Holographic Geometries of one-dimensional gapped quantum systems from Tensor Network States.

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Abstract: We investigate a recent conjecture connecting the AdS/CFT correspondence and entanglement renormalization tensor network states (MERA). The proposal interprets the tensor connectivity of the MERA states associated to quantum many body systems at criticality, in terms of a dual holographic geometry which accounts for the qualitative aspects of the entanglement and the correlations in these systems. In this work, some generic features of the entanglement entropy and the two point functions in the ground state of one dimensional gapped systems are considered through a tensor network state. The tensor network is builded up as an hybrid composed by a finite number of MERA layers and a matrix product state (MPS) acting as a cap layer. Using the holographic formula for the entanglement entropy, here it is shown that an asymptotically AdS metric can be associated to the hybrid MERA-MPS state. The metric is defined by a function that manages the growth of the minimal surfaces near the capped region of the geometry. Namely, it is shown how the behaviour of the entanglement entropy and the two point correlators in the tensor network, remains consistent with a geometric computation which only depends on this function. From these observations, an explicit connection between the entanglement structure of the tensor network and the function which defines the geometry is provided.

Keywords: AdS-CFT Correspondence, Holography and condensed matter physics (AdS/CMT), Renormalization Group, Field Theories in Lower Dimensions
1 Introduction

The best known concrete realization of the holographic principle is the AdS/CFT correspondence [1]. This duality is generally formulated as the equivalence between the generating functionals $Z$ of two very different theories: a classical theory of supergravity defined on a $D+1$ asymptotically Anti de Sitter spacetime and a QFT living on a $D$-dimensional spacetime that is the boundary of the AdS, i.e,

$$Z_{\text{QG}}[\text{AdS}_{D+1}, \mathcal{J}(x,z)] \equiv Z_{\text{QFT}}[\partial(\text{AdS})_D, J(x)], \quad (1.1)$$

and more explicitly,

$$e^{-S_{\text{QG}}[\mathcal{J}(x,z)]} \sim \mathcal{O}\left(\int_{\mathcal{J}(x,0)\equiv J(x)} e^{-\int J(x)O(x) d^Dx} d\mathcal{J}(x)\right), \quad (1.2)$$

where $x$ are the spacetime dimensions of the QFT, $z$ is the extra-dimension of the gravity theory on AdS, $O$ are operators of the quantum field theory on the boundary, and $\mathcal{J}(x,z)$ are the classical fields of the gravitational theory whose boundary values $\mathcal{J}(x,z)|_{z\to 0}$ act as the sources $J(x)$ for the correlation functions of the QFT.

Since the initial formulations of the correspondence, a huge amount of work has been carried out in order to generalize the AdS/CFT while providing numerous valuable models to study non-perturbative effects in quantum field theory such as confinement and quantum
phase transitions. However, still there is not a first principles derivation of the correspondence that allow us to explicitly construct a bulk gravity theory from a boundary QFT [2]. Nevertheless, it is widely accepted that the AdS/CFT is at heart a geometric formulation of the renormalization group (RG), in which the renormalization scale becomes the extra radial dimension \(z\) and the beta functions of the boundary field theory are the saddle point equations of motion of the bulk gravity theory [3].

In recent years, ideas coming from quantum information have been especially relevant in establishing new numerical real space quantum renormalization group methods such as density matrix renormalization group (DMRG) [4], and tensor network states (TNS) algorithms including matrix products states (MPS) [5], projected entangled-pair states (PEPS) [6], multi-scale renormalization ansatz (MERA) [9], tensor renormalization group (TRG) [7], and tensor-entanglement-filtering renormalization (TEFR) [8]. These algorithms provide a set of variational ansätze useful to characterize the low-energy (long distance) physics of quantum many-body systems. The TNS assumes a parameterization of a many-body wave-function by means of a collection of tensors connected into a network. The number of parameters required to specify these tensors is much smaller than the exponentially large dimension of the system’s Hilbert space, allowing for the efficient representation of very large (and even infinite) systems. It has been recently proposed [11] that tensor network states can be broadly classified into two categories according to the geometry of the underlying networks.

In the first category, the network mimics the physical geometry of the system, as specified by the pattern of interactions in the Hamiltonian. The MPS ansatz for one dimensional systems, which consists of a collection of tensors connected into a chain and its generalization for higher dimensional systems (PEPS), lie on this physical geometry category.

At variance, according to [11], the collection of tensors in the second category of TNS are connected so as to parametrize the different length scales (or, equivalently, energy scales) relevant to the description of the many-body wave-function. These tensor networks organize the quantum information contained in a state in terms of different scales through a characteristic tensor connectivity which spans an additional dimension related with the RG scale, as for instance in MERA. This has been used to define a generalized notion of holography inspired by the AdS/CFT duality [10]. There, the author has made the groundbreaking observation that the tensor networks in MERA, happens to be a realization of the AdS/CFT correspondence. In [10], the dual higher dimensional holographic geometry of the original system emerges when one realizes that the tensors in MERA corresponding to a one dimensional quantum critical point, are connected so as to reproduce a discrete version of the anti-de Sitter spacetime (AdS). After the initial proposal, a substantial amount of work has appeared supporting and extending the original idea [11–23].

In terms of the AdS/CFT correspondence, the most salient feature of the holographic dual to a system with a characteristic length (energy) scale is the capping or truncation of the geometry for values of the radial coordinate close to this length scale. While the geometry remains approximately AdS for values of the radial coordinate smaller than the characteristic length scale, the capping of the geometry implies an infrared (IR) fixed point.
at a finite energy scale for the original CFT lying at the boundary of the AdS region, thus yielding the dual for a massive deformation of the theory.

It is thus expected that the state at the IR scale will be a non entangled state if all the correlations vanish for it i.e there is no topological order. However, it also might be the case of nontrivially entangled IR states in some circumstances, which could be due to some underlying topological order. In [16], authors speculated, in terms of the AdS/MERA duality, about the structure of a tensor network that might suitably describe both possibilities.

The aim of this paper is to offer new insights on the connection between the structure of MERA states and their potential holographic descriptions by exploring some hints posed in [16]. Here it is shown that the entanglement entropy and the two point functions in a type of hybrid tensor network state (composed by a finite number of MERA layers and an MPS acting as a cap layer) representing a one dimensional quantum many body system, remain consistent with a geometric computation of both quantities once a sensible emergent metric associated to the tensor network is provided. This metric corresponds to an asymptotically AdS geometry with an IR capping region characterized in terms of a function which keeps track of the growth of minimal surfaces near the capped region of the geometry. From these observations, an explicit connection between the entanglement structure of the tensor network and the function which defines the IR geometry is provided.

The paper is organized as follows: In section two, the MPS and MERA tensor network states are briefly reviewed and the hybrid tensor network composed of a finite number of MERA layers and a capping MPS on the top layer is presented. Section three reviews, the AdS/MERA conjecture and its relation with the formula for the computation of the holographic entanglement entropy [24]. In section four, the generic features of the entanglement entropy and the two point functions computed with the hybrid tensor network presented in section two are constrained with the same quantities computed through holographic calculations in sensible-looking geometries that may be associated to the tensor network. Finally, in section five, results are summarized and future research problems are exposed.

