Peculiarities of mathematical modeling of contact interaction of massive bodies and shells

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Abstract. The limits of using a mathematical model of contact interaction of elastic bodies, based on the Hertz’s solution in engineering practice are defined. Numerical investigation of the contact of a massive body and a shell has been conducted when the parameters of the latter vary. Three-dimensional finite-element models of contacting bodies have been developed in the case of different curvatures and thicknesses of the shell walls. A solution of the normal contact problem is given by the force method for various configuration models of the shell. Ratios of the geometric parameters of models the finite-element schemes of the contacting bodies are determined where the problem is described with sufficient accuracy by the Hertz’s solution.

1. Introduction

A solution of the contact problem is very often required in engineering calculations [1]. In some cases, the contacting bodies have complex geometry and are thin-walled shells. Axisymmetric shells are widespread in engineering. Their median surfaces are rotational surfaces. Boilers, tanks, pipes, and cisterns can serve as examples. As shown in paper [2], where the contact problem for the roller and the cylindrical guidance has been solved, the stress-strain state in the roller support has a rather complex nature, which is due to the high compliance of the shell and local deformation of its wall. For these reasons, the use of analytical methods of solution is difficult, and they are used only for an approximate evaluation.

Most of the analytical approaches to static contact problem solving are based on Hertz's solution. Hertz's work [3] gives a solution of the normal contact problem for solid elastic entire bodies. The normal contact stresses in the contact spot area are distributed according to the equation of the ellipsoid, and according to Hertz's assumptions about the smallness of the sizes of the contact patch, the contacting bodies can be represented by half-spaces and the solution of the contact problem can be obtained on the basis of the Boussinesq’s solution. However, if one of the contacting bodies is a shell, the latter cannot be represented in the form of a half-space, which imposes substantial limitations on the Hertz’s solution.

This work is aimed at research peculiarities of mathematical modelling contact interaction of elastic bodies and revealing deviations of the parameters of the contact from the Hertz’s ones in the case when one of the contacting bodies is a shell. In addition, this article studies the nature of these deviations when the magnitude of a load of contacting bodies, the geometric shapes of the shell and the thickness of its wall vary. Based on the research, the ratios of the geometric sizes of bodies for which the problem is fairly accurately described by Hertz's solution are established.
2. Peculiarities of modeling the finite-element schemes of the contacting bodies

Due to the fact that the application of analytical methods for solving the contact problem in this situation is problematic, the most common numerical approaches are used. One of such approaches is the application of the finite element method (FEM), according to which finite element calculation schemes are built at the first stage [4]. In our case, it is necessary to study the change in the basic parameters of the contact when the shell geometry and the wall thickness are varied. A series of experiments with various geometric parameters of the contacting bodies was conducted with the help of computer modeling.

To conduct any multivariate study and to reveal common patterns, it is necessary to provide the identity of the initial conditions prior to the calculation and, first of all, the identity of the mathematical models of the studied bodies. When using FEM [5], all finite element models should have the same structure and the same boundary conditions, and in the process of solving the contact problem using the force method described in [6], in refining the calculation results, it is necessary to ensure that all allocated and then refined local areas, containing the contact spot have the same size.

Thus, when composing finite-element computational schemes of contacting bodies, it is necessary to satisfy three conditions of identity:

- to provide the same size and the same structure of rough finite element calculation schemes of contacting bodies with superimposed identical boundary conditions;
- for contact finite elements to provide the identity of the ratio of their linear sizes;
- to ensure their equal size and structure in the process of calculation by the force method when allocating local areas.

If these conditions are met, errors that always give finite element schemes will have the same influence on the calculation results of the contacting pairs of bodies with different geometric parameters. Thus, the errors in the FEM will not affect the common patterns of a particular phenomenon, thereby ensuring the purity of model experiments.

Because of the fact that contact spot size is very small in comparison with the sizes of the contacting bodies, it is necessary to solve the problem of sizes of the rough finite element calculation schemes that are used in the first stage of solving the contact problem by the force method. Considering a large number of the proposed calculations and the rigid requirements applied by the FEM to computer resources, the minimum size of the schemes and a small number of finite elements should be ensured. To reveal the acceptable size of finite element schemes, the following experiment was conducted. We considered the contact of a massive sphere with a radius of 1 m, which was presented taking into account the symmetry, by the calculation scheme 100 × 100 × 100 mm, and a shell with a radius of 1 m and a thickness of 20 mm.

