Lattice animals, fractality and criticality in hadronic and partonic systems

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Abstract. The cluster description of near coexistence phases (e.g. Fisher theory) requires an evaluation of cluster surface entropy. This surface degeneracy can be estimated with lattice models where clusters appear. The maximum probability lies near the maximum cluster surface. At low temperatures, clusters are forced to be nearly spherical by the surface energy and the associated Boltzmann factor. At higher temperatures and near criticality, the fractal dimension of clusters changes so that clusters become fractal. In the MIT bag model, where there is no surface energy, bags are always fractal.

1. Introduction
Phase transitions have a way to catch the attention of our collective scientific unconscious. The dramatic changes in structure, density and other parameters at times overwhelm our best judgment. We know but forget that, at least for first order transitions, these transitions can be described, and predicted, on the basis of the thermodynamics characterization of each individual infinite phase with the requirement of equal chemical potential. It is a case of myth hiding triviality.

The advent of mesoscopic physics made the waters murkier by introducing the additional complication of finiteness. Now finite size effects transformed, in the eyes of some, the general thermodynamics of infinite systems into an infinity of thermodynamics, one for each finite system.

In nuclear physics we should have known better. On one hand, we are used to finite nuclei, on the other we understand the danger of long range forces, such as the Coulomb force. Eventually, it was learned how to move elegantly from finite to infinite system trough the liquid drop model $A^{-1/3}$ expansion.

In this way, to do justice to our forefathers, some properties of $T = 0$ symmetric nuclear matter were quickly determined from the volume term of the liquid drop expansion. The guess of the existence of a dilute gas-like phase at $T > 0$ should have been obvious as neutrons started “evaporating” under the hands of experimentalists [1, 2].

2. Alleged signals of phase transitions
This should have been enough to predict and, to a large extent, characterize the liquid vapor phase transition and the attendant phase diagram. Unfortunately, the advent of heavy ion
reactions and their complexity spurred experimentalists to look, in the foggy environment of the theory and data, for “signals” of phase transitions.
Plateaus in caloric curves were first held to be such signals [3, 4], forgetting that these plateaus are just due to keeping the pressure constant, which was not obvious here.
Similarly, spurred by numerical calculations in finite systems, others saw in negative heat capacities the desired signals [5, 6, 7]. Unfortunately “negative” heat capacities can be produced at command in infinite systems and even in a single phase [8].
Bimodality, another finite size effect, was also considered as a possible signal of a phase transition [9].
It may be a fair assessment to say that all these experimental signals did not lead to any hint of a phase diagram, nor not to a characterization of the phases involved in the alleged phase transitions.

3. Thermodynamic frugality
Yet much more progress could have been achieved much earlier and with very little work. The Clapeyron equation shows the way, giving in differential form the $p, T$ coexistence line:
\[
\frac{dp}{dT}_{\text{coex}} = \frac{\Delta H_m}{T \Delta V_m}.
\] (1)
Assuming $\Delta V_m = V_{m}^{\text{vap}}$ and ideal behavior of the gas we obtain
\[
\frac{d \ln p}{dT} = \frac{\Delta H_m}{T^2},
\] (2)
and taking for $\Delta H_m = -a_v + T$,
\[
p = f(T) \exp \left( -\frac{a_v}{T} \right).
\] (3)

Figure 1. Reduced temperature-density phase diagram for several fluids.
Thus much was implicitly known and could have been made explicit before 1940.

Compare now the above equation with the expression used by Guggenheim [10] to fit the data in figure 1 for a variety of Van der Waals systems:

\[ p = p_c \exp \left[ -\frac{\Delta H_0}{T} \left( 1 - \frac{T}{T_c} \right) \right], \quad (4) \]

where \( p_c \) and \( T_c \) are the critical pressure and temperature respectively. The two equations are obviously equivalent. So, in order to obtain the full phase diagram of nuclear matter we must obtain the critical parameters from experiments.

4. Critical parameters

In order to obtain such information we must rely on the properties of the gas phase (vapor) in equilibrium with nuclear matter at \( T > 0 \). We do not have access to such a system. However we have the possibility of studying the particles and clusters “evaporated” from hot finite nuclei. Three problems are to be overcome:

1) No vapor is ever in equilibrium with a nucleus, but emission (evaporation) occurs in vacuum [11].
2) Coulomb distorts the relative abundance of fragments through the Coulomb barrier [12].
3) The finiteness of nuclei affects the rate of emission of fragments [13].

We have shown that:

a) We can easily correct the distortion of rates and abundances due to Coulomb effects [12].
b) We can correct for finite size effects with the complement approach [11].
c) The corrected rates can be transformed in the concentrations of the clusters in equilibrium [13].

