Reasoning about proof and knowledge

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Abstract

In previous work [Lewitzka, 2017], we presented a hierarchy of classical modal logics, along with algebraic semantics, for the reasoning about intuitionistic truth (i.e. proof), belief and knowledge. Interpreting \( \Box \) as a proof predicate, the systems also express properties of intuitionistic belief and knowledge established in [Artemov and Protopopescu, 2016] where epistemic principles are in line with Brouwer-Heyting-Kolmogorov (BHK) semantics. In this article, we further develop our approach and show that the S5-style systems of our hierarchy are complete w.r.t. a relational semantics based on intuitionistic general frames. This result can be seen as a formal justification of our modal axioms as adequate principles for the reasoning about proof combined with belief and knowledge. In fact, the semantics turns out to be a uniform framework able to describe also the intuitionistic epistemic logics of [Artemov and Protopopescu, 2016]. The relationship between intuitionistic epistemic principles and their representation by modal laws of our classical logics becomes explicit.

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1 Introduction

In \cite{5, 6, 7, 8} we investigated Lewis-style modal logics which are strong enough to interpret strict equivalence $\Box(\varphi \leftrightarrow \psi)$ as propositional identity $\varphi \equiv \psi$ in the sense that $\varphi \equiv \psi := \Box(\varphi \leftrightarrow \psi)$ satisfies the identity axioms of R. Suszko’s non-Fregean logic \cite{3} adapted to the language of modal logic\footnote{Read $\varphi \equiv \psi$ as “$\varphi$ and $\psi$ have the same meaning” or “$\varphi$ and $\psi$ denote the same proposition”.}. Modal logics satisfying that condition, such as S3 and some others in the vicinity of S1, have a natural algebraic semantics (see \cite{5}). Proceeding from the assumption that the axioms of strict implication express certain intuitive aspects of constructive reasoning, we introduced in \cite{7} the Lewis-style modal system L which contains a copy of intuitionistic propositional logic via the map $\varphi \mapsto \Box \varphi$ and combines, in a precise sense, classical and intuitionistic propositional logic (CPC and IPC, respectively). L is a classical modal system for the reasoning about intuitionistic truth.\footnote{We shall use the term “classical”, in the sense that the principle of tertium non datur is valid.} We shall work here with the slightly stronger (and technically more comfortable) logic $L_3$ based on Lewis system S3. Adding further axioms of strict implication coming from S4 and S5, we obtain a hierarchy of modal logics $L_3$–$L_5$ with principles of increasing strength for the reasoning about “proof”. Inspired by Intuitionistic Epistemic Logic, introduced by Artemov and Protopopescu \cite{2}, we study in \cite{8} epistemic extensions of those modal logics. Intuitionistic Epistemic Logic relies on the intuition that proof (i.e. intuitionistic truth) yields belief and knowledge. This is the principle of co-reflection, axiomatized by $\varphi \rightarrow K \varphi$. Reflection $K \varphi \rightarrow \varphi$, read “known propositions are (classically) true”, is a property of classical knowledge. In the intuitionistic setting, that principle is replaced by intuitionistic reflection $K \varphi \rightarrow \neg \neg \varphi$, read “known propositions cannot be false”.\footnote{Semantically, we have an extensional view on a modality, i.e. we represent a modality, such as truth or knowledge, as a set (property) of propositions of a model-theoretic propositional universe. The modality of classical knowledge is factive in the sense that it presupposes facticity (here: classical truth) of its elements, i.e. all known propositions are elements of the set of classically true propositions, which is logically expressed by $K \varphi \rightarrow \varphi$.}

Intuitionistic Epistemic Logic is in line with the Brouwer-Heyting-Kolmogorov (BHK) semantics of intuitionistic logic. If our classical modal logics $L_3$–$L_5$ are intended as systems for the reasoning about intuitionistic truth, then one may expect that their epistemic extensions reflect in some way the intuitionistic epistemic principles established in \cite{2}. In fact, the principle of (intuitionistic) reflection $K \varphi \rightarrow \neg \neg \varphi$ can be adopted directly. However, reflection together with tertium non datur and intuitionistic co-reflection $\varphi \rightarrow K \varphi$ would imply the equivalence
of knowledge and classical truth: $\varphi \leftrightarrow K\varphi$. That is, knowledge becomes classical truth whenever tertium non datur is added to intuitionistic epistemic logic. So if we want to use a classical modal logic to speak about proof and knowledge, intuitionistic co-reflection must be replaced by a weaker principle. We adopt the following classical version of co-reflection: $\Box \varphi \rightarrow \Box K\varphi$, “if there is a proof of $\varphi$, then there is a proof of $K\varphi$”, a statement that follows from the BHK interpretation of original intuitionistic co-reflection $\varphi \rightarrow K\varphi$.

All our modal systems and epistemic extensions are complete w.r.t. algebraic semantics based on special Heyting algebras (see [7]). Although Heyting algebras are models of intuitionistic logic, they hardly can be regarded as intuitive models of informal BHK interpretation. The question arises if there is a relational semantics based on intuitionistic frames that interprets the modal axioms in accordance with the informal BHK reading. Soundness w.r.t. such a semantics would prove that all our (modal) axioms are “intuitionistically acceptable” (i.e. they are true at each world of a frame), while completeness would provide a formal justification for the adequacy of our modal axioms as principles for the reasoning about proof in the sense of BHK interpretation. Are all our axioms justifiable by informal BHK semantics? Which logic of our hierarchy is strong enough to represent an adequate system for the reasoning about proof and (intuitionistic) epistemic concepts? Do we need other or stronger modal principles? A frame-based BHK-compatible semantics could help to give answers to these questions.

A main result of this paper is the development of a relational semantics, based on intuitionistic general frames, for those of our logics which contain the modal axioms of S5. We were unable to find any kind of possible worlds semantics for the weaker systems corresponding to S3 or S4. This suggests that the S5 modal axioms together with the intuitionistic disjunction law $\Box (\varphi \lor \psi) \rightarrow (\Box \varphi \lor \Box \psi)$ and the bridge axiom of co-reflection form a complete basis for the reasoning about proof and knowledge justifiable by BHK semantics. Moreover, it turns out that the intuitionistic epistemic logics of [2] can be described within our framework of relational semantics, too. In this sense, our S5-style modal systems and the intuitionistic epistemic logics of [2] are semantically compatible and can be compared with each other using the same semantic terms. The uniform semantic framework reveals the precise relationship between intuitionistic knowledge formalized in [2] and the kind of knowledge formalized in our S5-style modal logics.

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4When we speak of “relational semantics”, we always mean here a relational semantics based on intuitionistic general frames.
2 Axiomatization and Algebraic Semantics

The propositional modal epistemic language is inductively defined over an infinite set of variables $x_0, x_1, \ldots$, logical connectives $\land, \lor, \rightarrow, \bot$, the modal operator $\square$ and the epistemic operator $K$. $Fm$ is the set of all formulas, and $Fm_0 \subseteq Fm$ is the language of classical propositional logic, i.e. the set of those formulas of $Fm$ that contain neither the modal operator $\square$ nor the epistemic operator $K$. Finally, by $Fm_1$ we denote the set of those formulas which may contain the modal operator but not the epistemic operator. By $\varphi[x := \psi]$ we denote the formula that results from substituting all occurrences of variable $x$ in formula $\varphi$ with formula $\psi$. This notion of substitution can be formally defined by induction on the complexity of $\varphi$. Furthermore, we shall use the following abbreviations:

\[
\neg \varphi := \varphi \rightarrow \bot \\
\top := \neg \bot \\
\varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \land (\psi \rightarrow \varphi) \\
\varphi \equiv \psi := \square(\varphi \rightarrow \psi) \land \square(\psi \rightarrow \varphi) \quad \text{("propositional identity = strict equivalence")}
\]

\[
\square \Phi := \{ \square \psi \mid \psi \in \Phi \}, \text{ for } \Phi \subseteq Fm \\
\Diamond \varphi := \neg \square \neg \varphi
\]

We consider the following list of Axiom Schemes

(INT) all theorems of IPC and their substitution-instances\(^5\)

(A1) $\square(\varphi \lor \psi) \rightarrow (\square \varphi \lor \square \psi)$ (disjunction property)

(A2) $\square \varphi \rightarrow \varphi$

(A3) $\square(\varphi \rightarrow \psi) \rightarrow \square(\square \varphi \rightarrow \square \psi)$

(A4) $\square \varphi \rightarrow \square \square \varphi$

(A5) $\neg \square \varphi \rightarrow \square \neg \square \varphi$

(KBel) $K(\varphi \rightarrow \psi) \rightarrow (K \varphi \rightarrow K \psi)$ (distribution of belief)

(CoRe) $\square \varphi \rightarrow \square K \varphi$ (co-reflection)

(IntRe) $K \varphi \rightarrow \neg \neg \varphi$ (intuitionistic reflection)

(E4) $K \varphi \rightarrow KK \varphi$ (positive introspection)

(E5) $\neg K \varphi \rightarrow K \neg K \varphi$ (negative introspection)

(PNB) $K \varphi \rightarrow \square K \varphi$ (positive necessitation of belief)

(NNB) $\neg K \varphi \rightarrow \square \neg K \varphi$ (negative necessitation of belief)

\(^{5}\)A substitution-instance of $\varphi$ is the result of uniformly replacing variables in $\varphi$ by formulas of $Fm$. 

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and the following **Theorem Scheme** (TND) of *tertium non datur*

(TND) \( \varphi \lor \neg \varphi \)

The inference rules are Modus Ponens MP “From \( \varphi \) and \( \varphi \rightarrow \psi \) infer \( \psi \)”, and Axiom Necessitation AN “If \( \varphi \) is an axiom, then infer \( \Box \varphi \)".

The Substitution Principle SP is the following scheme:

(1) \[ (\varphi \equiv \psi) \rightarrow (\chi[x := \varphi] \equiv \chi[x := \psi]). \]

It turned out in previous research (see [5, 6, 7]) that SP is of particular interest for the study of Lewis-style modal logics. The systems S1–S3 were originally proposed by C.I. Lewis as formalizations of the concept of *strict implication* \( \Box(\varphi \rightarrow \psi) \) (see, e.g., [4] for a discussion). We propose an interpretation of *strict equivalence* \( \Box(\varphi \leftrightarrow \psi) \) as *propositional identity* \( \varphi \equiv \psi \) in the sense that the following identity axioms, inspired by R. Suszko’s *non-Fregean logic* (see, e.g, [3]), are satisfied:

(Id1) \( \varphi \equiv \varphi \)
(Id2) \( \varphi \equiv \psi \) \( \rightarrow \) \( \varphi \leftrightarrow \psi \)
(Id3) \( \varphi \equiv \psi \) \( \rightarrow \) \( (\chi[x := \varphi] \equiv \chi[x := \psi]) \)

Note that (Id1) and (Id2) are satisfied in all Lewis modal systems S1–S5. We showed in [5, 6] that S3 is the weakest Lewis modal system that also satisfies (Id3), i.e. Substitution Principle SP. This means that in S3–S5, strict equivalence can be regarded as propositional identity axiomatized by (Id1)–(Id3). Moreover, we showed in [5] that S1+SP, i.e. the system that results from S1 by adding SP as a scheme of additional theorems, has a natural algebraic semantics which, by imposing appropriate semantic conditions, extends to semantics for S3–S5, respectively.

The Transitivity Axiom (2) holds in all Lewis modal systems:

(2) \[ \Box(\varphi \rightarrow \psi) \rightarrow (\Box(\psi \rightarrow \chi) \rightarrow \Box(\varphi \rightarrow \chi)). \]

In the following, we present a hierarchy of modal systems for the reasoning about intuitionistic truth (interpreted as “proof”), belief and knowledge. Each system extends classical propositional logic CPC.
• The weakest system in the hierarchy is modal logic $L$ which is formalized in the object language $Fm_1$, i.e. the modal language without epistemic operator $K$. $L$ is given by the axiom schemes (INT), (A1), (A2) and (2), the two theorems schemes (TND) and SP and the rules of MP and AN.

• Modal system $L3$ is given by the axiom schemes (INT), (A1), (A2), (A3), the unique theorem scheme (TND) and the rules of MP and AN.

• Now, we consider the full language $Fm$ and define epistemic and modal extensions of $L3$ by adding further axiom schemes:

- $EL3^- = L3 + (KBel) + (CoRe)$
- $ELA^- = EL3^- + (A4)$
- $EL5^- = ELA^- + (A5)$
- $E4Ln^- = ELn^- + (E4), n \in \{3, 4, 5\}$
- $E5Ln^- = E4Ln^- + (E5), n \in \{3, 4, 5\}$
- $E6Ln^- = ELn^- + (PNB) + (NNB), n \in \{3, 4, 5\}$
- $ELn = ELn^- + (IntRe), n \in \{3, 4, 5\}$
- $EkLn = EkLn^- + (IntRe), k \in \{4, 5, 6\}$ and $n \in \{3, 4, 5\}$.

