Quantum coherence–classical uncertainty tradeoff relations

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We derive upper bounds for the quantum uncertainty of one-qudit states in terms of its classical uncertainty. Hilbert-Schmidt quantum coherence and Hellinger quantum coherence are used to measure quantum uncertainty while classical uncertainty is quantified by the linear entropy and von Neumann entropy of the corresponding closest incoherent state or of its square root. The obtained inequalities are also given as coherence-populations tradeoff relations.

Keywords: Quantum coherence; uncertainty; populations; entropy; Gell-Mann basis

I. INTRODUCTION

In Quantum Mechanics [1], the more general description of a system state is given by its density operator \(\rho = \sum p_m |\psi_m\rangle\langle\psi_m|\), where \(\{p_m\}\) is a probability distribution and \(\{|\psi_m\rangle\}\) are state vectors [2]. Because of this ensemble interpretation, the density operator is required to be a positive (semi-definite) linear operator, besides having trace equal to one [3]. If we consider, for instance, two-level systems whose density operator represented in the orthonormal basis \(\{|\beta_n\rangle\}_{n=1}^{2}\) reads

\[
\rho = \begin{bmatrix}
|\langle \beta_1 | \rho | \beta_1 \rangle |^2 & |\langle \beta_1 | \rho | \beta_2 \rangle| \\
|\langle \beta_2 | \rho | \beta_1 \rangle| & |\langle \beta_2 | \rho | \beta_2 \rangle| ^2
\end{bmatrix} =: \begin{bmatrix}
\rho_{1,1} & \rho_{1,2} \\
\rho_{2,1} & 1 - \rho_{1,1}
\end{bmatrix},
\]

these properties impose a well known restriction on the off-diagonal elements of \(\rho\), its coherences, by the product of its diagonal elements, its populations:

\[
\rho_{1,1}(1 - \rho_{1,1}) \geq |\rho_{1,2}|^2.
\]

The product of the populations of \(\rho\) can be seen as the classical uncertainty we have about measurements of an observable with eigenvectors \(\{|\beta_n\rangle\}_{n=1}^{2}\), since it is independent of the whole ensemble coherences. On the other hand, the presence of non-null off-diagonal elements of \(\rho\) implies that one or more members of the ensemble are a coherent superposition of the base states \(\{|\beta_n\rangle\}_{n=1}^{2}\). The associated uncertainty, coming from this superposition, exists only in the quantum realm and thus can be seen as a quantum uncertainty, which is intertwined with \(\rho\)’s coherences.

Recently researchers have been developing a resource theory framework to quantify the kind of quantum uncertainty we consider in this article, the so called resource theory of coherence. See e.g. Refs. [4, 5] and references therein. In this resource theory, given an orthonormal reference basis \(\{|\beta_n\rangle\}_{n=1}^{\dim \mathcal{H}}\) for a system with state space \(\mathcal{H}\), the free states are incoherent mixtures of these base states:

\[
\iota = \sum_{n=1}^{\dim \mathcal{H}} \iota_n |\beta_n\rangle\langle\beta_n|,
\]

where \(\{\iota_n\}_{n=1}^{\dim \mathcal{H}}\) is a probability distribution. A geometrical way of defining functions to quantify coherence is via the minimum distance from \(\rho\) to incoherent states:

\[
C_D(\rho) = \min_{\iota} D(\rho, \iota),
\]

where \(\iota\) is the closest incoherent state to \(\rho\) under the distance measure \(D\). If \(C_D\) does not increase under incoherent operations, which are those quantum operations mapping incoherent states to incoherent states, then it is dubbed a coherence monotone.

For quantum uncertainty measured using quantum coherence, it is an interesting mathematical, physical, and possibly practical problem to derive quantum-classical uncertainty tradeoff relations regarding general quantum systems.

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In this article, we obtain such tradeoffs for one-qudit \((d\text{-level})\) quantum systems. We first measure quantum uncertainty using the Hilbert-Schmidt coherence (HSC) function \([6]\). Although HSC has a convenient algebraic structure, it is known not to be a coherence monotone \([7]\). For that reason, we also consider Hellinger coherence \([8]\), which is costly to calculate numerically for high-dimension systems but is also algebraically friendly and, more importantly, is a coherence monotone. To quantify the classical uncertainty of a state \(\rho\), we employ linear entropy and von Neumann entropy of is closest incoherent state or of its square root, depending on the coherence measure utilized.

