Global well-posedness and I-method for the
fifth-order Korteweg-de Vries equation

Wengu Chen\textsuperscript{1}, Zihua Guo\textsuperscript{2}
\textsuperscript{1}Institute of Applied Physics and Computational Mathematics
P.O.Box 8009, Beijing 100088, China
\textsuperscript{2}LMAM, School of Mathematical Sciences, Peking University
Beijing 100871, China
E-mail: chenwg@iapcm.ac.cn, zihuaguo@math.pku.edu.cn

Abstract

We prove that the Kawahara equation is locally well-posed in $H^{-7/4}$ by using
the ideas of $\bar{F}^s$-type space [7]. Next we show it is globally well-posed in $H^s$ for
$s \geq -7/4$ by using the ideas of “I-method” [6]. Compared to the KdV equation,
Kawahara equation has less symmetries, such as no invariant scaling transform
and not completely integrable. The new ingredient is that we need to deal with
some new difficulties that are caused by the lack symmetries of this equation.

Keywords: Global Well-posedness, I-method, Kawahara equation

1 Introduction

This paper is mainly concerned with the global well-posedness of the Cauchy problem
for the Kawahara equation

\[
\begin{aligned}
& u_t + \alpha u_{xxx} + \beta u_{xxxxx} + uu_x = 0, \quad x, t \in \mathbb{R}, \\
& u(x, 0) = u_0(x),
\end{aligned}
\tag{1.1}
\]

where $\alpha$ and $\beta$ are real constants and $\beta \neq 0$. By a renormalizing of $u$, we may assume
$\beta = 1$. The fifth-order KdV type equations arise in modeling gravity-capillary waves on
a shallow layer and magneto-sound propagation in plasmas (see e.g. [14]).

The well-posedness on the fifth-order KdV type equations has attracted many atten-
tions. Ponce [22] proved an $H^4$ global well-posedness for the Cauchy problem of the
following general fifth-order KdV equation

\[
\begin{aligned}
& u_t + u_x + c_1 uu_x + c_2 u_{xxx} + c_3 u_x u_{xx} + c_4 uu_{xxx} + c_5 u_{xxxxx} = 0, \quad x, t \in \mathbb{R}.
\end{aligned}
\]

In [17, 18] Kenig, Ponce and Vega studied the following high-order dispersive equation

\[
u_t + \partial_x^{2j+1} u + P(u, \partial_x u, \cdots, \partial_x^{2j} u) = 0,
\]
where $P$ is a polynomial without constant or linear terms. For the Kawahara equation (1.1), Cui, Deng and Tao [1] proved $H^s$ LWP for $s > -1$, which is later improved by Wang, Cui and Deng [27] to $s \geq -7/5$. Their proofs are based on Kenig, Ponce and Vega’s work [19]. In [3], the authors proved local well-posedness in $H^s$ for $s > -7/4$ by following the ideas of $[k; Z]$-multiplier [23]. Modified Kawahara equation (with nonlinear terms $u^2u_x$ in (1.1) instead of $u u_x$) was also studied, for example see [24, 9].

The purpose of this paper is to address the following two issues: one is LWP at $H^{-7/4}$, the other is GWP in $H^s$ for $s < 0$. Our main motivation of this paper is inspired by [3] and [7]. These two problems arise naturally in view of the results for the Korteweg-de Vries equation. Compared to the KdV equation, we will encounter a new difficulty. The equation (1.1) doesn’t have an invariant scaling transform. We will use the following scaling transform: if $u(x, t)$ is a solution of (1.1), then for $\lambda > 0$, $u_\lambda(x, t) = \lambda^4u(\lambda x, \lambda^5t)$ is a solution to the following equation

$$u_t + \mu u_{xxx} + u_{xxxxx} + uu_x = 0, \quad u(x, 0) = \phi(x),$$

where $\mu = \lambda^2 \alpha$ and $\phi(x) = \lambda^4u_0(\lambda x)$. Thus we see from $\|\lambda^4u_0(\lambda x)\|_{H^s} = \lambda^{s+7/2}\|u_0\|_{H^s}$ that when $s > -7/2$ we can assume $\|\phi\|_{H^s} \ll 1$ by taking $0 < \lambda \ll 1$. Since $0 < \lambda \leq 1$, heuristically the equation (1.2) has a uniform propagation speed in high frequency. More generally, we study the following equation

$$\partial_t u + Lu + uu_x = 0, \quad u(x, 0) = u_0(x),$$

(1.3)

here $L$ is a Fourier multiplier

$$\widehat{Lf}(\xi) = -i\omega(\xi)$$

where the symbol $\omega : \mathbb{R} \to \mathbb{R}$ is an odd function, and smooth on $\mathbb{R} \setminus \{0\}$. To study the well-posedness for (1.3) in the Sobolev space $H^s$, we will see that the crucial things are related to the dispersive effect of the equation (1.3) in high frequency, since $H^s$ spaces have very good low frequency structure.

**Definition 1.1.** Assume $\omega : \mathbb{R} \to \mathbb{R}$ is an odd function, and smooth on $\mathbb{R} \setminus \{0\}$. For some $\alpha > 0$, $\omega$ is said to have $\alpha$-order dispersive effect at high frequency if for $|\xi| \gtrsim 1$

$$|\partial_\xi^k \omega(\xi)| \sim |\xi|^{-\alpha-k}, \quad k = 1, 2; \quad |\partial_\xi^j \omega(\xi)| \lesssim |\xi|^{-\alpha-j}, \quad j \geq 3.$$

Moreover, we denote $\omega \in D_{hi}(\alpha)$.

For example, the KdV equation corresponds to $\omega = \xi^3$, then $\omega \in D_{hi}(3)$, for the Kawahara equation (1.2) considered in this paper $\omega = \mu \xi^3 - \xi^5 \in D_{hi}(5)$ uniformly on $|\mu| \leq 1$. We consider first the L.W.P of (1.1) in $H^{-7/4}$. Then it suffices to consider the equation (1.2) under the condition

$$|\mu| \leq 1, \quad \|\phi\|_{H^{-7/4}} \ll 1.$$

We will use the $F^s$ type space that was first used recently by the second author [7]. But different from the KdV equation, here for the local well-posedness a very weak low frequency structure will work, since the dispersive effect of (1.1) is very strong in high frequency. However, in order to apply the I-method, we will use the same low frequency structure as the KdV structure. We prove the following
Theorem 1.2. The Cauchy problem (1.1) is locally well-posed in $H^{-7/4}$.

Next, we will extend the local solution to a global one, using the ideas of I-method [6]. Compared to the KdV equation, the Kawahara equation has less symmetries. We will use the ideas in [11] to estimate the pointwise bounds of the multipliers.

Theorem 1.3. The Cauchy problem (1.1) is globally well-posed in $H^s$ for $s \geq -7/4$.

In the end of this section we give the notations and definitions. In Section 2 we prove Theorem 1.2. In Section 3 we give the modified energy and pointwise multiplier estimates. In Section 4 we prove Theorem 1.3. In Section 5 we give an ill-posedness result.

Notation and Definitions. Throughout this paper we fix $0 < \mu \leq 1$. We will use $C$ and $c$ to denote constants which are independent of $\mu$ and not necessarily the same at each occurrence. For $x, y \in \mathbb{R}$, $x \sim y$ means that there exist $C_1, C_2 > 0$ such that $C_1|x| \leq |y| \leq C_2|x|$. For $f \in S'$ we denote by $\widehat{f}$ or $\mathcal{F}(f)$ the Fourier transform of $f$ for both spatial and time variables,

$$\widehat{f}(\xi, \tau) = \int_{\mathbb{R}^2} e^{-ix\xi} e^{-it\tau} f(x, t) dx dt.$$ 

We denote by $\mathcal{F}_x$ the Fourier transform on spatial variable and if there is no confusion, we still write $\mathcal{F} = \mathcal{F}_x$. Let $\mathbb{Z}$ and $\mathbb{N}$ be the sets of integers and natural numbers, respectively. $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. For $k \in \mathbb{Z}_+$ let

$$I_k = \{\xi : |\xi| \in [2^{k-1}, 2^{k+1}]\}, \quad k \geq 1; \quad I_0 = \{\xi : |\xi| \leq 2\}.$$ 

Let $\eta_0 : \mathbb{R} \to [0, 1]$ denote an even smooth function supported in $[-8/5, 8/5]$ and equal to 1 in $[-5/4, 5/4]$. We define $\psi(t) = \eta_0(t)$. For $k \in \mathbb{Z}$ let $\eta_k(\xi) = \eta_0(\xi/2^k) - \eta_0(\xi/2^{k-1})$ if $k \geq 1$ and $\eta_k(\xi) \equiv 0$ if $k \leq -1$. For $k \in \mathbb{Z}$ let $\chi_k(\xi) = \eta_0(\xi/2^k) - \eta_0(\xi/2^{k-1})$. Roughly speaking, $\{\chi_k\}_{k \in \mathbb{Z}}$ is the homogeneous decomposition function sequence and $\{\eta_k\}_{k \in \mathbb{Z}_+}$ is the non-homogeneous decomposition function sequence to the frequency space. For $k \in \mathbb{Z}$ let $P_k$ denote the operator on $L^2(\mathbb{R})$ defined by

$$\widehat{P_ku}(\xi) = \eta_k(\xi)\widehat{u}(\xi).$$

By a slight abuse of notation we also define the operator $P_k$ on $L^2(\mathbb{R} \times \mathbb{R})$ by the formula $\mathcal{F}(P_ku)(\xi, \tau) = \eta_k(\xi)\mathcal{F}(u)(\xi, \tau)$. For $l \in \mathbb{Z}$ let

$$P_{\leq l} = \sum_{k \leq l} P_k, \quad P_{\geq l} = \sum_{k \geq l} P_k.$$ 

Thus we see that $P_{\leq 0} = P_0$.

Let

$$\omega(\xi) = \mu \xi^3 - \xi^5$$

(1.4)
be dispersion relation associated to equation (1.2). For $\phi \in \mathcal{S}'(\mathbb{R})$, we denote by $W(t)\phi$ the linear solution of (1.2) which is defined by

$$\mathcal{F}_x(W(t)\phi)(\xi) = \exp[i\omega(\xi)t]\hat{\phi}(\xi), \ \forall \ t \in \mathbb{R}.$$  

We define the Lebesgue spaces $L^q_{t \in I}L^p_x$ and $L^p_xL^q_{t \in I}$ by the norms

$$\|f\|_{L^q_{t \in I}L^p_x} = \|\|f\|_{L^p_x}\|_{L^q_{t \in I}}\}, \quad \|f\|_{L^p_xL^q_{t \in I}} = \|\|f\|_{L^q_{t \in I}(t)}\|_{L^p_x}.$$  

(1.5)

If $I = \mathbb{R}$ we simply write $L^q_{t \in I}L^p_x$ and $L^p_xL^q_{t \in I}$. We will make use of the $X^{s,b}$ norm associated to equation (1.2) which is given by

$$\|u\|_{X^{s,b}} = \|\langle \tau - \omega(\xi)\rangle^{b}\hat{u}(\xi, \tau)\|_{L^2(\mathbb{R}^2)},$$

where $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$. The spaces $X^{s,b}$ turn out to be very useful in the study of low-regularity theory for the dispersive equations. These spaces were first used to systematically study nonlinear dispersive wave problems by Bourgain [1] and developed by Kenig, Ponce and Vega [19] and Tao [23]. Klainerman and Machedon [20] used similar ideas in their study of the nonlinear wave equation.

In applications we usually apply $X^{s,b}$ space for $b$ very close to $1/2$. In the case $b = 1/2$ one has a good substitute-$l^1$ type $X^{s,b}$ space. For $k \in \mathbb{Z}_+$ we define the dyadic $X^{s,b}$-type normed spaces $X_k = X_k(\mathbb{R}^2)$,

$$X_k = \left\{ f \in L^2(\mathbb{R}^2) : f(\xi, \tau) \text{ is supported in } I_k \times \mathbb{R} \text{ and } \|f\|_{X_k} = \sum_{j=0}^{\infty} 2^{j/2}\|\eta_j(\tau - \omega(\xi)) \cdot f\|_{L^2} \right\}.$$  

(1.6)

Then we define the $l^1$-analogue of $X^{s,b}$ space $F^s$ by

$$\|u\|^2_{F^s} = \sum_{k \geq 0} 2^{2sk}\|\eta_k(\xi)\mathcal{F}(u)\|^2_{X_k}.$$  

(1.7)

Structures of this kind of spaces were introduced, for instance, in [23], [13] and [12] for the BO equation. The space $F^s$ is better than $X^{s,1/2}$ in many situations for several reasons (see [7, 11]). From the definition of $X_k$, we see that for any $l \in \mathbb{Z}_+$ and $f_k \in X_k$ (see also [13]),

$$\sum_{j=0}^{\infty} 2^{j/2}\left\|\eta_j(\tau - \omega(\xi)) \int |f_k(\xi, \tau')|2^{-l} (1 + 2^{-l}|\tau - \tau'|)^{-4}d\tau' \right\|_{L^2} \lesssim \|f_k\|_{X_k}.$$  

(1.8)

Hence for any $l \in \mathbb{Z}_+$, $t_0 \in \mathbb{R}$, $f_k \in X_k$, and $\gamma \in \mathcal{S}(\mathbb{R})$, then

$$\|\mathcal{F}[\gamma(t_0 + t) \cdot f_k]\|_{X_k} \lesssim \|f_k\|_{X_k}.$$  

(1.9)

In order to avoid some logarithmic divergence, we need to use a weaker norm for the low frequency

$$\|u\|_{\hat{X}_0} = \|u\|_{L^2_xL^{2\infty}_t}.$$  

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Actually a weaker structure will suffice for LWP. However, we will need this strong structure to extend it. It is easy to see from Proposition 2.5 that

\[ \| \eta_0(t)P_{\leq 0}u \|_{X_0} \lesssim \| P_{\leq 0}u \|_{X_0}. \]  

(1.10)

On the other hand, for any \( 1 \leq q \leq \infty \) and \( 2 \leq r \leq \infty \) we have

\[ \| P_{\leq 0}u \|_{L^q_t L^r \cap L^q_t L^\infty} \lesssim \| P_{\leq 0}u \|_{L^2_t L^\infty}. \]  

(1.11)

For \( -7/4 \leq s \leq 0 \), we define the resolution spaces

\[ F^s = \{ u \in S'(\mathbb{R}^2) : \| u \|_{F^s}^2 = \sum_{k \geq 1} 2^{2sk} \| \eta_k(\xi)F(u) \|^2_{X_k} + \| P_{\leq 0}(u) \|^2_{X_0} < \infty \}. \]

For \( T \geq 0 \), we define the time-localized spaces \( \tilde{F}^s(T) \):

\[ \| u \|_{\tilde{F}^s(T)} = \inf_{w \in \tilde{F}^s} \{ \| P_{\leq 0}u \|_{L^2_t L^\infty} + \| P_{\geq 1}w \|_{\tilde{F}^s} : w(t) = u(t) \text{ on } [-T,T] \}. \]  

(1.12)

Let \( a_1, a_2, a_3 \in \mathbb{R} \). It will be convenient to define the quantities \( a_{\text{max}} \geq a_{\text{med}} \geq a_{\text{min}} \) to be the maximum, median, and minimum of \( a_1, a_2, a_3 \) respectively. Usually we use \( k_1, k_2, k_3 \) and \( j_1, j_2, j_3 \) to denote integers, \( N_i = 2^{k_i} \) and \( L_i = 2^j \) for \( i = 1, 2, 3 \) to denote dyadic numbers.

