The existential fragment of S1S over $(\varepsilon, s)$ is the co-Büchi languages

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Abstract

Büchi’s theorem, in establishing the equivalence between languages definable in S1S over $(\varepsilon, <)$ and the $\omega$-regular languages also demonstrated that S1S over $(\varepsilon, <)$ is no more expressive than its existential fragment. It is also easy to see that S1S over $(\varepsilon, <)$ is equi-expressive with S1S over $(\varepsilon, s)$. However, it is not immediately obvious whether it is possible to adapt Büchi’s argument to establish equivalence between expressivity in S1S over $(\varepsilon, s)$ and its existential fragment. In this paper we show that it is not: the existential fragment of S1S over $(\varepsilon, s)$ is strictly less expressive, and is in fact equivalent to the co-Büchi languages.

1 Preliminaries

1.1 Second order theory of one successor

Definition 1 (S1S syntax). We introduce the following components:

- A set of first order variables, denoted by lower case letters, possibly with subscripts.
- A set of second order variables, denoted by upper case letters, possibly with subscripts.

S1S$(\varepsilon, <)$ has the following set of well formed formulae:

$$\varphi ::= t < t \mid t \in X \mid \varphi \land \varphi \mid \neg \varphi \mid \exists x_i \varphi \mid \exists X_i \varphi.$$ 

$t$ is understood to range over terms, which in this case are just the first order variables, and $X$ to range over second order variables.
\( \exists S1S(\varepsilon, <) \) consists of those formulae with an initial block of second order quantifiers over a first order matrix:

\[
\varphi_1 := t < t \mid t \in X \mid \varphi \land \varphi \mid \neg \varphi \mid \exists x_i \varphi.
\]

\[
\varphi := \varphi_1 \mid \exists X_i \varphi.
\]

In \( S1S(\varepsilon, s) \) we have two cases for terms, letting \( x \) range over first order variables:

\[
t ::= x \mid st.
\]

\[
\varphi ::= t \in X \mid \varphi \land \varphi \mid \neg \varphi \mid \exists x_i \varphi \mid \exists X_i \varphi.
\]

\( \exists S1S(\varepsilon, s) \) follows analogously:

\[
t ::= x \mid st.
\]

\[
\varphi_1 ::= t \in X \mid \varphi \land \varphi \mid \neg \varphi \mid \exists x_i \varphi.
\]

\[
\varphi ::= \varphi_1 \mid \exists X_i \varphi.
\]

The abbreviations \( \lor, \rightarrow \) and \( \forall \) are defined in the usual way.

**Definition 2** (S1S semantics). A formula of S1S \( \varphi(\overline{x}_i, \overline{X}_i) \) with \( n \) free first order variables and \( m \) free second order variables will be evaluated on \( (\overline{a}_i, \overline{A}_i) \), with \( a_i \in \mathbb{N} \) and \( A_i \in 2^{\mathbb{N}} \). Satisfaction is defined inductively:

\[
(\overline{a}_i, \overline{A}_i) \models s^k x_i \in X_j \iff a_i + k \in A_j.
\]

\[
(\overline{a}_i, \overline{A}_i) \models x_i < x_j \iff a_i < a_j.
\]

\[
(\overline{a}_i, \overline{A}_i) \models \neg \varphi \iff (\overline{a}_i, \overline{A}_i) \not\models \varphi.
\]

\[
(\overline{a}_i, \overline{A}_i) \models \varphi \land \psi \iff (\overline{a}_i, \overline{A}_i) \models \varphi \text{ and } (\overline{a}_i, \overline{A}_i) \models \psi.
\]

\[
(\overline{a}_i, \overline{A}_i) \models \exists x_i \varphi \iff \exists b_i \in \mathbb{N} : (\overline{a}_i[a_i \leftarrow b_i], \overline{A}_i) \models \varphi.
\]

\[
(\overline{a}_i, \overline{A}_i) \models \exists X_i \varphi \iff \exists B_i \in 2^{\mathbb{N}} : (\overline{a}_i, \overline{A}_i[A_i \leftarrow B_i]) \models \varphi.
\]

A model will be represented by an infinite word over \( \{0, 1\}^{n+m} \), with the \( i \)th component being the characteristic word of the \( i \)th set (treating numbers as singleton sets). We will conflate a model with its representation in the sequel.

