Analysis of fuzzified boundary value problems for MHD Couette and Poiseuille flow

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In an uncertain atmosphere, the magnetohydrodynamics (MHD) flow in three principal flows of the third grade fluid across two parallel plates is presented. Fuzzy differential equations are constructed by manipulating dimensionless differential equations. The prime purpose of the current article is to use a semi-analytical approach fuzzy-based Adomian decomposition method to achieve numerical results for nonlinear FDEs with fuzzy boundary conditions. Triangular fuzzy numbers are used in fuzzy BCs with help of α-cut approach. This strategy is linked to the membership function. In a graphic and tabular depiction, the effect of α and other constraints on fuzzy velocity profiles is explored. The current findings are in good agreement with their previous numerical and analytical results in a crisp environment.

Scientists are paying close attention to non-Newtonian fluids because of their frequent practices in the industry, science, and engineering such as mayonnaise, soap, cosmetics, paints, biological solutions, blood, shampoos, glues, tars, syrups, yoghurt, and other industrial materials fall into this category. As a result, researchers have given differential type fluids a higher priority. We will focus on well known third grade fluid or differential types, which have been extensively researched in a variety of flow processes. The study of three basic streams (especially, Poiseuille, Couette, and generalized Couette flow) attracts investigators across several non-Newtonian fluids owing to their potential applications in industries and science. Injection moulding, continuous casting, die flow, plastic forming, extrusion, and asthenosphere flows are examples of unidirectional flows utilised in polymer engineering. The study of electrically conducting liquids moving in the presence of a magnetic field is known as MHD. MHD flow across infinite parallel plates has important applications in geophysical, geothermal reservoirs, metallurgical processing, mineral industries, pumps, astrophysical, MHD generators, polymer technology, and other fields. MHD liquid is a lubricant that prevents lubricant viscosity from changing suddenly with temperature in industrial and other applications. Kamran and Siddique investigated the three flow problems (Poiseuille, Couette, and generalized Couette flow) on MHD third grade fluid across the two parallel plates with help of ADM. There is a lot of literature on this topic, such as.

The ADM was first proposed by Adomian. ADM is a procedure for solving linear and nonlinear (DEs) that is both trustworthy and efficient. The ADM has several benefits over other analytical and numerical approaches, notably the absence of perturbation, linearization, discretization, or spatial translation. ADM was utilised by Siddiqui et al. to examine the parallel plates flow of a third grade fluid and the results were compared to numerical methods. Pirzada and Vakaaskar used fuzzy ADM to find the solution to the fuzzy heat equation. Paripour et al. evaluated the fuzzy ADM and predictor–corrector (PC) strategies for the numerical solutions of hybrid FDEs, concluding that the ADM is superior to the PC method. In addition, For squeezed flow between the two circular plates, Siddiquie et al. compared the ADM to the homotopy perturbation technique (HPM). ADM outperforms HPM, as per their observations. Biswal et al. investigated the spontaneous convection of nanofluid flow over two parallel plates by HPM in an uncertain atmosphere. TFN stands for nanoparticle volume percentage, as well as the fact that a fuzzy output is preferable to a crisp one.

In science and engineering, fluid flow is extremely important. An increase in a wide variety of issues such as magnetic effect, chemical diffusion, and heat transfer. These physical problems are then transformed into linear or nonlinear DEs after being governed. The solution of DEs is highly influenced by physical difficulties

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involving coefficients, geometry, initial, parameters, and boundary conditions. The coefficients, geometry, initial, parameters, and boundary conditions are not precise because of measurement errors, mechanical faults, confidence intervals, and other causes. As a result, fuzzy sets theory (FST) is a great resource for grasping the facts at hand, and it is more realistic than assuming crispness. FDEs, in particular, are useful for reducing ambiguity and clarifying physical difficulties.

Chang and Zadeh\(^\text{27}\) were the first to propose the concept of a fuzzy derivative (FD). On FNs, Dubois and Prade\(^\text{28}\) created arithmetic techniques. Trapezoidal, triangular, and Gaussian FNs are three different types of FNs. TFNs are included here for completeness’ sake. In 1987 Seikkala\(^\text{29}\) familiarized the concept of FD. After that, Kaleva\(^\text{30}\) obtained FD and integration. The geometric technique for solving SFDEs was devised by Gasilov et al.\(^\text{31}\). To solve the second-order FDE, Khastan and Nieto\(^\text{32}\) used an extended differentiability approach. There were several studies a few decades ago that revolved on the topic of FDEs. Many scientists have employed FDEs to obtain well-known technological and scientific breakthroughs\(^\text{33–42}\).

In the literature review for third grade problems, only crisp or classic cases were considered. As a result of the above-mentioned works, in three essential flow problems of a third-grade liquid across two parallel plates, we created a model to explain the fuzzy evaluation for unidirectional MHD flow. The primary purpose of this work is to use FDEs to demonstrate the uncertain flow mechanism.

The article is prepared as follows: second section contains the fundamental preliminaries; third section contains the main body of the article. In third section, the proposed study’s governing equations were developed, and the governing equations were transformed into a fuzzy form for the solution by a fuzzy ADM. Fourth section presents the results and discussion in graphical and tabular formats. Fifth section contains some conclusions.'

**Preliminaries**

**Definition**\(^\text{26}\) Fuzzy set \(Z\) is defined as set of ordered pairs such that \(Z = \{(x, \mu_Z(x)) : x \in U, \mu_Z(x) \in [0, 1]\}\), here \(U\) is the universal set, \(\mu_Z(x)\) is membership function of \(Z\) and mapping defined as \(\mu_Z(x) : U \rightarrow [0, 1]\).

**Definition**\(^\text{27}\) \(\alpha\)-cut or \(\alpha\)-level of a fuzzy set \(Z\), defined by \(Z_\alpha = \{x/\mu_Z(x) \geq \alpha\}\), where \(Z_\alpha\) is crisp set and \(0 \leq \alpha \leq 1\).

