WKB Method and Relativistic Potential Models

V.V. Rubish

Department of Theoretical Physics, Uzhhorod National University,
Voloshin str. 32, 88000 Uzhgorod, Ukraine
e-mail: vrubish@univ.uzhgorod.ua

Abstract

Based on the Dirac approach we have developed the relativistic vision of the WKB method for centrally symmetrical potential with mixed Lorenz structure. We have obtained relativistic wavefunctions of light quark and the new rule of quantization containing the spin-orbit interaction. These gives us the possibility of finding the energy levels and decay width of hydrogen-like quark-systems.

Keyword: WKB method, quark model, potential, meson masses

1. Introduction

The Dirac equation with potentials of a concrete type, for which possible to write the exact solution, meet enough seldom. Most often for searching the solutions either numerical or asymptotic methods are used. In many theoretical and applied problems a possibility of obtaining the asymptotic solution allows to carry out the fullest analysis of a problem. Therefore hardly there is a necessity in detail to explain importance of creating and investigating asymptotic methods of the solving the Dirac equation.

The Wentzel-Kramers-Brillouin quasi-classical approximation (or WKB method) is one of basic and most universal asymptotic methods of solving problems of theoretical and mathematical physics (see, for example, [1, 2, 3, 4]), for which the exact solutions are either unknown or rather onerous. As is known [1, 2, 3, 4], in case of a Coulomb field it has a high accuracy even for small values of quantum numbers. In contrast to the perturbation theory the given approach is not connected with a smallness of interaction and consequently has more wide applicability area allowing to study qualitative legitimacies in behaviour and properties of quantum mechanical systems. The new application of quasi-classical approach can be low-energy sector of QCD (the energy spectrum of hadrons, confinement study, decay widths of hadrons), where standard approaches based on perturbative theories are inapplicable in consequence of the fact that interaction between quarks in this area is not small.

2. The WKB method for the Dirac equation in a centrally symmetrical field

Obtain the formulae of the quasi-classical approximation for solutions of the Dirac equation with potential of centrally symmetry having mixed Lorenz structure. Corresponding to this we search wavefunctions of the stationary states (in standard representation) in the form of

\[ \Psi = r^{-\frac{1}{2}} \left( \begin{array}{c} F(r)\Omega_{jm} (n) \\ iG(r)\Omega_{j'n'} (n) \end{array} \right), \]  

(1)
where $\Omega$ – the spherical spinor, $j$ and $m$ – the total angular moment and projection of $j$ ($j = l \pm 1/2$), $l$ – the orbital moment \((l + l' = 2j)\), \(n = r/r\).

After separation the angular variable the Dirac equation with centrally symmetrical potential containing both vector $V(r)$ and scalar $S(r)$ parts, takes form \((\hbar = c = 1)\)

\[
\frac{dF}{dr} + \frac{k}{r} F - \frac{1}{\hbar} [(E - V(r)) + (m + S(r))] G = 0,
\]
\[
\frac{dG}{dr} - \frac{k}{r} G + \frac{1}{\hbar} [(E - V(r)) - (m + S(r))] F = 0,
\]

where $F$ and $G$ are the radial functions, $k = \mp (j + 1/2)$ for states with $j = l \pm 1/2$, $E$ – the energy of level.

For finding the quasi-classical solutions of the system \(2\) it is convenient to write equations \(2\) in the matrix form \(3\):

\[
\Psi' = \frac{1}{\hbar} D\Psi, \quad \Psi = \begin{pmatrix} F \\ G \end{pmatrix}, \quad D = \begin{pmatrix} -\hbar k/r & E - V(r) + m + S(r) \\ -E + V(r) + m + S(r) & \hbar k/r \end{pmatrix}.
\]

Here a Planck constant $\hbar$ is remained in the explicit view, the prime denotes the derivative with respect to $r$. The solution of the matrix equation \(3\) we shall look as the formal expansion in powers of $\hbar$:

\[
\Psi = \varphi \exp \left( \int y dr \right), \quad y(r) = \frac{1}{\hbar} y_{-1}(r) + y_0(r) + \hbar y_1(r) + \hbar^2 y_2(r) + ..., \quad \varphi(r) = \sum_{n=0}^{\infty} \hbar^n \varphi^{(n)}(r),
\]

where the upper (lower) component $\varphi^{(n)}_f$ ($\varphi^{(n)}_g$) of the vector $\varphi^{(n)}$ corresponds to the radial wavefunction $F(G)$. Having substituted \(4\) into \(3\) and equated to zero the coefficients of each power of $\hbar$, we arrive at the recurrent system

\[
(D - y_{-1}) \varphi^{(n)} = 0,
\]
\[
(D - y_{-1}) \varphi^{(n+1)} = \varphi^{(n)'} + \sum_{k=0}^{n} y_{n-k} \varphi^{k}, \quad n = 0, 1, \ldots
\]

From the first equation of system \(3\) follows that $y$ is eigenvalues, $\varphi^n \equiv \varphi_i$ is one of (right) eigenvectors of the matrixes $D$. Eigenvalues $y_{-1}(r) \equiv \lambda_i$ are roots of the secular equation $\det (D - y_{-1}) = 0$. Corresponding eigenvectors $\varphi_i$ can be find in explicit form at diagonalization of the matrix $D - \lambda_i$:

\[
y_{-1} = \pm \sqrt{(m + S)^2 - (E - V)^2 + \frac{k^2}{r^2}} = \pm q = \lambda_i,
\]
\[
\varphi_i = A_1 \left( \begin{array}{c} m + S + E - V \\ kr^{-1} + \lambda_i \end{array} \right) = A_2 \left( \begin{array}{c} \lambda_i - kr^{-1} \\ m + S - E + V \end{array} \right).
\]

Here index $i = \pm$, $A_1(r)$ and $A_2(r)$ are some functions, which will be determined below.

