Design of a nonscaling fixed field alternating gradient accelerator

D. Trbojevic,* E. D. Courant, and M. Blaskiewicz

BNL, Upton, New York 11973, USA

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We present a design of nonscaling fixed field alternating gradient accelerators (FFAG) minimizing the dispersion action function $H$. The design is considered both analytically and via computer modeling. We present the basic principles of a nonscaling FFAG lattice and discuss optimization strategies so that one can accelerate over a broad range of momentum with reasonable apertures. Acceleration schemes for muons are discussed.

II. PARTICLE MOTION AT VERY LARGE MOMENTUM OFFSET

We consider a particle of momentum $p$ and a reference particle with momentum $p_0$ and charge $q$; the momentum offset is $\delta = (p - p_0)/p_0$. The magnetic rigidity of the reference particle is $(Bp_0) = p_0/q$; the reference particle is on a reference orbit (assumed planar) with local radius of curvature $p_0$ and vertical field $B_0(s) = (Bp_0)/p_0(s)$. In the cases considered here the field $B_0(s)$ and with it the radius of curvature $p_0(s)$ are constant in each magnet; so that the reference orbit consists of circular arcs in the magnets and straight sections between the magnets; we also assume that the magnet edges are straight at right angles to the reference orbit. We assume that the magnetic field in the dipole magnets is linear:

$$B_y = B_0 + G_x,$$

FIG. 1. (Color) Basic half cell. The cell is symmetric with respect to $s = 0$. The total length of the cell equals $L_{cell} = L_{cavity} + 2 * L_{QF} + L_{QD} + 2 * L_{DF}$. There are two types of magnets, $QD$ (normal bend, field $B_{QD} = 4.95$ T) and $QF$ (negative bend, field $B_{QF} = -2.29$ T). Each has constant gradient, $G_{QF} = 57.19$ T/m and $G_{QD} = -34.27$ T/m.

A. The Basic Cell

We end up with the rather simple configuration of Fig. 1. The accelerator is composed of a large number (66 in our case) of identical unit cells. Each cell contains a magnet triplet, with a relatively long gradient bending magnet $QD$ ("combined function") having a strong central field and negative gradient (horizontally defocusing) at the center, flanked by a pair of negative bend magnets $QF$ that are horizontally focusing, and then a drift space about 2 m long to accommodate an accelerating cavity.

The cell is symmetric with respect to the center of the deflecting combined function dipole and the center of the long drift. The horizontal and vertical betatron functions $\beta_x$ and $\beta_y$ and the dispersion function in the basic cell, at the reference momentum, are shown in Fig. 2.

*Electronic address: dejan@bnl.gov

1In the recent 10–20 GeV scaling FFAG lattice for muon acceleration in KEK Japan the orbit offsets were reported [6] to be of the order of 40 cm. A design of the nonscaling FFAG with the same circumference [7] produces maximum orbit offsets between $\pm 42$ mm.

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with no higher order terms in \( x \) and \( y \). Here \( x \) and \( y \) are the horizontal and vertical deviations from the reference orbit; the central field \( B_0 \) is positive in QD and negative in QF while the gradient \( G \) is positive in QF and negative in QD.

For muon acceleration we have to accelerate from 10 to 20 GeV very quickly because of the short decay time of the muon. Suitable superconducting cavities for muon acceleration have been developed at Cornell University [15]; these require at least two meters of drift space. Therefore we have to provide this much space in each cell. The lengths, fields, and gradients of the magnets are adjusted so that the total bend angle of the reference orbit through one cell is

\[
B_0 = G_y,
\]

and the horizontal and vertical betatron oscillations are stable for all energies from 10 to 20 GeV, i.e., \( \delta \) ranging from \(-0.33\) to \(0.33\) with the central momentum at \( p_0 = 15 \text{ GeV}/c \). Because of the high acceleration rate it is permissible for the betatron tunes to vary over the energy range as long as they stay between about 0.05 and 0.45 per period.

