On Groups in Which Many Automorphisms Are Cyclic

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Abstract: Let G be a group. An automorphism \( \alpha \) of G is said to be a cyclic automorphism if the subgroup \( \langle x^g \rangle \) is cyclic for every element \( x \) of G. In [F. de Giovanni, M.L. Newell, A. Russo: On a class of normal endomorphisms of groups, J. Algebra and its Applications 13, (2014), 6pp] the authors proved that every cyclic automorphism is central, namely, that every cyclic automorphism acts trivially on the factor group \( G/Z(G) \). In this paper, the class \( FW \) of groups in which every element induces by conjugation a cyclic automorphism on a (normal) subgroup of finite index will be investigated.

Keywords: FC-groups; FW-groups; cyclic automorphisms; cyclicizer

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1. Introduction

Let G be a group. Following the work in [1], an automorphism \( \alpha \) of G is called a cyclic automorphism if the subgroup \( \langle x^g \rangle \) is cyclic for every element \( x \) of G. Clearly, any power automorphism of G (i.e., an automorphism which maps every subgroup onto itself) is cyclic; however, the multiplication by a rational number greater than 1 is a cyclic automorphism of the additive group of rational numbers which is not a power automorphism. Finally, it is easy to show that any cyclic automorphism of a periodic group is a power automorphism.

In [1], it was proved that any cyclic automorphism of a group \( G \) is central, i.e., it acts trivially on the factor group \( G/Z(G) \). Notice that this result is an extension to cyclic automorphisms of a renowned theorem by Cooper [2] for power automorphisms. It is not difficult to prove that the set \( CAut(G) \) of all cyclic automorphisms of G forms a normal abelian subgroup of the automorphism group \( Aut(G) \) of G. In [3], the structure of \( CAut(G) \) has been investigated in detail and some well-known properties of power automorphisms (see in [2]) has been extended to cyclic automorphisms. Moreover, the groups in which every automorphism is cyclic have been characterized there.

In the following, we will say that an element \( g \) of a group \( G \) induces by conjugation a weakly cyclic automorphism of G if there exists a normal subgroup \( W(g) \) of G such that the index \( |G:W(g)| \) is finite and the subgroup \( \langle x,x^g \rangle \) is cyclic for each element \( x \) of \( W(g) \). Let \( g_1 \) and \( g_2 \) be elements of G inducing weakly cyclic automorphisms and put \( W = W(g_1) \cap W(g_2) \). If \( x \) is an element of \( W \), then \( \langle x,x^{g_1} \rangle = \langle y \rangle \) for some \( y \in W \), and so \( \langle x,x^{g_2} \rangle \) is contained in the cyclic subgroup \( \langle y,y^{g_2} \rangle \). It follows that \( g_1g_2 \) induces a weakly cyclic automorphism of G and hence the set \( FW(G) \) of all elements of G inducing by conjugation weakly cyclic automorphisms of G is a subgroup of G. Moreover, if \( g \) is an element of \( FW \), \( x \) is an element of \( W(g) \) and \( y \) is an element of G, we have that \( \langle x^{y^{-1}},x^{y^{-1}g} \rangle^y \) is again a cyclic subgroup of \( W(g) \), so that \( FW(G) \) is a normal subgroup of G. We name this subgroup the \( FW\)-centre of G. A group which coincides with its \( FW\)-center will be called an \( FW\)-group.
Recall that the cyclic norm $C(G)$ of a group $G$ is defined as the intersection of the normalizers of every maximal locally cyclic subgroup of $G$. By [3], Lemma 2.1, any cyclic automorphism of $G$ fixes all maximal locally cyclic subgroups of $G$. It follows that $C(G)$ coincides with the set of all elements of $G$ inducing cyclic automorphisms of $G$. In particular, $C(G)$ is a subgroup of $FW(G)$.

In the first part of the article, the class $FW$ of groups in which every element induces by conjugation a weakly cyclic automorphism will be investigated. In particular, it will be proved that the class $FW$ coincides with the class $FP$ recently studied by De Falco et al. [4]. Recall here that a group $G$ is said to be an $FP$-group if every element of $G$ induces by conjugation a power automorphism on some subgroup of finite index of $G$. Clearly, the groups with finitely many conjugacy classes (the so-called $FC$-groups) are $FP$-groups, while every $FP$-group is an $FW$-group. The consideration of the infinite dihedral group $D_\infty$ shows that there are $FP$-groups which are not $FC$-groups.

