ETA FORMS FOR FIBREWISE DIRAC OPERATORS WITH
1-DIMENSIONAL KERNEL OVER A HYPERSURFACE

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Abstract. We generalize the transgression formula for the \( \tilde{\eta} \)-form of Bismut, Cheeger and Berline, Getzler, Vergne for vertical Dirac operators on a fibre bundle \( \pi: M \to B \) with odd-dimensional fibres where the Dirac operators have locally exactly one eigenvalue of multiplicity one crossing zero transversally.

CONTENTS

1. Introduction 1
2. Fibrations and the Bismut superconnection 3
3. Example of a \( S^1 \)-bundle 4
4. Transversal zero-crossing of a single eigenvalue 11
References 24

1. INTRODUCTION

The \( \tilde{\eta} \)-form was introduced by J.-M. Bismut and J. Cheeger in [BC89] as a tool to compute the adiabatic limit of \( \eta \)-invariants on the total space of a fibre bundle. On the other hand it can be seen as a generalization of the transgression forms introduced by D. Quillen in [Qui85]. Bismut and Cheeger studied the case where the fibrewise Dirac operators are invertible and in this case the differential of \( \tilde{\eta} \) makes the cohomological index exact

\[
d\tilde{\eta} = \int_{M/B} \hat{A}(\nabla^{M/B}) \operatorname{ch}(L, \nabla^L).
\]

[BGV04] generalized their result for Dirac operators where the dimension of the kernel is constant. The differential of \( \tilde{\eta} \) transgresses between the cohomological and the analytical index. We get a refinement on the level of differential forms for the cohomological formula

\[
\int_{M/B} \hat{A}(\nabla^{M/B}) \operatorname{ch}(L, \nabla^L) = \operatorname{ch}(\operatorname{ind} D) \in H^*_dR(B),
\]

which comes from applying the chern character to the Atiyah-Singer family index theorem in K-theory (see [AS71, Theorem 3.1] for the even-dimensional case or [APS76, Theorem 3.4] for the odd-dimensional). It turns out that if the dimension of the fibres is odd \( \tilde{\eta} \) still makes the cohomological index exact because of [Dai91,
Theorem 0.1

\[ \lim_{t \to \infty} \text{tr}^{\text{odd}}(\exp(-\mathcal{A}^2)) = \text{tr}^{\text{odd}}(\exp(-\mathcal{A}^2)) = 0. \]

Besides this explicit calculation there is a cohomological reason for this vanishing. For odd-dimensional fibres the family of Dirac operators define an index in $K^{-1}(B)$

\[ \chi(D) : B \to \hat{\mathcal{F}} \]

\[ b \mapsto D_b (1 + D_b^2)^{-1/2} \]

where $\hat{\mathcal{F}}$ is the classifying space for $K^{-1}$ of Atiyah and Singer [AS69]. It is shown that under the assumption that the dimension of the kernels is locally constant, the index $\chi(D) \in [B, \hat{\mathcal{F}}] \cong K^{-1}(B)$ vanishes. Hence, interesting classes in $K^{-1}(B)$ come exactly from operators with varying kernel.

However it is not clear that the differential form

\[ \hat{\eta} = \frac{1}{\sqrt{\pi}} \int_0^\infty \text{tr}^{\text{ev}}(\frac{d\mathcal{A}_t}{dt} \exp(-\mathcal{A}_t^2)) \, dt \in \Omega^\bullet(B) \]

is even defined if the kernel dimension varies. The proof of [BGV04] that the integral converges as $t \to \infty$, relies heavily on the fact that $\ker D \subset \pi_*V$ defines a vector bundle of finite rank over the base manifold $B$.

In the present article we want to take the next step. We will consider vertical Dirac bundles on fibre bundles with odd-dimensional fibres where one eigenvalue of the Dirac operators crosses zero transversally. We will also assume that this eigenvalue has multiplicity one. In this setting it turns out that $\hat{\eta}$ exists not as a differential form but as a current and that

\[ \lim_{t \to \infty} \frac{1}{\sqrt{\pi}} \text{tr}^{\text{odd}}(\exp(-\mathcal{A}_t^2)) = -\delta_{B_0} \text{ch}(\ker D \to B_0, \nabla^{\ker}) , \]

where $B_0 \subset B$ is the codimension 1 submanifold where the kernels of the Dirac operators $D_b$ form a vector bundle $\ker D \to B_0$ of rank 1. Therefore, if $V = \Sigma \otimes L$, the transgression formula becomes

\[ d\hat{\eta} = \int_{M/B} \hat{\mathcal{A}}(\nabla^{M/B}) \text{ch}(L, \nabla^L) + \delta_{B_0} \text{ch}(\ker D \to B_0, \nabla^{\ker}) . \]

We get a very nice representative for the analytical index $\delta_{B_0} \text{ch}(\ker D \to B_0, \nabla^{\ker})$ which is determined by the zero-locus of our particular eigenvalue and the kernel bundle over this hypersurface. To understand the analytical index just by the knowledge of the eigenvalues and eigenspaces was the main motivation for [DK10].

In contrary to our article, R. Douglas and J. Kaminker investigated the influence of the multiplicity of the eigenvalues on the $K^1$-index. Our formula for the analytical index also fits into the framework of [Cib11]. D. Cibotaru developed a model of the classifying space for $K^{-1}$ which allows to deal with unbounded operators.

We think that one could prove a similar result for even-dimensional fibres. Though in this case we know how to fix the problem from a cohomological point of view by [BGV04, Chapter 9.5].

In section 3 of this article we'll look at an example of a vertical Dirac bundle $V$ of rank 1 over a sphere bundle $S^1 \hookrightarrow M \xrightarrow{\pi} B$ where again we have one single eigenvalue of the Dirac operators crossing zero transversally. We explicitly calculate...
\( \tilde{\eta} \) as a differential form with \( L^1 \)-coefficients. In these calculations the Bernoulli polynomials will play an important role. The differential of \( \tilde{\eta} \) fulfills formula (1.5) as expected.

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2. **Fibrations and the Bismut superconnection**

In this chapter we will fix some notation and the situation of families of manifolds we’re working with. For more details have a look at [BC89, Chapter 4] or [BGV04, Chapter 9, 10].

Let \( X \hookrightarrow M \overset{\pi}{\to} B \) be an oriented Riemannian fibre bundle with closed odd-dimensional fibres \( X \) over a closed, oriented, connected Riemannian manifold \((B,g_B)\). We denote the vertical tangent bundle by \( T(M/B) = \ker d\pi \) and choose a horizontal distribution \( T_H M \sim = \pi^* T_B \) such that \( T_M = T(M/B) \oplus T_H M \). We will denote vertical local frames by \( e_i \) and horizontal ones by \( f_\alpha \). We take the metric \( g = g_{M/B} \oplus \pi^* g_B \) and the associated Levi-Civita-connection \( \nabla^M \). The projected connection onto \( T(M/B) \) is denoted by \( \nabla^{M/B} \) and we define a connection \( \nabla^\oplus = \nabla^{M/B} \oplus \pi^* \nabla^B \) which has torsion

\[
T(U,V) = \nabla^\oplus_U V - \nabla^\oplus V U - [U,V] \in T(M/B)
\]

for horizontal vectors \( U, V \in T_H M \).

For a vertical Dirac bundle \((V,g^V,\nabla^V,c)\) with associated fibrewise Dirac operator

\[
D = \sum_i c(e_i) \nabla^V_{e_i} : \Gamma (M,V) \to \Gamma (M,V)
\]

we get the associated vector bundle \( \pi_* V \to B \) whose infinite-dimensional fibres are the fibrewise smooth sections of \( V \). We will make use of the natural isomorphism \( \Gamma (B,\pi_* V) \cong \Gamma (M,V) \) without actually mentioning it. The induced connection

\[
\nabla^{\pi_* V} = \nabla^V + \frac{1}{2} k
\]

is Euclidean with respect to the \( L^2 \)-metric on \( \pi_* V \). The Bismut superconnection [Bis85, Definition 3.2] is then defined by

\[
\hat{\kappa}_t = \sqrt{t} D + \nabla^{\pi_* V} - \frac{1}{4\sqrt{t}} c(T) : \Omega^* (B,\pi_* V) \to \Omega^* (B,\pi_* V),
\]

where we assume that \( dy_\alpha \) and \( c(e_i) \) anticommute. It follows from the transgression formula, see for example [BC89, Eq. (4.38)], that

\[
d \int_T \text{tr}^\text{ev} \left( \frac{d\hat{\kappa}_t}{dt} \exp (-\hat{\kappa}_t^2) \right) dt = \text{tr}^{\text{odd}} (\exp (-\hat{\kappa}_T^2)) - \text{tr}^{\text{odd}} (\exp (-\hat{\kappa}_0^2)).
\]
If \( T(M/B) \) is spin, \( \Sigma \) denotes the spinor bundle for a chosen spin structure. Then we know by [BF86, Theorem 2.10] that for \( V = \Sigma \otimes L \)

\[
\lim_{T \to 0} \frac{1}{\sqrt{\pi}} \left( \sum_k (2\pi i)^{-k} \text{tr}^{\text{odd}} \left( \exp \left( -A_T^2 \right) \right) \right)_{[2k+1]} = \int_{M/B} \hat{A}(\nabla^{M/B}) \text{ch}(L, \nabla^L)
\]

which is a representative for the odd Chern class of the family \( \{D_b\}_{b \in B} \). One should notice that we use Chern-Weil forms of the form \( P(-F/2\pi i) \) for a curvature \( F \) of a connection.

