Gauge fields and quantum entanglement

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A B S T R A C T

The purpose of this letter is to explore the relation between gauge fields, which are at the base of our understanding of fundamental interactions, and the quantum entanglement. To this end, we investigate the case of SU(2) gauge fields. It is first argued that holonomies of the SU(2) gauge fields are naturally associated with maximally entangled two-particle states. Then, we provide some evidence that the notion of such gauge fields can be deduced from the transformation properties of maximally entangled two-particle states. This new insight unveils a possible relation between gauge fields and spin systems, as well as contributes to understanding of the relation between tensor networks (such as MERA) and spin network states considered in loop quantum gravity. In consequence, our results turn out to be relevant in the context of the emerging Entanglement/Gravity duality.

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1. Introduction

Entanglement is usually considered to be a solely quantum mechanical phenomenon. However, it has been broadly argued in the recent years that classical geometry provides a description of the structure of entanglement in some quantum systems. One way to uncover this relation is by using tensor networks (graphs constructed from contracted tensors), which represent wave functions of certain many-body quantum systems [1]. In particular, entanglement renormalization of a many-body system, performed in terms of tensor networks, leads to anti-de Sitter (AdS) spacetime geometry [2], which plays a central role in the AdS/CFT correspondence [3]. While the tensor networks are discrete objects, their continuous limits are possible to define [4]. This opens a possibility that continuous space-time may be considered as an approximation to a discrete tensor network representing the entanglement structure of a certain discrete (e.g. spin) system. In the holographic language, the quantum system under consideration represents the boundary and the bulk geometry is represented by the entanglement structure between quantum degrees of freedom at the boundary. Further evidence for such a viewpoint comes from considerations of the entanglement entropy [5] and quantum complexity [6].

While the bulk geometry in the above context is considered to be classical, theories such as loop quantum gravity (LQG) [7,8] provide the quantum viewpoint on the nature of spacetime. The question that arises is whether the quantum description of spacetime is consistent with the (classical) geometry describing the entanglement of a many-body quantum system or a field theory. However, at least in the LQG approach, a possibility to merge the two viewpoints occurs. Namely, the spatial geometry in LQG is described by a graph called the spin network, which is an object built from the holonomies of Ashtekar su(2) gauge fields [9]. A state of the spin network is constructed as a contraction of the holonomies, so that the obtained function is invariant under local gauge transformations (i.e. it is annihilated by the Gauss constraint).

It was argued in [10] that coarse-graining of spin network states leads to the emergence of a tensor network at the boundary of space. The underlying reason is that when open spin networks are considered, the endpoints of open links, which explicitly break the local gauge invariance, can be associated with the (boundary) degrees of freedom. On the other hand, (superpositions of) spin networks themselves can be treated as tensor networks [11] (see also [12] for a generalization of this relation to the group field theory framework). The links (holonomies) of spin networks are, therefore, expected to be inevitably related to entanglement [13] (in particular, maximally entangled states impose specific gluing conditions on spatial polyhedra associated with spin network vertices [14]).

The purpose of this letter is to explore these relationships further. However, let us stress that while our inspiration comes from LQG, the analysis will not be restricted to this theory – it concerns a general relation between holonomies and (maximally) entangled states.
2. Gauge fields

Let us begin our considerations from a su(2) gauge field $A_{\mu}^\alpha$, with algebraic indices $\alpha = 1, 2, 3$ and spatial indices $a = 1, 2, 3$. The field is characterized by the connection 1-form $A = A_{\mu}^\alpha \tau^\alpha \mathbf{d}x^\mu$, where $\tau_i$ generate the su(2) algebra $[\tau_i, \tau_j] = i\varepsilon_{ijk} \tau_k$. In the fundamental representation, $\tau_i$ are related to Pauli matrices via the relation $\tau_i = -\frac{i}{2} \sigma_i$.

