Abstract

General matterless–theories in 1+1 dimensions include dilaton gravity, Yang–Mills theory as well as non–Einsteinian gravity with dynamical torsion and higher power gravity, and even models of spherically symmetric $d = 4$ General Relativity. Their recent identification as special cases of 'Poisson–sigma–models' with simple general solution in an arbitrary gauge, allows a comprehensive discussion of the relation between the known absolutely conserved quantities in all those cases and Noether charges, resp. notions of quasilocal 'energy–momentum'. In contrast to Noether like quantities, quasilocal energy definitions require some sort of 'asymptotics' to allow an interpretation as a (gauge–independent) observable. Dilaton gravitation, although a little different in detail, shares this property with the other cases. We also present a simple generalization of the absolute conservation law for the case of interactions with matter of any type.

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1 wkummer@tph.tuwien.ac.at
2 Address after Oct. 1, 1994: Institut für Theoretische Physik, ETH-Zürich, Hönggerberg, CH-8093 Zürich, Switzerland
1 Introduction

The interest in two dimensional diffeomorphism invariant theories has many roots. Presumably the most basic one is the central role played by spherically symmetric models in d = 4 General Relativity (GR) as a consequence of Birkhoff’s theorem. Two dimensional models with ‘time’ and ‘radius’ possess an impressive history with promising recent developments [1]. On the other hand, the fact that the Einstein–Hilbert action of pure gravity in 1+1 dimensions is trivial, also has spurred the development of models with additional nondynamical [2] and dynamical (dilaton and tachyon) scalar fields [3], besides higher powers of the curvature [3, 4]. Especially the study of 2d–dilaton theories turned out to be an important spin–off from string theory, and has led to novel insights into properties of black holes [3, 4, 5]. For a scalar field, coupled more generally than a dilaton field, even more complicated singularity structures have been found [6] than in the ordinary dilaton–black hole [3].

Actually such structures were known already before in another branch of completely integrable gravitational theories which modify Einstein–relativity in 1+1 dimensions by admitting nonvanishing dynamical torsion [7, 8]. Here the introduction of the light–cone (LC) gauge led to the expression of the full solution in terms of elementary functions [9] and to an understanding of quantum properties of such a theory in the topologically trivial [10] and nontrivial [11] case.

Recently important progress has been made by the insight [12] that all theories listed above are but special cases of a ‘Poisson–sigma–model’ (PSM) with action

\[ L = \int_M (A_B \wedge dX^B + \frac{1}{2} P^{BC}(X) A_B \wedge A_C) . \]

The zero forms \( X^B \) are target space ‘coordinates’ with connection one forms \( A_B \). \( P \) expresses the (in general degenerate) Poisson–structure on the manifold \( M \), it has to obey a Jacobi–type identity, generalizing the Yang–Mills case, where \( P \) is linear in \( X \) and proportional to the structure constants. For the subclass of models describing 2d–covariant theories the \( A_B \) are identified with the zweibein \( e^a \), with the connection \( \omega^a_b = \epsilon^{ab} \omega \), and may include possibly further Yang–Mills fields \( A_i \). Introducing the Minkowskian frame metric \( \eta_{ab} = \text{diag} (1, -1) \), target coordinates \( X^A \) on the manifold will be denoted as \( \{X^a, X, X^i\} \). In order to have a generic model including also another gauge field we shall consider in our present paper occasionally \( X^i \to Y \) for a U(1)–connection \( A_i = A \). Of course, a nonabelian Yang–Mills field could be included as well, even in our simple explicit model.

Then the (matterless) dilaton, torsion, \( f(R) \)–gravity theories including U(1) gauge fields [28] and even spherically symmetric gravity are obtained as special cases [12] of an action of type (1), namely \( (\epsilon = \frac{1}{2} \epsilon_{ab} e^a \wedge e^b) \)

\[ L = \int_M (X_a D e^a + X d\omega + Y dA - \epsilon V) . \]

Appropriate fixing of \( V = V(X^a X_a, X, Y) \) yields all the models listed above (and many more). It is possible in principle to write down the full solution for (2) in an arbitrary gauge (coordinate system). As seen below, the solution has much of the shape of the LC
gauge solution [9], but in its present general form resembles more a generalization of H. Verlinde’s solution to the (torsionless) dilaton case [15]. It also shows that the form of the solution given in [16] and [17] for nonvanishing torsion may be generalized further. A crucial role for the integrability of theories (1) in the general case

play ‘Casimir–functions’ \( C_i(X^A) \) [12, 14] which on–shell become constants and thus (gauge–independent) ’observables’, also in the classical case [18]. These constants \( C_i \) together with other parameters in \( P \) (or \( V \), e.g. the cosmological constant) determine the almost limitless variety of

Penrose diagrams, characterizing the singularities of such theories [3, 8, 17, 19]. E.g. Schwarzschild or Reissner–Nordström black holes are just relatively simple members of that set. In a previous note [20] we have related \( C_1 \) for the special theory [7] to a global symmetry of the action which, of course, easily generalizes to the present framework. Within the context of dilaton and dilaton–like models, conservation laws of total energy have been discussed repeatedly [13, 17, 21, 22]. Although also some Noether charge concepts of GR have been used [23], the notion

‘ADM’–mass [24] seems to cover various approaches, which, however, always basically refer to variants of the Regge–Teitelboim argument [25] for canonically formulated theories in a finite space volume: Only the choice of appropriate surface terms allows to drop the requirement of vanishing variation on those surfaces. Terms of this type are built in automatically into carefully developed concepts of quasilocal energies (cf. e.g. [26]).

In 1+1 dimensions, of course, such a surface reduces to two points in a space–like direction. In addition, already for the relatively simple case of dilaton gravity, asymptotic flatness is achievable only in one ’space’–direction [3, 21] and the ’vacuum’ still contains a dilaton field. This problem even becomes more acute in the general framework of PSM-s with more general structure [1, 2], where usually no direction leads into flat asymptotics. Thus all GR approaches based upon requirements of this type become inapplicable. On the other hand, we now have a situation where the conserved quantities are known. Thus, in our present work, we are able to turn the problem around. We search a proper definition of a Noether ’charge’ or quasilocal ’energy’, if possible, at a finite ’distance’ which in a gauge independent manner should reproduce the (in our case available and completely known) conservation laws. We also need not just compare a certain prescription with few examples, like the mass of the Schwarzschild black hole and its relatives in GR, but with conserved quantities for any singularity in an (almost limitless) set of singularities which may be ’designed’ at will by a suitable choice of \( V \) in (2) (admittedly in \( d = 2 \) only).

