FULL CROSSED PRODUCTS BY HOPF $C^*$-ALGEBRAS

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Abstract. We show that when a co-involutive Hopf $C^*$-algebra $S$ coacts via $\delta$ on a $C^*$-algebra $A$, there exists a full crossed product $A \times_\delta S$, with universal properties analogous to those of full crossed products by locally compact groups. The dual Hopf $C^*$-algebra is then defined by $\hat{S} := \mathbb{C} \times_{id} S$.

INTRODUCTION.

Due to differences in notation and construction, it is often considered that coactions of locally compact groups on $C^*$-algebras lead to a type of crossed product different to that of actions of locally compact groups. But the definitions of the two types of full crossed product by their universal properties given by Raeburn [25, 26] are quite similar. With a slight change of notation, actions and coactions can look very similar indeed. By taking the point of view that these crossed products are essentially of the same type, we were able to give a short proof of the Imai-Takai duality theorem [9], the key ideas of which also worked for coactions, thus giving a short proof of Katayama’s duality theorem [10] as well [16].

The natural umbrella under which both actions and coactions fit is that of co-involutive Hopf $C^*$-algebras. Both the full group $C^*$-algebra $C^*(G)$ and the algebra of continuous functions on the group disappearing at infinity $C_0(G)$ are examples of co-involutive Hopf $C^*$-algebras. By using universal properties to define full crossed products of $C^*$-algebras by coactions of Hopf $C^*$-algebras, one should be able to obtain full crossed products by locally compact groups by actions and coactions as two special cases. Ng suggested that it should be possible to do this using Raeburn’s method [15, Prop 2.13].

That is what we accomplish in this paper. We will define a class of Hopf $C^*$-algebras, called co-involutive, for which full crossed products can be defined. At each stage of the construction we look at the examples $C^*(G)$ and $C_0(G)$, so that it is evident that our construction does give the usual full crossed products in those cases. We also give examples which do not arise from locally compact groups, namely those arising from certain amenable multiplicative unitaries of Baaj and Skandalis [3], so that it is clear that the full crossed products constructed here are a genuine generalization of those by locally compact groups. The fundamental difference between our crossed products and those of Baaj and Skandalis is that ours generalize full crossed products, whereas theirs generalize reduced crossed products [2, p11].

We simply define the dual of a co-involutive Hopf $C^*$-algebra $S$ to be the trivial crossed product of the complex numbers by $S$. Using its universal properties we can readily write down its comultiplication, making the dual object a Hopf $C^*$-algebra.

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as well. An integral part of our method is the construction of a distinguished pair $(\mu_S, V_S)$, where $\mu_S$ is a representation of $S$, and $V_S$ is a corepresentation of $S$, on a Hilbert space $H_S$. They are the analogue of the left regular representation of $C^*(G)$ and the representation of $C_0(G)$ by multiplication operators on $L^2(G)$.

In §4, we give some notation and summarize preliminaries concerning locally compact groups. In §3 we define Hopf $C^*$-algebras and show that $C^*(G)$ and $C_0(G)$ are Hopf $C^*$-algebras; we also provide other examples. In §4, we define and give examples of coactions of Hopf $C^*$-algebras. We define co-involutive Hopf $C^*$-algebras in §5 which are the class of Hopf $C^*$-algebras to which we can associate dual Hopf $C^*$-algebras. In this section a $*$-algebra $A(S)$ is defined. In §5 we go on to define full crossed products by co-involutive Hopf $C^*$-algebras. The dual Hopf $C^*$-algebra $\hat{S}$ of a given co-involutive Hopf $C^*$-algebra $S$ is defined in §6 and we show that $A(S)$ is dense in $\hat{S}$. Finally, we close with a discussion of the problems surrounding whether or not the double dual $\hat{\hat{S}}$ is isomorphic to $S$. These, and other issues, will be dealt with in §7.

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1. Preliminaries.

All representations of a $C^*$-algebra $A$ are involutive and nondegenerate, and so extend uniquely to unital $*$-representations of the multiplier algebra $M(A)$ [1, Lem 1.1]. All tensor products $\otimes$ will be minimal and all identity maps are denoted by $id$.

For each $f$ in the dual space $B^*$ of a $C^*$-algebra $B$, there is a well-defined map, called a slice map, $id \otimes f : A \otimes B \rightarrow A$ with the property that $id \otimes f(a \otimes b) = af(b)$. Using the Cohen factorization theorem one can show that slice maps extend to multiplier algebras [2, p115]; the extension is also denoted by $id \otimes f$. Tomiyama showed that slice maps $id \otimes f$ separate points of $A \otimes B$ [3, Th 1]. When extended to $M(A \otimes B)$, they still separate points because the multiplier algebra $M(A \otimes B)$ is the largest $C^*$-algebra containing $A \otimes B$ as an essential ideal [3, p82].

Let $C$ be a $C^*$-algebra. Given a nondegenerate homomorphism $\phi : A \rightarrow M(C)$, there is nondegenerate homomorphism $\phi \otimes id : A \otimes B \rightarrow M(C \otimes B)$ [2, Lem 1.1]. The flip isomorphism $\Sigma : A \otimes B \rightarrow B \otimes A$ satisfies $\Sigma(a \otimes b) = b \otimes a$.

Let $G$ be a locally compact group, $C^*(G)$ be the full group $C^*$-algebra with canonical embedding $i_G : G \rightarrow C^*(G)$, and $C_0(G)$ be the continuous functions on $G$ disappearing at infinity. The Fourier-Stieltjes algebra $B(G)$ of $G$ is the set of all finite, complex-linear combinations of continuous, positive definite functions on $G$, with pointwise operations [24, p21] [3, 32.10]. Eymard showed that $B(G)$ can be identified with $C^*(G)^*$ the space of linear functionals on $C^*(G)$, where a functional $f \in C^*(G)^*$ corresponds to the function on $G$ defined by $s \mapsto f(i_G(s))$ [6, p192]. This means that $C^*(G)^*$ can be considered to be contained in $C_0(G)$. Eymard also showed that $B(G)$ is a Banach $*$-algebra with the norm inherited from $C^*(G)^*$.

The Fourier algebra $A(G)$ is the $*$-ideal in $B(G)$, which is the norm closure of the set of functions of compact support [24, p21]. The spectrum of $A(G)$ is $G$ [3, 3.34], hence $A(G)$ is a dense $*$-subalgebra of $C_0(G)$ in the supremum norm.
The map \( w_G := s \mapsto i_G(s) \) is a continuous bounded function on \( G \) into \( M(C^*(G)) \), and since \( C_b(G, M(C^*(G))) \) is contained in \( M(C_0(G) \otimes C^*(G)) \) \cite[p751]{11}, \( w_G \) can be considered to be a unitary element of \( M(C_0(G) \otimes C^*(G)) \). The unitary \( v_G := \Sigma(w_G) \) is an element of \( M(C^*(G) \otimes C_0(G)) \). Note that if \( f \in \mathcal{A}(G) \), then

\[
\text{id} \otimes f(w_G) = \text{id} \otimes f(s \mapsto i_G(s)) = s \mapsto f(i_G(s)) = f \in C_0(G).
\]

If \( f \in C^*(G)^* \), then \( \text{id} \otimes f(w_G) = f \in M(C_0(G)) \). For each \( t \in G \), there is \(*\)-homomorphism \( \varepsilon_t : C_0(G) \rightarrow \mathbb{C} \), evaluation at \( t \). It is an element of \( C_0(G)^* \) and

\[
\text{id} \otimes \varepsilon_t(v_G) = \text{id} \otimes \varepsilon_t(s \mapsto i_G(s)) = i_G(t) \in M(C^*(G)).
\]

Also, for \( z \in L^1(G) \subseteq M_1(G) = C_0(G)^* \) \cite[7.1.2]{21}, \( \text{id} \otimes z(v_G) = z \in C^*(G) \).

Let \( \alpha \) be a strongly continuous action of a locally compact group \( G \) on a \( C^* \)-algebra \( A \). Because each map \( s \mapsto \alpha_s(a) \) is continuous, they are in \( C_b(G, A) \), and making use of the embedding of \( C_b(G, A) \) in \( M(A \otimes C_0(G)) \), we can consider \( \alpha \) to be a \(*\)-homomorphism from \( A \) to \( M(A \otimes C_0(G)) \). The fact that the original \( \alpha \) was a group homomorphism, gives an identity which the new \( \alpha \) satisfies:

\[
(\alpha \otimes \text{id}) \circ \alpha = (\text{id} \otimes \alpha_G) \circ \alpha,
\]

as maps on \( M(A \otimes C_0(G) \otimes C_0(G)) \), and where \( \alpha_G : C_0(G) \rightarrow M(C_0(G) \otimes C_0(G)) \) is the homomorphism given by

\[
\alpha_G(f)(s, t) = f(st).
\]

This map satisfies the comultiplication identity:

\[
(\alpha_G \otimes \text{id}) \circ \alpha_G = (\text{id} \otimes \alpha_G) \circ \alpha_G,
\]

as maps on \( M(C_0(G) \otimes C_0(G) \otimes C_0(G)) \) \cite[Rem 2.2(1)]{24}. Having expressed actions of \( G \) on \( A \) entirely in terms of \( C_0(G) \), to define coactions of \( G \), we simply replace \( C_0(G) \) by \( C^*(G) \). An injective \(*\)-homomorphism \( \delta : A \rightarrow M(A \otimes C^*(G)) \) is a coaction of \( G \) on \( A \) if it satisfies the coaction identity: \((\delta \otimes \text{id}) \circ \delta = (\text{id} \otimes \delta_G) \circ \delta\), where \( \delta_G : C^*(G) \rightarrow M(C^*(G) \otimes C^*(G)) \) is the injective \(*\)-homomorphism satisfying

\[
\delta_G(i_G(s)) = i_G(s) \otimes i_G(s).
\]

The map \( \delta_G \) satisfies a comultiplication identity like Equation \cite[p628]{26}. A system where \( G \) coacts on a \( C^* \)-algebra \( A \) via \( \delta \) is called a cosystem \cite{11, 17, 23}. In the special case where \( G \) is abelian, Pontryagin duality says that the characters form a locally compact abelian group \( \hat{G} \) \cite[Th 1.7.2]{28}. The Fourier transform gives an isomorphism between \( C^*(G) \) and \( C_0(\hat{G}) \), which carries \( \delta_G \) to \( \alpha_G \). Thus even when \( G \) is non-abelian, \( C_0(G) \) can be thought of as the dual of \( C^*(G) \).