2 Tensor Network States for 1D systems

Tensor network states (TNS) constitute a new class of numerical methods that aim to efficiently describe ground states of strongly correlated quantum systems. TNS implement techniques drawed from renormalization group methods and the knowledge about the structure of entanglement in the ground states of quantum many body systems. In this section we briefly review some issues concerning the two classes of tensor network states (MPS and MERA) used in this paper in order to offer further evidence on the connection between AdS/CFT and MERA.
2.1 Matrix Product States

Let us first consider a 1-D translationally invariant quantum many body system composed by \( N \) sites of dimension \( d \). Its ground state can be expressed as

\[
|\Psi\rangle = \sum_{s_1,s_2,\ldots,s_N=1}^{d} T_{s_1,s_2,\ldots,s_N} |s_1, s_2, \ldots, s_N\rangle , \tag{2.1}
\]

where for \( j \in \{1, \cdots, N\} \) and \( s_j \in \{1, \cdots, d\} \), the vectors \( |s_j\rangle \) form the computational basis of the \( j \)-th system site and where the type - \( (0_N) \) tensor \( T_{s_1,s_2,\ldots,s_N} = \langle s_1, s_2, \cdots, s_N|\Psi\rangle \) represents the associated probability amplitudes. The MPS representation of this state is given by the ansatz \([5]\),

\[
T_{s_1,s_2,\ldots,s_N} = \text{Tr} (A^{s_1} A^{s_2} \cdots A^{s_N}) , \tag{2.2}
\]

where \( A^{s_j} \) are matrices of dimension \( W \) with elements \( A^{s_j}_{\alpha,\beta} \) whose values are determined by means of a variational search procedure. Away from the critical point, a finite \( W \) suffices to exactly describe the ground state. Essentially, the so called bond dimension \( W \), controls how much entanglement between adjacent sites is retained during the variational procedure. Treating the matrices \( A^{s_j} \) as \( \left( \frac{1}{2} \right) \) tensors (one extra index is due to the physical site), the MPS ansatz can be pictorially described as in figure 1.

Regarding the computation of observables with the MPS ansatz, given the state (2.1) with coefficients (2.2), the expectation value of an operator \( \Theta = \mathcal{O}_1 \otimes \mathcal{O}_2 \otimes \cdots \otimes \mathcal{O}_N \) which is the tensor product of local operators \( \mathcal{O}_j \) for each site \( j \), can be expressed as (Fig.2),

\[
\langle \Psi|\Theta|\Psi\rangle = \text{Tr} (E_{\mathcal{O}_1} E_{\mathcal{O}_2} \cdots E_{\mathcal{O}_N}) . \tag{2.3}
\]

The transfer matrices \( E_{\mathcal{O}_j} \) are defined by,

\[
E_{\mathcal{O}_j} = \sum_{s_j,s'_{j}=1}^{d} \langle s'_j|\mathcal{O}_j|s_j\rangle \left( A^{s_j} \otimes \bar{A}^{s'_j} \right) , \tag{2.4}
\]
Figure 2. Pictorial representation of an expectation value computation through the MPS ansatz given by equations (2.3) and (2.4).

and $\bar{A}^{s\beta}$ is the hermitian conjugate of $A^{s\beta}$. As a result, as posed in equation (2.3), the computation of expectation values with MPS, may be viewed as the total contraction of a 1-dimensional array of tensors ($E_J$).

It is well known that an MPS with a finite bond dimension $W$, faithfully represents the ground state of a 1D gapped system and supports the exponential decay of correlations expected for these models. Considering a translational invariant matrix product state $|\Psi\rangle$, homogeneously defined by the matrices $A_s$, the two point correlation function of two local observables $\Theta_\alpha$ and $\Theta_\beta$, with support on sites $s_i$ and $s_j$ separated an arbitrary distance $\ell + 1$ is defined by,

$$C_{\ell+1}^{\alpha \beta} \equiv \langle \Psi| \Theta_\alpha(s_i) \Theta_\beta(s_j)|\Psi\rangle = \text{Tr} \left( E_{s_i}[s] \cdots E_{s_j}[s] \cdots E_{N}[1] \right). \quad (2.5)$$

Following [25], the correlator in the limit when $N \to \infty$ can be written as:

$$C_{\alpha \beta}^{\ell+1} = \sum_{\nu \geq 2} c_\nu (\lambda_\nu)^\ell, \quad (2.6)$$

where $\lambda_{\nu \geq 2}$ are the eigenvalues of $E_1$ for which it holds that $|\lambda_{\nu \geq 2}| < 1$. The coefficients $c_\nu$ are given by,

$$c_\nu = \langle L_1 | E_{\Theta_\alpha}^{[s_i]} | R_\nu \rangle \langle L_\nu | E_{\Theta_\beta}^{[s_j]} | R_1 \rangle, \quad (2.7)$$

where $|R_\nu\rangle$ and $\langle L_\nu|\rangle$ are the right and left eigenvectors of $E_1$ for the eigenvalues $|\lambda_\nu| < 1$ and $|R_1\rangle$ and $\langle L_1|\rangle$ are the right and left eigenvectors of $E_1$ for the eigenvalue $\lambda_1 = 1$.

Since every $|\lambda_{\nu \geq 2}| < 1$, then the correlator $C_{\alpha \beta}^{\ell+1}$ decay exponentially because it is possible to write its leading behaviour as a superposition of exponentials with decay lengths defined by,

$$\xi_\nu = -\frac{1}{\log |\lambda_\nu|}. \quad (2.8)$$

This implies that the MPS gapped nature dominates when probing long range correlations, which are ruled by the eigenvalues $\lambda_\nu$ of the identity transfer matrix $E_1$. 

Furthermore, if one computes the entanglement entropy of a subsystem $A$ within an 1D-MPS state, bipartitioning the system into two subsystems $A$ and $B$, and then taking the partial trace over the sites on $B$, it is easy to convince oneself that,

$$S_A \leq 2 \log W,$$

as one realizes that the bipartition in a 1-dimensional MPS only involves two bond indices, which maximal contribution to $S_A$ is given by $\log W$. Thus, $S_A$ does not scale with the size of the subsystem $A$ despite it can be made arbitrarily large by increasing $W$. As the entanglement entropy in 1D critical systems scales proportional to $\log \ell$, where $\ell$ is the size of the subsystem $A$ [26, 27], then, from the bound (2.9) one clearly sees that an MPS does not support the large amount of entanglement required in quantum critical systems.

2.2 Entanglement renormalization tensor network states. MERA

The MERA representation of the 1-D state (2.1) assumes a decomposition of $T_{s_1,s_2,\ldots,s_N}$ in terms of a collection of smaller, finite size tensors, which differently from the linear MPS structure, are organized in a two-dimensional layered graph. The sites of the graph represent tensors. They are divided in two groups: the type $\chi$ tensors with elements $\chi_{s_1,s_2}$ called disentanglers and the type $\Lambda$ tensors with elements $\Lambda_{s_1,s_2}$ called isometries (Fig. 3). For quantum critical systems, a MERA tensor network possess a characteristic scale invariant structure, i.e, a unique $\chi$ and a unique $\Lambda$ precisely define the MERA graph.

