

From Lattice Gauge Theories to Hydrogen Atoms

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Using canonical transformations we obtain a complete and most economical realization of the loop or physical Hilbert space of pure SU(2)λ,1 lattice gauge theory in terms of Wigner coupled Hilbert spaces of hydrogen atoms. One hydrogen atom is assigned to every plaquette of the lattice. The SU(2) gauge theory loop basis states over a plaquette are the bound energy eigenstates |nlm⟩ of the corresponding hydrogen atom. The Wigner couplings of these hydrogen atom energy eigenstates on different plaquettes provide a complete SU(2) gauge theory loop basis on the entire lattice. The loop basis is invariant under simultaneous rotations of all hydrogen atoms. The dual description of this basis diagonalizes all Wilson loop operators and is given in terms of hyperspherical harmonics on the SU(2) group manifold S3. The SU(2) loop dynamics is governed by a “SU(2) spin Hamiltonian” without any gauge fields. The relevance of the hydrogen atom basis and its dynamical symmetry group SO(4,2) in SU(2) loop dynamics in weak coupling continuum limit (g2 → 0) is emphasized.

I. INTRODUCTION

The idea that gauge theories should be reformulated as theory of loops and strings without any color degrees of freedom, is very old [1] and refuses to die. One hopes that eventually some appropriate loop description of gauge theories will provide a natural and elegant framework to compute low energy QCD effects leading to a better understanding of non-perturbative issues like color confinement. Since the work of Ashketak, loops carrying SU(2) fluxes have found their relevance in quantum gravity also. In loop quantum gravity they describe quantum excitations of geometry [2]. In condensed matter physics [3], many effective models are described in terms of non-abelian gauge theories. In view of above, the importance of developing new ideas and techniques to understand gauge theories better requires no emphasis. In recent past quest to develop real time quantum simulators for gauge theories better requires no emphasis. In recent past quest to develop real time quantum simulators for gauge theories better requires no emphasis. In view of above, the importance of developing new ideas and techniques to understand gauge theories better requires no emphasis.

In this letter we show that the gauge invariant sector or more precisely loop formulation of pure SU(2) lattice gauge theories in d=2 space dimension [4] can be completely and most economically realized in terms of the states and operators of Wigner coupled hydrogen atoms. One hydrogen atom is attached to each plaquette of the lattice and each hydrogen atom lives in its 3-dimensional space. However, following Fock [5], we describe them on their momentum hyperspheres S3 so that their hidden symmetries become manifest. On the other hand, the standard SU(2) lattice gauge theory [6] is reformulated in terms of SU(2) plaquette fluxes or plaquette holonomies and their conjugate electric fields through a series of iterative canonical transformations over the entire lattice. We show that the resulting SU(2) loop basis states, in the electric representation of the plaquette formulation, are also the Wigner coupled energy eigenstates |nlm⟩ of hydrogen atoms on different plaquettes (see (11)). This equivalence has its origin in the identification of the SU(2) group manifold S3 (associated with the plaquette holonomy) with the S3 of hydrogen atom wave functions on the momentum hypersphere obtained by Fock in 1935. In the last section we show that the loop Hamiltonian has a global SU(2) symmetry (no gauge fields) to be fixed by a single Gauss law constraint. We also briefly discuss the relevance of the hydrogen atom dynamical symmetry group SO(4,2) and weak coupling limit (g2 → 0) of SU(2) loop Hamiltonian. We start with a brief review of hydrogen atom [10] in a language which is in complete coherence with the prepotential language of lattice gauge theory [8] used in the next section to establish the above equivalence.

Since the work of Pauli, it is known that the hydrogen atom can be completely solved using group theoretical methods involving angular momentum (\(\vec{L}\)) and Runge Lenz (\(\vec{A}\)) vector operators [13]. They commute with the hydrogen Hamiltonian (\(\mathcal{H}\)) and satisfy \(\vec{L} \cdot \vec{A} \equiv 0\). This leads to SU(2)⊗SU(2) algebra [10] generated by \(\hat{J}_1 \equiv \frac{1}{2} (\vec{L} + \vec{A})\) and \(\hat{J}_2 \equiv \frac{i}{2} (\vec{L} - \vec{A})\) on the bound states of Hydrogen atom (\(E < 0\)). Further, for hydrogen atom [10] as \([\hat{J}_1, \hat{J}_2] = 0\) and \(\hat{J}_1 = \hat{J}_2 \equiv \hat{J}^2\), the complete set of commuting operators (CSCO) is either \([\hat{J}^2, \hat{J}_1, \hat{J}_2]\) (CSCO-1) in decoupled basis or \([\hat{J}^2, \hat{\vec{L}}^2, \hat{\vec{L}}_z]\) (CSCO-2) in the coupled basis. Following Wybourne [10], we define:

