A NOTE ON THE DIVISIBILITY OF THE WHITEHEAD SQUARE

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ABSTRACT. We show that if we suppose \( n \geq 4 \) and \( \pi^{S}_{2n-1} \) has no 2-torsion, then the Whitehead squares of the identity maps of \( S^{2n+1} \) and \( S^{4n+3} \) are divisible by 2. By applying the result of G. Wang and Z. Xu on \( \pi^{S}_{61} \), we find that the Kervaire invariant one elements in dimensions 62 and 126 exist.

1. Introduction

Let \([\iota_{n}, \iota_{n}]\in \pi_{2n-1}(S^{n})\) denote the Whitehead square of \( \iota_{n} \in \pi_{n}(S^{n}) \) where \( \iota_{n} \) is the homotopy class of the identity map of \( S^{n} \). For \( n \) odd \( \neq 1, 3, 7 \), it is well known that \([\iota_{n}, \iota_{n}]\) generates a subgroup of order 2 [2]. Furthermore, when \( n \) is not of the form \( 2^{r} - 1 \), this subgroup splits off as a direct summand [3]. Let \( n_{k} = 2^{k} - 1 \) and write \( w_{k} \) for \([\iota_{2n_{k}+1}, \iota_{2n_{k}+1}]\in \pi_{4n_{k}+1}(S^{2n_{k}+1})\). In this note we consider the divisibility of \( w_{k} \) by 2. But, since \( w_{1} = 0 \) and \( w_{2} = 0 \) [7], we assume here that \( k \geq 3 \). The main result is then the following

THEOREM. Suppose \( \pi^{S}_{2n_{k}-1} \) has no 2-torsion. Then \( w_{k} \) and \( w_{k+1} \) are divisible by 2.

From [7] and [8] we know that \( 2^{2} \pi^{S}_{13} = 0 \) and \( 2^{2} \pi^{S}_{61} = 0 \) where the subscript 2 represents the 2-primary part. By applying these two results to the theorem we obtain

COROLLARY. \( w_{3}, w_{4}, w_{5} \) and \( w_{6} \) are divisible by 2.

Because the Kervaire invariant one element \( \theta_{k} \in \pi^{S}_{2n_{k}} \) exists if and only if \( w_{k} \in 2^{2} \pi_{4n_{k}+1}(S^{2n_{k}+1}) \) [1], this corollary together with the fact that \( w_{1} = 0 \) and \( w_{2} = 0 \) shows that there exist \( \theta_{1}, \theta_{2}, \theta_{3}, \theta_{4}, \theta_{5} \) and \( \theta_{6} \). This settles the problem of determining whether or not there exists \( \theta_{6} \) which has been perhaps left unsolved. Based on the result of [4] that \( \theta_{k} \) does not exist for \( k \geq 7 \), it can be therefore concluded that the only \( \theta_{k} \) which exist are these six ones.

In order to prove the theorem we use an expression for \( w_{k} \) by means of the characteristic map of a principal bundle over a sphere. Let \( T_{n+1} : S^{n-1} \to SO(n) \) denote the characteristic map of the canonical principal \( SO(n) \)-bundle \( SO(n+1) \to S^{n} \) and let \( J \) be the \( J \)-homomorphism \( \pi_{n-1}(SO(n)) \to \pi_{2n-1}(S^{n}) \). Then from [5, p. 521] we know that, when \( n \) is odd \( \geq 9 \), \([\iota_{n}, \iota_{n}]\) can be written \([\iota_{n}, \iota_{n}] = J([T_{n+1}])\), so that

\[ w_{k} = J([T_{n_{k}+2}]) \quad (k \geq 3) \]

(the bracket [ ] denotes the homotopy class).

2010 Mathematics Subject Classification. 55Q15, 55Q50.
Let \( \mathbb{R}^n \) be euclidean space: \( x = (x_1, \ldots, x_n) \). Let \( S^{n-1} \subset \mathbb{R}^n \) be the unit sphere with base point \( x_0 = (0, \ldots, 0, 1) \). According to [1], \( T_{n+1} \) is then given by
\[
T_{n+1}(x) = (\delta_{ij} - 2x_ix_j) \begin{pmatrix} I_{n-1} & 0 \\ 0 & -1 \end{pmatrix} \quad (1 \leq i, j \leq n)
\]
where \( I_{n-1} \) is the identity matrix of dimension \( n - 1 \). Then obviously \( T_{n+1}(\pm x_0) = I_n \).

