On variants of $H$-measures and compensated compactness

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Abstract

We introduce new variants of $H$-measures defined on spectra of general algebra of test symbols and derive the localization properties of such $H$-measures. Applications for the compensated compactness theory are given. In particular, we present new compensated compactness results for quadratic functionals in the case of general pseudo-differential constraints. The case of inhomogeneous second order differential constraints is also studied.

Keywords: algebra of admissible symbols, $H$-measures, localization principles, compensated compactness

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1. Introduction

Let

$$F(u)(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x} u(x) dx, \quad \xi \in \mathbb{R}^n,$$

be the Fourier transformation extended as a unitary operator on the space $u(x) \in L^2(\mathbb{R}^n)$, let $S = S^{n-1} = \{ \xi \in \mathbb{R} \mid |\xi| = 1 \}$ be the unit sphere in $\mathbb{R}^n$. Denote by $u \to \overline{u}$, $u \in \mathbb{C}$ the complex conjugation.

The concept of an $H$-measure corresponding to some sequence of vector-valued functions bounded in $L^2(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is an open domain, was introduced by Tartar [9] and Gérard [4] on the basis of the following result. For $r \in \mathbb{N}$ let $U_r(x) = (U_1^r(x), \ldots, U_N^r(x)) \in L^2(\Omega, \mathbb{R}^N)$ be a sequence weakly convergent to the zero vector.

**Proposition 1.1** (see Theorem 1.1 in [9]). **There exists a family of complex Borel measures** $\mu = \{ \mu^{\alpha\beta} \}_{\alpha,\beta=1}^N$ **in** $\Omega \times S$ **and a subsequence of** $U_r(x)$ **(still denoted** $U_r$ **) such that**

$$\langle \mu^{\alpha\beta}, \Phi_1(x) \overline{\Phi_2(x)} \psi(\xi) \rangle = \lim_{r \to \infty} \int_{\mathbb{R}^n} F(U_1^r \Phi_1)(\xi) \overline{F(U_2^r \Phi_2)(\xi)} \psi \left( \frac{\xi}{|\xi|} \right) d\xi \quad (1.1)$$

**for all** $\Phi_1(x), \Phi_2(x) \in C_0(\Omega)$ **and** $\psi(\xi) \in C(S)$.

Here and in the sequel we use notations $C_0(\Omega)$ for the space of continuous functions on $\Omega$ with compact supports.

The family $\mu = \{ \mu^{\alpha\beta} \}_{\alpha,\beta=1}^N$ is called the $H$-measure corresponding to $U_r(x)$.

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In [1], the new concept of parabolic \( H \)-measures was suggested. This concept was extended in [6], where the notion of ultra-parabolic \( H \)-measures was introduced. Suppose that \( X \subset \mathbb{R}^n \) is a linear subspace, \( X^\perp \) is its orthogonal complement, \( P_1, P_2 \) are orthogonal projections on \( X, X^\perp \), respectively. We denote for \( \xi \in \mathbb{R}^n \), \( \xi = P_1 \xi, \xi = P_2 \xi \), so that \( \xi \in X, \xi \in X^\perp, \xi = \xi + \xi \).

Let \( S_X = \{ \xi \in \mathbb{R}^n \mid |\xi|^2 + |\xi|^4 = 1 \} \). Then \( S_X \) is a compact smooth manifold of codimension 1; in the case when \( X = \{0\} \) or \( X = \mathbb{R}^n \), it coincides with the unit sphere \( S = \{ \xi \in \mathbb{R}^n \mid |\xi| = 1 \} \). Let us define a projection \( \pi_X : \mathbb{R}^n \setminus \{0\} \to S_X \) by

\[
\pi_X(\xi) = \frac{\tilde{\xi}}{|\tilde{\xi}|^{1/2}} + \frac{\bar{\xi}}{|\bar{\xi}|^{1/4}}.
\]

Remark that in the case when \( X = \{0\} \) or \( X = \mathbb{R}^n \), \( \pi_X(\xi)/|\xi| \) is the orthogonal projection on the sphere. With the notations from Proposition 1.1, the following extension holds:

**Proposition 1.2** (see [6, 7]). There exists a family of complex Borel measures \( \mu = \{\mu_{\alpha\beta}\}_{\alpha, \beta = 1}^N \) in \( \Omega \times S_X \) and a subsequence \( U_r(x) = U_k(x), k = k_r \), such that

\[
\langle \mu_{\alpha\beta}, \Phi_1(x)\Phi_2(x)\psi(\xi) \rangle = \lim_{r \to \infty} \int_{\mathbb{R}^n} F(U_r^\alpha \Phi_1)(\xi)F(U_r^\beta \Phi_2)(\xi)\psi(\pi_X(\xi)) \, d\xi
\]

for all \( \Phi_1(x), \Phi_2(x) \in C_0(\Omega) \) and \( \psi(\xi) \in C(S_X) \).

The family \( \mu = \{\mu_{\alpha\beta}\}_{\alpha, \beta = 1}^N \) we shall call an ultra-parabolic \( H \)-measure corresponding to \( U_r(x) \).

In paper [7], the localization properties of ultra-parabolic \( H \)-measures were applied to extend the compensated compactness theory [5, 8] for weakly convergent sequences \( u_r \in L^p_{\text{loc}}(\Omega, \mathbb{R}^N) \) to the case when the differential constraints may contain second-order terms while all the coefficients are variable. We describe the results of [7] in the particular case \( p = 2 \). Thus, assume that a sequence \( u_r \in L^2_{\text{loc}}(\Omega, \mathbb{R}^N) \) converges weakly to a vector-function \( u(x) \) as \( r \to \infty \) and satisfies the condition that the sequences

\[
\sum_{\alpha=1}^N \sum_{k=1}^n \partial_{x_k} (a_{\alpha k} u_{\alpha r}) + \sum_{\alpha=1}^N \sum_{k,l=\nu+1}^n \partial_{x_k x_l} (b_{\alpha k l} u_{\alpha r}), \quad s = 1, \ldots, m
\]

are pre-compact in the anisotropic Sobolev space \( W^{-1,-2}_{2,\text{loc}}(\Omega) \) (the parameter \(-1\) corresponds to the first \( \nu \) variables \( x_1, \ldots, x_\nu \) while the parameter \(-2\) corresponds to the remaining variables \( x_{\nu+1}, \ldots, x_n \)). Here \( \nu \) is an integer number between 0 and \( n \), and the coefficients \( a_{\alpha k} = a_{s\alpha k}(x), b_{s k l} = b_{s k l}(x) \) are assumed to be continuous on \( \Omega \).

We introduce the set \( \Lambda \) (here \( i = \sqrt{-1} \)):

\[
\Lambda = \Lambda(x) = \left\{ \lambda \in \mathbb{C}^N \mid \exists \xi \in \mathbb{R}^n, \xi \neq 0 : \sum_{\alpha=1}^N \left( i \sum_{k=1}^\nu a_{\alpha k}(x) \xi_k - \sum_{k,l=\nu+1}^n b_{\alpha k l}(x) \xi_k \xi_l \right) \lambda_\alpha = 0 \forall s = 1, \ldots, m \right\}.
\]

Consider the quadratic form \( q(x, u) = Q(x) u \cdot u \), where \( Q(x) \) is a symmetric matrix with coefficients \( q_{\alpha\beta}(x) \in C(\Omega), \alpha, \beta = 1, \ldots, N \) and \( u \cdot v \) denotes the scalar multiplication on \( \mathbb{R}^N \).
The form $q(x,u)$ can be extended as Hermitian form on $\mathbb{C}^N$ by the standard relation

$$q(x,u) = \sum_{\alpha,\beta=1}^N q_{\alpha\beta}(x)u_{\alpha\beta}.$$ 

Now, let the sequence $q(x,u_r) \to v$ as $r \to \infty$ weakly in $\mathcal{D}'(\Omega)$. Since this sequence is bounded in $L^1_{loc}(\Omega)$ then, passing to a subsequence if necessary, we may claim that $v$ is a locally finite measure on $\Omega$ (i.e., $v \in M_{loc}(\Omega)$), and $q(x,u_r) \to v$ weakly in $M_{loc}(\Omega)$.

**Theorem 1.1.** Assume that $q(x,\lambda) \geq 0$ for all $\lambda \in \Lambda(x)$, $x \in \Omega$. Then $q(x,u(x)) \leq v$ (in the sense of measures).

In the case $\nu = n$ when the second order terms in (1.3) are absent and all the coefficients are constant the statement of Theorem 1.1 is the classical Tartar-Murat compensated compactness.

In this paper we generalize the result of Theorem 1.1 to the case when the degeneration subspaces $X_s$ in constraints (1.3) may depend on $s$ and give some applications.

For that, we introduce the general variant of $H$-measures by extension of a class of admissible test functions $\psi(\xi)$. We will describe this class in the next section.

2. Algebra of admissible symbols

Let us denote by $B_\Phi$ and $A_\psi$ the bounded pseudodifferential operators on $L^2(\mathbb{R}^n)$ with symbols $\Phi(x), \psi(\xi) \in L^\infty(\mathbb{R}^n)$, respectively, that is,

$$B_\Phi u(x) = \Phi(x)u(x), \quad F(A_\psi u)(\xi) = \psi(\xi)F(u)(\xi).$$

We introduce the subalgebra $A$ of the algebra $L^\infty(\mathbb{R}^n)$, consisting of bounded measurable functions $\psi(\xi)$ on $\mathbb{R}^n$ such that the commutators $[A_\psi, B_\Phi]$ are compact operators in $L^2(\mathbb{R}^n)$ for all $\Phi(x) \in C_0(\mathbb{R}^n)$. Let $A_0 = L^1_{loc}(\mathbb{R}^n)$ be a subspace of $L^\infty(\mathbb{R}^n)$ consisting of functions $\psi(\xi)$ vanishing at infinity: $\text{ess lim}_{|\xi| \to \infty} \psi(\xi) = 0$.

**Lemma 2.1.** For every $\Phi(x) \in C_0(\mathbb{R}^n)$, $\psi(\xi) \in A_0$ the operators $A_\psi B_\Phi$, $B_\Phi A_\psi$ are compact in $L^2(\mathbb{R}^n)$.

**Proof.** First, assume that $\psi(\xi) \in L^\infty(\mathbb{R}^n)$ is a function with compact support $K = \text{supp} \psi \subset \mathbb{R}^n$. Let $u_k, k \in \mathbb{N}$, be a sequence in $L^2(\mathbb{R}^n)$, weakly convergent to zero: $u_k \to 0$. We have to prove that $A_\psi B_\Phi u_k \to 0$ in $L^2(\mathbb{R}^n)$ (strongly). Since $B_\Phi u_k = \Phi(x)u_k(x) \to 0$ weakly in $L^1(\mathbb{R}^n)$, then

$$F(B_\Phi u_k)(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x} u_k(x)\Phi(x)dx \to 0$$

for all $\xi \in \mathbb{R}^n$, and

$$|F(B_\Phi u_k)(\xi)| \leq \|\Phi u_k\|_1 \leq C = \|\Phi\|_2 \sup_{k \in \mathbb{N}} \|u_k\|_2 < \infty.$$ 

Then, by the Lebesgue dominated convergence theorem, we claim that

$$\|A_\psi B_\Phi u_k\|_2 = \int_K |F(B_\Phi u_k)(\xi)|^2 d\xi \to 0$$
as \( k \to \infty \), that is, \( A_\psi B_\theta u_k \to 0 \) in \( L^2(\mathbb{R}^n) \). We see that the operator \( A_\psi B_\theta \) transforms weakly convergent sequences in \( L^2 \) to strongly convergent ones. Hence, this operator is compact.

In the general case \( \psi(\xi) \in A_0 \) we introduce the sequence \( \psi_m(\xi) = \psi(\xi)\theta(m - |\xi|) \), \( m \in \mathbb{N} \), where \( \theta(r) = \begin{cases} 0, & r \leq 0, \\ 1, & r > 0 \end{cases} \) is the Heaviside function. Then

\[
\|\psi_m - \psi\|_\infty = \underset{|\xi| \geq m}{\text{ess sup}} |\psi(\xi)| \to 0
\]

as \( m \to \infty \), and therefore the operator norms

\[
\|A_{\psi_m} - A_\psi\| = \|\psi_m - \psi\|_\infty \to 0.
\]

This implies that \( A_{\psi_m} B_\theta \to A_\psi B_\theta \) as \( m \to \infty \) in the algebra of bounded linear operators on \( L^2(\mathbb{R}^n) \). The functions \( \psi_m(\xi) \) have compact supports and it has been already proven that the operators \( A_{\psi_m} \) are compact. We conclude that \( A_\psi B_\theta \) is a compact operator, as the limit of the sequence of compact operators \( A_{\psi_m} \).

In order to prove compactness of \( B_\theta A_\psi \), observe that this operator is conjugate to \( A_\psi B_\theta = (A_\psi)^*(B_\theta)^* \). As we have already established, the operator \( A_\psi B_\theta \) is compact. Therefore, the operator \( B_\theta A_\psi = (A_\psi B_\theta)^* \) is compact as well. The proof is complete.

In view of Lemma 2.1 we find that for \( \psi(\xi) \in A_0 \) the commutator \( [A_\psi, B_\theta] = A_\psi B_\theta - B_\theta A_\psi \) is a compact operator in \( L^2(\mathbb{R}^n) \) for all \( \Phi(x) \in C_0(\mathbb{R}^n) \). In particular \( A_0 \subset A \). It is clear that \( A_0 \) is a closed ideal in \( A \). We denote by \( A = A/A_0 \) the correspondent quotient algebra. Clearly, \( A \) is a commutative Banach \( C^* \)-algebra (subject to the involution defined by complex conjugation) equipped with the factor-norm (we identify the class \([\psi] \in A \) with the corresponding representative function \( \psi(\xi) \))

\[
\|\psi\| = \underset{\xi \to \infty}{\text{ess lim sup}} |\psi(\xi)| = \lim_{R \to \infty} \underset{|\xi| > R}{\text{ess sup}} |\psi(\xi)|.
\]

Therefore, the Gelfand transform \( \psi(\xi) \to \hat{\psi}(\eta) \) is an isomorphism of \( A \) into the algebra \( C(S) \) of continuous functions on the spectrum \( S \) of \( A \).

We introduce the order in \( A \) generated by the cone of nonnegative functions, that is, a class \( a \geq 0 \) if and only if there exists a real nonnegative function \( \psi \in a \), i.e., \( a = [\psi] \). As is easy to verify, for \( a, b \in A \), \( a, b \geq 0 \), and \( \alpha, \beta \in [0, +\infty) \) \( \alpha a + \beta b \geq 0 \) \( \alpha b \geq 0 \). As usual, we say that \( a_1 \geq a_2 \) if \( a_1 - a_2 \geq 0 \). It turns out that the Gelfand transform is monotone, that is, the following statement is fulfilled.

**Lemma 2.2.** The class \( a = [\psi] \geq 0 \) if and only if \( \hat{\psi}(\eta) \geq 0 \) for all \( \eta \in S \).

**Proof.** If \( \hat{\psi}(\eta) \geq 0 \) for all \( \eta \in S \) then the function \( \alpha(\eta) = (\hat{\psi}(\eta))^{1/2} \) is well-defined and continuous on \( S \). Therefore, there exists a unique class \( b = [\beta(\xi)] \in A \) such that \( \alpha(\eta) = \beta(\eta) \). Since the Gelfand transform satisfies the property \( \hat{\psi}(\eta) = \overline{\hat{\psi}(\eta)} \), we see that \( b^2(\eta) = (\alpha(\eta))^2 = \hat{\psi}(\eta) \) and the equality \( a = [\psi] = b^2 = [\beta^2] \) follows. This equality implies that \( a \geq 0 \).

Conversely, let \( a = [\psi] \geq 0 \). Since \( a = \tilde{a} \), the function \( \hat{\psi}(\eta) \) is real. We define the real nonnegative functions \( \hat{\psi}^+(\eta) = \max(0, \pm \hat{\psi}(\eta)) \in C(S) \). Then, there exist classes \( a^\pm = [\psi^\pm] \) such that \( \hat{\psi}^+(\eta) = \hat{\psi}^-(\eta) \). As we have already established, \( a^\pm \geq 0 \). Since \( \hat{\psi}(\eta) = \hat{\psi}^+(\eta) - \hat{\psi}^-(\eta) \), and \( \hat{\psi}^+(\eta) \cdot \hat{\psi}^-(\eta) = 0 \), the same is true for the \( a^\pm \): \( a = a^+ - a^- \), \( a^+ a^- = 0 \). Therefore,
We divide the proof into 6 steps.

Proof.

