FRACTIONAL DIFFERENTIAL COUPLES
BY SHARP INEQUALITIES AND DUALITY EQUATIONS

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ABSTRACT. This paper presents a highly non-trivial two-fold study of the fractional differential couples - derivatives \((\nabla_0^{\nu} \leq s < 1 + (-\Delta)^{s/2})\) and gradients \((\nabla_0^{\nu} \leq s < 1 - (-\Delta)^{(s-1)/2})\) of basic importance in the theory of fractional advection-dispersion equations: one is to discover the sharp Hardy-Rellich \((s p < p < n)\) and Adams-Moser \((s p = n)\) inequalities for \(\nabla_0^{\nu} \leq s < 1\); the other is to handle the distributional solutions \(u\) of the duality equations \([\nabla_0^\nu u]^* = \mu\) (a nonnegative Radon measure) and \([\nabla_0^\nu - u]^* = f\) (a Morrey function).

INTRODUCTION

In his celebrated 1988 work [3], Adams extends the Moser inequality in [24] from the first order to the higher order gradients in the Euclidean space \(\mathbb{R}^n\geq 2\) - given the gradient
\[
\nabla = (\partial_{x_1}, \ldots , \partial_{x_n})
\]
and the Laplacian
\[
\Delta = \sum_{j=1}^n \partial^2_{x_j}
\]
as well as
\[
\nabla^m = \begin{cases} 
(-1)^{\frac{m}{2}} (-\Delta)^{\frac{m}{2}} & \text{for } m \text{ even } \\
(-1)^{\frac{m-1}{2}} \nabla (-\Delta)^{\frac{m-1}{2}} & \text{for } m \text{ odd}
\end{cases}
\]
and \(0 < m < n\),

there is a constant \(c_{0,m,n}\) such that
\[
\int_{\Omega} \exp \left( \frac{\beta |u(x)|}{\|\nabla^m u\|_{L^{\frac{n}{m}}} \cdot |\Omega|} \right) \frac{\|\nabla^m u\|_{L^{\frac{n}{m}}} \cdot |\Omega|}{c_{0,m,n}} \leq \forall \ u \in C^m_c(\Omega)
\]
holds, where:

- \(0 \leq \beta \leq \beta_{0,m,n} = \left( \frac{n}{\omega_{n-1}} \right)^{\frac{2m}{n}} \frac{\frac{2m}{n} \Gamma\left( \frac{2m}{n} \right)}{\Gamma\left( \frac{2m}{n} \right)} \) for \( m \text{ even } \) and \( 0 < m < n\);
- \(0 \leq \beta \leq \beta_{0,m,n} = \left( \frac{n}{\omega_{n-1}} \right)^{\frac{2m}{n}} \frac{\frac{2m}{n} \Gamma\left( \frac{2m}{n} \right)}{\Gamma\left( \frac{2m}{n} \right)} \) for \( m \text{ odd } \)

\(\Omega\) is a subdomain of \(\mathbb{R}^n\) with finite \(n\)-measure \(|\Omega|\) and its associate space \(C^m_c(\Omega)\) stands for all \(C^m\)-functions supported in \(\Omega\);
- \(\Gamma(\cdot)\) is the standard gamma function and induces \(\omega_{n-1} = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left( \frac{n}{2} \right)}\) - the area of the unit sphere \(S^{n-1}\) of \(\mathbb{R}^n\);

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• (0.1) is established through the Adams-Riesz potential inequality (just under [3, (23)])

\[ |u(x)| \leq \left( \frac{n}{\lambda (n-1)} \right)^{n-m} \beta_{0,m,n} \int_{\mathbb{R}^n} |y - x|^{m-n} |\nabla^m u(y)| \, dy \quad \forall \ u \in C^\infty_c. \]

Moreover, if \( \beta > \beta_{0,m,n} \) then there is \( u \in C^\infty_c(\Omega) \) such that the integral in (0.1) can be made as large as desired - in other words \( \beta_{0,m,n} \) is sharp.

Upon examining \( \|\nabla^m u\|_{L^\infty} \) in (0.1), we are automatically suggested to consider a variant of (0.1) for

\[ \|\nabla^m u\|_{L^{1,p} \times \mathbb{R}^m} \quad \text{or} \quad \|\nabla^m u\|_{L^{\infty,p} \times \mathbb{R}^m}. \]

• For the former, we use the \( m \)-form of [6, Corollary 1 & Theorem 4 (16)] to derive the sharp \( m \)-order Hardy-Rellich inequality

\[ \left( \int_{\mathbb{R}^n} \left( \frac{|u(x)|}{|x|^m} \right)^p \, dx \right)^{\frac{1}{p}} \leq c_{mp,n} \|\nabla^m u\|_{L^p} \quad \forall \ u \in C^\infty_c, \]

where

\[ c_{mp,n} = \begin{cases} 
\left( \frac{2^{-m} \Gamma(\frac{m-p}{2p}) \Gamma(\frac{n-m+1}{p})}{\Gamma(\frac{m-p}{2p} + \frac{n-m+1}{p})} \right) & \text{for } m \text{ even} \\
\left( \frac{(2^{1-m} n-p)}{n-p} \Gamma(\frac{m-p}{2p} + \frac{n-m+1}{p}) \Gamma(\frac{1}{p}) \right) & \text{for } m \text{ odd} \end{cases} \quad \text{and} \quad 0 < m < n. \]

Of course, the case \( m = 1 \) of (0.3) is the classical sharp Hardy inequality (cf. [14]).

• For the latter, we use the \( m \)-form of Theorem 2.1(iii) (viewed as a sharp Morrey-Riesz inequality) and (0.2) to discover the sharp \( m \)-order Morrey-Sobolev inequality

\[ \|u\|_{L^\infty} \leq \frac{(m(p-1))^{p-1}}{\beta_{0,m,n}} |\Omega|^\frac{mp-n}{mn} \|\nabla^m u\|_{L^p} \quad \forall \ u \in C^m_c(\Omega). \]

In particular, the case \( m = 1 \) of (0.4) is the classical sharp Morrey-Sobolev inequality (cf. [39, Theorem 2.1E]).

Clearly, (0.1), (0.3) and (0.4) give a complete structure on utilizing the higher derivatives and gradients to sharply dominate the size of a derivative/gradient-free function. However, upon recognizing the fractional vector calculus considerably used in both Herbst’s study of the Klein-Gordon equation for a Coulomb potential [15] and Meerschaert-Mortensen-Wheatcraft’s investigation of the particle mass density \( u(x,t) \) of a contaminant in some fluid at a point \( x \in \mathbb{R}^n \) at time \( t > 0 \) which solves the fractional advection-dispersion equation (with a constant average velocity \( \vec{v} \) of contaminant particles and a positive constant \( \kappa \))

\[ \partial_t u(x,t) = -\vec{v} \cdot \nabla u(x,t) - \kappa (-\Delta)^{\frac{1}{2}} u(x,t) \]
\[ = -\vec{v} \cdot \nabla p(x,t) + \kappa \text{div}^\gamma(\nabla u(x,t)) \]
\[ = -\vec{v} \cdot \nabla u(x,t) + \kappa \text{div}(\vec{v} \cdot u(x,t)) \]

combining a fractional Fick’s law for flux with a classic mass balance - and reversely- a fractional mass balance with a classic Fickian flux [22], in the forthcoming sections we are driven to work out versions of (0.1), (0.3) and (0.4) for the fractional differential couples - derivatives and gradients:

\[ \{\nabla_+^{0<s<1}, \nabla_-^{0<s<1}\} \quad \text{corresponding naturally to} \quad \{\nabla_{\text{even}}, \nabla_{\text{odd}}\}, \]

and their essential applications in the study of the distributional solutions to some fractional partial differential equations of dual character. More precisely,
• §1 collects some fundamental facts on

\[ \nabla_{\pm}^{0<s<1} \quad \text{and} \quad [\nabla_{\pm}^{0<s<1}]^* \]

through the Stein-Weiss-Hardy inequalities and the Fefferman-Stein type decompositions (cf. [10, 7, 21]).

• §2 utilizes Theorem 2.1 - an sharp embedding principle for the Riesz potentials to discover the fractional extensions of (0.1), (0.3) and (0.4) - Theorem 2.2.

• §3 discusses the fractional Hardy-Sobolev spaces

\[ H_{-}^{0<s<1,1<p<\infty} \quad \text{and} \quad H_{+}^{0<s<1,1<p<\infty} \]

and their dualities generated by \( \nabla_{\pm}^{0<s<1} \) - Theorems 3.1-3.2.

• §4 studies the distributional solutions of the duality equations

\[ [\nabla_{\pm}^{0<s<1}]^* u = \mu \]

for a nonnegative Radon measure \( \mu \) and their absolutely continuous forms

\[ [\nabla_{\pm}^{0<s<1}]^* u = f \]

under the hypothesis that \( f \) is in the Morrey space \( L^{1<p<\infty,0<s<n} \) (cf. [1]) - Theorems 4.1-4.3.

Notation. In what follows, \( U \lesssim V \) (resp. \( U \gtrsim V \)) means \( U \leq cV \) (resp. \( U \geq cV \)) for a positive constant \( c \) and \( U \approx V \) amounts to \( U \gtrsim V \approx U \).

1. Fractional differential couples \( \nabla_{\pm}^{0<s<1} \) and their dualities \([\nabla_{\pm}^{0<s<1}]^* \)

1.1. Fractional differential couples \( \nabla_{\pm}^{0<s<1} \). For \( (n, p) \in \mathbb{N} \times [1, \infty) \) let \( H^p \) be the real Hardy space of all functions \( u \) in the Lebesgue space \( L^p \) on the Euclidean space \( \mathbb{R}^n \) with

\[ \|u\|_{H^p} = \|u\|_{L^p} + \|\tilde{R}u\|_{L^p} < \infty, \]

where \( \tilde{R} = (R_1, \ldots, R_n) \) is the vector-valued Riesz transform on \( \mathbb{R}^n \), with

\[ \tilde{R} u = (R_1 u, \ldots, R_n u) \quad \text{and} \quad R_j u(x) = \left( \frac{\Gamma(n+1)}{\pi^{n/2}} \right) \text{p.v.} \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x - y|^{n+1}} u(y) \, dy \quad \text{a.e.} \ x \in \mathbb{R}^n. \]

Also, for a vector-valued function

\[ \bar{f} = (f_1, \ldots, f_n) \]

let

\[ \|\bar{f}\|_{L^p} = \|\bar{f}\|_{(L^p)^n} \approx \sum_{j=1}^n \|f_j\|_{L^p}. \]

Note that \( H^p \) coincides with the classical Lebesgue space \( L^p \) whenever \( p \in (1, \infty) \) and the \( (0, 1) \) \( s \)-th order Riesz singular integral operator \( I_s \) acting on a suitable function \( u \) is defined by

\[ I_s u(x) = \left( \frac{\Gamma(n-s)}{\pi^{n/2} \Gamma(\frac{s}{2})} \right) \int_{\mathbb{R}^n} \frac{1}{|x - y|^{n-s}} u(y) \, dy \quad \text{a.e.} \ x \in \mathbb{R}^n. \]

We refer the reader to Stein’s seminal texts [36, 37] for more about these basic notions. The Stein-Weiss-Hardy inequality (cf. [38] for \( p > 1 \) and (4.5) in §4 for \( p = 1 \)) states that under

\[ 0 < s < 1 \leq p < \frac{n}{s} \]
we have
\begin{equation}
(1.1) \quad \left( \int_{\mathbb{R}^n} (|x|^{-s} |I_s u(x)|)^p \, dx \right)^{\frac{1}{p}} \lesssim \|u\|_{L^p} + \|\tilde{R} u\|_{L^p} \approx \|u\|_{H^p} \quad \forall \ u \in H^p.
\end{equation}

