Notion of Random Domino Automaton revisited

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Abstract.
Inspired by a need of effective simulations of a system of Random Domino Automaton type defined for Bethe lattice, new variables are introduced. Main results obtained for 1-dimensional system — including a set of equations describing stationary state and relation to Motzkin numbers — are investigated in this new notion.

1. Introduction

Random Domino Automaton (RDA) was introduced as a straightforward totally discrete stochastic dynamical framework serving as a toy model of earthquakes [9, 10]. It may be considered as an extension of 1-dimensional Drossel and Schwabl forest fire model [14, 13, 17]. However, contrary to the Drossel-Schwabl model where parameters are constants, RDA allows some parameters to be functions of cluster’s size (forests’ size) and a good mathematical structure of the equations of the model is preserved. As a consequence, RDA generates bigger variety of distributions depending on respective parameters — form exponential distributions to inverse-power ones [10] and also quasi-periodic for a finite version [5]. Moreover, it is remarkable that the inverse problem of finding parameters (that define the dynamics of the system) from the given distribution can be solved [10, 4]. That last property is useful for explaining statistical properties of occurrence of earthquakes [6]. The extension also leads to a significant link to the Motzkin number recurrence [3, 7].

The published results are related to 1-dimensional RDA. Construction of RDA type systems on more complex geometries leads to more complicated equations, mainly due to loss of one-to-one correspondence between a size of a cluster and a size of its boundary. For 1-dimensional systems there are always just two end cells of a cluster, and the cluster may be enlarged only through them. For 2-dimensional systems the perimeter depends on a shape of a cluster. Thus, in order to extend the geometry of RDA preserving its good mathematical structure, one may choose Bethe lattice — an infinite loop free graph with fixed coordination number $K$, introduced by Hans Bethe in 1935 [12, 19, 15, 11]. In Bethe lattice there is a fixed relation between a cluster size and the size of its perimeter. This property allows to derive a set of equations using mean-field approximation in a stationary state of RDA type system on Bethe lattice [2].

Bethe lattice is infinite and homogenous everywhere, so any of its point can be regarded as an origin. However, when considering a finite part of a Bethe Lattice — for example in simulations — one may notice that majority of cells within that part are placed close to its boundary — see FIG. 1, where the boundary is created by fixing a certain radius. It follows, one can find many
clusters which cross the boundary and extend outside. When an avalanche happen, i.e. a cluster is removed from the system (or equivalently all cells of a clusters become empty) one can not estimate how many cells are actually responsible for the avalanche by observing that finite part only. The definition of RDA (and other similar systems) on Bethe lattice in natural way leads to the difficulty described above. To avoid this difficulty, a new variable $S_i$ (for $i = 1, 2, \ldots$) is introduced as follows [8]

$$S_i = \sum_{j=i}^{\infty} n_j,$$

where $n_j$ is the number of clusters of size $j$ (i.e. with exactly $j$ occupied cells) and $(S_i)$ is the number of clusters of size $\geq i$.

In the FIG. 1 it is visible, that the finite part contains only 10 cells in which 8 are occupied and 2 are empty. There are five clusters and they are extended through the boundary. So it is not possible to find out information about the distribution of clusters ($n_i$), because there is no cluster which lay just inside the finite part. By the definition of new variables one can say all 5 clusters contribute to $S_1$ and at least one cluster contribute to $S_4$.

In the present paper, this new variable is investigated for 1-dimensional RDA. For example, FIG. 2 presents a finite section of line where two clusters cross left and right boundaries of fixed finite part of the line. Thus it is impossible to know which $n_i$ they contribute to, but it is clear that they both contribute to $S_2$ and at least one to $S_3$. Main results obtained for 1-dimensional RDA are rewritten in the new notion. Study of the RDA on Bethe lattice — using both ”old” and ”new” notions — is a matter of a forthcoming paper.

