EMBEDDINGS OF GRASSMANN GRAPHS

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ABSTRACT. Let \( V \) and \( V' \) be vector spaces of dimension \( n \) and \( n' \), respectively. Let \( k \in \{2, \ldots, n-2\} \) and \( k' \in \{2, \ldots, n'-2\} \). We describe all isometric and \( l \)-rigid isometric embeddings of the Grassmann graph \( \Gamma_k(V) \) in the Grassmann graph \( \Gamma_{k'}(V') \).

1. INTRODUCTION

Let \( V \) be an \( n \)-dimensional vector space. Denote by \( G_k(V) \) the Grassmannian consisting of \( k \)-dimensional subspaces of \( V \). Two elements of \( G_k(V) \) are adjacent if their intersection is \((k - 1)\)-dimensional. The Grassmann graph \( \Gamma_k(V) \) is the graph whose vertex set is \( G_k(V) \) and whose edges are pairs of adjacent \( k \)-dimensional subspaces. By Chow’s theorem [3], if \( 1 < k < n - 1 \) then every automorphism of \( \Gamma_k(V) \) is induced by a semilinear automorphism of \( V \) or a semilinear isomorphism of \( V \) to the dual vector space \( V^* \) and the second possibility can be realized only in the case when \( n = 2k \) (if \( k = 1, n - 1 \) then any two distinct vertices of \( \Gamma_k(V) \) are adjacent and every bijective transformation of \( G_k(V) \) is an automorphism of \( \Gamma_k(V) \)). Some results closely related with Chow’s theorem can be found in [2, 7, 8, 9, 11, 12] and we refer [13] for a survey.

Let \( V' \) be an \( n' \)-dimensional vector space. We investigate isometric embeddings of \( \Gamma_k(V) \) in \( \Gamma_{k'}(V') \) under assumption that \( 1 < k < n - 1 \) and \( 1 < k' < n' - 1 \) (the case \( k = k' \) was considered in [9]). Then \( n, n' \geq 4 \) and the existence of such embeddings implies that the diameter of \( \Gamma_k(V) \) is not greater than the diameter of \( \Gamma_{k'}(V') \), i.e.

\[
\min\{k, n-k\} \leq \min\{k', n'-k'\}.
\]

In the case when \( k \leq n-k \), we show that every isometric embedding of \( \Gamma_k(V) \) in \( \Gamma_{k'}(V') \) is induced by a semilinear \((2k)\)-embedding (a semilinear injection such that the image of every independent \((2k)\)-element subset is independent) of \( V \) in \( V'/S \), where \( S \) is a \((k' - k)\)-dimensional subspace of \( V' \), or a semilinear \((2k)\)-embedding of \( V \) in \( U^* \), where \( U \) is a \((k' + k)\)-dimensional subspace of \( V' \) (Theorem 4.1). If \( k > n-k \) then there are isometric embeddings of \( \Gamma_k(V) \) in \( \Gamma_{k'}(V') \) which can not be induced by semilinear mappings of \( V \) to some vector spaces.

Our second result (Theorem 6.1) is related with so-called \( l \)-rigid embeddings. An embedding \( f \) of a graph \( \Gamma \) in a graph \( \Gamma' \) is rigid if for every automorphism \( g \) of \( \Gamma \) there is an automorphism \( g' \) of \( \Gamma' \) such that the diagram

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{f} & \Gamma' \\
\downarrow g & & \downarrow g' \\
\Gamma & \xrightarrow{f} & \Gamma'
\end{array}
\]
is commutative, roughly speaking, every automorphism of $\Gamma$ can be extended to an automorphism of $\Gamma'$. We say that an embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ is $l$-rigid if every automorphism of $\Gamma_k(V)$ induced by a linear automorphism of $V$ can be extended to the automorphism of $\Gamma_{k'}(V')$ induced by a linear automorphism of $V'$.

In the case when $n = 2k$, every isometric embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ is $l$-rigid. In general case (we do not require that $k \leq n - k$), every $l$-rigid isometric embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ is induced by a semilinear embedding (a semilinear injection transferring independent subsets to independent subsets) of $V$ in $V'/S$, where $S$ is a $(k' - k)$-dimensional subspace of $V'$, or a semilinear embedding of $V$ in $U^*$, where $U$ is a $(k' + k)$-dimensional subspace of $V'$. The proof of this result is based on a characterization of semilinear embeddings (Theorem 5.1).

Using [10], we establish the existence of isometric embeddings of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ which are not $l$-rigid (Example 6.4).

2. Basic facts and definitions

2.1. Let $\Gamma$ be a connected graph. A subset in the vertex set of $\Gamma$ formed by mutually adjacent vertices is called a clique. Using Zorn lemma, we can show that every clique is contained in a maximal clique. The distance $d(v, w)$ between two vertices $v$ and $w$ of $\Gamma$ is defined as the smallest number $i$ such that there exists a path of length $i$ (a path consisting of $i$ edges) connecting $v$ and $w$. The diameter of $\Gamma$ is the maximum of all distances $d(v, w)$.

An injective mapping of the vertex set of $\Gamma$ to the vertex set of a graph $\Gamma'$ is called an embedding of $\Gamma$ in $\Gamma'$ if vertices of $\Gamma$ are adjacent only in the case when their images are adjacent vertices of $\Gamma'$. Every surjective embedding is an isomorphism. An embedding is said to be isometric if it preserves the distance between any two vertices.

