The Judgment of Chaotic Detection System’s State Based on the Lyapunov Exponent

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Abstract

In order to improve the accuracy of the judgment of chaotic detection system’s state, the Lyapunov exponent was used for judging the state of the chaotic detection system in this paper. With the method the system’s state can be quantitatively judged, and based on the method this paper carried out the system simulation and the data analysis. Proved by the experiment result, it is feasible that uses Lyapunov exponent to judge the state of the chaotic detection system, which therefore can be available used on the weak signal detection and then ensure the reliability of the signal detection.

1. Introduction

Lyapunov exponent is an important index to measure the dynamic characteristics of dynamic system, denotes the convergent or divergent average exponential rate of the adjacent orbits in the phase space of the system. And it is an important parameter to judge whether a nonlinear time series is in chaos state or not, and is the most direct characteristic quantity of the chaos system [1].

The Duffing chaotic oscillator is extremely sensitive to the system parameters and immune to the various noises at the same time [2]. Therefore, the weak periodic signals introduced to the system can be detected based on the characteristics of the Duffing chaotic oscillator, and this method is feasible to detect
the weak periodic signal in the complex noises environment. And the essence of the aforesaid method is that uses the weak periodic signal to realize the chaos control, and then, the output state of the system in phase space jumps. Consequently, the accurate judgment of the output state is an important and critical content of the weak signal detection based on the Duffing chaos oscillator. It is only a qualitative analysis to judge the state by the observation of the change in the phase space, and it inevitably has the subjectivity because of the lack of the rigorous reasoning and proof according to the chaos theories. So far, the simple and intuitive method is still adopted to detect the weak signal in the Duffing system on some occasions, but the reliability of the detected result is quite difficult to ensure. In order to accurately detect the weak signal from the complex noises, it is essential to define an index to describe the state changes of the system in the phase space, and the index should meet the requirement that it is sensitive to the weak signal and has immunity to noises. And Lyapunov exponent just reflects the dynamic characteristics of dynamic system, and is of quite importance to quantitatively judge the state of chaos system. In the paper, quantitative analysis by the Lyapunov exponent is adopted to accurately judge the critical state of chaos system and further to improve the reliability of the result.

2. The method of system’s state judgment

2.1. Traditional method

The weak periodic signal under complex noises is able to make the output state of the chaotic system change from chaos to the great scale period; and this is the main basis for the weak signal detection by Duffing chaotic oscillator, and the Duffing equation is sensitive to the parameters. The theory equation is the Holmes Duffing equation [3]:

\[ x''(t) + kx'(t) - x^3(t) + x^3(t) = f_r \cos t \]

(1)

In the expression, \( k \) is the damping ratio, and \( f_r \) is the periodic force. And researches show that on the premise of \( k \) fixation and \( f_r \) gradual increase from 0, the solution of the Duffing equation experiences five states, which are subharmonic orbit, period-doubling bifurcation, chaotic orbit, critical periodic orbit and great periodic state. When \( f_r \) is over the threshold value, the state of system rapidly changes from chaos to the great period. And at this moment the system is sensitive to the weak periodic signal which frequency is equal to that of \( f_r \), and even quite weak signal with the same frequency introduced to the system can cause the obvious change of system state in the phase space (shown as Fig.1), but the signal with different frequency is adverse.

![Fig. 1. Phase change of the Duffing system](image-url)
In practice, when detecting the weak periodic signal from the strong noises based on the Duffing system, the accurate judgment of the state’s change is vital to the detection, and often adopts the method that changing the force \( f_r \) to have the detection system be in the critical state from chaos to great scale period. And then introduces the weak periodic signal to the system and observes the phase change in phase space, thus can detect the weak signal. However, there are some defaults of the method. Firstly, the step has negative effect on the accurate judgment to some extent when adjusts the force \( f_r \) to change the system’s state, and the size of the step is only chosen by the experience, and consequently the accuracy of the threshold value is difficult to ensure. On the other hand, the system’s state is difficult to be accurately judged, and inevitably exists subjectivity.

2.2. Method based on the Largest Lyapunov exponent

By analyzing the Lyapunov exponents of the system, the system’s state can be clearly described, and it is easy to decide the corresponding relationship between the amplitude of the signal and the system state. So the state of the system can be accurately judged by the Lyapunov exponent. For the two-dimensional system, one Lyapunov exponent is zero means that the system is in the chaos vital state, accordingly the two Lyapunov exponents are both negative means that the system is in the periodic state, at least one Lyapunov exponent is positive when the system is in the chaotic system [4]. Just as the Lyapunov exponents to the two-dimension system’s state, the conclusion can be drown for the three-dimension system [5], and the concrete content is shown as table 1.

| Lyapunov exponents | Exponent sign | System’s state          |
|--------------------|---------------|-------------------------|
| \( \lambda_1, \lambda_2, \lambda_3 \) | (-,-,-)        | Fixed point             |
|                    | (0,-,-)       | Limit cycle             |
|                    | (0,0,-)       | Two-dimension torus     |
|                    | (+,-,0)       | Unstable limit cycle    |
|                    | (+,0,0)       | Unstable two-dimension torus |
|                    | (+,0,-)       | Strange attractor       |

