Quantum Entanglement Induced by Mutual Gravitational Interaction

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Recent researches suggest that the emergence of spacetime is connected to entanglement. However, the connection is indirectly through the gauge/gravity or AdS/CFT correspondence. Motivated by searching for direct connection between entanglement and the geometry properties of gravity, we developed a generic formulation to calculate an entanglement measure for a bipartite system where the two subsystems interact through classical gravity. From numerical calculation, we found that the ground state of such quantum system is an entangled state. This result generalizes the conclusion in early studies that mutual gravitational interaction can induce entanglement between two masses. More importantly, the result suggests that the entanglement of the ground state of two masses is intrinsically connected to the curvatures of spacetime they create. This provides hint for a quantum gravity theory in the limit of very weak field and low relative velocity.

I. INTRODUCTION

Quantum entanglement has been a source of theoretical interest in probing the foundation of quantum mechanics. For instance, it was used in EPR thought experiment to argue that quantum mechanics is an incomplete physical theory. Quantum entanglement is considered a more fundamental property in some of the quantum mechanics interpretations such as decoherence theory and relational quantum mechanics. In recent decades, it is recognized that entanglement is connecting to the gravitational dynamics in the context of holography. Holography in high energy physics refers to the duality, or more specifically, the gauge/gravity or AdS/CFT correspondence. The duality proposes that the quantum gravity formulated in terms of string theory in an asymptotic Anti de Sitter (AdS) spacetime is equivalent to a standard conformance field theory without gravity defined on the boundary of AdS. The study of the AdS/CFT correspondence leads to the question on how the geometric picture emerges from the CFT dynamics. In other words, how the gravitational spacetime is constructed from CFT dynamics? Interestingly, latest researches suggest that the building block of the spacetime geometry is connected to the entanglement structure of the quantum state in the CFT.

Although the idea that entanglement is related to emergence of spacetime is inspiring, the connection is indirect via the AdS/CFT correspondence. In this setting, the emergence of geometry occurs in a $d + 1$ dimensional AdS with gravity, while the entanglement is between a spatial region in a $d$ dimensional boundary of AdS and the rest of the boundary without gravity. It is natural to ask whether there is direct connection between entanglement and the emergence of spacetime geometry without the need of the holographic correspondence. Furthermore, one may also wonder if the connection in the holographic context is a unique result from the string theory, or it is intrinsic to any plausible quantum gravity theory. Unfortunately, these questions cannot be easily answered as we do not have a truly unified quantum gravity theory. Practically, we can take one step back and ask a simpler question: can the traditional quantum theory provide similar hints on the connection between entanglement and gravity? To probe the answer to this question, one can investigate whether entanglement between two massive objects can be induced through interaction of classical gravity. However, given that the more general gravity theory is the General Relativity, is it still a value to study the systems using traditional quantum mechanics and classical gravity? We argue that the answer is yes due to Bohr’s Correspondence Principle. Supposed eventually a theory $M$ successfully unifies GR and QFT, in the limit of very weak field and low relative velocity, the gravity aspect of theory $M$ is approximated by the classical gravity field. In such a limit, theory $M$ should either reproduce the result calculated from the Schrödinger equation with gravitational potential, or give a reasonable explanation why the results may be different. In either cases, the classical result can be used as a check point for theory $M$ in the limit of very weak field and low relative velocity.

With this motivation, we first develop a generic formulation to calculate the reduced density matrix of a bipartite system with continuous variable. This allows us to derive an entanglement measure based on purity of the reduced density matrix. The formulation is then applied to calculate the entanglement measure of the ground state of a bipartite system where the two subsystems interact through classical gravity field. The wave functions of the ground state for such systems are well known and an explicitly formulation of their entangle-

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1 The Correspondence Principle was initially advocated by Bohr, which states that quantum mechanics must be reduced to classical mechanics in the limit of large quantum number. It is then generalized to require that a new physical theory should reproduce the results of older well-established theories under some conditions.
ment measure can be derived. However, since the formulation involves a twelve dimensional integral, the final calculation is carried out using Monte Carlo integration. The numerical results show that the ground state of the two masses is an entangled state, so as the first excited state. Thus, we confirm that the entanglement between the two masses is connected to the properties of the classical gravity field. This result, as mentioned earlier, can be a check point for theory $M$ in the limit of very weak field and low relative velocity. Furthermore, we speculate that the result indicates the entanglement manifests the curvature of spacetime if we take the general relativity perspective.

After this work was completed, we become aware that gravitationally induced entanglement was investigated earlier [14, 15], but in a very different context to assess whether gravity field is a quantum entity. Although with a very different motivation, our result confirms and generalizes the conclusion that gravity interaction can induce entanglement between two masses. More importantly, we provide new insight that such entanglement is intrinsically connected to the spacetime curvature the two masses create.

The paper is organized as following. In Section II we derive a generic formulation of entanglement measure for a bipartite system with continuous variable. The formulation is then applied to the ground state derived from the Schrödinger equation with gravity potential in Section III to obtain explicit expression of entanglement measure. The numerical calculations shown in Section IV clearly show that the ground state is an entangled state. Section V explores the conceptual implications of this finding. Section VI is dedicated to discussing the similarity and difference between the results in Refs [14, 15] and the results in this paper. Limitations and conclusive remarks are presented in Section VII and VIII respectively.

