SOLUTIONS WITHOUT SINGULARITIES IN
GAUGE THEORY OF GRAVITATION

G. ZET* and C. D. OPRISAN
Department of Physics,
"Gh. Asachi" Technical University,
Iasi 6600, Romania
*gzet@phys.tuiasi.ro
S. BABETI
Department of Physics,
"Politehnica" University,
Timisoara 1900, Romania

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Abstract
A de-Sitter gauge theory of the gravitational field is developed using a spherical symmetric Minkowski space-time as base manifold. The gravitational field is described by gauge potentials and the mathematical structure of the underlying space-time is not affected by physical events. The field equations are written and their solutions without singularities are obtained by imposing some constraints on the invariants of the model. An example of such a solution is given and its dependence on the cosmological constant is studied. A comparison with results obtained in General Relativity theory is also presented.

Keywords: gauge theory, gravitation, singularity, computer algebra

1 Introduction

The gauge theory of gravitation has been considered by many authors in order to describe the gravity in a similar way with other interactions (electromagnetic, weak or strong). As gauge groups there were chosen Poincaré group, de-Sitter group, affine group, etc. In this paper we use the de-Sitter (DS) group as gauge group in order to obtain a model with cosmological constant for the gravitational field. The Poincaré gauge theory is obtained as a limit of DS model when the cosmological constant vanishes.

In Sect.1 we introduce the gauge fields $e^a_\mu(x)$ (tetrad fields) and $\omega^{ab}_\mu(x)$ (spin connection). They are used to construct the field strengths $F^a_\mu$, and $F^{ab}_\mu$, and the invariants of the theory. Then the integral of the action is written and the
constraints for non-singular solutions are introduced in its expression by means of two Lagrange-multiplier fields $\varphi_1(t)$ and $\varphi_1(t)$.

The field equations are obtained in Sect. 2 for a particular form of spherically symmetric gauge fields. They contain the cosmological constant $\Lambda$ introduced into the model by using the $DS$ group as gauge symmetry. The calculations in this paper, especially in Sect. 3, have been performed using an analytical program written by us in the package GRTensor II running on the Maple V platform. This program allows to calculate the components of the field strengths, the invariants of the model and also the integral of the action. In the same time, it enables the obtaining of the field equations for the gravitational field in a region without matter.

An example of solution without singularities is presented in Sect. 3. This solution is a time-periodic one with frequency of the gravitational field depending on the cosmological constant ($\Lambda < 0$). The case with positive cosmological constant ($\Lambda > 0$) can be studied choosing the anti-de-Sitter group as gauge group.

In Sect. 4 some concluding remarks are presented and a comparison with other results based on the General Relativity theory is made.

### 2 Gauge theory of gravitation

We consider a gauge theory of gravitation having de-Sitter ($DS$) group as local symmetry. Let $X_A$, $A = 1, 2, ..., 10$, be a basis of $DS$ Lie algebra with the corresponding equations of structure given by:

$$[X_A, X_B] = i f_{AB}^C X_C, \quad (1)$$

where $f_{AB}^C = -f_{BA}^C$ are the constants of structures whose expressions will be given below [see Eq.(4)]. We envision space-time as a four-dimensional manifold $M_4$; at each point of $M_4$ we have a copy of $DS$ group (i.e., a fibre, in fibre-bundle terminology). Introduce, as usually, the gauge potentials $h_A^{\mu}(x)$, $A = 1, 2, ..., 10, \mu = 0, 1, 2, 3$, where $(x)$ denotes the local coordinates on $M_4$. Then, we calculate the field-strengths $F_{\mu\nu} = F_A^{\mu\nu} X_A$, which take values in Lie algebra of $DS$ group (Lie-algebra valued). The components $F_A^{\mu\nu}$ are given by:

$$F_A^{\mu\nu} = \partial_\mu h_A^\nu - \partial_\nu h_A^\mu + f_{BC}^A h_B^\mu h_C^\nu. \quad (2)$$

