MEAN LI-YORKE CHAOTIC SET ALONG POLYNOMIAL SEQUENCE WITH FULL HAUSDORFF DIMENSION FOR $\beta$-TRANSFORMATION

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Abstract. We construct a mean Li-Yorke chaotic set along polynomial sequences (the degree of this polynomial is not less than three) with full Hausdorff dimension and full topological entropy for $\beta$-transformation. An uncountable subset $C$ is said to be a mean Li-Yorke chaotic set along sequence $\{a_n\}$, if both

$$
\liminf_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} d(f^{a_j}(x), f^{a_j}(y)) = 0 \quad \text{and} \quad \limsup_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} d(f^{a_j}(x), f^{a_j}(y)) > 0
$$

hold for any two distinct points $x$ and $y$ in $C$.

1. Introduction. Let $(X, d)$ be a metric space and $f: X \to X$ be a continuous self-map. The pair $(X, f)$ is said to be a topological dynamical system. Since the Li-Yorke chaos was introduced by Li and Yorke [12] in 1975, various versions of chaos have been studied, such as Devaney chaos, three kinds of distributional chaos [23], Xiong chaos [25], topological mixing, and positive topological entropy. Besides, the mean form of Li-Yorke chaos was first considered in [6]. A subset $C$ of $X$ is called a mean Li-Yorke scrambled set if any two distinct points $x, y$ in $C$ satisfy

$$
\liminf_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} d(f^j(x), f^j(y)) = 0 \quad \text{and} \quad \limsup_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} d(f^j(x), f^j(y)) > 0.
$$

A system $(X, f)$ is called mean Li-Yorke chaotic, if it contains an uncountable mean Li-Yorke scrambled set. And the author also mentioned in [6] that mean Li-Yorke chaos is equivalent to DC2 chaos and positive topological entropy implies mean Li-Yorke chaos. Moreover, positive topological entropy also implies multivariant mean Li-Yorke chaos [11]. Afterwards, another condition that promises the existence of an uncountable mean Li-Yorke scrambled set was given. In [9], the authors showed that if a system is mean sensitive and contains a mean proximal pair $(x, y)$ with $x$ a transitive point and $y$ a periodic point, then the system is mean Li-Yorke chaotic. Not long ago, mean Li-Yorke chaos in some random dynamical systems with positive entropy was considered in [24]. Analogous to “sequence version” of other dynamical

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notions (e.g., sequence entropy), the sequence version of mean Li-Yorke chaos was introduced in [17]. Let \( A = \{a_1 < a_2 < \cdots \} \) be a sequence of positive integers. According to [17], a subset \( C \) of \( X \) is called mean Li-Yorke scrambled along the sequence \( A \) if any two distinct points \( x, y \) in \( C \) satisfy
\[
\liminf_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} d(f^{a_j}(x), f^{a_j}(y)) = 0 \quad \text{and} \quad \limsup_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} d(f^{a_j}(x), f^{a_j}(y)) > 0.
\]
If in addition, \( C \subset X \) is uncountable, then we say that the system \((X, f)\) is mean Li-Yorke chaotic along the sequence \( A \). By the definitions, the mean Li-Yorke chaos can be viewed as the mean Li-Yorke chaos along the sequence \( \{n\} \). Moreover, it is worth pointing out that with different sequences, the sequence version of mean Li-Yorke scrambled set may have different properties. Li and Qiao [17] showed that in the case when \( \{a_n\} \) is a “good” sequence in some sense, positive topological entropy implies multivariant mean Li-Yorke chaos along the sequence \( \{a_n\} \). Let \( \{a_n\} \) be a sequence of positive integers. By a “good” sequence, it means that \( \{a_n\} \) is increasing and satisfies the following two conditions,
\begin{enumerate}
\item For each measure preserving system \((X, \mathcal{B}, \mu, T)\) and any \( f \in L^2(\mu) \), the limit
\[
\frac{1}{N} \sum_{k=1}^{N} f(T^{a_k}x)
\]
converges for \( \mu \)-a.e. \( x \in X \);
\item For any \( L > 0 \),
\[
\lim_{N \to \infty} \frac{1}{N^2} \#\{(i, j) \in \{1, 2, \ldots, N\}^2 : |a_i - a_j| \leq L\} = 0
\]
where the symbol \( \#(\cdot) \) means the cardinality of a set.
\end{enumerate}
They also mentioned that the polynomial sequence like \( \{\lfloor p(k) \rfloor\}_{k=1}^{\infty} \), where \( p(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0 \) with \( m \geq 1, b_0, \ldots, b_m \in \mathbb{R} \), and \( b_m > 0 \), is “good”. Then by their main result, there exists an uncountable mean Li-Yorke scrambled set along \( \{\lfloor p(k) \rfloor\} \) for any topological dynamical system with positive entropy.

We focus on studying the size of the mean Li-Yorke scrambled set along polynomial sequences. As the definition of the Li-Yorke chaos or the mean Li-Yorke chaos only requires the cardinality of the scrambled set to be uncountable, it is natural to give more description about the size of the scrambled set. There have been two ways to characterize the size of the scrambled set: topological and measure-theoretic. Blanchard, Huang and Snoha studied the topological size of scrambled sets extensively in [2]. The Lebesgue measure of scrambled sets for continuous maps on the interval were surveyed (for example, [1] and [3]). Xiong [25] constructed a Xiong chaotic set with full Hausdorff dimension everywhere in the full shift over finite symbols. In [13], the author constructed a Xiong chaotic with full topological entropy everywhere for positively expansive systems with specification property. Recently, in [16], Liu and Li constructed a scrambled set with full Hausdorff dimension for the Gauss system.

Motivated by the ideas and results above, we study the Hausdorff dimension and the topological entropy of mean Li-Yorke chaotic set along polynomial sequences for any \( \beta \)-transformation. Let \( \beta > 1 \) be a real number. The \( \beta \)-transformation is defined by \( T_{\beta} : [0, 1) \to [0, 1) \) with \( T_{\beta}(x) = \beta x - \lfloor \beta x \rfloor \), where \( \lfloor z \rfloor \) denotes the integer part of \( z \). Li and Chen [15] proved that \( \beta \)-transformation is chaotic in the sense of Devaney
and Li-Yorke. They also studied the “size” of the set of the points having dense (resp. non-dense) orbit under the $\beta$-transformation. Afterwards in [18], the authors constructed a scrambled set in the Li-Yorke sense with full Hausdorff dimension for $\beta$-transformation. We will prove the following main result in this paper.

