BLOCH-WIGNER THEOREM OVER RINGS WITH MANY UNITS II

BEHROOZ MIRZAI AND FATEMEH YEGANEH

Abstract. In this article we study the Bloch-Wigner theorem over a domain with many units. Our version of Bloch-Wigner theorem is very close to Suslin’s version of the theorem over infinite fields.

Introduction

For a commutative ring $R$ with 1, there are two types of algebraic $K$-groups: Milnor $K$-groups and Quillen $K$-groups, denoted by $K^n_M(R)$ and $K^n(R)$ respectively. For any $n \geq 1$, there is a canonical homomorphism $\iota_n : K^n_M(R) \to K^n(R)$.

When $R$ is a field or a ring with many units, it is known that $\iota_1$ and $\iota_2$ are isomorphisms [7], [17], [11], [4]. For $n > 2$, $\iota_n$ is not an isomorphism most of the times. In fact the kernel of $\iota_n$ is annihilated by multiplication by $(n-1)!$ [11], [4] and the cokernel of $\iota_n$ can be very large [13]. The group

$$K_n(R)_{\text{ind}} := \text{coker}(K^n_M(R) \to K^n(R))$$

is called the indecomposable part of $K_n(R)$.

The Bloch-Wigner theorem studies the indecomposable part of $K_3(C)$ and asserts the existence of the exact sequence

$$0 \to \mathbb{Q}/\mathbb{Z} \to K_3(C)_{\text{ind}} \to \mathfrak{p}(C) \to \bigwedge^2_{\mathbb{Z}} C^* \to K_2(C) \to 0,$$

where $\mathfrak{p}(C)$ is the pre-Bloch group of $C$ [1], [3].

In a remarkable paper [16], Suslin has generalized this result to all infinite fields. In fact he proved that for any infinite field $F$ the sequence

$$0 \to \text{Tor}^\mathbb{Z}_1(\mu(F), \mu(F)) \to K_3(F)_{\text{ind}} \to \mathfrak{p}(F) \to \big(F^* \otimes_{\mathbb{Z}} F^*\big)_w \to K_2(F) \to 0$$

is exact, where $\text{Tor}^\mathbb{Z}_1(\mu(F), \mu(F))$ is the unique non-trivial extension of $\text{Tor}^\mathbb{Z}_1(\mu(F), \mu(F))$ by $\mu_2(F)$.

It was known for very long time that $K_3(C)_{\text{ind}} \simeq H_3(SL_2(C), \mathbb{Z})$. One can show that for an infinite field $F$, there is a canonical homomorphism

$$H_0(F^*, H_3(SL_2(F), \mathbb{Z})) \to K_3(F)_{\text{ind}}.$$

Suslin has asked that is this map is bijective [13, Question 4.4]? Recently Hutchinson and Tao have shown that it is in fact surjective [5, Lemma 5.1].

These facts and Suslin’s question raise the question that is there a version of the Bloch-Wigner exact sequence that involves a homology group very close to $H_0(F^*, H_3(SL_2(F), \mathbb{Z}))$, replacing $K_3(F)_{\text{ind}}$!
In [9], the first author studied this question. In the meantime he tried to give a general form of Bloch-Wigner exact sequence, valid over rings with many units. (We should mention that Theorem 5.1 in [9] is not correct. A correct formulation of that theorem, which is very close to Proposition 4.2 below, will appear in an erratum to [9].)

In the present article, we give a version of the Bloch-Wigner theorem over any ring with many units (Proposition 4.2). This recovers and also improves the main results of [9]. Our proof here is different than the one given in [9]. When $R$ is a domain with many units, e.g. an infinite field, we make our formulation of the Bloch-Wigner exact sequence more precise (Theorem 4.4). In fact we prove that there exists the exact sequence

$$0 \to \text{Tor}^\mathbb{Z}_1(\mu(R), \mu(R)) \to \tilde{H}_3(\text{SL}_2(R)) \to p(R) \to (R^* \otimes \mathbb{Z} R^*)_a \to K_2(R) \to 0,$$

where $\tilde{H}_3(\text{SL}_2(R))$ is the following quotient of the group $H_3(\text{GL}_2(R), \mathbb{Z})$,

$$\tilde{H}_3(\text{SL}_2(R)) := H_3(\text{GL}_2(R), \mathbb{Z})/\text{im}(H_3(\text{GL}_1(R), \mathbb{Z}) + R^* \cup H_2(\text{GL}_1(R), \mathbb{Z})).$$

This exact sequence and Suslin’s Bloch-Wigner exact sequence suggest that $K_3(F)^{\text{ind}}$ and $\tilde{H}_3(\text{SL}_2(F))$ should be isomorphism. But there is no natural homomorphism from one of these groups to the other one! But there is a natural maps from $H_0(F^*, H_3(\text{SL}_2(F), \mathbb{Z}))$ to both of them. These relation will be studied somewhere else.

**Notation.** In this paper by $H_i(G)$ we mean the homology of group $G$ with integral coefficients, namely $H_i(G, \mathbb{Z})$. By $\text{GL}_n$ (resp. $\text{SL}_n$) we mean the general (resp. special) linear group $\text{GL}_n(R)$ (resp. $\text{SL}_n(R)$), where $R$ is a commutative ring with 1. If $A \to A'$ is a homomorphism of abelian groups, by $A'/A$ we mean $\text{coker}(A \to A')$ and we take other liberties of this kind. For a group $A$, by $A_{\mathbb{Z}[1/2]}$ we mean $A \otimes_{\mathbb{Z}} \mathbb{Z}[1/2]$.

1. **Suslin’s block-Wigner exact sequence**

Let $R$ be a commutative ring with 1. Define the pre-Bloch group $p(R)$ of $R$ as the quotient of the free abelian group $Q(R)$ generated by symbols $[a], a, 1 - a \in R^*$, by the subgroup generated by elements of the form

$$[a] - [b] + \left[ \frac{b}{a} \right] - \left[ \frac{1 - a^{-1}}{1 - b^{-1}} \right] + \left[ \frac{1 - a}{1 - b} \right],$$

where $a, 1 - a, b, 1 - b, a - b \in R^*$. Define

$$\lambda' : Q(R) \to R^* \otimes R^*, \quad [a] \mapsto a \otimes (1 - a).$$

By a direct computation, we have

$$\lambda'([a] - [b] + \left[ \frac{b}{a} \right] - \left[ \frac{1 - a^{-1}}{1 - b^{-1}} \right] + \left[ \frac{1 - a}{1 - b} \right]) = a \otimes \left( \frac{1 - a}{1 - b} \right) + \left( \frac{1 - a}{1 - b} \right) \otimes a.$$
Let \((R^* \otimes R^*)_p := R^* \otimes R^*/(a \otimes b + b \otimes a : a, b \in R^*)\). We denote the elements of \(p(R)\) and \((R^* \otimes R^*)_p\) represented by \([a]\) and \(a \otimes b\) again by \([a]\) and \(a \otimes b\), respectively. Thus we have a well-defined map
\[
\lambda : p(R) \to (R^* \otimes R^*)_p, \quad [a] \mapsto a \otimes (1 - a).
\]
The kernel of \(\lambda\) is called the **Bloch group** of \(R\) and is denoted by \(B(R)\).

