Semilinear elliptic inequalities with nonlinear convolution terms in cone-like domains

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Abstract

We study the inequality
\[-\Delta u \geq (|x|^{-\alpha} * u^p) u^q\]
in an unbounded cone \(C^\rho_\Omega \subset \mathbb{R}^N\) generated by a proper subdomain \(\Omega\) of the unit sphere \(S^{N-1} \subset \mathbb{R}^N\), \(p, q, \rho > 0\), \(\alpha \in (0, N)\) and \(|x|^{-\alpha} * u^p\) denotes the standard convolution operator in the cone. We discuss the existence and nonexistence of a positive solution in terms of \(N, p, q, \alpha\) and \(\Omega\). The study is further extended to systems of inequalities with nonlinear convolution terms in cone-like domains.

Keywords: Semilinear elliptic inequalities; nonlinear convolution term; cone-like domains; a priori estimates

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1 Introduction and the main results

Let \(\Omega\) be a proper subdomain of the unit sphere \(S^{N-1} \subset \mathbb{R}^N\) (\(N \geq 2\)), that is, \(\Omega\) is connected, relatively open in \(S^{N-1}\) and \(\overline{\Omega} \neq S^{N-1}\). We set
\[C^r_\Omega = \{(r, \omega) \in \mathbb{R}^N : r > 0 \text{ and } \omega \in \Omega\}\]
and for any \(\rho > 0\) and \(0 \leq \rho_1 < \rho_2 < \infty\) we define
\[C^\rho_\Omega = \{(r, \omega) \in \mathbb{R}^N : r > \rho \text{ and } \omega \in \Omega\},\]
\[C^{\rho_1,\rho_2}_\Omega = \{(r, \omega) \in \mathbb{R}^N : \rho_1 < r < \rho_2 \text{ and } \omega \in \Omega\}.

We are concerned in this paper with the study of positive solutions to the semilinear elliptic inequality
\[-\Delta u \geq (|x|^{-\alpha} * u^p) u^q\text{ in } C^\rho_\Omega,\]
and related systems of inequalities, where \( p, q > 0, \alpha \in (0, N) \). The quantity \(|x|^{-\alpha} * u^p\) represents the convolution operation in \( C^\rho_\Omega \) given by

\[
|x|^{-\alpha} * u^p = \int_{C^\rho_\Omega} \frac{u^p(y)}{|x - y|^{\alpha}} dy, \quad x \in C^\rho_\Omega.
\]

We say that \( u \) is a positive solution of (1.1) in \( C^\rho_\Omega \) if:

- \( u \in W^{1,1}_{loc}(C^\rho_\Omega) \cap C(C^\rho_\Omega) \) and \( (|x|^{-\alpha} * u^p) u^q \in L^1_{loc}(C^\rho_\Omega) \).
- \( u > 0 \) in \( C^\rho_\Omega \) and
  \[
  \int_{C^\rho_\Omega} \frac{u^p(y)}{1 + |y|^{\alpha}} dy < \infty.
  \]
- for any \( \varphi \in C^\infty_c(C^\rho_\Omega), \varphi \geq 0 \), we have
  \[
  \int_{C^\rho_\Omega} \nabla u \cdot \nabla \varphi \geq \int_{C^\rho_\Omega} (|x|^{-\alpha} * u^p) u^q \varphi.
  \]

Condition (1.2) follows from the fact that \(|x|^{-\alpha} * u^p < \infty\) almost everywhere in \( C^\rho_\Omega \).

The prototype model

\[
- \Delta u = (|x|^{-1} * u^2) u \quad \text{in} \quad \mathbb{R}^3,
\]

has been around for nearly a century, being introduced by D.R. Hartree [12, 13, 14] in 1928 in the form of time-dependent equations given by

\[
i \psi_t + \Delta \psi + (|x|^{-1} * |\psi|^2) \psi = \psi \quad \text{in} \quad \mathbb{R}^3 \times (0, \infty).
\]

We may regard (1.5) in connection with the Schrödinger-Newton (or Schrödinger-Poisson) equation, namely

\[
i \psi_t + \Delta \psi + \phi \psi = \psi \quad \text{in} \quad \mathbb{R}^3 \times (0, \infty),
\]

where \( \phi \) is a gravitational potential representing the interaction of the particle with its own gravitational field and satisfies the Poisson equation

\[
\Delta \phi = 4\pi G|\psi|^2 \quad \text{in} \quad \mathbb{R}^3.
\]

Thus, one may write the gravitational potential as a convolution, namely

\[
\phi(x) = 4\pi G(|x|^{-1} * |\psi|^2),
\]

which plugged into (1.6) leads us (modulo a scaling of coefficients) to (1.5). Looking for a solution \( \psi \) of (1.5) in the form \( \psi(x, t) = e^{-it} u(x) \) we see that \( u \) satisfies (1.4).

More recently, the equation (1.4) is encountered in the literature under the name of Choquard or Choquard-Pekar equation. Precisely, (1.4) was introduced in 1954 by S.I. Pekar [25] as a model in quantum theory of a Polaron at rest (see also [5]) and
in 1976, P. Choquard used (1.4) in a certain approximation to Hartree-Fock theory of one component plasma (see [20]). The first mathematical study of (1.4) appeared in the mid 1970s and is due to E.H. Lieb [20], followed by P.-L. Lions’ works [21, 22].

The inequality
\[-\Delta u \geq (|x|^{-\alpha} * u^p)u^q \quad \text{in } \mathbb{R}^N \setminus B_1,
\]
was first discussed in [24]. Let us note that (1.7) corresponds to (1.1) in the cone $C^1_\Omega$ where $\Omega = S^{N-1}$. The approach in [24] relies on the so-called Agmon-Allegretto-Piepenbrink positivity principle. Quasilinear versions of (1.7) (again in exterior domains) are studied in [8] while the polyharmonic case is discussed in [9]( see also [3, 4, 6, 7, 10, 11] for more contexts in which (1.7) occurs).

To the best of our knowledge, this is the first work dealing with semilinear elliptic inequalities in cone-like domains featuring convolution terms. The local cases
\[-\Delta u = u^p \quad \text{and} \quad -\Delta u \geq u^p
\]
in a cone-like domain have been studied in [1, 2, 15, 16, 19]. In this work we rely on a different approach. Precisely, we devise an a priori estimate result in Proposition 3.1 based on specific test functions which we combine with Harnack inequalities. In order to state our main result regarding the inequality (1.1), let
\[
\lambda_1 = \lambda_1(\Omega) > 0
\]
denote the first eigenvalue of the Laplace-Beltrami operator $-\Delta_{S^{N-1}}$ in $\Omega$ subject to the homogeneous Dirichlet boundary condition. Denote also by $\phi$ the corresponding eigenfunction which we normalize as $\phi > 0$ in $\Omega$ and $\max_{\Omega} \phi = 1$. Let $\beta_* < 0 \leq \beta^*$ be the two real solutions of the quadratic equation
\[
\beta(\beta + N - 2) = \lambda_1.
\]
(1.8)

Our main result concerning (1.1) is stated below.

**Theorem 1.1.** Assume $p, q, \rho > 0$ and $\alpha \in (0, N)$.

(i) If
\[
p \leq \frac{N - \alpha}{|\beta_*|} \quad \text{or} \quad p + q \leq 1 + \frac{N - \alpha + 2}{|\beta_*|},
\]
then, inequality (1.1) has no positive solutions in the cone $C^\rho_\Omega$;

(ii) If $q > 1 + \frac{2 - \alpha}{|\beta_*|}$ and
\[
p > \frac{N - \alpha}{|\beta_*|} \quad \text{and} \quad p + q > 1 + \frac{N - \alpha + 2}{|\beta_*|},
\]
then, inequality (1.1) has a positive solution $u \in W^{1,1}_{loc}(C_\Omega) \cap C(C_\Omega \setminus \{0\})$ in the whole cone $C_\Omega$.

The existence and nonexistence regions for (1.1) in the positive quadrant of the $pq$-plane are depicted in Figure 1. We leave open the question of existence and nonexistence of positive solutions in the range $0 < q \leq 1 + (2 - \alpha)/|\beta_*|$ and $p + q > 1 + (N - \alpha + 2)/|\beta_*|$. 

