The Luneburg-Lissajous lens

HUIYAN PENG1,2, HUASHUO HAN1,2, PINCHAO HE1,2, KEQIN XIA1,2, JIAXIANG ZHANG1,2, XIAOCHAO LI1,2, QIAOLIANG BAO3, YING CHEN1,2(a) and HUANYANG CHEN1,2(b)

1 Institute of Electromagnetics and Acoustics and Key Laboratory of Electromagnetic Wave Science and Detection Technology, Xiamen University - Xiamen 361005, China
2 Department of Electrical and Electronics Engineering, Xiamen University Malaysia - 43900 Sepang, Selangor, Malaysia
3 Department of Materials Science and Engineering, ARC Centre of Excellence in Future Low-Energy Electronics Technologies (FLEET), Monash University - Clayton, Victoria 3800, Australia

received 29 December 2019; accepted in final form 1 April 2020
published online 10 April 2020
PACS 42.15.Dp – Wave fronts and ray tracing
PACS 42.15.Eq – Optical system design

Abstract – We design a new absolute optical instrument by composing Luneburg lens and Lissajous lens, and analyze its imaging mechanism from the perspective of simple harmonic oscillations. The imaging positions are determined by the periods of motions in the x and y directions. Besides, instruments composed with multipart is also devised, which can form imaging or self-imaging as long as the motion periods of the x and y directions are satisfied to similar conditions. We further investigate the imaging performance of such Luneburg-Lissajous lens in wave optics. Our work provides a new way to analyze the imaging of different lens by simply dissociating the equations of motions, and reveal the internal mechanism of some absolute optical instruments.

Introduction. – Absolute optical instruments are important lenses with perfect imaging or self-imaging in geometric optics [1–3]. It is very important in conformal cloaking designs [3–5]. Famous lenses are Maxwell’s fish-eye lens [6], Etaon lens [7] and Luneburg lens [8]. Most of them are of rotation symmetry. Recently, the Lissajous lens [9] has also been studied to be such kind of lenses, yet without rotation symmetry. Conformal transformation could also be performed on the Mikaelian lens [10] to obtain lenses without rotation symmetry [11]. A general lens could be achieved by the analogy of geometric optics and classical mechanics based on the Hamilton-Jacobi equation [12]. In fact, the Luneburg lens and the Lissajous lens share the same equations of motions, which are simply the harmony oscillations [2]. Hence, if we could separately construct different harmony oscillations in the x and y directions, by matching well each of their periods, a kind of new absolute optical instruments could be obtained.

In the paper, we simply study the compositions of Luneburg lens and/or Lissajous lens with matching boundaries, and study their imaging properties based on harmony oscillations. We give a very simple picture for this kind of complicated lenses.

Results. – Let us start from the well-known Luneburg lens expressed as \( n(x, y) = \sqrt{2 - x^2 - y^2} \), and the trajectory is shown in fig. 1(a), where a closed elliptic light path can be seen inside the Luneburg lens. Similarly, for the Lissajous lens \( n(x, y) = \sqrt{2 - x^2/a^2 - y^2/c^2} \), with \( a = 1 \), \( c = 2 \), as displayed in fig. 1(b), the index at the boundary of this elliptical lens (the green circular areas with semiaxes \( \sqrt{2} \) and \( \sqrt{8} \)) is zero, and the trajectory is also a closed one. As is known, the above two classical lenses are absolute optical instruments, namely, rays emitting from any point \((x_0, y_0)\) in space will later converge to another point and form imaging (or self-imaging). We next design a Luneburg-Lissajous (L-L) lens in fig. 1(c), with the refractive index profiles \( n(x, y) = \sqrt{2 - x^2 - y^2} \) in the upper space \((y > 0)\) and \( n(x, y) = \sqrt{2 - x^2/a^2 - y^2/c^2} \) in the lower space \((y < 0)\). The continual refractive index condition at the interface of \( y = 0 \) should be satisfied to keep light without either refraction or reflection at the interface. Figure 1(c) shows an example of such a L-L lens with \( a = 1 \) and \( c = 2 \).
Interestingly, a closed path can still be found in such a composite lens. Light emerging from the \((x_0, y_0)\) point will later return back to the original point after going through several periods. The closed path indicates that this L-L lens may be an effective absolute optical instrument.

