Minimum Size of Some Metrics for the Graph \( H(n) \), and the Line Graph of the Graph \( H(n) \)

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Abstract

For an arranged subset \( Q = \{q_1, q_2, \ldots, q_k\} \) of vertices in a connected graph \( G \) the metric representation of a vertex \( v \) in \( G \), is the \( k \)-vector \( r(v|Q) = (d(v, q_1), d(v, q_2), \ldots, d(v, q_k)) \) relative to \( Q \). Also, the subset \( Q \) is considered as resolving set for \( G \) if any pair of vertices of \( G \) is distinguished by some vertices of \( Q \). In the present article, we study the minimum size of resolving set, and doubly resolving set for the graph \( H(n) \), and the line graph of the graph \( H(n) \) is denoted by \( L(n) \). It is well known that these problems are NP hard.

Keywords: resolving set, doubly resolving set, strong resolving set

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1. Introduction

Suppose \( G \) is a finite, simple connected graph with vertex set \( V(G) \) and edge set \( E(G) \). We use \( d_G(p, q) \) to indicate the distance between two vertices \( p \) and \( q \) in graph \( G \) as the length of a shortest path between \( p \) and \( q \) in \( G \). We also, use \( L(G) \) to indicate the line graph of a graph \( G \), as the vertex set of \( L(G) \) is the edges of \( G \) and two vertices of \( L(G) \) are adjacent in \( L(G) \) if and only if they are incident in \( G \), see [4].

The study of graphs has been considered from different perspectives and is very interesting. In a graph based representation of a computer network, every vertex of graph may be seen as a location a special place. Therefore, determining the structure of a graph plays a very important role in solving related problems. According to these facts, it would be useful to uniquely recognize each vertex of graph. The metric dimension of graphs is very useful and play a significant role to solve such sorts of problems. Suppose \( Q = \{q_1, \ldots, q_k\} \) is a set of vertices in graph \( G \), for any vertex \( p \) in \( G \) we use the \( k \)-vector \( r(p|Q) = (d(p, q_1), \ldots, d(p, q_k)) \) to indicate the arranged list of distances and recall that the metric representation of \( p \) relative to \( Q \). A resolving set for a graph \( G \) is a set \( Q \) of vertices so that the vector of distances relative to vertices in \( Q \) is various for any \( p \in V(G) \). The metric dimension of \( G \), is indicated by \( \beta(G) \) defined as the minimum size over all resolving sets of \( G \). The concept and notation of the metric dimension problem, was first introduced by Slater [13] under the term locating set. Also, Harary and Melter studied these problems under the term metric dimension in [5], independently. Besides, one of useful tool for calculating the metric dimension of a graph is to find doubly resolving sets of a graph. The notion of a doubly resolvability of vertices in graphs introduced by Cáceres et al. [1] as follows. Suppose \( G \) is a connected graph with at least two vertices, a set \( S \subseteq V(G) \) is called doubly resolving set of \( G \), if for any various vertices \( p, q \) of \( G \) there are some two vertices of \( S \) say \( r, s \) so that \( d(p, r) - d(p, s) \neq d(q, r) - d(q, s) \). The minimum doubly resolving set of vertices of \( G \), is indicated by \( \psi(G) \), defined as the minimum size over all doubly resolving sets of \( G \). The notion of a strong metric dimension problem set of vertices of graph \( G \) introduced by A. Sebó and E. Tannier [12], indeed introduced a more restricted invariant than the metric dimension and this was further investigated by O. R. Oellermann and Peters-Fransen [11]. A set \( R \subseteq V(G) \) is called strong resolving set of \( G \), if for any various vertices

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Note that for simply we use refinement of the natural relabelling of the graph left-invariant in the graph \(H\)

resolving set for Theorem 3.1. Suppose that \(n\) is a natural number greater than or equal to \(n\) of size

components of two distinct vertices (\(n\) is not a regular graph. Now, it is obvious that \(G\)

\(s\) 3. Main Results

2. Preliminaries

Remark 2.1. Consider a graph \(G\). Then we have:

(i) \(\beta(G) \leq \psi(G)\).

(ii) \(\beta(G) \leq \text{sdim}(G)\).

Theorem 2.1. [3] Suppose \(G\) is connected graph of order greater than or equal to 5, then \([\log_2 \Delta(G)] \leq \beta(L(G)) \leq n - 2\), where \(\Delta(G)\) is the maximum degree of \(G\).

3. Main Results

3.1. Minimum Size of Some Metrics for the Graph \(H(n)\)

1). An interesting family of bipartite graphs of order \(n+\binom{n}{2}\) is provided by the \(H(n)\) as follows. Let \([n] = \{1, 2, \ldots, n\}\). The graph \(H(n)\) is a graph with vertex set \(V = V_1 \cup V_2\), where \(V_1 = \{v_1, v_2, \ldots, v_n\} = \{v_i | i \in [n]\}\), \(V_2 = \{(v_i, v_j) | i, j \in [n], i \neq j, j \leq i \leq n - 1, 2 \leq j \leq n\}\), and the edge set \(E = \{(v_r, v_j) | v_{i, v_r} \in V_1, (v_i, v_j) \in V_2, v_r = v_i \lor v_r = v_j\}\).