As follows from [6, Lemma 2], functions \( \psi(\pi_X(\xi)) \) belong to the algebra \( A \) for each \( \psi \in C(S_X) \). Hence, the algebra of quasi-homogeneous functions

\[
A_X = \{ \psi(\pi_X(\xi)) \mid \psi \in C(S_X) \}
\]

is a closed \( C^* \)-subalgebra of \( A \) and its spectrum coincides with \( S_X \). The embedding \( A_X \subset A \) yields the continuous projection of the spectra \( p_X : S \rightarrow S_X \). One of our aims is to formulate localization properties for \( H \)-measures corresponding to sequences satisfying general second order differential constraints. For this, we need to find simple necessary and sufficient conditions for a family of vectors \( \{ \xi_X \}_{X \subset \mathbb{R}^n} \) to satisfy the property \( \xi_X = p_X(\eta) \) for all \( X \subset \mathbb{R}^n \), where \( \eta \in S \). The following statement holds.

**Proposition 2.1.** Assume that \( \eta \in S \) and for \( X \subset \mathbb{R}^n \) let \( p_X(\eta) = (\xi_X, \bar{\xi}_X) \in X \oplus X^\perp \). Then there exist a unique orthonormal system \( \{ \zeta_1, \ldots, \zeta_m \} \) in \( \mathbb{R}^n \) and an integer \( d \in \{ m-1, m \} \) such that

(i) \( \bar{\xi}_X \neq 0 \Leftrightarrow X \supset \bar{X} = \mathcal{L}(\zeta_1, \ldots, \zeta_d) \) (this is a linear span of vectors \( \zeta_1, \ldots, \zeta_d \)). Besides, if \( \xi_X \neq 0 \), then \( \xi_X \uparrow \zeta_1 \);

(ii) \( \bar{\xi}_X \neq 0 \Leftrightarrow X \not\supset \bar{X} = \mathcal{L}(\zeta_1, \ldots, \zeta_m) \). Besides, if \( \xi_X \neq 0 \), then \( \xi_X \uparrow \text{pr}_X \cdot \zeta_k(\eta) \), where \( k(X) = \min \{ k = 1, \ldots, m \mid \zeta_k \notin X \} \).

**Proof.** We divide the proof into 6 steps.

1st Step.

We introduce the set \( \mathcal{L} \) of all subspaces \( X \subset \mathbb{R}^n \) such that \( \xi_X \neq 0 \). Let us show that \( \mathcal{L} \) contains the smallest space. For that, we first prove that the intersection \( X_1 \cap X_2 \) of two spaces \( X_1, X_2 \in \mathcal{L} \) lays in \( \mathcal{L} \) as well. We denote \( X_0 = X_1 \cap X_2, X_{10} = X_1 \oplus X_0 = \{ x \in X_1 : x \perp X_0 \}, X_{20} = X_2 \oplus X_0 \). Then we have the following representations

\[
\mathbb{R}^n = X_0 \oplus X_{10} \oplus X_{1}^\perp = X_0 \oplus X_{20} \oplus X_{2}^\perp. \tag{2.1}
\]

Let

\[
\xi = \xi_0 + \xi_1 + \xi_3 = \xi_0 + \xi_2 + \xi_4 \tag{2.2}
\]

be orthogonal decompositions of a vector \( \xi \in \mathbb{R}^n \) corresponding to (2.1). Here \( \xi_0 \subset X_0, \xi_1 \subset X_{10}, \xi_2 \subset X_{20}, \xi_3 \subset X_{1}^\perp, \) and \( \xi_4 \subset X_{2}^\perp \). We introduce the functions

\[
f_1(\xi) = \frac{|\xi_0|^2 + |\xi_1|^2}{|\xi_0|^2 + |\xi_1|^2 + |\xi_3|^2}, \quad f_1(\xi) = \frac{|\xi_0|^2 + |\xi_2|^2}{|\xi_0|^2 + |\xi_2|^2 + |\xi_4|^2}
\]

defined on \( \mathbb{R}^n \setminus \{0\} \). Obviously, \( f_1 \in A_{X_1} \subset A, f_2 \in A_{X_2} \subset A, \) and \( \tilde{f}_1(\eta) = |\xi_{X_1}|^2 \neq 0, \tilde{f}_2(\eta) = |\xi_{X_2}|^2 \neq 0 \). We define the subspace \( Y \subset X_{1}^\perp \oplus X_{2}^\perp \) consisting of pairs \( (\xi_3, \xi_4) \) such that \( \xi_1 + \xi_3 = \xi_2 + \xi_4 \) for some vectors \( \xi_1, \xi_2 \in X_{10} \), \( \xi_3, \xi_4 \in X_{20} \). Observe that the vectors \( \xi_1, \xi_2 \) are uniquely defined by the above equality. Indeed, if \( \xi_1 + \xi_3 = \xi_2 + \xi_4 \) for some other vectors \( \xi_1' \subset X_{10}, \xi_2' \subset X_{20} \) then \( \xi_1 - \xi_1' = \xi_2' - \xi_2 \in X_{10} \cap X_{20} = \{0\} \) and we conclude that \( \xi_1 = \xi_1', \xi_2 = \xi_2' \). Thus, we can define the linear maps \( A_1 : Y \rightarrow X_{10}, A_2 : Y \rightarrow X_{20} \) such that \( A_1(\xi_3, \xi_4) = \xi_1, A_2(\xi_3, \xi_4) = \xi_2 \). Since these maps are continuous, we can find a positive constant \( C \) such that

\[
|A_i(\xi_3, \xi_4)|^2 \leq C(|\xi_3|^2 + |\xi_4|^2) \quad \text{for all } (\xi_3, \xi_4) \in Y. \tag{2.3}
\]
Then
\[
\begin{align*}
    f_1(\xi) &\leq \frac{|\xi_0|^2 + C(|\xi_3|^2 + |\xi_4|^2)}{|\xi_0|^2 + |\xi_1|^2 + |\xi_3|^4} \leq \frac{|\xi_0|^2 + C|\xi_4|^2}{|\xi_0|^2 + |\xi_1|^2 + |\xi_3|^4} + \alpha_1(\xi), \\
    f_2(\xi) &\leq \frac{|\xi_0|^2 + C(|\xi_3|^2 + |\xi_4|^2)}{|\xi_0|^2 + |\xi_1|^2 + |\xi_4|^4} \leq \frac{|\xi_0|^2 + C|\xi_3|^2}{|\xi_0|^2 + |\xi_1|^2 + |\xi_4|^4} + \alpha_2(\xi),
\end{align*}
\]
where
\[
\begin{align*}
    \alpha_1(\xi) &= \frac{C|\xi_3|^2}{|\xi_0|^2 + |\xi_1|^2 + |\xi_3|^4} \xrightarrow{\xi \to 0} 0, \\
    \alpha_2(\xi) &= \frac{C|\xi_4|^2}{|\xi_0|^2 + |\xi_1|^2 + |\xi_4|^4} \xrightarrow{\xi \to 0} 0,
\end{align*}
\]
that is, \( \alpha_j(\xi) \in A_0, j = 1, 2 \). In view of (2.4)
\[
0 \leq f_1(\xi) \leq \frac{|\xi_0|^2 + C|\xi_4|^2}{|\xi_0|^2 + |\xi_1|^2 + |\xi_4|^4}, \\
0 \leq f_2(\xi) \leq \frac{|\xi_0|^2 + C|\xi_3|^2}{|\xi_0|^2 + |\xi_1|^2 + |\xi_4|^4},
\]
in \( A \), which implies that in this algebra
\[
0 \leq f_1(\xi)f_2(\xi) \leq \frac{(|\xi_0|^2 + C|\xi_4|^2)(|\xi_0|^2 + C|\xi_3|^2)}{|\xi_0|^2 + |\xi_1|^2 + |\xi_3|^4[|\xi_0|^2 + |\xi_1|^2 + |\xi_4|^4]}. \tag{2.5}
\]
Observe that
\[
(|\xi_0|^2 + C|\xi_4|^2)(|\xi_0|^2 + C|\xi_3|^2) \leq |\xi_0|^4 + C|\xi_1|^2(|\xi_0|^2 + C|\xi_3|^2) + C|\xi_0|^2|\xi_3|^2, \\
(|\xi_0|^2 + |\xi_1|^2 + |\xi_3|^4)(|\xi_0|^2 + |\xi_1|^2 + |\xi_4|^4) \geq (|\xi_0|^2 + |\xi_3|^4)(|\xi_0|^2 + |\xi_4|^4),
\]
and it follows from (2.5) that
\[
0 \leq f_1(\xi)f_2(\xi) \leq \frac{|\xi_0|^4}{(|\xi_0|^2 + |\xi_1|^2 + |\xi_4|^4)(|\xi_0|^2 + |\xi_3|^4)} + C|\xi_0|^2|\xi_3|^2
\]
\[
= \frac{|\xi_0|^2 |\xi_1|^2}{|\xi_0|^2 + |\xi_4|^4} \cdot \frac{|\xi_0|^4}{(|\xi_0|^2 + |\xi_3|^4)(|\xi_0|^2 + |\xi_4|^4)} + \beta(\xi), \tag{2.6}
\]
where \( \beta(\xi) \in A_0 \). Since
\[
(|\xi_1|^2 + |\xi_3|^2)^2 \leq (C(|\xi_1|^2 + |\xi_3|^2) + |\xi_3|^2)^2 \leq (C + 1)^2(|\xi_1|^2 + |\xi_3|^2)^2 \leq 2(C + 1)^2(|\xi_1|^4 + |\xi_3|^4),
\]
then
\[
(|\xi_0|^2 + |\xi_3|^4)(|\xi_0|^2 + |\xi_4|^4) \geq |\xi_0|^2(|\xi_0|^2 + |\xi_4|^4) \geq \frac{1}{2(C + 1)^2}|\xi_0|^2(|\xi_0|^2 + (|\xi_1|^2 + |\xi_3|^2)^2),
\]
and
and it follows from (2.6) that in $A$

$$0 \leq f_1(\xi)f_2(\xi) \leq f_3(\xi) = \frac{2(C+1)^2|\xi_0|^2}{|\xi_0|^2 + (|\xi_1|^2 + |\xi_2|^2)^2} \in A_{X_0},$$  \hspace{1cm} (2.7)$$

Taking into account monotonicity of the Gelfand transform (cf. Lemma 2.2), we derive from (2.7) that

$$0 < |\xi_X|^2|\tilde{\xi}_X|^2 = \tilde{f}_1(\eta)\tilde{f}_2(\eta) \leq \tilde{f}_3(\eta) = 2(C+1)^2|\tilde{\xi}_{X_0}|^2.$$ 

Hence, $\tilde{\xi}_{X_0} \neq 0$ and $X_0 = X_1 \cap X_2 \in \mathcal{L}$. Let $\tilde{X}$ be a subspace from $\mathcal{L}$ of minimal dimension. As was already established, for each $X \in \mathcal{L}$ the subspace $X_0 = X \cap \tilde{X} \in \mathcal{L}$. Since $X_0 \subset \tilde{X}$ while $\dim \tilde{X} \leq \dim X_0$, we obtain that $\tilde{X} = X_0 \subset X$. Thus, $X \supset \tilde{X} \forall X \in \mathcal{L}$. Let us demonstrate that, conversely, any subspace $X \supset \tilde{X}$ belongs to $\mathcal{L}$ and $\tilde{\xi}_X \parallel \tilde{\xi}_{\tilde{X}}$. For that, we introduce the space $X_1 = X \ominus \tilde{X}$, so that $\mathbb{R}^n = \tilde{X} \oplus X_1 \oplus X_\perp$. Denote by $\xi_0, \xi_1, \xi_2$ the orthogonal projections of a vector $\xi \in \mathbb{R}^n$ on the subspaces $\tilde{X}, X_1, X_\perp$, respectively. Then $\xi = \xi_0 + \xi_1 + \xi_2$. For arbitrary $u, v \in \mathbb{R}^n$ we find

$$\frac{u \cdot \xi_0}{(|\xi_0|^2 + (|\xi_1|^2 + |\xi_2|^2)^2)^{1/2}} = \frac{v \cdot (\xi_0 + \xi_1)}{(|\xi_0|^2 + (|\xi_1|^2 + |\xi_2|^2)^2)^{1/2}} = \frac{u \cdot (\xi_0 + \xi_1)}{(|\xi_0|^2 + (|\xi_1|^2 + |\xi_2|^2)^2)^{1/2}} + \gamma(\xi),$$

where $\gamma(\xi) = \frac{v \cdot \xi_1}{(|\xi_0|^2 + (|\xi_1|^2 + |\xi_2|^2)^2)^{1/2}} \in A_0$. Applying the Gelfand transform to the above equality, we obtain the equality

$$(u \cdot \tilde{\xi}_X)(v \cdot \tilde{\xi}_X) = (u \cdot \pr_{\tilde{X}}\tilde{\xi}_X)(v \cdot \tilde{\xi}_X).$$  \hspace{1cm} (2.8)$$

Taking $u = \tilde{\xi}_X, v \perp \tilde{X}$, we derive from (2.8) that $v \cdot \tilde{\xi}_X = 0$ for all $v \perp \tilde{X}$, which implies the inclusion $\tilde{\xi}_X \in \tilde{X}$. In particular, $\pr_{\tilde{X}}\tilde{\xi}_X = \tilde{\xi}_X$ and it follows from (2.8) that

$$(u \cdot \tilde{\xi}_X)(v \cdot \tilde{\xi}_X) = (u \cdot \tilde{\xi}_X)(v \cdot \tilde{\xi}_X) \quad \forall u, v \in \mathbb{R}^n.$$ 

In view of this relation we find that $\tilde{\xi}_X = c\tilde{\xi}_X$ for some real constant $c$. Further,

$$\frac{|\xi_0|^2}{|\xi_0|^2 + (|\xi_1|^2 + |\xi_2|^2)^2} = g(\xi) = \frac{|\xi_0|^2 + |\xi_1|^2}{|\xi_0|^2 + (|\xi_1|^2 + |\xi_2|^2)^2},$$  \hspace{1cm} (2.9)$$

Observe that

$$\frac{|\xi_0|^2 + |\xi_1|^2 + |\xi_2|^2}{|\xi_0|^2 + (|\xi_1|^2 + |\xi_2|^2)^2} = g(\xi) = \frac{|\xi_0|^2 + |\xi_1|^4}{|\xi_0|^2 + (|\xi_1|^2 + |\xi_2|^2)^2}$$

up to a term vanishing at infinity, and $g(\xi) \in A_{\tilde{X}}$. Hence, applying the Gelfand transform to (2.9), we obtain

$$0 < |\tilde{\xi}_X|^2 = |\tilde{\xi}_X|^2 g(\eta).$$

It follows from this relation that $\tilde{\xi}_X \neq 0$, and the constant $c \neq 0$. Finally, $c|\tilde{\xi}_X|^2 = \tilde{\xi}_X \cdot \tilde{\xi}_X = h(\eta)$, where

$$h(\xi) = \frac{|\xi_0|^2}{(|\xi_0|^2 + (|\xi_1|^2 + |\xi_2|^2)^2)^{1/2}(|\xi_0|^2 + (|\xi_1|^2 + |\xi_2|^2)^2)^{1/2}} \geq 0.$$
By the monotonicity of the Gelfand transform, we find that $c > 0$. Therefore, $\tilde{\xi}_X \| \tilde{\xi}_X$. Denote $\zeta_1 = \frac{\xi_1}{|\tilde{\xi}_X|} \in \tilde{X}$ (remark that $\zeta_1 = \tilde{\xi}_{2^n}$). Thus,

$$\tilde{\xi}_X \neq 0 \Leftrightarrow X \ni \tilde{X} \text{ and } \tilde{\xi}_X \| \zeta_1.$$  \hspace{1cm} (2.10)

2nd Step.