Note that the set of models satisfying a formula \( \varphi(\overline{x}_i, \overline{X}_i) \) thereby induce a language over \( \{0, 1\}^{n+m} \). The notion of an \( L \) definable language, with \( L \in \{S1S(\varepsilon, <), S1S(\varepsilon, s), \exists S1S(\varepsilon, <), \exists S1S(\varepsilon, s)\} \), thus follows in the obvious way: a language is \( L \) definable just if it describes the models of some formula \( \varphi(\overline{x}_i, \overline{X}_i) \) of \( L \).
Proposition 1. The $SIS(\varepsilon, <)$ definable languages are precisely the $SIS(\varepsilon, s)$ definable languages.

Proof. Every instance of $x < y$ can be replaced with $\forall X(\forall z(z \in X \rightarrow z \in X) \rightarrow (y \in X \rightarrow x \in X))$. In every formula involving $st$, $st$ can be replaced by $x$ and $\forall y(t < y \rightarrow (x < y \lor x = y))$. □

1.2 Automata on infinite words

Definition 3 (Büchi automata). A Büchi automaton is a 5-tuple $A = (Q, \Sigma, \Delta, q_{\text{initial}}, F)$, with $Q$ and $F$ finite sets, $\Delta \subseteq Q \times \Sigma \times Q$, $q_{\text{initial}} \in Q$ and $F \subseteq Q$.

The language accepted by $A$ is the set of all $\alpha \in \Sigma^\omega$ such that there exists a $\rho \in Q^\omega$ such that $\rho_0 = q_{\text{initial}}$, and $(\rho_i, \alpha_i, \rho_i + 1) \in \Delta$ such that for infinitely many $i$, $\rho_i \in F$. A language accepted by some Büchi automaton is called Büchi recognisable.

We remind the reader of the following three standard results:

Theorem 1. The Büchi recognisable languages are precisely the $\omega$-regular languages.

Theorem 2 (Büchi’s theorem 1\footnote{Strictly speaking, we are not referring to the result in [1], but rather to the way this result is presented in, for example, the University of Oxford course on Automata, Logic and Games.} ). Every $SIS(\varepsilon, <)$ definable language is Büchi recognisable.

Theorem 3 (Büchi’s theorem 2). Every Büchi recognisable language is $\exists SIS(\varepsilon, <)$ definable.

The crux of the proof is that for every $A = (Q, \Sigma, \Delta, q_{\text{initial}}, F)$ we can construct the following formula defining the same language:

$$\varphi_A(X_a) = \exists Y_1 \ldots \exists Y_n \left( \begin{array}{c} \text{partition}(Y_1, \ldots, Y_n) \\ \exists Z \exists x(x \in Z \land \forall y(sy \notin Z) \land x \in Y_1) \\ \forall x \bigvee_{(i, a, j) \in \Delta}(x \in Y_i \land x \in X_a \land x \in Y_j) \\ \bigvee_{q \notin F} \text{infinite}(Y_q) \end{array} \right).$$

Intuitively, $i \in Y_j$ just if the automaton is in state $j$ at step $i$. The first line states that the automaton is at precisely one state at every step, the
second that the automaton starts in the initial state, the third that the transition relation is respected and the last that some accepting state is visited infinitely often.

Definition 4 (co-Büchi automata). A co-Büchi automaton is a 5-tuple \( A = (Q, \Sigma, \Delta, q_{initial}, F) \), with \( Q \) and \( F \) finite sets, \( \Delta \subseteq Q \times \Sigma \times Q \), \( q_{initial} \in Q \) and \( F \subseteq Q \).

The language accepted by \( A \) is the set of all \( \alpha \in \Sigma^\omega \) such that there exists a \( \rho \in Q^\omega \) such that \( \rho_0 = q_{initial} \), and \( (\rho_i, \alpha_i, \rho_i + 1) \in \Delta \) such that \( \rho_i \notin F \) for finitely many \( i \). A language accepted by some co-Büchi automaton is called co-Büchi recognisable.

