A Shear Stress Reynolds’ Limit Formula

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Abstract

Historically, meteorological and climate studies have been prompted by the need for understanding precipitation to have better logistics in food production. Despite all efforts, nonlinearity in atmosphere dynamics is still a source of uncertainty. On the other hand, aeronautical science studies the boundary layer separation through the shear stress. In this work, a mathematical interpretation of methods in classical aerodynamics theory in terms of successive layers of diffeomorphisms over Lipschitz domains allows us to estimate the boundary layer’s shear stress, $\tau_d^*$ and $\tau_m^*$, in dry and humid atmospheric conditions without assuming that there is not a convective derivative term in the conservation of momentum equation or that the gaseous boundary layer is incompressible:

$$\tau_d^* = \frac{U}{h} \left(1 - \frac{U^2}{2c_{pd} T_0}\right)^{19/25}, \quad \tau_m^* = \frac{U}{h} \left(1 - \frac{U^2}{2c_{pm} T_0}\right)^{19/25},$$

where $h$ is the boundary layer’s height, $T_0$ is the surface temperature, $U$ is the free stream velocity; $c_{pd}$ is the specific heat at constant pressure for dry air and $c_{pm}$ is the specific heat at constant pressure for moist air. Furthermore, if $\hat{R}_m$ is a gas constant for moist air and $p_0$ is the pressure at the surface, the density

$$\rho \equiv p_0 T_0^{\frac{2b-1}{b}} \hat{R}_m^{-1} [1 - (U^2/2c_{ph} T_0)]^{1-b}$$

for $b = 1.405$. Moreover, this opens the possibility of finding a different deterministic family of atmosphere natural convection models.

Keywords: Gas dynamics, Boundary-layer theory, Reynolds’ Limit Formula

2010 MSC: 35Q30, 76N15, 76N20

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1. Introduction

This work suggests that a shear stress value obtained from limit formulas where the deterministic model is approximated but not oversimplified would give us a way to better evaluate the conditions under which there is an atmospheric boundary layer separation, and thus, convective clouds formation. Previous work by the author showed that a Reynolds’ Limit Formula could be deduced from Dorodnitzyn’s gaseous boundary layer model to overcome the inherited Navier-Stokes non-linearity in its convective derivative term [1] with the application of Bayada and Chambat’s diffeomorphism [2]. For more details on the statement of the small parameter problem in terms of an incompressible field, see [1].

Dorodnitzyn reduced the original system of seven equations for seven variables to a quasi-linear problem for a transformation of the shear stress in a new domain. This work offers a mathematical proof of Dorodnitzyn’s deduction in Theorem [2]. Surely, there exists a mathematical formalization preceding from the one given here, but the author could not find it in the literature. This might be a consequence of the fact that Dorodnitzyn’s work of the subsequent years is partially classified [3, p. 1973].

From the small parameter point of view, in 1886, Reynolds published: “On the Theory of Lubrication and Its Application to Mr. Beauchamp Tower’s Experiments, Including an Experimental Determination of the Viscosity of Olive Oil” [4], where he gave the formula to study fluid behavior when it moves in a narrow space between two plates. Reynolds’ Limit Formula was effectively used for a hundred years before Guy Bayada and Michèle Chambat [2] formally proved that this was indeed a limit formula for the Stokes’ Equations when the small parameter of a proportion of the boundary layer’s height and its length tend to zero, at 1986. In 2009, Chupin and Sart demonstrated that the compressible Reynolds equation is an approximation of compressible Navier-Stokes equations [5]. They mention that there seems to be only one noticeable work of this type of problem for a thin domain filled with gas [6] found in Eduard Marusic-Paloka and Maja Starcevic’s results [6] [7].

The proportion $L \gg h > 0$ allows the introduction of a small parameter $\epsilon = h/L$ and the application of Bayada’s change of variables [2] to obtain a Reynolds’ Limit Formula. Theorem [3] gives a demonstration of a
Reynolds’ Limit Formula for Dorodnitzyn’s shear stress quasi-linear problem. The corresponding Reynolds’ Limit Formula for the shear stress deduction was accepted for an Oral Presentation at the 10th European Nonlinear Dynamics Conference (ENOC 2020), accompanied by an extended abstract that will be published in its proceedings. The complete proofs and shear stress approximations for dry and humid atmospheric conditions are presented here, in Theorem 3, as a new additional result. In particular, it justifies the wide use of the free stream velocity as a good approximation of horizontal velocity near the Earth’s surface in meteorology.

