On some new hook-content identities

Michal Sedláčk\textsuperscript{a,b}, Alessandro Bisio\textsuperscript{c,d}

\textsuperscript{a}Institute of Physics, Slovak Academy of Sciences, Dúbravská cesta 9, 845 11 Bratislava, Slovakia
\textsuperscript{b}Faculty of Informatics, Masaryk University, Botanická 68a, 60200 Brno, Czech Republic
\textsuperscript{c}Dipartimento di Fisica, Università di Pavia, via Bassi 6, 27100 Pavia, Italy
\textsuperscript{d}INFN Sezione di Pavia, via Bassi 6, 27100 Pavia, Italy

Abstract

Based on the work of A. Vershik\cite{1}, we introduce two new combinatorial identities. We show how these identities can be used to prove a new hook-content identity. The main motivation for deriving this identity was a particular optimization problem in the field of quantum information processing.

1. Introduction

The representation theory of the symmetric group $S_n$ and of the general linear group $GL(d)$ are related by the so-called Schur-Weyl duality\cite{2}. This famous theorem proves the decomposition

$$V \otimes^n = \bigoplus_{\lambda \vdash n} V_{\lambda} \otimes S_{\lambda} \quad V = C^d$$

for the representation of $GL(d) \times S_n$, where $V_{\lambda}$ is either zero or a polynomial irreducible representation of $GL(d)$, $S_{\lambda}$ is an irreducible representation of $S_n$ and $\lambda$ runs over the partitions of $n$ and is conveniently represented by Young diagram. Both the symmetric group and $GL(d)$ (and especially its compact subgroups $U(d)$ and $SU(d)$) are of paramount importance in theoretical physics, especially in quantum mechanics. For example, $S_n$ is a fundamental symmetry of systems of identical particles and unitary groups represent the set of reversible (finite-dimensional) transformations. Therefore, it is not surprising that physics community keeps a steady interest in the representation theory of these groups and in the Shur-Weyl duality, from the early work of Weyl \cite{3} up to the most recent applications in quantum computing and quantum information processing. For example, Equation (1) denotes a subsystem decomposition (induced by the symmetry of the system-environment interaction) in which one can identify error free subsystems \cite{4,5}, or the relevant subsystems for quantum estimation \cite{6}. These are just a couple of examples of a much wider variety of applications (see e.g. Ref. \cite{7} for a review). In the light of this discussion it is clear that the dimensions of the irreducible spaces $V_{\lambda}$ and $S_{\lambda}$, are, more often than not, a crucial piece of information. The value of $\dim(S_{\lambda})$ and $\dim(V_{\lambda})$ are given by the hook length formula \cite{8} and the hook-content formula \cite{9} respectively.
Those celebrated equations have a nice combinatorial interpretation and, since their discovery, they have been generalized (see e.g. [10] and references therein) and applied in different fields like algebraic geometry [11] and probability [12]. Closely related are also the Littlewood-Richardson rules [13] in the expansion
\[ S_\lambda \otimes S_\mu = \bigoplus \nu S_\nu \]  
and the branching rules for restricting \( S_\lambda \) to \( S_{n-1} \) and inducing \( S_\lambda \) to \( S_{n+1} \) [14].

Our work introduces a new identity, represented by Eq. (14) in Proposition 3, which relates the dimensions \( \dim(S_\lambda), \dim(V_\lambda), \dim(S_\lambda(j)), \) and \( \dim(V_\lambda(j)) \) for any Young diagram \( \lambda \) consisting of \( n \) boxes and diagrams \( \lambda(j) \) that can be obtained from \( \lambda \) by adding a single box. The proof of our result relies on a couple of combinatorial identities, Equations (5) and (6) in Proposition 2, which can be of independent interest. Our approach is modeled after the seminal work [1] of A. Vershik, which provides the essential tools used in this work.

The main motivation for the presented results originates in the problem we were solving [ref!!!] within the field of quantum information processing. While trying to derive optimal success probability for a problem with an arbitrary number of uses of a unitary transformation, \( n \), and an arbitrary dimension of quantum systems, \( d \), we observed several identities involving \( n \) and \( d \), which the optimality of the solution required. By reducing the problem even further we arrived at the necessity to prove the hook content identity (12) that forms the core of this paper, and the needed identities for our original problem correspond to our final Proposition 3.

