A matrix description of weakly bipartitive and bipartitive families

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Abstract

The notions of weakly bipartitive and bipartitive families were introduced by Montgolfier (2003) as a general tool for studying some decomposition of graphs and other combinatorial structures. In this paper, we give a matrix description of these notions.

Keywords: Graph; Modular decomposition; Bipartitive families; matrices.

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1. Introduction

Modular decomposition has arisen as a technique that applies to many combinatorial structures such as graphs, tournaments, 2-structures, hypergraphs, and matroids, among others. It is based on module. For graphs, this notion goes back to Gallai [9]. More precisely, let $G = (V, E)$ be an undirected simple graph. A module of $G$ is a set $M \subseteq V$ such that for all $x \in V \setminus M$ either $N_G(x) \cap M = \emptyset$ or $M \subseteq N_G(x)$, where $N_G(x)$ is the neighborhood of $x$, that is, $N_G(x) := \{y \in V : \{x, y\} \in E\}$. For tournaments, the notion of module can be defined in a similar way. Recall that a tournament is a directed graph such that for every distinct vertices $x$ and $y$, either $x \rightarrow y$ or $y \rightarrow x$ and never both. Let $T$ be a tournament with vertex set $V$. The
out-neighborhood of a vertex $x \in V$ is the set $N^+_T(x) = \{y \in V : x \rightarrow y\}$ and the in-neighborhood is $N^-_T(x) = \{y \in V : y \leftarrow x\}$. A module of $T$ is a set $M \subseteq V$ such that for all $x \in V \setminus M$ either $N^+_T(x) \cap M = \emptyset$ or $M \subseteq N^-_T(x)$.

The split decomposition of graphs and the bi-join decomposition of graphs and of tournaments can be seen as a generalization of the modular decomposition. These decompositions were introduced respectively by Cunningham [3] and Montgolfier [10]. Let $G = (V, E)$ be an undirected simple graph and let $\{X, Y\}$ be a bipartition of $V$. We say that $\{X, Y\}$ is a split of $G$ if there exist $X_1 \subseteq X$ and $Y_1 \subseteq Y$ such that for all $x \in X_1$, $N_G(x) \cap Y = Y_1$ and for all $x \in X \setminus X_1$, $N_G(x) \cap Y = \emptyset$. We say that $\{X, Y\}$ is a bi-join of $G$ if there exist $X_1 \subseteq X$ and $Y_1 \subseteq Y$ such that for all $x \in X_1$, $N_G(x) \cap Y = Y_1$ and for all $x \in X \setminus X_1$, $N_G(x) \cap Y = Y \setminus Y_1$. Remark that if $X$ or $Y$ is a module of $G$ then $\{X, Y\}$ is both a split and a bi-join of $G$. The notion of bi-join can be also defined for tournaments in the following way. Let $T$ be a tournament with vertex set $V$. A bipartition $\{X, Y\}$ of $V$ is a bi-join of $T$ if there exist $X_1 \subseteq X$ and $Y_1 \subseteq Y$ such that for all $x \in X_1$ (resp. $x \in X \setminus X_1$), $N^+_T(x) \cap Y = Y_1$ and $N^-_T(x) \cap Y = Y \setminus Y_1$ (resp. $N^+_T(x) \cap Y = Y \setminus Y_1$ and $N^-_T(x) \cap Y = Y_1$).

Figure 1 : A split in a graph

Figure 2 : A bi-join in a graph and in a tournament

Bipartitive families are a general tool for studying both split decomposition and bi-join decomposition. They were introduced by Montgolfier [10] as follows. Let $V$ be a nonempty set. Two bipartitions $\{X, Y\}$ and $\{X', Y'\}$ of $V$ overlap if $X \cap Y$, $X \cap Y'$, $X' \cap Y$ and $X' \cap Y'$ are nonempty. A family $\mathcal{F}$ of bipartitions of $V$ is weakly bipartitive if:

Q1) for all $v \in V$, $\{\{v\}, V \setminus \{v\}\}$ is in $\mathcal{F}$.  


