Thermal and Resonant Emission of Dark Ages Halos in the Rotational Lines of HeH$^+$

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We present the main features of thermal and resonant emission for one of the first radiating molecules, the helium hydride ion (HeH$^+$), from dark ages halos on the stage of the early Universe. Evaluating the optical depth, thermal and resonant brightness temperatures and spectral fluxes of dark ages halos is based on computing the cross-sections and rate coefficients of excitation/de-excitation of the lowest five rotational states of HeH$^+$ by inelastic collisions with atomic hydrogen. It was shown that in Dark Ages the collisional excitation/de-excitation by atoms of neutral hydrogen and electrons are competitive, but in the denser regions, e.g. virialized halos, the contribution of collisions with atomic hydrogen is larger. We demonstrate the peak time dependence in the thermal and resonance luminosities evolution of halo in HeH$^+$ lines, and the optimization of observations can be made concerning for redshift.

Keywords: cosmology:

I. INTRODUCTION

In astrophysics, the interest in the helium hydride ion HeH$^+$ discovered in the lab in 1925 [22], is caused by its stability in the cosmic isolation and by its composition, since it consists of the most widespread atoms in the Universe - hydrogen and helium. It is not surprising that numerous discussions about the formation of this molecule and its search both in the Galaxy and the deep space have been undertaken since the 70s of the last century [10 11 13 27 28 30 31 34 52 57 58]. Unfortunately, the HeH$^+$ detection has not directly been confirmed for more than a half-century ago in [36]. Due to his results the detailed information about the internal structure of HeH$^+$ was obtained more than a half-century ago in [36].

Molecule HeH$^+$ plays a significant role in the emerging of first stars at the end of Dark Ages since it is one of the first molecules to appear in the early Universe among with $\text{H}_2$, HD, LiH, and their ions [6 8 14 19 32 41 46 47 49]. Molecules are known to be the unique coolers for baryonic matter in the early Universe at temperatures below $\sim 8000$ K [14]. Emissions of these molecules result in cooling, fragmentation, and collapse of molecular clouds. In particular, molecular ions HeH$^+$ is an effective cooler [5] due to the large value of their electric dipole moment, 1.722 D, emitting in the molecular lines in the ground rotational state. The population of the rotational levels in the epoch of Dark Ages is defined by collisions with photons of the cosmic microwave background (CMB), free electrons, ions and atoms. Calculations of coefficients for rotational and vibrational excitation/de-excitation rates by electrons are performed by [4 24 29 51]. However, up to date, there are no estimations for HeH$^+$ rotational excitation/de-excitation by collisions with atomic hydrogen.

The rest of the paper is organized as follows: In Section 2., the potential energy surface (PES) for H–HeH$^+$ collisions is obtained in analytical and numerical approximations. In Section 3. the state-to-state integral cross sections for rotational transitions during H–HeH$^+$ collisions are obtained and corresponding rate coefficients are calculated. The results for rate coefficients are presented in graphical form and analytical approximations. In Section 4. the role of collisional excitations of low rotational levels of HeH$^+$ in Dark Ages is discussed. In Section 5. the differential thermal and resonant brightness temperatures, as well as spectral fluxes from dark ages halos in rotational lines of the helium hydride ion HeH$^+$ are evaluated. The conclusions are in Section VI.

II. PES FOR H-HeH$^+$ INTERACTION

We treat the molecular ion HeH$^+$ as a rigid rotor and the interaction potential between the colliding partners is fully specified by two Jacobi coordinates $V(r, \theta)$ as it is showed in Fig. 1 where $r$ is the distance between the center of mass of the molecular ion and the atom H, and $\theta$ is the angle between the molecular symmetry axis, which coincides with the $z$-axis and the vector $\vec{r}$.

To write down the long-range interaction potential between atomic hydrogen and helium hydride ion, we need detail information about the internal structure of HeH$^+$. The quantum mechanical solutions for the wave function of the electrons at the ground and excited low-energy levels for ion molecule HeH$^+$ was obtained more than a half-century ago in [36]. Due to his results the $^1\Sigma$ ground two-electron wave function of HeH$^+$ can be written as $\psi(\xi_1, \eta_1; \xi_2, \eta_2) = \phi_A(\xi_1, \eta_1) \cdot \phi_B(\xi_2, \eta_2) + \phi_A(\xi_2, \eta_2) \cdot \phi_B(\xi_1, \eta_1)$, where $\phi_A(\xi, \eta) = \exp\{-\delta_A \xi - \zeta_A \eta\}$, $\phi_B(\xi, \eta) = \exp\{-\delta_B \xi - \zeta_B \eta\}$, labels 1 and 2 correspond
to the first and second electrons, ξ and η are the prolate ellipsoidal coordinates connected to spherical coordinates by means of the following relations [33]:

\[ \xi = \frac{r_A + r_B}{R}, \quad \eta = \frac{r_A - r_B}{R}, \]

so that \( 1 \leq \xi \leq \infty, -1 \leq \eta \leq 1 \) and 0 \( \leq \phi \leq 2\pi \), \( A \) and \( B \) denote the He and H nucleus in HeH\(^{+}\) ion molecule, \( r_A \) and \( r_B \) are radial distances from these nuclei. As it follow from [36], the minimal bound energy of HeH\(^{+}\) correspond to the interatomic distance \( R = 0.772 \) Å, for which the approximation parameters are as follows \( \delta_A = 1.5311143, \quad \zeta_A = 1.6848429, \quad \delta_B = 1.2168429, \quad \zeta_B = 0.1503286. \)

Since \( \psi(\xi_1, \eta_1; \xi_2, \eta_2) \) is symmetric under replacement of the electrons, we can obtain the probability density for electrons spatial localization as

\[ \psi^2(\xi, \eta) = 2\pi \frac{R}{2} \int \left[ \int_{-1}^{1} |\psi(\xi, \eta; \xi', \eta')|^2 (\xi'^2 - \eta'^2) d\xi' d\eta' \right] \frac{1}{R}, \]

where we take into account that the volume in ellipsoidal coordinates is as follows \( dV = (R/2)^3 (\xi^2 - \eta^2) d\xi d\eta d\phi \) and spherical coordinates are expressed through the ellipsoidal coordinates by means of the well-known expressions

\[ r_k = \frac{R}{2} (\xi + k\eta), \]

\[ \cos(\theta_k) = \frac{1 + k\xi\eta}{\xi + k\eta}, \]

\[ \sin(\theta_k) = \frac{\sqrt{({\xi}^2 - 1)(1 - \eta^2)}}{\xi + k\eta}. \]

where the value of \( k \) is equal to +1 if the center of coordinate system is located in the center \( A \) or -1 if it is located in the center \( B \). It is clear that [11] can be expressed as \( \psi^2(\xi, \eta) = c_1\phi_A(\xi, \eta) \cdot \phi_A(\xi, \eta) + c_2\phi_B(\xi, \eta) \cdot \phi_B(\xi, \eta) + 2c_3\phi_A(\xi, \eta) \cdot \phi_B(\xi, \eta) \), where \( c_1 = 4.995054, \quad c_2 = 6.339531 \) and \( c_3 = 4.492389. \)