In the design, the detection system is used as a three-dimension system. The \( f_r \) is in the interval \([0, 1]\), and chooses 0.02 as the size of the step considering the computing complexity. For the per \( f_r \), the time interval is 0.01 when computes the Lyapunov exponents, and the evolution step is 10. The unstable iteration is discarded when draws the figure of the Lyapunov exponent spectrum. Figure 2 shows the Lyapunov exponent spectrum of chaotic state and the periodic state respectively. In the two figures there is an unstable interval, and in the interval the sign of the exponent is uncertain, in order to accurately judge the system’s state by the Lyapunov exponent, should choose the value in the stable interval as the final Lyapunov exponent [6]. In the article the final Lyapunov exponent is the value at the point of \( n=900 \).

![Fig. 2. (a) Lyapunov exponent spectrum of chaotic state; (b) Lyapunov exponent spectrum of great scale period](image-url)
3. Algorithm of Lyapunov exponent

For the n-dimension continuous dynamic system \( x' = F(x) \), at the time of \( t=0 \), \( x_0 \) for the center, \( \|\Delta x(x_0,0)\| \) for the radius, makes an n-dimension sphere. With the change of time, the sphere becomes ellipsoid. Assumption the radius is \( \|\Delta x(x_0,t)\| \) in a certain direction and the corresponding Lyapunov exponent is defined as follows [7]:

\[
\lambda_i = \lim_{t \to \infty} \frac{1}{t} \ln \frac{\|\Delta x_i(x_0,t)\|}{\|\Delta x_i(x_0,0)\|}
\]

The expression (2) is the Lyapunov exponent definition of the continuous dynamic system. In practice, \( \|\Delta x(x_0,0)\| \) is replaced by a constant \( d \), \( x_0 \) for the center of sphere, and takes the orthogonal set \( \{e_1, e_2, \ldots, e_n\} \) which Euclidean norm is \( d \) as the initial sphere. And then computes the trajectory of \( x_0, x_0 + e_1, x_0 + e_2, \ldots, x_0 + e_n \) when the time \( t \) changes. Assume the corresponding end points are \( x_{00}, x_{01}, \ldots, x_{0n} \), and let \( \Delta x_1^{(i)} = x_{01} - x_{00}, \Delta x_2^{(i)} = x_{02} - x_{00}, \ldots, \Delta x_n^{(i)} = x_{0n} - x_{00}, \) so get a new orthogonal set \( \{\Delta x_1^{(i)}, \Delta x_2^{(i)}, \ldots, \Delta x_n^{(i)}\} \).

During the course of evolution, all the vectors closes up the direction of the largest Lyapunov exponent, in order to track the vector the method of Gram-Schmidt Renormalization is adopted. The computing course is shown as follows [5] [8]:

\[
\begin{align*}
\{v_1^{(1)} &= \Delta x_1^{(1)} \\
u_1^{(1)} &= v_1^{(1)} / \|v_1^{(1)}\| \\
v_2^{(1)} &= \Delta x_2^{(1)} - <\Delta x_2^{(1)}, u_1^{(1)}>, u_1^{(1)} > u_1^{(1)} \\
u_2^{(1)} &= v_2^{(1)} / \|v_2^{(1)}\| \\
\cdots \\
v_n^{(1)} &= \Delta x_n^{(1)} - <\Delta x_n^{(1)}, u_{n-1}^{(1)}>, u_{n-1}^{(1)} > u_{n-1}^{(1)} \\
u_n^{(1)} &= v_n^{(1)} / \|v_n^{(1)}\|
\end{align*}
\]

Subsequently, \( x_{00} \) for the center of sphere, new orthogonal set \( \{du_1^{(1)}, du_2^{(1)}, \ldots, du_n^{(1)}\} \) continue to carry out the same evolution. When \( N \) is enough, the Lyapunov exponent formula is expressed as follows:

\[
\begin{align*}
\lambda_i &= -\frac{\ln d}{T} + \frac{1}{NT} \sum_{t=1}^{T} \ln \|v_t^{(1)}\| \\
\cdots \\
\lambda_n &= -\frac{\ln d}{T} + \frac{1}{NT} \sum_{t=1}^{T} \ln \|v_t^{(n)}\|
\end{align*}
\]

4. Simulations

For the system shown as the expression (1), assumes the initial values \( x(0) = 1, x'(0) = 1 \), integration time interval \( \Delta t = 0.01 \), then computes the data by the fourth order Runge-Kutta equation and discards the unstable iterative. The Lyapunov exponent distribution is shown in the figure 3. From the figure, the relationship between the system’s state and the Lyapunov exponents can be clearly judged.
Fig. 3. Distribution of Lyapunov exponent according to $f_r$.

5. Conclusions

Proved by the research, the Lyapunov exponent can be used to analyze the chaotic characteristics of system. And the system’s state can be clearly judged according to the method, in addition, improve the accuracy of the system’s state.

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