2. **Hopf \( C^* \)-algebras.**

A **Hopf \( C^* \)-algebra** is a \( C^* \)-algebra \( S \) together with a nondegenerate injective \(*\)-homomorphism \( \delta_S : S \rightarrow M(S \otimes S) \) such that

\[
(\delta_S \otimes \text{id}) \circ \delta_S = (\text{id} \otimes \delta_S) \circ \delta_S,
\]

as maps of \( S \) into \( M(S \otimes S \otimes S) \). The homomorphism \( \delta_S \) is called the comultiplication on \( S \) and Equation \cite[11]{11} is called the comultiplication identity. A Hopf \( C^* \)-algebra is right-simplifiable if

\[
\delta_S(S)(1 \otimes S) = S \otimes S,
\]
and left-simplifiable if $\delta_S(S)(S \otimes 1) = S \otimes S$. A Hopf $C^*$-algebra is bi-simplifiable if it is both left and right-simplifiable \[2\] Defn 0.1.

A corepresentation of $S$ on a Hilbert space $H$ is a unitary $V \in M(K(H) \otimes S)$ such that
\[
\text{id} \otimes \delta_S(V) = V_{12}V_{13} \in M(K(H) \otimes S \otimes S),
\]
and where $V_{12}$ is the element of $M(K(H) \otimes S \otimes S)$ that is $V$ acting on the first and second factors (that is $V \otimes 1$), and where $V_{13}$ is $V$ acting on the first and third factors \[3\] Defn 0.3. Such a unitary $V$ also satisfies
\[
\delta_S \otimes \text{id}(\Sigma V) = (\Sigma V)_{13}(\Sigma V)_{23} \in M(S \otimes S \otimes K(H)).
\]

**Example 1(a).** Let $A$ be any $C^*$-algebra. It can be given the trivial right-simplifiable comultiplication $\delta: A \rightarrow M(A \otimes A)$ satisfying $\delta(a) = a \otimes 1$. The unitary $1 \otimes 1 \in M(K(H) \otimes A)$ is the only corepresentation because $\text{id} \otimes \delta(U) = U_{12}$ must equal $U_{12}U_{13}$.

**Example 2(a).** The $C^*$-algebra $C_0(G)$ is a Hopf $C^*$-algebra with right-simplifiable comultiplication $\alpha_G: C_0(G) \rightarrow M(C_0(G) \otimes C_0(G))$ (Equation (3) \[26\] Rem 2.2(2)). Let $\lambda$ be the left regular representation of $C^*(G)$ on $L^2(G)$. Since $\lambda: C^*(G) \rightarrow M(K(L^2(G)))$, \[29\] Lem 1.1 implies that the unitary $\lambda \otimes \text{id}(v_G)$ is in $M(K(L^2(G)) \otimes C_0(G))$. The following shows that $\lambda \otimes \text{id}(v_G)$ is a corepresentation of $C_0(G)$:
\[
\text{id} \otimes \alpha_G(\lambda \otimes \text{id}(v_G)) = \lambda \otimes \text{id} \otimes \text{id}(\text{id} \otimes \alpha_G(v_G))
= \lambda \otimes \text{id} \otimes \lambda((s,t) \mapsto i_G(st))
= \lambda \otimes \text{id} \otimes \lambda((s,t) \mapsto i_G(s)i_G(t))
= (\lambda \otimes \text{id}(v_G))_{13}(\lambda \otimes \text{id}(v_G))_{13},
\]
in $M(K(L^2(G)) \otimes C_0(G) \otimes C_0(G))$. Any $\ast$-representation $U: C^*(G) \rightarrow B(H)$ gives rise to a corepresentation of $C_0(G)$, namely $U \otimes \text{id}(v_G) \in M(K(H) \otimes C_0(G))$.

Conversely, a corepresentation $V$ of $C_0(G)$ is a unitary element of $M(K(H) \otimes C_0(G)) \subseteq C_0(G, B(H))$, and so $V$ corresponds to a continuous bounded function $\tilde{V}$ from $G$ into the unitary group of $B(H)$. We know that $\text{id} \otimes \alpha_G(V) = V_{12}V_{13}$. The left hand side is the function on $G \times G$ that takes $(s,t)$ to $\tilde{V}(st)$, and the right hand side is the function that takes $(s,t)$ to $\tilde{V}(s)\tilde{V}(t)$. This means that $\tilde{V}$ is a unitary group representation and so integrates up to a $\ast$-representation $\tilde{V}$ of $C^*(G)$. To know that all corepresentations of $C_0(G)$ are of the form $\text{id} \otimes \text{id}(v_G)$ for some $\ast$-representation $\tilde{V}$ of $C^*(G)$, we need to show that $\tilde{V} \otimes \text{id}(v_G) = V$. The point evaluation functionals $\epsilon_s$ on $C_0(G)$ are sufficient to separate points. So functionals of the form $\text{id} \otimes \epsilon_s$ separate points of $M(K(H) \otimes C_0(G))$ \[29\] Th 1 and the following calculation suffices:
\[
\text{id} \otimes \epsilon_s(\tilde{V} \otimes \text{id}(v_G)) = \tilde{V}(\text{id} \otimes \epsilon_s(v_G)) = \tilde{V}(s) = \text{id} \otimes \epsilon_s(V).
\]

**Example 3(a).** In \[26\] Ex 2.3(1)) Raeburn notes that $\delta_G$ (Equation (3)) is a right-simplifiable comultiplication on $C^*(G)$. Let $M$ be the representation of $C_0(G)$ on
$L^2(G)$ as multiplication operators. Then $M \otimes \text{id}(w_G)$ is a corepresentation of $C^*(G)$:

$$
\text{id} \otimes \delta_G(M \otimes \text{id}(w_G)) = M \otimes \text{id} \otimes \text{id}(\delta_G(w_G))
= M \otimes \text{id} \otimes (s \mapsto i_G(s) \otimes i_G(s))
= M \otimes \text{id} \otimes (s \mapsto i_G(s) \otimes 1 \cdot s \mapsto 1 \otimes i_G(s))
= M \otimes \text{id}(w_G) \otimes 1 \cdot \text{id} \otimes (M \otimes \text{id}(w_G) \otimes 1))
= (M \otimes \text{id}(w_G))_{12}(M \otimes \text{id}(w_G))_{13}.
$$

Any $*$-representation $\mu : C_0(G) \to B(H)$ gives rise to a corepresentation of $C^*(G)$, namely $\mu \otimes \text{id}(w_G) \in M(K(H) \otimes C^*(G))$.

Now for the converse. Given a corepresentation $U$ of $C^*(G)$ in $M(K(H) \otimes C^*(G))$, we can define a map $\bar{\mu}$ on $C^*(G)^*$ to $B(H)$ by $\bar{\mu}(f) := \text{id} \otimes f(U)$. We denote its restriction to the Fourier algebra $A(G)$ by $\mu$. We first show that $\mu$ is multiplicative:

$$
\mu(fg) = \text{id} \otimes (fg)(U) = \text{id} \otimes ((f \otimes g) \circ \delta_G)(U)
= \text{id} \otimes f \otimes g \circ \delta_G(U) \circ \text{id} = \text{id} \otimes f \otimes g(\Sigma(U_{12}U_{13}))
= \text{id} \otimes f(U) \otimes g(U) = \mu(f)\mu(g).
$$

An argument described in [14, p130] shows that $\mu$ is involutive. Thus $\mu$ is a $*$-homomorphism from the abelian Banach $*$-algebra $A(G)$ into $B(H)$. The fact that it is $*$-preserving implies that it is norm decreasing on $A(G)$ [13, Th 2.1.7]. Thus there exists a unique $*$-representation, also denoted by $\mu$, on the enveloping $C^*$-algebra $\mathcal{B}$. Since the spectrum of $A(G)$ is $G$ [4, 3.34], the enveloping $C^*$-algebra is $C_0(G)$.

To see that every corepresentation of $C^*(G)$ is of the form $\mu \otimes \text{id}(w_G)$, we show that $\mu \otimes \text{id}(w_G) = U$. By definition $\bar{\mu}(f) = \text{id} \otimes f(U)$ for all $f \in C^*(G)^*$. Using Equation (2), we have $\mu(\text{id} \otimes f(w_G)) = \text{id} \otimes f(U)$, which means $\text{id} \otimes f(\mu \otimes \text{id}(w_G)) = \text{id} \otimes f(U)$. Since functionals of the form $\text{id} \otimes f$ separate the points of $M(K(H) \otimes C^*(G))$ [24, Th 1], we deduce that $\mu \otimes \text{id}(w_G) = U$ (cf. [4, Lem 1.2]).

