Entanglement in coupled harmonic oscillators studied using a unitary transformation

Ahmed Jellal\textsuperscript{1,2,3}, Fethi Madouri\textsuperscript{1,4} and Abdeldjalil Merdaci\textsuperscript{1,5}

\textsuperscript{1} Physics Department, College of Science, King Faisal University, PO Box 380, Alahsa 31982, Saudi Arabia
\textsuperscript{2} Saudi Center for Theoretical Physics, Dhahran, Saudi Arabia
\textsuperscript{3} Theoretical Physics Group, Faculty of Sciences, Chouaïb Doukkali University, PO Box 20, 24000 El Jadida, Morocco
\textsuperscript{4} IPEIT, Rue J Lal Nehru, Montfleury, University of Tunis, Tunisia
\textsuperscript{5} Département des Sciences Fondamentales, Université 20.08.1955 Skikda, BP 26, DZ-21000 Skikda, Algeria

E-mail: jellal@pks.mpg.de and jellal.a@ucd.ac.ma

Received 20 June 2011
Accepted 18 August 2011
Published 20 September 2011

Online at stacks.iop.org/JSTAT/2011/P09015
doi:10.1088/1742-5468/2011/09/P09015

Abstract. We develop an approach for studying the entanglement of two coupled harmonic oscillators. We start by introducing a unitary transformation and end up with the solutions for the energy spectrum. These are used to construct the corresponding coherent states in the standard way. To evaluate the degree of the entanglement between the states obtained, we calculate the purity function in terms of the coherent and number states, separately. The result yielded is a two-parameter dependence of the purity, which can be controlled easily. Interesting results are derived by fixing the mixing angle of such a transformation as $\pi/2$. We compare our results with already published work and point out the relevance of these findings to a systematic formulation of the entanglement effect in two coupled harmonic oscillators.

Keywords: exact results, entanglement in extended quantum systems (theory)
1. Introduction

Entanglement is one of the most remarkable features of quantum mechanics, and does not have any classical counterpart. It is a notion which was initially introduced and named by Schrödinger [1] when quantum mechanics was still in its early stages of development. Its status has evolved throughout the decades and has undergone significant changes. Traditionally, entanglement has been related to the most exotic quantum mechanical concepts such as Schrödinger’s cat [1], the Einstein–Podolsky–Rosen paradox [2] and the violation of Bell’s inequalities [3]. Despite its conventional significance, entanglement has gained, in the past few decades, renewed interest mainly because of the development of quantum information science [4]. It has been revealed that it lies at the heart of various communication and computational tasks that cannot be implemented classically. It is believed that entanglement is the main ingredient of the quantum speed-up in quantum computation [4]. Moreover, several quantum protocols such as teleportation, quantum dense coding, and so on [5]–[11] are exclusively realized with the help of entangled states. In this respect, many interesting works have appeared dealing with the development of a quantitative theory of entanglement and the definition of its basic measure. These concern simultaneity, the arising of entanglement and linear entropy [12]–[15].

Entangled quantum systems can exhibit correlations that cannot be explained on the basis of classical laws, and entanglement in a collection of states is clearly a signature of non-classicality [16]. Furthermore, in the past few years it has become evident that quantum information may lead to further insights into other areas of physics [17]. This has led to a cross-fertilization between different areas of physics. It is worthy of note that the nonlinear Kerr effect [18] has been considered as the most famous source of physical realizations of photon pairs of entangled polarization states. However, it raises a number of difficulties as regards the control of photons that are traveling at the speed of light. This is why so much attention has been paid recently to the entangled states of massive particles, as they are viewed as being much easier to control [17, 19].

doi:10.1088/1742-5468/2011/09/P09015
On the other hand, the harmonic oscillator machinery plays a crucial role in many areas of physics. These include the Lee model in quantum field theory [20], the Bogoliubov transformation in superconductivity [21], two-mode squeezed states of light [22]–[24], the covariant harmonic oscillator model for the parton picture [25], and models in molecular physics [26]. There are also models in which one of the variables is not observed, including thermo-field dynamics [27], two-mode squeezed states [28,29], the hadronic temperature [30], and the Barnet–Phoenix version of information theory [31]. These physical models are examples of Feynman’s ‘rest of the universe’. In the case of two coupled harmonic oscillators, the first one is the universe and the second one is the rest of the universe. For sake of mathematical simplicity, the mixing angle (the rotation of the coordinate system), in the above mentioned references, is taken to be equal to $\pi/2$. This means that the system consists of two identical oscillators coupled together by a potential term.

In the context of the entangled massive particles, we cite the recently completed investigation of a specific realization of a model of two coupled harmonic oscillators by the authors of reference [19]. In fact, they calculated the interatomic entanglement for Gaussian and non-Gaussian pure states by using the purity function of the reduced density matrix. This allowed them to treat the cases of free and trapped molecules and heteronuclear and homonuclear molecules. Finally, they concluded that when the trap frequency and the molecular frequency are very different, and when the atomic masses are equal, the atoms are highly entangled for molecular coherent states and number states. Surprisingly, while the interatomic entanglement can be quite large even for molecular coherent states, the covariance of atomic position and momentum observables can be entirely explained by a classical model with appropriately chosen statistical uncertainty.

Motivated by the references mentioned above, and in particular [19], we undertake to develop a new approach for studying the entanglement of two coupled harmonic oscillators. It is based on a suitable transformation having the merit of reducing the relevant physical parameters to two: the coupling parameter $\eta$ and the mixing angle $\theta$. It turns out that we can easily derive the solutions corresponding to the energy spectrum. Then, the solutions obtained are used to construct the coherent states through the standard method. In order to characterize the degree of entanglement, we calculate, within the framework of the coherent states, the purity function. Then the final form of the purity is cast in terms of $\eta$ and $\theta$. Our finding shows two interesting results: the first one tells us that the present system is not entangled at $\eta = 0$, as expected, and highly entangled at large $\eta$ (figure 1). The second one is that when we fix $\theta = \pi/2$, the purity behaves like the inverse of $\cosh \eta$, and the corresponding plot (figure 2) shows that the purity ranges between 0 and 1. It is worthy of note that, in this case, the purity becomes dependent on one parameter, which means that it is easy to control.

Subsequently, we evaluate the purity in terms of the number states. In doing so, we use a well-known relation to express the number states $|n_1, n_2\rangle$ as functions of the corresponding coherent states $|\alpha, \beta\rangle$. Then after lengthy but straightforward algebra, we end up with the final form of the purity. To be much more concrete, we restrict ourselves to some interesting cases, namely $(n_1 = 1, n_2 = 0)$ and $(n_1 = 1, n_2 = 1)$. For the first configuration, the purity obtained is simply a ratio of hyperbolic and sinusoidal functions, which tells us that the entanglement is maximal at large $\eta$ for all $\theta$ (figure 3). In the particular case when $\theta = \pi/2$, the purity is typically a ratio of hyperbolic cosine.
Entanglement in coupled harmonic oscillators studied using a unitary transformation

Figure 1. Purity in terms of the coupling parameter $\eta$ and the mixing angle $\theta$.

Figure 2. Purity in terms of the coupling parameter $\eta$ for the mixing angle $\theta = \pi/2$.

functions, which shows clearly that the purity is positive, as it should be (figure 4). The second configuration also gives a mixing dependence between the hyperbolic and sinusoidal functions where the corresponding plots (figures 5 and 6) show some difference in form with respect to the first one. In both cases, we notice that the numerators are always hyperbolic cosines of even $\eta$ and the denominators are also powers of the function $\cosh \eta$.

doi:10.1088/1742-5468/2011/09/P09015
Figure 3. Purity $P_{01}$ as a function of the coupling parameter $\eta$ and the mixing angle $\theta$ for the quantum numbers $(n_1 = 0, n_2 = 1)$.

Figure 4. Purity $P_{01}$ as a function of $\eta$ measuring the entanglement between the ground state $n_1 = 0$ and the first excited state $n_2 = 1$ for $\theta = \pi/2$.

The present paper is organized as follows. In section 2, we review the derivation of the solutions of the energy spectrum for two coupled harmonic oscillators [32]. These will be used to build the corresponding coherent states and therefore evaluate the purity function of the reduced matrix elements in section 3. The final form of the purity function is subjected to different investigations where we underline its dependence on two physical parameters, $\eta$ and $\theta$. In section 4, we evaluate the purity in terms of the number states.
Figure 5. Purity $P_{11}$ as a function of the coupling parameter $\eta$ and the mixing angle $\theta$ for the quantum numbers $(n_1 = 1, n_2 = 1)$.

Figure 6. Purity $P_{11}$ as a function of $\eta$ measuring the entanglement between the first excited states $(n_1 = 1, n_2 = 1)$ for $\theta = \pi/2$.

after a series of transformations. Two interesting cases for the purity will be discussed in section 5. Finally, we give a conclusion and perspective for our work.