2. Dorodnitzyn’s shear stress statement of the problem

The quasi-linear statement of the original problem in terms of the shear stress is obtained by a series of two essential steps. First, Theorem 1 shows that the original problem has a simplified expression as a system of just one condition for the stream function $\psi$ taken over the polygon $\Pi = s(R)$ in terms of Dorodnitzyn’s change of coordinates $s(x, y) = (\ell, s)$ of the original rectangular domain $R$, where the convective derivative has an incompressible form. Second, Theorem 2 gives a formal proof of how this system can be written in terms of a transformation that takes the original shear stress to a new domain, an infinite strip band $S: = \{(\ell, z) \in \mathbb{R}^2 \mid (\ell, s) \in \Pi, \text{ and } z = s/\ell^{1/2} \in (0, \infty)\}$, following a composition of the original stream function with Dorodnitzyn’s diffeomorphism $s$, and with Blasius’ adapted height normalization $z = s/\ell^{1/2}$.

![Diagram](https://via.placeholder.com/150)

From this point forward, $W^{k,p}(D)$ denotes the Sobolev Space of elements in the Lebesgue Space $L^p(D)$ on a domain $D \subset \mathbb{R}^2$ with generalized derivatives up to the order $k$, all of which belong to $L^p(D)$. We might recall that $\[9, 10, 11\]$. 

3
Definition 1. A domain is an open and connected subset $D \subset \mathbb{R}^2$ of the Euclidean space $\mathbb{R}^2$. A distribution $g \in L^1(D)$ is a generalized derivative of $f$ with respect to $x$ —also called weak or distributional, if for all analytic functions $\varphi$ with compact support in $D$, $\varphi \in C^\infty_0(D)$, we have:

$$\iint_D f \frac{\partial \varphi}{\partial x} \, dx \, dy = -\iint_D g \varphi \, dx \, dy.$$ 

Analogously, it can be defined for other coordinate systems and orders. A necessary and sufficient condition for the density of $C^\infty(\bar{D})$ in a Sobolev Space $W^{k,2}(D)$ is unknown [10, p. 10]. However, it is enough for the domain $D$ to be a rectangle. Therefore, the following results can be stated for a $\hat{f} \in C^\infty(\bar{D})$ approximation of each distribution $f \in W^{k,p}(D)$.

Remark 1. As a particular case, Leibnitz Rule for product differentiation is valid in a non-empty open domain $D \subset \mathbb{R}^2$ when both factors and all the generalized derivatives involved are elements of $L^2(\mathbb{R})$ [9, p. 11]. Moreover, there is a generalized Green’s Theorem [12, p. 121] that is valid for elements of the Sobolev Spaces $W^{1,2}(D)$ in a bounded Lipschitz domain $D \subset \mathbb{R}^2$. This allows the existence of a stream function, defined in Theorem 1.

Definition 2. Let $L \gg \gg h > 0$, $R = [0, L] \times [0, \infty)$, and $\hat{R} = R \times [0, h]$. If $\hat{\rho} \in L^1(\hat{R} \times [0, \infty); (0, \infty))$ such that $\partial \hat{\rho}/\partial t = 0$, $\rho = \hat{\rho}|_{\hat{R}}$, $\rho \in L^2(\hat{R}; (0, \infty))$, $u \in L^2(\hat{R})$ with generalized derivatives $\partial u/\partial x$, $\partial u/\partial y$, $\partial^2 u/\partial y^2 \in L^2(\hat{R})$; $v \in L^2(\hat{R})$ and $T \in L^2(\hat{R}; (0, \infty))$ such that $\partial T/\partial y$, $\partial^2 T/\partial y^2 \in L^2(\hat{R})$; $\mu \in L^2(\hat{R})$, $p \in L^2(\hat{R})$, and $\kappa \in L^2(\hat{R})$, to be the stationary density, the horizontal and vertical velocity components, the absolute temperature, the dynamic viscosity, the pressure, and the thermal conductivity, respectively. Moreover, assume that both products $\rho u$, $\rho v \in L^2(\hat{R})$, and that all of them have first order generalized derivatives in $L^2(\hat{R})$. This is, $\rho$, $u$, $v$, $T$, $\mu$, $p$ and $\kappa$ are elements of the space $W^{1,2}(\hat{R})$.