Let \( \Sigma^n = \mathbb{R}^n \cup \{\infty\} \) be the one-point compactification of \( \mathbb{R}^n \) with \( \infty \) as base-point. Applying the Hopf construction to \( T_{n+1} \) we obtain a map
\[
\tau_n : \Sigma^n \wedge S^{n-1} \to \Sigma^n.
\]
By virtue of \( T_{n+1}(\pm x_0) = I_n \) it follows that
\[
\tau_n \mid \Sigma^n \wedge S^0 \simeq c_\infty
\]
where \( S^0 = \{x_0, -x_0\} \) and \( c_\infty \) denotes the constant map at \( \infty \). Since \( (\Sigma^n, \infty) \simeq (S^n, x_0) \), it is clear that \( [\tau_n] = J([T_{n+1}]) \in \pi_{2n-1}(S^n) \) and so \( w_k = [\tau_{nk+1}] \) for \( k \geq 3 \).

For \( 1 \leq s \leq n-1 \), if we set \( x' = (x_1, \ldots, x_s) \) and \( x'' = (x_{s+1}, \ldots, x_n) \), then the map \( (x', x'') \to (\pm x', x'') \) of \( \mathbb{R}^n \) defines involutions on \( \Sigma^n \) and \( S^{n-1} \), denoted by \( \bar{a}_{s,n-s} \) and \( a_{s,n-s} \), respectively. Let \( I_{s,n-s} = (-I_s) \times I_{n-s} \) be the diagonal matrix whose first \( s \) diagonal elements equal to \(-1\) and the remaining \( n-s \) diagonal elements equal to \( 1 \). Then we find that
\[
T_{n+1}(a_{s,n-s}(x)) = I_{s,n-s}T_{n+1}(x)I_{s,n-s} \quad (x \in S^{n-1}).
\]
This gives \( \tau_n(\bar{a}_{s,n-s}(v) \wedge a_{s,n-s}(x)) = \bar{a}_{s,n-s}(\tau_n(v \wedge x)) \) where \( v \in \Sigma^n, \ x \in S^{n-1}, \ i.e.
\[
\tau_n \circ (\bar{a}_{s,n-s} \wedge a_{s,n-s}) = \bar{a}_{s,n-s} \circ \tau_n \quad (0 \leq s \leq n-1).
\]
If \( s \) can be written as \( s = 2j + r \) \( (j, r \geq 0) \), then considering the above formula with \( I_{s,n-s}T_{n+1}(x)I_{s,n-s} \) replaced by
\[
(-I_r \times \nu(t) \times I_{n-s})T_{n+1}(x)(-I_r \times \nu(t)^{-1} \times I_{n-s})
\]
where \( \nu(t) \) is a path in \( SO(2j) \) from \(-I_{2j}\) to \( I_{2j} \), we obtain
\[
\tau_n \circ (\bar{a}_{r,n-r} \wedge a_{s,n-s}) \simeq \bar{a}_{r,n-r} \circ \tau_n \quad (0 \leq s \leq n-1).
\]
This is a homotopy relative to \( \Sigma^n \wedge S^0 \), so that it maintains the relation \( \tau_n \mid \Sigma^n \wedge S^0 \simeq c_\infty \). In particular, when \( s \) is even, it becomes a homotopy
\[
\tau_n \circ (1 \wedge a_{s,n-s}) \simeq \tau_n \quad (0 \leq s \leq n-1)
\]
where \( 1 \) denotes the identity map of \( \Sigma^n \). This exhibits a certain symmetry property of \( \tau_n \) about \( x_{n-1} \)-axis. In the case when \( s \) is odd, considering its suspension \( S\tau_n \) instead of \( \tau_n \) we have
\[
S\tau_n \circ (1 \wedge a_{s,n-s}) \simeq S\tau_n \quad (0 \leq s \leq n-1)
\]
where \( 1 \) is the identity map of \( \Sigma^{n+1} \) (here \( \Sigma^{n+1} \) is identified with the suspension \( S\Sigma^n \) of \( \Sigma^n \) in the usual way).

For \( 1 \leq s \leq n-1 \), let \( \mathbb{R}^{n-s} \subset \mathbb{R}^n \) be the subspace defined by \( x_1 = \cdots = x_s = 0 \) and let \( (\Sigma^{n-s}, S^{n-s-1}) \) be the pair defined for it above. Then according to the definition of \( \tau_n \) we see that the restriction of \( \tau_n \) to \( \Sigma^n \wedge S^{n-s-1} \) can be written as the composition
\[
\Sigma^n \wedge S^{n-s-1} \xrightarrow{\rho_{n+1}} \Sigma^{n-s} \wedge S^{n-s-1} \xrightarrow{\tau_{n-s}} \Sigma^{n-s} \xrightarrow{i} \Sigma^n
\]
where \( p \) and \( i \) denote the maps induced by the canonical projection of \( \mathbb{R}^n \) onto \( \mathbb{R}^{n-s} \) and inclusion of \( \mathbb{R}^{n-s} \) into \( \mathbb{R}^n \), respectively and 1 denotes the identity map of \( S^{n-s-1} \). This means that the restriction of \( \tau_n \) to \( \Sigma^n \wedge S^{n-s-1} \) coincides with the \( s \)-fold suspension of \( \tau_{n-s} \), i.e. \( \tau_n \mid \Sigma^n \wedge S^{n-s-1} = S^s \tau_{n-s} \). But, by abuse of notation, we use the notation \( \tau_n \mid \Sigma^n \wedge S^{n-s-1} \) to denote the composite of the first two maps, i.e. we write
\[
\tau_n \mid \Sigma^n \wedge S^{n-s-1} = \tau_{n-s} \circ (p \wedge 1).
\]

