|σ)\) at any \( σ ∈ B \).

**Proof.** It is enough to consider \( κ_σ, c_σ \) with \(|κ_σ| = 1 = |c_σ|\). Denote by

\[S(φ^*NC|σ), S(Λ^{(0,1)}(φ^*NC|σ)) \cong S^{2n−1}\]

the corresponding unit sphere bundles.

For each given \( σ ∈ B \) and \((κ_σ,c_σ) ∈ S(φ^*NC|σ) × S(Λ^{(0,1)}(φ^*NC|σ))\), we can choose an open subset

\[N(κ_σ,c_σ) ⊂ S(φ^*NC) × S(Λ^{(0,1)}(φ^*NC))|B = \bigcup_{σ \in B} S(φ^*NC) × S(Λ^{(0,1)}(φ^*TC))\]

and \( J′ ∈ V(J) \cap J(J,C) \) and its open neighborhood \( V′ ⊂ V(J) \cap J(J,C) \) such that \( 12.3 \) holds for all \((κ′,c′) ∈ N(κ_σ,c_σ)\) and \( J'' \in V′ \). We now fix a countable dense
subset of \((\kappa_i, c_i) \in S^1(\phi^* N C) \times S^1(\Lambda^{(0,1)}(\phi^* N C))|_{B}\) and apply the above procedure to each point starting from \(i = 1\) and obtain a sequence of open subsets of the form
\[
N(\kappa_i, c_i) \subset S(\phi^* N C) \times S(\Lambda^{(0,1)}(\phi^* N C))|_{B}
\]
and \(J_{i+1} \in \mathcal{V}'_i\)
\[
\mathcal{V}'_i \subset \mathcal{V}(J) \cap \mathcal{J}(J,C)
\]
and \(\mathcal{V}'_{i+1}\) is an open neighborhood of \(J_i\) such that
\[
\mathcal{V}'_{i+1} \subset \mathcal{V}'_i
\]
(12.3) holds for \(N(\kappa_i, c_i)\) and \(\mathcal{V}'_i\). Since \(S(\phi^* N C) \times S(\Lambda^{(0,1)}(\phi^* N C))|_{B}\) is compact there exists \(N_0 \in \mathbb{N}\) such that
\[
\bigcup_{i=1}^{N_0}N(\kappa_i, c_i) = S(\phi^* N C) \times S(\Lambda^{(0,1)}(\phi^* N C))|_{B}.
\]
Then \(\mathcal{V}'_{N_0} \subset \mathcal{V}(J) \cap \mathcal{J}(J,C)\) is nonempty open and (12.3) holds for all \((\kappa', c') \in S(\phi^* N C) \times S(\Lambda^{(0,1)}(\phi^* N C))|_{B}\).

By setting \(\mathcal{V}' = \mathcal{V}'_{N_0}\) and choosing a \(J' \in \mathcal{V}'\), we have finished the proof. \(\square\)

13. \(\mathcal{J}^{rigid}_{(G,K;L)}\) is dense

In this section we prove \(\mathcal{J}^{rigid}_{(G,K;L)}\) is dense in \(\mathcal{J}_J\). To prove this density, we need to prove that for any \(J \in \mathcal{J}_J\) and a neighborhood \(\mathcal{V}(J)\) of \(J\), there exists \(J' \in \mathcal{J}^{rigid}_{(G,K;L)}\) and their neighborhoods \(\mathcal{V}(J)\). Since \(\mathcal{J}^{nodal}_{(G,K)}\) is dense in \(\mathcal{J}_J\), it will be enough to consider \(J'\)’s coming from \(\mathcal{J}^{nodal}_{(G,K)}\).

Let \(J \in \mathcal{J}^{nodal}_{(G,K)}\) and a neighborhood \(\mathcal{V}(J) \subset \mathcal{J}_J\) of \(J\) be given. By shrinking \(\mathcal{V}(J)\) if necessary, we may choose \(\mathcal{V}(J)\) so that Corollary 11.2 holds for all \(J' \in \mathcal{V}(J)\).

Therefore we will study \(C_1\)’s separately on the open subset \(\mathcal{V}(J) \cap \mathcal{J}(J,C)\) of the (Frechet) manifold \(\mathcal{J}(J,C)\). (This is a place where we need to use the finiteness of \(#(M^{emb}_{(G,K;L)}(J))\).) We will suppress the subindex \(i\) of \(C_1\) until the end of this section.

