Control power of quantum channels is not multiplicative

Tie-jun Wang\(^1\)\(^,\)\(^2\) and Shohini Ghose\(^1\)\(^,\)\(^2\)\(^,\)\(^3\)\(^,\)\(^*\)

\(^1\) School of Science, Beijing University of Posts and Telecommunications, Beijing 100876, People’s Republic of China
\(^2\) Department of Physics and Computer Science, Wilfrid Laurier University, Waterloo, Ontario N2L 3C5, Canada
\(^3\) Institute for Quantum Computing, University of Waterloo, Ontario N2L 3G1, Canada

\(^*\) Author to whom any correspondence should be addressed.
E-mail: sghose@wlu.ca

Keywords: quantum channels, control power, teleportation, entanglement, multiplicativity

Abstract

We construct a family of parallel channels for controlled quantum communication, and show that the control power of a quantum channel can become larger when it is combined with a zero control power channel. As a consequence, we demonstrate for the first time that control power is non-multiplicative. In contrast to previous results which indicated that increasing control power requires the consumption of channel capacity or the consumption of additional non-local three-qudit entangled resources, in our construction the quantum capacity of the tensor product of two channels remain unchanged and it does not consume non-local three-qudit entangled resources. We show that this lossless channel capacity method is quite generically applicable: every channel can be embedded into our construction, and control power is increased whenever the given channel configuration matches the effect of local operations. Our protocol not only provides a feasible method for achieving higher control power for information transmission with less entangled resources, but also provides new understanding of the resource theory of quantum channels.

1. Introduction

Quantum Shannon theory \([1, 2]\) allows us to analyze the resources required to accomplish quantum information tasks effectively. Static resources contained in quantum states as well as dynamic resources due to quantum measurements have been considered in the literature \([3, 4]\). In multi-party quantum communication, due to the different number and roles of participants, it is necessary to use various quantities to analyze and quantify resources. These quantities must be comprehensively evaluated to understand and control the efficiency and fidelity of the communication, and to compare the performance of different resources for the same quantum task. In this paper, we analyze the effect of parallel channels on information transmission from the perspective of fidelity and control power.

A quantum channel is mathematically described by an isometric map from the input Hilbert space to the combined Hilbert space of the output and the environment \([5]\). The maps are linear and multiplicative \([6]\). Some corresponding quantities used to describe the channel characteristics also satisfy the multiplicative relationship. For example, the maximal output \(p\)-norm is multiplicative for \(p\) approaching 1 \([7]\). Quantum error correction can be analyzed using the multiplicative nature of quantum channels \([8, 9]\). Also, the multiplicative domain appears to be useful in the study of private algebras and complementary quantum channels \([10]\). On the other hand, some channel quantities will have dramatically different behaviors, and display fundamentally quantum synergies not present classically. For example, there exist quantum channels with non-multiplicative maximal output \(p\)-norms for all \(p > 1\) \([7]\). These counterexamples reflect the complexity of quantum channels and the ongoing challenge of fully characterizing their use for implementing quantum tasks.

Control power \([11]\) is an important figure of merit for quantifying quantum channels in quantum controlled communication schemes \([12–31]\). It is used to analyze the authority of a controller who can permit or restrict the successful quantum state transfer from the sender to the receiver. The control power \(P\)
is defined as the difference between the conditioned fidelity $f_{\text{CQT}}$ (with the controller’s permission) and the non-conditioned fidelity $f_{\text{NC}}$ (without the controller’s permission) [24]. In early studies, $P$ was theoretically calculated in perfect controlled teleportation (CT) schemes [11–14, 18, 27], in which the $f_{\text{CQT}}$ with the controller’s permission is equal to 1. The control power $P$ has been explored in a variety of two-dimensional non-optimal single-channel quantum schemes [18, 24–26]. In 2016, Kabgyun et al calculated the minimal control power of a general CT scheme with tripartite entangled pure states [24]. In 2019, Barasinski et al defined tight upper and lower bounds on the fidelity and the control power for given three-qubit pure and mixed states [25]. The lower bound on the control power is $P \geqslant 1/3$ for a perfect CT scheme [18, 25], and much effort has been devoted to improve control power using high-dimensional systems [18, 19]. For example, with a $(2d - 1)$-dimensional GHZ-class state channel, the control power can be improved from $1 - \frac{d}{2d+1}$ to $1 - \frac{d}{2d+1}$, to teleport a $d$-dimensional qudit [19]. In another GHZ-class quantum channel in which the dimension of the control qudit is larger than other qudits, the control power can be improved to $\frac{2d+1}{2d+3}$ [18, 19]. So far, there are two approaches to improving the control power for single-channel schemes. One approach is to increase the channel capacity between the sender and the receiver, and the other is to increase the dimension of the control qudit. This assumes that in order to improve the control power, the cost is either in channel capacity, or in non-local tripartite entangled resources.

In this paper, we construct a family of parallel channels, such that the control power of a quantum channel increases when combined with a zero control power channel. Hence, we can conclude for the first time that control power is non-multiplicative. In contrast to previous results, which indicated that increasing control power means increasing channel capacity or additional nonlocal entangled resources involving the control qudit, in our construction the quantum capacity of the tensor product of two channels remain unchanged and it does not require additional nonlocal three-party entangled resources. Our method is quite general so that any channel can be included in our construction. We show that control power increases whenever the given channel configuration matches the effect of local operations. Hence our scheme can achieve higher control power with less three-party entangled resources. Our results provide a new perspective to fundamental problems in quantum Shannon theory while providing effective tools to address concrete problems.

