On Integers that are Covering Numbers of Groups

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1. Introduction

A group $G$ is said to have a finite cover by subgroups if it is the union of finitely many proper subgroups. A cover of size $n$ of a group $G$ is called a minimal cover if no cover of $G$ has fewer than $n$ subgroups. Following Cohn [Cohn 94], we call the size of a minimal cover of a group $G$ the covering number, denoted by $\sigma(G)$. For a survey of results about the covering number of groups (and related results about analogously defined covering numbers of other algebraic structures), see [Kappe 14].

The parameter $\sigma(G)$ has received a great deal of attention in recent years. One reason for this is the connection to sets of pairwise generators of a group. For a finite noncyclic group $G$ that can be generated by two elements, define $\omega(G)$ to be the largest integer $n$ such that there exists a set $S$ of size $n$ consisting of elements of $G$ such that every pair of distinct elements of $S$ generates $G$. The covering number $\sigma(G)$ provides a natural and often tight upper bound for $\omega(G)$; see [Blackburn 06, Britnell et al. 08, Britnell et al. 11, Holmes and Maróti 10] for investigations on the relationship between these two parameters for various simple and almost simple groups.

It suffices to restrict our attention to finite groups when determining covering numbers, since, by a result of Neumann [Neumann 54], a group is the union of finitely many proper subgroups if and only if it has a finite noncyclic homomorphic image. Moreover, if a finite cover of a group $G$ exists, then we may realize a finite homomorphic image of $G$ by taking the quotient over the normal core of the intersection of the subgroups belonging to the finite cover. Determining the covering number of a group $G$ predates Cohn’s 1994 publication [Cohn 94]. It is easy to show that no group is the union of two proper subgroups. Already in 1926, Scorza [Scorza 26] characterized groups having covering number 3 as those groups which have a homomorphic image isomorphic to the Klein four-group, a result forgotten and rediscovered later.

Cohn conjectures in [Cohn 94] that the covering number of a noncyclic solvable group has the form $p^d + 1$, where $p$ is a prime and $d$ is a positive integer, and he shows that $\sigma(A_5) = 10$ and $\sigma(S_5) = 16$. In [Tomkinson 97], Tomkinson proves Cohn’s conjecture and shows that there is no group with $\sigma(G) = 7$. In addition, he conjectures that there are no groups with covering number 11, 13, or 15. Tomkinson’s conjecture is confirmed only for the case $n = 11$ in [Detomi and Lucchini 08]. In fact, in [Abdollahi et al. 07] it is shown that $\sigma(S_5) = 13$ and in [Bryce et al. 99] that
\( \sigma(\text{PSL}(2, 7)) = 15 \). Furthermore, Tomkinson suggests that it might be of interest to investigate minimal covers of nonsolvable and, in particular, simple groups. For an overview of the recent contributions addressing this question, we refer to [Kappe et al. 16]. The real question arising out of Tomkinson’s results is to find all integers that are covering numbers and ascertain whether there are infinitely many integers that are not covering numbers. The aim of this paper is to investigate which integers are covering numbers of groups.

To show that there are no groups \( G \) for which \( \sigma(G) = 7 \), Tomkinson [Tomkinson 97] proved that any group that can be covered by seven subgroups can actually be covered by fewer than seven subgroups. In the long run, this is not the way to attack this problem. In [Detomi and Lucchini 08], the authors observe that if a group \( G \) exists with \( \sigma(G) = n \), then there must exist a group \( H \) with \( \sigma(H) = n \) that has no homomorphic image with covering number \( n \), making \( H \) minimal in this sense. The authors of [Detomi and Lucchini 08] leverage this idea to show that no such group \( G \) can exist with \( \sigma(G) = 11 \). Originally introduced in [Detomi and Lucchini 08] and following [Garonzi 13b], we say that a group \( G \) is \( \sigma \)-elementary if \( \sigma(G) < \sigma(G/N) \) for every nontrivial normal subgroup \( N \) of \( G \). For convenience, we say that the covering number of a cyclic group is infinite. In [Garonzi 13b], the first author of this paper shows that a finite \( \sigma \)-elementary nonabelian group with covering number less than 26 is either of affine type or an almost simple group with socle of prime index, which can then be used to determine which integers less than or equal to 25 are covering numbers. From earlier results and those in [Garonzi 13b], we have that the integers less than 26 that are not covering numbers are 2, 7, 11, 19, 22, and 25.

In this paper, we extend this classification of integers that are covering numbers up to 129. We formulate this result by listing all integers between 26 and 129 that are not covering numbers.

**Theorem 1.1.** The integers between 26 and 129 which are not covering numbers are 27, 34, 35, 37, 39, 41, 43, 45, 47, 49, 51, 52, 53, 55, 56, 58, 59, 61, 66, 69, 70, 75, 76, 77, 78, 79, 81, 83, 87, 88, 89, 91, 93, 94, 95, 96, 97, 99, 100, 101, 103, 105, 106, 107, 109, 111, 112, 113, 115, 116, 117, 118, 119, 120, 123, 124, 125.

**Theorem 1.1** follows from **Theorem 4.5**, **Proposition 6.1**, and Table 1. To prove this result, we need to identify all potential candidates for \( \sigma \)-elementary groups with covering number between 26 and 129. Toward this end, we show in **Theorem 4.5** that the \( \sigma \)-elementary groups in question are among the groups with a unique minimal normal subgroup and degree of primitivity not exceeding 129. We say a finite group is primitive if it admits a maximal subgroup \( M \) with trivial normal core, and the index of \( M \) in \( G \) is called the primitivity degree of \( G \) with respect to \( M \). A finite group that has a unique minimal normal subgroup is called monolithic.

This characterization allows us to decide if a given integer in the range is a covering number or not. Using GAP [The GAP Group 14], we determine all nonsolvable monolithic groups with degree of primitivity between 26 and 129. According to **Theorem 4.5**, these are the candidates for nonabelian \( \sigma \)-elementary groups with covering number between 26 and 129 that are not solvable. It is then necessary to determine—or, at least, bound—the covering number of these groups. Either these results are known, or, if not, we have to attain them ourselves.

Computationally, there are two main methods that we use. The first is a method developed in [Kappe et al. 16], where a program in GAP creates a system of equations that can be solved (either partially or totally) by the linear optimization software GUROBI [Gurobi Optimization 14]. This method, which we refer to as Algorithm KNS, is detailed in **Section 5.1**. The second method is introduced for the first time in this paper and is presented in **Section 5.2**. It is essentially a greedy algorithm, which works roughly as follows. To build a cover, we take entire conjugacy classes of maximal subgroups. Given a group \( G \), we

| Covering number | Nonsolvable \( \sigma \)-elementary groups |
|----------------|--------------------------------------|
| 10             | \( A_5 \)                              |
| 13             | \( S_6 \)                              |
| 15             | \( \text{PSL}(2, 7) \)                  |
| 16             | \( S_5, A_5 \)                         |
| 23             | \( M_{10} \)                           |
| 29             | \( \text{PGL}(2, 7), \text{PTL}(2, 8) \) |
| 31             | \( A_6, \text{AGL}(4, 2) \)            |
| 36             | \( \text{PSL}(2, 8) \)                 |
| 40             | \( \text{ASL}(3, 3), \text{AGL}(3, 3) \) |
| 46             | \( M_{10}, \text{PGL}(2, 9) \)         |
| 57             | \( A_6 \wr 2 \)                        |
| 60             | \( \text{PTU}(3, 3) \)                 |
| 63             | \( \text{AGL}(5, 2) \)                 |
| 64             | \( S_6, S_5, \text{PSU}(3, 3), \text{PSp}(4, 3) : 2, \text{Sp}(6, 2) \) |
| 67             | \( \text{PSL}(2, 11), \text{PGL}(2, 11), \text{PSp}(4, 3) \) |
| 71             | \( A_8 \)                              |
| 73             | \( (A_6 \times A_2) : 4 \)             |
| 85             | \( \text{ASL}(3, 4), \text{AGL}(3, 4), \text{ASL}(3, 4) \) |
| 86             | \( \text{PTL}(2, 16), \text{PGL}(2, 16) \) |
| 92             | \( \text{PSL}(2, 13), \text{PGL}(2, 13) \) |
| 114            | \( \text{PSL}(2, 7) \wr 2 \)           |
| 121            | \( \text{ASL}(4, 3), \text{AGL}(4, 3) \) |
| 126            | \( (A_6 \times A_2) : 4 \)             |
| 127            | \( \text{AGL}(6, 2), \text{PGL}(2, 25) \) |
determine in GAP the minimum number of subgroups from a single conjugacy class of maximal subgroups needed to cover each conjugacy class of elements, and we choose the conjugacy class of elements that requires the maximum number of subgroups (among these minimums). Rather than spending time checking precisely which maximal subgroups are absolutely necessary (something that the first method will do), all subgroups from the entire class of subgroups are chosen to be part of a cover. All conjugacy classes of elements that are covered by subgroups in this class of maximal subgroups are removed, and this process is repeated until all elements of the group are covered. Perhaps surprisingly, the cover produced this way is frequently a minimal cover and can be verified as such quickly using a simple calculation detailed in Lemma 5.1. While the first method is extremely precise, it is often time and memory consuming, and it becomes impractical for groups of order more than half a million on a machine with a Core i7 processor and 16 GB of RAM. The second method, while cruder, runs much faster in practice, and can essentially always be used to provide both upper and lower bounds for the covering number when GAP can determine the maximal subgroups and conjugacy classes of a given group. Pseudocode for this method is provided in Algorithm GKS. When neither of these methods is totally effective, ad hoc methods are used; for this, see the Supplementary Material, which is available online.

It is a natural question to ask if every nonabelian \(\sigma\)-elementary group is a monolithic primitive group. In this respect, the authors of [Detomi and Lucchini 08] make the following conjecture.

**Conjecture 1.2.** ([Detomi and Lucchini 08]). Every nonabelian \(\sigma\)-elementary group is a monolithic primitive group.

So far, no counterexamples to Conjecture 1.2 are known. In Remark 4.3, it is observed that if \(\sigma(X) < 2\sigma^*(X)\) (see Definition 2.4) for all primitive monolithic groups \(X\) with a nonabelian socle, then Conjecture 1.2 is true. (See also the preceding Lemma 4.2, which represents how close we are to proving Conjecture 1.2.) For the proof of Theorem 1.1, we establish in the proof of Theorem 4.5 that the inequality \(\sigma(X) < 2\sigma^*(X)\) holds at least when \(\sigma(X) < 130\). The question arises if this inequality can be extended to bounds larger than 130. On the other hand, the techniques used to determine the covering numbers of the candidate groups with primitivity degree at most 129 were often pushed to the limit. (For instance, there is one group in particular whose covering number cannot be determined to any smaller range than between 138 and 166 using current methods; for this, see Table 2.)

Extending the bound of the primitivity degree may then necessitate new methods (or the use of extremely powerful computers) for determining covering numbers.

After the conjectures of Tomkinson [Tomkinson 97] were settled, i.e., precisely which integers up to 18 are not covering numbers, only three out of the 17 integers between 2 and 18 were found not to be covering numbers, around 18%. One possibility is that there are only finitely many integers that are not covering numbers. The statistics following Theorem 1.1 tell us that around fifty percent of the integers less than 130 are not covering numbers, leading us to make the following conjecture with confidence.

**Conjecture 1.3.** Let \(\delta\) be the set of all integers that are covering numbers. Then, the set \(\mathbb{N} - \delta\) is infinite, in other words there are infinitely many natural numbers that are not covering numbers.

