POINTWISE ESTIMATE FOR ELLIPTIC EQUATIONS IN PERIODIC PERFORATED DOMAINS

LI-MING YEH
Department of Applied Mathematics, National Chiao Tung University
Hsinchu, 30050, Taiwan, R.O.C.

(Communicated by Xuefeng Wang)

Abstract. Pointwise estimate for the solutions of elliptic equations in periodic perforated domains is concerned. Let $\epsilon$ denote the size ratio of the period of a periodic perforated domain to the whole domain. It is known that even if the given functions of the elliptic equations are bounded uniformly in $\epsilon$, the $C^{1,\alpha}$ norm and the $W^{2,p}$ norm of the elliptic solutions may not be bounded uniformly in $\epsilon$. It is also known that when $\epsilon$ closes to 0, the elliptic solutions in the periodic perforated domains approach a solution of some homogenized elliptic equation. In this work, the H"older uniform bound in $\epsilon$ and the Lipschitz uniform bound in $\epsilon$ for the elliptic solutions in perforated domains are proved. The $L^\infty$ and the Lipschitz convergence estimates for the difference between the elliptic solutions in the perforated domains and the solution of the homogenized elliptic equation are derived.

1. Introduction. Pointwise estimate for the solutions of elliptic equations in perforated domains with periodic microstructure is presented. This problem arises from contaminant transport in the subsurface, heat transfer in two-phase media, the stress in composite materials, and so on (see [3, 6, 15, 16]). Let $\Omega \subset \mathbb{R}^n$ ($n = 2$ or 3) be a bounded smooth simply-connected domain with boundary $\partial \Omega$; $Y \equiv (0,1)^n$ consists of a smooth sub-domain $Y_m$ completely surrounded by another connected sub-domain $Y_f (\equiv Y \setminus Y_m)$; $\epsilon \in (0,1)$; $\Omega(\epsilon) \equiv \{ x \in \Omega | \text{dist}(x, \partial \Omega) > \epsilon \}$; $\Omega_m \equiv \{ x | x \in \epsilon(Y_m + j) \subset \Omega(\epsilon) \text{ for some } j \in \mathbb{Z}^n \}$ with boundary $\partial \Omega_m$; $\Omega_f \equiv \Omega \setminus \Omega_m$ is a connected region. The problem that we consider is

$$
\begin{cases}
-\nabla \cdot (K_\epsilon \nabla U_\epsilon + T_\epsilon U_\epsilon + Q_\epsilon) + P_\epsilon \nabla U_\epsilon + E_\epsilon U_\epsilon = F_\epsilon & \text{in } \Omega_f^*, \\
(K_\epsilon \nabla U_\epsilon + T_\epsilon U_\epsilon + Q_\epsilon) \cdot \mathbf{n}_\epsilon = 0 & \text{on } \partial \Omega_m^*, \\
U_\epsilon = 0 & \text{on } \partial \Omega.
\end{cases}
$$

In (1), $K_\epsilon(x) = K(\frac{x}{\epsilon})$ is a symmetric positive definite matrix; $T_\epsilon(x) = T(\frac{x}{\epsilon})$ and $P_\epsilon(x) = P(\frac{x}{\epsilon})$ are vector functions; $E_\epsilon(x) = E(\frac{x}{\epsilon})$ is a scalar non-negative function; $K, T, P, E$ are smooth periodic functions in $\mathbb{R}^n$ with period $Y$; $Q_\epsilon, F_\epsilon$ are given functions; $\mathbf{n}_\epsilon$ is a unit normal vector on $\partial \Omega_m^*$.

Problem (1) consists of an elliptic equation with oscillatory periodic coefficients. One example shows that even the given functions $Q_\epsilon, F_\epsilon$ of the elliptic equation are

2000 Mathematics Subject Classification. Primary: 35J05, 35J15, 35J25.

Key words and phrases. Periodic perforated domains, homogenized elliptic equation, two-phase media.

This research is supported by the grant number NSC 101-2115-M-009-012 from the research program of National Science Council of Taiwan.
bounded uniformly in $\epsilon$, the $C^{1,\alpha}$ norm and the $W^{2,p}$ norm of the elliptic solution of (1) may not be bounded uniformly in $\epsilon$ (see Remark 1). It is also known that, to obtain an accurate approximation of the solution of (1) by classical numerical methods (see \cite{7, 10}), the computational mesh size in each space direction should be less than $\epsilon$ \cite{14}. So a direct numerical simulation of the solution of (1) requires large computational memory and computing time when $\epsilon$ is small.

If $T_\epsilon, P_\epsilon$ are small and $Q_\epsilon, F_\epsilon$ are bounded, (1) is uniquely solvable in $H^1$ and

$$\|U_\epsilon\|_{H^1(\Omega_\epsilon)} \leq c(\|Q_\epsilon\|_{L^2(\Omega_\epsilon)} + \|F_\epsilon\|_{L^2(\Omega_\epsilon^*)}),$$

where $c$ is a constant independent of $\epsilon$ \cite{13}. In addition that $Q_\epsilon$ is bounded in $H^1(\Omega_\epsilon^*)$, by tracing the proof of Theorem 2.7 \cite{3}, there exists a subsequence of $\{U_\epsilon, Q_\epsilon, F_\epsilon\}$ (same notation for subsequence) satisfying

$$\begin{cases}
  K_\epsilon \nabla U_\epsilon \chi_{\Omega_\epsilon^*} + T_\epsilon U_\epsilon \chi_{\Omega_\epsilon^*} + Q_\epsilon \chi_{\Omega_\epsilon^*} \\
  \to \hat{K} \nabla U + \hat{T}_1 U + |Y_f| \hat{Q}
\end{cases} \text{ in } L^2(\Omega) \text{ weakly as } \epsilon \to 0. \quad (3)$$

In (3), $\chi_{\Omega_\epsilon}$ denotes the characteristic function on $\Omega_\epsilon^*$; $\hat{K}$ is a symmetric positive definite matrix depending on $K$ and $Y_f$; $\hat{T}_1$ and $\hat{P}$ are constant vectors; $\hat{E}_{1,1}$ is a constant function; $|Y_f|$ is the volume of $Y_f$; $U \in H^1(\Omega)$; $\hat{Q}, \hat{F}, \hat{Q}_*, \hat{F}_* \in L^2(\Omega)$. $\hat{Q}$ depends on $K, Q_\epsilon$; $\hat{F}$ depends on $K, P_\epsilon, Q_\epsilon, F_\epsilon$. Explicit form of $\hat{K}, \hat{T}_1, \hat{P}, \hat{E}_{1,1}, \hat{Q}, \hat{F}$ can be found in (8) below. The function $U$ in (3) satisfies

$$\begin{cases}
  -\nabla \cdot (\hat{K} \nabla U) + (\hat{P} - \hat{T}_1) \nabla U + \hat{E}_{1,1} U = |Y_f| (\nabla \cdot \hat{Q} + \hat{F}) \quad \text{in } \Omega, \\
  U = 0 \quad \text{on } \partial \Omega. \quad (4)
\end{cases}$$

Similar results as (3)–(4) can also be found in \cite{5, 11, 15, 19}. So it seems that the solution of (4) is a good approximation of the solution of (1), especially when $\epsilon$ is small. Indeed in those works \cite{4, 14, 18}, the solution of (1) was regarded as the perturbation of the homogenized solution of (4). Furthermore, the homogenized solution of (4) was used to construct the solution of (1). It is also known that the solution of (1) converges to the solution of (4) with convergence rate $\epsilon$ in $L^2$ norm and with convergence rate $\sqrt{\epsilon}$ in $H^1$ norm as $\epsilon$ closes to 0 (see \cite{5, 11, 15, 19} and references therein). In \cite{8, 17}, higher order asymptotic expansion for the solutions of elliptic equations in perforated domains was given. Higher order convergence estimate for the solution of (1) for $T_\epsilon = P_\epsilon = E_\epsilon = 0$ case was also derived in Hilbert spaces (see \cite{5, 9, 19}).

In this work, we show that although the $C^{1,\alpha}$ norm and the $W^{2,p}$ norm of the elliptic solution of (1) may not be bounded uniformly in $\epsilon$, the Hölder uniform bound in $\epsilon$ and the Lipschitz uniform bound in $\epsilon$ for the elliptic solution can be obtained. We also present pointwise convergence estimate for the solution of (1). Precisely, the $L^\infty$ convergence estimate with convergence rate $\epsilon$ for the solution of (1) is derived. For problem (1) with periodic boundary conditions, Lipschitz convergence estimate with higher order convergence rate $\epsilon^k$ for $k \geq 1$ is also proved.

The rest of the work is organized as follows: Notation and main results are stated in section 2. In section 3, we derive a priori $W^{2,p}$ estimates for the solutions
of elliptic equations around the Neumann boundary. In section 4, a uniform Hölder estimate in $\epsilon$ for the elliptic solution of (1) is proved. In section 5, $L^\infty$ convergence estimate for the solution of (1) is given. Uniform Lipschitz bound in $\epsilon$ and Lipschitz convergence estimate for the solution of (1) with periodic boundary conditions are obtained in section 6.

2. Notation and main results. Denote by $C^{k,\alpha}$ the Hölder space with norm $\| \cdot \|_{C^{k,\alpha}}$ by $[\cdot]_{C^{k,\alpha}}$ the Hölder semi-norm, and by $W^{s,p}$ the Sobolev space with norm $\| \cdot \|_{W^{s,p}}$ for $k \geq 0, \alpha \in [0,1], s \in \mathbb{Z}$, and $p \in [1,\infty]$ (see [2]). $L^p \equiv W^{0,p}$ and $H^s \equiv W^{s,2}$. For any Banach space $B$, we define $\|g_1, \cdots, g_k\|_B = \|g_1\|_B + \cdots + \|g_k\|_B$. $B_r(x)$ is a ball centered at $x$ with radius $r$. For any domain $D$ and $r > 0$, $\partial D$ is the closure of $D$; $\partial D$ is the boundary of $D$; $D/r \equiv \{x | x \in D\}$; $|D|$ is the volume of $D$; $\tilde{X}_D$ is the characteristic function on $D$; $C^0_0(D)$ contains infinitely differentiable functions with compact support in $D$.

$$\int_D g(y)dy = \frac{1}{|D|} \int_D g(y)dy.$$  

For any $g \in L^1(\Omega)$,

$$(g)_{x,r} \equiv \int_{B_r(x) \cap \Omega} g(y)dy.$$  

Define $\tilde{A}_m \equiv \bigcup_{j \in \mathbb{Z}_+}(Y_m + j)$, $\tilde{A}_f \equiv \mathbb{R}^n \setminus \tilde{A}_m$, $\tilde{A}_m^c \equiv \epsilon \tilde{A}_m$, and $\tilde{A}_f^c \equiv \epsilon \tilde{A}_f$. $\mathbb{D} = (L^1, L)^n$ for $L \in \mathbb{N}$, $\mathbb{D}_r \equiv \tilde{A}_m^c \cap \mathbb{D}$, and $\mathbb{D}_f \equiv \tilde{A}_f^c \cap \mathbb{D}$. Let $C^0_{per}(\mathbb{R}^n)$ denote the space of infinitely differentiable $Y$-periodic functions in $\mathbb{R}^n$. If $D \in \mathbb{R}^n$ or $\tilde{A}_f$, $W^{s,p}_r(D)$ (resp. $C^{k,\alpha}(D)$) is the closure of $C^0_{per}(\mathbb{R}^n)$ under the $W^{s,p}_r(D)$ (resp. $C^{k,\alpha}$) norm and $\|g\|_c^{s,p}_{r}(D) = \|g\|_{W^{s,p}(D;Y)}$ (resp. $\|g\|_{c^{k,\alpha}(D)} = \|g\|_{C^{k,\alpha}(D;Y)}$) for $s \geq 1$, $p \in [1,\infty]$, $k \geq 0$, and $\alpha \in [0,1]$. $L^p_{per} \equiv W^{0,p}_{per}$ and $H^1_{per}(\tilde{A}_f) \equiv \{g \in W^{1,p}_{loc}(\tilde{A}_f) \big| \|g\|_{W^{1,p}(\tilde{A}_f)} = 0\}$. If $g(x) = G(x/\epsilon)$ and $G \in W^{1,p}(\tilde{A}_f)$, we define $\|g\|_{W^{1,p}_{loc}(\tilde{A}_f)} \equiv \sup_{y \in \mathbb{Z}_+} \|G\|_{W^{1,p}(Y_{\tilde{A}_f} \cap \mathbb{D}; \epsilon)}$.

For any $\lambda \in [0,1]$ and $i \in \{1, \cdots, n\}$, we find scalar functions $S(\lambda), Y(i) \in \tilde{H}^{1}_{per}(\tilde{A}_f)$ satisfying

$$\begin{cases}
\nabla \cdot (K \nabla S(\lambda) + \lambda T) = 0 & \text{in } Y_f,
(K \nabla S(\lambda) + \lambda T) \cdot \vec{n} = 0 & \text{on } \partial Y_m,
\end{cases} \quad (5)$$

$$\begin{cases}
\nabla \cdot (K \nabla Y(i) + \vec{e}_i) = 0 & \text{in } Y_f,
(K \nabla Y(i) + \vec{e}_i) \cdot \vec{n} = 0 & \text{on } \partial Y_m,
\end{cases} \quad (6)$$

where $\vec{n}$ denotes a unit normal vector on $\partial Y_m$ and $\vec{e}_i$ is the unit vector in the $i$-th coordinate direction.

For each $i \in \{1, \cdots, n\}$, we find scalar functions $X^{(0)}_1, X^{(0),i}_1 \in \tilde{H}^{1}_{per}(\tilde{A}_f)$ satisfying

$$X^{(0)}_1(y) \equiv S(1)(y),$$

$$\begin{cases}
\nabla \cdot (K(\nabla X^{(0),i}_1) + \vec{e}_i) = 0 & \text{in } Y_f,
K(\nabla X^{(0),i}_1 + \vec{e}_i) \cdot \vec{n} = 0 & \text{on } \partial Y_m.
\end{cases} \quad (7)$$
Denote by $\Xi_1$ (resp. $\Xi_2$) a $n \times n$ matrix function whose $(i,j)$-component is $\partial_i Y^{(j)}$ (resp. $\partial_i X_1^{(0,j)}$) for $i, j \in \{1, \cdots, n\}$ and define, for $\lambda, \omega \in [0, 1]$,

$$
\begin{align*}
\tilde{K} &= \int_{Y_f} K(I + \Xi_2)(y)dy, \\
\tilde{P} &= \int_{Y_f} P(I + \Xi_2)(y)dy, \\
\tilde{T}_\lambda &= \int_{Y_f} (K\nabla S^{(\lambda)} + \lambda T)(y)dy, \\
\tilde{E}_{\lambda, \omega} &= \int_{Y_f} (\lambda P\nabla S^{(\lambda)} + \omega E)(y)dy, \\
\tilde{Q} &= \frac{1}{|Y_f|} \int_{Y_f} (I + K\Xi_1)(y)dy \tilde{Q}_*, \\
\tilde{F} &= \tilde{F}_* - \frac{1}{|Y_f|} \int_{Y_f} P\Xi_1(y)dy \tilde{Q}_*.
\end{align*}
$$

In (8), $I$ is the identity matrix; $\tilde{P}, \tilde{T}_\lambda$ are constant vectors; $|Y_f|, \tilde{Q}_*$ and $\tilde{F}_*$ are from (3). Following the remark in page 90 [15], $\tilde{K}$ is a constant symmetric positive definite matrix.