In Section 2 we summarize the general solution, using for explicit demonstration a model including \( R^2 \)–gravity and dynamical torsion and containing also a U(1) Yang–Mills field. The reason for introducing the latter is to see how more than one ’Casimir–function’ may yield one ’energy’. The nonpositivity of that energy is evident in all generic models. Among the different special cases the torsionless limit (including dilaton gravity) will be of special interest. We then discuss the general Killing field which belongs to a metric formed by the solution in an arbitrary gauge. The analysis of different ways to determine the energy by means of surface terms in Section 3 starts with the concept of ’energy–momentum’ by analogy with the case involving matter in \( d = 4 \) for nonvanishing torsion [16, 28]. Then the ’classical’ Noether–procedure follows which we combine with the ADM–setting for space and time. We also apply the general formulation of Wald [28] which covers all covariant theories. In each
case dilaton gravity shows differences in detail, but not in principle. A common difficulty of all these concepts is the interpretation of the ‘mass–shell’, i.e. the manner the equations of motion have to be used. A second basic problem arises, if one tries to implement a quasilocal energy concept in a gauge–independent way, without having an asymptotically flat space at disposal. These problems can be addressed directly in terms of the Regge–Teitelboim argument. One approach follows the trick of recombining constraints \([22, 29]\). The identical result for the surface term can be obtained, however, also directly. Moreover, it is now possible to apply a simple consistent prescription which solves the ‘mass–shell’ problem in a very straightforward manner. Still, the quasilocal energy related to that surface term — as in the previous approaches — only yields the conserved quantity in some ‘asymptotic’ sense. It should be noted that all our results refer to the classical theory. Nevertheless, precisely the Casimir functions in the matterless case become the discrete quantum variables living on a compactified space \(S^1 \[11, 12, 14\] \). Sparse remarks on that and on the changes in the case of interactions with matter are included in the final outlook (Section 4).

2 PSM–Gravitation

2.1 The General Model

With \(V\) in (2) depending linearly on \(X^a X_a\)

\[
V = \frac{\alpha}{2} X^a X_a + v(X, Y) ,
\]

the equations of motion from (2) in a LC basis of the frame metric (\(\varepsilon_{++} = -1\), \(X^\pm = (X^0 \pm X^1)/\sqrt{2}\), \(\eta_{+-} = \eta_{-+} = 1\)) are

\[
\begin{align*}
dX^\pm \pm \omega X^\pm &= \pm e^\pm V \\
dX + X^- e^+ - X^+ e^- &= 0 \\
dY &= 0
\end{align*}
\]

and

\[
\begin{align*}
d e^\pm \pm \omega \wedge e^\pm &= -\alpha e^+ \wedge e^- X^\pm \\
d\omega &= -e^+ \wedge e^- \frac{\partial v}{\partial X} \\
dA &= -e^+ \wedge e^- \frac{\partial v}{\partial Y} .
\end{align*}
\]

Multiplying the first pair of equations in (4) with \(X^-\) and \(X^+\), respectively, the second one with \(V\) and adding yields

\[
d(X^+ X^-) + V dX = 0,
\]

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producing an absolutely conserved quantity \((dC = 0)\)

\[
C_1 = C = X^+ X^- e^{\alpha X} + w(X, Y)
\]  

(7)

\[
w(X) = \int_{X_0}^X v(y, Y) e^{\alpha y} dy.
\]  

(8)

Clearly the lower limit \(X_0 = \text{const.} \) must be determined appropriately so that (inside a certain patch) the integral exists for a certain range of the curvature \(X\). Eq. (7) generalizes \([12, 14]\) the previously known \([7, 9]\) analogous quantity for 2d gravity with dynamical torsion. However, the limit \(\alpha \to 0\) immediately also yields the conservation law for torsionless cases (F(R)–gravity, dilaton gravity \([13]\) etc.). With our additional U(1)–field, (7) depends on the ‘field–strength’ which according to (4) is the second conserved quantity \(C_2 = Y\).

Setting one LC component of the torsion, e.g. \(X^+\) identically zero, the first equation (5) (at \(e^+ \neq 0\)) may yield a constant curvature depending on \(Y\), if \(v(X, Y) = 0\) has a solution at all. Such 'de–Sitter' solutions are related to \(C = 0\) (within an appropriate convention for the integration constant in (8)). Representing discrete points in phase space they are notorious especially in the quantum case \([11]\) but, fortunately, need not be considered in our present context. If \(X^+ \neq 0\) the solution of (4) and (5) becomes

\[
e^+ = X^+ e^{\alpha X} df
\]

\[
e^- = \frac{dX}{X^+} + X^- e^{\alpha X} df
\]

\[
\omega = -\frac{dX^+}{X^+} + Ve^{\alpha X} df
\]

(9)

in terms of arbitrary functions \(f, X, X^+\). One simply solves one of the first eqs. (4) for \(\omega\), the second eq. for, say, \(e^-\) and inserts into (3), using (3) etc. This means that (4) is understood with \(X^-\) in \(e^-\) and in \(V\) to be reexpressed in terms of \(C\) by (7).

Of course, for \(X^- \neq 0\) the analogous solution exists with the roles of \(X^+ \leftrightarrow X^-\) exchanged. This is important for patching together general solutions \([13, 14]\).

The first terms in the first three eqs. for \(e^+ \to \delta e^+, \omega \to \delta \omega, df \to \delta \gamma\) are the on–shell extension of a global nonlinear symmetry of (2), \([20]\). It is related to the conservation \(\partial_\mu J^\mu_\nu = 0\) of a Noether current \(J^\mu_\nu = C(X^a Y_a, X, Y) \partial^\mu_\nu\) because under such a transformation the Lagrangian density in (2) changes by a total derivative only. Mathematically (4) coincides with the solution in the LC–gauge \([1, 27]\) where the curvature \(X\) is gauge–fixed to be linear in ‘time’. But (3) has the big advantage that it is valid in an arbitrary gauge, whereas solutions obtained in the literature to such theories had to rely on special gauges and sometimes on sophisticated mathematical methods to solve the respective equations (cf. e.g. \([4, 5, 6, 7, 22]\)). The line element from (3) generally reads

\[
(ds)^2 = 2e^{\alpha X} df \otimes (dX + X^+ X^- e^{\alpha X} df)
\]

(10)

with \(X^+ X^-\) to be expressed by (7) for fixed \(C\). For the case with torsion our generic model with a U(1) field may be chosen as

\[
V = \alpha X^+ X^- + \frac{L}{2} X^2 + \sigma XY + \frac{\tau}{2} Y^2 - \Lambda.
\]

(11)
This $V$ allows to produce $C$ by a simple integral according to (7). Integrating out $X$ and $X^\pm$ in (2) for $\sigma = \tau = 0$ leads to the model quadratic in curvature and torsion of [7, 9] which, in four dimensions, together with the Einstein–Hilbert term has been known as the 'Poincare–gauge theory' for some time [14, 27]. It only contains second derivatives in the field equations for the variables $e^a$ and $\omega$. However, higher derivative theories are to be treated with equal ease, when polynomials of higher degree in $X$ and $X^+X^-$ are admitted in (3). Of course, $V$ could even be a nonpolynomial function. This would only make the integration harder which leads to (7). As we shall recall shortly below, the zeros of (7) determine the singularity structure of the theory. Thus one could design such a structure by prescribing $C(X^\Lambda)$. From (7) the corresponding $V$ can be read off by differentiation, and the action for that structure follows immediately.