**Theorem 1.1.** Let $f(x) = d_m x^n + d_{m-1} x^{n-1} + \cdots + d_1 x + d_0$ with $m \geq 3$ and $d_m > 0$. Let $A = \{a_1 < a_2 < \ldots\} \subset \{[f(n)] : n \geq 1\}$ be a sequence of positive integers. For any $\beta > 1$, there exists a mean Li-Yorke scrambled set along $A$ with full Hausdorff dimension and full topological entropy for $(\{0,1\}, T_\beta)$.

The paper is organized as follows. In Section 2, we introduce some preliminaries. Section 3 is devoted to proving Theorem 1.1.

## 2. Preliminaries.

### 2.1. Mean Li-Yorke chaotic set along a sequence.

Let $A = \{a_1 < a_2 < \ldots\}$ be a sequence of positive integers. We say that a subset $C$ of $X$ is mean Li-Yorke scrambled along the sequence $A$ if any two distinct points $x, y$ in $C$ satisfy

$$\liminf_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} d(f^{a_j}(x), f^{a_j}(y)) = 0$$

and

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} d(f^{a_j}(x), f^{a_j}(y)) > 0.$$

If in addition, $C \subset X$ is uncountable, then we say that the system $(X, f)$ is mean Li-Yorke chaotic along the sequence $A$.

### 2.2. Hausdorff dimension.

In a metric space $(X, \rho)$, for a subset $Y$ of $X$, a real number $\delta > 0$, and $s \geq 0$, define

$$\mathcal{H}_s^\delta(Y) = \inf \left\{ \sum_{i \geq 1} \text{diam}(U_i)^s : Y \subset \bigcup_{i \geq 1} U_i \text{ and } \text{diam}(U_i) < \delta \text{ for any } i \geq 1 \right\}$$

where $\text{diam}(\cdot)$ denotes the diameter of a set. The $s$-dimension Hausdorff measure of $Y$ is given by

$$\mathcal{H}^s(Y) = \lim_{\delta \to 0} \mathcal{H}_s^\delta(Y)$$

and the Hausdorff dimension of $Y$ is

$$\text{dim}_H(Y) = \left\{ \begin{array}{ll} \inf \{s > 0 : \mathcal{H}^s(Y) = 0\}, & \text{if } \{s > 0 : \mathcal{H}^s(Y) = 0\} \neq \emptyset; \\ +\infty, & \text{otherwise.} \end{array} \right.$$  

The basic knowledge about Hausdorff dimension can be found in [7], which we refer the reader to.

Let $\alpha$ be a positive real number. We say that a map $f : X \to \mathbb{R}$ satisfies the locally $\alpha$-Hölder condition, if there exist a real number $r > 0$ and a constant $c > 0$ such that $|f(x) - f(y)| \leq c(\rho(x, y))^\alpha$ holds for any $x, y \in X$ with $\rho(x, y) < r$.

The following well known lemma can be easily deduced from the definitions of Hausdorff dimension and the locally $\alpha$-Hölder condition.

**Lemma 2.1.** Let $(X, \rho)$ be a metric space and $s, \alpha > 0$ be real numbers. If a map $f : X \to \mathbb{R}$ satisfies the locally $\alpha$-Hölder condition, then $\mathcal{H}^s(f(X)) \leq c^s \mathcal{H}^{s-\alpha}(X)$, where $c$ is the constant in the definition of the locally $\alpha$-Hölder condition. Moreover, $\alpha \text{dim}_H(f(X)) \leq \text{dim}_H(X)$. 


2.3. $\beta$-transformation. Let $\beta > 1$ be a real number. The $\beta$-transformation is defined by $T_\beta: [0, 1) \to [0, 1)$ such that $T_\beta(x) = \beta x - \lfloor \beta x \rfloor$, where $\lfloor z \rfloor$ denotes the integer part of $z$. For any $x$ in $[0, 1)$, the $\beta$-transformation determines the $\beta$-expansion of $x$ uniquely by

$$x = \sum_{n=1}^{\infty} \frac{\lfloor \beta T_\beta^{n-1}(x) \rfloor}{\beta^n}.$$  

It is easy to see that every integer coefficient $\lfloor \beta T_\beta^{n-1}(x) \rfloor$ is not negative and less than $\beta$ for any $i \geq 1$. Set

$$\gamma = \begin{cases} \beta - 1 & \text{if } \beta \text{ is an integer;} \\ \lfloor \beta \rfloor & \text{otherwise;} \end{cases}$$

and

$$\Sigma_\beta = \{ \omega \in \{0, 1, \ldots, \gamma\}^\mathbb{N} : \text{there exists } x \text{ in } [0, 1) \text{ such that} \sum_{n=1}^{\infty} \frac{\omega_n}{\beta^n} \text{ is the } \beta\text{-expansion of } x \}.$$  

It is natural to define a map $\pi_\beta: \Sigma_\beta \to [0, 1)$ such that $\pi_\beta(\omega) = \sum_{n=1}^{\infty} \frac{\omega_n}{\beta^n}$ in order to associate the $\beta$-transformation with another system $(\Sigma_\beta, \sigma)$, where $\sigma$ is the shift map such that $\sigma(\omega) = \omega_2 \omega_3 \cdots$ with $\omega = \omega_1 \omega_2 \omega_3 \cdots \in \Sigma_\beta$. It is well known that the metric on the symbolic space $\{0, 1, \ldots, \gamma\}^\mathbb{N}$ is given by

$$d(\omega, \eta) = \begin{cases} 0 & \text{if } \omega = \eta; \\ \frac{1}{\beta^{i-1}} & \text{otherwise;} \end{cases}$$

where $k = \min \{i \geq 1 : \omega_i \neq \eta_i\}$ for any $\omega = \omega_1 \omega_2 \cdots$ and $\eta = \eta_1 \eta_2 \cdots$ in $\{0, 1, \ldots, \gamma\}^\mathbb{N}$. If view $\Sigma_\beta$ as a subspace of the metric space $\{0, 1, \ldots, \gamma\}^\mathbb{N}$, it is easy to verify that $\pi_\beta \circ \sigma = T_\beta \circ \pi_\beta$.