Since matrix $D$ is not symmetrical, together with right eigenvectors $\varphi_i$, it is necessary to enter left eigenvectors $\varphi_i$. They define by conditions

\[
\varphi_i (D - \lambda_i) = 0, \quad \varphi_i \neq \varphi_i^T,
\]
\[
\varphi_i = B_1 \left( m + S - E + V, \ kr^{-1} + \lambda_i \right) = B_2 \left( \lambda_i - kr^{-1}, \ m + S + E - V \right).
\]
Furthermore, left and right eigenvectors the mutually orthogonal

\[(\hat{\varphi}_i, \varphi_j) = \sum_{\alpha=1}^{2} (\hat{\varphi}_i)_{\alpha} (\varphi_j)_{\alpha} = \text{const} \cdot \delta_{ij}.\] (10)

Remind that \(\hat{\varphi}_i\) there is vector-column, and \(\delta_{ij}\) is the Kronecker symbol. For determination \(y_0\), we take \(\varphi_0 = \varphi_i\) in the first equation of (5) and multiply both parts of this equation by \(\hat{\varphi}_i\) on the left. Then due to (8), the left part is zero and we get the equation for \(y_0\), from which

\[y_0(r) = -\frac{(\hat{\varphi}_i, \varphi_i')}{(\varphi_i, \varphi_i)}.\] (11)

We select functions \(A_{1,2}(r), B_{1,2}(r)\), in (7), (9) so that the condition

\[(\hat{\varphi}_i, \varphi_i') = (\hat{\varphi}_i', \varphi_i)\] (12)

is realized. Then

\[\int^r y_0 dr = \ln [(\hat{\varphi}_i, \varphi_i)^{-1/2}]\] (13)

and as a result we obtain

\[\Psi = \varphi_i (\hat{\varphi}_i, \varphi_i)^{-1/2} \exp \left( \frac{1}{\hbar} \int^r \lambda_i dr \right),\] (14)

that is similar usual expression for quasi-classical wavefunction in nonrelativistic quantum mechanics

\[\Psi \sim p^{-1/2} \exp \left( \pm i \int^r p dr \right).\] (15)

Using the first two equation of system (14) by means of left and right vectors technique we find the terms \(y_1, y_0\) and \(\varphi^{(0)}\). Solving next equations of the system (15) by the similar procedure one can sequentially find the terms \(y_2, y_3, ..., \varphi^{(2)}, \varphi^{(3)}, \ldots\) in the expansions (11). But formulae for them are rather cumbersome, therefore in applications ones usually restrict themselves to only first terms. Actually the reason of this is the fact that the expansions in powers of \(\hbar\) in the general case don’t convergent and are asymptotic series, the finite number of terms of which gives the good approximation for the wavefunction, if a parameter of an expansion (the Planck constant \(\hbar\)) is rather small.

It is always easy to satisfy the condition (12). By substituting the expression (7), (8), into (12) we come to the equation

\[\frac{A_1 B_1' - A_1' B_1}{A_1 B_1} = \frac{(m + S) V' + (E - V) S'}{q (q \pm kr^{-1})},\] (16)

where

\[\Psi = \left\{ \begin{array}{c} F \\ G \end{array} \right\} = \left[ 2q \left( q \pm \frac{k}{r} \right) \right]^{-1/2} \exp \left\{ \pm \int q dr + \right. \]

\[\left. \frac{1}{2} \int^r \frac{(m + S) V' + (E - V) S'}{q (q \pm kr^{-1})} dr \right\} \left( \begin{array}{c} m + S + E - V \\ kr^{-1} \pm q \end{array} \right).\] (17)
The wavefunction of state has the different form in the various regions. The wavefunction in classically allowed region with discrete spectrum is the quasi-stationary state can be normalized on a single particle localized in the region I, neglecting its penetrability into the classically forbidden regions $r < r_0$ and $r > r_−$.

3. The wavefunction of the Dirac particle in classically allowed and forbidden regions

The wavefunction of state has the different form in the various regions. The wavefunction in classically allowed region with discrete spectrum is the quasi-stationary state can be normalized on a single particle localized in the region I, neglecting its penetrability into the classically forbidden regions $r < r_0$ and $r > r_−$.

The effective potential

$$U (r, E) = \frac{E}{m} V + S + \frac{S^2 - V^2}{2m} + \frac{k^2}{2mr^2}$$

corresponds to the Dirac system. So $U (r, E)$ looks like a potential with a barrier. We consider the most general case, when the effective potential is barrier type. Then wavefunction has the different view in the three regions: 1) potential well $r_0 < r < r_−$ ($q^2 < 0$); 2) the below-barrier region $r_− < r < r_+$ ($q^2 > 0$); 3) the classically allowed region with continuum spectrum $r > r_+$ ($q^2 < 0$); 4) classically forbidden region $r < r_0$ ($q^2 > 0$), where $r_0$, $r_−$ and $r_+$ are turning points. In following section we consider behaviour of the solutions in these regions.

