New Type of Riesz Sequence Space of Non-absolute Type

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Abstract

The aim of this paper is to introduce the space $r_n^*(u, p)$, we show its completeness property, show that the spaces $r_n^*(u, p)$, are linearly isomorphic to the spaces $l(p)$, respectively and compute their $\alpha$-, $\beta$- and $\gamma$-duals.

Keywords: Sequence space of non-absolute type; Paranormed sequence space; $\alpha$-, $\beta$- and $\gamma$-duals

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Preliminaries, Background and Notation

We denote the set of all sequences with complex terms by $\omega$. It is a routine verification that $\omega$ is a linear space with respect to the co-ordinate wise addition and scalar multiplication of sequences which are defined, as usual, by

$$x + y = (x_k) + (y_k) = (x_k + y_k)$$

and

$$ax = a(x) = (ax_k),$$

respectively; where $x=(x_k), y=(y_k) \in \omega$ and $a \in \mathbb{C}$. By a sequence space we understand a linear subspace of $\omega$, i.e., the sequence space is the set of scalar sequences(real or complex) which is closed under co-ordinate wise addition and scalar multiplication. Throughout the paper $N, R$ and $C$ denotes the set of non-negative integers, the set of real numbers and the set of complex numbers, respectively. Let $l_q, c$ and $c_0$, respectively, denotes the space of all bounded sequences , the space of convergent sequences and the sequences converging to zero. Also, by $l_1, l(p)$, $c$ and $bs$ we denote the spaces of all absolutely, $p$-absolutely convergent, convergent and bounded series, respectively.

Let $X, Y$ be two sequence spaces and let $A=(a_{nk})$ be an infinite real or complex matrices, where $n, k \in N$. Then, the matrix $A$ defines the $A$-transformation from $X$ into $Y$, if for every sequence $x=(x_k) \in X$ the sequence $Ax=\{(Ax)_n\}$, the $A$-transform of $x$ exists and is in $Y$, where $(Ax)_n=\sum_{k} a_{nk}x_k$. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to $\infty$. By $A \in (X:Y)$ we mean the characterizations of matrices $X$ to $Y$ i.e., $AX \rightarrow Y$. A sequence $x$ is said to be $A$-summable to $l$ if $Ax$ converges to $l$ which is called as the $A$-limit of $x$.

For a sequence space $X$, the matrix domain $X_{A}$ of an infinite matrix $A$ is defined as

$$X_A = \{ x=(x_k) : Ax \in X \}. \tag{1}$$

The theory of matrix transformations is a wide field in summability; it deals with the characterizations of classes of matrix mappings between sequence spaces by giving necessary and sufficient conditions on the entries of the infinite matrices.

The classical summability theory deals with a generalization of convergence of sequences and series. One original idea was to assign a limit to divergent sequences or series. Toeplitz [1] was the first to study summability methods as a class of transformations of complex sequences by complex infinite matrices.

Let $A=(a_{nk})$ be any matrix. Then a sequence $x$ is said to be summable to $l$, written $x_k \rightarrow l$, if and only if $Ax=(a_{nk}x_k)$ exists for each $n$ and $A(x) \rightarrow l (n \rightarrow \infty)$. For example, if $I$ is the unit matrix, then $x_k \rightarrow l(I)$ means precisely that $x_k \rightarrow l (k \rightarrow \infty)$, in the ordinary sense of convergence.

We denote by $(A)$ the set of all sequences which are summable $A$. The set $(A)$ is called summability field of the matrix $A$. Thus, if $Ax=(a_{nk}(x))$, then $(A)=\{x:Ax \in c\}$, where $c$ is the set of convergent sequences. For example, $(I) = c$.

A infinite matrix $A=(a_{nk})$ is said to be regular [2] if and only if the following conditions hold:

(i) $\lim_{k \rightarrow \infty} \sum_{n=0}^{\infty} a_{nk} = 1$,

(ii) $\lim_{k \rightarrow \infty} q_{nk} = 0, (k = 0, 1, 2,...)$,

(iii) $\sum_{n=0}^{\infty} |a_{nk}| < M, (M > 0, n = 0, 1, 2,...)$.