**Example 4(a).** Let $\alpha$ be an action of $G$ on $A$. There is a dual coaction $\hat{\alpha}$ on the full crossed product $A \times_\alpha G$. Specifically, $\hat{\alpha} : A \times_\alpha G \to M((A \times_\alpha G) \otimes C^*(G))$ such that $\hat{\alpha}(k_A(a)) = k_A(a) \otimes 1$ and $\hat{\alpha}(k_G(s)) = k_G(s) \otimes i_G(s)$ [24, Ex 2.3(1)]. Composing this map with $\text{id} \otimes k_G$, where $k_G$ is the canonical homomorphism carrying $C^*(G)$ into $A \times_\alpha G$, gives a map $\delta_{A \times_\alpha G} : A \times_\alpha G \to M((A \times_\alpha G) \otimes (A \times_\alpha G))$ such that

$$
\delta_{A \times_\alpha G}(k_A(a)) = k_A(a) \otimes 1 \text{and } \delta_{A \times_\alpha G}(k_G(s)) = k_G(\delta_G(s)).
$$

This is a right-simplifiable comultiplication on $A \times_\alpha G$, so $A \times_\alpha G$ is a Hopf $C^*$-algebra.

Let $W$ be a corepresentation of $C^*(G)$. The following shows that $\text{id} \otimes k_G(W)$ is a corepresentation of $A \times_\alpha G$:

$$
\text{id} \otimes \delta_{A \times_\alpha G}(\text{id} \otimes k_G(W)) = \text{id} \otimes k_G \otimes k_G(\text{id} \otimes \delta_G(W))
= \text{id} \otimes k_G \otimes k_G(W_{12}W_{13})
= \text{id} \otimes k_G \otimes k_G(W_{12}) \text{id} \otimes k_G \otimes k_G(W_{13})
= \text{id} \otimes k_G(W)_{12} \text{id} \otimes k_G(W)_{13}.
$$

As yet we have not been able to show that all corepresentations of $A \times_\delta G$ are of this form, although we expect this to be the case.
In the case where $A$ is abelian, the crossed product is a transformation group $C^*$-algebra. These $C^*$-algebras are a type of groupoid $C^*$-algebra. So we have shown that at least a subclass of groupoid $C^*$-algebras are Hopf $C^*$-algebras.

In the case where $G$ is acting by right translation $\sigma$ on $C_0(G)$, the crossed product $C_0(G) \times_\sigma G$ is isomorphic to $K(L^2(G))$ \cite{27}, so we have that $K(L^2(G))$ is a Hopf $C^*$-algebra. When $G$ is a finite group of order $n$, this leads to identifying the $n \times n$ matrices $M_n(\mathbb{C})$ as a Hopf $C^*$-algebra.

**Example 5(a).** Let $\delta$ be a coaction of a locally compact group $G$ a $C^*$-algebra $A$. There is a dual action $\hat{\delta}$ on the full crossed product $A \times_\delta G$ \cite[Cor 2.14]{28}. We can compose this with $\text{id} \otimes j_{C_0(G)}$ to obtain a map $\delta_{A \times G}$ from $A \times_\delta G$ into $M\big((A \times_\delta G) \otimes (A \times_\delta G)\big)$ making $A \times_\delta G$ into a Hopf $C^*$-algebra. As in Example 4(a), if $W$ is a corepresentation of $C_0(G)$, then $\text{id} \otimes j_{C_0(G)}(W)$ is a corepresentation of $A \times_\delta G$.

**Example 6(a).** Let $V \in B(H \otimes H)$ be a regular multiplicative unitary on a Hilbert space $H$, as defined by Baaj and Skandalis \cite[Defn 1.1, 3.3]{3}. This implies, among other things, that $V$ is a unitary operator which satisfies the pentagonal identity:

$$V_{12} V_{13} V_{23} = V_{23} V_{12}.$$ 

In \cite[Cor A.6]{3} they associate to each such $V$ a pair of Hopf $C^*$-algebras, $S_p$ and $\hat{S}_p$. They also define a pair of Hopf $C^*$-algebras, $S_V := A(V)$ and $\hat{S}_V := \hat{A}(V)$, where

$$A(V) := \text{sp}\{f \otimes \text{id}(V)|f \in B(H)\} \text{ and } \hat{A}(V) := \text{sp}\{\text{id} \otimes f(V)|f \in B(H)\},$$

\cite[Defn 1.3, 1.5]{3}, which should be considered as the reduced version of the theory.

Baaj and Skandalis show \cite[\S 4]{3}, that the compact quantum groups of Woronowicz \cite{31} give rise to compact regular multiplicative unitaries. This provides examples of Hopf $C^*$-algebras which do not arise from locally compact groups.

The following examples are not actually examples – they are examples which one might think could be Hopf $C^*$-algebras, but aren’t.

**Example 7.** Let $(id, u)$ be a twisted action of a locally compact group $G$ on the complex numbers $\mathbb{C}$ \cite[Def 2.1]{3}. We recall that the twist $u$ is a Borel map from $G \times G$ to $\mathbb{T}$ satisfying $r_G(s) r_G(t) = u(s, t) r_G(st)$. The crossed product $\mathbb{C} \times_{id,u} G$ for this twisted dynamical system is called a twisted group $C^*$-algebra, sometimes denoted by $C^*_\text{tw}(G, u)$. We would like to define a comultiplication $\delta_\text{tw} : C^*_\text{tw}(G, u) \to M\big(C^*_\text{tw}(G, u) \otimes C^*_\text{tw}(G, u)\big)$ as we did in Example 2(a), but $\delta_\text{tw}$ is not a homomorphism:

$$\delta_\text{tw}(i_G(s) i_G(t)) = \delta_\text{tw}(u(s, t) i_G(st)) = u(s, t) \delta_\text{tw}(i_G(st)) = u(s, t)(i_G(st) \otimes i_G(st)), \text{and}$$

$$\delta_\text{tw}(i_G(s)) \delta_\text{tw}(i_G(t)) \neq \delta_\text{tw}(i_G(s)) \delta_\text{tw}(i_G(t)) = [i_G(s) \otimes i_G(s)] [i_G(t) \otimes i_G(t)] = i_G(s) i_G(t) \otimes i_G(s) i_G(t) = u(s, t) i_G(st) \otimes u(s, t) i_G(st).$$

So $C^*_\text{tw}(G, u)$ cannot be made into a Hopf $C^*$-algebra in this manner.

**Example 8.** Let $\Gamma$ be a totally ordered discrete abelian group with positive cone $\Gamma^+$. The semigroup $C^*$-algebra $C^*(\Gamma^+)$ has been defined \cite[\S 1]{3} and is the universal $C^*$-algebra whose representations correspond to the isometric representations of the semigroup $\Gamma^+$. Routine calculations show that we can define a comultiplication $\delta_{\Gamma^+}$
as in Example 3(a), so $C^*(G^+)$ is a Hopf $C^*$-algebra. We can again define $w_{1+} \in M(C_0(G^+) \otimes C^*(G^+))$, but since each $i_{1+}(s)$ is a not necessarily unitary isometry, $w_{1+}$ may not be unitary, and so elements of the form $\mu \otimes id(w_{1+})$ are not corepresentations of $C^*(G^+)$ in the sense we are using in this paper.

The function algebra $C_0(G^+)$ has comultiplication $\alpha_{1+}$ as in Example 2(a), but again it does not necessarily have any unitary corepresentations.

**Example 9.** Let $\mathcal{G}$ be a groupoid. In Example 4(a) we showed how groupoid $C^*$-algebras arising as transformation group $C^*$-algebras are Hopf $C^*$-algebras. But in general there does not seem to be an obvious way to make a groupoid $C^*$-algebra $C^*(\mathcal{G})$ into a Hopf $C^*$-algebra. Working with von Neumann algebras, Yamanouchi \cite{Yamanouchi} p12 shows that $L^\infty(\mathcal{G})$ admits a comultiplication $\alpha_{\mathcal{G}}$ satisfying

$$\alpha_{\mathcal{G}}(f)(s,t) = \begin{cases} f(st), & \text{if } (s,t) \text{ is a composable pair}, \\ 0, & \text{otherwise}. \end{cases}$$

This approach does not work for $C_0(\mathcal{G})$, as $\alpha_{\mathcal{G}}(f)$ need not be continuous on $\mathcal{G} \times \mathcal{G}$.

**Remark.** Another concept frequently seen in the literature is that of a co-unit \cite{Defn 1.2 Defn 1.5(b) 26}. We have no use for a co-unit in this paper. Actually, there is a problem with the name “co-unit”. It’s logical to think that a co-unit is something that would be used to define a unit in the dual object. But consider the fundamental example: $C^*(G)$ and its dual $C_0(G)$. The trivial representation of $G$ integrates up to a representation of $C^*(G)$, denoted by 1, and $(id \otimes 1) \circ \delta = id$ \cite{Lem 1.3}, which means $C^*(G)$ is co-unital. However, unless the group is compact, $C_0(G)$ is not unital. So, the existence of such a co-unit does not imply that the dual is unital.

## 3. Coactions of Hopf $C^*$-algebras.

A coaction of a Hopf $C^*$-algebra $S$ on a $C^*$-algebra $A$ is a nondegenerate injective *-homomorphism $\delta: A \to M(A \otimes S)$ such that $(\delta \otimes id) \circ \delta = (id \otimes \delta_S) \circ \delta$, as maps of $A$ into $M(A \otimes S \otimes S)$. If $\delta(A)(1 \otimes S) = A \otimes S$, then $\delta$ is called a nondegenerate coaction (cf. \cite{Defn 0.2, Defn 1.2} 3).

