THICK ISOTOPY PROPERTY AND THE MAPPING CLASS GROUPS OF HEegaard SPLITTINGS

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ABSTRACT. We give a necessary and sufficient condition for the fundamental group of the space of Heegaard splittings of an irreducible 3-manifold to be finitely generated. The condition is exactly the conclusion of the thick isotopy lemma proved by Colding, Gabai and Ketover, which says that any isotopy of a Heegaard surface is achieved by a 1-parameter family of surfaces with area bounded above by a universal constant and with some “thickness property”. We also prove that a Heegaard splitting of a hyperbolic or spherical 3-manifold satisfies the condition if it is topologically minimal (in the sense of Bachman) and its disk complex has finitely generated homotopy group. In conclusion, such a Heegaard splitting has finitely generated mapping class group.

1. Introduction

Let $M$ be a closed orientable 3-manifold. A Heegaard splitting is a decomposition of $M$ into two handlebodies along a closed embedded surface $\Sigma$. We will denote such a splitting of $M$ by $(M, \Sigma)$. In [21], Johnson and McCullough defined the space $\mathcal{H}(M, \Sigma)$ of Heegaard splittings equivalent to $(M, \Sigma)$ by $\text{Diff}(M)/\text{Diff}(M, \Sigma)$, where $\text{Diff}(M)$ is the space of self-diffeomorphisms of $M$ and $\text{Diff}(M, \Sigma)$ is its subspace consisting of maps that send $\Sigma$ to itself. For example, computing the 0-th homotopy group of $\mathcal{H}(M, \Sigma)$ is the same as classifying Heegaard splittings up to isotopy. Throughout the paper, we will focus only on the case that $M$ is irreducible. In [21], the $k$-th homotopy group of $\mathcal{H}(M, \Sigma)$ was computed for $k \geq 2$. On the other hand, $\pi_1(\mathcal{H}(M, \Sigma))$ is closely related to the mapping class group of a Heegaard splitting or the Goeritz group, and these groups are still mysterious. In this paper, we give a necessary and sufficient condition for $\pi_1(\mathcal{H}(M, \Sigma))$ to be finitely generated.

In [13], Colding, Gabai and Ketover found an effective algorithm to construct the complete list of Heegaard splittings of a non-Haken hyperbolic 3-manifold. A key of their argument is the thick isotopy lemma ([13 Lemma 2.10]), which allows us to turn the computation of the 0-th homotopy group of the space of Heegaard splittings into a purely combinatorial problem, involving the (crudely) almost normal surface theory. The same strategy is also useful in computing $\pi_1(\mathcal{H}(M, \Sigma))$ as stated below. From now on, we fix a Riemannian metric of $M$. Let $\delta > 0$. A surface $S$ in $M$ is said to be $\delta$-compressible if there exists a compressing disk $D$ for $S$ such that $\text{diam}\partial D \leq \delta$. Otherwise $S$ is said to be $\delta$-locally incompressible.

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Definition. We say \((M, \Sigma)\) satisfies the **thick isotopy property** if the following holds. There exist \(C > 0\) and \(\delta > 0\), depending only on \(\Sigma\) and the metric of \(M\), such that any isotopy \(\{\Sigma_t\}_{t \in I}\) with \(\Sigma_0 = \Sigma_1 = \Sigma\) can be deformed within its homotopy class (as a loop in \(\mathcal{H}(M, \Sigma)\)) so that afterward for all \(t \in I\),

- \(\text{Area}(\Sigma_t) < C\), and
- \(\Sigma_t\) is \(\delta\)-locally incompressible.

**Theorem 1.1.** The fundamental group of \(\mathcal{H}(M, \Sigma)\) is finitely generated if and only if \((M, \Sigma)\) satisfies the thick isotopy property.

Our second aim is to investigate what kind of Heegaard splitting satisfies the thick isotopy property. Let \(S\) be a closed embedded surface in \(M\) of genus at least 2. The **disk complex** \(\Gamma(S)\) of \(S\) is defined to be the simplicial complex whose vertices are the isotopy classes of compressing disks for \(S\), and whose \(i\)-simplices are \((i + 1)\)-tuples of vertices that admit disjoint representatives. In [3], Bachman introduced the concept of a topologically minimal surface as a generalization of several important classes of surfaces in a 3-manifold, including incompressible surfaces and strongly irreducible surfaces.

**Definition (Bachman [3]).** We say \(S\) is topologically minimal if \(\Gamma(S) = \emptyset\) or \(\pi_{d-1}(\Gamma(S)) \neq 1\) for some \(d \in \mathbb{N}\). If \(S\) is topologically minimal, the **topological index** of \(S\) is defined to be the smallest number \(d\) such that \(\pi_{d-1}(\Gamma(S)) \neq 1\).

**Theorem 1.2.** Let \(M\) be a hyperbolic or spherical 3-manifold but not \(S^3\). Let \((M, \Sigma)\) be a Heegaard splitting of \(M\). Suppose that \(\Sigma\) is a topologically minimal surface of index \(d\). Furthermore, suppose that \(\pi_{d-1}(\Gamma(\Sigma))\) is finitely generated if \(d > 1\). Then \((M, \Sigma)\) satisfies the thick isotopy property.

The above theorems have an application to the theory of the mapping class group of a Heegaard splitting. For a Heegaard splitting \((M, \Sigma)\), its **mapping class group** \(\text{MCG}(M, \Sigma)\) is defined to be \(\pi_0(\text{Diff}(M, \Sigma))\).