Let $(q_{nk})$ be a sequence of positive numbers and let us write, $Q_n = \sum_{k=0}^{n} q_{nk}$ for all $n \in N$. Then the matrix $R^*_n = (r^*_n)$ of the Riesz mean $(R, q_{nk})$ is given by

$$r^*_n = \begin{cases} \frac{u_n q_n}{Q_n}, & \text{if } 0 \leq n \leq n, \\ 0, & \text{if } k > n \end{cases}$$

The Riesz mean $(R, q_{nk})$ is regular if and only if $Q_n \rightarrow \infty$ as $n \rightarrow \infty$ [2].

The approach of constructing a new sequence space by means of matrix domain of a particular limitation method has been studied by
The Riesz Sequence Space \( r^c_s(u, p) \) of Non-absolute Type

In the present section, we introduce Riesz sequence space \( r^c_s(u, p) \), prove that these spaces are complete paranormed linear space and show that the \( r^c_s(u, p) \) are linearly isomorphic to the space \( l_1(p) \). We also compute \( \alpha-, \beta- \) and \( \gamma- \) duals of these spaces. Finally, we give basis for the spaces \( r^c_s(u, p) \), where \( u=(u_k) \) is a sequence such that \( u_k \neq 0 \) for all \( k \in \mathbb{N} \).

A linear Topological space \( X \) over the field of real numbers \( \mathbb{R} \) is said to be a paranormed space if there is a sub-additive function \( h: X \to \mathbb{R} \) and scalar multiplication is continuous, such that \( h(θ)=0, h(-x)=h(x) \) and \( h(ax) \leq |a|h(x) \) for all \( a \in \mathbb{R} \) and \( x \)’s in \( X \), where \( θ \) is a zero vector in the linear space \( X \). Assume here and after that \( (p_k) \) be a bounded sequence of strictly positive real numbers with \( \inf p_k > 0 \).

Following Basar and Altay [3], Choudhary and Mishra [4], Edermann [5], Mursaleen et al. [12-14], Neyaz and Hamid [15] we define the spaces \( r^c_s(u, p) \) as the set of all sequences whose \( R^c_s \)-transform is in the spaces \( l_1(p) \) i.e.,

\[
r^c_s(u, p) = \left\{ x=(x_k) \in E : \sup_i \left( \frac{1}{p_i} \sum_{j=0}^{i} |x_j|^p \right)^{1/p} < \infty \right\},
\]

with the notation of (1) that

\[
r^c_s(u, p) = \{l_1(p) \}^{s*}.
\]

Define the sequence \( y=(y_k) \), which will be used, by the \( R^c_s \)-transform of a sequence \( x=(x_k) \), i.e.,

\[
y_k = \frac{1}{p_k} \sum_{j=0}^{k} p_j x_j, \quad k \in \mathbb{N}.
\]

Now, we begin with the following theorem which is essential in the text.

**Theorem 1:** The spaces \( r^c_s(u, p) \) are complete linear metric space paranormed by \( g \) defined

\[
g(x) = \sup_i \left( \frac{1}{p_i} \sum_{j=0}^{i} |x_j|^p \right)^{1/p}.
\]

**Proof:** The linearity of \( r^c_s(u, p) \) with respect to the co-ordinate wise addition and scalar multiplication follows from the inequalities which are satisfied for \( x, x' \in r^c_s(u, p) \) [8].

\[
\sup_i \left( \frac{1}{p_i} \sum_{j=0}^{i} |x_j|^p \right)^{1/p} \leq \sup_i \left( \frac{1}{p_i} \sum_{j=0}^{i} |x_j|^p \right)^{1/p} + \sup_i \left( \frac{1}{p_i} \sum_{j=0}^{i} |x'_j|^p \right)^{1/p}
\]

and for any \( a \in \mathbb{R} \)

\[
|a|^p \leq \max(|a|, |a|^p).
\]

It is clear that, \( g(0)=0 \) and \( g(x)=g(-x) \) for all \( x \in r^c_s(u, p) \). Again the inequality (4) and (5), yield the subadditivity of \( g \) and

\[
g(ax) \leq \max(1, |a|) g(x).
\]

Let \( (x^n) \) be any sequence of points of the space \( r^c_s(u, p) \) such that \( g(x^n - x) \to 0 \) and \( (a^n) \) is a sequence of scalars such that \( a_n \to a \). Then, since the inequality,

\[
g(x^n - x) \leq g(x) + g(x - x^n)
\]

holds by subadditivity of \( g \), \( (g(x^n)) \) is bounded and we thus have

\[
g(ax^n - ax) = \sup_i \left( \frac{1}{p_i} \sum_{j=0}^{i} p_j (a^n x_j - ax_j) \right)^{1/p} \leq |a| g(x^n - x)
\]

which tends to zero as \( n \to \infty \). That is to say that the scalar multiplication is continuous. Hence, \( g \) is paranorm on the space \( r^c_s(u, p) \).

