ABSOLUTE PARALLELISM FOR 2-NONDEGENERATE CR STRUCTURES VIA BIGRADED TANAKA PROLONGATION

CURTIS PORTER AND IGOR ZELENKO

Abstract. This article is devoted to the local geometry of 2-nondegenerate CR manifolds $M$ of hypersurface type. An absolute parallelism for such structures was recently constructed independently by Isaev-Zaitsev, Medori-Spiro, and Pocchiola in the minimal possible dimension ($\dim M = 5$), and for $\dim M = 7$ in certain cases by the first author. In this paper, we develop a bigraded (i.e., $\mathbb{Z} \times \mathbb{Z}$-graded) analog of Tanaka’s prolongation procedure to construct a canonical absolute parallelism for these CR structures in arbitrary (odd) dimension with Levi kernel of arbitrary admissible dimension. We introduce the notion of a bigraded Tanaka symbol – a complex bigraded vector space – containing all essential information about the CR structure. Under the additional regularity assumption that the symbol is a Lie algebra, we define a bigraded analog of the Tanaka universal algebraic prolongation, endowed with an anti-linear involution, and prove that for any CR structure with a given regular symbol there exists a canonical absolute parallelism on a bundle whose dimension is that of the bigraded universal algebraic prolongation. Moreover, we show that there is a unique (up to local equivalence) such CR structure whose algebra of infinitesimal symmetries has maximal possible dimension, and the latter algebra is isomorphic to the real part of the bigraded universal algebraic prolongation of the symbol. In the case of 1-dimensional Levi kernel we classify all regular symbols and calculate their bigraded universal algebraic prolongations. In particular, we show that in this case all these prolongations are finite dimensional and, for the most interesting class of the regular CR symbols, the universal prolongation is isomorphic to the complex orthogonal algebra $\mathfrak{so}((m, \mathbb{C})$ where $m = \frac{1}{2}(\dim M + 5)$. For each such symbol we specify the real form of this algebra corresponding to the algebra of infinitesimal symmetries of the maximally symmetric model.

1. Introduction

1.1. Preliminaries. A CR structure on a manifold $M$ is a distribution $D \subset TM$ of even rank, together with an isomorphism field $J_x : D_x \to D_x$ which satisfies $J^2 = -\text{Id}$, such that the following integrability condition holds: if the complexification $\mathbb{C}D_x$ is split into the $i$-eigenspace $H_x$ of $J_x$ and the $(-i)$-eigenspace $\overline{H}_x$ (i.e., $\sqrt{-1}$-eigenspace), then the distribution $H - \overline{H}$ – and therefore $\mathcal{H}$ – is involutive. At their generic points, real, codimension-$c$ submanifolds of $\mathbb{C}^{n+c}$ are endowed with the natural CR structure determined by rank-$n$, complex subbundles of their tangent bundles. When $D$ has corank-1, the CR structure is said to be of hypersurface type. The Levi form is a $\mathbb{C}T M/\mathbb{C}D$-valued Hermitian form $\mathcal{L}$ defined on $H$ by

$$\mathcal{L}(X, Y) = i[X, Y] \mod \mathbb{C}D \quad X, Y \in \Gamma(H).$$

The Levi kernel $K \subset H$ consists of CR vectors which are degenerate for the Levi form. CR-structures with $K = 0$ are called Levi-nondegenerate.

The equivalence problem for Levi-nondegenerate CR structures of hypersurface type is classical: E. Cartan solved it for hypersurfaces in $\mathbb{C}^2$ [4], then Tanaka [24] and Chern and Moser [5] generalized the solution to hypersurfaces in $\mathbb{C}^{n+1}$ for $n \geq 1$. This case is well understood in the general framework of parabolic geometries [26] [2] [3]. If the Levi form has signature $(p, q)$ for $p + q = n$, then the maximally symmetric model is obtained as a complex projectivization of the cone of nonzero vectors in $\mathbb{C}^{n+2}$ which are isotropic with respect to a Hermitian form of signature $(p + 1, q + 1)$, and the algebra of infinitesimal symmetries of this model is isomorphic to $\mathfrak{su}(p + 1, q + 1)$.

2010 Mathematics Subject Classification. 32V05, 32V40, 53C10.

Key words and phrases. 2-nondegenerate CR structures, absolute parallelism, Tanaka symbol, Tanaka universal algebraic prolongation.

Research of I. Zelenko is supported by NSF grant DMS-1406193.
Now suppose that a CR structure is uniformly (i.e., at every point) Levi-degenerate and that its Levi kernel $K$ is a distribution of complex rank $r$. Freeman [10] introduced a special filtration inside of $K$ and $\mathcal{K}$ and showed that the CR manifold cannot be “straightened” – i.e., it is not equivalent to the direct product of a CR manifold of dimension $(\dim M - 2r)$ with $\mathbb{C}^r$ – if and only if the filtration terminates trivially. If the zero distribution first occurs in the $k\text{th}$ step of the filtration of $K$, the CR-structure is called $k$-nondegenerate. Since $K$ itself is the first step of the filtration, Levi-nondegenerate CR structures are exactly 1-nondegenerate in this terminology.

In the present paper we work with $k = 2$ only, so we restrict ourselves to the description of 2-nondegeneracy. For any fiber bundle $\pi : E \to M$, $E_x = \pi^{-1}(x)$ denotes the fiber of $E$ over $x \in M$ and $\Gamma(E)$ denotes the sheaf of smooth (local) sections of $E$. For $v \in K_x$, and $y \in \mathcal{H}_x$, take $V \in \Gamma(K)$ and $Y \in \Gamma(\mathcal{H})$ such that $V(x) = v$ and $y \equiv Y(x) \mod CD$, and define a linear map

$$\text{ad}_v : \mathcal{H}_x/K_x \to H_x/K_x,$$

$$y \mapsto [V,Y]|_x \mod K_x \oplus \mathcal{H}_x.$$  

One can similarly define a linear map $\text{ad}_v : H_x/K_x \to \mathcal{H}_x/K_x$ for $v \in K_x$ (or simply take complex conjugates). A Levi-degenerate CR structure is 2-nondegenerate at $x$ if there is no nonzero $v \in K_x$ (equivalently, no nonzero $v \in K_x$) such that $\text{ad}_v = 0$.

The lowest dimension in which 2-nondegeneracy can occur is when $\dim M = 5$ and the CR structure is of hypersurface type. A natural candidate for the maximally symmetric model is given by a tube hypersurface in $\mathbb{C}^3$ over the future light cone in $\mathbb{R}^3$,

$$M_0 = \{ (z_1, z_2, z_3) \in \mathbb{C}^3 : (\text{Re} z_1)^2 + (\text{Re} z_2)^2 - (\text{Re} z_3)^2 = 0, \text{Re} z_4 > 0 \}$$

and this model has been extensively studied in [8, 9, 10, 13]. In particular, its algebra of infinitesimal symmetries is equal to $\mathfrak{so}(3,2)$. However, the structure of absolute parallelism in this situation was constructed only recently and independently in the following three papers (preceded by the work [4] for a more restricted class of structures): by Isaev and Zaitsev [14], Medori and Spiro [16] and Pocchiola [20]. In the second paper the parallelism is also claimed to be a Cartan connection, while in the other two it is not. In order to address this discrepancy, Medori and Spiro compared their construction with others in [17].

The only result about an absolute parallelism for 2-nondegenerate, hypersurface-type CR structures of dimension higher than 5 was recently obtained by the first author [21] in the case of 7-dimensional CR manifolds with rank$_K K = 1$, under certain additional algebraic assumptions that are automatic in 5-dimensional case.

In this article we develop a unified framework for the construction of an absolute parallelism for 2-nondegenerate, hypersurface-type CR structures on manifolds of arbitrary odd dimension $\geq 5$ and with Levi kernel of arbitrary admissible dimension, under certain additional algebraic assumptions. These algebraic assumptions are automatic in dimension 5, and in dimension 7 they include all cases treated in [21], along with an additional case only mentioned therein. The algebraic assumption in dimension 7 also matches the homogeneous models with simple symmetry groups discussed recently by A. Santi [22]. Our method is a modification of Tanaka’s algebraic and geometric prolongation procedures from his 1970 paper [25].

Note that the method of Medori and Spiro [16] is also in a spirit of Tanaka theory, namely of the 1979 Tanaka paper [23] devoted to parabolic geometries. Although the geometries under consideration are not parabolic, from an a priori knowledge of the homogeneous model [1, 2] and its algebra of infinitesimal symmetries $\mathfrak{so}(3,2)$, and by introducing the notion of filtered structures with an additional semitone, the authors were able to construct the normal Cartan connection. However, their construction seems specific to the concrete case under consideration, using the properties of the algebra $\mathfrak{so}(3,2)$ in particular, and the method does not appear to be easily extended to 2-nondegenerate CR structures in higher dimensions.

1.2. **Tanaka’s algebraic and geometric prolongation procedures.** Our initial objective was to follow the scheme of Tanaka’s 1970 paper [25], in which he developed a deep generalization of the theory of $G$-structures that is ideally adapted to nonholonomic structures. Let us briefly describe the main steps.
of the Tanaka constructions relevant to us here (for details see the original paper \(^25\) and also \(^19\) \(^1\) \(^29\)). Here and in the sequel, “graded vector space” means \(\mathbb{Z}\)-graded.

First, a distribution \(D \subset TM\) generates a filtration of the tangent bundle by taking iterative Lie brackets of its sections. Passing from this filtered structure to the corresponding graded objects, Tanaka assigns to the distribution at a point \(x\) a negatively \((\mathbb{Z}_-)\) graded nilpotent Lie algebra \(\mathfrak{g}_-(x) = \bigoplus_{i<0} \mathfrak{g}_i(x)\), called the Tanaka symbol of \(D\) at \(x\), which contains the information about the principal parts of all commutators of vector fields taking values in \(D\). Tanaka considered distributions with constant symbols or “of constant type \(\mathfrak{g}_-\)”, when the graded Lie algebras \(\mathfrak{g}_-(x)\) are isomorphic for every \(x \in M\) to a fixed, graded, nilpotent Lie algebra \(\mathfrak{g}_-\), generated by \(\mathfrak{g}_-\). The flat distribution \(D(\mathfrak{g}_-)\) of constant type \(\mathfrak{g}_-\) is the left-invariant distribution on the simply connected Lie group \(G_-\) with Lie algebra \(\mathfrak{g}_-\) such that the fiber of \(D\) at the identity is \(\mathfrak{g}_-\).

Now let \(\text{Aut}(\mathfrak{g}_-)\) be the group of automorphisms of the graded Lie algebra \(\mathfrak{g}_-\), and \(\mathfrak{der}(\mathfrak{g}_-)\) be the Lie algebra of \(\text{Aut}(\mathfrak{g}_-)\), so that \(\mathfrak{der}(\mathfrak{g}_-)\) is the algebra of all derivations of \(\mathfrak{g}_-\) which preserve the grading. The vector space \(\mathfrak{g}_- \oplus \mathfrak{der}(\mathfrak{g}_-)\) is naturally endowed with the structure of a graded Lie algebra. To a distribution of type \(\mathfrak{g}_-\) one can assign a principal \(\text{Aut}(\mathfrak{m})\)-bundle \(P^0(\mathfrak{g}_-)\) over \(M\) whose fiber over \(x\) consists of all graded Lie algebra isomorphisms from \(\mathfrak{g}_-\) to \(\mathfrak{g}_-(x)\).

Additional structures on \(D\) can be encoded in the choice of a subgroup \(G_0 \subset \text{Aut}(\mathfrak{g}_-)\) with Lie algebra \(\mathfrak{g}_0\), leading to a \(G_0\)-reduction \(P^0\) of the bundle \(P^0(\mathfrak{g}_-)\). The bundle \(P^0\) is called the structure of type \(\mathfrak{g}_- \oplus \mathfrak{g}_0\), or of Tanaka symbol \(\mathfrak{g}_- \oplus \mathfrak{g}_0\). One can define the flat structure of type \(\mathfrak{g}_- \oplus \mathfrak{g}_0\) as the \(G_0\)-reduction \(P^0\) of the bundle \(P^0(\mathfrak{g}_-)\) \(\to \mathfrak{g}_-\) such that the fiber of \(P^0\) over \(x \in G_-\) consists of pullbacks of isomorphisms in the fiber over the identity of \(G_-\) under the left translation mapping \(x\) to the identity. Note that if \(G^0\) denotes a Lie group with Lie algebra \(\mathfrak{g}_- \oplus \mathfrak{g}_0\) such that \(G_0 \subset G^0\), then the flat structure of type \(\mathfrak{g}_- \oplus \mathfrak{g}_0\) is at least locally equivalent to the structure of type \(\mathfrak{g}_- \oplus \mathfrak{g}_0\) given by the bundle \(G^0 \to G^0/G_0\).

Next, Tanaka defines the universal algebraic prolongation
\begin{equation}
\mathcal{U}(\mathfrak{g}_- \oplus \mathfrak{g}_0) = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \bigoplus_{i>0} \mathfrak{g}_i(1.3)
\end{equation}
of \(\mathfrak{g}_- \oplus \mathfrak{g}_0\), which is the maximal (nondegenerate) graded Lie algebra containing \(\mathfrak{g}_- \oplus \mathfrak{g}_0\) as its nonpositive part. Nondegeneracy here means the adjoint action \(\text{ad}(y)|_{\mathfrak{g}_-}\) is nontrivial for any nonzero, nonpositively graded \(y \in \mathcal{U}(\mathfrak{g}_- \oplus \mathfrak{g}_0)\). The prolongation procedure to construct canonical frames for structures of type \(\mathfrak{g}_- \oplus \mathfrak{g}_0\) can be described uniformly by the following

**Theorem 1.1** (Tanaka. \(^25\)). Assume that \(\dim \mathcal{U}(\mathfrak{g}_- \oplus \mathfrak{g}_0) < \infty\). Then the following holds:

1. To any Tanaka structure of type \(\mathfrak{g}_- \oplus \mathfrak{g}_0\) one can assign the canonical structure of absolute parallelism on a bundle over \(M\) of dimension equal to \(\dim \mathcal{U}(\mathfrak{g}_- \oplus \mathfrak{g}_0)\).
2. The algebra of infinitesimal symmetries of the flat structure of type \(\mathfrak{g}_- \oplus \mathfrak{g}_0\) is isomorphic to \(\mathcal{U}(\mathfrak{g}_- \oplus \mathfrak{g}_0)\).
3. The dimension of the algebra of infinitesimal symmetries of a Tanaka structure of type \(\mathfrak{g}_- \oplus \mathfrak{g}_0\) is not greater than \(\dim \mathcal{U}(\mathfrak{g}_- \oplus \mathfrak{g}_0)\) and any Tanaka structure of type \(\mathfrak{g}_- \oplus \mathfrak{g}_0\) with the algebra of infinitesimal symmetries of dimension equal to \(\dim \mathcal{U}(\mathfrak{g}_- \oplus \mathfrak{g}_0)\) is locally equivalent to the flat structure of type \(\mathfrak{g}_- \oplus \mathfrak{g}_0\).

**Remark 1.2.** A few words about item (1) of the previous theorem: If \(\mathfrak{g}_-\) has \(\mu\) nonzero graded components and the positively graded part of \(\mathcal{U}(\mathfrak{g}_- \oplus \mathfrak{g}_0)\) consists of \(l\) nonzero graded components, then Tanaka recursively constructs a sequence of bundles \(\{P^i\}_{1 \leq i \leq \mu}\),
\begin{equation}
M \leftarrow P^0 \leftarrow P^1 \leftarrow P^2 \leftarrow \ldots
\end{equation}
where for \(i > 0\), \(P^i\) is a bundle over \(P^{i-1}\) whose fibers are affine spaces with modeling vector space \(\mathfrak{g}_i\) from (1.3). Observe that all \(P^i\) with \(i \geq l\) are identified with each other by the bundle projections, which are diffeomorphisms in those cases. The bundle \(P^{1+\mu}\) is an \(e\)-structure over \(P^{l+\mu-1}\); i.e., \(P^{l+\mu-1}\) is endowed with a canonical frame – a structure of absolute parallelism. It is important to note that for any \(0 < i \leq \mu - 1\) the recursive construction of the bundle \(P^{i+1}\) over \(P^i\) depends on a choice of normalization conditions. Algebraically, “normalization condition” refers to a choice of vector space complement to the
image of a certain Lie algebra cohomology differential. Therefore, the word “canonical” in item (1) of Theorem 1.1 means that for any Tanaka structure of type \( g_- \oplus g_0 \), the same fixed normalization conditions are applied in each step of the construction of the sequence (1.4).

The main advantage of this approach compared to Cartan’s original method of equivalence is that, independently of the choice of normalization conditions, the basic features of the prolongation procedure such as the dimension of the resulting bundles, at which step the canonical frame is achieved, and the algebra of infinitesimal symmetries of the maximally symmetric “flat” model can be described purely algebraically in terms of \( \Omega(g_- \oplus g_0) \). Questions concerning the most natural normalization conditions – or whether normalization conditions exist such that the resulting absolute parallelism is a \( \Omega(g_- \oplus g_0) \)-valued Cartan connection – are more subtle and are best understood in the framework of the parabolic geometries; i.e., when the nonnegatively graded part of \( \Omega(g_- \oplus g_0) \) is a parabolic subalgebra \([20, 4]\).

1.3. Bigraded Tanaka prolongation and description of main results. It is clear that the standard Tanaka approach will not work for Levi degenerate CR structures, because describing the \( k \)-nondegeneracy of a CR structure on the graded level requires the assignment nonnegative degree to vectors in the Levi kernel, while in the standard Tanaka theory the nonnegatively graded components of the universal prolongation algebra correspond to vertical vector fields on the appropriate bundle. Thus, the analog of a Tanaka symbol for such structure is not immediately obvious.

An attempt to define an analog of the Tanaka symbol for \( k \)-nondegenerate CR-structure, called an abstract core, was made by A. Santi in [22]. This notion is very natural and was used there toward the description of homogeneous models, but neither a functorial notion analogous to the universal prolongation of a Tanaka symbol nor a relation of the abstract core to a construction of an absolute parallelism was given there.

In the present paper, in the case of 2-nondegenerate, hypersurface-type CR structures, we first propose the analog of the Tanaka symbol (section 2). In contrast to the standard Tanaka theory, this symbol is not a Lie algebra in general. It is a graded and even bigraded complex vector space endowed with an anti-linear involution, and with bigrading-compatible Lie brackets defined for most pairs of bigraded components, except the pair corresponding to \( K \) and \( \overline{K} \). Here and in the sequel, “bigraded vector spaces” mean \((\mathbb{Z} \times \mathbb{Z})\)-graded ones.

Then we restrict ourselves to the class of symbols which are Lie algebras, calling them regular symbols (see Remark 1.3 below discussing the naturality of this restriction). Regular symbols have the structure of bigraded Lie algebras, so that the bigrading-compatible Lie brackets are defined on the whole symbol. We also define the notion of the flat CR structure with given regular symbol.

Our symbols and Santi’s abstract cores [22] in the considered situation are in fact equivalent: one notion is uniquely recoverable from the other one (see Remark 2.6 below for more detail). However, the main novelty here is that for our regular symbols we define the analog of the Tanaka universal algebraic prolongation in a functorial way. This analog is the maximal, nondegenerate, complex, bigraded Lie algebra, such that it contains the symbol as its part with nonnegative first weight and, in addition, the only possible non-vanishing bigraded components with first weight equal to 1 have biweights \((1, -1)\) and \((1, -1)\) (Definition 3.1). This bigraded algebra is endowed with an anti-linear involution (i.e., with a real form). The naturality of this notion is justified by our main theorem – Theorem 3.2 – on the existence of a canonical absolute parallelism for all CR structures with given regular 2-nondegenerate CR symbol, which shows that the real part of the bigraded universal algebraic prolongation plays exactly the same role for our structure as Tanaka’s universal prolongation for standard Tanaka structures. In other words, for every CR structure with a given regular, 2-nondegenerate CR symbol \( g^0 \) a canonical absolute parallelism exists on a bundle of dimension equal to the (complex) dimension of the bigraded universal algebraic prolongation. Moreover, among such structures the flat structure is the only one – up to local equivalence – whose algebra of infinitesimal symmetries has the latter dimension, and the algebra of infinitesimal symmetries of the flat structure of type \( g^0 \) is isomorphic to the real part of the bigraded universal Tanaka prolongation.