Note that for simply we use refinement of the natural relabelling of the graph \(H(n)\) which is defined in [10]. If one component of two distinct vertices \((v_i, v_j), (v_r, v_j) \in V_2\) are equal then we recall that the vertices \((v_i, v_j), (v_r, v_j)\) are left-invariant in the graph \(H(n)\). Also, we say that two distinct vertices \((v_i, v_j), (v_r, v_j) \in V_2\) are right-invariant in the graph \(H(n)\) if \(v_i = v_r\). It is not hard to see that if \(v \in V_1\), then \(\text{deg}(v) = n - 1\) whereas if \(v \in V_2\), then \(\text{deg}(v) = 2\). Hence \(H(n)\) is not a regular graph. Now, it is obvious that \(H(n)\) has \((n - 1)\) edges. We can see, by an easy argument that the graph \(H(n)\) is connected and its diameter is 4, see [10]. In this section, we consider the problem of determining the cardinality \(\psi(H(n))\) of minimal doubly resolving sets of \(H(n)\). First, we prove that if \(n\) is an integer and \(n \geq 5\) then the metric dimension of the graph \(H(n)\) is \(n - 2\). Also, we show that the cardinality of minimum doubly resolving set of the graph \(H(n)\) is \(n - 1\).

Theorem 3.1. Suppose that \(n\) is a natural number greater than or equal to 5. Then the minimum size of resolving set in the graph \(H(n)\) is \(n - 2\).

Proof. Suppose that \(V(H(n)) = V_1 \cup V_2\), where \(V_1 = \{v_1, v_2, \ldots, v_n\} = \{v_i | i \in [n]\}\), \(V_2 = \{(v_i, v_j) | i, j \in [n], i \neq j, j \leq i \leq n - 1, 2 \leq j \leq n\}\) which is defined already. Based on the following cases we can be concluded that the \(S = \bigcup_{i=1}^{n-1}\{(v_i, v_j)\}\), where \((v_i, v_j) \in V_2\) for \(2 \leq j \leq n - 1\); of vertices in the graph \(H(n)\) so that \(|S| = n - 2\) and all the vertices in \(S\) are left-invariant is a minimum resolving set for \(H(n)\) of size \(n - 2\).

Case 1. Suppose \(W_1\) is an arranged subset of \(V_1\) in the graph \(H(n)\) so that \(|W_1| < n - 2\), we can show that \(W_1\) can not be a resolving set for \(H(n)\). In particular if \(W_1\) is an arranged subset of \(V_1\) in the graph \(H(n)\) so that \(|W_1| = n - 2\), then we show that \(W_1\) can not be a resolving set for \(H(n)\). Without loss of generality one can assume that an arranged subset \(W_1\) of vertices in \(H(n)\) is \(W_1 = \{v_1, v_2, \ldots, v_{n-2}\}\). Hence \(V(H(n)) - W_1 = V_2 \cup \{v_{n-1}, v_n\}\), where \(V_2\) which is defined as before. So, the metric representation of the vertices \(v_{n-1}, v_n\) relative to \(W_1\) is the \(n - 2\)-vector \(r(v_{n-1}|W_1) = r(v_n|W_1) = (2, 2, \ldots, 2)\), because for every vertex \(v_i \in W_1\) we have \(d(v_i, v_{n-1}) = d(v_i, v_n) = 2\). Therefore, \(W_1\) can not be a resolving set for \(H(n)\).

Case 2. Suppose \(Z_1\) is an arranged subset of \(V_1\) in the graph \(H(n)\) so that \(|Z_1| = n - 1\), we show that \(Z_1\) is a resolving set for \(H(n)\). Without loss of generality one can assume that an arranged subset \(Z_1\) of vertices in \(H(n)\) is \(Z_1 = \{v_1, v_2, \ldots, v_{n-1}\}\). Hence \(V(H(n)) - Z_1 = V_2 \cup \{v_n\}\), where \(V_2\) which is defined as before. We show that all the
vertices in $V(H(n)) - Z_1$ have different representations relative to the subset $Z_1$. Because for every vertex $(v_s, v_t) \in V_2$, where $s < t$, $1 \leq s \leq n - 1$, $2 \leq t \leq n$ and every vertex $v_i \in Z_1$, $1 \leq i \leq n - 1$, if $s = i$ or $t = i$ then we have $d((v_s, v_t), v_i) = 1$, otherwise $d((v_s, v_t), v_i) = 3$. Also, for the vertex $v_n \in V_1$ and every vertex $v_i \in Z_1$, $1 \leq i \leq n - 1$, we have $d(v_n, v_i) = 2$. Therefore, all the vertices in $V(H(n)) - Z_1$ have different representations relative to the subset $Z_1$. This implies that this subset is a resolving set of $H(n)$.

