LATTICE MULTIVERSE MODELS

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ABSTRACT. Will the cosmological multiverse, when described mathematically, have easily stated properties that are impossible to prove or disprove using mathematical physics? We explore this question by constructing lattice multiverses which exhibit such behavior even though they are much simpler mathematically than any likely cosmological multiverse.

1. INTRODUCTION

We first describe our lattice multiverse models (precise definitions follow). Start with a fixed directed graph $G = (N^k, \Theta)$ (vertex set $N^k$, edge set $\Theta$) where $N$ is the set of nonnegative integers and $k \geq 2$. The vertex set $N^k$ of $G$ is the nonnegative $k$ dimensional integral lattice. If every $(x, y) \in \Theta$ satisfies $\max(x) > \max(y)$ where $\max(z)$ is the maximum coordinate value of $z$ then we call $G$ a downward directed lattice graph. The infinite lattice graph $G$ defines the set $\{G_D \mid D \subset N^k, D \text{ finite}\}$ of finite vertex induced subgraphs of $G$.

With each downward directed lattice graph $G$ we associate, in various ways, sets of functions $P_G = \{f \mid f : D \to N, D \subset N^k, D \text{ finite}\}$ (the finite set $D$ is the domain of $f$, and $N$ is the range of $f$). Infinite sets of the form $M = \{(G_D, f) \mid f \in P_G, \text{domain}(f) = D\}$, will be called lattice “multiverses” of $G$ and $P_G$; the sets $(G_D, f)$ will be the “universes” of $M$.

Our use of the terms “multiverse” and “universe” in this combinatorial lattice context is inspired by the analogous but much more complex structures of the same name in cosmology. The lattice multiverse is a geometric structure for defining the possible lattice universes, $(G_D, f)$, where $G_D$ represents the geometry of the lattice universe and $f$ the things that can be computed about that universe (roughly analogous to the physics of a universe). An example and discussion is given below, see Figure 1.

In this paper, we state some basic properties of our elementary lattice multiverses that provably cannot be proved true or false using the mathematical techniques of physics. Could the much more complex cosmological multiverses also give rise to conjectured properties provably out of the range of
mathematical physics? Our results suggest that such a possibility must be considered.

For the provability results, we rely on the important work of Harvey Friedman concerning finite functions and large cardinals [Fri97] and applications of large cardinals to graph theory [Fri98].

**Definition 1.1 (Vertex induced subgraph \(G_D\)).** For any finite subset \(D \subset N^k\) of vertices of \(G\), let \(G_D = (D, \Theta_D)\) be the subgraph of \(G\) with vertex set \(D\) and edge set \(\Theta_D = \{(x, y) \mid (x, y) \in \Theta, x, y \in D\}\). We call \(G_D\) the subgraph of \(G\) induced by the vertex set \(D\).

**Definition 1.2 (Path and terminal path in \(G_D\)).** A sequence of distinct vertices of \(G_D\), \((x_1, x_2, \ldots, x_t)\), is a *path* in \(G_D\) if \(t = 1\) or if \(t > 1\) and \((x_i, x_{i+1}) \in \Theta_D, i = 1, \ldots, t - 1\). This path is *terminal* if there is no path of the form \((x_1, x_2, \ldots, x_t, x_{t+1})\).

We refer to sets of the form \(E^k \equiv x^k E \subset N^k, E \subset N\), as *k-cubes* or simply as *cubes*. If \(x \in N^k\), then \(\min(x)\) is the minimum coordinate value of \(x\) and \(\max(x)\) is the maximum coordinate value (see discussion of Figure 1).

**Definition 1.3 (Terminal label function for \(G_D\)).** Consider a downward directed graph \(G = (N^k, \Theta)\) where \(N\) is the set of nonnegative integers and \(k \geq 2\). For any finite \(D \subset N^k\), let \(G_D = (D, \Theta_D)\) be the induced subgraph of \(G\). Define a function \(t_D\) on \(D\) by

\[
t_D(z) = \min(\{\min(x) \mid x \in T_D(z)\} \cup \{\min(z)\})
\]

where \(T_D(z)\) is the set of all last vertices of terminal paths \((x_1, x_2, \ldots, x_t)\) where \(z = x_1\). We call \(t_D\) the *terminal label function* for \(G_D\).