Let $B$ be a $C^*$-algebra. A covariant homomorphism of $(A,S,\delta)$ into $M(B)$ is a pair $(\phi,v)$, where $\phi: A \to M(B)$ is a nondegenerate *-homomorphism and $v \in M(B \otimes S)$ is a unitary such that,

1. $id \otimes \delta_S(v) = v_{12}v_{13} \in M(B \otimes S \otimes S)$, and
2. $\phi \otimes id(\delta(a)) = v(\phi(a) \otimes 1)v^* \in M(B \otimes S)$, for all $a \in A$.

A covariant representation of $(A,S,\delta)$ is a covariant homomorphism of $(A,S,\delta)$ into $M(K(H)) \cong B(H)$. Specifically, it is a pair $(\pi,V)$, where $\pi: A \to B(H)$ is a *-representation of $A$ and $V \in M(K(H) \otimes S)$ is a corepresentation of $S$, such that

$$\pi \otimes id(\delta(a)) = V(\pi(a) \otimes 1)V^* \in M(K(H) \otimes S),$$

\cite{Defn 0.2, 0.3}. Note that this implies that $ker((\pi \otimes id) \circ \delta) = ker(\pi)$.

**Example 10.** There is always the trivial coaction of a Hopf $C^*$-algebra $S$ on a $C^*$-algebra $A$, that is, $id: A \to M(A \otimes S)$ satisfying $id(a) = a \otimes 1$. Also, a right-simplifiable comultiplication $\delta_S: S \to M(S \otimes S)$ is a coaction of a Hopf $C^*$-algebra $S$ on itself.
Example 4(b). Let \((A, G, \alpha)\) be a dynamical system, so that, as described in §1, we have the map \(\alpha: A \rightarrow M(A \otimes C_0(G))\). This map is a coaction of the Hopf \(C^*\)-algebra \(C_0(G)\) on \(A\). A pair \((\pi, U)\) which is a covariant representation of the dynamical system \((A, G, \alpha)\), that is, \(\pi: A \rightarrow B(H)\) and \(U: C^*(G) \rightarrow B(H)\) are representations such that

\[
\pi \otimes \text{id}(\alpha(a)) = U \otimes \text{id}(v_G)(\pi(a) \otimes 1)U \otimes \text{id}(v_G^*) \in M(K(H) \otimes C_0(G)),
\]
gives rise to a covariant pair \((\pi, U \otimes \text{id}(v_G))\) of \((A, C_0(G), \alpha)\), because the covariance condition is precisely the same, and, as we showed in Example 2(a), \(U \otimes \text{id}(v_G)\) is a corepresentation of \(C_0(G)\).

Example 5(b). Let \((A, G, \delta)\) be a cosystem. Then \(\delta\) is a coaction of the Hopf \(C^*\)-algebra \(C^*(G)\) on \(A\). A pair \((\pi, \mu)\) which is a covariant representation of the cosystem \((A, G, \delta)\), that is, \(\pi: A \rightarrow B(H)\) and \(\mu: C_0(G) \rightarrow B(H)\) are representations such that

\[
\pi \otimes \text{id}(\delta(a)) = \mu \otimes \text{id}(w_G)(\pi(a) \otimes 1)\mu \otimes \text{id}(w_G^*) \in M(K(H) \otimes C^*(G)),
\]
gives rise to a covariant pair \((\pi, \mu \otimes \text{id}(w_G))\) of \((A, C^*(G), \delta)\).

Example 6(b). Baaj and Skandalis point out that if \((\pi, V)\) is a covariant representation for any coaction of a Hopf \(C^*\)-algebra of a \(C^*\)-algebra \(A\), then \(\text{id} \otimes \pi(V)\) is a multiplicative unitary \([3, \text{Ex } 1.2(5)]\). This is not difficult to see because the covariance condition implies that

\[
\text{id} \otimes \pi \otimes \text{id}(\otimes \delta(V))(1 \otimes V) = (1 \otimes V)(\text{id} \otimes \pi(V) \otimes 1),
\]
so using Equation \([3]\) gives \(\text{id} \otimes \pi \otimes \text{id}(V_{12}V_{13})V_{23} = V_{32}(\text{id} \otimes \pi \otimes \text{id}(V_{12}))\). By applying \((\text{id} \otimes \pi \otimes \pi)\) to both sides we obtain

\[
(\text{id} \otimes \pi(V))_{12} (\text{id} \otimes \pi(V))_{13} (\text{id} \otimes \pi(V))_{23} = (\text{id} \otimes \pi(V))_{23} (\text{id} \otimes \pi(V))_{12}.
\]

4. Co-involutive Hopf \(C^*\)-algebras.

Let \(\delta\) be a coaction of a Hopf \(C^*\)-algebra \(S\) on a \(C^*\)-algebra \(A\). Let \((\pi_1, W_1)\) and \((\pi_2, W_2)\) be covariant representations of \((A, S, \delta)\) on \(H_1\) and \(H_2\) respectively. The direct sum \(\pi_1 \oplus \pi_2\) is a representation of \(A\) on \(H_1 \oplus H_2\) and the unitary \(W_1 \oplus W_2\) can be considered an element of \(M(K(H_1 \oplus H_2) \otimes S)\). Furthermore, \(W_1 \oplus W_2\) is a corepresentation of \(S\) and \((\pi_1 \oplus \pi_2, W_1 \oplus W_2)\) is a covariant representation of \((A, S, \delta)\) on \(H_1 \oplus H_2\). Infinite direct sums are handled similarly. All these follow from Raeburn’s arguments for the group case \([26, \text{p634}]\). Now let \(W_1\) and \(W_2\) be corepresentations of \(S\) on \(H\). Then \(W_1\) is unitarily equivalent to \(W_2\) if there exists a unitary \(U\) in \(B(H)\) such that \(W_1 = (U \otimes 1)W_2(U^* \otimes 1)\) \([22, \text{Eq5.2}]\).

Remark. Example \([3]\) suggests that it is possible to have Hopf \(C^*\)-algebras which do not have any corepresentations, and consequently no covariant representations. From here on we will only work with Hopf \(C^*\)-algebras that admit at least one covariant representation.

Let \(S\) be a Hopf \(C^*\)-algebra. Let \(\Gamma\) be the smallest set of covariant representations of \((S, S, \delta_S)\) such that, for every covariant representation \((\nu, W)\) of \((S, S, \delta_S)\) on \(H\), there exists an element \((\mu_\gamma, V_\gamma)\) of \(\Gamma\), with \(\nu\) unitarily equivalent to \(\mu_\gamma\) and \(W\) unitarily equivalent to \(V_\gamma\). Let \(\mu_S := \oplus_\Gamma \mu_\gamma\), \(V_S := \oplus_\Gamma V_\gamma\) and \(H_S := \oplus_\Gamma H_\gamma\). The representation \(\mu_S\) of \(S\) is called the regular representation of \(S\) and the pair \((\mu_S, V_S)\) is called the regular covariant representation of \((S, S, \delta_S)\).
Define \(A_0(S)\) to be the pre-dual of the von Neumann algebra generated by \(\mu_S(S)\) in \(B(H_S)\). For each \(f \in A_0(S)\), define \(f_S \in S^*\) by \(f_S := f \circ \mu_S\). We may put operations on \(A_0(S)\) by pointwise addition and scalar multiplication, and multiplication * defined by

\[
f \ast g(\mu_S(x)) := f_S \otimes g_S(\delta_S(x)),
\]

[Eq 1.1]. Since \(\mu_S\) is part of a covariant pair, \(\ker \mu_S = \ker((\mu_S \otimes \mu_S) \circ \delta_S)\), and \(f \ast g\) is well-defined. To show that the product \(f \ast g\) is an element of \(A_0(S)\), we need to know that it is weakly continuous. Since \(f \otimes g\) is weakly continuous, it follows from the covariance of \((\mu_S, V_S)\) because

\[
f \ast g(\mu_S(x)) = f \otimes g(id \otimes \mu_S(V_S))(\mu_S(x) \otimes 1)(id \otimes \mu_S(V_S))^*).
\]

The associativity follows from the comultiplication identity Equation (6):

\[
f \ast (g \ast h)(\mu_S(x)) = f \otimes (g \ast h)(\mu_S \otimes \mu_S(\delta_S(x)))
= f_S \otimes g_S \otimes h_S(id \otimes \delta_S(\delta_S(x)))
= f_S \otimes g_S \otimes h_S(\delta_S \otimes id(\delta_S(x)))
= (f \ast g) \otimes h(\mu_S \otimes \mu_S(\delta_S(x))) = (f \ast g) \ast h(\mu_S(x)).
\]

Hence \(A_0(S)\) is an algebra. Let

\[
M := \{ f \in A_0(S) | id \otimes f_S(W) = 0 \forall \text{ corepresentations } W \text{ of } S \}.
\]

Using Equation [8], we see that \(M\) is an ideal in \(A_0(S)\):

\[
id \otimes (f \ast g)_S(W) = id \otimes f_S \otimes g_S(id \otimes \delta_S(W))
= id \otimes f_S \otimes g_S(W_{12}W_{13}) = id \otimes f_S(W) id \otimes g_S(W). \tag{9}
\]

Let \(A(S)\) be the quotient of \(A_0(S)\) by \(M\).

A Hopf \(C^*\)-algebra \(S\) is called **nondegenerate** if the ideal \(M\) is zero.