For $i_1, i_2 \in \{1, \cdots, n\}$, we find scalar functions $\tilde{X}_2^{(0)}, \tilde{X}_2^{(0,i_1)}, \tilde{X}_2^{(0,i_1,i_2)} \in \tilde{H}_{per}(\tilde{A}_f)$ satisfying

$$
\begin{align*}
\nabla \cdot (K\nabla \tilde{X}_2^{(0)} + \tilde{X}_2^{(0)}) &= P\nabla \tilde{X}_1^{(0)} + E + \tilde{N}_2^{(0)} \quad \text{in } Y_f, \\
(K\nabla \tilde{X}_2^{(0)} + \tilde{X}_1^{(0)}) \cdot \tilde{n} &= 0 \quad \text{on } \partial Y_m, \\
\nabla \cdot (K(\nabla \tilde{X}_2^{(0,i_1)} + \tilde{X}_1^{(0,i_1)})) &= -K\partial_{i_1} \tilde{X}_1^{(0)} - T^{(i_1)} + P^{(i_1)} + P\nabla \tilde{X}_1^{(0,i_1)} + \tilde{N}_2^{(0,i_1)} \quad \text{in } Y_f, \\
(K(\nabla \tilde{X}_2^{(0,i_1)} + \tilde{X}_1^{(0,i_1)})) \cdot \tilde{n} &= 0 \quad \text{on } \partial Y_m, \\
\nabla \cdot (K(\nabla \tilde{X}_2^{(0,i_1,i_2)} + \tilde{X}_1^{(0,i_1,i_2)})) &= -K(\partial_{i_1} \tilde{X}_1^{(0,i_1)} + \delta_{i_1,i_2}) + \tilde{N}_2^{(0,i_1,i_2)} \quad \text{in } Y_f, \\
(K(\nabla \tilde{X}_2^{(0,i_1,i_2)} + \tilde{X}_1^{(0,i_1,i_2)})) \cdot \tilde{n} &= 0 \quad \text{on } \partial Y_m,
\end{align*}
$$

where $\tilde{N}_2^{(0)} = \frac{-E_{i_1}}{|Y_f|}$, $\tilde{N}_2^{(0,i_1)} = \frac{\tilde{T}^{(i_1)} - \tilde{P}^{(i_1)}}{|Y_f|}$, $\tilde{N}_2^{(0,i_1,i_2)} = \frac{K^{(i_1,i_2)}}{|Y_f|}$, $\tilde{K}^{(i_1,i_2)}$ is the $(i_1,i_2)$-component of $\tilde{K}$, $\delta_{i_1,i_2} = \begin{cases} 
1 & \text{if } i_1 = i_2 \\
0 & \text{if } i_1 \neq i_2
\end{cases}$, and $P^{(i_1)}$ (resp. $T^{(i_1)}$, $\tilde{T}^{(i_1)}$, $\tilde{P}^{(i_1)}$) is the $i_1$-th component of the vector $P$ (resp. $T$, $\tilde{T}_1$, $\tilde{P}$). See (8) for $\tilde{K}, \tilde{T}_1, \tilde{P}, \tilde{E}_{1,1}$.
For $k \geq 3$, we find scalar functions $X_k^{(i_0,i_1,\cdots,i_s)} \in \tilde{H}^1_{per}(\tilde{A}_f)$ for $s \in \{0,1,\cdots,k\}$, $i_0 = 0$, and $i_1,\cdots,i_s \in \{1,\cdots,n\}$ in such a way that

(1) $X_k^{(0)}$ satisfies

$$\begin{cases}
\nabla \cdot (K \nabla X_k^{(0)}) = EX_k^{(0)} + N_k^{(0)} & \text{in } Y_f, \\
K \nabla X_k^{(0)} \cdot \nu = 0 & \text{on } \partial Y_m,
\end{cases}$$

(12)

(2) $X_k^{(i_0,i_1,\cdots,i_s)}$ satisfies, for $s \geq 1$,

$$\begin{cases}
\nabla \cdot (K(\nabla X_k^{(i_0,i_1,\cdots,i_s)} + X_k^{(i_0,i_1,\cdots,i_{s-1})} \epsilon_{i_s})) \\
\quad = -K(\partial_{x_{i_s}} X_k^{(i_0,i_1,\cdots,i_{s-1})} + \delta_{i_{s-1},i_s} X_{k-2}^{(i_0,i_1,\cdots,i_{s-2})}) \\
\quad \quad \quad + (1 - \delta_{s,k-1})(1 - \delta_{s,k})EX_{k-2}^{(i_0,i_1,\cdots,i_s)} + N_k^{(i_0,i_1,\cdots,i_s)} & \text{in } Y_f, \\
K(\nabla X_k^{(i_0,i_1,\cdots,i_s)} + X_k^{(i_0,i_1,\cdots,i_{s-1})} \epsilon_{i_s}) \cdot \nu = 0 & \text{on } \partial Y_m,
\end{cases}$$

(13)

where $N_k^{(i_0,i_1,\cdots,i_s)}$ is a constant defined as

$$\begin{aligned}
\hat{N}_k^{(i_0,i_1,\cdots,i_s)} &= \begin{cases}
\frac{1}{|Y_f|} \int_{Y_f} EX_k^{(0)} dy & \text{if } s = 0, \\
\frac{1}{|Y_f|} \int_{Y_f} K(\partial_{x_{i_s}} X_k^{(i_0,i_1,\cdots,i_{s-1})} + \delta_{i_{s-1},i_s} X_{k-2}^{(i_0,i_1,\cdots,i_{s-2})}) dy & \text{if } s \geq 1.
\end{cases}
\end{aligned}$$

In (13), we adopt the convention: If $s = 1$, then $\delta_{s-1,i_s} X_k^{(i_0,i_1,\cdots,i_{s-2})} = 0$; If $s \notin \{k - 1, k\}$, $\delta_{s-1,i_s} X_k^{(i_0,i_1,\cdots,i_{s-2})} = 0$. By Lax-Milgram Theorem [13], we know that if $\lambda \in [0,1]$, $K \geq \tau > 0$, $K,T,P,E \in L^\infty_{per}(\mathbb{R}^n)$, and $Y_m$ is a $C^{1,1}$ domain, then $S^{(k)}$ in (5)–(6) and $X_k^{(i_0,i_1,\cdots,i_s)}$ in (7) and (9)–(13) are solvable uniquely in $\tilde{H}^1_{per}(\tilde{A}_f)$. For any $\nu > 0, k \geq 1$, $s \in \{0,1,\cdots,k\}$, $i_0 = 0$, and $i_1,\cdots,i_s \in \{1,\cdots,n\}$, we define

$$\begin{cases}
\Xi \equiv (\Xi_1^{(0,1)},\cdots,\Xi_1^{(0,n)}), \\
X_k^{(i_0,i_1,\cdots,i_s)}(x) \equiv \chi_k^{(i_0,i_1,\cdots,i_s)}(x), \\
X_k^{(0,1)} \equiv \chi_k^{(0,1)}, \\
X_k^{(0,n)} \equiv \chi_k^{(0,n)}.
\end{cases}$$

(14)

We recall an extension result in [1].

**Lemma 2.1.** For any $\epsilon \in (0,1)$ and $p \in [1,\infty)$, there is a constant $c(Y_f,p)$ and a linear continuous extension operator $\Pi : W^{1,p}(\Omega_f) \to W^{1,p}(\Omega)$ such that

(1) If $g \in W^{1,p}(\Omega_f)$, then

$$\Pi g = g \text{ in } \Omega_f \text{ almost everywhere},$$

$$\|\Pi g\|_{L^p(\Omega)} \leq c(Y_f,p)\|g\|_{L^p(\Omega_f)},$$

$$\|\Pi g\|_{L^p(\Omega)} \leq c(Y_f,p)\|\nabla g\|_{L^p(\Omega_f)},$$

$$\|g\|_{C^{0,\alpha}(\Omega_f)} \leq c(Y_f,p)\|g\|_{C^{0,\alpha}(\Omega_f)},$$

if $g \in C^{0,\alpha}(\overline{\Omega_f}), \alpha \in (0,1)$,

$$\Pi g = G \text{ in } \Omega \text{ if } g = G|_{\Omega_f} \text{ for some linear function } G \text{ in } \Omega.$$  

(2) If $G(x) \equiv g(rx)$ in $B_1(z) \cap \Omega_f/r$ for any $z \in \mathbb{R}^n$ and any constant $r \geq \epsilon$, there is a $\theta \in (0,1)$ so that $\Pi_{f/r}G(x) = (\Pi g)(rx)$ in $B_\theta(z) \cap \Omega/r$.

Lemma 2.1 also holds if $\{\Omega_f, \Omega\}$ is replaced by $\{\tilde{A}_f, \mathbb{R}^n\}$. First, we state a H"older uniform estimate for problem (1).
Theorem 2.2. Suppose
A1. $\epsilon \in (0, 1)$, $\Omega$ and $Y_m$ are bounded $C^{1,1}$ simply-connected domains in $\mathbb{R}^n$ for $n \in \{2, 3\}$,
A2. $K, T \in C^{0,1}_\text{per}(\mathbb{R}^n)$ and $K$ is a symmetric positive definite matrix,
A3. $|T, \nabla T, P, E|_{L^\infty(\mathbb{R}^n)} \leq M; E - \frac{3}{4}(\sqrt{\nabla T} + |\sqrt{\nabla T} P|) > 0$ in $\mathbb{R}^n; P, E \in L^\infty(\mathbb{R}^n)$,
A4. $\delta \in (0, 3)$ and $Q, F \in L^{n+\delta}(\Omega'_f)$,
the solution of (1) exists uniquely in $H^1(\Omega'_f)$ and there is a $\mu_\ast \in (0, \frac{1}{n+\delta})$ (depending on $\delta, Y_f, K, M, \Omega$) such that
$$\|U_\epsilon\|_{C^{0,\mu_\ast}(\Omega'_f)} \leq c\|Q, F\|_{L^{n+\delta}(\Omega'_f)},$$
where $c$ is a constant independent of $\epsilon$. $\sqrt{K^{-1}}$ is the square root of the inverse matrix of $K$.

Proof of Theorem 2.2 is given in section 4. Next we give a $L^\infty$ convergence estimate. We find a scalar function $U_\epsilon \in H^1(\Omega'_f)$ satisfying
$$\begin{cases}
-\nabla \cdot (K, \nabla U_\epsilon + T, U_\epsilon) + P, \nabla U_\epsilon + E, U_\epsilon = F_\epsilon & \text{in } \Omega'_f, \\
(K, \nabla U_\epsilon + T, U_\epsilon) \cdot \tilde{n}_\epsilon = 0 & \text{on } \partial\Omega'_m, \\
U_\epsilon = 0 & \text{on } \partial\Omega.
\end{cases}$$
Under A1–A3 and $F_\epsilon \in L^2(\Omega'_f)$, (15) is solvable uniquely. By Lemma 2.1 and tracing the argument of Theorem 2.7 [3], there is a subsequence of $\{U_\epsilon, F_\epsilon\}$ (same notation for subsequence) satisfying
$$\begin{cases}
K, \nabla U_\epsilon, \tilde{X}_{\Omega'_f} + T, U_\epsilon, \tilde{X}_{\Omega'_f} \to \tilde{K}\nabla U + \tilde{T}_1 U & \\
P, \nabla U_\epsilon, \tilde{X}_{\Omega'_f} + E, U_\epsilon, \tilde{X}_{\Omega'_f} - F_\epsilon \tilde{X}_{\Omega'_f} \to \tilde{P}\nabla U + \tilde{E}_{1,1} U - |Y_f|\tilde{F}_\epsilon
\end{cases} \in L^2(\Omega) \text{ weakly as } \epsilon \to 0, \quad (16)$$
where $U \in H^1(\Omega), \tilde{F}_\epsilon \in L^2(\Omega)$. See (8) for $\tilde{K}, \tilde{T}_1, \tilde{P}, \tilde{E}_{1,1}$. The U in (16) satisfies
$$\begin{cases}
-\nabla \cdot (\tilde{K}\nabla U) + (\tilde{P} - \tilde{T}_1)\nabla U + \tilde{E}_{1,1} U = |Y_f|\tilde{F}_\epsilon & \text{in } \Omega, \\
U = 0 & \text{on } \partial\Omega.
\end{cases}$$

Theorem 2.3. Under A1–A3 and
A5. $\Omega$ is a $C^{2,1}$ domain, $\delta \in (0, 3)$, $T \cdot \tilde{n} = 0$ on $\partial Y_m, F_\epsilon$ is bounded in $W^{1,n+\delta}(\Omega)$, the solution $U_\epsilon$ of (15) and the solution $U$ of (17) satisfy
$$\|U_\epsilon - U\|_{L^\infty(\Omega'_f)} \leq c(\epsilon + \|F_\epsilon - \tilde{F}_\epsilon\|_{L^{n+\delta}(\Omega'_f)}),$$
where $c$ is a constant independent of $\epsilon$.

Proof of Theorem 2.3 is given in section 5. From Theorem 2.3, we note that if $F_\epsilon$ is close to $\tilde{F}_\epsilon$ in $\Omega'_f$ then the right hand side of (18) is small. In this case, the $U_\epsilon$ of (15) converges uniformly to the $U$ of (17) in $\Omega'_f$.

Next we state Lipschitz uniform bound results. Consider the following problem: Find a scalar function $U_\epsilon \in H^1(D'_f)$ satisfying
$$\begin{cases}
-\nabla \cdot (K, \nabla U_\epsilon + Q, U_\epsilon) + E, U_\epsilon = F_\epsilon & \text{in } D'_f, \\
(K, \nabla U_\epsilon + Q, U_\epsilon) \cdot \tilde{n}_\epsilon = 0 & \text{on } \partial D'_m,
\end{cases}$$
$U_\epsilon$ satisfies periodic boundary conditions on $\partial D$. 

Theorem 2.4. Suppose
A6. \( \epsilon \in (0, 1) \), \( Y_m \) is a \( C^{1,1} \) simply-connected domain in \( \mathbb{R}^n \) for \( n \in \{2, 3\} \),
A7. \( K \in C^0_{\text{per}}(\mathbb{R}^n) \) is a symmetric positive definite matrix,
A8. \( E \in C^0_{\text{per}}(\mathbb{R}^n) \) and \( E(x) \in (0, M) \) for all \( x \in \mathbb{R}^n \),
A9. \( \delta \in (0, 3) \), \( F_\epsilon \in L^{n+\delta}(\varOmega^r_f) \), and \( Q_\epsilon \in W^{1, n+\delta}(\varOmega^r_f) \) is a periodic function in \( \tilde{A}^r_f \) with period \( \varOmega \),
then there is a \( M_0 < 1 \) (depending on \( \delta, Y_f, K \)) such that if \( M \leq M_0 \), the solution of (19) exists uniquely in \( H^1(\varOmega^r_f) \) and satisfies
\[
\| \nabla U_\epsilon \|_{L^\infty(\varOmega^r_f)} \leq c \left( \| \epsilon^{\frac{\delta}{n+\delta}} Q_\epsilon F_\epsilon \|_{L^{n+\delta}(\varOmega^r_f)} + \epsilon^{-\frac{\mu}{n+\delta}} \| Q_\epsilon \|_{W^{1, n+\delta}_{\text{loc}}(\varOmega^r_f)} \right),
\]
where \( \mu \equiv \frac{\delta}{n+\delta} \) and \( c \) is a constant independent of \( \epsilon \).

Consider the following problem: Find a scalar function \( U_\epsilon \in H^1(\varOmega^r_f) \) such that
\[
\begin{align*}
-\nabla \cdot (K, \nabla U_\epsilon + Q_\epsilon) &= F_\epsilon & \text{in } \varOmega^r_f, \\
(K, \nabla U_\epsilon + Q_\epsilon) \cdot \vec{n}_\epsilon &= 0 & \text{on } \partial \varOmega^r_m, \\
U_\epsilon &\text{ satisfies periodic boundary conditions on } \partial \varOmega,
\end{align*}
\tag{20}
\]
See Lemma 2.1 for \( \Pi_\epsilon \). Then we have

Theorem 2.5. Besides A6–A7 and A9, if \( \int_{\varOmega^r_f} F_\epsilon(x) dx = 0 \), the solution of (20) exists uniquely in \( H^1(\varOmega^r_f) \) and satisfies
\[
\| \nabla U_\epsilon \|_{L^\infty(\varOmega^r_f)} \leq c \left( \| \epsilon^{\frac{\delta}{n+\delta}} Q_\epsilon F_\epsilon \|_{L^{n+\delta}(\varOmega^r_f)} + \epsilon^{-\frac{\mu}{n+\delta}} \| Q_\epsilon \|_{W^{1, n+\delta}_{\text{loc}}(\varOmega^r_f)} \right),
\]
where \( \mu \equiv \frac{\delta}{n+\delta} \) and \( c \) is a constant independent of \( \epsilon \).

Proofs of Theorem 2.4 and Theorem 2.5 are in subsection 6.1. Now we state Lipschitz convergence estimates. We find a scalar function \( U_\epsilon \in H^1(\varOmega^r_f) \) satisfying
\[
\begin{align*}
-\nabla \cdot (K, \nabla U_\epsilon) + E_\epsilon U_\epsilon &= F_\epsilon & \text{in } \varOmega^r_f, \\
K \nabla U_\epsilon \cdot \vec{n}_\epsilon &= 0 & \text{on } \partial \varOmega^r_m, \\
U_\epsilon &\text{ satisfies periodic boundary conditions on } \partial \varOmega.
\end{align*}
\tag{21}
\]
Under A6–A8 and \( F_\epsilon \in L^2(\varOmega^r_f) \), (21) is solvable uniquely. By Lemma 2.1 and tracing the argument of Theorem 2.7 [3], there is a subsequence of \( \{U_\epsilon, F_\epsilon\} \) (same notation for subsequence) satisfying
\[
\begin{align*}
K \nabla U_\epsilon \tilde{X}_{\varOmega^r_f} &\rightarrow \tilde{K} \nabla U & \text{in } L^2(\varOmega) \text{ weakly as } \epsilon \rightarrow 0, \\
E_\epsilon U_\epsilon \tilde{X}_{\varOmega^r_f} &\rightarrow \tilde{E}_{0,1} U & \text{in } L^2(\varOmega) \text{ weakly as } \epsilon \rightarrow 0, \\
F_\epsilon \tilde{X}_{\varOmega^r_f} &\rightarrow |Y_f| \tilde{F} & \text{in } L^2(\varOmega),
\end{align*}
\tag{22}
\]
where \( U \in H^1(\varOmega) \), \( \tilde{F} \in L^2(\varOmega) \). See (8) for \( \tilde{K}, \tilde{E}_{0,1} \). The \( U \) in (22) satisfies
\[
\begin{align*}
-\nabla \cdot (\tilde{K} \nabla U) + \tilde{E}_{0,1} U &= |Y_f| \tilde{F} & \text{in } \varOmega, \\
U &\text{ satisfies periodic boundary conditions on } \partial \varOmega.
\end{align*}
\tag{23}
\]
Suppose \( T = P = 0 \) in (5), (7), and (9)–(11), then \( \chi(\ell^{i_0,i_1,\cdots,i_s})_\ell, \tilde{N}^{(i_0,i_1,\cdots,i_s)}_\ell \) in (7)–(13) satisfy
where M is a constant independent of ɛ. Let us also consider the following problem: Find a scalar function $U_\epsilon$ (i.e., the solution of (23)) and $\tilde{\varphi}_j$ for $j \geq 1$ be the solution of

$$\begin{cases}
\sum_{s=0}^{k} \tilde{N}^s_i \partial_i \tilde{\varphi}_j = - \sum_{q=0}^{k-1} \sum_{s=0}^{k-1-j+2-\ell} \tilde{N}_j \partial_q \tilde{\varphi}_q & \text{in } D',
\tilde{\varphi}_j & \text{satisfies periodic boundary condition on } \partial D'.
\end{cases}$$

(24)

Here $\hat{\ell} \equiv (i_0, i_1, \ldots, i_s)$; $[\hat{\ell}] \equiv s; i_0 = 0; i_1, \ldots, i_s \in \{1, \ldots, n\}; \tilde{N}^s_i \equiv \tilde{N}^s_{i_0, i_1, \ldots, i_s}$ for $0 \leq s \leq \ell; \partial_i \varphi \equiv \partial_{i_1} \cdots \partial_{i_s} \varphi$. By A6–A8 and (8), equation (24) is an elliptic equation with constant coefficients and its solution $\tilde{\varphi}_j$ exists uniquely in $W^{m-\ell+2,p}(D)$ if $F_\epsilon \in W^{m,p}(D)$ for $m \geq j \geq 0, p \in (1, \infty)$. Let us define

$$\begin{cases}
\tilde{\Phi}_k \equiv \sum_{j=0}^{k-1} \epsilon^j \tilde{\varphi}_j & \text{in } D' \text{ for } k \geq 1,
\tilde{\mathcal{V}}_k \equiv \tilde{\Phi}_k + \sum_{\ell=1}^{k} \sum_{s=0}^{\ell} \chi^s_{\ell,s} \partial_\ell \tilde{\Phi}_k
\end{cases}$$

(25)

where $\chi^s_{\ell,s} \equiv \chi^{(i_0,i_1,\ldots,i_s)}_{\ell,s}$. See (7) and (9)–(14) for $\chi^{(i_0,i_1,\ldots,i_s)}_{\ell,s}$.