Case 3. Suppose $W_2$ is an arranged subset of $V_2$ in the graph $H(n)$ so that $|W_2| = n - 3$ and all the vertices in $W_2$ are left-invariant. Without loss of generality one can assume that an arranged subset $W_2$ of vertices in $H(n)$ is $W_2 = \cup_{j=2}^{n-2}((v_1, v_j))$, where $(v_1, v_j) \in V_2$. Hence,

$$V(H(n)) - W_2 = V_1 \cup ((v_1, v_{n-1}), (v_1, v_n), \cup_{j=3}^{n-2}((v_2, v_j)), \cup_{j=4}^{n-2}((v_3, v_j)), \ldots, \cup_{j=n-1}^{n-2}((v_{n-2}, v_j)), (v_{n-1}, v_n)).$$

Thus the metric representation of the vertices $v_{n-1}, v_n \in V_1$ relative to the subset $W_2$ is the $n - 3$-vector $r(v_{n-1}|W_2) = r(v_n|W_2) = (3, 3, \ldots, 3)$, because for every vertex $(v_1, v_j) \in W_2$ we have $d((v_1, v_j), v_{n-1}) = d((v_1, v_j), v_n) = 3$. Therefore, the subset $W_2$ cannot be a resolving set for $H(n)$. In this case note that, if $W_2$ is an arranged subset of $V_2$ in the graph $H(n)$ such that $|W_2| = n - 3$ and all the vertices in $W_2$ are not left-invariant then there exists a vertex such as $(v_i, v_j) \in W_2$ such that $v_i, v_j \in V_1$ and the metric representation of the vertices $v_i, v_j$ is identical $n - 3$-vector relative to $W_2$. Thus $W_2$ can not be a resolving set for $H(n)$.

Case 4. Now, suppose that $S$ is an arranged subset of $V_2$ in the graph $H(n)$ so that $|S| = n - 2$ and all the vertices in $S$ are left-invariant. Without loss of generality one can assume that an arranged subset $S$ of vertices in $H(n)$ is $S = \cup_{j=2}^{n-2}((v_1, v_j))$, where $(v_1, v_j) \in V_2$ for $2 \leq j \leq n - 1$. Hence,

$$V(H(n)) - S = V_1 \cup ((v_1, v_{n-1}), \cup_{j=3}^{n-2}((v_2, v_j)), \cup_{j=4}^{n-2}((v_3, v_j)), \ldots, \cup_{j=n-1}^{n-2}((v_{n-2}, v_j)), (v_{n-1}, v_n)).$$

We show that all the vertices in $V(H(n)) - S$ have different representations relative to the subset $S$. Because for every $k \in V_1$, $1 \leq k \leq n$ and $(v_1, v_j) \in S$, $2 \leq j \leq n - 1$, if $k = 1$ or $k = j$ then we have $d(k, (v_1, v_j)) = 1$, otherwise $d(k, (v_1, v_j)) = 3$. Also, for the vertices $(v_i, v_j) \in V(H(n)) - S$, where $s < t$, $2 \leq s \leq n - 1$, $3 \leq t \leq n$ and $(v_i, v_j) \in S$, $2 \leq j \leq n - 1$, if $s = j$ or $t = j$ then we have $d((v_i, v_j), (v_1, v_j)) = 2$, otherwise $d((v_i, v_j), (v_1, v_j)) = 4$. Moreover for the vertex $(v_n, v_n) \in V(H(n)) - S$ and $(v_1, v_j) \in S$, $2 \leq j \leq n - 1$, we have $d(((v_1, v_n), (v_1, v_j)) = 2$. Therefore, all the vertices in $V(H(n)) - S$ have different representations relative to the subset $S$. This implies that $S$ is a resolving set for $H(n)$.

Case 5. Suppose $Q$ is an arranged subset of vertices in the graph $H(n)$ such that $Q = Q_1 \cup Q_2$, where $Q_1$ is a subset of $V_1$ and $Q_2$ is a subset of $V_2$ so that $|Q_1 \cup Q_2| = n - 3$ and $|Q_1| \neq |Q_2|$ or may be $|Q_1| = |Q_2|$ if $n$ is odd integer). Then we can show that $Q$ is not a resolving set for the graph $H(n)$.

**Theorem 3.2.** Suppose that $n$ is a natural number greater than or equal to 5. Then the minimum size of doubly resolving set in the graph $H(n)$ is $n - 1$.

**Proof.** Let $V(H(n)) = V_1 \cup V_2$, where $V_1 = \{v_1, v_2, \ldots, v_n\} = \{v_i | i \in [n]\}$, $V_2 = \{(v_i, v_j) | i, j \in [n], i \neq j, i < j, 1 \leq i \leq n - 1, 2 \leq j \leq n\}$ which is defined already. By the following cases, we show that the minimum doubly resolving set of the graph $H(n)$ is $n - 1$.