In 1942, Dorodnitzyn put forward a stationary gaseous boundary layer problem [13] —Eq. (1), (2), (3), (4), (5), (6), (7), and boundary conditions —Eq. (8), (9), (10), (11), (12), (13), (14), in a long rectangle $R = (0, L) \times (0, h) \in \mathbb{R}^2$ that represents the boundary layer region for $L \gg \gg h > 0$. Dorodnitzyn’s model is based on three simplified stationary Conservation of Mass, Conservation of Momentum, and Conservation of Energy laws, Eq.
(1), (2) and (3),

\[
\frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} = 0; \quad (1)
\]

\[
\rho \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y} \left( \mu \frac{\partial u}{\partial y} \right); \quad y \quad (2)
\]

\[
\rho \left[ u \frac{\partial (c_p T)}{\partial x} + v \frac{\partial (c_p T)}{\partial y} \right] = \frac{\partial}{\partial y} \left[ \kappa \frac{\partial T}{\partial y} \right] + \mu \left( \frac{\partial u}{\partial y} \right)^2 + \frac{\partial p}{\partial t}, \quad (3)
\]

for a stationary density \( \rho \), a horizontal and vertical velocity components, \( u \) and \( v \), an absolute temperature \( T \), a dynamic viscosity \( \mu \), a pressure \( p \), and a thermal conductivity \( \kappa \) whose main assumptions as elements of the Lebesgue space \( L^2(R) \) are described in the Definition[2]. Under these assumptions, the complete system is made up of seven identities in the Lebesgue space \( L^1(R) \).

The value \( c_p \) is the specific heat at constant pressure. It is worth to notice that there is a considerable difference between values of a gas constant for dry air \( \hat{R}_d = 287 \ [J/Kg^{-1}] \) and a gas constant for saturated water vapor \([14, \ p. 1047] \hat{R}_v = 461.50 \ [J/Kg^{-1}] \) and the specific heat at constant pressure for dry air \([15] \ c_{pd} = 1004 \ [J/Kg^{-1}] \) and the specific heat at constant pressure for water vapor \([14] \ c_{pv} = 1875 \ [J/Kg^{-1}] \). Therefore, one question that arises is if each model’s solution will continuously vary with modifications of these constants and what consequences does it have on the boundary layer separation.

Furthermore, we have four Ideal Gas Thermodynamic Laws, Eq. (4), (5), (6), (7): the Prandtl number \( Pr = 1 \),

\[
Pr = \frac{c_p \mu}{\kappa} = 1; \quad (4)
\]

the Equation of State for the Universal Gas constant \( R^* \), the volume \( V \) of a rectangular prism \([0, L] \times [0, h] \times [0, h] \subset R^3 \) and the number of moles \( n \) of an ideal gas corresponding to the volume \( V \),

\[
pV = n R^* T; \quad (5)
\]

the adiabatic polytropic atmosphere \([16, \ p. 35] \) where \( b = 1.405 \) and \( c \) are constants,

\[
pV^b = c; \quad (6)
\]
and the \textit{Power-Law} \cite[p. 46]{6}

\begin{equation}
\frac{\mu}{\mu_h} = \left( \frac{T}{T_h} \right)^{\frac{19}{25}},
\end{equation}

with boundary conditions, Eq. (8), (9), (10), (11), (13), (12):

\begin{align}
(u, v)\big|\{(x,h): 0 \leq x \leq L\} &= (-U, 0), \\
(u, v)\big|\{(x,0): 0 \leq x \leq L\} &= (0, 0),
\end{align}

for a positive value of the \textit{free-stream velocity}, \(U > 0\), the \textit{no slip condition} at the surface, a \textit{free-stream temperature} \(T_h > 0\), a \textit{free-stream dynamic viscosity} \(\mu_h > 0\),

\begin{align}
T\big|\{(x,h): 0 \leq x \leq L\} &= T_h > 0, \\
\mu\big|\{(x,h): 0 \leq x \leq L\} &= \mu_h > 0,
\end{align}

and a Neumann condition:

\begin{equation}
\frac{\partial T}{\partial y}\big|\{(x,0): 0 \leq x \leq L\} = 0.
\end{equation}