Let \( C^\infty_c \) be the collection of all infinitely differentiable functions compactly supported in \( \mathbb{R}^n \). Note that \( C^\infty_c \cap H^p \) is dense in \( H^p \) for any \( p \in [1, \infty) \). For any \( u \in C^\infty_c \) let
\begin{equation}
(1.2) \quad (-\Delta)^s u(x) = \begin{cases} 
L_{-s} u(x) = c_{n,s} \int_{\mathbb{R}^n} \frac{u(x+y)}{|y|^{n+s}} \, dy & \text{as } s \in (-1, 0), \\
\alpha_{n,s} \text{ p.v.} \int_{\mathbb{R}^n} \frac{\alpha(x+y) - u(x)}{|y|^{n+s}} \, dy & \text{as } s = 0 \\
\end{cases}
\end{equation}
and
\begin{equation}
(1.3) \quad D^s u(x) = \left( \frac{\partial^s u}{\partial x_j^s} \right)_{j=1}^n = \tilde{R} (-\Delta)^s u(x) = c_{n,s,-} \int_{\mathbb{R}^n} \frac{\alpha(u(x) - u(x-y))}{|y|^{n+1+s}} \, dy,
\end{equation}
where (cf. [8, Definition 1.1, Lemma 1.4] for \( c_{n,s,+} \) and [21] for \( c_{n,s,-} \))
\begin{align*}
\begin{cases} 
\alpha_{n,s} = \frac{\Gamma(\frac{n}{2} + \frac{s}{2})}{\pi^{\frac{n+s}{2}} 2^{\frac{n}{2}} \Gamma(\frac{n}{4})} \\
\alpha_{n,s,+} = \frac{\Gamma(\frac{n}{2} + s) 
\alpha_{n,s,-} = \frac{s^{n-1} \Gamma(\frac{n+s}{2})}{\pi^{\frac{n+s}{2}} \Gamma(1 + \frac{s}{2})} \\
\alpha_{n,s,-} = \frac{2^{1-s} \Gamma(\frac{n}{2}+1-s)}{\pi^{\frac{n}{2}} \Gamma(\frac{n}{2}+1)}.
\end{cases}
\end{align*}

Especially, if \( 0 < s < n = 1 \) then there are two \( s \)-dependent constants \( c_s \) to make the following Liouville fractional derivative formulae (cf. [32]):
\begin{align*}
\begin{cases} 
(-\Delta)^s u(x) = c_s \left( \frac{\partial^s u}{\partial x_j^s} + \frac{\partial^s u}{\partial x_j^s} \right) u(x) \\
D^s u(x) = c_s \left( \frac{\partial^s u}{\partial x_j^s} - \frac{\partial^s u}{\partial x_j^s} \right) u(x) \\
\frac{\partial^s u}{\partial x_j^s} u(x) = \frac{s}{\Gamma(1-s)} \int_{\mathbb{R}^n} \frac{\alpha(u(x+y) - u(x))}{|y|^{n+s}} \, dy.
\end{cases}
\end{align*}
Hence it is natural and reasonable to adopt the notations
\begin{equation}
(1.4) \quad \nabla^s_+ u = (-\Delta)^s u \quad & \quad \nabla^s_- u = D^s u = \tilde{R} (-\Delta)^s u.
\end{equation}
The operators \( \nabla^s_+ \) and \( \nabla^s_- \) can be viewed as the fractional derivative and the fractional gradient due to
\[
id = - \sum_{j=1}^n R_j^s = -\tilde{R} \cdot \tilde{R}.
\]
Accordingly, for any \( s \in (0, 1) \), the Stein-Weiss-Hardy inequality (1.1) (cf. [29]) amounts to
\begin{equation}
(1.5) \quad \left( \int_{\mathbb{R}^n} (|x|^{-s} |u(x)|)^p \, dx \right)^{\frac{1}{p}} \lesssim \|\nabla^s_+ u\|_{L^p} + \|\nabla^s_- u\|_{L^p} \quad \forall \ u \in I_s(C^\infty_c \cap H^p).
\end{equation}
Here it is worth pointing out the following fundamentals:
\begin{itemize}
\item If \( 0 < s < 1 < p < \frac{n}{s-1} \), then the right-hand-side of (1.5) can be replaced by \( \|\nabla^s_+ u\|_{L^p} \). More precisely, on the one hand, the boundedness of \( \tilde{R} \) on \( L^{p+1} \) and (1.5) give (cf. [31, Lemma 2.4])
\[
\left( \int_{\mathbb{R}^n} (|x|^{-s} |u(x)|)^p \, dx \right)^{\frac{1}{p}} \lesssim \|\nabla^s_+ u\|_{L^p} \quad \forall \ u \in I_s(C^\infty_c \cap H^p).
\]
One the other hand, [31, Theorems 1.8-1.9] derives
\[
\left( \int_{\mathbb{R}^n} (|x|^{-s}|u(x)|)^p \, dx \right)^{\frac{1}{p}} \leq \|\nabla_x^s u\|_{L^p} \quad \forall \ u \in I_s(C^\infty_c \cap H^p).
\]

- If $0 < s < p = 1 \leq n$, then according to Spector’s [35, Theorem 1.4] the right-hand-side of (1.5) except $n = 1$ (cf. (4.6)) can be replaced by $\|\nabla_x^s u\|_{L^1}$ - i.e.
\[
\int_{\mathbb{R}^n} |x|^{-s}|u(x)| \, dx \leq \|\nabla_x^s u\|_{L^1} \quad \text{under } n \geq 2 \quad \forall \ u \in I_s(C^\infty_c \cap H^1).
\]

which may be viewed as a rough extension of Shieh-Spector’s [32, Theorem 1.2] and the classic sharp Hardy’s inequality (cf. [11]) under $n \geq 2$:
\[
\begin{cases}
\int_{\mathbb{R}^n} |x|^{-1}|u(x)| \, dx \leq (n-1)^{-1}\|u\|_{L^1} & \forall \ u \in C^\infty_c \\
\int_{\mathbb{R}^n} |x|^{-1}|I_{-s}u(x)| \, dx \leq (n-1)^{-1}\|\nabla_x^s u\|_{L^1} & \forall \ u \in I_{-s}(C^\infty_c).
\end{cases}
\]

However, the right-hand-side of (1.5) cannot be replaced by $\|\nabla_x^s u\|_{L^1}$ (cf. [36, p.119], [29, Section 3.3] & [32, Section 1.1]).

1.2. Dual fractional differential couples $[\nabla_x^s]^*$. Suppose that $C^\infty_c$ is the space of all infinitely differentiable functions on $\mathbb{R}^n$. Denote by $S$ the Schwartz class on $\mathbb{R}^n$, consisting of all functions $f$ in $C^\infty_c$ such that
\[
\rho_{N,\alpha}(f) = \sup_{x \in \mathbb{R}^n} (1 + |x|^N)|D^\alpha f(x)| < \infty \quad \text{holds for } \begin{cases} N \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\} \\
\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n \\
D^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} \end{cases}
\]

Also, write $S'$ for the Schwartz tempered distribution space - the dual of $S$ endowed with the weak-* topology. According to [33, 21], given $s \in (0, 1)$, if we let
\[
S_s = \left\{ f \in C^\infty_c : \rho_{N+s,\alpha}(f) = \sup_{x \in \mathbb{R}^n} (1 + |x|^{s+N})|D^\alpha f(x)| < \infty \quad \forall \ \alpha \in \mathbb{Z}^n_+ \right\}
\]
and $S'_s$ be the dual space of $S_s$ (i.e., the space of all continuous linear functionals on $S_s$), then for any $u \in S'_s \subseteq S'$ we can define below $\nabla_x^s u$ as a distribution in $S'$:
\[
\begin{align*}
\langle \nabla_x^s u, \phi \rangle &= \langle u, \nabla_x^s \phi \rangle \\
\nabla_x^s &= (\nabla_1^s, \ldots, \nabla_n^s) \\
\langle \nabla_j^s u, \phi \rangle &= -\langle u, \nabla_j^s \phi \rangle \quad \forall \ j \in \{1, \ldots, n\}
\end{align*}
\]
where the action of $\nabla_x^s$ on any function $\phi \in S$ is determined by the Fourier transform
\[
\hat{\phi}(\xi) = \int_{\mathbb{R}^n} \phi(x) e^{-2\pi i x \xi} \, dx \quad \forall \ \xi \in \mathbb{R}^n
\]
according to
\[
\begin{align*}
\langle \nabla_x^s \phi, \phi \rangle &= (2\pi |\xi|)^s \hat{\phi}(\xi) \\
\langle \nabla_j^s \phi, \phi \rangle &= -(2\pi i \xi_j)(2\pi |\xi|)^{j-1} \hat{\phi}(\xi) \quad \forall \ \xi \in \mathbb{R}^n.
\end{align*}
\]
If $\phi \in C^\infty_c$, then (1.7) goes back to (1.2)-(1.3)-(1.4) (cf. [33, 8, 21]). Moreover, the above equalities in (1.6) are well defined because $\nabla_x^s$ and $\nabla_j^s$ send $S$ to $S_s$ (cf. [33, 8] for $\nabla_x^s$ and [21, Lemma 2.6] for $\nabla_j^s$).
Based on the foregoing discussion, we may describe the dual/adjoint operators of $\nabla_x^\pm$ and one of their most important consequences.

- The adjoint operator $[(-\Delta)^\pm]^*$ of $(-\Delta)^\pm$ is itself, namely,
  $$[\nabla_x^\pm]^* = (-\Delta)^\pm,$$
  which can be understood in the sense of
  $$\langle [\nabla_x^\pm]^* f, \phi \rangle = \langle f, \nabla_x^\pm \phi \rangle = \langle \nabla_x^\pm f, \phi \rangle \quad \forall \ (f, \phi) \in S'_f \times S.$$
  This is reasonable, because for nice function pair $(f, \phi) \in (C_0^\infty)^2$ we have (cf. [34])
  $$\langle [\nabla_x^\pm]^* f, \phi \rangle = \int_{\mathbb{R}^n} ((-\Delta)^\pm f(x)) \phi(x) \, dx = \int_{\mathbb{R}^n} f(x)((-\Delta)^\pm \phi(x)) \, dx = \langle f, \nabla_x^\pm \phi \rangle$$
  and
  $$(-\Delta)^\pm ((-\Delta)^\pm u) = (-\Delta)^\pm u \quad \forall \ u \in C_c^\infty.$$

- Upon setting
  $$\text{div}^s \vec{g} = (-\Delta)^\pm \vec{R} \cdot \vec{g},$$
  then $-\text{div}^s$ exists as the adjoint operator $[\nabla_x^-]^*$ of $\nabla_x^-$ - in short -
  $$[\nabla_x^-]^* = -\text{div}^s.$$

Note that (cf. [31, Theorem 1.3])
  $$-\text{div}^s(\nabla_x^- u) = (-\Delta)^\pm u \quad \forall \ u \in C_c^\infty$$
  and (cf. [9, Lemma 2.5])
  $$\int_{\mathbb{R}^n} f(x)(-\text{div}^s \vec{g})(x) \, dx = \int_{\mathbb{R}^n} \vec{g}(x) \cdot \nabla_x^- f(x) \, dx \quad \forall \ (f, \vec{g}) \in C_c^\infty \times (C_c^\infty)^n.$$

- Recall that BMO stands for the John-Nirenberg class of all locally integrable functions $f$ on $\mathbb{R}^n$ with bounded mean oscillation (cf. [16])
  $$\|f\|_{\text{BMO}} = \sup_{B \subseteq \mathbb{R}^n} \frac{1}{|B|} \int_B |f(x) - f_B| \, dx < \infty$$
  where the supremum is taken over all Euclidean balls $B \subseteq \mathbb{R}^n$ with
  $$|B| = \int_B dx \quad \& \quad f_B = \frac{1}{|B|} \int_B f(x) \, dx.$$
  Of remarkable interest is that the Fefferman-Stein decomposition (cf. [10, 40])
  $$\text{BMO} = L^\infty + \vec{R} \cdot (L^\infty)^n$$
  can be shortened via $[\nabla_x^\pm, [\nabla_x^\pm]^*]$ to the Liu-Xiao’s form (cf. [21, Theorem 4.4(iii)])
  $$\text{BMO} = \vec{R} \cdot (L^\infty)^n \quad \text{under} \quad n \geq 2.$$

This decomposition provides a surprising solution to the Bourgain-Brazis’ open problem (cf. [7, p.396]) - *What are the function spaces $X$, $W^{1,n} \subseteq X \subseteq \text{BMO}$, such that every $F \in X$ has a decomposition $F = \sum_{j=1}^n R_j Y_j$ where $Y_j \in L^\infty$? Here $W^{1,n}$ is the conformal Sobolev space consisting of all functions $f$ with $\|\nabla f\|_{L^n} < \infty$ and has the following Bourgain-Brazis representation
  $$W^{1,n} = \vec{R} \cdot (L^\infty \cap W^{1,n})^n \quad \text{under} \quad n \geq 2.$$. 
2. Sharp fractional differential–integral inequalities

2.1. Optimal control for Riesz’s operator $I_{0<\alpha<n}$. The following is of independent interest.

**Theorem 2.1.** Let

$$\left\{ (p, \alpha) \in (1, \infty) \times (0, n) \right\}$$

$$I_\alpha = \left( \frac{\Gamma(\frac{\alpha}{2})}{2 \pi n \Gamma(\frac{\alpha}{2})} \right) I_\alpha = c_n \alpha I_\alpha$$

$$I_\alpha f = \int_{\mathbb{R}^n} |x-y|^{\alpha-n} f(y) \, dy.$$  

Then the following assertions are true.