\[
\hat{J}_1^a = \frac{a^+ (\sigma_1^a)}{2}; \quad \hat{J}_2^a = \frac{b^+ (\sigma_1^a)}{2} \quad (1)
\]

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In (14), \( (a_1^+, a_2^+) \) and \( (b_1^+, b_2^+) \) represent SU(2) doublets of Schwinger boson creation operators, \( \sigma^a \) (\( a = 1, 2, 3 \)) are the Pauli matrices. The condition \( \tilde{J}_Z = \tilde{J}_2 \) implies \( N_a = N_b \) where \( N_a = a^+ a \) and \( N_b = b^+ b \) are the total number operators. The orthonormal and complete basis diagonalizing CSKO-1 is given by (10):

\[
\langle j, m, m' | j, m \rangle = \frac{1}{(j + m)!(j - m)!} (j + m)! \langle j, m | j, m \rangle.
\]
The coupled hydrogen atom basis diagonalizing the CSKO-2 is given by:

\[
| n \, l \, m \rangle = \sum_{m', \tilde{m}} C_{j, m', j, m} | j, m' \rangle_a | j, \tilde{m} \rangle_b.
\]

In (3), \( n \equiv (2j + 1) \) and \( l = 0, 1, 2, \ldots ; m = -l, \ldots, +l; C_{j, m', j, m} \equiv \frac{1}{2m'!} (2m) \rangle (a = 1, 2, 3) \) are the Pauli coefficients. The hydrogen atom states \( | n \, l \, m \rangle \) are eigenstates \( J^2, L^2, L_z \) and also of the Hamiltonian (10) with energy \( E_n = -1/2m^2 \). Note that the bosonic representation (14) and the states (2, 9) have additional U(1) symmetry: \( \sigma_a \rightarrow e^{i\theta} \sigma_a, b_a \rightarrow e^{-i\theta} b_a \). In 1935 Fock (2) showed that the mysterious Runge Lenz symmetries can be made obvious for the bound states \( (p_0^2 \equiv -2mE > 0) \) by going to the momentum \( (p_1, p_2, p_3) \) space. In other words, the full SO(4) \( \sim SU(2) \otimes SU(2) \) symmetry of hydrogen atom becomes manifest if we transcribe the dynamics on hypersphere \( S^3 \) representing momentum coordinates \( (p_1, p_2, p_3) \) through stereographic projection in \( R^4 \) which consists of points \( (p_0, p_1, p_2, p_3) \). The Cartesian coordinates \( (q_0, \vec{q}) \) on \( S^3 \) of a momentum point \( \vec{p} \) are:

\[
\Omega(\vec{q}) \equiv q_0^a + i\vec{q} \cdot \sigma, \quad q_0^a + q_\vec{q}^2 = 1: \quad S^3, \quad q_\vec{q} \equiv \frac{2q_0p_0}{(p_0^2 + p_\vec{q}^2)}.
\]

In (4) \( \sigma_0 \) is 2x2 identity matrix, \( \vec{\sigma} \) denotes 3 Pauli matrices. The angles on \( S^3 \) are defined by:

\[
q_0 \equiv \cos \left( \frac{\varphi}{2} \right), \quad \vec{q} \equiv \sin \left( \frac{\varphi}{2} \right), \quad \varphi \equiv \tilde{q} (\vartheta, \varphi).
\]

The mapping (3) transforms (2) and (11) the hydrogen atom Schrodinger equation in momentum space into the integral equation for the 4-dimensional spherical harmonics \( Y_{n,l,m}(\vartheta, \varphi) \). It was later shown by Bargmann (2, 11) that the three angular momentum operators \( (L_1, L_2, L_3) \) and three Lenz operators \( (A_1, A_2, A_3) \) generate rotations in \( (q_0, q_1), (q_1, q_2), (q_0, q_3) \) and \( (q_0, q_1), (q_0, q_2), (q_0, q_3) \) phases respectively.