The MERA representation of (2.1) implements an efficient real space renormalization group procedure through a tensor network organized in different layers labelled by $\tau$. Each layer of MERA defines a RG transformation: prior to the renormalization of a block of typically two sites located at layer $\tau$ into a single site by means of a $\Lambda^\tau$-type tensor, short range entanglement between the sites is removed by means of the disentangler $\chi^\tau$. In other words, the MERA tensor networks, as a consequence of the locality of physical interactions in the ground state of many-body systems, at each layer, decouple the relevant low energy degrees of freedom from the high energy ones, which are then safely removed, by unitarily transforming with disentanglers small regions of space. The MERA coarse-graining transformation induces an RG map that can be applied arbitrarily many times keeping constant the computational cost for obtaining consecutive effective theories for the original system which are also labelled by $\tau$.

For 1D systems, the MERA computation of an observable $\Theta$ requires the contraction of a 2D tensor network, and this may be efficiently done by the isometricity requirements for $\chi$ and $\Lambda$ tensors (see figure 4). Indeed, in an scale invariant MERA, contracting an entire layer of MERA and its conjugate, maps the original expectation value problem into a new one i.e,

$$\langle \Psi | \Theta | \Psi \rangle = \langle \Psi' | S(\Theta) | \Psi' \rangle = \langle \Psi' | \Theta^{[1]} | \Psi' \rangle,$$

where the quantum state $| \Psi' \rangle$ is the original MERA state $| \Psi \rangle$ after the full contraction of the tensors at the bottom layer with their conjugates, the observable $\Theta^{[1]} = S(\Theta)$ is the effective tensor shown inside the rectangle in figure 4 (right), and $S$ is the scaling
Figure 3. Pictorial representation of two layers (τ and τ + 1) of an scale invariant MERA tensor network. Three legged triangles represent isometries Λ and four-legged circles represent disentanglers χ. At each layer τ, the tensors are chosen so as to fulfill χ†χ = I and Λ†Λ = I.

superoperator \(^1\) of the observable Θ. The total contraction of the tensor network proceeds by mapping the effective observable in a sequential way as,

\[ Θ[τ] → Θ[τ+1] = S(Θ[τ]) , \]  

until the top of the tensor network |C⟩ is reached, for which it holds that \(⟨Ψ|Θ|Ψ⟩ = ⟨C|Θ^h|C⟩\), with \(h = \log_2 N\) and \(N\) being the total size of the system.

One of the most salient features of an scale invariant MERA tensor network is that it naturally supports the well known power-law decaying of correlation functions in systems at the quantum critical point \([29]\). To show this, it is convenient to define the fixed point one site scaling superoperator \(S ≡ S(1)\) as,

\[ S(1)(Φ) = \sum μ_α Φ_α Tr(Φ_α ∘) , \]  

where the scaling dimensions of the scaling operators \(Φ_α\) of the theory are \(Δ_α ≡ −\log_2 μ_α\) and \(Tr(Φ_α Φ_β) = δ_{αβ}\). Let us consider the correlator of two scaling operators \(Φ_α(s_i)\) and \(Φ_β(s_j)\) initially located at sites \(s_i\) and \(s_j\) of the original lattice,

\[ C_{αβ} = ⟨Ψ|Φ_α(s_i) Φ_β(s_j)|Ψ⟩ = ⟨Φ_α(s_i) Φ_β(s_j)⟩ . \]  

After \(τ^* = \log_2 |s_i − s_j|\) contractions of the type described above, the sites supporting the original \(Φ_α\) and \(Φ_β\) become first neighbors \([29]\) and each iteration contributes a factor \(μ_α μ_β\) giving,

\[ C_{αβ} = ⟨C^*|(S(1)(Φ_α(s_i)))^{τ^*} (S(1)(Φ_β(s_j)))^{τ^*}|C^*⟩ = (μ_α μ_β)^{τ^*} ⟨C^*|Φ_α(0) Φ_β(1)|C^*⟩ , \]  

\(^1\)A superoperator is a linear operator acting on a linear space of linear operators. The prefix super- have no connection to supersymmetry or superalgebras.
Figure 4. To compute the expectation value of an observable $\Theta$ (blue squares) supported on an interval of sites, $\{s_1..s_k\}$ with a 1D MERA, the first layer of disentanglers $\chi^{[1]}$, for which, by definition, holds $\chi^\dagger \chi = I$ is initially considered. Those $\chi^{u_1..u_2}_{r_1..r_2}$ whose (both) physical indices do not belong to the support interval $\{s_1..s_k\}$, are trivially contracted with their adjoints. However, this contraction is not trivial for those disentanglers directly connected to the support. The same argument holds for the isometries $\Lambda$ belonging to the first layer of the MERA state. 

where the state $|C^*\rangle$ is the result of $\tau^*$ contractions of the original MERA state with its conjugate. Defining $\mathcal{C}^*_{\alpha\beta} \equiv \langle C^*|\Phi_{\alpha}(0)\Phi_{\beta}(1)|C^*\rangle$, noticing that $\Delta_{\alpha,\beta} \equiv -\log_2 \mu_{\alpha,\beta}$ and recalling the identity $a^{\log_2 b} = b^{\log_2 a}$, the two point function (2.13) can be written as,

$$\langle \Phi_{\alpha}(s_i) \Phi_{\beta}(s_j) \rangle = \frac{\mathcal{C}^*_{\alpha\beta}}{|s_i - s_j|^\eta}, \quad (2.15)$$

with $\eta = (\Delta_{\alpha} + \Delta_{\beta})\delta_{\alpha\beta} = 2\Delta_{\alpha}$.

2.3 Hybrid MERA-MPS networks

When considering the ground state $|\Psi\rangle$ of gapped 1D-Hamiltonian $\mathcal{H}$, the correlations decay exponentially at large distances due to the characteristic correlation length $\xi$, while typically keep power-law decaying for distances smaller than $\xi$. To emulate this behavior with a tensor network state, it is necessary to have an ansatz which suitably reproduces these features. A potential candidate for this, is a tensor network state with an MPS at the top of a finite number ($\tau_0 \sim \log_2 \xi$) of MERA layers [11]. Here it is assumed that these $\tau_0$ layers correspond to those in the scale invariant MERA describing a neighbouring critical point while the capping MPS accurately describes the behaviour of the long distance correlations in the gapped phase.

In the MERA representation of the ground state of a gapped Hamiltonian, after $\tau_0 \approx \log_2 \xi$ renormalization steps, the original ground state $|\Psi_0\rangle$ of the system has flowed into a state $|\Omega\rangle$ that can be well approximated by a product state with no entanglement.
between the different coarse grained lattice sites. This state describes the ground state at a fixed-point of the RG flow corresponding to a gapped phase without topological order. Nevertheless, the finite layered MERA may also be combined with another tensor network (an MPS in figure 5) in order to represent the ground states of gapped systems that flow towards non trivially entangled IR fixed point ground states $|\Phi\rangle$.

The MPS tensor at the top layer of this hybrid MPS-MERA state, accounts for the gapped character and non trivial long range entanglement of the IR fixed point in a natural way, i.e, it naturally arranges the exponential decay of the two point correlations at distances larger than $\xi$, while the MERA 'curtain' of $\tau_0$ layers implements the power-law decaying behaviour of correlators when the distance between the physical sites is smaller than $\xi$. The case of the trivial non-entangled product IR state $|\Omega\rangle$ is represented by an MPS with bond dimension $\mathcal{W} = 1$.