2. Energy spectrum solutions

In performing our task, we consider a system of two coupled harmonic oscillators parameterized by the planar coordinates $(X_1, X_2)$ and masses $(m_1, m_2)$. Accordingly,
the corresponding Hamiltonian is written as the sum of free and interacting parts \[33\]
\[
H_1 = \frac{1}{2m_1}P_1^2 + \frac{1}{2m_2}P_2^2 + \frac{1}{2}(C_1X_1^2 + C_2X_2^2 + C_3X_1X_2)
\]
where \(C_1, C_2\) and \(C_3\) are constant parameters. After rescaling the position variables
\[
x_1 = \mu X_1, \quad x_2 = \mu^{-1}X_2
\]
as well as the momenta
\[
p_1 = \mu^{-1}P_1, \quad p_2 = \mu P_2,
\]
\(H_1\) can be written as
\[
H_2 = \frac{1}{2m}(\hat{p}_1^2 + \hat{p}_2^2) + \frac{1}{2}(c_1x_1^2 + c_2x_2^2 + c_3x_1x_2)
\]
where the parameters are given by
\[
\mu = (m_1/m_2)^{1/4}, \quad m = (m_1m_2)^{1/2}, \quad c_1 = C_1\sqrt{m_2/m_1}, \quad c_2 = C_2\sqrt{m_1/m_2}, \quad c_3 = C_3.
\]

As the Hamiltonian (4) involves an interacting term, a straightforward investigation of the basic features of the system is not easy. Nevertheless, we can simplify this situation by means of a transformation to new phase space variables
\[
y_a = M_{ab}x_b, \quad \hat{p}_a = M_{ab}p_b
\]
where the matrix
\[
(M_{ab}) = \begin{pmatrix}
\cos \frac{\theta}{2} & -\sin \frac{\theta}{2} \\
\sin \frac{\theta}{2} & \cos \frac{\theta}{2}
\end{pmatrix}
\]
is a unitary rotation with the mixing angle \(\theta\). Inserting the mapping (6) into (4), one realizes that \(\theta\) should satisfy the condition
\[
\tan \theta = \frac{c_3}{c_2 - c_1}
\]
to get a factorizing Hamiltonian
\[
H_3 = \frac{1}{2m}(\hat{p}_1^2 + \hat{p}_2^2) + \frac{k}{2}\left(e^{2\eta}y_1^2 + e^{-2\eta}y_2^2\right)
\]
where we have introduced two parameters
\[
k = \sqrt{c_1c_2 - c_3^2/4}, \quad e^{2\eta} = \frac{c_1 + c_2 + \sqrt{(c_1 - c_2)^2 + c_3^2}}{2k}
\]
with the proviso that the condition \(4c_1c_2 > c_3^2\) must be fulfilled. The parameter \(\eta\) is actually measuring the strength of the coupling.
For later use, it is convenient to separate the Hamiltonian (9) into two commuting parts and then write $H_3$ as

$$H_3 = e^\eta \mathcal{H}_1 + e^{-\eta} \mathcal{H}_2$$

where $\mathcal{H}_1$ and $\mathcal{H}_2$ are given by

$$\mathcal{H}_1 = \frac{1}{2m} e^{-\eta} \dot{p}_1^2 + \frac{k}{2} e^{\eta} y_1^2, \quad \mathcal{H}_2 = \frac{1}{2m} e^{\eta} \dot{p}_2^2 + \frac{k}{2} e^{-\eta} y_2^2.$$  \hspace{1cm} (12)

One can see that the decoupled Hamiltonian

$$H_0 = \frac{1}{2m} \dot{p}_1^2 + \frac{k}{2} y_1^2 + \frac{1}{2m} \dot{p}_2^2 + \frac{k}{2} y_2^2.$$ \hspace{1cm} (13)

is obtained for $\eta = 0$, which is equivalent to setting $c_3 = 0$.

The Hamiltonian $H_3$ can simply be diagonalized by defining a set of annihilation and creation operators. These are

$$a_i = \sqrt{\frac{k}{\hbar \omega}} e^{\epsilon \eta/2} y_i + \frac{i}{\sqrt{2m \hbar \omega}} e^{-\epsilon \eta/2} \dot{p}_i, \quad a_i^\dagger = \sqrt{\frac{k}{\hbar \omega}} e^{\epsilon \eta/2} y_i - \frac{i}{\sqrt{2m \hbar \omega}} e^{-\epsilon \eta/2} \dot{p}_i$$ \hspace{1cm} (14)

with the frequency

$$\omega = \sqrt{\frac{k}{m}}$$ \hspace{1cm} (15)

and $\epsilon = \pm 1$ for $i = 1, 2$, respectively. They satisfy the commutation relations

$$[a_i, a_j^\dagger] = \delta_{ij}$$ \hspace{1cm} (16)

whereas other commutators vanish. Now we can map $H_3$ into a form in terms of $a_i$ and $a_i^\dagger$:

$$H_3 = \hbar \omega \left( e^\eta a_1^\dagger a_1 + e^{-\eta} a_2^\dagger a_2 + \cosh \eta \right).$$ \hspace{1cm} (17)

To obtain the eigenstates and the eigenvalues, one solves the eigenequation

$$H_3|n_1, n_2\rangle = E_{n_1,n_2}|n_1, n_2\rangle$$ \hspace{1cm} (18)

getting the states

$$|n_1, n_2\rangle = \frac{(a_1^\dagger)^n_1 (a_2^\dagger)^n_2}{\sqrt{n_1! n_2!}} |0, 0\rangle$$ \hspace{1cm} (19)

as well as the energy spectrum

$$E_{3,n_1,n_2} = \hbar \omega \left( e^n n_1 + e^{-n} n_2 + \cosh \eta \right).$$ \hspace{1cm} (20)

It is clear that these eigenvalues reduce to those of the decoupled harmonic oscillators, namely $\hbar \omega (n_1 + n_2 + 1)$. This shows clearly that the presence of the coupling parameter $\eta$ will make a difference and allow us to derive interesting results in the forthcoming analysis.
To show the correlation between variables, let us just focus on the ground state and write the corresponding wavefunction in the $y$-representation. This is

$$\psi_0(\vec{y}) \equiv \langle y_1, y_2 | 0, 0 \rangle = \sqrt{\frac{m\omega}{\pi \hbar}} \exp \left\{ -\frac{m\omega}{2\hbar} \left( e^{\eta} y_1^2 + e^{-\eta} y_2^2 \right) \right\}$$

which can easily be used to deduce the ground state wavefunction in terms of the variables $(x_1, x_2)$. Therefore, from the unitary representation we find

$$\psi_0(\vec{x}) \equiv \langle x_1, x_2 | 0, 0 \rangle = \sqrt{\frac{m\omega}{\pi \hbar}} \exp \left\{ -\frac{m\omega}{2\hbar} \left[ e^{\eta} \left( x_1 \cos \frac{\theta}{2} - x_2 \sin \frac{\theta}{2} \right)^2 
+ e^{-\eta} \left( x_1 \sin \frac{\theta}{2} + x_2 \cos \frac{\theta}{2} \right)^2 \right] \right\}. \quad (22)$$

We notice that (21) is separable in terms of the variables $y_1$ and $y_2$, which is not the case for (22) in terms of $x_1$ and $x_2$. We close this part by claiming that the results obtained so far can be used to study the entanglement in the present system.

3. Entanglement in coherent states

As we claimed above, we implement our approach to study the entanglement of two coupled harmonic oscillators. Actually, it can be seen as another alternative method for recovering the results obtained in [19] not only in a simpler way but also with fewer physical control parameters. To start, let us first introduce the coherent states corresponding to the eigenstates $|n_1, n_2\rangle$ given in (19). As usual, we can use the displacement operator to define the coherent states in terms of two complex numbers $\alpha$ and $\beta$. These are

$$|\alpha, \beta\rangle = D(\alpha_1, \alpha) D(\alpha_2, \beta) |0, 0\rangle,$$

(23)

giving the wavefunction

$$\Phi_{\alpha\beta}(y_1, y_2) = \left( \frac{\lambda_1 \lambda_2}{\pi} \right)^{1/2} \exp \left[ -\frac{\lambda_1^2}{2} y_1^2 - \frac{|\alpha|^2}{2} \right.
+ \sqrt{2} \alpha \lambda_1 y_1 - \frac{\lambda_2^2}{2} y_2^2 - \frac{|\beta|^2}{2} \left. - \sqrt{2} \beta \lambda_2 y_2 \right]$$

(24)

where we have set the quantities

$$\lambda_1 = e^{\eta/2} \left( \frac{mk}{\hbar^2} \right)^{1/4}, \quad \lambda_2 = e^{-\eta/2} \left( \frac{mk}{\hbar^2} \right)^{1/4}.$$ 

(25)

In terms of the original variables $(X_1, X_2)$, (24) reads as

$$\Phi_{\alpha\beta}(X_1, X_2) = \left( \frac{\lambda_1 \lambda_2}{\pi} \right)^{1/2} \exp \left[ -\frac{\lambda_1^2}{2} \left( \mu \cos \frac{\theta}{2} X_1 - \frac{1}{\mu} \sin \frac{\theta}{2} X_2 \right)^2 
- \frac{\lambda_2^2}{2} \left( \mu \sin \frac{\theta}{2} X_1 + \frac{1}{\mu} \cos \frac{\theta}{2} X_2 \right)^2 \right] \exp \left[ \sqrt{2} \alpha \lambda_1 \left( \mu \cos \frac{\theta}{2} X_1 - \frac{1}{\mu} \sin \frac{\theta}{2} X_2 \right) 
+ \sqrt{2} \beta \lambda_2 \left( \mu \sin \frac{\theta}{2} X_1 + \frac{1}{\mu} \cos \frac{\theta}{2} X_2 \right) \right] \exp \left[ -\frac{|\alpha|^2}{2} - \frac{|\beta|^2}{2} \right. 
+ \left. \frac{\alpha^2}{2} - \frac{\beta^2}{2} \right]. \quad (26)$$
As is clearly shown in the wavefunction (26), the non-separability of the variables will play a crucial role in discussing the entanglement in the present system. This statement will be clarified later on, when we come to the analysis of the role of the parameters involved.