Since we know now what happens as \( T \to 0 \) the next question would be "What happens to formula (2.3) as \( s \to \infty \)?" We already know that for constant kernel dimension we have an answer by [Dai91, Theorem 0.1] or rather [BGV04, Theorem 9.23]

\[
d\tilde{\eta} = \int_{M/B} \hat{A}(\nabla^{M/B}) \text{ch}(L, \nabla^L)
\]

The next step is the case where there is one eigenvalue of the Dirac operators crossing zero. This is the aim of the present article.

3. Example of a \( S^1 \)-bundle

Before we come to the more general case, we will consider one special example of a family of Dirac operators. We are following the requirements in [Zha94], where we adopt the construction of the fibre bundle but change the Dirac bundle.

Let \((E, g_E) \to (B, g_B)\) be a real, Euclidean, oriented vector bundle of rank 2 and denote by \( \nabla^E \) a Euclidean connection on it. We write \( T_H E \cong \pi^* TB \) for the horizontal bundle of \( TE \), which is specified by \( \nabla^E \). We define the metric \( g_{TE} = \pi^* g_E \oplus \pi^* g_B \) on \( TE = \pi^* E \oplus T_H E \).

Let \( M = \{v \in E \mid g_E(v, v) = 1\}, \)

\( T_H M = T_H E|_M, \)

\( TM = \ker d\pi \oplus T_H M = T(M/B) \oplus T_H M, \)

\( g = g_{TE}|_M = g_{M/B} \oplus \pi^* g_B. \)

\( M \to B \) is an oriented, Riemannian fibre bundle with fibres \( X = S^1 \). Let \( e \in \Gamma(M, T(M/B)) \) be the unique section which is positive oriented and of length \( g_{M/B}(e, e) = 1. \)

Let \( (V, g^V, \nabla^V) \to M \) be a Hermitian vector bundle of rank 1 with compatible connection. By setting \( c(e) = -1 \) we make it into a vertical Dirac bundle with Dirac operator \( D = -i\nabla^V_e \). The fibrewise holonomies \( e^{-2\pi i a} \) give rise to a smooth function \( a: B \to \mathbb{R}/\mathbb{Z} \).

3.1. Assumption. \( a: B \to \mathbb{R}/\mathbb{Z} \) crosses \([0]\) transversally.

We denote the codimension 1 submanifold \( a^{-1}(\{0\}) \subset B \) by \( B_0 \).

3.2. Remark. If the holonomies give rise to a non-constant \( a: B \to \mathbb{R}/\mathbb{Z} \) we can always modify the connection \( \nabla^V \) to fulfill assumption 3.1. Sard’s Theorem makes sure that there exists an element \([x] \in \text{im } a\) which is a regular value. The connection \( \hat{\nabla}^V = \nabla^V - ixe^* \)
then gives rise to
\[ \tilde{a} = a - [x] : B \to \mathbb{R} \setminus \mathbb{Z} \]
which crosses zero transversally.

3.3. **Lemma.** We have a vector bundle \( \ker D \to B_0 \) of rank 1 over the hypersurface \( B_0 \) and \( D_b \) is invertible for \( b \in B \setminus B_0 \).

**Proof:** We choose open neighbourhoods \( U_j \subseteq B \) where \( \pi^{-1}(U_j) \cong U_j \times S^1 \) such that we can find local trivializations \( \sigma_0 : U_j \times S^1 \to V|_{U_j \times S^1} \) of \( V \) on \( U_j \times S^1 \) coming from a local eigensection of \( D \)
\[ D \sigma_0 = f_j \sigma_0, \]
where \( f_j : U_j \to \mathbb{R} \) is smooth. For coordinates \( \varphi \) of \( S^1 \) such that \( \partial / \partial \varphi = e \) we can see that \( e^{ik \varphi} \sigma_0 \) is an eigensection of \( D \) corresponding to the eigenvalue \( k + f_j \). Therefore the spectrum of \( D_b \) for \( b \in U_j \) is \( (k + f_j) \), \( k \in \mathbb{Z} \). Since \( [f_j] = a \) the statement follows by [BGV04, Corollary 9.11].

In the following we will for simplicity just write \( f \) for \( f_j \). We orient \( B_0 \) such that
\[ (v_2, ..., v_n) \in \alpha_+(B_0) \Leftrightarrow (\text{grad} f, v_2, ..., v_n) \in \alpha_+(B). \]
Since \( D = -i \nabla^V \), the connection \( \nabla^V \) locally looks like
\[ \nabla^V = d + i f e^* + \gamma, \]
for some \( \gamma \in \Gamma(U, T_H^* M|_U \otimes \mathbb{R} \mathbb{C}) \). We will assume that \( \gamma = \pi^* \beta \).

3.4. **Lemma** ([Zha94, Lemma 1.3]). Let \( T \) be the torsion of \( \nabla^\oplus \) as in (2.1). Then
\[ g(T(U,V), e) = de^* (U,V) \]
an hence \( T \) defines a two-form which we will also denote by \( T \in \Omega^2(B) \).

3.5. **Lemma.** [Zha94, Lemma 1.6] The mean curvature \( k \) of the fibres vanishes and therefore (2.2) leads to
\[ \nabla^{\pi^*V} \sigma = \nabla^V \sigma. \]

3.6. **Remark.** To facilitate the computations for the next theorem we calculate the following summands of the curvature \( A^2_t \) of the Bismut superconnection. We write \([.,.]\) for the supercommutator with respect to the grading of \( \Omega^\bullet(B) \) and keep in mind that \( dy, c(e_j) \) anticommute.

\[ [c(T), \nabla_{\pi^*V}] = 0 \]
\[ [D, c(T)] = 2 D c(T) \]
\[ c(T)^2 = -T^2. \]

In our chosen trivialization
\[ [D, \nabla^{\pi^*V}] = df \]
\[ (\nabla^{\pi^*V})^2 = d\beta + ifT - T \nabla^V_e. \]

3.7. **Theorem.** Set
\[ \alpha(T) := \frac{1}{\sqrt{\pi}} \int_0^T \text{tr}^{\pi^*V} \left( \frac{dk_t}{dt} \text{exp} (-k_t^2) \right) dt \in \Omega^{2\bullet}(B). \]
For each $b \in B$ the differential form $\alpha(T)_b$ converges as $T \to \infty$ to

$$\hat{\eta}_b = \lim_{T \to \infty} \alpha(T)_b \in \Lambda^2 T_b \wedge B$$

and we get that

$$\tilde{\eta}_b = \sum_j \frac{1}{(2\pi i)^j} \tilde{\eta}^{2j}_b$$

$$= \exp \left( -\frac{d\beta + ifT}{2\pi i} \right) \left( \sum_{k=1}^{\infty} \frac{B_k(a)}{k!} \left( \frac{T}{2\pi} \right)^{k-1}, \text{ if } b \in B\setminus B_0 \right)$$

$$= \exp \left( -\frac{d\beta + ifT}{2\pi i} \right) \left( \frac{T/2\pi}{\exp(T/2\pi) - 1}, \text{ if } b \in B \right)$$

where $f : U \to \mathbb{R}$ describes a local eigenvalue of $D$, $\beta$ is the corresponding horizontal connection form of the Dirac bundle in this trivialization and $B_{2k}$ are the Bernoulli numbers and $B_k(a)$ the Bernoulli polynomials.

3.8. Remark. An easy computation shows that our formula for $\hat{\eta}$ corresponds to the one given in [Sav14, (5.23)] for $r = f$. The difference lies in the fact that in our case $f$ is a function depending on the parameter $b \in B$ such that we get a differential form which has jumps, whereas in [Sav14] $r \in \mathbb{R}$ is seen as a fixed integer and $\hat{\eta}$ is seen as a smooth differential form for each $r \in \mathbb{R}$.