Given \( f, g : \Sigma^n \wedge S^{n-1} \to \Sigma^n \), the sum \( f + g \) is given by the composition
\[
\Sigma^n \wedge S^{n-1} \xrightarrow{\Delta} (\Sigma^n \wedge S^{n-1}) \lor (\Sigma^n \wedge S^{n-1}) \xrightarrow{f \lor g} \Sigma^n \lor \Sigma^n \xrightarrow{\mu} \Sigma^n
\]
where \( \Delta \) is the inclusion induced by the diagonal map of \( S^{n-1} \) or \( \Sigma^n \) as necessary and \( \mu \) the folding map.

Let \( D^{n-1}_\pm = S^{n-1} \cap \{ x \mid \pm x_1 \geq 0 \} \). Then \( S^{n-1} = D^{n-1}_+ \cup D^{n-1}_- \) and \( S^{n-2} = D^{n-1}_+ \cap D^{n-1}_- \). Let \( S^{n-1}_\pm = D^{n-1}_\pm \cap S^{n-2} \). Then \( S^{n-1}_\pm \) becomes homeomorphic to \( S^{n-1}_+ \) or \( S^{n-1}_- \). Denote by \( \pi : \Sigma^n \wedge S^{n-1} \to (\Sigma^n \wedge S^{n-1}_+) \lor (\Sigma^n \wedge S^{n-1}_-) \) the composition of the quotient map \( \Sigma^n \wedge S^{n-1} \to \Sigma^n \wedge S^{n-1}/\Sigma^n \wedge S^{n-2} \) and the homeomorphism to \((\Sigma^n \wedge S^{n-1}_+) \lor (\Sigma^n \wedge S^{n-1}_-)\). Let \( \pi_\pm : S^{n-1} \to S^{n-1}_\pm \) denote the collapsing maps. We use also the same symbols \( \pi_\pm \) to denote the \( m \)-fold suspension \( S^n \pi_\pm \) (\( m \geq n \)).

The proof of the theorem proceeds in four steps. First, we consider the decomposition of \( S\tau_n \) into two homotopic maps.

**Lemma 1.** Suppose \( S\tau_n \mid \Sigma^{n+1} \wedge S^{n-2} \simeq c_\infty \) where \( S^{n-2} \) denotes the equator of \( S^{n-1} \) defined by \( x_1 = 0 \). Then there exist maps \( f_\pm : \Sigma^{n+1} \wedge S^{n-1}_\pm \to \Sigma^{n+1} \) such that
\[
S\tau_n \simeq f_+ \circ \pi_+ - f_- \circ \pi_-,
\]
so that \( f_- \circ \pi_- \simeq f_+ \circ (1 \wedge a_{1,n-1})_- \circ \pi_- \). Since clearly \( (1 \wedge a_{1,n-1})_- \circ \pi_- \simeq \pi_+ \circ (1 \wedge a_{1,n-1}) \) and \( \pi_+ \circ (1 \wedge a_{1,n-1}) \simeq -\pi_+ \), we have \( (1 \wedge a_{1,n-1})_- \circ \pi_- \simeq -\pi_+ \) and so \( f_+ \circ (1 \wedge a_{1,n-1})_- \circ \pi_- \simeq -f_+ \circ \pi_+ \). Substituting the relation obtained above in this formula we get \( f_- \circ \pi_- \simeq -f_+ \circ \pi_+ \), which completes the proof.
2. Decompositions of $\tau_{2n+1}$ and $\tau_{2n+2}$

From now on, let $\eta = n_k/2$ and assume that the assumption of the theorem is fulfilled, i.e., $2\pi^S_{2n-1} = 0$. Also we work modulo odd torsion since we consider the 2-primary homotopy decomposition of maps.

From the fact that the suspension homomorphism $E : \pi_{4n-1}(S^{2n}) \to \pi_{n}(S^{2n+1})$ of the EHP sequence is a surjection with kernel $\mathbb{Z}$, generated by $[\ell_{2n}, t_{2n}]$, we see that it induces an isomorphism $2\pi_{4n-2}(S^{2n}) \cong 2\pi^S_{2n-1}$ between their 2-primary parts. Hence from the assumption above we have

(*)

$$2\pi_{4n-1}(S^{2n}) = 0.$$ 

In the above, when we write $\mathbb{R}^n$ as $\mathbb{R}^n_\alpha$ by attaching a suffix $\alpha$, to denote its associated spaces and maps given above we use here the notations $\Sigma^n_\alpha$, $S^n_\alpha$, $(D^n_\alpha)\pm$, $(S^n_\alpha)\pm$, $(\tau_n)_\alpha$, $(\pi_\alpha)\pm$ with adding the suffix $\alpha$.

