Bosonic Realizations of $U_q(C_n^{(1)})$

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Abstract

We construct explicitly the quantum symplectic affine algebra $U_q(\hat{sp}_{2n})$ using bosonic fields. The Fock space decomposes into irreducible modules of level $-1/2$, quantizing the Feingold-Frenkel construction for $q = 1$.

1 Introduction

Quantum affine algebras play an important role in understanding the trigonometric solutions of the quantum Yang-Baxter equation [3]. Subsequent work shows that they also serve as symmetry for many statistical models [7]. In most of the theoretical and applicational tasks one needs to construct them explicitly by known infinite dimensional algebras.

The first such explicit bosonic construction was done for all the simply laced (ADE) untwisted types by Frenkel and Jing [4], and subsequently the twisted types (ADE)$^{(r)}$ [8] and type $B_n^{(1)}$ [2] were also completed. The fermionic constructions were furnished in [6]. The $q$-Wakimoto construction was also known [15, 1, 16] afterwards. However, the quantum symplectic affine algebra $U_q(C_n^{(1)})$ was left without an explicit construction except for $n = 1$ [12].

In the classical ($q = 1$) case Kac-Wakimoto [10, 11] have introduced an wider class of irreducible highest weight representations of affine Lie algebras called admissible representations. Unlike integrable highest weight representations these representations admit fractional levels. Feingold and Frenkel [4] constructed level $-1/2$ highest weight admissible representations of the symplectic affine Lie algebra.

In this paper we will construct explicit realizations of the quantum affine algebra $U_q(C_n^{(1)})$ at level $-1/2$. Our realization can be thought as a $q$-analog of Feingold-Frenkel construction, but it appeals to the equivalent form of homogeneous quantum Z-algebra construction considered in [9]. These are achieved in terms of the internal bosonic fields and auxiliary
bosonic fields unlike the classical $\beta\gamma$-system, which is inappropriate for quantization. Due to this new phenomenon we use certain screening operators to show that the Fock space contains four irreducible highest weight modules of level $-1/2$. These are $q$-analog of the admissible representations considered by Feingold-Frenkel in [4], where they used a different method to get the irreducibility.

2 Quantum affine algebras $U_q(C_n^{(1)})$

Let $A = (A_{ij}), i, j \in I = \{0, 1, \cdots, n\}$ be the generalized Cartan matrix of type $C_n^{(1)}$ so that

$$A = 2 \sum_{i=0}^{n} E_{ii} - \sum_{i=0}^{n-1} (E_{i,i+1} + E_{i+1,i}) - E_{10} - E_{n-1,n}, \quad (1)$$

where $E_{ij}$’s are the unit matrices in $\mathbb{Z}^{(n+1)\times(n+1)}$.

Let $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ ($i = 1, \cdots, n$) and $\alpha_n = 2\varepsilon_n$ be the simple roots of the simple Lie algebra $\mathfrak{sp}_{2n}$, and $\lambda_i = \varepsilon_1 + \cdots + \varepsilon_i$ ($i = 1, \cdots, n$) be the fundamental weights. Let $P = \mathbb{Z}\varepsilon_1 + \cdots + \mathbb{Z}\varepsilon_n$ and $Q = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_n$ be the weight and root lattices. We then let $\Lambda_i, i = 0, \cdots, n$ be the fundamental weights for the affine Lie algebra $\widehat{\mathfrak{sp}}_{2n}$, here $\Lambda_i = \lambda_i + \Lambda_0$. The nondegenerate symmetric bilinear form $(\cdot | \cdot)$ on $\mathfrak{h}^*$, the dual Cartan subalgebra of $\widehat{\mathfrak{sp}}_{2n}$, satisfies that

$$(\varepsilon_i | \varepsilon_j) = \frac{1}{2} \delta_{ij}, \quad (\alpha_i | \alpha_j) = d_i A_{ij}, \quad (\delta | \alpha_i) = (\delta | \delta) = 0 \quad \text{for all } i, j \in I, \quad (2)$$

where $(d_0, \cdots, d_n) = (1, 1/2, \cdots, 1/2, 1)$.

Let $q_i = q^{d_i} = q^{\frac{1}{2}(\alpha_i | \alpha_i)}, i \in I$. We now recall Drinfeld’s realization of the quantum affine algebra $U_q(C_n^{(1)})$ [3]. The quantum affine algebra $U_q(C_n^{(1)})$ is the associative algebra with $1$ over $\mathbb{C}(q^{1/2})$ generated by the elements $x_{ik}^\pm, a_{il}, K_i^{\pm1}, \gamma_{k/2}, q^{\pm d}$ ($i = 1, 2, \cdots, n, k \in \mathbb{Z}, l \in \mathbb{Z} \setminus \{0\}$) with the following defining relations :

$$[\gamma_{k/2}, u] = 0 \quad \text{for all } u \in U_q(C_n^{(1)}), \quad (3)$$

$$[a_{ik}, a_{jl}] = \delta_{k+l,0} \frac{[A_{ij}^+]_k \gamma^k - \gamma^{-k}}{k} \delta_{ij} - q_j^{-1}, \quad (4)$$

$$[a_{ik}, K_j^{\pm1}] = [q^{\pm d}, K_j^{\pm1}] = 0, \quad (5)$$

$$q^{d} x_{ik}^+ q^{-d} = q^x x_{ik}^+, \quad q^d a_i q^{-d} = q^l a_i, \quad (6)$$

$$K_i^{\pm1} x_{jk}^{\pm1} = q^{\pm(\alpha_i | \alpha_j)} x_{jk}^{\pm1}, \quad (7)$$