In order to write the constants of structure $f_{AB}^C$ in a compact form, we use the following notations for the index $A$:

$$A = \left\{ \begin{array}{c} a = 0, 1, 2, \\
[ab] = [01], [02], [03], [12], [13], [23]. \end{array} \right. \quad (3)$$

This means that $A$ can stand for a single index like 2 as well as for a pair of indices like $[01]$, $[12]$ etc. The infinitesimal generators $X_A$ are interpreted as: $X_a \equiv P_a$ (energy-momentum operators) and $X_{[ab]} \equiv M_{ab}$ (angular momentum
operators) with the property $M_{ab} = - M_{ba}$. The constants of structures $f^C_{AB}$ have then the following expressions:

\[
\begin{align*}
  f^a_{bc} &= f^{[ab]}_{c[de]} = f^a_{[bc][de]} = 0, \\
  f^{[ab]}_{cd} &= 4\lambda^2 \left( \delta^a_c \delta^b_d - \delta^a_d \delta^b_c \right) = - f^{[ab]}_{dc}, \\
  f^a_{b[cd]} &= - f^a_{[cd]b} = \frac{1}{2} \left( \eta_{bc} \delta^a_d - \eta_{bd} \delta^a_c \right), \\
  f^{[ef]}_{[ab][cd]} &= \frac{1}{4} \left( \eta_{bc} \delta^a_d \delta^f_e - \eta_{ac} \delta^b_d \delta^f_e + \eta_{ad} \delta^b_c \delta^f_e - \eta_{bd} \delta^a_c \delta^f_e \right) - e \leftrightarrow f,
\end{align*}
\]

where $\lambda$ is a real parameter, and $(\eta_{ab}) = \text{diag}(1, -1, -1, -1)$ is the Minkowski metric. In fact, here we have a deformation of de-Sitter Lie algebra having $\lambda$ as parameter. Considering the contraction $\lambda \to 0$ we obtain the Poincaré Lie algebra, i.e., the group DS contracts to the Poincaré group.

We denote the gravitational gauge fields (or potentials), $h^A_{\mu}(x)$, by $e^a_{\mu}(x)$ (tetrads fields) if $A = a$, and by $\omega^a_{\mu}(x) = - \omega^a_{\mu}(x)$ (spin connection) if $A = [ab]$. Then, introducing the Eqs. (4) into the definition (2), we obtain the expressions of the strength tensor components:

\[
F^a_{\mu\nu} = \partial_\mu e^a_\nu - \partial_\nu e^a_\mu + \left( \omega^a_{\mu} e^e_\nu - \omega^a_{\nu} e^e_\mu \right) \eta_{bc}
\]

\[
F^a_{\mu\nu} = \partial_\mu \omega^a_{\nu} - \partial_\nu \omega^a_{\mu} + \left( \omega^c_{\mu} \omega^b_{\nu} + \omega^b_{\mu} \omega^c_{\nu} \right) \eta_{cd} - 4\lambda^2 \left( e^a_\mu e^b_\nu - e^a_\nu e^b_\mu \right).
\]

The integral of action associated to the gravitational gauge fields, quadratic in the components $F^A_{\mu\nu}$, is written in the form:

\[
S_g = \int d^4x \varepsilon^{\mu\nu\rho\sigma} F^A_{\mu\nu} F^B_{\rho\sigma} Q_{AB},
\]

where $\varepsilon^{\mu\nu\rho\sigma}$ is the Levi-Civita symbol of rank four, with $\varepsilon^{0123} = 1$. This action is independent of any specific metric on $M_4$; indeed, the property of general covariance for action imposes the volume element $\sqrt{-g} d^4 x$ [with $g = \det(g_{\mu\nu})$] and the tensor Levi-Civita has the form $\varepsilon^{\mu\nu\rho\sigma} \sqrt{-\bar{g}}$, so that the $g_{\mu\nu}$-dependence of $S_g$ cancels.

