On the stress state of thin-walled isotropic building constructions of the shell type

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Abstract. The stress-strain state of elastic inhomogeneous isotropic shallow thin-walled shell constructions is studied in the framework of S.P. Timoshenko shear model. The stress-strain state of shell constructions is described by a system of the five equilibrium equations and by the five static boundary conditions with respect to generalized displacements. Equilibrium equations are second-order partial differential equations that are linear with respect to tangential displacements, rotation angles, and non-linear with respect to normal displacement (deflection). The aim of the work is to find generalized displacements from a system of equilibrium equations that satisfy given static boundary conditions. The research is based on integral representations for generalized displacements containing arbitrary holomorphic functions. Holomorphic functions are found so that the generalized displacements should satisfy five static boundary conditions. The integral representations constructed in this way allow to obtain a nonlinear operator equation with respect to the deflection. The solvability of the nonlinear equation with respect to deflection is established with the use of contraction mappings principle.

Keywords: building constructions of the shell type, stress-strain state, equilibrium equations, static boundary conditions, generalized displacements, integral representations.

1 Introduction

Currently, there is a large number of works devoted to the calculation of thin-walled building constructions of the shell type taking into account geometrical and (or) physical nonlinearity (see, for example, [1]-[9]). Nonlinear problems in very rare cases are solved in a closed form. For this reason, such problems are solved by a wide range of approximate methods using computers. This circumstance makes it particularly relevant to provide a strict qualitative study of the stress-strain state of thin-walled shell constructions. At present, this problem has been sufficiently fully studied in the framework of the Kirchhoff-Love model (see [10]-[17] and the literature quoted in them). Questions related to a qualitative study of the stress-strain state in the framework of more general models of the theory of thin-walled constructions that are not based on the Kirchhoff-Love hypotheses were included into the well-known list of unresolved problems of the mathematical theory of shells [10]. Today there is a number of works [18]-[23] in which the stress-strain state is studied in the framework of the shear model of S.P. Timoshenko. The studies in [18]-[23] are based on integral representations for the generalized displacements. The integral representations contain the arbitrary holomorphic functions. The holomorphic functions are defined so that the generalized displacements should satisfy the given boundary conditions. For their construction two approaches are used. The first approach is based on application of explicit representations of solutions of a problem of Riemann-Hilbert for the holomorphic functions in a unit disk. On this path, the strict study of the stress-strain state was carried out for shallow segments of spherical constructions [18] - [20], [22]. The second approach used for determination of the holomorphic functions uses the theory of integrals of Cauchy type with the real densities. The densities are defined as the solutions of system of one-dimensional singular integral equations. In this direction, the stress-strain state was investigated in the case of arbitrary isotropic shallow thin-walled constructions of the shell type. In the present work, the second approach is used.
The work is a direct development of the works [21], [23] for the case of isotropic shell constructions with variable principal curvatures.

2 Materials and methods

The following model of the theory of thin-walled constructions is considered:

1) strain-displacement relations [24, pp. 168-170, 269]:

\[ y_i^0 = w_{jai} - k_j w_j^i + w_{ji}^2 / 2 \quad (j = 1, 2), \]
\[ y_i^1 = \psi_{jai} \quad (j = 1, 2), \]
\[ y_i^2 = \psi_{ji} + \psi_{2ai}, \]
\[ y_{ji} = w_{3ai} + \psi_{ji} \quad (j = 1, 2), \]
\[ y_{ji}^0 = y_{ji}^1 = 0, \quad k = 1, 3, \]

(1)

where \( y_i^0 \) and \( y_i^j \) \((i, j = 1, 2)\) are the components of the tangential and bending strains of middle surface \( S_0 \) of the shell; \( y_{ji}^0 \) \((j = 1, 2)\) are the components strains of transverse shears; \( y_{ji}^0 \) is the strain of transverse compression; \( w_j \) \((j = 1, 2)\) are the tangential displacements of points \( S_0 \); \( \psi_{ji} \) \((j = 1, 2)\) are the rotation angles of normal cross-sections of middle surface \( S_0 \); \( k_j = k_j(\alpha') \) \((j = 1, 2)\) are the principal curvatures; \( \alpha' \) and \( \alpha'' \) are Cartesian coordinates of points of the flat domain \( \Omega \) with the boundary \( \Gamma \), where \( \Omega \) is homeomorphic to \( S_0 \); subscript \( \alpha' \) in relations (1) and further means differentiation with respect to variable \( \alpha' \); 2) defining relations:

\[ \sigma_{ij} = B^{00}_{ij} \gamma_{in} \quad (i \leq j, k \leq n, i, j, k, n = 1, 3), \]
\[ \gamma_{in} = y_{in}^0 + \alpha' \gamma_{in}^1, \]

(2)

where

\[ B^{111} = B^{222}_1 = E \frac{1}{1-v^2}, \quad B^{122}_1 = E \frac{vE}{1-v^2}, \quad B^{212}_1 = E \frac{E}{2(1+v)}, \quad B^{333} = B^{323} = E k^2 \frac{1}{2(1+v)}, \]

the remaining \( B^{00_1} \) are zero; \( v \) is the Poisson ratio; \( E \) is the Young modulus; \( k^2 \) is the shear coefficient;

3) mass forces \( \mathcal{F}(\alpha', \alpha'', \alpha^3) \) and surface forces \( \mathcal{F}(\alpha, \alpha', \alpha^3) \) act on the shell; forces \( \mathcal{F}(s, \alpha^3) \) are applied at the boundary of the shell \((s \) is the length of the arc of the curve \( \Gamma \)).