The outline of the paper is as follows. In section 2 we briefly review the definition of control power. We discuss the control power for teleporting 2 independent qubits by using two parallel states composed of an imperfect three-body pure quantum state and a Bell state in section 2.1. In section 2.2, we generalize the above result to the case of optimal high-dimensional parallel channels for teleporting 2 independent qudits, and improve the control power by using local operations. We demonstrated the non-multiplicativity of control power in this type of combined channel. Finally, in section 3, we present a discussion and conclusion.

2. Improving the control power by LOCC

First, we briefly review how control power is calculated [11, 25]. The control power $P$ is defined as the difference between two different fidelities, the conditioned fidelity $f_{\text{CQT}}$ and the non-conditioned fidelity $f_{\text{NC}}$. That is, $P = f_{\text{CQT}} - f_{\text{NC}}$. Here, $f_{\text{CQT}}$ is the so-called conditioned fidelity, which is the scheme fidelity with the controller’s complete involvement, and $f_{\text{NC}}$ is the non-conditioned fidelity, which is the scheme fidelity without the controller’s permission. In the CT scheme, $f_{\text{CQT}}$ is the fidelity averaged over all input states and is defined to be

$$f_{\text{CQT}} = \frac{\int_{x_0} \cdots \int_{x_n} F(\rho_i, \sigma) \, dx_0 \cdots dx_n}{\int_{x_0} \cdots \int_{x_n} dx_0 \cdots dx_n},$$

(1)

where, the state $|\psi_i\rangle = \sum_{|i|} |x_i\rangle i$ is the initial state to be transmitted ($x_i$ is the normalization coefficient, and $\sum_{|i|} |x_i|^2 = 1$), $\rho$ is the density matrix of the final state obtained by the receiver with the full cooperation of the controller, and $F(\rho_i, \sigma) = \langle \psi_i | \rho | \psi_i \rangle$. By integrating over all input states, one can obtain the average fidelity [19]. The corresponding $f_{\text{NC}}$ is

$$f_{\text{NC}} = \frac{\int_{x_0} \cdots \int_{x_n} F(\rho_{BC}, \sigma) \, dx_0 \cdots dx_n}{\int_{x_0} \cdots \int_{x_n} dx_0 \cdots dx_n},$$

(2)

where the density matrix $\rho_{BC}$ of the receiver’s qudit can be computed by $\rho_{BC} = \text{tr}_C(\rho_{BC})$, and $\rho_{BC}$ is the density matrix of the system shared by the receiver and controller after the Bell-state measurements performed by the sender. In the perfect CT scheme, $P = 1 - f_{\text{NC}}$.

The fidelity is not a metric on the space of density matrices, but it can be used to define the Bures metric on this space. In a cascaded quantum communication scheme, the overall transmission fidelity $f$ of the channel is equal to the product of the implementation fidelity $f_i$ of each step in the scheme, $f = \prod_i f_i$ and...
thus it is multiplicative. For two parallel perfect channels, if $f_i$ ($i = 1, 2$) is the non-conditioned fidelity of an optimal channel without the controller’s cooperation, we can ask whether the overall control power is equal to $1 - f_1f_2$. If the answer is no, we can say that control power is non-multiplicative. Below we will illustrate this property through a counterexample to multiplicativity.

2.1. The control power of non-optimal channels

In the quantum communication schemes, the quantum channel is the medium and means of quantum information transmission. We analyze the control power $P$ for teleporting 2 independent qubits with a combined channel composed of two different parallel states and bilateral CNOT operations. One is a non-optimal state with a non-zero control power, and the other one is a Bell-state channel with zero control power. The structure of the communication network is shown in figure 1. Assume that there are five particles $A_1$, $A_2$, $B_1$, $B_2$, and $C$. The particles $A_1$ and $A_2$ belong to Alice, the particles $B_1$ and $B_2$ belong to Bob who is far away from Alice, and the particle $C$ belongs to Charlie who is the controller of the communication network.

The state of the three-particle $A_1B_1C$ system is a two-dimensional three-qubit pure state described as

$$|\varphi\rangle = (t_0|00\rangle + t_1|11\rangle + t_2|01\rangle + t_3|10\rangle)_{A_1B_1}|0\rangle_C$$

(3)

Here, $0 \leq \theta_1(\theta_2) \leq 2\pi$, $\Sigma_i|t_i|^2 + \Sigma_j|t_j|^2 = 1$ ($i, j = 0, 1, 2, 3$). We note that any two-dimensional three-qubit pure state can be rewritten in this form with suitable bases. For simplicity, the parameters $t_i$ and $t_j'$ are taken to be real numbers greater than 0, and $t_0, t_1 > t_2, t_3, t_0', t_1', t_2', t_3'$. In order to make the control power of the channel as large as possible, we further assume that $\Sigma_i|t_i|^2 = \Sigma_j|t_j'|^2 = 1/2$ and $t_0', t_1' > t_2', t_3'$. The conditioned teleportation fidelity $f_{CQT}^C$ of the state $|\varphi\rangle$ for teleporting an independent qubit is equal to

$$f_{CQT}^C = \frac{1}{2} \left[ \frac{1}{3} \left( 1 + \frac{2(t_0 + t_1)^2}{3} \right) + \frac{1}{3} \left( 1 + \frac{2(t_0' + t_1')^2}{3} \right) \right].$$