We say that a commutative ring \(R\) is a **ring with many units** if for any \(n \geq 2\) and for any finite number of surjective linear forms \(f_i : R^n \to R\), there exists a \(v \in R^n\) such that, for all \(i\), \(f_i(v) \in R^*\). Important examples of rings with many units are semilocal rings with infinite residue fields.

If \(R\) is a ring with many units, then we obtain the exact sequence
\[
0 \to B(R) \to p(R) \to (R^* \otimes R^*)_p \to K_2^M(R) \to 0.
\]

We refer the reader to [4, Section 3.2] for the definition of Milnor’s \(K\)-groups \(K_n^M(R)\) of a ring \(R\). When \(R\) is a ring with many units, then
\[
K_2^M(R) \simeq R^* \otimes R^*/\langle a \otimes (1 - a) : a, 1 - a \in R^* \rangle
\]
\[
\simeq (R^* \otimes R^*)_p/\langle a \otimes (1 - a) : a, 1 - a \in R^* \rangle
\]

[8, Proposition 3.2.3], [4, Section 3.2].

The following remarkable theorem is due to Suslin [16, Theorem 5.2].

**Theorem 1.1** (Suslin). If \(F\) is an infinite field, then we have the following exact sequence
\[
0 \to \text{Tor}_1^Z(\mu(F), \mu(F)) \to K_3(F)^{\text{ind}} \to B(F) \to 0,
\]
where \(\text{Tor}_1^Z(\mu(F), \mu(F))\) is the unique nontrivial extension of the group \(\text{Tor}_1^Z(\mu(F), \mu(F))\) by \(\mathbb{Z}/2\) if \(\text{char}(F) \neq 2\) and is equal to \(\text{Tor}_1^Z(\mu(F), \mu(F))\) if \(\text{char}(F) = 2\).

The group \(\text{Tor}_1^Z(\mu(F), \mu(F))\) is studied in some details in the next section.

When \(F\) is algebraically closed, one can show that \(K_3(F)^{\text{ind}} \simeq H_3(\text{SL}_2(F))\) [13, Theorem 4.1], [8, Proposition 5.4] and
\[
\text{Tor}_1^Z(\mu(F), \mu(F)) \simeq \text{Tor}_1^Z(\mu(F), \mu(F))
\]
(see the next section). Thus we have the following corollary of Suslin’s theorem.

**Corollary 1.2** (Bloch-Wigner exact sequence). Let \(F\) be an algebraically closed field. Then we have the exact sequence
\[
0 \to \text{Tor}_1^Z(\mu(F), \mu(F)) \to H_3(\text{SL}_2(F)) \to B(F) \to 0.
\]

**Remark 1.3.** (i) Recently Hutchinson has shown that Theorem 1.1 is also valid for finite fields [6, Corollary 7.5].

(ii) When \(F = \mathbb{C}\), Corollary 1.2 was proved by Bloch [1, Lecture 6] and Wigner, independently and in somewhat different form. A proof of this corollary in case of \(\text{char}(F) = 0\) can be found in [3, App. A].
(iii) In case of positive characteristic, using techniques similar to the case of zero characteristic, one can show that the sequence
\[
\text{Tor}^Z_1(\mu(F), \mu(F)) \to H_3(\text{SL}_2(F)) \to B(F) \to 0
\]
is exact (see for example Proposition 4.2 below). The proof of the injectivity of \(\text{Tor}^Z_1(\mu(F), \mu(F)) \to H_3(\text{SL}_2(F))\) seems to be difficult, and as far as I know there are only two proofs available, both due to Suslin. One proof follows from the Suslin’s results on the \(K\)-theory of algebraically closed fields [14], [15], and the other proof uses the theory of Chern classes [16, Lemma 5.7].

2. Finite cyclic groups as extensions

In this section we will study the group \(\text{Tor}^Z_1(\mu(F), \mu(F))\) that appears in Theorem 1.1.

Let us first to calculate the groups \(H^2(\mathbb{Z}/n, \mathbb{Z}/2)\) and \(H^2(\mathbb{Z}/2, \mathbb{Z}/n)\). Using the Künneth formula [18, Exercise 6.1.5], Example 6.2.3 of [18] and [18, 3.3.2], we have

\[
H^2(\mathbb{Z}/2, \mathbb{Z}/n) \simeq \text{Ext}_\mathbb{Z}^1(\mathbb{Z}/2, \mathbb{Z}/n) \simeq \begin{cases} 0 & \text{if } 2 \nmid n \\ \mathbb{Z}/2 & \text{if } 2 \mid n \end{cases},
\]

\[
H^2(\mathbb{Z}/n, \mathbb{Z}/2) \simeq \text{Ext}_\mathbb{Z}^1(\mathbb{Z}/n, \mathbb{Z}/2) \simeq \begin{cases} 0 & \text{if } 2 \nmid n \\ \mathbb{Z}/2 & \text{if } 2 \mid n \end{cases}.
\]

For a group \(G\) and a \(G\)-module \(A\), the cohomology group \(H^2(G, A)\) classifies all the equivalence classes of group extensions of \(G\) by \(A\) (as an abelian group) [18, Theorem 6.6.3]. If \(2\mid n\), we have the following two non-split exact sequences,

\[
0 \to \mathbb{Z}/n \xrightarrow{r \to z^r} \mathbb{Z}/2n \xrightarrow{\bar{a} \to \bar{a}} \mathbb{Z}/2 \to 0,
\]

\[
0 \to \mathbb{Z}/2 \xrightarrow{1 \to z} \mathbb{Z}/2n \xrightarrow{\bar{a} \to \bar{a}} \mathbb{Z}/n \to 0.
\]

Therefore the first exact sequence is the only nontrivial extension of \(\mathbb{Z}/2\) by \(\mathbb{Z}/n\) and the second one is the only nontrivial extension of \(\mathbb{Z}/n\) by \(\mathbb{Z}/2\).