The elasticity modulus and the Poisson's ratio were chosen to be the same for both bodies and equal, respectively $E = 2 \times 10^3$ and $\mu = 0.25$. Three computational schemes of contacting bodies, differing in the size of the finite element calculation scheme of the shell, were prepared (Figure 1).

![Figure 1](image)

**Figure 1.** Calculation schemes of contacting bodies with varying sizes of the shell scheme:
(a) - 100 × 100 × 20 mm; (b) - 200 × 200 × 20 mm; (c) - 400 × 400 × 20 mm

Taking into account that finite element models are axisymmetric, this figure represents $\frac{1}{4}$ of the design, which was used for the subsequent calculation.

The links were superimposed accordingly in the direction of the symmetry axes and along the computational domain selection boundaries.
The results of calculations [7] when using the indicated schemes by the force method are presented in the table 1.

Table 1. The results of solving the contact problem for various shell models

| The type of shell scheme (Figure 1) | Number of elements in the scheme | Load $P/4$, (kN) | Maximum pressure in the contact $p_0$, (MPa) | Contact spot radius $a$, (mm) | $\sigma_{IV_{\text{max}}^M}$, (MPa) | $\sigma_{IV_{\text{max}}^S}$, (MPa) |
|------------------------------------|---------------------------------|------------------|-----------------------------------------------|-------------------------------|---------------------------------|---------------------------------|
| 1a                                 | 400                             | 24.954           | 980.3                                         | 4.21                          | 662.32                          | 596.9                           |
| 1b                                 | 1000                            | 24.956           | 977.8                                         | 4.27                          | 660.83                          | 577.6                           |
| 1c                                 | 6400                            | 24.968           | 976.4                                         | 4.30                          | 660.64                          | 574.9                           |

Here $\sigma_{IV_{\text{max}}^M}$ and $\sigma_{IV_{\text{max}}^S}$ are the maximum equivalent stresses calculated using the IV strength theory for a massive body and a shell, respectively.

Based on these results and taking into account that the difference between the two schemes (Figure 1a and 1c) differing fourfold in size, by the maximum pressure $p_0$ is 0.4%, by the size of the contact spot is 2%, and by the maximum equivalent stresses are 3.6%, quite small, then to achieve acceptable accuracy for further calculations the scheme 1a may be used. To carry out planned researches five configurations of shells with different radii of curvature were considered.

It is these dimensions of the chosen scheme that were used to construct finite element models of all subsequent calculation schemes with the variable geometry.

To carry out the planned researches the five configurations of shells with different radii of curvature $R_1^S$ and $R_2^S$ were considered (Table 2).

Table 2. Configurational shell models

| Designation of the shell configuration | Shell type     | $R_1^S$, (mm) | $R_2^S$, (mm) | Wall thickness of the shell $\delta$, (mm) |
|--------------------------------------|----------------|--------------|--------------|------------------------------------------|
| S1                                   | Spherical, convex | 1000         | 1000         |                                          |
| S2                                   | Spherical, convex | 2000         | 2000         |                                          |
| C1                                   | Cylindrical     | $\infty$     | 1000         | Varies from 10 to 100 with a pitch of 10 |
| P1                                   | Plate           | $\infty$     | $\infty$     |                                          |
| S2                                   | Spherical, concave | -2000        | -2000        |                                          |

Thus, up to 50 calculation schemes were prepared, for which the wall thickness and shell geometry varied. For each calculation scheme, a normal contact problem was solved for various implementations in order to select a given load, which amounted to several hundred calculations by the force method.

The choice of a correct small finite element model of contacting bodies and the use of an algorithm for improving the calculation results by isolating the local calculation scheme in the region of the contact spot and grinding of the latter described in [3] allowed to perform such a large number of calculations. All calculations were conducted using a software package FEMS developed by the author [4].

