Warped spherical compactifications in the gravity theory

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Abstract

We present new exact solutions of the warped spherical compactifications in the higher-dimensional gravitational theory coupled to scalar and several form field strengths. We find two classes of solutions. One has a de Sitter spacetime with a static warp factor. The other gives an accelerating universe in the non-Einstein conformal frame with a time-dependent warp factor.

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I. INTRODUCTION

Recently, many models of warped compactifications have been studied in the higher-dimensional gravitational theory. These models have shed light on many different aspects of cosmology, phenomenology and black hole physics. The purpose of the present paper is to obtain new classes of warped solutions and apply them to study of the four-dimensional cosmological dynamics, in particular a realization of a de Sitter or an accelerating universe. A motivation for studying a de Sitter or an accelerating universe is that it may give us a fairly direct explanation of the inflationary universe and/or the current acceleration of the universe from the purely gravitational point of view [1]. So far, most works devoted to compactifications to de Sitter or accelerating universes have been made in the context of the four-dimensional effective theory. However, straightforward derivations of such solutions in the higher-dimensional gravity have not been extensively explored so much in terms of finding the exact solutions of the field equations. In particular, we will present warped cosmological solutions in which the internal space geometry is given by a product of several spheres. These are similar to the solutions discussed in [2]. Although we consider a simple class of the ansatz for fields, this provides a good illustration to find the warped cosmological solution.

A warped spherical compactification has been used in [2]. The basic idea was to assume that the internal geometry is given by a product of $S^1$ and several other compact spaces, which are spheres in our case, and also that the warp factor depends only on the coordinate of $S^1$. Along such a way, with a more general class of the space as $S^1 \times \prod_{I} S^{L_i}$, we will study the compactification to a maximally symmetric spacetime, i.e., a de Sitter or an anti-de Sitter (AdS) universe. Because of the maximal symmetry of an external spacetime, it can be embedded into a warped spacetime where the warp factor and the internal space are independent of time. In fact, the model we will focus on has been studied, from a rather different vantage point in [3–5]. A number of important features were pointed out in [2], including not only the gravitational field but also matter fields. Such a model has also been investigated recently in the works [6, 7] which have overlap with our present work, but has still become interesting in light of other type of compactifications.

We also discuss an embedding of a more general expanding Friedmann-Lemaître-Robertson-Walker (FLRW) universe whose scale factor is given by a power-law function of
the cosmic time into the warped compactification on $S^1 \times \prod_I S^{L_I}$. Because of a less symmetry of an external geometry, it may be possible only into a higher-dimensional spacetime where the warp factor and the internal space become time-dependent. We will investigate whether and how cosmic expansion can be incorporated with a time-dependence of the warp and the internal space, and, if there are solutions, whether they could realize accelerating universes. Such solutions with time-dependence in the warp factor have been less investigated, and exceptional cases are the time-dependent brane solutions in the supergravity theories [8–32], which have direct connections with D-branes and M-branes in superstring and M-theory, respectively. Since these dynamical brane solutions provide accelerated expansion of the four-dimensional universe in the non-Einstein conformal frame, we will explore new cosmological solutions for the warped spacetime. We present an ansatz for a warped spacetime in the $D$-dimensional theory that realizes the four-dimensional de Sitter or AdS universe, and then we solve the higher-dimensional field equations.

We start by describing warped de Sitter solutions in section II. In section III, we discuss a time-dependent warped compactification with an infinite volume of the internal space. We summarize our results in section IV.

II. WARPED DE SITTER SOLUTIONS

In this section, we focus on an embedding of a maximally symmetric, a de Sitter or AdS, universe into a warped spacetime where the warp factor and the internal space are static.

We consider a gravitational theory with the metric $g_{MN}$, the scalar field $\phi$ and the antisymmetric tensor field of rank $p_I$. The action which we consider is given by

$$ S = \frac{1}{2\kappa^2} \int \left[ R * 1 - \frac{1}{2} d\phi \wedge d\phi - \sum_I \frac{1}{2 ! p_I !} \varepsilon^{c_I} F(p_I) \wedge * F(p_I) \right], $$

where $\kappa^2$ is the $D$-dimensional gravitational constant, and $*$ is the Hodge operator in $D$-dimensional spacetime, $F(p_I)$ is the $p_I$-form field strength, and $c_I$ is a constant, and $\varepsilon = \pm 1$.