2 L.W.P. at \( H^{-7/4} \)

To prove LWP by using \( X^{s,b} \)-method, the argument is standard. The first step is to prove a linear estimate, for its proof we refer the readers to [11].

**Proposition 2.1** (Linear estimates). (a) Assume \( s \in \mathbb{R} \) and \( \phi \in H^s \). Then there exists \( C > 0 \) such that

\[ \| \psi(t)W(t)\phi \|_{\tilde{F}^s} \leq C \| \phi \|_{H^s}. \]  

(2.13)

(b) Assume \( s \in \mathbb{R}, k \in \mathbb{Z}_+ \) and \( u \) satisfies \((i + \tau - \omega(\xi))^{-1}F(u) \in X_k \). Then there exists \( C > 0 \) such that

\[ \left\| \mathcal{F} \left[ \psi(t) \int_0^t W(t-s)(u(s))ds \right] \right\|_{X_k} \leq C \| (i + \tau - \omega(\xi))^{-1}F(u) \|_{X_k}. \]  

(2.14)

Then the remaining task is to show bilinear estimates. We will need symmetric estimates which will be used to prove bilinear estimates. For \( \xi_1, \xi_2 \in \mathbb{R} \) and \( \omega : \mathbb{R} \to \mathbb{R} \) as in [1,4] let

\[ \Omega(\xi_1, \xi_2) = \omega(\xi_1) + \omega(\xi_2) - \omega(\xi_1 + \xi_2). \]  

(2.15)

This is the resonance function that plays a crucial role in the bilinear estimate of the \( X^{s,b} \)-type space. See [23] for a perspective discussion. For compactly supported nonnegative functions \( f, g, h \in L^2(\mathbb{R} \times \mathbb{R}) \) let

\[ J(f, g, h) = \int_{\mathbb{R}^4} f(\xi_1, \mu_1)g(\xi_2, \mu_2)h(\xi_1 + \xi_2, \mu_1 + \mu_2 + \Omega(\xi_1, \xi_2))d\xi_1d\xi_2d\mu_1d\mu_2. \]

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We will apply to function \( f_{k_i,j_i} \in L^2(\mathbb{R} \times \mathbb{R}) \) are nonnegative functions supported in 
\([2^{k_i-1}, 2^{k_i+1}] \times \tilde{I}_{j_i}, \ i = 1, 2, 3 \). It is easy to see that \( J(f_{k_1,j_1}, f_{k_2,j_2}, f_{k_3,j_3}) \equiv 0 \) unless
\[
|k_{med} - k_{max}| \leq 5, \quad 2^{j_{max}} \sim \max(2^{j_{med}}, |\Omega(\xi_1, \xi_2)|).
\tag{2.16}
\]

We give an estimate on the resonance in the following proposition that follows from the fundamental calculus theorem.

**Proposition 2.2.** Assume \( \max(|\xi_1|, |\xi_2|, |\xi_1 + \xi_2|) \geq 10 \). Then
\[
|\Omega(\xi_1, \xi_2)| \sim |\xi|_{max}^4 |\xi|_{min},
\]
where
\[
|\xi|_{max} = \max(|\xi_1|, |\xi_2|, |\xi_1 + \xi_2|), \quad |\xi|_{min} = \min(|\xi_1|, |\xi_2|, |\xi_1 + \xi_2|).
\]

In \cite{8} the author actually proved the following proposition, also see the second author’s doctoral thesis (P33-34, \cite{9}).

**Lemma 2.3.** Let \( \alpha > 1 \). Assume \( \omega \in D_{hi}(\alpha) \) and \( k_i \in \mathbb{Z}, \ j_i \in \mathbb{Z}_+ \), \( N_i = 2^{k_i}, L_i = 2^{j_i} \) for \( i = 1, 2, 3 \). Let \( f_{k_i,j_i} \in L^2(\mathbb{R} \times \mathbb{R}) \) are nonnegative functions supported in 
\([2^{k_i-1}, 2^{k_i+1}] \times \tilde{I}_{j_i}, \ i = 1, 2, 3 \). Then
(a) For any \( k_1, k_2, k_3 \in \mathbb{Z} \) and \( j_1, j_2, j_3 \in \mathbb{Z}_+ \),
\[
J(f_{k_1,j_1}, f_{k_2,j_2}, f_{k_3,j_3}) \leq C 2^{j_{min}/2} 2^{k_{min}/2} \prod_{i=1}^{3} \| f_{k_i,j_i} \|_{L^2}.
\tag{2.17}
\]
(b) If \( N_{min} \ll N_{med} \sim N_{max} \) and \( (k_i, j_i) \neq (k_{min}, j_{max}) \) for all \( i = 1, 2, 3 \),
\[
J(f_{k_1,j_1}, f_{k_2,j_2}, f_{k_3,j_3}) \leq C 2^{(j_1+j_2+j_3)/2} 2^{-(\alpha-2)k_{max}/2} 2^{-(j_1+k_1)/2} \prod_{i=1}^{3} \| f_{k_i,j_i} \|_{L^2}.
\tag{2.18}
\]
(c) For any \( k_1, k_2, k_3 \in \mathbb{Z} \) with \( N_{min} \sim N_{med} \sim N_{max} \gg 1 \) and \( j_1, j_2, j_3 \in \mathbb{Z}_+ \)
\[
J(f_{k_1,j_1}, f_{k_2,j_2}, f_{k_3,j_3}) \leq C 2^{j_{min}/2} 2^{j_{med}/4} 2^{-(\alpha-2)k_{max}/4} \prod_{i=1}^{3} \| f_{k_i,j_i} \|_{L^2}.
\tag{2.19}
\]

In \cite{8} the authors proved a similar results for \( \omega(\xi) = \mu \xi^3 - \xi^5 \). However, there seems to be some error in the high × high → high case in their proof. The main reason is a wrong estimate on the measure of a set. Nevertheless, this error doesn’t change their LWP results for \( s > -7/4 \) because this case is not the worst case for the restriction. More explicitly, we give a counterexample below which shows part (c) in Lemma 2.3 is sharp for \( \omega(\xi) = \mu \xi^3 - \xi^5 \). It suffices to show that for some \( f_{k_1,j_1}, f_{k_2,j_2}, f_{k_3,j_3} \)
\[
J(f_{k_1,j_1}, f_{k_2,j_2}, f_{k_3,j_3}) \geq C 2^{j_{min}/2} 2^{j_{med}/4} 2^{-(\alpha-2)k_{max}/4} \prod_{i=1}^{3} \| f_{k_i,j_i} \|_{L^2}.
\tag{2.20}
\]
We use the ideas of “Knapp example” as in the KdV case \[23\]. We may assume $L_1 \leq L_2 \leq L_3$. Let
\begin{align*}
  f_1(\xi, \tau) &= 1_{|\xi - N_1| \leq N_1^{3/2} L_1^{1/2} \cdot 1_{|\tau - \omega(\xi)| \leq L_1}, \\
  f_2(\xi, \tau) &= 1_{|\xi - N_1| \leq N_1^{3/2} L_2^{1/2} \cdot 1_{|\tau - \omega(\xi)| \leq L_2}, \\
  f_3(\xi, \tau) &= 1_{|\xi - 2N_1| \leq N_1^{3/2} L_2^{1/2} \cdot 1_{|\tau - \omega(\xi)| \leq L_2}.
\end{align*}
Then we take $f_{k_i,j_i} = f_i$ for $i = 1, 2, 3$. It is easy to see that $f_i$ satisfy the support properties, and
\[
  \prod_{i=1}^3 \|f_{k_i,j_i}\|_{L^2} \sim N_1^{-9/4} L_1^{1/2} L_2^{7/4}.
\]
On the other hand, by the calculation we get
\[
  \xi_1^3 + \xi_2^3 = \frac{3(\xi_1 + \xi_2)^3}{4} + \frac{3(\xi_1 + \xi_2)(\xi_1 - \xi_2)^2}{4}
\]
and
\[
  \xi_1^5 + \xi_2^5 = \frac{5(\xi_1 + \xi_2)^5}{16} + \frac{5(\xi_1 + \xi_2)^3(\xi_1 - \xi_2)^2}{8} + \frac{5(\xi_1 + \xi_2)(\xi_1 - \xi_2)^4}{16},
\]
thus it is easy to see that
\[
  J(f_1, f_2, f_3) \gtrsim \int_{\mathbb{R}^4} f_1(\xi_1, \mu_1) f_2(\xi_2, \mu_2) d\xi_1 d\xi_2 d\mu_1 d\mu_2 \\
  \gtrsim N_1^{-3} L_1 L_2^2 \geq 2^{j_1/2} 2^{j_2/4} 2^{-3k_{\text{max}}/4} \prod_{i=1}^3 \|f_{k_i,j_i}\|_{L^2}
\]
which is as desired.

Next, we prove some dyadic bilinear estimates. It certainly work for general $\omega \in D_{hi}(\alpha)$ for $\alpha \geq 3$. But for our purpose we restrict ourselves to the case $\omega = \mu \xi^3 - \xi^5$. We will need the estimates for the linear solution to equation \((1.2)\).

**Lemma 2.4.** Let $I \subset \mathbb{R}$ be an interval with $|I| \leq 1$, $k \in \mathbb{Z}_+$ and $k \geq 10$. Then for all $\phi \in \mathcal{S}(\mathbb{R})$ we have
\[
  \|W(t)P_k\phi\|_{L^q_t L^r_x} \lesssim 2^{-3k/q}\|\phi\|_{L^2},
\]
\[
  \|W(t)P_{\leq k}\phi\|_{L^q_t L^r_x} \lesssim 2^{5k/4}\|\phi\|_{L^2},
\]
\[
  \|W(t)P_k\phi\|_{L^q_t L^2} \lesssim 2^{k/4}\|\phi\|_{L^2},
\]
\[
  \|W(t)P_k\phi\|_{L^q_t L^\infty} \lesssim 2^{-2k}\|\phi\|_{L^2},
\]
where $(q, r)$ satisfies $2 \leq q, r \leq \infty$ and $2/q = 1/2 - 1/r$.

**Proof.** For the first inequality, see [10] and also [5], for the second see [15]. For the third we use the results in [16], for the last we use the results in [15] by noting that $|\omega'(\xi)| \sim 2^{4k}$ if $|\xi| \sim 2^k$. \qed
Using the extension lemma in [7], then we get immediately that

**Lemma 2.5.** Let $I \subset \mathbb{R}$ be an interval with $|I| \lesssim 1, k \in \mathbb{Z}_+$ and $k \geq 10$. Then for all $u \in \mathcal{S}(\mathbb{R}^2)$ we have

\[
\|P_k u\|_{L^q_t L^r_x} \lesssim 2^{-3k/q} \|\hat{P}_k u\|_{X_k},
\]

\[
\|P_{\leq k} u\|_{L^1_t L_{r_1}^\infty} \lesssim 2^{5k/4} \|\hat{P}_{\leq k} u\|_{X_k},
\]

\[
\|P_k u\|_{L^q_t L^r_x} \lesssim 2^{k/4} \|\hat{P}_k u\|_{X_k},
\]

\[
\|P_k u\|_{L^{\infty}_t L^2_x} \lesssim 2^{-2k} \|\hat{P}_k u\|_{X_k},
\]

where $(q, r)$ satisfies $2 \leq q, r \leq \infty$ and $2/q = 1/2 - 1/r$.

**Proposition 2.6** (high-low). (a) If $k \geq 10, |k - k_2| \leq 5$, then for any $u, v \in F^0$

\[
\|(i + \tau - \omega(\xi))^{-1} \eta_k(\xi) i \xi P_{\leq 0} u * \psi(t) P_{k_2} v\|_{X_k} \lesssim \|P_{\leq 0} u\|_{L^2_t L^\infty_x} \|\hat{P}_{k_2} v\|_{X_{k_2}}.
\]

(b) If $k \geq 10, |k - k_2| \leq 5$ and $1 \leq k_1 \leq k - 9$. Then for any $u, v \in F^0$

\[
\|(i + \tau - \omega(\xi))^{-1} \eta_k(\xi) i \xi P_{0} u * \hat{P}_{k_2} v\|_{X_k} \lesssim k^3 2^{-7k/2} 2^{-k_1} \|P_{k_1} u\|_{X_{k_1}} \|\hat{P}_{k_2} v\|_{X_{k_2}}.
\]

**Proof.** For simplicity of notations we assume $k = k_2$. For part (a), it follows from the definition of $X_k$ that

\[
\|(i + \tau - \omega(\xi))^{-1} \eta_k(\xi) i \xi P_{0} u * \psi(t) P_{k_2} v\|_{X_k} \lesssim 2^k \sum_{j \geq 0} 2^{-j/2} \|P_{0} u * \psi(t) P_{k_2} v\|_{L^2_{t,x}},
\]

From Plancherel’s equality and Proposition 2.5 we get

\[
2^k \|P_{0} u * \psi(t) P_{k_2} v\|_{L^2_{t,x}} \lesssim 2^k \|P_{0} u\|_{L^2_t L^\infty_x} \|P_{k_2} v\|_{L^2_t L^\infty_x} \lesssim 2^{-k} \|P_{0} u\|_{L^2_t L^\infty_x} \|\hat{P}_{k_2} v\|_{X_k},
\]

which is part (a) as desired. For part (b), from the definition we get

\[
\|(i + \tau - \omega(\xi))^{-1} \eta_k(\xi) i \xi P_{0} u * \hat{P}_{k_2} v\|_{X_k} \lesssim 2^k \sum_{j_1, j_2 \geq 0} 2^{-j_3/2} \|1_{D_{k,j_3}} u_{k_1,j_1} * v_{k,j_2}\|_{2},
\]

where

\[
u_{k_1,j_1} = \eta_k(\xi) \eta_{j_1}(\tau - \omega(\xi)) \hat{u}, \quad v_{k,j_2} = \eta_k(\xi) \eta_{j_2}(\tau - \omega(\xi)) \hat{v}.
\]