Then we consider a smooth section
\[
(\phi, J') \mapsto \Psi(j, \phi; J', C); \quad C^\infty(\Sigma, C) \times \mathcal{V}(J) \cap \mathcal{J}(J,C) \to \text{Fred}_C.
\]
We recall that the definition of the operator \(\Psi(j, \phi; J, C)\) requires the presence of the embedded \(J\)-holomorphic curve \(C\) and so we need Corollary 11.2 to be able to localize the study of deformations of \(C\) and \(\Psi(j, \phi; J, C)\).

We will study the bundle \(\text{Fred}_C \to C^\infty(\Sigma, C) \times \mathcal{J}(J,C)\) and its subbundle \(\text{Fred}_{C,k} \to C^\infty(\Sigma, C) \times \mathcal{J}(J,C)\). At each given point \((\phi, J') \in Hol_{g+h}(C; d[C]) \times \mathcal{J}(J,C)\), we need to study transversality of the map
\[
(\psi, J') \mapsto \Psi(j, \phi \circ \psi; J', C); \quad Diff(\Sigma) \times \mathcal{V}(J) \to \text{Fred}_C
\]
against \(\text{Fred}_{C,k}\) in \(\text{Fred}_C\) under the action by \(\psi \in Diff(\Sigma)\). Here we denote \(k = \dim \text{Ker} \Psi(j, \phi; J, C)\).

In general the dimensions of the kernel and of the cokernel may change as \((\phi, J')\) varies. But they are upper semi-continuous. By parallelly transporting the elements of
\[
\text{Ker} \Psi(j, \phi; J, C) \times \text{Coker} \Psi(j, \phi; J, C)
\]
to each point in a small neighborhood
\[ \mathcal{U}(\phi) \times \mathcal{V}^\prime(J) \subset C^\infty(\Sigma, C) \times \mathcal{V}(J) \cap \mathcal{J}(J, C), \]
we define trivial bundles
\[ K, C \to \mathcal{U}(\phi) \times \mathcal{V}^\prime(J) \]
whose fibers at \((\phi', J')\) are given by
\[ K_{(\phi', J')} = \Pi^\phi_{\phi'}(\Ker \Psi(j, \phi; J, C)), \quad C_{(\phi', J')} = \Pi^\phi_{\phi'}(\Coker \Psi(j, \phi; J, C)) \]
(13.1)
respectively, where \(\Pi^\phi_{\phi'}\) is the parallel transport between \(\phi^*NC\) and \((\phi')^*NC\) along the short geodesics with respect to any given metric (e.g., the almost Kähler metric \(\phi\)). Thus \(\Pi^\phi_{\phi'}\) is the parallel transport between \(\phi(\sigma)\) to \(\phi'\) at each point \(\sigma \in \Sigma\). We would like to emphasize that the fiber is not chosen to be \(\Ker \Psi(j, \phi'; J', C) \times \Coker \Psi(j, \phi'; J', C)\) whose rank may vary even in a small neighborhood of the given \((\phi, J)\).

Since the parallel transport depends only on \(\phi\)'s, we have the identity
\[ K_{(\phi, J)} = K_{(\phi, J)} \subset \Gamma(\phi^*NC) \]
and similar identity \(C_{(\phi, J)} = C_{(\phi, J)}\) holds for \(C\). Denote by \(S(K)\) and \(S(C)\) the unit-sphere bundles of \(K\) and \(C\) respectively.

We denote the pull-back of \(K \times C\) to \(\mathcal{U}(\phi) \times \mathcal{V}^\prime(J)\) by the same notation \(K \times C\). Here \(\mathcal{U}(\phi)\) is the preimage of \(\mathcal{U}(\phi)\) under the forgetful map
\[ \mathcal{F}_{g+h,1}(\Sigma; d[C]) \rightarrow \mathcal{F}_{g+h}(C; d[C]). \]
We denote
\[ \mathcal{U}^{hol}(\phi) = \mathcal{U}(\phi) \cap \text{Hol}_{g+h}(C; d[C]) \]
\[ \mathcal{U}^{hol,1}(\phi) = \mathcal{U}(\phi) \cap \text{Hol}_{g+h+1}(\Sigma; d[C]) \]
\[ \mathcal{U}^{hol,0}(\phi) = \mathcal{U}(\phi) \cap \text{Hol}_{g+h+1}^0(\Sigma; d[C]). \]
We fix a sequence of distinct points
\[ B = \{\sigma_j\}_{j=1}^\infty \subset \Sigma \]
such that \(\sigma_j \to \sigma_\infty\) as \(j \to \infty\). We denote
\[ B_N = \{\sigma_1, \ldots, \sigma_N\} \subset B, \quad N \geq -i + 1 := N_0 \]
with \(i = 2(2d(g - 1) + 2(1 - g - h))\). Recall \(r = -i\) is the number (counted with multiplicity) of branch points of \(\phi \in \text{Hol}_{g+h}(C; d[C])\).

