A computable measure of nonclassicality for light

János K. Asbóth, John Calsamiglia, Helmut Ritsch

1 Institute of Theoretical Physics, University of Innsbruck, Technikerstrasse 25, A-6020 Innsbruck, Austria
2 Research Institute of Solid State Physics and Optics, Hungarian Academy of Sciences, H-1525 Budapest P.O. Box 49, Hungary

(Dated: October 31, 2018)

We propose the entanglement potential (EP) as a measure of nonclassicality for quantum states of a single-mode electromagnetic field. It is the amount of two-mode entanglement that can be generated from the field using linear optics, auxiliary classical states and ideal photodetectors. The EP detects nonclassicality, has a direct physical interpretation, and can be computed efficiently. These three properties together make it stand out from previously proposed nonclassicality measures. We derive closed expressions for the EP of important classes of states and analyze as an example the degradation of nonclassicality in lossy channels.

PACS numbers: 42.50.Dv, 03.67.Mn, 42.50.-p

Quantum mechanics, and in particular quantum electrodynamics, enjoys impeccable internal consistency and shows unmatched agreement with experimental observations. Something remaining unresolved is determining its borderline to the classical world, where quantum rules are not observed. A first step towards understanding the quantum to classical transition is to find a description of classical states within quantum theory. One can then ask how much non-classicality any given state possesses.

Coherent states $|\alpha\rangle$ are generally accepted to be the most classical states of optical fields. They reflect the wave-like nature of classical light fields, are generated by classical currents, and they are pointer states in realistic decohering environments. Here we will also adopt this notion and call a state classical if it is a coherent state, or a mixture thereof (see however $|\alpha\rangle$). All other quantum states are referred to as nonclassical.

With recent progress in quantum optics it is becoming possible to experimentally create an increasing wealth of light fields with properties deviating more and more from coherent states $|\alpha\rangle$. Such nonclassical states can be distinguished from classical ones by many different features. Examples are sub-Poissonian photon statistics, squeezing, photon number oscillations or negative values of the Wigner function. Most of these signatures can be quantified defining degrees of nonclassicality. This could quantify a resource for many applications. The leading role of quantum optics in the study of the foundations of quantum mechanics [15] and in the implementation of many of the QIT protocols has triggered in the recent years a lot of interest in the characterization, generation, and distillation of entanglement between optical fields [16 17].
This provides the EP with a direct physical meaning: single-mode nonclassicality as an entanglement resource. Moreover, it supplies a pool of results and methods from QIT to be used in the study of classicality.

At first glance, the computation of EP seems prohibitively complicated. For any nonclassical state $\sigma$ one has to find the optimal linear optical transformation and auxiliary states to create the most two-mode entanglement. However, as we show below, the optimal linear optics entangler is the same for any state, and consist of a single beamsplitter (BS) and an additional vacuum input.

A representation of the transformations that we allow in the definition of the EP is shown in Fig. 1 (top). A passive linear optical transformation can be modeled by a circuit of BS’s (including phase-shifters) $\sigma_{a}$. This transforms the input state and auxiliary states according to a linear unitary map $a' = \sum_{i}U_{ij}a_{j}$ between input and output mode annihilation operators. At the output two modes are sent to A(llice) and B(ob), all other modes are measured by ideal photodetectors. Note that nonlinear optical devices and also linear operations conditioned on a measurement outcome are excluded $\sigma_{a}$. Although such linear feedback schemes cannot create nonclassicality — they preserve positivity of P-distributions—, we explicitly disallow them since they can increase the amount of existing nonclassicality.

We first show that the auxiliary modes can be chosen to be in the vacuum state. The displacement of one ancilla mode, i.e. $D(\alpha)|0\rangle = |\alpha\rangle$ with $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$, amounts to (local) displacement of all output modes by amounts depending on the circuit of BS’s. Mixing the displaced input modes translates to local mixing of the output modes with additional classical communication. As these operations cannot increase entanglement, vacuum ancillas are optimal for entangling modes A and B.