We emphasize that Theorem 3.2 establishes the existence of a canonical absolute parallelism following a choice of normalization condition in each step of the prolongation – cf., Remark 1.2 above – but we do
not specify which normalization conditions are preferable, and in particular we do not investigate if such conditions exist to determine a canonical Cartan connection. However, we believe that the framework developed here will be very useful to address this question.

Further, we classify all regular, 2-nondegenerate symbols with 1-dimensional Levi kernel in section 3 (Theorem 3.3) and calculate their bigraded universal prolongations in section 5 (Theorem 5.1). Let us describe the results of these calculations for regular symbols with 1-dimensional kernel satisfying the additional assumption – called strong regularity – that the linear maps ad_ν from (1.1) are bijective. The bigraded universal algebraic prolongation of any strongly regular symbol is isomorphic to the complex orthogonal algebra so(m,C) with m = \frac{1}{2}(\text{dim} M + 5). To describe the algebras obtained as real parts of bigraded universal algebraic prolongations of strongly regular symbols, suppose the Hermitian form induced on H/K by the Levi form has signature (p,q) with p \geq q (note that dim M = 2(p + q + 3).

If p \neq q, there exists exactly one strongly regular symbol and the real part of its bigraded universal prolongation is isomorphic to so(p+2,q+2). Such symbol is called of type \text{I}_{p,q}. If p = q, there are two strongly regular symbols, and the real parts of their universal prolongations are isomorphic to so(p+2,p+2) or so^*(2p+4). Such symbols are called of type \text{I}_{p,p} and \text{I}_{p,p} respectively. The universal algebraic prolongation of a regular symbol with 1-dimensional Levi kernel which is not strongly regular is equal to the symbol itself.

Among all 2-nondegenerate, hypersurface-type CR structures with strongly regular CR symbol of type \text{I}_{p,q} the CR structure with the maximal possible algebra of infinitesimal symmetries is locally equivalent to that of the tube hypersurface in \mathbb{C}^{p+q+2} over the future light cone in \mathbb{R}^{p+q+2}, as in the model (1.2), but with signature (p+1,q+1). This can be shown easily using our main Theorem 3.2 and the same arguments as in [13] section 2 based on [23].

For dim M = 5 there exists exactly one CR symbol, which is automatically strongly regular and corresponds to p = 1 and q = 0, so that the real part of the bigraded universal algebraic prolongation is equal to so(3,2) as expected from [13] [17] [20].

For dim M = 7 the universal algebraic prolongation of the strongly regular symbol is isomorphic to so(6,C) \cong sl_4(\mathbb{C}). There are two possibilities for the signature (p,q): (p,q) = (2,0) or (p,q) = (1,1). In the first case, the real part of the bigraded universal algebraic prolongation is isomorphic to so(4,2) \cong su(2,2). In the second case it is isomorphic to either so(3,3) \cong sl_4(\mathbb{R}) or so^*(6) \cong su(3,1). Hence, we have three different strongly regular symbols here. Note that in [21], an absolute parallelism was constructed for two of these three cases: the one corresponding to so(4,2) \cong su(2,2) and another one corresponding to so^*(6) \cong su(3,1). In dim M = 7 there is only one 2-nondegenerate regular symbol with 1-dimensional kernel which is not strongly regular. Also, in this case there are no 2-nondegenerate CR structures with 2-dimensional Levi kernel.

Remark 1.3. Several words about the potential importance of CR structures with regular symbol. Starting with dim M = 7 the space of non-equivalent symbols of 2-nondegenerate CR structures with 1-dimensional Levi kernel is large and contains continuous parameters (moduli). As a matter of fact, the classification of such symbols is equivalent to the classification of pairs consisting of a real line \ell of nondegenerate Hermitian forms of arbitrary signature and a complex line \ell of self-adjoint, anti-linear operators in a complex vector space of dimension \frac{1}{2}(\text{dim} M − 3), up to a change of a basis in this space (see Proposition 1.11). As far as we know, this classification problem is solved in the case of positive-definite Hermitian forms only (13), and even in this case the normal forms contain continuous parameters related to the eigenvalues of \ell^2 – a line of linear self-adjoint operators. The case of sign-indefinite Hermitian forms is richer, as evidenced by the well known problem of normal forms for pairs consisting of a real line of nondegenerate Hermitian forms of arbitrary signature in a complex vector space and a complex line of self-adjoint linear operators on the same complex vector space (12 Theorem 5.1.1); these normal forms in the sign-indefinite case may feature Jordan blocks as well). So, the regular symbols classified in the present paper represent a very small subset of all symbols. However, among all symbols corresponding to CR structures for which the linear maps ad_ν from (1.1) are bijective, the strongly regular symbols are those that do not impose additional structures on H/K. This motivates the following conjecture: The strongly regular symbols have homogeneous models of maximal possible dimension within this large class of symbols.
2. CR symbol for 2-nondegenerate CR structures of hypepersurface type

As mentioned in the Introduction, we consider a 2-nondegenerate, hypersurface-type CR structure \((D, J)\) on \(M\). For any real vector bundle \(E \to M\), \(CE\) denotes the complexified vector bundle whose fiber over \(x \in M\) is \(CE_x = E_x \otimes_{\mathbb{R}} \mathbb{C} = E_x \oplus_{\mathbb{R}} iE_x\). Recall that \(CD_x = H_x \oplus \overline{H}_x\) splits into \(\pm\)-eigenspaces for \(J_x\) each of which we suppose to have complex dimension \(n+1\) and that the Levi kernel is a complex subbundle \(K \subset H\) with conjugate \(\overline{K} \subset H\). Define

\[
\mathfrak{g}_{-1}(x) = CD_x/(K_x + \overline{K}_x), \quad \mathfrak{g}_{-2}(x) = CT_x M/CD_x.
\]

Similar to the Levi form on the holomorphic subbundle \(H \subset CTM\), one can define a \(CT_x M/CD_x\)-valued alternating form \(\omega\) on \(g_{-1}(x)\): for \(y_1, y_2 \in g_{-1}\), let \(Y_i \in \Gamma(CD)\) be such that \(Y_i(x) = y_i \bmod K \oplus \overline{K}\), \((i = 1, 2)\), and set

\[
\omega(y_1, y_2) = [Y_1, Y_2](x) \bmod g_{-1}(x).
\]

Henceforth in this section we suppress the subscript \(x\) or argument \((x)\) when referring to pointwise structures like the fibers of bundles or components of the symbol algebra. This is no loss of specificity, as we will restrict our attention to CR structures of constant type. The complex vector space

\[
\mathfrak{g}_- = g_{-1} \oplus g_{-2}
\]

is endowed with the natural structure of a graded Lie algebra with the only nontrivial brackets coming from the form \(\omega\). This algebra is isomorphic to the Heisenberg algebra. Note that its algebra of derivations \(\mathfrak{der}(\mathfrak{g}_-)\) is isomorphic to the complex conformal symplectic algebra \(\mathfrak{osp}(\mathfrak{g}_{-1}, \omega)\). Let

\[
g_{-1,1} = H/K, \quad g_{-1,-1} = \overline{H}/\overline{K}, \quad g_{-2,0} = g_{-2}.
\]

We say that a bigrading is compatible with the grading of a vector space if the \(i\)th component of the grading is the direct sum of all bigraded components whose first weight is \(i\). Bearing in mind the involutivity of \(H\) and \(\overline{H}\), we see that \(\mathfrak{g}_-\) is a bigraded Lie algebra with respect to the bigraded splitting

\[
g_- = g_{-1,-1} \oplus g_{-1,1} \oplus g_{-2,0},
\]

which is compatible with the grading \((2.2)\). Also recall that \(\mathfrak{g}_-\) is endowed with the anti-linear involution given by the complex conjugation in \(CTM\). By construction, it satisfies

\[
\overline{g_{i,j}} = g_{-i,-j}.
\]

For \(v \in K\), take \(\text{ad}_v : \overline{H}/\overline{K} \to H/K\) to be as in \((1.1)\). Extending \(\text{ad}_v\) trivially to \(H/K\), we obtain a derivation \(\text{ad}_v \in \mathfrak{der}(\mathfrak{g}_-)\). Here, as before, \(\mathfrak{der}(\mathfrak{g}_-)\) is the Lie algebra of derivations of \(\mathfrak{g}_-\) preserving the grading \((2.2)\), but not necessarily the bigrading \((2.3)\). Referring to the collection of such \(\text{ad}_v\) operators as \(\text{ad}_K\), we identify \(\text{ad}_K\) with a complex subspace in \(\mathfrak{der}(\mathfrak{g}_-)\), which we denote by \(g\). Taking the complex conjugate, we define \(\overline{\text{ad}}_K\) and similarly identify it with a subspace in \(\mathfrak{der}(\mathfrak{g}_-)\), denoted \(g_{0,-2}\). By construction,

\[
[g_{0,2}, g_{-1,-1}] \subset g_{-1,1}, \quad [g_{0,2}, g_{-1,1}] = 0,
\]

\[
[g_{0,-2}, g_{-1,1}] \subset g_{-1,-1}, \quad [g_{0,-2}, g_{-1,1}] = 0.
\]

Finally, let \(g_{0,0}\) be the maximal subalgebra of \(\mathfrak{der}(\mathfrak{g}_-)\) such that

\[
[g_{0,0}, g_{-1,\pm1}] \subset g_{-1,\pm1}
\]

\[
[g_{0,0}, g_{0,\pm2}] \subset g_{0,\pm2}.
\]

Equivalently, \(g_{0,0} \subset \mathfrak{der}(\mathfrak{g}_-)\) is the maximal subalgebra of derivations preserving the bigrading \((2.3)\) and satisfying \((2.7)\). Collecting all bracket relations between the bigraded components defined so far, we have

\[
[g_{i_1,j_1}, g_{i_2,j_2}] \subset g_{i_1+j_2, j_1+j_2}, \quad i_1 \leq 0, \quad \{ (i_1, j_1), (i_2, j_2) \} \neq \{ (0, 2), (0, -2) \}.
\]

Note that complex conjugation of \(y \in \mathfrak{g}_-\) induces conjugation on \(f \in \mathfrak{der}(\mathfrak{g}_-)\) via

\[
\overline{f}(y) = f(\overline{y}).
\]
In this way, (2.10) extends to $g_{0,j}$, with $j = 0, \pm 2$. Thus, the anti-linear involution given by the operator of complex conjugation is defined on the space
\begin{equation}
(2.10) \quad g^0 = g_{-2,0} \oplus g_{-1,-1} \oplus g_{-1,1} \oplus g_{0,-2} \oplus g_{0,0} \oplus g_{0,2}, \quad \dim g_{0,\pm 2} = 1.
\end{equation}

**Definition 2.1.** The vector subspace $g^0$ of $g \oplus \mathfrak{Der}(g)$, having bigrading as in (2.10) and endowed with the anti-linear involution $\overline{\cdot}$ is called the *symbol of the CR structure $(D, J)$ at point $x$*, or more briefly, the CR symbol.

Collecting all properties that we used in the previous definition, we get the following, natural

**Definition 2.2.** Let $g_- = g_{-1} \oplus g_{-2}$ be the complex graded Heisenberg algebra. A vector space $g^0 = g_- \oplus g_0$ with $g_0 \subset \mathfrak{Der}(g_-)$ that has a bigrading like (2.10) compatible with the grading $g_- \oplus g_0$ is called an *abstract symbol* of 2-nondegenerate, hypersurface-type CR structure if it satisfies (2.8), $g_{0,0}$ is the maximal subalgebra of $\mathfrak{Der}(g_-)$ satisfying (2.6) and (2.7), and it is endowed with the anti-linear involution $\overline{\cdot}$ satisfying (2.4).

**Definition 2.3.** Let $g^0$ be an abstract 2-nondegenerate CR symbol. We say that a 2-nondegenerate CR structure of hypersurface type has *constant CR symbol* $g^0$ if its CR symbols at all points are isomorphic to $g^0$.

In contrast to the standard Tanaka symbol, an abstract 2-nondegenerate symbol is not a Lie algebra in general, because the brackets $[g_{0,-2}, g_{0,2}]$ may not belong to it.

**Definition 2.4.** An abstract 2-nondegenerate CR symbol $g^0$ is called *regular* if it is a Lie subalgebra of $g_- \oplus \mathfrak{Der}(g_-)$, which is equivalent to the condition
\begin{equation}
(2.11) \quad [g_{0,-2}, g_{0,2}] \subset g_{0,0}.
\end{equation}

Note that condition (2.11) endows $g^0$ with the structure of the *bigraded Lie algebra*, because the rest of the bigrading conditions follow from the constructions.

**Remark 2.5.** The component $g_{0,0}$ in the definition of 2-nondegenerate CR symbols can be recovered from the other components. However, we prefer to include it in the definition for brevity, and because it simplifies the notion of algebraic prolongation in the sequel.

**Remark 2.6.** In the situation considered here, the abstract core defined by A. Santi in [22] is in our notation the real part of the vector space
\begin{equation}
(2.12) \quad m = g_{-2,0} \oplus g_{-1,-1} \oplus g_{-1,1} \oplus g_{0,-2} \oplus g_{0,2}
\end{equation}
(i.e. consisting of all components of $g^0$ except $g_{0,0}$), together with:
- the operator $J$ defined on real parts of $g_{-1}$ and $g_{0,-2} \oplus g_{0,2}$ and satisfying $J^2 = -\text{Id}$,
- the structure of the Heisenberg algebra on $g_-$, and
- the identification of $g_{0,-2}$ and $g_{0,2}$ with elements of $\mathfrak{Der}(g_-)$ satisfying (2.6), although he does explicitly use the structure of a bigrading. Clearly this data is equivalent to that encoded in our symbol. Note that the real part of the symbol does not have any natural bigrading.

Finally, for a regular, 2-nondegenerate CR symbol $g^0$, by analogy with the standard Tanaka theory, one can define the notion of the flat structure of type $g^0$. Let $\mathfrak{R}G^0$ and $\mathfrak{R}G_{0,0}$ be the simply connected Lie groups corresponding to the real parts of the Lie algebras $g^0$ and $g_{0,0}$. Let $M_0 = \mathfrak{R}G^0/\mathfrak{R}G_{0,0}$. Since the tangent space to $M_0$ at the coset $o$ of the identity can be identified with the real part of the space $m$ defined by (2.12), the direct sum of real parts of $g_{-1}$ and $g_{0,-2} \oplus g_{0,2}$ define a subspace $D_0 \subset T_o M_0$, and the structures on $g^0$ determine the operator $J$ on $D_0$ satisfying $J^2 = -\text{Id}$. The action of $\mathfrak{R}G^0$ on $M_0$ induces the same structures on the tangent space at every point of $M$ from the structure on $D_0$ – i.e., a CR structure. From the properties of the bigraded components of $g^0$ it follows that this structure is a 2-nondegenerate CR structure of hypersurface type with CR symbol $g^0$.

**Definition 2.7.** The CR structure on $\mathfrak{R}G^0/\mathfrak{R}G_{0,0}$ constructed above is called the *flat CR structure of type* $g^0$. 
3. Universal algebraic prolongation of regular symbols and existence of absolute parallelism

In this section we fix a regular, abstract, 2-nondegenerate CR symbol \( g^0 \). The bigraded structure of \( g^0 \) is crucial for defining the correct universal algebraic prolongation so that it is suitable to the geometric prolongation procedure for the constructing the absolute parallelism.

**Definition 3.1.** The bigraded universal algebraic prolongation of the symbol \( g^0 \) is the maximal (nondegenerate) bigraded Lie algebra \( U_{\text{bigrad}}(g^0) \) of the form

\[
U_{\text{bigrad}}(g^0) = g^0 \oplus g_{1,-1} \oplus g_{1,1} \oplus \bigoplus_{i \geq 2, j \in \mathbb{Z}} g_{i,j},
\]

where, as in the standard Tanaka theory, nondegeneracy means that \( \text{ad}(y)_{|_{g_-}} \neq 0 \) for any nonzero \( y \in U_{\text{bigrad}} \) with a nonnegative first weight.

Let us compare this prolongation to the standard Tanaka universal algebraic prolongation \( U(g^0) \) of \( g^0 = g_- \oplus g_0 \) with respect to the first weight (here \( g_0 = \bigoplus_{j \in \{0, \pm 2\}} g_{0,j} \)). Let

\[
U(g^0) = g_- \oplus g_0 \oplus \bigoplus_{i > 0} g_i
\]

We use \( \tilde{\ } \) for the positively graded components to distinguish from the notation that will be used below for the bigraded Tanaka prolongation.

The spaces \( \tilde{g}_i \) have an explicit realization, which we now describe for \( \tilde{g}_1 \). Recall that an endomorphism \( f \) of a graded vector space has degree \( k \in \mathbb{Z} \) if the image of a vector of weight \( i \) is a vector of weight \( i + k \); we write \( \text{deg} f = k \) in this case. Similarly, we say that an endomorphism \( f \) of a bigraded space has bidegree \( (k, l) \) if the image of a vector of biweight \( (i, j) \) has biweight \( (i + k, j + l) \). Then

\[
\tilde{g}_1 = \{ f \in \text{End}(g^0) \mid \text{deg} f = 1, \ f([v_1, v_2]) = [f(v_1), v_2] + [v_1, f(v_2)] \ \forall \ v_1, v_2 \in g_- \}.
\]

Observe that \( f|_{g_0} = 0 \) for \( f \in \tilde{g}_1 \) as \( \text{deg} f = 1 \), so \( f \) can be considered an element of \( \text{Hom}(g_-, g^0) \) and one can define brackets between \( f_1 \in \tilde{g}_0 \) and \( f_2 \in \tilde{g}_1 \) in the following natural way,

\[
[f_1, f_2](v) = [f_1(v), f_2] + [f_1, f_2(v)] \ \forall v \in g_-.
\]

The bigrading \( (2.10) \) on \( g^0 \) induces a bigrading on \( \tilde{g}_k \) with components of biweight \( (k, l) \) consisting of the endomorphisms of \( g^0 \) of bidegree \( (k, l) \). Based on the biweights of \( g^0 \), the only nonzero bigraded components of \( \tilde{g}_1 \) have biweights \( (1, \pm 1), (1, \pm 3) \); i.e.,

\[
\tilde{g}_1 = \tilde{g}_{1,-3} \oplus \tilde{g}_{1,-1} \oplus \tilde{g}_{1,1} \oplus \tilde{g}_{1,3}.
\]

From the maximality condition for the Tanaka universal prolongation it follows that \( g_{1,-1} \) and \( g_{1,-1} \) can be realized as subspaces if \( \tilde{g}_1 \). The bigrading condition on the prolongation \( (3.1) \) requires \( [g_{1,-1}, g_{0,2}] \subset g_{1,-3} = 0 \), as well as \( [g_{1,1}, g_{0,2}] = 0 \). This implies

\[
g_{1,-1} = \{ f \in \tilde{g}_{1,-1} \mid [f, g_{0,-2}] = 0 \}, \quad g_{1,1} = \{ f \in \tilde{g}_{1,1} \mid [f, g_{0,2}] = 0 \}.
\]

So, the space \( g_{1} = g_{1,-1} \oplus g_{1,1} \) can be significantly smaller than \( \tilde{g}_1 \). For example, in the case \( \text{dim} M = 5 \), since \( g_- \) is a free truncated Lie algebra and \( g_0 = \text{der}(g_-) \cong g_1(g_-) \), \( g_1 \) can be identified with \( \text{Hom}(g_{-1}, g_0) \). Because \( \text{dim} g_{-1} = 2 \), \( \text{dim} \tilde{g}_1 = 8 \). On the other hand, from section \( 5.2 \) it will follow that \( \text{dim} g_1 = 2 \) in this case.