The quantities $Q_{AB}$ are constants, symmetric with respect to the indices $A, B$: $Q_{AB} = Q_{BA}$. If we chose

\[
Q_{AB} = \begin{cases} 
\varepsilon_{abcd}, & \text{for } A = [ab], \\
0, & \text{otherwise},
\end{cases}
\]

then we obtain the action integral of the General Relativity ($GR$). It is possible also to obtain the integral action of Teleparallel Gravity ($TG$) by an appropriate choice of $Q_{AB}$.

Now, we use the form given in Eq. (8) in order to obtain solutions without singularities of $DS$-gauge theory of gravitation. Namely, we impose some
restrictions\(^7\) on two invariants \(I_1\) and \(I_2\) of the theory. Introducing the Lagrange-multiplier \(\varphi_1(t)\) and \(\varphi_2(t)\), and using the choice (8), the integral of action (7) can be rewritten as:

\[
S_g = -\frac{1}{16\pi G} \int d^4x e [F + \varphi_1(t) f_1(I_1) + \varphi_2(t) f_2(I_2) + V(\varphi_1, \varphi_2)], \tag{9}
\]

where

\[
F = F_{\mu\nu}^{ab} \varepsilon_{\mu\nu}, \quad e = \det(e^a_\mu) \tag{10}
\]

and \(\varepsilon^\nu_b\) is the inverse of \(e^a_\mu\) defined by Eq. (21) below. The quantities \(f_i(I_i), i = 1, 2\) are functions which must be chosen in an appropriate form in order to obtain solutions without singularities of the corresponding field equations. Thus, the potential \(V(\varphi_1, \varphi_2)\) have to satisfy the constraint equations\(^7\):

\[
f_1(I_1) = -\frac{\partial V}{\partial \varphi_1}, \quad f_2(I_2) = -\frac{\partial V}{\partial \varphi_2}, \tag{11}
\]

The model can be simplified further if we assume:

\[
V(\varphi_1, \varphi_2) = V_1(\varphi_1) + V_2(\varphi_2), \tag{12}
\]

and chose the invariants \(I_1, I_2\) in the form

\[
I_1 = F - \sqrt{3} \left(4F_{\mu}^a F^\mu_a - F^2\right)^{1/2}, \tag{13}
\]

respectively

\[
I_2 = 4F_{\mu}^a F^\mu_a - F^2. \tag{14}
\]

In these expressions, the quantities \(F_{\mu}^a\) are defined by

\[
F_{\mu}^a = F_{\mu\nu}^{ab} \varepsilon_b^\nu. \tag{15}
\]

As an example, we chose the functions \(f_1\) and \(f_2\) in the simple form\(^7\):

\[
f_1(I_1) = I_1, \quad f_2(I_2) = -\sqrt{I_2}. \tag{16}
\]

Then, the action \(S_g\) in Eq. (9) becomes:

\[
S_g = -\frac{1}{16\pi G} \int d^4x e \left[F + \varphi_1 I_1 - \varphi_2 \sqrt{I_2} + V_1(\varphi_1) + V_2(\varphi_2)\right]. \tag{17}
\]

Now, all we have to do is to write the variational field equations which follow from (17) and search their solutions without singularities.
3 Field equations

We develop the DS gauge theory in a space-time Minkowski \( M_4 \) endowed with spherical symmetry:

\[
d s^2 = d t^2 - d r^2 - r^2 \left( d \theta^2 + \sin^2 \theta d \varphi^2 \right)
\]  