In the framework of this model, we obtain the equilibrium equations

\[ T_i'^{ij} + R_i' = 0, \quad i = 1, 2, \]
\[ T_i^{ij} + k_j T_i^{jk} + (T_i^{ik} w_{ji})_{\alpha'} + R_i^3 = 0, \]
\[ M_{ii'} - T_i^{i+3} + E_i' = 0, \quad i = 1, 2, \]

(3)

and static boundary conditions

\[ T_i^{ij} \frac{d \alpha^2}{ds} - T_i^{ij} \frac{d \alpha^3}{ds} = P_i^j(s), \quad j = 1, 2, \]
\[ T_i^{ij} \frac{d \alpha^2}{ds} + T_i^{ij} w_{ji} \frac{d \alpha^2}{ds} + T_i^{ij} w_{3ai} \frac{d \alpha^3}{ds} = P_i^j(s), \]
\[ M_i^{ij} \frac{d \alpha^2}{ds} - M_i^{ij} \frac{d \alpha^3}{ds} = N_i^j(s), \quad j = 1, 2, \]

(4)

where \( T_i^{ij} \) are the stresses, \( M_i^{ij} \) are the moments:

\[ T_i^{ij} = T_i^{ij} (a) = D_i^{0n} \gamma_{in}, \]
\[ M_i^{ij} = M_i^{ij} (a) = D_i^{0n} \gamma_{in}, \]
\[ D_i^{0n} = \int_{-S_0}^{S_0} \mathcal{B}(\alpha', \alpha'', \alpha^3) \frac{d \alpha^2}{ds}, \quad i, j, k, n = 1, 3, \]
\[ R_i^{ij} = F_i^{ij} (\alpha', \alpha'', \alpha^3) + F_i^{ij} (\alpha', \alpha'', \alpha^3) + \int_{-S_0}^{S_0} \mathcal{F}(\alpha', \alpha'', \alpha^3) d \alpha^3, \]

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\( L'(\alpha', \alpha^2) = \left[ F^{(1)}(\alpha', \alpha^2) - F^{(2)}(\alpha', \alpha^2) \right] h_b/2 + \int_{-h_b/2}^{h_b/2} F^{(3)}(\alpha', \alpha^2, \alpha^3) d\alpha^3, \)

\( P'(s) = \int_{-h_b/2}^{h_b/2} F^{(4)}(s, \alpha') d\alpha^3, N'(s) = \int_{-h_b/2}^{h_b/2} F^{(5)}(s, \alpha') d\alpha^3, \quad j = 1, 3, \quad k = 1, 2; \)

\( a = (w_i, w_z, w_{\psi i}, w_{\psi z}) \) is the generalized displacement vector, \( h_b = \text{const} \) is the shell thickness.

In (2)-(5) and further, in case of repeated Latin indices summation is carried out from 1 to 3, in case of repeated Greek indices summation is carried out from 1 to 2.

**Problem (3), (4).** Find a solution of system (3) satisfying the boundary conditions (4).

We study the boundary value Problem (3), (4) in the generalized setting. Let the following conditions be satisfied:

(a) the elastic characteristics \( B^{(i)}(\alpha', \alpha^2, \alpha^3) \) are the even functions of a variable \( \alpha^3 \in [-h_b/2, h_b/2] \) and \( B^{(i)} \in W^{(1)}_p(\Omega) \times L^1_{[-h_b/2, h_b/2]} \);

(b) \( k_j \in W^{(3)}_p(\Omega), \quad j = 1, 2; \)

(c) \( \overline{F} = (F^1, F^2, F^3) \in L^p_{w}(\Omega) \times L^1_{[-h_b/2, h_b/2]}, \quad \overline{F}^2 = (F^{11}, F^{12}) \in L^p_{w}(\Omega), \)

(d) \( \Omega \) is an arbitrary simply connected domain with boundary \( \Gamma \in C^1 \);

(e) the external load is self-balanced.

Here and further: \( 2 < p < 4/(2-\beta), \quad 0 < \beta < 1. \)

**Definition.** A vector \( a = (w_i, w_z, w_{\psi i}, w_{\psi z}) \) is called a generalized solution of Problem (3), (4) if \( a \in W^{(2)}_{w}(\Omega) \), satisfies the system (3) almost everywhere and satisfies the boundary conditions (4) pointwise.