In words, \(t_D(z)\) is gotten by finding all of the end vertices of terminal paths starting at \(z\), taking their minimum coordinate values, throwing in the minimum coordinate value of \(z\) itself and, finally, taking the minimum of all of these numbers.

Figure 1 shows an example of computing \(t_D\) where \(D = E \times E, E = \{0, \ldots, 14\}\). The graph \(G_D = (D, \Theta_D)\) has \(|D| = 225\) vertices and \(|\Theta_D| = 12\) edges (shown by arrows in Figure 1). Vertices not on any edge, such as the vertex \((6, 10)\), are called *isolated* vertices. A path in \(G_D\) will be denoted by a sequence of vertices \((x_1, x_2, \ldots, x_t)\), \(t \geq 1\). For example, \(((5, 9), (3, 8), (2, 6))\) is a path: \(x_1 = (5, 9), x_2 = (3, 8), x_3 = (2, 6)\). Note that the path \(((5, 9), (3, 8), (2, 6))\) can be extended to \(((5, 9), (3, 8), (2, 6), (3, 4))\), but this latter path is terminal (can’t be extended any farther, Definition 1.2). Note that there is another terminal path shown in Figure 1 that starts at \((5, 9)\): \(((5, 9), (3, 8), (5, 3))\).
As an example of computing $t_D(z)$, look at $z = (5,9)$ in Figure 1 where the value, $t_D(z) = 3$, of the terminal label function is indicated. From Definition 1.3, the set $T_D((5,9)) = \{(3,4), (5,3)\}$ and $\{\min(x) \mid x \in T_D(z)\} = \{3,3\} = \{3\}$. The set $\{\min(z)\} = \{5\}$ and, thus, $\{\min(x) \mid x \in T_D(z)\} \cup \{\min(z)\} = \{3,5\}$ and $T_D(z) = \min\{3,5\} = 3$. If $z$ is isolated, $t_D(z) = \min(z)$ (for example, $z = (4,2)$ in the figure is isolated, so $t_D(z) = 2$). Such trivial labels are omitted in the figure. For $z = (10,3)$, $T_D(z) = \{(6,5)\}$ so $t_D(z) = \min(z) = 3$.

**Definition 1.4 (Significant labels).** Let $t_D$ be the terminal label function for $G_D$ and let $S \subset D$. The set $\{t_D(z) \mid z \in S, t_D(z) < \min(z)\}$ is the set of $t_D$–significant labels of $S$ in $D$.

Referring to Figure 1 with $S = D$, $\{(5,9), (6,14), (8,10), (9,7), (10,6), (12,12)\}$ are vertices with significant labels, and the set of significant labels is $\{t_D(z) \mid z \in S, t_D(z) < \min(z)\} = \{2,3,5\}$. The terminology comes from the “significance” of these number with respect to order type equivalence classes and the concept of regressive regularity (e.g., Theorem 2.4). The set of significant labels also occurs in certain studies of lattice embeddings of posets [RW99].

In the next section, we study the set of significant labels.
We start with a definition and related theorem that we state without proof.

**Definition 2.1 (full, reflexive, jump-free).** Let $Q$ denote a collection of functions whose domains are finite subsets of $N^k$ and ranges are subsets of $N$.

1. **full:** We say that $Q$ is a full family of functions on $N^k$ if for every finite subset $D \subset N^k$ there is at least one function $f$ in $Q$ whose domain is $D$.

2. **reflexive:** We say that $Q$ is a reflexive family of functions on $N^k$ if for every $f$ in $Q$ and for each $x \in D$, $D$ the domain of $f$, $f(x)$ is a coordinate of some $y$ in $D$.

