Stochastic variational inequalities with oblique subgradients

Anouar M. Gassous\textsuperscript{a}, Aurel Răşcanu\textsuperscript{a,b}, Eduard Rotenstein\textsuperscript{a,*}

\textsuperscript{a}Faculty of Mathematics, “Al. I. Cuza” University, Bd. Carol I, no. 9-11, Iaşi, Romania
\textsuperscript{b}“Octav Mayer” Mathematics Institute of the Romanian Academy, Bd. Carol I, no. 8, Iaşi, Romania

Abstract

In this paper we will study the existence and uniqueness of the solution for the stochastic variational inequality with oblique subgradients of the following form:

$$
\begin{aligned}
\{ & dX_t + H(X_t) \partial \phi(X_t) (dt) \ni f(t, X_t) dt + g(t, X_t) dB_t, \quad t > 0, \\
& X_0 = x \in \text{Dom}(\phi).
\end{aligned}
$$

Here, the mixture between the monotonicity property of the subdifferential operator $\partial \phi$ and the Lipschitz property of matrix mapping $X \mapsto -H(X)$ leads to stronger difficulties comparing to the classical case of stochastic variational inequalities. The existence result is based on a deterministic approach: a differential system with singular input is first analyzed.

Keywords: oblique reflection, Skorohod problem, stochastic variational inequalities

AMS Classification subjects: 60H10, 60H15, 35K85

1. Introduction

Since the early sixties, research has paid increasing attention to the study of reflected stochastic differential equations, the reflection process being approached in different ways. Skorohod, for instance, considered the problem of reflection for diffusion processes into a bounded domain (see, e.g., [16]). Tanaka focused on the problem of reflecting boundary conditions into convex sets for stochastic differential equations (see [17]). This kind of problem became the interest of many other authors, who considered that the state process is reflected by one or two reflecting barriers (see, e.g., [2], [3], [10], [8] and the references therein). While, during the first studies, the trajectories of the system were reflected upon the normal direction, in 1984 Lions and Sznitman, in the paper [11], studied for the first time the following problem of oblique reflection in a domain:

$$
\begin{aligned}
\{ & dX_t + dK_t = f(t, X_t) dt + g(t, X_t) dB_t, \quad t > 0, \\
& X_0 = x, \quad K_t = \int_0^t 1_{\{X_s \in \text{Bd}(E)\}} \gamma(X_s) d\gamma K_{s}^{\uparrow},
\end{aligned}
$$

\textsuperscript{∗}The work for this paper was supported by the Grant Marie Curie Initial Training Network (ITN) FP7-PEOPLE-2007-1-1-ITN, no.213841.

\textsuperscript{*}Corresponding author.

Email addresses: masigassous@gmail.com (Anouar M. Gassous), aurel.rascanu@uaic.ro (Aurel Răşcanu), eduard.rotenstein@uaic.ro (Eduard Rotenstein)

Preprint submitted to Elsevier January 12, 2013
where, for the bounded oblique reflection $\gamma \in C^2(\mathbb{R}^d)$, there exists a positive constant $\nu$ such that $(\gamma(x), n(x)) \geq \nu$, for every $x \in Bd(E)$, $n(x)$ being the unit outward normal vector.

A generalization, with respect to the smoothness of the domain, of the result of Lions and Sznitman was given after by Depuis and Ishii in the paper [7]. They assumed that the domain in which we have the oblique reflection has some additional regularity properties.

The aim of our paper consists in extending the problem of oblique reflection in the framework of deterministic and stochastic variational inequalities. This kind of multivalued stochastic differential equations were introduced in the literature by Asiminoaei & Răşcanu in [1], Barbu & Răşcanu in [2] and Bensoussan & Răşcanu in [3]. They proved the existence and uniqueness result for the case of stochastic variational differential systems involving subdifferential operators and, even more, they provided approximation and splitting-up schemes for this type of equations. The general result, for stochastic differential equations governed by maximal monotone operators

$$\begin{cases}
    dX_t + A(X_t)(dt) \ni f(t, X_t) dt + g(t, X_t) dB_t,
    
    X_0 = \xi, \ t \in [0, T]
\end{cases}$$

was given by Răşcanu in [13], the approach for proving the existence and uniqueness being done via a deterministic multivalued equation with singular input.