In Fig. 1 we place the ion molecule HeH\(^{+}\) along z axis so that its center of mass coincide with the center of the coordinate system while helium and hydrogen nucleus are placed at \( z_{He} = -0.2R \) and \( z_{H^{+}} = 0.8R \) correspondingly. The center of the two-electron cloud is on the z-axis at

\[ z_{2e} = \int \int \int \frac{z}{V} \psi^2 dV \]

\[ = 2\pi \frac{R}{2} \int \int_{-1}^{1} (0.6 - \xi\eta) |\psi(\xi, \eta)|^2 (\xi^2 - \eta^2) d\xi d\eta, \]

where we take into account that \( z = r\cos(\theta) = r_A \cos(\theta_A) + z_{He} \). Integration gives \( z_{2e} = -0.032R \), that means that the center of electrons cloud placed between helium and hydrogen nucleus and is very close to the mass center of the ion molecule. Since the center of two-electrons cloud is only a bit shifted from the helium nucleus we can consider helium as a neutral atom which is polarized in the external electric field of hydrogen nucleus. The electric dipole moment of such polarized helium is equal to \( \mu_{He} = 2e(z_{He} - z_{2e}) \approx -0.259497 e\AA \). The total dipole moment of HeH\(^{+}\) is equal to \( \mu_{He+H^{+}} = \mu_{He} + ezB = 0.35810 e\AA \), or in an atomic unit \( \mu_{He+H^{+}} = 0.67694 ea_B \) and in a CGS unit \( \mu_{He+H^{+}} = 1.72165 D \). The obtained value for dipole moment of HeH\(^{+}\) coincides with the estimation obtained by [11] (use Table.3 in that paper and make interpolation for \(^4\)HeH\(^{+}\) at the equilibrium interatomic distance 1.455a\(_B\)).

So, the long-range interaction potential between
atomic hydrogen and helium hydride ion can be written as follows

\[ V_{L,H-HeH^+} = -\mu_H \vec{E}_{HeH^+}, \]  

(3)

where \( \mu_H = \frac{1}{2} \alpha_H \vec{E}_{HeH^+} \) is the induced dipole moment of atomic hydrogen in the external electric field of ion molecule \( \text{HeH}^+ \), and \( \alpha_H = 9/2 \) is the polarizability of neutral hydrogen. The external electric field of ion molecule \( \text{HeH}^+ \) we represent as the sum of two components \( \vec{E}_{HeH^+} = \vec{E}_{He} + \vec{E}_{H^+} \) where electric fields \( \vec{E}_{He} \) and \( \vec{E}_{H^+} \) have next components:

\[
\vec{E}_{He} = \left( \frac{\mu_H}{r_A^3} \right) \sin(\theta_A), \quad \vec{E}_{H^+} = \left( \frac{e}{r_B} \right) \left( \cos(\theta) - x \right),
\]

(4)

(5)

where \( r_B = r \sqrt{1 + x^2 - 2x \cos(\theta)} \), \( x = z_H/r \), and we can use approximations \( \theta_A \approx \theta \) and \( r_A \approx r \).

The expression (3) is a good approximation to long-range interaction potential for \( r \geq 10 \, \text{Å} \). To show this let us consider the long-range interaction potential between proton and hydrogen \( V_{L,HH^+} = -\mu_H \vec{E}_{H^+} \) well approximated at large distance \( r \) by well known expression for charge-induced dipole interaction \( V_L(r) = -\frac{r^2}{2} \alpha_H r^{-4} \). To describe deviations of this analytical potential from the results of accurate numerical calculations for proton-hydrogen interaction performed by [21] (see Table 1 of that paper) we introduce the screening function

\[ F(r) = (C_{00}(r) - C_{00}(\infty))/V_L(r), \]

(6)

that, as it is shown by the red dashed line in Fig. [2] at distances \( r < 10 \, \text{Å} \) deviates from unity. Therefore we have to modify approximation for potential interaction between proton and hydrogen in [3] as follows

\[ V_{HH^+}(r, \theta) = F(r_B(r, \theta)) \cdot V_{L,HH^+}(r, \theta). \]

Since the deviation of the screening function \( F \) from unity is a result of the internal reordering of charges

within atomic hydrogen caused predominantly by the inhomogeneity of an external electric field of the proton, we need to take this into account also for the dipole interaction of hydrogen and helium, i.e. \( V_{HH^+}(r, \theta) = F(r_B(r, \theta)) V_{L,HH^+}(r, \theta) \), where \( V_{HH^+} = -\mu_H \vec{E}_{He} \) is the long-range interaction between polarized hydrogen and helium. Thus, the modified long-range interaction between atomic hydrogen and the helium hydride ion take a form \( V_{HH^+}(r, \theta) \approx F(r_B(r, \theta)) V_{L,HH^+}(r, \theta) \).

In order to write down the interaction potential between atomic hydrogen and helium hydride ion that is valid at interatomic distances \( r_A < 5 \, \text{Å} \) and/or \( r_B < 5 \, \text{Å} \) (see Fig. [1]) we used the analytical approximation for the PES proposed in [26] to the data of numerical calculations performed in [35]:

\[
V(R, r_A, r_B) = V_{H-He}(R) + V_{H-he}(r_A) + V_{H-H^+}(r_B) + V_{L}(R, r_A, r_B),
\]

(7)

where two-atomic potential \( V_{H-H} \) (solid black line in Fig. [2]) exponentially suppressed at \( r > 5 \, \text{Å} \) (i.e. \( V_{H-H}/V_L \approx 0 \), see dashed black line in Fig. [2]) we replaced by our approximation \( V_{HH^+} \) (solid red line in Fig. [2]) that posses correct extrapolation \( V_{HH^+}/V_L \approx 1 \) at large \( r \) (dashed red line in Fig. [2]). The two-atomic potential \( V_{H-He} \) in [7] is also exponentially suppressed at \( r_A > 5 \, \text{Å} \), so at these interatomic distances we replaced it by our approximation \( V_{He} \) described above.

In Fig. [4] the resulting PES considered for \( \text{H--HeH}^+ \) interaction is shown as a contour map in the plane of two Jacobi coordinates, \( r \), and \( \theta \). A 0.72 eV deep van der Waals well is placed at approximately \( r \approx 1.83 \, \text{Å} \) and \( \theta = 0^\circ \) (see Fig. [3]). Potential has a repulsive barrier at \( r \approx 1.5 \, \text{Å} \) for all Jacobi angles.