3. **jump-free:** For $D \subset N^k$ and $x \in D$ define $D_x = \{ z \mid z \in D, \max(z) < \max(x) \}$. Suppose that for all $f_A$ and $f_B$ in $Q$, where $f_A$ has domain $A$ and $f_B$ has domain $B$, the conditions $x \in A \cap B$, $A_x \subset B_x$, and $f_A(y) = f_B(y)$ for all $y \in A_x$ imply that $f_A(x) \geq f_B(x)$. Then $Q$ will be called a jump-free family of functions on $N^k$.

Figure 2 may be helpful in thinking about the jump-free condition ($k = 2$). The square shown in the figure has sides of length $\max(x)$. The set $A_x$ is the intersection of the set $A$ with the set of lattice points interior to the square. The set $B_x$ is this same intersection for the set $B$.

![Figure 2. Jump-free “light cone”](image_url)

To prove our main result, we use a theorem of Harvey Friedman called the “jump-free theorem,” Theorem 2.2. The jump-free theorem is proved and
shown to be independent of the ZFC (Zermelo, Fraenkel, Choice) axioms of mathematics in Section 2 of [Fri97]. “Applications of Large Cardinals to Graph Theory,” October 23, 1997, No. 11 of Preprints, Drafts, and Abstracts. The proof uses results from [Fri98].

**Theorem 2.2 (Friedman’s jump-free theorem).** Let \( Q \) denote a full, reflexive, and jump-free family of functions on \( N^k \) (Definition 2.1). Given any integer \( p > 0 \), there is a finite \( D \subset N^k \) and a subset \( S = E^k \subset D \) with \( |E| = p \) such that for some \( f \in Q \) with domain \( D \), the set \( \{ f(z) \mid z \in S, f(z) < \min(z) \} \) has at most cardinality \( k^k \).

*Technical Note:* The function \( f \) of the jump-free theorem can be chosen such that for each order type \(^1\) \( \omega \) of \( k \)-tuples, either \( f(x) \geq \min(x) \) for all \( x \in E^k \) where \( x \) is of type \( \omega \) or \( f(x) = f(y) < \min(E) \) for all \( x \in E^k \) and \( y \in E^k \), \( x \) and \( y \) of order type \( \omega \). We call such a function * regressively regular* over \( E^k \). Note that for \( k \geq 2 \), the number of order type equivalence classes is always strictly less than \( k^k \).

**Definition 2.3 (Multiverse TL – terminal label multiverse).** Let \( G = (N^k, \Theta) \) be a downward directed graph where \( N \) is the nonnegative integers. Define \( M_{\text{TL}} \) to be the set \( \{(G_D, t_D) \mid D \subset N^k, D \text{ finite} \} \) where \( t_D \) is the terminal label function of the induced subgraph \( G_D \). We call \( M_{\text{TL}} \) a \( k \)-dimensional multiverse of type TL. We refer to the pairs \((G_D, t_D)\) as the universes of \( M_{\text{TL}} \).

We now use Friedman’s jump-free theorem to prove a basic structure theorem for Multiverse TL. Intuitively, this theorem (Theorem 2.4) states that for any specified cube size, no matter how large, there is a universe of Multiverse TL that contains a cube of that size with certain special properties.

**Theorem 2.4 (Multiverse TL).** Let \( M_{\text{TL}} \) be a \( k \)-dimensional lattice multiverse of type TL and let \( p \) be any positive integer. Then there is a universe \((G_D, t_D)\) of \( M_{\text{TL}} \) and a subset \( E \subset N \) with \(|E| = p\) and \( S = E^k \subset D \) such that the set of significant labels \( \{t_D(z) \mid z \in S, t_D(z) < \min(z)\} \) has size at most \( k^k \). In fact, \( t_D \) is regressively regular over \( E^k \).