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Figure 1: The existence region (dark shaded) and nonexistence region (light shaded) for positive solutions of (1.1).

**Corollary 1.2.** Assume \( p, \rho > 0, q > 1 + \frac{2 - \alpha}{|\beta_*|} \) and \( \alpha \in (0, N) \). The following statements are equivalent:

(i) The inequality

\[
-\Delta u \geq (|x|^{-\alpha} * u^p)u^q \quad \text{in } C^0_{\Omega_1},
\]

has a positive solution \( u \in W^{1,1}_{loc}(C^0_{\Omega_1}) \cap C(C^0_{\Omega_1}) \);

(ii) The inequality

\[
-\Delta u \geq (|x|^{-\alpha} * u^p)u^q \quad \text{in } C_{\Omega},
\]

has a positive solution \( u \in W^{1,1}_{loc}(C_{\Omega}) \cap C(C_{\Omega} \setminus \{0\}) \);

(iii) The exponents \( p \) and \( q \) satisfy \( p > \frac{N - \alpha}{|\beta_*|} \) and \( p + q > 1 + \frac{N - \alpha + 2}{|\beta_*|} \).

Assume next that \( \Omega = S^{N-1} \cap \{x_N > 0\} \) is the upper hemisphere, case in which \( C_{\Omega} = \mathbb{R}^N_+ = \mathbb{R}^{N-1} \times (0, \infty) \) is the half-space in \( \mathbb{R}^N \). In this case \( \lambda_1 = N - 1 \) is the first eigenvalue of \(-\Delta_{S^{N-1}}\). Solving (1.8) we find \( \beta_* = 1 - N < 0 < \beta^* = 1 \).

**Corollary 1.3.** Assume \( p > 0, q > \frac{N - \alpha + 1}{N - 1} \) and \( \alpha \in (0, N) \). Then, the inequality

\[
-\Delta u \geq (|x|^{-\alpha} * u^p)u^q \quad \text{in } \mathbb{R}^N_+,
\]

has a positive solution \( u \in W^{1,1}_{loc}(\mathbb{R}^N_+) \cap C(\mathbb{R}^N_+ \setminus \{0\}) \) if and only if

\[
p > \frac{N - \alpha}{N - 1} \quad \text{and} \quad p + q > \frac{2N - \alpha + 1}{N - 1}.
\]
We next turn to the study of the related system

$$\begin{align*}
-\Delta u &\geq (|x|^{-\alpha} \ast v^p)v^q \\
-\Delta v &\geq u^s
\end{align*}$$

in $C^\rho_\Omega$, \hspace{1cm} (1.11)

where $p, q > 0$ and $s > 1$. The system (1.11) contains a combination of the local nonlinearity $u^s$ and nonlocal nonlinearities given by the convolution operator. As in the case of a single inequality, we say that $(u, v)$ is a positive solution of (1.11) in $C^\rho_\Omega$, $\rho > 0$, if:

- $u, v \in W^{1,1}_{\text{loc}}(C^\rho_\Omega) \cap C(C^\rho_\Omega)$ and $(|x|^{-\alpha} \ast v^p)v^q \in L^1_{\text{loc}}(C^\rho_\Omega)$.
- $u, v > 0$ in $C^\rho_\Omega$ and
  $$\int_{C^\rho_\Omega} \frac{v^p(y)}{1 + |y|^\alpha} \, dy < \infty.$$ \hspace{1cm} (1.12)
- for any $\varphi \in C^\infty_c(C^\rho_\Omega)$, $\varphi \geq 0$, we have
  $$\begin{align*}
  \int_{C^\rho_\Omega} \nabla u \cdot \nabla \varphi &\geq \int_{C^\rho_\Omega} (|x|^{-\alpha} \ast v^p)v^q \varphi, \\
  \int_{C^\rho_\Omega} \nabla v \cdot \nabla \varphi &\geq \int_{C^\rho_\Omega} u^s \varphi.
  \end{align*}$$ \hspace{1cm} (1.13)

Semilinear elliptic systems of inequalities in cone-like domains were discussed in [17, 23] (see also [18] for time dependent inequalities). In the case of (1.11) we are able to identify a critical curve that separates the existence and nonexistence regions of positive solutions to (1.11); see equation (1.15). The main result on the system (1.11) is stated below.

**Theorem 1.4.** Assume $p, q, \rho > 0$ and $\alpha \in (0, N)$.

- (i) If $s > 1$, $p + q \geq 2$ and
  $$p \leq \frac{N - \alpha}{|\beta_*|} \quad \text{or} \quad p + q < \frac{1}{s} \left(1 + \frac{2}{|\beta_*|}\right) + \frac{N - \alpha + 2}{|\beta_*|},$$ \hspace{1cm} (1.14)

  then (1.11) has no positive solutions;

- (ii) If $1 < s \leq 1 + 2/|\beta_*|$, $p + q \geq 2$ and
  $$p + q = \frac{1}{s} \left(1 + \frac{2}{|\beta_*|}\right) + \frac{N - \alpha + 2}{|\beta_*|},$$ \hspace{1cm} (1.15)

  then (1.11) has no positive solutions;

- (ii) If $s > 1 + 2/|\beta_*|$, $q > 1 + (2 - \alpha)/|\beta_*|$ and
  $$p > \frac{N - \alpha}{|\beta_*|} \quad \text{and} \quad p + q > \frac{1}{s} \left(1 + \frac{2}{|\beta_*|}\right) + \frac{N - \alpha + 2}{|\beta_*|},$$ \hspace{1cm} (1.16)

  then (1.11) has positive solutions.
Theorem 1.4 yields a new critical curve given by (1.15) which defines the regions of existence and nonexistence of a positive solution to (1.11). We may regard the quantity \( p + q \) in (1.15) as the (global) exponent of \( v \) in the first inequality of (1.11).

The remaining of this work is organised as follows. In Section 2 we collect some preliminary results consisting of various estimates of the Riesz potential in the cone-like domains. In Section 3 we derive an a priori estimate for positive solutions of (1.1). This is one of the main tools in the proof of Theorem 1.1 which is presented in Section 4. Section 5 is devoted to the proof of Theorem 1.4. Throughout this article by \( c, C, C_1, C_2, \ldots \) we denote positive constants whose value may change on every occurrence.

2 Preliminary results

In this section we collect some preliminary results for our approach. We start with the following integral estimates for Riesz potentials in \( C^p_\Omega \).

Lemma 2.1. Let \( \Omega \subset S^{N-1} \) be a subdomain, \( \rho > 0 \) and \( \alpha \in (0, N) \).

(i) If \( f \in L^1_{loc}(C^p_\Omega) \), \( f \geq 0 \), then there exists \( C > 0 \) such that
\[
\int_{C^p_\Omega} \frac{f(y)}{|x-y|^\alpha} dy \geq C|x|^{-\alpha} \quad \text{for all } x \in C^{2p}_\Omega.
\]

(ii) If \( f \geq 0 \) in \( C^p_\Omega \) and there exists a subdomain \( \Omega_0 \subset \Omega \) and \( c, \gamma > 0 \) such that \( f(x) \geq c|x|^{-\gamma} \) in \( C^p_{\Omega_0} \), then
\[
\begin{cases}
|x|^{-\alpha} * f = \infty & \text{if } N \geq \alpha + \gamma \\
|x|^{-\alpha} * f \geq C|x|^{N-\alpha-\gamma} & \text{if } N < \alpha + \gamma
\end{cases}
\]
in \( C^p_{\Omega} \).

where \( C > 0 \) is a constant.

(iii) If \( f : C_\Omega \to [0, \infty) \) satisfies \( f \in L^1(C^0_{\Omega}) \) and \( f(x) \leq c|x|^{-\gamma} \) in \( C^p_\Omega \) for some \( c > 0 \) and \( \gamma > N - \alpha > 0 \), then there exists a positive constant \( C > 0 \) such that
\[
\int_{C_\Omega} \frac{f(y)}{|x-y|^\alpha} dy \leq \begin{cases}
C|x|^{N-\alpha-\gamma} & \text{if } \gamma < N \\
C|x|^{-\alpha} \log(1 + |x|) & \text{if } \gamma = N \\
C|x|^{-\alpha} & \text{if } \gamma > N
\end{cases}
\]
in \( C^{2p}_\Omega \).

