ENTROPY PRODUCTION ESTIMATES FOR THE POLYATOMIC ELLIPSOIDAL BGK MODEL

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Abstract. We study the entropy production estimate for the polyatomic ellipsoidal BGK model, which is a relaxation type kinetic model describing the time evolution of polyatomic particle systems. An interesting dichotomy is observed between $0 < \theta \leq 1$ and $\theta = 0$: In each case, a distinct target Maxwellians should be chosen to estimate the entropy production functional from below by the relative entropy. The time asymptotic equilibrium state toward which the distribution function stabilizes bifurcates accordingly.

1. INTRODUCTION

In this paper, we are interested in the entropy production property of the ellipsoidal BGK model for polyatomic molecules \cite{1 5 7}

$$\partial_t f + v \cdot \nabla_x f = A_{\nu,\theta}(M_{\nu,\theta}(f) - f),$$

$$f(0, x, v, I) = f_0(x, v, I).$$

(1.1)

The polyatomic velocity distribution function $f(t, x, v, I)$ represents the number density on phase space $(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$ with internal energy $I^{2/\delta}$ ($I \geq 0$) at time $t \geq 0$. Here, $\delta$ is the additional degree of freedom other than the translation motion. Such internal energy formulation can be traced back to \cite{6 11 12}. The collision frequency $A_{\nu,\theta}$ is given by

$$A_{\nu,\theta} = \frac{\rho T}{\mu (1 - \nu + \theta \nu)}$$

where $\mu > 0$ denotes the viscosity. To explain the polyatomic ellipsoidal Gaussian $M_{\nu,\theta}(f)$, we need to introduce several macroscopic quantities. We start with the definition of local density, bulk velocity, stress tensor and specific internal energy:

$$\rho(t, x) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} f(t, x, v, I)dv dI,$$

$$U(t, x) = \frac{1}{\rho} \int_{\mathbb{R}^3 \times \mathbb{R}^3} vf(t, x, v, I)dv dI,$$

$$\Theta(t, x) = \frac{1}{\rho} \int_{\mathbb{R}^3 \times \mathbb{R}^3} (v - U) \otimes (v - U) f(t, x, v, I)dv dI,$$

$$E_{\delta}(t, x) = \frac{1}{\rho} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \left( \frac{1}{2} |v - U|^2 + I^{2/\delta} \right) f(t, x, v, I)dv dI.$$

The specific internal energy $E_{\delta}$ is divided into the energy from the translational motion $E_{\text{tr}}$ and the energy due to the internal configuration $E_{\text{int}}$:

$$E_{\text{tr}} = \frac{1}{\rho} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{1}{2} |v - U|^2 f(t, x, v, I)dv dI,$$

$$E_{\text{int}} = \frac{1}{\rho} \int_{\mathbb{R}^3 \times \mathbb{R}^3} I^{2/\delta} f(t, x, v, I)dv dI,$$

which, as a consequence of equipartition theorem, are associated with the corresponding temperatures $T_{\delta}$, $T_{\text{tr}}$ and $T_{\text{int}}$ respectively:

$$E_{\delta} = \frac{3 + \delta}{2} T_{\delta}, \quad E_{\text{tr}} = \frac{3}{2} T_{\text{tr}}, \quad E_{\text{int}} = \frac{\delta}{2} T_{\text{int}}.$$

Note that $T_{\delta}$ is represented by a convex combination of $T_{\text{tr}}$ and $T_{\text{int}}$:

$$T_{\delta} = \frac{3}{3 + \delta} T_{\text{tr}} + \frac{\delta}{3 + \delta} T_{\text{int}}.$$

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For $0 \leq \theta \leq 1$, we define the relaxation temperature $T_\theta$ and the corrected temperature tensor $T_{\nu,\theta}$ by

$$T_\theta = \theta T_\delta + (1 - \theta)T_{\nu,mt}, \quad T_{\nu,\theta} = \theta T_\delta Id + (1 - \theta)\{ (1 - \nu)T_{\nu,Id} + \nu \Theta \}.$$ 