We have also tried other short-range PES approximations designed to describe chemical reaction \( \text{He} + \)
H\textsuperscript{+}\textsubscript{2} \rightarrow \text{HeH}^+ + \text{H}, which are valid at interatomic distances \( r_A, r_B < 5 \) \AA and are considered more accurate \textit{1,2,50,53,54}. However, as we found, with the exception of PES from \textit{1}, that approximates the same numerical data from \textit{25} and \textit{26} it does, the others PESs cannot be easily extrapolated on large interatomic distances by replacement of short-range two-atomic potential \( V_{H-H} \) by its long-range version from \textit{21} because of their difference at small interatomic distances. We believe that the approximation for PES used here is sufficiently accurate for our task.

III. STATE-TO-STATE INTEGRAL CROSS SECTIONS AND RATE COEFFICIENTS

The rovibrational energy structure of \(^4\text{HeH}^+\) (i.e. \text{HeH}^+) and its isotopes \(^3\text{HeH}^+, \text{HeD}^+, \) and \(^4\text{HeD}^+, \) were calculated previously in \textit{11} and results are available with corresponding Einstein coefficients at \texttt{www.exomol.com}. In Table \textit{I} we listed this data for six lowest rotational levels within the vibrational ground state of \text{HeH}^+. The MOLSCAT code (ver.14, \textit{23}) was used to calculate state-to-state cross sections for inelastic scattering of hydrogen atoms on \text{HeH}^+ molecular ions. The calculated inelastic scattering cross-section for low-level rotational transitions in \text{HeH}^+ ion as a function of collision energy is shown in Fig. \textit{5}.

TABLE I. The lowest rotational energy levels data for \text{HeH}^+.

| Frequency \( \nu_u \) [GHz] | Transitions \( j_u - j_l \) | \( E_u \) [K] | \( A_u \) [s\(^{-1}\)] |
|------------------------|----------------|---------|---------|
| 2010                   | 1 - 0          | 96      | 0.109   |
| 4009                   | 2 - 1          | 289     | 1.04    |
| 5984                   | 3 - 2          | 576     | 3.75    |
| 7925                   | 4 - 3          | 956     | 9.14    |
| 9821                   | 5 - 4          | 1428    | 18.1    |

By assuming a Maxwell–Boltzmann distribution over the collision energy, we calculated the state selected de-excitation rate coefficients as follows

\[
k_{u\rightarrow l}(T) = \left[ \frac{8}{\pi \mu k_B T^3} \right]^{1/2} \exp \left( \frac{\epsilon_u}{k_B T} \right) \times \int_0^\infty \sigma_{u\rightarrow l}(E)(E - \epsilon_u) \exp \left( -\frac{E}{k_B T} \right) dE, \tag{8}
\]

where \( \epsilon_u \) is the energy of the \( u \)-th rotational level, \( \mu \) is the reduced mass and \( E \) is the total energy equal to the sum of collision energy and energy of the initial state of the molecular ion, \( \epsilon_u \). The results of the calculations are shown on the upper panel in Fig. \textit{6}. The reverse transition rate (excitation) coefficients can be obtained through the expression

\[
k_{l\rightarrow u}(T) = \frac{2j_u + 1}{2j_l + 1} \exp \left\{ (\epsilon_u - \epsilon_l)/k_B T \right\} k_{u\rightarrow l}(T). \tag{9}
\]

IV. \text{HeH}^+ COLLISIONAL (DE)EXCITATION IN DARK AGES

To understand the role of collisional de-excitations of \text{HeH}^+ by atomic hydrogen we need to compare them with collisional de-excitations this ion molecule by free electrons as well as with spontaneous emission. For this in Table \textit{VII} in Appendix we present the estimation of the critical densities in \text{cm}^{-3} for collisional \( e^-\text{HeH}^+ \) and \( \text{H}^-\text{HeH}^+ \) de-excitations which are defined for a rotational level \( j \) as follows

\[
n_{e,cr}(j) = \sum_{j' < j} A(j \rightarrow j') \frac{k_e(j \rightarrow j')}{k_e(j' \rightarrow j'), \tag{11}
\]

\[
n_{HI,cr}(j) = \sum_{j' < j} A(j \rightarrow j') \frac{k_H(j \rightarrow j')}{k_H(j' \rightarrow j'). \tag{12}
\]

where we can assume that the Einstein \( A \)-coefficients are dominated for \( \Delta j = 1 \) transitions, the rate coefficients for rotational transitions induced by collisions with free
electrons, for $k_e(j \rightarrow j')$ we used the approximations given by [29]. Since, the rates of collisional de-excitations by electrons and atomic hydrogen are proportional to their number densities,

$$C^e_{ij} = n_e k_e, i,j, \quad C^H_{ij} = n_{HI} k_{HI,ij},$$

one can compare their values by ratio

$$\frac{C^e_{ij}}{C^H_{ij}} = \frac{n_{HI,cr}(j)}{n_e} \frac{n_e}{n_{HI}}.$$

The last term in the right hand side of expression in the conditions of dark ages halos is close to the value of degree of hydrogen ionization or electron-proton fraction $x_p \equiv x_e \equiv n_e/(n_{HI} + n_{HI})$. In Figure 7 we present the ratios of critical densities $n_{e,cr}/n_{HI,cr}$ for rotational levels with $j = 1 - 4$ and ratio $n_e/n_{HI}$ for different gas temperatures in the Dark Ages and in the halo which is virialized at $z \approx 30$. One can see that ratio $n_e/n_{HI}$ in Dark Ages at $z < 100$ is a bit lower than ratios $n_{e,cr}/n_{HI,cr}$ for levels with $j = 1, 2, 3$, and a bit larger for level with $j = 4$. It means that collisional excitation/de-excitation of the lowest rotational levels of ion molecule HeH$^+$ by electrons and atomic hydrogen are comparable and both must be taken into account.

The fraction of electrons in the halo decreases over time due to their recombinaton since the process of electron recombination is stronger in denser regions and less for areas with lower gas concentration [39]. So, de-excitations of the HeH$^+$ rotational levels through collisions with atomic hydrogen dominate over the de-excitations through collisions with electrons in more dense regions such as halos. As an example, the filled circles in Fig. 7 show the evolution of ratio $n_e/n_{HI}$ in the overdensity region that reaches of virialization at $z = 30$. One can see, that number density of electrons decreases after virialization despite the constant gas density (see for details Fig. 4 in [41]). As a result, at $z = 10$ de-excitations of the HeH$^+$ rotational levels through collisions with atomic hydrogen start to dominate over the de-excitations in the electron-HeH$^+$ inelastic collisions by two orders of magnitude.