Fix a point $\omega = \omega_1 \omega_2 \cdots$ in $\{0, 1, \ldots, \gamma\}^\mathbb{N}$, we say that the position of the word $\omega_n \omega_{n+1} \cdots \omega_m$ appeared in $\omega$ is $n$ for any $m \geq n \geq 1$. We use the notation

$$\omega = \omega_1 \omega_2 \cdots \omega_n \omega_{n+1} \cdots \frac{1}{n}$$

to specify that the number $\omega_n$ is at the $n$-th position of $\omega$. It is convenient to denote by $P(\omega_1 \omega_{n+1} \cdots \omega_m, \omega)$ the position of the word $\omega_n \omega_{n+1} \cdots \omega_m$ appeared in $\omega$ for the first time. For example, if $\xi = 0101011010110 \cdots$, then $P(0, \xi) = 1$ and $P(1, \xi) = 6$. For $\omega = \omega_1 \omega_2 \cdots \omega_n \omega_{n+1} \cdots$, denote by $\omega[n, m] = \omega_n \omega_{n+1} \cdots \omega_m$. A point $\omega = \omega_1 \omega_2 \cdots$ is called lexicographically less than another point $\eta = \eta_1 \eta_2 \cdots$ writing as $\omega <_{\text{lex}} \eta$, if $\omega_n < \eta_n$ with $n = \min \{i \geq 1 : \omega_i \neq \eta_i\}$. Define the lexicographic supremum of $\Sigma_\beta$ as a special sequence not contained in $\Sigma_\beta$. It is worth pointing out that the $\beta$-expansion of 1 can still be obtained by $T_\beta$ and the lexicographic supremum is closely related to the $\beta$-expansion of 1. Let $\varepsilon_\beta(1) = \eta_1(\beta) \eta_2(\beta) \cdots$ such that $\sum_{i=1}^{\infty} \frac{\eta_i(\beta)}{\beta^i} = 1$. (In fact, $\eta_1(\beta) = \gamma$ for any $\beta > 1$.) We denote the lexicographic supremum by

$$\varepsilon_\beta^* = \begin{cases} \varepsilon_\beta(1) & \text{if } \sup \{i \geq 1 : \eta_i(\beta) \geq 1\} = +\infty; \\ \eta_1(\beta) \eta_2(\beta) \cdots \eta_{m-1}(\beta)(\eta_m(\beta) - 1) & \text{if } m = \sup \{i \geq 1 : \eta_i(\beta) \geq 1\} < +\infty; \end{cases}$$

where $a_1 a_2 \cdots a_m$ means the point

$$a_1 a_2 \cdots a_n a_1 a_2 \cdots a_n \cdots a_1 a_2 \cdots a_n \cdots$$
Lemma 2.2. Let $\beta > 1$. Then $\omega \in \Sigma_\beta$ if and only if $\sigma^k(\omega) <_{lex} \varepsilon^*_\beta(1)$ for any $k \geq 0$.

Let $m, n \geq 1$ be two integers and $\beta > 1$ be a fixed real number. For $\omega_1 \omega_2 \cdots \omega_n$ in $\{0, 1, \ldots, b\}^n$, if there exists $x \in [0, 1)$ with $\beta$-expansion $x = \sum_{i=1}^{\infty} \frac{x_i}{\beta^i}$ such that $\omega_1 \omega_2 \cdots \omega_n = \xi_1 \xi_2 \cdots \xi_n$, then $\omega_1 \omega_2 \cdots \omega_n$ is called an admissible word with length $n$. The concatenation of two admissible words $\omega_1 \omega_2 \cdots \omega_m$ and $\xi_1 \xi_2 \cdots \xi_m$ is a word $\omega_1 \omega_2 \cdots \omega_m \xi_1 \xi_2 \cdots \xi_m$ in $\{0, 1, \ldots, b\}^{n+m}$. Sometimes we also use the symbol "⊔" to represent the concatenation operation, i.e., $\omega_1 \omega_2 \cdots \omega_m \sqcup \xi_1 \xi_2 \cdots \xi_m$. Since $\omega_1 \omega_2 \cdots \omega_m \sqcup \xi_1 \xi_2 \cdots \xi_m$ may be different from $\xi_1 \xi_2 \cdots \xi_m \sqcup \omega_1 \omega_2 \cdots \omega_m$, we require that the symbol $\bigcup_{1 \leq i \leq n} \omega_i$ "means" $\omega_1 \sqcup \omega_2 \sqcup \cdots \sqcup \omega_n$. Let $\varepsilon^*_{\beta}(1) = \eta^*_1(\beta) \eta^*_2(\beta) \cdots$ be the lexicographic supremum. For any $n \geq 1$, denote by $l_n(\beta) = \min\{i \geq 0: \eta^*_{n+i}(\beta) \geq 1\}$ the length of zeros behind the symbol $\eta^*_i(\beta)$. The following useful lemmas will be frequently used later.

Lemma 2.3. [18, Lemma 2.3 and Lemma 2.4] Let $\beta > 1$ and $\omega_1 \omega_2 \cdots \omega_n$ be an admissible word with length $n$. Set $M_n = \max\{l_n(\beta): 1 \leq k \leq n\}$. Then, for any integer $m > M_n$, any integer $p \geq 1$, and $\eta$ being any admissible word with length $p$, the concatenation $\omega_1 \omega_2 \cdots \omega_n \sqcup \eta^* m \eta$ is admissible.

In [14], Li and Wu showed that the set $\{\beta > 1: \{l_n(\beta)\} \text{ is bounded}\}$ has zero Lebesgue measure (while it has full Hausdorff dimension), which means that there still exist "many" $\beta > 1$ such that $\{l_n(\beta): n \geq 1\}$ is unbounded. Denote by $B_0 = \{\beta > 1: \{l_n(\beta)\} \text{ is bounded}\}$ and $B_1 = (1, +\infty) \setminus B_0$. Fix $\beta > 1$, for any $n \geq 1$ and any admissible word $\omega_1 \omega_2 \cdots \omega_m$, we call $I^\beta_n(\omega_1, \omega_2, \ldots, \omega_m) = \{x \in [0, 1]: [\beta T^\beta_n(x)] = \omega_i \text{ for any } 1 \leq i \leq n\}$ the $n$-th cylinder. Clearly, the Lebesgue measure of $I^\beta_n(i_1, i_2, \ldots, i_n)$ is not more than $\frac{1}{\beta^n}$ for any $n \geq 1$. The set $B_0$ can be characterized by the length of cylinders.

