\( \mathcal{N} \)-fold Supersymmetry and Quasi-solvability Associated with \( X_2 \)-Laguerre Polynomials

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Abstract

We construct a new family of quasi-solvable and \( \mathcal{N} \)-fold supersymmetric quantum systems where each Hamiltonian preserves an exceptional polynomial subspace of codimension 2. We show that the family includes as a particular case the recently reported rational radial oscillator potential whose eigenfunctions are expressed in terms of the \( X_2 \)-Laguerre polynomials of the second kind. In addition, we find that the two kinds of the \( X_2 \)-Laguerre polynomials are ingeniously connected with each other by the \( \mathcal{N} \)-fold supercharge.

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I. INTRODUCTION

Recently, a new type of polynomial systems called exceptional polynomials has attracted attention in mathematical physics. Roughly speaking, they are an infinite sequence of polynomials which are eigenfunctions of a second-order differential operator but which have a non-zero lowest degree \( m > 0 \). They are usually generated by an infinite sequence of exceptional polynomial subspaces of a fixed codimension \( m \). According to the definition in Refs. [1, 2], a \( k \)-dimensional subspace \( M_k \) of the \( N \)-dimensional type A monomial subspace \( \tilde{\mathcal{V}}^{(A)}_N \):

\[
M_k \subset \tilde{\mathcal{V}}^{(A)}_N = \langle 1, z, \ldots, z^{N-1} \rangle,
\]

is called an exceptional polynomial subspace of codimension \( m = N - k \) or an \( X_m \) subspace if the linear space \( D_2(M_k) \) of second-order linear differential operators which preserve \( M_k \) is not a subspace of the linear space \( D_2(\tilde{\mathcal{V}}^{(A)}_N) \) of those which preserve \( \tilde{\mathcal{V}}^{(A)}_N \), namely, \( D_2(M_k) \not\subset D_2(\tilde{\mathcal{V}}^{(A)}_N) \). Various mathematical results on the \( X_1 \) polynomials and \( X_1 \) subspaces are explored in Refs. [1–3].

On the other hand, some exactly solvable quantum mechanical potentials whose eigenfunctions are expressed in terms of \( X_m \) polynomials have been reported in the last few years [4–6]. All of them were constructed using the technique of supersymmetric (SUSY) quantum mechanics [7–10] and have shape invariance [11]. Those findings imply wide applicability of such mathematical concepts in various physics and mathematical sciences. Thus, it would be interesting to see whether there exist other exactly solvable and/or less restrictive quasi-exactly solvable models [12, 13] associated with \( X_m \) polynomials and \( X_m \) subspaces.

In this respect, the recent research developments have shown that the framework of \( \mathcal{N} \)-fold SUSY [14–17] is remarkably useful for investigating such issues (for a review of \( \mathcal{N} \)-fold SUSY see Ref. [18]). This originates from the important fact that \( \mathcal{N} \)-fold SUSY is essentially equivalent to weak quasi-solvability (for the precise definitions of the hierarchy of solvability, see, e.g., Refs. [18, 19]) proved in Ref. [15] (for the equivalence in specific models, see also references cited therein). Since weak quasi-solvability includes solvability, on the one hand, and shape invariance is a sufficient condition for solvability\(^1\), on the other hand, shape invariance always implies \( \mathcal{N} \)-fold SUSY,

\[
(\mathcal{N}\text{-fold SUSY}) \equiv (\text{Weak quasi-solvability}) \supset (\text{Solvability}) \supset (\text{Shape invariance}).
\]

Hence, all the shape invariant potentials reported in Refs. [4–6] must have \( \mathcal{N} \)-fold SUSY. The ones whose eigenfunctions are expressed in terms of the \( X_1 \)-Laguerre or \( X_1 \)-Jacobi polynomials are evidently preserve exceptional polynomial subspaces of codimension 1. On the other hand, according to Theorem 1.4 in Ref. [2], there is only one exceptional polynomial subspace of codimension 1 up to the projective equivalence and its representative can be chosen as type B monomial space \( \tilde{\mathcal{V}}^{(B)}_N \):

\[
\tilde{\mathcal{V}}^{(B)}_N \langle z \rangle = \langle 1, z, \ldots, z^{N-2}, z^N \rangle.
\]

\(^1\) It should be noted that shape invariance is not a sufficient condition for exact solvability; shape invariance, as well as solvability, is a local concept while exact solvability is a global one. For the importance of recognizing the difference between local and global concepts, see, e.g., Ref. [20].
This monomial space is first considered in Ref. [21] in the context of the classification of monomial spaces preserved by second-order linear ordinary differential operators. It was shown that any Hamiltonian which preserves the type B monomial space belongs to type B \(N\)-fold SUSY [17, 22]. Thus, we come to the conclusion that all of the models whose eigenfunctions are expressed in terms of the \(X_1\)-Laguerre or \(X_1\)-Jacobi polynomials in Refs. [4, 5] belong to type B \(N\)-fold SUSY up to the projective equivalence. In fact, the rational potential \(V(x)\) in Ref. [4], Eq. (8), whose eigenfunctions are written in terms of the \(X_1\)-Laguerre polynomials coincides with one of the type B \(N\)-fold SUSY models \(V_N^\pm(q)\) in Ref. [22], Eq. (4.2). Explicitly, they are identical to each other up to a multiplicative factor \(V(x) = 2V_N^\pm(x)\) with the following parameter relations:

\[
b_1 = \omega, \quad h_0 = \frac{2l + 1}{\omega}, \quad R = \frac{\omega N - \omega(2N - 2l - 1)}{4}.
\]

(1.3)

Hence, the framework of \(N\)-fold SUSY would provide a powerful tool for investigating and constructing solvable and quasi-solvable quantum systems associated not only with monomial spaces but also with exceptional polynomial subspaces. On the other hand, it has not been reported yet, to the best of our knowledge, any \(N\)-fold SUSY where a component Hamiltonian preserves any \(X_m\) subspace of codimension \(m > 1\).

In this paper, we construct for the first time a family of quasi-solvable and \(N\)-fold SUSY quantum systems where each Hamiltonian preserves an exceptional polynomial subspace of codimension 2. We rely on the algorithmic construction developed in Ref. [17]. The resulting \(N\)-fold SUSY systems turn to include as a particular case the rational shape invariant potential whose eigenfunctions are expressed in terms of the \(X_2\)-Laguerre polynomials of the second kind in Ref. [5]. Furthermore, we find, in particular, that the two \(X_2\) subspaces connected by the \(N\)-fold supercharge would generate the two different kinds of the \(X_2\)-Laguerre polynomials found in the latter reference.

We organize the paper as follows. In Section II, we introduce a finite-dimensional polynomial space which turns out to be an exceptional polynomial subspace of codimension 2. Then, we present quasi-solvable operators which leave the latter space invariant. With the set of the polynomial space and the quasi-solvable operators, we construct in Section III a new type of \(N\)-fold SUSY systems by applying the algorithm developed in Ref. [17]. We present the pair of \(N\)-fold SUSY Hamiltonians and the \(N\)-fold supercharges in closed form. As a by-product, we obtain another exceptional polynomial subspace of codimension 2 and a set of quasi-solvable operators which preserve it. In Section IV, we exhibit a couple of examples of quantum mechanical systems which possess the new \(N\)-fold SUSY. In particular, we show that one of them coincides with the rational potential model whose eigenfunctions are expressed in terms of the \(X_2\)-Laguerre polynomials of the second kind recently found in Ref. [5]. Finally, we summarize the results and discuss various perspectives of future issues in Section V.

II. AN \(X_2\) POLYNOMIAL SUBSPACE AND ITS QUASI-SOLVABLE OPERATORS

Our starting point is to consider a vector space \(\tilde{V}_N^{-}\) of finite dimension \(\mathcal{N}\),

\[
\tilde{V}_N^{-} = \tilde{V}_N^{(X_{2a})}[z; \alpha] = \langle \tilde{\varphi}_1(z; \alpha), \ldots, \tilde{\varphi}_\mathcal{N}(z; \alpha) \rangle,
\]

(2.1)
spanned by polynomials $\tilde{\varphi}_n(z; \alpha)$ of degree $n + 1$ in $z$ as follows:

$$
\tilde{\varphi}_n(z; \alpha) = (\alpha + n - 2)z^{n+1} + 2(\alpha + n - 1)(\alpha - 1)z^n + (\alpha + n)(\alpha - 1)\alpha z^{n-1}, \quad (2.2)
$$

where $\alpha(\neq 0, 1)$ is a parameter. When $\alpha = 1$, then $\tilde{\varphi}_n(z; 1) = (n - 1)z^{n+1}$ by definition. Hence, the vector space $\tilde{V}_N^{(X_{2a})}[z; \alpha]$ in (2.1) reduces to a space which is equivalent to type A monomial space $\tilde{V}_N^{(A)}$ (cf., Ref. [17]),

$$
\tilde{V}_N^{(X_{2a})}[z; 1] = z^3\langle 1, z, \ldots, z^{N-2} \rangle = z^3\tilde{V}_N^{(A)}[z]. \quad (2.3)
$$

Similarly, when $\alpha = 0$, then $\tilde{\varphi}_n(z; 0) = (n - 2)z^{n+1} - 2(n - 1)z^n$ by definition and, in particular, $2\tilde{\varphi}_1(z; 0) = \tilde{\varphi}_2(z; 0) = -2z^2$. On the other hand, we can inductively prove the following formula:

$$
(n - 1) \sum_{k=3}^{n} \frac{2^{n-k}}{(k-1)(k-2)} \tilde{\varphi}_k(z; 0) = z^{n+1} - (n - 1)2^{n-2}z^3. \quad (2.4)
$$

Hence, in the case of $\alpha = 0$ we have

$$
\tilde{V}_N^{(X_{2a})}[z; 0] = z^2 V_{N-1} = z^2\langle 1, z^2 - 4z, \ldots, z^{N-1} - (N - 1)2^{N-2}z \rangle. \quad (2.5)
$$

By Proposition 2.5 in Ref. [2] with $n = N - 1$, $\lambda = 1$, $\mu = 0$, and $\beta_j = -j2^{j-1}$, the $(N - 1)$-dimensional linear space $V_{N-1} \subset \tilde{V}_N^{(A)}[z]$ introduced above is a polynomial subspace of codimension 1 and its fundamental covariant $q_{V_{N-1}}(z)$ is given by

$$
q_{V_{N-1}}(z) = -(N - 1)(z - 2)^{N-2}. \quad (2.6)
$$

That is, it has a root of multiplicity $N - 2 = n - 1$ at $z = 2$ and thus the space $V_{N-1}$ is an exceptional polynomial subspace of codimension 1 which is projectively equivalent to type B monomial space [2]

$$
V_{N-1} \sim \tilde{V}_N^{(B)}[z] = \langle 1, z, \ldots, z^{N-3}, z^{N-1} \rangle. \quad (2.7)
$$

Therefore, the constraint $\alpha \neq 0, 1$ prevents the space $\tilde{V}_N^{(X_{2a})}[z; \alpha]$ from reducing to the well-studied monomial spaces of types A and B (for the latter issues, see Refs. [1, 21–24]). In addition, we can assume $N > 2$ without any loss of generality. Indeed, when $N = 1, 2$, the space $\tilde{V}_N^{(X_{2a})}[z; \alpha]$ is also essentially equivalent to type A monomial space as

$$
\tilde{V}_1^{(X_{2a})}[z; \alpha] = \tilde{\varphi}_1(z; \alpha)\tilde{V}_1^{(A)}[z], \quad \tilde{V}_2^{(X_{2a})}[z; \alpha] = \tilde{\varphi}_2(z; \alpha)\tilde{V}_2^{(A)}[w], \quad (2.8)
$$

where $w = \tilde{\varphi}_2(z; \alpha)/\tilde{\varphi}_1(z; \alpha)$. Thus, we hereafter assume $N > 2$.