We introduce the family $\tilde{\mathcal{L}} = \{ X \subset \mathbb{R}^n \ | \ \tilde{\xi}_X = 0 \}$. Let $X_1, X_2 \in \tilde{\mathcal{L}}$. We show that $X_0 = X_1 \cap X_2 \in \tilde{\mathcal{L}}$. For that, we denote $X_{10} = X_1 \ominus X_0$, $X_{20} = X_2 \ominus X_0$. Then representations (2.4) and (2.5) hold. We introduce the functions

$$g_1(\xi) = \frac{|\xi_3|^4}{|\xi_0|^2 + |\xi_1|^2 + |\xi_3|^2}, \quad g_2(\xi) = \frac{|\xi_4|^4}{|\xi_0|^2 + |\xi_2|^2 + |\xi_4|^2},$$

and remark that $\tilde{g}_1(\eta) = \tilde{g}_2(\eta) = 0$, in view of the condition $\tilde{\xi}_{X_1} = \tilde{\xi}_{X_2} = 0$. Since

$$h_1(\xi) = \frac{|\xi_3|^4}{|\xi_0|^2 + (|\xi_1|^2 + |\xi_3|^2)^2} = g_1(\xi) \frac{|\xi_0|^2 + |\xi_1|^2 + |\xi_3|^2}{|\xi_0|^2 + (|\xi_1|^2 + |\xi_3|^2)^2},$$

$$h_2(\xi) = \frac{|\xi_4|^4}{|\xi_0|^2 + (|\xi_2|^2 + |\xi_4|^2)^2} = g_2(\xi) \frac{|\xi_0|^2 + |\xi_2|^2 + |\xi_4|^2}{|\xi_0|^2 + (|\xi_2|^2 + |\xi_4|^2)^2},$$

then $0 \leq h_k(\xi) \leq 2g_k(\xi)$ for $k = 1, 2$ and sufficiently large $|\xi|$. Therefore, $0 \leq \tilde{h}_k(\eta) \leq 2\tilde{g}_k(\eta) = 0$, $k = 1, 2$, and we arrive at $\tilde{h}_k(\eta) = 0$ for $k = 1, 2$. Remark also that $|\xi_1|^2 + |\xi_3|^2 = |\xi_2|^2 + |\xi_4|^2 = |\xi - \xi_0|^2$, and, therefore,

$$p(\xi) \leq |\xi_0|^2 + (|\xi_1|^2 + |\xi_3|^2) = |\xi_0|^2 + (|\xi_2|^2 + |\xi_4|^2) = |\xi - \xi_0|^2.$$ \hspace{1cm} (2.11)

By estimates (2.13) we see that $|\xi_1|^2 \leq C(|\xi_3|^2 + |\xi_4|^2)$. This inequality together with (2.11) imply that

$$\frac{|\xi_1|^2}{(p(\xi))^{1/2}} \leq C \frac{|\xi_3|^2}{(p(\xi))^{1/2}} + C \frac{|\xi_4|^2}{(p(\xi))^{1/2}} = C(h_1(\xi))^{1/2} + C(h_2(\xi))^{1/2}.$$ 

Therefore,

$$h(\xi) \leq \frac{|\xi_1|^2 + |\xi_3|^2}{(p(\xi))^{1/2}} \leq (C + 1)h(\xi)^{1/2} + C(h_2(\xi))^{1/2}.$$ 

This implies that $|\tilde{\xi}_{X_0}|^2 = \tilde{h}(\eta) \leq (C + 1)(\tilde{h}_1(\eta))^{1/2} + C(\tilde{h}_2(\eta))^{1/2} = 0$. Hence, $\tilde{\xi}_{X_0} = 0$ and $X_0 \in \tilde{\mathcal{L}}$. This statement allows to establish existence of minimal element $\tilde{X}$ in $\tilde{\mathcal{L}}$, in the same way as for the family $\mathcal{L}$. Namely, let $\tilde{X}$ be an element in $\tilde{\mathcal{L}}$ of minimal dimension. Then for arbitrary $X \in \tilde{\mathcal{L}}$ the intersection $X_0 = \tilde{X} \cap X \in \tilde{\mathcal{L}}$. Since $X_0 \subset \tilde{X}$ while $\dim \tilde{X} \leq \dim X_0$, then $\tilde{X} = X_0 \subset X$. Hence $\tilde{X}$ is the smallest subspace in $\tilde{\mathcal{L}}$. Notice also that if a subspace $X \supset \tilde{X}$ then $X \in \tilde{\mathcal{L}}$. Indeed, $\mathbb{R}^n = \tilde{X} \oplus X_1 \oplus X_2$, where $X_1 = X \ominus \tilde{X}$. Therefore, $\xi = \xi_0 + \xi_1 + \xi_2$, with $\xi_0, \xi_1, \xi_2$ being the orthogonal projection of $\xi \in \mathbb{R}^n$ on the subspaces $X, X_1, X_2$, respectively. Let

$$\rho(\xi) = \frac{|\xi_0|^2 + |\xi_1|^2 + |\xi_2|^2}{|\xi_0|^2 + (|\xi_1|^2 + |\xi_2|^2)^2}.$$ 

Then

$$\tilde{\rho}(\eta) = |\tilde{\xi}_X|^2 + |\tilde{\xi}_X|^4 = |\tilde{\xi}_X|^2 = 1$$

because $\tilde{\xi}_X = 0$ while $|\tilde{\xi}_X|^2 + |\tilde{\xi}_X|^4 = 1$. 

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Let \( q(\xi) = \frac{|\xi_2|^4}{|\xi_0|^2 + |\xi_1|^2 + |\xi_2|^4} \). Since \( \rho(\xi)q(\xi) = \frac{|\xi_2|^4}{|\xi_0|^2 + (|\xi_1|^2 + |\xi_2|^2)^2} \), we find \( \hat{q}(\eta) = \frac{\hat{\rho}(\eta)q(\eta)}{\hat{\rho}(\eta)} = |pr_{X \perp}\hat{\xi}_X|^4 = 0 \), which implies that \( \hat{\xi}_X = 0 \). Thus, \( X \in \mathcal{L} \), and
\[
\mathcal{L} = \{ X \subset \mathbb{R}^n \mid X \supset X \}
\] (2.12)

Notice that \( |\hat{\xi}_X| = 1 \). Therefore, \( \hat{X} \in \mathcal{L} \) and, in view of (2.11), \( \hat{X} \supset \hat{X} \).

3rd Step.
Assume that \( X_1 \subset X_2 \subset \mathbb{R}^n \) and \( \hat{\xi}_{X_2} \neq 0 \). We claim that \( \hat{\xi}_{X_1} \neq 0 \) and \( \hat{\xi}_{X_2} \mid \hat{\xi} = pr_{X_2}\hat{\xi}_{X_1} \) (that is, \( \hat{\xi} = c\hat{\xi}_{X_2} \) for some \( c \geq 0 \)).

Indeed, if \( \hat{\xi}_{X_1} = 0 \) then \( X_1 \in \mathcal{L} \). By (2.12) we find \( X_2 \in \mathcal{L} \). But this contradicts to the assumption \( \hat{\xi}_{X_2} \neq 0 \). Further, let
\[
p_1(\xi) = (|\xi_1|^2 + (|\xi_2|^2 + |\xi_3|^2)^2)^{1/4}, \quad p_2(\xi) = (|\xi_1|^2 + |\xi_2|^2 + |\xi_3|^4)^{1/4},
\]
where \( \xi_1 = pr_{X_1}\xi \), \( \xi_2 = pr_{X_2 \oplus X_1}\xi \), \( \xi_3 = pr_{X_2'}\xi \). Evidently, for each \( u, v \in \mathbb{R}^n \)
\[
\frac{u \cdot \xi_3}{p_1(\xi)} \cdot \frac{v \cdot \xi_3}{p_2(\xi)} = \frac{v \cdot \xi_3}{p_1(\xi)} \cdot \frac{u \cdot \xi_3}{p_2(\xi)}.
\]

Applying the Gelfand transform to this identity, we find
\[
(u \cdot \hat{\xi})(v \cdot \hat{\xi}_{X_2}) = (v \cdot \hat{\xi})(u \cdot \hat{\xi}_{X_2}) \forall u, v \in \mathbb{R}^n,
\]
where \( \hat{\xi} = pr_{X_2'}\hat{\xi}_{X_1} \). It readily follows from (2.13) that \( \hat{\xi} = c\hat{\xi}_{X_2} \) for some constant \( c \in \mathbb{R} \).

Since \( \hat{\xi} \cdot \hat{\xi}_{X_2} \) coincides with the Gelfand transform of the nonnegative symbol \( \frac{|\xi_3|^2}{p_1(\xi)p_2(\xi)} \), we conclude that \( \hat{\xi} \cdot \hat{\xi}_{X_2} \geq 0 \), i.e. \( c \geq 0 \).

4th Step.
In this step we prove that for any \( X \subset \hat{X} \) the vector \( \hat{\xi}_X \in \hat{X} \). Moreover, in the case \( X \subset \hat{X} \), \( X \neq \hat{X} \) the vector \( \hat{\xi}_X \in \hat{X} \).

First, we notice that if \( X = \hat{X} \), then \( \xi_X = 0 \in \hat{X} \). In the remaining case \( X \neq \hat{X} \) we denote by \( \xi_1, \xi_2, \xi_3 \) the orthogonal projections of \( \xi \in \mathbb{R}^n \) onto the subspaces \( X, X \oplus X, X \perp X \), respectively, and introduce the symbols
\[
a(\xi) = \frac{|\xi_3|^4}{|\xi_1|^2 + (|\xi_2|^2 + |\xi_3|^2)^2} \in A_X, \quad b(\xi) = \frac{|\xi_3|^4}{|\xi_1|^2 + |\xi_2|^2 + |\xi_3|^4} \in A_{\hat{X}}.
\]
as is easy to verify, \( a(\xi) \leq 2b(\xi) \) for sufficiently large \( |\xi| \), which implies
\[
|pr_{X \perp}\hat{\xi}_X|^4 = a(\eta) \leq 2b(\eta) = 2|\xi_X|^4 = 0 \Rightarrow pr_{X \perp}\hat{\xi}_X = 0.
\]
This means that \( \hat{\xi}_X \in \hat{X} \), as was to be proved.

It remains only to consider the case when \( X \subsetneq \hat{X} \). Let \( \xi_1, \xi_2, \xi_3 \) be the orthogonal projections of \( \xi \in \mathbb{R}^n \) onto the subspaces \( X, X \oplus X, X \perp X \), respectively. We introduce the functions
\[
a(\xi) = \frac{|\xi_1|^2}{|\xi_1|^2 + (|\xi_2|^2 + |\xi_3|^2)^2} \in A_X, \quad b(\xi) = \frac{|\xi_1|^2}{|\xi_1|^2 + |\xi_2|^2 + |\xi_3|^2} \in A_{\hat{X}},
\]
\[
c(\xi) = \frac{|\xi_1|^2 + |\xi_2|^2 + |\xi_3|^4}{|\xi_1|^2 + (|\xi_2|^2 + |\xi_3|^2)^2} \sim \frac{|\xi_1|^2 + |\xi_3|^4}{|\xi_1|^2 + (|\xi_2|^2 + |\xi_3|^2)^2}.
\]
Since \( X \not\subseteq \hat{L} \), then \( \hat{\xi}_X = 0 \) and
\[
\hat{c}(\eta) = |\hat{\xi}_X|^2 + |\text{pr}_X \hat{\xi}_X|^4 = |\text{pr}_X \hat{\xi}_X|^4.
\] (2.14)

Further, \( \hat{a}(\eta) = |\hat{\xi}_X|^2 = 0, \hat{b}(\eta) = |\hat{\xi}_X|^2 = 0 \), and \( 0 = \hat{a}(\eta) = \hat{b}(\eta) \hat{c}(\eta) \). Therefore, \( \hat{c}(\eta) = 0 \), and it follows from (2.11) that \( \text{pr}_X \hat{\xi}_X = 0 \), that is, \( \hat{\xi}_X \in \hat{X} \).

5th Step.

Here we construct the orthonormal family \( \{\zeta_k\}_{k=1}^m \). First, we set \( \zeta_1 = \hat{\xi}_{\mathbb{R}^n} = \hat{\xi}_0 \). Assuming that the vectors \( \zeta_1, \ldots, \zeta_{k-1} \) have already known, we define
\[
\zeta_k = \hat{\xi}_{X_{k-1}} / |\hat{\xi}_{X_{k-1}}| \in X_{k-1}^\perp,
\] (2.15)
where \( X_{k-1} \) is a subspace spanned by the vectors \( \zeta_1, \ldots, \zeta_{k-1} \) (notice that \( X_0 = \{0\} \)). This definition is correct while \( X_{k-1} \not\subseteq \hat{X} \) because by (2.12) \( \hat{\xi}_{X_{k-1}} \neq 0 \). As was demonstrated in the 4th step, \( \zeta_k \in \hat{X} \). We see that the construction of \( \zeta_k \) may be continued until \( k = m = \dim \hat{X} \). The \( m \)-dimensional subspace \( X_m \subset \hat{X} \) must coincide with \( \hat{X} \): \( X_m = \hat{X} \), so that \( \hat{\xi}_{X_m} = 0 \). By the construction \( \{\zeta_k\}_{k=1}^m \) is an orthonormal basis in \( \hat{X} \). Let \( d = \dim \hat{X} \). Then \( 1 \leq d \leq m \). By the second statement proven in 4th Step \( \zeta_k \in \hat{X} \) while \( X_{k-1} \not\subseteq \hat{X} \). Since \( \zeta_1 \in \hat{X} \) then by induction \( X_k \subset \hat{X} \) for all \( 1 \leq k \leq d \). Comparing the dimension, we claim that \( \hat{X} = X_d = \mathcal{L}(\zeta_1, \ldots, \zeta_d) \). As was shown in 1st step, for \( \hat{\xi}_X \not\in 0 \) this vector is co-directed with \( \zeta_1 \). The proof of (i) is complete.

To complete the proof of statement (ii), we choose a subspace \( X \subseteq \mathbb{R}^n \) such that \( \hat{\xi}_X \neq 0 \). Then, in view of (2.12) \( \hat{X} \not\subseteq \hat{X} = \mathcal{L}(\zeta_1, \ldots, \zeta_m) \). Therefore, there exists the vector \( \zeta_k \not\in \hat{X} \). Let \( k = k(X) = \min \{ k = 1, \ldots, m \mid \zeta_k \not\in \hat{X} \} \). Then \( X_{k-1} \subset X \), \( \zeta_k \not\in \hat{X} \). Since \( \zeta_k \mid \hat{\xi}_X = 0 \), then by the assertion established in the 3rd Step we claim that \( \hat{\xi}_X \mid \hat{\xi}_{X_{k-1}} = 0 \), as was to be proved.

Remark also that by results of the 3rd Step requirement (ii) for \( X = X_{k-1} \) implies (2.15). This readily implies that the orthonormal family \( \zeta_k, k = 1, \ldots, m \) is uniquely defined by the point \( \eta \). The parameter \( d \) is also uniquely determined by the condition \( d = \dim \hat{X} \).

6th Step. It only remains to show that \( d \geq m - 1 \). Assuming the contrary \( d \leq m - 2 \), we see that the space \( X_1 \) spanned by the vectors \( \zeta_k, k = 1, \ldots, d + 1 \) is a proper subspace of \( \hat{X} \): \( \hat{X} \not\subseteq X_1 \not\subseteq \hat{X} \). We extend the system \( \zeta_k, k = 1, \ldots, m \) to an orthonormal basis \( \zeta_k, k = 1, \ldots, n \) in \( \mathbb{R}^n \). Let \( s_k = s_k(\xi) \), \( k = 1, \ldots, n \) be coordinates of a vector \( \xi \in \mathbb{R}^n \) in this basis: \( \xi = \sum_{k=1}^n s_k \zeta_k \).

We introduce the following functions
\[
p_1(\xi) = \frac{s_1^2}{\sum_{k=1}^d s_k^2 + (\sum_{k=d+1}^n s_k^2)^2}, \quad q_1(\xi) = \frac{s_1^4}{\sum_{k=1}^{d+1} s_k^2 + (\sum_{k=d+2}^n s_k^2)^2},
\]
\[
p_2(\xi) = \frac{s_1^2}{\sum_{k=1}^{d+1} s_k^2 + (\sum_{k=d+2}^n s_k^2)^2}, \quad q_2(\xi) = \frac{s_1^4}{\sum_{k=1}^d s_k^2 + (\sum_{k=d+1}^n s_k^2)^2}.
\]
Obviously, \( p_1, q_2 \in S_{\hat{X}}, p_2, q_1 \in S_{X_1}, \) and \( p_1 q_1 = p_2 q_2 \). Therefore,
\[
\hat{p}_1(\eta) \hat{q}_1(\eta) = \hat{p}_2(\eta) \hat{q}_2(\eta).
\] (2.16)

Now observe that \( \hat{p}_1(\eta) = |\hat{\xi}_X|^2 \neq 0, \hat{p}_2(\eta) = |\hat{\xi}_{X_1}|^2 \neq 0, \hat{q}_1(\eta) = |\hat{\xi}_{X_1}|^4 \neq 0 \) (because \( X_1 \not\subseteq \hat{X} \)), and \( \hat{q}_2(\eta) = |\hat{\xi}_X \cdot \zeta_{d+2}|^4 = 0 \) because \( \hat{\xi}_X \uparrow \xi_{d+1} \perp \xi_{d+2} \). Hence \( \hat{p}_1(\eta) \hat{q}_1(\eta) \neq 0, \hat{p}_2(\eta) \hat{q}_2(\eta) = 0 \), which contradicts (2.16). The proof is complete.
The statement of Proposition 2.1 is sharp, in the sense that for every orthonormal system \( \zeta_k, k = 1, \ldots, m, \) and an integer number \( d \in \{m - 1, m\}, \) one can find a point \( \eta \in S \) such that the statements (i), (ii) of Proposition 2.2 hold. To prove this assertion, we need the notion of an essential ultrafilter. We call sets \( A, B \subseteq \mathbb{R}^n \) equivalent: \( A \sim B \) if \( \mu(A \Delta B) = 0, \) where \( A \Delta B = (A \setminus B) \cup (B \setminus A) \) is the symmetric difference and \( \mu \) is the outer Lebesgue measure. Let \( \mathcal{F} \) be a filter in \( \mathbb{R}^n \). This filter is called essential if from the conditions \( A \in \mathcal{F} \) and \( B \sim A \) it follows that \( B \in A. \) It is clear that an essential filter cannot include sets of null measure, since such sets are equivalent to \( \emptyset. \) Using Zorn’s lemma, one can prove that any essential filter is contained in a maximal essential filter. Maximal essential filters are called essential ultrafilters.

