Extended Phase Space Thermodynamics for Black Holes in a Cavity

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Abstract

In this paper, we extend the phase space of black holes enclosed by a spherical cavity of radius $r_B$ to include $V \equiv 4\pi r_B^3/3$ as a thermodynamic volume. The thermodynamic behavior of Schwarzschild and Reissner-Nordstrom (RN) black holes is then investigated in the extended phase space. In a canonical ensemble at constant pressure, we find that the aforementioned thermodynamic behavior is remarkably similar to that of the anti-de Sitter (AdS) counterparts with the cosmological constant being interpreted as a pressure. Specifically, a first-order Hawking-Page-like phase transition occurs for a Schwarzschild black hole in a cavity. The phase structure of a RN black hole in a cavity shows a strong resemblance to that of the van der Waals fluid. Our results may provide a new perspective for the extended thermodynamics of AdS black holes by analogy with black holes in a cavity.

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I. INTRODUCTION

The area theorem of black holes [1] asserts that the total horizon area of black holes is a non-decreasing function of time in reasonable physical processes, which hints that black holes are endowed with thermodynamic properties. Since the area law bears a close resemblance to the second law of thermodynamics, Bekenstein postulated that black hole entropy is proportional to the horizon area [2, 3]. The analogy between usual thermodynamics and black hole thermodynamics was further confirmed by the discovery of Hawking radiation, assigning black holes a temperature [4, 5]. Analogous to the laws of thermodynamics, the four laws of black hole mechanics were established in [6].

Partly due to AdS/CFT correspondence [7], nearly fifty years after the discovery of black hole thermodynamics, understanding thermodynamic properties of various black holes, especially phase transitions of AdS black holes, is still a rather hot topic in the literature. Since AdS boundary plays a role of a reflecting wall, AdS black holes can be thermally stable in some cases, which makes it possible to study their phase behavior. In the seminal work [8], the Hawking-Page phase transition (i.e., a phase transition between the thermal AdS space and a black hole) was revealed in Schwarzschild-AdS black holes. Subsequently, there has been much interest in studying thermodynamics and phase structure of AdS black holes [9–16]. Interestingly, RN-AdS black holes were found to possess a van der Waals-like phase transition (i.e., a phase transition consisting of a first-order phase transition terminating
at a second-order critical point) in a canonical ensemble [11, 12] and a Hawking-Page-like phase transition in a grand canonical ensemble [17].

Later, the cosmological constant in AdS black holes is identified as a thermodynamic pressure [18, 19]. In this framework, the first law becomes consistent with the corresponding Smarr relation, and black hole mass is interpreted as a chemical enthalpy [20]. The phase behavior and $P$-$V$ criticality have been explored for various AdS black holes in the extended phase space [21–34], which discovered a broad range of new phenomena. For a recent review, see [35]. Specifically for a Schwarzschild-AdS black hole, the coexistence line of the Hawking-Page phase transition in the $P$-$T$ diagram is semi-infinite and reminiscent of the solid/liquid phase transition [36]. In the extended phase space, the analogy between a RN-AdS black hole and the van der Waals fluid becomes more complete, in that the coexistence lines in the $P$-$T$ diagram are both finite and terminate at critical points, and the $P$-$V$ criticality matches with one another [19].

In parallel with research on AdS black holes, studies of thermodynamics of black holes in a cavity have also attracted a lot attentions since York realized that Schwarzschild black holes can be thermally stable by placing them inside a spherical cavity, on the wall of which the metric is fixed [37]. Phase structure and transitions of Schwarzschild black holes in a cavity and Schwarzschild-AdS black holes were shown to be strikingly similar [37]. Afterwards, it was found that the phase behavior of RN black holes in a cavity and RN-AdS black holes also has extensive similarities in a grand canonical ensemble [38] and a canonical ensemble [39, 40]. Similar analysis has been extended to a broad class of brane backgrounds, including $Dp$-brane and NS5-brane configurations, and most of the brane systems were observed to undergo Hawking-Page-like or van der Waals-like phase transitions [41–46]. In addition, properties of boson stars and hairy black holes in a cavity were investigated [47–54], which were shown to closely resemble those of holographic superconductors in the AdS gravity. Lately, it was discovered that Gauss-Bonnet black holes in a cavity have quite similar phase structure and transitions to the AdS counterparts [55]. These observations lend support to the analogy between black holes in a cavity and AdS black holes.