II. ENTANGLEMENT MEASURE FOR CONTINUOUS VARIABLE SYSTEM

For a continuous variable bipartite system with subsystem 1 and 2, a pure state can be expressed as

$$|\Psi\rangle_{12} = \int \psi(x, y)|x\rangle|y\rangle dx dy.$$  \hspace{1cm} (1)

Here we assume one dimensional continuous variable $x, y$ for each subsystem 1 and 2, respectively. It is straightforward to extend to three dimensional variables. Normalization requires

$$\int \psi(x, y)\psi^*(x, y) dx dy = 1.$$  \hspace{1cm} (2)

Rewriting (1) in the form of density matrix, we have

$$\rho_{12} = |\Psi\rangle_{12}\langle \Psi| = \int \int \psi(x, y)\psi^*(x', y')|x\rangle|y\rangle\langle x'|\langle y'| dx dy dx' dy'.$$  \hspace{1cm} (3)

This gives the density matrix element

$$\rho_{12}(x, y; x', y') = \psi(x, y)\psi^*(x', y').$$  \hspace{1cm} (4)

The reduced density matrix for subsystem 1 can be derived by taking partial trace,

$$\hat{\rho}_1 = Tr_2(\hat{\rho}_{12}) = \int \int \int \psi(x, z)\psi^*(x', z) dx dz$$

$$= \int \int \int \psi(x, z)\psi^*(x', z) dx dz x' dx'.$$  \hspace{1cm} (5)

Here, the integration over variable $z$ is on subsystem 2. From (4), one obtains the reduced density matrix element for subsystem 1,

$$\rho_1(x; x') = \int \psi(x, z)\psi^*(x', z) dz.$$  \hspace{1cm} (6)

Due to the normalization property in (2), we have

$$Tr(\hat{\rho}_1) = \int \rho_1(x; x) dx = 1,$$  \hspace{1cm} (7)

as expected. To quantify the entanglement between subsystem 1 and 2, we use the following definition [16]

$$E = 1 - Tr(\hat{\rho}_1^2).$$  \hspace{1cm} (8)

Quantity $Tr(\hat{\rho}_1^2)$ is the purity of reduced density matrix $\rho_1$, given by

$$Tr(\hat{\rho}_1^2) = \int \rho_1^2(x, x) dx = \int \rho_1(x, x')\rho_1(x', x) dx dx'.$$  \hspace{1cm} (9)

Appendix A shows the justification on why $E$ can be considered as an entanglement measure. Substitute (6) into (9) and replace variable $z$ with $y$, we have

$$Tr(\hat{\rho}_1^2) = \int \psi(x, y)\psi^*(x', y)\psi(x', y')\psi^*(x, y') dy dy' dx' dx.$$  \hspace{1cm} (10)

Thus, given a continuous variable wave function of a bipartite system, $\psi(x, y)$, we can calculate the entanglement measure from (8) and (10). Suppose that the wave function can be factorized as $\psi(x, y) = \phi(x)\varphi(y)$, we have

$$Tr(\hat{\rho}_1^2) = \left( \int |\phi(x)|^2 dx \right) \left( \int |\varphi(y)|^2 dy \right) = 1.$$  \hspace{1cm} (11)

Thus, $E = 0$, there is no entanglement between the two subsystems. However, when $\psi(x, y) \neq \phi(x)\varphi(y)$, it is not obvious whether $E \neq 0$. A detailed calculation is needed.
We can extend (10) to three-dimensional system with continuous variables for subsystem 1 and 2 are \( \vec{r}_1 \) and \( \vec{r}_2 \), respectively, then
\[
Tr(\hat{\rho}_{12}^2) = \int \psi(\vec{r}_1, \vec{r}_2)\psi^*(\vec{r}_1, \vec{r}_2)\psi(\vec{r}_1, \vec{r}_2) \times \psi^*(\vec{r}_1, \vec{r}_2) d\vec{r}_2 d\vec{r}_1 d\vec{r}_1 ,
\]
where \( d\vec{r} := dx dy dz \). Note that (12) is a twelve dimensional integration.

III. ENTANGLEMENT OF TWO MASSES INTERACTING WITH CLASSICAL GRAVITY

Consider a three-dimensional bipartite system, and the interaction between two subsystems is described as a potential only depends on the distance between the two subsystems \( r_{12} = |\vec{r}_1 - \vec{r}_2| \). The stationary Schrödinger Equation is given by
\[
\left[ -\frac{\hbar^2}{2m_1} \nabla_1^2 - \frac{\hbar^2}{2m_2} \nabla_2^2 + V(r_{12}) \right] \psi(\vec{r}_1, \vec{r}_2) = E\psi(\vec{r}_1, \vec{r}_2) ,
\]
where \( \nabla^2 \) is the Laplacian operator in the three-dimensional coordinator, \( E \) is the total energy of the system, \( m_1 \) and \( m_2 \) are the masses of subsystem 1 and 2, respectively. The potential can be a Coulomb potential or a gravity potential. The formulation is constructed here in the context of traditional quantum mechanics. Nevertheless, this can be considered as a first order approximation of more general formulation. The question we want to answer is that given a solution of wave function from (13), whether the two subsystems are in entangled state. In particular, we are interested in the ground state.

A typical way to solve (13) is to introduce transformation (15)
\[
\vec{r}_{12} = \vec{r}_1 - \vec{r}_2 ,
\]
\[
\vec{R}_{12} = \vec{r}_1 + \vec{r}_2 ,
\]
where \( \vec{R}_{12} \) is the center of mass coordinate of the system. Omit the subscript “12” and denote
\[
\psi(\vec{r}_1, \vec{r}_2) = \phi(\vec{R}) \phi(\vec{r}) ,
\]
which is separated into two equations,
\[
-\frac{\hbar^2}{2M} \nabla_\vec{R}^2 \phi(\vec{R}) = E_c \phi(\vec{R}) ,
\]
\[
\left[ -\frac{\hbar^2}{2\mu} \nabla^2 + V(r) \right] \phi(\vec{r}) = E_r \phi(\vec{r}) ,
\]
where \( M = m_1 + m_2 \) is the total mass, \( \mu = m_1 m_2 / (m_1 + m_2) \) is the effective mass, \( E_c \) is the kinetic energy of the center mass, \( r = |\vec{r}_{12}| \), and \( E_r = E_t - E_c \).