Proof. (i) For any \( x \in C^{2p}_\Omega \) and \( y \in C^{3p/2,2p}_{\Omega} \) we have \( |x-y| \leq |x| + |y| \leq 2|x| \). Thus,
\[
|x|^{-\alpha} * f \geq \int_{C^{3p/2,2p}_{\Omega}} \frac{f(y)}{|x-y|^\alpha} dy \geq C \int_{C^{3p/2,2p}_{\Omega}} \frac{f(y)}{(2|x|)^\alpha} dy = C|x|^{-\alpha}.
\]
where $r > 0$. Because $\Omega_0$ is a subdomain of $S^{N-1}$, each $\theta_k$ $(1 \leq k \leq N-1)$ belongs to a nondegenerate interval $I_k \subset \mathbb{R}$.

Let $|y| \geq 2|x| \geq 2\rho$. Then $|x - y| \leq |x| + |y| \leq 3|y|/2$ so that

$$|x|^{-\alpha} \ast f \geq c \int_{\mathbb{R}^N} \frac{|y|^{-\gamma}}{|x - y|} dy \geq C \int_{\mathbb{R}^N} |y|^{-\alpha - \gamma} dy$$

and the conclusion follows.

(iii) Let $|x| \geq 2\rho$. We have

$$\int_{\mathbb{R}^N} f(y) \frac{1}{|x - y|} dy = \int_{\mathbb{R}^N} f(y) \frac{1}{|x|} dy + \int_{\mathbb{R}^N} f(y) \frac{1}{|y|} dy + \int_{\mathbb{R}^N} f(y) \frac{1}{|y - x|} dy.$$  \hspace{1cm} (2.2)

If $y \in \mathbb{R}^N$ then $|x - y| \geq |x| - |y| \geq |x|/2$ so

$$\int_{\mathbb{R}^N} \frac{f(y)}{|x - y|} dy \leq c|x|^{-\alpha} \int_{\mathbb{R}^N} f(y) dy \leq C|x|^{-\alpha}.$$  \hspace{1cm} (2.3)

Since $f(x) \leq c|x|^{-\gamma}$ in $\mathbb{R}$, we have

$$\int_{\mathbb{R}^N} \frac{f(y)}{|x - y|} dy \leq c \int_{\mathbb{R}^N} \frac{|y|^{-\gamma}}{|x - y|} dy.$$  \hspace{1cm} (2.4)

We next estimate the last integral in (2.3). We have

$$\int_{\mathbb{R}^N} \frac{|y|^{-\gamma}}{|x - y|} dy = \left\{ \begin{array}{l} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|y|^{-\gamma}}{|x|} \frac{1}{|y|} \frac{1}{|y - x|} dy \end{array} \right\} \int_{\mathbb{R}^N} \frac{|y|^{-\gamma}}{|x - y|} dy.$$  \hspace{1cm} (2.4)

For $|y| \geq 2|x|$ we have $|x - y| \geq |y| - |x| \geq |y|/2$ so that

$$\int_{\mathbb{R}^N} \frac{|y|^{-\gamma}}{|x - y|} dy \leq C \int_{\mathbb{R}^N} \frac{1}{|y|^{\alpha + \gamma}} dy = C|x|^{N-\alpha - \gamma}.$$  \hspace{1cm} (2.5)
Similarly, we estimate
\[
\int_{|x|/2, |y|} \frac{|y|^{-\gamma}}{|x-y|^\alpha} dy \leq c|x|^{-\gamma} \int_{|x|/2, |y|} \frac{dy}{|x-y|^\alpha}
\]
\[
\leq c|x|^{-\gamma} \int_{|y-x| \leq 3|x|} \frac{dy}{|x-y|^\alpha}
\]
\[
= C|x|^{N-\alpha-\gamma}.
\]

Finally, if \( \rho < |y| \leq |x|/2 \) then \( |x-y| \geq |x| - |y| \geq |x|/2 \). Hence,
\[
\int_{|x|/2, |y|} \frac{|y|^{-\gamma}}{|x-y|^\alpha} dy \leq C|x|^{-\gamma} \int_{|x|/2, |y|} |y|^{-\gamma} dy = C|x|^{-\gamma} \int_{\rho} t^{N-\gamma-1} dt,
\]
which yields
\[
\int_{|x|/2, |y|} \frac{|y|^{-\gamma}}{|x-y|^\alpha} dy \leq C \begin{cases} |x|^{N-\alpha-\gamma} & \text{if } \gamma < N, \\ |x|^{-\alpha} \log(1 + |x|) & \text{if } \gamma = N, \\ |x|^{-\alpha} & \text{if } \gamma > N. \end{cases}
\]
(2.7)

The result now follows by combining the estimates (2.4)-(2.7) with (2.3).

**Lemma 2.2.** Suppose \( N \geq 2 \) and \( u \in W^{1,1}_{loc}(C_{\rho}) \cap C(\overline{C_{\rho}}) \) is such that \( u > 0 \) and \( -\Delta u \geq 0 \) in \( C_{\rho} \). Then, there exists \( c > 0 \) such that
\[
u(x) \geq c \phi(|x|) |x|^\beta_* \quad \text{in } C_{2\rho}^2.
\]
(2.8)

**Proof.** Let \( w(x) = c \phi(|x|) |x|^\beta_* \) where \( c > 0 \) is small, such that \( u > w \) on \( C_{\rho} \cap \partial B_{2\rho} \).

By direct calculations one can check that \( w \) is harmonic in \( C_{\rho} \). Take \( \delta > 0 \). Then, for \( k > 2\rho \) large we have \( w \leq \delta \) on \( C_{\rho} \cap \partial B_k \). Then,
\[
u + \delta > w \quad \text{on } \partial C_{2\rho}^2.
\]

By the maximum principle for \( u + \delta \) and \( w \) in \( C_{2\rho}^2 \) we derive \( u + \delta \geq w \) in \( C_{2\rho}^2 \). Since \( k > 2\rho \) can be arbitrarily large, it follows that \( u + \delta \geq w \) on \( C_{\rho}^2 \). Now, letting \( \delta \to 0 \) we derive the conclusion.

**Lemma 2.3.** (see [19, Theorem 1.2]) Assume \( N \geq 2, q > 1 \) and \( \gamma \in \mathbb{R} \). Then, the inequality
\[
-\Delta u \geq |x|^\gamma u^q \quad \text{in } C_{\rho}^2,
\]
has positive solutions if and only if \( q > 1 + (\gamma + 2)/|\beta_*| \).

### 3 An a priori estimate

In this section we establish the following a priori estimate for positive solutions of (1.1) in \( C_{\rho}, \rho > 0 \).
Proposition 3.1. Let \( \rho > 0 \) and \( u \in W_{loc}^{1,1}(C_\Omega^\rho) \cap C(\overline{C_\Omega^\rho}) \) be a positive solution of \( (1.1) \). For \( R > \rho \) denote \( \eta_R(x) = \eta(|x|/R) \) where \( \eta \in C_\varepsilon^1(\mathbb{R}) \) is such that

\[
0 \leq \eta \leq 1, \quad \text{supp}\, \eta \subset [1,4] \quad \text{and} \quad \eta = 1 \text{ on } [2,3].
\]

Then, for any \( \lambda > 4 \) and \( 0 \leq m < 1 \), there exists \( C = C(N, m, \alpha, \lambda) > 0 \) such that

\[
\left( \int_{C_\Omega^\rho R} u^{(p+q-m)/2} \phi(\omega)^{1/2} \eta_R^{\lambda/2} \, dx \right)^2 \leq CR^{\alpha-2} \int_{C_\Omega^\rho R} u^{1-m} \phi(\omega) \eta_R^{\lambda-2} \, dx. \tag{3.1}
\]

