BASIC ANALYTIC NUMBER THEORY

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Abstract. We give an informal introduction to the most basic techniques used to evaluate moments on the critical line of the Riemann zeta-function and to find asymptotics for sums of arithmetic functions.

1. Introduction

The simplest way to compute a moment of the zeta-function is to approximate the zeta-function by Dirichlet polynomials and then compute the moment of the polynomials. In this paper we describe the most rudimentary techniques in this area. Along the way discuss the basic methods for finding asymptotics of sums of arithmetic functions, and we also compute the arithmetic factor in the standard conjectures for moments of the zeta-function. The methods described here are completely standard: our intention is to give a brief introduction to those who are new to the subject. The standard reference is Titchmarsh \cite{T} and we cite the specific sections where one can look for more details.

We assume a knowledge of the calculus of one complex variable. Readers unfamiliar with the big-$O$, little-$o$, and $\ll$ notation (and physicists, who may use the $\ll$ notation differently), should consult the Appendix.

2. The “first moment”

The simplest approximation to the zeta-function is

\begin{equation}
\zeta(s) = \sum_{1 \leq n \leq T} \frac{1}{n^s} - \frac{T^{1-s}}{1-s} + O(T^{-\sigma}),
\end{equation}

where $s = \sigma + it$ and the equation is valid for $|t| \leq T$. See \cite{T}, Section 4.11. Specializing to $s = \frac{1}{2} + it$, we have

\begin{equation}
\zeta\left(\frac{1}{2} + it\right) = \sum_{1 \leq n \leq T} \frac{1}{n^{1/2 + it}} + O \left( \frac{T^{1/2}}{1 + |t|} \right).
\end{equation}

Now we can find the average value of the zeta-function on the critical line. The justification of the steps follows the calculation.

\[
\int_0^T \zeta\left(\frac{1}{2} + it\right) dt = \sum_{1 \leq n \leq T} \frac{1}{\sqrt{n}} \int_0^T n^{-it} dt + O \left( T^{1/4} \int_0^T \frac{1}{1 + |t|} dt \right)
\]

\[
= T + \sum_{2 \leq n \leq T} \frac{1}{\sqrt{n}} \frac{1 - n^{-iT}}{\log n} + O \left( T^{1/4} \log T \right)
\]

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In the first step we switched the sum and the integral, which is justified because both are finite. The sum on the second line was estimated by

\[ \sum_{2 \leq n \leq T} \frac{1}{\sqrt{n}} \frac{1 - n^{-iT}}{\log n} \ll \sum_{2 \leq n \leq T} \frac{1}{\sqrt{n} \log n} \ll \int_2^T \frac{1}{\sqrt{x} \log x} \, dx \ll \frac{T^{3/2}}{\log T}, \]

which is smaller than the other error term.

Thus, the zeta-function on the critical line is 1 on average. This is not a particularly useful piece of information, for it is the magnitude of the zeta-function which is of interest. So we consider the moments of \( |\zeta(\frac{1}{2} + it)| \).

3. The 2nd moment

The simple fact that \( |z|^2 = z \bar{z} \) means that \( |\zeta(\frac{1}{2} + it)|^K \) is much more amenable to methods of complex analysis when \( K = 2k \) is an even integer. The easiest case is the 2nd moment, which we compute in this section.

For later use it will be helpful to first consider the 2nd moment of a general Dirichlet polynomial. Suppose

\[ P(s) = \sum_{1 \leq n \leq N} \frac{a_n}{n^s}. \]

We have

\[
\int_0^T |P(it)|^2 \, dt = \int_0^T \left| \sum_{1 \leq n \leq N} \frac{a_n}{n^it} \right|^2 \, dt \\
= \int_0^T \sum_n a_n \overline{a_m} \frac{1}{nm} \sum_{m} \frac{1}{m-it} \, dt \\
= \sum_{n,m} a_n \overline{a_m} \int_0^T \left( \frac{m}{n} \right) \frac{it}{n} \, dt \\
= T \sum_n |a_n|^2 + \sum_{n \neq m} \frac{a_n \overline{a_m} \left( \frac{m}{n} \right)^iT - 1}{\log(m/n)} \\
= T \mathcal{M}(N) + \mathcal{E}(N, T),
\]

say. We think of \( T \mathcal{M}(N) \) as the main term and \( \mathcal{E}(N, T) \) as an error term, so we want to understand when \( \mathcal{E}(N, T) \) will be smaller in magnitude than \( T \mathcal{M}(N) \).