**Remark.** It is tempting to suggest that a Hopf \(C^*\)-algebra \(S\) is nondegenerate if and only if \(S\) is bi-simplifiable. But we have only been able to show one direction of the implication, that \(M = \{0\}\) implies \(S\) is bi-simplifiable.

Katayama showed that, for a coaction \(\delta\) of a locally compact group \(G\) on a \(C^*\)-algebra \(A\), \(\delta\) is nondegenerate in Landstad’s sense if and only if \(\delta(A)(1 \otimes C^*(G)) = A \otimes C^*(G)\) [14, Th 5] (Landstad’s nondegeneracy is that if \((f \otimes g) \circ \delta = 0\) for all \(g \in B(G)\), then \(f = 0\) in \(A^*\)). For a comultiplication \(\delta_S\) one can show that it is bi-simplifiable if and only if it is nondegenerate in Landstad’s sense: \((f_S \otimes g_S) \circ \delta_S = 0\) for all \(g \in A(S)\) implies \(f = 0\) in \(A(S)\), and, \((g_S \otimes f_S) \circ \delta_S = 0\) for all \(g \in A(S)\) implies \(f = 0\) in \(A(S)\). We show that if \(M = \{0\}\), then \(\delta_S\) is nondegenerate in Landstad’s sense.

Suppose \(f \in A(S)\), \(M = \{0\}\) and \((f_S \otimes g_S) \circ \delta_S = 0\) for all \(g \in A(S)\). Then \(id \otimes f_S \otimes g_S(id \otimes \delta_S(W)) = 0\) for all corepresentations \(W\) of \(S\). By Equation [8] this means \(id \otimes f_S \otimes g_S(W_{12}W_{13}) = id \otimes f_S(W) id \otimes g_S(W) = 0\) for all \(g\) and \(W\). Thus \(id \otimes f_S(W) = 0\) for all \(W\), and \(f \in M\). Since \(M\) is zero, \(f = 0\) in \(A(S)\).

A Hopf \(C^*\)-algebra \(S\) is **co-involutive** if for each \(f \in A(S)\), there exists a functional \(f^* \in A(S)\) such that

\[
id \otimes (f^*)_S(W) = (id \otimes f_S(W))^*,
\]

for all corepresentations \(W\) of \(S\). Such an \(f^*\) is the unique functional in \(A(S)\) satisfying this relation because elements in the ideal \(M\) are zero in \(A(S)\).
Example 1(c). Consider $A$ from Example 1(a). Since the comultiplication is trivial, its only corepresentation is $1 \otimes 1$, and the pair $(\pi, 1 \otimes 1)$ is covariant for every representation $\pi: A \to B(H)$. So $\Gamma$ has an element from each equivalence class of representations of $A$ and $\mu_S$ is the universal representation. Thus $A_0(S)$ is simply the dual space $A^*$. The ideal $M$ in this case is
\[
\{ f \in A^* | \id \otimes f(1 \otimes 1) = 0 \} = \{ f \in A^* | f(1) = 0 \}.
\]
We can realize $A(S)$ as the complex numbers $\mathbb{C}$ via the isomorphism $f + M \mapsto f(1)$. Complex conjugation satisfies the requirements of a co-involution, so $A$ is a co-involutive Hopf $C^*$-algebra, but it is not nondegenerate.

Example 2(c). Consider the Hopf $C^*$-algebra $C_0(G)$. From Examples 2(a) and (b), we know that $\sigma_G$ is a coaction of $C_0(G)$ on itself. The covariant representations of $(C_0(G), C_0(G), \sigma_G)$ are the same as those of the dynamical system $(C_0(G), G, \sigma)$, where $\sigma$ is action by right translation. By [27], the covariant representations of this dynamical system are in one-to-one correspondence with the representations of $K(L^2(G))$. Now, $K(L^2(G))$ is a simple, Type I $C^*$-algebra, so it has only one equivalence class of representations, namely the faithful ones. So, in this example, there is only one element in $\Gamma$. The covariant representation $(M, \lambda)$ of $(C_0(G), G, \sigma)$ has $M$ faithful, and we choose it to be the element of $\Gamma$. That is, $\mu_S := M$ and $V_S := \lambda \otimes \id(v_G)$. Since $M$ is a faithful representation the weak closure of $M(C_0(G))$ is $L^\infty(G)$, whose predual is $L^1(G)$.

In Example 2(a) we showed that every corepresentation of $C_0(G)$ is of the form $U \otimes \id(v_G)$, where $U$ is a *-representation of $C^*(G)$. Thus the ideal $M$ is
\[
\{ f \in L^1(G) | \id \otimes f(U \otimes \id(v_G)) = 0 \forall U \} = \{ f \in L^1(G) | U(f) = 0 \forall U \} = \{ 0 \},
\]
so $A(C_0(G))$ is $L^1(G)$. Using Equation (2), for $f \in L^1(G)$, we have
\[
\id \otimes f^*(U \otimes \id(v_G)) = U(\id \otimes f^*(v_G)) = U(f^*) = U(f)^* = [U(\id \otimes f(v_G))]^* = [\id \otimes f(U \otimes \id(v_G))]^*.
\]
This shows that $C_0(G)$ is a nondegenerate co-involutive Hopf $C^*$-algebra.

This next calculation shows that the operations defined on $A(C_0(G)) = L^1(G)$ give group multiplication between the point masses in $L^1(G)$:
\[
\varepsilon_s \star \varepsilon_t(f) = \varepsilon_s \otimes \varepsilon_t(\alpha_G(f)) = \varepsilon_s \otimes \varepsilon_t((u, v) \mapsto f(uv)) = f(st) = \varepsilon_{st}(f).
\]
When an element of $L^1(G)$ is viewed as a measure on $G$ the multiplication defined here corresponds to convolution of measures [21, 7.1.2].

Example 3(c). Consider the Hopf $C^*$-algebra $C^*(G)$. The argument here is similar to that in Example 2(c). Again the covariant representations of $(C^*(G), C^*(G), \delta_G)$ are precisely the covariant representations of the cosystem $(C^*(G), G, \delta_G)$, which are in one-to-one correspondence with the representations of $K(L^2(G))$ [27, Ex 2.9]. So, we can choose $(\mu_S, V_S)$ to be $(\lambda, M \otimes \id(w_G))$. The group von Neumann algebra $VN(G)$ is defined to be the weak closure of the image of $C^*(G)$ under $\lambda$, that is $\lambda(C^*(G))^w = VN(G)$. The predual of $VN(G)$ is the Fourier algebra $A(G)$ [6, Th 3.10] [21, 7.2.2].

We showed in Example 3(a) that every corepresentation of $C^*(G)$ is of the form $\mu \otimes \id(w_G)$, where $\mu$ is a *-representation of $C_0(G)$. Thus the ideal $M$ is
\[
\{ f \in A(G) | \id \otimes f(\mu \otimes \id(w_G)) = 0 \forall \mu \} = \{ f \in A(G) | \mu(f) = 0 \forall \mu \} = \{ 0 \},
\]
so in this example $A(C^*(G))$ is $A(G)$ (hence the notation).
Using Equation \([1]\), we show that \(C'(G)\) is a nondegenerate co-involutive Hopf \(C^*\)-algebra:

\[
id \otimes f^*(\mu \otimes \id(w_G)) = \mu(\id \otimes f^*(w_G)) = \mu(f^*) = \mu(f)^* = \id \otimes f(\mu \otimes \id(w_G))^*.
\]

This next calculation shows that the multiplication defined on \(A(G)\) gives point-wise multiplication:

\[
f \ast g(s) = f \otimes g(\delta_G(s)) = f \otimes g(i_C(s) \otimes i_G(s)) = f(s)g(s) = fg(s).
\]

**Remark.** The weakness in the definition of the co-involutive property is that one must know all the corepresentations of a Hopf \(C^*\)-algebra before you can check whether or not it is co-involutive.

**Example 6(c).** Let \(V\) be an amenable multiplicative unitary which is part of a Kac triplet \([3\text{ Defn 6.4, p485}]\). We now show that Hopf \(C^*\)-algebra \(S_V\) (see Example 6(a)) is co-involutive. Firstly, \(V\) is a corepresentation of \((S_V, S_V, \delta)\), by a similar calculation as in Example 6(b). Baaj and Skandalis show that if \(V\) is part of a Kac triplet, the crossed product \(S_V \rtimes S_V\) is the compact operators on \(H\) \([3\text{ Prop 6.3, p485}]\), and thus there is only one unitary equivalence class or covariant representations of \((S_V, S_V, \delta)\). Thus \(\mu_S = \id \otimes V_S = V\). Define \(\phi: A_0(S_V) \to \hat{A}(V)\) by \(\phi(f) = \id \otimes f(V)\). This is well-defined because if \(f = 0\), this means that \(f\) is in the pre-annihilator of \(S_V\), so that \(f = 0\) on \(S_V\). Thus for all \(g \in B(H)_+\), \(f(\id \otimes f(V)) = g(\id \otimes f(V)) = 0\), which implies \(\id \otimes f(V) = 0\) in \(\hat{A}(V)\).

The ideal \(M = \ker \phi\) in \(A_0(S_V)\) is zero by the following argument. Suppose \(f \in A_0(S_V)\) is positive and \(\id \otimes f(W) = 0\) for all corepresentations \(W\) of \(S_V\). In particular then \(\id \otimes f(V) = 0\). So for all \(g \in B(H)_+\), \(\id \otimes f(V) = 0\), which implies \(\id \otimes f(V) = 0\) in \(\hat{A}(V)\).

