Further results on some singular linear stochastic differential equations

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ABSTRACT
A class of Volterra transforms, preserving the Wiener measure, with kernels of Goursat type is considered. Such kernels satisfy a self-reproduction property. We provide some results on the inverses of the associated Gramian matrices which lead to a new self-reproduction property. A connection to the classical reproduction property is given. Results are then applied to the study of a class of singular linear stochastic differential equations together with the corresponding decompositions of filtrations. The studied equations are viewed as non-canonical decompositions of some generalized bridges.

Keywords: Brownian motion; Canonical decomposition; Enlargement of filtrations; Goursat kernels; Gramian matrices; Self-reproducing kernels; Stochastic differential equations; Volterra transform.

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1. Introduction and preliminaries

Gaussian enlargement of filtrations has been extensively studied between the late 70’s and the early 90’s, see [10], [20], [21], [22] and the references therein. Results stemming from the Gaussian nature of the underlying generalized Gaussian bridges are of interest not only in probability, also in financial mathematics, since they have appeared in an insider trading model developed in [6] and [25]. Transforms of
Volterra type allow to construct interesting families of Gaussian processes. Volterra-transforms are classified, both from the theory and applications points of view, according to whether their kernels are square-integrable or not. Those with square-integrable kernels play a crucial role in the study of equivalent Gaussian measures, stochastic linear differential equations and the linear Kalman-Bucy filter, see [18] and [24]. To our knowledge, comparably, less interest was given to Volterra transforms with non-square-integrable kernels. Such transforms naturally appear, for instance, in non-canonical representations of some Gaussian processes. They also appear if one forces such transforms to preserve the Wiener measure. The most known examples have corresponding kernels of Goursat type. A few nontrivial ones originate from P. Lévy, see [26], [27], and serve as a standard reference for showing the importance of the canonical decomposition of semi-martingales. Such constructions have been enriched by people from the Japanese school, see [15], [17] and [19].

Let us now fix the mathematical setting and summarize results of this paper. We take $B := (B_t, t \geq 0)$ to be a standard Brownian motion, defined on a complete probability space $(\Omega, \mathcal{F}, P_0)$. Denote by $\{\mathcal{F}_t^B, t \geq 0\}$ the filtration it generates. Let $f = (f_1, \cdots, f_n)^* \in L^2_{\text{loc}}(\mathbb{R}_+) = \{h; \int_0^t h^2(s) ds < \infty, \text{ for all } t \in [0, \infty)\}$, where $*$ stands for the transpose operator and $n$ is a natural number. Although some of our results extend readily to the cases when $n = \infty$, to simplify the study, we only consider the cases where $n$ is finite. We assume that, for any fixed $t > 0$, the covariance matrix $m_t$, of the Gaussian random variable $\int_0^t f(s) dB_s$, is invertible, i.e., the Gramian matrix $m_t = \int_0^t f(s) \cdot f^*(s) ds$ has an inverse $\alpha_t$. We emphasize that, under the aforementioned condition, it is not difficult to see that $\alpha_t \to \alpha_\infty$, as $t \to \infty$, where $\alpha_\infty$ is a finite matrix. Furthermore, for any $i$, $(\alpha_\infty)_{ij} = 0$ for all $j$, if and only if $\|f_i\| := (\int_0^\infty f_i^2(s) ds)^{1/2} = \infty$. With $\phi(t) = \alpha_t \cdot f(t)$ for $t > 0$, we shall establish in Theorem 2.2 that $(\alpha_t, t > 0)$ is given in terms of $\phi$ by $\alpha_t = \int_t^\infty \phi(u) \cdot \phi^*(u) du + \alpha_\infty$, for any $t > 0$. This relation has its own right of importance in this work and may have interesting applications to other fields where Gramian matrices together with
their inverses are of prime importance, see for instance [8] and the references therein. In particular, we also refer to [3] for applications to the theory of special functions and to [4] and [5] for applications to reproducing kernel-Hilbert spaces and spline functions.

We define the Volterra transform \( \Sigma \), associated to a Volterra kernel \( k \), on the set of continuous semi-martingales \( X \) such that

\[
(1) \quad \lim_{\varepsilon \to 0} \int_{\varepsilon}^{t} \int_{0}^{v} k(u,v) \, dX_u \, dv < \infty, \quad 0 < t < \infty \quad \text{a.s.,}
\]

by

\[
(2) \quad \Sigma(X)_t = X_t - \int_{0}^{t} \int_{0}^{u} k(u,v) \, dX_v \, du, \quad 0 < t < \infty.
\]

Following [15], the kernel \( k(t,s) = \phi^*(t) \cdot f(s) \), for \( 0 < s \leq t < +\infty \) is a self-reproducing Volterra kernel. That is equivalent to saying that \( \Sigma \), when applied to the Brownian motion \( B \), satisfies the following two conditions:

(i) \( \Sigma(B) \) is a standard Brownian motion;

(ii) For any fixed \( t \geq 0 \), \( \mathcal{F}^{\Sigma(B)}_t \) is independent of \( \int_{0}^{t} f(u) \, dB_u \).

Existence of \( \Sigma(B) \) may be justified by using a generalized Hardy inequality discovered in [15], see Remark 2.3 given below. We call \( k \) and \( \Sigma \), respectively, a Goursat-Volterra kernel and transform, with reproducing basis \( f \). The dimension of \( \text{Span}\{f\} \) is called the order of the Goursat-Volterra kernel \( k \). This terminology is formally fixed in Definition 2.1.