Q2) for all \( \{X, Y\} \) and \( \{X', Y'\} \) in \( \mathcal{F} \) such that \( \{X, Y\} \) overlaps \( \{X', Y'\} \), the four bipartitions \( \{X \cap X', Y \cup Y'\}, \{X \cap Y', Y \cup X'\}, \{Y \cap X', X \cup Y'\} \) and \( \{Y \cap Y', X \cup X'\} \) are in \( \mathcal{F} \).

A weakly bipartitive family \( \mathcal{F} \) is bipartitive if it satisfies the following additional condition:

Q3) for all \( \{X, Y\} \) and \( \{X', Y'\} \) which overlap in \( \mathcal{F} \), \( \{X \Delta X', X \Delta Y'\} \) is in \( \mathcal{F} \).

Cunningham \([3]\) proved that the family of splits of a connected graph is bipartitive. The same result was obtained for the family of bi-joins of a graph by Montgolfier \([10]\). For tournaments, the family of bi-joins is only weakly bipartitive.

We will present now another important example of weakly bipartitive family which comes from the works of Hartfiel and Loewy \([5]\) and of Loewy \([8]\). Let \( A = [a_{ij}]_{1 \leq i,j \leq n} \) be a \( n \times n \) matrix with entries in a field \( K \) and let \( X, Y \) be two nonempty subsets of \( [n] \) (where \( [n] := \{1, \ldots, n\} \)). We denote by \( A[X,Y] \) the submatrix of \( A \) having row indices in \( X \) and column indices in \( Y \). The matrix \( A \) is irreducible if for any proper subset \( X \) of \( [n] \), both of matrices \( A[X,[n] \setminus X] \) and \( A[[n] \setminus X,X] \) are nonzero. An HL-bipartition of \( A \) is a partition \( \{X,Y\} \) of \( [n] \) such that both of matrices \( A[X,Y] \) and \( A[Y,X] \) have rank at most 1. The concept of HL-bipartitions is equivalent to that of HL-clan \([1]\). In the case when \( A \) is irreducible, the family of its HL-bipartitions is weakly bipartitive (see Lemma 1 of \([8]\)).

Splits and bi-joins can be interpreted in terms of HL-bipartitions. More precisely, we will prove in the next section that the splits (resp. the bi-joins) of an undirected simple graph \( G \) with vertex set \( [n] \), are exactly the HL-bipartitions of its adjacency matrix (resp. Seidel adjacency matrix). Likewise, the bi-joins of a tournament \( T \) with vertex set \( [n] \) are the HL-bipartitions of its Seidel adjacency matrix.

Throughout this paper, the family of HL-bipartitions of a matrix \( A \) is denoted by \( \mathcal{H}_A \). Our main result is the following theorem.

**Theorem 1.1.** If \( A \) is a symmetric and irreducible \( n \times n \) matrix over a field \( K \) then \( \mathcal{H}_A \) is bipartitive. Conversely, if \( \mathcal{F} \) is a weakly bipartitive family of \( [n] \) then there exists an irreducible matrix \( A \) with entries in \( \{-1,0,1\} \) such that \( \mathcal{F} = \mathcal{H}_A \). In the particular case when \( \mathcal{F} \) is bipartitive, the matrix \( A \) can be chosen symmetric.
2. Splits, bi-joins and HL-bipartitions

Let $G$ be a graph with $n$ vertices $v_1, ..., v_n$. The adjacency matrix of $G$ is the $n \times n$ real symmetric matrix $A(G) = [a_{ij}]_{1 \leq i,j \leq n}$ where $a_{ij} = 1$ if $\{v_i, v_j\}$ is an edge of $G$ and $a_{ij} = 0$ otherwise. The Seidel adjacency matrix of $G$ is the $n \times n$ symmetric matrix $S(G) = [s_{ij}]_{1 \leq i,j \leq n}$ in which $s_{ij} = 0$ if $i = j$ and otherwise is $-1$ if $\{v_i, v_j\}$ is an edge, $+1$ if it is not. The Seidel matrix was introduced by Van Lint and Seidel [11]. Adjacency matrix and Seidel matrix for a tournament are defined in the same way.