Proposition 2. Every co-Büchi recognisable language is \( \exists S1S(\varepsilon, s) \) definable.

Proof. Let \( A = (Q, \Sigma, \Delta, q_{initial}, F) \) be a co-Büchi automaton. Consider the following formula:

\[
\varphi_A(X_a) = \exists Y_1 \ldots \exists Y_n \left( \begin{array}{c}
\text{partition}(Y_1, \ldots, Y_n) \\
\exists Z \exists x (x \in Z \land \forall y (sy \notin Z) \land x \in Y_1) \\
\forall X \bigvee_{(i, a, j) \in \Delta} (x \in Y_i \land x \in X_a \land x \in Y_j) \\
\land_{q \notin F} \text{finite}(Y_q)
\end{array} \right).
\]

The finite predicate is defined as follows:

\[
\text{finite}(X) = \exists Y \left( \forall x (x \in Y \rightarrow sx \in Y) \land \forall x (x \in Y \rightarrow x \notin X) \right).
\]

Note that neither the finite nor the zero predicate in \( \varphi_A(X_a) \) is bound by a universal quantifier, so they can be pulled out front.

To conclude this section, in Figure 1 we illustrate both the known relations between the languages so far introduced and the result we will prove in the next section.

2 Proof

Theorem 4. The \( \exists S1S(\varepsilon, s) \) definable languages are precisely the co-Büchi recognisable languages.

Proof. We have already seen that every language accepted by a co-Büchi automaton can be defined with a \( \exists S1S(\varepsilon, s) \) formula. It remains to show
that for every $\exists S1S(\varepsilon, s)$ formula we can construct a co-Büchi automaton accepting the language it defines.

Predictably, the construction will be given by induction. However, in contrast to the usual nature of such proofs the brunt of the work will fall on the base cases, while the inductive cases will follow trivially from the closure properties of co-Büchi automata.

Let $\varphi = \exists X_1 \cdots \exists X_m \varphi'$ be a formula of $\exists S1S(\varepsilon, s)$, and $\varphi'$ be its first order matrix. Observe that $\varphi'$ is a formula of monadic first order logic. As such, we can without loss of generality assume that:

1. $\varphi'$ is in negation normal form.
2. Every occurrence of $\forall x_i$ has in its scope a quantifier-free disjunction of terms involving $x_i$ and no other variables.\footnote{This follows from the fact that every formula of monadic first order logic is equivalent to a formula where every universal quantifier appears only in subformulae of the form:

$$\forall x(F_1 x \lor \cdots \lor F_n x \lor \neg G_1 x \lor \cdots \lor \neg G_m x)$$

and every existential quantifier only in subformulae of the form:

$$\exists x(F_1 x \land \cdots \land F_n x \land \neg G_1 x \land \cdots \land \neg G_m x).$$}

As a result of this assumption we will consider the following base cases:
The inductive cases will be for $\lor$, $\land$, first order $\exists$ and second order $\exists\exists$; the aim of the induction being to establish that for every formula $\varphi$, there exists a co-Büchi automaton $A_\varphi$ that accept $\alpha$ if and only if $\alpha$ is a model satisfying $\varphi$. To avoid a lengthy check at the end, we will instead verify that this property holds at the end of each case in the proof.

$s^k x_i \in X_j$:
For a formula of the form $s^k x_i \in X_j$, if $k \geq 1$ construct an automaton as in Figure 2.

That is to say, construct an automaton with $k+3$ states. Add a transition from $q_0$ to itself for every vector with a 0 in the $i$th component, and a transition to $q_1$ for every vector with a 1 in the $i$th component. Add a transition from $q_l$ to $q_{l+1}$ for every vector. Add a transition from $q_k$ to $q_{final}$ for any vector with a 1 in the $n+j$th component, and a transition from $q_k$ to $q_{dead}$ for any vector with a 0 in the $n+j$th component. Add loops from $q_{dead}$ and $q_{accept}$ to themselves for any vector.

In the case of $k = 0$ construct an automaton with 2 states. $q_0$ has a transition to itself for every vector with a 0 in the $i$th component and a transition to $q_{accept}$ for every vector with a 1 in both the $i$th and $n+j$th component.