**Corollary 1.3.** If \(M\) and \(\Sigma\) are as in Theorem 1.2, then \(\text{MCG}(M, \Sigma)\) is finitely generated.

There have been many efforts to find a finite generating set for the mapping class group of a Heegaard splitting. Possibly the most interesting is \(\text{MCG}(S^3, \Sigma_g)\), where \((S^3, \Sigma_g)\) is a standard genus \(g\) Heegaard splitting of the 3-sphere. It is known that \(\text{MCG}(S^3, \Sigma_g)\) is finitely generated for \(g = 2\) by [18] (see also [23]), and for \(g = 3\) by [17]. However, it is not known if the same is true for \(g \geq 4\). On the other hand, a genus \(\geq 2\) Heegaard surface in \(S^3\) is topologically minimal by [2] or [4] (though the disk complex is not finite type). So there might be a good chance to improve our proof to remove the assumption that \(M \neq S^3\). In fact, much of our argument is still valid when \(M = S^3\): Lemma 5.5 below is the only place where the assumption \(M \neq S^3\) is used essentially.

As another example, any genus 2 weakly reducible Heegaard splitting has finitely presented mapping class group by [15] [10]. A finite generating set for a genus 3 Heegaard splitting of the 3-torus is also known by [20]. While little has been known about the mapping class group of a Heegaard splitting of genus greater than 3, an advantage of our approach is that it is applicable to arbitrarily high genus Heegaard splittings.
Organization of the paper. Section 2 is a preliminary towards the proof of Theorem 1.1 including the definition of a crudely almost normal surface. In Section 3, Theorem 1.1 will be proved. Section 4 is a quick introduction to min-max theory. In the final section, we will prove Theorem 1.2 and then conclude with the proof of Corollary 1.3.

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2. Normal surface theory

Throughout the paper, we will use the following notations:

- \( I := [0, 1] \).
- For \( r > 0 \), \( B^d_r := \{ x \in \mathbb{R}^d | |x| \leq r \} \).
- If \( K \) is a simplicial complex, we will denote by \( K^i \) its \( i \)-skeleton.

In this section, we recall some definitions and lemmas from [13]. Let \( M \) be a closed orientable 3-manifold. Let \( T \) be a triangulation of \( M \).

Definition. A closed embedded surface \( S \subset M \) is crudely almost normal (with respect to \( T \)) if the following are satisfied:

1. \( S \) is transverse to any simplex of \( T \).
2. If \( \tau \) is a 2-simplex of \( T \), \( S \cap \tau \) consists of finitely many arcs (with no circle component).
3. If \( \sigma \) is a 3-simplex of \( T \), \( S \cap \sigma \) consists of finitely many disks but possibly with one exception: there may be exactly one 3-simplex that contains exactly one unknotted annulus component.

A crudely almost normal surface will be called a crudely normal surface if it has no exceptional annulus component.

The weight of \( S \) is defined to be \( |S \cap T| \). Let \( S' \) be another crudely almost normal surface. Then, \( S \) and \( S' \) are said to be normally isotopic if they are isotopic through surfaces transverse to each simplex. We say \( S' \) is obtained from \( S \) by a pinch if it is obtained from \( S \) and a 2-sphere in \( M \) by connecting them with a tube. Such a move or its inverse will be called a pinch. Note that if \( M \) is irreducible, a pinch can be achieved by an isotopy that contracts the 2-sphere across a 3-ball in \( M \).

Lemma 2.1 ([13, Lemma 3.4]). There are only finitely many normal isotopy classes of crudely almost normal surfaces with weight at most \( L \).

Lemma 2.2 ([13, Lemma 3.2]). Let \( T \) be a triangulation of the Riemannian 3-manifold \( M \) with metric \( \rho \), \( L > 0 \) and \( \epsilon > 0 \). Then there exists \( K(T, L, \epsilon, \rho) > 0 \) such that if \( S \)
is a closed embedded surface with $\text{Area}(S) < C$, then $S$ is isotopic to a surface $S'$ such that $|S' \cap T^1| < KC$ and the diameter of the trace of any point of the isotopy is at most $\varepsilon$.

If $F : S \times [0, 1] \to M$ is a $C$-isotopy between surfaces $S_0$ and $S_1$ that are transverse to $T$ of weight at most $L$, then there exists a generic $K(C+1)\cdot T$-isotopy $G$ from $S_0$ to $S_1$ such that, for all $x \in S$ and $t \in [0, 1]$, $d(G(x,t), F(x,t)) < \varepsilon$.

3. The proof of Theorem 1.1

In this section, we prove Theorem 1.1. It is not hard to see the necessity of the theorem. First, recall that a path in $\mathcal{H}(M, \Sigma)$ can be identified with an isotopy of a Heegaard surface. More precisely, as $\text{Diff}(M) \to \text{Diff}(M)/\text{Diff}(M, \Sigma) = \mathcal{H}(M, \Sigma)$ is a fibration [21], any path $\alpha : I \to \mathcal{H}(M, \Sigma)$ lifts to $\hat{\alpha} : I \to \text{Diff}(M)$ and we can define an isotopy of a Heegaard surface by $\Sigma_t := \hat{\alpha}(t)(\Sigma)$. Conversely, if an isotopy of a Heegaard surface is given, it defines a path in $\mathcal{H}(M, \Sigma)$ via the isotopy extension theorem. Now if $\pi_1(\mathcal{H}(M, \Sigma))$ is finitely generated, we can find a finite collection of isotopies of $\Sigma$ such that any isotopy representing an element of $\pi_1(\mathcal{H}(M, \Sigma))$ can be expressed as the product of isotopies in the collection. Thus, $(M, \Sigma)$ satisfies the thick isotopy property.