At this level we have set out all the ingredients for studying the entanglement in the present system. All we need to do is to determine explicitly the purity function that is the trace of the density squared, corresponding to the eigenstates obtained. More precisely, we have

\[ P = \text{Tr} \rho^2 \]  

(27)

which in terms of the above coherent states reads as

\[ P_{\alpha\beta} = \int dX_1 dX'_1 dX_2 dX'_2 \Phi_{\alpha\beta} (X_1, X_2) \Phi^*_{\alpha\beta} (X'_1, X'_2) \Phi_{\alpha\beta} (X'_1, X'_2) \Phi^*_{\alpha\beta} (X_1, X_2). \]  

(28)

Upon substitution, we obtain the form

\[ P_{\alpha\beta} = \left( \frac{\lambda_1 \lambda_2}{\pi} \right)^2 \int dX_1 dX'_1 dX_2 dX'_2 \exp \left\{ -\frac{\mu^2}{2} \left( \lambda_1^2 \cos^2 \frac{\theta}{2} + \lambda_2^2 \sin^2 \frac{\theta}{2} \right) (X_1^2 + X'_1^2) \right. \]

\[ - \frac{1}{\mu^2} \left( \lambda_1^2 \sin^2 \frac{\theta}{2} + \lambda_2^2 \cos^2 \frac{\theta}{2} \right) (X_2^2 + X'_2^2) \]

\[ \left. \times e^{(1/2) \left( \lambda_1^2 - \lambda_2^2 \right) \sin \theta (X_1 X_2 + X_1 X'_2 + X'_1 X_2 + X'_1 X'_2)} \right. \]

\[ \times \exp \left\{ 2\mu \left( \frac{\alpha + \alpha^*}{\sqrt{2}} \lambda_1 \cos \frac{\theta}{2} + \frac{\beta + \beta^*}{\sqrt{2}} \lambda_2 \sin \frac{\theta}{2} \right) (X_1 + X'_1) \right\} \]

\[ \times \exp \left\{ -2\mu \left( \frac{\alpha + \alpha^*}{\sqrt{2}} \lambda_1 \sin \frac{\theta}{2} - \frac{\beta + \beta^*}{\sqrt{2}} \lambda_2 \cos \frac{\theta}{2} \right) (X_2 + X'_2) \right\} \]

\[ \times e^{-2|\alpha|^2 - 2|\beta|^2 - \alpha^2 - \beta^2 - \beta^2}. \]  

(29)

This integral can be easily evaluated by introducing an appropriate transformation. This can be done by making use of the following change of variables:

\[
\begin{pmatrix}
X_1 \\
X'_1 \\
X_2 \\
X'_2
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
\omega_1 & \sqrt{2} \omega_1 & \omega_1 & 0 \\
\mu \sqrt{1-2a} & \mu \sqrt{1-2a} & \omega_1 & 0 \\
\sqrt{2} \omega_1 & \omega_1 & \mu \sqrt{1+2a} & \sqrt{2} \mu \omega_2 \\
0 & \mu \sqrt{1-2a} & 0 & \mu \omega_2 \\
\mu \sqrt{1+2a} & \omega_1 & \mu \omega_2 & \sqrt{2} \mu \omega_2 \\
\sqrt{1-2a} & \omega_1 & \mu \sqrt{1+2a} & 0 \\
0 & \sqrt{1-2a} & 0 & \mu \omega_2 \\
0 & 0 & \sqrt{1+2a} & \mu \sqrt{1-2a}
\end{pmatrix} \begin{pmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4
\end{pmatrix}
\]  

(30)

where \( \omega_1, \omega_2 \) and \( a \) are given by

\[
\omega_1 = \frac{1}{\sqrt{\lambda_1^2 \cos^2 (\theta/2) + \lambda_2^2 \sin^2 (\theta/2)}},
\]

\[
\omega_2 = \frac{1}{\sqrt{\lambda_1^2 \sin^2 (\theta/2) + \lambda_2^2 \cos^2 (\theta/2)}},
\]

\[
a = -\frac{1}{4} (\lambda_1^2 - \lambda_2^2) \sin \theta \omega_1 \omega_2.
\]  

(31)
Having shown that the determinant of such a transformation is \( \omega_1 \omega_2 / \lambda_1 \lambda_2 \), it is easy to map \( P_{\alpha\beta} \) into a form in terms of the new variables:

\[
P_{\alpha\beta} = \frac{1}{\pi^2} \lambda_1 \lambda_2 \omega_1 \omega_2 e^{-2|\alpha|^2 - 2|\beta|^2 - \alpha^2 - \beta^2} \int_{-\infty}^{+\infty} du_2 \int_{-\infty}^{+\infty} du_4 e^{-u_2^2 - u_4^2} \times \int_{-\infty}^{+\infty} du_1 \exp \left\{ -u_1^2 + \frac{\sqrt{2}}{1 - 2a} \left[ \lambda_1 \left( \omega_1 \cos \frac{\theta}{2} + \omega_2 \sin \frac{\theta}{2} \right) (\alpha + \alpha^*) 
\right.
+ \lambda_2 \left( \omega_1 \sin \frac{\theta}{2} - \omega_2 \cos \frac{\theta}{2} \right) (\beta + \beta^*) \right\} u_1 

\times \int_{-\infty}^{+\infty} du_3 \exp \left\{ -u_3^2 + \frac{\sqrt{2}}{1 + 2a} \left[ \lambda_1 \left( \omega_1 \cos \frac{\theta}{2} - \omega_2 \sin \frac{\theta}{2} \right) (\alpha + \alpha^*) 
\right.
+ \lambda_2 \left( \omega_1 \sin \frac{\theta}{2} + \omega_2 \cos \frac{\theta}{2} \right) (\beta + \beta^*) \right\} u_3 \right\}. \tag{32}
\]

Performing the integration we end up with the result

\[
P_{\alpha\beta}(\eta, \theta) = \frac{1}{\sqrt{2 \cosh 2\eta \sin^2(\theta/2) \cos^2(\theta/2) + \cos^4(\theta/2) + \sin^4(\theta/2)}}. \tag{33}
\]

This is among the most interesting results derived so far in the present work. Indeed, it shows clearly that the purity depends on the physical parameters \( (\eta, \theta) \) rather than the complex displacements \( (\alpha, \beta) \) and hereafter it will be denoted by \( P(\eta, \theta) \). Furthermore, the purity obtained is dependent on two parameters, which means that it can be controlled easily. If one requires the decoupling case \( (\eta = 0) \), \( P(\eta, \theta) \) reduces to 1 as expected and therefore there is no entanglement.

To understand better the above results, we recall that the purity is related to linear entropy by the simple form

\[
L = 1 - P \tag{34}
\]

where \( P \) lies in the interval \([0, 1]\). Now let us proceed to plot the purity for a range of \( \eta \) and by considering \( \theta \in [0, \pi] \). From figure 1, it is clear that the purity, as a function of \( \eta \), is symmetric with respect to the decoupling case \( \eta = 0 \). It is maximal for \( \eta = 0 \), which actually shows that the system is disentangled. After that it decreases rapidly to reach zero and this indicates that the entanglement is maximal. More importantly, the purity becomes constant whenever \( \theta \) takes the value zero or \( \pi \). This behavior of the purity, sketched below, tells us that one can easily play with two parameters to control the degree of entanglement in the present system.

