On the existence of Auslander-Reiten \((d + 2)\)-angles in 
\((d + 2)\)-angulated categories

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Abstract

Let \(C\) be a \((d + 2)\)-angulated category. In this note, we show that if \(C\) is a locally finite, then \(C\) has Auslander-Reiten \((d + 2)\)-angles. This extends a result of Xiao-Zhu for triangulated categories.

Key words: \((d + 2)\)-angulated categories; Auslander-Reiten \((d + 2)\)-angles; locally finite.

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1 Introduction

Auslander-Reiten theory was introduced by Auslander and Reiten in [AR1, AR2]. Since its introduction, Auslander-Reiten theory has become a fundamental tool for studying the representation theory of Artin algebras. Later it has been generalized to these situation of exact categories [Ji], triangulated categories [H, RV] and its subcategories [AS, J] and some certain additive categories [L, J, S] by many authors. Extriangulated categories were recently introduced by Nakaoka and Palu [NP] as a simultaneous generalization of exact categories and triangulated categories. Hence, many results hold on exact categories and triangulated categories can be unified in the same framework. Iyama, Nakaoka and Palu [NP] introduced the notion of almost split extensions and Auslander-Reiten-Serre duality for extriangulated categories, and gave explicit connections between these notions and also with the classical notion of dualizing \(k\)-varieties. Xiao and Zhu [XZ1, XZ2] showed that if a triangulated category \(C\) is locally finite, then \(C\) has Auslander-Reiten triangles. Recently, Zhu and Zhuang [ZZ] proved that if an extriangulated category \(C\) is locally finite, then \(C\) has Auslander-Reiten \(E\)-triangles.

In [GKO], Geiss, Keller and Oppermann introduced \((d + 2)\)-angulated categories. These are generalizations of triangulated categories, in the sense that triangles are replaced by \((d + 2)\)-angles, that is, morphism sequences of length \(d + 2\). Thus a 1-angulated category is precisely a triangulated category. Iyama and Yoshino [IY] defined Auslander-Reiten \((d + 2)\)-angle in special \((d + 2)\)-angulated categories. Later, Fedele [F] defined Auslander-Reiten \((d + 2)\)-angles in additive subcategories of \((d + 2)\)-angulated categories closed under \(d\)-extensions, an example

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of which are wide subcategories. He also proved that there are Auslander-Reiten \((d+2)\)-angles in certain additive subcategories of \((d + 2)\)-angulated categories. Recently, the author [Z] showed that a \((d + 2)\)-angulated category \(C\) has Auslander-Reiten \((d + 2)\)-angles if and only if \(C\) has a Serre functor.

In this note, we continue to study Auslander-Reiten \((d + 2)\)-angles in \((d + 2)\)-angulated categories. We will generalise Xiao-Zhu’s result into \((d + 2)\)-angulated categories. Moreover, our proof is not far from the usual triangulated case.

Our main result is the following.

**Theorem 1.1.** (see Theorem 3.8 for details) Let \(C\) be a locally finite \((d+2)\)-angulated category. If \(X\) is an indecomposable, then there exists an Auslander-Reiten \((d + 2)\)-angle ending at \(X\), and if \(X\) is an indecomposable, then there exists an Auslander-Reiten \((d + 2)\)-angle starting at \(X\). Thus \(C\) has Auslander-Reiten \((d + 2)\)-angles.

This article is organised as follows: In Section 2, we review some elementary definitions that we need to use, including \((d + 2)\)-angulated categories and Auslander-Reiten \((d + 2)\) angles. In Section 3, we show our main result.

## 2 Preliminaries

In this section, we first recall the definition and basic properties of \((d + 2)\)-angulated categories from [GK]. Let \(C\) be an additive category with an automorphism \(\Sigma^d : C \rightarrow C\), and an integer \(d\) greater than or equal to one.

A \((d + 2)\)-\(\Sigma^d\)-sequence in \(C\) is a sequence of objects and morphisms

\[
A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{d-1}} A_n \xrightarrow{f_d} A_{d+1} \xrightarrow{f_{d+1}} \Sigma^d A_0.
\]

Its **left rotation** is the \((d + 2)\)-\(\Sigma^d\)-sequence

\[
A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{d-1}} A_{d+1} \xrightarrow{f_{d+1}} \Sigma^d A_0 \xrightarrow{(-1)^d \Sigma^d f_0} \Sigma^d A_1.
\]

A morphism of \((d + 2)\)-\(\Sigma^d\)-sequences is a sequence of morphisms \(\varphi = (\varphi_0, \varphi_1, \cdots, \varphi_{d+1})\) such that the following diagram commutes

\[
\begin{array}{cccccccc}
A_0 & \xrightarrow{f_0} & A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_d} & A_{d+1} & \xrightarrow{f_{d+1}} & \Sigma^d A_0 \\
\varphi_0 & & \varphi_1 & & \varphi_2 & & \cdots & & \varphi_{d+1} & & \Sigma^d \varphi_0 \\
B_0 & \xrightarrow{g_0} & B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & \cdots & \xrightarrow{g_d} & B_{d+1} & \xrightarrow{g_{d+1}} & \Sigma^d B_0
\end{array}
\]

where each row is a \((d + 2)\)-\(\Sigma^d\)-sequence. It is an **isomorphism** if \(\varphi_0, \varphi_1, \varphi_2, \cdots, \varphi_{d+1}\) are all isomorphisms in \(C\).

**Definition 2.1.** [GK] A \((d+2)\)-angulated category is a triple \((C, \Sigma^d, \Theta)\), where \(C\) is an additive category, \(\Sigma^d\) is an automorphism of \(C\) (\(\Sigma^d\) is called the \(d\)-suspension functor),
and $\Theta$ is a class of $(d+2)$-$\Sigma^d$-sequences (whose elements are called $(d+2)$-angles), which satisfies the following axioms:

**(N1)** (a) The class $\Theta$ is closed under isomorphisms, direct sums and direct summands.

(b) For each object $A \in \mathcal{C}$ the trivial sequence

$$A \xrightarrow{f_0} A \to 0 \to 0 \to \cdots \to 0 \to \Sigma^d A$$

belongs to $\Theta$.

(c) Each morphism $f_0: A_0 \to A_1$ in $\mathcal{C}$ can be extended to $(d+2)$-$\Sigma^d$-sequence:

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{d-1}} A_d \xrightarrow{f_d} A_{d+1} \xrightarrow{f_{d+1}} \Sigma^d A_0.$$

**(N2)** A $(d+2)$-$\Sigma^n$-sequence belongs to $\Theta$ if and only if its left rotation belongs to $\Theta$.

**(N3)** Each solid commutative diagram

$$\begin{array}{c}
A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_d} A_{d+1} \xrightarrow{f_{d+1}} \Sigma^d A_0 \\
\downarrow \varphi_0 \downarrow \varphi_1 \downarrow \varphi_2 \downarrow \varphi_{d+1} \downarrow \Sigma^d \varphi_0 \\
B_0 \xrightarrow{g_0} B_1 \xrightarrow{g_1} B_2 \xrightarrow{g_2} \cdots \xrightarrow{g_d} B_{d+1} \xrightarrow{g_{d+1}} \Sigma^d B_0
\end{array}$$

with rows in $\Theta$, the dotted morphisms exist and give a morphism of $(d+2)$-$\Sigma^d$-sequences.