Now, if the ancillas are vacuum, the circuit of BS’s inside the box can be simplified to a standard form. As shown in Fig. 1 (bottom), this consists of a single BS that splits the input mode in two modes corresponding to Alice and Bob, and then a series of additional BS’s further splits the signal into various modes in Bob’s side. All the measurements can be carried out in Bob’s auxiliary modes. Local operations cannot increase entanglement, and hence Bob can expect no advantage from splitting off and measuring a part of the beam. The optimal entangling device is therefore a single BS. Although we currently lack of a general proof, all examples that we checked analytically and numerically indicate that the transmissivity of the optimal BS is 1/2 independent of the input state. We will denote by $U_{BS}$ the 50:50 BS transformation, which induces the mapping $a = 2^{-\frac{1}{2}}(a_{A} + a_{B})$ on the input mode annihilation operator.

Clearly, the EP is zero for classical states. Moreover, any decomposition of a given output two-mode mixed state in terms of pure states $\rho = \sum_{i}p_{i}|\Psi_{i}\rangle\langle\Psi_{i}|$ must be consistent term-by-term with a pure input $|\psi_{i}\rangle|0\rangle = U_{BS}^{-1}|\Psi_{i}\rangle$: this is necessary for the corresponding input mixed state to have a vacuum auxiliary mode. In particular, we find that any separable output state must correspond to a convex combination of input states $|\alpha_{i}\rangle|0\rangle$, i.e. to a classical input state. In other words, all nonclassical input states, pure or mixed, will generate entanglement. This can be seen as an extension of the results of [20] that nonclassicality at the input is a prerequisite to entanglement at the output of a BS. Since coherent displacement and phase shifting can be realized on a single BS with an additional strong coherent beam, the EP is invariant with respect to “classical” operations, as defined in [8].

To obtain a specific measure of nonclassicality an entanglement measure has to be chosen. The value of EP of a state depends on this measure, and different choices may give rise to different orderings between states. Here we consider two alternatives.

A computable measure of nonclassicality for pure as well as for mixed single-mode states is obtained by taking the logarithmic negativity $E_{N}$ $[21]$ leading to the following definition for the Entanglement Potential,

$$EP(\sigma) \equiv E_{N}(\rho_{\sigma}) = \log_{2}||\rho_{\sigma}^{T_{A}}||_{1}.\quad(1)$$

Here $\rho_{\sigma} = U_{BS}(\sigma \otimes |0\rangle\langle 0|)U_{BS}^{\dagger}$, $\rho^{T_{A}}$ denotes the partial transpose of $\rho$, and $|| \cdot ||_{1}$ is the trace norm. A
nonzero value of $E_N$ reveals that the state is nonseparable, however, the converse is not true in general [21]. The so-called bound entangled states are not detected by the partial transposition criterion. Although examples of such entangled states exists for two-mode non-Gaussian states [22], it remains an open question whether bound entanglement can arise in our setup, i.e. whether or not $EP$ detects all nonclassical states.

An alternative entanglement measure that does detect all entangled states is the relative entropy of entanglement $E_R$. We will call the induced nonclassicality measure $Entropic Entanglement Potential$ ($EEP$), defined as

$$EEP(\sigma) \equiv \min_{\rho \in D} \text{Tr} \rho_{\sigma} (\log_2 \rho_{\sigma} - \log_2 \rho)$$

where the minimization is carried out over the set $D$ of all two-mode separable states. $EEP$ detects all nonclassical states, and can be calculated for important classes of states. For finite dimensional mixed states, it can be numerically computed by an iterative procedure [24]. For pure states the $EEP$ reduces to the von Neumann entropy generated in one of the output arms. Moreover, as we show in [25], $EEP$ gives a lower bound to the nonclassical relative entropy distance $\tilde{D}$. In the remainder of this Letter we calculate the $EP$ and $EEP$ for a variety of nonclassical states. A more comprehensive study and comparison with other nonclassicality measures will be presented elsewhere [25].