Let \(R\) be a domain. Let \(F\) be the quotient field of \(R\) and \(\bar{F}\) the algebraic closure of \(F\). Let \(\mu_{2\infty}(R) \subseteq \mu(R)\) be the following set

\[
\mu_{2\infty}(R) := \{ a \in R : \text{there exists an } m \in \mathbb{N}, \text{ s.t. } a^{2^m} = 1 \}.
\]

Here we are interested in the calculation of the cohomology groups

\[
H^2(\mathbb{Z}/2, \text{Tor}^Z_1(\mu(R), \mu(R))), \quad H^2(\text{Tor}^Z_1(\mu(R), \mu(R)), \mathbb{Z}/2).
\]

Applying the Künneth formula together with the facts that \(\text{Tor}^Z_1(\mu(R), \mu(R))\) is a direct limit of a family of finite cyclic groups and the homology functor commutes with direct limits, with directed set of indices [2, Chap. V.5, Exercise 3], we get the isomorphisms

\[
H^2(\mathbb{Z}/2, \text{Tor}^Z_1(\mu(R), \mu(R))) \simeq \text{Ext}_\mathbb{Z}^1(\mathbb{Z}/2, \text{Tor}^Z_1(\mu(R), \mu(R))),
\]
On the other hand, since \( \mu \)

\[
\text{Tor}_1^R(\mu(R), \mu(R)) \approx \text{Ext}_1^Z(\mu(R), \mu(R)) / 2
\]

It is easy to see that

\[
\text{Ext}_2^Z(\mathbb{Z}/2, \text{Tor}_1^R(\mu(R), \mu(R))) \approx \text{Tor}_1^R(\mu(R), \mu(R)) / 2
\]

\[
\approx \begin{cases} 
0 & \text{if } \mu_2(F) \subseteq \mu_2(\mathbb{R}) \\
\mathbb{Z}/2 & \text{if } \mu_2(\mathbb{R}) \text{ is finite, } \text{char}(\mathbb{R}) \neq 2.
\end{cases}
\]

On the other hand, since \( \mu(R) \) can be written as union \( \bigcup_{i=1}^{\infty} \mu_n(R) \), where \( 2|n_1|n_2| \cdots \), we have the following exact sequence \([18, 3.5.10]\)

\[
0 \to \text{lim}^1 \text{Ext}_Z^0(\mu_n(R), \mathbb{Z}/2) \to \text{Ext}_Z^1(\mu(R), \mathbb{Z}/2) \to \text{lim}^1 \text{Ext}_Z^1(\mu_n(R), \mathbb{Z}/2) \to 0.
\]

Since \( \mu_n(R) \) is finite and cyclic,

\[
\text{lim}^1 \text{Ext}_Z^0(\mu_n(R), \mathbb{Z}/2) \approx \text{lim}^1 \text{Hom}_Z(\mu_n(R), \mathbb{Z}/2) \approx \text{lim}^1 \mathbb{Z}/2 = 0,
\]

\([18, \text{Exercise } 3.5.2]\) and

\[
\text{lim} \text{Ext}_Z^1(\mu_n(R), \mathbb{Z}/2) \approx \begin{cases} 
0 & \text{if } \text{char}(\mathbb{R}) = 2 \\
\mathbb{Z}/2 & \text{if } \text{char}(\mathbb{R}) \neq 2.
\end{cases}
\]

Thus

\[
\text{Ext}_Z^1(\mu(R), \mathbb{Z}/2) \approx \begin{cases} 
0 & \text{if } \text{char}(\mathbb{R}) = 2 \\
\mathbb{Z}/2 & \text{if } \text{char}(\mathbb{R}) \neq 2.
\end{cases}
\]

Now by \([12, \text{Theorem } 10.86]\) we have

\[
\text{Ext}_Z^1(\text{Tor}_1^R(\mu(R), \mu(R)), \mathbb{Z}/2) \approx \text{Ext}_Z^1(\mu(R), \text{Ext}_Z^1(\mu(R), \mathbb{Z}/2))
\]

\[
\approx \begin{cases} 
0 & \text{if } \text{char}(\mathbb{R}) = 2 \\
\mathbb{Z}/2 & \text{if } \text{char}(\mathbb{R}) \neq 2.
\end{cases}
\]

Therefore if \( \mu_2(\mathbb{R}) \) is finite, then

\[
H^2(\mathbb{Z}/2, \text{Tor}_1^R(\mu(R), \mu(R))) \approx H^2(\text{Tor}_1^R(\mu(R), \mu(R)), \mathbb{Z}/2)
\]

\[
\approx \begin{cases} 
0 & \text{if } \text{char}(\mathbb{R}) = 2 \\
\mathbb{Z}/2 & \text{if } \text{char}(\mathbb{R}) \neq 2.
\end{cases}
\]

From this it follows that when \( \text{char}(\mathbb{R}) \neq 2 \) and \( \mu_2(\mathbb{R}) \) is finite, we have a unique nontrivial extension of \( \mathbb{Z}/2 \) by \( \text{Tor}_1^R(\mu(R), \mu(R)) \) and a unique nontrivial extension of \( \text{Tor}_1^R(\mu(R), \mu(R)) \) by \( \mathbb{Z}/2 \). If we denote these extensions by \( \text{Tor}_1^R(\mu(R), \mu(R)) \) and \( \text{Tor}_1^R(\mu(R), \mu(R)) \) respectively, then we have the non-split exact sequences

\[
0 \to \text{Tor}_1^R(\mu(R), \mu(R)) \to \text{Tor}_1^R(\mu(R), \mu(R)) \to \mathbb{Z}/2 \to 0,
\]

\[
0 \to \mathbb{Z}/2 \to \text{Tor}_1^R(\mu(R), \mu(R)) \to \text{Tor}_1^R(\mu(R), \mu(R)) \to 0.
\]

The calculation done at the beginning of this section, implies that the groups \( \text{Tor}_1^R(\mu(R), \mu(R)) \) and \( \text{Tor}_1^R(\mu(R), \mu(R)) \) must be isomorphic. Hence if we have one of the above exact sequences, we will get the other one too.
Proposition 2.1. Let $R$ be a commutative domain with $\text{char}(R) \neq 2$. Let $F$ be the quotient field of $R$ and $\bar{F}$ be the algebraic closure of $F$.

(i) There exists a unique nontrivial extension of $\text{Tor}_2^\mathbb{Z}(\mu(R), \mu(R))$ by $\mathbb{Z}/2$. We denote this group by $\text{Tor}_2^\mathbb{Z}(\mu(R), \mu(R))^\sim$.

(ii) If $\mu_2(\infty)(R)$ is finite, then there is a unique nontrivial extension of $\mathbb{Z}/2$ by $\text{Tor}_2^\mathbb{Z}(\mu(R), \mu(R))$, which is isomorphic to $\text{Tor}_2^\mathbb{Z}(\mu(R), \mu(R))^\sim$.

Remark 2.2. If $\text{char}(R) = 2$, then $\mu_2(\infty)(R)$ is trivial. In this case, we set $\text{Tor}_2^\mathbb{Z}(\mu(R), \mu(R))^\sim := \text{Tor}_1^\mathbb{Z}(\mu(R), \mu(R))$.

