ON THE CHAOTICITY OF DERIVATIVES

MARAT V. MARKIN

Abstract. We show that the \( n \)th derivative with maximal domain is a chaotic operator in the spaces \( C[a,b] \) and \( L_p(a,b) \) \((-\infty < a < b < \infty, 1 \leq p < \infty)\) for each \( n \in \mathbb{N} \).

In all chaos there is a cosmos, in all disorder a secret order.

Carl Yung

1. Introduction

The linear operator of differentiation

\[ Df := f' \]

with maximal domain has been considered in various settings and shown to be chaotic

- on the Fréchet space \( H(\mathbb{C}) \) of entire functions with the topology of uniform convergence on compact subsets \([8,13]\),
- in the Hardy space \( H^2 \) \([3]\), and
- in the Bargmann space \( F(\mathbb{C}) \) \([7]\),

being continuous on \( H(\mathbb{C}) \) while unbounded in \( H^2 \) and \( F(\mathbb{C}) \).

In the complex space \( L_2(\mathbb{R}) \), the unbounded differentiation operator \( D \) with maximal domain is shown to be non-hypercyclic because of being normal \([15,19]\).

In all the above cases, whenever \( D \) is chaotic and, provided the underlying space is complex, each \( \lambda \in \mathbb{C} \) is a simple eigenvalue for \( D \).

We prove chaoticity for the differentiation operators

\[ D^n f := f^{(n)} \]

\((n \in \mathbb{N}) (\mathbb{N} := \{1,2,\ldots\})\) is the set of natural numbers) with maximal domains in the (real or complex) Banach spaces \( C[a,b] \) and \( L_p(a,b) \) \((-\infty < a < b < \infty, 1 \leq p < \infty)\), the former equipped with the maximum norm

\[ C[a,b] \ni f \mapsto \|f\|_\infty := \max_{a \leq x \leq b} |f(x)|. \]

2020 Mathematics Subject Classification. Primary 47A16, 47B38; Secondary 47A10, 47B93.

Key words and phrases. Hypercyclic vector, periodic point, hypercyclic operator, chaotic operator, spectrum.
It is noteworthy that the setting offered by the foregoing spaces does not allow reducing the problem of the chaoticity for the derivatives to that of weighted backward shifts (cf. [2, 3, 7, 8, 11]).

2. Preliminaries

The subsequent preliminaries are essential for our discourse.

2.1. Hypercyclicity and Chaoticity.

For a (bounded or unbounded) linear operator $A$ in a (real or complex) Banach space $X$, a nonzero vector

$$f \in \mathcal{C}^{\infty}(A) := \bigcap_{n=0}^{\infty} D(A^n)$$

($D(\cdot)$ is the domain of an operator, $A^0 := I$, $I$ is the identity operator on $X$) is called hypercyclic if its orbit under $A$

$$\text{orb}(f, A) := \{A^n f\}_{n \in \mathbb{Z}^+}$$

($\mathbb{Z}^+ := \{0, 1, 2, \ldots \}$ is the set of nonnegative integers) is dense in $X$.

Linear operators possessing hypercyclic vectors are said to be hypercyclic.

If there exist an $N \in \mathbb{N}$ ($\mathbb{N} := \{1, 2, \ldots \}$ is the set of natural numbers) and a vector

$$f \in D(A^N) \quad \text{with} \quad A^N f = f,$$

such a vector is called a periodic point for the operator $A$ of period $N$. If $f \neq 0$, we say that $N$ is a period for $A$.

Hypercyclic linear operators with a dense in $X$ set $\text{Per}(A)$ of periodic points are said to be chaotic.

See [3, 5, 8].

Remarks 2.1.

- In the prior definition of hypercyclicity, the underlying space is necessarily infinite-dimensional and separable (see, e.g., [12]).
- For a hypercyclic linear operator $A$, the set $HC(A)$ of its hypercyclic vectors is necessarily dense in $X$, and hence, the more so, is the subspace $\mathcal{C}^{\infty}(A) \supseteq HC(A)$.
- Observe that

$$\text{Per}(A) = \bigcup_{N=1}^{\infty} \text{Per}_N(A),$$

where

$$\text{Per}_N(A) = \ker(A^N - I), \quad N \in \mathbb{N}$$

is the subspace of $N$-periodic points of $A$. 
• As immediately follows from the inclusions

\[ HC(A^n) \subseteq HC(A), \ Per(A^n) \subseteq Per(A), n \in \mathbb{N}, \]

if, for a linear operator \( A \) in an infinite-dimensional separable Banach space \( X \) and some \( n \geq 2 \), the operator \( A^n \) is hypercyclic or chaotic, then \( A \) is also hypercyclic or chaotic, respectively.

Prior to [3,4], the notions of linear hypercyclicity and chaos had been studied exclusively for continuous linear operators on Fréchet spaces, in particular for bounded linear operators on Banach spaces (for a comprehensive survey, see [1,12]).

The following statement, obtained in [14] by strengthening one of the hypotheses of a well-known sufficient condition for linear hypercyclicity [3, Theorem 2.1] is a shortcut for establish chaoticity for (bounded or unbounded) linear operators without explicitly constructing both hypercyclic vectors and a dense set periodic points for them.