3. Investigation of the contact of the sphere and a spherical shell

One of the simple analytical solutions that can be obtained according to Hertz is the contact of two elastic spheres. Let us consider how the parameters of the contact will change if one of the spheres is a spherical shell, the wall thickness of which will vary. We take the radius of the sphere equal to 1 m. We choose the elastic modulus and Poisson's ratio to be the same for the both bodies and equal, respectively, $E = 2\times10^11$ Pa and $\mu = 0.25$. The calculated finite element scheme of contacting bodies is shown in Figure 2.
The shell with a wall thickness of 40 mm is shown here. All other calculation schemes will be obtained by changing the wall thickness from 10 to 100 mm with a pitch of 10 mm, while the calculation scheme of the sphere and the finite element mesh on the contact surface will remain unchanged. According to the force method, it is necessary to specify the penetration of one body into another, according to which the load can be defined after solving the contact problem. Let us select such a penetration for each calculation scheme so that the load is \( P = 100 \text{kN} \). Taking into account that 1/4 part of the contacting body is taken as the calculation scheme, the load attributable to the finite element model will be \( \dot{P} = P/4 = 25 \text{kN} \). The results of the calculation by the force method with a variation of the shell wall thickness are accumulated in Table 3.

**Table 3. The results of solving the contact problem for two spherical bodies with a variation in the wall thickness of the shell**

| Sizes of the shell model, (mm) | The number of nodes in the contact | Load \( P/4 \), (kN) | Contact spot radius \( a \), (mm) | Maximum pressure in the contact \( P_c \), (MPa) | \( \sigma_{IV}^\mu \) (MPa) | \( \sigma_{IV}^\nu \) (MPa) |
|-------------------------------|-----------------------------------|----------------------|---------------------------------|---------------------------------|-----------------|-----------------|
| According to Hertz            |                                   | 25.00                | 7.06                            | 959.8                           | 643.5           | 642.3           |
| 100 \times 100 \times 100     | 52                               | 24.98                | 7.15                            | 962.1                           | 643.3           | 637.7           |
| 100 \times 100 \times 60      | 52                               | 24.97                | 7.25                            | 960.1                           | 642.5           | 625.4           |
| 100 \times 100 \times 40      | 52                               | 24.97                | 7.32                            | 958.6                           | 637.4           | 565.6           |
| 100 \times 100 \times 20      | 60                               | 24.99                | 7.64                            | 950.0                           | 524.8           | 746.2           |
| 100 \times 100 \times 10      | 62                               | 24.98                | 8.25                            | 780.9                           |                 |                 |

According to the presented data, the contact parameters smoothly change with a decrease of the shell thickness, and apparent deviations from the Hertz’s values are observed only with a wall thickness of less than 20 mm. With a shell wall thickness of 10 mm, the deviation of the maximum pressures from the values obtained by the Hertz’s solution is 18.6%, while the sizes of the contact patch increase by 13.9%. With a shell thickness of less than 10 mm, a dip in the central point of the ellipsoid of the distribution of contact pressures (Figures 3b and 3c) is observed, in comparison with the pressure distribution at a shell wall thickness of 100 mm (Figure 3a).

**Figure 3. Distribution of contact pressures when shell wall thickness is:**

(a) – 100 mm; (b) – 10 mm; (c) – 5 mm.

Thus, the nature of distribution of contact pressures in the area of the contact spot varies significantly for thin-walled shells. This affects the distribution of equivalent stresses in the vicinity of the contact spot. When
there is a dip of the central point of the pressure diagram, the arrangement of the points corresponding to the maximum equivalent stresses calculated according to the IV strength theory changes. They start to shift from the location below the central point of the pressure ellipsoid and are located under the corresponding points of maximum contact pressures. Moreover, when a thickness of the shell wall is small, they come out on the inner surface of the shell, which indicates large tension stresses in this area.

The behavior of another parameter of the contact - the approach of contacting bodies \( \alpha \) - will be considered when the wall thickness of the shell varies. Let us proceed to non-dimensional parameters and introduce the notation:

\[
\bar{\delta} = \delta K, \quad \bar{\alpha} = \frac{\alpha(a+b)}{2ab}, \\
\bar{p}_0 = \frac{p_0(k1+k2)(a+b)}{2abK},
\]

where \( \delta \) - is the shell wall thickness;

\( a,b \) - are dimensions of the semiaxes of the contact spot;

\( p_0 \) - is the maximum contact pressure.