Next, we bring our focus on conditions (i) and (ii) and think of them in terms of enlargement of filtrations and stochastic differential equations. Condition (ii) says that the orthogonal decomposition

\[
(3) \quad \mathcal{F}^{B}_t = \mathcal{F}^{\Sigma(B)}_t \otimes \sigma \left( \int_{0}^{t} f(u) \, dB_u \right)
\]

holds true, for any \( t \geq 0 \). Here, by \( \mathcal{F} \otimes \mathcal{G} \) we mean \( \mathcal{F} \vee \mathcal{G} \) with independence between \( \mathcal{F} \) and \( \mathcal{G} \). We shall show that, for Goursat-Volterra transforms, equation (3) can in
fact be rewritten as

(4) \[ \mathcal{F}^B_t = \mathcal{F}^{\Sigma(B)}_t \otimes \sigma \left( Y - \int_t^\infty \phi(u) \, d\Sigma(B)_u \right) \]

valid for any \( t \geq 0 \), where \( Y = (Y_1, \cdots, Y_N)^* \) is a Gaussian random vector which is independent of \( \mathcal{F}^{\Sigma(B)}_\infty \) with covariance matrix \( E[Y \cdot Y^*] = \alpha_\infty = \lim_{t \to \infty} \alpha_t \) in case \( \alpha_\infty \neq 0 \), and \( Y \equiv 0 \) otherwise. We allow here \( Y \) to have some null or constant components. Going back to condition (i), we observe that the determination of all continuous semi-martingales which satisfy it amounts to solving equation

(5) \[ X_t = W_t + \int_0^t \int_0^s \phi^*(s) \cdot f(u) \, dX_u \, ds, \quad X_0 = 0, \quad t > 0, \]

considered on a possibly enlarged probability space, where \( W \) is a standard Brownian motion. Note that we only assume

(6) \[ \lim_{\varepsilon \to 0} \int_0^t \int_0^\varepsilon \phi^*(v) \cdot f(u) \, dX_u \, dv < \infty, \quad 0 < t < \infty \quad \text{a.s.}, \]

and the latter is not absolutely convergent. Because of the singularity at time 0, we call (5) a singular linear stochastic differential equation. If we take \( W = \Sigma(B) \) then, by construction, the original Brownian motion \( B \) is one solution. A second one coincides with the associated \( f \)-generalized bridge on the interval of its finite life-time, introduced in [1]. It follows that the Goursat-Volterra transform \( \Sigma \), when defined as above, is not invertible in the sense that (5) has many solutions. This is not a surprising fact. Indeed, \( k \) being a self-reproducing kernel implies that it is not square-integrable, as seen in [11]. Next, Theorem 3.1 deals with the investigation of all continuous semi-martingale solutions to (5). In particular, we show that a necessary and sufficient condition for the existence of a strong solution that is Brownian and \( \mathcal{F}^B_\infty \)-measurable is \( \alpha_\infty \equiv 0 \). In that case \( \mathcal{F}^{\Sigma(B)}_\infty = \mathcal{F}^B_\infty \). When \( \alpha_\infty \neq 0 \), Theorem 3.1 concludes that there exists still a strong solution which is a Brownian motion, in an enlarged space, that involves an independent centered Gaussian vector \( Y \) with covariance matrix \( \alpha_\infty \). Another natural question is a characterization of all continuous semi-martingales that satisfy both conditions (i) and (ii). This is
partially solved in Theorem 4.1 for the case $\alpha_\infty \equiv 0$ and the analysis exhibits some connections to certain space-time harmonic functions. The latter are functions $h \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+)$ such that $h(\cdot, \int_0^s f^*(s) dB_s)$ is a continuous $(\mathbb{P}_0, \mathcal{F})$-martingale with expectation 1, where $\mathbb{P}_0$ stands for the Wiener measure.

The main results of this paper extend a part of the first chapter of [30] and some results found in [22]. Our work offers explicit examples of conditionings and conditioned stochastic differential equations introduced and studied in [7]. Furthermore, singular equations of type (5) and the progressive enlargement of filtration given in Corollary 3.2 can easily be applied to insider trading models elaborated in [6], [7] and [25].

2. Goursat-Volterra kernels and transforms

To a Brownian motion $B$ we associate the centered Gaussian process $\Sigma(B)$ defined by (2), which we assume is well-defined, where $k$ is a continuous Volterra kernel. That is to say that $k : \mathbb{R}^2_+ \to \mathbb{R}$ satisfies

$$k(u, v) = 0, \quad 0 < u \leq v < \infty$$

and is continuous on $\{(u, v) \in (0, +\infty) \times (0, +\infty) : u > v\}$. We know from [11] that $\Sigma$ preserves the Wiener measure, or $\Sigma(B)$ is a Brownian motion, if and only if $k$ satisfies the self-reproducing property

$$k(t, s) = \int_0^s k(t, u)k(s, u) du, \quad 0 < s \leq t < \infty. \quad (7)$$

For a connection with reproducing kernels, in the usual sense, we refer to end of this section. Observe that (2), when applied to $B$, can be viewed as the semi-martingale decomposition of $\Sigma(B)$ with respect to the filtration $(\mathcal{F}_t^B, t \geq 0)$. Now, as a consequence of the Doob-Meyer decomposition of $\Sigma(B)$ in its own filtration, we must have the strict inclusion

$$\mathcal{F}_t^{\Sigma(B)} \subsetneq \mathcal{F}_t^B, \quad 0 < t < \infty.$$
It is shown in [22] that the missing information, called the reproducing Gaussian space, is given in the orthogonal decomposition

\[ F^B_t = \mathcal{F}_t^{\Sigma(B)} \otimes \sigma(\Gamma^{(k)}_t), \]

where

\[ \Gamma^{(k)}_t = \left\{ \int_0^t f(u) \, dB_u; f \in L^2((0,t]), f(s) = \int_0^s k(s,u) f(u) \, du \quad \text{a.e.} \right\} \]

for any \( t > 0 \). Given a kernel \( k \), it is not an easy task to determine a basis of \( \Gamma^{(k)}_t \) for each fixed \( t > 0 \), because this amounts to solving explicitly the integral equation

\[ f(t) = \int_0^t k(t,u) f(u) \, du, \quad 0 < t < \infty. \]

It is easier to fix the family of spaces \( (\Gamma^{(k)}_t, t > 0) \) and work out the corresponding Volterra kernel. This procedure, in fact, corresponds to desintegrating the Wiener measure over the interval \([0,t]\), for any fixed \( t > 0 \), along \( \Gamma^{(k)}_t \). Recall that a Goursat kernel is a kernel of the form

\[ k(t,s) = \phi^*(t) \cdot f(s), \quad 0 < s \leq t < \infty, \]

where \( \phi = (\phi_1, \cdots, \phi_n)^* \) and \( f = (f_1, \cdots, f_n)^* \) are two vectors of functions defined on \((0,\infty)\) and \( n \in \mathbb{N} \). For such kernels it is natural to introduce the following definition.