3.9. Remark. We prove that the right hand side of the formula in Theorem 3.7 is independent of the chosen trivialization. Therefore we take another local eigensection $\sigma_1$ with

$$D\sigma_1 = f_1 \sigma_1.$$ 

Since the eigenvalues of $D$ differ by integers, there exists a $k \in \mathbb{Z}$ such that $f_1 = f + k$ and $\sigma_1 = e^{ik\varphi} \sigma_0$. The local horizontal connection 1-form $\beta_1$ in this trivialization is then defined by

$$\beta_1 = g^V (\nabla^V \sigma_1, \sigma_1) \frac{g^V (\sigma_1, \sigma_1)}{g^V (\nabla^V \sigma_1, \sigma_1)}$$

and we can conclude that

$$\beta_1 = d (e^{-ik\varphi}) e^{ik\varphi} + \beta$$

$$= -ike^* + \beta.$$ 

It follows that

$$d\beta_1 = -ikT + d\beta$$

and therefore

$$\exp \left( -\frac{d\beta + ifT}{2\pi i} \right) = \exp \left( -\frac{d\beta_1 + if_1 T}{2\pi i} \right).$$
Proof of Theorem 3.7:

\[ \hat{\eta} = \frac{1}{\sqrt{\pi}} \int_0^\infty \text{tr}^\text{ev} \left( \frac{d\kappa_t}{dt} \exp \left( -\kappa_t^2 \right) \right) dt \]

\[ = \frac{1}{\sqrt{\pi}} \int_0^\infty \text{tr}^\text{ev} \left( \left( D - \frac{iT}{4t} \right) \cdot \exp \left( -tD^2 - \sqrt{t}df - d\beta - ifT + T\nabla_e^V + \frac{Dc(T)}{2} + \frac{T^2}{16t} \right) \right) \frac{dt}{2\sqrt{t}}. \]

We see that \( df \) is the only odd differential form and because of \( df \wedge df = 0 \) it doesn’t contribute to \( \text{tr}^\text{ev} \). Since the eigenspaces of \( D \) are preserved by all occurring operators, we can write the trace as

\[ \hat{\eta} = \frac{1}{\sqrt{\pi}} \exp \left( -d\beta - ifT \right) \int_0^\infty \sum_{k \in \mathbb{Z}} \left( \left( k - f - \frac{iT}{4t} \right) \exp \left( -t(k + f)^2 + \frac{(k + f)iT}{2} + \frac{T^2}{16t} \right) \right) \frac{dt}{2\sqrt{t}} \]

\[ = \frac{1}{\sqrt{\pi}} \exp \left( -d\beta - ifT \right) \int_0^\infty \sum_{k \in \mathbb{Z}} \left( \left( k + f - \frac{iT}{4t} \right) \exp \left( \left( i\sqrt{t}(k + f) + \frac{T}{4\sqrt{t}} \right)^2 \right) \right) \frac{dt}{2\sqrt{t}}. \]

That’s why we have to calculate

\[ \sum_{k \in \mathbb{Z}} \left( k + f - \frac{iT}{4t} \right) \exp \left( \left( i\sqrt{t}(k + f) + \frac{T}{4\sqrt{t}} \right)^2 \right) \stackrel{\text{def}}{=} \sum_{k \in \mathbb{Z}} g(k + f). \]

We denote by \( \hat{g} \) the Fourier transform of \( g \) and use the generalized Poisson summation formula

\[ \sum_{k \in \mathbb{Z}} g(k + f) = \sum_{k \in \mathbb{Z}} \hat{g}(k) \cdot \exp \left( 2\pi ikf \right) \]

\[ = -\sum_{k \in \mathbb{Z}} ik \left( \frac{\pi}{t} \right)^{3/2} \exp \left( -\frac{\pi^2 k^2}{t} + 2\pi ikf + \frac{\pi kT}{2t} \right). \]
We insert that into the formula of $\hat{\eta}$ and get

\[
\hat{\eta} = \pi \exp (-d\beta - ifT) \int_0^\infty \sum_{k \in \mathbb{Z}} k \frac{1}{t^{3/2}} \exp \left(-\frac{\pi^2 k^2}{t}\right) \exp \left(2\pi ikf + \frac{\pi kiT}{2t}\right) \frac{dt}{t^{3/2}}
\]

\[
= -\pi \exp (-d\beta - ifT) \sum_{k=1}^\infty \int_0^\infty k \exp \left(-\frac{\pi^2 k^2}{t}\right) \sin \left(-2\pi kf + \frac{\pi kiT}{2t}\right) \frac{dt}{t^{3/2}}
\]

\[
= -\pi \exp (-d\beta - ifT) \sum_{k=1}^\infty k \int_0^\infty \exp \left(-\pi^2 k^2 x\right) \sin \left(-2\pi kf + \frac{\pi kiT}{2}\right) dx
\]

\[
= -\pi \exp (-d\beta - ifT) \sum_{k=1}^\infty \left(\frac{4k}{4\pi^2 k^2 - T^2} \sin (-2\pi kf) + i 4\pi^2 k^2 - T^2 \cos (-2\pi kf)\right)
\]

\[
= \exp (-d\beta - i fT) \left(\sum_{k=1}^\infty \sum_{n=0}^{\dim B} \frac{T^{2n}}{2^{2n+1}n^{2n+1}} \sin (2\pi kf) - i \sum_{k=1}^\infty \sum_{n=0}^{\dim B} \frac{T^{2n+1}}{2^{2n+1}n^{2n+2}2n+1} \cos (2\pi kf)\right).
\]

We define

\[
(3.3) \quad g_n(x) = \begin{cases} 
\sum_{k=1}^\infty (2^{n+1}k^{n+1})^{-1} \sin (2\pi kx), & \text{for } n \text{ even} \\
-i \sum_{k=1}^\infty (2^{n+1}k^{n+1})^{-1} \cos (2\pi kx), & \text{for } n \text{ odd}
\end{cases}
\]

such that

\[
(3.4) \quad \hat{\eta} = \exp (-\beta) \sum_n g_n(f) T^n.
\]

We see that the functions $g_n$ just depend on $a = f - |f| \in [0, 1)$.

First of all we look at the case $f(b) \in \mathbb{Z}$ and see immediately that $g_n = 0$ for $n \in 2\mathbb{N}$.

If $n = 2k + 1 \in 2\mathbb{N} + 1$ we compute

\[
g_n(f) = -\frac{i}{2^{n+1}n+1} \zeta(n+1) = -\frac{i}{2^{2k+1}2^{2k+2}} \zeta(2k + 2)
\]

and therefore

\[
\hat{\eta}|_{B_0} = -\exp (-d\beta - ifT) \sum_{k=0}^\infty \frac{i 2^{2k+1}2^{2k+2}}{\zeta(2k + 2)} T^{2k+1}
\]

\[
= -\exp (-d\beta - ifT) \sum_{k=0}^\infty \frac{i 2^{2k+1}2^{2k+2}}{(2k + 2)!} B_{2k+2} T^{2k+1},
\]
Fourier series of the Bernoulli polynomials

For points where \( f \) have

Theorem.

3.10. It follows that

where the \( B_{n+1} = 0 \) if \( k \geq 1 \) and get

\[
\hat{n}|_{B_0} = -\exp (-d \beta - ifT) (iT)^{-1} \sum_{k=0}^{\infty} \frac{d^{2k+2} h(x)}{dx^{2k+2}} \bigg|_{x=0} \frac{1}{(2k+2)!} (iT)^{2k+2}
\]

\[
= \exp (-d \beta - ifT) (-iT)^{-1} \left( \frac{iT}{e^{IT} - 1} - 1 + \frac{iT}{2} \right).
\]

For points where \( f \not\in \mathbb{Z} \) up to a constant the functions \( g_n : (0,1) \to \mathbb{R} \) are the Fourier series of the Bernoulli polynomials

\[
g_n(x) = \frac{(-1)^{n+1}}{i^n(n+1)!} B_{n+1}(x) = -\frac{i^n}{(n+1)!} B_{n+1}(x).
\]

For Bernoulli polynomials we know that

\[
B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k},
\]

where the \( B_k \) are again the Bernoulli numbers. So we get

\[
\hat{n}|_{B\setminus B_0} = -\exp (-d \beta - ifT) \sum_{n=0}^{\infty} \frac{1}{(n+1)!} B_{n+1}(a)(iT)^n
\]

\[
= -\exp (-d \beta - ifT) (iT)^{-1} \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n h(x)}{dx^n} \bigg|_{x=0} \frac{1}{(n+1-k)!} (iaT)^{n+1-k}
\]

\[
= -\exp (-d \beta - ifT) (iT)^{-1} \left( \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n h(x)}{dx^n} \bigg|_{x=0} (iT)^n \left( \sum_{n=0}^{\infty} \frac{1}{n!} (iaT)^n \right) - 1 \right)
\]

\[
= \exp (-d \beta - ifT) (-iT)^{-1} \left( \frac{iT}{e^{IT} - 1} \exp (iaT) - 1 \right)
\]

It follows that

\[
\hat{n} = \exp (-d \beta - ifT) (-iT)^{-1} \left\{ \frac{-iT}{\exp(-iT) - 1} - 1 - \frac{iT}{2}, \quad \text{for } b \in B_0 \right. \\
\left. \frac{iT}{\exp(IT) - 1} \exp (iaT) - 1, \quad \text{for } b \in B \setminus B_0 \right\}
\]

and

\[
\hat{n} = \sum_{k} \frac{1}{(2 \pi i)^k} \hat{n}[2k]
\]

\[
= \exp \left( -\frac{d \beta + ifT}{2 \pi i} \right) \left( -\frac{T}{2 \pi} \right)^{-1} \left\{ \frac{-T/2\pi}{\exp(-T/2\pi) - 1} - 1 - \frac{T}{4 \pi}, \quad b \in B_0 \\
\left. \frac{T/2\pi}{\exp(T/2\pi) - 1} \exp \left( \frac{T}{4 \pi} \right) - 1, \quad b \in B \setminus B_0 \right\}
\]

\[
\text{3.10. Theorem. We define } d\hat{n} : \Omega^+(B) \to \mathbb{R} \text{ by}
\]

\[
\int_B (d\hat{n}) \wedge \omega := - \int_B \hat{n} \wedge d\omega.
\]
The following formula for the differential holds

\[ d\tilde{\eta} = \int_{M/B} \text{ch}(V, \nabla^V) + \delta_{B_0} \text{ch}(\ker D \to B_0, \nabla^{\ker}) , \]

where \( \nabla^{\ker} = P_0 \nabla^{\pi, V} P_0 \) and \( P_0 \) is the projection onto the kernel of \( D \).