From Proposition 2.2 and (2.16) we may assume $j_{\max} \geq 4k + k_1 - 10$ in the summation on the right-hand side of (2.32). We may also assume $j_1, j_2, j_3 \leq 10k$, since otherwise we will apply the trivial estimates

\[
\|1_{D_{k,j_3}} u_{k_1,j_1} * v_{k,j_2}\|_2 \lesssim 2^{j_{\min}/2} 2^{k_{\min}/2} \|u_{k_1,j_1}\|_2 \|v_{k,j_2}\|_2,
\]
then there is a $2^{-5k}$ to spare which suffices to give the bound (2.30). Thus by applying (2.18) we get

\[
2^k \sum_{j_1, j_2, j_3 \geq 0} 2^{-j_3/2} \| 1_{D_{k,j_3}} u_{k_1,j_1} \ast v_{k,j_2} \|_2 \\
\lesssim 2^k \sum_{j_1, j_2, j_3 \geq 0} 2^{-j/2} 2^{\min/2} 2^{-3k/2} 2^{-k_1/2} 2^{\text{med}/2} \| u_{k_1,j_1} \|_2 \| v_{k,j_2} \|_2 \\
\lesssim 2^k \sum_{j_1, j_2, j_3 \geq 0} k^3 2^{-3k/2} 2^{-k_1/2} 2^{-\text{max}/2} \| \hat{P}_{k_1} u \|_{X_{k_1}} \| \hat{P}_{k} v \|_{X_{k}} \\
\lesssim k^3 2^{-7k/2} 2^{-k_1} \| \hat{P}_{k_1} u \|_{X_{k_1}} \| \hat{P}_{k} v \|_{X_{k}},
\] (2.34)

which completes the proof of the proposition. \qed

**Proposition 2.7.** If $k \geq 10$, $|k - k_2| \leq 5$ and $k - 9 \leq k_1 \leq k + 10$, then for any $u, v \in F^{-7/4}$

\[
\|(i + \tau - \omega(\xi))^{-1} \eta_{k_1}(\xi) i \xi \hat{P}_{k_1} u \ast \hat{P}_{k} v \|_{X_{k_1}} \lesssim 2^{-9k/4} \| \hat{P}_{k_1} u \|_{X_{k_1}} \| \hat{P}_{k} v \|_{X_{k_2}}.
\] (2.35)

**Proof.** As in the proof of Proposition 2.6 we assume $k = k_2 = k_1$ and it follows from the definition of $X_{k_1}$ that

\[
\|(i + \tau - \omega(\xi))^{-1} \eta_{k_1}(\xi) i \xi \hat{P}_{k_1} u \ast \hat{P}_{k} v \|_{X_{k_1}} \lesssim 2^{k_1} \sum_{j_1, j_2, j_3 \geq 0} 2^{-j_1/2} \| 1_{D_{k_1,j_1}} u_{k_1,j_2} \ast v_{k,j_3} \|_2,
\] (2.36)

where $u_{k_1,j_1}, v_{k,j_2}$ are as in (2.33) and we may assume $j_{\text{max}} \geq 5k - 20$ and $j_1, j_2, j_3 \leq 10k$ in the summation. Applying (2.19) we get

\[
2^{k_1} \sum_{j_1, j_2, j_3 \geq 0} 2^{-j_1/2} \| 1_{D_{k_1,j_1}} u_{k_1,j_2} \ast v_{k,j_3} \|_2 \\
\lesssim \left( \sum_{j_2 = j_{\text{max}}} + \sum_{j_2 = j_{\text{max}}} + \sum_{j_3 = j_{\text{max}}} \right) 2^{-j_1/2} 2^{k/4} 2^{\min/2} 2^{\text{med}/4} \| u_{k_1,j_2} \|_2 \| v_{k,j_3} \|_2 \\
:= I + II + III.
\]

For the contribution of $I$, since it is easy to get the bound, thus we omit the details. We only need to bound $II$ in view of the symmetry. We get that

\[
II \lesssim \left( \sum_{j_2 = j_{\text{max}}, j_1 \leq j_3} + \sum_{j_2 = j_{\text{max}}, j_1 \geq j_3} \right) 2^{-j_1/2} 2^{k/4} 2^{\min/2} 2^{\text{med}/4} \| u_{k,j_2} \|_2 \| v_{k,j_3} \|_2 \\
:= II_1 + II_2.
\]

For the contribution of $II_1$, by summing on $j_1$ we have

\[
II_1 \lesssim \sum_{j_2 = j_{\text{max}}, j_1 \leq j_3} 2^{-j_1/2} 2^{k/4} 2^{j_1/2} 2^{j_3/2} \| u_{k,j_2} \|_2 \| v_{k,j_3} \|_2 \\
\lesssim \sum_{j_2 \geq 5k - 20, j_3 \geq 0} 2^{k/2} 2^{j_1/2} \| u_{k,j_2} \|_2 \| v_{k,j_3} \|_2 \| \hat{P}_{k_1} u \|_{X_{k_1}} \| \hat{P}_{k_2} v \|_{X_{k_2}},
\]
which is acceptable. For the contribution of $II_2$, we have
\[
II_2 \lesssim \sum_{j_2 = j_{\text{max}},j_1 \geq j_3} 2^{-j_1/2} 2^{j_2/4} 2^{j_3/2} 2^{j_1/4} \|u_{k,j_2}\|_2 \|v_{k,j_3}\|_2 \\
\lesssim 2^{-9k/4} \|\hat{P}_k u\|_{X_k} \|\hat{P}_k v\|_{X_{k_2}}.
\]
Therefore, we complete the proof of the proposition. \qed

For the low-low interaction, it is the same as the KdV case [7].

**Proposition 2.8** (low-low). If $0 \leq k_1, k_2, k_3 \leq 100$, then for any $u, \ v \in F^8$
\[
\| (i + \tau - \omega(\xi))^{-1} \eta_{k_1}(\xi) i\xi \psi(t) \hat{P}_{k_1}(u) * \hat{P}_{k_2}(v) \|_{X_{k_1}} \lesssim \|P_{k_2} u\|_{L^1_t L^2_x} \|P_{k_3} v\|_{L^1_t L^2_x}. \tag{2.37}
\]

Now we consider the high-high interactions. This is the only case where the restriction comes from.

**Proposition 2.9** (high-high). If $k \geq 10$, $|k - k_2| \leq 5$ and $1 \leq k_1 \leq k - 9$, then for any $u, \ v \in F^0$
\[
\| (i + \tau - \omega(\xi))^{-1} \eta_{k_1}(\xi) i\xi \hat{P}_{k_1} u * \hat{P}_{k_2} v \|_{X_{k_1}} \lesssim (2^{-7k/2} + k 2^{-4k_1/2}) \|\hat{P}_k u\|_{X_k} \|\hat{P}_k v\|_{X_{k_2}}. \tag{2.38}
\]

**Proof.** We assume $k = k_2$ and it follows from the definition of $X_{k_1}$ that
\[
\| (i + \tau - \omega(\xi))^{-1} \eta_{k_1}(\xi) i\xi \hat{P}_{k_1} u * \hat{P}_{k_2} v \|_{X_{k_1}} \lesssim 2^{k_1} \sum_{j_1,j_2,j_3 \geq 0} 2^{-j_1/2} \|1_{D_{k_1,j_1}} u_{k,j_2} * v_{k,j_3}\|_2, \tag{2.39}
\]
where $u_{k,j_2}, v_{k,j_3}$ are as in (2.33). For the same reasons as in the proof of Proposition 2.6 we may assume $j_{\text{max}} \geq 4k + k_1 - 10$ and $j_1, j_2, j_3 \leq 10k$. We will bound the right-hand side of (2.39) case by case. The first case is that $j_1 = j_{\text{max}}$ in the summation. Then we apply (2.18) and get that
\[
2^{k_1} \sum_{j_1,j_2,j_3 \geq 0} 2^{-j_1/2} \|1_{D_{k_1,j_1}} u_{k,j_2} * v_{k,j_3}\|_2 \\
\lesssim 2^{k_1} \sum_{j_1 \geq 4k + k_1 - 10} \sum_{j_2,j_3 \geq 0} 2^{-j_1/2 - 3k/2} 2^{-k_1/2} 2^{j_2} 2^{(j_1+j)/2} \|u_{k,j_2}\|_2 \|v_{k,j_3}\|_2 \\
\lesssim 2^{-7k/2} \|\hat{P}_k u\|_{X_k} \|\hat{P}_k v\|_{X_{k_2}},
\]
which is acceptable. If $j_2 = j_{\text{max}}$, then in this case we have better estimate for the characterization multiplier. By applying (2.18) we get
\[
2^{k_1} \sum_{j_1,j_2,j_3 \geq 0} 2^{-j_1/2} \|1_{D_{k_1,j_1}} u_{k,j_2} * v_{k,j_3}\|_2 \\
\lesssim 2^{k_1} \sum_{j_2 \geq 4k + k_1 - 10} \sum_{j_1 \leq 10k} \sum_{j_3 \geq 0} 2^{-j_1/2 - 2k_1} 2^{(j_1+j)/2} \|u_{k,j_2}\|_2 \|v_{k,j_3}\|_2 \\
\lesssim k 2^{-4k_1/2} \|\hat{P}_k u\|_{X_k} \|\hat{P}_k v\|_{X_{k_2}},
\]
where in the last inequality we use $j_1 \leq 10k$. The last case $j_3 = j_{\text{max}}$ is identical to the case $j_2 = j_{\text{max}}$ from symmetry. Therefore, we complete the proof of the proposition. \qed
For $k_1 = 0$ we can prove a similar proposition but with $k^{1-k/2}$ instead of $2^{-k/2}$ on the right-hand side of (2.38). In order to avoid the logarithmic divergence, we prove the following

**Proposition 2.10** ($\bar{X}_0$ estimate). Let $|k_1 - k_2| \leq 5$ and $k_1 \geq 10$. Then we have for all $u, v \in \bar{F}^0$

$$\left\| \psi(t) \int_0^t W(t-s)P_{\leq 0}\partial_x[P_{k_1}u(s)P_{k_2}v(s)]ds \right\|_{L^1_x L^\infty_t} \lesssim 2^{-\frac{7}{2}k_1} \|P_{k_1}u\|_{X_{k_1}} \|P_{k_2}v\|_{X_{k_2}}.$$ 

**Proof.** Denote $Q(u, v) = \psi(t) \int_0^t W(t-s)P_{\leq 0}\partial_x[P_{k_1}u(s)P_{k_2}v(s)]ds$. By straightforward computations we get

$$\mathcal{F}[Q(u, v)](\xi, \tau) = C \int_{\mathbb{R}} \frac{\hat{\psi}(\tau - \tau') - \hat{\psi}(\tau - \omega(\xi))}{\tau' - \omega(\xi)} \eta_0(\xi)i\xi \times d\tau' \int_{\xi = \xi_1 + \xi_2, \tau' = \tau_1 + \tau_2} \hat{P}_{k_1}u(\xi_1, \tau_1)\hat{P}_{k_2}v(\xi_2, \tau_2).$$

Fixing $\xi \in \mathbb{R}$, we decompose the hyperplane $\Gamma := \{\xi = \xi_1 + \xi_2, \tau' = \tau_1 + \tau_2\}$ as following

$$\Gamma_1 = \{\xi \leq 2^{-4k_1}\} \cap \Gamma;$$

$$\Gamma_2 = \{\xi \gg 2^{-4k_1}, |\tau_i - \omega(\xi_i)| < 3 \cdot 2^{4k_1}|\xi_i|, i = 1, 2\} \cap \Gamma;$$

$$\Gamma_3 = \{\xi \gg 2^{-4k_1}, |\tau_1 - \omega(\xi_1)| \geq 3 \cdot 2^{4k_1}|\xi_1\} \cap \Gamma;$$

$$\Gamma_4 = \{\xi \gg 2^{-4k_1}, |\tau_2 - \omega(\xi_2)| \geq 3 \cdot 2^{2k_1}|\xi_2\} \cap \Gamma.$$

Then we get

$$\mathcal{F}\left[ \psi(t) \cdot \int_0^t W(t-s)P_{\leq 0}\partial_x[P_{k_1}u(s)P_{k_2}v(s)]ds \right] (\xi, \tau) = A_1 + A_2 + A_3 + A_4,$$

where

$$A_1 = C \int_{\mathbb{R}} \frac{\hat{\psi}(\tau - \tau') - \hat{\psi}(\tau - \omega(\xi))}{\tau' - \omega(\xi)} \eta_0(\xi)i\xi \int_{\Gamma_1} \hat{P}_{k_1}u(\xi_1, \tau_1)\hat{P}_{k_2}v(\xi_2, \tau_2)d\tau'.$$

We consider first the contribution of the term $A_1$. Using Proposition [2.5] and Proposition [2.1] (b), we get

$$\|\mathcal{F}^{-1}(A_1)\|_{L^2_x L^\infty_t} \lesssim \left\| (i + \tau' - \omega(\xi))^{-1}\eta_0(\xi)i\xi \int_{\Gamma_1} \hat{P}_{k_1}u(\xi_1, \tau_1)\hat{P}_{k_2}v(\xi_2, \tau_2) \right\|_{\bar{X}_0}.$$ 

Since in the area $A_1$ we have $|\xi| \lesssim 2^{-4k_1}$, thus we get

$$\left\| (i + \tau' - \omega(\xi))^{-1}\eta_0(\xi)i\xi \int_{A_1} \hat{P}_{k_1}u(\xi_1, \tau_1)\hat{P}_{k_2}v(\xi_2, \tau_2) \right\|_{\bar{X}_0} \lesssim \sum_{k_3 \geq -4k_1 + 10} \sum_{j_3 \geq 0} 2^{-j_3/2} \sum_{j_1 \geq 0, j_2 \geq 0} \|1_{D_{k_3,j_3}} \cdot u_{k_1,j_1} \ast v_{k_2,j_2}\|_{L^2}.$$
where
\[ u_{k_1,j_1}(\xi, \tau) = \eta_{k_1}(\xi)\eta_{j_1}(\tau - \omega(\xi))\widehat{u}(\xi, \tau), \]
\[ v_{k_1,j_1}(\xi, \tau) = \eta_{k_1}(\xi)\widehat{v}(\xi, \tau). \]

Using (2.17), then we get
\[
\|F^{-1}(A_1)\|_{L^2_t L^\infty_x} \lesssim \sum_{k_1 \leq -4k_1 + 10} \sum_{j_1 \geq 0} 2^{-j_1/2} k_1^{2j_1 - 4k_1 + 1} \|u_{k_1,j_1}\|_{L^2} \|v_{k_2,j_2}\|_{L^2} \\
\lesssim 2^{-6k_1} \|\widehat{P_{k_1}} u\|_{X_{k_1}} \|\widehat{P_{k_2}} u\|_{X_{k_2}},
\]
which suffices to give the bound for the term \( A_1 \).