**Definition 2.1.** A Goursat-Volterra transform \( \Sigma \) of order \((n_t, t > 0)\) is a Volterra transform preserving the Wiener measure such that, for any Brownian motion \( B \) and \( t > 0 \), \( \mathcal{F}_t^{\Sigma(B)} \) is independent of \( \int_0^t f(u) \, dB_u \) for some vector \( f \equiv (f_1, \cdots, f_{n_t})^* \) of \( n_t \) linearly independent \( L^2_{\text{loc}}(\mathbb{R}_+) \) functions. The associated kernel is called a Goursat-Volterra kernel. The objects \( f \), \( \text{Span}\{f\} \) and \( \text{Span}\{\int_0^s f(s) \, dB_s\} \) are called reproducing basis, space and Gaussian space, respectively.

Because for each fixed \( t > 0 \), \( m_t \) is positive definite, it can be seen that \( t \rightarrow n_t \) is nondecreasing. However, in our setting, we always take the order to be constant
and finite. The simplest known example of Goursat-Volterra kernels is \( k_1(t, s) = t^{-1} \) and this gives
\[
\Sigma(B) = B - \int_0^B \frac{B_u}{u} \, du.
\]
That corresponds to setting \( n = 1 \) and taking \( f_1 \equiv 1 \). It is observed in [22] that \( \Sigma \) when iterated takes a remarkably simple form. That is with \( \Sigma^{(0)} = \text{Id}, \Sigma^{(1)} = \Sigma \) and \( \Sigma^{(m)} = \Sigma^{(m-1)} \circ \Sigma \), for \( m \geq 2 \), where \( \circ \) stands for the composition operation, we have
\[
\Sigma^{(n)}(B) = \int_0^B L_n(\log \frac{1}{s}) \, dB_s,
\]
where \((L_n, n \in \mathbb{N})\) is the sequence of Laguerre polynomials. As a generalization of the above kernel, we quote the following result from [15].

**Theorem 2.1** (Hibino-Hitsuda-Muraoka, [15]). Let \( f \) be a vector of \( n \) functions of \( L^2_{\text{loc}}(\mathbb{R}_+) \) such that for any \( t > 0 \) the Gramian matrix \( m_t = \int_0^t f(s) \cdot f^*(s) \, ds \) has an inverse denoted by \( \alpha_t \). Then, with \( \phi(\cdot) = \alpha \cdot f(\cdot) \), the kernel \( k \), defined by \( k(t, s) = 0 \) if \( s > t \) and \( k(t, s) = \phi^*(t) \cdot f(s) \) otherwise, is a Goursat-Volterra kernel of order \( n \).

For a proof of this result, we refer to [15]. Some arguments of the proof are sketched in Remark [2.3] given below. In the remainder of this paper, unless otherwise specified, we work under the setting of Theorem 2.1. The objective of the next result is to obtain an expression of \( \alpha \) in terms of \( \phi(\cdot) \). As a straightforward application, we shall show that it allows to obtain a new self-reproducing property satisfied by the kernel \( k \). To our knowledge the following result is not known.

**Theorem 2.2.** \( \alpha_t \) converges to a finite matrix \( \alpha_\infty \) as \( t \to \infty \). Moreover, we have
\[
\alpha_t = \int_t^\infty \phi(u) \cdot \phi^*(u) \, du + \alpha_\infty, \quad 0 < t < \infty.
\]
Consequently, the self-reproduction property
\[
k(t, s) = \int_t^\infty k(u, t)k(u, s) \, du + f^*(t) \cdot \alpha_\infty \cdot f(s), \quad 0 < s \leq t < \infty,
\]
holds true.
Proof. Fix $t > 0$. Observe that the matrices $\alpha_t$ and $m_t$ are symmetric positive definite with absolutely continuous entries. Next, the identity $\alpha_t \cdot m_t = I_d = m_t \cdot \alpha_t$, when differentiated, yields $\alpha'_t \cdot m_t = -\alpha_t \cdot m'_t$. It follows that

$$
\phi(t) \cdot f^*(t) = \alpha_t \cdot f(t) \cdot f^*(t) = \alpha_t \cdot m'_t = -\alpha'_t \cdot m_t.
$$

Consequently, we have $\alpha'_t = -\phi(t) \cdot f^*(t) \cdot \alpha_t = -\phi(t) \cdot \phi^*(t)$. For any $1 \leq j \leq n$, $(\alpha'_t)_{j,j} = -\phi^2_j(t)$ is negative. Hence, $(\alpha_t)_{j,j}$ is decreasing. Because $(\alpha_t)_{j,j} > 0$ we get that $\int_r^\infty \phi^2_j(s) \, ds < \infty$, $r > 0$. Since, for $t \geq r$, we can write $\alpha_t = \alpha_r - \int_r^t \phi(s) \cdot \phi^*(s) \, ds$, by letting $t \to +\infty$, we find $\lim_{t \to \infty} \alpha_t = \alpha_r - \int_r^\infty \phi(s) \cdot \phi^*(s) \, ds = \alpha_\infty$. Thus, $\alpha_\infty$ is a matrix with finite entries. The last statement follows from $k(t, s) = f^*(t) \cdot \alpha_t \cdot f(s)$ where we use the expression of $\alpha_t$ given in (8).