*Proof.* Recall that \( t_D \) is the terminal labeling function of the induced subgraph \( G_D = (D, \Theta_D) \) of the graph \( G = (N^k, \Theta) \). We apply Theorem 2.2 to a “relaxed” version, \( \hat{t}_D \), of \( t_D \) defined by \( \hat{t}_D(z) = \max(z) \) if \( (z) \) is a terminal path in \( G_D \) and \( \hat{t}_D(z) = t_D(z) \) otherwise. \(^2\) If \( (z) \) is a terminal path in \( G_D \) then \( \hat{t}_D(z) = \max(z) \) by definition, and if \( (z) \) is not a terminal path in \( G_D \), the downward condition

\(^1\)Two \( k \)-tuples, \( x = (x_1, \ldots, x_k) \) and \( y = (y_1, \ldots, y_k) \), have the same order type if \( \{ (i, j) \mid x_i < x_j \} = \{ (i, j) \mid y_i < y_j \} \) and \( \{ (i, j) \mid x_i = x_j \} = \{ (i, j) \mid y_i = y_j \} \).

\(^2\)This clever idea is due to Friedman [Fri97].
implies that $\hat{t}_D(z) = t_D(z) < \max(z)$. Thus, $\hat{t}_D(z) \leq \max(z)$ with equality if and only if $(z)$ is terminal.

Let $Q$ denote the collection of functions $\hat{t}_D$ as $D$ ranges over all finite subsets of $N^k$. We will show that $Q$ is full, reflexive, and jump-free (Definition 2.1). Full and reflexive are obvious from the definition of $\hat{t}_D$. We want to show that for all $\hat{t}_A$ and $\hat{t}_B$ in $Q$ the conditions $x \in A \cap B, A_x \subset B_x$, and $\hat{t}_A(y) = \hat{t}_B(y)$ for all $y \in A_x$ imply that $\hat{t}_A(x) \geq \hat{t}_B(x)$.

Suppose that $(x)$ is terminal in $G_A$. Then $\hat{t}_A(x) = \max(x) \geq \hat{t}_B(x)$ from our observations above.

Suppose that $(x)$ is not terminal in $G_A$. Then $\hat{t}_A(x) = t_A(x)$ by definition of $\hat{t}_A$. From the definition of $t_A(x)$, there is a path, $x = x_1, x_2, \ldots, x_t$, with $t > 1$ such that $t_A(x) = \min(x_t)$ and $(x_t)$ is terminal in $G_A$. Thus, $\hat{t}_A(x_t) = \max(x_t)$. Our basic assumption is that $\hat{t}_A(y) = \hat{t}_B(y)$ for all $y \in A_x$ and hence for $y = x_t$. Thus, $\hat{t}_B(x_t) = \max(x_t)$ and hence, from our discussion above, $(x_t)$ is also terminal in $G_B$. Since $A_x \subset B_x$, the path $x = x_1, x_2, \ldots, x_t$ with $t > 1$ is also a terminal path in $G_B$. Thus, $x_t \in T_B(x)$ (Definition 1.3) and $t_B(x) \leq \min(x_t) = t_A(x)$. Since $(x)$ is not terminal in either $G_A$ or $G_B$, $\hat{t}_B(x) = t_B(x) \leq \min(x_t) = t_A(x) = \hat{t}_A(x)$ which completes the proof that $Q$ is jump-free.

From Theorem 2.2, given any integer $p > 0$, there is a finite $D \subset N^k$ and a subset $E^k \subset D$ with $|E| = p$ such that, for some $\hat{t}_D \in Q$, the set \{ $\hat{t}_D(z) \mid z \in S, \hat{t}_D(z) < \min(z)$ \} has at most cardinality $k^k$. In fact, $\hat{t}_D(z)$ is regressively regular on $E^k$ (See Theorem 2.2, technical note).

Finally, we note that if $t_D(z) < \min(z)$ then $(z)$ is not a terminal path in $G_D$ and hence $\hat{t}_D(z) = t_D(z) < \min(z)$. Thus, \{ $t_D(z) \mid z \in S, t_D(z) < \min(z)$ \} has at most cardinality $k^k$ also. In fact, $\hat{t}_D(z)$ is regressively regular on $E^k$ implies that $t_D(z)$ is regressively regular on $E^k$.