Eq. (16) corresponds to the Schrödinger equation of a free particle. Suppose the center mass of the bipartite system is moving with a constant momentum \( \vec{P}_c \), and the system is in a three-dimensional spatial box with length \( L \), each of the dimensional variable \( x, y, z \in \{-L/2, L/2\} \). The wave function \( \phi(\vec{R}) \) can be expressed as
\[
\phi(\vec{R}) = \lim_{L \to \infty} \sqrt{\frac{1}{L^3}} e^{i\vec{P}_c \cdot \vec{R}/\hbar} .
\]
Substitute this into (15) and then into (12), the purity of reduced density matrix is simplified as
\[
Tr(\hat{\rho}_{12}^2) = \lim_{L \to \infty} \left( \frac{1}{L} \right)^6 \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} \phi(\vec{r}_{12}) \times \phi^*(\vec{r}_{12'}) d\vec{r}_{12} d\vec{r}_{12'} .
\]
Once the wave function for the relative movement between the two subsystems, \( \varphi(\vec{r}_{12}) \), is solved from (17), one can compute the entanglement \( E \) from (19) and (8).

The solution of (17) depends on the actual form of central potential energy \( V(r_{12}) \) where \( r_{12} \) is the relative distance between the two subsystem. For gravitational potential energy,
\[
V(r_{12}) = -G\frac{m_1 m_2}{r_{12}} ,
\]
where \( G \) is the Newtonian constant of gravitation. Another example is the Coulomb potential energy of hydrogen atom, given by
\[
V(r_{12}) = -\frac{e^2}{r_{12}} .
\]
(17) with such potential energies can be solved analytically. In particular, we are interested the solution for the ground state. The wave function for the ground state is given by (18)
\[
\varphi_g(\vec{r}_{12}) = \sqrt{\frac{1}{\pi a^3}} e^{-r_{12}/a} .
\]
Here, constant \( a \) in the case of Coulomb potential energy given by (21) is \( h^2/(\mu e^2) \approx 0.83 \times 10^{-8} cm \), which is the famous Bohr Radius. In the case of gravity potential energy given by (20),
\[
a = \frac{\hbar^2}{G m_1 m_2} = \frac{\hbar^2}{G m_1^2 m_2} (1 + \frac{m_1}{m_2}) ,
\]
which is called the Gravitational Bohr Radius. We will discuss the practical meaning of this constant in section VI For the time being, just consider it as a constant in the solution for the relative wave function \( \varphi(\vec{r}_{12}) \).

Substituting (22) into (19), after some algebra, we obtain
\[
Tr(\hat{\rho}_{12}^2) = \lim_{L \to \infty} \frac{1}{\pi^2 a^6 L^6} \int_{-L/2}^{L/2} d\vec{r}_{12} d\vec{r}_{12'} d\vec{r}_{22'} e^{-2(r_{12}+r_{12'}+r_{22'})/a} .
\]
Replacing the position variable $\vec{r}$ with non-dimensional variable $\vec{\gamma} = 2\vec{r}/L$, for $\vec{r}_1, \vec{r}_1', \vec{r}_2, \vec{r}_2'$, and denoting $\alpha = L/a$, we rewrite (24) as

$$Tr(\hat{\rho}_1^2) = \lim_{\alpha \to \infty} \frac{1}{\pi^3} \left(\frac{\alpha}{4}\right)^6 \int_{-1}^{1} d\gamma_1' d\gamma_1'' d\gamma_2' d\gamma_2'' e^{-\alpha(\gamma_{12} + \gamma_{12}')}.$$

Explicitly written in Cartesian coordinate, variable $\gamma_{12} = 2\sqrt{(x_1 - x'_1)^2 + (y_1 - y'_1)^2 + (z_1 - z'_1)^2}/L$. Similar expressions can be written down for $\gamma_{12}', \gamma_{12}'$, and $\gamma_{12}'$. $d\gamma_1' = (2/L)^3 dx_1 dy_1 dz_1$, and similar expressions for $d\gamma_1$, $d\gamma_2$, and $d\gamma_2'$. As seen, (25) is a twelve dimensional integral. There is no analytic solution. Numerical calculation is needed.

IV. MONTE CARLO INTEGRATION

In this section, Monte Carlo integration method is utilized to estimate the twelve-dimensional integration in (25). We use the MISER algorithm [19] of GNU Scientific Library version 2.5. To validate the accuracy of the algorithm, we first calculate $Tr(\rho_1)$. It is expected to have $Tr(\rho_1) = 1$ due to the normalization requirement. Followed similar derivation steps in previous section, $Tr(\rho_1)$ is expressed as

$$Tr(\rho_1) = \lim_{\alpha \to \infty} \frac{1}{\pi^3} \left(\frac{\alpha}{4}\right)^6 \int_{-1}^{1} e^{-\alpha(\gamma_{12})} d\gamma_1' d\gamma_2'. \tag{26}$$

This is a six-dimensional integration. Table I shows the calculation results with different value of $\alpha$ using the Monte Carlo integration algorithm. The MISER algorithm is set to recursively calculate the integration till the statistical error is less than one percent. Double precision variables are used in the calculation. The number of Monte Carlo calls $N_{MC}$ increases significantly when $\alpha$ increases. Our calculation ends at $\alpha = L/a = 200$. From the results in Table I, it is reasonable to extrapolate that $Tr(\rho_1) \to 1$ when $\alpha \to \infty$. This confirms the MISER algorithm is fairly accurate.

| $L/a$ | $Tr(\rho_1)$ | Error | $N_{MC}$ (million) |
|-------|--------------|-------|-------------------|
| 10    | 0.786360     | 0.004336 | 1                 |
| 20    | 0.876379     | 0.008357 | 2                 |
| 40    | 0.955348     | 0.006011 | 16                |
| 100   | 0.981460     | 0.006937 | 128               |
| 150   | 0.984406     | 0.005142 | 512               |
| 200   | 0.994176     | 0.007497 | 1.024             |

We now proceed to calculate the purity of reduced density $\rho_1$ derived from the ground state, given in (25). The results are shown in Table II. Note that the calculation becomes very expensive when $\alpha$ increases, as the number of Monte Carlo calls increases to billions. To reduce the computation cost, the calculation is terminated whenever the error is <20% of the value of $Tr(\rho_1^2)$. From the results in Table II, $Tr(\rho_1^2) \to 0$ rapidly when $\alpha$ increases. It is reasonably to extrapolate that $E = 1 - Tr(\rho_1^2) = 1$ when $\alpha \to \infty$. This confirms that the ground state is an entangled state. The two masses are entangled due to the gravity interaction.