V. EMISSION OF DARK AGES HALOS IN THE ROTATIONAL LINES OF HeH$^+$

Dark ages hide the mystery of the formation of the first stars in the Universe, also known as the first-generation stars or Population III stars. The formation of stars is known occurs during the gravitational compression of protostar gas clouds, which can cool by converting the kinetic energy of atoms and molecules into electromagnetic radiation in the processes of their inelastic scattering. Gas cooling is inefficient when the energy of collisions between atoms and molecules becomes less than their lowest excitation threshold. The subsequent collapse of the protostar cloud would continue if its mass, thanks cooling, will be larger than the Jeans mass, which depends
on the gas temperature as \( \sim T^{3/2} \). Thus, the lowest energy of excitation of gas ingredients is connected with the lower mass bound of the forming stars. The lowest excitation energies correspond to lowest rotational and vibrational transitions of the molecules as well as the excitation of the external electrons in the heavy multi-electron atoms. However, gas in the Dark Ages does not contain heavy elements, since they were synthesized somewhat later in the nuclei of the first stars or during their supernova bursts. Therefore, only small-mass atomic molecules such as H, HD, HeH\(^{+}\), LiH, etc. were involved in the processes of gas cooling in the Dark Ages. Despite the small abundance of these molecules in the early Universe \([3,11,22]\), they played a crucial role in the formation of the first stars. Therefore, observation of these molecules in the Dark Ages is one of the most important tasks of modern cosmology. Below, we estimate the differential brightness temperatures caused by thermal collisions and resonant scattering of CMB quanta in the rotational lines of HeH\(^{+}\) ions.

### A. Halo formation and chemistry in the Dark Ages

The set of the physical conditions and chemistry of the halos in the Dark Ages we obtain by modeling the evolution of individual spherical perturbations in the four-component Universe (cold dark matter, baryon matter, dark energy, and thermal relict radiation) by integrating the appropriate system of equations described in \([10,11]\). The complete set of the cosmological parameters used in the paper is as follows: the Hubble constant \( H_0 = 70 \) \( \text{km/s/Mpc} \), the mean density of baryonic matter in the units of critical one \( \Omega_m = 0.05 \), the mean density of dark matter \( \Omega_m = 0.25 \), the mean density of dark energy \( \Omega_{de} = 0.7 \), its equation of state parameter \( w_{de} = -0.9 \), the effective sound speed \( c_s = 1 \) (in units of speed of light). All physical values and chemical composition of a halo with mass \( M_h = 5.3 \cdot 10^9 \, M_\odot \), which are necessary for computation of the excitations and brightness temperatures in the molecular rotational lines, are presented in Tab. II. The data for halos of other masses, \( 6.6 \cdot 10^8 \), \( 8.3 \cdot 10^8 \), \( 1.0 \cdot 10^9 \) and \( 1.3 \cdot 10^9 \, M_\odot \), are presented in Tab. VIII in Appendix.

For simplicity, we assume that halos are homogeneous top-hat spheres forming from the primordial cosmological density perturbations due to their gravitational instability at the moment of reaching the virial density \( \rho_{vir}(z_v) = \Delta_v \rho_m(z_v) \), where \( z_v \) is the redshift of halo virialization and we assume \( \Delta_v = 178 \). Since this moment, halos have treated as the static gravity-bound systems with constant densities, \( \rho_{vir}(z \leq z_v) = \rho_{vir}(z_v) \), where chemical kinetic continues. The mass of each halo \( M_h \) in the solar mass, its radius in comoving coordinates \( r_h \) [kpc] and the wave number \( k \) [Mpc\(^{-1}\)] of initial perturbation from which halo is formed, are connected by relations \([13]\):

\[
\frac{M_h}{M_\odot} = 1159 \Delta_v (1 + z_v)^3 \Omega_m h^2 r_h^3 = 4.5 \cdot 10^{12} \Omega_m h^2 k^{-3},
\]

where \( h \equiv H_0/100 \text{km/s/Mpc} \).

Since processes of virialization can take some time after virial density reached, we assume the adiabatic temperature of the gas inside newly formed halos. In this case, according to \([11]\), the number density of HeH\(^{+}\) inside virialized halos decreases with time due to a higher destruction rate of these molecular ions in a more dense medium. For halos with higher final temperatures of baryonic matter, the virial ones, the number density of HeH\(^{+}\) ions vanishes via dominating of destruction collisions \([11]\). Let us note that regardless of the assumption about the final baryonic gas temperature, there would be a peak-like time luminosity of halos in lines of molecular ion HeH\(^{+}\) with the maximum at the moment of virialization when the number density of HeH\(^{+}\) reached its maximum.

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**TABLE II.** Physical values and chemical composition of halos with mass \( M_h = 5.3 \cdot 10^9 \, M_\odot \) (wave number of corresponding linear perturbation is \( k = 5.0 \, \text{Mpc}^{-1} \)) virialized at different \( z_v \): \( C_k \) is the amplitude of initial curvature perturbation (seed of halo), \( z_v \) is the redshift of virialization, \( \rho_m \) is the matter density virialized halo, \( T_K \) is kinetic temperature of baryonic gas, \( n_{HI} \) is the number density of neutral hydrogen atoms, \( n_p, n_e \) are the number densities of protons and electrons at \( z = z_v/\Omega_m = 10 \), \( n_{HeH^+} \) is the number densities of ion molecule HeH\(^{+}\), \( r_h \) is the radius of halo in comoving coordinates, \( \theta_h \) is the angular radius of geometrically limited halo.

| \( C_k \) | \( z_v \) | \( \rho_m \) | \( T_K \) | \( n_{HI} \) | \( n_p \approx n_e \) | \( n_{HeH^+} \) | \( r_h \) | \( \theta_h \) |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| \( 10^{-4} \) | \[ \text{[g/cm}^3] \] | \[ \text{[K]} \] | \[ \text{[cm}^{-3} \] | \[ \text{[10}^{-6}\text{cm}^{-3} \] | \[ \text{[10}^{-15}\text{cm}^{-3} \] | \[ \text{kpc} \] | \[ \text{arcsec} \] |
| 3.0 | 30.41 | 1.52(-23) | 402.1 | 1.14 | 106.2/3.8 | 3.65 | 1.78 | 1.03 |
| 2.5 | 25.15 | 8.79(-24) | 298.9 | 0.66 | 66.1/4.0 | 2.63 | 2.14 | 1.05 |
| 2.0 | 19.90 | 4.49(-24) | 206.3 | 0.34 | 36.7/4.0 | 1.76 | 2.68 | 1.09 |
| 1.5 | 14.65 | 1.89(-24) | 124.3 | 0.14 | 17.0/4.4 | 1.05 | 3.60 | 1.15 |
| 1.0 | 9.41 | 5.55(-25) | 59.8 | 0.04 | 5.6/5.0 | 0.50 | 5.38 | 1.26 |
B. Population of HeH$^+$ excited states

The kinetic equations for populations of rotational levels of HeH$^+$ ions

$$\frac{dn_i}{dt} = \sum_{j \neq i} n_j R_{ji} - n_i \sum_{j \neq i} R_{ij}, \quad (13)$$

where indexes $i$ and $j$ mark rotational levels, $n_i$ – are number densities of molecules with state $i$, and $R_{ij}$ – are rate coefficients for up/down-levels transitions.