2.2. Let $k \in \{1, \ldots, n - 1\}$. Consider incident subspaces $S, U \subset V$ such that

$$\dim S < k < \dim U$$

and denote by $[S, U]_k$ the set formed by all $X \in G_k(V)$ satisfying $S \subset X \subset U$. In the case when $S = 0$ or $U = V$, this set will be denoted by $[U]_k$ or $[S]_k$, respectively. The Grassmann space $G_k(V)$ is the partial linear space whose point set is $G_k(V)$ and whose lines are subsets of type

$$[S, U]_k, \ S \in G_{k-1}(V), \ U \in G_{k+1}(V).$$

It is clear that $G_1(V) = \Pi_V$ and $G_{n-1}(V) = \Pi_V^*$ (we denote by $\Pi_V$ the projective space associated with $V$ and write $\Pi_V^*$ for the corresponding dual projective space). Two distinct points of $G_k(V)$ are collinear (joined by a line) if and only if they are adjacent vertices of the Grassmann graph $\Gamma_k(V)$.

If $1 < k < n - 1$ then there are precisely the following two types of maximal cliques of $\Gamma_k(V)$:

1. the top $\langle U \rangle_k, \ U \in G_{k+1}(V)$,
2. the star $[S]_k, \ S \in G_{k-1}(V)$.

The top $\langle U \rangle_k$ and the star $[S]_k$ together with the lines contained in them are projective spaces. The first projective space is $\Pi_V^*$ and the second can be identified with $\Pi_V^*/S$.

The distance $d(X, Y)$ between $X, Y \in G_k(V)$ in the graph $\Gamma_k(V)$ is equal to

$$k - \dim(X \cap Y).$$
and the diameter of $\Gamma_k(V)$ is equal to $\min\{k, n - k\}$. The annihilator mapping (which transfers every subspace $S \subset V$ to the annihilator $S^0 \subset V^*$) induces an isomorphism between $\Gamma_k(V)$ and $\Gamma_{n-k}(V^*)$.

2.3. Throughout the paper we will suppose that $V$ and $V'$ are left vector spaces over division rings $R$ and $R'$, respectively. An additive mapping $l : V \to V'$ is said to be semilinear if there exists a homomorphism $\sigma : R \to R'$ such that

$$l(ax) = \sigma(a)l(x)$$

for all $x \in V$ and all $a \in R$. If $l$ is non-zero then there is only one homomorphism satisfying this condition. Every non-zero homomorphism of $R$ to $R'$ is injective.

Every semilinear injection of $V$ to $V'$ induces a mapping of $G_1(V)$ to $G_1(V')$ which transfers lines of $\Pi_V$ to subsets in lines of $\Pi_{V'}$ (note that this mapping is not necessarily injective). We will use the following version of the Fundamental Theorem of Projective Geometry [4, 5, 6, see also [13, Theorem 1.4].

**Theorem 2.1** (C. A. Faure, A. Frölicher, H. Havlicek). Every mapping of $G_1(V)$ to $G_1(V')$ transferring lines of $\Pi_V$ to subsets in lines of $\Pi_{V'}$ is induced by a semilinear injection of $V$ to $V'$.

A semilinear mapping of $V$ to $V'$ is called a semilinear isomorphism if it is bijective and the associated homomorphism of $R$ to $R'$ is an isomorphism. If $v$ is a semilinear automorphism of $V$ then the mapping $u_k$ sending every $X \in G_k(V)$ to $u(X)$ is an automorphism of $\Gamma_k(V)$. If $n = 2k$ and $v : V \to V^*$ is a semilinear isomorphism then the bijection transferring every $X \in G_k(V)$ to the annihilator of $v(X)$ is an automorphism of $\Gamma_k(V)$.

**Theorem 2.2** (W. L. Chows [3]). Every automorphism of $\Gamma_k(V)$, $1 < k < n - 1$ is induced by a semilinear automorphism of $V$ or a semilinear isomorphism of $V$ to $V^*$; the second possibility can be realized only in the case when $n = 2k$.

3. General properties of embeddings

Let $k \in \{2, \ldots, n - 2\}$ and $k' \in \{2, \ldots, n' - 2\}$. Every embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ transfers maximal cliques of $\Gamma_k(V)$ to subsets in maximal cliques of $\Gamma_{k'}(V')$; moreover, every maximal clique of $\Gamma_{k'}(V')$ contains at most one image of a maximal clique of $\Gamma_k(V)$ (otherwise, the preimages of some adjacent vertices of $\Gamma_{k'}(V')$ are non-adjacent which is impossible).

**Proposition 3.1.** For any embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ the image of every maximal clique of $\Gamma_k(V)$ is contained in precisely one maximal clique of $\Gamma_{k'}(V')$.

**Proof.** Let $f$ be an embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$. Suppose that $\mathcal{X}$ is a maximal clique of $\Gamma_k(V)$ and $f(\mathcal{X})$ is contained in two distinct maximal cliques of $\Gamma_{k'}(V')$. Since the intersection of two distinct maximal cliques is empty or a one-element set or a line, there exist $S \in G_{k-1}(V')$ and $U \in G_{k'+1}(V')$ such that

$$f(\mathcal{X}) \subset [S, U]_{k'}.$$  

We take any maximal clique $\mathcal{Y} \neq \mathcal{X}$ of $\Gamma_k(V)$ which intersects $\mathcal{X}$ in a line and consider a maximal clique $\mathcal{Y}'$ of $\Gamma_{k'}(V')$ containing $f(\mathcal{Y})$. The inclusion $\mathcal{Y}'$ guarantees that the line $[S, U]_{k'}$ intersects $f(\mathcal{Y}) \subset \mathcal{Y}'$ in a set containing more than one element. Then $[S, U]_{k'} \subset \mathcal{Y}'$ (a line is contained in a maximal clique or intersects it in at most one element). So, the maximal clique $\mathcal{Y}'$ contains the images of both $\mathcal{X}$ and $\mathcal{Y}$ which are distinct maximal cliques of $\Gamma_k(V)$, a contradiction.  