The essential feature of gauge field theories is invariance with respect to local gauge transformations:

$$A_{\mu}^\alpha \rightarrow A_{\mu}^{\prime \alpha} = U^\dagger A_{\mu}U + U^\dagger \partial_\mu U ,$$

where $U$ is a certain unitary matrix. While the form of a transformation of the gauge field rather does not tell us too much directly, there are other objects that allow us to look at the gauge transformation from slightly different perspective. An example of such an object on which we are going to focus our attention is the holonomy of a gauge field, which is an SU(2) element defined as follows:

$$h_{e}[A] := \mathcal{P} \exp \int_e A ,$$

where $e$ is a path $e : \{0, 1\} \rightarrow \Sigma$, intermediating between the source $e(0) = s$ and target $e(1) = t$ points on a spatial hypersurface $\Sigma$, and $\mathcal{P}$ denotes the path ordering. The holonomy is clearly a non-local object, which under the transformation (1) transforms as

$$h_{e}[A] \rightarrow h_{e}^{\prime}[A] = U^\dagger (e(0)) h_{e}[A] U (e(1)) = U^\dagger h_{e}[A] U_t ,$$

where for further convenience we defined $U_s := U(e(0))$ and $U_t := U(e(1))$. The gauge transformation acts at the holonomy only at the endpoints. This property is widely used in the construction of lattice gauge theories, especially by introducing gauge-invariant Wilson loops $W_e[A] := \text{tr}(h_{e}[A])$, where $e$ is then a closed path.

3. Unitary map

In the gauge theory context, holonomies are parallel-transported elements of the gauge group. However, they can also be treated as isomorphisms between certain linear spaces. In order to see this explicitly, let us consider the holonomy (2) in the case of the fundamental representation of SU(2), i.e. spin-1/2. Then, the holonomy is given by a $2 \times 2$ SU(2) matrix, which belongs to the automorphism group of $\mathbb{C}^2$ (i.e. the space of non-relativistic spinors).

We will now make the qualitative jump to quantum mechanics, using the fact that $\mathbb{C}^2$ equipped with the natural scalar product becomes the Hilbert space of a qubit system. (Pure) physical states correspond to rays in $\mathbb{C}^2$, i.e. elements of $\mathbb{CP}^1$, which can be represented as the (SU(2)-invariant) unit sphere. Thus, the holonomy is an isomorphism between the two 2-dimensional (projective) Hilbert spaces $\mathcal{H}_s = \text{span} \{|0\rangle_s, |1\rangle_s\}$ and $\mathcal{H}_t = \text{span} \{|0\rangle_t, |1\rangle_t\}$, in which we choose orthonormal bases, i.e. $\langle 0|1\rangle_s = \delta_{11}$ and $\langle 0|1\rangle_t = \delta_{01}$, where $I, J = 0, 1$. $\mathcal{H}_s$ and $\mathcal{H}_t$ are assumed to be associated with two different points in space. Since now the holonomy (2) is represented by a unitary matrix, the corresponding map (isomorphism) $h$ is unitary as well. Let us now see whether a transformation (3) of such a map arises in some natural way in the quantum-mechanical context.

Employing the bases of the source and target Hilbert spaces ($\mathcal{H}_s$ and $\mathcal{H}_t$), it is convenient to express an arbitrary unitary map between $\mathcal{H}_s$ and $\mathcal{H}_t$ as