**Lemma 2.** Under the assumption of Theorem, there exist maps $g_\pm : \Sigma^{2n+q}\wedge S^{2n+q-1}_\pm \to \Sigma^{2n+q}$ for $q = 1, 2$ such that $\tau_{2n+q} \simeq g_+ \circ \pi_+ - g_- \circ \pi_-$, $g_+ \circ \pi_- \simeq -g_- \circ \pi_+$, so that $\tau_{2n+q} \simeq 2g_+ \circ \pi_+$.

**Proof of Case $q=1$.** We first consider the suspension $S\tau_{2n} : \Sigma^{2n+1} \wedge S^{2n-2} \to \Sigma^{2n+1}$ of $\tau_{2n}$. Then $S\tau_{2n} \mid \Sigma^{2n+1} \wedge S^{2n-2}$ represents a map from $\Sigma^{2n} \wedge S^{2n-1}$ to $\Sigma^{2n}$, so by (*) we have

$$S\tau_{2n} \mid \Sigma^{2n+1} \wedge S^{2n-2} \simeq c_\infty.$$ 

This shows that the null homotopy condition of Lemma 1 with $n$ replaced by $2n$ is satisfied and therefore we see that there exists a decomposition of $S\tau_{2n}$ such that

(2)

$$S\tau_{2n} \simeq f_+ \circ \pi_+ - f_- \circ \pi_-, \quad f_- \circ \pi_- \simeq -f_+ \circ \pi_+,$$

where $f_\pm : \Sigma^{2n+1} \wedge S^{2n-1}_\pm \to \Sigma^{2n+1}$.

Using polar coordinates for the first two variables $x_1, x_2$ we express $x \in \mathbb{R}^{2n+1}$ as

$$x_t = (r \cos t, r \sin t, x_3, \ldots, x_{2n+1}) \quad (0 \leq t < \pi, \ r \in \mathbb{R}).$$

For any fixed $t$, let $\mathbb{R}^{2n}_t \subseteq \mathbb{R}^{2n+1}$ denote the $2n$-dimensional subspace generated by the $x_t$. Regard $\mathbb{R}^{2n}$ as $\mathbb{R}^{2n}_t$ with $t = 0$ and put

$$M(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & I_{2n-1} \end{pmatrix}.$$ 

Then the map $x \to xM(t)^T$ gives a linear isomorphism $\mathbb{R}^{2n} \to \mathbb{R}^{2n}_t$ (the subscript $T$ denotes the transpose). This induces homeomorphisms $\bar{e}_t : \Sigma^{2n+1}_t \to \Sigma^{2n}_t$, $\bar{e}_t : S^{2n-1}_t \to S^{2n-1}_t$ and $e_{t\pm} : S^{2n-1}_\pm \to (S^{2n-1}_\pm)\pm$. Here $(S^{2n-1}_\pm)\pm = (D^{2n-1}_t)\pm / S^{2n-2}$ where $(D^{2n-1}_t)\pm = S^{2n-1}_t \cap \{x_1 \mid \pm r \geq 0\}$ and $S^{2n-2}$ is the unit sphere in $\mathbb{R}^{2n-1}$ in $\mathbb{R}^{2n}_t$ defined by $r = 0$.

Then clearly $\pi_{t\pm} \circ (\bar{e}_t \wedge e_t) = (\bar{e}_t \wedge e_{t\pm}) \circ \pi_{t\pm}$. Let $(\tau_{2n})_t = \tau_{2n+1} \mid \Sigma^{2n} \wedge S^{2n-1}_t$ and $f_{t\pm} : S^{2n}_t \wedge (S^{2n-1}_1)\pm \to S^{2n}_t$ be the maps defined by the formula $f_{t\pm} \circ (\bar{e}_t \wedge e_{t\pm}) = \bar{e}_t \circ f_{t\pm}$ where $f_{t\pm}$ are as in (2). Then we have

$$S(\tau_{2n})_t \circ (\bar{e}_t \wedge e_t) = \bar{e}_t \circ S\tau_{2n}, \quad (f_{t\pm} \circ \pi_{t\pm}) \circ (\bar{e}_t \wedge e_{t\pm}) = \bar{e}_t \circ (f_{t\pm} \circ \pi_{t\pm}),$$
respectively. Substituting $S\tau_{2n}$ and $f_{\pm} \circ \pi_{\pm}$ derived from these equalities by composing $\tilde{e}_t$ in (2) we get

$$(3) \quad S(\tau_{2n})_t \simeq f_{t+} \circ \pi_{t+} - f_{t-} \circ \pi_{t-}, \quad f_{t-} \circ \pi_{t-} \simeq -f_{t+} \circ \pi_{t+} \quad (0 \leq t < \pi).$$