In contrast to the case of \( \tilde{g}_1 \), for the subspaces \( g_i = \bigoplus_{k \in \mathbb{Z}} g_{i,k} \) of \( U_{\text{bigrad}}(g^0) \) with \( i \geq 2 \), the description is exactly the same as for the standard Tanaka universal prolongation of the space \( g_- \oplus g_0 \oplus g_1 \) (by the latter we mean the maximal nondegenerate graded Lie algebra such that its part corresponding to weights not greater than 1 is equal to \( g_- \oplus g_0 \oplus g_1 \)). Recursively, one has

\[
\tilde{g}_i = \left\{ f \in \text{End} \left( \bigoplus_{k \leq i-1} g_k \right) \mid \text{deg} f = i, f([v_1, v_2]) = [f(v_1), v_2] + [v_1, f(v_2)] \ \forall \ v_1, v_2 \in g_- \right\}.
\]

and the second weight is assigned in the obvious way. The fact that we prolong \( g_- \oplus g_0 \oplus g_1 \) and not \( g_- \oplus g_0 \oplus \tilde{g}_1 \) implies that \( g_i \) is in general a proper subspace of \( \tilde{g}_i \).
Since \( g_1 \) is a subspace of endomorphisms acting on a space endowed with an anti-linear involution satisfying (2.4), an anti-linear involution satisfying (2.4) is induced on \( g \) by the formula (2.9). This involution can be extended recursively to the whole algebra \( \mathfrak{U}_{\text{bigrad}}(g^0) \) using the formula (2.9). Let \( \mathfrak{R}_{\text{bigrad}}(g^0) \) be the real part of \( \mathfrak{U}_{\text{bigrad}}(g^0) \) with respect to this involution.

The following theorem is the main result of this paper:

**Theorem 3.2.** Assume that \( \dim \mathfrak{U}_{\text{bigrad}}(g^0) < \infty \).

1. To any 2-nondegenerate, hypersurface type CR structure with regular symbol \( g^0 \) one can assign the canonical structure of absolute parallelism on a bundle over \( M \) of dimension equal to \( \dim_{\mathbb{C}} \mathfrak{U}_{\text{bigrad}}(g^0) \);
2. The algebra of infinitesimal symmetries of the flat CR structure of type \( g^0 \) is isomorphic to the real part \( \mathfrak{R}_{\text{bigrad}}(g^0) \) of \( \mathfrak{U}_{\text{bigrad}}(g^0) \);
3. The dimension of the algebra of infinitesimal symmetries of a 2-nondegenerate, hypersurface type CR structure with regular symbol \( g^0 \) is not greater than \( \dim_{\mathbb{C}} \mathfrak{U}_{\text{bigrad}}(g^0) \), and any CR structure of type \( g^0 \) whose algebra of infinitesimal symmetries has dimension \( \dim \mathfrak{U}_{\text{bigrad}}(g^0) \) is locally equivalent to the flat CR structure of type \( g^0 \).

The proof of this theorem constitutes section 5. We classify all regular, abstract 2-nondegenerate CR symbols with 1-dimensional Levi kernel in section 4 (Theorem 4.3) and calculate the real parts of their bigraded universal algebraic prolongations in section 5 (Theorem 5.1), so that Theorem 3.2 can be formulated more explicitly in this case. In particular, in this case all bigraded universal algebraic prolongations are finite dimensional.

4. Classification of Regular 2-Nondegenerate CR Symbols with \( \dim_{\mathbb{C}} g_{0, \pm 2} = 1 \).

In this section we restrict our attention to the case of 1-dimensional Levi kernel. Let \( g^0 \) be a regular, abstract, 2-nondegenerate CR symbol with 1-dimensional Levi kernel. Then one can assign to it a pair: a canonical real line of nondegenerate Hermitian forms and a canonical complex line of anti-linear operators on the space \( g_{-1,1} \); see Remark 1.3 above.

A generator \( \ell \) of the canonical real line of Hermitian forms on \( g_{-1,1} \) is defined as follows: First, fix a basis vector \( e_0 \in i\mathfrak{R}g_{-2,0} \). Then \( \ell \) is defined via

\[
[\lambda_1, \lambda_2] = \lambda(y_1, y_2)e_0, \quad y_1, y_2 \in g_{-1,1}.
\]

Re-scaling \( e_0 \in i\mathfrak{R}g_{-2,0} \) effects re-scaling of \( \ell \).

A generator \( A \) of the canonical complex line of anti-linear operators is defined by choosing a basis vector \( f \in g_{0,2} \) and setting

\[
Ay = [f, \overline{y}], \quad y \in g_{-1,-1}.
\]

Re-scaling \( f \in g_{0,2} \) effects re-scaling of \( A \). Moreover, it is easy to show ([21 §2.3]) that \( A \) is self-adjoint with respect to \( \ell \), i.e.

\[
\ell(Ay_1, y_2) = \ell(Ay_2, y_1), \quad y_1, y_2 \in g_{-1,1}.
\]

Equivalence relations are naturally defined on the space \( \mathfrak{C} \) of abstract, 2-nondegenerate CR symbols and the space \( \mathfrak{P} \) of pairs consisting of a real line of nondegenerate Hermitian forms and a complex line of anti-linear, self-adjoint operators on \( g_{-1,1} \). Two CR symbols are equivalent if there exists a linear isomorphism between them preserving all their algebraic structures. Regarding \( \mathfrak{P} \), the standard action of the group \( \text{GL}(g_{-1,1}) \) induces an action – and therefore an equivalence relation – on it as well. Using Remark 2.3 and the fact that all nontrivial Lie brackets on the space \( \mathfrak{m} \) defined by (2.12) are given via the form \( \ell \) and the operator \( A \), the following is immediate.

**Proposition 4.1.** Two abstract 2-nondegenerate CR symbols \( g^0 \) and \( \overline{g}^0 \) are equivalent if and only if the corresponding pairs \( (\ell, A) \) and \( (\overline{\ell}, \overline{A}) \) from \( \mathfrak{P} \) are equivalent.

Normal forms for the pair \((\langle \ell \rangle_\mathbb{R}, \langle A \rangle_\mathbb{C})\) in \( \mathfrak{P} \) are known in the case of sign-definite \( \ell \) only ([13]). Without the assumption of sign-definiteness of \( \ell \), normal forms are still known for pairs \((\langle \ell \rangle_\mathbb{R}, \langle A^2 \rangle_\mathbb{R})\) ([12] Theorem...
Let us call the latter pair the \textit{reduced pair} of the original one. Note that the equivalence of pairs in \(\mathfrak{P}\) obviously implies the equivalence of the corresponding reduced pairs, but not vice versa. Both normal forms from \cite{[13]} and \cite{[12]} show that the space of equivalence classes of 2-nondegenerate CR symbols is very rich and contains moduli. In this paper we are interested in regular symbols only, and this space is small and discrete (see Remark \cite{[13]} above discussing why regular symbols are particularly important). The following lemma is important toward the classification of all regular 2-nondegenerate CR symbols:

\textbf{Lemma 4.2.} An abstract 2-nondegenerate CR symbol \(\mathfrak{g}^0\) described by a pair \(((\ell)_{\mathbb{R}}, (A)_{\mathbb{C}}) \in \mathfrak{P}\) is regular if and only if there exists the unique splitting

\begin{equation}
\mathfrak{g}_{-1,1} = Z \oplus W
\end{equation}

with

\begin{equation}
Z = \ker A, \quad W = \text{Im} A
\end{equation}

such that

\begin{equation}
A|_Z = 0, \quad A^2|_W = \alpha \mathbb{I}_W.
\end{equation}

for a nonzero \(\alpha \in \mathbb{R}\). Furthermore, the restrictions of the Hermitian form \(\ell\) to \(W\) and \(Z\) are nondegenerate.

\textbf{Proof.} Fix \(f \in \mathfrak{g}_{0,2}\) so that \(\mathcal{T} \in \mathfrak{g}_{0,-2}\). Both \(f, \mathcal{T}\) belong to \(\text{det}(\mathfrak{g}_-) \subset \mathfrak{gl}(\mathfrak{g}_{-1})\), and their Lie brackets are defined by the commutator in \(\mathfrak{gl}(\mathfrak{g}_{-1})\). Hence, the regularity condition \eqref{2.11} reads \(f \circ \mathcal{T} - \mathcal{T} \circ f \in \mathfrak{g}_{0,0}\), which by the definition of \(\mathfrak{g}_{0,0}\) implies

\begin{equation}
(f \circ \mathcal{T} - \mathcal{T} \circ f) \circ f - f \circ (f \circ \mathcal{T} - \mathcal{T} \circ f) = \alpha_0 f,
\end{equation}

for some constant \(\alpha_0\). Recall that the construction of \(\mathfrak{g}_{0,\pm 2}\) requires \(f^2 = 0\). Therefore, the left-hand side of \eqref{4.7} reduces to \(2f \circ \mathcal{T} \circ f\), whence

\begin{equation}
f \circ \mathcal{T} \circ f = \alpha f
\end{equation}

for some \(\alpha \neq 0\), which must be \(\mathbb{R}\)-valued by \cite{[18]}. Restricting the last identity to \(\mathfrak{g}_{-1,-1}\) and using \cite{[18]} we get that

\begin{equation}
A^3 = \alpha A.
\end{equation}

For subspaces \(Z\) and \(W\) as in \eqref{1.20}, we must show \(Z \cap W = 0\). Indeed, if \(y \in Z \cap W\), then \(Ay = 0\) and there exists \(w \in \mathfrak{g}_{-1,1}\) such that \(y = Aw\). Hence, \(0 = A^2 y = A^2 w = \alpha A w = \alpha y\) and \(y = 0\). Thus, \(Z \cap W = 0\) and we have \eqref{1.20}. Further \(A|_W\) is a bijection, so restricting \eqref{1.20} to \(W\) we get that the second relation of \eqref{1.20}.

Finally, since \(A\) is self-adjoint with respect to \(\ell\), we have that \(Z = W^\perp\), the orthogonal complement of \(W\) with respect to \(Z\). This and the fact that \(\ell\) is nondegenerate implies that also \(\ell|_W\) and \(\ell|_Z\) are nondegenerate. \qed

\textbf{Theorem 4.3.} Suppose a regular, abstract 2-nondegenerate CR symbol \(\mathfrak{g}^0\) is described by a pair \(((\ell)_{\mathbb{R}}, (A)_{\mathbb{C}})\), such that \(\ell\) has signature \((p, q)\) with \(p \geq q\), and \((p_1, q_1)\) is the signature of the restriction of \(\ell\) to \(W = \text{Im} A\) \((p_1 \leq p, q_1 \leq q)\). Re-scaling if necessary, we can assume that \(A^2 = \mathbb{I}_W\) or \(A^2 = -\mathbb{I}_W\), depending on the sign of \(\alpha\) in \eqref{4.6}. Then the following two statements hold:

\begin{enumerate}
\item If \(A^2|_W = \mathbb{I}_W\), then there are bases of \(W\) and \(Z\) \((= W^\perp)\) such that \(\ell|_W\), \(\ell|_Z\), \(A|_W\), and \(A|_Z\) are represented by the matrices

\begin{equation}
\ell|_W = \begin{pmatrix} 1_{p_1} & 0 \\ 0 & -1_{q_1} \end{pmatrix}, \quad \ell|_Z = \begin{pmatrix} 1_{p-p_1} & 0 \\ 0 & -1_{q-q_1} \end{pmatrix},
\end{equation}

\begin{equation}
A|_W = \mathbb{I}_W, \quad A|_Z = 0.
\end{equation}

In particular, a regular symbol with \(A^2|_W = \mathbb{I}_W\) is uniquely – up to isomorphism – determined by the signature \((p, q)\) of \(\ell\), and the signature \((p_1, q_1)\) of the restriction of \(\ell\) to \(W = \text{Im} A\).
(2) If \( A^2 |_W = -1 |_W \), then \( q_1 = p_1 \) and there are bases of \( W \) and \( Z (= W^\perp) \) such that \( \ell |_W, \ell |_Z, A |_W, \) and \( A |_Z \) are represented by the matrices

\[
\ell |_W = \begin{pmatrix} \mathbb{1}_{p_1} & 0 \\ 0 & -\mathbb{1}_{p_1} \end{pmatrix}, \quad \ell |_Z = \begin{pmatrix} \mathbb{1}_{p-p_1} & 0 \\ 0 & -\mathbb{1}_{q-p_1} \end{pmatrix},
\]

\[
A |_W = \begin{pmatrix} 0 & -\mathbb{1}_{p_1} \\ \mathbb{1}_{p_1} & 0 \end{pmatrix}, \quad A |_Z = 0.
\]

Thus, a regular symbol with \( A^2 |_W = -1 |_W \) exists only if \( \dim W \) is even, the signature of the restriction of \( \ell \) to \( W \) is \((p_1, p_1)\), and in this case it is uniquely – up to isomorphism – determined by the signature \((p, q)\) of \( \ell \), \( p \geq q \geq p_1 \).

Before proving this theorem, let us distinguish an important subclass of regular symbols and reveal the consequence of Theorem 4.3 for this subclass.

**Definition 4.4.** A regular, abstract 2-nondegenerate CR symbol is called **strongly regular** if \([\mathfrak{g}_{0,2}, \mathfrak{g}_{-1,-1}] = \mathfrak{g}_{-1,1} \). Equivalently, any nonzero \( f \in \mathfrak{g}_{0,2} \) maps \( \mathfrak{g}_{-1,1} \) onto \( \mathfrak{g}_{-1,1} \). A regular 2-nondegenerate CR symbol is called **weakly regular** if it is not strongly regular.

Strong regularity is equivalent to the condition \( Z = 0 \) in the splitting (4.4) or, equivalently, \( p_1 = p \) and \( q_1 = q \) in Theorem 4.3. Theorem 4.3 reads as follows in this particular case.

**Corollary 4.5.** Suppose a strongly regular, abstract, 2-nondegenerate CR symbol \( \mathfrak{g}^0 \) is described by a pair \((\ell, \langle \mathcal{A} \rangle)\), such that \( \ell \) has signature \((p, q)\) with \( p \geq q \). Re-scaling if necessary, we may assume \( A^2 = \mathbb{1}_{\mathfrak{g}_{-1,1}} \) or \( A^2 = -\mathbb{1}_{\mathfrak{g}_{-1,1}} \), depending on the sign of \( \alpha \) in (4.6). Then the following two statements hold:

1. If \( A^2 = \mathbb{1}_{\mathfrak{g}_{-1,1}} \), there is a basis of \( \mathfrak{g}_{-1,1} \) such that the generator \( \ell \) of the canonical line of Hermitian forms and a generator \( A \) of the canonical line of anti-linear operators are represented by the matrices

\[
\ell = \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix} \quad \text{and} \quad A = \mathbb{1}_{p+q}
\]

in this basis. Thus, in this case strongly regular symbols are uniquely – up to isomorphism – determined by the signature \((p, q)\), \( p \geq q \).

2. If \( A^2 = -\mathbb{1}_{\mathfrak{g}_{-1,1}} \), then \( q = p \) and there is a basis of \( \mathfrak{g}_{-1,1} \) such that a generator \( \ell \) of the canonical line of Hermitian forms and a generator \( A \) of the canonical line of anti-linear operators are represented by the matrices

\[
\ell = \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_p \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & -\mathbb{1}_p \\ \mathbb{1}_p & 0 \end{pmatrix},
\]

in this basis. Thus, in this case strongly regular symbols exist only if \( \dim \mathfrak{g}_{-1,1} \) is even and the signature the Hermitian form is \((p, p)\). Such a symbol is uniquely – up to an isomorphism – determined by \( p \) or \( \dim \mathfrak{g}_{-1,1} = 2p \).

**Definition 4.6.** The 2-nondegenerate CR symbol from item (1) of Corollary 4.5 is called the strongly regular symbol of type II \( \Pi_{p,q} \) or simply of type II if the specification of the signature is not essential. The 2-nondegenerate CR symbol from item (2) of Corollary 4.5 is called the strongly regular symbol of type II \( \Pi_p \) or simply of type II, if the specification of the signature is not essential.

**Proof of Theorem 4.3**

First consider \( A^2 = \mathbb{1}_W \). Following the proof of [27, Lemma 3.1], let

\[
W_+ = \{ x + Ax, x \in W \}, \quad W_- = \{ x - Ax, x \in W \}.
\]

Then \( W_\pm \) are real vector subspaces and \( Aw = \pm w \) for any \( w \in W_\pm \). Also, from anti-linearity, \( W_- = iW_+ \) and \( W = W_+ \oplus W_- \). Therefore, the restriction of \( A \) to \( W_\pm \) considered as a linear operator on a real space coincides with \( \mathbb{1}_{W_\pm} \). By the standard theory for nondegenerate Hermitian forms, we can choose an orthonormal basis in \( W_\pm \) with respect to which \( \ell |_W \) and \( A |_W \) are represented by matrices as in (4.10).
The two matrix representations for the restrictions of $\ell$ and $A$ to $Z$ follow from the facts that $Z = W^\perp$ and $Z = \ker A$ by choosing an $\ell$-orthonormal basis in $Z$.

Now consider the case $A^2 = -1_W$. Define positive and negative cones in $W$ by $C^+ = \{ w : \ell(w, w) \geq 0 \}$ and $C^- = \{ w : \ell(w, w) \leq 0 \}$. By (4.14),

$$\ell(Aw, Aw) = \ell(A^2 w, w) = -\ell(w, w).$$

Hence, the anti-linear map $A$ sends $C^+$ to $C^-$ and vice versa, which is possible only if the signature of $\ell|_W$ is $(p_1, p_1)$. Indeed, the positive (negative) index of $\ell|_W$ is the maximal dimension of subspaces belonging to $C^+$ ($C^-$). Since $A$ sends subspaces to subspaces and is nonsingular on $W$ (from the fact that $A^2 = -1_W$), the positive and negative indices of $\ell|_W$ must be equal.

Let $e \in W$ with $\ell(e, e) = 1$ so that $\ell(Ae, Ae) = -1$ and $\text{span}_C \{ e, Ae \}$ is an $A$-stable subspace of $W$. We can take $e, Ae$ to be orthogonal with respect to $\ell$ as follows. If $\ell(e, Ae) = re^{i\phi}$ for real $\phi, r > 0$ (and $e$ the natural exponential), replace $e$ with $e^{-i\phi}e$ so that $\ell(e, Ae) = r$.

Let $E = \text{span}_R \{ e, Ae \}$. Now set $\tilde{e} = a_1 e + a_2 Ae$ for $a_1, a_2 \in \mathbb{R}$ and let $z = a_1 + ia_2$. Invoking our hypothesis on $A, Ae = -a_2 e + a_1 Ae$, and it is easy to see that

$$\tilde{e} - i\tilde{e} = z(e - iAe).$$

The restriction of $\ell$ to $E$ defines a symmetric form that can be extended $C$-linearly to the symmetric form $(\cdot, \cdot)$ on $CE$. By construction, $(e - iAe, e - iAe) = 2 - 2r$ and orthonormality of $\{ e, Ae \}$ is equivalent to $(\tilde{e} - i\tilde{e}, \tilde{e} - i\tilde{e}) = 2$. From this and (4.14) it follows that $z$ must satisfy the equation $z^2 = \frac{1}{1-r^2}$, which has a solution.

This proves that an orthonormal basis of $E$ of the form $\{ e, Ae \}$ exists. Self-adjointness ensures that the orthogonal complement of the nondegenerate, $A$-stable subspace $E$ is also $A$-stable. Iterating the above procedure, we determine that there is an orthonormal basis $\{ e_{\alpha_1} \}_{\alpha_1 = 1}^{2p_1}$ of $W$ whose first $p_1$ vectors belong to $C^+$ and last $p_1$ vectors belong to $C^-$, such that $Ae_{\alpha} = e_{\alpha + a}$ for $1 \leq \alpha \leq p_1$. Therefore, the matrix representations of $\ell|_W$ and $A|_W$ in this basis are exactly (4.14). The proof is completed in the same manner as in the previous case by noticing that $Z = W^\perp$, $Z = \ker A$, and we can choose an $\ell$-orthonormal basis in $Z$ to get the rest of (4.11) concerning the restrictions to $Z$. \hspace{1cm} \Box

5. Calculation of Bigraded Universal Algebraic Prolongation of Regular Symbols

The purpose of the present section is to prove

**Theorem 5.1.** Let $\mathfrak{g}^0$ be a regular $CR$ symbol with 1-dimensional Levi kernel and refer to Definition 4.6.