Our results here about integers that are not covering numbers in a certain range were obtained by determining the complement in this range. Indications are that a proof of Conjecture 1.3 requires a similar approach. By [Detomi and Lucchini 08, Theorem 1], a \(\sigma\)-elementary group \(G\) with no abelian minimal normal subgroups is of one of two types:

| Group | Degree | Lower bound | Upper bound | Reference |
|-------|--------|-------------|-------------|-----------|
| PSL(2, 27) | 28 | 167 | 184 | Algorithm KNS |
| \(A_7\) wr 2 | 49 | 447 | 667 | Proposition A.2(i) |
| PSL(3, 4), 2 | 56 | 138 | 166 | Algorithm KNS |
| PSL(6, 2) | 63 | 56,313 | 57,010 | Algorithm GKS |
| \(A_7\) wr 2 | 81 | 10,978 | 30,178 | Algorithm GKS |
| PSL(2, 81), 2 | 82 | 621 | 731 | Algorithm KNS |
| PSp(4, 4), 2 | 85 | 196 | 222 | Proposition A.2(ii) |
| \(J_3\) | 100 | 1063 | 1121 | Algorithm KNS |
| HS : 2 | 100 | 11,859 | 22,375 | Proposition A.2(iii) |
| \((A_{10} \times A_{10})\), 4 | 100 | 22,746 | 30,377 | Proposition A.2(iv) |
| PSU(4, 3), 4 | 112 | 344 | 442 | Proposition A.2(iv) |
| PSU(4, 3), 2 | 112 | 256 | 554 | Proposition A.2(vii) |
| PSU(4, 3), 2 | 112 | 183 | 365 | Proposition A.2(vii) |
| PSU(4, 3), 2 | 112 | 412 | 554 | Proposition A.2(viii) |
| PSU(4, 3), 4 | 112 | 540 | 652 | Algorithm GKS |
| PSL(4, 3), 2 | 117 | 242 | 365 | Proposition A.2(ix) |
| O' (8, 2) | 119 | 25,706 | 26,283 | Proposition A.2(ix) |
| O' (8, 2) | 120 | 204 | 765 | Proposition A.2(x) |
| PSL(2, 11), wr 2 | 121 | 570 | 926 | Proposition A.2(xii) |
| PSL(5, 3) | 121 | 393,030,144 | 393,846,872 | Proposition A.3(i) |
| \((A_{11} \times A_{11})\), 4 | 121 | 213,444 | 213,444 | Proposition A.3(ii) |
| \(P\Gamma L(2, 121)\) | 122 | 671 | 794 | Algorithm GKS |
| \(A_7\) wr 3 | 125 | 216 | 342 | Algorithm GKS |
| \((A_8 \times A_8)\), 6 | 125 | 1000 | 1217 | Algorithm GKS |
| PGU(3, 5) | 126 | 6000 | 6526 | Algorithm GKS |
| \(P\Gamma L(2, 125)\) | 126 | 7750 | 7876 | Algorithm GKS |
| PSL(7, 2) | 127 | 184,308,203,520 | 184,308,203,520 | Proposition A.3(iii) |
1. $G$ is a primitive monolithic group such that $G/\text{soc}(G)$ is cyclic, or

2. $G/\text{soc}(G)$ is nonsovable, and all the nonabelian composition factors of $G/\text{soc}(G)$ are alternating groups of odd degree.

The following result represents progress toward proving Conjecture 1.3 by dealing with the groups of type (1).

**Theorem 1.4.** Let $\mathcal{G}$ be the family of primitive monolithic groups $G$ with nonabelian socle such that $G/\text{soc}(G)$ is cyclic. Then there exists a constant $c$ such that for every $x > 0$,

$$|\{\sigma(G) : G \in \mathcal{G}, \sigma(G) \leq x\}| \leq cx^{5/6}.$$

In particular, $\mathbb{N} - \{\sigma(G) : G \in \mathcal{G}\}$ is infinite.

Given Tomkinson’s result about the covering number of solvable groups and in light of Conjecture 1.2 and Theorem 1.4, it is perhaps probable even that almost all integers are not covering numbers; that is, we make the following conjecture.
Conjecture 1.5. Let $\delta(n) := \{m : m \leq n, \sigma(G) = m \text{ for some group } G\}$. Then,

$$\lim_{n \to \infty} \frac{|\delta(n)|}{n} = 0.$$ 

As a corollary of Theorem 1.4, we prove the following result, which can be viewed as additional evidence for the validity of Conjecture 1.3.

Corollary 1.6. Let $\mathcal{F}_1$ be the family of finite groups $G$ such that all proper quotients of $G$ are solvable. Then the set $\mathbb{N} - \{\sigma(G) : G \in \mathcal{F}_1\}$ is infinite.

Indeed, one may view Corollary 1.6 as a generalization of Tomkinson’s result on the covering numbers of solvable groups (see Proposition 6.1(ii)) in the following sense. Let $\mathcal{F}_n$ denote the set of all finite groups with at most $n$ nonsolvable quotients. Note that $\mathcal{F}_i \subset \mathcal{F}_j$ for all $i < j$, and, if we define $\mathcal{F} := \bigcup_{n=0}^{\infty} \mathcal{F}_n$, then $\mathcal{F}$ is the family of all finite groups. If for a collection of groups $C$ we define

$$\delta(C) := \{\sigma(G) : G \in C\},$$

then Conjecture 1.3 is that $\mathbb{N} - \delta(\mathcal{F})$ is infinite. A consequence of Tomkinson’s result is that $\mathbb{N} - \delta(\mathcal{F}_0)$ is infinite, and Corollary 1.6 above states that $\mathbb{N} - \delta(\mathcal{F}_1)$ is infinite. The next step toward proving Conjecture 1.3 would be to prove that $\mathbb{N} - \delta(\mathcal{F}_2)$ is infinite, and for this one needs to determine (among other things) the covering number of affine groups, i.e., primitive groups with an abelian socle. As a first step in this process, we prove the following.

Theorem 1.7. Let $q = p^d$, where $p$ is prime and $d \in \mathbb{N}$, and let $n \geq 1$, $n \neq 2$ be a positive integer. Then

$$\sigma(AGL(n,q)) = \sigma(ASL(n,q)) = \frac{q^{n+1} - 1}{q - 1}.$$  

In particular, for all $m \geq 2$, $m \neq 3$, $(q^m - 1)/(q - 1)$ is a covering number.

This result can also be viewed as a generalization of a result of Cohn [Cohn 94, Corollary to Lemma 17]. There it is shown that all integers of the form $(q^2 - 1)/(q - 1) = q + 1$, where $q$ is a prime power, are covering numbers.

Our hope is to obtain further density results along the lines of Theorem 1.4 for $\sigma$-elementary groups and eventually arrive at a proof of Conjecture 1.3. Other than the remaining primitive groups with abelian socle, we would like to highlight the family of wreath products $S \wr K$, where $S$ is a nonabelian simple group and $K$ is a transitive group of degree $k$. These groups are an archetypal case among those that remain, and it would be interesting to prove results for this family in particular.

The structure of this paper is as follows. Section 2 contains preparatory results for subsequent sections, especially the following two sections. Section 3 contains the proof of Theorem 1.4, the density result for a certain class of $\sigma$-elementary groups, and Section 4 contains the proof of the necessary condition that a nonabelian $\sigma$-elementary group $G$ with $\sigma(G) \leq 129$ is a monolithic primitive group with a degree of primitivity of at most 129. Section 5 details the two main computational methods used to determine covering numbers.

The covering numbers or estimates of them are known for some classes of monolithic primitive groups that are candidates to be $\sigma$-elementary groups with covering number at most 129, and these results are summarized in Section 6. The covering numbers for groups in two other families of monolithic primitive groups, the affine general linear groups and the affine special linear groups, are determined in Section 7, which also shows that every integer of the form $(q^m - 1)/(q - 1)$, where $q$ is a prime power and $n > 3$, is a covering number. All our results for calculations and bounds are summarized in a series of tables, which are included in the text. Table 1, which contains a list of the nonsolvable $\sigma$-elementary groups with $\sigma(G) \leq 129$. Table 5 contains a list of nonsolvable primitive groups for which the covering number has not been precisely determined, as well as bounds on the covering numbers for these groups. Tables 1, 2, 3, and 4 contain the exact covering numbers for various nonsolvable primitive groups. For further information about these tables and their contents, see Section 8. If $p$ is a prime and $d$ is a positive integer, then this table together with the integers in this range of the form $p^d + 1$ (that is, the covering numbers of solvable groups by [Tomkinson 97]) establish Theorem 1.1. Finally, the Supplementary Material, which is available online, contains calculations or bounds for the covering number for the various groups that could not be dealt with using the methods of the previous sections.

2. Background

In this section for the convenience of the reader we begin with some well-known concepts used throughout this paper and present some relevant definitions and lemmas which can be found in earlier publications addressing this topic. The uninitiated reader is encouraged to consult [Dixon and Mortimer 96] or [Robinson 96].

The socle of a finite group $G$, denoted $\text{soc}(G)$, is the subgroup of $G$ generated by the minimal normal subgroups of $G$, and, in fact, the socle of $G$ is a direct product of some minimal normal subgroups of $G$. A finite group $G$ is said to be monolithic if it admits a unique minimal normal subgroup, which therefore equals the socle of $G$. 

### Table 5. Covering numbers of various nonsolvable primitive groups of degree 81–117.

| Group | Degree | Covering number | Reference |
|-------|--------|-----------------|-----------|
| $3^4 : 2.S_6$, $3^3 : S_6$ | 81 | 13 | Lemma 2.11 |
| $3^3 : 2.A_5$, $3^4 : 4.A_5$, $3^4 : 8.A_5$ | 81 | 16 | Lemma 2.11 |
| $3^2 : 2.A_5$, $3^3 : 2.A_5$ | 81 | 16 | Lemma 2.11 |
| $2^3 : 3^3 : 2.15$ | 81 | 16 | Lemma 2.11 |
| $2^3 : 3^4 : (2 \times A_5)$ | 81 | 16 | Lemma 2.11 |
| $3^4 : S_3 : 4 \times S_3$ | 81 | 46 | Lemma 2.11 |
| $4.A_5, 2.3^3 : A_5, 2$ | 81 | 46 | Lemma 2.11 |
| $3^3 : 2.PSL(2, 9)$ | 81 | 46 | Lemma 2.11 |
| $3^3 : Sp(4, 3) : 2$ | 81 | 64 | Lemma 2.11 |
| $3^3 : Sp(4, 3)$ | 81 | 67 | Lemma 2.11 |
| $ASL(4, 3), AG(4, 3)$ | 81 | 121 | Theorem 1.7 |
| $PSL(2, 8) \wr 2$ | 81 | 586 | Proposition A.1(iv) |
| $PSL(2, 8) \times PSL(2, 8), 6$ | 81 | 586 | Algorithm GKS |
| $(A_9 \times A_9).4$ | 81 | 24,310 | Algorithm GKS |
| $A_9 wr 2$ | 81 | $\geq 10,978$ | Algorithm GKS |
| $PSL(2, 81), 2^2, PSL(2, 81)$ | 82 | 3 | $C_2 \times C_2$ |
| $PSL(2, 81)$ | 82 | 452 | Algorithm KNS |
| $PSL(2, 81), 4$ | 82 | 3322 | Algorithm KNS |
| $PSL(2, 81), 2$ | 82 | $\geq 621$ | Algorithm KNS |
| $PSp(4, 4)$ | 85 | 256 | Algorithm GKS |
| $PSL(4, 4)$ | 85 | 24,277 | Algorithm GKS |
| $PSL(4, 4)$ | 85 | 45,778 | Algorithm GKS |
| $PSp(4, 4), 2$ | 85 | $\geq 196$ | Proposition A.2(iii) |
| $PSL(3, 9)$ | 91 | 7652 | Algorithm GKS |
| $PSL(3, 9)$ | 91 | 155,661 | Algorithm GKS |
| $(A_9 \times A_9).2^2, S_5 \wr 2$ | 100 | 3 | $C_2 \times C_2$ |
| $(A_9 \times A_9).2^2$ | 100 | 3 | $C_2 \times C_2$ |
| $(A_9 \times A_9), (2 \times 4)$ | 100 | 3 | $C_2 \times C_2$ |
| $(A_9 \times A_9), (2 \times D_8)$ | 100 | 3 | $C_2 \times C_2$ |
| $PFTL(2, 9) \wr 2$ | 100 | 3 | $C_2 \times C_2$ |
| $(A_9 \times A_9).4$ (#16 in the list) | 100 | 1387 | Proposition A.1(iv) |
| $(A_9 \times A_9).4$ (#18 in the list) | 100 | 2026 | Algorithm GKS |
| $J_2, 2$ | 100 | 2921 | Algorithm GKS |
| $A_{10} \wr 2$ | 100 | 30,377 | Algorithm GKS |
| $A_7$ | 100 | $\geq 1063$ | Algorithm KNS |
| $JSF : 12$ | 100 | $\geq 11,859$ | Proposition A.2(iii) |
| $(A_9 \times A_9).4$ | 100 | $\geq 22,746$ | Proposition A.2(iv) |
| $PSL(3, 4).2^2, PSL(3, 4).D_12$ | 105 | 3 | $C_2 \times C_2$ |
| $PSL(3, 4).S_3$ | 105 | 4 | $S_3$ |
| $PSL(3, 4), 6$ | 105 | 386 | Algorithm GKS |
| $PSL(3, 4).2^2$ (all such groups) | 112 | 3 | $C_2 \times C_2$ |
| $PSU(3, 4).D_8$ | 112 | 3 | $C_2 \times C_2$ |
| $PSU(3, 4)$ | 112 | $\geq 344$ | Algorithm GKS |
| $PSU(3, 4).Z$ | 112 | 256 | Algorithm GKS |
| $PSU(3, 4).2_2$ | 112 | $\geq 183$ | Algorithm GKS |
| $PSU(3, 4), 2_3$ | 112 | $\geq 412$ | Algorithm GKS |
| $PSU(3, 4), 4$ | 112 | $\geq 540$ | Algorithm GKS |
| $PSL(3, 4).2$ | 117 | 170 | Algorithm KNS |