A. Accuracy of accelerator physics codes

The design of a fixed field accelerator requires design codes accurate for large momentum ranges. We have previously reported [16] a study of several accelerator physics lattice design codes for a large momentum range. The tunes and the Courant-Snyder amplitude functions were calculated by several different codes and compared to the analytical predictions for the case of a simple test lattice consisting of five identical 72° dipoles, with fixed linear fields. In addition, the FFAG examples designed using the program SYNCH [17] were cross checked by Forest with the code PTC [18], by Berg using the COSY INFINITY code [19] up to 8th order in \( \delta \), and by Méot with the ZGOUBI-Saclay code [20]. Results from the “ZGOUBI” test were shown at the KEK FFAG workshop [21]. A comparison between calculations using the codes PTC, COSY INFINITY, and SYNCH, in our FFAG lattice for muon acceleration over the whole momentum range of the tunes is presented in Fig. 3, and of the maximum of the radial orbit offsets in Fig. 4. A comparison of the path length calculations by the three codes is shown in Fig. 5.

The exact equations of the particle motion in a non-scaling FFAG for very large momentum offsets have been previously discussed in detail [9,22,23,25–27].
To minimize aperture it is advantageous to minimize the
understood by observing that the maximum of the vector
(normalized dispersion action function [28] (the Courant-Snyder invari-
main momentum range. The normalized dispersion coordinates are
shown is the Table I. Figure 6 shows the
The advantage of our triplet cell compared to the doublet
or FODO structures is explained by the significantly smaller values of \( \sqrt{H_{\text{max}}} \).
Muon acceleration is performed near the top of the
sinusoidal rf wave, with a small number of turns and a
large energy gain per turn. As discussed in the section on acceleration (see below) this necessitates a fixed rf frequency. Therefore it is desirable that the time of travel around the circumference vary as little as possible with
Since the muons are relativistic this is equivalent to having the orbit length vary as little as possible with
The variation in the particle path length around the ring is
\[
\Delta C = \int_C \left( \sqrt{1 + \frac{x^2}{\rho} + y^2} - 1 \right) \, ds
\]
\[
\equiv \left[ \int_C \frac{D(s)}{\rho} \, ds \right] \delta.
\]
The lattice with the smallest dispersion function gives both

III. CONTROL OF THE AMPLITUDE OF THE DISPERSION FUNCTION

The dependence of radial offset on momentum is given by
\( \Delta r = D_s \delta \), where \( D_s \) is the horizontal dispersion function. A small dispersion function makes for large momentum acceptance. If the aperture is to be less than 4 cm the dispersion function has to be \( D_s < 12 \) cm in the whole momentum range.

A useful measure of the variation of path length is the
(normalized dispersion action function [28] (the Courant-Snyder invari-
ant of the dispersion function \( D \)), defined as
\[
H(D, D') = \left( \frac{D}{\sqrt{\beta}} \right)^2 + \left( D' \sqrt{\beta} + \frac{\alpha D}{\sqrt{\beta}} \right)^2.
\]
The normalized dispersion coordinates are
\[
\xi = \frac{D}{\sqrt{\beta}} \quad \text{and} \quad \chi = D' \sqrt{\beta} + \frac{\alpha D}{\sqrt{\beta}};
\]
\[
H(D, D') = \xi^2 + \chi^2.
\]
To minimize aperture it is advantageous to minimize the function \( H \). It suffices to minimize the average function \( \langle H \rangle \) within the main bending element. This could be easily understood by observing that the maximum of the vector
\( \sqrt{H_{\text{max}}} \) is at the end of the major bend, as shown in the
normalized dispersion plot in Fig. 6. Figure 6 shows the
table of these three different cells
are shown is the Table I.