The number densities of e$^-$ and H species in halos formed in Dark Ages (see Tab. I and Tab. II) are much lower than their critical values given in Table. VII. By that means that the frequencies of e$^-$–HeH$^+$ and H–HeH$^+$ scattering excitation/de-excitation, $C_{ij}^e$ and $C_{ij}^H$, are much lower than the frequencies of CMB relic photon–HeH$^+$ excitation, $B_{nu}U_{nu}$, and spontaneous emission, $A_{ul}$. Besides, as can be seen from Fig. 8, all these values are essentially larger than the characteristic times of changes in the rate of expansion of the Universe and concentration of molecular ions HeH$^+$ within considered range of redshifts. Consequently, the values in the left-hand side of (13) can be ignored in comparison with values on the right side of these equations. So, to obtain the number densities of molecules in the ground and excited rotational states we can use the following system of the linear equations:

$$n_i \sum_{j \neq i} R_{ij} = \sum_{j} n_j R_{ji}, \quad \sum_{j} n_j = n_{HeH^+}, \quad (14)$$

where $R_{ij} = A_{ij} + B_{ij} U_{nu,ij} + C_{ij}$, $A_{ij}$ are not equal to zero only for transitions between adjacent levels from top to bottom and $C_{ij} = k_{ij}^H n_H + k_{ij}^e n_e$. The occupations of the rotational levels can be expressed in the terms of excitation temperatures as follows

$$T_{ex,ul} = \frac{h \nu_{ul}}{k_B} \left[ \ln \frac{g_{e} n_{e}}{g_{ul} n_{ul}} \right]^{-1},$$

where $u = l + 1$.

C. The brightness temperatures of dark ages halos

The optical depth of spherical homogeneous top-hat halo for rotational $\nu_{ul}$ line is as follows [38, 43]

$$\tau_{ul} = 1.55 \cdot 10^{50} n_i g_v A_{ul} \frac{g_{e} n_{e}}{g_{ul} \nu_{ul}} \sqrt{m_A T} \left[ 1 - \exp \left( - \frac{h \nu_{ul}}{k_B T_{ex}} \right) \right] r_h, \quad (15)$$

where $m_A$ is the mass of HeH$^+$ ion molecule in atomic units, and $r_h$ is the radius of the halo with mass $M_h$ which can be estimated as follows

$$r_h = 7 \cdot \left( \frac{M_h}{10^7 M_\odot} \right)^{1/3} \frac{1}{1 + z_v} \text{ kpc},$$

where $z_v$ is the moment of halo virialization. Results of computations $\tau$ for halos with mass $M_h = 5.3 \cdot 10^9 M_\odot$ are presented in Tab. III for other masses in Tab. IX in Appendix.

As it follows from the radiative transfer equation the Rayleigh-Jeans brightness temperature for thermal (th) emission can be obtained from the next expression

$$\delta T_{br,ul}^{th} = \frac{h \nu_{obs,ul}}{k_B} \left( \frac{1}{\nu_{hul}/k_B T_{ex} - 1} - \frac{1}{\nu_{hul}/k_B T_r - 1} \right) \left( 1 - e^{-\tau_{ul}} \right), \quad (16)$$

where $\nu_{obs,ul} = \nu_{ul}/(1 + z)$ is the observed frequency of $\nu_{ul}$ line emission from halo placed at redshift $z$. Results of computations $\delta T_{br,ul}^{th}$ for halos with mass $M_h = 5.3 \cdot 10^9 M_\odot$ are presented in Tab. IV for other masses in Tab. IX in Appendix.

Table III. The opacity of dark ages halos with mass $M = 5.3 \cdot 10^9 M_\odot$ in rotational lines of ion molecule HeH$^+$ formed at different redshifts $z = 10 - 30$. Powers of 10 are given in parentheses.

| $z$     | 0-1 | 1-2 | 2-3 | 3-4 |
|---------|-----|-----|-----|-----|
| 30.41   | 0.949(-6) | 0.815(-6) | 0.141(-6) | 0.681(-8) |
| 25.15   | 0.121(-5) | 0.787(-6) | 0.846(-7) | 0.206(-8) |
| 19.90   | 0.157(-5) | 0.687(-6) | 0.366(-7) | 0.321(-9) |
| 14.65   | 0.213(-5) | 0.491(-6) | 0.826(-8) | 0.133(-10) |
| 9.41    | 0.283(-5) | 0.195(-6) | 0.337(-9) | 0.183(-13) |

Table IV. The thermal brightness temperatures $\delta T_{br}^{th}$ [K] and spectral flux density $\delta F_{br,ul}^{th}$ [μJy] in rotational lines of ion molecule HeH$^+$ for dark ages halos with mass $M = 5.3 \cdot 10^9 M_\odot$ formed at $z \simeq 10 - 30$. Powers of 10 are given in parentheses.

| $z$     | 0-1 | 1-2 | 2-3 |
|---------|-----|-----|-----|
| 30.41   | 3.1(-15)/3.0(-11) | 9.5(-16)/3.7(-11) | 1.5(-16)/1.3(-11) |
| 25.15   | 2.7(-15)/4.0(-11) | 6.8(-16)/4.0(-11) | 7.7(-17)/1.0(-11) |
| 19.90   | 2.1(-15)/5.1(-11) | 3.8(-16)/3.8(-11) | 2.6(-17)/5.8(-12) |
| 14.65   | 1.3(-15)/6.3(-11) | 1.3(-16)/2.4(-11) | 3.3(-18)/1.4(-12) |
| 9.41    | 4.3(-16)/5.8(-11) | 7.6(-18)/4.0(-12) | 1.6(-20)/1.8(-14) |

Since the halos in the Dark Ages possess the peculiar velocities $\vec{v}_p$ there are resonant scattering (rs) of CMB quanta that leads to the differential brightness temperature in rotational lines of HeH$^+$ ions [38, 43]

$$\delta T_{br,ul}^{rs} = \frac{h^2 \nu_{ul}^2}{k_B^2 T_r} \left( \frac{\nu_{hul}/k_B T_r}{\nu_{hul}/k_B T_r - 1} \right)^2 \left( 1 - e^{-\tau_{ul}} \right) \left| \frac{\vec{v}_p}{c} \right|, \quad (17)$$
where \( \nu_{\text{ps}} \) is the projection of the vector of peculiar velocity on the line of sight of Earth observer. A reliable estimation of peculiar velocity of halo with mass \( M_h \) is the rms value \( V_{\text{rms}} \equiv \sqrt{\langle v_p^2 \rangle} \), where

$$
\langle v_p^2 \rangle = \frac{H^2(z)}{2\pi^2(1+z)^2} \int_0^\infty P(k;z)W^2(kR)dk
$$

(18)

where \( P(k;z) \) is the power spectrum of density perturbations, \( W(x) = 3(\sin x - x \cos x)/x^3 \) is the top-hat sphere in Fourier space, and \( R = (3M_h/4\pi \rho_m)^{1/3} \) is the comoving radius of the halo. Results of computations \( \delta T_{\text{th,ul}} \) for halos with mass \( M_h = 5.3 \cdot 10^9 \, M_\odot \) are presented in Tab. IV for other masses in Tab. IX in Appendix.