(4)

The non-conditioned teleportation fidelity $f_{NC}^C$ is

$$f_{NC}^C = \frac{1}{2} \left[ \frac{1}{3} \left( 1 + \frac{2(t_0 + t_1)^2}{3} \right) + \frac{1}{3} \left( 1 + \frac{2(t_0'^2 + 2\cos \theta_1 t_0' t_1' + t_1'^2)}{3} \right) \right].$$

(5)

The control power of the $|\varphi\rangle$-state is

$$P_0 = \frac{2}{3} (1 - \cos \theta_1)t_0't_1'.$$

(6)

When $t_0 = t_1 = t_0' = t_1' = \frac{1}{2}$ and $\theta_1 = \pi$, the form of the $|\varphi\rangle_{A_1B_1C}$ state is $\frac{1}{\sqrt{2}}(|0\rangle_C|\phi^+\rangle_{A_1B_1} + |1\rangle_C|\phi^-\rangle_{A_1B_1})$, the corresponding $f_{NC}^C = \frac{5}{12}$ and $P = \frac{1}{3}$ and this is the maximal control power for the two-dimensional case $|\phi^\pm\rangle_{A_1B_1} = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)_{A_1B_1}$.
The form of a Bell-state in the two-qubit system $A_2B_2$ is $|\phi^+\rangle_{A_2B_2}$. The teleportation fidelity of this Bell-state is $f_2 = 1$, and the corresponding $P = 0$. If we do not perform any local operations on these two states, the overall control power of these two parallel states for teleporting 2 independent qubits is equal to $1 - f_{NC}^2 = P_0$.

Now, we calculate the improved control power of a combined state composed of a $|\varphi\rangle_{A_1B_1C}$-state and a Bell-state $|\phi^+\rangle_{A_2B_2}$ with bilateral CNOT operations. Before Alice performs a Bell-state measurement on her qubits, Alice and Bob perform the bilateral local CNOT gates $C^{A_1}_{B_1}$ (on qubit $A_2A_1$) and $C^{B_1}_{B_2}$ (on qubits $B_2B_1$), respectively, as shown in Figure 1. Here, the upper subscript represents the control qubit and the lower subscript represents the target qubit of CNOT gates. The operation represented by a CNOT gate is that, when the state of the control qubit is $|0\rangle$, the target qubit remains unchanged; when the state of control qubit is $|1\rangle$, the target qubit is flipped. The combined state becomes

$$C^{A_1}_{B_1}C^{B_1}_{B_2}|\varphi\rangle_{A_1B_1C}|\phi^+\rangle_{A_2B_2}$$

$$= \frac{1}{\sqrt{2}}[(t_0|00\rangle + t_1|11\rangle + t_2|01\rangle + t_3|10\rangle)|000\rangle + (t_0|11\rangle$$

$$+ t_1|00\rangle + t_2|10\rangle + t_3|01\rangle)|011\rangle + (t_0|00\rangle + e^{\phi_1}t_1^*|11\rangle$$

$$+ t_2^*|01\rangle + e^{\phi_2}t_3^*|10\rangle)|100\rangle + (t_1|11\rangle + e^{\phi_1}t_2^*|00\rangle$$

$$+ t_3^*|10\rangle + e^{\phi_2}t_1^*|01\rangle)|111\rangle]_{A_1B_1C}\Delta_{t_0t_1t_2t_3}.$$ (7)

With the controller’s permission, the conditioned teleportation fidelity of the combined state is

$$f_{CQT} = \frac{1}{9}[2 + 3(t_0 + t_1)^2 + 3(t_0^2 + t_1^2) + 2(t_0t_1 + t_2t_3) + 2(t_0^2t_1 + t_2^2t_3)].$$ (8)

The non-conditioned teleportation fidelity $f_{NC}$ is

$$f_{NC} = \frac{1}{9}[2 + 3(t_0 + t_1)^2 + 3(t_0^2 + t_1^2) + 2(t_0t_1 + t_2t_3) + 2(t_0^2t_1 + t_2^2t_3)] + 2(\cos \theta_1 t_0^2 t_1^* + \cos \theta_2 t_2^2 t_3^*).$$ (9)

The control power of the $|\varphi\rangle \otimes |\phi^+\rangle$ state is improved to

$$P = \frac{8}{9}(1 - \cos \theta_1 t_0^2 t_1^*) + \frac{2}{9}(1 - \cos \theta_2) t_2^2 t_3^*.$$ (10)

After bilateral local CNOT operations, the increase in the control power of the parallel states is

$$\Delta P = P - P_0$$

$$= \frac{2(1 - \cos \theta_1) t_0^2 t_1^* + 2(1 - \cos \theta_2) t_2^2 t_3^*}{9}.$$ (11)

As the parameters $t$ are greater than 0 and $-1 \leq \cos \theta_1 \leq 1, \frac{1}{2} \geq \Delta P \geq 0$. Thus, in most cases, there will be a significant increase in control power. When $t_0 = t_1 = t'_2 = t'_3 = \frac{1}{2}$ and $\theta_1 = \pi$, the corresponding $\Delta P = \frac{1}{9}$. In the case of $\theta_1 = \theta_2 = 0$, the corresponding $\Delta P = 0$ with the $P_0 = P = 0$.

