An Abelian theorem with application to the conditional Gibbs principle

Zhansheng Cao

LSTA, Université Paris 6, France

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Abstract

Let $X_1, \ldots, X_n$ be $n$ independent unbounded real random variables which have common, roughly speaking, light-tailed type distribution. Denote by $S^n_1$ their sum and by $\pi^{a_n}$ the tilted density of $X_1$, where $a_n \to \infty$ as $n \to \infty$. An Abelian type theorem is given, which is used to approximate the first three centered moments of the distribution $\pi^{a_n}$. Further, we provide the Edgeworth expansion of $n$-convolution of the normalized tilted density under the setting of a triangular array of row-wise independent summands, which is then applied to obtain one local limit theorem conditioned on extreme deviation event ($S^n_1/n = a_n$) with $a_n \to \infty$.

Key words. Abelian theorem, Edgeworth expansion, extreme deviation, Gibbs principle

1 Introduction

It will be assumed that $P_X$, which is the distribution of $X_1$, has a density $p$ with respect to the Lebesgue measure on $\mathbb{R}$. The fact that $X_1$ has a light tail is captured in the hypothesis that $X_1$ has a moment generating function

$$
\Phi(t) := E \exp tX_1
$$

which is finite in a non void neighborhood $\mathcal{N}$ of 0. This fact is usually referred to as a Cramer type condition.

Defined on $\mathcal{N}$ are the following functions. The functions

$$
t \to m(t) := \frac{d}{dt} \log \Phi(t)
$$

$$
t \to s^2(t) := \frac{d}{dt} m(t)
$$

$$
t \to \mu_j(t) := \frac{d}{dt} s^2(t) \quad , \quad j = 3, 4
$$

1
are the expectation and the three first centered moments of the r.v. $X_t$ with density

$$\pi_t(x) := \frac{\exp tx}{\Phi(t)} p(x)$$

which is defined on $\mathbb{R}$ and which is the tilted density with parameter $t$. When $\Phi$ is steep, meaning that

$$\lim_{t \to t^+} m(t) = \infty$$

where $t^+ := \text{ess sup} N$ then $m$ parametrizes the convex hull of the support of $P_X$. We refer to Barndorff-Nielsen [2] for those properties. As a consequence of this fact, for all $a$ in the support of $P_X$, it will be convenient to define

$$\pi^a = \pi_t$$

where $a$ is the unique solution of the equation $m(t) = a$.

Let $X^n_1 := (X_1, ..., X_n)$ denote $n$ independent unbounded real valued random variables and $S^n_1 := X_1 + ... + X_n$ denote their sum.

A first contribution of this paper is the approximation of the first three moments of the tilted density, from which we obtain the Edgeworth expansion of $n$-convolution of the normalized tilted density. It is worthwhile to note that this expansion is under the setting of a triangular array of row-wise independent summands.

We now come to some remark on the Gibbs conditional principle in the standard above setting. A phrasing of this principle is: As $n$ tends to infinity the conditional distribution of $X_1$ given $(S^n_1/n = a)$ is $\Pi^a$, the distribution with density $\pi^a$ (See [10] and [9]).

Another contribution is that we obtain the local limit distribution of $(X_1, ..., X_k)$ for fixed integer $k$ conditioned on extreme deviations (ED) pertaining to $S^n_1$. By extreme deviation we mean that $S^n_1/n$ is supposed to take values which are going to infinity as $n$ increases (See also Borovkov and Mogul’skiĭ [4] [5], where the authors call this case “superlarge deviation”). It will be showed that for fixed $k$, any set of r.v.’s $X_{i_1}, ..., X_{i_k}$ are asymptotically independent given $S^n_1/n = a_n$ with $1 \leq i_1 \leq i_k \leq n$, which extends the events by Dembo and Zeitouni [9] to the extreme deviation case; see also Csiszár [8] for a similar result.

The paper is organized as follows. Notation and hypotheses are stated in Section 2 along with some necessary facts from asymptotic analysis in the context of light tailed densities. In Section 3 the approximations of the expectation and the two first centered moments of the tilted density are given. Section 4 states the Edgeworth expansion under extreme normalizing factors. Section 5 provides the Gibbs’ conditional limit theorem under extreme events.

The main tools to be used come from asymptotic analysis and local limit theorems, developed from [11] and [1]; we also have borrowed a number of arguments from [12]. A number of technical lemmas have been postponed to Section 6.

## 2 Notation and hypotheses

In this paper, we consider the uniformly bounded density function $p(x)$

$$p(x) = c \exp \left( - (g(x) - q(x)) \right) \quad x \in \mathbb{R}_+,$$

(2.1)
where $c$ is some positive normalized constant. Define $h(x) := g'(x)$. We assume that there exists some positive constant $\vartheta$, for large $x$, it holds

$$\sup_{|v-x|<\vartheta x} |g(v)| \leq \frac{1}{x^{1/2}h(x)} \quad (2.2)$$

The function $g$ is positive and satisfies

$$\frac{g(x)}{x} \to \infty, \quad x \to \infty. \quad (2.3)$$

Not all positive $g$’s satisfying (2.3) are adapted to our purpose. Regular functions $g$ are defined as follows. We define firstly a subclass $R_0$ of the family of slowly varying functions. A function $l$ belongs to $R_0$ if it can be represented as

$$l(x) = \exp \left( \int_{1}^{x} \frac{\epsilon(u)}{u} du \right), \quad x \geq 1, \quad (2.4)$$

where $\epsilon(x)$ is twice differentiable and $\epsilon(x) \to 0$ as $x \to \infty$.

We follow the line of Juszczak and Nagaev [12] to describe the assumed regularity conditions of $h$.

**Class $R_\beta$** : $h(x) \in R_\beta$, if, with $\beta > 0$ and $x$ large enough, $h(x)$ can be represented as

$$h(x) = x^\beta l(x),$$

where $l(x) \in R_0$ and in (2.4), $\epsilon(x)$ satisfies

$$\limsup_{x \to \infty} x|\epsilon'(x)| < \infty, \quad \limsup_{x \to \infty} x^2|\epsilon''(x)| < \infty. \quad (2.5)$$

**Class $R_\infty$** : Further, $l \in \tilde{R}_0$, if, in (2.4), $l(x) \to \infty$ as $x \to \infty$ and

$$\lim_{x \to \infty} \frac{x\epsilon'(x)}{\epsilon(x)} = 0, \quad \lim_{x \to \infty} \frac{x^2\epsilon''(x)}{\epsilon(x)} = 0, \quad (2.6)$$

and for some $\eta \in (0, 1/8)$

$$\liminf_{x \to \infty} x^{3\eta}\epsilon(x) > 0. \quad (2.7)$$

We say that $h \in R_\infty$ if $h$ is increasing and strictly monotone and its inverse function $\psi$ defined through

$$\psi(u) := h^- (u) := \inf \{ x : h(x) \geq u \} \quad (2.8)$$

belongs to $\tilde{R}_0$.

Denote $\mathfrak{R} := R_\beta \cup R_\infty$. In fact, $\mathfrak{R}$ covers one large class of functions, although, $R_\beta$ and $R_\infty$ are only subsets of Regularly varying and Rapidly varying functions, respectively.
Remark 2.1. The role of (2.4) is to make \( h(x) \) smooth enough. Under (2.4) the third order derivative of \( h(x) \) exists, which is necessary in order to use a Laplace method for the asymptotic evaluation of the moment generating function \( \Phi(t) \) as \( t \to \infty \), where

\[
\Phi(t) = \int_0^\infty e^{tx}p(x)dx = c \int_0^\infty \exp \left( K(x,t) + q(x) \right) dx, \quad t \in (0, \infty)
\]

in which

\[
K(x,t) = tx - g(x).
\]

If \( h \in \mathbb{R} \), \( K(x,t) \) is concave with respect to \( x \) and takes its maximum at \( \hat{x} = h^{+}(t) \). The evaluation of \( \Phi(t) \) for large \( t \) follows from an expansion of \( K(x,t) \) in a neighborhood of \( \hat{x} \); this is Laplace’s method. This expansion yields

\[
K(x,t) = K(\hat{x},t) - \frac{1}{2} h'(\hat{x})(x - \hat{x})^2 - \frac{1}{6} h''(\hat{x})(x - \hat{x})^3 + \varepsilon(x,t), \quad (2.9)
\]

where \( \varepsilon(x,t) \) is some error term. Conditions (2.6), (2.7) and (2.5) guarantee that \( \varepsilon(x,t) \) goes to 0 when \( t \) tends to \( \infty \) and \( x \) belongs to some neighborhood of \( \hat{x} \).

Example 2.1. Weibull Density. Let \( p \) be a Weibull density with shape parameter \( k > 1 \) and scale parameter 1, namely

\[
p(x) = kx^{k-1}\exp(-x^k), \quad x \geq 0
\]

\[
= k \exp \left( -(x^k - (k - 1) \log x) \right).
\]

Take \( g(x) = x^k - (k - 1) \log x \) and \( q(x) = 0 \). Then it holds

\[
h(x) = kx^{k-1} - \frac{k - 1}{x} = x^{k-1}(k - \frac{k - 1}{x^k}).
\]

Set \( l(x) = k - (k - 1)/x^k, x \geq 1 \), then (2.4) holds, namely,

\[
l(x) = \exp \left( \int_1^x \frac{\epsilon(u)}{u} du \right), \quad x \geq 1,
\]

with

\[
\epsilon(x) = \frac{k(k - 1)}{kx^k - (k - 1)}.
\]

The function \( \epsilon \) is twice differentiable and goes to 0 as \( x \to \infty \). Additionally, \( \epsilon \) satisfies condition (2.6). Hence we have shown that \( h \in \mathbb{R}_{k-1} \).

Example 2.2. A rapidly varying density. Define \( p \) through

\[
p(x) = c \exp(-e^{x-1}), \quad x \geq 0.
\]

Then \( g(x) = h(x) = e^{x-1} \) and \( q(x) = 0 \) for all non negative \( x \). We show that \( h \in \mathbb{R}_\infty \). It holds \( \psi(x) = \log x + 1 \). Since \( h(x) \) is increasing and monotone, it remains to show that \( \psi(x) \in \mathbb{R}_0 \). When \( x \geq 1 \), \( \psi(x) \) admits the representation of (2.4) with \( \epsilon(x) = 1/\log x + 1 \). Also conditions (2.6) and (2.7) are satisfied. Thus \( h \in \mathbb{R}_\infty \).
Throughout the paper we use the following notation. When a r.v. $X$ has density $p$ we write $p(X = x)$ instead of $p(x)$. This notation is useful when changing measures. For example $\pi^a(X = x)$ is the density at point $x$ for the variable $X$ generated under $\pi^a$, while $p(X = x)$ states for $X$ generated under $p$. This avoids constant changes of notation.

3 An Abelian-type theorem

We inherit of the definition of the tilted density $\pi^a$ defined in Section 1, and of the corresponding definitions of the functions $m$, $s^2$ and $\mu_3$. Because of (2.1) and the various conditions on $g$ those functions are defined as $t \to \infty$. The following Theorem is basic for the proof of the remaining results.