\textbf{Proof:} We have two different possibilities to calculate the differential of \( \tilde{\eta} \). On the one hand we have the transgression formula (2.3)

\[ d\int_s \text{tr}^{ev} \left( \frac{d\hat{\kappa}_t}{dt} e^{-\frac{\kappa_t^2}{4}} \right) = \text{tr}^{odd} \left( e^{-\frac{A_2}{2}} - e^{-\frac{\lambda_2}{2}} \right) \]

By [BF86, Theorem 2.10] we know the limit for \( s \to 0 \) is

\[ \lim_{s \to 0} \frac{1}{\sqrt{\pi}} \text{tr}^{odd} \left( e^{-\frac{\lambda_2}{2}} \right) = (2\pi)^{-1} \int_{M/B} \det \left( \frac{R^M/B}{\sinh (R^M/B/2)} \right) \frac{1}{2} \text{tr} \left( \exp \left( - (\nabla^V)^2 \right) \right) \]

and since \( \hat{\kappa} (\nabla^M/B) = \hat{\kappa} (TS^1) = 1 \) we get the first term. For the second we need to proof that

\[ \lim_{T \to \infty} \text{tr}^{odd} \left( e^{-\frac{\lambda_2}{2}} \right) = \sqrt{\pi} \delta_{B_0} \text{tr} \left( \exp \left( - (\nabla^{\ker})^2 \right) \right) . \]

For that we know that for all eigenvalues \( k + f, k \neq 0 \) and all \( C^\ell \)-norms

\[ \left\| \exp \left( -t(k + f)^2 - \sqrt{t} df - d\beta - i f T + i \frac{(k + f)T}{2} + \frac{T^2}{16t} \right) \right\|_{C^\ell} \leq Ce^{-ct} . \]

For \( k = 0 \) we see that we cannot take the limit as a differential form, we have to integrate over the normal direction of a tubular neighbourhood \( N = N \sim B_0 \times (-\varepsilon, \varepsilon) \) of \( B_0 \) where \( f(x, y) = y \). Let \( \omega \in \Omega^\ell(B) \) where \( \text{supp} \omega \subset N \)

\[ \int_{-\varepsilon}^{\varepsilon} \exp \left( -ty^2 - \sqrt{t} dy - d\beta - iyT + \frac{i y T}{2} + \frac{T^2}{16t} \right) \omega \]

\[ = \int_{-\varepsilon}^{\varepsilon} \exp \left( -y^2 - dy - f^*_t d\beta - i y f^*_T + \frac{f^*_T T^2}{2} + \frac{T^2}{16t} \right) f^*_t \omega \]

where \( f_t: (-\varepsilon, \varepsilon) \to (-\varepsilon, \varepsilon), x \mapsto \frac{x}{\sqrt{t}} \). Now we can see that we have a Gaussian bell curve and therefore

\[ \lim_{t \to \infty} \int_{-\varepsilon}^{\varepsilon} \exp \left( -ty^2 - \sqrt{t} dy - d\beta - \frac{i y T}{2} + \frac{T^2}{16t} \right) \omega \]

\[ = -\sqrt{\pi} i^* \exp \left( -d\beta \right) i^* \omega , \]

where \( i: B_0 \to B \) denotes the inclusion.

On the other hand we can directly calculate the formula for \( d\tilde{\eta} \) by the formula for
\( \tilde{\eta} \) of Theorem 3.7 and
\[
\int_{B} (d\tilde{\eta}) \omega = - \int_{B} \tilde{\eta} d\omega \\
= - \lim_{\varepsilon \to 0} \int_{B \setminus N} \tilde{\eta} d\omega \\
= \lim_{\varepsilon \to 0} \int_{B \setminus N} (d\tilde{\eta}) \omega - \lim_{\varepsilon \to 0} \int_{B \setminus N} d(\tilde{\eta} \omega) \\
= \lim_{\varepsilon \to 0} \int_{B \setminus N} (d\tilde{\eta}) \omega - \lim_{\varepsilon \to 0} \int_{B_0 - \varepsilon} i^* (\tilde{\eta} \omega) + \lim_{\varepsilon \to 0} \int_{B_0 + \varepsilon} i^* (\tilde{\eta} \omega),
\]
which will lead to the same formula as the reader may easily check.

\[\square\]

4. **Transversal zero-crossing of a single eigenvalue**

We will now turn to a more general setting. Let \( M \to B \) be a Riemannian fibre bundle and \( V \to M \) a vertical Dirac bundle as in section 2. The transgression formula in [BC89, Theorem 4.95] holds for invertible vertical Dirac operators, it was generalized by [BGV04, Theorem 10.32] for vertical Dirac operators with constant kernel dimension (see also [Da91, Theorem 0.1] for odd-dimensional fibres). We want to take the next step and give a generalization for a transversal zero-crossing of one eigenvalue of multiplicity one. For the proof we adopt many ideas of the proof of [Bis90, Theorem 3.2]. However, we have to be very careful which norms we use, since our operators are endomorphisms of an infinite rank vector bundle. We also use different contours as in [Bis90] which comes from the fact that we want to use holomorphic functional calculus of the form
\[
\exp \left( -\mathcal{A}_t^2 \right) = \frac{1}{2\pi i} \int_{\gamma} \frac{e^{-z}}{z - \mathcal{A}_t^2} \, dz
\]
rather than
\[
\exp \left( -\mathcal{A}_t^2 \right) = \frac{1}{2\pi i} \int_{\gamma} \frac{e^{-z^2}}{z - \mathcal{A}_t} \, dz.
\]

4.1. **Assumption.** We assume that we can find a covering \( \{U_i\}_{i \in \mathbb{N}} \) for \( B \) such that on each \( U_i \) either \( D_b \) is invertible or we have a smooth function \( f_i : U_i \to (-\varepsilon, \varepsilon) \) which has 0 as a regular value, such that \( \text{spec } D_b \cap (-\varepsilon, \varepsilon) = \{ f_i(b) \} \) and \( f_i(b) \) is of multiplicity 1.

4.2. **Remark.** We get a codimension 1 submanifold
\[
B_0 = \bigcup_{i \in \mathbb{N}} f_i^{-1} (\{0\}) \subset B
\]
where we have a complex vector bundle \( \ker D \to B_0 \) of rank 1 and \( D_b \) is invertible for all \( b \in B \setminus B_0 \). We denote by \( \iota : B_0 \to B \) the inclusion. As in section 3 we get an orientation on \( B_0 \) by
\[
(v_2, \ldots, v_m) \in o_x(B_0) \iff (\text{grad } f_x, v_2, \ldots, v_m) \in o_x(B).
\]
Let $N \cong B_0 \times (-\varepsilon, \varepsilon)$ be a tubular neighbourhood of $B_0$ in $B$. Without loss of generality we can assume that $\lambda_0(x,y) = f(x,y) = y$. The corresponding eigensection will be denoted by $\sigma_0$, on $N$ we can therefore decompose $\Gamma(M_b, V_b) = \langle \sigma_0 \rangle \oplus W$ where $\langle \sigma_0 \rangle|_{B_0} = \ker D$. We denote the projection onto $\langle \sigma_0 \rangle$ by $P$ and onto $W$ by $Q = 1 - P$. Later on we will also write $D^+$ for $QDQ$.

We take the pullback of the bundle $\ker D \rightarrow B_0$ of $\pi_1 : B_0 \times \mathbb{R} \rightarrow B_0$ with the connection $\pi_1^* \nabla_{\ker}$ which, by abuse of notation, will also be denoted by $\nabla_{\ker}$. We denote the second coordinate of $B_0 \times \mathbb{R}$ by $y$ and consider the superconnection $y + \nabla_{\ker}^* : \Omega^\bullet(B_0 \times \mathbb{R}, \pi_1^* \ker D) \rightarrow \Omega^\bullet(B_0 \times \mathbb{R}, \pi_1^* \ker D)$, where we assume that $y$ and 1-forms anticommute. Note that this differs slightly from the superconnection $B$ introduced in [Bis90, III.a].

4.3. Remark. We denote by

$$E_t := \kappa_t^2 - tD^2 = \sqrt{t}[D, \nabla_{\pi_*V}] + (\nabla_{\pi_*V})^2 - \frac{[D, c(T)]}{4} - \frac{[\nabla_{\pi_*V}, c(T)]}{4\sqrt{t}} + \frac{c(T)^2}{16t}.$$ 

By our assumption

$$\exists \tilde{K} > 0 : \sup_{(x,y) \in N} \lambda_k^2(x,y) + \tilde{K} = \varepsilon^2 + \tilde{K} \leq \inf_{(x,y) \in N} \lambda_k^2(x,y) \quad \forall k \neq 0.$$ 

Let $K := \varepsilon^2 + \frac{\tilde{K}}{2}$ and $K' \geq 1$ such that $\varepsilon^2K' \leq K$.