$$[a_{ik}, x_{jl}^\pm] = \pm \frac{[A_{ij}^+]_k}{k} \gamma^{-|k|/2} x_{j,k+l}^\pm, \quad (8)$$

$$(z - q^{\pm(\alpha_i | \alpha_j)} w) X_{i,k}^{\pm}(z) X_{j,l}^{\pm}(w) + (w - q^{\pm(\alpha_i | \alpha_j)} z) X_{j,l}^{\pm}(w) X_{i,k}^{\pm}(z) = 0 \quad (9)$$

$$[X_{i}^+(z), X_{j}^-(w)] = \frac{\delta_{ij}}{(q_i - q^{-1}_i) zw} \left( \psi_z(w^{1/2}) \delta(\frac{w}{z}) - \psi_z(w^{-1/2}) \delta(\frac{w^{-1}}{z}) \right) \quad (10)$$
where \( X_i^\pm(z) = \sum_{n \in \mathbb{Z}} z^{n-1} \psi_{im} \) and \( \varphi_{im} \) (\( m \in \mathbb{Z}_{\geq 0} \)) are defined by
\[
\psi_i(z) = \sum_{m=0}^{\infty} \psi_{im} z^{-m} = K_i \exp \left( (q_i - q_i^{-1}) \sum_{k=1}^{\infty} a_{ik} z^{-k} \right),
\]
\[
\phi_i(z) = \sum_{m=0}^{\infty} \varphi_{im} z^{-m} = K_i^{-1} \exp \left( -(q_i - q_i^{-1}) \sum_{k=1}^{\infty} a_{ik} z^{-k} \right),
\]
\[
\text{Sym}_{k_1, \ldots, k_m} \sum_{r=0}^{m=1-A_{ij}} (-1)^r \left[ \begin{array}{c} m \\ r \end{array} \right] X_i^\pm(z_1) \cdots X_i^\pm(z_r) \\
\times X_j^\pm(w) X_i^\pm(z_{r+1}) \cdots X_i^\pm(z_m) = 0
\] (11)

3 Fock representation of \( U_q(C_n^{(1)}) \)

Let \( a_i(m) \) be the operators satisfying the Heisenberg relations for \( U_q(C_n^{(1)}) \) at \( \gamma = q^{-1/2} \) and \( b_i(m) \) be the independent free bosonic operators:
\[
[a_i(m), a_j(l)] = \delta_{m+l,0} \frac{[mA_{ij}]}{m} q^{-m/2 - l/2}, \quad [b_i(m), b_j(l)] = m \delta_{ij} \delta_{m+l,0},
\]
\[
[a_i(m), b_j(l)] = 0.
\] (12)

We define the module \( \mathcal{F}_{\alpha, \beta} = \mathcal{F}_d^1 \otimes \mathcal{F}_d^2 \) for the Heisenburg algebra by the defining relations
\[
a_i(m)|\alpha, \beta\rangle = 0 \quad (m > 0), \quad b_i(m)|\alpha, \beta\rangle = 0 \quad (m > 0),
\]
\[
a_i(0)|\alpha, \beta\rangle = (\alpha_i|\alpha\rangle|\alpha, \beta\rangle, \quad b_i(0)|\alpha, \beta\rangle = (2\varepsilon_i|\beta\rangle|\alpha, \beta\rangle,
\]
where \( |\alpha, \beta\rangle = |\alpha\rangle \otimes |\beta\rangle \) (\( \alpha \in P + \frac{\mathbb{Z}}{2} \lambda_n, \beta \in P \)) is the vacuum vector. The grading operator \( d \) is defined by
\[
d.|\alpha, \beta\rangle = ((|\alpha\rangle - (|\beta\rangle - \lambda_n)|\alpha, \beta\rangle).
\]

We set the Fock space
\[
\tilde{\mathcal{F}} := \bigoplus_{\alpha \in P + \frac{\mathbb{Z}}{2} \lambda_n, \beta \in P} \mathcal{F}_{\alpha, \beta}.
\]

Let \( e^a \) and \( e^b \) be operators on \( \tilde{\mathcal{F}} \) given by:
\[
e^a|\alpha, \beta\rangle = |\alpha + \alpha_i, \beta\rangle, \quad e^b|\alpha, \beta\rangle = |\alpha, \beta + \varepsilon_i\rangle.
\]

Let : be the usual bosonic normal ordering defined by
\[
: a_i(m)a_j(l) := a_i(m)a_j(l) (m \leq l), \quad a_j(l)a_i(m) (m > l),
\]
\[
: e^a a_i(0) := a_i(0)e^a := e^a a_i(0),
\]
\[
: e^b b_i(0) := b_i(0)e^b := e^b b_i(0).
\]

and similar normal products for the \( b_i(m) \)'s.
Let $\partial = \partial_{q^{1/2}}$ be the $q$-difference operator:

$$\partial_{q^{1/2}}(f(z)) = \frac{f(q^{1/2}z) - f(q^{-1/2}z)}{(q^{1/2} - q^{-1/2})z}.$$ 

We introduce the following operators.

$$Y_{i}^{\pm}(z) = \exp(\pm \sum_{k=1}^{\infty} \frac{a_{i}(k)}{k} q^{\frac{\pm k}{2}} z^{k}) \exp(\mp \sum_{k=1}^{\infty} \frac{a_{i}(k)}{k} q^{\frac{\pm k}{2}} z^{-k}) e^{\pm a_{i} z^{2a_{i}(0)}},$$

$$Z_{i}^{\pm}(z) = \exp(\pm \sum_{k=1}^{\infty} \frac{b_{i}(k)}{k} z^{k}) \exp(\mp \sum_{k=1}^{\infty} \frac{b_{i}(k)}{k} z^{-k}) e^{\pm b_{i} z^{b_{i}(0)}}.$$ 