(i) If $\alpha p < n$, then

$$\sup_{0 \neq f \in L^p} \left( \frac{\int_{\mathbb{R}^n} (|x|^{-\alpha} |I_\alpha f(x)|)^p \, dx}{\|f\|_{L^p}^p} \right)^{\frac{1}{p}} = c_{\alpha p < n} = \frac{2^{-\alpha(p-1)} \pi^\frac{\alpha}{2} \Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{n}{2p} - \frac{\alpha}{2}\right) \Gamma\left(\frac{n(p-1)}{2p}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right) \Gamma\left(\frac{n}{2p}\right)}.$$  

(ii) If $\alpha p = n$, $\Omega \subseteq \mathbb{R}^n$ is a domain with volume $|\Omega| < \infty$ and $L^p(\Omega)$ stands for the class of all $f \in L^p$ with support contained in $\Omega$, then there is a constant $c_{\alpha p = n}$ depending only on $\alpha$ and $n$ such that

$$\sup_{f \in L^p(\Omega)} \int_{\Omega} \exp\left( \beta \frac{\|I_\alpha f(x)\|}{\|f\|_{L^p}} \right)^{\frac{n}{2p}} \, dx \leq c_{\alpha p = n} \quad \forall \ 0 \leq \beta \leq \frac{n}{\omega_{n-1}}.$$  

Here $\frac{n}{\omega_{n-1}}$ is sharp in the sense that if $\Omega$ is a Euclidean ball and $\beta > \frac{n}{\omega_{n-1}}$ then the last integral inequality cannot hold without forcing $c_{\alpha p = n}$ to depend only on $\alpha$ and $n$.

(iii) If $\alpha p > n$, $\Omega \subseteq \mathbb{R}^n$ is a domain with volume $|\Omega| < \infty$ and $L^p(\Omega)$ stands for the class of all $f \in L^p$ with support contained in $\Omega$, then

$$\sup_{f \in L^p(\Omega)} \frac{\|I_\alpha f\|_{L^\infty}}{\|f\|_{L^p} \|\Omega|} \leq c_{\alpha p > n} = \left( \frac{\omega_{n-1}}{n} \right) \left( \frac{n(p-1)}{\alpha p - n} \right)^{\frac{1}{p}}.$$  

Moreover, the constant $c_{\alpha p > n}$ is sharp in the sense that if $\Omega$ is a Euclidean ball then

$$\sup_{f \in L^p(\Omega)} \frac{\|I_\alpha f\|_{L^\infty}}{|\Omega|^{\frac{\alpha p}{n}} \|f\|_{L^p}} = c_{\alpha p > n}.$$  

**Proof.** (i) This is regarded as the sharp Stein–Weiss–Hardy inequality. The sharp constant $c_{\alpha p < n}$ is obtained in Herbst [15]; see also [6, 28, 13] for more information.

(ii) This is just the sharp Adams inequality in [3, Theorem 2] whose argument is still valid for $n = 1$ and $\frac{\alpha n}{n} = 2$.

(iii) This is totally brand-new. In the sequel let $p' = \frac{p}{n-1}$. For any $f \in L^p$ supported on $\Omega$ and for any $x \in \mathbb{R}^n$, we utilize the Hölder inequality to derive that

$$|I_\alpha f(x)| \leq \int_{\Omega} |f(y)||x-y|^{\alpha-n} \, dy \leq \|f\|_{L^p} \left( \int_{\Omega} |x-y|^{(\alpha-n)p'} \, dy \right)^{\frac{1}{p'}}.$$
Note that the Fubini theorem and \((\alpha - n)p' + n > 0\) imply
\[
\int_{\Omega} |x - y|^{(\alpha - n)p'} \, dy = (n - \alpha)p' \int_{0}^{\infty} \left( \int_{|x - y|}^{\infty} f^{(\alpha - n)p' - 1} \, dr \right) \, dy
\]
\[
= (n - \alpha)p' \int_{0}^{\infty} \left( \int_{B(x, r) \cap \Omega} dy \right) r^{(\alpha - n)p' - 1} \, dr
\]
\[
\leq (n - \alpha)p' \int_{0}^{\infty} \min \left\{ \frac{\omega_{n-1}}{n} |r^n|, |\Omega| \right\} r^{(\alpha - n)p' - 1} \, dr
\]
\[
= (n - \alpha)p' \left( \frac{\omega_{n-1}}{n} \int_{0}^{\frac{\omega_{n-1}}{n} - \frac{1}{n}} r^{(\alpha - n)p' + n - 1} \, dr + |\Omega| \int_{\frac{\omega_{n-1}}{n} - \frac{1}{n}}^{\infty} r^{(\alpha - n)p' - 1} \, dr \right)
\]
\[
= (n - \alpha)p' \left( \frac{\omega_{n-1}}{n} \int_{0}^{\frac{\omega_{n-1}}{n}} r^{(\alpha - n)p' + n} \, dr + \frac{1}{(n - \alpha)p'} \left( \frac{n(p - 1)}{n \alpha} \right)^{\frac{n-1}{p}} |\Omega| \right) \frac{(\alpha - n)p' + n}{n}.
\]
Thus we arrive at the desired inequality
\[
|I_{\alpha} f(x)| \leq \int_{\Omega} |f(y)||x - y|^{\alpha - n} \, dy \leq \|f\|_{L^p} \left( \frac{n(p - 1)}{n \alpha} \right)^{\frac{n-1}{p}} \frac{\omega_{n-1}}{n} |\Omega| \frac{(\alpha - n)p' + n}{n}.
\]
To prove that
\[
c_{\alpha > n} = \left( \frac{\omega_{n-1}}{n} \right)^{\frac{n-1}{p}} \left( \frac{n(p - 1)}{n \alpha} \right)^{\frac{n-1}{p}}
\]
is sharp, let us consider the case
\[
\Omega = B(x_0, r_0) \forall (x_0, r_0) \in \mathbb{R}^n \times (0, \infty)
\]
and the function
\[
\mathbb{R}^n \ni x \mapsto f_{\beta}(x) = 1_{B(x_0, r_0)} |x - x_0|^\beta,
\]
where \(\beta\) satisfies
\[
\beta + \frac{n}{p} > 0.
\]
On the one hand, a direct calculation gives
\[
\|f_{\beta}\|_{L^p} = \left( \int_{B(x_0, r_0)} |x - x_0|^\beta p \, dx \right)^{\frac{1}{p}}
\]
\[
= \left( \frac{\omega_{n-1}}{n} \int_{0}^{r_0} \frac{1}{r^\beta p + n} \, dr \right)^{\frac{1}{p}}
\]
\[
= \left( \frac{\omega_{n-1}}{n} \right)^{\frac{1}{p}} \left( \frac{n}{\beta p + n} \right)^{\frac{1}{p}} r_0^{\frac{\beta + \frac{n}{p}}{p}}.
\]
On the other hand, by the fact
\[
\alpha + \beta > \alpha - \frac{n}{p} > 0,
\]
we get
\[ |I_{\alpha}f_{\beta}(x_0)| = \int_{B(x_0, r_0)} |x - x_0|^{\alpha-n+\beta} \, dx = \omega_{n-1} \int_{r_0}^{r} r^{\alpha+\beta-1} \, dr = \frac{\omega_{n-1}}{\alpha+\beta} r_0^{\alpha+\beta}. \]

Combining the last two formulae gives
\[ c_{\alpha,p,n} \geq \sup_{x \in B(x_0, r_0)} \left( \frac{|I_{\alpha}f_{\beta}(x)|}{|B(x_0, r_0)|^{\alpha-n} \|f_{\beta}\|_{L^p}} \right) \geq \left( \frac{\omega_{n-1}}{n} \right)^{\frac{\alpha-n}{n-p}} \left( \frac{\omega_{n-1}^{\frac{n}{n-p}}}{n^{\frac{n}{n-p}}} \right)^{\frac{1}{p}} \|f_{\beta}\|_{L^p}. \]

Now the problem turns to calculate
\[ \sup_{\beta \in \left(-\frac{n}{p}, 0\right)} \frac{\beta p + n}{(\alpha + \beta)^p}. \]

Consider the function
\[ -\frac{n}{p} < \beta \mapsto h(\beta) = \frac{\beta p + n}{(\alpha + \beta)^p}. \]

Note that
\[ h'(\beta) = p(\alpha + \beta)^{-p} - p(\beta p + n)(\alpha + \beta)^{-p-1} = -p(\alpha + \beta)^{-p-1}((p - 1) + n - \alpha). \]

and
\[ \begin{cases} h'(\beta) \geq 0 & \text{if } \beta \leq -\frac{n-\alpha}{p-1} \\ h'(\beta) \leq 0 & \text{if } \beta \geq -\frac{n-\alpha}{p-1}. \end{cases} \]

So, this, combined with
\[ \lim_{\beta \to -\frac{n}{p}} h(\beta) = 0, \]

shows that \( h \) attains its sharp value at the point
\[ \beta = -\frac{n}{p} - \frac{\alpha}{p-1}. \]

Consequently,
\[ \sup_{\beta \in \left(-\frac{n}{p}, 0\right)} \frac{\beta p + n}{(\alpha + \beta)^p} = \left( \frac{\alpha n}{p^{p-1}} \right)^{1-p}. \]

This in turn implies
\[ c_{\alpha,p,n} \geq \sup_{x \in B(x_0, r_0)} \left( \frac{|I_{\alpha}f_{\beta}(x)|}{|B(x_0, r_0)|^{\alpha-n} \|f_{\beta}\|_{L^p}} \right) \geq \sup_{\beta \in \left(-\frac{n}{p}, 0\right)} \left( \frac{\omega_{n-1}}{n} \right)^{\frac{\alpha-n}{n-p}} \left( \frac{\omega_{n-1}^{\frac{n}{n-p}}}{n^{\frac{n}{n-p}}} \right)^{\frac{1}{p}} \|f_{\beta}\|_{L^p}. \]

This in turn implies
\[ = c_{\alpha,p,n}. \]
Accordingly, when $\Omega$ is a Euclidean ball of $\mathbb{R}^n$, it holds that
\[
\sup_{f \in L^p_c(\Omega)} \frac{||A_{\mu} f||_{L^\infty}}{||f||_{L^p}} = c_{\alpha p > n}.
\]

2.2. **Optimal domination for** $\nabla_{0 \leq s < 1}^\ast$. Interestingly and naturally, with
\[
\nabla^{m \in \{ \text{even} \}} = (-1)^{s/2} (-\Delta)^{m/2} \quad \text{or} \quad \nabla^{m \in \{ \text{odd} \}} = (-1)^{m-1} \nabla (-\Delta)^{m/2} = (-1)^{m-1} \tilde{R}(-\Delta)^{m/2}
\]
replaced by the fractional version
\[
\nabla_+^s = (-\Delta)^{s/2} \quad \text{or} \quad \nabla_-^s = \nabla(-\Delta)^{s/2} = \tilde{R}(-\Delta)^{s/2},
\]
Theorem 2.1 induces the following new assertion.

**Theorem 2.2.** Let $0 < s < 1 < p < \infty$ and
\[
\mathcal{F}^s_{s,n}(\Omega) = \begin{cases} 
I_s(C_c^\infty(\Omega)) & \text{for } \nabla_+^s \\
(-\Delta)^{s/2}(C_c^\infty(\Omega)) & \text{for } \nabla_-^s.
\end{cases}
\]

Then the following assertions are true.