(18)

and having the coordinates \((x^\mu) = (x^0, x^1, x^2, x^3) = (t, r, \theta, \varphi)\). In addition, we choose a particular form of spherically gauge fields \( e^a_\mu (x) \) and \( \omega^a_{\mu \nu} (x) \) given by the following ansatz:

\[
e^0_\mu = (N(t), 0, 0, 0), \quad e^1_\mu = \left( 0, \frac{a(t)}{\sqrt{1-k r^2}}, 0, 0 \right),
\]

\[
e^2_\mu = (0, 0, r a(t), 0), \quad e^2_\mu = (0, 0, 0, r a(t) \sin \theta),
\]

respectively

\[
\omega^{01}_\mu = \left( 0, -\frac{a'(t)}{N(t) \sqrt{1-k r^2}}, 0, 0 \right), \quad \omega^{02}_\mu = \left( 0, 0, -\frac{ra'(t)}{N(t)}, 0 \right),
\]

\[
\omega^{03}_\mu = \left( 0, 0, 0, -\frac{r a'(t) \sin \theta}{N(t)} \right), \quad \omega^{12}_\mu = \left( 0, 0, 0, \sqrt{1-k r^2} \right), \quad \omega^{23}_\mu = (0, 0, 0, \cos \theta)
\]

(20a)

(20b)

(20c)

where \( N(t) \) and \( a(t) \) are functions only of the time variable, \( k \) is a constant, and \( a' \) is the derivative of \( a(t) \) with respect to the variable \( t \). The choice (20) of gauge fields \( \omega^a_{\mu \nu} (x) \) assures that all components of the strength tensor \( F^a_{\mu \nu} \) vanish. If we remember the Riemann-Cartan theory of gravitation, then this result implies the vanishing of the torsion tensor \( T^a_{\mu \nu} = \delta^a_\nu F^a_{\mu \nu}, \) in accord with GR theory. Here, \( \tilde{e}^a_\mu \) denotes the inverse of \( e^a_\mu \) with the properties:

\[
e^a_\mu \tilde{e}^b_\mu = \delta^a_b, \quad e^a_\mu \tilde{e}^a_\nu = \delta^\nu_\mu.
\]

(21)

From this point to the end we performed all the calculations using an analytical program conceived by us which is presented in the final part of this Section.

Using the Eqs. (19) and (20), we obtain the following expressions of the invariants \( F, I_1 \) and \( I_2 \) above defined:

\[
F = -6 \frac{a a'' N - a a' N' + k N^3 + a'^2 N + 8 \lambda^2 a^2 N^3}{a^2 N^3},
\]

(22)

\[
I_1 = -12 \frac{k N^2 + a'^2 + 4 \lambda^2 a^2 N^2}{a^2 N^2},
\]

(23)
and respectively
\[ I_2 = 12 \frac{(kN^3 + a^2N - aa''N + aa'N')^2}{a^4N^6}. \] (24)

where \(a''\) is the second derivative of \(a(t)\) with respect to the variable \(t\). Introducing these expressions into Eq. (17) and imposing the variational principle \(\delta S_g = 0\) with respect to \(N(t), \varphi_1(t)\) and \(\varphi_2(t)\), we obtain the corresponding field equations. We write now these equations for the particular case \(N(t) = 1\) which is of interest in our model:

\[
\begin{aligned}
\left\{ \begin{array}{l}
-\frac{1}{2} (V_1 + V_2) + 3H^2 (1 - 2\varphi_1) + 3\frac{k}{a^2} (1 + 2\varphi_1) - 2\Lambda = \\
\sqrt{3} \left( \varphi'_2 + 3H \varphi_2 - \frac{k}{2a^2} \varphi_2 \right),
\end{array} \right.
\end{aligned}
\] (25)

\[
\frac{k}{a^2} + H^2 - \frac{\Lambda}{3} = \frac{1}{12} \frac{dV_1}{d\varphi_1}, \quad H = \frac{a'}{a},
\] (26)

\[
H' - \frac{k}{a^2} = -\frac{1}{2\sqrt{3}} \frac{dV_2}{d\varphi_2}, \quad H' = \frac{dH}{dt} = \frac{a''a - a'^2}{a^2},
\] (27)

where \(\varphi'_2\) is the derivative of \(\varphi_2(t)\) with respect to \(t\), and \(\Lambda = -12\lambda^2\) is interpreted as cosmological constant.\(^2\)\(^8\)