We shall look for a vector space of linear differential operators of (at most) second order which preserve the polynomial space $\tilde{V}_N^{-}$. One of the characteristic features of the set of polynomials $\tilde{\varphi}_n(z; \alpha)$ defined in (2.2) is the following factorization under the action of the operator $\partial_z - 1$:

$$
(\partial_z - 1)\tilde{\varphi}_n(z; \alpha) = -[\alpha + n - 2]z - (n - 1)(\alpha + n)z^{n-2}f(z; \alpha), \quad (2.9)
$$
where the common factor \( f(z; \alpha) \) is an \( n \)-independent polynomial of second degree in \( z \) given by

\[
f(z; \alpha) = z^2 + 2(\alpha - 1)z + (\alpha - 1)\alpha.
\] (2.10)

Taking into account the factorization property (2.9), we easily see that any (at most) second-order linear differential operator \( J \) of the following form:

\[
J = p_4(z)\partial_z^2 + p_3(z)\partial_z + p_2(z) + \frac{p_1(z)}{f(z; \alpha)} (\partial_z - 1),
\]

(2.11)

where \( p_i(z) \) \((i = 1, \ldots, 4)\) are all polynomials in \( z \), maps all the polynomials \( \tilde{\varphi}_n(z; \alpha) \) \((n = 1, 2, 3, \ldots)\) to other polynomials which are not necessarily elements of \( \tilde{V}_N^- \). The latter fact is rather considered as a necessary condition for an operator to preserve the vector space \( \tilde{V}_N^- \). Hence, we restrict ourselves to considering linear operators of the form (2.11). As we will show shortly, quasi-solvable operators of the form (2.11) which preserve \( \tilde{V}_N^- = \tilde{V}_N^{(X_{2n})}[z; \alpha] \subset \tilde{V}_{N+2}^{(A)}[z] \) consist of those which have non-trivial \( p_1(z) \), and thus do not preserve type A monomial space \( \tilde{V}_{N+2}^{(A)}[z] \), namely, \( D_2(\tilde{V}_N^{(X_{2n})}) \not\subseteq D_2(\tilde{V}_N^{(A)}) \). Hence, the polynomial subspace \( \tilde{V}_N^{(X_{2n})}[z; \alpha] \) belongs to an exceptional polynomial subspace of codimension 2 which is denoted by \( X_2 \) in Ref. [2]. For \( N > 2 \), we find that there are four linearly independent such quasi-solvable operators which leave the space \( \tilde{V}_N^- \) invariant. The first operator denoted by \( J_1 \) is given by

\[
J_1 = z\partial_z^2 - (z - \alpha + 3)\partial_z + \frac{4(\alpha - 1)(z + \alpha)}{f(z; \alpha)} (\partial_z - 1).
\] (2.12)

The action of \( J_1 \) on \( \tilde{\varphi}_n(z; \alpha) \) \((n = 1, 2, 3, \ldots)\) reads as

\[
J_1\tilde{\varphi}_n(z; \alpha) = -(n + 1)\tilde{\varphi}_n(z; \alpha) + (n - 1)(\alpha + n)\tilde{\varphi}_{n-1}(z; \alpha).
\] (2.13)

Hence, it preserves not only \( \tilde{V}_N^- \) for a specific value of \( N \) but also an infinite flag of the spaces

\[
\tilde{V}_1^- \subset \tilde{V}_2^- \subset \cdots \subset \tilde{V}_N^- \subset \cdots.
\] (2.14)

Therefore, the operator \( J_1 \) is not only quasi-solvable but also solvable\(^2\). The second operator, denoted by \( J_2 \), which preserves \( \tilde{V}_N^- \) is given by

\[
\begin{align*}
J_2 &= [z^2 + (\alpha - 1)(\alpha + N - 1)]\partial_z^2 - [z^2 + (N + 1)z + (\alpha - 1)(3\alpha + 3N - 7)]\partial_z \\
&\quad + (N + 1)z - 4(\alpha - 1)(2\alpha + N - 3)z + (\alpha - 1)(2\alpha + N - 1) \frac{f(z; \alpha)}{f(z; \alpha)} (\partial_z - 1).
\end{align*}
\] (2.15)

The action of \( J_2 \) on \( \tilde{\varphi}_n(z; \alpha) \) \((n = 1, 2, 3, \ldots)\) reads as

\[
\begin{align*}
(\alpha + n - 2)(\alpha + n - 1)J_2\tilde{\varphi}_n(z; \alpha) &= -(n - N)(\alpha + n - 2)^2\tilde{\varphi}_{n+1}(z; \alpha) + s^{-}(n, \alpha, N) \\
&\times \tilde{\varphi}_n(z; \alpha) - (n - 1)(\alpha - 1)(\alpha + N - 1)[3\alpha^2 + 6(n - 1)\alpha + 3n^2 - 6n + 4]\tilde{\varphi}_{n-1}(z; \alpha) \\
&+ (n - 1)(\alpha - 1)(\alpha + N - 1)(\alpha + n - 1)(\alpha + n)\tilde{\varphi}_{n-2}(z; \alpha),
\end{align*}
\] (2.16)

\(^2\) This significant characterization of solvability first appeared in Ref. [25] but unfortunately without sufficient appreciation of the difference between solvability and exact solvability.
where $s^{-}(n, \alpha, \mathcal{N})$ is given by

$$s^{-}(n, \alpha, \mathcal{N}) = 2\alpha^3 + [n^2 - (\mathcal{N} - 4)n - \mathcal{N} - 7]\alpha^2$$
$$+ [2n^3 - (2\mathcal{N} + 1)n^2 + (\mathcal{N} - 12)n + 5\mathcal{N} + 9]\alpha$$
$$+ n^4 - (\mathcal{N} + 3)n^3 + (2\mathcal{N} - 1)n^2 + (\mathcal{N} + 9)n - 4(\mathcal{N} + 1).$$

(2.17)

Hence, it certainly satisfies $J_2\tilde{\mathcal{V}}^{-}_{\mathcal{N}} \subset \tilde{\mathcal{V}}^{-}_{\mathcal{N}}$ only for a specific value of $\mathcal{N} \in \mathbb{N}$. The third operator, denoted by $J_3$, leaving $\tilde{\mathcal{V}}^{-}_{\mathcal{N}}$ invariant is given by

$$J_3 = (z + 2\alpha + \mathcal{N} - 1)z^2\partial_z^2 + \{(\alpha - \mathcal{N} - 2)z^2 + [3\alpha^2 + (\mathcal{N} - 2)\alpha - 2(\mathcal{N} + 1)]z$$
$$+ 4(\alpha - 1)(\alpha + \mathcal{N} + 1)\}\partial_z - (\mathcal{N} + 1)(\alpha - 2)z$$
$$- 4(\alpha - 1)(\alpha^2 + \mathcal{N}\alpha - 2\mathcal{N} - 2)z + \alpha(\alpha - 1)(\alpha + \mathcal{N} + 1)\) (\partial_z - 1).$$

(2.18)

The action of $J_3$ on $\tilde{\varphi}_n(z; \alpha) (n = 1, 2, 3, \ldots)$ admits a relatively simple form as

$$J_3\tilde{\varphi}_n(z; \alpha) = (n - \mathcal{N})(\alpha + n - 2)\tilde{\varphi}_{n+1}(z; \alpha) + \{(3n + 1)\alpha^2 + [2n^2 + (\mathcal{N} - 4)n$$
$$+ 3\mathcal{N} + 2]\alpha + (\mathcal{N} - 1)n(n - 1) - 4(\mathcal{N} + 1)\}\tilde{\varphi}_n(z; \alpha).$$

(2.19)

and thus it also satisfies $J_3\tilde{\mathcal{V}}^{-}_{\mathcal{N}} \subset \tilde{\mathcal{V}}^{-}_{\mathcal{N}}$ only for a specific value of $\mathcal{N} \in \mathbb{N}$. The fourth quasi-solvable operator denoted by $J_4$ has the most complicated form as follows:

$$(2\alpha + \mathcal{N} - 1)J_4 = [(2\alpha + \mathcal{N} - 1)z + 3\alpha^2 + (3\mathcal{N} - 2)\alpha + \mathcal{N}(\mathcal{N} - 1)]z^3\partial_z^2$$
$$- B^{-}_4(z; \alpha, \mathcal{N})\partial_z + \mathcal{N}(\mathcal{N} + 1)(2\alpha + \mathcal{N} - 1)z^2 + (\mathcal{N} + 1)(\alpha - 1)[3\alpha^2$$
$$+ 2(3\mathcal{N} - 8)\alpha + 2(\mathcal{N} - 4)(\mathcal{N} - 1)]z + 4(\alpha - 1)^2\frac{D^{-}_4(z; \alpha, \mathcal{N})}{f(z; \alpha)}(\partial_z - 1),$$

(2.20)

where $B^{-}_4(z; \alpha, \mathcal{N})$ and $D^{-}_4(z; \alpha, \mathcal{N})$ are, respectively, given by

$$B^{-}_4(z; \alpha, \mathcal{N}) = 2\mathcal{N}(2\alpha + \mathcal{N} - 1)z^3 + [3\alpha^3 + (9\mathcal{N} - 19)\alpha^2 + (5\mathcal{N}^2 - 18\mathcal{N} + 24)\alpha$$
$$+ (\mathcal{N} - 1)(\mathcal{N}^2 - 2\mathcal{N} + 8)]z^2 + (\alpha - 1)[7\alpha^3 + (14\mathcal{N} - 15)\alpha^2$$
$$+ (3\mathcal{N} + 1)(3\mathcal{N} - 8)\alpha + 2(\mathcal{N} - 4)(\mathcal{N}^2 - 1)]z + (\alpha - 1)[2\alpha^4 + (5\mathcal{N} + 1)\alpha^3$$
$$+ (4\mathcal{N}^2 + 3\mathcal{N} + 13)\alpha^2 + (\mathcal{N}^3 + 2\mathcal{N}^2 - 7\mathcal{N} - 36)\alpha - 8(\mathcal{N} - 1)(\mathcal{N} + 2)\]$$

(2.21)

and

$$D^{-}_4(z; \alpha, \mathcal{N}) = [\alpha^3 + (2\mathcal{N} + 3)\alpha^2 + (\mathcal{N}^2 - 5\mathcal{N} - 16)\alpha - 4(\mathcal{N} - 1)(\mathcal{N} + 2)]z$$
$$+ \alpha[\alpha^3 + (2\mathcal{N} + 3)\alpha^2 + (\mathcal{N}^2 - 2\mathcal{N} - 9)\alpha - 2(\mathcal{N} - 4)(\mathcal{N} + 2)]\]$$

(2.22)