Setting \( m = n + h \) we can rewrite the error term as

\[ \mathcal{E}(N, T) = \sum_n \sum_{h \neq 0} \frac{a_n \overline{a_{n+h}} \left( \frac{n+h}{n} \right)^iT - 1}{\log(1+h/n)}. \]

Now consider only the \( h = 1 \) term from the above sum:

\[ \sum_{1 \leq n \leq N} a_n a_{n+1} \cdot n \cdot (\text{something bounded}). \]
Without any information on \(a_n\), the above sum, which is just one part of \(E(N,T)\), could be about the same size as \(N \mathcal{M}(N)\). Thus, in general one should only expect \(E(N,T)\) to be smaller than \(T \mathcal{M}(N)\) if \(N < T\). And if \(N > T\) then one may need some detailed information about the coefficients \(a_n\) in order to extract something meaningful from \(E(N,T)\). Goldston and Gonek \([GG]\) have given a clear discussion of these issues.

One can carry through the above calculation to obtain a useful general mean value theorem for Dirichlet polynomials. See Titchmarsh \([T]\), Section 7.20. However, a stronger result is provided by the mean value theorem of Montgomery and Vaughan \([MV]\):

\[
\int_0^T \left| \sum_{1 \leq n \leq N} \frac{a_n}{n^{it}} \right|^2 dt = \sum_{1 \leq n \leq N} |a_n|^2 (T + O(n)).
\]

This result is best possible for general sequences \(a_n\). Note that if \(N = o(T)\) then the error term is smaller than the main term.

To use the mean value theorem to compute the second moment of the zeta-function, first write (2.2) as \(\zeta = S + E\). That is, \(S = \sum_{1 \leq n \leq T} \frac{1}{n^{1/2+it}}\) and \(E = O \left( \frac{T^{1/2}}{1 + |t|} \right)\).

Using
\[
|\zeta|^2 = |S + E|^2 = (S + E)(\overline{S} + \overline{E}) = |S|^2 + 2 \Re SE + |E|^2,
\]
we have
\[
\int_0^T |\zeta(1/2 + it)|^2 dt = \int_0^T |S|^2 dt + 2 \Re \int_0^T SE dt + \int_0^T |E|^2 dt.
\]

We want to evaluate the \(|S|^2\) integral as our main term and estimate the \(|E|^2\) integral as our error term, but what to do about the cross term? By the Cauchy-Schwartz inequality,
\[
2 \Re \int_0^T SE dt \ll \int_0^T |S||E| dt \ll \left( \int_0^T |S|^2 dt \right)^{1/2} \left( \int_0^T |E|^2 dt \right)^{1/2}.
\]

If \(\int_0^T |E|^2 dt\) is smaller than \(\int_0^T |S|^2 dt\), then so is the right side of the inequality above. We have the general principle that if the “main error term” is smaller than the main term, then so are the cross terms. It remains only to evaluate \(\int_0^T |S|^2 dt\) and estimate \(\int_0^T |E|^2 dt\).

For the main term use (3.5) with \(a_n = n^{-1/2} \):
\[
\int_0^T |S|^2 dt = \sum_{1 \leq n \leq T} \frac{1}{n} (T + O(n)) = T (\log T + O(1)) + \sum_{1 \leq n \leq T} O(1) = T \log T + O(T).
\]

For the error term we have
\[
\int_0^T |E|^2 dt \ll T \int_0^T \frac{1}{(1 + |t|)^2} dt \ll T.
\]
which is smaller than the main term. We have just proven the mean value result

\[(3.12) \quad \int_0^T |\zeta(\frac{1}{2} + it)|^2 dt = T \log T + O(T \log^{3/2} T).\]

Note that we do not obtain an error term of \(O(T)\) in (3.12) by these methods. However, a much better error term can be obtained. The first step toward this is given in the next section.

