Master Field on Fuzzy Sphere

Tsunehide Kuroki

Institute of Physics, University of Tokyo, Komaba, Meguro-ku, Tokyo 153, Japan

Abstract

The $O(N)$ symmetric vector model is considered on both ordinary and fuzzy sphere. It is shown that in both cases master fields exist and their explicit forms are presented. They are found to mix the internal symmetry and the (fuzzy) space-time symmetry. It is also argued that the cutoff brought by the fuzzy sphere plays an essential role in constructing the master field.
1 Introduction

Noncommutative nature is one of the remarkable features of recent developments in non-perturbative string theory. For example, space-time coordinates of N D-branes should be treated as non-commuting \( N \times N \) matrices [1]. In particular, this description of coordinates of D-particles exhibits the noncommutative nature of space-time at the sub-string scale [2] where our conventional ideas of space-time cease to make sense. Therefore, D-particles are considered to be inherently non-local or fuzzy objects. Thus it seems that noncommutative geometry [3] is a mathematical tool fit in the nonperturbative description of string theory.

The large-\( N \) limit is the crucial point other than the noncommutativity of space-time in the nonperturbative formulations of string theory. It was conjectured [4] that M-theory in the infinite momentum frame can be defined as the large-\( N \) limit of a matrix quantum mechanics obtained from the ten-dimensional SYM theory by means of the reduction to 0 + 1 dimensional theory. It was also proposed [5] that the constructive definition of type IIB superstring is given by another matrix model which is the large-\( N \) reduced model of ten-dimensional SYM theory to a point. In relation to the noncommutative structure of space-time, it is worth noticing that a similar model to the latter was presented [6] as a theory based on the space-time uncertainty principle which incorporates a minimal length beyond which the ordinary description of space-time by a commutative geometry breaks down. In both cases, the large-\( N \) limit plays an essential role in matching the degrees of freedom of SYM theories with those of M-theory or type IIB theory.[7]

Master field [8] is one of the most appealing ideas in the context of the large-\( N \) field theory. It is well-known that the large-\( N \) limit of a certain model is governed by a single classical field called master field. It has the following remarkable features:

1. The master field links space-time symmetries with the internal symmetry which becomes large in the large-\( N \) limit.

2. Physically, the master field may be considered to represent the internal collective motions.

Owing to the first feature, space-time dependence of the master field is encoded into the internal degrees of freedom by making use of the internal symmetry which generates

\footnote{Another conjecture is proposed [7] that the sector of DLCQ of M-theory is exactly described by a finite-\( N \) SYM theory.}
the space-time symmetry transformations \[9\]. In relation to string theory, it has been long pointed out that the reduced large-\(N\) gauge theories \[10, 11, 12\] may be solvable and provide a formalism for discussing certain aspects of string theory \[13\], and there the master field plays a central role \[14, 15\]. The reduced model proposed in \[7\] is based on the large-\(N\) reduction of SYM theory to a point, which is nothing other than the master field of the theory. As for the second feature, we recall that higher dimensional D-branes may be regarded as bound states of D0-branes \[16\] and collective motions of lower dimensional D-branes give rise to fluctuation of higher dimensional one. Thus master fields of SYM theory may be relevant to dynamics of D-branes.

These observations tempt us to formulate the large-\(N\) SYM theory on more generic noncommutative geometry and to examine its master field. Such a formulation is expected to provide new insights into the nonperturbative definition of string theory as in \[4, 5, 6\]. As a first step in this project, it is instructive to formulate a simple large-\(N\) field theory on the known noncommutative geometry and to examine the existence and the property of the master field. In particular, it seems interesting to clarify the relation mentioned above between the space-time and the internal symmetry because in this case the space-time structure is fuzzy. In this paper, we take the fuzzy sphere as a noncommutative geometry which is defined in \[17\] and consider the \(O(N)\) symmetric vector model on it.

Another motivation is from the field theoretic point of view. The fuzzy sphere can also be regarded as one of the regularization schemes which manifestly preserves the rotational symmetry unlike the lattice regularization. Moreover, the supersymmetric version of the fuzzy sphere (‘fuzzy supersphere’) is constructed recently \[18\]. It should be noticed in that it provides regularization which manifestly preserves the supersymmetry which is difficult to preserve in the lattice regularization. Moreover, also from string theory point of view, if we intend to define nonperturbative superstring theory as the large-\(N\) limit of a certain model as in \[4, 5\], we expect this model to be manifestly supersymmetric and to be regularized by a minimal length because string theory is ultraviolet finite due to a minimal length. Noncommutative geometry including the fuzzy sphere naturally incorporates such a minimal length. Thus noncommutative geometry could be again the geometrical framework in which nonperturbative superstring theory should be described. However, only a few examples have been explored for field theories on the noncommutative geometry. Thus we believe that our model is useful to deepen our understanding of the field theory on the noncommutative geometry and its role as regularization of the theory.
The organization of this paper is as follows: in the subsequent section, we give a brief review of the formulation based on the idea of the master field. In section 3 we review the definition of the fuzzy sphere and of a field theory defined on it. In section 4, we consider the $O(N)$ symmetric vector model on the ordinary sphere as a commutative warm-up for examining the same model on the fuzzy sphere. We will give the explicit representation for the master field. Section 5 is a main part of this paper. There the $O(N)$ symmetric vector model is considered on the fuzzy sphere. It is shown that this model also has the ‘master field’. It is also seen that regularization by means of the fuzzy sphere is essential for the construction of the master field. The last section contains the conclusions and the discussions of further extensions of our work.

2 Master Field in the Large-$N$ Field Theory

In this section, we consider the generic large-$N$ field theory and present a formulation of it based on the idea of the master field following [19].