$$h = h_{1J}[l]_{st} (J) \in \mathcal{H}_s \otimes \mathcal{H}_t^* ,$$

where $h_{1J}$ are matrix elements of $h$ and $\mathcal{H}_s^*$ is the space dual to $\mathcal{H}_s$. The action of this map can be either left-handed or right-handed, so that $h_l : \mathcal{H}_s^* \rightarrow \mathcal{H}_t^*$ and $h_r : \mathcal{H}_t \rightarrow \mathcal{H}_s$. Analogously, the Hermitian conjugation of $h$, $h^\dagger = h^\dagger_{1J}[l_{st}]_l \in \mathcal{H}_t \otimes \mathcal{H}_s^*$, acts as $h^\dagger_l : \mathcal{H}_s^* \rightarrow \mathcal{H}_s^*$ or $h^\dagger_r : \mathcal{H}_s \rightarrow \mathcal{H}_t$. For example, a basis state $|s\rangle \in \mathcal{H}_s^*$ at a point $s$ is mapped according to $|s\rangle h = h_{1J}[s]_s |t\rangle = h_{kJ}[s]_s |t\rangle \in \mathcal{H}_t^*$ into a certain state at a point $t$. We may proceed to the crucial point. The choice of basis in both the source and target Hilbert space is completely arbitrary. Therefore, we can ask how the map (4) behaves under the action of unitary transformations that change these bases. The transformations at the source and target can in general be different and we will distinguish them by using the indices $s$ and $t$. A unitary transformation of a basis state $|l\rangle$ can be written as $|l\rangle' = U_l |l\rangle$ or $|l\rangle' = U_t |l\rangle$. The transformation of bases at the source $s$ and the target $t$ has been pictorially presented in Fig. 1.

![Fig. 1. Graphical representation of the change of bases of the source and the target Hilbert spaces, associated with the space points $s$ and $t$ respectively.](image)

Applying such transformations to both source and target bases in (4), we find that

$$h_{IJ}[l]_{st} (J) = h^\dagger_{IJ}[l']_{st} (J) = U_s,k h_{sJ} U^\dagger_{l,t} |K\rangle_{st} (L) ,$$

which leads to the following transformation rule:

$$h \rightarrow h' = U^\dagger_l h U_t ,$$

which is graphically represented in Fig. 2.

![Fig. 2. Graphical representation of the transformation rule of holonomy under the change of bases.](image)

It is clear that the above change of $h$ under unitary transformations $U_l$ and $U_t$ in the source and target spaces is formally equivalent to the action of a SU(2) gauge transformation. At least in this sense, a map (4) shares properties of a holonomy along the curve connecting $s$ and $t$. Analysis in the next section will underscore important consequences of this simple fact. However, if and how a map (4) may inherit any information about the curve is a much more involved issue, which we do not consider here. In the next sections of this letter we will provide some evidence for the conjecture that SU(2) gauge transformations reflect the invariance of physics of the corresponding quantum system under unitary transformations of both bases.
4. Anti-linear map

Since the holonomy is associated with a pair of Hilbert spaces \( \mathcal{H}_s \) and \( \mathcal{H}_t \), it is natural to ask whether there are some interesting states belonging to \( \mathcal{H}_s \otimes \mathcal{H}_t \) that can be defined using \( h \). Following Refs. [16,17], let us consider a state

\[
|\Psi\rangle := \frac{1}{\sqrt{2}} h_{IJ}^\dagger |I\rangle_s |J\rangle_t \in \mathcal{H}_s \otimes \mathcal{H}_t ,
\]

where \( h_{IJ} \) are matrix elements of the \( SU(2) \) holonomy.

The fact that coefficients of the state (7) are given by components of a special unitary matrix has profound consequences. Namely, this implies that the state (7) belongs to the class of \textit{maximally entangled states}, for which the reduced density matrix is diagonal (as it is well known in quantum information theory; the space of such states for a 2-qubit system is actually \( SU(2)/Z_2 \) [18]). Explicitly, the density matrix associated with the state (7) has the form

\[
\hat{\rho} = |\Psi\rangle \langle \Psi | = \frac{1}{2} \left( |I\rangle_s \langle I| \otimes |J\rangle_t \langle J| \right) + \frac{1}{2} \left( |I\rangle_s \langle J| \otimes |J\rangle_t \langle I| \right).
\]

In consequence, the reduced density matrix \( \hat{\rho}_s := \text{tr}_t (\hat{\rho}) = \frac{1}{2} h_{IJ}^\dagger h_{IJ}^\ast \langle I|_s \langle J|_t \rangle = \frac{1}{2} I_s \), where unitarity of the matrix \( h_{IJ} \) has been used. The analogous formula can be obtained for \( \hat{\rho}_t := \text{tr}_s (\hat{\rho}) \).