**Exercise.** Approximate the sum by an integral to show that if \(A > 1\) then

\[(3.13) \quad \sum_{n>T} \frac{1}{n^A} = \frac{T^{1-A}}{A-1} + O(T^{-A}).\]

Conclude that if \(A > 1\) then

\[(3.14) \quad \sum_{n\leq T} \frac{1}{n^A} = \zeta(A) - \frac{T^{1-A}}{A-1} + O(T^{-A}).\]

**Exercise.** Show that if \(\frac{1}{2} < \sigma < 1\) then

\[(3.15) \quad \int_0^T |\zeta(\sigma + it)|^2 dt = T \left(\zeta(2\sigma) - \frac{T^{1-2\sigma}}{2\sigma - 1}\right) + O\left(T^{\frac{1}{2} - \sigma} \log^{\frac{1}{2}} T\right),\]

where the implied constant in the big-\(O\) term is independent of \(\sigma\).

Since the error term in (3.15) is uniform in sigma, we can let \(\sigma \to \frac{1}{2}^+\) to recover (3.12). This makes use of the fact that \(\zeta(s)\) has a simple pole with residue 1 at \(s = 1\).

Note that if \(\sigma > \frac{1}{2}\) is independent of \(T\) then the right side of (3.15) is of size \(\approx T\). On the other hand, the second moment on the \(\frac{1}{2}\)-line is of size \(T \log T\). Thus there is an abrupt change in the behavior of the zeta-function when one moves onto the critical line. Equation (3.15) illustrates that the transition occurs on the scale of \(1/\log T\) from the \(\frac{1}{2}\)-line.

### 4. Better 2nd moment

The methods of the previous section are not sufficient to evaluate the main term of the 2nd moment with an error term \(O(T^A)\) for \(A < 1\), nor are those methods sufficient to evaluate the 4th moment of the zeta-function. Evaluating the 4th moment by squaring (2.2) gives a Dirichlet polynomial of length \(T^2\), which cannot be handled by the Montgomery-Vaughan mean value theorem. So one needs either a shorter approximation to \(\zeta(s)\), or an approximation to \(\zeta^2(s)\) of length \(\leq T\), or a way to handle longer polynomials. In preparation for the 4th moment, we first evaluate the 2nd moment with a better error term.

The “approximate functional equation” of Hardy and Littlewood expresses the \(\zeta\)-function as a sum of two short Dirichlet polynomials:

\[(4.1) \quad \zeta(s) = \sum_{1 \leq n \leq x} \frac{1}{n^s} + \chi(s) \sum_{1 \leq n \leq y} \frac{1}{n^{1-s}} + O\left(x^{-\sigma} + |t|^\frac{1}{2} - \sigma y^{\sigma-1}\right),\]

where \(xy = t/2\pi\) and \(\chi(s)\) is the usual factor in the functional equation \(\zeta(s) = \chi(s)\zeta(1-s)\). The name “approximate functional equation” comes from the fact that the right side looks
like \( \zeta(s) \) when \( x \) is large and like \( \chi(s) \zeta(1 - s) \) when \( y \) is large. On the \( \frac{1}{2} \)-line we have

\[
\zeta(s) = \sum_{1 \leq n \leq N} \frac{1}{n^{s+i\tau}} + \chi(s) \sum_{1 \leq n \leq N} \frac{1}{n^{s-i\tau}} + O\left(N^{-\frac{1}{2}}\right),
\]

where we set \( x = y = N = \sqrt{t/2\pi} \).

Hardy and Littlewood used the approximate functional equation to evaluate the second moment of the zeta-function with a better error term. We will not carry out the calculation, but just give the flavor. See Titchmarsh \[T\] Section 7.4 for details. Writing (4.2) as \( \zeta(s) = S + \chi(s)S + E \), we have

\[
\int_{T}^{2T} |\zeta(\frac{1}{2} + it)|^2 dt = 2 \int_{T}^{2T} |S|^2 dt + 2 \Re \int_{0}^{T} \chi(\frac{1}{2} - it)S^2 dt + \text{error term}.
\]

In that calculation we used the fact that \(|\chi(\frac{1}{2} + it)| = 1\). Below we will use the more precise information

\[
\chi(s) = \left(\frac{t}{2\pi}\right)^{\frac{1}{2}-s} e^{it+\pi i/4} (1 + O(t^{-1})).
\]

