A class of global large, smooth solutions for the magnetohydrodynamics with Hall and ion-slip effects

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1 INTRODUCTION

In this paper, we consider the following incompressible magnetohydrodynamics (MHD) with Hall and ion-slip effects:

\[
\begin{align*}
    u_t + \nu \Lambda^a u + u \cdot \nabla u + \nabla p - b \cdot \nabla b &= 0, \\
    b_t + \mu \Lambda^\beta b + u \cdot \nabla b - b \cdot \nabla u + \sigma \nabla \times ((\nabla \times b) \times b) - \kappa \nabla \times ((\nabla \times b) \times b) &= 0, \\
    \nabla \cdot u &= 0, \quad \nabla \cdot b = 0, \\
    u|_{t=0} = u_0, \quad b|_{t=0} = b_0,
\end{align*}
\]

(1.1)

on the domain \((t,x) \in \mathbb{R}^+ \times \mathbb{R}^3\), where \(\Lambda = \sqrt{-\Delta}, \beta = 2, \alpha \in [0,2]\). Here, \(u = (u_1, u_2, u_3)^T, b = (b_1, b_2, b_3)^T \in \mathbb{R}^3\) denote the fluid velocity and magnetic fields respectively. The scalars \(p, \nu, \) and \(\mu\) are the pressure, viscosity, magnetic diffusivity, respectively (\(\nu \) and \(\mu\) are positive constants). \(\kappa\) and \(\sigma\) are constants and \(\kappa \geq 0\). The term \(\nabla \times ((\nabla \times b) \times b)\) is for the Hall effect, and \(\nabla \times ((\nabla \times b) \times b)\) is for the ion-slip effect. The functions \(u_0\) and \(b_0\) are initial data, and they satisfy

\[
\nabla \cdot u_0 = \nabla \cdot b_0 = 0.
\]

(1.2)

Equation (1.1) is very important to describe some physical phenomena, e.g., in the magnetic reconnection in space plasmas, star formation, neutron stars, and dynamo. In the case \(\sigma = \kappa = 0\), Equation (1.1) reduces to the standard MHD equations; in the case \(\kappa = 0\), Equation (1.1) reduces to Hall-MHD system. They have been extensively researched by a lot of excellent works.1–18
For the MHD system with Hall and ion-slip effects, there are some interesting results related to the local well-posedness theory; see previous studies. For global solutions, Fan et al. established global existence and time decay for small solutions. Very recently, Zhao and Zhu gave a proof of global existence for small solutions under weaker smallness conditions. However, none of results are known for MHD system with the Hall and ion-slip effects for general initial data without smallness conditions. It is quite rare to prove the existence of large, smooth, global solutions for quasilinear system. Under a class of large initial data, we found some results for incompressible Navier–Stokes equations, the incompressible MHD equations, and the incompressible Hall-MHD equations; see previous studies. Those motivate us to study the global well-posedness of the Cauchy problem of Equation (1.1) with large initial data. But the Hall and ion-slip term heightens the level of nonlinearity of the standard MHD system from a second-order semilinear to a second-order quasilinear level, significantly making its qualitative analysis more difficult. To the author's knowledge, it is quite rare to prove the existence of large, smooth, global solutions for quasilinear system.

The aim of this paper is to prove the existence of a unique, global smooth solution of MHD system with the Hall and ion-slip effects in $H^3(R^3)$. Our result completely drops the smallness condition on the initial data.

Before we state our main results, let us first give some notations. Let $\chi(x) \in C^\infty_0(R^3)$ be a cut off function satisfying

\begin{align}
\chi(x) &\equiv 1, \text{ for } |x| \leq 1; \chi(x) \equiv 0, \text{ for } |x| \geq 2, \\
|\nabla^k \chi(x)| &\leq 2, 0 \leq k \leq 5.
\end{align}

Denote

\begin{align}
\chi_M(x) := \chi \left( \frac{x}{M_0} \right).
\end{align}

Here, $M_0$ is a positive constant. Let $v_0$ be that constructed by Lei et al., and it has the following properties

\begin{align}
\nabla \cdot v_0 = 0, \quad \nabla \times v_0 = \sqrt{-\Delta} v_0,
\end{align}

\begin{align}
\text{supp} \hat{v}_0 &\subseteq \{ |\xi| 1 - \delta \leq |\xi| \leq 1 + \delta \}, \quad 0 < \delta \leq \frac{1}{2}, \\
\| \hat{v}_0 \|_{L^1} &\leq M_1, \quad |\nabla^k \hat{v}_0| \leq \frac{M_2}{1 + |x|}, \quad 0 \leq k \leq 5,
\end{align}

where $M_1$ and $M_2$ are positive constants, $\hat{v}_0$ is the Fourier transform of $v_0$, and the operator $\sqrt{-\Delta}$ is defined through the Fourier transform

\begin{align}
\sqrt{-\Delta} f(\xi) = |\xi| \hat{f}(\xi).
\end{align}

Our main result is as follows.

**Theorem 1.1.** Consider the Cauchy problem (1.1)–(1.2). Suppose that

\begin{align}
u_0 &= u_{01} + \chi_M u_{02}, \\
b_0 &= b_{01} + \chi_M b_{02}.
\end{align}

with

\begin{align}
\nabla \cdot u_{02} = \nabla \cdot b_{02} = 0, \\
u_{02} &= \alpha_1 v_0, \quad b_{02} = \alpha_2 v_0,
\end{align}
where $\chi_{M_0}$ and $v_0$ are stated as above. $a_1$, $a_2$ are two real constants. Then there exist constants $\delta^{-\frac{1}{2}} \geq M_0 \gg 1$ depending on $M_1, M_2, a_1, a_2, \mu, v, \sigma$, and $\kappa$ such that the problem (1.1)–(1.2) has a unique, global smooth solution provided that

$$\|u_0\|_{H^3} + \|b_0\|_{H^3} \leq M_0^{-\frac{1}{2}}.$$  \hspace{1cm} (1.12)

**Remark 1.1.** For

$$\|u_0\|_{L^\infty} + \|b_0\|_{L^\infty} \leq M_1,$$

$$\|u_0\|_{H^3} + \|b_0\|_{H^3} \leq \left( M_0^{-\frac{1}{2}} + (|a_1| + |a_2|) \sum_{k \geq 0} M_2 \right),$$

and the constant $M_1, M_2$ can be arbitrary large, thus our initial data can be arbitrary large. Comparing with Fan et al.\textsuperscript{22,23} and Zhao et al.\textsuperscript{22,23} our result can be seen as a nontrivial improvement of Fan et al.’s and Zhao et al.’s work, for we completely drop the smallness condition on the initial data.

**Remark 1.2.** The parameter $\alpha, \beta$ indicates the strength of dissipation for velocity and magnetic field, respectively. If the parameter $\alpha, \beta$ is larger, then the corresponding dissipation is stronger. When $\sigma = \kappa = 0$, the conclusion in Theorem 1.1 still holds for all $\alpha, \beta \in [0, 2]$. If $\sigma \neq 0$ or $\kappa \neq 0$, considering the quasilinear terms for magnetic field on (1.1), then the strong dissipative term $\mu \Delta b(\beta = 2)$ may be necessary to compensate for the loss of regularity in exploring large solutions.

**Remark 1.3.** In the limiting case $\delta = 0$, $\nabla \times u_{02} = u_{02}$, $\nabla \times b_{02} = b_{02}$, and the flow, magnetic field are called Beltrami flow and force-free fields, respectively. Let us also mention that the magnetic energy achieves the minimum value for force-free fields; one can refer Taylor\textsuperscript{37} for details.