Self-reproducing kernels, in particular Goursat-Volterra kernels, are different from but related to kernel systems and reproducing kernel Hilbert spaces. Our next objective is to outline this connection. For, let us start by fixing a time interval $[0, t]$, for some $t > 0$. Let the vector $q_t(u) := (q_{m,t}(u), 0 < u \leq t; 1 \leq m \leq n)$ be formed by the orthonormal sequence associated to $f_1, f_2, \cdots, f_n$ over the interval $[0, t]$. This system is uniquely characterized by

$$
\int_0^t q_{m,t}(r) q_{k,t}(r) \, dr = \delta_{m,k}, \quad 1 \leq m, k \leq n,
$$

with the requirement that for each integer $1 \leq m \leq n$, $q_{m,t}$ is a linear combination of $f_1, \cdots, f_m$ with a positive leading coefficient associated to $f_m$. We refer to Lemma 6.3.1, p. 294, in [3] for an expression of the latter in terms of a determinant. The classical kernel system is then given by the symmetric kernel

$$
\kappa_t(u, v) = q_t(u) \cdot q_t^*(v), \quad 0 < u, v \leq t.
$$

This is a reproducing kernel in the sense that

$$
\kappa_t(u, v) = \int_0^t \kappa_t(u, r) \kappa_t(v, r) \, dr, \quad 0 < u, v \leq t.
$$
For $1 \leq i, j \leq n$, $(\alpha_t)_{i,j}$ is seen to be the coefficient of $f_i(u)f_j(v)$ in the expansion of $\kappa_t$. To be more precise, $(\alpha_t)_{i,j} = (b_t \cdot b^*_t)_{i,j}$ where $b$ is an upper diagonal matrix whose entry $(b_t)_{i,k}$ is the coefficient of $f_i(u)$ in $q_{k,t}(u)$ for $i \leq k$. We clearly have $\phi_t^2(t) = -2(b'_t \cdot b^*_t)_{i,i}$ for all $i$ and it would be interesting to express the matrix $b_t$ in terms of $\phi(t)$. Now, we are ready to state the following result which proof is omitted.

**Proposition 2.1.** For each fixed $t > 0$, the kernel system associated to $f$, over the time interval $[0,t]$, is given by $\kappa_t(u,v) = \int_0^\infty k(r,u)k(r,v)dr + f^*(u) \cdot \alpha_\infty \cdot f(v)$ for $0 < u, v \leq t$. In particular, we have $k(t,s) = \kappa_t(t,s)$ for all $0 < s \leq t < \infty$.

**Proof.** As in the proof of Theorem 2.2, the first part of the result follows from the well-known relationship $\kappa_t(u,v) = f^*(u) \cdot \alpha_t \cdot f(v)$ for any $0 < u, v \leq t$. The second part follows by taking the limit and using continuity. □

**Remark 2.1.** To see an example where $\alpha_\infty \not\equiv 0$, let us discuss the case $n = 2$. Assume that $f_1$ and $f_2$ are two functions in $L^2_{loc}(\mathbb{R}^+)$. We distinguish four cases and three different forms for $\alpha_\infty$. The first corresponds to $\alpha_\infty \equiv 0$ when $\|f_1\| = \|f_2\| = +\infty$. The second corresponds to case when $\|f_1\|$ and $\|f_2\|$ are finite which implies that $\alpha_\infty$ is positive-definite. Observe that the off-diagonal entries are zero only when $\int_0^\infty f_1(s)f_2(s)ds = 0$. The latter integral is zero if, for instance, we take $f_1 = \varphi - \psi$ and $f_2 = \varphi + \psi$, where $\|\varphi\| = \|\psi\| < \infty$. In the third case, all the entries of $\alpha_\infty$ are zero but $(\alpha_\infty)_{1,1} = 1/\|f_1\|^2$ if $\|f_1\| < +\infty$ and $\|f_2\| = +\infty$. The remaining case is similar by symmetry.

**Remark 2.2.** We shall now discuss examples of kernels of order $n$, $n \in \mathbb{N}$, which reproducing spaces are Müntz spaces. We refer to [2] for proofs of results given below. Take $f_i(s) = s^{\lambda_i}$, $i = 1, 2, \cdots$, where $\Lambda = \{\lambda_1, \lambda_2, \cdots\}$ is a sequence of reals such that $\lambda_i \neq \lambda_j$ for $i \neq j$ and $\lambda_i > -1/2$. For a fixed $n < \infty$, the kernel $k_n$ defined
by \( k_n(t, s) = 0 \) if \( s > t \) and
\[
(10) \quad k_n(t, s) = t^{-1} \sum_{j=1}^{n} a_{j,n} (s/t)^{\lambda_j}, \quad a_{j,n} = \frac{\prod_{i=1}^{n} (\lambda_i + \lambda_j + 1)}{\prod_{i=1, i\neq j}^{n} (\lambda_i - \lambda_j)}, \quad j = 1, \ldots, n,
\]
if \( 0 < s \leq t \), is a Goursat-Volterra kernel of order \( n \). Its reproducing Gaussian space, at time \( t > 0 \), is \( \text{Span}\{ \int_0^t s^i dB_s; i = 1, 2, \ldots, n \} \). Going back to the Gramian matrix \((m_t, t \geq 0)\), observe that it has the entries
\[
(m_t)_{i,j} = (\lambda_i + \lambda_j + 1)^{-1} \frac{1}{t^{\lambda_i + \lambda_j + 1}}, \quad i, j = 1, \ldots, n.
\]
Thus if \( t = 1 \) then \( m_1 \) is a Cauchy matrix. When \( \lambda_i = ci \), for some constant \( c \neq 0 \), and \( n = \infty \), \( m_1 \) is the well-known Hilbert matrix. Note that because \( ||f_i|| = +\infty \), \( i = 1, \ldots, n \), we have \( \alpha_\infty \equiv 0 \). So we have \( \phi_i(t) = a_{i,n} t^{-\lambda_i - 1}, \ i = 1, 2, \ldots, n \).
Furthermore, the entries of \( \alpha_t \) are given by
\[
(\alpha_t)_{i,j} = a_{i,n} a_{j,n} (\lambda_i + \lambda_j + 1)^{-1} \frac{1}{t^{\lambda_i} - \lambda_j - 1}, \quad i, j = 1, \ldots, n,
\]
which follows from the expression of the kernels when compared with Theorem 2.2.