However, we recently studied Born-Infeld black holes enclosed in a cavity in a canonical ensemble [56] and a grand canonical ensemble [57] and revealed that their phase structure has dissimilarities from that of Born-Infeld-AdS black holes. Moreover, contrary to the phase behavior, we found that there exist significant differences between the thermodynamic
geometry of RN black holes in a cavity and that of RN-AdS black holes [58], and some dissimilarities between the two cases also occur for validities of the second thermodynamic law and the weak cosmic censorship [59]. These findings motivate us to further explore connections between thermodynamic properties of black holes and their boundary conditions. Additionally, it has been proposed [60] that the holographic dual of $T\bar{T}$ deformed CFT$_2$ is a finite region of AdS$_3$ with the wall at finite radial distance, which makes studying properties of black holes in a cavity more attractive. Note that thermodynamics and critical behavior of de Sitter black holes in a cavity were recently investigated in [61–63].

In the existing research on thermodynamic properties of asymptotically flat black holes surrounded by a cavity, thermodynamic quantities that can be interpreted as a pressure or volume are absent. To make analogies with the AdS counterparts and corresponding real-world systems more complete, it is highly desirable to introduce a thermodynamic pressure or volume for black holes in a cavity. To this end, we extend the phase space to include the cavity radius as a new thermodynamic variable, from which the thermodynamic pressure and volume of black holes in a cavity can be established. Particularly in this paper, we confine Schwarzschild and RN black holes in a cavity and study their thermodynamic behavior in the aforementioned extended phase space. In hindsight, the extended phase behavior of Schwarzschild and RN black holes in a cavity is discovered to bear a striking resemblance to that of the AdS counterparts.

The rest of this paper is organized as follows. Section II discusses phase structure of Schwarzschild black holes in a cavity in the extended phase space. In section III, we investigate the extended phase properties of RN black holes in a cavity in a canonical ensemble that maintains constant temperature, pressure and charge. We summarize our results with a brief discussion in section IV. For simplicity, we set $G = \hbar = c = k_B = 1$ in this paper.

II. SCHWARZSCHILD BLACK HOLE IN A CAVITY

In this section, we consider a thermodynamic system with a Schwarzschild black hole enclosed in a cavity. The 4-dimensional Schwarzschild black hole solution is described by

$$ds^2 = -f(r)\, dt^2 + \frac{dr^2}{f(r)} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right), \quad f(r) = 1 - \frac{r_+}{r},$$

(1)
where \( r_+ \) is the radius of event horizon. The Hawking temperature \( T_b \) of the Schwarzschild black hole is
\[
T_b = \frac{1}{4\pi r_+}.
\]  
(2)

Suppose the wall of the cavity enclosing the black hole is at \( r = r_B \geq r_+ \) and maintained at a temperature of \( T \). In [38], it showed that \( T \) can be related to \( T_b \) as
\[
T = \frac{T_b}{\sqrt{f(r_B)}}.
\]  
(3)

The thermal energy \( E \) and the Helmholtz free energy \( F \) were derived in [39]:
\[
E = r_B \left[ 1 - \sqrt{f(r_B)} \right],
\]
\[
F = r_B \left[ 1 - \sqrt{f(r_B)} \right] - T\pi r_+.
\]  
(4)

We now introduce a new thermodynamic variable, the thermodynamic volume of the system \( V \),
\[
V \equiv \frac{4}{3} \pi r_B^3,
\]  
(5)

which naturally gives the conjugate thermodynamic pressure,
\[
P = -\frac{\partial E}{\partial V} = -\frac{1}{4\pi r_B^2} \left( 1 - \sqrt{1-x} - \frac{x}{2\sqrt{1-x}} \right), \quad 0 \leq x \equiv \frac{r_+}{r_B} \leq 1.
\]  
(6)

Extending the phase space of black hole thermodynamics to include the pressure/volume conjugate pair leads to the first law of thermodynamics,
\[
dE = TdS - PdV,
\]  
(7)

where \( S = \pi r_+^2 \) is the entropy of black hole, and we use \( T = \partial E/\partial S \). If the system undergoes an isobaric process, the Gibbs free energy \( G \equiv F + PV \) is employed in the study of phases and phase transitions.