In \cite{6}, periodic conditions, such as the ones used in Chupin and Sart’s work \cite{5}, were included at the vertical sections of the \textit{topological boundary} \(\partial R\), such that for all \(y \in [0, h]\):

\begin{equation}
(u(0, y), 0) = (u(L, y), 0).
\end{equation}

It is worth to remark the fact that both these laws and their boundary conditions are satisfied in atmospheric conditions. For example, the no slip condition, Eq. (9) is verified for values of \(u\) under the speed of sound.

Dorodnitzyn took the first gaseous boundary layer model, stated by Busemann in 1935 \cite{18} and his idea to express absolute temperature \(T\) in terms of the horizontal component \(u\) of velocity, but included a term \(\partial p/\partial x\), not present in Busemann’s model, which could lead to a boundary layer separation. Busemann used a different power-law exponent in Eq. (7), which was later corrected in von Kármán and Tsien’s article of 1938 \cite{19}. Instead, he shows how to eliminate \(\partial p/\partial x\) when the \textit{free stream velocity}, \(U\) of Eq. (8), is constant. In order to do this, he expressed the Conservation of Energy Law,
Eq. (3), in terms of the total energy per unit mass, \( E = c_p T + u^2/2 \), as Luigi Crocco did in 1932 [20].

As a result, Eq. (3) is substituted by Eq. (14), and the system of equations becomes Eq. (1), (2), (14), (4), (5), (6), (7). Another consequence is that the constant \( E = c_p T + u^2/2 \), given in terms of (8) and (10), is a solution of Eq. (14) that satisfies its boundary conditions. This makes possible to express the absolute temperature \( T \) in terms of the horizontal velocity \( u \), and, to reduce the model to a system of two conditions for the stream function \( \tilde{\psi} \in C^1(\mathbb{R}) \), as it is proved in Theorem 1.

Luigi Crocco’s Procedure, described in the original article [20], can be applied to the distributions \( \rho, u, v, T, p, \kappa, \mu \) because the generalized derivatives of the variables are elements of the Lebesgue space \( L^2(\mathbb{R}) \), and we can proceed as we would with classical derivatives to apply a generalized Leibniz Rule for the product — as stated in Remark 1 and [9, p. 11], so that Eq. (3) is satisfied if and only if:

\[
\rho \left[ u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right] \left( c_p T + \frac{u^2}{2} \right) = \frac{\partial}{\partial y} \left[ \mu \frac{\partial}{\partial y} \left( c_p T + \frac{u^2}{2} \right) \right].
\]

Moreover, \( T(u) = T_0 \left( 1 - \frac{u^2}{2c_p T_0} \right) \) where \( T_0 = T_{th} + 1 - \frac{(U^2/2c_p)}{2} > 0 \) and \( i_0 = c_p T_0 > 0 \). If we take into account the atmospheric pressure expression \( p(x, \hat{y}) = g \int_{\hat{y}}^{\infty} \rho(x, y) \, dy \) for the standard gravity \( g \) and a linear decrease \( T(x, y) = T_0 - \beta y \) for a constant \( \beta > 0 \) for \((x, y) \in \mathbb{R} \), then:

\[
p \cong c_1 \left[ 1 - \frac{(U^2/2i_0)}{2} \right]^{b/17}
\]

and the density \( \rho(u) \cong c_2 \left[ 1 - \frac{(U^2/2i_0)}{2} \right]^{(b-1)/17} / \left[ 1 - \frac{(u^2(x, y)/2i_0)}{2} \right] \). From Eq. (3), the dynamic viscosity \( \mu(u) = c_3 \left[ 1 - \frac{(u^2/2i_0)}{2} \right]^{19/17} \) for a gas constant \( R = R^*/M \), the molecular weight \( M \), \( p_0 = (n R^* T_0) / V > 0 \), \( c_1 = p_0 T_0^{19/17} \), \( c_2 = c_1 \frac{1}{\tilde{T}_0} T_0^{-1} \), \( c_3 = \mu_h T_0^{19/17} \).