We can also compute the entanglement measure for an excited state. The relative wave function for the first spherical symmetry excited state is given by [18]

$$\varphi_e(\vec{r}_{12}) = \sqrt{\frac{1}{8\pi a^3}} (1 - \frac{r_{12}}{2a}) e^{-r_{12}/2a}. \tag{27}$$

Substituting this into (18), using the same notations as for (25), and denoting $\beta = L/(4a) = \alpha/4$, we obtain

$$Tr(\rho_1^2) = \lim_{\alpha \to \infty} \frac{1}{\pi^3} \left(\frac{\beta}{2}\right)^6 \int_{-1}^{1} d\gamma_1' d\gamma_1'' d\gamma_2' d\gamma_2'' \times (1 - \beta\gamma_{12}) (1 - \beta\gamma_{12}') (1 - \beta\gamma_{12}')(1 - \beta\gamma_{12}') e^{-\beta(\gamma_{12} + \gamma_{12}')} \tag{28}.$$ 

The Monte Carlo integration results for (28) are shown in Table III. The numerical results show that $Tr(\rho_1^2) \to 0$ asymptotically when $\alpha$ increases. Compared to Table II, the purity approaching zero slower when the bipartite system is in the excited state. However, it still shows that the two subsystems are in an entangled state.

| $L/a$ | $Tr(\rho_1^2)$ | Error | $N_{MC}$ (million) |
|-------|--------------|-------|-------------------|
| 10    | 4.94 x 10^{-2} | 1.64 x 10^{-4} | 1                 |
| 40    | 2.13 x 10^{-2} | 1.40 x 10^{-4} | 128               |
| 100   | 2.07 x 10^{-3} | 1.15 x 10^{-4} | 2,048             |
| 200   | 6.97 x 10^{-5} | 5.94 x 10^{-6} | 2,048             |
| 400   | 5.86 x 10^{-6} | 1.51 x 10^{-6} | 8,192             |

The results presented in this section are derived from numerical calculation rather than analytic calculation. There is always doubt that the extrapolation from numerical calculation is not the same as the asymptotic limit, particularly for a twelve-dimension integration. In Appendix B, we give an analytic calculation of the $Tr(\rho_1^2)$.
for a harmonic oscillator, and show that the numerical calculation using the same Monte Carlo algorithm is consistent with the analytic result. This further confirms the validity and reliability of the Monte Carlo method for the twelve-dimension integration.

V. CONCEPTUAL IMPLICATIONS

The observation that two masses $A$ and $B$ interacting through gravitational field are entangled in the ground state has some interesting conceptual implications. First of all, we need to emphasize that the entanglement calculated in section IV is different from the entanglement in the AdS/CFT correspondence. The later refers to the correlation between a spatial region in the boundary of AdS and the rest of the boundary without gravity defined. On the other hand, the entanglement calculated in this paper is for two masses that interacts through classical gravitational field.

Entanglement between $A$ and $B$ implies that there is correlation information between them. By knowing information on $A$, one can infer information on $B$. What information is correlated is in the case of gravity interaction? The degree of freedom in our calculation is the position of the masses. Thus, the correlation encoded in the entangled systems is about the position of each system. Each system is in a mixed state. Their positions are correlated and cannot be described independently. Subsystem $A$ has to be described relative to subsystem $B$ for completeness. This appears consistent with the principle of the relational quantum mechanics.

The root cause of such correlation is due to the intrinsic property of the gravitational potential, which is proportional to the inverse of relative distance between $A$ and $B$. The origin of such property can be better understood in the context of General Relativity. In GR, there is no gravitation field. Instead, the gravitation field in the Newtonian formulation is just the effect of spacetime curvature in the limit of very weak fields and low velocities. Here, the full spacetime metrics, which determine the curvature, is written as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $\eta_{\mu\nu}$ is the Minkowskian metrics and $h_{\mu\nu}$ is the small deviation on it. Only the $h_{00}$ component of the metrics is of important in the limit of very weak fields and low velocities, which turns out to be proportional to the inverse of relative distance between the two masses. Thus, the curvature of spacetime is manifested in such geometry property of the classical gravity potential. Since our calculation shows that the gravity with such property induced entanglement between two masses in their ground state, we argue that the curvature of spacetime causes the entanglement between the two masses. In other words, the entanglement at the ground state intrinsically connects to the curvature of spacetime. In fact, it is interested to investigate whether a quantum state for two masses interacting with gravity can even be separable. Such question is speculative only and needs a unified quantum gravity theory $\mathcal{M}$ for accurate treatment.

The above arguments are considered reasonable based on Bohr’s Correspondence Principle. Traditional quantum mechanics with classical gravity interaction, nevertheless, can be considered an approximation of theory $\mathcal{M}$. We expect when certain classical limit is imposed, theory $\mathcal{M}$ should either predict similar result as the less general physical theory $\mathcal{E}$, or explain why the results are different. In this notion, whether the ground state of two systems interacting through gravity is an entangled state can be a check point for theory $\mathcal{M}$ in the classical limit.