### Table 6. Covering numbers of various nonsolvable primitive groups of degree 117–129.

| Group | Degree | Covering number | Reference |
|-------|--------|-----------------|-----------|
| $PSL(4, 3), 2$ | 117 | $\geq 242$ | Proposition A.2(iv) |
| $PSO(8, 2)$ | 119 | 256 | Algorithm GKS |
| $O^+(8, 2)$ | 119 | $\geq 25,706$ | Proposition A.2(v) |
| $PSL(3, 4).2$ | 120 | 3 | $C_2 \times C_2$ |
| $Sp(8, 2)$ | 120 | 256 | Proposition A.1(vi) |
| $PSO^+(8, 2)$ | 120 | $\geq 204$ | Proposition A.2(vi) |
| $O^+(8, 2)$ | 120 | $\geq 204$ | Proposition A.2(vi) |
| $(A_9 \times A_9).2^2, S_5 \wr 2$ | 121 | 10 | Lemma 2.11 |
| $11^3 : (2.A_4), 11^2 : (5 \times 2.A_3)$ | 121 | 10 | Lemma 2.11 |
| $11^2 : (SL(2, 11) : 2), 11^2 : (5 \times SL(2, 11))$ | 121 | 67 | Lemma 2.11 |

A finite group $G$ is said to be primitive if it admits a maximal subgroup $M$ such that $M_G = \bigcap_{g \in G} g^{-1}Mg$ (the normal core of $M$) is trivial, and in this case the index $|G : M|$ is called a primitivity degree (or degree of primitivity) of $G$; a primitive group in general has many primitivity degrees. This definition for abstract groups is equivalent to the permutation group definition; that is, if $\Omega$ is a finite set, then $G \leq \text{Sym}(\Omega)$ is primitive on the set $\Omega$ if and only if $G$ is transitive on $\Omega$ and stabilizes no nontrivial partition of $\Omega$. (The equivalence can be seen by taking the set of right cosets of $M$ in $G$ to be $\Omega$ under the action of right multiplication.) It is well-known that a finite primitive group $G$ is either monolithic or it admits precisely two minimal normal subgroups, and, in this case, such minimal normal subgroups are nonabelian and isomorphic; see, for instance [Ballester-Bolinches and Ezquerro 06, Theorem 1.1.7].

Any minimal normal subgroup of a finite group $G$ is **characteristically simple**, so it has the form $S'$ for some simple group $S$ (which could be abelian) and some positive integer $r$. A minimal normal subgroup $N$ of $G$ is said to be **supplemented** if it admits a supplement in $G$, that is, a proper subgroup $H$ of $G$ such that $HN = G$, and complemented if it admits a complement in $G$, that is, a supplement $H$ such that $H \cap N = \{1\}$. The minimal normal subgroup $N$ is said to be
Frattini if it is contained in the Frattini subgroup of $G$, which is the intersection of the maximal subgroups of $G$, and non-Frattini, otherwise. (Note that a Frattini minimal normal subgroup is automatically abelian, since the Frattini subgroup is nilpotent.)

We now present some results regarding the Frattini subgroup and minimal normal subgroups. For a minimal normal subgroup $N$ of $G$, being non-Frattini is equivalent to being supplemented, where the supplement of $N$ is any maximal subgroup of $G$ which does not contain $N$. For an abelian minimal normal subgroup $N$ of $G$, being supplemented is equivalent to being complemented: if $H$ is a supplement of $N$ in $G$, then $H \cap N \neq N$, since $HN = G$; $H \cap N$ is normal in $H$, since $N$ normal in $G$; and $H \cap N$ is normal in $N$, since $N$ is abelian. Hence, we have $N \cap H \triangleleft NH = G$, and $N \cap H = \{1\}$, since $N$ a minimal normal subgroup. Also, observe that a monolithic group $G$ is primitive if and only if the Frattini subgroup of $G$ is trivial: indeed, if $G$ is primitive, then it is clear from the definition that the Frattini subgroup of $G$ is trivial; conversely, suppose that $G$ is monolithic with trivial Frattini subgroup. Since $G$ is monolithic, the socle of $G$ is contained in every nontrivial normal subgroup of $G$, and, since $G$ has trivial Frattini subgroup, there must exist a maximal subgroup with trivial normal core, since, otherwise, the Frattini subgroup would contain the socle. Hence, $G$ is primitive.

Following [Detomi and Lucchini 03], the primitive monolithic group $X_N$ associated to a non-Frattini minimal normal subgroup $N$ of a group $G$ is defined as follows, based on whether or not $N$ is abelian:

- If $N$ is abelian, then there exists a complement $H$ of $N$ in $G$. Then $C_H(N) \leq G$ and we define $X_N := G/C_H(N)$.
- If $N$ is nonabelian, then we define $X_N := G/C_G(N)$.

The following observations can be proved easily. In the case when $N$ is abelian the primitive monolithic group associated to $N$ depends on the choice of complement; however, choosing a different complement gives an isomorphic primitive group. In any case, $X_N$ is a primitive monolithic group with socle isomorphic with $N$ (the socle is $NC_H(N)/C_H(N)$ in the first case and $NC_G(N)/C_G(N)$ in the second case). Observe that if $G$ is itself primitive and monolithic, then $G$ coincides with the primitive monolithic group associated with its socle: if $G$ is primitive and monolithic with socle $N$, then the centralizer of $N$ in $G$ is trivial if $N$ is nonabelian and equals $N$ if $N$ is abelian; otherwise, a larger centralizer would give rise to a nontrivial normal subgroup not containing $N$ in both cases.

Let $\sigma(G)$ denote the covering number of $G$, with $\sigma(G) = \infty$ if $G$ is cyclic (with the convention that $n < \infty$ for all integers $n$). It is easy to see that $\sigma(G) \leq \sigma(G/N)$ for all normal subgroups $N$ of $G$, and, with this in mind, we have the following definition.

**Definition 2.1.** A finite noncyclic group $G$ is said to be $\sigma$-elementary if $\sigma(G) < \sigma(G/N)$ for all nontrivial normal subgroups $N$ of $G$.

In the following result, $\Phi(G)$ denotes the Frattini subgroup of $G$.

**Lemma 2.2** ([Detomi and Lucchini 08, Corollary 14]). Let $G$ be a finite $\sigma$-elementary group. If $G$ is abelian, then $G \cong C_p \times C_p$ for some prime $p$. If $G$ is nonabelian, then the following hold:

1. The Frattini subgroup $\Phi(G)$ of $G$ is trivial.
2. $G$ has at most one abelian minimal normal subgroup.
3. Let $\text{soc}(G) = N_1 \times \cdots \times N_n$ be the socle of $G$, where $N_1, \ldots, N_n$ are the minimal normal subgroups of $G$. Then $G$ is a subdirect product of the primitive monolithic groups $X_i$ associated to the $N_i$’s, and the natural map from $G$ to $X_1 \times \cdots \times X_n$ given by the natural projections of $G$ onto each $X_i$ is injective.

We also note the following additional structural result.

**Lemma 2.3** ([Cohn 94, Theorem 4]). If $G$ is a nonabelian $\sigma$-elementary group, then the center of $G$ is trivial.

The following definition provides a concept that is useful for bounding the covering number of a group from below.

**Definition 2.4** ([Detomi and Lucchini 08, Definition 15]). Let $X$ be a primitive monolithic group with socle $N$. If $\Omega$ is an arbitrary union of cosets of $N$ in $X$, define $\sigma_\Omega(X)$ to be the smallest number of supplements of $N$ in $X$ needed to cover $\Omega$. Define $\sigma^\ast(X) := \min \{ \sigma_\Omega(X) | \Omega = \cup_i \omega_i N, \langle \Omega \rangle = X \}$.

The values of $\sigma^\ast(X_i)$, where the $X_i$ are as in Lemma 2.2, provide a lower bound for $\sigma(G)$ when $G$ is a $\sigma$-elementary group in terms of the primitive monolithic groups associated to its minimal normal subgroups.

**Lemma 2.5** ([Detomi and Lucchini 08, Proposition 16]). Let $G$ be a nonabelian group, let $\text{soc}(G) = N_1 \times \cdots \times N_n$, and let $X_1, \ldots, X_n$ be the primitive
monolithic groups associated to \(N_1, \ldots, N_n\), respectively. If \(G\) is \(\sigma\)-elementary, then
\[
\sigma^*(X_1) + \cdots + \sigma^*(X_n) \leq \sigma(G).
\]

For a primitive monolithic group \(X\) with socle \(N\), we denote by \(\ell_X(N)\) the minimal index of a proper supplement of \(N\) in \(X\). In other words, \(\ell_X(N)\) is the smallest primitive degree of \(X\).

**Lemma 2.6** ([Detomi and Lucchini 08, Remark 17]). If \(X\) is a primitive monolithic group, then
\[
\sigma^*(X) \geq \ell_X(\text{soc}(X)).
\]

If \(G\) is a nonabelian \(\sigma\)-elementary group, \(N_1, \ldots, N_n\) are the minimal normal subgroups of \(G\), and \(X_1, \ldots, X_n\) are the primitive monolithic groups associated to \(N_1, \ldots, N_n\), respectively, then
\[
\sum_{i=1}^{n} \ell_X(N_i) \leq \sum_{i=1}^{n} \sigma^*(X_i) \leq \sigma(G).
\]

In particular, for every \(i \in \{1, \ldots, n\}\), the group \(X_i\) has primitivity degree at most \(\sigma(G)\).

The following lemmas from [Detomi and Lucchini 08] are critical in later proofs. Note that Lemma 2.9 includes information contained in the proof as well as the statement of [Detomi and Lucchini 08, Proposition 10].