**Theorem 3.1.** Let $p(x)$ be defined as in (2.1) and $h(x) \in \mathcal{R}$. Denote by

$$m(t) = \frac{d}{dt} \log \Phi(t), \quad s^2(t) = \frac{d}{dt} m(t), \quad \mu_3(t) = \frac{d^3}{dt^3} \log \Phi(t),$$

then with $\psi$ defined as in (2.8) it holds as $t \to \infty$

$$m(t) \sim \psi(t), \quad s^2(t) \sim \psi'(t), \quad \mu_3(t) \sim \frac{M_6 - 9}{6} \psi''(t),$$

where $M_6$ is the sixth order moment of standard normal distribution.

**Proof.** The proof of this result relies on a series of Lemmas. Lemmas (6.2), (6.3), (6.4) and (6.5) are used in the proof. Lemma (6.1) is instrumental for Lemma (6.5). The proof of Theorem 3.1 and these Lemmas are postponed to Section 6.1.

**Remark 3.1.** As a by-product of Theorem 3.1, we obtained the following Abel type result (see (6.42)):

$$\Phi(t) = c \sqrt{2\pi} \sigma e^{K(\hat{x},t)} (1 + o(1)),$$

where $K(\hat{x},t)$ is defined as in (2.9) and $\sigma$ is defined in Section 6.1. It is easily verified that this result is in accordance with Theorem 4.12.11 of [1], Theorem 3 of [3] and Theorem 4.2 of [12].

**Corollary 3.1.** Let $p(x)$ be defined as in (2.1) and $h(x) \in \mathcal{R}$. Then it holds as $t \to \infty$

$$\frac{\mu_3(t)}{s^3(t)} \to 0. \quad (3.1)$$

**Proof.** Its proof relies on Theorem 3.1 and is also put in Section 6.1.
4 Edgeworth expansion under extreme normalizing factors

With $\pi^{an}$ defined through
\[ \pi^{an}(x) = \frac{e^{tx}p(x)}{\Phi(t)}, \] (4.1)
and $t$ determined by $m(t) = a_n$, define the normalized density of $\pi^{an}$ by
\[ \bar{\pi}^{an}(x) = s\pi^{an}(sx + a_n), \]
where $s$ is defined in Section 1 (notice that it depends on $a_n$ here). Denote the $n$-convolution of $\bar{\pi}^{an}(x)$ by $\bar{\pi}^{an}_n(x)$, and denote by $\rho_n$ the normalized density of $n$-convolution $\bar{\pi}^{an}_n(x)$,
\[ \rho_n(x) := \sqrt{n}\bar{\pi}^{an}_n(\sqrt{n}x). \]
The following result extends the local Edgeworth expansion of the distribution of normalized sums of i.i.d. r.v.'s to the present context, where the summands are generated under the density $\bar{\pi}^{an}$. Therefore the setting is that of a triangular array of row-wise independent summands; the fact that $a_n \to \infty$ makes the situation unusual. We mainly adapt Feller’s proof (Chapter 16, Theorem 2 [11]).

**Theorem 4.1.** With the above notation, uniformly upon $x$ it holds
\[ \rho_n(x) = \phi(x)\left(1 + \frac{\mu_3}{6\sqrt{3}s^3}(x^3 - 3x)\right) + o\left(\frac{1}{\sqrt{n}}\right). \]
where $\phi(x)$ is standard normal density.

**Proof.** The proof of this Theorem is postponed to Section 6.2. \hfill \Box

5 Gibbs’ conditional principles under extreme events

We now explore Gibbs conditional principles under extreme events. The result obtained is a pointwise approximation of the conditional density $p_{an}(y_1^k)$ on $\mathbb{R}^k$ for fixed $k$.

Fix $y_1^k := (y_1, ..., y_k)$ in $\mathbb{R}^k$ and define $s^j_i := y_i + \ldots + y_j$ for $1 \leq i < j \leq k$. Define $t$ through $m(t) = a_n$, similarly, define $t_i$ through
\[ m(t_i) := \frac{na_n - s^j_i}{n - i}. \] (5.1)
For the sake of brevity, we write $m_i$ instead of $m(t_i)$, and define $s^2_i := s^2(t_i)$. Consider the following condition
\[ \lim_{n \to \infty} \frac{\psi(t)^2}{\sqrt{n}v'(t)} = 0, \] (5.2)
which can be seen as a growth condition on $a_n$, avoiding too large increases of this sequence.

For $0 \leq i \leq k - 1$, define $z_i$ through
\[ z_i = \frac{m_i - y_i + 1}{s_i\sqrt{n - i - 1}}. \]
Remark 5.1. Formula \[\text{(5.2)}\] states the precise behaviour of the sequence \(a_n\) which defines the present extended Gibbs principle. In the case when the common density \(p(x)\) is Weibull with shape parameter \(k\), using Theorem \[\text{(3.1)}\] we obtain \(\psi(t) \sim m(t) = a_n\) and \(\psi'(t) \sim a_n^{-k}\). Replace \(\psi(t)\) and \(\psi'(t)\) in \[\text{(5.2)}\] by these two terms, we have

\[
\lim_{n \to \infty} \frac{a_n^k}{\sqrt{n}} = 0.
\]

This rate controls the growth of \(a_n\) to infinity.

Lemma 5.1. Assume that \(p(x)\) satisfies \[\text{(2.1)}\] and \(h(x) \in \mathcal{R}\). Let \(t_i\) be defined in \[\text{(5.1)}\]. Assume that \(a_n \to \infty\) as \(n \to \infty\) and that \[\text{(5.2)}\] holds. Then as \(n \to \infty\)

\[
\lim_{n \to \infty} \sup_{0 \leq i \leq k-1} z_i = 0, \quad \text{and} \quad \lim_{n \to \infty} \sup_{0 \leq i \leq k-1} z_i^2 = o\left(\frac{1}{\sqrt{n}}\right).
\]

Proof. The proof of this Lemma is postponed in Section 6.3.

Theorem 5.1. With the same notation and hypotheses as in Lemma \[\text{(5.7)}\] it holds

\[
p_{a_n}(y^k_t) = p(X^k_t = y^k_t | S^n_t = n a_n) = g_m(y^k_t) \left(1 + o\left(\frac{1}{\sqrt{n}}\right)\right),
\]

with

\[
g_m(y^k_t) = \prod_{i=0}^{k-1} \pi^{m_i}(X_{i+1} = y_{i+1}).
\]

Proof. Using Bayes formula,

\[
p_{a_n}(y^k_t) := p(X^k_t = y^k_t | S^n_t = n a_n)
\]

\[
= p(X_1 = y_1 | S^n_1 = n a_n) \prod_{i=1}^{k-1} p(X_{i+1} = y_{i+1} | X_i = y_i, S^n_i = n a_n)
\]

\[
= \prod_{i=0}^{k-1} p(X_{i+1} = y_{i+1} | S^n_{i+1} = n a_n - s_i^1).
\]

We make use of the following invariance property: for all \(y^k_t\) and all \(\alpha > 0\)

\[
p(X_{i+1} = y_{i+1} | X^n_i = y^n_i, S^n_i = n a_n) = \pi^\alpha(X_{i+1} = y_{i+1} | X^n_i = y^n_i, S^n_i = n a_n)
\]

where on the LHS, the r.v’s \(X^n_i\) are sampled i.i.d. under \(p\) and on the RHS, sampled i.i.d. under \(\pi^\alpha\). It thus holds

\[
p(X_{i+1} = y_{i+1} | S^n_{i+1} = n a_n - S_1^1) = \pi^{m_i}(X_{i+1} = y_{i+1} | S^n_{i+1} = n a_n - s_i^1)
\]

\[
= \pi^{m_i}(X_{i+1} = y_{i+1}) \frac{\pi^{m_i}(S^n_{i+2} = n a_n - s_i^{i+1})}{\pi^{m_i}(S^n_{i+1} = n a_n - s_i^1)}
\]

\[
= \frac{\sqrt{n-i}}{\sqrt{n-i-1}} \pi^{m_i}(X_{i+1} = y_{i+1}) \frac{\pi_{n-i-1}^{m_i}(y_{i+1})}{\pi_{n-i}(0)}, \tag{5.4}
\]

7
where \( \widetilde{\pi}_{n-i-1} \) is the normalized density of \( S_{i+2}^n \) under i.i.d. sampling with the density \( \pi_{m_i} \); correspondingly, \( \widetilde{\pi}_{n-i} \) is the normalized density of \( S_{i+1}^n \) under the same sampling. Note that a r.v. with density \( \pi_{m_i} \) has expectation \( m_i \) and variance \( s_i^2 \).

Write \( z_i = \frac{m_i - y_{i+1}}{\sqrt{n} - 1} \), and perform a third-order Edgeworth expansion of \( \widetilde{\pi}_{n-i-1}(z_i) \), using Theorem 4. It follows

\[
\widetilde{\pi}_{n-i-1}(z_i) = \phi(z_i) \left( 1 + \frac{\mu_3^i}{6s_i^3\sqrt{n}-1}(z_i^3 - 3z_i) \right) + o\left( \frac{1}{\sqrt{n}} \right), \tag{5.5}
\]

The approximation of \( \widetilde{\pi}_{n-i}(0) \) is obtained from (5.3)

\[
\widetilde{\pi}_{n-i}(0) = \phi(0) \left( 1 + o\left( \frac{1}{\sqrt{n}} \right) \right). \tag{5.6}
\]

Put (5.5) and (5.6) into (5.4) to obtain

\[
p(X_{i+1} = y_{i+1} | S_{i+1}^n = na_n - S_1^n) = \frac{\sqrt{n-i}}{\sqrt{n-i-1}} \pi_{m_i}^n(X_{i+1} = y_{i+1}) \frac{\phi(z_i)}{\phi(0)} \left[ 1 + \frac{\mu_3^i}{6s_i^3\sqrt{n}-1}(z_i^3 - 3z_i) + o\left( \frac{1}{\sqrt{n}} \right) \right]
= \frac{\sqrt{2\pi(n-i)}}{\sqrt{n-i-1}} \pi_{m_i}^n(X_{i+1} = y_{i+1}) \phi(z_i) \left( 1 + R_n + o(1/\sqrt{n}) \right), \tag{5.7}
\]

where

\[
R_n = \frac{\mu_3^i}{6s_i^3\sqrt{n}-1}(z_i^3 - 3z_i).
\]

Under condition (5.2), using Lemma 5.1 it holds \( z_i \to 0 \) as \( a_n \to \infty \), and under Corollary (3.1), \( \mu_3^i/s_i^3 \to 0 \). This yields

\[
R_n = o(1/\sqrt{n}),
\]

which, combined with (5.7), gives

\[
p(X_{i+1} = y_{i+1} | s_{i+1}^n = na_n - S_1^n) = \frac{\sqrt{n-i}}{\sqrt{n-i-1}} \pi_{m_i}^n(X_{i+1} = y_{i+1}) (1 + o(1/\sqrt{n}))
= \frac{\sqrt{n-i}}{\sqrt{n-i-1}} \pi_{m_i}^n(X_{i+1} = y_{i+1}) (1 - z_i^2/2 + o(z_i^2)) (1 + o(1/\sqrt{n})),
\]

where we use a Taylor expansion in the second equality. Using once more Lemma 5.1 under conditions (5.2), we have as \( a_n \to \infty \)

\[
z_i^2 = o(1/\sqrt{n}),
\]

hence we get

\[
p(X_{i+1} = y_{i+1} | S_{i+1}^n = na_n - S_1^n) = \frac{\sqrt{n-i}}{\sqrt{n-i-1}} \pi_{m_i}^n(X_{i+1} = y_{i+1}) (1 + o(1/\sqrt{n})),
\]

8
which together with (5.3) yields

\[ p(X_k^k = y_n^k | S_n^k = na_n) = \prod_{i=0}^{k-1} \left( \frac{\sqrt{n - i}}{\sqrt{n - i - 1}} \pi_m(X_{i+1} = y_{i+1}) (1 + o(1/\sqrt{n})) \right) \]

\[ = \prod_{i=0}^{k-1} \left( \pi_m(X_{i+1} = y_{i+1}) \right) \prod_{i=0}^{k-1} \left( \frac{\sqrt{n - i}}{\sqrt{n - i - 1}} \right) \prod_{i=0}^{k-1} \left( 1 + o\left( \frac{1}{\sqrt{n}} \right) \right) \]

\[ = \left( 1 + o\left( \frac{1}{\sqrt{n}} \right) \right) \prod_{i=0}^{k-1} \left( \pi_m(X_{i+1} = y_{i+1}) \right), \]

which completes the proof.