Since the formulation and calculation in earlier sections are generic to any bipartite system where the interaction between two subsystems is described as a potential only depends on the distance between the two subsystems, the calculation is also applicable to well-studied systems such as a hydrogen atom where the interaction is described by the Coulomb potential, or a three-dimensional harmonic oscillator. In the case of a hydrogen atom, the entanglement in the ground state is an interesting new finding because traditionally one is only interested in the energy levels of each eigenstate. The entanglement information encoded in the wave function had been left unrecognized.

VI. LIMITATIONS

Although we show that two masses interacting through classical gravity potential field are entangled when they are in the ground state, there are practical limitations of this result.

The limitation can be examined from the only numerical parameter $\alpha = L/a$ in (25) which depends on physical parameter $a$. In the case of hydrogen atom, $a = 0.83 \times 10^{-8} cm$ is the Bohr Radius. The electron and the hydrogen nuclear are entangled when they are in the ground state. Our calculation here does not consider the electron spin and its coupling effect with the angular momentum. A refined calculation should include it in the total Hamiltonian. The spin-angular momentum coupling is considered perturbation to the ground state derived from just the Coulomb interaction. One question is that whether the electron-nuclear entanglement in the ground state is still maintained when considering the spin-angular momentum perturbation. Further numerical calculation must be performed to answer this

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2 More precisely, the bipartite system as a whole is in a pure state while subsystem $A$ is described relative to subsystem $B$.

3 For instance, Einstein’s general relativity predict practically identical results as the Newton’s law of gravitation (or its reformulation as classical gravitational field theory) in the limit of very weak fields and low velocities.
question. Certainly, a more accurate treatment of this problem should employ the quantum field theory.

We are more interested in the case that the two subsystems are interacting through gravity potential. In this case, parameter $a$ is given by (23) and is called the Gravitational Bohr Radius. Its value is strongly depending on the values of masses $m_1$ and $m_2$. Considered the case of the Sun and the Earth. Given that $G = 6.67 \times 10^{-11} m^3 kg^{-1} s^{-2}$, the mass of the Sun $m_1 = 1.99 \times 10^{30} kg$, and the mass of the Earth $m_2 = 5.97 \times 10^{24} kg$, and the Plank constant $\hbar = 1.06 \times 10^{-34} m^2 kg s^{-1}$, (23) gives $a = 2.35 \times 10^{-135} m$. This is much smaller than the Plank length $1.62 \times 10^{-35} m$ and becomes non-physical. Clearly the wave function given in (22) is not suitable to describe quantum state of the Sun-Earth system. Consequently, it is meaningless to discuss entanglement between the quantum states of the Sun and the Earth.

However, suppose the masses of the two subsystems are at the scale of $10^{-21} kg$ and $10^{-17} kg$, such as the mass of a Brome mosaic virus [20] and a vaccinia virus [21], respectively. One can estimate $a \approx 5 cm$. If a universe consists only two such viruses and they interact only through classical gravity, and if we further assume such virus can be considered as quantum systems, then the conclusion can be drawn that the two systems are entangled when they are in the ground state. Obviously these are very strict conditions. Due to such practical limitation, one must be very cautious to draw such a conclusion. Instead, the significance of our result comes from the conceptual implication as discussed earlier.

The Schrödinger Equation in the form of (1) implies that we have chosen a reference coordinate system such that the positions of the two systems are given by variable $\vec{r}_1$ and $\vec{r}_2$, respectively. The entanglement obtained in the calculation is with respect to such coordinate system [1]. To study the entanglement properties in a relativistic framework, one should use the QFT. It has been shown that under Lorentz transformation, the entanglement measure for the subsystem-to-subsystem partition is Lorentz invariant [10]. But it is not clear whether this conclusion is still true for systems in a curve spacetime. Furthermore, a recent study suggests that if we consider the reference system as quantum system as well, entanglement properties may depend on the choice a reference system [22]. For example, we have shown that in a hydrogen atom, the electron and the nuclear are entangled in ground state with the lab as a reference frame. But if instead, we choose the nuclear itself as the reference system, the electron can be in a pure state with respect to the nuclear. However, the study is still in the early stage and needs more investigations.

VII. ENTANGLEMENT THROUGH ADJACENT INTERFEROMETERS

In an effort to confirm that gravitational field is a quantum entity, Refs [14, 17] proposed a novel experiment to induce entanglement between two test masses interaction through gravity. The motivation there is that entanglement can only be generated through mediation of a quantum entity. By confirming that mutual gravitational interaction between test masses can entangle the states of two masses, one can conclude that gravity field necessarily obeys quantum mechanics principles. The experiment is briefly restated below. Two test masses with masses $m_1$ and $m_2$ are prepared in superposition of two spatial separated states $|L\rangle$ and $|R\rangle$. Suppose the distance between the center of the two state is $\Delta x$. Each state is a localized Gaussian wave packets with the width much smaller than $\Delta x$, so that $\langle L|R\rangle = 0$. The distance between the centers of the two masses is $d$. Essentially these initial conditions can be physically realized by two Mach-Zehnder interferometers separated at distance $d$. The initial state is then given by

$$|\Psi(t = 0)\rangle_{12} = \frac{1}{\sqrt{2}}(|L\rangle_1 + |R\rangle_1)\frac{1}{\sqrt{2}}(|L\rangle_2 + |R\rangle_2).$$

(29)