Now, the polyatomic ellipsoidal Gaussian $\mathcal{M}_{\nu,\theta}$ is given by

$$\mathcal{M}_{\nu,\theta}(f) = \frac{\rho \Lambda_\delta}{\sqrt{\det(2\pi T_{\nu,\theta})} T_\delta^2} \exp \left( -\frac{1}{2} (v - U)^\top T_{\nu,\theta}^{-1} (v - U) - I_{\nu}^2 \right).$$

Here, $\Lambda_\delta$ denotes $\Lambda_\delta = 1/\int_{\mathbb{R}^+} e^{-|I|^{2/3}} dI$. The relaxation operator satisfies the following cancellation property:

$$\int_{\mathbb{R}^3 \times \mathbb{R}^+} (\mathcal{M}_{\nu,\theta}(f) - f) \left\{ 1, v, \frac{1}{2} |v|^2 + I_{\nu}^2 \right\} dv dI = 0,$$

which leads to the conservation of mass, momentum and energy. The $H$-theorem for this model was established in [11] (See also [5]):

$$\int_{\mathbb{R}^3 \times \mathbb{R}^+} f(t) \ln f(t) dv dI \leq \int_{\mathbb{R}^3 \times \mathbb{R}^+} f_0 \ln f_0 dv dI, \quad (t \geq 0).$$

The original BGK model [3] for monatomic gases, which is widely used in place of the Boltzmann equation for practical purposes, has one well-known shortcoming that it gives incorrect Prandtl number in the Navier Stokes limit. To overcome this, Holway [10] introduced a free parameter $\nu$ and generalized the local Maxwellian into the anistropic Gaussian, which is well-defined in the range $-1/2 < \nu < 1$ (See [11, 14, 16, 17]). The resultant model is called the ellipsoidal BGK model (ES-BGK model). In generalizing this model further to cover the polyatomic case, however, we are confronted with the another incorrect physical coefficient: the relaxation collision number, which is defined as the number of collision needed to transform the rotational and vibrational internal energy into the translational energy. In this regard, another relaxation parameter $\theta$ is introduced (See [11, 5, 12, 13]), leading to the elliptical BGK model for polyatomic particles [11].

In this paper, we are concerned with the Cercignani type entropy-entropy production estimate for the polyatomic ellipsoidal BGK model [11]. Obtaining lower bounds of the entropy production functional for kinetic equations in terms of the relative entropy is important in that such estimates provide the coercivity (at least partial) that pushes the distribution function to the equilibrium state. It was first suggested by Cercignani [8] for the Boltzmann equation, and culminated in [14] where Villani proved the “almost true” version of the conjecture. In their proof, the entropy production estimate of the Landau equation established in [12] was crucially used (See [15] for recent improvement on this issue). In [18], the author proved that the elliptical BGK model for monatomic particle system ([11, 4, 11]) satisfies the Cercignani type entropy production estimate, implying that the entropy production mechanism of the elliptical BGK model resembles that of the linear Boltzmann equation, rather than that of the full Boltzmann equation. (See [2]). In this paper, we extend the result to the polyatomic ellipsoidal BGK model (See Theorem 1.1 below). Due to the presence of various types of temperatures in the polyatomic ellipsoidal Gaussian, the fine cancellation of the temperature function in the entropy comparison of various Maxwellians, which was crucially used in the proof in [18], is not available in the polyatomic case, and we need to keep track of the behavior of those temperatures carefully throughout the argument (See Lemma 2.1).

We also make an interesting observation that different target equilibrium states, to which the distribution function converges time asymptotically, should be chosen according to the value of $\theta$: when $0 < \theta \leq 1$, the relative entropy should be measured with respect to $\mathcal{M}_{0,1}$ where

$$\mathcal{M}_{0,1} = \frac{\rho \Lambda_\delta}{(2\pi T_\delta)^{2/3} (T_\delta)\pi} \exp \left( -\frac{|v - U|^2}{2T_\delta} - I_{\nu}^2 \right),$$

for $0 \leq \theta \leq 1$, we define the relaxation temperature $T_\theta$ and the corrected temperature tensor $T_{\nu,\theta}$ by