We begin by the Euclidean path integral representation for the partition function

$$Z = \int [d\phi] \exp \left( - \frac{1}{g^2} S[\phi] \right).$$  \hfill (2.1)

This can be formally written as

$$Z = \int dS \exp \left[ - \frac{1}{g^2} (S - g^2 \mathcal{J}(S)) \right],$$  \hfill (2.2)

$$\mathcal{J}(S) = \ln \int [d\phi] \delta(S - S[\phi]).$$  \hfill (2.3)

The entropy factor $\mathcal{J}(S)$ measures the volume of the action orbit $\mathcal{O}(S)$ which is a set of configurations with a given value ($=S$) of the action functional. Suppose that the action is invariant under some internal symmetry transformation whose degrees of freedom increase as $N$ becomes large. By definition, for any configurations $\tilde{\phi}$, the symmetry orbit $\mathcal{O}_I(\tilde{\phi})$ which consists of configurations given by the internal symmetry transformations of $\tilde{\phi}$ must be included in $\mathcal{O}(S[\tilde{\phi}])$. If we take the large-$N$ limit, the volume of the symmetry orbit increases in general and it is possible that the symmetry orbit effectively covers the whole action orbit for a suitable choice of $\tilde{\phi}$. Then the entropy can be computed only from the transformation property of the field:

$$\mathcal{J}(S[\tilde{\phi}]) \sim \ln \text{vol}(\mathcal{O}(S[\tilde{\phi}])) \sim \ln \text{vol}(\mathcal{O}_I(\tilde{\phi})).$$  \hfill (2.4)
We call such configurations \textit{maximal entropy configurations}. In many cases, one can take the large-$N$ limit so that $S_{\text{eff}} = S - g^2 J(S)$ increases in proportion to a positive power of $N$. There $\tilde{\phi}$ can be given as a solution to the saddle-point equation. This suggests the existence of a single dominant symmetry orbit for describing all amplitudes in the large-$N$ limit which is a candidate for the master field.

Based on the above idea, one can derive the equation of the master field $\tilde{\phi}$ \cite{19},

$$
\frac{\delta S[\tilde{\phi}]}{\delta \tilde{\phi}_i(x)} + g^2 \sum_{\alpha, \beta} M^{-1\alpha \beta}[\tilde{\phi}](\delta^\alpha \delta^\beta \tilde{\phi}(x))_i = 0, \quad (2.5)
$$

$$
M^{\alpha \beta}[\tilde{\phi}] = \int (\delta^\alpha \tilde{\phi}(x))_i (\delta^\beta \tilde{\phi}(x))_i dx, \quad (2.6)
$$

where $\delta^\alpha$ denotes the infinitesimal internal symmetry transformation.

3 \quad \textbf{The Fuzzy Sphere}

In this section, we briefly discuss the main characteristics of the fuzzy sphere and an example of a field theory on it. In what follows, we distinguish quantities defined on the ordinary sphere from those defined on the fuzzy sphere by denoting the former with the tilde. These arguments are largely based on \cite{17, 20}.

3.1 \quad \textbf{Definitions}

In this subsection we consider the sequence of algebras $\text{Mat}_M$ of $M \times M$ complex matrices and find that in the large-$M$ limit these geometries tend towards the algebra $\mathcal{C}(S^2)$ of smooth complex-valued functions on $S^2$. In other words, finite dimensional algebra $\text{Mat}_M$ for each $M$ can be regarded as finite truncation or regularization of infinite dimensional algebra $\mathcal{C}(S^2)$. We also see that this regularization corresponds to an ultraviolet cutoff. Since $\text{Mat}_M$ is of course a noncommutative algebra, the base space can be considered to be noncommutative. We refer such a space to \textit{the fuzzy sphere}.

Consider $\mathbb{R}^3$ with coordinates $\tilde{x}^a (a = 1, 2, 3)$ and the standard Euclidean metric $g_{ab} = \delta_{ab}$. The ordinary sphere is defined by

$$
\delta_{ab} \tilde{x}^a \tilde{x}^b = r^2. \quad (3.1)
$$

We associate these coordinate functions $\tilde{x}^a$ with the generators of $SU(2)$:

$$
\tilde{x}^a \mapsto x^a \equiv \kappa J^a, \quad (3.2)
$$

where $J^a$ be an $M$-dimensional irreducible representation\footnote{We need this irreducibility to make the derivations of $\text{Mat}_M$ complete. See below.} (spin $J = (M - 1)/2$ repre-
sentation) of the Lie algebra of $SU(2)$: $[J^a, J^b] = i\epsilon_{abc}J^c$, and $\kappa$ be a positive number defined by $4r^2 = (M^2 - 1)\kappa^2$ from which it follows that

$$\delta_{ab}x^a x^b = r^2 \cdot 1. \quad (3.3)$$

Let $\mathcal{P}$ be the algebra of analytic functions of $\bar{x}^a$ and $\mathcal{I}$ be its ideal consisting of all functions of a form $h(\bar{x}^a)(\sum_a \bar{x}^a)^2 - r^2)$. Then the quotient algebra $\mathcal{P}/\mathcal{I}$ is dense in the algebra $\mathcal{C}(S^2)$. Any element $\tilde{f} \in \mathcal{P}/\mathcal{I}$ can be represented as

$$\tilde{f} = \sum_{l=0}^{\infty} \frac{1}{l!} f_{a_1 \cdots a_l} \bar{x}^{a_1} \cdots \bar{x}^{a_l}, \quad (3.4)$$

where $f_{a_1 \cdots a_l}$ is traceless between any two indices and totally symmetric. Truncating the expansion at terms of order $M - 1$ and replacing $\bar{x}^a$ with $x^a$, we get an element of $\text{Mat}_M$ corresponding to $\tilde{f}$,

$$f = \sum_{l=0}^{M-1} \frac{1}{l!} f_{a_1 \cdots a_l} x^{a_1} \cdots x^{a_l} \in \text{Mat}_M. \quad (3.5)$$

This is the truncation map $\phi_M : \mathcal{C}(S^2) \to \text{Mat}_M$. If we define a quantity $k \equiv 2\pi \kappa r$ which has the dimension of (length)$^2$, it plays a role of a minimal length in the sense that the generators $x^a$ of the algebra $\text{Mat}_M$ satisfy

$$[x^a, x^b] = i \frac{k}{2\pi} C_{abc} x^c, \quad C_{abc} = \frac{1}{r} \epsilon_{abc}. \quad (3.6)$$

This means that for a finite $M$ the ‘coordinates’ $x^a$ of the fuzzy sphere are noncommutative, while in the limit $M \to \infty$, they commute with each other and all of the points of $S^2$ can be distinguished. So the ordinary $S^2$ can be recovered.