Take \(f \in A_0(S_V)\) and define \(f^*\) to be the unique functional satisfying \(\id \otimes f^*(V) = (\id \otimes f(V))^*\). This gives an element in \(\hat{A}(S_V)\) because \(V\) is regular so \(\hat{A}(V)\) is a \(*\)-algebra \([3\text{ Prop 3.5}]\).

A multiplicative unitary is **amenable** if \(\hat{S}_V = \hat{S}_p\) \([3\text{ Def 1.17}]\). \([3\text{ Prop 5.5}]\). So, since \(V\) is an amenable multiplicative unitary, it follows from \([3\text{ Lemma 2.6}]\) that every corepresentation \(W\) of \(S_V\) gives rise to a \(*\)-representation \(\mu_w\) of \(\hat{S}_V\) satisfying \(W = \mu_w \otimes \id(V)\). Thus the following calculation shows that \(S_V\) is co-involutive:

\[
id \otimes f^*(W) = \id \otimes f^*(\mu_w \otimes \id(V)) = \mu_w((\id \otimes f(V))^*) = \mu_w(\id(\otimes f^*(V)))^* = [\mu_w((\id \otimes f(V)))]^* = [(\id \otimes f(W))]^*.
\]

As an example, the multiplicative unitary associated to \(S_{\mu}(2)\) defined by Woronowicz is amenable and part of a Kac triplet \([4\ Ch 7]\), and so gives rise to a co-involutive Hopf \(C^*\)-algebra.

**Remark.** From the example we can see that the definition of co-involutive given here is suitable for full Hopf \(C^*\)-algebras rather than reduced.
Lemma 4.1. Let $S$ be a co-involutive Hopf C*-algebra. Then $A(S)$ is a *-algebra with involution satisfying $\text{id} \otimes f_S(W)^* = \text{id} \otimes (f^*)_S(W)$, for all corepresentations $W$ of $S$.

Proof. We need to verify that $f^{**} = f$ and $(f \ast g)^* = g^* \ast f^*$:

$$\text{id} \otimes (f^{**})_S(W) = \text{id} \otimes (f^*)_S(W)^* = (\text{id} \otimes f_S(W))^{**} = \text{id} \otimes f_S(W),$$

so that $(f^{**})_S = f_S$. Also,

$$\text{id} \otimes ((f \ast g)^*)_S(W) = \text{id} \otimes (f \ast g)_S(W)^* = [\text{id} \otimes f_S(W) \text{id} \otimes g_S(W)]^*$$

$$= \text{id} \otimes g_S(W)^* \text{id} \otimes f_S(W)^*$$

$$= \text{id} \otimes (g^*)_S(W) \text{id} \otimes (f^*)_S(W) = \text{id} \otimes (g^* \ast f^*)_S(W),$$

so that $((f \ast g)^*)_S = (g^* \ast f^*)_S$. We will be finished them both when we show that $f_S = g_S$ in $S^*$ implies $f = g$ in $A(S)$, or equivalently, $f_S = 0$ in $S^*$ implies $f = 0$ in $A(S)$. If $f_S = 0$ in $S^*$, then $\text{id} \otimes f_S(W) = 0$ for all corepresentations $W$ of $S$. This means $f$ is in the ideal $M$ of $A_0(S)$, and thus is zero in the quotient $A(S)$.

A Banach space $Z$ is a nondegenerate Banach $S$-module is a Banach $S$-module if $S$ is a C*-algebra and $Z$ is a Banach $S$-module such that there exists an approximate identity $\{e_\lambda\}$ in $S$ satisfying $e_\lambda \cdot z \to z$ for all $z \in Z$.

Lemma 4.2. Let $S$ be a Hopf C*-algebra. Then there is an action of $S$ on $A(S)$ satisfying $(f \cdot x)_S(y) = f_S(xy)$, and $A(S)$ is a nondegenerate Banach $S$-module with norm $\|f\| = \sup\{|f(x)|: x \in S, \|x\| \leq 1\}$.

Proof. Certainly $A_0(S)$ is complete with respect to the norm it inherits as a subspace of $(\mu_S(S)^*)_w$ in $B(H_S)$, that is, $\|f\|_\mu := \sup\{|f(y)|: y \in \mu(S), \|y\| \leq 1\}$. There is an isometric isomorphism between the subspace of functionals that annihilate $\ker \mu_S$ and $(S/(\ker \mu_S))^*$. This implies that $\|f\|$ is equal to $\|f\|_\mu$. So $A_0(S)$ is complete with respect to the norm $\|f\|$.

To see that $A(S)$ is complete we need to show that the ideal $M := \{f \in A_0(S) \mid \text{id} \otimes f_S(W) = 0 \forall \text{ corepresentations } W \text{ of } S\}$ is closed. It suffices to show that each $M_W := \{f \in A_0(S) \mid \text{id} \otimes f_S(W) = 0\}$. Well, $M_W$ is the kernel of the map $\mu_W': A_0(S) \to B(H)$ satisfying $\mu_W'(f) = \text{id} \otimes f_S(W)$, so if we show that $\mu_W'$ is continuous, we are done:

$$\|f\| = \|f_S\| = \|\text{id} \otimes f_S\| = \sup_{\|x\| \leq 1} \|\text{id} \otimes f_S(x)\| \geq \|\text{id} \otimes f_S(W)\| = \|\mu_W'(f)\|.$$
Before we show that such a crossed product exists, we will show that the vector space in (c) is actually a C*-algebra. The proof is modelled on [26, Lem 2.10].

**Lemma 5.1.** Let \((A, S, \delta)\) be a Hopf system, \(C\) be a C*-algebra, and \((j_A, u_S)\) be a covariant homomorphism into \(M(C)\). Then

\[
B := \sigma_p\{j_A(a) \otimes f_S(u_S) : a \in A, f \in A(S)\}
\]

is a C*-algebra.

**Proof.** We begin by showing that \(B\) is closed under multiplication. For \(f, g \in A(S)\),

\[
\begin{align*}
\text{id} \otimes f_S(w_S) \text{id} \otimes g_S(u_S) &= \text{id} \otimes f_S \otimes g_S(u_{12}v_{13}) \\
&= \text{id} \otimes f_S \otimes g_S(\text{id} \otimes \delta_S(u_S)) \\
&= \text{id} \otimes (f \ast g)_S(u_S).
\end{align*}
\]

(10)

Take \(a \in A\) and \(f \in A(S)\). From the covariance of \((j_A, u_S)\) we have that

\[
\text{id} \otimes f_S(j_A(a)) = \text{id} \otimes f_S(j_A \otimes \text{id}(\delta(a))u_S).
\]

Since the coaction satisfies \(\delta(A)1 \otimes S = A \otimes S\), we can approximate \((\delta(a))(1 \otimes x)\) by a sum of the form \(\sum a_i \otimes x_i\), so that

\[
\begin{align*}
\text{id} \otimes f_S(j_A \otimes \text{id}(\delta(a))u_S) &\sim \text{id} \otimes g_S(j_A \otimes \text{id}(\sum a_i \otimes x_i)u_S) \\
&= \text{id} \otimes g_S(\sum j_A(a_i) \otimes x_iu_S) \\
&= \sum j_A(a_i) (\text{id} \otimes (g \cdot x_i)_S(u_S)).
\end{align*}
\]

Thus \(B\) is closed under multiplication. We next show that

\[
\text{id} \otimes f_S(u_S)^* = \text{id} \otimes f_S^*(u_S).
\]

(11)

Let \(\pi : B \to B(H)\) be a faithful representation. Then \(\pi \otimes \text{id}(u_S)\) is a corepresentation of \(S\) and thus

\[
[\pi(\text{id} \otimes f_S(u_S))]^* = [\text{id} \otimes f_S(\pi \otimes \text{id}(u_S))]^* = \text{id} \otimes f_S^*(\pi \otimes \text{id}(u_S)) = \pi(\text{id} \otimes f_S^*(u_S)).
\]

Equation (11) follows from the fact that \(\pi\) is a faithful *-homomorphism. This equation shows that the adjoint of \(j_A(a) \otimes f_S(u_S)\) is in \(B\). Hence \(B\) is *-closed. \(\square\)

Let \((A, S, \delta)\) be a Hopf system. A covariant representation \((\pi, W)\) of \((A, S, \delta)\) on \(H\) is cyclic if there exists a vector \(\xi \in H\) such that

\[
H = \sigma_p\{\pi(a) \otimes f_S(W)(\xi) : a \in A, f \in A(S)\}
\]

[Defn 2.8(c)]. Let \((\mu_S, V_S)\) be the regular covariant representation of \((S, S, \delta_S)\) on \(H_S\) ([4]). We know there are non-trivial covariant representations of \((A, S, \delta)\) – given a representation \(\pi\) of \(A\), \((\pi \otimes \mu_S \circ \delta, 1 \otimes V_S)\) is covariant:

\[
[((\pi \otimes \mu) \circ \delta) \otimes \text{id}(\delta(a))] [1 \otimes V] = [\pi \otimes (\mu \otimes \text{id})(\delta(\delta(a)))][1 \otimes V] \\
= [\pi \otimes ((\mu \otimes \text{id}) \circ \delta_S)(\delta(a))][1 \otimes V] \\
= [\pi \otimes (\mu(\delta(a)))[1 \otimes V]
\]

Define \(\text{Ind} \pi : A \times_S S \to B(H) \otimes H_S\) by \(\text{Ind} \pi := (\pi \otimes \mu_S \circ \delta) \times (1 \otimes V_S)\). The following theorem is modelled on [26, Prop 2.13].
Theorem 5.2. Let \((A, S, \delta)\) be a Hopf system. Then there is a full crossed product \((B, j_A, j_S)\) for \((A, S, \delta)\), which is unique in the sense that if \((C, k_A, k_S)\) is another, then there is an isomorphism \(\phi\) of \(B\) onto \(C\) such that \(\phi \circ k_A = j_A\) and \(\phi \circ k_S = j_S\). The full crossed product is denoted by \(A \times_\delta S\).