1. If $\mathfrak{g}^0$ is strongly regular of Type $\Pi_{p,q}$, then $\mathfrak{u}_{\text{bigrad}}(\mathfrak{g}^0)$ is isomorphic to $\mathfrak{so}(p + q, q + 2)$;
2. If $\mathfrak{g}^0$ is strongly regular of Type $\Pi_{p,p}$, then $\mathfrak{u}_{\text{bigrad}}(\mathfrak{g}^0)$ is isomorphic to $\mathfrak{so}^*(2p + 4)$;
3. If $\mathfrak{g}^0$ is weakly regular, then $\mathfrak{u}_{\text{bigrad}}(\mathfrak{g}^0) = \mathfrak{g}^0$.

The first three subsections will be dedicated to proving parts (1) and (2) of Theorem 5.1. We begin by noting that the two strongly regular symbol Types have the following in common.

**Lemma 5.2.** When $\mathfrak{g}^0$ is strongly regular, $\dim \mathfrak{g}_{1,\pm 1} = \dim \mathfrak{g}_{-1,\pm 1}$.

**Proof.** $f \in \mathfrak{g}_{1,1}$ satisfies

$$f([y_1, y_2]) = [f(y_1), y_2] + [y_1, f(y_2)], \quad y_1, y_2 \in \mathfrak{g},$$

as well as

$$f([v, y]) = [v, f(y)], \quad v \in \mathfrak{g}_{0,2}, \quad y \in \mathfrak{g}_{-1}.$$

Any $f \in \text{Hom}(\mathfrak{g}_{-1,1}, \mathfrak{g}_{0,2})$ trivially satisfies when $y_1, y_2 \in \mathfrak{g}_{-1,1}$. In the strongly regular case, the adjoint action of nonzero $v \in \mathfrak{g}_{0,2}$ is a linear isomorphism from $\mathfrak{g}_{-1,1}$ to $\mathfrak{g}_{1,0}$, so $f \in \text{Hom}(\mathfrak{g}_{-1,1}, \mathfrak{g}_{0,2})$ extends to $\mathfrak{g}_{-1,1}$ by (5.2), and from $\mathfrak{g}_{-1}$ to $\mathfrak{g}_{-2,0}$ by (5.1). Thus, $\dim \mathfrak{g}_{1,1} = \dim \text{Hom}(\mathfrak{g}_{-1,1}, \mathfrak{g}_{0,2}) = \dim \mathfrak{g}_{-1,1}$. Arguments for $\mathfrak{g}_{1,-1}$ are the same. \hspace{1cm} \Box

**Proof of parts (1) and (2) of Theorem 5.1**
5.1. An adapted basis of $g^0$. Let $g^0$ be a strongly regular multiplicative symbol with $n = \dim C g_{-1, \pm 1}$. We introduce a convenient basis of $g_-$ and explicitly calculate bigraded components of $g^0$ with weight 0. Fix index ranges and constants

\begin{equation}
1 \leq a, b \leq p, \quad 1 \leq \alpha, \beta, \gamma \leq n, \quad \epsilon_\alpha = \begin{cases} 1, & \alpha \leq p; \\ -1, & \alpha > p. \end{cases}
\end{equation}

Let $\{e_\alpha\}$ be the basis of $g_{-1,1}$ from Corollary (4.11) so that the Hermitian form $\ell$ and operator $A$ are nicely presented by normal forms (4.12) or (4.13). Including $\epsilon_\alpha$ with $A$ according to the normal form for the anti-linear operator (5.8)

\begin{equation}
\text{Let } \sigma \text{ the matrix } [e_\alpha] \text{ adapted basis of } g_{-1,1}, \quad \{e_\alpha\} \subset g_{-1,1}, \quad \{\overline{e}_\alpha\} \subset g_{-1,-1},
\end{equation}

with bracket relations

\begin{equation}
\epsilon_\alpha e_0 = [e_\alpha, \overline{e}_\alpha],
\end{equation}

all others being trivial. The corresponding real basis for $Rg_-$ is

\begin{equation}
i e_0 \in iRg_{-2,0}, \quad \{e_\alpha\} \subset Rg_{-1,1}, \quad \{\overline{e}_\alpha\} \subset Rg_{-1,-1},
\end{equation}

A raised index $e^0, e^\alpha, \overline{e}^\alpha \in (g_-)^*$ refers to the dual basis vector $e^\alpha(e_\alpha) = 1$, etc. which vanishes on all other basis elements. The overline notation for complex conjugation is therefore consistent with (2.9).

For the symbol of Type I$_{p,q}$ we choose basis elements for $g_{0, \pm 2}$ by specifying how they act on $g_{-1,\pm 1}$ according to the normal form for the anti-linear operator $A$ from (4.12):

\begin{equation}
\text{(Type I)} \quad e_{n+1} = ad_v = \sum_\alpha e_\alpha \otimes \overline{e}^\alpha \in g_{0,2}, \quad \overline{e}_{n+1} = ad_\overline{v} = \sum_\alpha \overline{e}_\alpha \otimes e^\alpha \in g_{0,-2},
\end{equation}

whose corresponding bracket relations are

\begin{equation}
[e_{n+1}, \overline{e}_\alpha] = e_\alpha, \quad [\overline{e}_{n+1}, e_\alpha] = \overline{e}_\alpha, \quad [e_{n+1}, \overline{e}_{n+1}] = \sum_\alpha e_\alpha \otimes e^\alpha - \overline{e}_\alpha \otimes \overline{e}^\alpha.
\end{equation}

In the case of the symbol of Type II$_p$

\begin{equation}
e_{n+1} = ad_v = \sum_a e_{p+a} \otimes \overline{e}^a - e_a \otimes \overline{e}^{p+a} \in g_{0,2}, \quad \overline{e}_{n+1} = ad_\overline{v} = \sum_a \overline{e}_{p+a} \otimes e^a - \overline{e}_a \otimes e^{p+a} \in g_{0,-2},
\end{equation}

giving brackets

\begin{equation}
[e_{n+1}, \overline{e}_a] = e_{p+a}, \quad [e_{n+1}, \overline{e}_{p+a}] = -e_a, \quad [\overline{e}_{n+1}, e_a] = \overline{e}_{p+a}, \quad [\overline{e}_{n+1}, e_{p+a}] = -\overline{e}_a,
\end{equation}

\begin{equation}
[e_{n+1}, \overline{e}_{n+1}] = \sum_a -e_a \otimes e^a + \overline{e}_a \otimes \overline{e}^a.
\end{equation}

Both symbol types relate to their real forms via

\begin{equation}
(e_{n+1} + \overline{e}_{n+1}), i(e_{n+1} - \overline{e}_{n+1}) \in Rg_0.
\end{equation}

With $e_0 \in g_{-2,0}$ fixed, one can interpret the Lie bracket as a symplectic form $\sigma \in \Lambda^2(g_-)^*$ on the vector space $g_{-1,1} \oplus g_{-1,-1}$ via $[y_1, y_2] = \sigma(y_1, y_2)e_0$. Each bigraded component $g_{-1,\pm 1}$ determines a Lagrangian subspace. Identifying $\{e_\alpha, \overline{e}_\alpha\}$ with the standard basis of column vectors for $C^{2n}$, $\sigma(e_\alpha, \overline{e}_\beta) = \epsilon_\alpha \sigma[\overline{e}_\beta]$ for the matrix $[\sigma]$ divided into $n \times n$ blocks

\begin{equation}
[\sigma] = \begin{bmatrix} 0 & \epsilon \\ -\epsilon & 0 \end{bmatrix},
\end{equation}

with $\epsilon \in \text{Mat}_{n \times n, R}$ as the diagonal matrix $\epsilon_{\alpha,\beta} = \epsilon_\alpha \delta_{\alpha,\beta}$ (Kronecker delta). The symplectic group $Sp_{2n}C$ consists of $2n \times 2n$ matrices $S$ satisfying $S^t[\sigma]S = [\sigma]$, so the group $G_{0,0}$ of all bigraded algebra automorphisms of $g_-$ is represented by the subgroup of $Sp_{2n}C$ comprised of $n \times n$ block-diagonal matrices, along with the conformal scaling operation

\begin{equation}
\{e_0, e_\alpha, \overline{e}_\alpha\} \mapsto \{r^2e_0, re_\alpha, r\overline{e}_\alpha\}, \quad r \in R \setminus \{0\},
\end{equation}

On absolute parallelism for 2-nondegenerate CR structures via bigraded Tanaka prolongation 13
corresponding to a different choice of \( c_0 \in i\mathbb{R}_0^0 \).

The Lie algebra of \( Sp_{2n}\mathbb{C} \) is denoted \( sp_{2n}\mathbb{C} \), and its standard representation is discussed in [11]. The subalgebra \( sp_{2n}\mathbb{C} \cap \mathfrak{g}_{0,0} \) is represented by matrices of the form

\[
(5.10) \begin{bmatrix} \epsilon B & 0 \\ 0 & -\epsilon B^t \end{bmatrix}, \quad B \in \text{Mat}_{n \times n}\mathbb{C}.
\]

We can also represent the restrictions to \( \mathfrak{g}_- \) of \( \text{ad}_{e_{n+1}} \) and \( \text{ad}_{\overline{e}_{n+1}} \) as off-diagonal matrices in \( sp_{2n}\mathbb{C} \). For CR symbols of Type I,

\[
(5.11) \quad \text{ad}_{e_{n+1}} = \begin{bmatrix} 0 & 1_n \\ 0 & 0 \end{bmatrix}, \quad \text{ad}_{\overline{e}_{n+1}} = \begin{bmatrix} 0 & 0 \\ 1_n & 0 \end{bmatrix},
\]

where \( 1_n \) is the \( n \times n \) identity matrix. For Type II when \( n = 2p \), we subdivide into \( p \times p \) blocks

\[
\text{ad}_{e_{n+1}} = \begin{bmatrix} 0 & 0 & -1_p \\
0 & 0 & 1_p \\
0 & 0 & 0 \\
0 & 0 & 0 \end{bmatrix}, \quad \text{ad}_{\overline{e}_{n+1}} = \begin{bmatrix} 0 & 0 & 0 \\
0 & 0 & 0 \\
0 & -1_p & 0 \\
1_p & 0 & 0 \end{bmatrix}.
\]

Writing \([\epsilon B]\) to abbreviate the matrix \( (5.10) \), \( \mathfrak{g}_{0,0} \subset \mathfrak{der}(\mathfrak{g}_-) \) defined by \( (2.9)-(2.7) \) contains all such matrices which satisfy

\[
(5.12) \quad [\epsilon B] \text{ad}_{v} - \text{ad}_{\epsilon B} v = \text{cad}_{v}, \quad v = e_{n+1}, \overline{e}_{n+1}, \quad \epsilon \in \mathbb{C}.
\]

In particular, \([1_n]\) is in \( \mathfrak{g}_{0,0} \), and we name

\[
(5.13) \quad E = \sum_\alpha e_\alpha \otimes e^\alpha - \overline{e}_\alpha \otimes \overline{e}^\alpha,
\]

which is imaginary with respect to conjugation in \( \mathfrak{g}_0^0 \). In addition to \( (5.12) \), \( \mathfrak{g}_{0,0} \) includes the infinitesimal generator of the conformal scaling operation \( (5.9) \),

\[
(5.14) \quad \hat{E} = 2c_0 \otimes e^0 + \sum_\alpha e_\alpha \otimes e^\alpha + \overline{e}_\alpha \otimes \overline{e}^\alpha,
\]

which is real and clearly commutes with \( e_{n+1} \) and \( \overline{e}_{n+1} \). The real form \( \mathbb{R}\mathfrak{g}_0^0 \) will feature

\[
(5.15) \quad iE, \hat{E} \in \mathbb{R}\mathfrak{g}_{0,0}.
\]

We augment our bracket relations

\[
(5.16) \quad [E, e_\alpha] = e_\alpha = [\hat{E}, e_\alpha], \quad [E, \overline{e}_\alpha] = -\overline{e}_\alpha = -[\hat{E}, \overline{e}_\alpha], \quad [\hat{E}, c_0] = 2c_0,
\]

\[
[E, e_{n+1}] = 2e_{n+1}, \quad [E, \overline{e}_{n+1}] = -2\overline{e}_{n+1}, \quad [e_{n+1}, \overline{e}_{n+1}] = \pm E,
\]

where the last bracket depends on the CR symbol Type. We will consider the two symbol Types separately to fill out a basis of \( \mathfrak{g}_{0,0} \) and perform the algebraic prolongation procedure.

### 5.2. Type I CR Symbols and the Algebra \( sp(p+2, q+2) \)

In light of \( (5.11) \), condition \( (5.12) \) is equivalent to \( \epsilon(B + B^t) = c 1_n \). For diagonal \( B \), we have already seen in \( (5.13) \) that \( B = c 1_n \) corresponds to \( E \in \mathfrak{g}_{0,0} \), which leaves the case \( c = 0 \) when \( B \) is skew-symmetric. For each pair \( \alpha, \beta \) of indices \( \alpha < \beta \), define

\[
e_{\alpha \beta} = e_\alpha e_\beta \otimes e^\beta - e_\beta e_\alpha \otimes e^\alpha + e_\alpha \overline{e}_\beta \otimes \overline{e}^\beta - e_\beta \overline{e}_\alpha \otimes \overline{e}^\alpha \in \mathfrak{g}_{0,0},
\]

and denote

\[
e_{\beta \alpha} = -e_{\alpha \beta}, \quad e_{\alpha \alpha} = 0.
\]

Notice that by \( (2.9) \), \( \overline{e}_{\alpha \beta} = e_{\alpha \beta} \), whereby

\[
(5.17) \quad e_{\alpha \beta} \in \mathbb{R}\mathfrak{g}_{0,0}.
\]
Along with $E$ and $\hat{E}$ as in (5.13) and (5.14), these $\binom{n}{2}$ derivations complete a basis for $\mathfrak{g}_{0,0}$. They also have structure equations

$$[e_{\alpha \beta}, e_{\gamma}] = \delta^\beta_\gamma e_{\alpha} - \delta^\alpha_\gamma e_{\beta}, \quad [e_{\alpha \beta}, \overline{e}_{\gamma}] = \delta^\beta_\gamma e_{\alpha} - \delta^\alpha_\gamma \overline{e}_{\beta}, \quad [e_{\alpha \beta}, e_{\beta \gamma}] = e_{\beta e_{\alpha \gamma}},$$

all others being trivial.

By Lemma 5.2, the following $n$ derivations of bigraded degree $(1,1)$ span $\mathfrak{g}_{1,1}$,

$$E_{\alpha} = 2e_{\alpha} \otimes e^0 + 2e_{\alpha}e_{n+1} \otimes e^\alpha + e_{\alpha}(\hat{E} - E) \otimes \overline{e}^\alpha - 2e_{\alpha} \sum_{\beta} \beta e_{\alpha \beta} \otimes \overline{e}^\beta.$$ 

Similarly, $\mathfrak{g}_{1,-1}$ is spanned by

$$\overline{E}_{\alpha} = -2\overline{e}_{\alpha} \otimes e^0 + 2e_{\alpha}\overline{e}_{n+1} \otimes \overline{e}^\alpha + e_{\alpha}(\hat{E} + E) \otimes e^\alpha - 2e_{\alpha} \sum_{\beta} \beta e_{\alpha \beta} \otimes e^\beta.$$ 

The second prolongation is spanned by the imaginary derivation of bigraded degree $(2,0)$

$$E_0 = 2\hat{E} \otimes e^0 + \sum_{\alpha} (E_{\alpha} \otimes e^\alpha - \overline{E}_{\alpha} \otimes \overline{e}^\alpha),$$

and all higher prolongations are trivial. For the real form, we have

$$\{ (E_{\alpha} + \overline{E}_{\alpha}), i(E_{\alpha} - \overline{E}_{\alpha}) \} \subset \mathbb{R}\mathfrak{g}_1, \quad iE_0 \in \mathbb{R}\mathfrak{g}_2.$$

The final algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ has dimension $4n + 6 + \binom{n+4}{2}$. We will see that $\mathbb{R}\mathfrak{g}$ is isomorphic to a familiar matrix Lie algebra, so fix the constant and index ranges

$$N = n + 4, \quad 1 \leq i, j, k, l \leq N,$

and write $E^i_j$ for the $N \times N$ matrix which has entry 1 in the $i$th column and $j$th row and zeros elsewhere. Multiplication in $\text{Mat}_{N \times N} \mathbb{C}$ is written in terms of the basis $\{ E^i_j \}$ using the Kronecker delta,

$$E^i_j E^k_l = \delta^i_k \delta^j_l,$$

and $\mathfrak{g}_N \mathbb{R} = \text{Mat}_{N \times N} \mathbb{C}$ is equipped with the Lie bracket defined by the matrix commutator

$$[E^i_j, E^k_l] = \delta^k_i E^j_l - \delta^j_k E^i_l.$$

$\mathbb{R}^N$ admits a symmetric bilinear form $Q$ of signature $(p + 2, q + 2)$ defined by

$$Q = E_1^3 + E_3^1 + E_4^2 + E_4^3 + \sum_{\alpha} \epsilon_{\alpha} E^{4+\alpha} \otimes e^{4+\alpha}, \quad \text{i.e.,} \quad Q = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \end{bmatrix}.$$ 

Let $\mathfrak{so}(p + 2, q + 2)$ denote the Lie subalgebra of $\mathfrak{g}_N \mathbb{R}$ comprised of matrices $C$ which satisfy

$$C^* Q + QC = 0.$$

To show that $\mathbb{R}\mathfrak{g}$ is isomorphic to $\mathfrak{so}(p + 2, q + 2)$, we will exhibit a basis of the latter with the same names as that of the former such that the structure equations of each coincide. To begin, set

$$e_0 = E_1^1 - E_3^3, \quad e_\alpha = e_{\alpha} E_4^{4+\alpha} - E_4^1 - E_4^3, \quad \overline{e}_\alpha = e_{\alpha} E_4^{4+\alpha} - E_4^1 - E_4^3,$$

so that these replicate (5.5). Corresponding to $\mathfrak{g}_{0,\pm 2}$ we have

$$e_{n+1} = E_3^4 - E_1^1, \quad \overline{e}_{n+1} = E_4^3 - E_4^1,$$

brackets for which agree with (5.6). Analogous to basis elements of $\mathfrak{g}_{0,0}$ we name

$$E = -E_1^1 + E_2^2 + E_3^3 - E_4^4, \quad \hat{E} = -E_1^1 - E_2^2 + E_3^3 + E_4^4, \quad e_{\alpha \beta} = e_{\alpha} E_4^{4+\alpha} - E_4^1 - E_4^3,$$

whose brackets with (5.21) and (5.22) in $\mathfrak{so}(p + 2, q + 2)$ reproduce (5.10) and (5.13). Playing the same role as the first prolongation in $\mathfrak{g}$ are the matrices

$$E_{\alpha} = 2(e_{\alpha} E_4^{4+\alpha} - E_4^1 - E_4^3), \quad \overline{E}_{\alpha} = 2(e_{\alpha} E_4^{4+\alpha} - E_4^1 - E_4^3).$$
and a basis of $\mathfrak{so}(p + 2, q + 2)$ is completed by the matrix

$$E_0 = 2(E_2^3 - E_1^4).$$

A change of (real) basis according to (5.3), (5.8), (5.16), (5.17), and (5.19) now shows that $\mathfrak{g}$ is isomorphic to $\mathfrak{so}(p + 2, q + 2)$.

5.3. Type II CR Symbols and the Algebra $\mathfrak{so}^*(2p + 4)$. Recall that Type II is only possible for $p = q$ when $n = 2p$. In this case,

$$\epsilon = \begin{pmatrix} 1_p & 0 \\ 0 & -1_p \end{pmatrix},$$

and (5.10) is clarified by splitting $B \in \text{Mat}_{n \times n} \mathbb{C}$ into $p \times p$ blocks $B_1, B_2, B_3, B_4 \in \text{Mat}_{p \times p} \mathbb{C}$ to yield

$$\begin{bmatrix} B_1 & B_2 & 0 & 0 \\ -B_3 & -B_4 & 0 & 0 \\ 0 & 0 & -B_1^t & -B_2^t \\ 0 & 0 & B_2^t & B_1^t \end{bmatrix},$$

at which point (5.12) becomes

$$B_2 + B_2^t = 0 = B_3 + B_3^t, \quad B_1 - B_1^t = \pm c 1_p.$$ We have already seen that $c \neq 0$ corresponds to $E \in \mathfrak{g}_{0,0}$ when $B_1 = 1_p = -B_1^t$, which leaves matrices satisfying $B_4 = B_4^t$ while $B_2$ and $B_3$ are skew-symmetric.