The main difficulty is evaluating the \( \chi(\frac{1}{2} - it)S^2 \) term, which equals

\[
\sum_{n,m} \frac{1}{\sqrt{nm}} \int_{0}^{T} \chi(\frac{1}{2} - it)(nm)^{-it} dt.
\]

By (4.4), up to a negligible error that integral is of the form \( \int e^{if(t)} dt \). If \( f(t) \) has a stationary point in the range of integration then we can extract a main term, otherwise it will become an error term. In particular, that integral can be handled by the method of stationary phase. See Titchmarsh \[T\] Section 7.4 for details. Our point here is that by virtue of (4.4), integrals involving \( \chi(\frac{1}{2} + it) \) and Dirichlet polynomials can be handled. The result that can be obtained by the above argument is

\[
\int_{0}^{T} |\zeta(\frac{1}{2} + it)|^2 dt = T \log T + (2\gamma - 1)T + O(T^{\frac{1}{2}+\epsilon}).
\]

The error term can be improved by more sophisticated methods.

There are many applications of mean values of the zeta-function multiplied by a Dirichlet polynomial. Suppose

\[
M(s) = \sum_{1 \leq n < T^\theta} \frac{b_n}{n^s}
\]

with \( b_n \ll n^\varepsilon \). By the approximate functional equation, \( \zeta(s)M(s) \) can be approximated by Dirichlet polynomials of length \( T^{\frac{1}{2}+\theta} \). So by the above methods, if \( \theta < \frac{1}{2} \) then one should be able to find an asymptotic formula for \( \int_{0}^{T} |\zeta M(\frac{1}{2} + it)|^2 dt \). Evaluating such an integral, with \( \theta = \frac{1}{2} - \varepsilon \), was key to Levinson’s proof that more than one-third of the zeros of the zeta-function are on the critical line. Conrey made use of very deep and technical results to evaluate such an integral with \( \theta = \frac{4}{7} - \varepsilon \), leading to the result that more than two-fifths of the zeros are on the critical line.
5. The 4th moment

To evaluate the 4th moment of the zeta-function by the methods described above, one requires an approximation to \( \zeta(s)^2 \) of length less than \( T \). This is provided by the following approximate functional equation:

\[
\zeta(s)^2 = \sum_{1 \leq n \leq x} \frac{d(n)}{n^s} + \chi(s)^2 \sum_{1 \leq n \leq y} \frac{d(n)}{n^{1-s}} + O(x^{\frac{1}{2} - \sigma} \log t),
\]

where \( xy = (t/2\pi)^2 \) and \( d(n) \) is the number of divisors of \( n \). More generally one should expect an approximate functional equation of the form

\[
\zeta(s)^k = \sum_{1 \leq n \leq x} \frac{d_k(n)}{n^s} + \chi(s)^k \sum_{1 \leq n \leq y} \frac{d_k(n)}{n^{1-s}} + \text{error term}
\]

where \( xy \approx t^k \) and \( d_k(n) \) is the \( k \)-fold divisor function

\[
d_k(n) = \sum_{n_1 \cdots n_k = n} 1,
\]

which has generating function

\[
\zeta(s)^k = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s}, \quad \sigma > 1.
\]

Plugging (5.1) into the Montgomery-Vaughan mean value theorem leads to

\[
\int_0^T |\zeta(\frac{1}{2} + it)|^4 dt = \frac{1}{2\pi^2} T \log^4 T + \text{error term}.
\]

If you actually do the calculation, you will find that in order to determine the main term you need to evaluate sums like

\[
\sum_{1 \leq n \leq X} \frac{d(n)^2}{n}.
\]

There is a standard technique for finding the leading-order asymptotics of such sums, which is given in the next section.