In the present case, namely for fixed k, an equivalent statement is (its proof is similar to the proof of Theorem 5.1, we refer to Broniatowski and Cao [6] for more details.)

**Theorem 5.2.** Under the same notation and hypotheses as in the previous Theorem, it holds

\[ p_{an}(y_1^k) = p(X_k^k = y_n^k | S_n^k = na_n) = g_{an}(y_1^k) \left( 1 + o\left( \frac{1}{\sqrt{n}} \right) \right), \]

with

\[ g_{an}(y_1^k) = \prod_{i=1}^{k} \left( \pi_{am}(X_i = y_i) \right). \]

**Remark 5.2.** The above result shows that asymptotically the point condition \((S_n^k = na_n)\) leaves blocks of \(k\) of the \(X_i\)'s independent. Obviously this property does not hold for large values of \(k\), close to \(n\). A similar statement holds in the LDP range, conditioning either on \((S_n^k = na)\) (see Diaconis and Friedman 1988)), or on \((S_n^k \geq na)\); see Csiszar 1984 for a general statement on asymptotic conditional independence.

### 6 Proofs

#### 6.1 Proofs of Theorem 3.1 and Corollary 3.1

For density functions \(p(x)\) defined in (2.1) satisfying also \(h(x) \in \mathcal{R}\), denote by \(\psi(x)\) the reciprocal function of \(h(x)\) and \(\sigma^2(v) = (h'(v))^{-1}, v \in \mathbb{R}_+\). For brevity, we write \(\hat{x}, \sigma, l\) instead of \(\hat{x}(t), \sigma(t), l(t)\).

When \(t\) is given, \(K(x, t)\) attain its maximum at \(\hat{x} = \psi(t)\). The fourth order Taylor expansion of \(K(x, t)\) on \(x \in [\hat{x} - \sigma l, \hat{x} + \sigma l]\) yields

\[ K(x, t) = K(\hat{x}, t) - \frac{1}{2} h'(\hat{x}) (x - \hat{x})^2 - \frac{1}{6} h''(\hat{x}) (x - \hat{x})^3 + \varepsilon(x, t), \quad (6.1) \]
with some $\theta \in (0, 1)$

$$\varepsilon(x, t) = -\frac{1}{24} h'''(\hat{x} + \theta(x - \hat{x}))(x - \hat{x})^4. \quad (6.2)$$

For proving Theorem 3.1 and Corollary 3.1 we state firstly the following Lemmas.

**Lemma 6.1.** For $p(x)$ in $(2, 1)$, $h(x) \in \mathfrak{R}$, it holds when $t \to \infty$,

$$\frac{|\log \sigma(\psi(t))|}{\int_1^t \psi(u)du} \to 0. \quad (6.3)$$

**Proof.** If $h(x) \in R_\beta$, by Theorem (1.5.12) of [1], there exists some slowly varying function such that it holds $\psi(x) \sim x^{1/\beta} l_1(x)$. Hence as $t \to \infty$ (see [11], Chapter 8)

$$\int_1^t \psi(u)du \sim t^{1+\frac{1}{\beta}} l_1(t). \quad (6.4)$$

On the other hand, $h'(x) = x^{\beta-1} l(x)(\beta + \epsilon(x))$, thus we have as $x \to \infty$

$$|\log \sigma(x)| = |\log (h'(x))^{-\frac{1}{2}}| = \left| \frac{1}{2} ((\beta - 1) \log x + \log l(x) + \log(\beta + \epsilon(x))) \right| \leq \frac{1}{2} (\beta + 1) \log x,$n

set $x = \psi(t)$, then when $t \to \infty$, it holds $x < 2t^{1/\beta} l_1(t) < t^{1/\beta + 1}$, hence we get

$$|\log \sigma(\psi(t))| < \frac{(\beta + 1)^2}{2 \beta} \log t,$$n

which, together with (6.4), yields (6.3).

If $h(x) \in R_\infty$, then by definition $\psi(x) \in \hat{R}_0$ is slowly varying as $x \to \infty$, and as $t \to \infty$ (see [11], Chapter 8)

$$\int_1^t \psi(u)du \sim t \psi(t). \quad (6.5)$$

Additionally, we have $h'(x) = 1/\psi'(t)$ with $x = \psi(t)$, it follows

$$|\log \sigma(x)| = |\log (h'(x))^{-\frac{1}{2}}| = \frac{1}{2} |\log \psi'(t)|.$$

Since $\psi(t) \in \hat{R}_0$, it holds

$$|\log \sigma(\psi(t))| = \frac{1}{2} |\log \psi'(t)| = \frac{1}{2} |\log \left( \psi(t) \frac{\epsilon(t)}{t} \right)| \leq \log t + \frac{1}{2} |\log \epsilon(t)| \leq 2 \log t, \quad (6.6)$$

where last inequality follows from (2.7), (6.5) and (6.6) imply (6.3). This completes the proof. \qed
Lemma 6.2. For \( p(x) \) in \((2.1)\), \( h \in \mathcal{R} \), then for any varying slowly function \( l(t) \to \infty \) as \( t \to \infty \), it holds

\[
\sup_{|x| \leq \sigma l} h''(\hat{x} + x)\sigma^4 t^4 \to 0 \quad \text{as} \quad t \to \infty.
\] (6.7)

Proof. Case 1: \( h \in R_\beta \). We have \( h(x) = x^\beta l_0(x) \), \( l_0(x) \in R_0 \), \( \beta > 0 \). Then

\[
h''(x) = \beta(\beta - 1)x^{\beta - 2}l_0(x) + 2\beta x^{\beta - 1}l'_0(x) + x^{\beta}l''_0(x).
\] (6.8)

and

\[
h''(x) = \beta(\beta - 1)(\beta - 2)x^{\beta - 3}l_0(x) + 3\beta(\beta - 1)x^{\beta - 2}l'_0(x) + 3\beta x^{\beta - 1}l''_0(x) + x^{\beta}l''_0(x).
\] (6.9)

Since \( l_0 \in R_0 \), it is easy to obtain

\[
l'_0(x) = \frac{l_0(x)}{x} \epsilon(x), \quad l''_0(x) = \frac{l_0(x)}{x^2} \left( \epsilon(x) + x\epsilon'(x) - \epsilon(x) \right),
\] (6.10)

and

\[
l''_0(x) = \frac{l_0(x)}{x^3} \left( \epsilon(x) + 3x\epsilon'(x) - 3\epsilon(x) - 2x\epsilon'(x) + 2\epsilon(x) + x^2 \epsilon''(x) \right).
\]

Under condition \((2.3)\), there exists some positive constant \( Q \) such that it holds

\[
|l'_0(x)| \leq Q \frac{l_0(x)}{x^2}, \quad |l''_0(x)| \leq Q \frac{l_0(x)}{x^3},
\]

which, together with \((6.9)\), yields with some positive constant \( Q_1 \)

\[
|h''(x)| \leq Q_1 \frac{h(x)}{x^3}.
\] (6.11)

By definition, we have \( \sigma^2(x) = 1/h'(x) = x/(h(x)(\beta + \epsilon(x))) \), thus it follows

\[
\sigma^2 = \sigma^2(\hat{x}) = \frac{\hat{x}}{h(\hat{x})(\beta + \epsilon(\hat{x}))} = \frac{\psi(t)}{l(\beta + \epsilon(\psi(t)))} = \frac{\psi(t)}{l t}(1 + o(1)),
\] (6.12)

this implies \( \sigma l = o(\psi(t)) = o(\hat{x}) \). Thus we get with \((6.11)\)

\[
\sup_{|x| \leq \sigma l} |h''(\hat{x} + x)| \leq \sup_{|x| \leq \sigma l} Q_1 \frac{h(\hat{x} + x)}{(\hat{x} + x)^3} \leq Q_2 \frac{t^4}{\psi^3(t)},
\] (6.13)

where \( Q_2 \) is some positive constant. Combined with \((6.12)\), we obtain

\[
\sup_{|x| \leq \sigma l} |h''(\hat{x} + x)|\sigma^4 t^4 \leq Q_2 \frac{t^4}{\psi^3(t)} = \frac{Q_2 l^4}{\beta^2 t \psi(t)} \to 0,
\]
as sought.
Case 2: $h \in R_\infty$. Since $\hat{x} = \psi(t)$, we have $h(\hat{x}) = t$. Thus it holds
\[ h'(\hat{x}) = \frac{1}{\psi'(t)} \quad \text{and} \quad h''(\hat{x}) = -\frac{\psi''(t)}{(\psi'(t))^3}, \quad (6.14) \]

further we get
\[ h'''(\hat{x}) = -\frac{\psi''(t)\psi'(t) - 3(\psi''(t))^2}{(\psi'(t))^5}. \quad (6.15) \]

Notice if $h(\hat{x}) \in R_\infty$, then $\psi(t) \in \tilde{R}_0$. Therefore we obtain
\[ \psi'(t) = \frac{\psi(t)}{t} \epsilon(t), \quad (6.16) \]

and
\[ \psi''(t) = -\frac{\psi(t)}{t^2} \epsilon(t) \left(1 - \epsilon(t) - \frac{t\epsilon'(t)}{\epsilon(t)}\right) = -\frac{\psi(t)}{t^2} \epsilon(t) (1 + o(1)) \quad \text{as} \quad t \to \infty, \quad (6.17) \]

where last equality holds from (2.6). Using (2.6) once again, we have also $\psi'''(t)$
\[ \psi'''(t) = \frac{\psi(t)}{t^3} \epsilon(t) \left(2 + \epsilon^2(t) + 3t\epsilon'(t) - 3\epsilon(t) - \frac{2t\epsilon'(t)}{\epsilon(t)} + \frac{t^2\epsilon''(t)}{\epsilon(t)}\right) \]
\[ = \frac{\psi(t)}{t^3} \epsilon(t) (2 + o(1)) \quad \text{as} \quad t \to \infty. \quad (6.18) \]

Put (6.16), (6.17) and (6.18) into (6.15) we get
\[ h'''(\hat{x}) = \frac{t}{\psi^3(t)\epsilon^3(t)} (1 + o(1)) \]