We can also estimate the differential energy flux per unit frequency as follows

$$
\delta F_{\text{ul}} = 2\pi \left( \frac{\nu_{\text{obs,ul}}}{c} \right)^2 k_B < \delta T_{\text{br,ul}} > \theta_h^2
$$

(19)

where \( \theta_h = r_h(M_h)/D_A(z) \) is the angular radius of halo with mass \( M_h \), \( D_A(z) \) is the angular diameter distance to the halo placed at redshift \( z \). The halos analyzed here have angular radii \( \sim 0.06'' \). The results for thermal luminescence and resonant scattering for halos with mass \( M_h = 5.3 \cdot 10^9 \, M_\odot \) are presented in Tab. IV and Tab. V correspondingly.

In Fig. 9 we present the evolution of opacity (left column) and thermal brightness temperature (right column) in the lines of rotational transitions \( J = 1 \to J = 0 \) (upper row), \( J = 2 \to J = 1 \) (middle row) and \( J = 3 \to J = 2 \) (bottom row) for halos with mass \( M_h = 5.3 \cdot 10^9 \, M_\odot \), \( M_h = 6.6 \cdot 10^8 \, M_\odot \), \( M_h = 8.3 \cdot 10^7 \, M_\odot \), \( M_h = 1.0 \cdot 10^7 \, M_\odot \) and \( M_h = 1.3 \cdot 10^6 \, M_\odot \) (animated). Each line corresponds to the halo with different initial amplitude of curvature perturbation: \( C_k = 3 \cdot 10^{-4}, 2.5 \cdot 10^{-4}, 2 \cdot 10^{-4}, 1.5 \cdot 10^{-4}, 1 \cdot 10^{-4}, 0 \) (from top to bottom in the right hand side of each panel and from bottom to top in the left hand side of each panel) which correspond to different times of halos virializations. As can be seen from the figure, the thermal brightness temperature increases due to increasing of the temperature of baryonic matter during compression of the proto-halo and decreases after the moment of halo virialization due to decreasing in the number of HeH$^+$ molecular ions. The similar peak-like time dependence is also expected for the resonance brightness temperature, but with an amplitude of approximately eight orders of magnitude greater (see Tab. IX of electronic version of this paper). We also note that an increase in the resonance brightness temperature in time caused by the increase in the amplitude of the peculiar velocity according to (18), and do not connect to an increase in the temperature of the baryon gas.

VI. CONCLUSIONS

In order to analyze the thermal luminescence and resonant scattering of CMB quanta in the rotational lines of molecule HeH$^+$ in dark ages halos we have computed
FIG. 9. Evolution of opacity (left column) and thermal brightness temperature (right column) in the lines of transitions $J = 1 \rightarrow J = 0$ (upper row), $J = 2 \rightarrow J = 1$ (middle row) and $J = 3 \rightarrow J = 2$ (bottom row) of helium hydride ion for halos with mass $M_h = 5.3 \cdot 10^9 M_\odot$, $M_h = 6.6 \cdot 10^8 M_\odot$, $M_h = 8.3 \cdot 10^7 M_\odot$, $M_h = 1.0 \cdot 10^7 M_\odot$, and $M_h = 1.3 \cdot 10^6 M_\odot$ (animated). Each line corresponds to the halo with different initial amplitude of curvature perturbation: $C_k = 3 \cdot 10^{-4}$, $2.5 \cdot 10^{-4}$, $2 \cdot 10^{-4}$, $1.5 \cdot 10^{-4}$, $1 \cdot 10^{-4}$, 0 (from top to bottom in the right hand side of each panel).

The cross-sections and rate coefficients of excitation/de-excitation of the lowest five rotational energy states of HeH$^+$ by collisions with atomic hydrogen. We have shown also that in the conditions of Dark Ages the collision excitation/de-excitation of the lowest rotational levels of HeH$^+$ by atoms of neutral hydrogen and electrons are competitive.

It was shown, due to the small number density of residual electrons in the Universe after recombination in redshifts range from 100 to 10, collisional excitation/de-excitation of the lowest rotational levels of ion-molecule HeH$^+$ by electrons and atomic hydrogen are comparable and both must be taken into account. In the virialized halos, collisions with atomic hydrogen play a much more important role in the excitations and de-excitations of rotational levels of HeH$^+$ than collisions with electrons but both are subdominant in comparison with radiative activations/de-activations. Due to the peak time depen-
dence of the thermal and resonance luminosities of the halo in HeH$^+$ lines, we can only observe those that have just formed at a given redshift, while the observations in lines of HD and H$_2$ will give us an integral number of halos formed up to a given redshift. But extremely low differential brightness temperatures caused by thermal luminescence in rotational lines of HeH$^+$ caused only by electron and hydrogen collisional excitations do not leave a chance to detect them in the coming years. At the same time, the signal caused by resonant scattering of CMB quanta in the rotational lines of HeH$^+$ in dark ages halos may be detected by the next-generation telescopes.

Since, the rotational transitions in HeH$^+$ are more sensitive to the electromagnetic radiation than to collisions with electrons and hydrogen atoms the appearance in the early Universe of any other electromagnetic radiation with another color temperature will radically increase the luminescence of these molecules and will make visible of dark ages halos against the cosmic microwave background, that will be studied in other paper.