Among the models with vanishing torsion ($\alpha = 0$ in (3)), the Jackiw–Teitelboim model obtains for $V = \Lambda X$. Witten’s black hole [3] represents a special case of a class of more general torsionless theories involving the curvature scalar $R$ and one additional scalar field [4, 22] in a Lagrangian of the type

$$L = \sqrt{-g}[\partial_\alpha \varphi \partial_\beta \varphi g^{\alpha \beta} + A(\varphi) + RB(\varphi)]$$

(12)

with arbitrary functions $A$ and $B$. Matterless dilaton–gravity [3] is the special case $\varphi^2 = 4B = A/\lambda^2 = 4e^{-2\Phi}$

$$L_{dil} = \sqrt{-\tilde{g}}e^{-2\Phi}[4\partial_\alpha \Phi \partial_\beta \Phi g^{\alpha \beta} + 4\lambda^2 + R].$$

(13)

Using the conformal identity for $\tilde{g}_{\alpha \beta} = e^{-2\Phi}g_{\alpha \beta}$ (or $\tilde{e}^a = e^{-\Phi} e^a$)

$$\sqrt{-\tilde{g}}\tilde{R} = \sqrt{-g}R + 2\partial_\alpha(\sqrt{-g}g^{\alpha \beta}\partial_\beta \phi).$$

(14)

Eq. (14) allows the elimination of the kinetic term for $\varphi$ in (12). The resulting action may be written readily in the first order form (2) for $\tilde{e}^a = e^{-\Phi} e^a, V = 4\lambda^2, X = 2e^{-2\Phi}, Y = 0$. Going back from (9) for $\alpha = 0$ as $e^a = \tilde{e}^a/\sqrt{X/2}$ [12], simply leads to

$$e^+ = X^+ e^\Phi df$$

$$e^- = \frac{1}{X^+}[-4d\Phi e^{-\Phi} + df(C e^\Phi - 8\lambda^2 e^{-\Phi})]$$

(15)

$$\omega = \frac{dX^+}{X^+} + 4\lambda^2 df.$$

Here $\Phi, f$ and $X^+$ are arbitrary functions. E.g. the Kruskal form for the metric $(ds)^2 = 2e^+ \otimes e^-$ a dilaton black hole follows from the gauge–fixation ($X^+X^- = uv$)

$$8\lambda^2 e^{-2\Phi} = C - uv$$

$$4\lambda^2 f = \ln(u).$$

The mass of the dilaton black hole is related to $C$ by $C = 8\lambda M$. $X^+ \neq 0$ being still arbitrary, it may be used to gauge $\omega = 0$ which shows that the connection $\omega$ in (13) really has nothing to do with a curvature belonging to the metric derived from that equation. Now precisely the same procedure may be applied to the generalized theories of
type $[12]$. Here $\varphi$ can be eliminated $[1, 22]$ using $[14]$
\[
\mathcal{L} = \sqrt{-g}[A(\varphi)/F(\varphi) + \tilde{R}B(\varphi)]
\] (16)
with
\[
g_{\alpha\beta} = \tilde{g}_{\alpha\beta}/F(\varphi)
\ln F(\varphi) = \int dy/(\frac{dB}{dy}) .
\] (17)

The corresponding first order action $[2]$ becomes
\[
\tilde{L} = \int (X_a D\tilde{e}^a + 2Bd\omega - \tilde{\epsilon}A/F) .
\]

In $[2]$ for $\alpha = 0$ we have as a consequence
\[
X = 2B,
V = v = +\frac{A(B^{-1}(X/2))}{F(B^{-1}(X/2))},
\] (18)
and the conserved quantity for any theory of type $[12]$ is $[7]$ with $\alpha = 0$ and
\[
w(X) = +\int_{B^{-1}(X_0/2)}^{B^{-1}(X/2)} A(y)dy/F(y) .
\] (19)

The line–element in terms of coordinates $(f, X) = Y^\alpha$ reads for any such theory and in any gauge
\[
(ds)^2 = F^{-1}2df \otimes [dX + df(C - w(X))] .
\] (20)

Of course, in each application to a particular model a careful analysis of the range of validity of the mathematical manipulations is required in order to determine a patch, where those steps are justified: allowed ranges for transformations of fields, inversions of functions like $B^{-1}$, integrability of $F$, admissible gauges for $f$ and $X$ etc. It is precisely for this reason that the very comprehensive framework reviewed here $[12]$ does not invalidate approaches in special gauges for special models $[4, 17, 22]$ which allow a careful analysis of these points.

Another example is the action for the Schwarzschild black hole in 4d GR. $v(X) = -1/(2X^2)$ in $[2]$ is found to yield the correct line–element $[12]$.

In the literature also theories with vanishing curvature and dynamical torsion have been considered (‘teleparallelism’ theories $[16, 27]$). E.g. a pure $T^2$–action is the limit $\rho = \sigma = \tau = 0(\alpha \neq 0)$ in $[14]$. 


2.2 Killing Vector and Singularities

In our very general class of models the Killing vector can be found without fixing the gauge (coordinate–system). Using (10) we rewrite the line element (10) in a theory (2) as

$$(ds)^2 = df \otimes [2e^{\alpha X}dX + ldf]$$  \hspace{1cm} (21)

where

$$l = 2X^+X^-e^{2\alpha X} = 2e^{\alpha X}(C - w(X,Y)) .$$  \hspace{1cm} (22)

In terms of the variables $Y^\alpha = (f, X)$, resp. $\partial/\partial Y^\alpha, k^\alpha$ is the Killing vector with norm (22)

$$k^\alpha = (1, 0) \hspace{1cm} k^2 = k^\alpha k^\beta g_{\alpha\beta} = l .$$  \hspace{1cm} (23)

Hence $l > 0$ in (22) determines a timelike Killing field. It is instructive to consider a translation in the (‘on–shell’) Killing direction $-\delta \beta k^\mu$

$$\delta e^\alpha_\nu = (\delta \beta)_{,\nu} k^\mu e^\alpha_\mu + (\delta \beta k^\mu)_{,\nu} e^\alpha_\mu =$$

$$= (\delta \beta)_{,\nu} k^\mu e^\alpha_\mu ,$$  \hspace{1cm} (24)

and the analogous equation for $e^\alpha_\nu \rightarrow \omega_\nu$, inserting back $k^\mu e^\alpha_\mu$, resp. $k^\mu \omega_\nu$ from the solution (9):

$$\delta e^\alpha_\nu = (\delta \beta)_{,\nu} e^{\alpha X} X^a$$

$$\delta \omega_\nu = (\delta \beta)_{,\nu} V e^{\alpha X}$$  \hspace{1cm} (25)