3. Third Homology of Monomial Matrices of Rank Two

Let $R$ be a commutative ring. Let $\text{GM}_2$ denote the group of monomial matrices in $\text{GL}_2$. Let $T_2 := R^* \times R^* \subseteq \text{GM}_2$ and consider the extension

$$1 \longrightarrow T_2 \longrightarrow \text{GM}_2 \longrightarrow \Sigma_2 \longrightarrow 1,$$

where $\Sigma_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$. We often think of $\Sigma_2$ as the symmetric group of order two $\{1, \sigma\}$. Note that $\text{GM}_2 = T_2 \rtimes \Sigma_2$. The action of $\Sigma_2$ on $T_2$ is as follow:

$$1(a, b) = (a, b), \quad \sigma(a, b) = (b, a).$$

From this extension we obtain the first quadrant spectral sequence

$$E^2_{p,q} = H_p(\Sigma_2, H_q(T_2)) \Rightarrow H_{p+q}(\text{GM}_2).$$

Since the homology of finite cyclic groups is known [18, Theorem 6.2.2], one easily sees that

$$E^2_{p,0} = \begin{cases} \mathbb{Z} & \text{if } p = 0 \\ \mathbb{Z}/2 & \text{if } p \text{ is odd} \\ 0 & \text{if } p \text{ is even} \end{cases}, \quad E^2_{p,1} = H_p(\Sigma_2, T_2) \simeq \begin{cases} R^* & \text{if } p = 0 \\ 0 & \text{if } p \neq 0 \end{cases}.$$

From these, we obtain the isomorphism $E^\infty_{1,2} \simeq E^2_{1,2}$. The spectral sequence gives us a filtration

$$0 = F_{-1}H_3(\text{GM}_2) \subseteq F_0H_3(\text{GM}_2) \subseteq \ldots \subseteq F_3H_3(\text{GM}_2) = H_3(\text{GM}_2),$$
such that $E_{3,3}^\infty \simeq F_1 H_3(\text{GM}_2)/F_{i-1} H_3(\text{GM}_2)$, $0 \leq i \leq 3$. Thus we have

\[
E_{0,3}^\infty \simeq F_0 H_3(\text{GM}_2) = \text{im}(H_3(T_2)),
E_{1,2}^\infty \simeq E_{1,2}^2 \simeq F_1 H_3(\text{GM}_2)/F_0 H_3(\text{GM}_2),
E_{2,1}^\infty \simeq F_2 H_3(\text{GM}_2)/F_1 H_3(\text{GM}_2) = 0,
E_{3,0}^\infty \simeq F_3 H_3(\text{GM}_2)/F_2 H_3(\text{GM}_2) = H_3(\text{GM})/F_2 H_3(\text{GM}_2).
\]

From the natural inclusion $\Sigma_2 \subseteq \text{GM}_2$, one easily sees that the composition $H_3(\Sigma_2) \to H_3(\text{GM}_2) \to H_3(\Sigma_2)$ coincides with the identity map. Thus $E_{3,0}^\infty \simeq H_3(\Sigma_2)$. Now the above relations imply the following isomorphisms

\[
H_3(\text{GM}_2) \simeq F_2 H_3(\text{GM}_2) \oplus H_3(\Sigma_2), \quad E_{1,2}^2 \simeq F_2 H_3(\text{GM}_2)/H_3(T_2).
\]

Set $M := H_3(R^*) \oplus H_3(R^*) \oplus R \otimes H_2(R^*) \oplus H_2(R^*) \otimes R^* \subseteq H_3(T_2)$.

By applying the Snake lemma to the commutative diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & H_3(T_2) & \longrightarrow & \text{Tor}_1^Z(\mu(R), \mu(R)) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_2 H_3(\text{GM}_2) & \longrightarrow & F_2 H_3(\text{GM}_2) & \longrightarrow & 0 & \longrightarrow & 0
\end{array}
\]

we obtain the exact sequence

\[
\text{Tor}_1^Z(\mu(R), \mu(R)) \longrightarrow T_R \longrightarrow E_{1,2}^2 \longrightarrow 0,
\]

where $T_R = F_2 H_3(\text{GM}_2)/M$. Note that $E_{1,2}^2$ is a 2-torsion group and we have [2, Chap. III.1, Example 2]

\[
E_{1,2}^2 = H_1(\Sigma_2, H_2(T_2)) = H_2(T_2)^\sigma/(1 + \sigma)H_2(T_2) = (R^* \otimes R^*)^\sigma/(1 + \sigma)(R^* \otimes R^*) = H_1(\Sigma_2, R^* \otimes R^*),
\]

where the action of $\Sigma_2$ on $R^* \otimes R^*$ is as follow:

\[
1(a \otimes b) = a \otimes b, \quad \sigma(a \otimes b) = -b \otimes a.
\]

**Lemma 3.1.** Let $A$ be an abelian group and $\Sigma_2$ acts on $A \otimes A$ as above. Then $H_1(\Sigma_2, A \otimes A) \simeq H_1(\Sigma_2, A_2^\infty \otimes A_2^\infty)$, where $A_2^\infty := \{a \in A : \text{there exists } m \in \mathbb{N} \text{ s.t. } 2^m a = 0\}$. If $A$ is 2-divisible, then $H_1(\Sigma_2, A \otimes A) = 0$.

**Proof.** Let $\{A_i : i \in I\}$ be a family of finitely generated subgroups of $A$ such that $I$ is a directed set and $A \simeq \lim_{\to_I} A_i$. Then one can easily see that $A \otimes A \simeq \lim_{\to_I} A_i \otimes A_i$. So to prove the lemma we may assume that $A$ is
finitely generated. Let $A \simeq A_{2\infty} \oplus A_{\text{odd}} \oplus \mathbb{Z}^n$, where $A_{\text{odd}}$ is the subgroup of $A$ consisting of elements of odd order. Then we have

$$H_1(\Sigma_2, A \otimes A) \simeq H_1(\Sigma_2, A_{2\infty} \otimes A_{2\infty})$$

$$\oplus H_1(\Sigma_2, A_{\text{odd}} \otimes A_{\text{odd}})$$

$$\oplus H_1(\Sigma_2, \mathbb{Z}^n \otimes \mathbb{Z}^n)$$

$$\oplus H_1(\Sigma_2, A_{2\infty} \otimes A_{\text{odd}} \oplus A_{\text{odd}} \otimes A_{2\infty})$$

$$\oplus H_1(\Sigma_2, A_{2\infty} \otimes \mathbb{Z}^n \oplus \mathbb{Z}^n \otimes A_{2\infty})$$

$$\oplus H_1(\Sigma_2, A_{\text{odd}} \otimes \mathbb{Z}^n \oplus \mathbb{Z}^n \otimes A_{\text{odd}}).$$