In the next section we construct iterative canonical transformations which not only make the spurious gauge degrees of freedom decouple from the dynamics but also fuse the link flux operators into a set of (physical) plaquette or loop flux operators which are mutually independent as well as complete (10). These loop operators, all starting and ending at the origin, in turn lead to a complete orthonormal loop basis which are identified with Wigner coupled hydrogen atom energy eigenstates (2) and (3) (see (14)). To keep the discussion simple and short we always illustrate the ideas on a single plaquette and then directly generalize the results to the entire lattice.

**FIG. 1:** Three canonical transformations on the 4 link flux operators of a plaquette leading to the loop flux operators \( W_{n,a} \). Three \( \bullet \) at the end of a flux operators \( U(1), U(2) \) and \( U(3) \) represent their right electric fields which are also the Gauss law operators at A, B and C respectively.

**II. CANONICAL TRANSFORMATIONS, LOOP OPERATORS, LOOP STATES & HYDROGEN ATOMS**

We start with a plaquette OABC with four SU(2) flux operators \( U(I), I = 1, 2, 3, 4 \) attached to its four sides. The right rotations on \( U(I) \) are generated by the left, right electric fields \( E^a_\pm(I), E^a_\pm(I) \) (\( a = 1, 2, 3 \)) respectively. As operators on different links commute, we suppress link index I. The basic quantization rules on every link are:

\[
\begin{align*}
\left[ E^a_+, U_{a\beta} \right] &= -i \left( \frac{\sigma^a}{2} \right)_{\alpha\beta} E^c_\pm = i \epsilon^{abc} E^c_+ \\
\left[ E^a_-, U_{a\beta} \right] &= i \left( \frac{\sigma^a}{2} \right)_{\alpha\beta} E^c_\pm = i \epsilon^{abc} E^c_-.
\end{align*}
\]

One can also check that \( E^a_\pm = -R_{ab}(U)E^b_\pm \) where \( R_{ab}(U) = \frac{1}{2} Tr (U \sigma^a U^\dagger \sigma^b) \) is SO(3) rotation matrix implying \( (E^-)^2 = (E^+)^2 = 0 \) and \( E^a_\pm E^b_\pm = 0 \) on each of the 4 links. We now make iterative canonical transformations to fuse 4 link operators \( U(I) \) into three unphysical string flux operators \( U(1), U(2), U(3) \) and one physical Wilson loop operator \( W \) around OABC. The corresponding left, right string and loop electric fields are denoted by \( E^a_\pm(I) \) (\( I = 1, 2, 3 \)) and \( E^a_\pm \) respectively as shown in Figure 1. The first canonical transformation is:

\[
U(1) \equiv U(1), \quad U(2) \equiv U(1)U(2), \quad E^a_\pm(1) = E^a_\pm + E^a_\pm(2), \quad E^a_\pm(2) = E^a_\pm(2) \sim 0 \quad \text{on the}
\]

Note that the new sets are canonical and mutually independent and \( E^a_\pm(1) = E^a_\pm(1) + E^a_\pm(2) \sim 0 \) on the
physical Hilbert space. Further, only $U(1)$ changes under gauge transformations at $A$ and hence it decouples from the gauge invariant dynamics. We now iterate canonical transformation $\mathbf{[4]}$ with $U(1), U(2)$ replaced by $U(2), U(3)$ respectively:

$$
\mathcal{U}(2) \equiv \mathcal{U}(2), \quad \mathcal{U}(3) \equiv \mathcal{U}(2)U(3), \quad \mathcal{E}_+^a(2) = \mathcal{E}_+^a(2) + E_+^a(3), \quad \mathcal{E}_-^a(3) = E_+^a(3). \quad (7)
$$