TABLE I. A comparison between designs by the FODO, DOUBLET, or TRIPLET cells. The cells properties include the length of
the cells, gradient and magnet strengths, and bending angles

| TYPE       | \( L_{\text{cell}} \) (m) | \( G_{QF} \) (T/m) | \( G_{QD} \) (T/m) | \( \theta_{QF} \) (rad) | \( \theta_{QD} \) (rad) | \( B_{QF} \) (T) | \( B_{QD} \) (T) |
|------------|-----------------------|-----------------|-----------------|------------------|------------------|----------------|----------------|
| FODO       | 2.2237                | 64.792          | -68.214         | -0.020 87       | 0.124 40         | -1.80          | 7.323          |
| DOUBLET    | 2.2162                | 65.258          | -68.914         | -0.021 90       | 0.128 73         | -1.89          | 7.577          |
| TRIPLET    | 2.2162                | 66.464          | -70.112         | -0.021 79       | 0.128 50         | -1.88          | 7.564          |
the smallest orbit offsets and the shortest path length differences over the range of momenta. The average value of the dispersion action in the central dipole and the conditions for its minimum are given by the equations

$$\langle H \rangle = \frac{1}{L} \int_0^L H(s) ds, \quad \frac{\partial \langle H \rangle}{\partial D_0} = 0, \quad \frac{\partial \langle H \rangle}{\partial \beta_0} = 0. \quad (7)$$

The dispersion function, its slope, and the Courant-Snyder functions in the central dipole are

$$D(s) \equiv \frac{1}{\rho K} (\cosh \phi - 1) + D_0 \cosh \phi$$

$$= \left( D_0 + \frac{1}{\rho K} \right) \cosh \phi - \frac{1}{\rho K}, \quad (8)$$

$$D'(s) \equiv \left( D_0 \sqrt{K} + \frac{1}{\rho \sqrt{K}} \right) \sinh \phi + D_0' \cosh \phi$$

$$= \sqrt{K} \left( D_0 + \frac{1}{\rho K} \right) \sinh \phi, \quad (9)$$

$$\beta(s) \equiv \left( \beta_0 + \frac{\gamma_0}{K} \right) \cosh \phi - \frac{\gamma_0}{K} \cosh \phi - \frac{\gamma_0}{K}, \quad (10)$$

$$\alpha(s) \equiv -\sqrt{K} \left( \beta_0 + \frac{\gamma_0}{K} \right) \sinh \phi \cosh \phi, \quad (11)$$

$$\gamma(s) \equiv K \left( \beta_0 + \frac{\gamma_0}{K} \right) \cosh^2 \phi - K \beta_0, \quad (12)$$

where the defocusing strength $K$ is

$$K(s) = \frac{G}{B \rho} - \frac{1}{\rho^2(s)}, \quad (13)$$

and the betatron phase is $\phi = \sqrt{K} s$. The dispersion action function $H$ in the dipole is

$$H(s) = \frac{1}{\rho^2 K} \left( \beta_0 + \frac{\gamma_0}{K} \right) \cosh^2 \phi - \frac{2 \gamma_0}{\rho K} \left( D_0 + \frac{1}{\rho K} \right)$$

$$\times \cosh(\phi) - \frac{\beta_0}{\rho K} + \gamma_0 \left( D_0 + \frac{1}{\rho K} \right) \quad (14)$$

and its average value $\langle H \rangle$ is

$$\langle H \rangle = \frac{2}{L} \int_0^{L/2} H(s) ds, \quad \langle H \rangle = \frac{\theta^2}{2 q^2} \left( \beta_0 + \frac{L^2}{\beta_0 q^2} \right) \sinh q - 1 - \frac{4 \theta L}{\beta_0 q^2} \left( D_0 + \frac{L \theta}{q^2} \right)$$

$$\times \sinh \frac{q}{2} - \frac{\beta_0 \theta^2}{q^2} + \frac{1}{\beta_0} \left( D_0 + \frac{L \theta}{q^2} \right) \quad (15)$$