Case 1. From Theorem 3.1 case 4, we know that the arranged subset $S = \cup_{j=2}^{n-2}((v_1, v_j))$, where $(v_1, v_j) \in V_2$ for $2 \leq j \leq n - 1$; of vertices in the graph $H(n)$ so that $|S| = n - 2$ and all the vertices in $S$ are left-invariant is a minimal resolving set for the graph $H(n)$ of size $n - 2$. Now let $u = v_1$ and $v = v_n$, then for every elements $x, y \in S$ we have $0 = 1 - 1 = d(u, x) - d(u, y) = d(v, x) - d(v, y) = 3 = 3 = 0$. Therefore, the subset $S$ is not a doubly resolving set for the graph $H(n)$.

Case 2. From Theorem 3.1 case 2, we know that the arranged subset $Z_1 = \{v_1, v_2, \ldots, v_{n-1}\}$ of $V_1$ in the graph $H(n)$ is a resolving set for $H(n)$. In this case we show that $Z_1$ is a doubly resolving set for the graph $H(n)$. It is sufficient to show that for every two distinct vertices $u, v \in V(H(n)) - Z_1$ there are elements $x, y \in Z_1$ so that
\[ d(u, x) - d(u, y) \neq d(v, x) - d(v, y). \] Consider two distinct vertices \( u, v \in V(H(n)) - Z_1 \), then we have the following:

Case 2.1. Suppose, both vertices \( u, v \in V_2 \), so that \( u, v \) are left-invariant. So we can assume that \( u = (v_r, v_s) \) and \( v = (v_r, v_s) \), where \( i, r, s \in [n] \), and \( r \neq s, i < r, s \). In this case if we consider \( x = v_r \) and \( y = v_s \), then we have \( d(u, x) - d(u, y) \neq d(v, x) - d(v, y) \).

Case 2.2. Suppose, both vertices \( u, v \in V_2 \), so that \( u, v \) are right-invariant. So we can assume that \( u = (v_r, v_i) \) and \( v = (v_r, v_s) \), where \( i, j, r, s \in [n] \), and \( r \neq s, r, s < i \). In this case if we consider \( x = v_r \) and \( y = v_i \), then we have \( d(u, x) - d(u, y) \neq d(v, x) - d(v, y) \).

Case 2.3. Suppose, both vertices \( u, v \in V_2 \), so that \( u, v \) are not, left-invariant and right-invariant. So we can assume that \( u = (v_r, v_j) \) and \( v = (v_r, v_i) \), where \( i, j, r, s \in [n] \), and \( i \neq r, j \neq s \). In this case if we consider \( x = v_r \) and \( y = v_i \), then we have \( d(u, x) - d(u, y) \neq d(v, x) - d(v, y) \).

Case 2.4. Now, suppose that \( u = v_n \in V_1 \) and \( v = (v_r, v_i) \in V_2 \), where \( i, j \in [n] \), and \( i < j \). In this case may be \( j = n \) or \( j \neq n \). If we consider \( x = v_1 \) and \( y = v_r \), \( r < i \), then we have \( d(u, x) - d(u, y) \neq d(v, x) - d(v, y) \).

3.2. Minimum Size of Some Metrics for the Graph \( L(n) \)

Suppose \( n \) is a natural number greater than or equal to 5, and \([n] = \{1, ..., n\}\). Now, suppose that \( G \) is a graph with vertex set \( V_1 \cup ... \cup V_n \), where \( V_i = \{(i, [i, j]) \mid i, j \in [n], i \neq j \} \) for \( 1 \leq i \leq n \), and two various vertices \((i, [i, j])\) and \((k, [k, r])\) are adjacent in \( G \) if and only if \( i = k \) or \([i, j] = [k, r]\). It is not hard to see that this family of graphs is isomorphic with the line graph of the graph \( H(n) \), and hence is indicated by \( L(n) \), where \( H(n) \) which is defined as \([10]\). We can see that \( L(n) \) is a connected vertex transitive graph of valency \( n - 1 \), with diameter 3, and the order \( n(n - 1) \). It is easy to see that any \( V_i \) is a maximal clique of size \( n - 1 \) in \( L(n) \). We say that two maximal cliques \( V_i \) and \( V_j \) are adjacent in \( L(n) \), if there is a vertex \( v_i \) in maximal clique \( V_i \) so that \( v_i \) is adjacent to exactly one vertex of maximal clique \( V_j \), say \( v_j \), \( i, j \in [n] \), \( i \neq j \). Also, for any maximal clique \( V_i \) in \( G = L(n) \) we use \( N(V_i) = \bigcup_{v \in V_i} N_G(v) \) to indicate the vertices in the all maximal cliques \( V_k \), say \( v_k \), \( 1 \leq k \leq n \) and \( k \neq i \) so that \( v_k \) is adjacent one vertex of the maximal clique \( V_i \). In this section, we consider the minimum size of doubly resolving determination problem for the graph \( L(n) \). In Theorem 3.3, we prove that \( \beta(L(n)) = n - 2 \). In the following, we show that \( \psi(L(n)) = n - 1 \), especially, in Theorem 3.5, we prove that \( sdim(L(n)) = n \).

