Stable determination of a rigid inclusion in an anisotropic elastic plate

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Abstract

In this paper we consider the inverse problem of determining a rigid inclusion inside a thin plate by applying a couple field at the boundary and by measuring the induced transversal displacement and its normal derivative at the boundary of the plate. The plate is made by non-homogeneous linearly elastic material belonging to a general class of anisotropy. For this severely ill-posed problem, under suitable a priori regularity assumptions on the boundary of the inclusion, we prove a stability estimate of log-log type.

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1 Introduction

The present study continues a line of research on Non-Destructive Techniques (NDT) in the theory of elastic plates. Specifically, the paper deals
with the determination of a rigid inclusion embedded in elastic plates by measurements taken at the boundary. This inverse problem arises in damage assessment of plates which are possibly defective due to the presence of interior rigid inclusions induced during the manufacturing process. The most simple and common NDT is the visual inspection of the specimen. However, the visual inspection is unable to detect a rigid inclusion which is embedded in a plate or which is indistinguishable from the surrounding material. Other conventional NDTs, such as those based on thermal or ultrasonic analysis, X-rays methods and others, are local by nature. To be effective, these methods require the vicinity of the inclusion be known a priori and readily accessible for testing, whereas in many practical situations the boundary of the plate is the only accessible region. Vibration response-based identification methods were also recently developed to overcome these difficulties, see, for example, [F-Q].

The basic idea of most NDTs is that the inclusion changes the response of a plate and that the defect can be determined by comparing the response of a possibly defective specimen with a reference specimen, i.e. a plate without inclusion. In the case of inclusion made by (different) elastic material, in [M-Ro-Ve3], [M-Ro-Ve5] constructive upper and lower estimates of the area of the inclusion in terms of the difference between the work exerted in deforming the defective and the reference plate by applying a given couple field at the boundary were determined. These results give no indication on the position of the domain occupied by the inclusion and, unfortunately, they still have not been extended to the case of a rigid inclusion. In this paper we tackle these two issues and we establish a stability result for the inverse problem of determining a rigid inclusion in an anisotropic elastic plate by one boundary measurement.

From the mathematical point of view, [Fi], [Gu], we shall work within the Kirchhoff-Love theory of thin, elastic anisotropic plates under infinitesimal deformations. Let \( \Omega \) denote the middle plane of the plate and let \( h \) be its constant thickness. We assume that \( \Omega \) is a bounded domain of \( \mathbb{R}^2 \) of class \( C^{1,1} \). The rigid inclusion \( D \) is modelled as an open simply connected domain compactly contained in \( \Omega \). The transversal displacement \( w \in H^2(\Omega) \) of the plate satisfies the following mixed boundary value problem

\[
\begin{align*}
(1.1) \quad & \text{div}(\text{div}(P\nabla^2 w)) = 0, \quad \text{in } \Omega \setminus \overline{D}, \\
(1.2) \quad & (P\nabla^2 w)n \cdot n = -\hat{M}_n, \quad \text{on } \partial \Omega, \\
(1.3) \quad & \text{div}(P\nabla^2 w) \cdot n + ((P\nabla^2 w)n \cdot \tau)_{,s} = (\hat{M}_\tau)_{,s}, \quad \text{on } \partial \Omega, \\
(1.4) \quad & w|_{\Gamma} \in A, \quad \text{in } \overline{D}, \\
(1.5) \quad & \frac{\partial w}{\partial n} = \frac{\partial ^s w}{\partial n}, \quad \text{on } \partial D,
\end{align*}
\]
coupled with the equilibrium conditions for the rigid inclusion $D$

\begin{equation}
\int_{\partial D} \left( \text{div}(P \nabla^2 w) \cdot n + ((P \nabla^2 w) n \cdot \tau)_s \right) g - ((P \nabla^2 w) n \cdot n) g_n = 0,
\end{equation}

for every $g \in \mathcal{A}$,

where $\mathcal{A}$ denotes the space of affine functions. In the above equations, $n$ and $\tau$ are the unit outer normal and the unit tangent vector to $\Omega \setminus D$, respectively, and we have defined $w^e \equiv w|_{\Omega \setminus D}$ and $w^t \equiv w|_{\partial D}$. Moreover, $M_x, M_n$ are the twisting and bending components of the assigned couple field $\hat{M}$, respectively.

The plate tensor $P$ is given by $P = \frac{h^3}{12}C$, where $C$ is the elasticity tensor describing the response of the material.

Given any $\hat{M} \in H^{-\frac{1}{2}}(\partial \Omega, \mathbb{R}^2)$, satisfying the compatibility conditions $\int_{\partial \Omega} \hat{M}_i = 0$, for $i = 1, 2$, and for $P$ bounded and strongly convex, problem (1.1)–(1.6) admits a solution $w \in H^2(\Omega)$, which is uniquely determined up to addition of an affine function.

The uniqueness issue for the inverse problem under consideration is the following.

Given an open portion $\Sigma$ of $\partial \Omega$ and given two solutions $w_i$ to (1.1)–(1.6) for $D = D_i$, $i = 1, 2$, satisfying

\begin{align}
(1.7) & \quad w_1 = w_2, \text{ on } \Sigma, \\
(1.8) & \quad \frac{\partial w_1}{\partial n} = \frac{\partial w_2}{\partial n}, \text{ on } \Sigma,
\end{align}

does $D_1 = D_2$ hold?

Let us notice that, by (1.2), (1.3), (1.7), (1.8), $w_1$ and $w_2$ assume the same Cauchy data on $\Sigma$. Under the a priori assumption of $C^{3,1}$ regularity of the boundary of the inclusion, a positive answer to the uniqueness issue has been given in [M-Ro2], see also [M-Ro-Ve2] for a result of unique determination of a cavity in an elastic plate by two boundary measurements.

In order to face the stability issue, following the approach used for the case of rigid inclusions in elastic bodies in [M-Ro1], we found it convenient to replace each solution $w_i$ introduced above with $v_i = w_i - g_i$, where $g_i$ is the affine function which coincides with $w_i$ on $\partial D_i$, $i = 1, 2$. By this approach, maintaining the same letter to denote the solution, we rephrase the equilibrium problem (1.1)–(1.5) in terms of the following mixed boundary value problem with homogeneous Dirichlet conditions on the boundary of the rigid inclusion.
\[
\begin{align*}
(1.9) & \quad \text{div}(\text{div}(\mathbb{P} \nabla^2 w)) = 0, \quad \text{in } \Omega \setminus \overline{D}, \\
(1.10) & \quad (\mathbb{P} \nabla^2 w) n \cdot n = -\hat{M}_n, \quad \text{on } \partial \Omega, \\
(1.11) & \quad \text{div}(\mathbb{P} \nabla^2 w) \cdot n + ((\mathbb{P} \nabla^2 w) n \cdot \tau)_s = (\hat{M}_\tau)_s, \quad \text{on } \partial \Omega, \\
(1.12) & \quad w = 0, \quad \text{on } \partial D, \\
(1.13) & \quad \frac{\partial w}{\partial n} = 0, \quad \text{on } \partial D,
\end{align*}
\]

which has a unique solution \( w \in H^2(\Omega \setminus \overline{D}) \).

On the other hand, it is clear that the arbitrariness of this normalization, related to the fact that \( g_i \) is unknown, \( i = 1, 2 \), leads to the following formulation of the stability issue.

Given two solutions \( w_i \) to \((1.9) \,–\, (1.13), (1.6)\) when \( D = D_i, \ i = 1, 2 \), satisfying, for some \( \epsilon > 0 \),

\[
(1.14) \quad \min_{g \in \mathcal{A}} \left\{ \| w_1 - w_2 - g \|_{L^2(\Sigma)} + \left\| \frac{\partial}{\partial n} (w_1 - w_2 - g) \right\|_{L^2(\Sigma)} \right\} \leq \epsilon,
\]

to evaluate the rate at which the Hausdorff distance between \( D_1 \) and \( D_2 \) tends to zero as \( \epsilon \) tends to zero.

In the present paper, assuming \( C^{3,1} \) regularity of \( \partial D \), we prove the following constructive stability estimate of log-log type

\[
(1.15) \quad d_H(D_1, D_2) \leq C (\log |\log \epsilon|)^{-\eta},
\]

where \( C, \eta, C > 0 \) and \( 0 < \eta \leq 1 \), are constants only depending on the a priori data, see Theorem 3.1 for a precise statement.

The methods used to prove (1.15) are based essentially on quantitative estimates of unique continuation from Cauchy data for a solution of the mixed problem \((1.9) \,–\, (1.13), (1.6)\). These estimates involve a cornerstone result of unique continuation, namely the \textit{Three Spheres Inequality} (4.2) for solutions to the plate equation \((1.9)\), which has been determined in [M-Ro-Ve5] under the very general assumption that the elastic material of the plate obeys the \textit{dichotomy condition} (2.30a)-(2.30b).

The logarithmic character of the stability estimate (1.15) seems difficult to improve. First, the Cauchy problem up to the boundary is severely ill-posed and, even in the simpler context of the electrical impedance tomography which involves a second order elliptic equation instead of a fourth order one, the corresponding stability estimate with a single logarithm is the best possible result, see [Al-B-Ro-Ve] and also [Al-R-Ro-Ve] for a general discussion on the ill-posedness of the Cauchy problem. We also quote [DiC-R] for examples of exponential instability for the inverse inclusion problem in a conducting body. In addition, a further difficulty arises in our analysis. It is due to
the lack of quantitative estimates of the strong unique continuation property at the boundary in the form of either Three Spheres Inequality or Doubling Inequality. It has been shown in [Al-B-Ro-Ve] that this is a key ingredient in proving that the stability estimate for the corresponding inverse problem with unknown boundaries in the conductivity context is not worse than logarithm. This mathematical tool is not currently available for the plate operator, even in the simplest case of isotropic material, and this is the reason for the presence of a double logarithm in the stability estimate (1.15). We refer to [M-Ro1] for a discussion on the analogous problem in the determination of a rigid inclusion in an isotropic elastic body by boundary measurements. Finally, as remarked in [M-Ro-Ve4], it seems hopeless the possibility that solutions to (1.1) can satisfy even a strong unique continuation property at the interior, without any a priori assumption on the anisotropy of the material, see also [Ali]. Regarding this point, our dichotomy condition (2.30a)-(2.30b) basically contains the same assumptions under which the unique continuation property holds for a fourth order elliptic equation in two variables.

The paper is organized as follows. Some notation is presented in Section 2. In Section 3 we state two auxiliary propositions concerning the estimate of continuation from the interior (Proposition 3.2) and from Cauchy data (Proposition 3.3), and we give the proof of the main Theorem 3.1. Section 4 contains the proofs of Proposition 3.2 and Proposition 3.3. Proofs of regularity estimates of the solution to the mixed problem (1.9)–(1.13), (1.6) are presented in Section 5 and, in part, in the Appendix.

2 Notation

Let $P = (x_1(P), x_2(P))$ be a point of $\mathbb{R}^2$. We shall denote by $B_r(P)$ the disk in $\mathbb{R}^2$ of radius $r$ and center $P$ and by $R_{a,b}(P)$ the rectangle of center $P$ and sides parallel to the coordinate axes, of length $2a$ and $2b$, namely $R_{a,b}(P) = \{x = (x_1, x_2) \mid |x_1 - x_1(P)| < a, |x_2 - x_2(P)| < b\}$. To simplify the notation, we shall denote $B_r = B_r(O), R_{a,b} = R_{a,b}(O)$.

Given a bounded domain $\Omega$ in $\mathbb{R}^2$ we shall denote

$$\Omega_\rho = \{x \in \Omega \mid \text{dist}(x, \partial \Omega) > \rho\}.$$  

When representing locally a boundary as a graph, we use the following definition.

**Definition 2.1.** ($C^{k,1}$ regularity) Let $\Omega$ be a bounded domain in $\mathbb{R}^2$. Given $k \in \mathbb{N}$, we say that a portion $S$ of $\partial \Omega$ is of class $C^{k,1}$ with constants $\rho_0,$
$M_0 > 0$, if, for any $P \in S$, there exists a rigid transformation of coordinates under which we have $P = 0$ and

$$\Omega \cap R_{\frac{\rho_0}{M_0} \cdot \rho_0} = \{ x = (x_1, x_2) \in R_{\frac{\rho_0}{M_0} \cdot \rho_0} \mid x_2 > \psi(x_1) \},$$

where $\psi$ is a $C^{k,1}$ function on $(-\frac{\rho_0}{M_0}, \frac{\rho_0}{M_0})$ satisfying

$$\psi(0) = 0,$$

$$\psi'(0) = 0, \quad \text{when } k \geq 1,$$

$$\|\psi\|_{C^{k,1}}(-\frac{\rho_0}{M_0}, \frac{\rho_0}{M_0}) \leq M_0 \rho_0.$$

When $k = 0$ we also say that $S$ is of Lipschitz class with constants $\rho_0, M_0$.

**Remark 2.2.** We use the convention to normalize all norms in such a way that their terms are dimensionally homogeneous with the $L^\infty$ norm and coincide with the standard definition when the dimensional parameter equals one. For instance, the norm appearing above is meant as follows

$$\|\psi\|_{C^{k,1}}(-\frac{\rho_0}{M_0}, \frac{\rho_0}{M_0}) = \sum_{i=0}^{k+1} \rho_0^i \|\psi^{(i)}\|_{L^\infty}(-\frac{\rho_0}{M_0}, \frac{\rho_0}{M_0}),$$

where $\psi^{(i)}$ is the $i$-th derivative with respect to the $x_1$ variable. Similarly, denoting by $\nabla^i u$ the vector which components are the derivatives of order $i$ of a function $u$ defined in $\Omega$, we denote

$$\|u\|_{C^{k,1}(\Omega)} = \sum_{i=0}^{k+1} \rho_0^i \|\nabla^i u\|_{L^\infty(\Omega)},$$

$$\|u\|_{L^2(\Omega)} = \rho_0^{-1} \left( \int_{\Omega} u^2 \right)^{\frac{1}{2}},$$

$$\|u\|_{H^m(\Omega)} = \rho_0^{-1} \left( \sum_{i=0}^{m} \rho_0^{2i} \int_{\Omega} |\nabla^i u|^2 \right)^{\frac{1}{2}},$$

and so on for boundary and trace norms such as $\| \cdot \|_{H^{\frac{1}{2}}(\partial \Omega)}$, $\| \cdot \|_{H^{-\frac{1}{2}}(\partial \Omega)}$.

Notice also that, when $\Omega = B_R$, then $\Omega$ satisfies Definition 2.1 with $\rho_0 = R, M_0 = 2$ and therefore, for instance, we have

$$\|u\|_{H^m(B_R)} = R^{-1} \left( \sum_{i=0}^{m} R^{2i} \int_{B_R} |\nabla^i u|^2 \right)^{\frac{1}{2}},$$

and so on.
Given a bounded domain $\Omega$ in $\mathbb{R}^2$ such that $\partial \Omega$ is of class $C^{k,1}$, with $k \geq 1$, we consider as positive the orientation of the boundary induced by the outer unit normal $n$ in the following sense. Given a point $P \in \partial \Omega$, let us denote by $\tau = \tau(P)$ the unit tangent at the boundary in $P$ obtained by applying to $n$ a counterclockwise rotation of angle $\frac{\pi}{2}$, that is
\begin{equation}
(2.2)
\tau = e_3 \times n,
\end{equation}
where $\times$ denotes the vector product in $\mathbb{R}^3$, $\{e_1, e_2\}$ is the canonical basis in $\mathbb{R}^2$ and $e_3 = e_1 \times e_2$.

