Total vertex irregularity strength for trees with many vertices of degree two

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Abstract

For a simple graph $G = (V, E)$, a mapping $\phi : V \cup E \rightarrow \{1, 2, \ldots, k\}$ is defined as a vertex irregular total $k$-labeling of $G$ if for every two different vertices $x$ and $y$, $wt(x) \neq wt(y)$, where $wt(x) = \phi(x) + \sum_{xy \in E(G)} \phi(xy)$. The minimum $k$ for which the graph $G$ has a vertex irregular total $k$-labeling is called the total vertex irregularity strength of $G$. In this paper, we provide three possible values of total vertex irregularity strength for trees with many vertices of degree two. For each of the possible values, sufficient conditions for trees with corresponding total vertex irregularity strength are presented.

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1. Introduction

The concept of total vertex irregularity strength of graphs was first introduced by Baca et.al \cite{2} in 2007. They defined a mapping $\phi : V \cup E \rightarrow \{1, 2, 3, \ldots, k\}$ to be a vertex irregular total $k$-labeling of $G$ if for every two different vertices $x$ and $y$, $wt(x) \neq wt(y)$, where $wt(x) = \phi(x) + \sum_{xy \in E(G)} \phi(xy)$. The minimum $k$ for which the graph $G$ has a vertex irregular...
total $k$-labeling is called the total vertex irregularity strength of $G$, denoted by $tvs(G)$. Baca et al. determined the total vertex irregularity strength of some well-known classes of graphs, i.e. paths, cycles, and stars. Other authors (for instance, [1], [3]) determined the total vertex irregularity strength of some other classes of graphs, however results are still limited.

In the original paper of Baca et al. [2], it was proved that for a tree $T$ with $m$ pendant vertices and no vertex of degree 2, $\left\lceil \frac{m+1}{2} \right\rceil \leq tvs(T) \leq m$. In 2010, Nurdin et al. [4] settled the total vertex irregularity strength for a tree $T$ with $m$ pendant vertices and no vertices of degree 2, i.e. $tvs(T) = \left\lceil \frac{m+1}{2} \right\rceil$. They also improved the lower bound of Baca et al. as in the following.

**Theorem 1.1.** [4] Let $T$ be any tree having $n_i$ vertices of degree $i(i = 1, 2, \ldots, \Delta)$, where $\Delta$ is the maximum degree in $T$. Then

$$tvs(T) \geq \max \left\{ \left\lceil \frac{1+n_1}{2} \right\rceil, \left\lceil \frac{1+n_1+n_2}{3} \right\rceil, \ldots, \left\lceil \frac{1+n_1+n_2+\cdots+n_\Delta}{\Delta+1} \right\rceil \right\}.$$

The lower bound in Theorem 1.1 remains the most general bound known for trees. However, it was conjectured that the total vertex irregularity strength of a tree is only determined by the number of vertices of degrees at most 3.

**Conjecture 1.1.** [4] Let $T$ be a tree with maximum degree $\Delta$. Let $n_i$ be the number of vertices of degree $i(i = 1, 2, \ldots, \Delta)$ and $t_i = \left\lceil \frac{1+\sum_{k=1}^{i} n_k}{(i+1)} \right\rceil (i = 1, 2, \ldots, \Delta)$. Then

$$tvs(T) = \max \{t_1, t_2, t_3\}.$$

To date, the conjecture has been confirmed for some types of trees, i.e. paths and stars, trees with maximum degree up to 5 [4, 6, 7] and subdivision of some classes of trees [5, 8].

In this paper, our aim is to determine the total vertex irregularity strength of trees with many vertices of degree 2 which include subdivision of trees. This result could somewhat be viewed as generalization of our result in [8], where we presented sufficient conditions for subdivision of trees to admit total vertex irregularity strength of $t_2$.

Throughout the paper, we consider $T$ as a tree with maximum degree $\Delta$. We denote by $n_i$ the number of vertices of degree $i(i = 1, 2, \ldots, \Delta)$ and $t_i = \left\lceil \frac{1+\sum_{k=1}^{i} n_k}{(i+1)} \right\rceil (i = 1, 2, \ldots, \Delta)$.