**Remark 1.4.** We throughout use a notation $C$. It may be different from line to line, but it is a universal positive constant in this paper.

The proof of Theorem 1.1 is based on a perturbation argument along with a standard cut-off technique, and the perturbation is as large as the initial data. Compared with Hall-MHD equations, a part of the nonlinearities may not be small for Equation (1.1) (see 3.3). Fortunately, by combining the nonlinear structure of the term and commutator estimates, these terms can be estimated carefully.

This paper is organized as follows: In Section 2, we introduce commutator estimates and give some estimate of quadratic and cubic terms. Section 3 is devoted to prove the global existence and uniqueness of large smooth solutions for Equation (1.1).

2 | PRELIMINARIES

**Lemma 2.1** (3). Let $s > 0$. Let $p, p_2, p_3 \in (1, \infty)$ and $p_2, p_4 \in (1, \infty)$ satisfy

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

Then there exist two constants $C_1, C_2$,

$$\|\Lambda^s (fg)\|_{L^p} \leq C_1 \left( \|\Lambda^s f\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|\Lambda^s g\|_{L^{p_3}} \|f\|_{L^{p_4}} \right),$$

$$\|[\Lambda^s, f]g\|_{L^p} \leq C_2 \left( \|\Lambda^s f\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|\Lambda^s g\|_{L^{p_3}} \|\nabla f\|_{L^{p_4}} \right).$$

Let $f$ and $g$ satisfy

$$\begin{cases}
f_t + \nu \Lambda^s f = 0, \\
t = 0 : f = u_{02}.
\end{cases}$$

(2.1)
and
\[
\begin{aligned}
g_t - \mu \Delta g &= 0, \\
t = 0 : g &= b_{02}. 
\end{aligned}
\] (2.2)

Therefore, we have
\[
f = e^{-\nu t} u_{02}, \quad g = e^{\mu t} b_{02}.
\]

**Lemma 2.2.** Let \(f, g\) be defined in (2.1) and (2.2). It holds
\[
\nabla \cdot f = 0, \quad \nabla \times f = \sqrt{-\Delta} f, \\
\nabla \cdot g = 0, \quad \nabla \times g = \sqrt{-\Delta} g.
\]

|\[\nabla^k f| \leq \frac{|\alpha_1|M_2}{1 + |x|} e^{-\frac{|x|}{2}}, \quad |\nabla^k g| \leq \frac{|\alpha_2|M_2}{1 + |x|} e^{-\frac{|x|}{2}}, \quad 0 \leq |k| \leq 5.\]

**Proof.** By
\[
\nabla \cdot v_0 = 0, \quad \nabla \times v_0 = \sqrt{-\Delta} v_0, \\
f = e^{-\nu t} u_{02}, \quad g = e^{\mu t} b_{02},
\]
we can deduce that
\[
\nabla \cdot f = 0, \quad \nabla \times f = \sqrt{-\Delta} f, \\
\nabla \cdot g = 0, \quad \nabla \times g = \sqrt{-\Delta} g.
\]

We choose a \(C^\infty(\mathbb{R}^3)\) cut-off function \(\gamma(\xi)\) such that \(a \equiv 1\) on the support of \(v_0\), and \(\gamma(\xi) \equiv 0\) if \(|\xi| \geq 1 + 2\delta\) or \(|\xi| \leq 1 - 2\delta\). Then we have
\[
f(t, x) = a_1 e^{-\frac{|x|}{2}} F^{-1} \left( e^{-\nu |\xi|} \pi^{-\frac{1}{2}} \gamma(\xi) \right) * v_0, \\
g(t, x) = a_2 e^{-\frac{|x|}{2}} F^{-1} \left( e^{-\mu |\xi|} \pi^{-\frac{1}{2}} \gamma(\xi) \right) * v_0.
\]

In a result, we get
\[
|\nabla^k f| \leq \frac{|\alpha_1|M_2}{1 + |x|} e^{-\frac{|x|}{2}}, \quad |\nabla^k g| \leq \frac{|\alpha_2|M_2}{1 + |x|} e^{-\frac{|x|}{2}}, \quad 0 \leq |k| \leq 5. \quad \square
\]

**Lemma 2.3.** Set \(\tilde{f} := \chi_{M_0} f, \tilde{g} := \chi_{M_0} g\). Let \(f, g, \chi_{M_0}\) be defined in (2.1), (2.2), and (1.4) respectively. Then we have
\[
\|\tilde{f}\|_{W^{\infty, \infty}} + \|\tilde{g}\|_{W^{\infty, \infty}} \leq C \left( |\alpha_1|M_1 e^{-\frac{|x|}{2}} + |\alpha_2|M_1 e^{-\frac{|x|}{2}} \right),
\] (2.3)
\[
\|\nabla \times (\nabla \times f)\|_{H^2} + \|\nabla \times (\nabla \times g)\|_{H^2} \leq C \left( a_1^2 e^{-\frac{|x|}{2}} + a_2^2 e^{-\frac{|x|}{2}} \right) \left( \delta M_0^3 M_1^3 + M_0^{-1} M_2^3 \right),
\] (2.4)
\[
\|((\nabla \times \tilde{g}) \times \tilde{g})\|_{H^2} \leq C \left( \delta M_0^3 M_1^3 + M_0^{-1} M_2^3 \right) |\alpha_2|^3 e^{-\frac{|x|}{2}},
\] (2.5)
\[
\int_0^\infty \|\tilde{f} \times \tilde{g}\|_{H^1} dt \leq C M_0^\frac{3}{2} M_1^2 (1 + a) \delta.
\] (2.6)

**Proof.** First, we have \(|\nabla^k \chi_{M_0}| \leq C M_0^{-k}, k \leq 5\). Then
\[
\|\tilde{f}\|_{W^{\infty, \infty}} = \|\chi_{M_0} f\|_{W^{\infty, \infty}} \leq \|\chi_{M_0}\|_{W^{\infty, \infty}} \|f\|_{W^{\infty, \infty}} \leq C \|f\|_{W^{\infty, \infty}}.
\] (2.7)
Using $\hat{f} = e^{-v|\xi|}\hat{u}_{02}$ and $\text{supp}\hat{u}_{02} \subseteq \{\xi | 1 - \delta \leq |\xi| \leq 1 + \delta\}$, $0 < \delta \leq \frac{1}{2}$, we get

$$\|f\|_{W^{2,\infty}} \leq \|(1 + |\xi|)^3 \hat{f}\|_{L^2} \leq C\|e^{-v|\xi|}\hat{u}_{02}\|_{L^2} \leq C|a_1|M_1e^{-\frac{v}{2}}.$$  

Similarly, we have

$$\|g\|_{W^{2,\infty}} \leq C|a_2|M_1e^{-\frac{v}{2}}. \quad (2.8)$$

Adding (2.8) to (2.7), we obtain

$$\|\hat{f}\|_{W^{3,\infty}} + \|\hat{g}\|_{W^{3,\infty}} \leq C\left(|a_1|M_1e^{-\frac{v}{2}} + |a_2|M_1e^{-\frac{v}{2}}\right).$$

Secondly, we notice the fact

$$\nabla \times (\chi_{M_\delta}f) = \nabla \chi_{M_\delta} \times f + \chi_{M_\delta} \nabla \times f,$$

$$\nabla \times (\chi_{M_\delta}g) = \nabla \chi_{M_\delta} \times g + \chi_{M_\delta} \nabla \times g.$$