2. Basic identities

Let \( \{a_i\}_{i=1}^{2s}, s \geq 1 \), be elements of an arbitrary field and let the following coefficients be well defined:
\[
q_{j/m/n}^j := \prod_{i=m+1}^{j} \left( 1 - \frac{a_{2i-1}}{a_{2i-1} + a_{2i} + \cdots + a_{2j}} \right) \prod_{i=j+1}^{n} \left( 1 - \frac{a_{2i}}{a_{2j+1} + a_{2j+2} + \cdots + a_{2i}} \right)
\]
for \( 0 \leq m \leq j \leq n \leq s \).

In Ref. [1] the following result is proved:

**Proposition 1** (Vershik). Let \( q_{m/n}^j \) be defined as in Equation (2). Then, for any \( m, n, j, 0 \leq m \leq j \leq n \leq s \), the following identities hold
\[
C_{m,n} q_{m/n}^j = \sum_{k=m+1}^{j} a_{2k} q_{k/n}^j + \sum_{i=j}^{n-1} a_{2i+1} q_{m/i}^j
\]
\[
C_{m,n} := a_{2m+1} + a_{2m+2} + \cdots + a_{2n},
\]
\[
\sum_{j=m}^{n} q_{m/n}^j = 1.
\]

By rephrasing this result, we could say that \( q_{m/n}^j \) are the solution of the recursion relation (3). Proposition 1 is the main tool for the proof of the following result.
Proposition 2. The following identities hold:

\[
\sum_{j=m}^{n} \left( \sum_{i=m+1}^{j} a_{2i-1} - \sum_{i=j+1}^{n} a_{2i} \right) q_{m,n}^{j} = 0
\]  
(5)

\[
\sum_{j=m}^{n} \left( \sum_{i=m+1}^{j} a_{2i-1} - \sum_{i=j+1}^{n} a_{2i} \right)^2 q_{m,n}^{j} = \sum_{i=m+1}^{n} a_{2i} \sum_{i=m+1}^{j} a_{2i-1}
\]  
(6)

Proof of Equation (5). The proof is by induction on \(n - m\). Let us first consider the case \(n - m = 1\). If we fix an arbitrary \(m\) we have \(n = m + 1\) and Equation (5) gives:

\[-a_{2m+2} q_{m,m+1}^{m} + a_{2m+1} q_{m,m+1}^{m+1} = -a_{2m+2} \frac{a_{2m+1}}{a_{2m+1} + a_{2m+2}} + a_{2m+1} \frac{a_{2m+2}}{a_{2m+1} + a_{2m+2}} = 0.\]

Next, we fix arbitrary \(m\) and \(n\) with \(n - m > 1\) and let us suppose that the thesis holds for any \(m',n'\) such that \(m' < m\) and \(n' - m' < n - m\). By multiplying Equation (5) by \(C_{m,n}\) and by using the recursion formula (3) we obtain:

\[
C_{m,n} \sum_{j=m}^{n} \left( \sum_{i=m+1}^{j} a_{2i-1} - \sum_{i=j+1}^{n} a_{2i} \right) q_{m,n}^{j} = A_{m,n} + B_{m,n}
\]

\[
A_{m,n} := \sum_{j=m}^{n} \left( \sum_{i=m+1}^{j} a_{2i-1} - \sum_{i=j+1}^{n} a_{2i} \right) \sum_{k=m+1}^{j} a_{2k} q_{k,n}^{j}
\]

\[
B_{m,n} := \sum_{j=m}^{n} \left( \sum_{i=m+1}^{j} a_{2i-1} - \sum_{i=j+1}^{n} a_{2i} \right) \sum_{l=j}^{n-1} a_{2l+1} q_{l,m}^{j}
\]