In the MPS-MERA tensor network, the total system size $N$ is given by,

$$N = 2^{\tau_0} m, \quad (2.16)$$

where $m$, is the number of effective MPS sites located at the top of the $\tau_0$-th layer of the binary MERA 'curtain' hanging from the MPS tensor. Regarding the entanglement that may be supported by these type of tensor states, as each "site" in the top MPS represents a cluster of $\sim 2^{\tau_0}$ coarse grained strongly correlated sites of the original lattice, then the entanglement between different sites in the MPS top tensor represents the short-range entanglement between these clusters. If the MPS top tensor is not the trivial product state, its short-range entanglement then describes a particular pattern of long-range entanglement between distant sites of the original lattice. For finite $\mathcal{W} > 1$, this pattern of long-range entanglement, which relates with some kind of topological order [30, 31], clearly differs from the characteristic pattern appearing in quantum critical systems described by a non-capped scale invariant MERA network, while providing a non zero contribution to the total entanglement entropy of a region $A$ in the original lattice.
In the hybrid MERA-MPS tensor network, the local unitary transformations carried out by the disentanglers of the MERA "curtain" only remove the short range entanglement, between the neighboring coarse grained sites lying on the curtain. Nevertheless, the pattern of entanglement between distant sites of the original lattice encoded in the short range entanglement between clusters, is not removed by the MERA disentanglers. It is precisely this pattern of long range entanglement which entails the non trivial topological order of the IR fixed point state represented by the MPS at the top of the MERA curtain.

3 The AdS/MERA duality

In [10], it was firstly observed that MERA [9] happens to be a realization of the AdS/CFT correspondence [1]. As pointed out in [10], from the entanglement structure of a quantum critical many body system, is possible to define a higher dimensional geometry in which, apart from the coordinates labelling the position and the time \( t \), one may add a "radial" coordinate \( z \) labelling the hierarchy of scales. Then, the higher dimensional geometry emerging from MERA may be visualized by locating cells around all the sites of the tensor network representing the quantum state. The size of each cell is defined to be proportional to the entanglement entropy of the site in the cell. Through this procedure, a geometric dual picture of the tensor network arises quite naturally from the entanglement of the degrees of freedom of the critical system lying on the boundary [12, 20].

The discrete geometry emerging at the critical point is a discrete version of AdS.

For a one-dimensional quantum critical system with a space coordinate labelled by \( x \), the continuous isometry \( w \rightarrow w + \theta, \ x \rightarrow e^{\theta} x \) of the metric

\[
ds_{\text{AdS}}^2 \sim dw^2 + e^{-2w} (dx^2 - dt^2)
\]

(3.1)

is replaced by the MERA’s discretized version, \( w \rightarrow w + k, \ x \rightarrow 2^k x \) or \( x \rightarrow 3^k x \) depending on the binary or ternary implementation of the renormalization algorithm [9]. Here, the extra direction \( w \) in the AdS is identified with \( \tau \) (the variable labelling the number of renormalization steps) in MERA. To be more concrete, the \((D + 1)\)-dimensional AdS metric can be written in the form,

\[
ds_{\text{AdS}}^2 = \frac{L_{\text{AdS}}^2}{z^2} (dz^2 + dx^2 - dt^2)
\]

(3.2)

where, \( L_{\text{AdS}} \) is a constant called the AdS radius; it has the dimension of a length and it is related with the curvature of the AdS space. We have also made the change of variable \( \tau = \log z \), where \( z \) is the standard radial coordinate of the Poincaré AdS. With this choice of the spacetime coordinates, the one dimensional quantum critical system lies at the boundary \((z = 0)\) of the bulk geometry.

The most convenient way to establish the AdS/MERA connection is to compare the procedures to compute the entanglement entropy in both cases. In the classical gravity limit of AdS/CFT, Ryu and Takayanagi (RT) derived a celebrated formula yielding the entanglement entropy of a region \( A \) provided that the (boundary) conformal field theory describing the critical system admits an holographic gravity dual [24]. In the RT approach,
the entanglement entropy is obtained from the computation of a minimal surface in the
dual higher dimensional gravitational geometry (bulk theory); as a result, the entanglement
entropy $S_A$ in a CFT$_D$ is given by the area law relation,

$$S_A = \frac{\text{Area}(\gamma_A)}{4G_N^{(D+1)}},$$

(3.3)

where $D$ is the number of spacetime dimensions of the boundary CFT, $\gamma_A$ is the $D$-
dimensional static minimal surface in AdS$_{D+1}$ whose area is given by $\text{Area}(\gamma_A)$ and $G_N^{(D+1)}$
is the $D+1$ dimensional Newton constant. In the RT proposal, looking for the minimal
surface $\gamma_A$ separating the degrees of freedom contained in region $A$ from those contained
in the complementary region $B$, amounts to search for the severest entropy bound on the
information hidden in the AdS$_{D+1}$ region related with $B$. Despite the RT formula has not
been rigorously proven its validity is supported by very reassuring evidence [24, 32, 33].

In this paper, the interest will be mainly focused in the $D = 2 = (1+1)$ case, for which the equation (3.3) reduces to,

$$S_A = \frac{\text{Length}(\gamma)}{4G_N^{(3)}}.$$  

(3.4)

Namely, there is a striking similarity between the formula (3.4) and the computation of
the entanglement entropy of a region $A$ with MERA. The entanglement entropy in MERA
is fixed by the structure of the causal cone [9, 28]. The causal cone $\mathcal{CC}(A)$ of a region $A$
of $\ell$ sites, is determined by all the disentanglers and isometries along all the levels of
the tensor network, which are directly connected with the sites in the original region $A$, (see figure 6). As a result, to compute the entropy $S_A$ it is necessary to trace out any
site within the tensor network which does not lie in $\mathcal{CC}(A)$. The boundary of $\mathcal{CC}(A)$
is a curve $\tilde{\gamma}_A = \partial \mathcal{CC}(A)$. The length of $\tilde{\gamma}_A$ counts the number of bonds in MERA linking
the $\mathcal{CC}(A)$ with the rest of the sites and, when multiplied by the maximal contribution to
entanglement entropy given by each bond lying in $\tilde{\gamma}_A$, it provides an upper bound for the
entropy $S_A$ [9–11],

$$S_A \leq \text{Length}(\tilde{\gamma}_A) \log W,$$  

(3.5)

where $W$ is the dimension of the auxiliary space of disentanglers and isometries. For instance, for disentanglers $\chi_{v_1 v_2}, u, v_1, v_2 \in \{1, \cdots, W\}$ and $u_1, u_2 \in \{1, \cdots, W\}$.

The close connection with the RT formula emerges as one realizes that $\tilde{\gamma}_A$ can be
regarded as the minimal curve $\gamma_A$ in the RT proposal, since it counts the minimal number
of bonds which must be considered in order to define the bipartition in the MERA tensor
network. Indeed, the minimal curve $\tilde{\gamma}_A$ in an optimized scale invariant MERA network
[11] saturates the bound given in (3.5); this has been confirmed by explicit computation in
one dimensional critical systems, where it has been shown that [9],

$$S_A \approx k \cdot \log \ell.$$  

(3.6)

where $k$ is a constant of order one. Since $\tilde{\gamma}_A$ is defined as the boundary of the $\mathcal{CC}(A)$,
it can be interpreted as an holographic screen which optimally separates the region in
the MERA tensor network related with the degrees of freedom of $A$, from those in its complementary region $B$.