By dimensional analysis, we find that the thermodynamic quantities scale as powers of the pressure \( P \),
\[
T = \bar{T}\sqrt{\bar{P}}, \quad G = \bar{G}/\sqrt{\bar{P}}, \quad r_B = \bar{r}_B/\sqrt{\bar{P}},
\]  
(8)

where the tildes denote dimensionless quantities. Solving eqn. (3) for \( x \) in terms of \( \bar{T} \) gives \( x = x(\bar{T}) \), which is plotted in the left panel of FIG. 1. It shows that for \( T \geq T_{\text{min}} \), \( x = x(\bar{T}) \) is multivalued and consists of two branches, namely Small BH and Large BH.
FIG. 1: Plots of $r_+/r_B$ and $r_B\sqrt{P}$ against $T/\sqrt{P}$ for a Schwarzschild black hole in a cavity. The red and blue lines represent Small BH and Large BH, respectively. Black hole solutions do not exist when $T < T_{\text{min}}$.

FIG. 2: Plots of $C_P P$ and $G\sqrt{P}$ against $T/\sqrt{P}$ for a Schwarzschild black hole in a cavity. **Left Panel:** The heat capacity at constant pressure $C_P$ of Small/Large BH is negative/positive, which means that Small/Large BH is thermally unstable/stable in an isobaric process. **Right Panel:** As $T/\sqrt{P}$ increases from zero, a first-order phase transition from the thermal spacetime to Large BH occurs at the black dot. Above $T_p/\sqrt{P}$, Large BH is the thermodynamically preferred state.

When $T < T_{\text{min}}$, no black hole exists. Plugging $x(\tilde{T})$ into eqn. (6), one can express $\tilde{r}_B$ in terms of $\tilde{T}$: $\tilde{r}_B = \tilde{r}_B(\tilde{T})$. In the right panel of FIG. 1, we show that $\tilde{r}_B(\tilde{T})$ is also composed of the Small BH and Large BH branches. Note that the Small (Large) BH branch possesses smaller (larger) $r_+$ and $r_B$.

To study the thermodynamic stability of the two branches against thermal fluctuations
in an isobaric process, we consider the heat capacity at constant pressure,

\[ C_P = \frac{\tilde{C}_P}{P} = T \left( \frac{\partial S}{\partial T} \right)_P. \]  

Using \( x(\tilde{T}) \), we can rewrite \( \tilde{C}_P \) as

\[ \tilde{C}_P = \tilde{C}_P(\tilde{T}), \]  

which is presented in the left panel of FIG. 2. It shows that Small (Large) BH is thermally unstable (stable). To study phase transitions, we also need to consider the thermal flat spacetime as a phase of the system since it is a classical solution in the canonical ensemble. The Gibbs free energy \( \tilde{G}(\tilde{T}) \) of the two branches and the thermal flat spacetime is displayed in the right panel of FIG. 2. One finds that the thermal spacetime is the only phase when \( \tilde{T} < \tilde{T}_{\text{min}} \), and the black hole appears when \( \tilde{T} \geq \tilde{T}_{\text{min}} \). Similar to a Schwarzschild-AdS black hole, a first-order Hawking-Page-like phase transition between the thermal spacetime and Large BH occurs at \( \tilde{T} = \tilde{T}_p \), where these two phases are of equal Gibbs free energy. The thermal spacetime is globally stable for \( \tilde{T} < \tilde{T}_p \) while Large BH is globally preferred for \( \tilde{T} \geq \tilde{T}_p \). The coexistence line of thermal spacetime/Large BH phases is determined by \( \tilde{G} = 0 \), which reads

\[ T|_{\text{coexistence}} \simeq 1.257\sqrt{P}. \]  

The coexistence line is of infinite length and hence reminiscent of the solid/liquid phase transition. It is noteworthy that the coexistence line of thermal AdS spacetime/Large Schwarzschild-AdS BH was given by [35]

\[ T|_{\text{coexistence}} \simeq 0.921\sqrt{P}. \]  