**(N4)** In the situation of (N3), the morphisms $\varphi_2, \varphi_3, \ldots, \varphi_{d+1}$ can be chosen such that the mapping cone

$$A_1 \oplus B_0 \xrightarrow{\begin{pmatrix} -f_1 & 0 \\ \varphi_1 & g_0 \end{pmatrix}} A_2 \oplus B_1 \xrightarrow{\begin{pmatrix} -f_2 & 0 \\ \varphi_2 & g_1 \end{pmatrix}} \cdots \xrightarrow{\begin{pmatrix} -f_d & 0 \\ \varphi_d & g_{d+1} \end{pmatrix}} \Sigma^n A_0 \oplus B_{d+1} \xrightarrow{\begin{pmatrix} -\Sigma^d f_0 & 0 \\ \Sigma^d \varphi_1 & g_{d+1} \end{pmatrix}} \Sigma^d A_1 \oplus \Sigma^d B_0$$

belongs to $\Theta$.

Now we give an example of $(d+2)$-angulated categories.

**Example 2.2.** We recall the standard construction of $(d+2)$-angulated categories given by Geiss-Keller-Oppermann [GKO, Theorem 1]. Let $\mathcal{C}$ be a triangulated category and $\mathcal{T}$ a $d$-cluster tilting subcategory which is closed under $\Sigma^d$, where $\Sigma$ is the shift functor of $\mathcal{C}$. Then $(\mathcal{T}, \Sigma^d, \Theta)$ is a $(d+2)$-angulated category, where $\Theta$ is the class of all sequences

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{d-1}} A_d \xrightarrow{f_d} A_{d+1} \xrightarrow{f_{d+1}} \Sigma^d A_0$$

such that there exists a diagram

$$\begin{array}{c}
A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{d-1}} A_d \xrightarrow{f_d} A_{d+1} \\
\begin{array}{c}
A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_d} A_d \\
A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_d} A_d
\end{array}
\end{array}$$
with $A_i \in T$ for all $i \in \mathbb{Z}$, such that all oriented triangles are triangles in $C$, all non-oriented triangles commute, and $f_{d+1}$ is the composition along the lower edge of the diagram.

The following two lemmas are very useful which are needed later on.

**Lemma 2.3.** [F, Lemma 3.13] Let $C$ be a $(d+2)$-angulated category, and

$$A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} A_{d+1} \xrightarrow{\alpha_{d+1}} \Sigma^d A_0.$$  \hspace{1cm} (2.1)

a $(d+2)$-angle in $C$. Then the following are equivalent:

1. $\alpha_0$ is a section;
2. $\alpha_d$ is a retraction;
3. $\alpha_{d+1} = 0$.

If a $(d+2)$-angle (2.1) satisfies one of the above equivalent conditions, it is called split.

**Lemma 2.4.** [LZ, Corollary 3.4] Let $C$ be a $(d+2)$-angulated category, and

$$A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} A_{d+1} \xrightarrow{\alpha_{d+1}} \Sigma^d A_0.$$  \hspace{1cm} (2.1)

a $(d+2)$-angle in $C$. Then for any morphism $\varphi_0 : A_0 \to B_0$, there exists the following commutative diagram of $(d+2)$-angles

$$
\begin{array}{cccccccccccc}
A_0 & \xrightarrow{\alpha_0} & A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_{d-1}} & A_d & \xrightarrow{\alpha_d} & A_{d+1} & \xrightarrow{\alpha_{d+1}} & \Sigma^d A_0 \\
| & & | & & | & & | & & | & & | & & | \\
\varphi_0 & \xrightarrow{\beta_0} & \varphi_1 & \xrightarrow{\beta_1} & \varphi_2 & \xrightarrow{\beta_2} & \cdots & \xrightarrow{\varphi_d} & \beta_d & \xrightarrow{\beta_{d+1}} & \Sigma^d B_0 \\
B_0 & \xrightarrow{\beta_0} & B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & \cdots & \xrightarrow{\beta_d} & B_{d+1} & \xrightarrow{\beta_{d+1}} & \Sigma^d B_0 \\
\end{array}
$$

such that

$$A_0 \xrightarrow{(\alpha_0, \beta_0)} A_1 \oplus B_0 \xrightarrow{(\alpha_1, \beta_1)} A_2 \oplus B_1 \xrightarrow{(\alpha_2, \beta_2)} \cdots \xrightarrow{(\alpha_{d-1}, \beta_{d-1})} A_d \oplus B_{d-1} \xrightarrow{(\alpha_d, \beta_d)} \Sigma^d A_0 \text{ is a } (d+2)-\text{angle in } C.$$

Now we recall an Auslander-Reiten $(d+2)$ theory in $(d+2)$-angulated categories.

We denote by $\text{rad}_C$ the Jacobson radical of $C$. Namely, $\text{rad}_C$ is an ideal of $C$ such that $\text{rad}_C(A, A)$ coincides with the Jacobson radical of the endomorphism ring $\text{End}(A)$ for any $A \in C$.

**Definition 2.5.** [LY, Definition 3.8] and [F, Definition 5.1] Let $C$ be a $(d+2)$-angulated category. A $(d+2)$-angle

$$A_* : \hspace{0.5cm} A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} A_{d+1} \xrightarrow{\alpha_{d+1}} \Sigma^d A_0$$

in $C$ is called an Auslander-Reiten $(d+2)$-angle if $\alpha_0$ is left almost split, $\alpha_d$ is right almost split and when $d \geq 2$, also $\alpha_1, \alpha_2, \cdots, \alpha_{d-1}$ are in $\text{rad}_C$. 

Remark 2.6. [F, Remark 5.2] Assume $A_\bullet$ as in the above definition is an Auslander-Reiten $(d + 2)$-angle. Since $\alpha_0$ is left almost split implies that $\text{End}(A_0)$ is local and hence $A_0$ is indecomposable. Similarly, since $\alpha_d$ is right almost split, then $\text{End}(A_{d+1})$ is local and hence $A_{d+1}$ is indecomposable. Moreover, when $d = 1$, we have $\alpha_0$ and $\alpha_d$ in $\text{rad}_C$, so that $\alpha_d$ is right minimal and $\alpha_0$ is left minimal. When $d \geq 2$, since $\alpha_{d-1} \in \text{rad}_C$, we have that $\alpha_d$ is right minimal and similarly $\alpha_0$ is left minimal.

Remark 2.7. [F, Lemma 5.3] Let $C$ be a $(d + 2)$-angulated category and $A_\bullet : A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} A_{d+1} \xrightarrow{\alpha_{d+1}} \Sigma^d A_0$ be a $(d+2)$-angle in $C$. Then the following statements are equivalent:

1. $A_\bullet$ is an Auslander-Reiten $(d + 2)$-angle;
2. $\alpha_0, \alpha_1, \cdots, \alpha_{d-1}$ are in $\text{rad}_C$ and $\alpha_d$ is right almost split;
3. $\alpha_1, \alpha_2, \cdots, \alpha_d$ are in $\text{rad}_C$ and $\alpha_0$ is left almost split.