Systems containing axiom scheme (IntRe) are regarded as logics of knowledge and systems without that scheme are logics of belief.

In a first step, our basic modal logic $L3$ is extended to the basic epistemic logics $EL3^-$ and $EL3$. These contain only the basic epistemic axiom of distribution and the bridge axiom of co-reflection; the basic logic of knowledge $EL3$ contains additionally the axiom of intuitionistic reflection. Now, there are two ways to augment these two basic epistemic logics with stronger principles: we may add stronger epistemic principles, and we may add stronger modal laws. This situation is reflected in the notations. In $ELn^-, ELn, EkLn^-, EkLn, k = 4, 5, n = 3, 4, 5$, the index $k$ refers to corresponding epistemic extensions whereas index $n$ refers to modal extensions corresponding to S3–S5. Exceptions from that rule are the notations $E6Ln^-$ and $E6Ln$ where $n$ still refers to the corresponding modal extension but number 6 indicates the addition of the two bridge axioms (PNB) and (NNB) to $ELn^-$ and $ELn$, respectively.

\textsuperscript{6}Note that in the definition of our deductive systems we distinguish between axiom and theorem schemes. Rule AN applies to the axioms but not to the theorems of a given system.
Lemma 2.1. The logics of belief form the following hierarchies:

- $EL3 \subseteq EL4 \subseteq EL5$
- $ELn \subseteq E4Ln \subseteq E5Ln \subseteq E6Ln$, for $n = 3, 4, 5$
- $EkL3 \subseteq EkL4 \subseteq EkL5$, for $k = 4, 5, 6$.

Analogous hierarchies are formed by the logics of knowledge.

Proof. Most of the inclusions follow immediately from the definitions. It remains to prove $E5Ln \subseteq E6Ln$, for $n = 3, 4, 5$. It is enough to show that $\Box(K\varphi \rightarrow KK\varphi)$ and $\Box(\neg K\varphi \rightarrow K\neg K\varphi)$ are theorems of $E6Ln$. Observe that $K\varphi \rightarrow \Box K\varphi$ is an instance of (PQB), $K\varphi \rightarrow KK\varphi$ is an instance of (CoRe), and $KK\varphi$ is an instance of (A2). Then rule AN and the Transitivity Axiom (2) and MP yield $\Box(K\varphi \rightarrow KK\varphi)$. On the other hand, $\neg K\varphi \rightarrow \Box \neg K\varphi$ is an instance of (NNB), $\Box \neg K\varphi \rightarrow \Box KK\varphi$ is an instance of (CoRe) and $\Box K\neg K\varphi$ is an instance of (A2). In the same way as before, we derive $\Box(\neg K\varphi \rightarrow K\neg K\varphi)$.

The full Necessitation Rule of normal modal logics does not hold in our systems. However, that rule is valid for the intuitionistic part of $EL4$.

Lemma 2.2. Let $\mathcal{L}$ be one of our logics extending $EL4$. If $\varphi$ is a theorem of $\mathcal{L}$ derivable without scheme (TND), then $\Box \varphi$ is a theorem of $\mathcal{L}$.

Proof. We show the assertion by induction on the length of the derivation of $\varphi$. If the length is 0, then $\varphi$ is an axiom or a theorem of the form $\psi \lor \neg \psi$. The latter, however, is impossible by hypothesis. Thus, $\varphi$ is an axiom and rule AN is applicable. This yields the assertion. Now, suppose $\varphi$ is derived in $n + 1$ steps. We may assume that $\varphi$ is obtained by rule AN or by rule MP. In the former case, $\varphi = \Box \psi$ for some axiom $\psi$. Axiom (A4) and rule MP then yield $\Box \varphi$. In the latter case, there are formulas $\psi$ and $\psi \rightarrow \varphi$, both derived in at most $n$ steps. The induction hypothesis together with the distribution law and MP shows that $\Box \varphi$ is derivable.

System $L$ was introduced in [7] as a modal logic that combines classical and intuitionistic propositional logic: $L$ is a conservative extension of CPC and for any $\varphi \in Fm_0$ it holds that

$$\vdash_{IPC} \varphi \iff \vdash_L \Box \varphi.$$
Furthermore, $L$ combines basic laws of strict implication coming from Lewis’ modal logics with principles of Suszko’s non-Fregean logic in the sense that strict equivalence satisfies the axioms of propositional identity (Id1)--(Id3) above. The stronger logic $L_3$, introduced in [8], inherits those properties. Moreover, the definition of $L_3$ distinguishes explicitly between intuitionistic and classical principles: while all axioms are supposed to be both intuitonistically and classically acceptable, the unique theorem scheme of tertium non datur represents a classical law rejected from a constructive point of view. $L_3$ contains the S3-axiom (A3) which ensures that SP is derivable. Actually, one can show a stronger fact: all instances of SP prefixed by $\Box$ are derivable, i.e. $L_3$ contains $\Box$SP (see [7] for a proof where it is shown that S3 contains $\Box$SP).

The logics $EL_3^−$ and $EL_3^−EL_5$ were introduced in [8] as epistemic (and modal) extensions of $L_3$. Obviously, these classical systems reflect or represent in some sense intuitionistic epistemic principles inherent in the logics $IEL^−$ and $IEL$ introduced in [2]. However, the real nature of that relationship remained somewhat unclear in [8]. In this article, we shall see that if our systems are strong enough, then the epistemic concepts and principles coming from both approaches can be formalized and compared to each other within the same semantic framework. At least from that semantic point of view, the kind of relationship between the logics of [2] and our S5-based systems becomes explicit.

In the following, we present some properties which are shared by all our modal logics.

**Lemma 2.3.** [5, 8] The Substitution Principle SP holds in all our modal logics.

**Proof.** Of course, SP holds in $L$ where it is explicitly stated as a theorem scheme of the deductive system. Scheme SP involves the notion of a substitution $\varphi[x:=\psi]$ which is defined in the canonical way by induction on the complexity of formula $\varphi$. This means in particular that the validity of SP depends on the underlying language. We showed in [5] that in the language of modal logic, all instances of SP are theorems of S3. By the same arguments given there, SP is also valid in $L_3$. In [8], we showed that SP is still valid in $EL_3^−$, a logic in the extended epistemic language. Then SP also holds in all extensions of $EL_3^−$ given in the same language.

A substitution or replacement principle valid in CPC says that if two formulas $\varphi, \psi$ are logically equivalent, then replacing an occurrence of $\varphi$ with the occurrence of $\psi$ in a formula $\chi$ results in a formula which is logically equivalent to $\chi$. In the following, we present some properties which are shared by all our modal logics.
It is well-known that that principle is also valid in normal modal logics. It fails, however, in our modal logics which are of higher “intensional degree” in the sense that more propositions can be distinguished: classically equivalent formulas such as \( \varphi \) and \( \neg
eg
eg\varphi \) denote the same proposition in any model of a normal modal logic (apply the Necessitation Rule to \( \varphi \leftrightarrow \neg\neg\varphi \)) while such formulas may refer to distinct propositions in a model of \( L \) or \( L3 \) (nevertheless, intuitionistically equivalent formulas are always identified). This illustrates that we are actually working with non-Fregean logics where the Fregean Axiom \((\varphi \leftrightarrow \psi) \rightarrow (\varphi \equiv \psi)\), “Sentences with the same truth value have the same meaning”, is invalid.

**Lemma 2.4.** Suppose \( \mathcal{L} \) is one of our modal logics. Then

\[ \vdash_\mathcal{L} \varphi \leftrightarrow \psi \text{ does not generally imply } \vdash_\mathcal{L} \chi[x := \varphi] \leftrightarrow \chi[x := \psi]. \]

**Proof.** We anticipate here the fact that our logics are sound and complete w.r.t. the kind of algebraic semantics introduced in [8] and also used in the present article. Towards a counterexample, we consider the model constructed in the proof of [Theorem 4.4, [8]] which is a model of \( EL5 \) based on the linearly ordered Heyting algebra of the interval of reals \([0, 1]\) with ultrafilter \((0, 1]\). One easily recognizes that this is actually a model of \( E6L5 \), i.e. a model of all our modal logics. Of course, the formula \( y \leftrightarrow \neg\neg y \) is a classical tautology and therefore valid in our classical modal logics. \( y \) and \( \neg\neg y \) have always the same classical truth value, however, these formulas may denote distinct propositions in some model of our non-Fregean logics. For instance, one finds an element \( m \) of the considered model such that \( m \) is distinct from its double negation, and the double negation of \( m \) equals 1. In fact, every \( m \in (0, 1] \) has this property. So if the variable \( y \) denotes \( m \), then \( \varnothing y \) denotes 0 and \( \varnothing\neg\neg y \) denotes 1. Then that model together with \( \varphi = y \), \( \psi = \neg\neg y \) and \( \chi = \varnothing x \) represents a counterexample. □

We showed in [Lemma 2.2, [5]] that all biconditionals of the form \( \varnothing\varphi \leftrightarrow (\varphi \equiv \top) \) are valid in system \( S1+SP \), a Lewis-style modal logic in the vicinity of \( S1 \). The proof given there essentially relies on \( SP \). We present here a simpler proof.

**Theorem 2.5.** All biconditionals of the form

\[ (4) \quad \varnothing\varphi \leftrightarrow (\varphi \equiv \top) \]

are theorems of our modal logics.

\[ ^7 \text{Actually, normal modal logics are non-Fregean logics, too (see, e.g., [5, 6]). For a discussion on the motivations of non-Fregean logics, see, e.g., [9, 13].} \]
Proof. Of course, $\varphi \rightarrow \top$ is a theorem of IPC. AN then yields $\Box(\varphi \rightarrow \top)$. On the other hand, $\varphi \rightarrow (\top \rightarrow \varphi)$ is the substitution-instance of a theorem of IPC and AN yields $\Box(\varphi \rightarrow (\top \rightarrow \varphi))$. Applying distribution and MP, we get $\Box \varphi \rightarrow \Box(\top \rightarrow \varphi)$. Hence, $\Box \varphi \rightarrow \Box(\varphi \rightarrow \top)$ and $\Box(\top \rightarrow \varphi)$, i.e. $\Box \varphi \rightarrow (\varphi \equiv \top)$ is derivable. The other way round, $(\varphi \equiv \top)$ implies in particular $\Box(\top \rightarrow \varphi)$. By distribution and MP, $\Box \top \rightarrow \Box \varphi$. $\top$ is a theorem of ICP. Hence, AN and MP yield $\Box \varphi$. This shows that $(\varphi \equiv \top) \rightarrow \Box \varphi$ is derivable.

Under the assumption that $\equiv$ represents propositional identity, we called the scheme $(\Box \varphi \land \Box \psi) \rightarrow (\varphi \equiv \psi)$ the Collapse Axiom in [6]. It implies that there is exactly one necessary proposition, namely the proposition denoted by $\top$. This is also expressed by scheme (4) above, provided that $\equiv$ is the connective for propositional identity. However, we cannot expect that strict equivalence is propositional identity in any modal system. For instance, axiom (Id2) is not fulfilled in normal modal logic $K$ and (Id3) does not hold in Lewis’ systems $S1$ and $S2$ if we define $\varphi \equiv \psi := \Box(\varphi \leftrightarrow \psi)$ (see [6] [5] for a proof). If we introduce $\equiv$ as a primitive symbol (instead of an abbreviation of strict equivalence) and suppose that the axioms of propositional identity (Id1)–(Id3) are satisfied, then $\equiv$ refines the relation of strict equivalence. The Collapse Axiom is valid whenever both relations coincide. The logics $S3$–$S5$ as well as our modal logics are strong enough to ensure that strict equivalence can be interpreted as propositional identity axiomatized by (Id1)–(Id3).

There is a further interpretation of the biconditionals (4). Under the assumption that the proposition denoted by $\top$ stands for intuitionistic truth and $\equiv$ is propositional identity, we may read (4) as

“$\Box \varphi$ iff $\varphi$ is intuitionistically true”.

Under this interpretation, scheme (4) is an analogue to the Tarski biconditionals, sometimes called $T$-scheme, coming from A. Tarski’s truth theory. The modal operator $\Box$ plays here the role of a predicate for intuitionistic truth given in the object language. In fact, it will follow from the definition of our model-theoretic semantics (which is based on Heyting algebras) that the following holds: $\Box \varphi$ is satisfied in a given model iff $\varphi$ denotes the top element of the underlying Heyting

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8We show that propositional identity $\varphi \equiv \psi$ refines strict equivalence $\Box(\varphi \leftrightarrow \psi)$: By (Id3), $(\varphi \equiv \psi) \rightarrow \Box((\varphi \leftrightarrow x)[x := \varphi] \equiv \Box((\varphi \leftrightarrow x)[x := \psi]))$. Thus, $(\varphi \equiv \psi) \rightarrow (\Box(\varphi \leftrightarrow \varphi) \equiv \Box((\varphi \leftrightarrow \psi)))$. By (Id2), $(\varphi \equiv \psi) \rightarrow (\Box((\varphi \leftrightarrow \varphi) \rightarrow \Box((\varphi \leftrightarrow \psi))))$. We may assume that $\Box((\varphi \leftrightarrow \varphi)$ is a theorem. Hence, $(\varphi \equiv \psi) \rightarrow \Box(\varphi \leftrightarrow \psi)$ is a theorem. Hence, $(\varphi \equiv \psi) \rightarrow \Box(\varphi \leftrightarrow \psi)$ is a theorem.
lattice (i.e. \( \phi \) is intuitionistically true in the model).