It remains to prove the completeness of the space \( r^c_s(u, p) \). Let \( [x^n] \) be any Cauchy sequence in the space \( r^c_s(u, p) \), where \( x^n=(x_{0n}, x_{1n}, \ldots) \). Then, for a given \( \varepsilon > 0 \) there exists a positive integer \( n_\varepsilon \) such that

\[
g(x^n - x^{n'}) < \varepsilon
\]

for all \( i \geq n_\varepsilon (\varepsilon) \). Using definition of \( g \) and for each fixed \( k \in \mathbb{N} \) that

\[
\left| R^c_s(x^n_k) - R^c_s(x^{n'}_k) \right| \leq \sup_j \left| R^c_s(x^n_j) - R^c_s(x^{n'}_j) \right| < \varepsilon
\]

for \( i \geq n_\varepsilon (\varepsilon) \), which leads us to the fact that \( \left( (R^c_s(x^n_k), (R^c_s(x^{n'}_k), \ldots) \right) \) is a Cauchy sequence of real numbers for every fixed \( k \in \mathbb{N} \). Since \( R \) is complete, it converges, say, \( R^c_s(x^k_n) \to R^c_s(x^k) \) as \( i \to \infty \). Using these infinitely many limits \( (R^c_s(x^k_n), (R^c_s(x^k), \ldots) \), we define the sequence \( (R^c_s(x^k_n), (R^c_s(x^k), \ldots) \). From (6) for with \( j \to \infty \) we have

\[
g(R^c_s(x^k - R^c_s(x^{n'})) \leq \varepsilon,
\]

for all \( k \), i.e.,

\[
g(x^n - x^{n'}) \leq \varepsilon (i \geq n_\varepsilon (\varepsilon)).
\]

Finally, taking \( \varepsilon=1 \) in (7) and letting \( i \geq n_\varepsilon (1) \), we have by Minkowski’s inequality for each \( m \in \mathbb{N} \) that

\[
\left| R^c_s(x^m_n) \right|^p \leq g(x^m - x^{n'}) \leq 1 + g(x^n)
\]

which implies that \( x \in r^c_s(u, p) \). Since \( g(x-x^n) \leq \varepsilon \) for all \( i \geq n_\varepsilon (\varepsilon) \), it follows that \( x \to x \) as \( i \to \infty \), hence we have shown that \( r^c_s(u, p) \) is complete, hence the proof.

Note that one can easily see the absolute property does not hold on the space \( r^c_s(u, p) \), that is, \( g(x) \neq g(|x|) \) for at least one sequence in the spaces \( r^c_s(u, p) \) and consequently we see that the space \( r^c_s(u, p) \) is a sequence space of non-absolute type.

**Theorem 2:** The sequence spaces \( r^c_s(u, p) \) of non-absolute type is linearly isomorphic to the spaces \( l_1(p) \).

**Proof:** To prove the theorem, we should show the existence of a linear bijection between the spaces \( r^c_s(u, p) \) and \( l_1(p) \). With the notation of (3), define the transformation \( T \) from \( r^c_s(u, p) \) to \( l_1(p) \) by \( x \to y=Tx \). The linearity of \( T \) is trivial. Further, it is obvious that \( x=θ \) whenever \( Tx=θ \) and hence \( T \) is injective.

Let \( y \in l_1(p) \) and define the sequence \( x=(x_k) \) by

\[
x_k = \frac{1}{u_k q_k} (Q_k y_k - Q_{k-1} y_{k-1}) \quad \text{for } k \in \mathbb{N}.
\]
Then, 
\[ g(x) = \sup_x \left[ \frac{1}{Q_x} \sum_{n} a_n q_n x_n^+ \right]^2 = \sup_x \left[ \frac{1}{Q_x} \sum_{n} \beta_n y_n^2 \right] \]
\[ = \sup_{x} \left[ \frac{1}{Q_x} \right]^2 = g_{x}(x) < \infty . \]

Thus, we have \( x \in r_{\alpha}^\infty (u, p) \). Consequently, \( T \) is surjective and is paranorm preserving. Hence, \( T \) is a linear bijection and this says us that the spaces \( r_{\alpha}^\infty (u, p) \) and \( l_{\infty} (p) \) are linearly isomorphic, hence the proof.