5.1. Comments on approximate functional equations. The error term in (5.1) is rather large and leads to an error term of size \( O(T \log^2 T) \) in the 4th moment \( \int_0^T |\zeta(\frac{1}{2} + it)|^4 dt \). The large size of the error term is due to the fact that our sums have a sharp cut-off. The error term can be much reduced by having a smooth weight in the sums. That is,

\[
\zeta(s)^k = \sum_n \frac{d_k(n)}{n^s} \varphi(n, t) + \chi(s)^k \sum_n \frac{d_k(n)}{n^{1-s}} \varphi^*(n, t) + \text{small error term},
\]

where \( \varphi \) and \( \varphi^* \) are particular functions that are approximately 1 for \( n < t^\frac{k}{2} \) and decay for \( n > t^\frac{k}{2} \). As our previous discussion should suggest, it is the length of the sums, and not the size of the error term, which provides the true difficulty when \( k > 2 \).
Here is the problem: you have an arithmetical function \(a_n\) and you want to find the asymptotics of

\[
S(X) = \sum_{1 \leq n \leq X} a_n.
\]

This problem can often be solved by the most basic methods of analytic number theory.

First note that for integers \(N \geq 1,\)

\[
\frac{1}{2\pi i} \int_{1-iY}^{1+iY} A^s \frac{ds}{s^N} = \begin{cases} 
\log N - 1 + \text{error term} & \text{if } A > 1 \\
0 + \text{error term} & \text{if } A < 1 
\end{cases}
\]

To see this, consider the integral

\[
\frac{1}{2\pi i} \int_{C_1} A^s \frac{ds}{s^N}
\]

where the integration is over the closed rectangular path connecting the points

\[
C_1 = \begin{cases} 
[1 - iY, 1 + iY, -B + iY, -B - iY], & \text{if } A > 1 \\
[1 - iY, 1 + iY, B + iY, B - iY] & \text{if } A < 1,
\end{cases}
\]

where \(B\) is a large positive number. In both cases the main term comes from the residue of the pole at 0, which is or is not inside the path of integration.

**Exercise.** Bound the error term in (6.2) by estimating the integral along the three segments of \(C_1\) other than \([1 - iY, 1 + iY]\). You should find that if \(N \geq 2,\) then you can let \(Y \to \infty\) and the error vanishes. See Section 3.12 of [T] if you aren’t sure how to begin.

To evaluate (6.1), let

\[
F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},
\]

and suppose that the sum converges absolutely for \(\sigma > \sigma_0\). Using (6.2) and supposing \(\sigma > \sigma_0,\) we have

\[
\frac{1}{2\pi i} \int_{\sigma-iY}^{\sigma+iY} F(s) X^s \frac{ds}{s} = \sum_{n=1}^{\infty} a_n \frac{1}{2\pi i} \int_{\sigma-iY}^{\sigma+iY} \left( \frac{X}{n} \right)^s \frac{ds}{s} = \sum_{n=1}^{\infty} a_n \begin{cases} 
1 + \text{error term} & \text{if } X > n \\
0 + \text{error term} & \text{if } X < n 
\end{cases} = S(X) + \text{error term}.
\]

This is known as Perron’s formula. If we can learn enough about the function \(F(s)\) so that the integral in (6.6) can be evaluated in another way, then we will have a formula for \(S(X)\).

Suppose (as is frequently the case) that \(F(s)\) has a pole at \(\sigma_0\) and no other poles in the half-plane \(\sigma > \sigma_1\) for some \(\sigma_1 < \sigma_0\). Then consider

\[
\frac{1}{2\pi i} \int_{C_2(\epsilon)} F(s) X^s \frac{ds}{s},
\]
where for \( \varepsilon > 0 \) the integration is over the rectangular path with vertices
\[
C_2(\varepsilon) = [\sigma - iY, \sigma + iY, \sigma_1 + \varepsilon + iY, \sigma_1 + \varepsilon - iY].
\]
We can evaluate (6.7) by finding the residue at the pole \( s = \sigma_0 \) and at \( s = 0 \) (if \( 0 \) is inside the path of integration. And in the same way as you estimated the error term in (6.2), we find that (6.7) equals the integral in Perron’s formula plus an error term. The final step of bounding the integral on the 3 other segments requires the additional ingredient of a bound for \( F(\sigma + it) \) as \( t \to \infty \), uniformly for \( \sigma > \sigma_1 \).

For example, at the end of the previous section we wanted to evaluate the sum of \( d(n)^2/n \).