While distinguishing the state (7) might seem arbitrary, it has been shown in [17] that (7) can be used to define the holonomy map between the (dual) source space and target space. Namely, the idea is to consider an appropriate anti-linear map (each such a map can be decomposed into a linear map and the complex conjugation \( C \); in the physical context, \( C \) can arise e.g. via the CPT transformation), which in our notation acts on basis states as:

\[
\mathcal{H}_s^* \ni s \rightarrow \sqrt{2} |\Psi\rangle \circ C \left( \left| \langle I| \rightangle \right) = h_{IJ}^\dagger \left| \langle J| \rightangle \in \mathcal{H}_t ,
\]

where the state \( |\Psi\rangle \) is given by (7). Applying the operation \( Q \equiv \sqrt{2} |\Psi\rangle \circ C \) to an arbitrary state \( c_i |i\rangle_s \), we have:

\[
\mathcal{H}_s^* \ni c_i |i\rangle_s \rightarrow \langle i| \rightarrow \sqrt{2} \left( U^\ast_{s,I} \right)^\ast s \langle j| \Psi \rangle =: U^\ast_{s,I} h_{IJ} U_{s,I}^\dagger |j\rangle_t ,
\]

The transformation rule of coefficients \( h_{IJ} \) under the basis change is therefore:

\[
h_{IJ} \rightarrow h'_{IJ} = U^\ast_{s,I} h_{IJ} U_{s,I}^\dagger ,
\]

which is consistent with the gauge transformation of the holonomy (cf. (3) and (6)).

5. Spatial entanglement from holonomies

The discussion presented so far indicates the existence of a non-trivial relation between holonomies of gauge fields and maximally entangled quantum states. Ref. [16] has introduced the concept of \textit{entanglement holonomies}, used to define the quantum version of parallel transport between reference frames. This is achieved by the quantum teleportation of an auxiliary state via a maximally entangled state shared by the frames. Here, we would like to present a different perspective.

Let us consider the following situation. Initially, we have two qubits at the source (the analogous reasoning can be done for the target) and a total state of the system can be written as:

\[
|\phi_s\rangle = S_{KL} |K\rangle_s |L\rangle_s \in \mathcal{H}_s \otimes \mathcal{H}_s ,
\]

where \( S_{KL} \) are some coefficients. Next, we would like to map one of the qubits from the source to the target. This means that we take one of the qubits and move it (in the gauge field \( A \)) from \( s \) to another space point \( t \). If we choose the second qubit to be the one that is moved, the corresponding unitary map is:

\[
\mathbb{I} \otimes h^\ast_{KL} : \mathcal{H}_s \otimes \mathcal{H}_s \rightarrow \mathcal{H}_s \otimes \mathcal{H}_t
\]

\[
(11)
\]

\( h^\ast_{KL} \) is the right-hand action of the Hermitian conjugation of the holonomy \( h \), which leads to

\[
|\phi_s\rangle \rightarrow S_{KL} |K\rangle_s h^\ast |L\rangle_s = S_{KL} h^\ast_{KL} |K\rangle_s |L\rangle_t = C_{KL} |K\rangle_s |L\rangle_t =: |\phi_t\rangle \in \mathcal{H}_s \otimes \mathcal{H}_t .
\]

(12)

Coefficients of the obtained state \( |\phi_t\rangle \) are related to coefficients of \( |\phi_s\rangle \) through the relation \( C_{KL} = S_{KL} h^\ast_{KL} \). If the state \( |\phi_s\rangle \) is equivalent to (7), then \( C_{KL} = \frac{1}{\sqrt{2}} h^\ast_{KL} \) and in consequence \( S_{KL} = \frac{1}{\sqrt{2}} \delta_{KL} \).

In such a case, \( |\phi_t\rangle \) is a Bell state: \( |\phi_t\rangle = \frac{1}{\sqrt{2}} (|0\rangle_s |0\rangle_t + |1\rangle_s |1\rangle_t) = |\Phi^\ast\rangle \).