Now the two masses go through time evolution with mutual gravitational interaction for a period $\tau$. The time evolution introduces an additional phase shift in the probability amplitude, given by $\phi_{ij} = V_{ijkl}r = \frac{Gm_im_j}{hr^2}$, where $i \in \{L, R\}$ is index for mass 1, $j \in \{L, R\}$ is index for mass 2, and $r_{ij}$ is the distance between the distinct components of the superposition state of the two masses. Since $r_{ij}$ is different for the four possible combinations of spatial states of the two masses, the final state is

$$|\Psi(t = \tau)\rangle_{12} = \frac{e^{i\phi_{LL}}}{2}|L\rangle_1(|L\rangle_2 + e^{i(\phi_{LR} - \phi_{LL})}|R\rangle_2) + \frac{e^{i\phi_{RL}}}{2}|R\rangle_1(|L\rangle_2 + e^{i(\phi_{RR} - \phi_{RL})}|R\rangle_2).$$

(30)

$|\Psi(t = \tau)\rangle_{12}$ can be factorized if the following condition is met,

$$\Delta \phi = \phi_{LR} - \phi_{LL} + \phi_{RR} - \phi_{RL} = 2n\pi.$$

(31)

For the two masses interact with classical gravity in the interferometers, $r_{LL} = r_{RR} = d$, $r_{LR} = d + \Delta x$, and $r_{RL} = d - \Delta x$, we obtain

$$\Delta \phi = \frac{Gm_1m_2}{h} \left(\frac{2}{d} - \frac{1}{d + \Delta x} - \frac{1}{d - \Delta x}\right),$$

(32)

which is in general not equal to $2n\pi$ if proper parameters $d$ and $\Delta x$ are chosen. Thus, $|\Psi(t = \tau)\rangle_{12}$ cannot be factorized and entanglement between two test masses can be created. We now proceed to discuss the similarity and difference between this result and our result.

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4 For instance, in the case of hydrogen atom, we can choose the lab where the hydrogen atom is prepared as a reference system. The coordinate system is at rest with respect to the lab.
First, both results show that gravitational interaction can induce entanglement between two masses, and consequently confirm that gravity is a quantum entity if we acknowledge the reasoning described in Refs [14, 15]. However, the entanglement in the interferometer approach is generated through a very specific experimental design, the test masses are prepared in specific initial condition, while the result presented in this paper is generic and derived rigorously from first principle, i.e., from the Schrödinger Equation. In this sense, our result generalizes the finding in Refs [14, 15] since it is not depending on specific experimental setup.

Second, both results show that the entanglement can be generated through other interaction such as Coulomb interaction. The key is that the interaction potential energy cannot be factorized into two independent terms with respective to the position degree of freedoms. To see this, suppose \( V(r_{12}) = U(r_1) + W(r_2) \), it is easy to verify that \( \Delta \phi = 0 \). One crucial example is the gravitational field from the Earth acting on the two masses, which can be approximated as \( V(z_i) = m_1 g z_i + m_2 g z_j \), where \( z \) is the distance between the surface of the Earth and the masses. For this gravitational field, \( \Delta \phi = 0 \) and cannot induce entanglement. The same conclusion can be drawn from Eq. (13). When \( V(r_{12}) = U(r_1) + W(r_2) \), \( \Delta \phi \) can be separated two independent equations and admits \( \psi(r_1, r_2) = \phi(r_1) \phi(r_2) \) as a solution, which is a product state. Thus, the origin of the entanglement strongly depends on the geometry properties of the interacting field, although this is not pointed out in Refs [14, 15].

Third, most importantly, the motivation of our work is to search for direct connection between entanglement and gravity. Since the entanglement strongly depends on the geometry properties of the gravitational field and such properties attributes to the spacetime curvature from the perspective of GR, it leads us to argue that there is a connection between the entanglement and the spacetime curvature. This is a new insight not presented in Refs [14, 15].

The significance of the experiments proposed in Refs [14, 15] is that the gravitationally induced entanglement is practically detectable, while our work is mostly theoretical and conceptual.

VIII. CONCLUSIONS

Motivated by searching for direct connection between entanglement and gravity, we developed a generic formulation to calculate the entanglement measure for a bipartite system where the two subsystems interact through classical gravity in the context of non-relativistic quantum mechanics. Through numerical calculation, we found that the ground state of such quantum system is an entangled state.

Although the result cannot be practically applied to cosmic systems such as the Sun-Earth system, its significance comes from the conceptual implications. It confirms and generalizes the idea that entanglement can be induced by classical gravitational field. Since the gravitational field is an approximation of the curvature of spacetime in the limit of very weak fields and low velocities, it is reasonable to argue that there is an intrinsic connection between the entanglement of two masses and the curvature of the spacetime they create. We speculate that a quantum state for two masses interacting with gravity cannot be separable. Lastly, a quantum gravity theory should either predict similar result in the weak field and low velocity limit, or gives a reasonable explanation why the results may be different. Thus, the result presented here can be used to test a quantum gravity theory in such limit.

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Appendix A: Justification of Entanglement Measure

Using the entanglement measure defined in Eq. (3), we say that two subsystems are entangled when $E > 0$, which corresponds to $Tr(\hat{\rho}_2^2) < 1$. On the other hand, the two subsystems are separable when $E = 0$, which corresponds to $Tr(\hat{\rho}_1^2) = 1$. We will show that $E$ indeed measures the entanglement of the bipartite system. Suppose the state vector of the bipartite system can be decomposed with a set of orthogonal basis $\{\tilde{\phi}_i, \tilde{\varphi}_i\}$ where $i = 0, 1, 2, \ldots, d - 1$, and $d$ could be infinite,

$$|\Psi\rangle_{12} = \sum_i \lambda_i |\phi_i\rangle |\varphi_i\rangle$$

$$= \sum_i \lambda_i \int \phi_i(x)|x dx\rangle \int \varphi_i(y)|y dy\rangle$$

$$= \int \{\sum_i \lambda_i \{\phi_i(x)\varphi_i(y)\}|x\rangle|y\rangle dx dy,$$

where $\sum_i |\lambda_i|^2 = 1$. Compared to Eq. (11) gives

$$\psi(x, y) = \sum_i \lambda_i \phi_i(x)\varphi_i(y).$$

The orthogonal relations are given by

$$\int \phi_i(x)\phi^*_i(x) dx = \delta_{ij}, \int \varphi_k(y)\varphi^*_k(y) dy = \delta_{kl}.$$  \hspace{1cm} (A3)