**Definition**\(^\text{27}\) Let \(Z = (\delta, \chi, \eta)\) with membership function \(\mu_Z(x)\) is called a TFNs if

\[
\mu_Z(x) = \begin{cases} 
1 - \frac{x - \delta}{\delta - \eta}, & \text{for } \delta \leq x \leq \chi, \\
1 - \frac{\chi - x}{\chi - \eta}, & \text{for } \chi \leq x \leq \eta, \\
0, & \text{otherwise}.
\end{cases}
\]

The TFNs with peak (center) \(\chi\), right width \(\eta - \chi > 0\), left width \(\chi - \delta > 0\), and these TFNs are transformed into interval numbers through \(\alpha\)-cut approach, is written as \(\tilde{Z} = [\nu(x; \alpha), u(x; \alpha)] = [\delta + \alpha(\chi - \delta), \eta - \alpha(\eta - \chi)]\), where \(0 \leq \alpha \leq 1\). TFNs satisfy the following conditions: (1) \(\nu(x; \alpha)\) is non-decreasing on \([0, 1]\). (ii) \(u(x; \alpha)\) is non-increasing on \([0, 1]\). (iii) \(u(x; \alpha) \geq \nu(x; \alpha)\) on \([0, 1]\). (iv) \(\nu(x; \alpha)\) and \(u(x; \alpha)\) are bounded on left continuous and right continuous at \([0, 1]\) respectively.

**Definition**\(^\text{28}\) Let \(I^+\) be an interval and \(I^+ \subseteq R\). A mapping \(\pi : I^+ \rightarrow F^+\) is called a fuzzy process, defined as \(\pi(x; \alpha) = [\nu(x; \alpha), u(x; \alpha)], x \in I^+\) and \(0 \leq \alpha \leq 1\). The derivative \(\frac{d\pi(x; \alpha)}{dx} \in F^+\) of a fuzzy process \(\pi(x; \alpha)\) is defined as

\[
\frac{d\pi(x; \alpha)}{dx} = \left[\frac{d\nu(x; \alpha)}{dx}, \frac{du(x; \alpha)}{dx}\right].
\]

**Definition**\(^\text{28}\) Let \(I^+ \subseteq R\) and \(\Pi(x)\) be a fuzzy valued function define on \(I^+\). Let \(\Pi(x; \alpha) = [\nu(x; \alpha), u(x; \alpha)]\) for all \(\alpha\)-cut. Assume that \(\nu(x; \alpha)\) and \(u(x; \alpha)\) have continuous derivatives or differentiable, for all \(x \in I^+\) and \(0 \leq \alpha \leq 1\) then

\[
\frac{d\Pi(x; \alpha)}{dx}_\alpha = \left[\frac{d\nu(x; \alpha)}{dx}_\alpha, \frac{du(x; \alpha)}{dx}_\alpha\right].
\]

Similarly, we can define higher-order derivatives in the same way. Then \(\frac{d\Pi(x; \alpha)}{dx}_\alpha\) satisfy the following conditions: (i) \(\frac{d\nu(x; \alpha)}{dx}_\alpha\) and \(\frac{du(x; \alpha)}{dx}_\alpha\) are continuous on \([0, 1]\). (ii) \(\frac{d\nu(x; \alpha)}{dx}_\alpha\) is non-decreasing on \([0, 1]\). (iii) \(\frac{du(x; \alpha)}{dx}_\alpha\) is non-increasing on \([0, 1]\). (iv) \(\frac{d\nu(x; \alpha)}{dx}_\alpha \geq \frac{du(x; \alpha)}{dx}_\alpha\) on \([0, 1]\).

**Basic equations**

The following equations describe the flow of an incompressible unidirectional third-grade fluid with MHD effects:

\[
\nabla \cdot \mathbf{V} = 0.
\]

\[
\rho \frac{\partial \mathbf{V}}{\partial t} = \mathbf{J} \times \mathbf{B} - \nabla p + \text{div} \mathbf{\tau}^{**},
\]

where the density \(\rho\), stress tensor \(\mathbf{\tau}^{**}\), pressure \(p\), velocity vector \(\mathbf{V}\), viscosity \(\mu\), electric current \(\mathbf{J}\), and total magnetic field \(\mathbf{B}\), \(\mathbf{B} = \mathbf{B}_0 + \mathbf{b}\), where induced magnetic field \(\mathbf{b}\) and imposed magnetic field \(\mathbf{B}_0\). The modified Ohm’s law and Maxwell’s equations\(^\text{43}\) are applicable in the absence of displacement currents.

\[
\mathbf{J} = \sigma^{**} [\mathbf{E} + \mathbf{V} \times \mathbf{B}],
\]
\[ \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{B} = \mu_m \mathbf{J}, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (4) \]

where magnetic permeability \((\mu_m)\), electric field \((\mathbf{E})\), and electrical conductivity \((\sigma^{**})\). The MHD force in Eq. (2) can be expressed as follows:\(^1\)

\[ \mathbf{J} \times \mathbf{B} = -\sigma \mathbf{B}_0^2 \nabla. \quad (5) \]

The \(\tau^{**}\) is given by\(^1\–^3\)

\[ \tau^{**} = \sum_{i=1}^{3} \mathbf{S}_i \quad (6) \]

where \(\mathbf{S}_1 = \mu A_1, \mathbf{S}_2 = \alpha_1 A_2 + \alpha_2 A_3^2, \mathbf{S}_3 = \beta_1 A_1 + \beta_2 (A_2 A_1 + A_1 A_2) + \beta_3 A_1, \) where \(\mu\) is coefficient of viscosity, \(\alpha_1, \alpha_2, \beta_1, \beta_2\) and \(\beta_3\) are material constants.

Define velocity profile for one-dimensional flows as:

\[ \mathbf{V} = (u(x), 0, 0). \quad (7) \]

\[ \frac{dp^*}{dx} = 0. \quad (8) \]

\[ -\frac{dp}{dx} + (2\alpha_1 + \alpha_2) \frac{d}{dx} \left( \left( \frac{du}{dx} \right)^2 \right) = 0, \quad (9) \]

and modified pressure \(p^\tau\) is

\[ p^\tau = -p + (2\alpha_1 + \alpha_2) \left( \left( \frac{du}{dx} \right)^2 \right). \quad (10) \]

For simplicity, the momentum Eq. (2) along with Eqs. (5)–(10) reduces to,

\[ -\frac{1}{\mu} \frac{dp^\tau}{dy} + \frac{d^2 u}{dx^2} + \frac{6}{\mu} \left( \frac{\beta_2 + \beta_3}{\mu} \right) \left( \frac{du}{dx} \right)^2 \frac{d^2 u}{dx^2} = \frac{\sigma B_0^2 u}{\mu} = 0. \quad (11) \]

Equation (11) is a second-order non-linear ODE.