Lemma 2.4. [18, Proposition 2.5] Let $\beta > 1$ and $L$ be the Lebesgue measure on $[0, 1)$. $\beta \in B_0$ if and only if there exists a constant $C$ such that

$$\frac{C}{\beta^n} \leq L(I^\beta_n(i_1, i_2, \ldots, i_n)) \leq \frac{1}{\beta^n}$$

holds for any admissible word $i_1 i_2 \cdots i_n$ and any $n \geq 1$.

2.4. Bowen dimension of topological entropy. Let $(X, f)$ be a topological dynamical system equipped with the metric $d$. We denote the $n$-step Bowen ball with center $x$ and radius $r$ by $B_n(x, r)$. If $y$ is in $B_n(x, r)$, it means that $d(x, y) < r$, $d(f(x), f(y)) < r$, $\ldots$, and $d(f^{n-1}(x), f^{n-1}(y)) < r$. Let $Y$ be a subset of $X$. For any $\varepsilon > 0$ and $N > 0$, let

$$\Gamma = \left\{ \{B_n(x_i, \varepsilon) \cap Y: \min\{n_i: i \geq 1\} > N\}: i \geq 1 \right\}.$$ 

Put

$$m(Y; s, N, \varepsilon) = \inf \sum_i e^{-s n_i}.$$
Proposition 1. Let \( m(Y; s, \varepsilon) = \lim_{N \to \infty} m(Y; s, N, \varepsilon) \), since \( m(Y; s, N, \varepsilon) \) is not decreasing with respect to \( N \). Now, we define
\[
h_t(f, Y; \varepsilon) = \begin{cases} 
+\infty, & \text{if } \inf \{ s : m(Y; s, \varepsilon) < +\infty \} = 0; \\
\inf \{ s : m(Y; s, \varepsilon) < +\infty \}, & \text{otherwise.}
\end{cases}
\]

And the Bowen dimension of topological entropy of the subset \( Y \) is \( h_t(f, Y) = \lim_{\varepsilon \to 0} h_t(f, Y; \varepsilon) \). The limit exists because \( h_t(f, Y; \varepsilon) \) is not increasing with respect to \( \varepsilon \). If \( Y = X \), then the value \( h_t(f, X) \) is exactly the topological entropy of the system \((X, f)\).

**Lemma 2.5.** [8, Lemma 5.4] Let \((X, f)\) be a topological dynamical system with a metric \( d \) and \( Y \) be a subset of \( X \). If there exists \( \varepsilon > 0 \) and \( L > 0 \) such that \( d(f(x), f(y)) \geq Ld(x, y) \) with any \( d(x, y) < \varepsilon \), then \( \dim_h(Y) \leq h_t(f, Y) \).

**Remark 1.** It is necessary to notice that the \( \beta \)-transformation \((\{0, 1\}, T_\beta)\) is not continuous. So we use the Bowen dimension of topological entropy. By \([22], [21], [10]\) and \([5]\), we know that the topological of \( \beta \)-transformation is \( \ln \beta \).

3. **The mean Li-Yorke scrambled set along polynomial sequence.** In this section, we give the proof of our main result. The main tool is inspired by the method in \([18]\). We first construct a mean Li-Yorke chaotic set along some polynomial sequence with Full Hausdorff dimension for the \( \beta \)-transformation with \( \beta \in B_0 \), then the case when \( \beta \) in \( B_1 \) will be considered. At the beginning, we present a useful lemma.

**Lemma 3.1.** Fix \( \beta > 1 \). Let \( x = x_1x_2\cdots \) and \( y = y_1y_2\cdots \) in \( \Sigma_\beta \). If \( d(x, y) = \frac{1}{\beta^n} \), then \( |\pi_\beta(x) - \pi_\beta(y)| \leq \frac{\gamma + 2}{\beta^2} d(x, y) \); If \( d(x, y) = \frac{1}{\beta^n} \) and \( x_{n+2} = y_{n+2} \), then \( |\pi_\beta(x) - \pi_\beta(y)| \geq \frac{\beta - 1}{\beta^2} d(x, y) \).

**Proof.** If \( d(x, y) = \frac{1}{\beta^n} \), then
\[
|\pi_\beta(x) - \pi_\beta(y)| \leq \left| \frac{x_{n+1}}{\beta^{n+1}} - \frac{y_{n+1}}{\beta^{n+1}} \right| + \left| \frac{x_{n+2}}{\beta^{n+2}} - \frac{y_{n+2}}{\beta^{n+2}} + \frac{x_{n+3}}{\beta^{n+3}} + \cdots \right| + \left| \frac{y_{n+2}}{\beta^{n+2}} + \frac{y_{n+3}}{\beta^{n+3}} + \cdots \right| \\
\leq \frac{\gamma}{\beta^{n+1}} + \frac{2}{\beta^{n+1}} = \frac{\gamma + 2}{\beta^2} d(x, y),
\]
as both \( \frac{x_{n+2}}{\beta^2} + \frac{x_{n+3}}{\beta^2} + \cdots \) and \( \frac{y_{n+2}}{\beta^2} + \frac{y_{n+3}}{\beta^2} + \cdots \) are less than 1. If \( d(x, y) = \frac{1}{\beta^n} \) with \( x_{n+1} > y_{n+1} \) and \( x_{n+2} = y_{n+2} \), then
\[
\pi_\beta(x) - \pi_\beta(y) \geq \left( \frac{x_{n+1}}{\beta^{n+1}} - \frac{y_{n+1}}{\beta^{n+1}} + \frac{x_{n+3}}{\beta^{n+3}} + \cdots \right) - \left( \frac{y_{n+3}}{\beta^{n+3}} + \frac{y_{n+4}}{\beta^{n+4}} + \cdots \right) \\
\geq \frac{1}{\beta^{n+1}} - \frac{1}{\beta^{n+2}} = \frac{\beta - 1}{\beta^2} d(x, y).
\]