Substituting Eq. (A2) into Eq. (11) and applying the orthogonal properties, we obtain

$$Tr(\hat{\rho}_2^2) = \sum_{i=0}^{d-1} (|\lambda_i|^2)^2.$$  \hspace{1cm} (A4)

Since $\sum_i |\lambda_i|^2 = 1$, it can be shown that

$$\frac{1}{d^2} \leq Tr(\hat{\rho}_2^2) \leq 1.$$  \hspace{1cm} (A5)

$Tr(\hat{\rho}_2^2)$ is 1 if and only if $d = 1$, which implies $\psi(x, y) = \phi(x)\varphi(y)$ and consequently, $|\Psi\rangle_{12}$ is a separable state. On the other hand, when $\psi(x, y) \neq \phi(x)\varphi(y)$, $d > 1$ hence $Tr(\hat{\rho}_2^2) < 1$, we get $E > 0$. Thus, Eq. (3) is a proper quantity to measure on whether the two subsystems are entangled. When $d \to \infty$, the lower bound of $Tr(\hat{\rho}_2^2)$ can be 0.

Eq. (A1) is essentially the Schmidt decomposition of the state vector for the bipartite system. However, it is not clear whether the decomposition is applicable when $d \to \infty$. Furthermore, it is very difficult to find the analytic solution of the decomposition in order to use Eq. (A4). Practically, we still rely on numerical method to calculate $Tr(\hat{\rho}_2^2)$.

With Eq. (11), $Tr(\hat{\rho}_2^2)$ is calculated in the $|x\rangle$ position basis, which strictly speaking is not a Hilbert space, because the norm is a delta function, i.e., $|x|x\rangle = \delta(x_i - x_j)$. We can calculate $Tr(\hat{\rho}_2^2)$ in the basis $\{|\phi_i\rangle\}$ instead, where the new basis form a truly Hilbert space since the norm is 1 by definition. Since $|\Psi\rangle_{12} = \sum_i \lambda_i |\phi_i\rangle |\varphi_i\rangle$, we can derive the reduced density operator

$$\hat{\rho}_1 = Tr_2(|\Psi\rangle_{12}\langle\Psi|) = \sum_i |\lambda_i|^2 |\phi_i\rangle \langle\phi_i|.$$  \hspace{1cm} (A6)

From this one can derive the same expression of $Tr(\hat{\rho}_2^2)$ as Eq. (A4). In other words, quantity $Tr(\hat{\rho}_2^2)$, and consequently the entanglement measure $E$, are invariant in either the position basis or the basis $\{|\phi_i\rangle\}$. This is not surprised since the transformation between the two basis are unitary. The transform matrix element for variable $x$ can be written as

$$M_{ij} = \langle x_i |\phi_j \rangle$$

$$= \{ \int \delta(x - x_i) \langle x| dx\} \int \phi_j(x')|x'\rangle dx' \}$$

$$= \int \phi_j(x) \delta(x - x_i) dx = \phi_j(x_i).$$

Similarly, $M_{ki}^\dagger = M_{ik}^\ast = \phi_k^\ast(x_i)$. Then,

$$(M^\dagger M)_{kj} = \sum_i M_{ki}^\dagger M_{ij} = \int \phi_k^\ast(x_i) \phi_j(x_i) dx_i = \delta_{kj}.$$  \hspace{1cm} (A8)

This confirm $M^\dagger M = I$ and $M$ is unitary.

Appendix B: Harmonics Oscillator

For a bipartite system that behaves like a harmonic oscillator, the potential energy is given by $V(r_1 - r_2) = \frac{1}{2} \omega^2 r_1^2 r_2^2$, where $\omega$ describes the strength of the potential.
Using the center of mass coordinate system, the wave function for ground state is given by (13) and

$$\varphi_0(\vec{r}_{12}) = \sqrt{\frac{1}{\pi^{3/2} a^3}} e^{-\frac{1}{2} \left(\frac{\vec{r}_{12}}{a}\right)^2},$$  \hspace{1cm} (B1)$$

where \(a\) is constant determined by the masses and \(\omega\). Substituting this into (13), using the same notations as for (25), and denoting \(\beta = L/(4a) = \alpha/4\), we obtain

$$Tr(\rho_1^2) = \lim_{\beta \to \infty} \left\{ \frac{\beta^6}{\pi^3} \int_{-1}^{1} d\vec{r}'_1 d\vec{r}'_2 d\vec{x}_2 d\vec{x}_2 \times e^{-2\beta^2(\gamma_1^2 + \gamma_2^2 + \gamma_1^2 + \gamma_2^2)} \right\}$$  \hspace{1cm} (B2)$$

We can perform similar Monte Carlo calculation on this twelve-dimensional integration. But fortunately, this integration can be calculated analytically, so that the result can be used to check the accuracy of Monte Carlo integration. First, we expand \(\gamma_2^2 = (\vec{x}_1 - \vec{x}_2)^2 + (\vec{y}_1 - \vec{y}_2)^2\), where we denote dimensionless variables \(\vec{x}_1 = x_1/L, \vec{x}_2 = x_2/L\) and so on. Also noted that \(d\vec{r}'_1 = d\vec{x}_1 d\vec{y}_1 d\vec{z}_1\), (B2) can be simplified into

$$Tr(\rho_1^2) = \lim_{\beta \to \infty} \left\{ \frac{\beta^6}{\pi^3} \int_{-1}^{1} d\vec{r}'_1 d\vec{r}'_2 d\vec{x}_2 d\vec{x}_2 \times e^{-2\beta^2(\gamma_1^2 + \gamma_2^2 + \gamma_1^2 + \gamma_2^2)} \right\}$$  \hspace{1cm} (B3)$$

where

$$J(\vec{x}_2, \vec{x}_2') = \int_{-1}^{1} e^{-2\beta^2((\vec{x}_1 - \vec{x}_2)^2 + (\vec{y}_1 - \vec{y}_2)^2)} d\vec{x}_1.$$  \hspace{1cm} (B4)$$