**Lemma 2.7** ([Detomi and Lucchini 08, Lemma 18]). Let \(N\) be a normal subgroup of a group \(X\). If a set of subgroups of \(X\) covers a coset \(yN\) of \(N\) in \(X\), then it also covers every coset \(\alpha z \cdot N\) with \(\alpha\) prime to \(|y|\).

**Lemma 2.8** ([Detomi and Lucchini 08, Proposition 21]). Let \(G\) be a nonabelian \(\sigma\)-elementary group. If a proper quotient of \(G\) is solvable, then it is cyclic.

**Lemma 2.9** ([Detomi and Lucchini 08, Proposition 10]). Let \(G\) be a group. If \(V\) is a complemented normal abelian subgroup of \(G\) and \(V \cap \text{Z}(G) = \{1\}\), then \(\sigma(G) \leq 2|V|-1\). In particular, if \(V\) is a minimal normal subgroup, where \(q = |\text{End}_G(V)|\) and \(|V| = q^n\), and \(H\) is a complement of \(V\) in \(G\), then the collection
\[
\{H' : v \in V\} \cup \{C_{H}(W)V : W \subseteq V, \dim_{\text{GF}(q)}(W) = 1\}
\]
is a cover for \(G\) and
\[
\sigma(G) \leq 1 + q + \cdots + q^n = \frac{q^{n+1}-1}{q-1}.
\]

Finally, the following lemmas prove to be extremely useful when calculating covering numbers. The first lemma is a straightforward criterion for showing that a maximal subgroup is contained in any minimal cover containing only maximal subgroups.

**Lemma 2.10** ([Garonzi 13b, Lemma 1]). If \(H\) is a maximal subgroup of a group \(G\) and \(\sigma(H) > \sigma(G)\), then \(H\) appears in every minimal cover of \(G\) containing only maximal subgroups. In particular, if \(H\) is maximal and non-normal then \(\sigma(H) < |G : H|\) implies \(\sigma(G) \geq \sigma(H)\).

The final lemma of this section is due to Detomi and Lucchini (also proved independently by Jafarian Amiri [Jafarian Amiri 10]) and is useful when proving results about covering numbers of primitive groups with an elementary abelian minimal normal subgroup.

**Lemma 2.11** ([Detomi and Lucchini 08, Corollary 6]). If \(G\) is a primitive group with stabilizer \(H\) and unique abelian minimal normal subgroup \(N\), then \(\sigma(G) \geq |N| + 1\) or \(\sigma(G) = \sigma(H)\).

### 3. A density result

One of the main problems about group coverings is the following: what does the set \(\mathcal{E}\) of numbers of the form \(\sigma(G)\), where \(G\) is a finite group, look like?

Recall that Conjecture 1.3 hypothesizes that there are infinitely many natural numbers that are not covering numbers. A good strategy to approach this conjecture is the following: first, find a specific (and “easy” to handle) family \(\mathcal{F}\) of groups such that \(\{\sigma(G) : G \in \mathcal{F}\} = \mathcal{E}\) and then deal with the family \(\mathcal{F}\). If Conjecture 1.2 is true, then, because we clearly can choose as \(\mathcal{F}\) the family of \(\sigma\)-elementary groups, we may choose as \(\mathcal{F}\) the family of primitive monolithic groups. An important subfamily of it is the family of primitive monolithic groups whose quotient over the socle is cyclic, since the proper solvable quotients of \(\sigma\)-elementary groups are cyclic. In this section, we show that the density of the values \(\sigma(G)\) for \(G\) a primitive monolithic group with \(G/\text{soc}(G)\) cyclic is zero; specifically, setting \(\mathcal{G}\) to be the family of such primitive monolithic groups, we show that
\[
\left|\{\sigma(G) : G \in \mathcal{G}, \sigma(G) \leq x\}\right| \leq cx^{5/6}
\]
for some constant \(c\). This implies that \(\mathbb{N} - \{\sigma(G) : G \in \mathcal{G}\}\) is infinite.

Before we can proceed with the proof of the main theorem in this section, we need some preparatory results. The first can be considered part of the O’Nan–Scott Theorem (see [Ballester-Bolinches and Ezquerro 06, Remark 1.140]). Let \(G\) be a primitive monolithic group with nonabelian socle \(N = T_1 \times T_2 \times \cdots \times T_m \cong T^{m}\), where \(T\) is a nonabelian finite simple group. Let \(H\) be a maximal subgroup of \(G\).
such that $N \nsubseteq H$, i.e., $HN = G$. Suppose $H \cap N \neq \{1\}$, i.e., $H$ does not complement $N$. Since $N$ is a minimal normal subgroup of $G$ and $H$ is a maximal subgroup of $G$ not containing $N$, $H = N_G(H \cap N)$. In the following, let $X := N_G(T_i)/C_G(T_i)$, which is an almost simple group with socle $T_iC_G(T_i)/C_G(T_i) \cong T$. There are two possibilities for the intersection $H \cap N$, and the primitive group $G$ is described as having one of two types, depending on the possibility. These two types and some basic properties of each are described as follows.

1. **Product type.** In this case, the projections $H \cap N \to T_i$ are not surjective. This implies that there exists a subgroup $M$ of $T$, which is an intersection of $T$ with a maximal subgroup of $X$, such that $N_x(M)$ supplements $T$ in $X$, and there exist elements $a_2, \ldots, a_m \in T$ such that $H \cap N$ equals
   \[
   M \times M^{a_1} \times \cdots \times M^{a_m}.
   \]

2. **Diagonal type.** In this case, the projections $H \cap N \to T_i$ are surjective. This implies that there exists a minimal $H$-invariant partition $P$ of $\{1, \ldots, m\}$ into imprimitivity blocks of the action of $H$ on $\{1, \ldots, m\}$ such that $H \cap N$ equals
   \[
   \prod_{D \in P} (H \cap N)^x_D,
   \]
   and, for each $D \in P$, the projection $(H \cap N)^x_D$ is a full diagonal subgroup of $\prod_{i \in D} T_i$. (Following [Ballester-Bolívar and Esquerro 06, Definition 1.1.37], a subgroup $H$ of $\prod_{i \in I} T_i$ is said to be full diagonal if each projection $\pi_i : H \to T_i$ is an isomorphism.)

For a finite nonabelian simple group $T$, denote by $m(T)$ the minimal index of a proper subgroup of $T$, which is equal to the minimal degree of a transitive permutation representation of $T$. Recall that $\ell_G(N)$ denotes the minimal index of a proper supplement of $N$ in $G$. The following lemma provides information about $\ell_G(N)$ for primitive monolithic groups $G$ with socle $N$.

**Lemma 3.1.** Let $G$ be a primitive monolithic group with socle $N$. If $N$ is abelian, then $\ell_G(N) = |N|$. If $N$ is nonabelian, then write $N = T^r$ with $T$ a nonabelian simple group. Let $H$ be a maximal subgroup of $G$ supplementing $N$.

i. If $H$ complements $N$, then $|G : H| = |N| = |T|^r$.

ii. If $H$ has product type, then $H = N_G(M \times M^{a_1} \times \cdots \times M^{a_m})$ for some subgroup $M$ of $T$ of the form $Y \cap T$, where $Y$ is a maximal subgroup of $X$ supplementing $T$, and $|G : H| = |T : M|^r$.

iii. If $H$ has diagonal type, then $H = N_G(\Delta)$, where $\Delta$ is a product of nilpotent diagonal subgroups (in the sense of the above description of diagonal type) with $c$ a prime divisor of $r$ larger than 1, and $|G : H| = |T|^{r-1/c}$. Moreover $\ell_G(N) \geq m(T)^r$.

**Proof.** Suppose $N$ is abelian. Since $G$ is primitive, $N$ is non-Frattini, so it is complemented and each of its complements have index $\ell_G(N) = |N|$.

Suppose $N$ is nonabelian. The three listed facts in the statement follow easily from the fact that $|G : H| = |N : H \cap N|$. Now let us prove that $\ell_G(N) \geq m(T)^r$. Since $m(T)^r \leq |T : M|^r$ for every proper subgroup $M$ of $T$, it suffices to show that $m(T)^r \leq |T|^{r-1/c}$ for every divisor $c > 1$ of $r$. For this, it is enough to show that $m(T)^r \leq |T|^{r/2}$, i.e., $m(T)^2 \leq |T|$. This is true by inspection, using [Damian and Lucchini 06].

Next, we will prove a technical lemma which basically says under the right conditions that, if a set of numbers is "small," then the set of all possible powers of those numbers is also small.

**Lemma 3.2.** Let $A$ be a subset of $\mathbb{N}$, and for $x \in \mathbb{R}$ let
   \[
   \theta(x) := |\{n \in A : n \leq x\}|.
   \]
   If there exists a constant $c$ such that
   \[
   \log(x) \theta(x^c) \leq c \theta(x)
   \]
   for every $x > 0$, then there exists a constant $d$ such that
   \[
   \left|\left\{n^k : n \in A, k \in \mathbb{N}, n^k \leq x\right\}\right| \leq d \theta(x)
   \]
   for every $x > 0$.

**Proof.** Let $N(x)$ be the smallest natural number such that $2^{N(x)} > x$. Clearly there exists a constant $b$ such that $N(x) \leq b \log(x)$, and
   \[
   \left|\left\{n^k : n \in A, k \in \mathbb{N}, n^k \leq x\right\}\right| \leq b \log(x) \log(2x) + \cdots + \log(x^{1/N(x)})
   \]
   \[
   \leq b \log(x) + N(x) \log(x^{1/2}) \leq d \theta(x),
   \]
   where $d = 1 + bc$. □

The following result is due to Frobenius; see [Cossey et al. 80, Section 5] for a modern treatment. Here, if $H$ and $K$ are two groups and $K \leq S_n$, then $H \wr K$ denotes the wreath product between $H$ and
Let $K$, i.e., the semidirect product $H^n \rtimes K$, where $K$ acts on $H^n$ by permuting the coordinates.

**Theorem 3.3.** Let $H$ be a subgroup of the finite group $G$, let $x_1, \ldots, x_n$ be a right transversal for $H$ in $G$, and let $\xi$ be any homomorphism with domain $H$. Then the map $G \to \xi(H)$ wr $S_n$ given by

$$x \mapsto \left(\xi(x_1x_1^{-1}), \ldots, \xi(x_nx_n^{-1})\right)\pi,$$

where $\pi \in S_n$ satisfies $x_i x_n^i \in H$ for all $i = 1, \ldots, n$, is a well-defined homomorphism with kernel equal to the normal core $(\ker \xi)_G$.

Next, in the following remark we establish some notation and basic results about monolithic groups with a nonabelian socle.

**Remark 3.4.** Let $G$ be a monolithic group with socle $N = \soc(G) = T_1 \times \cdots \times T_m$, where $T_1, \ldots, T_m$ are pairwise isomorphic nonabelian simple groups. We also define $X := N_G(T_1)/C_G(T_1)$, which is an almost-simple group with socle $T := T_1C_G(T_1)/C_G(T_1) \cong T_1$. The minimal normal subgroups of $T^n = T_1 \times \cdots \times T_m$ are precisely its factors $T_1, \ldots, T_m$. Since automorphisms send minimal normal subgroups to minimal normal subgroups, it follows that $G$ acts on the $m$ factors of $N$. Let $\rho : G \to S_m$ be the homomorphism induced by the conjugation action of $G$ on the set $\{T_1, \ldots, T_m\}$. The group $K := \rho(G)$ is a transitive permutation group of degree $m$. Choosing $H := N_G(T_1)$ and $\xi : H \to \Aut(T_1)$, the homomorphism given by the conjugation action of $H$ on $T_1$, by Theorem 3.3, we see that $G$ embeds in the wreath product $X \wr K$.