In fig. 2, we first analyze the imaging mechanisms of the Lissajous lens from the perspective of simple harmonic oscillations (SHO), based on the analogy of light to classical mechanics [2]. The emission point \((x_0, y_0)\) is set at \((0.5, 0.5)\) in the first quadrant for example, as shown by the grey points. For the refractive index distribution of a two-dimensional (2D) Lissajous lens \(n(x, y) = \sqrt{2 - x^2/a^2 - y^2/c^2}\), the solved motion curves of SHO along the \(x\) and \(y\) directions can be written as

\[
\begin{align*}
x &= x_0 \cos(t/a) + A \sin(t/a), \\
y &= y_0 \cos(t/c) + B \sin(t/c),
\end{align*}
\]

where \(t\) is the time parameter from classical mechanics, \(A\) and \(B\) are constants depending on the initial emission angle \(\theta\) (i.e., \(\arctan(B/A)\) as the tangential angle \(\theta\)), and \((x_0, y_0)\) is the coordinate of the source. We can see that the motions in the \(x\) and \(y\) directions have the period of \(2\pi\) and \(2\pi\), respectively. The full period is then equal to \(2N\pi\), where \(N\) is the least common multiple of \(a\) and \(c\). When \(a = 1, c = 2\), the ray trajectories in the lens can be seen in fig. 2(a), where three rays are emitted from the original point and imaged at \((x_0, -y_0)\), following mirror symmetry along the \(x\)-axis. We also plot two other cases with \(a = 1, c = 3\) and \(a = 2, c = 1\), and the optical paths are displayed in figs. 2(c) and (e). Interestingly, different imaging positions are obtained. For the Lissajous lens with \(a = 1, c = 3\), the centrosymmetric imaging is achieved at \((-x_0, -y_0)\), while for that with \(a = 2, c = 1\), imaging of mirror symmetry along the \(y\)-axis is shown. Therefore, three kinds of imaging position are observed in the Lissajous lens. In order to explore the relationships between the imaging mechanism and parameters \(a\) and \(c\), we analytically plot the curves of SHO along the \(x\) and \(y\) directions based on eq. (1), as shown in figs. 2(b), (d) and (f). The solid curves indicate the motions in the \(x\)-direction, while the dashed curves represent those in the \(y\)-direction. Firstly, for the SHO of Lissajous lens with \(a = 1, c = 2\) (see fig. 2(b)), when emitting from the grey point \((x_0, y_0)\) at \(t = 0\), the motions in the \(x\)-direction have a period of \(2\pi\) for the three paths, and that in the \(y\)-direction is \(4\pi\). We find that these intersections locating at \(x\) and \(y\) curves are exactly the coordinate points \((x_0, -y_0)\) and \((x_0, y_0)\), corresponding to the results in fig. 2(a). The imaging time parameter is at \(2\pi\), while the full period is \(4\pi\). When it comes to the case of \(a = 1\) and \(c = 3\) in eq. (1), the oscillation periods in the \(x\)- and \(y\)-direction are \(2\pi\) and \(6\pi\), respectively, as shown in fig. 2(d). The imaging point appears at the time of \(3\pi\), i.e., \((-x_0, -y_0)\) and the full period is \(6\pi\), coming back to \((x_0, y_0)\). Similarly, the light oscillations for the case of \(a = 2\) and \(c = 1\) are shown in fig. 2(f), with the imaging time of \(2\pi\), i.e., at the black dots related to \((-x_0, y_0)\). We can see that the analysis of SHO gives a nice explanation of the imaging mechanism, which is determined by the periodicities of SHO in both \(x\) and \(y\) directions. The position...
of imaging time is at $t = N\pi$. Again, $N$ is the least common multiple of $a$ and $c$.

We next come to the general L-L lens and study the ray propagation inside it. As is shown in fig. 3, there are two parts composing it, with different refractive index profiles:

$$n_{\text{L-L}}(x, y) = \begin{cases} \sqrt{2 - x^2/a^2 - y^2/b^2}, & y > 0, \\ \sqrt{2 - x^2/a^2 - y^2/c^2}, & y < 0. \end{cases} \quad (2)$$

The refractive index at the boundary of $y = 0$ is matched for every $x$. In terms of the SHO, the period in the $x$-direction is still $2a\pi$, while that in the $y$-direction has changed to $(b + c)\pi$. The full period is then equal to $M\pi$, where $M$ is the least common multiple of $2a$ and $(b + c)$.

