\( \mathcal{R}(p, q) \) – deformed super Virasoro \( n \)– algebra

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**Abstract**

In this paper, we construct the super Witt algebra and super Virasoro algebra in the framework of the \( \mathcal{R}(p, q) \)– deformed quantum algebras. Moreover, we perform the super \( \mathcal{R}(p, q) \)– deformed Witt \( n \)– algebra, the \( \mathcal{R}(p, q) \)– deformed Virasoro \( n \)– algebra and discuss the super \( \mathcal{R}(p, q) \)– Virasoro \( n \)– algebra \((n \text{ even})\). Besides, we define and construct another super \( \mathcal{R}(p, q) \)– deformed Witt \( n \)– algebra and study a toy model for the super \( \mathcal{R}(p, q) \)– Virasoro constraints. Relevant particular cases induced from the quantum algebras known in the literature are deduced from the formalism developed.

**keyword** \( \mathcal{R}(p, q) \)– calculus, Super Virasoro algebra, super-Virasoro constraints.

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**1 Introduction**

The nature of the Virasoro algebra was described by Kupershmidt [25]. Its applications in mathematics and physics, such that in conformal field theory and string theory
were also presented \[6, 25, 31\]. Many generalizations and deformations (one or two parameters) of the Virasoro algebra were investigated in the literature \[1, 8, 18\]. The generalization of Kupershmidt’s work was provided in \[25\]. The relation between the Korteweg-de Vries (KdV) equation and the Virasoro algebra was described by Gervais \[12\] and Kupershmidt \[25\]. Moreover, Huang and Zhdanov presented the realizations of Witt and Virasoro algebras. Their connection with integrable equations was determined \[22\].

The construction of $\alpha^k$ derivation and a representation theory were investigated \[2\]. Also, the cohomology complex of Hom-Lie superalgebras was furnished and the central extensions was computed. As application, the derivations and the second cohomology group of a twisted $osp(1,2)$ superalgebra were calculated. Moreover, Curtright and Zachos introduced the $q-$ deformed Witt algebra \[10\] and from this results Ding et al determined a nontrivial $q-$ deformed Witt $n-$ algebras. It is a generalization of the Lie algebra also called sh-n-Lie algebra \[11\]. Wang et al \[32\] investigated the two different $q-$ deformed Witt algebra and constructed their $n-$ algebras. In one case, the super version is also presented. Moreover the central extensions is provided and the super $q-$ deformed Virasoro $n-$ algebra for the $n$ even case is furnished.

The two parameters deformation of the Virasoro algebra with conformal dimension was studied in \[8\]. Also, the central charge term for the Virasoro algebra and the associated deformed nonlinear equation (Korteweg-de Vries equation) were determined.

Moreover, the generalizations of $(p,q)$- deformed Heisenberg algebras, called $\mathcal{R}(p,q)$- deformed quantum algebras were investigated in \[17\]. Houkonnou and Melong \[19\] constructed the $\mathcal{R}(p,q)$- deformed conformal Virasoro algebra, derived the $\mathcal{R}(p,q)$- deformed Korteweg-de Vries equation for a conformal dimension $\Delta = 1$, and presented the energy-momentum tensor from the $\mathcal{R}(p,q)$- deformed quantum algebras for the conformal dimension $\Delta = 2$.

Recently, the generalizations of Witt and Virasoro algebras were performed, and the associated Korteweg-de Vries equations from the $\mathcal{R}(p,q)$- deformed quantum algebras were derived. Related relevant properties were investigated and discussed. Furthermore, the $\mathcal{R}(p,q)$- deformed Witt $n-$ algebra constructed, and the Virasoro constraints for a toy model, which play an important role in the study of matrix models was presented \[20\].

The aim of this paper is to construct the super Witt $n-$ algebra, Virasoro $2n-$ algebra, and super Virasoro $n-$ algebra ($n$ even) from the quantum deformed algebra \[17\]. As application, we construct another super $\mathcal{R}(p,q)$- deformed Witt $n-$ algebra and investigate a toy model for the super $\mathcal{R}(p,q)$- Virasoro constraints. Furthermore, we deduce particular cases associated to quantum algebra presented in the literature.

This paper is organized as follows: Section 2 is reserved to some notations, definitions and results used in the sequel. In section 3, we investigate the super Witt algebra and super Witt $n-$ algebra induced by the $\mathcal{R}(p,q)$- deformed quantum algebra. Moreover, we construct the $\mathcal{R}(p,q)$- deformed Virasoro $2n-$ algebra and deduce particular cases. In section 4, we furnished the super $\mathcal{R}(p,q)$- deformed Jacobi identity. Besides, we construct the super $\mathcal{R}(p,q)$- deformed Virasoro algebra and perform the super $\mathcal{R}(p,q)$- deformed Virasoro $n-$ algebra. Particular cases are deduced. Section 5 is dedicated to the application. We construct another super $\mathcal{R}(p,q)$- deformed Witt $n-$ algebra and study a toy model. We end with the concluding remarks in section 6.
2 Basics definitions and notations

Let us recall some definitions, notations, and known results used in this work. For that, let \( p \) and \( q \), two positive real numbers such that \( 0 < q < p \leq 1 \), and a meromorphic function \( \mathcal{R} \) defined on \( \mathbb{C} \times \mathbb{C} \) by [16]:

\[
\mathcal{R}(s, t) = \sum_{u, v = -l}^{\infty} r_{uv} s^u t^v,
\]

where \( r_{uv} \) are complex numbers, \( l \in \mathbb{N} \cup \{0\} \), \( \mathcal{R}(p^n, q^n) > 0 \), \( \forall n \in \mathbb{N} \), and \( \mathcal{R}(1, 1) = 0 \) by definition. The bidisk \( \mathbb{D}_R \) is defined by:

\[
\mathbb{D}_R = \{ a = (a_1, a_2) \in \mathbb{C}^2 : |a_j| < R_j \},
\]

where \( R \) is the convergence radius of the series [11] defined by Hadamard formula [29]:

\[
\lim_{s+t \to \infty} \sup \sqrt[s+t]{|r_{st}|} R_1^s R_2^t = 1.
\]

We also consider \( \mathcal{O}(\mathbb{D}_R) \) the set of holomorphic functions defined on \( \mathbb{D}_R \). Define the \( \mathcal{R}(p, q) \)-deformed numbers [16]:

\[
[n]_{\mathcal{R}(p, q)} = \mathcal{R}(p^n, q^n), \quad n \in \mathbb{N} \cup \{0\},
\]

the \( \mathcal{R}(p, q) \)-deformed factorials

\[
[n]!_{\mathcal{R}(p, q)} = \begin{cases} 1 & \text{for } n = 0 \\ \mathcal{R}(p, q) \cdots \mathcal{R}(p^n, q^n) & \text{for } n \geq 1, \end{cases}
\]

and the \( \mathcal{R}(p, q) \)-binomial coefficients

\[
\binom{m}{n}_{\mathcal{R}(p, q)} := \frac{[m]!_{\mathcal{R}(p, q)}}{[n]!_{\mathcal{R}(p, q)} [m-n]!_{\mathcal{R}(p, q)}}, \quad m, n \in \mathbb{N} \cup \{0\}, \quad m \geq n.
\]

Consider the following linear operators defined on \( \mathcal{O}(\mathbb{D}_R) \), (see [17] for more details),

\[
Q : \psi \mapsto Q\psi(z) : = \psi(qz),
\]

\[
P : \psi \mapsto P\psi(z) : = \psi(pz),
\]

\[
\partial_{p, q} : \psi \mapsto \partial_{p, q}\psi(z) : = \frac{\psi(pz) - \psi(qz)}{z(p - q)},
\]

and the \( \mathcal{R}(p, q) \)-derivative

\[
\partial_{\mathcal{R}(p, q)} := \partial_{p, q} \frac{p - q}{P - Q} \mathcal{R}(P, Q) = \frac{p - q}{p^P - q^Q} \mathcal{R}(p^P, q^Q) \partial_{p, q}.
\]

The algebra associated with the \( \mathcal{R}(p, q) \)-deformation is a quantum algebra, denoted \( \mathcal{A}_{\mathcal{R}(p, q)} \), generated by the set of operators \( \{1, A, A^\dagger, N\} \) satisfying the following commutation relations [17]:

\[
AA^\dagger = [N + 1]_{\mathcal{R}(p, q)}, \quad A^\dagger A = [N]_{\mathcal{R}(p, q)}.
\]
\[
[N, A] = -A, \quad [N, A^\dagger] = A^\dagger
\]
with the realization on \(O(\mathbb{D}_R)\) given by:
\[
A^\dagger := z, \quad A := \partial_{R(p,q)}, \quad N := z\partial_z,
\]
where \(\partial_z := \frac{\partial}{\partial z}\) is the derivative on \(\mathbb{C}\).

The super multibracket of order \(n\) is defined as [14]:
\[
\left[ A_1, A_2, \ldots, A_n \right] := \epsilon_{12}^{i_1 \cdots i_n} (-1)^{\sum_{k=1}^{n-1} |A_k| |\sum_{i=k+1}^{n} |A_i|} A_{i_1} A_{i_2} \cdots A_{i_n},
\]
where the symbol \(|A|\) is to be understood as the parity of \(A\) and \(\epsilon_{12}^{i_1 \cdots i_n}\) is the Lévi-Civitá symbol defined by:
\[
\epsilon_{12}^{i_1 \cdots i_p} := \det \begin{pmatrix}
\delta_{i_1}^{j_1} & \cdots & \delta_{i_p}^{j_1} \\
\vdots & \ddots & \vdots \\
\delta_{i_1}^{j_p} & \cdots & \delta_{i_p}^{j_p}
\end{pmatrix}.
\]

Moreover, the \(q-\)deformed generalized Jacobi identity is given by [34]:
\[
\epsilon_{n_1, \ldots, n_{2n-1}}^{i_1, \ldots, i_{2n-1}} \left[ \left[ l_{i_1}, \ldots, l_{i_{2n-1}} \right]_q, l_{i_{n+1}}, \ldots, l_{i_{2n-1}} \right]_q = 0.
\]

### 3 Super \(\mathcal{R}(p, q)\) – deformed Witt \(n–\)algebra

In this section, we construct the super Witt algebra and the super Witt \(n–\)algebra from the \(\mathcal{R}(p, q)\) – deformed quantum algebra.

Let \(\mathcal{B} = \mathcal{B}_0 \oplus \mathcal{B}_1\) be the super-commutative associative superalgebra such that \(\mathcal{B}_0 = \mathbb{C}[z, z^{-1}]\) and \(\mathcal{B}_1 = \theta \mathcal{B}_0\), where \(\theta\) is the Grassman variable with \(\theta^2 = 0\) [32].

We define the algebra endomorphism \(\sigma\) on \(\mathcal{B}\) as follows:
\[
\sigma(t^n) := (\phi(p, q))^{\star} t^n \quad \text{and} \quad \sigma(\theta) := \phi(p, q) \theta,
\]
where \(\phi(p, q)\) is a function depending on the parameters \(p\) and \(q\) such that \(\phi(p, q) \rightarrow 1\) as \((p, q) \rightarrow (1, 1)\).