Thus, we get

$$\|\hat{f} \times (\nabla \times \hat{f})\|_{H^1} + \|\hat{g} \times (\nabla \times \hat{g})\|_{H^1} \leq C\|\chi_{M_\delta}f \times (\nabla \times (\chi_{M_\delta}f))\|_{H^1} + \|\chi_{M_\delta}g \times (\nabla \times (\chi_{M_\delta}g))\|_{H^1} \leq C\|\chi_{M_\delta}f\|_{H^3} \left(\|f\|_{H^2} + \|\nabla f\|_{H^1}\right) + C\|\nabla (\chi_{M_\delta}^2)\|_{W^{3,\infty}} \left(\|f\|_{H^1}^2 + \|g\|_{H^3}^2\right)$$

and

$$\|(\nabla \times \hat{g}) \times \hat{g}\|_{H^1} \leq C\left(\|\nabla \chi_{M_\delta}^3\|_{H^1} \|g\|_{W^{3,\infty}} + \|\nabla (\chi_{M_\delta}^3)\|_{W^{3,\infty}}\right). \quad (2.9)$$

We calculate that

$$\|\chi_{M_\delta}f\|_{H^3} + \|\chi_{M_\delta}^3\|_{H^1} \leq C\sum_{l=0}^{3} M_{0}^{-l} M_{0}^{3} \leq CM_{0}^{3} \quad (2.10)$$

$$\|\nabla (\chi_{M_\delta}^3)\|_{W^{3,\infty}} \leq C\sum_{l=0}^{3} M_{0}^{-l} \leq CM_{0}^{-1}.$$  

For $f \times f = 0$, $g \times g = 0$, then we have

$$\|f \times (\nabla \times f)\|_{W^{3,\infty}} + \|g \times (\nabla \times g)\|_{W^{3,\infty}} \leq C\sum_{l=0}^{3} M_{0}^{-l} M_{0}^{3} \leq CM_{0}^{3} \quad (2.11)$$

and

$$\|(\nabla \times g) \times g\|_{W^{3,\infty}} \leq C\delta \|g\|_{W^{3,\infty}} \leq C\delta M_{1} |a_2|^3 e^{-\frac{v}{2}}. \quad (2.12)$$
Then we complete the proof of Lemma 2.3.

\[ |||f|||_{H^s}^{2} + |||g|||_{H^s}^{2} \leq C(|||f|||_{L^2}^{2} + |||g|||_{L^2}^{2}) \leq C \left( a_1^2 e^{-\frac{m}{3}} M_2^2 + a_2^2 e^{-\frac{n}{3}} M_2^2 \right), \]  

(2.13)

and

\[ |||g|||_{H^s}^{3} \leq C|||g|||_{L^2}^{3} \leq C|||g|||_{L^2}^{3} \leq C[a_2]^3 M_2^3 e^{-\frac{n}{3}}. \]  

(2.14)

Combining the inequalities (2.10), (2.11), and (2.13), we get

\[ \| \hat{f} \times (V \times \hat{f}) \|_{H^s} + \| \hat{g} \times (V \times \hat{g}) \|_{H^s} \leq C \left( a_1^2 e^{-\frac{m}{3}} + a_2^2 e^{-\frac{n}{3}} \right) \left( \delta M_0^3 M_1^3 + M_0^{-1} M_2^3 \right). \]  

Combining inequalities (2.9), (2.12), and (2.14), we deduce that

\[ \| (V \times \hat{g}) \times \hat{g} \|_{H^s} \leq C \left( \delta M_0^3 M_1^3 + M_0^{-1} M_2^3 \right) |a_2|^3 e^{-\frac{n}{3}}. \]

In what follows, we will estimate \( \int_0^t \| \hat{f} \times \hat{g} \|_{H^s}(t) dt \). On one hand,

\[ \| \hat{f} \times \hat{g} \|_{H^s} \leq \| \chi_{M_0} f \times (\chi_{M_0} g) \|_{H^s} \leq \| \chi_{M_0} f \|_{H^s} \| f \times g \|_{W^{3,\infty}}. \]  

(2.15)

On the other hand, \( \supp \hat{f} \times g \subseteq \{ |\xi| \leq 2 + 2\delta \} \), \( 0 < \delta \leq \frac{1}{2} \). Then, we have

\[ \| \hat{f} \times \hat{g} \|_{H^s} \leq CM_0^3 \| \hat{f} \times \hat{g} \|_{L^2}. \]  

(2.16)

Calculate

\[ \hat{f} \times \hat{g} = a_1 a_2 \int_{\mathbb{R}^3} e^{-\mu|\xi-\eta|^2} \hat{v}_0(\xi-\eta) \times e^{-\mu|\eta|^2} \hat{v}_0(\eta) d\eta \]  

(2.17)

and

\[ |e^{-\mu|\xi-\eta|^2} - e^{-\mu|\eta|^2}| \leq C e^{-\mu|\eta|^2} \frac{|\xi - \eta|^2}{|\xi - \eta|^a} + C e^{-\mu|\eta|^2} \frac{|\xi - \eta|^a}{|\xi - \eta|^2} \]  

(2.18)

In the support of \( \hat{v}_0(\xi-\eta) \times \hat{v}_0(\eta) \), we have

\[ \frac{|\xi - \eta|^2}{|\xi - \eta|^a} \leq 3^{1-a} \delta, \quad \frac{|\xi - \eta|^a}{|\xi - \eta|^2} \leq 8a \delta. \]  

(2.19)

Therefore, we conclude that

\[ \int_0^\infty \| \hat{f} \times \hat{g} \|_{H^s}(t) dt \leq CM_0^3 M_1^3 (1 + \alpha) \delta \]  

(2.20)

Then we complete the proof of Lemma 2.3.
3 | THE PROOF OF THEOREM 1.1

In this section, we will prove Theorem 1.1 using a perturbation argument along with a standard cut-off technique.

Proof of Theorem 1.1. Let \( \tilde{f} = \chi_{M_0} \tilde{f}, \tilde{g} = \chi_{M_0} \tilde{g} \), and \( u = U + \tilde{f}, b = B + \tilde{g} \). Then, \( U, B \) satisfy

\[
U_t + \nu \Lambda^n U + \nabla \left( p + \frac{1}{2} |\tilde{f}|^2 - \frac{1}{2} |\tilde{g}|^2 \right) = -U \cdot \nabla U - \tilde{f} \cdot \nabla U - U \cdot \nabla \tilde{f} + B \cdot \nabla B + \tilde{g} \cdot \nabla B + B \cdot \nabla \tilde{g} + F, \tag{3.1}
\]

\[
B_t - \mu \Delta B - \nabla \times ((\nabla \times B) \times B) = -U \cdot \nabla B - \tilde{f} \cdot \nabla B - U \cdot \nabla \tilde{g} + B \cdot \nabla U + \tilde{g} \cdot \nabla U + B \cdot \nabla \tilde{f} - \sigma \nabla \times ((\nabla \times B) \times B) - \sigma \nabla \times ((\nabla \times \tilde{g}) \times \tilde{g}) + G \tag{3.2}
\]

where

\[
F := \tilde{f} \times (\nabla \times \tilde{f}) - \tilde{g} \times (\nabla \times \tilde{g}) - \nu \Delta \chi_{M_0} f + 2\nu \nabla \cdot (\chi_{M_0} f),
\]

\[
G := \nabla \times (\tilde{f} \times \tilde{g}) - \mu \Delta \chi_{M_0} g + 2\mu \nabla \cdot (\chi_{M_0} g) + \frac{1}{2} \tilde{f} \cdot \nabla \chi_{M_0} \tilde{g} - \frac{1}{2} \tilde{g} \cdot \nabla \chi_{M_0} \tilde{f} - \sigma \nabla \times ((\nabla \times \tilde{g}) \times \tilde{g}) - \kappa \nabla \times ((\nabla \times \tilde{g}) \times B) + \kappa \tilde{g} \times ((\nabla \times \tilde{g}) \times B). \tag{3.3}
\]

In what follows, we will derive some energy estimates of \( U \) and \( B \).