The wavefunction of state has the different form in the various regions. The wavefunction in classically allowed region with discrete spectrum is the quasi-stationary state can be normalized on a single particle localized in the region I, neglecting its penetrability into the classically forbidden regions $r < r_0$ and $r > r_−$.

The wavefunction of state has the different form in the various regions. The wavefunction in classically allowed region with discrete spectrum is the quasi-stationary state can be normalized on a single particle localized in the region I, neglecting its penetrability into the classically forbidden regions $r < r_0$ and $r > r_−$.

The wavefunction of state has the different form in the various regions. The wavefunction in classically allowed region with discrete spectrum is the quasi-stationary state can be normalized on a single particle localized in the region I, neglecting its penetrability into the classically forbidden regions $r < r_0$ and $r > r_−$.

The wavefunction of state has the different form in the various regions. The wavefunction in classically allowed region with discrete spectrum is the quasi-stationary state can be normalized on a single particle localized in the region I, neglecting its penetrability into the classically forbidden regions $r < r_0$ and $r > r_−$.
Here $\cos^2 \Theta_i (r)$ can be replaced with average value 1/2:

$$
|C^\pm_i| = \left\{ \int_{r_0}^{r_-} \frac{E - V (r)}{p (r)} dr \right\}^{-1/2} \left( \frac{2}{T} \right)^{1/2},
$$

(24)

where $T$— the frequency period of a relativistic particle inside a potential well. We note that in turning points $r_0$ and $r_-$ equation

$$
E - V = [(m + S)^2 + k/r^2]^{1/2}
$$

is true and $E - V > m + S$ in region I.

II. The below-barrier region $r_- < r < r_+$ is classically forbidden. Here $p = iq$, and quantities $q$, $y_1$ and $y_0$ are real. As known [1] the wavefunction should exponentially decreases inside of this region. So the solutions of the Dirac system (2) in the below-barrier region for $k < 0$ are

$$
\Psi = \frac{C^+_2}{\sqrt{qQ}} \exp \left[ \int_{r^+_1}^{r} \left( -q - \frac{(m + S) V' + (E - V) S'}{2qQ} \right) dr \right] \left( \begin{array}{c} -Q \\ m + S - E + V \end{array} \right),
$$

(25)

for $k > 0$

$$
\Psi = \frac{C^-_2}{\sqrt{qQ}} \exp \left[ \int_{r^+_1}^{r} \left( -q + \frac{(m + S) V' + (E - V) S'}{2qQ} \right) dr \right] \left( \begin{array}{c} m + S + E - V \\ -Q \end{array} \right).
$$

(26)

III. The result for region of continuum ($r > r_+$) is the most interesting. Here the wavefunction corresponds to the divergent wave (taking off particle):

for $k > 0$

$$
\Psi = \frac{C^+_3}{\sqrt{pP}} \exp \left[ \int_{r^+_1}^{r} \left( ip + \frac{(m + S) V' + (E - V) S'}{2pP} \right) dr \right] \left( \begin{array}{c} iP \\ m + S - E + V \end{array} \right),
$$

(27)

for $k < 0$

$$
\Psi = \frac{C^-_3}{\sqrt{pP}} \exp \left[ \int_{r^+_1}^{r} \left( ip - \frac{(m + S) V' + (E - V) S'}{2pP} \right) dr \right] \left( \begin{array}{c} m + S + E - V \\ iP \end{array} \right),
$$

(28)

where $P = p + i |k| / r$.

IV. In classically forbidden region ($r < r_0$) the wavefunctions are of the form:

for $k < 0$

$$
\Psi = \frac{C^-_4}{\sqrt{q(q - k/r)}} \exp \left[ \int_{r_0}^{r} \left( q + \frac{(m + S) V' + (E - V) S'}{2q(q - k/r)} \right) dr \right] \left( \begin{array}{c} E - V + m + S \\ q - k/r \end{array} \right),
$$

(29)

for $k > 0$

$$
\Psi = \frac{C^+_4}{\sqrt{q(q + k/r)}} \exp \left[ \int_{r_0}^{r} \left( q - \frac{(m + S) V' + (E - V) S'}{2q(q + k/r)} \right) dr \right] \left( \begin{array}{c} q + k/r \\ m + S - E + V \end{array} \right).
$$

(30)
The formulae (20)-(30) include the whole range of values of $r$, except for neighborhoods of turning points. For bypass of these points and matching the solutions we use the usual method \cite{2}. Closely to the $r_-$ and $r_+$ the system (2) reduces to the Schrödinger equation with the effective potential linearly depending on $r - r_{\pm}$, the solution of which expressed through the Airy function; one can match by the more elegant Zwaan method \cite{1}, \cite{3}. So the relation between the constants in various regions is of the form

$$C_2^\pm = i C_3^\pm = \mp \frac{C_1^\pm}{2} \left[ \frac{E - V(r_-) + m + S(r_-)}{|k| r_-^{-1}} \right]^{1/2} \exp \left[ - \int_{r_-}^{r_+} \left( q \pm \frac{(m+S)V' + (E-V)S'}{2qQ} \right) dr \right] ,$$

$$2C_4^\pm \left[ \frac{|k| r_0^{-1}}{E - V(r_0) + m + S(r_0)} \right]^{\text{sgn} \frac{k}{r}} = (-1)^n C_1^\pm . \quad (31)$$

Let us find the equation determining the real part of the level energy $E$ and width $\Gamma$ of quasi-stationary levels $E = E_r - i \Gamma/2$. Neglecting the barrier penetrability from (23) we obtain the quantization condition:

$$\int_{r_0}^{r_-} \left( p + kw \right) dr = \left( n + \frac{1}{2} \right) \pi , \quad (32)$$

where $n = 0, 1, 2, \ldots$ is the radial quantum number.