Given any connected component $C$ of $\partial \Omega$ and fixed a point $P \in C$, let us define as positive the orientation of $C$ associated to an arclength parametrization $\varphi(s) = (x_1(s), x_2(s))$, $s \in [0, l(C)]$, such that $\varphi(0) = P$ and $\varphi'(s) = \tau(\varphi(s))$, where $l(C)$ denotes the length of $C$.

Throughout the paper, we denote by $\partial_i u$, $\partial_s u$, and $\partial_n u$ the derivatives of a function $u$ with respect to the $x_i$ variable, to the arclength $s$ and to the normal direction $n$, respectively, and similarly for higher order derivatives.

We denote by $\mathbb{M}^2$ the space of $2 \times 2$ real valued matrices and by $\mathcal{L}(X,Y)$ the space of bounded linear operators between Banach spaces $X$ and $Y$.

For every $2 \times 2$ matrices $A$, $B$ and for every $L \in \mathcal{L}(\mathbb{M}^2, \mathbb{M}^2)$, we use the following notation:
\begin{align}
(2.3) & \quad (\mathbb{L}A)_{ij} = L_{ijkl}A_{kl}, \\
(2.4) & \quad A \cdot B = A_{ij}B_{ij}, \\
(2.5) & \quad |A| = (A \cdot A)^{\frac{1}{2}}, \\
(2.6) & \quad A^{sym} = \frac{1}{2} (A + A^t),
\end{align}
where $A^t$ denotes the transpose of the matrix $A$. Notice that here and in the sequel summation over repeated indexes is implied.

Finally, let us introduce the linear space of the affine functions on $\mathbb{R}^2$
\[ A = \{g(x_1, x_2) = ax_1 + bx_2 + c, \ a, b, c \in \mathbb{R} \}. \]

2.1 A priori information

i) A priori information on the domain.
Let us consider a thin plate $\Omega \times [-\frac{h}{2}, \frac{h}{2}]$ with middle surface represented by a bounded domain $\Omega$ in $\mathbb{R}^2$ and having uniform thickness $h, h << \text{diam}(\Omega)$.

We shall assume that, given $\rho_0, M_1 > 0$,

\begin{equation}
|\Omega| \leq M_1 \rho_0^2, \tag{2.7}
\end{equation}

where $|\Omega|$ denotes the Lebesgue measure of $\Omega$. We shall also assume that $\Omega$ contains an open simply connected rigid inclusion $D$ such that

\begin{equation}
\text{dist}(D, \partial \Omega) \geq \rho_0. \tag{2.8}
\end{equation}

Moreover, we denote by $\Sigma$ an open portion within $\partial \Omega$ representing the part of the boundary where measurements are taken.

Concerning the regularity of the boundaries, given $M_0 > 0$, we assume that

\begin{equation}
\partial \Omega \text{ is of class } C^{2,1} \text{ with constants } \rho_0, M_0, \tag{2.9}
\end{equation}

\begin{equation}
\Sigma \text{ is of class } C^{3,1} \text{ with constants } \rho_0, M_0. \tag{2.10}
\end{equation}

\begin{equation}
\partial D \text{ is of class } C^{3,1} \text{ with constants } \rho_0, M_0. \tag{2.11}
\end{equation}

Moreover, we shall assume that for some $P_0 \in \Sigma$ and some $\delta_0, 0 < \delta_0 < 1$,

\begin{equation}
\partial \Omega \cap R_{M_0 \rho_0}(P_0) \subset \Sigma, \tag{2.12}
\end{equation}

and that

\begin{equation}
|\Sigma| \leq (1-\delta_0)|\partial \Omega|. \tag{2.13}
\end{equation}

\textit{ii) Assumptions about the boundary data.}

On the Neumann data $\widehat{M}$ we assume that

\begin{equation}
\widehat{M} \in L^2(\partial \Omega, \mathbb{R}^2), \quad (\widehat{M}_n, (\widehat{M}_r)_{ss}) \neq 0, \tag{2.14}
\end{equation}

\begin{equation}
\text{supp}(\widehat{M}) \subset \subset \Sigma, \tag{2.15}
\end{equation}

the (obvious) compatibility condition

\begin{equation}
\int_{\partial \Omega} \widehat{M}_i = 0, \quad i = 1, 2, \tag{2.16}
\end{equation}

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and that, for a given constant $F > 0$,

\begin{equation}
\frac{\|\hat{M}\|_{L^2(\partial\Omega, \mathbb{R}^2)}}{\|\hat{M}\|_{H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)}} \leq F.
\end{equation}

\textit{iii) Assumptions about the elasticity tensor.}

Let us assume that the plate is made of nonhomogeneous linear elastic material with plate tensor

\begin{equation}
P = \frac{h^3}{12} C,
\end{equation}

where the elasticity tensor $C(x) \in \mathcal{L}(\mathbb{M}^2, \mathbb{M}^2)$ has cartesian components $C_{ijkl}$ which satisfy the following symmetry conditions

\begin{equation}
C_{ijkl} = C_{klij} = C_{klji} \quad i, j, k, l = 1, 2, \text{ a.e. in } \Omega.
\end{equation}

We recall that (2.19) are equivalent to

\begin{equation}
CA = CA^\text{sym},
\end{equation}

\begin{equation}
CA \text{ is symmetric,}
\end{equation}

\begin{equation}
CA \cdot B = CB \cdot A,
\end{equation}

for every $2 \times 2$ matrices $A, B$.

In order to simplify the presentation, we shall assume that the tensor $C$ is defined in all of $\mathbb{R}^2$.

Condition (2.19) implies that instead of 16 coefficients we actually deal with 6 coefficients and we denote

\begin{equation}
\begin{cases}
C_{1111} = A_0, & C_{1122} = C_{2211} = B_0, \\
C_{1112} = C_{1211} = C_{1211} = C_{2111} = C_0, \\
C_{2212} = C_{2221} = C_{1222} = C_{2122} = D_0, \\
C_{1212} = C_{1221} = C_{2112} = C_{2121} = E_0, \\
C_{2222} = F_0,
\end{cases}
\end{equation}

and

\begin{equation}
a_0 = A_0, \ a_1 = 4C_0, \ a_2 = 2B_0 + 4E_0, \ a_3 = 4D_0, \ a_4 = F_0.
\end{equation}
Let $S(x)$ be the following $7 \times 7$ matrix

\[
S(x) = \begin{pmatrix}
a_0 & a_1 & a_2 & a_3 & a_4 & 0 & 0 \\
0 & a_1 & a_2 & a_3 & a_4 & 0 & 0 \\
0 & 0 & a_0 & a_1 & a_2 & a_3 & a_4 \\
4a_0 & 3a_1 & 2a_2 & a_3 & 0 & 0 & 0 \\
0 & 4a_0 & 3a_1 & 2a_2 & a_3 & 0 & 0 \\
0 & 0 & 4a_0 & 3a_1 & 2a_2 & a_3 & 0 \\
0 & 0 & 0 & 4a_0 & 3a_1 & 2a_2 & a_3 \\
\end{pmatrix},
\]

and

\[
D(x) = \frac{1}{a_0} | \det S(x) |.
\]

On the elasticity tensor $C$ we make the following assumptions:

I) Regularity

\[
C \in C^{1,1}(\mathbb{R}^2, L(M^2, M^2)),
\]

with

\[
\sum_{i,j,k,l=1}^{2} \sum_{m=0}^{2} \rho_0^m \| \nabla^m C_{ijkl} \|_{L^\infty(\mathbb{R}^2)} \leq M,
\]

where $M$ is a positive constant;

II) Ellipticity (strong convexity) There exists $\gamma > 0$ such that

\[
CA \cdot A \geq \gamma |A|^2, \quad \text{in } \mathbb{R}^2,
\]

for every $2 \times 2$ symmetric matrix $A$.

III) Dichotomy condition

\[
\text{either } D(x) > 0, \quad \text{for every } x \in \mathbb{R}^2, \\
\text{or } D(x) = 0, \quad \text{for every } x \in \mathbb{R}^2,
\]

where $D(x)$ is defined by (2.26).

Remark 2.3. Whenever (2.30a) holds we denote

\[
\delta_1 = \min_{\mathbb{R}^2} D.
\]

We emphasize that, in all the following statements, whenever a constant is said to depend on $\delta_1$ (among other quantities) it is understood that such dependence occurs only when (2.30a) holds.
We shall refer to the set of constants $M_0, M_1, \delta_0, F, \gamma, M, \delta_1$ as the \textit{a priori data}. The dependence on the thickness parameter $h$ will be omitted.

In the sequel we shall consider the following boundary value problem of mixed type

\begin{align}
\text{(2.32)} & \quad \text{div}(\text{div}(P\nabla^2 w)) = 0, & & \text{in } \Omega \setminus \overline{D}, \\
\text{(2.33)} & \quad (P\nabla^2 w) n \cdot n = -\hat{M}_n, & & \text{on } \partial \Omega,
\end{align}

\begin{align}
\text{(2.34)} & \quad \text{div}(P\nabla^2 w) \cdot n + ((P\nabla^2 w) n \cdot \tau)_{s,s} = (\hat{M}_\tau)_{s,s}, & & \text{on } \partial \Omega,
\text{(2.35)} & \quad w = 0, & & \text{on } \partial D,
\text{(2.36)} & \quad \frac{\partial w}{\partial n} = 0, & & \text{on } \partial D,
\end{align}

coupled with the \textit{equilibrium conditions} for the rigid inclusion $D$

\begin{align}
\text{(2.37)} & \quad \int_{\partial D} \left( \text{div}(P\nabla^2 w) \cdot n + ((P\nabla^2 w) n \cdot \tau)_{s,s} \right) g - ((P\nabla^2 w) n \cdot n) g_n = 0,
& & \text{for every } g \in \mathcal{A}.
\end{align}

By standard variational arguments, it is easy to see that problem (2.32)–(2.37) admits a unique solution $w \in H^2(\Omega \setminus \overline{D})$ such that

\begin{align}
\text{(2.38)} & \quad \|w\|_{H^2(\Omega \setminus \overline{D})} \leq C \rho_0^2 \|\hat{M}\|_{H^{-\frac{1}{2}}(\partial \Omega, \mathbb{R}^2)},
\end{align}

where $C > 0$ only depends on $\gamma, M_0,$ and $M_1$.

\section{Statement and proof of the main result}

Here and in the sequel we shall denote by $G$ the connected component of $\Omega \setminus (D_1 \cup D_2)$ such that $\Sigma \subset \partial G$.

\textbf{Theorem 3.1} (Stability result). \textit{Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ satisfying (2.7) and (2.9). Let $D_i, i = 1, 2,$ be two simply connected open subsets of $\Omega$ satisfying (2.8) and (2.11). Moreover, let $\Sigma$ be an open portion of $\partial \Omega$ satisfying (2.10), (2.12) and (2.13). Let $\hat{M} \in L^2(\partial \Omega, \mathbb{R}^2)$ satisfy (2.14)–(2.17) and let the plate tensor $P$ given by (2.18) satisfy the symmetry conditions (2.19), the regularity condition (2.28), the strong convexity condition (2.29) and the dichotomy condition. Let $w_1 \in H^2(\Omega \setminus \overline{D_i})$ be the solution to (2.32)–(2.37), when $D = D_i, i = 1, 2$. If, given $\epsilon > 0$, we have

\begin{align}
\text{(3.1)} & \quad \min_{g \in \mathcal{A}} \left\{ \|w_1 - w_2 - g\|_{L^2(\Sigma)} + \rho_0 \left\| \frac{\partial}{\partial n} (w_1 - w_2 - g) \right\|_{L^2(\Sigma)} \right\} \leq \epsilon,
\end{align}

\textit{then $w_1 = w_2.$}
then we have

\begin{align}
(3.2) \quad d_H(\partial D_1, \partial D_2) & \leq \rho_0 \omega \left( \frac{\epsilon}{\rho_0^2 \| \hat{M} \|_{H^{-\frac{1}{2}}(\partial \Omega, \mathbb{R}^2)}} \right), \\
(3.3) \quad d_H(D_1, D_2) & \leq \rho_0 \omega \left( \frac{\epsilon}{\rho_0^2 \| \hat{M} \|_{H^{-\frac{1}{2}}(\partial \Omega, \mathbb{R}^2)}} \right),
\end{align}

where \( \omega \) is an increasing continuous function on \([0, \infty)\) which satisfies

\begin{align}
(3.4) \quad \omega(t) & \leq C (\log |\log t|)^{-\eta}, \quad \text{for every } t, \ 0 < t < e^{-1},
\end{align}

and \( C, \eta, C > 0, 0 < \eta \leq 1 \), are constants only depending on the a priori data.

The proof of Theorem 3.1 is obtained from the following sequence of Propositions.

**Proposition 3.2** (Lipschitz Propagation of Smallness). Let \( \Omega \) be a bounded domain in \( \mathbb{R}^2 \) satisfying (2.7) and (2.9). Let \( D \) be an open simply connected subset of \( \Omega \) satisfying (2.8), (2.11). Let \( w \in H^2(\Omega \setminus D) \) be the solution to (2.32)–(2.37), where the plate tensor \( \mathbb{P} \) given by (2.18) satisfies (2.19), (2.28), (2.29), and the dichotomy condition. Let the couple field \( \hat{M} \) satisfy (2.14)–(2.17).

There exists \( s > 1 \), only depending on \( \gamma, M, \delta_1, M_0 \) and \( \delta_0 \), such that for every \( \rho > 0 \) and every \( \bar{x} \in (\Omega \setminus D)_{s\rho} \), we have

\begin{align}
(3.5) \quad \int_{B_{s}(\bar{x})} |\nabla^2 w|^2 & \geq \frac{C \rho_0^2}{\exp \left[ A \left( \frac{\rho}{\rho_0} \right)^B \right]} \| \hat{M} \|_{H^{-\frac{1}{2}}(\partial \Omega, \mathbb{R}^2)}^2,
\end{align}

where \( A > 0, B > 0 \) and \( C > 0 \) only depend on \( \gamma, M, \delta_1, M_0, M_1, \delta_0 \) and \( F \).

**Proposition 3.3** (Stability Estimate of Continuation from Cauchy Data). Let the hypotheses of Theorem 3.1 be satisfied. We have

\begin{align}
(3.6) \quad \int_{D_2 \setminus D_1} |\nabla^2 w_1|^2 & \leq \rho_0^2 \| \hat{M} \|_{H^{-\frac{1}{2}}(\partial \Omega, \mathbb{R}^2)}^2 \omega \left( \frac{\epsilon}{\rho_0^2 \| \hat{M} \|_{H^{-\frac{1}{2}}(\partial \Omega, \mathbb{R}^2)}} \right),
\end{align}
(3.7) \[ \int_{D_1 \setminus D_2} |\nabla^2 w_2|^2 \leq \rho_0^2 \| \hat{M} \|^2_{H^{-\frac{1}{2}}(\partial \Omega, \mathbb{R}^2)} \omega \left( \frac{\epsilon}{\rho_0^2 \| \hat{M} \|^2_{H^{-\frac{1}{2}}(\partial \Omega, \mathbb{R}^2)}} \right), \]

where \( \omega \) is an increasing continuous function on \([0, \infty)\) which satisfies

(3.8) \( \omega(t) \leq C(|\log |\log t||^{-\frac{1}{2}}, \quad \text{for every } t < e^{-1}, \)

with \( C > 0 \) only depending on \( \gamma, M, \delta_1, M_0, M_1 \) and \( \delta_0 \).