**Theorem 2.6.** Under A6–A8 and

**A10.** $\delta \in (0, 3), k \geq 1, F_\epsilon$ is bounded in $W^{2k,n+\delta}(D)$ and is a periodic function in $\mathbb{R}^n$ with period $D$,

there is a $M_0 < 1$ (depending on $\delta, Y_f, K$) such that if $M \leq M_0$, the solution $U_\epsilon$ of (21) and the $\tilde{\mathcal{V}}_k$ in (25) satisfy

$$\|\nabla (U_\epsilon - \tilde{\mathcal{V}}_k)\|_{L^{\infty}(D')} \leq c(\epsilon^k + \|F_\epsilon\|_{L^{n+\delta}(D')}),$$

where $c$ is a constant independent of $\epsilon$. See (23) for $\tilde{F}$.

Let us also consider the following problem: Find a scalar function $U_\epsilon \in H^1(D')$ satisfying

$$\begin{cases}
-\nabla \cdot (K_\epsilon \nabla U_\epsilon) = F_\epsilon & \text{in } D',
K_\epsilon \nabla U_\epsilon \cdot \tilde{n}_\epsilon = 0 & \text{on } \partial D_{\tau},
U_\epsilon \text{ satisfies periodic boundary conditions on } \partial D,
\int_D \Pi U_\epsilon \, dx = 0.
\end{cases}$$

(26)

Under A6–A7 and $F_\epsilon \in L^2(D')$ with $\int_{D'} F_\epsilon(x) \, dx = 0$, (26) is solvable uniquely. By Lemma 2.1 and tracing the argument of Theorem 2.7 [3], there is a subsequence of
\{U_\epsilon, F_\epsilon\} \text{ (same notation for subsequence) satisfying}
\begin{align*}
\begin{cases}
\Pi_{\epsilon} U_\epsilon \to U & \text{in } L^2(\mathbb{D}) \text{ strongly} \\
K_\epsilon \nabla U_\epsilon \tilde{X}_{D_\epsilon} \to \tilde{K} \nabla U & \text{in } L^2(\mathbb{D}) \text{ weakly as } \epsilon \to 0, \\
F_\epsilon \tilde{X}_{D_\epsilon} \to |Y_j| \tilde{F} & \text{in } L^2(\mathbb{D}) \text{ weakly}
\end{cases}
\end{align*}

where \(U \in H^1(\mathbb{D}), \tilde{F} \in L^2(\mathbb{D})\). See (8) for \(\bar{K}\). The \(U\) in (27) satisfies
\begin{align*}
\begin{cases}
-\nabla \cdot (\tilde{K} \nabla U) = |Y_j| \tilde{F} & \text{in } \mathbb{D}, \\
U \text{ satisfies periodic boundary conditions on } \partial \mathbb{D}, \\
\int_{\mathbb{D}} U dx = 0 = \int_{\mathbb{D}} \tilde{F} dx.
\end{cases}
\end{align*}

Let \(\tilde{\varphi}_0 = U\) (i.e., the solution of (28)) and let \(\tilde{\varphi}_j\) for \(j \geq 1\) be the solution of
\begin{align*}
\begin{cases}
\sum_{|i| = 2} \tilde{N}_{ij} \partial_1 \tilde{\varphi}_j = - \sum_{q=0}^{j-1} \sum_{|i| = j+2-q} \tilde{N}_{ij} \partial_1 \tilde{\varphi}_q & \text{in } \mathbb{D}, \\
\tilde{\varphi}_j \text{ satisfies periodic boundary condition on } \partial \mathbb{D}, \\
\int_{\mathbb{D}} \tilde{\varphi}_j(y) dy = 0.
\end{cases}
\end{align*}

By A6–A7 and (8), equation (29) is an elliptic equation with constant coefficients and its solution \(\tilde{\varphi}_j\) exists uniquely in \(W^{m-j+2, p}(\mathbb{D})\) if \(F_\epsilon \in W^{m, p}(\mathbb{D})\) for \(m \geq j \geq 0, p \in (1, \infty)\), and \(\int_{D_\epsilon} F_\epsilon(x) dx = 0\). Let us define
\begin{align*}
\begin{cases}
\tilde{\Phi}_k = \sum_{j=0}^{k-1} \epsilon^j \tilde{\varphi}_j & \text{in } D_\epsilon^c, \\
\tilde{V}_k = \tilde{\Phi}_k + \sum_{\ell=1}^{k+1} \sum_{|i| = \ell} \tilde{N}_{ij} \partial_1 \tilde{\varphi}_k
\end{cases}
\end{align*}

\textbf{Theorem 2.7.} Besides A6–A7 and A10, if \(\int_{D_\epsilon} F_\epsilon dx = 0\), the solution \(U_\epsilon\) of (26) and the \(\tilde{V}_k\) in (30) satisfy
\[
\|\nabla (U_\epsilon - \tilde{V}_k)\|_{L^\infty(D_\epsilon^c)} \leq c (\epsilon^k + \|F_\epsilon - \tilde{F}\|_{L^{\infty}(D_\epsilon^c)}),
\]
where \(c\) is a constant independent of \(\epsilon\). See (28) for \(\tilde{F}\).

Proofs of Theorem 2.6 and Theorem 2.7 are given in subsection 6.2.

3. \textbf{A priori } \(W^{2,p}\) \textbf{estimates.} This section is to derive a priori \(W^{2,p}\) estimates for the solutions of elliptic equations around the Neumann boundary. Let \(\mathbb{R}^n_+ = \{(x_1, \cdots, x_n) | x_n > 0\}, \partial \mathbb{R}^n_+ = \{(x_1, \cdots, x_n) | x_n = 0\}\), and \(\bar{n}\) is a unit vector normal to \(\partial \mathbb{R}^n_+\). Consider the following problem
\begin{align*}
\begin{cases}
\nabla \cdot (K \nabla \Psi + Q) + P \nabla \Psi + E \Psi = G & \text{in } B_1(0) \cap \mathbb{R}^n_+, \\
(K \nabla \Psi + Q) \cdot \bar{n} = 0 & \text{on } B_1(0) \cap \partial \mathbb{R}^n_+.
\end{cases}
\end{align*}

\textbf{POINTWISE ESTIMATE 1969}
Lemma 3.1. If $K \in W^{1,\infty}(B_1(0) \cap \mathbb{R}^n_+)$, $K > \tau > 0$ is a symmetric positive definite matrix, $Q \in W^{1,p}(B_1(0) \cap \mathbb{R}^n_+)$, $\mathbb{P}, \mathbb{E} \in L^\infty(B_1(0) \cap \mathbb{R}^n_+)$, and $G \in L^p(B_1(0) \cap \mathbb{R}^n_+)$ for some $p \in (1, \infty)$, then any solution of (31) satisfies

$$
\|\Psi\|_{W^2,p(B_{1/2}(0) \cap \mathbb{R}^n_+)} \leq c(\|\Psi\|_{L^p(B_1(0) \cap \mathbb{R}^n_+)} + \|Q\|_{W^{1,p}(B_1(0) \cap \mathbb{R}^n_+)})
$$

where $c$ is a constant depending on $p, \tau, \|K, \nabla K, \mathbb{P}, \mathbb{E}\|_{L^\infty(B_1(0) \cap \mathbb{R}^n_+)}$.

Proof. By Theorem 7.53 [2], there exists a $\Phi \in W^2,p(\mathbb{R}^n_+)$ satisfying

$$
\begin{cases}
(K \nabla \Phi + Q) \cdot \vec{n} = 0 & \text{on } B_{3/4}(0) \cap \partial \mathbb{R}^n_+,

\|\Phi\|_{W^2,p(B_{3/4}(0) \cap \mathbb{R}^n_+)} \leq c\|Q\|_{W^{1,p}(B_1(0) \cap \mathbb{R}^n_+)} \leq c\|Q\|_{W^{1,p}(B_1(0) \cap \mathbb{R}^n_+)},
\end{cases}
\tag{32}
$$

where $c$ is a constant. If we define $\psi \equiv \Psi - \Phi$, then

$$
\nabla \cdot (K \nabla \psi) + \mathbb{P} \nabla \psi + \mathbb{E} \psi = G - \nabla \cdot (K \nabla \Phi + Q) - \mathbb{P} \nabla \Phi - \mathbb{E} \Phi \equiv \tilde{G} \quad \text{in } B_{3/4}(0) \cap \mathbb{R}^n_+,
$$

$$
K \nabla \psi \cdot \vec{n} = 0 \quad \text{on } B_{3/4}(0) \cap \partial \mathbb{R}^n_+.
$$

We do odd extension for $\mathbb{P}$ and even extension for $K, \psi, \mathbb{E}, \tilde{G}$. Same notation is used for the extended function. Then $\psi$ in the domain $B_{3/4}(0)$ satisfies

$$
\nabla \cdot (K \nabla \psi) + \mathbb{P} \nabla \psi + \mathbb{E} \psi = \tilde{G} \quad \text{in } B_{3/4}(0).
$$

By Theorem 9.11 [13], we obtain $\|\psi\|_{W^2,p(B_{1/2}(0))} \leq c\|\psi\|_{L^p(B_{3/4}(0))}$, where $c$ is a constant depending on $p, \tau, \|K, \nabla K, \mathbb{P}, \mathbb{E}\|_{L^\infty(B_1(0) \cap \mathbb{R}^n_+)}$. Together with (32), we prove the lemma. \hfill \Box

Assume $x_0 \in \partial Y_m$ and $B_{1/4}(x_0) \subset Y$.

Lemma 3.2. Consider the following problem

$$
\begin{cases}
\sum_{i=1}^n \partial_i(\sum_{j=1}^n K^{i,j} \partial_j \Psi + Q^{(i)}) + \mathbb{E} \Psi = G & \text{in } Y_f,

\sum_{i=1}^n (\sum_{j=1}^n K^{i,j} \partial_j \Psi + Q^{(i)}) \vec{n}^{(i)} = 0 & \text{on } \partial Y_m,
\end{cases}
\tag{33}
$$

where $\vec{n} = (\vec{n}^{(1)}, \vec{n}^{(2)}, \ldots, \vec{n}^{(n)})$ is a unit vector normal to $\partial Y_m$. If $Y_m$ is a $C^{1,1}$ domain, $K^{i,j} \in W^{1,\infty}(Y_f)$, $K \equiv (K^{i,j}) > \tau > 0$ is a symmetric positive definite matrix, $Q \equiv (Q^{(1)}, \ldots, Q^{(n)}) \in W^{1,p}(Y_f)$, $\mathbb{E} \in L^\infty(Y_f)$, and $G \in L^p(Y_f)$, then any solution of (33) satisfies

$$
\|\Psi\|_{W^2,p(B_{1/8}(x_0) \cap \mathbb{R}^n_+)} \leq \min(\|\Psi, G\|_{L^p(Y_f)} + \|Q\|_{W^{1,p}(Y_f)}) \quad \text{for } p \in (1, \infty),
$$

where $c$ is a constant depending on $p, \tau, Y_m, \|K, \nabla K, \mathbb{E}\|_{L^\infty(Y_f)}$.

Proof. By assumption, at each point $x_0 \in \partial Y_m$ there exist a neighborhood $B_{1/8}(x_0)$ of $x_0$ and a $C^{1,1}$ diffeomorphism $\hat{L}$ that straightens the boundary $\partial Y_m$ in $B_{1/8}(x_0)$. Set $D = B_{1/8}(x_0) \cap Y_f$, $\hat{D} = \hat{L}(D)$, $S = B_{1/8}(x_0) \cap \partial Y_m \subset \partial D$, and $\hat{S} = \hat{L}(S) \subset \partial \hat{D}$. Under the mapping $y = \hat{L}(x) = (\hat{L}^{(1)}(x), \ldots, \hat{L}^{(n)}(x))$, if we define $\hat{\Psi}(y) \equiv \Psi(x)$,
then problem (33) is transformed to
\[
\begin{align*}
\left\{ \begin{array}{l}
\sum_{j=1}^{n} \partial_j \left( \sum_{i=1}^{n} K(i,j) \partial_i \tilde{\psi} + \tilde{Q}(i) \right) + \sum_{i=1}^{n} \tilde{P}(i) \partial_i \tilde{\psi} + \tilde{E} \tilde{\psi} = \tilde{G} \quad \text{in } \tilde{D}, \\
\sum_{j=1}^{n} \left( \sum_{i=1}^{n} \tilde{K}(i,j) \partial_j \tilde{\psi} + \tilde{Q}(i) \right) n(y_j) = 0 \quad \text{on } \tilde{S},
\end{array} \right.
\end{align*}
\]
where \( \mathbf{n} = (n(y_1), n(y_2), \ldots, n(y_n)) \) is a unit vector normal to \( \tilde{S} \) and
\[
\begin{align*}
\tilde{K}(i,j)(x) &= \sum_{k,m=1}^{n} K^{(k,m)} \partial_k \tilde{L}^{(i)} \partial_m \tilde{L}^{(j)}(x), \\
\tilde{Q}(i)(x) &= \sum_{k=1}^{n} \tilde{Q}^{(k)} \partial_k \tilde{L}^{(i)}(x), \\
\tilde{P}(i)(x) &= - \sum_{j,k,m=1}^{n} \tilde{K}^{(m,j)} \partial_k \tilde{L}^{(k)} \partial_j \tilde{L}^{(i)}(x), \\
\tilde{E}(x) &= \tilde{E}(x), \\
\tilde{G}(x) &= \tilde{G}(x) + \sum_{k,m=1}^{n} \tilde{Q}(m) \partial_k \tilde{L}^{(k)}(x).
\end{align*}
\]
Clearly, \( \tilde{K}(i,j), \tilde{Q}(i), \tilde{P}(i), \tilde{E}, \tilde{G} \) satisfy the requirements of Lemma 3.1. This lemma follows from Lemma 3.1. \( \square \)

If \( K, T \in C_{per}^{0,1}([\mathbb{R}^n]) \), \( K \) is a symmetric positive definite matrix, \( P, E \in L^\infty_{per}([\mathbb{R}^n]) \), and \( Y_m \) is a \( C^{1,1} \) domain, the \( \mathcal{S}(\lambda), \mathcal{Y}(i) \) in (5)–(6) and the \( \mathcal{X}^{k(i_1,i_2,\ldots,i_s)}_k \) in (7) and (9)–(13) satisfy, by Theorem 9.11 [13] and Lemma 3.2,
\[
\| \mathcal{S}(\lambda), \mathcal{Y}(i), \mathcal{X}^{k(i_1,i_2,\ldots,i_s)}_k \|_{W^{2,p}(Y_f)} \leq c(\| K, T \|_{C^{0,1}(Y_f)} + \| P, E \|_{L^\infty(Y_f)}).
\]
In (34), \( p \in (1, \infty) \); \( \lambda \in [0, 1] \); \( i_0 = 0 \); \( i, i_1, \ldots, i_s \in \{ 1, \ldots, n \} \); \( s \in \{ 0, 1, \ldots, k \} \); \( k \geq 1 \); \( c \) is a constant independent of \( \lambda, k \). (34) implies the functions in (8) satisfy
\[
|K|, |\tilde{P}|, |\tilde{T}_f|, |\tilde{E}_{\lambda,\omega}| \leq c(\| K, T \|_{C^{0,1}(Y_f)} + \| P, E \|_{L^\infty(Y_f)}),
\]
where \( c \) is a constant independent of \( \lambda, \omega \in [0, 1] \).