The following Proposition gives a description of splits and bi-joins in terms of HL-bipartitions.

**Proposition 2.1.** Let $G$ be a graph with vertex set $[n]$ let $\{X, Y\}$ be a bi-partition of $[n]$. Then

i) $\{X, Y\}$ is a split of $G$ if and only if $\{X, Y\}$ is an HL-bipartition of $A(G)$.

ii) $\{X, Y\}$ is a bi-join of $G$ if and only if $\{X, Y\}$ is an HL-bipartition of $S(G)$.

**Proof.** For positive integers $r$ and $s$, we denote by $0_{r,s}$ the $r \times s$ zero matrix and by $J_{r,s}$ the $r \times s$ matrix of ones.

i) Let $|X| := p$ and $|Y| := q$. It is easy to see that $\{X, Y\}$ is a split of $G$ if and only if we can reorder rows and columns of $A(G)[X,Y]$ so that the resulting matrix is $0_{p,q}$, $J_{p,q}$ or one of the following matrices:

\[
\begin{pmatrix}
J_{r,s} & 0_{r,q-s} \\
0_{p-r,s} & 0_{p-r,q-s}
\end{pmatrix}
\begin{pmatrix}
J_{r,q} \\
0_{p-r,q}
\end{pmatrix}
\begin{pmatrix}
J_{p,s} & 0_{p,q-s}
\end{pmatrix}
\]

These are the only possible forms (up to permutation of rows and columns) of a $p \times q$ $(0,1)$-matrices having rank at most 1.

ii) The argument is the same as in i). It suffices to check that $\{X, Y\}$ is a bi-join of $G$ if and only if we can reorder rows and columns of $S(G)[X,Y]$ so that the resulting matrix is $J_{p,q}$, $-J_{p,q}$ or one of the following matrices:

\[
\begin{pmatrix}
J_{r,s} & -J_{r,q-s} \\
-J_{p-r,s} & J_{p-r,q-s}
\end{pmatrix}
\begin{pmatrix}
J_{r,q} \\
-J_{p-r,q}
\end{pmatrix}
\begin{pmatrix}
J_{p,s} & -J_{p,q-s}
\end{pmatrix}
\]
The results of Cunningham and Montgolfier mentioned in the introduction can be deduced from the first assertion of our main theorem and the previous proposition.

A similar result of Proposition 2.1 holds for tournaments. More precisely, we have the following.

**Proposition 2.2.** Let $T$ be a tournament with vertex set $[n]$ and let $\{X, Y\}$ be a bipartition of $[n]$. Then $\{X, Y\}$ is a bi-join of $T$ if and only if $\{X, Y\}$ is an HL-bipartition of $S(T)$.

3. Clans of $l_2$-structures and their relationship with HL-bipartitions

Let $V$ be a nonempty set and let $\hat{V}^2 := \{(x, y) / x \neq y \in V\}$. Following a labelled 2-structure on $V$, or a $l_2$-structure, for short, is a function $g$ from $\hat{V}^2$ to a set of labels $C$. With each subset $X$ of $V$ associate the $l_2$-substructure $g[X]$ of $g$ induced by $X$ defined on $X$ by $g[X](x, y) := g(x, y)$ for any $x \neq y \in X$. A $l_2$-structure $g$ on a set $V$ is symmetric if $g(x, y) = g(y, x)$ for every $x \neq y \in V$.

Let $g$ be a $l_2$-structure on $[n]$ whose set of labels is a field $K$. We associate to $g$ the $n \times n$ matrix $M(g) = [m_{ij}]_{1 \leq i, j \leq n}$ in which $m_{ij} = 0$ if $i = j$ and $m_{ij} = g(v_i, v_j)$ otherwise. Conversely, let $A = [a_{ij}]_{1 \leq i, j \leq n}$ be a matrix with entries in a field $K$. We associated to $A$ the $l_2$-structure $g_A$ on $[n]$ and set of labels $K$ such that $g_A(i, j) = a_{ij}$ for $i \neq j \in [n]$.