The action of $\Sigma_2$ on $A_{2\infty} \otimes A_{\text{odd}} \oplus A_{\text{odd}} \otimes A_{2\infty}$, $A_{2\infty} \otimes \mathbb{Z}^n \oplus \mathbb{Z}^n \otimes A_{2\infty}$ and $A_{2\infty} \otimes \mathbb{Z}^n \oplus \mathbb{Z}^n \otimes A_{2\infty}$ is as follow

$$\sigma(x \otimes a, b \otimes y) = -(y \otimes b, a \otimes x).$$

Since $A_{\text{odd}} \otimes A_{\text{odd}}$ has no element of even order, $H_1(\Sigma_2, A_{\text{odd}} \otimes A_{\text{odd}}) = 0$. By an easy direct commutation one can see that the homology groups $H_1(\Sigma_2, A_{2\infty} \otimes A_{\text{odd}} \oplus A_{\text{odd}} \otimes A_{2\infty})$, $H_1(\Sigma_2, A_{2\infty} \otimes A_{\text{odd}} \otimes \mathbb{Z}^n \oplus \mathbb{Z}^n \otimes A_{2\infty})$ and $H_1(\Sigma_2, A_{\text{odd}} \otimes \mathbb{Z}^n \otimes \mathbb{Z}^n \otimes A_{\text{odd}})$ are trivial. Also one can show that

$$(\mathbb{Z}^n \otimes \mathbb{Z}^n)^{\sigma} = (1 + \sigma)(\mathbb{Z}^n \otimes \mathbb{Z}^n) = \langle e_i \otimes e_j - e_j \otimes e_i : 1 \leq i < j \leq n \rangle,$$

where $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{Z}^n$. Hence $H_1(\Sigma_2, \mathbb{Z}^n \otimes \mathbb{Z}^n) = 0$. This completes the proof of the first part of the lemma. When $A$ is 2-divisible, $A_{2\infty}$ is 2-divisible too. Therefore $A_{2\infty} \otimes A_{2\infty}$ is trivial. Now the claim follows from the first part of the lemma.

By the previous lemma we have the isomorphism

$$E_{1,2}^2 = H_1(\Sigma_2, H_2(T_2)) \simeq H_1(\Sigma_2, \mu_{2\infty}(R) \otimes \mu_{2\infty}(R)).$$

Combining this with the above exact sequence, we obtain the exact sequence

$$\text{(3.1)} \quad \text{Tor}_1^G(\mu(R), \mu(R)) \longrightarrow T_R \longrightarrow H_1(\Sigma_2, \mu_{2\infty}(R) \otimes \mu_{2\infty}(R)) \longrightarrow 0.$$ 

For later use, we need to give an explicit description of the map

$$H_2(T_2)^{\sigma} \longrightarrow E_{1,2}^2 \longrightarrow F_2H_3(GM_2)/H_3(T_2) \subseteq H_3(GM_2)/H_3(T_2).$$

To do this we need to introduce certain notations.

For an arbitrary group $G$, let $B_*(G) \rightarrow \mathbb{Z}$ denote the bar resolution of $G$. So $B_n(G)$, $n \geq 0$, is the free left $G$-module with basis consisting of elements of the form $[g_1] \cdots [g_n]$, $g_i \in G$, where $[g_1] \cdots [g_i] = 0$ if $g_i = 1$ for some $i$. The differential $\partial^G_n : B_n(G) \rightarrow B_{n-1}(G)$ is defined as follow

$$\partial^G_n([g_1] \cdots [g_n]) = g_1[g_2] \cdots [g_n] + \sum_{i=1}^{n-1} (-1)^i [g_1] \cdots [\widehat{g_i}] [g_i g_{i+1}] [\widehat{g_{i+1}}] \cdots [g_n]$$

$$+ (-1)^n [g_1] \cdots [g_{n-1}].$$

Note that $B_0(G) = \mathbb{Z}G[\cdot]$ and $\varepsilon : \sum n_g[\cdot] \mapsto \sum n_g$. We turn the $B_n(G)$ into a right $G$-module in usual way. For any left $G$-module $M$, the homology
Consider the commutative diagram
\[ H \]
\[ \text{group } H_i(G, M) \text{ coincides with the homology of the complex } B_\bullet(G) \otimes_G M. \]
In particular
\[ H_n(G) = H_n(B_\bullet(G) \otimes_G \mathbb{Z}) = H_n(B_\bullet(G)_G). \]

For simplicity we denote the element of \( B_n(G)_G \) represented by \([g_1] \cdots [g_n]\) again by \([g_1] \cdots [g_n]\). Any element \( g \in G \), determines an automorphism of the complex \( B_\bullet(G) \otimes_G M \) given by
\[ [g_1] \cdots [g_n] \otimes m \mapsto [gg_1g^{-1}] \cdots [gg_ng^{-1}] \otimes gm. \]
This automorphism is homotopic to the identity, with the corresponding homotopy given by the formula
\[ \rho_g : B_n(G) \otimes_G M \to B_{n+1}(G) \otimes_G M, \]
\[ [g_1] \cdots [g_n] \mapsto \sum_{j=0}^{n} (-1)^j [g_1] \cdots [g_j] g^{-1} [gg_{j+1}g^{-1}] \cdots [gg_ng^{-1}] \otimes m. \]

Lemma 3.2. Let \( u \in H_2(T_2) \) and \( h \in B_2(T_2)_\Sigma_2 \) a representing cycle for \( u \). Let \( \tau \) be the automorphism of transposition of terms and let \( \tau(h) = h = \delta_3 T_2(b), b \in B_3(T_2)_\Sigma_2 \). Then the image of \( u \) under the map
\[ H_2(T_2) \to H_3(GM_2)/H_3(T_2) \]
coincides with the homology class of the cycle \( b - \rho_s(h) \), where \( s := \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \) and
\[ \rho_s([g_1|g_2]) := [s|sg_1s^{-1}|sg_2s^{-1}] - [g_1|s|sg_2s^{-1}] + [g_1|g_2|s]. \]

Proof. Let \( B_\bullet(GM_2) \to \mathbb{Z} \) and \( B_\bullet(\Sigma_2) \to \mathbb{Z} \) be the bar resolutions of \( GM_2 \) and \( \Sigma_2 \) respectively. We know that
\[ B_\bullet(GM_2)_\Sigma_2 \simeq (B_\bullet(GM_2)_T_2)_{\Sigma_2} \simeq \mathbb{Z} \oplus \Sigma_2 B_\bullet(GM_2)_T_2. \]
The spectral sequence \( E_{p,q}^2 \) is one of the two corresponding spectral sequences of the double complex \( B_\bullet(\Sigma_2) \otimes_{\Sigma_2} B_\bullet(GM_2)_T_2 \), with \( E_1 \)-terms of the form
\[ E_{p,q}^1 = H_q(B_p(\Sigma_2) \otimes_{\Sigma_2} B_\bullet(GM_2)_T_2). \]
[2, Chap. VII, Sections 5, 6]. Let \( h = [g_1|g_2] - [g_2|g_1] \in B_2(T_2) \). Then
\[ \tau(h) = [sg_1s^{-1}|sg_2s^{-1}] - [sg_2s^{-1}|sg_1s^{-1}]. \]

Consider the commutative diagram
\[ \begin{array}{c}
\mathbb{Z} \otimes_{\Sigma_2} B_3(GM_2)_T_2 \leftarrow B_0(\Sigma_2) \otimes_{\Sigma_2} B_3(GM_2)_T_2 \leftarrow B_1(\Sigma_2) \otimes_{\Sigma_2} B_3(GM_2)_T_2 \\
\downarrow \quad \downarrow \quad \downarrow \\
\mathbb{Z} \otimes_{\Sigma_2} B_2(GM_2)_T_2 \leftarrow B_0(\Sigma_2) \otimes_{\Sigma_2} B_2(GM_2)_T_2 \leftarrow B_1(\Sigma_2) \otimes_{\Sigma_2} B_2(GM_2)_T_2.
\end{array} \]
Now the necessary computations are collected in the following diagram
\[
\begin{array}{c}
\begin{array}{c}
b - \rho_s(h) \\
\downarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\rightarrow s[ ] \otimes b - [ ] \otimes \rho_s(h)
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\rightarrow s[ ] \otimes h - [ ] \otimes h
\end{array}
\end{array}
\begin{array}{c}
\rightarrow [s] \otimes h.
\end{array}
\end{array}
\]