In order to clearly illustrate the performance of the combined state, we numerically calculated the conditioned teleportation fidelity $f_{CQT}$ and the change $\Delta P$ in control power as a function of the parameters $t_0$ and $\gamma = \frac{\Delta P}{9}$. The results are shown in Figure 2. When $t_0 = 0$ or $t_0 = \frac{1}{\sqrt{2}}$, the state $|\varphi\rangle_{A_1B_1C}$ collapses to the direct product state of three qubits. The corresponding conditioned fidelity $f_{CQT}$ is not 0 but it is already lower than $2/3$ (the control power is meaningful when $f_{CQT} \geq \frac{2}{3}$), and the corresponding control power $P = 0$ and $\Delta P = 0$. When $t_0 \neq 0$ and $t_0 \neq \frac{1}{\sqrt{2}}$, $\Delta P > 0$. Note that the maximum values of $f_{CQT}$ and $\Delta P$ occur at the same point $t_0 = t_1$ for different $\gamma$. In the case of $\gamma < 1$, even if the maximum value of $f_{CQT}$ is less than 1, the $\Delta P$ can reach its maximum value $\frac{1}{3} \Delta P$. Only in the case of the optimal state, when $t_0 = t_1 = t'_2 = t'_3 = \frac{1}{2}$, do $f_{CQT}$ and $\Delta P$ reach the maximum value 1 and $1/9$, respectively, at the same time.

By comparing equations (3) and (11), we note that the increase of control power mainly comes from the second part of the state $|\varphi\rangle$, that is, the part with the phase terms $e^{\phi_1}t_1^*$ and $e^{\phi_2}t_3^*$. When the other parameters of the state $|\varphi\rangle$ remain unchanged, from equation (11), one can see that the $\Delta P$ is monotonic with $\theta_1$ and $\theta_2$ as shown in Figure 3 ($\theta_1, \theta_2 \in [0, \pi]$). When $\theta_1 = \theta_2 = 0$, $\Delta P = 0$, and when $\theta_1 = \theta_2 = \pi$, $\Delta P$ reaches...
Figure 2. The conditioned teleportation fidelity $f_{\text{CQT}}$ and the change $\Delta P$ in control power as a function of the parameter $t_0$ and $\gamma$ in the case of $t_i = t_i'$ ($i = 0, 1, 2, 3$), $\theta_1 = \theta_2 = \pi$, and $t_{0(3)} = \sqrt{\gamma t_0}$ ($1 \geq \gamma \geq 0$).

Figure 3. $\Delta P$ as a function of the parameter $\theta_1$ and $\theta_2$ for (a) $\gamma = 0$; (b) $\gamma = 0.2$; (c) $\gamma = 0.5$; and (d) $\gamma = 1$ $[t_i = t_i' (i = 0, 1, 2, 3), t_0 = t_1$, and $t_{0(3)} = \sqrt{\gamma t_0(1)}]$. The maximum value. Therefore, $\Delta P$ is caused by the part of state $|\varphi\rangle$ with the phase terms. Even if there is no phase term, such as in the state, $|\varphi_b\rangle = (t_0|00\rangle + t_1|11\rangle)_{A_1B_1C} + (t_0'|01\rangle + t_1'|10\rangle)_{A_1B_1C}$, (12)

by choosing appropriate local operations to enhance the association between parallel states, the control power of parallel states can be increased. For example, with Hadamard operations on qubits $A_1B_1$, the state $|\varphi'_b\rangle$ becomes

$$|\varphi'_b\rangle = \frac{1}{2} \left\{ [(t_0 + t_1)(|00\rangle + |11\rangle) + (t_0 - t_1)(|01\rangle + |10\rangle)]_{A_1B_1C} + [(t_0' + t_1')(|00\rangle - |11\rangle) - (t_0' - t_1')(|01\rangle - |10\rangle)]_{A_1B_1C} \right\}.$$

(13)

which is in a similar form as $|\varphi\rangle$. Then, by using bilateral CNOT operations, the control power of the state $|\varphi'_b\rangle$ can be increased when used in parallel with a Bell-state. Our results show that except for the case of $P = 0$, this lossless channel capacity method can generally improve the control power. As a consequence, the control power of a $|\varphi\rangle_{A_1B_1C}$ state will become larger than the previous control capability, when combined with a Bell-state (zero control power), and suitable local operations by Alice and Bob. That is,

$$\Delta P = P_t - P_0 \geq 0,$$

(14)
where $P_0$ is the control power of the combined channel achieved without any local operation, and $P_1$ is the control power of the same combined state achieved with suitable local operations. When $P_0 = P_1 = 0$, the equal sign in this formula holds. In the case of optimal states, the overall control power is

$$P \geq 1 - f_{NC1}.$$  

When $P = 0$ the equal sign holds. Thus, the non-multiplicativity of control power is proved.