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APPENDIX

TABLE VI. Parameters $a_0$, $a_1$, $a_2$, $a_3$, $a_4$ and $a_5$ of the polynomial approximation (10) for de-excitation rate coefficients of rotational transitions in molecular ion $\text{HeH}^+$ driven by collisions with atomic hydrogen.

| $j_u \rightarrow j_l$ | $\epsilon_{l\rightarrow j_u}$ | $\rho_{l\rightarrow j_u}$ | $a_0$ | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ |
|---------------------|---------------------------|------------------------|-------|-------|-------|-------|-------|-------|
| 1 – 0               | 96                        | 2.49784520             | 14.5294049 | -19.9804699 | 9.64689660 | -2.04783110 | 0.16224520 | 0.16224520 |
| 2 – 0               | 289                       | 1.66245387             | 1.71265602 | -2.14429979 | 0.64225028 | -0.05337602 | -0.00165880 |
| 3 – 0               | 576                       | 1.16269324             | 0.48945033 | -1.14011353 | 0.49379212 | -0.08539511 | 0.00524559 |
| 4 – 0               | 956                       | 0.16737696             | 0.20624841 | -0.36864051 | 0.19007230 | -0.04252340 | 0.00355176 |
| 2 – 1               | 192                       | 7.56499250             | 0.18096010 | -3.56233510 | 1.32790730 | -0.04252340 | 0.00355176 |
| 3 – 1               | 480                       | 3.68632342             | 1.04451383 | -2.55252704 | 0.88008568 | -0.09152202 | 0.00052455 |
| 4 – 1               | 860                       | 0.74968815             | -0.48554430 | 0.19317081 | -0.05466120 | 0.00633272 | 0.00052455 |
| 3 – 2               | 287                       | 4.32938021             | 2.08802240 | -3.28381213 | 1.01243704 | -0.09737233 | 0.00052455 |
| 4 – 2               | 668                       | 2.74075205             | -1.11344116 | -0.12951747 | 0.10764670 | -0.01207553 | 0.00052455 |
| 4 – 3               | 380                       | 3.7476127              | -3.26422964 | 1.64299466 | -0.44879104 | 0.04624444 | 0.00052455 |

TABLE VII. Critical electron/hydrogen density (cm$^{-3}$) as a function of gas temperature for rotational levels $j = 1, 2, 3, 4$. Powers of 10 are given in parentheses.

| $T$, [K] | $j = 1$ | $j = 2$ | $j = 3$ | $j = 4$ |
|----------|--------|--------|--------|--------|
| 10       | 9.3(4)/2.29(8) | 5.89(5)/1.95(9) | 2.28(6)/9.17(9) | 6.94(6)/5.21(10) |
| 20       | 1.37(5)/3.05(8) | 9.04(5)/2.47(9) | 3.33(6)/1.10(10) | 9.44(6)/6.38(10) |
| 30       | 1.72(5)/3.79(8) | 1.16(6)/2.87(9) | 4.16(6)/1.25(10) | 1.14(7)/7.19(10) |
| 50       | 2.29(5)/5.25(8) | 1.59(6)/3.54(9) | 5.52(6)/1.50(10) | 1.45(7)/8.29(10) |
| 100      | 3.36(5)/8.90(8) | 2.44(6)/5.01(9) | 8.13(6)/1.95(10) | 2.04(7)/9.80(10) |
| 200      | 4.97(5)/1.52(9) | 3.73(6)/7.95(9) | 1.20(7)/2.70(10) | 2.90(7)/1.17(11) |
| 300      | 6.24(5)/2.00(9) | 4.78(6)/1.08(10) | 1.51(7)/3.44(10) | 3.59(7)/1.35(11) |
| 500      | 8.34(5)/2.66(9) | 6.51(6)/1.60(10) | 2.03(7)/4.88(10) | 4.73(7)/1.70(11) |
| 1000     | 1.24(6)/3.91(9) | 9.87(6)/2.65(10) | 3.03(7)/7.84(10) | 6.95(7)/2.51(11) |
| 2000     | 1.84(6)/6.30(9) | 1.49(7)/4.54(10) | 4.56(7)/1.24(11) | 1.04(8)/4.01(11) |
TABLE VIII. Physical values and chemical composition of halos virialized at different $z_v$: $M$ is the total mass, $C_k$ is the amplitude of initial curvature perturbation (seed of halo), $z_v$ is the redshift of virialization, $\rho_m$ is the matter density virialized halo, $T_K$ is kinetic temperature of baryonic gas, $n_H$ is the number density of neutral hydrogen atoms, $n_p$, $n_e$ are the number densities of protons and electrons at $z = z_v$, $n_H^+$ is the number density of molecular ion $H_2^+$, $r_h$ is the radius of halo in comoving coordinates, $\theta_h$ is the angular radius of geometrically limited halo.

| $M_h$   | $k$  | $C_k$ | $z_v$ | $\rho_m$ [g/cm$^3$] | $T_K$ [K] | $n_H$ [cm$^{-3}$] | $n_p \approx n_e$ [cm$^{-3}$] | $n_{H=H^+}$ [cm$^{-3}$] | $r_h$ [kpc] | $\theta_h$ [arcsec] |
|---------|------|-------|-------|----------------------|-----------|-------------------|-----------------------------|-------------------------|-------------|-------------------|
| 6.6 $\cdot 10^8$ | 10   | 2.5 $\cdot 10^{-4}$ | 20.41 1.38 $\cdot 10^{-23}$ | 382.8 1.04 97.4/3.8 | 3.428 0.92 0.52 |
| 8.3 $\cdot 10^7$ | 20   | 2.0 $\cdot 10^{-4}$ | 23.28 7.05 $\cdot 10^{-24}$ | 266.3 0.53 54.2/3.9 | 2.287 1.15 0.53 |
| 1 $\cdot 10^7$  | 40   | 1.5 $\cdot 10^{-4}$ | 17.17 2.95 $\cdot 10^{-24}$ | 162.1 0.22 25.1/4.2 | 1.359 1.54 0.56 |
| 1.3 $\cdot 10^6$ | 80   | 1.0 $\cdot 10^{-4}$ | 11.05 8.60 $\cdot 10^{-25}$ | 78.2 0.06 8.3/4.7 | 0.644 2.32 0.61 |

TABLE IX: The optical depths, brightness temperatures and spectral fluxes in the three lowest rotational lines of molecular ion $H_2^+$ in the dark ages halos of different masses $M_h$ virialized at different redshift $z_v$. Marking (th) means the thermal emission, marking (rs) means the resonant scattering. Powers of 10 are given in parentheses.