**Theorem 3.3.** Suppose \( n \) is a natural number greater than or equal to 5, then the minimum size of resolving set in the graph \( L(n) \) is \( n - 2 \).

**Proof.** Suppose that \( V(L(n)) = V_1 \cup ... \cup V_n \), where \( V_i = \{(i, [i, j]) \mid i, j \in [n], i \neq j \} \), \( 1 \leq i \leq n \); as the vertices in \( L(n) \) to indicate a maximal clique in \( L(n) \). For any maximal clique \( V_i \) in \( L(n) \) we prove that an arranged subset \( Q_1 \) of vertices in \( L(n) \) so that \( Q_1 \subset N(V_i) \) and \( |Q_1| = n - 2 \) is a minimum resolving set for \( L(n) \). Since \( L(n) \) is vertex transitive, this implies that we can be consider the maximal clique \( V_i \) in \( L(n) \), and hence we see that \( N(V_i) = \{y_1, ..., y_n\} \), where \( y_k = (k, [1, k]) \in V_k \) for \( 2 \leq k \leq n \). Based on the following cases we can be concluded that the subset \( Q_1 = N(V_i) - y_n \) of vertices in \( L(n) \) is a minimum resolving set for \( L(n) \) of size \( n - 2 \).

Case 1. Suppose \( W_1 = \{1, ...., n-1\} \) is an arranged subset of vertices in the maximal clique \( V_1 \), where \( y_k = (1, [1, k + 1]) \in V_1 \) for \( 1 \leq k \leq n - 1 \). In this case we prove that the subset \( W_1 \) can not be a resolving set for \( L(n) \). Since \( W_1 = \{1, ...., n-1\} \), it follows that \( V(L(n)) - W_1 = \{V_2, ..., V_n\} \). On the other hand, we know that the maximal cliques \( V_1 \) and \( V_r \), are adjacent in \( L(n) \) for \( 2 \leq r \leq n \), and hence there are various vertices such as \( y, z \in V_r \) such that \( y, z \notin N(V_1) \) and \( r(y|W_1) = r(z|W_1) \). Thus, \( W_1 \) can not be a resolving set for \( L(n) \).

Case 2. Suppose \( W_2 = \{1, ...., n-2\} \) is an arranged subset of vertices in \( L(n) \) so that \( x_k = (k, [k, k + 1]) \in V_k \) for \( 1 \leq k \leq n - 2 \), and for two consecutive members of \( W_2 \) we have \( d(x, x_{r+1}) = 2 \) for \( 1 \leq r \leq n - 3 \), also for each element \( y \in W_2 \) so that \( x_{r+1} \neq y \) we have \( d(x, y) = 3 \). Hence, there are exactly two various vertices \( (1, [1, n - 1]), (1, [1, n]) \in V_1 \) so that \( r((1, [1, n - 1])|W_2) = r((1, [1, n])|W_2) = (1, 3, ..., 3) \). Thus, \( W_2 \) can not be a resolving set for \( L(n) \). Suppose
Given $z \in V_{n-1}$ and $W_2 = \{x_1, ..., x_{n-2}\}$ which is defined already. Note that, in this case may be $d(z, x_{n-2}) = 2$ or $d(z, x_{n-2}) = 1$. Now, suppose that $T_1 = (W_2 \cup z) = \{x_1, ..., x_{n-2}, z\}$ is an arranged subset of vertices in $L(n)$. We prove that $T_1$ is a resolving set for $L(n)$. Because in this case if $z \in V_{n-1}$ then the vertex $(1, [1, n-1]) \in V_1$ is adjacent to a vertex of $V_{n-1}$, and hence $r((1, [1, n-1])T_1) \neq r((1, [1, n])T_1)$. So, the metric representations of all the vertices $x_1 \neq v \in V_1$ in $L(n)$ is not identical relative to $T_1$. Also, all the vertices $x_i \neq v \in V_2$, $2 \leq t \leq n - 1$ have various metric representations relative to $T_1$ because $L(n)$ is a vertex transitive graph. In particular, all the vertices in the maximal clique $V_n$ have various metric representations relative to $T_1$ because each vertex in the maximal clique $V_n$ is adjacent to exactly one vertex of a maximal clique $V_s$, $1 \leq s \leq n - 1$. According to the above discussion we deduce that the arranged subset $T_1$ of vertices in $L(n)$ is a resolving set for $L(n)$ of size $n - 1$.