The remaining values included in the presented expressions are determined by Hertz’s solution [2]:

\[
K = A + B = \left( \frac{1}{R^M_1} + \frac{1}{R^M_2} + \frac{1}{R^I_1} + \frac{1}{R^I_2} \right), \quad k_1 = 1 - \frac{\mu_1^2}{\pi E_1}, \quad k_2 = 1 - \frac{\mu_2^2}{\pi E_2},
\]

where \( R^M_1, R^M_2, R^I_1 \) and \( R^I_2 \) - are the radii of curvature of the massive body and shell, \( E_1, E_2, \mu_1 \) and \( \mu_2 \) - are elastic characteristics of the contacting bodies, respectively.

If the contacting bodies have a spherical surface of the same radius \( a = b \); \( R = R^M_1 = R^M_2 = R^I_1 = R^I_2 \); \( k = k_1 = k_2 \); \( \mu = \mu_1 = \mu_2 \) and \( K = A + B = \frac{1}{4R} \), the entered parameters, in this case, can be represented as follows:

\[
\bar{\delta} = \frac{\delta}{4R}, \quad \bar{\alpha} = \frac{\alpha}{a}, \quad \bar{p}_0 = \frac{2p_0k}{aK}.
\]

The dependence of the \( \bar{\alpha} \) parameter on \( \bar{\delta} \) in the case of constant curvature of the contacting bodies according to the relations (2) is presented in Figure 4a.

**Figure 4.** Dependences of \( \bar{\alpha} \) and \( \bar{p}_0 \) on the thickness of the shell wall at constant curvature and load \( P = 10000 \) kg in case of the contact of two spherical bodies: 1 – theoretical curve (according to Hertz); 2 – calculated curve
According to this dependence, the fracture point of the calculated curve corresponds to the value $\bar{\delta}$, equal to 0.008, and consequently, the shell wall thickness $\delta \approx 20$ mm, characterizes the sharp change in the contact parameters for the load $P = 100$ kN. Consequently, when the shell thickness decreases from 20 mm or less, the deviation of the value $\alpha$ from the corresponding value according to the Hertz’s solution sharply rises (from 40% or more). In addition, the approach of the contacting bodies increases quite dramatically, which can lead to a loss of stability of the shell, and therefore a verification calculation of the stability for a thin-walled shell is required.

Another dependence (Figure 4b) characterizes the behavior of the maximum pressure in the contact with a decrease in the shell thickness. Here, the fracture of the calculated curve 2 is observed practically in the same parameter range $\bar{\delta}$, as for the previous dependence. However, at this point, corresponding to $\bar{\delta} = 0.08$, the deviation of the $p_0$ parameter from the theoretical value is 8.5 %. And only in the case of $\bar{\delta} = 0.04$ the deviation reaches 30 %, which corresponds to the shell wall thickness of 10 mm.

Thus, for a given load $P = 100$ kN, when two spherical bodies of the same curvature are contacting, the most significant deviations of the contact parameters from the theoretical ones are observed when a shell wall thickness is 10 mm or less. It is the thickness at which a sharp change in the distribution of contact pressures is observed (Figure 3). In this case, the application of the Hertz’s solution gives sufficiently significant errors and it is necessary to use numerical methods of calculation.

To establish similar limitations of the Hertz’s solution for other loads, we will perform a series of calculations for identical spherical bodies in the case of various penetrations. We introduce the notation $\bar{P} = \frac{P(k_1 + k_2)}{ab}$, where $P$ – is an external load on the contacting bodies. In this case, the results of calculations can be represented by the dependences, shown in Figure 5.

The points of fracture of the given curves indicate a sharp change in contact parameters in the case of varying the thickness of the shell. Then curve 5 (Figure 5) constructed from these points will reflect the nature of the change in the contact parameters for varying the given approach of contacting bodies according to the force method and known values $\bar{P}$ and $\bar{\delta}$. This curve can approximately be considered as the boundary of two sections (Figure 5): I - characterizes the behavior of contacting bodies, as massive, where the assumptions are valid and the contact parameters obtained by Hertz’s solution can be used with sufficient accuracy; II - characterizes the influence of shell properties on the contact parameters, accordingly, the increasing deviation from the Hertz’s solution. However, section II can be considered as a transitional from thick-walled to thin-walled shells. Significant deviations of the contact parameters and the form of the distribution of contact pressures from the Hertz values are observed when the thickness is sufficiently small, from 10 mm or less. In this case, the application of Hertz's solution for an approximate evaluation is impossible.