Here \( W^{(i)}_p(\Omega) (i = 1, 2) \) are the Sobolev spaces. By the embedding theorems for the Sobolev spaces \( W^{(2)}_p(\Omega) \) with \( p > 2 \), a generalized solution is \( a \in C^1_\text{loc}(\Omega) \). Here and further \( \alpha = (p-2)/p \). Let us notice what if the condition \( 2 < p < 4/(2-\beta) \) is satisfied, the inequality \( \alpha < \beta/2 \) holds true.

Let us introduce the two complex-valued functions \( \omega_j = \omega_j(z) = D_i^{(1)}(f_{jau} + f_{jau}) + i f_i^{(2)}(f_{jau} - f_{jau}) \quad (j = 1, 2), \quad z = \alpha^1 + i\alpha^2, \quad f_{jau} = w_j, \quad f_{jau} = \psi_j, \quad j = 1, 2. \)

In relation to functions \( \omega_j(z) (j = 1, 2) \) and normal displacement \( w_i(z) \) we consider the equations

\[ \omega_{j} = \rho^{j}, \quad j = 1, 2, \quad D_i^{(3)}w_{3au} + D_0^{(3)}w_{3au} = \rho^{j}, \quad \omega_{j} = (\omega_{j} + i\omega_{j})/2. \]

The first two equations in (6) are inhomogeneous Cauchy-Riemann equations. Therefore, their general solutions are given by formulas \([25, p. 29]\)

\[ \omega_j(z) = \Phi_j(z) + T\rho_j(z), \quad T\rho_j = \frac{1}{\pi} \int_{C_\alpha}(\zeta - z) d\zeta \]

where \( \Phi_j(z) \) are arbitrary holomorphic functions belonging to the space \( C_\alpha(\Omega) \).

It is known \([25, p. 29]\) that \( T \) is a completely continuous operator in spaces \( L^p_{w}(\Omega) \) and \( C^1_\alpha(\Omega) \), and it carries out mapping these spaces in \( C^1_\alpha(\Omega) \) and \( C^{1+1}_\alpha(\Omega) \), respectively. In addition, there exist generalized derivatives

\[ \frac{\partial T \phi}{\partial \xi} = f, \quad \frac{\partial T \phi}{\partial \eta} = Sf = \frac{1}{\pi} \int_{C_\alpha}(\zeta - z)^2 d\zeta \]

where \( S \) is a linear bounded operator in \( L^p_{w}(\Omega), \quad p > 1 \) and in \( C^1_\alpha(\Omega) \).
So, using the functions $\omega_0^j = w_j + iw_j^*$, $\omega_0^k = \psi_j + i\psi_j^*$, the relations (7) can be written in the form of the inhomogeneous Cauchy-Riemann equations

$$\omega_0^j = i(d_j^j \omega_j + d_j^j \bar{\omega}_j) = i d_j^j [\omega_j], \quad d_j^j = \frac{1}{4} \left( \frac{1}{D_{2(j-1)}^{(111)}} + \frac{(-1)^j}{D_{2(j-1)}^{(112)}} \right), \quad j, k = 1, 2. \tag{9}$$

General solutions of equations (9) have the form

$$\omega_0^j(z) = \Psi_j(z) + iTd_j^j [\omega_j](z), \quad j = 1, 2, \tag{10}$$

where $\Psi_j(z)$ are arbitrary holomorphic functions of the space $C_{\mu}^2(\Omega)$.

The general solution of the third equation in (6), taking into account $D_0^{(111)} = D_0^{(112)}$, is obtained in the form

$$w_j(z) = \text{Re} \Phi_j(z) - \tilde{T} \tilde{\rho}_j, \quad \tilde{T} \tilde{\rho}_j = -\frac{1}{\pi} \iint \tilde{\rho}_j(\zeta) \ln \frac{1}{|z - \zeta|} d\zeta d\eta, \quad \tilde{\rho}_j = \frac{\rho_j}{2D_0^{(111)}}, \tag{11}$$

where $\Phi_j(z) \in C_{\mu}^2(\Omega)$ is an arbitrary holomorphic function.

Relations (10) and (11) are the desired integral representations for generalized displacements.

Integral representations (10) and (11) for generalized displacements $a = (w_j, w_j^*, \psi_j, \psi_j^*)$ contain arbitrary holomorphic functions $\Phi_j(z) (j = 1, \overline{3})$, $\Psi_j(z) (k = 1, 2)$ and arbitrary functions $\rho_j(z) (j = 1, \overline{3})$.