Specifically at \( \theta = \pi/2 \), we obtain a simple form

\[
P(\eta, \theta = \pi/2) = \frac{1}{\cosh \eta} \tag{35}
\]

which is dependent on one parameter and can be adjusted only by varying the coupling \( \eta \) to control the degree of the entanglement. To be much more accurate, we underline such behavior by plotting (35) in figure 2. From figure 2, one can deduce two interesting conclusions. The first one tells us that \( P(\eta, \theta) \) is bounded, i.e. \( 0 \leq P \leq 1 \), as expected. The second one shows clearly that the purity goes to zero for a strong coupling, which indicates that the entanglement is maximal.

doi:10.1088/1742-5468/2011/09/P09015
4. Entanglement in number states

To gain more information about the behavior of the present system, we evaluate the degree of the entanglement between interstates. For this, we consider the inverse relation, to express the number states in terms of the coherent states. This is

\[ |n_1, n_2\rangle = \frac{1}{\sqrt{n_1!n_2!}} \frac{\partial^{n_1}}{\partial \alpha^{n_1}} \frac{\partial^{n_2}}{\partial \beta^{n_2}} e^{\alpha|\beta|^2} |\alpha, \beta\rangle \Big|_{\alpha=0, \beta=0}. \]  \hspace{1cm} (36)

In the \( y \)-representation, (36) leads to the wavefunction

\[ \tilde{\Phi}_{n_1n_2} (y_1, y_2) = \Phi_{n_1n_2} \left( \mu \cos \frac{\theta}{2} X_1 - \frac{1}{\mu} \sin \frac{\theta}{2} X_2, \mu \sin \frac{\theta}{2} X_1 + \frac{1}{\mu} \cos \frac{\theta}{2} X_2 \right) \equiv \tilde{\Phi}_{n_1n_2} (X_1, X_2) \]

\[ = \frac{1}{\sqrt{n_1!n_2!}} \frac{\partial^{n_1}}{\partial \alpha^{n_1}} \frac{\partial^{n_2}}{\partial \beta^{n_2}} e^{\alpha|\beta|^2/2} e^{\beta|\beta|^2/2} \Phi_{n_1n_2} (X_1, X_2) \Big|_{\alpha=0, \beta=0} \]  \hspace{1cm} (37)

where \( \Phi_{n_1n_2} (X_1, X_2) \) is given in (26). This will be implemented to study the purity in terms of the number states and discuss different issues.

Returning to the purity definition, we have

\[ P_{n_1n_2} = \int dX_1 dX'_1 dX_2 dX'_2 \tilde{\Phi}_{n_1n_2} (X_1, X_2) \tilde{\Phi}_{n_1n_2}^* (X'_1, X'_2) \tilde{\Phi}_{n_1n_2} (X_1, X'_2). \]  \hspace{1cm} (38)

We use (37) to obtain the form

\[ P_{n_1n_2} = \int dX_1 dX'_1 dX_2 dX'_2 \left( \frac{1}{n_1!n_2!} \right)^2 \frac{\partial^{n_1}}{\partial \alpha^{n_1}} \frac{\partial^{n_2}}{\partial \beta^{n_2}} e^{\alpha|\beta|^2/2} e^{\beta|\beta|^2/2} \Phi_{n_1n_2} \]

\[ \times \left( \mu \cos \frac{\theta}{2} X_1 - \frac{1}{\mu} \sin \frac{\theta}{2} X_2, \mu \sin \frac{\theta}{2} X_1 + \frac{1}{\mu} \cos \frac{\theta}{2} X_2 \right) \Big|_{\alpha, \beta=0} \]

\[ \times \frac{\partial^{n_1}}{\partial \alpha^{n_1}} \frac{\partial^{n_2}}{\partial \beta^{n_2}} e^{\alpha|\beta|^2/2} e^{\beta|\beta|^2/2} \Phi_{n_1n_2} \]

\[ \times \left( \mu \cos \frac{\theta}{2} X'_1 - \frac{1}{\mu} \sin \frac{\theta}{2} X'_2, \mu \sin \frac{\theta}{2} X'_1 + \frac{1}{\mu} \cos \frac{\theta}{2} X'_2 \right) \Big|_{\alpha, \beta=0} \]

\[ \times \frac{\partial^{n_1}}{\partial \alpha^{n_1}} \frac{\partial^{n_2}}{\partial \beta^{n_2}} e^{\alpha|\beta|^2/2} e^{\beta|\beta|^2/2} \Phi_{n_1n_2} \]

\[ \times \left( \mu \cos \frac{\theta}{2} X'_1 - \frac{1}{\mu} \sin \frac{\theta}{2} X'_2, \mu \sin \frac{\theta}{2} X'_1 + \frac{1}{\mu} \cos \frac{\theta}{2} X'_2 \right) \Big|_{\alpha, \beta=0} \]

\[ \times \frac{\partial^{n_1}}{\partial \alpha^{n_1}} \frac{\partial^{n_2}}{\partial \beta^{n_2}} e^{\alpha|\beta|^2/2} e^{\beta|\beta|^2/2} \Phi_{n_1n_2} \]

\[ \times \left( \mu \cos \frac{\theta}{2} X'_1 - \frac{1}{\mu} \sin \frac{\theta}{2} X'_2, \mu \sin \frac{\theta}{2} X'_1 + \frac{1}{\mu} \cos \frac{\theta}{2} X'_2 \right) \Big|_{\alpha, \beta=0}. \]  \hspace{1cm} (39)
After some algebra, we show that the purity takes the form

\[
P_{n_1n_2} = \left( \frac{\lambda_1 \lambda_2}{\pi n_1! n_2!} \right)^2 \prod_{i=1}^{2} \frac{\partial^{n_1}}{\partial \beta_i^{n_1}} \frac{\partial^{n_2}}{\partial \beta_i^{n_2}} \int dX_1 \, dX_1' \, dX_2 \, dX_2' \, e^{-(1/2)(\alpha_1^2 + \beta_1^2 + \alpha_2^2 + \beta_2^2 + \alpha_3^2 + \beta_3^2 + \alpha_4^2 + \beta_4^2)}
\]

\[
\times \exp \left\{ -\frac{\lambda_1^2}{2} \left[ \left( \mu \cos \frac{\theta}{2} X_1 - \frac{1}{\mu} \sin \frac{\theta}{2} X_2 \right)^2 + \left( \mu \cos \frac{\theta}{2} X_1' - \frac{1}{\mu} \sin \frac{\theta}{2} X_2' \right)^2 \right] \right\}
\]

\[
+ \left( \mu \cos \frac{\theta}{2} X_1' - \frac{1}{\mu} \sin \frac{\theta}{2} X_2' \right)^2 + \left( \mu \cos \frac{\theta}{2} X_1 - \frac{1}{\mu} \sin \frac{\theta}{2} X_2 \right)^2 \right\} \right\}
\]

\[
\times \exp \left\{ -\frac{\lambda_2^2}{2} \left[ \left( \mu \sin \frac{\theta}{2} X_1 + \frac{1}{\mu} \cos \frac{\theta}{2} X_2 \right)^2 + \left( \mu \sin \frac{\theta}{2} X_1' + \frac{1}{\mu} \cos \frac{\theta}{2} X_2' \right)^2 \right] \right\}
\]

\[
+ \left( \mu \sin \frac{\theta}{2} X_1 + \frac{1}{\mu} \cos \frac{\theta}{2} X_2 \right)^2 + \left( \mu \sin \frac{\theta}{2} X_1' + \frac{1}{\mu} \cos \frac{\theta}{2} X_2' \right)^2 \right\} \right\}
\]

\[
\times \exp \left\{ \sqrt{2} \mu \left[ \left( \lambda_1 (\alpha_1 + \alpha_4) \cos \frac{\theta}{2} + \lambda_2 (\beta_1 + \beta_4) \sin \frac{\theta}{2} \right) X_1 \right. \right.
\]

\[
+ \left( \lambda_1 (\alpha_2 + \alpha_3) \cos \frac{\theta}{2} + \lambda_2 (\beta_2 + \beta_3) \sin \frac{\theta}{2} \right) X_1' \left. \right] \right\}
\]

\[
\times \exp \left\{ -\sqrt{2} \mu \left[ \left( \lambda_1 (\alpha_1 + \alpha_2) \sin \frac{\theta}{2} - \lambda_2 (\beta_2 + \beta_1) \cos \frac{\theta}{2} \right) X_2 \right. \right.
\]

\[
+ \left( \lambda_1 (\alpha_3 + \alpha_4) \sin \frac{\theta}{2} - \lambda_2 (\beta_3 + \beta_4) \cos \frac{\theta}{2} \right) X_2' \left. \right] \right\} \right\} . \tag{40}
\]

This can be written, in a compact form, as

\[
P_{n_1n_2} = \left( \frac{\lambda_1 \lambda_2}{\pi n_1! n_2!} \right)^2 \prod_{i=1}^{2} \frac{\partial^{n_1}}{\partial \beta_i^{n_1}} \frac{\partial^{n_2}}{\partial \beta_i^{n_2}} \int d^4Z \, e^{-Z^\dagger \cdot A \cdot Z + B^\dagger \cdot Z + C} \tag{41}
\]

where \( z^i = (X_1 \, X_1' \, X_2 \, X_2') \), the matrix \( A \) is given by

\[
A = \begin{pmatrix}
A_{11} & 0 & A_{13} & A_{13} \\
0 & A_{11} & A_{13} & A_{13} \\
A_{13} & A_{13} & A_{33} & 0 \\
A_{13} & A_{13} & 0 & A_{33}
\end{pmatrix} \tag{42}
\]

such that the components read as

\[
A_{11} = \mu^2 \left( \lambda_1^2 \cos^2 \frac{\theta}{2} + \lambda_2^2 \sin^2 \frac{\theta}{2} \right),
\]

\[
A_{33} = \frac{1}{\mu} \left( \lambda_1^2 \sin^2 \frac{\theta}{2} + \lambda_2^2 \cos^2 \frac{\theta}{2} \right), \quad A_{13} = \frac{\lambda_2^2 - \lambda_1^2}{4 \sin \theta} \tag{43}
\]
and the matrix $B$ takes the form

$$B = \sqrt{2} \begin{pmatrix}
\mu \lambda_1 (\alpha_1 + \alpha_4) \cos \frac{\theta}{2} + \mu \lambda_2 (\beta_1 + \beta_4) \sin \frac{\theta}{2} \\
\mu \lambda_1 (\alpha_2 + \alpha_3) \cos \frac{\theta}{2} + \mu \lambda_2 (\beta_2 + \beta_3) \sin \frac{\theta}{2} \\
\frac{1}{\mu} \lambda_2 (\beta_1 + \beta_2) \cos \frac{\theta}{2} - \frac{1}{\mu} \lambda_1 (\alpha_1 + \alpha_2) \sin \frac{\theta}{2} \\
\frac{1}{\mu} \lambda_2 (\beta_3 + \beta_4) \cos \frac{\theta}{2} - \frac{1}{\mu} \lambda_1 (\alpha_3 + \alpha_4) \sin \frac{\theta}{2}
\end{pmatrix}.$$  