**Remark 1.** The \( C^{1,\alpha} \) norm and the \( W^{2,p} \) norm of the solution of (1) in general are not bounded uniformly in \( \epsilon \) even if \( \| Q_\epsilon \|_{W^{1,p}(\Omega_f)}, \| F_\epsilon \|_{L^p(\Omega_f)} \) are bounded uniformly in \( \epsilon \). For example, suppose \( B_1(0) \subset \Omega \) and \( \eta \) is a bell-shaped smooth function satisfying \( \eta \in C_0^\infty(B_1(0)) \), \( \eta \in [0, 1] \), and \( \eta(x) = 1 \) in \( B_{1/2}(0) \). By (7) for \( i = 1 \) and (14) for \( X_{1,\epsilon}^{(0,1)} \) and \( \epsilon \in (0, 1) \), we know
\[
\begin{align*}
-\nabla \cdot (K_\epsilon (\nabla (\eta X_{1,\epsilon}^{(0,1)} + \nabla \eta + \eta \tilde{e}_1)) = -K_\epsilon (\nabla X_{1,\epsilon}^{(0,1)} + \tilde{e}_1) \nabla \eta \quad \text{in } \Omega_f, \\
K_\epsilon (\nabla (\eta X_{1,\epsilon}^{(0,1)} + \nabla \eta + \eta \tilde{e}_1)) = 0 \quad \text{on } \partial \Omega_f, \\
|\eta X_{1,\epsilon}^{(0,1)}| = 0 \quad \text{on } \partial \Omega.
\end{align*}
\]
If \( K \in C_{per}^{0,1}(\mathbb{R}^n) \) and \( Y_m \) is a \( C^{1,1} \) domain, by (34), we see that
\[
\| \nabla \eta \|_{W^{1,\infty}(\Omega_f)} + \| K_\epsilon (\nabla X_{1,\epsilon}^{(0,1)} + \tilde{e}_1) \nabla \eta \|_{L^\infty(\Omega_f)}
\]

is bounded uniformly in $\epsilon$. But $\|\eta_{x_1}^{(0,1)}\|_{C^{1,\alpha}(\mathbb{D}_1^\nu)}$ for $\alpha \in (0,1)$ and $\|\eta_{x_1}^{(0,1)}\|_{W^{2,p}(\mathbb{D}_1^\nu)}$ for $p \in (1,\infty)$ are not bounded uniformly in $\epsilon$.

4. Uniform Hölder estimate. We shall prove a uniform Hölder estimate in $\epsilon$ for problem (1) (that is, Theorem 2.2). The Hölder estimate in the interior region of $\Omega$ is considered in subsection 4.1 and the estimate around the boundary $\partial \Omega$ is in subsection 4.2. A1–A3 are assumed in this section.

4.1. Interior Hölder estimate. For convenience, $\overline{B_1(0)} \subset \Omega$.

Lemma 4.1. For any $\delta > 0$, there are $\theta_1, \theta_2 \in (0,1)$ (depending on $\delta, Y_f, K, M$) satisfying $\theta_1 < \theta_2^2$ and there is a $\epsilon_0 \in (0,1)$ (depending on $\theta_2, \delta, Y_f, K, M$) so that if

$$
\begin{align*}
-\nabla \cdot (K \nabla \psi') + \lambda \psi' - \psi' + \theta_1 & \in B_1(0) \cap \Omega_f, \\
+ \lambda \psi' - \psi' + \theta_1 & = 0 \quad \text{in } B_1(0) \cap \Omega_f^c, \\
(\mu \nabla \psi') + \lambda \psi' + \theta_1 & = 0 \quad \text{on } \partial \Omega_f,
\end{align*}
$$

(36)

and if

$$
\begin{align*}
\nu & \in (0, \epsilon_0), \quad \theta \in [\theta_1, \theta_2], \quad \lambda, \omega, \theta \in [0,1], \\
|\|\psi'\|_{L^2(B_1(0) \cap \Omega_f^c)} \leq \epsilon_0, |Q_{\psi'}|_{L^2(B_1(0) \cap \Omega_f^c)} \leq 1,
\end{align*}

(37)

then

$$
\int_{B_2(0)} |\Pi_{\nu} \psi' - (\Pi_{\nu} \psi')_{0,\theta}|^2 dx \leq \theta^{2\mu},
$$

(38)

where $\mu \equiv \frac{\delta}{n+3}$. See section 2 for the definitions of $\Pi_{\nu}, (\Pi_{\nu} \psi')_{0,\theta}$.

Proof. Consider

$$
-\nabla \cdot (K \nabla \psi) + (\lambda \hat{\mathcal{P}} - \tilde{T}_\lambda) \nabla \psi + \tilde{E}_\lambda \omega \psi = 0 \quad \text{in } B_{3/4}(0),
$$

(39)

where $\tilde{K}, \tilde{T}_\lambda, \tilde{P}, \tilde{E}_\lambda \omega$ are defined in (8) and $\lambda, \omega \in [0,1]$. Note that $\tilde{K}$ is a symmetric positive definite matrix. By energy method, (35), and Theorem 9.11 [13],

$$
\|\psi\|_{C^{1,\alpha}(\overline{B_{1/2}(0)})} \leq c \|\psi\|_{L^2(B_{3/4}(0))},
$$

where $c$ is a constant depending on $Y_f, K, M$. If $\mu' \in (\mu,1)$, then, by Theorem 1.2 in page 70 [12],

$$
\int_{B_2(0)} |\psi - (\psi)_{0,\theta}|^2 dx \leq \theta^{2\mu'} \int_{B_{3/4}(0)} \psi^2 dx
$$

(40)

for $\theta$ sufficiently small (depending on $\delta, Y_f, K, M$). Fix $\theta_1, \theta_2 \in (0,1/2)$ such that (1) $\theta_1 < \theta_2^2$ and (2) equation (40) holds for any $\theta \in [\theta_1, \theta_2]$.

Now we claim (38). If not, there is a sequence $\{\theta_\nu, \lambda_\nu, \omega_\nu, \psi_\nu, Q_{\psi'}, G_{\psi'}\}$ satisfying (36) and

$$
\begin{align*}
\theta_\nu & \in [\theta_1, \theta_2], \quad \lambda_\nu, \omega_\nu \in [0,1], \\
|\|\psi_\nu\|_{L^2(B_1(0) \cap \Omega_f^c)} \leq 1, \\
\lim_{\nu \to 0} |Q_{\psi'}, G_{\psi'}|_{L^2(B_1(0) \cap \Omega_f^c)} = 0, \\
\int_{B_2(0)} |\Pi_{\nu} \psi_\nu - (\Pi_{\nu} \psi_\nu)_{0,\theta}|^2 dx & > \theta^{2\mu}_\nu.
\end{align*}
$$

(41)
By tracing the proof of Theorem 2.7 [3] and employing Lemma 2.1, we can extract a subsequence (same notation for subsequence) such that, as \( \nu \to 0 \),

\[
\begin{align*}
\lambda_\nu, \omega_\nu &\to \lambda, \omega \in [0, 1], \\
\Pi_\nu \psi_\nu &\to \psi \\
(K_\nu \nabla \psi_\nu + \lambda_\nu T_\nu \psi_\nu) \tilde{X}_{\Omega_\nu^1} &\to \tilde{K} \nabla \psi + \tilde{T}_\lambda \psi \\
(\lambda_\nu P_\nu \nabla \psi_\nu + \omega_\nu E_\nu \psi_\nu) \tilde{X}_{\Omega_\nu^1} &\to \lambda \tilde{P} \nabla \psi + \tilde{E}_{\lambda, \omega} \psi
\end{align*}
\]

in \( L^2(B_{3/4}(0)) \) strongly, \((42)\) weakly,

where \( \tilde{K}, \tilde{T}_\lambda, \tilde{P}, \tilde{E}_{\lambda, \omega} \) are defined in \((8)\). The \( \psi \) in \((42)\) satisfies \((39)\). By \((40)-(42)\),

\[
\theta^{2\mu} = \lim_{\nu \to 0} \theta^{2\mu}_\nu \leq \lim_{\nu \to 0} \int_{B_{\nu}(0)} |\Pi_\nu \psi_\nu - (\Pi_\nu \psi_\nu)_{0, \theta}|^2 dx \leq \theta^{2\mu'} \int_{B_{1/4}(0)} \psi^2 dx.
\]

If \( \theta_2 \) is small enough, the right hand side of \((43)\) is less than \( \theta^{2\mu''} \) for \( \mu'' \in (\mu, \mu') \). So we have \( \theta^{2\mu} \leq \theta^{2\mu''} \) for \( \mu'' \in (\mu, \mu') \), which is impossible. So we prove \((38)\) \( \square \)

**Lemma 4.2.** For any \( \delta \in (0, 3) \), there are \( \theta_1, \theta_2 \in (0, 1) \) (depending on \( \delta, Y_\nu, K, M \)) satisfying \( \theta_1 < \theta_2 \) and there is a \( \epsilon_0 \in (0, 1) \) (depending on \( \theta_2, \delta, Y_\nu, K, M \)) so that if

\[
\begin{align*}
-K_\nu \nabla \Psi_\varepsilon + \beta_\varepsilon T_\varepsilon \Psi_\varepsilon + Q_\varepsilon \\
+ \beta_\varepsilon P_\varepsilon \nabla \Psi_\varepsilon + \ell_\varepsilon E_\varepsilon \Psi_\varepsilon = G_\varepsilon \\
\text{in } B_1(0) \cap \Omega_\varepsilon
\end{align*}
\]

and if \( \epsilon \in (0, \epsilon_0) \), \( \theta \in [\theta_1, \theta_2] \), and \( \beta_\varepsilon, \ell_\varepsilon \in [0, 1] \), then, for any \( k \) satisfying \( \epsilon/\theta^k \leq \epsilon_0 \),

\[
\int_{B_{\varepsilon k}(0)} |\Pi_\varepsilon \Psi_\varepsilon - (\Pi_\varepsilon \Psi_\varepsilon)_{0, \theta}|^2 dx \leq \theta^{2\mu} J_\varepsilon^2,
\]

where \( \mu \equiv \frac{\delta}{n+\delta} \) and \( J_\varepsilon \equiv \frac{\delta}{\nu} \left( \|\Psi_\varepsilon\|_{L^2(B_1(0) \cap \Omega_\varepsilon)} + \|Q_\varepsilon, G_\varepsilon\|_{L^2(B_1(0) \cap \Omega_\varepsilon)} \right) \).

**Proof.** Let \( \epsilon \) denote a constant independent of \( \epsilon, \beta_\varepsilon, \ell_\varepsilon \). This is done by induction on \( \varepsilon \). For \( \varepsilon = 1 \), we define \( \psi_\varepsilon \equiv \frac{\Psi_\varepsilon}{\varepsilon}, \quad Q_\varepsilon \equiv \frac{Q_\varepsilon}{\varepsilon}, \quad G_\varepsilon \equiv \frac{G_\varepsilon}{\varepsilon}, \quad \ell_\varepsilon \equiv \beta_\varepsilon, \omega_\varepsilon \equiv \ell_\varepsilon, \quad \lambda_\varepsilon \equiv \beta_\varepsilon, \omega_\varepsilon \equiv \ell_\varepsilon \). Then these functions satisfy \((36)\) and \((37)\) with \( \nu = \epsilon \). By Lemma 4.1,

\[
\int_{B_\varepsilon(0)} |\Pi_\varepsilon \psi_\varepsilon - (\Pi_\varepsilon \psi_\varepsilon)_{0, \theta}|^2 dx \leq \theta^{2\mu}.
\]

This implies \((45)\) for \( \varepsilon = 1 \). By A2–A3, energy method, Theorem 7.26 [13], and Lemma 2.1, any solution of \((44)\) satisfies

\[
\|\Pi_\varepsilon \Psi_\varepsilon\|_{L^\infty(B_{\varepsilon k}(0))} + \|\Psi_\varepsilon\|_{H^1(B_{\varepsilon k}(0) \cap \Omega_\varepsilon)} \leq c \|\Psi_\varepsilon\|_{L^2(B_1(0) \cap \Omega_\varepsilon)}.
\]

If \((45)\) holds for some \( k \) satisfying \( \epsilon/\theta^k \leq \epsilon_0 \), we define

\[
\begin{align*}
\psi_\varepsilon/\theta^k(x) &\equiv J_\varepsilon^{-1} \theta^{-k\mu} \left( \Psi_\varepsilon(\theta^k x) - (\Pi_\varepsilon \Psi_\varepsilon)_{0, \theta} \right) \\
Q_\varepsilon/\theta^k(x) &\equiv J_\varepsilon^{-1} \theta^{k(1-\mu)} \left( Q_\varepsilon(\theta^k x) + \beta_\varepsilon T_\varepsilon(\theta^k x)(\Pi_\varepsilon \Psi_\varepsilon)_{0, \theta} \right) \\
G_\varepsilon/\theta^k(x) &\equiv J_\varepsilon^{-1} \theta^{k(2-\mu)} \left( G_\varepsilon(\theta^k x) - \ell_\varepsilon E_\varepsilon(\theta^k x)(\Pi_\varepsilon \Psi_\varepsilon)_{0, \theta} \right) \\
\lambda_\varepsilon/\theta^k &\equiv \theta^k \beta_\varepsilon \\
\omega_\varepsilon/\theta^k &\equiv \theta^{2k} \ell_\varepsilon
\end{align*}
\]
See section 2 for the definition of $\Omega_f^k/\theta^k$. Then these functions satisfy
\[
\begin{cases}
-\nabla \cdot (K_{c/\theta^k} \nabla \psi_{c/\theta^k} + \lambda_{c/\theta^k} T_{c/\theta^k} \psi_{c/\theta^k} + Q_{c/\theta^k}) \\
\quad + \lambda_{c/\theta^k} P_{c/\theta^k} \nabla \psi_{c/\theta^k} + \omega_{c/\theta^k} E_{c/\theta^k} \psi_{c/\theta^k} = G_{c/\theta^k} & \text{in } B(0) \cap \Omega_f^k/\theta^k, \\
(K_{c/\theta^k} \nabla \psi_{c/\theta^k} + \lambda_{c/\theta^k} T_{c/\theta^k} \psi_{c/\theta^k} + Q_{c/\theta^k}) \cdot \mathbf{n}_{c/\theta^k} = 0 & \text{on } B(0) \cap \partial \Omega_m^k/\theta^k,
\end{cases}
\]
where $\mathbf{n}_{c/\theta^k}$ is a unit vector normal to $\partial \Omega_m^k/\theta^k$. By A3 and A4,
\[
\|Q_{c/\theta^k}\|_{L^{n+\delta}(B(0) \cap \Omega_f^k/\theta^k)} \leq J^{-1}\|Q_{c}\|_{L^{n+\delta}(B(0) \cap \Omega_f^k)} + J^{-1} \theta^{k(1-\mu)} M \|\Pi_c \psi_c\|_{0, \theta^k} \leq J^{-1}\|Q_{c}\|_{L^{n+\delta}(B(0) \cap \Omega_f^k)} + c J^{-1} \theta^{k(2-\mu-n/2)} M \|\Pi_c \psi_c\|_{L^{2\mu}(B(0))}.
\]
By induction, (46), $\delta \in (0, 3)$, and small $\theta_2$, we see
\[
\begin{aligned}
&\|Q_{c/\theta^k}\|_{L^{2n}(B(0) \cap \Omega_f^k/\theta^k)} \leq J^{-1}\|Q_{c}\|_{L^{2n}(B(0) \cap \Omega_f^k)} + c J^{-1} \theta^{k(2-\mu-n/2)} M \|\Pi_c \psi_c\|_{L^{2\mu}(B(0))} \\
&\leq J^{-1}\|Q_{c}\|_{L^{2n}(B(0) \cap \Omega_f^k)} + c J^{-1} \theta^{k(2-\mu-n/2)} M \|\Pi_c \psi_c\|_{L^{2\mu}(B(0))} \leq 1.
\end{aligned}
\]
By Lemma 4.1 (take $\nu = c/\theta^k$), we obtain
\[
\int_{B(0)} \|\Pi_{c/\theta^k} \psi_{c/\theta^k} - (\Pi_{c/\theta^k} \psi_{c/\theta^k})_{0, \theta^k}\|^2 dx \leq \theta^{2\mu}.
\]
Note, by Lemma 2.1,
\[
\int_{B(0)} \|\Pi_{c/\theta^k} \psi_{c/\theta^k} - (\Pi_{c/\theta^k} \psi_{c/\theta^k})_{0, \theta^k}\|^2 dx = \int_{B(0)} \|\Pi_{c/\theta^k} \psi_{c/\theta^k} - (\Pi_{c/\theta^k} \psi_{c/\theta^k})_{0, \theta^k}\|^2 dx.
\]
Equations (47)–(48) imply (45) for $k + 1$ case.

**Lemma 4.3.** For any $\delta \in (0, 3)$, there is a $\mu_* \in (0, \frac{\delta}{\mu-n+\delta})$ (depending on $\delta, \gamma_f, K, M$) such that if $\epsilon \in (0, 1)$ and $\beta, \ell \in (0, 1)$, then any solution of (44) satisfies
\[
\|\Psi_c\|_{C^0, \mu_* (B_{r/2}(0) \cap \Omega_f^k)} \leq c(\|\Psi_c\|_{L^2(B(0) \cap \Omega_f^k)} + \|Q_c, G_c\|_{L^{n+\delta}(B(0) \cap \Omega_f^k)}),
\]
where $c$ is a constant independent of $\epsilon, \beta, \ell, \epsilon$. 