We find them so that the generalized displacements should satisfy the system (3) of equilibrium equations and the boundary conditions (4). To this end, first we replace the stresses $T^{\alpha}$ and moments $M^{\alpha}$ in the system (3) by their expressions from (5) and then, instead of generalized displacements, we substitute (10) and (11). As a result, the system of Eqs. (3) takes the form

$$\rho_j(z) + h_j \rho_j(z) + h_{jz}(\Phi_j(z)) = f_j^z(z), \quad j = 1, \overline{3}, \tag{12}$$

where

$$h_j = \frac{D_0^{(111)} \omega_j^{(11)} - D_0^{(112)} \omega_j^{(12)}}{2} + \frac{i \left( D_0^{(111)} \omega_j^{(12)} - D_0^{(112)} \omega_j^{(11)} \right) \omega_j}{2} - \frac{1}{2} \left( D_0^{(111)} \omega_j^{(12)} \omega_j^{(12)} + D_0^{(112)} \omega_j^{(11)} \omega_j^{(11)} \right) / 2,$$

$$h_{jz} = \frac{D_0^{(111)} \omega_j^{(12)} - D_0^{(112)} \omega_j^{(11)}}{2} + \frac{i \left( D_0^{(111)} \omega_j^{(11)} - D_0^{(112)} \omega_j^{(12)} \right) \omega_j}{2} - \frac{1}{2} \left( D_0^{(111)} \omega_j^{(11)} \omega_j^{(11)} + D_0^{(112)} \omega_j^{(12)} \omega_j^{(12)} \right) / 2,$$

$$f_j^z = \left( f_j + if_j^* \right) / 2, \quad f_j^* = \left( f_j - if_j^* \right) / 2, \quad f^2 = f_j, \quad f_j^* = f_j^* \quad (j = 1, \overline{3});$$

$$\hat{f}_j = \hat{f}_j(w_j) = -\left( D_0^{(12)} \hat{X}_0^0 \right), \quad \hat{f}_j = -\left( D_0^{(12)} \hat{X}_0^0 \right) / \delta, \quad j = 1, 2, \quad \hat{f}_3 = \hat{f}_3(a) = -k J^1 - (J^{12} w_j^{(12)} \omega_j^{(12)}),$$

$$\hat{f}_j = \hat{f}_j(\omega_j) = J^1 \omega_j^{(12)} - \hat{f}_j(\omega_j), \quad \hat{f}_j = \hat{f}_j(\omega_j), \quad J^1 = T^{\alpha} T^{\alpha}, \quad T^{\alpha} = T^{\alpha} + T^{\alpha};$$

$\epsilon_0^0$ and $\chi_0^0$ denote linear and nonlinear summands of the strains components $\gamma_0^0, \gamma_0^0 \in \epsilon_0^0 + \chi_0^0$, $\delta, \beta = 1, 2$; $w_j = w_j^1 + w_j^2$, $j = 1, 3, \quad \psi_j = \psi_j^1 + \psi_j^2$, $k = 1, 2$; here the summands with a superscript “1” contain only the functions $\rho = (\rho^1, \rho^2, \rho^3)$, and the summands with a superscript “2” contain only holomorphic functions $\Phi = (\Phi_1, \Phi_2, \Phi_3); \Psi_j(z)$.

The boundary conditions (4) are transformed to the form

$$\text{Re} \{i \left[ T + (I - 1)\hat{a}_{2k}(t) \omega_{2j}(t) \omega_{2j}(t) - i \hat{f}_{2k}(t) \hat{f}_{2j}(t) + i S_{2k}^j [\omega_{2j}(t)] \right] \}$$

$$+ (2 - k) \left( \hat{f}_{2j}(w_j)(t) \varphi_{j, 2k - 2j}(t) \right), \quad j, k = 1, 2, \quad D_0^{(111)} \text{Im} \{i \left[ \Phi_j^2(t) + T \tilde{\rho}_j(t) \right] \} + \text{Im} \{\varphi_j(a)(t) \} = \varphi_j(a)(t), \tag{13}$$

where
\[ l_j(w)(t) = (-1)^j D_{\alpha^j}^2 k_j w_j \frac{d\alpha^j}{ds}, j = 1, 2, \quad l_j(\psi)(t) = D_{\alpha^1}^{11} \psi_1 \frac{d\alpha^2}{ds} - D_{\alpha^2}^{233} \psi_2 \frac{d\alpha^1}{ds}, \]

\[ \psi = (\psi_1, \psi_2); \quad \phi_j(\tau) = P^j(\psi) + (-1)^j \left[ D_{\alpha^j}^{\tau} k_j^{\tau} \frac{d\alpha_{\tau-j}}{d\tau} - D_{\alpha_{\tau-j}}^{232} \frac{d\alpha^j}{d\tau} \right], j = 1, 2, \]

\[ \phi_j(\tau) = P^j(\psi) \left( T^{11} w_{3\alpha} \frac{d\alpha^2}{d\tau} - T^{22} w_{3\alpha} \frac{d\alpha^1}{d\tau} + T^{12} \left( w_{3\alpha} \frac{d\alpha^2}{d\tau} - w_{3\alpha} \frac{d\alpha^1}{d\tau} \right) \right), \]

Here and below, the symbol $\psi'(t)$ denotes the limit of a function $\psi(z)$ as $z \to t \in \Gamma$ from the interior of the domain $\Omega$.