**Step 1: Energy inequalities of** \( \tilde{B} \)

Taking the derivatives \( \Lambda^k, 0 \leq k \leq 3 \) on Equation (3.2) and \( L^2 \) inner product with \( \Lambda^k B \), we get

\[
\frac{1}{2} \frac{d}{dt} \|B\|_{H^1}^2 + \|\nabla B\|_{H^1}^2 = I_1 + I_2 + I_3 + I_4 + I_5 + I_6
\]

\[
+ J_1 + J_2 + J_3 + \int_{\mathbb{R}^3} G \cdot B dx
\]

\[
+ K_0 + K_1 + K_2 + K_3 + K_4 + K_5 + K_6,
\]

where

\[
I_1 = -\sum_{0 \leq k \leq 3} \int_{\mathbb{R}^3} \Lambda^k (U \cdot \nabla B) \cdot \Lambda^k B dx,
I_2 = -\sum_{0 \leq k \leq 3} \int_{\mathbb{R}^3} \Lambda^k (\tilde{f} \cdot \nabla B) \cdot \Lambda^k B dx,
I_3 = -\sum_{0 \leq k \leq 3} \int_{\mathbb{R}^3} \Lambda^k (U \cdot \nabla \tilde{g}) \cdot \Lambda^k B dx,
I_4 = \sum_{0 \leq k \leq 3} \int_{\mathbb{R}^3} \Lambda^k (B \cdot \nabla U) \cdot \Lambda^k B dx,
I_5 = \sum_{0 \leq k \leq 3} \int_{\mathbb{R}^3} \Lambda^k (\tilde{g} \cdot \nabla U) \cdot \Lambda^k B dx,
I_6 = \sum_{0 \leq k \leq 3} \int_{\mathbb{R}^3} \Lambda^k (B \cdot \nabla \tilde{f}) \cdot \Lambda^k B dx,
\]

\[
J_1 = -\sigma \sum_{0 \leq k \leq 3} \int_{\mathbb{R}^3} \Lambda^k ((\nabla \times B) \times B) \cdot (\nabla \times \Lambda^k B) dx,
J_2 = -\sigma \sum_{0 \leq k \leq 3} \int_{\mathbb{R}^3} \Lambda^k ((\nabla \times B) \times \tilde{g}) \cdot (\nabla \times \Lambda^k B) dx,
J_3 = -\sigma \sum_{0 \leq k \leq 3} \int_{\mathbb{R}^3} \Lambda^k ((\nabla \times \tilde{g}) \times B) \cdot (\nabla \times \Lambda^k B) dx,
\]
\[ K_0 = \kappa \sum_{0 \leq k \leq 3} \int_{\mathbb{R}^3} \Lambda^k((\nabla \times B) \times B) \cdot (\nabla \times \Lambda^k B) dx, \]

\[ K_1 = \kappa \sum_{0 \leq k \leq 3} \int_{\mathbb{R}^3} \Lambda^k((\nabla \times B) \times \tilde{g}) \cdot (\nabla \times \Lambda^k B) dx, \]

\[ K_2 = \kappa \sum_{0 \leq k \leq 3} \int_{\mathbb{R}^3} \Lambda^k((\nabla \times B) \times \tilde{g}) \times (\nabla \times \Lambda^k B) dx, \]

\[ K_3 = \kappa \sum_{0 \leq k \leq 3} \int_{\mathbb{R}^3} \Lambda^k((\nabla \times B) \times \tilde{g}) \cdot (\nabla \times \Lambda^k B) dx, \]

\[ K_4 = \kappa \sum_{0 \leq k \leq 3} \int_{\mathbb{R}^3} \Lambda^k((\nabla \times \tilde{g}) \times B) \cdot (\nabla \times \Lambda^k B) dx, \]

\[ K_5 = \kappa \sum_{0 \leq k \leq 3} \int_{\mathbb{R}^3} \Lambda^k((\nabla \times \tilde{g}) \times B) \cdot (\nabla \times \Lambda^k B) dx, \]

\[ K_6 = \kappa \sum_{0 \leq k \leq 3} \int_{\mathbb{R}^3} \Lambda^k((\nabla \times \tilde{g}) \times B) \cdot (\nabla \times \Lambda^k B) dx. \]

Firstly, we estimate \( I_1, I_2 \) in the following:

\[
|I_1 + I_2| \leq \sum_{0 \leq k \leq 3} \left( \int_{\mathbb{R}^3} (\Lambda^k(U \cdot \nabla B) - (U \cdot \nabla \Lambda^k B) \cdot \Lambda^k B) dx + \int_{\mathbb{R}^3} (\Lambda^k(\tilde{f} \cdot \nabla B - \tilde{f} \cdot \nabla \Lambda^k B)) \cdot \Lambda^k B dx \right) \\
+ \sum_{0 \leq k \leq 3} \left( \int_{\mathbb{R}^3} (U \cdot \nabla \Lambda^k B) \cdot \Lambda^k B dx \right) \\
\leq \sum_{0 \leq k \leq 3} \left( \int_{\mathbb{R}^3} (\Lambda^k(U \cdot \nabla B) - (U \cdot \nabla \Lambda^k B) \cdot \Lambda^k B dx) + \int_{\mathbb{R}^3} (\Lambda^k(\tilde{f} \cdot \nabla B - \tilde{f} \cdot \nabla \Lambda^k B)) \cdot \Lambda^k B dx \right) \\
\leq C \left( \| \nabla U \|_{L^\infty} \| \nabla B \|_{H^3} + \| \nabla B \|_{L^4} \| U \|_{W^{3,2}} \right) \| B \|_{H^3} \\
+ C \left( \| \nabla B \|_{H^3} \| \nabla \tilde{f} \|_{L^\infty} + \| \nabla B \|_{L^4} \| \tilde{f} \|_{W^{3,1}} \right) \| B \|_{H^3}. \]

By Sobolev's inequality, we deduce that

\[
|I_1 + I_2| \leq C \left( \| \Lambda^3 \check{U} \|_{H^3} \| \nabla B \|_{H^3} \| B \|_{H^3} + \| B \|_{H^3}^2 (\| \tilde{f} \|_{W^{2,\infty}} + \| \tilde{f} \|_{W^{3,1}}) \right). \tag{3.5} \]

For \( I_3 \), it is easy for us to get

\[
|I_3| \leq C \| U \|_{H^3} \| B \|_{H^3} \| \tilde{g} \|_{W^{2,\infty}}. \tag{3.6} \]

For \( I_5 \) and \( I_4 \), we derive that

\[
|I_4 + I_5| \leq C \left( \| \nabla U \|_{H^3} \| \nabla B \|_{L^\infty} + \| \nabla U \|_{L^2} \| B \|_{W^{3,2}} \right) \| B \|_{H^3} \\
+ C \left( \| \nabla U \|_{H^3} \| \nabla \tilde{f} \|_{L^\infty} + \| \nabla U \|_{L^2} \| \tilde{f} \|_{W^{3,1}} \right) \| B \|_{H^3} \tag{3.7} \]

\[
\leq C \left( \| \Lambda^3 \check{U} \|_{H^3} \| \nabla B \|_{H^3} \| B \|_{H^3} + \| U \|_{H^3} \| B \|_{H^3} (\| \tilde{f} \|_{W^{2,\infty}} + \| \tilde{f} \|_{W^{3,1}}) \right). \]