(44)

To go further in evaluating the purity, we perform a method for simplifying our calculation. This can be done by introducing the change of variables

$$\left(\begin{array}{c}
X_1 \\
X_1' \\
X_2 \\
X_2'
\end{array}\right) = \left(\begin{array}{cccc}
\frac{\omega_1}{2\mu\sqrt{1-2a}} & \frac{\sqrt{2}}{2\mu} & \frac{\omega_1}{2\mu\sqrt{1+2a}} & 0 \\
\frac{\omega_1}{2\mu\sqrt{1-2a}} & -\frac{\sqrt{2}}{2\mu} & \frac{\omega_1}{2\mu\sqrt{1+2a}} & 0 \\
-\frac{\mu\omega_2}{2\sqrt{1-2a}} & 0 & \frac{\mu\omega_2}{2\sqrt{1+2a}} & \frac{\sqrt{2}}{2\mu\omega_2} \\
-\frac{\mu\omega_2}{2\sqrt{1-2a}} & 0 & \frac{\mu\omega_2}{2\sqrt{1+2a}} & -\frac{\sqrt{2}}{2\mu\omega_2}
\end{array}\right) \left(\begin{array}{c}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{array}\right)$$

(45)

where the corresponding measure is $dX_1 dX_1' dX_2 dX_2' = J dx_1 dx_1 dx_2 dx_2$ and the Jacobian reads as

$$J = \frac{1}{\lambda_1 \lambda_2 \sqrt{(\lambda_1^2 \cos^2(\theta/2) + \lambda_2^2 \sin^2(\theta/2))}}.$$  

(46)

This performance allows us to map (41) into the form

$$P_{n_1 n_2} = \left(\frac{\lambda_1 \lambda_2}{\pi n_1! n_2!}\right)^2 J \prod_{i=1}^4 \left(\frac{\omega_1}{2\sqrt{1-2a}} \frac{\omega_2}{2\sqrt{1+2a}}ight) \left(\frac{\omega_1}{2\sqrt{1-2a}} \frac{\omega_2}{2\sqrt{1+2a}}ight)^{-1} \left(\frac{\omega_1}{2\sqrt{1-2a}} \frac{\omega_2}{2\sqrt{1+2a}}\right)^{n_1} \left(\frac{\omega_1}{2\sqrt{1-2a}} \frac{\omega_2}{2\sqrt{1+2a}}\right)^{n_2} e^{-(1/2)(\alpha_1^2+\beta_1^2)} \int d^4Q e^{-Q^2+D^t Q}$$

(47)

where $Q^t = (x_1 \ x_2 \ x_3 \ x_4)$ and $D^t$ is the transpose of $D$, which is given by

$$D = \sqrt{2} \times \begin{pmatrix}
\frac{\omega_1 \cos \frac{\theta}{2} + \omega_2 \sin \frac{\theta}{2}}{2\sqrt{1-2a}} \lambda_1 (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) + \frac{\omega_1 \sin \frac{\theta}{2} - \omega_2 \cos \frac{\theta}{2}}{2\sqrt{1-2a}} \lambda_2 (\beta_1 + \beta_2 + \beta_3 + \beta_4) \\
\frac{\sqrt{2}}{2\omega_1 \lambda_1 (\alpha_1 + \alpha_4 - \alpha_2 - \alpha_3) \cos \frac{\theta}{2} + \sqrt{2}}{2\omega_1 \lambda_2 (\beta_1 + \beta_4 - \beta_2 - \beta_3) \sin \frac{\theta}{2}} \\
\frac{\omega_1 \cos \frac{\theta}{2} - \omega_2 \sin \frac{\theta}{2}}{2\sqrt{1+2a}} \lambda_1 (\alpha_1 + \alpha_4 + \alpha_2 + \alpha_3) + \frac{\omega_1 \sin \frac{\theta}{2} + \omega_2 \cos \frac{\theta}{2}}{2\sqrt{1+2a}} \lambda_2 (\beta_1 + \beta_4 + \beta_2 + \beta_3) \\
-\frac{\sqrt{2}}{2\omega_2 \lambda_1 (\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4) \sin \frac{\theta}{2} + \sqrt{2}}{2\omega_2 \lambda_2 (\beta_1 + \beta_2 - \beta_3 - \beta_4) \cos \frac{\theta}{2}}
\end{pmatrix}.$$  

(48)
Since the above integral is Gaussian, after some algebra we end up with the form

\[ P_{n_1n_2} = \left( \frac{\lambda_1 \lambda_2}{\pi n_1! n_2!} \right)^2 \int \prod_{i=1}^{4} \frac{\partial^{n_1}}{\partial \alpha_i^{n_1}} \frac{\partial^{n_2}}{\partial \beta_i^{n_2}} \exp \left[ \frac{u}{\rho} \alpha_1 + \frac{2v}{\rho} \alpha_1 \alpha_2 - \frac{2u}{\rho} \alpha_1 \alpha_3 + \frac{2w}{\rho} \alpha_1 \alpha_4 \right] \]

\[ + \frac{2s}{\rho} \alpha_1 \beta_1 - \frac{2t}{\rho} \alpha_1 \beta_2 + \frac{2s}{\rho} \alpha_1 \beta_3 - \frac{2t}{\rho} \alpha_1 \beta_4 + \frac{u}{\rho} \alpha_2 + \frac{2w}{\rho} \alpha_2 \beta_1 - \frac{2u}{\rho} \alpha_2 \beta_3 + \frac{2v}{\rho} \alpha_2 \beta_4 \]

\[ - \frac{2s}{\rho} \alpha_3 \beta_1 + \frac{2t}{\rho} \alpha_3 \beta_2 + \frac{2s}{\rho} \alpha_3 \beta_3 - \frac{2t}{\rho} \alpha_3 \beta_4 + \frac{u}{\rho} \alpha_4 + \frac{2w}{\rho} \alpha_4 \beta_1 - \frac{2u}{\rho} \alpha_4 \beta_3 + \frac{2v}{\rho} \alpha_4 \beta_4 \]

\[ - \frac{2s}{\rho} \alpha_4 \beta_1 + \frac{2t}{\rho} \alpha_4 \beta_2 + \frac{2s}{\rho} \alpha_4 \beta_3 - \frac{2t}{\rho} \alpha_4 \beta_4 + \frac{u}{\rho} \beta_2 + \frac{2v}{\rho} \beta_2 \beta_1 - \frac{2u}{\rho} \beta_2 \beta_3 + \frac{2v}{\rho} \beta_2 \beta_4 \]

\[ - \frac{2s}{\rho} \beta_3 + \frac{2t}{\rho} \beta_3 \beta_1 - \frac{2u}{\rho} \beta_3 \beta_2 + \frac{2v}{\rho} \beta_3 \beta_4 - \frac{u}{\rho} \beta_4 \]

\[ (49) \]

where we have set the parameters involved as

\[ \rho = 4 \frac{mk}{\hbar^2} \left[ 2 \cosh(2\eta) + \cot^2 \frac{\theta}{2} + \tan^2 \frac{\theta}{2} \right], \quad u = 2mk \sinh 2\eta \]

\[ v = 2 \frac{mk}{\hbar^2} \left[ \cosh(2\eta) + \tan^2 \frac{\theta}{2} \right], \quad w = 2 \frac{mk}{\hbar^2} \left[ \cosh(2\eta) + \cot^2 \frac{\theta}{2} \right] \]

\[ t = 4 \frac{mk}{\hbar^2} \cot \eta, \quad s = -4 \frac{mk}{\hbar^2} \sin \frac{\cos \theta}{\sin \theta}. \]