**Exercise.** Check that
\[
(6.9) \sum_n \frac{d(n)^2}{n^s} = \frac{\zeta^4(s)}{\zeta(2s)}.
\]
Hint: both sides have an Euler product. The factors on the right can be found from the Euler product for the zeta-function. Those on the left require summing \( \sum_{j=0}^\infty d(p^j)^2 p^{-js} \).

Thus, we apply Perron’s formula (6.6) with
\[
(6.10) F(s) = \frac{\zeta^4(s+1)}{\zeta(2s+2)}.
\]
To determine the analytic properties of \( F(s) \), use the fact that \( \zeta(s) \) is entire except for a simple pole at \( s = 1 \), where we have the Laurent expansion
\[
(6.11) \zeta(s) = \frac{1}{s-1} + \gamma + \cdots.
\]
Also \( \zeta(s) \) has no zeros in \( \sigma > 1 \), and no zeros in \( \sigma > \frac{1}{2} \) assuming the Riemann Hypothesis. In these calculations one frequently needs that \( \zeta(2) = \pi^2/6 \) and
\[
(6.12) \zeta(s) = -\frac{1}{2} - \frac{1}{2} \log(2\pi)s + \cdots
\]
for \( s \) near 0.

To estimate the error terms, one can use the convexity estimate
\[
(6.13) \zeta(\sigma + it) \ll \begin{cases} 1 & \sigma > 1 \\ |t|^\frac{1}{2} & 0 < \sigma < 1 \\ |t|^\frac{1}{2} - \sigma & \sigma < 0, \end{cases}
\]
along with \( \zeta(s) \gg 1 \) for \( \sigma > 1 \). Also, assuming RH we have \( t^{-\varepsilon} \ll \zeta(\sigma + it) \ll t^\varepsilon \) for \( \sigma > \frac{1}{2} \). All of these estimates are for fixed \( \sigma \) as \( t \to \infty \).

Assembling the pieces we find
\[
(6.14) \sum_{1 \leq n \leq X} \frac{d(n)^2}{n} \sim \frac{\log^4 X}{4\pi^2}.
\]

**Exercise.** Argue that (6.14) is of the form \( XP_4(\log X) + O(X^B) \) where \( P_4 \) is a polynomial of degree 4 and \( B < 1 \). Find the next-to-leading coefficient of \( P_4 \), and estimate \( B \) both with and without assuming the Riemann Hypothesis.
Exercise. Deduce the following asymptotics:
\[
\begin{align*}
\sum_{1 \leq n \leq X} d_k(n) & \sim \frac{1}{k!} X \log^k X \\
\sum_{1 \leq n \leq X} \varphi(n) & \sim \frac{3}{n^2} X^2,
\end{align*}
\]
(6.15)
where \(\varphi(n)\) is the Euler totient function. In addition to the generating function (5.4), you should use (and prove):
\[
\sum_{n} \varphi(n) n^s = \zeta(s-1) \zeta(s).
\]
(6.16)

Exercise. Find the next-to-leading order terms in the previous exercise. Also determine the shape of the main terms and estimate the size of the error terms, both with and without the Riemann Hypothesis.

Note that there are interesting and important cases where the above analysis is inadequate. For example, in the proof of the prime number theorem \(a_n = \Lambda(n)\), the von Mangoldt function. Then \(F(s)\) has a pole at \(s = 1\) as well as poles at the zeros of the \(\zeta\)-function and one must use a more complicated path of integration as well as nontrivial estimates for \(\zeta(s)\) in the critical strip.

7. The conjecture for moments of the zeta-function

Much recent work on the relationship between \(L\)-functions and Random Matrix Theory was motivated by the problem of finding conjectures for the \(2k\)th moment of the Riemann zeta-function on the critical line. Conrey and Ghosh [CG] formulated it as follows: for each integer \(k \geq 0\) there exists an integer \(g_k\) such that
\[
\begin{align*}
\int_0^T |\zeta(\frac{1}{2} + it)|^{2k} dt & \sim g_k \frac{a_k}{k^{2k}} T \log^{2k} T,
\end{align*}
\]
(7.1)
where
\[
a_k = \prod_p \left(1 - \frac{1}{p}\right)^{k^2} \sum_{m=0}^{\infty} \binom{k + m - 1}{m} p^{-m}.
\]
(7.2)
In this conjecture the only missing ingredient is the integer \(g_k\). Keating and Snaith [KS] computed the moments of characteristic polynomials of unitary matrices and used the result to conjecture
\[
g_k = k^2! \prod_{j=0}^{k-1} \frac{j!}{(k+j)!}.
\]
(7.3)
It is not trivial to show that this \(g_k\) is actually an integer [CF].