Given a $l_2$-structure $g$ on $V$, a subset $X$ of $V$ is a clan (4, Subsection 3.2) of $g$ if for any $a, b \in X$ and $x \in B \setminus X$, we have $g(a, x) = g(b, x)$ and $g(x, a) = g(x, b)$.

**Remark 1.**

i) Graphs and tournaments can be seen as special classes $l_2$-structure. Moreover, the notion of clan generalizes that of module.

ii) let $A$ be a matrix. if $I$ is a proper clan of $g_A$ then $\{I, [n] \setminus I\}$ is an HL-bipartition of $A$.

The following Proposition appears in another form in [1] (see Lemma 2.2). It describes the HL-bipartitions of a particular type of matrices called normalized matrices. Let $A = [a_{ij}]_{1 \leq i, j \leq n}$ be a matrix and let $v \in [n]$. We say that $A$ is $v$-normalized if $a_{vj} = a_{jv} = 1$ for every $j \in [n] \setminus \{v\}$. 


Proposition 3.1. Let $A = [a_{ij}]_{1 \leq i,j \leq n}$ be a $v$-normalized matrix for some $v \in [n]$ and let $I \subseteq [n] \setminus \{h\}$. Then $\{I, [n] \setminus I\}$ is an HL-bipartition of $A$ if and only if $I$ is a clan of $g_A([n] \setminus \{v\})$.

Proof. In order to prove the necessary condition, let $i, j \in I$ and $k \in ([n] \setminus \{v\}) \setminus I$. Since $\{I, [n] \setminus I\}$ is an HL-bipartition of $A$, both of matrices $A([n] \setminus I, I)$ and $A(I, [n] \setminus I)$ have rank at most 1. It follows that $\det(A([v, k], \{i, j\})) = \det(A(\{i, j\}, \{v, k\})) = 0$ and so $g(k, i) = a_{ki} = a_{kj} = g(k, j)$ and $g(i, k) = a_{ik} = a_{jk} = g(j, k)$. We conclude that $I$ is clan of $g_A([n] \setminus \{h\})$. Conversely, let $I$ be a clan of $g_A([n] \setminus \{v\})$. Since $A$ is $v$-normalized, $I$ is a clan of $g_A$ and then, by Remark 1 $\{I, [n] \setminus I\}$ is an HL-bipartition of $A$. $\square$

Let $V$ be a nonempty set $V$ and let $g$ be a $l^2$-structure on $V$. We denote by $\text{Cl}(g)$ the family of nonempty clans of $g$. This family satisfies the following well-known properties (see, for example, Subsection 3.3 of [4]).

P1) $V \in \mathcal{P}$, $\emptyset \notin \text{Cl}(g)$ and for all $v \in V$ , $\{v\} \in \text{Cl}(g)$;

P2) Given $X, Y \in \text{Cl}(g)$; if $X$ and $Y$ overlap, that is $X \cap Y, X \setminus Y$ and $Y \setminus X$ are all nonempty, then $X \cap Y \in \text{Cl}(g), X \setminus Y \in \text{Cl}(g), Y \setminus X \in \text{Cl}(g)$ and $X \cup Y \in \text{Cl}(g)$.

Moreover, if $g$ is symmetric then $\text{Cl}(g)$ satisfies the additional property:

P3) Given $X, Y \in \text{Cl}(g)$; if $X$ and $Y$ overlap then $X \triangle Y = (X \setminus Y) \cup (Y \setminus X) \in \text{Cl}(g)$.

Let $\mathcal{P}$ be a family of subsets of $V$. We say that $\mathcal{P}$ is weakly partitive if $\textbf{P1}$ and $\textbf{P2}$ hold. If also $\textbf{P3}$ holds, we say that $\mathcal{P}$ is partitive. Partitive and weakly partitive families were introduced in [2]. They are closely related to partitive families as shown in the following lemma.

Lemma 3.2. Let $\mathcal{B}$ be a family of bipartitions of $V$ and let $v \in V$. We denote by $\mathcal{P}$ the family of subsets $X$ of $V \setminus \{v\}$ such that $\{X, V \setminus X\} \in \mathcal{B}$. Then $\mathcal{B}$ is weakly bipartitive (resp. bipartitive) if and only if $\mathcal{P}$ is weakly partitive (resp. partitive).