Having reached this point, we attempt the characterization of the phase coexistence line through the Fisher cluster model, its formula having been modified to account for finite size effect [13]:

\[ n(A, T) = q_0 A^{-\tau} \exp \left( \frac{\epsilon c_0 A^\sigma}{T} \right) \quad \text{(infinite system)}, \]
\[ n(A, A_0, T) = q_0 \left( \frac{A(A_0 - A)}{A_0} \right)^{-\tau} \exp \left( \frac{\epsilon c_0 (A^\sigma + (A_0 - A)^\sigma - A_0^\sigma)}{T} \right) \quad \text{(finite system)}, \quad (5) \]

where \( n(A, T) \) and \( n(A, A_0, T) \) are the clusters abundances, \( A_0 \) is the emitter size, \( \tau \) and \( \sigma \) are critical exponents, and \( \epsilon = \frac{T_c - T}{T_c} \), where \( T_c \) is the critical temperature.

This approach can be tested on a finite size Ising Model as well as on a finite size Lennard Jones system and it allows one to recover accurately the critical parameters [13]. In figure 2 we show the Fisher plot, scaled fragment abundancies versus the surface energy divided by the temperature, for a variety of energies, nuclear systems and fragments [14]. The excellent scaling observed is equivalent to a coexistence line and allows us to obtain the critical parameters for infinite nuclear matter.

5. Partonic transitions and the bag model

The encouraging results described above lead us to consider another phase transition that has been studied and looked for in the past decades: the hadronic to partonic phase transition. We concentrate here on the simplest model, the bag model [15], which describes the transition in terms of two vacua: the hadronic vacuum in which the hadrons live, and the partonic vacuum.
in which quarks and gluons move unconfined. A finite volume of the partonic vacuum, a bag, is confined by a bag wall which exerts a constant pressure, $B$, on it and which must be balanced by the partonic bag content. In the case of no conserved charges, as in a thermal gas of the various quarks and gluons [15],

$$ p = g \frac{\pi^2}{90} T_H^4 = B, \quad (6) $$

where $g$ is the number of partonic degrees of freedom and $B$ is the bag pressure. This leads to a universal, unique bag temperature:

$$ T_H = \left( \frac{B}{g \frac{\pi^2}{90}} \right)^{1/4}. \quad (7) $$

We have shown that the bag at equilibrium can neither be cooled nor heated [16, 17]. Although its total energy can be varied by increasing or decreasing the size of the bag, the energy density remains constant:

$$ \epsilon = g \frac{\pi^2}{30} T_H^4 + B. \quad (8) $$

It follows that the entropy of the bag is extensive and proportional to its mass, $m$:

$$ S = \frac{m}{T_H}, \quad (9) $$

which leads to a level density or bag degeneracy

$$ \rho = e^{m/T_H}. \quad (10) $$

This extraordinary result implies that the bag is a perfect thermostat, which will impart its unique temperature $T_H$ to any system in equilibrium with it [16, 17]. Let us consider now
an amount of energy, $E$, to be shared between a bag and an ideal gas in which the mass of the particles of degeneracy $g(m)$ is derived also from the total energy. The total level density, $P(E, \epsilon)$, is the product of the gas and bag level densities,

$$P(E, \epsilon) = \rho_H(E - \epsilon) \rho_{iv}(\epsilon) = g(m) \frac{V^N}{N!} \frac{1}{\Gamma\left(\frac{3}{2}N\right)} \left(\frac{2\pi m}{\hbar}\right)^{\frac{3}{2}N} \epsilon^{\left(\frac{3}{2}N-1\right)} \exp\left(-\frac{E - mN - \epsilon}{T_H}\right). \quad (11)$$

Finding the average energy per particle from the above probability distribution, we obtain:

$$\frac{\epsilon}{N} = \frac{3}{2}T_H. \quad (12)$$

In other words, the gas is isothermal with the bag. Furthermore, the gas density is:

$$\frac{N}{V} = g(m) \left(\frac{2\pi m T_H}{\hbar}\right)^{3/2} e^{-m/T_H}. \quad (13)$$

This means that the concentration of the gas is volume independent. It is a saturated vapor in equilibrium with a bag, like a saturated vapor in equilibrium with its liquid. This is the phase coexistence we were looking for. At temperatures lower than $T_H$ we have a gas of those hadronic particles whose degeneracy is $g < e^{m/T_H}$.

