LINE TRANSVERSALS IN FAMILIES OF CONNECTED SETS IN THE PLANE

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Abstract. We prove that if a family of compact connected sets in the plane has the property that every three members of it are intersected by a line, then there are three lines intersecting all the sets in the family. This answers a question of Eckhoff from 1993 [3], who proved that, under the same condition, there are four lines intersecting all the sets. In fact, we prove a colorful version of this result, under weakened conditions on the sets. A triple of sets $A, B, C$ in the plane is said to be a tight if $\text{conv}(A \cup B) \cap \text{conv}(A \cup C) \cap \text{conv}(B \cap C) \neq \emptyset$. This notion was first introduced by Holmsen in [6], where he showed that if $F$ is a family of compact convex sets in the plane in which every three sets form a tight triple, then there is a line intersecting at least $\frac{1}{8}|F|$ members of $F$.

Here we prove that if $F_1, \ldots, F_6$ are families of compact connected sets in the plane such that every three sets, chosen from three distinct families $F_i$, form a tight triple, then there exists $1 \leq j \leq 6$ and three lines intersecting every member of $F_j$. In particular, this improves $\frac{1}{8}$ to $\frac{1}{3}$ in Holmsen’s result.

1. Introduction

Let $F$ be a family of sets in the plane. We say that $F$ has property $T(r)$ if every $r$ or fewer sets in $F$ admit a line transversal, that is, there exists a line intersecting these sets. We say that $F$ is pierced by $k$ lines if there are $k$ lines in the plane whose union intersects all the sets in $F$. The line-piercing number of the family is the minimum $k$ so that $F$ is pierced by $k$ lines.

The problem of bounding the line-piercing numbers of families of compact convex sets in the plane with the $T(r)$ property has been investigated since the 1960’s. In 1964 Eckhoff [1] proved that if a family of compact convex sets satisfies the $T(4)$ property then it can be pierced by two lines. In 1974 he gave an example of a family of compact convex sets satisfying the $T(3)$ property that is not pierced by two lines [2]. Various upper bounds on the line-piercing numbers were proved when further restrictions on the sets are imposed (see [3] for more details).

For a while it was not clear whether the $T(3)$ property implies a finite universal upper bound on the line-piercing number. This question was eventually resolved in 1975 by Kramer [9], who showed that a family of compact convex sets in $\mathbb{R}^2$ with the $T(3)$ property is pierced by 5 lines. Finally, in 1993 Eckhoff [3] proved that such families are pierced by 4 lines, and asked whether this bound can be improved to 3.

Quantitative versions have been studied too. In 1980, Katchalski and Liu [7] showed the existence of a constant $0 < \alpha(r) < 1$, so that every finite family $F$ of...
convex sets in the plane with the \( T(r) \) property admits a line intersecting \( \alpha(r)|\mathcal{F}| \) of its members. In 2010 Holmsen \cite{Holmsen} showed that \( \left(\frac{2}{r(r-1)}\right)^{\frac{1}{r-1}} \leq \alpha(r) \leq \frac{r^2}{r-1} \), and in particular, \( \frac{1}{3} \leq \alpha(3) \leq \frac{1}{2} \).

In \cite{Holmsen}, Holmsen introduced the notion of tight triples. Three compact, connected sets in the plane \( A, B, C \) are said to be a \textit{tight triple} if

\[
\text{conv}(A \cup B) \cap \text{conv}(A \cup C) \cap \text{conv}(B \cap C) \neq \emptyset.
\]

We will call a family of sets in the plane a \textit{family of tight triples} if every three sets in the family form a tight triple. Note that if \( A, B, C \) has a line transversal, then it is a tight triple. Holmsen \cite{Holmsen} proved that if \( \mathcal{F} \) is a family of tight triples in which every set is compact and convex, then there is a line intersecting at least \( \frac{1}{6}|\mathcal{F}| \) members of \( \mathcal{F} \).