We organized the remainder of this article in the following manner. In Sec. II, we present the Gell-Mann matrix basis (GMB), defined using any vector basis of \(\mathbb{C}^d\), and discuss the representation of general and diagonal matrices in the GMB. We obtain tradeoff relations between classical and quantum uncertainties measuring the last with Hilbert-Schmidt coherence in Sec. IIIA and with Hellinger coherence in Sec. IIIIB. We show how to write these inequalities as coherence-populations tradeoff relations in Sec. IV. We give our conclusions in Sec. V.

II. GELL-MANN BASIS FOR \(\mathbb{C}^{d \times d}\)

Let \(\{|\beta_m\rangle\}_{m=1}^d\) be any given vector basis for \(\mathbb{C}^d\). Using this basis, we can define the generalized Gell-Mann matrices as \([9]\):

\[
\Gamma^d_j := \sqrt{\frac{2}{j(j+1)}} \sum_{m=1}^{j+1} (-j)^{j_{m,j+1}} |\beta_m\rangle \langle \beta_m|,
\]

\[
\Gamma^s_{k,l} := |\beta_k\rangle \langle \beta_l| + |\beta_l\rangle \langle \beta_k|,
\]

\[
\Gamma^a_{k,l} := -i(|\beta_k\rangle \langle \beta_l| - |\beta_l\rangle \langle \beta_k|),
\]

where we use the following possible values for the indexes \(j, k, l\) throughout this article:

\[
j = 1, \ldots, d-1 \text{ and } 1 \leq k < l \leq d.
\]

One can easily see that these matrices are Hermitian and traceless. Besides, if we use \(\Gamma^d_0\) for the \(d \times d\) identity matrix, it is not difficult to verify that under the Hilbert-Schmidt inner product,

\[
\langle A|B \rangle_{hs} := \text{Tr}(A^\dagger B),
\]

\[
\left\{ \frac{\Gamma^d_0}{\sqrt{d}}, \frac{\Gamma^d_j}{\sqrt{2}}, \frac{\Gamma^s_{k,l}}{\sqrt{2}} \right\},
\]

\[
X = \frac{\text{Tr}(X)}{d} \Gamma^d_0 + \frac{1}{2} \sum_j \langle \Gamma^d_j | X \rangle \Gamma^d_j + \frac{1}{2} \sum_{k,l,\tau} \langle \Gamma^s_{k,l} | X \rangle \Gamma^s_{k,l}.
\]

We observe that the most general decomposition in GMB of a matrix \(X_d \in \mathbb{C}^{d \times d}\) which is diagonal in the basis \(\{|\beta_m\rangle\}_{m=1}^d\) shall be given by:

\[
X_d = \frac{\text{Tr}(X_d)}{d} \Gamma^d_0 + \frac{1}{2} \sum_j \langle \Gamma^d_j | X_d \rangle \Gamma^d_j,
\]

i.e., only the diagonal elements of the GMB can have non-null components in this decomposition.

III. TRADEOFF RELATIONS BETWEEN QUANTUM COHERENCE AND CLASSICAL UNCERTAINTY

A. Upper bound for the Hilbert-Schmidt coherence

The Hilbert-Schmidt coherence (HSC) of a quantum state \(\rho\) is defined as \([6]\)

\[
C_{hs}(\rho) := \min_i \|\rho - \iota\|_{hs}^2,
\]

\[
\]
with the Hilbert-Schmidt norm of a matrix $A \in \mathbb{C}^{d \times d}$ being defined as

$$||A||_{hs} := \sqrt{\langle A | A \rangle_{hs}},$$

and here the minimization is taken over the incoherent states of Eq. (3). For general one-qudit states, using the decompositions in GMB:

$$\rho = \frac{1}{d} \Gamma_0^d + \frac{1}{2} \sum_j \langle \Gamma_j^d | \rho | \Gamma_j^d \rangle, \quad t = \frac{1}{d} \Gamma_0^d + \frac{1}{2} \sum_{k,l,\tau} \langle \Gamma_{k,l,\tau}^d | \rho | \Gamma_{k,l,\tau}^d \rangle,$$

the analytical formulas for the HSC and for the associated closest incoherent state were obtained in Ref. [6] and read:

$$C_{hs}(\rho) = \frac{1}{2} \sum_{k,l,\tau} \langle \Gamma_{k,l,\tau}^d | \rho | \Gamma_{k,l,\tau}^d \rangle^2,$$

$$\rho_{hs}^t = \frac{1}{d} \Gamma_0^d + \frac{1}{2} \sum_j \langle \Gamma_j^d | \rho | \Gamma_j^d \rangle.$$