Let $G$ be a group and denote by $Cyc(G)$ the set of all elements $x$ of $G$ such that $\langle x, y \rangle$ is cyclic for every $y$ in $G$. It is easy to show that $Cyc(G)$ is a central, characteristic subgroup of $G$ called the cyclicizer of $G$ (see [5,6]). Clearly, $Cyc(G)$ is locally cyclic and hence every automorphism of $G$ induces a cyclic automorphism on $Cyc(G)$. In the last part of the article, groups with non-trivial cyclicizer will be investigated extending to the infinite case some results in [6–8]. In particular, it is shown that any torsion-free or primary generalized soluble group with non-trivial cyclicizer is an $FW$-group. Moreover, the well-known characterization of finite $p$-groups with only one subgroup of order $p$ (see, for instance, [9], 5.3.6) will be extended to locally finite groups. Finally, it is proved that the factor group $G/Cyc(G)$ is finite if and only if $G$ has a finite covering of locally cyclic subgroups.

Most of our notation is standard and can be found in [10].

2. $FW$-Groups

Our first result is an easy remark concerning cyclic automorphisms of finite order.

**Lemma 1.** Let $G$ be a group. Every periodic cyclic automorphism of $G$ is a power automorphism.

**Proof.** Let $\alpha$ be a cyclic automorphism of $G$, let $g$ be an element of $G$, and consider a maximal locally cyclic subgroup $M$ of $G$ such that $g \in M$. As one can easily see that $M^\alpha = M$ (see, for instance, in [3], Lemma 2.1), then the normal closure $(x)^{(\alpha)}$ is locally cyclic and hence there exists an element $x$ of $G$ such that $(g)^{(\alpha)} = (x)$. Clearly, $(x)^{(\alpha)} = (x)$ and we may suppose that $g$ has infinite order. Therefore, $x^\alpha = x^{-1}$ and $g^\alpha = g^{-1}$. Thus, $\alpha$ induces a power automorphism on $G$. \qed

Let $G$ be a group. A normal subgroup $W$ of $G$ is said to be weakly central if every element of $G$ induces by conjugation a cyclic automorphism of $W$. Clearly, if $G$ contains a weakly central subgroup of finite index, then $G$ is an $FW$-group.

**Proposition 1.** Let $G$ be a group. If $W$ is a weakly central subgroup of finite index of $G$, then every subgroup of $W$ is normal in $G$. In particular, $G$ is an $FP$-group.

**Proof.** First, assume that every inner automorphism of $G$ is cyclic. Then, $G$ coincides with its cyclic norm and hence every maximal locally cyclic subgroup of $G$ is normal. Let $g$ be an element of $G$ and consider a maximal locally cyclic subgroup $M$ containing $g$. As $G$ is an $FC$-group (see [3], Theorem 4.2), then the normal closure $(g)^G$ of $g$ in $G$ is a finitely generated subgroup of $M$. Therefore, $(g)^G$ is normal in $G$ and thus $G$ is a Dedekind group.

The above argument shows that $W$ is a Dedekind group. Since a cyclic automorphism of a periodic group is a power automorphism (see in [3], Lemma 2.3), we may suppose that $W$ is abelian. It follows that the factor group $G/C_G(W)$ is finite and hence every element $g$ of $G$ induces on $W$ a cyclic automorphism of finite order. The statement now follows from Lemma 1. \qed
**Corollary 1.** Let $G$ be a group all of whose inner automorphisms are cyclic automorphisms. Then $G$ is a Dedekind group.

Let $G$ be a group. We denote here with $FP(G)$ the FP-centre of $G$, namely the subgroup of all elements of $G$ inducing by conjugation power automorphisms on some subgroup of finite index of $G$. Clearly, $FP(G)$ is a subgroup of $FW(G)$.

Recall that a non-periodic group is said to be weak if it can be generated by its elements of infinite order, while it is said to be strong otherwise. In particular, all non-periodic abelian groups are weak.