Pass to calculation of the level width

$$\Gamma = -2 \text{Im} \left[ G^* F \right]_{r \to \infty} . \quad (33)$$

Having used the explicit expressions (27), (28) for $F$ and $G$ functions, relation between normalization constants (31) and expression for $C_1^\pm$ (24), we get the tunnelling probability:

$$\Gamma = \frac{1}{T} \exp \left[ -2 \int_{r_-}^{r_+} \left( q - kw \right) q r dr \right] . \quad (34)$$

Though the formulae (23)-(34) essentially differ from the formulae of nonrelativistic quasi-classics (in particular, by relativistic expression for quasimomentum $p$, taking into account spin-orbit interaction, and additional pre-exponent multiplier) and more complicated from them. Their application to concrete problems does not meet difficulties, since all quantities in functions $F$ and $G$ are expressed in quadratures.

4. Approbation of results obtained

4.1. For testing elaborated by us version of the WKB method, get the known in atomic physics the Bohr-Sommerfeld formula. We choose $V(r) = -\frac{\alpha Z}{r}$ and $S(r) = 0$ ($\alpha \approx 1/137$ – the fine structure constant). Then the formulae (21), (22) are of the form

$$p(r) = \left[ \left( E + \frac{\alpha Z}{r} \right)^2 - m^2 - (k/r)^2 \right]^{1/2} , \quad w = \frac{1}{2} \left[ \frac{\alpha Z/r^2}{E + \frac{\alpha Z}{r} + m} - \frac{1}{r} \right] .$$

Having calculated the integral from $r_0$ to $r_-$

$$r_{0,-} = \frac{E\alpha Z \mp \sqrt{(E\alpha Z)^2 - (m^2 - E^2)(k^2 - (\alpha Z)^2)}}{m^2 - E^2}$$
in quantization condition (32), we arrive at the expression
\[
\frac{E\alpha Z}{\sqrt{m^2 - E^2}} = n + \sqrt{k^2 - (\alpha Z)^2},
\]
whence we get expression, which is similar to the Bohr-Sommerfeld formula
\[
E = \pm m \left[ 1 + \left( \frac{\alpha Z}{n + \sqrt{k^2 - (\alpha Z)^2}} \right)^2 \right]^{-1/2}.
\]

4.2. One more rather interesting case, in which it is possible to get exact solution of the Dirac equation with potential of the type \(V(r) = -\frac{\alpha}{r}, S(r) = -\frac{\alpha'}{r}\), exists. In the same way that the Coulomb potential \(V(r)\) is derived from the exchange of massless photon between the nucleus and the leptons orbiting around it, the scalar potential \(S(r)\) is created by the exchange of massless scalar mesons. The \(\sigma\) meson frequently quoted in the literature has a very high mass and therefore the corresponding potential has a very short range. In this case the formulae (21), (22) take the form
\[
p(r) = \left[ (E + \frac{\alpha}{r})^2 - \left( m - \frac{\alpha'}{r} \right)^2 - (k/r)^2 \right]^{1/2}, \quad w = \frac{1}{2} \left[ \frac{(\alpha - \alpha')/r^2}{E + \frac{\alpha - \alpha'}{r} + m} - \frac{1}{r} \right].
\]
Quantization condition (32) gives following result
\[
\frac{E\alpha + ma'}{\sqrt{m^2 - E^2}} = n + \gamma, \quad \gamma = \pm \sqrt{k^2 - \alpha^2 + \alpha'^2},
\]
there was the integration from \(r_0\) to \(r_-\)
\[
r_{0,-} = \frac{E\alpha + ma'}{\sqrt{(E\alpha + ma')^2 - (m^2 - E^2)\gamma^2}}.
\]
Thus we find the formula coinciding with the expression obtained in [6]:
\[
E = m \left\{ \frac{-\alpha\alpha'}{\alpha^2 + (n + \gamma)^2} + \left[ \left( \frac{\alpha\alpha'}{\alpha^2 + (n + \gamma)^2} \right)^2 - \frac{\alpha^2 - (n + \gamma)^2}{\alpha^2 + (n + \gamma)^2} \right]^{1/2} \right\}.
\]