In the following, we prove the sufficiency of the theorem. Let $C > 0$ and $\delta > 0$ be the constants given in the definition of the thick isotopy property. Fix a triangulation $T$ of $M$ such that

- $\Sigma$ is crudely normal with respect to $T$, and
- any simplex of $T$ has the diameter at most $\delta$.

Let $\{\Sigma_t\}_{t \in I}$ be any isotopy with $\Sigma_0 = \Sigma_1 = \Sigma$.

Claim 1. $\{\Sigma_t\}_{t \in I}$ can be deformed within its homotopy class (as the loop in the space $\mathcal{H}(M, \Sigma)$) so that afterward for all $t$ but finitely many points in $I$, $\Sigma_t$ is a crudely almost normal surface with weight bounded above by a universal constant.

This will be proved along the same line as [13, Lemma 3.6]. For our present purpose, an important thing is that the deformation preserves the homotopy class, and this is what we will care about in the proof.
Proof. By assumption, we may assume that for $t \in I$ the area of $\Sigma$ is less than $C$ and $\Sigma_t$ is $\delta$-locally incompressible. After perturbing $\{\Sigma_t\}_{t \in I}$ slightly, $\Sigma_t$ is transverse to every simplex of $T$ for all $t$ but finitely many points $0 < t_1 < \cdots < t_l < 1$. By Lemma 2.2 we may assume that $\Sigma_t$ has the weight at most $L := K(C + 1)$ for $t \in I \setminus \{t_1, \ldots, t_l\}$.

For $t \in I \setminus \{t_1, \ldots, t_l\}$, $\Sigma_t$ can be turned into a crudely normal surface as follows. Fix a small neighborhood $N_t$ of $T^2$. Note that $\partial N_t$ consists of 2-spheres, each corresponding to a 3-simplex of $T$. Put $P_t := \partial N_t$. Choose a disk $D \subset P_t$ such that $\partial D \subset \Sigma_t \cap P_t$ and $\text{int} D \cap \Sigma_t = \emptyset$. Since $\Sigma_t$ is $\delta$-locally incompressible and any simplex of $T$ has diameter at most $\delta$, $\partial D$ must be inessential in $\Sigma_t$. So if we compress $\Sigma_t$ across $D$, the result is a genus $g$ surface with a 2-sphere. In other words, the surface is obtained from $\Sigma_t$ by a pinch, which is achieved by an isotopy as $M$ is irreducible. If we repeat the above process until all the intersection circles in $\Sigma_t \cap P_t$ are removed, the resulting surface will be crudely normal with respect to $T$. Moreover, the weight of the surface is at most $L$ because the normalizing process does not increase the weight.

We need to show that the above normalizing process can be done simultaneously for $t$. The basic idea in the following construction of the isotopy is borrowed from [19].

Let $\{I_i \mid 0 \leq i \leq l\}$ be a cover of $I$ by closed intervals such that

- $I_i \cap I_j \neq \emptyset$ iff $|i - j| \leq 1$,
- $I_i \cap \{t_1, \ldots, t_l\} = \{t_i\}$.

Let $\{I_i\}$ be a cover of $I$ obtained by enlarging $I_i$ slightly. For $i = 0, \ldots, l$, choose a neighborhood $N_i$ of $T^2$ so that

- the boundary, denoted by $P_i$, of $N_i$ is transverse to $\Sigma_t$ for $t \in I_i$,
- $P_i$’s are mutually disjoint.

For a simplex $\sigma$ of $T$, put $N_i^\sigma := \sigma \cap N_i$. By choosing $N_i$ to be small enough, we may assume that any component of $\Sigma_t \cap N_i^\sigma$ is an annulus, but a single component that is either a disk or a pair of pants, corresponding to the tangency in $\Sigma_t \cap \partial \sigma$.

For $t \in I$, let $C_t$ denote the set of intersecting circles in $\Sigma_t \cap \bigcup P_t$, where the union is taken for $i$ such that $t \in I_i$. For each $c \in C_t$, choose a disk $D$ in $\bigcup P_t$ with $\partial D = c$, across which we will compress $\Sigma_t$ to remove $c$ from $\Sigma_t \cap \bigcup P_t$. Denote by $D_t$ the collection of such disks. Pick a family of functions $\{\varphi_t : D_t \to [0, 2] \mid t \in I\}$ satisfying the following:

- If $D, D' \in D_t$ and $D \subset D'$, then $\varphi_t(D) < \varphi_t(D')$.
- $\varphi_t(D) > 1$ iff $t \in I_i \setminus I_i$.