**Theorem 2.1** (Sufficient Condition for Linear Chaos [14, Theorem 3.2]).

Let \((X, \| \cdot \|)\) be a (real or complex) infinite-dimensional separable Banach space and \( A \) be a densely defined linear operator in \( X \) such that each power \( A^n \) \((n \in \mathbb{N})\) is a closed operator. If there exists a set

\[ Y \subseteq C^\infty(A) := \bigcap_{n=1}^\infty D(A^n) \]

dense in \( X \) and a mapping \( B : Y \to Y \) such that

1. \( \forall f \in Y : ABf = f \) and
2. \( \forall f \in Y \exists \alpha = \alpha(f) \in (0,1) , \exists c = c(f, \alpha) > 0 \ \forall n \in \mathbb{N} : \max (\|A^n f\|, \|B^n f\|) \leq c \alpha^n , \)

or equivalently,

\[ \forall f \in Y : \max (r(A,f),r(B,f)) < 1 , \]

where

\[ r(A,f) := \limsup_{n \to \infty} \|A^n f\|^{1/n} \text{ and } r(B,f) := \limsup_{n \to \infty} \|B^n f\|^{1/n} , \]

then the operator \( A \) is chaotic.

We also need the subsequent

**Corollary 2.1** (Chaoticity of Powers [14, Corollary 4.5]).

For a chaotic linear operator \( A \) in a (real or complex) infinite-dimensional separable Banach space \((X, \| \cdot \|)\) subject to the Sufficient Condition for Linear Chaos (Theorem 2.1), each power \( A^n \) \((n \in \mathbb{N})\) is chaotic.
2.2. Resolvent Set and Spectrum.

For a linear operator $A$ in a complex Banach space $X$, the set

$$\rho(A) := \{ \lambda \in \mathbb{C} \mid \exists (A - \lambda I)^{-1} \in L(X) \}$$

($L(X)$ is the space of bounded linear operators on $X$) and its complement $\sigma(A) := \mathbb{C} \setminus \rho(A)$ are called the operator’s resolvent set and spectrum, respectively.

The spectrum of $A$ contains the set $\sigma_p(A)$ of all its eigenvalues, called its point spectrum of $A$ (see, e.g., [6, 16]).

Remark 2.1. For an unbounded linear operator $A$ in a complex Banach space $X$ with a nonempty resolvent set $\rho(A) \neq \emptyset$ (i.e., $\sigma(A) \neq \mathbb{C}$), all powers $A^n$ ($n \in \mathbb{N}$) are closed operators [6].

2.3. $L_p$ Spaces.

Henceforth, the notations $f$ and $f(\cdot)$ are used to designate an equivalence class in $L_p(a,b)$ ($1 \leq p < \infty$, $-\infty < a < b < \infty$) and its representative, respectively.

The inclusions

$$C[a,b], C^\infty[a,b], C^\infty_0[a,b], P \subseteq L_p(a,b),$$

where

(2.2) $C^\infty_0[a,b] := \left\{ f(\cdot) \in C^\infty[a,b] \mid f^{(n)}(a) = f^{(n)}(b) = 0, n \in \mathbb{Z}_+ \right\}$

and

(2.3) $P := \left\{ \sum_{k=0}^n c_k x^k \mid n \in \mathbb{Z}_+, c_k \in \mathbb{F}, k = 0, \ldots, n, x \in [a,b] \right\}$

($\mathbb{F} := \mathbb{R}$ or $\mathbb{F} := \mathbb{C}$) is the subspace of polynomials, are understood in the sense of the natural embedding

$$C[a,b] \ni f \mapsto i(f) := f \in L_p(a,b),$$

where $i(f)$ is the equivalence class represented by the continuous function $f$.

3. Chaoticity of Derivatives

3.1. Derivatives in $C[a,b]$.

Proposition 3.1 (Closedness and Spectrum of Derivatives in $C[a,b]$).

In the (real or complex) space $(C[a,b], \| \cdot \|_\infty)$ ($-\infty < a < b < \infty$), the $n$th derivative

$$D^n f := f^{(n)}$$

with maximal domain $D(D^n) := C^n[a,b]$ is a densely defined unbounded closed linear operator for every $n \in \mathbb{N}$.

Furthermore, provided the underlying space is complex,

(3.1) $\sigma(D^n) = \sigma_p(D^n) = \mathbb{C}$, $n \in \mathbb{N}$,

each $\lambda \in \mathbb{C}$ being an eigenvalue of geometric multiplicity $n$, i.e.,

$$\dim \ker(D^n - \lambda I) = n, \lambda \in \mathbb{C}.$$
Proof. Let $n \in \mathbb{N}$ be arbitrary.

Observe that $D^n$ is the $n$th power of $D$, and hence, is linear.

The fact that $D^n$ is densely defined follows from the inclusion

$$P \subseteq C^\infty[a, b] = C^\infty(D),$$

where $P$ is the subspace of polynomials (see (2.3)), which, by the Weierstrass approximation theorem, is dense in $(C[a, b], \| \cdot \|_\infty)$ (see, e.g., [18]).