We minimize $\langle H \rangle$ with respect to the initial values of $\beta_0$ and $D_0$ at the center of the defocusing central bending element, by setting the partial derivatives $\partial \langle H \rangle/\partial D_0$ and $\partial \langle H \rangle/\partial \beta_0$, equal to 0. The solutions are

$$D_0 = \frac{2 \theta q^2 - \theta^2}{q^2}, \quad (16)$$

$$\beta_0 = \frac{L}{q} \left( \frac{4}{q^2} \sinh^2 \frac{q}{2} - \frac{1}{2} \left( 1 + \sinh^2 \frac{q}{2} \right) \right), \quad (17)$$

where $q = L/\sqrt{K}$ and $\theta = L/\rho$ as previously defined [29]. With the magnet lengths and strengths as given in Fig. 1 we use Eqs. (16) and (17) to set the central values of $\beta_0$ and $D_0$ and, using the SYNCH code, solve for the focusing strengths and gap dimensions to obtain zero slope at the symmetry points. We then calculate the lattice functions for a range of momenta from $\delta = -0.33$ to $\delta = +0.33$. But we find that this lattice is unstable below 14.5 GeV when the central energy is 15 GeV. Since we need stability in the whole range between 10 and 20 GeV, this lattice is clearly not acceptable. Therefore the next step is to search for stable solutions in a larger momentum range by setting larger initial values of $\beta_{x_0}$ and $D_{x_0}$. Unfortunately, with larger value of the initial $\beta_{x_0}$ and $D_{x_0}$ the maximum orbit offsets, maximum values of the $H$ function, dispersion, and variation of path length around the ring with momentum, also increase. The best solution with a stable lattice and the required range in muon energy from 10 to 20 GeV is the one of Fig. 1.

### A. A search for a large acceleration range in $\delta p/p$

A lattice cell is constructed at the central muon energy of 15 GeV with the horizontal and vertical tunes to be in the middle of the range 0.1 $< \nu_x, \nu_y < 0.5$. A search for the closed orbit solutions within a large momentum range is performed. It starts for a lattice with the initial conditions of $\beta_{x_0}$ and $D_{x_0}$ obtained by Eqs. (16) and (17). It is found that stable solutions exist only for a range of $\delta p/p$ in the positive direction from the central momentum up to +33%. As the values of the initial conditions for $\beta_{x_0}$ and $D_{x_0}$ are sequentially raised, the stable closed orbit solutions start to show for larger and larger negative range in $\delta p/p$. This procedure is described in Fig. 7. It is important to note that our analysis is done for muon acceleration, with an additional constraint for the path length variation on momentum to be as small as possible. This is optimized when the minimum of the path length occurs in the middle of the momentum range. This condition is not necessary for other applications (such as proton or electron acceleration) and the minimum of the $\langle H \rangle$ function may well come with the reference momentum at the lowest energy. This would allow larger ranges of stable horizontal and vertical betatron tunes. The dependence of the maximum of the orbit offsets on momentum for different values of the initial $\beta_{x_0}$ and $D_{x_0}$ is presented in Fig. 8. The dependence of the path length variation (units are centimeters) and the maximum orbit offsets on the maximum value of the $H$ function are
both presented in Fig. 9. The dependence of the normalized dispersion function (amplitude of $\sqrt{H}$) on momentum, for our lattice, is presented in Fig. 10. The nonscaling triplet lattice FFAG design has many possible applications. This report concentrates on muon acceleration. Muon acceleration needs to be very fast because of the short muon life time; this imposes special requirements on the lattice design, principally a small variation of the path length. This restriction is not necessary for other applications such as proton acceleration (high intensity proton FFAG, proton therapy), heavy ion acceleration (for carbon therapy or RIA-radioactive ion beams), electron acceleration, etc.

There are a few important restrictions on the lattice design for muon acceleration:

**B. Small path length difference during acceleration**

The difference in the path length as the energy increases has to be small ($\Delta L \sim 13$ cm). The path length dependence on momentum is a parabolic function [24]. It is best to have the same values of the path lengths at the beginning and the end of the acceleration. The path length minimization, for lattices with the same circumference $C_0 = 328$ m, was studied by varying the length of $QD$, the major combined function magnet, between $L_d = 8-15$ m. The relationship between the path length difference and the magnet length is presented in Fig. 11. A complete optimization has to consider additional parameters such as muon decay time (the longer the circumference the larger muon loss due to the decay), initial emittance of the muon beam to determine more precisely the magnet aperture and cost, number of turns, cavity voltage limita-
With the approximately parabolic dependence of path length on momentum the minimum value occurs near the central energy. The dependence of momentum compaction on energy during acceleration is to the first order a linear function as shown in Fig. 12. The zero value of $\alpha_e$ occurs near, but not exactly at, the reference momentum of $p_0 = 15 \text{ GeV}/c$. This is a result of the optimization for the path length dependence.