We now have $\mathcal{E}_+^a(2) = \mathcal{E}_+^a(2) + E_+^a(3) = E_+^a(2) + E_+^a(3) \sim 0$ and $\mathcal{U}(2)$ decouples. Last canonical transformation is:

$$
\mathcal{U}(3) \equiv \mathcal{U}(3), \quad \mathcal{W} \equiv \mathcal{U}(3)U(4), \quad \mathcal{E}_+^a(3) = \mathcal{E}_+^a(3) + E_+^a(4) \sim 0, \quad \mathcal{E}_+^a = E_+^a(4). \quad (8)
$$

We now ignore the unphysical 3 canonical string pairs $[\mathcal{U}(I), \mathcal{E}_+^a(I)] \mathbf{|}_{I=1,2,3}$ and focus only on the canonical loop or plaquette pair $[\mathcal{W}, \mathcal{E}_+^a]$. They transform covariantly under SU(2) transformations at the origin O. The Gauss law constraint also takes a simple form:

$$
\mathcal{E}_+^a = \mathcal{W}_a \mathcal{L}_a = \mathcal{L}_a \mathcal{W}_a^\dagger, \quad \mathcal{W} \to \mathcal{W}, \quad \mathcal{E}_+^a = \mathcal{E}_+^a + \mathcal{E}_-^a = 0, \quad a = 1, 2, 3. \quad (9)
$$

Writing loop operator $W$ and electric fields $\mathcal{E}_+^a$ in terms of SU(2) prepotentials $\mathbf{[3]}$:

$$
\mathcal{E}_+^a = a^i \left( \frac{\sigma^a}{2} \right) a \mathcal{W}_a^\dagger = b^i \left( \frac{\sigma^a}{2} \right) b \mathcal{W}_a, \quad \mathcal{E}_-^a = a^i \mathcal{W}_a^\dagger = b^i \mathcal{W}_a \quad (10)
$$

In $\mathbf{[11]}$ the left, right electric fields satisfy $(\mathcal{E}_-^2)^2 = (\mathcal{E}_+^2)^2$ $(\mathcal{E}_- = -\mathcal{W} \mathcal{E}_+, \mathcal{W}^\dagger)$ implying $N_a = N_b \equiv N$ and $a_\alpha = e_{\alpha\beta} a_\beta$, with $e_{12} = e_{21} = 1, e_{11} = e_{22} = 0$. We also have a new $U(1)$ gauge invariance $\mathbf{[3]}$ where the prepotential operators $a^i, b^i$ of canonical loop pair $[\mathcal{E}_-^a, \mathcal{W}_a\beta]$ undergo opposite phase changes.

The correspondence between hydrogen atoms and lattice gauge theories starts with identifying $\mathbf{[1]}$ and $\mathbf{[11]}$. More precisely, the basis states in the electric representation of lattice gauge theory defined by the diagonalization of CSOC $[\mathcal{E}^2, \mathcal{E}_+^2, \mathcal{E}_-^2]$ correspond to CSOC-1 of hydrogen atom. Therefore, in one plaquette case with the above identification, an orthonormal and complete SU(2) basis in the electric representation is also the hydrogen atom basis $[j, m, \bar{m}]$ in $\mathbf{[2]}$. Further, the Gauss law constraint $\mathbf{[10]}$ becomes trivial in the coupled basis $\mathbf{[3]}$ leading to states $[n l m \bar{m}]$ as a complete orthonormal loop basis for one plaquette case. The addition of SU(2) fluxes and the hydrogen atom state $[n l m]$ over a plaquette is pictorially shown by a tadpole diagram in Figure-2b. The loop of the tadpole represents left-right self intertwining $(a_1, b_1)^{(2j-1)}$ of the SU(2) plaquette flux operator $W$ in $\mathbf{[12]}$. The vertical line of the tadpole represents the net SU(2) flux $\langle l, m \rangle$ emerging out of that plaquette. In a single plaquette case $\mathbf{[10]}$ implies $l = 0, m = 0$. This tadpole picture over one plaquette can be trivially extended to the entire lattice by first constructing tadpoles over each plaquette and then coupling their emerging fluxes $(l, m)$ in a sequential manner starting from plaquette at the origin. In case of a lattice with $p$ plaquettes, the Gauss law at origin O is simple:

$$
G^a = \sum_{i=1}^p L^a(i) = \sum_{i=1}^p \left( \mathcal{E}_-^a(i) + \mathcal{E}_+^a(i) \right) = L^a_{\text{total}} = 0. \quad (13)
$$