We start with the coefficient \(A_{m,n}\). Since the term with \(j = m\) is zero, we have

\[
A_{m,n} = \sum_{j=m+1}^{n} \sum_{k=m+1}^{j} \left( \sum_{i=m+1}^{j} a_{2i-1} - \sum_{i=j+1}^{n} a_{2i} \right) a_{2k} q_{k,n}^{j} = \sum_{k=m+1}^{n} \sum_{j=m+1}^{k} \left( \sum_{i=m+1}^{j} a_{2i-1} - \sum_{i=j+1}^{n} a_{2i} \right) a_{2k} q_{k,n}^{j} =
\]

\[
= \sum_{k=m+1}^{n} a_{2k} \left\{ \sum_{j=k}^{k} a_{2i-1} \sum_{j=k}^{n} q_{k,n}^{j} + \sum_{j=k}^{n} \left( \sum_{i=k+1}^{j} a_{2i-1} - \sum_{i=j+1}^{n} a_{2i} \right) q_{k,n}^{j} \right\} = \sum_{k=m+1}^{n} a_{2k} \sum_{i=m+1}^{k} a_{2i-1}.
\]

where we used Equation (4) and the inductive hypothesis in the last equality. Now we evaluate the coefficient \(B_{m,n}\). Since the term with \(j = n\) is zero we have

\[
B_{m,n} := \sum_{j=m}^{n-1} \sum_{l=j}^{n-1} \left( \sum_{i=m+1}^{j} a_{2i-1} - \sum_{i=j+1}^{n} a_{2i} \right) a_{2l+1} q_{l,m}^{j} = \sum_{l=m+1}^{n-1} \sum_{j=m}^{l} \left( \sum_{i=m+1}^{j} a_{2i-1} - \sum_{i=j+1}^{n} a_{2i} \right) a_{2l+1} q_{l,m}^{j} =
\]

\[
= \sum_{l=m}^{n-1} \sum_{i=m+1}^{l} \left( \sum_{j=m+1}^{i} a_{2i-1} - \sum_{j=i+1}^{l} a_{2i} \right) a_{2l+1} q_{l,m}^{j} = \sum_{l=m}^{n+1} a_{2l+1} \left\{ \sum_{j=m+1}^{l} \sum_{i=m+1}^{j} a_{2i-1} - \sum_{i=j+1}^{l} a_{2i} \right\} q_{l,m}^{j} = \sum_{l=m}^{n-1} n \sum_{i=m+1}^{l} a_{2l+1} \sum_{i+l+1}^{n} q_{l,m}^{j} = \sum_{l=m}^{n+1} n \sum_{i=m+1}^{l} a_{2l+1} \sum_{i+l+1}^{n} a_{2i}.
\]
Finally, we have
\[ A_{m,n} + B_{m,n} = \sum_{k=m+1}^{n} \sum_{i=m+1}^{k} a_{2i-1}a_{2k} = \sum_{l=m}^{n-1} \sum_{i=l+1}^{n} a_{2l+1}a_{2i} = \]
\[ = \sum_{k=m+1}^{n} \sum_{i=m+1}^{k} a_{2i-1}a_{2k} = \sum_{l'=m+1}^{n} \sum_{i=l'+1}^{n} a_{2l'-1}a_{2i} = \]
\[ = \sum_{k=m+1}^{n} \sum_{i=m+1}^{k} a_{2i-1}a_{2k} = \sum_{i=m+1}^{n} \sum_{l'=m+1}^{i} a_{2l'-1}a_{2i} = 0 \]
where we defined \( l' = l + 1 \).