At $T = T_H$ there is the appearance of bags with $g = e^{m/T_H}$ in equilibrium with the gas. As we attempt to increase the temperature by feeding energy to the system, the temperature stays fixed at $T_H$ and the bag(s) grows in size until all the space is occupied by the bag. No higher temperature can be achieved without imposing an external pressure, which would force the disappearance of the vapor. Thus coexistence is signaled by fixed composition of the vapor hadronic constituents at fixed temperature $T_H$ [16, 17].

6. A gas of bags
One may wonder whether a gas of bags (resonance gas) can exist. If it does, what are its properties? First of all, it is clear that a bag is intrinsically indifferent to fragmentation or coalescence:

$$\exp\left(\frac{m}{T_H}\right) = \exp\left(\sum \frac{m_i}{T_H}\right), \quad \text{with } M = \sum m_i. \quad (14)$$

What about the translational degrees of freedom? The answer is given by the equation 13, with $g(m) = e^{m/T_H}$:

$$\frac{N}{V} = \left(\frac{2\pi m T_H}{\hbar}\right)^{3/2}. \quad (15)$$

The gas of bags has the above mass distribution, which has a maximum at $m = \infty$. The bags in the gas tend to coalesce into the largest possible bag. Is there criticality in the system described so far, and if so what are the critical parameters?

7. Criticality, surface entropy and fractal dimension
We preface the answer to this question with an apparently trivial question: What is the most probable shape of a bag? Should the bag have surface energy, then the bag would tend to be compact and near spherical. But in the bag model considered here, all terms are extensive, meaning proportional to the bag volume. There is no $V^{2/3}$ term that would signify the presence of a surface energy. So what is the shape of the bag? We can give a precise answer in a lattice model. Occupied sites can cluster into objects, called animals, of different sizes. Let us consider a given animal of size $A$. In how many shapes can it exist? The logarithm of this number is the
shape entropy. In three dimensions, the surface entropy has a maximum very near the maximum possible surface. This corresponds to a highly dendritic, fractal shape and not to a sphere-like shape. We would conclude that the bag’s most probable shape is fractal. What does this have to do with criticality? In Fisher’s model, criticality is wholly dependent on surface entropy. We can rewrite Fisher’s formula, as follows:

\[ n(A) = g(A) \exp \left( -\frac{c_0 A^\sigma}{T_c} \right), \]  

(16)

where \( \sigma \approx 2/3 \) and \( \ln[g(A)] \) is the surface entropy. Fisher guesses

\[ g(A) = A^{-\tau} \exp(k A^\sigma) = A^{-\tau} \exp \left( \frac{c_0 A^\sigma}{T_c} \right). \]  

(17)

This guess is apparently strange because for lattice animals \( \ln[g(A)] \approx kA \). In order to understand this, let us use equation 16 and the actual \( g(A) \) to calculate an effective \( \sigma \) at various temperatures. Figure 3 and 4 show that at low temperatures the Boltzmann factor kills the highly fractal shapes, retaining only the compact ones. Accordingly, \( \sigma \approx 2/3 \). As we approach the critical temperature, there is a rapid transition from to \( \sigma \approx 2/3 \) to \( \sigma \approx 1 \). The clusters go fractal. This would suggest that the bag(s), which are inherently fractal due to the lack of surface energy, portray critical features at their unique temperature \( T_H \). A rather embarrassing mix of 1st order and criticality. One may wonder whether one can cure this problem by introducing an ad hoc surface term to the bag model. This is equivalent to introducing an additional volume dependent pressure to the bag constant,

\[ p = \frac{1}{3} f(T) - B + \frac{2}{3} a_s V^{-1/3} = 0. \]  

(18)

This gives a \( T \) as an implicit function of the bag volume, \( V \). More specifically, the smaller the bag, the larger its temperature. This is a rather disastrous result because a gas of bags of various sizes cannot be isothermal, and the only way out is to coalesce the bags into a single one.

8. Conclusion

There is a liquid-gas phase transition for symmetric, uncharged infinite nuclear matter, and the relevant coexistence curve has been extracted from the analysis of experimental cluster distributions. The bag model supports a 1st order phase transition between non-Hagedorn hadrons and a bag at a constant, unique temperature \( T_H \). A gas of bags is entropically unstable towards coalescence. The lack of surface energy entropically drives the bag towards fractal shapes, thus simulating criticality. An ad hoc introduction of surface energy makes the bags non-isothermal.

**Figure 3.** Probability of a lattice animal to have a given surface, \( S \), multiplied by Boltzmann factors at different temperatures.  

**Figure 4.** Effective surface dimensionality, \( \sigma \), for lattice animals on a cubic grid as a function of temperature.
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