If we assume, in addition, that there exist \( L > 0 \) and \( \tilde{\rho}_0 \), \( 0 < \tilde{\rho}_0 \leq \rho_0 \), such that \( \partial G \) is of Lipschitz class with constants \( \tilde{\rho}_0, L \), then (3.6)–(3.7) hold with \( \omega \) given by

(3.9) \( \omega(t) \leq C|\log t|^{-\sigma}, \quad \text{for every } t < 1, \)

where \( \sigma > 0 \) and \( C > 0 \) only depend on \( \gamma, M, \delta_1, M_0, M_1, \delta_0, L \) and \( \tilde{\rho}_0 \).

Proof of Theorem 3.1. Let us denote, for simplicity, \( d = d_{H}(\partial D_1, \partial D_2) \). Let us see that, if \( \eta > 0 \) is such that

(3.10) \[ \int_{D_2 \setminus D_1} |\nabla^2 w_1|^2 \leq \frac{\eta}{\tilde{\rho}_0^2}, \quad \int_{D_1 \setminus D_2} |\nabla^2 w_2|^2 \leq \frac{\eta}{\rho_0^2}, \]

then we have

(3.11) \[ d \leq C \rho_0 \left[ \log \left( \frac{C \rho_0^4 \| \hat{M} \|^2_{H^{-\frac{1}{2}}(\partial \Omega, \mathbb{R}^2)}}{\eta} \right) \right]^{-\frac{1}{4}}, \]

where \( B > 0 \) and \( C > 0 \) only depend on \( \gamma, M, \delta_1, M_0, M_1, \delta_0 \) and \( F \).

We may assume, with no loss of generality, that there exists \( x_0 \in \partial D_1 \) such that \( \text{dist}(x_0, \partial D_2) = d \). Let us distinguish two cases:

i) \( B_d(x_0) \subset D_2 \);

ii) \( B_d(x_0) \cap D_2 = \emptyset \).

In case i), by the regularity assumptions made on \( \partial D_1 \), there exists \( x_1 \in D_2 \setminus D_1 \) such that \( B_{td}(x_1) \subset D_2 \setminus D_1 \), with \( t = \frac{1}{1 + \sqrt{1 + M_0^2}} \).

By (3.10) and by Proposition 3.2 with \( \rho = \frac{td}{\epsilon} \), we have

(3.12) \[ \eta \geq \frac{C \rho_0^4}{\exp \left[ A \left( \frac{\rho_0}{td} \right)^B \right] \| \hat{M} \|^2_{H^{-\frac{1}{2}}(\partial \Omega, \mathbb{R}^2)}}, \]

where \( A > 0, B > 0 \) and \( C > 0 \) only depend on \( \gamma, M, \delta_1, M_0, M_1, \delta_0 \) and \( F \).
By (3.12) we easily find (3.11).

Case ii) can be treated similarly by substituting $w_1$ with $w_2$.

Hence, by Proposition 3.3 and assuming $\epsilon < e^{-\epsilon \hat{\rho}_0^2 \| \tilde{M} \|_{H^{-\frac{1}{2}(\partial \Omega, \mathbb{R}^2)}}}$ we obtain

$$d \leq C \hat{\rho}_0 \left\{ \log \left[ \log \frac{\epsilon}{\rho_0^2 \| \tilde{M} \|_{H^{-\frac{1}{2}(\partial \Omega, \mathbb{R}^2)}}} \right] \right\}^{-\frac{1}{2}},$$

where $B > 0$ and $C > 0$ only depend on $\gamma$, $M$, $\delta_1$, $M_0$, $M_1$, $\delta_0$ and $F$.

Thus we have obtained a stability estimate of log-log-log type.

In order to prove an analogous estimate for the Hausdorff distance between $D_1$ and $D_2$, let us now set $d = d_H(D_1, D_2)$ and let us assume, with no loss of generality, that there exists $x_0 \in D_1$ such that $\text{dist}(x_0, D_2) = d$. If $B_d(x_0) \subset D_1$, then $B_d(x_0) \subset D_1 \setminus D_2$ and (3.12) follows with $t$ replaced by 1. If, otherwise, $B_d(x_0) \not\subset D_1$, then $\text{dist}(x_0, \partial D_1) \leq d$, and we can distinguish two cases:

i) $\text{dist}(x_0, \partial D_1) > \frac{d}{2}$,

ii) $\text{dist}(x_0, \partial D_1) \leq \frac{d}{2}$.

When i) holds, then $B_{\frac{d}{2}}(x_0) \subset D_1 \setminus D_2$ and again (3.12) follows with $t$ replaced by $\frac{1}{2}$. When ii) holds, there exists $y_0 \in \partial D_1$ such that $|y_0 - x_0| \leq \frac{d}{2}$. Therefore there exists $y_1 \in D_1$ such that $B_{\frac{d}{4}}(y_1) \subset D_1 \setminus D_2$, and (3.12) follows with $t$ replaced by $\frac{1}{2}$. From (3.12), arguing as above, we obtain (3.13) for $d = d_H(D_1, D_2)$.

Next, by this rough estimate, we can apply Proposition 3.6 in [Al-B-Ro-Ve] which ensure that we can find $\epsilon_0 > 0$, only depending on $\gamma$, $M$, $\delta_1$, $M_0$, $M_1$, $\delta_0$ and $F$, such that if $\epsilon \leq \epsilon_0$ then $\partial G$ is of Lipschitz class with constants $\tilde{\rho}_0$, $L$, with $L$ and $\tilde{\rho}_0$ only depending on $M_0$. By the second part of Proposition 3.3, the log-log type estimates (3.2), (3.3), (3.4) follow.

4 Proofs of Propositions 3.2 and 3.3

We need to premise some auxiliary results. The first one is the basic tool of our approach, the three spheres inequality.

**Proposition 4.1** (Three Spheres Inequality ([M-Ro-Ve5], Proposition 5.1)). Let $\Omega$ be a domain in $\mathbb{R}^2$, and let the plate tensor $P$ given by (2.18) satisfy
and the dichotomy condition. Let $w \in H^2(\Omega)$ be a weak solution to the equation

\[(4.1) \quad \text{div}(\text{div}(\mathbb{P} \nabla^2 w)) = 0, \quad \text{in } \Omega.\]

For every $r_1, r_2, r_3, r_0$, $0 < r_1 < r_2 < r_3 \leq r_0$, and for every $x \in \Omega_r$ we have

\[(4.2) \quad \int_{B_{r_2}(x)} |\nabla^2 w|^2 \leq C \left( \int_{B_{r_1}(x)} |\nabla^2 w|^2 \right)^\delta \left( \int_{B_{r_3}(x)} |\nabla^2 w|^2 \right)^{1-\delta},\]

where $C > 0$ and $\delta$, $0 < \delta < 1$, only depend on $\gamma$, $M$, $\delta_1$, $\frac{r_1}{r_2}$ and $\frac{r_3}{r_1}$.

Let us define, for $\rho \leq \rho_0$,

\[(4.3) \quad \mathcal{U}^\rho = \{ x \in \Omega \mid \text{dist}(x, \partial\Omega) \leq \rho \},\]

that is $\mathcal{U}^\rho = \Omega \setminus \Omega_\rho$.

The following three lemmas state regularity estimates.

**Lemma 4.2** (Global $H^3$ regularity for the mixed problem). Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ satisfying (2.7) and (2.9). Let $D$ be a simply connected open subset of $\Omega$ satisfying (2.8) and (2.11). Let $\widehat{M} \in H^\frac{1}{2}(\partial\Omega, \mathbb{R}^2)$ satisfy (2.16). Let the plate tensor $\mathbb{P}$ be defined by (2.18) and satisfying (2.19), (2.28), (2.29). Let $w \in H^2(\Omega \setminus \overline{D})$ be the solution to (2.32)–(2.37). We have

\[(4.4) \quad \|w\|_{H^3(\Omega \setminus \overline{D})} \leq C\rho_0^2 \|\widehat{M}\|_{H^\frac{1}{2}(\partial\Omega, \mathbb{R}^2)},\]

where $C > 0$ only depends on $M_0$, $M_1$, $\gamma$, $M$.

**Lemma 4.3** ($H^4$ regularity up to the boundary of the rigid inclusion). Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ satisfying (2.7) and (2.9). Let $D$ be a simply connected open subset of $\Omega$ satisfying (2.8) and (2.11). Let $\widehat{M} \in H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)$ satisfy (2.16). Let the plate tensor $\mathbb{P}$ be defined by (2.18) and satisfying (2.19), (2.28), (2.29). Let $w \in H^2(\Omega \setminus \overline{D})$ be the solution to (2.32)–(2.37). We have

\[(4.5) \quad \|w\|_{H^4(\Omega \setminus \overline{D})} \leq C\rho_0^2 \|\widehat{M}\|_{H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)},\]

where $C > 0$ only depends on $M_0$, $M_1$, $\gamma$, $M$.  

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Lemma 4.4 \((C^{2,\alpha}\text{ regularity up to the boundary of the rigid inclusion})\).

Let \(\Omega\) be a bounded domain in \(\mathbb{R}^2\) satisfying (2.7) and (2.9). Let \(D\) be a simply connected open subset of \(\Omega\) satisfying (2.8) and (2.11). Let \(\widehat{M} \in H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)\) satisfy (2.16) and let the plate tensor \(P\) given by (2.18) satisfy (2.19), (2.28) and (2.29). Let \(w \in H^2(\Omega \setminus D)\) be the solution to (2.32)–(2.37). We have

\[
\|w\|_{C^{2,\alpha}(\Omega \setminus D)} \leq C\rho_0^2\|\widehat{M}\|_{H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)},
\]

for every \(\alpha < 1\), where \(C > 0\) only depends on \(M_0, M_1, \gamma, M, \alpha\).

Proof of Lemma 4.4. The above Lemma follows by Lemma 4.3 and by Sobolev embedding theorem, [Ad].

Remark 4.5. We have stated the above regularity lemmas under the regularity assumptions made for our main Lemma 3.1, but we point out that, as can be seen from the proof given in Section 5 for the validity of Lemma 4.2 it suffices to assume \(C^{2,1}\) regularity of \(\partial D\) and \(C^{0,1}\) regularity of the coefficients of the elasticity tensor \(C\).

We also need the following trace-type inequality.

Lemma 4.6 \((\text{Trace-type inequality})\). Let \(\Omega\) be a bounded domain in \(\mathbb{R}^2\), satisfying (2.7) and (2.9), and let \(D\) be a simply connected open subset of \(\Omega\) satisfying (2.8) and (2.11). Let the fourth order tensor \(P\) be defined by (2.18) and satisfying (2.19), (2.28) and (2.29). Let \(\widehat{M} \in H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)\) satisfy (2.15)–(2.16), with \(\Sigma\) satisfying (2.13). Let \(w \in H^2(\Omega \setminus D)\) be the unique weak solution of the problem (2.32)–(2.37). We have

\[
\|\widehat{M}\|_{H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)} \leq C\|\nabla^2 w\|_{L^2(\Omega \setminus D)};
\]

where \(C > 0\) only depends on \(M_0, M_1, \delta_0\) and \(M\).

Remark 4.7. The proof of the above lemma can be obtained by adapting to the mixed problem the proof of the analogous Lemma 7.1 in [M-Ro-Ve1], which was established for the solutions to a Neumann problem. Indeed, the above result holds under weaker assumptions: it suffices that the coefficients of the elasticity tensor \(C\) are of class \(L^\infty\) and that \(\partial D\) is of class \(C^{2,1}\).

Next lemma ensures regularity of the boundaries of a family of domains approximating the sets \(D^\rho_i = \{x \in \Omega \mid \text{dist}(x, \partial D_i) < \rho\}\).
Lemma 4.8 (Regularized domains [Al-B-Ro-Ve], Lemma 5.3). Let $\Omega$ be a domain with boundary of class $C^{1,1}$ with constants $\rho_0$, $M_0$ and satisfying (2.7). Let $D$ be a simply connected open subset of $\Omega$ with boundary of class $C^{1,1}$ with constants $\rho_0$, $M_0$ and satisfying (2.8). One can construct a family of regularized domains $\tilde{D}^h \subset \Omega$, for $0 < h \leq a \rho_0$, having boundary of class $C^1$ with constants $\tilde{\rho}_0$ and $\tilde{M}_0$, such that

\begin{align}
\tag{4.8} & D \subset \tilde{D}^{h_1} \subset \tilde{D}^{h_2}, \quad 0 < h_1 \leq h_2, \\
\tag{4.9} & \gamma_0 h \leq \text{dist}(x, \partial D) \leq \gamma_1 h, \quad \text{for every } x \in \partial \tilde{D}^h, \\
\tag{4.10} & |\tilde{D}^h \setminus D| \leq \gamma_2 M_1 \rho_0 h, \\
\tag{4.11} & |\partial \tilde{D}^h|_1 \leq \gamma_3 M_1 \rho_0,
\end{align}

for every $x \in \partial \tilde{D}^h$ there exists $y \in \partial D$ such that

\begin{align}
\tag{4.12} & |y - x| = \text{dist}(x, \partial D), \quad |\nu(x) - \nu(y)| \leq \gamma_4 \frac{h}{\rho_0},
\end{align}

where $\nu(x)$, $\nu(y)$ denote the outer unit normals to $\tilde{D}^h$ at $x$ and to $D$ at $y$ respectively, and $a$, $\gamma_j$, $j = 0, 1, \ldots, 4$, and the ratios $\tilde{\rho}_0$ and $\tilde{M}_0$ only depend on $M_0$. Here $|\cdot|_1$ denotes the 1-dimensional measure.

Proof of Proposition 3.2. The proof of this Proposition is rather technical and follows the lines of the proof of Proposition 4.2 in [M-Ro-Ve5], which refers to the Neumann problem for the plate equation under the same general assumptions on the plate tensor and on the couple field. For this reason, in order to simplify the present exposition, we illustrate the main ideas underlying the proof of the first part of the result which is essentially the same as in the proof of Proposition 4.2 in [M-Ro-Ve5].

The main idea is that of applying the three spheres inequality (4.2) over a chain of disks centered at points of a path connecting $\bar{x}$ to any point $y \in (\Omega \setminus \overline{D})_{sp}$, thus estimating $\int_{B_{\rho}(y)} |\nabla^2 w|^2$ in terms of $\int_{B_{\rho}(\bar{x})} |\nabla^2 w|^2$. In doing this, in order to reduce the number of iterations, which is responsible of the deterioration of the estimate, we exploit the regularity of the boundary components $\partial \Omega$ and $\partial D$ of $\Omega \setminus \overline{D}$ (in fact Lipschitz regularity suffices for this step), constructing suitable cones near the boundary, inside which one can apply the three spheres inequality over a chain of disks tangent to the cones.

It is not restrictive to assume that $\rho_0 = 1$. 

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By a covering argument, one obtains the following estimate, which is the analogous of (5.36) in [M-Ro-Ve5]:

\[(4.13)\]

\[
\int_{B_\rho(\Omega)} |\nabla^2 w|^2 \geq \int_{\Omega\setminus\gamma} |\nabla^2 w|^2 \left( \frac{C' \rho^2 \int_{(\Omega\setminus\gamma)\setminus(\Omega\setminus\gamma)(s+1)\rho} |\nabla^2 w|^2}{\int_{\Omega\setminus\gamma} |\nabla^2 w|^2} \right)^{\delta_\chi}, \forall \rho \leq \bar{\rho}
\]

where \(\delta_\chi, 0 < \delta_\chi < 1, \bar{\rho}, B_1\) and \(C'\) are positive constants only depending on \(\gamma, M, \delta_1\) and \(M_0\), whereas \(A_1 > 0\) only depends on \(\gamma, M, \delta_1, M_0\) and \(M_1\).