The sets investigated in all the above results are assumed to be convex, but the results apply also for families of connected sets. This follows from the fact that if \( S \) is a connected set in \( \mathbb{R}^2 \) and \( \ell \) is a line intersecting \( \text{conv}(S) \) then \( \ell \) must intersect \( S \). Similarly, three connected sets \( A, B, C \) form a tight triple if and only if \( \text{conv}(A), \text{conv}(B), \text{conv}(C) \) form a tight triple.

In this paper we show that the line-piercing number of a family of compact connected tight triples is at most 3. This improves the \( \frac{1}{5} \) in Holmsen’s result to \( \frac{1}{3} \).

In fact, we show a colorful version of this fact.

\textbf{Theorem 1.1.} Let \( \mathcal{F}_1, \ldots, \mathcal{F}_6 \) be families of compact connected sets in \( \mathbb{R}^2 \). If every three sets \( A_1 \in \mathcal{F}_{i_1}, A_2 \in \mathcal{F}_{i_2}, A_3 \in \mathcal{F}_{i_3}, 1 \leq i_1 < i_2 < i_3 \leq 6, \) form a tight triple, then there exists \( i \in [6] \) such that the line-piercing number of \( \mathcal{F}_i \) is at most 3.

In particular, if \( \mathcal{F} \) is a family with the \( T(3) \) property and we take \( \mathcal{F}_i = \mathcal{F} \) for \( 1 \leq i \leq 6 \), then Theorem 1.1 implies that \( \mathcal{F} \) has line-piercing number at most three. This gives an affirmative answer to Eckhoff’s question. This also gives another proof to Holmsen’s result \( \alpha(3) \geq 1/3 \).

Our main tool is the colorful version of the topological KKM theorem \cite{Gale} due to Gale \cite{Gale}. Let \( \Delta^{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geq 0, \sum_{i=1}^n x_i = 1\} \) denote the \((n-1)\)-dimensional simplex in \( \mathbb{R}^n \), whose vertices are the canonical basis vectors \( e_1, \ldots, e_n \). Let \( S_n \) be the group of permutations on \([n]\).

\textbf{Theorem 1.2} (The colorful KKM theorem \cite{Gale}). Let \( A_1^i, \ldots, A_n^i, i \in [n], \) be open sets of \( \Delta^{n-1} \), such that for every \( i \in [n] \) and for every face \( \sigma \) of \( \Delta^{n-1} \) we have \( \sigma \subset \bigcup_{j \in \sigma} A_j^i \). Then there exists a permutation \( \pi \in S_n \) such that \( \cap_{i=1}^n A_{\pi(i)}^i \neq \emptyset \).

2. Proof of Theorem 1.1

Throughout the proof, addition in integers is taken modulo 6. For \( a, b \in \mathbb{R}^2 \), let \([a, b] = \text{conv}\{a, b\}\) be the line segment connecting \( a, b \).

As is explained in \cite{Gale}, the compactness of the sets in each \( \mathcal{F}_j \) allows us to assume that \( \mathcal{F}_j \) is finite. Thus we may scale the plane so that every set in \( \mathcal{F}_j \) is contained in the unit disk \( D \) for each \( j \). Denote by \( U \) the unit circle. Let \( f(t) \) be a parameterization of \( U \) defined by \( f(t) = (\cos(2\pi t), \sin(2\pi t)) \).

A point \( x = (x_1, \ldots, x_6) \in \Delta^5 \) corresponds to 6 points on \( U \) given by \( f_1(x) = f(\sum_{j=1}^{i} x_j) \) for \( 1 \leq i \leq 6 \). Let \( l_1(x) = l_4(x) = [f_1(x), f_4(x)], l_2(x) = l_5(x) = [f_2(x), f_5(x)], \) and \( l_3(x) = l_6(x) = [f_3(x), f_6(x)] \).
For \( i = 1, \ldots, 6 \) let \( R^i_x \) be the interior of the region bounded by \( l_{i-1}(x), l_i(x) \) and the arc on \( U \) connecting \( f_{i-1}(x) \) and \( f_i(x) \) (see Figure 1). Notice that \( R^i_x = \emptyset \) when \( x_i = 0 \). Also, it is possible that some of the regions \( R^i_x \) intersect.