We claim that this automaton has the following property: if $\alpha$ is a valid specification of a model $^3$, then the automaton accepts $\alpha$ if and only if it is

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$^3$By which we mean the first order components contain one and only one 1 each.
a model satisfying \( s^k x_i \in X_j \).

Suppose \( \alpha \) is a word with well formed first order components. Upon reading the unique 1 in the \( i \)th component the automaton will transition to \( q_1 \), and \( k \) steps later will enter the accepting state if it reads a 1 in the \( n+j \)th component, i.e. if \( x_i + k \) is in \( X_j \), as desired. In like manner, the only way the automaton can reach \( q_{\text{accept}} \) is by reading a 1 in the \( n+j \)th component exactly \( k \) steps after reading a 1 in the \( i \)th component, which would mean that if the input word is a model, it is a model satisfying \( s^k x_i \in X_j \).

To transform this automaton into the \( A_\varphi \) of the inductive hypothesis, simply intersect it with an automaton that accepts all valid model specifications (with \( n \) first order and \( m \) second order variables).

\( s^k x_i \notin X_j \):

For \( k \geq 1 \), construct the same automaton as before but make \( q_{\text{dead}} \) the accepting state. For \( k = 0 \) allow the transition to \( q_{\text{accept}} \) only upon reading a 1 in the \( i \)th component and a 0 in the \( n+j \)th. Replicating the argument before, we obtain our result.

\( \forall x_i \):

This is the bulk of the proof, so we shall first offer some intuition.

Suppose we have a formula \( \varphi \) of the form:

\[ \forall x_i (s^{k_1} x_i \in X_{j_1} \lor \cdots \lor s^{k_p} x_i \in X_{j_p} \lor s^{k'_1} x_i \notin X_{j'_1} \lor \cdots \lor s^{k'_q} x_i \notin X_{j'_q}) \]

Let \( r \) be the number of distinct values of \( k \) and \( k' \), and label these values \( \kappa_1, \ldots, \kappa_r \) in increasing order. We wish to verify that every natural number satisfies \( \varphi \), so let us take the perspective of the automaton and consider how we should verify whether 0 satisfies it. For this we must first wait for \( \kappa_1 \) steps, at which point there will be a number of terms of the form \( s^{\kappa_1} x_i \in X_j \) or \( s^{\kappa_1} x_i \notin X_j \). If the input word satisfies any one of these requirements, then 0 satisfies \( \varphi \). If not, we cannot safely reject yet and have to wait for \( \kappa_2 - \kappa_1 \) to check the next set of terms. Only if we reach \( \kappa_r \) without finding any term the input word satisfies can we conclude that 0 does not satisfy \( \varphi \) and hence reject the word.

There is no problem, then, in designing an automaton that verifies whether any given integer satisfies \( \varphi \). The trouble is that we cannot pause or backtracks. While we are waiting on the outcome of 0, the entries corresponding to 1, 2, 3, will have been read and we need to verify them concurrently, and na"ively constructing an automaton for each natural number will mean an infinite number of automata. However, we do not need an automaton for every single natural number: observe that if a number does satisfy \( \varphi \), we shall know it within at most \( \kappa_r \) states, as such we only need to concern ourselves with finitely many (precisely, \( \kappa_r \)) numbers at any given step.
This leads us to the proof idea. We shall construct \( \kappa_r \) automata, such that the \( s \)th automaton verifies that \( s \mod \kappa_r \) satisfies \( \varphi \). By taking the intersection of all these automata the resulting machine will verify that all natural numbers satisfy \( \varphi \).

We will thus construct an automaton verifying \( s \mod \kappa_r \). This is illustrated in Figure 3.

The automaton has states \( \{q_0, \ldots, q_{\kappa_r - 1}\}, \{q_1, \ldots, q_{\kappa_r}\} \) and a dead state. All states sans the dead state are accepting. The initial state is \( q_0 \). There is a transition from \( q_{i+1}' \) to \( q_i' \) for any vector, a transition from \( q_0' \) to \( q_1 \) for any vector and a transition from \( q_{u+1} \neq \kappa_v \) to \( q_{u+1} \) for any vector.