We are still looking for the final form of the purity, which can be obtained by calculating the partial derivatives. These calculations can be performed in different ways and it may be easier to proceed step by step. Indeed, we factorize the exponential function and then map each factor into a series expansion. This operation has been postponed to the appendix and the result yielded is

\[ P_{n_1n_2}(\eta, \theta) = \frac{2 \rho^{2(n_1+n_2)} (n_1! n_2!)^2}{\sin(\theta) \sqrt{2 \cosh 2\eta + \tan^2 (\theta/2) + \cot^2 (\theta/2)}} \]

\[ \times \sum_{i+j+k+l+r=2(n_1+n_2)} C_{n_1n_2} (i, j, k, l, r) u^i v^j w^k t^l s^r \]

\[ (51) \]

where the coefficients \( C_{n_1n_2} \) are given by

\[ C_{n_1n_2} = \left( \prod_{i=1}^{4} \sum_{i=0}^{n_i} \right) \left( \prod_{j=1}^{3} \sum_{j=0}^{n_j} \right) \left( \prod_{k=1}^{3} \sum_{k=0}^{n_k} \right) \]

\[ \times \left( \prod_{l=1}^{7} \sum_{l=0}^{n_l} \frac{1}{(l-1)!(l-1)!} \right) \left( \prod_{r=1}^{7} \sum_{r=0}^{n_r} \frac{1}{(r-1)!(r-1)!} \right) \]

\[ \times \frac{2^{-i_1-i_2-i_3-i_4} i_1 i_2 i_3 i_4 i_5 i_6 i_7 i_8 i_9 i_{10}}{(i-1)! (i_1-i_2)! (i_2-i_3)! (i_3-i_4)! (i_4-i_5)! (i_5-i_6)! (i_6-i_7)! (i_7-i_8)! (i_8-i_9)! (i_9-i_{10})!} \]

\[ (52) \]
It is clear that the final form of the purity is actually only dependent on two parameters, i.e. $\eta$ and $\theta$. On the other hand, it is easy to check that $P_{n_1n_2}$ is symmetric under the change of the quantum numbers $n_1$ and $n_2$.

5. Two special cases

To be much more accurate, let us illustrate some particular cases. With these we will be able to get more information from the above purity about the degree of entanglement. At the beginning, let us choose the configuration $(n_1 = 0, n_2 = 1)$, which means that we are considering now the entanglement between the ground state of the first oscillator and the first excited state of the second one. In this case, (51) reduces to the form

$$P_{01}(\eta, \theta) = \frac{2(2/\rho)^2}{\sin \theta \sqrt{2 \cosh(2\eta) + \tan^2(\theta/2) + \cot^2(\theta/2)}} \times \sum_{l+r+j+k+i=2} C_{01}(i, j, k, l, r) u^i v^j w^k l^r s^r$$ (53)

which can be evaluated to obtain

$$P_{01}(\eta, \theta) = \frac{2(2/\rho)^2}{\sin \theta \sqrt{2 \cosh(2\eta) + \tan^2(\theta/2) + \cot^2(\theta/2)}} (u^2 + v^2 + w^2)$$ (54)

and after replacing the different parameters, one gets the final result

$$P_{01}(\eta, \theta) = \frac{3 \cosh(4\eta) + 4(\tan^2(\theta/2) + \cot^2(\theta/2)) \cosh(2\eta) + 2\tan^4(\theta/2) + 2\cot^4(\theta/2) + 1}{\sin \theta \left(2 \cosh(2\eta) + \tan^2(\theta/2) + \cot^2(\theta/2)\right)^{5/2}}.$$ (55)

This is a nice form, which can be worked with more, since it is only function of two physical parameters $\eta$ and $\theta$. Indeed, we plot it in figure 3. Here we reach the same conclusion as for figure 1, except that the present plot shows some deformation at the point $\eta = 0$. Otherwise, for certain values of $\theta$ the purity does not always hold a maximum value at $\eta = 0$. More precisely, at this point it decreases to reach $1/2$ at $\theta = \pi/2$ and then increases to attain $1$ at $\theta = \pi$. This is because in the present case the masses are equal and the same conclusion is obtained in [19].

Now let us look at some interesting situations by fixing the mixing angle $\theta$ and varying the coupling parameter $\eta$. In particular when $\theta = \pi/2$, $P_{01}$ reduces to the form

$$P_{01}\left(\eta, \theta = \frac{\pi}{2}\right) = \frac{3 \cosh(4\eta) + 8 \cosh(2\eta) + 5}{32 \cosh^5 \eta}.$$ (56)

This can be plotted to obtain figure 4. Comparing to figure 2, we notice that the behavior of the purity in terms of the coupling parameter $\eta$ is marked. As long as $\eta$ is large, the entanglement will hold the maximum value. This shows clearly the role played by $\eta$ and thus allows easy control of the degree of the entanglement. This may give some hint as regards an experiment realization of the present case.
Now let us look at the case of the entanglement between the two first excited states of the two oscillators, i.e. \( n_1 = n_2 = 1 \). This result is

\[
P_{11} (\eta, \theta) = \frac{2 (2/\rho)^4}{\sin \theta \sqrt{2 \cosh(2\eta)} + \tan^2(\theta/2) + \cot^2(\theta/2)} \times \sum_{i+j+k+l+r=4} C_{11} (i, j, k, l, r) u^i v^j w^k t^l s^r
\]

and after lengthy but simple calculations, we find

\[
P_{11} = \frac{2 (2/\rho)^4}{\sin \theta \sqrt{2 \cosh(2\eta)} + \tan^2(\theta/2) + \cot^2(\theta/2)} (u^4 + v^4 + w^4 + 2s^2 + 2t^4 + 2t^2s^2 + 2v^2w^2).
\]

Finally, we obtain

\[
P_{11} (\eta, \theta) = \frac{1}{4 \sin \theta [2 \cosh(2\eta) + \tan^2(\theta/2) + \cot^2(\theta/2)]^{9/2}} \times \left[ 9 \cosh (8\eta) + 16 \left( \tan^2 \frac{\theta}{2} + \cot^2 \frac{\theta}{2} \right) \cosh 6\eta 
+ 96 \tan^4 \frac{\theta}{2} + 96 \cot^4 \frac{\theta}{2} - 36 \right] \cosh (4\eta) 
+ 240 \left( \tan^2 \frac{\theta}{2} + \cot^2 \frac{\theta}{2} \right) \cosh (2\eta)
+ 8 \tan^8 \frac{\theta}{2} + 8 \cot^8 \frac{\theta}{2} - 64 \tan^4 \frac{\theta}{2} - 64 \cot^4 \frac{\theta}{2} + 459 \right].
\]

Comparing this with (55), we notice that the numerators in both cases contain a hyperbolic cosine function of an even coupling parameter \( \eta \) and that the denominators are powers of \( \cosh \eta \). To go further, we plot (58) in figure 5. Clearly, we see that for certain values of \( \theta \) the purity does not always hold a maximum value in the decoupling case, i.e. \( \eta = 0 \). At this point, the purity decreases to reach 1/2 at \( \theta = \pi/2 \) and then increases to attain 1 at \( \theta = \pi \).

Furthermore, (59) can be worked with much more, to underline its behavior. The simplest way to do so is to fix the mixing angle \( \theta \) and play with the coupling parameter \( \eta \). For instance, by requiring \( \theta = \pi/2 \) we end up with the form

\[
P_{11} (\eta, \theta) = \frac{9 \cosh(8\eta) + 32 \cosh(6\eta) + 156 \cosh(4\eta) + 480 \cosh(2\eta) + 347}{2048 \cosh^9 \eta}.
\]

This shows clearly that \( P_{11}(\eta, \theta) \) is dependent on one parameter and therefore it can be manipulated easily. For more precision, we plot (60) in figure 6. This shows a difference...
with respect to figure 4. It is clear that as long as \( \eta \) is small, the purity increases rapidly to reach its maximal value. Also it decreases rapidly to attain zero for large \( \eta \), which means that the system is strongly entangled.

### 6. Conclusion

The present work is devoted to studying the entanglement of two coupled harmonic oscillators, by adopting a new approach. For this, a Hamiltonian describing the system is considered and a unitary transformation is introduced. With this latter, the corresponding solutions of the energy spectrum are obtained in terms of the coupling parameter \( \eta \) and the mixing angle \( \theta \). It is clearly seen that when \( \eta = 0 \), the system becomes decoupled, and therefore nothing new except a harmonic oscillator in two dimensions.

To study the entanglement of the present system, we have introduced the purity function to evaluate its degree. At the beginning, we realized the corresponding coherent states by using the standard method based on the displacement operator. These are used to determine explicitly the form of the purity in terms of the physical parameters \( \eta \) and \( \theta \). Also, the result obtained confirmed the range of the purity, that is \( 0 \leq P \leq 1 \). Moreover, we have clearly shown that the purity is easy to control and can also be cast in a simple form when we fix \( \theta = \pi/2 \). In such a case the purity is obtained as the inverse of the hyperbolic function \( \cosh \eta \) and the disentanglement simply corresponds to switching off \( \eta \).

Subsequently, we have the inverse relation between the number of states and the coherent states to determine the purity. After making different changes of variable, we obtained a tractable Gaussian form, which was integrated easily. The final result showed that the purity is dependent on two parameters. This allowed us to illustrate our finding by restricting ourselves to two particular cases. We considered in the first configuration the entanglement between the ground state and the excited state, i.e. \( (n_1 = 0, n_2 = 1) \), where the purity is exactly obtained. We studied in the second configuration the entanglement between the states \( (n_1 = 1, n_2 = 1) \). In both cases, we analyzed the case where \( \theta = \pi/2 \), which showed a strong dependence of the purity on the hyperbolic cosine function of an even coupling parameter.