Figure 5. Dependencies that determine the nature of load change from the thickness of the pipe with constant curvature for the penetrations:
1 – 0.05 mm; 2 – 0.07 mm;
3 – 0.1 mm; 4 – 0.14 mm;
5 – boundary curve
4. The study of the effect of shell configuration on contact parameters

Let us consider the effect of shell configuration on contact parameters taking into account the fact that most contacting bodies in real engineering problems have complex geometry. One of the contacting bodies we will represent as a sphere with a radius of 1 m, and the other in the form of a shell, the geometry of which will vary. We will choose five types of shells (Table 2). For all types of shells, we will select the value of penetration of one body into another to obtain a load \( P = 40 \text{kN} \). Considering that significant deviations from the Hertz’s solution are observed when the shell thickness is 10 mm or less, we will choose the shell thickness \( \delta = 10 \text{ mm} \) in all calculation schemes.

The calculation results using the force method when the shell curvature varies are summarized in Table 4. The presented data indicate that the parameters of the contact change substantially when the shell curvature changes. The deviation of the parameters values from the Hertz’s solution increases when curvature decreases.

Table 4. The results of solving the contact problem for varying the shell curvature and the constant thickness of its wall \( \delta = 10 \text{ mm} \).

| Type of the shell (Table 2) | Number of nodes in the contact | Load \( P/4 \), (kN) | Sizes of contact spot semi-axes \( a/b \), (mm) | Maximum pressure in the contact \( P_0 \), (MPa) | Deviation of \( P_0 \) from the Hertz’s solution, (%) |
|-----------------------------|-------------------------------|---------------------|-----------------------------------------------|-----------------------------------------------|-----------------------------------------------|
| S1                          | 31                            | 10.007              | 5.2/5.2                                       | 614.22                                        | 13.1                                          |
| S2                          | 33                            | 10.002              | 5.6/5.6                                       | 494.95                                        | 15.1                                          |
| C1                          | 36                            | 10.002              | 7.1/4.2                                       | 476.12                                        | 15.9                                          |
| P1                          | 43                            | 10.009              | 6.2/6.2                                       | 364.94                                        | 17.9                                          |
| S2                          | 65                            | 10.006              | 8.1/8.1                                       | 216.97                                        | 22.6                                          |

In addition, a study of the deviation of parameters values determined by the relation (1) from the Hertz’s ones when the shell curvature changes was conducted for all types of shells. The results of this study are shown in Figure 6.

Figure 6. Dependences \( \bar{\alpha} \) and \( \bar{p}_0 \) on the shell curvature when the thickness of its wall \( \delta = 10 \text{ mm} \) and the load \( P = 40 \text{ kN} \): 1 – theoretical curve (according to Hertz); 2 – calculated curve.

On the given diagram (Figure 6a) for the shell wall thickness \( \delta = 10 \text{ mm} \), when its curvature decreases, the deviation \( \bar{\alpha} \) from the corresponding value according to the Hertz’s solution decreases and deviation \( \bar{P}_0 \) increases.

Thus, according to the obtained results, a change in the shell curvature with a small thickness of its wall also has an effect on the values of main contact characteristics and their deviation from the Hertz’s
solution. In the case of a four-fold increase in the curvature for the shell from 0.01 to 0.04 when the parameter $\delta$ changes, the deviation of the $\alpha$ value from the corresponding value according to the Hertz’s solution increases by 3.5% and the deviation of the value $p_0$ decreases by 8.1%.

5. Conclusions

The presented work considers the behavior of contact parameters and quite completely analyzes the deviations of the latter from the corresponding values according to the Hertz’s solution when the shell characteristics such as curvature and wall thickness change. Thus, the boundaries of the Hertz’s solution are determined in the case when one of the contacting bodies is a shell. The presented dependences study the behavior of such contact characteristics as the approach of contacting bodies $\alpha$ and the maximum contact pressure $p_0$ when the geometry of the shell and the thickness of its wall change. The dependence (Figure 4a) allows one to control $\alpha$ when the thickness of the shell wall changes, and accordingly, to carry out verification calculations for stability. The dependence (Figure 5) shows the possibility of applying Hertz’s solution for an approximate evaluation of the contact characteristics for known values of the load, curvature, and shell wall thickness.

Thus, the presented study of the case of contact between a massive body and a shell demonstrates when it is sufficient to use the Hertz’s solution to solve the contact problem and when it is necessary to use numerical methods. However, any model studies need an appropriate experimental verification.

6. References

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