A CONDITION FOR THE EXISTENCE OF ZERO COEFFICIENTS IN THE POWERS OF THE DETERMINANT POLYNOMIAL

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ABSTRACT. We discuss the existence of zero coefficients in the powers of the determinant polynomial of order $n$. D. G. Glynn proved that the coefficients of the $m$th power of the determinant polynomial are all nonzero, if $m = p - 1$ with a prime $p$. We show that the converse also holds, if $n \geq 3$. The proof is quite elementary.

1. Introduction

We consider the expansion of the powers of the determinant polynomial, and discuss the existence of zero coefficients. D. G. Glynn proved that the coefficients of the $m$th power of the determinant polynomial of order $n$ are all nonzero, if $m = p - 1$ with a prime $p$. This result is remarkable because this leads a proof of the Alon–Tarsi conjecture in dimension $p - 1$. In this article, we show that the converse of Glynn’s result also holds, if $n \geq 3$. The proof is quite elementary.

Let us explain the assertion precisely. Let $X = (x_{ij})_{1 \leq i,j \leq n}$ be an $n$ by $n$ matrix whose entries are indeterminates. We define the coefficients $C_L$ by the following expansion of $(\det X)^m$:

$$ (\det X)^m = \sum_{L \in \Psi(m)} C_L x^L. $$

Here $\Psi(m)$ is the set of all $n$ by $n$ matrices of nonnegative integers with each row and column summing to $m$:

$$ \Psi(m) = \left\{ (l_{ij})_{1 \leq i,j \leq n} \mid l_{ij} \in \mathbb{Z}_{\geq 0}, \sum_{i=1}^{n} l_{ij} = m \text{ for any } j = 1, 2, \ldots, n, \sum_{j=1}^{n} l_{ij} = m \text{ for any } i = 1, 2, \ldots, n \right\}. $$

Moreover, we put $x^L = \prod_{1 \leq i,j \leq n} x_{ij}^{l_{ij}}$ for $L = (l_{ij})_{1 \leq i,j \leq n}$.

D. G. Glynn proved the following theorem for these coefficients $C_L$:

**Theorem 1.1.** If $p$ be prime, we have $C_L \neq 0$ for all $L \in \Psi(p - 1)$.

**Remark.** Actually, Glynn proved a stronger theorem, namely, that we have $L! C_L \equiv (-1)^n \pmod{p}$ for all $L \in \Psi(p - 1)$. Here we put $L! = \prod_{i,j=1}^{n} l_{ij}!$ for $L = (l_{ij})_{1 \leq i,j \leq n}$.

In the present article, we prove that the inverse of Theorem 1.1 also holds, when $n \geq 3$:

**Theorem 1.2.** Assume that $n \geq 3$. Let $m$ be a natural number which cannot be expressed as $m = p - 1$ with a prime $p$. Then, there exists $L \in \Psi(m)$ satisfying $C_L = 0$.

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Thus, we see that the following two conditions on $m$ are equivalent, when $n \geq 3$:

- We have $C_L \neq 0$ for all $L \in \Psi(m)$.
- There is a prime $p$ satisfying $m = p - 1$.

2. **The coefficients are all nonzero, when $m = p - 1$**

Theorem 1.1 was shown by D. G. Glynn in [G2]. The key of the proof is the hyperdeterminant introduced by Glynn himself in [G1] (this hyperdeterminant occurs only for fields of prime characteristic $p$). Nowadays, a proof of Theorem 1.1 without hyperdeterminant is also known [K1], [K2].

Theorem 1.1 is remarkable, because this leads to a special case of the Alon–Tarsi conjecture on Latin squares. Let $\text{els}(n)$ and $\text{ols}(n)$ denote the numbers of even and odd Latin squares of size $n$, respectively. We can easily show $\text{els}(n) = \text{ols}(n)$ when $n$ is an odd number greater than 1. In contrast, on the case that $n$ is even, the following conjecture was proposed by N. Alon and M. Tarsi [AT]:

**Conjecture 2.1.** When $n$ is even, we have $\text{els}(n) \neq \text{ols}(n)$.

Glynn proved that this conjecture is true, when $n = p - 1$ with a prime $p$:

**Theorem 2.2.** For any prime $p$, we have $\text{els}(p - 1) \neq \text{ols}(p - 1)$.

This is deduced from Theorem 1.1 by looking at the coefficient corresponding to the all-ones matrix $J_n = (1)_{1 \leq i,j \leq n} \in \Psi(n)$. Indeed we have the relation

$$C_{J_n} = (-)^{n(n-1)/2}(\text{els}(n) - \text{ols}(n))$$

as a corollary of the relation between the ordinary parity and the symbol parity of Latin squares given in [J].

**Remark.** The Alon–Tarsi conjecture is also proved when $n = p + 1$ with an odd prime $p$ [D]. This conjecture is related to various problems including Rota’s basis conjecture. See [FM] for the results related to this conjecture.

3. **There exists a zero coefficient, when $m \neq p - 1$**

Let us prove Theorem 1.2. This theorem was first found in Master’s thesis of the second author [S].