**Theorem 1.** Let $G$ be a group. Then, FW-centre and FP-centre of $G$ coincide.

**Proof.** As the FP-centre of $G$ is a subgroup of $FW(G)$, we just have to show that every element of $G$ inducing a weakly cyclic automorphism of $G$ induces a weakly power automorphism of $G$. Therefore, let $g$ be an element of $FW(G)$ and let $W(g)$ be a normal subgroup of finite index of $G$ such that $g$ induces on $W(g)$ a cyclic automorphism. By Lemma 1, we may assume that $g$ induces an aperiodic automorphism on $W(g)$. Clearly, $g^n \in W(g)$ for some positive integer $n$ and $g^n \neq 1$. If $W(g)$ is weak, then $g$ acts universally on $W(g)$ (see [3], Theorem 3.5) and then $[W(g), g] = \{1\}$ as $g^n$ belongs to $W(g)$, so we may further assume that $W(g)$ is strong. If we let $W$ be the subgroup of $G$ generated by every element of infinite order of $G$, by Theorem 3.5 in [3], $g$ fixes $W$ and $G/W$ elementwise. Let now $x$ be an element of finite order of $W(g)$ and let $m$ be the order of $x$. As $(x)$ and $(x^g)$ are both subgroups of order $m$ of the cyclic group $(x, x^g)$, they coincide and this shows that $g$ acts as a power automorphism on every finite cyclic subgroup of $W(g)$. As $g$ centralizes every element of infinite order of $G$, it follows that $g$ induces a power automorphism on $W(g)$ and our thesis is proved. 

**Corollary 2.** Let $G$ be a group. Then, $G$ is an FW-group if and only if $G$ is an FP-group.

Recall that a subgroup $X$ of a group $G$ is said to be pronormal if the subgroups $X$ and $X^g$ are conjugate in the subgroup $(X, X^g)$ for all elements $g$ of $G$. As any subnormal and pronormal subgroup of a group is normal, it follows that a group all of whose subgroups are pronormal is a $T$-group (i.e., a group in which normality is a transitive relation in every subgroup). However, the converse is false, as an example due to Kuzennyi and Subbotin [11] shows. We point out incidentally that in the universe of groups with no infinite simple sections the property $T$ for a group $G$ is equivalent to saying that every subgroup of $G$ is weakly normal (see [12]). A tool which is useful to control pronormal subgroups of a group $G$ is the pronorm of $G$, which is defined as the set $P(G)$ of all elements $g$ of $G$ such that $X$ and $X^g$ are conjugate in $(X, X^g)$ for any subgroup $X$ of $G$. The consideration of the alternating group $A_5$ shows that the pronorm of a group need not be in general a subgroup. On the other hand, the pronorm of a $T$-group $G$ with no infinite simple sections is a subgroup of $G$ which coincides with the set $L(G)$ consisting of all elements $g \in G$ such that, if $H$ is a subgroup of $G$, then $g$ normalizes a subgroup of finite index of $H$ (see [13], Theorem 2.2). The last result of this section shows in particular that a $T$-group $G$ with no infinite simple sections has all subgroups pronormal whenever $G$ belongs to the class $FW$.

**Corollary 3.** Let $G$ be a group. Then, $FW(G)$ is contained in $L(G)$. In particular, if $G$ is a $T$-group with no infinite simple sections, $FW(G)$ is a subgroup of $P(G)$.

**Proof.** By Theorem 1, for every element $g$ of $FW(G)$ we may find a normal subgroup $W(g)$ of finite index of $G$ on which $g$ acts as a power automorphism. If we let $H$ be a subgroup of $G$, then the subgroup $H \cap W(g)$ of $W(g)$ is normalized by $g$, has finite index in $H$ and this proves our claim. 


3. Groups with Non-Trivial Cyclicizer

It is straightforward to see that a group with non-trivial cyclicizer is either torsion-free or periodic. Therefore, it is natural to inspect the cases in which the groups are either torsion-free or primary groups. As some arguments can be unified, in the following elements of infinite order will be said *elements of order 0* and torsion-free groups will be called 0-groups.

**Lemma 2.** Let $G$ be a $p$-group where $p$ is a prime or 0. If the cyclicizer $\text{Cyc}(G)$ of $G$ is not trivial, then it coincides with the centre $Z(G)$ of $G$.

**Proof.** Assume for a contradiction that $\text{Cyc}(G)$ is a proper subgroup of $Z(G)$. Then, we may find an element $x$ of $G$ and an element $y \in Z(G)$ such that $\langle x, y \rangle = \langle x \rangle \times \langle y \rangle$. Let now $c$ be a non-trivial element of $\text{Cyc}(G)$. As the subgroups $\langle x, c \rangle$ and $\langle y, c \rangle$ are cyclic, there is a power of $c$ which belongs to $\langle x \rangle \cap \langle y \rangle = \{1\}$. It follows that $\text{Cyc}(G)$ is periodic, so that also $G$ is periodic and hence the subgroups $\langle x, c \rangle$ and $\langle y, c \rangle$ have a unique subgroup of order $p$ for a prime $p$ dividing the order of, say, $\langle x, c \rangle$. In particular, the intersection $\langle x \rangle \cap \langle y \rangle$ is not trivial. This contradiction completes the proof. □

The consideration of the direct product of a group of order 3 and a dihedral group of order 8 shows that there exists a (finite) group $G$ whose order is divided by only two primes and such that $\{1\} \neq \text{Cyc}(G) < Z(G)$.