| $M_h$ [M$_\odot$] | $z_v$ | $\nu_{obs}$ [Hz] | $\Delta \nu_{obs}$ [Hz] | $\tau_v$ | $\delta T_{br}^{(th)}$ [K] | $\delta T_{br}^{(rs)}$ [K] | $\delta F_{br}^{(th)}$ [$\mu$Jy] | $\delta F_{br}^{(rs)}$ [$\mu$Jy] |
|------------------|-------|-----------------|-----------------|---------|-----------------|-----------------|----------------|----------------|
| 1.29($+6$) | 50.33 | 0.392($+11$) | 0.361($+06$) | 0.284($-07$) | 0.230($-15$) | 0.296($-14$) | 0.103($-07$) | 0.133($-06$) |
| 0.781($+11$) | 0.720($+06$) | 0.429($-07$) | 0.101($-15$) | 0.518($-14$) | 0.139($-07$) | 0.713($-06$) | 0.591($-06$) |
| 0.117($+12$) | 0.107($+07$) | 0.192($-07$) | 0.291($-16$) | 0.323($-14$) | 0.516($-08$) | 0.909($-06$) | 0.909($-06$) |
| 41.52 | 0.473($+11$) | 0.381($+06$) | 0.370($-07$) | 0.214($-15$) | 0.471($-14$) | 0.109($-07$) | 0.214($-06$) | 0.214($-06$) |
| 0.943($+11$) | 0.759($+06$) | 0.462($-07$) | 0.836($-16$) | 0.650($-14$) | 0.116($-07$) | 0.898($-06$) | 0.898($-06$) | 0.898($-06$) |
| 0.141($+12$) | 0.113($+07$) | 0.151($-07$) | 0.197($-16$) | 0.341($-14$) | 0.290($-08$) | 0.502($-06$) | 0.502($-06$) | 0.502($-06$) |
| 32.65 | 0.597($+11$) | 0.403($+06$) | 0.502($-07$) | 0.185($-15$) | 0.607($-14$) | 0.114($-07$) | 0.373($-06$) | 0.373($-06$) |
| 0.119($+12$) | 0.803($+06$) | 0.474($-07$) | 0.606($-16$) | 0.791($-14$) | 0.824($-08$) | 0.108($-05$) | 0.108($-05$) | 0.108($-05$) |
| 0.178($+12$) | 0.120($+07$) | 0.963($-08$) | 0.105($-16$) | 0.306($-14$) | 0.112($-08$) | 0.325($-06$) | 0.325($-06$) | 0.325($-06$) |
| 24.40 | 0.791($+11$) | 0.427($+06$) | 0.746($-07$) | 0.153($-15$) | 0.946($-14$) | 0.119($-07$) | 0.738($-06$) | 0.738($-06$) | 0.738($-06$) |
| 0.158($+12$) | 0.852($+06$) | 0.469($-07$) | 0.374($-16$) | 0.922($-14$) | 0.472($-08$) | 0.116($-05$) | 0.236($+12$) | 0.127($+07$) | 0.458($-08$) | 0.405($-17$) | 0.223($-14$) | 0.240($-09$) | 0.132($-06$) |
| 15.39 | 0.123($+12$) | 0.454($+06$) | 0.114($-06$) | 0.748($-16$) | 0.129($-13$) | 0.944($-08$) | 0.163($-05$) | 0.245($+12$) | 0.905($+06$) | 0.294($-07$) | 0.824($-17$) | 0.566($-14$) | 0.910($-09$) | 0.625($-06$) |
| 0.365($+12$) | 0.135($+07$) | 0.608($-09$) | 0.263($-18$) | 0.402($-15$) | 0.491($-11$) | 0.751($-08$) | 0.751($-08$) | 0.751($-08$) |

Continued on next page
| $M_h$ | $z_v$ | $\nu_{obs}$ | $\Delta \nu_{obs}$ | $\Delta \nu_{obs}^{(s)}$ | $\delta T_{br}^{(s)}$ | $\delta T_{br}^{(rs)}$ | $\mu Jy$ | $K$ | $\mu Jy$ | $K$ | $\mu Jy$ | $K$ |
|-------|-----|-------------|-----------------|-----------------|-----------------|-----------------|-------|-----|-------|-----|-------|-----|
| 1.04(+7) | 45.72 | 0.430(+11) | 0.371(+06) | 0.669(-07) | 0.462(-15) | 0.293(-07) | 0.139(-05) | 8.28(+7) | 40.57 | 0.384(+06) | 0.158(-06) | 0.882(-15) | 0.455(-07) | 0.149(-04) |
| 0.858(+11) | 0.740(+06) | 0.923(-07) | 0.192(-15) | 0.485(-13) | 0.263(-07) | 0.665(-05) |
| 0.128(+12) | 0.110(+07) | 0.355(-07) | 0.501(-16) | 0.282(-13) | 0.813(-08) | 0.458(-05) |
| 37.85 | 0.517(+11) | 0.389(+06) | 0.867(-07) | 0.421(-15) | 0.406(-13) | 0.231(-07) | 0.223(-05) |
| 0.163(+12) | 0.776(+06) | 0.979(-07) | 0.155(-15) | 0.593(-13) | 0.214(-07) | 0.821(-05) |
| 0.154(+12) | 0.116(+07) | 0.269(-07) | 0.325(-16) | 0.278(-13) | 0.431(-08) | 0.369(-05) |
| 29.92 | 0.650(+11) | 0.411(+06) | 0.118(-06) | 0.371(-15) | 0.589(-13) | 0.241(-07) | 0.383(-05) |
| 0.130(+12) | 0.820(+06) | 0.993(-07) | 0.113(-15) | 0.712(-13) | 0.149(-07) | 0.941(-05) |
| 0.194(+12) | 0.122(+07) | 0.165(-07) | 0.172(-16) | 0.242(-13) | 0.155(-08) | 0.218(-05) |
| 22.02 | 0.873(+11) | 0.434(+06) | 0.170(-06) | 0.389(+06) | 0.867(-07) | 0.223(-05) |
| 0.174(+12) | 0.866(+06) | 0.889(-07) | 0.595(-16) | 0.736(-13) | 0.726(-08) | 0.699(-06) |
| 0.260(+12) | 0.129(+07) | 0.653(-08) | 0.523(-17) | 0.144(-13) | 0.243(-09) | 0.671(-06) |
| 14.00 | 0.134(+12) | 0.459(+06) | 0.257(-06) | 0.834(+06) | 0.979(-07) | 0.223(-05) |
| 0.267(+12) | 0.915(+06) | 0.533(-07) | 0.115(-16) | 0.389(-13) | 0.120(-08) | 0.406(-05) |
| 0.399(+12) | 0.137(+07) | 0.738(-09) | 0.251(-18) | 0.358(-11) | 0.155(-08) | 0.270(-07) |

Continued on next page
| $M_h$  | $z_v$  | $\nu_{obs}$ | $\Delta \nu_{obs}$ | $\tau_\nu$ | $\delta T_{br}^{(th)}$ | $\delta F^{(th)}_{br}$ | $\delta T_{br}^{(rs)}$ | $\delta F^{(rs)}_{br}$ |
|--------|--------|-------------|-------------------|----------|-------------------|-------------------|-------------------|-------------------|
| [M$_\odot$] | [Hz] | [Hz] | [K] | [K] | [\mu Jy] | [K] | [\mu Jy] |
| 0.385(+12) | 0.951(+06) | 0.195(-06) | 0.755(-17) | 0.402(-11) | 0.783(-09) | 0.417(-03) |
| 0.575(+12) | 0.142(+07) | 0.337(-09) | 0.155(-19) | 0.183(-13) | 0.106(-12) | 0.126(-06) |