(i) If $sp < p < n$, then
\[
\sup_{g \in C_c^\infty} \left( \int_{\mathbb{R}^n} \frac{||x||^s |g(x)|^p}{\Omega^s} \, dx \right)^{1/p} = \kappa_{sp < n, \pm} = \begin{cases} 
2^{-s/2} \left( \frac{\Gamma(\frac{sp}{2})}{\Gamma(\frac{n+1}{2})} \right)^{1/p} \Gamma\left(\frac{n+1}{2p} + \frac{1}{2}\right) & \text{for } \nabla_+^s \\
\left( \frac{1}{n-p} \right)^{1/p} \left( \frac{\Gamma\left(\frac{sp}{2}-\frac{1}{2}\right)}{\Gamma\left(\frac{n+1}{2p} + \frac{1}{2}\right)} \right) & \text{for } \nabla_-^s.
\end{cases}
\]

(ii) If $sp = n$ and $\Omega \subseteq \mathbb{R}^n$ is a domain with volume $|\Omega| < \infty$, then exists a positive constant $c_{sp = n, \pm}$ depending only on $s$ and $n$ such that
\[
\sup_{g \in \mathcal{F}_{s,n}(\Omega)} \int_{\Omega} \exp\left( \frac{\kappa |g(x)|}{||\nabla_+^s g||_{L^\infty}} \right)^{n/p} \, dx \leq c_{sp = n, \pm} \quad \forall \ 0 \leq \kappa \leq \kappa_{sp = n, \pm}.
\]

Here
\[
\kappa_{sp = n, \pm} = \begin{cases} 
\left( \frac{\Gamma\left(\frac{sp}{2} + \frac{1}{2}\right)}{\Gamma\left(\frac{n}{2p} + \frac{1}{2}\right)} \right)^{1/p} & \text{for } \nabla_+^s \\
\left( \frac{\Gamma\left(\frac{sp}{2} - \frac{n}{2p} - \frac{1}{2}\right)}{\Gamma\left(\frac{n}{2p} + \frac{1}{2}\right)} \right)^{1/p} & \text{for } \nabla_-^s.
\end{cases}
\]
is sharp in the sense that if $\Omega$ is a Euclidean ball and $\kappa > \kappa_{sp = n, \pm}$ then the last integral inequality cannot hold without forcing $c_{sp = n, \pm}$ to depend only on $s$ and $p$.

(iii) If $sp > n$ and $\Omega \subseteq \mathbb{R}^n$ is a domain with volume $|\Omega| < \infty$ then
\[
\sup_{g \in \mathcal{F}_{s,n}(\Omega)} \frac{|\Omega|^{\frac{n-p}{pm}} ||g||_{L^\infty}}{||\nabla_+^s g||_{L^p}} \leq \kappa_{sp > n, \pm} = \begin{cases} 
c_{sp > n} \left( \frac{\Gamma\left(\frac{sp}{2} - \frac{n}{2p} - \frac{1}{2}\right)}{2^s \pi \Gamma\left(\frac{sp}{2} + \frac{1}{2}\right)} \right)^{1/p} & \text{for } \nabla_+^s \\
c_{sp > n} \left( \frac{\Gamma\left(\frac{sp}{2} - \frac{n}{2p} - \frac{1}{2}\right)}{2^s \pi \Gamma\left(\frac{sp}{2} + \frac{1}{2}\right)} \right)^{1/p} & \text{for } \nabla_-^s.
\end{cases}
\]
Moreover, the constant $\kappa_{sp > n, \pm}$ is sharp in the sense that if $\Omega$ is a Euclidean ball then
\[
\sup_{g \in \mathcal{F}_{s,n}(\Omega)} \frac{||g||_{L^\infty}}{|\Omega|^{\frac{n-p}{pm}} ||\nabla_+^s g||_{L^p}} = \kappa_{sp > n, \pm}.
\]
Proof. The sharp inequalities in (i), (ii) and (iii) are suitably called the sharp Hardy-Rellich, Adams-Moser and Morrey-Sobolev inequalities for the fractional order twin gradients $\nabla_{\pm}^{s}$, respectively. Since (i) follows readily from [6, Corollary 1 & Theorem 4 (16)], the definition of $\nabla_{\pm}^{s}$ and $I_{s} = (-\Delta)^{-\frac{s}{2}}$, it remains to verify (ii)-(iii).

Case - $\nabla_{+}^{s}$. Under this situation we have
\[
g \in I_{s}(C_{c}^{\infty}(\Omega)) \iff \exists u \in C_{c}^{\infty}(\Omega) \text{ such that } g = I_{s}u
\]
and
\[
\nabla_{+}^{s}g = (-\Delta)^{\frac{s}{2}}I_{s}u = u \in C_{c}^{\infty}(\Omega).
\]
This, along with Theorem 2.1(ii)/(iii), directly gives the desired conclusion in (ii)/(iii) for $\nabla_{+}^{s}$ and the corresponding sharp case.

Case - $\nabla_{-}^{s}$. From the hypothesis
\[
g \in (-\Delta)^{\frac{s}{2}}(C_{c}^{\infty})
\]
it follows that
\[
g = (-\Delta)^{\frac{s}{2}}u \quad \text{for some } u \in C_{c}^{\infty},
\]
and hence
\[
\nabla_{-}^{s}g = \nabla_{-}^{\frac{s}{2}}(-\Delta)^{\frac{s}{2}}u = \nabla u.
\]

Also, according to [34, (5.6)&(4.4)] we have
\[
\begin{cases}
-(-\Delta)^{\frac{s}{2}}u = \text{div}^{-s}\nabla u = \kappa_{s} \int_{\mathbb{R}^{n}} \frac{h|\nabla u(x+y)|}{|h|^{s-1}} \, dh & \\
\kappa_{s} = \frac{1}{2^s \pi 1^{\frac{s}{2}}} \end{cases}
\]
thereby finding
\[
|g(x)| = |(-\Delta)^{\frac{s}{2}}u(x)| \leq \kappa_{s} \int_{\mathbb{R}^{n}} |x-y|^{-s} |\nabla u(y)| \, dy = \kappa_{s} I_{s}|\nabla u|(x),
\]
which exists as a fractional variant of (0.2). In light of (2.2) and Theorem 2.1(ii)/(iii), we obtain the desired inequality in Theorem 2.2(ii)/(iii).

To see that $\kappa_{sp\geq n}$ is sharp, we consider two situations below.

- $sp = n$. Without loss of generality we may assume that $\Omega$ is the origin-centered unit ball $\mathbb{B}^{n}$. If for some $\kappa > \kappa_{sp\geq n}$, it holds that
\[
\sup_{u \in C_{c}^{\infty}(\mathbb{B}^{n})} \int_{\mathbb{B}^{n}} \exp \left( \frac{\kappa(-\Delta)^{\frac{s}{2}}u(x)}{\|\nabla u\|_{L^{p}}} \right) \frac{dx}{|\mathbb{B}^{n}|} = \sup_{g \in F_{s,c}(\mathbb{B}^{n})} \int_{\mathbb{B}^{n}} \exp \left( \frac{\kappa|g(x)|}{\|\nabla_{-}^{s}g\|_{L^{p}}} \right) \frac{dx}{|\mathbb{B}^{n}|} \leq c_{sp\geq n},
\]
then we are about to construct suitable functions $u$ to show that (2.3) forces $\kappa \leq \kappa_{sp\geq n}$, thereby revealing that $\kappa_{sp\geq n}$ is the sharp number to guarantee Theorem 2.2(ii).

Being somewhat motivated by [3, pp.391-392] and [12, p.7], for $r \in (0,1)$ we let $\mathbb{B}_{r}^{n}$ be the origin-centered ball with radius $r$ and
\[
u_{r}(x) = \frac{|x|^{1-s}1_{\mathbb{B}_{r}\setminus\mathbb{B}_{r}^{n}}(x)}{(1-s)\omega_{n-1} \log \frac{1}{r}}.
\]
Then
\[
\begin{cases}
\nabla u_{r}(x) = \frac{|x|^{1-s}1_{\mathbb{B}_{r}\setminus\mathbb{B}_{r}^{n}}(x)}{\omega_{n-1} \log \frac{1}{r}} \\
\|\nabla u_{r}\|_{L^{p}} = \left( \omega_{n-1} \log \frac{1}{r} \right)^{\frac{s}{p}} = \left( \omega_{n-1} \log \frac{1}{r} \right)^{\frac{sp}{n}}
\end{cases}
\]
Consequently, we use the first equation in (2.4), (2.1) and the polar-coordinate-system to achieve that if $x \in \mathbb{B}_{r}^{n}$ then
\[ -(-\Delta)^{\frac{\alpha}{2}} u_r(x) = \kappa_{-s} \int_{\mathbb{R}^n} \frac{h \cdot \nabla u_r(x+h)}{|h|^{n+1}} \, dh \]

\[ = \kappa_{-s} \int_{\mathbb{R}^n \setminus B_r} \frac{(z-x) \cdot \nabla u_r(z)}{|z-x|^{n+1-s}} \, dz \]

\[ = \left( \frac{\kappa_{-s}}{\omega_{n-1} \log \frac{1}{r}} \right) \int_{\mathbb{R}^n \setminus B_r} \frac{(z-x) \cdot \nabla u_r(z)}{|z-x|^{n+1-s}} \, dz \]

\[ = \left( \frac{\kappa_{-s}}{\log \frac{1}{r}} \right) \int_{|\xi|}^{|\omega_n|} \left( \int_{S^{n-1}} \frac{(\theta - \frac{\xi}{|\xi|}) \cdot \theta}{|\theta - \frac{\xi}{|\xi|}|^{n+1-s}} d\theta \right) \frac{dt}{t} \]

\[ = \left( \frac{\kappa_{-s}}{\log \frac{1}{r}} \right) \int_{|\xi|}^{|\omega_n|} U(t) \frac{dt}{t}, \]

where

\[ U(t) = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \frac{(\theta - t \frac{\xi}{|\xi|}) \cdot \theta}{|\theta - t \frac{\xi}{|\xi|}|^{n+1-s}} d\theta \]

is independent of the variable \( x \) after a rotation. Since \( U(0) = 1 \), we write

\[ \int_{|\xi|}^{|\omega_n|} U(t) \frac{dt}{t} = \int_{|\xi|}^{|\omega_n|} U(0) \frac{dt}{t} + \int_{|\xi|}^{|\omega_n|} (U(t) - U(0)) \frac{dt}{t} = \log \frac{1}{r} + T(|\xi|, r). \]

For the error term \( T(|\xi|, r) \), observing that

\[ \int_0^1 |U(t) - U(0)| \frac{dt}{t} \leq \int_0^{1/2} t \sup_{t \in (0,1/2)} |\nabla U(t)| \frac{dt}{t} + \int_{1/2}^1 |U(t) - 1| \frac{dt}{t} \leq 1, \]

we therefore derive that, for \( \varepsilon > 0 \) there is a sufficiently small \( r_0 > 0 \) such that

\[ \sup_{x \in B_r} |T(|\xi|, r)(\log \frac{1}{r})^{-1}| \leq \sup_{x \in B_r} \left| \frac{1}{\log \frac{1}{r}} \int_0^1 |U(t) - U(0)| \frac{dt}{t} \right| < \varepsilon \quad \forall \ 0 < r \leq r_0. \]

So, we have

\[ |(-\Delta)^{\frac{\alpha}{2}} u_r(x)| \geq \kappa_{-s}(1 - \varepsilon) \quad \forall \ 0 < r \leq r_0. \]

This, along with (2.3) and the second formula of (2.4), gives

\[ c_{sp,n,-} \geq \int_{\mathbb{R}^n} \exp \left( \frac{k|(-\Delta)^{\frac{\alpha}{2}} u_r(x)|}{\int_{\mathbb{B}^n} |\nabla u_r|_{L^p}}^{1/p} \right) \, dx \geq \rho^n \exp \left( \frac{\kappa_{-s}(1 - \varepsilon)}{(\omega_{n-1} \log \frac{1}{r})^{\frac{n}{n-1}}} \right), \]

which in turns implies that if \( 0 < r \leq r_0 \) then

\[ \kappa_{-s}(1 - \varepsilon) \leq \left( \log \frac{c_{sp,n,-}}{\rho^n} \right)^{\frac{n}{n-1}} \left( \omega_{n-1} \log \frac{1}{r} \right)^{\frac{n}{n-1}} = \left( \log \frac{c_{sp,n,-}}{\rho^n} \right)^{\frac{n}{n-1}} \left( \omega_{n-1} \log \frac{1}{r} \right)^{\frac{n}{n-1}}. \]
Letting $\epsilon \downarrow 0$ and $r \downarrow 0$ yields

$$\kappa \kappa_{-s} \leq \left( \frac{n}{\omega_{n-1}} \right)^{\frac{1}{\alpha}}$$

i.e. $\kappa \leq \kappa_{sp=n,-} = \left( \frac{n}{\omega_{n-1}} \right)^{-1} \left( \frac{n}{\omega_{n-1}} \right)^{\frac{1}{\alpha}}$,

as desired.