It was noted above that the intersection of two distinct maximal cliques of $\Gamma_k(V)$ is empty or a one-element set or a line. The latter possibility can be realized only in the case when the maximal cliques are of different types — one of them is a star and the other is a top. For any distinct maximal cliques $X, Y$ of $\Gamma_k(V)$ there is a sequence of maximal cliques of $\Gamma_k(V)$

$$X = \mathcal{X}_0, \mathcal{X}_1, \ldots, \mathcal{X}_i = Y$$

such that $\mathcal{X}_{j-1} \cap \mathcal{X}_j$ is a line for every $j \in \{1, \ldots, i\}$. This implies the following.

**Proposition 3.2.** For every embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ one of the following possibilities is realized:

(A) stars go to subsets of stars and tops go to subsets of tops,
(B) stars go to subsets of tops and tops go to subsets of stars.

We say that an embedding is of type (A) or (B) if the corresponding possibility is realized.

If an embedding $f$ of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ is of type (A) then the embedding of $\Gamma_k(V)$ in $\Gamma_{n'−k'}(V''')$ sending every $X \in \mathcal{G}_k(V)$ to the annihilator of $f(X)$ is of type (B).

4. ISOMETRIC EMBDDBINGS

A semilinear injection of $V$ to $V'$ is said to be a semilinear $m$-embedding if the image of every independent $m$-element subset is independent. The existence of such mappings implies that $m \leq \min\{n, n'\}$. In the case when $n \leq n'$, semilinear $n$-embeddings of $V$ in $V'$ will be called semilinear embeddings.

**Remark 4.1.** By $[10]$, there exist fields $F$ and $F'$ such that for any natural numbers $p$ and $q$ there is a semilinear $p$-embedding of $F^{p+q}$ in $F'^p$. It is clear that such semilinear $p$-embeddings can not be $(p + 1)$-embeddings.

Let $l : V \to V'$ be a semilinear $m$-embedding. For every $p \in \{1, \ldots, m\}$ and every $p$-dimensional subspace $X \subset V$ the dimension of the subspace spanned by $l(X)$ is equal to $p$. So, we have the mapping

$$l_p : \mathcal{G}_p(V) \to \mathcal{G}_p(V')$$

$$X \to \langle l(X) \rangle.$$ 

By $[9]$ Proposition 2], if $2p \leq \min\{n, n'\}$ and $l : V \to V'$ is a $(2p)$-embedding then $l_p$ is an isometric embedding of $\Gamma_p(V)$ in $\Gamma_p(V')$.

Let $k \in \{2, \ldots, n - 2\}$ and $k' \in \{2, \ldots, n' - 2\}$. The existence of isometric embeddings of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ implies that

$$\min\{k, n - k\} \leq \min\{k', n' - k'\}$$

(the diameter of $\Gamma_k(V)$ is not greater than the diameter of $\Gamma_{k'}(V')$). In the next three examples we suppose that $k \leq n - k$, i.e.

$$k \leq \min\{k', n - k, n' - k'\}. \quad (4.1)$$

**Example 4.1.** Let $S \in \mathcal{G}_{n'−k}(V')$. By $[4.1]$, 

$$\dim(V'/S) = n' - k' + k \geq 2k.$$ 

If $l : V \to V'/S$ is a semilinear $(2k)$-embedding then $l_k$ is an isometric embedding of $\Gamma_k(V)$ in $\Gamma_k(V'/S)$. Let $\pi$ be the natural isometric embedding of $\Gamma_k(V'/S)$ in
\( \Gamma_k(V) \) (which transfers every \( k \)-dimensional subspace of \( V'/S \) to the corresponding \( k' \)-dimensional subspace of \( V' \)). Then \( \pi l_k \) is an isometric embedding of \( \Gamma_k(V) \) in \( \Gamma_{k'}(V') \) of type (A).

**Example 4.2.** Let \( U \in \mathcal{G}_{k'+k}(V') \) (by Example 4.1), we have \( k'+k \leq n' \). If \( v : V \to U^* \) is a semilinear \( (2k) \)-embedding then \( v_k \) is an isometric embedding of \( \Gamma_k(V) \) in \( \Gamma_k(U^*) \). By duality, it can be considered as an isometric embedding of \( \Gamma_k(V) \) in \( \Gamma_{k'}(U) \). Since \( U \) is contained in \( V' \), we get an isometric embedding of \( \Gamma_k(V) \) in \( \Gamma_{k'}(V') \) of type (B).

**Example 4.3.** Suppose that \( n = 2k \) and \( S \in \mathcal{G}_{k'-k}(V') \), \( U \in \mathcal{G}_{k'+k}(V') \) are incident. Then

\[
\dim(U/S) = 2k = n.
\]

By Example 4.1 every semilinear embedding of \( V \) in \( U/S \subset V'/S \) induces a isometric embedding of \( \Gamma_k(V) \) in \( \Gamma_{k'}(V') \). If \( w : V \to (U/S)^* \) is a semilinear embedding then \( w_k \) is an isometric embedding of \( \Gamma_k(V) \) in \( \Gamma_k((U/S)^*) \) and, by duality, it can be considered as an isometric embedding of \( \Gamma_k(V) \) in \( \Gamma_{k'}(U/S) \). As in Example 4.1 we construct an isometric embedding of \( \Gamma_k(V) \) in \( \Gamma_{k'}(V') \).