A different approach for solving these type of equations was introduced by Răşcanu & Rotenstein in the paper [15]. They reduced the existence problem for multivalued stochastic differential equations to a minimizing problem of a convex lower semicontinuous function. The solutions of these equations were identified with the minimum points of some suitably constructed convex lower semicontinuous functionals, defined on well chosen Banach spaces.

As the main objective of this paper we prove the existence and uniqueness of the solution for the following stochastic variational inequality

$$\begin{cases}
    dX_t + H(X_t)\partial\varphi(X_t)(dt) \ni f(t, X_t) dt + g(t, X_t) dB_t, \quad t > 0,

    X_0 = x_0,
\end{cases}$$

where $B$ is a standard Brownian motion defined on a complete probability space and the new quantity that appears acts on the set of subgradients and it will be called, from now on, oblique subgradient. The problem becomes challenging due to the presence of this new term, which impose the use of some specific approaches because this new term preserve neither the monotony of the subdifferential operator nor the Lipschitz property of the matrix involved.

First, we will focus on the deterministic case, considering a generalized Skorohod problem with oblique reflection of the form

$$\begin{cases}
    x(t) + \int_0^t H(x(s)) \, dk(s) = x_0 + \int_0^t f(s, x(s)) \, ds + m(t), \quad t \geq 0,

    dk(s) \in \partial\varphi(x(s))(ds),
\end{cases}$$

where the singular input $m : \mathbb{R}_+ \to \mathbb{R}^d$ is a continuous function. The existence results are obtained via Yosida penalization techniques.
The paper is organized as follows. Section 2 presents the notations and assumptions that will be used along this article and, also, a deterministic generalized Skorohod problem with oblique reflection is constructed. The existence and uniqueness result for this problem can also be found here. Section 3 is dedicated to the main result of our work; more precisely, the existence of a unique strong solution for our stochastic variational inequality with oblique subgradients is proved. The last part of the paper groups together some useful results that are used throughout this article.

2. Generalized convex Skorohod problem with oblique subgradients

2.1. Notations. Hypotheses

We first study the following deterministic generalized convex Skorohod problem with oblique subgradients:

\[
\begin{cases}
  dx(t) + H(x(t)) \partial \varphi(x(t)) (dt) \ni dm(t), & t > 0, \\
  x(0) = x_0,
\end{cases}
\]

where

\[
\begin{align*}
(i) & \quad x_0 \in \text{Dom} \varphi \overset{def}= \{x \in \mathbb{R}^d : \varphi(x) < \infty\}, \\
(ii) & \quad m \in C_+ (\mathbb{R}^d; \mathbb{R}^d), & m(0) = 0,
\end{align*}
\]

\[H = (h_{i,j})_{d \times d} \in C^2_b (\mathbb{R}^d, \mathbb{R}^d \times \mathbb{R}^d)\]

is a matrix, such that for all \(x \in \mathbb{R}^d\),

\[
\begin{cases}
(i) & \quad h_{i,j}(x) = h_{j,i}(x), & \text{for every } i, j \in \overline{1,d}, \\
(ii) & \quad \frac{1}{c} |u|^2 \leq \langle H(x) u, u \rangle \leq c |u|^2, & \forall u \in \mathbb{R}^d \text{ (for some } c \geq 1),
\end{cases}
\]

and

\[\varphi : \mathbb{R}^d \to ]-\infty, +\infty] \text{ is a proper l.s.c. convex function.}\]

Denote by \(\partial \varphi\) the subdifferential operator of \(\varphi\):

\[\partial \varphi(x) \overset{def}= \{\dot{x} \in \mathbb{R}^d : \langle \dot{x}, y - x \rangle + \varphi(x) \leq \varphi(y), \text{ for all } y \in \mathbb{R}^d\}\]

and \(\text{Dom}(\partial \varphi) = \{x \in \mathbb{R}^d : \partial \varphi(x) \neq \emptyset\}\). We will use the notation \((x, \dot{x}) \in \partial \varphi\) in order to express that \(x \in \text{Dom}(\partial \varphi)\) and \(\dot{x} \in \partial \varphi(x)\).

The vector defined by the quantity \(H(x) h\), with \(h \in \partial \varphi(x)\), will be called in what follows \textit{oblique subgradient}.