\textbf{Proof.} Let \( \{ \Omega_\varepsilon \}_{\varepsilon > 0} \) be a sequence of smooth subdomains that exhaust \( \Omega \). Denote by \( \phi_\varepsilon > 0 \) the first eigenfunction of \( -\Delta_{S^{N-1}} \) in \( \Omega_\varepsilon \) which satisfies \( \max_{\Omega_\varepsilon} \phi_\varepsilon = 1 \). We take the test function \( \varphi \) in \( (1.3) \) given by

\[
\varphi = u^{-m} \phi_\varepsilon \eta_R^\lambda. \tag{3.2}
\]

Observe that \( \text{supp}\, \varphi = C_{\Omega_\varepsilon}^{RAR} \subset C_\Omega^\rho \). We find

\[
\int_{C_\Omega^\rho} (|x|^{-\alpha} * u^p) u^{q-m} \phi_\varepsilon \eta_R^\lambda + m \int_{C_\Omega^\rho} u^{-m-1} \phi_\varepsilon \eta_R^\lambda |\nabla u|^2 \leq \int_{C_\Omega^\rho} u^{-m} \nabla u \cdot \nabla (\phi_\varepsilon \eta_R^\lambda) \tag{3.3}
\]

\[
= \frac{1}{1-m} \int_{C_\Omega^{RAR}} \nabla u_{1-m} \cdot \nabla (\phi_\varepsilon \eta_R^\lambda). \tag{3.4}
\]

By the divergence theorem we further compute

\[
\int_{C_\Omega^{RAR}} \nabla u_{1-m} \cdot \nabla (\phi_\varepsilon \eta_R^\lambda) = \int_{\partial C_\Omega^{RAR}} u_{1-m} \nabla (\phi_\varepsilon \eta_R^\lambda) \cdot \nu \, d\sigma - \int_{C_\Omega^{RAR}} u^{1-m} \Delta (\phi_\varepsilon \eta_R^\lambda) \, dx, \tag{3.5}
\]

where \( \nu \) is the outer unit normal vector at \( \partial C_\Omega^{RAR} \). We claim that

\[
\nabla (\phi_\varepsilon \eta_R^\lambda) \cdot \nu \leq 0 \quad \text{on } \partial C_\Omega^{RAR}. \tag{3.6}
\]

Indeed, let \( x \in \partial C_\Omega^{RAR} \) which we write \( x = (r, \omega) \) where \( r = |x|, \omega = x/|x| \). Then \( x \in \partial C_\Omega^{RAR} \) implies

\[
(r, \omega) \in (\{ R \} \times \overline{\Omega}_\varepsilon) \cup (\{ 4R \} \times \overline{\Omega}_\varepsilon) \cup ((R, 4R) \times \partial \Omega_\varepsilon).
\]

By the properties of \( \eta_R^\lambda \) we have that

\[
\nabla (\phi_\varepsilon \eta_R^\lambda)(x) = 0 \quad \text{at } x = (r, \omega) \in (\{ R \} \times \overline{\Omega}_\varepsilon) \cup (\{ 4R \} \times \overline{\Omega}_\varepsilon).
\]

Let now \( x = (r, \omega) \in (R, 4R) \times \partial \Omega_\varepsilon \). Then \( \phi_\varepsilon(\omega) = 0 \) and so

\[
\nabla (\phi_\varepsilon \eta_R^\lambda)(x) = \eta_R^\lambda(x) \nabla \phi_\varepsilon(\omega) = \frac{1}{|x|} \eta_R^\lambda(x) \nabla_{S^{N-1}} \phi_\varepsilon(\omega). \tag{3.7}
\]

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Recall that $\nabla_{S^{N-1}} \phi(x)(\omega)$ lies in the tangent plane to $S^{N-1}$ at $x = \frac{x}{|x|} \in \partial \Omega$. Further, $\nu(x)$ and $\nabla_{S^{N-1}} \phi(x)(\omega)$ are both located in the tangent plane to $S^{N-1}$ at $x = \frac{x}{|x|} \in \partial \Omega$ and have opposite direction. This comes from the fact that $\partial \Omega = \{ \phi = 0 \}$ is a level set of $\phi$ and $\nabla \phi(x) = \frac{1}{|x|} \nabla_{S^{N-1}} \phi(x)$ points towards inside of $\Omega$. Thus,

$$\nu(x) = -\frac{\nabla_{S^{N-1}} \phi(x)(\omega)}{|\nabla_{S^{N-1}} \phi(x)|}.$$ 

Hence, by \((3.6)\) we find

$$\nabla (\phi \eta^\lambda_R)(x) \cdot \nu(x) = -\frac{\eta^\lambda_R(x)}{|x|} \nabla_{S^{N-1}} \phi \leq 0.$$

Combining the above estimates we achieve the claim \((3.5)\). Using \((3.5)\) in \((3.4)\) and then in \((3.3)\) we derive

$$\int_{C^\alpha_{\Omega}} (|x|^{-\alpha} \ast u^p) u^{q-m} \phi \eta^\lambda_R \leq -\frac{1}{1-m} \int_{C^\alpha_{R,AR}} u^{1-m} \Delta (\phi \eta^\lambda_R).$$

Now, letting $\varepsilon \to 0$ we deduce

$$\int_{C^\alpha_{\Omega}} (|x|^{-\alpha} \ast u^p) u^{q-m} \phi \eta^\lambda_R \leq -\frac{1}{1-m} \int_{C^\alpha_{R,AR}} u^{1-m} \Delta (\phi \eta^\lambda_R), \tag{3.7}$$

where $\phi > 0$ is the first eigenfunction of the Laplace-Beltrami operator $-\Delta_{S^{N-1}}$ in $\Omega$ with $\max_{\Omega} \phi = 1$. We have

$$\Delta (\phi \eta^\lambda_R) = \eta^\lambda_R \Delta \phi + 2 \nabla \phi \cdot \nabla \eta^\lambda_R + \phi \Delta \eta^\lambda_R.$$ 

By direct calculation on $C^\alpha_{\Omega}$ we deduce

$$|\eta^\lambda_R \Delta \phi| = \frac{\lambda_1}{|x|^2} \eta^\lambda_R \Delta \phi \leq CR^{-2} \eta^\lambda_R,$$

$$|\Delta \eta^\lambda_R| = \lambda \eta^\lambda_R \left( |\nabla \eta^\lambda_R|^2 + \eta_R \Delta \eta^\lambda_R \right) \leq CR^{-2} \eta^\lambda_R.$$ 

Also,

$$\nabla \phi \cdot \nabla \eta^\lambda_R = \frac{\lambda}{R} \eta' \left( \frac{|x|}{R} \right) \eta^\lambda_R \nabla_{S^{N-1}} \phi \cdot \frac{x}{|x|} = 0,$$

since $\nabla_{S^{N-1}} \phi$ lies in the tangent plane to the unit sphere $S^{N-1}$ at $x/|x| \in \Omega$, and it is orthogonal to $x$. Combining the above estimates we arrive at

$$|\Delta (\phi \eta^\lambda_R)| \leq CR^{-2} \eta^\lambda_R \phi(\omega) \quad \text{in} \quad C^\alpha_{R,AR},$$

so that \((3.7)\) yields

$$\int_{C^\alpha_{\Omega}} (|x|^{-\alpha} \ast u^p) u^{q-m} \phi(\omega) \eta^\lambda_R \leq CR^{-2} \int_{C^\alpha_{R,AR}} u^{1-m} \phi(\omega) \eta^\lambda_R.$$ \tag{3.8}
Let us next estimate the right-hand side of (3.8). For \( x, y \in C^\rho_\Omega \supset C^\rho_\Omega \cap \text{supp} \eta_R \) we have \( |x - y| \leq |x| + |y| \leq 8R \) so
\[
(|x|^{-\alpha} * u^p)(x) \geq \int_{C^\rho_\Omega} \frac{u^p(y)}{|x - y|^\alpha} \, dy \geq (8R)^{-\alpha} \int_{C^\rho_\Omega} u^p(y) \, dy.
\]
Hence, (3.8) yields
\[
\int_{C^\rho_\Omega} u^{1-m} \phi(\omega) \eta_R^{\lambda-2} \geq CR^{2-\alpha} \left( \int_{C^\rho_\Omega} u^p \right) \left( \int_{C^\rho_\Omega} \phi(\omega) \eta_R^\lambda \right).
\]
(3.9)

We now apply the Hölder’s inequality in the right-hand side of (3.9) in order to deduce (3.1).