Comparing (23) to the global symmetry in such theories (21)

$$\delta e^\alpha_\nu = (\delta \gamma)_{,\nu} e^{\alpha X} X^a$$

$$\delta \omega_\nu = (\delta \gamma)_{,\nu} V e^{\alpha X}$$  \hspace{1cm} (26)

for a global variation $\delta \gamma_\mu = \epsilon_{\mu\nu}\delta \gamma^\nu$, we find complete agreement, as long as the components $X^a = (X^+, X^-)$ in (4) are taken to be independent functions (and not related by $C$ according to (9)). For the discussion of the singularity structure of (9) a (partial) gauge fixing is useful. If $l > 0$ in (24) we choose coordinates time ($t$) and space ($r$) in $f = f(t, r), X = X(r)$ with $\dot{f} = T(t)$ and

$$X' e^{\alpha X} + f' l(X) = 0 ,$$  \hspace{1cm} (27)

where $f' = \partial f/\partial r , \hspace{0.5cm} \dot{f} = \partial f/\partial t$. Introducing

$$K(z) = - \int_{z_o}^z dy e^{\alpha y} l^{-1}(y) ,$$  \hspace{1cm} (28)
Equation (27) implies
\[ f = \int T(t')dt' + K(X(r)) \]  \hspace{1cm} (29)

In such a gauge \( g_{tr} \), the off–diagonal part of the metric vanishes, so that ‘space’ and ‘time’ are separated. In order to avoid zeros in the norm of the Killing–vector field \( k \) it is obvious to restrict \( z \) and \( z_0 \) to a suitable interval of \( y = X(r) \) where \( k \) exists. The remaining elements of \( g_{\alpha\beta} \) are:
\[ g_{tt} = \dot{f}^2 l \]
\[ g_{rr} = -(f')^2 l \]  \hspace{1cm} (30)

Requiring a 'Schwarzschild'–form of the metric, i.e. \( \det g = -1 \), eliminates the arbitrary functions \( T(t) \) and \( X(r) \) altogether,
\[ T = 1 \]
\[ \alpha X = \ln(\alpha r) \quad (\alpha \neq 0) \]
\[ X = r \quad (\alpha = 0) , \]  \hspace{1cm} (31)

dropping a multiplicative constant \( a \) in \( f \), and \( 1/a \) together with \( r \), and two further constants for the zero points of \( t \) and \( r \). Now
\[ g_{tt} = -g_{rr}^{-1} = l(X(r)) \]  \hspace{1cm} (32)

follows with \( X(r) \) from (29). Especially (32) clarifies the remark above, how an action may be reconstructed for a given singularity in the metric, proceeding backwards through (22) to (2).

We note that for a (generalized) dilaton theory, besides \( \alpha = 0 \), because of the additional factor \( 1/F \) in (20) there is a corresponding change to \( l \) in (30) etc. Thus the singularity structure is determined by \( l/F \).

The Katanaev–Volovich model with (9) at \( \sigma = \tau = 0 \) is sufficiently general to show the intrinsic singularity structure by an analysis of completeness of geodesics. \( C^2 \) global completeness was first shown in [8] within the conformal gauge. The more suitable LC–gauge allows the extension to \( C^\infty \) completeness and a discussion of possible compactifications [17, 19]. In that model altogether 11 types of Penrose diagrams appear (G1, . . . G11 in the classification of [8]). Some show similarities to Schwarzschild and to Reissner–Nordström types, but there are many more. In the more complicated cases they are obtained by the possibility to successively gluing together LC–solutions (I) and (II) with 'complementary' gauges (LCI: \( X^+ = 1, X = t, \omega_t = 0, X^- = X^- (C, X = t); \) LCII: \( X^- = 1, X = r, \omega_r = 0, X^+ = X^+ (C, X = r) \) ) in appropriate lozenges. The diffeomorphisms for doing that is essentially (28) again. For further details we refer to the relevant work [6, 8, 17, 19].

It is sufficient for our present purposes to note that for all types of singularities (including also e.g. naked ones) there are space–like directions allowing the study of surfaces (points) between such singularities at finite (incomplete case) or infinite (complete) distances. Also a second point is obvious from this section: In all two dimensional theories the conserved quantity (-ies) never has (have) a well-defined sign. Thus any hope to find a positive ‘energy’ must be in vain. Therefore, also adding matter to the theories (2) is not likely to improve this situation.

8
3 Conserved Quantities and Surface Terms

3.1 Energy–Momentum

An intrinsic Palatini–type formulation of a covariant theory like (2)
precludes the immediate application of those well–tested concepts from GR which are
based upon $g_{\alpha\beta}$ as dynamical variables. The (off–shell) expression ($e = det e^\mu_\mu$)

$$\mathcal{T}^{\mu\nu} := e^{\alpha\nu} T^\mu_\alpha = \frac{e^{\alpha\nu}}{e} \frac{\delta L}{\delta e^a_\mu}$$

(33)
clearly could be considered the analogue of the ‘energy momentum tensor’, but the sym-
metry $\mu \leftrightarrow \nu$, needed e.g. to prove $(T^{\mu\nu} k_\nu)_;\mu = 0$ in the presence of a Killing–vector $k_\mu$
cannot be true if applied naively [17]. One more sophisticated approach to a conserved
‘energy’, also related to an argument involving the Killing–vector, has been suggested in
connection with theories involving torsion in $d = 4$ (cf. the second ref. [16], [27]).

For that purpose let us add to (2) an additional piece $L^{(m)}$ containing, say, scalar
fields. Then still $\delta L^{(m)}/\delta \omega^a_\mu = 0$, and the only change in the field equations occurs on the
r.h.s. of the first eq. (4) where $T^{(m)}_a$ (analogous to $T^\mu_\alpha$ in (33)) may be written as 1-form
$a = \pm, -$:

$$dX^a - \varepsilon^b_\mu X^b \omega + e^b_\varepsilon \varepsilon^b_\mu V = T^{(m)}_a$$

(34)
The rest of (4) remains unchanged. By analogy to GR we may assume that matter is
concentrated somewhere in space so that at the point at which one considers the energy,
the matterless equations (4) and (5) hold ‘asymptotically’. Although this background
for the effect of matter is by no means flat, it possesses (within certain regions) a time–
like ($\ell > 0$) Killing vector (23) which does not depend on space ($X$), despite the possible
singularity structure of such models, as reviewed at the end of Section 2. Then it is natural
to interpret ‘energy’ — for the matter part on the r.h.s. of (33) but then also for the l.h.s.
— as the projection of $T^{(m)}_a$ onto $k^a$, the matter–less, and hence also ‘asymptotical’ Killing
field $k^a$ measured in terms of the inverse zweibein ($\hat{e}^a_\alpha e^\alpha_\beta = \delta^a_\beta$)

$$k^a = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = k^a \hat{e}^a_\alpha.$$