Let us set

\[(4.14)\]

\[
\frac{\int_{(\Omega\setminus\gamma)\setminus(\Omega\setminus\gamma)(s+1)\rho} |\nabla^2 w|^2}{\int_{\Omega\setminus\gamma} |\nabla^2 w|^2} = 1 - \frac{\int_{(\Omega\setminus\gamma)\setminus(\Omega\setminus\gamma)(s+1)\rho} |\nabla^2 w|^2}{\int_{\Omega\setminus\gamma} |\nabla^2 w|^2}
\]

By Hölder inequality

\[(4.15)\]

\[
\|\nabla^2 w\|_{L^2((\Omega\setminus\gamma)\setminus(\Omega\setminus\gamma)(s+1)\rho)} \leq \left( (\Omega \setminus \gamma) \setminus (\Omega \setminus \gamma)(s+1)\rho \right)^{\frac{1}{2}} \|\nabla^2 w\|_{L^4((\Omega\setminus\gamma)\setminus(\Omega\setminus\gamma)(s+1)\rho)}
\]

and by Sobolev inequality [Ad]

\[(4.16)\]

\[
\|\nabla^2 w\|_{L^4((\Omega\setminus\gamma)} \leq C \|\nabla^2 w\|_{H^2((\Omega\setminus\gamma)}
\]

we have

\[(4.17)\]

\[
\|\nabla^2 w\|_{L^2((\Omega\setminus\gamma)\setminus(\Omega\setminus\gamma)(s+1)\rho)} \leq C \left( (\Omega \setminus \gamma) \setminus (\Omega \setminus \gamma)(s+1)\rho \right)^{\frac{1}{2}} \|w\|_{H^2((\Omega\setminus\gamma)}
\]

where \(C\) only depends on \(M_0, M_1\).

By interpolating the estimates (2.38) and (4.4), we have

\[(4.18)\]

\[
\|w\|_{H^2((\Omega\setminus\gamma)} \leq C \|M\|_{L^2(\partial\Omega,\mathbb{R}^2)}
\]

where \(C\) only depends on \(M_0, M_1, \gamma\) and \(M\).

Moreover

\[(4.19)\]

\[
\left( (\Omega \setminus \gamma) \setminus (\Omega \setminus \gamma)(s+1)\rho \right) \leq C \rho
\]

with \(C\) only depending on \(M_0\) and \(M_1\), see for details (A.3) in [Al-Ro]. From (4.17), (4.18) and (4.19) we have

\[(4.20)\]

\[
\int_{(\Omega\setminus\gamma)\setminus(\Omega\setminus\gamma)(s+1)\rho} |\nabla^2 w|^2 \leq C \rho^2 \|M\|_{L^2(\partial\Omega,\mathbb{R}^2)}^2
\]
where $C$ only depends on $M_0$, $M_1$, $\gamma$, $M$.

By (4.14), (4.20) and (4.7) we obtain that there exists $\tilde{\rho} > 0$, only depending on $M_0$, $M_1$, $\gamma$, $M$, such that for every $\rho \leq \tilde{\rho}$ we have

$$\int_{(\Omega \setminus \mathcal{D})_{\rho+1}} |\nabla^2 w|^2 \geq \frac{1}{2}. \tag{4.21}$$

By (4.13), (4.21) and (4.7), we have that for $\rho \leq \min\{\bar{\rho}, \tilde{\rho}\}$,

$$\int_{B_{\rho}(\bar{x})} |\nabla^2 w|^2 \geq \left(\tilde{C}\rho^2\right) \delta_{x}^{-A_1-1} \|\hat{M}\|_{H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)}^2, \tag{4.22}$$

where $\tilde{C} > 0$ only depends on $\gamma$, $M$, $\delta_1$, $M_0$, $M_1$ and $\delta_0$. Let us take $\rho \leq \tilde{\rho}$.

Noticing that $|\log \rho| \leq \frac{1}{\rho}$, for every $\rho > 0$, and that $\tilde{\rho} < 1$, by straightforward computations we obtain that (3.5) holds with $A = 3 \exp(A_1 |\log \delta_x|)$, $B = |\log \delta_x| B_1 + 1$ for every $\rho \leq \rho^*$ with $\rho^* = \min\{\bar{\rho}, \frac{\tilde{\rho}}{s+1}, \tilde{C}\}$, $\rho^*$ only depending on $\gamma$, $M$, $\delta_1$, $M_0$, $M_1$, $\delta_0$, and $F$.

If $\rho > \rho^*$ and $\bar{x} \in (\Omega \setminus \mathcal{D})_{\rho} \subset (\Omega \setminus \mathcal{D})_{\rho^*}$, then, by applying (3.5) with $\rho = \rho^*$, we have

$$\int_{B_{\rho}(\bar{x})} |\nabla^2 w|^2 \geq \int_{B_{\rho^*}(\bar{x})} |\nabla^2 w|^2 \geq C^* \|\hat{M}\|_{H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)}^2, \tag{4.23}$$

where $C^*$ only depends on $\gamma$, $M$, $\delta_1$, $M_0$, $M_1$, $\delta_0$ and $F$.

Since $\bar{x} \in (\Omega \setminus \mathcal{D})_{\rho^*}$, we have that

$$\text{diam}(\Omega \setminus \mathcal{D}) \geq 2s \rho, \tag{4.24}$$

and, on the other hand,

$$\text{diam}(\Omega \setminus \mathcal{D}) \leq C_2, \tag{4.25}$$

with $C_2$ only depending on $M_0$ and $M_1$, so that

$$\frac{2s}{C_2} \leq \frac{1}{\rho}. \tag{4.26}$$

By (4.23) and (4.26), we have

$$\int_{B_{\rho}(\bar{x})} |\nabla^2 w|^2 \geq \frac{C}{\exp \left[A \left(\frac{1}{\rho}\right) B\right]} \|\hat{M}\|_{H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)}^2, \tag{4.27}$$

with $C = C^* \exp \left[A \left(\frac{2s}{C_2}\right) B\right]$. \qed
Proof of Proposition 3.3. Let $G$ be the connected component of $\Omega \setminus (D_1 \cup D_2)$ such that $\Sigma \subset \partial G$. Let us prove (3.6)–(3.8), the proof of (3.7)–(3.8) being analogous.

Let $\overline{\gamma} \in \mathcal{A}$ the affine function realizing the minimum in (3.1) and let us set

$$w = w_1 - w_2 - \overline{\gamma}.$$  

(4.28)

**Step 1.** Let $\overline{\gamma} \equiv 0$.

In this case $w = w_1 - w_2$.

It is not restrictive to assume $\epsilon < \rho_0^2 \|M\|_{H^{-1/2}(\partial \Omega, \mathbb{R}^2)} \tilde{\mu}$, where $\tilde{\mu}, 0 < \tilde{\mu} < e^{-1}$ is a constant only depending on $\gamma, M, \delta_1, M_0, M_1$, which will be chosen later on.

Let $\vartheta = \min \left\{ \frac{3}{4 \gamma}, \frac{h_0}{4 \sqrt{1 + M_0^2}} \right\}$, where $h_0, 0 < h_0 < 1$, only depending on $M_0$, is such that $\Omega_\vartheta$ is connected for every $\vartheta \leq h_0$ (see Prop. 5.5 in [Al-R-Ro-Ve]), and where $a, \gamma_0, \gamma_1$, have been introduced in Lemma 4.8. Let $\rho = \vartheta \rho_0$ and let $\rho \leq \overline{\rho}$. Let us denote by $\overline{\Gamma}_\rho$ the connected component of $\overline{\Omega} \setminus (\overline{D}_1^\rho \cup \overline{D}_2^\rho)$ which contains $\partial \Omega$, where $\overline{D}_i^\rho$ are the regularized domains introduced in Lemma 4.8. We have

$$D_2 \setminus \overline{D}_1^\rho \subset \left( (\overline{D}_1^\rho \setminus \overline{D}_1^\vartheta) \setminus \overline{G} \right) \cup \left( (\Omega \setminus \overline{\nu}_\vartheta) \setminus \overline{D}_1^\rho \right),$$

(4.29)

$$\partial \left( (\Omega \setminus \overline{\nu}_\vartheta) \setminus \overline{D}_1^\rho \right) = \overline{\Gamma}_1^\rho \cup \overline{\Gamma}_2^\rho,$$

(4.30)

where $\overline{\Gamma}_2^\rho = \partial \overline{D}_2^\rho \cap \partial \overline{\nu}_\vartheta$ and $\overline{\Gamma}_1^\rho \subset \partial \overline{D}_1^\rho$.

We have

$$\int_{D_2 \setminus \overline{D}_1^\rho} |\nabla^2 w_1|^2 \leq \int_{(\overline{D}_1^\rho \setminus \overline{D}_1^\vartheta) \setminus \overline{G}} |\nabla^2 w_1|^2 + \int_{(\Omega \setminus \overline{\nu}_\vartheta) \setminus \overline{D}_1^\rho} |\nabla^2 w_1|^2.$$  

(4.31)

By (4.6) and (4.10) we have

$$\int_{(\overline{D}_1^\rho \setminus \overline{D}_1^\vartheta) \setminus \overline{G}} |\nabla^2 w_1|^2 \leq C \rho_0^2 \|\overline{M}\|_{H^{-1/2}(\partial \Omega, \mathbb{R}^2)} \frac{\rho}{\rho_0},$$

(4.32)

with $C > 0$ only depending on $\gamma, M, M_0$ and $M_1$. 

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By applying the divergence theorem and by (2.29) we have

\[(4.33) \quad \int_{(\Omega \setminus \tilde{V}_\rho) \setminus \tilde{D}^\rho_1} |\nabla^2 w_1|^2 \leq \gamma^{-1} \int_{\Gamma^1_1} B_2(w_1)w_{1,\nu} + \tilde{B}_2(w_1)w_{1,s} + B_3(w_1)w_1 + \gamma^{-1} \int_{\Gamma^2_2} B_2(w_1)w_{1,\nu} + \tilde{B}_2(w_1)w_{1,s} + B_3(w_1)w_1,\]

where

\[(4.34) \quad B_2(w_1) = (\mathbb{P} \nabla^2 w_1) \nu \cdot \nu,\]

\[(4.35) \quad \tilde{B}_2(w_1) = (\mathbb{P} \nabla^2 w_1) \nu \cdot \tau,\]

\[(4.36) \quad B_3(w_1) = -\text{div} (\mathbb{P} \nabla^2 w_1) \cdot \nu,\]

with \(\nu\) denoting the unit outer normal to \((\Omega \setminus \tilde{V}_\rho) \setminus \tilde{D}^\rho_1\), and \(s\) an arclength with associated parametrization \(\varphi(s)\), such that \(\varphi'(s) = \tau(\varphi(s))\), where \(\tau = e_3 \times \nu\).

Let \(x \in \Gamma^1_1\). By (4.9), \(\text{dist}(x, \partial D_1) \leq \gamma_1 \rho\). Let \(y \in \partial D_1\) such that \(|y - x| = \text{dist}(x, \partial D_1) \leq \gamma_1 \rho\). Since \(\gamma_1 \rho \leq \frac{2}{3} \rho_0\) and by (2.8) we have that \(x \in (\Omega \setminus D_1) \setminus U^{\rho_0/8}\) and (4.6) applies. Recalling also that \(w_1 \equiv 0\), \(\nabla w_1 \equiv 0\) on \(\partial D_1\), we have that

\[(4.37) \quad |w_1(x)| = |w_1(x) - w_1(y)| \leq C \rho_0^2 \|\tilde{M}\|_{H^{-\frac{1}{2}}(\partial \Omega, \mathbb{R}^2)} \frac{\rho}{\rho_0},\]

\[(4.38) \quad |w_{1,\nu}(x)| = |(\nabla w_1(x) - \nabla w_1(y)) \cdot \nu(x)| \leq C \rho_0 \|\tilde{M}\|_{H^{-\frac{1}{2}}(\partial \Omega, \mathbb{R}^2)} \frac{\rho}{\rho_0},\]

\[(4.39) \quad |w_{1,s}(x)| = |(\nabla w_1(x) - \nabla w_1(y)) \cdot \tau(x)| \leq C \rho_0 \|\tilde{M}\|_{H^{-\frac{1}{2}}(\partial \Omega, \mathbb{R}^2)} \frac{\rho}{\rho_0},\]

where \(C\) only depends on \(\gamma, M, M_0\) and \(M_1\).

From (4.6), (4.38), (4.39) and (4.11) we have

\[(4.40) \quad \left| \int_{\Gamma^1_1} B_2(w_1)w_{1,\nu} + \tilde{B}_2(w_1)w_{1,s} \right| \leq C \rho_0^2 \|\tilde{M}\|_{H^{-\frac{1}{2}}(\partial \Omega, \mathbb{R}^2)}^2 \frac{\rho}{\rho_0},\]

where \(C\) only depends on \(\gamma, M, M_0\) and \(M_1\).
Moreover, by (4.37) and (4.5), we have

\[
\begin{align*}
(4.41) \quad \left| \int_{\Gamma_2^l} B_3(w_1)w_1 \right| & \leq C \rho_0 \| B_3(w_1) \|_{L^2(\Gamma_2^l)} \| w_1 \|_{L^2(\Gamma_2^l)} \\
& \leq C \rho_0^2 \| w_1 \|_{H^3(\Gamma_2^l)} \| w_1 \|_{L^\infty(\Gamma_2^l)} \leq C \rho_0^2 \| w_1 \|_{H^4(\Omega) \setminus \mathcal{U}^{\rho_0/8}} \| w_1 \|_{L^\infty(\Gamma_2^l)} \\
& \leq C \rho_0^2 \| \tilde{M} \|_{H^{1/2}(\partial \Omega, \mathbb{R}^2)}^2 \rho_0,
\end{align*}
\]

where $C$ only depends on $\gamma$, $M$, $M_0$ and $M_1$.

Let $x \in \Gamma_2^l$. By (4.9), $\text{dist}(x, \partial D_2) \leq \gamma_1 \rho$. Let $y \in \partial D_2$ such that $|y - x| = \text{dist}(x, \partial D_2) \leq \gamma_1 \rho$. Since $\gamma_1 \rho \leq \frac{3}{4} \rho_0$ we have that $x \in (\Omega \setminus D_2) \setminus \mathcal{U}^{\rho_0/8}$ and (4.16) applies. Recalling also that $w_2 \equiv 0, \nabla w_2 \equiv 0$ on $\partial D_2$, we have that

\[
(4.42) \quad |w_1(x)| = |w(x) + w_2(x) - w_2(y)| \leq |w(x)| + C \rho_0 \| \tilde{M} \|_{H^{-1/2}(\partial \Omega, \mathbb{R}^2)} \rho_0,
\]

\[
(4.43) \quad |w_{1,\nu}(x)| = |w_{1,\nu}(x) + (\nabla w_2(x) - \nabla w_2(y)) \cdot \nu(x)| \leq |\nabla w(x)| + C \rho_0 \| \tilde{M} \|_{H^{-1/2}(\partial \Omega, \mathbb{R}^2)} \rho_0,
\]

\[
(4.44) \quad |w_{1,\tau}(x)| = |w_{1,\tau}(x) + (\nabla w_2(x) - \nabla w_2(y)) \cdot \tau(x)| \leq |\nabla w(x)| + C \rho_0 \| \tilde{M} \|_{H^{-1/2}(\partial \Omega, \mathbb{R}^2)} \rho_0,
\]

where $C$ only depends on $\gamma$, $M$, $M_0$ and $M_1$.

From (4.6) (4.43), (4.44) and (4.11) we have

\[
(4.45) \quad \left| \int_{\Gamma_2^l} B_2(w_1)w_{1,\nu} + \tilde{B}_2(w_1)w_{1,\tau} \right| \leq C \rho_0^2 \| \tilde{M} \|_{H^{-1/2}(\partial \Omega, \mathbb{R}^2)}^2 \rho_0 + C \rho_0 \| \tilde{M} \|_{H^{-1/2}(\partial \Omega, \mathbb{R}^2)} \max_{\Gamma_2^l} |\nabla w|,
\]

where $C$ only depends on $\gamma$, $M$, $M_0$ and $M_1$.