The Adomian decomposition method. “Write the basic non-linear equation and discuss the basic sketch of ADM.

\[ L_1 u^\tau(x) + N_1 u^\tau(x) = q(x), \quad (12) \]

where \(L_1, q,\) and \(N_1\) are linear, source term, and non-linear operators respectively. Also, the operator \(L_1\) can be written as

\[ L_1 = R_1 + \tilde{L}, \quad (13) \]

here, \(\tilde{L}\) is the highest order derivative in \(L_1\) and \(R_1\) is the remaining operator in \(L_1\) whose order is less than the order of \(\tilde{L}\). From (12) and (13) we have

\[ \tilde{L} u^\tau(x) = q(x) - R_1 u^\tau(x) - N_1 u^\tau(x). \quad (14) \]

Applying \(\tilde{L}^{-1}\)

\[ u^\tau(x) = -\tilde{L}^{-1} R_1 u^\tau(x) - \tilde{L}^{-1} N_1 u^\tau(x) + g(x), \quad (15) \]

where \(g(x; \alpha)\) signifies the terms arising after integration of \(q(x; \alpha)\) and calculate constants of integration with the help of boundary conditions. So, \(u^\tau(x; \alpha)\) and \(N_1 u^\tau(x; \alpha)\) can be written as\(^{18–21}\)

\[ u^\tau(x; \alpha) = \sum_{n=0}^{\infty} u_n^\tau(x; \alpha), \quad (16) \]

\[ N_1 u(x; \alpha) = \sum_{n=0}^{\infty} A_n^\tau(x; \alpha), \quad (17) \]

where \(A_n^\tau\) are called Adomian polynomials\(^{18,20}\).

The algorithm of the general ADM can be communicated as
In a fuzzy sense, we employ the ADM to three flow problems. 

**Couette flow.** Let a third grade fluid flow steadily between two parallel plates at \(x = 0\) and \(x = d\). The upper plate at \(x = d\) is moving with constant velocity \(U\) while the lower plate is fixed. The magnetic field is applied vertically upward in a non-conducting manner to both plates. Also, assume that the normal flow is in \(y\)-direction while the \(x\)-axis is engaged as the way of flow (see in Fig. 1). When there is no pressure gradient, the consequent DE (21) with BCs (22) is\(^4\),

\[
d^2u \over dx^2 + 6 \left( \beta_2 + \beta_3 \right) \left( {du \over dx} \right)^2 d^2u \over dx^2 - {\sigma B_0^2 \mu} u = 0, 
\]

\[
u(x) = 0 \text{ at } x = 0, \]
\[
u(x) = U \text{ at } x = d. \]

Offering the dimensionless variables

\[
x^\mp = {x \over d}, \quad \beta_2^\mp = {\beta_2 \mu d^2 / U^2}, \quad u^\mp = {u \over U}, \quad m^\mp = {\sigma B_0^2 \mu / d^2}, \quad \beta_3^\mp = {\beta_3 \mu d^2 / U^2}. 
\]

Dropping the ‘\(\mp\)’ the (21) and (22) becomes

\[
d^2u \over dx^2 + 6 \beta \left( {du \over dx} \right)^2 d^2u \over dx^2 - m^2 u = 0, 
\]

with the BCs

\[
u(x) = 0, \text{ at } x = 0, \]
\[
u(x) = 1 \text{ at } x = 1. \]

**For fuzzy solution.** To deal with these problems, we used TFNs \((\delta, \chi, \eta)\) and \((d, e, f)\). Because the boundary of the parallel plates is treated as fuzzified, this discretization is applied in the boundary for certain flow behaviour. The Eqs. (24 and 25) are converted to an FDE is given below

\[
d^2\bar{u}(x; \alpha) \over dx^2 + 6 \odot \beta \odot \left( {d\bar{u}(x; \alpha) \over dx} \right)^2 \odot d^2\bar{u}(x; \alpha) \over dx^2 - m^2 \odot \bar{u}(x; \alpha) = 0, 
\]

\[
\bar{u}(x; \alpha) = \left[ \frac{1}{100} (\alpha - 1), \frac{1}{100} (1 - \alpha) \right] \text{ at } x = 0, 
\]
\[
\bar{u}(x; \alpha) = \left[ \frac{1}{50} (\alpha - 1), \frac{1}{50} (1 - \alpha) \right] \text{ at } x = 1, 
\]
here lower $v(x; \alpha)$ and upper $u(x; \alpha)$ fuzzy velocity profiles, while (27) are fuzzy BCs\textsuperscript{28} are

\[
\frac{d^2v(x; \alpha)}{dx^2} + 6\beta \left( \frac{dv(x; \alpha)}{dx} \right)^2 \frac{d^2v(x; \alpha)}{dx^2} - m^2v(x; \alpha) = 0,
\]

(28)

\[
v(x; \alpha) = (\alpha - 1) \frac{1}{100} \text{ at } x = 0,
\]

\[
v(x; \alpha) = (13\alpha - 2) \frac{1}{250} \text{ at } x = 1,
\]

(29)

\[
\frac{d^2u(x; \alpha)}{dx^2} + 6\beta \left( \frac{du(x; \alpha)}{dx} \right)^2 \frac{d^2u(x; \alpha)}{dx^2} - m^2u(x; \alpha) = 0,
\]

(30)

\[
u(x; \alpha) = (1 - \alpha) \frac{1}{100} \text{ at } x = 0,
\]

\[
u(x; \alpha) = (9 - 5\alpha) \frac{1}{50} \text{ at } x = 1.
\]

(31)