Let \(\phi \in \text{Hol}_{g+h}(C; d[C])\) satisfy \(B_0 = B_{N_0} \subset \Sigma \setminus \text{br}(\phi)\) for \(I_0 = \{1, \ldots, N_0\} \subset \mathbb{N}\). For any subset \(B'\) with \(|B'| = N_0\) of \(B\) with \(B' \subset \Sigma \setminus \text{br}(\phi)\) we define an open subset of \(S(K(\phi, J)) \times S(C(\phi, J))\) by
\[ V_{B_0; B'}^{(\phi, J)} = \{(c, \kappa) \in K(\phi, J) \times C(\phi, J) | \exists 1 \leq j \leq N_0, \kappa(\sigma_{i_j}) \neq 0, c(\sigma_{i_j}), \sigma_{i_j} \in B'\}. \]
(13.2)

The following lemma is a useful lemma, which enables us to apply Theorem 12.3.

**Lemma 13.1.** Let \((\phi, J)\) be given and let \(\mathcal{V}(J)\) be an open neighborhood of \(J\) in \(\mathcal{J}_\omega\). Let \((\kappa, c) \in K(\phi, J) \times C(\phi, J)\) be a non-zero pair. Then there exists a sufficiently large integer \(N_1(\phi, J) > 0\) and a collection
\[ I = \{i_1, \ldots, i_{N_0}\} \subset \mathbb{N} \]
with \( i_{N_0} \leq N_1 \) such that

\[ B_I = \{ \sigma_{i_1}, \ldots, \sigma_{i_{N_0}} \} \subset B_{N_1} \subset C \setminus \text{br}(\phi) \tag{13.3} \]

and \((\kappa, c) \in V_{B_0; B_1}'\).

Proof. We recall by the unique continuation that the zero sets of \( \kappa \neq 0 \) or \( c \neq 0 \) cannot have an accumulation point in \( \Sigma \) and so contain only a finite number of elements among the given sequence \( B \). Therefore for any given \((c, \kappa)\) with \( \kappa \neq 0 \), \( c \neq 0 \) we can find a sufficiently large integer \( N_1 = N_1(\phi, J; \kappa, c) \) such that there are at least \( N_0 \) elements \( I_0 = \{ \sigma_{i_1}, \ldots, \sigma_{i_{N_0}} \} \) out of \( B_{N_1} \) and \( \sigma_{i_j} \) such that

\[ \sigma_{i_j} \in C \setminus \text{br}(\phi) \quad \text{for some } 1 \leq j \leq N_0 \]

and

\[ c(\sigma_{i_j}) \neq 0, \kappa(\sigma_{i_j}) \neq 0. \]

This finishes the proof. \( \square \)

The following proposition is an important one

**Proposition 13.2.** There exists \( J' \in \mathcal{V}(J) \cap \mathcal{J}(J,C) \) and a nonempty open neighborhood \( \mathcal{V}'(J; \phi) \subset \mathcal{V}(J) \cap \mathcal{J}(J,C) \) of \( J' \) and an open neighborhood \( \mathcal{U}'(\phi) \subset \mathcal{U}(\phi) \) in \( C^\infty(\Sigma, C) \) of \( \phi \in \text{Hol}_{\phi h}(C; d[C]) \) such that

\[ \left\langle c, R_{J'; J''}(\cdot) \kappa \right\rangle \neq 0 \tag{13.4} \]

for all \((\kappa, c) \in K_{(\phi', J')} \times C_{(\phi', J')}, J'' \in \mathcal{V}'(J; \phi) \).

Proof. It is enough to consider the pairs \((\kappa, c)\) lying in the product

\[ S(\mathcal{K}(\phi, J')) \times S(\mathcal{C}(\phi, J')) = S(\mathcal{K}(\phi, J)) \times S(\mathcal{C}(\phi, J)) \]

of \( L^2 \)-unit spheres.