2. Basic Properties of Trees

In this section, we shall provide properties of trees, in term on $n_1$, $n_2$, and $n_3$, having $t_1$, $t_2$ or $t_3$ as the maximum among all $t_i$s. We start by quoting a useful property proved in [2].

**Lemma 2.1.** [2]

$$n_1 = 2 + \sum_{i \geq 2} (i-2)n_i.$$
Lemma 2.2. If \( n_1 \geq 2n_2 - 1 \) and \( n_2 = n_3 \) then \( t_1 \geq \max\{t_1, t_2, \ldots, t_\Delta\} \).

Proof. Utilising Lemma 2.1 in the definition of \( t_i \), we have
\[
t_i = \left\lceil \frac{3 + \sum_{k=2}^{i} (k-1)n_k + \sum_{j=i+1}^{\Delta} (j-2)n_j}{i+1} \right\rceil.
\]
Consider \( t_1 - t_2 = \left\lceil \frac{1+n_1}{2} \right\rceil - \left\lceil \frac{1+n_1+n_2}{3} \right\rceil = \left\lceil \frac{(2+2n_1+2n_2+2n_3)+(2n_1+2-2n_3-2n_2)}{8} \right\rceil - \left\lceil \frac{2+2n_1+2n_2+2n_3}{8} \right\rceil. \)
Since \( n_1 \geq 2n_2 - 1 \), we have \( n_1 + 1 - 2n_2 \geq 0 \) and thus \( t_1 \geq t_2 \).

On the other hand,
\[
t_1 - t_3 = \left\lceil \frac{1+n_1}{2} \right\rceil - \left\lfloor \frac{1+n_1+n_2+n_3}{4} \right\rfloor
= \left\lceil \frac{(2+2n_1+2n_2+2n_3)+(2n_1+2-2n_3-2n_2)}{8} \right\rceil - \frac{2+2n_1+2n_2+2n_3}{8}.
\]
Since \( n_1 \geq 2n_2 - 1 \) and \( n_2 = n_3 \) then \( 2n_1 + 2 - 2n_3 - 2n_2 \geq 0 \), which yields \( t_1 \geq t_3 \).

For \( i \geq 4 \),
\[
t_1 - t_i = \left\lceil \frac{1+n_1}{2} \right\rceil - \left\lfloor \frac{3 + \sum_{k=2}^{i} (k-1)n_k + \sum_{j=i+1}^{\Delta} (j-2)n_j}{i+1} \right\rfloor
\geq \left\lceil \frac{5+5n_1}{2(i+1)} \right\rceil - \frac{6+2n_2+4n_3+6n_4+2\sum_{j=5}^{\Delta} (j-2)n_j}{2(i+1)}.
\]
Since \( n_1 \geq 2n_2 - 1 \) and \( n_2 = n_3, 9+n_3+4n_4+3\sum_{i=5}^{\Delta} (i-2)n_i-2n_2 \geq 6+2n_4+2\sum_{i=5}^{\Delta} (i-2) > 0 \), which leads to \( t_1 - t_i \geq 0 \).

Using similar proof of Lemma 2.2, we could prove the following lemmas.

Lemma 2.3. If \( n_2 \geq \frac{1}{2}(n_1 + 1) \) and \( n_1 \geq 2n_3 - 1 \) then \( t_2 \geq \max\{t_1, t_2, \ldots, t_\Delta\} \).

Lemma 2.4. If \( n_2 = n_1 \) and \( n_3 \geq \frac{1}{3}(2n_2 + 1) \) then \( t_3 \geq \max\{t_1, t_2, \ldots, t_\Delta\} \).

3. Trees with Many Vertices of Degree 2

In this section, we provide sufficient conditions, in term on \( n_1, n_2, \) and \( n_3 \), for a tree \( T \) with many vertices of degree 2 admitting \( tvs(T) = t_1, t_2 \) or \( t_3 \).