We define also the two linear maps by:
\[
\begin{cases}
\partial_t(t^n) := [n]_{\mathcal{R}(p,q)} t^n, & \partial_t(\theta t^n) := [n]_{\mathcal{R}(p,q)} \theta t^n, \\
\partial_\theta(t^n) := 0, & \partial_\theta(\theta t^n) := (\phi(p, q))^{\star} t^n.
\end{cases}
\]

**Lemma 1** The linear map \(\Delta = \partial_t + \theta \partial_\theta\) on \(\mathcal{B}\) is an even \(\sigma\)-derivation. Then:
\[
\Delta(x y) = \Delta(x) y + \sigma(x) \Delta(y), \\
\Delta(t^n) = [n]_{\mathcal{R}(p,q)} t^n \quad \text{and} \quad \Delta(\theta t^n) = ([n]_{\mathcal{R}(p,q)} + (\phi(p, q))^{\star}) \theta t^n.
\]

**Proof 2** By direct computation. \(\square\)
Similarly, we have:

Taking $R_{x1} = (q - 1)^{-1}(x - 1)$ and $\phi(q) = q$, we obtained the result given in [32].

The super $R_{x1}$ - deformed Witt algebra is generated by bosonic and fermionic operators $l_{m}^{R(p,q)} = -t^{m} \Delta$ of parity 0 and $G_{m}^{R(p,q)} = -\theta t^{m} \Delta$ of parity 1.

**Proposition 3** The operators $l_{m}^{R(p,q)}$ and $G_{m}^{R(p,q)}$ satisfy the following relations:

\[
\begin{align*}
\left[ l_{m_{1}}^{R(p,q)}, l_{m_{2}}^{R(p,q)} \right] &= \left( [m_{1}]_{R(p,q)} - [m_{2}]_{R(p,q)} \right) l_{m_{1}+m_{2}}^{R(p,q)}, \\
\left[ l_{m_{1}}^{R(p,q)}, G_{m_{2}}^{R(p,q)} \right] &= \left( [m_{1}]_{R(p,q)} - [m_{2}+1]_{R(p,q)} \right) G_{m_{1}+m_{2}}^{R(p,q)}, \\
\left[ G_{m_{1}}^{R(p,q)}, l_{m_{2}}^{R(p,q)} \right] &= 0,
\end{align*}
\]

where

\[
\begin{align*}
\dot{x} &= \chi_{m_{1}m_{2}}(p,q), \quad \dot{y} = (\phi(p,q))^{m_{2}-m_{1}} \chi_{m_{1}m_{2}}(p,q), \\
x &= \tau_{m_{1}m_{2}}(p,q), \quad y = (\phi(p,q))^{1+m_{2}-m_{1}} \tau_{m_{1}m_{2}}(p,q), \\
\chi_{m_{1}m_{2}}(p,q) &= \frac{[m_{1}]_{R(p,q)} - [m_{2}]_{R(p,q)}}{(\phi(p,q))^{m_{2}-m_{1}} [m_{1}]_{R(p,q)} - [m_{2}]_{R(p,q)}}, \\
\tau_{m_{1}m_{2}}(p,q) &= \frac{[m_{1}]_{R(p,q)} - [m_{2}+1]_{R(p,q)}}{(\phi(p,q))^{1+m_{2}-m_{1}} [m_{1}]_{R(p,q)} - [m_{2}]_{R(p,q)} - (\phi(p,q))^{m_{2}}}.
\end{align*}
\]

**Proof 4** From the definition of the deformed commutators, we get:

\[
\begin{align*}
\left[ l_{m_{1}}^{R(p,q)}, l_{m_{2}}^{R(p,q)} \right] &= \dot{x} \ l_{m_{1}}^{R(p,q)} l_{m_{2}}^{R(p,q)} - \dot{y} \ l_{m_{2}}^{R(p,q)} l_{m_{1}}^{R(p,q)}. \\
\end{align*}
\]

Thus,

\[
\begin{align*}
\dot{x} \ l_{m_{1}}^{R(p,q)} l_{m_{2}}^{R(p,q)} &= -t^{m_{1}} \Delta \ l_{m_{2}}^{R(p,q)} \\
&= -x \ [m_{2}]_{R(p,q)} l_{m_{1}+m_{2}}^{R(p,q)} - x (\phi(p,q))^{m_{2}} l_{m_{1}+m_{2}}^{R(p,q)} \Delta.
\end{align*}
\]

Similarly, we have:

\[
\begin{align*}
\dot{y} \ l_{m_{2}}^{R(p,q)} l_{m_{1}}^{R(p,q)} &= -\dot{y} \ [m_{1}]_{R(p,q)} l_{m_{1}+m_{2}}^{R(p,q)} - \dot{y} (\phi(p,q))^{m_{1}} l_{m_{1}+m_{2}}^{R(p,q)} \Delta.
\end{align*}
\]

Then, the relation (11) takes the following form:

\[
\begin{align*}
\left[ l_{m_{1}}^{R(p,q)}, l_{m_{2}}^{R(p,q)} \right] &= \dot{y} \ [m_{1}]_{R(p,q)} - \dot{x} \ [m_{2}]_{R(p,q)} \ l_{m_{1}+m_{2}}^{R(p,q)} \\
&\quad + \dot{y} (\phi(p,q))^{m_{1}} - \dot{x} (\phi(p,q))^{m_{2}} l_{m_{1}+m_{2}}^{R(p,q)} \Delta.
\end{align*}
\]

We need to get

\[
\begin{align*}
\left[ l_{m_{1}}^{R(p,q)}, l_{m_{2}}^{R(p,q)} \right] &= \left( [m_{1}]_{R(p,q)} - [m_{2}]_{R(p,q)} \right) l_{m_{1}+m_{2}}^{R(p,q)}.
\end{align*}
\]

Thus, we obtain the system:

\[
\begin{align*}
\dot{y} \ [m_{1}]_{R(p,q)} - \dot{x} \ [m_{2}]_{R(p,q)} &= [m_{1}]_{R(p,q)} - [m_{2}]_{R(p,q)} \\
\dot{x} (\phi(p,q))^{m_{1}} - \dot{x} (\phi(p,q))^{m_{2}} &= 0.
\end{align*}
\]
Solving the above system, we obtain:

\[
\hat{x} = \frac{[m_1]_{\mathcal{R}(p,q)} - [m_2]_{\mathcal{R}(p,q)}}{(\phi(p,q))^{m_2-m_1} [m_1]_{\mathcal{R}(p,q)} - [m_2]_{\mathcal{R}(p,q)}} := \chi_{m_1m_2}(p,q).
\]

After computation, we get

\[
\hat{y} = (\phi(p,q))^{m_2-m_1} \chi_{m_1m_2}(p,q).
\]

Moreover,

\[
x_{m_1}^{\mathcal{R}(p,q)} G_{m_2}^{\mathcal{R}(p,q)} = -x([m_2]_{\mathcal{R}(p,q)} + (\phi(p,q))^{m_2}) G_{m_1+m_2}^{\mathcal{R}(p,q)} - x(\phi(p,q))^{m_2+1} G_{m_1+m_2}^{\mathcal{R}(p,q)} \Delta
\]

and

\[
y G_{m_2}^{\mathcal{R}(p,q)} x_{m_1}^{\mathcal{R}(p,q)} = -y [m_1]_{\mathcal{R}(p,q)} G_{m_1+m_2}^{\mathcal{R}(p,q)} - y (\phi(p,q))^{m_1} G_{m_1+m_2}^{\mathcal{R}(p,q)} \Delta.
\]

Thus, we get

\[
[m_{m_1}^{\mathcal{R}(p,q)},G_{m_2}^{\mathcal{R}(p,q)}]_{x,y} = \left( y [m_1]_{\mathcal{R}(p,q)} - x ([m_2]_{\mathcal{R}(p,q)} + (\phi(p,q))^{m_2}) \right) G_{m_1+m_2}^{\mathcal{R}(p,q)} + (y (\phi(p,q))^{m_1} - x(\phi(p,q))^{m_2+1}) G_{m_1+m_2}^{\mathcal{R}(p,q)} \Delta
\]

and

\[
\begin{cases}
{y [m_1]_{\mathcal{R}(p,q)} - x ([m_2]_{\mathcal{R}(p,q)} + (\phi(p,q))^{m_2}) = [m_1]_{\mathcal{R}(p,q)} - [m_2 + 1]_{\mathcal{R}(p,q)}} \\
y (\phi(p,q))^{m_1} - x(\phi(p,q))^{m_2+1} = 0.
\end{cases}
\]

Solving the above system, we obtain:

\[
x = \frac{[m_1]_{\mathcal{R}(p,q)} - [m_2 + 1]_{\mathcal{R}(p,q)}}{(\phi(p,q))^{1+m_2-m_1} [m_1]_{\mathcal{R}(p,q)} - [m_2]_{\mathcal{R}(p,q)} - (\phi(p,q))^{m_2}} := \tau_{m_1m_2}(p,q)
\]

and

\[
y = (\phi(p,q))^{1+m_2-m_1} \tau_{m_1m_2}(p,q).
\]

Let us now construct the super $\mathcal{R}(p,q)$—deformed Witt $n$—algebra. We define the $\mathcal{R}(p,q)$—deformed $n$— bracket ($n \geq 3$) as follows:

\[
[i_1^{\mathcal{R}(p,q)}, \ldots, i_n^{\mathcal{R}(p,q)}] := \left( \frac{-2 \sum_{j=1}^{n} m_j [\mathcal{R}(p,q)]}{-2 \sum_{j=1}^{n} m_j [\mathcal{R}(p,q)]} \right)^{\alpha_{i_1}i_{i_2} \cdots i_n} \times (\phi(p,q))^{\sum_{j=1}^{n} \left( \frac{1+(-1)^j}{2} \right) m_j [\mathcal{R}(p,q)]} \cdots i_{m_n^{\mathcal{R}(p,q)}}, \quad (12)
\]

where $\alpha = \frac{1+(-1)^n}{2}$, $[n] = \text{Max}\{ m \in \mathbb{Z} \mid m \leq n \}$ is the floor function.
Introducing the operator $i_m^{R(p,q)} = -i^n m \Delta$ into the relation (12), the $R(p,q)$—deformed $n$—bracket can be reduced in the simpler form as follows:

$$\left\{[m_1]_{R(p,q)}, [m_2]_{R(p,q)}, \ldots, [m_n]_{R(p,q)}\right\} = \frac{(q-p)^{\binom{n-1}{2}}}{(\phi(p,q))^{\frac{n-1}{2} \sum_{i=1}^{n} m_i} \left(\frac{-2 \sum_{i=1}^{n} m_i}{2 \sum_{i=1}^{n} m_i} \right)\left(\frac{1}{\mathcal{R}(p,q)}\right)} \times \prod_{1 \leq i < j \leq n} \left([m_i]_{R(p,q)} - [m_j]_{R(p,q)}\right)\sum_{i=1}^{n} m_i.$$ 

Now, we investigate the super $R(p,q)$—deformed Witt $n$—algebra.