If the state of qubits $A_2B_2$ is not a standard Bell state, but a partially entangled state $|\psi\rangle_{A_2B_2} = \alpha |00\rangle_{A_2B_2} + \beta |11\rangle_{A_2B_2}$, one can easily see that the control power of the combined state $|\varphi\rangle \otimes |\psi\rangle$ can also be improved with bilateral local CNOT gates. Next, we extend our analysis to the high-dimensional case and explain the reasons for this non-multiplicativity.

### 2.2. The control power of optimal high-dimensional parallel channels

We now generalize the above result to the case of optimal high-dimensional parallel channels for teleporting 2 independent qudits, and achieve the maximum value of the control power improved by using LOCC. For a combined channel composed of a $d$-dimensional standard GHZ state and a $d$-dimensional Bell state, the total state can be rewritten as,

$$\frac{1}{\sqrt{d}}(|000\rangle + |111\rangle + \cdots + |d-1, d-1, d-1\rangle)_{A_1B_1C} \otimes \frac{1}{\sqrt{d}}(|00\rangle + |11\rangle + \cdots + |d-1, d-1\rangle)_{A_2B_2} = \frac{1}{\sqrt{d}}|B(0, 0)\rangle_{A_2B_2} \sum_{u=0}^{d-1} \sum_{l=0}^{d-1} |\hat{u}\rangle_C |B(u, 0)\rangle_{A_1B_1},$$  

(16)

where $|m\rangle = \frac{1}{\sqrt{d}}(\sum_{u=0}^{d-1} e^{2\pi i um}|\hat{u}\rangle)$, $m = 0, 1, 2, \ldots, d-1$. $|B(u, v)\rangle = \frac{1}{\sqrt{d}}\sum_{l=0}^{d-1} e^{2\pi i lu}|A\rangle \otimes |l+v\rangle_B$. Here, $v = 0, 1, 2, \ldots, d-1$ (where $l + v$ have a maximum value modulo $d$) denotes the bit information of the two-particle state and $u = 0, 1, 2, \ldots, d-1$ represents the relative phase information. When qudit $C$ is traced out, the reduced state of the remaining system is a mixture of $d$ different pure entangled states consisting of 2 parallel and independent Bell states. In these Bell states, only one can perform perfect teleportation, that is $|B(0, 0)\rangle_{A_1B_1}|B(0, 0)\rangle_{A_2B_2}$, with the probability of $\frac{1}{d}$. For the remaining $(d-1)$ combined channels, if the information of qudit $C$ is not known, there will be a phase error in the channel of qudits $A_1B_1$ and the fidelity for information transmission is $\frac{1}{d^{1/2}}$. Therefore, the total average non-conditional teleportation fidelity of this combined state is $f_{NC} = \frac{1}{d^{1/2}}$.

With the high-dimensional CF gates $CF_{A_1}^d$ on qudits $A_1A_2$, and $CF_{B_1}^d$ on qudits $B_1B_2$, as shown in figure 5(a), the state of the combined state will become,

$$\frac{1}{\sqrt{d}}\sum_{u=0}^{d-1} |\hat{u}\rangle_C |B(u, 0)\rangle_{A_1B_1} |B(u, 0)\rangle_{A_2B_2}.$$  

(17)

Here, the upper subscript represents the control qudit and the lower subscript represents the target qudit of CF gates. The operation represented by a CF gate is that when the state of the control qudit is $|\hat{u}\rangle$, the state of the target qudit is changed as $|i\rangle \rightarrow |j - i\rangle$. From equation (17), one can see that the role of the bilateral high-dimensional CF gates used here is to copy the phase information $u$ of the GHZ-state into the parallel $|B(0, 0)\rangle$ state. Except for the state $|B(0, 0)\rangle_{A_1B_1}|B(0, 0)\rangle_{A_2B_2}$, in the order $(d-1)$ combined states, there are phase errors in both states $|B(u, 0)\rangle_{A_1B_1}|B(u, 0)\rangle_{A_2B_2}$ without Charlie’s cooperation, and the fidelity for information transmission is $\frac{1}{d^{1/2}}$. Therefore, the total average teleportation fidelity of this combined state can be changed from $\frac{1}{d^{1/2}}$ to $\frac{d-1}{d+1}$ without Charlie’s cooperation in the communication process.

Correspondingly, the control power changes from $P_1 = \frac{d-1}{d+1}$ to $P_2 = \frac{d^2-d-2}{d+1}$. If the dimension of the standard GHZ-state $d_1$ is bigger than the dimension of the Bell-state $d_2$, after the asymmetric CF gate, the control power of the combined state is $P_2(d_1, d_2) = \left[1 - \frac{d^2+d-2}{(d+1)(d+1)}\right] < P_2(d_1, d_1)$.

If we increase the dimension of qudit $C$ from $d$ to $d^2$, the three-party entangled state becomes more complicated than the standard GHZ state. It can be written as a direct product of a superposition of $d^2$ different Bell state and the states of qudits $C_1C_2$,

$$|GHZ^2\rangle = \frac{1}{d} \sum_{i=0}^{d-1} |\hat{u}\rangle_C |\hat{v}\rangle_{C_2} |B(i, j)\rangle_{A_1B_1}.$$  

(18)
Here the control qudit of Charlie is constructed using two d-dimensional qudits \( C_1 \) and \( C_2 \). The total average non-conditional teleportation fidelity of this state is \( \frac{1}{d} \).