Despite a complete characterization of the AdS/MERA duality is still lacking, very recent advances point to fulfill this objective. In [20], authors have used a continuous version of MERA (cMERA, [21]), the holographic formula for the entanglement entropy and the concept of distance between quantum states to address one of the major issues concerning the AdS/MERA conjecture, i.e., how gravity duals arise from the structure of the quantum entanglement of the quantum critical systems under consideration. The question about the role of the large $N$ limit in the AdS/MERA duality is also considered in [20] as well as in [22], where it is shown how a number of features of the holographic duality in the large $N$ limit emerge naturally from entanglement renormalization tensor networks.

4 Holographic geometries for hybrid MERA-MPS networks

It is widely accepted that, non-local observables such as two-point functions, entanglement entropy and Wilson loops turn out to be crucial for elucidating the subtleties of the AdS/CFT correspondence [1]. In this section, we propose a set of sensible ans"atze geometries which provide a consistent behaviour between the holographic entanglement entropy (3.4) and the entanglement entropy computed through the hybrid MERA-MPS tensor network presented in section (2.3).

The main features of the truncated hybrid MERA-MPS network, are the characteristic length scale $\xi$ which fixes the depth of the curtain and the MPS tensor at the top of the curtain which accounts for the gapped nature and the non-trivial entanglement of the IR fixed point. On the other hand, the most salient feature of the holographic geometry dual to a system with a characteristic length scale $z_0$, is the capping or truncation of the geometry for values of the radial coordinate close to this length scale. The geometry remains approximately AdS for values of the radial coordinate smaller than the characteristic length.
scale and the capping of the geometry implies an infrared (IR) fixed point at a finite energy scale ($\sim 1/z_0$).

### 4.1 A geometric ansatz for the hybrid tensor network

In the context of AdS/MERA, it is natural that the ansatz metric to be associated to the hybrid tensor network corresponds to an asymptotically AdS$_3$ spacetime with an IR capped region located at $z_0$. In the Poincaré patch, this metric, in the constant time slice, can be written as,

$$ds_0^2 = \frac{dz^2}{A(z)^2} + B(z)^2 dx^2. \quad (4.1)$$

In these coordinates, the boundary of the spacetime lies at $z = 0$ and, requiring that (4.1) to be asymptotically AdS$_3$ amounts to impose that, the functions $A(z)$ and $B(z)$ which only depend on the radial coordinate $z$, behave as,

$$A(z)^2 \sim \frac{z^2}{L^2_{\text{AdS}}} \quad B(z)^2 \sim \frac{L^2_{\text{AdS}}}{z^2}, \quad (4.2)$$

when $z \to 0$. The capping of the metric at a finite length scale $z_0$ is accomplished through the function $f(z)$,

$$f(z) = 1 + Q \left( \frac{z}{z_0} \right) \log \left( \frac{z}{z_0} \right) - \left( \frac{z}{z_0} \right)^2, \quad (4.3)$$

which defines (4.1) by making,

$$A(z)^2 = \frac{z^2 f(z)}{L^2_{\text{AdS}}} \quad B(z)^2 = L^2_{\text{AdS}} \frac{f(z)}{z^2}. \quad (4.4)$$

The function $f(z)$ is always positive and does not posses any singularity or zero as long as $0 < z < z_0$ and the values of the parameter $Q$ are restricted to lie within $0 \leq Q \leq 2$. The asymptotic requirements for (4.1) are immediately fulfilled by noting that $f(z) \to 1$ when $z \to 0$, while the capping of the geometry follows from $f(z_0) = 0$.

As it will be shown, the function $f(z)$ controls the growth of the minimal surfaces appearing in the holographic computations of entanglement entropy and two point functions near the capped region of the geometry. In other words, different values of the parameter $Q$ give rise to different IR behaviours for the entropy and the correlators.

As commented in section (2.3), different configurations of the hybrid MERA-MPS tensor network result in significant distinct behaviours of the long range entanglement and the correlations. In our approach, once a sensible ansatz such as (4.1) is provided, the problem is to show that the behaviour of the entropy and correlations in different arrangements of the tensor network are consistent with a geometric computation which only depends on $f(z)$ [22].

This approach differs from the one posed in [20]. Instead of our consistency-inverse-like problem, authors take a more direct approach and use cMERA [21] and the concept of distance between quantum states, to compute a quantum metric which then is identified with
an holographic metric. The identification uses the connection between the entanglement entropy in MERA and the holographic entanglement entropy commented in section (3). In the case of a free scalar theory with mass $m$, this procedure yields a spatial part for the associated bulk metric given by,

$$
 ds_0^2 = \frac{dz^2}{4z^2} + \frac{1}{z^2} \left[ 1 - \left( \frac{z}{z_0} \right)^2 \right] dx^2 ,
$$  \hspace{1cm} (4.5)

with $z_0 \equiv 1/m$. The geometry is capped off at $z_0$, which is consistent with the mass gap in the scalar field theory, and the vanishing of the metric at $z_0$ is controlled by a function which coincides with our anstaz (4.3) for $Q = 0$. It is necessary to recall that in the case of such a weakly coupled theory, it might be expected that the gravity dual (4.5) cannot be related to a solution of Einstein gravity in three dimensions. Nonetheless, the procedure provides a metric which describes the cMERA structure in a qualitative way.

Finally, it is worth to note that in [34], it has been proposed an algorithm to find stationary solutions to 3D Einstein gravity with a self-interacting scalar field. The algorithm bootstraps the solutions by requiring a geometric input in terms of a function which is close in spirit to (4.3). The appropriate choices for the input function are subjected to severe physical constraints and it would be the matter of a future study to use the algorithm to investigate if (4.3) may be related with some of these solutions.

### 4.2 Geometric computation of Entanglement Entropy and Correlators

In order to obtain the entanglement entropy of an static region $A$ lying at the boundary of an asymptotically AdS spacetime, the Ryu and Takayanagi proposal [24] accomplishes the task through the computation of an static surface of minimal area in the bulk geometry. In AdS$_3$, this means a static curve of minimal length $\gamma$, i.e a geodesic. Thus, let us consider the region $A$ in the boundary of (4.1) given by an interval with length $\ell$ along $x$ direction. Choosing the origin in the center of this interval, the symmetry of the problem allows us to restrict to curves described by an even function $z = z(x)$ embedded in the constant time slice of (4.1). The metric induced on those curves reads

$$
 ds_{\text{ind}}^2 = \left( \frac{\dot{z}^2}{A(z)^2} + B(z)^2 \right) dx^2 ,
$$  \hspace{1cm} (4.6)

where $\dot{z} = dz/dx$. To compute the length of the minimal curve $\gamma$ amounts to integrate $\sqrt{\det (g_{\text{ind}})}$ over the interval $A$ i.e,

$$
 \text{Length}(\gamma) = 2 \int_0^{\ell/2} \sqrt{\det (g_{\text{ind}})} \, dx = 2 \int_0^{\ell/2} dx \sqrt{\frac{\dot{z}^2}{A(z)^2} + B(z)^2} ,
$$  \hspace{1cm} (4.7)

where $g_{\text{ind}}$ is the induced metric on the static curve $\gamma$.