On the other hand, the system of two coupled oscillators can serve as an analog computer for many of the physical theories and models. Therefore, one can extend the method developed here to study the entanglement in other interesting systems such as those illustrating the Feynman’s ‘rest of the universe’. Furthermore, one immediate extension is to consider the case of coupled systems subjected to an external magnetic field. This work and related matters are actually under consideration.

### Acknowledgments

The authors acknowledges financial support from King Faisal University. The present work was done under Project Number 110135, ‘Quantum information and Entangled Nano Electron Systems’. The authors would like to thank E B Choubabi for numerical help and are indebted to a referee for his or her constructive comments.

doi:10.1088/1742-5468/2011/09/P09015
Appendix: The final form of the purity

In this appendix, we show how to derive the final form of the purity given in (51). Indeed from (49), we obtain the result

\[
P_{n_1 n_2} = \left( \frac{\lambda_1 \lambda_2}{\pi n_1! n_2!} \right)^2 J \sum_{i,j,k,l,r=0}^{\infty} \left( \prod_{e=1}^{j_e-1} \frac{1}{(i_e-1 - i_e)!} \right) \times \left( \prod_{e=1}^{j_e-1} \frac{1}{(j_e-1 - j_e)!} \right) \times \left( \prod_{e=1}^{k_e-1} \frac{1}{(k_e-1 - k_e)!} \right) \times \left( \prod_{e=1}^{l_e-1} \frac{1}{(l_e-1 - l_e)!} \right) \times \left( \prod_{e=1}^{r_e-1} \frac{1}{(r_e-1 - r_e)!} \right) (u \rho)^i (2v \rho)^j (2w \rho)^k (2t \rho)^l \times \left( \frac{\partial^{a_1}}{\partial \alpha_1^{a_1}} \right) \left( \frac{\partial^{a_2}}{\partial \alpha_2^{a_2}} \right) \left( \frac{\partial^{a_3}}{\partial \alpha_3^{a_3}} \right) \left( \frac{\partial^{a_4}}{\partial \alpha_4^{a_4}} \right) \left( \frac{\partial^{a_5}}{\partial \alpha_5^{a_5}} \right) \left( \frac{\partial^{a_6}}{\partial \alpha_6^{a_6}} \right) \left( \frac{\partial^{a_7}}{\partial \alpha_7^{a_7}} \right) \left( \frac{\partial^{a_8}}{\partial \alpha_8^{a_8}} \right) \bigg|_{(a_1, a_2, \ldots, a_8) = (0, 0)}
\]

where the different parameters are given by

\[
a_1 = 2i_{11} + i_3 - i_4 + l_1 - l_2 + l - l_4 + r_1 - r_2 + r - r_1 + j_3 + k - k_1 \\
a_2 = 2i_{10} - 2i_{11} + i_2 - i_3 + l_7 - l_6 - l_7 + r_6 - r_7 + r_5 - r_6 + j_3 + k_3 \\
a_3 = 2i_9 - 2i_{10} + i_3 - i_4 + l_5 - l_6 + l_5 + r_5 + r_4 - r_5 + j_2 - j_3 + k_3 \\
a_4 = i_2 - i_3 + 2i_8 - 2i_9 + j_2 - j_3 + l_3 - l_4 + l_2 - l_3 + k - k_1 + r_3 - r_4 + r_2 - r_3 \\
a_5 = r_1 - r_2 + i_1 - i_2 + l_6 - l_7 + l_3 - l_4 + 2 l_7 - 2 i_8 + r_4 - r_5 + j_1 - j_2 + k_2 - k_3 \\
a_6 = l_5 - l_6 + i - i_1 + l_1 - l_2 + 2 i_6 - 2i_7 + r_5 - r_6 + r_3 - r_4 + j - j_1 + k_2 - k_3 \\
a_7 = l_2 - l_3 + l_7 + i_1 - i_2 + r_7 + r - r_1 + j - j_1 + k_1 + k_2 + 2 l_5 - 2 i_6 \\
a_8 = l - l_1 + l_4 - l_5 + i - i_1 + r_6 - r_7 + r_2 + r_3 + 2i_4 - 2i_5 + k_1 - k_2 + j_1 - j_2
\]

and for the coherence of notation, \((i_0, j_0, k_0, l_0, r_0) \equiv (i, j, k, l, r)\) has to be understood. Making use of the well-known formula

\[
\frac{\partial}{\partial x^n} x^l \bigg|_{x=0} = n! \delta_{l,n}
\]

we end up with the form

\[
P_{n_1 n_2} = \left( \frac{\lambda_1 \lambda_2}{\pi n_1! n_2!} \right)^2 J \sum_{i,j,k,l,r=0}^{\infty} \left( \prod_{e=1}^{j_e-1} \frac{1}{(i_e-1 - i_e)!} \right) \times \left( \prod_{e=1}^{j_e-1} \frac{1}{(j_e-1 - j_e)!} \right) \times \left( \prod_{e=1}^{k_e-1} \frac{1}{(k_e-1 - k_e)!} \right) \times \left( \prod_{e=1}^{l_e-1} \frac{1}{(l_e-1 - l_e)!} \right) \times \left( \prod_{e=1}^{r_e-1} \frac{1}{(r_e-1 - r_e)!} \right) \left( \frac{u \rho}{\lambda_1 \lambda_2} \right)^i \left( \frac{2v \rho}{\lambda_1 \lambda_2} \right)^j \left( \frac{2w \rho}{\lambda_1 \lambda_2} \right)^k \left( \frac{2t \rho}{\lambda_1 \lambda_2} \right)^l \times \left( \frac{n_1! n_2!}{\lambda_1 \lambda_2} \right) \left( \frac{n_1! n_2!}{\lambda_1 \lambda_2} \right)
\]

doi:10.1088/1742-5468/2011/09/P09015
Entanglement in coupled harmonic oscillators studied using a unitary transformation

\[
\times \left( \prod_{c=1}^{7} \sum_{e=1}^{l_e-1} \frac{1}{(l_e-1 - l_e)!} \right) \left( \prod_{c=1}^{7} \sum_{r_e=0}^{r_e-1} \frac{1}{(r_e-1 - r_e)!} \right)
\times \left( \frac{4}{p} \right)^i \left( \frac{2v}{p} \right)^j \left( \frac{2w}{p} \right)^k \left( \frac{2t}{p} \right)^l \left( \frac{2s}{p} \right)^r
\times \frac{2^{i-r}}{i_1! j_3! k_1! l_1! r_1!} \delta_b_{1,n_1} \delta_b_{2,n_2} \delta_b_{3,n_1} \delta_b_{4,n_2} \delta_b_{5,n_2} \delta_b_{6,n_2} \delta_b_{7,n_2} \delta_b_{8,n_2}.
\] (A.3)

This shows clearly that a nonvanishing purity should satisfy a set of constraints on different quantum numbers. These are

\[
\begin{align*}
b_1 - n_1 &= 2i_1 + i_3 - i_4 - l_2 + l - r_2 + r + j_3 + k - k_1 - n_1 = 0 \\
b_2 - n_1 &= 2i_{10} - 2i_9 + i_2 - i_3 + l_6 - r_7 + r_5 + j_3 + k_3 - n_1 = 0 \\
b_3 - n_1 &= 2i_9 - 2i_{10} + i_3 - i_4 - l_6 + l_4 + r_7 + r_4 - r_5 + j_2 - j_3 + k_3 - n_1 = 0 \\
b_4 - n_1 &= i_2 - i_3 + 2i_8 + 2i_9 + j_2 - j_3 - l_4 + l_2 + k - k_1 - r_4 + r_2 - n_1 = 0 \\
b_5 - n_2 &= l_5 - l_6 + i - i_1 + l_1 - l_2 + 2i_6 - 2i_7 + r_5 - r_6 + r_3 - r_4 \\
&\quad + j - j_1 + k_2 - k_3 - n_2 = 0 \quad \text{(A.4)}
\end{align*}
\]

We arrange the labels into two sets that we refer to as the principal and secondary ones. The so-called secondary ones disappear upon summation of the eight constraints and we get

\[
i + j + k + l + r = 2(n_1 + n_2) \tag{A.5}
\]

which is the constraint on the principal labels. The main result that emerges is that the purity only depends on two parameters, as follows:

\[
P_{n_1,n_2}(\eta, \theta) = \frac{2 (2/\rho)^{2(n_1+n_2)} (n_1! n_2)!}{\sin \theta \sqrt{2} \cosh(2\eta) + \tan^2(\theta/2) + \cot^2(\theta/2)}
\times \sum_{i+j+k+l+r=2(n_1+n_2)} C_{n_1,n_2}(i,j,k,l,r) u^iv^jw^ku^lw^r.
\] (A.6)