Thus, to determine the functions $\rho_j^j \in L_p(\Omega)$, $\phi_j \in C_\alpha(\Omega)$, $k = 1, 2$, we have the system of Eqs. (12) and (13). We will find the holomorphic functions in the form of Cauchy type integrals with real densities:

\[ \Phi_j(z) = \theta(\mu_j)(z), \quad \psi_j'(z) = \theta(\mu_j)(z) \]

\[ \psi_j'(z) = i\theta(\mu_j)(z), \quad \theta(f)(\tau) = \frac{1}{2\pi i} \int \frac{f(\tau) d\tau}{\tau - z}, \]

where $\mu_j(\tau) \in C_\alpha(\Gamma)$ ($j = \overline{1, 5}$) are arbitrary real functions, $\tau' = d\tau / d\sigma$, $d\sigma$ is an element of the arc length of the curve $\Gamma$.

For functions $\psi_j(z)$ ($j = 1, 2$), $\Phi_j(z)$, we have representations:

\[ \psi_j(z) = -\frac{1}{2\pi i} \int \frac{\mu_j(\tau)}{\tau'} \ln \left( 1 - \frac{z}{\tau} \right) d\tau + c_{j-1} + ic_{j}, \quad \psi_j'(z) = \psi_j(\mu_j)(z) + c_{j-1} + ic_{j}, \quad j = 1, 2, \]

\[ \Phi_j(z) = -\frac{1}{2\pi i} \int \frac{\mu_j(\tau)}{\tau'} \ln \left( 1 - \frac{z}{\tau} \right) d\tau + c_0 + ic_0 = \Phi_j(\mu_j)(z) + c_0 + ic_0, \]

where $c_j (j = \overline{1, 6})$ are arbitrary real constants, where $\ln(1 - z/\tau)$ is a one-valued branch that vanishes when condition $z = 0$.

Using formulas of Sokhotsky [26, p. 66], we find $\Phi_j(t)$, $\psi_j'(t)$, $j = 1, 2$, $\Phi_j(t)$, $t \in \Gamma$. We substitute their expressions, as well as (15) into (12) and (13), taking into account relations (7) - (9) and formulas (4.7), (4.9) from [25, p. 28]. As a result, after simple transformations, we arrive at a system of equations with respect to functions $\mu_j = (\mu_j, \mu_j, \mu_j, \mu_j, \mu_j) \in C_\alpha(\Gamma)$ and $\rho = (\rho, \rho', \rho') \in L_p(\Omega)$:

\[ \sum_{j=1}^{5} \left[ a_j(t) \mu_j(t) + b_j(t) \frac{\mu_j(\tau) d\tau}{\tau - t} \right] + K_j \mu_j(t) + H_j \rho(t) = g_j(a)(t), \quad t \in \Gamma, \quad j = \overline{1, 5}, \]

\[ \rho^j(z) + h_j(\rho)(z) + h_j(\mu_j)(z) = f^j(z) - h_j(c)(z), \quad z \in \Omega, \quad j = \overline{1, 5}. \]

In the system of Eqs. (16): $h_j(\rho)(j = \overline{1, 3})$ are linear completely continuous operators in $L_p(\Omega)$; $h_j(\mu_j)(j = \overline{1, 3})$ are linear completely continuous operators from $C_\alpha(\Gamma)$ in $L_p(\Omega)$, where $v$ is any number from the interval (0, 1); $K_j \mu_j(t) (j = \overline{1, 5})$ are linear completely continuous operators from $C_\alpha(\Gamma)$ in $C_\alpha(\Gamma)$ for $\alpha' < \alpha$ and are bounded operators from $C_\alpha(\Gamma)$ in $C_\alpha(\Gamma)$; $H_j \rho(t) (j = \overline{1, 5})$ are linear completely continuous operators from $L_p(\Omega)$ in $C_\alpha(\Gamma)$ for $\alpha' < \alpha$ and are bounded operators from $L_p(\Omega)$ in $C_\alpha(\Gamma)$; we use $h_j(c)(z)$ to denote parts $h_j$ containing arbitrary constants; $a_j, b_j \in C_\alpha(\Gamma)$ are known functions; $f^j(z) \in L_p(\Omega)$ ($j = \overline{1, 3}$) and $g_j(a)(t) \in C_\alpha(\Gamma)$ ($j = \overline{1, 5}$) are known expressions which depend on the deflection $w_j$.
We investigate the solvability of the system of Eqs. (16) in the space \( L_2(\Omega) \times \mathbb{C}_w(\Gamma) \). Direct calculations show that the index of the system \( (16) \chi = \frac{1}{2\pi i} \left[ \arg \frac{\det(a - \pi ib)}{\det(a + \pi ib)} \right] = 0 \), where \( a = (a_{\nu})_{\nu=5} \) and \( b = (b_{\nu})_{\nu=5} \) are fifth-order matrices; the symbol \( \left[ \arg \phi \right]_R \) means the increment of function argument \( \phi \) when traversing the curve \( \Gamma \) once in the positive direction. Consequently, the Fredholm alternative is applicable to system \( (16) \). Let \( (\rho, \mu_0) \in L_2(\Omega) \times \mathbb{C}_w(\Gamma) \) be a solution of system \( (16) \) with \( f^j - h^j(c) = 0, \quad j = 1,3, \quad g^j(a) = 0, \quad j = 1,5 \). This solution \( (\rho, \mu_0) \), according to formulas \((14) \) and \((15) \), where the constants \( c_j (j = 1,6) \) are zero, corresponds to the holomorphic functions \( \Phi_j(z) (j = 1,3), \Psi_j(z) (k = 1,2) \). Functions \( \Phi_j(z) (j = 1,3), \Psi_j(z) (k = 1,2) \) together with \( \rho(z), \psi_j(z) (k = 1,2) \) using formulas \((10), (11) \). It is easy to see that these displacements satisfy the homogeneous system of linear equations \((f_j = 0, j = 1,5) \) in \((3) \) and the homogeneous linear boundary conditions. We multiply these equations in \((3) \) by \( w_1, w_2, w_3, \psi_1, \psi_2, z \), respectively, integrate the resulting relations over the domain \( \Omega \) and add up the results of integrations. Taking into account homogeneous linear boundary conditions, we obtain system \( e_{\nu 0} = 0, \quad \gamma_{\nu 0} = 0, \gamma_{\nu 2} = 0, \lambda, \mu = 1,2, \) the solution of which is obtained in the form \( w_i = -c_{\nu_i 1} \alpha^2 + c_{\nu_i 1}, \quad w_j = c_{\nu_j 1} \alpha^3 + c_{\nu_j 2}, \quad w_k = \psi_1 = 0, \quad k = 1,2, \) where \( c_{\nu_0}, c_{\nu_1}, c_{\nu_2} \) are arbitrary real constants. Then we have \( \alpha_0(z) = 2ic_0D_0^{121}, \quad \alpha_1(z) = 0 \) and
\[
\rho^j(z) = 2ic_0D_0^{121}, \quad \rho^1(z) = 0, \quad \rho^3(z) = 0. \tag{17}
\]