**Proof.** Let $\theta_1, \theta_2, \epsilon_0, \mu, J_\epsilon$ be same as those in Lemma 4.2, define $\epsilon_* \equiv \epsilon_0 \theta_2/2$, and let $\epsilon \leq \epsilon_*$. Denote by $c(\cdot)$ a constant independent of $\epsilon, \beta, \ell, \epsilon$. Because of $\theta_1 < \theta_2^2$, for any $r \in [\epsilon/\epsilon_0, \theta_2]$, there are $\theta \in [\theta_1, \theta_2]$ and $k \in \mathbb{N}$ satisfying $r = \theta^k$. Lemma 4.2 implies, for any $r \in [\epsilon/\epsilon_0, \theta_2]$,
\[
\int_{B(0)} \|\Pi_c \psi_c - (\Pi_c \psi_c)_{0, \theta^k}\|^2 dx \leq r^{2\mu} J_\epsilon^2.
\]
Since $2\epsilon/\epsilon_0 \in [\epsilon/\epsilon_0, \theta_2]$, we define, for any $\tilde{\mu} \in (0, \mu),$
\[
\begin{aligned}
\psi_1(x) &\equiv J_\epsilon^{-1} e^{-\tilde{\mu}} (\Psi_c(\epsilon x) - (\Pi_c \psi_c)_{0, \epsilon}) \\
Q_1(x) &\equiv J_\epsilon^{-1} e^{-\tilde{\mu}} (Q_c(\epsilon x) + \beta \mathbf{T}_c(\epsilon x)(\Pi_c \psi_c)) \\
G_1(x) &\equiv J_\epsilon^{-1} e^{-\tilde{\mu}} (G_c(\epsilon x) - \ell \mathbf{E}_c(\epsilon x)(\Pi_c \psi_c)) \\
\lambda_1 &\equiv \epsilon \beta \\
\omega_1 &\equiv \epsilon^2 \ell 
\end{aligned}
\]
Then those functions satisfy
\[
\begin{cases}
-\nabla \cdot (K \nabla \psi_1 + \lambda_1 T \psi_1 + Q_1) + \lambda_1 P \nabla \psi_1 + \omega_1 E \psi_1 = G_1 & \text{in } B_{r/2}(0) \cap \Omega_f^k/\epsilon, \\
(K \nabla \psi_1 + \lambda_1 T \psi_1 + Q_1) \cdot \mathbf{n}_{c/\epsilon} = 0 & \text{on } B_{r/2}(0) \cap \partial \Omega_m^k/\epsilon,
\end{cases}
\]
Lemma 4.4. For any $\delta > 0$, there are $\bar{\theta}_1, \bar{\theta}_2 \in (0, 1)$ (depending on $\delta, Y_f, K, M, \Omega$) satisfying $\bar{\theta}_1 < \bar{\theta}_2^2$ and there is an $\epsilon_0 \in (0, 1)$ (depending on $\bar{\theta}_2, \delta, Y_f, K, M, \Omega$) satisfying $\epsilon_0 < \epsilon_0$ (where $\epsilon_0$ is that in Lemma 4.1) such that if

$$\begin{aligned}
-\nabla \cdot (K_{\epsilon, \nu} \nabla \psi_{\epsilon, \nu} + \lambda_{\epsilon, \nu} \mathbf{T}_{\epsilon, \nu} \psi_{\epsilon, \nu} + Q_{\epsilon, \nu}) + \lambda_{\epsilon, \nu} \mathbf{P}_{\epsilon, \nu} \nabla \psi_{\epsilon, \nu} + \omega_{\epsilon, \nu} \mathbf{E}_{\epsilon, \nu} \psi_{\epsilon, \nu} = \mathbb{G}_{\epsilon, \nu} 
& \text{in } B_1(0) \cap \Omega_f'/\nu, \\
\psi_{\epsilon, \nu} &= 0 
& \text{on } B_1(0) \cap \partial \Omega_m'/\nu,
\end{aligned}$$

(53)

and if

$$\begin{aligned}
\epsilon, \epsilon/\nu & \in (0, \epsilon_0), \quad \nu \in (0, 1), \quad \bar{\theta} \in [\bar{\theta}_1, \bar{\theta}_2], \\
\lambda_{\epsilon, \nu}, \omega_{\epsilon, \nu} & \in [0, 1], \\
\|\psi_{\epsilon, \nu}\|_{L^2(B_1(0) \cap \Omega_f'/\nu)}, & \epsilon_0^{-1} \|Q_{\epsilon, \nu}, \mathbb{G}_{\epsilon, \nu}\|_{L^\infty(B_1(0) \cap \Omega_f'/\nu)} \leq 1,
\end{aligned}$$

then

$$\int_{B_\nu(0) \cap \Omega_f/\nu} |\Pi_{\nu, \epsilon} \psi_{\epsilon, \nu}|^2 d\nu \leq \tilde{\theta}^{2\nu},$$

(54)

where $\mu = \frac{\delta}{n+\gamma}$.

Proof. Consider the following problem

$$\begin{aligned}
-\nabla \cdot (\tilde{K} \nabla \psi) + (\lambda \tilde{P} - \tilde{T}_\lambda) \nabla \psi + \tilde{E}_{\lambda, \omega} \psi &= 0 
& \text{in } B_{3/4}(0) \cap \Omega/t, \\
\psi &= 0 
& \text{on } B_{3/4}(0) \cap \partial \Omega/t,
\end{aligned}$$

(55)
where $\tilde{K}, \tilde{T}_\lambda, \tilde{P}, \tilde{E}_{\lambda, \omega}$ are defined in (8), $t \in [0, 1]$, and $\lambda, \omega \in [0, 1]$. By (35) and Theorem 9.13 [13],
\[
\|\psi\|_{C^{0,0}(B_{3/4}(0) \cap \Omega/t)} \leq c \|\psi\|_{L^2(B_{3/4}(0) \cap \Omega/t)},
\]
(56)
where $c$ is a constant depending on $Y_f, K, M, \Omega$ but independent of $t$. If $\mu' \in (\mu, 1)$, then, by (56),
\[
\int_{B_{3/4}(0) \cap \Omega/t} \psi^2 dx \leq \tilde{\theta}^{2\mu'} \int_{B_{3/4}(0) \cap \Omega/t} \psi^2 dx,
\]
(57)
for sufficiently small $\tilde{\theta}$ (depending on $\delta, Y_f, K, M, \Omega$ but independent of $t$). Fix $\tilde{\theta}_1, \tilde{\theta}_2 \in (0, 1/2)$ such that (1) $\tilde{\theta}_1 < \tilde{\theta}_2$ and (2) equation (57) holds for any $\tilde{\theta} \in [\tilde{\theta}_1, \tilde{\theta}_2]$.

We claim (54). If not, there is a sequence $\{\nu, \tilde{\theta}, \nu, \tilde{\theta}, \lambda, \omega, \epsilon, \epsilon, \omega, \psi, \epsilon, \nu, \tilde{Q}, \epsilon, \nu, \tilde{G}, \epsilon, \nu, \}$ satisfying (53) and
\[
\begin{aligned}
&\epsilon, \epsilon/\nu \to 0, \quad \nu \in (0, 1], \quad \tilde{\theta}, \nu, \tilde{\theta}, \nu, \in [\tilde{\theta}_1, \tilde{\theta}_2], \quad \lambda, \omega \in [0, 1], \\
&\lim_{\epsilon, \epsilon/\nu \to 0} \|Q_{\epsilon, \nu}, G_{\epsilon, \nu}\|_{L^{\infty}(B_{1/2}(0) \cap \Omega/f/\nu)} = 0, \\
&\lim_{\epsilon, \epsilon/\nu \to 0} \|\Pi_{\epsilon, \nu} \psi_{\epsilon, \nu}\|^2 dx > |\tilde{\theta}, \epsilon, \nu, |^{2\mu}.
\end{aligned}
\]
(58)
By compactness principle and by tracing the proof of Theorem 2.7 [3], we can extract a subsequence (same notation for subsequence) such that, as $\epsilon, \epsilon/\nu \to 0$,
\[
\begin{aligned}
&\tilde{\theta}, \epsilon, \nu, \tilde{\theta}, \epsilon, \nu, \to \tilde{\theta} \in [\tilde{\theta}_1, \tilde{\theta}_2], \\
&\nu, \lambda, \omega \in [0, 1], \\
&\Pi_{\epsilon, \nu} \psi_{\epsilon, \nu} \to \psi \\
&\to \tilde{K} \nabla \psi + \tilde{T}_\lambda \psi \\
&\to \tilde{P} \nabla \psi + \tilde{E}_{\lambda, \omega} \psi
\end{aligned}
\]
(59)
in $L^2(B_{3/4}(0) \cap \Omega/\nu)$ strongly.
\[
\begin{aligned}
(K_{\epsilon, \nu} \nabla \psi_{\epsilon, \nu} + \lambda_{\epsilon, \nu} T_{\epsilon, \nu} \psi_{\epsilon, \nu}) \tilde{X}_{\epsilon, \nu} \\
\to \tilde{K} \tilde{X}_{\lambda, \omega} \psi \\
\to \lambda \tilde{P} \tilde{X}_{\lambda, \omega} \psi
\end{aligned}
\]
(59)
in $L^2(B_{3/4}(0) \cap \Omega/\nu)$ weakly,
where $\tilde{K}, \tilde{T}_\lambda, \tilde{P}, \tilde{E}_{\lambda, \omega}$ are defined in (8). The $\psi$ in (59) is a solution of (55) with $t = \nu$. By (57)–(59),
\[
\hat{\theta}^{2\mu} = \lim_{\epsilon, \epsilon/\nu \to 0} |\tilde{\theta}, \epsilon, \nu, |^{2\mu} \leq \lim_{\epsilon, \epsilon/\nu \to 0} \int_{B_{3/4}(0) \cap \Omega/\nu} |\Pi_{\epsilon, \nu} \psi_{\epsilon, \nu}|^2 dx
\]
(60)
\[
\int_{B_{3/4}(0) \cap \Omega/\nu} \psi^2 dx \leq \tilde{\theta}^{2\mu'} \int_{B_{3/4}(0) \cap \Omega/\nu} \psi^2 dx.
\]
If $\tilde{\theta}_2$ is small enough, the right hand side of (60) is less than $\hat{\theta}^{2\mu''}$ for $\mu'' \in (\mu, \mu')$. So we have $\tilde{\theta}^{2\mu} \leq \hat{\theta}^{2\mu''}$ for $\mu'' \in (\mu, \mu')$, which is impossible. Therefore, there is a $\tilde{\epsilon}_0$ such that (54) holds for $\epsilon, \epsilon/\nu < \tilde{\epsilon}_0$. Clearly, $\tilde{\epsilon}_0$ can be chosen so that $\tilde{\epsilon}_0 < \epsilon_0$.

**Lemma 4.5.** For any $\delta > 0$, there are $\tilde{\theta}_1, \tilde{\theta}_2 \in (0, 1)$ (depending on $\delta, Y_f, K, M, \Omega$) satisfying $\tilde{\theta}_1 < \tilde{\theta}_2$ and there is a $\epsilon_0 \in (0, 1)$ (depending on $\delta, Y_f, K, M, \Omega$) satisfying $\epsilon_0 < \epsilon_0$ (Lemma 4.3) that is satisfied in Lemma 4.2) such that if
\[
\begin{aligned}
&-\nabla \cdot (K_{\epsilon} \nabla \Psi_{\epsilon} + T_{\epsilon} \Psi_{\epsilon} + Q_{\epsilon}) + P_{\epsilon} \nabla \Psi_{\epsilon} + E_{\epsilon} \Psi_{\epsilon} = G_{\epsilon} \quad \text{in } B_1(0) \cap \Omega/\nu, \\
&(K_{\epsilon} \nabla \Psi_{\epsilon} + T_{\epsilon} \Psi_{\epsilon} + Q_{\epsilon}) \cdot \mathbf{n} = 0 \quad \text{on } B_1(0) \cap \partial \Omega/\nu, \\
&\Psi_{\epsilon} = 0 \quad \text{on } B_1(0) \cap \partial \Omega,
\end{aligned}
\]
(61)
and if $\epsilon \in (0, \varepsilon_0)$ and $\tilde{\theta} \in [\tilde{\theta}_1, \tilde{\theta}_2]$, then, for any $k$ satisfying $\epsilon/\tilde{\theta}^k \leq \varepsilon_0$,
\[
\int_{B_{\tilde{\theta}^k}(0) \cap \Omega} |\Pi_{\varepsilon}\Psi_{\varepsilon}|^2 \, dx \leq \tilde{\theta}^{2k \mu} \tilde{J}_{\varepsilon},
\]
where $\mu \equiv \frac{\delta}{n + 2}$ and $\tilde{J}_{\varepsilon} \equiv \frac{1}{\varepsilon_0} \left( \| \Psi_{\varepsilon} \|_{L^2(B_1(0) \cap \Omega_1')} + \| Q_{\varepsilon}, G_{\varepsilon} \|_{L^{n+4}(B_1(0) \cap \Omega_1')} \right)$.

**Proof.** The proof is similar to that of Lemma 4.2 and is done by induction on $k$. Define $\psi_{\varepsilon,1} \equiv \frac{\Psi_{\varepsilon}}{\tilde{\theta}^k}$, $Q_{\varepsilon,1} \equiv \frac{Q_{\varepsilon}}{\tilde{\theta}^k}$, $G_{\varepsilon,1} \equiv \frac{G_{\varepsilon}}{\tilde{\theta}^k}$, $\lambda_{\varepsilon,1} = \omega_{\varepsilon,1} = 1$. Then these functions satisfy (53) with $\nu = 1$. (62) for $k = 1$ is deduced from Lemma 4.4. Suppose (62) holds for some $k$ satisfying $\epsilon/\tilde{\theta}^k \leq \varepsilon_0$, then we define
\[
\begin{align*}
\psi_{\varepsilon,\tilde{\theta}^k}(x) & \equiv \tilde{J}_{\varepsilon}^{-1} \tilde{\theta}^{-k\mu} \Psi_{\varepsilon}(\tilde{\theta}^k x) \\
Q_{\varepsilon,\tilde{\theta}^k}(x) & \equiv \tilde{J}_{\varepsilon}^{-1} \tilde{\theta}^{k(1-\mu)} Q_{\varepsilon}(\tilde{\theta}^k x) \\
G_{\varepsilon,\tilde{\theta}^k}(x) & \equiv \tilde{J}_{\varepsilon}^{-1} \tilde{\theta}^{k(2-\mu)} G_{\varepsilon}(\tilde{\theta}^k x) \quad \text{in } B_1(0) \cap \Omega_1' / \tilde{\theta}^k, \\
\lambda_{\varepsilon,\tilde{\theta}^k} & \equiv \tilde{\theta}^k \\
\omega_{\varepsilon,\tilde{\theta}^k} & \equiv \tilde{\theta}^{2k}
\end{align*}
\]
Then these functions satisfy
\[
\begin{align*}
-\nabla \cdot (K_{\varepsilon,\tilde{\theta}^k} \nabla \psi_{\varepsilon,\tilde{\theta}^k} + \lambda_{\varepsilon,\tilde{\theta}^k} T_{\varepsilon,\tilde{\theta}^k} \psi_{\varepsilon,\tilde{\theta}^k} + Q_{\varepsilon,\tilde{\theta}^k}) \\
\quad + \lambda_{\varepsilon,\tilde{\theta}^k} P_{\varepsilon,\tilde{\theta}^k} \nabla \psi_{\varepsilon,\tilde{\theta}^k} + \omega_{\varepsilon,\tilde{\theta}^k} E_{\varepsilon,\tilde{\theta}^k} \psi_{\varepsilon,\tilde{\theta}^k} = G_{\varepsilon,\tilde{\theta}^k} 
\end{align*}
\]
in $B_1(0) \cap \Omega_1' / \tilde{\theta}^k$, on $B_1(0) \cap \Omega_1' / \tilde{\theta}^k$, $\tilde{\theta}^{2k}$, where $\tilde{n}_{\varepsilon,\tilde{\theta}^k}$ is a unit normal vector to $\partial \Omega_1' / \tilde{\theta}^k$. By induction, we see
\[
\begin{align*}
\lambda_{\varepsilon,\tilde{\theta}^k}, \omega_{\varepsilon,\tilde{\theta}^k} & \in [0,1], \\
\| \psi_{\varepsilon,\tilde{\theta}^k} \|_{L^2(B_1(0) \cap \Omega_1' / \tilde{\theta}^k)} & \leq \varepsilon_0^{-1} \| Q_{\varepsilon,\tilde{\theta}^k}, G_{\varepsilon,\tilde{\theta}^k} \|_{L^{n+4}(B_1(0) \cap \Omega_1')} \leq 1.
\end{align*}
\]
Then we follow the argument of Lemma 4.2 and employ Lemma 4.4 with $\nu = \tilde{\theta}^k$ to obtain (62) with $k + 1$ in place of $k$.

**Lemma 4.6.** For any $\delta \in (0, 3)$, there is a $\tilde{\mu}_* \in (0, \frac{\delta}{4+\delta})$ (depending on $\delta, Y_f, K, M, \Omega$) such that if $\epsilon \in (0, 1)$, any solution of (61) satisfies
\[
\| \Pi_{\varepsilon} \Psi_{\varepsilon} \|_{L^{\frac{2}{n-2}}(B_{\tilde{\theta}^k}(0) \cap \Omega_1')} \leq c(\| \Pi_{\varepsilon} \Psi_{\varepsilon} \|_{L^2(B_1(0) \cap \Omega_1')} + \| Q_{\varepsilon}, G_{\varepsilon} \|_{L^{n+4}(B_1(0) \cap \Omega_1')}),
\]
where $c$ is a constant independent of $\epsilon$.