Next we consider the contribution of the term \( A_3 \). As for the term \( A_1 \), using Proposition 2.5 and Proposition 2.1 (b), we get
\[
\|F^{-1}(A_3)\|_{L^2_t L^\infty_x} \lesssim \|\tau' - \omega(\xi)\|^{-1} \eta_0(\xi) i\xi \int_{\Gamma_3} \widehat{P_{k_1}} u(\xi_1, \tau_1) \widehat{P_{k_2}} v(\xi_2, \tau_2) \|_{X_0} \\
\lesssim \sum_{k_1 \leq 0} \sum_{j_1 \geq 0} 2^{-j_1/2} k_1^{2j_1 - 3k_1} \sum_{j_1 \geq 0,j_2 \geq 0} \|1_{D_{k_1,j_3}} \cdot u_{k_1,j_1} \cdot v_{k_2,j_2}\|_{L^2} 
\]
Clearly we may assume \( j_3 \leq 10k_1 \) in the summation above. Using (2.18), then we get
\[
\|F^{-1}(A_3)\|_{L^2_t L^\infty_x} \lesssim \sum_{k_1 \leq 0} \sum_{j_1 \geq k_1 + 10} k_1^{2j_1 - 3k_1} \|u_{k_1,j_1}\|_{L^2} \|v_{k_2,j_2}\|_{L^2} \\
\lesssim k_1^{2-5k_1} \|\widehat{P_{k_1}} u\|_{X_{k_1}} \|\widehat{P_{k_2}} u\|_{X_{k_2}},
\]
which suffices to give the bound for the term \( A_3 \). From symmetry, the bound for the term \( A_4 \) is the same as \( A_3 \).

Now we consider the contribution of the term \( A_2 \). From the proof of the dyadic bilinear estimates, we know this term is the main contribution. By computation we get
\[
F_t^{-1}(A_2) = \psi(t) \int_0^t e^{i(t-s)\omega(\xi)} \eta_0(\xi) i\xi \int_{\mathbb{R}^2} e^{i(s(\tau_1 + \tau_2))} \int_{\xi = \xi_1 + \xi_2} u_{k_1}(\xi_1, \tau_1) v_{k_2}(\xi_2, \tau_2) \, d\tau_1 d\tau_2 ds
\]
where
\[
\begin{align*}
&u_{k_1}(\xi_1, \tau_1) = \eta_{k_1}(\xi_1) 1_{\{|\tau_1 - \omega(\xi_1)| \leq 3 \cdot 2^{4k_1 + 1} \xi_1\}} \widehat{u}(\xi_1, \tau_1), \\
v_{k_2}(\xi_2, \tau_2) = \eta_{k_2}(\xi_2) 1_{\{|\tau_2 - \omega(\xi_2)| \leq 3 \cdot 2^{4k_1 + 1} \xi_2\}} \widehat{v}(\xi_2, \tau_2).
\end{align*}
\]
By a change of variable \( \tau'_1 = \tau_1 - \omega(\xi_1), \tau'_2 = \tau_2 - \omega(\xi_2), \) we get
\[
F_t^{-1}(A_2) = \psi(t) e^{i\omega(\xi)} \eta_0(\xi) \int_{\mathbb{R}^2} e^{i(\tau_1 + \tau_2)} \int_{\xi = \xi_1 + \xi_2} \frac{e^{i(t(\omega(\xi_1) + \omega(\xi_2) - \omega(\xi)))} - e^{-i(\tau_1 + \tau_2)}}{\tau_1 + \tau_2 - \omega(\xi) + \omega(\xi_1) + \omega(\xi_2)} \\
\times u_{k_1}(\xi_1, \tau_1 + \omega(\xi_1)) v_{k_2}(\xi_2, \tau_2 + \omega(\xi_2)) \, d\tau_1 d\tau_2 \\
= F_t^{-1}(I) - F_t^{-1}(II).
\]
For the contribution of the term \(II\), we have

\[
\mathcal{F}_t^{-1}(II) = \int_{\mathbb{R}^2} \psi(t) e^{it\omega(\xi)} \eta_0(\xi) \int_{\xi = \xi_1 + \xi_2} \frac{u_{k_1}(\xi_1, \tau_1 + \omega(\xi_1)) v_{k_2}(\xi_2, \tau_2 + \omega(\xi_2))}{\tau_1 + \tau_2 - \omega(\xi) + \omega(\xi_1) + \omega(\xi_2)} \, d\tau_1 d\tau_2.
\]

Since in the support of \(u_{k_1}\) and \(u_{k_2}\) we have \(|\tau_1 + \tau_2 - \omega(\xi) + \omega(\xi_1) + \omega(\xi_2)| \sim 2^{4k_1}|\xi|\), then we get from Lemma 2.4 that

\[
\|\mathcal{F}_t^{-1}(II)\|_{L^2_x L^{\infty}_t} \lesssim \int_{\mathbb{R}^2} \left\| \int_{\xi = \xi_1 + \xi_2} \frac{u_{k_1}(\xi_1, \tau_1 + \omega(\xi_1)) v_{k_2}(\xi_2, \tau_2 + \omega(\xi_2))}{\tau_1 + \tau_2 - \omega(\xi) + \omega(\xi_1) + \omega(\xi_2)} \right\|_{L^2_x} d\tau_1 d\tau_2
\]

\[
\lesssim 2^{-\frac{7k_1}{2}} \| \hat{P}_{k_1} u \|_{x_{k_1}} \| \hat{P}_{k_2} u \|_{x_{k_2}}.
\]

To prove the proposition, it remains to prove the following

\[
\|\mathcal{F}_t^{-1}(I)\|_{L^2_x L^{\infty}_t} \lesssim 2^{-\frac{7k_1}{2}} \| \hat{P}_{k_1} u \|_{x_{k_1}} \| \hat{P}_{k_2} u \|_{x_{k_2}}.
\]

Compare the term \(I\) with the following term \(I'\):

\[
\mathcal{F}_t^{-1}(I') = \psi(t) e^{it\omega(\xi)} \eta_0(\xi) \int_{\mathbb{R}^2} e^{it(\tau_1 + \tau_2)} \int_{\xi = \xi_1 + \xi_2} \frac{e^{it(\omega(\xi_1) + \omega(\xi_2) - \omega(\xi))}}{-\omega(\xi) + \omega(\xi_1) + \omega(\xi_2)} \times u_{k_1}(\xi_1, \tau_1 + \omega(\xi_1)) v_{k_2}(\xi_2, \tau_2 + \omega(\xi_2)) \, d\tau_1 d\tau_2.
\]

Since on the hyperplane \(\xi = \xi_1 + \xi_2\) one has

\[-\omega(\xi + \xi) + \omega(\xi_1) + \omega(\xi_2) = \xi_1 \xi_2 \xi(\xi_1^2 + \xi_2^2 + \xi^2 - \lambda \alpha) = C \xi_1 \xi_2 \xi(-2\xi_1 \xi_2 + 2\xi^2 - \lambda \alpha)\]

In the integral area, we have \(|2\xi^2 - \lambda \alpha| \ll |\xi_1 \xi_2|\), thus we get

\[
\frac{1}{-2\xi_1 \xi_2 + 2\xi^2 - \lambda \alpha} = \frac{1}{-2\xi_1 \xi_2} \sum_{n=0}^{\infty} \left( \frac{2\xi^2 - \lambda \alpha}{2\xi_1 \xi_2} \right)^n
\]

Inserting this into \(I'\) we have

\[
\mathcal{F}_t^{-1}(I') = \psi(t) \eta_0(\xi) \sum_{n=0}^{\infty} \int_{\mathbb{R}^2} e^{it(\tau_1 + \tau_2)} \int_{\xi = \xi_1 + \xi_2} \frac{e^{it(\omega(\xi_1) + \omega(\xi_2))}}{(\xi_1 \xi_2)^{n+2}} \times u_{k_1}(\xi_1, \tau_1 + \omega(\xi_1)) v_{k_2}(\xi_2, \tau_2 + \omega(\xi_2)) \, d\tau_1 d\tau_2.
\]

Since it is easy to see that (actually we need a smooth version of \(1_{\{\xi \gg \lambda\}}\)): \(\forall \lambda > 0\),

\[
\|\mathcal{F}_x^{-1}(\{\xi \gg \lambda\}) \mathcal{F}_x u\|_{L^2_x L^{\infty}_t} \lesssim \|u\|_{L^2_x L^{\infty}_t},
\]

and setting

\[
\mathcal{F}(f_{\tau_1})(\xi) = \hat{P}_{k_1} u(\xi, \tau_1 + \omega(\xi)), \quad \mathcal{F}(g_{\tau_2})(\xi) = \hat{P}_{k_2} v(\xi, \tau_2 + \omega(\xi)),
\]

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thus we get from Lemma 2.4 that we set
\[ \|F^{-1}(I')\|_{L^2_x L^\infty_t} \lesssim \sum_{n=0}^{\infty} C^n \int_{\mathbb{R}^2} \|W(t) \partial_x^{-(n+2)} f_{\tau_1} W(t) \partial_x^{-(n+2)} g_{\tau_2}\|_{L^2_x L^\infty_t} d\tau_1 d\tau_2 \]
\[ \lesssim \sum_{n=0}^{\infty} C^n \int_{\mathbb{R}^2} \|W(t) \partial_x^{-(n+2)} f_{\tau_1}\|_{L^2_x L^\infty_t} \|W(t) \partial_x^{-(n+2)} g_{\tau_2}\|_{L^2_x L^\infty_t} d\tau_1 d\tau_2 \]
\[ \lesssim 2^{-7k_1/2} \|\widehat{P_{k_1}} u\|_{X_{k_1}} \|\widehat{P_{k_2}} u\|_{X_{k_2}}, \]
which gives the bound for the term $I''_1$.

To prove the proposition, it remains to prove the following
\[ \|F^{-1}(I - I')\|_{L^2_x L^\infty_t} \lesssim 2^{-7k_1/2} \|\widehat{P_{k_1}} u\|_{X_{k_1}} \|\widehat{P_{k_2}} u\|_{X_{k_2}}. \]
Since in the integral area we have $|\tau_i| \ll 2^{4k_1} |\xi|$, $i = 1, 2$, thus on the hyperplane $\xi = \xi_1 + \xi_2$ we have
\[
\tau_1 + \tau_2 - \omega(\xi) + \omega(\xi_1) + \omega(\xi_2) = \sum_{n=1}^{\infty} \frac{1}{-\omega(\xi) + \omega(\xi_1) + \omega(\xi_2)} \left( \frac{\tau_1 + \tau_2}{-\omega(\xi) + \omega(\xi_1) + \omega(\xi_2)} \right)^n \]
\[ = C \sum_{n=1}^{\infty} \frac{1}{(\xi_1 \xi_2)^2} \sum_{k=0}^{\infty} \left( \frac{2\xi_2 - \lambda \alpha}{2\xi_1 \xi_2} \right)^k \left( \frac{\tau_1 + \tau_2}{(\xi_1 \xi_2)^2} \right)^n \sum_{j_1, \ldots, j_n = 0}^{\infty} \prod_{i=1}^{n} \left( \frac{2\xi_2 - \lambda \alpha}{2\xi_1 \xi_2} \right)^{j_i}. \]
The purpose of decomposing this is to make the variable separately, thus then we can apply Lemma 2.4. Then by decomposing low frequency we get
\[ F_t^{-1}(I - I') = \sum_{n=1}^{\infty} \psi(t) \eta_0(\xi) \int_{\mathbb{R}^2} e^{it(\tau_1 + \tau_2)} \sum_{\tau_3, \tau_4, \tau_5, \tau_6 \text{ max}(|\tau_1|, |\tau_2|)} C_{3} \chi_{3}(\xi) \]
\[ \times \int_{\xi = \xi_1 + \xi_2} e^{it(\omega(\xi_1) + \omega(\xi_2))} u_{k_1}(\xi_1, \tau_1 + \omega(\xi_1)) u_{k_2}(\xi_2, \tau_2 + \omega(\xi_2)) \frac{1}{(\xi_1 \xi_2)^2} \]
\[ \times \sum_{k=0}^{\infty} \left( \frac{2\xi_2 - \lambda \alpha}{2\xi_1 \xi_2} \right)^k \left( \frac{\tau_1 + \tau_2}{(\xi_1 \xi_2)^2} \right)^n \sum_{j_1, \ldots, j_n = 0}^{\infty} \prod_{i=1}^{n} \left( \frac{2\xi_2 - \lambda \alpha}{2\xi_1 \xi_2} \right)^{j_i} d\tau_1 d\tau_2. \]
Using the fact that $\chi_{k_3}(\xi/2^{k_3})^{-n}$ is a multiplier for the space $L^2_x L^\infty_t$ and as for the term $I'$, we get
\[ \|F^{-1}(I - I')\|_{L^\infty_x L^\infty_t} \]
\[ \lesssim \sum_{n=1}^{\infty} \int_{\mathbb{R}^2} 2^{k_3} \geq 2^{-4k_1} \max(|\tau_1|, |\tau_2|) C^n |\tau_1 + \tau_2|^{n+2nk_3} 2^{-4nk_1} \]
\[ \times 2^{-7k_1/2} \|\mathcal{F}(f_{\tau_1})\|_{L^2} \|\mathcal{F}(g_{\tau_2})\|_{L^2} d\tau_1 d\tau_2 \]
\[ \lesssim 2^{-7k_1/2} \|P_{k_1} u\|_{X_{k_1}} \|P_{k_2} u\|_{X_{k_2}}. \]
Therefore, we complete the proof of the proposition. \(\square\)
For \( u, v \in \bar{F}^s \) we define the bilinear operator

\[
B(u, v) = \psi(t) \frac{d}{dt} \left( \int_0^t W(t - \tau) \partial_x (\psi^2(\tau) u(\tau) \cdot v(\tau)) d\tau \right).
\]

(2.40)

In order to apply a fixed point argument, all the issues are then reduced to show the boundness of \( B : \bar{F}^s \times \bar{F}^s \to \bar{F}^s \). Then Theorem 1.2 follows from standard arguments.

**Proposition 2.11 (Bilinear estimates).** Assume \(-7/4 \leq s \leq 0\). Then there exists \( C > 0 \) such that

\[
\|B(u, v)\|_{\bar{F}^s} \leq C(\|u\|_{\bar{F}^s} \|v\|_{\bar{F}^s}^{7/4} + \|u\|_{\bar{F}^{-7/4}} \|v\|_{\bar{F}^s})
\]

(2.41)

hold for any \( u, v \in \bar{F}^s \).