4.3. We consider the potential \(V(r) = S(r) = ar^2/4\), \(a\) is the force of an oscillator. In this case the formulae (21), (22) is given by
\[
p(r) = \left[ E^2 - m^2 - (E + m)\frac{a}{2}r^2 - (k/r)^2 \right]^{1/2}, \quad w = -\frac{1}{2r}.
\]
We integrate in quantization condition (32) from \(r_0\) to \(r_-\)
\[
r_{0,-} = \frac{1}{\sqrt{a}} \left[ \left( E - m \mp \sqrt{(E - m)^2 - \frac{2ak^2}{E + m}} \right) \right]^{1/2}
\]
and gives following result
\[
(E - m)\sqrt{\frac{E + m}{2a}} - |k| + \frac{1}{2} \text{sgn} k = 2n,
\]
for $k < 0$ get

$$(2 |k| + 1) \sqrt{a} - (E - m) \sqrt{2 (E + m)} = -4n \sqrt{a}.$$  

The last equation for energy is cubic, its real solution is

$$E_{n,k} = \left(2m + 8 \cdot 2^{2/3} m^2 A^{-1/3} + 2^{2/3} A^{1/3}\right) / 6,$$

where $A = -B + \sqrt{B^2 - 1024 m^6}$, $B = 32m^3 - 27a (1 + 2k + 4n)$. This expression completely coincides with result obtained in [7].

5. Description of the energy spectrum in two-quarks systems with Cornell potential

Relativistic description of the bounded states always was an important problem of nuclear physics and elementary particle physics, in which relativistic characteristics of the light quarks play the important role. Our world mainly consists of the light $u$ and $d$ quarks (the protons - $u\bar{u}d$ and neutrons - $u\bar{d}d$).

Quantum chromodynamics (QCD) based on principles of the quantum field theory is justly considered as the most consequent approach to solving this problem. However, standard perturbative QCD gives rather reliable recipes of the calculation of various characteristics only for description so named ”hard” processes which are characterized by large transmitted momentum, and not applicable for calculation of the characteristics which are defined by ”soft” processes (the mass spectrum, confinement of quarks, decay widths of hadrons). In the same time nonperturbative effects basically define the nature of forming bound states of interacting particles. Confinement is the result of circumstance that, unlike quantum electrodynamics in which interaction mediators - photons - are electroneutral, exchange particles in quantum chromodynamics - gluons - possess non-zero colour charge and therefore can interact one with other. Thus confinement is not nested in the framework of the perturbation theory. On account of this in the given time from principles of quantum chromodynamics the structure of interquark forces cannot completely be defined.

Just study of relativistic properties of quarks systems, namely of effects caused by spins of quarks, enables to improve both the form of confinement part of potential, and Lorenz structure of potential.

In the work [8] starting from the QCD Lagrangian and taking into account both perturbative and nonperturbative effects, using the method of vacuum correlators the Dirac equation is obtained (rigorously for the Coulomb interaction and heuristically for the confining potential) for a system consisting of a heavy antiquark (quark) $Q$ ($b$ and $c$ quarks) with the mass $m_2$ and light quark (antiquark) $q$ ($u, d, s$ - quarks) with mass much less $m_1$.

Such quark systems ($B^+ = \bar{b}u$, $B^0 = \bar{b}d$, $B^0_s = \bar{b}s$, $D^0 = \bar{c}u$, $D^- = \bar{c}d$, $D^-_s = \bar{c}s$) are QCD analogues of the hydrogen atom and thus its study is of fundamental importance. From the theoretical viewpoint the interest in heavy-light systems stems from several considerations.

First, in the limit of one infinitely heavy quark, one hopes to get the dynamics of a light quark in the external field of a heavy one. That would be similar to the picture of the hydrogen atom.

Second, since the external field is time-independent, one may hope to obtain a static potential in QCD together with spin-dependent forces, as has been done for heavy quarkonia.

Third, in the heavy-light system one may study how the chiral symmetry breaking (CSB) affects the spectrum. When one quark mass is vanishing, in the chiral symmetric case the spectrum would consist of parity doublets, and the CSB would lift the degeneracy.

Fourth, using the Dirac equation we implement explicitly relativistic dynamics and can study relativistic properties of the spectrum, e.g., in the case of a vanishing quark mass, and also relativistic spin splitting in the spectrum.
In the work \[8\] it is explicitly shown that a reasonable spectrum in Dirac equation occurs only when the confining potential is a Lorentz scalar and the Coulomb potential is the fourth component of a 4-vector. In this case the scalar potential breaks explicitly chiral symmetry, and states with opposite parities are not degenerate. In the case of a vector confining part, only quasi-stationary states are found.

We consider the most general type of the confining part, which taking into account also cases considered in \[8\]. Let us suggest that the static quark potential has vector and scalar properties of Lorenz transformation:

\[ V_{NR}(r) = V_v(r) + V_s(r). \] (35)

The quantity \( V \) means that the potential is a 4-th component of operator \( \hat{p}_\mu \), \( S \) means that the potential has scalar nature.

Following many authors \[9, 10\] we assume the mixture of vector-scalar quark interaction potential

\[ V(r) = V_{OGE} + \varepsilon V_{conf}, \quad V_s(r) = (1 - \varepsilon) V_{conf}, \] (36)

where \( V_{OGE} \) is the one-gluon exchange (OGE) term, \( V_{conf} \) is the confinement part of the potential, \( \varepsilon \) is mixing constant. Here the Lorenz nature of the one-gluon part of potential and the confining potential is different, the one-gluon potential is totally vector, while the confinement is a vector-scalar mixture. Very interesting reviews were done in \[11, 12\] concerning the choice of interaction potential.

To simplify the calculations we consider the simplest case when \( \varepsilon = 0 \). This corresponds to the one-gluon potential being totally vector and the confinement potential being totally scalar.