Lemma 2.8. [F, Lemma 5.4] Let $C$ be a $(d + 2)$-angulated category and $A_\bullet : A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} A_{d+1} \xrightarrow{\alpha_{d+1}} \Sigma^d A_0$ be a $(d+2)$-angle in $C$. Assume that $\alpha_d$ is right almost split and if $d \geq 2$, also that $\alpha_1, \alpha_2, \cdots, \alpha_{d-1}$ are in $\text{rad}_C$. Then the following are equivalent:

1. $A_\bullet$ is an Auslander-Reiten $(d + 2)$-angle;
2. $\text{End}(A_0)$ is local;
3. $\alpha_{d+1}$ is left minimal;
4. $\alpha_0$ is in $\text{rad}_C$.

In the case $d = 1$, so in the case of a triangulated category, a morphism can be extended to a triangle in a unique way up to isomorphism. On the other hand, for $d \geq 2$, a morphism can be extended to a $(d + 2)$-angle in different non-isomorphic ways. However, we still have a unique “minimal” $(d + 2)$-angle extending any given morphism.

Lemma 2.9. [OT, Lemma 5.18] and [F, Lemma 3.14] Let $d \geq 2$ and $h : A_{d+1} \to \Sigma^d A_0$ be any morphism in a $(d + 2)$-angulated category $C$. Then, up to isomorphism, there exists a unique $(d + 2)$-angle of the form

$$A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-2}} A_{d-1} \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} A_{d+1} \xrightarrow{\alpha_{d+1}} \Sigma^d A_0$$

with $\alpha_1, \alpha_2, \cdots, \alpha_{d-1}$ in $\text{rad}_C$. 
3 Proof of main result

In this section, let $k$ be a field. We always assume that $C$ is a $k$-linear Hom-finite Krull-Schmidt $(d+2)$-angulated category. We denote by $\text{ind}(C)$ the set of isomorphism classes of indecomposable objects in $C$. For any $X \in \text{ind}(C)$, we denote by $\text{Supp}\text{Hom}_C(X, -)$ the subcategory generated by objects $Y$ in $\text{ind}(C)$ with $\text{Hom}_C(X,Y) \neq 0$. Similarly, $\text{Supp}\text{Hom}_C(-, X)$ denotes the subcategory generated by objects $Y$ in $\text{ind}(C)$ with $\text{Hom}_C(Y,X) \neq 0$. If $\text{Supp}\text{Hom}_C(X, -)$ (respectively) contains only finitely many indecomposables, we say that $|\text{Supp}\text{Hom}_C(X, -)| < \infty$ (respectively).

Based on the definition of locally finite triangulated categories [XZ1, XZ2], we define the notion of locally finite $(d+2)$-angulated categories.

**Definition 3.1.** A $(d+2)$-angulated category $C$ is called locally finite if $|\text{Supp}\text{Hom}_C(X, -)| < \infty$ and $|\text{Supp}\text{Hom}_C(-, X)| < \infty$, for any object $X \in \text{ind}(C)$.

We know that the derived categories of finite dimensional hereditary algebras of finite type and the stable module categories of finite dimensional self-injective algebras of finite type are examples of locally finite triangulated categories, see [XZ1, XZ2]. In those locally finite triangulated categories, we take a $d$-cluster tilting subcategory which is closed under the $d$-th power of the shift functor. By Example 2.2, we can obtain locally finite $(d+2)$-angulated categories.

**Definition 3.2.** Let $C$ be a $(d+2)$-angulated category and $X \in \text{ind}(C)$. We define a set of $(d+2)$-angles as follows:

$$S(X) := \left\{ A_\bullet : A \xrightarrow{a_0} A_1 \xrightarrow{a_1} \cdots \xrightarrow{a_{d-1}} A_d \xrightarrow{a_d} X \xrightarrow{\alpha_{d+1}} \Sigma^d A_0 \mid A_\bullet \text{ is a non-split } (d+2)-\text{angle with } A \in \text{ind}(C), \text{ and when } d \geq 2, \alpha_1, \alpha_2, \cdots, \alpha_{d-1} \text{ in } \text{rad}_C \right\}$$

Dually, we can define a set of $(d+2)$-angles as follows:

$$T(X) := \left\{ A_\bullet : X \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} A \xrightarrow{\alpha_{d+1}} \Sigma^d A_0 \mid A_\bullet \text{ is a non-split } (d+2)-\text{angle with } A \in \text{ind}(C), \text{ and when } d \geq 2, \alpha_1, \alpha_2, \cdots, \alpha_{d-1} \text{ in } \text{rad}_C \right\}$$

**Lemma 3.3.** Let $C$ be a $(d+2)$-angulated category and $X \in \text{ind}(C)$. Then $S(X)$ and $T(X)$ are non-empty.

**Proof.** We only show that $S(X)$ non-empty, dually one can show that $T(X)$ is non-empty.

Since $X \in \text{ind}(C)$, there is an object $A \in C$ such that $\text{Hom}_C(X, \Sigma^d A) \neq 0$. Thus there exists a non-split $(d+2)$-angle:

$$B_\bullet : A \xrightarrow{a_0} B_1 \xrightarrow{a_1} B_2 \xrightarrow{a_2} \cdots \xrightarrow{a_{d-2}} B_{d-1} \xrightarrow{a_{d-1}} B_d \xrightarrow{a_d} X \xrightarrow{h} \Sigma^d A.$$

We decompose $A$ into a direct sum of indecomposable objects $A = \bigoplus_{i=1}^n A_i$. Without loss of generality, we can assume that $A = U \oplus V$ where $U$ and $V$ are indecomposable. By Lemma
We claim that the at least one of the following two \((d+2)\)-angles

\[
\begin{array}{ccccccccc}
U \oplus V & \xrightarrow{(u, v)} & A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_{d-1}} & A_d & \xrightarrow{\alpha_d} & X & \xrightarrow{h} & \Sigma^d U \oplus \Sigma^d V \\
| & & | & & | & & | & & | & & & & \\
(1, 0) & \beta_0 & C_1 & \beta_1 & C_2 & \beta_2 & \cdots & \beta_{d-1} & C_d & \beta_d & X & \beta_{d+1} & \Sigma^d U.
\end{array}
\]

Similarly, for the morphism \((0, 1) : U \oplus V \rightarrow V\), there exists the following commutative diagram of \((d+2)\)-angles

\[
\begin{array}{ccccccccc}
U \oplus V & \xrightarrow{(u, v)} & A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_{d-1}} & A_d & \xrightarrow{\alpha_d} & X & \xrightarrow{h} & \Sigma^d U \oplus \Sigma^d V \\
| & & | & & | & & | & & | & & & & \\
(0, 1) & \gamma_0 & D_1 & \gamma_1 & D_2 & \gamma_2 & \cdots & \gamma_{d-1} & D_d & \gamma_d & X & \gamma_{d+1} & \Sigma^d V.
\end{array}
\]

We claim that the at least one of the following two \((d+2)\)-angles is non-split

\[
\begin{array}{ccccccccc}
U & \xrightarrow{\beta_0} & C_1 & \xrightarrow{\beta_1} & C_2 & \beta_2 & \cdots & \beta_{d-1} & C_d & \beta_d & X & \beta_{d+1} & \Sigma^d U, \\
| & & | & & | & & | & & | & & & & \\
V & \xrightarrow{\gamma_0} & D_1 & \gamma_1 & D_2 & \gamma_2 & \cdots & \gamma_{d-1} & D_d & \gamma_d & X & \gamma_{d+1} & \Sigma^d V.
\end{array}
\]