From the super multibracket of order $n$ (3), we define another $R(p,q)$—deformed $n$—bracket as follows:

$$\left\{[m_1]_{R(p,q)}, [m_2]_{R(p,q)}, \ldots, [m_n]_{R(p,q)}\right\} = \frac{(-2 \sum_{i=1}^{n} m_i - 1)_{\mathcal{R}(p,q)}}{2 \left(\sum_{i=1}^{n} m_i - 1\right)_{\mathcal{R}(p,q)}} \sum_{j=0}^{n-1} (-1)^{n-1+j} \prod_{1 \leq i \leq j}^{n-1} \mathcal{R}(p,q)$$ 

$$\times (\phi(p,q))^{\beta} i_{m_j+1}^{R(p,q)} \ldots i_{m_i+1}^{R(p,q)} \frac{1}{\mathcal{R}(p,q)} i_{m_1}^{R(p,q)} \ldots i_{m_n}^{R(p,q)},$$

where $\beta = \sum_{k=1}^{n} \left(\left\lfloor \frac{n}{2} \right\rfloor - k \right) m_i + \left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right) (m_n + 1) + \sum_{k=j+1}^{n} \left(\left\lfloor \frac{n}{2} \right\rfloor - k \right) m_i.$

Using the bosonic and fermionic operators, the $R(p,q)$—deformed $n$—bracket (13) can be rewritten as:

$$\left\{[m_1]_{R(p,q)}, [m_2]_{R(p,q)}, \ldots, [m_n]_{R(p,q)}\right\} = \frac{(q-p)^{\binom{n-1}{2}}}{(\phi(p,q))^{\frac{n-1}{2} \sum_{i=1}^{n} m_i} \left(\frac{-2 \sum_{i=1}^{n} m_i - 1}{2 \sum_{i=1}^{n} m_i} \right)\left(\frac{1}{\mathcal{R}(p,q)}\right)} \times \prod_{1 \leq i < j \leq n} \left([m_i]_{\mathcal{R}(p,q)} - [m_j]_{\mathcal{R}(p,q)}\right)\sum_{i=1}^{n} m_i.$$ 

**Proposition 5** The super $R(p,q)$—deformed Witt $n$—algebras are generated by the operators $i_m^{R(p,q)}$ and $G_m^{R(p,q)}$ satisfying the following commutation relations:

$$\left\{[m_1]_{R(p,q)}, [m_2]_{R(p,q)}, \ldots, [m_n]_{R(p,q)}\right\} = \frac{(q-p)^{\binom{n-1}{2}}}{(\phi(p,q))^{\frac{n-1}{2} \sum_{i=1}^{n} m_i} \left(\frac{-2 \sum_{i=1}^{n} m_i - 1}{2 \sum_{i=1}^{n} m_i} \right)\left(\frac{1}{\mathcal{R}(p,q)}\right)} \times \prod_{1 \leq i < j \leq n} \left([m_i]_{\mathcal{R}(p,q)} - [m_j]_{\mathcal{R}(p,q)}\right)\sum_{i=1}^{n} m_i.$$ 

and

$$\left\{[m_1]_{R(p,q)}, [m_2]_{R(p,q)}, \ldots, [m_n]_{R(p,q)}\right\} = \frac{(q-p)^{\binom{n-1}{2}}}{(\phi(p,q))^{\frac{n-1}{2} \sum_{i=1}^{n} m_i} \left(\frac{-2 \sum_{i=1}^{n} m_i - 1}{2 \sum_{i=1}^{n} m_i} \right)\left(\frac{1}{\mathcal{R}(p,q)}\right)} \times \prod_{1 \leq i < j \leq n} \left([m_i]_{\mathcal{R}(p,q)} - [m_j]_{\mathcal{R}(p,q)}\right)\sum_{i=1}^{n} m_i.$$ 

and other anti-commutators are zeros.
Taking $n = 3$ in the relations (14) and (15), we obtain the super $\mathcal{R}(p, q)$–deformed Witt 3–algebra:

$$
\left[ \mathcal{R}(p, q), \mathcal{R}(p, q), \mathcal{R}(p, q) \right] = \frac{(q - p)(|m_1| \mathcal{R}(p, q) - |m_2| \mathcal{R}(p, q))}{(\phi(p, q))^{m_1 + m_2 + m_3}} \left( \frac{[m_1] \mathcal{R}(p, q) - [m_2] \mathcal{R}(p, q)}{2(\phi(p, q))^{m_1 + m_2 + m_3 + 3}} \right) \left( [m_1] \mathcal{R}(p, q) - [m_3 + 1] \mathcal{R}(p, q) \right)
$$

and other anti-commutators are zeros.

Now, we investigate the Virasoro $2n$–algebra in the framework of the $\mathcal{R}(p, q)$–deformed quantum algebras. The Virasoro algebra

$$
\mathcal{V}ir = \bigoplus_{n \in \mathbb{Z}} \mathbb{K} L_n \oplus \mathbb{K} C
$$

is the Lie algebra which satisfies the commutation relations (23):

$$
[L_m, L_n] = (m - n)L_{m+n} + \frac{1}{12}m(m - 1)(m + 1)\delta_{m+n,0} C,
$$

$$
[\mathcal{V}ir, C] = \{0\},
$$

where $\delta_{i,j}$ denotes the Kronecker delta and $C$ the central charge.

The $\mathcal{R}(p, q)$–deformed operators $L_n$ defined as

$$
L_n := -t^n \bar{D}_{\mathcal{R}(p, q)}
$$

satisfy the $\mathcal{R}(p, q)$–deformed Witt $n$–algebra given by (14). From the skewsymmetry and the $\mathcal{R}(p, q)$–deformed generalized Jacobi identity, we have:

**Lemma 6** The $\mathcal{R}(p, q)$–deformed Virasoro $2n$–algebra is generated by the following relation:

$$
\left[ \mathcal{R}(p, q), \mathcal{R}(p, q), \mathcal{R}(p, q) \right] = g_{\mathcal{R}(p, q)}(m_1, \cdots, m_{2n}) + C_{\mathcal{R}(p, q)}(m_1, \cdots, m_{2n}), \quad (16)
$$

where

$$
g_{\mathcal{R}(p, q)}(m_1, \cdots, m_{2n}) = \frac{(q - p)(2^{n-1})}{(\phi(p, q))^{(n-1)} \sum_{i=1}^{2n} m_i} \left( \frac{-2 \sum_{i=1}^{2n} m_i \mathcal{R}(p, q)}{2[- \sum_{i=1}^{2n} m_i \mathcal{R}(p, q)]} \right)
$$

$$
\times \prod_{1 \leq i < j \leq 2n} \left( [m_i] \mathcal{R}(p, q) - [m_j] \mathcal{R}(p, q) \right) L^{m_1}_{\sum_{i=1}^{2n} m_i} \quad (17)
$$

and

$$
C_{\mathcal{R}(p, q)}(m_1, \cdots, m_{2n}) = \frac{c(p, q) \epsilon_1^{1 \cdots 2n}}{6 \times 2^n \times n!} \prod_{i=1}^{n} \frac{[m_{2i-1} - 1] \mathcal{R}(p, q) [m_{2i-1}] \mathcal{R}(p, q)}{(\phi(p, q))^{m_{2i-1}} [2m_{2i-1} \mathcal{R}(p, q)]}
$$

$$
\times \frac{[m_{2i-1}] \mathcal{R}(p, q) [m_{2i-1} + 1] \mathcal{R}(p, q) \delta_{m_{2i-1}, m_{2i}}}{6 \times 2^n \times n!} \quad (18)
$$

is the $\mathcal{R}(p, q)$–deformed central extension.
Example 7 Some examples are given for \( n = 2 \) and \( n = 3 \).

(a) Taking \( n = 2 \) in the realtions (16), (17), and (18), we obtain the \( \mathcal{R}(p, q) \) - deformed Virasoro \( 4 \) - algebra:

\[
[L_{m_1}, L_{m_2}, L_{m_3}, L_{m_4}]_{\mathcal{R}(p, q)} = g_{\mathcal{R}(p, q)}(m_1, m_2, m_3, m_4) + C_{\mathcal{R}(p, q)}(m_1, \ldots, m_4),
\]

where

\[
g_{\mathcal{R}(p, q)}(m_1, m_2, m_3, m_4) = \frac{(q - p)^3}{(\phi(p, q))^{m_1 + m_2 + m_3 + m_4}} \left( \frac{-2 \sum_{i=1}^{4} m_i|_{\mathcal{R}(p, q)}}{2| - \sum_{i=1}^{4} m_i|_{\mathcal{R}(p, q)}} \right)
\times \prod_{1 \leq i < j \leq 4} \left( [m_i]_{\mathcal{R}(p, q)} - [m_j]_{\mathcal{R}(p, q)} \right) L_{\mathcal{R}(p, q)}^{m_1}
\]

and

\[
C_{\mathcal{R}(p, q)}(m_1, \ldots, m_4) = \frac{c(p, q)e^{q_{1\ldots4}}}{48} \sum_{l=1}^{2} \left( \phi(p, q) \right)^{-m_{2l-1}} \left[ \frac{m_{2l-1}|_{\mathcal{R}(p, q)}}{2m_{2l-1}|_{\mathcal{R}(p, q)}} \right]
\times \prod_{1 \leq i < j \leq 4} \left( [m_i]_{\mathcal{R}(p, q)} - [m_j]_{\mathcal{R}(p, q)} \right) L_{\mathcal{R}(p, q)}^{m_1}
\]

(b) The \( \mathcal{R}(p, q) \) - deformed Virasoro \( 6 \) - algebra is deduced from the generalization by taking \( n = 3 \):

\[
[L_{m_1}, \ldots, L_{m_6}]_{\mathcal{R}(p, q)} = g_{\mathcal{R}(p, q)}(m_1, \ldots, m_6) + C_{\mathcal{R}(p, q)}(m_1, \ldots, m_6),
\]

where

\[
g_{\mathcal{R}(p, q)}(m_1, \ldots, m_6) = \frac{(q - p)^{10}}{(\phi(p, q))^{2\sum_{i=1}^{6} m_i}} \left( \frac{-2 \sum_{i=1}^{6} m_i|_{\mathcal{R}(p, q)}}{2| - \sum_{i=1}^{6} m_i|_{\mathcal{R}(p, q)}} \right)
\times \prod_{1 \leq i < j \leq 6} \left( [m_i]_{\mathcal{R}(p, q)} - [m_j]_{\mathcal{R}(p, q)} \right) L_{\mathcal{R}(p, q)}^{m_1}
\]

and

\[
C_{\mathcal{R}(p, q)}(m_1, \ldots, m_6) = \frac{c(p, q)e^{q_{1\ldots6}}}{288} \sum_{l=1}^{3} \left( \phi(p, q) \right)^{-m_{3l-1}} \left[ \frac{m_{3l-1}|_{\mathcal{R}(p, q)}}{2m_{3l-1}|_{\mathcal{R}(p, q)}} \right]
\times \prod_{1 \leq i < j \leq 6} \left( [m_i]_{\mathcal{R}(p, q)} - [m_j]_{\mathcal{R}(p, q)} \right) L_{\mathcal{R}(p, q)}^{m_1}
\]

4 Super\( \mathcal{R}(p, q) \) - deformed Virasoro \( n \) - algebra

In this section, we determine the super \( \mathcal{R}(p, q) \) - deformed Jacobi identity. Furthermore, we discuss the super \( \mathcal{R}(p, q) \) - deformed Virasoro algebra and derive the super \( \mathcal{R}(p, q) \) - deformed Virasoro \( n \) - algebra (\( n \) even).

Lemma 8 The \( \mathcal{R}(p, q) \) - deformed superalgebra (7), (8), and (9) satisfies the super \( \mathcal{R}(p, q) \) - deformed Jacobi identity:

\[
\sum_{(i,j,l) \in \mathbb{C}(n,m,k)} (-1)^{|A_i||A_l|} [\rho(A_i), [A_j, A_l]]_{\mathcal{R}(p, q)} = 0,
\]

\[ (19) \]
where $\rho(l_m^{R(p,q)}) = \frac{[2m]_{2(p,q)}}{[m]_{2(p,q)}} l_m^{R(p,q)}$, $\rho(G_m^{R(p,q)}) = \frac{[2(m+1)]_{2(p,q)}}{[m+1]_{2(p,q)}} G_m^{R(p,q)}$ and $C(n,m,k)$ denotes the cyclic permutation of $(n,m,k)$.