Using formulas \((7), (10) \) and \((11) \), we find
\[
\Phi_1(z) = c_0 \alpha_0(z), \quad \Psi_1(z) = c_0 \beta_0(z), \quad \Phi_2(z) = \Psi_1(z) = \Phi_1(z) = 0, \quad \alpha_0(z) = \frac{1}{\pi i} \left[ \frac{D_0^{121}(t)}{t - z} \right] dt, \quad \beta_0(z) = \frac{1}{2\pi i} \int \frac{d t}{t - z}. \tag{18}
\]

We substitute functions \((18) \) into \((14) \). We obtain
\[
\frac{\mu_j(t)}{t' - 2ic_0D_0^{121}(t)} = F_j (t), \quad \frac{\mu_j(t)}{t' - c_0(\bar{t})^2} = F_j (t), \quad \frac{\mu_j(t)}{t'} = F_j (t), \quad j = 1,5. \tag{19}
\]

where \( F_j (t) \) are the boundary values of the function \( F_j (z) \), which is holomorphic function in the exterior \( \Omega \) and disappears at infinity. Therefore, for the function \( F_j (z) \) in the exterior \( \Omega \), we arrive at the Riemann-Hilbert problem with the boundary condition \( \text{Re} [itF_j (t)] = f_j (t), \quad j = 1,5, \) where \( f_1(t) = 2c_0D_0^{121}(t) \text{Re}(t'), \quad f_2(t) = c_0 \text{Re}(t'), \quad f_3(t) = 0, \quad j = 3,5 \). The solution of this problem has the form [27, p. 253] \( F_j (z) = c_\nu f_j (z) + b_\nu f_j (z), \quad j = 1,2, \quad F_j (z) = b_\nu f_j (z), \quad j = 3,5 \); here \( f_j (z) \) are the known holomorphic functions in the exterior \( \Omega \), \( c_\nu, \beta_\nu \) are arbitrary real constants. Then for the functions \( \mu_j (t) \) we obtain equalities
\[
\mu_j(t) = c_\nu \mu_j (t) + b_\nu \mu_j (t), \quad j = 1,2, \quad \mu_j(t) = b_\nu \mu_j(t), \quad j = 3,5, \tag{20}
\]
in which \( \mu_j (t) \) are the known real functions.

Solutions \((17) \) and \((19) \) show, that the homogeneous system of Eqs. \((16) \) has six linearly independent solutions. Then the system of equations, which is adjoint with system \((16) \), will also have six linearly independent solutions and it can be shown that its general solution has the form:
\[
\begin{align*}
v_1 &= c_\nu \bar{k}_1 (\alpha^2 - k_1 (\bar{\alpha}^2)) - c_\nu \alpha^2 k_1 (\bar{\alpha}^2) + c_\nu k_1 (\alpha - \bar{\alpha}) + c_\nu \bar{\alpha}^2 + c_\nu, \\
v_2 &= c_\nu \alpha^2 k_1 (\bar{\alpha}^2) + c_\nu \bar{k}_1 (\alpha - \bar{\alpha}) + c_\nu k_1 (\alpha - \bar{\alpha}) + c_\nu \bar{\alpha}^2 + c_\nu, \\
v_3 &= -c_\nu \bar{\alpha} \frac{\alpha^2}{2} + c_\nu \frac{\alpha^2}{2}, \quad v_4 = c_\nu, \quad v_5 = c_\nu, \quad v_6 = -v_3, \quad j = 1,2,4,5; \quad v_1 = -v_3,
\end{align*}
\]
where

\[ k_j^m(x) = \int_0^{x^m} k_j(x) dx , \quad m = 0,1 , \quad \bar{k}_j(x) = \int_0^{x^j} k_j(x) dx , \quad j = 1,2 ; \]

c_j are arbitrary real constants.