**Theorem 1.** Let \( \rho, u, v, T, p, \kappa, \mu \) be as in Definition 2. Assume \( p = c_1 \left[ 1 - \frac{(U^2/2i_0)}{2} \right]^{(b-1)/17}, \partial u/\partial x = 0 \), and that the variables verify Eq. (1), (2), (14), (4), (5), (6), (7) and (8), (9), (10), (11), (12), (13), (14). Consider \( R \rightarrow \Pi, (x, y) \rightarrow (l, s) \), where:

\[
\ell(\hat{x}, \hat{y}) \rightarrow \int_{0}^{\hat{x}} p(x, \hat{y}) \, dx
\]
and
\[ s(\hat{x}, \hat{y}) = \int_{0}^{\hat{y}} \rho(\hat{x}, y) \, dy. \]

Denote \( \Pi = s(R) \) and \( \sigma_0 = 1 - U^2/(2i_0) \). Then, there is a stream-function \( \tilde{\psi} \in W^{2,2}(R) \) such that \( \partial \tilde{\psi}/\partial x = -\rho v, \partial \tilde{\psi}/\partial y = \rho u \), and a \( \tilde{\sigma} = 1 - (u^2/2i_0) \in W^{1,2}(R; (0, \infty)) \), such that \( \psi = \tilde{\psi} \circ s^{-1} \in W^{2,2}(\Pi) \) and \( \sigma = \tilde{\sigma} \circ s^{-1} \in W^{1,2}(\Pi; (0, \infty)) \) satisfy:

\[
\frac{\partial \psi}{\partial s} \frac{\partial^2 \psi}{\partial \ell \partial s} - \frac{\partial \psi}{\partial \ell} \frac{\partial^2 \psi}{\partial s^2} = c_1^{-1} c_2 c_3 \sigma_0^{\frac{1}{(b-1)^{-1}}} \frac{\partial}{\partial s} \left[ \sigma_0^{\frac{1}{2b-1}} \frac{\partial^2 \psi}{\partial s^2} \right].
\]

(15)

Proof. First, we describe Dorodnitzyn’s diffeomorphism: The new domain’s, \( \Pi \), extremes are \( \ell_M = \ell(0, L) = c_1 \sigma_0 L \) and \( s(x, h) = c_2 \sigma_0^{b/(b-1)-1} h \). The partial derivatives of \( \ell \) over \( R \) are \( \partial \ell/\partial x = c_1 \sigma_0 \) and \( \partial \ell/\partial y = 0 \). Given that \( \partial u/\partial x = 0 \), the Dominated Convergence Theorem [11, p. 44] implies that \( \partial s/\partial x = 0 \). Moreover, \( \partial s/\partial y = \rho \). This may distinguish that \( s \) is the entropy [21, p. 432] and that Dorodnitzyn’s statement of the problem is, in fact, an entropy method. The Jacobian determinant \( |D s| = c_1 \sigma_0 \rho > 0 \). Thus, the Inverse Function Theorem implies that \( s \) is a diffeomorphism that takes the rectangle \( R \) into a polygonal domain \( \Pi \). In this coordinate system, von Kármán’s Integral Formula for a compressible fluid in \( R \) has an incompressible form in \( \Pi \) [22, p. 258].

Because of the zero divergence given in Eq. (11), the generalized Green Theorem for Sobolev Spaces \( W^{1,2}(R) \) on a rectangular domain \( R \) [12, p. 121] and the Poincaré Lemma allow us to define a stream function \( \tilde{\psi} \in W^{2,2}(R) \), \( \tilde{\psi}_{(0,0)}(x, y) = \int_{(0,0)}^{(x,y)} -\rho v \, dx \). The stream function \( \tilde{\psi} \) is regarded in \( \Pi \) as \( \psi \in W^{2,2}(\Pi) \). Once more, over the rectangular domain \( R \), we can apply the Leibniz Rule for \( L^2 \)-distributions of Remark 1 to see that, in terms of \( \psi \), the system has an incompressible non-linear expression for the convective derivative term in the left hand side of Eq. (2) in \( \Pi \) as:

\[
\rho \left( \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = c_1 \sigma_0 \rho \left( \frac{\partial \psi}{\partial \ell} \frac{\partial^2 \psi}{\partial \ell \partial s} - \frac{\partial \psi}{\partial s} \frac{\partial^2 \psi}{\partial s^2} \right).
\]