$$T_\theta = \theta T_\delta + (1 - \theta)T_{\nu,mt}, \quad T_{\nu,\theta} = \theta T_\delta Id + (1 - \theta)\{ (1 - \nu)T_{\nu,Id} + \nu \Theta \}.$$
while it is $\mathcal{M}_{0,0}$ when $\theta = 0$ for

$$
\mathcal{M}_{0,0} = \frac{\rho A_0}{(2\pi T_{tr})^{\frac{3}{2}} (T_{int})^{\frac{3}{2}}} \exp \left( -\frac{|v - U|^2}{2T_{tr}} - \frac{f^2}{T_{int}} \right).
$$

This is because, when $\theta = 0$, the translational energy and the internal energy is split, making the equation essentially, but not exactly, monatomic. More precisely, the internal energy part is cancelled out in measuring the difference of $H$-functional of various Maxwellians (see Lemma 3.1 in Section 3). This implies a dichotomy in the time asymptotic state of the distribution function $H_{\text{out}}$ in measuring the difference of equation essentially, but not exactly, monatomic. More precisely, the internal energy part is cancelled out in measuring the difference of $H$-functional of various Maxwellians (See Lemma 3.1 in Section 3).

Let us define the $H$-functional $H(f)$, the relative entropy $H(f|g)$ and the entropy production functional $D_{\nu,\theta}(f)$:

$$
H(f) = \int_{\mathbb{R}^3 \times \mathbb{R}^+} f \ln f \, dvdI, \quad H(f|g) = \int_{\mathbb{R}^3 \times \mathbb{R}^+} f \ln(f/g) \, dvdI,
$$

$$
D_{\nu,\theta}(f) = -\int_{\mathbb{R}^3 \times \mathbb{R}^+} A_{\nu,\theta} \{ \mathcal{M}_{\nu,\theta}(f) - f \} \ln f \, dvdI.
$$

Our main result is as follows:

**Theorem 1.1.** For $0 \leq \theta \leq 1$ and $-1/2 < \nu < 1$, the entropy production functional $D_{\nu,\theta}(f)$ of the ES-BGK model satisfies

1. In the case $0 < \theta \leq 1$

   $$
   D_{\nu,\theta}(f) \geq \theta A_{\nu,\theta} H(f|\mathcal{M}_{0,1}).
   $$

2. In the case $\theta = 0$

   $$
   D_{\nu,\theta}(f) \geq \min\{1 - \nu, 1 + 2\nu\} A_{\nu,\theta} H(f|\mathcal{M}_{0,0}).
   $$

These entropy production estimates readily give the asymptotic behavior of $f$ in the homogeneous case:

**Corollary 1.1.** The distribution function for the spatially homogeneous polyatomic ellipsoidal BGK model stabilizes exponentially fast to the equilibrium states:

1. In the case $0 < \theta \leq 1$

   $$
   \|f(t) - \mathcal{M}_{0,1}\|_{L_{v,I}^1} \leq e^{-\frac{A_{\nu,\theta} t}{2}} \sqrt{2H(f_0|\mathcal{M}_{0,1})}.
   $$

2. In the case $\theta = 0$

   $$
   \|f(t) - \mathcal{M}_{0,0}\|_{L_{v,I}^1} \leq e^{-\frac{A_{\nu,\theta} \min\{1 - \nu, 1 + 2\nu\} t}{2}} \sqrt{2H(f_0|\mathcal{M}_{0,0})},
   $$

where $\|f(t)\|_{L_{v,I}^1} = \int_{\mathbb{R}^3 \times \mathbb{R}^+} |f(v, t, I)| \, dvdI$.