Let $A = \langle a \rangle$ be a cyclic group of order 4, let $B$ be a group of type $2^n$ and let $b$ be an element of order 4 of $B$. Consider the semidirect product $H = A \rtimes B$ where $a$ acts as the inversion on $B$. Take $K = \langle a^2b^2 \rangle$ and put $G = H/K$. Clearly, every finite non-abelian subgroup of $G$ is a generalized quaternion group. Therefore, in analogy with the locally dihedral 2-group $D_{2^8}$, we call $G$ a *locally generalized quaternion group* and we denote it with $Q_{2^8}$.

Here we give a first extension of Theorem 8 in [5].

**Lemma 3.** Let $G$ be a locally finite $p$-group for some prime $p$. Then, the cyclicizer of $G$ is not trivial if and only if

1. $G$ is locally cyclic or
2. $G$ is isomorphic with a subgroup of $Q_{2^8}$.

In particular, if $G$ is finite and non-abelian, then $G$ is a generalized quaternion group.

**Proof.** Assume that the cyclicizer $C$ of $G$ contains a non-trivial element $c$ of order $p$. If $G$ is abelian, then Lemma 2 yields that $G$ coincides with its cyclicizer and then $G$ is locally cyclic. Assume thus that there exists a finite non-abelian subgroup $H$ of $G$ and let $x$ be an element of $\langle H, c \rangle$ of order $p$. As $\langle x, c \rangle$ is cyclic, one has that $x$ is a power of $c$, namely $\langle H, c \rangle$ contains a unique subgroup of order $p$. By a well-known characterization (see, for instance, [9], 5.3.6) we have that $\langle H, c \rangle$ is a generalized quaternion group. As this property holds for every finite subgroup of $G$ containing $(H, c)$ and the set of finite subgroups of $G$ containing $\langle H, c \rangle$ is a direct system of $G$, we can clearly assume that $G$ is infinite. Therefore, it is possible to find in $G$ a subgroup $Q$ which is isomorphic with $Q_{2^8}$. Let $g$ be any element of $G$, let $P$ be the Prüfer 2-subgroup of $Q$ and let $y$ be an element of order $n > 4$ of $P$. As $\langle g, y \rangle = \langle g, y, c \rangle$ is either a cyclic or a generalized quaternion group, we have in any case that $\langle y \rangle$ is normalized by $g$ and hence the whole $P$ is normalized by $g$. Moreover, $\langle g \rangle$ has non-trivial intersection with $P$, as both must contain $c$. Then, $g$ has to be contained in $Q$, otherwise $\langle g, Q \rangle$ would contain a direct product of two cyclic subgroups of order 2. From this it immediately follows that $G$ is isomorphic with $Q_{2^8}$.

Let us prove the converse. If $G$ is locally cyclic the result is clear. On the other hand, take $G$ to be a subgroup of $Q_{2^8}$ which is not locally cyclic. Then, $G$ is not abelian, so that it is either the whole $Q_{2^8}$ or a generalized quaternion group. In both cases $Z(G)$ is the only subgroup of $G$ of order 2 and therefore it coincides with the cyclicizer of $G$, which is then non-trivial. □

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[5] T. Y. Lam, *A First Course in Noncommutative Ring Theory*, Cambridge University Press, Cambridge, 2015.
This result gives a generalization to the locally finite case of the already quoted result about finite $p$-groups [9], 5.3.6.

**Corollary 4.** Let $p$ be a prime. A locally finite $p$-group $G$ contains exactly one subgroup of order $p$ if and only if it satisfies one of the following conditions:

1. $G$ is locally cyclic;
2. $G$ is isomorphic with a generalized quaternion group;
3. $G$ is isomorphic with $Q_{2^n}$.

In [7], it is proved that if $G$ is a torsion-free group such that cyclicizer $Cyc(G)$ is not trivial, then $Cyc(G) = Z(G)$ and if $Z(G)$ is divisible, then $G$ is locally cyclic. One may ask whether a torsion-free or a $p$-group with non-trivial cyclicizer is locally cyclic. In general, these questions can be answered in the negative because of two results by Olšanskiı̆ (see in [14], Theorem 31.4 and Theorem 31.5). On the other hand, our next result shows that for a wide class of generalized soluble groups the statement is true.

A group $G$ is said to be weakly radical if it contains an ascending (normal) series all of whose factors are either locally soluble or locally finite.