**Figure 1.** New variable in Bethe lattice with coordination number 3

**Figure 2.** New variable in line
2. Evolution rules of extended RDA

We shortly recall evolution rules of RDA. Assume that any cell may be in one of two states: empty or occupied by a ball (portion energy). The state of automaton in the discrete time $t$ is characterized by instantaneous density $\rho_t$ equal to number of occupied cells in a given time $t$ divided by the size of the system $N$.

Rules of evolution include parameters $c$ and $\mu$ and are as follows. In each time step $t$ one cell is chosen (we assume that each has the same probability).

The probability that the chosen cell is empty is equal to $(1-\rho_t)$. Then the empty cell becomes occupied with the probability $c$, or it remains empty (the ball is rebounded) with probability $(1-c)$ and the state of automaton is unchanged.

If the chosen cell is already occupied by energy (the respective probability is $\rho_t$), then either the energy will be rebounded with the probability $(1-\mu)$ or it triggers an avalanche with the probability $\mu$. The avalanche size is equal to the number of cells changing their state.

Then the update procedure repeats in next time step.

\[
\begin{align*}
time &= t & \cdots & | & c & c & c & c & c & | & \downarrow & c & c & \cdots \\
time &= t + 1 & \cdots & | & \bullet & | & | & | & | & \downarrow & \downarrow & \downarrow & | & \bullet & \bullet & \cdots \\
& & & & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \cdots
\end{align*}
\]

3. Equations for RDA

3.1. Equations for RDA with the variable $n_i$

The stationary state of the system may be described by the distribution of clusters $n_i$. The number of all clusters $n$ and and the density $\rho$ are

\[
n = \sum_{i \geq 1} n_i, \quad \rho = \frac{1}{N} \sum_{i \geq 1} i n_i.
\]

The set of equations for the distribution of clusters $n_i$ and the distribution of empty clusters $n_i^0$ for $i = 1, 2, \ldots$ in stationary state is [10]

\[
n_1 = \frac{1}{m} \left( (1-\rho)N - 2n + n_1^0 \right), \quad (2)
\]

\[
n_2 = \frac{2n - 2n_1^0}{4 + \frac{2}{c m} \sum_{i \geq 1} \mu_i i n_i}, \quad (3)
\]

\[
n_i = \frac{1}{c i} \left[ 2n_{i-1} \left( 1 - \frac{n_1^0}{n} \right) + \frac{n_1^0}{n^2} \sum_{k=1}^{i-2} n_k n_{i-k-1} \right], \quad \text{for } i \geq 3 \quad (4)
\]
and

\[ n_1^0 = \frac{2n}{\left(3 + \frac{2}{c_0} \sum_{i \geq 1} \mu_i i n_i\right)}, \quad (5) \]

\[ n_2^0 = \frac{2n - 2n_1^0}{\left(4 + \frac{2}{c_0} \sum_{i \geq 1} \mu_i i n_i\right)}, \quad (6) \]

\[ n_k^0 = \frac{2(n - \sum_{i=1}^{k-1} n_i^0) + \sum_{j=1}^{k-1} \sum_{l=1}^{k-j} \frac{\mu_j}{c} j n_j \frac{n_i^0 n_{k-j-l}^0}{n}}{2 + k + \frac{2}{c_0} \sum_{i \geq 1} \mu_i i n_i}, \quad \text{for } k \geq 3. \quad (7) \]

The balance equation for the total number of clusters \( n \) and for the density \( \rho \) reads \[10\]

\[ (1 - \rho)N - 2n = \sum_{i \geq 1} \frac{\mu_i}{c} n_i i, \quad (8) \]

\[ (1 - \rho)N = \sum_{i \geq 1} \frac{\mu_i}{c} n_i i^2. \quad (9) \]

### 3.2. Equations for RDA with the variable \( S_i \)

**Some properties of \( S_i \):**

1. By the basic definition of \( S_i \), \( S_1 \) can be written as the total number of clusters.

\[ S_1 = n = \sum_{i \geq 1} n_i. \quad (10) \]

2. \( S_i \) is decreasing function i.e.

\[ S_1 \geq S_2 \geq S_3 \geq ...... \quad (11) \]