Some remarks are in order. First, these results are a priori estimates, which means that they hold when everything is fine: For this to be mathematically rigorous, integrability of the distribution function should be good enough to justify all the integral in the proof, and the strict positivity of the temperatures $T_{int}, T_{tr}$ should be assumed, which should be checked at the level of existence theory. These issues were checked for monatomic ES-BGK model in [16] [18]. The investigation on the existence theory for the polyatomic case is in progress. Secondly, these results can be generalized in a straightforward manner to general $d$-dimensions. For this, the temperatures should be redefined as $E_\delta = \frac{d^2 + 3}{2} T_\delta, E_{tr} = \frac{d^2}{2} T_{tr}$ and the constant 3 in several places, for example in (2.1) and (2.3), should be replaced by $d$. The argument then goes in the exactly same manner, giving the essentially same result with the constants adjusted according to $d$. Instead of treating the most general case, however, we restrict ourselves to three dimensional case for clarity of the proof.
This paper is organized as follows. In section 2, we prove the entropy production estimate in the case $0 < \theta \leq 1$. It is also shown that the spatially homogeneous distribution function converges exponentially fast to $M_{0,1}$. In section 3, analogous result is proved for the case $\theta = 0$, with the target Maxwellian replaced by $M_{0,0}$.

2. Entropy Production Estimate in the Case: $0 < \theta \leq 1$

We need to introduce the following multi-variate Gaussian with the stress tensor as its covariance matrix, which plays an important role in the proof of our main theorem:

$$
\mathcal{M}_\Theta(f) = \frac{\rho \Lambda_\delta}{\sqrt{\text{det} 2\pi \Theta(T_{int})^2}} \exp \left( -\frac{1}{2} (v - U)^T \Theta^{-1} (v - U) - \frac{1}{2} \frac{f^2}{\text{det} T_{int}} \right).
$$

Note that $\mathcal{M}_\Theta$ corresponds to $\mathcal{M}_{1,0}$. We start with the following lemma connecting the $H$-functionals of $\mathcal{M}_{\nu,\theta}$, $\mathcal{M}_\Theta$ and $\mathcal{M}_{0,\theta}$.

**Lemma 2.1.** The $H$-functionals for $\mathcal{M}_{0,1}$, $\mathcal{M}_\Theta$ and $\mathcal{M}_{\nu,\theta}$ satisfy

$$
H(\mathcal{M}_{0,1}) - H(\mathcal{M}_{\nu,\theta}) \geq (1 - \theta) \{ H(\mathcal{M}_{0,1}) - H(\mathcal{M}_\Theta) \}, \ (0 < \theta \leq 1)
$$

**Proof.** A straightforward calculation gives

$$
\begin{align*}
H(\mathcal{M}_{\nu,\theta}) &= \rho \ln \rho \Lambda_\delta - \frac{1}{2} \rho \ln \text{det}(2\pi T_{\nu,\theta}) - \frac{\delta}{2} \rho \ln T_{\theta} - \frac{3 + \delta}{2} \rho, \\
H(\mathcal{M}_\Theta) &= \rho \ln \rho \Lambda_\delta - \frac{1}{2} \rho \ln \text{det}(2\pi \Theta) - \frac{\delta}{2} \rho \ln T_{int} - \frac{3 + \delta}{2} \rho, \\
H(\mathcal{M}_{0,1}) &= \rho \ln \rho \Lambda_\delta - \frac{3}{2} \rho \ln(2\pi \Theta_3) - \frac{\delta}{2} \rho \ln T_{\delta} - \frac{3 + \delta}{2} \rho,
\end{align*}
$$

so that

$$
H(\mathcal{M}_{0,1}) - H(\mathcal{M}_{\nu,\theta}) = \frac{\rho}{2} \{ \ln \text{det} T_{\nu,\theta} + \delta \ln T_{\theta} - (3 + \delta) \ln T_{\delta} \} \equiv I_{\delta,\theta}.
$$

Due to the symmetry of $\Theta$, there exists an orthogonal matrix $P$ such that $P^T \Theta P$ is a diagonal matrix. We denote its eigenvalues by $\Theta_i$ ($i = 1, 2, 3$) to compute

$$
\begin{align*}
\text{det} T_{\nu,\theta} &= \text{det} \left\{ P^T T_{\nu,\theta} P \right\} \\
&= \text{det} \left\{ (1 - \nu) (1 - \nu) T_{\nu} I_3 + (1 - \nu) T_{\nu} \nu \Theta P + \theta T_{\delta} I_3 \right\} \\
&= \prod_{1 \leq i \leq 3} \left\{ (1 - \theta) (1 - \nu) T_{\nu} + \nu \Theta_i + \theta T_{\delta} \right\}.
\end{align*}
$$