Our last topic in this paper is to show how the factor \(a_k\) arises naturally. From the approximate functional equation (5.22) it is reasonable to consider
\[
\int_0^T \left| \sum_{n< T} \frac{d_k(n)}{n^{\frac{3}{2}+it}} \right|^2 dt.
\]
(7.4)
Note that the sum has length $T$. This is good because we can use the mean value theorem. But it is bad because the polynomial is not long enough to fully approximate $\zeta(\frac{1}{2} + it)^k$. We cannot expect this mean value to equal the $2k$th moment of the zeta-function, but how far will it be off? It would be nice if it were off by some simple factor, so one possible interpretation of $g_k$ is “the number of length $T$ polynomials needed to capture the $2k$th moment of the $\zeta$-function.” We do not claim that this was the original reasoning of Conrey and Ghosh.

By the Montgomery-Vaughan mean value theorem the above integral has main term

$$\sum_{n<T} \frac{d_k(n)^2}{n}.$$  

(7.5)

By Perron’s formula, to evaluate this we need to find the leading pole of

$$F(s) = \sum_{n<T} \frac{d_k(n)^2}{n^s}.$$  

(7.6)

If $k > 2$ then $F(s)$ is not a simple expression involving known functions, but fortunately we do not require complete information about $F(s)$.

First note that

$$\zeta(s) = \prod_p \left(1 + \frac{1}{p^s} + \cdots \right)$$  

(7.7)

$$= \frac{1}{s-1} + \cdots,$$

so

$$\zeta(s)^N = \prod_p \left(1 + \frac{N}{p^s} + \cdots \right)$$  

(7.8)

$$= \frac{1}{(s-1)^N} + \cdots.$$  

Thus, if the coefficients of $p^{-js}$ in an Euler product are integers that only depend on $j$, then the coefficient of $p^{-s}$ tells you the order of the pole at $s = 1$. Since

$$d_k(p) = \sum_{n_1 \cdots n_k = p} 1$$  

(7.9)

$$= \frac{1}{k},$$

we see that $F(s)$ has a pole of order $k^2$ at $s = 1$, that is, $F(s) = a_k(s-1)^{-k^2} + \cdots$, where we will show that $a_k$ is as given above. To see that $F(s)$ has no other poles in $\sigma > \frac{1}{2}$, note that

$$\zeta^{-k^2}(s)F(s) = \prod_p \left(1 + \frac{\beta_2}{p^{2s}} + \frac{\beta_3}{p^{3s}} + \cdots \right),$$  

(7.10)

where the $\beta_j$ are certain integers that do not grow too fast. In particular, the above Euler product converges absolutely for $\sigma > \frac{1}{2}$ so it represents a regular function that is bounded in $\sigma > \frac{1}{2} + \varepsilon$. We have all of the pieces to apply the methods of the previous section, giving

$$\int_0^T \left| \sum_{n<T} \frac{d_k(n)}{n^{\frac{1}{2} + it}} \right|^2 dt \sim \frac{a_k}{k^2!} T \log k^2 T,$$  

(7.11)
where
\[
\begin{align*}
    a_k &= \lim_{s \to 1} (s - 1)^{k^2} F(s) \\
    &= \lim_{s \to 1} \zeta(s)^{-k^2} F(s) \\
    &= \prod_p \left(1 - \frac{1}{p}ight)^{k^2} \sum_{m=0}^{\infty} d_k(p^m)p^{-m}.
\end{align*}
\]
(7.12)

Note that the product converges because \(d_k(p) = k\) and \(d_k(n) \ll n^\varepsilon\).