Moreover, by (4.42) and (4.5), we have, similarly to above,

\[
(4.46) \quad \left| \int_{\Gamma_2^l} B_3(w_1)w_1 \right| \leq C \rho_0^2 \| \tilde{M} \|_{H^{-1/2}(\partial \Omega, \mathbb{R}^2)}^2 \rho_0 + C \| \tilde{M} \|_{H^{-1/2}(\partial \Omega, \mathbb{R}^2)} \max_{\Gamma_2^l} |w|,
\]

where $C$ only depends on $\gamma$, $M$, $M_0$ and $M_1$.

By (4.31) (4.33), (4.40), (4.11), (4.45) and (4.46) we have

\[
(4.47) \quad \int_{D_2 \setminus \Omega_{1}} |\nabla^2 w_1|^2 \leq C \rho_0^2 \| \tilde{M} \|_{H^{-1/2}(\partial \Omega, \mathbb{R}^2)}^2 \rho_0 + C \| \tilde{M} \|_{H^{-1/2}(\partial \Omega, \mathbb{R}^2)} \| w \|_{C^1(\partial \mathcal{N} \cap \partial \Omega)},
\]
where $\|w\|_{C^1(\partial \tilde{\Omega} \setminus \partial \Omega)} = \max_{\partial \tilde{\Omega} \setminus \partial \Omega} |w| + \rho_0 \max_{\partial \tilde{\Omega} \setminus \partial \Omega} |\nabla w|$, and $C$ only depends on $\gamma, M, M_0$ and $M_1$.

Let us notice that $w \in H^2(G)$ satisfies

\begin{align}
(4.48) & \quad \text{div}(\text{div}(P \nabla^2 w)) = 0, \quad \text{in } G, \\
(4.49) & \quad (P \nabla^2 w)n \cdot n = 0, \quad \text{on } \Sigma, \\
(4.50) & \quad \text{div}(P \nabla^2 w) \cdot n + ((P \nabla^2 w)n \cdot \tau)_s = 0, \quad \text{on } \Sigma, \\
(4.51) & \quad \|w\|_{L^2(\Sigma)} + \rho_0 \|w, n\|_{L^2(\Sigma)} \leq \epsilon.
\end{align}

By the regularity of $w$ near $\Sigma$ (see Theorem 5.1) and by standard interpolation inequalities, we can apply to $w$ the stability estimate for the Cauchy problem contained in [M-Ro-Ve4, Theorem 3.8]. Therefore, noticing that, by (2.38), $\|w\|_{L^2(G)} \leq \|w_1\|_{L^2(\Omega, \mathbb{R}^1)} + \|w_2\|_{L^2(\Omega, \mathbb{R}^1)} \leq C \rho_0^2 \|\tilde{M}\|_{H^{-\frac{1}{2}}(\partial \Omega, \mathbb{R}^2)}$, with $C$ only depending on $\gamma, M_0, M_1$, we have

\begin{equation}
(4.52) \quad \|w\|_{L^2(R_{\rho_0 \frac{\rho_0}{2M_0} \rho_0}(P_0) \cap \Omega)} \leq C \epsilon^\tau \rho_0^{2(1-\tau)} \|\tilde{M}\|_{H^{-\frac{1}{2}}(\partial \Omega, \mathbb{R}^2)}
\end{equation}

where $C > 0, \tau \in (0, 1)$ only depend on $\gamma, M, \delta_1, M_0$ and $M_1$.

Let

\begin{equation}
(4.53) \quad z_0 = P_0 - \frac{\rho_0}{4} n,
\end{equation}

where $n$ denotes as usual the outer unit normal to $\partial \Omega$ at $P_0$, and notice that $B_{\rho_0 \frac{\rho_0}{2M_0} \rho_0}(P_0) \cap \Omega$.

By (4.52),

\begin{equation}
(4.54) \quad \|w\|_{L^2(B_{\rho_0 \frac{\rho_0}{2M_0} \rho_0}(z_0))} \leq C \epsilon^\tau \rho_0^{2(1-\tau)} \|\tilde{M}\|_{H^{-\frac{1}{2}}(\partial \Omega, \mathbb{R}^2)}.
\end{equation}

Let us consider the set $\tilde{V}_\rho \cap \Omega_{\rho_0 \frac{\rho_0}{4} \rho_0}$. By the choice of $\tilde{\rho}$ and recalling that $h_0 < 1$, we have by straightforward computations that this set is connected and contains $z_0$. If $x \in \tilde{V}_\rho \cap \Omega_{\rho_0 \frac{\rho_0}{4} \rho_0}$ then, by the choice of $\tilde{\rho}$, $\text{dist}(x, \partial \Omega) > \frac{\rho_0}{4\sqrt{1+M_0^2}} \geq \gamma_0 \rho$ and by (4.9) $\text{dist}(x, \partial D_1 \cup \partial D_2) \geq \gamma_0 \rho$. Therefore any disk of center in $\tilde{V}_\rho \cap \Omega_{\rho_0 \frac{\rho_0}{4} \rho_0}$ and radius $\gamma_0 \rho$ is contained in $G$.

Let $x$ be any point in $\partial \tilde{V}_\rho \setminus \partial \Omega$ and let $\gamma$ be a path in $\tilde{V}_\rho \cap \Omega_{\rho_0 \frac{\rho_0}{4} \rho_0}$ joining $x$ to $z_0$. Let us define $\{x_i\}, i = 1, \ldots, s$, as follows: $x_1 = z_0, x_{i+1} = \gamma(t_i)$,
where \( t_i = \max \{ t \text{ s. t. } |g(t) - x_i| = \frac{70\varphi}{2} \} \), if \( |x_i - x| > \frac{70\varphi}{2} \), otherwise let \( i = s \) and stop the process. By construction, the disks \( B_{\frac{70\varphi}{2}}(x_i) \) are pairwise disjoint, \( |x_{i+1} - x_i| = \frac{70\varphi}{2} \), for \( i = 1, \ldots, s - 1 \), \( |x_s - x| \leq \frac{70\varphi}{2} \). Hence we have

\[
\rho \leq S \left( \frac{\hat{\varphi}}{\rho} \right)^2, \quad \text{with } S = \frac{16M}{\pi^{\frac{1}{2}}\gamma_0} \text{ only depending on } M_0 \text{ and } M_1.
\]

An iterated application of the three spheres inequality (4.2) to \( w \) over the disks of center \( x_i \) and radii \( r_1 = \frac{70\varphi}{4}, \ r_2 = \frac{2\varphi}{4}, \ r_3 = \gamma_0\rho \), gives that for every \( \rho, \ 0 < \rho \leq \bar{\rho} \),

\[
(4.55) \quad \int_{B_{\frac{70\varphi}{4}}(x)} |w|^2 \leq C \left( \int_{G} |w|^2 \right)^{1-\delta^s} \left( \int_{B_{\frac{70\varphi}{4}}(z_0)} |w|^2 \right)^{\delta^s},
\]

where \( \delta > 0, \ 0 < \delta < 1, \ C \geq 1, \) only depend on \( \gamma, M \) and \( \delta_1 \).

Since \( B_{\frac{70\varphi}{4}}(z_0) \subset B_{\frac{70\varphi}{4}} \left( \frac{\sqrt{\gamma_0}\rho}{2} \right) \), by (4.54), (4.55) and (2.38) we have

\[
(4.56) \quad \int_{B_{\frac{70\varphi}{4}}(x)} |w|^2 \leq C \rho_0^2 \| \hat{M} \| H^{-\frac{1}{2}}_{L^2(\partial \Omega, \mathbb{R}^2)} \tilde{\epsilon}^2 \rho_0 \tilde{\epsilon},
\]

where

\[
(4.57) \quad \tilde{\epsilon} = \frac{\epsilon}{\rho_0^2 \| \hat{M} \| H^{-\frac{1}{2}}_{L^2(\partial \Omega, \mathbb{R}^2)}},
\]

Let us recall the following interpolation inequality

\[
(4.58) \quad \| v \|_{L^\infty(B_{t_1})} \leq C \left( \left( \int_{B_{t_1} \setminus B_{t_2}} |v|^2 \right)^{1 \over 2} \| \nabla v \|_{L^\infty(B_{t_1})} + {1 \over t} \left( \int_{B_{t_1} \setminus B_{t_2}} |v|^2 \right)^{1 \over 2} \right),
\]

which holds for any function \( v \in W^{1,\infty}(B_{t_1}) \).

By applying (4.58) to \( w \) in \( B_{\frac{70\varphi}{4}}(x) \) and by (4.56) and (4.6), we have

\[
(4.59) \quad \| w \|_{L^\infty(B_{\frac{70\varphi}{4}}(x))} \leq C \rho_0^2 \| \hat{M} \| H^{-\frac{1}{2}}_{L^2(\partial \Omega, \mathbb{R}^2)} \frac{\rho_0 \tilde{\epsilon} \tau t^{4s}}{\rho}.
\]

By the following interpolation inequality (see for instance [G-T])

\[
(4.60) \quad \| v \|_{C^1(B_{t_1})} \leq C \| v \|_{C^0(B_{t_1})} \| v \|_{C^2(B_{t_1})}^{1 \over 2},
\]

we have, recalling (4.60), that

\[
(4.61) \quad \| w \|_{C^1(B_{\frac{70\varphi}{4}}(x))} \leq C \rho_0^2 \| \hat{M} \| H^{-\frac{1}{2}}_{L^2(\partial \Omega, \mathbb{R}^2)} \left( \frac{\rho_0}{\rho} \right)^{1 \over 2} \tilde{\epsilon}^2 t^{4s},
\]

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so that

\( \|w\|_{C^1(\partial \Omega \cap \partial \Omega)} \leq C \rho_0^2 \|\tilde{M}\|_{H^{-\frac{1}{2}}(\partial \Omega, \mathbb{R}^2)} \left( \frac{\rho_0}{\rho} \right)^{\frac{1}{2}} \tilde{e}^{\frac{\tau}{\tilde{\tau}}} \).

From (4.47) and (4.62) we have that, for every \( \rho \leq \bar{\rho} \),

\( \int_{D_2 \setminus D_1} |\nabla^2 w_1|^2 \leq C \rho_0^2 \|\tilde{M}\|^2_{H^{-\frac{1}{2}}(\partial \Omega, \mathbb{R}^2)} \left( \frac{\rho}{\rho_0} + \frac{\rho_0}{\rho} \tilde{e}^{\frac{\tau}{\tilde{\tau}}} \right). \)

Let us set \( \bar{\mu} = \exp \left\{ -\frac{4}{\tau} \log \left( \frac{2S \log \delta}{\log \delta} \right) \right\}, \) \( \tilde{\mu} = \min \left\{ \bar{\mu}, \exp \left( -\frac{16}{\tau^2} \right) \right\}. \) We have that \( \tilde{\mu}, \bar{\mu} < e^{-1} \), only depends on \( \gamma, M, \delta_1, M_0 \) and \( M_1 \). Let \( \tilde{c} \leq \tilde{\mu} \), and let

\( \rho(\tilde{c}) = \rho_0 \left( \frac{2S |\log \delta|}{\log |\log \tilde{c}|} \right)^{\frac{\tau}{2}}. \)

Since \( \rho(\tilde{c}) \) is increasing in \( (0, e^{-1}) \) and since \( \rho(\tilde{\mu}) \leq \rho(\bar{\mu}) = \rho_0 \theta = \bar{\rho} \), we can apply inequality (4.63) with \( \rho = \rho(\tilde{c}) \), obtaining

\( \int_{D_2 \setminus D_1} |\nabla^2 w_1|^2 \leq C \rho_0^2 \|\tilde{M}\|^2_{H^{-\frac{1}{2}}(\partial \Omega, \mathbb{R}^2)} \left( \log |\log \tilde{c}| \right) \tilde{c}^{\frac{\tau}{\tilde{\tau}}}, \)

where \( C \) only depends on \( \gamma, M, \delta_1, M_0 \) and \( M_1 \).

Since \( \tilde{c} \leq \exp(-\frac{16}{\tau^2}) \), we have that \( \log \frac{\tau}{4} \geq -\frac{1}{2} \log |\log \tilde{c}| \), so that

\( \log |\log \tilde{c}| \geq \frac{1}{2} \log |\log \tilde{c}|. \)

From (4.65) and (4.66) we have

\( \int_{D_2 \setminus D_1} |\nabla^2 w_1|^2 \leq \rho_0^2 \|\tilde{M}\|^2_{H^{-\frac{1}{2}}(\partial \Omega, \mathbb{R}^2)} \omega(\tilde{c}), \)

with

\( \omega(t) = C \left( \log |\log t| \right)^{-\frac{1}{2}} \) for every \( t < e^{-1} \),

where \( C > 0 \) is a constant only depending on \( \gamma, M, \delta_1, M_0 \) and \( M_1 \).

**Step 2.** Let \( \tilde{\gamma} \neq 0. \)

Let

\( \tilde{\rho} = \rho(\tilde{c}), \)

where \( \rho(\tilde{c}) \) is given by (4.64).

We have two cases:
I) \( \partial \tilde{D}_1^\rho \cap \tilde{\Gamma}_2^\rho = \emptyset \);

II) \( \partial \tilde{D}_2^\rho \cap \tilde{\Gamma}_2^\rho \neq \emptyset \).

When I) holds, there are three possible subcases:

Ia) \( \tilde{D}_1^\rho \cap \tilde{D}_2^\rho = \emptyset \),

Ib) \( \tilde{D}_1^\rho \subset \tilde{D}_2^\rho \),

Ic) \( \tilde{D}_2^\rho \subset \tilde{D}_1^\rho \).

In case Ia) we have that \( (\Omega \setminus \tilde{V}_\rho) \setminus \tilde{D}_1^\rho = \tilde{D}_2^\rho \) and, therefore, \( \partial \left((\Omega \setminus \tilde{V}_\rho) \setminus \tilde{D}_1^\rho \right) = \partial \tilde{D}_2^\rho \), whereas in case Ib) \( (\Omega \setminus \tilde{V}_\rho) \setminus \tilde{D}_1^\rho = \tilde{D}_2^\rho \setminus \tilde{D}_1^\rho \).

For both cases, by applying the estimates of continuation from Cauchy data \( (4.62) \) obtained in the above step to the function \( w \) defined by \( (4.28) \), we have

\[
\int_{\partial \tilde{D}_2^\rho} (\text{div}(\mathbb{P}\nabla^2 w_1) \cdot \nu + ((\mathbb{P}\nabla^2 w_1)\nu \cdot \tau)_s) \bigg|_{\partial \tilde{D}_2^\rho} \bigg) g - (\mathbb{P}\nabla^2 w_1)\nu \cdot \nu \bigg|_{\partial \tilde{D}_2^\rho} = 0,
\]

for every \( g \in A \), with \( \nu \) denoting the unit outer normal to \( \tilde{D}_2^\rho \), and \( s \) an arclength with associated parametrization \( \varphi(s) \), such that \( \varphi'(s) = \tau(\varphi(s)) \), where \( \tau = \epsilon_3 \times \nu \).