Hence,

$$
\ln \text{det} T_{\nu,\theta} = \sum_{1 \leq i \leq 3} \ln \left\{ (1 - \theta) (1 - \nu) T_{\nu} + \nu \Theta_i + (1 - \theta) \nu \Theta \right\}.
$$

Now, we divide the remaining argument into the following two cases:

(1) $0 \leq \nu < 1$: Recalling the concavity of $\ln$, we have

$$
\begin{align*}
\ln \text{det} T_{\nu,\theta} &\geq \sum_{1 \leq i \leq 3} \left\{ (1 - \theta) (1 - \nu) \ln T_{\nu} + (1 - \theta) \nu \ln \Theta_i + \theta \ln T_{\delta} \right\} \\
&= 3(1 - \theta) (1 - \nu) \ln T_{\nu} + (1 - \theta) \nu \ln \Theta_1 \Theta_2 \Theta_3 + 3 \theta \ln T_{\delta} \\
&= 3(1 - \theta) (1 - \nu) \ln T_{\nu} + (1 - \theta) \nu \ln \text{det} \Theta + 3 \theta \ln T_{\delta}.
\end{align*}
$$
Inserting this, we estimate (2.2) as
\[
I_{\delta, \theta} \geq \frac{\rho}{2} \left\{ (1 - \theta)(1 - \nu) \ln T_{tr} + (1 - \theta)\nu \ln \det \Theta + 3\theta \ln T_{\delta} + \delta \ln T_\theta - (3 + \delta) \ln T_\delta \right\}
\]
\[
= \frac{\rho}{2} \left\{ (1 - \theta)(3(1 - \nu) \ln T_{tr} + \nu \ln \det \Theta - 3 \ln T_\delta) + \delta \ln T_\theta - \delta \ln T_\delta \right\}.
\]
We then employ \( \ln T_\theta \geq (1 - \theta) \ln T_{int} + \theta \ln T_\delta \) to see that
\[
\delta \ln T_\theta - \delta \ln T_\delta \geq \delta(1 - \theta) \left\{ \ln T_{int} - \ln T_\delta \right\}
\]
from which we get
\[
I_{\delta, \theta} \geq (1 - \theta) \frac{\rho}{2} \left\{ (1 - \nu) \ln T_{tr} + \nu \ln \det \Theta - 3 \ln T_\delta + \delta \left( \ln T_{int} - \ln T_\delta \right) \right\}.
\]

Then, in regard of the following relation between \( T_{tr} \) and \( \Theta [1] \):
\[
3 \ln T_{tr} = \ln \left( \frac{3 \Theta_1 + \Theta_2 + \Theta_3}{3} \right) \geq \ln \Theta_1 \Theta_2 \Theta_3 = \ln \det \Theta,
\]
which is a direct consequence of arithmetic-geometric inequality, we can proceed further as
\[
I_{\delta, \theta} \geq (1 - \theta) \frac{\rho}{2} \left\{ (1 - \nu) \ln \det \Theta + \nu \ln \det \Theta - 3 \ln T_\delta + \delta \left( \ln T_{int} - \ln T_\delta \right) \right\}
\]
\[
= \frac{1}{2} \left\{ \ln \det \Theta + \delta \ln T_{int} - (3 + \delta) \ln T_\delta \right\}.
\]

Another explicit computation using (2.1) shows that this is exactly \((1 - \theta) \{H(M_{0,1}) - H(M_{\theta})\}\), which gives the desired estimate for positive \( \nu \).