Therefore, a maximally entangled state given by Eq. (7) can be obtained via the following map:

\[
|\Psi\rangle = (\mathbb{I} \otimes h^\ast_{KL}) |\Phi^\ast\rangle .
\]

(13)

A maximally entangled state of two qubits at two space points \( s \) and \( t \) can be considered the result of the holonomy acting on one of the two qubits initially located at the same space point. It is, however, necessary that a 2-qubit state is initially the \( |\Phi^\ast\rangle \) Bell state. Therefore, the above construction requires the initial entanglement of qubits, which can be generated from a non-entangled state as e.g. \( |\Phi^\ast\rangle = \text{CNOT} \circ (H \otimes I) |0\rangle_s |0\rangle_t \), where CNOT and \( H \) denote respectively the controlled-NOT and Hadamard quantum gates.

6. Gauge transformation from entanglement

We have shown that maximally entangled 2-particle quantum states emerge from considerations of holonomies of gauge fields. One may now ask whether this relation could also work in the opposite direction. As an inspiration, it is worth to mention that systems of spins (which, in the quantum information context, describe qubits) have been widely studied as the infrared approximation of SU(2) Yang-Mills theory [15]. The first thing we recall here is that any maximally entangled 2-qubit state is characterized by a \( SU(2) \) matrix (up to the \( Z_2 \) degeneracy, i.e. the overall sign).

For simplicity, and to give an explicit example of the employed procedure, let us consider the following 2-qubit singlet state:

\[
|\Psi\rangle = |\Phi^\ast\rangle := \frac{1}{\sqrt{2}} (|0\rangle_s |1\rangle_t - |1\rangle_s |0\rangle_t) .
\]

(14)

Comparing this state with (7), we observe that its coefficients correspond to the \( SU(2) \) matrix:

\[
h = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i \sigma_y = e^{i \frac{\pi}{2} \sigma_y} .
\]

(15)

\( h \) is again interpreted as a unitary map (holonomy) between the \( \mathcal{H}_s \) and \( \mathcal{H}_t \) Hilbert spaces, which transforms under the change of their bases according to (6). In general, we have a three-parameter family of unitary basis transformations. Here, for simplicity, let us consider a one-parameter family of rotations:

\[
U(\theta) = e^{-i \theta \sigma_y} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} .
\]

(16)

The parameter \( \theta \) is assumed to change smoothly between the values \( \theta_s := \theta(e(0) = s) \) and \( \theta_t := \theta(e(1) = t) \), corresponding to \( U_s \) and \( U_t \), as well as to satisfy the limiting condition \( \lim_{t \rightarrow s} \theta_t = \theta_s \)
(see also the discussion around (21)). Under such a transformation of both bases, the map (15) transforms as follows:
\[
\hat{h} \rightarrow \hat{h}' = U^\dagger_t U_I = e^{i \lambda \sigma_y - i (\hat{a}^\dagger - \hat{a}) \sigma_y} = he^{-i \sigma_y \int_{t}^{0} \hat{a} \hat{a}^\dagger dx^t}.
\] (17)

One can, therefore, conclude that the matrix (15) can be written as exponentiation of a certain integral, such that its integrand under the change of bases transforms as
\[
\text{integrand} \rightarrow \text{integrand} - i \hat{a} \hat{a}^\dagger \theta.
\] (18)

which has the same form as a U(1) gauge transformation. The integrand is what we can call a gauge field. This is how the concepts of a gauge symmetry and gauge field can be deduced from considerations on maximally entangled states.