Next let us see that this truncation provides an ultraviolet cutoff for theories on the fuzzy sphere. If we define a norm on $\mathcal{C}(S^2)$ as

$$||\tilde{f}||^2 = \frac{1}{4\pi r^2} \int_{S^2} |\tilde{f}|^2,$$

where $f_{S^2}$ is an abbreviation for $\int r^2 \sin \theta d\theta d\varphi$, then one can take as an orthonormal basis the usual spherical harmonics $\{\hat{Y}_{lm}(\theta, \varphi)\}$ ($l \geq 0, -l \leq m \leq l$) defined by

$$\hat{Y}_{lm}(\theta, \varphi) \equiv \sqrt{4\pi} Y_{lm} \equiv \sqrt{4\pi N_{lm}} P_{l|m|}(\cos \theta) e^{im\varphi}, \quad (3.8)$$

where $Y_{lm}(\theta, \varphi)$ is the standard spherical harmonics, $N_{lm}$ is its normalization constant,

$$N_{lm} \equiv (-)^{\frac{m+|m|}{2}} \sqrt{\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}} \quad (3.9)$$
and $P_{lm}(\cos \theta)$ is an associated Legendre function. Here we note that solid harmonics $r^l \hat{Y}_{lm}(\theta, \varphi)$ is homogeneous polynomial of degree $l$ in variables $\tilde{x}^1 = r \sin \theta \cos \varphi$, $\tilde{x}^2 = r \sin \theta \sin \varphi$, and $\tilde{x}^3 = r \cos \varphi$. Therefore, $\{\hat{Y}_{lm}\}$ ($0 \leq l \leq M - 1, -l \leq m \leq l$) form a basis of the vector space which consists of functions on $S^2$ which can be expanded in terms of $\tilde{x}^a$ of order up to and including $M - 1$. Thus we have observed that the truncation to $\text{Mat}_M$ is nothing but a cutoff of higher angular momenta $l \geq M$.

For later convenience, we define an orthonormal basis of $\text{Mat}_M$ corresponding to $\hat{Y}_{lm}$. A norm on $\text{Mat}_M$ corresponding to (3.7) is defined as

$$||f||^2_M \equiv \frac{1}{M} \text{Tr}(f^\dagger f).$$

Then there exists an orthonormal basis $\{T_{lm}\}$ ($0 \leq l \leq M - 1, -l \leq m \leq l$)

$$||T_{lm}T_{lm'}||_M = \frac{1}{M} \text{Tr}(T_{lm}^\dagger T_{lm'}) = \delta_{ll'} \delta_{mm'},$$

which tends to $\hat{Y}_{lm}$ in the large-$M$ limit.

So far we have discussed that $\text{Mat}_M$ determines noncommutative geometry (the fuzzy sphere) and provides regularization of the ultraviolet divergences like a lattice field theory. However, unlike lattice regularization, it manifestly preserves the rotational symmetry $SO(3)$ of space-time, because $T_{lm}$ for fixed $l$ and $-l \leq m \leq l$ form a $(2l + 1)$-dimensional representation of $SO(3)$. Namely, for an $SO(3)$ rotation $R$,

$$U(R)T_{lm}U(R)^{-1} = \sum_{m'=-l}^l T_{lm'} R^l_{mm'}(R),$$

where $U(R)$ is a $M$-dimensional representation of $R$ and $R^l_{mm'}(R)$ are the rotation matrices for angular momentum $l$.

On $S^2$ we have vector fields

$$\tilde{e}_a = -C_{abc} \tilde{x}^b \partial_c,$$

which satisfy

$$\tilde{\Delta} = \delta^{ab} \tilde{e}_a \tilde{e}_b,$$

where $\tilde{\Delta}$ is the ordinary Laplacian on $S^2$ of radius $r$. It is argued in [17] that we can correspondingly define vector fields on the fuzzy sphere, i.e., derivations of $\text{Mat}_M$ as

$$e_a = \frac{2\pi}{ik} \text{ad}(x^a),$$

and Laplacian on the fuzzy sphere as

$$\Delta = \delta^{ab} e_a e_b,$$
which are shown to become $\tilde{e}_a$ and $\tilde{\Delta}$ in the large-$M$ limit respectively. The set of three derivations $e_a$ are complete in the sense that if $e_a f = 0$, then $f$ must be proportional to the unit matrix, which follows from the irreducibility of the representation of $SU(2)$ which defines $\text{Mat}_M$. As $\hat{Y}_{lm} (-l \leq m \leq l)$ are eigenfunctions of $\hat{\Delta}$:
\begin{equation}
\hat{\Delta} \hat{Y}_{lm} = \frac{l(l+1)}{r^2} \hat{Y}_{lm},
\end{equation}
so $T_{lm} (-l \leq m \leq l)$ are eigenmatrices of $\Delta$:
\begin{equation}
\Delta T_{lm} = \frac{l(l+1)}{r^2} T_{lm}.
\end{equation}

### 3.2 Field Theory

The ordinary real scalar field theory on the Euclidean sphere is defined by the partition function
\begin{equation}
\tilde{Z}[\tilde{J}] = \int_{C(S^2)} [d\tilde{f}] \, e^{-\tilde{S}(\tilde{f}, \tilde{J})},
\end{equation}
where $\tilde{J}$ is an appropriate external source and $\tilde{S}$ is an action
\begin{equation}
\tilde{S}(\tilde{f}, \tilde{J}) = \int_{S^2} \left( -\frac{1}{2} \tilde{f} \tilde{\Delta} \tilde{f} + \frac{1}{2} \mu_0^2 \tilde{f}^2 + V_{\text{int}}(\tilde{f}) + \tilde{J} \tilde{f} \right).
\end{equation}
The functional integral $\int_{C(S^2)} [d\tilde{f}]$ is performed over the infinite-dimensional space $C(S^2)$ and requires some regularization to make it well-defined. As discussed in the previous subsection, the fuzzy sphere is considered as a way of the ultraviolet regularization. Namely, replacing $C(S^2)$ with $\text{Mat}_M$ by the truncation map $\phi_M$, we consider corresponding to (3.19),
\begin{equation}
Z_M[J] = \int_{\text{Mat}_M} df \, e^{-S(f, J)},
\end{equation}
where $f, J$ are Hermitian $M \times M$ matrices which are the image of $\tilde{f}, \tilde{J}$ by $\phi_M$ respectively, and $\int_{\text{Mat}_M} df$ is understood to denote the integration over all of the components of the matrix $f$. Then for each $M$, $Z_M[J]$ is well-defined and is expected to become (3.19) in the large-$M$ limit. Moreover, $Z_M$ is in itself regarded as the definition of the quantum field theory on the fuzzy sphere.