### 4 Proof of Theorem 1.1

(i) Assume that (1.1) has a positive solution \( u \in C^\rho_\Omega \). From Lemma 2.2 we derive \( u \geq c \phi(\omega)|x|^{\beta_*} \) in \( C^\rho_\Omega \) for some \( c > 0 \). If \( p \leq (N - \alpha)/|\beta_*| \), then, by Lemma 2.1(ii) it follows that \( |x|^{-\alpha} * u^p = \infty \) in \( C^\rho_\Omega \) which contradicts condition (1.2).

In order to deal with the second condition in (1.9) we proceed in two steps.

**Step 1:** If \( p + q \leq 1 \) then (1.1) has no positive solutions.

Assume by contradiction that (1.1) has a positive solution \( u \).

If \( p + q = 1 \), then, by taking \( m = q \in (0, 1) \) in (3.9) we find
\[
\int_{C^\rho_\Omega} u^p \geq \int_{C^\rho_\Omega} \phi(\omega) \eta_R^\lambda \eta_R^{\lambda-2}
\]
\[
\geq CR^{2-\alpha} \left( \int_{C^\rho_\Omega} \phi(\omega) \eta_R^\lambda \right) \left( \int_{C^\rho_\Omega} u^p \right),
\]
where \( \eta_R \) is the test function defined in the statement of Proposition 3.1. The above estimate implies
\[
C \geq R^{2-\alpha} \int_{C^\rho_\Omega} \phi(\omega) \eta_R^\lambda \geq CR^{2-\alpha+N},
\]
which yields a contradiction as \( R \to \infty \).

Assume now \( p + q < 1 \). By letting \( m = p + q \in (0, 1) \) in (3.1) we find
\[
\int_{C^\rho_\Omega} u^{1-m} \geq \int_{C^\rho_\Omega} u^{1-m} \phi(\omega) \eta_R^{\lambda-2}
\]
\[
\geq CR^{2-\alpha} \left( \int_{C^\rho_\Omega} u^{(p+q-m)/2} \phi(\omega)^{1/2} \eta_R^{\lambda/2} \, dx \right)^2
\]
\[
= CR^{2-\alpha} \left( \int_{C^\rho_\Omega} \phi(\omega)^{1/2} \eta_R^{\lambda/2} \, dx \right)^2
\]
\[
\geq CR^{2N-\alpha+2}.
\]
Let $\Omega_0 \subset \subset \Omega$ be a subdomain of $\Omega$ and $x \in C_{\Omega_0}$, $|x| = R > \rho_0 > \max\{1, \rho\}$. We can find a real number $\delta \in (0,1)$ depending only on $\Omega$, $\Omega_0$ such that $B_{R\delta}(x) \subset C_{\Omega}$. Replacing eventually $\Omega_0$ with a smaller subdomain, we may assume that $\text{diam}(\Omega_0) < \delta/6$ and for any $x \in C_{\Omega_0}$, $|x| = R$ one has (see Figure 2 below):

$$D := C_{\Omega_0}^{R-R\delta/6, R+R\delta/6} \subset B_{R\delta/3}(x) \subset C_{\Omega}.$$

(4.2)

Figure 2: The domain $D$ and the ball $B_{R\delta/3}(x)$ in (4.2) on which one applies the Harnack inequality.

With a similar argument as in the Proposition 3.1 in which we replace the pair $(R, 4R)$ by $(R - R\delta/6, R + R\delta/6)$ and $\Omega$ by $\Omega_0$, we derive as in (4.1) that

$$\int_D u^{1-m} \geq CR^{2N-\alpha+2}. \quad (4.3)$$

By the weak Harnack inequality (see [26, Theorem 1.2]) and (4.3) we find

$$u(x) \geq \inf_{B_{R\delta/3}(x)} u \geq C \left( \frac{1}{(R\delta/3)^N} \int_{B_{R\delta/3}(x)} u^{1-m} \right)^{1/(1-m)} \geq C \left( \frac{1}{(R\delta/3)^N} \int_D u^{1-m} \right)^{1/(1-m)} \geq CR^{(N-\alpha+2)/(1-m)}.$$
Hence, \( u(x) \geq C|x|^\zeta \) in \( C_{\Omega_0}^{\rho_0} \) for some \( \rho_0 > \max\{1, \rho\} \), where \( \zeta = \frac{N-\alpha+2}{1-m} > 0 \). This contradicts (1.2) since
\[
\int_{C_{\Omega_0}^{\rho_0}} \frac{u^p(y)}{1 + |y|^\alpha} dy \geq C \int_{C_{\Omega_0}^{\rho_0}} \frac{|y|^{p\zeta}}{2|y|^\alpha} dy = C \int_{C_{\Omega_0}^{\rho_0}} |y|^{p\zeta - \alpha} dy = \infty,
\]
by using the fact that \( \zeta > 0 \) and \( \alpha \in (0, N) \).

**Step 2:** If \( p + q > 1 \) and (1.1) has a positive solution, then \( p + q > 1 + \frac{N-\alpha+2}{|\beta^*|} \).

Let \( m \in (0, 1) \) be close to 1 such that \( p + q > 2 - m \). Let
\[
\mu = \frac{p + q - m}{2 - 2m} > 1 \quad \text{and} \quad \mu' = \frac{\mu}{\mu - 1} > 1. \tag{4.4}
\]
With the same notations as in Proposition 3.1 and using Hölder's inequality we find
\[
\int_{C_{\Omega_0}^{\rho}} u^{1-m} \phi(\omega) \eta_R^{\lambda - 2} \leq \left( \int_{C_{\Omega_0}^{\rho}} u^{(p+q-m)/2} \phi(\omega)^{1/2} \eta_R^{\lambda/2} \right)^{1/\mu} \left( \int_{C_{\Omega_0}^{\rho}} \phi(\omega)^{\gamma} \eta_R^{\sigma} \right)^{1/\mu'}, \tag{4.5}
\]
where
\[
\gamma = \frac{2\mu - 1}{2(\mu - 1)} > 1 \quad \text{and} \quad \sigma = \frac{2(\lambda - 2)\mu - \lambda}{2(\mu - 1)} > 0.
\]
Combining (4.5) and (3.1) we deduce
\[
\left( \int_{C_{\Omega_0}^{\rho}} u^{(p+q-m)/2} \phi(\omega)^{1/2} \eta_R^{\lambda/2} \right)^{2-1/\mu} \leq CR^{\alpha-2} \left( \int_{C_{\Omega_0}^{\rho}} \phi(\omega)^{\gamma} \eta_R^{\sigma} \right)^{1/\mu'} \tag{4.6}
\]
\[
\leq CR^{\alpha-2+N/\mu'}.
\]
Take a subdomain \( \Omega_0 \subset \subset \Omega \). Then, for some constant \( c > 0 \) we have
\[
\phi > c > 0 \quad \text{in} \quad \Omega_0, \tag{4.7}
\]
and the estimate (4.6) yields
\[
\left( \int_{C_{\Omega_0}^{\rho}} u^{(p+q-m)/2} \eta_R^{\lambda/2} \right)^{2-1/\mu} \leq CR^{\alpha-2+N/\mu'} \quad \text{for} \quad R > 2\rho \text{ large}. \tag{4.8}
\]
We next use the estimate (2.8) of Lemma 2.2 together with (4.7) to derive
\[
u \geq c_0 |x|^{\beta_*} \quad \text{in} \quad C_{\Omega_0}^{2\rho}, \tag{4.9}
\]
for some \( c_0 > 0 \). Then, from (4.8) we deduce
\[
CR^{\alpha - 2 + N/\mu'} \geq \left( \int_{\mathcal{C}_{\Omega_0}^p} u(p+q-m)/2 \eta_R^{\lambda/2} \right)^{2-1/\mu} \\
\geq \left( c \int_{\mathcal{C}_{\Omega_0}^p} |x|^{(p+q-m)\beta_*/2} \eta_R^{\lambda/2} \right)^{2-1/\mu} \\
\geq c \left( R^{(p+q-m)\beta_*)/2} \int_{\mathcal{C}_{\Omega_0}^p} \eta_R^{\lambda/2} \right)^{2-1/\mu} \\
\geq c R^{(N+(p+q-m)\beta_*/2)/(2-1/\mu)} \quad \text{for } R > 2 \rho \text{ large.}
\]
This implies
\[
\left( N + \frac{p + q - m}{2} \beta_* \right) \left( 2 - \frac{1}{\mu} \right) \leq \alpha - 2 + \frac{N}{\mu'},
\]
where \( \mu, \mu' \) are given in (4.4). The above estimate yields
\[
p + q \geq 1 + \frac{N - \alpha + 2}{|\beta_*|}. \tag{4.10}
\]