We start by defining several notions that will be frequently utilized in our labeling algorithms. Let \( v \) be a vertex of \( T \). A branch of \( T \) at \( v \) is defined as maximal subtree of \( T \) containing \( v \) as an end point. That is, a branch of \( T \) at \( v \) is the subgraph induced by \( v \) and one of the components of \( T - v \). If the degree of \( v \) is \( k \), then \( v \) has \( k \) different branches. A branch of \( T \) at \( v \) which isomorphic to a path will be called a branch path at \( v \), provided that the degree of \( v \) is at least 3. The vertex \( v \), in this case, will be called a stem of the branch path at \( v \). We define an interior path in \( T \) as a path whose both of end vertices are stem vertices. A vertex of degree one in \( T \) is called a pendant vertex. A vertex incident to a pendant vertex in \( T \) is called an exterior vertex. The vertices other than exterior and pendant vertices are called interior vertices. An edge incident with a pendant vertex is called a pendant edge. We denote by \( E^p(v) \) the set of pendant edges incident to an exterior vertex \( v \).
**Theorem 3.1.** If \( n_1 \geq 2n_2 - 1 \) and \( n_2 = n_3 > 0 \) then \( \text{tvs}(T) = t_1 \).

**Proof.** By Lemma 2.2 and Theorem 1.1, \( \text{tvs}(T) \geq t_1 \). We define a total labeling \( \alpha : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, t_1\} \) according to the following algorithm.

**Algorithm 1:** Labeling \( \alpha \) with tvs \( t_1 \)

1. Let \( W = \{w_1, w_2, \ldots, w_k\} \) be the set of exterior vertices in \( T \) such that either \( d(w_i) \geq d(w_{i+1}) \) or \( |E^p(w_i)| \geq |E^p(w_{i+1})| \).
2. Let \( V_1 = \{w_{ij}|i = 1, 2, \ldots, k \text{ and } j = 1, 2, \ldots, |E^p(w_i)|\} \) be the ordered set of pendant vertices adjacent to all exterior vertices. Label the first \( t_1 \) pendant vertices in \( V_1 \) with 1 and the remaining \( (n_1 - t_1) \) pendant vertices with \( 2, 3, \ldots, n_1 - t_1 + 1 \), respectively.
3. Let \( E_1 = \{e_{ij}|i = 1, 2, \ldots, k, j = 1, 2, \ldots, |E^p(w_i)|\} \) be the ordered set of pendant edges incident to \( w_{ij} \). Label the first \( t_1 \) pendant edges in \( E_1 \) with \( \{1, 2, \ldots, t_1\} \) and the remaining edges with \( t_1 \).
4. Let \( y_1, y_2, \ldots, y_N \) be vertices in \( V \backslash V_1 \). For all \( y \in V \backslash V_1 \), define \( \text{wt}'(y) = \alpha(y) + \sum_{y \in E(T)} \alpha(yz) \), as a temporary weight of a vertex \( y \), where \( \text{wt}'(y_i) \leq \text{wt}(y_{i+1}) \). Label \( y_1 \) with \( \text{wt}(y_1) \). For \( 2 \leq i \leq N \), label \( y_i \) with \( \text{max}\{1, \text{wt}(y_{i-1}) + 1 - \text{wt}'(y_i)\} \).

We observe that \( \alpha \) is a labeling from \( V(T) \cup E(T) \) into \( \{1, 2, \ldots, t_1\} \) where the weights of \( n_1 \) pendant vertices are \( 2, 3, \ldots, n_1 + 1 \) and the weights of all remaining vertices are \( n_1 + 2 = \text{wt}(y_1) < \text{wt}(y_2) < \text{wt}(y_3) < \cdots < \text{wt}(y_N) \) where \( N = \sum_{i=2}^{\Delta} n_i \). Therefore, \( \text{tvs}(T) \leq t_1 \). \( \Box \)

**Theorem 3.2.** If \( n_2 \geq \frac{1}{2}(n_1 + 1) \) and \( n_1 \geq 2n_3 - 1 \) then \( \text{tvs}(T) = t_2 \).

**Proof.** By Lemma 2.3 and Theorem 1.1, \( \text{tvs}(T) \geq t_2 \). We show that \( \text{tvs}(T) \leq t_2 \) through a total labeling \( \beta : V(T) \cup E(T) \rightarrow \{1, 2, \ldots, t_2\} \) according to the following algorithm.
Algorithm 2: Labeling $\beta$ with tvs $t_2$