The ADM is now being used in fuzzy boundary value problems Eqs. (28)–(31) and Eq. (28) becomes

\[
L_1v(x; \alpha) = m^2v - 6\beta \left( \frac{dv(x; \alpha)}{dx} \right)^2 \frac{d^2v(x; \alpha)}{dx^2}.
\]

(32)

where $L_1 = d^2/dx^2$ and inverse operator is $\hat{L}^{-1} = \iint (\cdot) dx dx$. Using $\hat{L}^{-1}$ in Eq. (32) we have

\[
v(x; \alpha) = c_1x + c_2 - 6\beta\hat{L}^{-1} \left( \frac{dv(x; \alpha)}{dx} \right)^2 \frac{d^2v(x; \alpha)}{dx^2} + \hat{L}^{-1}[m^2v(x; \alpha)],
\]

(33)

where the constants of integration are $c_1$ and $c_2$. Given Eqs. (16) and (17), Eq. (33) provides

\[
\sum_{n=0}^{\infty} v_n(x; \alpha) = c_1x + c_2 - 6\beta\hat{L}^{-1} \left( \frac{dv(x; \alpha)}{dx} \right)^2 \frac{d^2v(x; \alpha)}{dx^2} + \hat{L}^{-1}[m^2v(x; \alpha)],
\]

(34)

zeroth component as

\[
v_0(x; \alpha) = c_1x + c_2,
\]

(35)

and the recurrence relation as,

\[
v_{n+1}(x; \alpha) = \hat{L}^{-1}[m^2v_n(x; \alpha)] - 6\beta\hat{L}^{-1}[A^*_n(x; \alpha)],
\]

(36)

where $A^*_n$ are

\[
A^n_0(x; \alpha) = \frac{d^2v_0(x; \alpha)}{dx^2} \left( \frac{dv_0(x; \alpha)}{dx} \right)^2,
\]

\[
A^n_1(x; \alpha) = \left( \frac{dv_0(x; \alpha)}{dx} \right)^2 \frac{d^2v_1(x; \alpha)}{dx^2} + 2 \frac{dv_0(x; \alpha)}{dx} \frac{dv_1(x; \alpha)}{dx} \frac{d^2v_0(x; \alpha)}{dx^2},
\]

\[
A^n_2(x; \alpha) = \left( \frac{dv_1(x; \alpha)}{dx} \right)^2 \frac{d^2v_0(x; \alpha)}{dx^2} + 2 \frac{dv_0(x; \alpha)}{dx} \frac{dv_2(x; \alpha)}{dx} \frac{d^2v_0(x; \alpha)}{dx^2} + 2 \frac{d^2v_1(x; \alpha)}{dx^2} \frac{dv_0(x; \alpha)}{dx} \frac{dv_1(x; \alpha)}{dx}
\]

\[+ \left( \frac{dv_0(x; \alpha)}{dx} \right)^2 \frac{d^2v_2(x; \alpha)}{dx^2},
\]

(37)

The fuzzy BCs become

\[
v_0(x; \alpha) = (\alpha - 1) \frac{1}{100}, \text{ at } x = 0,
\]

\[
v_0(x; \alpha) = (13\alpha - 2) \frac{1}{250}, \text{ at } x = 1,
\]

(38)

\[
v_n(x; \alpha) = 0 \text{ at } x = 0,
\]

\[
v_n(x; \alpha) = 0 \text{ at } x = 1, \quad n > 0.
\]

(39)

\[
v(x; \alpha) = v_0(x; \alpha) + v_1(x; \alpha) + v_2(x; \alpha) + \cdots,
\]

(40)

Solving Eqs. (35) to (39) and putting all values of $v_0(x; \alpha)$, $v_1(x; \alpha)$, \ldots in Eq. (40) we have

\[
v(x; \alpha) = A_1x + A_4 + \frac{m^2}{6} (A_1x^3 + 3A_4x^2 - A_2x) (1 - \beta A_1^2) + \frac{m^4}{360} (3A_1x^5 + 15A_4x^4 - 10A_2x^3 + A_3x) + \cdots,
\]

(41)
similarly, \( u(x; \alpha) \) is,

\[
u(x; \alpha) = B_1 x + B_4 + \frac{m^4}{6} (B_1 x^3 + 3B_4 x^2 - B_2 x)(1 - \beta B_1^2) + \frac{m^4}{360} (3B_1 x^5 + 15B_4 x^4 - 10B_2 x^3 + B_3 x) + \cdots. \tag{42}\]

**Plane Poiseuille flow.** Under a constant pressure gradient, we consider the continuous laminar flow of third grade fluid among two fixed infinite parallel plates (see in Fig. 2). The gap between adjacent plates is \( 2d \), are at \( x = -d \) and \( x = d \). As a result, with the transversal magnetic field and continuous pressure gradient, the governing Eq. (11) we have

\[
d\frac{d^2 u}{dx^2} + \frac{6(\beta_3 + \beta_2)}{\mu} \left( \frac{du}{dx} \right)^2 \frac{d^2 u}{dx^2} - \frac{\sigma B_0^2}{\mu} u = \frac{1}{\mu} \frac{dp^*}{dy}, \tag{43}\]

with the BCs

\[
\begin{align*}
u(x) &= 0 \text{ at } x = d, \\
u(x) &= 0 \text{ at } x = -d.
\end{align*} \tag{44}\]

The dimensionless parameters are presented as

\[
\beta_2^\pm = \frac{\beta_2}{\mu U^2/d}, \quad \beta_3^\pm = \frac{\beta_3}{\mu U^2/d}, \quad x^\pm = \frac{x}{d}, \quad m^\pm = \frac{\sigma B_0^2}{\mu U^2/d}, \quad p^\pm = \frac{p}{\mu U/d}, \quad u^\pm = \frac{u}{U}, \quad y^\pm = \frac{y}{d}; \tag{45}\]

After dropping ‘\(^\pm\)’ we have

\[
d\frac{d^2 u}{dx^2} + 6\beta \left( \frac{du}{dx} \right)^2 \frac{d^2 u}{dx^2} - m^2 u = \frac{d p}{dy}, \tag{46}\]

where \( \beta = \beta_2 + \beta_3 \), with the BCs

\[
\begin{align*}
u(x) &= 0 \text{ at } x = 1, \\
u(x) &= 0 \text{ at } x = -1. \tag{47}\]