First we fix \( \phi \). Let \((\kappa, c) \in \mathcal{S}(\mathcal{K}(\phi, J)) \times \mathcal{S}(\mathcal{C}(\phi, J)) \). If \((\phi, J)\) already holds for \((\phi', J')\), existence of such neighborhoods is an immediate consequence of continuity. Otherwise we apply Lemma 13.1 to find a compact subset \( B \subset \Sigma \setminus \text{br}(\phi) \) such that there is at least one point \( \sigma \in B \) such that \( \kappa(\sigma) \neq 0 \neq c(\sigma) \). Then we apply Theorem \( 12.3 \) to find \( J' \) so that \( 12.3 \) holds at the point \( \sigma \). Then by continuity \( 12.3 \) will continue to hold on an open neighborhood \( U(\sigma) \subset \Sigma \). Then again by continuity, we can find open neighborhoods

\[ A(\kappa, c) \subset \mathcal{S}(\mathcal{K}(\phi, J)) \times \mathcal{S}(\mathcal{C}(\phi, J)) \]

of \((\kappa, c)\) and \( \mathcal{V}' \) of \( J' \) such that \( 13.4 \) holds for all \((\kappa', c') \in A(\kappa, c) \) and \( J'' \in \mathcal{V}' \).

We have only to choose \( V_{B_0; B_1}' \) for \( A(\kappa, c) \) provided in Lemma 13.1 and then by continuity we can choose an open neighborhood of \( J' \).

We now fix a countable dense subset of \((\kappa_i, c_i) \in \mathcal{S}(\mathcal{K}(\phi, J)) \times \mathcal{S}(\mathcal{C}(\phi, J)) \) and apply the arguments in the above paragraph and Lemma 13.1 to each point \((\kappa_i, c_i)\) with \( J, \mathcal{V}(J) \) replaced by \( J_i, \mathcal{V}'_i \) starting from \( i = 1 \) and obtain a sequence of open subsets of the form

\[ A(\kappa_i, c_i) \subset \mathcal{S}(\mathcal{K}(\phi, J)) \times \mathcal{S}(\mathcal{C}(\phi, J)), \quad \mathcal{V}'_i \subset \mathcal{V}(J) \cap \mathcal{J}(J_i, C) \]

such that \( \mathcal{V}'_{i+1} \) is a neighborhood of \( J_i \),

\[ \mathcal{V}'_{i+1} \subset \mathcal{V}'_i \]
and (13.4) holds for all \((\kappa, c) \in A(\kappa_i, c_i)\) and \(J' \in V'_i\). Since \(S(K(\phi, J)) \times S(C(\phi, J))\) is compact there exists \(N_0 \in \mathbb{N}\) such that
\[
\bigcup_{i=1}^{N_0} A(\kappa_i, c_i) = S(K(\phi, J)) \times S(C(\phi, J)).
\]
Then \(V'_{N_0} \subset V(\mathcal{J}) \cap \mathcal{J}(J, C)\) is nonempty open and (13.4) holds for all \((\kappa, c) \in S(K(\phi, J)) \times S(C(\phi, J))\).

Now we vary \(\phi'\) in \(U^{hol}(\phi)\). Since (12.3) (and so (13.4)) is an open condition in \(C^\infty\) and \(S(N_C), S(A_{(0,1)}^1(N_C))\) are compact, we can find an open neighborhood \(W(\phi, J) \subset U(\phi) \times (\mathcal{V}(\mathcal{J}) \cap \mathcal{J}(J, C))\) on which (13.4) continues to hold for \((\phi', J')\) \(\in W(\phi, J)\). Then by the local compactness \(\text{Hol}_{g+1}(C;d[C])\) we can find an open subset \(W' \subset V(\mathcal{J}) \cap \mathcal{J}(J, C)\) and an open neighborhood \(U'(\phi) \subset U(\phi)\) such that
\[
U(\phi) \times W' \subset W(\phi, J)
\]
and hence (13.4) holds thereon. By setting \(V'(\mathcal{J}; \phi) = V'\), we have finished the proof. \(\square\)

By shrinking \(V'(\mathcal{J}; \phi)\) and \(U'(\phi)\) if necessary, we get the following as an easy corollary of the upper semi-continuity of the dimension of kernel. We would like to point out the difference between the content of Proposition 13.2 and that of the following corollary: The product \(K(\phi, J') \times C(\phi', J')\) in the above lemma, which is the parallel transport of \(\Psi(j, \phi, J, C) \times \text{Coker } \Psi(j, \phi, J, C)\) and so \(\text{whose dimension is fixed}\), is replaced by \(\text{Ker } \Psi(j, \phi', J, C) \times \text{Coker } \Psi(j, \phi', J, C)\), whose dimension may vary depending on \((\phi', J')\).