The $D$-dimensional action (I) gives the field equations:

1. $R_{MN} = \frac{1}{2} \partial_M \phi \partial_N \phi + \sum_I \frac{\varepsilon^{c_I} \phi}{2 ! p_I !} p_I (F(p_I))_{M N}^2 - \frac{p_I - 1}{D - 2} g_{M N} F(p_I)^2$,  

2. $d \left[ \varepsilon^{c_I} F(p_I) \right] = 0$,  

3. $d * d\phi - \sum_I \frac{\varepsilon c_I}{2 ! p_I !} \varepsilon^{c_I} F(p_I) \wedge * F(p_I) = 0$.  


where \((F_{pI})_{MN}^2 = F_{MA_1...A_{pI-1}} F_{N}^{A_1...A_{pI-1}}\). In this paper, the derivation of the solutions completely follows the method developed in the previous works devoted to finding the solutions of the warped de Sitter compactifications \[2, 3, 4\]. To solve the field equations, we look for solutions whose spacetime metric has the form

\[
d s^2 = e^{2A(y)} \left[ q_{\mu\nu}(X) dx^\mu dx^\nu + dy^2(Y) + \sum_I u_{a_I b_I}(Z_I) dz^{a_I} dz^{b_I} \right],
\]

where \(q_{\mu\nu}(X)\) is the \(n\)-dimensional metric which depends only on the \(n\)-dimensional coordinates \(x^\mu\), and \(u_{a_I b_I}(Z_I)\) is the \(L_I\)-dimensional metric which depends only on the \(L_I\)-dimensional coordinates \(z^{a_I}\), the function \(A(y)\) depends only on the coordinate \(y\), respectively. Hence, the warp factor is limited to the function \(A\). The number of dimensions is related as \(D = n + \sum_L^i L_I + 1\). Although we consider the particular class of solutions, they provide us a direct route to warped de Sitter solutions.

Concerning the gauge fields, we adopt the following assumptions

\[
F_{(n)} = f_n \Omega(X) ,
\]

\[
F_{(L_I)} = \ell_I \omega_I ,
\]

\[
F_{(0)} = m,
\]

where \(f_n\), \(\ell_I\) and \(m\) are constants, and \(c_0\) is the coupling constant for \(F_{(0)}\), and \(\Omega(X)\) and \(\omega_I\) denote the volume \(n\)-form and the \(L_I\)-form for each \(Z_I\), respectively. Under the \(D\)-dimensional metric and form fields given above, we first reduce the Einstein equations other than the gauge and scalar field equations to a simple set of equations

\[
R_{\mu\nu}(X) - \frac{1}{2} \partial_\mu \phi \partial_\nu \phi - \left[ U - \frac{1}{2(D-2)} \left\{ (D - n - 3)f_n^2 e^{-2(n-1)A + \varepsilon c_n \phi} 
\right.
\right.
\]

\[
+ \left( D - 1 \right) \ell_I^2 e^{-2(L_I - 1)A + \varepsilon c_I \phi} - m^2 e^{2A + \varepsilon c_0 \phi} \right] q_{\mu\nu}(X) = 0 ,
\]

\[
-(D - 1) \partial_y^2 A - \frac{1}{2} \left( \partial_y \phi \right)^2 - \frac{1}{2(D-2)} \left[ (n - 1) f_n^2 e^{-2(n-1)A + \varepsilon c_n \phi} 
\right.
\]

\[
- \sum_I (L_I - 1) \ell_I^2 e^{-2(L_I - 1)A + \varepsilon c_I \phi} + m^2 e^{2A + \varepsilon c_0 \phi} \right] = 0 ,
\]

\[
R_{a_I b_I}(Z_I) - \frac{1}{2} \partial_{a_I} \phi \partial_{b_I} \phi - \left[ U + \frac{1}{2(D-2)} \left\{ (n - 1) f_n^2 e^{-2(n-1)A + \varepsilon c_n \phi} 
\right.
\right.
\]

\[
+ \sum_I (D - L_I - 3) \ell_I^2 e^{-2(L_I - 1)A + \varepsilon c_I \phi} + m^2 e^{2A + \varepsilon c_0 \phi} \right] u_{a_I b_I}(Z_I) = 0 ,
\]
where $R_{\mu\nu}(X)$, $R_{a_ib_j}(Z_I)$ are the Ricci tensors of the metrics $q_{\mu\nu}(X)$, $u_{a_ib_j}(Z_I)$, and $U$ is defined by

$$U = \partial_y^2 A + (D - 2) (\partial_y A)^2 .$$

(6)