It remains to rule out the equality case in (4.10). Assume by contradiction that the equality holds in (4.10). Using \( u \geq c |x|^{\beta_*} \phi(\omega) \) in \( C_{\Omega}^{2\rho} \) and Lemma 2.1(ii) we find
\[
|x|^{-\alpha} u^p \geq C |x|^{N-\alpha + p \beta_*} \quad \text{in } C_{\Omega}^{2\rho}.
\]
Thus, \( u \) satisfies
\[
- \Delta u \geq C |x|^{N-\alpha + p \beta_*} q^q \quad \text{in } C_{\Omega}^{2\rho}. \tag{4.11}
\]

If \( q > 1 \), then by Lemma 2.3 the above inequality has no positive solutions.

It remains to find a contradicting argument for the case \( 0 < q \leq 1 \). Using the assumption \( p + q = 1 + (N - \alpha + 2)/|\beta_*| \) and the estimate (2.8) in (4.11) we deduce
\[
- \Delta u \geq C \phi(\omega)|x|^{\beta_*-2} \quad \text{in } C_{\Omega}^{2\rho}. \tag{4.12}
\]
Replacing \( \rho \) by \( \max \{ \rho, 1 \} \) we may assume \( \rho \geq 1 \). Let \( 0 < \kappa < 1 \) and
\[
v(x) = \phi(\omega)|x|^{\beta_*} \left( \log |x| \right)^{\kappa} \quad \text{for all } \ x \in C_{\Omega}^{2\rho}.
\]
Recall that
\[
\Delta v = \frac{\partial^2 v}{\partial \mathbf{n}^2} + \frac{N - 1}{|x|} \frac{\partial v}{\partial \mathbf{n}} + \frac{1}{|x|^2} \Delta s^{N-1} v, \tag{4.13}
\]
where \( \mathbf{n} = x/|x| \). We find
\[
- \Delta v = \kappa \phi(\omega)|x|^{\beta_*-2} \left( \log |x| \right)^{\kappa-2} \left( 2|\beta_*| - N + 2 \right) \log |x| + 1 - \kappa \quad \text{in } C_{\Omega}^{2\rho}.
\]
Since \( |\beta_*| > N - 2, \kappa \in (0, 1) \) and \( 0 < q \leq 1 \) we deduce
\[
- \Delta v \leq c \phi(\omega)|x|^{\beta_*-2} \leq \phi(\omega)|x|^{\beta_*-2} \quad \text{in } C_{\Omega}^{2\rho}. \tag{4.14}
\]
By taking $\tilde{c} > 0$ small, from (4.12), (4.14) we find
\[- \Delta (u - \tilde{c} v) \geq 0 \quad \text{in} \quad C^2_{\tilde{\Omega}}.\]  (4.15)

By the maximum principle in the same way as we did in Lemma 2.2 it follows that
\[u \geq \tilde{c} \phi(\omega) |x|^\beta (\log |x|)^\kappa \quad \text{in} \quad C^2_{\tilde{\Omega}},\]  (4.16)
for some constant $\tilde{c} > 0$. We are now able to improve the estimate in Proposition 3.1 and thus to raise a contradiction as follows. Take $m \in (0, 1)$ such that $p + q > 2 - m$ and let $\mu, \mu' > 1$ be given by (4.4). Then, in the same way as above, we derive the estimate (4.5) and then (4.8) which in light of (4.16) yields
\[R^{\alpha - 2 + N/\mu'} \geq C R^{\left(\frac{N + \beta_* (p + q - m)}{2}(2 - 1/\mu') \left(\log R\right)^{\kappa (p + q - m)/2} \right) 2^{-1/\mu}},\]  (4.17)
for $R > 2\rho$ large. By the choice of $\mu > 1$ in (4.4) we have
\[\alpha - 2 + \frac{N}{\mu'} = \left(\frac{N + \beta_* (p + q - m)}{2}\right) \left(2 - \frac{1}{\mu}\right)\]
and this contradicts the inequality (4.17) due to the presence of the log term in the right-hand side whose exponent is positive.

(ii) Assume $q > 1 + \frac{2 - \alpha}{|\beta|}$ and that (1.10) holds. We shall construct a solution $u \in W^{1,1}_{\text{loc}}(\tilde{\Omega}) \cap C(\overline{\tilde{\Omega}} \setminus \{0\})$ of (1.1).

Let $\hat{\Omega}$ be a smooth domain such that $\Omega \subset \subset \tilde{\Omega} \subset S^{N-1}$. Denote by $\tilde{\lambda}_1 > 0$ the first eigenvalue of $-\Delta_{S^{N-1}}$ on $\tilde{\Omega}$ and let $\tilde{\phi}$ be the corresponding eigenfunction which we normalize as $\tilde{\phi} > 0$ in $\tilde{\Omega}$ and $\max_{\tilde{\Omega}} \tilde{\phi} = 1$. Thus,
\[\tilde{\phi} > \tilde{c} > 0 \quad \text{in} \quad \tilde{\Omega},\]  (4.18)
for some constant $\tilde{c} > 0$. By virtue of (1.10), one may find $\beta \in (\beta_*, 0)$, close to $\beta_*$ such that
\[q > 1 + \frac{2 - \alpha}{|\beta|}, \quad p > \frac{N - \alpha}{|\beta|}, \quad p + q > 1 + \frac{N - \alpha + 2}{|\beta|}\]  (4.19)
and 
\[ \tilde{\lambda}_1 > \beta(\beta + N - 2). \] 
(4.20)

By slightly changing the value of \( \beta \), we may assume \( p|\beta| \neq N \). This will allow us to avoid the logarithmic terms when using Lemma 2.1(iii).

Finally, let \( w(x) = (A + |x|^2)^{1/2} \), where \( A > 1 \) is large. We shall construct a solution \( u \) of (1.1) in the form

\[ u = cv, \quad \text{where} \quad v(x) = \tilde{\phi}(\omega)w(x)^\beta. \] 
(4.21)

Observe first that by (4.19) we have

\[ \int_{\Omega} \frac{u^p(y)}{1 + |y|^\alpha} dy \leq c \int_{\gamma_1} \frac{w(y)^p}{1 + |y|^\alpha} dy \leq C \int_{B_1} \frac{A^{pβ/2}}{1 + |y|^\alpha} dy + C \int_{R^N \setminus B_1} |y|^{βp - α} dy < \infty, \]

so \( u \) satisfies condition (1.2). Using (4.13), (4.18) and (4.20) we find

\[ -\Delta v = \tilde{\phi}(\omega)w^{β - 4}\left\{ \tilde{\lambda}_1\frac{w^4}{|x|^2} - \beta(\beta + N - 2)|x|^2 + N A |\beta| \right\} \]
\[ \geq \tilde{c}w^{β - 4}\left\{ \tilde{\lambda}_1 w^2 - \beta(\beta + N - 2)|x|^2 + N A |\beta| \right\} \quad \text{in} \quad \Omega. \]

Thus, by (4.20) it follows that for some constant \( c > 0 \) we have

\[ -\Delta v \geq cw^{β - 2} \quad \text{in} \quad \Omega. \] 
(4.22)

The functions \(-\Delta v \) and \((|x|^{-α} \ast u^p)u^q\) are positive, bounded and continuous in \( C_0^{0,1} \). Hence, there exists \( C > 0 \) such that

\[ -\Delta v \geq C(|x|^{-α} \ast u^p)u^q \quad \text{in} \quad C_0^{0,1}. \]