We define the following contours in $\mathbb{C}$:
Since
\[(z - A_t^2)^{-1} = \sum_{n=0}^{\dim B} (z - tD^2)^{-1} \left( E_t (z - tD^2)^{-1} \right)^n \]
the spectrum of $A_t^2$ equals the spectrum of the rescaled Dirac operator. On $B_0 \times (-\varepsilon, \varepsilon)$ we have $\sigma\left(A_t^2\right) = \sigma\left(tD^2\right) = \{t\lambda_k\}_{k \in \mathbb{Z}}$. By holomorphic functional calculus [GGK90, Chapter XV, Proposition 1.1] we know that
\[
\exp\left(-A_t^2\right) = \frac{1}{2\pi i} \int_{\Gamma_t} \exp\left(-z\right) (z - A_t^2)^{-1} \, dz
\]
\[
= \frac{1}{2\pi i} \int_{\Omega_t} \exp\left(-z\right) (z - A_t^2)^{-1} \, dz + \frac{1}{2\pi i} \int_{\Gamma_t} \exp\left(-z\right) (z - A_t^2)^{-1} \, dz
\]
\[
= \mathbb{P}\left(\exp\left(-A_t^2\right)\right) + \left(1 - \mathbb{P}\right) \left(\exp\left(-A_t^2\right)\right).
\]

If we scale the normal direction by $t^{-1/2}$
\[f_t: B_0 \times (-\varepsilon \sqrt{t}, \varepsilon \sqrt{t}) \to B_0 \times (-\varepsilon, \varepsilon), (x, y) \mapsto \left(x, \frac{y}{\sqrt{t}}\right)\]
the smallest eigenvalue of $f_t^*(tD^2)$ becomes $y^2$. Therefore, if $y \neq 0$, we can also write
\[
\mathbb{P}\left(\exp\left(-f_t^*A_t^2\right)_{(x,y)}\right) = \frac{1}{2\pi i} \int_{\Omega_y} \exp\left(-z\right) (z - f_t^*A_t^2)^{-1} \, dz
\]

**Notation.** We will need different kinds of norms in the following statements and proofs which we’ll introduce here. See also [RS75, Appendix of IX.4, Example 2].

- For $1 \leq p < \infty$ the $p$-Schatten norm is defined by
\[
\|A\|_p = (\text{tr}(|A|^p))^{1/p}.
\]
- For a smooth differential form $\omega \in \Omega^\bullet(B)$ we denote by $\|\omega\|_{C^\ell}$ the $C^\ell$-norm.

4.4. **Lemma.** Let $z \in \Gamma_t$ or $z \in \Omega_t$ or $z \in \Omega_y$, $k \geq \dim M_b + 1$ and $t$ big enough, then we have the following estimates:
\[
\left\| (z - tD^2)^{-1} \right\|_{0,0} \leq 1
\]
\[
\left\| (z - tD^2)^{-1} \right\|_k \leq C_1 \left(1 + \frac{|z|}{t}\right)
\]
\textbf{Proof:} (4.11) follows from the choice of the contours $\Gamma_t$, $\Omega_t$ and $\Omega_y$. Inequality (4.13) is clear from the definition of $E_t = A_t^2 - tD^2$.

So let’s prove estimate (4.12). We use the well-known fact, which follows for example by [Roe98, Remark 5.32, Proposition 8.9], that for $k \geq \dim M_b + 1$ we have a constant $C > 0$ such that

$$\| (i - D^2)^{-1} \|_k \leq C.$$ 

Then we use the same formula as in [BG00, Eq. (7.6)]

$$ (z - tD^2)^{-1} = t^{-1} (i - D^2)^{-1} - (i - D^2)^{-1} \left( \frac{z}{t} - i \right) (z - tD^2)^{-1}$$

and therefore we get

$$\| (z - tD^2)^{-1} \|_k \leq t^{-1} \| (i - D^2)^{-1} \|_k + \left( \frac{|z|}{t} + 1 \right) \| (i - D^2)^{-1} \|_k \| (z - tD^2)^{-1} \|_{0,0}$$

$$\leq C \left( \frac{1}{t} + \frac{|z|}{t} + 1 \right)$$

$$\leq C_1 \left( 1 + \frac{|z|}{t} \right).$$

$\square$

\textbf{4.5. Proposition.} \textit{On the tubular neighbourhood $N \cong B_0 \times (-\varepsilon, \varepsilon)$ of $B_0$ in $B$ we have the following estimate}

\begin{equation}
\| \text{tr}^{\text{odd}} ((1 - P) (\exp (-A_t^2))) \|_{C^t} \leq cf(t) \exp (-Ct)
\end{equation}

where $f(t) \in \mathbb{R}[t, t^{-1}]$ is polynomial in $t$ and $t^{-1}$.

\textbf{Proof:} Combining the estimates (4.11), (4.12) and (4.13) we get

$$\| (z - A_t^2)^{-k} \|_1 \leq \| (z - A_t^2)^{-1} \|_k^n$$

$$\leq \left( \sum_{n=0}^{\dim B} \| (z - tD^2)^{-1} \|_k \| E_t \|_{0,0} \| (z - tD^2)^{-1} \|_{0,0}^n \right)^k$$

$$\leq \left( \sum_{n=0}^{m} C \left( 1 + \frac{|z|}{t} \right) t^{n/2} \right)^k$$

$$\leq C \left( 1 + \frac{|z|}{t} \right)^k t^{mk/2},$$
where $m = \text{dim } B$ and constants $C$ varying from line to line. It follows that
\[
\left\| \text{tr}^{\text{odd}} \left( (1 - \mathcal{P}) \left( \exp \left( -A_i^2 \right) \right) \right) \right\|_C \leq \left\| \left( 1 - \mathcal{P} \right) \left( \exp \left( -A_i^2 \right) \right) \right\|_1
\]
\[
= \left\| \frac{1}{2\pi i} \int_{B} \frac{\exp(-z)}{z - A_i^2} \, dz \right\|_1
\]
\[
= \frac{1}{2\pi k!} \left\| \int_{\Omega} \frac{\exp(-z)}{(z - A_i^2)^k} \, dz \right\|_1
\]
\[
\leq \frac{1}{2\pi k!} \int_{\Omega} |\exp(-z)| C \left( 1 + \frac{|s|}{t} \right)^k t^{nk/2} \, dz
\]
\[
\leq C f(t) \exp(-Kt),
\]
where $f \in \mathbb{R}[t, t^{-1}]$. \hfill \Box

4.6. **Remark.** We have the following facts:

1. \[
\left\| \int_{\mathbb{R} \setminus B \sqrt{t}} \text{tr} \left( \exp \left( -\left( y + \nabla^{\text{ker}} \right)^2 \right) \right) \right\|_0 \leq C t^{-1/2} \exp(-Ct)
\]
   for some $C > 0$.

2. For $|y| \leq \varepsilon \sqrt{t}$ we have
   \[
   \exp \left( -\left( y + \nabla^{\text{ker}} \right)^2 \right) = \frac{1}{2\pi i} \int_{\Omega} \frac{\exp(-z)}{z - \left( y + \nabla^{\text{ker}} \right)^2} \, dz.
   \]
   If $t$ is big enough and $1 \leq |y| \leq \varepsilon \sqrt{t}$ we have
   \[
   \exp \left( -\left( y + \nabla^{\text{ker}} \right)^2 \right) = \frac{1}{2\pi i} \int_{\Omega} \frac{\exp(-z)}{z - \left( y + \nabla^{\text{ker}} \right)^2} \, dz.
   \]

3. We define
   \[
   g : B_0 \times \left( -\varepsilon \sqrt{t}, \varepsilon \sqrt{t} \right) \rightarrow B_0, (x, y) \mapsto x
   \]
   such that
   \[
   i \circ g : B_0 \times B_{\varepsilon \sqrt{t}} \rightarrow B_0 \times B_{\varepsilon}, (x, y) \mapsto (x, 0).
   \]
   If $\omega$ is a differential form on $B$ with support in $B_0 \times B_{\varepsilon}$ then
   \[
   \left( i \circ g \right)^* \omega(x, y) - f_t^* \omega(x, y) \leq \frac{C}{\sqrt{t}} \|\omega\|_{C^1(B)} (1 + |y|).
   \]

4.7. **Lemma.** Let $z \in \Omega_t$ or $z \in \Omega_y$. All the estimates are for $(x, y) \in B_0 \times \left( -\varepsilon \sqrt{t}, \varepsilon \sqrt{t} \right)$.