**Lemma 2.3.** Let \( \mathcal{U} \) be an essential ultrafilter. Then for each \( A \subseteq \mathbb{R}^n \) either \( A \in \mathcal{U} \) or \( \mathbb{R}^n \setminus A \in \mathcal{U}. \)

**Proof.** Assuming that \( A \notin \mathcal{U}, \) we introduce

\[ \mathcal{F} = \{ B \subseteq \mathbb{R}^n \mid B \cup A \in \mathcal{U} \}. \]

Obviously, \( \mathcal{F} \) is an essential filter, \( \mathbb{R}^n \setminus A \in \mathcal{F}, \) and \( \mathcal{U} \subseteq \mathcal{F}. \) Since the filter \( \mathcal{U} \) is maximal, we obtain that \( \mathcal{U} = \mathcal{F}. \) Hence, \( \mathbb{R}^n \setminus A \in \mathcal{U}. \) The proof is complete.

The property indicated in Lemma 2.3 is the characteristic property of ultrafilters, see for example, [2]. Therefore, we have the following statement.

**Corollary 2.1.** Any essential ultrafilter is an ultrafilter, i.e. a maximal element in a set of all filters.

**Lemma 2.4.** Let \( \mathcal{U} \) be an essential ultrafilter, and \( f(\xi) \) be a bounded function in \( \mathbb{R}^n. \) Then there exists \( \lim_{\mathcal{U}} f(\xi). \) If a function \( g(\xi) = f(\xi) \) almost everywhere on \( \mathbb{R}^n, \) then there exists \( \lim_{\mathcal{U}} g(\xi) = \lim_{\mathcal{U}} f(\xi). \)

**Proof.** By Corollary 2.1 \( \mathcal{U} \) is an ultrafilter. By the known properties of ultrafilters, the image \( f, \mathcal{U} \) is an ultrafilter on the compact \( [-M, M], \) where \( M = \sup \{f(\xi)\}, \) and this ultrafilter converges to some point \( x \in [-M, M]. \) Therefore, \( \lim_{\mathcal{U}} f(\xi) = \lim_{\mathcal{U}} f, x. \) Further, suppose that a function \( g = f \) a.e. on \( \mathbb{R}^n. \) Then the set \( E = \{ \xi \in \mathbb{R}^n \mid g(\xi) \neq f(\xi) \} \) has null Lebesgue measure. Let \( V \) be a neighborhood of \( x. \) Then \( g^{-1}(V) \supset f^{-1}(V) \setminus E. \) By the convergence of the ultrafilter \( f, \mathcal{U} \) the set \( f^{-1}(V) \subseteq \mathcal{U}. \) Since \( \mathcal{U} \) is an essential ultrafilter while \( f^{-1}(V) \setminus E \sim f^{-1}(V) \), then \( f^{-1}(V) \setminus E \in \mathcal{U}. \) This set is contained in \( g^{-1}(V), \) and we claim that \( g^{-1}(V) \in \mathcal{U}. \) Since \( V \) is an arbitrary neighborhood of \( x, \) we conclude that \( \lim_{\mathcal{U}} g(\xi) = x. \) The proof is complete.

By the statement of Lemma 2.4 the functional \( f \rightarrow \lim_{\mathcal{U}} f(\xi) \) is well-defined on \( L^\infty(\mathbb{R}^n) \) and it is a linear multiplicative functional on \( L^\infty(\mathbb{R}^n). \) In other words, this functional belongs to the spectrum of algebra \( L^\infty(\mathbb{R}^n) \) (actually, this spectrum coincides with the space of such functionals).

Now we are ready to prove the sharpness of Proposition 2.1.

**Proposition 2.2.** Let \( \zeta_k, k = 1, \ldots, m \) be an orthonormal system in \( \mathbb{R}^n, 1 \leq d \in \{m - 1, m\}. \) Then there exists a point \( \eta \in S \) such that the statements (i), (ii) of Proposition 2.1 hold.
PROOF. We extend vectors $\zeta_k$, $k = 1, \ldots, m$ to a basis $\zeta_k$, $k = 1, \ldots, n$ in $\mathbb{R}^n$. Let $\sigma_k$, $k = 2, \ldots, n$ be a decreasing family of positive numbers such that $1 > \sigma_2 > \cdots > \sigma_d > 1/2 \geq \sigma_{d+1} > \cdots > \sigma_n > 0$, and $\sigma_{d+1} = 1/2$ only if $d = m - 1$. We introduce the sets

$$B_r = \left\{ \xi = \sum_{k=1}^{n} s_k \zeta_k \mid |\xi| > r, s_1 > 0, \text{ and } s_1^{\sigma_1} < s_k < 2s_1^{\sigma_1} \forall k = 2, \ldots, n \right\},$$

$r > 0$. It is clear that $B_r$ are nonempty open sets in $\mathbb{R}^n$, which form the base of some essential filter $\mathfrak{F}$. Let $\mathfrak{U}$ be an essential ultrafilter such that $\mathfrak{F} \leq \mathfrak{U}$. Since the limit along $\mathfrak{U}$ is a linear multiplicative functional on $A$ vanishing on the ideal $A_0$, it forms a linear multiplicative functional on $A$, and there exists a unique element $\eta \in \mathcal{S}$ such that $\hat{a}(\eta) = \lim_{\mathfrak{U}} a(\xi)$ for each $a \in A$. We will demonstrate that the element $\eta$ satisfies conditions (i), (ii) of Proposition 2.2. Assume that a subspace $X \not\supset X = \mathcal{L}(\zeta_1, \ldots, \zeta_d)$. Then there exists $k$, $1 \leq k \leq d$ such that $\zeta_k \not\in X$. Let $k = k(X)$ be the minimal one among such $k$. We denote by $P_1$, $P_2$ the orthogonal projections onto the spaces $X$, $X^\perp$, respectively, and set $v_i = P_2 \zeta_i$. Then $v_k \neq 0$ while $v_i = 0$ for $1 \leq i < k$. If $\xi \in B_r$, then

$$r^2 < |\xi|^2 = \sum_{i=1}^{n} s_i^2 \leq s_1^2 + 2 \sum_{i=2}^{n} s_i^{2\sigma_i} \leq C_r s_1^2,$$

(2.17)

where $C_r \to 1$ as $r \to \infty$. Here we take into account the condition $\sigma_i < 1$. In particular, it follows from (2.17) that $s_1 > r/2$ for large $r$. Denote, as above, $\hat{\xi} = P_1 \xi$, $\bar{\xi} = P_2 \xi$. Since $\sigma_i < \sigma_k$ for $i > k$, $s_1 > r/2 \to \infty$, and $|v_k| > 0$, we find that for sufficiently large $r$

$$|\bar{\xi}| = \sum_{i=k}^{n} s_i |v_i| \geq s_k |v_k| - \sum_{i=k+1}^{n} s_i |v_i| \geq s_1^{\sigma_1} |v_k| - 2 \sum_{i=k+1}^{n} s_1^{\sigma_1} |v_i| \geq c s_1^{\sigma_1},$$

(2.18)

where $c = \text{const} > 0$. It follows from (2.17), (2.18) that for $\xi \in B_r$, where $r$ is sufficiently large

$$a(\xi) = \frac{|\bar{\xi}|^2}{|\xi|^2 + |\bar{\xi}|^4} \leq \frac{|\bar{\xi}|^2}{|\xi|^4} \leq \frac{C_r}{c^4} s_1^{2-4\sigma_k} \to 0,$$

(2.19)

as $r \to \infty$ because $\sigma_k > 1/2$ and $s_1 \to \infty$ as $r \to \infty$. It follows from (2.19) that

$$|\bar{\xi}_X(\eta)|^2 = \hat{a}(\eta) = \lim_{\mathfrak{U}} a(\xi) = \lim_{\mathfrak{U}} a(\xi) = 0.$$

We claim that $\bar{\xi}_X(\eta) = 0$.

If $X \supset X$, then $\xi = \sum_{i=d+1}^{n} s_i v_i$, where $v_i = P_2 \zeta_i$, and for $\xi \in B_r$

$$s_1^2 \leq |\xi|^2 \leq \sum_{i=1}^{n} s_i^2 \leq C_1 s_1^2,$$

(2.20)

$$|\bar{\xi}| \leq \sum_{i=d+1}^{n} s_i \leq C_2 s_1^{\sigma_{d+1}}, \quad C_1, C_2 = \text{const}. \quad (2.21)$$
Since $\sigma_{d+1} \leq 1/2$, then it follows from (2.20), (2.21) that for sufficiently large $r$

$$a(\xi) = \frac{|\xi|^2}{|\xi|^2 + |\xi|^4} \geq (C_1 + C_2^2)^{-1} > 0,$$

which implies that $|\hat{\xi}_X(\eta)|^2 = \hat{a}(\eta) = \lim_{\eta \to \eta} a(\xi) > 0$. Hence $\hat{\xi}_X = \hat{\xi}_X(\eta) \neq 0$. Observe also that, as follows from (2.20), (2.21),

$$\frac{s_i}{(|\xi|^2 + |\xi|^4)^{1/2}} \leq cs_i^2 - 1 \implies 0, \quad i = 2, \ldots, n,$$

and, therefore,

$$\hat{\xi}_X = \lim_{\eta \to \eta} \frac{\xi}{(|\xi|^2 + |\xi|^4)^{1/2}} = \left(\lim_{\eta \to \eta} \frac{s_1}{(|\xi|^2 + |\xi|^4)^{1/2}}\right) \zeta_1 \uparrow \zeta_1.$$ 

We conclude that condition (i) is satisfied.

To prove (ii), assume that $X \supset \hat{X} = \mathcal{L}(\zeta_1, \ldots, \zeta_m)$. Then $\hat{\xi} = \sum_{i=m+1}^n s_i v_i, v_i = P_2 \zeta_i$, which implies the estimate

$$|\hat{\xi}| \leq 2 \sum_{i=m+1}^n s_i^2 \leq Cs_1^{\sigma_{m+1}}, \quad C = \text{const},$$

for all $\xi \in B_r$ with sufficiently large $r$. On the other hand, $|\hat{\xi}| \geq s_1$. Therefore, for $\xi \in B_r$

$$\frac{|\hat{\xi}|^4}{|\hat{\xi}|^2 + |\hat{\xi}|^4} \leq C^4 s_1^{4\sigma_{m+1} - 2} \to 0 \quad r \to \infty$$

because $\sigma_{m+1} < 1/2$. Hence,

$$|\hat{\xi}_X|^4 = \lim_{\eta \to \eta} \frac{|\hat{\xi}|^4}{|\hat{\xi}|^2 + |\hat{\xi}|^4} = 0,$$

that is, $\hat{\xi}_X = 0$.

Now, suppose that $X \supset \hat{X}$. Then there exists $\zeta_k \notin X$, where $1 \leq k \leq m$. We chose $k$ being the minimal one. Then $\xi \in X, 1 \leq i < k$, and $\hat{\xi} = \sum_{i=k}^n s_i v_i, v_i = P_2 \zeta_i$, which implies the estimate

$$|\hat{\xi}| \geq s_k |v_k| - \sum_{i=k+1}^n s_i |v_i| \geq s_1^{\sigma_k} |v_k| - 2 - \sum_{i=k+1}^n s_i^{\sigma_k} |v_i| \geq cs_1^{\sigma_k}, \quad c = |v_k|/2 > 0, \quad (2.22)$$

for all $\xi \in B_r$ with sufficiently large $r$. We use here that $v_k \neq 0$ and $\sigma_k > \sigma_i$ for $i > k$. Further,

$$|\hat{\xi}|^2 \leq |\xi|^2 = \sum_{i=1}^n s_i^2 \leq s_1^2 + 2 \sum_{i=2}^n s_i^{2\sigma_i} \leq 2 s_1^2 \quad (2.23)$$

for all $\xi \in B_r$ with large $r$. It follows from (2.22), (2.23) and from the condition $\sigma_k \geq \sigma_m \geq 1/2$ that

$$cs_1^{\sigma_k} \leq (|\hat{\xi}|^2 + |\hat{\xi}|^4)^{1/4} \leq Cs_1^{\sigma_k}, \quad C = \text{const}. \quad (2.24)$$
In view of (2.24) for all $\xi \in B_r$ with sufficiently large $r$

$$\frac{s_i}{(|\xi|^2 + |\xi|^4)^{1/4}} \leq \frac{2}{c} \frac{s_i}{s_i^\sigma} \to 0, \quad k + 1 \leq i \leq n,$$

$$\frac{1}{C} \leq \frac{s_k}{(|\xi|^2 + |\xi|^4)^{1/4}} \leq \frac{2}{c}.$$ 

This implies that

$$\bar{\xi}(\eta) = av_k \upharpoonright \text{pr}_{X\perp} \xi_k,$$

where

$$a = \lim_{k \to \infty} \frac{s_k}{(|\xi|^2 + |\xi|^4)^{1/4}} > 0,$$

and $k = k(\xi) = \min\{ k = 1, \ldots, m \mid \xi_k \notin X \}$.

We see that requirement (ii) of Proposition 2.4 is also satisfied. The proof is complete.

**Remark 2.1.** For each real $t \neq 0$ the map $h_t(\psi)(\xi) = \psi^t \hat{=} \psi(t\xi)$ is an isomorphism of algebra $A$. Indeed, it is easy to verify that

$$A_{\psi^t} = Q_t^{-1} A_{\psi} Q_t, \quad B_{\psi} = Q_t^{-1} B_{\psi} Q_t \quad \forall \psi(\xi) \in A, \Phi(x) \in C_0(\mathbb{R}^n),$$

where $\Phi(x) = \Phi(tx)$, and the operator $Q_t$ in $L^2(\mathbb{R}^n)$ is defined by the equality $Q_t u(x) = u^t = u(tx)$. Therefore, the operators $[A_{\psi^t}, B_{\psi}] = Q_t^{-1} [A_{\psi}, B_{\psi}] Q_t$ are compact in $L^2$ for all $\Phi(x) \in C_0(\mathbb{R}^n)$. This implies that $h_t$ is well-defined on $A$ and evidently transfers the ideal $A_0$ into itself. This allows to define the operator $h_t$ on the quotient algebra $A = A/A_0$. It is clear that $h_t$ is invertible and $h_t^{-1} = h_{1/t}$. Therefore, the operator $h_t$ generates the corresponding homeomorphism of the spectrum $h_t : S \to S$, so that $\psi(h_t(\eta)) = h_t(\psi)(\eta)$. We denote $h_t(\eta) = t\eta$. This determines an action of the multiplicative group of $\mathbb{R}$ on the space $S$. If $X$ is a subspace of $\mathbb{R}^n$, and $(\xi(\eta), \xi(\eta)) = \prod X(\eta)$, then it is directly verified that for each $t \neq 0$

$$\bar{\xi}(t\eta) = a(t, \eta) \bar{\xi}(t\eta), \quad \bar{\xi}(\eta) = b(t, \eta) \bar{\xi}(\eta),$$

where

$$a(t, \eta) = t(t^2 |\xi(\eta)|^2 + t^4 |\xi(\eta)|^4)^{-1/2}, \quad b(t, \eta) = t(t^2 |\xi(\eta)|^2 + t^4 |\xi(\eta)|^4)^{-1/4}.$$

In particular, $(b(t, \eta))^2 = ta(t, \eta)$.

**3. H-measures and the localization property**

Now, let $\Omega \subset \mathbb{R}^n$ be an open domain and $U_r(x) \in L^2_{loc}(\Omega, \mathbb{C}^N)$ be a sequence of generally complex-valued vector functions weakly convergent to the zero vector. Denote by $B\lim$ a generalized Banach limit (see [2]), that is, a linear functional on the Banach space $l_\infty$ of bounded sequences such that for each real sequence $x = \{x_r\}_{r=1}^\infty \in l_\infty$

$$\lim_{r \to \infty} x_r \leq B\lim x_r \leq \lim_{r \to \infty} x_r$$

(we use the customary notation $B\lim x_r$ for the the Banach limit of the sequence $x$).