- $sp > n$. Let

\[
\begin{aligned}
(x_0, r_0) &\in \mathbb{R}^n \times (0, \infty) \\
\Omega &= B(x_0, r_0) \\
\beta &= -\frac{n-s}{p-1} \\
\nu_{\beta}(x) &= (\beta + 1)^{-1}1_{B(x_0, r_0)}|x - x_0|^{\beta+1} \\
g_{\beta}(x) &= (-\Delta)^{\frac{1}{2}} u_{\beta}(x).
\end{aligned}
\]

Notice that $u_{\beta}$ can be approximated by functions in $C^\infty_c$ and

$$\nabla^s_{-} g_{\beta}(x) = \nabla u_{\beta}(x) = 1_{B(x_0, r_0)}|x - x_0|^{\beta} \frac{x - x_0}{|x - x_0|}.$$ 

So, by (2.1) and the calculations in the proof of Theorem 2.1(iii), we obtain

$$\|\nabla^s_{-} g_{\beta}\|_{L^p} = \left( \int_{B(x_0, r_0)} |x - x_0|^{\beta p} \, dx \right)^{\frac{1}{p}} = \left( \frac{\omega_{n-1}}{n} \right)^{\frac{1}{p}} \left( \frac{n}{\beta p + n} \right)^{\frac{1}{p}} r_0^{\beta + \frac{p}{n}}$$

and

$$|g_{\beta}(x_0)| = \kappa_{-s} \left| \int_{\mathbb{R}^n} h \cdot \nabla u_{\beta}(x_0 + h) \, dh \right| = \kappa_{-s} \int_{|h|<r_0} |h|^{\beta + s - n} \, dh = \kappa_{-s} \left( \frac{\omega_{n-1}}{\beta + s} \right) r_0^{\beta + s}.$$ 

This in turn implies

$$\kappa_{sp=n,-} \geq \sup_{g \in C^\infty_c(B(x_0, r_0))} \frac{\|g\|_{L^p(B(x_0, r_0))}}{|B(x_0, r_0)|^{\frac{1}{\alpha}} \|\nabla^s_{-} g\|_{L^p}} \geq \frac{|g_{\beta}(x_0)|}{|B(x_0, r_0)|^{\frac{1}{\alpha}} \|\nabla^s_{-} g_{\beta}\|_{L^p}} = \kappa_{-s} c_{sp=n} = \kappa_{sp=n,-},$$

and so $\kappa_{sp=n,-}$ is sharp.

\[\square\]

3. Fractional Hardy-Sobolev spaces and their dualities

3.1. Fractional Hardy-Sobolev spaces $H^{s,p}$ and $H^{s,p}_\pm$. Suppose $0 < s < 1 \leq p < \infty$. Since both $\nabla^s_u$ and $\nabla^- u$ are well defined when $u \in S'$, the study for the case $p = 1$ of (1.5) in [21] motivates us to consider the fractional Hardy-Sobolev space

$$H^{s,p} = \left\{ u \in S' : [u]_{H^{s,p}} = \|(-\Delta)^{\frac{s}{2}} u\|_{H^p} < \infty \right\}.$$ 

Note that

$$u_1 - u_2 = \text{constant} \iff [u_1]_{H^{s,p}} = [u_2]_{H^{s,p}}.$$ 

So, $[\cdot]_{H^{s,p}}$ is properly a norm on quotient space of $H^{s,p}$ modulo the space of all real constants, and consequently this quotient space is a Banach space.

Upon introducing

$$H^{s,p}_\pm = \left\{ u \in S' : [u]_{H^{s,p}_\pm} = \|\nabla^s_u u\|_{L^p} < \infty \right\},$$
we find immediately
\[ H^{s,p} = H_{+}^{s,p} \cap H_{-}^{s,p}. \]
Indeed, as shown in the next theorem, when \( s \in (0, 1) \) and \( p \in (1, \infty) \), these three spaces are equal to each other and they all have the Schwartz class \( S \) and
\[ S_{\infty} = \{ \phi \in S : \text{the Fourier transform of } \phi \text{ is 0 near the origin} \} \]
as dense subspaces.

**Theorem 3.1.** Let \( 0 < s < 1 < p < \infty \). Then
\[ S_{\infty} \subseteq S \subseteq H^{s,p} = H_{+}^{s,p} = H_{-}^{s,p}. \]
Moreover, both \( S_{\infty} \) and \( S \) are dense in \( H^{s,p} \) and \( H_{\pm}^{s,p} \).

**Proof.** Notice that any \( u \in S \) satisfies \((-\Delta)^{s/2} u \in S_s\) (cf. [33]). Of course, any function in \( S_s \) belongs to \( L^{1<p<\infty} \). We therefore obtain
\[ S \subseteq H^{s,p}. \]

Given \( p \in (1, \infty) \), upon recalling boundedness of the Riesz transforms \( R_j \) on \( L^p \) (cf. [36]) and the identity
\[ \text{id} = - \sum_{j=1}^{n} R_j^2 \text{ in } L^p, \]
we achieve
\[ \|f\|_{L^p} + \|\vec{R} f\|_{L^p} \approx \|f\|_{L^p} \approx \|\vec{R} f\|_{L^p} \quad \forall \ f \in L^p, \]
thereby reaching
\[ \|(-\Delta)^{s/2} u\|_{L^p} + \|\vec{R}(-\Delta)^{s/2} u\|_{L^p} \approx \|(-\Delta)^{s/2} u\|_{L^p} \approx \|\vec{R}(-\Delta)^{s/2} u\|_{L^p}. \]
This in turn implies
\[ [u]_{H^{s,p}} \approx [u]_{H_{+}^{s,p}} \approx [u]_{H_{-}^{s,p}}. \]
Consequently, we obtain
\[ H^{s,p} = H_{+}^{s,p} = H_{-}^{s,p}. \]

It suffices to show the density of \( S_{\infty} \) in \( H_{+}^{s,p} \). If \( u \in H_{+}^{s,p} \), then
\[ u \in S_s' \text{ & } (-\Delta)^{s/2} u \in L^p. \]
Due to the density of \( S_{\infty} \) in \( L^p \) (cf. [21, Lemma 2.9(iii)]), we can find a sequence \( \{f_j\}_{j \in \mathbb{N}} \) in \( S_{\infty} \) such that
\[ \lim_{j \to \infty} \|f_j - (-\Delta)^{s/2} u\|_{L^p} = 0. \]
For any \( j \in \mathbb{N} \), we write
\[ u_j = I_s f_j \in S_{\infty}. \]
Upon noticing
\[ f_j = (-\Delta)^{s/2} u_j, \]
we obtain
\[ [u_j - u]_{H_{+}^{s,p}} = \|(-\Delta)^{s/2} (u_j - u)\|_{L^p} = \|f_j - (-\Delta)^{s/2} u\|_{L^p} \to 0 \quad \text{as } \ j \to \infty. \]
Thus, any \( u \in H_{+}^{s,p} \) can be approximated by the \( S_{\infty} \)-functions \( \{u_j\}_{j \in \mathbb{N}} \). \( \square \)
3.2. **Dual Hardy-Sobolev spaces** $[H^{s,p}]^*$ and $[H_\pm^{s,p}]^*$. In this subsection, we are about to show that these dual spaces can be characterized by

$$(T_0, T_1, \ldots, T_n) \in (L^{p'})^n+1$$

solving the fractional differential equation

$$[\nabla^s_+]^*T_0 = T \quad \text{or} \quad [\nabla^s_-]^*(T_1, \ldots, T_n) = T.$$

**Theorem 3.2.** Let $0 < s < 1 < p < \infty$ and $p' = \frac{p}{p-1}$. Then for any distribution $T \in S'$ the following three assertions are equivalent:

(i) $T \in [H_+^{s,p}]^* = [H_-^{s,p}]^* = [H_0^{s,p}]^*$;

(ii) $\exists T_0 \in L^{p'}$ such that $T = [\nabla^s_+]^*T_0 \text{ in } S'$;

(iii) $\exists (T_1, \ldots, T_n) \in (L^{p'})^n$ such that $T = [\nabla^s_-]^*(T_1, \ldots, T_n) \text{ in } S'$.

**Proof.** Note that Theorem 3.1 implies

$$[H_+^{s,p}]^* = [H_-^{s,p}]^* = [H_0^{s,p}]^*.$$ 

So, we begin with showing that (ii) implies (i) by considering $H_+^{s,p}$. If (ii) is valid, i.e., if

$$T = [\nabla^s_+]^*T_0 \text{ in } S \text{ for some } T_0 \in L^{p'},$$

then

$$\langle T, \phi \rangle = \langle [\nabla^s_+]^*T_0, \phi \rangle = \langle T_0, \nabla^s_+\phi \rangle = \langle T_0, (-\Delta)^{\frac{s}{2}}\phi \rangle \quad \forall \ \phi \in S,$$

and hence

$$|\langle T, \phi \rangle| \leq ||T_0||_{L^{p'}}||(-\Delta)^{\frac{s}{2}}\phi||_{L^p} \Rightarrow ||T_0||_{L^{p'}}[\phi]_{H_+^{s,p}} \quad \forall \ \phi \in S.$$ 

Accordingly, using the density of $S$ in $H_+^{s,p}$, we see that $T$ induces a bounded linear functional on $H_+^{s,p}$. This proves that

$$T \in [H_+^{s,p}]^*$$

and (i) holds due to (3.1).

Conversely, in order to show that (i) implies (ii), upon assuming

$$T \in [H_+^{s,p}]^*,$$

we are required to find

$$T_0 \in L^{p'} \text{ such that } T = [\nabla^s_+]^*T_0 \text{ in } S'.$$

Inspiring by [7, Proposition 1, pp. 399-400], we consider the operator

$$A_+ : H_+^{s,p} \rightarrow L^p \text{ via } u \mapsto A_+u = (-\Delta)^{\frac{s}{2}}u.$$ 

Note that $A_+$ is bounded and closed. So, if

$$u \in H_+^{s,p} \text{ enjoys } ||(-\Delta)^{\frac{s}{2}}u||_{L^p} = 0,$$

then

$$(-\Delta)^{\frac{s}{2}}u = 0 \text{ almost everywhere on } \mathbb{R}^n,$$

and hence

$$u = I_s(-\Delta)^{\frac{s}{2}}u \equiv 0 \text{ on } \mathbb{R}^n.$$ 

This in turn implies that the operator $A_+$ is injective. Moreover, due to

$$||A_+u||_{L^p} = ||(-\Delta)^{\frac{s}{2}}u||_{L^p} = [u]_{H_+^{s,p}},$$
the operator \( A_+ \) has actually a continuous inverse from \( L^p \) to \( H^{s,p}_+ \). Accordingly, by the closed range theorem (see [41, p. 208, Corollary 1]), we know that the adjoint operator

\[
A_+^* : L^{p'} \to [H^{s,p}_+]^*
\]
defined by \( \langle A_+^* F, u \rangle = \langle F, A_+ u \rangle \quad \forall (F, u) \in L^{p'} \times H^{s,p}_+ \),
is surjective. In particular, if \( T \in [H^{s,p}_+]^* \),
then there exists

\[
T_0 \in L^{p'}
\]
such that \( A_+^* T_0 = T \).

Consequently, for any \( \phi \in S \), we have

\[
\langle A_+^* T_0, \phi \rangle = \langle T_0, A_+ \phi \rangle = \langle T_0, (-\Delta)^{\frac{s}{2}} \phi \rangle = \langle [\nabla^{s}_+]^* T_0, \phi \rangle,
\]
namely,

\[
T = A_+^* T_0 = [\nabla^{s}_+]^* T_0 \text{ in } S'.
\]

This completes the argument for that (i) implies (ii).

Next, we show that (iii) implies (i) by considering \( H^{s,p}_- \). If

\[
T = [\nabla^{s}_-]^* \vec{T} \text{ in } S' \text{ for some } \vec{T} = (T_1, \ldots, T_n) \in (L^p)^n,
\]
then for any \( \phi \in S \) we have

\[
\langle T, \phi \rangle = \langle [\nabla^{s}_-]^* \vec{T}, \phi \rangle
\]

\[
= - \sum_{j=1}^{n} \langle (-\Delta)^{\frac{s}{2}} R_j T_j, \phi \rangle
\]

\[
= - \sum_{j=1}^{n} \langle R_j T_j, (-\Delta)^{\frac{s}{2}} \phi \rangle
\]

\[
= \sum_{j=1}^{n} \langle T_j, R_j (-\Delta)^{\frac{s}{2}} \phi \rangle
\]

\[
= \sum_{j=1}^{n} \langle T_j, \nabla^{s}_j \phi \rangle,
\]

whence

\[
|\langle T, \phi \rangle| \leq \sum_{j=1}^{n} ||T_j||_{L^{p'}} ||\nabla^{s}_j \phi||_{L^p} \quad \forall \ \phi \in S.
\]

Since \( S \) is dense in \( H^{s,p}_- \), it follows that \( T \) induces a bounded linear functional on \( H^{s,p}_- \). This shows (iii)\( \implies \) (i).