Next we consider the scalar field. We require that the scalar field satisfies the condition

$$2A + \varepsilon c_0 \phi = 0, \quad -2(n-1)A + \varepsilon c_n \phi = 0, \quad -2(L_I - 1)A + \varepsilon c_{I} \phi = 0.$$  

(7)

With these expressions, we immediately see that the scalar field $\phi$ depends on the linear function of the warp factor and the exponents on the exponential functions in the Einstein equations (5) vanish separately. Hence, (7) is equivalent to

$$\phi = -\frac{2A}{\varepsilon c_0}, \quad c_n = -(n-1)c_0, \quad c_I = -(L_I - 1)c_0,$$

(8)

where $c_n$ and $c_I$ are the coupling constants for $F(n)$ and $F(I)$, respectively. Under the assumptions (4) and (8), the Bianchi identities and the equations of motion for the gauge fields are automatically satisfied.

We should not choose $m = 0$ because of the ansatz of the scalar field (7). Thus, we focus on the case of $m \neq 0$ and come back to the case of $m = 0$ later. The choice of the ansatz for the metric and matter fields used to solve field equations is essential, and a particular coupling constant is preferred in order to make both the equations of motion and the form fields for the internal spaces including the 0-form simple.

Substituting Eqs. (3), (4) and (8) into Eq. (2c), the scalar field equation gives

$$U + \frac{c_0^2}{4} (K + m^2) = 0,$$

(9)

where $K$ is defined as

$$K = (n - 1)f_n^2 - \sum_I (L_I - 1)f_I^2 .$$

(10)

Using the scalar field equation (9), the Einstein equations (5) are rewritten as

$$R_{\mu\nu}(X) - \frac{1}{2} \left( 2U - f_n^2 + \frac{K + m^2}{D - 2} \right) q_{\mu\nu}(X) = 0,$$

(11a)

$$-(D - 2) \partial_y^2 A + \left( D - 2 - \frac{2}{c_0^2} \right) (\partial_y A)^2 - \frac{1}{2} \left( 2U + \frac{K + m^2}{D - 2} \right) = 0,$$

(11b)

$$R_{a_ib_j}(Z_I) - \frac{1}{2} \left( 2U + \ell_I^2 + \frac{K + m^2}{D - 2} \right) u_{a_ib_j}(Z_I) = 0.$$  

(11c)
From Eqs. (9) and (11b), we get

\[ A = A_0 + \frac{1}{D-2} \ln \left[ \cos \left\{ \sqrt{\frac{K + m^2}{2(D-1)} (y - y_0)} \right\} \right], \]

(12a)

\[ c_0^2 = \frac{2}{(D-1)(D-2)}, \]

(12b)

where \( A_0 \) is a constant. In terms of Eqs. (12), the field equations reduce to

\[ R_{\mu\nu}(X) - \frac{1}{2} \left( -f_n^2 + \frac{K + m^2}{D-1} \right) q_{\mu\nu}(X) = 0, \]

(13a)

\[ R_{a_I b_I}(Z_I) - \frac{1}{2} \left( \ell_I^2 + \frac{K + m^2}{D-1} \right) u_{a_I b_I}(Z_I) = 0. \]

(13b)

Hence, the metric of \( D \)-dimensional spacetime can be written as

\[ ds^2 = \bar{A}^2 \left[ \cos (\bar{y} - \bar{y}_0) \right]^{2/(D-2)} \left[ q_{\mu\nu}(X)dx^\mu dx^\nu + \frac{2(D-1)}{K + m^2} d\bar{y}^2 + \sum_I u_{a_I b_I}(Z_I)dz^a dz^b \right], \]

(14)

where \( \bar{A} = e^{A_0} \) is a constant, and \( \bar{y}, \bar{y}_0 \) are defined by

\[ \bar{y} = \sqrt{\frac{K + m^2}{2(D-1)} y}, \quad \bar{y}_0 = \sqrt{\frac{K + m^2}{2(D-1)} y_0}. \]