The main tool we use to obtain the results reported in this article is a condition for matrix positivity. The eigenvalues of a matrix $A \in \mathbb{C}^{d \times d}$, let us call them $\lambda$, can be obtained from [10]:

$$0 = \det(A - a \Gamma_0^d)$$

$$= \sum_{(j_1, j_2, \cdots, j_d)} \text{sgn}(j_1, j_2, \cdots, j_d) (A_{1, j_1} - a \delta_{j_1}) (A_{2, j_2} - a \delta_{j_2}) \cdots (A_{d, j_d} - a \delta_{j_d})$$

$$= (-1)^d c_d a^d + (-1)^{d-1} c_{d-1} a^{d-1} + (-1)^{d-2} c_{d-2} a^{d-2} + \cdots + c_2 a^2 - c_1 a + c_0.$$

By Descartes rule of signs (see e.g. Ref. [11] and references therein), we see that for $A$ to be a positive matrix, we have to have non-negativity for all the coefficients $\{c_m \geq 0\}_{m=0}^d$. In this article, we shall look at the positivity of

$$c_{d-2} = \sum_{(j_1, j_2)} \text{sgn}(j_1, j_2, 3, \cdots, d) A_{1, j_1} A_{2, j_2} + \cdots + \sum_{(j_1, j_d)} \text{sgn}(j_1, 2, \cdots, d-1, j_d) A_{1, j_1} A_{d, j_d}$$

$$+ \sum_{(j_2, j_3)} \text{sgn}(1, j_2, j_3, 4, \cdots, d) A_{2, j_2} A_{3, j_3} + \cdots + \sum_{(j_2, j_d)} \text{sgn}(1, j_2, 3, \cdots, d-1, j_d) A_{2, j_2} A_{d, j_d}$$

$$+ \cdots + \sum_{(j_{d-1}, j_d)} \text{sgn}(1, 2, \cdots, d-2, j_{d-1}, j_d) A_{d-1, j_{d-1}} A_{d, j_d}$$

$$= \sum_{m=1}^{d-1} \sum_{n=m+1}^d (A_{m,m} A_{n,n} - A_{m,n} A_{n,m})$$

$$= \frac{1}{2} \left( \text{Tr}(A^2) - \text{Tr}(A^2) \right) \geq 0.$$

Using the orthonormality of GMB, i.e., the inner product between different elements of the GMB is zero and $\langle \Gamma_0^d | \Gamma_0^d \rangle = d$ and $\langle \Gamma_j^d | \Gamma_j^d \rangle = \langle \Gamma_{k,l,\tau}^d | \Gamma_{k,l,\tau}^d \rangle = 2$, the positivity condition for the coefficient in Eq. (24) applied to the density matrix of Eq. (15), as $\text{Tr}(\rho^2) = 1$, can be rewritten as

$$0 \leq \frac{d-1}{d} - \frac{1}{d} \sum_j \langle \Gamma_j^d | \rho | \Gamma_j^d \rangle^2 - \frac{1}{2} \sum_{k,l,\tau} \langle \Gamma_{k,l,\tau}^d | \rho | \Gamma_{k,l,\tau}^d \rangle^2.$$

Now, if we use the formula for the HSC in Eq. (17), this inequality can be cast as a restriction to the quantum uncertainty of $\rho$ as quantified by HSC:

$$C_{hs}(\rho) \leq \frac{d-1}{d} - \frac{1}{2} \sum_j \langle \Gamma_j^d | \rho | \Gamma_j^d \rangle^2.$$
If we utilize again the orthonormality of the GMB, the right hand side of this inequality is easily seen to be the classical uncertainty of the state $\rho$ measured using the linear entropy of its closest incoherent state (Eq. (18)), i.e.,

$$S_l(i^h_\rho) = 1 - \text{Tr} \left( (i^h_\rho)^2 \right) = \frac{d-1}{d} - \frac{1}{2} \sum_j \langle \Gamma^d_j | \rho \rangle^2.$$  \hfill (28)

Now, using $-\ln x \geq 1 - x [12]$, we can get an upper bound for the linear entropy in terms of von Neumann entropy as follows:

$$S_{vn}(x) := \text{Tr} (x (-\ln x)) \geq \text{Tr} (x(1-x)) = \text{Tr}(x) - \text{Tr}(x^2) = S_l(x) + \text{Tr}(x) - 1.$$  \hfill (29)