**Proof.** The proof is similar to the Proposition 4.2 [7]. We omit the details. \( \square \)

### 3 Modified energies

In this section we follow I-method [6] to extend the local solution. Let \( m : \mathbb{R}^k \to \mathbb{C} \) be a function. We say \( m \) is symmetric if \( m(\xi_1, \cdots, \xi_k) = m(\sigma(\xi_1, \cdots, \xi_k)) \) for all \( \sigma \in S_k \), the group of all permutations on \( k \) objects. The symmetrization of \( m \) is the function

\[
[m]_{\text{sym}}(\xi_1, \xi_2, \cdots, \xi_k) = \frac{1}{k!} \sum_{\sigma \in S_k} m(\sigma(\xi_1, \xi_2, \cdots, \xi_k)).
\]

(3.42)

We define a \( k \)-linear functional associated to the function \( m \) (multiplier) acting on \( k \) functions \( u_1, \cdots, u_k \),

\[
\Lambda_k(m; u_1, \cdots, u_k) = \int_{\xi_1 + \cdots + \xi_k = 0} m(\xi_1, \cdots, \xi_k) \hat{u}_1(\xi_1) \cdots \hat{u}_k(\xi_k).
\]

(3.43)

We will often apply \( \Lambda_k \) to \( k \) copies of the same function \( u \). \( \Lambda_k(m; u, \cdots, u) \) may simply be written \( \Lambda_k(m) \). By the symmetry of the measure on hyperplane, we have \( \Lambda_k(m) = \Lambda_k([m]_{\text{sym}}) \). The following proposition may be directly verified by using the Kawahara equation (1.2).

**Proposition 3.1.** Suppose \( u \) satisfies the Kawahara equation (1.2) and that \( m \) is a symmetric function. Then

\[
\frac{d}{dt} \Lambda_k(m) = \Lambda_k(mh_k) - \Lambda_k(mv_k) - \frac{i}{2} \Lambda_{k+1}(m(\xi_1, \cdots, \xi_{k-1}, \xi_k + \xi_{k+1})(\xi_k + \xi_{k+1})),
\]

(3.44)

where

\[
h_k = i\mu(\xi_1^3 + \xi_2^3 + \cdots + \xi_k^3), \quad v_k = i(\xi_1^5 + \xi_2^5 + \cdots + \xi_k^5).
\]

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We then follow the I-method [6] to define a set of modified energies. Let \( m : \mathbb{R} \rightarrow \mathbb{R} \) be an arbitrary even \( \mathbb{R} \)-valued function and define the operator by

\[
\hat{I}f(\xi) = m(\xi)\hat{f}(\xi),
\]

where the multiplier \( m(\xi) \) is smooth, monotone, and of the form for \( N \gg 1 \)

\[
m(\xi) = \begin{cases} 
1, & |\xi| < N, \\
N^{-s}|\xi|^s, & |\xi| > 2N.
\end{cases}
\]  

(3.45)

We define the modified energy \( E^2_I(t) \) by

\[
E^2_I(t) = \|Iu(t)\|_{L^2}^2.
\]

Using Plancherel’s identity and that \( m \) and \( u \) are \( \mathbb{R} \)-valued, and \( m \) is even, we get

\[
E^2_I(t) = \Lambda_2(m(\xi_1)m(\xi_2)).
\]

Using (3.44) then we have

\[
\frac{d}{dt}E^2_I(t) = \Lambda_3(-i[m(\xi_1)m(\xi_2 + \xi_3)(\xi_2 + \xi_3)]_{sym}).
\]

The first two terms vanish. We symmetrize the third term to get

\[
\frac{d}{dt}E^2_I(t) = \Lambda_3(-i[m(\xi_1)m(\xi_2 + \xi_3)(\xi_2 + \xi_3)]_{sym}).
\]

Let us denote

\[
M_3(\xi_1, \xi_2, \xi_3) = -i[m(\xi_1)m(\xi_2 + \xi_3)(\xi_2 + \xi_3)]_{sym}.
\]

Form the new modified energy

\[
E^3_I(t) = E^2_I(t) + \Lambda_3(\sigma_3)
\]

where the symmetric function \( \sigma_3 \) will be chosen momentarily to achieve a cancellation. Applying (3.44) gives

\[
\frac{d}{dt}E^3_I(t) = \Lambda_3(M_3) + \Lambda_3(\sigma_3h_3) - \Lambda_3(\sigma_3v_3) - \frac{3}{2}i\Lambda_4(\sigma_3(\xi_1, \xi_2, \xi_3 + \xi_4)(\xi_3 + \xi_4)).
\]  

(3.46)

Compared to the KdV case [6], there is one more term to cancel, so we choose

\[
\sigma_3 = -\frac{M_3}{h_3 - v_3}
\]

to force the three \( \Lambda_3 \) terms in (3.46) to cancel. Hence if we denote

\[
M_4(\xi_1, \xi_2, \xi_3, \xi_4) = -i\frac{3}{2}[\sigma_3(\xi_1, \xi_2, \xi_3 + \xi_4)(\xi_3 + \xi_4)]_{sym}
\]
then
\[ \frac{d}{dt}E_1^3(t) = \Lambda_4(M_4). \]

Similarly defining
\[ E_1^4(t) = E_1^3(t) + \Lambda_4(\sigma_4), \]
with
\[ \sigma_4 = -\frac{M_4}{h_4 - v_4}, \]
we obtain
\[ \frac{d}{dt}E_1^4(t) = \Lambda_5(M_5) \]
where
\[ M_5(\xi_1, \ldots, \xi_5) = -2i[\sigma_4(\xi_1, \xi_2, \xi_3, \xi_4 + \xi_5)(\xi_4 + \xi_5)]_{\text{sym}}. \]

In order to prove the pointwise estimates for the multiplier \( \sigma_3, \sigma_4 \), we need the following lemma which is crucial. It just follows from simple calculations, thus we do not give the proof.

**Lemma 3.2.**

(a) Assume \( \xi_1 + \xi_2 + \xi_3 + \xi_4 = 0 \), then
\[ \xi_1^5 + \xi_2^5 + \xi_3^5 + \xi_4^5 = -\frac{5}{2}(\xi_1 + \xi_2)(\xi_1 + \xi_3)(\xi_2 + \xi_3)(\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2), \]

and
\[ \xi_1^3 + \xi_2^3 + \xi_3^3 + \xi_4^3 = -3(\xi_1 + \xi_2)(\xi_1 + \xi_3)(\xi_2 + \xi_3). \]

(b) Assume \( \xi_1 + \xi_2 + \xi_3 = 0 \), then
\[ \xi_1^5 + \xi_2^5 + \xi_3^5 = \frac{5}{2}\xi_1\xi_2\xi_3(\xi_1^2 + \xi_2^2 + \xi_3^2), \]

and
\[ \xi_1^3 + \xi_2^3 + \xi_3^3 = 3\xi_1\xi_2\xi_3. \]

Now we turn to give the pointwise estimates of the multiplier. It is easy to see that if \( m \) is of the form (3.45), then \( m^2 \) satisfies
\[ \begin{align*}
  m^2(\xi) &\sim m^2(\xi') \text{ for } |\xi| \sim |\xi'|, \\
  (m^2)'(\xi) &= O\left(\frac{m^2(\xi)}{|\xi|^2}\right), \\
  (m^2)''(\xi) &= O\left(\frac{m^2(\xi)}{|\xi|^3}\right). 
\end{align*} \]

(3.47)

We will need two mean value formulas which follow immediately from the fundamental theorem of calculus. If \(|\eta|, |\lambda| \ll |\xi|\), then we have
\[ |a(\xi + \eta) - a(\xi)| \lesssim |\eta| \sup_{|\xi'| \sim |\xi|} |a'(\xi')|, \]

(3.48)
and the double mean value formula that
\[
|a(\xi + \eta + \lambda) - a(\xi + \eta) - a(\xi + \lambda) + a(\xi)| \lesssim |\eta| |\lambda| \sup_{|\xi'| \sim |\xi|} |a''(\xi')|.
\]  
(3.49)

In order to use the formulas, we extend the surface supported multiplier \(\sigma_3\) to the whole space as in [21].

**Proposition 3.3.** If \(m\) is of the form (3.45), then for each dyadic \(\lambda \leq \eta\) there is an extension of \(\sigma_3\) from the diagonal set

\[
\{(\xi_1, \xi_2, \xi_3) \in \Gamma_3(\mathbb{R}), |\xi_1| \sim \lambda, |\xi_2|, |\xi_3| \sim \eta\}
\]
to the full dyadic set

\[
\{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3, |\xi_1| \sim \lambda, |\xi_2|, |\xi_3| \sim \eta\}
\]
which satisfies

\[
|\partial_1^{\beta_1} \partial_2^{\beta_2} \partial_3^{\beta_3} \sigma_3(\xi_1, \xi_2, \xi_3)| \leq C m^2(\lambda) \eta^{-4} \lambda^{-\beta_1} \eta^{-\beta_2 - \beta_3}.
\]  
(3.50)

**Proof.** We may assume \(\max(|\xi_1|, |\xi_2|, |\xi_3|) \gg 1\), otherwise \(\sigma_3 \equiv 0\). Since on the hyperplane \(\xi_1 + \xi_2 + \xi_3 = 0\),

\[
v_3 = i(\xi_1^5 + \xi_2^5 + \xi_3^5) = \frac{5i}{2} \xi_1 \xi_2 \xi_3 (\xi_1^2 + \xi_2^2 + \xi_3^2)
\]
is with a size about \(\lambda \eta^4\) and

\[
M_3(\xi_1, \xi_2, \xi_3) = -i[m(\xi_1) m(\xi_2 + \xi_3)(\xi_2 + \xi_3)]_{sym} = i[m^2(\xi_1) \xi_1 + m^2(\xi_2) \xi_2 + m^2(\xi_3) \xi_3],
\]
if \(\lambda \sim \eta\), we extend \(\sigma_3\) by setting

\[
\sigma_3(\xi_1, \xi_2, \xi_3) = -\frac{i(m^2(\xi_1) \xi_1 + m^2(\xi_2) \xi_2 + m^2(\xi_3) \xi_3)}{\frac{5i}{2} \xi_1 \xi_2 \xi_3 (\xi_1^2 + \xi_2^2 + \xi_3^2 - \frac{6}{7} \mu)},
\]  
(3.51)

and if \(\lambda \ll \eta\), we extend \(\sigma_3\) by setting

\[
\sigma_3(\xi_1, \xi_2, \xi_3) = -\frac{i(m^2(\xi_1) \xi_1 + m^2(\xi_2) \xi_2 - m^2(\xi_1 + \xi_2)(\xi_1 + \xi_2))}{\frac{5i}{2} \xi_1 \xi_2 \xi_3 (\xi_1^2 + \xi_2^2 + \xi_3^2 - \frac{6}{7} \mu)}.
\]  
(3.52)

From (3.48) and (3.47), we see that (3.50) holds.

Now we give the pointwise bounds for \(\sigma_4\) which is key to estimate the growth of \(E^4_I(t)\).

**Proposition 3.4.** Assume \(m\) is of the form (3.45). In the region where \(|\xi_1| \sim N_i, |\xi_j + \xi_k| \sim N_{jk}\) for \(N_i, N_{jk}\) dyadic and \(N_1 \geq N_2 \geq N_3 \geq N_4\),

\[
\frac{|M_4(\xi_1, \xi_2, \xi_3, \xi_4)|}{|h_4 - v_4|} \lesssim m^2(\min(N_i, N_{jk})) \sim (N + N_1)^2 (N + N_2)^2 (N + N_3)^2 (N + N_4).
\]  
(3.53)
Proof. We will use the ideas in [11]. From Lemma 3.2 it suffices to prove
\[
\frac{|M_4(\xi_1, \xi_2, \xi_3, \xi_4)|}{|v_4|} \lesssim \frac{m^2(\min(N_i, N_{jk}))}{N_1^4(N + N_3)^3(N + N_4)}.
\]

Since \(\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0\), then \(N_1 \sim N_2\). We can also assume that \(N_1 \sim N_2 \gtrsim N\), otherwise \(M_4\) vanishes, since \(m^2(\xi) = 1\) if \(|\xi| \leq N\). If \(\max(N_{12}, N_{13}, N_{14}) \ll N_1\), then \(\xi_3 \approx -\xi_1, \xi_4 \approx -\xi_1\), which contradicts that \(\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0\). Hence we get \(\max(N_{12}, N_{13}, N_{14}) \sim N_1\). We rewrite the right-hand side of (3.53) as
\[
m^2(\min(N_i, N_{jk}))
\]
\[
N_1^4(N + N_3)^3(N + N_4).
\]

From Lemma 3.2 we get if \(\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0\) then
\[
v_4 = \xi_1^5 + \xi_2^5 + \xi_3^5 + \xi_4^5 = -\frac{5}{2}(\xi_1 + \xi_2)(\xi_1 + \xi_3)(\xi_2 + \xi_3)(\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2)
\]
is with size \(N_{12}N_{13}N_{14}N_1^2\). From the construction of \(M_4\) we get
\[
CM_4(\xi_1, \xi_2, \xi_3, \xi_4) = [\sigma_3(\xi_1, \xi_2, \xi_3 + \xi_4)(\xi_3 + \xi_4)]_{sym}
\]
\[
= \sigma_3(\xi_1, \xi_2, \xi_3 + \xi_4)(\xi_3 + \xi_4) + \sigma_3(\xi_1, \xi_3, \xi_2 + \xi_4)(\xi_2 + \xi_4)
\]
\[
+ \sigma_3(\xi_1, \xi_4, \xi_2 + \xi_3)(\xi_2 + \xi_3) + \sigma_3(\xi_2, \xi_3, \xi_1 + \xi_4)(\xi_1 + \xi_4)
\]
\[
+ \sigma_3(\xi_2, \xi_4, \xi_1 + \xi_3)(\xi_1 + \xi_3) + \sigma_3(\xi_3, \xi_4, \xi_1 + \xi_2)(\xi_1 + \xi_2)
\]
\[
= |\sigma_3(\xi_1, \xi_2, \xi_3 + \xi_4) - \sigma_3(-\xi_3, -\xi_4, \xi_1 + \xi_4)|\xi_3 + \xi_4
\]
\[
+ |\sigma_3(\xi_1, \xi_3, \xi_2 + \xi_4) - \sigma_3(-\xi_2, -\xi_4, \xi_2 + \xi_4)|\xi_2 + \xi_4
\]
\[
+ |\sigma_3(\xi_1, \xi_4, \xi_2 + \xi_3) - \sigma_3(-\xi_2, -\xi_3, \xi_2 + \xi_3)|\xi_2 + \xi_3
\]
\[
:= I + II + III.
\]

The bound (3.53) will follow from case by case analysis.

Case 1. \(|N_4| \gtrsim \frac{N}{2}\).

Case 1a. \(N_{12}, N_{13}, N_{14} \gtrsim N_1\).

For this case, we just use (3.50), then we get
\[
\frac{|M_4(\xi_1, \xi_2, \xi_3, \xi_4)|}{|v_4|} \lesssim \frac{m^2(N_4)}{N_1^8},
\]
which suffices to give the bound (3.53).

Case 1b. \(N_{12} \ll N_1, N_{13} \gtrsim N_1, N_{14} \gtrsim N_1\).