**Proof of Equation (6).** The proof is again by induction and is very similar to the proof of Equation (5). Let us first fix an arbitrary \( m \) and \( n = m + 1 \). Then, Equation (6) becomes:
\[ a^{2m+2} q^m_{m+1} + a^{2m+2} q^{m+1}_{m+1} = a^{2m+2} \frac{a_{2m+1}}{a_{2m+1} + a_{2m+2}} a^{2m+2} \frac{a_{2m+2}}{a_{2m+1} + a_{2m+2}} = a_{2m+2} a_{2m+1}. \]
We now fix arbitrary \( m \) and \( n \) with \( n - m > 1 \) and let us suppose that the thesis holds for any \( m', n' \) such that \( m' \leq n' \) and \( n' - m' < n - m \). By multiplying the left hand side of Equation (6) by \( C_{m,n} \) and by inserting the recursion formula (3) we obtain:
\[ C_{m,n} \sum_{j=m}^{n} \left( \sum_{i=m+1}^{j} a_{2i-1} - \sum_{i=j+1}^{n} a_{2i} \right)^2 q^j_{m,n} = \alpha_{m,n} + \beta_{m,n} \]
\[ \alpha_{m,n} := \sum_{j=m}^{n} \left( \sum_{i=m+1}^{j} a_{2i-1} - \sum_{i=j+1}^{n} a_{2i} \right)^2 \sum_{k=m+1}^{j} a_{2k} q^j_{k,n} \]
\[ \beta_{m,n} := \sum_{j=m}^{n} \left( \sum_{i=m+1}^{j} a_{2i-1} - \sum_{i=j+1}^{n} a_{2i} \right)^2 \sum_{l=1}^{n-1} \sum_{i=m+1}^{n} a_{2l-1} q^j_{m,l} \]
Next we rewrite coefficient \( \alpha_{m,n} \). Since the term with \( j = m \) is zero we have
\[ \alpha_{m,n} = \sum_{j=m+1}^{n} \sum_{k=m+1}^{j} \left( \sum_{i=m+1}^{j} a_{2i-1} - \sum_{i=j+1}^{n} a_{2i} \right)^2 a_{2k} q^j_{k,n} = \]
\[ = \sum_{k=m+1}^{n} \sum_{j=k}^{n} a_{2k} \left( \sum_{i=m+1}^{k} a_{2i-1} + \sum_{i=k+1}^{j} a_{2i-1} - \sum_{i=j+1}^{n} a_{2i} \right)^2 q^j_{k,n} = \]
\[ = \sum_{k=m+1}^{n} a_{2k} \left( \sum_{i=m+1}^{k} a_{2i-1} \right)^2 + \sum_{i=k+1}^{n} \sum_{j=i+1}^{n} a_{2i-1} a_{2j} \right)^2 q^j_{k,n} = \]
\[ = \sum_{k=m+1}^{n} a_{2k} \left( \sum_{i=m+1}^{k} a_{2i-1} \right)^2 + \sum_{i=k+1}^{n} \sum_{l=k+1}^{i} a_{2l-1} a_{2i} \right) \]
where we used the inductive hypothesis, Equation (4) and Equation (5) that was previously proved. Similarly, we rewrite the coefficient $\beta_{m,n}$. Since the term with $j = n$ is zero we have

$$\beta_{m,n} = \sum_{l=m}^{n-1} \sum_{j=m}^{n-1} \left( \sum_{i=m+1}^{j} a_{2i-1} - \sum_{i=m+1}^{n} a_{2i} \right)^2 a_{2l+1} q_{m,l}^j =$$

$$= \sum_{l=m}^{n-1} \sum_{j=m}^{n-1} \left( \sum_{i=m+1}^{j} a_{2i-1} - \sum_{i=m+1}^{l} a_{2i} \right)^2 a_{2l+1} q_{m,l}^j =$$

$$= \sum_{l=m}^{n-1} \sum_{j=m}^{n-1} \left( \sum_{i=m+1}^{j} a_{2i-1} - \sum_{i=1}^{l} a_{2i} \right)^2 \left( \sum_{i=m+1}^{l} a_{2i} \right) \left( \sum_{i=m+1}^{l} a_{2i-1} - \sum_{i=1}^{l} a_{2i} \right)$$

$$+ \left( \sum_{i=m+1}^{l} a_{2i} \right)^2 q_{m,l}^j = \sum_{l=m}^{n-1} \sum_{j=m}^{n-1} \left( \sum_{i=m+1}^{l} a_{2i-1} - \sum_{i=1}^{l} a_{2i} \right) \left( \sum_{i=m+1}^{l} a_{2i} \right) \left( \sum_{i=m+1}^{l} a_{2i-1} - \sum_{i=1}^{l} a_{2i} \right)$$

where we have used the inductive hypothesis, Equation (4) and Equation (5). Let us inspect the product of the right hand side of Equation (6) and $C_{m,n}$. We obtain

$$C_{m,n} \left( \sum_{i=m+1}^{n} a_{2i} \sum_{l=m+1}^{n} a_{2l-1} \right) = \gamma_{m,n}^{(1)} + \gamma_{m,n}^{(2)},$$

$$\gamma_{m,n}^{(1)} := \sum_{i=m+1}^{n} \sum_{l=m+1}^{n} a_{2i} a_{2l-1} a_{2k-1}, \quad \gamma_{m,n}^{(2)} := \sum_{i=m+1}^{n} \sum_{l=m+1}^{n} a_{2i} a_{2l-1} a_{2k}.$$