Considering \( I_6 \), we have

\[
|I_6| \leq C \| B \|_{H^3}^2 \| \tilde{f} \|_{W^{2,\infty}}. \tag{3.8} \]
Next step, we will estimate $J_1, J_2, J_3$. For

$$J_1 = \sum_{0 \leq k \leq 3} \sigma \int_{\mathbb{R}^3} \Lambda^k (\nabla \times B) \cdot \Lambda^k ((\nabla \times B) \times B) \, dx$$

$$= \sigma \sum_{0 \leq k \leq 3} \int_{\mathbb{R}^3} \{ \Lambda^k ((\nabla \times B) \times B) - \Lambda^k ((\nabla \times B) \times B) \} \cdot \Lambda^k (\nabla \times B) \, dx.$$ 

Using Lemma 2.1, we deduce that

$$|J_1| \leq C|\sigma| \sum_{0 \leq k \leq 3} ||\Lambda^k (\nabla \times B)||_{L^2} ||\Lambda^k ((\nabla \times B) \times B) - \Lambda^k ((\nabla \times B) \times B)||_{L^2}$$

$$\leq C|\sigma| \|\nabla B\|_{H^1} (||\nabla \times B||_{H^1} ||\nabla B||_{L^\infty} + ||\nabla \times B||_{L^\infty} ||B||_{H^1})$$

$$\leq C|\sigma| \|\nabla B\|^2_{H^1} ||B||_{H^1}. \quad (3.9)$$

For $J_2$, we calculate

$$J_2 = \sum_{0 \leq k \leq 3} \sigma \int_{\mathbb{R}^3} \Lambda^k (\nabla \times B) \cdot \Lambda^k ((\nabla \times B) \times \tilde{g}) \, dx$$

$$= \sigma \sum_{0 \leq k \leq 3} \int_{\mathbb{R}^3} \Lambda^k (\nabla \times B) \cdot (\Lambda^k ((\nabla \times B) \times \tilde{g}) - \Lambda^k ((\nabla \times B) \times \tilde{g}) \, dx$$

$$+ \sigma \sum_{0 \leq k \leq 3} \int_{\mathbb{R}^3} (\nabla \times \Lambda^k B) \cdot ((\nabla \times \Lambda^k B) \times \tilde{g}) \, dx$$

$$= \sigma \sum_{0 \leq k \leq 3} \int_{\mathbb{R}^3} \Lambda^k (\nabla \times B) \cdot (\Lambda^k ((\nabla \times B) \times \tilde{g}) - \Lambda^k ((\nabla \times B) \times \tilde{g}) \, dx,$$

for we use the fact that $\int_{\mathbb{R}^3} (\nabla \times \Lambda^k B) \cdot ((\nabla \times \Lambda^k B) \times \tilde{g}) \, dx = 0$. Using Lemma 2.1, we then derive that

$$|J_2| \leq C|\sigma| \|\nabla B\|_{H^1} \|\nabla B\|_{H^1} \|\tilde{g}\|_{W^{3,\infty}}$$

$$\leq C|\sigma| \|\nabla B\|_{H^1} \|B\|_{H^1} \|\tilde{g}\|_{W^{3,\infty}}$$

$$\leq \frac{\mu}{16} \|\nabla B\|^2_{H^1} + C \|B\|^2_{H^1} \|\tilde{g}\|^2_{H^{4,\infty}}. \quad (3.10)$$

For $J_3$, we could estimate it directly that

$$|J_3| \leq C \|\tilde{g}\|_{W^{4,\infty}} \|B\|_{H^1} \|\nabla B\|_{H^1}$$

$$\leq \frac{\mu}{16} \|\nabla B\|^2_{H^1} + C \|B\|^2_{H^1} \|\tilde{g}\|^2_{H^{4,\infty}}. \quad (3.11)$$

For some quadratic $\nabla \Lambda^3 B$ (highest derivatives) in $K_1, K_2$, we could not get good estimate of $K_1, K_2$ when $\tilde{g}$ is large. We also should not neglect $K_0, K_3$ containing some positive items. Therefore, we then find that it’s a effective way to estimate $K_1, K_2, K_0$ and $K_3$ together.

$$K_0 + K_1 + K_2 + K_3$$

$$= -\kappa \sum_{0 \leq k \leq 3} \int_{\mathbb{R}^3} |(\nabla \times \Lambda^k B) \times B|^2 + 2((\nabla \times \Lambda^k B) \times B) \cdot (\tilde{g} \times (\nabla \times \Lambda^k B))$$

$$+ F_0 + F_1 + F_2 + F_3 - (\nabla \times \Lambda^k B) \times \tilde{g}|^2 dx$$

$$\leq -\kappa \sum_{0 \leq k \leq 3} \int_{\mathbb{R}^3} |(\nabla \times \Lambda^k B) \times B| - |(\nabla \times \Lambda^k B) \times \tilde{g}|^2 + F_0 + F_1 + F_2 + F_3, \quad (3.12)$$
where

\[
F_0 = \kappa \sum_{0 \leq k \leq 3} \sum_{|\beta| \leq 2, \beta + \gamma = \kappa} \int_{\mathbb{R}^3} (((\nabla \times \Lambda^\beta B) \times \Lambda^\gamma B) \times (\nabla \times \Lambda^\kappa B)) dx, \tag{3.13}
\]

\[
F_1 = \kappa \sum_{0 \leq k \leq 3} \sum_{|\beta| \leq 2, \beta + \gamma = \kappa} \int_{\mathbb{R}^3} (((\nabla \times \Lambda^\beta B) \times \Lambda^\gamma \tilde{g}) \times \Lambda^\kappa B) \cdot (\nabla \times \Lambda^\kappa B) dx, \tag{3.14}
\]

\[
F_2 = \kappa \sum_{0 \leq k \leq 3} \sum_{|\beta| \leq 2, \beta + \gamma = \kappa} \int_{\mathbb{R}^3} (((\nabla \times \Lambda^\beta B) \times \Lambda^\gamma \tilde{g}) \times \Lambda^\kappa \tilde{g}) \cdot (\nabla \times \Lambda^\kappa B) dx, \tag{3.15}
\]

\[
F_3 = \kappa \sum_{0 \leq k \leq 3} \sum_{|\beta| \leq 2, \beta + \gamma = \kappa} \int_{\mathbb{R}^3} (((\nabla \times \Lambda^\beta B) \times \Lambda^\gamma \tilde{g}) \times \Lambda^\kappa \tilde{g}) \cdot (\nabla \times \Lambda^\kappa B) dx. \tag{3.16}
\]

For \( F_0 \), we have

\[
F_0 \leq C \left( \|\nabla B\|_{L^2} \|\nabla B\|_{L^\infty} \|B\|_{L^2} + \|\nabla B\|_{L^2} \|B\|_{L^2} + \frac{3}{2} \|\nabla B\|_{L^2} \right) \|\nabla B\|_{H^3}
\leq C \|\nabla B\|_{H^2}^2. \tag{3.17}
\]

For \( F_1 \), we derive that

\[
F_1 \leq C \|\tilde{g}\|_{W^{4,\infty}} \|B\|_{H^3}^2 \|\nabla B\|_{H^3}
\leq \frac{\mu}{16} \|\nabla B\|_{H^3}^2 + C \|\tilde{g}\|_{W^{4,\infty}}^2 \|B\|_{H^3}^4. \tag{3.18}
\]