Note that \( \alpha_t \), for \( t \neq 1 \), can easily be constructed from \( \alpha_1 \) which is known and can be found in [29]. Finally, we mention that some results are obtained about infinite order kernels in the Müntz case, see [2] and [16].

**Remark 2.3.** Observe that we can write
\[
\Sigma(B_t) = \int_0^\infty (I - K_f^*) 1_{[0,t]}(u) dB_u, \quad 0 < t < \infty
\]
where \( K_f^* \) is the adjoint of the bounded integral operator \( K_f \) defined on \( L^2_{\text{loc}}(\mathbb{R}_+) \) by
\[
K_f \alpha(t) = \int_0^t k(t, r) \alpha(r) dr, \quad \alpha \in L^2_{\text{loc}}(\mathbb{R}_+).
\]
That \( I - K_f \) is a partial isometry, with initial subspace \( L^2_{\text{loc}}(\mathbb{R}_+) \ominus \text{Span}\{f\} \) and final subspace \( L^2_{\text{loc}}(\mathbb{R}_+) \), follows from the generalized Hardy inequality
\[
||K_f \alpha|| \leq 2 ||\alpha||, \quad \alpha \in L^2_{\text{loc}}(\mathbb{R}_+).
\]
Consequently, the operator $I - K^*_f$, when defined on $L^2_{loc}(\mathbb{R}_+)$, is isometric which implies the statement of Theorem 2.1. For the above results, we refer to [15]. We also refer to the comments of Section 3 therein because here we are working with $L^2_{loc}(\mathbb{R}_+)$ instead of $L^2_{loc}([0,1])$.

**Remark 2.4.** Many authors work under the condition

\begin{equation}
\int_0^t \left( \int_0^u k^2(u,v) \, dv \right)^{1/2} \, du < \infty
\end{equation}

for all $t > 0$, which is sufficient for $\Sigma(B)$, where $B$ is a standard Brownian motion, to be well-defined, see for instance [11]. However, condition (11) is too strong for $\Sigma(B)$ to be well-defined. To see that let us fix $b \in L^2_{loc}(\mathbb{R}_+)$. The associated Goursat-Volterra kernel of order 1 is then found to be

$$k(t, v) = b(t)b(v) \sqrt{\int_0^t b^2(r) \, dr}.$$ 

It satisfies (11) if and only if $\int_0^t |b(s)|/\left(\int_0^s b^2(r) \, dr\right)^{1/2} \, ds < \infty$ for all $t < \infty$. For example, the kernel associated to $b(t) = t^{-1}e^{-1/t}$ fails to satisfy (11).

**3. On some singular linear stochastic differential equations**

Consider the singular linear stochastic equation (5). Our interest lies in the set of all its continuous semi-martingale solutions which may be defined on a possibly enlarged space. For a particular solution $X$, we recall that (5) is well-defined in the sense that (6) holds. If we set $W = \Sigma(B)$, where $B$ is a Brownian motion, then the set includes at least two solutions which one shall now briefly describe. First, $B$ is a solution. Second, there is a solution which is defined on $\mathbb{R}_+$ and coincides with the $f$-generalized bridge over its life time. The latter process, denoted by $(B_u^y, u \leq t_1)$, for some $t_1 > 0$ and a column vector of reals $y$, is defined by

$$B_u^y = B_u - \psi^*(u) \cdot \int_0^{t_1} f(s) \, dB_s + \psi^*(u) \cdot y, \quad 0 < u < t_1,$$
where $\psi$ is the unique solution to the linear system

$$
\int_0^u f(s) \, ds = \psi(u) \cdot \int_0^{t_1} f(s) \cdot f^*(s) \, ds = \psi(u) \cdot m_{t_1}, \quad 0 < u < t_1.
$$

Thus $\psi(u) = \alpha_{t_1} \cdot \int_0^u f(s) \, ds$ which implies that $\int_0^{t_1} f(s) \, dB^y_s = y$, since $\alpha_{t_1}$ is the inverse of $m_{t_1}$. This is the reason why the above process is called an $f$-generalized bridge over $[0, t_1]$ with endpoint $y$. Now, we have $\Sigma(B^y) = \Sigma(B)$ which is true because $\Sigma$ is linear and $\Sigma(\int_0^t f(r) \, dr) \equiv 0$ since $f(t) = \int_0^t k(t, v)f(v) \, dv$ for all $0 < t < \infty$. This shows that $B^y$ is also a solution to (5) which, in fact, is a non-canonical decomposition. For further results on these processes, such as their canonical decomposition in their own filtrations, we refer to [1]. Now, we consider equation (5) where the driving Brownian motion $W$ is taken to be arbitrary.

**Theorem 3.1.** 1) $X$ solves equation (5) if and only if there exists a random vector $Y = (Y_1, \cdots, Y_n)^*$ such that

$$
X = X^0 + \int_0^u f^*(u) \, du \cdot Y
$$

where

$$
X^0 = W - \int_0^\infty \int_u^\infty \phi^*(v) \cdot f(u) \, dW_v \, du.
$$

In terms of $X$, $Y$ is given by $Y = \lim_{t \to \infty} \alpha_t \cdot \int_0^t f(u) \, dX_u$.

2) $X^0$ is a Brownian motion if and only if $\alpha_\infty \equiv 0$. In case $\alpha_\infty \neq 0$, a process $X$ solving equation (5) is a Brownian motion if and only if $Y$ is centered Gaussian with covariance matrix $\alpha_\infty$ and is independent $\mathcal{F}_X^{X^0}$. 