Let \( R \) be a domain and \( \xi \in R \) a primitive \( n \)-th root of unity. Therefore \( \mu_n(R) = \langle \xi \rangle \). Let \( \langle \xi, n, \xi \rangle \in \text{Tor}_1^Z(\mu_n(R), \mu_n(R)) \) be the image of \( \xi \) under the following composition
\[
\mu_n(R) \longrightarrow \text{Tor}_1^Z(\mu_n(R), \mu_n(R)) \hookrightarrow \text{Tor}_1^Z(\mu_n(R), \mu_n(R)).
\]

We have the following lemma.

**Lemma 3.3.** Let \( R \) be a domain. Then we have the canonical decomposition
\[
H_3(T_2) = \bigoplus_{i+j=3} H_i(R^*) \otimes H_j(R^*) \oplus \text{Tor}_1^Z(\mu_n(R), \mu_n(R)),
\]
where a splitting map \( \text{Tor}_1^Z(\mu_n(R), \mu_n(R)) \longrightarrow H_3(T_2) \) is defined by the formula
\[
\langle \xi, n, \xi \rangle \mapsto \chi(\xi),
\]
\[
\chi(\xi) := \sum_{i=1}^n \left( [(\xi, 1)](1, \xi)[(1, \xi^i)] - [(1, \xi)](\xi, 1)[(1, \xi^i)] + [(1, \xi)](1, \xi^i)[(\xi, 1)] \right.
\]
\[
+ [(\xi, 1)](\xi^i, 1)[(1, \xi)] - [(\xi, 1)](1, \xi)[(\xi^i, 1)] + [(1, \xi)](\xi, 1)[(\xi^i, 1)] \right).
\]

**Proof.** For a proof, see Section 4 of [8]. \(\square\)

### 4. Bloch-wigner exact sequence

The following theorem is due to Suslin [16, Theorem 2.1].

**Theorem 4.1.** Let \( R \) be a commutative ring with many units. Then
\[
B(R) \simeq H_3(\text{GL}_2)/H_3(\text{GM}_2).
\]

**Proof.** Suslin has proved this theorem for infinite fields [16, Theorem 2.1]. But his arguments without any change also works for any ring with many units. For example Lemmas 2.1 and 2.2 in [16] and their proofs are still true (see for example [9, Section 1]). For a proof of Lemma 2.4 in [16], see the proof of Lemma 4.1 in [9]. The rest of Suslin’s argument goes through without any change. \(\square\)

From Theorem 4.1 we obtain the exact sequence
\[
H_3(\text{GM}_2) \longrightarrow H_3(\text{GL}_2) \longrightarrow B(R) \longrightarrow 0.
\]
Proposition 4.2. Let $R$ be a commutative ring with many units and set $\tilde{H}_3(\text{SL}_2(R)) := H_3(\text{GL}_2)/(H_3(\text{GL}_1) + R^* \cup H_3(\text{GL}_1))$.

(i) Then we have the following exact sequences
\[
T_R \longrightarrow \tilde{H}_3(\text{SL}_2(R)) \longrightarrow B(R) \longrightarrow 0,
\]
\[
\text{Tor}_1^R(\mu(R), \mu(R)) \longrightarrow T_R \longrightarrow H_1(\Sigma_2, \mu_2^\infty(R) \otimes \mu_2^\infty(R)) \longrightarrow 0,
\]
(ii) We have the exact sequence
\[
\text{Tor}_1^R(\mu(R), \mu(R)) \otimes_{\mathbb{Z}[1/2]} \longrightarrow H_3(\text{SL}_2(R), \mathbb{Z}[1/2]) \longrightarrow B(R) \otimes_{\mathbb{Z}[1/2]} \longrightarrow 0.
\]
(iii) If $R^* = R^{*2}$, then we have the exact sequence
\[
\text{Tor}_1(\mu(R), \mu(R)) \longrightarrow H_3(\text{SL}_2(R)) \longrightarrow B(R) \longrightarrow 0.
\]

Proof. Let $B_2 := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, d \in R^*, b \in R \right\}$. A theorem of Suslin claims that the inclusion $T_2 \subseteq B_2$ induces the isomorphism $H_n(T_2) \simeq H_n(B_2)$, $n \geq 0$, [4, Theorem 2.2.2]. The matrix $s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is similar to the matrix $\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \in B_2$, and hence
\[
\text{im}(H_3(\Sigma_2)) \subseteq \text{im}(H_3(B_2)) = \text{im}(H_3(T_2)) \subseteq H_3(\text{GL}_2).
\]
Thus Suslin’s exact sequence (4.1) together with $H_3(\text{GM}_2) \simeq F_2 H_3(\text{GM}_2) \oplus H_3(\Sigma_2)$, implies the exact sequence
\[
F_2 H_3(\text{GM}_2) \longrightarrow H_3(\text{GL}_2) \longrightarrow B(R) \longrightarrow 0.
\]
Now from the commutative diagram
\[
\begin{array}{ccc}
M & \longrightarrow & M \\
\downarrow & & \downarrow \\
F_2 H_3(\text{GM}_2) & \longrightarrow & H_3(\text{GL}_2) \longrightarrow B(R) \longrightarrow 0,
\end{array}
\]
we obtain the exact sequence
\[
T_R \longrightarrow H_3(\text{GL}_2)/M \longrightarrow B(R) \longrightarrow 0.
\]
This together with exact sequence (3.1) proves (i). The parts (ii) and (iii) follow from (i), Lemma 3.1 and the fact that we have the isomorphism
\[
\tilde{H}_3(\text{SL}_2(R)) \otimes_{\mathbb{Z}[1/2]} \simeq H_3(\text{SL}_2(R), \mathbb{Z}[1/2]) \otimes
\]
for any ring with many units and the isomorphism
\[
\tilde{H}_3(\text{SL}_2(R)) \simeq H_3(\text{SL}_2(R))
\]
if moreover $R^* = R^{*2}$ [9, Corollary 5.4].
Remark 4.3. (i) Proposition 4.2 recovers and also improves one of the main result of [9]. We should menion that Theorem 5.1 in [9] is not correct. The correct formulation of that theorem, which is almost the above proposition, is stated in an erratum to [9].