**Theorem 3.1** The Fock space $\tilde{F}$ is a $U_{q}$-module of level $-\frac{1}{2}$ under the action defined by $\gamma \mapsto q^{-\frac{1}{2}}$, $K_{i} \mapsto q^{a_{i}(0)}$, $a_{im} \mapsto a_{i}(m)$, $q^{d} \mapsto q^{d}$, and

$$X_{i}^{+}(z) \mapsto \partial Z_{i+1}^{+}(z) Y_{i}^{+}(z), \quad i = 1, \ldots, n - 1$$

$$X_{i}^{-}(z) \mapsto Z_{i}^{-}(z) \partial Z_{i+1}^{-}(z) Y_{i}^{-}(z), \quad i = 1, \ldots, n - 1$$

$$X_{n}^{+}(z) \mapsto \left(\frac{1}{q^{2} + q^{-2}} : Z_{n}^{+}(z) \partial_{z}^{2} Z_{n}^{+}(z) : - : \partial_{z} Y_{b_{i}}^{+}(q^{\frac{1}{2}} z) \partial_{z} Y_{b_{i}}^{+}(q^{-\frac{1}{2}} z) : \right) Y_{a_{i}}^{+}(z)$$

$$X_{n}^{-}(z) \mapsto \frac{1}{q^{2} + q^{-2}} : Z_{n}^{-}(q^{\frac{1}{2}} z) Z_{n}^{-}(q^{-\frac{1}{2}} z) : Y_{n}^{-}(z).$$

We now prove the theorem by checking that the defined action satisfy the Drinfeld relations. It is easy to see that the relation (3) is true by the construction. The relation (4) follows from the expression of $Y_{i}^{\pm}(z)$ and the commutativity between $Y_{i}^{\pm}(z)$ and $Z_{j}^{\pm}(z)$. We are left to show the relations (5).

To this end we first list the relations for $Y_{i}^{\pm}(z)$ and $Z_{j}^{\pm}(z)$:

$$Y_{i}^{\pm}(z) Y_{j}^{\mp}(w) =: Y_{i}^{\pm}(z) Y_{j}^{\mp}(w) :$$

$$\times \begin{cases} 
1, & \text{if } (\alpha_{i} | \alpha_{j}) = 0 \\
(z - q^{1/2} w), & \text{if } (\alpha_{i} | \alpha_{j}) = -1/2 \\
((z - q^{1/2} w)(z - w))^{(\alpha_{i} | \alpha_{j})}, & \text{if } (\alpha_{i} | \alpha_{j}) = \pm 1 \\
((z - w)(z - q w)(z - q^{-1} w)(z - q^{2} w))^{-1}, & \text{if } (\alpha_{i} | \alpha_{j}) = 2 
\end{cases}$$

$$Y_{i}^{\pm}(z) Y_{j}^{\mp}(w) =: Y_{i}^{\pm}(z) Y_{j}^{\mp}(w) :$$

$$\times \begin{cases} 
1, & \text{if } (\alpha_{i} | \alpha_{j}) = 0 \\
(z - w)^{-1}, & \text{if } (\alpha_{i} | \alpha_{j}) = -1/2 \\
((z - q^{1/2} w)(z - q^{1/2} w))^{(\alpha_{i} | \alpha_{j})}, & \text{if } (\alpha_{i} | \alpha_{j}) = \pm 1 \\
(z - q^{-1/2} w)(z - q^{3/2} w)(z - q^{3/2} w), & \text{if } (\alpha_{i} | \alpha_{j}) = 2 
\end{cases}$$

$$Z_{i}^{\epsilon}(z) Z_{i}^{\epsilon}(w) =: Z_{i}^{\epsilon}(z) Z_{i}^{\epsilon}(w) : (z - w)^{\epsilon \delta_{ij}}.$$
where the contraction factors such as \((z - w)^{-1}\) are understood as power series in \(w/z\).

For \(\epsilon = \pm = \pm 1\) we define

\[
X^{+}_{i\epsilon}(z) = Z^{+}_{i}(q^{\epsilon}/z)Z^{+}_{i+1}(z)Y^{+}_{i}(z)
\]
\[
X^{-}_{i\epsilon}(z) = Z^{-}_{i}(z)Z^{-}_{i+1}(q^{\epsilon}/z)Y^{-}_{i}(z)
\]
\[
X^{\pm}_{i\epsilon}(z) = : Z^{\pm}_{n}(q^{\epsilon+\epsilon}z)Z^{\pm}_{n}(q^{\epsilon^{-1}+\epsilon}z)Y^{\pm}_{n}(z) : ,
\]
where \(i = 1, \ldots, n - 1\) and we allow \(\epsilon = 0\) in \(X^{\pm}_{n\epsilon}(z)\), and \(q_{\pm} = q^{1/2}, q_{0} = q\). Then we have

\[
X^{\pm}_{i}(z) = \frac{1}{(q^{\epsilon}/2q^{-\epsilon}/2)z} \left( X^{\pm}_{i+1}(z) - X^{\pm}_{i}(z) \right)
\]
\[
X^{\pm}_{n}(z) = \frac{1}{(q^{\epsilon^{-1}}q^{-\epsilon^{-1}})/2)z^{2}} \left( q^{1/2}X^{\pm}_{n+1}(z) + q^{-1/2}X^{\pm}_{n-1}(z) - (q^{1/2} + q^{-1/2})X^{\pm}_{n0}(z) \right)
\]
where \(i = 1, \ldots, n - 1\).