Let us define

$$\alpha_{m,n}^{(1)} := \sum_{k=m+1}^{n} \sum_{i=m+1}^{k} a_{2i-1} a_{2j-1} a_{2k}, \quad \beta_{m,n}^{(1)} := \sum_{l=m}^{n-1} \sum_{i=m+1}^{l} a_{2k-1} a_{2l+1},$$

$$\alpha_{m,n}^{(2)} := \sum_{k=m+1}^{n} \sum_{i=m+1}^{k} a_{2i-1} a_{2j} a_{2k}, \quad \beta_{m,n}^{(2)} := \sum_{l=m}^{n-1} \sum_{i=m+1}^{l} a_{2j} a_{2l+1}$$

$$\beta_{m,n} = \alpha_{m,n}^{(1)} + \alpha_{m,n}^{(2)}.$$
In a similar way we have
\[ a_{m,n}^{(2)} + \beta_{m,n}^{(2)} = \sum_{j=m+1}^{n} \sum_{i=j+1}^{n} a_{2j}a_{2i}a_{2(l-1)} + \sum_{j=m+1}^{n} \sum_{i=1}^{j-1} a_{2l-1}a_{2j} = \]
\[ = \sum_{j=m+1}^{n} \sum_{i=1}^{j-1} a_{2j}a_{2(l-1)} + \sum_{i=m+1}^{n} \sum_{j=i+1}^{n} a_{2j}a_{2(l-1)}a_{2j} = \]
\[ = \sum_{i=m+1}^{n} \sum_{j=m+1}^{n} a_{2j}a_{2l-1}a_{2j} = \sum_{i=m+1}^{n} \sum_{j=i+1}^{n} a_{2j}a_{2l-1}a_{2j} = \gamma_{m,n}^{(2)}. \]
Therefore \( \alpha_{m,n}^{(1)} + \alpha_{m,n}^{(2)} + \beta_{m,n}^{(1)} + \beta_{m,n}^{(2)} = \gamma_{m,n}^{(1)} + \gamma_{m,n}^{(2)} \) which finally proves the thesis. ■

3. Hook-content identities and \( GL(n) \)

For a natural number \( n \), we denote with \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \) \( \lambda_i \in \mathbb{Z}, \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0 \), \( \sum_{i=1}^{k} \lambda_i = n \) a partition of \( n \) and we write \( \lambda \vdash n \). Any partition corresponds to a Young diagram, which is an array of boxes, in the plane, left-justified, with \( \lambda_i \) cells in the \( i \)-th row from the top (English convention).

A greek letter \( \lambda \) denotes both the partition and the corresponding Young diagram. A box \( b \in \lambda \) of a Young diagram can be labeled by a pair of integer numbers \( b = (i,j) \) where \( i \) denotes the row and \( j \) denotes the column. We denote with \( h(b) \) the hook length of the box \( b = (i,j) \), i.e. the number of boxes \( b' = (i',j') \) such that \( i = i' \) and \( j \geq j' \) or \( j = j' \) and \( i \geq i' \). For example, if \( \lambda = (4,3,1) \) and \( b = (1,2) \), we have \( h(b) = 4 \).