Our modal systems are classical logics and formulas are interpreted classically. Symbol \( \Box \), however, is intended as a proof predicate where “proof” is interpreted in the sense of BHK semantics of intuitionistic logic.

- \( \Box \phi \) reads “there is a proof of \( \phi \)”, i.e. “\( \phi \) is proved”.
- \( \neg \Box \phi \) reads “\( \phi \) is not proved”, “no proof of \( \phi \) is available”.
- \( \Diamond \phi \), which abbreviates \( \neg \Box \neg \phi \), reads “there is no proof of \( \neg \phi \)”, i.e. “\( \neg \phi \) is not true from the intuitionistic point of view”, i.e. “a proof of \( \phi \) is possible”. We interpret this sometimes in the following way: “\( \phi \) is consistent with the given set of hypotheses”, or “\( \phi \) is consistent” for short.
- \( \Box \neg \phi \) reads “\( \neg \phi \) is proved”, i.e. “\( \phi \) is intuitionistically false”. The logically equivalent formula \( \neg \Diamond \phi \), i.e. \( \neg \neg \Box \neg \phi \), reads “a proof of \( \phi \) is impossible”.
- Intuitively, a proof of \( K \phi \) is possible, i.e. \( \Diamond K \phi \), if and only if classical truth of \( K \phi \) can be established in some model. In such a model, \( \phi \) then would denote a believed (a known) proposition. In this sense, we may read \( \Diamond K \phi \) as “\( \phi \) is believable (knowable)”.

We present a few examples of derivable statements.

**Theorem 2.6.**

1. \( \vdash_{EL3^-} \neg \Box \bot \). “There is no proof of the contradictory proposition, i.e. the underlying system is consistent.”

2. \( \vdash_{EL3^-} \Box \neg \bot \). “There is a proof that the underlying system is consistent.”

3. \( \vdash_{EL3^-} \phi \rightarrow \Diamond \phi \). “Classical truth implies consistency, i.e. the possibility of a proof.”

4. \( \vdash_{EL3^-} \Box \phi \rightarrow \neg \Diamond \neg \phi \). “If \( \phi \) is proved, then a proof of \( \neg \Box \phi \) is impossible”.

5. \( \vdash_{EL3^-} K \phi \rightarrow \Diamond K \phi \). “Believed (known) propositions are believable (knowable), respectively.”

\(^9\)Note that the assertion refers to a particular (intuitionistic) system described by a given model of \( EL3^- \). It does not refer to the system \( EL3^- \) itself.
(vi) \( E_{6L5} \varphi \rightarrow \lozenge K \varphi \). “There may exist true propositions that are unknowable (unbelievable).”

(vii) \( E_{6L3} \lozenge K \varphi \rightarrow K \varphi \). “All believable (knowable) propositions are already believed (known).”

**Proof.** (i): By (A2), \( \Box \bot \rightarrow \bot \) is an axiom. By contraposition, \( \top \rightarrow \neg \Box \bot \) is derivable. MP yields \( \neg \Box \bot \).

(ii): \( \neg \Box \bot \rightarrow \top \) is an axiom (a substitution-instance of axiom \( x \rightarrow \top \) of IPC). Rule AN yields \( \Box (\neg \Box \bot \rightarrow \top) \). On the other hand, by (A2), \( \Box \bot \rightarrow \bot \) is an axiom. Furthermore, \( (\Box \bot \rightarrow \bot) \rightarrow (\top \rightarrow \neg \Box \bot) \) is an axiom (an substitution-instance of axiom \( (x \rightarrow y) \rightarrow (\neg y \rightarrow \neg x) \) of IPC. Then AN, distribution and MP yield \( \Box (\top \rightarrow \neg \Box \bot) \). Thus, \( \neg \Box \bot \equiv \top \) is a theorem. By Theorem 2.5, \( \Box \neg \Box \bot \leftrightarrow \neg \Box \bot \equiv \top \) is derivable, and so also \( \Box \neg \Box \bot \).

(iii): By (A2), \( \Box \neg \varphi \rightarrow \neg \varphi \). Apply contraposition.

(iv): \( \Box \varphi \rightarrow \neg \neg \Box \varphi \) is a substitution-instance of the axiom \( x \rightarrow \neg \neg x \) of IPC. Applying AN, we derive \( \Box (\Box \varphi \rightarrow \neg \neg \Box \varphi) \). By distribution and MP, we get \( \Box \Box \varphi \rightarrow \Box \neg \neg \Box \varphi \). Finally, by (A5) and transitivity of implication, \( \Box \varphi \rightarrow \neg \Box \neg \Box \varphi \). Obviously, by transitivity of implication, then also \( \Box \varphi \rightarrow \neg \Box \neg \Box \varphi \) is derivable, i.e. \( \Box \varphi \rightarrow \neg \lozenge \neg \Box \varphi \).

(v): This is a special case of (iii).

(vi): Again, we anticipate algebraic semantics given below and prove the assertion by presenting an \( E6L5 \)-model that satisfies a formula \( \varphi \) but not \( \lozenge K \varphi \). Such a model should contain a proposition \( m \) such that \( m \) is true and unknown. If variable \( x \) denotes \( m \), the model then satisfies \( x \) and \( \neg Kx \). Since the model validates axiom (NNB), \( \Box \neg Kx \) is true, i.e. \( \lozenge Kx \) is false. In the proof of [Theorem 4.4, [8]], a specific model of logic \( EL5 \) is constructed which turns out to be a model of logic \( E6L5 \) with the desired properties as one easily verifies.

(vii): This is the contrapositive of axiom (NNB).

In the proof of item (iv), we have derived the formula \( \Box \varphi \rightarrow \Box \neg \neg \Box \varphi \) in \( EL4 \) using (A4). Thus, the formula is weaker than (A4). Interestingly, we are able to derive that formula in \( EL5 \) without using (A4): We have \( \Box \varphi \rightarrow \lozenge \Box \varphi \), by item (iii). By (A5) and transitivity of implication, \( \Box \varphi \rightarrow \Box \lozenge \Box \varphi \). By (A5), (A2) and AN, \( \Box \neg \Box \varphi \equiv \neg \Box \varphi \). SP applied to \( \neg x \) yields \( \neg \Box \neg \Box \varphi \equiv \neg \neg \Box \varphi \), i.e. \( \lozenge \Box \varphi \equiv \neg \neg \Box \varphi \). Then, by SP, we may replace \( \lozenge \Box \varphi \) with \( \neg \neg \Box \varphi \) in every formula, in particular in \( \Box \varphi \rightarrow \Box \lozenge \Box \varphi \). This results in \( \Box \varphi \rightarrow \Box \neg \neg \Box \varphi \). Nevertheless, we are unable to derive (A4) in \( EL5 \) without the use of (A4) itself.
The systems $IEL^-$ and $IEL$ of Intuitionistic Epistemic Logic, introduced by S. Artemov and T. Protopopescu [2], are axiomatized by IPC, distribution of belief $K(\varphi \rightarrow \psi) \rightarrow K\varphi \rightarrow K\psi$, the axiom of intuitionistic co-reflection $\varphi \rightarrow K\varphi$ ("proof yields knowledge") and, only in case of $IEL$, the axiom of intuitionistic reflection $K\varphi \rightarrow \neg\neg\varphi$. Modus Ponens is the only inference rule. Principles of Intuitionistic Epistemic Logic are reflected in some sense in our classical modal logics. For instance, our axiom (CoRe) of co-reflection, $\Box\varphi \rightarrow \Box K\varphi$, can be seen as a classical counterpart of intuitionistic co-reflection $\varphi \rightarrow K\varphi$. In the following Theorem, we list some further examples of derivable modal laws that are related in some way to principles of Intuitionistic Epistemic Logic. Observe that (i) and (ii) refer to intuitionistic reflection $K\varphi \rightarrow \neg\neg\varphi$ while (iii)-(v) are, in a sense, classical variants of intuitionistic co-reflection $\varphi \rightarrow K\varphi$.

**Theorem 2.7.**

(i) $\vdash_{EL3} \Box K\varphi \rightarrow \neg\Diamond \neg\varphi$. "If it is proved that $\varphi$ is known, then it is impossible to prove $\neg\varphi$.”

(ii) $\vdash_{EL3} \Box K\varphi \rightarrow \Diamond \varphi$. “If it is proved that $\varphi$ is known, then a proof of $\varphi$ is possible.”

(iii) $\vdash_{EL5-} \neg\Box K\varphi \rightarrow \neg\Diamond \Box \varphi$. “If $K\varphi$ has no proof, then a proof of $\Box \varphi$ is impossible (i.e. $\Box \varphi$ is inconsistent with the hypotheses).”

(iv) $\vdash_{ELA-} \Box \varphi \rightarrow K \Box \varphi$. “If $\varphi$ is proved, then it is believed (known) that $\varphi$ is proved.”

(v) $\vdash_{EL5-} \neg\Box \varphi \rightarrow K \neg\Box \varphi$. “If $\varphi$ has no proof, then it is believed (known) that $\varphi$ has no proof.”

**Proof.** (i): We derive $\Box (K\varphi \rightarrow \neg\neg\varphi)$ by AN applied to (IntRe). Distribution and MP yield $\Box K\varphi \rightarrow \Box \neg\neg\varphi$. Observe that $\Box \neg\neg\varphi$ is equivalent to $\neg\neg \Box \neg\neg\varphi$ which in turn can be abbreviated by $\neg\Diamond \neg\varphi$. The remaining assertions are proved in [Theorem 2.5, [8]].

We adopt the model-theoretic algebraic semantics from [7, 8] where models are given as Heyting algebras with a designated ultrafilter and some additional structure. For convenience, we recall in the following some facts about Heyting algebras as well as some definitions from [8].

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Lemma 2.8. Let \( \mathcal{H} \) be a Heyting algebra with universe \( H \).

(a) \( U \subseteq H \) is an ultrafilter iff there is an Heyting algebra homomorphism \( h \) from \( \mathcal{H} \) to the two-element Boolean algebra \( \mathcal{B} \) such that the top element of \( \mathcal{B} \) is precisely the image of \( U \) under \( h \).
(b) If \( U \subseteq H \) is an ultrafilter, then for all \( m, m' \in H \):

- \( f_\vee(m, m') \in U \) iff \( m \in U \) or \( m' \in U \) (i.e. \( U \) is a prime filter)
- \( m \in U \) or \( f_\wedge(m) \in U \)
- \( f_\rightarrow(m, m') \in U \) iff \( \{ m \notin U \text{ or } m' \in U \} \) or \( f_\wedge(f_\rightarrow(m), m') \in U \).

(c) Every filter is the intersection of a set of prime filters.
(d) Let \( m_1, m_2 \in H \) and let \( P \) be a prime filter. Then we have \( f_\rightarrow(m_1, m_2) \in P \) if, and only if, for all prime filters \( P' \supseteq P \), \( m_1 \in P' \) implies \( m_2 \in P' \).

Proof. (a)–(d) are known properties of Heyting algebras which are not hard to prove. The right-to-left implication of (d), however, might be less familiar. We present a detailed proof. Let \( m_1, m_2 \in H \) and let \( P \) be a prime filter. We consider the quotient Heyting algebra \( \mathcal{H}' \) of \( \mathcal{H} \) modulo \( P \). That is, the elements of \( \mathcal{H}' \) are the equivalence classes \( \overline{m} \) of \( m \in M \) modulo the equivalence relation \( \sim \) defined by \( m \sim m' \Leftrightarrow \{ f_\rightarrow(m, m') \in P \text{ and } f_\rightarrow(m', m) \in P \} \). Then one easily checks that \( P \) is the equivalence class of \( f_\top \) modulo \( \sim \), and it is the top element \( f_\top' \) of \( \mathcal{H}' \).

Claim1: Let \( m, m' \in H \). If \( \overline{m} \in F' \) implies \( \overline{m'} \in F' \), for all filters \( F' \) of \( \mathcal{H}' \), then \( \overline{m} \leq' \overline{m'} \), where \( \leq' \) is the lattice order of \( \mathcal{H}' \).

Proof of Claim1. Suppose \( \overline{m} \notin \overline{m'} \). Consider the filter \( G = \{ \overline{m'} \mid \overline{m} \leq' \overline{m'} \} \). Then \( \overline{m} \in G \) and \( \overline{m'} \notin G \). We have proved the Claim.

Claim2: Let \( m, m' \in H \). If \( \overline{m} \in F' \) implies \( \overline{m'} \in F' \), for all prime filters \( F' \) of \( \mathcal{H}' \), then \( \overline{m} \leq' \overline{m'} \), where \( \leq' \) is the lattice ordering of \( \mathcal{H}' \).