(ii) If $R$ is a noncommutative ring with many units [8, Section 2], [4, Section 1], then with minor modification the groups $p(R)$, $B(R)$, $K_2^M(R)$ can be defined in a similar way [9, Remark 5.2]. Now most of the above arguments also works in this noncommutative setting and thus we obtain the exact sequences

\[ \begin{align*}
T_R & \to \tilde{H}_3(\text{SL}_2(R)) \to B(R) \to 0, \\
\text{Tor}_1^Z(K_1(R), K_1(R)) & \to T_R \to H_1(\Sigma_2, K_1(R) \otimes K_1(R)) \to 0.
\end{align*} \]

This corrects and also improves Remark 5.2 in [9].

**Theorem 4.4.** Let $R$ be a domain with many units. Then we have the exact sequence

\[ 0 \to \text{Tor}_1^Z(\mu(R), \mu(R)) \to \tilde{H}_3(\text{SL}_2(R)) \to B(R) \to 0, \]

where $\text{Tor}_1^Z(\mu(R), \mu(R))$ is the unique nontrivial extension of the group $\text{Tor}_1^Z(\mu(R), \mu(R))$ by $\mathbb{Z}/2$ if $\text{char}(R) \neq 2$ and is equal to $\text{Tor}_1^Z(\mu(R), \mu(R))$ if $\text{char}(R) = 2$.

**Proof.** By Proposition 4.2, we have the following exact sequences

\[ T_R \to \tilde{H}_3(\text{SL}_2(R)) \to B(R) \to 0, \]

\[ \text{Tor}_1^Z(\mu(R), \mu(R)) \to T_R \to H_1(\Sigma_2, \mu_2(\mu(R)) \otimes \mu_2(\mu(R))) \to 0. \]

Let $F$ be the quotient field of $R$ and $\bar{F}$ be the algebraic closure of $F$. Then

\[ H_1(\Sigma_2, \mu_2(\mu(R)) \otimes \mu_2(\mu(R))) \simeq \frac{(\mu_2(\mu(R)) \otimes \mu_2(\mu(R)))^\sigma}{(1 + \sigma)(\mu_2(\mu(R)) \otimes \mu_2(\mu(R)))} \]

\[ \simeq (\mu_2(\mu(R)) \otimes \mu_2(\mu(R)))^\sigma \]

\[ = 2(\mu_2(\mu(R)) \otimes \mu_2(\mu(R))) \]

\[ \simeq \begin{cases} 
0 & \text{if } \mu_2(\mu(R)) = \mu_2(\bar{F}) \subseteq R \\
\mathbb{Z}/2 & \text{if } \mu_2(\mu(R)) \text{ is finite and } \text{char}(R) \neq 2.
\end{cases} \]

Using the isomorphism $\tilde{H}_3(\text{SL}_2(\bar{F})) \simeq H_3(\text{SL}_2(\bar{F}))$ [9, Corollary 5.4] and the Bloch-Wigner exact sequence, Corollary 1.2, from the above exact sequences we get the isomorphism $\text{Tor}_1^Z(\mu(\bar{F}), \mu(\bar{F})) \simeq T_{\bar{F}}$. From the commutative diagram

\[ \begin{array}{ccc}
\text{Tor}_1^Z(\mu(R), \mu(R)) & \longrightarrow & T_R \\
\downarrow & & \downarrow \\
\text{Tor}_1^Z(\mu(\bar{F}), \mu(\bar{F})) & \longrightarrow & T_{\bar{F}}.
\end{array} \]

and the injectivity of $\text{Tor}_1^Z(\mu(R), \mu(R)) \to \text{Tor}_1^Z(\mu(\bar{F}), \mu(\bar{F}))$, the injectivity of $\text{Tor}_1^Z(\mu(R), \mu(R)) \to T_R$ follows.
If char($R$) = 2, then $\mu_{2\infty}(R) = \mu_{2\infty}(F) = \{1\}$, and so $H_1(\Sigma_2, \mu_{2\infty}(R) \otimes \mu_{2\infty}(R)) = 0$. Hence the map $\text{Tor}_1^R(\mu(R), \mu(R)) \rightarrow T_R$ is surjective as well and thus

$$\text{Tor}_1^R(\mu(R), \mu(R)) \simeq T_R.$$ 

Now assume char($R$) $\neq 2$ and $\mu_{2\infty}(R)$ is finite. Then we have the exact sequence

$$(4.2) \quad 0 \rightarrow \text{Tor}_1^R(\mu(R), \mu(R)) \rightarrow T_R \rightarrow \mathbb{Z}/2 \rightarrow 0.$$ 

We show that this exact sequence does not split. This exact sequence is, in fact, the exact sequence

$$0 \rightarrow H_3(T_2)/M \rightarrow F_2H_3(GM_2)/M \rightarrow F_2H_3(GM_2)/H_3(T_2) \rightarrow 0.$$ 

Let $\mu_{2\infty}(R) = \langle \xi \rangle$, $n = 2m = 2^r$. Under the inclusion

$$\{0, (-1) \otimes \xi\} = 2(\mu_{2\infty}(R) \otimes \mu_{2\infty}(R)) \hookrightarrow H_2(T_2),$$

the image of $(-1) \otimes \xi$ in $H_2(T_2)$ is represented by the cycle

$$h := \left[\left(-1, 1\right)\left(1, \xi\right) - \left(1, \xi\right)\left(-1, 1\right)\right] \in B_2(T_2).$$

Now

$$\tau(h) - h = \left[\left(1, -1\right)\left(1, \xi\right) - \left(1, 1\right)\left(1, -1\right) - \left(-1, 1\right)\left(1, \xi\right) + \left(1, \xi\right)\left(-1, 1\right)\right],$$

$$\sigma(h) = \left[\left(1, 1\right)\left(1, -1\right) - \left(1, -1\right)\left(1, -1\right) - \left(1, 1\right)\left(-1, -1\right) - \left(1, 1\right)\left(-1, 1\right)\right].$$

If $b := \chi_1(\xi) + \chi_3(\xi)$, where

$$\chi_1(\xi) := \sum_{i=1}^{m-1} \left[\left(1, \xi\right)\left(1, \xi^i\right) - \left(1, \xi\right)\left(1, \xi^i\right) + \left(1, \xi\right)\left(1, \xi^i\right)\left(1, \xi\right)\right]$$

$$\chi_3(\xi) := \left[\left(1, 1\right)\left(1, -1\right) - \left(1, -1\right)\left(1, -1\right) - \left(1, 1\right)\left(1, -1\right)\right],$$

then, $\partial_3^{T_2}(b) = \tau(h) - h$. Now by Lemma 3.2, the cycle $b - \rho_s(h)$ represents the image of $(-1) \otimes \xi$ in $F_2H_3(GM_2)/H_3(T_2) \subseteq H_3(GM_2)/H_3(T_2)$. Now let

$$\omega(\xi) := b - \rho_s(h)$$

represents the element of $F_2H_3(GM_2)/M$ generated by the cycle $b - \rho_s(h)$. To show that our exact sequence does not split, it is sufficient to show that $2\omega(\xi)$ is equal to the image of $\langle \xi, n, \xi \rangle \in \text{Tor}_1^R(\mu(R), \mu(R))$ under the inclusion $\text{Tor}_1^R(\mu(R), \mu(R)) \simeq H_3(T_2)/M \hookrightarrow F_2H_3(GM_2)/M$. By Lemma 3.3, the image of $\langle \xi, n, \xi \rangle$ is equal to $\chi(\xi)$. Therefore we should show
that \( \overline{2\omega(\xi)} = \chi(\xi) \). If