Thus by (2.7) as $t \to \infty$
\[ \sup_{|v| \leq t/4} h'''(\psi(t + v)) = \sup_{|v| \leq t/4} \frac{t + v}{\psi^3(t + v)\epsilon^3(t + v)} (1 + o(1)) \leq 4t^{11/8} \psi^3(t)^{-1/8} \leq 4t^{11/8} \psi^3(t)^{-1/8}, \quad (6.19) \]

where last inequality holds from the slowly varying propriety: $\psi(t + v) \sim \psi(t)$. Using $\sigma = (h'(\hat{x}))^{-1/2}$, it holds
\[ \sup_{|v| \leq t/4} h'''(\psi(t + v)) \sigma^4 \leq \frac{4t^{11/8}}{\psi^3(t) (h'(\hat{x}))^2} \leq \frac{4t^{11/8}}{\psi^3(t) t^2} \leq \frac{4\epsilon^2(t)}{\psi(t) t^{5/8}} \to 0. \]

Hence for any slowly varying function $l(t) \to \infty$ it holds as $t \to \infty$
\[ \sup_{|v| \leq t/4} h'''(\psi(t + v)) \sigma^4 t^4 \to 0. \]
Consider $\psi(t) \in \tilde{R}_0$, thus $\psi(t)$ is increasing, we have the relation
\[
\sup_{|v| \leq t/4} h''(\psi(t + v)) = \sup_{|\zeta| \leq \hat{\zeta}_1, \hat{\zeta}_2} h''(\hat{x} + \zeta),
\]
where
\[
\zeta_1 = \psi(3t/4) - \hat{x}, \quad \zeta_2 = \psi(5t/4) - \hat{x}.
\]
Hence we have showed
\[
\sup_{|\zeta| \leq \hat{\zeta}_1, \hat{\zeta}_2} h''(\hat{x} + \zeta) \sigma^4 t^4 \to 0.
\]

For completing the proof, it remains to show
\[
\sigma l \leq \min(|\zeta_1|, \zeta_2) \quad \text{as} \quad t \to \infty. \tag{6.20}
\]

Perform first order Taylor expansion of $\psi(3t/4)$ at $t$, for some $\alpha \in [0, 1]$, it holds
\[
\zeta_1 = \psi(3t/4) - \hat{x} = \psi(3t/4) - \psi(t) = -\psi'(t - \alpha t/4) \frac{t}{4} = -\frac{\psi(t - \alpha t/4)}{4 - \alpha} \epsilon(t - \alpha t/4),
\]
thus using (2.7) and slowly varying propriety of $\psi(t)$ we get as $t \to \infty$
\[
|\zeta_1| \geq \frac{\psi(t - \alpha t/4)}{4} \epsilon(t - \alpha t/4) \geq \frac{\psi(t)}{5} \epsilon(t - \alpha t/4) \geq \frac{\psi(t)}{5t^{1/8}}. \tag{6.21}
\]

On the other hand, we have $\sigma = (h'(\hat{x}))^{-1/2} = (\psi(t) \epsilon(t)/t)^{1/2}$, which, together with (6.21), yields
\[
\frac{\sigma}{|\zeta_1|} \leq 5 \sqrt{\frac{\epsilon(t)}{\psi(t) \sqrt{t}}} \to 0 \quad \text{as} \quad t \to \infty,
\]
which implies for any slowly varying function $l(t)$ it holds $\sigma l = o(|\zeta_1|)$. By the same way, it is easy to show $\sigma l = o(\zeta_2)$. Hence (6.20) holds, as sought. \hfill \Box

**Lemma 6.3.** For $p(x)$ in (2.1), $h \in \mathfrak{H}$, then for any varying slowly function $l(t) \to \infty$ as $t \to \infty$, it holds
\[
\sup_{|x| \leq \sigma t} \frac{h'''(\hat{x} + x)}{h''(\hat{x})} \sigma l \to 0 \quad \text{as} \quad t \to \infty, \tag{6.22}
\]
and
\[
h''(\hat{x}) \sigma^3 l \to 0, \quad h''(\hat{x}) \sigma^4 \to 0. \tag{6.23}
\]
Proof. **Case 1:** \( h \in R_\beta \). Using (6.8) and (6.10), we get \( h''(x) = (\beta(\beta - 1) + o(1))x^{\beta - 2}l_0(x) \) as \( x \to \infty \), where \( l_0(x) \in R_0 \). Hence it holds

\[
h''(\hat{x}) = (\beta(\beta - 1) + o(1))\psi(t)^{\beta - 2}l_0(\psi(t)),
\]

which, together with (6.12) and (6.13), yields with some positive constant \( Q_3 \)

\[
\sup_{|x| \leq \sigma} \frac{|h'''(\hat{x} + x) - h''(\hat{x})\sigma l|}{h''(\hat{x})\sigma l} \leq \frac{t}{\psi^2(t)} \frac{1}{\psi(t)^{\beta - 2}l_0(\psi(t))} \left( \frac{\psi(t)}{\beta t} \right)^{3/2} l
\]

\[
= \left( \frac{\beta(\beta - 1) + o(1)}{\beta^{3/2} \beta^{1/2} l_0(\psi(t))} \right)^{3/2} \frac{\beta - 1}{\beta^{1/2}} \frac{l_1(t)^{\beta - 1/2}}{t^{1/2 + 1/2} l_0(\psi(t))} l
\]

This implies the first formula of (6.22) holds.

**Case 2:** \( h \in R_\infty \). Using (6.14) and (6.17), we obtain

\[
h''(\hat{x}) = -\frac{\psi''(t)}{(\psi'(t))^3} = \frac{t}{\psi^2(t) e^2(t)} (1 + o(1)).
\]

Combine (6.19) and (6.26), using \( \sigma = (h'(\hat{x}))^{-1/2} \), we have as \( t \to \infty \)

\[
\sup_{|v| \leq t/4} \frac{h'''(\psi(t + v))}{h''(\hat{x})} \sigma \leq \frac{5\epsilon(t)^{2/3}}{\psi(t)^{1/3} h'(\hat{x})} \frac{1}{\sqrt{h'(\hat{x})}} = \frac{5\epsilon(t)^{5/2}}{t^{1/3} \sqrt{\psi(t)}} \to 0,
\]

where \( \epsilon(t) \to 0 \) and \( \psi(t) \) varies slowly. Hence for arbitrarily slowly varying function \( l(t) \) it holds as \( t \to \infty \)

\[
\sup_{|v| \leq t/4} \frac{h'''(\psi(t + v))}{h''(\hat{x})} \sigma l \to 0.
\]

Define \( \zeta_1, \zeta_2 \) as in Lemma 6.2, we have showed

\[
\sup_{|\zeta| \leq |\zeta_1, \zeta_2|} \frac{h'''(\hat{x} + \zeta)}{h''(\hat{x})} \sigma l \to 0.
\]

(6.22) is obtained by using (6.20). Using (6.20), for any slowly varying function, it holds

\[
h''(\hat{x})\sigma^3 l \sim \frac{l}{\sqrt{\psi(t)\epsilon(t)t}} \to 0.
\]

By the same method as proving \( h''(\hat{x})\sigma^3 l \to 0 \), it is easy to get for **Case 1** and **Case 2**

\[
h''(\hat{x})\sigma^4 \to 0.
\]

Hence the proof. \( \square \)
Lemma 6.4. For \( p(x) \) in (2.1), \( h \in \mathfrak{R} \), then for any slowly varying function \( l(t) \rightarrow \infty \) as \( t \rightarrow \infty \), it holds

\[
\sup_{y \in [-l, l]} \frac{|\xi(\sigma y + \hat{x}, t)|}{h''(\hat{x})\sigma^3} \rightarrow 0,
\]

where \( \xi(x, t) = \varepsilon(x, t) + q(x) \).

Proof. A close look to the proof of Lemma 6.3, it is straightforward that (6.22) can be slightly modified as

\[
\sup_{|x| \leq \sigma t} \frac{h''(\hat{x} + x)}{h''(\hat{x})} \sigma t^4 \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty.
\]

Hence for \( y \in [-l, l] \), by (6.22) and Lemma 6.3 it holds as \( t \rightarrow \infty \)

\[
\left| \frac{\varepsilon(\sigma y + \hat{x}, t)}{h''(\hat{x})\sigma^3} \right| \leq \sup_{|x| \leq \sigma t} \left| \frac{h''(\hat{x} + x)}{h''(\hat{x})} \right| \sigma t^4 \rightarrow 0. \tag{6.27}
\]

Under condition (2.2), set \( x = \psi(t) \), we get

\[
\sup_{|v - \psi(t)| \leq \psi(t)} |q(v)| \leq \frac{1}{t \sqrt{\psi(t)}}.
\]

Then we show

\[
\left| \frac{q(\sigma y + \hat{x})}{h''(\hat{x})\sigma^3} \right| \rightarrow 0. \tag{6.28}
\]

Case 1: \( h \in R_{\beta} \). We have \( h(x) = x^\beta l_0(x), l_0(x) \in R_0, \beta > 0 \). Hence

\[
h'(x) = x^{\beta - 1} l_0(x) (\beta + \epsilon(x)).
\]

Notice \( \psi'(t) = 1/h'(\psi(t)) \), it holds as \( t \rightarrow \infty \)

\[
\frac{\sigma l}{\psi(t)} = \frac{l}{\psi(t) \sqrt{h'(\psi(t))}} = \frac{l}{\psi(t)^{(\beta+1)/2} l_0(\psi(t))^{1/2} (\beta + \epsilon(\psi(t)))^{1/2}} \rightarrow 0.
\]

It follows

\[
\sup_{|v - \psi(t)| \leq \sigma t} |q(v)| \leq \frac{1}{t \sqrt{\psi(t)}}.
\]

Using the above inequality and by the second line of (6.22), when \( y \in [-l, l] \), it holds as \( t \rightarrow \infty \)

\[
\left| \frac{q(\sigma y + \hat{x})}{h''(\hat{x})\sigma^3} \right| \sim q(\sigma y + \hat{x}) \left( \frac{\beta - 1}{\sqrt{\beta}} \frac{1}{t^{3/2} l_0(\psi(t))} \right)^{-1} \leq 2 \frac{\sqrt{\beta}}{|\beta - 1|} \sup_{|v - \psi(t)| \leq \sigma t} |q(v)| \frac{t^{3/2}}{\psi(t)^{\beta - 1/2} l_0(\psi(t))} \leq 2 \frac{\sqrt{\beta}}{|\beta - 1|} \frac{\sqrt{t}}{\psi(t)^{\beta} l_0(\psi(t))} \rightarrow 0,
\]

\[15\]
where last step holds since \( \psi(t) \sim t^{1/\beta} l_1(t) \) for some slowly varying function \( l_1 \).