The action of $J_4$ on $\tilde{\varphi}_n(z; \alpha) (n = 1, 2, 3, \ldots)$ reads as

$$(\alpha + n - 1)(\alpha + n)J_4\tilde{\varphi}_n(z; \alpha) = (n - \mathcal{N})(n - \mathcal{N} + 1)(\alpha + n - 2)(\alpha + n - 1)\tilde{\varphi}_{n+2}(z; \alpha)$$
$$- (n - \mathcal{N})t^{-}_1(n, \alpha, \mathcal{N})\tilde{\varphi}_{n+1}(z; \alpha) - (\alpha - 1)t^{-}_2(n, \alpha, \mathcal{N})\tilde{\varphi}_n(z; \alpha)$$
$$- (n - 1)\alpha(\alpha - 1)(\alpha + \mathcal{N} - 1)(\alpha + \mathcal{N})(\alpha + n)^2\tilde{\varphi}_{n-1}(z; \alpha),$$

(2.23)
and thus it surely preserves the space $\tilde{V}_N^-$ for a specific value of $N \in \mathbb{N}$. In the above, $t_1^{-}(n, \alpha, N)$ and $t_2^{-}(n, \alpha, N)$ are, respectively, given by

$$
(2\alpha + N - 1)t_1^{-}(n, \alpha, N) = 3 \alpha^5 + (3n + 6N - 20)\alpha^4 - [3n^2 - (9N - 26)n + 2N^2 + 27N - 50] \alpha^3 - [3n^3 + 4n^2 - (3N^2 - 35N + 52)n + 7N^2 - 49N + 52] \alpha^2
$$

$$
- [(3N - 2)n^3 + (7N - 10)n^2 + 2(4N^2 - 21N + 18)n - 2(N - 1)(5N - 12)] \alpha
$$

$$
- (N - 1)(n - 1)[Nn^2 + 2(N - 2)n - 4(N - 1)]
$$

(2.24)

and

$$
(2\alpha + N - 1)t_2^{-}(n, \alpha, N) = (7n + 1)\alpha^5 + 2[7n^2 + (7N - 8)n + 2N + 6] \alpha^4
$$

$$
+ [7n^3 + 28(N - 1)n^2 + (9N^2 - 19N + 48)n + 3N^2 - 19N - 49] \alpha^3
$$

$$
+ [(14N - 11)n^3 + 18(N^2 - 2N + 2)n^2 + (2N^3 - 7N^2 + N - 97)n - 19N^2 + 7N + 52] \alpha^2
$$

$$
+ [(N - 1)(9N - 4)n^3 + 2(2N - 7)(N^2 + 3)n^2 - (2N^3 + 15N^2 + 13N - 70)n - 2(N - 1)(N^2 - 5N - 8)] \alpha
$$

$$
+ 2(N - 1)n(n - 1)[N(N - 1)n - 4(N + 2)].
$$

(2.25)

Hence, the most general quasi-solvable operator $\tilde{H}^-$ of the form (2.11) which preserves the space $\tilde{V}_N^-(N > 2)$ is given by

$$
\tilde{H}^- = - \sum_{i=1}^{4} a_i J_i - c_0
$$

$$
= -A(z) \frac{d^2}{dz^2} - \left[ \tilde{B}(z) + \frac{4(\alpha - 1)D(z)}{f(z; \alpha)} \right] \frac{d}{dz} - \tilde{C}(z) + \frac{4(\alpha - 1)D(z)}{f(z; \alpha)},
$$

(2.26)

where $a_i (i = 1, \ldots, 4)$ and $c_0$ are constants, while $A(z)$, $\tilde{B}(z)$, $\tilde{C}(z)$, and $D(z)$ are polynomials in $z$ given by

$$
A(z) = a_4 z^4 + \left[ \frac{3 \alpha^2 + (3N - 2) \alpha + N(N - 1)}{2 \alpha + N - 1} a_4 + a_3 \right] z^3 + [(2\alpha + N - 1)a_3 + a_2] z^2
$$

$$
+ a_1 z + (\alpha - 1)(\alpha + N - 1)a_2,
$$

(2.27)

$$
\tilde{B}(z) = - \frac{B_4^{-}(z; \alpha, N)}{2 \alpha + N - 1} a_4 + [(\alpha - N - 2)a_3 - a_2] z^2
$$

$$
+ \left\{ [3 \alpha^2 + (N - 2) \alpha - 2(N + 1)]a_3 - (N + 1)a_2 - a_1 \right\} z
$$

$$
+ 4(\alpha - 1)(\alpha + N + 1)a_3 - (\alpha - 1)(3 \alpha + 3N - 7)a_2 + (\alpha - 3)a_1,
$$

(2.28)

$$
\tilde{C}(z) = N(N + 1)a_4 z^2 + (N + 1) \left[ \frac{(\alpha - 1) 3 \alpha^2 + 2(3N - 8) \alpha + 2(N - 4)(N - 1)}{2 \alpha + N - 1} a_4
$$

$$
- (\alpha - 2)a_3 + a_2 \right\} z + c_0,
$$

(2.29)

$$
D(z) = \frac{(\alpha - 1)D_4^{-}(z; \alpha, N)}{2 \alpha + N - 1} a_4 - [(\alpha^2 + N \alpha - 2N - 2)a_3 + (2\alpha + N - 3)a_2 - a_1] z
$$

$$
- \alpha(\alpha - 1)(\alpha + N + 1)a_3 - (\alpha - 1)(2\alpha + N - 1)a_2 + \alpha a_1.
$$

(2.30)

The operator $\tilde{H}$ becomes solvable only when

$$
a_4 = a_3 = a_2 = 0.
$$

(2.31)
III. CONSTRUCTION OF $\mathcal{N}$-FOLD SUPERSYMMETRIC SYSTEMS

Now that we have constructed the set of quasi-solvable operators which leave the space $\mathcal{V}_N$ invariant, we are in a position to construct $\mathcal{N}$-fold supersymmetric systems associated with the latter space by applying the systematic algorithm developed in Ref. [17]. The first step is to construct an $\mathcal{N}$th-order linear differential operator $\tilde{P}_N^-$ which annihilate the space $\mathcal{V}_N$ and which has the following form (cf., Eq. (2.29) in Ref. [17]):

$$\tilde{P}_N^- = z'(q)^N \left( \frac{d^N}{dq^N} + \sum_{k=0}^{N-1} w_k^{[\mathcal{N}]}(z) \frac{d^k}{dz^k} \right).$$ (3.1)

where $z(q)$ denotes the change of variable connecting the variable $z$ with a physical coordinate $q$, and $z'(q)$ is the first derivative of $z(q)$ with respect to $q$. To construct the operator $\tilde{P}_N^-$, we first note that the following first-order linear differential operator $\tilde{A}^-(\alpha)$ plays a role of lowering operator for all the polynomials $\tilde{\varphi}_n(z;\alpha)$,

$$\tilde{A}^-(\alpha) = \frac{f(z;\alpha+1)}{f(z;\alpha)} \left( \frac{d}{dz} - \frac{f'(z;\alpha+1)}{f(z;\alpha+1)} \right),$$ (3.2)

where $f(z;\alpha)$ is the polynomial introduced in (2.10) and the prime denotes derivative with respect to $z$. In fact, the action of $\tilde{A}^-(\alpha)$ on $\tilde{\varphi}_n(z;\alpha)$ for an arbitrary $n = 1, 2, 3, \ldots$ reads as

$$\tilde{A}^-(\alpha)\tilde{\varphi}_n(z;\alpha) = (n-1)\tilde{\varphi}_{n-1}(z;\alpha+1).$$ (3.3)

Using the latter formula, we easily prove the following formula by induction:

$$\left( \prod_{k=0}^{m-1} \tilde{A}^-(\alpha+k) \right) \tilde{\varphi}_n(z;\alpha) = \frac{\Gamma(n)}{\Gamma(n-m)} \tilde{\varphi}_{n-m}(z;\alpha+m),$$ (3.4)

where $\Gamma$ denotes the gamma function and the product of operators is defined by

$$\prod_{k=k_0}^{k_1} A_k \equiv A_{k_1}A_{k_1-1} \ldots A_{k_0}.$$ (3.5)

Then, we immediately know that the properly normalized operator $\tilde{P}_N^-$ of the form (3.1) whose kernel is the polynomial space $\tilde{V}_N$ is given by

$$\tilde{P}_N^- = z'(q)^N \frac{f(z;\alpha)}{f(z;\alpha+N)} \prod_{k=0}^{N-1} \tilde{A}^-(\alpha+k)$$

$$= z'(q)^N \frac{f(z;\alpha)}{f(z;\alpha+N)} \prod_{k=0}^{N-1} \frac{f(z;\alpha+k+1)}{f(z;\alpha+k)} \left( \frac{d}{dz} - \frac{f'(z;\alpha+k+1)}{f(z;\alpha+k+1)} \right).$$ (3.6)

The next task is to calculate the coefficient $\tilde{w}^{[\mathcal{N}]}_{N-1}(z)$ of the $(\mathcal{N}-1)$th-order differential operator $\partial_z^{\mathcal{N}-1}$ in $\tilde{P}_N^-$ defined by (3.1). For the latter purpose, we first derive a formula for the calculation of $\tilde{w}^{[\mathcal{N}]}_{N-1}(z)$ in a general setting,

$$\prod_{k=0}^{N-1} f_k(z) \left( \frac{d}{dz} + g_k(z) \right) = \left( \prod_{k=0}^{N-1} f_k(z) \right) \left( \frac{d^N}{dz^N} + \tilde{w}^{[\mathcal{N}]}_{N-1}(z) \frac{d^{N-1}}{dz^{N-1}} + \ldots \right).$$ (3.7)
It is an easy task to show inductively with respect to $\mathcal{N}$ the following recursion relation:

$$\tilde{w}_{\mathcal{N}}^{[\mathcal{N}+1]}(z) = \tilde{w}_{\mathcal{N}-1}^{[\mathcal{N}]}(z) + g_{\mathcal{N}}(z) + \sum_{k=0}^{\mathcal{N}-1} \frac{f'_k(z)}{f_k(z)}, \quad \tilde{w}_{0}^{[1]}(z) = g_0(z).$$  \hspace{1cm} (3.8)

Its general solution is given by

$$\tilde{w}_{\mathcal{N}-1}^{[\mathcal{N}]}(z) = \sum_{k=0}^{\mathcal{N}-1} \left[ g_k(z) + (\mathcal{N} - 1 - k) \frac{f'_k(z)}{f_k(z)} \right].$$  \hspace{1cm} (3.9)

In our present case, we read from (3.2) and (3.6) that the functions $f_k(z)$ and $g_k(z)$ are given by

$$f_k(z) = \frac{f(z; \alpha + k + 1)}{f(z; \alpha + k)}, \quad g_k(z) = -\frac{f'(z; \alpha + k + 1)}{f(z; \alpha + k)}.$$  \hspace{1cm} (3.10)

Then, for the latter $f_k(z)$ and $g_k(z)$, we have

$$\sum_{k=0}^{\mathcal{N}-1} \frac{f'_k(z)}{f_k(z)} = \frac{d}{dz} \ln \left( \prod_{k=0}^{\mathcal{N}-1} \frac{f(z; \alpha + k + 1)}{f(z; \alpha + k)} \right) = \frac{f'(z; \alpha + \mathcal{N})}{f(z; \alpha + \mathcal{N})} - \frac{f'(z; \alpha)}{f(z; \alpha)}.$$  \hspace{1cm} (3.11)

and

$$\sum_{k=0}^{\mathcal{N}-1} k \frac{f'_k(z)}{f_k(z)} = \frac{d}{dz} \ln \left( \prod_{k=0}^{\mathcal{N}-1} \frac{f(z; \alpha + k + 1)}{f(z; \alpha + k)} \right)^k = (\mathcal{N} - 1) \frac{f'(z; \alpha + \mathcal{N})}{f(z; \alpha + \mathcal{N})} - \sum_{k=1}^{\mathcal{N}-1} \frac{f'(z; \alpha + k)}{f(z; \alpha + k)}.$$  \hspace{1cm} (3.12)