Proof of Claim2. Claim2 follows from Claim1 together with (a).

Claim3: If \( F' \) is a (prime) filter of \( \mathcal{H}' \), then \( F = \{ m \mid \overline{m} \in F' \} \) is a (prime) filter of \( \mathcal{H} \) extending \( P \).

Proof of Claim3. Suppose \( m \in F \) and \( m \leq' m' \). Then \( f_\rightarrow(m, m') = f_\top \). Thus, \( f_\rightarrow(m, m') = P = f_\top' \). That is, \( f_\rightarrow(\overline{m}, \overline{m'}) = f_\top' \) and therefore \( \overline{m} \leq' \overline{m'} \). It follows that \( \overline{m} \in F' \) and \( \overline{m'} \in F' \). The remaining filter properties follow straightforwardly. \( m \in F' \) implies \( \overline{m} = P = f_\top' \in F' \) implies \( m \in F \). Thus, \( P \subseteq F \) and Claim3 holds true.

Now suppose for all prime filters \( P' \supseteq P \), \( m_1 \in P' \) implies \( m_2 \in P' \). We show that this implies \( f_\rightarrow(m_1, m_2) \in P \). Let \( F' \) be any prime filter of \( \mathcal{H}' \) and \( \overline{m_1} \in F' \). Then, by Claim3, \( m_1 \in F = \{ m \mid \overline{m} \in F' \} \) and \( F \) is a prime filter of \( \mathcal{H} \) with
$P \subseteq F$. By hypothesis of (d), $m_2 \in F$. Thus, $\overline{m_2} \in F'$. By Claim2, $\overline{m_1} \leq \overline{m_2}$. Then $\overline{f(m_1, m_2)} = f'_\top = P$. That is, $f_\rightarrow(m_1, m_2) \in P$. 

**Definition 2.9.** [8] An $\text{EL3}^-$-model (also called model of belief or epistemic model) is a Heyting algebra

$$M = (M, \text{TRUE}, BEL, f_\bot, f_\top, f_\lor, f_\land, f_\rightarrow, f_\Box, f_K)$$

with universe $M$, a designated ultrafilter $\text{TRUE} \subseteq M$, a set $BEL \subseteq M$ and additional unary operations $f_\Box$ and $f_K$ such that for all $m, m', m'' \in M$ the following truth conditions are fulfilled (as before, $\leq$ denotes the lattice order):

1. $f_\Box(f_\lor(m, m')) \leq f_\lor(f_\Box(m), f_\Box(m'))$
2. $f_\Box(m) \leq m$
3. $f_\Box(f_\rightarrow(m, m')) \leq f_\Box(f_\rightarrow(f_\Box(m), f_\Box(m')))$
4. $f_\Box(m) \in \text{TRUE} \Leftrightarrow m = f_\top$
5. $f_K(m) \in \text{TRUE} \Leftrightarrow m \in BEL$
6. $f_K(f_\rightarrow(m, m')) \leq f_\rightarrow(f_K(m), f_K(m'))$
7. $f_\Box(m) \leq f_\Box(f_K(m))$

$M$ is the universe of all propositions and $\text{TRUE} \subseteq M$ is the subset of classically true propositions. The propositions $f_\top, f_\bot$ represent intuitionistic truth and intuitionistic falsity, respectively. $BEL$ is the set of believed propositions. A believed proposition is known if it is an element of $\text{TRUE}$.

We shall tacitly make use of the equivalence $m \leq m' \Leftrightarrow f_\rightarrow(m, m') = f_\top$ which holds in all Heyting algebras. Note that truth conditions (i) and (iv) ensure that every model has the following Disjunction Property DP: for all $m, m' \in M$, $f_\lor(m, m') = f_\top$ iff $m = f_\top$ or $m' = f_\top$. That is, the smallest lattice filter $\{f_\top\}$ is a prime filter.

**Definition 2.10.** Let $M$ be an $\text{EL3}^-$-model. We say that

- $M$ is an $\text{ELA}^-$-model if for all $m \in M$: $f_\Box(m) \leq f_\Box(f_\Box(m))$. 

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• \( \mathcal{M} \) is an \( \text{EL}5^-\)-model if for all \( m \in \mathcal{M} \):

\[
f_\square(m) = \begin{cases} f_\top, & \text{if } m = f_\top \\ f_\bot, & \text{else} \end{cases}
\]

• \( \mathcal{M} \) is an \( \text{EALn}^-\)-model, for \( n = 3, 4, 5 \), if \( \mathcal{M} \) is an \( \text{ELn}^-\)-model and for all \( m \in \mathcal{M} \):

\[
f_K(m) \leq f_K(f_K(m)).
\]

• \( \mathcal{M} \) is an \( \text{E}5\text{Ln}^-\)-model, \( n = 3, 4, 5 \), if \( \mathcal{M} \) is an \( \text{E}4\text{Ln}^-\)-model and for all \( m \in \mathcal{M} \):

\[
f_\neg(f_K(m)) \leq f_K(f_\neg(f_K(m))).
\]

• \( \mathcal{M} \) is an \( \text{E}6\text{Ln}^-\)-model, \( n = 3, 4, 5 \), if \( \mathcal{M} \) is an \( \text{ELn}^-\)-model and for all \( m \in \mathcal{M} \):

\[
f_K(m) \leq f_\Box(f_K(m))
\]

\[
f_K(m) = \begin{cases} f_\top, & \text{if } m \in \text{BEL} \\ f_\bot, & \text{else}. \end{cases}
\]

• If \( \mathcal{M} \) is an \( \text{ELn}^-\)-model or an \( \text{EkLn}^-\)-model, for \( k \in \{4, 5, 6\} \) and \( n \in \{3, 4, 5\} \), and \( \mathcal{M} \) satisfies the additional truth condition

\[
(5) \quad f_K(m) \leq f_\neg(f_\neg(m)) \quad \text{for all propositions } m,
\]

then we omit the superscript \(^-\) in the notation and refer to \( \mathcal{M} \) as an \( \text{ELn}^-\)-model or an \( \text{EkLn}^-\)-model, respectively. We refer to models with that additional truth condition also as models of knowledge.

One easily verifies that every \( \text{EL}5^-\)-model is an \( \text{ELA}^-\)-model, every \( \text{E}5\text{Ln}^-\)-model is an \( \text{E}4\text{Ln}^-\)-model and every \( \text{E}6\text{Ln}^-\)-model is an \( \text{E}5\text{Ln}^-\)-model. The latter follows from the fact that \( f_\top \in \text{BEL} \). This is shown in item (i) of the next Lemma.

**Lemma 2.11.** Let \( \mathcal{M} \) be an epistemic model. Then for all propositions \( m, m' \in \mathcal{M} \), the following hold:

• \( f_\top \in \text{BEL} \).

• If \( m, m' \in \text{BEL} \), then \( f_\land(m, m') \in \text{BEL} \).

• If \( m \in \text{BEL} \) and \( m \leq m' \), then \( m' \in \text{BEL} \).
If $\mathcal{M}$ is a model of knowledge, i.e. $\mathcal{M}$ satisfies truth condition (5) above, then all believed propositions are known propositions, i.e. $\text{BEL} \subseteq \text{TRUE}, f_\bot \notin \text{BEL}$ and the set of known propositions $\text{BEL}$ is a filter of the underlying Heyting lattice.

**Proof.** By the truth conditions (iv), (vii), (ii) and (v) of a model, $f_\top \in \text{BEL}$. Suppose $m, m' \in \text{BEL}$, i.e. $f_K(m), f_K(m') \in \text{TRUE}$. As in every Heyting algebra, $f_\rightarrow(m, f_\rightarrow(m', f_\wedge(m, m'))) = f_\top$. Since $f_\top \in \text{BEL}$, we conclude that $f_K(f_\rightarrow(m, f_\rightarrow(m', f_\wedge(m, m')))) = f_\top$. Since $f_\top \in \text{BEL}$, we conclude that $f_K(f_\rightarrow(m, f_\rightarrow(m', f_\wedge(m, m')))) = f_\top$. Applying two times truth condition (vi) of a model, we obtain $f_K(f_\wedge(m, m')) \in \text{TRUE}$, i.e. $f_\wedge(m, m') \in \text{BEL}$. We have shown the second item of the Lemma. Now, let $m \in \text{BEL}$ and $m \leq m'$. It follows that $f_K(m) \in \text{TRUE}$ and $f_\rightarrow(m, m') = f_\top$. The latter implies $f_K(f_\rightarrow(m, m')) \in \text{TRUE}$. Then by truth condition (vi), $f_K(m') \in \text{TRUE}$ and $m' \in \text{BEL}$. Finally, we suppose that $\mathcal{M}$ is a model of knowledge, i.e. $f_K(m) \leq f_\rightarrow(f_\neg(m))$ for all $m \in M$. Applying Lemma 2.23 we have $m \in \text{BEL} \Rightarrow f_K(m) \in \text{TRUE} \Rightarrow f_\neg(f_\neg(m)) \in \text{TRUE} \Rightarrow m \in \text{TRUE}$. Hence, $\text{BEL} \subseteq \text{TRUE}$. By definition, $\text{TRUE}$ is a filter. Thus, $f_\bot \notin \text{BEL}$. Then $f_\bot \notin \text{BEL}$ and $\mathcal{B}$ satisfies all filter axioms.

**Definition 2.12.** An assignment in an epistemic model $\mathcal{M}$ is a function $\gamma : V \rightarrow M$ which extends in the canonical way to a function $\gamma : Fm \rightarrow M$. More specifically, we have $\gamma(\bot) = f_\bot, \gamma(\top) = f_\top, \gamma(\Box \varphi) = f_\Box(\gamma(\varphi)), \gamma(K \varphi) = f_K(\gamma(\varphi))$ and $\gamma(\varphi \ast \psi) = f_\ast(\gamma(\varphi), \gamma(\psi)), \ast \in \{\lor, \land, \rightarrow\}$. If $\mathcal{L}$ is one of our epistemic logics, then an $\mathcal{L}$-interpretation is a tuple $(\mathcal{M}, \gamma)$ consisting of an $\mathcal{L}$-model and an assignment $\gamma \in M^V$. The relation of satisfaction is defined by

$$(\mathcal{M}, \gamma) \models \varphi :\Leftrightarrow \gamma(\varphi) \in \text{TRUE}.$$  

If $(\mathcal{M}, \gamma) \models \varphi$, then we say that $\varphi$ is true in $\mathcal{M}$ under assignment $\gamma \in M^V$. If $\varphi$ is true in $\mathcal{M}$ under all assignments $\gamma \in M^V$, then we write $\mathcal{M} \models \varphi$ and say that $\varphi$ is valid in $\mathcal{M}$. A formula $\varphi$ is valid in logic $\mathcal{L}$ if $\varphi$ is valid in all $\mathcal{L}$-models. The defined notions extend in the usual way to sets of formulas.

Logical consequence of logic $\mathcal{L}$ is defined by $\Phi \models_{\mathcal{L}} \varphi :\Leftrightarrow (\mathcal{M}, \gamma) \models \varphi$ implies $(\mathcal{M}, \gamma) \models \Phi$, for every $\mathcal{L}$-interpretation $(\mathcal{M}, \gamma)$.

The intended meaning of the identity connective is: $\varphi \equiv \psi$ is true iff $\varphi$ and $\psi$ denote the same proposition. Indeed, the following holds (see also [6, 5]):

**Lemma 2.13.**

$$(\mathcal{M}, \gamma) \models \varphi \equiv \psi \iff \gamma(\varphi) = \gamma(\psi).$$
Proof. Recall that propositional identity $\varphi \equiv \psi$ is defined as strict equivalence $\Box(\varphi \rightarrow \psi) \land \Box(\psi \rightarrow \varphi)$. Suppose $\gamma(\varphi) = m \in M$ and $\gamma(\psi) = m' \in M$. Then $\gamma(\varphi \equiv \psi) \in \text{TRUE}$ iff $f(\Box(\varphi \rightarrow \psi)) \in \text{TRUE}$ and $f(\Box(\psi \rightarrow \varphi)) \in \text{TRUE}$ iff $f(\varphi \rightarrow \psi) = \top$ and $f(\psi \rightarrow \varphi) = \top$, iff $m \leq m'$ and $m' \leq m$ iff $m = m'$.

Soundness and completeness of the logics $\mathcal{L} = EL3^-, EL3, ELA^-, EL4, EL5^-, EL5$ w.r.t. to the class of all $\mathcal{L}$-interpretations, respectively, follow from corresponding proofs presented in [8]. We may extend the arguments straightforwardly to the remaining logics and conclude the following.

Theorem 2.14 (Strong completeness w.r.t. algebraic semantics). Let $\Phi \cup \{\varphi\} \subseteq Fm$ and let $\mathcal{L}$ be any of our epistemic logics. Then $\Phi \vdash_{\mathcal{L}} \varphi \iff \Phi \models_{\mathcal{L}} \varphi$.