The $EP$ and the $EEP$ of a Fock state $|n\rangle$ are

$$EP(n) = -n + 2 \log_2 \sum_{k=0}^{n} \sqrt{\binom{n}{k}},$$

$$EEP(n) = n - 2^{-n} \sum_{k=0}^{n} \binom{n}{k} \log_2 \binom{n}{k}. \tag{4}$$

In the large-$n$ limit, the two diverge logarithmically and differ only in an additive constant: $EP(n) \approx \frac{1}{2} \log_2(2\pi n)$; $EEP(n) \approx EP(n) - (1 - 1/\ln 4)$. The entanglement potential can thus detect the nonclassicality of Fock states, and shows increasing nonclassicality with increasing photon number.

For any finite superposition of coherent states $|\Psi\rangle = \sum_k c_k |\alpha_k\rangle$ both the $EP$ and the $EEP$ can be calculated exactly in the nonorthogonal basis subtended by the $|\alpha_k\rangle$, by making use of the “metric tensor” as in [26]. This yields complicated formulas, a notable exception being the “odd coherent state” $|\alpha\rangle = |-\alpha\rangle$, for which both the $EP$ and $EEP$ are 1, independent of $\alpha$. If the coherent states are truly distinct $|\alpha_i - \alpha_k| > \epsilon$, the nonclassicality is determined by the probability amplitudes $c_i$, and the coherent amplitudes $\alpha_i$ bring only an exponentially small correction. Such superpositions are often called Schrödinger cats (SC), and are typical examples of nonclassical states. Neglecting the correction arising from the overlaps, we obtain for the $EP$ and the $EEP$ of a Schrödinger cat state:

$$EP(\Psi) = 2 \log_2 \sum_k |c_k|$$

$$EEP(\Psi) = - \sum_k |c_k|^2 \log_2 |c_k|^2. \tag{6}$$

The most nonclassical superposition of $N$ coherent states is the “odd coherent state” $\sum_k c_k |\alpha_k\rangle$ where the probability amplitudes are of equal magnitude: in that case $EP = EEP = \log_2 N$.

A Gaussian state can be written as a displaced squeezed thermal state: $\rho(\alpha, r, \phi, \bar{n}) = \hat{D}(\alpha) \hat{S}(r, \phi) \rho_{\bar{n}} \hat{S}(r, \phi)^\dagger \hat{D}(\alpha)^\dagger$. Here $\rho_{\bar{n}}$ is a thermal state of average photon number $\bar{n}$, and $\hat{S}(r, \phi) = \exp(\frac{1}{2} r [\hat{a}^\dagger e^{i \phi} \hat{a}^2 - e^{-i \phi} \hat{a}^2 - 2])$ is the squeezing operator. It is well known [27] that Gaussian states are classical for $r \leq r_c$, where the nonclassicality threshold is $r_c = \ln(2\bar{n} + 1)/2$. Using the results of Wolf et al. [17], it immediately follows that the $EP$ of a general Gaussian state is given by the nonclassical part of the squeezing,

$$EP(\rho(\alpha, r, \phi, \bar{n})) = \frac{r - r_c}{\ln 2}. \tag{7}$$

For Gaussian states $EP$ detects nonclassicality [28], and it is a monotonous function of the nonclassical depth $\tilde{D}$.

To calculate the $EEP$ we restrict ourselves to pure Gaussian states. Displacement operators in the input map to local displacements at the output, and similarly the squeezing maps to two-mode squeezing followed by additional (local) squeezing in each mode. Hence, $EEP$ reduces to the single-mode entropy of the two-mode squeezed vacuum,

$$EEP(\rho(\alpha, r, \phi, 0)) = \cosh^2(r/2) \log_2 \cosh^2(r/2) - \sinh^2(r/2) \log_2 \sinh^2(r/2). \tag{8}$$

For strong $r \gg 1$ squeezing we again find approximate equality with the $EP$ up to a constant: $EEP(\rho_{\alpha, r, \phi, \epsilon}) \approx r/\ln 2 - (2 - 1/\ln 4)$. For weak $r \ll 1$ squeezing, however, the $EEP$ increases quadratically with $r$: $EEP(\rho_{\alpha, r, \phi, \epsilon}) \approx -r^2/2 \log_2(r/2)$.

A quantitative measure allows us to investigate how much nonclassicality is lost in a physical process. As an example, we study photon dissipation, which is the dominant decoherence process for states propagating in an optical fiber. Any coherent state is decreased in amplitude by the factor $\xi = \exp(-\gamma t)$, whereby the Glauber P function changes as $P'(\alpha) = \xi^{-1} P(\xi^{-1/2} \alpha)$ [29].