Now Eqs. (46) and (47) convert in FDE as

\[
d\frac{d^2 \pi(x; \alpha)}{dx^2} + 6 \odot \beta \odot \left( \frac{d \pi(x; \alpha)}{dx} \right)^2 \odot \frac{d^2 \pi(x; \alpha)}{dx^2} - m^2 \odot \pi(x; \alpha) = \frac{d p}{dy}, \tag{48}\]

with the fuzzy BCs

\[
\begin{align*}
\pi(x; \alpha) &= \left[ \frac{\alpha - 1}{\alpha}, \frac{\alpha}{\alpha} \right] \text{ at } x = 1, \\
\bar{\pi}(x; \alpha) &= \left[ \frac{\alpha}{\alpha}, \frac{\alpha}{\alpha} \right] \text{ at } x = -1.
\end{align*} \tag{49}\]

Using the above, the solution of \( v(x, \alpha) \) and \( u(x, \alpha) \) are
with the fuzzy BCs 

$$\bar{v}(x; \alpha) = \left[ \frac{1}{100}, \frac{1}{100}, \frac{1}{100} \right] \quad \text{at} \quad x = 0,$$

$$\bar{u}(x; \alpha) = \left[ \frac{1}{50}, \frac{1}{50}, \frac{1}{50} \right] \quad \text{at} \quad x = 1,$$

The Solutions of $v(x, \alpha)$ and $u(x, \alpha)$ are


\[ v(x; \alpha) = \frac{px^2}{2} + \frac{\beta}{2p} \left[ D^4 - \left( D^4 - (p + D)^4 \right)x - (px + D)^4 \right] + \frac{6\beta^2}{5p} (px + D)^5 + \frac{6\beta^2}{5p} (px + D)^6 + Dx \]

\[ + E + m^3 \left[ - \frac{\beta p^2 (2p + 3)x^6}{30} - \frac{\beta p^2 D_1 x^5}{24} + \frac{p - 36\beta p^2 E_1 - 42\beta D_1^2}{24} x^4 - \frac{D - 24\beta p E_1 - 6D^3}{6} x^3 \right] \]

\[ + E + m^3 \left[ - \frac{\beta D^6 + px^6}{60p^3} - \frac{\beta (D + px)^6}{60p^3} \right] \]

\[ + m^4 \left[ \frac{px^6}{720} + \frac{D_1 x^5}{120} + \frac{E_1 x^4}{24} - \frac{p + 4D + 12E}{144} x^3 \right] + \frac{3\beta^2}{2p^2} (px + D)^3 \left( D^4 - (p + D)^4 \right) \]

\[ - 6\beta Lx - 6\beta K + \cdots, \]

\( m = 0, \beta = 0.75 \)

\( \alpha = 0, \beta = 0.75 \)

Figure 3. Fuzzy velocity profiles for the impact of \( m \).

Results and discussion

In a fuzzy environment, discuss the three elementary flow problems of a third-grade fluid such as plane Poiseuille, Couette, and generalised Couette flow. The governing equations convert into FDEs for the analytical solutions using fuzzy ADM to find the fuzzy velocity profiles of a third-grade differential type fluid among two parallel plates with MHD (magnetic parameter \( m \)) effect under a constant pressure gradient \( (dp/dy = p) \). Figures 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42 and 43 display the obtained fuzzy flow rate for various \( \alpha \)-cut levels (\( \alpha = 0, 0.3, 0.7, 1 \)).

Tables 1, 2 and 3 show the comparison of the crisp velocity profile with Siddiqui et al.⁴ and Yurisoy⁷. The validation of the present study findings was determined to be in excellent agreement.
Figure 4. Fuzzy velocity profiles for the impact of $m$.

Figure 5. Fuzzy velocity profiles for the influence of $m$.

Figure 6. Fuzzy velocity profiles for the impact of $m$. 
Figure 7. Fuzzy velocity profiles for the impact of $\beta$. 

Figure 8. Fuzzy velocity profiles for the impact of $\beta$. 

Figure 9. Fuzzy velocity profiles for the impact of $\beta$. 
Figure 10. Fuzzy velocity profiles for the impact of $\beta$.

Figure 11. Fuzzy velocity profiles for different values of $\alpha$-cut.

Figure 12. Triangular MF of fuzzy velocity profiles for the impact of $m$. 
Plane Couette flow. At distinct values of $\alpha$-cut ($0 \leq \alpha \leq 1$), the upshot of a magnetic parameter ($m$) on the $v(x, \alpha)$ and $u(x, \alpha)$ with non-Newtonian fluid, parameters ($\beta$) is exposed in Figs. 3, 4, 5 and 6. For various values of $\alpha$-cut, the crisp velocity profile is generalised so the $v(x, \alpha)$ and $u(x, \alpha)$ reduce gradually near the centre of the plates as the $m$ upsurges. The influence of $\beta$ on the $v(x, \alpha)$ and $u(x, \alpha)$ for $m$ at varying values of the $\alpha$-cut is shown in Figs. 7, 8, 9 and 10. At different values of $\alpha$, the fuzzy flow rates grow gradually near the middle of the plates with rising $\beta$. It has a favourable influence in Figs. 6 and 10 to give a classic solution in which the $v(x, \alpha)$

Figure 13. Triangular MF of fuzzy velocity profiles for impact of $\beta$.

Figure 14. Fuzzy velocity profiles for the impact of $\beta$.

Figure 15. Fuzzy velocity profiles for the impact of $\beta$. 
and \(u(x, \alpha)\) are the same at \(\alpha = 1\). Figure 11 shows the \(v(x, \alpha)\) and \(u(x, \alpha)\) for various \(\alpha\) values. Because the crisp velocity profile lies between the \(v(x, \alpha)\) and \(u(x, \alpha)\), the fuzzy velocity drops into the crisp velocity profile when \(\alpha = 1\), indicating that the current problem is a expansion of Kamran and Siddique\(^{14}\). Figures 12 and 13 depict the uncertain response of the TFN memberships function with the triangle fuzzy plot when \(\beta\) and \(m\) are varied. Figure 12 shows the fuzzy width declines through growing input \(m\), but Fig. 13 demonstrates how the uncertain

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**Figure 16.** Fuzzy velocity profiles for the impact of \(\beta\).