Next let us give the definition of the correlation functions of the theory (3.22). On the analogy of the commutative case, we adapt the following definitions for the correlation
functions and the propagator:

\[
\langle f_1 f_2 \cdots f_n \rangle_J \equiv Z[J]^{-1} \frac{\delta^n Z[J]}{k \delta J \cdots k \delta J} \in \text{Mat}_M \otimes \cdots \otimes \text{Mat}_M, \tag{3.23}
\]

\[
G \equiv \frac{\delta^2 \ln W[J]}{k \delta J k \delta J} \in \text{Mat}_M \otimes \text{Mat}_M. \tag{3.24}
\]

Note that differentiating \( n \) times the function of \( \text{Mat}_M \) with respect to its element yields an element of the direct product of \( n \) copies of \( \text{Mat}_M \). For example, if \( V_{\text{int}} = 0 \), the free propagator (Green function) is given as

\[
G_0 = \frac{1}{k M} \sum_{l=0}^{M-1} \sum_{m=-l}^{l} \frac{T_{lm} \otimes T_{lm}^\dagger}{l(l+1) + \mu_0^2}, \tag{3.25}
\]

which tends to the Green function on the sphere

\[
\tilde{G}_0(x, y) = \frac{1}{4 \pi r^2} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{\hat{Y}_{lm}(x) \hat{Y}_{lm}^*(y)}{l(l+1) + \mu_0^2}, \tag{3.26}
\]

in the large-\( M \) limit.

Let us conclude this section by a comparison of two theories regularized by a naive angular momentum cutoff and by the fuzzy sphere. Although both theories preserve the rotational symmetry manifestly, it makes an important difference that the former apparently fails to preserve the angular momentum while the ‘fuzzy angular momentum’ conservation law holds in the latter in the sense that \( \{T_{lm}\} (0 \leq l \leq M - 1, -l \leq m \leq l) \) form a closed algebra.

4 \( O(N) \) Symmetric Vector Model on the Sphere

In this section, the \( O(N) \) symmetric vector model is considered on the sphere of the radius \( r \). Using the results in section 2, we construct the large-\( N \) master field explicitly and show that it in fact reproduces the \( O(N) \) invariant two-point function. Of course, these results can be regarded as the commutative limit of those of the \( O(N) \) symmetric vector model on the fuzzy sphere considered in the next section.

4.1 Large-\( N \) Limit

The action we consider is

\[
S = \int_{S^2} \left( -\frac{1}{2} \phi_i \Delta \phi_i + \frac{1}{2} \mu_0^2 \phi_i^2 + \frac{g_0}{4N} (\phi_i^2)^2 \right), \tag{4.1}
\]
where \( \tilde{\Delta} \) is the Laplacian on the sphere given in (3.14) and \( \phi_i \) is a real scalar field which transforms as a component of a \( N \)-dimensional vector under the \( O(N) \) rotation. Note that the sphere is imbedded in \( \mathbb{R}^3 \) with the Euclidean metric so that it is possible to apply the formulation introduced in section 2 and to extend it to the fuzzy sphere.\(^1\)

Introducing the auxiliary field \( \sigma(\theta, \varphi) \) and performing the Gaussian integration over the \( N \)-component field \( \phi_i \), we obtain the following expression for the partition function,

\[
Z = \int \mathcal{D}\sigma e^{-\frac{N}{2}S_{\text{eff}}}, \tag{4.2}
\]

\[
S_{\text{eff}} = \int_{S^2} \left( -\frac{1}{2}g_0\sigma^2 + \frac{1}{4\pi r^2} \log \frac{\det(-\tilde{\Delta} + \mu_0^2 + g_0\sigma)}{\det(-\tilde{\Delta} + \mu_0^2)} \right), \tag{4.3}
\]

where we added a constant to \( S_{\text{eff}} \) so that \( S_{\text{eff}}|_{\sigma=0} = 0 \). In the large-\( N \) limit, the dominant contribution to the integral comes from the rotationally invariant saddle point \( \sigma(\theta, \varphi) = \sigma \) which is a solution of the saddle point equation

\[
0 = \frac{\partial S_{\text{eff}}}{\partial \sigma} = -g_0\sigma + \frac{g_0}{4\pi r^2} \text{Tr} \frac{1}{-\tilde{\Delta} + \mu_0^2 + g_0\sigma}, \tag{4.4}
\]

and hence the gap equation

\[
\sigma = \frac{1}{4\pi r^2} \sum_{l=0}^{L} \sum_{m=-l}^{l} \frac{1}{l(l+1)/r^2 + \mu_0^2 + g_0\sigma} = \frac{1}{4\pi r^2} \sum_{l=0}^{L} \frac{2l+1}{l(l+1)/r^2 + \mu_0^2 + g_0\sigma}, \tag{4.5}
\]

where we have introduced the cutoff \( L \in \mathbb{Z} \) for the angular momentum.

Next let us calculate the two-point function (propagator) \( \tilde{G}_{ij}(x, y; \mu_0^2) = \langle \phi_i(x)\phi_j(y) \rangle \) \( (x, y \in S^2) \) in the large-\( N \) limit. Making the Feynman graph expansion shows that the leading contributions to the propagator are “tree chain diagrams”, namely, randomly branching polymers. Thus let \( \sigma_0 \) be the sum of all connected tree chain diagrams, then we can express \( \tilde{G}_{ij}(x, y; \mu_0^2) \) as

\[
\tilde{G}_{ij}(x, y; \mu_0^2) = \frac{\delta_{ij}}{4\pi r^2} \sum_{l=0}^{L} \sum_{m=-l}^{l} \frac{\hat{Y}_{lm}(x)\hat{Y}_{lm}(y)}{l(l+1)/r^2 + \mu_0^2 + g_0\sigma_0}. \tag{4.6}
\]

Consistency condition on \( \sigma_0 \) requires that \( \sigma_0 \) is given as a solution to the gap equation.

### 4.2 Master Field on the Sphere

Now let us apply the formulation developed in section 2 to our model and derive the master field. In the present case, the master field equation of motion eq.(2.5) becomes

\[
0 = (-\tilde{\Delta} + 2V'(\phi(x)^2))\phi_i(x) + \sum_{\alpha,\beta} M^{-1}_{\alpha\beta} [\tilde{\phi}] (\tau^\alpha \tau^\beta \phi(x))_i, \tag{4.7}
\]

\[
M^{\alpha\beta} = -\int_{S^2} \phi_i(x)(\tau^\alpha \tau^\beta \phi(x))_i; \quad \tau^\alpha : \text{infinitesimal generator of } O(N)
\]

\(^1\)There is at present no satisfactory noncommutative version of Minkowski space.
\[ V(x) = \frac{1}{2}\mu_0^2 x + \frac{g_0}{4N} x^2. \]  

(4.8)