Let $\Gamma$ be the graph in $I \times [0, 2]$ given by taking the union of images of $\varphi_t$’s. For $t \in I$, the line $t \times [0, 2]$ hits $\Gamma$ in finitely many points, say $(t, u_1), \ldots, (t, u_m)$. Going up along $t \times [0, 2]$, we find a sequence of disks $\varphi_t^{-1}(u_1), \ldots, \varphi_t^{-1}(u_m)$. Define the isotopy $\{\Sigma_{tu}\}_{u \in [0, 2]}$ to be sequence of pinches corresponding to these disks. Note that if $(t, u_i)$ is a vertex of $\Gamma$, $\varphi_t^{-1}(u_i)$ is a collection of disks. But the definition of $\varphi_t$ guarantees that there is no inclusion relation among such disks. This implies that the corresponding pinches are commutative with each other, and thus $\{\Sigma_{tu}\}$ is well-defined around vertices of $\Gamma$. The restriction of $\{\Sigma_{tu}\}$ on $I \times I$ can be taken to be continuous (though $\{\Sigma_{t2}\}$ cannot be continuous for $t$ because $C_t$ is not constant around $\partial I_i$). In conclusion, we obtain a path of isotopies $\{\Sigma_{tu}\}$ with $\Sigma_{t0} = \Sigma_t$. Moreover, if $t \in I_i$, $\Sigma_{t1}$ is obtained from $\Sigma_t$ by successive pinches until all the intersecting circles in $\Sigma_t \cap P_t$ is removed.
It remains to check that $\Sigma_{t_1}$ is a crudely almost normal surface for $t \in I \setminus \{t_1, \ldots, t_l\}$. Let $\sigma$ be any 3-simplex of $T$. Recall that $N_{\sigma}^g = \sigma \cap N_{\sigma}^g$. Let $S$ be a component of $\Sigma_{t_1} \cap N_{\sigma}^g$. By the choice of $N_{\sigma}$, any component of $\Sigma_{t_1} \cap N_{\sigma}^g$ is either a disk, an annulus or a pair of pants. As $\Sigma_{t_1}$ is the result of compressing $\Sigma_t$ across disks in $\bigcup P_t$, $S$ must be either a disk or an annulus. If $S$ is a disk, there is nothing to prove. If $S$ is an annulus, it suffices to see that $S$ is unknotted. Note that such an annulus must be the result of compressing a pair of pants, which corresponds to the saddle tangency in $\Sigma_{t_1} \cap \partial \sigma$. Thus, in this case $S$ can be $\partial$-compressed to create a disk, which means that $S$ is unknotted. \hfill $\square$

We now finish the proof of Theorem 1.1. Consider the graph $G$ such that each vertex of $G$ corresponds to a normal isotopy class of crudely almost normal surfaces w.r.t. $T$ with weight at most $L$, and each edge corresponds to one of the moves $(0 - 2\Pi)$ shown in Figure 1 or a pinch. By Lemma 2.1, $G$ is a finite graph. In particular, $\pi_1(G)$ is finitely generated. By Claim 1, the natural homomorphism $\pi_1(G) \to \pi_1(H(M, \Sigma))$ is a surjection. Thus, we conclude that $\pi_1(H(M, \Sigma))$ is finitely generated. \hfill $\square$

4. THE MIN-MAX THEOREM

This section is a quick introduction to the min-max theory of Simon-Smith [24], which will be used in the next section. One can consult e.g. [12, 14] for more details on this subject.

Let $M$ be a closed, orientable, Riemannian 3-manifold. We will denote by $\mathcal{H}^2(\cdot)$ the 2-dimensional Hausdorff measure on $M$.

Definition. Let $X^k$ be a manifold. A family $\{\Sigma_t\}_{t \in X}$ of closed subsets of $M$ is called a (genus $g$) sweep-out if it satisfies the following conditions:

- $\Sigma_t$ converges to $\Sigma_{t_0}$ in Hausdorff topology when $t \to t_0$.
- $\mathcal{H}^2(\Sigma_t) \to \mathcal{H}^2(\Sigma_{t_0})$ when $t \to t_0$.
- $\Sigma_t$ is a closed genus $g$ surface in $M$ if $t \in \text{int } X$. On the other hand, if $t \in \partial X$, $\Sigma_t$ is a closed surface of genus $\leq g$ plus finitely many arcs.
- $\Sigma_t$ varies smoothly for $t \in \text{int } X$.

For later use, we restrict ourselves to the case that $X = I \times B^d$. Consider the subspace $\mathcal{J}$ of $C^\infty(M \times (I \times B^d), M)$ consisting of those maps $\psi$ such that

(i) $\psi(\cdot, t)$ is a self-diffeomorphisms of $M$ for $t \in I \times B^d$, and
(ii) $\psi(\cdot, t) = \text{id}_M$ for $t \in \partial I \times B^d$.

Let $\mathcal{J}_0 \subset \mathcal{J}$ be the component containing the map $\psi_0$ given by $\psi_0(x, t) := x$. Given a sweep-out $\{\Sigma_t\}_{t \in I \times B^d}$, define the collection $\Pi(\Sigma_t)$ of sweep-outs by

$$\Pi(\Sigma_t) := \left\{ \psi(\Sigma_t, t) \mid \psi \in \mathcal{J}_0 \right\}.$$ 

The width of $\Pi(\Sigma_t)$ is defined by

$$W(\Pi(\Sigma_t), M) := \inf_{\{\Lambda_t\} \in \Pi(\Sigma_t)} \sup_{t \in I \times B^d} \mathcal{H}^2(\Lambda_t).$$

A sequence $\{\Sigma_{t_i}\}_{i \in X}$ (where $i \in N$) of sweep-outs in $\Pi(\Sigma_t)$ is a minimizing sequence if $W(\Pi(\Sigma_{t_i}), M) = \lim_{i \to \infty} \sup_{t \in I \times B^d} \mathcal{H}^2(\Sigma_t)$. Furthermore, a sequence $\{\Sigma_{t_i}\}_{i \in N}$ is a min-max sequence if $W(\Pi(\Sigma_{t_i}), M) = \lim_{i \to \infty} \mathcal{H}^2(\Sigma_{t_i})$. 


Simon-Smith’s min-max theorem is the following theorem. (The following statement can be found in [13] with minor modification, see [12, 15, 22] for the proof and also [13, Appendix] for the multi-parameter case.)