\textbf{Remark 1.} If \(E\) is a closed convex subset of \(\mathbb{R}^d\), then

\[\varphi(x) = I_E(x) = \begin{cases} 0, & \text{if } x \in E, \\
+\infty, & \text{if } x \notin E \end{cases}\]
is a convex l.s.c. function and, for \( x \in E \),
\[
\partial I_E (x) = \{ \hat{x} \in \mathbb{R}^d : \langle \hat{x}, y - x \rangle \leq 0, \forall y \in E \} = N_E (x),
\]
where \( N_E (x) \) is the closed external normal cone to \( E \) at \( x \). We have \( N_E (x) = \emptyset \) if \( x \notin E \) and \( N_E (x) = \{0\} \) if \( x \in \text{int} (E) \) (we denoted by \( \text{int} (E) \) the interior of the set \( E \)).

**Remark 2.** A vector \( \nu_x \) associated to \( x \in \text{Bd} (E) \) (we denoted by \( \text{Bd} (E) \) the boundary of the set \( E \)) is called an external direction if there exists \( \rho_0 > 0 \) such that \( x + \rho \nu_x \notin E \) for all \( 0 < \rho \leq \rho_0 \). In this case there exists \( c' > 0 \), \( n_x \in N_E (x) \), \( |n_x| = 1 \), such that \( \langle n_x , \nu_x \rangle \geq c' \).

Remark that, if we consider the symmetric matrix
\[
H (\nu_x ) = n_x \otimes n_x - n_x \otimes \nu_x + \frac{2}{\langle \nu_x , n_x \rangle} \nu_x \otimes \nu_x ,
\]
then
\[
\nu_x = M (\nu_x ) n_x , \text{ for all } x \in \text{Bd} (E).
\]

Let \( [H (x)]^{-1} \) be the inverse matrix of \( H (x) \). Then \( [H (x)]^{-1} \) has the same properties as \( H (x) \). Denote
\[
b = \sup_{x,y \in \mathbb{R}^d} \frac{|H (x) - H (y)|}{|x - y|} + \sup_{x,y \in \mathbb{R}^d} \frac{|[H (x)]^{-1} - [H (y)]^{-1}|}{|x - y|},
\]
where \( |H (x)| \overset{\text{def}}{=} \left( \sum_{i,j=1}^{d} |h_{i,j} (x)|^2 \right)^{1/2} \).

We shall call oblique reflection directions of the form
\[
\nu_x = H (x) n_x, \text{ with } x \in \text{Bd} (E),
\]
where \( n_x \in N_E (x) \).

If \( E = \overline{E} \subset \mathbb{R}^d \) and \( E^c = \mathbb{R}^d \setminus E \), then we denote by
\[
E_\varepsilon = \{ x \in E : \text{dist} (x, E^c) \geq \varepsilon \} = \{ x \in E : B (x, \varepsilon) \subset E \}
\]
the \( \varepsilon \)–interior of \( E \).

We impose the following supplementary assumptions
\[
\begin{cases}
(i) & D = \text{Dom} (\varphi) \text{ is a closed subset of } \mathbb{R}^d, \\
(ii) & \exists r_0 > 0, D_{r_0} \neq \emptyset \text{ and } h_0 = \sup_{z \in D} \text{dist} (z, D_{r_0}) < \infty, \\
(iii) & \exists L \geq 0, \text{ such that } |\varphi (x) - \varphi (y)| \leq L + L |x - y|, \text{ for all } x, y \in D.
\end{cases}
(9)
\]

For example, condition \((ii)(iii)\) is verified by functions \( \varphi : \mathbb{R}^d \to \mathbb{R} \) of the following type:
\[
\varphi (x) = \varphi_1 (x) + \varphi_2 (x) + I_D (x),
\]
where \( D \) is a convex set satisfying \((ii)\), \( \varphi_1 : \mathbb{R}^d \to \mathbb{R} \) is a convex lower semicontinuous function, \( \varphi_2 : D \to \mathbb{R} \) is a Lipschitz function and \( I_D \) is the convex indicator of the set \( D \).
2.2. A generalized Skorohod problem

In this section we present the notion of solution for the generalized convex Skorohod problem with oblique subgradients (4) and, also, we provide full proofs for its existence and uniqueness.