For each pair $a, b$ of indices (5.3), define

$$E_{ab} = e_a \otimes e^b - e_{p+b} \otimes e^{p+a} - e_b \otimes e^a + e_{p+a} \otimes e^b,$$

so that $E_{ab} - E_{ba}$ is real and $E_{ab} + E_{ba}$ is imaginary with respect to conjugation (2.20) in $\mathfrak{g}^0$. For each pair of indices with $a < b$,

$$e_{ab} = e_a \otimes e^{p+b} - e_b \otimes e^{p+a} + e_{p+b} \otimes e^a - e_{p+a} \otimes e^b,$$

$$e_{ba} = e_a \otimes e^{p+b} - e_b \otimes e^{p+a} + e_{p+b} \otimes e^a - e_{p+a} \otimes e^b,$$

These $p^2 + 2(p_2) = (2p_2)$ derivations complete $E, \hat{E}$ to a basis of $\mathfrak{g}_{0,0}$. A corresponding basis for the real form is furnished by

(5.23) $$\{E_{ab} - E_{ba}, i(E_{ab} + E_{ba}), (e_{ab} + e_{ba}), i(e_{ab} - e_{ba})\} \in \mathfrak{g}_{0,0}.$$ For the first prolongation, Lemma (5.22) shows that $\mathfrak{g}_{1,1}$ is spanned by $n$ derivations

$$E_a = 2e_a \otimes e^0 + 2e_{n+1} \otimes e^{p+a} + (\hat{E} - E) \otimes e^a - 2 \sum_b E_{ab} \otimes e^b + 2 \sum_b e_{ab} \otimes e^{p+b},$$

$$E_{p+a} = 2e_{p+a} \otimes e^0 + 2e_{n+1} \otimes e^a + (\hat{E} - E) \otimes e^{p+a} - 2 \sum_b E_{ba} \otimes e^{p+b} + 2 \sum_b e_{ab} \otimes e^b,$$

while $\mathfrak{g}_{1,-1}$ is the span of

$$\bar{E}_a = -2e_a \otimes e^0 + 2e_{n+1} \otimes e^a + (\hat{E} + E) \otimes e^a + 2 \sum_b E_{ab} \otimes e^b + 2 \sum_b e_{ab} \otimes e^{p+b},$$

$$\bar{E}_{p+a} = -2e_{p+a} \otimes e^0 + 2e_{n+1} \otimes e^a - (\hat{E} + E) \otimes e^{p+a} + 2 \sum_b E_{ab} \otimes e^{p+b} + 2 \sum_b e_{ab} \otimes e^b.$$ As in the case of Type I, the second prolongation is spanned by the imaginary derivation of bigraded degree $(2, 0)$

$$E_0 = 2\hat{E} \otimes e^0 + \sum_a (E_a \otimes e^a - \bar{E}_a \otimes e^a),$$

and all higher prolongations are trivial.
Again, the real form of the final algebra \( \mathfrak{g} = \mathfrak{g}^1 \oplus \mathfrak{g}_2 \) is isomorphic to a real form of the complex, special orthogonal algebra \( \mathfrak{so}_{n+4} \mathbb{C} \). Using the same representation as in \( \mathfrak{so}_{2(p+2)} \mathbb{C} \) is realized as complex matrices of the form

\[
(5.24) \begin{bmatrix}
A & B \\
C & -A^t
\end{bmatrix}, \quad A, B, C \in \text{Mat}_{(p+2) \times (p+2)} \mathbb{C}; \quad B + B^t = 0 = C + C^t.
\]

The real form \( \mathfrak{so}^*(2p + 4) \subset \mathfrak{so}_{2(p+2)} \mathbb{C} \) is the subalgebra of matrices \( (5.24) \) which additionally satisfy

\[
\begin{bmatrix}
\mathcal{A} & \mathcal{C} \\
\mathcal{B} & -\mathcal{A}
\end{bmatrix} \begin{bmatrix}
0 & \mathbb{I}_{p+2} \\
-\mathbb{I}_{p+2} & 0
\end{bmatrix} + \begin{bmatrix}
0 & \mathbb{I}_{p+2} \\
-\mathbb{I}_{p+2} & 0
\end{bmatrix} \begin{bmatrix}
A & B \\
C & -A^t
\end{bmatrix} = 0,
\]

so that \( \mathcal{A} = A, \mathcal{B} = -B, \) and \( \mathcal{C} = -C \). This is a real Lie algebra with the matrix commutator bracket \( (5.20) \), and we use the same matrices \( E_i \) as in \( (5.2) \) to construct a basis of \( \mathfrak{so}^*(2p + 4) \) whose structure equations are the same as those of \( \mathfrak{g} \). To begin, set

\[
e_0 = iE_{p+2}^{p+2} - iE_{n+4}^{p+1}, \quad e_a = iE_{p+2+a}^{p+1} - iE_{n+3}^{a}, \quad e_{p+a} = aE_{p+2+a}^{p+1} - E_{n+3}^{p+2+a}, \quad \mathcal{e}_a = E_{p+2+a}^{p+1} - E_{n+4}^{a}, \quad \mathcal{e}_{p+a} = iE_{p+2+a}^{p+1} - iE_{n+4}^{a},
\]

so that brackets of these matrices agree with \( (5.3) \). Adding

\[
e_{n+1} = E_{p+2}^{p+1} - E_{n+3}^{n+4}, \quad \mathcal{e}_{n+1} = E_{n+3}^{n+4} - E_{p+1}^{p+2},
\]

yields equations \( (5.7) \). To replicate \( (5.10) \), set

\[
E = -E_{p+1}^{p+1} + E_{p+2}^{p+2} + E_{n+3}^{n+3} - E_{n+4}^{n+4}, \quad \mathcal{E} = -E_{p+1}^{p+1} - E_{p+2}^{p+2} + E_{n+3}^{n+3} + E_{n+4}^{n+4},
\]

along with

\[
E_{ab} = -E_a^b + E_{p+2+b}^{p+2}, \quad e_{ab} = iE_{p+2+a}^b - iE_{p+2+b}^a, \quad \mathcal{E}_{ab} = iE_{p+2+a}^{p+1} - iE_{p+2+a}^{p+2}.\]

completing a basis corresponding to that of \( \mathfrak{g}_{0,0} \). Playing the role of the first prolongation, we have

\[
E_a = 2E_{p+2}^a - 2E_{p+2+a}^{n+4}, \quad \mathcal{E}_a = 2E_{p+1}^{p+2} - 2E_{n+3}^{a}, \quad E_{p+a} = 2E_{a}^{n+4} - 2E_{p+2}^{p+2+a}, \quad \mathcal{E}_{p+a} = 2E_{p+1}^{p+2} - 2E_{p+2+a}^{n+3},
\]

and a basis for \( \mathfrak{so}^*(2p + 4) \) is completed by

\[
E_0 = 2iE_{p+2}^{n+3} - 2iE_{p+1}^{n+4}.
\]

A change of (real) basis according to \( (5.5), (5.8), (5.15), (5.23), \) and \( (5.19) \) now shows that \( \mathfrak{N} \mathfrak{g} \) is isomorphic to \( \mathfrak{so}^*(2p + 4) \).

This concludes the proof of parts (1) and (2) of Theorem \( 5.1 \).

5.4. Proof of part (3) of Theorem \( 5.1 \) Recalling Lemma \( 4.2 \) we have a splitting \( \mathfrak{g}_{-1,1} = Z \oplus W \) where

\[
Z = \ker \text{ad}_{\mathcal{N}}, \quad W = \text{Im}(\text{ad}_K).
\]

By \( (5.3) \), it is straightforward to confirm that derivations in \( \mathfrak{g}_{0,0} \) preserve the splitting of \( \mathfrak{g}_{-1,1} \). We augment \( (5.3) \) with index ranges

\[
1 \leq a_1, \beta_1 \leq p_1 + q_1 = n_1, \quad \epsilon_{a_1} = \begin{cases} 1, & a_1 \leq p_1; \\ -1, & a_1 > p_1; \end{cases} \quad \epsilon_{\alpha} = \begin{cases} 1, & n_1 < \alpha \leq n_1 + p_1 - p_1; \\ -1, & \alpha > n_1 + p_1 - p_1; \end{cases}
\]

Let \( e_0 \in \mathfrak{N} \mathfrak{g}_{-2}, \{ e_{a_1} \} \subset W, \{ e_\alpha \} \subset \mathfrak{g}_{-1,1} \) be a basis as described in Theorem \( 1.3 \) and identify \( \{ e_\alpha, \mathcal{e}_\alpha \} \) with the standard basis of \( \mathbb{C}^{2n} \) as in \( (5.3) \) so that \( \mathfrak{g}_0 \) may once again be interpreted as a subalgebra of the conformal symplectic algebra. In contrast to \( (5.11) \), basis vectors \( e_{n+1} \in \mathfrak{g}_{0,2} \) and \( \mathcal{e}_{n+1} \in \mathfrak{g}_{0,-2} \) for Type I are now represented

\[
ad_{e_{n+1}} = \begin{bmatrix} 0 & \delta_{n_1} \\ 0 & 0 \end{bmatrix}, \quad \nad_{\mathcal{e}_{n+1}} = \begin{bmatrix} 0 & 0 \\ \delta_{n_1} & 0 \end{bmatrix},
\]

\[
E = -E_{p+1}^{p+1} + E_{p+2}^{p+2} + E_{n+3}^{n+3} - E_{n+4}^{n+4}, \quad \mathcal{E} = -E_{p+1}^{p+1} - E_{p+2}^{p+2} + E_{n+3}^{n+3} + E_{n+4}^{n+4},
\]

along with

\[
E_{ab} = -E_a^b + E_{p+2+b}^{p+2}, \quad e_{ab} = iE_{p+2+a}^b - iE_{p+2+b}^a, \quad \mathcal{E}_{ab} = iE_{p+2+a}^{p+1} - iE_{p+2+a}^{p+2},
\]

completing a basis corresponding to that of \( \mathfrak{g}_{0,0} \). Playing the role of the first prolongation, we have

\[
E_a = 2E_{p+2}^a - 2E_{p+2+a}^{n+4}, \quad \mathcal{E}_a = 2E_{p+1}^{p+2} - 2E_{n+3}^{a}, \quad E_{p+a} = 2E_{a}^{n+4} - 2E_{p+2}^{p+2+a}, \quad \mathcal{E}_{p+a} = 2E_{p+1}^{p+2} - 2E_{p+2+a}^{n+3},
\]

and a basis for \( \mathfrak{so}^*(2p + 4) \) is completed by

\[
E_0 = 2iE_{p+2}^{n+3} - 2iE_{p+1}^{n+4}.
\]

A change of (real) basis according to \( (5.5), (5.8), (5.15), (5.23), \) and \( (5.19) \) now shows that \( \mathfrak{N} \mathfrak{g} \) is isomorphic to \( \mathfrak{so}^*(2p + 4) \).

This concludes the proof of parts (1) and (2) of Theorem \( 5.1 \).
where \( \delta_{n_i} \) is the \( n \times n \) diagonal matrix which is 1 in its first \( n_1 \) diagonal entries and zero elsewhere. For Type II, we subdivide into \( p \times p \) blocks

\[
\text{ad}_{\epsilon_{n+1}} = \begin{bmatrix}
0 & 0 & -\delta_{p_1} \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad \text{ad}_{\epsilon_{n+1}} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
-\delta_{p_1} & 0 & 0
\end{bmatrix}.
\]

Aside from the conformal scaling generator \( \hat{E} \) as in (5.13), \( g_{0,0} \) is represented by matrices

\[
\begin{bmatrix}
\epsilon B_W & 0 & 0 \\
0 & \epsilon B_Z & 0 \\
0 & 0 & -\epsilon B_W & -\epsilon B_Z
\end{bmatrix}, \quad B_W \in \text{Mat}_{n_1 \times n_1} \mathbb{C}, \quad B_Z \in \text{Mat}_{(n-n_1) \times (n-n_1)} \mathbb{C},
\]

where the analog of condition (5.12) requires that \( B_W \) is as in the strongly regular case of Type I, \( p_{1,q_1} \) or \( \Pi_{p_1} \). Therefore, the subalgebra of \( g^0 \) comprised of \( g_{-2} \oplus W \oplus \overline{W} \oplus g_{0,2} \oplus g_{0,-2} \) along with \( \hat{E} \) and derivations in \( g_{0,0} \) corresponding to matrices \( B_W \) satisfying (5.12) determines a strongly regular symbol of Type I, \( p_{1,q_1} \) or \( \Pi_{p_1} \), which we refer to as the underlying strongly regular symbol of \( g^0 \).

Suppose \( f \in g_{1,1} \) and take \( z \in Z, \overline{y} \in g_{-1,-1} \) so that

\[
f([z, \overline{y}]) = [f(z), \overline{y}] + [z, f(\overline{y})].
\]

If \( \overline{y} \in \overline{W} \), we get \( 0 = [f(z), \overline{y}] + [z, f(\overline{y})] \) which shows that both terms are zero, because \( f(\overline{y}) \in g_{0,0} \Rightarrow [z, f(\overline{y})]\) is zero. If \( f(z) \in g_{0,2} \Rightarrow [f(z), \overline{y}] \in W \). If \( f(z) \in g_{0,2} \), then \( f(z) = 0 \). Thus, \( f(z) \in g_{0,2} \) acts trivially on all of \( g_{-1} \), whence \( f(z) = 0 \).

Now let \( y_1, y_2 \in g_{-1} \) such that \( 0 \neq [y_1, y_2] = y \in g_{-2} \). If \( y_1 \in Z \) and \( y_2 \in \overline{Z} \), then the fact that \( f(y_1) = 0 \) implies

\[
f(y) = [y_1, f(y_2)],
\]

which lies in \( Z \) since \( f(y_2) \in g_{0,0} \). On the other hand, if \( y_1 \in W \) and \( y_2 \in \overline{W} \),

\[
f(y) = [f(y_1), y_2] + [y_1, f(y_2)],
\]

which lies in \( W \). Thus \( f(y) = 0 \). The Lie bracket pairs \( Z \) nondegenerately with \( \overline{Z} \), so the fact that \( f \) acts trivially on \( Z \) and \( g_{-2} \) implies \( f \) vanishes on \( \overline{Z} \) as well.

Similar arguments show \( f \in g_{1,-1} \) acts trivially on \( Z \oplus \overline{Z} \). In this way we see that \( g_{1} \) is a subspace of the first prolongation of the underlying strongly regular symbol of \( g^0 \). However, no nontrivial degree-1 derivations of such a strongly regular symbol vanish on \( g_{-2} \), so it must be that \( g_{1} = 0 \), and Theorem 5.1 is proved.

\section{Proof of Theorem 6.2}

\subsection{Proof of Part (1): Geometric Prolongation}

Consider a symbol \( g^0 \) whose bigraded universal algebraic prolongation \( \mathcal{X}_{\text{bigrad}}(g^0) \) has the bigraded splitting (3.1), such that \( g_i = \bigoplus_{j \in \mathbb{Z}} g_{i,j} \) consists of all elements with first weight \( i \). Let \( l \) be the nonnegative integer such that \( g_l \neq 0 \) but \( g_{l+1} = 0 \). For instance, when \( \dim_{\mathbb{C}} g_{0,2} = 1 \) Theorem 5.1 shows \( l = 2 \) for strongly regular symbols and \( l = 0 \) for weakly regular symbols.

As in the standard Tanaka theory (see Remark 1.2 and set \( \mu = 2 \)), we will recursively construct a sequence of bundles

\[
(6.1) \quad P^{-1} = M \leftarrow P^0 \leftarrow P^1 \leftarrow P^2 \leftarrow \ldots
\]

such that \( P^0 \) is a principal bundle whose structure group has Lie algebra \( g_{0,0} \) and for \( i > 0 \), \( P^i \) is a bundle over \( P^{i-1} \) whose fibers are affine spaces with modeling vector space \( g_i \). In each step of the inductive procedure, the construction of the bundle \( P^{i+1} \rightarrow P^i \) produces a concomitant bundle \( \mathbb{R}P^{i+1} \rightarrow \mathbb{R}P^i \).

Here, \( \mathbb{R}P^0 \) is a principal bundle whose structure group has Lie algebra isomorphic to the real part \( \mathbb{R}g_{0,0} \) of \( g_{0,0} \), and for \( i > 0 \) \( \mathbb{R}P^i \) is a bundle over \( \mathbb{R}P^{i-1} \) whose fibers are affine spaces with modeling vector
space equal to the real part $\mathbb{R}g_i$ of $g_i$ (see the discussion in the paragraph before Theorem 2.2). The bundle $\mathbb{R}P^{i+2}$ is endowed with a canonical frame – a structure of absolute parallelism.

6.1.1. **Bigraded Frame Bundle $P^0$.** Let $x \in M$. Recalling definitions (2.1), we name the canonical quotient projections

\[
 q_{-1} : CD_x \to g_{-1}(x), \quad q_{-2} : CT_x M \to g_{-2}(x).
\]

Regarding the splitting $CD = H \oplus \overline{P}$, we also name the summand projections

\[
 p_+ : CD \to H, \quad p_- : CD \to \overline{P},
\]

furnishing linear projections onto the bigraded components

\[
 q_{-1,1} = q_{-1} \circ p_+ : CD_x \to g_{-1,1}(x), \quad q_{-1,-1} = q_{-1} \circ p_- : CD_x \to g_{-1,-1}(x).
\]

For the remaining bigraded components of the CR symbol algebra at $x$ (Definition 2.1) that also lie in $m(x)$ (see (2.12)), recall

\[
 \text{ad}_{K_x} = g_{0,2}(x), \quad \text{ad}_{\overline{K}_x} = g_{0,-2}(x).
\]

Fix an abstract symbol (Definition 2.2) $g^0$, which is isomorphic to $g^0(x)$ for every $x \in M$. For any vector subspace $s \subset g^0$ defined as a sum of specified bigraded components of $g^0$, we will denote by $I(s)$ the indexing set of all bidegrees of the components defining $s$. For example, $I(m) = \{(i,j) \mid (i,j) = (-2,0), (-1,1), (0,2)\}$.

We now define a bundle $\pi : P^0 \to M$ whose fiber over $x \in M$ is comprised of all adapted frames, or bigraded Lie algebra isomorphisms,

\[
P^0_x = \left\{ \varphi_x : g_- \to g_-(x) \left| \begin{array}{c} \varphi_x(g_{i,j}) = g_{i,j}(x) \\ \varphi_x^{-1} \circ g_{0,\pm 2}(x) \circ \varphi_x = g_{0,\pm 2} \\ \varphi_x([y_1,y_2]) = [\varphi_x(y_1),\varphi_x(y_2)] \\ y_1, y_2 \in g_- \end{array} \right. \right\}.\]

The Lie algebra $g_{0,0} \subset g^0$ is tangent to the Lie group $G_{0,0} \subset \text{AUT}(g_-)$ of bigraded algebra isomorphisms of $g_-$ whose adjoint action on $\text{der}(g_-)$ preserves the spaces $g_{0,\pm 2}$. $P^0$ is a principal $G_{0,0}$-bundle, where the right principal action $R_g : P^0 \to P^0$ of $g \in G_{0,0}$ on each fiber is given by

\[
 R_g(\varphi_x) = \varphi_x \circ g : g_- \to g_-(x).
\]

$\mathbb{R}P^0 \to M$ will denote the subbundle of $P^0$ whose fiber over $x$ consists of those $\varphi_x$ as in (6.5) which map $\mathbb{R}g_- \to g_{-}(x)$. $\mathbb{R}P^0$ is a principal $\mathbb{R}G_{0,0}$-bundle (see Definition 2.4 and the comments preceding it).