In order to justify the notion of $H$-measures, we will need the following result on representation of bilinear functionals.
Lemma 3.1. Let $X,Y$ be locally compact Hausdorff spaces, and $F(f,g)$ be a bilinear functional on $C_0(X) \times C_0(Y)$ such that for every compact subsets $K_1 \subset X$, $K_2 \subset Y$

$$|F(f,g)| \leq C(K_1,K_2)||f||\|g\|_\infty \quad \forall f \in C_0(K_1), g \in C_0(K_2), \quad \text{(continuity)},$$

where the constant $C(K_1,K_2)$ depends only on compacts $K_1$, $K_2$, and $F(f,g) \geq 0 \quad \forall f,g \geq 0 \quad \text{(nonnegativity)}.$

Then there exists a unique locally finite nonnegative Radon measure $\mu = \mu(x,y)$ on $X \times Y$ such that

$$F(f,g) = \int_{X \times Y} f(x)g(y)d\mu(x,y).$$

PROOF. First, we consider the case when $X, Y$ are compact sets of Euclidean spaces: $X \subset \mathbb{R}^m$, $Y \subset \mathbb{R}^l$. In this case the statement of Lemma 3.1 was established in [9, Lemma 1.10]. For completeness we reproduce below the proof. Assuming that $m \geq l$, we may suppose that $X,Y$ are compact subsets of the same Euclidean space: $X,Y \subset \mathbb{R}^m$. We choose a function $K(z) \in C_0(\mathbb{R}^m)$ such that $K(z) \geq 0$, $\text{supp} \ K \subset B_1 = \{ z \in \mathbb{R}^m \mid |z| \leq 1 \}$, $\int K(z)dz = 1$, and set $K_r(z) = r^mK(rz)$, where $r \in \mathbb{N}$. Obviously, the sequence $K_r(z)$ converges as $r \to \infty$ to the Dirac $\delta$-measure $\delta(z)$ weakly in $D'((\mathbb{R}^m))$. For $f(x) \in C(\mathbb{R}^m)$ we introduce the averaged functions $f_r(p) = f \ast K_r(p)$ where $K_r(p) = \int f(x)K_r(p-x)dx$. By the known properties of averaged functions, $f_r \to f$ as $r \to \infty$ uniformly on any compact. This together with the continuity assumption implies that

$$F(f,g) = \lim_{r \to \infty} F_r(f,g),$$

where $F_r(f,g) = F(f_r,g_r)$, and the averaged functions

$$f_r(p) = \int_{\mathbb{R}^m} f(x)K_r(p-x)dx, \quad g_r(q) = \int_{\mathbb{R}^m} g(y)K_r(q-y)dy$$

are reduced to the sets $X$ and $Y$, respectively. As it follows from the continuity of $F$,

$$F_r(f,g) = F(f_r,g_r) = \int_{\mathbb{R}^m \times \mathbb{R}^m} f(x)g(y)\alpha_r(x,y)dxdy,$$

where

$$\alpha_r(x,y) = F(K_r(p-x),K_r(q-y)).$$

It is easy to verify that $\alpha_r(x,y) \in C_0(\mathbb{R}^m \times \mathbb{R}^m)$, supp $\alpha_r \subset X_r \times Y_r$, where $X_r = X + B_{1/r}$, $Y_r = Y + B_{1/r}$, $r \in \mathbb{N}$, and by $B_{\rho}$ we denotes the closed ball of radius $\rho$ centered at zero: $B_{\rho} = \{ z \in \mathbb{R}^m \mid |z| \leq \rho \}$. Moreover, by the nonnegativity of $F$ we see that the functionals $F_r$ are also nonnegative: $F_r(f,g) \geq 0$ whenever $f,g \geq 0$, and this readily implies that the kernels $\alpha_r(x,y) \geq 0$. Besides,

$$\int_{\mathbb{R}^m \times \mathbb{R}^m} \alpha_r(x,y)dxdy = F_r(1,1) \leq C,$$

where $C = C(X,Y)$ is the constant from (3.1). Therefore, the sequence of nonnegative measures $\mu_r = \alpha_r(x,y)dxdy$ weakly converges as $r \to \infty$ to a finite nonnegative Radon measure $\mu = \mu(x,y)$. Since $X \times Y = \cap_{r=1}^{\infty} X_r \times Y_r$, we see that supp $\mu \subset X \times Y$. For $f \in C(X)$, $g \in C(Y)$ let
\[ \tilde{f}, \tilde{g} \in C(\mathbb{R}^m) \] be continuous extensions of these functions on the whole space. Then, in view of [3.2], [3.3]

\[
F(f, g) = \lim_{r \to \infty} F_r(\tilde{f}, \tilde{g}) = \lim_{r \to \infty} \int_{\mathbb{R}^m \times \mathbb{R}^m} \tilde{f}(x)\tilde{g}(y)\alpha_r(x, y)dx dy = \int_{X \times Y} \tilde{f}(x)\tilde{g}(y) d\mu(x, y) = \int_{X \times Y} f(x)g(y) d\mu(x, y),
\]

and representation [3.3] follows. Observe that the measure \( \mu \) is finite and uniquely defined by [3.3] because linear combinations of the functions \( f(x)g(y) \) are dense in \( C(X \times Y) \). Thus, the proof in the case of compacts \( X \subseteq \mathbb{R}^m, Y \subseteq \mathbb{R}^l \) is complete.

In the case of arbitrary Hausdorff compacts \( X, Y \), we introduce the set \( \mathfrak{A} \), consisting of pairs \( (A, B) \) of finite subsets \( A \subset C(X), B \subset C(Y) \). The set \( \mathfrak{A} \) is ordered by the inclusion order: \( \alpha = (A_1, B_1) \leq \beta = (A_2, B_2) \) if \( A_1 \subset A_2, B_1 \subset B_2 \). It is clear, that for each \( \alpha, \beta \in \mathfrak{A} \) there exists \( \gamma \in \mathfrak{A} \) such that \( \alpha \leq \gamma, \beta \leq \gamma \), that is, \( \mathfrak{A} \) is a directed set. Let \( \alpha = (A, B) \in \mathfrak{A}, A = \{f_1(x), \ldots, f_m(x)\} \subset C(X), B = \{g_1(y), \ldots, g_l(y)\} \subset C(Y), m, l \in \mathbb{N}, \) and let \( F : X \mapsto \mathbb{R}^m, G : Y \mapsto \mathbb{R}^l \) be continuous mapping such that \( F(x) = (f_1(x), \ldots, f_m(x)), G(y) = (g_1(y), \ldots, g_l(y)) \). Then \( \tilde{X} = F(X), \tilde{Y} = G(Y) \) are compact subsets of Euclidean spaces \( \mathbb{R}^m \) and \( \mathbb{R}^l \), respectively.

We introduce the bilinear functional \( F_\alpha(\phi, \psi) \) on \( C(\tilde{X}) \times C(\tilde{Y}) \), setting

\[
F_\alpha(\phi, \psi) = F(\phi(F(x)), \psi(G(y))). \tag{3.6}
\]

Clearly, this functional satisfies both the continuity and the nonnegativity conditions. Then, as we have already established, there exists a unique nonnegative Radon measure \( \nu_\alpha = \nu_\alpha(p, q) \) on \( \tilde{X} \times \tilde{Y} \) such that

\[
F_\alpha(\phi, \psi) = \int_{\tilde{X} \times \tilde{Y}} \phi(p)\psi(q) d\nu_\alpha(p, q). \tag{3.7}
\]

Moreover, \( \nu_\alpha(\tilde{X} \times \tilde{Y}) = F(1, 1) \leq C \), where \( C = C(\tilde{X}, \tilde{Y}) \) is the constant from condition [3.1].

We consider the linear functional

\[
\varphi_\alpha(h) = \int \tilde{h}(p, q) d\nu_\alpha(p, q), \tag{3.8}
\]

defined on the subspace \( H_\alpha \) of \( C(X \times Y) \), consisting of functions \( h(x, y) = \tilde{h}(F(x), G(y)), \tilde{h}(p, q) \in C(\tilde{X} \times \tilde{Y}) \). This functional satisfies the property

\[
\varphi_\alpha(h) \leq p(h) = C \max_{X \times Y} h^+(x, y), \quad h^+ = \max(h, 0) \tag{3.9}
\]

for all real function \( h \in H_\alpha \). Observe that \( p(h) \) is a sub-linear functional on \( C(X \times Y) \). Hence, by Hahn-Banach theorem the functional \( \varphi_\alpha \) can be extended to a linear functional \( \hat{\varphi}_\alpha \) on the whole space \( C(X \times Y) \), satisfying estimate [3.9] for real continuous functions on \( X \times Y \). In particular, for each real \( h(x, y) \in C(X \times Y) \)

\[
-C \max_{X \times Y} h^-(x, y) \leq -p(-h) \leq \hat{\varphi}_\alpha(h) \leq p(h) = C \max_{X \times Y} h^+(x, y),
\]

which implies, firstly, that \( \hat{\varphi}_\alpha(h) \geq 0 \) whenever \( h \geq 0 \) and, secondly, that \( |\hat{\varphi}_\alpha(h)| \leq C \max_{X \times Y} |h(x, y)| = C \|h\|_\infty \). We see that \( \hat{\varphi}_\alpha \) is a nonnegative continuous functional on
\[ C(X \times Y), \text{ and } \| \varphi_\alpha \| \leq C. \] By Riesz-Markov representation theorem there exists a unique nonnegative Radon measure \( \mu_\alpha \) on \( X \times Y \) such that

\[ \varphi_\alpha(h) = \int_{X \times Y} h(x, y) d\mu_\alpha(x, y), \tag{3.10} \]

and \( \mu_\alpha(X \times Y) \leq C. \) Observe also that in view of (3.6), (3.7), (3.8), and (3.10)

\[ F(f, g) = \varphi_\alpha(f(x)g(y)) = \int_{X \times Y} f(x)g(y) d\mu_\alpha(x, y) \tag{3.11} \]

for all \( f \in A, \ g \in B. \) Since the space \( M(X \times Y) \) of bounded Radon measures on \( X \times Y \) (with the total variation as a norm) is dual to \( C(X \times Y) \), then bounded sets in \( M(X \times Y) \) are weakly precompact. Therefore, there exists an accumulation point \( \mu \) of a net \( \mu_\alpha, \alpha \in \mathfrak{A} \) with respect to the weak topology in \( M(X \times Y) \). Let \( f(x) \in C(X), \ g(y) \in C(Y), \) and \( \alpha_0 = (\{ f \}, \{ g \}) \in \mathfrak{A}. \) Since \( \mu \) is an accumulation point of a net \( \mu_\alpha, \) then there exists a increasing sequence \( \alpha_n = (A_n, B_n) \in \mathfrak{A}, \ n \in \mathbb{N}, \) such that \( \alpha_n > \alpha_0 \) and in view of (3.11)

\[ F(f, g) = \varphi_\alpha((f(x)g(y))) = \int_{X \times Y} f(x)g(y) d\mu_{\alpha_n}(x, y) \to_{n \to \infty} \int_{X \times Y} f(x)g(y) d\mu(x, y). \]

This relation implies the desired representation (3.3) with the finite nonnegative Radon measure \( \mu. \) Uniqueness of the measure \( \mu \) follows again from the density in \( C(X \times Y) \) of linear combinations of the functions \( f(x)g(y). \)

Now, we consider the general case of locally compact Hausdorff spaces \( X, Y. \) We introduce the directed set \( \mathfrak{R} \) consisting of pairs \( \alpha = (K, L) \) of compacts \( K \subset X, \ L \subset Y \) and ordered by the inclusion order, i.e., \( \alpha = (K, L) \leq \alpha_1 = (K_1, L_1) \) if \( K \subset K_1, \ L \subset L_1. \) For each \( \alpha = (K, L) \in \mathfrak{R} \) there exist functions \( a_\alpha(x) \in C_0(X), \ b_\alpha(y) \in C_0(Y) \) with the following properties 0 \( \leq a_\alpha(x) \leq 1, \ 0 \leq b_\alpha(y) \leq 1, \) and \( a_\alpha(x) = b_\alpha(y) = 1 \) for all \( x \in K, \ y \in L. \) We denote \( X_\alpha = \text{supp} a_\alpha, \ Y_\alpha = \text{supp} b_\alpha \) and define the bilinear functional \( F_\alpha : C(X_\alpha) \times C(Y_\alpha) \to \mathbb{C} \) by the identity \( F_\alpha(f, g) = \int_{X_\alpha \times Y_\alpha} f(x)g(y) d\mu_\alpha(x, y). \) It is assumed that the functions \( f(a_\alpha)(x), \ (g b_\alpha)(y) \) are extended on the whole spaces \( X, Y, \) being zero outside of \( X_\alpha, Y_\alpha, \) respectively. In particular, these functions have compact supports and the functional \( F_\alpha \) is well-defined. Obviously,

\[ F_\alpha(f, g) \leq C(X_\alpha, Y_\alpha)\| f a_\alpha \| \| g b_\alpha \| \leq C_\alpha\| f \| \| g \|, \quad C_\alpha = C(X_\alpha, Y_\alpha) \]

and \( F_\alpha(f, g) = \int_{X_\alpha \times Y_\alpha} f(x)g(y) d\mu_\alpha(x, y). \) This measure may be considered as a Radon measure on the space \( X \times Y \) with the support in \( X_\alpha \times Y_\alpha. \) Then, for every \( f = f(x) \in C(X_\alpha), \ g = g(y) \in C(Y_\alpha) \)

\[ F_\alpha(f, g) = \int_{X \times Y} f(x)g(y) d\mu_\alpha(x, y). \tag{3.12} \]

Let \( K \subset X, \ L \subset Y \) be compact subsets, \( \beta = (K, L) \in \mathfrak{R}. \) Suppose that \( \alpha \in \mathfrak{R}. \) Since the nonnegative function \( a_\beta(x) b_\beta(y) \equiv 1 \) on \( K \times L, \) then

\[ \mu_\alpha(K \times L) \leq \int_{X \times Y} a_\beta(x) b_\beta(y) d\mu_\alpha(y) = F_\alpha(a_\beta, b_\beta) \]

\[ F(a_\alpha a_\beta, b_\alpha b_\beta) \leq F(a_\beta, b_\beta) = F_\beta(1, 1) \leq C_\beta, \tag{3.13} \]

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where we use the nonnegativity of $F$, that implies the monotonicity of this functional on the sets of nonnegative functions: $F(f_1, g_1) \geq F(f_2, g_2)$ for all $f_1, f_2 \in C_0(X), g_1, g_2 \in C_0(Y)$ such that $0 \leq f_1(x) \leq f_2(x), 0 \leq g_1(y) \leq g_2(y)$ (indeed, $F(f_2, g_2) - F(f_1, g_1) = F(f_2 - f_1, g_2) + F(f_1, g_2 - g_1) \geq 0$).

In view of estimates (3.13), the net $\mu_\alpha, \alpha \in \mathbb{R}$ is bounded in locally convex space $M_{loc}(X \times Y)$ of locally finite Radon measures (with topology generated by seminorms $p_\alpha(\mu) = |\mu|(K \times L), \alpha = (K, L) \in \mathbb{R}, |\mu|$ stands for the variation of measure $\mu$.

Since the bounded sets of the space $M_{loc}(X \times Y)$ (which is dual to $C_0(X \times Y)$) are compact, there exists a weak accumulation point $\mu \in M_{loc}(X \times Y)$ of the net $\mu_\alpha, \alpha \in \mathbb{R}$. Since $\mu_\alpha \geq 0$ for all $\alpha \in \mathbb{R}$, we claim that $\mu \geq 0$. Let $f(x) \in C_0(X), g(y) \in C_0(Y)$, and $\alpha_0 = \sup f, \sup g \in \mathbb{R}$. Since $\mu$ is an accumulation point of the net $\mu_\alpha, \alpha \in \mathbb{R}$, there exists an increasing sequence $\alpha_n = (K_n, L_n) \in \mathbb{R}, n \in \mathbb{N}$, such that $\alpha_n > \alpha_0$ and

$$F(f, g) = \int_{X \times Y} f(x)g(y)d\mu_{\alpha_n}(x, y) \rightarrow_{n \rightarrow \infty} \int_{X \times Y} f(x)g(y)d\mu(x, y).$$

This implies representation (3.3) and conclude the proof.

The following statement, analogous to the assertions of Propositions 1, 11, 1.2 holds.