To see this latter point, suppose that $\hat{t}_D(z) \geq \min(z)$ for all $z \in E^k$ of order type $\omega$. If $(z)$ is terminal, $t_D(z) = \min(z)$. If $(z)$ is not terminal, $t_D(z) = \hat{t}_D(z) \geq \min(z)$. Thus, $\hat{t}_D(z) \geq \min(z)$ for all $z \in E^k$ of order type $\omega$ implies $t_D(z) \geq \min(z)$ for all $z \in E^k$ of order type $\omega$.

Now suppose that for all $z,w \in E^k$ of order type $\omega$, $\hat{t}_D(z) = \hat{t}_D(w) < \min(E)$. This inequality implies that $\hat{t}_D(z) < \min(z)$ and $\hat{t}_D(w) < \min(w)$ and thus $(z)$ and $(w)$ are not terminal. Hence, $t_D(z) = \hat{t}_D(z) = \hat{t}_D(w) = t_D(w) < \min(E)$.

Summary: We have proved that given an arbitrarily large cube, there is some universe $(G_D,t_D)$ of $M_{TL}$ for which the “physics,” $t_D$, has a simple structure over a cube of that size. To prove this large-cube property, we have used a theorem independent of ZFC. We do not know if this large-cube property can be proved in ZFC. The mathematical techniques of physics lie within the ZFC axiomatic system.
3. Lattice Multiverse SL

We now consider a class of multiverses where the “physics” is more complicated than in Multiverse TL. For us, this means that the label function is more complicated than in Multiverse TL. For us, this means that the label function is more complicated than \( t_D \). Again, we consider a fixed downward directed graph \( G = (N^k, \Theta) \) where \( N \) is the set of nonnegative integers.

**Definition 3.1 (Partial selection).** A function \( F \) with domain a subset of \( X \) and range a subset of \( Y \) will be called a partial function from \( X \) to \( Y \) (denoted by \( F : X \to Y \)). If \( z \in X \) but \( z \) is not in the domain of \( F \), we say \( F \) is not defined at \( z \). A partial function \( F : (N^k \times N)^r \to N \) will be called a partial selection function if whenever \( F((y_1,n_1),(y_2,n_2),\ldots,(y_r,n_r)) \) is defined we have \( F((y_1,n_1),(y_2,n_2),\ldots,(y_r,n_r)) = n_i \) for some \( 1 \leq i \leq r \).

For \( x \in N^k \), let \( G^x = \{ y \mid (x,y) \in \Theta \} \). \( G^x \) is the set of vertices adjacent to \( x \) in \( G \).

For \( x \in N^k \) a vertex of \( G \) and \( r \geq 1 \), let \( F_r^x : (G^x \times N)^r \to N \) denote a partial function. Let \( \mathcal{F}_G = \{ F_r^x \mid x \in N^k, r \geq 1 \} \) be the set of partial functions for \( G \).

Let \( G_D^x \) denote the set of vertices of \( G_D \) that are adjacent to \( x \) in \( G_D \).

**Definition 3.2 (Selection labeling function \( s_D \) for \( G_D \)).** For \( D \subset N^k \), \( x \in D \), we define \( s_D(x) \) by induction on \( \max(x) \). For each \( x \in N^k \), let

\[
\Phi_x^D = \{ F_r^x((y_1,n_1),(y_2,n_2),\ldots,(y_r,n_r)) \mid y_1,\ldots,y_r \in G_D^x, r \geq 1, F_r^x \in \mathcal{F}_G \}
\]

be the set of defined values of \( F_r^x \) where \( s_D(y_1) = n_1, \ldots, s_D(y_r) = n_r \). If \( \Phi_x^D \neq \emptyset \), let \( s_D(x) \) be the minimum over \( \Phi_x^D \); otherwise, let \( s_D(x) = \min(x) \).

**Definition 3.3 (Multiverse SL – selection label multiverse).** Let \( G = (N^k, \Theta) \) be a downward directed graph where \( N \) is the nonnegative integers. Define \( M_{SL} \) to be the set of universes \( \{ (G_D,s_D) \mid D \subset N^k, D \text{ finite} \} \) where \( s_D \) is the selection labeling function of the induced subgraph \( G_D \). We call \( M_{SL} \) a k-dimensional multiverse of type SL and \( (G_D,s_D) \) a universe of type SL.