(15)

If \( f_n, \ell_I \) and \( m \) satisfy the relation

\[ -f_n^2 + \frac{K + m^2}{D-1} > 0, \]

(16)

the solution leads to an \( n \)-dimensional de Sitter spacetime whose expansion rate \( H \) is given by

\[ H^2 = \frac{1}{2(n-1)} \left( -f_n^2 + \frac{K + m^2}{D-1} \right). \]

(17)

Using Eq. (13b) and the relation (16), we have

\[ R(Z_I) = \frac{L_I}{2} \left( \ell_I^2 + \frac{K + m^2}{D-1} \right) > \frac{L_I}{2} (\ell_I^2 + f_n^2) > 0. \]

(18)

Hence, for the de Sitter case \( R(X) > 0 \), all the internal spaces \( Z_I \) must be positively curved.

From Eq. (13b), the \( I \)th internal space can be a sphere \( S' \), if

\[ \ell_I^2 + \frac{K + m^2}{D-1} > 0. \]

(19)
The $D$-dimensional metric (14) implies that there are curvature singularities at $\bar{y} = \bar{y}_0 + (n + \frac{1}{2})\pi$ ($n$ is integer) because the Kretschmann invariant of the metric (14) is given by

$$R_{ABCD}R^{ABCD} = \left[ \cos(\bar{y} - \bar{y}_0) \right]^{-\frac{D}{n-2}} [n(n-2) + \frac{2(D-2)}{\cos^4(\bar{y} - \bar{y}_0)}].$$

(20)

The scalar field has the bounce configuration and diverges at $\bar{y} = \bar{y}_0 + (n + \frac{1}{2})\pi$, which gives the singularities in the $D$-dimensional background. Thus it is impossible to extend the spacetime across such a point and we should restrict $\bar{y}$ to be for a period, e.g., $\bar{y}_0 - \pi/2 < \bar{y} < \bar{y}_0 + \pi/2$. Then, unless we modify the ansatz of fields and add the matter fields, the solutions considered in this section provide neither realistic cosmological models nor compactification schemes that lead to the finite Newton constant.

We would like to mention the relation of our solutions with the supergravity theories. The coupling constants of the scalar field in the supergravity theory are severely restricted. Actually, coupling constants take the particular values, which are written as (27, 33)

$$c_n^2 = N_n - \frac{2(n-1)(D-n-1)}{D-2}, \quad (21a)$$

$$c_I^2 = N_I - \frac{2(L_I-1)(D-L_I-1)}{D-2}, \quad (21b)$$

$$c_0^2 = N_0 + \frac{2(D-1)}{D-2}, \quad (21c)$$

where $N_n$, $N_I$ and $N_0$ are constants. In terms of Eqs. (8) and (12b), these constants become

$$N_n = \frac{2(n-1)(D-n)}{D-1}, \quad (22a)$$

$$N_I = \frac{2(L_I-1)(D-L_I)}{D-1}, \quad (22b)$$

$$N_0 = -\frac{2D}{D-1}. \quad (22c)$$

Though the cases of $N_i = 4$, $(i = n, I, 0)$ in the ten- or eleven-dimensional theory correspond to supergravities, we cannot choose it due to $N_0 < 0$. These supergravity solutions always give the X space to be an AdS spacetime. In our limited ansatz of fields, it is difficult to obtain the four-dimensional de Sitter spacetime in the supergravity models.

We then discuss the case of $m = 0$. We assume that the $D$-dimensional metric and the gauge field strengths take the same form as (3), (4a) and (4b). Then, the Einstein equations
become

\[
R_{\mu\nu}(X) - \left[ U - \frac{1}{2(D-2)} \left\{ (D - n - 3) f_n^2 e^{-2(n-1)A + \varepsilon c_n \phi} + \sum_I (L_I - 1) \ell_I^2 e^{-2(L_I-1)A + \varepsilon c_I \phi} \right\} \right] q_{\mu\nu}(X) = 0, 
\]

\[
- (D - 1) \partial_y^2 A - \frac{2}{\ell_0^2} (\partial_y A)^2 - \frac{1}{2(D-2)} \left[ (n - 1) f_n^2 e^{-2(n-1)A + \varepsilon c_n \phi} - \sum_I (L_I - 1) \ell_I^2 e^{-2(L_I-1)A + \varepsilon c_I \phi} \right] = 0, 
\]