Proof. Let \(\Gamma\) be a set of cyclic covariant representations of \((A, S, \delta)\) such that, for every covariant representation \((\nu, W)\) of \((A, S, \delta)\), there exists a member \((\mu, V)\) of \(\Gamma\), with \(\nu\) unitarily equivalent to \(\mu\), and \(W\) unitarily equivalent to \(V\). Let \(j_S := \oplus \nu \circ \mu_S\) and \(H := \oplus \nu H_S\). Let \(B\) be the closed linear span of \(\{j_A(a) \id \otimes f_S(u_S) : a \in A, f \in A(S)\}\). By Lemma 5.1, \(B\) is a C*-algebra and \(j_A(a)\) and \(\id \otimes f_S(u_S)\) are both multipliers of \(B\). As in [26, p635], \((j_S, u_S)\) is a covariant homomorphism into \(B\), so condition (a) is satisfied. Condition (c) holds by the definition of \(B\).

As in [26, Cor 2.12], it follows from Lemma 5.1 that any covariant representation is a direct sum of cyclic representations. It is enough to check condition (b) for a cyclic covariant representation \((\mu, V)\). But \((\mu, V)\) is equivalent to a member of \(\Gamma\) and we can construct \(\mu \times V\) by compressing \(B\) to the appropriate summand of \(H\) (as in proof of [26, Prop 2.13]). The uniqueness of the full crossed product follows immediately from its universal properties. \(\square\)

Example 4(c). Now we show that given a dynamical system \((A, G, \alpha)\), the crossed product \(A \times_\alpha G\) is a crossed product for the Hopf system \((A, C_0(G), \alpha)\). The embeddings are \((k_A, k_G \otimes \id(v(G)))\), so condition (a) is satisfied. Let \((\pi, W)\) be a covariant representation of \((A, C_0(G), \alpha)\). In Example 2(a) we showed that there exists a representation \(V\) of \(C^*(G)\) satisfying \(V(s) = \id \otimes \varepsilon_s(W)\).

Since \(W = V \otimes \id(v(G)), (\pi, V)\) is a covariant representation of \((A, G, \alpha)\). Thus \(\pi \times V\) exists, \((\pi \times V) \circ k_A = \pi\) and

\[
(\pi \times V) \otimes \id(k_G \otimes \id(v(G))) = V \otimes \id(v(G)) = W,
\]

so condition (b) is satisfied. It remains to verify condition (c). Since \(A(C_0(G)) = L^1(G)\) is dense in \(C^*(G)\), we have

\[
A \times_\alpha C_0(G) = \overline{\span}\{k_A(a) \id \otimes z(k_G \otimes \id(v(G))) : a \in A, z \in L^1(G)\} = \overline{\span}\{k_A(a) k_G(z) : a \in A, z \in L^1(G)\} = A \times_\alpha G.
\]

Example 5(c). For a cosystem \((A, G, \delta)\), the argument that \(A \times_\delta G\) is a crossed product for the Hopf system \((A, C^*(G), \delta)\) follows closely that of Example 4(c), so that \(A \times_\delta C^*(G) = \overline{\span}\{j_A(a) j_{C_0(G)}(f) : a \in A, f \in A(G)\}\).

Theorem 5.3. Let \((A, S, \delta)\) be a Hopf system, \(B\) be a C*-algebra and \((\pi, W)\) be a covariant homomorphism into \(M(B)\). Then there exists a unique nondegenerate homomorphism \(\pi \times W : A \times_\delta S \to M(B)\) such that

\[
(\pi \times W) \circ j_A = \pi \quad \text{and} \quad (\pi \times W) \otimes \id(u_S) = W.
\]

Proof. Represent \(B\) faithfully on Hilbert space via its universal representation \(\phi : B \to B(H) = M(K(H))\), so that \((\phi \circ \pi, \phi \circ \id(W))\) is a covariant representation of \((A, S, \delta)\). Then there is a representation \(\sigma := (\phi \circ \pi) \times (\phi \circ \id(W))\) of \(A \times_\delta S\) on \(H\) such that \(\sigma \circ j_A = \widetilde{\phi} \circ \pi \sigma\otimes \id(u_S) = \widetilde{\phi} \otimes \id(W)\). Since the image of \(\sigma\) is contained in the image of \(\phi, \phi^{-1} \circ \sigma : A \times_\delta S \to M(B)\) is a homomorphism such that \(\phi^{-1} \circ \sigma) \circ j_A = \widetilde{\phi}^{-1} \circ \widetilde{\phi} \circ \pi = \pi, \text{and} (\phi^{-1} \circ \sigma) \otimes \id(u_S)) = W\). The argument of [20] Lem 1.3 gives the nondegeneracy. \(\square\)
Remark. Let \((A,S,\delta)\) be a Hopf system. One would like to define the reduced crossed product \(A \times_{\delta, r} S\) as the quotient of \(A \times S\) by \(\ker(\text{Ind} \pi)\) where \(\pi\) is a faithful representation of \(A\). But for this to work, we need to know that \(\text{Ind}\) is well-defined on ideals. We have not been able to show this is the case without further assumptions on \(S\). This will be discussed further in [18].

6. The Dual Hopf C*-Algebra.

We noted in Example [14] that any Hopf C*-algebra \(S\) coacts trivially on the complex numbers \(C\). For a co-involutive Hopf C*-algebra \(S\), define
\[
\hat{S} := C \times_{\text{id}} S.
\]
From the universal properties of the crossed product we have canonical unitaries \(w_S \in M(\hat{S} \otimes S)\) and \(v_S := \Sigma(w_S) \in M(S \otimes \hat{S})\), satisfying
\[
\text{id} \otimes \delta_S(w_S) = (w_S)_{12}(w_S)_{13} \in M(\hat{S} \otimes S \otimes S), \quad \text{and}
\]
\[
\delta_S \otimes \text{id}(v_S) = (v_S)_{13}(v_S)_{23} \in M(S \otimes S \otimes \hat{S}).
\]

**Theorem 6.1.** Let \(S\) be a co-involutive Hopf C*-algebra. Define
\[
\psi: A(S) \to \hat{S} \quad \text{by} \quad \psi(f) := \text{id} \otimes f_S(w_S),
\]
where \(w_S\) is the canonical unitary in \(M(\hat{S} \otimes S)\). Then \(\psi\) is an injective *-homomorphism from \(A(S)\) into \(\hat{S}\) with dense range.

If \(W\) is a corepresentation of \(S\), then \(\nu_w\), defined by \(\nu_w(\psi(f)) := \text{id} \otimes f_S(W)\), is a *-representation of \(\hat{S}\) such that \(\nu_w \otimes \text{id}(w_S) = W\).

Furthermore, \(\hat{S}\) is a Hopf C*-algebra with comultiplication \(\delta_{\hat{S}}: \hat{S} \to M(\hat{S} \otimes \hat{S})\) satisfying
\[
\delta_{\hat{S}}(\psi(f)) = \text{id} \otimes \text{id} \otimes f_S((w_S)_{13}(w_S)_{23}).
\]

**Proof.** The operations on \(A(S)\) are defined in [4]. Equation (10) shows that \(\psi\) is multiplicative. The norm on \(\hat{S} = C \times_{\text{id}} S\) is the supremum of the covariant representations of \((C,S,\text{id})\), which are exactly the corepresentations of \(S\), so
\[
\|\psi(f)\| = \sup_W \|\text{id} \otimes f_S(w_S)\| = \sup_W \|\text{id} \otimes f_S((\text{id} \times W(\text{id} \otimes f_S(w_S)))\| = \sup_W \|\text{id} \otimes f_S((\text{id} \times W) \otimes \text{id}(w_S))\| = \sup_W \|\text{id} \otimes f_S(W)\|.
\]
Now if \(\psi(f) = 0\), the above shows that \(\text{id} \otimes f_S(W) = 0\) for all corepresentations \(W\) of \(S\), which means that \(f = 0\) in \(A(S)\). Thus \(\psi\) is injective. The image of \(\psi\) is dense in \(\hat{S}\) because \(\hat{S}\) is spanned by \(\{\text{id} \otimes f_S(w_S): f \in A(S)\}\).

Let \(W\) be a corepresentation of \(S\) on \(H\); define \(\nu_w: \hat{S} \to B(H)\) by
\[
\nu_w(\psi(f)) := \text{id} \otimes f_S(W).
\]
A calculation like that of Equation (13) shows that it is multiplicative, and it is involutive because \(S\) is a co-involutive Hopf C*-algebra. It is norm decreasing since the norm in \(\hat{S}\) is the supremum of the corepresentations of \(S\). Since the slice maps \(\text{id} \otimes f\) for \(f \in S^*\) separate points, it follows from the definition of \(\nu_w\) that \(\nu_w \otimes \text{id}(w_S) = W\).