The resulting orthonormal and complete SU(2) gauge invariant loop basis on a lattice with $p$ plaquettes is:

$$
\mathbf{\{ n_1 n_2 \cdots n_p \mid l_1 l_2 \cdots l_p \mid l_{12} l_{13} \cdots \cdots l_{p-1} = l_p, l_{123} \cdots = 0 \}}_{m_{\text{total}}=0} = \mathbf{\{ n_1 n_2 \cdots n_p l_1 l_2 \cdots l_p m_1 m_2 \cdots m_{p-1} = 0 \}_{m_{\text{total}}=0}}. \quad (14)
$$

**FIG. 2:** The loop operators $\mathcal{W}_{a\beta}$ in single plaquette case $[a]$, four plaquette case $[c]$. The SU(2) loop network or hydrogen atom states $[n l m]$ in single plaquette case $[b]$, four plaquette case $[d]$. The $\bullet$ and $\circ$ in (b), (d) represent $jj$ and $ll$ couplings in $\mathbf{[3]}$ and $\mathbf{[14]}$ respectively.

The above loop basis will be briefly denoted by $[n l \mid l \mid l]$ where the symbols $[n l \mid l]$ stand for the sets $[n_1 n_2 \cdots n_p]; \{ l_1, l_2, \cdots, l_p \}$ and $(l_{12}, l_{13}, \cdots, l_{p-1} = l_p, l_{123} \cdots = 0)$ respectively. Each hydrogen atom state $[n l m]$ in $\mathbf{[14]}$ is explicitly constructed in $\mathbf{[2]}$ and $\mathbf{[3]}$. We further note that the loop states $\mathbf{[14]}$, are characterized by $N = 3(p - 1)$ gauge invariant angular momentum quantum numbers. This, as expected, is also the dimension of the quotient space $N = \frac{\text{Volume SU}(2)}{\text{Volume SU}(1)}$ in space dimension $d = 2$. 

![Figure 2](image-url)
It is interesting and illuminating to analyze this equivalence further in the (dual) magnetic representation. This duality also enables us to understand the continuum weak coupling ($g^2 \rightarrow 0$) limit of loop dynamics of non-abelian gauge theories in a completely new framework (section III). We first construct an orthonormal and complete magnetic basis which diagonalizes all Wilson loops and is dual to the electric basis \( [11] \). We again start with single plaquette basis states \(|nlm\rangle\) and make a duality or Fourier transform to states on SU(2) group manifold \( S^3 \):

\[
|\Omega_W\rangle = \sum_{j=0}^{\infty} \sum_{m,n=-j} \begin{pmatrix} j \end{pmatrix} D_{nm}^j(\Omega_W) \left| j, m, n \right\rangle 
\]

In \([13]\) \( \left| j, m \right\rangle \equiv (2j + 1)^{1/2} \) and the Wigner matrix \( D_{nm}^j(\Omega_W) \) is characterized by SU(2) group manifold \( S^3 \):

\[
\Omega_W (w_0, \vec{w}) = w_0 \sigma_0 + i \vec{w} \cdot \hat{\sigma}, \quad w_0^2 + \vec{w}^2 = 1 : S^3.
\]

Again, the angles on \( S^3 \) are defined by \( w_0 \equiv \cos \frac{\theta}{2}, \quad \vec{w}(\theta, \phi) \equiv \sin \frac{\theta}{2} \vec{w} \). The recursion relations of the Wigner matrices show \([13]\) that the orthonormal and complete angular states \([15]\) also diagonalize the magnetic loop operators \( W_{\alpha \beta} \) in \([12]\):

\[
W_{\alpha \beta} |\Omega_W\rangle = (\Omega_W (\omega, \vec{w}))_{\alpha \beta} |\Omega_W\rangle.
\]

Defining equation \([15]\) shows that under \( SU(2) \otimes SU(2) \) rotations \( \Lambda_L, \Lambda_R \) generated by \( (\xi_n^e, \xi_n^a) \) respectively:

\[
|\Omega_W\rangle \rightarrow |\Lambda_L \Omega_W \Lambda_R^+\rangle.
\]