Clearly, if \( x_{n+1} < y_{n+1} \), we can also obtain \( \pi_\beta(y) - \pi_\beta(x) \geq \frac{\beta - 1}{\beta^2} d(x, y) \). This ends the proof. □

**Proposition 1.** Let \( f(x) = d_mx^m + d_{m-1}x^{m-1} + \cdots + d_1x + d_0 \) with \( m \geq 3 \) and \( d_m > 0 \). Let \( A = \{a_1 < a_2 < \ldots\} \subset \{\lfloor f(n)\rfloor : n \geq 1\} \) be a sequence of positive integers. For any \( \beta \) in \( B_0 \), there exists a mean Li-Yorke scrambled set along \( A \) with full Hausdorff dimension for \((\{0, 1\}, T_\beta)\).
Proof. Fix $\beta \in B_0$. We first assume that $a_n = |f(n)|$ for any $n \geq 1$. At the end of this proof, we will explain the situation where $\{a_n\}$ is a subsequence of $|f(n)|$. It is not hard to see that there exists some positive integer $K$ such that $a_{n+1} - a_n > 4n$ for any $n \geq K$. Set $W = \sup \{t_n(\beta) : n \geq 1\}$ and $M = \max \{K, W\}$. Let $\{q_n\}_{n=1}^{\infty}$, $\{k_n\}_{n=1}^{\infty}$ and $\{p_n\}_{n=1}^{\infty}$ be three strictly increasing sequences of positive integers satisfying the following requirements.

(R1) $q_0 = k_0 = p_0 = 0$ and $t_0 = t_1 = 0$.
(R2) Choose $q_n$ such that $q_n$ is larger than both $(n - 1)(M + t_n)$ and

$$\frac{n(\gamma + 2)}{\beta(\beta - 1)\beta^M + t_n} - M - t_n$$

for any $n \geq 1$.
(R3) $k_n > M + t_n + q_n$ for any $n \geq 1$.
(R4) Set $t_n = \sum_{i=0}^{n-1} q_i + k_i + p_i$ and require $t_n + M + 3 > 2n$ for any $n \geq 2$.

To avoid frequently using sub-subscripts, in the rest of this article, we denote $a_n$ by $a(n)$ for any $n \geq 1$. Choose a point $b = b_1 b_2 \cdots$ in $\Sigma_\beta$ with $b_1 \neq 0$ and define a map $\Delta_\beta(b, \cdot) : \Sigma_\beta \to \Sigma_\beta$ by two steps. First, Let $V(x)_{i,j}$ be a word related to the point $x$ for any $i, j \geq 1$. We construct the map $\Delta_\beta(b, x)$ as follow,

$$\Delta_\beta(b, x) = \bigcup_{n \geq 1} \left( \bigcup_{1 \leq j \leq q_n} V(x)_{n,j} \bigcup \bigcup_{q_n+1 \leq j \leq q_n+k_n} V(x)_{n,j} \bigcup \bigcup_{q_n+k_n+1 \leq j \leq q_n+k_n+p_n} V(x)_{n,j} \right).$$

Until now, the definition of the map $\Delta_\beta(b, x)$ is still not clear because all the $V(x)$ are unknown. In order to finish the construction, let $V(x)_{i,j}$ be an admissible word such that the point $\bigcup_{n \geq 1} \bigcup_{1 \leq j \leq q_n+k_n+p_n} V(x)_{n,j}$ is exactly the point $x$ for any $x = x_1 x_2 \cdots$ in $\Sigma_\beta$. We claim that the subset $\pi_\beta \circ \Delta_\beta(b, \Sigma_\beta)$ is a mean Li-Yorke chaotic subset along the sequence $\{a(n)\}$ for $([0, 1), T_\beta)$. By Lemma 2.2 and Lemma 2.3, it is easy to see that the map $\Delta_\beta(b, \cdot)$ is well defined and injective. Arbitrarily pick two distinct points $\pi_\beta \circ \Delta_\beta(b, x)$ and $\pi_\beta \circ \Delta_\beta(b, y)$ in $\pi_\beta \circ \Delta_\beta(b, \Sigma_\beta)$. For any $n \geq 2$, it can be estimated by (R2) that

$$\frac{1}{t_n + q_n + M} \sum_{j=1}^{t_n+q_n+M} \left| T^{a(j)}_\beta \circ \pi_\beta \circ \Delta_\beta(b, x) - T^{a(j)}_\beta \circ \pi_\beta \circ \Delta_\beta(b, y) \right|$$

$$\leq \frac{1}{t_n + q_n + M} \left( M + t_n + \frac{\gamma + 2}{\beta} \sum_{i=t_n+1+M}^{t_n+q_n+M} d(\sigma^{a(i)}(x), \sigma^{a(i)}(y)) \right)$$

$$\leq \frac{1}{t_n + q_n + M} \left( \frac{1}{\beta^{M+t_n+1}} + \frac{1}{\beta^{M+t_n+2}} + \cdots + \frac{1}{\beta^{M+t_n+q_n}} \right)$$

$$= \frac{1}{t_n + q_n + M} \left( \frac{1}{\beta^{M+t_n}} + \frac{1}{\beta^{M+t_n+1}} + \cdots + \frac{1}{\beta^{M+t_n+q_n}} \right)$$

$$\leq \frac{1}{n} + \frac{1}{n}.$$
Thus, \( \liminf_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} |T_{\beta}^{a(j)} \circ \pi_{\beta} \circ \Delta_{\beta}(b, x) - T_{\beta}^{a(j)} \circ \pi_{\beta} \circ \Delta_{\beta}(b, y)| \leq \lim_{n \to \infty} \frac{2}{n} = 0. \) For another part, assume that \( s \) is the least positive integer such that \( x_s \neq y_s \), that is, \( x_i \) is equal to \( y_i \) for any \( 1 \leq i \leq s - 1 \), but \( x_s \neq y_s \). By (R3), for any \( n \geq s \), it can be calculated that