We need some consequences of the classification of finite simple groups (henceforth CFSG); see [Gorenstein et al. 94]. For $T$ a finite nonabelian simple group, recall that $m(T)$ denotes the minimal index of a proper subgroup of $T$. Clearly $m(A_n) = n$, and the value of $m(T)$ when $T$ is a group of Lie type can be found in [Damian and Lucchini 06, Table 1].

**Lemma 3.5.** Let $T$ be a nonabelian finite simple group.

1. There exists a constant $c$ such that $|\Out(T)| \leq c \log(m(T))$.
2. If $T$ is non-alternating and not of the form $\mathrm{PSL}(2, q)$, then there are at most $c \sqrt{x}/\log(x)$ groups $T$ such that $m(T) \leq x$, where $c$ is a constant.

**Proof.** Item (1) follows from CFSG by inspection. For (2), by CFSG, if $q$ is the size of the base field and $T$ is not $\mathrm{PSL}(2, q)$, then there always is a constant $b$ such that $bq^2 \leq m(T)$ so that $m(T) \leq x$ implies $q \leq (x/b)^{2}$, and by the prime number theorem and

**Lemma 3.2,** there are at most $cx^{2}/\log(x)$ choices for $q$, where $c$ is a constant. For a given $q$, we need only consider the constant number of families where $m(T) \approx q^2$. Indeed, if $bq^2 \leq m(T)$, then we may replace the square root by a cube root, and there are on the order of $\log(x)$ possible values of $n$, and so there are at most $(d \sqrt{x}/\log(x)) \cdot \log(x)$ total possibilities outside the families when $m(T) \approx q^2$, where $d$ is a constant. The result follows.

Let $\mathcal{G}$ be a family of monolithic $\sigma$-elementary groups with nonabelian socle, and for $G \in \mathcal{G}$, let $\soc(G) = T^k$ for $T$ a nonabelian simple group and let $n_{\sigma}(G) := m(T)^k$. The following lemma provides bounds for the number of integers that are covering integers outside the families when $m(T) \approx q^2$, where $d$ is a constant. The result follows.

**Lemma 3.6.** Let $\mathcal{H}$ be a subfamily of $\mathcal{G}$. Define

$$A := \{\sigma(G) : G \in \mathcal{H}\}, \quad B := \{n_{\sigma}(G) : G \in \mathcal{H}\}.$$ 

Let $g(x)$ be a function such that, for all $n \leq x$,

$$|\{G \in \mathcal{H} : n_{\sigma}(G) = n\}| \leq g(x).$$

Then

$$|\{n \in A : n \leq x\}| \leq g(x) \cdot |\{n \in B : n \leq x\}|.$$

**Proof.** Indeed, if $G \in \mathcal{H}$, then $G$ is monolithic, so by Lemmas 3.1 and 2.6, respectively,

$$n_{\sigma}(G) \leq \ell_{\sigma}(\soc(G)) \leq \sigma(G),$$

which means that

$$|\{n \in A : n \leq x\}| \leq |\{G \in \mathcal{H} : \sigma(G) \leq x\}|$$

$$\leq \sum_{n \leq x} \left|\{G \in \mathcal{H} : n_{\sigma}(G) = n\}\right|$$

$$\leq g(x) \cdot |\{n \in B : n \leq x\}|.$$


Table 7. Covering numbers of symmetric groups.

| Group | \(m(G)\) | Covering number | Citation |
|-------|----------|----------------|----------|
| \(S_5\) | 5 | 16 | [Cohn 94] |
| \(S_6\) | 6 | 13 | [Abbott and al. 07] |
| \(S_7\) | 8 | 64 | [Kappe et al. 16] |
| \(S_8\) | 9 | 256 | [Kappe et al. 16] |
| \(S_{10}\) | 10 | 221 | [Kappe et al. 16] |
| \(S_{12}\) | 12 | 761 | [Kappe et al. 16] |
| \(S_{14}\) | 14 | 3096 | [Oppenheim and Swartz 19] |
| \(S_{18}\) | 18 | 36,773 | [Swartz 16] |

\(S_{12k}, k \geq 4\) \(6k + \frac{1}{2} \left( \frac{6k}{3k} \right) + \sum_{i=0}^{2k-1} \left( \frac{6k}{i} \right)\) [Swartz 16]

\(S_{12k}, k \neq 4\) \(2k + 1\) \(2^{2k}\) [Maróti 05]

\(S_{12k}, k \geq 16\) \(2k\) \(\geq \frac{1}{2} \left( \frac{2k}{k} \right)\) [Maróti 05]

Table 9. Covering numbers of 2-dimensional linear groups.

| Group | \(m(G)\) | Covering number | Citation |
|-------|----------|----------------|----------|
| \(PSL(2, 5)\) | 6 | 10 | [Cohn 94] |
| \(PGL(2, 5)\) | 6 | 16 | [Cohn 94] |
| \(PSL(2, 7)\) | 7 | 15 | [Bryce et al. 99] |
| \(PGL(2, 7)\) | 8 | 29 | [Bryce et al. 99] |
| \(PSL(2, 9)\) | 10 | 16 | [Bryce et al. 99] |
| \(PGL(2, 9)\) | 10 | 46 | [Bryce et al. 99] |
| \(PGL(2, 11)\) | 10 | 29 | [Garonzi 13b] |
| \(PGL(2, 3), PGL(2, q), q + 1\) | \(\frac{1}{2}q(q + 1)\) | [Bryce et al. 99]

Claim now follows from the fact that the table has a constant number of entries.

Proof of Theorem 1.4. Let us use the notation established in Remark 3.4. Let \(g\) be an element of \(G\) which generates \(G\) modulo soc\(\left(G\right)\). We know that \(G\) embeds in the wreath product \(X\) wr \(K\), so \(g\) has the form \((y_1, \ldots, y_k)t\), where \(y_1, \ldots, y_k \in X\) and \(t \in K\) is a \(k\)-cycle in \(S_k\) that generates \(K\). Moreover, without loss of generality, we may assume that \(t\) is the \(k\)-cycle (12...\(k\)). Observe that conjugating \(t\) by \((1, 1, \ldots, 1, y)\) for any \(y \in X\) gives \((1, 1, \ldots, 1, y^{-1})\), and conjugating elements of \(T^n\) by \(t\) has the effect of “cycling” the coordinates. This implies that up to replacing \(G\) by a conjugate of \(G\) in \(X\) wr \(K\) we may assume that \(g = (1, 1, \ldots, 1, y)t\), where \(y\) is a generator of \(X\) modulo \(T\); such a generator \(y\) must exist since \(G/\text{soc}\left(G\right)\) is cyclic. Since \(X/T \leq \text{Out}(T)\), this implies that in \(G\) there are at most \(|\text{Out}(T)|\) isomorphism classes of groups \(G\) with given socle \(T^n\).

We claim that for fixed \(j \leq x\) the number of simple groups \(T\) with \(m(T) = j\) is at most \(cx^j\), where \(c\) is a constant. This is transparent in the case of sporadic and alternating groups. Groups of Lie type are parametrized by two numbers, \(q\) and \(n\), where \(q\) is the size of the base field and \(n\) is the dimension of the vector space. A simple inspection using [Damian and Lucchini 06, Table 1] shows that, if \(m(T) = j \leq x\), then \(q^{n-1} \leq x\), and for any \(n \leq 3\) we have exactly one choice for \(q\). This, in turn, gives a bounded number of choices for \(T\) since the table is finite. For \(n \geq 4\) we have \(q \leq x^4\) and at most \(d \log(x)\) choices for \(n\), so, using Lemma 3.2 and the prime number theorem, we find at most \(d \log(x)x^4/\log(x^4)\) choices for \(T\), where \(d\) is a constant, giving an upper bound of \(3dx^7\). This holds for every table entry, and the claim now follows from the fact that the table has a constant number of entries.

By Lemma 3.5(1) there exists a positive constant \(d\) such that, when setting \(g(x) = d \log(x)x^4\), we have
\[
\left| \left\{ G \in \mathcal{G} : n_{\sigma}(G) = n \right\} \right| \leq g(n) \leq g(x)
\]
for every \(n \leq x\), observing that there are \(o(1) \log(x)\) choices for \(m(T)\), and hence \(k\) is uniquely determined. We are going to use this function \(g(x)\) below when we apply Lemma 3.6.

Let \(A\) be the family of the alternating groups \(A_n\) with \(n \geq 5\), and let \(P\) be the set of simple groups isomorphic to \(PSL(2, q)\) with \(q\) a prime power. As in Lemma 3.5(2), let \(S\) be the family of the nonabelian simple groups not in \(P \cup A\). Observe that \(G\) is a disjoint union \(\bigcup_{i=1}^{\infty} G_i\) where
\[
G_1 := \left\{ G \in \mathcal{G} : k \geq 1, T \in S \right\},
G_2 := \left\{ G \in \mathcal{G} : k = 1, T \in A \right\},
G_3 := \left\{ G \in \mathcal{G} : k = 2, T \in A \right\},
G_4 := \left\{ G \in \mathcal{G} : k \geq 3, T \in A \right\},
G_5 := \left\{ G \in \mathcal{G} : k = 1, T \in P \right\},
G_6 := \left\{ G \in \mathcal{G} : k \geq 2, T \in \mathcal{P} \right\}.
\]

By Lemmas 3.1 and 3.7, \(n_\sigma(G) \leq \ell_{G}(\text{soc}(G)) \leq \sigma(G)\). Using Lemma 3.2 and Lemma 3.5(2), we see that
\[
\left| \left\{ n_{\sigma}(G) : G \in G_1, \sigma(G) \leq x \right\} \right| \leq c_1x^7/\log(x).
\]

Using that \(n_\sigma \leq \sigma(G)\), \(\sigma(S_n)\) for large \(n\) (see [Maróti 05, Theorem 3.1] and [Lucchini and Maróti 09, Theorem 9.2]) and that \(\text{Aut}(A_n) = S_n\) for large \(n\), we see that
| Group | \(m(G)\) | Covering number | Citation |
|-------|-------------|----------------|----------|
| \(M_{11}\) | 11 | 23 | [Holmes 06] |
| \(M_{12}\) | 12 | 208 | [Kappe et al. 16] |
| \(M_{13}\) | 22 | 771 | [Holmes 06] |
| \(M_{23}\) | 23 | 41,079 | [Holmes 06] |
| \(M_{24}\) | 24 | 3336 | [Epstein and Magliveras 16] |
| \(J_2\) | 100 | 1376 | [Holmes and Maróti 10] |
| \(J_4\) | 100 | 380 | [Holmes and Maróti 10] |

\[
|\{ G \in G_2 : \sigma(G) \leq x\}| \leq c_2 x^{3/2}.
\]

Since \(m(A_n) = n\) and \(\text{Aut}(A_n \times A_n) = \text{Aut}(A_n) \wr C_2\), we clearly have

\[
|\{ G \in G_3 : \sigma(G) \leq x\}| \leq c_3 x^{3/2},
\]

\[
|\{ n_6(G) : G \in G_4, \sigma(G) \leq x\}| \leq c_4 x^{3/2}.
\]

If \(G \in G_5\) and \(q\) is the size of the base field, then \(q \leq x\). If \(q\) is not a prime, then by the Prime Number Theorem there are at most \(o(1)x^{1/2}\) such \(q\), since, if \(q = p^f\) with \(p\) prime and \(f > 1\), then \(p^{f-1} < p^f = q\) gives at most \(o(1)x^{1/2}/\log(x)\) choices for \(p\) and at most \(o(1)\log(x)\) choices for \(f\). Using that for a \(p\) a large prime we have \(p^2/2 \leq \sigma(\text{PSL}(2,p)) = \sigma(\text{PGL}(2,p))\) (see [Bryce et al. 99]) and that \(\text{Aut}(\text{PSL}(2,p)) = \text{PGL}(2,p)\), together with the prime number theorem, we see that

\[
|\{ G \in G_5 : \sigma(G) \leq x\}| \leq c_5 x^{3/2}.
\]