The content of the box \( b = (i,j) \) is defined as \( c(b) := j - i \). A Young tableau of shape \( \lambda \) is a Young diagram \( \lambda \) in which each box is filled with an integer number. A semi-standard Young tableau of parameters \( (d, \lambda) \), is a Young tableau of shape \( \lambda \) such that the entries are positive integers no greater that \( d \) and they weakly increase along rows and strictly increase along columns. For example the following tableau

\[
\begin{array}{cccc}
1 & 2 & 2 & 3 \\
2 & 3 \\
4 \\
\end{array}
\]

is a semistandard Young tableau of parameters \( (4, (4,2,1)) \). The Stanley’s hook-content formula gives the number of semistandard Young tableaux of parameters \( (\lambda, d) \) (denoted with \( SSYT(\lambda, d) \)), namely

\[ SSYT(\lambda, d) = \prod_{b \in \lambda} \frac{d - c(b)}{h(b)}. \tag{9} \]

The number \( SSYT(\lambda, d) \) is the dimension \( \dim(V_{\lambda}) \) of the vector space \( V_{\lambda} \) which is the irreducible polynomial representation of \( GL(d) \) labelled by the partition \( \lambda \).

A standard Young tableau of shape \( \lambda \vdash n \) is semistandard Young tableau of parameters \( (n, \lambda) \) such that the filling is a bijective assignment of \( 1, 2, \ldots, n \). For example the following tableau

\[
\begin{array}{cccc}
1 & 2 & 5 & 7 \\
3 & 4 \\
6 \\
\end{array}
\]
is a standard Young tableau of shape \((4,2,1)\). The number of standard Young tableaux of shape \(\lambda \vdash n\) (denoted with \(\text{SYT}(\lambda)\)) is given by the 
\[\text{SYT}(\lambda) = \prod_{b \in \lambda} \frac{n!}{h(b)},\]
(10)

The number \(\text{SYT}(\lambda)\) is the dimension \(\dim(S_\lambda)\) of the Specht module \(S_\lambda\), i.e. the irreducible representation of the symmetric group \(S_n\) labelled by the partition \(\lambda\).

We now introduce the step coordinates for a Young diagram \(\lambda\). First, let us define the following notation for a partition \(\lambda = ((\lambda'_1, k'_1), (\lambda'_2, k'_2), \ldots, (\lambda'_s, k'_s))\), where \(k'_i\) denotes the multiplicity of the number \(\lambda'_1\) and \(\lambda'_1 > \lambda'_2 > \cdots > \lambda'_s\). For example we have \(\lambda = (4,4,3,3,1,1,1,1)\) = \(((4,3), (3,2), (1,3))\) Then we define \(p_1 = \lambda'_s\), \(p_i := \lambda'_{s-i+1} - \lambda'_{s-i+2} - 1\) for \(i = 2, \ldots s\) and \(k_i := k'_{s-i+1} - 1\) for \(i = 1, \ldots s\). The numbers \((p_1, k_1, p_2, k_2, \ldots, p_s, k_s)\) are the step coordinates of the Young diagram \(\lambda\), the reason for this notation is clear by looking at the example in Figure 1.

![Figure 1: The Young diagram \(\lambda = (8,8,4,4,1,1,1)\) has step coordinates \((1,3,3,2,4,2)\)](image)

For a given Young diagram \(\lambda\) we denote with \(\lambda^{(j)}\) the Young diagram that can be obtained from \(\lambda\) by the addition of the box \(b^{(j)}\), for example, if \(\lambda = (3,3,2,2)\) we have

\[
\lambda = \begin{array}{cccc}
1 & 2 & \square & \\
\square & 3 & 4 & \\
\end{array}
\quad
\lambda^{(0)} = \begin{array}{cccc}
1 & 2 & 5 & \\
\square & 3 & 4 & \\
\end{array}
\quad
\lambda^{(1)} = \begin{array}{cccc}
1 & 2 & 5 & 6 \\
\square & 3 & 4 & \\
\end{array}
\quad
\lambda^{(2)} = \begin{array}{cccc}
1 & 2 & 5 & 6 \\
\square & 3 & 4 & 7 \\
\end{array}
\]