To explore the ray behavior inside such L-L lens and verify its imaging features, we first consider a L-L lens with $a = 1$, $b = 1$ and $c = 2$ (the lens in fig. 1(c)). The trajectories of three light paths are shown in fig. 3(a), where light emitting from the source position $(x_0, y_0)$ with different angles later focuses on the mirror point $(-x_0, y_0)$ without aberration, illustrating the property of absolute optical instrument. In fig. 3(b) the analytic harmonic motions along the $x$ and $y$ directions are also exhibited.

We clearly see that the intersections of $x$ and $y$ rays happen at the position of $(-x_0, y_0)$ when $t = M\pi/2 = 3\pi$, and at the position of $(x_0, y_0)$ when $t = M\pi = 6\pi$, forming $y$-axis symmetric imaging and self-imaging. Secondly, when $a = 1$, $b = 1$ and $c = 3$, the closed ray trajectories and dissociated $x$- and $y$-motion paths are shown in figs. 3(c) and (d). Obviously, the perfect intersections only appear at the integer time of $t = M\pi = 4\pi$, corresponding to the coordinate $(x_0, y_0)$, which illustrates only the self-imaging phenomenon in fig. 3(c). Note that for $t = M\pi/2 = 2\pi$, there is no perfect intersection, which means that there is no other image. In the last case of $a = 4$, $b = 1$ and $c = 3$ in figs. 3(e) and (f), the intersections are at the position of $(-x_0, y_0)$ for $t = M\pi/2 = 4\pi$, and the position of $(x_0, y_0)$ for $t = M\pi = 8\pi$, i.e., the $y$-axis symmetric imaging and self-imaging. Therefore, by composing two lenses together, a new absolute optical instrument is designed with various imaging positions, which are determined by the periods of SHO in the $x$ and $y$ directions.

We will further focus on the properties of these general L-L lenses in fig. 3 in wave optics, as shown in fig. 4. The full-wave simulations are performed by the commercial software COMSOL Multiphysics. For comparison, we take the same parameters in the general L-L lens as those in fig. 3, and use a current source with electric component (with unit V/m) out of plane of electromagnetic waves (with a frequency of 2 GHz). In fig. 4(a) of $(a) a = 1, b = 1$ and $c = 2$, when the source is placed at $(x_0, y_0)$, as the red arrow shows, an obvious image can be found at the position of $(-x_0, y_0)$, indicating the wave phenomenon. The $y$-axis symmetric imaging and self-imaging are consistent with the phenomena in geometrical optics in fig. 3(a). Besides, for $a = 1, b = 1, c = 3$ in fig. 4(b), only self-imaging exists, and the centrosymmetric imaging in the third quadrant is not a perfect imaging, which can also be found in fig. 3(c) with a small aberration. In the same way, we study the case of $a = 4, b = 1, c = 3$ in the general L-L lens, and the imaging performance in wave optics is shown in fig. 4(c), where the self-imaging and $y$-axis symmetric
imaging are both observed. Therefore, the results in fig. 4 verify the effective retention of imaging in wave optics.

Lastly, we introduce a complicated L-L lens composed by four Lissajous lens, as denoted by four different colors in fig. 5(a). The boundary of \( n = 0 \) is determined by the following refractive index profiles:

\[
\begin{align*}
n_{L-L}(x,y) &= \begin{cases} 
\sqrt{2 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}, & x > 0, \ y > 0, \\
\sqrt{2 - \frac{x^2}{d^2} - \frac{y^2}{b^2}}, & x < 0, \ y > 0, \\
\sqrt{2 - \frac{x^2}{d^2} - \frac{y^2}{c^2}}, & x < 0, \ y < 0, \\
\sqrt{2 - \frac{x^2}{a^2} - \frac{y^2}{c^2}}, & x > 0, \ y < 0,
\end{cases}
\end{align*}
\]