Case 3. Now, suppose $W_3$ is an arranged subset of vertices in $L(n)$ so that $W_3 \subset N(V_1)$, and $|W_3| = n - 3$; indeed for every two vertices $x, y \in W_3$ we have $d(x, y) = 3$. Without lack of theory we can assume that $W_3 = \{y_1, ..., y_{n-2}\}$, where $y_k = (k, [1, k]) \in V_1$ for $2 \leq k \leq n - 2$. Hence, there are exactly two vertices $(1, [1, n-1]), (1, [1, n]) \in V_1$ so that $r((1, [1, n-1])|W_3) = r((1, [1, n])|W_3) = (2, ..., 2)$. Thus, $W_3$ can not be a resolving set for $L(n)$. Now, suppose $z \in V_{n-1}$ so that $z \in N(V_1)$ and $W_3 = \{y_2, ..., y_{n-2}\}$ which is defined already. In this case we can assume that $z = y_{n-1}$, where $y_{n-1} = (n - 1, [1, n - 1]))$, and hence if $Q_1 = N(V_1) - y_n = (W_3 \cup z) = \{y_2, ..., y_{n-1}\}$ is an arranged subset of vertices in $L(n)$, then we prove that $Q_1$ is a resolving set for $L(n)$. Because in this case the vertex $(1, [1, n - 1]) \in V_1$ is adjacent to the vertex $y_{n-1} \in V_{n-1}$, and hence $r((1, [1, n-1])Q_1) \neq r((1, [1, n])Q_1)$. In particular every vertex $v$ in the maximal clique $V_s$ is adjacent to exactly a vertex of one maximal clique $V_j$, $2 \leq j \leq n$. So, all the vertices $v \in V_1$ have various metric representations relative to the subset $Q_1$. Also, for every vertex $v \in V_s$, $2 \leq r \leq n - 1$ so that $v \in N(V_1)$ and each $y_s \in Q_1$, $2 \leq s \leq n - 1$, if $v, y_s$ lie in a maximal clique $V_s$, $2 \leq s \leq n - 1$, then we have $d(v, y_s) = 1$; otherwise $d(v, y_s) \geq 2$. In particular, all the vertices in the maximal clique $V_n$ have various metric representations relative to the subset $Q_1$ because for every vertex $v$ in the maximal clique $V_n$ so that $v$ is not equal to the vertex $(n, [1, n])$ in the maximal clique $V_n$, there is a exactly one $y_s \in Q_1$ such that $d(v, y_s) = 2$; otherwise $d(v, y_s) > 2$, $2 \leq s \leq n - 1$. In particular, for the vertex $(n, [1, n])$ in the maximal clique $V_n$ and every $y_s \in Q_1$, we have $d(v, y_s) = 3$. Thus, the arranged subset $Q_1 = \{y_2, ..., y_{n-1}\}$ of vertices in $L(n)$ is a resolving set for $L(n)$ of size $n - 2$.

**Lemma 3.1.** Consider the graph $L(n)$ with vertex set $V_1 \cup ... \cup V_n$ for $n \geq 5$. Then the subset $Q_1 = N(V_1) - y_n = \{y_2, ..., y_{n-1}\}$ of vertices in the graph $L(n)$, where $y_k = (k, [1, k]) \in V_k$ for $2 \leq k \leq n$ can not be a doubly resolving set for $L(n)$.

**Proof.** From Theorem 3.3 case 3, it follows that for the maximal clique $V_1$ in $L(n)$, the subset $Q_2 = N(V_1) - y_n = \{y_2, ..., y_{n-1}\}$ of vertices in $L(n)$ is a minimum resolving set for $L(n)$ of size $n - 2$. Now, by consider various vertices $p = (1, [1, n]) \in V_1$ and $y_n = (n, [1, n]) \in V_n$, we see that $d(p, r) - d(p, s) = d(q, r) - d(q, s)$ for elements $r, s \in Q_1$, because for each element $z \in Q_1$ we have $d(p, z) = 2$ and $d(q, z) = 3$. Thus, the subset $Q_1 = N(V_1) - y_n = \{y_2, ..., y_{n-1}\}$ of vertices in $L(n)$ can not be a doubly resolving set for $L(n)$.

**Theorem 3.4.** Suppose $n$ is a natural number greater than or equal to 5, then the minimum size of doubly resolving set in the graph $L(n)$ is $n - 1$.