The combined state \( \ket{\text{GHZ}}_T \ket{A_1 B_1 C_1} \otimes \ket{B(0,0)}_B \) of applying the high-dimensional CF gates \( U^A_{\lambda_i} \) on qudits \( A_1 A_2 \) and \( U^B_{\lambda_i} \) on qudits \( B_1 B_2 \), will become,

\[
\frac{1}{d} \sum_{\mu=0}^{d-1} |\mu\rangle_{C_1} |v\rangle_{C_2} \langle B(u,v)| A_1 B_1 \rangle \langle B(u,0)| A_2 B_2 \rangle.
\]  

When qudit \( C \) is traced out, the remaining system is in a mixed state composed of \( d^2 \) different pure entangled states each consisting of two parallel and independent Bell states. In these Bell states, only one can perform perfect teleportation, that is \( \ket{B(0,0)}_A \langle B(0,0)|_B \) with a probability of \( \frac{1}{d} \). Of the remaining combined states, some have errors on two parallel states simultaneously. These states have the form \( \ket{B(u,0)}_A \langle B(0,0)|_B \) or \( \ket{B(u,v)}_A \langle B(u,0)|_B \) (\( u \neq 0 \)). The corresponding average fidelity for teleporting 2 independent qudits is \( \frac{1}{d+1} \). We will call this type of state a \( T_1 \) state. The number of \( T_1 \) states is \( k = (d-1) + (d-1)^2 \). The other \( (d-1) \) combined states have errors on only one state. These states have the form \( \ket{B(0,0)}_A \langle B(u,0)|_B \) (\( v \neq 0 \)). The corresponding average teleportation fidelity is \( \frac{1}{d+1} \). We call this type of state a \( T_2 \) state. Therefore, the average non-conditional teleportation fidelity of this \( \text{GHZ} - \text{Bell} \) state can be changed from \( \frac{1}{d} \) to \( \frac{1}{d(d+1)} \) if Charlie does not participate in the communication process. Correspondingly, the control power changes from \( P_3 = \frac{d-1}{d} \) to \( P_4 = 1 - \frac{1+3d}{d(d+1)^2} \).

The fidelity of the two different combined states described above can be summed up in one formula,

\[
f = \frac{1}{N} + \frac{N-k-1}{d+1} + \frac{k}{N(d+1)^2}.
\]  

Here, \( N \) is the dimension of the control qudit and \( k \) is the number of \( T_1 \) states.

To evaluate the performance of different combined states after bilateral local CF operations, we numerically calculated the control power \( P \) using different quantum states. The results are shown in figure 4. Here, \( P_1 = 1 - \frac{1}{d} \) is the control power of the single \( d \)-dimensional standard GHZ state, and \( P_2 = 1 - \frac{3d-1}{d(d+1)^2} \) is the improved control power of the \( d \)-dimensional combined GHZ-Bell state with bilateral local CF operations. \( P_3 = 1 - \frac{1}{d(d+1)^2} \) is the control power of two parallel \( d \)-dimensional standard GHZ state channels used to teleport 2 parallel \( d \)-dimensional qudits without any additional local operations. It is smaller than \( P_4 = 1 - \frac{1+3d}{d(d+1)^2} \), which is the control power of the \( \text{GHZ}' - \text{Bell} \) type state with a \( d^2 \)-dimensional control qudit \( C \).

It should be noted that it takes 2 \( d \)-dimensional nonlocal Bell states to construct a \( \text{GHZ}' \)-state. For example, when the four-dimensional qudit \( C \) is encoded by 2 qudits, the quantum state \( \ket{\text{GHZ}}_T \ket{A_1 B_1 C} \) can be

![Figure 4](image_url). Control power of four different high-dimensional channels as a function of the dimension parameter \( d \) (\( 2 \leq d \leq 12 \)).
rewritten in the following two-dimensional form,

\[
|\text{GHZ}'\rangle_{A_1B_1C_1} = |\phi^+\rangle_{A_1C_1}|\psi^+\rangle_{B_1C_1}
\]

\[
= \frac{1}{2}(|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle)_{A_1B_1C_1}
\]

\[
= \frac{1}{2} \left(|\phi^+\rangle_{C_1C_2}|\phi^+\rangle_{A_1B_1} + |\phi^-\rangle_{C_1C_2}|\phi^-\rangle_{A_1B_1}
\right.

\[
+ |\psi^+\rangle_{C_1C_2}|\psi^+\rangle_{A_1B_1} + |\psi^-\rangle_{C_1C_2}|\psi^-\rangle_{A_1B_1}\right),
\]

where \( |\phi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle) \). Constructing a standard \( d \)-dimensional GHZ state should consume \( 2d \)-dimensional nonlocal Bell states. Therefore it should take the same nonlocal entanglement resources (\( 3d \)-dimensional nonlocal Bell states) to construct a combined \( \text{GHZ}' \)--Bell state as shown in figure 5(a) or a combined \( \text{GHZ} \)--Bell state as shown in figure 5(b). However, with local operations, the former has a larger control power with the same quantum capacity. In particular, using only bilateral local CF operations, the control power of the combined \( \text{GHZ}' \)--Bell type state \((P_3)\) is bigger than the combined \( \text{GHZ}--\text{GHZ} \) type state shown in figure 5(c) \((P_1)\). Thus, from the view of control power, we can choose a more economical state to transfer information and build quantum networks. The non-multiplicativity of the average teleportation fidelities allows us to enhance the control power with suitable local operations and a zero control power state.