At \( q_{\kappa_v} \) add a transition to \( q_{\kappa_r - \kappa_v} \) for any vector that satisfies one of the terms with \( \kappa_v \) successor operations. For example, if the terms with \( \kappa_v \) successor terms in \( \varphi \) are \( \kappa_v x_i \in X_3 \) and \( \kappa_v x_i \notin X_4 \) then add a transition to \( q_{\kappa_r - \kappa_v} \) for any vector with a 1 in the \( n + 3 \)th component or a 0 in the \( n + 4 \)th. For any other vector, add a transition from \( q_{\kappa_v} \) to \( q_{\kappa_v+1} \). If \( \kappa_v = \kappa_r \), then add a transition to \( q_{\text{dead}} \) instead.

It is easy to see that if \( \alpha \) satisfies \( \varphi \), then since each \( s \mod \kappa_r \) satisfies \( \varphi \) the associated automaton will never leave the accepting zone, hence the
intersection automaton will accept $\alpha$. If however $\alpha$ does not satisfy $\varphi$ then there must exist some number which will force an automaton to enter the dead state, causing the intersection automaton to reject.

$\lor$:
Given a formula of the form $\varphi = \psi_1 \lor \psi_2$, obtain $A_{\psi_1}$ and $A_{\psi_2}$ via the inductive hypothesis and let $A_\varphi = A_{\psi_1} \cup A_{\psi_2}$.

Now, should $\alpha$ be a model satisfying $\varphi$ we can without loss of generality suppose it satisfies $\psi_1$. By inductive hypothesis $A_{\psi_1}$ accepts $\alpha$, hence $A_\varphi$ accepts $\alpha$. Similarly, should $\alpha$ be accepted by $A_\varphi$, it must be accepted by either $A_{\psi_1}$ or $A_{\psi_2}$ as required.

$\land$:
Given a formula of the form $\varphi = \psi_1 \land \psi_2$, obtain $A_{\psi_1}$ and $A_{\psi_2}$ via the inductive hypothesis and let $A_\varphi = A_{\psi_1} \cap A_{\psi_2}$. The desired property follows, mutatis mutandis, in the same manner as above.

$\exists x_i$:
Given a formula of the form $\varphi = \exists x_i \psi$, obtain $A_\psi$ via the inductive hypothesis and project away the $i$th component to obtain $A_\varphi$. That is, $A_\varphi$ accepts $\alpha$ if and only if there is a way to reinsert the $i$th component into $\alpha$ such that the resulting $\alpha'$ is accepted by $A_\psi$.

Suppose $\alpha$ is a model satisfying $\varphi$. It follows there must be a way to instantiate $x_i$ to satisfy $\psi$, meaning $\alpha$ can be expanded into $\alpha'$, which is accepted by $A_\psi$ by the induction hypothesis; in the case, $A_\varphi$ accepts $\alpha$ as desired. If $A_\varphi$ accepts $\alpha$, then by definition of projection it must be the case that $\alpha$ can be expanded into an $\alpha'$ accepted by $A_\psi$, meaning $\alpha'$ is a model satisfying $\psi$ and hence $\exists \psi$ is satisfied by $\alpha'$ with the $x_i$ component ignored, namely $\alpha$.

$\exists X_i$:
As before, but this time projecting away a second order variable.

**Corollary 1.** The languages definable by monadic first order logic over $(\varepsilon, s)$ on the natural numbers are included in the intersection of co-Büchi and deterministic Büchi recognisable languages.

**Proof.** The reader will first observe that non-determinism is only introduced by the existential quantifiers. In the case of first order $\exists$, this could be avoided by giving a similar construction as in the $\lor$ case. Next, note that in the universal construction the rejecting state is a sink, whereas in the existential construction the accepting state would be a sink. Because of this, there is no difference between the Büchi and co-Büchi acceptance conditions.
References

[1] J.Richard Büchi. On a decision method in restricted second order arithmetic. In Saunders Mac Lane and Dirk Siefkes, editors, The Collected Works of J. Richard Büchi, pages 425–435. Springer New York, 1990.