\[
\frac{1}{M + t_n + q_n + k_n} \sum_{j=1}^{M + t_n + q_n + k_n} |T_{\beta}^{a(j)} \circ \pi_{\beta} \circ \Delta_{\beta}(b, x) - T_{\beta}^{a(j)} \circ \pi_{\beta} \circ \Delta_{\beta}(b, y)| \\
\geq \frac{\beta - 1}{\beta^2} \frac{1}{M + t_n + q_n + k_n} \sum_{j=M + t_n + q_n + k_n}^{M + t_n + q_n + k_n} d(\sigma^{a(j)} \circ \Delta(x), \sigma^{a(j)} \circ \Delta(y)) \\
\geq \frac{\beta - 1}{\beta^2} \frac{1}{M + t_n + q_n + k_n} \beta^{2s-1} \\
= \frac{\beta - 1}{\beta^{2s+1}} > 0,
\]

which implies \( \limsup_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} |T_{\beta}^{a(j)} \circ \pi_{\beta} \circ \Delta_{\beta}(b, x) - T_{\beta}^{a(j)} \circ \pi_{\beta} \circ \Delta_{\beta}(b, y)| > 0. \) So far, we have shown that \( \pi_{\beta} \circ \Delta_{\beta}(b, \Sigma_{\beta}) \) is a mean Li-Yorke chaotic set along \( \{a_n\} \) for \( (0, 1), T_{\beta} \) with \( \beta \) in \( B_{0}. \)

Now, it turns to show that \( \pi_{\beta} \circ \Delta_{\beta}(b, \Sigma_{\beta}) \) has full Hausdorff dimension. To this end, define a map \( \varphi_{\beta}: \pi_{\beta} \circ \Delta_{\beta}(b, \Sigma_{\beta}) \to [0, 1) \) such that \( \varphi_{\beta}(x) = \pi_{\beta} \circ \Delta_{\beta}(b, \cdot)^{-1} \circ \pi_{\beta}^{-1}(x). \) For any fixed \( \varepsilon > 0 \), we are going to show that the map \( \varphi_{\beta} \) satisfies the locally \( \frac{1}{1+\varepsilon} \) Hölder condition. The point \( \Delta_{\beta}(b, x) \) can be viewed as the result for inserting some pieces into the point \( x \). Then, define \( \eta(n) \) as the sum of the length of all the inserted pieces before the element in the position of \( a(n) \) for any \( n \geq 1 \). Pick a large \( N > a(t_1 + 1 + q_1 + M) + 1 \) such that for any \( k \geq N \),

\[
a(k + 1) - a(k) + \eta(k) + 1 < \varepsilon \\
a(k) - \eta(k)
\]

and

\[
\frac{\eta(k) - \log_{\beta} C}{a(k + 1) - \eta(k)} < \varepsilon
\]

are satisfied. Set

\[
r = \min \left\{ |I^{\beta}_{a(N)}(i_1, i_2, \ldots, i_{a(N)})| : i_1 i_2 \cdots i_{a(N)} \text{ is admissible} \right\}.
\]

Pick two distinct points \( u = \pi_{\beta} \circ \Delta_{\beta}(b, x) \) and \( v = \pi_{\beta} \circ \Delta_{\beta}(b, y) \) in \( \pi_{\beta} \circ \Delta_{\beta}(b, \Sigma_{\beta}) \) with \( |u - v| < r \). Set \( \lambda = \min \{i \geq 1 : \Delta_{\beta}(b, x)_i \neq \Delta_{\beta}(b, y)_i\} \), then \( \lambda \) is larger than \( a(N) \).

**Case 1.** If \( |\Delta_{\beta}(b, x)_\lambda - \Delta_{\beta}(b, y)_\lambda| \geq 2 \), then

\[
|u - v| = |\pi_{\beta} \circ \Delta_{\beta}(b, x) - \pi_{\beta} \circ \Delta_{\beta}(b, y)| \\
\geq |\Delta_{\beta}(b, x)_\lambda - \Delta_{\beta}(b, y)_\lambda| \geq \frac{1}{\beta^\lambda} \geq \frac{1}{\beta^{\lambda+1}}.
\]

There exists \( k \geq N \) such that \( a(k) < \lambda < a(k + 1) \). And we can estimate that

\[
|\varphi_{\beta}(u) - \varphi_{\beta}(v)| \leq \beta^{-\lambda - \eta(k)} \leq \beta^{-a(k) - \eta(k)} \leq \left( \frac{1}{\beta^{\lambda+1}} \right)^{\frac{1}{1+\varepsilon}}
\]

since

\[
\beta^{-\lambda-1} = \beta^{-\lambda+1-a(k)+\eta(k)} \beta^{-a(k) - \eta(k)} \geq \beta^{-\lambda+1-a(k)} \beta^{-a(k) - \eta(k)}.
\]
It turns out that $|\varphi_\beta(u) - \varphi_\beta(v)| \leq |u - v|^{1/(1+\varepsilon)}$, which means that the map $\varphi_\beta$ satisfies the locally $\frac{1}{1+\varepsilon}$-Hölder condition.

**Case 2.** If $|\Delta_\beta(b, x)_1 - \Delta_\beta(b, y)_1| = 1$, we claim that there must exists an interval contained in $(u, v)$ or $(v, u)$. Assume $\Delta_\beta(b, x)_1 > \Delta_\beta(b, y)_1$. Observe that there exists some $\Gamma = \mathbf{P}(x_1, x_2, 0 \cdots 0 x_{\lambda+1}, \Delta_\beta(b, x)) + t_{\lambda+1} + 2 + M + q_{\lambda+1}$ such that $\Delta_\beta(b, x)_1 = \Delta_\beta(b, y)_1 = 1 > 0$. Clearly,

$$\Gamma < a(t_{\lambda+1} + 1 + M + q_{\lambda+1}) + 4(t_{\lambda+1} + 1 + M + q_{\lambda+1}) < a(k+1)$$

and

$$\Delta_\beta(b, y)_1 \Delta_\beta(b, y)_2 \cdots \Delta_\beta(b, y)_\Gamma \leq \mathbf{P}(x_1, x_2, 0 \cdots 0 x_{\lambda+1}, \Delta_\beta(b, x)) + t_{\lambda+1} + 2 + M + q_{\lambda+1}$$