Using the $df$ components in (10)

$$k^a = \delta^a_\alpha k^\alpha = e^\alpha X^a$$

follows. Thus with (34) we have

$$\mathcal{T}^{(m)} := k^a \cdot T^{(m)}_a = (X^a dX_a + e^b_\varepsilon \varepsilon^b_\mu X^a V) e^\alpha X$$

$$= (d(X^+ X^-) + V dX) e^\alpha X = dC$$

(35)

where the definition of $C$ in (7) has been inserted. However, here $C$ still must be considered
to be a function of $X^+ X^-$, $\dot{X}$ and $Y$ as in (4) and not a constant, although this is implied
by the full field equations which contain (2)! Disregarding this problem for the time being
(for the matter–less case) which we shall encounter
again below, from (35) for the matter part \( dT^{(m)} = 0 \) or

\[
T^{(m)} = -dJ
\]

(36)

follows. Thus a conversation law holds,

\[
d(C + J) = 0
\]

(37)

involving the matter–less (‘background’) Killing projection in \( J \). The r.h.s. of (35) for the matter–less case implies – what we really already know from (7) – that the ‘Noether charge’ \( C = \text{const} \). Although we have obtained a ‘gauge–independent’ definition of energy, our argument is not conclusive for vanishing matter (cf. e.g. also [17]), because the r.h.s. of (35) with \( C \) vanishes identically.

According to Sect.2, dilaton gravity and its generalizations differ from the
other models by a redefinition of the zweibein by a factor involving the dilaton field according to (15) or a factor \( \sqrt{F} \) in [12] with (16). This, however, does not change the preceding argument: \( T_a^{(m)} \) in (34) just acquires a further factor \( \sqrt{F} \) upon variation with respect to the ‘true’ zweibeins \( e^a \), instead of \( \tilde{e}^a = \sqrt{F} e^a \), but the projection of the Killing–vector onto \( e^a \) contains a compensating \( 1/\sqrt{F} \).

Although starting from the consideration of a local object like \( T_a^{(m)} \), we thus arrive generically at surface–related quantities like \( C \) and \( J \). For 1 + 1 dimensional theories the space–like boundary reduces to two points in space attached to two time–like curves. For any application to a singularity in ‘space’, enclosed by such a surface, those two space points also sit on a spacelike curve which may even be separated completely into two disconnected pieces precisely by the singularity (and/or a horizon) under consideration. Therefore, a ‘surface integral ’ consisting of the difference of the values at those points seems to be of little significance. It is rather the value at each point that is important. As we have seen already for the matter part in the preceding example, a ‘superpotential’ \( J \) automatically leads to such surface terms as \( Q^0 = \int dr \epsilon^{01} \partial_r J \) receives contributions from the boundary only.

Another important point is the question of the background [26]. In \( d = 4 \) GR for the black hole the latter is represented naturally by flat asymptotic space which also determines a corresponding time–like Killing vector. In the case with matter in \( d = 2 \) the corresponding background turned out to have a generically very complicated intrinsic singularity structure with no direct relation to the matter contained in it, the Killing vector being related just to that (matter–less) background. This clearly points towards a possible basic weakness of any analysis based upon 2d covariant singular models, when a comparison with GR in \( d = 4 \) is intended. Still, within the \( d = 2 \) matterless models and their conserved quantity \( C \) the freedom persists to subtract yet another background according to some guiding principle. We shall return to that point in the next subsection.

### 3.2 Noether–Currents

Among the pseudo–tensor approaches we take in (2) the simplest one with

text–book Noether currents for the (global) \( \text{SO}(1,1) \) Lorentz invariance \( \delta a^\pm = \pm \delta \gamma a^\pm \) and (global) translation invariance \( \delta x^\mu = \delta a^\mu = \text{const} \). In coordinates \( x^\mu = (t, r) \) the
Lorentz–current becomes

\[ J^{\mu} = \varepsilon^{\mu \nu} \partial_{\nu} X \]  

(38)

and the 'energy momentum tensor'

\[ T_{\nu}^{\mu} = \varepsilon^{\mu \alpha} \partial_{\alpha} K_{\nu} \]  

(39)

\[ K_{\nu} = X^{+} e^{-\nu} + X^{-} e^{+\nu} + X\omega_{\nu} + YA_{\nu} \]  

(40)

Both expressions follow from the use of the field equations (the second eq. (3) in (39) and the first and last eq.(5) in (40)) in the standard definitions of such currents, and both turn out to be expressible in terms of superpotentials. The corresponding densities \( J_{0} = \partial_{1} X \), \( T_{0}^{0} = \partial_{1} K_{0} \), upon integration yield surface terms \( X \), resp. \( K_{0} \). Without any physically motivated choice of coordinates in (39) and (40) no relation to an 'energy ' concept can be expected. Their possible significance may be checked in specific gauges, e.g. in LC gauge \( e_{0}^{+} = \omega_{0} = A_{0} = 0 \), \( e_{0}^{-} = 1 \): In that case the solution in (3) implies \( X^{+} = A(r) \), \( X = A(r)t + B(r) \). However, a residual gauge–fixing [9] allows to set \( A = 1, B = 0 \) for which \( \partial_{1} X = \partial_{1} K_{0} = 0 \) follows identically.

In a general gauge with \( \nu = 0 = \text{‘time’}, (40) \) with (3) is easily evaluated with the typical \( V \) of (10) (e.g. for \( \alpha \neq 0 \)):

\[ K_{0} = \dot{X} - X \frac{\dot{X}^{+}}{X^{+}} + \dot{Y} + \dot{f} \tilde{E} \]  

(41)

\[ \tilde{E} = 2C(1 + \frac{\alpha}{2} X) + \frac{e^{\alpha X}}{\alpha} (-\frac{\rho}{\alpha^2} + \frac{\rho X + \sigma Y}{\alpha} + 2\Lambda) \]  

(42)

In an ADM–setting

\[ (ds)^{2} = N^{2} d^{2}t - h(dr + N_{1} dt)^{2} \]  

(43)

the situation for the surface term (11) is less trivial than in the LC gauge. With mutually orthogonal time–like and space–like vectors

\[ n^{\pm \mu} n^{\pm}_{\mu} = \lambda_{\pm} = \pm 1 \quad n^{+ \mu} n^{-}_{\mu} = 0 \]  

(44)

on 'hypersurfaces' (curves) with internal metric

\[ \gamma^{\pm \mu \nu} = g_{\mu \nu} - \lambda_{\pm} n^{\pm \mu} n^{\pm}_{\nu} \]  