Substituting (3.10)–(3.12) into (3.9), we obtain

$$\tilde{w}_{\mathcal{N}-1}^{[\mathcal{N}]}(z) = -(\mathcal{N} - 1) \frac{f'(z; \alpha)}{f(z; \alpha)} - \frac{f'(z; \alpha + \mathcal{N})}{f(z; \alpha + \mathcal{N})}.$$  \hspace{1cm} (3.13)

With the obtained function $\tilde{w}_{\mathcal{N}-1}^{[\mathcal{N}]}(z)$, the $\mathcal{N}$-fold SUSY pair of gauged Hamiltonians $\tilde{H}^-$ and $\tilde{H}^+$ are expressed as (cf., Eq. (2.45) in Ref. [17])

$$\tilde{H}^\pm = -A(z) \frac{d^2}{dz^2} + \left( \frac{\mathcal{N} - 2}{2} A'(z) \pm Q(z) \right) \frac{d}{dz} - C(z)$$

$$- (1 \pm 1) \left( \frac{\mathcal{N} - 1}{2} Q'(z) - \frac{A'(z) \tilde{w}_{\mathcal{N}-1}^{[\mathcal{N}]}(z)}{2} - A(z) \tilde{w}_{\mathcal{N}-1}^{[\mathcal{N}']} \right),$$  \hspace{1cm} (3.14)

where $Q(z)$ and $C(z)$ in the present case read as

$$Q(z) = \frac{\mathcal{N} - 2}{2} A'(z) + \tilde{B}(z) + \frac{4(\alpha - 1)D(z)}{f(z; \alpha)},$$  \hspace{1cm} (3.15)

$$C(z) = \tilde{C}(z) - \frac{4(\alpha - 1)D(z)}{f(z; \alpha)}.$$  \hspace{1cm} (3.16)

9
It is worth studying another linear space $\mathcal{V}^+_\mathcal{N}$ preserved by the partner gauged Hamiltonian $H^+$. The latter space is characterized by the kernel of another $\mathcal{N}$th-order linear differential operator $P^+_{\mathcal{N}}$, namely, through the relation $\ker P^+_{\mathcal{N}} = \mathcal{V}^+_\mathcal{N}$. The operator $P^+_{\mathcal{N}}$ is obtained from $P^+_{\mathcal{N}}$ in (3.1) by (cf., Eqs. (2.30)–(2.32) in Ref. [17])

$$
P^+_{\mathcal{N}} = (-1)^{N} q^{N-1} (P^+_{\mathcal{N}})^T q^{N-1}$$

where the subscript T stands for the transposition in the physical $q$-space. In the present case where $P^+_{\mathcal{N}}$ is given by (3.6), it reads as

$$
P^+_{\mathcal{N}} = q^{N} \sum_{k=0}^{N-1} (-1)^{N-k} \frac{d^k}{dz^{N-k}} q^{N-k} (z),$$

where the subscript $T$ stands for the transposition in the physical $q$-space. The latter space is characterized by the kernel of another $\mathcal{N}$th-order linear differential operator $P^+_{\mathcal{N}}$, namely, through the relation $\ker P^+_{\mathcal{N}} = \mathcal{V}^+_\mathcal{N}$. The operator $P^+_{\mathcal{N}}$ is obtained from $P^+_{\mathcal{N}}$ in (3.1) by (cf., Eqs. (2.30)–(2.32) in Ref. [17])

$$
P^+_{\mathcal{N}} = z'(q)^N \sum_{k=0}^{N-1} (-1)^{N-k} \frac{d^k}{dz^{N-k}} q^{N-k} (z),$$

(3.17)

where the subscript $T$ stands for the transposition in the physical $q$-space. In the present case where $P^+_{\mathcal{N}}$ is given by (3.6), it reads as

$$
P^+_{\mathcal{N}} = z'(q)^N \prod_{k=0}^{N-1} \left( \frac{d}{dz} + \frac{f'(z; \alpha + \mathcal{N} - k)}{f(z; \alpha + \mathcal{N})} \right) \frac{f(z; \alpha + \mathcal{N} - k)}{f(z; \alpha + \mathcal{N})}$$

$$
= z'(q)^N \prod_{k=0}^{N-1} \left( \frac{d}{dz} + \frac{f'(z; \alpha + \mathcal{N} - k)}{f(z; \alpha + \mathcal{N})} \right) \frac{f(z; \alpha + \mathcal{N})}{f(z; \alpha + \mathcal{N} - 1)},$$

(3.18)

where $\tilde{A}^+(\alpha)$ is a first-order linear differential operator defined by

$$
\tilde{A}^+(\alpha) = \frac{f(z; \alpha + \mathcal{N})}{f(z; \alpha + \mathcal{N} - 1)} \left( \frac{d}{dz} + \frac{f'(z; \alpha + \mathcal{N})}{f(z; \alpha + \mathcal{N})} \right).$$

(3.19)

The vector space $\mathcal{V}^+_\mathcal{N}$ annihilated by $P^+_{\mathcal{N}}$ is obtained by integrating inductively the differential equation $P^+_{\mathcal{N}}(z) = 0$. The result is

$$
f(z; \alpha)f(z; \alpha + \mathcal{N})\mathcal{V}^+_\mathcal{N} = \mathcal{V}^+_\mathcal{N} = \langle \tilde{\chi}_1(z; \alpha + \mathcal{N}), \ldots, \tilde{\chi}_\mathcal{N}(z; \alpha + \mathcal{N}) \rangle,$$

(3.20)

where each $\tilde{\chi}_\mathcal{N}(z; \alpha)$ is a polynomial of degree $\mathcal{N} + 1$ defined by

$$
\tilde{\chi}_\mathcal{N}(z; \alpha) = (\alpha - \mathcal{N}) (\alpha - \mathcal{N} + 1) z^{\mathcal{N}+1} + 2(\alpha - \mathcal{N} - 1) (\alpha - \mathcal{N} + 1) (\alpha - 1) z^\mathcal{N} + (\alpha - \mathcal{N} - 1) (\alpha - 1) \alpha z^{\mathcal{N}-1}.
$$

(3.21)

We find that the obtained linear space $\mathcal{V}^+_\mathcal{N}$ in (3.20) is such a space on which the operator $\tilde{A}^+(\alpha)$ introduced in (3.19) acts essentially as a lowering operator. Indeed, we easily derive the following formula:

$$
\tilde{A}^+(\alpha) \frac{\tilde{\chi}_\mathcal{N}(z; \alpha)}{f(z; \alpha - 1) f(z; \alpha)} = (\alpha - \mathcal{N} + 1) \frac{\tilde{\chi}_{\mathcal{N}-1}(z; \alpha - 1)}{f(z; \alpha - 2) f(z; \alpha - 1)}.
$$

(3.22)

With repeated applications of the latter formula, we obtain

$$
\left( \prod_{k=0}^{m-1} \tilde{A}^+(\alpha + \mathcal{N} - k) \right) \frac{\tilde{\chi}_\mathcal{N}(z; \alpha + \mathcal{N})}{f(z; \alpha + \mathcal{N} - 1) f(z; \alpha + \mathcal{N})} = \frac{\Gamma(\mathcal{N})}{\Gamma(\mathcal{N} - m)} \frac{\tilde{\chi}_{\mathcal{N}-m}(z; \alpha + \mathcal{N} - m)}{f(z; \alpha + \mathcal{N} - m - 1) f(z; \alpha + \mathcal{N} - m)}. \quad (3.23)
$$
Then, combining (3.18), (3.20), and (3.23), we eventually have

\[
P_N^+ \hat{V}_N^+ \propto \left( \prod_{k=0}^{N-1} A^+(\alpha + N - k) \right) \frac{\langle \tilde{\chi}_1(z; \alpha + N), \ldots, \tilde{\chi}_N(z; \alpha + N) \rangle}{f(z; \alpha + N - 1)f(z; \alpha + N)} = 0. \tag{3.24}
\]

As the gauged Hamiltonian \( \tilde{H}^+ \) preserves the vector space \( \tilde{V}_N^+ \) defined by the relation in (3.20), namely, \( \hat{H}^+ \tilde{V}_N^+ \subset \tilde{V}_N^+ \), it is evident that the linear operator

\[
\hat{H}^+ = f(z; \alpha)f(z; \alpha + N)\tilde{H}^+f(z; \alpha + N)^{-1}f(z; \alpha)^{-1}, \tag{3.25}
\]

preserves the polynomial subspace \( \tilde{V}_N^{(X_{2\theta})}[z; \alpha + N] \subset \tilde{V}_N^{(A)}[z] \), namely, \( \hat{H}^+ \tilde{V}_N^{(X_{2\theta})}[z; \alpha + N] \subset \tilde{V}_N^{(X_{2\theta})}[z; \alpha + N] \). On the other hand, it immediately follows from the form of \( \hat{H}^+ \) given by (3.14)–(3.16) that the operator \( \tilde{H}^+ \) does not preserve the monomial space \( \tilde{V}_N^{(A)} \). In other words, \( \mathcal{D}_2(\tilde{V}_N^{(X_{2\theta})}) \not\subset \mathcal{D}_2(\tilde{V}_N^{(A)}) \). Hence, the linear space \( \tilde{V}_N^{(X_{2\theta})} \) spanned by the polynomials \( \tilde{\chi}_n \) in (3.21) provides another exceptional polynomial subspace of codimension 2. From (3.14) and (3.25), the form of the operator \( \tilde{H}^+ \) reads as

\[
\tilde{H}^+ = -A(z) \frac{d^2}{dz^2} - B^+(z) \frac{d}{dz} - C^+(z), \tag{3.26}
\]

where \( B^+(z) \) and \( C^+(z) \) are, respectively, given by

\[
B^+(z) = -(\mathcal{N} - 2)\mathcal{A}'(z) - \tilde{B}(z) - 2A(z)f'(z; \alpha) + 2(\alpha - 1)D(z)\frac{f(z; \alpha)}{f(z; \alpha + N)}
\]

\[
- 2A(z) \frac{f'(z; \alpha + N)}{f(z; \alpha + N)}, \tag{3.27}
\]

and

\[
C^+(z) = (\mathcal{N} - 1) \left[ \frac{\mathcal{N} - 2}{2} A''(z) + \tilde{B}'(z) \right] + \tilde{C}(z) + \frac{1}{f(z; \alpha)} \left\{ 2(\mathcal{N} - 3)A(z) + [2(\mathcal{N} - 3)A'(z) + \tilde{B}(z)]f'(z; \alpha) - 4(\alpha - 1)[D(z) - (\mathcal{N} - 1)D'(z)] \right\}
\]

\[
+ \frac{2A(z) + [(\mathcal{N} - 1)A'(z) + \tilde{B}(z)]f'(z; \alpha + N)}{f(z; \alpha + N)} - 2A(z) \frac{f'(z; \alpha) + 2(\alpha - 1)D(z)}{f(z; \alpha)} \frac{f'(z; \alpha + N)}{f(z; \alpha + N)} \] \tag{3.28}