**Figure 17.** Fuzzy velocity profiles for the impact of \(\beta\).

**Figure 18.** Fuzzy velocity profiles for the impact of \(\beta\).
width grows with rising $\beta$. It was also discovered that when $\alpha$ grows, the $v(x, \alpha)$ usurges and the $u(x, \alpha)$ drops, implying that the solutions are powerful. The width between the $v(x, \alpha)$ and $u(x, \alpha)$ narrows as $\alpha$ grows, and at $\alpha = 1$, they coherent with the original answer. The analysis of $v(x, \alpha)$, mid, and $u(x, \alpha)$ for various values of $x$ with $\beta = 0.75$ and $m = 1$ are presented in Table 4. The crisp value of the original problem agrees with the TFN's
Figure 22. Fuzzy velocity profiles for the impact of $m$.

Figure 23. Fuzzy velocity profiles for the impact of $m$.

Figure 24. Fuzzy velocity for different values of $\alpha$-cut.
mid-value. Furthermore, for every set $\alpha$-cut, fuzzy velocity profiles always change within a particular range, and the range steadily diminishes as the $\alpha$-cut values improve.

**Plane Poiseuille flow.** At varying values of $\alpha$, the impact of $\beta$ on the $v(x, \alpha)$ and $u(x, \alpha)$ with relentless pressure gradient ($dp/dy = p$) was seen in Figs. 14, 15, 16, 17 and 18. With rising $\beta$, the $v(x, \alpha)$ and $u(x, \alpha)$
Figure 28. Fuzzy velocity profiles for the impact of $\beta$.

Figure 29. Fuzzy velocity profiles for the impact of $\beta$.

Figure 30. Fuzzy velocity for the impact of $\beta$. 
Figure 31. Fuzzy velocity for the impact of $\beta$.

Figure 32. Fuzzy velocity for the impact of $m$.

Figure 33. Fuzzy velocity for the impact of $m$. 
Figure 34. Fuzzy velocity for the impact of $m$.

Figure 35. Fuzzy velocity for the impact of $m$.

Figure 36. Fuzzy velocity for the impact of $p$. 
Figure 37. Fuzzy velocity for the impact of $p$.

Figure 38. Fuzzy velocity for the impact of $p$.

Figure 39. Fuzzy velocity for the impact of $p$. 

$m = 0.1, \beta = 0.2, \alpha = 0.3$

$m = 0.1, \beta = 0.2, \alpha = 0.7$

$m = 0.1, \beta = 0.2, \alpha = 1$
Figure 40. Fuzzy velocity for numerous values of $\alpha$-cut.

Figure 41. Triangular MF of the fuzzy velocity for the impact of $m$.

Figure 42. Triangular MF of fuzzy velocity for different values of $\beta$. 
Figure 43. Triangular MF of fuzzy velocity for the impact of $p$.

### Table 1. Comparison of analytical results for the crisp velocity profile of Couette flow when $\beta = 0.1$, $m = 0$, $p = -0.5$ and $\alpha$-cut = 1.

| $x$  | Siddiqui et al. | Yüresoy et al. | Kamran and Siddique | ADM present results |
|------|-----------------|----------------|---------------------|---------------------|
| 0    | 0               | 0              | 0                   | 0                   |
| 0.1  | 0.0814178       | 0.091211       | 0.06018             | 0.076017            |
| 0.2  | 0.1618119       | 0.181128       | 0.151479            | 0.151478            |
| 0.3  | 0.2618251       | 0.301244       | 0.249147            | 0.249145            |
| 0.4  | 0.3410156       | 0.374915       | 0.330156            | 0.330151            |
| 0.5  | 0.4721819       | 0.467189       | 0.476186            | 0.476182            |
| 0.6  | 0.5215171       | 0.572820       | 0.518252            | 0.518251            |
| 0.7  | 0.6214168       | 0.667189       | 0.610428            | 0.610425            |
| 0.8  | 0.7514268       | 0.771810       | 0.741514            | 0.741516            |
| 0.9  | 0.8810148       | 0.882185       | 0.871813            | 0.871816            |
| 1    | 1               | 1              | 1                   | 1                   |

### Table 2. Comparison of analytical results for the crisp velocity profile of Poiseuille flow when $\beta = 0.1$, $m = 0$, $p = -0.5$ and $\alpha$-cut = 1.

| $x$  | Siddiqui et al. | Yüresoy et al. | Kamran and Siddique | ADM present results |
|------|-----------------|----------------|---------------------|---------------------|
| 0    | 0.049141        | 0.049792       | 0.048813            | 0.048812            |
| 0.1  | 0.048818        | 0.049211       | 0.048315            | 0.048314            |
| 0.2  | 0.046781        | 0.047801       | 0.045145            | 0.045140            |
| 0.3  | 0.045168        | 0.045314       | 0.044143            | 0.044141            |
| 0.4  | 0.041015        | 0.041830       | 0.040018            | 0.040019            |
| 0.5  | 0.037188        | 0.037351       | 0.036126            | 0.036124            |
| 0.6  | 0.031417        | 0.031876       | 0.031103            | 0.031109            |
| 0.7  | 0.025001        | 0.025404       | 0.024916            | 0.024914            |
| 0.8  | 0.017912        | 0.017934       | 0.017815            | 0.017819            |
| 0.9  | 0.009214        | 0.009466       | 0.009167            | 0.009164            |
| 1    | 0               | 0              | 0                   | 0                   |
Table 3. Comparison of analytical results for the crisp velocity profile of Couette–Poiseuille flow when \( \beta = 0.1, m = 0, p = -0.5 \) and \( \alpha\text{-cut} = 1. \\

| x   | Siddiqui et al.4 | Yürüsoy et al.9 | Kamran and Siddique14 | ADM present results |
|-----|-----------------|-----------------|------------------------|--------------------|
| 0.1 | 0.081018        | 0.091401        | 0.044183               | 0.044181           |
| 0.2 | 0.121481        | 0.020762        | 0.109166               | 0.109168           |
| 0.3 | 0.194415        | 0.300141        | 0.191622               | 0.191620           |
| 0.4 | 0.281510        | 0.411359        | 0.266406               | 0.266405           |
| 0.5 | 0.380151        | 0.521415        | 0.361792               | 0.361791           |
| 0.6 | 0.481141        | 0.591618        | 0.471814               | 0.471819           |
| 0.7 | 0.601486        | 0.691619        | 0.599162               | 0.599160           |
| 0.8 | 0.721412        | 0.791819        | 0.714993               | 0.714991           |
| 0.9 | 0.881514        | 0.900410        | 0.871680               | 0.871680           |
| 1   | 1               | 1               | 1                      | 1                  |