In the considered example, the apparent U(1) gauge symmetry is actually a remnant of the SU(2) gauge symmetry restricted to the case of a one-parameter family of transformations. This can be seen by substituting (16) into the transformation rule (1). From the form of (15), we see that the only non-vanishing component of the gauge field is \( A^2_y \) (which is contracted with \( \sigma_2 = \sigma_y \)). In consequence, the gauge transformation reduces to:
\[
A^2_x \rightarrow A^2_x' = A^2_x - i \hat{a} \hat{a}^\dagger \theta
\] (19)

and, making a comparison with (17), one can conclude that
\[
\int_{t}^{0} A^2_x dx^t = -2 \pi.
\]

The above observations should be confirmed by analysis of the general case but such considerations will be rather computationally involving and are beyond the scope of this letter. Therefore, instead of this, let us make some crucial remarks and later provide another simple example. A general SU(2) transformation of basis of either \( \mathcal{H}_s \) or \( \mathcal{H}_t \) can be obviously expressed in the form:
\[
U(\hat{a}) = e^{i \hat{a} \cdot \hat{\sigma}} = \cos \alpha \mathbb{I} + i \sin \alpha (\hat{n} \cdot \hat{\sigma}),
\] (20)

parametrized by a vector \( \hat{a} = a \hat{n} \), where \( a = \sqrt{\hat{a} \cdot \hat{a}} \) and \( \hat{n} \) is a versor. In order to be able to obtain the correspondence between a maximally entangled 2-particle state and a holonomy, we first have to restrict the allowed transformations to pairs of \( U_s(\hat{a}_s) \) and \( U_t(\hat{a}_t) \) that satisfy the conditions
\[
\lim_{t \rightarrow s} U_t U_s \Leftrightarrow \lim_{t \rightarrow s} \hat{a}_t = \hat{a}_s, \quad \forall a : (\hat{a}_s \cdot t) \in C^1(\Sigma),
\] (21)

which means in particular that \( U_s \) and \( U_t \) are not independent. The next step is to make an extension to families of transformations \( U(\hat{a}(\tau)) \) acting in a family of Hilbert spaces \( \mathcal{H}_\tau \), such that \( U(\hat{a}(0)) = U_s \) and \( U(\hat{a}(1)) = U_t \), as a generalization of (16). Subsequently, one should check whether \( \hat{h} \) is indeed equivalent to a holonomy of the gauge field.

As an illustration, we will now consider the second example, given by the state
\[
|\Psi\rangle = |\Phi^+\rangle := \frac{1}{\sqrt{2}} (|0\rangle_s |0\rangle_t + |1\rangle_s |1\rangle_t).
\] (22)

for which holonomy is simply an identity operator, \( h = \mathbb{I} \). If we choose the vectors \( \hat{a}_s, \hat{a}_t \) at the source and target, respectively, the transformation (6) takes the form:
\[
\hat{h} \rightarrow \hat{h}' = U^\dagger_t U_I = e^{iu N \hat{\sigma}},
\] (23)

where
\[
\cos \alpha = \cos a_s \cos a_t + \sin a_s \sin a_t \hat{n}_s \cdot \hat{n}_t,
\] (24)
\[
\sin \alpha \hat{N} = \sin a_s \sin a_t (\hat{n}_s \times \hat{n}_t) + \cos a_s \sin a_t \hat{n}_t - \sin a_t \cos a_s \hat{n}_s.
\] (25)

While in general, \( \hat{a}_s \) and \( \hat{a}_t \) are different, the case with \( \hat{a}_s = \hat{a}_t =: \hat{n} \) is especially illustrative and corresponds to rotations around a fixed axis. Then, (24) reduces to \( \cos \alpha = \cos(a_s - a_t) \) and (25) becomes \( \hat{N} = \hat{n} \). In consequence, the transformation (23) simplifies to
\[
\hat{h} \rightarrow \hat{h}' = U^\dagger_t U_I = e^{i (\hat{a} \cdot \hat{n} \cdot \hat{\alpha}) \sigma} = e^{i (\int_{t}^{0} \hat{a} \cdot \hat{n} dx^t) \sigma}.
\] (26)