By Wick’s theorem we can easily derive operator product expansions for \(X^{\pm}_{i}(z)\). For instance we have

\[
X^{\pm}_{i\epsilon}(z)X^{\pm}_{i\epsilon}(w) = : X^{\pm}_{i\epsilon}(z)X^{\pm}_{i\epsilon}(w) : (q^{\epsilon}/2z - q^{\epsilon}/2w)(z - q^{\epsilon+1}w)^{-1}
\]
\[
X^{\pm}_{i}(z)X^{-\epsilon}_{i}(w) = : X^{\pm}_{i}(z)X^{-\epsilon}_{i}(w) : (q^{\epsilon}/2z - w)^{-1}(z - q^{-\epsilon}/2w)
\]
\[
X^{-\epsilon}_{i}(z)X^{\pm}_{i}(w) = : X^{-\epsilon}_{i}(z)X^{\pm}_{i}(w) : (z - q^{-\epsilon}/2w)(q^{\epsilon}/2z - w)^{-1}
\]
\[
X^{\pm}_{i}(z)X^{\pm}_{i+1\epsilon}(w) = : X^{\pm}_{i}(z)X^{\pm}_{i+1\epsilon}(w) : (z - q^{-\epsilon}/2w)^{-1}(z - q^{\epsilon}/2w)
\]
\[
X^{\pm}_{i}(z)X^{-\epsilon}_{i+1}(w) = : X^{\pm}_{i}(z)X^{-\epsilon}_{i+1}(w) : (z - q^{-\epsilon}/2w)^{-1}(z - q^{\epsilon}/2w)
\]
\[
X^{\pm}_{i+1\epsilon}(z)X^{-\epsilon}_{i}(w) = : X^{\pm}_{i+1\epsilon}(z)X^{-\epsilon}_{i}(w) : (q^{\epsilon}/2z - q^{-\epsilon}/2w)(z - w)^{-1}
\]
\[
X^{\pm}_{n\epsilon}(z)X^{\pm}_{n}(w) = : X^{\pm}_{n\epsilon}(z)X^{\pm}_{n}(w) : \left( \frac{q^{-3\epsilon/2}(z - q^{3\epsilon}/2w)}{q^{-\epsilon/2}z - w} \right)^{|\epsilon|}
\]

where one can also read \(X^{\pm}_{i+1\epsilon}(z)X^{\pm}_{i\epsilon}(w) = : X^{\pm}_{i+1\epsilon}(z)X^{\pm}_{i\epsilon}(w) : (z - q^{\epsilon}/2w)^{-1}(z - q^{1/2}w)\) etc.

**Proof of (9).** Most of + cases and − cases are similar. We show the representative cases in the following.

\[
(z - q^{-1/2}w)X^{+}_{i\epsilon}(z)X^{+}_{i+1\epsilon}(w)
\]
\[
= : X^{+}_{i\epsilon}(z)X^{+}_{i+1\epsilon}(w) : (z - q^{\epsilon}/2w)^{-1}(z - q^{-1/2}w)(z - q^{1/2}w)
\]
\[
= (q^{-1/2}z - w)X^{+}_{i\epsilon}(z)X^{+}_{i+1\epsilon}(w)X^{+}_{i\epsilon}(z)
\]

from which (9) follows in the case of \((\alpha_i)|\alpha_j| = -1/2\). The case of \((\alpha_{1}|\alpha_{1}) = 1\) follows from (13).

For the case of \((\alpha_{i}|\alpha_{j}) = -1\), we have

\[
(z - q^{-1}w)X^{+}_{n-1\epsilon}(z)X^{+}_{n\epsilon}(w)
\]
\[
= : X^{+}_{n-1\epsilon}(z)X^{+}_{n\epsilon}(w) : (z - q^{1/2}w)^{-1}(z - q^{-1}w)^{1}(z - q^{-1}w)(z - q^{1/2}w)
\]
\[
= (q^{-1}z - w)X^{+}_{n\epsilon}(w)X^{+}_{n-1\epsilon}(z),
\]
since it has no singularities.

The $-$ case of $(\alpha_n|\alpha_n) = 2$ is shown as above, but the $+$ case requires a different approach.

\[
(z - q^2w)X^+_n(z)X^+_{n'}(w)
= X^+_n(z)X^+_{n'}(w) : q^2(z - q^2q^2z)\delta(\frac{qw}{z}) - \delta(\frac{w}{qz})[2]_{q^{1/2}}^2
\]

which implies that

\[
\left((z - q^2w)X^+_n(z)X^+_{n'}(w) - (w - q^2z)X^+_n(z)X^+_{n'}(w)\right)(q - q^{-1})^2(q^{1/2} - q^{-1/2})^2z^2w^2
\]

\[
= \sum_{\epsilon = \epsilon'} q^{\epsilon'} : X^+_n(z)X^+_{n'}(w) : q^2(z - w)\delta(\frac{qw}{z}) - \delta(\frac{w}{qz})[2]_{q^{1/2}}^2
\]

\[
+ \sum_{\epsilon = -\epsilon'} : X^+_n(z)X^+_{n'}(w) : q^2(z - w)\delta(\frac{qw}{z}) - \delta(\frac{w}{qz})[2]_{q^{1/2}}^2
\]