1. If $T$ has more interior paths than branch paths then
   (a) Let $W = \{w_1, w_2, \ldots, w_k\}$ be the set of stem vertices where $d(w_i) \geq d(w_{i+1})$.
   (b) Let $V_1 = \{w_{ij} | i = 1, 2, \ldots, k, j = 1, 2, \ldots, j_i\}$ be the set of all pendant vertices $w_{ij}$ in
       the branch path of $w_i$. Label $n_1$ pendant vertices in $V_1$ with $\lceil \frac{i}{2} \rceil$.
   (c) Let $E_1 = \{e_{ij}\}$ be the set of all pendant edges $e_{ij}$ incident to $w_{ij}$. Label $n_1$ pendant
       edges $e \in E_1$ with $\lceil \frac{i+1}{2} \rceil$.
   (d) Label all edges incident to stem vertices with $t_2$.
   (e) Let $E_2 = \{e_1, e_2, \ldots, e_k\}$ be the set of edges where both of end vertices of $e_i$ are of
       degree two. Label $e_i$ with $\lceil \frac{n_1+1+i}{3} \rceil$.

   else
   (a) Let $P = \{P^1, P^2, \ldots, P^k\}$ be the ordered set of branch paths, where $|P^i| \geq |P^{i+1}|$.
   (b) Let $W = \{w_1, w_2, \ldots, w_k\}$ be the set of stem vertices where $d(w_i) \geq d(w_{i+1})$.
   (c) Let $E_1 = \bigcup_{i=1}^k E(w_i)$ be an ordered set of all pendant edges in the path $P^i$. Label $n_1$
       pendant edges in $E_1$ with $\lceil \frac{i+1}{2} \rceil$.
   (d) Label $n_1$ pendant vertices incident to $e_i$ with $\lceil \frac{i}{2} \rceil$.
   (e) Label all edges incident to stem vertices with $t_2$.
   (f) Let $E_2 = \{e_1, e_2, \ldots, e_k\}$ be the ordered set of edges in $P^1 \cup P^2 \cup \cdots \cup P^k$. Label
       $e_i \in E_2$ with $\beta(e_i) = \lceil \frac{1+n_i+i}{3} \rceil$.
   (g) Let $L = \{L_1, L_2, \ldots, L_k\}$ be the set of interior paths where $|L_i| \geq |L_{i+1}|$.
   (h) Let $E_3 = \{f_1, f_2, \ldots, f_k\}$ be the ordered set of edges in path $L_1 \cup L_2 \cup \cdots \cup L_k$.
       Label $f_i \in E_3$ with $\lceil \frac{n_1+1+i}{3} \rceil$.

2. Denote all vertices not in $V_1$ by $y_1, y_2, \ldots, y_N$ such that $wt'(y_1) \leq wt'(y_2) \leq \cdots \leq wt'(y_N)$,
   where $wt'(y) = \sum_{y \in E} \beta(yz)$ can be considered as a temporary weight of $y$. Label $y_1$
   with $n_1 + 2 - s(y_1)$. For $2 \leq i \leq N$, label $y_i$ with $\max\{1, wt(y_i + 1 - s(y_i))\}$.

We observe that $\beta$ is a labeling from $V(T) \cup E(T)$ into $\{1, 2, \ldots, t_2\}$, the weight of all pendant
vertices form a sequence $1, 2, 3, \ldots, n_1 + 1$, and the weight of all remaining vertices are $n_1 + 2 = wt(y_1) \leq wt(y_2) \leq \cdots < wt(y_N)$. Therefore, $tvs(T) \leq t_2$. 

Examples of families of trees admitting total vertex irregularity strength of $t_2$ are special cases of subdivision of tress that could be found in [8].

Theorem 3.3. If $n_2 = n_1 > 0$ and $n_3 \geq \frac{1}{3}(2n_2 + 1)$ then $tvs(T) = t_3$. 

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We observe that $\gamma$ is a labeling from $V(T) \cup E(T)$ into $\{1, 2, \ldots, t_3\}$ where the weights of $n_1$ pendant vertices are $\{2, 3, 4, \ldots, n_1 + 1\}$ and the weights of all the remaining vertices are $n_1 + 2 = wt(y_1) < wt(y_2) < \cdots < wt(y_N)$. This yields $t_3 \leq tvs(T)$. \qed
4. Conclusion

Our results provide sufficient conditions for trees containing many vertices of degree 2 where the total vertex irregularity strength is either $t_1$, $t_2$ or $t_3$. These results strengthens the conjecture Nurdin et al.

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