To show that there exists a nondegenerate *-homomorphism \(\delta_{\hat{S}}\) satisfying Equation (13), we will apply Theorem 6.3. It will suffice to show that there exists a
unitary $u$ in $M(\hat{S} \otimes \hat{S} \otimes S)$ such that $\id \otimes \id \otimes \delta_\hat{S}(u) = u_{123} u_{124}$. A routine calculation shows that $u := w_{13} w_{23}$ does the job. It follows from the definition of $\delta_\hat{S}$ that

$$\delta_\hat{S} \otimes \id\(w_S\) = (w_S)_{13}(w_S)_{23} \in M(\hat{S} \otimes \hat{S} \otimes S) : (14)$$

The following shows that the comultiplication identity is satisfied:

$$\id \otimes \delta_\hat{S}(\delta_\hat{S}(f)) = \id \otimes \delta_\hat{S}(\id \otimes \delta_\hat{S}(f)) \otimes \id(w_{13}w_{23})$$

$$= \id \otimes \id \otimes \delta_\hat{S}(f) \otimes \id(w_{14}w_{24}w_{34})$$

$$= \id \otimes \id \otimes \id \otimes \id \otimes \id(w_{13}w_{23})$$

$$= \delta_\hat{S} \otimes \id(\delta_\hat{S}(f)) = \delta_\hat{S} \otimes \id(\delta_\hat{S}(f)).$$

Remark. Note that if $V$ and $W$ are unitarily equivalent corepresentations of $S$, then the representations $\nu_v$ and $\nu_w$ of $\hat{S}$ are unitarily equivalent.

From now on the injection $\psi: A(S) \to \hat{S}$ given by Theorem 6.1 will be used implicitly. So, for example, we can consider $f \in A(S)$ to be an element of $\hat{S}$, and the canonical unitary $w_S$ in $M(\hat{S} \otimes S)$ satisfies

$$\id \otimes f_S(w_S) = f \in A(S).$$

For $f \in S^*$, $\id \otimes f_S(w_S) = f \in M(\hat{S})$.

The Hopf $C^*$-algebra $\hat{S}$ is called the dual Hopf $C^*$-algebra for $S$. It follows from Equation (14) that

$$\id \otimes \delta_\hat{S}(v_S) = (v_S)_{12}(v_S)_{13} \in M(S \otimes \hat{S} \otimes \hat{S}).$$

Thus given a representation $\nu$ of $S$, $\nu \otimes \id(v_S)$ is a corepresentation of $\hat{S}$.

Let $(A, S, \delta)$ be a Hopf system. By definition there is a (not necessarily faithful) *-homomorphism $j_A: A \to M(A \times_A S)$. Now define $j_\hat{S}: \hat{S} \to M(A \times_A S)$ by $j_\hat{S}(f) := \id \otimes f(u_S)$, where $w_S$ is the canonical unitary in $M((A \times_A S) \otimes S)$. Calculations just like those in Theorem 6.3 show that $j_\hat{S}$ is an injective *-homomorphism. Thus

$$A \times_A S = \varprojlim \{j_A(a)j_\hat{S}(f): a \in A, f \in A(S)\}.$$
dual Hopf C*-algebra \( \hat{C}(G) \) is \( C_0(G) \) with comultiplication \( \alpha_G \):
\[
\delta_S(f) = \text{id} \otimes \text{id} \otimes f((w_G)_{13}(w_G)_{23})
\]
\[
= \text{id} \otimes \text{id} \otimes f((s \mapsto i_G(s))(t \mapsto i_G(t)))
\]
\[
= \text{id} \otimes \text{id} \otimes f((s, t) \mapsto i_G(s)i_G(t))
\]
\[
= \text{id} \otimes \text{id} \otimes f((s, t) \mapsto i_G(st)) = (s, t) \mapsto f(st) = \alpha_G(f).
\]

**Example 6(d).** Let \( V \) be an amenable multiplicative unitary which is part of a Kac triplet. We argued in Example 6(c) that \( \hat{S}_V \) is the C*-algebra whose representations are in one-to-one correspondence with the corepresentations of \( S_V \). The formula given for the comultiplication in Theorem 6.1 agrees with that given by Banaj and Skandalis in [3, cor A.6], and thus \( \hat{S}_V \) is the same as the dual constructed here.

7. The double dual.

Let \( S \) be a co-involutive Hopf C*-algebra and let \( (\mu, V) \) be a covariant representation of \( (\hat{S}, S, \delta_S) \) on \( H \). Define
\[
\hat{\mu} : \hat{S} \to B(H_S) \quad \text{by} \quad \hat{\mu}(f) := \text{id} \otimes f(V), \quad \text{and}
\]
\[
\hat{V} \in M(K(H_S) \otimes \hat{S}) \quad \text{by} \quad \hat{V} := \mu \otimes \text{id}(v_S).
\]
By Theorem 6.1, \( \hat{\mu} \) is a *-representation of \( \hat{S} \). Using Equation (16) we can show that \( \hat{V} \) is a corepresentation of \( \hat{S} \):
\[
\text{id} \otimes \delta_S(\hat{V}) = \text{id} \otimes \delta_S(\mu \otimes \text{id}(v_S))
\]
\[
= \mu \otimes \text{id}(\text{id} \otimes \delta_S(v_S)) = \mu \otimes \text{id}(v_S)_{12}(v_S)_{13}
\]
\[
= \mu \otimes \text{id}(v_S)_{12} \mu_S \otimes \text{id}(v_S)_{13} = (V)_{12} = (V)_{13}.
\]

Now, it is natural to ask whether or not \((\hat{\mu}, \hat{V})\) is a covariant representation of \((\hat{S}, \hat{S}, \delta_{\hat{S}})\). Both Quigg and Raeburn have sought such results [26, Ex 2.9], [22, Prop 2.5]. But it doesn’t always work. For example, it doesn’t work for \((C^*(G), C^*(G), \delta_G)\) and \((\mu, M \otimes \text{id}(w_G))\), since \((M, \lambda \otimes \text{id}(v_G))\) is not covariant for \((C_0(G), C_0(G), \alpha)\). In order to construct the dual of a Hopf C*-algebra we need to know that there exists at least one covariant representation, and as we have just seen, \( \hat{S} \) does not automatically have them.

So, in order to build the double dual \( \hat{S} \) we need to know a number of things about \((\hat{S}, \hat{S}, \delta_{\hat{S}})\), in particular, that it is a co-involutive Hopf C*-algebra. For now, we suppose that \( S \) is a Hopf C*-algebra such that \( \hat{S} \) is co-involutive, and denote its regular covariant representation by \((\mu_S, V_S)\). Is that enough to ensure that \( \hat{S} \) is isomorphic to \( S \)?

Given \( x \in S \), evaluation at \( x \) is a functional on \( S^* \); denote it by \( \varepsilon_x \). This functional can be viewed as a functional on \( A(S) \) if and only if \( S \) is nondegenerate, that is, if the ideal \( M \) as \( A_0(S) \), defined in [4, 6], is zero. In this case \( \varepsilon_x \) extends to a functional on \( \hat{S} \) and \( f(x) = \varepsilon_x(f) = \varepsilon_x(\text{id} \otimes f(w_S)) = f(\varepsilon_x \otimes \text{id}(w_S)) \), for all \( f \in S^* \), \((\text{Equation 13})\) which implies that \( \varepsilon_x \otimes \text{id}(w_S) = x \), or equivalently,
\[
\text{id} \otimes \varepsilon_x(v_S) = x. \tag{17}
\]

One way to show that the double dual is isomorphic to \( S \), would be to show that \( S \) is a crossed product for the Hopf system \((\mathbb{C}, \hat{S}, \text{id})\). We would need to check the conditions of the definition of a crossed product. For a homomorphism of \( \mathbb{C} \) into
$M(S)$ we just map $z$ to $z1$. For the unitary in $M(S \otimes \hat{S})$ we just use $v_S$. This pair is certainly covariant, so condition (a) is satisfied.

Let $W$ be a corepresentation of $\hat{S}$ on $H$. We would need to show that there exists a representation $\nu$ of $S$ such that $\nu \otimes \text{id}(v_S) = W$. Well, define $\nu_w : S \to B(H)$ by $\nu_w(x) := \text{id} \otimes \varepsilon_x(W)$, where $\varepsilon_x$ is evaluation at $x$. Then, if $S$ is nondegenerate, then $\text{id} \otimes \varepsilon_x(\nu_w \otimes \text{id}(v_S)) = \nu_w(\text{id} \otimes \varepsilon_x(v_S)) = \nu_w(x) = \text{id} \otimes \varepsilon_x(W)$, for all $x \in S$ (Equation [4]). Since $A(S)$ is dense in $\hat{S}$, the point evaluation functionals $\varepsilon_x$ are sufficient to separate points of $\hat{S}$. Thus functionals of the form $\text{id} \otimes \varepsilon_x$ separate points of $M(K(H) \otimes S)$, $\nu_w \otimes \text{id}(v_S) = W$ and condition (b) is satisfied.

For condition (c) we need to show that $S$ is equal to $\prod\{\text{id} \otimes \varepsilon_S(v_S) : g \in A(S)\}$. This is the problem. We have nothing that relates the covariant pairs of $(S,S,\delta_S)$ to the covariant pairs of $(\hat{S},\hat{S},\delta_{\hat{S}})$, so we have no way of relating the elements of $A(S)$ to elements of $\hat{S}$.

To be able to show that the double dual $\hat{S}$ is isomorphic to $S$, we need a prescribed correspondence between the covariant pairs of $(S,S,\delta_S)$ and the covariant pairs of $(\hat{S},\hat{S},\delta_{\hat{S}})$. For this we will define a Kac system, which will be modelled on the relationship between the representations, $\lambda$ and $M$, associated to a locally compact group $G$, and the unitary operator $S \in B(L^2(G))$ defined by $S(\xi)(s) = \xi(s^{-1})$. There are some similarities between our approach and that of Baaj and Skandalis [3, §6]. This issue will be investigated in [18].

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