(2) \(-1/2 < \nu \leq 0\): In this case, \((1 - \nu) T_{tr} + \nu \Theta_i\) is not a convex combination of \( T_{tr} \) and \( \Theta_i \). Instead, we use \( \Theta_1 + \Theta_2 + \Theta_3 = 3T_{tr} \) to see
\[
(1 - \nu) T_{tr} + \nu \Theta_i = (1 + 2\nu) T_{tr} - \nu \sum_{j \neq i} \Theta_j
\]
so that
\[
\det T_{\nu, \theta} = \prod_{1 \leq i < 3} \left\{ (1 - \theta) \left( (1 + 2\nu) T_{tr} - \nu \sum_{j \neq i} \Theta_j \right) + \theta T_\delta \right\}.
\]

Taking \( \ln \) on both sides and using the concavity inequality, we get
\[
\ln \det T_{\nu, \theta} = \sum_{1 \leq i < 3} \ln \left\{ (1 - \theta) \left( (1 + 2\nu) T_{tr} - \nu \sum_{j \neq i} \Theta_j \right) + \theta T_\delta \right\}
\]
\[
\geq \sum_{1 \leq i < 3} \left\{ (1 - \theta) \left( (1 + 2\nu) \ln T_{tr} - \nu \sum_{j \neq i} \ln \Theta_j \right) + \theta \ln T_\delta \right\}
\]
\[
= (1 - \theta) \left\{ 3(1 + 2\nu) \ln T_{tr} - 2\nu \ln \det \Theta \right\} + 3\theta \ln T_\delta.
\]

Now, we can compute similarly as in the previous case as
\[
\ln \det T_{\nu, \theta} + \delta \ln T_\theta - (3 + \delta) \ln T_\delta
\]
\[
\geq \left\{ (1 - \theta) \left( 3(1 + 2\nu) \ln T_{tr} - 2\nu \ln \det \Theta \right) + 3\theta \ln T_\delta \right\} + \delta \ln T_\theta - (3 + \delta) \ln T_\delta
\]
\[
= (1 - \theta) \left\{ 3(1 + 2\nu) \ln T_{tr} - 2\nu \ln \det \Theta - 3 \ln T_\delta \right\} + \delta \ln T_{int} - \delta \ln T_\delta.
\]

We recall (2.3) to bound the last line from below by
\[
(1 - \theta) \left\{ 3(1 + 2\nu) \ln T_{tr} - 2\nu \ln \det \Theta - 3 \ln T_\delta \right\} + \delta(1 - \theta) \left\{ \ln T_{int} - \ln T_\delta \right\}
\]
\[
= (1 - \theta) \left\{ 3(1 + 2\nu) \ln T_{tr} - 2\nu \ln \det \Theta - 3 \ln T_\delta \right\} + \delta \left( \ln T_{int} - \ln T_\delta \right)
\]
\[
= (1 - \theta) \left\{ 3(1 + 2\nu) \ln T_{tr} - 2\nu \ln \det \Theta + \delta \ln T_{int} - (3 + \delta) \ln T_\delta \right\}.
\]

Therefore, by making another use of (2.3), we obtain
\[
I_{\delta, \theta} \geq (1 - \theta) \frac{\rho}{2} \left\{ \ln \det \Theta + \delta \ln T_{int} - (3 + \delta) \ln T_\delta \right\},
\]
which, again from (2.1), can be shown to be \((1 - \theta) \{ H(\mathcal{M}_{0,1}) - H(\mathcal{M}_\Theta) \} \). This completes the proof. \( \square \)

The following lemma can be found in [1].

**Lemma 2.2.** [1] The \( H \)-functionals of the \( f, \mathcal{M}_\Theta \) and \( \mathcal{M}_{0,1} \) are related by
\[
H(\mathcal{M}_{0,1}) \leq H(\mathcal{M}_\Theta) \leq H(f).
\]

**2.1. Proof of Theorem 1.1 (1).** We first recall \( F'(x)(x - y) \geq F(x) - F(y) \) satisfied by any convex function \( F \), which, in view of the convexity of \( x \ln x \) implies
\[
D_{\nu,\theta}(f) = -\int_{\mathbb{R}^+} A_{\nu,\theta}\{\mathcal{M}_{\nu,\theta}(f) - f\} \ln f \, dv dI
= A_{\nu,\theta} \int_{\mathbb{R}^+} \{f - \mathcal{M}_{\nu,\theta}(f)\} H'(f) \, dv dI
\geq A_{\nu,\theta}\{H(f) - H(\mathcal{M}_{\nu,\theta})\}.
\]