Finally, we discuss the equation of state for the Schwarzschild black hole in a cavity. By rewriting the pressure equation (6) whilst using the temperature equation (3), we can obtain the equation of state in terms of \( VT^3 \) and \( PT^{-2} \), which is depicted as solid lines in FIG. 3. With fixed \( PT^{-2} \), the equation of state has two branches, corresponding to Small BH and Large BH. In the limit of large \( VT^3 \), the equation of state becomes

\[ VT^3 \simeq \frac{1}{48\pi^2} \left( \frac{\pi}{2PT^{-2}} \right)^{3/4} \]  

for Small BH, and

\[ VT^3 \simeq \frac{1}{48\pi^2} \left( \frac{2\pi}{PT^{-2}} \right)^3 \]  

for Large BH. (13)
Defining a specific volume \( v \equiv (6V/\pi)^{1/3} \), the Large BH equation of state in eqn. (13) gives the ideal gas law, \( P v \simeq T \). We also plot the equation of state for a Schwarzschild-AdS black hole, which is represented by dashed lines in FIG. 3. Similarly, there exist two branches (i.e., Small BH and Large BH) for a fixed value of \( PT^{-2} \). However, only Large BH exists when \( VT^3 \) is large enough. It is worth noting that, in the limit of large \( VT^3 \), the equation of state for the Schwarzschild-AdS black hole reduces to

\[
VT^3 \simeq \frac{1}{48\pi^2} \left( \frac{2\pi}{PT^{-2}} \right)^3,
\]

which is the same as that for the Large BH branch of the Schwarzschild black hole in a cavity.

### III. RN BLACK HOLE IN A CAVALTY

In this section, we discuss phase structure and transitions of a RN black hole in a cavity in the extended phase space. The 4-dimensional RN black hole solution is

\[
ds^2 = -f(r) \, dt^2 + \frac{dr^2}{f(r)} + r^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right),
\]

\[
f(r) = \left( 1 - \frac{r_+}{r} \right) \left( 1 - \frac{Q_b^2}{r_+ r} \right), \quad A = A_t (r) \, dt = -\frac{Q_b}{r} \, dt,
\]
where $Q_b$ is the black hole charge, and $r_+$ is the radius of the outer event horizon. The Hawking temperature $T_b$ of the RN black hole is given by

$$T_b = \frac{1}{4\pi r_+} \left( 1 - \frac{Q_b^2}{r_+^2} \right). \quad (16)$$

In a canonical ensemble, the wall of the cavity, which is located at $r = r_B$, is maintained at a temperature of $T$ and a charge of $Q$. It was showed in [38] that the system temperature $T$ and charge $Q$ can be related to the black hole temperature $T_b$ and charge $Q_b$ as

$$Q = Q_b \text{ and } T = \frac{T_b}{\sqrt{f(r_B)}}, \quad (17)$$

respectively. For this system, the Helmholtz free energy $F$ and the thermal energy $E$ were also given in [39]

$$F = r_B \left[ 1 - \sqrt{f(r_B)} \right] - \pi T r_+^2, \quad (18)$$

$$E = r_B \left[ 1 - \sqrt{f(r_B)} \right].$$

The physical range of $r_+$ is constrained by

$$\frac{r_e}{r_B} \leq x \equiv \frac{r_+}{r_B} \leq 1, \quad (19)$$

where $r_e = Q$ is the horizon radius of the extremal black hole.

By introducing $V \equiv 4\pi r_B^3/3$ as a thermodynamic variable, the pressure of the system is

$$P = -\frac{\partial E}{\partial V} = \frac{2r_+r_B - Q^2 - r_+^2}{8\pi r_B^3 r_+ \sqrt{\left(1 - \frac{Q^2}{r_+ r_B}\right) \left(1 - \frac{r_+}{r_B}\right)}} - \frac{1}{4\pi r_B^2}. \quad (20)$$