We finish this section with a discussion on self-referential propositions. Having in mind the intended meaning of the identity connective (see Lemma 2.13), we are able to express self-referential statements by means of equations. For instance, the equation
\begin{equation}
\label{eq:liar}
x \equiv (x \rightarrow \bot)
\end{equation}
defines a version of the liar proposition. In fact, if the equation is satisfied in a given model, then the proposition denoted by $x$ says “This proposition implies the absurdum” or “This proposition is false” or “I’m lying”. Fortunately, equations defining such paradoxical self-referential statements are unsatisfiable (for essentially the same reasons as $\varphi \leftrightarrow \neg \varphi$ is unsatisfiable in two-valued classical logic). The liar can be described, claimed or declared syntactically in the object language by means of a self-referential equation. However, the liar proposition, as a semantic object, does not exist.

Is there a proposition saying “I am proved”? Since we identify proof with intuitionistic truth, we are actually asking for the existence of a truth-teller proposition “I am true”, where true here has an intuitionistic meaning. This is a non-paradoxical proposition which can be defined by the equation
\begin{equation}
\label{eq:truth-teller}
x \equiv \Box x.
\end{equation}

\footnote{We could easily extend the language by an operator $T$, add the axiom $T\varphi \leftrightarrow \varphi$ (Tarski biconditionals) to our systems, and consider an additional function $f_T$ on the universe of our models such that the condition $f_T(m) \in \text{TRUE} \leftrightarrow m \in \text{TRUE}$ is satisfied for all $m \in M$. $T$ then would be a total truth predicate in the object language for classical truth such that the Tarski biconditionals hold and semantic antinomies such as the liar paradox are avoided.}
Any proposition \( m \) that solves the equation is an intuitionistic truth-teller saying “I am proved”. A truth-teller \( m \) may be classically true (i.e. \( m \in TRUE \)) or classically false (i.e. \( m \notin TRUE \)). In our specific example, there is only one potential true truth-teller, namely the top element \( f \top \) of a model (because of truth condition (iv) of Definition 2.9). In fact, the top element of each \( EL4 \)-model is a truth-teller, since \( f \Box (m) = f \top \Leftrightarrow m = f \top \), i.e. \( f \top \) is the unique fixed point of operation \( f \Box \). On the other hand, in every \( EL5 \)-model, the proposition \( f \bot \) is a false truth-teller. In models which are not \( EL5 \), there might exist further truth-tellers that are classically false, i.e. fixed points of \( f \Box \) not belonging to \( TRUE \).

Is there a proposition asserting its own consistency? We are asking for a solution of the equation

\[
(8) \quad x \equiv \Diamond x.
\]

One easily checks that in any \( EL4 \)-model, the bottom element \( f \bot \) is a solution; and in an \( EL5 \)-model, the top element \( f \top \) is a solution as well. So (8) is a further example of an equation allowing both true and false propositions as solutions in suitable models. There might exist further solutions distinct from \( f \top \) and \( f \bot \). Finally, a proposition that asserts its own inconsistency is described by the equation

\[
(9) \quad x \equiv \neg \Diamond x.
\]

An alternative version of that self-referential statement is given by the equation

\[
(10) \quad x \equiv \Box \neg x.
\]

The difference between both equations is subtle. In fact, \( \neg \Diamond x = \neg \neg \Box \neg x \) and \( \Box \neg x \) are logically equivalent formulas in our classical modal logics. From an intuitionistic point of view, however, they express different intensions and therefore may denote different propositions. Suppose equation (10) is true in a given model. Then the proposition \( m \) denoted by \( x \) says “There is a proof that I am false” or, in other words, “I am inconsistent with the given hypotheses”. Suppose \( m \) is classically true, i.e. \( m = f \Box (f \neg ((m))) \in TRUE \), then, by truth condition (iv) of a model, \( f \neg (m) = f \top \), i.e. \( m = f \bot \notin TRUE \). This contradiction shows that \( m \) cannot be classically true. So whenever equation (10) is satisfied, the proposition \( m \) denoted by \( x \) must be classically false. Furthermore, \( m \) cannot be the proposition \( f \bot \) for otherwise \( m = f \Box (f \neg (m)) = f \Box (f \neg (f \bot)) = f \Box (f \top) \in TRUE \), contradicting \( m \notin TRUE \). We conclude that \( m \) is a false proposition distinct
from \( f_\bot \). In particular, a proof of \( m \) is possible (i.e. \( m \) is consistent with the hypotheses). Hence, \( m \) is a consistent proposition asserting its own inconsistency. This sounds paradoxical, but actually it is not because \( m \) is classically false. As a non-paradoxical proposition, a solution of (10) should exist in some model. The construction of a model that satisfies equations (10) or (9) goes beyond the scope of this discussion.

The simplest self-referential statements involving knowledge are described by the equations

\[
(11) \quad x \equiv Kx
\]

\[
(12) \quad x \equiv \neg Kx.
\]

Obviously, if (11) is true, then the proposition denoted by \( x \) says “I am known”, and if (12) is true, then \( x \) denotes a proposition saying “I am unknown”. (11) is true in every \( E6Ln \)-model with \( x \) denoting \( f_\top \) or \( f_\bot \). Of course, the same holds true if we consider belief instead of knowledge. There may exist further solutions. For a discussion of equation (12), let us assume that \( K \) stands for knowledge as true belief, i.e. scheme \( K\varphi \rightarrow \varphi \) is valid. Then one recognizes that a solution of (12) must be a proposition that is classically true and unknown. If \( K \) refers to belief that can be false, i.e. \( K\varphi \rightarrow \varphi \) is not valid, then (12) may have classically false propositions as solutions. Such a proposition then says something like “Nobody believes in me”. The equations

\[
(11) \quad x \equiv \Box Kx
\]

\[
(12) \quad x \equiv \neg \Box Kx.
\]

define propositions asserting something like “I’m believable (knowable)” and “I’m unbelievable (unknowable)”, respectively. These are further examples of non-paradoxical self-referential statements, i.e. the corresponding equations are satisfiable.

### 3 Relational semantics for logics which are at least as strong as \( EL5^- \) or \( EL5 \)

In [8] we introduced a hierarchy of modal logics, namely \( EL^-3, EL3, ELA4 \) and \( EL5 \), and showed that these logics are sound and complete w.r.t. algebraic semantics. In the preceding section, we extended that hierarchy by further logics which
can also be described by the same kind of algebraic semantics. In this section, we show that the S5-style modal logics of our hierarchy, i.e. those containing the axioms (A4) and (A5), are strong enough to be complete w.r.t. a relational semantics based on intuitionistic general frames. We are unable to develop any kind of possible worlds semantics for the weaker logics of our hierarchy. Interestingly, the presented semantic framework can also be used to describe the intuitionistic epistemic logics $IEL^-$ and $IEL$ presented in [2], as we shall see in the next section.

**Definition 3.1.** An $EL5^-$-frame $\mathcal{F} = (W, R, P, E, w_T)$ is given by

- a non-empty set $W$ of worlds
- a partial ordering $R \subseteq W \times W$, called accessibility relation, such that $W$ has an $R$-smallest element $w_\perp$ (the bottom of the frame) and every $R$-chain has an upper bound in $W$ (Zorn’s Lemma then implies that each $w \in W$ accesses an $R$-maximal element); for $w \in W$ let $R(w) := \{w' \in W \mid wRw'\}$; and let $\text{Max}(W)$ be the set of all $R$-maximal elements
- a set $P \subseteq \text{Pow}(W)$ of upper sets (recall that $A \in \text{Pow}(W)$ is an upper set if for all $w, w' \in W$: $w \in A$ and $wRw'$ implies $w' \in A$)
- a function $E: W \rightarrow \text{Pow}(P)$ such that
  - for all $w \in W$, $E(w) \subseteq P$ is a non-empty set with the following properties: if $A \in E(w)$ and $B \in E(w)$, then $A \cap B \in E(w)$; and if $A \in E(w)$ and $A \subseteq B \in P$, then $B \in E(w)$
  - for all $w, w' \in W$: $wRw'$ implies $E(w) \subseteq E(w')$
- a designated $R$-maximal element $w_T \in W$.

Furthermore, we require that $P$ is closed under the following conditions:

(a) $\emptyset, W \in P$

(b) If $A, B \in P$, then the following sets are elements of $P$:

$A \cap B$

$A \cup B$

$A \triangleright B := \{w \in W \mid \text{for all } w' \in R(w), w' \in A \text{ implies } w' \in B\}$

$KA := \{w \in W \mid A \in E(w)\}$. 

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$P$ is the set of all propositions, and the elements of $E(w) \subseteq P$ are the propositions believed at world $w$. If $A \in P$, then, intuitively, $KA = \{ w \in W \mid A \in E(w) \}$ is the proposition saying “$A$ is believed (known)”. Note that $KA$ is an upper set because $E$ is a monotonic function on $W$. Also note that $W \in P$ implies that $W \in E(w)$, for any $w \in W$.

**Definition 3.2.** Let $\mathcal{F} = (W, R, P, E, w_T)$ be an EL5$^-$-frame. $\mathcal{F}$ is

- an E4L5$^-$-frame if $A \in E(w)$ implies $\{ w' \in W \mid A \in E(w') \} \subseteq E(w)$, for any $w \in W$ and $A \in P$.

- an E5L5$^-$-frame if $\mathcal{F}$ is an E4L5$^-$-frame and for every $w \in W$ and every $A \in P$: if $A \notin E(w')$ for all $w' \in R(w)$, then the proposition $\{ w'' \in W \mid A \in E(w'') \} \supset \emptyset$ belongs to $E(w)$.

- an E6L5$^-$-frame if for all $w \in W$, $E(w) = E(w_\bot)$, where $w_\bot$ is the bottom world. That is, $E : W \rightarrow \text{Pow}(P)$ is a constant function and there is only one (global) set of believed propositions $E(w_\bot)$ which we simply denote by $E := E(w_\bot)$.

- an EkL5-frame, for $k \in \{4, 5, 6\}$, if $\mathcal{F}$ is an EkL5$^-$-frame and for every $w \in W$, each element of $E(w)$ contains all those maximal worlds which are accessible from $w$:

$$\text{Max}(W) \cap R(w) \subseteq A, \text{ for each } A \in E(w)$$

The condition of an E4L5$^-$-frame says that whenever a proposition $A$ is believed at $w$, then the proposition “$A$ is believed” is believed at $w$. The E5L5$^-$-condition says that if proposition $A$ is unbelievable from the point of view of world $w$, then the proposition “$A$ is unbelievable” is believed at $w$. In an E6L5$^-$-frame, a proposition $A$ is believed at some world iff $A$ is believed at all worlds iff $A \in E$. Consequently, for any $A \in P$, the proposition $KA = \{ w \in W \mid A \in E(w) \}$, “$A$ is believed”, is either given by the whole set $W$ or by the empty set. Finally, the condition of an EkL5-frame says that if a proposition $A$ is believed at world $w$, then $A$ is true at all maximal worlds accessible from $w$. This is equivalent to the following: If a proposition $A$ is believed at world $w$, then for any world $w'$ accessible from $w$, $A$ cannot be false at $w'$. This is a semantic counterpart of intuitionistic

\[\text{It follows in particular that } \emptyset \notin E(w). \text{ Thus, } E(w) \text{ is a filter on } P.\]
reflection, axiom (IntRe). Under this condition, belief becomes knowledge.

It is clear by the definition that every $E5L5^-$-frame is an $E4L5^-$-frame. Furthermore:

**Lemma 3.3.** Every $E6L5^-$-frame is an $E5L5^-$-frame.

**Proof.** Suppose we are given an $E6L5^-$-frame and $A \in E$, for some $A \in P$. For all $w \in W$, we have $E = E(w)$. It follows that $\{w \in W \mid A \in E(w)\} = W$. From the properties of $E = E(w)$ it follows that $W \in E$. Hence, the condition of an $E4L5^-$-frame is satisfied. Now, assume $A \notin E$. That is, $A \notin E(w)$ for all $w \in W$. Hence, $\{w \in W \mid A \in E(w)\} = \emptyset$ and $(\emptyset \supset \emptyset) = W \in E$. Thus, the condition of an $E5L5^-$-frame holds, too. \qed

An assignment (or valuation) in a given frame $F = (W, R, P, E, w_T)$ is a function $g: V \to P$. Given a frame $F$ and an assignment $g$ in $F$, we call the tuple $(F, g)$ a relational model based on frame $F$. Given a relational model $K = (F, g)$, the relation of satisfaction $w \models \varphi$, read: “$\varphi$ is true at $w$”, between worlds and formulas is defined by induction on the complexity of formulas, simultaneously for all worlds of the underlying frame $F$:

1. $w \not\models \bot$
2. $w \models x :\iff w \in g(x)$
3. $w \models \varphi \lor \psi :\iff w \models \varphi$ or $w \models \psi$
4. $w \models \varphi \land \psi :\iff w \models \varphi$ and $w \models \psi$
5. $w \models \varphi \rightarrow \psi :\iff$ for all $w' \in R(w)$, $w' \models \varphi$ implies $w' \models \psi$
6. $w \models \Box \varphi :\iff w_\bot \models \varphi$
7. $w \models K\varphi :\iff \varphi^* \in E(w)$, where $\varphi^* := \{w' \in W \mid w' \models \varphi\}$.