We divide the last term as
\[
H(f) - H(\mathcal{M}_{\nu,\theta}) = H(f) - H(\mathcal{M}_{0,1}) + H(\mathcal{M}_{0,1}) - H(\mathcal{M}_{\nu,\theta}).
\]

Then, by Lemmas 2.1 and 2.2, we obtain
\[
H(f) - H(\mathcal{M}_{\nu,\theta}) \geq H(f|\mathcal{M}_{0,1}) + (1 - \theta)\{H(\mathcal{M}_{0,1}) - H(\mathcal{M}_\Theta)\}
\geq H(f|\mathcal{M}_{0,1}) + (1 - \theta)\{H(\mathcal{M}_{0,1}) - H(f)\}
= H(f|\mathcal{M}_{0,1}) + (\theta - 1)H(f|\mathcal{M}_{0,1})
= \theta H(f|\mathcal{M}_{0,1}).
\]

**2.2. The proof of Corollary 1.1 (1).** From Theorem 1.1 (1), we get
\[
\frac{d}{dt} H(f|\mathcal{M}_{0,1}) = -D_{\nu,\theta}(f) \leq -\theta A_{\nu,\theta} H(f|\mathcal{M}_{0,1}).
\]

Note that \( A_{\nu,\theta} \) is a constant since \( \rho \) and \( T_\delta \) is constant in the homogeneous case. Then Gronwall’s lemma gives
\[
H(f|\mathcal{M}_{0,1}) \leq e^{-\theta A_{\nu,\theta} t} H(f_0|\mathcal{M}_{0,1}).
\]

Hence, the application of the Kullback inequality:
\[
\|f - g\|_{L^1} \leq \sqrt{2H(f|g)} \quad \text{if} \quad \int f = \int g
\]
gives the desired result.

**3. Entropy production estimate in the case: \( \theta = 0 \)**

In this case, we see that \( T_\theta = T_{\text{int}} \) and \( T_{\nu,0} = (1 - \nu)T_{\text{int}}I + \nu \Theta \) to get
\[
\mathcal{M}_{\nu,0}(f) = \frac{\rho A_\delta}{\sqrt{\det(2\pi T_{\nu,0})(T_{\text{int}})^2}} \exp \left( \frac{1}{2}(v - U)^\top T_{\nu,0}^{-1}(v - U) - I_{\text{w}} \right).
\]

**Lemma 3.1.** The \( H \)-functional for \( \mathcal{M}_{0,0}, \mathcal{M}_\Theta \) and \( \mathcal{M}_{\nu,0} \) satisfies
\[
H(\mathcal{M}_{0,0}) - H(\mathcal{M}_{\nu,0}) \geq \max\{\nu, -2\nu\} \{H(\mathcal{M}_{0,0}) - H(\mathcal{M}_\Theta)\}.
\]

**Proof.** An explicit computation, which is almost identical with the one given in [18] gives
\[
H(\mathcal{M}_{0,0}) - H(\mathcal{M}_{\nu,0})
= \begin{cases} 
(1/2)\rho \{3(1 - \nu) \ln T_{\text{int}} + \nu \ln \det \Theta - 3 \ln T_{\text{int}} \} & \text{for} \ 0 \leq \nu < 1, \\
(1/2)\rho \{3(1 + 2\nu) \ln T_{\text{int}} - 2\nu \ln \det \Theta - 3 \ln T_{\text{int}} \} & \text{for} \ -1/2 < \nu \leq 0.
\end{cases}
\]