If we consider the limit \(\lambda \to 0\), or equivalently \(\Lambda = 0\), we obtain the results in Ref.[7]; but, for \(\Lambda \neq 0\) we can study in addition the dependence on the cosmological constant of the solutions (without singularities) obtained by solving the Eqs.(25)-(27). We make also the mention that the Eqs. (26) and (27) are identically with the constraints (11) introduced into the integral of the Lagrange-multiplier fields \(\varphi_1(t)\) and \(\varphi_2(t)\).

We can also add matter to the previous model considering the integral of action:
\[
S_m = \int d^4xL_m,
\] (28)

where \(L_m\) is the matter density of Lagrangian. In this paper we restrict ourselves to the case without matter. In Section 4 we will obtain a particular solution with fixed cosmological constant \(\Lambda = \text{const}\). Of course, there are possible also solutions with variable cosmological ”constant” depending on time. The solution presented below is inspired from the results of Ref. [7] and we show that our cosmological constant \(\Lambda\) is related with the constant \(H_0\) in that work and which is expected to be Planck scale.

The calculations in this paper, especially in Sect. 3, were performed using an analytical program written by us in the package GRTensor II running on the Maple V platform. This program allows to calculate the components \(F^a_{\mu\nu}\) and \(F_{\mu\nu}^{ab}\) of the strength tensor field \(F_{\mu\nu}\), the expressions of the quantities \(F^a_{\mu}\) defined in Eq. (15), and the invariants \(F, I_1, I_2\). We calculated also the integrand in the action \(S_g\) and obtained the field Eqs. (25)-(27).

It is important to emphasize that in our gauge model of gravitation we do not use a metric, but only the gauge fields (potentials) \(e^a_{\mu}(x)\) and \(\omega^{ab}_{\mu}(x)\) of the
gravitational field. In program we used the notations: einv \{a \, \mu\} = e^\mu_a for the inverse of e^\mu_a, and de = det (e^\mu_a) for the determinant of e^\mu_a. The symbols for other quantities are introduced analogously.

Below, we list down the part of program which allows to define and to calculate the quantities needed in obtaining of Eqs. (25)-(27).

Program " DS GAUGE THEORY.MWS"

```
restart:
grtw( );
grload(minkowski, 'd:/maple/sferice.mpl');
grdef('ev{\hat{a} \mu}'); grcalc(ev(up,dn));
grdef('\eta1{a \, b}'); grcalc(\eta1(dn,dn));
grdef('\omega{\hat{a} \, \hat{b} \, \mu}'); grcalc(\omega(up,up,dn));
Famn{\hat{a} \mu \, \nu} := ev{\hat{a} \nu, \mu} - ev{\hat{a} \mu, \nu}
+ omega{\hat{a} \, \hat{b} \mu}*ev{\hat{c} \nu}*\eta1{b \, c}
- omega{\hat{a} \, \hat{b} \mu}*ev{\hat{c} \mu}*\eta1{b \, c}
); grcalc(Famn(up,dn,dn));
Fabmn{\hat{a} \, \hat{b} \mu \, \nu} := omega{\hat{a} \, \hat{b} \nu, \mu} - omega{\hat{a} \, \hat{b} \mu, \nu}
+ 4*lambda^2*( ev{\hat{a} \mu}*ev{\hat{b} \nu} - ev{\hat{b} \mu}*ev{\hat{a} \nu})
); grcalc(Rabmn(up,up,dn,dn));
R:=Rabmn{\hat{a} \, \hat{b} \mu \nu}*einv{\hat{b} \nu} *einv{\hat{a} \mu}
); grcalc(R);
F{\hat{a} \mu}:=Rabmn{\hat{a} \, \hat{b} \mu \nu}*einv{\hat{b} \nu}
); grcalc(F(up,dn));
I2:=4*F{\hat{a} \mu} *Finv{\hat{a} \mu} -(R)^2
); grcalc(I2);
I1:=R-sqrt(3)*sqrt(I2)
); grcalc(I1);
de
); grcalc(de);
Lg:=(R+phi1(t)*I1-phi2(t)*sqrt(I2)+V1(phi1)+V2(phi2)) *de
); grcalc(Lg); grdisplay( );
```