In this extend phase space, the first law of thermodynamics becomes

$$dE = TdS - PdV + \Phi dQ, \quad (21)$$

where the system’s potential $\Phi$ is defined as

$$\Phi \equiv \frac{A_t(r_B) - A_t(r_+)}{\sqrt{f(r_B)}}. \quad (22)$$

A system under constant pressure in the canonical ensemble is best described by the Gibbs free energy, $G = F + PV$. Two phases in equilibrium have equal Gibbs free energy. Moreover, the heat capacity at constant pressure $C_P = T (\partial S/\partial T)_P$ is a thermodynamic quantity.
FIG. 4: Plots of $r_B/r_B$ and $r_B\sqrt{\mathcal{P}}$ against $T/\sqrt{\mathcal{P}}$ for a RN black hole in a cavity with $Q = 0.05/\sqrt{\mathcal{P}} < Q_c$, in which three phases can coexist. The green, red and blue lines represent Small BH, Intermediate BH and Large BH. Black hole solutions do not exist when $T < T_{\text{min}}$.

measuring the stability of a phase in an isobaric process. For later convenience, we can express the thermodynamic quantities in units of $\sqrt{\mathcal{P}}$

$$
\tilde{Q} \equiv Q\sqrt{\mathcal{P}}, \quad \tilde{r}_B \equiv r_B\sqrt{\mathcal{P}}, \quad \tilde{T} \equiv T/\sqrt{\mathcal{P}}, \quad \tilde{G} \equiv G\sqrt{\mathcal{P}}, \quad \tilde{C}_P \equiv C_P\mathcal{P}.
$$

(23)

Our strategy to discuss the system’s phase structure and transitions under constant pressure, temperature and charge will thus be the following: we start from eqns. (17) and (20) to express $\tilde{T}$ as a function of $\tilde{r}_B$ and $\tilde{Q}$, $\tilde{T} = \tilde{T}(\tilde{r}_B, \tilde{Q})$. Similarly, using eqn. (20), one can rewrite the Gibbs free energy and heat capacity as $\tilde{G} = \tilde{G}(\tilde{r}_B, \tilde{Q})$ and $\tilde{C}_P = \tilde{C}_P(\tilde{r}_B, \tilde{Q})$, respectively. After $\tilde{T} = \tilde{T}(\tilde{r}_B, \tilde{Q})$ is solved for $\tilde{r}_B$ in terms of $\tilde{T}$ and $\tilde{Q}$, we can express $\tilde{G}$ and $\tilde{C}_P$ with respect to $\tilde{T}$ and $\tilde{Q}$, $\tilde{G} = \tilde{G}(\tilde{T}, \tilde{Q})$ and $\tilde{C}_P = \tilde{C}_P(\tilde{T}, \tilde{Q})$. A critical point occurs at the inflection point of $\tilde{T}$ as a function of $\tilde{r}_B$, where

$$
\frac{\partial \tilde{T}}{\partial \tilde{r}_B} = 0 \quad \text{and} \quad \frac{\partial^2 \tilde{T}}{\partial \tilde{r}_B^2} = 0.
$$

(24)

Solving the above equations gives quantities evaluated at the critical point

$$(\tilde{r}_{\text{be}}, \tilde{Q}_c, \tilde{T}_c, x_c) \simeq (0.222, 0.107, 1.100, 0.911).$$

(25)

The critical ratio, $P_c v_c/T_c \simeq 0.405$, where $v \equiv 2r_B$ is the specific volume. Note that $P_c v_c/T_c = 3/8$ for the van der Waals fluid and a RN-AdS black hole.

With fixed $\tilde{T}$ and $\tilde{Q}$, the number of the phases, corresponding to the branches of $\tilde{r}_B(\tilde{T}, \tilde{Q})$, depends on the value of $\tilde{Q}$. When $\tilde{Q} < \tilde{Q}_c$, three phases, namely Small BH, Intermediate BH
and Large BH, coexist for some range of $T$. For $\tilde{Q} = 0.05 < \tilde{Q}_c$, we plot $x$ and $\tilde{r}_B$ against $\tilde{T}$ in FIG. 4, where different colored lines represent different phases. The first thing to note is that there exists a nonzero minimum $\tilde{T}_{\text{min}}$ for Small BH, which means that, unlike a RN-AdS black hole, a RN black hole in a cavity can not become extremal if the pressure is not zero. In other words, eqn. (20) gives that an extremal RN black hole with $r_+ = Q$ always has $P = 0$. At $\tilde{T} = \tilde{T}_{\text{min}}$, it shows that $x = 1$, which corresponds to the black hole with the horizon merging with the wall of the cavity. As $\tilde{T}$ increases, the horizon radius to cavity radius ratio increases toward 1 for Large BH while decreases for Small BH and Intermediate BH. The cavity enclosing Small BH and Large BH (Intermediate BH) expands (contracts) when $\tilde{T}$ increases. Although it is not shown in the paper, we find that the horizon radius has quite similar behavior to the cavity radius. In FIG. 5, we display $x$ and $\tilde{r}_B$ against $\tilde{T}$ with $\tilde{Q} = 0.15 > \tilde{Q}_c$, for which there exists only one phase. As one increases $\tilde{T}$ from $\tilde{T}_{\text{min}}$, the size of cavity keeps growing. Meanwhile, the horizon radius to cavity radius ratio first decreases from 1 and then increases towards 1.