1. \[
\left\| \left( z - t \left( D^+ \right)^2 \left( x, \frac{y}{\sqrt{t}} \right) \right)^{-1} + t^{-1} \left( D^+ \right)^{-2} (x, 0) \right\|_{0,0} \leq C t^{-3/2} \left( |y| + |z| + |z|^3 \right)
\]
(2) \[ \left\| \left( z - t \left( D^+ \right)^2 \left( x, \frac{y}{\sqrt{t}} \right) \right)^{-1} \right\|_{0,0} \leq \frac{C}{t} \left( 1 + |z| + |z|^3 \right) \]

(3) [Bis90, Proposition 3.5] For \( x \in B_0 \) we have
\[
\left( \nabla^{\ker} \right)^2_x = P \left( \nabla^{\pi \cdot V} \right)^2 P - P \nabla^{\pi \cdot V}(D) \left( D^+ \right)^{-2} \nabla^{\pi \cdot V}(D)P.
\]

Proof: The proof of the first estimate is basically the first part of the proof of [Bis90, Proposition 3.4]. Our constants \( C > 0 \) may vary from line to line but they are all independent of \( t, x, y \) and \( z \).

We write
\[
\left( z - t \left( D^+ \right)^2 \left( x, \frac{y}{\sqrt{t}} \right) \right)^{-1} = -t^{-1}(D^+)^{-2} \left( x, \frac{y}{\sqrt{t}} \right) \left( \text{id}_W - \frac{z}{t} (D^+)^{-2} \left( x, \frac{y}{\sqrt{t}} \right) \right)^{-1}
\]
and estimate with the mean value theorem (we can make the estimate for each eigenvalue \( \lambda_k, k \neq 0 \) and therefore get an estimate for the operator norm)

(4.16) \[ \left\| \left( \text{id}_W - \frac{z}{t} (D^+)^{-2} \left( x, \frac{y}{\sqrt{t}} \right) \right)^{-1} - \text{id}_W \right\|_{0,0} \]

(4.17) \[ \leq \frac{|z|}{t} \left\| (D^+)^{-2} \left( x, \frac{y}{\sqrt{t}} \right) \right\|_{0,0} \sup_{0, ce \in [0,1]} \left\| \left( \text{id}_W - \frac{z}{t} (D^+)^{-2} \left( x, \frac{y}{\sqrt{t}} \right) \right)^{-1} \right\|_{0,0}^2.
\]

Since for \( |\text{Im} \, z| = 1 \)
\[
\inf_{c \in [0,1]} |1 - c^2|^2 = \inf_{c \in \mathbb{R}} |1 - c^2|^2 = \frac{1}{|z|^2},
\]
and we find a constant \( C > 0 \) such that for all \( \left( x, \frac{y}{\sqrt{t}} \right) \in B_0 \times (-\varepsilon, \varepsilon) \)
\[
\left\| (D^+)^{-1} \left( x, \frac{y}{\sqrt{t}} \right) \right\|_{0,0} \leq C
\]
we get

(4.18) \[ \left\| \left( \text{id}_W - \frac{z}{t} (D^+)^{-2} \left( x, \frac{y}{\sqrt{t}} \right) \right)^{-1} - \text{id}_W \right\|_{0,0} \leq C \frac{|z|^3}{t}.
\]

Also by the mean value theorem we get

(4.19) \[ \left\| \left( D^+ \left( x, \frac{y}{\sqrt{t}} \right) \right)^{-2} - (D^+(x,0))^{-2} \right\|_{0,0} \leq C \frac{|y|}{\sqrt{t}}.
and therefore

\[ (4.20) \quad \left\| (z - t (D^+)^2 \left( x, \frac{y}{\sqrt{t}} \right) \right\|^{-1} + t^{-1} (D^+)^{-2} (x, 0) \right\|_{0,0} \]

\[ (4.21) \quad \leq t^{-1} \left\| D^+ \left( x, \frac{y}{\sqrt{t}} \right) \right\|^{-2} \left\| \left( \text{id}_W - \frac{z}{t} (D^+)^{-2} \left( x, \frac{y}{\sqrt{t}} \right) \right)^{-1} - \text{id}_W \right\|_{0,0} \]

\[ (4.22) \quad + t^{-1} \left\| - (D^+)^{-2} \left( x, \frac{y}{\sqrt{t}} \right) + (D^+)^{-2} (x, 0) \right\|_{0,0} \]

\[ (4.23) \quad \leq C_1 t^{-2} |z|^3 + C_2 t^{-3/2} |y| \]

for all \( z \) on our contours that have \( |\text{Im} z| = 1 \).

If \( |\text{Im} z| \neq 1 \), we know that either \( \text{Re} z = Kt \) or \( \text{Re} z = My^2 \) or \( \text{Re} z = -1 \). In each of the three cases we get for \( (x, y \sqrt{t}) \in B_0 \times (-\varepsilon, \varepsilon) \) and \( t \) big enough

\[ (4.24) \quad \left\| \frac{c}{t} \text{Re}(z) (D^+)^{-2} \left( x, \frac{y}{\sqrt{t}} \right) \right\|_{0,0} \leq C \]

and therefore

\[ (4.25) \quad \sup_{c \in [0, 1]} \left\| \left( \text{id}_W - \frac{c}{t} z (D^+)^{-2} \left( x, \frac{y}{\sqrt{t}} \right) \right)^{-1} - \text{id}_W \right\|_{0,0} \leq C. \]

Combining this with estimate 4.17 we get

\[ (4.26) \quad \left\| \left( \text{id}_W - \frac{z}{t} \left( D^+ \right)^{-2} \left( x, \frac{y}{\sqrt{t}} \right) \right)^{-1} - \text{id}_W \right\|_{0,0} \leq C \frac{|z|}{t}. \]

By the same calculations as for \( |\text{Im} z| = 1 \) it follows that

\[ (4.27) \quad \left\| (z - t (D^+)^2 \left( x, \frac{y}{\sqrt{t}} \right) \right\|^{-1} + t^{-1} (D^+)^{-2} (x, 0) \right\|_{0,0} \leq \frac{C}{t^{3/2}} \left( |z| + |z|^3 + |y| \right) \]

for all \( z \) on one of our contours that have \( |\text{Im} z| \neq 1 \).

For the second statement we write again

\[ \left( z - t (D^+)^2 \left( x, \frac{y}{\sqrt{t}} \right) \right)^{-1} = - \frac{(D^+)^{-2} \left( x, \frac{y}{\sqrt{t}} \right)}{t} \left( \text{id}_W - \frac{z}{t} \left( D^+ \right)^{-2} \left( x, \frac{y}{\sqrt{t}} \right) \right)^{-1}. \]

By equations (4.18) and (4.26) we know that

\[ (4.28) \quad \left\| \left( \text{id}_W - \frac{z}{t} \left( D^+ \right)^{-2} \left( x, \frac{y}{\sqrt{t}} \right) \right)^{-1} \right\|_{0,0} \leq 1 + \frac{C}{t} \left( |z| + |z|^3 \right). \]

By the choice of our operator we know that

\[ \left\| (D^+)^{-2} \left( x, \frac{y}{\sqrt{t}} \right) \right\|_{0,0} \leq C. \]
for all \((x, \frac{y}{\sqrt{t}})\) \in B_0 \times (-\varepsilon, \varepsilon). Combining these yields to

\[
(4.29) \quad \left\| (z - t (D^+)^2 \left( x, \frac{y}{\sqrt{t}} \right) )^{-1} \right\|_{0,0} \leq \frac{C}{t} \left( 1 + |z| + |z|^3 \right).
\]

\[\Box\]

4.8. **Proposition.** We define for \((x, y) \in B_0 \times (-\sqrt{t}, \sqrt{t}), z\) in one of our contours and \(t\) big enough the operator \(\alpha\) by

\[
\left( P f^*_t E_i P + P f^*_t E_i Q \left( z - tf^*_t D^2 \right)^{-1} Q f^*_t E_i P - g^* \left( dy + (\nabla \ker^2) \right) \right)_{(x,y)} = g^* \left( dy + (\nabla \ker^2) \right)_{(x,y)} + \alpha (x, y, z, t)
\]

then

\[
(4.30) \quad \| \alpha (x, y, z, t) \|_{0,0} \leq C t^{-1/2} \left( 1 + |y| + |z| + |z|^3 \right).
\]

**Proof:** First we use Lemma 4.7.3 to see that

\[
\left\| P f^*_t E_i P + P f^*_t E_i Q \left( z - tf^*_t D^2 \right)^{-1} Q f^*_t E_i P - g^* \left( dy + (\nabla \ker^2) \right) \right\|_{0,0}
\]

\[
= \left\| P f^*_t E_i P + P f^*_t E_i Q \left( z - tf^*_t D^2 \right)^{-1} Q f^*_t E_i P - g^* \left( dy + P \left( \nabla \pi V \right)^2 P - P \nabla \pi V (D^+) (D^+) \nabla \pi V (D) P \right) \right\|_{0,0}
\]

\[
\leq \left\| P f^*_t E_i P - g^* \left( dy + P \left( \nabla \pi V \right)^2 P \right) \right\|_{0,0}
\]

\[
+ \left\| P f^*_t E_i Q \left( z - tf^*_t D^2 \right)^{-1} Q f^*_t E_i P + g^* \left( P \nabla \pi V (D^+) (D^+) \nabla \pi V (D) P \right) \right\|_{0,0}.
\]

By definition and equation (4.15)

\[
(4.31) \quad \left\| P f^*_t E_i P - g^* \left( dy + P \left( \nabla \pi V \right)^2 P \right) \right\|_{0,0} \leq \frac{C}{\sqrt{t}} \left( 1 + |y| \right).
\]