This way, it is possible to cancel the density \( \rho \) factor with with its correspondent right hand side of Eq. (2) written in \( \Pi \) as:

\[
\frac{\partial}{\partial y} \left( \mu \frac{\partial u}{\partial y} \right) = c_2 c_3 \sigma_0^{b/(b-1)} \rho \frac{\partial}{\partial s} \left[ \sigma_0^{1/(2b-1)} \frac{\partial^2 \psi}{\partial s^2} \right].
\]
where $\partial p/\partial x = 0$ because $p$ is constant in $R$, and $\sigma$ quantifies the amount of kinetic energy is transformed into heat [22]. As a distribution, $\sigma(u) \in W^{1,2}(R; (0, \infty))$ and $\partial^2 \sigma/\partial y^2 \in L^2(R)$ directly from $T \in W^{1,2}(R)$ and the generalized derivative of order two, $\partial^2 T/\partial y^2 \in L^2(R)$. Therefore, under the hypothesis of Definition [2] over the variables, the original problem of Eq. (1), (2), (3), (4), (5), (6), (7), is transformed into the condition given by Eq. (15) with inherited boundary conditions.

At this point, Dorodnitzyn adapts Blasius’ normalization $z$ to express Eq. (15) as an Ordinary Differential Eq. (17), which he transforms into the Quasi-Linear Parabolic Eq. (16).

**Theorem 2.** Under the same hypotheses of Theorem [7], let

$$S = \{ (\ell, z) \in \mathbb{R}^2 \mid (\ell, s) \in \Pi \text{ and } z = s/\ell^{1/2} \in (0, \infty) \},$$

$$\Pi \xrightarrow{z} S, \ (\ell, s) \xrightarrow{z} (\ell, z), \ z (\ell, s) \xrightarrow{z} s/\sqrt{T}, \ \Psi = \tilde{\psi} \circ s^{-1} \circ z^{-1} \in W^{4,1}(S)$$

such that $\Psi = f(z) g(\ell)$, $u_s = u \circ s^{-1} \circ z^{-1}$, and

$$\tau_s^{-1} = (1 - u_s^2/(2i_0))^{-6/25} \partial^2 f/\partial z^2.$$

Then,

$$\tau_s \frac{\partial^2 \tau_s}{\partial u_s^2} = -\frac{1}{2} c_1 c_2^{-1} c_3^{-1} \sigma_0^{1 - \nu/4} \partial \left( 1 - \frac{u_s^2}{2i_0} \right)^{-6/25}.$$

**Proof.** First of all, the Jacobian determinant $|Dz| = \ell^{-1/2} > 0$ for all $\ell > 0$. Therefore, $z$ is a diffeomorphism from $\Pi$ to $S$. Suppose $\Psi = g \cdot f$ is separable as the product of two distributions, independently determined by the variables $\ell$ and $z$, such that $\partial \Psi/\partial z = l^{1/2} \partial f/\partial z$. Then, the Leibniz Rule for a product [23, p. 149] of $g \in C^1(S)$ and an integrable distribution $f \in L^1_{\text{loc}}(S)$ over an open set $S \neq \varnothing$, applied to $\partial (g \cdot f)/\partial z$ and the condition $\partial \Psi/\partial z = l^{1/2} \partial f/\partial z$ imply that $g(\ell) = \ell^{1/2}.$

Second, if $\psi \in W^{2,2}(\Pi)$ is a weak solution of Eq. (15), then $f \in W^{4,1}(S)$, such that $\partial^k \Psi/\partial z^k = l^{1/2} \partial^k f/\partial z^k$ for $k \in \{1, 2, 3, 4\}$, is a weak solution to the Ordinary Differential Equation:

$$-\frac{1}{2} f \frac{\partial^2 f}{\partial z^2} = c_1^{-1} c_2 c_3 \sigma_0^{1 - \nu/4} \frac{\partial}{\partial z} \left( \sigma_s^{-\nu/25} \frac{\partial^2 f}{\partial z^2} \right),$$