By applying the estimates of continuation from Cauchy data \( (4.62) \) obtained in the above step to the function \( w \) defined by \( (4.28) \), we have

\[
\|w\|_{C^1(\partial\tilde{V}_\rho \setminus \partial A)} \leq C \rho_0^2 \|\hat{M}\|_{H^{-\frac{3}{2}}(\partial\Omega, \mathbb{R}^2)} \left( \frac{\rho_0}{\rho} \right)^{\frac{1}{2}} e^{\frac{s}{4}}.
\]

By recalling that \( w_i = 0, \nabla w_i = 0 \) on \( \partial D_i, i = 1, 2 \), and by arguing similarly to Step 1, we have, for both cases, that

\[
\int_{D_1} |\nabla^2 w_1|^2 \leq \int_{\tilde{D}_2} |\nabla^2 w_1|^2 \leq \gamma^{-1} \int_{\partial \tilde{D}_2^\rho} (\text{div}(\mathbb{P}\nabla^2 w_1) \cdot \nu + ((\mathbb{P}\nabla^2 w_1)\nu \cdot \tau)_s) w_1 - (\mathbb{P}\nabla^2 w_1)\nu \cdot \nu w_{1,\nu} = \gamma^{-1} \int_{\partial \tilde{D}_2^\rho} (\text{div}(\mathbb{P}\nabla^2 w_1) \cdot \nu + ((\mathbb{P}\nabla^2 w_1)\nu \cdot \tau)_s) w_2 - (\mathbb{P}\nabla^2 w_1)\nu \cdot \nu w_{2,\nu} = \gamma^{-1} \int_{\partial \tilde{D}_2^\rho} (\text{div}(\mathbb{P}\nabla^2 w_1) \cdot \nu + ((\mathbb{P}\nabla^2 w_1)\nu \cdot \tau)_s) w - (\mathbb{P}\nabla^2 w_1)\nu \cdot \nu w_{,\nu} \leq C \rho_0^2 \|\hat{M}\|_{H^{-\frac{3}{2}}(\partial\Omega, \mathbb{R}^2)} \left( \frac{\rho_0}{\rho} + \frac{\rho_0}{\rho} \epsilon^{\frac{2}{3}} \right) \|\hat{M}\|_{H^{-\frac{3}{2}}(\partial\Omega, \mathbb{R}^2)} \omega(\epsilon).
\]

In case Ic), by using \( (4.10) \), we have

\[
|D_2 \setminus \Gamma_1| \leq |\tilde{D}_1^\rho \setminus D_1| \leq C \rho_0 \hat{\rho},
\]

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with $C$ only depending on $M_0, M_1$. By (4.73) and by (4.6), we have

\begin{equation}
(4.74) \quad \int_{\Omega_1} |\nabla^2 w_1|^2 \leq C \rho_0 \|M\|^2_{H^{-\frac{1}{2}}(\partial\Omega_1, \mathbb{R}^2)},
\end{equation}

with $C$ only depending on $\gamma, M, M_0, M_1$, so that the thesis follows.

Let us consider now case II). In view of the above arguments,

\begin{equation}
(4.75) \quad \|w\|_{C^1(\partial\Omega_1 \setminus \partial\Omega)} \leq C \rho_0^2 \|\hat{M}\|_{H^{-\frac{1}{2}}(\partial\Omega_1, \mathbb{R}^2)} \omega(\tilde{\epsilon}),
\end{equation}

where $C > 0$ only depends on $\gamma, M, \delta_1, M_0, M_1$.

Let $z \in \partial \hat{D}_I \cap \hat{\Gamma}_2$. We have that

\begin{equation}
(4.76) \quad |w(z)| + \rho_0 |\nabla w(z)| \leq C \rho_0^2 \|\hat{M}\|_{H^{-\frac{1}{2}}(\partial\Omega_1, \mathbb{R}^2)} \omega(\tilde{\epsilon}),
\end{equation}

where $C > 0$ only depends on $\gamma, M, \delta_1, M_0, M_1$. On the other hand, by using the homogeneous Dirichlet conditions on the boundaries of the rigid inclusions,

\begin{equation}
(4.77) \quad |w_1(z)| + \rho_0 |\nabla w_1(z)| \leq C \rho_0^2 \|\hat{M}\|_{H^{-\frac{1}{2}}(\partial\Omega_1, \mathbb{R}^2)} \frac{\rho_0}{\rho_0} \leq C \rho_0^2 \|\hat{M}\|_{H^{-\frac{1}{2}}(\partial\Omega_1, \mathbb{R}^2)} \omega(\tilde{\epsilon}),
\end{equation}

where $C > 0$ only depends on $\gamma, M, \delta_1, M_0, M_1$. By (4.76)–(4.77), we get

\begin{equation}
(4.78) \quad |\bar{g}(z)| + \rho_0 |\nabla \bar{g}(z)| \leq C \rho_0^2 \|\hat{M}\|_{H^{-\frac{1}{2}}(\partial\Omega_1, \mathbb{R}^2)} \omega(\tilde{\epsilon}).
\end{equation}

Let $\bar{g}(x) = ax_1 + bx_2 + c$. Then, by (4.78), we have

\begin{equation}
(4.79) \quad |a| \leq C \rho_0 \|\hat{M}\|_{H^{-\frac{1}{2}}(\partial\Omega_1, \mathbb{R}^2)} \omega(\tilde{\epsilon}),
\end{equation}

\begin{equation}
(4.80) \quad |b| \leq C \rho_0 \|\hat{M}\|_{H^{-\frac{1}{2}}(\partial\Omega_1, \mathbb{R}^2)} \omega(\tilde{\epsilon}),
\end{equation}

\begin{equation}
(4.81) \quad |c| \leq |\bar{g}(z)| + |a||z| + |b||z| \leq C \rho_0^2 \|\hat{M}\|_{H^{-\frac{1}{2}}(\partial\Omega_1, \mathbb{R}^2)} \omega(\tilde{\epsilon}),
\end{equation}

where we are assuming for simplicity that the origin belongs to $\Omega$. It follows that

\begin{equation}
(4.82) \quad \|\bar{g}\|_{C^1(\Omega)} \leq C \rho_0^2 \|\hat{M}\|_{H^{-\frac{1}{2}}(\partial\Omega_1, \mathbb{R}^2)} \omega(\tilde{\epsilon}).
\end{equation}

By repeating the arguments of Step 1 for $w = w_1 - w_2 - \bar{g}$, we have the additional term $|\bar{g}|$ which is controlled by (4.82), and the thesis follows similarly.

The proof of the better rate of convergence (3.39), under Lipschitz regularity condition on the connected components of $\partial G$ can be obtained by merging the geometrical construction illustrated in [M-Ro1] and the arguments seen above.

\[\square\]
5 Proof of Lemma 4.2 and Lemma 4.3

Let us denote by
$$B^+_{\sigma} = \{(y_1, y_2) \in \mathbb{R}^2 | y_1^2 + y_2^2 < \sigma^2, \ y_2 > 0\}$$
the hemidisk of radius $\sigma$, $\sigma > 0$, and let
$$\Gamma_\sigma = \{(y_1, y_2) \in \mathbb{R}^2 | -\sigma \leq y_1 \leq \sigma, \ y_2 = 0\},$$
$$\Gamma^+_\sigma = \partial B^+_{\sigma} \setminus \Gamma_\sigma.$$ 

Moreover, let
$$H^2_{\Gamma^+_\sigma}(B^+_\sigma) = \left\{ g \in H^2(B^+_\sigma) | g = 0, \ \frac{\partial g}{\partial n} = 0 \text{ on } \Gamma^+_\sigma \right\}.$$

**Proof of Lemma 4.2.** It is not restrictive to assume $\rho_0 = 1$. By the regularity of $\partial \Omega$ and $\partial D$, and by (2.8), we can construct a finite collection of open sets $\Omega_0, \{\Omega_i\}_{i=1}^N, \{\Omega'_i\}_{i=1}^{N'}$ such that
$$\Omega_j \cap \Omega = \emptyset, \ i = 1, ..., N', \ j = 1, ..., N,$$
$$\Omega_j \cap \Omega'_i = \emptyset, \ i = 1, ..., N', \ j = 1, ..., N,$$
$$\Omega \setminus \overline{D} = \Omega_0 \cup \left( \bigcup_{j=1}^N T^{-1}_{(j)}(B^+_\frac{\sigma}{2}) \right) \cup \left( \bigcup_{i=1}^{N'} T^{-1}_{(i)}(B^+_\frac{\sigma}{2}) \right),$$
$$\Omega_0 \subset (\Omega \setminus \overline{D})_{\delta_0},$$
where $\delta_0$ only depends on $M_0$. Here, $T_{(j)}$, $j = 1, ..., N$, is a homeomorphism of $C^{2,1}$ class which maps $\Omega_j \cap \Omega$ into $B^+_\sigma$, $\overline{\Omega} \cap \partial \Omega$ into $\Gamma_1$ and $\partial \Omega_j \cap (\Omega \setminus \overline{D})$ into $\Gamma^+_1$. Similarly, $T'_{(i)}$ is an homeomorphism of $C^{2,1}$ class which maps $\Omega'_i \cap \Omega$ into $B^+_\sigma$, $\overline{\Omega}' \cap \partial D$ into $\Gamma_1$ and $\partial \Omega'_i \cap (\Omega \setminus \overline{D})$ into $\Gamma^+_1$, $i = 1, ..., N'$. It can be shown that every mapping $T_{(j)}$, $T'_{(i)}$, $i = j, ..., N$, $i = 1, ..., N'$, can be chosen such that the Jacobian of the transformation is identically equal to one, see [A3] (p. 129). By the regularity of $\partial \Omega$ and $\partial D$ and by (2.7), the numbers $N, N'$ are controlled by a constant only depending on $M_0$ and $M_1$.

By covering $\Omega_0$ with a finite number of spheres contained in $\Omega \setminus \overline{D}$ and using local interior regularity results (see, for instance, [M-Ro-Ve II], Theorem 8.3), we have that $w \in H^3(\Omega_0)$ and
$$\|w\|_{H^3(\Omega_0)} \leq C\|w\|_{H^2(\Omega \setminus \overline{D})},$$
where the constant $C > 0$ only depends on $M_1$, $\|P\|_{C^{0,1}(\Omega)}$ and $\gamma$.

Let us fix $j$, $1 \leq j \leq N$, and let us show that an analogous estimate holds true for $\Omega_j \cap \Omega$ near the boundary $\partial \Omega$ where the Neumann data $\hat{M}$ is given.

The function $w$, solution to (2.32)-(2.37), satisfies

\begin{equation}
\int_{\Omega_j \cap \Omega} P \nabla^2 w \cdot \nabla^2 \varphi = - \int_{\partial \Omega \cap \partial \Omega} \left( \hat{M}_n \varphi, n + (\hat{M}_r)_s \varphi \right) ds,
\end{equation}

for every $\varphi \in H^2_{\partial \Omega_j \cap \Omega}(\Omega_j \cap \Omega)$.

We define

\begin{equation}
y = T(j)(x), \quad y \in B_1^+,
\end{equation}

\begin{equation}
x = T^{-1}(y), \quad x \in \Omega_j \cap \Omega,
\end{equation}

\begin{equation}
u(y) = w(T^{-1}(y)).
\end{equation}

Then, changing the variables in (5.10), the function $u$ belongs to $H^2(B_1^+)$ and satisfies

\begin{equation}
a_+(u, \psi) = l_+(\psi), \quad \text{for every } \psi \in H^2_{\Gamma_1^+}(B_1^+),
\end{equation}

where

\begin{equation}
a_+(u, \psi) =
\end{equation}

\begin{equation}
= \int_{B_1^+} \left( Q \nabla^2 u \cdot \nabla^2 \psi + A \nabla^2 u \cdot \nabla \psi + B \nabla u \cdot \nabla^2 \psi + D \nabla u \cdot \nabla \psi \right) \left| \det \frac{\partial T(j)}{\partial x} \right|^{-1} dy,
\end{equation}

\begin{equation}
l_+(\psi) = - \int_{\Gamma_1} \left( \hat{M}_n \frac{\partial T(j)}{\partial x} \left( \frac{\partial T(j)}{\partial x} \right)^T \nabla \psi \cdot \nu \left| \frac{\partial T(j)}{\partial x} \right|^T n + \hat{M}_r \left| \frac{\partial T(j)}{\partial x} \right|^T \psi \right) \left| \frac{\partial T(j)}{\partial x} \right|^{-1} d\xi,
\end{equation}

with

\begin{equation}
S_{kr} = \frac{\partial T(j)_k}{\partial x_r}, \quad R_{ksr} = \frac{\partial^2 T(j)_k}{\partial x_s \partial x_r},
\end{equation}

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\[ Q_{ikmn} = \sum_{i,j,r,s=1}^{2} P_{ijrs} S_{ls} S_{kr} S_{mi} S_{nj}, \]  
(5.18)

\[ A_{ikn} = \sum_{i,j,r,s=1}^{2} P_{ijrs} S_{ks} R_{nij}, \]  
(5.19)

\[ B_{kmn} = \sum_{i,j,r,s=1}^{2} P_{ijrs} S_{mi} S_{nj} R_{ksr}, \]  
(5.20)

\[ D_{kn} = \sum_{i,j,r,s=1}^{2} P_{ijrs} R_{ksr} R_{nij}, \]  
(5.21)

and

\[ \psi(y) = \varphi(T^{-1}_j(y)), \quad \psi \in H^{2}_1(B^+_1), \]  
(5.22)

\[ \hat{M}_n(y) = \hat{M}_n(T^{-1}_j(y)), \quad \hat{M}_r(y) = \hat{M}_r(T^{-1}_j(y)) \]  
(5.23)

and where \( \nu \) is the unit outer normal to \( B^+_1 \). By (5.18) and (2.19), (2.29), the components of \( Q \) satisfy the symmetry conditions

\[ Q_{\alpha\beta\gamma\delta} = Q_{\gamma\delta\alpha\beta} = Q_{\gamma\delta\beta\alpha}, \quad \alpha,\beta,\gamma,\delta = 1,2, \quad \text{in } B^+_1, \]  
(5.24)

and the strong convexity condition

\[ Q A \cdot A \geq \gamma_0 |A|^2, \quad \text{in } B^+_1, \]  
(5.25)

for every 2 \( \times \) 2 symmetric matrix \( A \), where \( \gamma_0, \gamma_0 > 0 \), is a constant only depending on \( \gamma \) and \( M_0 \).

By the regularity assumptions on \( P \), the tensors \( Q, A, B, D \) belong to \( C^{0,1}(B^+_1) \). Moreover, \( \hat{M}_n \in H^1无线树(\Gamma_1) \) and \( \hat{M}_\tau,\xi \in H^{-1/2}(\Gamma_1) \).

Now, we use the following regularity result up to the boundary for the solution \( u \) to the problem (5.14), which is proved in the Appendix.

**Theorem 5.1** (Boundary regularity for non-homogeneous Neumann conditions). Let \( u \in H^2(B^+_1) \) defined by (5.13) be the solution to (5.14), where the tensors \( Q, A, B, D \) and the couple field \( \hat{M} \) are defined as above. Then, \( u \in H^3\left(B^+_1\right) \) and we have

\[ \|u\|_{H^3\left(B^+_1\right)} \leq C \left( \|\hat{M}\|_{H^1无线树(\Gamma_1)} + \|u\|_{H^2(B^+_1)} \right), \]  
(5.26)

where \( C > 0 \) only depends on \( M_0, \|P\|_{C^{0,1}(\Omega)} \) and \( \gamma \).
By applying the homeomorphism $\mathcal{T}_{(j)}$ to (5.26) we have

\begin{equation}
\|w\|_{H^3(\mathcal{T}_{(j)}^{-1}(B^+_1))} \leq C \left( \|\mathcal{M}\|_{H^{3/2}(\Omega_j \cap \partial \Omega)} + \|w\|_{H^2(\Omega_j \cap \partial \Omega)} \right),
\end{equation}

where the constant $C > 0$ only depends on $M_0$, $\gamma$ and $\|\mathcal{P}\|_{C^{0,1}(\Omega)}$.