**Theorem 4.1.** Let \( f \) be an isometric embedding of \( \Gamma_k(V) \) in \( \Gamma_{k'}(V') \). If \( k \leq n - k \) then one of the following possibilities is realized:

(A) there is \( S \in \mathcal{G}_{k'-k}(V') \) such that \( f \) is induced by a semilinear \( (2k) \)-embedding of \( V \) in \( V'/S \), see Example 4.1,

(B) there is \( U \in \mathcal{G}_{k'+k}(V') \) such that \( f \) is induced by a semilinear \( (2k) \)-embedding of \( V \) in \( U' \), see Example 4.2.

In the case when \( n = 2k \), there are incident \( S \in \mathcal{G}_{k'-k}(V') \) and \( U \in \mathcal{G}_{k'+k}(V') \) such that \( f \) is induced by a semilinear embedding of \( V \) in \( U/S \) or a semilinear embedding of \( V \) in \( (U/S)^* \), see Example 4.3.

**Remark 4.2.** Suppose that \( f \) is an isometric embedding of \( \Gamma_k(V) \) in \( \Gamma_{k'}(V') \) and \( k > n - k \). Consider the mapping which sends every \( X \in \mathcal{G}_{n-k}(V^*) \) to \( f(X^0) \) (recall that \( X^0 \) is the annihilator of \( X \)). This is an isometric embedding of \( \Gamma_{n-k}(V^*) \) in \( \Gamma_{k'}(V') \). Since \( n-k < n-(n-k) \), it is induced by a semilinear \( 2(n-k) \)-embedding of \( V^* \) in one of vector spaces described above.

**Proof of Theorem 4.1.** Suppose that \( f \) is an embedding of type (A). By Subsection 3, there exists an injective mapping

\[
f_{k-1} : \mathcal{G}_{k-1}(V) \to \mathcal{G}_{k'-1}(V')
\]

such that

\[
f([X]_k) \subset [f_{k-1}(X)]_{k'} \quad \forall X \in \mathcal{G}_{k-1}(V).
\]

Then

\[
f_{k-1}([Y]_{k-1}) \subset [f(Y)]_{k'-1} \quad \forall Y \in \mathcal{G}_k(V).
\]

Since for any two adjacent vertices there is a top containing them, the latter inclusion implies that \( f_{k-1} \) is adjacency preserving. Thus for any \( X, Y \in \mathcal{G}_{k-1}(V) \) we have

\[
d(X,Y) \geq d(f_{k-1}(X), f_{k-1}(Y)).
\]

We prove the inverse inequality.

The condition \( 2k \leq n \) implies the existence of \( X', Y' \in \mathcal{G}_k(V) \) such that \( X \subset X' \), \( Y \subset Y' \) and

\[
X \cap Y = X' \cap Y'.
\]
Then
\[ (4.2) \quad d(X, Y) = d(X', Y') - 1 \]
(Indeed, \( d(X, Y) = k - 1 - \dim(X \cap Y) = k - 1 - \dim(X' \cap Y') = d(X', Y') - 1 \)).

Since \( f_{k-1} \) is induced by \( f \), we have
\[ f_{k-1}(X) \subset f(X') \quad \text{and} \quad f_{k-1}(Y) \subset f(Y') \]
which guarantees that
\[ (4.3) \quad \dim(f_{k-1}(X) \cap f_{k-1}(Y)) \leq \dim(f(X') \cap f(Y')). \]

Using (4.2) and (4.3), we get the following
\[ d(X, Y) = d(X', Y') - 1 = d(f(X'), f(Y')) - 1 = k' - 1 - \dim(f(X') \cap f(Y')) \leq \]
\[ k' - 1 - \dim(f_{k-1}(X) \cap f_{k-1}(Y)) = d(f_{k-1}(X), f_{k-1}(Y)). \]

So, \( f_{k-1} \) is an isometric embedding of \( \Gamma_{k-1}(V) \) in \( \Gamma_{k'}(V') \). This is an embedding of type (A) (it was established above that \( f_{k-1} \) sends tops to subsets of tops).

Step by step, we construct a sequence of isometric embeddings
\[ f_i : \mathcal{G}_i(V) \to \mathcal{G}_{k'-k+i}(V'), \quad i = k, \ldots, 1, \]
of \( \Gamma_i(V) \) in \( \Gamma_{k-k+i}(V') \) such that \( f_k = f \) and we have
\[ f_i([X_i]) \subset [f_{i-1}(X)]_{k'-k+i} \quad \forall \ X \in \mathcal{G}_{i-1}(V) \]
and
\[ (4.4) \quad f_{i-1}([X_{i-1}]) \subset \langle f_i(Y) \rangle_{k'-k+i-1} \quad \forall \ Y \in \mathcal{G}_i(V) \]
if \( i > 1 \).

The image of \( f_1 \) is a clique of \( \Gamma_{k'-k+1}(V') \). This clique can not be contained in any top (otherwise, there is \( X' \in \mathcal{G}_{k'-k+2}(V') \) such that \( f_2(X) = X' \) for every \( X \in \mathcal{G}_2(V) \) and \( f_2 \) is not injective). Therefore, there is \( S \in \mathcal{G}_{k'-k}(V') \) such that the image of \( f_1 \) is contained in the star \( [S]_{k'-k+1} \).