\[
\chi_4(\xi) := \left[ (-1, 1) | (-1, 1) | (1, \xi) - (-1, 1) | (1, \xi) | (1, 1) \right] \\
+ \left[ (1, \xi) | (-1, 1) | (1, 1) \right],
\]

\[
\eta(\xi) := [s | (1, -1) | (1, 1) | (1, \xi) - s | (1, -1) | (1, \xi) | (1, 1) \\
+ [(-1, 1) | s | (\xi, 1) | (1, 1) - (-1, 1) | s | (\xi, 1) | (1, 1) \\
+ [(-1, 1) | (\xi, 1) | (1, 1) s - [(1, \xi) s | (1, 1) | (1, 1) \\
+ [s | (\xi, 1) | (1, 1) | (1, 1) - (-1, 1) | s | (\xi, 1) | (1, 1) \\
+ [(1, \xi) | (1, 1) s | (\xi, 1) - [(-1, 1) | (1, 1) | (1, \xi) | (1, 1) s,
\]

then by a direct computation we have

\[
\partial_4(\eta(\xi)) = -2\rho_s(h) + \chi_3(\xi) - \chi_4(\xi) \in B_3(GM_2)_{GM_2},
\]

and thus

\[
2\omega(\xi) = 2b - 2\rho_s(h) = 2\chi_1(\xi) + \chi_3(\xi) + \chi_4(\xi) + \partial_4(\eta(\xi)).
\]

We have \( \chi(\xi) = \chi_1(\xi) + \chi_5(\xi) + \chi_6(\xi) + \chi_2(\xi) \), where

\[
\chi_2(\xi) := \sum_{i=1}^{m-1} \left( [(\xi, 1) | (1, \xi) | (1, -\xi^i) - [(1, \xi) | (\xi, 1) | (1, -\xi^i)] + [(1, \xi) | (1, -\xi^i)] | (\xi, 1) | (1, -\xi^i) \right),
\]

\[
\chi_5(\xi) := [(\xi, 1) | (1, \xi) | (1, -1)] - [(1, \xi) | (\xi, 1) | (1, -1)] + [(\xi, 1) | (1, -1)] | (\xi, 1),
\]

\[
\chi_6(\xi) := [(\xi, 1) | (1, -1)] | (1, \xi) - [(\xi, 1) | (1, \xi) | (1, -1)] + [(1, \xi) | (\xi, 1) | (1, -1)],
\]

Now if

\[
v(\xi) := \sum_{i=0}^{m-1} \left( [(\xi, 1) | (1, \xi) | (1, \xi^i) | (1, -1)] - [(1, \xi) | (\xi, 1) | (1, \xi^i) | (1, -1)] \\
- [(1, \xi) | (1, \xi^i) | (1, -1)] | (\xi, 1) + [(1, \xi) | (1, \xi^i) | (\xi, 1) | (1, -1)] \\
+ [(1, \xi) | (\xi, 1) | (\xi^i, 1) | (1, -1)] - [(\xi, 1) | (1, \xi) | (\xi^i, 1) | (1, -1)] \\
- [(\xi, 1) | (\xi^i, 1) | (1, \xi) + [(\xi, 1) | (\xi^i, 1) | (1, \xi) | (1, -1)] \right),
\]

then \( \partial_4(v(\xi)) = \chi_1(\xi) - \chi_2(\xi) + \chi_3(\xi) - \chi_5(\xi) - \chi_6(\xi) + \chi_4(\xi) \). Therefore

\[
2\omega(\xi) = 2\chi_1(\xi) + \chi_3(\xi) + \chi_4(\xi) + \partial_4(\eta(\xi))
\]

\[
= \chi_1(\xi) + \chi_2(\xi) + \chi_5(\xi) + \chi_6(\xi) + \partial_4(\eta(\xi)) + v(\xi)
\]

\[
= \chi(\xi) + \partial_4(\eta(\xi) + v(\xi)).
\]

This shows that \( 2\omega(\xi) = \chi(\xi) \) and therefore the exact sequence (4.2) does not split.

Thus Proposition 2.1 implies the existence of the non-split exact sequence

\[
0 \longrightarrow \mathbb{Z}/2 \longrightarrow T_R \longrightarrow \text{Tor}_1^\mathbb{Z}(\mu(R), \mu(R)) \longrightarrow 0.
\]
Therefore $T_R$ is the unique nontrivial extension of $\text{Tor}_1^\mathbb{Z}(\mu(R), \mu(R))$ by $\mathbb{Z}/2$. Now from the commutative diagram
\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & T_R & \longrightarrow & \text{Tor}_1^\mathbb{Z}(\mu(R), \mu(R)) & \longrightarrow & 0 \\
& & \downarrow \text{inc} & & \downarrow & & \downarrow \text{inc} & & \\
0 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & T_F & \longrightarrow & \text{Tor}_1^\mathbb{Z}(\mu(\bar{F}), \mu(\bar{F})) & \longrightarrow & 0,
\end{array}
\]
we see that $T_R \longrightarrow T_F$ is injective.

From this result and the Bloch-Wigner exact sequence, Corollary 1.2, we obtain the injectivity of the map $T_R \longrightarrow \tilde{H}_3(\text{SL}_2(R))$. Therefore we have the exact sequence
\[
0 \longrightarrow T_R \longrightarrow \tilde{H}_3(\text{SL}_2(R)) \longrightarrow B(R) \longrightarrow 0,
\]
where $T_R$ is the unique nontrivial extension of $\text{Tor}_1^\mathbb{Z}(\mu(R), \mu(R))$ by $\mathbb{Z}/2$. This completes the proof of the theorem. □

**Remark 4.5.** For an infinite field $F$, by Theorems 1.1 and 4.4 we have two exact sequences which look very similar
\[
0 \longrightarrow \text{Tor}_1^\mathbb{Z}(\mu(F), \mu(F)) \longrightarrow K_3(F) \longrightarrow B(F) \longrightarrow 0,
\]
\[
0 \longrightarrow \text{Tor}_1^\mathbb{Z}(\mu(F), \mu(F)) \longrightarrow \tilde{H}_3(\text{SL}_2(F)) \longrightarrow B(F) \longrightarrow 0.
\]
This suggest that $K_3(F)$ and $\tilde{H}_3(\text{SL}_2(F))$ should be isomorphism. But there is no direct map from one of these groups to the other one! But there is a natural maps from $H_3(\text{SL}_2(F))_{\bar{F}^*}$ to both of them. These relation will be studied in another paper.

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Department of Mathematics,
Institute for Advanced Studies in Basic Sciences,
P. O. Box. 45195-1159, Zanjan, Iran.
email: bmirzai@insbs.ac.ir
email: f.mokari61@gmail.com