Considering the integrand of (4.7) as a Lagrangian density $\mathcal{L}[\dot{z}, z]$, one notices that it does not explicitly depends on $x$. Namely, the independence of $\mathcal{L}[\dot{z}, z]$ on $x$ leads to the conserved quantity $\mathcal{H} = p_z \dot{z} - \mathcal{L}$, where $p_z = \partial \mathcal{L}/\partial \dot{z}$. In particular, one gets
\[ H = -\frac{B(z)}{\sqrt{1 + (\dot{z}/A(z)^2 B(z)^2)}}. \] (4.8)

Since at the maximum value of the radial coordinate \( z_{\text{max}} \), we have \( \dot{z} = 0 \), the constancy of \( H \) can be expressed by settling that \( H^2 = B(z_{\text{max}})^2 \) which allow us to write (4.7) as,

\[ \text{Length}(\gamma) = -2 \int_{z_{\text{max}}}^{\epsilon} \frac{B(z) \, dz}{A(z) \sqrt{B(z)^2 - B(z_{\text{max}})^2}}, \] (4.9)

where \( \epsilon \) is an UV-cutoff such that the integration domain of the radial coordinate constricts to \((\epsilon, z_{\text{max}})\). Furthermore, it is worth to express the relation between the length of the interval \( A \) as a function of \( z_{\text{max}} \) given by,

\[ \ell_{\text{max}} \equiv \ell(z_{\text{max}}) = -2 \int_{z_{\text{max}}}^{\epsilon} \frac{B(z_{\text{max}}) \, dz}{A(z) B(z) \sqrt{B(z)^2 - B(z_{\text{max}})^2}}. \] (4.10)

The equation (4.9) allow us to compute the entanglement entropy of the boundary interval \( A \) using the RT formula (3.4). In addition, the two point functions of scaling operators \( \Phi \) inserted at two spacelike separated points on the boundary of AdS are also considered. According to [1], this correlator \( \mathcal{C}_\Phi^{[D]} \), is obtained from the spacetime propagator \( \mathcal{C}_\text{Holo}^{[D]} \) between the corresponding points on the UV-cutoff boundary lying at \( z \sim \epsilon \) by simply writting,

\[ \mathcal{C}_\Phi^{[D]} = \epsilon^{-2\Delta} \mathcal{C}_\text{Holo}^{[D]}, \] (4.11)

where \( D \) is the distance between the inserted operators on the boundary and \( \Delta \) is the scaling dimension of the operator \( \Phi \). The holographic propagator \( \mathcal{C}_\text{Holo}^{[D]} \), in the leading order of its semi-classical approximation, is given by a sum over geodesics \( \Upsilon_D \) connecting the points on the cut-off boundary,

\[ \mathcal{C}_\text{Holo}^{[D]} = \sum_{\Upsilon_D} \exp \left[ -m \text{Length}(\Upsilon_D) \right] \sim \exp \left[ -m \text{Length}(\gamma_D) \right]. \] (4.12)

Here, \( \text{Length}(\gamma_D) \) represents the length of the shortest geodesic connecting the two boundary points and \( m \) is the mass of the fields \( \phi \) dual to the operators \( \Phi \). Namely, in the AdS/CFT correspondence, the scaling dimensions of boundary operators are related to the masses \( m \) of their dual fields. For instance, when \( \Phi \) is an scalar operator of large scaling dimension \( \Delta \), then it holds that \( \Delta \approx m L_{\text{AdS}} \) [1].

### 4.3 Entanglement and Correlators in the ansatz geometry

In this subsection, we use the generic holographic formulas (4.9) and (4.10) particularized for (4.3) and (4.4). A simple substitution yields,

\[ \text{Length}(\gamma) = -2 L_{\text{AdS}} \int_{z_{\text{max}}}^{\epsilon} \frac{z_{\text{max}} \, dz}{z \sqrt{z_{\text{max}}^2 f(z) - z^2 f(z_{\text{max}})}}, \] (4.13)

and

\[ \ell_{\text{max}} = -2 \sqrt{f(z_{\text{max}})} \int_{z_{\text{max}}}^{\epsilon} \frac{z \, dz}{f(z) \sqrt{z_{\text{max}}^2 f(z) - z^2 f(z_{\text{max}})}}. \] (4.14)
Our main interest here, lies in characterizing the entanglement entropy contributions given by the IR regions of the geometry \[16, 35, 36\]. Thus, we are mainly concerned with those regimes for which \( z_{\text{max}} = z_0 \). When \( z_{\text{max}} \ll z_0 \), the values of the radial coordinate \( z \) keep close to the AdS boundary and \( f(z) \to 1 \). For these regimes one gets,

\[
\text{Length}(\gamma) \approx 2L_{\text{AdS}} \log \frac{\ell_{\text{max}}}{\epsilon}, \quad \ell_{\text{max}} \approx 2z_{\text{max}},
\]

(4.15)
i.e., one obtains the scaling of the entanglement entropy in a pure AdS spacetime, which is something to be expected as far as the geometry \(4.1\) reduces to pure AdS when \( z \ll z_0 \).

Contrarily, when \( z \to z_0 \) the function \( f(z) \) can be written as,

\[
f(z)|_{z \to z_0} \approx \frac{1}{z_0^2} \left( z_0^2 - Q z_0 z + (Q - 1) z^2 \right).
\]

(4.16)

As \( z_{\text{max}} \equiv z_0 \) and \( f(z_0) = 0 \), the geodesic length and \( \ell_{\text{max}} \) read as,

\[
\text{Length}(\gamma) = -2L_{\text{AdS}} \int_{z_0}^{\epsilon} \frac{z_0 \, dz}{z \sqrt{z_0^2 - Q z_0 z + (Q - 1) z^2}} = 2L_{\text{AdS}} \log \frac{\ell_{\text{max}}(\delta)}{\epsilon},
\]

(4.17)

and

\[
\ell_{\text{max}}(\delta) = -2\sqrt{h(z_0)} z_0 \int_{z_0}^{\epsilon} \frac{z \, dz}{(z_0^2 - Q z_0 z + (Q - 1) z^2)^{3/2}} = \frac{4z_0}{\delta},
\]

(4.18)

with \( \delta = 2 - Q \) and \( h(z) = z_0^2 - Q z_0 z + (Q - 1) z^2 \).

These results imply that the geodesic length in the \(\text{ansätz}e\) geometries parametrized by \( \delta \), saturate when considering intervals \( A \) whose length \( \ell > \ell_{\text{max}}(\delta) = 4z_0/\delta \).

For the sake of forthcoming discussions, let us illustrate the behaviour of the geodesic curves by simply recasting their length as,

\[
\text{Length}(\gamma) = 2L_{\text{AdS}} \left( \log \frac{2z_0}{\epsilon} + \log \frac{2}{\delta} \right).
\]

(4.19)

The first term in (4.19) amounts to the geodesic distance between the boundary points of \( A \) located at \( z \sim \epsilon \) and the capping region lying at \( z_0 \). In other words, this term is related with the piece of the curve connecting the boundary of the geometry and the IR capping region. On the other hand, the second term, may be visualized as follows: the geodesics, once arriving at the capping region, instead of finishing, stroll over the IR region to a maximum allowed extent quantified by this term.