For \( F_2 \), we could get

\[
F_2 \leq C \|\nabla B\|_{H^3} \left( \|\nabla B\|_{L^2} \|\tilde{g}\|_{L^\infty} + \|B\|_{L^2} \|\tilde{g}\|_{W^{4,\infty}} \right) \|\nabla B\|_{H^3}
+ C \|\nabla B\|_{W^{1,6}} \left( \|\nabla B\|_{W^{1,4}} \|\tilde{g}\|_{W^{1,6}} + \|B\|_{W^{1,6}} \|\tilde{g}\|_{W^{1,6}} \right) \|\nabla B\|_{H^3}
\leq \frac{\mu}{16} \|\nabla B\|_{H^3}^2 + C \left( \|\tilde{g}\|_{W^{1,6}}^2 + \|\tilde{g}\|_{W^{1,6}}^2 \right) \|B\|_{H^3}^4. \tag{3.19}
\]

For \( F_3 \), it is easy for us to get

\[
F_3 \leq C \|\nabla B\|_{H^3} \|\tilde{g}\|_{L^2}^2 \|\nabla B\|_{H^3} + \|\nabla B\|_{W^{2,4}} \|\tilde{g}\|_{W^{2,4}}^2 \|\nabla B\|_{H^3}
+ C \|\nabla B\|_{L^2} \|\tilde{g}\|_{W^{2,4}}^2 \|\nabla B\|_{H^3}
\leq \frac{\mu}{16} \|\nabla B\|_{H^3}^2 + C \left( \|\tilde{g}\|_{W^{2,4}}^4 + \|\tilde{g}\|_{W^{2,4}}^4 \right) \|B\|_{H^3}^2. \tag{3.20}
\]

To estimate \( K_4, K_5, K_6 \), we have

\[
K_4 \leq C \|\tilde{g}\|_{W^{4,\infty}} \|B\|_{H^3} \|\tilde{g}\|_{W^{3,\infty}} \|\nabla B\|_{H^3}
\leq \frac{\mu}{16} \|\nabla B\|_{H^3}^2 + C \|\tilde{g}\|_{W^{4,\infty}}^4 \|B\|_{H^3}^2, \tag{3.21}
\]

\[
K_5 \leq C \|\tilde{g}\|_{W^{4,\infty}} \|B\|_{H^3}^2 \|\nabla B\|_{H^3}
\leq \frac{\mu}{16} \|\nabla B\|_{H^3}^2 + C \|\tilde{g}\|_{W^{4,\infty}}^2 \|B\|_{H^3}^4, \tag{3.22}\]
\[ K_6 \leq C \lVert (\nabla \times \tilde{g}) \times \tilde{g} \rVert_{W^{3,\infty}} + \lVert B \rVert_{H^s} \lVert \nabla B \rVert_{H^{s+1}} \]
\[ \leq \frac{\mu}{16} \lVert \nabla B \rVert_{H^s}^2 + C \lVert \tilde{g} \rVert_{W^{3,\infty}} \lVert B \rVert_{H^s}. \]

(3.23)

At last, we consider the term \( \int_{\mathbb{R}^3} \Lambda^k G \Lambda^k B \, dx \). Recalling the expression of \( G \), we have

\[
\begin{aligned}
\sum_{0 \leq k \leq 3} \left| \int_{\mathbb{R}^3} \Lambda^k G \Lambda^k B \, dx \right| \\
= \sum_{0 \leq k \leq 3} \left| \int_{\mathbb{R}^3} \Lambda^k (\nabla \times (\tilde{f} \times \tilde{g}) + 2\nu \nabla \cdot (\nabla \chi_M g) - \nu \Delta \chi_M g \\
+ \frac{1}{2} f \cdot \nabla \chi_M^2 g - \frac{1}{2} g \cdot \nabla \chi_M^2 f \} \cdot \Lambda^k B \, dx \right| \\
&\quad + \sum_{0 \leq k \leq 3} \left| \int_{\mathbb{R}^3} \Lambda^k (\sigma (\nabla \times \tilde{g}) \times \tilde{g}) - \kappa (\nabla \times \tilde{g}) \times \tilde{g}) \cdot \Lambda^k (\nabla \times B) \, dx \right| \\
&\leq C \lVert \tilde{f} \times \tilde{g} \rVert_{H^s} \lVert \nabla B \rVert_{H^s} + \lVert \nabla \chi_M g \rVert_{H^s} \lVert \nabla B \rVert_{H^s} + C \lVert \Delta \chi_M g \rVert_{W^{3,\infty}} \lVert B \rVert_{W^{3,\infty}} \\
&\quad + C \left( \| f \cdot \nabla \chi_M^2 g \|_{W^{3,\infty}} + \| g \cdot \nabla \chi_M^2 f \|_{W^{3,\infty}} \right) \| B \|_{W^{3,\infty}} \\
&\quad + C \left( \| (\nabla \times \tilde{g}) \times \tilde{g} \|_{H^s} + C \| (\nabla \times \tilde{g}) \times \tilde{g} \|_{H^s} \| \nabla B \|_{H^s} \right) \\
&\leq C \left( \| \tilde{f} \times \tilde{g} \|_{H^s} + M_0^{-\frac{1}{2}} \| f \|_{H^s} + M_0^{-\frac{1}{2}} \| g \|_{H^s} \right) \| \nabla B \|_{H^s} \\
&\quad + C |\alpha_1 | \alpha_2 | M_0^{-\frac{1}{2}} \| f \|_{H^s} e^{-\frac{|x|^2}{2}} \| \nabla B \|_{H^s} \\
&\quad + C \left( \| (\nabla \times \tilde{g}) \times \tilde{g} \|_{H^s} + C \| (\nabla \times \tilde{g}) \times \tilde{g} \|_{H^s} \| \nabla B \|_{H^s} \right) \| \nabla B \|_{H^s}. \\
\end{aligned}
\]

(3.24)

**Step 2: Energy inequalities of \( U \).**

Operating Equation (3.1) with \( \Lambda^k, 0 \leq k \leq 3 \), and taking \( L^2 \) on Equation (3.1) yields

\[
\frac{1}{2} \frac{d}{dt} \| U \|_{H^s}^2 + \mu \| \nabla U \|_{H^s}^2 = H_1 + H_2 + H_3 + H_4 + H_5 + H_6 \\
+ \sum_{0 \leq k \leq 3} \int_{\mathbb{R}^3} \Lambda^k F \Lambda^k U \, dx - \sum_{0 \leq k \leq 3} \int_{\mathbb{R}^3} \Lambda^k \nabla \left( p + \frac{1}{2} | \tilde{f} |^2 - \frac{1}{2} | \tilde{g} |^2 \right) \Lambda^k U \, dx,
\]

where

\[
\begin{aligned}
H_1 &= - \sum_{0 \leq k \leq 3} \int_{\mathbb{R}^3} \Lambda^k (U \cdot \nabla U) \cdot \Lambda^k U \, dx, \\
H_2 &= - \sum_{0 \leq k \leq 3} \int_{\mathbb{R}^3} \Lambda^k (\tilde{f} \cdot \nabla U) \cdot \Lambda^k U \, dx, \\
H_3 &= - \sum_{0 \leq k \leq 3} \int_{\mathbb{R}^3} \Lambda^k (U \cdot \nabla \tilde{f}) \cdot \Lambda^k U \, dx, \\
H_4 &= \sum_{0 \leq k \leq 3} \int_{\mathbb{R}^3} \Lambda^k (B \cdot \nabla B) \cdot \Lambda^k U \, dx, \\
H_5 &= \sum_{0 \leq k \leq 3} \int_{\mathbb{R}^3} \Lambda^k (\tilde{g} \cdot \nabla B) \cdot \Lambda^k U \, dx, \\
H_6 &= \sum_{0 \leq k \leq 3} \int_{\mathbb{R}^3} \Lambda^k (B \cdot \nabla \tilde{g}) \cdot \Lambda^k U \, dx.
\end{aligned}
\]
First, we have