**Proposition 3.1.** There exists a family of Radon measures $\mu = \{\mu^{\alpha \beta}\}_{\alpha, \beta = 1}^N$ on $\Omega \times \mathcal{A}$ such that for all $\Phi_1(x), \Phi_2(x) \in C_0(\Omega), \psi(\xi) \in \mathcal{A},$ and $\alpha, \beta = 1, \ldots, N$

$$\langle \mu^{\alpha \beta}(x, \eta), \Phi_1(x)\overline{\Phi_2(x)}\psi(\eta) \rangle = B \lim_{r \rightarrow \infty} \int_{\mathbb{R}^n} F(\Phi_1 U^\alpha r \Phi_1)(\xi)F(\Phi_2 U^\beta r \Phi_2)(\xi)\psi(\xi)d\xi. \tag{3.14}$$

The matrix-valued measure $\mu$ is Hermitian and positive semi-definite, i.e., for every $\zeta = (\zeta_1, \ldots, \zeta_N) \in \mathbb{C}^n$

$$\mu \zeta \cdot \zeta = \sum_{\alpha, \beta = 1}^N \mu^{\alpha \beta} \zeta_\alpha \overline{\zeta_\beta} \geq 0.$$

**Proof.** Denote for $\Phi_1(x), \Phi_2(x) \in C_0(\Omega), \psi(\xi) \in \mathcal{A}$

$$I^{\alpha \beta} = \lim_{r \rightarrow \infty} \int_{\mathbb{R}^n} F(\Phi_1 U^\alpha r \Phi_1)(\xi)F(\Phi_2 U^\beta r \Phi_2)(\xi)\psi(\xi)d\xi \tag{3.15}$$

and observe that, by the Buniakowskii inequality and the Plancherel identity,

$$|I^{\alpha \beta}| \leq \|\Phi_1\|_\infty \|\Phi_2\|_\infty \|\psi\|_\infty \cdot \lim_{r \rightarrow \infty} \left[ \|U^\alpha r\|_{L^2(K)} \|U^\beta r\|_{L^2(K)} \right], \tag{3.16}$$

where $K \subset \Omega$ is a compact containing supports of $\Phi_1$ and $\Phi_2$. In view of the weak convergence of sequences $U^\alpha r$ in $L^2(K)$ these sequences are bounded in $L^2(K)$. Therefore, for some constant $C_K$ we have $\|U^\alpha r\|_{L^2(K)} \leq C_K$ for all $r \in \mathbb{N}, \alpha = 1, \ldots, N$. Then, it follows from (3.16) that

$$|I^{\alpha \beta}(\Phi_1, \Phi_2, \psi)| \leq C_K \|\Phi_1\|_\infty \|\Phi_2\|_\infty \|\psi\|_\infty \tag{3.17}$$

with $K = \sup \Phi_1 \cup \sup \Phi_2$. If $\psi(\xi) \in A_0$, then by Lemma 2.1, the operator $B_\psi A_{\Phi_1}$ is compact in $L^2(\mathbb{R}^n)$. Hence, the sequences $B_\psi A_{\Phi_1}(U^\alpha r) = B_\psi A_{\Phi_1}(U^\alpha r \chi_K)$ converge to $0$ in $L^2(\mathbb{R}^n)$. Here $\chi_K(x)$ is the indicator function of the compact $K = \sup \Phi_1$. We see that for all $\alpha = 1, \ldots, N$

$$F(\Phi_1 U^\alpha r)(\xi)\psi(\xi) = F(B_\psi A_{\Phi_1}(U^\alpha r))(\xi) \rightarrow 0 \text{ in } L^2(\mathbb{R}^n),$$

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and, therefore, for all \( \alpha, \beta = 1, \ldots, N \)
\[
\lim_{r \to \infty} \int_{\mathbb{R}^n} F(\Phi_1 U^\alpha_r) (\xi) F(\Phi_2 U^\beta_r) (\xi) \psi(\xi) d\xi = 0.
\]
In view of (3.15) \( I^{\alpha \beta}(\Phi_1, \Phi_2, \psi) = 0 \) for \( \Phi_1(x), \Phi_2(x) \in C_0(\Omega) \) and all \( \psi(\xi) \in A_0 \). We see that the linear with respect to \( \psi \) functional \( I^{\alpha \beta}(\Phi_1, \Phi_2, \psi) \) is well-defined on factor-algebra \( \mathcal{A} = A/A_0 \) and, in view of (3.17), for all \( \psi_0 \in A_0 \)
\[
|I^{\alpha \beta}(\Phi_1, \Phi_2, \psi)| = |I^{\alpha \beta}(\Phi_1, \Phi_2, \psi - \psi_0)| \leq C_K \| \Phi_1 \|_\infty \| \Phi_2 \|_\infty \| \psi - \psi_0 \|_\infty.
\]
Therefore,
\[
I^{\alpha \beta}(\Phi_1, \Phi_2, \psi) \leq C_K \| \Phi_1 \|_\infty \| \Phi_2 \|_\infty \inf_{\psi_0 \in A_0} \| \psi - \psi_0 \|_\infty = C_K \| \Phi_1 \|_\infty \| \Phi_2 \|_\infty \| \psi \|_\mathcal{A}, \tag{3.18}
\]
where \( \| \psi \|_\mathcal{A} = \inf_{\psi_0 \in A_0} \| \psi - \psi_0 \|_\infty \) is the factor-norm of \( [\psi] \) in \( \mathcal{A} \). Now, we observe that
\[
\int_{\mathbb{R}^n} F(\Phi_1 U^\alpha_r)(\xi) F(\Phi_2 U^\beta_r)(\xi) \psi(\xi) d\xi = (A_\psi(\Phi_1 U^\alpha_r), \Phi_2 U^\beta_r)_2, \tag{3.19}
\]
where \((\cdot, \cdot)_2\) is the scalar product in \( L^2 = L^2(\mathbb{R}^n) \). Let \( \omega(x) \in C_0(\mathbb{R}^n) \) be a function such that \( \omega(x) \equiv 1 \) on supp \( \Phi_1 \). Then
\[
A_\psi(\Phi_1 U^\alpha_r) = A_\psi B_{\Phi_1}(\omega U^\alpha_r) = B_{\Phi_1} A_\psi (\omega U^\alpha_r) + [A_\psi, B_{\Phi_1}](\omega U^\alpha_r), \tag{3.20}
\]
By the definition of algebra \( \mathcal{A} \), the operator \([A_\psi, B_{\Phi_1}]\) is compact on \( L^2 \) and since \( \omega U^\alpha_r \to 0 \) as \( r \to \infty \) weakly in \( L^2 \), we claim that \([A_\psi, B_{\Phi_1}](\omega U^\alpha_r) \to 0 \) as \( r \to \infty \) strongly in \( L^2 \). Since the sequence \( \Phi_2 U^\beta_r \) is bounded in \( L^2 \), we conclude that \((\omega U^\alpha_r, \Phi_2 U^\beta_r)_2 \to 0 \) as \( r \to \infty \). It follows from this limit relation and (3.19), (3.20) that
\[
I^{\alpha \beta}(\Phi_1, \Phi_2, \psi) = B \lim_{r \to \infty} (B_{\Phi_2} A_\psi(\omega U^\alpha_r), \Phi_2 U^\beta_r)_2 = B \lim_{r \to \infty} \int_{\mathbb{R}^n} \Phi_1(x) \Phi_2(x) A_\psi(\omega U^\alpha_r)(x) U^\beta_r(x) dx.
\]
We claim that
\[
I^{\alpha \beta}(\Phi_1, \Phi_2, \psi) = \tilde{I}^{\alpha \beta}(\Phi, \hat{\psi}),
\]
where
\[
\tilde{I}^{\alpha \beta}(\Phi, \hat{\psi}) = B \lim_{r \to \infty} \int_{\mathbb{R}^n} \Phi(x) A_\psi(\omega U^\alpha_r)(x) U^\beta_r(x) dx
\]
is a bilinear functional on \( C_0(\Omega) \times C(\mathcal{S}) \) for each \( \alpha, \beta = 1, \ldots, N \) (\( \omega(x) \in C_0(\mathbb{R}^n) \) is an arbitrary function equalled 1 on support of \( \Phi(x) \)), and \( \hat{\psi}(\eta) \) being the Gelfand transform of \( \psi(\xi) \). Taking in the above relation \( \Phi_1 = \Phi(x) / \sqrt{\Phi(x)} \) (we set \( \Phi_1(x) = 0 \) if \( \Phi(x) = 0 \), \( \Phi_2 = \sqrt{\Phi(x)} \), where \( \Phi(x) \in C_0(\Omega) \), we find with the help of (3.18) that
\[
|\tilde{I}^{\alpha \beta}(\Phi, \hat{\psi})| = |I^{\alpha \beta}(\Phi_1, \Phi_2, \psi)| \leq C_K \| \Phi_1 \|_\infty \| \Phi_2 \|_\infty \| \psi \|_\mathcal{A} = C_K \| \Phi \|_\infty \| \hat{\psi} \|_\mathcal{A}, K = \text{supp} \Phi.
\]
This estimate shows that the functionals in (3.22) are continuous on $C_0(\Omega) \times C(\mathcal{S})$. Now, we observe that for nonnegative $\Phi(x)$ and $\psi(\eta)$ the matrix $\hat{I} = \{\hat{I}^{\alpha\beta}(\Phi, \psi)\}_{\alpha, \beta=1}^N$ is Hermitian and positive definite. First, we remark that by Lemma 2.2 $\hat{I}^{\alpha\beta}(\Phi, \psi) \geq 0$ if and only if $\psi(\xi) \geq 0$. Taking $\Phi_1(x) = \Phi_2(x) = \sqrt{\Phi(x)}$, we find

$$\hat{I}^{\alpha\beta}(\Phi, \psi) = I^{\alpha\beta}(\Phi_1, \Phi_2, \psi) = B \lim_{r \to \infty} \int_{\mathbb{R}^n} F(\Phi_1 U_1^\alpha)(\xi) F(\Phi_1 U_2^\beta)(\xi) \psi(\xi) d\xi,$$  \hspace{1cm} (3.21)

For $\zeta = (\zeta_1, \ldots, \zeta_N) \in \mathbb{C}^N$ we have, in view of (3.21),

$$\hat{I} \zeta \cdot \zeta = \sum_{\alpha, \beta = 1}^N \hat{I}^{\alpha\beta}(\Phi, \psi) \zeta_\alpha \overline{\zeta_\beta} = B \lim_{r \to \infty} \int_{\mathbb{R}^n} |F(\Phi_1 V_r)(\xi)|^2 \psi(\xi) d\xi \geq 0,$$

where $V_r(x) = \sum_{\alpha=1}^N U_r^{\alpha} \zeta_\alpha$. The above relation proves that the matrix $\hat{I}$ is Hermitian and positive definite.

We see that for any $\zeta \in \mathbb{C}^n$ the bilinear functional $\hat{I}(\Phi, \psi) \zeta \cdot \zeta$ is continuous on $C_0(\Omega) \times C(\mathcal{S})$ and nonnegative, that is, $\hat{I}(\Phi, \psi) \zeta \cdot \zeta \geq 0$ whenever $\Phi(x) \geq 0$, $\psi(\eta) \geq 0$. By Lemma 3.1 such a functional is represented by integration over some unique locally finite non-negative Radon measure $\mu = \mu_{\zeta}(x, \eta) \in M_{loc}(\Omega \times \mathcal{S})$:

$$\hat{I}(\Phi, \psi) \zeta \cdot \zeta = \int_{\Omega \times \mathcal{S}} \Phi(x) \psi(\eta) d\mu_{\zeta}(x, \eta).$$

As a function of the vector $\zeta$, $\mu_{\zeta}$ is a measure valued Hermitian form. Therefore,

$$\mu_{\zeta} = \sum_{\alpha, \beta = 1}^N \mu^{\alpha\beta} \zeta_\alpha \overline{\zeta_\beta} \hspace{1cm} (3.22)$$

with measure valued coefficients $\mu^{\alpha\beta} \in M_{loc}(\Omega \times \mathcal{S})$, which can be expressed as follows

$$\mu^{\alpha\beta} = [\mu_{\epsilon_\alpha \epsilon_\beta} + i \mu_{\epsilon_\alpha + \epsilon_\beta}] / 2 - (1 + i)(\mu_{\epsilon_\alpha} + \mu_{\epsilon_\beta}) / 2,$$

where $\epsilon_1, \ldots, \epsilon_N$ is the standard basis in $\mathbb{C}^N$, and $i^2 = -1$.

By (3.22)

$$\hat{I}(\Phi, \psi) \zeta \cdot \zeta = \sum_{\alpha, \beta = 1}^N \langle \mu^{\alpha\beta}, \Phi(x) \psi(\eta) \rangle \zeta_\alpha \overline{\zeta_\beta}$$

and since

$$\hat{I}(\Phi, \psi) \zeta \cdot \zeta = \sum_{\alpha, \beta = 1}^N \hat{I}^{\alpha\beta}(\Phi, \psi) \zeta_\alpha \overline{\zeta_\beta},$$

then, comparing the coefficients, we find that

$$\langle \mu^{\alpha\beta}, \Phi(x) \psi(\eta) \rangle = \hat{I}^{\alpha\beta}(\Phi, \psi). \hspace{1cm} (3.23)$$

In particular,

$$\langle \mu^{\alpha\beta}, \Phi_1(x) \Phi_2(x) \psi(\eta) \rangle = I^{\alpha\beta}(\Phi_1, \Phi_2, \psi) = B \lim_{r \to \infty} \int_{\mathbb{R}^n} F(\Phi_1 U_1^\alpha)(\xi) F(\Phi_2 U_2^\beta)(\xi) \psi(\xi) d\xi.$$
To complete the proof, observe that for each \( \zeta \in \mathbb{C}^N \) the measure

\[
\sum_{\alpha, \beta = 1}^{N} \mu^{\alpha \beta} \zeta_\alpha \overline{\zeta}_\beta = \mu_\zeta \geq 0.
\]

Hence, \( \mu \) is Hermitian and positive definite.

The usage of generalized Banach limit instead of extraction of a subsequence is connected with the fact that the algebra \( \mathcal{A} \) is not separable. Therefore, the extraction of a subsequence of \( U_r \) such that relation (6.14) holds, with replacement of the Banach limit to the usual one, is not always possible. Certainly, the \( \mathcal{H} \)-measure \( \mu \) depend on the choice of the generalized Banach limit (this resembles the dependence of the Tartar \( \mathcal{H} \)-measure on the choice of a subsequence).

If \( X \) is a subspace of \( \mathbb{R}^n \) and \( p_X : \mathcal{S} \to S_X \) is the projection defined before Proposition 2.1 above, then the image of the measures \( \mu^{\alpha \beta} \) under the map \( (x, \eta) \to (x, p_X(\eta)) \) is exactly the ultraparabolic \( \mathcal{H} \)-measure corresponding to the subspace \( X \).

Evidently, if the sequence \( U_r \) converges as \( r \to \infty \) to the zero vector strongly in \( L^2_0(\Omega, \mathbb{C}^N) \), then \( \mathcal{H} \)-measure is trivial: \( \mu = 0 \). Conversely, if \( \mu = 0 \) then for each \( \Phi(x) \in C_0(\Omega) \)

\[
\lim_{r \to \infty} \int_{\Omega} |U_r(x)\Phi(x)|^2 \, dx = 0 \quad \forall \Phi(x) \in C_0(\Omega). \tag{3.24}
\]

This implies that

\[
\lim_{r \to \infty} \int_{\Omega} |U_r(x)\Phi(x)|^2 \, dx = 0 \quad \forall \Phi(x) \in C_0(\Omega).
\]

We can choose the sequence of real nonnegative functions \( \Phi_k(x) \in C_0(\Omega) \) such that \( \Phi_{k+1}(x) \geq \Phi_k(x) \) for all \( k \in \mathbb{N} \), and \( \lim_{k \to \infty} \Phi_k(x) = 1 \) for all \( x \in \Omega \). It follows from (3.24) that there exists a strictly increasing sequence \( r_k \in \mathbb{N} \) such that \( \int_{\Omega} |U_{r_k}(x)\Phi_k(x)|^2 \, dx < 1/k \). Then the subsequence

\[
U_{r_k}(x) \to 0 \quad \text{in} \quad L^2_0(\Omega, \mathbb{C}^N).
\]

Let \( \mu = \{\mu^{\alpha \beta}\}_{\alpha, \beta = 1}^{N} \) be an \( \mathcal{H} \)-measure corresponding to a sequence \( U_r = \{U_r^{\alpha}\}_{\alpha = 1}^{N} \in L^2_0(\Omega, \mathbb{C}^N) \). We define \( \mu_0 = \text{Tr} \mu = \sum_{\alpha = 1}^{N} \mu^{\alpha \alpha} \). As follows from Proposition 3.1, \( \mu_0 \) is a locally finite non-negative Radon measure on \( \Omega \times \mathcal{S} \). We assume that this measure is extended on \( \sigma \)-algebra of \( \mu_0 \)-measurable sets, and in particular that this measure is complete.