**Proof** 9 Taking respectively, $A_i = l_m^{R(p,q)}$, $A_j = l_m^{R(p,q)}$, $A_k = l_k^{R(p,q)}$, and by computation, the result follows.

The super $\mathcal{R}(p,q)$—deformed Virasoro algebra is generated by bosonic and fermionic operators $l_m^{R(p,q)} = -t^m \Delta$ of parity 0 and $G_m^{R(p,q)} = -\theta t^m \Delta$ of parity 1.

**Proposition 10** The operators $l_m^{R(p,q)}$ and $G_m^{R(p,q)}$ satisfy the following commutation relations:

$$[[l_{m_1}^{R(p,q)}, l_{m_2}^{R(p,q)}]]_{x,y} = ([m_1]_{R(p,q)} - [m_2]_{R(p,q)}) l_{m_1 + m_2}^{R(p,q)} + C_{R(p,q)}(m_1) \delta_{m_1 + m_2, 0}, \quad (20)$$

and

$$[[l_{m_1}^{R(p,q)}, G_{m_2}^{R(p,q)}]]_{x,y} = ([m_1]_{R(p,q)} - [m_2 + 1]_{R(p,q)}) G_{m_1 + m_2}^{R(p,q)} + C_{R(p,q)}(m_1) \delta_{m_1 + m_2 + 1, 0}, \quad (21)$$

where $\hat{x}$, $\hat{y}$, $x$, $y$ are given by the relation (10).

$$C_{R(p,q)}(m_1) = \frac{c(p,q)\phi(p,q)^{m_1}}{6[2m_1]_{R(p,q)}} [m_1 + 1]_{R(p,q)} [m_1 - 1]_{R(p,q)}$$

is the $\mathcal{R}(p,q)$—deformed central extension and other anti-commutators are zeros.

Note that, the super $q$—deformed Virasoro algebra proposed by Ammar et al [23] can be recovered by taking $\mathcal{R}(x,1) = (q-1)^{-1}(x-1)$.

Following the same procedure used to construct the $\mathcal{R}(p,q)$—deformed Virasoro $2n$—algebra (16), we can also derive the super $\mathcal{R}(p,q)$—deformed Virasoro $2n$—algebra. It’s generated by the bosonic and fermionic operators $L_m^{R(p,q)} = -t^m \Delta$ of parity 0 and $G_m^{R(p,q)} = -\theta t^m \Delta$ of parity 1 satisfying the following relations:

$$[L_{m_1}^{R(p,q)}, \ldots, L_{m_2}^{R(p,q)}] = g_{R(p,q)}(m_1, \ldots , m_{2n}) + C_{R(p,q)}(m_1, \ldots , m_{2n}),$$

$$[L_{m_1}^{R(p,q)}, L_{m_2}^{R(p,q)}, \ldots, G_{m_2}^{R(p,q)}]_{R(p,q)} = f_{R(p,q)}(m_1, m_2, \ldots , m_{2n}) + CS_{R(p,q)}(m_1, \ldots , m_{2n}),$$

where $g_{R(p,q)}(m_1, \ldots , m_{2n})$ and $C_{R(p,q)}(m_1, \ldots , m_{2n})$ are given by the relations (17), (18).

$$f_{R(p,q)}(m_1, m_2, \ldots , m_{2n}) = \frac{(q-p)^{2n-1}}{(\phi(p,q))^{(n-1)\sum_{i=2}^{2n} m_i + 1} \left( -2 \sum_{i=1}^{2n} m_i - 1 \right)_{R(p,q)}} \times \prod_{1 \leq i < j \leq 2n-1} (|m_i|_{R(p,q)} - |m_j|_{R(p,q)}) \times \prod_{i=1}^{2n-1} (|m_i|_{R(p,q)} - |m_{2n} + 1|_{R(p,q)}) G_{\sum_{i=1}^{2n} m_i},$$

10
\[ CS_{\mathcal{R}(p,q)}(m_1, m_2, \ldots m_{2n}) = \sum_{k=1}^{2n-1} \frac{(-1)^{k+1}c(p,q)(\phi(p,q))^{-m_k} |m_k|_{\mathcal{R}(p,q)}}{6 \times 2^{n-1}(n-1)!} [2m_k]_{\mathcal{R}(p,q)} \times [m_k + 1]|_{\mathcal{R}(p,q)} |m_k|_{\mathcal{R}(p,q)} [m_k - 1]|_{\mathcal{R}(p,q)} \delta_{m_k + m_{2n} + 1, 0} \]
\[ \times \epsilon_{i_1 \cdots i_{2n-2}}^{i_1 \cdots i_{2n-2}} \prod_{p=1}^{n-1} \frac{(\phi(p,q))^{-i_{2p-1}i_{2p-1}}[i_{2p-1}]}{[2i_{2p-1}]_{\mathcal{R}(p,q)}} \]
\[ \times [i_{2p-1} + 1]|_{\mathcal{R}(p,q)} [i_{2p-1}]|_{\mathcal{R}(p,q)} [i_{2p-1} - 1]|_{\mathcal{R}(p,q)} \delta_{i_{2p-1} + i_{2p-1}, 0}, \]

with \( \{j_1, \ldots, j_{2n-2}\} = \{1, \ldots, \hat{k}, \ldots, 2n-1\} \) and other anti-commutators are zeros.

5 A toy model for the super \( \mathcal{R}(p, q) \) — Virasoro constraints

In this section, we construct another super Witt \( n \)— algebra from the \( \mathcal{R}(p, q) \) — deformed quantum algebra. We use the super \( \mathcal{R}(p, q) \) — Virasoro constraints to study a toy model.

We consider the operators defined by:
\[ T_{\mathcal{R}^{R}(p^{a}, q^{a})}^{m} := \Delta z^{m} \] (22)
\[ T_{\mathcal{R}^{R}(p^{a}, q^{a})}^{m} := -\theta \Delta z^{m}. \] (23)

The operators (22) and (23) can be rewritten as:
\[ T_{m}^{\mathcal{R}(p^{a}, q^{a})} = -[m]_{\mathcal{R}(p^{a}, q^{a})} z^{m} \]
\[ T_{m}^{\mathcal{R}(p^{a}, q^{a})} = -\theta [m]_{\mathcal{R}(p^{a}, q^{a})} z^{m}. \]

The \( \mathcal{R}(p, q) \) — deformed numbers (2) can be rewritten as [20]:
\[ [n]_{\mathcal{R}(p,q)} = \frac{\tau_{1}^{n} - \tau_{2}^{n}}{\tau_{1} - \tau_{2}}, \quad \tau_{1} \neq \tau_{2}, \]

where \( \tau_{i}, i \in \{1, 2\} \) are the functions depending on the deformation parameters \( p \) and \( q \). For illustration, we have some particular cases [20]:

(i) \( q \) — Arick-Coon-Kuryskin deformation [5][26]
\[ \tau_{1} = 1, \quad \tau_{2} = q \quad \text{and} \quad [n]_{q} = \frac{1 - q^{n}}{1 - q}; \]

(ii) \( (p, q) \) — Jagannathan-Srinivasa deformation [24]
\[ \tau_{1} = p, \quad \tau_{2} = q \quad \text{and} \quad [n]_{p,q} = \frac{p^{n} - q^{n}}{p - q}. \]
Lemma 11 The following products hold.

\[
T_m^{R(p^n,q^a)} \cdot T_n^{R(p^b,q^b)} = \left(\frac{\tau_1^{a+b} - \tau_2^{a+b}}{\tau_1 - \tau_2}\right) \cdot \frac{\tau_1^{-m} - \tau_2^{-m}}{\tau_1 - \tau_2} T_{m+n}^{R(p^{a+b},q^{a+b})}
\]

\[
+ \frac{\tau_2^{-n} - \tau_1^{-n}}{\tau_1 - \tau_2} T_{m+n}^{R(p^{a+b},q^{a+b})}
\]

and

\[
T_m^{R(p^n,q^a)} \cdot T_n^{R(p^{a+b},q^b)} = \left(\frac{\tau_1^{a+b} - \tau_2^{a+b}}{\tau_1 - \tau_2}\right) \cdot \frac{\tau_1^{-m} - \tau_2^{-m}}{\tau_1 - \tau_2} T_{m+n}^{R(p^{a+b},q^{a+b})}
\]

\[
+ \frac{\tau_2^{-n} - \tau_1^{-n}}{\tau_1 - \tau_2} T_{m+n}^{R(p^{a+b},q^{a+b})}
\]

Proposition 12 The operators (22) and (23) satisfy the following commutation relations:

\[
\left[ T_m^{R(p^n,q^a)} , T_n^{R(p^{a+b},q^b)} \right] = \left(\frac{\tau_1^{a+b} - \tau_2^{a+b}}{\tau_1 - \tau_2}\right) \cdot \frac{\tau_1^{-m} - \tau_2^{-m}}{\tau_1 - \tau_2} T_{m+n}^{R(p^{a+b},q^{a+b})}
\]

\[
+ \frac{\tau_2^{-n} - \tau_1^{-n}}{\tau_1 - \tau_2} T_{m+n}^{R(p^{a+b},q^{a+b})}
\]

\[
\left[ T_m^{R(p^n,q^a)} , T_n^{R(p^b,q^b)} \right] = \left(\frac{\tau_1^{a+b} - \tau_2^{a+b}}{\tau_1 - \tau_2}\right) \cdot \frac{\tau_1^{-m} - \tau_2^{-m}}{\tau_1 - \tau_2} T_{m+n}^{R(p^{a+b},q^{a+b})}
\]

\[
+ \frac{\tau_2^{-n} - \tau_1^{-n}}{\tau_1 - \tau_2} T_{m+n}^{R(p^{a+b},q^{a+b})}
\]

where

\[
f(m, n) = -\left(\frac{\tau_1^{a+b} - \tau_2^{a+b}}{\tau_1 - \tau_2}\right) \cdot \frac{\tau_1^{-m} - \tau_2^{-m}}{\tau_1 - \tau_2} T_{m+n}^{R(p^{a+b},q^{a+b})}
\]

and other anti-commutators are zeros.

Setting \(a = b = 1\), we obtain:

\[
\left[ T_m^{R(p,q)} , T_n^{R(p,q)} \right] = \left(\frac{\tau_1^{-n} - \tau_2^{-m}}{\tau_1 - \tau_2}\right) \cdot \frac{2R(p,q)}{2R(p,q)} T_{m+n}^{R(p^2,q^2)}
\]

\[
+ \frac{\tau_2^{m+n} - \tau_1^{m+n}}{\tau_1 - \tau_2} T_{m+n}^{R(p^2,q^2)}
\]
\[
\left[ T^R_m(p,q), T^R_n(p,q) \right] = \frac{(\tau_1^{-m} - \tau_1^{-m+1})}{\tau_1 - \tau_2} [2]^{R(p,q)} T^R_{m+n}(p^2,q^2) + f(m,n) \\
+ \frac{\tau_2^{m+n}}{\tau_1 - \tau_2} \left( (\tau_1^{-m} \tau_1^{-n}) - (\tau_1^{-m} \tau_2 - \tau_2^{-n}) \right) T^R_{m+n}(p,q),
\]

where

\[
f(m,n) = -\frac{\tau_1^{-m-1} \tau_2^{2(m+n)}}{(\tau_1 - \tau_2)^2} [2]^{R(p,q)} T^R_{m+n}(p^2,q^2) + \frac{\tau_2^{m+n}}{\tau_1 - \tau_2} \left( \frac{\tau_1^{-m} + \tau_2^{m+n-1} \tau_2}{\tau_1 - \tau_2} \right) T^R_{m+n}(p,q)
\]

and other anti-commutators are zeros.