Conversely, in order to show (i)\( \implies \) (iii), assuming \( \quad T \in [H^{s,p}_-]^* \),
we are about to verify that

\[
T = [\nabla^{s}_-]^* \vec{T} \text{ in } S' \text{ for some } \vec{T} \in (L^p)^n.
\]

To this end, we consider the operator

\[
A_- : H^{s,p}_- \to (L^p)^n \text{ via } u \mapsto \nabla^{s}_- u.
\]

Now we validate that the just-defined operator \( A_- \) is injective. If \( \quad u \in H^{s,p}_- \) satisfies \( \nabla^{s}_- u = 0 \) in \( (L^p)^n \),
then, for any $\psi \in S_\infty$, we apply the Fourier transform to derive

$$\psi = - \sum_{j=1}^{n} \nabla_j^s I_j R_j \psi \text{ with } I_j R_j \psi \in S_\infty \subseteq L^p,$$

thereby giving

$$|\langle u, \psi \rangle| = \left| \sum_{j=1}^{n} \langle u, \nabla_j^s I_j R_j \psi \rangle \right| = \left| \sum_{j=1}^{n} \langle \nabla_j^s u, I_j R_j \psi \rangle \right| \leq \sum_{j=1}^{n} \| \nabla_j^s u \| L^p \| I_j R_j \psi \| L^p = 0.$$

This, along with the density of $S_\infty$ in $L^p$ (cf. [21, Lemma 2.9(iii)]), further gives

$$u = 0 \text{ in } L^p \implies u = 0 \text{ in } H^s_{-p}.$$

Accordingly, $A_-$ is an injective map from $H^s_{-p}$ onto $A_-(H^s_{-p})$ (the closed range of $A_-$) $\subseteq (L^p)^n$.

This, along with

$$\| A_- u \| L^p = \| \nabla^s u \| L^p = [u]_{H^s_{-p}},$$

ensures that $A_-$ has a continuous inverse from $A_-(H^s_{-p})$ to $H^s_{-p}$. Upon applying the closed range theorem (see [41, p. 208, Corollary 1]) we get that the adjoint operator

$$A^*_- : [A_-(H^s_{-p})]^* \to [H^s_{-p}]^* \text{ via } \langle A^*_- \vec{F}, u \rangle = \langle \vec{F}, A_- u \rangle \quad \forall \ (\vec{F}, u) \in [A_- (H^s_{-p})]^* \times H^s_{-p}$$

is surjective, thereby finding

$$\vec{T}_0 \in [A_- (H^s_{-p})]^* \text{ such that } A^*_- \vec{T}_0 = T.$$

Upon utilizing the Hahn-Banach theorem to extend $\vec{T}_0$ to

$$\vec{T} = (T_1, \ldots, T_n) \in (L^p)^n = [(L^p)^n]^*$$

we have

$$\langle T, \phi \rangle = \langle A^-_\phi \vec{T}_0, \phi \rangle = \langle \vec{T}_0, A_- \phi \rangle = \langle \vec{T}, \nabla^s \phi \rangle = \langle [\nabla^s]^* \vec{T}, \phi \rangle \quad \forall \ \phi \in S,$$

whence

$$T = A^*_- \vec{T} = [\nabla^s]^* \vec{T} \text{ in } S'.$$

This completes the argument for (i)$\implies$(iii). \qed

Let $\text{div}$ be the classical divergence operator whose action on a vector-valued function $\vec{Y}$ is given by

$$\text{div}\vec{Y} = \nabla \cdot \vec{Y}.$$

As a limiting case $s \uparrow 1$ of Theorem 3.2, we have the following conclusion.

**Proposition 3.3.** Let $p \in (1, \infty)$. Then $L^p = \vec{R} \cdot (L^p)^n$ - namely -

$$f \in L^p \iff \exists (f_1, \ldots, f_n) \in (L^p)^n \text{ such that } f = \sum_{j=1}^{n} R_j f_j \text{ in } L^p.$$

Consequently, for any $Y \in L^p$, there exist $(Y_0, Y_1, \ldots, Y_n) \in (L^p)^{1+n}$ such that

$$\text{div}((\Delta)^{-\frac{1}{2}} Y_1, \ldots, (-\Delta)^{-\frac{1}{2}} Y_n) = Y = (-\Delta)^{\frac{1}{2}} Y_0 \text{ in } L^p.$$
Proof. Given $1 < p < \infty$. Thanks to the boundedness of $\vec{R}$ on $L^p$ and the identity
\[ \vec{R} \cdot \vec{R} = -\text{id} \text{ in } L^p, \]
we have that any $f \in L^p$ enjoys the desired property
\[ f_j = -R_j f \in L^p \text{ & } f = \sum_{j=1}^{\infty} R_j f \in L^p. \]

As a consequence, for any $Y \in L^p$ we can find a vector-valued function
\[ \vec{Y} = (Y_1, \ldots, Y_n) \in (L^p)^n \]
such that
\[ Y = \sum_{j=1}^{n} R_j Y_j = \nabla \cdot ((-\Delta)^{\frac{1}{2}} \vec{Y}) = \text{div}((-\Delta)^{\frac{1}{2}} Y_1, \cdots, (-\Delta)^{\frac{1}{2}} Y_n) \text{ in } S'. \]

Also, if
\[ Y_0 = I_1 Y, \]
then
\[ Y = (-\Delta)^{\frac{1}{2}} I_1 Y = (-\Delta)^{\frac{1}{2}} Y_0 \text{ in } S'. \]

Since $S$ is dense in $[L^p]^* = L^{\frac{n}{p-1}}$, we deduce that the last two equalities hold in $L^p$. □

Remark 3.4. Whenever $s = p = 1$ we define
\[ H^{1,1} = \left\{ f \in S' : [f]_{H^{1,1}} = \|(-\Delta)^{\frac{1}{2}} f\|_{H^1} < \infty \right\}. \]

Just like $S_\infty$ is dense in $H^1$, we have also the density of $S_\infty$ in $H^{1,1}$ (cf. [21, Proposition 2.12]). But for functions in $S_\infty$ the Fourier transform easily derives
\[ [f]_{H^{1,1}} = \|(-\Delta)^{\frac{1}{2}} f\|_{L^1} + \|\nabla f\|_{L^1}. \]

Thus, $H^{1,1}$ can be equivalently defined to be the space of all locally integrable functions on $\mathbb{R}^n$ satisfying $[f]_{H^{1,1}} < \infty$. In analogy to Theorem 3.2 and Proposition 3.3, we have:

(i) $H^{1,1} = \vec{R} \cdot (H^{1,1})^n$ - namely -
\[ Z \in H^{1,1} \iff \exists (Z_1, \ldots, Z_n) \in (H^{1,1})^n \text{ such that } Z = \sum_{j=1}^{n} R_j Z_j. \]

This is due to the fact that any $Z \in H^{1,1}$ can be written as
\[ Z = \sum_{j=1}^{n} R_j Z_j \text{ where } Z_j = -R_j Z \in H^{1,1}. \]

(ii) Given a distribution $T \in S'$,
\[ T \in [H^{1,1}]^* \iff \exists (T_0, T_1, \ldots, T_n) \in (L^\infty)^{1+n} \text{ such that } T = (-\Delta)^{\frac{1}{2}} T_0 - \text{div}(T_1, \ldots, T_n) \text{ in } S'. \]

This follows from the endpoint $s = 1$ of [21, Theorem 4.3(i)] (cf. [26, Lemma 4.1] for the dual of the endpoint Sobolev space $W^{1,1}$) and the basic formula
\[ [\nabla_+]^* = (-\Delta)^{\frac{1}{2}} \& \ [\nabla_-]^* = -\text{div}. \]
Let 

\( \tilde{Z} = (Z_1, \ldots, Z_n) \in (H^{1,1})^n \)

satisfies

\[
\sum_{j=1}^{n} R_j Z_j = \nabla \cdot ((-\Delta)^{-\frac{1}{2}} \tilde{Z}) = \text{div}((-\Delta)^{-\frac{1}{2}} Z_1, \ldots, (-\Delta)^{-\frac{1}{2}} Z_n),
\]

we get that

\[ \forall \; Z \in H^{1,1} \; \exists \; (Z_1, \ldots, Z_n) \in (H^{1,1})^n \text{ such that } \text{div}((-\Delta)^{-\frac{1}{2}} Z_1, \ldots, (-\Delta)^{-\frac{1}{2}} Z_n) = Z. \]

4. DISTRIBUTIONAL SOLUTIONS OF DUALITY EQUATIONS

4.1. Distributional solutions to \( [\nabla_x^s]^* u = \mu \). For any \( \alpha \in (0, n) \) and nonnegative Radon measure \( \mu \) on \( \mathbb{R}^n \), define

\[ I_{\alpha} \mu(x) = c_{n,\alpha} \int_{\mathbb{R}^n} |x-y|^{\alpha-n} \, d\mu(y) \quad \forall \; x \in \mathbb{R}^n \]

and

\[ |||\mu|||_{n-\alpha} = \sup_{(x,r) \in \mathbb{R}^n \times (0, \infty)} \frac{\mu(B(x,r))}{r^{n-\alpha}}. \]

Observe that

\[ I_{\alpha} \mu(x) \geq c_{n,\alpha} \int_{\mathbb{R}^n} |x-y|^{\alpha-n} \, d\mu(y) \geq c_{n,\alpha} \mu(B(0,r))(|x|+r)^{\alpha-n} \quad \forall \; (x,r) \in \mathbb{R}^n \times (0, \infty). \]

As a straightforward application of Theorem 3.2, we can characterize distributional solutions to the following fractional duality equations

\[ [\nabla_x^s]^* u_0 = \mu \quad \& \quad [\nabla_x^s]^*(u_1, \ldots, u_n) = \mu. \]

Upon extending [25, Theorems 3.1-3.2-3.3] - if \( \mu \) is a nonnegative Radon measure on \( \mathbb{R}^{n+2} \) then

\[
\begin{cases}
\exists \tilde{F} \in (L^{\frac{p}{n+s}})^n \text{ such that } \text{div}\tilde{F} = \mu \iff I_1 \mu \in L^p \\
\exists \tilde{F} \in (L^{1 \leq p \leq \frac{n}{n+s}})^n \text{ such that } \text{div}\tilde{F} = \mu \iff \mu = 0 \\
\exists \tilde{F} \in (L^{n})^n \text{ such that } \text{div}\tilde{F} = \mu \iff |||\mu|||_{n-1} < \infty,
\end{cases}
\]

we obtain

**Theorem 4.1.** Let \( 0 < s < 1 < p \leq \infty \) and \( \mu \) be a nonnegative Radon measure on \( \mathbb{R}^n \). Then either

\[ \exists \; u_0 \in L^p \text{ such that } [\nabla_x^s]^* u_0 = \mu \text{ in } S' \]

or

\[ \exists \; (u_1, \ldots, u_n) \in (L^p)^n \text{ such that } [\nabla_x^s]^*(u_1, \ldots, u_n) = \mu \text{ in } S' \]

holds if and only if

\[
\begin{cases}
\mu = 0 & \text{if } p \in (1, \frac{n}{n+s}) \\
I_{\alpha} \mu \in L^p & \text{if } p \in (\frac{n}{n+s}, \infty).
\end{cases}
\]