Expanding the exponent in the integral for the \(J\) function, \((\vec{x}_1 - \vec{x}_2)^2 + (\vec{y}_1 - \vec{y}_2)^2 = 2(\vec{x}_1 - \vec{x})^2 + (\vec{x}_2 - \vec{x}_2')^2/2\) where \(\vec{x} = (\vec{x}_1 + \vec{x}_2)/2\),

$$J(\vec{x}_2, \vec{x}_2') = e^{-\beta^2((\vec{x}_2 - \vec{x}_2)^2 + (\vec{y}_2 - \vec{y}_2)^2)} e^{\frac{-\beta^2}{2} (\vec{x}_2 - \vec{x}_2)^2} \times$$  \hspace{1cm} (B5)$$

$$[Erf(2\beta(\vec{x}_2 - \vec{x})) - Erf(\beta(1 + \vec{x})].$$

Here \(Erf(x) = \pi^{-1/2} \int_{-\infty}^{x} e^{-t^2} dt\) is the error function. Since \(-1 < Erf(x) < 1\), the difference of the error function at two arbitrary values \(a\) and \(b\) is, \(-2 < \{Erf(a) - Erf(b)\}\). Thus, \(J^2(\vec{x}_2, \vec{x}_2') < \frac{1}{\alpha} e^{-2\beta^2((\vec{x}_2 - \vec{x}_2)^2)}\). Plug this into (B3),

$$Tr(\rho_1^2) < \lim_{\beta \to \infty} \left\{ \frac{1}{4} \int_{-1}^{1} e^{-2\beta^2((\vec{x}_2 - \vec{x}_2')^2)} d\vec{x}'_2 d\vec{x}_2 \right\}.$$  \hspace{1cm} (B6)$$

Recall \(\beta = L/(4a) = \alpha/2\), \(\vec{x}'_2 = \alpha \vec{x}/2\), we rewrite (B6) as

$$Tr(\rho_1^2) < \lim_{\alpha \to \infty} K^3(\alpha),$$  \hspace{1cm} (B7)$$

where

$$K(\alpha) = \frac{1}{\alpha^2} \int_{-\alpha/\sqrt{2}}^{\alpha/\sqrt{2}} e^{-\frac{1}{\alpha^2} (t^2 - \vec{x}'_2)^2} \alpha\sqrt{\pi} \int_{-\alpha/\sqrt{2}}^{\alpha/\sqrt{2}} erf(s) ds$$  \hspace{1cm} (B8)$$

Denote \(t = (\vec{x}_2 - \vec{x}_2')/\sqrt{\alpha}, \alpha = (\alpha/2 - \vec{x}_2)/\sqrt{\alpha}, \) and using the error function again,

$$K(\alpha) = \frac{\sqrt{\pi}}{\alpha^2} \int_{-\alpha/\sqrt{2}}^{\alpha/\sqrt{2}} \int_{-\alpha/\sqrt{2}}^{\alpha/\sqrt{2}} e^{-t^2} dt \alpha\sqrt{\pi} \int_{-\alpha/\sqrt{2}}^{\alpha/\sqrt{2}} erf(s) ds$$  \hspace{1cm} (B9)$$

Note that \(Erf(s)\) is a monotonically increasing function and approaches 1 rapidly when \(s\) is a large enough number, e.g., \(Erf(2) = 0.9953, Erf(3) = 0.9999\), we can approximate the integral \(\int_{0}^{\alpha/\sqrt{2}} erf(s) ds \approx \alpha/\sqrt{2} - \delta\) where \(\delta\) is some positive finite number. Thus,

$$K(\alpha) \approx \frac{\sqrt{\pi}}{\alpha^2} \frac{\alpha}{\sqrt{2} - \delta}.$$  \hspace{1cm} (B10)$$

We ignore \(\delta\) when \(\alpha \to \infty\), and finally get the upper bound of the purity

$$Tr(\rho_1^2) < \lim_{\alpha \to \infty} \left(\frac{2\pi^{3/2}}{\alpha^3}\right) \to 0.$$  \hspace{1cm} (B11)$$

However, \(Tr(\rho_1^2) \geq 0\), we conclude that \(Tr(\rho_1^2) = 0\).

Table IV shows the Monte Carlo estimation of the twelve-dimensional integration in (B2). The last column of the table lists the value of \(K^3(\alpha)\). These results confirm that the Monte Carlo integration is consistent with the analytic results.

| \(L/\alpha\) | \(Tr(\rho_1^2)\) | Error\(\times 10^{-5}\) | FYMC \(\times 10^5\) | \((\sqrt{2}\pi/\alpha)^3\) |
|------|-------------|----------------|----------------|----------------|
| 10   | 1.25 \times 10^{-2} | 4.99 \times 10^{-4} | 8 | 1.57 \times 10^{-2} |
| 20   | 1.06 \times 10^{-3} | 6.83 \times 10^{-5} | 128 | 1.97 \times 10^{-3} |
| 40   | 1.50 \times 10^{-5} | 2.75 \times 10^{-6} | 256 | 2.46 \times 10^{-4} |
| 60   | 1.19 \times 10^{-6} | 1.41 \times 10^{-7} | 512 | 7.29 \times 10^{-5} |
| 100  | 1.31 \times 10^{-7} | 2.45 \times 10^{-8} | 1024 | 1.57 \times 10^{-5} |