Therefore, the solution \((v,v)\), \(v = (v_1,v_2,v_3,v_4,v_5)\) of the adjoint system can be represented in the form

\[ (v,v) = c_1\gamma_1^* + c_2\gamma_2^* + c_3\gamma_3^* + c_4\gamma_4^* + c_5\gamma_5^* , \]

where \( \gamma_1^* = (\gamma_{11},\gamma_{12},...\gamma_{110}) \) are linearly independent solutions of the adjoint system. Then, for the solvability of system (16), it is necessary and sufficient that conditions

\[ \int_{\Omega} [\text{Re}(f^j - h_j)k \gamma_{1k} + \text{Re}(f^j - h_j)k \gamma_{1j} + (f^j - h_j)\gamma_{13}] d\alpha_1 d\alpha_2 ; \]

\[ + \int_{r} (g_2\gamma_{16} + g_3\gamma_{17} + g_4\gamma_{18} + g_5\gamma_{19} + g_6\gamma_{110}) ds , \quad k = 1,6 \]

should be satisfied. After simple transformations, these conditions take the form

\[ \int_{\Omega} R^j d\alpha_1 d\alpha_2 + \int_{r} P^i ds = 0 , \quad j = 1,2 , \int_{\Omega} (R^j - R^3) d\alpha_1 d\alpha_2 + \int_{r} (P^i - P^2) d\alpha_1 d\alpha_2 = 0 , \]

\[ \int_{\Omega} [R^j k_j^*(\alpha_1) + R^j k_j^*(\alpha_2) + R^3] d\alpha_1 d\alpha_2 + \int_{r} [P^i k_i^*(\alpha_1) + P^i k_i^*(\alpha_2) + P^3] ds = 0 , \]

\[ \int_{\Omega} [R^j k_j^*(\alpha_1) - R^j k_j^*(\alpha_2) - R^3] d\alpha_1 d\alpha_2 + \int_{r} [P^i k_i^*(\alpha_1) - P^i k_i^*(\alpha_2) - P^3] ds = 0 , \]

\[ \int_{\Omega} [R^j k_j^*(\alpha_2) - R^j k_j^*(\alpha_1) + R^3] d\alpha_1 d\alpha_2 + \int_{r} [P^i k_i^*(\alpha_2) - P^i k_i^*(\alpha_1) + P^3] ds = 0 , \]

\[ \text{where} \quad R^j , \quad P^i \quad (j = 1,3), \quad L^k , \quad N^i \quad (k = 1,2) \text{are defined in (5)}. \]

If conditions (20) are satisfied, then the general solution of system (16) can be represented in the form

\[ \rho^j(z) = \Re_j (f(w_j) - f^j(z) + \hat{\rho}^j(z) , \quad j = 1,3 , \quad \mu_j(t) = \Re_{j+1} (f(w_j) - f^j(t) + \hat{\mu}_j(t) , \quad k = 1,5 , \]

where

\[ \hat{\rho}^j(z) = 2i\epsilon D_\gamma^{j+2} , \quad \hat{\rho}^j = \hat{\rho}^j_0 = 0 , \quad \hat{\mu}_j(t) = c_j \mu_j^0(t) + \beta_j \mu_j^0(t) , \quad j = 1,2 , \quad \beta_j(t) = \beta_j(t) , \quad j = 3,5 ; \]

\[ f_j = (h_j(c),h_{2j}(c),h_{3j}(c),h_{4j}(c),h_{5j}(c),h_{6j}(c),h_{7j}(c),h_{8j}(c),h_{9j}(c),h_{10j}(c),h_{11j}(c),h_{12j}(c)) ; \]

\[ \Re_j (j = 1,3) \text{and} \Re_k (k = 4,8) \text{are linear bounded operators from} \quad L_p(\Omega) \times C_n(\Gamma) \quad \text{in} \quad L_p(\Omega) \quad \text{and in} \quad C_n(\Gamma) \quad \text{respectively;} \]

\[ c_j \quad \text{and} \quad \beta_j \quad \text{are arbitrary real constants}. \]

Now, if we substitute relations (21) into (10) and (11), then for tangential displacements and for rotation angles we obtain expressions through the deflection in the form

\[ \omega_j^0(z) = \omega_j^0(w_j(z) + \omega_j^0(z) , \quad j = 1,2 , \quad \omega_j^0 = w_2 + iw_1 , \quad \omega_j^0 = \psi_2 + i\psi_1 . \]

And with respect to the deflection \( w_j \), we arrive at equation of the form

\[ w_j - Gw_j = w_{jk} , \quad w_{jk} = -c_4 \alpha^1 - c_5 \alpha^2 + c_6 , \quad c_4 , \quad c_5 \quad \text{and} \quad c_6 \quad \text{are arbitrary real constants}. \]

Let us study the solvability of Eq. (23) in space \( W_{p}^{(2)}(\Omega) \), which we write in the form

\[ \hat{w}_j = G\hat{w}_j = 0 , \quad \text{(24)} \]
where \( \hat{G}\hat{w}_3 = G(\hat{w}_3 + w_{k_e}) \), \( \hat{w}_3 = w_3 - w_{k_e} \).