\[ |H_1 + H_2| \leq \sum_{0 \leq k \leq 3} \left( \int_{\mathbb{R}^3} (\Lambda^k (U \cdot \nabla U) - (U \cdot \nabla \Lambda^k U) \cdot \Lambda^k U) dx + \int_{\mathbb{R}^3} (\Lambda^k (f \cdot \nabla U - \bar{f} \cdot \nabla \Lambda^k U) \cdot \Lambda^k U) dx \right) \]

\[ + \sum_{0 \leq k \leq 3} \left( \int_{\mathbb{R}^3} ((U \cdot \nabla U) \cdot \Lambda^k U + (\bar{f} \cdot \nabla \Lambda^k U) \cdot \Lambda^k U) dx \right) \]

\[ \leq \sum_{0 \leq k \leq 3} \left( \int_{\mathbb{R}^3} (\Lambda^k (U \cdot \nabla U) - (U \cdot \nabla \Lambda^k U) \cdot \Lambda^k U) dx + \int_{\mathbb{R}^3} (\Lambda^k (\bar{f} \cdot \nabla U - \bar{f} \cdot \nabla \Lambda^k U) \cdot \Lambda^k U) dx \right) \]

\[ + \sum_{0 \leq k \leq 3} \left( \int_{\mathbb{R}^3} (u \cdot \nabla \Lambda^k U) \cdot \Lambda^k U dx \right) \]

\[ \leq \sum_{0 \leq k \leq 3} \left( \int_{\mathbb{R}^3} (\Lambda^k (U \cdot \nabla U) - (U \cdot \nabla \Lambda^k U) \cdot \Lambda^k U) dx \right) + \left( \int_{\mathbb{R}^3} (\Lambda^k (\bar{f} \cdot \nabla U - \bar{f} \cdot \nabla \Lambda^k U) \cdot \Lambda^k U) dx \right) \]

\[ \leq C \left( \|U\|_{L^6} \|\nabla U\|_{H^s} + \|\nabla U\|_{L^6} \|\|U\|_{W^{1,6}} \right) \|U\|_{H^s} \]

\[ + C \left( \|\nabla U\|_{H^s} \|\nabla \bar{f}\|_{L^6} + \|\nabla U\|_{L^6} \|\bar{f}\|_{W^{1,6}} \right) \|U\|_{H^s} \]

By using Sobolev inequality, we deduce that

\[ |H_1 + H_2| \leq C \left( \|\Lambda^\frac{2}{3} U\|_{H^s}^2 \|U\|_{H^s} + \|U\|_{H^s}^2 \|\bar{f}\|_{W^{1,6}} + \|\bar{f}\|_{W^{1,6}} \right) \]. \hspace{1cm} (3.25) \]

For \( H_3 \), it is easy for us to get

\[ |H_3| \leq C \|U\|_{H^s} \|B\|_{H^s} \|\bar{f}\|_{W^{1,6}}. \] \hspace{1cm} (3.26) \]

Using the similar way to estimate \( H_1 + H_2 \), we have

\[ |H_4 + H_5| \leq C \left( \|\nabla B\|_{H^s} \|\nabla B\|_{L^6} + \|\nabla B\|_{L^6} \|\|B\|_{W^{1,6}} \right) \|U\|_{H^s} \]

\[ + C \left( \|\nabla B\|_{H^s} \|\nabla g\|_{L^6} + \|\nabla B\|_{L^6} \|\|g\|_{W^{1,6}} \right) \|U\|_{H^s} \]

\[ \leq C \left( \|\nabla B\|_{H^s}^2 \|B\|_{H^s} + \|U\|_{H^s} \|B\|_{H^s} \|\|g\|_{W^{1,6}} + \|\|g\|_{W^{1,6}} \right) \] \hspace{1cm} (3.27) \]

For \( H_6 \), we have

\[ |H_6| \leq C \|U\|_{H^s} \|B\|_{H^s} \|\bar{g}\|_{W^{1,6}}. \] \hspace{1cm} (3.28) \]

As for \( \int_{\mathbb{R}^3} \Lambda^k F \Lambda^k U dx \), it suffices for us to have

\[ \sum_{0 \leq k \leq 3} \left| \int_{\mathbb{R}^3} \Lambda^k F \Lambda^k U dx \right| = \sum_{0 \leq k \leq 3} \left| \int_{\mathbb{R}^3} \Lambda^k (\bar{f} \times (\nabla \times \bar{f}) - \bar{g} \times (\nabla \times \bar{g}) - \nu \Delta \chi_{M_0} f \right. \]

\[ + 2 \nu \nabla \cdot (\nabla \chi_{M_0} f) \cdot \Lambda^k U dx \] \]

\[ \leq C \left( \|\bar{f} \times (\nabla \times \bar{f})\|_{H^s} + \|\bar{g} \times (\nabla \times \bar{g})\|_{H^s} \right) \|U\|_{H^s} \]

\[ + C \left( \|\nabla \chi_{M_0} f\|_{W^{1,6}} + \|\Delta \chi_{M_0} f\|_{W^{1,6}} \right) \|U\|_{W^{1,6}} \] \hspace{1cm} (3.29) 

\[ \leq C \left( \|\bar{f} \times (\nabla \times \bar{f})\|_{H^s} + \|\bar{g} \times (\nabla \times \bar{g})\|_{H^s} \right) \|U\|_{H^s} \]