The following theorem (Theorem 3.4) asserts that the “large-cube” property is valid for Multiverse SL.

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3 Our students called this the “committee labeling function.” The graph \( G_D \) describes the structure of an organization. The function \( \Phi_x^D = F_r^x((y_1,n_1),(y_2,n_2),\ldots,(y_r,n_r)) = n_i \) represents the committee members \( y_j \) with individual reports \( n_j \). The boss, \( x \) (an ex officio member), makes a decision \( n_i \) after taking into account the committee and its inputs.
Theorem 3.4 (Multiverse SL). Let $M_{\text{SL}}$ be a $k$-dimensional multiverse of type SL and let $p$ be any positive integer. Then there is a universe $(G_D, s_D)$ of $M_{\text{SL}}$ and a subset $E \subset N$ with $|E| = p$ and $S = E^k \subset D$ such that the set of $s_D$-significant labels $\{s_D(z) \mid z \in S, s_D(z) < \min(z)\}$ has size at most $k^k$. In fact, $s_D$ is regressively regular over $E^k$.

Proof. Define $\hat{s}_D(x) = s_D(x)$ if $\Phi_x^D \neq \emptyset$. Otherwise, define $\hat{s}_D(x) = \max(x)$. Induction on $\max(x)$ shows that $\hat{s}_D(x) \leq \max(x)$ with equality if and only if $\Phi_x^D = \emptyset$.

Let $Q$ denote the collection of functions $\hat{s}_D$ as $D$ ranges over all finite subsets of $N^k$. We will show that $Q$ is full, reflexive, and jump-free (Definition 2.1). Full and reflexive are obvious from the definition of $\hat{s}_D$. We want to show that for all $\hat{s}_A$ and $\hat{s}_B$ in $Q$ the conditions $x \in A \cap B$, $A_x \subset B_x$, and $\hat{s}_A(y) = \hat{s}_B(y)$ for all $y \in A_x$ imply that $\hat{s}_A(x) \geq \hat{s}_B(x)$.

If $\Phi_x^A = \emptyset$, then $\hat{s}_A(x) = \max(x)$ and thus $\hat{s}_A(x) = \hat{s}_B(x)$. Suppose $\Phi_x^A \neq \emptyset$. The conditions $x \in A \cap B$ and $A_x \subset B_x$ imply that $G_x^A \subset G_x^B$ and hence, using $s_B(y_i) = s_A(y_i)$, $1 \leq i \leq r$, that

$$\Phi_x^A = \{F_r^x((y_1, s_A(y_1)), \ldots (y_r, s_A(y_r))) \mid y_1, \ldots, y_r \in G_x^A, r \geq 1\}$$

equals

$$\{F_r^x((y_1, s_B(y_1)), \ldots (y_r, s_B(y_r))) \mid y_1, \ldots, y_r \in G_x^B, r \geq 1\}$$

which is contained in

$$\Phi_x^B = \{F_r^x((y_1, s_B(y_1)), \ldots (y_r, s_B(y_r))) \mid y_1, \ldots, y_r \in G_x^B, r \geq 1\}.$$ 

Thus, we have $\emptyset \neq \Phi_x^A \subset \Phi_x^B$ and hence $s_A(x) = \min(\Phi_x^A) \leq \min(\Phi_x^B) = s_B(x)$. Since both $\Phi_x^A$ and $\Phi_x^B$ are nonempty, we have $\hat{s}_A(x) = s_A(x) \geq s_B(x) = \hat{s}_B(x)$.

This shows that $Q = \{\hat{s}_D : D \subset N^k, D \text{ finite}\}$ is jump-free. From Theorem 2.2 and the Technical Note, given any integer $p > 0$, there is a finite $D \subset N^k$ and a subset $E^k \subset D$ with $|E| = p$ such that, for some $\hat{s}_D \in Q$, the set $\{\hat{s}_D(z) \mid z \in S, \hat{s}_D(z) < \min(z)\}$ has at most cardinality $k^k$. In fact, $\hat{s}_D$ is regressively regular over $E^k$. Finally, we must show that $s_D$ itself satisfies the conditions just stated for $\hat{s}_D$.