Gathering the results above, as $\text{Tr}(i^h_\rho) = 1$, we have obtained the following quantum-classical uncertainty tradeoff relation:

$$C_{hs}(\rho) \leq S_l(i^h_\rho) \leq S_{vn}(i^h_\rho),$$  \hfill (31)

which is valid for any one-qudit state. The verification of these inequalities using random states is presented in Fig. 1. We observe that the upper bound given by linear entropy is tight for qubits ($d = 2$). However, as the dimension increases, and typicality is approached, the upper bounds get less and less tight.

### B. Upper bound for the Hellinger coherence

The Hellinger coherence (HC) of a quantum state $\rho$ is defined as [8]:

$$C_{he}(\rho) := \min_i ||\sqrt{\rho} - \sqrt{i}||^2_{hs},$$  \hfill (32)

For convenience, regarding our main purpose in this article, we shall obtain the analytical expression for the HC using the following decompositions in the GMB:

$$\sqrt{\rho} = \frac{\text{Tr}(\sqrt{\rho})}{d} \Gamma_0^d + \frac{1}{2} \sum_j (\Gamma^d_j |\sqrt{\rho}\rangle \Gamma_j \Gamma^d_j + \frac{1}{2} \sum_{k,l,\tau} (\Gamma_{k,l,\tau}^d |\sqrt{\rho}\rangle \Gamma^d_{k,l,\tau},$$  \hfill (33)

$$\sqrt{i} = \frac{\text{Tr}(\sqrt{i})}{d} \Gamma_0^d + \frac{1}{2} \sum_j (\Gamma^d_j |\sqrt{i}\rangle \Gamma_j \Gamma^d_j.$$

Applying the orthonormality of GMB under Hilbert-Schmidt inner product, we shall have

$$C_{he}(\rho) = \min_i \text{Tr} \left( (\sqrt{\rho} - \sqrt{i})^2 \right)$$

$$= \min_i \text{Tr} \left( \left( \frac{\text{Tr}(\sqrt{\rho}) - \text{Tr}(\sqrt{i})}{d} \Gamma_0^d + \frac{1}{2} \sum_j ((\Gamma^d_j |\sqrt{\rho}\rangle - (\Gamma^d_j |\sqrt{i}\rangle \Gamma_j \Gamma^d_j + \frac{1}{2} \sum_{k,l,\tau} (\Gamma_{k,l,\tau}^d |\sqrt{\rho}\rangle \Gamma^d_{k,l,\tau} \right)^2 \right)$$

$$= \min_i \left( \frac{\text{Tr}(\sqrt{\rho}) - \text{Tr}(\sqrt{i})}{d} \right)^2 + \frac{1}{2} \sum_j ((\Gamma^d_j |\sqrt{\rho}\rangle - (\Gamma^d_j |\sqrt{i}\rangle \Gamma_j \Gamma^d_j + \frac{1}{2} \sum_{k,l,\tau} (\Gamma_{k,l,\tau}^d |\sqrt{\rho}\rangle \Gamma^d_{k,l,\tau} \right)^2 \right)$$

$$= \frac{1}{2} \sum_{k,l,\tau} ((\Gamma_{k,l,\tau}^d |\sqrt{\rho}\rangle \Gamma^d_{k,l,\tau} \right)^2.$$  \hfill (36)

That is to say, the closest incoherent state, $i^{he}_\rho$, in this case is such that

$$\sqrt{i^{he}_\rho} = \frac{\text{Tr}(\sqrt{\rho})}{d} \Gamma_0^d + \frac{1}{2} \sum_j (\Gamma^d_j |\sqrt{\rho}\rangle \Gamma_j \Gamma^d_j.$$

At this point, it is worthwhile mentioning that in general $\sqrt{i^{he}_\rho}$ is not a quantum state.
Figure 1: (color online) Verification of the Hilbert-Schmidt quantum coherence–classical uncertainty tradeoff relations of Eq. (31) for one thousand random density matrices generated for each value of the system dimension $d$. The random states were created using the standard method described in Refs. [13, 14]. The $y$ axis is for $C_{hs}(\rho)$ and the $x$ axis is for $S_{l}(\chi_{hs}\rho)$ or $S_{vn}(\chi_{hs}\rho)$, with $\chi_{hs}$ given in Eq. (18). The black lines stand for $C_{hs}(\rho) = S_{l}(\chi_{hs}\rho)$ and for $C_{hs}(\rho) = S_{vn}(\chi_{hs}\rho)$.