By (4.4), \( f_1 \) transfers lines of \( \Pi_V \) to subsets of lines contained in \( [S]_{k'-k+1} \). It was noted in Subsection 2.2 that the star \( [S]_{k'-k+1} \) (together with all lines contained in it) can be identified with the projective space \( \Pi_{V'/S} \). Theorem 2.1 shows that \( f_1 \) is induced by a semilinear injection \( l : V \to V'/S \).

Using (4.4), we establish that
\[ f_1([X_1]) \subset \langle f(X) \rangle_{k'-k+1} \quad \forall \ X \in \mathcal{G}_k(V). \]

On the other hand,
\[ f_1([X_1]) \subset \langle \pi((l(X))) \rangle_{k'-k+1} \quad \forall \ X \in \mathcal{G}_k(V), \]
where \( \pi \) is the mapping which transfers every subspace of \( V'/S \) to the corresponding subspace of \( V' \). Since the intersection of two distinct tops contains at most one element, we get
\[ f(X) = \pi((l(X))) \quad \forall \ X \in \mathcal{G}_k(V) \]
which means that \( f = \pi l_k \).

Every independent \((2k)\)-element subset \( A \subset V \) can be presented as the disjoint union of two independent \( k \)-element subsets \( A_1 \) and \( A_2 \). Then
\[ d(f([A_1]), f([A_2])) = d([A_1], [A_2]) = k \]
which means that \( (l(A_1)) \) and \( (l(A_2)) \) are \( k \)-dimensional subspaces of \( V'/S \) intersecting in 0. Hence the subspace spanned by \( l(A_1 \cup A_2) = l(A) \) is \((2k)\)-dimensional.
So, \( l \) is a semilinear \((2k)\)-embedding of \( V \) in \( V'/S \) and \( f \) is as in Example 4.1. In the case when \( n = 2k \), \( l \) is a semilinear embedding and the image of \( l \) is contained in \( U/S \), where \( U \in \mathcal{G}_{k'+k}(V') \).

Now suppose that \( f \) is an embedding of type (B). By duality, \( f \) can be considered as an isometric embedding of \( \Gamma_k(V) \) in \( \Gamma_{n'-k'}(V'') \) (this embedding sends every \( X \in \mathcal{G}_k(V) \) to the annihilator of \( f(X) \)). We get an embedding of type (A) and its image is contained in
\[
[S']_{n'-k'}, \ S' \in \mathcal{G}_{n'-k'-k}(V'');
\]
in the case when \( n = 2k \), the image is contained in
\[
[S', U']_{n'-k'}, \ S' \in \mathcal{G}_{n'-k'-k}(V''), \ U' \in \mathcal{G}_{n'-k'+k}(V'').
\]
The image of \( f \) is contained in \( \langle U \rangle_{k'} \), where \( U \in \mathcal{G}_{k'+k}(V') \) is the annihilator of \( S' \). Thus \( f \) is an isometric embedding of \( \Gamma_k(V) \) in \( \Gamma_k(U) \). By duality, \( f \) can be considered as an isometric embedding of \( \Gamma_k(V) \) in \( \Gamma_k(U^*) \) of type (A). Hence it is induced by a semilinear \((2k)\)-embedding of \( V \) in \( U^* \), i.e. \( f \) is as in Example 4.2.

If \( n = 2k \) then the image of \( f \) is contained in \( [S, U]_{k'} \), where \( S \in \mathcal{G}_{k'-k}(V') \) and \( U \in \mathcal{G}_{k'+k}(V') \) are the annihilators of \( U' \) and \( S' \), respectively. This means that \( f = \pi f' \), where \( f' \) is an isometric embedding of \( \Gamma_k(V) \) in \( \Gamma_k(U/S) \) of type (B) and \( \pi \) is the mapping which transfers every subspace of \( V'/S \) to the corresponding subspace of \( V' \). Since
\[
\dim(U/S) = 2k,
\]
\( f' \) can be considered as an isometric embedding of \( \Gamma_k(V) \) in \( \Gamma_k((U/S)^*) \) of type (A). The latter embedding is induced by a semilinear embedding of \( V \) in \( (U/S)^* \) and \( f \) is as in Example 4.3.

**Remark 4.3.** Using the same idea, the author describes the images of isometric embeddings of Johnson graphs in Grassmann graphs [14, Theorem 4].

5. Characterization of semilinear embeddings

**Theorem 5.1.** Let \( l : V \to V' \) be a semilinear injection. Then \( l \) is a semilinear embedding if and only if for every linear automorphism \( u \in \text{GL}(V) \) there is a linear automorphism \( u' \in \text{GL}(V') \) such that the diagram
\[
\begin{array}{ccc}
V & \xrightarrow{l} & V' \\
\downarrow u & & \downarrow u' \\
V & \xrightarrow{u} & V'
\end{array}
\]
is commutative.

To prove Theorem 5.1, we use the following result.

**Theorem 5.2** (M. Pankov [14]). For a finite subset \( \mathcal{X} \subset \mathcal{G}_1(V) \) the following conditions are equivalent:
- every permutation on \( \mathcal{X} \) is induced by a semilinear automorphism of \( V \),
- \( \mathcal{X} \) is a simplex or an independent subset of \( \Pi_V \).