3. \( n_i \) can be written as the difference between \( S_i \) and \( S_{i+1} \)

\[ n_i = S_i - S_{i+1}. \quad (12) \]

4. Density can be written in terms of \( S_i \) as follows.

\[ \rho = \frac{1}{N} \sum_{i=1}^{\infty} i n_i = \frac{1}{N} \sum_{i=1}^{\infty} S_i. \quad (13) \]

5. 2nd order moment of \( n_i \) is

\[ \frac{1}{N} \sum_{i=1}^{\infty} i^2 n_i = \frac{1}{N} \left(2 \sum_{i=1}^{\infty} i S_i - \sum_{i=1}^{\infty} S_i\right). \quad (14) \]
3.3. Equations for extended RDA with the variable $S_i$

The system of the equations ((2)-(4)) can be reconstructed in terms of $S_i$ as follows.

\[
\sum_{i \geq 1} \frac{\mu_i}{c} n_i + 2S_1 = (1 - \rho)N, \quad (15)
\]

\[
\sum_{i \geq 2} \frac{\mu_i}{c} n_i + 2S_2 = 2S_1 - n_1^0, \quad (16)
\]

\[
\sum_{i \geq 3} \frac{\mu_i}{c} n_i + 2S_3 = 2S_2 - \left( \frac{2S_2}{S_1} - 1 \right)n_1^0, \quad (17)
\]

\[
\sum_{i \geq j} \frac{\mu_i}{c} n_i + 2S_j = (2 - n_1^0 S_j)S_{j-1} + \frac{n_1^0}{S_1} \sum_{k=1}^{j-2} \left[ S_k - S_{k+1} \right] S_{j-1-k} \quad \text{for } i \geq 4. \quad (18)
\]

In general it is very difficult to solve the equation (18) because for the 1st term. Here we will investigate the solution for special choice of rebound parameter.

4. Special case: $\mu_i = \delta_i$, where $\theta = \text{constant}$

4.1. Solution for $n_i$ with special setting of rebound parameter

Choosing this setting of rebound parameters, equations (2)-(4) gives

\[
n_1 = \frac{1}{\theta + 2} \left( (1 - \rho)N - 2n + n_1^0 \right), \quad (19)
\]

\[
n_2 = \frac{2}{\theta + 2} \left( 1 - \frac{n_1^0}{n} \right) n_1, \quad (20)
\]

\[
n_i = \frac{1}{\theta + 2} \left[ 2n_{i-1} \left( 1 - \frac{n_1^0}{n} \right) + n_1^0 \sum_{k=1}^{i-2} n_k n_{i-1-k} \right] \quad (21)
\]

for $i \geq 3$, where $\theta = \frac{\delta_i}{\theta}, \theta \in [0, \infty)$.

We define new variables $M_i$ for $i = 0, 1, \ldots$, by

\[
M_i = \frac{\beta}{\alpha^{i+1}} n_{i+1}, \quad (22)
\]

where

\[
\alpha = \frac{2(1 - n_1^0)}{\theta + 2}, \quad (23)
\]

\[
\beta = \frac{n_1^0}{2(1 - n_1^0/n)}. \quad (24)
\]

Then, the equation (21) can be rewritten in the form

\[
M_{m+2} = M_{m+1} + \sum_{k=0}^{m} M_k M_{m-k}. \quad (25)
\]
which is valid for \( m \geq 0 \) \((m = i - 2)\). Initial data \( M_0 \) and \( M_1 \) are easily obtained from equations (19)-(20), when it is transformed according the rule of equation (22), namely

\[
M_0 = M_1 = \frac{n_0^1 [ (1 - \rho) N - 2n + n_1^0 ]}{4(1 - \frac{n_1^0}{n})^2} = \frac{n_1^0 (\theta + \frac{n_1^0}{n})}{4(1 - \frac{n_1^0}{n})^2}. \quad \text{(Using (8))} \tag{26}
\]

Notice, \( M_0 \) and \( M_1 \) are equal.