(17)
where \( \sigma_s = \sigma \circ s^{-1} \circ z^{-1} \). In order to verify this, we write the left and right side of Eq. (15) in terms of the new coordinates. The left side becomes:

\[
\frac{\partial \psi}{\partial s} \frac{\partial^2 \psi}{\partial s \partial \ell} - \frac{\partial \psi}{\partial \ell} \frac{\partial^2 \psi}{\partial s^2} = -\frac{1}{2} \ell^{-1} f \frac{\partial^2 f}{\partial z^2};
\]

and, the right side is:

\[
\frac{\partial}{\partial s} \left( \sigma^{-6/25} \frac{\partial^2 \psi}{\partial s^2} \right) \frac{\partial}{\partial z} = \ell^{-1} \frac{\partial}{\partial z} \left( \sigma^{-6/25} \frac{\partial^2 f}{\partial z^2} \right).
\]

This way, the factor \( \ell^{-1} \) is nullified when Eq. (18) is equal to Eq. (19) and we obtain Eq. (17).

Third, let \( u_s = u \circ s^{-1} \circ z^{-1} \), then \( f(z) = \int_0^z u_s(l, z') \, dz' \). From the stream-function's separation of the first step, we have \( \partial f/\partial z = \ell^{-1/2} \partial \Psi/\partial z \). Moreover, if \( f \in W^{1,1}(0, \infty) \), then \( f(z) = f(0) + \int_0^z \partial f/\partial z \, (z') \, dz' \). Because of \( \tilde{\psi}(0, 0) = 0, \psi(0, 0) = \Psi(0, 0) = f(0) = 0 \) and \( f(z) = \int_0^z \partial f/\partial z \, (z') \, dz' \).

In addition, for each \((\ell, z) \in S\): \( \partial \Psi/\partial z \, (\ell, z) = \ell^{1/2} \partial \psi/\partial s \, (s^{-1}(z^{-1}(\ell, z))) = \ell^{1/2} (1/\rho) \partial \tilde{\psi}/\partial y \, (s^{-1}(z^{-1}(\ell, z))) = \ell^{1/2} u \circ s^{-1} \circ z^{-1} \, (\ell, z) \). This is, \( \partial \Psi/\partial z \, (\ell, z) = \ell^{1/2} u_s \, (\ell, z) \).

As a direct consequence of both relations, \( f(z) = \int_0^z u_s(l, z') \, dz' \) and \( \partial u_s/\partial z = \partial^2 f/\partial z^2 \). Finally, if \( \tau_s = (1 - u_s^2/(2i_0))^{-6/25} \partial^2 f/\partial z^2 \), then:

\[
\partial^2 f/\partial z^2 = \left( 1 - u_s^2/(2i_0) \right)^{6/25} \tau_s;
\]

the left side of Eq. (17) is:

\[
-\frac{1}{2} f \frac{\partial^2 f}{\partial z^2} = -\frac{1}{2} \left( \int_0^z u_s(l, z') \, dz' \right) \left( 1 - \frac{u_s^2}{2i_0} \right)^{6/25} \tau_s;
\]

and the right side of Eq. (17) becomes:

\[
\frac{\partial \tau_s}{\partial z} = \frac{\partial \tau_s}{\partial u_s} \frac{\partial u_s}{\partial z} = \left( 1 - \frac{u_s^2}{2i_0} \right)^{6/25} \tau_s \frac{\partial \tau_s}{\partial u_s}.
\]

Thus, Eq. (17), in terms of \( \tau_s \) and \( u_s \), allows the elimination of the factor \( (\sigma^{6/25} \tau_s) \), present on both sides:

\[
-\frac{1}{2} \int_0^z u_s(l, z') \, dz' \, (\sigma^{6/25} \tau_s) = c_1^{-1} c_2 c_3 \sigma_0^{b-1} (\sigma^{6/25} \tau_s) \frac{\partial \tau_s}{\partial u_s}.
\]

A derivation with respect to \( z \) on both sides of Eq. (20) gives Eq. (16). □
3. Reynolds’ Shear Stress Limit for Dorodnitzyn’s Boundary Layer