**D. Orbit offsets during acceleration**

The emittance of the incoming muon beam to be accelerated by the nonscaling FFAG depends on the efficiency and properties of the previous stages of acceleration and cooling. The needed magnet size is mostly determined by the beam size, but it is also important to reduce the beam offsets during acceleration. The major advantage of the nonscaling FFAG design with respect to scaling version is the dramatically smaller orbit offset during acceleration. The dependence of orbit offset on the maximum value of the $H$ function is shown in Fig. 8. The small orbit offset during acceleration as shown by Fig. 13 represents the major advantage of the nonscaling design compared to the scaling one. Orbits at different momenta are parallel to each other within the drift between the two focusing combined function magnets, but the orbits are not parallel to each other in the rest of the cell due to the strong focusing and magnet edge effects.

**E. Small circumference**

The smaller the circumference and orbit offsets, the lower is the cost for the whole machine. The scaling FFAG design requires longer negative bend magnets: In [2], the relationship between the negative vs positive bend is $\theta_2/\theta_1 = 2/3$. This ratio means that the total magnet length needed, in this case, is $(1.5 + 1)/1.5 = 5$ times that needed just for bending, so the circumference has to be large. The negative bend also affects the disper-

\[ \alpha_e \equiv \frac{1}{C} \frac{d\Delta C}{d\delta} = \frac{1}{C} \int \frac{D(s)}{\rho} \, ds. \]  

(18)

### TABLE II. Major magnet length, number of periods, magnetic fields, and gradients in the $QF$ and $QD$ magnets.

| $L_d$ (m) | Periods | $B_f$ (T) | $B_d$ (T) | $G_d$ (T/m) | $G_f$ (T/m) |
|-----------|---------|-----------|-----------|-------------|-------------|
| 0.80      | 75      | -1.70     | 7.704     | -70.83      | 63.16       |
| 0.90      | 74      | -1.95     | 7.233     | -65.66      | 65.38       |
| 1.00      | 72      | -2.00     | 6.686     | -56.65      | 62.43       |
| 1.20      | 68      | -2.00     | 5.852     | -44.09      | 56.68       |
| 1.28      | 67      | -2.15     | 5.614     | -40.28      | 56.43       |
| 1.40      | 66      | -2.20     | 5.225     | -36.77      | 56.63       |
| 1.50      | 66      | -2.29     | 4.946     | -34.27      | 57.19       |
A. Amplitude functions dependence on momentum

The dependence on momentum of the horizontal and vertical amplitude functions $\sqrt{\beta_x}$ and $\sqrt{\beta_y}$, and of the dispersion function $D_x$ are shown in Figs. 15–17.

G. Transverse tracking

The nonscaling FFAG presented here does not include any nonlinear fields. Particle tracking in the transverse plane at the muon central energy of 15 GeV was performed for 1500 turns, with different initial conditions of the horizontal and vertical positions and slopes. A stable motion in the horizontal phase space is shown in Fig. 18.

A study of the particle motion during acceleration is shown in the next section, while particle tracking during acceleration including effects of magnet misalignment and gradient errors have been studied elsewhere [31,32].
time, \( q \) is the charge per particle, and \( T(E) \) is the revolution period as a function of energy. To get the equation for \( V(t) \) we model a cavity as a parallel \( LC \) resonator. The current through the capacitor is \( I_C = C dV/dt \) and the current through the inductor is \( I_L = \int dV/L \). The total current driving the cavity is then \( I_L + I_C \). The net current is the sum of the contributions from the power amplifier and the beam. Since the fill time of the cavity is long compared with the acceleration time we take the voltage due the amplifier to be \( V_a = \hat{V} \sin(\omega_{rf}t + \phi) \) where \( \hat{V} \) and \( \phi \) are constants. The voltage due to the beam current, \( V_b(t) \) is identically zero before the beam is injected at \( t = 0 \). The beam image currents on the cavity walls charge the capacitive gap so the negative of the beam current generates the beam induced voltage,