The geodesic length (4.19) thus permit us to identify two different contributions in the holographic entanglement entropy. Indeed, by inserting (4.19) into (3.4), one gets,

\[
S_A = S_A^{\text{UV}} + S_A^{\text{IR}}(\delta), \quad S_A^{\text{UV}} \propto \log \frac{2z_0}{\epsilon}, \quad S_A^{\text{IR}}(\delta) \propto \log \frac{2}{\delta}.
\]

(4.20)

The term \( S_A^{\text{IR}}(\delta) \) only depends on the parameter which characterizes the geometry at the capping region and its behaviour is quite rich within the range \( 0 < \delta \leq 2 \) (i.e, \( 0 \leq Q < 2 \)); while it remains finite for \( 0 < \delta < 2 \), vanishes for \( \delta = 2 \) and it diverges when \( \delta \to 0 \).
As the geodesic length is saturated for intervals $A$ with $\ell > \ell_{\text{max}}$, then, according to (4.11, 4.12), the two point correlators of operators inserted at the boundary of the geometry also show a similar saturation phenomena as shown by,

$$\mathcal{C}^{[\ell]}_{\text{Holog}} = \epsilon^{-2\Delta} \exp \left[ -2\Delta \log \frac{\ell_{\text{max}}(\delta)}{\epsilon} \right] = \ell_{\text{max}}(\delta)^{-2\Delta}.$$  (4.21)

This result implies that in a capped geometry defined by $\delta$, all the correlators vanish when the points at the boundary are separated by a distance $\ell > \ell_{\text{max}}(\delta)$. Namely, it is convenient to interpret $\ell_{\text{max}}$ as an effective correlation length, $\zeta_{\text{Holog}} = \ell_{\text{max}}(\delta)$. (4.22)

This effective correlation length diverges as $\delta \to 0$ while is minimal for $\delta = 2$, i.e $\min \{\zeta_{\text{Holog}}\} = \ell_{\text{max}}(\delta = 2) = 2z_0$.

### 4.4 Entanglement in MERA-MPS networks and geometry

In this subsection, we show how the features of the long range entanglement in an hybrid MERA-MPS state are consistently represented by the ansatz geometry (4.1). With this aim, we consider a tensor network composed of a MERA curtain with $\tau_0 = \log \xi$ layers so that $\xi = 2z_0$. In addition, on the top of the curtain, we place an MPS tensor state $|\mathcal{C}_{\text{MPS}}\rangle$ with bond dimension $W$.

Let us first to compute the entanglement entropy of a region $A$ with $\ell \gg \xi$. In the MERA curtain, the causal cone of the region $A$ is connected to the rest of the MERA sites through a number $n(A)$ of traced out bonds which is proportional to $\log \xi$ and not depends on $\ell$. This yields a saturated value for the contribution of the MERA layers to the entanglement entropy of the block $A$ given by,

$$S_{\text{MERA}}^A \propto \log \xi = \log 2z_0.$$  (4.23)

The second contribution to the entanglement entropy of $A$ comes from the top MPS tensor. This IR state provides an additional contribution $S_{\text{MPS}}^A$ which is bounded by $\log W$ (see eq.(2.9)). Thus, the total entanglement entropy amounts to,

$$S_A = S_{\text{MERA}}^A + S_{\text{MPS}}^A \leq \log \xi + 2 \log W.$$  (4.24)

Attending to (4.20) one realizes that: first, the term $S_{\text{UV}}^A$ is equal to the contribution given by the MERA curtain. On the other hand, the MPS contribution to the entanglement entropy may be interpreted in terms of the associated holographic geometry if one identifies this contribution with $S_{\text{IR}}^A$ in (4.20). This relates the parameter $\delta$ managing the growth of the geodesic length in the capping region of the geometry (4.1), with the bond dimension $W$ which characterizes the long range entanglement in the tensor network, i.e,

$$W \geq \frac{2}{\delta}.$$  (4.25)

The last expression allow us to consistently relate different configurations of the tensor network with different values of the geometric parameter $\delta$. As an example, if the top tensor
is a trivial non entangled product state with bond dimension $W = 1$, the contribution to the entropy it yields is identically zero. This pattern of long range entanglement is geometrically represented by taking $\delta = 2$. Conversely, one might say that there is no tensor network representation when taking $\delta \to 0$ as this requires a $W \to \infty$.

The next thing we would like to consider is the two point function of an scaling operator $\Phi$ inserted at two sites in the bottom of the MERA curtain and separated by a distance $\ell \gg \xi$. Remarkably, the analysis of the two point functions also provides a significative connection between the IR entanglement structure of the tensor network and the ansatz geometry (4.1). Assuming the translational invariance of the system, this correlator can be written as

$$ C^{\ell}_{\text{TNS}} = \langle \Psi | \Phi(-\ell/2) \Phi(\ell/2) | \Psi \rangle , \quad (4.26) $$

with $|\Psi\rangle$ being the tensor decomposition of the system given by the hybrid MERA-MPS network. To perform the computation, one needs to carry out $\tau_0 = \log \xi$ contractions in the MERA curtain and then an expectation value with the top MPS state $|C_{\text{MPS}}\rangle$. Noticing that $\Delta \equiv -\log \mu$, and (2.6) one obtains,

$$ C^{\ell}_{\text{TNS}} = \xi^{-2\Delta} \langle C_{\text{MPS}} | \Phi(-\ell/2 \xi) \Phi(\ell/2 \xi) | C_{\text{MPS}} \rangle = \xi^{-2\Delta} \sum_{|\lambda_\nu| < 1} c_\nu (\lambda_\nu)^{\ell/\xi} . \quad (4.27) $$

Here, it is reasonable to assume that the leading term controlling the decay of the correlator (4.27) is given by the biggest $\lambda_\nu < 1$, i.e $\lambda_2$. Furthermore, the leading decaying term can be expressed as an exponential with an effective correlation length given by,

$$ \zeta^{\ell}_{\text{TNS}} = -\frac{1}{\log |\lambda_\Gamma|} , \quad \lambda_\Gamma = \lambda_2^{1/\xi} \implies \zeta^{\ell}_{\text{TNS}} = -\frac{2 z_0}{\log |\lambda_2|} . \quad (4.28) $$

Regarding the formal equivalence between the entanglement entropy in the MERA-MPS network and the holographic entanglement entropy in the ansatz geometry (4.1) it is reasonable to connect their effective correlation lengths. The simplest assumption is to establish an equivalence up to a constant term $\kappa$ so that,

$$ \zeta^{\ell}_{\text{Holo}} = \frac{4 z_0}{\delta} = \kappa \zeta^{\ell}_{\text{TNS}} = -2 \frac{\kappa z_0}{\log |\lambda_2|} , \quad \lambda_2 = \exp \left( -\frac{\kappa}{2} \right) . \quad (4.29) $$

These expressions relate the piece of Spec ($E_1$) controlling the leading behaviour in the decay of correlators, with the holographic parameter $\delta$ in a consistent way. Indeed, if $\delta \to 0$ then $\lambda_2 \to 1$, reflecting the divergence in $\zeta^{\ell}_{\text{TNS}}$ which one might expect for an MPS with $W \to \infty$. On the other hand, when $\delta = 2$, $\lambda_2$ can be made parametrically small if $\kappa \sim 1$, i.e $\lambda_2 = e^{-\kappa}$. This is consistent with an MPS with a bond dimension $W \sim 1$, i.e close to be a non entangled product state.