Similarly, through $S(\tau_{2n})_t \circ (\tilde{e}_t \land e_t) = \tilde{e}_t \circ S\tau_{2n}$, the formula (1) can be rewritten as

$$S(\tau_{2n})_t \mid S\Sigma_2^n \land S^{2n-2} \simeq c_\infty.$$ 

Also we inherit $S(\tau_{2n})_t \circ (1 \land a_{1,2n-1}) \simeq S(\tau_{2n})_t$ from $S\tau_{2n} \circ (1 \land a_{1,2n-1}) \simeq S\tau_{2n}$. Considering (3) together with these two formulas we can determine the behavior of maps $f_{t\pm}$ when, for any fixed $t$, rotating along the region of $t$ by $\pi$ degrees.

According to the definition of the sum of maps we have a homotopy

$$f_{t+} \circ \pi_{t+} \simeq -f_{t-} \circ \pi_{t+}$$

On the other hand, according to the definition of $f_{t-}$, we have

$$f_{t+} \circ \pi_{t+} \simeq -f_{t-} \circ \pi_{t-},$$

so that it follows that

$$f_{t+} \circ \pi_{t+} \simeq f_{t-} \circ \pi_{t-}.$$ 

By applying this to the former formula of (3) we get $S(\tau_{2n})_t \mid S\Sigma_2^n \land S^{2n-1} \simeq c_\infty$. So identifying $S\Sigma_2^n$ with $\Sigma^{2n+1}$ in the usual way we obtain

$$(4) \quad S_{\tau_{2n+1}} \mid \Sigma^{2n+1} \land S^{2n-1} \simeq c_\infty \quad (0 \leq t < \pi).$$

Take $t = \pi/2$. Then $S^{2n}/S^{2n-1} \simeq S^{2n}_+ \lor S^{2n}_-$ where $S^{2n}_+ = S^{2n} \cap \{x_t \mid \pm x_1 \geq 0\}$. Thereby the null homotopy of (4), as in the case of $S\tau_{2n}$, yields a factorization of $S_{\tau_{2n+1}}$ into the composition

$$\Sigma^{2n+1} \land S^{2n} \xrightarrow{\gamma} (\Sigma^{2n+1} \land S^{2n}_+) \lor (\Sigma^{2n+1} \land S^{2n}_-) \xrightarrow{g_+ \lor g_- \circ \pi} \Sigma^{2n+1} \lor \Sigma^{2n-1} \xrightarrow{\mu} \Sigma^{2n+1},$$

so that a decomposition $\tau_{2n+1} \simeq g_+ \circ \pi_+ - g_- \circ \pi_-$ where $g_{\pm} : \Sigma^{2n+1} \lor \Sigma^{2n}_- \rightarrow \Sigma^{2n+1}$. But in fact, $g_{\pm}$ can be obtained by unifying $f_{t\pm}$ according to the former formula of (3) under the null homotopy of (4), respectively, and therefore we have $g_- \circ \pi_- \simeq -g_+ \circ \pi_+$ from the latter of (3).

**Proof of Case q=2.** In order to use the results in the proof of the case $q = 1$, we adopt spherical polar coordinate representation for the first three variables $x_1, x_2, x_3$ and express $x \in \mathbb{R}^{2n+2}$ as

$$x_{\theta t} = (r \cos \theta, r \sin \theta \cos t, r \sin \theta \sin t, x_4, \cdots, x_{2n+2}) \quad (0 \leq \theta \leq \pi, \ 0 \leq t < \pi, \ r \in \mathbb{R}).$$

For a fixed $0 < \theta < \pi$, let $\mathbb{R}^{2n+1}_\theta \subset \mathbb{R}^{2n+2}$ denote the $(2n+1)$-dimensional subspace generated by the $x_{\theta t}$. In addition, fix $t$, then these elements define the $2n$-dimensional subspace $\mathbb{R}^{2n+1}_{\theta t} \subset \mathbb{R}^{2n+1}_\theta$. For any fixed $t$, let $\mathbb{R}^{2n}_t \subset \mathbb{R}^{2n+2}$ denote the subspace generated by the $x_{\theta t}$ with $\theta = \pi/2$ and put