(45)

orthogonal to \( n^{+ \mu} \), resp. \( n^{- \mu} \), a finite region in 1 + 1 space–time may be surrounded (as in the case \( d = 4 \) [20]). From (13) we obtain

\[ n^{-\mu} = \frac{1}{\sqrt{h}} \delta^{\mu}_{1} \quad n^{-\mu} = -\sqrt{h} \left( \begin{array}{c} N_{1} \\ 1 \end{array} \right) \]  

(46)

The boundary with normal in the space–like direction simply becomes

\[ \gamma^{\mu \nu} = \left( \begin{array}{cc} N^{2} & 0 \\ 0 & 0 \end{array} \right) \]  

(47)
For our solution (9) we conclude from (10) with (22)

\[ \sqrt{h}N = e^{\alpha X}(f'X' - f'X) \]

\[ -h = 2f'e^{\alpha X}(X' + f'\frac{1}{2}e^{-\alpha X}) \]

\[ -hN_1 = e^{\alpha X}(f'X' + f'X + f'f'e^{-\alpha X}) . \]

Again dots and primes refer to derivatives with respect to \( t \) and \( r \). Here both \( f \) and \( X \) are functions of the (arbitrary) coordinates \( t \) and \( r \). Without restricting the generality of (48) we may decide to measure 'space' in terms of (curvature) \( X \), i.e. \( \dot{X} = 0 \). Still, (48) expresses altogether three functions \( N, h \) and \( N_1 \) in terms of \( X(r) \) and \( f(t,r) \). For the proper definition of a quasilocal quantity on the surface, \( \delta/\delta N \) will be needed at fixed \( h \) and \( N_1 \). As there must be thus a relation between the three ADM–functions in our model it is cumbersome to perform that derivative in the general case. We, therefore, fix the gauge in part to simplify matters, but in such a way that gauge–in)dependence of results may still be checked. The most natural gauge fixing is \( N_1 = 0 \), i.e. (27), as in section 2.2, yielding (29). The differentiation with respect to \( N \) at fixed \( h(r) \) implies differentiation at constant \( r \). By analogy with the quasi–local definition of energy in ref. [26] in that case the projection

\[ E = \frac{2n_0^+n_0^-}{\sqrt{\gamma}} \frac{\delta K_0}{\delta \gamma_{00}} , \]

seen by a physical (geodesic) observer [30] should be related to the 'energy' at fixed \( r \). Note that \( \gamma \) is the determinant of the submatrix in (17), i.e. \( N^2 \) again. We are, of course, aware of the fact that this approach only finds its justification using the full quasilocal approach [28]. This we defer to Section 3.3. Still, the problems will remain essentially the same ones in our present naive application. Thus from (13), (14), (15) we obtain

\[ E = \frac{\delta K_0}{\delta N} \bigg|_r \left[ \bar{E} \right] = \frac{\bar{E} \sqrt{h}e^{-\alpha X}}{X'} . \]

Eq. (50) refers to coordinate values \( r \), the physical distance \( \lambda \) may be obtained by integrating

\[ d\lambda = \sqrt{h(r)}dr \]

wherever this is possible \( (h > 0) \). Thus (51) may also be written as

\[ E = \bar{E}(\lambda) e^{-\alpha X} \left( \frac{dX}{d\lambda} \right)^{-1} . \]

Neither \( \bar{E}(\lambda) \) nor the factor in (52) are gauge–independent. They still contain the arbitrary function \( X(r(\lambda)) \), even when we assume that a background with \( C = 0 \) is to be subtracted. The situation is slightly improved in the torsion–less case \( \alpha = 0 \), where (12) is to be replaced by

\[ \bar{E} = 2(C - w) + vX + Y \frac{\partial w}{\partial Y} \int_{X_o}^{X(r)} v(y,Y)dy . \]

\[ w = \int_{X_0}^{X(r)} v(y,Y)dy . \]
Here the aforementioned subtraction of a background with $C = 0$ produces a gauge–independent result $\tilde{E} = 2C$ for any $V = v(X, Y)$. In any case, the problem of the gauge–dependent factor of $\tilde{E}$ in (52) remains.

Another way to see the same problem results from eliminating $X$ by $f$ with (27) in (52)

$$E = \tilde{E} \frac{f'}{\sqrt{h}} = \tilde{E} \frac{df}{d\lambda} ,$$

again measuring space by the physical distance $\lambda$. In the present coordinate system with a Killing field (23) the factor of $\tilde{E}(\lambda)$ reflects the dependence of flat field on the space, in a way still to be determined by some additional principle. We shall come back to that point in the next section.

As compared to the general torsion–less situation with (53), dilaton gravity in connection with a Noether current (40) needs a few comments, although the basic problems remain the same in that case. Here $Y = 0, v = v_0 = \text{const.}$ in (53), i.e. $\tilde{E}_{\text{dil}} = C - v_0 X$. From the preceding discussion for $\alpha \neq 0$ another modification comes from the fact that the metric to be used in the ADM–setting is given by (20), i.e. contains another factor $(X/2)^{-1}$. Hence differentiation with respect to $N$ as in (50) or (52) provides a factor $X/2$, i.e.

$$E_{\text{dil}} = \frac{X}{X'} \frac{\sqrt{h}}{2} (C - v_0 X) .$$

Again a subtraction of a background with $C = 0$ yields a gauge–independent $\tilde{E}_{\text{dil}}$. Now, for dilaton theory it is well–known that the mass (prop. $C$) can be obtained by an appropriate limit relative to the dilaton vacuum with linearly rising (or decreasing) $\Phi$ at space–like distances [3]. In fact, with $h(\lambda) = 1$ in (53)

$$\lim_{\lambda \to \infty} \frac{d}{d\lambda} \ln X \propto \lim_{\lambda \to \infty} \frac{d\Phi}{d\lambda} = \text{const.}$$

in an asymptotic sense determines (up to a constant factor) the mass. We shall take this as a hint for a possible extrapolation to the general case in Sect. 3.3.

The ‘naive’ Noether current approach still did require some asymptotic limit, but it, at least, avoided the mass–shell difficulty in our first attempt (10). Therefore, one may hope that a direct application of the very general approach e.g. of Wald [28] for Noether charges in covariant theories may avoid those problems: The Lagrangian $L$ in our action (2) is already a two–form as required in [28]. Its variation by a diffeomorphism with Lie–derivative $\Lambda_\xi$ produces a surface term

$$\delta L = \text{e.o.m.} + d\Theta$$

$$\Theta = X^B \Lambda_\xi A_B ,$$

using the comprehensive notation for connections and target space coordinates of [1]. Following the argument of [28], $\Theta$ can be related to a Noether current one–form
\[ j = \Theta - \xi \cdot \mathcal{L} \]  
(58)

where the dot indicates a contraction with the first index of the two-form \( \mathcal{L} \). It can be verified easily from (57, 58) with (2) that \( j \) is exact on–shell:

\[
\begin{align*}
  j &= dQ + e.o.m. \\
  Q &= X^B \xi \cdot A_B 
\end{align*}
\]  
(59)