Recall that \( Q_1, \ldots, Q_m \in \mathcal{G}_1(V) \) form an independent subset of \( \Pi_V \) if non-zero vectors \( x_1 \in Q_1, \ldots, x_m \in Q_m \) form an independent subset of \( V \). An \((m + 1)\)-element subset \( \mathcal{X} \subset \mathcal{G}_1(V) \) is called an \( m \)-simplex of \( \Pi_V \) if it is not independent and every \( m \)-element subset of \( \mathcal{X} \) is independent [1 Section III.3].
Remark 5.1. If \( x_1, \ldots, x_m \in V \) and \( \langle x_1, \ldots, x_m \rangle \) form an \((m - 1)\)-simplex then \( x_1, \ldots, x_{m-1} \) are linearly independent and \( x_m = \sum_{i=1}^{m-1} a_i x_i \), where every scalar \( a_i \) is non-zero.

Proof of Theorem 5.1. Suppose that \( l : V \to V' \) is a semilinear embedding. Let \( \{x_i\}_{i=1}^n \) be a base of \( V \). For every vector \( x = \sum_{i=1}^n a_i x_i \) and every linear automorphism \( u \in \text{GL}(V) \) we have
\[
l(x) = \sum_{i=1}^n \sigma(a_i)l(x_i) \quad \text{and} \quad l(u(x)) = \sum_{i=1}^n \sigma(a_i)l(u(x_i)),
\]
where \( \sigma : R \to R' \) is the homomorphism associated with \( l \). Since \( \{l(x_i)\}_{i=1}^n \) and \( \{l(u(x_i))\}_{i=1}^n \) both are independent subsets of \( V' \), the diagram (5.1) is commutative for any linear automorphism \( u' \in \text{GL}(V') \) transferring every \( l(x_i) \) to \( l(u(x_i)) \).

Conversely, suppose that for every linear automorphism \( u \in \text{GL}(V) \) there is a linear automorphism \( u' \in \text{GL}(V') \) such that the diagram (5.1) is commutative. Let \( B = \{x_i\}_{i=1}^n \) be a base of \( V \). Every permutation on the associated base of \( \Pi_V \) is induced by a linear automorphism of \( V \). Then, by our assumption, every permutation on the set
\[
\mathcal{X}(B) := \{\langle l(x_i) \rangle\}_{i=1}^n
\]
is induced by a linear automorphism of \( V' \). Theorem 5.1 implies that \( \mathcal{X}(B) \) is an \((n-1)\)-simplex or an independent subset of \( \Pi_{V'} \). In the second case, \( l(x_1), \ldots, l(x_n) \) are linearly independent and \( l \) is a semilinear embedding.

If \( \mathcal{X}(B) \) is an \((n-1)\)-simplex then \( \mathcal{X}(B') \) is an \((n-1)\)-simplex for every base \( B' \subset V \) (indeed, if there is a base \( B' \subset V \) such that \( \mathcal{X}(B') \) is an independent subset of \( \Pi_{V'} \) then \( l \) is a semilinear embedding and \( \mathcal{X}(B') \) is an independent subset for every base \( B' \subset V \)). We need to show that this possibility can not be realized.

So, suppose that \( \mathcal{X}(B) \) is an \((n-1)\)-simplex. By Remark 5.1 \( \{l(x_i)\}_{i=1}^{n-1} \) is an independent subset of \( V' \) and
\[
l(x_n) = \sum_{i=1}^{n-1} a_i l(x_i),
\]
where every scalar \( a_i \) is non-zero. Let \( u \) be an automorphism of \( V \). Then \( u(B) \) is a base of \( V \) and \( \mathcal{X}(u(B)) \) is an \((n-1)\)-simplex. The latter implies that \( \{l(u(x_i))\}_{i=1}^{n-1} \) is an independent subset of \( V' \) and
\[
l(u(x_n)) = \sum_{i=1}^{n-1} b_i l(u(x_i)),
\]
where every scalar \( b_i \) is non-zero. If \( u' \) is a linear automorphism of \( V' \) such that the diagram (5.1) is commutative then it transfers every \( l(x_i) \) to \( l(u(x_i)) \) and (5.2), (5.3) show that \( a_i = b_i \) for every \( i \), i.e.
\[
l(u(x_n)) = \sum_{i=1}^{n-1} a_i l(u(x_i)).
\]
Consider a linear automorphism \( v \in \text{GL}(V) \) such that \( u(x_i) = v(x_i) \) if \( i \leq n - 1 \) and \( u(x_n) \neq v(x_n) \). By the arguments given above,
\[
l(v(x_n)) = \sum_{i=1}^{n-1} a_i l(v(x_i)) = \sum_{i=1}^{n-1} a_i l(u(x_i)) = l(u(x_n))
\]
which contradicts the injectivity of $l$. \hfill \Box

6. $l$-Rigid Isometric Embeddings

As above, we suppose that $k \in \{2, \ldots, n - 2\}$ and $k' \in \{2, \ldots, n' - 2\}$. Let $f$ be an embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$. We say that $f$ is $l$-rigid if for every linear automorphism $u \in \text{GL}(V)$ there is a linear automorphism $u' \in \text{GL}(V')$ such that the diagram

$$
\begin{array}{ccc}
\Gamma_k(V) & \xrightarrow{f} & \Gamma_{k'}(V') \\
\downarrow u_k & & \downarrow u'_{k'} \\
\Gamma_k(V) & \xrightarrow{f} & \Gamma_{k'}(V')
\end{array}
$$

is commutative, i.e. every automorphism of $\Gamma_k(V)$ induced by a linear automorphism of $V$ can be extended to the automorphism of $\Gamma_{k'}(V')$ induced by a linear automorphism of $V'$.