**Theorem 4.1** (cf. [13, Theorem 2.1]). Given a sweep-out \( \{ \Sigma_t \}_{t \in I \times B^d} \) of genus-\( g \) surfaces, if

\[
W(\Pi_{\{\Sigma_t\}}, M) > \sup_{t \in \partial I \times B^d} \mathcal{H}^2(\Sigma_t),
\]

then there exists a min-max sequence \( \Sigma_i := \Sigma_i^t \) such that

\[
\Sigma_i \to \sum_{i=1}^k n_i \Gamma_i \text{ as varifolds},
\]

where \( \Gamma_i \) are smooth closed embedded minimal surfaces and \( n_i \) are positive integers. Moreover, after performing finitely many compressions on \( \Sigma_i \) and discarding some components, each connected component is isotopic to one of the \( \Gamma_i \) or to a double cover of one of the \( \Gamma_i \). We have the following genus bounds with multiplicity:

\[
\sum_{i \in \mathcal{O}} n_i g(\Gamma_i) + \frac{1}{2} \sum_{i \in \mathcal{N}} n_i (g(\Gamma_i) - 1) \leq g,
\]

where \( \mathcal{O} \) denotes the subcollection of \( \Gamma_i \) that is orientable and \( \mathcal{N} \) denotes those \( \Gamma_i \) that are nonorientable, and where \( g(\Gamma_i) \) denotes the genus of \( \Gamma_i \) if it is orientable, and the number of cross-caps that one attaches to a sphere to obtain a homeomorphic surface if \( \Gamma_i \) nonorientable.

**Lemma 4.2.** Let \( M \) be a hyperbolic or spherical 3-manifold. If \( \{ \Sigma_t \}_{t \in I \times B^d} \) is a genus \( g \) sweep-out satisfying \( W(\Pi_{\{\Sigma_t\}}, M) > \sup_{t \in \partial I \times B^d} \mathcal{H}^2(\Sigma_t) \), then \( W(\Pi_{\{\Sigma_t\}}, M) \leq 8\pi(g+1) \).

**Proof.** If \( M \) is hyperbolic, \( W(\Pi_{\{\Sigma_t\}}, M) \leq 4\pi(g-1) \) by [14, Lemma 9.4]. So, we prove the lemma when \( M \) is spherical. Theorem 4.1 shows that \( W(\Pi_{\{\Sigma_t\}}, M) = \sum_{i=1}^k n_i \text{Area}(\Gamma_i) \) for some embedded minimal surfaces \( \Gamma_i \) (1 \( \leq i \leq k \)). By Frankel’s theorem [16], the min-max limit is in fact connected and we can express the width as \( W(\Pi_{\{\Sigma_t\}}, M) = n\text{Area}(\Gamma) \). If \( \Gamma \) is non-orientable, its double cover is stable by [25, Theorem 7.2]. But this is impossible because \( S^3 \) with the standard metric and thus its quotient cannot contain a stable minimal surface. So \( \Gamma \) must be orientable and again by [25, Theorem 7.2], the multiplicity \( n \) must be one. This together with Choi-Schoen’s area bound [11] for a minimal surface in a spherical 3-manifold implies

\[
W(\Pi_{\{\Sigma_t\}}, M) = \text{Area}(\Gamma) \leq 8\pi \left( \frac{2}{|\pi_1(M)|} - \frac{\chi(\Gamma)}{2} \right) \leq 8\pi(g+1).
\]

\[ \square \]

5. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. Suppose that \( M \) is a hyperbolic or spherical 3-manifold but not \( S^3 \), and \( \Sigma \) is a genus \( g \) Heegaard surface of topological index \( d > 0 \). Furthermore, if \( d > 1 \), we assume that \( \pi_{d-1}(\Gamma(\Sigma)) \) is finitely generated.
Step 1: The definitions of $C$ and $\delta$. Let $\{\varphi_k : S^{d-1} \to \Gamma(\Sigma) \mid k = 1, \ldots, n\}$ be a collection maps that represents a finite generating set of $\pi_{d-1}(\Gamma(\Sigma))$. Put $B = B^d_1$. For each $\varphi_k$, we consider the sweep-out $\{\Sigma^k_s\}$ parametrized by $B$ and given below. (The same construction can be found in [3].)

Take a triangulation $K$ of $S^{d-1}$ for which $\varphi_k$ is simplicial. For a vertex $s$ of $K$, we will denote by $D_s$ a compressing disk for $\Sigma$ that represents $\varphi_k(s) \in \Gamma(\Sigma)$. Let $K'$ be the barycentric subdivision of $K$, and let $\tilde{K}$ be the cone over $K'$. So, $\tilde{K}$ is a triangulation of $B$. For the center 0 of $B$, define $\Sigma^k_0 = \Sigma$. For a vertex $s$ of $K'$, $\Sigma^k_s$ is defined as follows. Let $\sigma$ be the smallest simplex of $K$ that contains $s$. Let $s_0, \ldots, s_l \in K_0$ be the vertices of $K$ that spans $\sigma$. For $0 \leq i \leq l$, take a cylinder $N(D_{s_i}) \cong D_{s_i} \times [-1,1]$ in $M$ such that $D_{s_i} \times 0 = D_{s_i}$ and $N(D_{s_i}) \cap \Sigma = \partial D_{s_i} \times [-1,1]$. Furthermore, $N(D_{s_i})$'s can be chosen to be mutually disjoint. Define $\Sigma^k_s$ to be the result of shrinking each annulus $\partial D_{s_i} \times [-1,1]$ into an arc across $N(D_{s_i})$. By definition, if $s, s' \in B$ are vertices lying in the same simplex of $K$, then $N(D_s) \cap N(D_{s'}) = \emptyset$. Thus, the above construction can be extended linearly over any simplex of $\tilde{K}$, and defining the sweep-out $\{\Sigma^k_s\}_{s \in B}$.