If not like this, by Lemma 2.3 we obtain that \(\beta_{d+1} = 0 = \gamma_{d+1}\). By (N3), we have the following commutative diagram of \((d+2)\)-angles:

\[
\begin{array}{ccccccccc}
U \oplus V & \xrightarrow{(u, v)} & A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_{d-1}} & A_d & \xrightarrow{\alpha_d} & X & \xrightarrow{h} & \Sigma^d U \oplus \Sigma^d V \\
| & & | & & | & & | & & | & & & & \\
U \oplus V & \xrightarrow{\delta_0} & C_1 & \oplus & D_1 & \xrightarrow{\delta_1} & C_2 & \oplus & D_2 & \delta_2 & \cdots & \delta_{d-1} & C_d & \oplus & D_d & \delta_d & X & \oplus & X & \delta_{d+1} & \Sigma^d U \oplus \Sigma^d V
\end{array}
\]

where \(\delta_i = \left(\begin{smallmatrix} \beta_i \\ 0 \\ \gamma_i \end{smallmatrix}\right)\). It follows that \(h = 0\). This is a contradiction since \(B_\bullet\) is non-split.

For the morphism \(\beta_{d+1} \neq 0\) or \(\gamma_{d+1} \neq 0\), by Lemma 2.9 we can find a \((d+2)\)-angle as we want. This shows that \(S(X)\) is nonempty.

\[
\square
\]

**Definition 3.4.** Let \(C\) be a \((d+2)\)-angulated category, and

\[
A_\bullet : A \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} X \xrightarrow{\alpha_{d+1}} \Sigma^d A_0
\]

\[
B_\bullet : B \xrightarrow{\beta_0} B_1 \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_{d-1}} B_d \xrightarrow{\beta_d} X \xrightarrow{\beta_{d+1}} \Sigma^d B_0
\]

two \((d+2)\)-angles in \(S(X)\). We say that \(A_\bullet > B_\bullet\) if there are morphisms \(\varphi_i \in \text{Hom}_C(A_i, B_i)\), \((i = 0, 1, \ldots, d)\) such that the following diagram commutative:

\[
\begin{array}{ccccccccc}
A & \xrightarrow{\alpha_0} & A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_{d-1}} & A_d & \xrightarrow{\alpha_d} & X & \xrightarrow{\alpha_{d+1}} & \Sigma^d A_0 \\
| & \varphi_0 & | & \varphi_1 & | & \varphi_2 & | & \cdots & | & \varphi_d & | & | \\
B & \xrightarrow{\beta_0} & B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & \cdots & \xrightarrow{\beta_{d-1}} & B_d & \xrightarrow{\beta_d} & X & \xrightarrow{\beta_{d+1}} & \Sigma^d B_0.
\end{array}
\]
We say that \( A_\bullet \sim B_\bullet \) if \( \varphi_0 \) is an isomorphism.

Dually, let
\[
A_\bullet : X \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} A \xrightarrow{\alpha_{d+1}} \Sigma^d A_0
\]
\[
B_\bullet : X \xrightarrow{\beta_0} B_1 \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_{d-1}} B_d \xrightarrow{\beta_d} B \xrightarrow{\beta_{d+1}} \Sigma^d B_0
\]
be two \((d+2)\)-angles in \( T(X) \). We say that \( A_\bullet > B_\bullet \) if there are morphisms \( \varphi_i \in \text{Hom}_C(A_i, B_i) \), \((i = 1, 2, \cdots, d + 1)\) such that the following diagram commutative:
\[
\begin{array}{c}
X \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} A \xrightarrow{\alpha_{d+1}} \Sigma^d X \\
\downarrow \varphi_1 \downarrow \varphi_2 \downarrow \cdots \downarrow \varphi_d \downarrow \varphi_{d+1} \\
X \xrightarrow{\beta_0} B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_{d-1}} B_d \xrightarrow{\beta_d} B \xrightarrow{\beta_{d+1}} \Sigma^d X.
\end{array}
\]
We say that \( A_\bullet \sim B_\bullet \) if \( \varphi_{d+1} \) is an isomorphism.

**Lemma 3.5.** \( S(X) \) is a direct ordered set with the relation defined in Definition 3.4, and \( T(X) \) is a direct ordered set with the relation defined in Definition 3.4.

**Proof.** We just prove the first statement, the second statement proves similarly.

Assume that
\[
A_\bullet : A \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} X \xrightarrow{\alpha_{d+1}} \Sigma^d A_0
\]
\[
B_\bullet : B \xrightarrow{\beta_0} B_1 \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_{d-1}} B_d \xrightarrow{\beta_d} X \xrightarrow{\beta_{d+1}} \Sigma^d B_0
\]
belong to \( S(X) \).

We first show that if \( A_\bullet > B_\bullet \) and \( B_\bullet > A_\bullet \), then \( A_\bullet \sim B_\bullet \).

Since \( A_\bullet > B_\bullet \) and \( B_\bullet > A_\bullet \), we have the following two commutative diagrams
\[
\begin{array}{c}
A \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} X \xrightarrow{\alpha_{d+1}} \Sigma^d A_0 \\
\downarrow \varphi_0 \downarrow \varphi_1 \downarrow \varphi_2 \downarrow \varphi_d \downarrow \varphi_{d+1} \\
B \xrightarrow{\beta_0} B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_{d-1}} B_d \xrightarrow{\beta_d} X \xrightarrow{\beta_{d+1}} \Sigma^d B_0,
\end{array}
\]
\[
\begin{array}{c}
B \xrightarrow{\beta_0} B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_{d-1}} B_d \xrightarrow{\beta_d} X \xrightarrow{\beta_{d+1}} \Sigma^d B_0 \\
\downarrow \psi_0 \downarrow \psi_1 \downarrow \psi_2 \downarrow \psi_d \downarrow \psi_{d+1} \\
A \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} X \xrightarrow{\alpha_{d+1}} \Sigma^d A_0,
\end{array}
\]
Since \( A \) is an indecomposable, we have that \( \text{End}(A) \) is local implies that \( \psi_0 \varphi_0 \) is nilpotent or an isomorphism. If \( \psi_0 \varphi_0 \) is nilpotent, there exists a positive integer \( m \) such that \( (\psi_0 \varphi_0)^m = 0 \). We write \( \omega_1 = \psi_0 \varphi_1 \). Thus we have the following commutative diagram
\[
\begin{array}{c}
A \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-2}} A_{d-1} \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} X \xrightarrow{\alpha_{d+1}} \Sigma^d A_0 \\
\downarrow (\psi_0 \varphi_0)^m \downarrow (\omega_1)^m \downarrow (\omega_2)^m \downarrow (\omega_{d-1})^m \downarrow (\varphi_d)^m \downarrow \Sigma^d (\psi_0 \varphi_0)^m \\
A \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-2}} A_{d-1} \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} X \xrightarrow{\alpha_{d+1}} \Sigma^d A_0.
\end{array}
\]
Then \( \alpha_{d+1} = \Sigma^d (\psi_0 \varphi_0)^m \alpha_{d+1} = 0 \). This is a contradiction since \( A_\bullet \) is non-split. Hence \( \psi_0 \varphi_0 \) is
an isomorphism. By a similar argument we obtain that \( \varphi_0 \psi_0 \) is an isomorphism. This shows that \( \varphi_0 \) is isomorphism. So \( A_\bullet \sim B_\bullet \).