For a Schrödinger’s cat undergoing photon loss the $EP$ can be calculated exactly using the metric tensor [26]. For weak dissipation, decoherence dominates power loss, and the constituent coherent states $|\xi \alpha_i\rangle$ can still be considered approximately orthogonal. We then obtain

$$EP(t) \approx \log_2 \left( 1 + 2 \sum_{i<k} |\langle \alpha_i |\alpha_k \rangle|^{1 - \xi(t)} |c_i| |c_k| \right). \tag{9}$$
Although the $EP$ or $EEP$ of SC states is independent of the phase-space distance of the constituent coherent states, this distance determines the decoherence behaviour. For intermediate times $|\alpha_1 - \alpha_2|^2 < \gamma t \ll 1$ the $EP$ of dissipated SC is given by $EP(t) \approx \exp(-t/T_D)/2 \gamma_1 \gamma_2 / \ln 2$, with the well-known decoherence timescale $T_D = 2 |\alpha_1 - \alpha_2|^2 \gamma^{-1}$.

The compact formula (17) can be used to study the nonclassicality of Gaussian states in Gaussian channels. For example, for linear coupling to a heat bath of mean photon number $n_T$, we find

$$EP(t) = -\frac{1}{2} \log_2 \left[ e^{-\gamma t} e^{-2(r-r_c)} + (1-e^{-\gamma t})(2n_T+1) \right],$$

where $r$ is the initial squeezing, and $r_c$ is the initial classicality threshold (as defined above). We now concentrate on photon dissipation, i.e., $n_T=0$. For weak squeezing, $r - r_c \ll 1$, the above formula reduces to an exponential decay, $EP(t) \approx (2 \ln 2)^{-1} (r - r_c) e^{-\gamma t}$. For strong squeezing, $r-r_c \gg 1$, we find a different decoherence behaviour. Initially, $EP$ decreases linearly with time,

$$\gamma t \ll e^{-2(r-r_c)} : \quad EP(t) \approx \frac{r - r_c}{\ln 2} - \frac{e^{2(r-r_c)}}{2 \ln 2} \gamma t. \quad (10)$$

Note that the loss rate of $EP$ is exponentially large in the initial squeezing. After an exponentially short time $\tau = \gamma^{-1} e^{-2(r-r_c)}$ the $EP$ of the initially highly squeezed Gaussian state falls onto a general curve:

$$\gamma t > e^{-2(r-r_c)} : \quad EP(t) \approx -\frac{1}{2} \log_2 (1 - e^{-\gamma t}). \quad (11)$$

The initial squeezing now only adds a minor correction, exponentially small in $r - r_c$, to the $EP$. On longer timescales, $\gamma t \gg 1$, the $EP$ of strongly squeezed states also decreases exponentially, $EP(t) = (2 \ln 2)^{-1} (1 - e^{-2(r-r_c)}) e^{-\gamma t}$. Here we explicitly included the small correction in $r$, which is the only memory of the initial parameters.

Figure 2 shows the time dependence of $EP$ during photon loss for various states. Squeezed states retain their $EP$ better than other nonclassical states; although the weakly squeezed state $(r - r_c = 0.1)$ initially has less $EP$ than either the single-photon Fock state or the two examples of SC states, after $t > 2\gamma$ it is more valuable than these for entanglement generation. The most fragile states are the SC’s, which lose $EP$ on short timescales given by the $T_D$ mentioned above. We also notice that (9) gives an excellent approximation of the $EP$ for SC’s.

Based on a broad and physically motivated definition we have introduced a nonclassicality measure that grasps the essential feature of single-mode nonclassical states: their potential to generate entanglement by linear-optical means. We have reduced the definition to a simple and operational form, involving only a single beamsplitter. The $EP$ can be calculated analytically for a variety of pure and mixed states, and efficient numerical methods exist for general states. We have illustrated its use through a set of examples; no other nonclassicality measure can be computed to cover all of these. The definition of $EP$ in (11) has brought up the still open question of whether bound entanglement can be obtained from single-mode nonclassical light, auxiliary classical states, linear optics and photodetectors. The study of the additivity properties of the Entanglement Potential—some examples show that it is super-additive—, and the extension of the concept to multi-mode nonclassical fields are subjects of further research.