For simplicity, given a surface $T$ in $M$, we define

$$\gamma(T) := \min\{\text{diam } \partial D \mid D \text{ is a compressing disk for } T\}.$$ 

We write $\text{inj}(M)$ for the injectivity radius of $M$. Now define

$$C := \max\left\{\max_{s \in B} H^2(\Sigma^1_s), \ldots, \max_{s \in B} H^2(\Sigma^n_s), 8\pi(g+1)\right\} + 1,$$

and

$$\delta := \min\{3^{-(1+2+\ldots+d+(d+1))} \cdot \text{inj}(M), \gamma(\Sigma)/2\}.$$

Step 2: Lemmas. Let $k \in \{1, \ldots, n\}$. Note that by the isotopy extension theorem, we can fix diffeomorphisms $f_s : \Sigma \to \Sigma^k_s$ for $s \in \text{int } B$ simultaneously. In the next lemma, which follows from the definition of $\{\Sigma^k_s\}_{s \in B}$, we identify $\Gamma(\Sigma)$ with $\Gamma(\Sigma^k_s)$ through these diffeomorphisms.

Lemma 5.1. Let $S = \partial B^d_1$. If $\epsilon$ is small enough, $\Sigma^k_s$ is $\delta$-compressible for $s \in S$. Furthermore, if $S$ is a triangulation of $S$ such that the diameter of each simplex is small enough, and if we define $\psi : S^0 \to \Gamma(\Sigma)^0$ by sending $s \in S^0$ to one of $\delta$-compressing disks for $\Sigma_s$ (and applying $f_s^{-1}$), then $\psi$ determines the simplicial map $S \to \Gamma(\Sigma)$ homotopic to $\varphi_k$.

Define $U_0 \subset B$ to be the set of points $s$ such that $\Sigma^k_s$ is $\delta$-compressible.

Lemma 5.2. $B \setminus U_0$ is a star-shaped region.

Proof. The proof is by contradiction. Suppose that there exists a line segment $\ell \subset B$ connecting 0 with $s \in B \setminus U_0$ such that $\ell$ contains a point $u \in U_0$. We can find a compressing disk $D$ for $\Sigma^k_s$ with $\text{diam } \partial D \leq \delta$. Let $Q$ be a 3-ball of diameter $\leq \delta$ that contains $\partial D$. After perturbing $Q$, $\partial Q$ intersects $\Sigma^k_s = \Sigma, \Sigma^k_s$ and $\Sigma^k_s$ transversely. If $Q \cap \Sigma_s$ contains circles that are essential in $\Sigma^k_s$, one of such circles bounds a compressing for $\Sigma^k_s$. As $\Sigma^k_s$ is $\delta$-locally incompressible, this case cannot occur. Thus, all the circles in $Q \cap \Sigma^k_s$ are inessential in $\Sigma^k_s$. Similarly, all the circles in $Q \cap \Sigma$ are inessential in $\Sigma$. By the innermost disk argument, we can isotope $Q$ so that $Q$ is contained in the region.
between $\Sigma$ and $\Sigma^k$, which is diffeomorphic to $\Sigma \times I$. Note that $Q$ still intersects $\Sigma^k_0$ so that one of the circles in $Q \cap \Sigma^k_0$ is essential in $\Sigma^k_0$. Thus, $\Sigma^k_\ell$ is compressible in the product region. This is a contradiction because $\Sigma^k_\ell$ is isotopic to a level surface in the product region.

\[\Box\]

**Step 3: Extending the sweep-out.** Fix a nontrivial map $\varphi : S^{d-1} \to \Gamma(\Sigma)$ once. We may assume that $\varphi = \varphi_1$ and set $\{\Sigma_{0s}\}_{s \in B} := \{\Sigma_1\}_{s \in B}$.

Let $h_t : \Sigma \to M$ be any isotopy with $h_0(\Sigma) = h_1(\Sigma) = \Sigma$. To prove Theorem 1.2 we must show that there exists an isotopy $h_t'$ equivalent to $h_t$ such that for $t \in I$

- $\text{Area}(h_t'(\Sigma)) < C$, and
- $h_t'(\Sigma)$ is $\delta$-locally incompressible.

By the isotopy extension theorem, $h_t : \Sigma \to M$ extends to $\tilde{h}_t : M \to M$. Now define $\Sigma_{ts}$ for $t \in I$ and $s \in B$ by

$$
\Sigma_{ts} := \tilde{h}_t(\Sigma_{0s}).
$$

**Lemma 5.3.** $\{\Sigma_{ts}\}_{(t,s) \in I \times B}$ extends to $\{\Sigma_{ts}\}_{(t,s) \in [0,2] \times B}$ such that

1. $\Sigma_{0t} = \Sigma$ for $t \in [1,2]$,
2. $\text{Area}(\Sigma_{2s}) < C$ for $s \in B$,
3. $B \setminus U_2$ is a star-shaped region, where $U_2$ is the set of those points $s$ such that $\Sigma_{2s}$ is $\delta$-compressible.

**Proof.** First, note that there is a natural action of $\text{Diff}(M, \Sigma)$ on $\Gamma(\Sigma)$, which induces the action on $[S^{d-1}, \Gamma(\Sigma)]$. By construction, $\{\Sigma_{1s}\}_{s \in B}$ is the image of $\{\Sigma_{0s}\}_{s \in B}$ by $h_1$. In other words, $\{\Sigma_{1s}\}_{s \in B}$ can be recovered from $h_1 \cdot \varphi$ as follows. We repeat the same construction as in Step 1. Take a triangulation $K$ of $S^{d-1}$ such that $h_1 \cdot \varphi : S^{d-1} \to \Gamma(\Sigma)$ is simplicial. For each vertex $s$ of $K$, $\Sigma_{1s}$ is the result of compressing $\Sigma$ across the disk representing $h_1 : \varphi(s)$. We regard $B$ as a cone over $K$ and extend the construction of $\Sigma_{1s}$ linearly over each simplex, recovering the sweep-out $\{\Sigma_{1s}\}_{s \in B}$.