**Lemma 3.2.** The \( \mathcal{H} \)-measure \( \mu \) is absolutely continuous with respect to \( \mu_0 \), more precisely, \( \mu = H(x, \eta)\mu_0 \), where \( H(x, \eta) = \{h^{\alpha \beta}(x, \eta)\}_{\alpha, \beta = 1}^{N} \) is a bounded \( \mu_0 \)-measurable function taking values in the cone of nonnegative definite Hermitian \( N \times N \) matrices, moreover \( |h^{\alpha \beta}(x, \eta)| \leq 1 \).

**Proof.** Remark firstly that \( \mu^{\alpha \alpha} \leq \mu_0 \) for all \( \alpha = 1, \ldots, N \). Now, suppose that \( \alpha, \beta \in \{1, \ldots, N\}, \alpha \neq \beta \). By Proposition 3.1 for any compact set \( B \subset \Omega \times \mathcal{S} \) the matrix

\[
\begin{pmatrix}
\mu^{\alpha \alpha}(B) & \mu^{\alpha \beta}(B) \\
\mu^{\beta \alpha}(B) & \mu^{\beta \beta}(B)
\end{pmatrix}
\]

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Putting relations (3.27), (3.28) together, we find ones, which implies that

\[ \mu^{\alpha\beta}(B) \leq (\mu^{\alpha\alpha}(B)\mu^{\beta\beta}(B))^{1/2} \leq \mu_0(B). \]

By regularity of measures \( \mu^{\alpha\beta} \) and \( \mu_0 \) this estimate is satisfied for all Borel sets \( B \). This easily implies the inequality \( \text{Var} \mu^{\alpha\beta} \leq \mu_0 \). In particular, the measures \( \mu^{\alpha\beta} \) are absolutely continuous with respect to \( \mu_0 \), and by the Radon-Nykodim theorem \( \mu^{\alpha\beta} = h^{\alpha\beta}(x,\eta)\mu_0 \), where the densities \( h^{\alpha\beta}(x,\eta) \) are \( \mu_0 \)-measurable and, as follows from the inequalities \( \text{Var} \mu^{\alpha\beta} \leq \mu_0 \), \( |h^{\alpha\beta}(x,\eta)| \leq 1 \) \( \mu_0 \)-a.e. on \( \Omega \times \mathcal{S} \). We denote by \( H(x,\eta) \) the matrix with components \( h^{\alpha\beta}(x,\eta) \). Recall that the \( H \)-measure \( \mu \) is nonnegative definite. This means that for all \( \zeta \in \mathbb{C}^N \)

\[ \mu\zeta \cdot \zeta = (H(x,\eta)\zeta \cdot \zeta)\mu_0 \geq 0. \] (3.25)

Hence \( H(x,\eta)\zeta \cdot \zeta \geq 0 \) for \( \mu_0 \)-a.e. \( (x,\eta) \in \Omega \times \mathcal{S} \). Choose a countable dense set \( E \subset \mathbb{C}^N \). Since \( E \) is countable, then it follows from (3.25) that for a set \( (x,\eta) \in \Omega \times \mathcal{S} \) of full \( \mu_0 \)-measure \( H(x,\eta)\zeta \cdot \zeta \geq 0 \) \( \forall \zeta \in E \), and since \( E \) is dense we conclude that actually \( H(x,\eta)\zeta \cdot \zeta \geq 0 \) for all \( \zeta \in \mathbb{C}^N \). Thus, the matrix \( H(x,\eta) \) is Hermitian and nonnegative definite for \( \mu_0 \)-a.e. \( (x,\eta) \). After an appropriate correction on a set of null \( \mu_0 \)-measure, we can assume that the above property is satisfied for all \( (x,\eta) \in \Omega \times \mathcal{S} \), and also \( |h^{\alpha\beta}(x,\eta)| \leq 1 \) for all \( (x,\eta) \in \Omega \times \mathcal{S} \), \( \alpha,\beta = 1, \ldots, N \). The proof is complete.

Now we assume that for all \( \Phi(x) \in C_0^\infty(\Omega) \) the sequence

\[ \sum_{\alpha=1}^{N} \sum_{k=1}^{M} c_{\alpha k}(x)p_{\alpha k}(\partial/\partial x)(\Phi(x)U_\alpha^\beta(x)) \to 0 \text{ in } L^2_{\mu_0}(\Omega), \] (3.26)

where \( p_{\alpha k}(\partial/\partial x) \) denotes the pseudo-differential operator with symbol \( p_{\alpha k}(\xi) \in A \) and \( c_{\alpha k}(x) \in C(\Omega) \). Then the \( H \)-measure corresponding to the sequence \( U_\alpha \) satisfy the following localization property.

**Theorem 3.1.** For each \( \beta = 1, \ldots, N \)

\[ \sum_{\alpha=1}^{N} \sum_{k=1}^{M} c_{\alpha k}(x)\bar{p}_{\alpha k}(\eta)\mu^{\alpha\beta}(x,\eta) = 0. \]

**Proof.** In view of (3.26) for all \( \Phi_1(x), \Phi(x) \in C_0^\infty(\Omega) \) a sequence

\[ \sum_{\alpha=1}^{N} \sum_{k=1}^{M} \Phi(x)c_{\alpha k}(x)p_{\alpha k}(\partial/\partial x)(\Phi_1(x)U_\alpha^\beta(x)) \to 0 \text{ in } L^2(\mathbb{R}^n). \] (3.27)

Since \( p_{\alpha k} \in A \), then the commutator \([A_{p_{\alpha k}}, B_{\Phi_{c_{\alpha k}}}]\) is a compact operator in \( L^2(\mathbb{R}^n) \). Therefore, this operators transform the weakly convergent sequence \( \Phi_1(x)U_\alpha^\beta(x) \) to the strongly convergent ones, which implies that

\[ \sum_{\alpha=1}^{N} \sum_{k=1}^{M} [A_{p_{\alpha k}}, B_{\Phi_{c_{\alpha k}}}])(\Phi_1(x)U_\alpha^\beta(x)) \to 0 \text{ in } L^2(\mathbb{R}^n). \] (3.28)

Putting relations (3.27), (3.28) together, we find

\[ \sum_{\alpha=1}^{N} \sum_{k=1}^{M} p_{\alpha k}(\partial/\partial x)(\Phi(x)\Phi_1(x)c_{\alpha k}(x)U_\alpha^\beta(x)) \to 0 \text{ in } L^2(\mathbb{R}^n). \]
Hence,

\[ \sum_{\alpha=1}^{N} \sum_{k=1}^{M} p_{\alpha k}(\partial/\partial x)(\Phi_{1}(x)c_{\alpha k}(x)U_{r}^{\alpha}(x)) \rightarrow 0 \quad \text{in } L^{2}(\mathbb{R}^{n}). \]

Applying the Fourier transformation, we obtain

\[ \sum_{\alpha=1}^{N} \sum_{k=1}^{M} \hat{p}_{\alpha k}(\xi)F(\Phi_{1}c_{\alpha k}U_{r}^{\alpha})(\xi) \rightarrow 0 \quad \text{in } L^{2}(\mathbb{R}^{n}). \] (3.29)

We multiply (3.29) by the bounded sequence \( F(\Phi_{2}U_{r}^{\beta})(\xi) \psi(\xi) \), where \( \Phi_{2}(x) \in C_{0}^{\infty}(\Omega) \), \( \psi(\xi) \in A \), and \( 1 \leq \beta \leq N \). Integrating over \( \xi \in \mathbb{R}^{n} \), we arrive at the relation

\[ \lim_{r \rightarrow \infty} \sum_{\alpha=1}^{N} \sum_{k=1}^{M} \int_{\mathbb{R}^{n}} p_{\alpha k}(\xi)F(\Phi_{1}c_{\alpha k}U_{r}^{\alpha})(\xi)F(\Phi_{2}U_{r}^{\beta})(\xi)\psi(\xi) d\xi = 0. \]

On the other hand, by the definition of \( H \)-measure this limit coincides with

\[ \sum_{\alpha=1}^{N} \sum_{k=1}^{M} \hat{p}_{\alpha k}(\xi)F(\Phi_{1}c_{\alpha k}U_{r}^{\alpha})(\xi)F(\Phi_{2}U_{r}^{\beta})(\xi)\psi(\xi) d\xi = \sum_{\alpha=1}^{N} \sum_{k=1}^{M} \int_{\mathbb{R}^{n}} \langle \mu_{\alpha \beta}, c_{\alpha k}(x)\Phi_{1}(x)\Phi_{2}(x)\hat{p}_{\alpha k}(\eta)\hat{\psi}(\eta) \rangle. \]

Hence,

\[ \sum_{\alpha=1}^{N} \sum_{k=1}^{M} \langle \mu_{\alpha \beta}, c_{\alpha k}(x)\Phi_{1}(x)\Phi_{2}(x)\hat{p}_{\alpha k}(\eta)\hat{\psi}(\eta) \rangle = 0. \]

This relation can be written in the form

\[ \left\langle \sum_{\alpha=1}^{N} \sum_{k=1}^{M} c_{\alpha k}(x)\hat{p}_{\alpha k}(\eta)\mu_{\alpha \beta}(x, \eta), \Phi_{1}(x)\Phi_{2}(x)\hat{\psi}(\eta) \right\rangle = 0. \] (3.30)

Since the test functions \( \Phi_{1}(x), \Phi_{2}(x) \in C_{0}^{\infty}(\Omega) \), \( \hat{\psi}(\eta) \in C(\mathcal{S}) \) are arbitrary, the statement of Theorem 3.1 follows from (3.30).

4. **Compensated compactness**

Assume that a sequence \( u_{r}(x) \in L^{2}_{loc}(\Omega, \mathbb{R}^{N}) \) weakly converges to a vector-function \( u(x) \) as \( r \rightarrow \infty \) while for each \( \Phi(x) \in C_{0}^{\infty}(\Omega) \) the sequences

\[ \sum_{\alpha=1}^{N} \sum_{k=1}^{M} c_{\alpha k}(x)\hat{p}_{\alpha k}(\partial/\partial x)(\Phi(x)u_{r}^{\alpha}(x)) \quad \text{are precompact in } L^{2}_{loc}(\Omega), \] (4.1)
where \( p_{sak}(\partial/\partial x) \) are pseudo-differential operators with symbols \( p_{sak}(\xi) \in A, c_{sak}(x) \in C(\Omega) \), and \( s = 1, \ldots, m \). Introduce the set
\[
\Lambda = \Lambda(x) = \left\{ \lambda \in \mathbb{C}^N \mid \exists \eta \in S : \sum_{\alpha=1}^N \sum_{k=1}^M c_{sak}(x) \hat{p}_{sak}(\eta) \lambda_\alpha = 0 \ \forall s = 1, \ldots, m \right\}.
\]

Now, suppose that
\[
q(x, u) = Q(x)u \cdot u = \sum_{\alpha, \beta=1}^N q_{\alpha\beta}(x)u_\alpha \overline{u_\beta}
\]
is an Hermitian form with the matrix \( Q(x) \) of coefficients \( q_{\alpha\beta}(x) \in C(\Omega) \).

Let the sequence \( q(x, u_r) \to u \) as \( r \to \infty \) weakly in \( M_{\text{loc}}(\Omega) \). The following theorem is analogous to Theorem \ref{thm:1.1}.

**Theorem 4.1.** If \( q(x, \lambda) \geq 0 \) for all \( \lambda \in \Lambda(x), x \in \Omega \), then \( q(x, u(x)) \leq u \) \( (\text{in the sense of measures}) \).

**Proof.** Let \( \mu = \{\mu^{\alpha\beta}\}_{\alpha, \beta=1}^N \) be the \( H \)-measure corresponding to the sequence \( U_r = u_r - u \).

By Lemma \ref{lem:3.2} this \( H \)-measure admits the representation \( \mu = H(x, \eta)\mu_0 \), where \( \mu_0 = \text{Tr} \mu \geq 0 \) and \( H(x, \eta) \) is a \( \mu_0 \)-measurable map from \( \Omega \times S \) into the cone of nonnegative definite \( N \times N \) Hermitian matrices. As readily follows from \ref{eq:1.1}, for each \( \Phi(x) \in C_0^\infty(\Omega) \), \( s = 1, \ldots, m \)
\[
\sum_{\alpha=1}^N \sum_{k=1}^M c_{sak}(x) \hat{p}_{sak}(\eta) H^{\alpha\beta}(x, \eta) \mu_0 = \sum_{\alpha=1}^N \sum_{k=1}^M c_{sak}(x) \hat{p}_{sak}(\eta) \mu^{\alpha\beta}(x, \eta) = 0.
\]

This implies that for \( \mu_0 \)-a.e. \( (x, \eta) \) the image \( \text{Im} H(x, \eta) \subset \Lambda(x) \). Since the matrix \( H(x, \eta) \geq 0 \), there exists a unique Hermitian matrix \( R = R(x, \eta) = (H(x, \eta))^{1/2} \) such that \( R \geq 0 \) and \( H = R^2 \). By the known properties of Hermitian matrices \( \ker R = \ker H \), which readily implies that also \( \text{Im} R = \text{Im} H \). In particular, \( \text{Im} R(x, \eta) \subset \Lambda(x) \) for \( \mu_0 \)-a.e. \( (x, \eta) \in \Omega \times S \). Let \( \Phi(x) \in C_0(\Omega) \) be a real test function. Then
\[
\lim_{r \to \infty} \int \langle \Phi(x), q(x, U_r(x)) \rangle dx = \sum_{\alpha, \beta=1}^N \lim_{r \to \infty} \int q_{\alpha\beta}(x) \Phi(x) U_r^\alpha(x) \overline{U_r^\beta(x)} dx =
\]
\[
\sum_{\alpha, \beta=1}^N \lim_{r \to \infty} \int F(q_{\alpha\beta} U_r^\alpha)(\xi) \overline{F(U_r^\beta)(\xi)} d\xi = \sum_{\alpha, \beta=1}^N \langle \mu^{\alpha\beta}, q_{\alpha\beta}(\Phi(x))^2 \rangle =
\]
\[
\int_{\Omega \times S} \langle \Phi(x), q_{\alpha\beta}(x, \eta) \rangle d\mu_0(x, \eta).
\]

Since \( H = R^2 \) then
\[
H^{\alpha\beta} = \sum_{k=1}^N r_{\alpha k} r_{\beta k} \ \forall \alpha, \beta = 1, \ldots, N,
\]
\[
H^{\alpha\beta} = \sum_{k=1}^N r_{\alpha k} r_{\beta k} \ \forall \alpha, \beta = 1, \ldots, N,
\]
\[
H^{\alpha\beta} = \sum_{k=1}^N r_{\alpha k} r_{\beta k} \ \forall \alpha, \beta = 1, \ldots, N,
\]
where \( r_{ij} = r_{ij}(x,\eta) \), \( i,j = 1,\ldots,N \) are components of the matrix \( R \). Therefore,

\[
\sum_{\alpha,\beta=1}^{N} q_{\alpha\beta}(x)\mathcal{h}^{\alpha\beta}(x,\eta) = \sum_{k=1}^{N} \sum_{\alpha,\beta=1}^{N} q_{\alpha\beta}(x)r_{\alpha k}r_{\beta k} = \sum_{k=1}^{N} q(x,Re_k),
\]

(4.3)

where \( e_k, k = 1,\ldots,N \), is the standard basis in \( \mathbb{C}^N \). Since \( R(x,\eta)e_k \in \text{Im } R(x,\eta) \subset \Lambda(x) \) for \( \mu_0 \)-a.e. \( (x,\eta) \in \Omega \times \mathcal{S} \), then \( q(x,R(x,\eta)e_k) \geq 0 \) for \( \mu_0 \)-a.e. \( (x,\eta) \) and it follows from (4.2), (4.3) that

\[
\lim_{r \to \infty} \int (\Phi(x))^2 q(x,U_r(x)) \geq 0.
\]

(4.4)

for all real \( \Phi(x) \in C_0(\Omega) \). In view of the weak convergence \( u_r \rightharpoonup u, q(x,u_r) \to v \) as \( r \to \infty \),

\[
q(x,U_r(x)) = q(x,u_r(x)) + q(x,u(x)) - 2\Re(Q(x)u_r \cdot u) \to v - q(x,u(x)).
\]

in \( M_{\text{loc}}(\Omega) \). Now, it follows from (4.4) that

\[
\langle v - q(x,u(x))dx, (\Phi(x))^2 \rangle = \lim_{r \to \infty} \int (\Phi(x))^2 q(x,U_r(x)) \geq 0
\]

and since the real test function \( \Phi(x) \) is arbitrary, \( v \geq q(x,u(x)) \). The proof is complete.

**Corollary 4.1.** Suppose that \( q(x,\lambda) = 0 \) for all \( \lambda \in \Lambda(x) \). Then \( v = q(x,u(x)) \), that is, the functional \( u \to q(x,u) \) is weakly continuous.