We now derive an estimate analogous to (5.27) near the boundary of the rigid inclusion $D$. Let us fix $i$, $1 \leq i \leq N'$. The function $w$, solution to (2.32)-(2.37), satisfies

\begin{equation}
\int_{\Omega'_i \cap \Omega} \mathcal{P} \nabla^2 w \cdot \nabla^2 \varphi = 0, \quad \text{for every } \varphi \in H^2_0(\Omega'_i \cap \Omega).
\end{equation}

Then, by introducing the transformation

\begin{equation}
y = \mathcal{T}'_{(i)}(x), \quad y \in B^+_1,
\end{equation}

\begin{equation}x = \mathcal{T}'_{(i)}^{-1}(y), \quad x \in \Omega'_i \cap \Omega,
\end{equation}

\begin{equation}v(y) = w(\mathcal{T}'_{(i)}^{-1}(y)),
\end{equation}

and changing the variables in (5.28), the function $v$ belongs to $H^2_B(\Omega'_1)$ and satisfies

\begin{equation}a'^+(v, \psi) = 0, \quad \text{for every } \psi \in H^2_0(B^+_1),
\end{equation}

where

\begin{equation}a'^+(v, \psi) = \int_{B^+_1} \left( Q' \nabla^2 v \cdot \nabla^2 \psi + A' \nabla v \cdot \nabla \psi + B' \nabla^2 v \cdot \nabla \psi + D' \nabla v \cdot \nabla \psi \right) \left| \det \frac{\partial \mathcal{T}'_{(i)}}{\partial x} \right|^{-1} \, dy
\end{equation}

and the tensors $Q'$, $A'$, $B'$, $D'$ and the function $\psi$ are defined as in (5.17)-(5.21) and (5.22), respectively, with $\mathcal{T}_{(j)}$ replaced by $\mathcal{T}'_{(i)}$. We note that the tensor $Q'$ satisfies the conditions (5.24), (5.25) and all the tensors $Q'$, $A'$, $B'$, $D'$ belong to $C^{0,1}(B^+_1)$.

To this point we make use of the following regularity result up to the boundary $\partial D$ for the solution $v$ to the problem (5.32).
Theorem 5.2 (Boundary regularity for homogeneous Dirichlet conditions). Let $v \in H^2_1(B_1^+)$ defined by (5.31) be the solution to (5.32), where the tensors $Q', A', B', D'$ are defined as above. Then, $v \in H^3 \left( B_2^+ \right)$ and we have

\begin{equation}
\|v\|_{H^3 \left( B_2^+ \right)} \leq C \|v\|_{H^2 \left( B_1^+ \right)},
\end{equation}

where $C > 0$ only depends on $M_0$, $\|P\|_{C^{0,1}(\overline{\Omega})}$ and $\gamma$.

The proof of Theorem 5.2 follows the same guidelines of the proof of Theorem 5.1, see also [Ag] for details.

By applying the homeomorphism $T'(i)$ to (5.34) we have

\begin{equation}
\|w\|_{H^3 \left( T'(i)^{-1} \left( B_2^+ \right) \right)} \leq C \|w\|_{H^2 \left( \Omega' \cap \Omega \right)},
\end{equation}

where the constant $C > 0$ only depends on $M_0$, $\gamma$ and $\|P\|_{C^{0,1}(\overline{\Omega})}$.

Finally, estimate (4.4) follows from (5.7), (5.9), (5.27), (5.35) and from the estimate (2.38).

The proof of Lemma 4.3 follows from the following local version of the $H^4$-regularity near a boundary with homogeneous Dirichlet data for the solution to the problem (5.32).

Theorem 5.3. Let $v \in H^2_1(B_1^+)$ defined by (5.31) be the solution to (5.32). Then, $v \in H^4 \left( B_2^+ \right)$ and we have

\begin{equation}
\|v\|_{H^4 \left( B_2^+ \right)} \leq C \|v\|_{H^2 \left( B_1^+ \right)},
\end{equation}

where $C > 0$ only depends on $M_0$, $\|P\|_{C^{0,1}(\overline{\Omega})}$ and $\gamma$.

Proof. By Theorem 5.2, the function $v \in H^3 \left( B_2^+ \right)$ satisfies the estimate (5.34), e.g. $\|v\|_{H^3 \left( B_2^+ \right)} \leq C \|v\|_{H^2 \left( B_1^+ \right)}$, where $C > 0$ only depends on $M_0$, $\|P\|_{C^{0,1}(\overline{\Omega})}$ and $\gamma$.

In the first step of the proof we estimate the tangential derivative $\frac{\partial v}{\partial y_1}$. By the weak formulation of the problem (5.32) it can be shown that the function $v_{,\alpha} = \frac{\partial v}{\partial y_\alpha} \in H^2 \left( B_2^+ \right), \alpha = 1, 2$, satisfies the following equation

\begin{equation}
a'_+ (v_{,\alpha}, \psi) = l'_\alpha (\psi), \quad \text{for every } \psi \in H^2_0 \left( B_2^+ \right),
\end{equation}

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where

\[(5.38) \quad a'_+(v, \psi) = \int_{B^+_\frac{1}{2}} \left( (Q'_\alpha \nabla^2 v, \nabla^2 \psi + A'_\alpha \nabla v, \nabla \psi + \right. \]
\[+ B'_\alpha \nabla v, \nabla \psi + D'_\alpha \nabla v, \nabla \psi \right) \left| \det \frac{\partial T'_(i)}{\partial x} \right|^{-1} dy \]

\[(5.39) \quad l'_\alpha(\psi) = \int_{B^+_\frac{1}{2}} \left( \text{div} \left( (Q'_\alpha \nabla^2 v) \cdot \nabla^2 \psi - A'_\alpha \nabla^2 v \cdot \nabla \psi + \right. \]
\[+ (B'_{kmn,\alpha} v, m \nabla v, n - \nabla v, m \nabla \psi) \left| \det \frac{\partial T'_(i)}{\partial x} \right|^{-1} dy, \]

By the regularity assumptions on \( P \) and by (5.34) we have

\[(5.40) \quad |l'_\alpha(\psi)| \leq C \|v\|_{L^2(B^+_\frac{1}{2})} \|\psi\|_{L^2(B^+_\frac{1}{2})}, \]

\( \alpha = 1, 2 \), where the constant \( C > 0 \) only depends on \( M_0, \|P\|_{C^{1,1}(\Omega)} \) and \( \gamma \).

To this point, the arguments used in the proof of (6.35) for the Neumann case can be adapted to the Dirichlet boundary condition case to obtain

\[(5.41) \quad \| \frac{\partial}{\partial y_1} \nabla^2 v_\alpha \|_{L^2(B^+_\frac{1}{2})} \leq C \|v\|_{H^2(B^+_1)}, \]

\( \alpha = 1, 2 \), where the constant \( C > 0 \) only depends on \( M_0, \|P\|_{C^{1,1}(\Omega)} \) and \( \gamma \).

The estimate of the normal derivative \( \frac{\partial}{\partial y_2} \nabla^2 v_\alpha \), \( \alpha = 1, 2 \), can be obtained by adapting the proof of (6.44). We obtain

\[(5.42) \quad \| \frac{\partial}{\partial y_2} \nabla^2 v_\alpha \|_{L^2(B^+_\frac{1}{2})} \leq C \|v\|_{H^2(B^+_1)}, \]

\( \alpha = 1, 2 \), where the constant \( C > 0 \) only depends on \( M_0, \|P\|_{C^{1,1}(\Omega)} \) and \( \gamma \). By (5.41) and (5.42) we have the thesis.

\[\square\]

6 Appendix

In this Appendix we prove Theorem 5.1.
Proof of Theorem 5.1. As a first step of the proof, let us estimate the partial derivative \( \frac{\partial}{\partial \vartheta} \nabla^2 u \) in the direction parallel to the flat boundary \( \Gamma_1 \) of \( B_1^+ \).

Let \( \vartheta \in C_0^\infty(\mathbb{R}^2) \) be a function such that \( 0 \leq \vartheta(y) \leq 1 \) for every \( y \in \mathbb{R}^2 \), \( \vartheta \equiv 1 \) in \( B_{\rho} \), \( \vartheta \equiv 0 \) in \( \mathbb{R}^2 \setminus B_{\sigma_0} \) and \( |\nabla^k \vartheta| \leq C \), \( k = 1, \ldots, 4 \), where \( \rho = \frac{3}{4} \), \( \sigma_0 = \frac{1}{2}(\rho + 1) = \frac{7}{8} \) and \( C \) is an absolute constant.

For every function \( \psi \in H^2(B_1^+) \), we still denote by \( \psi \in H^2(\mathbb{R}_+^2) \) its extension to \( \mathbb{R}^2_+ \) obtained by taking \( \psi \equiv 0 \) in \( \mathbb{R}^2_+ \setminus B_1^+ \).

Let \( s \) be a real number different from zero. The difference operator in the \( y_1 \)-direction is defined by

\[
(\tau_{1,s}f)(y) = \frac{f(y + se_1) - f(y)}{s},
\]

for any function \( f \). In what follows we shall assume that \( |s| \leq \frac{1}{10} \). Let us notice that if \( u \in H^2(B_1^+) \), then \( \tau_{1,s}(\vartheta u) \in H^2(B_1^+) \).

We evaluate the bilinear form \( a_+(\cdot, \psi) \) with \( u \) replaced by \( \tau_{1,s}(\vartheta u) \) and \( \psi \in H^2(B_1^+) \). Since

\[
\nabla^k(\tau_{1,s}(\vartheta u)) = \tau_{1,s}(\nabla^k(\vartheta u)), \quad \text{in } B_1^+, \ k = 1, 2,
\]

we have

\[
a_+(\tau_{1,s}(\vartheta u), \psi) = \int_{B_1^+} (Q \tau_{1,s}(\nabla^2(\vartheta u)) \cdot \nabla^2 \psi + A \tau_{1,s}(\nabla^2(\vartheta u)) \cdot \nabla \psi + B \tau_{1,s}(\nabla(\vartheta u)) \cdot \nabla \psi + D \tau_{1,s}(\nabla(\vartheta u)) \cdot \nabla \psi) \ dy,
\]

where we have taken into account that \( \det \frac{\partial^2 \tau_{1,s}}{\partial x^2} = 1 \).

Let us consider the leading term of \( (6.3) \). We elaborate the corresponding expression by moving the difference operator from \( \vartheta u \) to \( \vartheta \psi \). We have

\[
\int_{B_1^+} Q \tau_{1,s}(\nabla^2(\vartheta u)) \cdot \nabla^2 \psi = \int_{B_1^+} Q \tau_{1,s}(\vartheta \nabla^2 u) \cdot \nabla^2 \psi + \int_{B_1^+} Q \tau_{1,s}(\vartheta \nabla \vartheta \nabla u + \nabla u \otimes \nabla \vartheta) \cdot \nabla^2 \psi + \int_{B_1^+} Q \tau_{1,s}(\vartheta \nabla \vartheta) \cdot \nabla^2 \psi \equiv I_1 + I_2 + I_3.
\]

The last two terms on the right hand side of \( (6.4) \) can be estimated as follows

\[
|I_2| \leq C \|u\|_{H^2(B_1^+)} \|\nabla^2 \psi\|_{L^2(B_1^+)},
\]

\[
|I_3| \leq C \|u\|_{H^1(B_1^+)} \|\nabla^2 \psi\|_{L^2(B_1^+)},
\]

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where the constant $C > 0$ only depends on $\|\mathbb{P}\|_{L^\infty(B^+_1)}$ and $M_0$.

The term $I_1$ can be written as

$$(6.7) \quad I_1 = \int_{B^+_1} \tau_{1,s}(Q(\vartheta \nabla^2 u)) \cdot \nabla^2 \psi - \int_{B^+_1} (\tau_{1,s} Q)(\vartheta \nabla^2 u)(y + s e_1) \cdot \nabla^2 \psi \equiv I'_1 + I''_1,$$

where

$$(6.8) \quad |I''_1| \leq C \|\nabla^2 u\|_{L^2(B^+_1)} \|\nabla^2 \psi\|_{L^2(B^+_1)},$$

with $C > 0$ constant only depending on $\|\mathbb{P}\|_{C^0,1(B^+_1)}$ and $M_0$.

The remaining term $I'_1$ can be elaborated as follows

$$(6.9) \quad I'_1 = \int_{B^+_1} \tau_{1,s}(Q(\vartheta \nabla^2 u)) \cdot \nabla^2 \psi = - \int_{B^+_1} Q(\vartheta \nabla^2 u) \cdot (\tau_{1,-s} \nabla^2 \psi) = - \int_{B^+_1} Q \nabla^2 u \cdot (\vartheta \tau_{1,-s} \nabla^2 \psi) + \int_{B^+_1} Q \nabla^2 u \cdot \nabla^2 (\vartheta \tau_{1,-s} \psi) + \int_{B^+_1} Q \nabla^2 u \cdot ((\nabla^2 \vartheta) \tau_{1,-s} \psi + \nabla \vartheta \otimes \nabla (\tau_{1,-s} \psi) + \nabla (\tau_{1,-s} \psi) \otimes \nabla \vartheta).$$

The last two terms of (6.9) can be estimated as follows:

$$(6.10) \quad \left| \int_{B^+_1} Q \nabla^2 u \cdot ((\nabla^2 \vartheta) \tau_{1,-s} \psi) \right| \leq C \|\nabla^2 u\|_{L^2(B^+_1)} \|\nabla \psi\|_{L^2(B^+_1)},$$

$$(6.11) \quad \left| \int_{B^+_1} Q \nabla^2 u \cdot (\nabla \vartheta \otimes \nabla (\tau_{1,-s} \psi) + \nabla (\tau_{1,-s} \psi) \otimes \nabla \vartheta) \right| \leq C \|\nabla^2 u\|_{L^2(B^+_1)} \|\nabla^2 \psi\|_{L^2(B^+_1)},$$

where the constant $C > 0$ only depends on $\|\mathbb{P}\|_{L^\infty(B^+_1)}$ and $M_0$.

Therefore, by (6.5), (6.6), (6.8), (6.9), (6.10) and (6.11), the left hand side of (6.4) can be written as

$$(6.12) \quad \int_{B^+_1} Q \tau_{1,s}(\nabla^2 (\vartheta u)) \cdot \nabla^2 \psi = - \int_{B^+_1} Q \nabla^2 u \cdot \nabla^2 (\vartheta \tau_{1,-s} \psi) + r_Q,$$

where, by the Poincaré inequality on $H^2_{11}(B^+_1)$, we have

$$(6.13) \quad |r_Q| \leq C \|u\|_{H^2(B^+_1)} \|\nabla^2 \psi\|_{L^2(B^+_1)},$$
where the constant $C > 0$ only depends on $\|\mathbb{P}\|_{C^{0,1}(\overline{B^+_1})}$ and $M_0$.