This is clearly equivalent to say that, for a Young diagram \(\lambda \vdash n\), \(\lambda^{(j)}\) is a partition of \(n + 1\) such that the Young diagram of \(\lambda\) fits inside that of \(\lambda^{(j)}\), and we write \(\lambda^{(j)} \leftarrow \lambda\).
The connection between combinatorial identities studied in Section 2 and the representation theory is provided by the following observation. Let us now consider an arbitrary Young diagram $\lambda$ with step coordinates $(p_1, k_1, p_2, k_2, \ldots, p_s, k_s)$ and let us apply Equation (2) to the sequence $a_{2i-1} := p_i$, $a_{2i} := k_i$ $i = 1, \ldots, s$. Then, using the definition of hook-length, we have
\[
q_0 \cdot s = \prod_{b \in \lambda(j)} h(b) \quad \text{for } j = 0, \ldots, s, \tag{11}
\]
for any Young diagram $\lambda \vdash n$, and $\lambda(j) \vdash n+1$, $\lambda(j) \leftarrow \lambda$. Applying this observation to combinatorial identities (5) and (6) and realizing that
\[
\sum_{i=1}^{j} a_{2i-1} - \sum_{i=j+1}^{s} a_{2i} = \sum_{i=1}^{j} p_i - \sum_{i=j+1}^{s} k_i = c(b(j))
\]
we obtain a proof of the following hook-content identities:
\[
\sum_{\lambda(j) \leftarrow \lambda} c(b(j)) \prod_{b \in \lambda(j)} h(b) = 0 \quad \tag{12}
\]
\[
\sum_{\lambda(j) \leftarrow \lambda} (c(b(j)))^2 \prod_{b \in \lambda(j)} h(b) = n \quad \tag{13}
\]
As consequence of the above Equations (12) and (13) we also have:

**Proposition 3.** For any Young diagram $\lambda \vdash n$, and $\lambda(j) \vdash n+1$, $\lambda(j) \leftarrow \lambda$ we have
\[
\sum_{\lambda(j) \leftarrow \lambda} \left( \frac{SSYT(\lambda(j), d)}{SSYT(\lambda, d)} \right)^2 \frac{SYT(\lambda)}{SYT(\lambda(j))} = \frac{n + d^2}{n + 1} \quad \tag{14}
\]

**Proof.** By expanding Equation (14) we obtain
\[
(n+1) \sum_{\lambda(j) \leftarrow \lambda} \left( \frac{SSYT(\lambda(j), d)}{SSYT(\lambda, d)} \right)^2 \frac{SYT(\lambda)}{SYT(\lambda(j))} = \sum_{\lambda(j) \leftarrow \lambda} (d - c(b(j)))^2 \prod_{b \in \lambda(j)} h(b) = \sum_{\lambda(j) \leftarrow \lambda} (d^2 - 2dc(b(j)) + (c(b(j)))^2) \prod_{b \in \lambda(j)} h(b) = d^2 + n,
\]
where the last equality follows from Equations (4), (11), (12) and (13). ■

We conclude this section by noticing that Equation (12), can be alternatively proved by combining the hook content formula (9) and easy consideration of representation theory. Indeed, let us consider the trivial identity
\[
d \dim(V_{\lambda}) = \sum_{\lambda(j) \leftarrow \lambda} \dim(V_{\lambda(j)}) \quad \tag{15}
\]

\(^{1}\)This observation appears in the work of Vershik \[1\] (see Eq. (17) therein), with different notation for the step coordinates.
where $V_\lambda$ an irreducible polynomial representation of $GL(d)$, $V$ is the fundamental (or defining) representation of $GL(d)$, and $V_{\lambda, j}$ are the irreducible inequivalent polynomial representations in the decomposition $V_\lambda \otimes V = \sum_{\lambda, j} V_{\lambda, j}$. From the hook content formula Equation (15) becomes

$$d = \sum_{\lambda, j} \frac{\dim(V_{\lambda, j})}{\dim(V_\lambda)} = \sum_{\lambda, j} \left( d - c(j) \right) \prod_{b \in \lambda} h(b) \sum_{\lambda, j} \left( d - c(j) \right) \prod_{b \in \lambda} h(b) = 0$$

where we used Equations (11) and (14) in the final step.

4. Conclusion

In this paper we proved some new combinatorial identities, Equations (5) and (6), which can be proved following the techniques of Ref. [1]. These identities lead to a couple of hook content identities (Equations (12) and (13)). The first identity can be proved with easy arguments from representation theory, and our approach provides an alternative proof. On the other hand, Equation (13) and Equation (14) are new.