By assumption, there is a homotopy $\Phi : [1,2] \times S^{d-1} \to \Gamma(\Sigma)$ such that $\Phi_1 = h_1 \cdot \varphi$ and $\Phi_2$ is a product of $\varphi_k$'s. After passing to a subdivision of $K$ if necessary, we can extend $K$ to a triangulation $\mathcal{L}$ of $[1,2] \times S^{d-1} (= [1,2] \times \partial B)$ such that $\Phi$ is simplicial with respect to $\mathcal{L}$. Define $\Sigma_{0t} = \Sigma$ for $t \in [1,2]$. For a vertex $(t, s)$ of $\mathcal{L}$, define $\Sigma_{ts}$ as the result of compressing $\Sigma$ across the disk representing $\Phi(t, s)$. Again regard $[1,2] \times B$ as a cone over $\partial([1,2] \times B)$ and extend the construction of $\Sigma_{ts}$ linearly over each simplex, obtaining the sweep-out $\{\Sigma_{ts}\}_{(t,s) \in [1,2] \times B}$.

This sweep-out satisfies the desired property. Indeed, if $\ell \subset B$ is a radius of $B$, $\{\Sigma_{2s}\}_{s \in \ell}$ appears in some $\{\Sigma^k_s\}_{s \in B}$ as a subfamily. Thus (2) and (3) hold. (1) is obvious from the construction.

By Theorem 4.1 we have the following lemma.

**Lemma 5.4.** $\{\Sigma_{ts}\}_{(t,s) \in [0,2] \times B}$ can be modified so that afterward $\mathcal{H}^2(\Sigma_{ts}) < C$ for $t \in [0,2]$ and $s \in B$. 

Lemma 5.5. Let $\delta$ be any $\{\text{simplicial map}\}$ $ts$ through this identification. We say $\Sigma$ $\in \{\text{triangulation of}\}$ $Y$ $\triangledown$ triangulation of $\{\text{Y}\}$ $\triangledown$ $\text{extends this identification.}$ We say $\Sigma$ $\in \{\text{such that if}\}$ $\delta$ $\in \{\text{Y}\}$ $\triangledown$ $\text{are} \epsilon$-close if $d_M(f_{ts}(x), f_{ts'}(x)) < \epsilon$ for any $x \in \Sigma_0$. Finally, define $U \subset [0, 2] \times B$ to be the set of all points $(t, s)$ such that $\Sigma_{ts}$ is $\delta$-compressible.

**Proof.** To extend $\psi$ to a simplicial map $\tilde{\psi}$, it suffices to find a collection $\{D_y \mid y \in \Sigma_0\}$ of disks with the following property:

- For $y \in \Sigma_0$, $D_y$ is a compressing disk for $\Sigma_y$.
- If $y$ and $y'$ are in the same simplex of $\Sigma'$, then $f_y^{-1}(D_y)$ and $f_{y'}^{-1}(D_{y'})$ are disjoint (i.e. they span a 1-simplex in $\Gamma(\Sigma_0)$).

Indeed, if such a collection of disks exists, we can define $\tilde{\psi}$ by $\tilde{\psi}(y) = f_y^{-1}(D_y)$. The proof is by induction: we will show

**Claim.** If $D_y$ has already been defined for $y'$ the barycenter of any $(i-1)$-simplex of $\Sigma'$ and $\text{diam} \partial D_{y'} < 3^{i-\cdots+1} \cdot \delta$ holds, then we can find $D_y$ for $y$ the barycenter of any $i$-simplex $\Sigma'$ such that $\text{diam} \partial D_{y'} < 3^{i-\cdots+i+1} \cdot \delta$.

Let $\sigma$ be an $i$-simplex of $\Sigma'$ and let $y$ be the barycenter of $\sigma$. Let $y_1, \ldots, y_{2^{i+1}-1} = y$ be the vertices of $\Sigma'$ that are contained in $\sigma$. By induction, for $1 \leq j \leq 2^{i+1} - 2$, $D_{y_j}$ has already been defined and the diameter of $\partial D_{y_j}$ is less than $3^{1+i+\cdots+i} \cdot \delta$. As $\Sigma_{y_j}$ and $\Sigma_y$ are $\epsilon$-close, the image of $\partial D_{y_j}$ on $\Sigma_y$ has diameter less than $3^{i+\cdots+i} \cdot \delta + 2\epsilon$. In what follows, we work on a single surface, say $\Sigma_y$, rather than multiple surfaces. We will not distinguish between $D_{y_j}$ and its image on $\Sigma_y$ from their notation.

After relabeling $y_j$'s if necessary, we can assume that there exists a number $k$ ($1 \leq k \leq 2^{i+1} - 2$) satisfying the following: there exists a metric ball $Q$ with $\text{diam} Q < k(3^{1+i+\cdots+i} \cdot \delta + 2\epsilon)$ such that $\bigcup_{j=1}^{k} \partial D_{y_j}$ is contained in $Q$ while $\bigcup_{j=k+1}^{2^{i+1}-2} \partial D_{y_j}$ is in the complement of $Q$. Note that $Q$ is a genuine 3-ball because

$$\text{diam} Q < k(3^{1+i+\cdots+i} \cdot \delta + 2\epsilon) \leq (2^{i+1} - 2) \cdot (3^{1+i+\cdots+i} \cdot \delta + 2\epsilon) < 3^{i+\cdots+i+1} \cdot \delta \leq \text{inj}(M).$$