In the above, the functions \( A(z), \tilde{B}(z), \tilde{C}(z), \) and \( D(z) \) are given by (2.27)–(2.30). Substituting them into the expression (3.27), we obtain

\[
B^+(z) = \tilde{B}^+(z; \alpha + \mathcal{N}, \mathcal{N}) + \frac{4(\alpha + \mathcal{N} - 1)D_1^+(z; \alpha + \mathcal{N}, \mathcal{N})}{f(z; \alpha + \mathcal{N})}, \tag{3.29}
\]
where

\[
\tilde{B}^+(z; \alpha + \mathcal{N}, \mathcal{N}) = -\frac{B_4^+(z; \alpha + \mathcal{N}, \mathcal{N})}{2\alpha + \mathcal{N} - 1} a_4 - \left[(\alpha + 2\mathcal{N})a_3 - a_2 \right]z^2
- \left\{[3\alpha^2 + (5\mathcal{N} - 2)\alpha + 2(\mathcal{N} - 1)^2]a_3 + (\mathcal{N} + 3)a_2 - a_1 \right\}z
+ 4(\alpha + \mathcal{N} - 1)(\alpha + 1)a_3 + (\alpha + \mathcal{N} - 1)(3\alpha + 1)a_2 - (\alpha + \mathcal{N} + 3)a_1,
\]

\[
D_1^+(z; \alpha + \mathcal{N}, \mathcal{N}) = \frac{\alpha + \mathcal{N} - 1}{2\alpha + \mathcal{N} - 1} D_{14}^+(z; \alpha + \mathcal{N}, \mathcal{N}) a_4 - \left[(\alpha^2 + \mathcal{N}\alpha + 2\mathcal{N} - 2)a_3
+ (2\alpha + \mathcal{N} - 3)a_2 - a_1 \right]z - (\alpha + \mathcal{N})(\alpha + \mathcal{N} - 1)(\alpha + 1)a_3
- (\alpha + \mathcal{N} - 1)(2\alpha + \mathcal{N} - 1)a_2 + (\alpha + \mathcal{N})a_1,
\]

Similarly, substituting (2.27)–(2.30) into expression (3.28), we obtain

\[
C^+(z) = \tilde{C}^+(z; \alpha + \mathcal{N}, \mathcal{N}) + \frac{4D_2^+(z; \alpha + \mathcal{N}, \mathcal{N})}{f(z; \alpha + \mathcal{N})},
\]

with

\[
\tilde{C}^+(z; \alpha + \mathcal{N}, \mathcal{N}) = \mathcal{N}(\mathcal{N} + 1)a_4 z^2
- (\mathcal{N} + 1) \left[3\alpha^3 + (\mathcal{N}^2 - 3)\alpha^2 - (\mathcal{N}^2 + 4\mathcal{N} - 4)\alpha - \mathcal{N}^2(\mathcal{N} - 1) \right] a_4
- (\alpha + \mathcal{N})a_3 + a_2 \right]z + c_0^+,
\]

\[
D_2^+(z; \alpha + \mathcal{N}, \mathcal{N}) = \frac{\alpha + \mathcal{N} + 1}{2\alpha + \mathcal{N} - 1} D_{24}^+(z; \alpha + \mathcal{N}, \mathcal{N}) a_4
- \left[(\alpha + \mathcal{N})(\alpha + \mathcal{N} - 1)\right]a_3 + \alpha + \mathcal{N} - 1)(2\alpha + \mathcal{N} - 1)a_2 - (\alpha + \mathcal{N})a_1 \right]z
- (\alpha + \mathcal{N} - 1) \left\{\alpha(\alpha + \mathcal{N})^2 a_3 + [2\alpha^2 + 3(\mathcal{N} - 1)\alpha + \mathcal{N}^2 - \mathcal{N} + 2]a_2
- (\alpha + \mathcal{N})a_1 \right\},
\]

where \(c_0^+\) is an irrelevant constant which depends on \(\alpha\) and \(\mathcal{N}\), and \(D_{24}^+(z; \alpha + \mathcal{N}, \mathcal{N})\) is given by

\[
D_{24}^+(z; \alpha + \mathcal{N}, \mathcal{N}) = [\alpha^3 + (\mathcal{N} + 3)\alpha^2 + (8\mathcal{N} - 9)\alpha + (\mathcal{N} - 1)(3\mathcal{N} - 4)]z
+ (\alpha + \mathcal{N})(\alpha + \mathcal{N} - 1)[\alpha^2 + 3\alpha + 2(\mathcal{N} - 1)].
\]
From expressions (3.26)–(3.27), it is evident that the operator \( \hat{H}^+ \) consists of four linearly independent operators, each of which preserves the linear space \( \tilde{V}_N^{(X_{2b})}[z; \alpha + \mathcal{N}] \). Setting all but one of the parameters \( a_i \) \( (i = 1, \ldots, 4) \) to be 0 and replacing \( \alpha + \mathcal{N} \) by \( \alpha \), we can extract the four second-order linear differential operators which preserve the second exceptional polynomial subspace \( \tilde{V}_N^{(X_{2b})}[z; \alpha] \). The first operator \( K_1 \) associated with \( a_1 \) is given by

\[
K_1 = z\partial_z^2 + (z - \alpha - 3)\partial_z + \frac{4}{f(z; \alpha)}[(\alpha - 1)(z + \alpha)\partial_z + \alpha(z + \alpha - 1)]. \tag{3.38}
\]

The action of \( K_1 \) on \( \tilde{\chi}_n(z; \alpha) \) \( (n = 1, 2, 3, \ldots) \) reads as

\[
K_1\tilde{\chi}_n(z; \alpha) = (n + 1)\tilde{\chi}_n(z; \alpha) - (n - 1)(\alpha - n - 1)\tilde{\chi}_{n-1}(z; \alpha). \tag{3.39}
\]

Hence, it preserves not only \( \tilde{V}_N^{(X_{2b})}[z; \alpha] \) for a specific value of \( \mathcal{N} \) but also an infinite flag of the spaces

\[
\tilde{V}_1^{(X_{2b})}[z; \alpha] \subset \tilde{V}_2^{(X_{2b})}[z; \alpha] \subset \cdots \subset \tilde{V}_N^{(X_{2b})}[z; \alpha] \subset \cdots. \tag{3.40}
\]

Therefore, the operator \( K_1 \) is not only quasi-solvable but also solvable. The second operator \( K_2 \) associated with the parameter \( a_2 \) is given by

\[
K_2 = [z^2 + (\alpha - 1)(\alpha - \mathcal{N} - 1)]\partial_z^2 + [z^2 - (\mathcal{N} + 3)z + (\alpha - 1)(3\alpha - 3\mathcal{N} + 1)]\partial_z
- (\mathcal{N} + 1)z - \frac{4(\alpha - 1)}{f(z; \alpha)}\{(2\alpha - \mathcal{N} - 3)z + (\alpha - 1)(2\alpha - \mathcal{N} - 1)]\partial_z
+ (2\alpha - \mathcal{N} - 1)z + 2\alpha^2 - (\mathcal{N} + 3)\alpha + 2(\mathcal{N} + 1)\}. \tag{3.41}
\]

The action of \( K_2 \) on \( \tilde{\chi}_n(z; \alpha) \) \( (n = 1, 2, 3, \ldots) \) reads as

\[
(\alpha - n + 2)(\alpha - n + 1)(\alpha - n)(\alpha - n - 1)K_2\tilde{\chi}_n(z; \alpha) = (n - \mathcal{N})(\alpha - n + 2)
\times (\alpha - n + 1)^2(\alpha - n)\tilde{\chi}_{n+1}(z; \alpha) - (\alpha - n + 2)(\alpha - n + 1)s^+(n, \alpha, \mathcal{N})\tilde{\chi}_n(z; \alpha)
+ (n - 1)(\alpha - 1)(\alpha - \mathcal{N} - 1)(\alpha - n - 1)^2[3\alpha^2 - 6(n - 1)\alpha + 3n^2 - 6n + 4]\tilde{\chi}_{n-1}(z; \alpha)
+ (n - 1)(n - 2)(\alpha - 1)(\alpha - \mathcal{N} - 1)(\alpha - n)^2(\alpha - n - 1)^2\tilde{\chi}_{n-2}(z; \alpha), \tag{3.42}
\]

where \( s^+(n, \alpha, \mathcal{N}) \) is given by

\[
s^+(n, \alpha, \mathcal{N}) = 2\alpha^3 - [n^2 - (\mathcal{N} - 2)n - \mathcal{N} + 3]\alpha^2
+ [2n^3 - (2\mathcal{N} + 1)n^2 - (3\mathcal{N} + 2)n + \mathcal{N} + 3]n
- n^4 + (\mathcal{N} + 1)n^3 + (2\mathcal{N} + 3)n^2 + (\mathcal{N} + 1)n - 2(\mathcal{N} + 1). \tag{3.43}
\]

Hence, it certainly satisfies \( K_2\tilde{V}_N^{(X_{2b})}[z; \alpha] \subset \tilde{V}_N^{(X_{2b})}[z; \alpha] \) only for a specific value of \( \mathcal{N} \in \mathbb{N} \). The third operator \( K_3 \) associated with \( a_3 \) is given by

\[
K_3 = (z + 2\alpha - \mathcal{N} - 1)z^2\partial_z^2 - \{(\alpha + \mathcal{N})z^2 + [3\alpha^2 - (\mathcal{N} + 2)\alpha - 2(\mathcal{N} - 1)]z
- 4(\alpha - 1)(\alpha - \mathcal{N} + 1)\}\partial_z + (\mathcal{N} + 1)\alpha
\frac{4(\alpha - 1)}{f(z; \alpha)}\{(\alpha^2 - \mathcal{N}\alpha + 2\mathcal{N} - 2)z + \alpha(\alpha - 1)(\alpha - \mathcal{N} + 1)\partial_z
+ \alpha[(\alpha - \mathcal{N} + 1)z + \alpha(\alpha - \mathcal{N})]\}. \tag{3.44}
\]
The action of $K_3$ on $\bar{\chi}_n(z; \alpha)$ admits a relatively simple form as in the case of $J_3$ as follows:

\[
K_3 \bar{\chi}_n(z; \alpha) = - (n - N)(\alpha - n + 1) \bar{\chi}_{n+1}(z; \alpha) - [(3n + 1)\alpha^2 \\
- (2n^2 + Nn + 3N - 2)\alpha + (N + 1)n(n - 1)]\bar{\chi}_n(z; \alpha),
\]

(3.45)

and thus it also satisfies $K_3 \bar{\chi}_n(z; \alpha) \in \bar{\chi}_n(z; \alpha)$ only for a specific value of $N \in \mathbb{N}$. The fourth operator $K_4$ associated with the parameter $a_4$ is given by

\[
(2\alpha - N - 1)K_4 = [(2\alpha - N - 1)z + 3\alpha^2 - (3N + 2)\alpha + N(N + 1)]z^3\partial_z^2 \\
- B_4^+(z; \alpha, N)\partial_z + N(N + 1)(2\alpha - N - 1)z^2 \\
- (N + 1)[3\alpha^3 - 3(2N + 3)\alpha^2 + 2(N^2 + 7N + 2)\alpha - 4N(N + 1)]z \\
+ \frac{4(\alpha - 1)}{f(z; \alpha)} \left[ (\alpha - 1)D_{14}^+(z; \alpha, N)\partial_z + \alpha D_{24}^+(z; \alpha, N) \right],
\]