\[
- \sum_{\epsilon = 0, \epsilon'} q^{\epsilon'}[2]_{q^{1/2}} : X^+_n(z)X^+_{n'}(w) : \left((z - q^{1/2}w) + q^{1/2}(w - q^{-1/2}z)\right)
\]

\[
- \sum_{\epsilon = \pm, \epsilon' = 0} q^{\epsilon'}[2]_{q^{1/2}} : X^+_n(z)X^+_{n'}(w) : \left((q^{1/2}z - w) + (w - q^{1/2}z)\right)
\]

\[
= \sum_{\epsilon \neq 0} : X^+_n(z)X^+_{n-\epsilon}(w) : q^2(z - q^\epsilon w)(z - q^{-2\epsilon w})\delta(q^\epsilon w/z)
\]

\[
+ : X^+_n(z)X^+_{n0}(w) : \frac{(z - w)(z - q^2w)(z - q^{-2w})}{zw(q - q^{-1})zw} \delta(\frac{qw}{z}) - \delta(\frac{w}{qz})[2]_{q^{1/2}}^2
\]

\[
= 0
\]

since : $X^+_n(z)X^+_{n0}(q^\epsilon w) := X^+_n(z)X^+_{n-\epsilon}(q^\epsilon w)$ : for $\epsilon = \pm$.

The case of $X^+_n(z)X^+_{n}(w)$ is simpler than the "$+$" case, and is omitted.

Proof of (11). We only need to show the case when $\alpha_i$ and $\alpha_j$ are connected in the Dynkin diagram.

\[
(q^{-1} - q)(q^{1/2} - q^{-1/2})^2z^2w[X^+_n(z), X^+_{n-1}(w)]
\]

\[
= \sum_{\epsilon, \epsilon'} q^{\epsilon'/2}(-[2]_{q^{1/2}})^{1-|\epsilon|}q' : X^+_n(z)X^+_{n-1, \epsilon'}(w) : \left(\frac{q^\epsilon + z - q^{1/2}w}{z - q^{1/2}w}(q^{1/2}w - q^{-1/2}z) - \frac{q^{1/2}w - q^{-1/2}z}{w - q^{1/2}z}(z - q^{-1/2}w)\right)
\]

\[
= \sum_{\epsilon \neq \epsilon'} q^{\epsilon'/2}(-[2]_{q^{1/2}})^{1-|\epsilon|}q' : X^+_n(z)X^+_{n-1, \epsilon'}(w) : \left(\frac{q^{1/2}w - q^{-1/2}z}{q - q^{-1}}zw\delta(\frac{q^{1/2}w}{z}) - \delta(\frac{q^{-1/2}w}{z})\right)
\]
\[
= - \sum_{\epsilon = \pm} q^{t/2}\epsilon : X_{n\epsilon}^+(z)X_{n-1,-\epsilon}^-(q^{-t/2}z) : \frac{1 - q^{-}\epsilon}{q^{-}\epsilon z} \delta(\frac{q^{t/2}w}{z})
\]

\[
- \sum_{\epsilon = \pm} : X_{n0}^+(z)X_{n-1,\epsilon}^-(q^{-t/2}z) : \frac{q^{-1/2}(1 - q)}{q^{-\epsilon}z} \delta(\frac{q^{t/2}w}{z})
\]

\[
= 0
\]

since we have the following identities.

\[
: X_{n,\pm}^+(z)X_{n-1,\mp}^-(q^{-1/2}z) := X_{n0}^+(z)X_{n-1,\pm}^-(q^{-1/2}z) : \tag{20}
\]

If follows from (14) that for \(i = 1, \ldots, n - 1\)

\[
[X_i^+(z), X_i^-(w)] = \frac{1}{(q^{1/2} - q^{-1/2})^2zw} \sum_{\epsilon, \epsilon'} q^{t/2} : X_{i\epsilon}^+(z)X_{i\epsilon'}^-(w) : \left(\frac{z - q^{-t/2}w}{q^{t/2}z - w} - \frac{w - q^{-t/2}z}{q^{-t/2}w - z}\right)
\]

\[
= -1 \frac{1}{(q^{1/2} - q^{-1/2})^2zw} \sum_{\epsilon = e'} : X_{i\epsilon}^+(z)X_{i\epsilon'}^-(w) : \frac{z - q^{t/2}w}{w} \delta(\frac{q^{t/2}z}{w})
\]

\[
= \frac{1}{(q^{1/2} - q^{-1/2})^2zw} \left(\psi_i(zq^{1/4})\delta(\frac{q^{1/2}z}{w}) - \phi_i(zq^{-1/4})\delta(\frac{z}{q^{1/2}w})\right)
\]

Similarly using (14) and noting that \(\epsilon = 0\) does not contribute to the commutator, we have

\[
[X_n^+(z), X_n^-(w)] = \frac{-1}{(q - q^{-1})^2z^2} \sum_{\epsilon = \pm} q^{t/2} : X_{n\epsilon}^+(z)X_{n\epsilon}^-(w) : \left(\frac{q^{-3\epsilon/2}(z - q^{3\epsilon/2}w)}{q^{3/2}z - w} - \frac{q^{-3\epsilon/2}(q^{3\epsilon/2}w - z)}{w - q^{3\epsilon/2}z}\right)
\]

\[
= \frac{-1}{(q - q^{-1})^2z^2} \sum_{\epsilon = \pm} q^{t/2} : X_{n\epsilon}^+(z)X_{n\epsilon}^-(w) : \frac{q^{-\epsilon}(z - q^{3\epsilon/2}w)}{w} \delta(\frac{q^{t/2}z}{w})
\]

\[
= \frac{1}{(q - q^{-1})zw} \left(\psi_n(zq^{1/4})\delta(\frac{q^{1/2}z}{w}) - \phi_n(zq^{-1/4})\delta(\frac{zq^{-1}}{w})\right)
\]