After perturbing $Q$, we assume that $\partial Q$ intersects $\Sigma_y$ transversely. We can find a circle in $\partial Q \cap \Sigma_y$ that is essential in $\Sigma_y$. Indeed, if all the circles in $\partial Q \cap \Sigma_y$ were inessential, by the innermost disk argument, $\Sigma_y$ could be isotoped so that afterward $\Sigma_y \subset Q$. This is impossible because $\Sigma_y$ is a Heegaard surface and $M$ is not a 3-sphere. So one of the
circles in $\partial Q \cap \Sigma_y$ bounds a compressing disk for $\Sigma_y$. Define $D_y$ as such a disk. By definition, $D_y \cap D_{y_j} = \emptyset$ for $1 \leq j \leq 2^i + 1 - 2$ and $\text{diam} \partial D_y < 3^{1+\ldots+(i+1)} \delta$, which prove the claim. \hfill $\Box$

**Step 5: The conclusion.** We now finish the proof of Theorem 1.2. If $(0, 0)$ and $(2, 0)$ can be connected by a path in $[0, 2] \times B$ without meeting $U$, it defines an isotopy $h'$ with the desired property, proving the theorem. Thus, it suffices to show that $[0, 2] \times B \setminus U$ is path-connected. We will prove this by contradiction. Recall that $S = \partial B^d_{1-\epsilon}$.

**Claim 2.** There exists a compact orientable $d$-manifold $Y$ in $U$ with $\partial Y = \emptyset \times S$.

**Proof.** Consider the map $f : [0, 2] \times \text{int} B \to \mathbb{R}$ given by $f(t, s) := \gamma(\Sigma_{ts})$. Since $\Sigma_{ts}$ varies smoothly for $(t, s) \in [0, 2] \times \text{int} B$, $f$ is a continuous function. By the smooth approximation theorem, $f$ is approximated by a smooth map $f'$. Let $r \in \mathbb{R}$ be a regular value of $f'$ just below $\delta$. By assumption, one of the components of $f'^{-1}(r)$, say $Y'$, separates $(0, 0)$ from $(2, 0)$. On the other hand, by construction, if $(t, s)$ is close enough to $[0, 2] \times \partial B$, then $f'(t, s) < r$. This implies that $\partial Y' \subset \{0, 2\} \times B$. By Lemmas 5.2 and 5.3 (3), $Y'$ extends to a $d$-manifold $Y$ in $U$ with $\partial Y = \emptyset \times S$. \hfill $\Box$

Pick a triangulation $\mathcal{Y}$ of $Y$ such that the diameter of any simplex of $\mathcal{Y}$ is small enough. Let $\mathcal{Y}'$ be the barycentric subdivision of $\mathcal{Y}$. By Lemma 5.5 we can find a simplicial map $\bar{\psi} : \mathcal{Y}' \to \Gamma(\Sigma)$. By Lemma 5.1 the restriction of $\bar{\psi}$ on $\partial Y$ must be homotopic to $\varphi$. Thus, $\varphi$ is homologically trivial. If $d \neq 2$, the Hurewicz theorem implies that $\varphi$ is homotopically trivial, contradicting the choice of $\varphi$.

If $d = 2$, we can deduce a contradiction as follows. Let $V$ and $W$ be the handlebodies in $M$ bounded by $\Sigma$. Denote by $\Gamma_V(\Sigma)$ (resp. $\Gamma_W(\Sigma)$) the subcomplex of $\Gamma(\Sigma)$ spaned by compressing disks for $\Sigma$ that lie in $V$ (resp. $W$). Furthermore, denote by $\Gamma_{VW}(\Sigma)$ the union of all simplices that contain vertices in both $\Gamma_V(\Sigma)$ and $\Gamma_W(\Sigma)$. Thus, $\Gamma(\Sigma) = \Gamma_V(\Sigma) \cup \Gamma_{VW}(\Sigma) \cup \Gamma_W(\Sigma)$. Recall that the choice of $\varphi$ is arbitrary as long as it is homotopically nontrivial. By Claim 2.7 in [3], we can assume that $\varphi$ is represented by a loop $\gamma$ in $\Gamma(\Sigma)$ with the following properties:

(a) $\gamma$ can be expressed as $e \cup \gamma_V \cup e' \cup \gamma_W$, where $e$, $e'$ are edges in $\Gamma_{VW}(\Sigma)$ while $\gamma_V$, $\gamma_W$ are paths in $\Gamma_V(\Sigma)$ and $\Gamma_W(\Sigma)$, respectively.

(b) $e$ is in the different component of $\Gamma_{VW}(\Sigma)$ from $e'$.

By definition, $\tilde{\psi}(\partial Y) = \gamma$. Note that $\bar{\psi}^{-1}(\Gamma_V(\Sigma)) \cap \bar{\psi}^{-1}(\Gamma_{VW}(\Sigma)) = \emptyset$. This along with (a) implies that there exists an arc in $\bar{\psi}^{-1}(\Gamma_{VW}(\Sigma))$ connecting $\bar{\psi}^{-1}(e)$ and $\bar{\psi}^{-1}(e')$. This contradicts (b) and completes the proof of Theorem 1.2. \hfill $\Box$

**Proof of Corollary 1.3.** To conclude the paper, we finally prove Corollary 1.3. By Theorems 1.1 and 1.2 $\pi_1(\mathcal{H}(M, \Sigma))$ is finitely generated, and this group projects onto $\text{Isot}(M, \Sigma)$, the subgroup of $\text{MCG}(M, \Sigma)$ that consists of maps $(M, \Sigma) \to (M, \Sigma)$ isotopic to $\text{id}_M$. Since $\text{MCG}(M)$ is a finite group, $\text{Isot}(M, \Sigma)$ has the finite index in $\text{MCG}(M, \Sigma)$. Thus, $\text{MCG}(M, \Sigma)$ is also finitely generated.

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