For the contribution of I, we just use (3.50), then we get
\[
\frac{|I|}{|v_4|} \lesssim \frac{m^2(\min(N_4, N_{12}))}{N_1^8},
\]
which suffices to give the bound (3.53).
Contribution of II. We first write

\[ II = \sigma_3(\xi_1, \xi_3, \xi_2 + \xi_4) - \sigma_3(-\xi_2, -\xi_4, \xi_2 + \xi_4)(\xi_2 + \xi_4) \]
\[ = [\sigma_3(\xi_1, \xi_3, \xi_2 + \xi_4) - \sigma_3(-\xi_2, \xi_3, \xi_2 + \xi_4)](\xi_2 + \xi_4) \]
\[ + [\sigma_3(-\xi_2, \xi_3, \xi_2 + \xi_4) - \sigma_3(-\xi_2, -\xi_4, \xi_2 + \xi_4)](\xi_2 + \xi_4) \]
\[ = II_1 + II_2. \]

If \( N_{12} \gtrsim N_3 \), then using (3.48), (3.50) for \( II_1 \) and using (3.50) for \( II_2 \), we get

\[ \frac{|II|}{|v_4|} \lesssim \frac{m^2(N_4)}{N_1^2 N_3}. \]

If \( N_{12} \ll N_3 \), using (3.48), (3.50) for both \( II_1 \) and \( II_2 \), then we get

\[ \frac{|II|}{|v_4|} \lesssim \frac{m^2(N_4)}{N_1^2 N_3}. \]

Contribution of III. This is identical to II.

**Case 1c.** \( N_{12} \ll N_1, N_{13} \ll N_1, N_{14} \gtrsim N_1 \).

Since \( N_{12} \ll N_1, N_{13} \ll N_1 \), then \( N_1 \sim N_2 \sim N_3 \sim N_4 \).

Contribution of I. We first write

\[ I = [\sigma_3(\xi_1, \xi_2, \xi_3 + \xi_4) - \sigma_3(-\xi_3, \xi_2, \xi_3 + \xi_4)](\xi_3 + \xi_4) \]
\[ + [\sigma_3(-\xi_3, \xi_2, \xi_3 + \xi_4) - \sigma_3(-\xi_3, -\xi_4, \xi_3 + \xi_4)](\xi_3 + \xi_4) \]
\[ := I_1 + I_2. \]

Using (3.50), (3.48) for both \( I_1 \) and \( I_2 \), then we get

\[ \frac{|I|}{|v_4|} \lesssim \frac{m^2(N_{12})}{N_1^8}. \]

Contribution of II. This is identical to I.

Contribution of III. We first write

\[ III = [\sigma_3(\xi_1, \xi_4, \xi_2 + \xi_3) - \sigma_3(-\xi_2, -\xi_3, \xi_2 + \xi_3)](\xi_2 + \xi_3) \]
\[ = \frac{1}{2} [\sigma_3(\xi_1, \xi_4, \xi_2 + \xi_3) - \sigma_3(-\xi_2, -\xi_3, \xi_2 + \xi_3) \]
\[ - \sigma_3(-\xi_3, -\xi_2, \xi_2 + \xi_3) + \sigma_3(\xi_4, \xi_1, \xi_2 + \xi_3)(\xi_2 + \xi_3). \]

Using (3.49) five times, we have

\[ \frac{|III|}{|v_4|} \lesssim \frac{m^2(N_1)}{N_1^8}. \]

**Case 1d.** \( N_{12} \ll N_1, N_{13} \gtrsim N_1, N_{14} \ll N_1 \).

This case is identical to Case 1c.
Case 2. \( N_4 \ll N/2 \).
In this case we have \( m^2(\min(N_i, N_{jk})) = 1 \), and \( N_{13} \sim |\xi_1 + \xi_4| = |\xi_2 + \xi_4| \sim N_1 \). We discuss this case in the following two subcases.

Case 2a. \( N_{1}/4 > N_{12} \geq N/2 \).
Since \( N_4 \ll N/2 \) and \(|\xi_3 + \xi_4| = |\xi_1 + \xi_2| \geq N/2 \), then \( N_3 \geq N/2 \). From \(|v_4| \sim N_{12} N_4^4\), then we bound the six terms in (3.53) respectively, and get
\[
|M_4| \lesssim \frac{1}{N_1^4 N_3^3 N},
\]
which suffices to give the bound (3.53).

Case 2b. \( N_{12} \ll N/2 \).
Since \( N_{12} = N_{34} \ll N/2 \) and \( N_4 \ll N/2 \), then we must have \( N_3 \ll N/2 \), and \( N_{13} \sim N_{14} \sim N_1 \).

Contribution of I. Since \( N_3, N_4, N_{34} \ll N/2 \), then we have \( \sigma_3(-\xi_3, -\xi_4, \xi_3 + \xi_4) = 0 \). Thus it follows from (3.50) that
\[
\frac{|I|}{|v_4|} \lesssim \frac{|\sigma_3(\xi_1, \xi_2, \xi_3 + \xi_4)|}{N_1^4} \lesssim \frac{1}{N_1^8}.
\]

Contribution of II and III. We have two items of \( N_3, N_4, N_{12} \) in the denominator, which will cause a problem. Thus we can’t deal with II and III separately, but we need to exploit the cancelation between II and III. We rewrite
\[
II + III = [\sigma_3(\xi_1, \xi_3, \xi_2 + \xi_4) - \sigma_3(-\xi_2, -\xi_4, \xi_2 + \xi_4)](\xi_2 + \xi_4) \\
+ [\sigma_3(\xi_1, \xi_4, \xi_2 + \xi_3) - \sigma_3(-\xi_2, -\xi_3, \xi_2 + \xi_3)](\xi_2 + \xi_3) \\
= [\sigma_3(\xi_1, \xi_3, \xi_2 + \xi_4) - \sigma_3(-\xi_2, -\xi_4, \xi_2 + \xi_4)]\xi_4 \\
+ [\sigma_3(\xi_1, \xi_4, \xi_2 + \xi_3) - \sigma_3(-\xi_2, -\xi_3, \xi_2 + \xi_3)]\xi_3 \\
+ [\sigma_3(\xi_1, \xi_3, \xi_2 + \xi_4) - \sigma_3(-\xi_2, -\xi_4, \xi_2 + \xi_4)]\xi_2 \\
= J_1 + J_2 + J_3.
\]

We first consider \( J_1 \). From
\[
\frac{|J_1|}{|v_4|} \lesssim \frac{|[\sigma_3(\xi_1, \xi_3, \xi_2 + \xi_4) - \sigma_3(-\xi_2, -\xi_4, \xi_2 + \xi_4)]\xi_4|}{N_{12} N_4^4}
\]
if \( N_{12} \ll N_3 \) (in this case, \( N_3 \sim N_4 \)), using (3.48) twice, otherwise using (3.48) once and (3.50), then we get
\[
\frac{|J_1|}{|v_4|} \lesssim \frac{1}{N_1^8}.
\]

The term \( J_2 \) is identical to the term \( J_1 \). Now we consider \( J_3 \). We first assume that \( N_{12} \geq N_3 \). Then by the symmetry of \( \sigma_3 \), we get
\[
J_3 = [\sigma_3(\xi_1, \xi_3, \xi_2 + \xi_4) - \sigma_3(-\xi_2 - \xi_3, \xi_3, \xi_2) \\
+ \sigma_3(\xi_1, \xi_4, \xi_2 + \xi_3) - \sigma_3(-\xi_2 - \xi_4, \xi_4, \xi_2)]\xi_2.
\]
From (3.48) and $N_{12} \gtrsim N_3$, we get

$$\frac{|J_3|}{|v_4|} \lesssim \frac{1}{N_1^8}.$$  

If $N_{12} \ll N_3$, then $N_3 \sim N_4$. We first write

$$J_3 = \sigma_3(-\xi_2, -\xi_2, \xi_3 + \xi_4) - \sigma_3(-\xi_2, -\xi_4, \xi_2 + \xi_4)$$

$$+ \sigma_3(\xi_1, \xi_3, \xi_2 + \xi_3) - \sigma_3(\xi_1, -\xi_3, \xi_2 + \xi_3) \xi_2$$

$$+ [\sigma_3(\xi_1, \xi_3, \xi_2 + \xi_4) - \sigma_3(-\xi_2, -\xi_3, \xi_2 + \xi_4)$$

$$+ \sigma_3(\xi_1, -\xi_3, \xi_2 + \xi_3) - \sigma_3(-\xi_2, -\xi_3, \xi_2 + \xi_3)] \xi_2$$

$$= J_{31} + J_{32}.$$  

It follows from (3.48) that

$$\frac{|J_{32}|}{|v_4|} \sim \frac{1}{N_1^8}.$$  

It remains to bound $J_{31}$. It follows from (3.50) and $m^2(\xi_3) = m^2(\xi_4) = 1$ that

$$CJ_{31} = C[\sigma_3(-\xi_2, \xi_3, \xi_2 + \xi_4) - \sigma_3(-\xi_2, -\xi_4, \xi_2 + \xi_4)$$

$$- \sigma_3(\xi_1, -\xi_3, \xi_2 + \xi_3) + \sigma_3(\xi_1, \xi_3, \xi_2 + \xi_3)] \xi_2$$

$$= \frac{-m^2(\xi_2)\xi_2 + m^2(\xi_2 + \xi_4)(\xi_2 + \xi_4)}{-\xi_2\xi_3(\xi_2 + \xi_4)[\xi_2^2 + \xi_3^2 + (\xi_2 + \xi_4)^2]} \xi_2$$

$$- \frac{-m^2(\xi_2)\xi_2 - \xi_4 + m^2(\xi_2 + \xi_4)(\xi_2 + \xi_4)}{\xi_2\xi_3(\xi_2 + \xi_4)[\xi_2^2 + \xi_3^2 + (\xi_2 + \xi_4)^2]} \xi_2$$

$$+ \frac{m^2(\xi_1)\xi_1 + \xi_4 + m^2(\xi_2 + \xi_3)(\xi_2 + \xi_3)}{\xi_1\xi_4(\xi_2 + \xi_3)[\xi_1^2 + \xi_3^2 + (\xi_2 + \xi_3)^2]} \xi_2$$

$$- \frac{m^2(\xi_1)\xi_1 - \xi_3 + m^2(\xi_2 + \xi_3)(\xi_2 + \xi_3)}{-\xi_1\xi_3(\xi_2 + \xi_3)[\xi_1^2 + \xi_3^2 + (\xi_2 + \xi_3)^2]} \xi_2.$$  

We rewrite $J_{31}$ as following

$$J_{31} = \left\{ \frac{-m^2(\xi_2)\xi_2 + m^2(\xi_2 + \xi_4)(\xi_2 + \xi_4)}{-\xi_2\xi_3(\xi_2 + \xi_4)[\xi_2^2 + \xi_3^2 + (\xi_2 + \xi_4)^2]} \xi_2 - \frac{-m^2(\xi_2)\xi_2 + m^2(\xi_2 + \xi_4)(\xi_2 + \xi_4)}{-\xi_1\xi_3(\xi_2 + \xi_3)[\xi_1^2 + \xi_3^2 + (\xi_2 + \xi_3)^2]} \xi_2$$

$$- \frac{m^2(\xi_1)\xi_1 + m^2(\xi_2 + \xi_3)(\xi_2 + \xi_3)}{-\xi_1\xi_3(\xi_2 + \xi_3)[\xi_1^2 + \xi_3^2 + (\xi_2 + \xi_3)^2]} \xi_2$$

$$+ \left\{ \frac{\xi_1(\xi_2 + \xi_3)[\xi_1^2 + \xi_3^2 + (\xi_2 + \xi_3)^2]}{1} - \frac{\xi_1(\xi_2 + \xi_3)[\xi_1^2 + \xi_3^2 + (\xi_2 + \xi_3)^2]}{1} \right\} \xi_2$$

$$+ \left\{ \frac{\xi_1(\xi_2 + \xi_3)[\xi_1^2 + \xi_3^2 + (\xi_2 + \xi_3)^2]}{1} - \frac{\xi_1(\xi_2 + \xi_3)[\xi_1^2 + \xi_3^2 + (\xi_2 + \xi_3)^2]}{1} \right\} \xi_2 \right\}$$

$$:= J_{311} + J_{312}.$$  

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We consider first the term $J_{312}$.

\[
J_{312} = \left\{ \frac{\xi_2(\xi_3 - \xi_4)(\xi_3 + \xi_4)}{\xi_1(\xi_2 + \xi_3)[\xi_1^2 + \xi_2^2 + (\xi_2 + \xi_3)^2]} \right\}
+ \frac{(\xi_3 - \xi_4)(\xi_3 + \xi_4)}{(\xi_2 + \xi_4)[\xi_2^2 + \xi_3^2 + (\xi_2 + \xi_4)^2][\xi_2^2 + \xi_3^2 + (\xi_2 + \xi_4)^2]}\}
\]

Thus we get

\[
\frac{|J_{312}|}{|v_4|} \lesssim \frac{1}{N_1^8}.
\]

It remains to bound the term $J_{311}$. We will compare it with the following term denoted by $J'_{311}$:

\[
\left\{ \frac{-m^2(\xi_2)\xi_2 + m^2(\xi_2 + \xi_4)(\xi_2 + \xi_4)}{\xi_2(\xi_2 + \xi_4)[\xi_2^2 + \xi_4^2 + (\xi_2 + \xi_4)^2]} \right\} \xi_2 \quad \frac{-m^2(\xi_2)\xi_2 + m^2(\xi_2 + \xi_4)(\xi_2 + \xi_4)}{\xi_2(\xi_2 + \xi_4)[\xi_2^2 + \xi_4^2 + (\xi_2 + \xi_4)^2]} \xi_2
\]
\[
+ \frac{m^2(\xi_1)\xi_1 + m^2(\xi_2 + \xi_3)(\xi_2 + \xi_3)}{\xi_3(\xi_2 + \xi_3)[\xi_1^2 + \xi_2^2 + (\xi_2 + \xi_3)^2]} \xi_2
\]
\[
+ \frac{m^2(\xi_1)\xi_1 + m^2(\xi_2 + \xi_3)(\xi_2 + \xi_3)}{\xi_3(\xi_2 + \xi_3)[\xi_1^2 + \xi_2^2 + (\xi_2 + \xi_3)^2]} \xi_2\}
\]

It is easy to see that as for the term $J_{312}$ we have

\[
\frac{|J_{311} - J'_{311}|}{|v_4|} \lesssim \frac{1}{N_1^8}.
\]

Thus it remains to show that

\[
\frac{|J'_{311}|}{|v_4|} \lesssim \frac{1}{N_1^8}.
\]

We rewrite $J'_{311}$ as following

\[
- \frac{\xi_3 + \xi_4 - m^2(\xi_2)\xi_2 + m^2(\xi_2 + \xi_4)(\xi_2 + \xi_4) + m^2(\xi_1)\xi_1 + m^2(\xi_2 + \xi_3)(\xi_2 + \xi_3)}{\xi_3(\xi_2 + \xi_3)[\xi_1^2 + \xi_2^2 + (\xi_2 + \xi_3)^2]} \xi_2
\]
\[
+ \frac{\xi_3 + \xi_4}{\xi_3(\xi_2 + \xi_3)[\xi_1^2 + \xi_2^2 + (\xi_2 + \xi_3)^2]} \left[ \frac{1}{\xi_1(\xi_2 + \xi_3)[\xi_1^2 + \xi_2^2 + (\xi_2 + \xi_3)^2]} + \frac{1}{\xi_2(\xi_2 + \xi_4)[\xi_2^2 + \xi_4^2 + (\xi_2 + \xi_4)^2]} \right] \xi_2.
\]

Therefore, we use (3.49) for the first term, and (3.48) for the second term, and finally we conclude that

\[
\frac{|J'_{311}|}{|v_4|} \lesssim \frac{1}{N_1^8},
\]

which completes the proof of the proposition. \qed

From the estimates of $\sigma_4$ we can immediately get the following

**Proposition 3.5.** Assume $m$ is of the form (3.45), then

\[
|M_5(\xi_1, \ldots, \xi_5)| \lesssim \left[ \frac{m^2(N_{*45})N_{45}}{(N + N_1)^2(N + N_2)^2(N + N_3)^3(N + N_{45})} \right]_{sym},
\]

where

\[
N_{*45} = \min(N_1, N_2, N_3, N_{45}, N_{12}, N_{13}, N_{23}).
\]
4 G.W.P. of fifth order KdV on $\mathbb{R}$

In this section we extend the local solution to a global one. We will rely on a variant well-posedness result which can be proved similarly as for the Theorem 1.2.