Again using the prime number theorem and Lemma 3.2, we have that

\[
|\{ n_6(G) : G \in G_6, \sigma(G) \leq x\}| \leq c_6 x^{3/2}/\log(x).
\]

Combining the above with Lemma 3.6, where \(g(x) = d(x) \log(x) x^{1/2}\), we obtain that

\[
\begin{align*}
|\{ G \in G : \sigma(G) \leq x\}| & \leq c_1 dx_1^{7/2} + c_2 x^{3/2} + c_3 dx_1^{3/2} + c_4 dx_1^{5/2} \log(x) \\
& + c_5 x^{3/2} + c_6 dx_1^{3/2},
\end{align*}
\]

which is at most \(c x^{3/2}\) with \(c\) a constant, completing the proof. \(\square\)

**Proof of Corollary 1.6.** Since the covering numbers of solvable groups are of the form \(q^2 + q + 1\) with \(q\) a prime power (by Tomkinson’s result; see Proposition 6.1(ii)), we know that there are infinitely many natural numbers that are not the covering number of a solvable group. Let now \(G \in \mathcal{F}\) be nonsolvable. Up to replacing \(G\) with a suitable \(\sigma\)-elementary quotient \(G_0\) of \(G\) such that \(\sigma(G) = \sigma(G_0)\), we may assume that \(G\) is \(\sigma\)-elementary. The group \(G\) must have a unique minimal normal subgroup \(N\), where \(N\) is nonabelian; otherwise, if \(N\) and \(L\) are two minimal normal subgroups of \(G\), then \(N\) is isomorphic to a subgroup of \(G/L\), which is solvable, and \(L\) is isomorphic to a subgroup of \(G/N\), which is solvable as well, contradicting the fact that \(G\) is nonsolvable. Since \(G\) is \(\sigma\)-elementary and \(G/N\) is solvable, \(G/N\) is cyclic. Moreover, \(\Phi(G) = \{1\}\) by Lemma 2.2. This implies that \(G\) is a primitive monolithic group with \(G/\text{soc}(G)\) cyclic, and now the result follows by Theorem 1.4. \(\square\)

### 4. Nonabelian \(\sigma\)-elementary groups whose covering number is at most 129

In this section, we prove that any nonabelian \(\sigma\)-elementary group with covering number at most 129 is both primitive and monolithic. When combined with the calculations of Section 8 and the Supplementary Material, this allows us to determine precisely which integers less than or equal to 129 are covering numbers of finite groups. The main theorem in this section is an easy consequence of the following lemmas.

**Lemma 4.1.** Let \(G\) be a primitive monolithic group with nonabelian socle \(N\). Then there exists a set \(\{g_i N, \ldots, g_k N\}\) generating \(G/N\) with the property that

\[
\sigma(\langle g_i, N \rangle) \leq \sigma^*(G) + \omega(\{g_i N|G/N\})
\]

for every \(i \in \{1, \ldots, k\}\), where \(\{g_i N|G/N\}\) denotes the order of \(g_i N\) in \(G/N\) and \(\omega(m)\) denotes the number of distinct prime divisors of \(m\).

**Proof.** There exists a set \(\{g_1 N, \ldots, g_k N\}\) generating \(G/N\) with the property that \(\sigma^*(G) = \sigma_0(G)\), where \(\Omega = g_1 N \cup \cdots \cup g_k N\). In particular, for \(i \in \{1, \ldots, k\}\) we have \(\sigma_{g_i N}(G) \leq \sigma^*(G)\). If \(H\) is a proper supplement of \(N\) in \(G\), then \(H \cap \langle g_i, N \rangle\) is a proper supplement of \(N\) in \(\langle g_i, N \rangle\), and therefore \(g_i N\) is contained in a union of \(\sigma_{g_i N}(G)\) proper subgroups of \(\langle g_i, N \rangle\). By Lemma 2.7, in order to cover \(\langle g_i, N \rangle\) with proper subgroups it suffices to use a family of proper subgroups covering \(g_i N\) and the maximal subgroups of \(\langle g_i, N \rangle\) containing \(N\). This implies that

| \(q\) | \(q^2 + q + 1\) | Covering number? | \(\sigma\)-elementary groups |
|------|----------------|-----------------|---------------------------|
| 2    | 7              | No              | \(\emptyset\)            |
| 3    | 13             | Yes             | \(S_6\)                   |
| 4    | 21             | No              | \(\emptyset\)            |
| 5    | 31             | Yes             | \(A_5\) AGL(4, 2)        |
| 7    | 57             | Yes             | \(A_6\) wr 2             |
| 8    | 73             | Yes             | \(A_6 \times A_6 : 4\)   |
| 9    | 91             | No              | \(\emptyset\)            |
| 11   | 133            | ?               | ?                         |
\[ \sigma((g_i, N)) \leq \sigma_{g,N}(G) + \omega\left(|g_iN|_{G/N}\right) \leq \sigma^*(G) + \omega\left(|g_iN|_{G/N}\right), \]

concluding the proof. \hfill \Box

**Lemma 4.2.** Let \( n \) be a fixed positive integer. Let \( \mathcal{F} \) be the family of monolithic primitive groups \( X \) of primitivity degree at most \( n \), with nonabelian solv N, where \( X/N \) is either nonsolvable or cyclic, and where \( \sigma^*(X) \leq 2n + 1 \). If for all \( X \in \mathcal{F} \) we have \( \sigma(X) < 2\sigma^*(X) \), then every nonabelian \( \sigma \)-elementary group \( G \) with \( \sigma(G) \leq 2n + 1 \) is primitive and monolithic.

**Proof.** Let \( G \) be a nonabelian \( \sigma \)-elementary group, and let \( \text{soc}(G) = N_1 \times \cdots \times N_t \). By Lemma 2.2 we know that at most one of the \( N_i \)'s is abelian. So we may assume that \( N_i \) is nonabelian whenever \( i \geq 2 \). We need to show that \( t = 1 \), so assume for the purpose of contradiction that \( t \geq 2 \). Let \( X_i \) be the primitive monolithic group associated with \( N_i \) for all \( i = 1, \ldots, t \). By Lemma 2.5 we know that \( \sum_{i=1}^t \sigma^*(X_i) \leq \sigma(G) \leq 2n + 1 \) for all \( i \); in particular, \( \sigma^*(X_i) \leq 2n + 1 \) for all \( i \). We consider the two possible cases.

Assume first that \( N_1 \) is abelian. In this case, since the center of \( G \) is trivial by Lemma 2.3, \( \sigma(G) < 2|N_1| \) by Lemma 2.9, and, since \( \ell_{X_1}(N_1) = |N_1| \), we have

\[ \frac{1}{2} \sigma(G) + \ell_{X_1}(N_2) \leq \sum_{i=1}^t \sigma^*(X_i) \leq \sigma(G). \]

This means \( 2\ell_{X_1}(N_2) \leq \sigma(G) \leq 2n + 1 \), and therefore \( \ell_{X_1}(N_2) \leq n \). Since \( X_2 \) is a quotient of \( G \), we conclude that \( X_2/N_2 \) is either nonsolvable or cyclic by Lemma 2.8, implying \( \sigma(X_2) < 2\sigma^*(X_2) \) by hypothesis. Since \( X_2 \) is a quotient of \( G \), we have \( \sigma(G) \leq \sigma(X_2) < 2\sigma^*(X_2) \), and, by Lemma 2.6, we have \( \ell_{X_2}(N_2) \leq \sigma^*(X_2) \). Combining this with Lemma 2.9 yields

\[ \ell_{X_1}(N_1) + \ell_{X_2}(N_2) \leq |N_1| + \sigma^*(X_2) \leq \sigma(G) < \text{min}\left\{2|N_1|, 2\sigma^*(X_2)\right\}. \]

But \( |N_1| + \sigma^*(X_2) < 2|N_1| \) implies \( \sigma(X_2) < |N_1| \) and \( |N_1| + \sigma^*(X_2) < 2\sigma^*(X_2) \) implies \( |N_1| < \sigma^*(X_2) \), a contradiction.

We may thus assume \( N_1 \) is nonabelian. In this case, we may assume that

\[ \text{min}\{|\sigma^*(X_i) : i = 1, \ldots, n\} = \sigma^*(X_1). \]

Therefore,

\[ 2\ell_{X_1}(N_1) \leq 2\sigma^*(X_1) \leq \sum_{i=1}^t \sigma^*(X_i) \leq \sigma(G) \leq 2n + 1, \]

which implies \( \ell_{X_1}(N_1) \leq n \), and so, since \( X_1 \) is a quotient of \( G \), we have that \( X_1/N_1 \) is either nonsolvable or cyclic by Lemma 2.8, we have \( \sigma(X_1) < 2\sigma^*(X_1) \) by hypothesis. Hence

\[ t \cdot \sigma^*(X_1) \leq \sum_{i=1}^t \sigma^*(X_i) \leq \sigma(G) \leq \sigma(X_1) < 2\sigma^*(X_1), \]

which contradicts the fact that \( t \geq 2 \), completing the proof. \hfill \Box

**Remark 4.3.** If \( \sigma(X) < 2\sigma^*(X) \) for all primitive monolithic groups \( X \) with a nonabelian solc, then Lemma 4.2 implies that Conjecture 1.2 is true.

**Lemma 4.4.** Let \( X \) be a primitive monolithic group with nonabelian solc \( N \). If \( X/N \) is a cyclic \( p \)-group for some prime \( p \) then \( \sigma(X) \leq \sigma^*(X) + 1 < 2\sigma^*(X) \).

**Proof.** Since \( X/N \) is a cyclic \( p \)-group, it admits exactly one maximal subgroup. Therefore a union \( \Omega \) of cosets of \( N \) in \( X \) generates \( X \) if and only if it contains a coset \( xN \), where \( x \) does not belong to the unique maximal subgroup of \( X \) containing \( N \). It follows that there exists some \( x \) such that \( \sigma^*(X) = \sigma_{xN}(X) \). Observe that since \( X/N \) is a \( p \)-group, we may choose such an \( x \) of \( p \)-power order. Now we can cover \( xN \) with a family \( K \) consisting of \( \sigma^*(X) \) supplements of \( N \), which therefore cover all the cosets \( x^KN \) with \( k \) coprime to \( p \) by Lemma 2.7. What is left to cover is every coset \( x^kN \) for \( k \geq 1 \). Thus adding \( \langle N, x^k \rangle \neq X \), we conclude that \( \sigma(X) \leq \sigma^*(X) + 1. \)

Our main result in this section is now an easy consequence of the above lemmas.

**Theorem 4.5.** Let \( G \) be a nonabelian \( \sigma \)-elementary group with \( \sigma(G) \leq 129 \). Then \( G \) is primitive and monolithic with primitivity degree at most 129.