We write $(F, w) \models \varphi$ instead of $w \models \varphi$ when we want to emphasize the ambient model $F$.

We observe that the meaning of logical connectives is given as in IPC whereas the necessity operator behaves as in modal logic S5: $(w, g) \models \Box \varphi \iff$ for all $w' \in W$: $(w', g) \models \varphi$. The latter follows from the fact that we are dealing with rooted frames in which the usual monotonicity condition of intuitionistic frames holds: formulas true at some world remain true at accessible worlds (see the next Remark).
Remark 3.4. Let $\mathcal{K} = (\mathcal{F}, g)$ be a relational model with set of propositions $P$. We extend the assignment $g: V \rightarrow P$ to a function $g: \text{Fm} \rightarrow P$ defining recursively $g(\bot) := \emptyset$, $g(\varphi \lor \psi) := g(\varphi) \cup g(\psi)$, $g(\varphi \land \psi) := g(\varphi) \cap g(\psi)$, $g(\varphi \rightarrow \psi) := g(\varphi) \supset g(\psi)$, and $g(K\varphi) := \{w \mid g(\varphi) \in E(w)\}$.

By closure properties of $P$, it follows inductively that $g$ is well-defined, i.e. $g(\varphi)$ is an element of $P$ for all $\varphi \in \text{Fm}$. Also by induction on the complexity of formulas, simultaneously for all worlds $w \in W$, one shows that for all $w \in W$ and all $\varphi \in \text{Fm}$, $w \models \varphi \iff w \in g(\varphi)$. That is, $g(\varphi) = \{w \in W \mid w \models \varphi\} = \varphi^*$, for any $\varphi \in \text{Fm}$. In particular, each $\varphi^*$ is a proposition, i.e. an element of $P$. Since all propositions are upper sets, the usual monotonicity condition of intuitionistic models follows: if $w \models \varphi$ and $wRw'$, then $w' \models \varphi$.

Definition 3.5. Let $\mathcal{K} = (\mathcal{F}, g)$ be a relational model with designated maximal world $w_T$, and let $\varphi \in \text{Fm}$. We say that $\mathcal{K}$ is a model of $\varphi$ (or $\varphi$ is true in $\mathcal{K}$) and write $\mathcal{K} \models \varphi$ if

$$(\mathcal{K}, w_T) \models \varphi.$$ 

This notion extends in the usual way to sets of formulas. Let $\mathcal{L}$ be the logic $\text{EL5}^-$, $\text{EL5}$, $\text{EkL5}^-$, or $\text{EkL5}$ for $k \in \{4, 5, 6\}$, and let $\Phi \subseteq \text{Fm}$. We denote by $\text{Mod}_L^r(\Phi)$ the class of all relational models of $\Phi$ which are based on $\mathcal{L}$-frames. Then we consider the following relation of logical consequence:

$$\Phi \vdash_L^r \psi :\Leftrightarrow \text{Mod}_L^r(\Phi) \subseteq \text{Mod}_L^r(\{\psi\}),$$

where $\Phi \cup \{\varphi\} \subseteq \text{Fm}$.

Theorem 3.6 (Soundness). Let $\mathcal{L}$ be the logic $\text{EL5}^-$, $\text{EL5}$, $\text{EkL5}^-$ or $\text{EkL5}$, for $k \in \{4, 5, 6\}$. Then for any set of formulas $\Phi \cup \{\varphi\} \subseteq \text{Fm}$,

$$\Phi \vdash_L \varphi \Rightarrow \Phi \vdash_L^r \varphi.$$ 

Proof. First, we consider logic $\mathcal{L} = \text{E5L5}$. Let $\mathcal{K}$ be a model based on an $\text{E5L5}$-frame. It suffices to show that $\mathcal{K} \models \Box \varphi$ for all axioms $\varphi$ of logic $\text{E5L5}$ (i.e., all
axioms along with rule AN are sound), and $K \models \psi \lor \neg \psi$ for all formulas $\psi$ (i.e., *tertium non datur* is sound). The latter follows immediately from the definition of satisfaction at a maximal world. For the former, we have to show that $w_\bot \models \varphi$, for each axiom $\varphi$, where $w_\bot$ is the bottom world. This is clear in case of theorems of IPC and their substitution-instances (the frame is also a frame for IPC). Also the cases of (A1) and (A2) follow easily.

(A3): It is enough to show that $w_\bot \models \varphi \to \psi$ implies $w_\bot \models \Box \varphi \to \Box \psi$. This follows easily from the definition of satisfaction.

(A4): It is enough to show that $w_\bot \models \varphi$ implies $w_\bot \models \Box \varphi$. Again, this is clear by the definition of satisfaction.

(A5): Truth of $\neg \Box \varphi$ at some world implies truth of $\neg \Box \varphi$ at all worlds.

(KBel): $K(\varphi \to \psi) \to (K\varphi \to K\psi)$. Let $w \in W$. We have to show:

If $(\varphi \to \psi)^* \in E(w)$ and $\varphi^* \in E(w')$, for any $w' \in R(w)$, then $\psi^* \in E(w')$.

Suppose the premises hold true. Since $wRw'$, we have $E(w) \subseteq E(w')$. Thus, $(\varphi \to \psi)^* \in E(w')$. Recall that $E(w')$ is a filter. The filter properties then imply $(\varphi \to \psi)^* \cap \varphi^* \subseteq \psi^* \in E(w')$.

(CoRe): $\Box \varphi \to \Box K\varphi$. It is enough to show that $w_\bot \models \varphi$ implies $w_\bot \models K\varphi$.

Suppose $w_\bot \models \varphi$. Then $\varphi^* = W$. Moreover, $W \in E(w)$ for every $w \in W$. In particular, $\varphi^* \in E(w_\bot)$. Thus, $w_\bot \models K\varphi$.

(IntRe): $K\varphi \to \neg \neg \varphi$. Suppose $w \models K\varphi$. Then $\varphi^* \in E(w)$. Since we are dealing with an $E5L5$-frame, $\varphi^*$ contains all maximal worlds accessible from $w$, i.e. $w' \models \varphi$ for all $w' \in Max(W) \cap R(w)$. Then for all $w'' \in W$ accessible from $w$, we have $w'' \models \varphi$. Hence, $w \models \neg \neg \varphi$.

(E4): Let $w \models K\varphi$. Then $\varphi^* \in E(w)$. By the property of an $E4L5$-frame, \{w' \in W | \varphi^* \in E(w')\} = \{w' \in W | w' \models K\varphi\} = (K\varphi)^* \in E(w)$. Thus, $w \models KK\varphi$.

(E5): Let $w \models \neg K\varphi$. Then for all $w' \in R(w)$, $\varphi^* \notin E(w')$. By properties of an $E5L5$-frame, $A := \{w'' \in W | \varphi^* \in E(w'')\} \supset \emptyset \in E(w)$.

Claim: $A = (\neg K\varphi)^*$.

Proof of Claim. We have $(\neg K\varphi)^* = \{w'' \in W | w'' \models \neg K\varphi\} = \{w'' \in W | \varphi \notin E(w''), w'' \in R(w'')\}$. Now, by the definition of a proposition of the form $B_1 \supset B_2$, one easily checks that the Claim is true.

So by the Claim, $(\neg K\varphi)^* \in E(w)$. That is, $w \models K\neg K\varphi$.

Finally, we consider the case of logic $L = E6L5$. It remains to show that the axiom schemes (PNB) and (NNB) are valid in the class of all models based on $E6L5$-frames. But this is clear since in an $E6L5$-frame there is only one global
set $E$ of known propositions: a proposition $A$ is known at some world of the frame iff $A$ is known at all worlds of the frame.

Towards the completeness theorem, we show that for any algebraic model of some of our S5-style logics there is a relational model that satisfies precisely the same set of formulas. Completeness w.r.t. relational semantics then will follow from completeness w.r.t. algebraic semantics.

**Theorem 3.7.** Let $\mathcal{L} \in \{EL5^-, EL5, EkL5^-, EkL5 \mid k \in \{4, 5, 6\}\}$ and let $(\mathcal{M}, \gamma)$ be an algebraic $\mathcal{L}$-interpretation. Then there is a relational model $\mathcal{K} = (\mathcal{F}, g)$, based on an $\mathcal{L}$-frame $\mathcal{F}$, such that for all $\varphi \in Fm$:

$$(\mathcal{M}, \gamma) \models \varphi \iff \mathcal{K} \models \varphi.$$  

**Proof.** We prove the assertion in detail for the case $\mathcal{L} = E5L5$. The remaining cases then follow straightforwardly. Suppose we are given an $\mathcal{L}$-interpretation $(\mathcal{M}, \gamma)$ with ultrafilter $TRUE \subseteq M$ of true propositions and filter $BEL \subseteq TRUE$ of known propositions. Let $W$ be the set of all prime filters of the Heyting algebra reduct of $\mathcal{M}$. For $w, w' \in W$, we define $wRw' \iff w \subseteq w'$. Then $W$ is partially ordered by $R$, $w_T := TRUE$ is a maximal element and $w_\bot := \{f_\bot\}$ is the smallest element. Recall that the union of a non-empty chain of prime filters is again a prime filter. Thus, every $R$-chain in $W$ has an upper bound in $W$. For $w \in W$ put

$$BEL(w) := \{m \in M \mid f_K(m) \in w\}.$$  

Obviously, $BEL = BEL(TRUE)$. For $m \in M$ we define

$$m^+ := \{w \in W \mid m \in w\}.$$  

The set of propositions of the desired frame is

$$P := \{m^+ \subseteq W \mid m \in M\}$$  

and the set of propositions known at world $w$ is

$$E(w) := \{m^+ \subseteq W \mid m \in BEL(w)\}.$$  

In the following, we show that $\mathcal{F} = (W, R, P, E, w_T)$ is an $EL5^-$-frame. We have to check that $\mathcal{F}$ satisfies all conditions of Definition 3.1. It is clear that the elements $m^+ \in P$ are upper sets under inclusion, i.e. under $R$. Suppose $wRw'$, i.e. $w \subseteq w'$. Then clearly $BEL(w) \subseteq BEL(w')$ and thus $E(w) \subseteq E(w')$. The
mapping $m \mapsto m^+$ defines a one-to-one correspondence between the propositions $m \in M$ of the algebraic model and the propositions $m^+ \in P$ of the frame.

From properties of prime filters (see also item (d) of Lemma 2.8) and from the definitions it follows that for all $m, m' \in M$:

$$
m^+ \cap m'^+ = f_\wedge(m, m')^+
m^+ \cup m'^+ = f_\vee(m, m')^+
m^+ \supset m'^+ = f_\rightarrow(m, m')^+
K(m^+) = \{ w \in W \mid m^+ \in E(w) \} = f_K(m)^+
$$

Of course, $P$ also contains $\emptyset = (f_\bot)^+$ and $W = (f_\top)^+$ and thus satisfies the closure conditions as required in Definition 3.1. Furthermore, it follows that $(P, \cup, \cap, \supset, \emptyset, W)$ forms, in the obvious way, a Heyting algebra with least and greatest elements $\emptyset, W$, respectively. Although not necessary for this proof, we may consider the following additional operations on that Heyting algebra:

$$
f_K^P(m^+) := K(m^+) = \{ w \in W \mid m^+ \in E(w) \} = f_K(m)^+
$$

$$
f_\Box^P(m^+) := \begin{cases} 
W = (f_\top)^+ = f_\Box(m)^+ & \text{if } m = f_\top \\
\emptyset = (f_\bot)^+ = f_\Box(m)^+ & \text{if } m \neq f_\top 
\end{cases}
$$

for all $m \in M$, and observe that this results in a structure that is isomorphic to the original $EL_5^-$-model. In fact, one easily recognizes that the map $m \mapsto m^+$ is an isomorphism between Heyting algebras. Suppose $m = f_\top$. Since we are dealing with an $EL_5^-$-model, we have $f_\Box(m) = f_\top$ and thus, by truth conditions (vii) and (ii) of an algebraic model, $f_K(m) = f_\top \in w$, as $w$ is a filter. By definition of $BEL(w)$, $m = f_\top \in BEL(w)$. We have shown that for every $w \in W$, $BEL(w)$ contains the top element $f_\top$ of the underlying Heyting lattice, and $E(w) \neq \emptyset$.