Let $m$ be a natural number which cannot be expressed as $m = p - 1$ with a prime $p$. We can specifically find $L \in \Psi(m)$ satisfying $C_L = 0$ as follows. When $n = 3$, we consider the following 3 by 3 matrix:

$$L_3(a,b) = \begin{pmatrix} ab + b - 1 & a & 1 \\ a & ab & b \\ 1 & b & ab + a - 1 \end{pmatrix}.$$ 

Here, $a$ and $b$ are natural numbers satisfying $(a+1)(b+1) = m+1$ (there exist such $a$ and $b$, because $m+1$ is a composite number). For general $n \geq 3$, we consider the following $n$ by $n$ matrix:

$$L_n(a,b) = \begin{pmatrix} L_3(a,b) \\ & \ddots \\ & & m \end{pmatrix}.$$
Then the coefficient corresponding to this matrix is zero:

**Proposition 3.1.** When \( n \geq 3 \), we have \( C_{L_n(a,b)} = 0 \).

This proposition follows from the following two lemmas. Firstly, \( C_{L_n(a,b)} \) is expressed as the difference of two multinomial coefficients:

**Lemma 3.2.** We have

\[
C_{L_n(a,b)} = (-)^{a+b+1} \begin{pmatrix} m \\ (ab - 1, 0, 0, a, b, 1) \end{pmatrix} + (-)^{a+b} \begin{pmatrix} m \\ (ab, 1, 1, a - 1, b - 1, 0) \end{pmatrix},
\]

where

\[
\begin{pmatrix} m \\ (m_1, m_2, m_3, m_4, m_5, m_6) \end{pmatrix} = \frac{m!}{m_1! m_2! m_3! m_4! m_5! m_6!}.
\]

Secondly, these two multinomial coefficients are equal to each other:

**Lemma 3.3.** We have

\[
\begin{pmatrix} m \\ (ab - 1, 0, 0, a, b, 1) \end{pmatrix} = \begin{pmatrix} m \\ (ab, 1, 1, a - 1, b - 1, 0) \end{pmatrix}.
\]

Lemma 3.3 follows by a direct calculation, and Lemma 3.2 is proved as follows:

**Proof of Lemma 3.2** First we consider the case of \( n = 3 \). We put

\[
\begin{align*}
\alpha_1 &= x_{11}x_{22}x_{33}, & \alpha_2 &= x_{12}x_{25}x_{31}, & \alpha_3 &= x_{13}x_{21}x_{32}, \\
\beta_1 &= x_{12}x_{21}x_{33}, & \beta_2 &= x_{11}x_{25}x_{32}, & \beta_3 &= x_{13}x_{22}x_{31},
\end{align*}
\]

such that

\[
\det X = \alpha_1 + \alpha_2 + \alpha_3 - \beta_1 - \beta_2 - \beta_3,
\]

and \((\det X)^m\) is expanded as follows:

\[
(\det X)^m = \sum_{k_1 + k_2 + k_3 + l_1 + l_2 + l_3 = m} (-)^{l_1 + l_2 + l_3} \begin{pmatrix} m \\ (k_1, k_2, k_3, l_1, l_2, l_3) \end{pmatrix} \alpha_1^{k_1} \alpha_2^{k_2} \alpha_3^{k_3} \beta_1^{l_1} \beta_2^{l_2} \beta_3^{l_3}.
\]

Let us determine all 6-tuples \((k_1, k_2, k_3, l_1, l_2, l_3)\) of nonnegative integers satisfying the following relation:

\[
\alpha_1^{k_1} \alpha_2^{k_2} \alpha_3^{k_3} \beta_1^{l_1} \beta_2^{l_2} \beta_3^{l_3} = x^{L_3(a,b)}.
\]

Since the left hand side can be expressed as

\[
\alpha_1^{k_1} \alpha_2^{k_2} \alpha_3^{k_3} \beta_1^{l_1} \beta_2^{l_2} \beta_3^{l_3} = x_{11}^{k_1 + l_1 + k_2 + l_2 + k_3 + l_3} x_{12}^{k_1 + l_1 + k_3 + l_3} x_{13}^{k_1 + l_1 + k_3 + l_3} x_{21}^{k_2 + l_2 + k_3 + l_3} x_{22}^{k_2 + l_2 + k_3 + l_3} x_{23}^{k_2 + l_2 + k_3 + l_3} x_{31}^{k_2 + l_2 + k_3 + l_3} x_{32}^{k_2 + l_2 + k_3 + l_3} x_{33}^{k_2 + l_2 + k_3 + l_3},
\]

this relation is equivalent with the following system of nine linear equations:

\[
\begin{pmatrix}
(k_1 + l_2, k_2 + l_1, k_3 + l_3) \\
(k_3 + l_1, k_1 + l_3, k_2 + l_2) \\
(k_2 + l_3, k_3 + l_2, k_1 + l_1)
\end{pmatrix} = L_3(a,b) = \begin{pmatrix}
ab + b - 1 & a & 1 \\
ab & b & \frac{ab}{a - b + 1} \\
1 & b & ab + a - 1
\end{pmatrix}.
\]

Solving this, we see that \((k_1, k_2, k_3, l_1, l_2, l_3) \in \mathbb{Z}_{\geq 0}^6\) satisfying (3.2) are

\((ab - 1, 0, 0, a, b, 1), (ab, 1, 1, a - 1, b - 1, 0)\).
Therefore, comparing (1.1) and (3.1), we have

\[
C_{L_3(a,b)} = (-)^{a+b+1} \left( \frac{m}{ab - 1, 0, 0, a, b, 1} \right) + (-)^{(a-1)+(b-1)+0} \left( \frac{m}{ab, 1, 1, a - 1, b - 1, 0} \right),
\]

namely the assertion in the case of \( n = 3 \).

The case of \( n > 3 \) is also almost the same. Indeed, to calculate \( C_{L_n(a,b)} \), we need to look at the following relation instead of (3.2):

\[
\prod_{1 \leq i \leq m} x_{1\sigma_i(1)} x_{2\sigma_i(2)} \cdots x_{n\sigma_i(n)} = x^{L_n(a,b)}.
\]

Since \( \sigma_1, \ldots, \sigma_m \) satisfying this relation belong to

\[
\{ \sigma \in S_n \mid \sigma(k) = k \text{ for any } k = 4, 5, \ldots, n \} \simeq S_3,
\]

the proof is reduced to the case of \( n = 3 \). \( \square \)

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