Table 4. Fuzzy solution of \( v(x; \alpha) \), mid and \( u(x; \alpha) \) at \( m = 1, \alpha\text{-cut} = 0, \) and \( \beta = 0.75 \) with varing of \( x \).

| x   | v(x; \alpha) Mid values | u(x; \alpha) Mid values |
|-----|-------------------------|-------------------------|
| 0   | 0.01000000              | 0                       |
| 0.1 | 0.00229948              | 0.00807392              |
| 0.2 | 0.00357698              | 0.01622165              |
| 0.3 | 0.01310203              | 0.02706633              |
| 0.4 | 0.02094899              | 0.03593537              |
| 0.5 | 0.02899275              | 0.04513585              |
| 0.6 | 0.03731059              | 0.05475324              |
| 0.7 | 0.04598305              | 0.06487813              |
| 0.8 | 0.05509474              | 0.07607020              |
| 0.9 | 0.06473526              | 0.08704432              |
| 1   | 0.07500000              | 0.11250000              |

Table 5. Fuzzy solution of \( v(x; \alpha) \), mid and \( u(x; \alpha) \) at fixed values of \( \alpha\text{-cut} = 0, m = 0.2, \beta = 0.3, \) and \( p = 0.2 \), with distinct values of \( x \).

| x   | v(x; \alpha) Mid values | u(x; \alpha) Mid values |
|-----|-------------------------|-------------------------|
| -1  | 0.0007543               | 0.0058772               |
| -0.9| 0.010613                | 0.022587                |
| -0.8| 0.028613                | 0.038751                |
| -0.7| 0.041062                | 0.052969                |
| -0.6| 0.053360                | 0.067464                |
| -0.5| 0.063706                | 0.079154                |
| -0.4| 0.072098                | 0.088036                |
| -0.3| 0.078533                | 0.094103                |
| -0.2| 0.083007                | 0.097005                |
| -0.1| 0.085517                | 0.097775                |
fall. At various values of $\alpha$, the influence of $m$ on the $v(x, \alpha)$ and $u(x, \alpha)$ with constant pressure, a gradient ($dp/dy = p$) is shown in Figs. 19, 20, 21, 22 and 23. With increasing $m$, the $v(x, \alpha)$ and $u(x, \alpha)$ diminish. At $\alpha = 1$, the $v(x, \alpha)$ and $u(x, \alpha)$ are the same in Figs. 18 and 23. It has a good impact on providing a classical or crisp solution. Figure 24 shows the $v(x, \alpha)$ and $u(x, \alpha)$ for various $\alpha$ values. As a result, when $\alpha = 1$, the fuzzy velocity becomes a crisp velocity profile, demonstrated that the current problem is an extension of Kamran and Siddique\(^14\). Figures 25, 26 and 27 depict the uncertain behaviour of the TFN membership function with the triangle fuzzy plot when the values of $\beta$, $m$, and $\alpha$ are varied. In Figs. 25 and 26, the uncertain width progressively reduces as the input parameters $m$ and $\beta$ are increased, however in Fig. 27, the uncertain width suddenly grows when the value of $p$ is increased. It was also discovered that when $\alpha$ grows, the $v(x, \alpha)$ increases and the higher drops, implying that the solutions are strong. The breadth between the $v(x, \alpha)$ and $u(x, \alpha)$ grows as $\alpha$ grows, and at $\alpha = 1$, they coherent with the traditional solution. The evaluation of $v(x, \alpha)$, mid, and $u(x, \alpha)$ at various $x$ values using $p = 0.2, \beta = 0.3$, and $m = 0.2$ are shown in Table 5. Furthermore, every fixed $\alpha$-cut, fuzzy velocity profiles always shift within a particular range, and the range steadily declines as $\alpha$-cut values increase.

**Table 6.** Fuzzy solution of $v(x; \alpha)$, mid and $u(x; \alpha)$ at $\alpha$-cut = 0, $p=0.015$, $\beta = 0.2$, and $m = 0.01$, with various values of $x$.

| $x$ | $v(x; \alpha)$ | Mid values | $u(x; \alpha)$ |
|-----|-----------------|------------|----------------|
| 0   | -0.01015000     | 0          | 0.01049999     |
| 0.1 | -0.02030000     | -0.00000001| 0.02030000     |
| 0.2 | -0.01545276     | 0.01496601 | 0.04544596     |
| 0.3 | 0.00438940      | 0.04498324 | 0.08557708     |
| 0.4 | 0.03921724      | 0.08994659 | 0.14067595     |
| 0.5 | 0.08901401      | 0.14986606 | 0.21071810     |
| 0.6 | 0.13375484      | 0.22471308 | 0.29567132     |
| 0.7 | 0.23340559      | 0.31445006 | 0.39549452     |
| 0.8 | 0.32921567      | 0.41902902 | 0.51013648     |
| 0.9 | 0.43724582      | 0.53838998 | 0.63953414     |
| 1   | 0.56130727      | 0.67245899 | 0.78361071     |