\[
R_{\alpha\beta}(Z_I) - \left[ U + \frac{1}{2(D-2)} \left\{ (n - 1) f_n^2 e^{-2(n-1)A + \varepsilon c_n \phi} + \sum_I (D - L_I - 3) \ell_I^2 e^{-2(L_I-1)A + \varepsilon c_I \phi} \right\} \right] u_{\alpha\beta}(Z_I) = 0, 
\]

where \( R_{\mu\nu}(X) \), \( R_{\alpha\beta}(Z_I) \) are the Ricci tensors of the metrics \( q_{\mu\nu}(X) \), \( u_{\alpha\beta}(Z_I) \), and \( U \) is defined by (6). Let us next consider the scalar field. For coupling constants of the scalar field, we choose

\[- 2(n - 1)A + \varepsilon c_n \phi = 0, \quad - 2(L_I - 1)A + \varepsilon c_I \phi = 0. \]

Among these equations, the first together with the second assumption in (24) reads

\[ c_I = (n - 1)^{-1} (L_I - 1) c_n. \]

Upon setting the assumptions of fields and the coupling constants (25), the Bianchi identities and the equations of motion for the gauge fields are again automatically satisfied. In terms of the assumptions (3) and the field equations reduce to Eqs. (13) and (14) with \( m = 0 \). The \((13a)\) with \( m = 0 \) illustrates that de Sitter compactification is not allowed (see [34] for the general cases). On the other hand, for \( m \neq 0 \), the X space can be an Einstein space with a positive curvature where the existence of de Sitter solutions is clear. Each internal space \( Z_I \) can be sphere \( S^I \), if (19) is satisfied.

In the case of \( m = 0 \), there are solutions of supergravity in ten dimensions. From (21) and (22), the solution of \( (D, n, L_1, L_2) = (10, 4, 1, 4) \) with \( m = 0 \) and \( \ell_1 = 0 \) (here \( I = 1, 2 \)) gives that of ten-dimensional type IIA supergravity. Similarly, the solution of \( (D, n, L_1) = (10, 2, 7) \) with \( f_2 = 0 \) and \( m = 0 \) (here \( I = 1 \)) gives that of ten-dimensional type IIB supergravity. However, these solutions only lead to AdS compactifications.
III. TIME-DEPENDENT INTERNAL SPACE

In this section, we investigate an embedding of an expanding FLRW universe into a time-dependent warped spacetime.

Let us consider a gravitational theory with the metric \( g_{MN} \), the scalar field \( \phi \), the cosmological constant \( \Lambda \), and the antisymmetric tensor field of rank \( p_I \). The action which we consider is given by

\[
S = \frac{1}{2\kappa^2} \int \left[ (R - 2e^{\alpha\phi}\Lambda) * 1 - \frac{1}{2} d\phi \wedge * d\phi - \sum_I \frac{1}{2! p_I!} e^{\alpha_I\phi} F_{(p_I)} \wedge * F_{(p_I)} \right],
\]

(26)

where \( \kappa^2 \) is the \( D \)-dimensional gravitational constant, and * is the Hodge operator in the \( D \)-dimensional spacetime, \( F_{(p_I)} \) is the \( p_I \)-form field strength, and \( \alpha, \alpha_I \) are constants.

The \( D \)-dimensional action (26) gives the field equations:

\[
R_{MN} = \frac{1}{2} \partial_M \phi \partial_N \phi + \frac{2}{D - 2} e^{\alpha\phi} \Lambda g_{MN} + \sum_I \frac{e^{\alpha_I\phi}}{2! p_I!} \left[ p_I \left( F_{(p_I)} \right)_{MN}^2 - \frac{p_I - 1}{D - 2} g_{MN} F_{(p_I)}^2 \right],
\]

(27a)

\[
d \left[ e^{\alpha_I\phi} * F_{(p_I)} \right] = 0,
\]

(27b)

\[
d * d\phi - 2\alpha e^{\alpha\phi} \Lambda * 1 - \sum_I \frac{\alpha_I}{2! p_I!} e^{\alpha_I\phi} F_{(p_I)} \wedge * F_{(p_I)} = 0,
\]