For the second summand we have

\[
\left\| P f^*_t E_i Q \left( z - t D^2 \left( x, \frac{y}{\sqrt{t}} \right) \right)^{-1} Q f^*_t E_i P + g^* \left( P \nabla \pi V (D^+) (D^+) \nabla \pi V (D) P \right) \right\|_{0,0}
\]

\[
\leq \left\| P f^*_t E_i Q \left( z - t D^2 \left( x, \frac{y}{\sqrt{t}} \right) \right)^{-1} Q \left( f^*_t E_i - \sqrt{t} \nabla \pi V (D) \right) P \right\|_{0,0}
\]

\[
+ \left\| P f^*_t E_i Q \left( z - t D^2 \left( x, \frac{y}{\sqrt{t}} \right) \right)^{-1} + t^{-1} (D^+)^{-2} (x, 0) \right\|_{0,0}
\]

\[
+ \left\| P \left( f^*_t E_i + \sqrt{t} \nabla \pi V (D) \right) t^{-1} (D^+)^{-2} (x, 0) \sqrt{t} \nabla \pi V (D) P \right\|_{0,0}
\]

\[
\leq C_1 t^{-1/2} \left( 1 + |z| + |z|^3 \right) + C_2 t^{-1/2} \left( |y| + |z| + |z|^3 \right) + C_3 t^{-1/2}
\]

where we used Lemma 4.7.2, equation (4.13) and the definition of \(E_i\) for the first term, Lemma 4.7.1 for the second.

\[\Box\]
4.9. Proposition. Let \((x, y) \in B_0 \times (-\varepsilon \sqrt{t}, \varepsilon \sqrt{t})\), \(z\) in one of our contours and \(t\) big enough. We define

\[(4.32) \quad (z - f^*_t A_t^2)^{-1} - \left( z - (y + \nabla^{\text{ker}})^2 \right)^{-1} =: \gamma (x, y, z, t).\]

Then there exist constants \(C_1, C_2, C_3, C_4 > 0\) and polynomials \(p_1, p_2, p_3, p_4, p_5\) such that

\[
\|P\gamma P\|_{0,0} \leq C_1 t^{-1/2} (1 + p_1 (|y|)) + p_2 (|z|))
\]
\[
\|P\gamma Q\|_{0,0} \leq C_2 t^{-1/2} (1 + p_3 (|z|))
\]
\[
\|Q\gamma P\|_{0,0} \leq C_3 t^{-1/2} (1 + p_4 (|z|))
\]
\[
\|Q\gamma Q\|_{0,0} \leq C_4 t^{-1} (1 + p_5 (|z|))
\]

Proof: Throughout the proof we will denote by \(p\) some polynomial in \(|z|\) or \(|y|\) which may vary from line to line but is independent of \(x, t\) and \(y\) or \(z\) respectively. The constants \(C > 0\) may also vary but again are independent of \(x, y, z\) and \(t\). For simplicity but by abuse of notation we define just for this proof \(A := \left( z - t f^*_t D^2 \right)^{-1}, \ B := f^*_t E_t, \ X := \left( z - y^2 \right)^{-1} \) and \(Y := dy + (\nabla^{\text{ker}})^2 \). Then we know that

\[
(z - f^*_t A_t^2)^{-1} - \left( z - (y + \nabla^{\text{ker}})^2 \right)^{-1} = \sum_{n \geq 0} A(BA)^n - X(YX)^n
\]

where the sum is finite.

Let us first look at

\[
P \left( \sum_{n \geq 0} A(BA)^n - X(YX)^n \right) P = \sum_{n \geq 0} XP(BA)^n P - X(YX)^n
\]
\[
= \sum_{n \geq 0} XP((PB P + PBQ + QBP + QBQ)A)^n P - X(YX)^n
\]

Since \(PQ = QP = 0\) the only combination in which \(QBQ\) can occur is of the following form

\[
PBQA(QBQA)^k QBP.
\]

But since we know from the previous lemma 4.7.2. that

\[
\|QAQ\|_{0,0} = \left\| \left( z - t (D^+)^2 \left( x, \frac{y}{\sqrt{t}} \right) \right)^{-1} \right\|_{0,0} \leq \frac{C}{t} \left( 1 + |z| + |z|^3 \right)
\]

and by equation (4.13) that

\[
\|B\|_{0,0} = \|f^*_t E_t\|_{0,0} \leq C \sqrt{t}.
\]

it follows that

\[
\|PBQA(QBQA)^k QBP\|_{0,0} \leq Ct^{-k/2} (1 + p(|z|)).
\]

By the same argument as above, \(PBQ\) and \(QBP\) can only occur as

\[
PBQAQBP.
\]
Combining these yields to
\[
\left\| P \left( (z - f_1^* A_2^2)^{-1} - (z - (y + \nabla^{ker}))^{-1} \right) P \right\|_{0,0} \\
\leq \left\| \sum_{n \geq 0} X P((PBP + PBQ + QBP)A)^n P - X(YX)^n \right\|_{0,0} + Ct^{-1/2} (1 + p(|z|)) \\
\leq \sum_{n \geq 0} \|X((PBP + PBQABP)X)^n - X(YX)^n\|_{0,0} + Ct^{-1/2} (1 + p(|z|)) \\
\leq Ct^{-1/2} (1 + p_1 (|y|) + p_2 (|z|)),
\]
where we used Proposition 4.8 in the last step.
For the other estimates we don’t need \(X(YX)^n\), since \(PX(YX)^nP = X(YX)^n\).
We know that
\[
\|A\| = \begin{pmatrix} (z - y^2)^{-1} & 0 \\ 0 & (z - tf_t^* (D^+)^2)^{-1} \end{pmatrix}.
\]
By the choice of our contours we know that
\[
\left\| (z - y^2)^{-1} \right\|_{0,0} \leq C
\]
and by Lemma 4.7.2.
\[
\left\| (z - tf_t^* (D^+)^2)^{-1} \right\|_{0,0} \leq Ct^{-1} (1 + p (|z|)).
\]
We know that \(\|B\|_{0,0} = \|f_t^* F_t\|_{0,0} \leq Ct^{1/2}\) but for \(PBP\) we even get
\[
\|PBP\|_{0,0} \leq C,
\]
since the only summand involving \(t\) with a positive exponent is
\[
\sqrt{tf_t^* P\nabla^{x+y} (D)P} = \sqrt{tf_t^* \omega} = dy.
\]
Now one can easily check inductively that
\[
\|PA(BA)^nQ\|_{0,0} \leq Ct^{-1/2} (1 + p (|z|)) \\
\|QA(BA)^nP\|_{0,0} \leq Ct^{-1/2} (1 + p (|z|)) \\
\|QA(BA)^nQ\|_{0,0} \leq Ct^{-1} (1 + p (|z|))
\]
which proves the other three estimates in the statement. \(\square\)

4.10. **Theorem.** There exist constants \(C, c > 0\) depending on \(\ell\), such that for \(t\) big enough we get the following estimates. On \(B \setminus N\)
\[
\|\text{tr} (\exp (-A_t^2))\|_{B \setminus N} \leq Ce^{-ct}, \quad (4.33)
\]
for all \(C^\ell\)-norms on \(\Omega^*(B \setminus N)\). On \(N \cong B_0 \times (-\varepsilon, \varepsilon)\) and for all \(\omega \in \Omega^*(B)\)
\[
\left\| \left( \int_{-\varepsilon}^{\varepsilon} \text{tr} (\exp (-A_t^2)) \right) \omega + \sqrt{\pi} \text{tr} (\exp (- (\nabla^{ker})^2)) i^* \omega \right\|_{C^\ell} \leq Ct^{-1/2} \|\omega\|_{C^{\ell+1}}. \quad (4.34)
\]
for all $C^\ell$-norms on $\Omega^\bullet(B_0)$. If we combine the estimates we have

$$
\left| \int_B \text{tr}^{\text{odd}} \left( \exp \left( -A^2 t \right) \right) \omega + \sqrt{\pi} \int_{B_0} \text{tr} \left( \exp \left( -\left( \nabla^{\text{ker}} \right)^2 \right) \right) i^* \omega \right| \leq \frac{C}{\sqrt{\ell}} \|\omega\|_{C^1}.
$$

**Proof:** In the following we have constants $C > 0$ which may vary from line to line and depend on $\ell$ but not on $t, y, z$ and $x$.

Since $D_b$ is invertible for all $b \in B \setminus N$, we know that

$$
\left| \text{tr} \left( \exp \left( -A^2 t \right) \right) \right|_{B \setminus N} \leq C e^{-ct}
$$
on $B \setminus N$ for all $C^\ell$-norms.