So we choose the scalar and vector parts of the potential (36) in the form \[9, 10, 11, 12\]

\[ V_{OGE}(r) = -\frac{\beta}{r}, \quad V_{conf}(r) = \mu r \] (37)

where \( \beta = \frac{4}{3} \alpha_s, \mu = 0.18 \text{ GeV}^2, \alpha_s \) is the constant of the strong coupling with asymptotic freedom:

\[ \alpha_s(r) = \frac{12\pi}{33 - 2N} \cdot \frac{1}{\ln(Q/\Lambda)^2}, \] (38)

where \( Q \) is the transmitted momentum, \( \Lambda \) is taken to be equal to \( \Lambda = 0.14 \text{ GeV}, N \) is the quarks flavours number.

In work \[13\] the WKB method was used for potential (37) in the Schrödinger equation. For solving of the Dirac equation system (2) with potential (37), we shall use early elaborated by us version of the WKB approximation.

Due to the confinement of quarks we are interested in only classically allowed region \( (r_0 < r < r_-, q^2 < 0, \text{ a potential well}) \) that corresponds to only the discrete energy spectrum of the quark-antiquark system. Thus the quasi-classical momentum (21) is of the form (\( \hbar = c = 1 \)):

\[ p(r) = \left( \left( E + \frac{\beta}{r} \right)^2 - (m + \mu r)^2 - \left( k/r \right)^2 \right)^{1/2}. \] (39)

We represent left part of the quantization condition (32) in the form of sum of two integrals \( I_1 \) and \( I_2 \)

\[ I_1 = \int_0^a p(r)dr, \quad I_2 = \int_0^a \frac{k}{r}w dr. \] (40)
The integration is between the two classical turning points \( r_0 = b \) and \( r_- = a \), which are real and positive roots \((a > b > 0)\) of the equation

\[
r^4 + \frac{2m}{\mu} r^3 - \frac{E^2 - m^2}{\mu^2} r^2 - \frac{2E\beta}{\mu^2} r + \frac{k^2 - \beta^2}{\mu^2} = 0.
\] (41)

Two roots of this equation are real and negatives \((d < c < 0)\). Formulae (40) can now be re-expressed in terms of \(a, b, c\) and \(d\) as

\[
I_1 = -\mu \int_{b}^{a} \frac{(r^3 + \frac{2m}{\mu} r^2 - \frac{E^2 - m^2}{\mu^2} r - \frac{2E\beta}{\mu^2} r + \frac{k^2 - \beta^2}{\mu^2})}{[(a-r)(r-b)(r-c)(r-d)]^{1/2}} dr
\] (42)

and

\[
I_2 = -\frac{k}{\mu (A-B)} \left[ \left( A + \frac{E + m}{2\mu} \right) \int_{b}^{a} \frac{dr}{(r-A)[(a-r)(r-b)(r-c)(r-d)]^{1/2}} - \left( B + \frac{E + m}{2\mu} \right) \int_{b}^{a} \frac{dr}{(r-B)[(a-r)(r-b)(r-c)(r-d)]^{1/2}} \right],
\] (43)

where \(A\) and \(B\), which are roots of the quadratic equation \(\mu r^2 + (E + m) r + \beta = 0\). These equations can be represented in terms of complete elliptic integrals. Then quantization condition (32) gives following result

\[
-\frac{2}{\sqrt{(a-c)(b-d)}} \left[ \frac{(b-c)^2}{(1-\nu)(\chi^2-\nu)} \left[ N_1 \cdot F(\chi) + N_2 \cdot E(\chi) + N_3 \cdot \Pi(\nu,\chi) + N_4 \cdot \Pi \left( \frac{c}{b\nu}, \chi \right) \right] + \frac{k}{\mu} \left[ \frac{b-c}{A-B} \left( N_5 \cdot \Pi(\nu_1,\chi) - N_6 \cdot \Pi(\nu_1,\chi) \right) + N_7 \cdot F(\chi) \right] \right] = \left( n + \frac{1}{2} \right) \pi,
\] (44)

where \(F(\chi), E(\chi)\) and \(\Pi(\nu,\chi)\) are complete elliptic integrals of the first, second and third kinds and

\[
N_1 = \left[ \frac{\chi^2}{4} - \frac{3\nu^2 - 2\nu - 2\nu\chi^2 + 3\chi^2}{8(1-\nu)} (b-c) - \left( m + \frac{3}{2} c\mu \right) (\chi^2 - \nu) + \frac{(1-\nu)(\chi^2 - \nu)}{(b-c)^2} \left( 2c^2 m + c^2 \mu + \frac{k^2 - \beta^2}{c\mu} - \frac{2\beta E}{\mu} - \frac{c(E^2 - m^2)}{\mu} \right) \right],
\]

\[
N_2 = -\nu \left[ m + \frac{3}{2} c\mu + \frac{3\nu^2 - 2\nu - 2\nu\chi^2 + 3\chi^2}{8(1-\nu)} (b-c) \right],
\]