**Example 6.1.** Suppose that $n = 2k$ and $V$ is a subspace of $V'$. We also require that the associated division ring is isomorphic to the opposite division ring. This implies the existence of semilinear isomorphisms of $V$ to $V^*$. Then the natural embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ is not rigid, since every automorphism of $\Gamma_k(V)$ induced by a semilinear isomorphism of $V$ to $V^*$ can not be extended to an automorphism of $\Gamma_{k'}(V')$. It is clear that this embedding is $l$-rigid.

Let $u$ be a linear automorphism of $V$ and let $u^*$ be the adjoint linear automorphism of $V^*$. The linear automorphism $\tilde{u} := (u^*)^{-1}$ is called the contragradient of $u$. It transfers the annihilator of every subspace $S \subset V$ to the annihilator of $u(S)$ \cite{13}. Section 1.3.3.

**Lemma 6.1.** If $f$ is an $l$-rigid embedding of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$ then the same holds for the following two embeddings:

- the embedding of $\Gamma_k(V)$ in $\Gamma_{n' - k'}(V'^*)$ transferring every $X \in \mathcal{G}_k(V)$ to the annihilator of $f(X)$,
- the embedding of $\Gamma_{n - k}(V^*)$ in $\Gamma_{k'}(V')$ transferring every $X \in \mathcal{G}_{n - k}(V^*)$ to $f(X^0)$.

**Proof.** Let $u$ be a linear automorphism of $V$. Then $u_k$ is an automorphism of $\Gamma_k(V)$. Suppose that it is extendable to the automorphism of $\Gamma_{k'}(V')$ induced by a linear automorphism $v \in \text{GL}(V')$. Then $\tilde{v}$ defines the extension of $u_k$ to an automorphism of $\Gamma_{n' - k'}(V'^*)$.

Consider the second embedding. The contragradient $\tilde{u}$ is a linear automorphism of $V^*$ and $\tilde{u}_{n - k}$ is an automorphism of $\Gamma_{n - k}(V^*)$. It is clear that $\tilde{u}_{n - k}$ is extendable to the automorphism of $\Gamma_{k'}(V')$ induced by $v$. This completes our proof, since every linear automorphism of $V^*$ is the contragradient of a linear automorphism of $V$. \hfill \Box

We give two examples of $l$-rigid isometric embeddings of $\Gamma_k(V)$ in $\Gamma_{k'}(V')$. In contrast with the previous section, we do not require that $k \leq n - k$.

**Example 6.2.** Suppose that $k \leq k'$ and $n - k \leq n' - k'$. If $S \in \mathcal{G}_{k' - k}(V')$ then $\dim V'/S = n' - k' + k \geq n$. 

For every semilinear embedding \( l : V \to V' / S \) the mapping \( l_k \) is an \( l \)-rigid isometric embedding of \( \Gamma_k(V) \) in \( \Gamma_k(V' / S) \). As in the previous section, we denote by \( \pi \) the mapping which transfers every subspace of \( V' / S \) to the corresponding subspace of \( V' \). Then \( \pi l_k \) is an \( l \)-rigid isometric embedding of \( \Gamma_k(V) \) in \( \Gamma_{k'}(V') \) of type (A).

**Example 6.3.** Suppose that
\[
0 < k + k' 
\]
and \( U \in G_{k + k'}(V') \). Let \( v : V \to U^* \) be a semilinear embedding. Then \( v_k \) is an \( l \)-rigid isometric embedding of \( \Gamma_k(V) \) in \( \Gamma_k(U^*) \). By duality, it can be considered as an isometric embedding of \( \Gamma_k(V) \) in \( \Gamma_k(U) \). Lemma 5.1 guarantees that the latter embedding is \( l \)-rigid. Since \( U \) is contained in \( V' \), we get an \( l \)-rigid isometric embedding of \( \Gamma_k(V) \) in \( \Gamma_{k'}(V') \) of type (B).

It follows from Theorem 6.1 and the examples given above that **every isometric embedding of \( \Gamma_k(V) \) in \( \Gamma_{k'}(V') \) is \( l \)-rigid if \( n = 2k \).**

**Example 6.4.** Suppose that \( R = F \) and \( R' = F' \), where \( F \) and \( F' \) are the fields from Remark 4.1 and \( n > n' \). Let \( l : V \to V' \) be a semilinear \( n' \)-embedding. If \( n' \geq 2k \) then \( l_k \) is an isometric embedding of \( \Gamma_k(V) \) in \( \Gamma_k(V') \). By Theorem 6.1 this embedding is not \( l \)-rigid.

**Proof of Theorem 6.1.** Let \( f \) be an \( l \)-rigid isometric embedding of \( \Gamma_k(V) \) in \( \Gamma_{k'}(V') \).

**Case** \( k < n - k \).

If \( f \) is an embedding of type (A) then, by Theorem 6.4, there exist \( S \in G_{k' - k}(V') \) and \( \Gamma_k(V) \) and a semilinear \((2k)\)-embedding \( l : V \to V' / S \) such that \( f \) transfers every \( X \in \Gamma_k(V) \) to the subspace of \( V' \) corresponding to \( \langle l(X) \rangle \). Since \( f \) is \( l \)-rigid, \( l_k \) is \( l \)-rigid and Theorem 5.1 implies that \( l \) is a semilinear embedding.

In the case when \( f \) is an embedding of type (B), Theorem 6.4 implies the existence of \( U \in G_{k' + k}(V') \) and a semilinear \((2k)\)-embedding \( v : V \to U^* \) such that \( f \) transfers every \( X \in \Gamma_k(V) \) to the annihilator of \( \langle v(X) \rangle \) in \( U \). Since \( f \) is an \( l \)-rigid embedding of \( \Gamma_k(V) \) in \( \Gamma_{k'}(V') \), Lemma 6.1 implies that \( v_k \) is \( l \)-rigid. By Theorem 5.1 \( v \) is a semilinear embedding.