Finally, \(d_k(p^m) = \binom{k+m-1}{k} \), as can be seen by the following argument. Since \(d_k(p^m)\) is the number of ways of writing \(e_1 + \cdots + e_k = m\), we can select the \(e_j\) by writing down \(m + k - 1\) circles \(\circ\) and filling in \(k - 1\) of them to make a dot \(\bullet\). Then the \(e_j\) are the number of circles between the dots, including the circles before the first and after the last dot. For example, here is one configuration that arises from \(d_5(p^3)\):

\[
\begin{array}{cccccc}
\bullet & \circ & \circ & \bullet & \bullet & \bullet \\
p^3 &=& 1 & p^2 & p & 1 & 1
\end{array}
\]
(7.13)

8. The Estermann phenomenon

The idea behind formula (7.10) can be generalized to show that if \(c(n)\) is a multiplicative function such that \(c(n) \ll n^\varepsilon\) and \(c(p^j)\) is an integer that is independent of the prime \(p\), then
\[
F(s) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s} = Z_f(s) \prod_{j<j} \zeta(js)^{C(j)},
\]
(8.1)

where the \(C(j)\) are integers and \(Z(s)\) is regular and bounded in \(\sigma > 1/J\). Thus \(F(s)\), which is originally defined for \(\sigma > 1\), has a meromorphic continuation to \(\sigma > 0\).

Note that \(F(s)\) cannot be continued past \(\sigma = 0\) unless \(C(j) = 0\) for almost all \(j\). This is because the zeros of the zeta-function lead to zeros or poles of \(F(s)\) that accumulate along the \(\sigma = 0\) line, giving a natural boundary. This is known as “the Estermann phenomenon”.

9. Appendix: big-O and \(\ll\) notation

The statement
\[
f(x) = O(g(x)) \quad \text{as} \quad x \to \infty
\]
is pronounced “\(f(x)\) is big oh of \(g(x)\).” It is equivalent to
\[
f(x) \ll g(x) \quad \text{as} \quad x \to \infty,
\]
which is pronounced “\(f(x)\) is less than less than \(g(x)\).” The symbol \(\ll\) is typed as \textbackslash\ll in \TeX. Both of the above statements mean the following: there exists a constant \(C\) such that if \(x\) is sufficiently large then \(|f(x)| \leq C g(x)\). The number \(C\) is called “the implied constant.”

Note:

- \(f(x) \ll g(x)\) does not mean that \(f(x)\) is much smaller than \(g(x)\). It is more accurate to say that \(f(x)\) does not grow faster than \(g(x)\).
• the above statements have the condition “as $x \to \infty$”. It is also common to use the big-$O$ and $\ll$ notation to describe the behavior of a function as $x \to 0$. Then the definition is modified to “if $x$ is sufficiently small”. Usually context makes it clear which behavior is being considered.

• Both notations are useful: the $\ll$ does not require parentheses, and the big-$O$ can be used as one term in a formula.

Here are some examples. Below, $A$ and $\varepsilon$ are arbitrary fixed positive numbers.

Examples assuming $x \to \infty$:

\[
x^3 \ll x^4 \\
\log(x) = O(x) \\
\log(x) \ll x^\varepsilon \\
x^4 \ll O(x^\varepsilon) \\
\sin(x) \ll x \\
(x + 2)^{10} \ll x^{10} \\
(x + 2)^{10} = x^{10} + O(x^9).
\]

Examples assuming $x \to 0$:

\[
x^4 \ll x^3 \\
\log(1 + x) = O(x) \\
\log(1 + x) = x + O(x^2) \\
\sin(x) \ll x \\
(x + 2)^{10} = 1024 + O(x).
\]

9.1. Little-$o$ notation. The statement

\[(9.5) \quad f(x) = o(g(x)) \quad \text{as} \quad x \to \infty\]

is pronounced “$f(x)$ is little oh of $g(x)$.” It means

\[(9.6) \quad \lim_{x \to \infty} \frac{f(x)}{g(x)} = 0.\]

Equivalently, for all $C > 0$, if $x$ is sufficiently large then $|f(x)| \leq Cg(x)$. It is like big-$O$ where the implied constant can be made arbitrarily small.

Note that $f(x) \sim g(x)$, “$f(x)$ is asymptotic to $g(x)$” is equivalent to $f(x) = (1 + o(1))g(x)$.

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