In order to solve this equation, we adopt a following ansatz:

in an appropriate base, the master field (i.e. a solution to the above equation) \( \bar{\phi}_i(x) \) is given by

for \( l^2 < i < (l+1)^2 \) \( (l \in \mathbb{N} \cup \{0\}) \),

\[
\bar{\phi}_i(\theta, \varphi) = \begin{cases}
\sqrt{2}c_l \text{Re} \hat{Y}_{l} \frac{1}{2}((l+1)\cdot \text{i}) (\theta, \varphi) & \text{if } i - l^2 \text{ : odd} \\
\sqrt{2}c_l \text{Im} \hat{Y}_{l} \frac{1}{2}((l+1)\cdot (i-1)) (\theta, \varphi) & \text{if } i - l^2 \text{ : even},
\end{cases}
\]

(4.9)

and for \( i = (l+1)^2 \),

\[
\bar{\phi}_{(l+1)^2}(\theta, \varphi) = c_l \hat{Y}_{l0}(\theta, \varphi),
\]

(4.10)

where \( \hat{Y}_{lm}(\theta, \varphi) \) is spherical harmonics defined in section 3. Note that \( \hat{Y}_{l0} = \hat{Y}^{*}_{l0} \in \mathbb{R} \). To be more explicit,

\[
\bar{\phi} = \sqrt{2} \begin{pmatrix}
\frac{c_0}{\sqrt{2}} \hat{Y}_{00} \\
c_1 \text{Re} \hat{Y}_{11} \\
c_1 \text{Im} \hat{Y}_{11} \\
\frac{c_1}{\sqrt{2}} \hat{Y}_{10} \\
c_2 \text{Re} \hat{Y}_{22} \\
c_2 \text{Im} \hat{Y}_{22} \\
c_2 \text{Re} \hat{Y}_{21} \\
c_2 \text{Im} \hat{Y}_{21} \\
\vdots \\
\end{pmatrix}.
\]

(4.11)

Note that \( \bar{\phi}_i(\theta, \varphi) \) is a rotational invariant field in the sense that there exists an orthogonal matrix \( O(\theta', \varphi') \in O(N) \) such that

\[
\bar{\phi}_i(\theta + \theta', \varphi + \varphi') = O_{ij}(\theta', \varphi') \bar{\phi}_j(\theta, \varphi).
\]

(4.12)

Substituting the ansatz into the master field equation of motion (4.7) and picking up the \( l^2 < i \leq (l+1)^2 \) \( (l \in \mathbb{N} \cup \{0\}) \) component yields

\[
0 = \left( \frac{l(l+1)}{r^2} + 2V'(\bar{\phi}^2) \right) \bar{\phi}_i + \sum_{\alpha\beta} M^{-1\alpha\beta} [\bar{\phi}] (\tau^\alpha \tau^\beta \bar{\phi})_i.
\]

(4.13)

Here it is worth noticing that \( \bar{\phi}_i^2(\theta, \varphi) \) is in fact independent of \( \theta, \varphi \):

\[
\bar{\phi}_i^2(\theta, \varphi) = \sum_l c_l^2 (2l + 1),
\]

(4.14)
where the well-known formula for the spherical harmonics \( \tilde{Y}_{lm}(\theta, \varphi) \) is employed:

\[
\sum_{m=-l}^{l} \tilde{Y}_{lm}^*(\theta, \varphi) \tilde{Y}_{lm}(\theta, \varphi) = 2l + 1. \tag{4.15}
\]

This relation enables us to rewrite the master field equation of motion as

\[
\left( \frac{l(l+1)}{r^2} + 2V'(\sum_{l}(2l+1)c_l^2) \right) \tilde{\phi}_i + \sum_{\alpha\beta} M^{-1\alpha\beta}[\tilde{\phi}](\tau^\alpha \tau^\beta \tilde{\phi})_i = 0. \tag{4.16}
\]

To make the field given by the ansatz a maximum-entropy configuration, the set of \((l, m)\) must span the whole angular-momentum number and magnetic quantum number space without degeneracy in the large-\(N\) limit. Then we have

\[
M^{\alpha\beta} = -\int S^2 \tilde{\phi}_i(\theta, \varphi)(\tau^\alpha \tau^\beta \tilde{\phi})_i(\theta, \varphi) = -4\pi r^2(c_l^2 + c_k^2)\delta^{\alpha\beta}, \tag{4.17}
\]

where the index \(\alpha\) is understood to denote the \(O(N)\) rotation in the \(ij\)-plane, and \(l^2 < i \leq (l+1)^2, k^2 < j \leq (k+1)^2\). Thus picking up the component for \(l^2 < i \leq (l+1)^2\), eq.(4.16) takes the form

\[
\left( \frac{l(l+1)}{r^2} + 2V'((\sum_{k}(2k+1)c_k^2)) \right) c_l = \frac{1}{4\pi r^2} \left( \sum_{k} \frac{(2k+1)c_k}{c_l^2 + c_k^2} - \frac{1}{2c_l} \right). \tag{4.18}
\]

We find in the large-\(N\) limit, \(c_l\) is given by

\[
c_l^2 = \frac{N}{4\pi r^2} \frac{1}{\frac{l(l+1)}{r^2} + 2V'((\sum_{l}(2l+1)c_l^2))}. \tag{4.19}
\]

Thus we obtain the gap equation in the following form:

\[
\sigma = \frac{1}{4\pi r^2} \sum_{l=0}^{L} \frac{1}{\frac{l(l+1)}{r^2} + 2V'(N\sigma)} = \frac{1}{4\pi r^2} \sum_{l=0}^{L} \frac{2l + 1}{\frac{l(l+1)}{r^2} + \mu_0^2 + g_0\sigma}. \tag{4.20}
\]

where \(\sigma = \sum_l(2l+1)c_l^2/N\). This indeed coincides with the gap equation (4.3) derived by means of the saddle point method. Using (4.3), (4.4), and (4.19), we can also verify that the field given by (4.11) reproduces the \(O(N)\) invariant two-point function

\[
\frac{1}{N} \sum_{i} \langle \phi_i(x)\phi_i(y) \rangle = \frac{1}{4\pi r^2} \sum_{l=0}^{L} \sum_{m=-l}^{l} \frac{\hat{Y}_{lm}(x)\hat{Y}_{lm}^*(y)}{\frac{l(l+1)}{r^2} + \mu_0^2 + g_0\sigma_0} = \frac{1}{N} \sum_{i} \tilde{\phi}_i(x)\tilde{\phi}_i(y). \tag{4.21}
\]

Thus (4.11) explicitly realizes the master field of the \(O(N)\) symmetric vector model on the sphere. To the best of our knowledge, it is the first example of the master field of linear \(\sigma\)-model on the sphere.
We mention some remarks concerning the master field (4.11). As we mentioned in
the introduction, it mixes the internal and space-time rotational symmetry. In fact, we
have one-to-one correspondence between the internal symmetry index $i$ and the pair of
the angular momentum and the magnetic quantum number $(l, m)$ within the master field
(4.9), (4.10). It can be regarded as one of the consequence of (4.12). Next we note the
relationship between the existence of the master field on the sphere and the cutoff of the
theory. In view of the form of (4.11), it is clear that the angular momentum cutoff $L$
is closely related to $N$. Therefore, we encounter a subtlety in taking the large-$N$
limit and the limit $L \to \infty$. In the next section, we show that considering the theory on the fuzzy
sphere settles this problem.