We consider the \( n \)– bracket defined by:

\[
\left[ T^R_{m_1}(p^{a_1},q^{a_1}), \ldots, T^R_{m_n}(p^{a_n},q^{a_n}) \right] := \epsilon_{1 \cdots t n} T^R_{m_1}(p^{a_1},q^{a_1}) \ldots T^R_{m_n}(p^{a_n},q^{a_n}),
\]

where \( \epsilon_{1 \cdots t n} \) is the Lévi-Civitá symbol defined by (4). Our study is focused in the case with the same \( R(p^a, q^a) \) leads to

\[
\left[ T^R_{m_1}(p^a, q^a), \ldots, T^R_{m_n}(p^a, q^a) \right] = \epsilon_{1 \cdots t n} T^R_{m_1}(p^a, q^a) \ldots T^R_{m_n}(p^a, q^a).
\]

Putting \( a = b \) in the relation (26), we obtain:

\[
\left[ T^R_{m}(p^a, q^a), T^R_{n}(p^a, q^a) \right] = \frac{(\tau_1^{-na} - \tau_1^{-ma})}{(\tau_1^{-a} - \tau_2^{-a})} [2]^{R(p^a,q^a)} T^R_{m+n}(p^{2a},q^{2a}) \\
+ \frac{\tau_2^{(m+n)a}}{\tau_1^{-a} - \tau_2^{-a}} \left( (\tau_1^{-na} - \tau_1^{-ma}) + (\tau_2^{-na} - \tau_2^{-ma}) \right) T^R_{m+n}(p^a,q^a).
\]

The \( n \)– bracket takes the following form:

\[
\left[ T^R_{m_1}(p^a, q^a), \ldots, T^R_{m_n}(p^a, q^a) \right] = \frac{(-1)^{n+1}}{(\tau_1^{-a} - \tau_2^{-a})} \left( M^n_m [n] R(p^a,q^a) T^R_{m+n}(p^{a(n)},q^{a(n)}) \right) \\
- \frac{n+1}{\tau_2^{-a} \left( \sum_{k=1}^{m} \right)} \left( M^n_a + C^n_a \right) T^R_{m_1 + \ldots + m_n}(p^{(n-1)a},q^{(n-1)a}),
\]

where

\[
M^n_a = \tau_1^{-a(n-1)} \sum_{k=1}^{m \leq n} \left( \frac{\tau_1^{-a} - \tau_2^{-a}}{2} \right)^2 \prod_{1 \leq j < k \leq n} \left( [m_k] R(p^a,q^a) - [m_j] R(p^a,q^a) \right) \\
+ \prod_{1 \leq j < k \leq n} \left( \frac{\tau_1^{-a} m_k - \tau_2^{-a} m_j}{\tau_2^{-a} \tau_1^{-a}} \right)
\]

and

\[
C^n_a = \tau_2^{-a(n-1)} \sum_{k=1}^{m \leq n} \left( \frac{\tau_1^{-a} - \tau_2^{-a}}{2} \right)^2 \prod_{1 \leq j < k \leq n} \left( [m_k] R(p^a,q^a) - [m_j] R(p^a,q^a) \right) \\
+ (-1)^{n+1} \prod_{1 \leq j < k \leq n} \left( \frac{\tau_1^{-a} m_k - \tau_1^{-a} m_j}{\tau_1^{-a}} \right),
\]

13
From the super multibracket of order $n$, we define the $\mathcal{R}(p,q)$ - deformed super $n$ - bracket as follows:

$$\left[ T_{m_1}^{\mathcal{R}(p,q^n)}, T_{m_2}^{\mathcal{R}(p,q^n)}, \ldots, T_{m_n}^{\mathcal{R}(p,q^n)} \right] := \sum_{j=0}^{n-1} \left(-1\right)^{n-1+j} \epsilon_{12 \cdots n-1} T_{m_1}^{\mathcal{R}(p,q^n)} \cdots T_{m_j}^{\mathcal{R}(p,q^n)} \times T_{m_{j+1}}^{\mathcal{R}(p,q^n)} \cdots T_{m_n}^{\mathcal{R}(p,q^n)}.$$ 

From the relation (27) with $a = b$, we obtain:

$$\left[ T_{m_1}^{\mathcal{R}(p,q^n)}, T_{m_2}^{\mathcal{R}(p,q^n)} \right] = \left( \tau_1^{a-m} - \tau_1^{-a-m} \right) \left[ 2 \right]_{\mathcal{R}(p,q^n)} T_{m+n}^{\mathcal{R}(p,q^n)} + f(m, n) + \tau_2^{(m+n)a} \left( \tau_2^{-a+m} - \tau_1^{-a} \right) T_{m+n}^{\mathcal{R}(p,q^n)},$$

where

$$f(m, n) = -\frac{\tau_1^{(m+1)a} - \tau_2^{(m+1)a}}{\tau_1^a - \tau_2^a} \left( \tau_2^{am} \left[ 2 \right]_{\mathcal{R}(p,q^n)} T_{m+n}^{\mathcal{R}(p,q^n)} - \frac{\left[ 2(m+1) \right]_{\mathcal{R}(p,q^n)}}{\tau_1} T_{m+n}^{\mathcal{R}(p,q^n)} \right).$$

Thus, the super $n$ - bracket can be rewritten as follows:

$$\left[ T_{m_1}^{\mathcal{R}(p,q^n)}, \ldots, T_{m_n}^{\mathcal{R}(p,q^n)} \right] = \left( \tau_1^{a-n} \sum_{s=1}^{n} \left( \frac{\alpha_1}{\tau_1^a - \tau_2^a} \right)^2 \prod_{1 \leq j < k \leq n} \left( m_{j} - 1 \right)_{\mathcal{R}(p,q^n)} - m_j \right)_{\mathcal{R}(p,q^n)} + \prod_{1 \leq j < k \leq n} \left( \tau_2^{a(m-1)} - \tau_2^{a(m)} \right),$$

where

$$A_a^m = \tau_1^{a(n-1)} \sum_{s=1}^{n} \left( \frac{\alpha_1}{\tau_1^a - \tau_2^a} \right)^2 \prod_{1 \leq j < k \leq n} \left( m_{j} - 1 \right)_{\mathcal{R}(p,q^n)} - m_j \right)_{\mathcal{R}(p,q^n)},$$

$$F_a^m = \tau_1^{a(n-1)} \sum_{s=1}^{n} \prod_{1 \leq j < k \leq n} \left( m_{j} - 1 \right)_{\mathcal{R}(p,q^n)} - m_j \right)_{\mathcal{R}(p,q^n)} \tau_2^{a(m)} \left( \tau_2^{a(m-1)} - \tau_2^{a(m)} \right),$$

$$S_a^m = \tau_1^{a(n-1)} \sum_{s=1}^{n} \prod_{1 \leq j < k \leq n} \left( m_{j} - 1 \right)_{\mathcal{R}(p,q^n)} - m_j \right)_{\mathcal{R}(p,q^n)} \tau_2^{a(m-1)} \left( \tau_2^{a(m)} - \tau_2^{a(m-1)} \right) + \left( -1 \right)^{n-1} \prod_{1 \leq j < k \leq n} \left( \tau_2^{a(m-1)} - \tau_2^{a(m)} \right).$$

14
Let us consider the generating function with infinitely many parameters presented by \([28]\):

\[
Z_{toy}(t) = \int x^\gamma \exp \left( \sum_{s=0}^{\infty} t_s s! x^s \right) dx.
\]

We assume that the following relation holds for the linear maps \(\Delta\) given by the relation \((6)\)

\[
\int \Delta f(x) dx = 0.
\]

Taking \(f(x) = x^{m+\gamma} \exp \left( \sum_{s=0}^{\infty} t_s s! x^s \right)\), we have

\[
\int_{-\infty}^{+\infty} \Delta \left( x^{m+\gamma} \exp \left( \sum_{s=0}^{\infty} t_s s! x^s \right) \right) dx = 0.
\]

We consider the following expression

\[
\exp \left( \sum_{s=0}^{\infty} t_s s! x^s \right) = \sum_{n=0}^{\infty} B_n(t_1, \ldots, t_n) \frac{x^n}{n!},
\]

where \(B_n\) is the Bell polynomials. Then

\[
\Delta \left( x^{m+\gamma} \exp \left( \sum_{s=0}^{\infty} t_s s! x^s \right) \right) = x^{m+\gamma} \exp \left( \sum_{s=0}^{\infty} t_s s! x^s \right) + \frac{(\phi(p, q))^{m+\gamma}}{(\tau_1^m - \tau_2^m)} \sum_{k=1}^{\infty} \frac{B_k(t^a_1, \ldots, t^a_k)}{k!} \exp \left( \sum_{s=0}^{\infty} \frac{t_s s!}{s!} x^{s+\gamma} \right),
\]

where \(t^a_k = (\tau^a_1 - \tau^a_2) t_k\). Then, from the relation

\[
T_{m}^R(\mu^n, \nu^n) Z_{toy}(t) = 0, \quad m \geq 0,
\]

the operator \([22]\) takes the following form:

\[
T_{m}^R(\mu^n, \nu^n) = [m + \gamma]_{R(\mu^n, \nu^n)} m! \frac{\partial}{\partial t_m} + \frac{(\phi(p, q))^{m+\gamma}}{\tau_1^m - \tau_2^m} \sum_{k=1}^{\infty} \frac{(k + m)!}{k!} B_k(t^a_1, \ldots, t^a_k) \frac{\partial}{\partial t_{k+m}}.
\]

Similarly, we obtain

\[
\pi_{m}^R(\mu^n, \nu^n) Z_{toy}(t) = 0, \quad m \geq 0,
\]
The results obtained here can be deduced from the general formalism by setting $\mathcal{R}(x, 1) = (q - 1)^{-1}(x - 1)$. Then, the $q$-deformed operators given by:

\[
\mathcal{T}_m^q = \Delta z^m \tag{28}
\]

\[
\mathcal{T}_n^q = -\theta \Delta z^n \tag{29}
\]

satisfy the products

\[
\mathcal{T}_m^q \cdot \mathcal{T}_n^q = -\frac{(q^{a+b} - 1)}{(q^a - 1)(q^b - 1)} \mathcal{T}_{m+n}^{a+b} + \frac{1}{q^b - 1} \mathcal{T}_m^q + \frac{q^{-m}b}{q^a - 1} \mathcal{T}_n^q \tag{30}
\]

and

\[
\mathcal{T}_m^q \cdot \mathcal{T}_n^b = -\frac{(q^{a+b} - 1)}{(q^a - 1)(q^b - 1)} \mathcal{T}_{m+n+1}^{a+b} + \frac{q^{-n}b}{q^a - 1} \mathcal{T}_m^q \tag{31}
\]