Proposition 4.1. Let $-7/4 \leq s \leq 0$. Assume $\phi$ satisfy $\|I\phi\|_{L^2(\mathbb{R})} \leq 2\epsilon_0 \ll 1$. Then equation (1.2) has a unique solution on $[-1,1]$ such that

$$
\|Iu\|_{F^0(1)} \leq C\epsilon_0,
$$

where $C$ is independent of $N$ and $0 < \lambda \leq 1$.

From Proposition 4.1 we see it suffices to control the growth of $E^2_7(t)$. It is better controlling directly the growth of $E^2_1(t)$, and then using the following proposition we can that of $E^2_7(t)$.

Proposition 4.2. Let $I$ be defined with the multiplier $m$ of the form (3.45) and $s = -7/4$. Then

$$
|E^1_1(t) - E^2_1(t)| \lesssim \|Iu(t)\|_{L^2}^3 + \|Iu(t)\|_{L^2}^4.
$$

Proof. Since $E^1_1(t) = E^2_1(t) + \Lambda_3(\sigma_3) + \Lambda_4(\sigma_4)$, then it suffices to show

$$
|\Lambda_3(\sigma_3; u_1, u_2, u_3)| \lesssim \prod_{i=1}^3 \|Iu_i\|_{L^2},
$$

$$
|\Lambda_4(\sigma_4; u_1, u_2, u_3)| \lesssim \prod_{i=1}^4 \|Iu_i\|_{L^2}.
$$

We may assume that $\hat{u}_i$ are non-negative. To prove (4.58), it suffices to prove

$$
\left| \Lambda_3 \left( \frac{m^2(\xi_1)\xi_1 + m^2(\xi_2)\xi_2 + m^2(\xi_3)\xi_3}{\xi_1\xi_2\xi_3(\xi_1^2 + \xi_2^2 + \xi_3^2)m(\xi_1)m(\xi_2)m(\xi_3)}; u_1, u_2, u_3 \right) \right| \lesssim \prod_{i=1}^3 \|u_i\|_2.
$$

By Littlewood-Paley decomposition, we get the left-hand side of (4.60) is bounded by

$$
\sum_{k_i \geq 0} \left| \Lambda_3 \left( \frac{m^2(\xi_1)\xi_1 + m^2(\xi_2)\xi_2 + m^2(\xi_3)\xi_3}{\xi_1\xi_2\xi_3(\xi_1^2 + \xi_2^2 + \xi_3^2)m(\xi_1)m(\xi_2)m(\xi_3)}; P_{k_1} u_1, P_{k_2} u_2, P_{k_3} u_3 \right) \right|.
$$

Let $N_i = 2^{k_i}$. From symmetry we may assume $N_1 \geq N_2 \geq N_3$, and hence $N_1 \sim N_2 \ll N$.

Case 1. $N_3 \ll N$.

In this case $m(N_3) = 1$, then we get

$$
(4.61) \lesssim \sum_{k_i \geq 0} \left| \Lambda_3 \left( \frac{N^s N^s}{N_1^{1+s} N_1^{1+s} N_1^{1+s}}; P_{k_1} u_1, P_{k_2} u_2, P_{k_3} u_3 \right) \right|
$$

$$
\lesssim \sum_{k_i \geq 0} \left| \Lambda_3 \left( \frac{N^{-1/4} N^{-1/4}}{N_1^{1+s} N_1^{1+s} N_1^{1+s}}; P_{k_1} u_1, P_{k_2} u_2, P_{k_3} u_3 \right) \right|.
$$
It suffices to prove
\[
\sum_{k_i \geq 0} \int_{\xi_1+\xi_2+\xi_3=0, |\xi_i| \sim N_i} N_1^{-1/2} \prod_{i=1}^{3} \eta_{k_i}(\xi_i) \hat{u}_i(\xi_i) \lesssim \prod_{i=1}^{3} \|u_i\|_{L^2}.
\]

Define \( v_i(x) \) as following:
\[
\hat{v}_i(\xi) = N_i^{-1/6} \hat{u}_i(\xi) \chi_{\{\xi|\sim N_i\}}(\xi).
\]

By Sobolev embedding inequality we have \( \|v_i\|_{L^3} \lesssim \|u_i\|_{L^2} \), thus using Hölder’s inequality we get
\[
\sum_{k_i \geq 0} \int_{\xi_1+\xi_2+\xi_3=0, |\xi_i| \sim N_i} N_1^{-1/2} \prod_{i=1}^{3} \eta_{k_i}(\xi_i) \hat{v}_i(\xi_i)
\]
\[
\lesssim \sum_{k_i \geq 0} N_1^{-1/6} N_3^{1/6} \prod_{i=1}^{3} \|v_i\|_{L^3} \lesssim \prod_{i=1}^{3} \|u_i\|_{L^2}.
\]

Case 2. \( N_3 \gtrsim N \). It is obvious that
\[
(4.61) \lesssim \sum_{k_i \geq 0} \left| \Lambda_3 \left( \frac{N_3^{-7/4} N^{-7/4}}{N_1^{1/2}} ; P_{k_1} u_1, P_{k_2} u_2, P_{k_3} u_3 \right) \right| \lesssim \prod_{i=1}^{3} \|u_i\|_{L^2}.
\]

Thus we get \((4.58)\).

Next we show \((4.59)\). It suffices to prove
\[
\left| \Lambda_4 \left( \frac{\sigma_4}{m(\xi_1)m(\xi_2)m(\xi_3)m(\xi_4)} ; u_1, u_2, u_3, u_4 \right) \right| \lesssim \prod_{i=1}^{4} \|u_i\|_{L^2}.
\]

(4.62)

By Littlewood-Paley decomposition we get the left-hand side of \((4.62)\) is dominated by
\[
\sum_{k_i \geq 0} \left| \Lambda_4 \left( \frac{\sigma_4}{m(\xi_1)m(\xi_2)m(\xi_3)m(\xi_4)} ; P_{k_1} u_1, P_{k_2} u_2, P_{k_3} u_3, P_{k_4} u_4 \right) \right|.
\]

(4.63)

Let \( N_i = 2^{k_i} \). From symmetry we may assume \( N_1 \geq N_2 \geq N_3 \geq N_4 \), hence we may assume \( N_1 \sim N_2 \gtrsim N \). Since
\[
\left| \frac{\sigma_4}{m(\xi_1)m(\xi_2)m(\xi_3)m(\xi_4)} \right| \lesssim \prod_{i=1}^{4} \frac{1}{N_i^{1/4}} \lesssim \frac{N^{-7}}{\prod_{i=1}^{4} N_i^{1/4}},
\]

using Hölder’s inequality we get
\[
(4.63) \lesssim \sum_{k_i \geq 0} \frac{N_i^{-7}}{\prod_{i=1}^{4} N_i^{1/4}} \|P_{k_1} u_1\|_{L^2} \|P_{k_2} u_2\|_{L^2} \|P_{k_3} u_3\|_{L^\infty} \|P_{k_4} u_4\|_{L^\infty}
\]
\[
\lesssim \prod_{i=1}^{4} \|u_i\|_{L^2}.
\]

Therefore, we complete the proof of the proposition. \( \square \)
Since $E_1^t(t)$ is very close to $E_1(t)$, then we will control $E_1^t(t)$ and hence control $E_1^2(t)$. In order to control the increase of $E_1^t(t)$, we need to control its derivative
\[
\frac{d}{dt} E_1^t(t) = \Lambda_5(M_5),
\]
where
\[
M_5(\xi_1, \ldots, \xi_5) = -2i[\sigma_4(\xi_1, \xi_2, \xi_3 + \xi_5)(\xi_4 + \xi_5)]_{\text{sym}}.
\]

**Proposition 4.3.** Assume $I \subset \mathbb{R}$ with $|I| \lesssim 1$. Let $0 \leq k_1 \leq k_2 \leq k_3 \leq k_4 \leq k_5$ and $k_4 \geq 10$. Then we have
\[
\left| \int_I \int P_{k_1}(P_{k_2}(w_1)P_{k_2}(w_2)) \prod_{i=3}^5 P_{k_i}(w_i)(x,t) dx dt \right| \lesssim 2^{k_1} 2^{k_2/4} 2^{k_3/4} 2^{-2k_4} 2^{-2k_5} \prod_{j=1}^5 \|P_{k_j}(w_j)\|_{X_{k_j}},
\]
where if $k_j = 0$ then $X_{k_j}$ is replaced by $\bar{X}_{k_j}$ on the right-hand side.

**Proof.** From Hölder’s inequality the left-hand side of (4.64) is dominated by
\[
\|P_{k_1}(w_1)\|_{L^2_t L^\infty_x} \|P_{k_2}(w_2)\|_{L^2_t L^\infty_x} \|P_{k_2}(w_2)\|_{L^6_t L^6_x} \|P_{k_4}(w_4)\|_{L^\infty_t L^2_x} \|P_{k_5}(w_5)\|_{L^\infty_t L^2_x}.
\]
Then the proposition follows immediately from Proposition 2.5. \hfill \Box

**Proposition 4.4.** Let $\delta \lesssim 1$. Assume $m$ is of the form (3.45) with $s = -7/4$, then
\[
\left| \int_0^\delta \Lambda_5(M_5) dt \right| \lesssim N^{-\frac{35}{4}} \|u\|_{F^0(\delta)}^5.
\]

**Proof.** By the definitions, it suffices to prove that
\[
\left| \int_0^\delta \Lambda_5 \left( \frac{\sigma_4(\xi_1, \xi_2, \xi_3, \xi_4 + \xi_5)(\xi_4 + \xi_5)}{m(\xi_1)m(\xi_2)m(\xi_3)m(\xi_4)m(\xi_5)} \right) dt \right| \lesssim N^{-\frac{35}{4}} \|u\|_{F^0(\delta)}^5.
\]
By the Littlewood-Paley decomposition $u = \sum_{k \geq 0} P_k u$, it suffices to prove
\[
\sum_{N_1, \ldots, N_5, N_{45} \geq 0} \left| \int_0^\delta \Lambda_5 \left( \frac{\sigma_4(\xi_1, \xi_2, \xi_3, \xi_4 + \xi_5)(\xi_4 + \xi_5)}{m(\xi_1)m(\xi_2)m(\xi_3)m(\xi_4)m(\xi_5)} ; P_{k_1}u, \ldots, P_{k_5}u \right) dt \right| \lesssim N^{-\frac{35}{4}} \|u\|_{F^0(\delta)}^5,
\]
where $N_i = 2^{k_i}$ and $|\xi_4 + \xi_5| \sim N_{45}$ for $N_{45}$ dyadic. From symmetry we may assume $N_1 \geq N_2 \geq N_3$ and $N_4 \geq N_5$ and two of the $N_i \gtrsim N$. We fix the extension $\tilde{u}_i$ such that $\|\tilde{u}_i\|_{F^0} \lesssim \|u_i\|_{F^0(\delta)}$. For simplicity, we still denote $u_i$.

**Case 1.** $N_1 \sim N_2 \gtrsim N$ and $N_4 \lesssim N_2$.

**Case 1a.** $N_{45} \gtrsim N_3$. 

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From the form (3.45) with \( s = -7/4 \) we get that \( \frac{1}{m(N_3)m(N_4)m(N_5)} \leq N^{-\frac{7}{4} N_1^{-1/4}} \) and \( \frac{1}{m(N_3)m(N_4)m(N_5)} \leq N^{-\frac{21}{4} N_3^{7/4} N_4^{7/4} N_5^{7/4}} \). Thus we have

\[
\left| \frac{\sigma_4(\xi_1, \xi_2, \xi_3, \xi_4 + \xi_5)(\xi_4 + \xi_5)}{m(\xi_1)m(\xi_2)m(\xi_3)m(\xi_4)m(\xi_5)} \right| \leq \frac{N^{-35/4} N_1^{-1/2} N_3^{7/4} N_4^{7/4} N_5^{7/4} m^2(\min(N_1, N_{j,k})) N_{45}}{(N + N_3)(N + N_{45})^3}
\]

Therefore in this case we need to control

\[
N^{-\frac{35}{4}} \sum_{N_i, N_{45}} \int_0^\delta \Lambda_5 \left( \frac{N_1^{-1/2} N_3^{7/4} N_4^{7/4} N_5^{7/4} N_{45}}{(N + N_3)(N + N_{45})^3}; P_{k_1}u, \ldots, P_{k_5}u \right) dt.
\]

We consider the worst case \( N_1 \geq N_2 \geq N_4 \geq N_5 \geq N_3 \). From (4.64) we get

\[
N^{-\frac{35}{4}} \sum_{N_i} \frac{N_1^{-1/2} N_3^{7/4} N_4^{7/4} N_5^{7/4} N_{45}}{(N + N_3)(N + N_{45})^3} N_1^{-1/2} N_3^{1/4} N_4^{1/4} N_5^{1/4} \prod_{i=1}^5 \| \widetilde{P}_{k_i}u \|_{X_{k_i}} \leq N^{-\frac{35}{4}} \| u \|_{\dot{F}^{0}(\delta)}. \]

**Case 1b.** \( N_{45} \ll N_3 \).

We have

\[
\left| \frac{\sigma_4(\xi_1, \xi_2, \xi_3, \xi_4 + \xi_5)(\xi_4 + \xi_5)}{m(\xi_1)m(\xi_2)m(\xi_3)m(\xi_4)m(\xi_5)} \right| \leq \frac{N^{-35/4} N_1^{-1/2} N_3^{7/4} N_4^{7/4} N_5^{7/4} m^2(\min(N_1, N_{j,k})) N_{45}}{(N + N_{45})(N + N_3)^3}
\]

Therefore in this case we need to control

\[
N^{-\frac{35}{4}} \sum_{N_i, N_{45}} \int_0^\delta \Lambda_5 \left( \frac{N_1^{-1/2} N_3^{7/4} N_4^{7/4} N_5^{7/4}}{(N + N_3)^3}; P_{k_1}u, \ldots, P_{k_5}u \right) dt.
\]

We get from Proposition 4.3 that (still consider the worst case \( N_1 \geq N_2 \geq N_4 \geq N_5 \geq N_3 \))

\[
N^{-\frac{35}{4}} \sum_{N_i} \frac{N_1^{-1/2} N_3^{7/4} N_4^{7/4} N_5^{7/4}}{(N + N_3)^3} N_1^{-1/2} N_3^{1/4} N_4^{1/4} N_5^{1/4} \prod_{i=1}^5 \| \widetilde{P}_{k_i}u \|_{X_{k_i}} \leq N^{-\frac{35}{4}} \| u \|_{\dot{F}^{0}(\delta)}. \]

The rest cases \( N_4 \sim N_5 \leq N, N_1 \leq N_5 \) or \( N_1 \sim N_4 \leq N \) follow in a similar ways. We omit the details.