The representation theory of the symmetric group and of the general linear group play a significant role in many areas of quantum information, as discussed e.g. in Ref. [15]. In particular, Equation (14) is directly linked to the optimal solution of the perfect probabilistic storing and retrieving of an unknown unitary transformation [16], which was the problem that led us to prove the presented hook-content identities.

Acknowledgement

A. B. is supported by the John Templeton Foundation under the project ID# 60609 Quantum Causal Structures. The opinions expressed in this publication are those of the author and do not necessarily reflect the views of the John Templeton Foundation. M.S. acknowledges the support by the QuantERA project HIPHOP (project ID 731473), projects QETWORK (APVV-14-0878), MAXAP (VEGA 2/0173/17), GRUPIK (MUNI/G/1211/2017) and the support of the Czech Grant Agency (GAR) project no. GA16-22211S. The authors are grateful to M. Ziman for fruitful discussions and collaborative work, which lead to formulation of identity, which is proved in this manuscript.

References

[1] A. M. Vershik, Hook formula and related identities, Journal of Soviet Mathematics 59 (5) (1992) 1029–1040. doi:10.1007/BF01480684

[2] I. Schur, Über eine Klasse von Matrizen, die sich einer gegebenen Matrix zuordnen lassen, Dieterich in Göttingen, 1901.

[3] H. Weyl, The theory of groups and quantum mechanics, Courier Corporation, 1950.

[4] E. Knill, R. Laflamme, L. Viola, Theory of quantum error correction for general noise, Phys. Rev. Lett. 84 (2000) 2525–2528. doi:10.1103/PhysRevLett.84.2525

[5] M. Junge, P. T. Kim, D. W. Kribs, Universal collective rotation channels and quantum error correction, Journal of Mathematical Physics 46 (2) (2005) 022102. arXiv:https://doi.org/10.1063/1.1824213 doi:10.1063/1.1824213

[6] G. Chiribella, G. M. D’Ariano, P. Perinotti, M. F. Sacchi, Efficient use of quantum resources for the transmission of a reference frame, Phys. Rev. Lett. 93 (2004) 180503. doi:10.1103/PhysRevLett.93.180503
[7] D. J. Rowe, M. J. Carvalho, J. Repka, Dual pairing of symmetry and dynamical groups in physics. Rev. Mod. Phys. 84 (2012) 711–757. doi:10.1103/RevModPhys.84.711
URL https://link.aps.org/doi/10.1103/RevModPhys.84.711

[8] J. S. Frame, G. d. B. Robinson, R. M. Thrall, et al., The hook graphs of the symmetric group, Canad. J. Math 6 (316) (1954) C324.

[9] R. P. Stanley, S. Fomin, Enumerative Combinatorics, Vol. 2 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1999. doi:10.1017/CBO9780511609589

[10] I. Ciocan-Fontanine, M. Konvalinka, I. Pak, The weighted hook length formula. Journal of Combinatorial Theory, Series A 118 (6) (2011) 1703–1717. doi:https://doi.org/10.1016/j.jcta.2011.02.004
URL http://www.sciencedirect.com/science/article/pii/S0097316511000288

[11] G. Fløystad, T. Johnsen, A. L. Knutsen, Combinatorial Aspects of Commutative Algebra and Algebraic Geometry: The Abel Symposium 2009, Vol. 6, Springer Science & Business Media, 2011.

[12] C. Greene, A. Nijenhuis, H. S. Wilf, A probabilistic proof of a formula for the number of young tableaux of a given shape, Advances in Mathematics 31 (1) (1979) 104–109.

[13] W. Fulton, Young tableaux: with applications to representation theory and geometry, Vol. 35, Cambridge University Press, 1997.

[14] J. Patera, D. Sankoff, Tables of branching rules for representations of simple lie algebras.

[15] A. W. Harrow, Applications of coherent classical communication and the schur transform to quantum information theory, PhD thesis (2005) arXiv:quant–ph/051225.

[16] M. Sedlak, A. Bisio, M. Ziman, Optimal probabilistic storage and retrieval of unitary channels, in preparation.