Moreover, under the condition \( p = \infty \) and \( n \geq 2 \), it holds that

\[ (4.3) \iff |||\mu|||_{n-\alpha} < \infty. \]
Proof. Let us start with the case \( p \in (1, \frac{n}{n-1}] \). Clearly, if \( \mu = 0 \), then
\[
 u_0 = u_1 = \cdots = u_n = 0
\]
ensures
\[
 [\nabla_+^s]^* u = 0
\]
and
\[
 [\nabla_-^s]^*(u_1, \ldots, u_n) = 0.
\]
Thus it is enough to show the only-if-part.
Consider first the operator \([\nabla_+^s]^*\) and assume that (4.2) holds. For any \( \phi \in \mathcal{S}_\infty \), we utilize the Fourier transform to derive
\[
 \phi = (-\Delta)^{\frac{s}{2}} I_s \phi
\]
and hence
\[
 \langle u_0, \phi \rangle = \langle u_0, (-\Delta)^{\frac{s}{2}} I_s \phi \rangle = \langle [\nabla_+^s]^* u_0, I_s \phi \rangle = \int_{\mathbb{R}^n} I_s \phi(x) d\mu(x) = \int_{\mathbb{R}^n} (I_s \mu(x)) \phi(x) dx,
\]
which, along with the fact that \( \mathcal{S}_\infty \) is dense in
\[
 [L^p]^* = L^{\frac{p}{n}},
\]
gives
\[
 I_s \mu = u_0 \text{ in } L^p.
\]
From this and the observation (4.1) it follows that
\[
 \int_{\mathbb{R}^n} \left( \mu(B(0, r)) (|x| + r)^{(s-n)} \right)^p dx < \infty \text{ under } (n-s)p \leq n.
\]
However, this is impossible unless \( \mu = 0 \).
Consider next the operator \([\nabla_-^s]^*\). Assume that (4.3) holds, that is,
\[
 \vec{u} = (u_1, \ldots, u_n) \in (L^p)^n
\]
is a distributional solution of
\[
 [\nabla_-^s]^* \vec{u} = \mu.
\]
For any \( \psi \in \mathcal{S}_\infty \), by the fact \( I_s \psi \in \mathcal{S}_\infty \), the definition of
\[
 [\nabla_-^s]^* = -\text{div}^s = -(-\Delta)^{\frac{s}{2}} \vec{R}
\]
and the self-adjointness of \((-\Delta)^{s}\), we obtain
\[
\langle I_s \mu, \psi \rangle = \langle I_s (\nabla_s^+ u), \psi \rangle = \langle \nabla_s^+ u, I_s \psi \rangle = -\int_{\mathbb{R}^n} \text{div}^s u(x) I_s \psi(x) \, dx
\]
\[
= - \sum_{j=1}^n \int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} R_j \mu_j(x) I_s \psi(x) \, dx
\]
\[
= - \sum_{j=1}^n \int_{\mathbb{R}^n} R_j \mu_j(x)(-\Delta)^{\frac{s}{2}} I_s \psi(x) \, dx
\]
\[
= - \sum_{j=1}^n \int_{\mathbb{R}^n} R_j \mu_j(x) \psi(x) \, dx
\]
\[
= - \sum_{j=1}^n \langle R_j \mu_j, \psi \rangle,
\]
which, together with the aforementioned density of \(S_{\infty}\) in
\[
[L^p]^* = L^{p^*} = L^{\frac{p}{p-1}}
\]
and the boundedness of \(R_j\) on \(L^p\), yields
\[
I_s \mu = - \sum_{j=1}^n R_j \mu_j \quad \text{in} \quad L^p.
\]
Similarly to the argument for the operator \([\nabla_s^+]^*\), the fact \(I_s \mu \in L^p\) and (4.1) again derive \(\mu = 0\).

Next, we handle the case \(p \in (\frac{n}{n-s}, \infty)\). Clearly, the only-if-part follows from the same argument as the case \(p \in (1, \frac{n}{n-s}]\). So, it remains to verify the if-part under \(I_s \mu \in L^p\) for \((n-s)p > n\).

According to Theorem 3.2, we only need to validate that such a measure \(\mu\) induces a bounded linear functional on \(H^{s,p'}\), where \(p' = \frac{p}{p-1}\). To this end, for any \(\phi \in S\), by the fact
\[
\phi = I_s(-\Delta)^{\frac{s}{2}} \phi
\]
and the Fubini theorem, we write
\[
\int_{\mathbb{R}^n} \phi \, d\mu = \int_{\mathbb{R}^n} I_s(-\Delta)^{\frac{s}{2}} \phi(x) \, d\mu(x) = \int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} \phi(x) I_s \mu(x) \, dx,
\]
so the Hölder inequality gives
\[
\left| \int_{\mathbb{R}^n} \phi \, d\mu \right| \leq ||I_s \mu||_{L^p} \|(-\Delta)^{\frac{s}{2}} \phi\|_{L^{p'}} = ||I_s \mu||_{L^p} [\phi]_{H^{s,p'}}.
\]
Combining this with the density of \(S\) in \(H^{s,p'}_r\) (cf. Theorem 3.1) leads to that \(\mu\) can be extended to a bounded linear functional on \(H^{s,p'}_r\).

Finally, we deal with the case \(p = \infty\). According to [19, Proposition 3.2], the dual of the space
\[
W^{s,1} = \left\{ f \in L^1_{\text{loc}} : [f]_{W^{s,1}} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|}{|x - y|^{n+s}} \, dx \, dy < \infty \right\}
\]
coincides with the collection of all nonnegative Radon measures $\mu$ on $\mathbb{R}^n$ with

$$||\mu||_{n-s} < \infty.$$  

Upon recalling

$$\hat{H}_s^{1,1} = \text{closure of } S \text{ in } H_s^{1,1} \text{ under } [\cdot]_{H_s^{1,1}}$$

and (cf. [21, Theorem 4.4(i)])

$$[\hat{H}_s^{1,1}]^* = [W_s^{1,1}]^* \text{ under } n \geq 2,$$

we utilize [21, Theorem 4.3(iv)] to derive that condition (4.3) is equivalent to that

$$\mu \in [\hat{H}_s^{1,1}]^* \Rightarrow ||\mu||_{n-s} < \infty.$$  

\[ \square \]

**Remark 4.2.** Regarding the case $p = \infty$ and $n \in \mathbb{N}$ in Theorem 4.1, we know from [21, Theorem 4.3] that either (4.2) or (4.3) implies that $\mu \in [W_s^{1,1}]^*$, which always gives $||\mu||_{n-s} < \infty$. But in general the converse is not known, because we know neither

$$[W_s^{1,1}]^* = [\hat{H}_s^{1,1}]^* \text{ when } n = 1$$

nor

$$[W_s^{1,1}]^* = [\hat{H}_s^{1,1}]^* \text{ for general } n \in \mathbb{N},$$

where $\hat{H}_s^{1,1}$ denotes the closure of $S$ in $H_s^{1,1}$.

### 4.2. Morrey’s regularity for distributional solutions of $[\nabla_s^\alpha]^* u = f$.

In accordance with the basic identity

$$[\nabla_s^\alpha]^* (\nabla_s^\beta u) = -[\nabla_s^{2\alpha}]^* u \quad \forall \quad u \in C_c^{\infty}$$

and [30, Theorem 1.1] if $\Omega$ is an open subset of $\mathbb{R}^n$,

$$(p, s) \in (2 - n^{-1}, \infty) \times (0, 1],$$

and $u \in H^{s,p}$ is a distributional solution to the following fractional $p$-Laplace equation with a natural variation structure

$$\text{div}^s(|\nabla_s^\alpha u|^{p-2} \nabla_s^\alpha u) = 0 \quad \text{in } \Omega,$$

i.e.,

$$\int_{\mathbb{R}^n} |\nabla_s^\alpha u|^{p-2} \nabla_s^\alpha u \cdot \nabla_s^\beta \phi \, dx = 0 \quad \forall \quad \phi \in C_c^{\infty}(\Omega),$$

then $u \in C^{s+\alpha}_{\text{loc}}(\Omega)$ for some positive constant $\alpha$ depending on $p$ only, we are led to settle Morrey’s regularity for the distributional solutions of the fractional duality equations

$$[\nabla_s^\alpha]^* u = f.$$  

For any $(p, \kappa) \in [1, \infty) \times (0, n]$, the Morrey space $L^{p,\kappa}$ was introduced by Morrey [23] and used to study the solution of some quasi-linear elliptic partial differential equations, where $L^{p,\kappa}$ comprises all Lebesgue measurable functions $f$ on $\mathbb{R}^n$ with

$$||f||_{L^{p,\kappa}} = \sup_{(x,r) \in \mathbb{R}^n \times (0,\infty)} \left( \frac{r^\kappa}{r^{n}} \int_{B(x,r)} |f(y)|^p \, dy \right)^{\frac{1}{p}} < \infty.$$  

In particular, when $(p, \kappa) \in [1, \infty) \times \{n\}$, the space $L^{p,n}$ is just the classical Lebesgue space $L^p$.

For $(p, \kappa) \in (1, \infty) \times (0, n)$, let $H^{p,\kappa}$ be the space of all Lebesgue measurable functions $f$ on $\mathbb{R}^n$ such that

$$||f||_{H^{p,\kappa}} = \inf_{\omega} \left( \int_{\mathbb{R}^n} |f(x)|^p (\omega(x))^{1-p} \, dx \right)^{\frac{1}{p}} < \infty,$$
where the infimum is taken over all nonnegative functions \( \omega \) on \( \mathbb{R}^n \) satisfying
\[
||\omega||_{L^1(\mathcal{H}_n^{(\infty)})} = \int_0^{\infty} \mathcal{H}_n^{(\infty)}(\{x \in \mathbb{R}^n : \omega(x) > t\}) \, dt \leq 1.
\]

Here and hereafter, for any given \( \alpha \in (0, n) \), the symbol \( \mathcal{H}_n^{(\infty)}(E) \) denotes the \( \alpha \)-th order Hausdorff capacity of a subset \( E \subseteq \mathbb{R}^n \), given by
\[
\mathcal{H}_n^{(\infty)}(E) = \inf \left\{ \sum_j r_j^\alpha : E \subseteq \bigcup_j B(x_j, r_j) \text{ with } x_j \in \mathbb{R}^n \text{ and } r_j \in (0, \infty) \right\}.
\]

According to [5], we have the duality
\[
[H^{p', \infty}]^* = L^{p, \infty}.
\]

From [27, (5.1)] and [2, Corollary & Proposition 5], we have that if
\[
|||\mu|||_{n-k} < \infty
\]
then
\[
(4.4) \quad \int_{\mathbb{R}^n} |I_ku| \, d\mu \leq |||I_ku|||_{L^1(\mathcal{H}_n^{(\infty)})} \leq ||u||_{H^1} \quad \forall \ u \in H^1.
\]

Consequently, if
\[
d\nu_k(x) = |x|^{-k} \, dx
\]
then
\[
|||\nu_k|||_{n-k} < \infty
\]
and hence (4.4) is used to produce the Stein-Weiss-Hardy inequality at the endpoint \( p = 1 \):
\[
(4.5) \quad \int_{\mathbb{R}^n} |x|^{-k}|u(x)| \, dx \leq ||u||_{H^1} \quad \forall \ u \in H^1.
\]

This, along with (cf. [21, (1.3)-(1.4)])
\[
[u]_{H^{1, 1}} \lesssim [u]_{W^{1, 1}} \quad \forall \ u \in \mathcal{S},
\]

derives
\[
(4.6) \quad \int_{\mathbb{R}^n} |x|^{-j}|u(x)| \, dx \leq [u]_{H^{1, 1}} \leq [u]_{W^{1, 1}} \quad \forall \ u \in \mathcal{S},
\]

which may be viewed as an improvement of the case \( p = 1 \) of [11, Theorem 1.1].

Upon taking a function \( \varphi \) satisfying
\[
\begin{align*}
0 \leq \varphi & \in \mathcal{S} \\
\int_{\mathbb{R}^n} \varphi(x) \, dx & = 1 \\
\varphi(t) & = r^{-n} \varphi(t^{-1}x) \quad \forall \ (t, x) \in (0, \infty) \times \mathbb{R}^n,
\end{align*}
\]

we extend the real Hardy space \( H^p \) from \( p \in [1, \infty) \) to \( p \in (0, \infty) \) via defining (cf. [37])
\[
H^p = \left\{ f \in \mathcal{S}' : ||f||_{H^p} = \left\| \sup_{r \in (0, \infty)} |\varphi_r \ast f| \right\|_{L^p} < \infty \right\} \quad \text{under } 0 < p < \infty.
\]

Then (cf. [10, 37])
\[
[H^p]' = \begin{cases} 
\text{BMO} & \text{as } p = 1 \\
\text{Lip}_p & \text{as } p \in (\frac{\alpha}{n-1}, 1).
\end{cases}
\]
Here and henceforth, $\text{Lip}_{0<\alpha<1}$ is the $(0, 1) \ni \alpha$-Lipschitz space of all functions $f$ on $\mathbb{R}^n$ satisfying
\[
\|f\|_{\text{Lip}_\alpha} = \sup_{x,y \in \mathbb{R}^n, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty.
\]

**Theorem 4.3.** Let
\[
\begin{cases}
0 < s < 1 < n \\
0 < \kappa \leq n \\
1 \leq p < \frac{n}{s} \\
1 < q < \frac{n}{p-s} \\
1 < \left\{ \begin{array}{ll}
q \leq \frac{n}{p-s} & \text{as } 1 < p < \frac{n}{s} \\
q < \frac{n}{n-s} & \text{as } 1 = p < \frac{n}{s}.
\end{array} \right.
\end{cases}
\]
If $f \in L^{p,\kappa}$, then
\[
\exists (F_0, F_1, \ldots, F_n) \in \begin{cases}
(L^{1+n}) \quad \text{as } f \in L^{p>\frac{n}{s},\kappa} \\
\text{BMO} \times (L^\infty)^n \quad \text{as } f \in L^{p=\frac{n}{s},\kappa} \\
(L^{q,\kappa} (\frac{n}{s-s})^{1+n}) \quad \text{as } f \in L^{p<\frac{n}{s},\kappa}
\end{cases}
\]
such that
\[
[\nabla^s_+] F_0 = f = [\nabla^s_] (F_1, ..., F_n)
\]
holds in the sense of
\[
\int_{\mathbb{R}^n} ( [\nabla^s_+] F_0 - f)(x) \phi(x) \, dx = 0 = \int_{\mathbb{R}^n} ( [\nabla^s_+] (F_1, ..., F_n) - f)(x) \phi(x) \, dx \quad \forall \phi \in \mathcal{S}.
\]

**Proof.** Suppose $f \in L^{p,\kappa}$. Note that the desired regularity for
\[
[\nabla^s_+] F_0 = (-\Delta)^s F_0 = f \quad \text{in } \mathcal{S}'
\]
follows from [18, Theorem 1.2] with $F_0 = I_s f$. So, it remains to check the desired regularity for
\[
[\nabla^s_+] (F_1, ..., F_n) = f \quad \text{in } \mathcal{S}'.
\]

To this end, we define the measure $\mu_f$ by
\[
d\mu_f(x) = |f(x)| \, dx.
\]
Then, for any $(x, r) \in \mathbb{R}^n \times (0, \infty)$, we utilize the Hölder inequality to derive
\[
\mu_f(B(x,r)) = \int_{B(x,r)} |f(y)| \, dy \leq \left( \int_{B(x,r)} |f(y)|^p \, dy \right)^{\frac{1}{p}} |B(x,r)|^{\frac{p-1}{p}} \leq \|f\|_{L^{p,r}} |B(x,r)|^{\frac{p-1}{p}},
\]
thereby achieving
\[
\|\|\mu_f\|_{L^{n-s}} \leq \|f\|_{L^{p,\kappa}} < \infty.
\]

The forthcoming demonstration consists of essentially three components.