First we state some lemmas which are needed in proving the theorems.

**Lemma 1** [7]: \( A \in (l_\infty (p):l_1) \) if and only if there exists an integer \( B>1 \) such that
\[ \sup_{a} \sum_{k} |a_k| B_k < \infty . \]  
(8)

**Lemma 2** [7]: Let \( 0 < p_k < \infty \) for every \( k \in \mathbb{N} \). Then \( A \in (l(p):l_\infty) \) if and only if there exists an integer \( B > 1 \) such that
\[ \sup_{a} \sum_{k} |a_k| B_k < \infty . \]  
(9)

**Lemma 3** [7]: Let \( 0 < p_k < \infty \) for every \( k \in \mathbb{N} \). Then \( A \in (l_{\infty}(p):c) \) if and only if \( (8) \) and \( (9) \) hold and
\[ \lim_{n \to \infty} a_n = b_n \quad \forall k \in \mathbb{N} . \]  

**Theorem 3**: Let \( 1 < p \leq H < \infty \) for every \( k \in \mathbb{N} \). Define the sets \( D_1(u, p), D_2(u, p) \) and \( D_3(u, p) \), as follows
\[ D_1(u, p) = \left\{ x = (a_k) \in c_s : \sup \sum_{k} \left[ x_k^{-1} - \frac{a_k}{u_k} Q_k B_k \right] \right\} \]
\[ D_2(u, p) = \left\{ x = (a_k) \in c_s : \sup \sum_{k} \left[ a_k \frac{Q_k B_k}{u_k} \right] \right\} \]
\[ D_3(u, p) = \left\{ x = (a_k) \in c_s : \sup \sum_{k} \left[ a_k \frac{Q_k B_k}{u_k} \right] \right\} . \]

Thus, \( x \in r_{\alpha}^\infty (u, p) \) if and only if \( x \in D_1(u, p) \).

**Proof**: Let us take any \( a=(a_k) \in c_s \). We can easily derive with (3) that \( a x_n = \sum_{n=1}^{\infty} \left[ x_n^{(1)} - \frac{a_n}{u_n} Q_n \right] y_n \) \( (Cy_k) \) \( (Cy_k) \) \( (Cy_k) \)
(11)

For \( n \in \mathbb{N} \) where, \( C_{\alpha} = (c_{\alpha,n}) \) is defined as
\[ c_{\alpha,n} = \left\{ \frac{(-1)^{k-1} a_{n-k}}{u_n} Q_{n-k} \right\} \quad \text{if} \quad n-1 \leq k \leq n, \]
\[ 0 \quad \text{if} \quad 0 \leq k < n-1 \text{ or } k > n, \]
where \( n,k \in \mathbb{N} \). Thus we observe by combining (10) with (i) of Lemma 2 that \( ax=(a x_n) \in l_{1} \) whenever \( x=(x_n) \in r_{\alpha}^\infty (u, p) \) if and only if \( Cy \in l_{1} \) whenever \( y \in l(p) \). This gives the result that \( [r_{\alpha}^\infty (u, p)]' = D_1(u, p) \). Consider the equation,
\[ \sum_{n=1}^{\infty} a_n x_n = \sum_{n=1}^{\infty} a_n Q_{n} y_n + \frac{a_n}{u_n} Q_{n} y_n = (Dy) \quad \text{for } n \in \mathbb{N} \]  
(12)

where, \( D=(d_{nk}) \) is defined as
\[ d_{nk} = \left\{ \frac{a_k}{u_n} Q_{n-k} \right\} \quad \text{if} \quad 0 \leq k \leq n-1, \]
\[ 0 \quad \text{if} \quad k = n, \]
\[ \left\{ \frac{a_k}{u_n} Q_{n-k} \right\} \quad \text{if} \quad k > n. \]

Where \( n,k \in \mathbb{N} \). Thus we deduce from Lemma 3 with (11) that \( ax=(a x_n) \in c_s \) whenever \( x=(x_n) \in r_{\alpha}^\infty (u, p) \) if and only if \( Dy \in c_s \) whenever \( y \in l(p) \). Therefore, we derive from (8) that
\[ \sum_{n=1}^{\infty} \left[ \frac{a_k}{u_n} Q_{n-k} \right] y_n < \infty \quad \text{and \sup}_{k} \frac{a_k}{u_n} Q_{n-k} y_n < \infty \]  
(13)

which shows that that \( [r_{\alpha}^\infty (u, p)]' = D_1(u, p) \).