For any fixed \( u_0 \in H^{-7/4} \) and time \( T > 0 \), our goal is to construct the solution to (1.1) on \( \tau \in [0, T] \). If \( u \) is a solution to (1.1) with initial data \( u_0 \), then for any \( \lambda > 0 \), \( u_{\lambda}(x, t) = \lambda^4 u(\lambda x, \lambda^2 t) \) is a solution to (1.2) with initial data \( u_{0,\lambda} = \lambda^4 u_0(\lambda x) \). By simple calculation we know

\[
\| I u_{0,\lambda} \|_{L^2} \leq \lambda^{7/4} N_{1/4} \| u_0 \|_{H^s}.
\]

For fixed \( N \) (\( N \) will be determined later), we take \( \lambda \sim N^{-1} \) such that

\[
\lambda^{7/4} N_{1/4} \| \phi \|_{H^{-7/4}} = \epsilon_0 < 1.
\]
For the simplicity of notations, we still denote $u_\lambda$ by $u$, $u_{0,\lambda}$ by $u_0$, and assume $\|Iu_0\|_{L^2} \leq \epsilon_0$, then the goal is to construct the solution to (1.2) on $[0, \lambda^{-5}T]$. According to Proposition 4.1 we get a local solution $u$ on $t \in [0, 1]$, then we need to control the modified energy $E_{I^2}(t) = \|Iu\|^2_{L^2}$.

First we see the control of $E_{I^2}(t)$ for $t \in [0, 1]$, we will prove that $E_{I^2}(t) < 4\epsilon_0^2$. Using bootstrap we may assume $E_{I^2}(t) < 5\epsilon_0^2$, then from Proposition 4.2 we get

$$E_{I^2}(0) = E_{I^2}(0) + O(\epsilon_0^3)$$

and

$$E_{I^2}(t) = E_{I^2}(t) + O(\epsilon_0^3).$$

Thus from Proposition 4.4 we get for $t \in [0, 1]$

$$E_{I^2}(t) \leq E_{I^2}(0) + C\epsilon_0^5 N^{-35/4}.$$

Therefore

$$\|Iu(1)\|^2_{L^2} = E_{I^2}(1) + O(\epsilon_0^3) \leq E_{I^2}(0) + C\epsilon_0^5 N^{-35/4} + O(\epsilon_0^3) = \epsilon_0^2 + C\epsilon_0^5 N^{-35/4} + O(\epsilon_0^3) < 4\epsilon_0^2.$$

Thus $u$ can be extended to $t \in [0, 2]$. Extending as this $M$-steps, we get for $t \in [0, M+1]$

$$E_{I^2}(t) \leq E_{I^2}(0) + CM\epsilon_0^5 N^{-35/4}.$$  

Thus as long as $MN^{-35/4} \leq 1$, then we have

$$E_{I^2}(M) = E_{I^2}(M) + O(\epsilon_0^3) = \epsilon_0^2 + O(\epsilon_0^3) + CM\epsilon_0^5 N^{-35/4} < 4\epsilon_0^2.$$  

Therefore, the solution can be extended to $t \in [0, N^{35/4}]$. Taking $N(T)$ sufficiently large such that

$$N^{35/4} > \lambda^{-5}T \sim N^{5}T.$$  

Thus $u$ is extended to $[0, \lambda^{-5}T]$, then for the original equation (1.1), using the scaling, we prove its solution extends to $[0, T]$.

At the end of this section, we see what we know about the global solution. Using the scaling we get

$$\sup_{t \in [0,T]} \|u(t)\|_{H^{-7/4}} \sim \lambda^{-7/4} \sup_{t \in [0,\lambda^{-5}T]} \|u_\lambda(t)\|_{H^{-7/4}} \leq \lambda^{-7/4} \sup_{t \in [0,\lambda^{-5}T]} \|Iu_\lambda(t)\|_{L^2},$$

$$\|I\phi_\lambda\|_{L^2} \leq N^{7/4} \|\phi_\lambda\|_{H^{-7/4}} \sim N^{7/4} \lambda^{7/4} \|\phi\|_{H^{-7/4}}.$$  

From the previous proof we know

$$\sup_{t \in [0,\lambda^{-5}T]} \|Iu_\lambda(t)\|_{L^2} \leq \|I\phi_\lambda\|_{L^2},$$

thus we get

$$\sup_{t \in [0,T]} \|u(t)\|_{H^{-7/4}} \leq N^{7/4} \|\phi\|_{H^{-7/4}}.$$  

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Take \( \lambda \) such that \( \| I\phi_\lambda \|_{L^2} \sim \epsilon_0 \ll 1 \), then
\[
\lambda = \lambda(N, \epsilon_0, \| \phi \|_{H^{-7/4}}) \sim \left( \frac{\| \phi \|_{H^{-7/4}}}{\epsilon_0} \right)^{-4/7} N^{-1}.
\]

We will choose \( N \) such that \( N^{25} > \lambda^5 T \sim c\| \phi \|_{H^s, \epsilon_0} N^5 T \), then we get \( N \sim T^{4/15} \). Therefore, we get that the obtained global solution \( u(x, t) \) satisfies
\[
\| u(t) \|_{H^{-7/4}} \lesssim (1 + |t|)^{7/15}\| \phi \|_{H^{-7/4}}^{4/3}.
\]

We can prove a similar results for \( s > -7/4 \).

5 Ill-posedness of the equation

In this section, we prove an ill-posedness results by following the method of Bourgain [2].

**Theorem 5.1.** For \( s < -\frac{9}{4} \), the solution map of the Cauchy problem (1.1) is not \( C^3 \) smooth at zero, namely, there is no \( T > 0 \) such that the solution map of:

\[
u_0 \in H^s(\mathbb{R}) \mapsto u \in C([0, T]; H^s(\mathbb{R}))
\]

is \( C^3 \) at zero.

**Proof.** Under \( s < -\frac{9}{4} \) for contradiction we assume that the solution map

\[
u_0 \in H^s(\mathbb{R}) \mapsto u \in C([0, T]; H^s(\mathbb{R}))
\]

is \( C^3 \) at zero. According to [2], we must have
\[
\sup_{t \in [0, T]} \| A_3(f) \|_{H^s} \lesssim \| f \|_{H^s}^3 \quad \text{for all } f \in H^s(\mathbb{R}),
\]

where
\[
A_3(f)(x, t) = \int_0^t W(t - \tau) \left( \partial_x (A_1(f) \cdot A_2(f)) \right)(\tau) d\tau;
\]
\[
A_2(f)(x, t) = \int_0^t W(t - \tau) \left( \partial_x (A_1(f)^2) \right)(\tau) d\tau;
\]
\[
A_1(f)(x, t) = W(t) f = S_t \ast f(x),
\]

where \( S_t = e^{it\omega(\xi)} \) with \( \omega(\xi) = \mu \xi^3 - \xi^5 \), or
\[
S_t(x) = \int e^{i(x\xi + t\omega(\xi))} d\xi.
\]

Motivated by the selection of a test function in [2] and [26], we choose an \( H^s(\mathbb{R}) \)-function \( f \) with
\[
\| f \|_{H^s} \sim 1 \quad \text{and} \quad \hat{f}(\xi) = \frac{1}{r^{1/2}} N^{-s} \chi_{[-r, r]}(|\xi| - N),
\]

where \( \chi_{[-r, r]} \) is the characteristic function of the interval \( [-r, r] \).
Thus this tells us that the major contribution to (5.1) is obviously gotten from the second
where

and

where

Setting

The key issue is to control \( \| A_3(f) \|_{H^s} \) from below. To proceed, we make the following estimates:

\[
A_1(f)(x, t) \sim r^{-1/2} N^{-s} \int_{|\xi| < r} e^{it\omega(\xi) + ix\xi} d\xi
\]

and

\[
A_2(f)(x, t) \sim F_1(x, t) - F_2(x, t)
\]

where

\[
F_1(x, t) = r^{-1} N^{-2s} \int_{\max_{j=1, 2} |\xi_j| < r} \frac{(\xi_1 + \xi_2) e^{ix(\xi_1 + \xi_2) + it(\omega(\xi_1) + \omega(\xi_2))}}{\omega(\xi_1) + \omega(\xi_2) - \omega(\xi_1 + \xi_2)} d\xi_1 d\xi_2
\]

and

\[
F_2(x, t) = r^{-1} N^{-2s} \int_{\max_{j=1, 2} |\xi_j| < r} \frac{(\xi_1 + \xi_2) e^{ix(\xi_1 + \xi_2) + it\omega(\xi_1 + \xi_2)}}{\omega(\xi_1) + \omega(\xi_2) - \omega(\xi_1 + \xi_2)} d\xi_1 d\xi_2.
\]

The contribution of \( F_1 \) to \( A_3(f) \) is comparable with

\[
r^{-3/2} N^{-3s} \int_{\max_{j=1, 2, 3} |\xi_j| < r} \frac{e^{ix(\xi_1 + \xi_2 + \xi_3) + it\omega(\xi_1 + \xi_2 + \xi_3)}}{Q_1(\xi_1, \xi_2, \xi_3)^{-1} Q_2(\xi_1, \xi_2, \xi_3)^{-1}} d\xi_1 d\xi_2 d\xi_3, \tag{5.1}
\]

where

\[
Q_1(\xi_1, \xi_2, \xi_3) := \frac{(\xi_1 + \xi_2 + \xi_3)(\xi_2 + \xi_3)}{\omega(\xi_1) + \omega(\xi_2) - \omega(\xi_1 + \xi_2 + \xi_3)}
\]

and

\[
Q_2(\xi_1, \xi_2, \xi_3) := \frac{e^{it(\omega(\xi_1) + \omega(\xi_2) + \omega(\xi_3) - \omega(\xi_1 + \xi_2 + \xi_3))} - 1}{\omega(\xi_1) + \omega(\xi_2) - \omega(\xi_1 + \xi_2 + \xi_3)}.
\]

Setting

\[
\theta = \omega(\xi_1) + \omega(\xi_2) + \omega(\xi_3) - \omega(\xi_1 + \xi_2 + \xi_3),
\]

we employ \( \omega(\xi) = -\xi^5 + \mu \xi^3 \) to get

\[
\theta = 5(\xi_1 + \xi_2)(\xi_1 + \xi_3)(\xi_2 + \xi_3) \left( \frac{\xi_1^2 + \xi_2^2 + \xi_3^2}{2} + \frac{(\xi_1 + \xi_2 + \xi_3)^2}{2} - \frac{3\mu}{5} \right).
\]

Thus

\[
|\theta| \sim N^5 \quad \text{or} \quad |\theta| \lesssim r N^3 \sim \frac{1}{(\log N)^2}.
\]

This tells us that the major contribution to (5.1) is obviously gotten from the second alternative, in which case we get

\[
G_1(x, t) = r^{-3/2} N^{-3s} \int_{\max_{j=1, 2, 3} |\xi_j| < r, |\theta| \lesssim r^2 N^3} \frac{e^{ix(\xi_1 + \xi_2 + \xi_3) + it\omega(\xi_1 + \xi_2 + \xi_3)}}{Q_1(\xi_1, \xi_2, \xi_3)^{-1}} d\xi_1 d\xi_2 d\xi_3
\]

with

\[
\| G_1 \|_{H^s} \sim r^{-1} N^{-3s} N^{-4} N^{1+s} r^2 \sim N^{-2s-9/2} (\log N)^{-1}.
\]
On the other hand, the contribution of $F_2$ to $A_3(f)$ is comparable with

$$G_2(x, t) = r^{-3/2} N^{-3s} \left| \int_{\max|\xi_j|<r \mid j=1,2,3} e^{i(x(\xi_1+\xi_2+\xi_3)+\omega(\xi_1+\xi_2+\xi_3))} \frac{e^{i(x(\xi_1+\xi_2+\xi_3)+\omega(\xi_1+\xi_2+\xi_3))}}{Q_1(\xi_1, \xi_2, \xi_3)Q_3(\xi_1, \xi_2, \xi_3)} d\xi_1d\xi_2d\xi_3, \right.$$

where

$$Q_3(\xi_1, \xi_2, \xi_3) := \frac{e^{i[(\omega(\xi_1)+\omega(\xi_2+\xi_3) - \omega(\xi_1+\xi_2+\xi_3) - 1)}}}{\omega(\xi_1) + \omega(\xi_2 + \xi_3) - \omega(\xi_1 + \xi_2 + \xi_3)}$$

and

$$\|G_2\|_{H^s} \lesssim r^{-3/2} N^{-2s-5} \left| \int_{\max|\xi_j|<r \mid j=1,2,3} e^{ix(\xi_1+\xi_2+\xi_3)} \frac{e^{ix(\xi_1+\xi_2+\xi_3)}}{\xi_2 + \xi_3 + N^{-4}} d\xi_1d\xi_2d\xi_3 \right|_{L^2_x(\mathbb{R})}$$

$$\lesssim r^{-1} N^{-2s-5} \left| \int_{\max|\xi_j|<r \mid j=1,2,3} (|\xi_2 + \xi_3| + N^{-4})^{-1} d\xi_2d\xi_3 \right|$$

$$\lesssim N^{-2s-5} \log N.$$

Consequently, we get

$$\frac{N^{-2s-9/2}}{\log N} \left( 1 - \left( \frac{\log N}{N^{1/4}} \right)^2 \right) \lesssim \|G_1\|_{H^s} - \|G_2\|_{H^s} \lesssim \|A_3(f)\|_{H^s} \lesssim 1$$

whence deriving $s \geq -9/4$ (via letting $N \to \infty$) – a contradiction to $s < -9/4$. This completes the proof of Theorem 5.1.

\[\square\]

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