The gauge transformation \( (\Lambda_L = \Lambda_R = \Lambda_0) \) generates \( L^1, L^2, L^3 \) in \([10]\) rotate \((w_0w_3), (w_0w_1), (w_1w_2)\) planes respectively leaving \( w_0 \) invariant. Defining “Lenz operators” \( A^0 = \xi_n^e = \xi_n^a \) in lattice gauge theory, we see that \( (A^1, A^2, A^3) \) generate rotations in \((w_0w_1), (w_0w_2), (w_1w_3)\) planes. Therefore, we identify \( S^3 \) of hydrogen atom with \( S^3 \) in the electric, magnetic representations respectively. The SO(4,2) generators in terms of SU(2) loop electric fields, loop flux operators form SO(4,2) algebra. We have defined \( \hat{W}^\pm \equiv e W^\pm \xi_n^a \) and \( a, b = 1, 2, 3; \mu, \nu = 1 \cdots 4 \). The 15 generators of SO(4,2) follow \([10]\):

\[
[L_{ab}, L_{cd}] = -i (g_{ad} L_{bc} + g_{ac} L_{bd} + g_{bd} L_{ac} + g_{bc} L_{ad}).
\]

In \([22]\), \( a, b = 1, 2 \cdots 6 \) and \( g_{ab} \) denotes the metric \((- \cdot - + +\cdot -). \)

The 15 generators of SO(4,2) follow \([10]\):

\[
H = g^2 \sum_{l=1}^4 \hat{E}^2(l) + \frac{1}{g^2} (2 - Tr(U(1)U(2)U(3)U(4)))
\]

III. SU(2) LOOP DYNAMICS & HYDROGEN ATOM DYNAMICAL SYMMETRY GROUP SO(4,2)

The “hydrogen atom basis” is a complete basis in the physical Hilbert space \( \mathcal{H}^0 \) of pure SU(2) lattice gauge theory. Therefore, we already expect that the SU(2) loop Hamiltonian will be in terms of the generators of the dynamical symmetry groups SO(4,2) of hydrogen atoms. We establish this starting again with a single plaquette case. The SO(4,2) generators in terms of SU(2) loop electric field and SU(2) loop flux operators are constructed in Table 1:

\[
L_{ab} = e_{abc} (\xi_n^e + \xi_n^a)
\]

\[
L_{ab} = e_{abc} (\xi_n^e - \xi_n^a)
\]

\[
L_{ab} = e_{abc} (\xi_n^e + \xi_n^a)
\]

\[
L_{ab} = e_{abc} (\xi_n^e - \xi_n^a)
\]

\[
|\Omega_W\rangle \equiv |\Omega_W\rangle \equiv |\Omega_W\rangle.
\]

Further, in the coupled basis \( [12] \) dual states \([15]\) takes even simpler form:

\[
|\Omega_W\rangle = \sqrt{2\pi^2} \sum_{n=0}^{n=\infty} \sum_{l=0, n-l=0}^{l=\infty} Y_{nlm} (\Omega_W) \left| n, l, m \right\rangle
\]

In \([19]\), \( n = (2j + 1) \) and \( Y_{nlm}(\Omega) \) are the hyperspherical harmonics obtained by relating \([11]\):

\[
Y_{nlm}(\Omega) = \sqrt{\frac{n}{2\pi^2}} \sum_{m=-n}^{m=n} C_{jm^l, jm^l} D_{m^l m^l}^j(\Omega).
\]

The hyperspherical harmonics \( Y_{nlm}(\Omega) \) in \([19]\) are the loop basis wave functions on \( S^3 \) in the magnetic representation. With \([15]\), they are also energy eigenfunctions of hydrogen atom on the momentum hypersphere \([11]\) founded by Fock in 1935. The Gauss law \([10]\) projects out \( |j\rangle \equiv |n = 2j + 1, l, l, m = 0 \rangle \) states:

\[
|\omega\rangle = \sum_j \begin{pmatrix} j \end{pmatrix} \chi_j (\omega) | j \rangle
\]