It is clear that if \( A_\bullet > B_\bullet \) and \( B_\bullet > C_\bullet \), then \( A_\bullet \sim C_\bullet \).

Now we show that if \( A_\bullet, B_\bullet \in S(X) \), then there exists \( C_\bullet \in S(X) \) such that \( A_\bullet > C_\bullet \) and \( B_\bullet \sim C_\bullet \).

For the morphism \( \beta_d: B_d \to X \), by the dual of Lemma 2.4, there exists the following commutative diagram of \((d + 2)\)-angles

\[
\begin{array}{cccccccc}
A & \xrightarrow{\gamma_0} & D_1 & \xrightarrow{\gamma_1} & D_2 & \cdots & \xrightarrow{\gamma_{d-1}} & D_d & \xrightarrow{\gamma_d} & B_d & \xrightarrow{\gamma_{d+1}} & \Sigma^d A \\
\downarrow{\psi_1} & & \downarrow{\psi_2} & & \downarrow{\psi_d} & & \downarrow{\psi_d} & & \downarrow{\psi_d} & & \downarrow{\psi_d} & \\
A & \xrightarrow{\alpha_0} & A_1 & \xrightarrow{\alpha_1} & A_2 & \cdots & \xrightarrow{\alpha_{d-1}} & A_d & \xrightarrow{\alpha_d} & X & \xrightarrow{\alpha_{d+1}} & \Sigma^d A
\end{array}
\]

such that

\[ M_\bullet: D_1 \to M_1 \to M_2 \to \cdots \to M_{d-1} \to B_d \oplus A_d \xrightarrow{(\beta_d, \alpha_d)} X \xrightarrow{h} \Sigma^d D_1 \]

is a \((d + 2)\)-angle in \( C \), where \( M_i = D_{i+1} \oplus A_i \) (\( i = 1, 2, \ldots, d - 1 \)). Since \( \beta_d \) and \( \alpha_d \) are not retraction, we have that \((\beta_d, \alpha_d)\) is also not retraction. If not like this, there exists a morphism \( (\iota) : X \to B_d \oplus A_d \) such that \((\beta_d, \alpha_d)(\iota) = 1_X\) and then \( \beta_d s + \alpha_d t = 1_X \). Since \( X \) is an indecomposable, we have that \( \text{End}(X) \) is local implies that either \( \beta_d s \) or \( \alpha_d t \) is an isomorphism. Thus either \( \beta_d \) or \( \alpha_d \) is a retraction, a contradiction. That is, \( M_\bullet \) is non-split.

Without loss of generality, we can assume that \( D_1 = U \oplus V \) where \( U \) and \( V \) are indecomposable. By Lemma 2.4 for the morphism \((1, 0): U \oplus V \to U\), there exists the following commutative diagram of \((d + 2)\)-angles

\[
\begin{array}{cccccccc}
U \oplus V & \xrightarrow{(u, v)} & M_1 & \xrightarrow{\gamma_1} & M_2 & \cdots & \xrightarrow{\gamma_{d-1}} & M_d & \xrightarrow{\gamma_d} & X & \xrightarrow{\gamma_{d+1}} & \Sigma^d U \oplus \Sigma^d V \\
\downarrow{(1, 0)} & & \downarrow{\varphi_1} & & \downarrow{\varphi_1} & & \downarrow{\varphi_1} & & \downarrow{\varphi_1} & & \downarrow{(1, 0)} \\
U & \xrightarrow{\delta_0} & L_1 & \xrightarrow{\delta_1} & L_2 & \cdots & \xrightarrow{\delta_{d-1}} & L_d & \xrightarrow{\delta_d} & X & \xrightarrow{\delta_{d+1}} & \Sigma^d U.
\end{array}
\]

Similarly, for the morphism \((0, 1): U \oplus V \to V\), there exists the following commutative diagram of \((d + 2)\)-angles

\[
\begin{array}{cccccccc}
U \oplus V & \xrightarrow{(u, v)} & M_1 & \xrightarrow{\gamma_1} & M_2 & \cdots & \xrightarrow{\gamma_{d-1}} & M_d & \xrightarrow{\gamma_d} & X & \xrightarrow{\gamma_{d+1}} & \Sigma^d U \oplus \Sigma^d V \\
\downarrow{(0, 1)} & & \downarrow{\varphi_1} & & \downarrow{\varphi_1} & & \downarrow{\varphi_1} & & \downarrow{\varphi_1} & & \downarrow{(0, 1)} \\
V & \xrightarrow{\eta_0} & N_1 & \xrightarrow{\eta_1} & N_2 & \cdots & \xrightarrow{\eta_{d-1}} & N_d & \xrightarrow{\eta_d} & X & \xrightarrow{\eta_{d+1}} & \Sigma^d V.
\end{array}
\]

Using similar arguments as in the proof of Lemma 3.3 we conclude that the at least one of the following two \((d + 2)\)-angles is non-split

\[
U \xrightarrow{\delta_0} L_1 \xrightarrow{\delta_1} L_2 \cdots \xrightarrow{\delta_d} L_d \xrightarrow{h} \Sigma^d U, \\
V \xrightarrow{\eta_0} N_1 \xrightarrow{\eta_1} N_2 \cdots \xrightarrow{\eta_d} N_d \xrightarrow{X} \Sigma^d V.
\]
Without loss of generality, we assume that
\[ U \xrightarrow{\delta_0} L_1 \rightarrow L_2 \rightarrow \cdots \rightarrow L_d \rightarrow X \xrightarrow{h} \Sigma^d U, \]
is non-split. By Lemma 2.9, we can find a non-split \((d + 2)\)-angle
\[ C_\bullet : U \xrightarrow{\lambda_0} C_1 \xrightarrow{\lambda_1} C_2 \xrightarrow{\lambda_2} \cdots \rightarrow C_d \xrightarrow{\lambda_{d-1}} X \xrightarrow{h} \Sigma^d U \]
with \(\lambda_1, \lambda_2, \cdots, \lambda_{d-1}\) in \(\text{rad}_C\). By (N3), we have the following commutative diagram

\[
\begin{array}{ccccccccccc}
A & \xrightarrow{\alpha_0} & A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_{d-1}} & A_d & \xrightarrow{\alpha_d} & X & \xrightarrow{\alpha_{d+1}} & \Sigma^d A_0 \\
\downarrow & & \downarrow & & \downarrow & & \cdots & & \downarrow & & \downarrow & & \downarrow \\
U \oplus V & \xrightarrow{(u, v)} & M_1 & \xrightarrow{\varphi_1} & M_2 & \xrightarrow{\varphi_1} & \cdots & \xrightarrow{\varphi_1} & B_d \oplus A_d & \xrightarrow{\beta_d, \alpha_d} & X & \xrightarrow{h} & \Sigma^d U \\
\downarrow & & \downarrow & & \downarrow & & \cdots & & \downarrow & & \downarrow & & \downarrow \\
U & \xrightarrow{\delta_0} & L_1 & \xrightarrow{\varphi_1} & L_2 & \xrightarrow{\varphi_1} & \cdots & \xrightarrow{\varphi_1} & L_d & \xrightarrow{\varphi_1} & X & \xrightarrow{h} & \Sigma^d U \\
\downarrow & & \downarrow & & \downarrow & & \cdots & & \downarrow & & \downarrow & & \downarrow \\
U & \xrightarrow{\lambda_0} & C_1 & \xrightarrow{\lambda_1} & C_2 & \xrightarrow{\lambda_2} & \cdots & \xrightarrow{\lambda_{d-1}} & C_d & \xrightarrow{\lambda_d} & X & \xrightarrow{h} & \Sigma^d U \\
\end{array}
\]
of \((d + 2)\)-angles. This shows that \(A_\bullet > C_\bullet\).