Thus, for \( c_1 = C^{1/(p + q - 1)} > 0 \) we find

\[ -\Delta (c_1 v) \geq (|x|^{-α} \ast (c_1 v)^p)(c_1 v)^q \quad \text{in} \quad C_0^{0,1}. \] 
(4.23)

We next apply Lemma 2.1(iii) for \( f(x) = v^p(x) \leq |x|^p \) in \( C_1^{1/2} \) and \( γ = p|β| \neq N \). By Lemma 2.1(iii) we find

\[ |x|^{-α} \ast u^p \leq C \begin{cases} |x|^{N - α + pβ} & \text{if} \ p|β| < N \quad \text{in} \quad C_1^1. \\ |x|^{N - 1} & \text{if} \ p|β| > N \quad \text{in} \quad \Omega. \end{cases} \]

Hence,

\[ (|x|^{-α} \ast u^p)v^q \leq C \begin{cases} |x|^{N - α + (p + q)β} & \text{if} \ p|β| < N \quad \text{in} \quad C_1^1. \\ |x|^{-α + qβ} & \text{if} \ p|β| > N \quad \text{in} \quad \Omega. \end{cases} \] 
(4.24)

Using next (4.19)3 (if \( p|β| < N \)) and (4.19)1 (if \( p|β| > N \)) together with (4.22) and (4.24) we find

\[ -\Delta v \geq cw^{β - 2} \geq c|x|^{β - 2} \geq C(|x|^{-α} \ast u^p)v^q \quad \text{in} \quad C_1^1. \]

Hence, for \( c_2 = C^{1/(p + q - 1)} > 0 \) we find

\[ -\Delta (c_2 v) \geq (|x|^{-α} \ast (c_2 v)^p)(c_2 v)^q \quad \text{in} \quad C_1^1. \] 
(4.25)

This shows that \( u = c_2 v \) is a solution of (1.1) in \( C_1^1 \). Further, by letting \( c = \min\{c_1, c_2\} \), from (4.23) and (4.25) it follows that \( u = cv \) is a solution of (1.1) in \( C_1^1 \).
5 Proof of Theorem 1.4

(i) Assume (1.11) has a positive solution \((u,v)\) and that (1.14) holds. Using Lemma 2.2 we find \(v \geq c\phi(\omega)|x|^\beta_* \) in \(C^2_{\beta_*}\). If \(p \leq (N - \alpha)/|\beta_*|\), it follows by Lemma 2.1(ii) that \(|x|^{-\alpha} * \phi^p = \infty \) in \(C^2_{\beta_*}\) which contradicts condition (1.12) in the definition of a solution to (1.11).

It remains to raise a contradiction if \(p,q > 0\) satisfy \(p + q \geq 2\) and

\[
p + q < \frac{1}{s} \left(1 + \frac{2}{|\beta_*|} \right) + \frac{N - \alpha + 2}{|\beta_*|}.
\]

With the same notations as in Proposition 3.1 let us take \(\varphi = \phi \eta_R^\lambda\) as a test function in (1.13); this corresponds to \(m = 0\) in (3.2). We use the same approach as in Proposition 3.1 to find

\[
\left( \int_{C^\rho_R} v^{(p+q)/2} \phi(\omega)^{1/2} \eta_R^{\lambda/2} \right)^2 \leq CR^{\alpha-2} \int_{C^\rho_R} u\phi(\omega)\eta_R^{\lambda-2}
\]

and

\[
\int_{C^\rho_R} u^s \phi(\omega) \eta_R^\lambda \leq CR^{-2} \int_{C^\rho_R} v\phi(\omega) \eta_R^{\lambda-2}.
\]

We next claim that

\[
\int_{C^\rho_R} v\phi(\omega) \eta_R^{\lambda-2} \leq CR^{N \left(1-2/(p+q)\right) + (\alpha-2)/(p+q)} \left( \int_{C^\rho_R} u\phi(\omega)\eta_R^{\lambda-2} \right)^{1/(p+q)}.
\]

If \(p + q = 2\), the above estimate follows directly from (5.2). Indeed, using the fact that \(0 < \phi, \eta_R \leq 1\) in \(C^\rho_R\) and \(\lambda - 2 > \lambda/2\), from (5.2) we have

\[
\int_{C^\rho_R} v\phi(\omega) \eta_R^{\lambda-2} \leq \int_{C^\rho_R} v^{(p+q)/2} \phi(\omega)^{1/2} \eta_R^{\lambda/2} \leq CR^{(\alpha-2)/2} \left( \int_{C^\rho_R} u\phi(\omega) \eta_R^{\lambda-2} \right)^{1/2},
\]

which yields (5.4) in the case \(p + q = 2\).

If \(p + q > 2\), then by Hölder’s inequality and (5.2) we have

\[
\int_{C^\rho_R} v\phi(\omega) \eta_R^{\lambda-2} \leq \left( \int_{C^\rho_R} v^{(p+q)/2} \phi(\omega)^{1/2} \eta_R^{\lambda/2} \right)^{2/(p+q)} \left( \int_{C^\rho_R} \phi(\omega)^{\tau_1} \eta_R^{\tau_2} \right)^{1-2/(p+q)}
\]

\[
\leq CR^{N \left(1-2/(p+q)\right)} \left( \int_{C^\rho_R} v^{(p+q)/2} \phi(\omega)^{1/2} \eta_R^{\lambda/2} \right)^{2/(p+q)}
\]

\[
\leq CR^{N \left(1-2/(p+q)\right)} \left( R^{\alpha-2} \int_{C^\rho_R} u\phi(\omega) \eta_R^{\lambda-2} \right)^{1/(p+q)}
\]

\[
= CR^{N \left(1-2/(p+q)\right) + (\alpha-2)/(p+q)} \left( \int_{C^\rho_R} u\phi(\omega) \eta_R^{\lambda-2} \right)^{1/(p+q)},
\]

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where
\[ \tau_1 = \frac{p + q - 1}{p + q - 2} > 1 \quad \text{and} \quad \tau_2 = \frac{(p + q)(\lambda - 2) - \lambda}{p + q - 2} > 1. \]
This concludes the proof of our claim (5.4). By (5.3) and Hölder’s inequality we have
\[
\int_{C^{p \eta} \Omega} u \phi(\omega) \eta_R^{\lambda - 2} \leq \left( \int_{C^{p \eta} \Omega} u^s \phi(\omega) \eta_R^{\lambda} \right)^{1/s} \left( \int_{C^{p \eta} \Omega} \phi(\omega) \eta_R^{\lambda - 2s/(s-1)} \right)^{1-1/s} 
\leq C R^{N(1-1/s)} \left( \int_{C^{p \eta} \Omega} u^s \phi(\omega) \eta_R^{\lambda} \right)^{1/s} 
\leq C R^{N(1-1/s)-2/s} \left( \int_{C^{p \eta} \Omega} \phi(\omega) \eta_R^{\lambda - 2} \right)^{1/s}. 
\]
Combining (5.4) and (5.5) we find
\[
\left( \int_{C^{R,4R} \Omega} v \phi(\omega) \eta_R^{\lambda - 2} \right)^{1-1/(s(p+q))} \leq C R^\tau \quad \text{for } R > \rho \text{ large, (5.6)}
\]
where
\[
\tau = \frac{1}{p + q} \left( N \left( 1 - \frac{1}{s} \right) - \frac{2}{s} + N(p + q - 2) + \alpha - 2 \right). \tag{5.7}
\]
Let \( \Omega_0 \subset \subset \Omega \) be a smooth subdomain. Then, there exists \( c_0 > 0 \) such that
\[
\phi > c_0 > 0 \quad \text{in } \Omega_0 \tag{5.8}
\]
and that the inequality (5.6) still holds when the integral on the left-hand side is taken over the smaller subdomain \( C^{R,4R} \Omega_0 \). Thus, using (5.6), (5.8) and Lemma 2.2 we find
\[
R^{N+\beta s} \left( 1-1/(s(p+q)) \right) \leq C R^\tau \quad \text{for } R > \rho \text{ large.}
\]
This yields
\[
\frac{N + \beta s}{s} \left( s(p + q) - 1 \right) \leq N \left( 1 - \frac{1}{s} \right) - \frac{2}{s} + N(p + q - 2) + \alpha - 2,
\]
which implies
\[
p + q \geq \frac{1}{s} \left( 1 + \frac{2}{|\beta|} \right) + \frac{N - \alpha + 2}{|\beta|}.
\]
and this contradicts condition (5.1).
(ii) Assume \( 1 < s \leq 1 + 2/|\beta| \). Then, from Lemma 2.2 we have \( u(x) \geq C \phi(\omega)|x|^{\beta s} \) in \( C^{2\rho}_{\Omega} \). Thus,
\[-\Delta v \geq u^s \geq \left( c \phi(\omega)|x|^{\beta s} \right)^s = C \phi(\omega)^s |x|^{s\beta s} \geq C \phi(\omega)^s |x|^{\beta s - 2} \quad \text{in } C^{2\rho}_{\Omega}.
\]
Let \( \Omega_0 \subset \subset \Omega \) be a proper subdomain and \( c_0 > 0 \) be a positive constant such that (5.8) holds. Then, from the above estimate we deduce
\[-\Delta v \geq C \phi(\omega)^s |x|^{\beta s - 2} \geq C c_0^{s-1} \phi(\omega)|x|^{\beta s - 2} \quad \text{in } C^{2\rho}_{\Omega_0}.
\]
Replacing $\rho$ by $\max\{1, \rho\}$ we may assume in the following that $\rho \geq 1$. Let $m \in (0, 1)$. With the same arguments as in the proof of Theorem 1.1 (see (4.12)-(4.16)) we find