(27c)

where \( \left( F_{(p_I)} \right)_{MN}^2 = F_{MA_1 \cdots A_{p_I - 1}} F_N^{A_1 \cdots A_{p_I - 1}} \). To solve the field equations, we assume that the \( D \)-dimensional metric takes the form

\[
ds^2 = e^{2[A_0(x) + A_1(y)]} \left[ q_{\mu\nu}(X)dx^\mu dx^\nu + dy^2(Y) + \sum_I u_{a_I b_I}(Z_I)dz^{a_I} dz^{b_I} \right],
\]

(28)

where \( q_{\mu\nu}(X) \) is the \( n \)-dimensional metric which depends only on the \( n \)-dimensional coordinates \( x^\mu \), and \( u_{a_I b_I}(Z_I) \) is the \( L_I \)-dimensional metric which depends only on the \( L_I \)-dimensional coordinates \( z^{a_I} \), the functions \( A_0(x) \) and \( A_1(y) \) depend only on the coordinates \( x^\mu \) and \( y \), respectively.

Furthermore, we consider the field strength \( F_{(L_I)} \). The scalar field \( \phi \) and the gauge field strengths are assumed to be

\[
\phi = \frac{-2}{\alpha} \left[ A_0(x) + A_1(y) \right],
\]

(29a)

\[
F_{(L_I)} = f_I \omega_I,
\]

(29b)
where \( f_I \) is a constant, and \( \omega_I \) denotes the volume \( L_I \)-form of the internal space \( Z_I \). In the following, we assume that the parameter \( \alpha_I \) is given by

\[
\alpha_I = -\alpha (L_I - 1) .
\]

Let us first consider the Einstein Eqs. (27a). Using the assumptions (28) and (29), the Einstein equations are given by

\[
R_{\mu\nu}(X) - (D - 2) D_{\mu} D_{\nu} A_0 + \left( D - 2 - \frac{2}{\alpha^2} \right) \partial_{\mu} A_0 \partial_{\nu} A_0 - (\bar{U} + \bar{K}) q_{\mu\nu}(X) = 0 , \tag{31a}
\]

\[
\left( D - 2 - \frac{2}{\alpha^2} \right) \partial_{\mu} A_0 \partial_{\nu} A_1 = 0 , \tag{31b}
\]

\[
(D - 2) \partial_\mu^2 A_1 - \left( D - 2 - \frac{2}{\alpha^2} \right) (\partial_\mu A_1)^2 + \bar{U} + \bar{K} = 0 , \tag{31c}
\]

\[
R_{aibj}(Z_I) - \left( \frac{f_I^2}{2} + \bar{U} + \bar{K} \right) u_{aibj}(Z_I) = 0 , \tag{31d}
\]

where \( D_\mu \) is the covariant derivative with respect to the metric \( q_{\mu\nu} \) and \( \triangle_X \) is the Laplace operator on the space of \( X \), and \( R_{\mu\nu}(X), R_{aibj}(Z_I) \) are the Ricci tensors of the metrics \( q_{\mu\nu} \), \( u_{aibj} \), respectively, and \( \bar{U}, \bar{K} \) are defined as

\[
\bar{U} = \triangle_X A_0 + (D - 2) q^{\rho\sigma} \partial_\rho A_0 \partial_\sigma A_0 + \partial_\mu^2 A + (D - 2) (\partial_\mu A)^2 , \tag{32a}
\]

\[
\bar{K} = \frac{1}{2(D - 2)} \left[ 4\Lambda - \sum_I (L_I - 1) f_I^2 \right] . \tag{32b}
\]

From (31b), the parameter \( \alpha \) must be in the form

\[
\alpha = \pm \sqrt{\frac{2}{D - 2}} . \tag{33}
\]

Next we consider the gauge fields. Under the assumption (29b), the Bianchi identities and the equations of motion for the gauge fields are automatically satisfied. Substituting Eqs. (28), (29) and (33) into Eq. (27c), the scalar field equation gives

\[
\bar{U} + \bar{K} = 0 . \tag{34}
\]
In terms of (29) and (33), the field equations reduce to

\[ R_{\mu\nu}(X) - (D - 2) D_\mu D_\nu A_0 = 0, \]  
\[ (D - 2) \partial_y^2 A_1 = 0, \]  
\[ R_{aib_1}(Z_I) - \frac{f_I^2}{2} u_{aib_1}(Z_I) = 0, \]  
\[ \Delta_X A_0 + (D - 2) q^{\mu\nu} \partial_\mu A_0 \partial_\nu A_0 + \partial_y^2 A + (D - 2) (\partial_y A)^2 \]
\[ + \frac{2}{2(D - 2)} \left[ 4\Lambda - \sum_I (L_I - 1) f_I^2 \right] = 0. \]