$$M(\theta t) = \begin{pmatrix}
\sin \theta & \cos \theta \cos t & \cos \theta \sin t & 0 \\
-\cos \theta \cos t & \sin \theta & 0 & 0 \\
-\cos \theta \sin t & 0 & \sin \theta & 0 \\
0 & 0 & 0 & I_{2n-1}
\end{pmatrix} \quad (0 < \theta < \pi).$$
Then the map \( x \to xM(\theta t)^T \) gives a linear isomorphism \( \mathbb{R}^{2n}_t \to \mathbb{R}^{2n}_t \). Denote by \( \tilde{e}_\theta t : S\Sigma^{2n}_t \to X\Sigma^{2n}_t, e_\theta t : S^{2n}_\theta t \to S^{2n}_\theta t^{-1} \) and \( (\tilde{e}_\theta t)_\pm : (S^{2n}_\theta t^{-1})_\pm \to (S^{2n}_\theta t^{-1})_\pm \) the homeomorphisms induced by this isomorphism. Here \( (S^{2n}_\theta t^{-1})_\pm = (D^{2n}_\theta t^{-1})_\pm / S^{2n}_\theta t^{-2} \) where \( (D^{2n}_\theta t^{-1})_\pm = S^{2n}_\theta t^{-1} \cap \{x_\theta t \mid x \geq 0 \} \) and \( S^{2n}_\theta t^{-2} = S^{2n}_\theta t^-2 \cap \{x_\theta t \mid r = 0 \} \). Then putting \( (\tau_2n)_\theta t = \tau_{2n+2} \mid S^{2n}_\theta t \cap S^{2n}_\theta t^{-1} \) we have

\[
S(\tau_2n)_\theta t \circ (e_\theta t \wedge e_\theta t) = \tilde{e}_\theta t \circ S(\tau_2n)_t.
\]

We also have

\[
((f_\theta t)_\pm \circ (\pi_\theta t)_\pm) \circ (\tilde{b}_\theta t \wedge b_\theta t) = \tilde{b}_\theta t \circ (f_\theta t \pm \circ \pi_\theta t \pm)
\]

where \( (f_\theta t)_\pm : S\Sigma^{2n} \wedge (S^{2n}_\theta t^{-1})_\pm \to S\Sigma^{2n} \theta t \) denote the maps defined by the formula \( (f_\theta t)_\pm \circ (e_\theta t \wedge (e_\theta t)_\pm) = \tilde{e}_\theta t \circ f_\theta t \pm \). Substituting \( S(\tau_2n)_t \) and \( f_\theta t \pm \circ \pi_\theta t \pm \) derived from these equalities by composing \( \tilde{b}_\theta t \) in \( (3) \) we obtain

\[
(5) \quad S(\tau_2n)_\theta t \simeq (f_\theta t)_+ \circ (\pi_\theta t)_+ - (f_\theta t)_- \circ (\pi_\theta t)_-, \quad (f_\theta t)_+ \circ (\pi_\theta t)_+ \simeq -(f_\theta t)_- \circ (\pi_\theta t)_-
\]

where \( 0 < \theta < \pi \).

Similarly, inheriting \( (1) \) through \( S(\tau_2n)_\theta t \circ (e_\theta t \wedge e_\theta t) = \tilde{e}_\theta t \circ S(\tau_2n)_t \) we have

\[
S(\tau_2n)_\theta t \mid S^{2n}_\theta t + 1 \wedge S^{2n}_\theta t \simeq c_\infty \quad (0 < \theta < \pi).
\]

We look at \( (1) \) from another angle. Replacing formally the domain \( \Sigma^{2n+1} \wedge S^{2n-1}_\theta \) of the null homotopy of \( S(\tau_2n)_t \) given there by \( \Sigma^{2n+1} \wedge S^{2n-1} \) where \( S^{2n-1} = S^{2n+1} \cap \{x_\theta t \mid \theta = 0 \} \), we have a null homotopy of \( S(\tau_2n)_\theta t \) over \( \Sigma^{2n+1} \wedge S^{2n-1} \). Combining this with the null homotopy obtained above we can form a homotopy

\[
(6) \quad \tau_{2n+2} \mid \Sigma^{2n+2} \wedge S^{2n} \simeq c_\infty
\]

where \( S^{2n} = S^{2n+1} \cap \{x_\theta t \mid t = \pi / 2 \} \). In the case here, differently from the case \( q = 1 \), imposing the coordinate condition \( \pm x_1 \geq 0 \) on \( x_2 \), we let \( D^{2n+1}_\pm = S^{2n+1} \cap \{x_\theta t \mid \pm x_2 = 0 \} \) and put \( S^{2n+1}_\pm = D^{2n+1}_\pm / S^{2n} \). Then according to the former formula of \( (5) \) under the null homotopy of \( (4) \), in the same way as in the case of \( \tau_{2n+1} \) we obtain a decomposition \( \tau_{2n+2} \simeq g_+ \circ \pi_+ \circ g_- \circ \pi_- \) of \( \tau_{2n+2} \) where \( g_+ : \Sigma^{2n+2} \wedge S^{2n+1}_\theta \to \Sigma^{2n+2} \). Then \( g_+ \circ \pi_+ \simeq -g_- \circ \pi_- \) follows from the latter formula of \( (5) \) as in the case of \( q = 1 \) above. This completes the proof of the lemma.