**Case 2:** \( h \in R_{\infty} \). For any slowly varying function \( l(t) \) as \( t \to \infty \)

\[
\frac{\sigma l}{\vartheta \psi(t)} = \sqrt{\frac{\psi'(t)}{\vartheta \psi(t)}} = \sqrt{\frac{\epsilon(t)}{t \vartheta \psi(t)}} \to 0,
\]

hence

\[
\sup_{|v-\psi(t)| \leq \sigma_1} |q(v)| \leq \frac{1}{t \sqrt{\psi(t)}}.
\]

Using this inequality and (6.26), when \( y \in [-l, l] \), it holds as \( t \to \infty \)

\[
\left| \frac{q(\sigma y + \hat{x})}{\hat{h}^\alpha(\hat{x}) \sigma^\alpha} \right| \leq 2 |q(\sigma y + \hat{x})| \sqrt{\psi(t) \epsilon(t)} t \leq 2 \sup_{|v-\psi(t)| \leq \sigma_1} |q(v)| \sqrt{\psi(t) \epsilon(t)} t \leq \sqrt{\epsilon(t)/t} \to 0.
\]

(6.28), together with (6.29), completes the proof. \( \square \)

**Lemma 6.5.** For \( p(x) \) belonging to (2.1), \( h(x) \in \mathbb{R}, \alpha \in \mathbb{N} \), denote by

\[
\Psi(t, \alpha) := \int_0^\infty (x - \hat{x})^\alpha e^{tx} p(x) dx,
\]

then there exists some slowly varying function \( l(t) \) such that it holds as \( t \to \infty \)

\[
\Psi(t, \alpha) = c \sigma^{\alpha+1} e^{K(\hat{x},t)} T_1(t, \alpha) (1 + o(1)),
\]

where

\[
T_1(t, \alpha) = \int_{-\frac{l^{1/3} \sqrt{2}}{\sqrt{2}}}^{\frac{l^{1/3} \sqrt{2}}{\sqrt{2}}} y^\alpha \exp \left( - \frac{y^2}{2} \right) dy - \frac{h''(\hat{x}) \sigma^3}{6} \int_{-\frac{l^{1/3} \sqrt{2}}{\sqrt{2}}}^{\frac{l^{1/3} \sqrt{2}}{\sqrt{2}}} y^{3+\alpha} \exp \left( - \frac{y^2}{2} \right) dy.
\]

**Proof.** By (6.2) and Lemma 6.2 for any slowly varying function \( l(t) \) it holds as \( t \to \infty \)

\[
\sup_{|x-\hat{x}| \leq \sigma_1} |\epsilon(x, t)| \to 0.
\]

Given a slowly varying function \( l \) with \( l(t) \to \infty \) and define the interval \( I_t \) as follows

\[
I_t := \left( - \frac{l^{1/3} \sigma}{\sqrt{2}}, \frac{l^{1/3} \sigma}{\sqrt{2}} \right).
\]

For large enough \( \tau \), when \( t \to \infty \) we can partition \( \mathbb{R}_+ \) as

\[
\mathbb{R}_+ = \{ x : 0 < x < \tau \} \cup \{ x : x \in \hat{x} + I_t \} \cup \{ x : x \geq \tau, x \notin \hat{x} + I_t \},
\]

where \( \tau \) large enough such that it holds for \( x > \tau \)

\[
p(x) < 2 c e^{-g(x)}.
\]
Obviously, for fixed $\tau$, $\{x : 0 < x < \tau\} \cap \{x : x \in \hat{x} + I_t\} = \emptyset$ since for large $t$ we have $\min(x : x \in \hat{x} + I_t) \to \infty$ as $t \to \infty$. Hence it holds

$$
\Psi(t, \alpha) = \int_0^T (x - \hat{x})^\alpha e^{tx} p(x) dx + \int_{x \in \hat{x} + I_t} (x - \hat{x})^\alpha e^{tx} p(x) dx + \int_{x \notin \hat{x} + I_t, x > \tau} (x - \hat{x})^\alpha e^{tx} p(x) dx
:= \Psi_1(t, \alpha) + \Psi_2(t, \alpha) + \Psi_3(t, \alpha).
$$

(6.30)

We estimate sequentially $\Psi_1(t, \alpha), \Psi_2(t, \alpha), \Psi_3(t, \alpha)$ in Step 1, Step 2 and Step 3.

**Step 1:** Using (6.29), for $\tau$ large enough, we have

$$
|\Psi_1(t, \alpha)| \leq \int_0^T |x - \hat{x}|^\alpha e^{tx} p(x) dx \leq 2c \int_0^T |x - \hat{x}|^\alpha e^{tx - g(x)} dx \leq 2c t^{-1} \hat{x}^\alpha e^{tr}. (6.31)
$$

We show it holds for $h \in \mathbb{R}$ as $t \to \infty$

$$
t^{-1} \hat{x}^\alpha e^{tr} = o(\sigma^\alpha e^{K(\hat{x}, t)\hat{x}''} \hat{x}^3). (6.32)
$$

(6.32) is equivalent to

$$
\sigma^{-\alpha - 4} t^{-1} \hat{x}^\alpha e^{tr} (t''(\hat{x}))^{-1} = o(e^{K(\hat{x}, t)}),
$$

which is implied by

$$
\exp(-(\alpha + 4) \log \sigma - \log t + \alpha \log \hat{x} + \tau t - \log h''(\hat{x})) = o(e^{K(\hat{x}, t)}).
$$

Since $\hat{x} = \psi(t)$, it holds

$$
K(\hat{x}, t) = t \psi(t) - g(\psi(t)) = \int_1^t \psi(u) du + \psi(1) - g(1), (6.33)
$$

where the second equality can be easily verified by the change of variable $u = h(\psi)$. By Lemma (6.1), we know $\log \sigma = o(e^{K(\hat{x}, t)})$ as $t \to \infty$. So it remains to show $t = o(e^{K(\hat{x}, t)})$, $\log \hat{x} = o(e^{K(\hat{x}, t)})$ and $\log h''(\hat{x}) = o(e^{K(\hat{x}, t)})$.

If $h(\hat{x}) \in R_B$, by Theorem (1.5.12) of [1], it holds $\psi(x) \sim x^{1/\beta} l_1(x)$ with some slowly varying function $l_1(x)$. (6.34) and (6.33) yield $t = o(e^{K(\hat{x}, t)})$. In addition, $\log \hat{x} = \log \psi(t) \sim (1/\beta) \log t + \log l_1(t)) = o(e^{K(\hat{x}, t)})$. By (6.22), it holds $\log h''(\hat{x}) = o(t)$. Thus (6.32) holds.

If $h(\hat{x}) \in R_\infty$, $\psi(x) \in R_0$ is slowly varying as $x \to \infty$. Therefore, by (6.5) and (6.33), it holds $t = o(e^{K(\hat{x}, t)})$ and $\log \hat{x} = \log \psi(t) = o(e^{K(\hat{x}, t)})$. Using (6.26), we have $\log h''(\hat{x}) \sim \log t - 2 \log \hat{x} - 2 \log \epsilon(t)$. Under condition (2.7), $\log \epsilon(t) = o(t)$, thus it holds $\log h''(\hat{x}) = o(t)$. We get (6.32).

(6.31) and (6.32) yield together

$$
|\Psi_1(t, \alpha)| = o(\sigma^{\alpha + 1} e^{K(\hat{x}, t)\hat{x}''} \hat{x}^3). (6.34)
$$

**Step 2:** Notice $\min(x : x \in \hat{x} + I_t) \to \infty$ as $t \to \infty$, which implies both $\varepsilon(x, t)$ and $q(x)$ go to 0 when $x \in \hat{x} + I_t$. By (2.1) and (6.1), as $t \to \infty$

$$
\Psi_2(t, \alpha) = \int_{x \in \hat{x} + I_t} (x - \hat{x})^\alpha c \exp(K(x, t) + q(x)) dx
:= \int_{x \in \hat{x} + I_t} (x - \hat{x})^\alpha c \exp\left(K(\hat{x}, t) - \frac{1}{2} h''(\hat{x})(\hat{x} - \hat{x})^2 - \frac{1}{6} h''(\hat{x})(\hat{x} - \hat{x})^3 + \xi(x, t)\right) dx,
$$

17
where \( \xi(x, t) = \varepsilon(x, t) + q(x) \). Make the change of variable \( y = (x - \hat{x})/\sigma \), it holds

\[
\Psi_2(t, \alpha) = \alpha \exp \left( K(\hat{x}, t) \right) \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} y^\alpha \exp \left( -\frac{y^2}{2} - \frac{h''(\hat{x}) \sigma^3}{6} y^3 + \xi(y + \hat{x}, t) \right) dy. \tag{6.35}
\]

On \( y \in \left( -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \), by (6.24), \( |h''(\hat{x})\sigma^3| \leq |h''(\hat{x})\sigma^3| \to 0 \) as \( t \to \infty \). Perform the first order Taylor expansion, as \( t \to \infty \)

\[
\exp \left( -\frac{h''(\hat{x}) \sigma^3}{6} y^3 + \xi(y + \hat{x}, t) \right) = 1 - \frac{h''(\hat{x}) \sigma^3}{6} y^3 + \xi(y + \hat{x}, t) + o_1(t, y),
\]

where

\[
o_1(t, y) = o \left( -\frac{h''(\hat{x}) \sigma^3}{6} y^3 + \xi(y + \hat{x}, t) \right). \tag{6.36}
\]

Hence we obtain

\[
\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} y^\alpha \exp \left( -\frac{y^2}{2} - \frac{h''(\hat{x}) \sigma^3}{6} y^3 + \xi(y + \hat{x}, t) \right) dy = \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \left( 1 - \frac{h''(\hat{x}) \sigma^3}{6} y^3 + \xi(y + \hat{x}, t) + o_1(t, y) \right) y^\alpha \exp \left( -\frac{y^2}{2} \right) dy = T_1(t, \alpha) + T_2(t, \alpha),
\]

where \( T_1(t, \alpha) \) and \( T_2(t, \alpha) \) are defined as follows

\[
T_1(t, \alpha) := \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} y^\alpha \exp \left( -\frac{y^2}{2} \right) dy - \frac{h''(\hat{x}) \sigma^3}{6} \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} y^{3+\alpha} \exp \left( -\frac{y^2}{2} \right) dy,
\]

\[
T_2(t, \alpha) := \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \left( \xi(y + \hat{x}, t) + o_1(t, y) \right) y^\alpha \exp \left( -\frac{y^2}{2} \right) dy. \tag{6.37}
\]

For \( T_2(t, \alpha) \), using (6.36) we have

\[
|T_2(t, \alpha)| \leq \sup_{y \in [-1, 1]} |\xi(y + \hat{x}, t)| \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} |y|^\alpha \exp \left( -\frac{y^2}{2} \right) dy \
+ \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \left( |o \left( \frac{h''(\hat{x}) \sigma^3}{6} y^3 \right) | + |o \left( \xi(y + \hat{x}, t) \right) | \right) |y|^\alpha \exp \left( -\frac{y^2}{2} \right) dy \
\leq 2 \sup_{y \in [-1, 1]} |\xi(y + \hat{x}, t)| \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} |y|^\alpha \exp \left( -\frac{y^2}{2} \right) dy + |o \left( h''(\hat{x}) \sigma^3 \right) | \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} |y|^{3+\alpha} \exp \left( -\frac{y^2}{2} \right) dy \
= |o \left( h''(\hat{x}) \sigma^3 \right) | \left( \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} |y|^\alpha \exp \left( -\frac{y^2}{2} \right) dy + \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} |y|^{3+\alpha} \exp \left( -\frac{y^2}{2} \right) dy \right),
\]
where last equality holds from Lemma 6.4. Since the integrals in the last equality are both bounded, it holds as \( t \to \infty \)

\[
T_2(t, \alpha) = o(h''(\hat{x})\sigma^3).
\]