(3.46)

where the functions $B_4^+(z; \alpha, N)$ and $D_{14}^+(z; \alpha, N)$ ($i = 1, 2$) are introduced in (3.32), (3.33), and (3.37), respectively. The action of $K_4$ on $\bar{\chi}_n(z; \alpha)$ ($n = 1, 2, 3, \ldots$) reads as

\[
(\alpha - n - 2)(\alpha - n)(\alpha - n + 1)(\alpha - n + 1)K_4 \bar{\chi}_n(z; \alpha) = (n - N)(n - N + 1) \\
\times (\alpha - n)^2(\alpha - n + 1)^2\bar{\chi}_{n+2}(z; \alpha) + (n - N)(\alpha - n + 1)^2t_1^+(n, \alpha, N)\bar{\chi}_{n+1}(z; \alpha) \\
+ (\alpha - 1)(\alpha - n - 2)(\alpha - n - 1)t_2^+(n, \alpha, N)\bar{\chi}_n(z; \alpha) + (n - 1)\alpha(\alpha - 1) \\
\times (\alpha - N - 1)(\alpha - N)(\alpha - n + 2)(\alpha - n - 1)^2(\alpha - n)\bar{\chi}_{n-1}(z; \alpha),
\]

(3.47)

and thus it surely preserves the space $\bar{\chi}_n(z; \alpha)$ for a specific value of $N \in \mathbb{N}$. In the above, $t_1^+(n, \alpha, N)$ and $t_2^+(n, \alpha, N)$ are, respectively, given by

\[
(2\alpha - N - 1)t_1^+(n, \alpha, N) = 3\alpha^5 - (3n + 6N + 20)\alpha^4 + [3n^2 - (9N + 26)n \\
- (2N^2 + 27N + 50)]\alpha^3 + [3n^3 - 4n^2 - (3N^2 + 35N + 52)n - 7N^2 - 49N - 52]\alpha^2 \\
- [(3N + 2)n^3 - (7N + 10)n^2 - 2(4N^2 + 21N + 18)n - 2(N + 1)(5N + 12)]\alpha \\
+ (N + 1)[Nn^3 - (N + 4)n^2 - 2(3N + 4)n - 4(N + 1)]
\]

(3.48)

and

\[
(2\alpha - N - 1)t_2^+(n, \alpha, N) = (7n + 1)\alpha^5 - 2[7n^2 + (7N + 3)n + 2N - 11]\alpha^4 \\
+ [7n^3 + 4(7N + 4)n^2 + (9N^2 + 7N - 42)n + 3N^2 - 21N + 1]\alpha^3 \\
- [(14N + 11)n^3 + 2(9N^2 + 8N - 10)n^2 + (2N^3 + N^2 - 29N - 7)n \\
- (N + 1)(11N - 8)]\alpha^2 + (N + 1)(n - 1)[(9N + 4)n^2 + (4N^2 + 2N - 10)n \\
+ 2N(N + 1)]\alpha - 2N(N + 1)^2n^2(n - 1).
\]

(3.49)

It is evident from the resulting actions of $K_i$ ($i = 1, \ldots, 4$) on $\bar{\chi}_n(z; \alpha)$ ($n = 1, 2, 3, \ldots$) that the gauged Hamiltonian $\bar{H}^+$, and thus $\bar{H}^+$ as well, are not only quasi-solvable but also solvable if and only if the condition (2.31), which is the solvability condition for $\bar{H}^-$, is satisfied. That is, $\bar{H}^-, \bar{H}^+$, and $\bar{H}^+$ get solvable only simultaneously.

The fact that the four operators $K_i$ ($i = 1, \ldots, 4$) leave the polynomial subspace $\bar{\chi}(X; \alpha)$ invariant in spite of the existence of the fractional coefficients $1/f(z; \alpha)$ is partially explained by factorization properties of the polynomials $\bar{\chi}_n(z; \alpha)$ for all $n = 1, 2, 3, \ldots$.
with the common factor $f(z; \alpha)$ under the actions of two first-order linear differential operators $O_1^+$ and $O_2^+$, which are analogous to (2.9),

$$O_1^+ \bar{\chi}_n(z; \alpha) = -\left(\alpha - n - 1\right)\left(\alpha - n\right)\left(\alpha - n + 1\right)z^{n-1}f(z; \alpha)$$

and

$$O_2^+ \bar{\chi}_n(z; \alpha) = \left[(\alpha - n)(\alpha - n + 1)z^2 + 2(\alpha - n - 1)(\alpha - n + 1)(\alpha - 1)zight.
\left.+(n - 1)(\alpha - n - 1)(\alpha - n)(\alpha - 1)\right]z^{n-2}f(z; \alpha),$$

where $O_1^+$ and $O_2^+$ are given by

$$O_1^+ = z\partial_z - \alpha, \quad O_2^+ = (\alpha - 1)\partial_z + z + 2\alpha - 2. \quad (3.52)$$

It is easy to check that all the fractional parts having the factor $1/f(z; \alpha)$ in $K_i$ ($i = 1, \ldots, 4$) are expressed as linear combinations of $O_1^+$ and $O_2^+$ and thus they map all the polynomials $\bar{\chi}_n(z; \alpha)$ to other polynomials in $z$. The latter fact is inevitable for the operators $K_i$ ($i = 1, \ldots, 4$) to preserve the polynomial subspace $\tilde{\mathcal{V}}^{(X, q)}_d[z; \alpha]$.

Finally, the $\mathcal{N}$-fold SUSY pair of Hamiltonians $H^\pm$ in the physical space is obtained by gauge transformations

$$H^\pm = e^{-W^\pm_{\mathcal{N}} z} H^\pm e^{W^\pm_{\mathcal{N}} z} \bigg|_{z = z(q)},$$

where the gauge potentials $W^\pm_{\mathcal{N}}$ are given by

$$W^\pm_{\mathcal{N}}(q) = \frac{\mathcal{N} - 1}{4} \ln |2A(z)| \pm \int dz \frac{Q(z)}{2A(z)} \bigg|_{z = z(q)}, \quad (3.54)$$

and the change of variable $z(q)$ is determined by

$$z'(q)^2 = 2A(z(q)). \quad (3.55)$$

If we introduce two functions $E(q)$ and $W(q)$ in the physical space as

$$E(q) = \frac{z''(q)}{z'(q)}, \quad W(q) = -\frac{Q(z(q))}{z'(q)}, \quad (3.56)$$

following the other types of $\mathcal{N}$-fold SUSY [17, 22, 24], the $\mathcal{N}$-fold SUSY Hamiltonians $H^\pm$ are expressed as

$$H^\pm = -\frac{1}{2} \frac{d^2}{dq^2} + \frac{1}{2} W(q)^2 - \frac{\mathcal{N} - 1}{4} \left(E'(q) - \frac{\mathcal{N} - 1}{2}E(q)^2 - 2W'(q) - 2E(q)W(q)\right)$$

$$- C(z(q)) \pm \frac{\mathcal{N}}{2} W'(q) + \frac{1 \pm 1}{2} \left(z''(q)w_{\mathcal{N}-1}^{[\mathcal{N}]}(z(q)) + z'(q)^2 \tilde{w}_{\mathcal{N}-1}^{[\mathcal{N}]}(z(q))\right), \quad (3.57)$$

where $C(z)$ and $\tilde{w}_{\mathcal{N}-1}^{[\mathcal{N}]}(z)$ are, respectively, given by (3.16) and (3.13). The pair of gauge potentials $W^\pm_{\mathcal{N}}(q)$ in (3.54) is expressed in terms of $E(q)$ and $W(q)$ as

$$W^\pm_{\mathcal{N}}(q) = \frac{\mathcal{N} - 1}{2} \int dq E(q) \mp \int dq W(q). \quad (3.58)$$
Similarly, the components of $\mathcal{N}$-fold supercharges $P^\pm_N$ in the $q$-space are also obtained by the same gauge transformations,

$$P^\pm_N = e^{-W_N^\pm} e^{W_N^+} \bigg|_{z=z(q)}. \quad (3.59)$$

Making the repeated use of the identity

$$\left( \frac{d}{dz} \pm \frac{f'(z; \alpha)}{f(z; \alpha)} \right) z'(q)^{-k} = z'(q)^{-k-1} \left( \frac{d}{dq} \pm F_\alpha(q) - k E(q) \right), \quad (3.60)$$

where the function $F_\alpha(q)$ is defined by

$$F_\alpha(q) = \frac{f'(z(q); \alpha)}{f(z(q); \alpha)} z'(q), \quad (3.61)$$

we immediately have

$$P^-_N = \frac{f(z(q); \alpha)}{f(z(q); \alpha + \mathcal{N})} \prod_{k=0}^{N-1} \left( \frac{f(z(q); \alpha + k + 1)}{f(z(q); \alpha + k)} \left( \frac{d}{dq} + W(q) - F_{\alpha+k+1}(q) + \frac{\mathcal{N} - 1 - 2k}{2} E(q) \right) \right)$$

and

$$P^+_N = \left[ \prod_{k=0}^{N-1} \left( \frac{d}{dq} - W(q) + F_{\alpha+N-k}(q) + \frac{\mathcal{N} - 1 - 2k}{2} E(q) \right) \right] \frac{f(z(q); \alpha + \mathcal{N} - k)}{f(z(q); \alpha + \mathcal{N} - k - 1)} \frac{f(z(q); \alpha)}{f(z(q); \alpha + \mathcal{N})}. \quad (3.62)$$

It is easy to check that they are connected by the correct relation $P^+_N = (-1)^N (P^-_N)^T$. The algorithm automatically ensures [17] that the obtained pair of Hamiltonians $H^\pm$ in (3.57) and the $\mathcal{N}$-fold supercharges $P^\pm_N$ in (3.62) and (3.63) satisfy the intertwining relations

$$P^-_N H^- = H^+ P^-_N, \quad P^+_N H^+ = H^- P^+_N, \quad (3.64)$$

and thus, in particular, the two Hamiltonians $H^-$ and $H^+$ are almost isospectral.

By the construction it is evident that the $\mathcal{N}$-fold SUSY Hamiltonians $H^\pm$ respectively preserve the vector spaces $V^\pm_N$ defined by

$$V^-_N = \tilde{V}^-_N e^{-W_N^-} \bigg|_{z=z(q)} = \langle \tilde{\varphi}_1(z(q); \alpha), \ldots, \tilde{\varphi}_N(z(q); \alpha) \rangle e^{-W_N^-(q)}, \quad (3.65a)$$

$$V^+_N = \tilde{V}^+_N e^{-W_N^+} \bigg|_{z=z(q)} = \langle \tilde{x}_1(z(q); \alpha + \mathcal{N}), \ldots, \tilde{x}_N(z(q); \alpha + \mathcal{N}) \rangle \frac{f(z(q); \alpha)}{f(z(q); \alpha + \mathcal{N})} e^{-W_N^+(q)}. \quad (3.65b)$$

Hence, if, for instance, both $H^-$ and $H^+$ are Hermitian in a Hilbert space $L^2(S)$ with $S \subset \mathbb{R}$, and $V^-_N$ and/or $V^+_N$ are subspaces of $L^2(S)$, then $H^-$ and/or $H^+$ are not only quasi-solvable but also quasi-exactly solvable on $S$. In the latter cases, the solvable sectors $V^-_N$ and/or $V^+_N$ provide parts of the eigenfunctions of $H^-$ and/or $H^+$ defined in $L^2(S)$. 

\[16\]
| $A(z)$ | Types of $V^\pm(q)$ |
|-------|------------------|
| $a_1 \neq 0, a_2 = a_3 = a_4 = 0$ | Rational |
| $a_4 \neq 0, a_1 = a_2 = a_3 = 0$ | Exponential |
| $a_1 a_2 \neq 0, a_3 = a_4 = 0$ | Trigonometric or Hyperbolic |
| $a_2 \neq 0, a_1 = a_3 = a_4 = 0$ | Elliptic |
| $a_3 \neq 0, a_1 = a_2 = a_4 = 0$ | |
| Other cases | |

TABLE I: The relations between the forms of $A(z)$ and the types of $V^\pm(q)$.