3. Proof of Theorem

The theorem follows immediately from Lemma 2, Case \( q = 1 \), and the following

**Lemma 3.** *Under the assumption of Theorem, there exist maps \( g_\pm : \Sigma^{4n+3} \wedge S^{4n+2} \pm \to \Sigma^{4n+3} \) such that \( \tau_{4n+3} \simeq g_+ \circ \pi_+ \circ g_- \circ \pi_- \), so that \( \tau_{4n+3} \simeq 2g_+ \circ \pi_+ \).*

*Proof.* Put \( m = 2n + 1 \) and let \( \phi = (\phi_1, \ldots, \phi_m) \) where \( 0 \leq \phi_i < \pi \) \((1 \leq i \leq m) \). We express \( x \in \mathbb{R}^{2m+1} \) as

\[
x_\phi = (x_1 \cos \phi_1, \cdots, x_m \cos \phi_m, x_1 \sin \phi_1, \cdots, x_m \sin \phi_m, x_{2m+1})
\]

where \( x_1, \cdots, x_m, x_{2m+1} \in \mathbb{R} \). For a fixed \( \phi \), let \( \mathbb{R}^{m+1}_\phi \subset \mathbb{R}^{2m+1} \) be the \((m+1)\)-dimensional subspace generated by the \( x_\phi \). Let

\[
c(\phi) = \text{diag} (\cos \phi_1, \cos \phi_2, \cdots, \cos \phi_m), \quad s(\phi) = \text{diag} (\sin \phi_1, \sin \phi_2, \cdots, \sin \phi_m)
\]
be the diagonal matrices whose \((i,i)\)-entries are \(\cos \phi_i\) and \(\sin \phi_i\), respectively, and set

\[
M(\phi) = \begin{pmatrix}
c(\phi) & -s(\phi) & 0 \\
s(\phi) & c(\phi) & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

We regard \(\mathbb{R}^{m+1}_\phi\) with \(\phi = (0, \cdots, 0)\) as \(\mathbb{R}^{m+1}\), so \(x_\phi = (x_1, \cdots, x_m, 0, \cdots, 0, x_{2m+1}) \in \mathbb{R}^{m+1}\) can be taken to be equal to \(x = (x_1, \cdots, x_m, x_{2m+1}) \in \mathbb{R}^{m+1}\). Here the first three variables \(x_1, x_2, x_3\) are supposed to have the same expression in polar form as those of \(x_{\theta 1}\) above. Then, for any fixed \(\phi\), the map \(x \to xM(\phi)^T\) defines a linear isomorphism \(\mathbb{R}^{m+1} \to \mathbb{R}^{m+1}\). Let \(\bar{e}_\phi : \Sigma^m+1 \to \Sigma^m+1\) and \(e_\phi : S^m \to S^m\) denote the homeomorphisms induced by this isomorphism. If we put \((\tau_{m+1})_\phi = \tau_{2m+1} | \Sigma^m+1 \land S^m\), then we have

\[(\tau_{m+1})_\phi \circ (\bar{e}_\phi \land e_\phi) = \bar{e}_\phi \circ \tau_{m+1}.
\]

Similarly as in the case \(q = 2\) above, taking \((D^m)_{\pm} = S^m_{\phi} \cap \{x_\phi \mid \pm x_2 \geq 0\}\) and \(S^{m-1}_\phi = S^m_{\phi} \cap \{x_\phi \mid x_2 = 0\}\) we put \((S^m_{\phi})_{\pm} = (D^m)_{\pm} / S^{m-1}_\phi\). Let \(e_{\phi \pm} : S^m_{\pm} \to (S^m_{\phi})_{\pm}\) be the homeomorphisms induced by \(e_\phi\) and we define the maps \(f_{\phi \pm} : \Sigma^m+1 \land (S^m_{\phi})_{\pm} \to \Sigma^m+1\) by the formula \(f_{\phi \pm} \circ (\bar{e}_\phi \land e_{\phi \pm}) = \bar{e}_\phi \circ g_{\pm}\) where \(g_{\pm} : \Sigma^m+1 \land S^m_{\pm} \to \Sigma^m+1\) are as the maps occurring in the decomposition of \(\tau_{m+1}\) above. Then clearly

\[(f_{\phi \pm} \circ \pi_{\phi \pm}) \circ (\bar{e}_\phi \land e_{\phi \pm}) = \bar{e}_\phi \circ (g_{\pm} \circ \pi_{\pm}).
\]