**Proof.** Suppose that $V(L(n)) = V_1 \cup ... \cup V_n$, where $V_i = \{(i, [i, j]) \mid i, j \in [n], i \neq j\}$ for $1 \leq i \leq n$. From Theorem 3.3 case 3, it follows that for the maximal clique $V_1$ in $L(n)$, the subset $Q_1 = N(V_1) - y_n = \{y_2, ..., y_{n-1}\}$ of vertices in $L(n)$ is a minimum resolving set for $L(n)$ of size $n - 2$, where $y_k = (k, [1, k]) \in V_k$ for $2 \leq k \leq n$. Also, from Lemma 3.1, can be concluded that $Q_1$ is not a doubly resolving set for $L(n)$, and hence the minimum size of doubly resolving set in $L(n)$ must be greater than or equal to $n - 1$. Now, suppose that $Q_2 = Q_1 \cup y_n = N(V_1) = \{y_2, ..., y_{n-1}\}$ is an arranged subset of vertices in $L(n)$. We can prove that $Q_2$ is a resolving set for $L(n)$ of size $n - 1$. We show that $Q_2$ is a doubly resolving set for $L(n)$. So this is enough, prove that for any two various vertices $p$ and $q$ in $L(n)$ there exist elements $r, s \in Q_2$ so that $d(p, r) - d(p, s) = d(q, r) - d(q, s)$. Consider two vertices $p$ and $q$ in $L(n)$. Then the result can be deduced from the following cases:

Case 1. Suppose, both vertices $p$ and $q$ lie in the maximal clique $V_1$. Hence, there exists a element $r \in Q_2$ so that $r \in V_n$ and $r$ is adjacent to $p$, also, there exists a element $s \in Q_2$ so that $s \in V_1$ and $s$ is adjacent to $q$ for some
a, b ∈ [n] − 1, a ≠ b; and hence −1 = 1 − 2 = d(p, r) − d(p, s) − d(q, r) − d(q, s) = 2 − 1 = 1.

Case 2. Suppose, both vertices p and q lie in the maximal clique \( V_a \), a ∈ [n] − 1, so that \( p, q \not\in Q_2 \). Hence, there exists an element \( r \in Q_2 \) so that \( r \in V_a \) and \( d(p, r) = d(q, r) = 1 \), also there exists an element \( s \in Q_2 \) so that \( s \in V_b \), \( a ≠ b \), and \( d(p, s) = 2, d(q, s) = 3 \) or \( d(p, s) = 3, d(q, s) = 2 \). Thus, \( d(p, r) − d(p, s) ≠ d(q, r) − d(q, s) \).

Case 3. Suppose, both vertices \( p \) and \( q \) lie in the maximal clique \( V_a \), a ∈ [n] − 1, so that \( p \in Q_2, q \not\in Q_2 \). Hence, by considering \( p = r \) and each element \( s \in Q_2 \) so that \( s \in V_b, a ≠ b \), we have \( d(p, r) − d(p, s) ≠ d(q, r) − d(q, s) \).

Case 4. Suppose that \( p \) and \( q \) are two vertices in \( L(n) \) so that \( p \in V_1 \) and \( q \in V_a, a ∈ [n] − 1 \). Hence, \( d(p, q) = t, 1 ≤ t ≤ 3 \). If \( t = 1 \), then \( q \in Q_2 \). So, if we consider \( r = q \) and \( q ≠ s ∈ Q_2 \), then we have \( d(p, r) − d(p, s) ≠ d(q, r) − d(q, s) \). If \( t = 2 \), then \( q \not\in Q_2 \), and hence there exists a \( r \in V_a \) so that \( r \in Q_2 \) and \( d(p, r) = d(q, r) = 1 \), also there exists an element \( s \in Q_2 \) so that \( s \in V_b, b ∈ [n] − \{1, a\} \), and \( d(p, s) = 2, d(q, s) = 3 \) or \( d(p, s) = 3, d(q, s) = 2 \), and hence we have \( d(p, r) − d(p, s) ≠ d(q, r) − d(q, s) \). If \( t = 3 \), then there exists an \( r \in V_a \) so that \( r \in Q_2 \) and \( d(p, r) = 2, d(q, r) = 1 \), also there exists an element \( s \in Q_2 \) so that \( s \in V_b, b ∈ [n] − \{1, a\} \), and \( d(p, s) = d(q, s) = 2 \), and hence we have \( d(p, r) − d(p, s) ≠ d(q, r) − d(q, s) \).

Case 5. Suppose that \( p \) and \( q \) are two vertices in \( L(n) \) so that \( p \in V_a \) and \( q \in V_b, a, b ∈ [n] − 1, a ≠ b \). If both two vertices \( p \) and \( q \) lie in \( Q_2 \), or exactly one vertex \( q \) lie in \( Q_2 \) then there is nothing to prove. Now, suppose that both two vertices \( p, q \not\in Q_2 \). Hence, there exists a vertex \( r \in V_a \) so that \( r \in Q_2 \), also there exists a vertex \( s \in V_b \) so that \( s ∈ Q_2 \), and hence we have \( d(p, r) − d(p, s) ≠ d(q, r) − d(q, s) \).