**Part 1 - the case $sp = \kappa$.**

Just like proving Theorem 4.1, we apply [19, Proposition 3.2] to deduce
\[
\left| \int_{\mathbb{R}^n} f(x) \phi(x) \, dx \right| \leq \int_{\mathbb{R}^n} |\phi(x)| \, d\mu_f(x) \leq \|\mu_f\|_{L^{n-s}} \|\phi\|_{W^{s,1}} \quad \forall \phi \in W^{s,1},
\]
whence
\[
f \in [W^{s,1}]'.
\]
This, along with [21, Theorem 4.4(i)] and [21, Theorem 4.3(iv)], yields that
\[
[\nabla^s_+] \hat{F} = f
\]
has a vector-valued distributional solution

\[ \tilde{F} = (F_1, \ldots, F_n) \in (L^\infty)^n \] under \( n \geq 2 \).

Part 2 - the case \( sp > \kappa \).

Under this condition we have

\[ \left[ \frac{H_{sp}}{(s + p)^n} \right] = \text{Lip}_{s-p}. \]

We are inspired by the proof of [7, Proposition 1, pp. 399-400] (cf. [25, Theorem 3.2]) to set

\[ Y = \left( \frac{H_{sp}}{s + p} \right)^n = \frac{H_{sp}}{s + p} \times \cdots \times \frac{H_{sp}}{s + p} \]

and

\[ X = \left\{ u \in S'_s : \nabla^j_u \in \frac{H_{sp}}{s + p} \text{ for } j = 1, 2, \ldots, n \right\}, \]

developed with the norm

\[ \|u\|_X = \sum_{j=1}^n \|\nabla^j_u\|_{\frac{H_{sp}}{s + p}}. \]

Note that \( \|u\|_X = 0 \) if and only if \( u \) is a constant function on \( \mathbb{R}^n \). So, \( X \) is treated as a quotient space modulo the space of constant functions.

Consider the operator

\[ A : X \to Y \text{ via } u \mapsto A(u) = \nabla^s_u. \]

This operator is well defined in that the action of the operator \( \nabla^s \) can be defined on the distribution space \( S'_s \). Moreover, it is easy to see that \( A \) is a bounded linear operator.

We can also show that the operator \( A \) is injective. To this end, assuming that \( u \in X \) satisfies

\[ \nabla^s_u = 0 \text{ in } \left( \frac{H_{sp}}{s + p} \right)^n, \]

we are required to show

\[ u = \text{constant} \implies u = 0 \text{ in } X. \]

Note that

\[ u \in X \implies u \in S'_s \quad \& \quad \nabla^j_u \in \frac{H_{sp}}{s + p}. \]

Thus, for any \( \psi \in S_{\infty} \), we use the Fourier transform to derive

\[ \psi = -\sum_{j=1}^n \nabla^j_i I_s R_j \psi \text{ with } I_s R_j \psi \in S_{\infty} \subseteq \text{Lip}_{s-p}, \]

thereby finding

\[ \langle u, \psi \rangle = \left| \sum_{j=1}^n \langle u, \nabla^j_i R_j \psi \rangle \right| = \sum_{j=1}^n \|\nabla^j_i u, I_s R_j \psi\|_{H_{sp}} \lesssim \sum_{j=1}^n \|\nabla^j_i u\|_{H_{sp}} \|I_s R_j \psi\|_{\text{Lip}_{s-p}} = 0. \]

This shows

\[ u = 0 \text{ in } S'/P. \]

In other words, \( u \) is a polynomial on \( \mathbb{R}^n \). However, if a polynomial \( u \) is a bounded linear functional on \( S_s \), then \( u \) must be a constant function, as desired.

The above analysis shows that the operator \( A \) is injective and has a continuous inverse from \( A(X) \subseteq Y \to X \). Upon applying the closed range theorem (see [41, p. 208, Corollary 1]), we deduce that the adjoint operator

\[ A^* : [A(X)]^* \to X^* \text{ via } \langle A^* \tilde{F}, u \rangle = \langle \tilde{F}, A u \rangle \forall (\tilde{F}, u) \in [A(X)]^* \times X. \]
is surjective.

Next, we validate that any \( f \in L^{p,\kappa} \) belongs to \( X^* \). Indeed, for any \( \phi \in S \cap X \), we apply [31, Theorem 1.12] to write

\[
(4.7) \quad \phi = I_s \left( \sum_{j=1}^{n} R_j \nabla_j^s \phi \right).
\]

Also, using \( \phi \in S \), we derive from [21, Lemma 2.6] that \( \nabla_j^s \in S_s \), which easily implies that \( R_j \nabla_j^s \phi \) is continuous on \( \mathbb{R}^n \). From the fact

\[
\frac{k}{p} < s < 1 \leq n - 1
\]

it follows that

\[
(4.8) \quad \frac{n}{n + s - \frac{k}{p}} < \frac{n}{s} \quad \text{and} \quad \frac{n - \frac{sn}{n + s - \frac{k}{p}}}{n - \frac{k}{p}} < 1
\]

while the second inequality of (4.8) holds because after a change of variable

\[
0 < t = \frac{k}{p} < s
\]

the function

\[
\psi(t) = t(n + s - t) - sn
\]

is strictly increasing on the interval \( (0,s) \) and \( \psi(s) = 0 \). By (4.8), [17, Theorem 1.1] and its remark, we can derive the continuity of the mapping

\[
I_s : H^{n+s-p} \cap \{ \text{all continuous functions} \} \to L^1_{\mu_f},
\]

with operator norm at most a constant multiple of \( \|\mu_f\|_{n-p} \). Combining these and boundedness of \( R_j \) on \( H^{n+s-p} \), yields

\[
\left| \int_{\mathbb{R}^n} \phi(x) f(x) \, dx \right| \leq \sum_{j=1}^{n} \int_{\mathbb{R}^n} |I_s (R_j \nabla_j^s \phi)(x)| f(x) \, dx
\]

\[
\leq \sum_{j=1}^{n} \int_{\mathbb{R}^n} |I_s (R_j \nabla_j^s \phi)(x)| \, d\mu_f(x)
\]

\[
\leq \sum_{j=1}^{n} \|\mu_f\|_{n-p} \|R_j \nabla_j^s \phi\|_{H^{n+s-p}}
\]

\[
\approx \|\phi\|_X \|f\|_{L^{p,\kappa}}.
\]

Due to the density of \( S \cap X \) in \( X \), we arrive at the conclusion that \( f \) induces a bounded linear functional on \( X \).

To continue, like proving Theorem 3.2(iii) we use the surjective property of \( A^* \) and the Hahn-Banach extension theorem to obtain

\[
\tilde{F} = (F_1, \ldots, F_n) \in Y^* = (\text{Lip}_{s-\frac{k}{p}})^n
\]

such that

\[
\langle f, \phi \rangle = \langle A^* \tilde{F}, \phi \rangle = \langle \tilde{F}, A\phi \rangle = \langle \tilde{F}, \nabla_j^s \phi \rangle = \langle [\nabla_j^s]^* \tilde{F}, \phi \rangle \quad \forall \phi \in S,
\]
whence

\[ \left[ \nabla_\perp \right]^* \vec{F} = A^* \vec{F} = f \quad \text{in} \quad \mathcal{S}'. \]

**Part 3 - the case** \( sp < \kappa \).

This part is similar to the case \( sp > \kappa \). To be precise, we take

\[ Y = \left( H^{q'.q(\frac{\kappa}{p} - s)} \right)^n. \]

Define

\[ X = \left\{ u \in \mathcal{S}' : \nabla_j^u \in H^{q'.q(\frac{\kappa}{p} - s)} \text{ for } j = 1, \ldots, n \right\} \]

endowed with the norm

\[ ||u||_X = \sum_{j=1}^n ||\nabla_j^u||_{H^{q'.q(\frac{\kappa}{p} - s)}}. \]

Again, observing that \( ||u||_X = 0 \) if and only if \( u \) is a constant, we also understood this \( X \) as a quotient space. Though we do not know if \( \mathcal{S} \cap X \) is dense in \( X \), we use the space \( \check{X} \) which is the closure of \( \mathcal{S} \cap X \) in \( X \).

Still we consider the operator

\[ A : \check{X} \to Y \text{ via } u \mapsto A(u) = \nabla_\perp^u, \]

and can show that \( A \) is injective and has a continuous inverse from \( A(\check{X}) \) (the range of \( A \)) to \( \check{X} \). Consequently, the closed range theorem (cf. [41, p. 208, Corollary 1]) can be applied to derive that the adjoint operator

\[ A^* : [A(\check{X})]^* \to (\check{X})^* \text{ via } \langle A^* \vec{F}, u \rangle = \langle \vec{F}, Au \rangle \quad \forall \ (\vec{F}, u) \in [A(\check{X})]^* \times \check{X} \]

is surjective.

Next, we validate that any \( f \in L^{p,\kappa} \) belongs to \( (\check{X})^* \). Applying [20, Proposition 5.1] gives the continuity of the mapping

\[ I_\phi : L^{p,\kappa} \to L^{q'.q(\frac{\kappa}{p} - s)}. \]

Note that the boundedness of \( R_j \) on \( H^{q'.q(\frac{\kappa}{p} - s)} \) was given in [4, Chapter 8]. By these, (4.7) and the Fubini theorem, we derive that any \( \phi \in \mathcal{S} \cap X \) satisfies

\[
\left| \int_{\mathbb{R}^n} \phi(x) f(x) \, dx \right| = \left| \sum_{j=1}^n \int_{\mathbb{R}^n} I_\phi \left( R_j \nabla_j^\phi \right) f(x) \, dx \right|
= \left| \sum_{j=1}^n \int_{\mathbb{R}^n} R_j \nabla_j^\phi f(x) \, dx \right|
\leq \sum_{j=1}^n ||R_j \nabla_j^\phi||_{H^{q'.q(\frac{\kappa}{p} - s)}} ||I_\phi f||_{L^{q'.q(\frac{\kappa}{p} - s)}}
\leq \sum_{j=1}^n ||\nabla_j^\phi||_{H^{q'.q(\frac{\kappa}{p} - s)}} ||f||_{L^{p,\kappa}}
\approx ||\phi||_X ||f||_{L^{p,\kappa}}.
\]

This implies that \( f \) can be extended to a bounded linear functional on \( \check{X} \), that is, \( f \in (\check{X})^* \).

Because of \( f \in (\check{X})^* \) and the surjective property of \( A^* \), we can borrow the idea of verifying Theorem 3.2(iii) and use the Hahn-Banach extension theorem to find a vector-valued function

\[ \vec{F} = (F_1, \ldots, F_n) \in Y^* = \left( L^{q'.q(\frac{\kappa}{p} - s)} \right)^n. \]
such that
\[ \langle A^* \vec{F}, \phi \rangle = \langle \vec{F}, A\phi \rangle = \langle \nabla_s^\perp \vec{F}, \phi \rangle \forall \phi \in \mathcal{S}, \]
thereby reaching
\[ [\nabla_s^\perp]^* \vec{F} = A^* \vec{F} = f \text{ in } \mathcal{S}'. \]
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