\[ + C M_0 \frac{1}{2} M_2 |\alpha_1| e^{-\frac{c}{2}} \|\Lambda^\frac{s}{2} U\|_{H^s}. \]
We could estimate the pressure term in the same way with in Zhang. Since
\[
p = (-\Delta)^{-1} \text{div} (u \cdot \nabla u - b \cdot \nabla b) = \sum_{ij} (-\Delta)^{-1} \partial_i \partial_j (u_i u_j - b_i b_j) + (-\Delta)^{-1} \nabla \cdot (U \cdot \nabla \tilde{f} - B \cdot \nabla \tilde{g}) + (-\Delta)^{-1} \nabla \cdot (\tilde{f} \times (\nabla \times \tilde{f}) - \tilde{g} \times (\nabla \times \tilde{g})) - \frac{1}{2} |\tilde{f}|^2 + \frac{1}{2} |\tilde{g}|^2,
\]
then we have
\[
\Pi := \left| - \sum_{k \geq 3} \int_{\mathbb{R}^3} \Lambda^k \nabla \left( p + \frac{1}{2} |\tilde{f}|^2 - \frac{1}{2} |\tilde{g}|^2 \right) \Lambda^k U dx \right| \\
\leq \left( \|u \otimes U\|_{W^{3,2}}^2 + \|h \otimes B\|_{W^{3,2}}^2 \right) \|\nabla \delta \chi M_1\|_{W^{3,2}} + \|\nabla \delta \chi B\|_{H^6} \|U\|_{H^6} + \|\nabla \delta \chi \tilde{g}\|_{H^3} \|U\|_{H^6}.
\]
By Hölder's inequality, we furthermore derive that
\[
\Pi \leq C \left( \|U\|_{H^6}^2 + \|B\|_{H^6}^2 \right) \left( \|\nabla \delta \chi M_1\|_{W^{3,2}} + \|\nabla \delta \chi B\|_{H^6} \right) \alpha_1 M_2 M_0^{-1} e^{-\frac{\mu}{\nu}} \left| a_2 \right| M_2 M_0^{-1} e^{-\frac{\mu}{\nu}} \left| a_1 \right| M_1 e^{-\frac{\mu}{\nu}} \left| a_2 \right| M_1 e^{-\frac{\mu}{\nu}} \left| a_1 \right| M_1 e^{-\frac{\mu}{\nu}} \left( \delta M_0^{-1} M_1^2 + M_0^{-1} M_2^2 \right) \|U\|_{H^6}.
\]
In a result, we get
\[
\Pi \leq C \left( \|a_1 \right| M_1 + M_2) e^{-\frac{\mu}{\nu}} \|U\|_{H^6}^2 + \left| a_1 \right| M_2 e^{-\frac{\mu}{\nu}} \|B\|_{H^6}^2 \right) + C \left| a_2 \right| M_1 e^{-\frac{\mu}{\nu}} \|U\|_{H^6} \|B\|_{H^6}^2 + C \left( a_2^2 e^{-\frac{\mu}{\nu}} + a_2^2 e^{-\frac{\mu}{\nu}} \right) \left( \delta M_0^{-1} M_1^2 + M_0^{-1} M_2^2 \right) \|U\|_{H^6}^2 + C \left| a_1 a_2 \right| M_0^{-1} M_2^2 e^{-\frac{\mu}{\nu}} \|\nabla B\|_{H^6}^2.
\]
Step 3: Energy estimates of $U$ and $B$.
Gathering above estimates in Steps 1 and 2, we obtain
\[
\frac{1}{2} \frac{d}{dt} \left( \|U\|_{H^6}^2 + \|B\|_{H^6}^2 \right) + \frac{\nu}{2} \|\nabla U\|_{H^3}^2 + \frac{\mu}{2} \|\nabla B\|_{H^3}^2 + P(t) \leq C \sum_{i=1}^8 I_i,
\]
where
\[
P(t) = \kappa \sum_{k \geq 3} \int_{\mathbb{R}^3} \left( |(\nabla \times \Lambda^k B) \times B| + |(\nabla \times \Lambda^k \tilde{g}) \times \tilde{g}| \right)^2 dx,
\]
\[
J_1 = \left( \|U\|_{H^6} + \|B\|_{H^6} \right) \left( \|\nabla U\|_{H^3}^2 + \|\nabla B\|_{H^3}^2 \right),
\]
\[
J_2 = \left( \|\tilde{f}\|_{W^{4,\infty}} + \|\tilde{g}\|_{W^{4,\infty}} \right) \left( \|U\|_{H^6}^2 + \|B\|_{H^6}^2 \right),
\]
\[ J_3 = \left( \| \tilde{f} \times \tilde{g} \|_{H^3} + M_0^{-\frac{1}{2}}M_2|a_2|e^{-\frac{u_i}{\tau}} + |a_1a_2|M_0^{-\frac{1}{2}}M_2^2e^{-\frac{(u_i^2 + v_i^2)}{4\tau}} \right) \| \nabla B \|_{H^4}. \]

\[ J_4 = (|\sigma|\|\nabla \times \tilde{g} \times \tilde{g}\|_{H^5} + \kappa\|((\nabla \times \tilde{g}) \times \tilde{g})\|_{H^5}) \| \nabla B \|_{H^5} \]

\[ J_5 = |a_1|\|M_1 + M_2\|e^{-\frac{u_i}{\tau}}\|U\|_{H^2} + \|\tilde{g}\|_{W^{3,\infty}} \| \nabla B \|_{H^5}. \]

\[ J_6 = |a_1|M_2e^{-\frac{u_i}{\tau}} \| \nabla B \|_{H^2} + |a_2|\|M_1e^{-\frac{u_i}{\tau}}\|U\|_{H^5} \| \nabla B \|_{H^5}. \]

\[ J_7 = \left( a_1^2e^{-\frac{u_i}{\tau}} + a_2^2e^{-\frac{v_i}{\tau}} \right) \left( \delta M_0^2M_1^3 + M_0^{-1}M_2^3 \right) \| U \|_{H^1}. \]

\[ J_8 = |a_1a_2|M_0^{-\frac{1}{2}}M_2^2e^{-\frac{(u_i^2 + v_i^2)}{4\tau}} \| \nabla B \|_{H^4}. \]

\[ J_9 = \left( \| \tilde{f} \times (\nabla \times \tilde{f})\|_{H^5} + \| \tilde{g} \times (\nabla \times \tilde{g})\|_{H^5} \right) \| U \|_{H^5}. \]

\[ J_{10} = \sigma\|\nabla B\|_{H^t}\|\nabla B\|_{H^4}\|\tilde{g}\|_{W^{3,\infty}} + M_0^{-\frac{3}{2}}M_2e^{-\frac{u_i}{\tau}} \| \Lambda^{\frac{3}{2}}U \|_{H^5}. \]

\[ J_{11} = C\|B\|_{H^t}^4 \left( \| \tilde{g} \|_{W^{3,\infty}} + \| \tilde{g} \|_{W^{2,\infty}} \right) + C\|B\|_{H^t}^2 \left( \| \tilde{g} \|_{W^{4,\infty}} + \| \tilde{g} \|_{W^{3,\infty}} \right). \]

Using Lemma 2.3, Young's inequality and \( \delta \geq M_0 \gg 1 \), we derive that

\[
\frac{d}{dt} \left( \| U \|_{H^t}^2 + \| B \|_{H^t}^2 \right) + \left( \frac{\nu}{2} - C\| U \|_{H^t} - | C \| B \|_{H^t} \right) \| \Lambda^{\frac{3}{2}}U \|_{H^5}^2 + \left( \frac{\mu}{2} - C\| U \|_{H^t} - | C \| B \|_{H^t} \right) \| \nabla B \|_{H^t}^2 \leq C \left( e^{-\frac{u_i}{\tau}} + e^{-\frac{v_i}{\tau}} \right) \left( \| U \|_{H^t}^2 + \| B \|_{H^t}^2 \| B \|_{H^t} \right) + C \left( M_0^{-1} + \delta^2M_0^2 \right) \left( e^{-\frac{u_i}{\tau}} + e^{-\frac{v_i}{\tau}} \right)
\]

for some constant \( C \) depending on \( M_1, M_2, \mu, \nu, \sigma, \kappa, a_1, a_2 \).

For \( t \in [0, \infty) \), we assume that

\[ \| U(t) \|_{H^t}^2 + \| B(t) \|_{H^t}^2 \leq \frac{\min\{\mu, \nu\}}{4C}. \]

In case \( t = 0 \), the above estimate holds. Applying differential inequality (3.33), Gronwall's inequality and \( \delta \leq M_0^{-2} \), we have

\[ \| U(t) \|_{H^t} + \| B(t) \|_{H^t} \leq M_0^{-\frac{1}{2}}. \]

In a result,

\[ \| U(t) \|_{H^t} + \| B(t) \|_{H^t} \leq M_0^{-\frac{1}{2}} \]

for all \( t \in [0, \infty) \). Therefore, we complete the proof of Theorem 1.1.

\[ \square \]

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**CONFLICT OF INTEREST**

The authors declared that this work does not have any conflicts of interest.
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