When \( \alpha \) is even, the second term of \( T_1(t, \alpha) \) vanishes. When \( \alpha \) is odd, the first term of \( T_1(t, \alpha) \) vanishes. \( h''(\hat{x})\sigma^3 \to 0 \) by \( 6.23 \), thus \( T_1(t, \alpha) \) is at least the same order than \( h''(\hat{x})\sigma^3 \). It follows as \( t \to \infty \)

\[
T_2(t, \alpha) = o(T_1(t, \alpha)). \tag{6.38}
\]

Using \( 6.35, 6.37 \) and \( 6.38 \) we get

\[
\Psi_2(t, \alpha) = c\sigma^{\alpha+1} \exp(K(\hat{x}, t))T_1(t, \alpha)(1 + o(1)). \tag{6.39}
\]

**Step 3:** Given \( h \in \mathbb{R} \), for any \( t \), \( K(x, t) \) as a function of \( x \) \( (x > \tau) \) is concave since

\[
K''(x, t) = -h'(x) < 0.
\]

Thus we get for \( x \notin \hat{x} + I_t \) and \( x > \tau \)

\[
K(x, t) - K(\hat{x}, t) \leq \frac{K(\hat{x} + \frac{1/3\sigma}{\sqrt{2}}sgn(x - \hat{x}), t) - K(\hat{x}, t)}{\frac{1/3\sigma}{\sqrt{2}}sgn(x - \hat{x})}(x - \hat{x}), \tag{6.40}
\]

where

\[
sgn(x - \hat{x}) = \begin{cases} 
1 & \text{if } x \geq \hat{x}, \\
-1 & \text{if } x < \hat{x}.
\end{cases}
\]

Using \( 6.11 \), we get

\[
K(\hat{x} + \frac{1/3\sigma}{\sqrt{2}}sgn(x - \hat{x}), t) - K(\hat{x}, t) \leq -\frac{1}{8}h'(\hat{x})t^{2/3}\sigma^2 = -\frac{1}{8}t^{2/3},
\]

which, combined with \( 6.40 \), yields

\[
K(x, t) - K(\hat{x}, t) \leq -\frac{\sqrt{2}}{8}t^{1/3}\sigma^{-1} |x - \hat{x}|.
\]

We obtain

\[
|\Psi_3(t, \alpha)| \leq 2c \int_{x \notin \hat{x} + I_t, x > \tau} |x - \hat{x}|^\alpha \exp(K(x, t)) \, dx
\]

\[
\leq 2ce^{K(\hat{x}, t)} \int_{|x - \hat{x}| > \frac{1/3\sigma}{\sqrt{2}}} |x - \hat{x}|^\alpha \exp\left(-\frac{\sqrt{2}}{8}t^{1/3}\sigma^{-1} |x - \hat{x}| \right) \, dx
\]

\[
= 2ce^{K(\hat{x}, t)} \sigma^{\alpha+1} \int_{|y| > \frac{1/3\sigma}{\sqrt{2}}} |y|^\alpha \exp\left(-\frac{\sqrt{2}}{8}t^{1/3}|y| \right) \, dy
\]

\[
= 2ce^{K(\hat{x}, t)} \sigma^{\alpha+1} \left(2e^{-t^{2/3}/8}(1 + o(1)) \right),
\]

19
where last equality holds when $l \to \infty$ (see e.g. Theorem 4.12.10 of [1]). With (6.39), we obtain

$$\left| \frac{\Psi_3(t, \alpha)}{\Psi_2(t, \alpha)} \right| \leq 8e^{-l^2/3}/|T_1(t, \alpha)|.$$  

In Step 2, we know $T_1(t, \alpha)$ has at least the order $h''(\tilde{x})\sigma^3$. Hence there exists some positive constant $Q$ and some slowly varying function $l_2$ with $l_2(t) \to \infty$ such that it holds as $t \to \infty$

$$\left| \frac{\Psi_3(t, \alpha)}{\Psi_2(t, \alpha)} \right| \leq \frac{Qe^{-l^2/8}}{h''(\tilde{x})\sigma^3}.$$  

For example, we can take $l_2(t) = (\log t)^3$.

If $h \in R_3$, one close look to (6.20), it is easy to know $h''(\tilde{x})\sigma^3 \geq 1/t^{1+1/(2\beta)}$, with the choice of $l_2$ as above, we have

$$\left| \frac{\Psi_3(t, \alpha)}{\Psi_2(t, \alpha)} \right| \leq Q \exp \left( -l_2^{2/3} + (1 + 1/(2\beta)) \log t \right) \to 0.$$  

If $h \in R_{\infty}$, using (6.20), then it holds as $t \to \infty$

$$\left| \frac{\Psi_3(t, \alpha)}{\Psi_2(t, \alpha)} \right| \leq 2Q \exp \left( -l_2^{2/3} + \log \sqrt{t}\psi(t)\epsilon(t) \right) = 2Q \exp \left( -l_2^{2/3} + (1/2) \left( \log t + \log \psi(t) + \log \epsilon(t) \right) \right) \to 0,$$  

where last step holds since $\log \psi(t) = O(\log t)$. The proof is completed by combining (6.39), (6.40), (6.39) and (6.41).

Proof of Theorem 3.1. By Lemma 6.5, if $\alpha = 0$, it holds $T_1(t, 0) \sim \sqrt{2\pi}$ as $t \to \infty$, hence for $p(x)$ defined in (2.1), we can approximate X’s moment generating function $\Phi(t)$

$$\Phi(t) = \int_{-\infty}^{\infty} e^{tx}p(x)dx = c\sqrt{2\pi}\sigma e^{K(\tilde{x},t)}(1 + o(1)).$$  

(6.42)

If $\alpha = 1$, it holds as $t \to \infty$,

$$T_1(t, 1) = -\frac{h''(\tilde{x})\sigma^3}{6} \int_{\sqrt{\frac{1}{2}}}^{\sqrt{\frac{1}{2}}} y^4 \exp \left( -\frac{y^2}{2} \right) dy = -\frac{\sqrt{2\pi}h''(\tilde{x})\sigma^3}{2}(1 + o(1)),$$

hence we have with $\Psi(t, \alpha)$ defined in Lemma 6.5

$$\Psi(t, 1) = -c\sqrt{2\pi}\sigma^2 e^{K(\tilde{x},t)}\frac{h''(\tilde{x})\sigma^3}{2}(1 + o(1)) = -\Phi(t)\frac{h''(\tilde{x})\sigma^4}{2}(1 + o(1)),$$  

(6.43)

which, together with the definition of $\Psi(t, \alpha)$, yields

$$\int_{0}^{\infty} xe^{tx}p(x)dx = \Psi(t, 1) + \hat{x}\Phi(t) = \left( \hat{x} - \frac{h''(\tilde{x})\sigma^4}{2}(1 + o(1)) \right) \Phi(t).$$  

(6.44)
Hence we get
\[ m(t) = \frac{d \log \Phi(t)}{dt} = \frac{\int_0^\infty xe^{tx}p(x)dx}{\Phi(t)} = \hat{x} - \frac{h''(\hat{x})\sigma^4}{2} (1 + o(1)). \] (6.45)

By (6.23), as \( t \to \infty \)
\[ m(t) \sim \hat{x} = \psi(t). \] (6.46)

Set \( \alpha = 2 \), as \( t \to \infty \), it follows
\[ \Psi(t, 2) = c\sigma^3 e^{K(\hat{x}, t)} \int_{\frac{-y^2}{2}}^{\frac{1}{\sqrt{2}}} y^2 \exp \left(-\frac{y^2}{2}\right) dy (1 + o(1)) = \sigma^2 \Phi(t)(1 + o(1)). \] (6.47)

Using (6.43), (6.45) and (6.47), we have
\[
\int_0^\infty (x - m(t))^2 e^{tx}p(x)dx = \int_0^\infty (x - \hat{x} + \hat{x} - m(t))^2 e^{tx}p(x)dx
\]
\[
= \int_0^\infty (x - \hat{x})^2 e^{tx}p(x)dx + 2(\hat{x} - m(t)) \int_0^\infty (x - \hat{x})e^{tx}p(x)dx + (\hat{x} - m(t))^2 \Phi(t)
\]
\[
= \Psi(t, 2) + 2(\hat{x} - m(t))\Phi(t, 1) + (\hat{x} - m(t))^2 \Phi(t)
\]
\[
= \sigma^2 \Phi(t)(1 + o(1)) - h''(\hat{x})\sigma^4 \left( \Phi(t) \frac{h''(\hat{x})\sigma^4}{2} (1 + o(1)) + \left( \frac{h''(\hat{x})\sigma^4}{2} \right)^2 \Phi(t)(1 + o(1)) \right)
\]
\[
= \sigma^2 \Phi(t)(1 + o(1)) - \frac{(h''(\hat{x})\sigma^4)^2}{4} \sigma^2 \Phi(t)(1 + o(1)) = \sigma^2 \Phi(t)(1 + o(1)),
\]
where last equality holds since \( h''(\hat{x})\sigma^4 \) goes to 0 by (6.23), thus as \( t \to \infty \)
\[ s^2(t) = \frac{d^2 \log \Phi(t)}{dt^2} = \frac{\int_0^\infty (x - m(t))^2 e^{tx}p(x)dx}{\Phi(t)} \sim \sigma^2 = \psi'(t). \] (6.48)

Set \( \alpha = 3 \), the first term of \( T_1(t, 3) \) vanishes, we obtain as \( t \to \infty \)
\[ \Psi(t, 3) = -c\sqrt{2\pi} \sigma^4 e^{K(\hat{x}, t)} \frac{h''(\hat{x})\sigma^3}{6} \int_{\frac{-y^2}{2}}^{\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{2\pi}} y^6 \exp \left(-\frac{y^2}{2}\right) dy
\]
\[
= -cM_6 \sqrt{2\pi} \sigma^6 e^{K(\hat{x}, t)} \frac{h''(\hat{x})\sigma^7}{6} (1 + o(1)) = -M_6 \frac{h''(\hat{x})\sigma^6}{6} \Phi(t)(1 + o(1)),
\] (6.49)
where \( M_6 \) denotes the sixth order moment of standard normal distribution. Using (6.43), (6.45),
Proof of Theorem 4.1

By (6.49), we have as \( t \to \infty \)
\[
\int_0^\infty (x - m(t))^3 e^{tx} p(x) dx = \int_0^\infty (x - \hat{x} - m(t))^3 e^{tx} p(x) dx \\
= \int_0^\infty \left( (x - \hat{x})^3 + 3(x - \hat{x})^2 (\hat{x} - m(t)) + 3(x - \hat{x})(\hat{x} - m(t))^2 + (\hat{x} - m(t))^3 \right) e^{tx} p(x) dx \\
= \Psi(t, 3) + 3(\hat{x} - m(t))\Psi(t, 2) + 3(\hat{x} - m(t))^2 \Psi(t, 1) + (\hat{x} - m(t))^3 \Phi(t) \\
= -M_6 \frac{h''(\hat{x})}{6} \Phi(t)(1 + o(1)) + (3/2) h''(\hat{x}) \sigma^4 (\sigma^2 \Phi(t))(1 + o(1)) \\
- 3 \left( \frac{h''(\hat{x})}{2} \right)^2 \Phi(t) \frac{h''(\hat{x})}{2} (1 + o(1)) + \left( \frac{h''(\hat{x})}{2} \right)^3 \Phi(t)(1 + o(1)) \\
= 9 - \frac{M_6 h''(\hat{x})}{6} \sigma^6 \Phi(t)(1 + o(1)) - h''(\hat{x}) \sigma^6 \Phi(t) \left( \frac{h''(\hat{x})}{4} \right)^2 (1 + o(1)) \\
= \frac{9 - M_6 h''(\hat{x})}{6} \sigma^6 \Phi(t)(1 + o(1)),
\]
where last equality holds since \( h''(\hat{x}) \sigma^3 \to 0 \) by (6.23). Hence we get as \( t \to \infty \)
\[
\mu_3(t) \sim \frac{d^3 \log \Phi(t)}{dt^3} = \frac{\int_0^\infty (x - m(t))^3 e^{tx} p(x) dx}{\Phi(t)} \\
\sim \frac{9 - M_6 h''(\hat{x})}{6} \sigma^6 = -\frac{9 - M_6 \psi''(t)}{6} \psi'(t)^3 = \frac{M_6 - 9}{6} \psi''(t). \tag{6.50}
\]

The proof is completed by combining (6.46) - (6.49) with (6.50).