IV. RESULTING $\mathcal{N}$-FOLD SUSY PAIRS OF QUASI-SOLVABLE POTENTIALS

In this section, we shall construct explicitly $\mathcal{N}$-fold SUSY pairs of potentials in the physical $q$-space associated with the two exceptional polynomial subspaces $\tilde{\mathcal{V}}^{(X_{2a})}_\mathcal{N}$ and $\tilde{\mathcal{V}}^{(X_{2b})}_\mathcal{N}$. As in the case of type B $\mathcal{N}$-fold SUSY [22], they have no covariance under the linear fractional transformations $GL(2, \mathbb{C})$. Hence, we can at most consider projective equivalence classes as was done in Ref. [1] for the $X_1$ subspace, or equivalently, the type B monomial space; the forms of potentials are not invariant under all projective transformations in general. In this paper, we shall rather content ourselves with exhibiting a couple of particular examples since the complete presentation of all the cases would involve many complicated formulas.

The functional types of potentials such as rational, exponential, hyperbolic, and so on are determined by the function $A(z)$ through Eq. (3.55). In Table I, we just show the relations between the form of $A(z)$ and the types of the potentials $V^\pm(q)$. In this paper, we shall just consider the two simplest cases where only $a_1$ (Example 1) or $a_2$ (Example 2) among the four parameters introduced in (2.26) is non-zero. In such restricted analyses, we can still reduce the freedom of parameters by considering the scale transformation as was done in Ref. [17] for the classification of type C $\mathcal{N}$-fold SUSY potentials. That is, by (2.27) and (3.55) a rescaling of $a_i$ ($i = 1, \ldots, 4$) and $c_0$ by an overall non-zero constant $\nu$ affects the change of variable $z(q)$ as

$$z(q; \nu a_i, \nu c_0) = z(\sqrt{\nu} q; a_i, c_0).$$

(4.1)

From this and Eqs. (3.54) and (3.56), we easily obtain the scaling relations:

$$E(q; \nu a_i, \nu c_0) = \sqrt{\nu} E(\sqrt{\nu} q; a_i, c_0), \quad F_\alpha(q; \nu a_i, \nu c_0) = \sqrt{\nu} F_\alpha(\sqrt{\nu} q; a_i, c_0),$$

(4.2a)

$$W(q; \nu a_i, \nu c_0) = \sqrt{\nu} W(\sqrt{\nu} q; a_i, c_0), \quad W_N^\pm(q; \nu a_i, \nu c_0) = W_N^\pm(\sqrt{\nu} q; a_i, c_0),$$

(4.2b)

Then, by formula (3.57) the potential terms are scaled as

$$V^\pm(q; \nu a_i, \nu c_0) = \nu V^\pm(\sqrt{\nu} q; a_i, c_0).$$

(4.3)

Hence, we can fix the value of parameters $a_1$ or $a_2$ without any loss of generality. In what follows, we shall exhibit for each case the change of variable $z = z(q)$, the functions $E(q)$, $W(q)$, and $W_N^\pm(q)$ which determine the gauge factors, the pair of potentials $V^\pm(q)$, and the pair of solvable sectors $\mathcal{V}_N^\pm$. They are obtained by the calculations of (3.55), (3.56), (3.54)
or (3.58), (3.57), and (3.65), respectively.

**Example 1.** \( A(z) = 2z \ [a_1 = 2] \).

Change of variable: \( z(q) = q^2 \).

Gauge factors:

\[
E(q) = \frac{1}{q}, \quad W(q) = q - \frac{2\alpha + \mathcal{N} - 8}{2q} - \frac{4(\alpha - 1)(q^2 + \alpha)}{f(q^2; \alpha) q},
\]

\[
\mathcal{W}_{\mathcal{N}}^\pm(q) = \pm \frac{q^2}{2} \pm \frac{2\alpha + \mathcal{N} \pm \mathcal{N}}{2} \ln |q| \mp \ln |f(q^2; \alpha)|.
\]

Potentials:

\[
V^-(q) = \frac{q^2}{2} + \frac{4\alpha^2 - 1}{8q^2} + 4 \left[ \frac{q^2 - \alpha + 1}{f(q^2; \alpha)} - \frac{4(\alpha - 1)q^2}{f(q^2; \alpha)^2} \right] - \alpha + 3 - \alpha_0,
\]

\[
V^+(q) = \frac{q^2}{2} + \frac{4(\alpha + \mathcal{N})^2 - 1}{8q^2} + 4 \left[ \frac{q^2 - \alpha - \mathcal{N} + 1}{f(q^2; \alpha + \mathcal{N})} - \frac{4(\alpha + \mathcal{N} - 1)q^2}{f(q^2; \alpha + \mathcal{N})^2} \right] - \alpha + \mathcal{N} + 3 - \alpha_0.
\]

Solvable sectors:

\[
\mathcal{V}^-_\mathcal{N} = \langle \tilde{\phi}_1(q^2; \alpha), \ldots, \tilde{\phi}_\mathcal{N}(q^2; \alpha) \rangle \frac{q^{\alpha+1/2}e^{-q^2/2}}{f(q^2; \alpha)},
\]

\[
\mathcal{V}^+_\mathcal{N} = \langle \tilde{\chi}_1(q^2; \alpha + \mathcal{N}), \ldots, \tilde{\chi}_\mathcal{N}(q^2; \alpha + \mathcal{N}) \rangle \frac{q^{-\alpha-N+1}e^{q^2/2}}{f(q^2; \alpha + \mathcal{N})}.
\]

Since the discriminant of \( f(z; \alpha) \) is \( 1 - \alpha \), both of the potentials \( V^\pm(q) \) only have a common unique pole at \( q = 0 \) for \( \alpha > 1 \). Hence, the \( \mathcal{N} \)-fold SUSY system is naturally defined on the half line \( q \in \mathbb{S} = (0, \infty) \). On the latter domain \( \mathbb{S} \), it is evident from (4.7) that \( \mathcal{V}^-_\mathcal{N}(\mathbb{S}) \subset L^2(\mathbb{S}) \) and \( \mathcal{V}^+_\mathcal{N}(\mathbb{S}) \not\subset L^2(\mathbb{S}) \). Therefore, it manifests unbroken \( \mathcal{N} \)-fold SUSY of the system. In addition, the solvability condition (2.31) is satisfied in the system, and thus the Hamiltonian \( H^- \) preserves the infinite flag of the subspaces of \( L^2(\mathbb{S}) \),

\[
\mathcal{V}_1(\mathbb{S}) \subset \mathcal{V}_2^-(\mathbb{S}) \subset \cdots \subset \mathcal{V}_\mathcal{N}^-((\mathcal{N} \rightarrow \infty) \subset L^2(\mathbb{S})).
\]

Hence, the Hamiltonian \( H^- \) is not only quasi-solvable but also *exactly solvable* on \( \mathbb{S} \) provided that the infinite flag constitutes a complete set of the Hilbert space \( L^2(\mathbb{S}) \), namely,

\[
\mathcal{V}_\mathcal{N}^-(\mathbb{S}) \rightarrow L^2(\mathbb{S}) \quad (\mathcal{N} \rightarrow \infty).
\]

The fulfilment of the solvability condition (2.31) also guarantees that the other Hamiltonian \( H^+ \) preserves the infinite flag of the spaces \( \mathcal{V}^\pm_\mathcal{N} : (\mathcal{N} = 1, 2, 3, \ldots) \) which are not subspaces of \( L^2(\mathbb{S}) \),

\[
\mathcal{V}_1^+(\mathbb{S}) \subset \mathcal{V}_2^+(\mathbb{S}) \subset \cdots \subset \mathcal{V}_\mathcal{N}^+(\mathbb{S}) \not\subset L^2(\mathbb{S})).
\]

On the other hand, Eqs. (4.6) tell us that the other Hamiltonian \( H^+ \) has the same form as its partner Hamiltonian \( H^- \). Indeed, \( H^+ \) is identical, up to an additive constant, to \( H^- \) with
its parameter $\alpha$ replaced by $\alpha + \mathcal{N}$. Hence, the $\mathcal{N}$-fold SUSY system has shape invariance for all $\mathcal{N} \in \mathbb{N}$. Combining it with the fact that $H^\pm \mathcal{V}^\pm_{\mathcal{N}} \subset \mathcal{V}^\pm_{\mathcal{N}'}$ for all $\mathcal{N} = 1, 2, 3, \ldots$, we come to the conclusion that both $H^-$ and $H^+$ preserve the two different infinite flags (4.8) and (4.10) with suitable parameters. In particular, $H^+$ is also exactly solvable on $S$ if the completeness (4.9) is assured, although its $\mathcal{N}$-fold SUSY sector $\mathcal{V}^+_{\mathcal{N}'}$ does not belong to the Hilbert space $L^2(S)$.

Finally, we note that the potential $2V^-(q)$ coincides, up to the scaling factor $\nu = \omega/2$ determining the scaling relations (4.1)–(4.3), with case II rational radial oscillator potential $V^-(x)$ in Ref. [5], Eq. (2.17), of which the eigenfunctions are expressed in terms of the $X_2$-Laguerre polynomials of the second kind $\tilde{L}^{(a)}_{2,n+2}(z)$.

**Example 2.** $A(z) = (z^2 + \zeta^2)/2$ [$a_2 = 1/2$, $\zeta^2 = (\alpha - 1)(\alpha + \mathcal{N} - 1) > 0$].

Change of variable: $z(q) = \zeta \sinh q$.