The above equation (25) has the form of Motzkin numbers recurrence \([18, 1, 16, 3]\). From the Motzkin numbers: \(1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, \ldots\), etc. we can find the condition \( M_0 = M_1 = 1 \) which gives,

\[
\frac{n_0^1}{n^2} [(1 - \rho)N - 2n + n_1^0] = 4\left(1 - \frac{n_1^0}{n}\right)^2 = 1. \tag{27}
\]

The solution of the equation (25) is

\[
M_m = \frac{1}{2} \sum_{j=0}^{m+2} \frac{(2M_0 - \frac{1}{2})^j}{(m-j+2)2^{m-j} \binom{2(m-j)+1}{m-j+1} \binom{m-j+2}{j}}. \tag{28}
\]

Thus formula (22) gives the explicit solution of the equation (19)-(21) for the distribution of \( n_i \) for any values of \( \theta \).

By the above choice of \( \mu_i \) (8) and (9) give the values of \( \rho \), \( n \) and \( n_1^0 \) \([10]\) as follows.

\[
\rho = \frac{1}{\theta + 1}, \tag{29}
\]
\[
n = \frac{N\theta}{(\theta + 1)(\theta + 2)}, \tag{30}
\]
\[
n_1^0 = \frac{2n}{2\theta + 3}. \tag{31}
\]

4.2. Solution for \( S_i \) with special setting of rebound parameter

By this chosen setting of rebound parameter, the set of equations (15)-(16) transform into the following set of equations.

\[
(\theta + 2)S_1 = (1 - \rho)N, \tag{32}
\]
\[
(\theta + 2)S_2 = 2S_1 - n_1^0, \tag{33}
\]
\[
(\theta + 2)S_3 = 2S_2 - \left(\frac{2S_2}{S_1} - 1\right)n_1^0, \tag{34}
\]
\[
(\theta + 2)S_j = (2 - \frac{n_1^0}{S_1})S_{j-1} + \frac{n_1^0}{S_1} \sum_{k=1}^{j-2} \left[S_k - S_{k+1}\right]S_{j-1-k} \quad \text{for } i \geq 4 \tag{35}
\]

To solve (35) we define new variables \( M_i^* \) for \( i = 0, 1, \ldots \), by

\[
M_i^* = \frac{B}{A_{i+1}}S_{i+1}, \tag{36}
\]
where
\[ A = \frac{2 - \frac{n_0^1}{S_1}}{\theta + 2}, \]  
(37)
\[ B = \frac{n_0^1}{S_1(\theta + 2)}. \]  
(38)

Then, the equation (35) can be rewritten in the form
\[ M^*_{m+2} = M^*_{m+1} + \frac{1}{A} \sum_{k=0}^{m} M^*_k M^*_{m-k} - \sum_{k=0}^{m} M^*_k M^*_{m-k}. \]  
(39)

This recurrence is equivalent to Motzkin number recurrence [3]. Initial data \( M^*_0 \) and \( M^*_1 \) are easily obtained from equations (32)-(33).

\[ M^*_0 = M^*_1 = \frac{n_0^1}{2 - \frac{n_0^1}{S_1}}. \]  
(40)

Notice, \( M^*_0 \) and \( M^*_1 \) are equal.

We will solve (39) by using generating function [3]. Let us consider the generating function
\[ D(z) = \sum_{m \geq 0} M^*_m z^m. \]  
(41)

By this generating function the recurrence (39) gives
\[ z \left( \frac{z}{A} - 1 \right) D^2(z) + \left( z + zM^*_0 - 1 \right) D(z) + \left( (M^*_1 - M^*_0)z + M^*_0 \right) = 0. \]

Since \( M^*_0 = M^*_1 \) it gives,
\[ z \left( \frac{z}{A} - 1 \right) D^2(z) + \left( z + zM^*_0 - 1 \right) D(z) + M^*_0 = 0. \]  
(42)