Proof of Corollary 3.1

The proof is immediate by (6.23) of Lemma 6.3 from which we get \( h''(\hat{x}) \sigma^3 \to 0 \) since \( l(t) \to \infty \) as \( t \to \infty \). By (6.48) and (6.50), it holds as \( t \to \infty \)
\[
\frac{\mu_3}{x^3} \sim \frac{9 - M_6 h''(\hat{x})}{6} \sigma^3 \to 0. \tag{6.51}
\]

6.2 Proof of Theorem 4.1

Proof. Step 1: Denote by
\[
G(x) := \rho_n(x) - \phi(x) - \frac{\mu_3}{6 \sqrt{n} s^3} (x^3 - 3x) \phi(x).
\]

Let \( \varphi_{an}(\tau) \) be the characteristic function (c.f) of \( \tilde{\pi}_{an} \); the c.f of \( \rho_n \) is \( \varphi_{an}(\tau/\sqrt{n}) \). Hence it holds by Fourier inversion theorem
\[
G(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix\tau} \left( \varphi_{an}(\tau/\sqrt{n}) \right)^n - e^{-\frac{1}{2} \tau^2} - \frac{\mu_3}{6 \sqrt{n} s^3} \left( i\tau \right)^3 e^{-\frac{1}{2} \tau^2} d\tau.
\]
We obtain
\[ G(x) \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \varphi^{a_n}(\tau/\sqrt{n}) \right)^n e^{-\frac{1}{2}x^2} - \frac{\mu_3}{6\sqrt{n}s^3} (i\tau)^3 e^{-\frac{1}{2}x^2} d\tau. \] \hspace{1cm} (6.52)

**Step 2:** In this step we show that the characteristic function \( \varphi^{a_n} \) of \( \hat{\pi}^{a_n}(x) \) satisfies
\[ \sup_{a_n \in \mathbb{R}^+} \int |\varphi^{a_n}(\tau)|^2 d\tau < \infty \quad \text{and} \quad \sup_{a_n \in \mathbb{R}^+} |\varphi^{a_n}(\tau)| < 1, \] \hspace{1cm} (6.53)
for any positive \( \rho \).

It is easy to verify that \( r \)-order \( (r \geq 1) \) moment \( \mu^r \) of \( \pi^{a_n}(x) \) satisfies
\[ \mu^r(t) = \frac{dt^r \log \Phi(t)}{dt}, \quad \text{with} \quad t = m^r(a_n), \]

By Parseval identity
\[ \int |\varphi^{a_n}(\tau)|^2 d\tau = 2\pi \int (\hat{\pi}^{a_n}(x))^2 dx \leq 2\pi \sup_{x \in \mathbb{R}} \hat{\pi}^{a_n}(x). \] \hspace{1cm} (6.54)

For the density function \( p(x) \) in (2.1), Theorem 5.4 of Juszczak and Nagaev [12] states that the normalized conjugate density of \( p(x) \), namely, \( \hat{\pi}^{a_n}(x) \) has the propriety
\[ \lim_{a_n \to \infty} \sup_{x \in \mathbb{R}} |\hat{\pi}^{a_n}(x) - \phi(x)| = 0. \]

Thus for arbitrary positive \( \delta \), there exists some positive constant \( M \) such that it holds
\[ \sup_{a_n \geq M} \sup_{x \in \mathbb{R}} |\hat{\pi}^{a_n}(x) - \phi(x)| \leq \delta, \]

which entails that \( \sup_{a_n \geq M} \sup_{x \in \mathbb{R}} \hat{\pi}^{a_n}(x) < \infty \). When \( a_n < M \), \( \sup_{a_n < M} \sup_{x \in \mathbb{R}} \hat{\pi}^{a_n}(x) < \infty \); hence we have
\[ \sup_{a_n \geq M} \sup_{x \in \mathbb{R}} \hat{\pi}^{a_n}(x) < \infty, \]
which, together with (6.53), gives the first inequality of (6.53). Furthermore, \( \varphi^{a_n}(\tau) \) is not periodic, hence the second inequality of (6.53) holds from Lemma 4 (Chapter 15, section 1) of [11].

**Step 3:** We complete the proof by showing that for \( n \) large enough
\[ \int_{-\infty}^{\infty} \left( \varphi^{a_n}(\tau/\sqrt{n}) \right)^n e^{-\frac{1}{2}x^2} - \frac{\mu_3}{6\sqrt{n}s^3} (i\tau)^3 e^{-\frac{1}{2}x^2} d\tau = o \left( \frac{1}{\sqrt{n}} \right). \] \hspace{1cm} (6.55)

For arbitrarily positive sequence \( a_n \), we have
\[ \sup_{a_n \in \mathbb{R}^+} |\varphi^{a_n}(\tau)| = \sup_{a_n \in \mathbb{R}^+} \int_{-\infty}^{\infty} e^{itx} \pi^{a_n}(x) dx \leq \sup_{a_n \in \mathbb{R}^+} \int_{-\infty}^{\infty} e^{itx} \bar{\pi}^{a_n}(x) dx = 1. \]
In addition, $\pi^a(x)$ is integrable, by Riemann-Lebesgue theorem, it holds when $|\tau| \to \infty$

$$\sup_{a_n \in \mathbb{R}^+} |\varphi^{a_n}(\tau)| \to 0.$$  

Thus for any strictly positive $\omega$, there exists some corresponding $N_\omega$ such that if $|\tau| > \omega$, it holds

$$\sup_{a_n \in \mathbb{R}^+} |\varphi^{a_n}(\tau)| < N_\omega < 1. \quad (6.56)$$

We now turn to (6.55) which is splitted on $|\tau| > \omega \sqrt{n}$ and on $|\tau| \leq \omega \sqrt{n}$.

It holds

$$\sqrt{n} \int_{|\tau| > \omega \sqrt{n}} \left| (\varphi^{a_n}(\tau/\sqrt{n}))^n - e^{\frac{i}{2}\tau^2} - \frac{\mu_3}{6\sqrt{n}s^3} (i\tau)^3 e^{-\frac{i}{2}\tau^2} \right| d\tau$$

$$\leq \sqrt{n} N_\omega^{n-2} \int_{|\tau| > \omega \sqrt{n}} \left| (\varphi^{a_n}(\tau/\sqrt{n}))^2 \right|^2 d\tau + \sqrt{n} \int_{|\tau| \leq \omega \sqrt{n}} e^{\frac{i}{2}\tau^2} \left(1 + \frac{\mu_3 \tau^3}{6\sqrt{n}s^3}\right) d\tau. \quad (6.57)$$

where the first term of the last line tends to 0 for $n$ large enough, since

$$\sqrt{n} N_\omega^{n-2} \int_{|\tau| > \omega \sqrt{n}} \left| (\varphi^{a_n}(\tau/\sqrt{n}))^2 \right|^2 d\tau$$

$$= \exp \left( \frac{1}{2} \log n + (n-2) \log N_\omega + \log \int_{|\tau| > \omega \sqrt{n}} \left| (\varphi^{a_n}(\tau/\sqrt{n}))^2 \right|^2 d\tau \right) \to 0, \quad (6.58)$$

where the last step holds from Lemma (6.53) and (6.56). As for the second term of (6.57), by Corollary (3.1), for $n$ large enough, we have $|\mu_3/s^3| \to 0$. Hence it holds for $n$ large enough

$$\sqrt{n} \int_{|\tau| > \omega \sqrt{n}} e^{\frac{i}{2}\tau^2} \left(1 + \frac{\mu_3 \tau^3}{6\sqrt{n}s^3}\right) d\tau$$

$$\leq \sqrt{n} \int_{|\tau| > \omega \sqrt{n}} e^{\frac{i}{2}\tau^2} |\tau|^3 d\tau = \sqrt{n} \int_{|\tau| > \omega \sqrt{n}} \exp \left\{ -\frac{1}{2} \tau^2 + 3 \log |\tau| \right\} d\tau$$

$$= 2\sqrt{n} \exp \left( -\omega^2 n/2 + o(\omega^2 n/2) \right) \to 0, \quad (6.59)$$

where the second equality holds from, for example, Chapter 4 of \cite{1}. (6.57), (6.58) and (6.59) implicate that, for $n$ large enough

$$\int_{|\tau| > \omega \sqrt{n}} \left| (\varphi^{a_n}(\tau/\sqrt{n}))^n - e^{\frac{i}{2}\tau^2} - \frac{\mu_3}{6\sqrt{n}s^3} (i\tau)^3 e^{-\frac{i}{2}\tau^2} \right| d\tau = o\left(\frac{1}{\sqrt{n}}\right). \quad (6.60)$$

If $|\tau| \leq \omega \sqrt{n}$, it holds

$$\int_{|\tau| \leq \omega \sqrt{n}} \left| (\varphi^{a_n}(\tau/\sqrt{n}))^n - e^{\frac{i}{2}\tau^2} - \frac{\mu_3}{6\sqrt{n}s^3} (i\tau)^3 e^{-\frac{i}{2}\tau^2} \right| d\tau$$