**Generalized Couette flow.** The impact of $\beta$, $m$, and $\alpha$ on the $v(x, \alpha)$ and $u(x, \alpha)$ with a pressure gradient is shown in Figs. 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38 and 39. The $v(x, \alpha)$ and $u(x, \alpha)$ drop gradually with growing $\beta$ in Figs. 28, 29, 30 and 31. The $v(x, \alpha)$ and $u(x, \alpha)$ grow steadily with upsurging $m$ in Figs. 32, 33, 34 and 35. The $v(x, \alpha)$ and $u(x, \alpha)$ grow fast with swelling $p$ in Figs. 36, 37, 38 and 39. Figures 32, 35, and 39 show that when $\alpha$ grows, the $v(x, \alpha)$ and $u(x, \alpha)$ are the same. Figure 40 shows the $v(x, \alpha)$ and $u(x, \alpha)$ for various $\alpha$ values. Because the crisp velocity profile lies between the $v(x, \alpha)$ and $u(x, \alpha)$, when $\alpha = 1$, the fuzzy velocity profile becomes crisp or classical, signifying that the current article is a modification of Kamran and Siddique\(^14\). Figures 41, 42 and 43 depict the uncertain behaviour of the TFN membership function expressed as a function of the triangle fuzzy plot for various values of $\beta$, $p$, and $m$. Now Figs. 41 and 42, the ambiguous width steadily grows as the input parameters $m$ and $\beta$ are raised, however in Fig. 43, the fuzzy width quickly rises as $p$ is enhanced. Table 6 presents the assessment of $v(x, \alpha)$, mid, and $u(x, \alpha)$ velocity profiles at various $x$ values using $p = 0.015$, $m = 0.1$ and $\beta = 0.2$, as fixed values. Moreover, each fixed $\alpha$-cut, fuzzy velocity profile always shift within a particular range, and the range steadily reduces as $\alpha$-cut values increase.

The solutions are well-suited in the aforementioned discussions; the crisp solution is sandwiched between the fuzzy solutions (lower and upper-velocity profiles), and $\alpha$ approaching one position the fuzzy solutions are close to the crisp solution. The fuzzy velocity profile of the fluid is a better choice than the crisp or classical velocity profile of the fluid, according to the conclusion of the entire discussion. The single flow situation of fluid is represented by a crisp or classical velocity profile, but the interval flow situation is represented by a fuzzy velocity profile, which has lower and higher boundaries. In addition, the model described a new feature at various $\alpha$ values and gave accurate solution intervals (lower and upper-velocity profiles) for better dynamic analysis judgment.

**Plane Couette flow**
Figures 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13 and Table 4.

**Plane Poiseuille flow**

Figures 14, 15, 16, 17, 18, 20, 21, 22, 23, 24, 25, 26, 27 and Table 5.

**Generalized Couette flow**

Figure 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43 and Table 6.

**Conclusions**

The three fundamental flow phenomena that inevitably arise in the study of fluid dynamics, especially plane Poiseuille, plane Couette, and generalised Couette flow of a non-Newtonian fluid under the impact of MHD force in a fuzzy environment, have been investigated in this work. The dimensionless governing DEs are discretized into FDEs with fuzzified BCs, and ADM is used to resolve them. When compared to previous results, the current crisp results acquired by ADM are shown to be in excellent agreement. The TFNs are utilised for uncertainties on the dynamic behaviour of the said problem. The velocity profiles (lower and higher) grow when the β and α-cut increase, whereas the fuzzy velocity profile decrease as the m increases in three flow situations. The range of predicted lower and upper-velocity profiles is dependent on the α-cut, according to the findings. The end outcome is always an envelope of solutions with a crisp solution in the middle. As a result, As a result, fuzzy velocity fields are the modification of the crisp velocity field of a third grade fluid flowing between two parallel plates.

**Data availability**

The datasets used and/or analysed during the current study available from the corresponding author on reasonable request.

**Appendix**

\[
A_1 = \frac{11\alpha}{200}, \quad A_2 = \frac{17\alpha - 6}{200}, \quad A_3 = \frac{107\alpha - 30}{200}, \quad A_4 = \frac{\alpha - 1}{100},
\]
\[
B_1 = \frac{41 - 21\alpha}{400}, \quad B_2 = \frac{53 - 33\alpha}{400}, \quad B_4 = \frac{1 - \alpha}{100}, \quad B_3 = \frac{347 - 207\alpha}{400},
\]
\[
E = \frac{(p + 1)(\alpha - 1)}{100}, \quad D = \frac{(p + 1)(\alpha - 1) - 50p}{100},
\]
\[
K = \frac{\beta D^5}{5p} + \frac{\beta D^3}{4p^2} \left( D^4 - (p + D)^4 \right) + \frac{\beta D^6}{5p} + \frac{m^2 D^3(p + 4D + 12E)}{72p},
\]
\[
R = \frac{\beta(p + D)^6}{30D^3} + \frac{\beta(p + D)^6}{12p} - \frac{D^4}{4p} - \frac{\beta D^6}{60p^2} - \frac{m^2}{720} \left( 6D + p + 30E - 5p - 20D - 60E \right),
\]
\[
L = \frac{\beta(p + D)^5}{5p} + \frac{\beta(p + D)^6}{5p} + \frac{3\beta(p + D)^3}{4p^2} \left( D^4 - (p + D)^4 \right) - \frac{\beta D^5}{5p} - \frac{\beta D^3(p + D + D^4)}{4p^2},
\]
\[
E_1 = \frac{(1 + p)(1 - \alpha)}{100}, \quad D_1 = \frac{(1 + p)(1 - \alpha) - 50p}{100},
\]
\[
K_1 = \frac{\beta D_1^5}{5p} + \frac{\beta D_1^3}{4p^2} \left( D_1^4 - (p + D_1)^4 \right) + \frac{\beta D_1^6}{5p} + \frac{m^2 D_1^3(p + 4D_1 + 12E_1)}{72p}.
\]
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**Author contributions**

Conceptualization; I.S, M.N and R.N.J. methodology; A.A and I.S, writing—original draft preparation; M.N and A.A, Investigation; R.N.J, writing—review and editing, I.S, I.K and M.A.S. Funding acquisition; I.K and M.A.S. Software; M.N and A.A, Validation, I.K and M.A.S, supervision; I.S. All authors contributed equally. All authors have read and approved the final manuscript.

**Competing interests**

The authors declare no competing interests.

**Additional information**

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