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Proof. 1) We proceed by checking first that $X^0_t$ is a particular solution to (5). Using the stochastic Fubini theorem, found for instance in [28], we perform the decompositions

$$X^0_t - \int_0^t \int_0^u k(u, v) dX^0_v du = W_t - \int_0^t \int_u^\infty k(v, u) dW_v du - \int_0^t \int_0^u k(u, v) \left( dW_v - \int_v^\infty k(\rho, v) dW_\rho d\rho \right) dv$$

Since $k$ is self-reproducing, the last four terms in the last equation cancel showing that $X^0_t$ solves (5). Next, if $X$ is a solution then by setting $X = X^0 + Z$ we see that $Z$ has to satisfy

$$dZ_r = \int_0^r k(r, v) dZ_v dv, \quad 0 < r < \infty.$$ 

Multiplying both sides by $f(r)$ and integrating with respect to $r$, along $[0, t]$, yields

$$\int_0^t f(v) dZ_v = \int_0^t f(v) \phi^*(v) \cdot \int_0^v f(r) dZ_r dv$$

where we used the expression of $\alpha'$ given in the proof of Theorem 2.2 to obtain the last equality. Because $\alpha$ is the inverse of $m$, the latter relation can be written as

$$\frac{d}{dt} \alpha_t \cdot \int_0^t f(s) dZ_s = 0.$$ 

This, when integrated, yields $\alpha_t \cdot \int_0^t f(s) dZ_s = Y$ for some random vector $Y$. Hence $\int_0^t f(r) dZ_r = m_t \cdot Y$ which implies that $Z_t = Y^* \cdot \int_0^t f(s) ds$. This completes the proof of the first part of the first assertion. For the second part,
by using Theorem 2.2 we obtain

\[ \phi(t)\,dW_t = \phi(t)\,dX_t - \phi(t)\phi^*(t) \cdot \int_0^t f(u)\,dX_u\,dt \]
\[ = \alpha_t \cdot d\left( \int_0^t f(u)\,dX_u \right) - \phi(t)\phi^*(t) \cdot \int_0^t f(u)\,dX_u\,dt \]
\[ = d\left( \alpha_t \cdot \int_0^t f(u)\,dX_u \right). \]

Integrating on both sides over \([s,t]\) we obtain

\[ \int_s^t \phi(u)\,dW_u = \alpha_t \cdot \int_s^t f(u)\,dX_u - \alpha_s \cdot \int_s^t f(u)\,dX_u. \]

Next, observe that as \(t \to \infty\) the left hand side converges almost surely. So the right hand side converges as well to some limit which we denote by \(\tilde{Y}\). To be more precise, setting

\[ \tilde{Y} = \lim_{t \to \infty} \alpha_t \cdot \int_0^t f(u)\,dX_u, \]

we have shown that

\[ (13) \quad \int_t^\infty \phi(u)\,dW_u = \tilde{Y} - \alpha_t \cdot \int_0^t f(u)\,dX_u, \quad 0 < t \leq \infty. \]

Consequently, we have

\[ \int_0^t \int_u^{\infty} f^*(u) \cdot \phi(v)\,dW_v\,du = \tilde{Y}^* \cdot \int_0^t f(u)\,du \]
\[ = \int_0^t f^*(u) \cdot \alpha(u) \cdot \int_u^{\infty} f(v)\,dX_v\,du \]
\[ = \int_0^t \int_0^u \phi^*(u) \cdot f(v)\,dX_v\,du. \]

Thus, we have

\[ \int_0^t \int_u^{\infty} k(v,u)\,dW_v\,du = -\tilde{Y}^* \cdot \int_0^t f(u)\,du \]
\[ = -\int_0^t \int_0^u k(u,v)\,dX_v\,du \]
\[ = W_t - X_t. \]
Comparing with previous calculations yields $Y = \tilde{Y}$, $\mathbb{P}_0$-almost surely.

2) Theorem 2.2 implies that

$$(14) \quad E[X^0_0 X^0_t] = s \wedge t - \int_0^{s \wedge t} \int_0^t f^* s(r) \cdot \alpha_\infty \cdot f(v) \, dv \, dr.$$  

This clearly shows that $X^0$ is a Brownian motion if and only if $\alpha_\infty \equiv 0$. Next, if $X$ is as prescribed then by virtue of (14), and the fact that $\alpha_\infty$ is the covariance matrix of $Y$, we have

$$E[X_0 X_t] = s \wedge t - \int_0^s \int_0^t f^*(u) \cdot \alpha_\infty \cdot f(v) \, dv \, du$$

$$+ \int_0^s \int_0^t E[(Y^* \cdot f(u))(Y^* \cdot f(v))] \, dv \, du$$

$$= s \wedge t.$$  

Because $X$ is a continuous Gaussian process we conclude that it is a Brownian motion. Conversely, if $X$ is a Brownian solution to (5) then it has to be of the form (12). By virtue of the orthogonal properties of Goursat-Volterra transform, we see that $Y$ is independent of $F_{\Sigma(X)} = F_W^t$ for any fixed $t > 0$. Next, by letting $t$ go to $\infty$, we get that $Y$ is independent of $F_{X^0} \subseteq F^W$. Thus, $Y$ is Gaussian vector, with covariance matrix $\alpha_\infty$, which is independent of $F_{X^0_\infty}$ as required.  

Thanks to the importance of the symmetric matrix $\alpha_\infty$, for instance in Theorem 2.2, it is natural to look for a description of its structure. The following result, which is hidden in the proof of Theorem 3.1, gives a necessary and sufficient condition for a column or a row to be zero.

**Corollary 3.1.** For $1 \leq i \leq n$, $(\alpha_\infty)_{i,j} = (\alpha_\infty)_{j,i} = 0$ for all $j$, if and only if $\|f_i\| = \infty$.