The CR filtration of $TM$ induces a filtration on $TP^0$ via the inverse image of the pushforward $\pi_*$, so we name

\[
 T^{-2}P^0 = T^0P^0, \quad T^{-1}P^0 = \pi_*^{-1}(D), \quad T^0P^0 = \pi_*^{-1}(\mathbb{R}(K \oplus \overline{K})),
\]

with $CT^iP^0$ being the inverse image of the corresponding complexification $(i = -2, -1, 0)$. The complexified filters have subbundles

\[
 T^{-1,1}P^0 = (\pi_*)^{-1}(H) \subset CT^{-1}P^0, \quad T^{-1,-1}P^0 = (\pi_*)^{-1}(\overline{H}) \subset CT^{-1}P^0,
\]

\[
 T^{0,2}P^0 = (\pi_*)^{-1}(K) \subset CT^{0}P^0, \quad T^{0,-2}P^0 = (\pi_*)^{-1}(\overline{K}) \subset CT^{0}P^0,
\]

\[
 T^{0,0}P^0 = \ker \pi_* \subset T^{0,\pm 2}P^0.
\]

$P^0$ is equipped with intrinsically defined soldering forms forms taking values in $g^0$. In the lowest graded degree, we have a $C$-linear one-form defined at the frame $\varphi_x \in P^0$ using the quotient projection (5.2),

\[
 \theta_{-2} = \theta_{-2,0} : CT^{-2}P^0 \to g_{-2}; \quad \theta_{-2}|_{\varphi_x} = (\varphi_x)^{-1} \circ q_{-2} \circ \pi_*.
\]

We will use the same names for the soldering forms when descending to a quotient of their domain by any subbundle of their kernel. By its definition,

\[
 \ker \theta_{-2} = CT^{-1}P^0, \quad \theta_{-2} : T^{-2}P^0 / T^{-1}P^0 \cong \mathbb{R}g_{-2}.
\]
where \( \simeq \) indicates a linear isomorphism which extends by \( C \)-linearity to the complexification. The remaining soldering forms are not true one-forms; i.e., they are not defined on all of \( \mathbb{C}TP^0 \), but only on individual filters. For instance, the quotient projection \( (6.2) \) provides

\[
\theta_{-1} : \mathbb{C}T^{-1}P^0 \to \mathfrak{g}_{-1}; \quad \theta_{-1}|_{\varphi_x} = (\varphi_x)^{-1} \circ q_{-1} \circ \pi_*.
\]

In this case,

\[
\ker \theta_{-1} = \mathbb{C}T^0P^0, \quad \theta_{-1} : T^{-1}P^0/T^0P^0 \xrightarrow{\simeq} \mathfrak{g}_{-1}.
\]

The definitions of the soldering forms, along with that of the bracket and the fact that \( \varphi_x \) is an algebra isomorphism, ensure the following bracket commutation relation, known as “regularity” in the study of parabolic geometries (cf. [3 Ch.3]),

\[
(6.7) \quad \theta_{-2}([Y_1, Y_2]) = [\theta_{-1}(Y_1), \theta_{-1}(Y_2)], \quad Y_1, Y_2 \in \Gamma(\mathbb{C}T^{-1}P^0).
\]

Incorporating the projections \( (6.3) \) splits \( \theta_{-1} \) into bigraded components,

\[
\theta_{-1, \pm 1} : \mathbb{C}T^{-1}P^0 \to \mathfrak{g}_{-1, \pm 1}, \quad \theta_{-1, \pm 1}|_{\varphi_x} = (\varphi_x)^{-1} \circ q_{-1, \pm 1} \circ \pi_*,
\]

which satisfy

\[
\ker \theta_{-1, \pm 1} = T^{-1, \mp 1}P^0, \quad \theta_{-1, \pm 1} : T^{-1, \pm 1}P^0/T^{0, \pm 2}P^0 \xrightarrow{\simeq} \mathfrak{g}_{-1, \pm 1},
\]

the latter being a \( C \)-linear isomorphism. Composing the \( \text{ad}_K, \text{ad}_R \) operators with the projections \( (6.3) \),

\[
\theta_{0, \pm 2} : \mathbb{C}T^0P^0 \to \mathfrak{g}_{0, \pm 2}, \quad \theta_{0, \pm 2}|_{\varphi_x}(v) = (\varphi_x)^{-1} \circ \text{ad}_{p_{\varphi_x}(v)} \circ \varphi_x, \quad v \in \mathbb{C}T^0P^0_0,
\]

and we have

\[
\ker \theta_{0, \pm 2} = T^{0, \mp 2}P^0, \quad \theta_{0, \pm 2} : T^{0, \pm 2}P^0/T^{0, 0}P^0 \xrightarrow{\simeq} \mathfrak{g}_{0, \pm 2}.
\]

By their definitions and \( (1.1) \), these satisfy a bigraded version of the regularity condition \( (6.7) \),

\[
(6.8) \quad \theta_{-1, \pm 1}([V, Y]) = [\theta_{0, 2}(V), \theta_{-1, \mp 1}(Y)], \quad V \in \Gamma(T^{0, \pm 2}P^0_0), \quad Y \in \Gamma(T^{-1, \mp 1}P^0_0).
\]

At this point, we have defined soldering forms taking values in each of the bigraded components of \( \mathfrak{m} \). These satisfy an important equivariance property with respect to the principal \( G_{0, 0} \) action \( (6.6) \) on the fibers of \( P^0 \), which determines a diffeomorphism of \( P^0 \).

**Lemma 6.1.** Let \( g \in G_{0, 0} \) and \( \varphi_x \in P^0_0 \). For any \( (i, j) \in I(\mathfrak{m}) \),

\[
R^*_g \theta_{i, j} = \text{Ad}_{g^{-1}} \circ \theta_{i, j}.
\]

**Proof.** We restate as follows. For \( (i, j) \in I(\mathfrak{g}_{-1}) \) and \( Y \in \Gamma(\mathbb{C}T^iP^0) \),

\[
\left( R^*_g \theta_{i, j} |_{R_g(\varphi_x)} \right)(Y|_{\varphi_x}) = g^{-1} \theta_{i, j}|_{\varphi_x}(Y|_{\varphi_x}),
\]

and for \( V \in \Gamma(\mathbb{C}T^0P^0) \),

\[
\left( R^*_g \theta_{0, 2} |_{R_g(\varphi_x)} \right)(V|_{\varphi_x}) = g^{-1} \theta_{0, 2}|_{\varphi_x}(V|_{\varphi_x}) g.
\]

First observe that \( x = \pi \circ R_g(\varphi_x) = \pi(\varphi_x) \Rightarrow (\pi \circ R_{g})* = \pi_* \). By definition of \( \theta_{-2} \) and \( R_g(\varphi_x) \),

\[
\left( R^*_g \theta_{-2} |_{R_g(\varphi_x)} \right)(Y|_{\varphi_x}) = (\varphi_x \circ g)^{-1} \circ q_{-2} \circ \pi_*(R_g)_* Y
\]

\[
= g^{-1} \circ (\varphi_x)^{-1} \circ q_{-2} \circ \pi_*(Y)
\]

\[
= g^{-1} \theta_{-2}|_{\varphi_x}(Y).
\]

The \(-1\)-graded forms may be treated similarly, and the analogous arguments for \( \theta_{0, 2} \) are also immediate from their definition. \( \square \)
Recall that the vertical bundle of a principal bundle is trivialized by fundamental vector fields which are associated to vectors in \( g_{0,0} \) by way of the principal action \([5, 6]\) and the Lie algebra exponential map \( \exp: g_{0,0} \to G_{0,0} \). Specifically, to \( v \in g_{0,0} \) we associate the vertical vector field

\[
\zeta_v|_{\varphi_x} = \frac{d}{dt} \bigg|_{t=0} R^{\exp(tv)}(\varphi_x).
\]

(6.9)

Thus we obtain another \( g_{0,0} \)-valued form \( \theta_{0,0}: T^{0,0}P^0 \to g_{0,0} \), which acts isomorphically by the linear extension of \( \theta \). Derivative of a one-form \( \alpha \) which acts isomorphically by the linear extension of \( \theta \)

which satisfy

\[
\exp(\theta_{i,j}|_V(y)) = \exp(\theta_{i,j}(y)) = [\theta_{0,0}(V), \theta_{i,j}(Y)].
\]

Proof. It suffices to consider fundamental vector fields \( V = \zeta_v \) for \( v \in g_{0,0} \). When \( (i,j) = (0,0) \), we can also assume \( Y \) is fundamental, and the result follows from the definition of (brackets of) fundamental vector fields. Therefore, we treat the case \( (i,j) \in I(m) \).

Let \( L_{\zeta_v} \) denote the Lie derivative along the vector field \( \zeta_v \) and recall Cartan’s formula for the Lie derivative of a one-form \( \alpha \in \Omega(P^0) \),

\[
(L_{\zeta_v}\alpha)(Y) = d\alpha(\zeta_v, Y) + d(\alpha(\zeta_v))(Y).
\]

\( \theta_{i,j} \) is not necessarily a true one-form, so the exterior derivative \( d\theta_{i,j} \) does not make sense in general. However, since \( \theta_{i,j} \) vanishes identically on \( \zeta_v \in \Gamma(T^{0,0}P^0) \) and takes constant value \( y \in g_{i,j} \) along the vector field \( Y \in \Gamma(T^{i,j}P^0) \), we can use the definition of the exterior derivative to interpret

\[
(L_{\zeta_v}\theta_{i,j})(Y) = d\theta_{i,j}(\zeta_v, Y) = -\theta_{i,j}([\zeta_v, Y]).
\]

On the other hand, \( R^{\exp(tv)}(\varphi_x) \) is the integral curve of \( \zeta_v \) which goes through \( \varphi_x \) when \( t = 0 \), so we have the definition of the Lie derivative given by

\[
L_{\zeta_v}\theta_{i,j} = \frac{d}{dt} \bigg|_{t=0} R^{\exp(tv)}\theta_{i,j}.
\]

Applying equivariance as in Lemma \([5, 1]\) to the right-hand-side of this, we see

\[
-\theta_{i,j}([\zeta_v, Y]) = \frac{d}{dt} \bigg|_{t=0} \text{Ad}_{\exp(-tv)} \circ \theta_{i,j}(Y)
\]

\[
= -\text{Ad}_v \circ \theta_{i,j}(Y)
\]

\[
= -[v, y].
\]

6.1.2. First Prolongation. We now begin the process of extending the soldering forms so that they are true one-forms. This cannot be done canonically, so we must incorporate into our construction the ambiguity of the choice of such extensions.

Definition 6.3. \( \tilde{\pi}^1: \tilde{P}^1 \to P^0 \) is the bundle whose fiber over \( \varphi \in P^0 \) is composed of maps

\[
\varphi_1 \in \text{Hom}(g_{-2}, \mathbb{C}T_{\varphi}^{-2}P^0/\mathbb{C}T_{\varphi}^0P^0) \oplus \bigoplus_{(i,j) \in I(g_{-1}\oplus g_0)} \text{Hom}(g_{i,j}, T_{\varphi}^{i,j}P^0),
\]

which satisfy

\[
\theta_{i,j} \circ \varphi_1|_{g_{i,j}} = 1_{g_{i,j}}, \quad (i,j) \in I(g^0),
\]

where \( 1_{g_{i,j}} \) is the identity map on \( g_{i,j} \). The subbundle \( \mathbb{R}\tilde{P}^1 \to P^0 \) is determined by those \( \varphi_1 \in \tilde{P}^1 \) with

\[
\varphi_1(y) = \tilde{\varphi}(y), \quad \forall y \in g^0.
\]
Observe that $\varphi|_{g_0,0}$ is simply the inverse of the isomorphism $\theta_{0,0}|_{\varphi}$, so this component is the same for each $\varphi_1 \in \hat{P}^1$. It therefore does not contribute to the affine structure of the fibers of $\hat{P}^1$, which is characterized by the following

**Proposition 6.4.** The fibers of $\hat{p}^1 : \hat{P}^1 \to P^0$ are affine spaces whose modelling vector space $\hat{g}_1 \subset \text{End}(g^0)$ is that of $\mathbb{C}$-linear endomorphisms which have degree-1 on $g_-$ and map

$$g_{0,\pm 2} \to g_{0,0}, \quad g_{0,0} \to 0.$$

$f \in \hat{g}_1$ can be visualized by the diagram

\[
\begin{array}{ccc}
& g_{-1,1} & \\
\phi : g_{0,2} & \rightarrow & g_{0,0} \\
& g_{1,1} & \\
\end{array}
\]

so that the solid and dashed lines have bidegrees $(1, 1)$ and $(1, -1)$, respectively. Letting $1_{g^0}$ denote the identity map on $g^0$ and $\hat{G}_1$ denote the abelian Lie group of linear isomorphisms whose Lie algebra is $\hat{g}_1$, the bundle $\hat{P}^1$ admits a right principal $\hat{G}_1$ action

$$R_{1+f}(\varphi_1) = \varphi_1 \circ (1_{g^0} + f) = \varphi_1 + \varphi_1 \circ f.$$

Fibers of the subbundle $\mathbb{R}\hat{P}^1 \to P^0$ correspond to maps $f \in \hat{g}_1$ for which $\bar{f} = f$ as defined in (2.2), whereby such $f$ comprise $\mathbb{R}\hat{g}_1$.

**Proof.** It is straightforward to see that the fiber over $\varphi \in P^0$ is nonempty: for a basis $b$ of $g_{i,j}$, there is a basis of $T^*_{\varphi}P^0$ that $\theta_{i,j}$ maps to $b$, and one can simply take $\varphi_1|_{g_{i,j}}$ to invert $\theta_{i,j}$ on $b$. Fix $\varphi_1$ and let $\hat{\varphi}_1$ be any other map in the fiber over $\varphi \in P^0$. We consider their difference as maps restricted to each bigraded component of $g^0$, beginning with $g_{-2,0}$,

$$0 = 1_{g_{-2,0}} - 1_{g_{-2,0}} = \theta_{-2,0} \circ (\hat{\varphi}_1 - \varphi_1)|_{g_{-2,0}},$$

so $(\hat{\varphi}_1 - \varphi_1)|_{g_{-2,0}}$ takes values in ker $\theta_{-2}$, i.e.,

$$(\hat{\varphi}_1 - \varphi_1)|_{g_{-2,0}} : g_{-2,0} \to \mathbb{C}T^{-1}_{\varphi}P^0/\mathbb{C}T^0_{\varphi}P^0.$$

We compose with $\theta_{-1}$ to define a linear map

$$\theta_{-1} \circ (\hat{\varphi}_1 - \varphi_1)|_{g_{-2,0}} = f^1_{-2,0} : g_{-2,0} \to g_{-1}.$$

Taking into account the splitting of $\theta_{-1}$ into $\theta_{-1, \pm 1}$, we identify bigraded components

\[
\begin{array}{ccc}
& g_{-1,1} & \\
\phi^1_{-2,0} : g_{-2,0} & \rightarrow & g_{-1,1} \\
& g_{-1,-1} & \\
\end{array}
\]

Since $\theta_{-1}$ is an isomorphism on $\mathbb{C}T^{-1}_{\varphi}P^0/\mathbb{C}T^0_{\varphi}P^0$ and $\varphi|_{g_{-1, \pm 1}}$ inverts $\theta_{-1, \pm 1}$,

$$(\hat{\varphi}_1 - \varphi_1)|_{g_{-2,0}} = \varphi_1|_{g_{-1,1}} \circ f^1_{-2,0} + \varphi_1|_{g_{-1,-1}} \circ f^1_{-2,0},$$

or more succinctly,

$$\hat{\varphi}_1|_{g_{-2,0}} = \varphi_1|_{g_{-2,0}} + \varphi_1 \circ f_{-2,0}.$$

Moving on, we aduce Definition 6.3 again to write

$$0 = \theta_{-1, \pm 1}((\hat{\varphi}_1 - \varphi_1)|_{g_{-1, \pm 1}}) \Rightarrow (\hat{\varphi}_1 - \varphi_1)|_{g_{-1, \pm 1}} : g_{-1, \pm 1} \to \mathbb{T}_{\varphi}^{0, \pm 2}P^0.$$
Composing the latter with \( \theta_{0, \pm 2} \) determines \( f_{-1, \pm 1}^1 : \mathfrak{g}_{-1, \pm 1} \to \mathfrak{g}_{0, \pm 2} \) with components

\[
\begin{array}{c}
\mathfrak{g}_{-1, 1} \xrightarrow{f_{-1, 1}^1} \mathfrak{g}_{0, 2} \\
\oplus \mathfrak{g}_{0, 0} \\
\mathfrak{g}_{-1, -1} \xrightarrow{f_{-1, -1}^1} \mathfrak{g}_{0, -2}
\end{array}
\]

By its definition, the map

\[
(\hat{\varphi}_1 - \varphi_1)|_{\mathfrak{g}_{-1, \pm 1}} = \varphi_1|_{\mathfrak{g}_{0, \pm 2}} \circ f_{-1, \pm 1}^1 : \mathfrak{g}_{-1, \pm 1} \to \mathbb{C}T^{-1}_\varphi P^0
\]

takes values in the kernel of \( \theta_{0, \pm 2} \), so evaluating the isomorphism \( \theta_{0, 0} \) on its image yields

\[
\begin{array}{c}
\mathfrak{g}_{0, 2} \\
\mathfrak{g}_{-1, 1} \xrightarrow{f_{-1, 1}^1} \mathfrak{g}_{0, 0} \\
\oplus \mathfrak{g}_{0, 0} \\
\mathfrak{g}_{-1, -1} \xrightarrow{f_{-1, -1}^1} \mathfrak{g}_{0, -2}
\end{array}
\]

Next we have that \( (\hat{\varphi}_1 - \varphi_1)|_{\mathfrak{g}_{0, \pm 2}} \) takes values in \( \ker \theta_{0, \pm 2} = T^0_\varphi P^0 \), so this difference factors through the isomorphism \( \varphi_1|_{\mathfrak{g}_{0, 0}} \) by a linear map

\[
f_{0, \pm 2}^0 : \mathfrak{g}_{0, \pm 2} \to \mathfrak{g}_{0, 0}.
\]

We have already remarked that \( \hat{\varphi}_1|_{\mathfrak{g}_{0, 0}} = \varphi_1|_{\mathfrak{g}_{0, 0}} = (\theta_{0, 0})^{-1} \), so the last component of \( f : \mathfrak{g}^0 \to \mathfrak{g}^0 \) is the trivial map \( f : \mathfrak{g}_{0, 0} \to 0 \).

Conversely, if \( f \) is any map satisfying the given hypotheses, then the image of

\[
\varphi_1 \circ f|_{\mathfrak{g}_{i, j}}
\]

lies in the kernel of \( \theta_{i, j} \) so \( \varphi_1 + \varphi_1 \circ f \) also defines an element in the fiber over \( \varphi \).

Finally, we observe that \( \varphi_1, \hat{\varphi}_1 \in \mathbb{R}\hat{P}^1 \) with \( \hat{\varphi}_1 = \varphi_1 + \varphi_1 \circ f \) implies \( f(\overline{y}) = \overline{f(y)} \), so \( \overline{f} = f \) as defined in (2.14) and \( f \in \mathbb{R}\hat{g}_1 \).

The bundle \( \hat{\pi}^1 : \hat{P}^1 \to P^0 \) inherits a filtration as before,

\[
CT^i \hat{P}^1 = (\hat{\pi}^1)^{-1}(CT^i P^0), \quad T^{i,j} \hat{P}^1 = (\hat{\pi}^1)^{-1}(T^{i,j} P^0), \quad (i, j) \in I(\mathfrak{g}^0),
\]

where each subbundle contains the vertical bundle

\[
CT^1 \hat{P}^1 = \ker \hat{\pi}^1.
\]

\( \hat{P}^1 \) also admits tautologically defined, \( \mathfrak{g}^0 \)-valued soldering forms which may be thought of as extending those on \( P^0 \). In particular, the forms of a given degree have “graduated” in terms of the filters on which they are defined.