\[ \text{Lemma 3.2. Consider the graph } L(n) \text{ with vertex set } V_1 \cup \ldots \cup V_n \text{ for } n ≥ 5. \text{ Then the subset } Q_2 = Q_1 \cup y_n = N(V_1) = \{y_2,...,y_n\} \text{ of vertices in the graph } L(n), \text{ where } y_k = (k,\{k,0\}) \in V_i \text{ for } 2 ≤ k ≤ n, \text{ can not be a strong resolving set for } L(n). \]

\[ \text{Proof. From Theorem 3.3 case 3, we know that for the maximal clique } V_1 \text{ in } L(n), \text{ the subset } Q_1 = N(V_1) - y_n = \{y_2,...,y_{n-1}\} \text{ of vertices in } L(n) \text{ is a minimum resolving set for } L(n) \text{ of size } n - 2. \text{ Now, by considering various vertices } p \in V_a \text{ and } q \in V_b, 1 < a, b < n - 1, a ≠ b, \text{ so that } d(u, v) = 3, \text{ there is not a } r \in Q_1 \text{ so that } p \text{ belongs to a shortest } q - r \text{ path or } q \text{ belongs to a shortest } p - r \text{ path. Thus, } Q_1 = N(V_1) - y_n = \{y_2,...,y_{n-1}\} \text{ can not be a strong resolving set for } L(n), \text{ and hence by this way we can prove that } Q_2 = Q_1 \cup y_n = N(V_1) = \{y_2,...,y_n\} \text{ can not be a strong resolving set for } L(n). \]

\[ \text{Lemma 3.3. Consider the graph } L(n) \text{ with vertex set } V_1 \cup \ldots \cup V_n \text{ for } n ≥ 5. \text{ Then the subset } T_1 = (W_2 \cup z) = \{x_1,\ldots,x_{n-2},z\}, \text{ where } x_k = (k,\{k,k+1\}) \in V_i \text{ for } 1 ≤ k ≤ n - 2, \text{ and } z \in V_n - 1 \text{ such that } d(z, x_{n-2}) = 2 \text{ or } d(z, x_{n-2}) = 1; \text{ can not be a strong resolving set for } L(n). \]

\[ \text{Proof. From Theorem 3.3 case 2, we know that for the maximal clique } V_1 \text{ in the graph } L(n), \text{ the subset } T_1 = (W_2 \cup z) = \{x_1,\ldots,x_{n-2},z\} \text{ of vertices in } L(n) \text{ is a resolving set for } L(n) \text{ of size } n - 1. \text{ Now, by considering various vertices } p \in V_1 \text{ and } q \in V_n, \text{ such that } d(p, q) = 3 \text{ and } p \not\in T_1, \text{ there is not a } r \in T_1 \text{ so that } p \text{ belongs to a shortest } q - r \text{ path or } q \text{ belongs to a shortest } p - r \text{ path. Thus, } T_1 = (W_2 \cup z) = \{x_1,\ldots,x_{n-2},z\} \text{ can not be a strong resolving set for } L(n). \]

\[ \text{Theorem 3.5. Consider the graph } L(n) \text{ with vertex set } V_1 \cup \ldots \cup V_n \text{ for } n ≥ 5. \text{ Then the minimum size of strong resolving set in the graph } L(n) \text{ is } n. \]

\[ \text{Proof. Based on the Lemmas 3.2 and 3.3, we know that the minimum size of strong resolving set for the graph } L(n) \text{ must be greater than or equal to } n. \text{ Suppose } T_2 = \{x_1,\ldots,x_{n-1}\} \cup z \text{ is a subset of vertices in } L(n), \text{ where } x_k = (k,\{k,k+1\}) \in V_i \text{ for } 1 ≤ k ≤ n - 1, \text{ and } z = (n,\{1,0\}) \in V_0. \text{ In the following, we prove that for every two various vertices } p \text{ and } q \in L(n), \text{ there is a } r \in T_2 \text{ so that } p \text{ belongs to a shortest } q - r \text{ path or } q \text{ belongs to a shortest } p - r \text{ path. Consider vertices } p, q \in L(n), \text{ suppose both vertices } p, q \text{ lie in the maximal clique } V_1. \text{ Hence there is a } r \in T_2 \text{ so that } d(r, p) = 2, \text{ and hence } d(r, q) = 3. \text{ Thus } p \text{ belongs to a shortest } q - r \text{ path. Therefore, if both vertices } p, q \text{ lie in the maximal clique } V_j, 1 ≤ j ≤ n \text{ then } T_2 \text{ is a strong resolving set for } L(n). \text{ Because } L(n) \text{ is vertex transitivg graph. Now, suppose that } p \text{ and } q \text{ are two vertices in } L(n) \text{ so that } p \in V_k \text{ and } q \in V_j, k, j \in [n] \text{ and } k ≠ j. \text{ Hence, } d(p, q) = t, \text{ and}\]
1 ≤ t ≤ 3. If t = 1, and both vertices p, q ∈ T_2, then there is a r ∈ T_2 so that r ∈ V_k, and hence p belongs to a shortest q − r path. If t ≥ 2, and both vertices p, q ∈ T_2, then there is a r ∈ T_2 so that r ∈ V_k or r ∈ V_j, and hence p belongs to a shortest q − r path or q belongs to a shortest p − r path.

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