**Corollary 13.3.** Let \(J' \in V'(\mathcal{J}; \phi)\) and \(U'(\phi)\) be the ones chosen in Proposition 13.2 for \(\phi \in \text{Hol}_{g+h}(C;d[C])\). Shrink \(U'(\phi)\) if necessary. Then as a one-form on \(\Sigma\)
\[
\left\langle c, R_{(j, \phi', J'; C)}^{(0,1)}(\cdot)K_\kappa \right\rangle \neq 0 \tag{13.5}
\]
for all \((\kappa, c) \in \text{Ker } \Psi(j, \phi', J', C) \times \text{Coker } \Psi(j, \phi', J', C)\), \(\phi' \in U'(\phi)\).

**Proof.** Suppose to the contrary that there exists a point \(\phi_i \to \phi\) and \((\kappa_i, c_i) \in S(\text{Ker } \Psi(j, \phi_i, J', C)) \times S(\text{Coker } \Psi(j, \phi_i, J', C))\) such that
\[
\left\langle c_i, R_{(j, \phi_i, J', C)}^{(0,1)}(\cdot)K_{\kappa_i} \right\rangle = 0
\]
for all \(i\). By compactness of the set \(S(K(\phi, J)) \times S(C(\phi, J))\) and upper semi-continuity of kernel, we obtain a subsequence of \((\kappa_i, c_i)\) converging to an element \((\kappa_\infty, c_\infty) \in S(K(\phi, J')) \times S(C(\phi, J'))\) such that
\[
\left\langle c_\infty, R_{(j, \phi, J', C)}^{(0,1)}(\cdot)K_{\kappa_\infty} \right\rangle = 0
\]
which contradicts to (13.4) at \((\phi', J') = (\phi, J')\). This finishes the proof. \(\square\)

In fact the proof of Proposition 13.2 proves the following explicit rigidity criterion of a given \((j, \phi, J, C)\). It states that the curvature, or more precisely its \((0,1)\)-component, of the normal bundle is ‘fat’ at an infinite number of points in \(\Sigma\), then \((j, \phi, J, C)\) is rigid.

**Theorem 13.4.** Suppose \(R_{(j, \phi, J, C)}^{(0,1)}(d_\sigma \phi(\cdot))\) satisfies
\[
\left\langle c_\sigma, R_{(j, \phi, \sigma, J, C)}^{(0,1)}(\cdot)K_{\sigma} \right\rangle \neq 0 \tag{13.6}
\]
for all \((\kappa, c) \in S^1(\gamma^* N C|_\sigma) \times S^1(A^{(0,1)}_{J,\psi} (\gamma^* N C|_\sigma))\) for a subset \(B \subset \Sigma\) of infinite cardinality. Then

\[
\ker \Psi(j, \phi; J, C) = \{0\}.
\]

**Proof.** By compactness of \(\Sigma\), any infinite subset contains a convergent sequence in it. Denote this convergent sequence by \(B\). Then the arguments used in the proof of Proposition \[13.2\] and Corollary \[13.3\] proves \([13.9]\) holds for all non-zero pair \((\kappa, c) \in \ker \Psi(j, \phi; J, C) \neq \{0\} \times \coker \Psi(j, \phi; J, C) \neq \{0\}\). Therefore the orbit of \(\text{Diff}(\Sigma)\) of \(\Psi(j, \phi; J, C)\) in \(\text{Fred}_C\) is transversal to \(\text{Fred}_{C,k}\) by Proposition \[8.2\]. But then Corollary \[8.4\] implies the theorem. \(\Box\)

Combining the above discussion, Corollary \[8.4\] and \[13.3\] we obtain

**Proposition 13.5.** There exists a nonempty Baire set \(\mathcal{V}'(J; \phi) \subset \mathcal{V}(J) \cap \mathcal{J}(J, C)\) such that

\[
\ker \Psi(j, \phi; J, C) = \{0\}
\]

for all \(\phi \in U_{g+h,d}^L\) for some \(J' \in \mathcal{V}'(J; \phi)\).