$$= \int_{|\tau| \leq \omega \sqrt{n}} e^{\frac{i}{2}\tau^2} \exp \left\{ n \log \varphi^{a_n}(\tau/\sqrt{n}) + \frac{1}{2} \right\} - 1 - \frac{\mu_3}{6\sqrt{n}s^3} (i\tau)^3 d\tau. \quad (6.61)$$
The integrand in the last display is bounded through
\[ |e^\alpha - 1 - \beta| = |(e^\alpha - e^\beta) + (e^\beta - 1 - \beta)| \leq (|\alpha - \beta| + \frac{1}{2} \beta^2) e^\gamma, \]
(6.62)
where \( \gamma \geq \max(|\alpha|, |\beta|) \); this inequality follows replacing \( e^\alpha, e^\beta \) by their power series, for real or complex \( \alpha, \beta \). Denote by
\[ \gamma(\tau) = \log \varphi^{(n)}(\tau) + \frac{1}{2} \tau^2. \]
Since \( \gamma'(0) = \gamma''(0) = 0 \), the third order Taylor expansion of \( \gamma(\tau) \) at \( \tau = 0 \) yields
\[ \gamma(\tau) = \gamma(0) + \gamma'(0)\tau + \frac{1}{2} \gamma''(0)\tau^2 + \frac{1}{6} \gamma'''(\xi)\tau^3 = \frac{1}{6} \gamma'''(\xi)\tau^3, \]
where \( 0 < \xi < \tau \). Hence it holds
\[ \left| \gamma(\tau) - \frac{\mu_3}{6s^3}(i\tau)^3 \right| = \left| \gamma'''(\xi) - \frac{\mu_3}{s^3} i^3 \right| \left| \frac{\tau}{\sqrt{n}} \right|^3. \]
Here \( \gamma''' \) is continuous; thus we can choose \( \omega \) small enough such that \( |\gamma'''(\xi)| < \rho \) for \( |\tau| < \omega \). Meanwhile, for \( n \) large enough, according to Corollary (5.1), we have \( |\mu_3/s^3| \to 0 \). Hence it holds for \( n \) large enough
\[ \left| \gamma(\tau) - \frac{\mu_3}{6s^3}(i\tau)^3 \right| \leq \left( |\gamma'''(\xi)| + \rho \right) \left| \frac{\tau}{\sqrt{n}} \right|^3 < \rho \left| \frac{\tau}{\sqrt{n}} \right|^3. \]
(6.63)
Choose \( \omega \) small enough, such that for \( n \) large enough it holds for \( |\tau| < \omega \)
\[ \left| \frac{\mu_3}{6s^3}(i\tau)^3 \right| \leq \frac{1}{4} \tau^2, \quad |\gamma(\tau)| \leq \frac{1}{4} \tau^2. \]
For this choice of \( \omega \), when \( |\tau| < \omega \) we have
\[ \max \left( \left| \frac{\mu_3}{6s^3}(i\tau)^3 \right|, |\gamma(\tau)| \right) \leq \frac{1}{4} \tau^2. \]
(6.64)
Replacing \( \tau \) by \( \tau/\sqrt{n} \), it holds for \( |\tau| < \omega \sqrt{n} \)
\[ \left| n \log \varphi^{(n)}(\tau/\sqrt{n}) + \frac{1}{2} \tau^2 - \frac{\mu_3}{6\sqrt{n}s^3}(i\tau)^3 \right| = n \left| \gamma \left( \frac{\tau}{\sqrt{n}} \right) - \frac{\mu_3}{6s^3} \left( \frac{i\tau}{\sqrt{n}} \right)^3 \right| < \frac{\rho |\tau|^3}{\sqrt{n}}, \]
(6.65)
where the last inequality holds from (6.63). In a similar way, with (6.64), it also holds for \( |\tau| < \omega \sqrt{n} \)
\[ \max \left( \left| n \log \varphi^{(n)}(\tau/\sqrt{n}) + \frac{1}{2} \tau^2 \right|, \left| \frac{\mu_3}{6\sqrt{n}s^3}(i\tau)^3 \right| \right) \]
\[ = n \max \left( \left| \gamma \left( \frac{\tau}{\sqrt{n}} \right) \right|, \left| \frac{\mu_3}{6s^3} \left( \frac{i\tau}{\sqrt{n}} \right)^3 \right| \right) \leq \frac{1}{4} \tau^2. \]
(6.66)
Apply (6.62) to estimate the integrand of last line of (6.61), with the choice of \( \omega \) in (6.63) and (6.64), using (6.65) and (6.66) we have for \( |\tau| < \omega \sqrt{n} \)

\[
\left| \exp \left\{ n \log \varphi^{\alpha_n}(\tau/\sqrt{n}) + \frac{1}{2} \tau^2 \right\} - 1 - \frac{\mu_3}{6\sqrt{n}s^3}(i\tau)^3 \right|
\leq \left( n \log \varphi^{\alpha_n}(\tau/\sqrt{n}) + \frac{1}{2} \tau^2 - \frac{\mu_3}{6\sqrt{n}s^3}(i\tau)^3 \right) + \left| \frac{1}{2} \frac{\mu_3}{6\sqrt{n}s^3}(i\tau)^3 \right|^2
\times \exp \left[ \max \left( \left| n \log \varphi^{\alpha_n}(\tau/\sqrt{n}) + \frac{1}{2} \tau^2 \right|, \left| \frac{\mu_3}{6\sqrt{n}s^3}(i\tau)^3 \right| \right] \right)
\leq \left( \frac{\rho |\tau|^3}{\sqrt{n}} + \frac{1}{2} \frac{\mu_3}{6\sqrt{n}s^3}(i\tau)^3 \right) \exp \left( \frac{\tau^2}{4} \right) = \left( \frac{\rho |\tau|^3}{\sqrt{n}} + \frac{\mu_3^2 \tau^6}{72n s^6} \right) \exp \left( \frac{\tau^2}{4} \right).
\]

Use this upper bound to (6.61), we obtain

\[
\int_{|\tau| \leq \omega \sqrt{n}} \left| (\varphi^{\alpha_n}(\tau/\sqrt{n}))^n - e^{-\frac{1}{2} \tau^2} - \frac{\mu_3}{6\sqrt{n}s^3}(i\tau)^3 e^{-\frac{1}{2} \tau^2} \right| d\tau
\leq \int_{|\tau| \leq \omega \sqrt{n}} \exp \left( - \frac{\tau^2}{4} \right) \left( \frac{\rho |\tau|^3}{\sqrt{n}} + \frac{\mu_3^2 \tau^6}{72n s^6} \right) d\tau
\leq \frac{\rho}{\sqrt{n}} \int_{|\tau| \leq \omega \sqrt{n}} \exp \left( - \frac{\tau^2}{4} \right) |\tau|^3 d\tau + \frac{\mu_3^2}{72n s^6} \int_{|\tau| \leq \omega \sqrt{n}} \exp \left( - \frac{\tau^2}{4} \right) \tau^6 d\tau,
\]

where both the first integral and the second integral are finite, and \( \rho \) is arbitrarily small; additionally, by Corollary (3.1), \( \mu_2 / s^6 \to 0 \) when \( n \) large enough, hence it holds for \( n \) large enough

\[
\int_{|\tau| \leq \omega \sqrt{n}} \left| (\varphi^{\alpha_n}(\tau/\sqrt{n}))^n - e^{-\frac{1}{2} \tau^2} - \frac{\mu_3}{6\sqrt{n}s^3}(i\tau)^3 e^{-\frac{1}{2} \tau^2} \right| d\tau = o \left( \frac{1}{\sqrt{n}} \right), \tag{6.67}
\]

Now (6.60) and (6.61) give (6.59). Further, using (6.55) and (6.52), we obtain

\[
\left| \rho_n(x) - \phi(x) - \frac{\mu_3}{6\sqrt{n}s^3} (x^3 - 3x) \phi(x) \right| = o \left( \frac{1}{\sqrt{n}} \right),
\]

which concludes the proof. \( \square \)

### 6.3 Proof of Lemma 5.1

**Proof.** When \( n \to \infty \), it holds

\[
z_i \sim m_i / s_i \sqrt{n} - i - 1 \sim m_i / (s_i \sqrt{n}).
\]

From Theorem 3.1 it holds \( m(t) \sim \psi(t) \) and \( s(t) \sim \sqrt{\psi'(t)} \). Hence we have

\[
z_i \sim \frac{\psi(t_i)}{\sqrt{n} \psi'(t_i)}. \tag{6.68}
\]

By (5.1), \( m_i \sim m(t) \) as \( n \to \infty \). Then

\[
m_i \sim \psi(t) = a_n,
\]

26
In addition, $m_i \sim \psi(t_i)$ by Theorem 3.1, this implies
\[ \psi(t_i) \sim \psi(t). \]  

**Case 1:** if $h(x) \in R_\beta$. We have $h(x) = x^\beta l_0(x), l_0(x) \in R_0, \beta > 0$. Hence
\[ h'(x) = x^{\beta-1} l_0(x)(\beta + \epsilon(x)), \]
set $x = \psi(u)$, we get
\[ h'(\psi(u)) = (\psi(u))^{\beta-1} l_0(\psi(u))(\beta + \epsilon(\psi(u))). \]

Notice $\psi'(u) = 1/h'(\psi(u))$, combine (6.69) with (6.70), we obtain
\[ \frac{\psi'(t_i)}{\psi'(t)} = \frac{h'(\psi(t))}{h'(\psi(t_i))} = \frac{(\psi(t))^{\beta-1} l_0(\psi(t))(\beta + \epsilon(\psi(t)))}{(\psi(t_i))^{\beta-1} l_0(\psi(t_i))(\beta + \epsilon(\psi(t_i)))} \rightarrow 1, \]
where we use the slowly varying propriety of $l_0$. Thus it holds
\[ \psi'(t_i) \sim \psi'(t), \]
which, together with (6.69), is put into (6.68) to yield
\[ z_i \sim \frac{\psi(t)}{\sqrt{n\psi'(t)}}. \]  
Hence we have under condition (5.2)
\[ z_i^2 \sim \frac{\psi(t)^2}{m\psi'(t)} = \frac{\psi(t)^2}{\sqrt{n\psi'(t)}} \frac{1}{\sqrt{n}} = o\left(\frac{1}{\sqrt{n}}\right), \]
which implies further $z_i \to 0$. Note that the final step is used in order to relax the strength of the growth condition on $a_n$.

**Case 2:** if $h(x) \in R_\infty$. By (5.1), it holds $m(t_i) \geq m(t)$ as $n \to \infty$. Since the function $t \to m(t)$ is increasing, we have $t \leq t_i$.

Notice the function $x \to \psi(x)$ is also increasing, we get
\[ \psi(t_i) \geq \psi(t). \]

The function $x \to \psi'(x)$ is decreasing, since
\[ \psi''(x) = -\frac{\psi(x)}{x^2}(1 + o(1)) < 0 \quad \text{as} \quad x \to \infty. \]
Therefore as $n \to \infty$,
\[ \psi'(t) \geq \psi'(t_i) > 0. \]
Perform one Taylor expansion of $\psi(t_i)$ for some $\theta_1 \in (0, 1)$

$$
\psi(t_i) - \psi(t) = \psi'(t)(t_i - t) + \frac{1}{2} \psi''(t + \theta_1(t_i - t))(t_i - t)^2
$$

$$
= \frac{\psi(t)\epsilon(t)}{t}(t_i - t) + \frac{1}{2} \psi''(t + \theta_1(t_i - t))(t_i - t)^2. \quad (6.75)
$$

By (6.69)

$$
\frac{\psi(t_i) - \psi(t)}{\psi(t)} \to 0,
$$

which together with (6.74) and (6.75) yields

$$
\frac{\epsilon(t)}{t}(t_i - t) \to 0. \quad (6.76)
$$

Perform one Taylor expansion of $\psi'(t_i)$ for some $\theta_2 \in (0, 1)$

$$
\psi'(t_i) - \psi'(t) = \psi''(t)(t_i - t) + \frac{1}{2} \psi'''(t + \theta_2(t_i - t))(t_i - t)^2
$$

$$
= -\frac{\psi(t)\epsilon(t)}{t^2}(t_i - t)(1 + o(1)) + \frac{1}{2} \psi''(t + \theta_2(t_i - t))(t_i - t)^2,
$$

where the first term goes to 0 as $n \to \infty$ by (6.76), and the second term is infinitely small with respect to the first term (see Section 3, e.g. (6.13)). Hence

$$
\psi'(t_i) \sim \psi'(t).
$$

The proof is completed by repeating steps (6.72) and (6.73). \qed

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