A quantum state, with spectral decomposition $\rho = \sum_{m=1}^{d} r_{m}\ket{r_{m}}\bra{r_{m}}$, is a positive matrix [12], i.e., $\{r_{m} \geq 0\}_{m=1}^{d}$. So, $\sqrt{\rho} = \sum_{m=1}^{d} \sqrt{r_{m}}\ket{r_{m}}\bra{r_{m}}$ is also a positive matrix. If we apply the positivity condition of Eq. (24) to $\sqrt{\rho}$ decomposed in GMB as in Eq. (33), the following inequality is obtained:

$$0 \leq (\text{Tr}(\sqrt{\rho}))^2 - \text{Tr}(\sqrt{\rho}^2) = \text{Tr}(\sqrt{\rho})^2 \left(1 - \frac{1}{d}\right) - \frac{1}{2} \sum_{j} \langle \Gamma_{d}^{j} | \sqrt{\rho} \rangle^2 - \frac{1}{2} \sum_{k,l,\tau} \langle \Gamma_{k,l,\tau}^{\tau} | \sqrt{\rho} \rangle^2. \quad (40)$$

Using Hellinger coherence in Eq. (38) and the linear and von Neumann entropies of the matrix in Eq. (39), we obtain the following upper bounds for the quantum uncertainty:

$$C_{he}(\rho) \leq S_{l} \left(\sqrt{\chi_{he}}\rho\right) + (\text{Tr}(\sqrt{\rho}))^2 - 1 =: \Upsilon \quad (42)$$

$$\leq S_{vn} \left(\sqrt{\chi_{he}}\rho\right) + \text{Tr}(\sqrt{\rho}) (\text{Tr}(\sqrt{\rho}) - 1) =: \Omega. \quad (43)$$

In the particular case of pure states, $\rho = \ket{\psi}\bra{\psi}$, we have $\text{Tr}(\sqrt{\rho}) = 1$ and these inequalities take a simpler form, becoming similar to those inequalities obtained for the Hilbert–Schmidt coherence in Eq. (31), i.e.,

$$C_{he}(\psi\bra{\psi}) \leq S_{l} \left(\sqrt{\chi_{he}}|\psi\rangle\langle\psi|\right) \leq S_{vn} \left(\sqrt{\chi_{he}}|\psi\rangle\langle\psi|\right). \quad (44)$$
The upper bounds for the Hellinger coherence in Eq. (39) were also verified using random states, as shown in Fig. 2. Here the restrictiveness of those upper bounds is also seen to diminish with the increase of the system dimension.

IV. COHERENCE-POPULATIONS TRADEOFF RELATIONS

In this section, we start rewriting the upper bound for Hilbert-Schmidt coherence given in Eq. (27) by expressing the components of the so called Bloch vector corresponding to the diagonal elements of Gell-Mann basis, $(\Gamma^d_j|\rho)$ with $j = 1, \cdots, d-1$, in terms of the density matrix populations, $\rho_{m,m} = \langle \beta_m | \rho | \beta_m \rangle$ with $m = 1, \cdots, d$. For that purpose, after some algebraic manipulations, one can infer that for $m = 2, \cdots, d-1$:

$$
\rho_{m,m} = \frac{1}{d} - \sqrt{\frac{m-1}{2m}} \langle \Gamma^d_{m-1} | \rho \rangle + \sum_{j=1}^{d-1} \frac{\langle \Gamma^d_j | \rho \rangle}{\sqrt{2j(j+1)}} \sum_{n=m}^{d-1} \delta_{n,j}.
$$

For $m = d$ and $m = 1$ we can use this same expression for the populations, but without the last and second terms, respectively. By inverting the expressions in Eq. (45) iteratively, we obtain the general expression we need to rewrite
the tradeoffs in Eq. (31) in terms of $\rho$’s populations:

$$\langle \Gamma_d^{d-j}\rho \rangle = \sqrt{\frac{2}{(d-j+1)(d-j)}} \left( 1 - \sum_{n=1}^{j} (d-n+1)^{k_{j,n}} \rho_{d-n+1,d-n+1} \right).$$

(46)