**Proof.** As in the proof of Proposition \[13.2\] we fix a countable dense subset \(\phi_i \in U_{g+h}^L\) and apply the above procedure to each point of \(\phi_i \in U_{g+h}^L\) and obtain a family of open subsets of the form

\[
\mathcal{U}(\phi_i) \times \mathcal{V}(\phi_i; J_i), \quad \phi_i \in \text{Hol}_{g+h}(C; \mathcal{I}(C))
\]

such that

\[
\left\langle c, R_{(J, \phi', J', C)}^{(0,1)}(\cdot) \kappa \right\rangle \neq 0
\]

for all

\[
(\phi', J') \in \mathcal{U}(\phi_i) \times \mathcal{V}(\phi_i; J_i)
\]

with \(J_i \in \mathcal{J}(G, K) \cap \mathcal{J}(J, C), \mathcal{V}(\phi_{i+1}; J_{i+1}) \subset \mathcal{V}(J) \cap \mathcal{J}(J, C)\) and

\[
\mathcal{V}(\phi_{i+1}; J_{i+1}) \subset \mathcal{V}(\phi_i; J_i).
\]

Since \(U_{g+h}^L\) is compact there exists \(N_1 \in \mathbb{N}\) such that

\[
\bigcup_{i=1}^{N_1} \mathcal{U}(\phi_i) \supset U_{g+h,d}^L.
\]

Therefore as a one-form on \(\Sigma\)

\[
\left\langle c, R_{(J, \phi', J', C)}^{(0,1)}(\cdot) \kappa \right\rangle \neq 0
\]

for all

\[
(\phi, J') \in U_{g+h,d}^L \times \mathcal{V}(\phi_{N_1}; J_{N_1})
\]

with \(\mathcal{V}(\phi_{N_1}; J_{N_1}) \subset \mathcal{V}(J) \cap \mathcal{J}(J, C)\). Then Corollary \[8.4\] implies the proposition for \(\mathcal{V}'(\phi; J) := \mathcal{V}(\phi_{N_1}; J_{N_1})\). This finishes the proof. \(\Box\)

We would like to emphasize that the open set \(\mathcal{V}'\) may not contain the initially given \(J\) although we can find some Baire subset of the given neighborhood \(\mathcal{V}(J) \cap \mathcal{J}(J, C)\) of \(J\) in \(\mathcal{J}(J, C)\). This is the subtle point in finding a \(J'\) that provides the super-rigidity for the given curve \(C\).

Now we are ready to wrap up the proof of the following main theorem of this section.
Theorem 13.6. For any given $J \in \mathcal{J}_\omega^{\text{nodal}}$ and a neighborhood $\mathcal{V}(J)$ in $\mathcal{J}_\omega$, there exists $J' \in \mathcal{V}(J)$ such that

$$\text{Ker } \Psi^1(j, \phi; J', C) = \{0\}$$

for all $(j, \phi) \in \mathcal{T}_{g+h,d}^r$ with $g+h \leq G$, $\omega(d[C]) \leq K$. In other words $J'$ is $(G, K; L)$-rigid. Therefore $J' \in \mathcal{J}_\omega^{\text{rigid}}$ is dense in $\mathcal{J}_\omega$.

Proof. Since $J \in \mathcal{J}_\omega^{\text{nodal}}$, Lemma 13.1 implies that $\mathcal{M}_{G,K,L}^{\text{emb}}(J)$ is compact and all elements therein are Fredholm regular. Therefore $\mathcal{M}_{G,K,L}^{\text{emb}}(J)$ is a compact zero dimensional manifold and hence comes the finiteness of $\#(\mathcal{M}_{G,K,L}^{\text{emb}}(J))$.

Denote $m = \#(\mathcal{M}_{G,K,L}^{\text{emb}}(J))$. Furthermore the images of $\mathcal{M}_{G,K,L}^{\text{emb}}(J)$ are disjoint from one another by Theorem 13.6. Therefore when we apply the proofs and the semi-continuity arguments used in the previous section to each element $C \in \mathcal{M}_{G,K,L}^{\text{emb}}(J)$, we can localize the perturbation obtained by Theorem 13.6 near each $B_C$, so that $J' \equiv J$ outside the union $B = \cup_{i=1}^m B_C$. This enables us to choose a single perturbation $J'$ of $J$ lying in

$$\mathcal{V}(J) \cap \mathcal{J}(J,C)$$

for which Proposition 13.2 holds uniformly over all $C \in \mathcal{M}_{G,K,L}^{\text{emb}}(J)$.

This finishes the proof. \hfill \square

We now take

$$\mathcal{J}^{\text{rigid}}_\omega = \bigcap_{(G,K) \in \mathbb{N}^2} \bigcap_{L \in \mathbb{N}} \mathcal{J}^{\text{rigid}}_{(G,K,L)}.$$  

Since each $\mathcal{J}^{\text{rigid}}_{(G,K,L)}$ is open and dense and $\mathbb{N}^3$ are countable, $\mathcal{J}^{\text{rigid}}_\omega$ is a Baire set of $\mathcal{J}_\omega$.