[1] R. J. Glauber, Phys. Rev. 131, 2766 (1963).
[2] L. M. Johansen, Phys. Lett. A 329, 184 (2004).
[3] Q. A. Turchette et al., Phys. Rev. A 58, 4056 (1998); A. I. Lvovsky, J. Mlynek, Phys. Rev. Lett. 88, 250401 (2002); N. Treps et al., ibid. 88, 203601 (2002); K. McKenzie et al., ibid. 93, 161105 (2004); A. Zavatta, S. Viciani, and M. Bellini, Science 306, 660 (2004).
[4] A hierarchy of necessary and sufficient criteria (not measures) for nonclassicality has been recently presented in T. Richter, W. Vogel, Phys. Rev. Lett. 89, 283601 (2002).
[5] M. Hillery, Phys. Rev. A 35, 725 (1987).
[6] A. Wünsche et al., Fortschr. Phys. 49, 1117 (2001); V. V. Dobanov et al., J. Mod. Opt. 47, 633 (2000).
[7] P. Marian, T. A. Marian, H. Scutaru, Phys. Rev. A 69, 022104 (2004).
[8] P. Marian, T. A. Marian and H. Scutaru, Phys. Rev. Lett. 88, 153601 (2002).
[9] C. T. Lee, Phys. Rev. A 44, R2775 (1991).
[10] N. Lütkenhaus and S. M. Barnett, Phys. Rev. A 51, 3340 (1995).
[11] Y. Aharonov et al., Ann. Phys. 39, 498 (1966).
[12] M. S. Kim, et al., Phys. Rev. A 65, 032323 (2002); W. Xiang-bin, *ibid.* 66, 024303 (2002).
[13] *Quantum Theory and Measurement*, ed. by J. A. Wheeler and W. H. Zurek (Princeton Univ. Press, 1984).
[14] J. S. Bell, Physics 1, 195 (1964).
[15] A. Aspect, J. Dalibard, and G. Roger, Phys. Rev. Lett. 49, 1804 (1982); G. Weihs et al., *ibid.* 81, 5039 (1998).
[16] G. Giedke et al., Phys. Rev. Lett. 87, 167904 (2001); S. J. van Enk, *ibid.* 91, 017902 (2003); V. Giovannetti et al., *ibid.* 91, 047901 (2003); G. Adesso, A. Serafini, and F. Illuminati, *ibid.* 92, 087901 (2004); J. Fiurasek, N. J. Cerf, *ibid.* 93, 063601 (2004); *Quantum Information Theory with Continuous Variables*, ed. by S. L. Braunstein and A. K. Pati (Kluwer Academic, Dordrecht, 2002).
[17] M. M. Wolf, J. Eisert, and M. B. Plenio, Phys. Rev. Lett. 90, 047904 (2003).
[18] M. Reck et al., Phys. Rev. Lett. 73, 58 (1994).
[19] J. Calsamiglia et al., Phys. Rev. A 64, 043814 (2001).
[20] G. Vidal, R. F. Werner, Phys. Rev. A 65, 032314 (2002).
[21] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. 80, 5239 (1998).
[22] P. Horodecki and M. Lewenstein, Phys. Rev. Lett. 85, 2657 (2000).
[23] V. Vedral et al., Phys. Rev. Lett. 78, 2275 (1997).
[24] J. Rehacek and Z. Hradil, Phys. Rev. Lett. 90, 127904 (2003).
[25] J. K. Asboth, J. Calsamiglia and H. Ritsch (unpublished).
[26] J. Janszky et al., Fortschr. Phys. 51, 157 (2003).
[27] P. Marian, T. A. Marian, Phys. Rev. A 47, 4487 (1993).
[28] R. Simon, Phys. Rev. Lett., 84, 2726 (2000).
[29] U. Leonhardt, *Measuring the Quantum State of Light* (Cambridge University Press, 1997).