As this, from Lemma 2 together with (11) that \( ax=(a x_n) \in b_s \) whenever \( x=(x_n) \in r_{\alpha}^\infty (u, p) \) if and only if \( Dy \in l_1 \) whenever \( y=(y_n) \in l(p) \). Therefore, we again obtain the condition (12) which means that \( [r_{\alpha}^\infty (u, p)]' = D_1(u, p) \) and the proof of the theorem is complete [19-21].

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**References**

1. Toepfiz O (1991) Uber allegemeine Lineare mittelbildungen. Prace Math Fiz 22: 113-119.
2. Petersen GM (1966) Regular matrix transformations. Mc Graw-Hill, London.
3. Aydin C, Basar F (2005) Some new sequence spaces which include the classes of sequences. Proc Camb Phil Soc 68: 99-104.
4. Kizmaz H (1981) On certain sequence. Canad Math Bull 24: 169-176.
5. Maddox IJ (1968) Paranormed sequence spaces generated by infinite matrices. Proc Camb Phil Soc 64: 641-656.
6. Kizmaz H (1981) On the Euler sequence spaces which include the spaces \( l_1 \) and \( l_\infty \). Demonst Math 30: 223-238.
7. Lascarides CG, Maddox IJ (1970) Matrix transformations between the sequence spaces of Maddox. J Math Anal Appl 180: 223-238.
8. Lascarides CG, Maddox IJ (1970) Matrix transformations between the sequence spaces of Maddox. J Math Anal Appl 180: 223-238.
9. Maddox IJ (1986) Paramepared sequences generated by infinite matrices. Proc Camb Phil Soc 64: 335-340.
10. Maddox IJ (1986) Paramepared sequences generated by infinite matrices. Proc Camb Phil Soc 64: 335-340.
11. Maddox IJ (1986) Paramepared sequences generated by infinite matrices. Proc Camb Phil Soc 64: 335-340.
12. Maddox IJ (1986) Paramepared sequences generated by infinite matrices. Proc Camb Phil Soc 64: 335-340.
13. Maddox IJ (1986) Paramepared sequences generated by infinite matrices. Proc Camb Phil Soc 64: 335-340.
14. Maddox IJ (1986) Paramepared sequences generated by infinite matrices. Proc Camb Phil Soc 64: 335-340.
15. Maddox IJ (1986) Paramepared sequences generated by infinite matrices. Proc Camb Phil Soc 64: 335-340.
16. Maddox IJ (1986) Paramepared sequences generated by infinite matrices. Proc Camb Phil Soc 64: 335-340.
17. Maddox IJ (1986) Paramepared sequences generated by infinite matrices. Proc Camb Phil Soc 64: 335-340.
18. Maddox IJ (1986) Paramepared sequences generated by infinite matrices. Proc Camb Phil Soc 64: 335-340.
19. Maddox IJ (1986) Paramepared sequences generated by infinite matrices. Proc Camb Phil Soc 64: 335-340.
20. Maddox IJ (1986) Paramepared sequences generated by infinite matrices. Proc Camb Phil Soc 64: 335-340.
21. Maddox IJ (1986) Paramepared sequences generated by infinite matrices. Proc Camb Phil Soc 64: 335-340.
16. Ng PN, Lee PY (1978) Cesáro sequences spaces of non-absolute type. Comment Math Prace Mat 20: 429-433.
17. Sengonul M, Basar F (2005) Some new Cesáro sequences spaces of non-absolute type, which include the spaces $c_0$ and $c$. Soochow J Math 1: 107-119.
18. Simmons S (1965) The sequence spaces $l(p_v)$ and $m(p_v)$. Proc London Math Soc 15: 422-436.
19. Wang CS (1978) On Norlund sequence spaces. Tamkang J Math 9: 269-274.
20. Wilansky A (1984) Summability through Functional Analysis. North Holland Mathematics Studies, Amsterdam, New York.
21. Yosida K (1966) Functional Analysis. Springer-Verlag, Berlin heidelberg, New York.