The unboundedness of $D^n$ instantly follows from the fact that, for $e_k(x) := \left(\frac{x - a}{b - a}\right)^k$, $k \in \mathbb{N}$, $x \in [a, b]$,

we have:

$$e_k \in D(D^n) \text{ and } \|e_k\|_\infty = e_k(b) = 1, \; k \in \mathbb{N},$$

and, for $k \geq n$,

$$\|D^n e_k\|_\infty = \left\| \prod_{j=0}^{n-1} (k - j) \left(\frac{1}{b - a}\right)^n \left(\frac{x - a}{b - a}\right)^{k-n} \right\|_\infty$$

$$= \frac{k!}{(k - n)!} \left(\frac{1}{b - a}\right)^n \left(\frac{x - a}{b - a}\right)^{k-n} \bigg|_{x=b}$$

$$= \frac{k!}{(k - n)!} \left(\frac{1}{b - a}\right)^n \geq k \left(\frac{1}{b - a}\right)^n \to \infty, \; k \to \infty.$$

For the complex space $C[a, b]$, (3.1) follows from the fact that, for an arbitrary $\lambda \in \mathbb{C}$, the equation

$$D^n f = \lambda f$$

has $n$ lineally independent solutions

$$f_k(x) := x^{k-1}, \; k = 1, \ldots, n, \; x \in [a, b],$$

when $\lambda = 0$ or

$$f_k(x) := e^{\lambda_k x}, \; k = 1, \ldots, n, \; x \in [a, b],$$

where $\lambda_k$, $k = 1, \ldots, n$, are the distinct values of $\sqrt[n]{\lambda}$, when $\lambda \neq 0$. Hence, $\lambda \in \sigma_p(A)$, with the corresponding eigenspace

$$\ker(D^n - \lambda I) = \text{span} \{f_1, \ldots, f_n\}$$

being $n$-dimensional.

Due to (3.1), the closedness of the operator $D^n$ is not automatic (see Remark 2.1), and hence, is to be shown.

Consistently with the Riesz representation theorem (see, e.g., [6, 9, 18]), there is a continuous embedding $E$ of $C[a, b]$ into the dual space $C^*[a, b]$, which relates the corresponding vectors $g \in C[a, b]$ and $g^* := Eg \in C^*[a, b]$ as follows:

$$(3.2) \quad g^* f = \int_a^b f(x)g(x) \, dx, \; f \in C[a, b],$$

with

$$\|g^*\| = \int_a^b |g(x)| \, dx =: ||g||_1 \leq ||g||_\infty.$$
\[ \|g\|_1 \text{ being the total variation of the antiderivative } \int_a^x g(t) \, dt, \; x \in [a, b]. \]

In \( C[a, b] \), the linear operator
\[
D^0_n f := f^{(n)}
\]
with domain
\[
D(D^n_0) := C^\infty_0 [a, b],
\]
is not densely defined, and hence, has no adjoint.

In \( C^* [a, b] \), let us consider the linear operator \((D^n_0)'\) defined as follows:

\[
(D^n_0)' E := E D^n_0,
\]
i.e., via the commutative diagram
\[
\begin{array}{ccc}
C^* [a, b] & \supseteq & D((D^n_0)') \xrightarrow{(D^n_0)' \, \, E} C^* [a, b] \\
E & \uparrow & \uparrow E \\
C[a, b] & \supseteq & D(D^n_0) \xrightarrow{D^n_0} C[a, b]
\end{array}
\]

for which
\[
D((D^n_0)') := E(D(D^n_0)),
\]
and
\[
\forall f \in C[a, b] \; \forall g \in D(D^n_0) : \; ((D^n_0)' g^*) f = \int_a^b f(x) g^{(n)}(x) \, dx,
\]
where \( g^* := E g \in D((D^n_0)'). \)

The domain \( D((D^n_0)') \) is a total subspace of \( C^* [a, b] \), i.e., a set separating points in \( C[a, b] \), [10, Definition II.2.9]. Indeed, let \( f \in C[a, b] \) and suppose that
\[
\forall g^* \in D((D^n_0)') : \; g^* f = \int_a^b f(x) g(x) \, dx = 0,
\]
where \( g := E^{-1} g^* \in D(D^n_0) \), which implies that
\[
\forall g \in C^\infty_0 [a, b] = D(D^n_0) : \; \int_a^b f(x) g(x) \, dx = 0,
\]
and hence, \( f = 0 \) (see, e.g., [21]).

Thus, for \((D^n_0)’\), well defined in \( C[a, b] \) is the pre-adjoint (or preconjugate) operator
\[
D’((D^n_0)') := \{ f \in C[a, b] \mid \exists h \in C[a, b] \; \forall g^* \in D((D^n_0)') : ((D^n_0)' g^*) f = g^* h \} \ni f \mapsto ((D^n_0)’ f := h,
\]
[10, Definition VI.1.1].