5 Master Field on the Fuzzy Sphere

In this section, we consider the master field for the $O(N)$ symmetric vector model on the
fuzzy sphere and discuss its large-$N$ limit. Then following the formulation in section 2, it
is shown that this model also has a master field similar to (4.11).

5.1 $O(N)$ Symmetric Vector Model on the Fuzzy Sphere

Combining the field theory (3.22) presented in section 3 and the vector model (4.1) in
section 4, we consider the $O(N)$ symmetric vector model on the fuzzy sphere defined by
the action

$$S = k \text{Tr} \left[ -\frac{1}{2} \phi_i \Delta \phi_i + \frac{1}{2} \mu^2 \phi_i^2 + \frac{g_0}{4N} (\phi_i^2)^2 \right], \quad (5.1)$$

where $\phi_i$ is an $M \times M$ Hermitian matrix for each $i$ and transforms under the $O(N)$ rotation as

$$\phi_i \to \phi_i' = O_{ij} \phi_j, \quad O \in O(N). \quad (5.2)$$

This action has an above $O(N)$ rotational symmetry as well as an $SU(2)$ symmetry

$$\phi_i \to \phi_i' = \phi_i + \varepsilon e_a \phi_i = \phi_i + \frac{2\pi \varepsilon}{ik} [x_a, \phi_i], \quad (5.3)$$

which is a remnant of the conformal symmetry on the sphere.

The equation of motion is

$$(\Delta - \mu^2) \phi_i = \frac{g_0}{2N} (\phi_i (\phi_j^2) + (\phi_j^2) \phi_i), \quad (5.4)$$

and the free propagator is $G_{0ij} = \delta_{ij} G_0$ where $G_0$ is given in (3.25).
Let us calculate the two-point function in the large-$N$ limit according to the definition in section 3. For this purpose, introducing an external source of Hermitian matrix $J_i \in \text{Mat}_M$ for each $i$ and an auxiliary Hermitian matrix $\sigma \in \text{Mat}_M$ in the same way as in sections 3, 4, the action becomes

$$S(\phi_i, \sigma, J_i) = k\text{Tr} \left[ -\frac{1}{2} \phi_i \Delta \phi_i + \frac{1}{2} \mu_0^2 \phi_i^2 + J_i \phi_i - \frac{1}{4} Ng_0 \sigma^2 + \frac{1}{2} g_0 \phi_i^2 \sigma \right].$$

(5.5)

The classical field $\phi^c_i$ is a solution to the equation of motion derived from this action

$$(\Delta - \mu_0^2)\phi^c_i = J_i + \frac{1}{2} g_0 (\phi^c_i \sigma + \sigma \phi^c_i).$$

(5.6)

Expanding the action around $\phi^c_i$ and integrating over fluctuations, we obtain the partition function in the presence of the source

$$Z[J] = (2\pi)^{N M^2 / 2} \int \text{d}\sigma \left( \text{det}(-k(-\Delta + \mu_0^2 + g_0 \sigma)) \right)^{-N / 2} \exp \left[ -k\text{Tr} \left( \frac{1}{2} J_i \phi^c_i - \frac{1}{4} Ng_0 \sigma^2 \right) \right].$$

(5.7)

In the large-$N$ limit, $\sigma$ in (5.7) is reduced to a matrix proportional to the unity which is determined as a ‘rotationally invariant’ saddle point of an effective action (see below) in the sense that $e_a \sigma = 0$. Note that as mentioned in section 3, the irreducibility of the representation of $x^a$ is important for the existence of such a matrix. Thus given such a scalar matrix $\sigma_0$, $\phi^c_i$ is the solution to the equation of motion in the large-$N$ limit,

$$(\Delta - \mu_0^2)\phi^c_i = J_i + g_0 \sigma_0 \phi^c_i,$$

(5.8)

being given by

$$\phi^c_i = -k\text{Tr}_2 (\delta_{ij} G_{\sigma_0} \cdot 1 \otimes J_j),$$

(5.9)

where the subscript on the trace indicates that it operates on the second factor in the tensor product and

$$G_{\sigma_0} \equiv \frac{1}{k M} \sum_{l=0}^{M-1} \sum_{m=-l}^{l} \frac{T_{lm} \otimes T_{lm}^\dagger}{\mu_0^2 + g_0 \sigma_0 + \mu_0^2 + g_0 \sigma_0}.$$ 

(5.10)

Thus using (5.9), we obtain in the large-$N$ limit,

$$W[J] = -\frac{k^2}{2} \text{Tr}_{1,2} (\delta_{ij} G_{\sigma_0} \cdot J_i \otimes J_j) + W[J = 0],$$

(5.11)

which leads to the two-point function

$$-\frac{\delta W[J]}{k \delta J_i k \delta J_j} = \delta_{ij} G_{\sigma_0}.$$

(5.12)
Next let us consider the gap equation. Setting $J_i = 0$, the partition function (5.7) becomes

$$
Z = \int d\sigma \exp(-\frac{N}{2} S_{\text{eff}}),
$$

(5.13)

$$
S_{\text{eff}} = k \text{Tr} \left[ -\frac{1}{2} g_0 \sigma^2 + \frac{1}{kM} \log \frac{\det(k(-\Delta + \mu_0^2 + g_0 \sigma))}{\det(k(-\Delta + \mu_0^2))} \right].
$$

(5.14)

Let $\sigma_0$ be the above-mentioned scalar matrix, it satisfies the saddle point equation

$$
0 = \frac{\delta S_{\text{eff}}}{\delta \sigma} \bigg|_{\sigma = \sigma_0} = -g_0 \sigma_0 + \frac{g_0}{kM} \text{Tr} \frac{1}{-\Delta + \mu_0^2 + g_0 \sigma_0} \cdot 1.
$$

(5.15)

Thus we have the gap equation

$$
\sigma_0 = \frac{1}{kM} \text{Tr} \frac{1}{-\Delta + \mu_0^2 + g_0 \sigma_0} \cdot 1.
$$

(5.16)

This equation can be regarded as the noncommutative analog of the gap equation (4.5) of the vector model on the sphere. In fact, it is easy to find that (5.16) leads to (4.5) in the commutative limit.