**Proof of Serre relations (14).** Again we show the representative ones in the following.

\[
X_{i\epsilon_1}^+(z_1)X_{i\epsilon_2}^+(z_2)X_{i+1,\epsilon}(w) = : X_{i\epsilon_1}^+(z_1)X_{i\epsilon_2}^+(z_2)X_{i+1,\epsilon}^+(w) : \frac{(q^{1/2}z_1 - q^{3/2}z_2)(z_1 - q^{1/2}w)(z_2 - q^{1/2}w)}{(z_1 - q^{1/2}w)(z_2 - q^{1/2}w)(z_2 - q^{3/2}w)}
\]

We have that

\[
X_{i\epsilon_1}^+(z_1)X_{i\epsilon_2}^+(z_2)X_{i+1,\epsilon}(w) - (q^{1/2} + q^{-1/2})X_{i\epsilon_1}^+(z_1)X_{i+1,\epsilon}^+(w)X_{i\epsilon_2}^+(z_2)
\]
implies the Serre relation

Observe that the contracting factor in (21) is antisymmetric under $(z_1, \epsilon_1) \leftrightarrow (z_2, \epsilon_2)$, which implies the Serre relation $X_i^+(z_1)X_i^+(z_2)X_i^{+1}(w) + \cdots + (z_1 \leftrightarrow z_2) = 0$. Similarly we obtain that $X_i^+(z_1)X_i^+(z_2)X_i^{+1}(w) - (q^{1/2} + q^{-1/2})X_i^+(z_1)X_i^{+1}(w)X_i^+(z_2) + \cdots = 0$.

The Serre relation

\[ X_n^-(z_1)X_n^-(z_2)X_{n-1}^-(w) = [2]q X_n^- (z_1)X_{n-1}^- (w)X_n^- (z_2) + X_{n-1}^- (w)X_n^- (z_1)X_n^- (z_2) \]

is proved similarly by considering $X_n^-(z_1)X_n^-(z_2)X_{n-1}^- (w)$ and using the identity (22) with $a = q^{-1}$.

Finally let’s check another representative Serre relation with $A_{ij} = -2$:

\[ Sym_{z_1, z_2, z_3}(X_{n-1}^-(z_1)X_{n-1}^+(z_2)X_{n-1}^+(z_3))X_n^+(w) - [3]q^{1/2}X_{n-1}^+(z_1)X_{n-1}^+(z_2)X_n^+(w)X_{n-1}^+(z_3) + [3]q^{1/2}X_{n-1}^+(z_1)X_n^+(w)X_{n-1}^+(z_2)X_{n-1}^+(z_3) - X_n^+(w)X_{n-1}^+(z_1)X_{n-1}^+(z_2)X_{n-1}^+(z_3)) = 0 \]

First we have

\[ X_{n-1, \epsilon_1}^+(z_1)X_{n-1, \epsilon_2}^+(z_2)X_{n-1, \epsilon_3}^+(z_3)X_{n, \epsilon}^+(w) = : X_{n-1, \epsilon_1}^+(z_1)\cdots : \prod_{1 \leq i < j \leq 3} \frac{q^{i/2}z_i - q^{j/2}z_j}{z_i - qz_j} \prod_{i=1}^{3} \frac{(z_i - qw)(z_i - w)}{(z_i - q^{1+\epsilon}w)(z_i - q^{1+\epsilon}w)} \]

Moving $X_{n, \epsilon}^+(w)$ around, we have that

\[ X_{n-1, \epsilon_1}^+(z_1)X_{n-1, \epsilon_2}^+(z_2)X_{n-1, \epsilon_3}^+(z_3)X_{n, \epsilon}^+(w) - [3]q^{1/2}X_{n-1, \epsilon_1}^+(z_1)X_{n-1, \epsilon_2}^+(z_2)X_{n, \epsilon}^+(w)X_{n-1, \epsilon_3}^+(z_3) + [3]q^{1/2}X_{n-1, \epsilon_1}^+(z_1)X_{n, \epsilon}^+(w)X_{n-1, \epsilon_2}^+(z_2)X_{n-1, \epsilon_3}^+(z_3) - X_{n, \epsilon}^+(w)X_{n-1, \epsilon_1}^+(z_1)X_{n-1, \epsilon_2}^+(z_2)X_{n-1, \epsilon_3}^+(z_3)X_{n, \epsilon}^+(w) \]
\[ = X_{n-1}^{+, \epsilon_1}(z_1) \cdots \prod_{1 \leq i < j \leq 3} q^{\epsilon_j/2} z_i - q^{\epsilon_j/2} z_j \sum_{i=1}^{3} \frac{z_i - w}{z_i - qz_j} \cdot \left( (z_1 - qw)(z_2 - qw)(z_3 - qw) + [3]_{q^{1/2}} (z_1 - qw)(z_2 - qw)(w - qz_3) \right) \]

The expression in the parenthesis is simplified to be
\[
(q^{-1} - q) \left( w^2(z_1 - (q + q^{-1})z_2 + q^3 z_3) + w(z_1z_2 - (q + q^{-1})z_1z_3 + q^3 z_2z_3) \right)
\]

Applying symmetrization of \( S_3 \) on \( z_1, z_2, z_3 \) (and on \( \epsilon_1, \epsilon_2, \epsilon_3 \) accordingly) and factoring out the symmetric part, we see that the proof of Serre relation in the case is reduced to show the following identity
\[
\sum_{\sigma \in S_3} sgn(\sigma) \sigma.(z_1 - (q + q^2) z_2 + q^3 z_3) \prod_{i < j} (qz_i - z_j) = 0 \tag{23}
\]
where the symmetric group \( S_3 \) acts on the ring of functions in \( z_i \) \((i = 1, 2, 3)\) in the natural way: \( \sigma z_i = z_{\sigma(i)} \).