In view of (3.5), (3.6) acquires the form
\[
D’((D^n_0)') := \left\{ f \in C[a, b] \mid \exists h \in C[a, b] \; \forall g \in D(D^n_0) : \int_a^b f(x) g^{(n)}(x) \, dx = \int_a^b h(x) g(x) \, dx \right\} \ni f \mapsto ((D^n_0)’ f := h,
\]
\[
\int_a^b f(x) g^{(n)}(x) \, dx = \int_a^b h(x) g(x) \, dx \}
\]

\[
\forall f \in C[a, b] \; \forall g \in D(D^n_0) : ((D^n_0)’ g^*) f = \int_a^b f(x) g^{(n)}(x) \, dx.
\]
From (3.7), we infer that $f \in D'(D_n^0) \subseteq C[a,b]$ iff its $n$th distributional derivative $(-1)^{n} h$ belongs to $C[a,b]$, which is the case iff

$$g(x) = (-1)^{n} \int_{a}^{x} \cdots \int_{a}^{t_2} h(t_1) \, dt_1 \cdots \, dt_n + \sum_{k=0}^{n-1} c_k x^k, \ x \in [a,b],$$

with some $c_k \in \mathbb{F}$, $k = 0, \ldots, n-1$, the latter being equivalent to the fact that $g \in C^n[a,b]$ and $g^{(n)}(x) = (-1)^{n} h(x), \ x \in [a,b]$.

Thus, we conclude that

$$D'(D_n^0) = C^n[a,b] = D(D_n)$$

and

$$'(D_n^0) f = (-1)^{n} D^n f, \ f \in C^n[a,b];$$

(see, e.g., [20, 21]), i.e.,

$$D^n = (-1)^{n}'(D_n^0)'.$$

By the closedness of a pre-adjoint operator [10, Lemma VI.1.2], we infer that the operator $D^n$ is closed, which completes the proof. 

**Theorem 3.1 (Chaoticity of Derivatives in $C[a,b]$).**

In the (real or complex) space $(C[a,b], \| \cdot \|_\infty)$ ($-\infty < a < b < \infty$), the $n$th derivative

$$D^n f := f^{(n)}$$

with maximal domain $D(D^n) := C^n[a,b]$ is a chaotic operator for every $n \in \mathbb{N}$.

**Proof.** By the prior proposition, $D^n$ is a densely defined unbounded closed linear operator for all $n \in \mathbb{N}$.

Let

$$Y := P = \bigcup_{n=1} \ker D^n \subseteq C^\infty[a,b] = C^\infty(D^n),$$

where $P$ is the dense in $(C[a,b], \| \cdot \|_\infty)$ subspace of polynomials (see (2.3)) and

$$\ker D^n = \{ f \in P \mid \deg f \leq n - 1 \}, \ n \in \mathbb{N}.$$

The mapping $B : Y \to Y$ is the restriction to $Y$ of the Volterra integration operator

$$[Bf](x) := \int_{a}^{x} f(t) \, dt, \ f \in C[a,b], x \in [a,b],$$

which is a quasinilpotent bounded linear operator on $C[a,b]$, i.e.,

$$(3.8) \quad \lim_{n \to \infty} \| B^n \|^{1/n} = 0$$

(here and henceforth, $\| \cdot \|$ also stands for the operator norm) (see, e.g., [16]).

Also,

$$ABf = f, \ f \in C[a,b].$$

Let $f \in Y$ be arbitrary. For all $n \geq \deg f + 1,$

$$D^n f = 0.$$
Further, by (3.8), we infer that
\[ \forall f \in C[a, b] : 0 \leq \limsup_{n \to \infty} \|B^n f\|^{1/n} \leq \limsup_{n \to \infty} (\|B^n\| \|f\|)^{1/n} = \lim_{n \to \infty} \|B^n\|^{1/n} \lim_{n \to \infty} \|f\|^{1/n} = 0 < 1 \]
(cf. (2.1)).

Thus, by the Sufficient Condition for Linear Chaos (Theorem 2.1) and the Chaoticity of Powers (Corollary 2.1), for each \( n \in \mathbb{N} \), the power \( D^n \) is chaotic. \( \square \)

Remarks 3.1.

• Since the natural embedding
\[ H(\mathbb{C}) \ni f \mapsto i(f) := f|_{[a, b]} \in C^\infty[a, b] \subseteq C[a, b] \]
(\( \cdot | \cdot \) is the restriction of a function (left) to a set (right)) is continuous and, due to the denseness of the subspace \( P \) of polynomials in \( (C[a, b], \| \cdot \|_\infty) \), has a dense range (see, e.g., [18]) and
\[ \forall f \in D(D_H) = H(\mathbb{C}) : i(f) \in C^1[a, b] = D(D_C) \quad \text{and} \quad DCi(f) = i(D_H f) \]
\( (D_H \text{ and } D_C \text{ stand for the differentiation operators in } H(\mathbb{C}) \text{ and } C[a, b], \text{ respectively}) \), i.e., the diagram
\[ \begin{array}{ccc}
C[a, b] & \ni D(D_C) & C[a, b] \\
\downarrow & i & \downarrow i \\
H(\mathbb{C}) & \ni D_H & H(\mathbb{C})
\end{array} \]
commutes, the image \( i(f) \) of any hypercyclic vector \( f \) for \( D_H \) is a hypercyclic vector for \( D_C \). Also, the image \( i(f) \) of an \( N \)-periodic point \( f \) for \( D_H \) is an \( N \)-periodic point for \( D_C \).