As examples, let us start considering qubit and qutrit systems. For $d = 2$, $\langle \Gamma_d^{d}\rho \rangle = 1 - 2 \rho_{2,2}$ and, from Eq. (27), we get $C_{hs}(\rho) = 2|\rho_{1,2}|^2 \leq 2 \rho_{1,1} \rho_{2,2}$, which is equivalent to Eq. (2). For $d = 3$, $\langle \Gamma_d^{d}\rho \rangle = 1 - \rho_{3,3} - 2 \rho_{2,2}$, $\langle \Gamma_d^{3}\rho \rangle = (1 - 3 \rho_{3,3})/\sqrt{3}$ and

$$C_{hs}(\rho) \leq 2(\rho_{1,1} \rho_{2,2} + \rho_{1,1} \rho_{3,3} + \rho_{2,2} \rho_{3,3}).$$

(47)

As this same pattern appears also for $d = 4$ and for $d = 5$, we conjecture that for any one-qudit state the following inequality will be satisfied:

$$C_{hs}(\rho) \leq 2 \sum_{m=1}^{d} \sum_{n=m+1}^{d} \rho_{m,m} \rho_{n,n}.$$

(48)

We could not give limitations for the Hellinger coherence of a state $\rho$ directly in terms of the density matrix populations. Notwithstanding, relations identical to the ones above shall follow for this quantum uncertainty if we replace $\rho$ by $\sqrt{\rho}$ in Eqs. (45) and (46) and on the right hand side of Eq. (48).

V. CONCLUSIONS

Quantum coherence (QC) is an important resource in Quantum Information Science [15–31]. In this article we proved upper bounds for the quantum uncertainty of a general one-qudit state $\rho$, quantified using Hilbert-Schmidt QC and Hellinger QC, by its associated classical uncertainty measured using the linear entropy and von Neumann entropy of the closest incoherent mixture or of its square root. We also wrote these quantum-classical uncertainty tradeoff relations with the upper bound given in terms of the populations of the density matrix or of its square root. For convenience regarding later use, we repeat here the main results obtained in this article:

$$C_{hs}(\rho) \leq S_l(l_{\rho}^{hs}) \leq S_{vn}(l_{\rho}^{hs}),$$

(49)

$$C_{hs}(\rho) \leq 2 \sum_{m=1}^{d} \sum_{n=m+1}^{d} \rho_{m,m} \rho_{n,n},$$

(50)

$$C_{he}(\rho) \leq S_l\left(\sqrt{I_{\rho}^{he}}\right) + (\text{Tr}(\sqrt{I_{\rho}^{he}}))^2 - 1 \leq S_{vn}\left(\sqrt{I_{\rho}^{he}}\right) + \text{Tr}(\sqrt{I_{\rho}^{he}})\left(\text{Tr}(\sqrt{I_{\rho}^{he}}) - 1\right),$$

(51)

$$C_{he}(\rho) \leq 2 \sum_{m=1}^{d-1} \sum_{n=m+1}^{d} (\sqrt{\rho})_{m,m}(\sqrt{\rho})_{n,n} + ((\text{Tr}(\sqrt{\rho}))^2 - 1)\left(1 - \frac{1}{d}\right),$$

(52)

$$C_{he}(|\psi\rangle\langle\psi|) \leq S_l\left(\sqrt{t_{\psi}^{he}}\right) \leq S_{vn}\left(\sqrt{t_{\psi}^{he}}\right).$$

(53)

We have performed numerical tests of the proven inequalities using random quantum states. These tests showed that the given upper bounds are tight for qubits and that they have their restrictiveness progressively weakened as the system dimension grows. So, in the future it would be interesting to investigate if the positivity of coefficients in Eq. (21) others than the one considered in this article may be used to obtain similar but more generally stronger upper bounds for quantum coherence.

Finding applications to the reported inequalities is another natural continuation for the present research. One possibility for investigation is regarding coherence generation via quantum operations with restrictions on the possible density matrix populations changes [32, 33]. Other promising candidate area for application of quantum-classical uncertainty tradeoff relations reported here is quantum thermodynamics [34–37]. In this scenario, if the reference basis is the energy basis, restrictions on populations changes shall be related with restrictions on energy changes. And these restrictions may be useful for analyzing thermodynamical processes that consume or create quantum coherence.
Acknowledgments

This work was supported by the Brazilian National Institute for the Science and Technology of Quantum Information (INCT-IQ), process 465469/2014-0, and by the Fundação de Amparo à Pesquisa do Estado do Rio Grande do Sul (FAPERGS).

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