**Proof.** We show that \( G \) is primitive and monolithic. By Lemma 4.2, to do so it is enough to show that \( \sigma(X) < 2\sigma^*(X) \) whenever \( X \) is a primitive monolithic group of degree at most 64 satisfying each of the following three conditions: (1) \( X \) has nonabelian solc \( N \), (2) \( X/N \) is either nonsolvable or cyclic, and (3) \( \sigma^*(X) \leq 129 \). Let \( X \) be such a group. If \( X/N \) is a cyclic \( p \)-group for some prime \( p \), then Lemma 4.4 implies \( \sigma(X) < 2\sigma^*(X) \). Now assume \( X/N \) is not a cyclic \( p \)-group. A GAP check shows that the only possibility is \( X \cong \text{Aut}(PSL(2,27)) \), in which case \( X/N \cong C_9 \) and \( \ell_X(N) = 28 \). In this case, Lemma 4.1 implies that either \( \sigma(X) \leq \sigma^*(X) + 2 < 2\sigma^*(X) \), or, for one of the \( g_i \)'s in this lemma, \( \langle N, g_i \rangle \cong \text{PGL}(2,27) \), and so \( \sigma^*(X) \geq \sigma(\text{PGL}(2,27)) - 1 = 378 \) holds, a contradiction to \( \sigma^*(X) \leq 129 \). Thus \( \sigma(X) < 2\sigma^*(X) \). Lemma 2.6
implies that the smallest primitivity degree of $G$ is at most $\sigma(G)$. \qed

5. Computational methods

In this section, we outline the computational methods used to prove Theorem 1.1. By Tomkinson’s result (see Proposition 6.1), it suffices to consider nonsolvable $\sigma$-elementary groups, and by Theorem 4.5, any nonabelian $\sigma$-elementary group $G$ with $\sigma(G) \leq 129$ is primitive and monolithic with a primitivity degree of at most 129. Using GAP, we are able to list every nonsolvable primitive group with degree of primitivity at most 129. The covering number of many of these groups is known; see Section 6. Moreover, the covering number of affine general linear groups and affine special linear groups when $n \geq 3$ are determined in Section 7.

All remaining groups, i.e., those groups not explicitly discussed in Sections 6 and 7, are listed in Tables 2–6, along with a reference as to how the computation was completed for each group. In many cases, the group has a noncyclic solvable homomorphic image whose covering number is the same as the original group. In these cases, the homomorphic image is listed in the reference column. For many primitive groups of affine type—that is, those that have a unique elementary abelian minimal normal subgroup—a result due to Detomi and Lucchini (also proved independently by Jafarian Amiri [Jafarian Amiri 10]) can be used: if $G$ is such a primitive group with elementary abelian minimal normal subgroup $N$ and point stabilizer $H$ in the primitive action and $\sigma(H) < |N|$, then $\sigma(G) = \sigma(H)$; see Lemma 2.11.

The covering number of many other groups can be computed exactly using either linear programming methods or other computational techniques. The details are discussed in Sections 5.1 and 5.2, respectively. There are only a few groups whose covering number cannot be determined using these methods, and they are considered on an ad hoc basis in the Supplementary Material.

5.1. Linear programing methods

In [Kappe et al. 16], the authors created a program in GAP [The GAP Group 14] that takes as input a group $G$, a list $E$ of elements of $G$, a list $M$ of maximal subgroups of $G$, and the name of a file of type .lp to which output is written. This output file is read by the linear optimization software GUROBI [Gurobi Optimization 14], which then determines the least number of subgroups conjugate to one of the subgroups in $M$ needed to cover the elements conjugate to the elements of $E$. This function is referred to in the remainder of the paper as “Algorithm KNS,” and the GAP code for this program can be found in [Kappe et al. 16].

For a group of order approximately 500,000 or less, Algorithm KNS generally will return a .lp file within 24 h. The calculations done here were completed with a laptop that has a Core i7 processor and 16 GB of RAM. The optimization software GUROBI sometimes is able to determine the exact covering number within seconds; other times, the program runs out of memory, but is still able to provide good bounds. For instance, the previous bounds on the covering number of $J_2$ were $380 \leq \sigma(J_2) \leq 1220$, given in [Holmes and Maróti 10]. With the aid of Algorithm KNS and GUROBI, we have improved these bounds to $1063 \leq \sigma(J_2) \leq 1121$.

5.2. A verification method for minimal covers and a greedy algorithm

Maróti introduced the following technique for showing that a cover is minimal. Following [Maróti 05], if $\Pi \subseteq G$, we define $\sigma(\Pi)$ to be the least integer $m$ such that $\Pi$ is a subset of the set-theoretic union of $m$ subgroups of $G$; clearly, $\sigma(\Pi) \leq \sigma(G)$. A set $\mathcal{H} = \{H_1, ..., H_m\}$ of $m$ proper subgroups of $G$ is definitely unbeatable on $\Pi$ if both of the following conditions hold:

i. the elements of $\Pi$ are partitioned among the subgroups in $\mathcal{H}$, and

ii. for all subgroups $K \subseteq G$ that are not contained in $\mathcal{H}$, we have $|K \cap \Pi| < |H_i \cap \Pi|$ for each $i, 1 \leq i \leq m$.

If $\mathcal{H}$ is definitely unbeatable on $\Pi$, then $|\mathcal{H}| = \sigma(\Pi) \leq \sigma(G)$.

However, definite unbeatability is often too stringent a condition. With this in mind, a more complicated but more generally applicable condition was introduced in [Swartz 16]. The following lemma is a slight modification of that condition (in that the parameter $c(M)$ may equal 1 here) and is useful in cases when the minimal cover is not unique.

Lemma 5.1. Let $\Pi$ be a union of conjugacy classes of elements of $G$; let $I \subseteq I_G$, where $I_G$ is an index set for the conjugacy classes of maximal subgroups of $G$; and let $C = \bigcup_{i \in I} \mathcal{M}_i$ be a cover of $\Pi$ such that each $\mathcal{M}_i$ denotes a conjugacy class of maximal subgroups, the elements of $\Pi$ are partitioned among the subgroups in $C$, and each subgroup in $C$ contains elements of $\Pi$. For a maximal subgroup $M \notin C$, define
The collection $\mathcal{C}$ consists only of subgroups from classes $\mathcal{M}_i$, where $i \in I$, and we let $a_i$ be the number of subgroups from $\mathcal{M}_i$ in $\mathcal{C}$. Similarly, the collection $\mathcal{B}'$ consists only of subgroups from classes $\mathcal{M}_j$, where $j \notin I$, and we let $b_j$ be the number of subgroups from $\mathcal{M}_j$ in $\mathcal{B}'$. Note that, since $\mathcal{B}$ is a different cover, for some $j \notin I$, we have $b_j > 0$.

By removing $a_i$ subgroups in class $\mathcal{M}_i$ from $\mathcal{C}$, the new subgroups in $\mathcal{B}'$ must cover the elements of $\mathcal{C}$ that were in these subgroups. Hence, for all $i \in I$, if $M_i$ denotes a subgroup in class $\mathcal{M}_i$ for each $k$, $a_i |M_i \cap \Pi_i| \leq \sum_{j \in I} b_j |M_j \cap \Pi_i|$, which in turn implies that, for all $i \in I$, we have $a_i \leq \sum_{j \in I} b_j |M_j \cap \Pi_i| |M_i \cap \Pi_i|$. This means that $|\mathcal{C}'| = \sum_{i \in I} a_i \leq \sum_{i \in I} \sum_{j \in I} b_j |M_j \cap \Pi_i| = \sum_{j \in I} \sum_{i \in I} |M_i \cap \Pi_i| b_j \leq \sum_{j \in I} b_j |\mathcal{B}'|$, which shows that $|\mathcal{C}| = |\mathcal{C}'| + |\mathcal{C} \cap \mathcal{B}'| \leq |\mathcal{B}'| + |\mathcal{C} \cap \mathcal{B}| = |\mathcal{B}|$.

Hence, any other cover of the elements of $\mathcal{C}$ using only maximal subgroups contains at least as many subgroups as $\mathcal{C}$. Therefore, $\mathcal{C}$ is a minimal cover of the elements of $\mathcal{C}$.

**Algorithm. GKS CoveringNumberBounds**

**Input:** A finite group $G$.

**Output:** A triple $(\ell, u, c)$, where $\ell \leq \sigma(G) \leq u$ and $c$ is True if it is verified that $\sigma(G) = u$ and False otherwise.

1: $\text{max:}$ list of representatives of each class of maximal subgroups of $G$
2: $\text{eltM:}$ for each subgroup $M$ in max, a list of representatives of each conjugacy class of elements of $M$
3: $\text{conj:}$ list of nonidentity conjugacy classes of elements of $G$
4: $\text{u:} = 0$
5: $\text{minlist:}$ an empty list
6: $\text{cvalues:}$ list with every entry 0 of length the size of max
7: while $\text{conj}$ is nonempty do
8: $\text{elts:}$ for each class $x^G$ left in conj, the elements of eltM that are in $x^G$
9: $\text{ints:}$ for each class $x^G$ left in conj, a list of the sizes of the intersection of $x^G$ with each subgroup $M$ in max, created using the list elts by summing the sizes of the conjugacy classes in $M$ over the set of elements in eltM that are in $x^G$
10: $\text{mins:}$ for each class $x^G$ left in conj, the minimum number of subgroups needed to cover $x^G$, calculated by dividing the size of $x^G$ by the maximum intersection size from ints corresponding to $x^G$
11: $\text{best:}$ maximum of mins, which can be thought of as the minimum number of subgroups needed at this stage to get a cover
12: add best to minlist
13: $x_0^G :=$ the conjugacy class in conj that needed best subgroups to be covered
14: $M_0 :=$ a maximal subgroup from max from the class used to cover $x_0^G$ with best subgroups
15: $\text{valueupdate:}$ list with entry $|M_0 \cap x_0^G|/|M_0 \cap x_0^G|$ for each $M \in \text{max}$
16: $\text{value:}$ $\text{value} + \text{valueupdate}$ (addition is entrywise)
17: if best $\neq |G : M_0|$ then
18: $c :=$ False
19: $u := u + |G : M_0|$
In practice, there is often a union of conjugacy classes \( \Pi \) of elements of \( G \) and a minimal cover \( C \) of the elements of \( \Pi \) that satisfies the hypotheses of Lemma 5.1. We can design an algorithm exploiting this idea that works roughly as follows: each conjugacy class of elements and representatives for each conjugacy class of maximal subgroups are computed in GAP. Next, the conjugacy class \( x^G \) of elements that requires the most maximal subgroups to cover is determined. Greedily, we take as part of a cover all subgroups from a conjugacy class \( \mathcal{M} \) of maximal subgroups that most efficiently covers \( x^G \). All elements that are covered by the subgroups of \( \mathcal{M} \) are removed, and this process is repeated again and again until all elements are covered. Often, the cover produced this way is a minimal cover, and this can typically be verified by using Lemma 5.1. Even if the cover is not verifiably minimal, the function returns upper and lower bounds for \( \sigma(G) \). The steps of this procedure are listed in Algorithm GKS, which is written in pseudocode.

We remark that, while it would be “simpler” to calculate \( \text{ints} \) in Step 10 of Algorithm GKS by taking the intersection of class \( x^G \) with each subgroup in \( \text{max} \), for many groups the sizes of the conjugacy classes are quite large, and it is much faster for such groups to calculate the intersection sizes as described in the pseudocode. Algorithm GKS can also be altered to return additional information, such as the classes \( x_0^G \) and subgroups \( M_0 \) chosen in various iterations of the while loop, which is useful for ad hoc calculations like those in the Supplementary Material.

6. Known bounds on and values of covering numbers

We collect in this section a list of known results regarding the covering number of specific families of groups. We use the notation \( S_n \) to refer to the symmetric group of degree \( n \) and \( A_n \) to refer to the alternating group of degree \( n \). The first proposition combines the results of Cohn and Tomkinson and completely solves the problem of which integers are covering numbers of solvable groups. In the tables we also indicate the smallest primitivity degree \( m(G) \) of any given primitive group \( G \).

**Proposition 6.1.**

i. [Cohn 94, Corollary to Lemma 17] For every prime \( p \) and positive integer \( d \), there exists a group \( G \) with covering number \( p^d + 1 \).

ii. [Tomkinson 97, Theorem 2.2] Let \( G \) be a finite solvable group and let \( H/K \) be the smallest chief factor of \( G \) having more than one complement in \( G \). Then \( \sigma(G) = |H/K| + 1 \). In particular, the covering number of any (noncyclic) solvable group has the form \( p^d + 1 \), where \( p \) is a prime and \( d \) is a positive integer.

Table 7 summarizes what is currently known about covering numbers of symmetric groups.