$$v(x) \geq c \phi(\omega)|x|^{\beta_*} \left( \log |x| \right)^m \quad \text{in} \quad C^{2\rho}_{\Omega_0}.$$ 

Using this last estimate in (5.6) in which we replace $C^R, 4R, \Omega$ with $C^R, 4R, \Omega_0$ we deduce

$$\left(R^{N+\beta_*} \log^m R\right)^{1-1/(s(p+q))} \leq CR^\tau \quad \text{for} \quad R > 2\rho \quad \text{large},$$

where $\tau$ is given by (5.7). This leads to a contradiction for $R > 2\rho$ large, since the exponents of $R$ in the above inequality are equal and $m > 0$.

(iii) Assume $s > 1 + 2/|\beta_*|$, $q > 1 + (2 - \alpha)/|\beta_*|$ and that (1.16) holds, which we write as

$$p|\beta_*| > N - \alpha \quad \text{and} \quad \left(p + q - \frac{1}{s}\right)|\beta_*| > N - \alpha + 2 + \frac{2}{s}.$$ 

Thus, we may find $b \in (\beta_*, 0)$ close to $\beta_*$ such that

$$s > 1 + \frac{2}{|b|}, \quad q > 1 + \frac{2 - \alpha}{|b|}, \quad p|b| > N - \alpha \quad \text{(5.9)}$$

and

$$\left(p + q - \frac{1}{s}\right)|b| > N - \alpha + 2 + \frac{2}{s} \quad \text{(5.10)}$$

By moving $b$ closer to $\beta_*$, we may also assume $p|b| \neq N$. As in the proof of Theorem 1.1 this will help us to avoid the log-term in Lemma 2.1(iii). From (5.10) we deduce

$$\frac{b - 2}{s} > N - \alpha + 2 + (p + q)b \quad \text{(5.11)}$$

On the other hand, from (5.9) we have $s > 1 + 2/|b|$ so

$$\frac{b - 2}{s} > b \quad \text{(5.12)}$$

Thus, from (5.11) and (5.12) we deduce

$$\frac{b - 2}{s} > \max \left\{b, N - \alpha + 2 + (p + q)b \right\}. \quad \text{(5.13)}$$

We thus may find $a \in (\beta_*, b)$ such that

$$0 > \frac{b - 2}{s} > a > \max \left\{b, N - \alpha + 2 + (p + q)b \right\}. \quad \text{(5.13)}$$

Let now $\tilde{\Omega}$ be a subdomain such that $\Omega \subset \subset \tilde{\Omega} \subset S^{N-1}$ and

$$\tilde{\lambda}_1 > \max \left\{a(a + N - 2), b(b + N - 2) \right\}. \quad \text{(5.14)}$$

where $\tilde{\lambda}_1 > 0$ and $\tilde{\phi} > 0$ denote the first eigenvalue and eigenfunction of $-\Delta_{S^{N-1}}$ on $\tilde{\Omega}$. As before, we normalize $\tilde{\phi} by \max_{\tilde{\Omega}} \tilde{\phi} = 1.$
Let now \( u(x) = \tilde{\phi}(\omega)|x|^a \) and \( v(x) = \tilde{\phi}(\omega)|x|^b \). Using (5.9), we estimate

\[
\int_{C_\Omega}^{} v^p(y) \, |y|^a \, dy \leq c \int_{C_\Omega}^{} |y|^b \, dy \leq c \int_{\mathbb{R}^N \setminus B_\rho} \, |y|^b \, dy < \infty,
\]

and thus, \( v \) satisfies (1.12). Also, since by (5.13) we have \((b-2)/s > a\), from (4.13) and (5.14) we find

\[
-\Delta v = (\tilde{\lambda}_1 - b(b + N - 2))\tilde{\phi}(\omega)|x|^{b-2}
\geq C|x|^{b-2} \geq C|x|^a \tilde{\phi}(\omega)^s
= C u^s \quad \text{in } C_\Omega^p.
\]

It remains to check that \( u \) satisfies the first inequality of the system (1.11). To this aim, let us first estimate the nonlocal term \(|x|^{-\alpha} \cdot v^p\) in \( C_\Omega^p \). We apply Lemma 2.1(iii) for \( f = v^p \leq |x|^{p}\) in \( C_\Omega^p \) and \( \gamma = p|b| \). We obtain

\[
|x|^{-\alpha} \cdot v^p \leq C \begin{cases} |x|^{N-\alpha + p}\ b & \text{if } |p|b| < N \\ |x|^{-\alpha} & \text{if } |p|b| > N \end{cases} \text{ in } C_\Omega^p
\]

and then

\[
(|x|^{-\alpha} \cdot v^p) v^q \leq C \begin{cases} |x|^{N-\alpha + (p+q)b}\ b & \text{if } |p|b| < N \\ |x|^{-\alpha + q}\ b & \text{if } |p|b| > N \end{cases} \text{ in } C_\Omega^p.
\]

**Case 1:** \( p|b| < N \). Let us observe that by (5.13) we have \( a - 2 > N - \alpha + (p+q)b \). Using this fact together with (4.13), (5.14) and (5.16) we find

\[
-\Delta u = (\tilde{\lambda}_1 - a(a + N - 2))\tilde{\phi}(\omega)|x|^{a-2}
\geq C|x|^{a-2}
\geq C|x|^{N-\alpha + (p+q)b}
\geq C(|x|^{-\alpha} \cdot v^p) v^q \quad \text{in } C_\Omega^p.
\]

**Case 2:** \( p|b| > N \). From (5.13) we have \( 0 > a > b \) and from (5.9) we know that \( q > 1 + (2 - \alpha)/|b| \). Thus, \( a - 2 > b - 2 > -\alpha + qb \). As in Case 1 above, by (4.13), (5.14) and (5.16) we deduce

\[
-\Delta u \geq C|x|^{a-2} \geq C|x|^{-\alpha + qb} \geq C(|x|^{-\alpha} \cdot v^p) v^q \quad \text{in } C_\Omega^p.
\]

From (5.15) and (5.17)-(5.18) there exists a constant \( C > 0 \) such that

\[
\begin{cases}
-\Delta u \geq C(|x|^{-\alpha} \cdot v^p) v^q \\
-\Delta v \geq C u^s
\end{cases} \quad \text{in } C_\Omega^p.
\]

Let now \( C_1, C_2 > 0 \) be given by

\[
C_1^{-1/(p+q)} = C_2^{1+1/(p+q)} \quad \text{and} \quad C_2^{p+q-1/s} = C_2^{1+1/s}.
\]

Then \((U, V) = (C_1 u, C_2 v)\) satisfies (1.11) and this finishes the proof of our result. \( \square \)
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