To see this latter point, suppose that $\hat{s}_D(x) \geq \min(x)$ for all $x \in E^k$ of order type $\omega$. If $\Phi_x^D = \emptyset$ then $s_D(x) = \min(x)$. If $\Phi_x^D \neq \emptyset$ then $s_D(x) = \hat{s}_D(x) \geq \min(x)$. Thus, $s_D(x) \geq \min(x)$ for all $x \in E^k$ of order type $\omega$.

Now suppose that for all $z, w \in E^k$ of order type $\omega$, $\hat{s}_D(z) = \hat{s}_D(w) < \min(E)$. This inequality implies that $\Phi_z^D \neq \emptyset$ and $\Phi_w^D \neq \emptyset$ and thus $s_D(z) = \hat{s}_D(z) = \hat{s}_D(w) = s_D(w) < \min(E)$.

Thus, the set $s_D$ is regressively regular over $E^k$. And $\{s_D(z) \mid z \in S, s_D(z) < \min(z)\}$ has at most cardinality $k^k$. \[\square\]
An example of a universe of type SL is given in Figure 3 where we show how $s_D(x)$ was computed for $x = (7, 11)$ using Definition 3.2.

Note that the directed graph structure, $G = (N^k, \Theta)$, which parameterizes Theorem 3.4 can be thought of as the “geometry” of Multiverse SL. This geometry has intuitive value in constructing examples but is a nuisance in proving independence. If we take $\Theta = \{(x, y) \mid \max(x) > \max(y)\}$ to be the maximal possible set of edges then the graph structure can be subsumed in the partial selection functions. We refer to the resulting “streamlined” multiverse $M_{SL}$ as Multiverse HF. Theorem 3.5 below then follows from Theorem 3.4.

**Theorem 3.5 (Multiverse HF).** Let $M$ be the $k$-dimensional multiverse of type $S$ where $\Theta = \{(x, y) \mid \max(x) > \max(y)\}$. Let $p$ be any positive integer.
Then there is a universe \((G_D, s_D)\) of \(M\) and a subset \(E \subset N\) with \(|E| = p\) and \(S = E^k \subset D\) such that the set of significant labels \(\{s_D(z) \mid z \in S, s_D(z) < \min(z)\}\) has size at most \(k^k\). In fact, \(s_D\) is regressively regular over \(E^k\).

A special case of Theorem 3.5 above (where the parameter \(r\) is fixed in defining the \(s_D\)) is equivalent to Theorem 4.4 of [Fri97]. Theorem 4.4 has been shown by Friedman to be independent of the ZFC axioms of mathematics (see Theorem 4.4 through Theorem 4.15 [Fri97] and Lemma 5.3, page 840, [Fri98]).

**Summary:** We have proved that given an arbitrarily large cube, there is some universe \((G_D, s_D)\) of \(M_{SL}\) for which the “physics,” \(s_D\), has a simple structure over a cube of that size. To prove this large-cube property, we have used a theorem independent of ZFC. No proof using just the ZFC axioms is possible. All of the mathematical techniques of physics lie within the ZFC axiomatic system.

4. Final Remarks

For a summary of key ideas involving multiverses, see Linde [Lin95] and Tegmark [Teg09]. Tegmark describes four stages of a possible multiverse theory and discusses the mathematical and physical implications of each. For a well written and thoughtful presentation of the multiverse concept in cosmology, see Sean Carroll [Car10].

Could foundational issues analogous to our assertions about large cubes occur in the study of cosmological multiverses? The set theoretic techniques we use in this paper are fairly new and not known to most mathematicians and physicists, but a growing body of useful ZFC–independent theorems like the jump-free theorem, Theorem 2.2, are being added to the set theoretic toolbox. The existence of structures in a cosmological multiverse corresponding to our lattice multiverse cubes (and requiring ZFC–independent proofs) could be a subtle artifact of the mathematics, physics, or geometry of the multiverse.

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