Note that \( T_{\text{int}} \) is cancelled out in this case, which reduces the remaining computation to that of the monatomic case carried out in [18]. Therefore, by Lemma 2.2 in [18], we get the desired result. \( \square \)
3.1. **Proof of Theorem 1.1 (2).** As in the previous case, we have from the convexity of $x \ln x$:

$$D_{\nu,0}(f) \geq A_{\nu,0} \left\{ H(f) - H(M_{\nu,0}) \right\}.$$ 

We split the last term as

$$H(f) - H(M_{\nu,0}) = H(f) - H(M_{0,0}) + H(M_{0,0}) - H(M_{\nu,0})$$

and apply Lemma 3.1 to get the desired result:

$$H(f) - H(M_{\nu,0}) \geq H(f|M_{0,0}) + \max \{ \nu, -2\nu \} \{ H(M_{0,0}) - H(M_{\Theta}) \}$$

$$\geq H(f|M_{0,0}) + \max \{ \nu, -2\nu \} \{ H(M_{0,0}) - H(f) \}$$

$$= H(f|M_{0,0}) - \max \{ \nu, -2\nu \} H(f|M_{0,0})$$

$$= \min \left\{ 1 - \nu, 1 + 2\nu \right\} H(f|M_{0,0}),$$

where we used $H(M_{\Theta}) \leq H(f)$.

The proof for the Corollary 1.1 (2) is identical to the previous case. We omit it.

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**References**

1. Andries, P., Le Tallec, P., Perlat, J.-P., Perthame, B.: The Gaussian-BGK model of Boltzmann equation with small Prandtl number. Eur. J. Mech. B Fluids 19 (2000), no. 6, 813–830.
2. Bisi, M., Cañizo, J. A., Lods, B.: Entropy dissipation estimates for the linear Boltzmann operator. J. Funct. Anal. 269 (2015), no. 4, 1028-1069.
3. Bhatnagar, P. L., Gross, E. P. and Krook, M.: A model for collision processes in gases. Small amplitude process in charged and neutral one-component systems, Physical Reviews, 94 (1954), 511-525.
4. Brull, S., Schneider, J.: A new approach for the Ellipsoidal Statistical Model. Cont. Mech. Thermodyn. 20 (2008), no.2, 63-74.
5. Brull, S., Schneider, J.: On the ellipsoidal statistical model for polyatomic gases. Contin. Mech. Thermodyn. 20 (2009), no. 8, 489-508.
6. Brun R.: Transport et Relaxation dans les écoulements Gazeux, Masson, 1986.
7. Cai, Z., Li, R.: The NRxx method for polyatomic gases. J. Comput. Phys. 267 (2014), 6391.
8. Cercignani, C.: $H$-theorem and trend to equilibrium in the kinetic theory of gases. Arch. Mech. (Arch. Mech. Stos.) 34 (1982), no. 3, 231-241 (1983).
9. Desvillettes, L., Villani, C.: On the spatially homogeneous Landau equation for hard potentials. I. Existence, uniqueness and smoothness. Comm. Partial Differential Equations 25 (2000), no. 1-2, 179-259 & 261–298.
10. Holway, L.H.: Kinetic theory of shock structure using and ellipsoidal distribution function. Rarefied Gas Dynamics, Vol. I (Proc. Fourth Internat. Sympos., Univ. Toronto, 1964), Academic Press, New York, (1966), pp. 193–215.
11. Khobalatte B., Perthame B.: Maximum principle on the entropy and minimal limitations for kinetic schemes, Math. Comp. 62 (205) (1994) 119-135.
12. Pitaevski L.P., Lifschitz E.M.: Physical Kinetics, Pergamon Press, Oxford, 1981.
13. Shen, C.: Rarefied Gas Dynamics: Fundamentals, Simulations and Micro Flows, Springer, 2005.
14. Villani, C.: Cercignani’s conjecture is sometimes true and always almost true. Comm. Math. Phys. 234 (2003), no. 3, 455-490.
15. Wu, K.-C.: Global in time estimates for the spatially homogeneous Landau equation with soft potentials. J. Funct. Anal. 266 (2014), no. 5, 3134-3155.
16. Yun, S.-B.: Classical solutions for the ellipsoidal BGK model with fixed collision frequency. J. Differential Equations 259 (2015), no. 11, 6009-6037.
17. Yun, S.-B.: Ellipsoidal BGK model near a global Maxwellian. SIAM J. Math. Anal. 47 (2015), no. 3, 2324-2354.
18. Yun, S.-B.: Entropy production for the ellipsoidal BGK model of the Boltzmann equation. submitted.

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