**Proof.** For a fixed $t > 0$, $\alpha_t$ is the covariance matrix of $\alpha_t \cdot \int_0^t f(s) \, dB_s$. Furthermore, due to Theorem 3.1 we conclude that $\alpha_t \cdot \int_0^t f(s) \, dB_s$ converges to a Gaussian vector $Y$, possibly with some null components, such that $E(Y \cdot Y^*) = \alpha_\infty$. Thus, $Y_i \equiv 0$ for some $i$ if and only if $(\alpha_\infty)_{i,i} = 0$ and if and only if $\|f_i\| = \infty$. Now, $(\alpha_\infty)_{i,i} = 0$
if and only if \((\alpha_\infty)_{i,j} = 0\) for all \(j\). In order to see that, we let \(t \to \infty\) and use continuity in the well-known inequality \(\left| (\alpha_t)_{i,j} \right|^2 \leq (\alpha_t)_{i,i}(\alpha_t)_{j,j}\) valid for symmetric positive definite matrices.

Now, we take a look at the orthogonal decompositions of filtrations which arise from Goursat-Volterra transforms and provide their interpretation.

**Corollary 3.2.** The orthogonal decomposition given by [4] holds true. Furthermore, the progressive decomposition

\[
\mathcal{F}_t^B = \mathcal{F}_t^{\Sigma(B)} \otimes \sigma \left( Y - \int_0^t \phi(u) d\Sigma(B)_u \right), \quad 0 < t < \infty
\]

holds true, where \(Y \equiv 0\) if \(\alpha_\infty \equiv 0\) and \(Y\) is a Gaussian vector independent of \(\mathcal{F}_\infty^{\Sigma(B)}\) with covariance matrix \(\alpha_\infty\) otherwise. Thus, we have \(\mathcal{F}_\infty^B = \mathcal{F}_\infty^{\Sigma(B)}\) in case \(\alpha_\infty \equiv 0\) and \(\mathcal{F}_\infty^B = \mathcal{F}_\infty^{\Sigma(B)} \vee \sigma\{Y\}\) otherwise.

**Proof.** For a fixed \(t > 0\), Theorem 3.1 implies that

\[
B_t = \Sigma(B)_t - \int_0^t \int_u^\infty k(v,u) d\Sigma(B)_v du + Y^* \cdot \int_0^t f(u) du
\]

where \(Y\) is a Gaussian vector with covariance \(\alpha_\infty\) which is independent of \(\mathcal{F}_\infty^{\Sigma(B)}\). Hence, we have

\[
\int_0^t f(u) dB_u = m_t \cdot \left( Y - \int_t^\infty \phi(u) d\Sigma(B)_u \right)
\]

which gives

\[
\sigma \left\{ \int_0^t f(u) dB_u \right\} = \sigma \left\{ Y - \int_t^\infty \phi(u) d\Sigma(B)_u \right\}.
\]

This implies the first assertion while the last one follows by letting \(t\) tend to \(+\infty\). \(\Box\)

**Remark 3.1.** Recall that \(\mathcal{F}_0^B\) and \(\mathcal{F}_0^{\Sigma(B)}\) are trivial. So by letting \(t\) converge to 0, in Corollary 3.2 we see that \(\phi^* \in L^2([\varepsilon, \infty)^n)\) for all \(\varepsilon > 0\) but \(\phi_i \notin L^2((0, +\infty))\), for \(i = 1, \cdots n\). This fact can also be shown by a combination of Theorem 2.2 and the inequality

\[
(\alpha_t)_{i,i} \geq 1/(m_t)_{i,i} = 1/\|f_i\|^2
\]
which follows from the orthogonal diagonalization of $m_t$ and may be found in Exercise 8, p. 274, in [12].

**Remark 3.2.** It is clear that if the choice of the vector $f$ allows the use of integration by parts for the integrand in the right hand side of (5) then we obtain a stochastic differential equation which does not involve a stochastic integral. For instance, that is the case for the examples given by P. Lévy, found in [26] and [27]. These go back to around the middle of last century when stochastic integration was not yet world-widely developed.

### 4. Connections to some positive martingales

Let $(k(t,s), t \geq s > 0)$ be a Goursat-Volterra kernel of order $n$, where $n$ is a natural number. Assume that $f$ is a reproducing basis for $k$, or for the associated Volterra transform $\Sigma$, and let us keep the notations used in the Introduction. Consider the singular stochastic differential equation (5) associated to $k$ and driven by a given standard Brownian motion $W$. Our aim here is to describe the set

$$
\Upsilon(k) = \{ \mathbb{P} \text{ is the probability law of a continuous semi-martingale } X \\
on (C([0,\infty), \mathbb{R}), \mathcal{F}_\infty^*) \text{ solving (5)} \text{ s.t. } \Sigma(X) \text{ is a Brownian motion} \\
\text{ and } \mathcal{F}^{\Sigma(X)}_t \text{ is independent of } \int_0^t f(s) \, dX_s, \text{ for all } 0 < t < \infty \}.
$$

We read from Corollary 3.1 that $\alpha_\infty \equiv 0$ if and only if $||f_i|| = \infty$ for all $i$. Now, we are ready to state the following unified characterization of the set $\Upsilon(k)$.

**Theorem 4.1.** If $\alpha_\infty \equiv 0$ then the following assertions are equivalent

1. $\mathbb{P} \in \Upsilon(k)$.
2. $\mathbb{P}$ is the law of $B + Y^* \cdot \int_0^t f(s) \, ds$, where $B$ is a standard Brownian motion and $Y$ is a vector of random variables which is independent of $\mathcal{F}_\infty^B$. 

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There exists a positive function \( h \in C^{1,2} (\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+) \) such that \( h(., \int_0^t f^*(s) \, dB_s) \) is a continuous \((\mathbb{P}_0, \mathcal{F})\)-martingale with expectation 1, and \( \mathbb{P} = \mathbb{P}_0^h \) with

\[
\mathbb{P}_0^h |_{\mathcal{F}_t} = h \left( t, \int_0^t f^*(s) \, dB_s \right) \cdot \mathbb{P}_0 |_{\mathcal{F}_t}, \quad 0 < t < \infty,
\]

where \( \mathbb{P}_0 \) stands for the Wiener measure.