**Proof.** Let $\tilde{\theta}_1, \tilde{\theta}_2, \varepsilon_0, \mu, \tilde{J}_\varepsilon$ be those in Lemma 4.5, define $\varepsilon_* \equiv \min\{\varepsilon_0 \tilde{\theta}_2^2 / 3, \varepsilon_*\}$ where $\varepsilon_*$ is the one in the proof of Lemma 4.3, and let $\varepsilon \leq \varepsilon_*$. Denote by $c$ a constant independent of $\varepsilon$. By energy method, A2–A3, Theorem 7.26 [13], and Lemma 2.1, any solution of (61) satisfies
\[
\| \Pi_{\varepsilon} \Psi_{\varepsilon} \|_{L^{\frac{2}{n-2}}(B_{\tilde{\theta}^k}(0) \cap \Omega_1')} \leq c(\| \Pi_{\varepsilon} \Psi_{\varepsilon}, Q_{\varepsilon}, G_{\varepsilon} \|_{L^2(B_1(0) \cap \Omega_1')} + \| Q_{\varepsilon}, G_{\varepsilon} \|_{L^{n+4}(B_1(0) \cap \Omega_1')}). \]
For any $x \in B_{\tilde{\theta}_2^2 / 3}(0) \cap \Omega_1'$, define $\rho(x) \equiv |x - x_0|$ where $x_0 \in \partial \Omega$ satisfying $|x - x_0| = \min_{y \in \partial \Omega} |x - y|$. Then we have case (1) $\rho(x) > \frac{4\tilde{\theta}_2}{\delta + 4}$ or case (2) $\rho(x) \leq \frac{4\tilde{\theta}_2}{\delta - 4}$. We now consider case (1). Because of $\tilde{\theta}_1 < \tilde{\theta}_2^2$, for any $r \in [\varepsilon \tilde{\theta}_0, \tilde{\theta}_2^2]$, there are $\tilde{\theta} \in [\tilde{\theta}_1, \tilde{\theta}_2]$ and $\kappa \in \mathbb{N}$ satisfying $r = \tilde{\theta}^k$. Since $\rho(x) \in \left[ \frac{2\tilde{\theta}_2}{\delta - 4}, \frac{4\tilde{\theta}_2}{\delta - 4} \right]$, by Lemma 4.5,
\[
\int_{B_r(x_0) \cap \Omega} |\Pi_{\varepsilon} \Psi_{\varepsilon}|^2 \, dy \leq r^{2\mu} \tilde{J}_{\varepsilon}^2 \quad \text{for } r \in \left[ \frac{3}{2} \rho(x), \frac{4}{3} \rho(x) \right].
\]
This implies

\[
\text{Shift the coordinate system so that the } \tilde{B} \text{ is located at the origin. For any } \tilde{\mu} < \mu, \text{ we define, in } B_1(x) \cap \Omega_f^\epsilon / \rho(x),
\]

\[
\begin{aligned}
\psi_{\epsilon, \rho}(x)(y) &\equiv \tilde{J}_e^{-1} \rho - \tilde{\mu}(x) (\Psi_\epsilon(x)y - (\Pi_\epsilon \Psi_\epsilon)_{x, \rho}(x)), \\
Q_{\epsilon, \rho}(x)(y) &\equiv \tilde{J}_e^{-1} \rho - \tilde{\mu}(x) (Q_\epsilon(x)y + T_\epsilon(x)(\Pi_\epsilon \Psi_\epsilon)_{x, \rho}(x)), \\
G_{\epsilon, \rho}(x)(y) &\equiv \tilde{J}_e^{-1} \rho - \tilde{\mu}(x) (G_\epsilon(x)y - E_\epsilon(x)(\Pi_\epsilon \Psi_\epsilon)_{x, \rho}(x)), \\
\lambda_{\epsilon, \rho}(x) &\equiv \rho(x), \\
\omega_{\epsilon, \rho}(x) &\equiv \rho^2(x).
\end{aligned}
\]

Then these functions satisfy

\[
\begin{aligned}
-\nabla \cdot (K_{\epsilon, \rho}(x) \nabla \psi_{\epsilon, \rho}(x) + \lambda_{\epsilon, \rho}(x) T_{\epsilon, \rho}(x) \psi_{\epsilon, \rho}(x) + Q_{\epsilon, \rho}(x)) \\
+ \lambda_{\epsilon, \rho}(x) P_{\epsilon, \rho}(x) \nabla \psi_{\epsilon, \rho}(x) + \omega_{\epsilon, \rho}(x) E_{\epsilon, \rho}(x) \psi_{\epsilon, \rho}(x) \\
= \mathbb{G}_{\epsilon, \rho}(x) \text{ in } B_1(x) \cap \Omega_f^\epsilon / \rho(x), \\
\end{aligned}
\]

(67)

where \(\tilde{n}_{\epsilon, \rho}(x)\) is a unit vector normal to \(\partial \Omega_m^\epsilon / \rho(x)\). Take \(s = \rho(x)\) in (65) to see, by (64),

\[
\begin{aligned}
\lambda_{\epsilon, \rho}(x), \omega_{\epsilon, \rho}(x) &\in [0, 1], \\
\|\psi_{\epsilon, \rho}(x)\|_{L^2(B_1(x) \cap \Omega_f^\epsilon / \rho(x))} &+ \frac{1}{\epsilon_0} \|Q_{\epsilon, \rho}(x), \mathbb{G}_{\epsilon, \rho}(x)\|_{L^{n+1}(B_1(x) \cap \Omega_f^\epsilon / \rho(x))} \leq \tilde{c},
\end{aligned}
\]

where \(\tilde{c}\) is independent of \(\tilde{\mu}, \epsilon\). Apply Lemma 4.3 to (67) to obtain

\[
|\psi_{\epsilon, \rho}(x)|_{C^{0, \tilde{s}}(B_{1/2}(x) \cap \Omega_f^\epsilon / \rho(x))} \leq c \quad \text{for some } \tilde{\mu}_1 < \mu.
\]

(68)

Note the \(\tilde{\mu}_1\) in (68) is independent of the \(\tilde{\mu}\) in (66). If the \(\tilde{\mu}\) in (66) is taken to be the \(\tilde{\mu}_1\) in (68), by Theorem 1.2 in page 70 [12] and Lemma 2.1, we obtain

\[
\int_{B_1(x) \cap \Omega_f^\epsilon / \rho(x)} |(\Pi_\epsilon \Psi_\epsilon - (\Pi_\epsilon \Psi_\epsilon)_{x, \rho}|^2 \, dy \leq c s^2 \tilde{\mu}_1 \tilde{J}_e^2 \quad \text{for } s < \frac{\tilde{\Theta}(x)}{2}.
\]

(69)

Now we consider case (2). Again because of \(\tilde{\Theta}_1 < \tilde{\Theta}_2\), for any \(r \in [\epsilon / \tilde{\epsilon}_0, \tilde{\Theta}_2]\), there are \(\tilde{\epsilon} \in [\tilde{\Theta}_1, \tilde{\Theta}_2]\) and \(k \in \mathbb{N}\) satisfying \(r = \tilde{\epsilon}^k\). By Lemma 4.5,

\[
\int_{B_{1}(x_0) \cap \Omega_f^\epsilon / \rho(x)} |(\Pi_\epsilon \Psi_\epsilon)|^2 \, dy \leq \epsilon^{2\mu} \tilde{J}_e^2 \quad \text{for } r \in [\epsilon / \tilde{\epsilon}_0, \tilde{\Theta}_2].
\]

(70)

This implies

\[
\int_{B_1(x) \cap \Omega_f^\epsilon / \rho(x)} |(\Pi_\epsilon \Psi_\epsilon - (\Pi_\epsilon \Psi_\epsilon)_{x, \rho}|^2 \, dy \leq c s^{2\mu} \tilde{J}_e^2 \quad \text{for } s \in [\frac{\tilde{\epsilon}_0}{3}, \frac{\tilde{\epsilon}_0}{3}]\).
\]

(71)

Shift the coordinate system so that the \(x\) is located at the origin. For any \(\tilde{\mu} < \mu\), we define

\[
\begin{aligned}
\psi_1(y) &\equiv \tilde{J}_e^{-1} \rho - \tilde{\mu}(x) (\Psi_\epsilon(\tilde{y}) - (\Pi_\epsilon \Psi_\epsilon)_{x, \tilde{\epsilon}_0}), \\
Q_1(y) &\equiv \tilde{J}_e^{-1} \rho - \tilde{\mu}(x) (Q_\epsilon(\tilde{y}) + T_\epsilon(\tilde{y})(\Pi_\epsilon \Psi_\epsilon)_{x, \epsilon / \tilde{\epsilon}_0}), \\
G_1(y) &\equiv \tilde{J}_e^{-1} \rho - \tilde{\mu}(x) (G_\epsilon(\tilde{y}) - E_\epsilon(\tilde{y})(\Pi_\epsilon \Psi_\epsilon)_{x, \epsilon / \tilde{\epsilon}_0}), \\
\lambda_1 &\equiv \epsilon, \\
\omega_1 &\equiv \epsilon^2.
\end{aligned}
\]

(72)
and
\[ \psi_{b_r}(y) = -\tilde{J}_r^{-1}\epsilon^{-\tilde{\mu}}(\Pi,\Psi_x)_{x,\epsilon/\epsilon_0} \text{ in } B_{1/\epsilon_0}(x) \cap \Omega/\epsilon. \]

By (70), \(|\psi_{b_r}|\) is a constant independent of \(\epsilon, \tilde{\mu}\). Functions in (72) satisfy
\[
\begin{cases}
-\nabla \cdot (\mathbf{K} \nabla \psi_1 + \lambda_1 \mathbf{T} \psi_1 + \mathbf{Q}_1) + \lambda_1 \mathbf{P} \nabla \psi_1 + \omega_1 \mathbf{E} \psi_1 = G_1 & \text{in } B_{1/\epsilon_0}(x) \cap \Omega_f/\epsilon, \\
(\mathbf{K} \nabla \psi_1 + \lambda_1 \mathbf{T} \psi_1 + \mathbf{Q}_1) \cdot \mathbf{n}_{\epsilon/\epsilon} = 0 & \text{on } B_{1/\epsilon_0}(x) \cap \partial \Omega_m/\epsilon, \\
\psi_1 = \psi_{b_r} & \text{on } B_{1/\epsilon_0}(x) \cap \partial \Omega/\epsilon,
\end{cases}
\]
where \(\mathbf{n}_{\epsilon/\epsilon}\) is a unit vector normal to \(\partial \Omega_m/\epsilon\). Since \(\tilde{\mu} \in [\epsilon/\epsilon_0, \tilde{\theta}_2]\), we take \(s = \frac{\epsilon}{\epsilon_0}\) in (71) to see, by (64),
\[
\left\{ \begin{array}{l}
\lambda_1, \omega_1 \in [0, 1], \\
\|\psi_1\|_{L^2(B_{1/\epsilon_0}(x) \cap \Omega_f/\epsilon)} + \|\mathbf{Q}_1, \mathbf{G}_1\|_{L^{n+1}(B_{1/\epsilon_0}(x) \cap \Omega_f/\epsilon)} + \|\psi_{b_r}\|_{C^{0,1}(B_{1/\epsilon_0}(x) \cap \Omega_f/\epsilon)} \leq \tilde{c},
\end{array} \right.
\]
where \(\tilde{c}\) is independent of \(\tilde{\mu}, \epsilon\). By Theorem 8.29 [13] and tracing the proof of Theorem 8.24 [13],
\[
\|\psi_1\|_{C^{0,\tilde{\mu}}(B_{1/2s_\epsilon}(x) \cap \Omega_f/\epsilon)} \leq c \text{ for some } \tilde{\mu}^1 < \mu. \tag{73}
\]
Note the \(\tilde{\mu}^1\) in (73) is independent of the \(\tilde{\mu}\) in (72). If the \(\tilde{\mu}\) in (72) is taken to be the \(\tilde{\mu}^1\) in (73), Theorem 1.2 in page 70 [12] and Lemma 2.1 imply that
\[
\int_{B_{s}(x) \cap \Omega} |\Pi_x \Psi_r - (\Pi_x \Psi_r)_{x,s}|^2 \, dy \leq c s^{2\tilde{\mu}^1} \tilde{J}_r^2 \text{ for } s \leq \frac{\epsilon}{2\epsilon_0}. \tag{74}
\]
Now define \(\tilde{\mu}_* \equiv \min\{\mu, \tilde{\mu}^1, \tilde{\mu}^1\}\). Then (65), (69), (71), (74), and Theorem 1.2 in page 70 [12] imply
\[
\text{For any } \delta \in (0, 3), \text{ there are } \tilde{\mu}_* \in (0, \frac{\delta}{n+\delta}) \text{ and } \tilde{\epsilon}_* \in (0, 1) \text{ (depending on } \delta, \mathbf{Y}_f, \mathbf{K}, \mathbf{M}, \Omega) \text{ such that if } \epsilon \in (0, \tilde{\epsilon}_*), \text{ then any solution of } (61) \text{ satisfies } (63).\]

By Theorem 8.29 [13] and tracing the proof of Theorem 8.24 [13], we also see
\[
\text{Let } \delta, \tilde{\epsilon}_*, \tilde{\mu}_* \text{ be same as above. If } \epsilon \in [\tilde{\epsilon}_*, 1), \text{ then any solution of } (61) \text{ satisfies } (63).\]

Combining the above two results, we prove Lemma 4.6. □

By A1–A4, energy method, partition of unity, Lemma 4.3, and Lemma 4.6, we obtain Theorem 2.2.

5. \(L^\infty\) convergence estimate. We now prove Theorem 2.3. By A5, Theorem 2.2, (8), (35), and Theorem 9.19 [13], the solution \(U\) of (17) satisfies
\[
\|U\|_{W^{3, n+\delta}(\Omega)} \leq c\|\tilde{F}_s\|_{W^{0, n+\delta}(\Omega)}, \tag{75}
\]
where \(c\) is a constant depending on \(\tilde{K}, \mathbf{M}, \Omega\). If \(U_\epsilon\) is the solution of (15) and if \(U\) is the solution of (17), we define
\[
\Psi_\epsilon(x) \equiv U_\epsilon(x) - U(x) - \sum_{\ell=1}^{\ell} \sum_{s=0}^{\ell} \sum_{i=s}^{\ell} \mathbf{X}_\ell,\epsilon(x) \partial_i U(x) \text{ for } x \in \Omega_f. \tag{76}
\]
In (76), \( \overline{i} \equiv (i_0, i_1 \cdots, i_s); \overline{i} = i; i_0 = 0; i_1, \cdots, i_s \in \{1, \cdots, n\}; X_{i_0,i_1,\cdots,i_s}(\ell, \epsilon) = X^{(i_0,i_1,\cdots,i_s)}_{\ell, \epsilon}; \partial_1 U \equiv \partial_{i_1, \cdots, i_s} U. \) See (7), (9)–(11), (14) for \( X_{\ell, \epsilon}^{(i_0,i_1,\cdots,i_s)} \). By (34) and (75),

\[
\begin{align*}
-\nabla \cdot \left( K_\epsilon \nabla \psi + T_\epsilon \psi + \hat{O}_1(\epsilon^2) \right) \\
+ P_\epsilon \nabla \psi + e_\epsilon \psi = F - \hat{F}_s + \hat{O}_1(\epsilon) \quad \text{in } \partial \Omega_f,
\end{align*}
\]

\[
\begin{align*}
(K_\epsilon \nabla \psi + T_\epsilon \psi + \hat{O}_1(\epsilon^2)) \cdot \hat{n}_\epsilon = 0 \quad \text{on } \partial \Omega_m,
\end{align*}
\]

where \( \hat{O}_1(\nu) \) denotes a function satisfying \( \| \hat{O}_1(\nu) \|_{L^{n+1}(\Omega_f)} \leq c \nu \), \( \hat{O}_2(\nu) \) is a function satisfying \( \| \hat{O}_2(\nu) \|_{L^\infty(\Omega_f)} \leq c \nu \), and \( c \) is a constant independent of \( \nu \). Decompose \( \psi = \tilde{\psi} + \hat{\psi} \), where \( \tilde{\psi} \) satisfies

\[
\begin{align*}
-\nabla \cdot \left( K_\epsilon \nabla \tilde{\psi} + T_\epsilon \tilde{\psi} + \hat{O}_1(\epsilon^2) \right) \\
+ P_\epsilon \nabla \tilde{\psi} + e_\epsilon \tilde{\psi} = F - \hat{F}_s + \hat{O}_1(\epsilon) \quad \text{in } \partial \Omega_f,
\end{align*}
\]

\[
\begin{align*}
(K_\epsilon \nabla \tilde{\psi} + T_\epsilon \tilde{\psi} + \hat{O}_1(\epsilon^2)) \cdot \hat{n}_\epsilon = 0 \quad \text{on } \partial \Omega_m,
\end{align*}
\]

and \( \hat{\psi} \) satisfies

\[
\begin{align*}
-\nabla \cdot (K_\epsilon \nabla \hat{\psi} + T_\epsilon \hat{\psi}) + P_\epsilon \nabla \hat{\psi} + e_\epsilon \hat{\psi} = 0 \quad \text{in } \Omega_f,
\end{align*}
\]

\[
\begin{align*}
(K_\epsilon \nabla \hat{\psi} + T_\epsilon \hat{\psi}) \cdot \hat{n}_\epsilon = 0 \quad \text{on } \partial \Omega_m,
\end{align*}
\]

By Theorem 2.2, the solution of (77) satisfies

\[
\| \tilde{\psi}_\epsilon \|_{L^\infty(\Omega_f)} \leq c(\epsilon + \| F_s - \hat{F}_s \|_{L^{n+1}(\Omega_f)}),
\]

where \( c \) is independent of \( \epsilon \). Next we claim that the solution of (78) satisfies

\[
\| \hat{\psi}_\epsilon \|_{L^\infty(\Omega_f)} = \| \hat{\psi}_\epsilon \|_{L^\infty(\partial \Omega)} \leq c, \]

where \( c \) is a constant independent of \( \epsilon \).