Using relations in (5) and representations (22) for tangential displacements \( \omega^\alpha \) and rotation angles \( \alpha^\alpha \) it is easy to show that \( G \) is a nonlinear bounded operator in space \( W^{(2)}_p(\Omega) \). Moreover, for any \( \hat{w}_j^j \in W^{(2)}_p(\Omega) \) \( (j = 1, 2) \) belonging to the ball \( \| \hat{w}_j^j \|_{W^{(2)}_p(\Omega)} < r \), we have the estimate

\[
\| \hat{G}\hat{w}_3^3 - \hat{G}\hat{w}_3^3 \|_{W^{(2)}_p(\Omega)} \leq q \| \hat{w}_3^3 - \hat{w}_3^3 \|_{W^{(2)}_p(\Omega)},
\]

where \( q = c[\| R\|_{\alpha, \alpha} + \| R\|_{\alpha, \alpha} + (r + |c_4| + |c_5|)(1 + r) + c_4^2 + c_5^2] \); \( c \) is the known positive constant depending on the physico-geometric characteristics of shell; \( c_4 \) and \( c_5 \) are the constants included in the expression of function \( w_{k_e} \).

### Results

Suppose that the radius \( r \) of the ball, the external forces \( R^j \) \( (j = 1, 2) \), the constants \( c_4 \) and \( c_5 \) are such that inequalities

\[
q < 1, \quad \| \hat{G}\hat{w}_3^3 \|_{W^{(2)}_p(\Omega)} < (1 - q)r
\]

are satisfied. Then we can apply the contraction mappings principle [28, p. 146] to the Eq. (24); as a result, in the ball \( \| \hat{w}_3^3 \|_{W^{(2)}_p(\Omega)} < r \), for fixed constants \( c_4 \) and \( c_5 \) using conditions (25), Eq. (24) has a unique solution \( \hat{w}_3 \in W^{(2)}_p(\Omega) \), which can be represented as \( \hat{w}_3 = R\hat{G}\hat{0}_0 \), where \( R \) is the resolvent of operator \( \hat{G}\hat{0}_0 - \hat{G}\hat{0} \). Therefore, the deflection \( w_3 \) has the form \( w_3 = \hat{w}_3 + w_{k_e} \). Knowing \( w_{k_e} \), according to formulas (22) we find the tangential displacements \( \omega^\alpha \) and rotation angles \( \alpha^\alpha \), which, as it can be easily verified, belong to the space \( W^{(2)}_p(\Omega) \). As a result, we obtain the generalized solution \( a = (w_i, w_{k_e}, w_j, \varphi_i, \varphi_j) \) of Problem (3), (4), which can be written as \( a = a_0 + a_\alpha \), where \( a_0 \) is the vector with components \( \text{Im}\omega^\alpha, \text{Re}\omega^\alpha, \varphi_i, \varphi_j \); \( a = (w_i, w_{k_e}, \varphi_i, \varphi_j) \) is the vector with components determined by the formulas \( \omega^\alpha_{\alpha j} = \omega^\alpha_{\alpha j}(w_{k_e}) + \omega^\alpha_{\alpha j}, \quad j = 1, 2, \quad w_{k_e} = w_{k_e}, \quad \omega^\alpha_{\alpha j} = w_{k_e} + iw_{k_e}, \quad \alpha^\alpha_{\alpha j} = \varphi_{\alpha j} + i\varphi_{\alpha j} \).

### Discussions

Let us notice that \( a_\alpha \) is the vector of rigid displacements of the shell as an absolutely rigid body.

Note that in the last two conditions in (20), \( w_j \) means \( w_j = \hat{w}_j + w_{k_e} \). In the case of linear problems, these summands containing \( w_{k_e} \) are absent.

It is easy to see that conditions (20) are not only sufficient, but also necessary conditions for the solvability of Problem (3), (4). Note that they mean that external load acting on the shell is self-balanced.

Thus, if conditions (a), (b), (c), (d), (e), (20) and inequalities (25) are satisfied, then Problem (3), (4) has a solution \( a = (w_i, w_{k_e}, w_j, \varphi_i, \varphi_j) \) up to rigid displacements \( a_\alpha \) of the shell as an absolutely rigid body. Knowing the generalized displacements \( w_j, \quad j = 1, 2, \quad \varphi_k, \quad k = 1, 2 \) and using formulas (1) and (2), we find all the components of the deformations and stresses, thereby the stress-strain state of the shell constructions is completely determined.

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