**Proposition 6.5.** There exist intrinsically defined forms

\[
\hat{\theta}_{-2}^1 : CT^{-2} \hat{P}^1 \to \mathfrak{g}_{-2}, \quad \hat{\theta}_{-1}^1 : CT^{-2} \hat{P}^1 \to \mathfrak{g}_{-1}, \quad \hat{\theta}_0^1 : CT^{-1} \hat{P}^1 \to \mathfrak{g}_0,
\]

with \( \mathfrak{g}_{i, j} \)-valued bigraded components \( \hat{\theta}_{i, j}^1 \) which restrict to give the pullback of the soldering forms on \( P^0 \),

\[
\hat{\theta}_{i, j}^1|_{T^{0, 0} \mathfrak{p}^1} = (\hat{\pi}^1)^* \theta_{0, 0} \quad \hat{\theta}_{i, j}^1|_{CT^i \mathfrak{p}^1} = (\hat{\pi}^1)^* \theta_{i, j} \quad (i, j) \in I(\mathfrak{m}).
\]
Furthermore, if $\varphi_1, \varphi_2 \in \hat{P}^1$ are two elements in the fiber over $\varphi \in P^0$ which are related by $\hat{\varphi}_1 = R_{1+f}(\varphi_1)$ for $f \in \hat{g}_1$ as in Proposition 6.4, then

$$(R_{1+f})^*\hat{\theta}_{-1}|_{\varphi_1} = (\hat{\theta}_{-1} - f \circ \hat{\theta}_{-2})|_{\varphi_1},$$

$$(R_{1+f})^*\hat{\theta}_{-1}|_{\varphi_1} = (\hat{\theta}_{-1} - f \circ \hat{\theta}_{-2})|_{\varphi_1},$$

$$(R_{1+f})^*\hat{\theta}_{1,0}|_{\varphi_1} = (\hat{\theta}_{1,0} - (f - f|_{g_0} \circ f) \circ \hat{\theta}_{1} - f \circ \hat{\theta}_{1,0})|_{\varphi_1},$$

so that the zero-graded form transforms according to

$$(R_{1+f})^*\hat{\theta}_{0}|_{\varphi_1} = (\hat{\theta}_{0} - (f - f \circ f) \circ \hat{\theta}_{-1} - f \circ \hat{\theta}_{0}|_{\varphi_1}.$$
Next we observe
\[ \hat{\pi}_1^1(Y) - \varphi_1 \circ \theta_{-1} \circ \hat{\pi}_1^1(Y) - \varphi_1 \circ \hat{\theta}_0^1|_{\varphi_1}(Y) - \varphi_1 \circ \hat{\theta}_0^1|_{\varphi_1}(Y) \in \ker \theta_{0,2} \cap \ker \theta_{0,-2} = T_{\varphi}^{0,0} P^0, \]
and define
\[ \hat{\theta}_{0,0}^1|_{\varphi_1}(Y) = \theta_{0,0}(\hat{\pi}_1^1(Y) - \varphi_1 \circ \theta_{-1} \circ \hat{\pi}_1^1(Y) - \varphi_1 \circ \hat{\theta}_0^1|_{\varphi_1}(Y) - \varphi_1 \circ \hat{\theta}_0^1|_{\varphi_1}(Y)). \]
When \( Y \in T_{\varphi}^{0,0} \hat{P}^1 \), each term vanishes except \( \theta_{0,0}(\hat{\pi}_1^1(Y)) \) as claimed. Showing the transformation property for \( \hat{\theta}_{0,0}^1 \) requires that of the other zero-graded components,
\[
(R_{1+f})^* \hat{\theta}_{0,0}^1|_{\varphi_1}(Y) = \theta_{0,0} \left( \hat{\pi}_1^1(Y) - \varphi_1 \circ \theta_{-1} \circ \hat{\pi}_1^1(Y) - \sum_{j=\pm 2} \varphi_1 \circ (R_{1+f})^* \hat{\theta}_{0,j}^1|_{\varphi_1}(Y) \right)
\]
\[
= \theta_{0,0} \left( \hat{\pi}_1^1(Y) - (\varphi_1|_0 + \varphi_1|_0 \circ f) \circ \theta_{-1} \circ \hat{\pi}_1^1(Y) \right)
\]
\[
- \sum_{j=\pm 2} (\varphi_1|_{0,0} + \varphi_1|_{0,0} \circ f|_{0,0}) (\hat{\theta}_{0,j}^1|_{\varphi_1}(Y) - \hat{\theta}_{0,j}^1|_{\varphi_1}(Y))
\]
\[
= \hat{\theta}_{0,0}^1|_{\varphi_1}(Y) - \theta_{0,0} \left( \hat{\pi}_1^1(Y) - \varphi_1 \circ \theta_{-1} \circ \hat{\pi}_1^1(Y) - \sum_{j=\pm 2} \varphi_1 \circ \hat{\theta}_{0,j}^1|_{\varphi_1}(Y) \right)
\]
\[
- \sum_{j=\pm 2} f|_{0,0} (\hat{\theta}_{0,j}^1 - \hat{\theta}_{0,j}^1|_{\varphi_1}(Y))
\]
\[
= \hat{\theta}_{0,0}^1|_{\varphi_1}(Y) - \theta_{0,0} \left( \hat{\pi}_1^1(Y) - \varphi_1 \circ \hat{\theta}_{-1}^1|_{\varphi_1}(Y) - \sum_{j=\pm 2} f|_{0,0} (\hat{\theta}_{0,j}^1 - \hat{\theta}_{0,j}^1|_{\varphi_1}(Y)) \right),
\]
so \( \hat{\theta}_{0,0}^1 \) transforms as indicated. Putting all of the zero-graded forms together, we obtain
\[
\hat{\theta}_{0}^1 = \sum_{j=0,\pm 2} \hat{\theta}_{0,j}^1 : CT^{-1} \hat{P}^1 \to g_0
\]
whose kernel when restricted to \( CT^0 \hat{P}^1 \) is the vertical bundle \( CT^1 \hat{P}^1 \).

To these soldering forms we now add one which acts isomorphically on the vertical bundle \( CT^1 \hat{P}^1 \). This form is defined via the principal \( \hat{G}_1 \) action on \( \hat{P}^1 \) in exactly the same manner that \( \theta_{0,0} \) was defined on \( P^0 \). Namely, \( CT^1 \hat{P}^1 \) is trivialized by fundamental vector fields \( \zeta_f \) associated to each \( f \in \mathfrak{g}_1 \) and defined point-wise at \( \varphi_1 \in \hat{P}^1 \) by
\[
\zeta_f|_{\varphi_1} = \frac{d}{dt} \bigg|_{t=0} R_{1+tf}(\varphi_1).
\]
Thus, we define
\[
\hat{\theta}_1^1 : CT^1 \hat{P}^1 \to \mathfrak{g}_1,
\]
by the linear extension of
\[
\hat{\theta}_1^1(\zeta_f) = f.
\]
The transformation properties of the \( g^0 \)-valued soldering forms described in Proposition 6.5 now provide the following.

**Corollary 6.6.** Let \( V \in \Gamma(CT^1 \hat{P}^1) \) and \( i = -2, -1 \). For \( Y \in \Gamma(CT^i \hat{P}^1) \) such that \( \hat{\theta}_1^1(Y) = y \in \mathfrak{g}_1 \) and \( \hat{\theta}_0^1(Y) = 0 \) when \( i = -1 \),
\[
\hat{\theta}_1^1([V,Y]) = 0, \quad \hat{\theta}_1^{i+1}([V,Y]) = \hat{\theta}_1^1(V)(y),
\]
where the latter indicates the action of \( \hat{\theta}_1^1(V) \in \mathfrak{g}_1 \subset \text{Hom}(g^0, g^0) \) on \( y \in g_0 \). Furthermore, for \( Y \in \Gamma(CT^0 \hat{P}^1) \) such that \( \hat{\theta}_0^1(Y) = y \in g_0 \),
\[
\hat{\theta}_{0,\pm 2}^1([V,Y]) = 0, \quad \hat{\theta}_{0,0}^1([V,Y]) = \hat{\theta}_1^1(V)(y).\]
Proof. The proof is analogous to that of Lemma 6.1 so we will leave out some of the details this time. Once again, it suffices to consider fundamental vector fields \( V = \zeta_f \) for \( f \in \mathfrak{g}_1 \). The definition of the Lie derivative is

\[
L_{\zeta_f} \hat{\theta}^i = \left. \frac{d}{dt} \right|_{t=0} R^*_1 + tf \hat{\theta}^i,
\]

and Cartan’s formula shows

\[
(L_{\zeta_f} \hat{\theta}^i)(Y) = -\hat{\theta}^i([\zeta_f, Y]).
\]

By Proposition 6.3 when \( i = -2 \),

\[
-\hat{\theta}^i_{-2}([\zeta_f, Y]) = \left. \frac{d}{dt} \right|_{t=0} R^*_1 + tf \hat{\theta}^i_{-2}(Y) = \left. \frac{d}{dt} \right|_{t=0} \hat{\theta}^i_{-2}(Y) = \left. \frac{d}{dt} \right|_{t=0} y = 0,
\]

while

\[
-\hat{\theta}^i_{-1}([\zeta_f, Y]) = \left. \frac{d}{dt} \right|_{t=0} R^*_1 + tf \hat{\theta}^i_{-1}(Y) = \left. \frac{d}{dt} \right|_{t=0} (\hat{\theta}^i_{-1} - tf \circ \hat{\theta}^i_{-2})(Y) = -f(y).
\]

Similarly, when \( i = -1 \) and \( Y \in \Gamma(\mathbb{C}T^{-1}P^1) \) as prescribed,

\[
-\hat{\theta}^i_{-1}([\zeta_f, Y]) = \left. \frac{d}{dt} \right|_{t=0} (\hat{\theta}^i_{-1} - tf \circ \hat{\theta}^i_{-2})(Y) = -f \circ \hat{\theta}^i_{-2}(Y) = 0,
\]

while

\[
-\hat{\theta}^i_{0}([\zeta_f, Y]) = \left. \frac{d}{dt} \right|_{t=0} (\hat{\theta}^i_{0} - (tf - t^2f \circ f) \circ \hat{\theta}^i_{-1} - tf \circ \hat{\theta}^i_{0})(Y) = -f(y).
\]

The latter equation also proves the result for \( Y \in \Gamma(\mathbb{C}T^0\hat{P}^1) \), as \( f \circ \hat{\theta}^i_{0} \) only takes values in \( \mathfrak{g}_{0,0} \). \( \square \)

The regularity conditions (6.7), (6.8), and Lemma 6.2 dictate how the soldering forms on \( P^0 \) relate the bracket of vector fields in \( \Gamma(\mathbb{C}T^0P^1) \) to the Lie algebra bracket on \( \mathfrak{g}^0 \). The soldering forms on \( \hat{P}^1 \) maintain this property when restricted to the subbundles of \( \mathbb{C}T\hat{P}^1 \) where they are the \( \hat{\pi}^1 \)-pullbacks of the soldering forms on \( P^0 \), but on their newly extended domains their interaction with the bracket of vector fields is measured by a torsion tensor associated to each \( \varphi_1 \in \hat{P}^1 \).

**Lemma 6.7.** For \( \varphi_1 \in \hat{P}^1 \), define the torsion tensor \( \tau_1 \in \text{Hom}(\mathfrak{g}^0 \wedge \mathfrak{g}_{1}, \mathfrak{g}_{-}) \) as follows. Let \( i_2 = -1, -2 \leq i_1 \leq 0 \), and \( y_\ell \in \mathfrak{g}_{i_\ell} \) for \( \ell = 1, 2 \). Take local vector fields \( Y_\ell \in \Gamma(\mathbb{C}T^\ell \hat{P}^1) \) such that

\[
\hat{\theta}^i_{\ell}(Y_\ell) = y_\ell, \quad \hat{\theta}^i_{\ell+1}(Y_\ell) = 0 \text{ (if } i_\ell < 0),
\]

and set

\[
\tau_1(y_1, y_2) = \hat{\theta}^i_{m|\varphi_1}([Y_1, Y_2]), \quad m = \min\{i_1, -1\}.
\]

Similarly, define \( \tau_\pm \in \text{Hom}(\mathfrak{g}_{0,\pm 2} \wedge \mathfrak{g}_{-1,\mp 1}, \mathfrak{g}_{0,\pm 2}) \) for \( y_1 \in \mathfrak{g}_{0,\pm 2} \) and \( y_2 \in \mathfrak{g}_{-1,\mp 1} \) by taking \( Y_1, Y_2 \) as above and setting

\[
\tau_\pm(v, y) = \hat{\theta}^i_{0,\pm 2|\varphi_1}([Y_1, Y_2]).
\]

Then \( \tau_1 \) and \( \tau_\pm \) are well-defined.

**Proof.** We must confirm that the definition is independent of the choices of vector fields, so suppose \( Y_\ell \in \Gamma(\mathbb{C}T^\ell \hat{P}^1) \) is an alternative choice for each \( i_\ell \). If \( i_\ell < 0 \),

\[
0 = y_\ell - y_\ell = \hat{\theta}^i_{i_\ell}(Y_\ell - Y_\ell), \quad 0 = \hat{\theta}^i_{i_\ell+1}(Y_\ell - Y_\ell),
\]

respectively.
which together imply
\[(6.10) \quad \tilde{Y}_\ell = Y_\ell + Z_\ell, \quad Z_\ell \in \Gamma(CT^{i+2}\hat{P}^1).\]

Similarly, when \(i_\ell = 0\),
\[(6.11) \quad \tilde{Y}_\ell = Y_\ell + Z_\ell, \quad Z_\ell \in \Gamma(CT^{1}\hat{P}^1).\]

With these vector fields, the first component of torsion is given by
\[(6.12) \quad \tau_1(y_1, y_2) = \hat{\theta}_m^1|_{\varphi_1}([\tilde{Y}_1, \tilde{Y}_2]) = \hat{\theta}_m^1([Y_1, Y_2] + [Z_1, Y_2] + [Y_1, Z_2] + [Z_1, Z_2]).\]

By Corollary 6.6, \([Z_1, Y_2] + [Y_1, Z_2] + [Z_1, Z_2] \in \ker \hat{\theta}_m^1\), so \(\tau_1\) is well-defined. Using \(\tilde{Y}_1 \in \Gamma(T^{0,\pm 2}\hat{P}^1)\) and \(\tilde{Y}_2 \in \Gamma(T^{-1,\mp 1}\hat{P}^1)\) to define \(\tau_\pm\) will also result in an expression of the form \((6.12)\), except in this case \(m = (0, \pm 2)\). The result will follow once again by Corollary 6.6 noting that \(y_2 \in \mathfrak{g}_{-1,\mp 1}\) implies \(\hat{\theta}_m^1(Z_1)(y_2) \in \mathfrak{g}_{0,0} \oplus \mathfrak{g}_{0,\mp 2}\), which implies \([Z_1, Y_2] \in \ker \hat{\theta}_m^{0,\pm 2}\). The rest is immediate from the Corollary. \(\square\)

**Proposition 6.8.** Let \(\varphi_1, \tilde{\varphi}_1\) in the same fiber of \(\hat{P}^1 \to P^0\) be related by \(\tilde{\varphi}_1 = R_{1+f}(\varphi_1)\) for \(f \in \hat{g}_1\) according to Proposition 6.4. The torsion tensor \(\tilde{\tau}_1\) associated to \(\tilde{\varphi}_1\) is related to \(\tau_1\) of \(\varphi_1\) by
\[(\tilde{\tau}_1)(y_1, y_2) = \tau_1(y_1, y_2) + [f(y_1), y_2] + [y_1, f(y_2)] - f([y_1, y_2]), \quad y_1 \in \mathfrak{g}_{-1}, y_2 \in \mathfrak{g}_0^{0},\]
and
\[(\tilde{\tau}_\pm)(y_1, y_2) = \tau_{\pm}(y_1, y_2) + [y_1, f_{\pm 1,\pm 1}(y_2)] - f_{\pm 1,\mp 1}([y_1, y_2]), \quad y_1 \in \mathfrak{g}_{0,\pm 2}, y_2 \in \mathfrak{g}_{-1,\mp 1}.\]

**Proof.** Take local vector fields \(Y_\ell \in \Gamma(CT^{i}\hat{P}^1)\) in a neighborhood of \(\varphi_1\) satisfying
\[\hat{\theta}_m^i(Y_\ell) = y_\ell, \quad \hat{\theta}_m^{i+1}(Y_\ell) = 0 \quad (if \ i_\ell < 0),\]
so that
\[\tau_1(y_1, y_2) = \hat{\theta}_m^1|_{\varphi_1}([Y_1, Y_2]), \quad m = \min\{i_1, -1\}.\]

We will produce local vector fields \(\tilde{Y}_\ell \in \Gamma(CT^{i+1}\hat{P}^1)\) in a neighborhood of \(\tilde{\varphi}_1\) with which to express \(\tilde{\tau}_1(y_1, y_2)\). To declutter notation, we abbreviate the diffeomorphism \(R_{1+f}\) as \(R\). For \(i_\ell < 0\), take \(Z_\ell \in \Gamma(CT^{i+1}\hat{P}^1)\) in a neighborhood of \(\varphi_1\) such that
\[\hat{\theta}_m^{i+1}(Z_\ell) = f(y_\ell) \in \mathfrak{g}_{i+1},\]
and set
\[\tilde{Y}_\ell = R_*Y_\ell + R_*Z_\ell\]
in a neighborhood of \(\tilde{\varphi}_1\). Note that when \(i_1 = -2\),
\[\hat{\theta}_m^{i+2}(\tilde{Y}_1) = R^*\hat{\theta}_m^{i+2}(Y_1 + Z_1) = \hat{\theta}_m^{i+2}(Y_1) = y_1\]
while
\[\hat{\theta}_m^{i+1}(\tilde{Y}_1) = R^*\hat{\theta}_m^{i+1}(Y_1 + Z_1) = (\hat{\theta}_m^1 - f \circ \hat{\theta}_m^{i+2})(Y_1 + Z_1) = f(y_1) - f(y_1) = 0,\]
and when \(i_\ell = -1\),
\[
\hat{\theta}^1_{-1}(Y_\ell) = R^*\hat{\theta}^1_{-1}(Y_\ell + Z_\ell) \\
= (\hat{\theta}^1_{-1} - f \circ \hat{\theta}^1_{-2})(Y_\ell) \\
= y_\ell,
\]
while
\[
\hat{\theta}^1_0(Y_\ell) = R^*\hat{\theta}^1_0(Y_\ell + Z_\ell) \\
= (\hat{\theta}^1_0 - (f - f \circ f) \circ \hat{\theta}^1_{-1} - f \circ \hat{\theta}^1_0)(Y_\ell + Z_\ell) \\
= f(y_\ell) - (f - f \circ f)(y_\ell) - f f(y_\ell) \\
= 0.
\]
Thus we see that when \(i_1 < 0\),
\[
\tilde{\tau}_1(y_1, y_2) = \hat{\theta}^1_{i_1, \varphi_1}([Y_1, Y_2]).
\]
Suppose \(i_1 = -2\),
\[
\tilde{\tau}_1(y_1, y_2) = \hat{\theta}^1_{-2, \varphi_1}([Y_1, Y_2]) \\
= R^*\hat{\theta}^1_{-2, \varphi_1}([Y_1 + Z_1, Y_2 + Z_2]) \\
= \hat{\theta}^1_{-2, \varphi_1}([Y_1, Y_2] + [Z_1, Y_2] + [Y_1, Z_2] + [Z_1, Z_2]) \\
= \tau_1(y_1, y_2) + \hat{\theta}^1_{-2, \varphi_1}([Z_1, Y_2] + [Y_1, Z_2] + [Z_1, Z_2]).
\]
Because \([Z_1, Z_2] \in \Gamma(CT^{-1}\hat{P}^1)\), \(\hat{\theta}^1_{-2}\) vanishes on this term. Here and in the sequel, we will use \(\tilde{=}\) to indicate any equality that follows from one of the regularity conditions expressed in [6.7], [6.8], or Lemma [6.2]. For example,
\[
[\hat{\theta}^1_{-2}(Y_1), \hat{\theta}^1_0(Z_2)] = [\hat{\theta}^1_{-2}(Y_1), \hat{\theta}^1_{0,0}(Z_2)]
\]
since \([g_{-2,0}, g_{0,\pm 2}] = 0\), so to continue our calculation we write
\[
\tilde{\tau}_1(y_1, y_2) = \tau_1(y_1, y_2) + \hat{\theta}^1_{-1, \varphi_1}([Z_1, Y_2] + [Y_1, Z_2]) \\
\tilde{=\;} \tau_1(y_1, y_2) + [\hat{\theta}^1_{-1}(Z_1), \hat{\theta}^1_{-1}(Y_2)] + [\hat{\theta}^1_{-2}(Y_1), \hat{\theta}^1_{0,0}(Z_2)] \\
= \tau_1(y_1, y_2) + [f(y_1), y_2] + [y_1, f(y_2)],
\]
and the proposition follows in this case from the fact that \([g_{-2}, g_{-1}] = 0\).