The last expression has been inferred under the assumption that \( U_s, \ U_t \) are the initial and final elements of a family of operators constructed by continuously changing the parameter vector \( \hat{a} \) of the group element, in accordance with the postulates (21) and below. This allows to associate a curve \( e \) in space to any such family. It turns out that (26) agrees with what is expected from considerations of gauge transformations (1). Since for the example \( h = \mathbb{I} \) we have \( A_0 = 0 \), the pure gauge obtained by applying to \( h \) a gauge transformation (1) leads to the following contribution to the exponent of the holonomy:
\[
\int_{e} U^\dagger \partial_{dx^a} U dx^a = i \bar{\sigma} \cdot \int_{e} (\cos \theta \partial_a \hat{n} + \hat{n} \partial_a \cos \theta + \hat{n} \times \partial_a \hat{n}) dx^a,
\] (27)

where \( \hat{n} := \sin \theta \hat{n}. \) It is straightforward to check that the above formula gives the exponent in (26) for \( \hat{n} = \text{const}. \)

7. Spin networks and tensor networks

We are now ready to briefly discuss application of the prior discussion to spin networks and tensor networks. Since full discussion of the issue goes beyond the scope of this letter, we will only refer to some essential observations.

Firstly, spin networks (without open links), which span the Hilbert space of LQP, by virtue of the loop transformation can be expressed as sums of products of the Wilson loops for the case of the fundamental representation of SU(2) [19]. As it is clear from the definition (7), components of a holonomy are obtained by projecting the state \( |\Psi\rangle \) on the basis states:
\[
h_{IJ} = \sqrt{2} ((i_{J}|i_{I})|\Psi\rangle)^* = \sqrt{2} (|\Psi\rangle (|I\rangle_{I})_{I})
\] (28)

and a Wilson loop (for which \( s = t \)) can be expressed as
\[
W_{e}[h] = \sum_{I} h_{IJ} = \sqrt{2} \sum_{I} (|\Psi\rangle (|I\rangle_{I})_{I}).
\] (29)

Therefore, the Wilson loop is a certain amplitude associated with the state \( |\Psi\rangle \). In consequence, the full spin network state can be expressed in terms of amplitudes of its constituent loops described by the corresponding states \( |\Psi\rangle \).

Secondly, using (13), one can define the unitary map:
\[
U_{\Psi} := (I \otimes h^\dagger_{E})(\text{CNOT})(H \otimes I).
\] (30)

which takes the state \( |0\rangle_s |0\rangle_t \) and generates the maximally entangled state (7) between two particles (qubits) at the space points \( s \) and \( t \). The map can be used as a building block for the construction of states of multiple particles located at different space points. Such a construction is in the spirit of tensor networks, an example of which are quantum circuits. The part a) of Fig. 3 contains a graphical representation of (30).

The map, assuming that one of the inputs is an ancilla qubit in the state \( |0\rangle \), allows us to build a MERA tensor network representing a state of several particles located at different positions.
However, not necessary the MERA type structures have to be considered. An example is shown in the part b) of Fig. 3, which depicts a circuit generating a state of four particles moved to space points \(x_1, x_2, x_3, x_4\) from their initial location at \(x_1\). The part c) of Fig. 3 represents the stages of distributing qubits to different space points.

Thirdly, tensor networks built with the use of blocks (30) may correspond to the spin network states. However, because Gauss constraint has to be satisfied at the nodes of spin networks, the gauge invariance (which is equivalent to the Gauss constraint) has to be imposed afterward. One possibility to achieve this is by projecting the obtained state onto the spin network basis.

8. Summary

The purpose of this letter was to emphasize the relation between (holonomies of) gauge fields and maximally entangled 2-particle states (belonging to the Hilbert space shared by subsystems at two different points in space). The analysis has been performed for the case of fundamental representation of the SU(2) gauge group but a generalization to the arbitrary \(j\) representation is straightforward. In such a case, the state \(|\Psi\rangle\) generalizes to

\[
|\Psi\rangle := \frac{1}{\sqrt{2j+1}} H_{ij} |l_{1}\rangle |j\rangle,
\]

with \(l, j = 0, \ldots, 2j\). Since an example of the gauge field is the Ashtekar connection of the gravitational

field, the presented study provides in particular a new perspective on the correspondence between the structure of entanglement and the (quantum) geometry of spacetime.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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