Let \( D(z) \) be the solution of (42). Then,
\[ D(z) = \frac{-(1 + z + zM^*_0) - \sqrt{1 - 2z(p + qz)}}{2zM^*_0 \left( \frac{z}{A} - 1 \right)}, \]  
(43)
where,
\[ p = (1 - M^*_0), \]  
(44)
\[ q = \frac{1}{2} \left( \frac{4M^*_0}{A} - (1 + M^*_0)^2 \right). \]  
(45)

We choose negative sign in order \( D(z) \) to be a power series of \( z \).
We are interested to find out the coefficient of $z^m$ of $D(z)$.

$$D(z) = \left[ 1 - \frac{q}{2M_0} z - \frac{1}{2M_0^2} \sum_{n \geq 2} \left( \frac{-1}{n^2} \right) \frac{(2n - 3)}{(n - 1)} z^n (p + qz)^n \right] \left( 1 - \frac{z}{A} \right)^{-1}$$

$$= \left[ 1 - \frac{q}{2M_0} z - \frac{1}{2M_0^2} \sum_{n \geq 2} \frac{p^n}{n2^{n-2}} (2n - 3) z^{n-1} \sum_{j=0}^{n} \left( \binom{n}{j} \left( \frac{q}{p} \right)^j \right) z^j \right] \left[ \sum_{n \geq 0} \left( \frac{z}{A} \right)^n \right]. \quad (46)$$

So, coefficient of $z^m$ is

$$M_m^* = \frac{M_0^*}{A^m} - \frac{q}{2A^{m-1}} - \frac{1}{2} \left[ \sum_{x=2}^{m+1} \sum_{y=0}^{m+1-x} \frac{p^x}{x2^{x-2}} \binom{2x-3}{x-1} \left( \frac{q}{p} \right)^y \frac{1}{A^{m+1-x-y}} \right] \sum_{j=0}^{m} \left( \binom{n}{j} \frac{q}{p} \right)^j \frac{1}{A^{m+1-x-y}} \sum_{x=\lfloor \frac{m+1}{2} \rfloor}^{m+1-x} \sum_{y=0}^{m+1-x} \frac{p^x}{x2^{x-2}} \binom{2x-3}{x-1} \left( \frac{q}{p} \right)^y \frac{1}{A^{m+1-x-y}} \right], \quad (47)$$

where

$$p = (1 - M_0^*),$$

$$q = \frac{1}{2} \left( \frac{4M_0^*}{A} - (1 + M_0^*)^2 \right),$$

$$A = 2 - \frac{\alpha i}{\theta + 2}.$$

By the above formula of $\rho$, $n$ and $n_1^0$ ($(29)-(31)$) we can calculate the values of $S_i$, for $i = 1, 2, \ldots$ as follows.

$$S_i = \begin{cases} \frac{N \theta}{\theta + 1}, & i = 1 \\ \frac{N \theta}{\theta + 2}, & i = 2 \\ \frac{\theta \theta^2 \cdots (\theta + i - 1)}{\theta^2 \cdots (\theta + i - 1)} M_{i-1}^* & i \geq 3 \end{cases} \quad (48)$$

Now we will find out the distribution of $n_i$ by $(22)$ and $(36)$ and finally we will show that both distributions are same.

From the equation $(22)$ we get,

$$n_i = \frac{\alpha^i}{\beta} M_{i-1}$$

$$= \begin{cases} \frac{(\theta + 1)^n}{\theta + 2}, & i = 1 \\ \frac{(2(1 - \frac{\alpha i}{\theta})^{i+1}}{2^{\frac{\alpha i}{\theta}}(\theta + 2)} \sum_{j=0}^{\lfloor \frac{i+1}{2} \rfloor} \frac{(2M_0 - \frac{1}{2})^j}{(i+j+1)2^{i+j}} \binom{2(i-j) - 1}{i-j} \binom{i-j+1}{j} & i \geq 2 \end{cases} \quad (49)$$
Again from the equation (12) and (36) we get

\[ n_i = S_i - S_{i-1} = \frac{A^i}{B} \left[ M^*_{i-1} - M^*_i \right] \]