Moreover, the following commutation relations hold:

\[
\left[ \mathcal{T}_m^q, \mathcal{T}_n^b \right] = \frac{(q^{a+b} - 1)}{(q^a - 1)(q^b - 1)} \mathcal{T}_{m+n}^{a+b} - \frac{(q^{-n}a - q^{-m}b)}{q^a - 1} \mathcal{T}_m^q + \frac{(q^{-m}b - 1)}{q^a - 1} \mathcal{T}_n^q \tag{32}
\]

\[
\left[ \mathcal{T}_m^q, \mathcal{T}_n^q \right] = \frac{(q^{a+b} - 1)}{(q^a - 1)(q^b - 1)} \mathcal{T}_{m+n}^{a+b} - \frac{(q^{-n}a - q^{-m}b)}{q^a - 1} \mathcal{T}_m^q + \frac{(q^{-m}b - 1)}{q^a - 1} \mathcal{T}_n^q + f(m, n) \tag{33}
\]

where

\[
f(m, n) = -\frac{(q^{a+b} - 1)}{(q^a - 1)(q^b - 1)} \mathcal{T}_{m+n}^{a+b} + \frac{q^{-m}b - b}{q^a - 1} \mathcal{T}_m^q + \frac{q^{-m}b - b}{q^a - 1} \mathcal{T}_n^q + f(m, n)
\]

and other anti-commutators are zeros. Setting $a = b = 1$, we obtain:

\[
\left[ \mathcal{T}_m^q, \mathcal{T}_n^q \right] = \frac{(q^{-n} - q^{-m})}{(q - 1)} \left[ 2q \mathcal{T}_{m+n}^2 - \frac{1}{q - 1} \left( (q^{-n} - 1) - (q^{-m} - 1) \right) \mathcal{T}_{m+n}^q \right]
\]
and other anti-commutators are zeros. We study the case with the same $q^n$. Then, putting $a = b$ in the relation \text{(32)}, we obtain:

$$
\left[ T_m, T_n \right] = \frac{(q^{-a} - q^{-m+1})}{q - 1} [2]_q T_m^2 q^n + \frac{1}{q - 1} \left( (q - q^{-m}) - (q^{-m} - 1) \right) T_m^q + f(m, n),
$$

where

$$
f(m, n) = -\frac{q^{-m-1}}{q - 1} [2]_q T_1^q + \frac{1 + q^{-m-1}}{q - 1} T_1^q
$$

From the super multibracket of order $n$ \text{(3)}, we define the $q$–deformed $n$–bracket as follows:

$$
\left[ T_{m_1}, T_{m_2}, \ldots, T_{m_n} \right] := \sum_{j=0}^{n-1} (-1)^{n-j+1} \epsilon_{j_1 \ldots j_n} T_{m_1}^{m_{j_1}} \ldots T_{m_j}^{m_{j+1}} \ldots T_{m_n}^{m_{n-1}}.
$$

From the relation \text{(33)} with $a = b$, we obtain:

$$
\left[ T_m, T_n \right] = \frac{(q^{-a} - q^{-m+1})}{q - 1} [2]_q T_m^2 q^n + \frac{1}{q - 1} \left( (q - q^{-m}) - (q^{-m} - 1) \right) T_m^q + f(m, n),
$$

where

$$
f(m, n) = -\frac{q^{-m-a-1}}{q - 1} [2]_q T_1^q + \frac{1 + q^{-m-a}}{q - 1} T_1^q.
$$
Thus, the super \( n \)– bracket takes the form:

\[
\left[ T^{n}_{m_1, \ldots, m_n} \right] = \frac{(-1)^{n+1}}{(q^n - 1)^{n-1}} \left( \prod_{1 \leq j < k \leq n} \left[ m_k - 1 \right] q^n \right) + f(m_1, \ldots, m_n),
\]

where

\[
A^n_a = q^{-a(n-1) \sum_{s=1}^{n}(m_s - 1)} \left( q^n - 1 \right)^{\binom{n}{2}} \prod_{1 \leq j < k \leq n} \left[ m_k - 1 \right] q^n,
\]

\[
F^n_a = q^{-a(n-1) \sum_{s=1}^{n} m_s} \left( q^n - 1 \right)^{\binom{n}{2}} \prod_{1 \leq j < k \leq n} \left[ m_k - 1 \right] q^n,
\]

\[
S^n_a = \left( q^n - 1 \right)^{\binom{n}{2}} \prod_{1 \leq j < k \leq n} \left[ m_k - 1 \right] q^n + (-1)^{n-1} \prod_{1 \leq j < k \leq n} \left( q^{m_k} - q^{m_j} q^{\binom{n}{2}} \right)
\]

and

\[
f(m_1, \ldots, m_n) = \frac{(-1)^{n+1} q^{-(m+1)a}}{(q^n - 1)^{n-1}} \left( \prod_{1 \leq j < k \leq n} \left[ m_k - 1 \right] q^n \right) - \frac{2(m+1)}{[m+1] q^n} \frac{\prod_{1 \leq j < k \leq n} \left[ m_k - 1 \right] q^n}{\left( q^n - 1 \right)^{n-1}}.
\]

The operators (28) and (29) take the following forms:

\[
T^{\alpha}_{m} = \left[ m + \gamma \right] q^m m! \frac{\partial}{\partial m} + \frac{q^{m+\gamma}}{q^a - q^{-a}} \sum_{k=1}^{\infty} \frac{(k+m)!}{k!} B_k(t_1^\alpha, \ldots, t_k^\alpha) \frac{\partial}{\partial \delta_{k+m}}
\]

\[
T^{\alpha}_{m} = \theta \left[ m + \gamma \right] q^m m! \frac{\partial}{\partial m} + \frac{q^{m+\gamma}}{q^a - q^{-a}} \sum_{k=1}^{\infty} \frac{(k+m)!}{k!} B_k(t_1^\alpha, \ldots, t_k^\alpha) \frac{\partial}{\partial \delta_{k+m}}
\]

Putting \( \tilde{m} = m + \gamma \), \( \tilde{n} = n + \gamma \), and by changing \( n! \frac{\partial}{\partial m} \leftrightarrow x^n \), we show directly that the products \( T^{\alpha}_{m} \cdot T^{\beta}_{n} \) and \( T^{\alpha}_{m} \cdot T^{\beta}_{n} \) are respectively equivalent to (30) and (31).

6 Relevant particular cases

Particular cases of super Virasoro \( n \)– algebra and application associated to different quantum algebras in the literature are deduced as follows:

6.1 Jagannathan- Srinivasa deformation [24]

Taking \( \mathcal{R}(x, y) = \frac{x-y}{p-q} \), we obtain: the algebra endomorphism \( \sigma \) on \( B \) is defined by:

\[
\sigma(t^n) := (p q)^n t^n \quad \text{and} \quad \sigma(\theta) := (p q) \theta.
\]

We define also the two \( (p, q) \)– deformed linear maps by:

\[
\left\{ \begin{array}{l}
\partial_1(t^n) = [n]_{p,q} t^n, \quad \partial(t^n) = [n]_{p,q} \theta t^n, \\
\partial_2(t^n) = 0, \quad \partial_2(\theta t^n) = (p q)^n t^n.
\end{array} \right.
\]
The linear map $\Delta = \partial_t + \theta \partial_\theta$ on $B$ is an even $\sigma$-derivation, and satisfy the following relations:

$$\Delta(x y) = \Delta(x) y + \sigma(x) \Delta(y),$$
$$\Delta(t^n) = [n]_{p,q} t^n \quad \text{and} \quad \Delta(\theta t^n) = ([n]_{p,q} + (pq)^n) \theta t^n.$$

It is generated by bosonic and fermionic operators $\mathcal{P}_{m_1}^{p,q} = -t^m \Delta$ of parity 0 and $\mathcal{G}_{m_1}^{p,q} = -\theta t^m \Delta$ of parity 1 verifying the following commutations relations:

$$\left[ \mathcal{P}_{m_1}^{p,q}, \mathcal{P}_{m_2}^{p,q} \right]_{\hat{x}, \hat{y}} = \left( [m_1]_{p,q} - [m_2]_{p,q} \right) \mathcal{P}_{m_1+m_2}^{p,q};$$
$$\left[ \mathcal{P}_{m_1}^{p,q}, \mathcal{G}_{m_2}^{p,q} \right]_{x,y} = \left( [m_1]_{p,q} - [m_2+1]_{p,q} \right) \mathcal{G}_{m_1+m_2}^{p,q};$$
$$\left[ \mathcal{G}_{m_1}^{p,q}, \mathcal{G}_{m_2}^{p,q} \right] = 0,$$

where

$$\begin{align*}
\hat{x} &= \chi_{m_1 m_2} (p,q), \quad \hat{y} = (pq)^{m_2-m_1} \chi_{m_1 m_2} (p,q), \\
x &= \tau_{m_1 m_2}, \quad y = (pq)^{m_2-m_1} \tau_{m_1 m_2}, \\
\chi_{m_1 m_2} (p,q) &= \frac{[m_1]_{p,q} - [m_2]_{p,q}}{(pq)^{m_2-m_1} [m_1]_{p,q} - [m_2]_{p,q}}, \\
\tau_{m_1 m_2} (p,q) &= \frac{[m_1]_{p,q} - [m_2+1]_{p,q}}{(pq)^{m_2-m_1} [m_1]_{p,q} - [m_2]_{p,q} - (pq)^{m_2}}.
\end{align*}$$

The $(p,q)$–deformed $n$–bracket $(n \geq 3)$ are defined as follows:

$$\left[ t_{m_1}^{p,q}, \ldots, t_{m_n}^{p,q} \right] := \left( \frac{p - \sum_{i=1}^{n-1} m_i + q - \sum_{i=1}^{n-1} m_i}{2} \right)^{\alpha_1 \cdots \alpha_n} \epsilon_{12 \cdots n} \times (pq)^{\sum_{j=1}^{n-1} \left( \left\lfloor \frac{m_j}{2} \right\rfloor - j \right) m_j} t_{m_1}^{p,q} \cdots t_{m_n}^{p,q},$$

and

$$\left[ t_{m_1}^{p,q}, \ldots, t_{m_n}^{p,q} \right] := \left( \frac{p - \sum_{i=1}^{n-1} m_i + q - \sum_{i=1}^{n-1} m_i}{2} \right)^{\alpha_{n-1} \cdots \alpha_1} \epsilon_{12 \cdots n-1} \times (pq)^{\beta_{p,q} m_{m_1}^{p,q} \cdots p_{m_j}^{p,q} \mathcal{G}_{m_{m_j+1}}^{p,q} \cdots p_{m_{m_j+1}}^{p,q} \cdots p_{m_{n-1}}^{p,q}},$$

where $\beta = \sum_{k=1}^{n} \left( \left\lfloor \frac{m_k}{2} \right\rfloor - k+1 \right) m_k + \left( \left\lfloor \frac{m_n}{2} \right\rfloor - 1 \right) m_n + 1 + \sum_{k=j+1}^{n-1} \left( \left\lfloor \frac{m_k}{2} \right\rfloor - k \right) m_k,$