Thus, the case of the differentiation operator \( D \) in the complex space \( C[a, b] \) follows from the chaoticity of MacLane’s operator \( D \) in \( H(\mathbb{C}) \) (see Introduction).

• For the complex space \( C[a, b] \ (\infty < a < b < \infty) \), equality (3.1) and the fact that all \( \lambda \in \mathbb{C} \) are eigenvalues for \( D^n \ (n \in \mathbb{N}) \) of geometric multiplicity \( n \) are consistent with [14, Theorem 4.1].

3.2. Derivatives in \( L_p(a, b) \).

The following statement has a value of its own and is not to be used to prove the chaoticity of derivatives in \( L_p(a, b) \) (\( 1 \leq p < \infty, \ -\infty < a < b < \infty \)).

Proposition 3.2 (Closedness and Spectrum of Derivatives in \( L_p(a, b) \)).

In the (real or complex) space \( L_p(a, b) \ (1 \leq p < \infty, \ -\infty < a < b < \infty) \), the \( n \)th derivative
\[ D^n f := f^{(n)} \]
with maximal domain

\[ D(D^n) := W^n_p(a, b) := \{ f \in L_p(a, b) \mid f(\cdot) \in C^{n-1}[a, b], \ f^{(n-1)}(\cdot) \in AC[a, b], \ f^{(n)} \in L_p(a, b) \} \]

is a densely defined unbounded closed linear operator for every \( n \in \mathbb{N} \).

Furthermore, provided the underlying space is complex,

\[ \sigma(D^n) = \sigma_p(D^n) = \mathbb{C}, \ n \in \mathbb{N}, \]

each \( \lambda \in \mathbb{C} \) being an eigenvalue of geometric multiplicity \( n \).

**Proof.** Let \( n \in \mathbb{N} \) be arbitrary.

Observe that \( D^n \) is the \( n \)th power of \( D \), and hence, is linear.

The fact that \( D^n \) is densely defined follows from the inclusion

\[ P \subseteq C^\infty[a, b] = C^\infty(D), \]

where \( P \) is the subspace of polynomials (see (2.3)), which, by the Weierstrass approximation theorem, is dense in \( (C[a, b], \| \cdot \|_\infty) \) (see, e.g., [18]), and hence, in view of the denseness of \( C[a, b] \) in \( L_p(a, b) \) (see, e.g., [17]), also in \( L_p(a, b) \).

The unboundedness of \( D^n \) follows from the fact that, for

\[ e_k(x) := \left( \frac{kp + 1}{b - a} \right)^{1/p} \left( \frac{x - a}{b - a} \right)^k, \ k \in \mathbb{N}, x \in [a, b], \]

we have:

\[ e_k \in D(D^n) \text{ and } \|e_k\|_p = 1, \ k \in \mathbb{N}, \]

and, for \( k \geq n \),

\[ \|D^n e_k\|_p = \left\| \left( \frac{kp + 1}{b - a} \right)^{1/p} \prod_{j=0}^{n-1} \left( \frac{k - j}{b - a} \right)^n \left( \frac{x - a}{b - a} \right)^{k-n} \right\|_p \]

\[ = \left( \frac{kp + 1}{b - a} \right)^{1/p} \frac{k!}{(k-n)!} \left( \frac{1}{b - a} \right)^n \left\| \frac{x - a}{b - a} \right\|_p \]

\[ = \left( \frac{kp + 1}{b - a} \right)^{1/p} \frac{k!}{(k-n)!} \left( \frac{1}{b - a} \right)^n \left( \frac{b - a}{(k-n)p + 1} \right)^{1/p} \]

\[ \geq \left( \frac{kp + 1}{(k-n)p + 1} \right)^{1/p} k \left( \frac{1}{b - a} \right)^n \to \infty, \ k \to \infty. \]

For the complex space \( L_p(a, b) \), (3.9) follows from the fact that, for an arbitrary \( \lambda \in \mathbb{C} \), the equation

\[ D^n f = \lambda f \]

has \( n \) lineally independent solutions \( f_k, \ldots, f_n \) represented by the powers

\[ f_k(x) := x^{k-1}, \ k = 1, \ldots, n, \ x \in [a, b], \]

when \( \lambda = 0 \) or by the exponentials

\[ f_k(x) := e^{\lambda x}, \ k = 1, \ldots, n, \ x \in [a, b], \]
where $\lambda_k$, $k = 1, \ldots, n$, are the distinct values of $\sqrt[2]{\lambda}$, when $\lambda \neq 0$. Hence, $\lambda \in \sigma_p(A)$, with the corresponding eigenspace
\[
\ker(D^n - \lambda I) = \text{span} \left\{ f_1, \ldots, f_n \right\}
\]
being $n$-dimensional.

Let us prove that the operator $D^n$ is closed, which, due to (3.9), is not a given (see Remark 2.1).

Let $q := p/(p - 1) \in (1, \infty]$ be the conjugate to $p$ index.