On $N$ we know by Proposition 4.5 that

$$
\left| \text{tr} \left( (1 - P) \left( \exp \left( -A^2 t \right) \right) \right) \right|_{C^\ell} \leq C f(t) \exp (-Kt)
$$

where $f(t) \in \mathbb{R}[t, t^{-1}]$ is a polynomial in $t$ and $t^{-1}$. It remains to show that

$$
\left( \int_{-\varepsilon}^\varepsilon \text{tr} \left( P \left( \exp \left( -A^2 t \right) \right) \right) \omega \right) \leq \varepsilon \sqrt{t} \left( \int_{-\varepsilon}^\varepsilon \text{tr} \left( \exp \left( -\left( \nabla^{\text{ker}} \right)^2 \right) \right) i^* \omega \right) \in \Omega^\bullet(B_0)
$$
is of $O \left( t^{-1/2} \right)$ for all $C^\ell$-norms on $\Omega^\bullet(B_0)$.

$$
\left( \int_{-\varepsilon}^\varepsilon \text{tr} \left( P \left( \exp \left( -A^2 t \right) \right) \right) \omega \right) \leq \left( \int_{-\varepsilon}^\varepsilon \text{tr} \left( \exp \left( -\left( \nabla^{\text{ker}} \right)^2 \right) \right) i^* \omega \right) + C t^{-1/2} e^{-ct}
$$

We write the projection $P$ via holomorphic functional calculus. We use the contour $\Omega_t$ for $|y| \leq 1$ and the contour $\Omega_y$ for $1 \leq |y| \leq \varepsilon \sqrt{t}$. Since $P$ projects our operators onto a 1-dimensional subspace we make our estimates in the operator instead of the $\|\|_{1}$-norm.
The top cohomology class of our representative
of length $\alpha$ and similarly for a multiindex $\ell$ of length $\alpha$

\[ |f_i^* \omega - g^* i^* \omega| \leq C t^{-1/2} \| \omega \|_{C^{t+1}}. \]

4.11. Remark. D. Cibotaru calculated explicitly $\lim_{t \to \infty} \text{ch}(A_t)$ for superconnections $A_t = \nabla + tA$ on finite rank vector bundles $E \to B$, see [Cib14, Theorem 9.4, 9.5]. Theorem 4.10 can be seen as a generalization to infinite dimensions. In exchange we restrict ourselves to a vector bundle of rank one $\ker D \to B_0$. In any case the currents we obtain are not surprising considering what we know from finite dimensions.

The top cohomology class of our representative $-\delta_{B_0} \text{ch}(\ker D \to B_0, \nabla^{\ker})$ of the
analytical index also agrees with the formula given in [Cib11, Proposition 1.1] for $\dim B = 3$.

4.12. Proposition. There exists a constant $C > 0$ such that for $t \to \infty$ and for all $\omega \in \Omega^\bullet(B)$ we have the following estimate on $N$

$$
\left\| \int_{-\varepsilon}^{\varepsilon} \text{tr}^{ev} \left( \frac{d\mathcal{A}_t}{dt} \exp \left( -\mathcal{A}_t^2 \right) \right) \omega \right\|_{C^t} \leq C t^{-3/2} \|\omega\|_{C^{t+1}}.
$$

Proof: Let $S = (1/2, \infty)$ and consider the fibre bundle $\tilde{M} = M \times S \to \tilde{B} = B \times S$ as in the proof of [BGV04, Theorem 10.32]. We denote the extra coordinate in $S$ by $s$ and define the vertical metric $g_{\tilde{M}/\tilde{B}} = s^{-1} g_{M/B}$. The vertical Dirac bundle will be $\tilde{V} = V \times S \to \tilde{M}$, where we take the natural extensions of the given connections. We will write $\sim$ over all induced objects on this family. So let $\tilde{\mathcal{A}}$ be the Bismut superconnection in this situation which we scale again by a parameter $t$ as follows

$$
\tilde{\mathcal{A}}_t = \sqrt{t} \tilde{\mathcal{D}} + \tilde{\nabla}_s \tilde{V} - \frac{1}{4t} \mathcal{C}(\mathcal{T}).
$$

We made assumption 4.1 for the Dirac operators $\mathcal{D}$, but

$$
\tilde{\mathcal{D}}_{(b,s)} = \sqrt{s} \mathcal{D}_b
$$

implies that it also holds for $\tilde{\mathcal{D}}$. We have a bundle $\ker \tilde{\mathcal{D}} \to \tilde{B}_0 = B_0 \times S$ which is just the pullback of $\ker \mathcal{D} \to B_0$. So now we can apply Theorem 4.10

$$
\left\| \left( \int_{-\varepsilon}^{\varepsilon} \text{tr}^{odd} \left( \exp \left( -\tilde{\mathcal{A}}_t^2 \right) \right) \right) \omega + \sqrt{\pi} \text{tr} \left( \exp \left( -\left( \tilde{\nabla}_{\text{ker}} \right)^2 \right) \right) i^* \omega \right\|_{C^t} \leq \frac{C}{\sqrt{t}} \|\omega\|_{C^{t+1}}.
$$

Then we know by [BGV04, Lemma 10.31] or by a straightforward calculation that

$$
\text{tr}^{\text{odd}} \left( \exp \left( -\tilde{\mathcal{A}}_t^2 \right) \right) \big|_{s=1} = \text{tr}^{\text{odd}} \left( \exp \left( -\tilde{\mathcal{A}}_t^2 \right) \right) - t \text{tr}^{ev} \left( \frac{d\mathcal{A}_t}{dt} \exp \left( -\mathcal{A}_t^2 \right) \right) ds.
$$

Since $\left( \tilde{\nabla}_{\text{ker}} \right)^2$ is just a pullback from $B_0$ and therefore doesn’t involve $ds$ equation (4.40) implies that

$$
\left\| \int_{-\varepsilon}^{\varepsilon} t \text{tr}^{ev} \left( \frac{d\mathcal{A}_t}{dt} \exp \left( -\mathcal{A}_t^2 \right) \right) \omega \right\|_{C^t} \leq \frac{C}{\sqrt{t}} \|\omega\|_{C^{t+1}}.
$$

for all $C^\bullet$-norms on $\Omega^\bullet(B_0)$ which proves equation (4.39). \[\square\]

4.13. Definition. We define the current $\hat{\eta}$ by

$$
\hat{\eta} : \Omega^\bullet(B) \to \mathbb{R}
$$

$$
\omega \mapsto \frac{1}{\sqrt{\pi}} \int_0^\infty \left( \int_B \text{tr}^{ev} \left( \frac{d\mathcal{A}_t}{dt} e^{-\mathcal{A}_t^2} \right) \omega \right) dt.
$$
This is well-defined because of Proposition 4.12 and [BC89, Eq. (4.92)].
We define \( \tilde{\eta} \) by
\[
\tilde{\eta} : \Omega^k(B) \to \mathbb{R} \\
\omega \mapsto (2\pi i)^{(k-\dim B)/2} \tilde{\eta}(\omega)
\]
and \( d\tilde{\eta} \) by
\[
d\tilde{\eta}(\omega) = -\tilde{\eta}(d\omega).
\]

4.14. Theorem. We assume that \( T(M/B) \) admits a spin structure and denote by \( \Sigma \) the corresponding spinor bundle. If the Dirac bundle \( V \) is of the form \( \Sigma \otimes L \) then
\[
d\tilde{\eta} = \int_{M/B} \hat{A} \left( \nabla^{M/B} \right) \text{ch} (L, \nabla^L) + \delta_{B_0} \text{ch} (\ker D \to B_0, \nabla^{\ker}) ,
\]
where \( \delta_{B_0} \) is the current of integration over the hypersurface \( B_0 \).

Proof. Equation (4.43) follows from the transgression formula (2.3)
\[
d \int_{M/B} \text{tr}^\text{ev} \left( \frac{d\delta_A}{dt} e^{-\lambda_t^2} \right) = \text{tr}^\text{odd} \left( e^{-\lambda_t^2} \right) - \text{tr}^\text{odd} \left( e^{-\lambda_t^2} \right)
\]
since we know by [BC89, Theorem 4.95] that for \( l = \dim M_b \)
\[
\lim_{T \to 0} \frac{1}{\sqrt{\pi}} \text{tr}^\text{odd} \left( e^{-\lambda_t^2} \right)
= (2\pi i)^{-(l+1)/2} \int_{M/B} \det \left( \frac{R^{M/B}/2}{\sinh (R^{M/B}/2)} \right)^{1/2} \text{tr} \left( \exp \left( - \left( \nabla^L \right)^2 \right) \right)
\]
and by Theorem 4.10 that
\[
\lim_{T \to \infty} \frac{1}{\sqrt{\pi}} \text{tr}^\text{odd} \left( e^{-\lambda_t^2} \right) = -\delta_{B_0} \text{tr} \left( \exp \left( - \left( \nabla^{\ker} \right)^2 \right) \right).
\]
If we define the \( 2\pi i \)-scaling as above the resulting formula is
\[
d\tilde{\eta} = \int_{M/B} \det \left( \frac{R^{M/B}/4\pi i}{\sinh (R^{M/B}/4\pi i)} \right)^{1/2} \text{tr} \left( \exp \left( - \left( \nabla^L \right)^2 /2\pi i \right) \right)
+ \delta_{B_0} \text{tr} \left( \exp \left( - \left( \nabla^{\ker} \right)^2 /2\pi i \right) \right)
= \int_{M/B} \hat{A} \left( \nabla^{M/B} \right) \text{ch} (L, \nabla^L) + \delta_{B_0} \text{ch} (\ker D \to B_0, \nabla^{\ker}) .
\]

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