Thus the interval $I^{\beta}_{\Gamma}(\Delta_\beta(b, x)_1, \Delta_\beta(b, x)_2, \cdots, \Delta_\beta(b, x)_{\Gamma-1}, 0)$ is contained in $(v, u)$ and

$$|u - v| \geq \left| I^{\beta}_{\Gamma}(\Delta_\beta(b, x)_1, \Delta_\beta(b, x)_2, \cdots, \Delta_\beta(b, x)_{\Gamma-1}, 0) \right| \geq \frac{C}{\beta^\Gamma} \geq \frac{C}{\beta^{a(k+1)}}.$$ Continue to estimate that

$$|\varphi_\beta(u) - \varphi_\beta(v)| \leq \beta^{-(\lambda - \eta(k))} \leq \left( \frac{C}{\beta^{a(k+1)}} \right)^{\frac{1}{\lambda+\varepsilon}}$$

because $C \cdot \beta^{-(a(k+1))} = \beta^{\log_\beta C - a(k+1)} = \beta^{-(\log C - \log \beta) + (k+1)} \geq \beta^{-(\log C - \log \beta) + (k+1)}$, which also implies that the map $\varphi_\beta$ satisfies the locally $\frac{1}{1+\varepsilon}$-Hölder condition. Applying Lemma 2.1 and note that $\varphi_\beta$ is surjective, $\pi_\beta \circ \Delta_\beta(b, \Sigma_\beta)$ is a mean Li-Yorke chaotic set along $\{ [f(n)] \}$ with full Hausdorff dimension for $\{ (0, 1), T_\beta \}$ with $\beta$ in $B_0$.

In the case when $\{a(n)\}$ is a subsequence of $\{ [f(n)] \}$, there exists a sequence of positive integers $\{a(n)' \}$ such that $\bigcup_{n \geq 1} \{a(n), a(n)' \}$ is exactly $\{ [f(n)] \}$. We can construct a map $\Delta'_\beta(b, \cdot)$ with respect to the sequence $\bigcup_{n \geq 1} \{a(n), a(n)' \}$. If we slightly change the requirements about the three sequences $\{p_n\}$, $\{q_n\}$ and $\{k_n\}$, by the same method, $\pi_\beta \circ \Delta'_\beta(b, \Sigma_\beta)$ can be a mean Li-Yorke chaotic set along $\{a(n)\}$. Since these three sequences $\{p_n\}$, $\{q_n\}$ and $\{k_n\}$ are not relevant to the estimation of Hausdorff dimension, the set $\pi_\beta \circ \Delta'_\beta(b, \Sigma_\beta)$ still has full Hausdorff dimension. □

**The proof of Theorem 1.1.** It is sufficient to consider the case when $\beta$ is contained in $B_1$. Let $\{a(n)\} = [f(n)]$ for any $n \geq 1$. Let $\varepsilon^*_\beta(1) = \eta_1^\ast(\beta) \eta_2^\ast(\beta) \cdots$ be the lexicographic supremum. According to the definition of $\varepsilon^*_\beta(1)$, the set $\{ n \geq 1 : \eta_n(\beta)^\ast \geq 1 \}$ is infinite. For every $m \in \{ n \geq 1 : \eta_n(\beta)^\ast \geq 1 \}$, define a number $\beta_m > 1$ satisfying the equation

$$1 = \eta_1^m(\beta) \beta_m + \eta_2^m(\beta) \beta_m^2 + \cdots + \eta_m^m(\beta) \beta_m^m.$$ It is not hard to verify the following properties.

- $\{\beta_m\}$ is increasing to $\beta$;
- For any $m \in \{ n \geq 1 : \eta_n^m(\beta) \geq 1 \}$, we have

$$\varepsilon^*_{m, \beta}(1) = (\eta_1(\beta) \eta_2(\beta) \cdots \eta_{m-1}(\beta))(\eta_m^m(\beta) - 1).$$

Thus, $\beta_m$ is contained in $B_0$ and $\Sigma_{\beta_m}$ is a proper subset of $\Sigma_{\beta_q}$ for any two integers $m < q$ in $\{ n \geq 1 : \eta_n^m(\beta) \geq 1 \}$. 
Let $W_m = \sup \{ l_n(\beta_m) : n \geq 1 \} < +\infty$ for all $m$ in $\{ n \geq 1 : \eta^*_n(\beta) \geq 1 \}$. Then, 
\{W_m\} is increasing to infinity.

Now, for convenience, assume that $\beta_1 < \beta_2 < \cdots < \beta$ and $\Sigma_{\beta_m}$ is a proper subset of $\Sigma_{\beta_{m+1}}$ for any $m \geq 1$. Let $M_m = \max \{W_m, K\}$ for any $m \geq 1$ with $K$ defined in the proof of Proposition 1. In the following, we assume that $M_{m+1} = M_1 + m$ for any $m \geq 1$ in order to keep the notations as simple as possible, since the general case can be easily deduced from this special case by a notation adjustment. Choose $b^{(m)} = b^{(m)}_1 b^{(m)}_2 \cdots \in \Sigma_{\beta_m} \setminus \Sigma_{\beta_{m-1}}$ for any $m \geq 1$ where $\Sigma_{\beta_0} = \emptyset$. Then, for any fixed $m \geq 1$, there exists some $t \geq 1$ such that $\sigma^t(b^{(m)}) <_{lex} \varepsilon_{\beta_m}(1)$ but $\sigma^t(b^{(m)}) \notin_{lex} \varepsilon_{\beta_{m-1}}(1)$. This means that $b^{(m)}_t$ must be nonzero and $\sigma^t(b^{(m)})$ is still contained in $\Sigma_{\beta_m} \setminus \Sigma_{\beta_{m-1}}$. So, we can assume $b^{(m)}_t \neq 0$ for any $m \geq 1$.

Let $\{ q_n \}_{n=1}^\infty, \{ k_n \}_{n=1}^\infty$ and $\{ p_n \}_{n=1}^\infty$ be three strictly increasing sequences of positive integers with the following requirements.

(R1) Set $t_n = \sum_{i=0}^{n-1} q_i + k_i + p_i$ with $q_0 = k_0 = p_0 = 0$ and require $t_n + 3 + M_1 > 2n$ for any $n \geq 1$.