Finally we prove the following corollary of Theorem 13.6 concerning the super-rigidity.

Theorem 13.7. Let $n = 3$ and $c_1 = 0$ and $\Sigma$ is smooth. For $J \in \mathcal{J}_\omega^{\text{rigid}}$, any somewhere injective $J$-holomorphic map $u : (\Sigma, j) \to (M, J)$ is (embedded and) super-rigid. And for such $J$, there are only a finite number of elements in $\mathcal{M}_{g}^{\text{emb}}(M, J; \beta)$ for all $(g, \beta)$.

Proof. Let $J \in \mathcal{J}_\omega^{\text{rigid}}$. Denote by $C$ the image locus of $u$. Consider any holomorphic map $\phi : \Sigma \to C$ with $g_C = g + h$ and $\deg \phi = d$ for any $g \geq 0$, $d \geq 1$. Then the composition map $\tilde{u} = u \circ \phi$ is a $J$-holomorphic map with genus $g + h$ and $\omega(\tilde{u}) = d(u)$. Set $G = g + h$, $K = d(\tilde{u})$. Furthermore $\phi \in \mathcal{T}_{g+h,d}$ for some $L \in \mathbb{N}$. Then $\tilde{J} \in \mathcal{J}_{G,K,L}$.

By the $(G, K; L)$-rigidity of $J$, we have $\text{Ker } \Psi^1(j, \phi; J', C) = \{0\}$. Therefore $u$ is $(d, h)$-rigid.

Since this is true for all $d \geq 1$ and $h \geq 0$, $u$ is super-rigid.

Then the finiteness of the number of elements in $\mathcal{M}_{g}^{\text{emb}}(M, J; \beta)$ follows by Corollary 13.2 and the Fredholm regularity of $\mathcal{M}_{g}^{\text{emb}}(M, J; \beta)$ as before. This finishes the proof. \hfill \square

An examination of the above proof shows that the following local perturbation result, which holds for any dimension, is the crucial ingredient in the proof of density result of $\mathcal{J}_\omega^{\text{rigid}}$ and in turn in the proofs of super-rigidity and finiteness results.
Theorem 13.8. Let $C = \text{Im} u_0$ for an embedded $J$-holomorphic curve $u_0 : \Sigma_0 \to M$ with $c_1(M)([C]) = 0$, and let $V(J)$ be a given neighborhood of $J$ in $\mathcal{J}_\omega$. Suppose that $u_0$ is Fredholm-regular. Then there exists $J' \in V(J) \cap \mathcal{J}_{(J,C)} C$ is super-rigid with respect to $J'$.

14. Embedded count and multiple cover contribution

The finiteness theorem Theorem 13.7 allows us to define an integer counting of embedded $J$-holomorphic curves for $J \in \mathcal{J}_{\omega}^{\text{rigid}}$. We first recall

$$L(J, C) := D(\mathcal{F}(J,C)) : \Omega^0(\mathcal{C} \cap \mathcal{J}_h) \to \Omega^{(0,1)}(\mathcal{C} \cap \mathcal{J}_h)$$

has the form $L = \mathcal{D} + (\mathcal{D}_\gamma)^{(0,1)}$ where $(\mathcal{D}_\gamma)^{(0,1)}$ is a real operator in general. By considering the path from $L$ to $\mathcal{D}$ defined by

$$t \in [0,1] \mapsto \mathcal{D} + t(\mathcal{D}_\gamma)^{(0,1)}$$

and the corresponding determinant bundle of $\mathcal{D}$ over the path we can associate a sign to each element $C \in \mathcal{M}_g^\text{emb}(M, J; \beta)$.

$$\varepsilon(L, J_0) \in \{-1, +1\}$$

depending on the parity of eigenvalue crossing number.

Definition 14.1 (Embedded count). For any $J \in \mathcal{J}_{\omega}^{\text{rigid}}$, we define

$$\bar{n}_g^\beta(J) = \sum_{C \in \mathcal{M}_g^\text{rig}(J, \beta)} \varepsilon(L(J, C))$$

and call it the embedded counting of $J$-curves in class $\beta$ and of genus $g$.

At this point, we do not know whether this number depends on $J$ or not.

Question 14.2. Let $J_1, J_2 \in \mathcal{J}_{\omega}^{\text{rigid}}$. Do we have

$$\bar{n}_g^\beta(J_1) = \bar{n}_g^\beta(J_2)$$
or do we have the wall crossing phenomenon?