The most important feature of our result is that the function \( C_{n_1,n_2}(i,j,k,l,r) \) can now be derived exactly for any \( n_1 \) and \( n_2 \). This is

\[
C_{n_1,n_2} = \left( \prod_{c=1}^{7} \sum_{e=1}^{l_e-1} \frac{1}{(l_e-1 - i_e)!} \right) \left( \prod_{c=1}^{7} \sum_{j_e=0}^{j_e-1} \frac{1}{(j_e-1 - j_e)!} \right) \left( \prod_{c=1}^{7} \sum_{k_e=0}^{k_e-1} \frac{1}{(k_e-1 - k_e)!} \right)
\times \left( \prod_{c=1}^{7} \sum_{e=1}^{l_e-1} \frac{1}{(l_e-1 - l_e)!} \right) \left( \prod_{c=1}^{7} \sum_{r_e=0}^{r_e-1} \frac{1}{(r_e-1 - r_e)!} \right)
\times \frac{2^{i_4}(1-2^{i_8})}{i_1! j_3! k_1! l_1! r_1!}(1-r_{r_1}+r_{r_3}-r_5+r_6-r_7)(-1)^{l_1-l_3+l_4-l_5+l_6-l_7}. \tag{A.7}
\]

doi:10.1088/1742-5468/2011/09/P09015
Using the above constraints, we show that $C_{n_1n_2}$ can be reduced to the form

$$C_{n_1n_2} = \frac{\left(\prod_{c=1}^{i_c-1} \sum_{c=1}^{j_c} \sum_{c=1}^{k_c} \sum_{c=1}^{l_c-1} \prod_{r=1}^{r_c-1} \frac{1}{(r_c-1 - r_c)!} \right) \prod_{r=1}^{r_c-1} \left(\prod_{c=1}^{i_c} \sum_{c=1}^{j_c} \sum_{c=1}^{k_c} \sum_{c=1}^{l_c} \prod_{r=1}^{r_c} \frac{1}{(r_c-1 - r_c)!} \right)}{\left(\prod_{c=1}^{i_c} \sum_{c=1}^{j_c} \sum_{c=1}^{k_c} \sum_{c=1}^{l_c} \prod_{r=1}^{r_c} \frac{1}{(r_c-1 - r_c)!} \right) \prod_{r=1}^{r_c} \left(\prod_{c=1}^{i_c} \sum_{c=1}^{j_c} \sum_{c=1}^{k_c} \sum_{c=1}^{l_c} \prod_{r=1}^{r_c} \frac{1}{(r_c-1 - r_c)!} \right)}$$

(A.8)

where the parameters involved are fixed as

$$c_1 = \frac{1}{2} \left[ 2n_1 - (i_2 - i_3) - (j_2 - j_3) - (k - k_1) - (l_2 - l_3) - (l_3 - l_4) - (r_2 - r_3) - (r_3 - r_4) \times (i_3 - i_4) - (r_4 - r_5) - (j_2 - j_3) - (l_4 - l_5) - (l_5 - l_6) - (r_7 - r_3) - (r_3 - r_4) \times (l_4 - l_5) - (r_7 - r_6) - (r_5 - r_7) - (r_7 - r_5) - (r_5 - r_7) - (l_6 - l_7) - (l_7 - l_8) - (l_8 - l_9) - (l_9 - l_10) \right]$$

$$c_2 = \frac{1}{2} \left[ n_1 - (i_2 - i_3) - (r_5 - r_6) - (r_6 - r_7) - (r_6 - r_7) - (l_6 - l_7) - (r_7 - r_7) - (l_7 - l_8) - (l_8 - l_9) - (l_9 - l_10) \right]$$

$$c_3 = \left( \frac{n_1 - (i_3 - i_4) - (r_4 - r_5) - (j_2 - j_3) - (l_4 - l_5) - (l_5 - l_6) - (r_7 - r_7) - (l_7 - l_8) - (l_8 - l_9) - (l_9 - l_10)}{2} \right)!$$

$$c_4 = \left( \frac{n_1 - (i_2 - i_3) - (r_5 - r_6) - (r_6 - r_7) - (l_6 - l_7) - (l_7 - l_8) - (l_8 - l_9) - (l_9 - l_10)}{2} \right)!$$

$$c_5 = \left( \frac{n_1 - (i_3 - i_4) - (l - l_1) - (l_1 - l_2) - (r - r_1) - (r_1 - r_2) - j_3 - (k - k_1)}{2} \right)!$$

$$c_6 = \left( \frac{n_1 - (i_2 - i_3) - (j_2 - j_3) - (k - k_1) - (l_2 - l_3) - (l_3 - l_4) - (l_4 - l_5) - (r_2 - r_3) - (r_3 - r_4)}{2} \right)!$$

$$c_7 = \left( \frac{n_2 - (r_1 - r_2) - (i_1 - i_2) - (l_6 - l_7) - (l_7 - l_8) - (l_8 - l_9) - (l_9 - l_10) - (r_4 - r_5) - (j_1 - j_2) - (k_2 - k_3)}{2} \right)!$$

$$c_8 = \left( \frac{n_2 - (l - l_1) - (l_1 - l_2) - (l_2 - l_3) - (l_3 - l_4) - (l_4 - l_5) - (l_5 - l_6) - (i - i_1) - (j_1 - j_2)}{2} \right)!$$

$$c_9 = \left( \frac{n_2 - (1 - l_1) - (l_1 - l_2) - (l_2 - l_3) - (j_1 - j_2) - (r_6 - r_7) - (r_7 - r_3) - (r_3 - r_4)}{2} \right)!$$

$$c_{10} = \left( \frac{n_2 - (l_1 - l_2) - (l_2 - l_3) - (i_1 - i_2)}{2} \right)!$$

References

[1] Schrödinger E, 1935 Naturwissenschaften 23 807
[2] Einstein A, Podolsky B and Rosen N, 1935 Phys. Rev. 47 777
[3] Bell J S, 1987 Speakable and Unspeakable in Quantum Mechanics (Cambridge: Cambridge University Press)
[4] Bennett C H and Shor P W, 1998 IEEE Trans. Inf. Theory 44 2724
[5] Bennett C H, Brassard G, Crépeau C, Josza R, Peres A and Wootters W K, 1993 Phys. Rev. Lett. 70 1895
[6] Bennett C H and Wiesner S J, 1992 Phys. Rev. Lett. 69 2881
[7] Ekert A K, 1991 Phys. Rev. Lett. 67 661
[8] Murao M, Jonathan D, Plenio M B and Vedral V, 1999 Phys. Rev. A 59 156
[9] Fuchs C A, 1997 Phys. Rev. Lett. 79 1162
[10] Raussendorf R and Briegel H, Quantum computing via measurements only, 2000 quant-ph/0010033
[11] Gottesman D and Chuang I, 1999 Nature 402 390
[12] Rungta P, Busek V, Caves C M, Hillery M and Milburn G J, 2001 Phys. Rev. A 64 042315
[13] Bennett C H, DiVincenzo D P, Smolin J and Wootters W K, 1996 Phys. Rev. A 54 3824
[14] Wootters W K, 1998 Phys. Rev. Lett. 80 2245

doi:10.1088/1742-5468/2011/09/P09015
Entanglement in coupled harmonic oscillators studied using a unitary transformation

[15] Coffman V, Kundu J and Wootters W K, 2000 Phys. Rev. A 61 052306
[16] Markham D and Vedral V, 2003 Phys. Rev. A 67 042113
[17] Bargatin I V, Grishanin B A and Zadkov V N, 2001 Phys.—Usp. 44 597
[18] Clauser J F and Shimony A, 1978 Rep. Prog. Phys. 41 1881
[19] Harshman N L and Flynn W F, 2011 Quantum Inf. Comput. 11 278
[20] Schweber S S, 1961 An Introduction to Relativistic Quantum Field Theory (New York: Row-Peterson)
[21] Fetter A L and Walecka J D, 1971 Quantum Theory of Many Particle Systems (New York: McGraw-Hill)
[22] Han D, Kim Y S and Noz M E, 1990 Phys. Rev. A 41 6233
[23] Dirac P A M, 1963 J. Math. Phys. 4 901
[24] Caves C M and Schumaker B L, 1985 Phys. Rev. A 31 3068
Schumaker B L and Caves C M, 1985 Phys. Rev. A 31 3093
[25] Kim Y S, 1989 Phys. Rev. Lett. 63 348
[26] Iachello F and Oss S, 1991 Phys. Rev. Lett. 66 2976
[27] Umezawa H, Matsumoto H and Tachiki M, 1982 Thermo Field Dynamics and Condensed States (Amsterdam: North-Holland)
[28] Yurke B and Potasek M, 1987 Phys. Rev. A 36 3464
[29] Ekert A K and Knight P L, 1989 Am. J. Phys. 57 692
[30] Han D, Kim Y S and Noz M E, 1989 Phys. Lett. A 144 111
[31] Barnett S M and Phoenix S J D, 1991 Phys. Rev. A 44 535
[32] Jellal A, El Kinani E H and Schreiber M, 2005 Int. J. Mod. Phys. A 20 1515
[33] Han D, Kim Y S and Noz M E, 1999 Am. J. Phys. 67 61

doi:10.1088/1742-5468/2011/09/P09015