\[
N_3 = \frac{3\mu}{8} \frac{(b-c)(\nu^2 - 2\nu - 2\nu\chi^2 + 3\chi^2)^2}{(1-\nu)(\chi^2 - \nu)} + \frac{(1-\nu)(\chi^2 - \nu)}{(b-c)} \left( 4cm + 3c^2 \mu - \frac{E^2 - m^2}{\mu} \right) + \left( m + \frac{3}{2} c\mu \right) \left( \nu^2 - 2\nu - 2\nu\chi^2 + 3\chi^2 \right) - \frac{\mu}{2} (b-c) \left( 3\chi^2 - (\chi^2 + 1) \nu \right),
\]

\[
N_4 = -\frac{(1-\nu)(\chi^2 - \nu)(k^2 - \beta^2)}{(b-c)} \frac{b\nu c}{c\mu}.
\]
\[ N_5 = \frac{A + \frac{E^{+m}}{2\mu}}{(b - A) (A - c)}, \quad N_6 = \frac{B + \frac{E^{+m}}{2\mu}}{(b - B) (B - c)}, \quad N_7 = \frac{c + \frac{E^{+m}}{2\mu}}{(A - c) (B - c)}, \]

\[ \chi = \sqrt{\frac{(a - b) (c - d)}{(a - c) (b - d)}}, \quad \nu = \frac{a - b}{a - c}, \quad \nu_1 = \frac{A - c}{A - b}, \quad \nu_2 = \frac{B - c}{B - b}. \]

Equation (44) is an implicit relation for the energy \( E_{\nu k} \).

Accordingly wave functions of light quarks in classically allowed region are defined by formulae (20), where normalization constant is of the form

\[ \left| C_{11}^\pm \right| = \left[ \frac{\mu \sqrt{(a - c) (b - d)}}{2 \left( (Ec + \beta) \cdot F(\chi) + E (b - c) \cdot \Pi (\nu, \chi) \right)} \right]^{1/2} \]

(45)

and phases \( \Theta_1 \) and \( \Theta_2 \) are

\begin{align*}
\Theta_1 &= -\frac{2}{\sqrt{(a - c) (b - d)}} \left[ \frac{(b - c)^2}{(1 - \nu) (\chi^2 - \nu)} \times \right.
\times \left[ N_1 \cdot F(\phi, \chi) + N_2 \cdot E(\phi, \chi) + N_3 \cdot \Pi(\phi, \nu, \chi) + N_4 \cdot \Pi\left(\phi, \frac{c}{b} \nu, \chi\right) + L \right] + \\
&\quad \frac{k}{\mu} \left[ \frac{b - c}{A - B} \left( N_5 \cdot \Pi(\nu, \nu_1, \chi) - N_6 \cdot \Pi(\phi, \nu_2, \chi) + N_7 \cdot F(\phi, \chi) \right) \right] + \frac{\pi}{4},
\end{align*}

(46)

\begin{align*}
\Theta_2 &= -\frac{2}{\sqrt{(a - c) (b - d)}} \left[ \frac{(b - c)^2}{(1 - \nu) (\chi^2 - \nu)} \times \right.
\times \left[ N_1 \cdot F(\phi, \chi) + N_2 \cdot E(\phi, \chi) + N_3 \cdot \Pi(\phi, \nu, \chi) + N_4 \cdot \Pi\left(\phi, \frac{c}{b} \nu, \chi\right) + L \right] - \\
&\quad \frac{k}{\mu} \left[ \frac{b - c}{A - B} \left( N_8 \cdot \Pi(\phi, \nu_1^*, \chi) - N_9 \cdot \Pi(\phi, \nu_2^*, \chi) + N_{10} \cdot F(\phi, \chi) \right) \right] + \frac{\pi}{4}.
\end{align*}

(47)

Here

\[ \phi = \arcsin \sqrt{\frac{(a - c) (r - b)}{(a - b) (r - c)}}, \]

\[ A_1, B_1 \] are roots of the quadratic equation \( \mu r^2 - (E - m) r - \beta = 0, \)

\[ L = \frac{\sin \phi \cdot \cos \phi \cdot \sqrt{1 - \chi^2 \sin^2 \phi}}{1 - \nu \sin^2 \phi} \times \]

\[ \times \left[ \frac{3 (b - c) (\nu^2 - 2 \nu - 2 \nu \chi^2 + 3 \chi^2)^2}{8 (1 - \nu) (\chi^2 - \nu)} + \frac{1}{4 \left( 1 - \nu \sin^2 \phi \right)} + \frac{m}{\mu (b - c)} \right], \]

\[ N_8 = \frac{A_1 - \frac{E^{-m}}{2\mu}}{b - A_1) (A_1 - c)}, \quad N_9 = \frac{B_1 - \frac{E^{-m}}{2\mu}}{(b - B_1) (B_1 - c)}, \quad N_{10} = \frac{c - \frac{E^{-m}}{2\mu}}{(A_1 - c) (B_1 - c)}, \]

\[ \nu_1^* = \frac{A_1 - c}{A_1 - b}, \quad \nu_2^* = \frac{B_1 - c}{B_1 - b}. \]
6. Quasi-independent quark model of hadrons

One of simple of meson quarks models, which, despite of singleness, gives the good agreement with experimental data, is the quasi-independent quark model [14, 15]. Instead of to consider a problem on the bound states of system of two quarks, intensive interacting one another, we shall consider quarks in a meson as independent particles, which move in mean or self-consistent field.