We can estimate the remaining terms appearing on the right hand side of (6.3) similarly. Concerning the term involving the tensor $A$, for instance,

\begin{equation}
(6.14) \int_{B^+_1} A\tau_{1,s}(\nabla^2(\vartheta u)) \cdot \nabla \psi = \int_{B^+_1} A\tau_{1,s}(\vartheta \nabla^2 u) \cdot \nabla \psi + \int_{B^+_1} A\tau_{1,s}(\nabla \vartheta \otimes \nabla u + \nabla u \otimes \nabla \vartheta) \cdot \nabla \psi + \int_{B^+_1} A\tau_{1,s}(u \nabla^2 \vartheta) \cdot \nabla \psi \equiv J_1 + J_2 + J_3,
\end{equation}

where, by the Poincaré inequality on $H^2_{\Gamma_1}(B^+_1)$ we have

\begin{equation}
(6.15) \quad |J_2 + J_3| \leq C \|u\|_{H^2(B^+_1)} \|\nabla \psi\|_{L^2(B^+_1)},
\end{equation}

where the constant $C > 0$ only depends on $\|\mathbb{P}\|_{L^\infty(B^+_1)}$ and $M_0$. The term $J_1$ can be elaborated as it was done for $I_1$, obtaining

\begin{equation}
(6.16) \quad J_1 = \int_{B^+_1} \tau_{1,s}(A(\nabla^2 u)) \cdot \nabla \psi - \int_{B^+_1} (\tau_{1,s}A)(\nabla \vartheta^2 u)(y + se_1) \cdot \nabla \psi \equiv J'_1 + J''_1,
\end{equation}

where

\begin{equation}
(6.17) \quad |J''_1| \leq C \|\nabla^2 u\|_{L^2(B^+_1)} \|\nabla \psi\|_{L^2(B^+_1)},
\end{equation}

with $C > 0$ constant only depending on $\|\mathbb{P}\|_{C^{0,1}(\overline{B^+_1})}$ and $M_0$. Concerning the term $J'_1$ we have

\begin{equation}
(6.18) \quad J'_1 = \int_{B^+_1} \tau_{1,s}(A(\nabla^2 u)) \cdot \nabla \psi = \int_{B^+_1} A\nabla^2 u \cdot (\nabla \tau_{1,-s} \nabla \psi) = -\int_{B^+_1} A\nabla^2 u \cdot (\nabla \tau_{1,-s} \psi) + \int_{B^+_1} A\nabla^2 u \cdot \nabla (\tau_{1,-s} \psi),
\end{equation}

where the last term of (6.18) can be estimated as

\begin{equation}
(6.19) \quad \left| \int_{B^+_1} A\nabla^2 u \cdot \nabla (\tau_{1,-s} \psi) \right| \leq C \|\nabla^2 u\|_{L^2(B^+_1)} \|\nabla \psi\|_{L^2(B^+_1)},
\end{equation}

with $C > 0$ constant only depending on $\|\mathbb{P}\|_{L^\infty(B^+_1)}$ and $M_0$. 

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Therefore, by (6.15), (6.17), (6.18) and (6.19), the left hand side of (6.14) can be written as

\[(6.20) \int_{B_1^+} A_{\tau_1,s}(\nabla^2 (\partial u)) \cdot \nabla \psi = - \int_{B_1^+} A (\nabla^2 u) \cdot \nabla (\partial \tau_{1,-s}) + r_\lambda,\]

where

\[(6.21) |r_\lambda| \leq C \|u\|_{H^2(B_1^+)} \|\nabla \psi\|_{L^2(B_1^+)},\]

with a constant \(C > 0\) only depending on \(\|P\|_{C^0,1(B_1^+)}\) and \(M_0\).

By similar procedure we obtain

\[(6.22) \int_{B_1^+} B_{\tau_1,s}(\nabla (\partial u)) \cdot \nabla u = - \int_{B_1^+} B (\nabla^2 u) \cdot \nabla (\partial \tau_{1,-s}) + r_B,\]

\[(6.23) \int_{B_1^+} D_{\tau_1,s}(\nabla (\partial u)) \cdot \nabla \psi = - \int_{B_1^+} D (\nabla^2 u) \cdot \nabla (\partial \tau_{1,-s}) + r_D,\]

where

\[(6.24) |r_B| \leq C \|u\|_{H^2(B_1^+)} \|\nabla^2 \psi\|_{L^2(B_1^+)},\]

\[(6.25) |r_D| \leq C \|u\|_{H^2(B_1^+)} \|\nabla \psi\|_{L^2(B_1^+)},\]

where the constant \(C > 0\) only depends on \(\|P\|_{C^0,1(B_1^+)}\) and \(M_0\). Collecting the above results and by using the Poincaré inequality on \(H^2(\Gamma_1^+ (B_1^+))\), by (6.3), (6.12), (6.13), (6.20)-(6.25), we have

\[(6.26) a_+ (\tau_1,s (\partial u), \psi) = -a_+ (u, \partial \tau_{1,-s}) + r_+,\]

where

\[(6.27) |r_+| \leq C \|u\|_{H^2(B_1^+)} \|\nabla^2 \psi\|_{L^2(B_1^+)},\]

with \(C > 0\) constant only depending on \(\|P\|_{C^0,1(B_1^+)}\) and \(M_0\). Since \(\psi \in H^2_{\Gamma_1} (B_1^+)\), the function \(\partial \tau_{1,-s} \psi \in H^2_{\Gamma_1} (B_1^+)\) is a test function and then, by the weak formulation of the problem (5.14), we have

\[(6.28) a_+ (u, \partial \tau_{1,-s} ) = l_+ (\partial \tau_{1,-s}) =
= - \int_{\Gamma_1} \left( \tilde{\mathcal{M}} n \left( \frac{\partial T_{(j)}}{\partial x} \right) \left( \frac{\partial T_{(j)}}{\partial x} \right)^T \nabla (\partial \tau_{1,-s}) \cdot \nu \right) \left( \frac{\partial T_{(j)}}{\partial x} \right)^{-T} n + \left[ \tilde{\mathcal{M}}_{\tau,\xi} \left| \frac{\partial T_{(j)}}{\partial x} \right| \left( \partial \tau_{1,-s} \right) \right]^{1} d\xi.\]
By trace inequalities and Poincarè inequality we have

\[
(6.29) \quad |I_+ (\varphi \tau_{1,-s} \psi)| \leq C \left( \| \hat{M} \|_{H^{\frac{3}{2}}(\Gamma_1)} \| \nabla (\varphi \tau_{1,-s} \psi) \|_{H^{-\frac{1}{2}}(\Gamma_1)} + \| \hat{M} \|_{H^{\frac{3}{2}}(\Gamma_1)} \| \varphi \tau_{1,-s} \psi \|_{H^{\frac{3}{2}}(\Gamma_1)} \right) \leq C \left( \| \hat{M} \|_{H^{\frac{3}{2}}(\Gamma_1)} \| \nabla (\varphi \tau_{1,-s} \psi) \|_{L^2(B^+_1)} + \| \hat{M} \|_{H^{\frac{3}{2}}(\Gamma_1)} \| \varphi \tau_{1,-s} \psi \|_{H^1(B^+_1)} \right) \leq C \| \hat{M} \|_{H^{\frac{3}{2}}(\Gamma_1)} \| \nabla^2 \psi \|_{L^2(B^+_1)},
\]

where the constant \( C > 0 \) only depends on \( M_0 \). By (6.26)–(6.29) we have

\[
(6.30) \quad a_+ (\tau_{1,s} (\varphi u), \psi) \leq C_1 \left( \| \hat{M} \|_{H^{\frac{3}{2}}(\Gamma_1)} + \| u \|_{H^2(B^+_1)} \right) \| \nabla^2 \psi \|_{L^2(B^+_1)},
\]

for every \( \psi \in H^{2,1}_1 (B^+_1) \), where the constant \( C_1 > 0 \) only depends on \( \| \varphi \|_{C^{0,1}(\overline{B^+_1})} \) and \( M_0 \).

Let \( \psi \in H^{2,1}_1 (B^+_1) \) and let us estimate from below \( a_+ (\psi, \psi) \). For every \( \epsilon > 0 \) and for every \( \psi \in H^{2,1}_1 (B^+_1) \) we have

\[
(6.31) \quad \left| \int_{B^+_1} \hat{A} \nabla^2 \psi \cdot \nabla \psi + \hat{B} \nabla \psi \cdot \nabla^2 \psi + \hat{D} \nabla \psi \cdot \nabla \psi \right| \leq C \left( \epsilon \| \nabla^2 \psi \|_{L^2(B^+_1)}^2 + \left( 1 + \frac{1}{\epsilon} \right) \| \nabla \psi \|_{L^2(B^+_1)}^2 \right),
\]

where the constant \( C > 0 \) only depends on \( \| \varphi \|_{C^{0,1}(\overline{B^+_1})} \) and \( M_0 \). Therefore, by the strong convexity of \( Q \) and choosing \( \epsilon \) small enough in (6.31) we have

\[
(6.32) \quad a_+ (\psi, \psi) \geq C_2 \| \nabla^2 \psi \|_{L^2(B^+_1)}^2 - C_3 \| \nabla \psi \|_{L^2(B^+_1)}^2,
\]

where \( C_2 > 0, C_3 > 0 \) are constants only depending on \( \| \varphi \|_{C^{0,1}(\overline{B^+_1})} \), \( \gamma \) and \( M_0 \). Now, by taking \( \psi = \tau_{1,s} (\varphi u) \) in (6.30) and (6.32), we obtain

\[
(6.33) \quad C_2 \| \nabla^2 (\tau_{1,s} (\varphi u)) \|_{L^2(B^+_1)} \leq C_3 \| \nabla (\tau_{1,s} (\varphi u)) \|_{L^2(B^+_1)}^2 + \left( C_1 \left( \| \hat{M} \|_{H^{\frac{3}{2}}(\Gamma_1)} + \| u \|_{H^2(B^+_1)} \right) \right) \| \nabla^2 (\tau_{1,s} (\varphi u)) \|_{L^2(B^+_1)},
\]

where the constants \( C_i > 0, i = 1, 2, 3 \), only depend on \( \| \varphi \|_{C^{0,1}(\overline{B^+_1})} \), \( \gamma \) and \( M_0 \). Therefore, recalling that \( \| \nabla (\tau_{1,s} (\varphi u)) \|_{L^2(B^+_1)} \leq c \| u \|_{H^2(B^+_1)} \), where \( c \) is an absolute constant, and by Poincarè inequality we have

\[
(6.34) \quad \| \nabla^2 (\tau_{1,s} (\varphi u)) \|_{L^2(B^+_1)} \leq C \left( \| \hat{M} \|_{H^{\frac{3}{2}}(\Gamma_1)} + \| u \|_{H^2(B^+_1)} \right),
\]

38
where the constant $C > 0$ only depends on $\|P\|_{C^{0,1}(\Omega)}$, $\gamma$ and $M_0$. Taking the limit as $s \to 0$ we finally have

\[(6.35) \quad \left\| \frac{\partial}{\partial y_1} \nabla^2 u \right\|_{L^2(B^+_N)} \leq C \left( \|\widehat{\mathcal{M}}\|_{H^2(\Gamma_1)} + \|u\|_{H^2(B^+_N)} \right), \]

where the constant $C > 0$ only depends on $\|P\|_{C^{0,1}(\Omega)}$, $\gamma$ and $M_0$.

To obtain an analogous estimate for the normal derivative $\frac{\partial}{\partial y_2} \nabla^2 u$, we shall use the following Lemma.

**Lemma 6.1.** ([Ag, Lemma 9.3]) Assume that the function $g \in L^2(B^+_\rho)$ has weak tangential derivative $\frac{\partial g}{\partial y_1} \in L^2(B^+_\rho)$ and that there exist a constant $K > 0$, such that

\[(6.36) \quad \left| \int_{B^+_\rho} g \frac{\partial^2 \psi}{\partial y_2^2} \right| \leq K \|\psi\|_{H^1(B^+_\rho)}, \quad \text{for every } \psi \in C_0^\infty(B^+_\rho). \]

Then, for every $\rho < \sigma$, $g \in H^1(B^+_\rho)$ and

\[(6.37) \quad \|g\|_{H^1(B^+_\rho)} \leq C \left( K + \|g\|_{L^2(B^+_\rho)} + \sigma \left\| \frac{\partial g}{\partial y_1} \right\|_{L^2(B^+_\rho)} \right), \]

where $C > 0$ only depends on $\frac{\rho}{\sigma}$.

Throughout this part let $\psi \in C_0^\infty(B^+_\sigma)$. The arguments above show that, without loss of generality, we may assume that $A = B = D = 0$. Therefore, we can write

\[(6.38) \quad a_+(u, \psi) = \int_{B^+_\sigma} (Q_{22,\gamma,\delta} u,_{\gamma,\delta} \psi,_{22} + 2Q_{12,\gamma,\delta} u,_{\gamma,\delta} \psi,_{12} + Q_{11,\gamma,\delta} u,_{\gamma,\delta} \psi,_{11}) \]

and, by integration by parts, we have

\[(6.39) \quad \int_{B^+_\sigma} Q_{22,\gamma,\delta} u,_{\gamma,\delta} \psi,_{22} = a_+(u, \psi) + 2 \int_{B^+_\sigma} (Q_{12,\gamma,\delta} u,_{\gamma,\delta} \psi,_{12} + Q_{12,\gamma,\delta} u,_{\gamma,\delta} \psi,_{22}) + \]

\[+ \int_{B^+_\sigma} (Q_{11,\gamma,\delta} u,_{\gamma,\delta} \psi,_{11} + Q_{11,\gamma,\delta} u,_{\gamma,\delta} \psi,_{11}). \]

Since $\psi \in C_0^\infty(B^+_\sigma)$, we have $a_+(u, \psi) = 0$. Let us define

\[(6.40) \quad g = \sum_{\gamma,\delta=1}^2 Q_{22,\gamma,\delta} u,_{\gamma,\delta}. \]
Clearly $g \in L^2(B_{\sigma_0}^+)$ and $\frac{\partial g}{\partial y_1} \in L^2(B_{\sigma_0}^+)$ by (6.35). By (6.39) and by estimate (6.35) we have

\begin{equation}
(6.41) \quad \left| \int_{B_{\sigma_0}^+} g \psi, 22 \right| \leq C \left( \| \mathcal{M} \|_{H^\infty(\Gamma_1)} + \| u \|_{H^2(B_{\sigma_0}^+)} \right) \| \psi \|_{H^1(B_{\sigma_0}^+)};
\end{equation}

with $C > 0$ constant only depending on $\| P \|_{C^{0,1}(B_{\sigma_0}^+)}$, $\gamma$ and $M_0$, so that, in $B_{\sigma_0}^+$, the function $g$ satisfies the conditions of Lemma 6.1. Hence, $g \in H^1(B_{\sigma_0}^+)$ and we have

\begin{equation}
(6.42) \quad \| g \|_{H^1(B_{\sigma_0}^+)} \leq C \left( \| \mathcal{M} \|_{H^\infty(\Gamma_1)} + \| u \|_{H^2(B_{\sigma_0}^+)} \right),
\end{equation}

where the constant $C > 0$ only depends on $\| P \|_{C^{0,1}(B_{\sigma_0}^+)}$, $\gamma$ and $M_0$. By the ellipticity of $Q$, by (6.35) and (6.42), it follows that

\begin{equation}
(6.43) \quad u_{22} = Q_{2222}^{-1} \left( g - \sum_{(\gamma, \delta) \neq (2, 2)} Q_{22, \gamma \delta} u_{\gamma \delta} \right) \in H^1(B_{\rho}^+),
\end{equation}

and therefore

\begin{equation}
(6.44) \quad \| u_{22} \|_{L^2(B_{\rho}^+)} \leq C \left( \| \mathcal{M} \|_{H^\infty(\Gamma_1)} + \| u \|_{H^2(B_{\rho}^+)} \right),
\end{equation}

where the constant $C > 0$ only depends on $\| P \|_{C^{0,1}(B_{\rho}^+)}$, $\gamma$ and $M_0$.

By (6.35) and (6.44) we have

\begin{equation}
(6.45) \quad \| u \|_{H^3(B_{\rho}^+)} \leq C \left( \| \mathcal{M} \|_{H^\infty(\Gamma_1)} + \| u \|_{H^2(B_{\rho}^+)} \right),
\end{equation}

where the constant $C > 0$ only depends on $\| P \|_{C^{0,1}(B_{\rho}^+)}$, $\gamma$ and $M_0$. This completes the proof.

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