From equations (17) and (19), one can see that the role of the high-dimensional CF gates used here is to copy the phase information of the GHZ-state into the \( |B(0, 0)\rangle \) state (the bit information \( v \) cannot be copied to the parallel \( |B(0, 0)\rangle \) state). It will thereby increase the complexity of the combined state, reduce the non-conditioned fidelity \( f_{NC} \), and extract additional control power of the GHZ-state. Although the bit-flip error cannot be distilled using bilateral CF gates operations, by choosing appropriate local operations the bit-flip error can be changed into phase error, and the control power of parallel states can be increased with the parallel zero control power Bell-state.

The improvement of control power is partly due to the superadditivity \([32–37]\) of quantum capacity of parallel states. It should be easy to evaluate the number of qubits being transferred for each teleportation fidelity. One can also achieve a larger increment of control power using the number of correct qubits teleported to Bob to define control power. This increase in the control power is a reflection of superadditivity of quantum capacity related to the controller. We note that the bilateral local operations in our scheme are similar to the encoding and decoding processes of some error-correction codes, and hence the fidelity for noisy channels can generally be formulated in a way that is zero-error in the asymptotic limit where error correction can be utilized.

3. Discussion and summary

In the previous section, we discussed the control power for teleporting 2 independent qudits by using a combined state which is composed of a three-body pure state and a Bell state. We demonstrated the non-multiplicativity of control power in this type of combined state. In fact, non-multiplicativity of control power also holds for two parallel GHZ-type states or two three-body pure states.

In addition to the Bell state, there are other entangled states that can be used to improve control. For example, by applying a Hadamard operation on each qubit, the two-dimensional standard GHZ-state can be changed into \( |\text{GHZx}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |011\rangle + |101\rangle + |110\rangle) = \frac{1}{\sqrt{2}}(|\phi^+\rangle|0\rangle + |\psi^+\rangle|1\rangle) \). The non-conditioned fidelity of a single two-dimensional GHZx state is \( f_{NC} = \frac{1}{2} \), and the control power is \( P = \frac{1}{2} \). Without any local operations on the combined GHZ--GHZx states, the overall control power of...
these two parallel states for teleporting 2 independent qubits is $1 - \left(\frac{2}{3}\right)^2 = \frac{5}{9}$. With bilateral CNOT gates, the overall control power of the parallel GHZ–GHZ states for teleporting 2 independent qubits is equal to $\frac{1}{3}$, and the increment is $\Delta P = \frac{1}{3}$. This result can be generalized to the optimal high-dimensional parallel GHZ–GHZ state case. With the bilateral CE gates, the overall control power of the parallel $d$-dimensional GHZ–GHZ states for teleporting 2 independent $d$-dimensional qudits is equal to $1 - \left(\frac{1}{d+d+1}\right)^2 > 1 - \left(\frac{1}{d+1}\right)^2$. The increment $\Delta P_{GG} = \frac{d(d+1)}{d+1}$, which shows the non-multicativity. However, we also note that, with the bilateral CE gates, the increment of control power $\Delta P_{GG}$ of the GHZ–GHZ state is lower than the GHZ–Bell state as $\Delta P_{GB} = \frac{d(d+1)}{d+1} = \Delta P_{GG}$ for $d > 2$.

Another question that we consider is whether Bob can do some local operations to increase fidelity and hence reduce Charlie’s control power. For example, here we need bilateral operations $C_{A_1}^B$ and $C_{B_1}^A$. If Bob does not perform the CNOT gate $C_{B_1}^A$ on his qubits, after the Bell-state measurement, the state of the whole system will evolve into

$$
\frac{1}{\sqrt{2}} \left[ |0\rangle_C (c|0\rangle + C_{B_1}^{B_2} d|1\rangle)_{B_2} (a|0\rangle + b|1\rangle)_{B_2} \\
+ |1\rangle_C (c|0\rangle - C_{B_1}^{B_2} d|1\rangle)_{B_2} (a|0\rangle - b|1\rangle)_{B_2} \right],
$$

(22)

where $a$, $b$, $c$, and $d$ are the normalization coefficients of the transmitted states. Note that once Alice performs the CNOT gate $C_{A_1}^B$ on her qubits, Bob must perform the same operation to recover the correct form of the teleported information. Otherwise, additional errors will occur and the non-conditional fidelity will be reduced, which can introduce a higher control power.

To summarize, we have identified a family of parallel states, and demonstrated that the control power of a quantum state can become larger when it is combined with a zero control power state. This proved that control power is non-multicative. Instead of changing the channel capacity or increasing the coding dimension of the control qudit, we can improve the control power of the parallel quantum states simply by applying suitable local operations, and making sure the quantum capacity of the tensor product of two states remain unchanged. Our protocol provides a practical method for improving control power for information transmission without using additional entangled resources that include the control qudit. Hence this approach allows us to design more economical channels to transfer information and build a quantum network.