This is a subject of future study, which we hope to come back elsewhere in the future.

Next we examine contributions of multiple covers. We recall that $\overline{M}_{g+h}(C; d[C])$ is the Deligne-Mumford stack [FP]. We point out that Theorem 13.7 implies that

$$\text{Coker } \Psi(j, J, C) \mapsto M_{g+h}(C; d[C])$$
defines a genuine real vector bundle over smooth moduli space $M_{g+h}(C; d[C]) \subset \overline{M}_{g+h}(C; d[C])$ with its rank the same as the dimension of $\overline{M}_{g+h}(C; d[C])$. After a suitable stabilization of this bundle, if necessary, we can extend the bundle coker $\Psi(j, \phi; J, C) \mapsto M_{g+h}(C; d[C])$ to the full $\overline{M}_{g+h}(C; d[C])$ as a virtual obstruction bundle of a Kuranishi neighborhood [FO], [CT] of $\overline{M}_{g+h}(M, J; \beta)$ at each stable map $\phi \in \overline{M}_{g+h}(C; d[C]) \subset \overline{M}_{g+h}(M, J; \beta)$.

We denote the virtual Euler class of the bundle [FO], [CT] by $E(\Psi(j, \phi; J, C))$ and its evaluation over the virtual fundamental cycle $\overline{M}_{g+h}(C; d[C])^{\text{virt}}$ by the rational number

$$e_g(h, d; (J, C)) := \int_{\overline{M}_{g+h}(C; d[C])^{\text{virt}}} E(\Psi(\cdot; J, C)).$$

(14.2)
Then we have the following structure theorem of Gromov-Witten invariant $N^g_\beta(M)$ defined by Ruan and Tian [RT].

**Theorem 14.3.** Let $J \in \mathcal{J}_w^{rigid}$. For any $\beta \in H_2(M, \mathbb{Z})$ and $g \in \mathbb{Z}$, we have

$$N^g_\beta(M) = \sum_{h=0}^g \sum_{\beta = d\gamma, \gamma \geq 1} \left( \sum_{C \in M^{vir}_{g,h}(J,M;\gamma)} e_{g-h}(h, d; (J,C)) \right).$$

(14.3)

The above definition (14.2) of $e_g(h, d; (J,C))$ is the real analog to the local Gromov-Witten invariants defined by Pandharipande [P] and Bryan-Pandharipande [BP1]

$$C_g(h, d; C \subset M) = \int_{[\mathcal{M}_{g+h}(C,d[C])]^{vir}} c(R^1\pi_*\phi^*(N_C)).$$

**Question 14.4.** Suppose $J \in \mathcal{J}_w^{rigid} \cap \mathcal{J}_{(J,C)}$. Is the number $e_g(h, d; (J,C))$ independent of $(J, C)$ but does it depend only on $g$, $h$ and $d$?

It seems to the authors that this question is tied to Question 14.2 and also related to the recent study of wall-crossing phenomenon of Donaldson-Thomas invariants [Kon., KS].

The definition (14.2) of $e_g(h, d; J, C)$ is the real analog to the local Gromov-Witten invariants defined by Pandharipande [P] and Bryan-Pandharipande [BP1]

$$C_g(h, d; C \subset M) = \int_{[\mathcal{M}_{g+h}(C,d[C])]^{vir}} c(R^1\pi_*\phi^*(N_C)).$$

Under the assumption $H^0(\Sigma, \phi^*N_C) = 0$, this local contribution $C_g(h, d; C \subset M)$ becomes

$$C_g(h, d; C \subset M) := \int_{[\mathcal{M}_{g+h}(C,d[C])]^{vir}} c(-R\pi_*f^*(N_C))$$

(14.4)

where $-R\pi_*f^*(N_C)$ is a $K$-theory element of $\mathcal{M}_{g+h}(C,d[C])$

A priori $C_{g-h}(h, d; C \subset M)$ depends on the embedding $C \subset M$ or on the normal bundle $N_C$. Bryan and Pandharipande [BP1] however showed that $N_C$ can be deformed to $\mathcal{O}_C \oplus \omega_C$ where $\omega_C$ is the canonical bundle of $C$. They denote the idealized local contribution by

$$C_g(h, d) = \int_{[\mathcal{M}_{g+h}(C,d[C])]} c(-R\pi_*f^*(\mathcal{O} \oplus \omega_C))$$

which depends only on $g$, $d$ and $h$. It would be interesting to see what the relationship between $e_g(h, d; (J,C))$ and $C_g(h, d; C \subset M)$ in general.

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