By (N3), we have the following commutative diagram

\[
\begin{array}{ccccccccccc}
B & \xrightarrow{\beta_0} & B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & \cdots & \xrightarrow{\beta_{d-1}} & B_d & \xrightarrow{\beta_d} & X & \xrightarrow{\beta_{d+1}} & \Sigma^d B_0 \\
\downarrow & & \downarrow & & \downarrow & & \cdots & & \downarrow & & \downarrow & & \downarrow \\
U \oplus V & \xrightarrow{(u, v)} & M_1 & \xrightarrow{\varphi_1} & M_2 & \xrightarrow{\varphi_1} & \cdots & \xrightarrow{\varphi_1} & B_d \oplus A_d & \xrightarrow{\beta_d, \alpha_d} & X & \xrightarrow{h} & \Sigma^d U \\
\downarrow & & \downarrow & & \downarrow & & \cdots & & \downarrow & & \downarrow & & \downarrow \\
U & \xrightarrow{\delta_0} & L_1 & \xrightarrow{\varphi_1} & L_2 & \xrightarrow{\varphi_1} & \cdots & \xrightarrow{\varphi_1} & L_d & \xrightarrow{\varphi_1} & X & \xrightarrow{h} & \Sigma^d U \\
\downarrow & & \downarrow & & \downarrow & & \cdots & & \downarrow & & \downarrow & & \downarrow \\
U & \xrightarrow{\lambda_0} & C_1 & \xrightarrow{\lambda_1} & C_2 & \xrightarrow{\lambda_2} & \cdots & \xrightarrow{\lambda_{d-1}} & C_d & \xrightarrow{\lambda_d} & X & \xrightarrow{h} & \Sigma^d U \\
\end{array}
\]
of \((d + 2)\)-angles. This shows that \(B_\bullet > C_\bullet\). \qed

**Lemma 3.6.** Let \(\mathcal{C}\) be a locally finite \((d + 2)\)-angulated category and \(X \in \text{ind}(\mathcal{C})\). Then \(S(X)\) has a minimal element, and \(T(X)\) has a minimal element.

**Proof.** We just prove the first statement, the second statement proves similarly.

Since \(X \in \text{ind}(\mathcal{C})\), there is an object \(A \in \mathcal{C}\) such that \(\text{Hom}_{\mathcal{C}}(X, \Sigma^d A) \neq 0\). Then there exists a non-split \((d + 2)\)-angle:

\[ A_\bullet : A \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-2}} A_{d-1} \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} X \xrightarrow{h} \Sigma^d A. \]

We decompose \(B_d\) into a direct sum of indecomposable objects \(A_d = \bigoplus_{k=1}^n B_k\). Thus \(A_\bullet\) can be
written as
\[ A_\bullet : A \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{d-2}} A_{d-1} \xrightarrow{\alpha_{d-1}} \bigoplus_{k=1}^{n} B_k \xrightarrow{(b_1,b_2,\ldots,b_n)} X \xrightarrow{h} \Sigma^d A \]
where \( b_k \in \text{rad}_C(B_k, X), k = 1, 2, \ldots, n. \)

Since \( C \) is locally finite, there are only finitely many objects \( X_i \in \text{ind}(C), i = 1, 2, \ldots, m \) such that \( \text{Hom}_C(X_i, X) \neq 0 \). We assume that \( \lambda_{ij}, 1 \leq j \leq q_i \) form a basis of the \( k \)-vector space \( \text{rad}_C(B_k, X) \). Put \( M := \left( \bigoplus_{k=1}^{n} B_k \bigoplus \bigoplus_{i=1}^{m} (X_i \oplus q_i) \right) \), we consider the morphism
\[ \delta := (b_1, b_2, \ldots, b_n, \lambda_{11}, \ldots, \lambda_{ij}, \ldots, \lambda_{mq_m}) \in \text{rad}_C(M, X) \]
which is not retraction, it can be embedded in a \((d+2)\)-angle:
\[ M_\bullet : B \rightarrow M_1 \rightarrow M_2 \rightarrow \cdots \rightarrow M_{d-1} \rightarrow M \xrightarrow{\delta} X \rightarrow \Sigma^d B. \]
Thus \( M_\bullet \) is non-split since \( \delta \) is not retraction. Without loss of generality, we can assume that \( B = U \oplus V \) where \( U \) and \( V \) are indecomposable. By Lemma 2.4 for the morphism \((1, 0) : U \oplus V \rightarrow U \), there exists the following commutative diagram of \((d+2)\)-angles
\[ \begin{array}{cccccccc}
U \oplus V & \xrightarrow{(u, v)} & M_1 & \xrightarrow{(1, 0)} & M_2 & \xrightarrow{\varphi_1} & \cdots & \xrightarrow{\varphi_1} & M & \xrightarrow{(1, 0)} & X & \xrightarrow{\Sigma^d U \oplus \Sigma^d V} \\
| & & | & & | & & | & & | & & | & & |
U & \xrightarrow{\theta_0} & L_1 & \xrightarrow{\varphi_1} & L_2 & \xrightarrow{\varphi_1} & \cdots & \xrightarrow{\varphi_1} & L_d & \xrightarrow{\varphi_1} & X & \xrightarrow{f} & \Sigma^d U.
\end{array} \]
Similarly, for the morphism \((0, 1) : U \oplus V \rightarrow V \), there exists the following commutative diagram of \((d+2)\)-angles
\[ \begin{array}{cccccccc}
U \oplus V & \xrightarrow{(u, v)} & M_1 & \xrightarrow{(0, 1)} & M_2 & \xrightarrow{\varphi_1} & \cdots & \xrightarrow{\varphi_1} & M & \xrightarrow{(0, 1)} & X & \xrightarrow{\Sigma^d U \oplus \Sigma^d V} \\
| & & | & & | & & | & & | & & | & & |
V & \xrightarrow{\eta_0} & N_1 & \xrightarrow{\varphi_1} & N_2 & \xrightarrow{\varphi_1} & \cdots & \xrightarrow{\varphi_1} & N_d & \xrightarrow{\varphi_1} & X & \xrightarrow{f} & \Sigma^d V.
\end{array} \]
Using similar arguments as in the proof of Lemma 3.3, we conclude that the at least one of the following two \((d+2)\)-angles is non-split
\[ \begin{array}{cccccccc}
U & \xrightarrow{\theta_0} & L_1 & \xrightarrow{\varphi_1} & L_2 & \xrightarrow{\varphi_1} & \cdots & \xrightarrow{\varphi_1} & L_d & \xrightarrow{f} & \Sigma^d U, \\
V & \xrightarrow{\eta_0} & N_1 & \xrightarrow{\varphi_1} & N_2 & \xrightarrow{\varphi_1} & \cdots & \xrightarrow{\varphi_1} & N_d & \xrightarrow{f} & \Sigma^d V.
\end{array} \]
Without loss of generality, we assume that
\[ \begin{array}{cccccccc}
U & \xrightarrow{\theta_0} & L_1 & \xrightarrow{\varphi_1} & L_2 & \xrightarrow{\varphi_1} & \cdots & \xrightarrow{\varphi_1} & L_d & \xrightarrow{f} & \Sigma^d U, \\
V & \xrightarrow{\eta_0} & N_1 & \xrightarrow{\varphi_1} & N_2 & \xrightarrow{\varphi_1} & \cdots & \xrightarrow{\varphi_1} & N_d & \xrightarrow{f} & \Sigma^d V.
\end{array} \]
is non-split. By Lemma 2.9 we can find a non-split \((d+2)\)-angle
\[ C_\bullet : U \xrightarrow{\omega_0} C_1 \xrightarrow{\omega_1} C_2 \xrightarrow{\omega_2} \cdots \xrightarrow{\omega_{d-1}} C_d \xrightarrow{\omega_d} X \xrightarrow{f} \Sigma^d U \]
with $\omega_1, \omega_2, \ldots, \omega_{d-1}$ in $\text{rad}_C$. Then $C_\bullet \in S(X)$. By (N3), we have the following commutative diagram