$\alpha = \frac{1+(n-1)}{2},$ and $[n] = Max\{m \in \mathbb{Z} \mid m \leq n\}$ is the floor function. Then, the generators $\mathcal{P}_{m_1}^{p,q}$ and $\mathcal{G}_{m_1}^{p,q}$ satisfy the commutation relations:

$$\left[ t_{m_1}^{p,q}, \ldots, t_{m_n}^{p,q} \right] = \left( \frac{q-p}{(pq)^{\frac{n-1}{2}}} \right)^{\alpha_{n-1} \cdots \alpha_1} \times (pq)^{\sum_{j=1}^{n-1} \left( \left\lfloor \frac{m_j}{2} \right\rfloor - j \right) m_j} \epsilon_{12 \cdots n-1} \times \prod_{1 \leq i < j \leq n} \left( [ been chosen. Please let me know if you need any further assistance.
\[
[p^{p,q}, \cdots, G_{m_n}^{p,q}] = \frac{(q-p)^{\binom{n-1}{2}}}{(pq)^{\sum_{i=1}^{n} m_i + 1}} \left( \frac{p^{\sum_{i=1}^{n-1} m_i + 1} + \sum_{i=1}^{n-1} m_i}{2} \right)
\times \prod_{1 \leq i < j \leq n-1} \left( [m_i]_{p,q} - [m_j]_{p,q} \right) \prod_{i=1}^{n-1} \left( [m_i]_{p,q} - [m_n + 1]_{p,q} \right) G_{\sum_{i=1}^{n} m_i}^{p,q}
\]
and other anti-commutators are zeros. Furthermore, the corresponding Virasoro \(2n\)– algebra is deduced as:

\[
[L_{m_1}, \cdots, L_{m_n}] = g_{p,q}(m_1, \cdots, m_{2n}) + C_{p,q}(m_1, \cdots, m_{2n}),
\]

where

\[
g_{p,q}(m_1, \cdots, m_{2n}) = \frac{(q-p)^{\binom{2n-1}{2}}}{2(pq)^{\sum_{i=1}^{2n} m_i}} \left( \frac{p^{\sum_{i=1}^{2n} m_i + 1} + \sum_{i=1}^{2n} m_i}{2} \right)
\times \prod_{1 \leq i < j \leq 2n} \left( [m_i]_{p,q} - [m_j]_{p,q} \right) L_{\sum_{i=1}^{2n} m_i}^{2n}
\]

and

\[
C_{p,q}(m_1, \cdots, m_{2n}) = \frac{c(p,q)\varepsilon_{i_1 \cdots i_{2n}}} {6 \times 2^n \times n!} \prod_{l=1}^{n} \left( \frac{m_{i_l - 1} - 1}{p} \right)_{p,q} \cdot m_{i_l - 1} \delta_{m_{i_l - 1} + m_{i_l}, 0}.
\]

Several examples are deduced as follows:

(a) Taking \(n = 2\) in the relations (36) and (37), we obtain the \((p,q)\)– deformed Virasoro \(4\)– algebra:

\[
[L_{m_1}, L_{m_2}, L_{m_3}, L_{m_4}] = g_{p,q}(m_1, m_2, m_3, m_4) + C_{p,q}(m_1, \cdots, m_4),
\]

where

\[
g_{p,q}(m_1, m_2, m_3, m_4) = \frac{(q-p)^3}{(pq)^{m_1 + m_2 + m_3 + m_4}} \left( \frac{p^{\sum_{i=1}^{4} m_i} + \sum_{i=1}^{4} m_i}{2} \right)
\times \prod_{1 \leq i < j \leq 4} \left( [m_i]_{p,q} - [m_j]_{p,q} \right) L_{\sum_{i=1}^{4} m_i}^{4}
\]

and

\[
C_{p,q}(m_1, \cdots, m_4) = \frac{c(p,q)\varepsilon_{i_1 \cdots i_4}} {48} \prod_{l=1}^{2} \left( \frac{m_{i_l - 1} - 1}{p} \right)_{p,q} \cdot m_{i_l - 1} \delta_{m_{i_l - 1} + m_{i_l}, 0}.
\]

(b) The \((p,q)\)– deformed Virasoro \(6\)– algebra is deduced from the generalization by taking \(n = 3\):

\[
[L_{m_1}, \cdots, L_{m_6}] = g_{p,q}(m_1, \cdots, m_6) + C_{p,q}(m_1, \cdots, m_6),
\]

20
where
\[ g_{\rho,q}(m_1,\ldots,m_6) = \frac{(q-p)^{10}}{(pq)^{\sum_{i=1}^6 m_i}} \left( p - \sum_{i=1}^6 m_i + q - \sum_{i=1}^6 m_i \right) \]
\[ \times \prod_{1 \leq i < j \leq 6} \left( [m_i]_{\rho,q} - [m_j]_{\rho,q} \right) L_{\sum_{i=1}^6 m_i}^q \]
and
\[ C_{\rho,q}(m_1,\ldots,m_6) = \frac{c(p,q)\epsilon_1\cdots\epsilon_4}{288} \prod_{l=1}^3 \frac{\left( m_{2l-1} \right)_{\rho,q}}{2\left( m_{2l-1} \right)_{\rho,q}} \]
\[ \times \left[ m_{2l-1} - 1 \right]_{\rho,q} \left[ m_{2l-1} \right]_{\rho,q} \delta_{m_{2l-1} + 1,0}. \]

The \((p,q)\) deformed super Jacobi identity is given by:
\[ \sum_{(i,j,l) \in \mathcal{C}(n,m,k)} (-1)^{|A_i||A_l|} \left[ \rho(A_i), [A_j, A_l] \right]_{\rho,q} = 0, \]
where \(\rho(L_{m_1}) = (p^{m_1} + q^{m_1}) L_{m_1}, \rho(G_{m_1,q}) = (p^{m_1+1} + q^{m_1+1}) G_{m_1,q}\) and \(\mathcal{C}(n,m,k)\) denotes the cyclic permutation of \((n,m,k)\).

Moreover, the operators \(L_{m_1,q}^p\) and \(G_{m,q}^p\) satisfy the following commutation relations:
\[ [L_{m_1,q}^p, L_{m_2,q}^p]_{\hat{x},\hat{y}} = \left( [m_1]_{\rho,q} - [m_2]_{\rho,q} \right) L_{m_1+\hat{m}_2}^q + \frac{c(p,q)(pq)^{m_1} [m_1]_{\rho,q}}{6[2m_1]_{\rho,q}} \]
\[ \times \left[ m_1 + 1 \right]_{\rho,q} \left[ m_1 \right]_{\rho,q} \delta_{m_1+\hat{m}_2,0}, \]
and
\[ [L_{m_1,q}^p, G_{m_2,q}^p]_{\hat{x},\hat{y}} = \left( [m_1]_{\rho,q} - [m_2]_{\rho,q} \right) G_{m_1+\hat{m}_2}^q + \frac{c(p,q)(pq)^{m_1} [m_1]_{\rho,q}}{6[2m_1]_{\rho,q}} \]
\[ \times \left[ m_1 + 1 \right]_{\rho,q} \left[ m_1 \right]_{\rho,q} \delta_{m_1+\hat{m}_2,0}, \]
where \(\hat{x}, \hat{y}, x,\) and \(y\) are given by the relation (35). The super Virasoro \(2n-\) algebra is presented as follows:
\[ \left[ L_{m_1,q}^p, \cdots, L_{m_{2n},q}^p \right] = g_{p,q}(m_1,\cdots,m_{2n}) + C_{p,q}(m_1,\cdots,m_{2n}), \]
\[ \left[ F_{m_1,q}^p, \cdots, F_{m_{2n},q}^p \right] = f_{p,q}(m_1,\cdots,m_{2n}) + CS_{p,q}(m_1,\cdots,m_{2n}), \]
where \(g_{p,q}(m_1,\cdots,m_{2n})\) and \(C_{p,q}(m_1,\cdots,m_{2n})\) are given by the relations (36), (37).

\[ f_{p,q}(m_1,\cdots,m_{2n}) = \frac{(q-p)^{2n-1}}{2(pq)^{(n-1)\sum_{i=1}^{2n} m_i + 1}} \left( p^{\sum_{i=1}^{2n} m_i - 1} + q^{\sum_{i=1}^{2n} m_i - 1} \right) \]
\[ \times \prod_{1 \leq i < j \leq 2n-1} \left( [m_i]_{\rho,q} - [m_j]_{\rho,q} \right) \prod_{i=1}^{2n-1} \left( [m_i]_{\rho,q} - [m_{2n} + 1]_{\rho,q} \right) G_{\sum_{i=1}^{2n} m_i}. \]
The following products hold.

\[
\begin{align*}
C_{p,q}(m_1, m_2, \ldots, m_{2n}) &= \sum_{k=1}^{2n-1} \frac{(-1)^{k+1} c[p,q](pq)^{-m_k}}{6 \times 2^{n-1}(n-1)!} \frac{1}{p^{m_k} + q^{m_k}} \\
&\times [m_k + 1]_{p,q} [m_k]_{p,q} [m_k - 1]_{p,q} \delta_{m_k + m_{2n+1}, 0} \\
&\times \sum_{s=1}^{j_1 - j_2 - 2} \prod_{i=1}^{n-1} \frac{(pq)^{-i_{2s+1}}}{p^{2i_{2s+1} - 1} + q^{2i_{2s+1}}}
\end{align*}
\]

with \( \{j_1, \ldots, j_{2n-2}\} = \{1, \ldots, \hat{k}, \ldots, 2n-1\} \) and other anti-commutators are zeros.

Now, we construct another \((p, q)\)-deformed super Witt \(n\)-algebra. We consider the operators defined by:

\[
\begin{align*}
\mathcal{T}_{p,q}^{a^m} &= \Delta z^m, \\
\mathcal{T}_{\bar{p},q}^{a^m} &= -\theta \Delta z^m.
\end{align*}
\]

The operators (38) and (39) can be rewritten as:

\[
\begin{align*}
\mathcal{T}_{p,q}^{a^m} &= -[m]_{p,q} a^m z^m, \\
\mathcal{T}_{\bar{p},q}^{a^m} &= -\theta [m]_{p,q} a^m z^m.
\end{align*}
\]

The following products hold.

\[
\begin{align*}
\mathcal{T}_{p,q}^{a^m} \cdot \mathcal{T}_{\bar{p},q}^{a^b} &= \frac{(p^{a+b} - q^{a+b}) p^{-mb}}{(p^a - q^a) (p^b - q^b)} \mathcal{T}_{m+n}^{a^b, a^{m+b}} \\
&+ \frac{q^{-nb}}{p^b - q^b} \mathcal{T}_{m+n}^{a^m, a^b} + \frac{q^{(m+n)a} p^{-mb}}{p^a - q^a} \mathcal{T}_{m+n}^{a^b, a^m} (40)
\end{align*}
\]

and

\[
\begin{align*}
\mathcal{T}_{p,q}^{a^m} \cdot \mathcal{T}_{\bar{p},q}^{a^b} &= \frac{(p^{a+b} - q^{a+b}) p^{-(m+1)b}}{(p^a - q^a) (p^b - q^b)} \mathcal{T}_{m+n+1}^{a^b, a^{m+b}} \\
&+ \frac{q^{-nb}}{p^b - q^b} \mathcal{T}_{m+n+1}^{a^m, a^b} + \frac{q^{(m+1)a} p^{-(m+1)b}}{p^a - q^a} \mathcal{T}_{m+n+1}^{a^b, a^m} (41)
\end{align*}
\]

and the operators satisfy the following commutation relations

\[
\begin{align*}
\left[ \mathcal{T}_{p,q}^{a^m}, \mathcal{T}_{\bar{p},q}^{a^b} \right] &= \frac{(p^{a+b} - q^{a+b}) (p^{a} - q^{a}) (p^{b} - q^{b})}{p^a - q^a} \mathcal{T}_{m+n}^{a^{m+b}, a^{m+b}} \\
&- \frac{q^{(m+n)b} (p^{a} - q^{a}) (p^{b} - q^{b})}{p^a - q^a} \mathcal{T}_{m+n}^{a^m, a^b} + \frac{q^{(m+n)a} (p^{a} - q^{a}) (p^{b} - q^{b})}{p^a - q^a} \mathcal{T}_{m+n}^{a^b, a^m} (42)
\end{align*}
\]

\[
\begin{align*}
\left[ \mathcal{T}_{p,q}^{a^m}, \mathcal{T}_{\bar{p},q}^{a^b} \right] &= \frac{(p^{a+b} - q^{a+b}) (p^{a} - q^{a}) (p^{b} - q^{b})}{p^a - q^a} \mathcal{T}_{m+n}^{a^{m+b}, a^{m+b}} \\
&+ \frac{q^{(m+n)b} (p^{a} - q^{a}) (p^{b} - q^{b})}{p^a - q^a} \mathcal{T}_{m+n}^{a^m, a^b} + \frac{q^{(m+n)a} (p^{a} - q^{a}) (p^{b} - q^{b})}{p^a - q^a} \mathcal{T}_{m+n}^{a^b, a^m} + f(m,n)
\end{align*}
\]
where

\[ f(m, n) = - \frac{(p^{a+b} - q^{a+b}) p^{-m} b - b q^{(a+b)(m+n)}}{(p^a - q^a)(p^b - q^b)} T_1^{a+b, q^{a+b}} + \frac{q^{(m+n)a} q^n b}{p^b - q^b} T_1^{p^a, q^n} + \frac{q^{(m+n)(a+b)} p^{-m} b - b q^a}{p^a - q^a} T_1^{b, q^b} \]

and other anti-commutators are zeros.