For \( \xi \) we may choose the Killing vector \( k \) of (23), i.e. simply selecting the \( df \) components in the solution (9). Within a patch where \( k \) is a time–like Killing vector, \( Q \) may be evaluated at each point of 'space' \( X^3 = X \). By comparing with the (variation of the) symplectic one–form serving as a Hamiltonian [28] the (variation of the) Wald’s 'energy'–density on the (here zero–dimensional) boundary becomes

\[
\delta E_{(W)} = \delta X^B k \cdot A_B = \delta X^B (A_B)_f .
\]  
(60)

However, we arrive at the desired gauge–independent result \( E_{(W)} \propto C \) again only by not fully going on–shell: Note that the \( f \)–components of the solution (9) may be expressed formally as

\[
(A_B)_f = \frac{\partial C(X^A)}{\partial X^B}
\]  
(61)

as long as \( C \) in (7) is taken to be a function of the \( X^A \) and not a constant. Then (60) may be integrated to yield

\[
E_{(W)} = C ,
\]  
(62)

but we have the same mass–shell problem as with the approach leading to \( T \) above. The present argument following [28] also does not change in the case of dilaton gravity, the Noether charge being the constant \( C \) and hence essentially coinciding with the black–hole mass everywhere (and not only asymptotically).

### 3.3 Regge–Teitelboim Surface Term

The problems encountered in the preceding Sections 3.1 and 3.2 may be summarized as follows:

Firstly, energy momentum in the Killing direction (23), and the closely related general Noether charge (59) in the sense of [28], yield the gauge–independent conserved quantity \( C \), however in each of these cases full use of the field equations (including \( C = \text{const.} \)) would not give any result.

By contrast the 'naive' Noether current (40) could be evaluated with the full equations of motion yielding again a surface term. However, that surface term showed already gauge–dependence which could be compensated by subtraction of a (non-flat) background (in the torsion–less case only).

Secondly, a problem arose if – by analogy with a quasilocal energy [28] — it was tried to
calculate the 'energy' referring to a proper ADM–setting from such a surface term. At best 'asymptotically' a relation to $C$ could be obtained in the special case of dilaton gravity. What is usually called 'ADM–mass', especially within numerous applications in 2d theories, explicitly or implicitly is based upon the argument of Regge and Teitelboim [25] for a Hamiltonian formulation in a finite region of space: An appropriate surface term with (spacelike normal vector) must be added to the Hamiltonian so that the equations of motion are reproduced without the requirement of the vanishing variation on that surface. Clearly the choice of that surface determines the result in a profound way. Actually this argument can be considered to be somehow hidden in the first and third approaches of Sect. 3.1 and 3.2, as applied to our present $d = 2$ theories. Here we use it explicitly to isolate the origin of the mass–shell problem and to propose a simple remedy. In our case the Hamiltonian with momenta $X^A = (X^-, X^+, X, Y)$ may be simply read off from the first order action (2)

$$H = \int_a^b dr [e^t G^- + \epsilon^- G^+ + \omega t G + A t \tilde{G} + \partial_r O ] \ , \quad (63)$$

fixing $t$ to be the Hamiltonian time. $H$ consists of the constraints

$$G^\pm = - \partial_r X^\pm \mp \omega_r X^\pm \pm \epsilon^\pm V$$

$$G = - \partial_r X + X^+ \epsilon^- - X^- \epsilon^+ \quad (64)$$

and of a surface density $O$, which should be related to the conserved quantity. With the abbreviations $\bar{q}_A = \{ e^t, \epsilon^+, \omega_t, A_t \}$ the straightforward steps from (2) to (63) show that in this case with

$$O^{(0)} = \bar{q}_A X^A \quad (65)$$

the validity of the Hamilton–Jacobi equations with nonvanishing variations on the surface for the $X^A$ from (63) are verified easily. However, (65) exactly coincides with the $K_0$ of (40) and (41) in the preceding subsection which has been found to be an unlikely candidate. There are, however, other ways to implement this argument. One is the proposal to (linearly) recombine the constraints in (63) so that a surface term appears naturally [22, 29]. In our present completely general case the proper combination of constraints can be deduced by a simple algebraic argument. The algebra of (first class) constraints $G^A$ and $X^A$ closes [31], and its center has two elements, $Q(X)$ which as a function of $X^A$ coincides with $C$, but can be considered not to be constant for the present, because the e.o.m. are not fulfilled, and

$$U(X) = (V G + X^+ G^- + X^- G^+) e^{\alpha X} \ . \quad (66)$$

The trivial further quantity $\tilde{U} = Y \tilde{G}$ involving the $U(1)$–field is not important here. Actually (66) generates the diffeomorphisms in 'space' [13] and may be expressed (still off–shell) as

$$U = - \partial_r Q(X) \ . \quad (67)$$

15
Following the steps of [22] in our completely general case (63), we eliminate the Lorentz–
constraint $G$ in favour of $C$ by (66) with (67):

$$H = \int_a^b dr \left[ \omega_t (c^r - X^e V^e) G^e + \frac{\omega_t e^{-\alpha X}}{V} (-\partial_r Q) + \partial_r O^{(1)} \right]$$

(68)

Now $O^{(1)}$ is determined in such a way that with the Killing direction as 'time' the canonical
equations of motion are reproduced for the finite end–points $a$ and $b$ of 'space' $r$. Varying
the first terms in (68), only expressions involving $\partial X^3, \partial X^a$ are critical. In order to
compensate those by a total derivative, for the correct equations for $X$ an appropriate
$O^{(1)}$ must be chosen as

$$O^{(1)} = \frac{\omega_t e^{-\alpha X}}{V} Q(X) .$$

(69)

In the next step we consider the on–shell limit $Q(X) = C = \text{const.}$, expressing also the
factor in front of $Q$ by means of the solution for $\omega_t$ in (3):

$$O^{(1)} = \dot{f} C$$

(70)

Actually, the same result obtains simply requiring that in (63) the proper equations of
motion for the $X$ should be reproduced — without any recombination of constraints. From the Regge–Teitelboim argument then the necessary condition for $O$ obviously is

$$\delta O = \bar{q}_A \delta X^A$$

(71)

in the short–hand notation already used above. Relaxing again for the moment the
on–shell condition that $X^-$ is related to $X^+$ by (3) at fixed constant $C$, and treating
$C = C(X^A)$ as a function, we again use the identity (61) which in our present notation reads

$$\bar{q}_A = \dot{f} \frac{\partial Q(X)}{\partial X^A} .$$

(72)

Inserting (72) into (71) allows the immediate integration leading to (70) again.