The left-hand side (LHS) of (23) is a polynomial in \( q \) of degree 5. It is easy to see that \( q = \pm 1, 0, \infty \) are roots of the polynomial. Let \( \omega \) be a 3rd primitive root of unity. We claim that \( \omega \) and \( \omega^2 \) are both roots of the LHS of (23). In fact, plugging in \( q = \omega \) reduces it into an equivalent identity
\[
\sum_{\sigma \in S_3} sgn(\sigma) \prod_{i < j} (\omega z_i - z_j)
= \sum_{\sigma \in S_3} sgn(\sigma) \left( (z_1^2 z_2 + z_1^2 z_2 + z_1^2 z_2 - z_2 z_3) + (1 + 2\omega) z_1 z_2 z_3 + \omega(z_1^2 z_3 + z_1 z_3^2 + z_1 z_2^2 + z_2 z_3) \right) = 0
\]
This proves the Serre relation and ends the proof of Theorem 3.1.

## 4 Irreducible Representations

In this section, we will see the irreducible representations are realized in \( \tilde{F} \). We treat the irreducible highest weight representations whose highest weights are the following four weights.
\[
\mu_1 = -\frac{1}{2} \Lambda_0, \quad \mu_2 = -\frac{3}{2} \Lambda_0 + \Lambda_1 - \frac{\delta}{2}, \quad \mu_3 = -\frac{1}{2} \Lambda_n + \frac{n \delta}{8}, \quad \mu_4 = \Lambda_{n-1} - \frac{3}{2} \Lambda_n + \frac{n \delta}{8}.
\]
These weights are admissible and their characters are given by we the Weyl-Kac-Wakimoto character formula [10]. In our case, the characters of the irreducible highest weight \( \tilde{sp}_{2n} \)-modules \( L(\lambda) \) have the following forms [13]:
\[
\text{ch } L(\mu_1) + \text{ch } L(\mu_2) = \frac{e^{-\frac{1}{2} \Lambda_0}}{\prod_{i=1}^{n} (p^{2} e^{\epsilon_i}; p)_{\infty} (p^{2} e^{-\epsilon_i}; p)_{\infty}} .
\]
\[
\text{ch } L(\mu_3) + \text{ch } L(\mu_4) = \frac{e^{-\frac{1}{2}n_\lambda p^{-\frac{1}{2}}}}{\prod_{i=1}^n (pe^{\varepsilon_i}; p)_\infty (e^{-\varepsilon_i}; p)_\infty},
\]
where \( p = e^{-\delta} \) and \( (a; p)_\infty = \prod_{n=0}^\infty (1 - ap^n) \).

Define the operators \( Q_i^- \) on \( \mathcal{F}_\beta^2 \) (the 2nd component) by
\[
Q_i^- := \oint Z_i^-(z) \frac{dz}{2\pi \sqrt{-1}}.
\]

We set subspaces \( \mathcal{F}_i \) \( (i = 1, 2, 3, 4) \) of \( \tilde{\mathcal{F}} \) as follows.
\[
\begin{align*}
\mathcal{F}_1 &= \bigoplus_{\alpha \in Q} \mathcal{F}_{\alpha, \alpha}, \\
\mathcal{F}_2 &= \bigoplus_{\alpha \in Q} \mathcal{F}_{\alpha + \varepsilon_1, \varepsilon_1}, \\
\mathcal{F}_3 &= \bigoplus_{\alpha \in Q} \mathcal{F}_{\alpha - \frac{1}{2}n, \alpha}, \\
\mathcal{F}_4 &= \bigoplus_{\alpha \in Q + \varepsilon_n} \mathcal{F}_{\alpha - \frac{1}{2}n, \alpha},
\end{align*}
\]
where
\[
\mathcal{F}_{\alpha, \beta} = \mathcal{F}_{\alpha} \otimes \prod_{j=1}^n \text{Ker } \mathcal{F}_{ij, \varepsilon_j} Q_j^-,
\]
for \( \beta = l_1 \varepsilon_1 + \cdots + l_n \varepsilon_n \).

**Theorem 4.1** Each \( \mathcal{F}_i \) \( (i = 1, 2, 3, 4) \) is an irreducible highest weight \( U_q \)-module isomorphic to \( V(\mu_i) \). The highest weight vectors are given by \( |\mu_1⟩ = |0, 0⟩ \), \( |\mu_2⟩ = b_1(-1)|\lambda_1, \lambda_1⟩ \), \( |\mu_3⟩ = |−\frac{1}{2}n, 0⟩ \), \( |\mu_4⟩ = |−\frac{1}{2}n - \varepsilon_n, -\varepsilon_n⟩ \).

**Proof.** Note that \( [X_j^+(z), Q_n^-] = 0 \) is given in [12], we can see similarly that each \( Q_i^- \) \( (i = 1, \ldots, n) \) commutes or anticommutes with \( X_j^+(z) \) for all \( j = 1, \ldots, n \). Therefore, each \( \mathcal{F}_i \) is a \( U_q \)-submodule of \( \tilde{\mathcal{F}} \).