With $\tilde{r}_B(\tilde{T}, \tilde{Q})$, we can obtain the heat capacity $\tilde{C}_P(\tilde{T}, \tilde{Q})$ and the Gibbs free energy $\tilde{G}(\tilde{T}, \tilde{Q})$ to discuss the stability of phases and phase transitions. FIG. 6 shows $\tilde{C}_P(\tilde{T}, \tilde{Q})$ and $\tilde{G}(\tilde{T}, \tilde{Q})$ in the left and right panels, respectively, for a RN black hole in a cavity with $\tilde{Q} = 0.05 < \tilde{Q}_c$. The heat capacity of Small BH and Large BH is positive, which means that Small BH and Large BH are thermally stable. On the other hand, Intermediate BH is a thermally unstable phase. Large BH, Small BH and Intermediate BH coexist when $\tilde{T}_1 < \tilde{T} < \tilde{T}_2$, and there is a first-order phase transition between Small BH and Large BH.
FIG. 6: Plots of $C_pP$ and $G\sqrt{P}$ against $T/\sqrt{P}$ for a RN black hole in a cavity with $Q = 0.05/\sqrt{P} < Q_c$.

**Left Panel:** Positive $C_p$ means that Large BH and Small BH are thermally stable while negative $C_p$ gives Intermediate BH is thermally unstable. **Right Panel:** As $T/\sqrt{P}$ increases from $T_{\text{min}}/\sqrt{P}$, a first-order phase transition from Small BH to Large BH occurs at $T = T_p$.

FIG. 7: Plots of $C_pP$ and $G\sqrt{P}$ against $T/\sqrt{P}$ for a RN black hole in a cavity with $Q = 0.15/\sqrt{P} > Q_c$.

**Left Panel:** The only phase has positive heat capacity and hence is thermally stable. **Right Panel:** There is no phase transition occurring at $\tilde{T} = \tilde{T}_p$ with $\tilde{T}_1 < \tilde{T}_p < \tilde{T}_2$. The globally stable phase is Large BH for $T > T_p$ and Small BH for $T < T_p$. For the system with $\tilde{Q} = 0.15 > \tilde{Q}_c$, there is only one phase, which is always thermally stable (see FIG. 7).

The left and right panels of FIG. 8 display the globally stable phase of a RN black hole in a cavity in the $PQ^2-TQ$ and $\tilde{Q}-\tilde{T}$ planes, respectively. Blue lines represent Large BH/Small BH first-order transition lines. These coexistence lines terminate at the critical point, where a second-order phase transition occurs. These phase diagrams show that the
FIG. 8: Phase diagrams of a RN-AdS black hole in a cavity in the $PQ^2$-$TQ$ (Left Panel) and $Q\sqrt{T}$-$T/\sqrt{P}$ (Right Panel) planes. The first-order phase transition lines separating Large BH and Small BH are displayed by blue lines and terminate at the critical point, marked by black dots. The coexistence lines are of finite length and reminiscent of the liquid/gas phase transition. No BH regions correspond to $T < T_{\text{min}}$, and hence no black hole solutions exist in light red regions.

Large BH/Small BH phase transition is analogous to the liquid/gas phase transition of the van der Waals fluid. Light red regions mark no BH regions, where $T < T_{\text{min}}$ and hence no black hole solutions exist. Except the existence of no BH regions, these phase diagrams in the cavity case are rather similar to those in the AdS case.