Now, in the same way as in the proof of Lemma 2.11 with $TRUE$ replaced by $w$, one shows the following two facts:

$$
m, m' \in BEL(w) \Rightarrow f_\wedge(m, m') \in BEL(w),
m \in BEL(w) \text{ and } m \leq m' \Rightarrow m' \in BEL(w).
$$

\footnote{Surjectivity is clear. Towards injectivity suppose $m^+ = m'^+$, i.e. $m$ and $m'$ are contained in exactly the same prime filters. Item (d) of Lemma 2.8 then implies, $f_\rightarrow(m, m') = f_\top$ and $f_\rightarrow(m', m) = f_\top$ (recall that $\{f_\top\}$ is the smallest prime filter, contained in all prime filters). But this means that $m \leq m'$ and $m' \leq m$, i.e. $m = m'$.}
Since $m \mapsto m^+$ is an isomorphism between Heyting algebras, we get
\[
m^+, m'^+ \in E(w) \Rightarrow m^+ \cap m'^+ \in E(w), \\
m^+ \in E(w) \text{ and } m^+ \subseteq m'^+ \Rightarrow m'^+ \in E(w).
\]
Thus, $\mathcal{F} = (W, R, P, E, w_T)$ is an $EL5^-$-frame. Let us show that $\mathcal{F}$ is an $E5L5^-$-frame according to Definition 3.2. Suppose $w \in W$ and $m^+ \in E(w)$. Then $m \in \text{BEL}(w)$ and $f_K(m) \in w$. Because $\mathcal{M}$ is an $E4L5$-model, we have $f_K(m) \leq f_K(f_K(m))$ and thus $f_K(f_K(m)) \in w$ (w is a filter). It follows that $f_K(m)^+ = K(m^+) = \{w \in W \mid m^+ \in E(w)\} \in E(w)$. Now suppose $w \in W$ and for all $w' \in R(w)$, $m^+ \notin E(w')$. Then $f_K(m) \notin w'$ for all prime filters $w'$ extending prime filter $w$. That is, $f_-(f_K(m)) \in w$ (see Lemma 2.8). Using the fact that $\mathcal{M}$ is an $E5L5^-$-model, we conclude $f_K(f_-(f_K(m))) \in w$. By definition of the sets $\text{BEL}(w)$ and $E(w)$, we get $f_-(f_K(m)) \in \text{BEL}(w)$ and thus
\[
f_-(f_K(m))^+ = f_-(f_K(m), f_\perp)^+ = f_K(m)^+ \supset (f_\perp)^+ = \{w \in W \mid m^+ \in E(w)\} \supset \emptyset \in E(w).
\]
Hence, $\mathcal{F}$ is an $E5L5^-$-frame. Moreover, since $\mathcal{M}$ is a model of knowledge, we have $f_K(m) \leq f_-(f_K(m))$ for all $m \in M$. So if $m \in \text{BEL}(w)$, then $f_K(m) \in w$ and thus $f_-(f_K(m)) \in w$. That is, $m \in \text{BEL}(w)$ implies that $m$ belongs to all maximal worlds, i.e. ultrafilters, accessible from $w$:
\[
m \in \text{BEL}(w) \Rightarrow m \in \text{Max}(W) \cap R(w).
\]
Hence,
\[
m^+ \in E(w) \Rightarrow \text{Max}(W) \cap R(w) \subseteq m^+.
\]
In particular, $\emptyset \notin E(w)$ and $E(w)$ is a filter on $P$, for every $w \in W$. Thus, $\mathcal{F} = (W, R, P, E, w_T)$ is an $E5L5$-frame in the sense of Definition 3.2. Now, we define the following assignment $g : V \rightarrow P$ in $\mathcal{F}$:
\[
g(x) := \gamma(x)^+,
\]
for each $x \in V$. Using induction, Remark 3.4 and the previous results, one shows that
\[
g(\varphi) = \gamma(\varphi)^+,
\]
for all $\varphi \in Fm$. For instance, using the induction hypothesis, we have $\gamma(K\psi)^+ = f_K(\gamma(\psi))^+ = \{w \in W \mid \gamma(\psi)^+ \in E(w)\} = \{w \in W \mid g(\varphi) \in E(w)\} = \ldots$

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We consider the relational model $\mathcal{K} = (\mathcal{F}, g)$. By Remark 3.4, $g(\varphi) = \varphi^* = \gamma(\varphi)^+$. So we have for all $w \in W$ and all $\varphi \in \mathcal{F}$:

$$w \models \varphi \iff w \subseteq \varphi^* \iff w \subseteq \gamma(\varphi)^+ \iff \gamma(\varphi) \subseteq w.$$  

In particular, for the designated maximal world $w_T = \text{TRUE}$:

$$\mathcal{K} \models \varphi \iff w_T \models \varphi \iff \gamma(\varphi) \subseteq w_T = \text{TRUE} \iff (\mathcal{M}, \gamma) \models \varphi.$$  

We have proved the assertion of the Theorem for the case $\mathcal{L} = E5L5$ and, implicitly, also for the cases $\mathcal{L} \in \{E5L5^-, E4L5^-, E4L5\}$. Finally, let us consider the cases $\mathcal{L} \in \{E6L5^-, E6L5\}$. We suppose that $\mathcal{M}$ is an $E6L5$-model. Applying the above construction, it suffices to show that the resulting function $E: W \rightarrow \text{Pow}(P)$ is constant, i.e. $E(w) = E(w_\bot)$ for all $w \in W$. By definition of an $E6L5^-$-model, for each $m \in M$, there are only two possibilities: either $f_K(m) = f_\top$ or $f_K(m) = f_\bot$. It follows that for any $w \in W$: $m \in \text{BEL}(w) \iff f_K(m) \subseteq w \iff f_K(m) = f_\top \iff m \in \text{BEL}$. That is, $\text{BEL}(w) = \text{BEL}$ for all $w \in W$ and function $E$ is constant on $W$.

**Corollary 3.8** (Completeness w.r.t. relational semantics). Let $\mathcal{L}$ be $EL5^-$, $EL5$, $EkL5^-$ or $EkL5$, for $k \in \{4, 5, 6\}$. Then for any $\Psi \cup \{\chi\} \subseteq \mathcal{F}$,

$$\Psi \not\models^{r}_{\mathcal{L}} \chi \Rightarrow \Psi \not\models_{\mathcal{L}} \chi.$$  

**Proof.** Suppose $\Psi \not\models_{\mathcal{L}} \chi$. By standard arguments, the set $\Psi \cup \{\neg \chi\}$ is consistent in classical logic $\mathcal{L}$. By algebraic completeness, we know that there is some algebraic $\mathcal{L}$-interpretation $(\mathcal{M}, \gamma)$ satisfying that set. By Theorem 3.7 there is a relational model $\mathcal{K}$ based on an $\mathcal{L}$-frame such that $\mathcal{K} \models \Psi \cup \{\neg \chi\}$. Then $\mathcal{K}$ witnesses $\Psi \not\models^{r}_{\mathcal{L}} \chi$.

## 4 New relational semantics for $IEL^-$ and $IEL$

In this section, we work with the reduced epistemic language $Fm_e = \{\varphi \in \mathcal{F} \mid \text{symbol } \square \text{ does not occur in } \varphi\}$. As already mentioned above, the intuitionistic epistemic logics $IEL^-$ and $IEL$, introduced in [2], can be axiomatized in $Fm_e$ by the axioms (INT), distribution of belief (KBel) $K(\varphi \rightarrow \psi) \rightarrow (K\varphi \rightarrow K\psi)$, intuitionistic co-reflection $\varphi \rightarrow K\varphi$, and – only in case of $IEL$ – intuitionistic reflection (IntRe) $K\varphi \rightarrow \neg \neg \varphi$. The only reference rule is Modus Ponens MP. In [2]
it is shown that $IEL^-$ and $IEL$ are sound and complete w.r.t. possible worlds semantics based on intuitionistic Kripke models. In this section, we show that these logics are sound and complete w.r.t. relational semantics of the kind presented in the preceding section. More precisely, $IEL^-$ and $IEL$ are complete w.r.t. classes of special $EL5^-$- and $EL5$-frames, respectively, which are now interpreted from an intuitionistic instead of the classical point of view. Consequently, the systems of Intuitionistic Epistemic Logic introduced in [2] and the modal logics $EL5^-$ and $EL5$ (and their extensions) can be described within the same semantic framework. The relationship between the intuitionistic epistemic systems of [2] on the one hand and the classical epistemic systems $EL5^-$ and $EL5$ on the other hand now becomes explicit and can be formalized in terms of our relational semantics.

**Definition 4.1.** An $IEL^-$-frame (an $IEL$-frame) $F = (W, R, P, E)$ is defined in exactly the same way as an $EL5^-$-frame (an $EL5$-frame), respectively (see Definition 3.1), but without a designated maximal world and with the following additional condition of intuitionistic co-reflection:

(IntCo) For every $w \in W$ and for all propositions $A \in P : w \models A \Rightarrow A \in E(w)$.

Intuitively, (IntCo) says that whenever a proposition $A$ is true at some world $w$, then $A$ is known at $w$. This is a rather strong condition which, in particular, implies positive and negative introspection, as the next result shows.

**Lemma 4.2.** Every $IEL^-$-frame ($IEL$-frame) is an $E5L5^-$-frame ($E5L5$-frame), respectively. That is, the axioms of positive and negative introspection, $(E4)$ and $(E5)$, are valid.

**Proof.** Let $F = (W, R, P, E)$ be an $IEL^-$-frame. It remains to show that the conditions of an $E5L5^-$-frame of Definition 3.2 are satisfied. Let $w \in W$ and suppose $A \in E(w)$. Then $w \in KA = \{w' \in W \mid A \in E(w')\}$, i.e. proposition $KA$ is true at $w$. By condition (IntCo), $KA \in E(w)$. Thus, the condition of an $E4L5^-$-frame is satisfied. Now, suppose $A \not\in E(w')$ for all $w' \in R(w)$. Then $w \in \neg KA = (\{w' \in W \mid A \in E(w')\} \supset \emptyset)$. By condition (IntCo), $\neg KA \in E(w)$. Thus, the condition of an $E5L5$-frame is satisfied. □

As before, an assignment (or valuation) in an $IEL^-$-frame $F = (W, R, P, E)$ is a function $g : V \rightarrow P$. A relational $IEL^-$-model ($IEL$-model) is a tuple $K = (F, g)$ where $F$ is an $IEL^-$-frame ($IEL$-frame), respectively, and $g$ is a corresponding assignment. Also the relation of satisfaction $w \models \varphi$ between worlds $w \in W$ and formulas $\varphi \in \text{Fm}_e$ is defined as before (of course, without the clause
Of course, the concept of intuitionistic truth in a frame-based model should differ from the concept of classical truth given in such a model. Instead of a designated maximal world, we define truth in a frame relative to the bottom world.

**Definition 4.3.** Let $\mathcal{F}$ be an $IEL^-$-frame with bottom world $w_\bot$ and let $g : V \to P$ be an assinment. The notion of “formula $\varphi \in \text{Fm}_e$ is true in model $(\mathcal{F}, g)$” is defined as follows:

$$(\mathcal{F}, g) \models \varphi :\iff w_\bot \models \varphi.$$  

We call $\mathcal{K} = (\mathcal{F}, g)$ a relational $IEL^-$-model of $\varphi$ if $\varphi$ is true in $\mathcal{K}$.

**Theorem 4.4 (Soundness).** Every theorem of $IEL^-$ is true in all relational $IEL^-$-models, and every theorem of $IEL$ is true in all relational $IEL$-models.

**Proof.** We consider logic $IEL$. Theorems of IPC and their substitution-instances are true in relational models because such models are based on intuitionistic Kripke frames. Let us show that intuitionistic co-reflection $\varphi \rightarrow K\varphi$ is valid. Suppose we are given a relational model based on an $IEL$-frame and $w \models \varphi$, for some world $w \in W$. Then $w \in \varphi^*$ and the semantic condition (IntCo) of an $IEL$-frame yields $\varphi^* \in E(w)$, i.e. $w \models K\varphi$. Validity of intuitionistic reflection (IntRe) and distribution of knowledge (KBel) is shown in exactly the same way as in the proof of Theorem 3.6. \hfill \Box

Towards completeness, we follow a similar strategy as before. That is, we reduce completeness w.r.t. relational semantics to completeness w.r.t. algebraic semantics. We proved in [Theorem 5.3, [8]] that $IEL^-$ and $IEL$ are sound and complete w.r.t. corresponding algebraic semantics. For convenience, we quote here the definition of that algebraic semantics from [8]:

**Definition 4.5.** [8] An algebraic $IEL^-$-model is a Heyting algebra

$$\mathcal{M} = (M, \text{BEL}, f_\bot, f_\top, f_\lor, f_\land, f_\rightarrow, f_K)$$

with propositional universe $M$, a set $\text{BEL} \subseteq M$ of believed propositions and an additional unary operation $f_K$ such that for all propositions $m, m' \in M$ the following truth conditions hold:

(i) $f_\top \in \text{BEL}$

(ii) $f_K(m) = f_\top \iff m \in \text{BEL}$
(iii) \( m \leq f_K(m) \)

(iv) \( f_K(f_{\rightarrow}(m, m')) \leq f_{\rightarrow}(f_K(m), f_K(m')) \)

(v) \( f_\lor(m, m') = f_\top \Rightarrow (m = f_\top \text{ or } m' = f_\top) \)

If additionally \( f_K(m) \leq f_\neg(f_\neg(m)) \) holds for all \( m \in M \), then we call \( \mathcal{M} \) an IEL-model and \( \text{BEL} \) is the set of known propositions.