$$
\begin{array}{cccccccccc}
U \oplus V & \xrightarrow{(u, v)} & M_1 & \xrightarrow{\delta_1} & M_2 & \cdots & \xrightarrow{\delta_{d-1}} & M & \xrightarrow{\delta} & X & \xrightarrow{\sum^dU} & \sum^dV \\
U & \xrightarrow{\delta_0} & L_1 & \xrightarrow{\varphi_1} & L_2 & \cdots & \xrightarrow{\varphi_{d-1}} & L_d & \xrightarrow{\varphi} & X & \xrightarrow{\sum^dU} & \sum^dU \\
U & \xrightarrow{\omega_0} & C_1 & \xrightarrow{\omega_1} & C_2 & \cdots & \xrightarrow{\omega_{d-1}} & C_d & \xrightarrow{\omega} & X & \xrightarrow{\sum^dU} & \sum^dU \\
\end{array}
$$

of $(d + 2)$-angles.

For any $D_\bullet \in S(X)$, it can be written as

$$
D_\bullet : D \rightarrow D_1 \rightarrow D_2 \rightarrow \cdots \rightarrow D_{d-1} \rightarrow \bigoplus_{i=1}^p L_i \xrightarrow{d=\sum_{i=1}^p d_i} X \rightarrow \sum^d D
$$

with $d_i \in \text{rad}_C(L_i, X), i = 1, 2, \ldots, p$. Since $D_\bullet \in S(X)$ is non-split, $d$ is not retraction implies that $d \in \text{rad}_C\bigoplus_{i=1}^p L_i, X$. By the definitions of $\lambda_{ij}$ and $\delta$, there exists a morphism $\rho : \bigoplus_{i=1}^p L_i \rightarrow M$ such that $d = \delta \rho$. By (N3), we have the following commutative diagram

$$
\begin{array}{cccccccccc}
D & \xrightarrow{} & D_1 & \xrightarrow{} & D_2 & \cdots & \xrightarrow{} & D_{d-1} & \xrightarrow{} & \bigoplus_{i=1}^p L_i & \xrightarrow{d} & X & \xrightarrow{} & \sum^d D \\
B & \xrightarrow{} & M_1 & \xrightarrow{} & M_2 & \cdots & \xrightarrow{} & M_{d-1} & \xrightarrow{} & M & \xrightarrow{\delta} & X & \xrightarrow{} & \sum^d B \\
\end{array}
$$

of $(d + 2)$-angles, where $B = U \oplus V$. Thus we get the following commutative diagram

$$
\begin{array}{cccccccccc}
D & \xrightarrow{} & D_1 & \xrightarrow{} & D_2 & \cdots & \xrightarrow{} & D_{d-1} & \xrightarrow{} & \bigoplus_{i=1}^p L_i & \xrightarrow{d} & X & \xrightarrow{} & \sum^d D \\
U & \xrightarrow{\omega_0} & C_1 & \xrightarrow{\omega_1} & C_2 & \cdots & \xrightarrow{\omega_{d-2}} & C_{d-1} & \xrightarrow{\omega_{d-1}} & C_d & \xrightarrow{\omega} & X & \xrightarrow{\sum^d U} \\
\end{array}
$$

of $(d + 2)$-angles. This shows that $C_\bullet$ is a minimal element in $S(X)$.

**Remark 3.7.** If there exists a minimal element $S(X)$ or $T(X)$, then it is unique up to isomorphism by Lemma [2.9]

We are now ready to state and prove our main result.

**Theorem 3.8.** Let $C$ be a locally finite $(d + 2)$-angulated category. If $X \in \text{ind}(C)$, then there exists an Auslander-Reiten $(d + 2)$-angle ending at $X$, and if $X \in \text{ind}(C)$, then there exists an Auslander-Reiten $(d + 2)$-angle starting at $X$. Thus $C$ has Auslander-Reiten $(d + 2)$-angles.

**Proof.** Since $X \in \text{ind}(C)$, we know that $S(X)$ is non-empty by Lemma [3.3] By Lemma [3.6]
there exists a \((d+2)\)-angle

\[
A_\bullet : A \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{d-1}} A_d \xrightarrow{\alpha_d} X \xrightarrow{\alpha_{d+1}} \Sigma^d A_0
\]

which is a minimal element in \(S(X)\). Since \(A_\bullet \in S(X)\), we have that \(\alpha_1, \alpha_2, \cdots, \alpha_{d-1} \in \text{rad}_C\) and \(A\) is an indecomposable. Then \(\text{End}(A)\) is local.

We want to prove that \(A_\bullet\) is an Auslander-Reiten \((d+2)\)-angle, by Lemma 2.8 it suffices to show that \(\alpha_d\) is right almost split.

Assume that \(g : M \to X\) is not retraction. By the dual of Lemma 2.4 there exists the following commutative diagram of \((d+2)\)-angles

\[
\begin{array}{ccc}
A & \xrightarrow{\gamma_0} & B_1 & \xrightarrow{\gamma_1} B_2 & \cdots & \xrightarrow{\gamma_{d-1}} B_d & \xrightarrow{\gamma_d} M & \xrightarrow{\gamma_{d+1}} \Sigma^d A \\
\downarrow \psi_1 & & \downarrow \psi_2 & & & & \downarrow \psi_d & & \\
A & \xrightarrow{\alpha_0} & A_1 & \xrightarrow{\alpha_1} A_2 & \cdots & \xrightarrow{\alpha_{d-1}} A_d & \xrightarrow{\alpha_d} X & \xrightarrow{\alpha_{d+1}} \Sigma^d A \\
\end{array}
\]

such that

\[
N_\bullet : B_1 \to N_1 \to N_2 \to \cdots \to N_{d-1} \to M \oplus A_d \xrightarrow{(g, \alpha_d)} X \xrightarrow{h} \Sigma^d B_1
\]
is a \((d+2)\)-angle in \(C\), where \(N_i = B_{i+1} \oplus A_i, \ i = 1, 2, \cdots, d-1\). Since \(g\) and \(\alpha_d\) are not retraction, we have that \((g, \alpha_d)\) is also not retraction by using similar arguments as in the proof of Lemma 3.5. That is, \(N_\bullet\) is non-split.