### 4.5 Mapping MERA to MPS networks

In this subsection we analyze the decomposition of the geodesic length (4.17) into the two terms shown in (4.19) from a tensor network perspective. With this aim, we note that the holographic entanglement entropy in terms of (4.17) given by,
\[ S_A \propto \log \frac{\ell_{\text{max}}(\delta)}{\epsilon}, \quad (4.30) \]

may be generically represented by a tensor network composed only by a finite number of MERA layers, \( \hat{\tau} = \log \xi(\delta) \) so that,

\[ \hat{\xi}(\delta) = \ell_{\text{max}}(\delta) = \frac{4 \zeta_0}{\delta}. \quad (4.31) \]

Nevertheless, as being concerned with 1-D gapped phases possessing a non trivial exponential decay of correlations for distances \( \ell \gg 2 \zeta_0 \), it is thus important to note here that, for a finite range MERA network, is not possible, in general, to yield such a behaviour. Namely, only a non homogeneous finite range MERA with \( \hat{\tau} \) layers can match (4.30) while implementing exponentially decaying correlations for \( \ell \gg 2 \zeta_0 \). This network is composed by a \textit{curtain} of \( \tau_0 = \log 2 \zeta_0 \) layers with identical disentanglers and isometries and a fixed number \( \Delta \tau = \log 2/\delta \) of top layers where the tensors are allowed to be different from those composing the \textit{curtain} [11]. In other words, the layered structure of the non homogeneous finite range MERA can be naturally decomposed as,

\[ \hat{\tau} = \tau_0 + \Delta \tau, \]

\[ \hat{\tau} = \log 2 \zeta_0 + \log \frac{2}{\delta}. \quad (4.32) \]

This is the tensor network justification for considering the arbitrary decomposition of the geodesic length given in (4.19). However, it is still difficult to explicitly show how this MERA network implements exponentially decaying correlations at long distances. To this end it is worth to recall that for a 1-D finite range MERA made of \( \Delta \tau \) layers of tensors with a bond dimension \( W^* \), there is map by which it may be re-expressed as an MPS with bond dimension \( W \). This map read as,

\[ W = (W^*)^{\Delta \tau}. \quad (4.33) \]

Assuming that \( \log(W^*) \sim \mathcal{O}(1) \), if one applies this map to the \( \Delta \tau \) top layers of the inhomogeneous finite range MERA, one gets the hybrid MPS-MERA network of section 4.4 with \( \mathcal{W} \sim 2/\delta \) (Eq. 4.25) and the ability to use the analytical bonanza for exponentially decaying correlations offered by MPS which also lead to establish the relationship (4.29).

5 Discussion and Concluding Remarks

By exploring some hints posed in [16], new insights on the connection between the structure of MERA states and their potential geometric descriptions have been provided. The striking relationship between the MERA and AdS/CFT, initially posed in [10], manifests through the use of the holographic entanglement entropy [24]. Both proposals establish a connection between entanglement and geometry. While in the former, one tries to ascribe a geometry to a known entanglement structure in the tensor network, in the latter, it is the behaviour of the entanglement which comes determined by the known dual geometry of the boundary quantum system.
Here, we have shown that the entanglement entropy and the two point functions computed through a tensor network state representing a one dimensional gapped quantum many body system, remains consistent, at least at the qualitative level, with a geometric computation of both quantities once a sensible metric associated to the tensor network is provided. The metric corresponds to an asymptotically AdS geometry with an infrared capping region parameterized in terms of a function. This function characterizes the growth of minimal surfaces near the capped region of the geometry and the holographic computation depends only on it. As different arrangements of the network provide significant distinct behaviours of the entropy and correlations, the ansätze metrics are proposed in a kind of inverse-like problem in order to obtain some qualitative matches between both computations.

From these matches, we have presented some expressions relating the entanglement structure of the tensor network and the parametrizing function of the associated geometry. Indeed, the equation (4.29) relates the spectrum of the transfer matrices of the top MPS tensor with the parameter $Q$ defining the IR behaviour of the geometry. This kind of result may be useful in order to, via AdS/TNS, open the possibility to classify the gapped phases of 1D-quantum many-body systems in terms of their associated geometries. As posed in [31, 37, 38], different phases (including symmetry protected topological phases) of 1D gapped systems with an MPS representation, can be characterized by the spectrum of the MPS transfer matrices.

It is necessary to recall that, despite the consistency requirements commented above, the ansätze metrics cannot be related to any solution of Einstein gravity. Indeed, our approach only provides a metric which describes the entanglement structure in the tensor network in a qualitative way. In this sense, in [34], it has been proposed an algorithm to find stationary solutions of the three dimensional Einstein gravity with an interacting scalar field. The algorithm bootstraps the solutions by requiring a geometric input in terms of function which is close in spirit to our parametrizing function. It would be the matter of a future study to use similar ideas in order to investigate if our ansätze may be related with solutions of a sensible gravitational theory.

Finally, the proposal of this paper could be extendable to higher dimensions. Namely, there are 2D generalizations of MERA [39] and MPS (PEPS) [6] to reasonably construct 2D versions of the hybrid network proposed here. For instance, in [40], authors presented the MERA tensor network for the Kitaev’s toric code model [41], the simplest topologically ordered phase in two spatial dimensions. The MERA tensor network for the toric code model is capped off by a top tensor which encodes the information about the topological order. Furthermore, and in a closer analogy to the tensor networks considered here, the toric code also possess a very simple representation in terms of a PEPS with a bond dimension $W = 2$ [42]. Indeed, for any PEPS with a finite bond dimension, it is always simple to calculate the topological entanglement entropy defined in [43], which is one of the major markers of topological order in gapped systems.

In addition to the toric code, there are more complex (2+1) topological phases which admit a tensor network description such as the Levin-Wen (LW) string-net states [44–46]. String-net states are lattice models representing a large class of doubled topological
phases i.e, topological phases with time reversal symmetry preserved. The ground state of a LW state can be viewed as the fixed point state of some renormalization group flow and it is conjectured that its properties can be described by a \emph{doubled} Chern-Simons theory. Recently, a supersymmetric version of a doubled Chern-Simons theory has attracted a strong attention in string theory \cite{47}. The model is a CS theory with a $U(N)_k \times U(N)_{-k}$ gauge group in addition to bifundamental matter fields (scalars and fermions) of $U(N)$. These theories have been argued to be dual to M-theory on $AdS_4 \times S^7/Z_k$. In this context, it is thus interesting to seek for ansatz geometries that qualitatively describe the entanglement structure of the tensor networks associated to doubled topological phases.

An equally challenging proposal runs conversely. In \cite{48}, authors provide the gravity dual of a (2+1) dimensional field theory which in its IR limit flows to a Chern-Simons theory. Since the theory is gapped, the geometry is capped off at IR. To complete the model, D7-branes are located lying on the capping region of the geometry, making the IR behavior different from that of the ”naked” geometry. Concretely, the D7-branes are the source of a Chern-Simons term in the action, which gives rise to a topological entanglement entropy term through (3.3). It is thus tempting to apply the ideas presented here in order to find 2D tensor network states with top tensors whose entanglement properties match the features observed in the geometric duals of \cite{48}.

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