A three-dimensional three-particle GHZ state has recently been experimentally created using photons [38, 39]. The first experimental results on qutrit teleportation have also been recently published [40]. Therefore, the quantum state and the main elements of our scheme (high-dimensional Bell-state measurements) in a three-dimensional control teleportation setting have been realized experimentally. We expect that further laboratory implementations of high-dimensional operations will be published in the near future. Our results are thus of relevance both from a theoretical and an applied perspective.

Acknowledgments

This work is supported in part by the Ministry of Science and Technology (MOST) of China under Grant 2016YFA0301304, in part by the National Natural Science Foundation of China under Grant 61671083, in part by the Fundamental Research Funds for the Central Universities of China under Grant 2019XD-A02, and in part by the Fund of State Key Laboratory of Information Photonics and Optical Communications, Beijing University of Posts and Telecommunications, China, and the Natural Sciences and Engineering Council of Canada. Wilfrid Laurier University is located in the traditional territory of the Neutral, Anishnawbe and Haudenosaunee peoples. We thank them for allowing us to conduct this research on their land.

References

[1] Nielsen M A and Chuang I L 2000 Quantum Computation and Quantum Information (Cambridge: Cambridge University Press)
[2] Shannon C E 1948 Bell Syst. Tech. J. 27 379—423
[3] Li L, Bu K F and Liu Z W 2020 Phys. Rev. A 101 022335
[4] Liu Y, He B and Chen P 2020 J. Phys.: Conf. Ser. 1600 012035
[5] Chen K, Winter A, Zou X B and Guo G C 2009 Phys. Rev. Lett. 103 120501
[6] Rahaman M 2017 J. Phys. A: Math. Theor. 50 345302
[7] Amosov G G, Holevo A S and Werner R F 2000 Probl. Peredachi Inf. 36 25–34
[8] Choi M-D, Johnston N and Kribs D W 2009 J. Phys. A: Math. Theor. 42 245303
[9] Johnston N and Kribs D W 2011 Proc. Am. Math. Soc. 139 627
[10] Levick J 2016 An uncertainty principle for completely positive maps (arXiv:1611.06352v1 [quant-ph])
[11] Li X H and Ghose S 2014 Phys. Rev. A 90 052305
[12] Hillery M, Bužek V and Berthiaume A 1999 Phys. Rev. A 59 1829
[13] Karlsson A, Koashi M and Imoto N 1999 Phys. Rev. A 59 162
[14] Xiao L, Long G L, Deng F G and Pan J W 2004 Phys. Rev. A 69 052307
[15] Biham E, Huttner B and Mor T 1996 Phys. Rev. A 54 2651
[16] Townsend P D 1997 Nature 385 47
[17] Aoun B and Tariff M 2004 Quantum networking (arXiv:0401076[quant-ph])
[18] Li X H and Ghose S 2015 Phys. Rev. A 91 012320
[19] Wang T J, Yang G Q and Wang C 2020 Phys. Rev. A 101 012323
[20] Deng F G, Li C Y, Li Y S, Zhou H Y and Wang Y 2005 Phys. Rev. A 72 022338
[21] Zhang W, Liu Y m, Zhang Z-j and Cheung C-Y 2010 Opt. Commun. 283 628–32
[22] Li X-H, Zhou P, Li C-Y, Zhou H-Y and Deng F-G 2006 J. Phys. B: At. Mol. Opt. Phys. 39 1975
[23] Yang C-P and Han S 2005 Phys. Lett. A 343 267–75
[24] Jeong K, Kim J and Lee S 2016 Phys. Rev. A 93 032328
[25] Barasiński A and Svozílk J 2019 Phys. Rev. A 99 012306
[26] Barasiński A, Arkhipov I I and Svozílk J 2018 Sci. Rep. 8 15209
[27] Sen(De) A, Sen U and Żukowski M 2003 Phys. Rev. A 68 032309
[28] Tittel W, Zbinden H and Gisin N 2001 Phys. Rev. A 63 042301
[29] Chen Y A, Zhang A N, Zhao Z, Zhou X Q, Lu C Y, Peng C Z, Yang T and Pan J W 2005 Phys. Rev. lett. 95 200502
[30] Shukla C and Pathak A 2013 Phys. Lett. A 377 1357
[31] Mishra S, Shukla C, Pathak A, Srikanth R and Venugopalan A 2015 Int. J. Theor. Phys. 54 3143
[32] DiVincenzo D P, Shor P W and Smolin J A 1998 Phys. Rev. A 57 830
[33] Smith G and Yard J 2008 Science 321 1812–5
[34] Hastings M B 2009 Nat. Phys. 5 255
[35] Yu S et al 2020 Phys. Rev. Lett. 125 060502
[36] Li K, Winter A, Zou X B and Guo G C 2009 Phys. Rev. Lett. 103 120501
[37] Zhu E Y, Zhuang Q and Shor P W 2017 Phys. Rev. Lett. 119 040503
[38] Erhard M, Malik M, Krenn M and Zeilinger A 2018 Nat. Photon. 12 759
[39] Lu L L et al 2020 npj Quantum Inf. 6 30
[40] Luo Y H et al 2019 Phys. Rev. Lett. 123 070505