The next Theorem of gives relationship between weakly partitive family and clans family.
Theorem 3.3. Let $\mathcal{P}$ be a weakly partitive family on $V$, then there exists an $l_2$-structure $g$ on $V$ with labels in a set of size at most 3 such that $\mathcal{P} = \text{Cl}(g)$. Moreover if $\mathcal{P}$ is partitive family on a set $V$, then $g$ can be chosen symmetric.

The first part of this theorem was proved by Ehrenfeucht, Harju, and Rozenberg (see [4], Theorem 5.7), and later by Ille and Woodrow [6]. As noted by Ille [7], the method given in [6] can also be used to prove the second part.

4. Proof of main theorem

We start with the following result.

Proposition 4.1. Let $A = [a_{ij}]_{1 \leq i,j \leq n}$ be an irreducible $n \times n$ matrix with entries in a field $\mathbb{K}$. Then for every $v \in [n]$ there is a $v$-normalized matrix $\hat{A}$ with non zero entries in a field $\hat{\mathbb{K}}$ containing $\mathbb{K}$ such that $A$ and $\hat{A}$ have the same HL-bipartitions. Moreover, if $A$ is symmetric then $\hat{A}$ can be chosen symmetric.

For the proof of this proposition, we use the following lemma.

Lemma 4.2. Let $A = [a_{ij}]_{1 \leq i,j \leq n}$ be a irreducible matrix. Let $x_1, x_2, \ldots, x_n$ be (independent) indeterminates, $\chi = \text{diag}(x_1, x_2, \ldots, x_n)$. Then we have the following statements:

i) the matrix $A + \chi$ is invertible in $\mathbb{K}(x_1, x_2, \ldots, x_n)$.

ii) all entries of $(A + \chi)^{-1}$ are nonzero.

iii) $A$, $A + \chi$ and $(A + \chi)^{-1}$ have the same HL-bipartitions.

For assertions i) and ii) of this lemma, see Theorem 1 of [5]. The third assertion is a direct consequence of the following Proposition.

Proposition 4.3. Let $T$ be an invertible matrix over $\mathbb{K}$, and suppose it has a block form

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

where $T_{11}$ is an invertible $k \times k$ matrix. Let $W = T^{-1}$, and partition $W$ conformably with $T$, so

$$W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}$$

Then rank($W_{12}$) = rank($T_{12}$) and rank($W_{21}$) = rank($T_{21}$).
Proof of Proposition 4.1. We will use the notations of Lemma 4.2. Let 
\((A + \chi)^{-1} := [b_{ij}]_{i,j \leq n}, D := [d_{i}]_{1 \leq i \leq n}\) and
\(D' := [d'_{i}]_{1 \leq i \leq n}\) where \(d_{i} = \frac{1}{b_{vi}}, d'_{i} = \frac{1}{b_{vi}}\) for \(i \neq v\) and \(d_{v} = d'_{v} = 1\). Clearly, the matrix \(A := D(A + X)^{-1}D'\)
is \(v\)-normalized and its entries are in \(\hat{\mathbb{K}} = \mathbb{K}(x_{1}, x_{2}, \ldots, x_{n})\). Moreover, if \(A\) is 
symmetric then \(A + \chi\) and \((A + \chi)^{-1}\) are also symmetric. It follows that 
\(D = D'\) and hence \(\hat{A}\) is symmetric. We conclude by applying iii) of Lemma 
4.2 and the following lemma.

Lemma 4.4. Let \(M\) be a \(n \times n\) matrix and let \(D_{1}, D_{2}\) be two \(n \times n\) diagonal 
and invertible matrices. Then, the matrices \(M\) and \(D_{1}MD_{2}\) have the same 
HL-bipartitions.