As is known (see, e.g., [6]), there is an isometric isomorphism $E_{q,p}$ between the dual space $L_q^* (a,b)$ and $L_p(a,b)$, which relates the corresponding vectors $g^* \in L_q^* (a,b)$ and $g := E_{q,p} g^* \in L_p(a,b)$ as follows:
\[
g^* f = \int_a^b f(x)g(x) \, dx, \quad f \in L_q(a,b).
\]

In $L_q(a,b)$, consider the linear operator
\[
D^n_0 f := f^{(n)}
\]
with domain
\[
D(D^n_0) := C_0^\infty [a,b].
\]

Suppose first that $1 < p < \infty$. Then $1 < q < \infty$ and operator $D_0$ is densely defined due to the denseness of $C_0^\infty [a,b]$ (see (2.2)) in $L_q(a,b)$ (see, e.g., [21]). Therefore, for $D^n_0$, well defined in $L_q^* (a,b)$, is the adjoint (or conjugate) operator
\[
D((D^n_0)^*) := \{ g^* \in L_q^* (a,b) \mid \exists h^* \in L_q^* (a,b) : g^* D_0^n = h^* D(D^n_0) \} \ni g^* \mapsto (D^n_0)^* g^* := h^*
\]
[6,10].

Due to the isometric isomorphism $E_{q,p}$ between the dual space $L_q^* (a,b)$ and $L_p(a,b)$ (see (3.10)), $(D^n_0)^*$ can be identified with a linear operator $(D^n_0)^{'}$ in $L_p(a,b)$ defined as follows:
\[
(D^n_0)^{'} E_{q,p} := E_{q,p}(D^n_0)^* ,
\]
i.e., via the commutative diagram
\[
\begin{array}{ccc}
L_q^* (a,b) & \supseteq & D((D^n_0)^*) \\
\downarrow_{\text{E}_{q,p}} & \nearrow \text{E}_{q,p}^{'} \\
L_p(a,b) & \supseteq & D((D^n_0)^{'}
\end{array}
\]
for which
\[
(D^n_0)^{'} := E_{q,p}((D^n_0)^*) = \left\{ g \in L_p(a,b) \mid \exists h \in L_p(a,b) \forall f \in D(D^n_0) : \int_a^b f^{(n)}(x)g(x) \, dx = \int_a^b f(x)h(x) \, dx \right\} \ni f \mapsto (D^n_0)^{'} g := h.
\]

By the closedness of an adjoint operator (see, e.g., [10]), we conclude that the operator $(D^n_0)^{'}$ is closed along with $(D^n_0)^*$. 
From (3.13), we infer that \( g \in D((D^n_0)') \subseteq L_p(a, b) \) iff its \( n \)th distributional derivative \((-1)^n h\) belongs to \( L_p(a, b)\), which is the case iff the equivalence class \( g \) admits the representative

\[
g(x) = (-1)^n \int_a^x \cdots \int_a^{t_2} h(t_1) \, dt_1 \cdots dt_n + \sum_{k=0}^{n-1} c_k x^k, \quad x \in [a, b],
\]

with some \( c_k \in \mathbb{F}, \ k = 0, \ldots, n - 1\), the latter being equivalent to the fact that \( g \in W^n_p(a, b) \) and \( g^{(n)}(x) = (-1)^n h(x) \) on \([a, b]\) \((\text{mod } \mu)\).

\( (\mu\) is the Lebesgue measure on \( \mathbb{R} \)).

Thus, we conclude that

\[
D((D^n_0)') = W^n_p(a, b) = D(D^n)
\]

and

\[
(D^n_0)' f = (-1)^n D^n f, \quad f \in W^n_p(a, b),
\]

(see, e.g., [20, 21]), which implies that

\[
D^n = (-1)^n (D^n_0)'.
\]

Whence, by the closedness of the operator \((D^n_0)'\), we infer that the operator \( D^n \) is closed.

For \( p = 1, \ q = \infty \) and the linear operator \( D^n_0 \) in \( L_\infty(a, b) \) (see (3.11)–(3.12) with \( q = \infty \)) is not densely defined, and hence, has no adjoint.

In \( L_1^*(a, b)\), let us consider a linear operator \((D^n_0)'\) defined as follows:

\[
(D^n_0)' E_{1, \infty}^{-1} := E_{1, \infty}^{-1} D^n_0,
\]

where \( E_{1, \infty}\) the isometric isomorphism between the dual space \( L_1^*(a, b)\) and \( L_\infty(a, b)\) (see (3.10) with \( q = 1 \) and \( p = \infty \)), i.e., via the commutative diagram

\[
\begin{array}{ccc}
L_1^*(a, b) & \supseteq & D((D^n_0)'), \\
\uparrow & \nearrow & \\
E_{1, \infty}^{-1} & & E_{1, \infty}^{-1}, \\
\downarrow & & \\
L_\infty(a, b) & \supseteq & D(D^n_0),
\end{array}
\]

for which

\[
D((D^n_0)') := E_{1, \infty}^{-1}(D(D^n_0)),
\]

and

\[
\forall f \in L_1(a, b) \forall g \in D(D^n_0) : ((D^n_0)'g^*)f = \int_a^b f(x) g^{(n)}(x) \, dx,
\]

where \( g^* := E_{1, \infty}^{-1}g \in D((D^n_0)').\)

The domain \( D((D^n_0)')\) is a total subspace of \( L_1^*(a, b)\), i.e., a set separating points in \( L_1(a, b)\), [10, Definition II.2.9]. Indeed, let \( f \in L_1(a, b) \) and suppose that

\[
\forall g^* \in D((D^n_0)') : g^* f = \int_a^b f(x) g(x) \, dx = 0,
\]
where \( g := E_{1,\infty}g^* \in D(D^0_n) \), which implies that
\[
\forall g \in \mathcal{C}_0^\infty[a,b] = D(D^0_0) : \int_a^b f(x)g(x) \, dx = 0,
\]
and hence, \( f = 0 \) (see, e.g., [21]).