If \( k : [0, T] \rightarrow \mathbb{R}^d \) and \( \mathcal{D}[0, T] \) is the set of the partitions of the time interval \([0, T]\), of the form \( \Delta = (0 = t_0 < t_1 < ... < t_n = T) \), we denote
\[
S_\Delta (k) = \sum_{i=0}^{n-1} |k(t_{i+1}) - k(t_i)|
\]
and \( \uparrow k \uparrow_T \overset{\text{def}}{=} \sup_{\Delta \in \mathcal{D}} S_\Delta (k) \). In the sequel we consider the space of bounded variation functions \( \text{BV}([0, T] ; \mathbb{R}^d) = \{ k \mid k : [0, T] \rightarrow \mathbb{R}^d, \uparrow k \uparrow_T < \infty \} \). Taking on the space of continuous functions \( C ([0, T] ; \mathbb{R}^d) \) the usual norm
\[
\|y\|_T \overset{\text{def}}{=} \|y\|_{C([0,T];\mathbb{R}^d)} = \sup \{ |y(s)| : 0 \leq s \leq T \},
\]
then \( (C([0, T] ; \mathbb{R}^d))^* = \{ k \in BV([0, T] ; \mathbb{R}^d) : k(0) = 0 \} \). The duality between these spaces is given by the Riemann–Stieltjes integral \((y, k) \mapsto \int_0^T \langle y(t), dk(t) \rangle \). We will say that a function \( k \in BV_{loc}([0, +\infty[; \mathbb{R}^d) \) if, for every \( T > 0 \), \( k \in BV([0, T] ; \mathbb{R}^d) \).

**Definition 1.** Given two functions \( x, k : \mathbb{R}_+ \rightarrow \mathbb{R}^d \), we say that \( dk(t) \in \partial\varphi(x(t)) (dt) \) if
\[
\begin{align*}
(a) & \quad x, k : \mathbb{R}_+ \rightarrow \mathbb{R}^d \text{ are continuous}, \\
(b) & \quad x(t) \in \text{Dom}(\varphi), \\
(c) & \quad k \in BV_{loc} ([0, +\infty[; \mathbb{R}^d), k(0) = 0, \\
(d) & \quad \int_s^t \langle y(r) - x(r), dk(r) \rangle + \int_s^t \varphi(x(r)) \, dr \leq \int_s^t \varphi(y(r)) \, dr, \quad \text{for all } 0 \leq s \leq t \leq T \text{ and } y \in C ([0, T] ; \mathbb{R}^d) .
\end{align*}
\]

We state that

**Definition 2.** A pair of functions \((x, k)\) is a solution of the Skorohod problem with \( H-\)oblique subgradients (4) (and we write \((x, k) \in \mathcal{SP} (H\partial\varphi; x_0, m)\)) if \( x, k : \mathbb{R}_+ \rightarrow \mathbb{R}^d \) are continuous functions and
\[
\begin{align*}
(i) & \quad x(t) + \int_0^t H(x(r)) \, dk(r) = x_0 + m(t), \quad \forall \ t \geq 0, \\
(ii) & \quad dk(r) \in \partial\varphi(x(r)) (dr) .
\end{align*}
\]

In Annex, Section 4.1., we present some lemmas with a priori estimates of the solutions \((x, k) \in \mathcal{SP} (H\partial\varphi; x_0, m)\). We here recall the result from Lemma 12.
Proposition 1. If \((x, k) \in \mathcal{SP} (H \partial \varphi; x_0, m)\) then, under assumptions (5), (6), (7) and (9) there exists a constant \(C_T (\|m\|_T) = C (T, \|m\|_T, b, c, r_0, h_0)\), increasing function with respect to \(\|m\|_T\), such that, for all \(0 \leq s \leq t \leq T\),

\[
\begin{align*}
(a) \quad & \|x\|_T + \|k^\#_T\| \leq C_T (\|m\|_T), \\
(b) \quad & |x(t) - x(s)| + \|k^\#_t - k^\#_s\| \leq C_T (\|m\|_T) \times \sqrt{t - s + m_m(t - s)},
\end{align*}
\]

where \(m_m\) represents the modulus of continuity of the continuous function \(m\).