Proof of the claim: Assume \( \hat{\psi}_\epsilon \) is not a constant in \( \Omega_f \) (otherwise (80) is clear), so \( \| \hat{\psi}_\epsilon \|_{L^\infty(\Omega_f)} > 0 \). Theorem 3.5 [13] imply that the maximum value of \( |\hat{\psi}_\epsilon| \) in \( \overline{\Omega_f} \) cannot appear on any interior point of \( \Omega_f \). So it is on the boundary \( \partial \Omega_m \) of \( \Omega_f \). If there is a \( x_0 \in \partial \Omega_m \) satisfying \( \| \hat{\psi}_\epsilon(x_0) \|_{L^\infty(\Omega_f)} > 0 \), then either \( \hat{\psi}_\epsilon(x_0) > 0 \) or \( \hat{\psi}_\epsilon(x_0) < 0 \). If \( \hat{\psi}_\epsilon(x_0) > 0 \), Lemma 3.4 [13] and A5 imply \( K_\epsilon \nabla \hat{\psi}_\epsilon \cdot \hat{n}(x_0) > 0 \), which is inconsistent with (78)2. If \( \hat{\psi}_\epsilon(x_0) < 0 \), we define \( \zeta = -\hat{\psi}_\epsilon \). Then \( \zeta \) satisfies (78) and \( \zeta(x_0) = \| \zeta \|_{L^\infty(\Omega_f)} > 0 \). Lemma 3.4 [13] and A5 imply \( K_\epsilon \nabla \zeta \cdot \hat{n}(x_0) > 0 \), which is again inconsistent with (78)2. So we know \( \| \hat{\psi}_\epsilon \|_{L^\infty(\Omega_f)} = \| \hat{\psi}_\epsilon \|_{L^\infty(\partial \Omega)} \). Clearly \( \| \hat{\psi}_\epsilon \|_{L^\infty(\partial \Omega)} \leq c \) by (78)3. So we prove the claim.

Theorem 2.3 follows from (34), (75), (79), and (80).

6. Uniform Lipschitz estimate. This section consists of two subsections. The first subsection is to prove a uniform Lipschitz bound in \( \epsilon \) for the solutions of elliptic equations (that is, Theorem 2.4 and Theorem 2.5). The second subsection is to derive Lipschitz convergence estimates for the elliptic solutions (that is, Theorem 2.6 and Theorem 2.7).
6.1. Uniform Lipschitz bound in $\epsilon$. Let us assume $A6$–$A8$ and $B_1(0) \subset \mathbb{D}$ in this subsection.

**Lemma 6.1.** For any $\delta \in (0, 3)$, there are $\tilde{\theta} \in (0, 1)$ (depending on $\delta, Y_f, K$) and $\tilde{\epsilon}_0 \in (0, 1)$ (depending on $\tilde{\theta}, \delta, Y_f, K$) such that if

$$
\begin{align*}
\begin{cases}
-\nabla \cdot (K_\nu \nabla \psi_\nu + Q_\nu) + \omega_\nu E_\nu \psi_\nu = G_\nu & \text{in } B_1(0) \cap \mathbb{D}_f, \\
(K_\nu \nabla \psi_\nu + Q_\nu) \cdot \mathbf{n}_\nu = 0 & \text{on } B_1(0) \cap \partial \mathbb{D}_f,
\end{cases}
\end{align*}
$$

and if

$$
\begin{align*}
\nu \in (0, \tilde{\epsilon}_0), \quad \omega_\nu \in [0, 1], \quad M \leq 1,
\|\psi_\nu\|_{L^\infty(B_1(0) \cap \mathbb{D}_f)}, \tilde{\epsilon}_0^{-1}\|Q_\nu, G_\nu\|_{L^{n+4}(B_1(0) \cap \mathbb{D}_f)} \leq 1,
\end{align*}
$$

then

$$
\sup_{|x| < \theta} |\Pi_\nu \psi_\nu(x) - \Pi_\nu \psi_\nu(0) - (x + \Pi_\nu X_\nu(x)) b_\nu| \leq \tilde{\theta}^{1+\frac{\mu}{2}},
$$

where $\mu \equiv \frac{\delta}{n + \sigma}$, $b_\nu \equiv \frac{\tilde{K}^{-1}}{B_4(0)} \int_{B_4(0) \cap \mathbb{D}_f} K_\nu \nabla \psi_\nu dx$, and $\tilde{K}^{-1}$ is the inverse matrix of the symmetric positive definite matrix $\tilde{K}$ in (8). See $A8$ for $M$, (14) for $X_\nu$, and Lemma 2.1 for $\Pi_\nu$.

**Proof.** Let $\omega \in [0, 1]$, $\tilde{E}_{0,1} \leq 1$ (see (8)), and $\psi$ satisfy

$$
-\nabla \cdot (\tilde{K} \nabla \psi) + \omega \tilde{E}_{0,1} \psi = 0 \quad \text{in } B_3(0),
$$

where $\tilde{K}$ is defined in (8). By (35) and Theorem 6.2 [13],

$$
\|\psi\|_{C^{2,1}(\overline{B_{3/4}(0)})} \leq c \|\psi\|_{L^\infty(B_{3/4}(0))},
$$

where $c$ is a constant depending on $\tilde{K}$. If $\mu' \in (\mu, 1)$, by Taylor’s expansion,

$$
\sup_{|x| < \theta} \left| \psi(x) - \psi(0) - x(\nabla \psi)_0,0 \right| \leq \theta^{1+\frac{\mu'}{2}} \|\psi\|_{L^\infty(B_{3/4}(0))}
$$

for $\theta$ (depending on $\tilde{K}$) sufficiently small.

Now we fix a $\tilde{\theta} < \frac{1}{2}$ such that (85) holds. We claim (83). If not, there is a sequence $\{\omega_\nu, \psi_\nu, Q_\nu, G_\nu\}$ satisfying (81) and

$$
\begin{align*}
\omega_\nu & \in [0, 1], \\
\|\psi_\nu\|_{L^\infty(B_1(0) \cap \mathbb{D}_f)} & \leq 1, \\
\lim_{\nu \to 0} \|Q_\nu, G_\nu\|_{L^{n+4}(B_1(0) \cap \mathbb{D}_f)} & = 0, \\
\sup_{|x| < \theta} |\Pi_\nu \psi_\nu(x) - \Pi_\nu \psi_\nu(0) - (x + \Pi_\nu X_\nu(x)) b_\nu| & > \tilde{\theta}^{1+\frac{\mu}{2}}.
\end{align*}
$$

After extraction of a subsequence (same notation for subsequence), we have, by Theorem 2.7 [3] and Lemma 4.3,

$$
\begin{align*}
\omega_\nu & \to \omega \in [0, 1] \\
\Pi_\nu \psi_\nu & \to \psi \quad \text{in } L^\infty(B_{3/4}(0)) \text{ strongly} \\
K_\nu \nabla \psi_\nu \tilde{X}_D & \to \tilde{K} \nabla \psi \quad \text{in } L^2(B_{3/4}(0)) \text{ weakly as } \nu \to 0, \\
\omega_\nu E_\nu \psi_\nu \tilde{X}_D & \to \omega \tilde{E}_{0,1} \psi \quad \text{in } L^2(B_{3/4}(0)) \text{ weakly}
\end{align*}
$$
where $\tilde{K}, \tilde{E}_{0,1}$ are defined in (8) and $\tilde{E}_{0,1} \leq 1$ by $(82)_1$. The $\psi$ in (87) is a solution of (84). (34) and (85)–(87) imply

$$\tilde{\theta}^{1+\frac{a}{2}} \leq \lim_{\nu \to 1} \sup_{|x|<\tilde{\theta}} |\Pi_\nu \psi_\nu(x) - \Pi_\nu \psi_\nu(0) - (x + \Pi_\nu X_\nu(x))b_\nu|$$

$$= \sup_{|x|<\tilde{\theta}} |\psi(x) - \psi(0) - x(\nabla \psi)_{0,\tilde{\theta}}| \leq \tilde{\theta}^{1+\frac{a}{2}} \|\psi\|_{L^\infty(B_{1/4}(0))},$$

which is impossible if $\tilde{\theta}$ is small. So (83) holds.

\[\square\]

**Lemma 6.2.** Let $\delta, \tilde{\theta}, \tilde{\varepsilon}_0, \mu$ be same as those in Lemma 6.1 and $M_{\tilde{\varepsilon}_0^{-1}} < 1$. For any $\varepsilon \in (0, \tilde{\varepsilon}_0)$ and $k$ satisfying $\varepsilon/\tilde{\theta}^k \leq \tilde{\varepsilon}_0$, there are constants $a_k^\varepsilon, b_k^\varepsilon$ such that any solution of

$$\begin{cases}
-\nabla \cdot (K_\varepsilon \nabla \psi_\varepsilon + Q_\varepsilon) + E_\varepsilon \psi_\varepsilon = G_\varepsilon & \text{in } B_1(0) \cap \mathbb{D}_f^\varepsilon \\
(K_\varepsilon \nabla \psi_\varepsilon + Q_\varepsilon) \cdot \bar{n}_\varepsilon = 0 & \text{on } B_1(0) \cap \partial \mathbb{D}_m^\varepsilon
\end{cases}
$$

(88)

satisfies

$$\begin{cases}
\sup_{|x|<\tilde{\theta}^k} |\Pi_\varepsilon \psi_\varepsilon(x) - \Pi_\varepsilon \psi_\varepsilon(0) - \varepsilon a_k^\varepsilon - (x + \Pi_\varepsilon X_\varepsilon(x))b_k^\varepsilon| \leq \tilde{\theta}^{1+\frac{a}{2}} k \tilde{\varepsilon}_0,

|a_k^\varepsilon| + |b_k^\varepsilon| \leq c Y_k,
\end{cases}
$$

(89)

where $c$ is a constant independent of $\varepsilon$ and

$$\tilde{Y}_k = 4\tilde{\varepsilon}_0^{-1}(\|\psi_\varepsilon\|_{L^\infty(B_1(0) \cap \mathbb{D}_f^\varepsilon)} + \|\tilde{\varepsilon}_0^{-1} Q_\varepsilon, G_\varepsilon\|_{L^{\infty}(B_1(0) \cap \mathbb{D}_f^\varepsilon)}).$$

**Proof.** This is proved by induction on $k$. Let $c$ denote a constant independent of $\varepsilon$. Set $\psi_k \equiv \frac{\psi}{\tilde{\varepsilon}_0}$, $Q_k \equiv \frac{Q_\varepsilon}{\tilde{\varepsilon}_0^k}$, $G_k \equiv \frac{G_\varepsilon}{\tilde{\varepsilon}_0^k}$, $\omega_k = 1$. Then they satisfy (81) and (82) with $\nu = \varepsilon$. By Lemma 6.1, we obtain (89) for $k = 1$ case, where $a_1^\varepsilon = 0$ and

$$b_1^\varepsilon = \int_{B_\varepsilon(0) \cap \mathbb{D}_f} K_\varepsilon \nabla \psi_\varepsilon \, dx.$$ By energy method and A7–A8, any solution of (88) satisfies

$$\|\psi_\varepsilon\|_{H^1(B_1(0) \cap \mathbb{D}_f^\varepsilon)} \leq c(\|\psi_\varepsilon\|_{L^2(B_1(0) \cap \mathbb{D}_f^\varepsilon)}, G_\varepsilon, Q_\varepsilon),$$

(90)

implies that (89) for $k = 1$ also holds. If (89) holds for some $k$ satisfying $\varepsilon/\tilde{\theta}^k \leq \tilde{\varepsilon}_0$, we define, in $B_1(0) \cap \mathbb{D}_f^\varepsilon/\tilde{\theta}^k$,

$$\psi_\varepsilon/\tilde{\varepsilon}_0(x) \equiv \frac{\psi_\varepsilon(\tilde{\theta}^k x) - \Pi \psi_\varepsilon(0) - \varepsilon a_k^\varepsilon - (\tilde{\theta}^k x + \Pi_k X_\varepsilon(\tilde{\theta}^k x))b_k^\varepsilon}{\tilde{\theta}^{1+\frac{a}{2}} k \tilde{\varepsilon}_0},

Q_\varepsilon/\tilde{\varepsilon}_0(x) \equiv \frac{Q_\varepsilon(\tilde{\theta}^k x)}{\tilde{\theta}^{1+\frac{a}{2}} k \tilde{\varepsilon}_0},

G_\varepsilon/\tilde{\varepsilon}_0(x) \equiv \frac{G_\varepsilon(\tilde{\theta}^k x) - E_\varepsilon(\tilde{\theta}^k x)(\Pi_\varepsilon \psi_\varepsilon(0) + \varepsilon a_k^\varepsilon + (\tilde{\theta}^k x + X_\varepsilon(\tilde{\theta}^k x))b_k^\varepsilon)}{\tilde{\theta}^{1+\frac{a}{2}} k \tilde{\varepsilon}_0},

\omega_k \equiv \tilde{\theta}^{1+\frac{a}{2}} k \tilde{\varepsilon}_0.$$

Then they satisfy, by (7) and (14),

$$\begin{cases}
-\nabla \cdot (K_\varepsilon/\tilde{\varepsilon}_0 \nabla \psi_\varepsilon/\tilde{\varepsilon}_0 + Q_\varepsilon/\tilde{\varepsilon}_0) + \omega_k \varepsilon\varepsilon/\tilde{\varepsilon}_0 E_\varepsilon/\tilde{\varepsilon}_0 \psi_\varepsilon/\tilde{\varepsilon}_0 = G_\varepsilon/\tilde{\varepsilon}_0 & \text{in } B_1(0) \cap \mathbb{D}_f^\varepsilon/\tilde{\theta}^k,

(K_\varepsilon/\tilde{\varepsilon}_0 \nabla \psi_\varepsilon/\tilde{\varepsilon}_0 + Q_\varepsilon/\tilde{\varepsilon}_0) \cdot \bar{n}_\varepsilon/\tilde{\varepsilon}_0 = 0 & \text{on } B_1(0) \cap \partial \mathbb{D}_m^\varepsilon/\tilde{\theta}^k,
\end{cases}
$$

(91)
where $\mathbf{n}_{\epsilon/\hat{\theta} k}$ is a unit vector normal to $\partial \Omega_{m} / \hat{\theta}^k$. By (89)_1 and A8,
\[
\|G_{\epsilon/\hat{\theta} k} \|_{L^{n+i}(B_{1}(0) \cap \Omega_{\epsilon/\hat{\theta} k})} \leq \hat{\theta}^{k\mu/2} J_{\epsilon}^{-1} \|G_{\epsilon} \|_{L^{n+i}(B_{1}(0) \cap \Omega_{\epsilon/\hat{\theta} k})} + \hat{\theta}^{k(1-\mu/2)} J_{\epsilon}^{-1} M(\|\Pi_{*} \Psi_{\epsilon} \|_{L^{\infty}(B_{\hat{\theta} k}(0))} + \hat{\theta}^{k(1+\mu/2)} J_{\epsilon}).
\]
By induction, $M_{\hat{\theta}^{-1}} < 1$, (90), Lemma 4.3, and small $\hat{\theta}$, we have
\[
\left\{
\begin{array}{l}
\omega_{\epsilon/\hat{\theta} k} \in [0,1], \\
\|\psi_{\epsilon/\hat{\theta} k} \|_{L^{\infty}(B_{1}(0) \cap \Omega_{\epsilon/\hat{\theta} k})}, \|Q_{\epsilon/\hat{\theta} k}, G_{\epsilon/\hat{\theta} k} \|_{L^{n+i}(B_{1}(0) \cap \Omega_{\epsilon/\hat{\theta} k})} \leq 1.
\end{array}
\right.
\]  
(92)
Apply Lemma 6.1 (take $\nu = \epsilon/\hat{\theta}^k$) to obtain
\[
\sup_{|x| < \hat{\theta}} |\Pi_{\epsilon/\hat{\theta} k} \psi_{\epsilon/\hat{\theta} k}(x) - \Pi_{\epsilon/\hat{\theta} k} \psi_{\epsilon/\hat{\theta} k}(0) - (x + \Pi_{\epsilon/\hat{\theta} k} X_{\epsilon/\hat{\theta} k}(x)) b_{\epsilon/\hat{\theta} k}| \leq \hat{\theta}^{1+\frac{\mu}{2}},
\]  
(93)
where $b_{\epsilon/\hat{\theta} k} \equiv \frac{K_{\epsilon/\hat{\theta} k}^{-1}}{|B_{\hat{\theta} k}(0) \cap \Omega_{\epsilon/\hat{\theta} k}} \int_{B_{\hat{\theta} k}(0) \cap \Omega_{\epsilon/\hat{\theta} k}} K_{\epsilon/\hat{\theta} k} \nabla \psi_{\epsilon/\hat{\theta} k} dx$. By Lemma 2.1, (93) can be written as
\[
\sup_{|x| < \hat{\theta}} |\Pi_{\epsilon} \Psi_{\epsilon}(\hat{\theta}^k x) - \Pi_{\epsilon} \Psi_{\epsilon}(0) + c \Pi_{\epsilon} X_{\epsilon}(0) b_{\epsilon}^k - (\hat{\theta}^k x + \Pi_{\epsilon} X_{\epsilon}(\hat{\theta}^k x)) b_{\epsilon}^k - \hat{J}_{\epsilon} \hat{\theta}^{k(1+\mu/2)} (x + \hat{\theta}^{-k} \Pi_{\epsilon} X_{\epsilon}(\hat{\theta}^k x)) b_{\epsilon/\hat{\theta} k}| \leq \hat{J}_{\epsilon} \hat{\theta}^{k(1+\mu/2)}. 
\]  
(94)
See (14) for $X$. Define
\[
a_{\epsilon}^{k+1} = -\Pi_{\epsilon} X_{\epsilon}(0) b_{\epsilon}^k \quad \text{and} \quad b_{\epsilon}^{k+1} = b_{\epsilon}^k + \hat{J}_{\epsilon} \hat{\theta}^{k\mu/2} b_{\epsilon/\hat{\theta} k}. 
\]  
(95)
Substituting (95) into (94) and making the change of variables $\hat{\theta}^k x$ to $x$, we obtain (89)_1. By (91) and (92), $|\psi_{\epsilon/\hat{\theta} k}|$ is bounded uniformly in $\epsilon, k$. So (89)_2 holds by (95).