Without loss of generality, we can assume that \(B_1 = U \oplus V\) where \(U\) and \(V\) are indecomposable. By Lemma 2.4 for the morphism \((1, 0) : U \oplus V \to U\), there exists the following commutative diagram of \((d+2)\)-angles

\[
\begin{array}{ccc}
U \oplus V & \xrightarrow{(u, v)} & N_1 & \xrightarrow{\varphi_1} N_2 & \cdots & \xrightarrow{\varphi_{d-1}} N_d & \xrightarrow{\varphi_d} M \oplus A_d & \xrightarrow{\varphi_{d+1}} X & \xrightarrow{\varphi_{d+2}} \Sigma^d U \oplus \Sigma^d V \\
\downarrow \delta_1 & & \downarrow \delta_2 & & & & \downarrow \delta_d & & \\
U & \xrightarrow{\delta_0} & L_1 & \xrightarrow{\delta_1} L_2 & \cdots & \xrightarrow{\delta_{d-1}} L_d & \xrightarrow{\delta_d} X & \xrightarrow{\delta_{d+1}} \Sigma^d U \\
\end{array}
\]

Similarly, for the morphism \((0, 1) : U \oplus V \to V\), there exists the following commutative diagram of \((d+2)\)-angles

\[
\begin{array}{ccc}
U \oplus V & \xrightarrow{(u, v)} & N_1 & \xrightarrow{\varphi_1} N_2 & \cdots & \xrightarrow{\varphi_{d-1}} N_d & \xrightarrow{\varphi_d} M \oplus A_d & \xrightarrow{\varphi_{d+1}} X & \xrightarrow{\varphi_{d+2}} \Sigma^d U \oplus \Sigma^d V \\
\downarrow \eta_0 & & \downarrow \eta_1 & & & & \downarrow \eta_d & & \\
V & \xrightarrow{\eta_0} & Q_1 & \xrightarrow{\eta_1} Q_2 & \cdots & \xrightarrow{\eta_{d-1}} Q_d & \xrightarrow{\eta_d} X & \xrightarrow{\eta_{d+1}} \Sigma^d V \\
\end{array}
\]

Using similar arguments as in the proof of Lemma 3.3 we conclude that the at least one of the following two \((d+2)\)-angles is non-split

\[
\begin{array}{ccc}
U & \xrightarrow{\delta_0} & L_1 & \xrightarrow{\delta_1} L_2 & \cdots & \xrightarrow{\delta_{d-1}} L_d & \xrightarrow{\delta_d} X & \xrightarrow{\delta_{d+1}} \Sigma^d U, \\
V & \xrightarrow{\eta_0} & Q_1 & \xrightarrow{\eta_1} Q_2 & \cdots & \xrightarrow{\eta_{d-1}} Q_d & \xrightarrow{\eta_d} X & \xrightarrow{\eta_{d+1}} \Sigma^d V.
\end{array}
\]
Without loss of generality, we assume that
\[
U \xrightarrow{d_0} L_1 \xrightarrow{d_1} L_2 \cdots \xrightarrow{d_d} X \xrightarrow{f} \Sigma^d U,
\]
is non-split. By Lemma 2.9, we can find a non-split \((d+2)\)-angle
\[
C_\bullet : U \xrightarrow{\lambda_0} C_1 \xrightarrow{\lambda_1} C_2 \xrightarrow{\lambda_2} \cdots \xrightarrow{\lambda_{d-1}} C_d \xrightarrow{\lambda_d} X \xrightarrow{f} \Sigma^d U
\]
with \(\lambda_1, \lambda_2, \cdots, \lambda_{d-1}\) in \(\text{rad}\, C\). By (N3), we have the following commutative diagram

\[
\begin{array}{ccccccccc}
A & \xrightarrow{\alpha_0} & A_1 & \xrightarrow{\alpha_1} & A_2 & \cdots & \xrightarrow{\alpha_d} & A_d & \xrightarrow{\alpha_{d+1}} & \Sigma^d A_0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\begin{array}{ccc}
U \oplus V & \xrightarrow{(u, v)} & N_1 & \xrightarrow{\gamma_1} & N_2 & \cdots & \xrightarrow{(g, \alpha_d)} & M \oplus A_d & \xrightarrow{X} & \Sigma^d U \oplus \Sigma^d V \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
U & \xrightarrow{d_0} & L_1 & \xrightarrow{\lambda_0} & L_2 & \cdots & \xrightarrow{\lambda_{d-1}} & L_d & \xrightarrow{\lambda_d} & X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
U & \xrightarrow{\lambda_0} & C_1 & \xrightarrow{\lambda_1} & C_2 & \cdots & \xrightarrow{\lambda_{d-1}} & C_d & \xrightarrow{\lambda_d} & X
\end{array}
\end{array}
\]

of \((d+2)\)-angles. We obtain that \(A_\bullet > C_\bullet\) implies that \(A_\bullet \sim C_\bullet\) since \(A_\bullet\) is the minimal element in \(S(X)\). Thus there exists the following commutative diagram

\[
\begin{array}{ccccccccc}
U & \xrightarrow{\lambda_0} & C_1 & \xrightarrow{\lambda_1} & C_2 & \cdots & \xrightarrow{\lambda_{d-1}} & C_d & \xrightarrow{\lambda_d} & X \xrightarrow{f} \Sigma^d U \\
A & \xrightarrow{\alpha_0} & A_1 & \xrightarrow{\alpha_1} & A_2 & \cdots & \xrightarrow{\alpha_d} & A_d & \xrightarrow{\alpha_{d+1}} & \Sigma^d A_0
\end{array}
\]

of \((d+2)\)-angles. Hence we get the following commutative diagram

\[
\begin{array}{ccccccccc}
U \oplus V & \xrightarrow{(u, v)} & N_1 & \xrightarrow{\gamma_1} & N_2 & \cdots & \xrightarrow{(g, \alpha_d)} & M \oplus A_d & \xrightarrow{X} & \Sigma^d U \oplus \Sigma^d V \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
A & \xrightarrow{\alpha_0} & A_1 & \xrightarrow{\alpha_1} & A_2 & \cdots & \xrightarrow{\alpha_d} & A_d & \xrightarrow{\alpha_{d+1}} & \Sigma^d A_0
\end{array}
\]

of \((d+2)\)-angles. It follows that \(g = a\alpha_d\). This shows that \(\alpha_d\) is right almost split.

Similarly, we can show that if \(X \in \text{ind}(C)\), then there exists an Auslander-Reiten \((d+2)\)-angle starting at \(X\). Thus \(C\) has Auslander-Reiten \((d+2)\)-angles.

\[\square\]

Remark 3.9. As a special case of Theorem 3.8 when \(d = 1\), that is, if \(C\) is a locally finite triangulated category, then \(C\) has Auslander-Reiten triangles, see [XZ1] [XZ2].

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