![Figure 1](image)

**Figure 1.** A point \( x \in \Delta^5 \) corresponds to six regions \( R^i_x \). The regions \( R^1_p, R^3_p, R^5_p \) are pairwise disjoint (on the right) or the regions \( R^2_p, R^4_p, R^6_p \) are pairwise disjoint (on the left), depending on the orientation of the triangle bounded by the lines \( l_1(x), l_2(x), l_3(x) \).

Set \( 1 \leq j \leq 6 \) and let \( A^j_i \) be the set of points \( x \in \Delta^5 \) so that \( R^i_x \) contains a set \( F \in \mathcal{F}_j \). Since the sets \( F \in \mathcal{F}_j \) are closed, \( A^j_i \) is open. If there is some \( x \in \Delta^5 \) for which \( x \notin \bigcup_{i=1}^6 A^j_i \), then since the sets in \( \mathcal{F}_j \) are connected, every set in \( \mathcal{F}_j \) must intersect \( \bigcup_{i=1}^5 l_i(x) \), and we are done. So we assume for contradiction that \( \Delta^5 = \bigcup_{i=1}^6 A^j_i \) for all \( j \). Observe that if \( x \in \text{conv}\{e_i : i \in I\} \) for some \( I \subset [6] \) then \( R^k_x = \emptyset \) for \( k \notin I \), and therefore, \( x \in \bigcup_{i \in I} A^j_i \) for all \( j \). This shows that the conditions of Theorem 1.2 hold.

Thus, by Theorem 1.2 there exists some permutation \( \pi \in S_6 \) and a point \( p = (p_1, \ldots, p_6) \in \bigcap_{i=1}^6 A^{{\pi(i)}}_i \). Therefore, each of the open regions \( R^i_p \) contains a set \( S_i \in \mathcal{F}_{\pi(i)} \), \( i = 1, \ldots, 6 \), and in particular \( R^i_p \neq \emptyset \) and thus \( p_i \neq 0 \) for all \( i \). We claim that at least one of the triples \( \{S_1, S_3, S_5\} \) or \( \{S_2, S_4, S_6\} \) is not a tight triple. To see this, note that the regions \( R^2_p, R^3_p, R^5_p \) are pairwise disjoint or the regions \( R^2_p, R^4_p, R^6_p \) are pairwise disjoint (depending on the orientation of the triangle bounded by the lines \( l_1, l_2, l_3 \), see Figure 1). Without loss of generality, we assume \( R^1_p, R^3_p, R^5_p \) are pairwise disjoint, and in this case, the three sets \( S_1, S_3, S_5 \) is not a tight triple. This is a contradiction.

3. Concluding Remarks

The proof of Theorem 1.2 implies a slightly stronger result: when each \( \mathcal{F}_i \) is finite, one can fix a point lying on one of the three piercing lines of \( \mathcal{F}_i \), as long as this point is outside \( \text{conv}(\cup_i \mathcal{F}_i) \).

A similar proof can be used to prove a colorful version of Eckhoff’s result that \( T(4) \) families are pierced by two lines.

**Theorem 3.1.** Let \( \mathcal{F}_1, \ldots, \mathcal{F}_4 \) be families of compact, connected sets in the plane such that any collection of four sets, one from each \( \mathcal{F}_i \), has a line transversal. Then for some \( i \in [4] \), \( \mathcal{F}_i \) has line piercing number at most 2.
This can be proved by associating a point in $\Delta^3$ with two lines and applying a similar argument as in the proof of Theorem 1.1.

When each of the families $\mathcal{F}_j$ is finite, one may drop the condition that the sets are compact. This is because we may replace each set $S \in \mathcal{F}_j$ with a compact, convex set $S' \subset \text{conv}(S)$ such that the resulting family is still a family of tight triples.

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