Finally, we show further analogies with the van der Waals fluid in FIG. 9, where the equations of state with $QT$ ranging from below (red) to above (blue) $Q_cT_c (= \tilde{Q}_c\tilde{T}_c \simeq 0.118$, black) for a RN black in a cavity are plotted in the $VT^3$-$PT^2$ plane. It is observed that these isotherms in the $VT^3$-$PT^2$ plane are strikingly similar to those of van der Waals fluid. In fact, for a RN black in a cavity with $T < \tilde{Q}_c\tilde{T}_c/Q$, the isotherm in the $VT^3$-$PT^2$ plane exhibits the oscillating part, in which a fixed $P$ corresponds to three $V$ solutions, namely Small BH, Intermediate BH and Large BH. Such oscillation is reminiscent of the liquid/gas phase transition. Note that the Maxwell’s equal area law can determine the Large BH/Small BH phase transition by defining a certain isobar to eliminate the oscillating part, such that the areas above and below the isobar are equal. When $T > \tilde{Q}_c\tilde{T}_c/Q$, the oscillating behavior of the isotherm disappears, which is reminiscent of the ideal gas law.
FIG. 9: The equations of state for a RN black hole in a cavity for different values of QT. The isotherms with fixed Q bear a striking resemblance to those of the Van der Waals fluid. The oscillating behavior below $Q_cT_c$ is reminiscent of the liquid/gas phase transition of the Van der Waals fluid.

IV. DISCUSSION AND CONCLUSION

In this paper, we extended the phase space of black holes enclosed in a cavity to include $4\pi r_B^3/3$ as a thermodynamic volume, where $r_B$ is the cavity radius. Consequently, the thermodynamic pressure $P$ is defined by $P = -\partial E/\partial V$, where $E$ is the thermal energy. Such extension is largely motivated by the extended phase space of AdS black holes, in which the cosmological constant is interpreted as a pressure. We showed that, in these extended phase spaces, the thermodynamic behavior of black holes in a cavity closely resembles that of the AdS counterparts, both exhibiting well-known phenomena found in real-world systems.

Specifically, we first investigated phase structure of a Schwarzschild black hole surrounded by a cavity in the extended phase space. We observed two branches of black holes, Large BH and Small BH, above some temperature $T_{\text{min}}$, below which only the thermal flat spacetime exists. The Large BH phase is thermally stable and has larger cavity and horizon radii. FIG. 2 revealed that there is a first-order Hawking-Page-like phase transition between the thermal spacetime and Large BH at $T_p > T_{\text{min}}$. Moreover, the coexistence line in the $P-T$ diagram is semi-infinite and hence reminiscent of a solid/liquid phase transition. All these observations indicate that the phase behavior of Schwarzschild black holes in a cavity and
Schwarzschild-AdS black holes is quite similar in the extended phase space. However, we noticed in FIG. 3 that the isotherms of Small BH in the $P$-$V$ diagram are significantly different in the two cases.

Considering a RN black hole enclosed by a cavity, we demonstrated that the first-order phase transition between Large BH and Small BH is similar to the van der Waals phase transition between liquid and gas, in that the Large BH/Small BH coexistence lines in the $T$-$P$ and $T$-$Q$ diagrams (see FIG. 8) are of finite length and end at a second-order critical point. FIG. 9 lent further evidence by showing that the isotherms of the RN black hole in a cavity with fixed $Q$ in the $P$-$V$ diagram behave as those of a van der Waals fluid system. Unlike RN-AdS black holes, RN black holes in a cavity are not admissible in some $(Q, P, T)$-parameter region (e.g., RN black holes in a cavity with nonzero $P$ can not become extremal). In spite of this difference, the thermodynamic properties of RN black holes in a cavity and RN-AdS black holes are very much alike.

In existing studies, thermodynamic properties of black holes in a cavity were investigated in the normal phase space, in which the cavity radius is fixed, and found to closely resemble those of the AdS counterparts for various black holes. In this paper, we extended the phase space to include the thermodynamic volume and showed that analogies with the AdS counterparts still hold for Schwarzschild and RN black holes in the extended phase space. On the other hand, AdS black holes have been observed to possess a lot richer phase structure and transitions, e.g., reentrant phase transitions and triple points, in the extended phase space. It would be very interesting to explore these phenomena in the context of black holes in a cavity and check whether analogies with the AdS counterparts can go beyond Schwarzschild and RN black holes.

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