Note that the (35a) is the differential equation with respect to the coordinates \( x^\mu \) while Eqs. (35b) are the equations of \( y \). Thus, we can treat them separately. We also find that from Eq. (35c), each internal space \( Z_I \) is positively curved if the corresponding field strength is nonvanishing along \( Z_I \).

Let us first consider the (35b). The solution of \( A_1 \) is

\[ A_1(y) = \ell (y - y_0), \]

where \( \ell \) is a constant.

Next we consider the (35a). We set the function \( A_0 \) and the \( n \)-dimensional spacetime metric \( q_{\mu\nu} \) to be a spatially homogeneous function and a spatially flat FLRW universe, respectively. With the scale factor \( a \) turned on, it turns out that the metric is of the form

\[ A_0 = A_0(t), \]
\[ q_{\mu\nu}(X) dx^\mu dx^\nu = -dt^2 + a^2(t) \delta_{mn} dx^m dx^n, \]

where \( \delta_{mn} \) is the metric of \( (n - 1) \)-dimensional Euclidean space.

Using the \( n \)-dimensional metric (37), the Eq. (31a) is rewritten by

\[ (n - 1) \left[ \left( \frac{\dot{a}}{a} \right)^2 + \left( \frac{\dot{a}}{a} \right)^2 \right] + (D - 2) \ddot{A}_0 = 0, \]
\[ \left( \frac{\dot{a}}{a} \right)^2 + (n - 1) \left( \frac{\dot{a}}{a} \right)^2 + (D - 2) \ddot{A}_0 = 0, \]

where “dot” denotes the ordinary derivative with respect to the coordinate \( t \). We assume that the functions \( a(t) \) and \( A_0(t) \) are given by

\[ \frac{\dot{a}}{a} = c_1 t^{-1}, \quad \dot{A}_0 = c_2 t^{-1}, \]
where \( c_1 \) and \( c_2 \) are constants. Thus, the Einstein equations (38) reduce to

\[
(n - 1)c_1 (c_1 - 1) - (D - 2)c_2 = 0, \tag{40a}
\]

\[
[-1 + (n - 1)c_1 + (D - 2)c_2]c_1 = 0. \tag{40b}
\]

Except for the trivial solution \( c_1 = c_2 = 0 \), we find

\[
c_1 = \pm(n - 1)^{-1/2}, \quad c_2 = \frac{n(n - 1) + \sqrt{n - 1}}{(D - 2)\sqrt{n - 1}}. \tag{41}
\]

From the Eq. (39), we have

\[
a(t) = a_0 (t - t_0)^{c_1}, \quad A_0(t) = c_2 \ln [\psi_0 (t - t_0)], \tag{42}
\]

where \( \psi_0 \) and \( a_0 \) are constants. Then the metric of the \( D \)-dimensional spacetime can be written as

\[
ds^2 = e^{2A_1(y)} \left[ -d\tau^2 + a_0^2 \left( \frac{\tau}{\tau_0} \right)^{2(c_1 + c_2)} \delta_{mn}dx^m dx^n + \left( \frac{\tau}{\tau_0} \right)^{2(c_2 + 1)} \left( dy^2 + u_{ab}(Z)dz^a dz^b \right) \right], \tag{43}
\]

where the cosmic time \( \tau \) and the parameter \( \tau_0 \) are given by

\[
\frac{\tau}{\tau_0} = t^{c_2 + 1}, \quad \tau_0 = \frac{1}{c_2 + 1}. \tag{44}
\]

We find power-law cosmological evolutions. The power of the expansion of the external \( (n - 1) \)-dimensional space is given by

\[
\lambda = \frac{c_1 + c_2}{c_2 + 1}. \tag{45}
\]