4 Example of solution without singularities

The solution of Eqs. (25)-(27) includes a dependence on the cosmological constant \( \Lambda \). We suppose that the Lagrange-multiplier function \( \phi_1(t) \) is absent, and consider the case when \( k = 0 \). Then, denoting \( \phi_2(t) = \phi(t) \) and \( V_2(\phi_2) = V(\phi) \), the Eqs. (25)-(27) become:

\[
H' = -\frac{1}{2\sqrt{3}} \frac{dV}{d\phi},
\]

\[
\phi' = -3H\phi + \sqrt{3}H - \frac{1}{2\sqrt{3}H} V - \frac{2\Lambda}{\sqrt{3}H}.
\]

We consider the potential \( V(\phi) \) of the simple form:

\[
V(\phi) = 2\sqrt{3}\lambda^2 \left( \frac{\phi^2}{1 + \phi^2} + \frac{24}{\sqrt{3}} \right),
\]

7
where $\lambda$ is the real parameter that determines the cosmological constant $\Lambda$. This parameter coincides with the constant $H_0$ in Ref. [7] that has been interpreted as a Planck scale of the model. Therefore, in our example the Planck scale is related to the cosmological constant $\Lambda$. For small values of $H$ and $\varphi$, the Eqs. (29) can be written as:

$$H' \simeq -2\lambda^2 \varphi,$$

$$\varphi' (t) \simeq \frac{\sqrt{3}H^2 - \lambda^2 \varphi^2}{H}.$$ (31)

These equations have the periodic solution

$$\varphi (t) = \varphi_0 \sin (\omega t), \quad H (t) = \frac{\omega \varphi_0}{2\sqrt{3}} [\cos (\omega t) - 1],$$ (32)

where $\varphi_0$ is an integration constant and $\omega = 2 \times 3^{1/4} \lambda$ is the frequency of oscillation of the corresponding gravitational field described by the gauge potentials $e_\mu^a (x)$ and $\omega_\mu^{ab} (x)$. This solution has no singularities and it is valid if the cosmological constant is negative ($\Lambda < 0$). The case with positive cosmological constant ($\Lambda > 0$) can be studied choosing the anti-de-Sitter group as gauge group. But, the deformation parameter $\lambda$ will be then pure imaginary.

We emphasize that there are possible also periodic solutions if we suppose a time-dependent cosmological "constant". In particular, we can consider a cosmological "constant" which is itself a periodic function on time. It will be also of interest to apply the previous method in obtaining non-singular solutions of the gauge theories with internal groups of symmetry.

5 Concluding remarks

We developed a de-Sitter gauge theory of gravitation on a spherical symmetric Minkowski space-time as base manifold. This theory allows a complementary description of the gravitational effects in which the mathematical structure of the underlying space-time is not affected by physical events. Only the gauge potentials $e_\mu^a (x)$ and $\omega_\mu^{ab} (x)$ of the gravitational field change as functions of coordinates. This is important when we consider a quantum gauge theory of gravitation.

In order to obtain solutions without singularities, we imposed constraints on some invariants of the gauge model we considered. The solutions in this paper are time-periodic and correspond to a fixed (negative) cosmological constant $\Lambda$ whose value is related to the Planck scale and that determines the frequency of the corresponding gravitational field.

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