Next we calculate the character of \( \mathcal{F}_i \). We may understand \( Q_j^- \) as the zero mode \( \eta_0 \) of the fermionic ghost system \( (\xi, \eta) \) of dimension \((0, 1)\).
\[
\xi(z) = Z_j^+(z) = \sum_{n \in \mathbb{Z}} \xi_n z^{-n}, \quad \eta(z) = Z_j^-(z) = \sum_{n \in \mathbb{Z}} \eta_n z^{-n-1}.
\]

Since we have \( \eta_0^2 = 0 \) and \( \xi_0 \eta_0 + \eta_0 \xi_0 = 1 \), we obtain the following exact sequence:
\[
0 \rightarrow \text{Ker } \mathcal{F}_{ij, \varepsilon_j} Q_j^- \rightarrow \mathcal{F}_{ij, \varepsilon_j}^2 Q_j^- \mathcal{F}_{ij} \mathcal{F}_{ij-1, \varepsilon_j} Q_j^- \mathcal{F}_{ij} \mathcal{F}_{ij-2, \varepsilon_j} \cdots
\]

Using this exact sequence, we can compute the character of \( \mathcal{F}_i \). Since \( a_j(n) \) and \( b_j(n) \) have weight \( n \delta \), we get
\[
\text{ch } \mathcal{F}^1_{\alpha} = e^{\frac{1}{2}p^{-\frac{1}{2}}} p^{-\frac{|\alpha|}{2}}, \quad \text{ch } \mathcal{F}^2_{l} = \frac{p^l(l^2-1)}{(p;p)_\infty}.
\]
The proof of the last equality was given in [12]. Similarly, we have

\[ \text{ch} (F_1 \oplus F_2) = \text{ch} \left( \bigoplus_{\alpha \in P} F'_{\alpha, \alpha} \right) \]

\[ = \text{ch} \left( \bigoplus_{l_1, \ldots, l_n \in \mathbb{Z}} F^1_{l_1 \epsilon_1 + \cdots + l_n \epsilon_n, l_1 \epsilon_1 + \cdots + l_n \epsilon_n} \right) \]

\[ = \text{ch} \left( \bigoplus_{l_1, \ldots, l_n \in \mathbb{Z}} F^1_{l_1 \epsilon_1 + \cdots + l_n \epsilon_n} \otimes \bigotimes_{j=1}^{n} \text{Ker} F_{l_j \epsilon_j, Q_j}^{j} \right) \]

\[ = \sum_{l_1, \ldots, l_n \in \mathbb{Z}} \left( \text{ch} (F^1_{l_1 \epsilon_1 + \cdots + l_n \epsilon_n}) \prod_{j=1}^{n} \text{ch} (\text{Ker} F_{l_j \epsilon_j, Q_j}^{j}) \right) \]

\[ = \sum_{l_1, \ldots, l_n \in \mathbb{Z}} \prod_{j=1}^{n} \left( (p; p)_\infty^{-\epsilon l_j} \sum_{k \leq l_j} (-1)^{l_j-k} p^{-\frac{1}{2} (l_j^2 - k^2 + k)} \right) \]

\[ = \prod_{l_1, \ldots, l_n \in \mathbb{Z}} \left( (p; p)_\infty^{-2 \epsilon l_j} \sum_{k \leq l_j} (-1)^{l_j-k} p^{-\frac{1}{2} (l_j^2 - k^2 + k)} \right) \]

\[ = \prod_{j=1}^{n} \left( (p; p)_\infty^{-2 \epsilon l_j} \sum_{k \leq l_k} (-1)^{l_j-k} p^{-\frac{1}{2} (l_j^2 - k^2 + k)} \right) \]

\[
= \frac{e^{-\frac{1}{2} \Lambda_0}}{\prod_{i=1}^{n} (p^{e^{-\epsilon_i}}; p)_\infty (p^{e^{-\epsilon_i}}; p)_\infty}.
\]

The proof of the last equality was given in [12]. Similarly, we have

\[ \text{ch} (F_3 \oplus F_4) = \frac{e^{-\frac{1}{2} \Lambda_0}}{\prod_{i=1}^{n} (p^{e^{-\epsilon_i}}; p)_\infty (p^{e^{-\epsilon_i}}; p)_\infty}, \]

Hence, by the above character formulas,

\[ \text{ch} F_i = \text{ch} L(\mu_i). \]

As in Lusztig([2]), \( F_i \) becomes a certain \( \widehat{sp}_{2n} \)-module in the classical limit \( q \to 1 \). Since the dimension of each weight space is invariant in the limit, the \( \widehat{sp}_{2n} \)-module in the classical limit is irreducible. Therefore \( F_i \) is irreducible for a generic \( q \).

Finally, it can be checked immediately that each \( \{\mu_i\} \) is a weight vector of weight \( \mu_i \), and belongs to \( F_i \). \( \mu_i \) is the highest weight of \( F_i \) by the character. Hence \( \{\mu_i\} \) is a highest weight vector of \( F_i \).

These representations become the ones constructed by Feingold and Frenkel [1] in the classical limit \( q \to 1 \), since the latter is equivalent to the \( \beta - \gamma \) system of \( \phi_j^1(z) \), \( \phi_j^2(z) \) (cf [12] for \( n=1 \) case). Our \( a_j(n), b_j(n) \) are related to them under \( q \to 1 \):

\[ Y_j^\pm(z) \longrightarrow e^{\pm e_j(z)}: \quad Z_j^\pm(z) \longrightarrow e^{\pm e_j^2(z)}. \]
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