(R2) Set
\[
\gamma_1 = \begin{cases} 
\beta_1 - 1 & \text{if } \beta_1 \text{ is an integer;} \\
\lceil \beta_1 \rceil & \text{otherwise.}
\end{cases}
\]

Choose $q_n$ such that $q_n$ is larger than $(n-1)(M_1 + t_n)$ and
\[
\frac{n(\gamma_1 + 2)}{\beta_1 (\beta_1 - 1) \beta^{M_1+t_n} - M_1 - t_n}
\]
for any $n \geq 1$.

(R3) $k_n > M_1 + t_n + q_n$ for any $n \geq 1$.

(R4) $p_n > M_1 + t_n + q_n + k_n$ for any $n \geq 1$.

For a large enough fixed $m$, there exists some $n$ and $1 \leq j \leq q_n + k_n + p_n$ such that $t_n + j + M_1 = M_1 + m$. We need to recall that the map $\Delta_{\beta}(b, \cdot)$ defined in the proof of Proposition 1 can be understood as an operator with respect to $\beta$. Now, define a map $\Theta_m : \Sigma_{\beta_m} \rightarrow \Sigma_{\beta_m}$ by defining a map $\Theta_m$ first,
\[
\Theta_m(x) = \bigcup_{1 \leq i \leq n-1} \bigcup_{1 \leq s \leq q_n-1+k_n-1+p_n-1} V(x)_{i,s} \cup V(x)_{n,1} V(x)_{n,2} \cup \ldots \\
\cup V(x)_{n,j} 00 \cdots 0 0 \\
\mathcal{P}(0^{t_n+j+M_1}, \Delta_{\beta_1}(b^{(1)}), x) + \infty \\
\Delta_{\beta_1}(b^{(1)}, x) [t_n + j + M_1 + \mathcal{P}(0^{t_n+j+M_1}, \Delta_{\beta_1}(b^{(1)}, x), +\infty]
\]
where each $V(x)_{i,s}$ is exactly the one appeared in the definition of the map $\Delta_{\beta_1}(b^{(1)}, \cdot)$ for any $(i, s) \in \{1, n-1\} \times \{1, q_n-1+k_n-1+p_n-1\} \cup \{n\} \times \{1, 2, \ldots, j\}$. Next, in order to define the map $\Theta_m$, we need to change all the $V(x)$ in the point
\[
\Delta_{\beta_1}(b^{(1)}, x) [t_n + j + M_1 + \mathcal{P}(0^{t_n+j+M_1}, \Delta_{\beta_1}(b^{(1)}, x), +\infty]
\]
such that $\bigcup_{n \geq 1} \bigcup_{1 \leq j \leq q_n + k_n + p_n} V(x)_{n,j}$ is exactly the point $x$ for any $x = x_1 x_2 \ldots$ in $\Sigma_{\beta}$.

Similar to the proof of Proposition 1, it is not hard to see that the subset $\pi_{\beta} \circ \Theta_m (\Sigma_{\beta_m})$ is a mean Li-Yorke chaotic set in $(0, 1)$ for any $m \geq 1$. Now, set $C = \bigcup_{m=1}^\infty \pi_{\beta} \circ \Theta_m (\Sigma_{\beta_m})$. We claim that $C$ is a mean Li-Yorke chaotic set in $(0, 1)$ with $\beta \in B_1$. It only needs to consider the case that when $H = \pi_{\beta} \circ \Theta_n(x)$ in $\pi_{\beta} \circ \Theta_n (\Sigma_{\beta_n})$ and $I = \pi_{\beta} \circ \Delta_m(y)$ in $\pi_{\beta} \circ \Delta_m (\Sigma_{\beta_m})$ with $m > n \geq 1$ and $H \neq I$. 
such that the definition of mean Li-Yorke scrambled set along a sequence. Let $r \geq 1$ be the least number that $b_i^{(m)}$ is not equal to $b_i^{(s)}$. There exists a positive number $s \geq 1$ such that $b_i^{(m)} \in \mathcal{B}_{0}^{r} \cdots \mathcal{B}_{i+s}^{r}$ appears for $p_{r+s}$ times. For any $N \geq r+s$,

$$
\frac{1}{q_N + k_N + p_N} \sum_{j=1}^{q_N+k_N+p_N} |T_{\beta}^{a(j)}(x) - T_{\beta}^{a(j)}(y)| \geq \frac{\beta - 1}{\beta^2} \frac{1}{q_N + k_N + p_N} \sum_{j=1}^{q_N+k_N+p_N} d(\sigma^{a(j)}(x), \sigma^{a(j)}(y))
$$

That implies that $\limsup_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} |T_{\beta}^{a(j)}(H) - T_{\beta}^{a(j)}(I)| > 0$. Since the map $\Theta_m$ and $\Delta_{\beta}(b, \cdot)$ has same structure, we can apply the same method to show that the set $\pi_{\beta_m} \circ \Theta_m(\Sigma_{\beta_m})$ has the same Hausdorff dimension as the set $\pi_{\beta_m}(\Sigma_{\beta_m})$ has. From [4, Lemma 3.5], which says that the Hausdorff dimension of $\pi_{\beta_m}(\Sigma_{\beta_m})$ is $\frac{\ln \beta_m}{\ln \beta}$, it can be deduced that $\dim_H(\pi_{\beta_m} \circ \Theta_m(\Sigma_{\beta_m})) = \frac{\ln \beta_m}{\ln \beta}$. Thus, $\dim_H(C) = \lim_{m \to \infty} \dim_H(\pi_{\beta_m} \circ \Theta_m(\Sigma_{\beta_m})) = 1$. According to Lemma 2.5 and the fact that the topological entropy of the $\beta$-transformation is $\ln \beta$, we know that the chaotic set constructed is of full topological entropy.

We are unable to find the answer of the following question but we conjecture that the answer is negative.

**Problem.** For the polynomial sequence like $\{p(k)\}$ where $p(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0$ with $1 \leq m \leq 2$, $b_0, \ldots, b_m \in \mathbb{R}$, and $b_m > 0$, do there exist a mean Li-Yorke scrambled set along $\{p(k)\}$ with full Hausdorff dimension for $\beta$-transformation?

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