Table 8 summarizes what is currently known about covering numbers of alternating groups.

Table 9 summarizes what is currently known about covering numbers of projective linear groups of dimension 2.

By [Lucido 03], if \( q = 2^{2m+1} \) for some \( m \in \mathbb{N} \), then \( \sigma(S_2(q)) = \frac{1}{2} q^2 (q^2 + 1) \).

Table 10 summarizes what is currently known in the literature regarding covering numbers of projective linear groups of dimension 2.

The following result was the main application of Lemma 2.11 in [Jafarian Amiri 10] and proves results about 2-dimensional affine general linear groups.

**Lemma 6.2** [Jafarian Amiri 10]. Let \( p > 3 \) be a prime. Then \( \sigma(\text{AGL}(2,p)) = p(p + 1)/2 + 1 \).

However, when combined with the results about \( \text{PSL}(2,q) \) and \( \text{PGL}(2,q) \) when \( q \) is not a prime (see Table 9), Lemma 2.11 can be used to prove the following stronger result.

**Lemma 6.3.** Let \( q \) be a prime power, \( q \geq 4 \). Then \( \sigma(\text{AGL}(2,q)) = \sigma(\text{ASL}(2,q)) = \sigma(\text{PSL}(2,q)) \), and consequently \( \text{AGL}(2,q) \) and \( \text{ASL}(2,q) \) are never \( \sigma \)-elementary when \( q \geq 4 \).
Finally, we have the following result about the covering numbers of two other primitive groups.

**Proposition 6.4** [Garonzi 13a]. For $A_5 \wr 2$ and $(A_5 \times A_5) : 4$ we have:

i. $\sigma(A_5 \wr 2) = 57$;

ii. $\sigma((A_5 \times A_5) : 4) = 126$, where $(A_5 \times A_5) : 4$ is the preimage of the normal cyclic subgroup of order 4 in $D_8$ via $\text{Aut}(A_5 \times A_5) \to \text{Out}(A_5 \times A_5) \cong D_8$.

### 7. Covering affine general linear groups and affine special linear groups with proper subgroups

This section is dedicated to proving Theorem 1.7, which can be interpreted as a generalization of Cohn’s result [Cohn 94] listed in Proposition 6.1.

**Proof of Theorem 1.7.** Since the proof is analogous for $\text{ASL}(n, q)$, we will only show the result in each case for $\text{AGL}(n, q)$.

First, if $n = 1$, then we have $\sigma(\text{AGL}(1, q)) = q + 1 = (q^2 - 1)/(q - 1)$. This follows from [Tomkinson 97, Lemma 2.1].

When $n \geq 3$, we consider the group $\text{AGL}(n, q)$. By Lemma 2.9,

$$\sigma(\text{AGL}(n, q)) \leq (q^{n+1} - 1)/(q - 1).$$

It remains to show that there is no smaller cover.

Let $\text{AGL}(n, q) = V \rtimes H$, where $V$ is the underlying vector space over $\text{GF}(q)$ (defined additively) and $H$, which is isomorphic to $\text{GL}(n, q)$, is the stabilizer of $0 \in V$. Define the cover $C = C_1 \cup C_2$ to be the union of two classes $C_1$ and $C_2$ of maximal subgroups. Let $C_1$ consist of all point stabilizers of $\text{AGL}(n, q)$ in its natural primitive action on $q^n$ points; that is, $C_1$ consists of all conjugates of $H$ in $\text{AGL}(n, q)$. In this case, $|C_1| = q^n$. Let $C_2$ consist of all subgroups isomorphic to $V \rtimes K$, where $V$ is the unique minimal normal subgroup of $\text{AGL}(n, q)$ of order $q^n$ and $K$ is the stabilizer of a 1-dimensional subspace. Since there are $(q^n - 1)/(q - 1)$ lines through the origin, there are $(q^n - 1)/(q - 1)$ such groups in $C_2$. This implies that $|C_2| = (q^{n+1} - 1)/(q - 1)$. By Lemma 2.9, the collection $C$ is a cover of the elements of $\text{AGL}(n, q)$.

We will show that no smaller collection of subgroups can cover the elements of $\text{AGL}(n, q)$. First, by a result of Kantor [Kantor 80], the only maximal subgroups of $\text{GL}(n, q)$ containing Singer cycles are field extension groups isomorphic to $\text{GL}(n/b, q^b)$, where $b > 1$ is a divisor of $n$. Moreover, the Singer cycles are partitioned among these subgroups. Since

$$q^{n(n-1)}(q-1)^n = \prod_{i=0}^{n-1} (q^n - q^{n-1}) \leq |\text{GL}(n, q)|$$

$$= \prod_{i=0}^{n-1} (q^n - q^i) < \prod_{i=0}^{n-1} q^i = q^{n^2},$$

we have

$$\sigma(\text{GL}(n, q)) \geq \frac{|\text{GL}(n, q)|}{|\text{GL}(n/b, q^b)|} \geq \frac{q^{n(n-1)}(q^i-1)}{(q^b)^{n2}}$$

when $n \geq 4$ and when $n = 3$ and $q = 2$, we have $\text{GL}(3, 2) \cong \text{PSL}(2, 7)$, and hence $\sigma(\text{GL}(3, 2)) = 31$ by Table 9. By Lemma 2.10, the groups in $C_1$ must appear in any minimal cover of the elements of $\text{AGL}(n, q)$.

In $V$, there is a natural bijection between 1-dimensional subspaces and hyperplanes, and so for any nonzero $v \in V$ we define $\phi(v)$ to be the hyperplane of $V$ such that $V = \langle v \rangle \oplus \phi(v)$. Let $v \in V, v \neq 0$, and, if $s$ is a Singer cycle on $\phi(v)$ that fixes both $0 \in \phi(v)$ and $v$, then define $g$ to be element of $\text{AGL}(n, q)$ that corresponds to $s$ followed by a translation by $v$. The element $g$ fixes no points of $V$. To see this, consider $w \in V$. Since $\phi(v)$ complements $\langle v \rangle$, there exist a unique $a \in \text{GF}(q)$ and $u \in \phi(v)$ such that $w = av + u$. If $w = w$, this implies that $(a + 1)v + u = av + u$, that is, we have $v = u - u' \in \phi(v)$, a contradiction. Hence $g$ is not contained in any group in $C_1$.

Moreover, $|g| = p \cdot (q^n - 1)/(q - 1)$, since translation by $v$ has order $p$ and the Singer cycle on $\phi(v)$ fixes $v$ and hence commutes with translation by $v$. For nonzero vectors $v_1, v_2 \in V$, let $g_1$ be an element that corresponds to a Singer cycle on the hyperplane $\phi(v_1)$ followed by translation by $v_1$, and let $g_2$ be an element that corresponds to a Singer cycle on the hyperplane $\phi(v_2)$ followed by translation by $v_2$. Then $g_1^p$ and $g_2^p$ are Singer cycles of hyperplanes, and both $g_1^p$ and $g_2^p$ are elements of $H$. By [Britnell et al. 08, Theorem 4.1(2)] (see also [Britnell et al. 11]), if $n \geq 4, (n, q) \neq (4, 2), (11, 2)$, and a maximal subgroup $M$ of $H$ contains $g_1^p$ and $g_2^p$, then both $g_1^p$ and $g_2^p$ leave the same 1-dimensional subspace (and hyperplane) fixed; that is, $\langle v_1 \rangle = \langle v_2 \rangle$. When $n = 3$, we can derive the same result using [Bray et al. 13, Tables 8.3–8.4] by considering a primitive prime divisor of $q^3 - 1$ when $q \neq 4$. Furthermore, when $(n, q) = (3, 4), (4, 2)$, the result follows by computation in GAP, and, when $(n, q) = (11, 2)$, the result follows by considering [Bray et al. 13, Tables 8.70–8.71]. We deduce from the above that, if $\langle v_1 \rangle \neq \langle v_2 \rangle$, then $(g_1^p, g_2^p) = H$. Since $g_1$ fixes no points of $V$ and $H$ is the stabilizer of $0 \in V$, it follows
that $g_1 \notin H$. Because $H$ is a maximal subgroup of $\text{AGL}(n, q)$ and $g_1 \notin H$, this implies that
\[
\text{AGL}(n, q) = \langle g_1, H \rangle = \langle g_1, g_2^p \rangle \leq \langle g_1, g_2 \rangle,
\]
and in this case $g_1$ and $g_2$ generate all of $\text{AGL}(n, q)$. Hence, if $g_1$ and $g_2$ do not stabilize the same $1$-dimensional subspace, they pairwise generate all of $\text{AGL}(n, q)$. Therefore, we need at least as many subgroups as there are $1$-dimensional subspaces to cover the elements of this type, i.e., we need at least $(q^n-1)/(q-1)$ subgroups in addition to those from $C_1$, and the result follows.

It is unclear which integers of the form $(q^n-1)/(q-1)$ are powers, are covering numbers. What is known so far is summarized in Table 11, and there does not appear to be any clear pattern. The smallest open case is when $q = 11$, and no group has been found yet with covering number $133 = 11^2 + 11 + 1$.

8. Tables and computational results

The purpose of this section is to provide a summary of the calculations of the covering numbers of the primitive monolithic groups with a degree of primitivity at most $129$. We remark that Theorem 4.5 follows from Theorem 4.3, Proposition 6.1, and Table 1. Table 1 contains the complete list of nonsolvable $\sigma$-elementary groups $G$ where $\sigma(G) \leq 129$, which summarizes the information about nonsolvable $\sigma$-elementary groups. Table 1 follows from the known results in Section 6 and the results of calculations which are listed in Tables 3–6.

Tables 1, 2, 3, and 4 list groups and their covering numbers, along with references. Excluded are groups whose covering number was determined previously. Specifically, we have excluded the groups $S_m, A_n, \text{PSL}(2, q), \text{PGL}(2, q), \text{PGU}(2, 8), \text{Sz}(q), M_{11}, M_{12}, M_{22}, M_2^3, M_2^4, \text{HS}, \text{AGL}(2, q), A_5 \wr 2, \text{and } (A_5 \times A_5) : 4$ since these are dealt with in Section 6. In the reference column, when a specific group $H$ is listed, it means that the group has $H$ as a homomorphic image and the same covering number as $H$. For instance, “$S_3$” means the group projects onto the symmetric group $S_3$ and has covering number 4, since $\sigma(S_3) = 4$. If it says “Algorithm KNS” in the reference column, this means that Algorithm KNS was used with representatives of all conjugacy classes of elements and representatives of all conjugacy classes of maximal subgroups as inputs to generate a .lp file that was then optimized using GUROBI [Gurobi Optimization 14]. Different groups in GAP can be given the same name; for instance, $(A_6 \times A_6).2^2$ means some extension of $A_6 \times A_6$ by a Klein 4-group. When it says “(all such groups)” in a table, we mean that all such groups listed by GAP with that name and degree of primitivity have the same covering number. In Table 5, there are two groups listed as $(A_6 \times A_6).4$, and they are distinguished by saying that one is “(#16 in the list)” and the other is “(#18 in the list).” This is referring to the position in the list of all primitive groups of degree 100 that is generated by GAP. With the exception of the groups labeled “PSU(4, 3).2,” for $1 \leq j \leq 3$, the labeling of groups such as “PSL(2, 121).2” (with a subscript after the name of the group) comes from GAP; on the other hand, the groups labeled “PSU(4, 3).2,” $1 \leq j \leq 3$, refer to groups $U_4(3).2$, in the ATLAS [Conway et al. 85]. Finally, “Proposition A.x” in the reference column refers to material in the Supplementary Material, which is available online.

The last of the main tables is Table 5, which lists the primitive monolithic groups $G$ such that $G$ has a degree of primitivity at most 129 and $\sigma(G)$ has not been determined exactly, as well as bounds on $\sigma(G)$ for each such group $G$.

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