**Proof.** We split the proof into several steps where we show that (1) \( \iff \) (2) and (2) \( \iff \) (3). Let us show that (1) \( \Rightarrow \) (2). Let \( \mathbb{P} \in \Upsilon^{(k)} \). Theorem 3.1 implies that there exist a vector \( Y \) such that \( \mathbb{P} \) is the law of \( X_t^0 + Y^* \cdot \int_0^t f(u) \, du \). That combined with the assumption \( \alpha_\infty \equiv 0 \) leads to the fact that \( X^0 \) is a Brownian motion. Hence, it suffices to show that \( Y \) is independent of \( X^0 \). From (13) we see that \( Y = \int_t^\infty \phi(u) \, dB_u + \alpha_t \cdot \int_0^t f(u) \, dX_u \), the vector \( \int_0^t f(u) \, dX_u \) is independent of \( B \) and, consequently, it is also independent of \( X^0 \). Thus, whenever \( Z \in L^2(\mathcal{F}_X^\infty) \), for any fixed \( t \geq 0 \), we have

\[
E \left[ E \left[ Z | \mathcal{F}_X^t \right] \phi \left( Y - \int_t^\infty \phi(u) \, dB_u \right) \right] = E \left[ \phi \left( Y - \int_t^\infty \phi(u) \, dB_u \right) \right]
\]

for any bounded function \( \phi : \mathbb{R}^n \to \mathbb{R} \). By letting \( t \to \infty \) we conclude that \( E[Z \cdot \phi(Y)] = E[Z]E[\phi(Y)] \) which implies the required independence. We shall now show that (2) \( \implies \) (1). To this end, let \( k \) be a Goursat-Volterra kernel. Denote by \( f \) a reproducing basis associated to \( k \) and put \( X_t = B_t + Y^* \cdot \int_0^t f(u) \, ds \) for \( t > 0 \). For a fixed \( t > 0 \), because \( \int_0^t f(u) \, dB_u \in \Gamma_t^{(k)} \), we can write

\[
X_t - \int_0^t \int_0^u k(u, v) \, dX_v \, du = B_t - \int_0^t \int_0^u k(u, v) \, dB_v \, du = \Sigma(B)_t
\]

which, of course, a Brownian motion. Furthermore, using once more the above argument we can easily see that \( \int_0^t f(u) \, dX_u \) is independent of \( \mathcal{F}_t^{\Sigma(B)} \). Next, we deal with (2) \( \implies \) (3). Denote by \( \nu(dy) \) the distribution of \( Y \). For any measurable
functional $\phi$, we then have that
\[
E\left[ \phi \left( B_s + Y^* \cdot \int_0^s f(u) \, du : s \leq t \right) \right]
\]
\[
= \int_{\mathbb{R}^n} E\left[ \phi \left( B_s + y^* \cdot \int_0^s f(u) \, du : s \leq t \right) \right] \nu(dy)
\]
\[
= \int_{\mathbb{R}^n} E\left[ \exp \left( \int_0^t y^* \cdot f(u) \, dB_u - \frac{1}{2} \int_0^t (y^* \cdot f(u))^2 \, du \right) \phi(B_s : s \leq t) \right] \nu(dy)
\]
where the last equality is obtained by Girsanov theorem. The required space-time harmonic function is thus given on $\mathbb{R}^+ \times \mathbb{R}^n$ by
\[
h(t, x) = \int_{\mathbb{R}^n} \exp \left( y^* \cdot x - \frac{1}{2} \int_0^t (y^* \cdot f(s))^2 \, ds \right) \nu(dy).
\]
It remains to show that $(3) \implies (2)$. For fixed $0 < u \leq t < +\infty$, set $\psi(u, t) = \alpha_t \cdot \int_0^u f(s) \, ds$. Let us write the obvious decomposition
\[
B_u = \left( B_u - \psi^*(u, t) \cdot \int_0^t f(s) \, dB_s \right) + \psi^*(u, t) \cdot \int_0^t f(s) \, dB_s
\]
and denote by $H^t_u$ the first term of its right hand side. We observe that the process $(H^t_u, u < t)$ has has then the same law under $\mathbb{P}_0$ as under $\mathbb{P}^h_0$. Next, to simplify notations, write
\[
\hat{H}^t_u = \psi^*(u, t) \cdot \int_0^t f(s) \, dB_s = Y^*_t \cdot \int_0^u f(s) \, ds,
\]
where we set $Y^*_t = \alpha_t \cdot \int_0^t f(r) \, dB_r$. For any $0 \leq s \leq u \leq t$, we have
\[
E[H^t_u H^t_u] = s - \psi^*(s, t) \cdot \int_0^u f(v) \, dv
\]
and
\[
\psi^*(s, t) \cdot \int_0^u f(v) \, dv = \int_0^u f^*(v) \, dv \cdot \alpha_t \cdot \int_0^s f(r) \, dr \to 0
\]
as $t \to \infty$ because $\alpha_\infty \equiv 0$. We conclude that the convergence in distribution $H^t_u \to B^{(h)}$ holds, where $B^{(h)}$ is a $\mathbb{P}^h_0$-Brownian motion. That implies the convergence of $\hat{H}^t_u$ as well to a finite limit. But that can happen if and only if $Y^*_t$ converges to a finite limit which we denote by $Y^*$. Finally, from the above arguments we see that $Y^*$ is independent of $\mathcal{F}^{B^{(h)}}_\infty$ which ends the proof. 

\[\square\]

**Remark 4.1.** Unfortunately, for the case $\alpha_\infty \neq 0$, the second statement in the above theorem is too strong. For example, $X^0$ satisfies the assertion $(1)$ but it is easily seen that it does not satisfy $(2)$. The implications $(2) \implies (1)$ and $(2) \implies (3)$
still work in this case. We also can replace $B$ by $X^0$ in statement (2) and prove that (1) $\iff$ (2) still holds true. However, (2) $\implies$ (3) fails.

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