Within this formulation, it is now easy to see how we are able to solve at least the problem
of a gauge–independent surface term: We just interpret that term containing (72) as given
'Independently' from the full field equations (which would require $C = \text{const.}$). We ’guess’
it so as to yield the proper Hamiltonian equations of motion for the $X^A$, because that
compensation works before the equations of motion are obtained and before they are
used to fix $C = \text{const.}$. In retrospect we see that a similar argument should have been
introduced within the context of the approaches leading to (38) and (41), however it seems
to be difficult to find an equally obvious way to do this.

Eq.(70) immediately allows to make contact with the projections of Noether charges
according to (34) and (36) above. Fixing our coordinate system so that $df$ is ‘time’ $(dt)$
in (70) precisely corresponds to that projection producing a result like (22).

For the quasilocal energy the problem, encountered in Sect. 3.2 remains, although now
instead of (50) and (52) in

$$E = \beta C$$

$$|\beta| = \frac{e^{-\alpha X}}{(dX/d\lambda)} = \frac{e^{-\alpha X}}{X'} \sqrt{h(r)}$$

(73)
the gauge–dependence reduces to the factor $\beta$, which persists even measuring length by the physical scale $\lambda$. In 2d covariant models only in exceptional cases an asymptotically flat direction exists, e.g. in dilaton gravity where $\beta \to \text{const.}$ asymptotically for the dilaton vacuum. However, already in that case a more careful definition would read for $r > r_c$ (or $\lambda > \lambda_c$)

$$|\beta| - c | < |p|^\alpha,$$

implying that above a certain space–like distance (again in a diagonal metric $N_1 = 0$ for simplicity) the deviation from some constant $c$ is much smaller than the smallest parameter in the solution with a power $\alpha$.

To match the dimension of the l.h.s. in (74). We observe that (74) may be assumed to hold in a general 2d theory as well, but for a finite interval (in ‘physical space’ $\lambda$) so that $r_0 < r < r_1$. The conditions to be fulfilled for such a region are most easily seen in Schwarzschild coordinates (31). From $|\beta(r)| \sim c$

$$|c| \simeq \left\langle \frac{1}{\sqrt{l}} \right\rangle$$

follows where the average value for the norm of the Killing vector in the interval $[r_0, r_1]$ appears, as indicated by the r.h.s. Of course, for (generalized) dilaton theories $|l|$ has to be corrected by the proper field by definition. In both cases the ‘average’ metric becomes $g_{\alpha\beta} \sim \langle g_{\alpha\beta} \rangle = \text{diag}(c^{-2}, -c^2)$, i.e. we have, as expected, flat space.

Obviously (75) only holds far from the zeros (horizons, singularities) of $|l|$ and may be well satisfied e.g. near (flat) extrema, saddle points etc. Actually precisely those properties for a function which essentially coincides with $l(r)/X(r)$ in our present notation has been the basis of the general discussion of singularity structures in ref. [19] so that details for the existence of an appropriate interval $[r_0, r_1]$ in a specific theory can be deduced easily for any model with any given set of parameters.

4 Conclusion and Outlook

Exploiting the properties of conserved charges in 2d covariant matterless theories in a comprehensive framework, we are able to critically analyze pseudo–tensor and ADM–methods to obtain quasilocal conserved quantities. In two dimensions the lack of asymptotic flatness is compensated by the fact that all (gauge–independent) conserved quantities are known, together with the general solution in arbitrary gauges. E.g. the mass of the black hole in dilaton gravity is just one particular very simple example for such a conserved quantity. In our present work we explicitly used for illustration an example where one conservation law $C_1 = \text{const.}$ refers to ‘energy’, and a second one is related to another U(1) gauge group (‘field strength’).

Trying to determine Noether–type charge definitions and to apply the concept to quasilocal energy, we encountered two basic problems which are typical for matter-less $d = 2$ covariant theories in the generic case, including theories with dynamical torsion as well. One problem arose with the use of the full equations of motion, an additional one appeared together with a quasilocal energy definition. Solutions to both problems could be
obtained by proper interpretation of the argument of Regge and Teitelboim for compensating terms on surfaces (in a finite distance) with a space–like normal in the Hamiltonian approach although that argument is not naively applicable in d = 2 theories: It is possible to ‘guess’ that surface term from the solution without actually going fully on–shell. This provided the solution for a (gauge–independent) Noether–charge, to be identified with $C$. By analogy with their asymptotic definition of quasilocal energy in dilaton gravity we propose a similar definition at finite distance, avoiding in this way the problem that, in general, the singularity structure has no 'flat' direction in 2d–theories.

It is obvious from the present approach that no positivity for any such ‘energy’ can be expected. From the mathematical point of view any sign of $C_1$ is equally acceptable, albeit belonging to very different singularity structures. Of course, any statement about the (classical) stability of such structures is possible only when additional interactions with 'genuine' matter are considered. A single scalar field, allowing always the reinterpretation as a dilaton is not enough. This not only holds for the original dilaton black hole [1, 2, 3] but for any covariant 2d theory. In our class of models the second conserved quantity influences the singularity structure at the same level as e.g. the cosmological constant. It naturally appears as part of the ‘energy’ $C_1$. The special role of one single conserved quantity — as opposed to the PSM-s [14] — clearly is typical for the peculiar choice of target space appropriate for 2d covariant theories.

In the course of our analysis we also encountered in one case (Sect. 3.1) the special role of the ‘background’ singularity structure in 2d covariant theories interacting with matter. Therefore it seems that precisely that structure assumes the place of an asymptotically flat ‘background’ in d = 4. If this were true in general, all 2d–modelling of GR involving time–dependent singularity structures in the interaction with matter etc. may suffer from a serious inherent defect. Fortunately known [3] and more recent [19] results at least for the dilaton black hole do not support such a pessimistic view, at least in the case of theories of that type.

In the quantum case a genuine field theory also only arises in interaction with matter. Without that only on a suitable compactified space isotopic to $S^1$ the finite number of zero modes precisely of the $C$-s covers

a quantum mechanical theory with a finite number of degrees of freedom.

Although $C_1$ (in our generic case) turns into a ‘energy density’, not necessarily constant in space and time anymore when matter is present [33], it retains its physical aspects related to the geometrical part of the action — very much like the mass parameter in the so far very most prominent case, the example of the dilaton black hole interacting with matter: E.g. in the energy momentum approach from eq.(14), generalizing even to an $\omega$–dependent $L^{(m)}$ with the r.h.s. of the second eq.(4), containing a one–form $S^{(m)}$, the steps leading to (35) imply a relation

$$\tilde{T} = T + 2V S^{(m)} = dC$$

and thus again a generalization of the absolute conservation law for $C$ to a relation of type (37). $C$ now will vary in space and time, in general.

We should emphasize again that the definition of $\tilde{T}$ involves a matter–less Killing direction i.e. refers to a background which assumes the place of asymptotically flat space in customary applications to $d = 4$ GR.

We believe that our present work considerably broadens the range of possible starting
points in that direction, at least by the almost limitless increase in conceivable singularity structures to be analyzed within this context.

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