\[
-(\frac{R}{Q})I_b(t) = \frac{1}{\omega_{rf}} \frac{dV_b}{dt} + \omega_{rf} \int_0^t V_b(t_1) \, dt_1,
\]

where \( I_b(t) \) is the beam current as a function of time at the rf cavity, and \( R/Q = \omega_{rf} L = 1/\omega_{rf} C \). The total voltage is then given by \( NV = N(V_a + V_b) \). We consider the three cases shown in Fig. 19. The lattice and rf parameters are given in Table III.

Figures 20–22 show the initial and final rf voltage and the bunch evolution for a total beam charge of 2.5 \( \mu \)C for the three cases in Table III. For all cases the initial emittance was 0.2eV \( \cdot \)s per bunch with an initial energy spread of \( \pm 400 \) MeV. For a lattice with the \( L_d = 0.8 \) m of the major bend, 40\% of the energy stored in the cavities is transmitted to the beam. For \( L_d = 1.5 \) m lattice this fraction is 32\%. However, the 960 MV required for \( L_d = 1.5 \) m requires more Cornell cavities than lattice periods, requiring the technology improvements.
V. CONCLUSIONS

We have presented an alternative “nonscaling triplet design” of the fixed field alternating gradient accelerator. Some disadvantages like the large momentum offsets, large circumference, large magnets sizes of the standard [2] scaling FFAG are improved or eliminated. The acceleration of muons for the “Neutrino Factory Project” or the “Muon Collider Project” has to be in a very short time due to a short muon lifetime. We have shown a lattice design for muon acceleration from 10 to 20 GeV. The properties of the lattice and magnet elements are shown at the reference momentum and through the acceleration range. The best efficiency with respect to particle stability, cost of the rf cavities and magnets is obtained when the number of muon bunch passages during acceleration is less than 15. This conclusion comes from the acceleration simulations where significant bunch distortions occur at the last few turns, especially if a larger number of turns is applied (part of it is shown in Figs. 20–22, in the bunch shape during the last few turns). The maximum orbit offsets during acceleration are within $-30 < x < 48$ mm. This reduces the aperture and the magnet size compared to the scaling design. We have studied and compared other possible applications like proton [33] and electron acceleration [34]. The proton and electron designs have no restrictions imposed by the muon lifetime and have shown many advantages with respect to other ways of accelerations such as multturn through the linear accelerator with many arcs for each energy, or ramping magnetic field in synchrotrons. Other examples of muon [35], proton, and electron acceleration for the proof of principle 10 MeV electrons and for electron acceleration up to 10 GeV for e-RHIC (future proposal for the electron-proton or heavy ion Relativistic Heavy Ion Collider) are shown in other publications [34].

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TABLE III. Parameters for the rf simulations.

| Parameter                              | 0.8  | 1.5  | 0.8  |
|----------------------------------------|------|------|------|
| $L_d$ (m)                              | 0.8  | 1.5  | 0.8  |
| Circumference (m)                      | 344  | 328  | 328  |
| Harmonic number                        | 229  | 220  | 220  |
| $f_{rf}$ (MHz)                         | 199.56 | 201.07 | 201.07 |
| Number of Cornell cavities             | 60   | 80   | 60   |
| Voltage per turn (MV)                  | 720  | 960  | 720  |
| Number of turns                        | 16   | 12   | 16   |
| Lattice periods                        | 80   | 66   | 75   |

FIG. 20. (Color) Initial (dash) and final (full) rf voltage, and evolution of the bunch on turns 0, 2, … 16, for $L_d = 0.8$ m and $C_0 = 344$ m.

FIG. 21. (Color) Initial (dash) and final (full) rf voltage, and evolution of the bunch on turns 0, 2, … 12, for $L_d = 1.5$ m and $C_0 = 328$ m.

FIG. 22. (Color) Initial (dash) and final (full) rf voltage, and evolution of the bunch on turns 0, 2, … 16, for $L_d = 0.8$ m and $C_0 = 328$ m.
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