Setting \( a = b = 1 \), we obtain:

\[
\begin{align*}
\left[ T_{m_1}^{p, q}, T_{m_2}^{p, q} \right] &= \frac{(p^{-n} - p^{-m})}{(p - q)} [2_{p, q} T_{m_1+n}^{p^2, q^2} - \frac{q^{m+n}}{p-q} \left( (p^{-n} - q^{-m}) - (p^{-m} - q^{-n}) \right) T_{m+n}^{p, q}, \\
\left[ T_{m_1}^{p, q}, T_{m_2}^{p, q} \right] &= \frac{(p^{-n} - p^{-m+1})}{p-q} [2_{p, q} T_{m_1+n}^{p^2, q^2} + f(m, n) \\
&+ \frac{q^{m+n}}{p-q} \left( (q^{-m} p - p^{-n}) - (p^{-m} q - q^{-n}) \right) T_{m+n}^{p, q}],
\end{align*}
\]

where

\[
f(m, n) = - \frac{p^{-m-1} q^{2(m+n)}}{(p-q)} [2_{p, q} T_{m+n}^{p^2, q^2} + \frac{q^{m+n} (q^n + q^{m+n} p^{-m-1} q)}{p-q} T_{m+n}^{p^2, q^2}]
\]

and other anti-commutators are zeros.

We consider the \( n \)- bracket defined by:

\[
\left[ T_{m_1}^{p^a, q^a}, \ldots, T_{m_n}^{p^a, q^a} \right] := \varepsilon_{1 \cdots n} T_{m_1}^{p^a, q^a} \cdots T_{m_n}^{p^a, q^a}.
\]

We study the case with the same \((p^a, q^a)\). Then,

\[
\left[ T_{m_1}^{p^a, q^a}, \ldots, T_{m_n}^{p^a, q^a} \right] = \varepsilon_{1 \cdots n} T_{m_1}^{p^a, q^a} \cdots T_{m_n}^{p^a, q^a}.
\]

Putting \( a = b \) in the relation (42), we obtain:

\[
\begin{align*}
\left[ T_{m}^{p^a, q^a}, T_{m}^{p^a, q^a} \right] &= \frac{(p^{-a} - p^{-m a})}{(p^a - q^a)} [2_{p^a, q^a} T_{m+n}^{2 a^2 a} \\
&- \frac{\tau_a^{m+n}}{p^a - q^a} \left( (p^{-a} - p^{-m a}) + (q^{-a} - q^{-m a}) \right) T_{m+n}^{p^a, q^a}.
\end{align*}
\]

The \( n \)- bracket takes the following form:

\[
\begin{align*}
\left[ T_{m_1}^{p^a, q^a}, \ldots, T_{m_n}^{p^a, q^a} \right] &= \frac{(-1)^{n+1}}{(p^a - q^a)^n} \left( M_n^m [n] p^a q^a T_{m_1+m+n}^{p^a, q^a} \right. \\
&- \frac{[n-1]}{q^a} \left( M_n^m + C_n^m \right) T_{m_1+m+n}^{(n-1) a, p^a q^a} \right).
\end{align*}
\]
where
\[
M_a^n = p^{-a(n-1) \sum_{s=1}^{n} m_s \left( (p^a - q^a)^2 \right)} \prod_{1 \leq j < k \leq n} \left( [m_k]_{p^a, q^a} - [m_j]_{p^a, q^a} \right) \\
+ \prod_{1 \leq j < k \leq n} \left( q^a m_k - q^a m_j \right)
\]
and
\[
C_a^n = q^{-a(n-1) \sum_{s=1}^{n} m_s \left( (p^a - q^a)^2 \right)} \prod_{1 \leq j < k \leq n} \left( [m_k]_{p^a, q^a} - [m_j]_{p^a, q^a} \right) \\
+ (-1)^{n-1} \prod_{1 \leq j < k \leq n} \left( p^a m_k - p^a m_j \right).
\]

From the super multibracket of order \( n \), we define the \((p, q)\)—deformed super \( n \)—bracket as follows:
\[
[T_{m_1}^{p^n, q^n}, T_{m_2}^{p^n, q^n}, \ldots, T_{m_n}^{p^n, q^n}] := \sum_{j=0}^{n-1} (-1)^{n-1-j} \prod_{i \neq j} T_{m_i}^{p^n, q^n} \\
\times T_{m_j}^{p^n, q^n} \prod_{1 \leq j < k \leq n} \left( p^a m_k - p^a m_j \right),
\]

Using the relation (43) with \( a = b \), we obtain:
\[
[T_{m_1}^{p^n, q^n}, T_{m_2}^{p^n, q^n}] = \left( \frac{p^{-n} - p^{-(m-1)n}}{(p^a - q^a)^2} \right) [2]_{p^n, q^n} T_{m_1}^{2, q^n} + f(m, n) \\
+ \frac{q^{(m+n)a}}{p^a - q^a} \left( (q^{-n} - p^{-n}) + (p^{-a} q^n - q^{-a}) \right) T_{m+n}^{a^n, q^n}.
\]

where
\[
f(m, n) = \frac{-p^{-m-a} q^{2(a+m+n)}}{(p^a - q^a)^2} [2]_{p^n, q^n} T_{m_1}^{2, q^n} + \frac{q^{(m+n)a}}{p^a - q^a} \left( q^{-a} + \frac{q^{(m+n+1)a}}{p^a + q^a} \right) T_{m+n}^{a^n, q^n}.
\]

Thus, the super \( n \)—bracket takes the form:
\[
[T_{m_1}^{p^n, q^n}, \ldots, T_{m_n}^{p^n, q^n}] = \left( \frac{(-1)^{n+1}}{(p^a - q^a)^{n-1}} \right) A^n_{a\{m\}_{p^a, q^a}^{p^n, q^n} - q^{a\{m\}_{p^a, q^a}^{p^n, q^n}}} \\
- \frac{[n-1]_{p^{-a} q^n} a^{n-1}}{q^{-a} \left( \sum_{j=1}^{n} m_j \right)} \left( F_a^n + S_a^n \right) T_{m_1}^{(n-1), q^{(n-1)a}} + f(m_1, \ldots, m_n),
\]

where
\[
A_a^n = p^{-a(n-1) \sum_{s=1}^{n} (m_s-1) \left( (p^a - q^a)^2 \right)} \prod_{1 \leq j < k \leq n} \left( [m_k - 1]_{p^a, q^n} - [m_j]_{p^a, q^n} \right) \\
+ \prod_{1 \leq j < k \leq n} \left( q^{a(m_k-1)} - q^{a(m_j)} \right),
\]
\[
F_a^n = p^{-a(n-1) \sum_{s=1}^{n} m_s \left( (p^a - q^a)^2 \right)} \prod_{1 \leq j < k \leq n} \left( [m_k]_{p^a, q^n} - [m_j]_{p^a, q^n} q^{(2)} \right) \\
+ \prod_{1 \leq j < k \leq n} \left( q^{a m_k} - q^{a m_j} q^{(2)} \right),
\]

24
\[ S_n^m = q^{-a(n-1) \sum_{i=1}^n m_i} \left( \prod_{1 \leq j < k \leq n} (m_{ik} p^a \cdot q^a p^a) \right) \]

and

\[ f(m_1, \ldots, m_n) = \frac{(pq)^{n-1} q^{\sum_{i=1}^n m_i}}{(p^a - q^a)^{n-1}} \left( q^{am}[n] p^a \cdot q^a \bar{T}_1^{m} \right) \]

Furthermore, the operators (38) and (39) are presented as follows:

\[ T_{m}^{\mathcal{R}(p^a, q^a)} = (m + \gamma) p^a q^a m \frac{\partial}{\partial t_m} + \frac{(pq)^{m+\gamma}}{p^a - q^a} \sum_{k=1}^{\infty} \frac{(k + m)!}{k!} B_k(t_1^a, \ldots, t_k^a) \frac{\partial}{\partial t_{k+m}} \]

\[ T_{n}^{p^a, q^a} = \theta \left[ (m + \gamma) p^a q^a m \frac{\partial}{\partial t_m} + \frac{(pq)^{m+\gamma}}{p^a - q^a} \sum_{k=1}^{\infty} \frac{(k + m)!}{k!} B_k(t_1^a, \ldots, t_k^a) \frac{\partial}{\partial t_{k+m}} \right] \]

Putting \( \bar{m} = m + \gamma \), \( \bar{n} = n + \gamma \), and by changing \( n! \frac{\partial}{\partial t_m} \leftarrow x^n \), we show directly that the products \( T_{m}^{p^a, q^a} \cdot T_{n}^{p^b, q^b} \) and \( T_{m}^{p^a, q^a} \cdot T_{n}^{p^b, q^b} \) are respectively equivalent to (40) and (41).

### 6.2 Chakrabarti and Jagannathan deformation [8]

Setting \( \mathcal{R}(x, y) = \frac{(1-xy)}{(x-y)x} \), we deduce the \( (p^{-1}, q) - \) deformed super Virasoro \( n \) - algebra and application.

### 6.3 Hounkonnou-Ngome generalized \( q \) – Quesne deformation [21]

The results corresponding here are obtained by taking \( \mathcal{R}(x, y) = \frac{(xy-1)}{(q-p)xy} \).

### 6.4 Biedenharn-Macfarlane deformation [7][27]

Putting \( \mathcal{R}(x) = \frac{x-x^{-1}}{q-q^{-1}} \), we obtain the \( q \) – deformed super Virasoro \( n \) – algebra.

### 7 Concluding and remarks

We have constructed a super Witt \( n \) and Virasoro \( 2n \) – algebras from quantum algebras. Moreover, we have generalized this study to investigate the super \( \mathcal{R}(p, q) \) – deformed Witt \( n \) – algebra, and super \( \mathcal{R}(p, q) \) – deformed Virasoro \( n \) – algebra and discuss a toy model. Particular cases have been investigated. For further, the super Virasoro algebra with a conformal dimension is in preparation for the future work.
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