**Lemma 6.3.** For any $\delta \in (0,3)$, there is a $\tilde{\epsilon}_{0} \in (0,1)$ (depending on $\delta, Y_{f}, K_{\epsilon}$) such that if $\epsilon \in (0, \tilde{\epsilon}_{0})$ and $M_{\hat{\theta}^{-1}} < 1$, then any solution of (88) satisfies
\[
\|\nabla \Psi_{\epsilon} \|_{L^{\infty}(B_{1}(0) \cap \Omega_{\epsilon/\hat{\theta} k})} \leq c(\hat{J}_{\epsilon} + \epsilon^{-\mu/2} \|Q_{\epsilon} \|_{W_{loc}^{1,n+k}(\Omega_{\epsilon/\hat{\theta} k})}) \equiv c J_{\epsilon}^*,
\]  
(96)
where $c$ is a constant independent of $\epsilon$. See Lemma 6.2 for $\mu, \hat{J}_{\epsilon}$ and see section 2 for $\|\cdot\|_{W_{loc}^{1,n+k}(\Omega_{\epsilon/\hat{\theta} k})}$.

**Proof.** Denote by $c$ a constant independent of $\epsilon$ and let $\hat{\theta}, \tilde{\epsilon}_{0}$ be those in Lemma 6.2. Let $k \in \mathbb{N}$ such that $c/\hat{\theta}^k \leq \tilde{\epsilon}_{0} < c/\hat{\theta}^{k+1}$. By Lemma 6.2,
\[
\sup_{|x| < \frac{\tilde{\epsilon}_{0}}{\epsilon}} |\Pi_{\epsilon} \Psi_{\epsilon}(x) - \Pi_{\epsilon} \Psi_{\epsilon}(0) - c a_{\epsilon}^k - (x + \Pi_{\epsilon} X_{\epsilon}(x)) b_{\epsilon}^k| \leq c \left|\frac{\epsilon}{\tilde{\epsilon}_{0}}\right|^{1+\frac{\mu}{2}} \hat{J}_{\epsilon}.
\]  
(97)
Define, in $B_{1/\tilde{\epsilon}_{0}}(0) \cap \Omega_{\epsilon/\hat{\theta} k}$,
\[
\left\{
\begin{array}{l}
\psi_{1}(x) = \frac{\Psi_{\epsilon}(\epsilon x) - \Pi_{\epsilon} \Psi_{\epsilon}(0) - c a_{\epsilon}^k - (\epsilon x + X_{\epsilon}(\epsilon x)) b_{\epsilon}^k}{J_{\epsilon}^{1+\mu/2}}, \\
Q_{1}(x) = \frac{Q_{\epsilon}(\epsilon x)}{J_{\epsilon}^{\mu/2}}, \\
G_{1}(x) = \frac{G_{\epsilon}(\epsilon x) - E(x)(\Pi_{\epsilon} \Psi_{\epsilon}(0) + c a_{\epsilon}^k + (\epsilon x + X_{\epsilon}(\epsilon x)) b_{\epsilon}^k)}{J_{\epsilon}^{1+\mu/2-1}}, \\
\omega_{1} = \epsilon^2.
\end{array}
\right.
\]
Then these functions satisfy, by (7) and (14),
\begin{align*}
\begin{cases}
-\nabla \cdot (K \nabla \psi_1 + Q_1) + \omega_1 E \psi_1 = G_1 \quad &\text{in } B_{\frac{1}{10}}(0) \cap \mathbb{D}'_f / \epsilon, \\
(K \nabla \psi_1 + Q_1) \cdot \vec{n}_{\epsilon / \epsilon} = 0 \quad &\text{on } B_{\frac{1}{10}}(0) \cap \partial \mathbb{D}'_m / \epsilon.
\end{cases}
\end{align*}
By (97), A8, $M_{\epsilon 0}^{-1} < 1$, and Lemma 4.3,
\begin{align*}
\begin{cases}
\omega_1 \in [0, 1], \\
\|\psi_1\|_{L^\infty(B_{1/10}(0) \cap \mathbb{D}'_f / \epsilon)} \leq c, \\
\|Q_1\|_{W^{1,n+\delta}(B_{1/10}(0) \cap \mathbb{D}'_f / \epsilon)} + \|G_1\|_{L^{\infty}(B_{1/10}(0) \cap \mathbb{D}'_f / \epsilon)} \leq c.
\end{cases}
\end{align*}
By Lemma 3.2, we see
\begin{equation}
\|\psi_1\|_{W^{2,n+\delta}(B_{\frac{1}{10}}(0) \cap \mathbb{D}'_f / \epsilon)} \leq c. \tag{98}
\end{equation}
Since $\nabla \psi_1(x) = \frac{\nabla \psi_s(x) - (I + \nabla \chi_s(x)) \delta_0}{\sqrt{1 + \nabla \chi_s(x)^2}}$, $|\nabla \psi_s(x)| \leq c |J_\epsilon|$ for $x \in B_{\frac{1}{10}}(0) \cap \mathbb{D}'_f / \epsilon$ by (34), (98), and Lemma 6.2. By shifting the coordinate system and repeating above argument, we see that this lemma holds.

Let $\epsilon_0$ be same as that in Lemma 6.3. From Lemma 3.2, we also see
\begin{align*}
\text{For any } \delta \in (0, 3) \text{ and } \epsilon \in [\epsilon_0, 1), \text{ any solution of (88) satisfies (96).
\end{align*}
Together with Lemma 6.3, we obtain

**Lemma 6.4.** For $\delta \in (0, 3)$, $\epsilon \in (0, 1)$, and small $M$ (depending on $\delta, Y_f, K$), any solution of (88) satisfies (96).

Theorem 2.4 and Theorem 2.5 follow from energy method, Poincaré inequality, Lemma 4.3, and Lemma 6.4.

### 6.2. Proofs of Theorem 2.6 and Theorem 2.7

In this subsection, we first prove Theorem 2.6 and then Theorem 2.7.

6.2.1. *Proof of Theorem 2.6.* Let $\hat{\Phi}_k, \hat{V}_k$ be those defined in (25). By assumptions of Theorem 2.6, (35), and Theorem 9.19 [13], the solution $U$ of (23) satisfies
\begin{equation}
\|U\|_{W^{2k+2,n+\delta}(\mathbb{D})} \leq c\|\hat{F}\|_{W^{2k,n+\delta}(\mathbb{D})} \quad \text{for } k \geq 1, \delta \in (0, 3), \tag{99}
\end{equation}
where $c$ is a constant depending on $\hat{K}, \hat{E}_{0,1}$. (99) implies
\begin{equation}
\begin{cases}
\|\hat{V}_k\|_{W^{2,n+\delta}(\mathbb{D})} + \|\hat{\Phi}_k\|_{W^{k+3,n+\delta}(\mathbb{D})} \leq c\|\hat{F}\|_{W^{2k,n+\delta}(\mathbb{D})} \quad &\text{for } k \geq 1, \delta \in (0, 3), \tag{100}
\end{cases}
\end{equation}
where $c$ is a constant depending on $K, E$. By (34) and (99)–(100),
\begin{align*}
\begin{cases}
\nabla \cdot (K \nabla \hat{V}_k + \tilde{O}_3(\epsilon^{k+1})) - E_k \hat{V}_k = \sum_{\ell=2}^{k+1} \sum_{s=0}^{\ell} \epsilon^{\ell-2} \tilde{N}_\ell \partial \tilde{\Phi}_k + \tilde{O}_4(\epsilon^k) &\text{in } \mathbb{D}', \\
(K \nabla \hat{V}_k + \tilde{O}_3(\epsilon^{k+1})) \cdot \vec{n}_s = 0 &\text{on } \partial \mathbb{D}'_m, \\
\hat{V}_k \text{ satisfies periodic boundary conditions on } \partial \mathbb{D},
\end{cases}
\end{align*}
where $\tilde{O}_3(\epsilon^{k+1}), \tilde{O}_4(\epsilon^k)$ are periodic functions with period $\mathbb{D}$ and satisfy
\begin{align*}
\begin{cases}
\|\tilde{O}_3(\epsilon^{k+1})\|_{L^{n+\delta}(\mathbb{D}')} + \|\tilde{O}_3(\epsilon^{k+1})\|_{W^{1,n+\delta}(\mathbb{D}')} \leq c \epsilon^{k+1}, \\
\|\tilde{O}_4(\epsilon^k)\|_{L^{n+\delta}(\mathbb{D}')} \leq c \epsilon^k.
\end{cases}
\end{align*}
for some constant $c$ independent of $\epsilon$. By (23)–(25),
\[
\sum_{\ell=2}^{k+1} \sum_{s=0}^{\ell} \epsilon^{\ell-2} \mathcal{N}_1^{s} \partial_1^s \tilde{\Phi}_k = \sum_{\ell=2}^{k+1} \sum_{j=0}^{k-1} \epsilon^{\ell-2+j} \left( \sum_{s=0}^{\ell} \sum_{|\mathbf{i}|=s} \mathcal{N}_1^{s} \partial_1^{\mathbf{s}} \right)
\]
\[
= \sum_{s=0}^{k+1} \sum_{|\mathbf{i}|=s} \mathcal{N}_1^{s} \partial_1^s \tilde{\Phi}_0 + \sum_{r=1}^{k+1} \epsilon^r \left( \sum_{s=0}^{\ell} \sum_{|\mathbf{i}|=s} \mathcal{N}_1^{s} \partial_1^{\mathbf{s}} + \sum_{q=0}^{r-1} \sum_{s=0}^{\ell} \sum_{|\mathbf{i}|=s} \mathcal{N}_1^{r} \partial_1^{\mathbf{s}+r} \right) + \mathcal{O}_1(\epsilon^k)
\]
\[
= -\mathcal{F} + \mathcal{O}_1(\epsilon^k).
\]
If $U_\epsilon$ solves (21) and if we define $\Psi_\epsilon \equiv U_\epsilon - \tilde{V}_k$ in $\mathbb{D}_f$, then
\[
\begin{aligned}
&-\nabla \cdot \left( \mathbf{K}_\epsilon \nabla \Psi_\epsilon + \partial_3(\epsilon^{k+1}) \right) + \mathbf{E}_\epsilon \Psi_\epsilon = F_\epsilon - \mathcal{F} + \mathcal{O}_1(\epsilon^k) \quad \text{in } \mathbb{D}_f, \\
\left( \mathbf{K}_\epsilon \nabla \Psi_\epsilon + \partial_3(\epsilon^{k+1}) \right) \cdot \mathbf{n}_\epsilon = 0 \quad \text{on } \partial \mathbb{D}_m, \\
\Psi_\epsilon \text{ satisfies periodic boundary conditions on } \partial \mathbb{D}.
\end{aligned}
\]
By Theorem 2.4,
\[
\|\nabla \Psi_\epsilon\|_{L^\infty(\mathbb{D}_f)} \leq c(\epsilon^k + \|F_\epsilon - \mathcal{F}\|_{L^{n+4}(\mathbb{D}_f)}),
\]
where $c$ is a constant independent of $\epsilon$. Which implies Theorem 2.6.

6.2.2. Proof of Theorem 2.7. The proof is similar to that of Theorem 2.6. Let $\tilde{\Phi}_k, \tilde{V}_k$ be those defined in (30). By assumptions of Theorem 2.7, and Theorem 9.19 [13], the solution $U$ of (28) satisfies
\[
\|U\|_{W^{2k+2,n+\delta}(\mathbb{D})} \leq c\|\mathcal{F}\|_{W^{2k,n+\delta}(\mathbb{D})} \quad \text{for } k \geq 1, \delta \in (0, 3),
\]
where $c$ is a constant depending on $\tilde{K}$. Which implies
\[
\|\tilde{V}_k\|_{W^{2, n+\delta}(\mathbb{D})} + \|\tilde{\Phi}_k\|_{W^{2k, n+\delta}(\mathbb{D})} \leq c\|\mathcal{F}\|_{W^{2k,n+\delta}(\mathbb{D})} \quad \text{for } k \geq 1, \delta \in (0, 3),
\]
where $c$ is a constant depending on $\mathbf{K}$. By (30) and (34),
\[
\begin{aligned}
&\nabla \cdot \left( \mathbf{K}_\epsilon \nabla \tilde{V}_k + \partial_3(\epsilon^{k+1}) \right) = \sum_{\ell=2}^{k+1} \sum_{|\mathbf{i}|=\ell} \epsilon^{\ell-2} \mathcal{N}_1^{s} \partial_1^{s} \tilde{\Phi}_k + \mathcal{O}_1(\epsilon^k) \quad \text{in } \mathbb{D}_f, \\
&\left( \mathbf{K}_\epsilon \nabla \tilde{V}_k + \partial_3(\epsilon^{k+1}) \right) \cdot \mathbf{n}_\epsilon = 0 \quad \text{on } \partial \mathbb{D}_m, \\
&\tilde{V}_k \text{ satisfies periodic boundary conditions on } \partial \mathbb{D},
\end{aligned}
\]
where $\partial_3(\epsilon^{k+1}), \mathcal{O}_1(\epsilon^k)$ are defined same as those in §6.2.1. By (28)–(30),
\[
\sum_{\ell=2}^{k+1} \sum_{|\mathbf{i}|=\ell} \epsilon^{\ell-2} \mathcal{N}_1^{s} \partial_1^{s} \tilde{\Phi}_k = \sum_{\ell=2}^{k+1} \sum_{j=0}^{k-1} \epsilon^{\ell-2+j} \sum_{|\mathbf{i}|=\ell} \mathcal{N}_1^{s} \partial_1^{\mathbf{s}}
\]
\[
= \sum_{|\mathbf{i}|=2} \mathcal{N}_1^{s} \partial_1^{s} \tilde{\Phi}_0 + \sum_{r=1}^{k+1} \epsilon^r \left( \sum_{|\mathbf{i}|=2} \mathcal{N}_1^{s} \partial_1^{s} + \sum_{q=0}^{r-1} \sum_{|\mathbf{i}|=r+2-q} \mathcal{N}_1^{s} \partial_1^{s+q} \right) + \mathcal{O}_1(\epsilon^k)
\]
\[
= -\mathcal{F} + \mathcal{O}_1(\epsilon^k).
\]
If $U_\epsilon$ solves (26), we define $\Psi_\epsilon \equiv U_\epsilon - \tilde{V}_k$ in $\mathbb{D}_f$. Theorem 2.5 implies Theorem 2.7.
Acknowledgments. The author would like to thank the anonymous referee’s suggestions for improving the presentation of this paper.

REFERENCES

[1] E. Acerbi, V. Chiado Piat, G. Dal Maso and D. Percivale, An extension theorem from connected sets, and homogenization in general periodic domains, *Nonlinear Analysis*, 18 (1992), 481–496.

[2] R. A. Adams, *Sobolev Spaces*, Academic Press, 1975.

[3] Grégoire Allaire, Homogenization and two-scale convergence, *SIAM J. Math. Anal.*, 23 (1992), 1482–1518.

[4] R. C. Morgan and I. Babuska, An approach for constructing families of homogenized equations for periodic media. I: An integral representation and its consequences, *SIAM J. Math. Anal.*, 22 (1991), 1–15.

[5] N. Bakhvalov and G. Panasenko, *Homogenisation: Averaging Processes in Periodic Media: Mathematical Problems in the Mechanics of Composite Materials*, Kluwer Academic Publishers, 1989.

[6] B. Berkowitz, et al., Physical pictures of transport in heterogeneous media: Advection–dispersion, random walk, and fractional derivative formulations, *Water resources Research*, 38 (2002), 1191–1194.

[7] Susanne C. Brenner and L. Ridgway Scott, *The Mathematical Theory of Finite Element Methods*, Springer, 2008.

[8] Alain Bensoussan, Jacques-Louis Lions and George Papanicolaou, *Asymptotic Analysis for Periodic Structures*, Elsevier North-Holland, 1978.

[9] Li-Qun Cao, Asymptotic expansions and numerical algorithms of eigenvalues and eigenfunctions of the Dirichlet problem for second order elliptic equations in perforated domains, *Numerische Mathematik*, 103 (2006), 11–45.

[10] Philippe G. Ciarlet, *The Finite Element Method for Elliptic Problems*, Amsterdam: North-Holland, 1978.

[11] D. Cioranescu, A. Damlamian, G. Grisoa and D. Onofrei, The periodic unfolding method for perforated domains and Neumann sieve models, *J. Math. Pures Appl.*, 89 (2008), 248–277.

[12] M. Giaquinta, *Multiple integrals in the calculus of variations*, Study 105, Annals of Math. Studies, Princeton Univ. Press., 1983.

[13] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, second edition, 1983.

[14] Thomas Y. Hou, Xiao-Hui Wu and Zhiqiang Cai, Convergence of a multiscale finite element method for elliptic problems with rapidly oscillating coefficients, *Math. Comp.*, 68 (1999), 913–943.

[15] V.V. Jikov, S. M. Kozlov and O. A. Oleinik, *Homogenization of Differential Operators and Integral Functions*, Springer-Verlag, 1994.

[16] Viviane Klein and Malgorzata Peszyńska, Adaptive Double-Diffusion Model and Comparison to a Highly Heterogeneous Micro-Model, *Journal of Applied Mathematics*, (2012), 26 pages.

[17] J. L. Lions, Asymptotic expansions in perforated media with a periodic structure, *The Rocky Mountain J. Math.*, 10 (1980), 125–144.

[18] N. Neuss, W. Jäger and G. Wittum, Homogenization and multigrid, *Computing*, 66 (2001), 1–26.

[19] O. A. Oleinik, A. S. Shamaev and G. A. Tosifan, *Mathematical Problems in Elasticity and Homogenization*, North-Holland, Amsterdam, 1992.

Received November 2014; revised February 2015.

E-mail address: liming@math.nctu.edu.tw