Above \( \omega \) is a gauge invariant angle and \( \chi_j (\omega) \equiv (\langle \sum_{m=-j}^{m=j} \langle \Omega_W | \sum_{m=-j}^{m=j} D_{m^l m^l}^j (\Omega_W) = \sin (j + 1/2) \omega / \sin (\omega/2) \rangle | \langle \sum_{m=-j}^{m=j} | \rangle ) \) is SU(2) character in the \( j \) representation. On a finite lattice the gauge invariant loop states are:

\[
|\omega\rangle; |\vec{w}_1 \cdot \vec{w}_2\rangle = k \sum_{\{(n)\}} Y_{[\{n\}][\{\mu\}]} (\langle \omega\rangle; |\vec{w}_1 \cdot \vec{w}_2\rangle)
\]
In loop space (after canonical transformations \([9, 14]\) and \([8]\) the Kogut Susskind Hamiltonian \([24]\) is \([4]\):

\[
H = 4g^2 \mathbf{E}^2 + \frac{1}{g^2} (2 - TrW) = g^2 (L_{56})^2 + \frac{1}{g^2} (L_{46})
\]

along with the Gauss law \([10]\) \(L^a = \epsilon^{abc} L_{bc} = 0\). In \([24]\) we have ignored purely \(g^2\) dependent constant terms. Similarly, on a finite lattice \(\Lambda\) with \(p = N \times N\) plaquettes, Kogut Susskind Hamiltonian in the loop space is \([9]\):

\[
H = \sum_{m,n \in \Lambda} \left\{ g^2 \left[ \mathcal{E}_- (m,n) + \mathcal{E}_+ (m,n-1) + \Delta_{XY} (m,n) \right] \right\}^2 \\
+ g^2 \left[ \mathcal{E}_+ (m,n) + R(W) \mathcal{E}_- (m,n-1) + \Delta_Y (m,n) \right] \right\}^2 \\
+ \frac{1}{g^2} (2 - Tr W(m,n)) \right\}.
\]

The electric field, flux operators \(\mathcal{E}_\pm (m,n), W(m,n)\) are located at \((m,n)\) as shown in Figure 2-c. We have defined

\[
\Delta_{XY}^a (m,n) \equiv \delta_{n,0} \sum_{r=m+1}^N \sum_{s=0}^N L^a(r,s),
\]

\[
\Delta_Y^a (m,n) \equiv \sum_{s=(n+1)}^N L^a(m,s), \quad R(W) \equiv \prod_{q=0}^{n-1} R(W(m,q))
\]

The loop Hamiltonian \([25]\) has a global SU(2) symmetry:

\[
\mathcal{E}_\pm (m,n) \rightarrow L_0 \mathcal{E}_\pm (m,n) \Lambda_0^\dagger, \\
W(m,n) \rightarrow L_0 W(m,n) \Lambda_0^\dagger.
\]

This global invariance is fixed by the Gauss law \([13]\) at the origin. The Hamiltonian \([25]\) describes dynamics of Wilson loops and has many useful novel features. The magnetic term, which dominates in \(g^2 \rightarrow 0\) weak coupling continuum limit, has been cast in its simplest possible (abelian) form. In standard loop formulation the action of this magnetic field term is extremely complicated (18-j, 30-j for \(d=2,3\)) and a stumbling block towards weak coupling loop perturbation theory \([3, 8]\). Now in magnetic basis \([21]\) it is \(\cos(q)\) for each plaquette in any dimension. Further, the magnetic representation \([21]\), unlike electric representation \([14]\), is free of notorious Mandelstam as well as triangular constraints. All interactions are in electric field terms which are proportional to \(g^2\). They can be handled in weak coupling loop perturbation theory exploiting SO(4,2) tensor properties while treating the magnetic term exactly. The non-local terms \(\Delta_{XY}, \Delta_Y, R(W)\) can be made local by using auxiliary fields. These issues will be addressed elsewhere. From experimental angle, hydrogen atom interpretation of \(H^E\) and absence of local gauge invariance should bypass the challenging task of imposing non-trivial and exotic non-abelian Gauss law constraints all over the lattice \([12]\).

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[14] We consider \(d = 2\) for technical convenience. The equivalence can be generalized to arbitrary \(d\) dimensions.

[15] We follow Wybourne for hydrogen atom discussions. The Runge Lenz vector has been scaled by \(\sqrt{2}\) for bound states \([10]\).

[16] Thus the highly non-trivial problem of non-local Mandelstam constraints and their solution \([2]\) in the space of all Wilson loops does not even arise.