**Theorem 3.** Under the same hypotheses of Theorem 2, let $R \xrightarrow{\phi^\epsilon} R^\epsilon$ for $\epsilon = h/L > 0$, where $(x,y) \xrightarrow{\phi^\epsilon} (x/L, y/(L\epsilon))$, $(x/L, y/(L\epsilon)) = (x^*, y^*)$. Furthermore, assume that $\partial u/\partial y(x,y) > 0$ for each Lebesgue point $(x,y) \in R$. Then, there is a limit $u^* \in W^{1,2}(R)$, $u^* = \lim_{\epsilon \to 0} u^\epsilon$ of $u^\epsilon = (1/L)u$, such that:

$$\frac{\partial}{\partial u^s} \left(1 - \frac{(u^s)^2}{2i_0}\right)^{19/25} = 0.$$ (21)

**Proof.** Let $\sigma^\epsilon = 1 - ((Lu^\epsilon)^2/2i_0)$, $\tau_s = (1 - u^2_s/(2i_0))^{(19/25)^{-1}} \partial u_s/\partial z = \tilde{c} x^{1/2} \tau$, where $\partial u_s/\partial z = \ell^{1/2}/\rho^{1/2} \partial u/\partial y$, $\tilde{c} = c_1^2 c_2^{-1} \sigma_0^{1/2-b/(b-1)}$, and $\tau = \mu \partial u/\partial y$. Thus, Eq. (16), in terms of $\epsilon$, becomes:

$$\epsilon \tilde{c} x^{1/2} (\sigma^\epsilon)^{19/25} \frac{\partial \tau_s}{\partial y} \left(\frac{\partial u^\epsilon}{\partial y^s}\right)^{1-\epsilon^2} \left(\frac{\partial \tau_s}{\partial y}\right)^2 \left(\frac{\partial u^\epsilon}{\partial y^s}\right)^{-2} = -\frac{1}{2} c_1 c_2^{-1} c_3^{-1} \sigma_0^{1-b/(b-1)} u_s \left(1 - \frac{u^2_s}{2i_0}\right)^{-6/25}. \tag{22}$$

In a previous article [1], we showed that, under these circumstances, $\|\nabla u^\epsilon\|_{L^2(R)} \lesssim (c_2 U^3)/(2C)$ for a constant $C$ that is independent of the parameter $\epsilon$. This way, the sequence $(u^\epsilon)$ is bounded in the Sobolev Space $W^{1,2}(R)$. Then, the Rellich-Kondrachov compactness theorem [23, p. 173, 178] implies that there is a subsequence that converges strongly in $L^2(R)$, and the sequence $\partial u^\epsilon/\partial y^s$ converges weakly in $L^2(R)$ to a generalized derivative $\partial u^s/\partial y^s$ of the limit $u^* \in L^2(R)$. Hence, $u^*$ is a weak solution of Eq. (22), in $L^2(R)$ when the parameter $\epsilon$ tends to 0. \hfill \Box

**Corollary 1.** The limit $u^* = U^*$ is a constant solution of Eq. (22). Moreover, if $\partial u/\partial y \cong U/h$, then

$$\tau_s \cong c_1^{1/2} c_2^{-1} U \frac{h}{x^{1/2}} \left(1 - \frac{U^2}{2c_p T_0}\right)^{1-\frac{b-1}{b} + \frac{1}{2}}.$$ 

and we obtain the shear stress estimate:

$$\tau^* = \frac{U}{h} \left(1 - \frac{U^2}{2c_p T_0}\right)^{19/25}.$$
Conclusion

It is possible to deduce approximate shear stress formulas from the Dorodnitzyn’s gaseous boundary layer model and a Reynolds’ Limit Formula developed through a small parameter statement of the problem without taking away the convective derivative non-linear term of the conservation of momentum equation. These estimates provide a new family of deterministic atmospheric boundary layer separation indicators to be tested. The simplicity of its calculations and interpretations in terms of its boundary conditions and variation of specific heat at constant pressure coefficients may allow a wide range of analyses with local data and free from computational time requirements.

Acknowledgements

I would like to express my deepest and sincere gratitude to Dr. Valeri Kucherenko, who trained me, taught me, showed me the light at the end of the tunnel in many occasions, supported and encouraged me to become a scientist and a mathematician.

Conflict of interest

The author declares no conflict of interest.

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