Thus, for \( (D^0_n)' \), well defined in \( L_1(a,b) \) is the pre-adjoint (or preconjugate) operator
\[
(D^0_n)' := \left\{ f \in L_1(a,b) \mid \exists h \in L_1(a,b) \forall g^* \in D((D^0_n)') : ((D^0_n)')(g^*)f = g^*h \right\} \ni f \mapsto (D^0_n)'f := h,
\]
[10, Definition VI.1.1].

In view of (3.15), (3.16) acquires the form
\[
D((D^0_n)') := \left\{ f \in L_1(a,b) \mid \exists h \in L_1(a,b) \forall g \in D(D^0_n) : \int_a^b f(x)g^{(n)}(x) \, dx = \int_a^b h(x)g(x) \, dx \right\} \ni f \mapsto (D^0_n)'f := h,
\]
(3.17)

From (3.17), we infer that \( f \in D((D^0_n)') \subseteq L_1(a,b) \) iff its \( n \)th distributional derivative \((-1)^nf\) belongs to \( L_1(a,b) \). Whence, by reasoning as in regard to (3.14), we can conclude that
\[
D((D^0_n)') = W^1_1(a,b) = D(D^n)
\]
and
\[
(D^0_n)'f = (-1)^nD^n f, \ f \in W^1_1(a,b),
\]
which implies that
\[
D^n = (-1)^n(D^0_n)'.
\]

Whence, by the closedness of a pre-adjoint operator [10, Lemma VI.1.2], we infer that the operator \( D^n \) is closed, which completes the proof. □

Since the Volterra integration operator
\[
[Bf](x) := \int_a^x f(t) \, dt, \ f \in L_p(a,b), x \in [a,b],
\]
is also a quasinilpotent bounded linear operator on \( L_p(a,b) \) \( (1 \leq p < \infty, -\infty < a < b < \infty) \) (see, e.g., [12]), the following statement can be proved by replicating the proof of Theorem 3.1 verbatim.

**Theorem 3.2** (Chaoticity of Derivatives in \( L_p(a,b) \)).
*In the (real or complex) space \( L_p(a,b) \) \( (1 \leq p < \infty, -\infty < a < b < \infty) \), the \( n \)th derivative
\[
D^n f := f^{(n)}
\]
with maximal domain \( D(D^n) := W^p(a,b) \) is a chaotic operator for every \( n \in \mathbb{N} \).*
Remarks 3.2.

- Since, for $1 \leq p < \infty$, the natural embedding
  $$(C[a, b], \| \cdot \|_{\infty}) \ni f \mapsto i(f) := f \in L_p(a, b)$$
  is continuous and has a dense range (see, e.g., [6, 17]) and, for $n \in \mathbb{N}$,
  $$\forall f \in D(D^p_n) = C^n[a, b] : i(f) \in W^n_p(a, b) = D(D^p_{\ell_p})$$
  and $D^p_{\ell_p} i(f) = i(D^p_n f)$
  ($D^p_n$ and $D^p_{\ell_p}$ stand for the $n$th derivative operators with maximal domain
  in $C[a, b]$ and $L_p(a, b)$, respectively), i.e., the diagram

  $\begin{array}{ccc}
  L_p(a, b) & \ni \ni & L_p(a, b) \\
  \downarrow & & \uparrow \\
  C[a, b] & \ni \ni & C[a, b]
  \end{array}$

  commutes, the image $i(f)$ of any hypercyclic vector $f$ for $D^p_n$ is a hypercyclic
  vector for $D^p_{\ell_p}$. Also, the image $i(f)$ of an $N$-periodic point $f$ for $D^p_n$ is an
  $N$-periodic point for $D^p_{\ell_p}$.

  Thus, the prior theorem can also be considered as a corollary of Theorem
  3.1.

- For the complex space $L^p[a, b]$ ($1 \leq p < \infty$, $-\infty < a < b < \infty$), equality
  (3.9) and the fact that all $\lambda \in \mathbb{C}$ are eigenvalues for $D^n$ ($n \in \mathbb{N}$) of geometric
  multiplicity $n$ are consistent with [14, Theorem 4.1].