\[ = \begin{cases} 
\frac{A}{B} (1 - A) M^*_0, & i = 1 \\
\frac{A^{i+1}}{B} \left[ \sum_{k=\left\lfloor \frac{i+2}{2} \right\rfloor}^{i+1} \frac{p^k}{k^{2k-2}} \left( 2^{k-3} \left( \frac{k}{i+1-k} \right) \left( \frac{q}{p} \right)^{i+1-k} \right) \right], & i \geq 2
\end{cases} \]

\[ = \begin{cases} 
\frac{(\theta + n^0_0) n}{(\theta + 2)^i}, & i = 1 \\
\frac{(2 - n^0_0)^{i+1}}{2^{i+1}(\theta + 2)^i} \left[ \sum_{j=0}^{\left\lfloor \frac{i+2}{2} \right\rfloor} \frac{p^{i+1}}{j \cdot (i-j+1)2^{i-j-1} - 1} \left( \frac{2(i-j) - 1}{i-j+1} \right) \left( \frac{q}{p^2} \right)^j \right], & i \geq 2
\end{cases} \quad (50) \]

We need to show that (49) and (50) are equivalent. If we substitute \( k = i+1-j \) and \( S_1 = n \) in the equation (50) then we get

\[ n_i = \frac{(2 - n^0_0)^{i+1}}{2^{i+1}(\theta + 2)^i} \left[ \sum_{j=0}^{\left\lfloor \frac{i+2}{2} \right\rfloor} \frac{p^{i+1}}{j \cdot (i-j+1)2^{i-j-1} - 1} \left( \frac{2(i-j) - 1}{i-j+1} \right) \left( \frac{q}{p^2} \right)^j \right], \quad \text{for} \quad i \geq 2 \quad (51) \]

Now if we proof the following hypothesis then we can tell that (49) and (50) are equivalent.

**Hypothesis:**

\[ 2^{i+1} \left( 1 - \frac{n^0_0}{n} \right)^{i+1} \left( 2M_0 - \frac{1}{2} \right)^i = p^{i+1} \left( 2 - \frac{n^0_0}{n} \right)^{i+1} \left( \frac{q}{p^2} \right)^j \quad \forall \; j \]

\[ \Rightarrow \left( \frac{2(1-n^0_0)}{2-n^0_0} \right)^{i+1} \left( 2C_0 - \frac{1}{2} \right)^i = p^{i+1} \left( \frac{q}{p^2} \right)^j \quad \forall \; j \quad (52) \]

We need to prove

\[ p = \frac{2(1-n^0_0)}{2-n^0_0} \quad (53) \]

and

\[ (2M_0 - \frac{1}{2}) = \frac{q}{p^2} \Rightarrow \frac{n^0_0}{2(1-n^0_0)} - \frac{1}{2} = \frac{1}{2} \left( \frac{4M_0 \left( \frac{1}{4} - \frac{1}{1-M_0^*} \right)}{\left( 1-M_0^* \right)^2 - 1} \right) \quad (54) \]

By the values of \( p, M_0^* \) and \( A \) form the equation (44), (37) and (40) we can easily prove (53) and (54).
5. Conclusions

We pointed out a difficulty in analysis of finite part of the Random Domino Automaton related to clusters crossing boundary of the system. The problem is present for any geometry, but essential for Bethe lattice — a natural candidate for extension RDA beyond dimension 1.

Following the remark [8] we introduce a new variable \( S_i = \sum_{j \geq i} n_j \) and repeated main results obtained for 1-dimensional Random Domino Automaton using the new notion. Obtained formulas are in the form convenient to comparisons with numerical simulations. Moreover, we solve the system by choosing special rebound parameters by generating function method which produce a counterpart of Motzkin number recurrence.

Obtained results will be useful for extension and investigation of Random Domino Automaton type system defined on Bethe lattice.

Acknowledgments

This work was partially supported by Institute of Geophysics, Polish Academy of Science, project 500-10-28E and partially supported within statutory activities No. 3841/E-41/S/2017 of the Ministry of Science and Higher Education of Poland.

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