Proof. Let \(X, Y\) be two subset of \([n]\). We have the following equalities:
\[
(D_{1}MD_{2})[X,Y] = (D_{1}[X])(M[X,Y])(D_{2}[Y])
\]
\[
(D_{1}MD_{2})[Y,X] = (D_{1}[Y])(M[Y,X])(D_{2}[X])
\]

It follows \((D_{1}MD_{2})[X,Y]\) and \((M[X,Y])\) (resp. \((D_{1}MD_{2})[Y,X]\) and 
\((M[Y,X])\) have the same rank because the matrices \(D_{1}[X], D_{2}[X], D_{1}[Y]\) and 
\(D_{2}[Y]\) are invertible. Thus, \([X,Y]\) is an HL-bipartition of \(M\) if and 
only if it is one for \(D_{1}MD_{2}\).

Proof of Theorem 4.1. The fact that \(\mathcal{H}_{A}\) is weakly bipartite follows from 
Lemma 1 of [8]. To complete the proof it suffices to check that \(\mathcal{H}_{A}\) 
satisfies the condition Q3. For this, let \(\{X,Y\}, \{X',Y'\} \in \mathcal{H}_{A}\) which 
overlap. Then \([n] \setminus (X \cup X') = Y \cap Y' \neq \emptyset\). Let \(i \in [n] \setminus (X \cup X')\). By 
Proposition 4.1 there is a symmetric and \(i\)-normalized matrix \(\hat{A}\) such that 
\(\mathcal{H}_{A} = \mathcal{H}_{\hat{A}}\). So it suffices to prove that \(\{X \Delta X', X \Delta Y'\} \in \mathcal{H}_{\hat{A}}\). By the choice 
of \(i\), we have \(i \notin X\) and \(i \notin X'\) and then by Lemma 3.1 \(X\) and \(X'\) are clans of 
g_{\hat{A}}[[n] \setminus \{i\}]\). Moreover, \(X\) and \(X'\) overlap because \(\{X,Y\}, \{X',Y'\} \in \mathcal{H}_{A}\) 
overlap. Now, since \(\hat{A}\) is symmetric, \(g_{\hat{A}}[[n] \setminus \{i\}]\) is symmetric and then by 
P3, \(X \Delta X'\) is a clan of \(g_{\hat{A}}[[n] \setminus \{i\}]\). By applying again Lemma 3.1 we 
deduce that \(\{X \Delta X', X \Delta Y'\} \in \mathcal{H}_{\hat{A}}\).

Conversely, let \(\mathcal{F}\) be a weakly bipartitive family on a set \([n]\). We will 
construct an irreducible matrix \(A\) with entries in \([-1,0,1]\) such that \(\mathcal{F} = \mathcal{H}_{A}\). From Lemma 
3.2 the family \(\mathcal{P} := \{X \subseteq [n-1] : \{X,[n] \setminus X\} \in \mathcal{F}\}\) 
is weakly partitive, then by Theorem 3.3 there exists an \(l^{2}\)-structure \(g\) on
\( [n - 1] \) with labels in \( \{-1, 0, 1\} \) such that \( P = Cl(g) \). Consider the following matrix

\[
A = \begin{pmatrix}
M(g) & 1 & \\
\vdots & \ddots & \ddots & 1 \\
1 & \cdots & 1 & 0
\end{pmatrix}
\]

Clearly, this matrix is \( n \)-normalized and then it is irreducible. To prove that \( F = H_A \), let \( \{X, [n] \setminus X\} \) be a bipartition of \([n]\) and assume for example that \( n \notin X \). By Lemma 3.1, \( \{X, [n] \setminus X\} \in H_A \) if and only if \( X \) is a clan of \( g_A[1, \ldots, n - 1] = g \). Then \( \{X, [n] \setminus X\} \in H_A \) if and only if \( X \in P \) or equivalently \( \{X, [n] \setminus X\} \in F \) because \( P = Cl(g) \).

Now if \( F \) is bipartitive, then the family \( P := \{X \subseteq [n - 1] : \{X, [n] \setminus X\} \in F\} \) is partitive. By Theorem 3.3, we can choose \( g \) symmetric, which implies that \( A \) is symmetric.

\[
\square
\]

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