where the continual refractive index conditions are satisfied at both \( x = 0 \) and \( y = 0 \) for the above similar reason. Here we take \( a = 1, \ b = 2, \ c = 3, \ d = 4 \) for example, the L-L lens is shown in fig. 5(a). We can see that the light can still return to the emission point and forms a closed trajectory. For such a special lens, the self-imaging character is displayed in fig. 5(b), which can be comprehended easily through the period of SHO above. For the oscillation in the \( x \)-direction, the period is \((a + d)\pi\), while for the \( y \)-direction, the period is \((b + c)\pi\). The self-imaging happens at \( t = M\pi \), where \( M \) is the least common multiple of \((a + d)\) and \((b + c)\). For this case, \( M = 5 \). In addition, we can also construct other lenses using reflection mirrors. For example in figs. 5(c) and (d), two cases with reflection mirrors are discussed to further study the imaging properties based on the Lissajous lenses in figs. 2(a) and (c). As is shown in fig. 5(c), when a \( x \)-axis mirror \((y = 0)\) is added, the previous imaging at \((x_0, -y_0)\) is mapped to the emission point and only self-imaging is observed. Meanwhile, when two mirrors at both \( x = 0 \) and \( y = 0 \) are added, the rays are reflected back with self-imaging as well. However, for the case in fig. 2(c), the \( x \)-axis mirror at \( y = 0 \) can map the imaging at the point of \((-x_0, -y_0)\) to the point \((-x_0, y_0)\), as denoted by the black dot of fig. 5(d). However, for mirrors at both \( x = 0 \) and \( y = 0 \), there is only self-imaging effect.

Finally, we remark that, although the parameters in the L-L lens we discussed above are integers, the imaging mechanisms are applicable for rational parameters \( a, b, c \) and \( d \). While for the irrational parameters in the refractive index (if they cannot be reduced), the trajectories inside the lens are not closed and there is no imaging or self-imaging existed.

**Conclusion.** – In this work, we propose a composed Luneburg-Lissajous (L-L) lens with perfect imaging, i.e., an absolute instrument. The imaging mechanisms of Lissajous lens and L-L lens are studied by harmonic oscillations in the \( x \) and \( y \) directions. The self-imaging happens at the parameter time of the least common multiple time of the periods of the \( x \) and \( y \) directions. The imaging happens at half of the self-imaging time, which have to be verified from the oscillations in the \( x \) and \( y \) directions. A complicated L-L lens and the effect of different reflection mirrors on imaging are further discussed. This method can be easily extended to three dimensions. Our study therefore enriches the family of absolute optical instruments, especially for Luneburg lens and Lissajous lens. They can be realized by either from curve surfaces, such as geodesic lenses [13,14], or gradient index lenses with required dielectric profiles [15,16].

This work was financially supported by the National Natural Science Foundation of China (Grant No. 11874311) and the Fundamental Research Funds for the Central Universities (Grant No. 20720170015). HH, PH, KX, and JZ are undergraduate students in Xiamen University, Malaysia. We thank the support from Prof. ZHONG CHEN, Prof. HUIQIONG WANG, Mr. JINGFENG CHEN, and Miss YULING ZHENG.

REFERENCES

[1] Tyc T., Herzanova L., Sarbort M. and Bering K., New J. Phys., 13 (2011) 115004.
[2] Leonhardt U. and Philbin T., Geometry and Light: The Science of Invisibility (Courier Corporation, Chelmsford, Mass.) 2012.
[3] Xu L. and Chen H., Nat. Photon., 9 (2014) 15.
[4] Leonhardt U., Science, 312 (2006) 1777.
[5] Leonhardt U., New. J. Phys., 8 (2006) 118.
[6] Maxwell J. C., Cambridge Dublin Math. J., 8 (1854) 188.
[7] Eaton J. E., Trans. IRE Antennas Propag., 4 (1952) 66.
[8] LUNEBURG R. K., *Mathematical Theory of Optics* (University of California Press, Berkeley, Cal.) 1964.

[9] DANNER A., DAO H. L. and TYC T., *Opt. Express*, 23 (2015) 5716.

[10] MIKAELIAN A. and PROKHOROV A., *Prog. Opt.*, 17 (1980) 279.

[11] CHEN H. Y., *Phys. Rev. A*, 98 (2018) 043843.

[12] TYC T. and DANNER A. J., *Phys. Rev. A*, 96 (2017) 053838.

[13] SARBORT M. and TYC T., *J. Opt.*, 14 (2012) 075705.

[14] XU L., WANG X., TYC T., SHENG C., ZHU S., LIU H. and CHEN H. Y., *Photon. Res.*, 7 (2019) 1266.

[15] WANG X., CHEN H. Y., LIU H., XU L., SHENG C. and ZHU S., *Phys. Rev. Lett.*, 119 (2017) 033902.

[16] LI S., ZHOU Y., DONG J., ZHANG X., CASSAN E., HOU J., YANG C., CHEN S., GAO D. and CHEN H. Y., *Optica*, 5 (2018) 1549.