Substituting \(\tau_{m+1}\) and \(g_{\pm} \circ \pi_{\pm}\) derived from the above two equalities by composing \(\bar{e}_\phi^{-1}\) in the decomposition formula of \(\tau_{m+1}\) obtained in Lemma ??, Case \(q = 2\), we have

\[(\tau_{m+1})_\phi \simeq f_{\phi +} \circ \pi_{\phi +} - f_{\phi -} \circ \pi_{\phi -}; \quad f_{\phi +} \circ \pi_{\phi +} \simeq -f_{\phi -} \circ \pi_{\phi -}.
\]

In addition, in a similar way we have the null homotopy

\[(\tau_{m+1})_\phi | \Sigma^{2m+1} \land S^m_{\phi-1} \simeq c_\infty
\]

which inherits the null homotopy of \(\tau_{m+1} | \Sigma^m+1 \land S^m\) used for proving the case \(q = 2\) above through the equality relating \((\tau_{m+1})_\phi\) and \(\tau_{m+1}\) given above. By using a similar argument to the case \(q = 1\), i.e. to the proof of \(\tau_m | \Sigma^m \land S^{m-2} \simeq c_\infty\), together with (7) and (8) we can determine the behavior of \(f_{\phi \pm}\) in rotating along the regions of \(\phi_i\) by \(\pi\) degrees.

Consider the composite \(\alpha = a_{m,1} \circ a_{1,m} \circ a_{2,m-1} : S^m \to S^m\) which maps \(x_i\) to itself if \(i = 2, m+1\) and to \(-x_i\) otherwise where \(a_{s,m-s}\) are as above and \(x = (x_1, x_2, \cdots, x_m, x_{2m+1}) \in S^m\). If we write \(\alpha_\phi\) for \(e_\phi \circ \alpha \circ e_\phi^{-1}\), then we have

\[(\tau_{m+1})_\phi \circ (1 \land \alpha_\phi) \simeq (\tau_{m+1})_\phi
\]

where 1 denotes the identity on \(\Sigma^m_{\phi+1}\). Since a number of \(x_i\) whose sign is converted by \(\alpha\) in reverse is \(m - 1 = 2n\) and so is even, this implies that the null homotopy (8) of \(\tau_{2m+1}\) over \(\Sigma^{2m+1} \land S^{m-1}\) can be extended over the subspace of \(\Sigma^{2m+1} \land S^{2m}\) consisting of the elements of \(\Sigma^{2m+1} \land S^{m-1}\) when \(\phi_1, \phi_3, \cdots, \phi_m\) vary over the full range and only the second variable \(\phi_2\) remains fixed.

Under this null homotopy, rotating \(\phi_2\) by \(\pi\) we have

\[(\tau_{m+1})(\phi_1, \phi_2 + \pi, \phi_3, \cdots, \phi_m) \simeq - (\tau_{m+1})_\phi.
\]
If we take $\phi_2 = 0$ and set $S^{2m-1} = S^{2m} \cap \{x_\phi \mid \phi_2 = 0\}$, then considering this null homotopy together with the extension of (8) obtained above we get a homotopy

$$\tau_{2m+1} \mid \Sigma^{2m+1} \land S^{2m-1} \simeq c_\infty.$$  

Let $S^{2m}_\pm = D^{2m}_\pm / S^{2m-1}$ where $D^{2m}_\pm = S^{2m} \cap \{x_\phi \mid \pm x_2 \geq 0\}$. Then, as in the cases $q = 1, 2$ above, considering (7) under the null homotopy now obtained we see that $f_{\pi \pm}$ ($0 \leq \phi < \pi$) can be integrated into the maps $g_{\pm} : \Sigma^{2m+1} \land S^{2m}_\pm \to \Sigma^{2m+1}$, respectively, and therefore the desired decomposition formula can be satisfied. This completes the proof of the lemma.

References

[1] M.G. Barratt, J.D.S. Jones, M.E. Mahowald, The Kervaire invariant problem, Contemp. Math. AMS 19 (1983) 9–23.
[2] F.R. Cohen, A course in some aspects of classical homotopy theory, Springer, Lecture Notes in Mathematics 1286, 1987.
[3] M. Gilmore, Some Whitehead Products on Odd Spheres, Proc. Amer. Math. Soc. 20 (1969) 375–377.
[4] M. A. Hill, M. J. Hopkins, D. C. Ravenel, On the nonexistence of elements of Kervaire invariant one, Ann. of Math. 184 (2016) 1–262.
[5] K. Morisugi, J. Mukai, Lifting to mod 2 Moore spaces, J. Math. Soc. Japan 52 (2000) 515–533.
[6] N. E. Steenrod, The Topology of Fibre Bundles, Princeton University Press, 1951.
[7] H. Toda, Composition methods in homotopy groups of spheres, Ann. of Math. Studies 49, 1962.
[8] G. Wang, Z. Xu, The triviality of the 61-stem in the stable homotopy groups of spheres, Ann. of Math. 186 (2017) 501–580.

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