4. Concluding Remarks

- Since $C^n[a, b]$ ($n \in \mathbb{N}$) is a Banach space relative to the norm

  $$\| f \|_n := \sum_{k=0}^{n} \| f^{(k)} \|_{\infty}, \ f \in C^n[a, b],$$

  (see, e.g., [16]), the closedness of the $n$th derivative operator $D^n$ with
  $D(D^n) = C^n[a, b]$ in $C[a, b]$ implies that $C^n[a, b]$ is also a Banach space
  relative to the weaker graph norm

  $$\| f \|_{D^n} := \| f \|_{\infty} + \| f^{(n)} \|_{\infty} \leq \| f \|_n, \ f \in C^n[a, b],$$

  and hence, as follows from the Inverse mapping theorem, the norms $\| \cdot \|_n$ and $\| \cdot \|_{D^n}$ on $C^n[a, b]$ are equivalent, i.e.,

  $$\exists C > 0 \ \forall f \in C^n[a, b] : \| f \|_n \leq C \left[ \| f \|_{\infty} + \| f^{(n)} \|_{\infty} \right]$$

  (see, e.g., [16]).

- Since the Sobolev space $W^n_p(a, b)$ ($1 \leq p < \infty$, $n \in \mathbb{N}$, $-\infty < a < b < \infty$)
  is a Banach space relative to the norm

  $$\| f \|_{p,n} := \sum_{k=0}^{n} \| f^{(k)} \|_p, \ f \in W^n_p(a, b),$$
(see, e.g., [20, 21]), the closedness of the \( n \)th derivative operator \( D^n \) with 
\[ D(D^n) = W^n_p(a, b) \] in \( L_p(a, b) \) similarly implies that the norm \( \| \cdot \|_{p,n} \) on 
\( W^n_p(a, b) \) is equivalent to the weaker graph norm 
\[ \| f \|_{D^n} := \| f \|_p + \| f^{(n)} \|_p \leq \| f \|_{p,n}, \quad f \in W^n_p(a, b), \]
i.e.,
\[ \exists C > 0 \forall f \in W^n_p(a, b) : \| f \|_{p,n} \leq C \left( \| f \|_p + \| f^{(n)} \|_p \right). \]

REFERENCES

[1] F. Bayart and É. Matheron, Dynamics of Linear Operators, Cambridge University Press, Cambridge, 2009.
[2] T. Bermúdez, A. Bonilla, and J.L. Torrea, Chaotic behavior of the Riesz transforms for Hermite expansions, J. Math. Anal. Appl. 337 (2008), 702–711.
[3] J. Bès, K.C. Chan, and S.M. Seubert, Chaotic unbounded differentiation operators, Integral Equations Operator Theory 40 (2001), no. 3.
[4] R. deLaubenfels, H. Emamirad, and K.-G. Grosse-Erdmann, Chaos for semigroups of unbounded operators, Math. Nachr. 261/262 (2003), 47–59. 257–267.
[5] R.L. Devaney, An Introduction to Chaotic Dynamical Systems, 2nd ed., Addison-Wesley, New York, 1989.
[6] N. Dunford and J.T. Schwartz with the assistance of W.G. Bade and R.G. Bartle, Linear Operators. Part I: General Theory, Interscience Publishers, New York, 1958.
[7] H. Emamirad and G.S. Heshmati, Chaotic weighted shifts in Bargmann space, J. Math. Anal. Appl. 308 (2005), 36–46.
[8] G. Godefroy and J.H. Shapiro, Operators with dense, invariant, cyclic vector manifolds, J. Funct. Anal. 98 (1991), 229–269.
[9] C. Goffman and G. Pedrick, First Course in Functional Analysis, 2nd ed., Chelsea Publishing Co., New York, 1983.
[10] S. Goldberg, Unbounded Linear Operators: Theory and Applications, Dover Publications, Inc., New York, 1985.
[11] K.-G. Grosse-Erdmann, Hypercyclic and chaotic weighted shifts, Studia Math. 139 (2000), no. 1, 47–68.
[12] K.-G. Grosse-Erdmann and A.P. Manguillot, Linear Chaos, Universitext, Springer-Verlag, London, 2011.
[13] G.R. MacLane, Sequences of derivatives and normal families, J. Analyse Math. 2 (1952/53), 72–87.
[14] M.V. Markin, On sufficient and necessary conditions for linear hypercyclicity and chaos, arXiv:2106.14872.
[15] ________, On the non-hypercyclicity of scalar type spectral operators and collections of their exponentials, Demonstr. Math. 53 (2020), no. 1, 352–359.
[16] ________, Elementary Operator Theory, De Gruyter Graduate, Walter de Gruyter GmbH, Berlin/Boston, 2020.
[17] ________, Real Analysis. Measure and Integration, De Gruyter Graduate, Walter de Gruyter GmbH, Berlin/Boston, 2019.
[18] ________, Elementary Functional Analysis, De Gruyter Graduate, Walter de Gruyter GmbH, Berlin/Boston, 2018.
[19] M.V. Markin and E.S. Sichel, On the non-hypercyclicity of normal operators, their exponentials, and symmetric operators, Mathematics 7 (2019), no. 10, Article no. 903, 8 pp.
[20] S.L. Sobolev, Applications of Functional Analysis in Mathematical Physics, Translations of Mathematical Monographs, vol. 7, American Mathematical Society, Providence, Rhode Island, 1963.
[21] W.P. Ziemer, Weakly Differentiable Functions. Sobolev Spaces and Functions of Bounded Variation, Graduate Texts in Mathematics, vol. 120, Springer-Verlag, New York, 1989.