### 5.2 Master Matrix

In this subsection we construct the master matrix which reproduces the two-point function calculated above. Applying the master field equation (2.5) to the present case, the master matrix is a solution to

$$
0 = -\Delta \phi_i + \mu_0^2 \phi_i + \frac{g_0}{2N} (\phi_i \phi_j^2 + \phi_j^2 \phi_i) + \sum_{\alpha,\beta} M^{-\alpha\beta} [\phi](\tau^\alpha \tau^\beta \phi)_i,
$$

(5.17)

$$
M^{\alpha\beta} = -k \text{Tr} \left[ \phi_i (\tau^\alpha \tau^\beta \phi) \right], \quad \tau^\alpha : \text{infinitesimal generator of } O(N).
$$

In order to solve this equation, we adopt a following ansatz:

the master matrix is obtained by replacing $\hat{Y}_{lm}$ with $T_{lm}$ in the master field constructed in section 4. Namely, in an appropriate base, the master matrix $\tilde{\phi}_i$ is given by

for $l^2 < i < (l + 1)^2$ (0 ≤ $l$ ≤ $M - 1$),

$$
\tilde{\phi}_i = \begin{cases} 
\sqrt{2}c_l \left( T_i \frac{1}{2}(l+1)^2 - i \right) + T_i^\dagger \frac{1}{2}(l+1)^2 - i \right) \bigg/ 2 & \text{if } i - l^2 : \text{ odd,} \\
\sqrt{2}c_l \left( T_i \frac{1}{2}(l+1)^2 - (i-1) \right) - T_i^\dagger \frac{1}{2}(l+1)^2 - (i-1) \right) \bigg/ (2i) & \text{if } i - l^2 : \text{ even,}
\end{cases}
$$

(5.18)

for $i = (l + 1)^2$,

$$
\tilde{\phi}(l+1)^2 = c_l T_{l0},
$$

(5.19)

and for $i > M^2$,

$$
\tilde{\phi}_i = 0.
$$

(5.20)
where \( \{T_{lm}\} \) is the orthonormal basis defined in section 3. Note that \( T_{00} \) is a Hermitian matrix. To be more explicit, for large \( N \),

\[
\bar{\phi} = \sqrt{2} \begin{pmatrix}
\frac{c_0 T_{00}}{\sqrt{2}} \\
\frac{c_1 (T_{11} + T_{11}^\dagger)}{2} \\
\frac{c_1 (T_{11} - T_{11}^\dagger)}{(2i)} \\
\frac{c_1 T_{10}}{\sqrt{2}} \\
\frac{c_2 (T_{22} + T_{22}^\dagger)}{2} \\
\frac{c_2 (T_{22} - T_{22}^\dagger)}{(2i)} \\
\frac{c_2 (T_{21} + T_{21}^\dagger)}{2} \\
\frac{c_2 (T_{21} - T_{21}^\dagger)}{(2i)} \\
\frac{c_2 T_{20}}{\sqrt{2}} \\
\vdots \\
\frac{c_{M-1} (T_{M-11} + T_{M-11}^\dagger)}{2} \\
\frac{c_{M-1} (T_{M-11} - T_{M-11}^\dagger)}{(2i)} \\
\frac{c_{M-1} T_{M-10}}{\sqrt{2}} \\
0 \\
\vdots \\
\vdots \\
\vdots \\
\end{pmatrix}.
\] (5.21)

It is important that the components of this master matrix for \( i > M \) are zero as explicitly shown in (5.21). Therefore, it makes difference between the vector models on the ordinary and the fuzzy sphere in such a way that in the latter case we do not encounter the subtlety in taking the large-\( N \) limit mentioned in the last paragraph in section 4.

\( \bar{\phi}_i \) is a 'rotational invariant' matrix in the sense that for an \( SO(3) \) rotation \( R \), there exists an orthogonal matrix \( O \in O(N) \) such that

\[
U(R) \bar{\phi}_i U(R)^{-1} = O_{ij} \bar{\phi}_j,
\] (5.22)

where \( U(R) \) is an \( M \)-dimensional representation of the rotation \( R \). This follows from (3.12) by picking up the \( l^2 < i \leq (l + 1)^2 \) components for each \( l \). Substituting this ansatz into (5.17) and considering the \( l^2 < i \leq (l + 1)^2 \) component, we have

\[
0 = -\Delta \bar{\phi}_i + \mu_0^2 \bar{\phi}_i + \frac{g_0}{2N} (\bar{\phi}_i \bar{\phi}_j^2 + \bar{\phi}_j^2 \bar{\phi}_i) + \sum_{\alpha,\beta} M^{-1\alpha\beta}[\bar{\phi}](\sigma^\alpha \sigma^\beta \bar{\phi})_i.
\] (5.23)

It is important that, similarly to (4.14), \( \bar{\phi}_i^2 \) satisfies the following equation:

\[
\bar{\phi}_i^2 = \sum_{l=0}^{M-1} c_l^2 (2l + 1) \cdot 1,
\] (5.24)
where we used the fact $T_{lm}^\dagger = (-)^m T_{l-m}$ and the following formula analogous to (4.13):

$$\sum_{m=-l}^l T_{lm}^\dagger T_{lm} = (2l + 1) \cdot 1.$$  \hspace{1em} (5.25)

Then the master field equation leads to the following equation for $c_l$:

$$\left( \frac{l(l+1)}{r^2} + 2V' \sum_{k=0}^{M-1} (2k + 1)c_k^2 \right) c_l = \frac{1}{kM} \left( \sum_k \frac{(2k + 1)c_k}{c_l^2 + c_k^2} - \frac{1}{2c_l} \right),$$  \hspace{1em} (5.26)

where it is used that

$$M^{\alpha\beta} = -k \text{Tr} \bar{\phi}_i (\tau^\alpha \tau^\beta \phi)_i = -kM(c_l^2 + c_k^2)\delta^{\alpha\beta},$$  \hspace{1em} (5.27)

which can be shown in the same way as (4.17) and $V(x)$ is given by (4.8). We find in the large-$N$ limit, $c_l$ is given by

$$c_l^2 = \frac{N}{kM} \frac{1}{\frac{l(l+1)}{r^2} + 2V'(\sum_l (2l + 1)c_l^2)}.$$  \hspace{1em} (5.28)