Gauge factors:

$$E(q) = \tanh q,$$
$$W(q) = \frac{\zeta}{2} \cosh q + \frac{3}{2} \tanh q + \frac{(\alpha - 1)(\alpha + \mathcal{N} - 3)}{\zeta \cosh q} + \frac{2(\alpha - 1)}{f(\zeta \sinh q; \alpha) \zeta \cosh q},$$
$$W^\pm_{\mathcal{N}}(q) = \mp \frac{\zeta}{2} \sinh q \mp \zeta \tanh q + \frac{\mathcal{N} - 1 \pm 1}{2} \ln |\cosh q| \mp \ln |f(\zeta \sinh q; \alpha)|.$$  

Potentials:

$$V^-(q) = \frac{\zeta^2}{8} \cosh^2 q + \frac{\mathcal{N} - 1}{4} \zeta \sinh q + \frac{4\alpha^2 + 4(\mathcal{N} - 4)\alpha + \mathcal{N}^2 + 16}{8} - c_0$$
$$+ \frac{1}{8 \cosh^2 q} \left[ 4(\mathcal{N} - 1) \zeta \sinh q + 4\alpha^2 + 4(\mathcal{N} - 2)\alpha - \mathcal{N}^2 - 2N + 4 \right]$$
$$- 2(\alpha - 1) \left[ \frac{\zeta \sinh q - \alpha - \mathcal{N} + 3}{f(\zeta \sinh q; \alpha)} - 2(\alpha - 1) \frac{2\zeta \sinh q - \mathcal{N} + 1}{f(\zeta \sinh q; \alpha)} \right],$$
$$V^+(q) = \frac{\zeta^2}{8} \cosh^2 q + \frac{3\mathcal{N} - 1}{4} \zeta \sinh q + \frac{4\alpha^2 + 4(\mathcal{N} - 4)\alpha + \mathcal{N}^2 + 16}{8} - c_0$$
$$- \frac{1}{8 \cosh^2 q} \left[ 4(\mathcal{N} + 1) \zeta \sinh q - 4\alpha^2 - 4(\mathcal{N} - 2)\alpha + \mathcal{N}^2 + 6\mathcal{N} - 4 \right]$$
$$- 2(\alpha + \mathcal{N} - 1) \left[ \frac{\zeta \sinh q - \alpha + 3}{f(\zeta \sinh q; \alpha + \mathcal{N})} - 2(\alpha + \mathcal{N} - 1) \frac{2\zeta \sinh q + \mathcal{N} + 1}{f(\zeta \sinh q; \alpha + \mathcal{N})^2} \right].$$

Solvable sectors:

$$\mathcal{V}^-_{\mathcal{N}} = \langle \hat{\varphi}_1(\zeta \sinh q; \alpha), \ldots, \hat{\varphi}_N(\zeta \sinh q; \alpha) \rangle \frac{e^{-\zeta(\sinh q)/2 - \zeta \tanh q}}{(\cosh q)^{\mathcal{N}/2 - 1} f(\zeta \sinh q; \alpha)},$$
$$\mathcal{V}^+_{\mathcal{N}} = \langle \hat{x}_1(\zeta \sinh q; \alpha + \mathcal{N}), \ldots, \hat{x}_N(\zeta \sinh q; \alpha + \mathcal{N}) \rangle \frac{e^{\zeta(\sinh q)/2 + \zeta \tanh q}}{(\cosh q)^{\mathcal{N}/2} f(\zeta \sinh q; \alpha + \mathcal{N})}.$$
In the above, \( gd \, q = \arctan(\sinh q) \) is the Gudermann function. The Hamiltonians are only quasi-solvable but not solvable since \( a_0 \neq 0 \) and the solvability condition (2.31) is not satisfied. The resulting potentials are of the generalized Pöschl-Teller types.

For \( \alpha > 1 \), both of the potentials \( V^\pm(q) \) have no singularities at any finite \(|q|\) and thus would be naturally defined on the full real axis \( \mathbb{R} \). However, neither \( \mathcal{V}_{\mathcal{N}}(\mathbb{R}) \) nor \( \mathcal{V}_{\mathcal{N}^+}(\mathbb{R}) \) belongs to the Hilbert space \( L^2(\mathbb{R}) \), which means that both \( H^- \) and \( H^+ \) are only quasi-solvable but are not quasi-exactly solvable on \( \mathbb{R} \). Hence, \( \mathcal{N} \)-fold SUSY of the system is dynamically broken in this example.

For \( \zeta^2 < 0 \), that is, for \(-\mathcal{N} + 1 < \alpha < 1\), the change of variable is given by \( z(q) = |\zeta| \cosh q \). The form of the potentials in the latter case is similar to the above system for \( \zeta^2 > 0 \) with \( \zeta, \cosh q, \) and \( \sinh q \) replaced by \(|\zeta|, \sinh q, \) and \( \cosh q \), respectively. However, there exists a pole at \( q = 0 \) in both of the potentials \( \mathcal{V}_{\mathcal{N}^+}(q) \) irrespective of the value of \( \alpha \). In addition, at \( z = 1 - \alpha(>0) \) we have \( f'(1 - \alpha; \alpha) = 0 \) and \( f(1 - \alpha; \alpha) = \alpha - 1 < 0 \), which means that the function \( f(z; \alpha) \) has a positive root for all \( \alpha < 1 \). Hence, a natural domain of the system may be a half line \( S = (0, \infty) \) with \( f(|\zeta| \cosh q_0; \alpha) = 0 \).

V. DISCUSSION AND SUMMARY

In this paper, we have constructed a family of quasi-solvable and \( \mathcal{N} \)-fold SUSY quantum systems where each Hamiltonian preserves an exceptional polynomial subspace of codimension 2. We started with the \( X_2 \) space \( \mathcal{V}_{\mathcal{N}}^{(X_{2a})} [z; \alpha] \) and constructed the four linearly independent second-order differential operators \( J_i \) \((i = 1, \ldots, 4)\) which preserve it. We then constructed the \( \mathcal{N} \)-fold SUSY quantum systems by applying the algorithm developed in Ref. \[17\]. As a by-product, we have automatically obtained the other \( X_2 \) space \( \mathcal{V}_{\mathcal{N}}^{(X_{2b})} [z; \alpha] \) and the four linearly independent second-order differential operators \( K_i \) \((i = 1, \ldots, 4)\) which preserve the latter space. This shows one of the advantageous and powerful aspects of the framework of \( \mathcal{N} \)-fold SUSY. We presented the two particular examples of the \( \mathcal{N} \)-fold SUSY systems. The one is the pair of rational-type potentials which coincide with the rational shape invariant potentials in Ref. \[5\] and thus are not only quasi-solvable but also solvable. In addition, it turned out that they admit two linearly independent analytic local solutions. The other is the pair of hyperbolic-type potentials both of which are only quasi-solvable. Dynamical \( \mathcal{N} \)-fold SUSY breaking would take place in the second example but not in the first example.

The polynomial parts of eigenfunctions of the rational potential \( V^-(q) \) in (4.6a) are, on the one hand, given by the infinite flag of the spaces \( \mathcal{V}_{\mathcal{N}}^{(X_{2a})} \) spanned by the polynomials \( \tilde{\varphi}_n(z; \alpha) \) in (2.2) and are, on the other hand, expressed in terms of the \( X_2 \)-Laguerre polynomials of the second kind \( \bar{L}_{2, \nu+2}^{(a)}(z) \) in Ref. \[5\]. Hence, the former polynomials \( \tilde{\varphi}_n(z; \alpha) \) and the latter \( \bar{L}_{2, \nu+2}^{(a)}(z) \) must be connected by linear transformations. In fact, the first few \( X_2 \)-Laguerre polynomials are expressed by linear combinations of the polynomials \( \tilde{\varphi}_n(z; \alpha) \) as follows:

\[
(\alpha - 1)\bar{L}_{2, 2}^{(a)}(z) = \tilde{\varphi}_1(z; \alpha), \\
\alpha \bar{L}_{2, 3}^{(a)}(z) = -\tilde{\varphi}_2(z; \alpha) + (\alpha + 2)\tilde{\varphi}_1(z; \alpha), \\
2(\alpha + 1)\bar{L}_{2, 4}^{(a)}(z) = \tilde{\varphi}_3(z; \alpha) - 2(\alpha + 3)\tilde{\varphi}_2(z; \alpha) + (\alpha + 2)(\alpha + 3)\tilde{\varphi}_1(z; \alpha).
\]

It also indicates that the polynomial system \( \{\bar{L}_{2, \nu+2}^{(a)}(z)\}_{\nu=0}^{\infty} \) would be obtained by the Gram–Schmidt orthogonalization of the base system \( \{\tilde{\varphi}_n(z; \alpha)\}_{n=1}^{\infty} \) if it constitutes an orthogonal
polynomial system with respect to a certain inner product.

Similarly, the first few $X_2$-Laguerre polynomials of the first kind $\tilde{L}^{(\alpha)}_{1,\nu+2}(z)$ in Ref. [5] are expressed by linear combinations of the polynomials $\bar{\chi}_n(z;\alpha)$ in (3.21) as follows:

$$\alpha(\alpha+1)\tilde{L}^{(\alpha)}_{1,2}(z) = \bar{\chi}_1(-z; -\alpha), \quad (5.4)$$
$$\alpha(\alpha+1)(\alpha+2)\tilde{L}^{(\alpha)}_{1,3}(z) = \bar{\chi}_2(-z; -\alpha) + (\alpha+3)\bar{\chi}_1(-z; -\alpha), \quad (5.5)$$
$$2(\alpha+2)(\alpha+3)\tilde{L}^{(\alpha)}_{1,4}(z) = \bar{\chi}_3(-z; -\alpha) + 2(\alpha+4)\bar{\chi}_2(-z; -\alpha) + (\alpha+3)(\alpha+4)\bar{\chi}_1(-z; -\alpha). \quad (5.6)$$

Hence, we arrive at the following conjecture:

**Conjecture 1** The $X_2$-Laguerre polynomial system of the first kind $\{\tilde{L}^{(\alpha)}_{1,\nu+2}(z)\}^{\infty}_{\nu=0}$ would be obtained by the Gram–Schmidt orthogonalization of the base system $\{\bar{\chi}_n(z;\alpha)\}^{\infty}_{n=1}$ with respect to a certain inner product, while the $X_2$-Laguerre polynomial system of the second kind $\{\tilde{L}^{(\alpha)}_{2,\nu+2}(z)\}^{\infty}_{\nu=0}$ would be obtained by the same orthogonalization scheme of the base system $\{\bar{\phi}_n(z;\alpha)\}^{\infty}_{n=1}$ with respect to another certain inner product.

Another important remaining issue is to establish a systematic algorithm to calculate the characteristic polynomials $P_N$ of the superHamiltonian $H$ in the solvable sectors $V^+_N$ which appear in the anti-commutators of the $N$-fold supercharges $Q^+_N$ [16],

$$\{Q_N, Q^+_N\} = 2^N P_N(H). \quad (5.7)$$

In the cases of type A and C $N$-fold SUSYs, it was shown [17, 24] that they are given by the generalized Bender–Dunne polynomials of critical degrees [26] and thus are systematically calculated via recursion relations. It would be interesting to examine whether a similar approach also works for the present $N$-fold SUSY systems.

One of possible continuations of the present work is to construct $N$-fold SUSY associated with the $X_2$-Jacobi polynomials. In this respect, we note that, in contrast to the $X_2$-Laguerre cases where two different kinds were found, the two sets of the Jacobi-type polynomial systems associated with the two different extended Scarf I potentials were found to be identical with each other [5]. The latter fact together with the present result that the two different kinds of the $X_2$-Laguerre polynomial systems would be connected by $N$-fold SUSY (cf., Eqs. (3.65)) leads to another conjecture:

**Conjecture 2** The pair of solvable sectors of $N$-fold SUSY Hamiltonians $H^\pm$ associated with the $X_2$-Jacobi polynomials $\tilde{P}^{(\alpha,\beta)}_{1,\nu+2}(z)$ are both spanned by a common polynomial system (with possibly different values of parameters) which would generate the system $\{\tilde{P}^{(\alpha,\beta)}_{1,\nu+2}(z)\}^{\infty}_{\nu=0}$ through its Gram–Schmidt orthogonalization.

We note that the classification of $X_2$ subspaces has remained unsolved yet. We expect that the present work, together with investigation on quasi-solvability and $N$-fold SUSY associated with $X_2$-Jacobi polynomials, would provide us crucial clues to it.

Another possible research direction is to explore quasi-solvability and $N$-fold SUSY associated with $X_m$ subspaces for $m > 2$. It would tell us how the shape invariant potentials associated with the $X_m$ ($m > 2$) polynomials in Refs. [5, 6] are realized in the more general framework of $N$-fold SUSY.
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