Next we consider \(i_1 = i_2 = -1\), and again we have
\[
\tilde{\tau}_1(y_1, y_2) = \hat{\theta}^1_{-1, \varphi_1}([Y_1, Y_2]) \\
= R^*\hat{\theta}^1_{-1, \varphi_1}([Y_1 + Z_1, Y_2 + Z_2]) \\
= (\hat{\theta}^1_{-1} - f \circ \hat{\theta}^1_{-2})|_{\varphi_1}([Y_1, Y_2] + [Z_1, Y_2] + [Y_1, Z_2] + [Z_1, Z_2]).
\]
Among these terms,
\[
[Z_1, Y_2] + [Y_1, Z_2] \in \Gamma(CT^{-1}\hat{P}^1) = \ker \hat{\theta}^1_{-2}, \quad [Z_1, Z_2] \in \Gamma(CT^0\hat{P}^1) = \ker \hat{\theta}^1_{-1} \subset \ker \hat{\theta}^1_{-2},
\]
and we proceed,
\[
\tilde{\tau}_1(y_1, y_2) = \tau_1(y_1, y_2) + \hat{\theta}^1_{-1}([Z_1, Y_2] + [Y_1, Z_2]) - f \circ \hat{\theta}^1_{-2}([Y_1, Y_2]) \\
\tilde{=\;} \tau_1(y_1, y_2) + \hat{\theta}^1_{-1}(Z_1) \hat{\theta}^1_{-1}(Y_2) + [\hat{\theta}^1_{-1}(Y_1), \hat{\theta}^1_0(Z_2)] - f([\hat{\theta}^1_{-1}(Y_1), \hat{\theta}^1_{-1}(Y_2)]) \\
= \tau_1(y_1, y_2) + [f(y_1), y_2] + [y_1, f(y_2)] - f([y_1, y_2]).
\]
The final component of \(\tau_1\) concerns \(i_1 = 0, i_2 = -1\). To treat this case we take \(Z_1 \in \Gamma(T^{0,0}\hat{P}^1)\) in a neighborhood \(\varphi_1\) so that
\[
\hat{\theta}^1_{0,0}(Z_1) = f(y_1),
\]
and define in a neighborhood of \( \tilde{\varphi} \)
\[
\tilde{Y}_1 = R_s Y_1 + R_s Z_1,
\]
which shows
\[
\theta^i_0|_{\tilde{\varphi}_1}(\tilde{Y}_1) = R^s \hat{\theta}^i_0|_{\tilde{\varphi}_1}(Y_1 + Z_1)
\]
\[
= (\hat{\theta}^i_0 - f \circ \hat{\theta}^i_{-1} - f \circ \hat{\theta}^i_0)|_{\tilde{\varphi}_1}(Y_1 + Z_1)
\]
\[
= y_1 + f(y_1) - f(y_1)
\]

since \( f|_{\mathfrak{g}_{0,0}} = 0 \) implies
\[
f \circ \hat{\theta}^i_0 = f \circ \hat{\theta}^i_{0,2} + f \circ \hat{\theta}^i_{0,-2}.
\]
Thus we see that
\[
\bar{\theta}_1(y_1, y_2) = \hat{\theta}^i_{-1}|_{\tilde{\varphi}_1}([\tilde{Y}_1, \tilde{Y}_2]),
\]
and we compute
\[
\hat{\theta}^i_{-1}|_{\tilde{\varphi}_1}([\tilde{Y}_1, \tilde{Y}_2]) = R^s \hat{\theta}^i_{-1}|_{\tilde{\varphi}_1}((Y_1 + Z_1, Y_2 + Z_2))
\]
\[
= (\hat{\theta}^i_{-1} - f \circ \hat{\theta}^i_{-2})|_{\tilde{\varphi}_1}((Y_1, Y_2) + [Z_1, Y_2] + [Y_1, Z_2] + [Z_1, Z_2])
\]
\[
= \hat{\theta}^i_{-1}|_{\tilde{\varphi}_1}((Y_1, Y_2) + [Z_1, Y_2])
\]
\[
= \hat{\theta}^i_{0}|_{\tilde{\varphi}}(\tilde{Y}_1, Y_2) + [\hat{\theta}^i_0(Z_1), \hat{\theta}^i_{-1}(Y_2)]
\]
\[
= \hat{\theta}^i_{-1}|_{\tilde{\varphi}_1}((Y_1, Y_2) + f(y_1), y_2).
\]

The claim for \( \tau_\pm \) is proved with similar arguments.

Proposition 3.8 may be more succinctly stated \( \bar{\tau} = \tau + \partial_1 f \) where the Lie algebra cohomology differential \( \partial_1 : \text{Hom}(\mathfrak{g}^0, \mathfrak{g}^0) \to \text{Hom}(\mathfrak{g}^0 \wedge \mathfrak{g}^0, \mathfrak{g}^0) \) is given by
\[
\partial_1 f(y_1, y_2) = [f(y_1), y_2] + [y_1, f(y_2)] - f([y_1, y_2]), \quad y_1, y_2 \in \mathfrak{g}^0.
\]

The normalization condition discussed in Remark 3.2 is best expressed in relation to this differential.

**Definition 6.9.** Fix \( \mathcal{N}_1 \subset \mathbb{R}(\text{Hom}(\Lambda^2 \mathfrak{g}^0, \mathfrak{g}^0)) \) such that \( \mathbb{C}\mathcal{N}_1 = \mathcal{N}_1 \otimes_{\mathbb{R}} \mathbb{C} \) a subspace complement to the image of \( \tilde{\theta}_1 \) under \( \partial_1 \),
\[
\text{Hom}(\Lambda^2 \mathfrak{g}^0, \mathfrak{g}^0) = \partial_1(\mathfrak{g}_1) \oplus \mathcal{N}_1.
\]

We call \( \mathcal{N}_1 \) the *first normalization condition*. Define \( \mathbb{R}P^1 \subset \mathbb{R}\tilde{P}^1 \) to be the subbundle of those \( \varphi_1 \) whose torsion tensor \( \tau_1 + \tau_\pm \) is *normalized*: i.e., \( \tau_1, \tau_\pm \in \mathcal{N}_1 \). By extension, \( P^1 \subset \tilde{P}^1 \) is the subbundle of frames whose torsion tensor lies in \( \mathcal{N}_1 \). Let \( \hat{\theta}_1 = \hat{\theta}_1|_{\tilde{P}^1} \).

By Proposition 3.8 and Definition 3.9, the fibers of \( \pi^1 : P^1 \to P^0 \) are isomorphic to \( \mathfrak{g}_1 = \mathfrak{g}_{1,-1} \oplus \mathfrak{g}_{1,1} \) as in 3.9, and we partially extend the bracket of \( \mathfrak{g}^0 \) to \( \mathfrak{g}^1 = \mathfrak{g}^0 \oplus \mathfrak{g}_1 \) according to 3.4. Letting \( \iota_1 : P^1 \hookrightarrow \tilde{P}^1 \) denote the inclusion map, we pull back the \( \mathfrak{g}_i \)-valued soldering forms to \( P^1 \),
\[
\theta_i^1 = \iota^* \hat{\theta}_i^1, \quad -2 \leq i \leq 1.
\]

### 6.1.3 Higher geometric prolongations.

As mentioned in the sentence before formula (3.7), for \( \kappa \geq 2 \) the space \( \mathfrak{g}_\kappa = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_{\kappa,j} \) of all elements in \( \mathfrak{U}_{\text{bigrad}}(\mathfrak{g}^0) \) of first weight \( \kappa \) is exactly the same as the degree-\( \kappa \) component of the standard Tanaka algebraic prolongation of \( \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \). With the bundle \( P^1 \) having been constructed in the previous subsection, the construction of remaining bundles \( P^\kappa \) in the chain (6.1) is also the same as in the standard Tanaka theory, and we will not repeat it, instead referring the reader to the Tanaka’s original paper [25] or to the second author’s [29]. For completeness, we briefly describe the nature of the bundles \( P^\kappa \) and indicate how their construction ultimately produces an absolute parallelism.

We proceed by induction. Fix \( \kappa \geq 2 \) and suppose we have a bundle \( \pi^{\kappa-1} : P^{\kappa-1} \to P^{\kappa-2} \) with graded and bigraded filters
\[
\mathbb{C}T^i P^{\kappa-1}, \quad T^i j P^{\kappa-1}, \quad (i, j) \in J(\mathfrak{g}^{\kappa-1}),
\]
and in particular
\[ \mathcal{C}T^{\kappa-1} P^{\kappa-1} = \ker \pi^\kappa. \]
Suppose further that there are soldering forms
\[
\theta_i^{-1} : \mathcal{C}T^{-2} P^{\kappa-1} \rightarrow \mathfrak{g}_i, \quad \theta_{i-1}^{-1} : \mathcal{C}T^{-1} P^{\kappa-1} \rightarrow \mathfrak{g}_k, \quad \theta_{i-1}^{-1} : \mathcal{C}T^{\kappa-1} P^{\kappa-1} \rightarrow \mathfrak{g}_{i-1},
\]
such that \( \theta_{i-1}^{-1} \) is an isomorphism and the others restrict and descend to isomorphisms
\[
(-2 \leq i \leq \kappa - 2)
\]
\[
\theta_i^{-1} \mid_{\mathcal{C}T^i P^{\kappa-1}} : \mathcal{C}T^i P^{\kappa-1} / \mathcal{C}T^{i+1} P^{\kappa-1} \rightarrow \mathfrak{g}_i.
\]
Define the bundle \( \hat{\pi}^\kappa : \hat{P}^\kappa \rightarrow P^{\kappa-1} \) whose fiber over \( \varphi \in P^{\kappa-1} \) is composed of maps
\[
\varphi^\kappa \in \mathrm{Hom}(\mathfrak{g}_{-2}, \mathcal{C}T_{\varphi}^{-2} P^{\kappa-1} / \mathcal{C}T_{\varphi}^{\kappa-1} P^0) \oplus \bigoplus_{(i,j) \in \mathcal{I}(\mathfrak{g}_{-1} \oplus \cdots \oplus \mathfrak{g}_{i-1})} \mathrm{Hom}(\mathfrak{g}_{i,j}, T_{\varphi^i,j} P^{\kappa-1}),
\]
which satisfy
\[
\begin{align*}
\theta_i^{-1} \circ \varphi^\kappa | \mathfrak{g}_i & = \mathbb{1}_\mathfrak{g}_i; \\
\theta_i^{-1} \circ \varphi^\kappa | \mathfrak{g}_i & = 0, \quad \mathfrak{g}_i \subset \mathfrak{g}^{\kappa-1}, \\
(-2 \leq i \leq \kappa - 3) \
\end{align*}
\]
where \( \mathbb{1}_\mathfrak{g}_i \) is the identity map on \( \mathfrak{g}_i \). The subbundle \( \mathbb{R} \hat{P}^\kappa \subset \hat{P}^\kappa \) is that of frames \( \varphi^\kappa \) which additionally satisfy \( \varphi^\kappa(y) = \varphi^\kappa(y) \) for \( y \in \mathfrak{g}^{\kappa-1} \).

The fibers of \( P^\kappa \) are affine spaces whose modeling vector space \( \mathfrak{g}_\kappa \subset \mathrm{Hom}(\mathfrak{g}^{\kappa-1}, \mathfrak{g}^{\kappa-1}) \) is that of linear maps \( f \) given by sums of bigraded maps
\[
f_{i,j}^{m,j_2} : \mathfrak{g}_{i,j} \rightarrow \mathfrak{g}_{i+m,j_1+j_2}, \quad (i,j_1,j_2, (i_1,m,j_1+j_2) \in I(\mathfrak{g}^{\kappa-1}), \quad m = \min\{\kappa, \kappa - 1 - i_1\},
\]
which vanish on \( \mathfrak{g}_{-1} \). The fibers of \( \mathbb{R} P^\kappa \) are modeled by those \( f \in \mathbb{R} \mathfrak{g}_\kappa \) with \( \mathcal{I} = f \) as defined by (2.9).

The bundle \( \hat{\pi}^\kappa : \hat{P}^\kappa \rightarrow P^{\kappa-1} \) inherits a filtration as before,
\[
\mathcal{C}T^i \hat{P}^\kappa = (\hat{\pi}^\kappa)^{-1}(\mathcal{C}T^i P^{\kappa-1}),
\]
\[
T^{i,j} \hat{P}^\kappa = (\hat{\pi}^\kappa)^{-1}(T^{i,j} P^{\kappa-1}), \quad (i,j) \in I(\mathfrak{g}^{\kappa-1}),
\]
where each subbundle contains the vertical bundle
\[
\mathcal{C}T^\kappa \hat{P}^\kappa = \ker \hat{\pi}^\kappa.
\]
Minimicking the proof of Proposition 6.5 and the constructions of \( \theta_0^0 \) and \( \hat{\theta}_1^1 \), we can show that there exist intrinsically defined forms
\[
(6.13)
\]
\[
\hat{\pi}^\kappa : \mathcal{C}T^{-2} \hat{P}^\kappa \rightarrow \mathfrak{g}_i, \quad (2 \leq i \leq \kappa - 2);
\]
\[
\hat{\pi}^\kappa_{i-1} : \mathcal{C}T^{-1} \hat{P}^\kappa \rightarrow \mathfrak{g}_{i-1}, \quad \hat{\theta}_i^\kappa : \mathcal{C}T^{\kappa} \hat{P}^\kappa \rightarrow \mathfrak{g}_{i},
\]
the first \( \kappa + 2 \) of which restrict to give the pullbacks of the soldering forms on \( P^{\kappa-1} \) while the last is an isomorphism.

One can define the torsion tensor \( \tau^\kappa \in \mathrm{Hom}(\mathfrak{g}^{\kappa-2} \wedge \mathfrak{g}_{-1}, \mathfrak{g}^{\kappa-2}) \) associated to \( \varphi^\kappa \in \hat{P}^\kappa \) by analogy with \( \tau_i \), \( i < \kappa \). Note that no analog of \( \tau_1 \) from Lemma 6.7 appears in higher prolongations. A normalization condition for the \( \kappa \text{th geometric prolongation} \) is choice of a subspace \( \mathcal{N}_\kappa \subset \mathcal{R}(\mathrm{Hom}(\Lambda^2 \mathfrak{g}^{\kappa-1}, \mathfrak{g}^{\kappa-1})) \) whose complexification is complementary to the image of \( \mathfrak{g}^\kappa \) under
\[
(\text{as defined in [23]})
\]
\[
\partial^\kappa : \mathrm{Hom}(\mathfrak{g}^{\kappa-1}, \mathfrak{g}^{\kappa-1}) \rightarrow \mathrm{Hom}(\mathfrak{g}^{\kappa-1} \wedge \mathfrak{g}_{-1}, \mathfrak{g}^{\kappa-1}),
\]
so that
\[
\mathrm{Hom}(\Lambda^2 \mathfrak{g}^{\kappa-1}, \mathfrak{g}^{\kappa-1}) = \partial_\kappa(\mathfrak{g}_\kappa) \oplus \mathcal{N}_\kappa.
\]

Once \( \mathcal{N}_\kappa \) is chosen, \( \mathbb{R} P^1 \subset \mathbb{R} P^1 \) is the subbundle of those \( \varphi^\kappa \) whose torsion tensors are normalized; i.e., \( \tau^\kappa \in \mathcal{N}_\kappa \), while \( P^\kappa \subset \hat{P}^\kappa \) is the subbundle whose torsion tensors lie in \( \mathcal{C}N_\kappa \). Set \( \pi^\kappa = \hat{\pi}^\kappa |_{P^\kappa} \), pull back \( \hat{\theta}^\kappa \) along the inclusion \( P^\kappa \hookrightarrow \hat{P}^\kappa \) to get \( \theta^\kappa \) on \( P^\kappa \), and iterate.

Note that the fibers of \( \pi^\kappa : \hat{P}^\kappa \rightarrow P^{\kappa-1} \) are isomorphic to \( \mathfrak{g}_\kappa \). Hence, the fibers are trivial for \( \kappa \geq l + 1 \), and \( \pi^\kappa \) is a diffeomorphism. By (6.13), all of the nontrivial soldering forms \( \theta^\kappa_i \) for \( i \leq l \) are true one
forms – defined on all of $\mathbb{CTP}^n$ – as of $\kappa = l + 2$. Thus, they may be assembled into a $\mathfrak{U}_{\text{bigrad}}(\mathfrak{g}^0)$-valued parallelism on $P^{l+2}$, or by restriction, a $\mathbb{R}\mathfrak{U}_{\text{bigrad}}(\mathfrak{g}^0)$-valued parallelism on $\mathbb{R}P^{l+2}$.

6.2. Sketch of the proof of part 2 of Theorem 3.2 Consider the manifold $M_0$ with the flat CR structure of type $\mathfrak{g}^0$ as described in the paragraph before Definition 2.7 and let $D$ be the underlying distribution with Levi kernel $K$. Name $\mathcal{A}$ the Lie algebra of germs of infinitesimal symmetries of this CR structure at the point $o$ corresponding to the coset of the identity in the group $\mathbb{R}G^0$ (in fact, the choice of this point is not important here). The vector space $\mathcal{A}$ has the following natural, decreasing filtration $\{A_\kappa\}_{\kappa \in \mathbb{Z}, \kappa \geq -2}$ with $A_{-2} = A$, $A_{-1} = \{X \in \mathcal{A} : X(o) \in D_o\}$, $A_0 = \{X \in \mathcal{A} : X(o) \in \mathbb{R}(K_o \oplus \overline{K}_o)\}$, and $A_\kappa$ with $\kappa > 0$ are defined recursively by $A_\kappa = \{X \in A_{\kappa-1} : [X, A_{-1}] \subset A_{\kappa-1}\}$.

Now let $A$ be the corresponding graded Lie algebra so that $A_\kappa = A_\kappa/A_{\kappa+1}$ for all $\kappa \in \mathbb{Z}, \kappa \geq 2$. Then in direct analogy with [25, section 6] (see also [28, subsections 2.2-2.3]), there is a natural isomorphism between the bigraded universal prolongation $\mathfrak{U}_{\text{bigrad}}(\mathfrak{g}^0)$ and $A$, and – when $\dim \mathfrak{U}_{\text{bigrad}}(\mathfrak{g}^0) < \infty$ – also between $\mathfrak{U}_{\text{bigrad}}(\mathfrak{g}^0)$ and $A$ itself.

6.3. Proof of part 3 of Theorem 3.2 The proof is a standard consequence of the fact that the real part of the universal bigraded algebraic prolongation $\mathfrak{U}_{\text{bigrad}}(\mathfrak{g}^0)$ contains a grading element $E$ with respect to the first weight, i.e. such that $[E, y] = iy$ if $y \in \mathfrak{g}^0$; e.g., (5.14) and (5.16) (where the grading element is actually $-\overline{E}$). If a 2-nondegenerate CR structure with CR symbol $\mathfrak{g}^0$ has infinitesimal symmetry algebra $\mathfrak{g}$ of dimension $\dim \mathfrak{U}_{\text{bigrad}}(\mathfrak{g}^0)$, then the canonical absolute parallelism assigned to it by part (1) of Theorem 3.2 has constant structure functions; i.e., the vector fields dual to the parallelism form a Lie algebra. By construction, this Lie algebra is naturally filtered and the associated graded Lie algebra is isomorphic to $\mathfrak{U}_{\text{bigrad}}(\mathfrak{g}^0)$. It is well-known (see [8, Lemma 3.3]) that the existence of the grading element in $\mathfrak{U}_{\text{bigrad}}(\mathfrak{g}^0)$ implies $\mathfrak{g}$ and $\mathfrak{U}_{\text{bigrad}}(\mathfrak{g}^0)$ are isomorphic as filtered Lie algebras, which proves the claim.

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DEPARTMENT OF MATHEMATICS, NORTH CAROLINA STATE UNIVERSITY RALEIGH, NC 27695-8205, E-MAIL: CWPORTER@NCSU.EDU

DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TX 77843-3368, USA; E-MAIL: ZELENKO@MATH.TAMU.EDU