**Theorem 2.** Let $G$ be a locally weakly radical group such that $|\pi(G)| \leq 1$. Then, $G$ has non-trivial cyclicizer if and only if

1. $G$ is locally cyclic or
2. $G$ is isomorphic with a subgroup of $Q_{2^n}$.

**Proof.** Let $C$ be the cyclicizer of $G$. If $C \neq \{1\}$, it follows from Lemma 2 that $C = Z(G)$. Moreover, as already pointed out, $G$ is either torsion-free or periodic. By Lemma 3, we may also suppose that $G$ is torsion-free. Let $c$ be a non-trivial element of $C$. If $x$ is an element of $G$, then the subgroup $E = \langle x, c \rangle$ of $G$ is cyclic and hence there exists a positive integer $n$ such that $x^n$ belongs to $\langle c \rangle$. Thus the factor group $G/C$ is periodic and so even locally finite since $G$ is locally weakly radical. Now an easy application of a famous theorem by Schur (see, for instance, Corollary to Theorem 4.12 in [10]) shows that the commutator subgroup of $G$ is locally finite and hence $G$ is abelian. In particular, $G$ is locally cyclic.

The converse is an immediate consequence of Lemma 3.

**Corollary 5.** Let $G$ be a locally weakly radical group such that $|\pi(G)| \leq 1$. If $G$ has non-trivial cyclicizer, then it is an FW-group.

A straightforward application of Theorem 2 and of [9], 12.1.1 is the following.

**Corollary 6.** Let $G$ be a locally nilpotent group. Then $G$ has non-trivial cyclicizer if and only if either it is locally cyclic or $G$ is periodic and there is a prime number $p$ such that the $p$-component $G_p$ of $G$ either is locally cyclic or is isomorphic with a subgroup of $Q_{2^n}$.

A well-known result of Baer (see, for instance, in [10], Theorem 4.16) states that a group is central-by-finite if and only if it has a finite covering consisting of abelian subgroups. Furthermore, we have already quoted the theorem by Schur that ensures that a central-by-finite group is finite-by-abelian. In the following we rephrase these results replacing thecentre $Z(G)$ of $G$ by the cyclicizer $Cyc(G)$. Recall that a collection $\Sigma$ of subgroups of a group $G$ is said to be a covering of $G$ if each element of $G$ belongs to at least one subset in $\Sigma$.

**Theorem 3.** Let $G$ be a group and let $C$ be the cyclicizer of $G$. Then, the following hold:

1. If $C$ has finite index in $G$, then $G$ is finite-by-(locally cyclic);
2. The factor group $G/C$ is finite if and only if $G$ has a finite covering consisting of locally cyclic subgroups.
Proof. (1) As $C \leq Z(G)$, then $G$ is central-by-finite and hence the commutator subgroup $G'$ of $G$ is finite. Clearly, we may assume that $G$ is infinite, so that $C$ too is infinite and, by replacing $G$ with $G/G'$, we may suppose that $G$ is abelian. Moreover, as $C$ is non-trivial, then $G$ is either torsion-free or periodic. In the former case, $G$ is locally cyclic by Proposition 2. Assume hence that $G$ is periodic. In this case, as we aim to show that $G$ is locally cyclic, we may also suppose that $G$ is a $p$-group for a prime $p$. However, $C$ is locally cyclic and hence of type $p^\infty$. It follows that $G$ can be decomposed as $G = C \times H$ where $H$ is a subgroup of $G$. If $c$ and $h$ are elements of order $p$ of $C$ and $H$, respectively, then the subgroup $\langle c, h \rangle$ is not cyclic. This contradiction shows that $H$ is trivial and hence $G = C$ is locally cyclic.

(2) First assume that the factor group $G/C$ is finite. Choose a (left) transversal to $C$ in $G$, say $\{x_1, \ldots, x_n\}$. Then, for any element $g$ of $G$, we can write $g = x_i c$ where $c$ is an element of $C$. Therefore, $g$ belongs to $\langle x_i, C \rangle$, which is locally cyclic, and $G$ is covered by the subgroups $\langle x_i, C \rangle$ with $i = 1, \ldots, n$.

Conversely, assume that $G$ is covered by finitely many locally cyclic subgroups. Then by a result of Neumann (see in [10], Lemma 4.17) $G$ is covered by finitely many locally cyclic subgroups of finite index. Let $L$ be their intersection. Clearly, $L$ is contained in $C$ and $|G : L|$ is finite. It follows that $G/C$ is finite. □

We remark that the cyclicizer of the direct product of $\mathbb{Z}_2 \times \mathbb{Q}$ is trivial, so that the converse of point (1) of Theorem 3 is not true.

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