Thus again we obtain the gap equation in the following form:

$$\sigma = \frac{1}{kM} \sum_{i=1}^{N} \frac{1}{\frac{l(l+1)}{r^2} + 2V'(N\sigma)}$$

$$= \frac{1}{kM} \sum_{l=0}^{M-1} \frac{2l + 1}{\frac{l(l+1)}{r^2} + \mu_0^2 + g_0\sigma},$$  \hspace{1em} (5.29)

where $\sigma = \sum_{l=0}^{M-1} (2l + 1)c_l^2/N$. This equation again agrees with the gap equation (5.16). Given the solution to the gap equation $\sigma_0$, the master matrix (5.21) indeed reproduces the $O(N)$ invariant two-point function

$$\frac{1}{N} \sum_i \langle \phi_i \phi_i \rangle \equiv -\frac{1}{2N} \sum_i \frac{\delta W[J]}{k\delta J_i k\delta J_i} = G_{\sigma_0} = \frac{1}{N} \sum_i \bar{\phi}_i \otimes \phi_i,$$  \hspace{1em} (5.30)

where we used eqs. (5.12), (5.28). Moreover, it can be checked that the rules analogous to the Feynman diagram expansion hold even in the noncommutative case.\footnote{\hspace{30pt}For example, performing the perturbative expansion in terms of $g_0$, it is found that the ‘connected’ two-point function $\langle \phi_i \phi_j \rangle_{pq,rs}$ includes a following term in the order of $g_0^2$,

$$-2g_0 kM \delta_{ij} \left( \frac{1}{M} \text{Tr} (p(G_0)) \right) (G_0 * G_0)_{pq,rs},$$

where $G_0$ is a free propagator (3.25), $\ast$ is defined as $(X \ast Y)_{pq,rs} = 1/M \sum_{tu} X_{pq,tu} Y_{ut,rs}$ and $p$ is a map $p: \text{Mat}_M \otimes \text{Mat}_M \ni f \otimes g \mapsto fg \in \text{Mat}_M$. It tends to in the commutative limit,

$$-2g_0 \delta_{ij} \Delta_F(0) \int_{S^2} dz \Delta_F(x_1 - z) \Delta_F(z - x_2).$$}

Therefore,
by means of the standard argument based on the Feynman diagram expansion, we can show that the theory considered in this section also has the factorization property and that all $O(N)$ invariant correlation functions can be made from the two-point function (5.30). Thus we can conclude that (5.21) is a master field of the theory.

Finally we wish to mention some important points as to the master matrix. In view of (5.21), we find that the internal $O(N)$ symmetry index $i$ is again associated with the pair of $(l, m)$. Although $(l, m)$ is now no more than the label of the matrices which form the basis of the representation of $SU(2)$, it can be regarded as the ‘fuzzy space-time’ index which tends to the angular momentum in the commutative limit. Thus the master matrix mixes the internal symmetry and the fuzzy space-time rotational symmetry.

As stated in section 3, the fuzzy sphere is regularization which manifestly preserves the rotational symmetry. This is crucial for the construction of the master matrix because the master matrix (5.21) is composed of the representations of the $SO(3)$ rotation. Therefore, it seems that the fuzzy sphere provides a sort of regularization scheme suited to formulate the master field.

### 6 Conclusions and Discussions

We considered the large-$N$ limit of the $O(N)$ symmetric vector model both on ordinary and fuzzy sphere and found that both theories have master fields. We also gave the explicit formulas of them and observed that they connect the internal $O(N)$ symmetry with (fuzzy) space-time symmetry. We also pointed out that the ultraviolet cutoff brought by the fuzzy sphere is essential for the existence of the master field in the large-$N$ limit.

We can think of several possible extensions and applications of our work. It is interesting to define our model on other fuzzy surfaces and observe the relation between the internal symmetry and the fuzzy space-time symmetry via the master field. Among others, we would like to mention the extension to the fuzzy torus. A fuzzy version of the torus [21] is constructed by introducing matrices $U, V$ which satisfy the Weyl relation

$$UV = qVU,$$  \hspace{1cm} (6.1)

as well as the constraints

$$U^M = 1, \quad V^M = 1, \quad q = e^{2\pi i/M}. \hspace{1cm} (6.2)$$

Like the fuzzy sphere, the fuzzy torus provides an ultraviolet cutoff of the space-time momentum. It is well known that the master field of the $O(N)$ symmetric vector model
defined in a box with periodic boundary conditions exists and links the internal symmetry index to the space-time momentum \([19, 22]\). Since Fourier modes in the two-dimensional box correspond to the powers of the matrices \(U, V\) on the fuzzy sphere, we can expect that there exists the master field of the \(O(N)\) vector model on the fuzzy torus and that it associates the internal symmetry index with the powers of the matrices \(U, V\).

Unfortunately, it is only in the particular cases of the sphere and the torus that the fuzzy versions of compact surfaces are known. It is important to search for a matrix description of the fuzzy surface of higher genus.

We have considered the model with \(M\) fixed and \(N \to \infty\) for the purpose of the search for the master fields on the noncommutative geometry. However, from the field theoretic point of view, we should take the limit \(M \to \infty\) as \(M\) plays a role of the cutoff. Therefore, it is worthwhile to investigate the model in taking the large-\(N\) and large-\(M\) limit simultaneously in a suitable way and to compare it with the double scaling limit in \(O(N)\) vector model in two dimension \([25]\).

Recently, ‘fuzzy supersphere’ is proposed in \([18]\). It is remarkable that the fuzzy supersphere is found to provide the regularization which manifestly preserves supersymmetry. It would be interesting to define our models on the fuzzy supersphere and to examine the existence of a ‘master superfield' and the connection between the internal and the space-time symmetry, in particular, supersymmetry.

In relation to the nonperturbative formulation of string theory, it is shown that from the Matrix theory \([4, 23]\) or IIB matrix models \([3, 3, 24]\) point of view, the noncommutative torus appears naturally on the same footing as the standard torus \([26]\). In \([27]\), it is also shown that gauge theories on noncommutative tori naturally appear as D-brane world-volume theories. It would be useful to apply our idea of the master field to these theories.

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