In such scheme not only wavefunction of all system of two quarks has a sense, but also individual one-particle wavefunctions of each quark. These one-particle wavefunctions are described by the usual Dirac equation for particle in some field. Thus, quarks, which are included in a structure of a meson, occupy one-particle energy levels. To simplify calculations it is assumed that this mean field is spherically symmetric and quark motion in space is determined by motion of its center. Each of constituents interacting with the spherical mean field gets the state with a definite value of its energy. On the phenomenological ground the mass formula for $J^{PC}$ meson, which is $n^{2S+1}L_J$ state of $qq$-system, can be represented in the following form:

$$M = \sqrt{p_1^2 + m_1^2} + \sqrt{p_2^2 + m_2^2}.$$  \hspace{1cm} (48)

In a system of the centre of mass ($|p_1| = |p_2| = p$) after simple mathematical transformation we have obtained the relation

$$\frac{M^2 + m_1^2 - m_2^2}{2M} = \hat{\alpha} p + \beta m_1 = E,$$  \hspace{1cm} (49)

from which

$$M = E + \sqrt{E^2 - m_1^2 + m_2^2}.$$  \hspace{1cm} (50)

Here $E$ is the eigenvalue of the one-particle Dirac equation [2].

7. Results and conclusions

The Dirac equation gives the spin-orbit splitting only into two levels while the experiment gives three levels. It is due to the fact that the individual quark spins can be composed in 0 or 1 and the total angular moment can be 0, 1 or 2 in the case $l = 1$. But in the limit of an infinitely heavy quark mass $m_2 \to \infty$ degrees of freedom of quarks system are determined by the quantum states of the light quark with the total angular moment $j = L + s$. This gives two sets of levels $j = 1/2$, $j = 3/2$. For example, in the hydrogen atom, we don’t take into account the nuclear spin: it is entered only as a hyperfine effect. For comparing our data with other data of therms with total moment we hold the following rule: $P_{1/2}$ was considered as being an averaged mixture of $^1P_1$ and $^3P_0$, while $P_{3/2}$ was considered as averaged mixture of $^3P_1$ and $^3P_2$ masses. This problem can have different solutions and it will be discussed in our next papers.

Everyone dealing with the Dirac equation encounters the problem of Lorenz structure of the potential. Our calculations demonstrate that considered one-gluon potential being totally vector and the confinement potential being totally scalar lead to incorrect spin-orbit splitting of $P$-levels: $M(P_{3/2}) < M(P_{1/2})$. For correct description of sign of spin-orbit splitting in the Dirac equation it is necessary to take a part of confinement potential as vector that corresponds to [36]. Hereinafter we will consider the case $\epsilon \neq 0$. This conclusion concerning the Lorenz character of the confinement agrees with other authors [16, 17].
References

[1] Landau L.D., Lifshitz E.M. Course of Theoretical Physics, Vol 3, Quantum mechanics. Non-relativistic theory (London-Paris: Pergamon Press, 1958).

[2] Migdal A.B. Qualitative methods in quantum theory (Moscow: Nauka, 1975).

[3] Heading J. An introduction to phase-integral methods (London: Methuen, 1962).

[4] Fröman N. and Fröman O. JWKB Approximation (Amsterdam: North-Holland Publishing Company, 1965).

[5] Mur V. and Popov V. Semiclassical approximation for the Dirac equation in strong fields // Yad. Fiz. – 1978. – 28. – P. 837.

[6] Greiner W., Müller B., and Rafelski J. Quantum Electrodynamics of Strong Fields (Berlin: Springer, 1985).

[7] Qiang Wen-Chao Bound states of the Klein-Gordon and Dirac equations for scalar and vector harmonic oscillator potentials // Chinese Physics,– 2002.– 11, No. 8.– P. 757.

[8] Mur V.D., Popov V.S., Simonov Yu.A. and Yurov V.P. Description of relativistic heavy-light quark-antiquark systems via Dirac equation // Zh. Eksp. Teor. Fiz.– 1994. – 105.– P. 3.

[9] Ebert D., Faustov R. N., Galkin V.O. Proc. Intern. Conf. ”Heavy Quark Physics 5”, Dubna, Russian, April 6–8 (2000).

[10] Lengyel V., Rubish V., Shpenik A. Use of the configuration interaction method to describe ”fine”-splitting in the bound two-quark systems // Ukr. Phys. J.– 2002.– 47, No. 5.– P. 508.

[11] Nora Brambilla Quark confinement and the hadron spectrum // Preprint HEPHY-PUB 696/98 UWThPh-1998-33.

[12] Simonov Yu. A. Confinement // Uspekhi Fiz. Nauk– 1996.– 166, No. 5.– P. 337.

[13] Seetharaman M., Raghavan S., Vasan S. Analytic WKB energy expression for the linear plus Coulomb potential // J. Phys. A.– 1983.– 16.– P. 455.

[14] Bogoliubov P. Equations for bound states (of quarks) // Elem. Chast. I Atom. Yad. – 1972.– 3, No. 1.– P. 144.

[15] Khurschev V.V., Savrin V.I. and Semenov S.V. On the parameters of the QCD-motivated potential in the relativistic independent quark model, hep-ph/0111055

[16] Crater H., Van Alstine // Phys.Rev. D – 1988. – 37. – P. 1982.

[17] Haysak I., Lengyel V., Shpenik A. Fine splitting in a potential model incorporating relativistic kinematics // Jour. Phys. Stad. – 1996.– 1, No. 1.– P. 42.