**Proof.** Applying Theorem 4.1 to the quadratic forms \( \pm q(x,u) \), we obtain the inequalities \( \pm v \geq \pm q(x,u(x)) \), which readily imply that \( v = q(x,u(x)) \).

### 4.1. The case of second order differential constraints

Now we assume that the sequences

\[
\sum_{\alpha=1}^{N} \sum_{k=1}^{n} \partial_{x_k}(a_{sak}u_{\alpha r}) + \sum_{\alpha=1}^{N} \sum_{k,l=1}^{n} \partial_{x_kx_l}(b_{sakl}u_{\alpha r}), \quad s = 1,\ldots,m,
\]

(4.5)

are pre-compact in the Sobolev space \( H_{\text{loc}}^{-1}(\Omega) \cong W_{2,\text{loc}}^{-1}(\Omega) \), where the coefficients \( a_{sak} = a_{sak}(x), b_{sakl} = b_{sakl}(x) \) are supposed to be (generally complex-valued) continuous functions on \( \Omega \), and \( b_{sakl} = b_{sakl}, s = 1,\ldots,m, \alpha = 1,\ldots,N, k,l = 1,\ldots,n \).

We denote by \( A_{sa} = A_{sa}(x) \) the vector \( \{a_{sak}\}_{k=1}^{n} \in \mathbb{C}^n \) and by \( B_{sa} = B_{sa}(x) \) the symmetric matrices with components \( \{b_{sakl}\}_{k,l=1}^{n} \). Let \( X_s \) be the maximal linear subspace of \( \mathbb{R}^n \) contained in \( \mathbb{R}^n \cap \ker B_{sa}(x) \) for all \( \alpha = 1,\ldots,N \) and \( x \in \Omega \). The following statement easily follows from the definition of the subspace \( X_s \):

**Lemma 4.1.** For each \( \alpha = 1,\ldots,N \), \( \Phi(x) \in C_0(\Omega) \)

\[
F(\Phi B_{sa})(\xi)\hat{\xi} = 0 \quad \forall \xi \in \mathbb{R}^n, \quad \hat{\xi} \in X_s.
\]

(4.6)

**Proof.** Equality (4.6) readily follows from the relation

\[
F(\Phi B_{sa})(\xi)\hat{\xi} = \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x} \Phi(x)B_{sa}(x)\xi dx = 0
\]

because \( B_{sa}(x)\hat{\xi} = 0 \) for all \( x \in \mathbb{R}^n \) by the definition of the subspace \( X_s \).
We introduce the set
\[ \Lambda = \Lambda(x) = \left\{ \lambda \in \mathbb{C}^N \mid \exists \eta \in \mathcal{S} : \right\} \]
\[
\sum_{\alpha=1}^{N} (iA_{s\alpha} \cdot \tilde{\xi}^s(\eta) - B_{s\alpha} \tilde{\xi}^s(\eta) \cdot \tilde{\xi}^s(\eta))\lambda_\alpha = 0 \ \forall s = 1, \ldots, m \} ,
\] (4.7)
where \( \tilde{\xi}^s(\eta) \in X_s, \tilde{\xi}^s(\eta) \in X_s^+ \) are such that \( (\tilde{\xi}^s(\eta), \tilde{\xi}^s(\eta)) = p_{X_s}(\eta) \), and \( p_{X_s} : \mathcal{S} \to S_{X_s} \) is the projection defined in section 2. Let
\[
\sum_{\alpha=1}^{N} \sum_{k,l=1}^{n} \partial_{x^k} \Phi_{x^l}(a_{s\alpha} u_{\alpha r}) - 2 \sum_{l=1}^{n} b_{s\alpha k l} \Phi_{x^l}(a_{s\alpha} u_{\alpha r}) + \sum_{\alpha=1}^{N} \sum_{k,l=1}^{n} \partial_{x^k x^l} (b_{s\alpha k l} u_{\alpha r}) = \Phi \left( \sum_{\alpha=1}^{N} \sum_{k,l=1}^{n} \partial_{x^k} (a_{s\alpha} u_{\alpha r}) + \sum_{\alpha=1}^{N} \sum_{k,l=1}^{n} \partial_{x^k x^l} (b_{s\alpha k l} u_{\alpha r}) \right) + \sum_{\alpha=1}^{N} \sum_{k,l=1}^{n} a_{s\alpha} \Phi_{x^l} u_{\alpha r} - \sum_{\alpha=1}^{N} \sum_{k,l=1}^{n} b_{s\alpha k l} \Phi_{x^k x^l} u_{\alpha r} \] (4.8)
are pre-compact in the Sobolev space \( H^{-1}(\mathbb{R}^n) \cong W_2^{-1}(\mathbb{R}^n) \). Relation (4.8) implies that the sequence
\[
(1 + |\xi|^2)^{-\frac{1}{2}} \left( \sum_{\alpha=1}^{N} \sum_{k=1}^{n} 2\pi i \xi_k \left( F(a_{s\alpha} \Phi u_{\alpha r})(\xi) - 2 \sum_{l=1}^{n} F(b_{s\alpha k l} \Phi_{x^l}(a_{s\alpha} u_{\alpha r}))(\xi) \right) - \sum_{\alpha=1}^{N} \sum_{k,l=1}^{n} 4\pi^2 \xi_k \xi_l F(b_{s\alpha k l} \Phi u_{\alpha r})(\xi) \right) ,
\] (4.9)
r \( \in \mathbb{N} \), is compact in \( L^2(\mathbb{R}^n) \). Denote \( \hat{\xi} = P_1 \xi, \tilde{\xi} = P_2 \xi \), where \( P_1, P_2 \) are the orthogonal projections onto the subspaces \( X_s, X_s^+ \), respectively. Multiplying (4.9) by the bounded function \( (1 + |\xi|^2)^{\frac{1}{2}} (1 + |\xi|^2 + |\xi|^4)^{-\frac{1}{2}} \), we obtain that the sequence
\[
(1 + |\hat{\xi}|^2 + |\tilde{\xi}|^4)^{-\frac{1}{2}} \sum_{\alpha=1}^{N} (2\pi i (F(A_{s\alpha} \Phi u_{\alpha r}))(\xi) - 2F(B_{s\alpha} \nabla \Phi u_{\alpha r}))(\xi)) \cdot \xi
- 4\pi^2 F(B_{s\alpha} \Phi u_{\alpha r})(\xi) \xi \cdot \xi ,
\] (4.10)
\( r \in \mathbb{N} \), is compact in \( L^2(\mathbb{R}^n) \).

By Lemma 4.11 and symmetry of the matrix \( F(B_{s_0} \Phi u_{\alpha r})(\xi) \) we find that

\[
F(B_{s_0} \Phi u_{\alpha r})(\xi) \cdot \xi = F(B_{s_0} \Phi u_{\alpha r})(\xi) \cdot \xi + 2F(B_{s_0} \Phi u_{\alpha r})(\xi) \cdot \xi = F(B_{s_0} \Phi u_{\alpha r})(\xi) \cdot \xi = F(B_{s_0} \Phi u_{\alpha r})(\xi) \cdot (\xi + \xi) = F(B_{s_0} \Phi u_{\alpha r})(\xi) \cdot \xi.
\]

(4.11)

\[
F(B_{s_0} \nabla \Phi u_{\alpha r})(\xi) \cdot \xi = F(B_{s_0} \nabla \Phi u_{\alpha r})(\xi) \cdot \xi.
\]

(4.12)

Notice also that the sequences

\[
(1 + |\xi|^2 + |\xi|^4) - \frac{1}{2} F(A_{s_0} \Phi u_{\alpha r})(\xi) \cdot \xi,
\]

(4.13)

(1 + |\xi|^2 + |\xi|^4) - \frac{1}{2} F(B_{s_0} \nabla \Phi u_{\alpha r})(\xi) \cdot \xi, \quad r \in \mathbb{N}, \text{ are compact in } L^2(\mathbb{R}^n),

since the functions \((1 + |\xi|^2 + |\xi|^4) - \frac{1}{2} \xi_k, k = 1, \ldots, n\) lay in the ideal \( A_0 \). It now follows from (4.10), (4.13) that the sequence of distributions

\[
l_b^r = (1 + |\xi|^2 + |\xi|^4) - \frac{1}{2} \sum_{a=1}^{N} (2\pi i F(A_{s_0} \Phi u_{\alpha r})(\xi) \cdot \xi - 4\pi^2 F(B_{s_0} \Phi u_{\alpha r})(\xi) \cdot \xi),
\]

(4.14)

\( r \in \mathbb{N}, \) is compact in \( L^2(\mathbb{R}^n) \). The distributions \( l_b^r \) can be represented as

\[
l_b^r = \sum_{a=1}^{N} \left( \sum_{k,l=1}^{n} 2\pi i p_{sak}(\xi) F(a_{sak} \Phi u_{\alpha r})(\xi) - \sum_{k,l=1}^{n} 4\pi^2 q_{sakl}(\xi) F(b_{sakl} \Phi u_{\alpha r})(\xi) \right),
\]

where

\[
p_{sak}(\xi) = (1 + |\xi|^2 + |\xi|^4) - \frac{1}{2} \xi_k \equiv (|\xi|^2 + |\xi|^4) - \frac{1}{2} \xi_k \mod A_0,
\]

\[
q_{sakl}(\xi) = (1 + |\xi|^2 + |\xi|^4) - \frac{1}{2} \xi_k \xi_l \equiv (|\xi|^2 + |\xi|^4) - \frac{1}{2} \xi_k \xi_l \mod A_0.
\]

In particular, we see that \( p_{sak}(\xi), q_{sakl}(\xi) \in \mathcal{A} \).

Taking into account compactness of commutators \([A_{s_0}, B_0]\) in \( L^2(\mathbb{R}^n) \), where \((\psi, \phi) = (p_{sak}(\xi), \chi(x) a_{sak}(x)), (\psi, \phi) = (q_{sakl}(\xi), \chi(x) b_{sakl}(x))\), and \( \chi(x) \in C_0(\Omega) \) is a function such that \( \chi(x) \Phi(x) = \Phi(x) \), we find that the sequence

\[
\chi(x) \sum_{a=1}^{N} \left( \sum_{k=1}^{n} a_{sak}(x) p_{sak}(\partial/\partial x) - 4\pi^2 \sum_{k,l=1}^{n} b_{sakl}(x) q_{sakl}(\partial/\partial x) \right)(\Phi u_{\alpha r}),
\]

(4.15)

\( r \in \mathbb{N}, \) is compact in \( L^2(\mathbb{R}^n) \), that is, the sequence

\[
\sum_{a=1}^{N} \left( \sum_{k=1}^{n} a_{sak}(x) p_{sak}(\partial/\partial x) - 4\pi^2 \sum_{k,l=1}^{n} b_{sakl}(x) q_{sakl}(\partial/\partial x) \right)(\Phi u_{\alpha r}),
\]

(4.16)

\( r \in \mathbb{N}, \) is compact in \( L^2_{loc}(\Omega) \). Here \( p_{sak}(\partial/\partial x), q_{sakl}(\partial/\partial x) \) are pseudodifferential operators with symbols

\[
(|\xi|^2 + |\xi|^4) - \frac{1}{2} \xi_k, (|\xi|^2 + |\xi|^4) - \frac{1}{2} \xi_k \xi_l
\]

laying in \( \mathcal{A} \). Since \( s = 1, \ldots, m, \Phi(x) \in C_0^\infty(\Omega) \) are arbitrary, we see that our sequence \( u_r \) satisfies constraints of the kind (4.15).

Since

\[
p_{sak}(\eta) = \xi_k(\eta), q_{sakl}(\eta) = \xi_k(\eta) \xi_l(\eta),
\]

we have
the set $\Lambda = \Lambda(x)$ corresponding to these constraints is

$$\Lambda = \Lambda(x) = \left\{ \lambda \in \mathbb{C}^N \mid \exists \eta \in \mathcal{S} : \sum_{\alpha=1}^{N} (2\pi i A_{\alpha \alpha} \cdot \xi^\alpha(\eta) - 4\pi^2 B_{\alpha \alpha} \xi^\alpha(\eta) \cdot \xi^\alpha(\eta)) \lambda_{\alpha} = 0 \forall s = 1, \ldots, m \right\}.$$  

Since, in accordance with Remark 2.1, $\xi^\alpha(t) = a(t, \eta)\xi^\alpha(t), \hat{x}^\alpha(t) = b(t, \eta)\hat{x}^\alpha(t)$, $v(t, \eta) = ta(t, \eta)$, then after the transformation $\eta = (2\pi)^{-1}t$ the set $\Lambda$ will coincide with (4.7). Then the assertion of Theorem 4.2 readily follows from Theorem 4.1. The proof is complete.

4.2. One example

Let us consider the sequence $u_r = (u_{r,1}, u_{r,2}, u_{r,3}) \in L^2_{\text{loc}}(\Omega, \mathbb{C}^3)$, $\Omega \subset \mathbb{R}^3$ weakly convergent to $u = (u_1, u_2, u_3)$ such that the sequences

$$i (\partial_{x_3} u_{r,2} - \partial_{x_2} u_{r,3}) + \partial^2_{x_2} u_{r,1}; \quad i (\partial_{x_1} u_{r,3} - \partial_{x_3} u_{r,1}) + \partial^2_{x_3} u_{r,2}; \quad i (\partial_{x_2} u_{r,1} - \partial_{x_1} u_{r,2}) + \partial^2_{x_1} u_{r,3}$$

are pre-compact in $H^2_{\text{loc}}(\Omega)$.

**Theorem 4.3.** For every pair $(k, l)$, $1 \leq k < l \leq 3$ we have

$$u_{rk} u_{rl} \xrightarrow{r \to \infty} u_k u_l.$$

**Proof.** In the notations of Theorem 4.2 we find that $X_i = \{ \xi \in \mathbb{R}^3 : \xi_i = 0 \}$, $i = 1, 2, 3$, while the set $\Lambda$ is determined by the relations

$$\lambda_2 \xi^2_1(\eta) - \lambda_3 \xi^2_2(\eta) + \lambda_1 (\xi^1_1(\eta))^2 = \lambda_3 \xi^2_1(\eta) - \lambda_1 \xi^2_3(\eta) + \lambda_2 (\xi^3_1(\eta))^2 = \lambda_1 \xi^2_2(\eta) - \lambda_2 \xi^2_1(\eta) + \lambda_3 (\xi^3_1(\eta))^2 = 0 \quad (4.15)$$

for some $\eta \in \mathcal{S}$. For $\gamma \in \mathbb{C}$ we introduce the Hermitian form

$$Q_\gamma(\lambda) = \text{Re} \gamma u_k u_l = \gamma \frac{1}{2} u_k u_l + \gamma \frac{1}{2} u_l u_k.$$

Let $\lambda \in \Lambda$. Then there exists $\eta \in \mathcal{S}$ such that (4.15) holds. Observe that the space $\mathcal{X}$ from Proposition 2.1 may be included at most in two subspaces $X_i$. If the set $I = \{ \alpha \in \mathbb{R}^3 : \mathcal{X} \not\subset X_i \}$ contains two different indexes $j, k$, then $\xi^j(t) = \xi^k(t) = 0, |\xi^j| = |\xi^k|$; $|\xi^j| = 1$ by Proposition 2.1 and it follows from (4.15) that $\lambda_k = \lambda_j = 0 \Rightarrow Q_\gamma(\lambda) = 0$.

In the remaining case there exists only one index $i$ such that $\mathcal{X} \not\subset X_i$. For definiteness, we assume that $j = 1$. Then again $\xi^j(\eta) = \xi^j(\eta) = 0, |\xi^j| = 1$ by Proposition 2.1 and it follows from (4.15) that $\lambda_k = \lambda_j = 0 \Rightarrow Q_\gamma(\lambda) = 0$.

Let $\xi^2(\eta) = (0, p, 0), \xi^3(\eta) = (0, 0, q)$. By (4.15)

$$a \lambda_3 + p^2 \lambda_2 = -b \lambda_2 + q^2 \lambda_3 = 0.$$

Since the determinant of this system $\Delta = p^2 q^2 + ab > 0$ we conclude $\lambda_2 = \lambda_3 = 0$. Thus, $\lambda = 0$ and $Q_\gamma(\lambda) = 0$. By Theorem 4.2 we see that $Q_\gamma(u_r) \to Q_\gamma(u)$. Therefore,

$$u_{rk} u_{rl} = Q_1(u_r) - iQ_i(u_r) \xrightarrow{r \to \infty} Q_1(u) - iQ_i(u) = u_k u_l,$$

as was to be proved.

Observe that in the notations of Theorem 4.1 the set $\Lambda = \{ \lambda \in \mathbb{R}^3 : \lambda_1 \lambda_2 \lambda_3 = 0 \}$ and this theorem does not allow to derive the statement of Theorem 4.3.
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