For \( D = 10 \) and \( n = 4 \), we find that the fastest expanding case has the power \( \lambda = 0.53478 \) in terms of the proper time \( \tau \), while the lower branch one gives \( \lambda = -0.175805 \). Since we have obtained \( \lambda < 1 \) for \( n = 4 \), these solutions do not lead to accelerated expansion. However, by taking \( \tau = \bar{\tau}_c - \bar{\tau} \), we get

\[
ds^2 = e^{2A_1(y)} \left[ -d\bar{\tau}^2 + a_0^2 \left( \frac{\bar{\tau}_c - \bar{\tau}}{\bar{\tau}_0} \right)^{2(c_1 + c_2)} \delta_{mn}dx^m dx^n + \left( \frac{\bar{\tau}_c - \bar{\tau}}{\bar{\tau}_0} \right)^{2(c_2 + 1)} \left( dy^2 + u_{ab}(Z)dz^a dz^b \right) \right], \tag{46}
\]

where \( \bar{\tau}_c \) is a constant. For \( \bar{\tau} < \bar{\tau}_c \) we have accelerated expansion. Since \( g_{\mu\nu}(X) \) is not the Einstein-frame metric, we consider the cosmic expansion in the Einstein frame. In order to
analyze in the Einstein frame, therefore we perform the following conformal transformation to make the non-Einstein conformal frame into the Einstein frame:

\[ q_{\mu\nu}(X) = \left( \frac{\tau}{\tau_0} \right)^{-4c_2(D-n)/(c_2+1)(n-2)} q_{\mu\nu}(\bar{X}), \quad (47) \]

where \( q_{\mu\nu}(\bar{X}) \) is the \( n \)-dimensional metric in the Einstein frame. Then, the metric of \( n \)-dimensional spacetime is given by

\[ q_{\mu\nu}(\bar{X})dx^\mu dx^\nu = -d\bar{\tau}^2 + \left( \frac{\bar{\tau}}{\bar{\tau}_0} \right)^{2c_1(n-2)+c_2(2D-n-2)/(c_2+1)(n-2)} \delta_{mn} dx^m dx^n, \quad (48) \]

where \( \bar{\tau}_0 \) is a constant and \( \bar{\tau} \) is defined by

\[ \bar{\tau} = \left( \frac{\tau}{\tau_0} \right)^{1+\frac{2c_2(D-n)}{(c_2+1)(n-2)}}. \quad (49) \]

The power of time-dependence in the scale factor is given by

\[ 0 < \frac{c_1 + c_2(D-3)}{1 + c_2(D-3)} < 1, \quad \text{For } n = 4, \, D > 5. \quad (50) \]

Hence, the Einstein frame metric \( q(\bar{X}) \) yields the decelerating universe.

**IV. DISCUSSIONS**

In this work, we have presented exact solutions of the warped compactification on \( S^1 \times \prod_I S^{L_I} \), as in our recent study on the warped compactification on \( S^1 \times S^{D-n-1} \) [2].

We gave a new class of warped de Sitter solutions and illustrated how the matter fields, in particular the 0-form field strength, provide a de Sitter spacetime. The Einstein equations have led to a de Sitter spacetime in the presence of the 0-form field strength and constant fluxes. If there is no 0-form, the Einstein equations require AdS compactifications. We note that it is easy to find a solution if flux terms are not allowed to depend on the extra dimensional coordinate \( y \). This solution is different from solutions found in [2]. However, the assumptions [1] and [8] may play an important role in constructing a de Sitter solution. We also presented warped cosmological solutions where the warp factor is time-dependent. Although we could find solutions in which the time-dependent warp factor allows the accelerated expansion in the non-Einstein conformal frame in the higher-dimensional spacetime, the solutions presented here could not give accelerated expansion in the Einstein frame.
However, it would be very interesting to explore such solutions and to see whether and how the time-dependence with the warped structure of spacetime could realize accelerated expansion.

Our result for the cosmological solutions consists of two pieces. The first is a singular solution with a spherical internal space; the second is a nonsingular solution with an infinite volume internal space. The solution which we desire should have a compact internal space, where the conventional dimensional reduction scheme can be applied. Although the examples illustrated in this paper provide neither realistic cosmological models nor compactification schemes, the feature of the solutions or the method to obtain them could open new directions to study how to construct accelerated expansion of the universe as well as an appropriate higher-dimensional cosmological model.

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