The notion of an assignment \( \gamma: V \rightarrow M \) in an IEL-model is given as usual. We refer to a tuple \((\mathcal{M}, \gamma)\) as an IEL-interpretation (IEL-interpretation) if \( \mathcal{M} \) is an algebraic IEL-model (IEL-model), respectively, and \( \gamma \) is a corresponding assignment. Satisfaction (truth) of a formula \( \varphi \in F_{me} \) in an IEL-interpretation \((\mathcal{M}, \gamma)\) is defined as follows:

\[(\mathcal{M}, \gamma) \models \varphi : \Leftrightarrow \gamma(\varphi) = f_\top.\]

We quote the soundness and completeness results (in weak form) from [8]:

**Theorem 4.6** ([8]). Let \( \varphi \in F_{me} \). Then \( \varphi \) is a theorem of IEL\(^{-}\) (of IEL) iff \( \varphi \) is true in all algebraic IEL\(^{-}\)-interpretations (in all algebraic IEL-interpretations), respectively.

The next result is an analogue to Theorem 3.7 above.

**Theorem 4.7.** Let \( \mathcal{M} \) be an algebraic IEL-model and let \( \gamma \in M^V \) be an assignment. There is a relational IEL-model \( \mathcal{K} = (\mathcal{F}, g) \) such that for all \( \varphi \in F_{me} \):

\[(\mathcal{M}, \gamma) \models \varphi \Leftrightarrow \mathcal{K} \models \varphi.\]

**Proof.** Let \( \mathcal{M} \) be an algebraic IEL-model with set \( \text{BEL} \) of believed propositions, and let \( \gamma \in M^V \) be an assignment in \( \mathcal{M} \). The construction of an IEL-frame \( \mathcal{F} \) from the given algebraic IEL-model works nearly in the same way as in the proof of Theorem 3.7, where an EL5-frame is constructed from a given algebraic EL5-model. The role of world \( w_T = \text{TRUE} \) now is played by world \( w_\perp = \{ f_\perp \} \). Also note that \( \text{BEL} = \text{BEL}(w_\perp) \). The frame \( \mathcal{F} = (W, R, P, E) \) then is given in exactly the same way as in the proof of Theorem 3.7 but without designated maximal world \( w_T \). By definition of an algebraic IEL-model, \( f_\top \in \text{BEL} \). Since \( \text{BEL} = \text{BEL}(w_\perp) \), that element is contained in all sets \( \text{BEL}(w) \). Hence, the sets \( E(w) \) are non-empty. All the remaining conditions of an EL5-frame are checked as in the proof of Theorem 3.7 (we may skip the part of the proof where the
additional conditions of an $E5L5^-$-frame are verified). Then, as before, we arrive at the following conclusions. For all $w \in W$ and all $\varphi \in Fm_e$:

$$w \models \varphi \iff w \in \varphi^* \iff w \in \gamma(\varphi)^+ \iff \gamma(\varphi) \in w.$$  

In particular, for the bottom world $w_\perp$, assignment $g(x) := \gamma(x)^+$ and relational $IEL$-model $K = (\mathcal{F}, g)$:

$$K \models \varphi \iff w_\perp \models \varphi \iff \gamma(\varphi) \in w_\perp = \{f_\perp\} \iff (\mathcal{M}, \gamma) \models \varphi.$$  

It remains to show that the $EL5$-frame $\mathcal{F}$ satisfies the additional condition (IntCo) of an $IEL$-frame. Let $w \in W$ be a prime filter of the algebraic model and let $m^+ \in P$ be a proposition such that $w \in m^+$, i.e. $m \in w$. By truth condition (iii) of an algebraic model (see Definition 4.5), it follows that $f_K(m) \in w$, since $w$ is a filter. Then, by the definitions, $m \in BEL(w)$ and $m^+ \in E(w)$. We have shown that (IntCo) holds.

It is clear that the assertion of Theorem 4.7 remains true if we replace $IEL$ with $IEL^-$. Finally, we obtain (weak) soundness and completeness of the intuitionistic epistemic systems of [2] w.r.t. our relational semantics.

**Corollary 4.8** (Completeness of $IEL^-$ and $IEL$ w.r.t. relational semantics). A formula $\varphi \in Fm_e$ is a theorem of $IEL^-$ (of $IEL$) iff $\varphi$ is true in all relational $IEL^-$-models ($IEL$-models), respectively.

**Proof.** Soundness is shown in Theorem 4.4. Completeness follows from the Theorems 4.6 and 4.7.

5 Final remarks and conclusions

We have further investigated a hierarchy of classical modal logics, originally presented in [8], for the reasoning about proof (i.e. intuitionistic truth), belief and knowledge. As a main result, we have developed a relational semantics, based on intuitionistic general frames, for the S5-style logics of our hierarchy. The existence of such a semantics not only confirms that all axioms are intuitionistically acceptable (they hold at each world of an intuitionistic frame) but also justifies the S5-style and epistemic axioms as adequate principles for the reasoning about proof (and belief and knowledge) in the sense of BHK reading. Moreover, we have seen that the same semantical framework can be used to describe the intuitionistic
epistemic logics introduced by Artemov and Protopopescu \[2\]. The relationship
between our classical S5-style systems and the intuitionistic logics of \[2\] becomes
explicit. In fact, both concepts, intuitionistic knowledge formalized in logic \(IEL\)
of \[2\] and (classical) knowledge formalized in \(EL5\) are modeled in a frame by
exactly the same sets \(E(w)\), for each world \(w \in W\). What makes knowledge
intuitionistic is the additional constraint of intuitionistic co-reflection (IntCo): if
\(w \in A\), then \(A \in E(w)\), i.e. if proposition \(A\) is true at \(w\), then \(A\) is known at \(w\).
This condition corresponds to the BHK reading of axiom \(\varphi \rightarrow K\varphi\) coming from
\[2\]. Our relational semantics reveals that that condition is strictly stronger than its
classical counterpart (CoRe) \(\square \varphi \rightarrow \square K\varphi\). Indeed, classical co-reflection (CoRe)
is inherent in our intuitionistic frames. (CoRe) corresponds to the implication
\[w_\bot \models \varphi \Rightarrow w_\bot \models K\varphi,\]
where \(w_\bot\) is the bottom world. This is a property of frames, warranted by the
definitions (recall that \(W \in E(w)\), for any world \(w\), by definition of a frame). On
the other hand, validity of intuitionistic co-reflection \(\varphi \rightarrow K\varphi\) is equivalent to the
following condition:
\[\text{for all } w \in W, w \models \varphi \text{ implies } w \models K\varphi.\]
This, however, is not a general property of frames. It must be forced by the
additional condition (IntCo). Now one clearly recognizes that intuitionistic co-
reflection (IntCo) is strictly stronger than classical co-reflection (CoRe).

As pointed out above, we regard our relational semantics as a formal justification
for our modal and epistemic axioms as laws for the reasoning about proof and
knowledge. Finally, let us justify our axioms applying informal BHK semantics
directly.

(A1) \(\square(\varphi \lor \psi) \rightarrow (\square \varphi \lor \square \psi)\). “For a proof of \(\varphi \lor \psi\) it is necessary to prove
\(\varphi\) or to prove \(\psi\).” This statement follows immediately from the BHK semantics of
disjunction\[13\]

(A2) \(\square \varphi \rightarrow \varphi\). “A proposition is classically true whenever it has a proof.” This
is plausible provided the underlying system is consistent (an inconsistent system
would prove false propositions, too). In the present approach, we consider only
consistent deductive systems.

\[13\text{The converse as well as } \square(\varphi \land \psi) \leftrightarrow (\square \varphi \land \square \psi) \text{ follow from the axioms along with rule AN.}\]
(A4) \( \Box \varphi \rightarrow \Box \Box \varphi \). We assume that a proof \( s \) of \( \varphi \) involves a confirmation of the fact that \( s \) is a proof of \( \varphi \). That is, we assume that proofs are “checked” in the sense that a given proof \( s \) of \( \varphi \) gives rise to a (unique) proof \( t_s \) confirming \( s \) as a proof of \( \varphi \). This justifies scheme (A4).\(^{14}\)

(A3) \( \Box (\varphi \rightarrow \psi) \rightarrow \Box (\Box \varphi \rightarrow \Box \psi) \). This says that whenever \( \varphi \rightarrow \psi \) has a proof, there is also a proof that the existence of a proof of \( \varphi \) implies the existence of a proof of \( \psi \). Suppose \( \Box (\varphi \rightarrow \psi) \), i.e. there is a proof of \( \varphi \rightarrow \psi \). By BHK semantics, there is a function \( \Box (\varphi \rightarrow \psi) \), i.e. the classical reading of \( \Box (\Box \varphi \rightarrow \Box \psi) \) is justified. This, however, is weaker than the BHK interpretation of \( \Box \varphi \rightarrow \Box \psi \) which implies the existence of a function \( s' \) that converts a proof of \( \Box \varphi \) into a proof of \( \Box \psi \). Given function \( s \), how can we justify the existence of function \( s' \)? Let \( t \) be a proof of \( \Box \varphi \). \( t \) proves the existence of a proof of \( \varphi \). That is, \( t \) witnesses a concrete proof of \( \varphi \) which, by means of function \( s \), can be converted into a proof of \( \psi \). Relying on the previous justification of (A4), from that proof of \( \psi \), a proof \( t' \) of \( \Box \psi \) can be produced. We have described a procedure that transforms a proof \( t \) of \( \Box \varphi \) into a proof \( t' \) of \( \Box \psi \). This procedure or function is a proof of \( \Box \varphi \rightarrow \Box \psi \).

(A5) \( \neg \Box \varphi \rightarrow \Box \neg \Box \varphi \). We justify this scheme by the following intuition. In a given ‘model of constructive reasoning’, the proof predicate \( \Box \) refers only to the current stage of the reasoning process, and there is exactly one such stage represented in the model. Suppose the current stage is \( S \) and \( \neg \Box \varphi \) is classically true in the model, i.e. “\( \varphi \) is not proved at stage \( S \)”. Of course, the statement \( \Box \varphi \), “\( \varphi \) is proved at stage \( S \)”, then cannot be true at any possible stage represented in the model. Consequently, the statement \( \Diamond \Box \varphi \), “a proof of \( \Box \varphi \) is possible”, must be false. Thus, \( \neg \Diamond \Box \varphi = \neg \neg \Box \neg \Box \varphi \) is classically true. Obviously, the latter is classically equivalent to \( \Box \neg \Box \varphi \). We have justified that \( \neg \Box \varphi \) implies \( \Box \neg \Box \varphi \). Note that \( \neg \Diamond \Box \varphi \) does not exclude that a proof of \( \varphi \) can eventually be established at some future stage given in the model, i.e. both \( \neg \Diamond \Box \varphi \) and \( \Diamond \varphi \) can be true.

(CoRe) \( \Box \varphi \rightarrow \Box K \varphi \) (co-reflection). We justify this axiom of classical co-

\(^{14}\)A similar intuition seems to underlie the idea of a “proof checker” justifying an explicit form of scheme (A4) as an axiom scheme of Justification Logics (see, e.g., [1]). Alternatively, one may argue that a given entity trivially proves its own existence. A proof of \( \Box \varphi \) is a witness of a concrete proof \( s \) of \( \varphi \). It might be acceptable to regard a proof \( s \) as a witness of itself.
reflection using the BHK interpretation of intuitionistic co-reflection $\varphi \rightarrow K\varphi$ coming from Intuitionistic Epistemic Logic [2]. Suppose $\Box\varphi$ is classically true, i.e. “there is a proof of $\varphi$”. By intuitionistic co-reflection, there is a function that converts the proof of $\varphi$ into a proof of $K\varphi$. Thus, $\Box K\varphi$ is classically true.

Note that (A4) relies on assumptions that we also use to justify (A3). Moreover, the justification of axiom (A5) is plausible and stands for itself. These observations suggest that a S3- or a S4-style axiomatization is not strong enough to formalize in a complete way the concept of “reasoning about proof” in the sense of BHK reading. Also note that the justification of classical co-reflection above relies on the BHK reading of intuitionistic co-reflection. All these observations are in line with the formal results regarding soundness and completeness of our S5-style modal logics w.r.t. BHK-compatible relational semantics. The formal completeness results as well as the informal observations above confirm our S5-style modal logics as adequate systems for the reasoning about proof, belief and knowledge. Finally, we notice that there seems to be no plausible BHK justification for the axioms (PNB) $K\varphi \rightarrow \Box K\varphi$ and (NNB) $\neg K\varphi \rightarrow \Box \neg K\varphi$. We conclude that these axioms are examples of optional stronger reasoning principles.

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