**Case** \( k > n - k \).

Let \( f \) be an embedding of type (A). Consider the embedding of \( \Gamma_{n - k}(V^*) \) in \( \Gamma_{k'}(V') \) transferring every \( X \in G_{n - k}(V^*) \) to \( f(X^0) \). By Lemma 6.1 this embedding is \( l \)-rigid; moreover, it is an embedding of type (B). Therefore, there exist \( U \in G_{k' + n - k}(V') \) and a semilinear embedding \( v : V^* \to U^* \) such that \( f \) transfers every \( X \in \Gamma_k(V) \) to the annihilator of \( \langle v(X^0) \rangle \) in \( U \). Since \( v \) is a semilinear embedding, the image of \( v \) spans an \( n \)-dimensional subspace of \( U^* \). Denote by \( S \) the annihilator of this subspace in \( U \). Then \( S \in G_{k' - k}(V') \) and the image of \( f \) is contained in \( \langle S \rangle_k \).
Consider the mapping $g$ which sends every subspace $X \subset V$ to the annihilator of $\langle v(X^0) \rangle$ in $U$. This is an injection of the set of all subspaces of $V$ to the set of all subspaces of $U$ containing $S$. The image of every $G_i(V)$ is contained in $\langle S \rangle_{k' - k + 1}$ and the restriction of $g$ to $G_k(V)$ coincides with $f$. The mapping $g$ is inclusions preserving: for any subspaces $X, Y \subset V$

$$X \subset Y \implies g(X) \subset g(Y).$$

This implies that the restriction of $g$ to $G_1(V)$ transfers every line of $\Pi_V$ to a subset in a line of the projective space $\langle S \rangle_{k' - k + 1}$. By Theorem 2.1 this restriction is induced by a semilinear injection $l : V \to V'/S$. We need to show that for every $X \in G_k(V)$

$$g(X) = \pi(\langle l(X) \rangle),$$

where $\pi$ transfers every subspace of $V'/S$ to the corresponding subspace of $V'$.

Since $v$ is a semilinear embedding, $g$ sends every base of $\Pi_V$ to an independent subset of the projective space $\langle S \rangle_{k' - k + 1}$. This implies that $l$ is a semilinear embedding. Let $X \in G_k(V)$. We take $P_1, \ldots, P_k \in G_1(V)$ such that

$$X = P_1 + \cdots + P_k.$$ 

Then

$$\pi(\langle l(X) \rangle) = g(P_1) + \cdots + g(P_k) \subset g(X)$$

and we get the required equality (since our subspaces both are $k'$-dimensional).

Now, let $f$ be an embedding of type (B). We consider $f$ as an embedding of $\Gamma_k(V)$ in $\Gamma_{n' - k'}(V'^*)$. This is an embedding of type (A) and, as in the proof of Theorem 4.1 we establish that the image of $f$ is contained in

$$\langle U \rangle_{k'}, \ U \in G_{k' + k}(V').$$

So, $f$ is an $l$-rigid isometric embedding of $\Gamma_k(V)$ in $\Gamma_k(U')$. By duality, it can be considered as an isometric embedding of $\Gamma_k(V)$ in $\Gamma_k(U^*)$ of type (A). Lemma 6.1 implies that the latter embedding is $l$-rigid. Thus it is induced by a semilinear embedding of $V$ in $U^*$.  

\[\Box\]

References

[1] Baer R., Linear algebra and projective geometry, Academic Press, New-York 1952.
[2] Blunck A., Havlicek H., On bijections that preserve complementarity of subspaces, Discrete Math. 301(2005), 46–56.
[3] Chow W.L., On the geometry of algebraic homogeneous spaces, Ann. of Math. 50(1949), 32–67.
[4] Faure C. A., Frölicher A., Morphisms of Projective Geometries and Semilinear maps, Geom. Dedicata, 53 (1994), 237–262.
[5] Faure C. A., An Elementary Proof of the Fundamental Theorem of Projective Geometry, Geom. Dedicata, 90 (2002), 145–151.
[6] Havlicek H., A Generalization of Brauner’s Theorem on Linear Mappings, Mitt. Math. Sem. Univ. Giessen, 215 (1994), 27–41.
[7] Huang W.-L., Adjacency preserving transformations of Grassmann spaces, Abh. Math. Sem. Univ. Hamburg 68(1998), pp. 65–77.
[8] Huang W.-L., Havlicek H., Diameter preserving surjections in the geometry of matrices, Linear Algebra Appl. 429(2008), 376–386.
[9] Kosioruk J., Matras A., Pankov M., Distance preserving mappings of Grassmann graphs, Beiträge Algebra Geom. 49(2008), 233–242.
[10] Kreuzer A., Projective embeddings of projective spaces, Bull. Belg. Math. Soc. Simon Stevin 5(1998), 363–372.
[11] Lim M. H., *Surjections on Grassmannians preserving pairs of elements with bounded distance*. Linear Algebra Appl. 432(2010), 1703–1707.

[12] Pankov M., *Chow's theorem and projective polarities*, Geom. Dedicata 107(2004), 17–24.

[13] Pankov M., *Grassmannians of classical buildings*, Algebra and Discrete Math. Series 2, World Scientific, 2010.

[14] Pankov M., *Isometric embeddings of Johnson graphs in Grassmann graphs*, J. Algebraic Combin. 33(2011), 555–570.

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