We renounce now at the restriction that the function \(f\) is identically 0 and we consider the equation written under differential form

\[
\left\{
\begin{array}{l}
dx(t) + H(x(t)) \partial \varphi(x(t)) (dt) \ni f(t, x(t)) dt + dm(t), \quad t > 0, \\
x(0) = x_0,
\end{array}
\right.
\]

where

\[
(i) \quad (t, x) \mapsto f(t, x) : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d
\]

is a Carathéodory function (i.e. measurable w.r. to \(t\) and continuous w.r. to \(x\)),

\[
(ii) \quad \int_0^T (f^\#(t))^2 dt < \infty, \quad \text{where} \quad f^\#(t) = \sup_{x \in \text{Dom}(\varphi)} |f(t, x)|.
\]

The estimates (11) hold too for a solution of Eq. (12), but, now, the constant \(C_T (\|m\|_T)\) depends also on the quantity \(\int_0^T f^\#(t)dt\). We are now able to formulate the main result of this section.

Theorem 2. Let the assumptions (5), (6), (7), (9) and (13) be satisfied. Then the differential equation (12) has at least one solution in the sense of Definition 2, i.e. \(x, k : \mathbb{R}_+ \to \mathbb{R}^d\) are continuous functions and

\[
\left\{
\begin{array}{l}
x(t) + \int_0^t H(x(r)) dr = x_0 + \int_0^t f(r, x(r)) dr + m(t), \quad \forall \ t \geq 0,
\end{array}
\right.
\]

Proof. We will divide the proof in two separate steps. First we will analyze the case of the regular function \(m\) and, in the sequel, we consider the situation of the singular input \(m\).

Step 1. Case \(m \in C^1 (\mathbb{R}_+: \mathbb{R}^d)\)

It is sufficient to prove the existence of a solution on an interval \([0, T]\) arbitrary, fixed. Let \(n \in \mathbb{N}^*, \ n \geq T\), fixed, consider \(\varepsilon = \frac{T}{n}\) and the extensions \(f(s, x) = 0\) and \(m(s) = s \cdot m'(0+)\) for \(s < 0\). Based on the notations from Annex 4.2., we consider the penalized problem

\[
\begin{align*}
x_\varepsilon(t) &= x_0, \quad \text{if} \ t < 0, \\
x_\varepsilon(t) + \int_0^t H(x_\varepsilon(s)) dk_\varepsilon(s) &= x_0 + \int_0^t \left[ f(s - \varepsilon, \pi_D(x_\varepsilon(s - \varepsilon))) + m'(s - \varepsilon) \right] ds,
\end{align*}
\]

\(t \in [0, T]\),
or, equivalent,
\[
x_\varepsilon(t) = x_0, \quad \text{if } t < 0,
\]
\[
x_\varepsilon(t) + \int_0^t H(x_\varepsilon(s)) \nabla \varphi_\varepsilon(x_\varepsilon(s)) ds = x_0 + \int_{-\varepsilon}^{t-\varepsilon} \left[ f(s, \pi_D(x_\varepsilon(s))) + m'(s) \right] ds, \quad t \in [0, T],
\]
where
\[
k_\varepsilon(t) = \int_0^t \nabla \varphi_\varepsilon(x_\varepsilon(s)) ds
\]
and \( \pi_D(x) \) is the projection of \( x \) on the set \( D = \overline{\text{Dom}(\varphi)} = \text{Dom}(\varphi) \), uniquely defined by \( \pi_D(x) \in D \) and \( \text{dist}(x, D) = |x - \pi_D(x)| \).

Since \( x \mapsto H(x) \nabla \varphi_\varepsilon(x) : \mathbb{R}^d \rightarrow \mathbb{R}^d \) is a sublinear and locally Lipschitz continuous function and, for \( s \leq t - \varepsilon \),
\[
\|f(s, \pi_D(x_\varepsilon(s)))\| \leq f^\#(s),
\]
then, recursively, on the intervals \([i\varepsilon, (i + 1)\varepsilon]\) the approximating equation admits a unique solution \( x_\varepsilon \in C([0, T]; \mathbb{R}^d) \). The regularity of the function \( x \mapsto |x - a|^2 + \varphi_\varepsilon(x) \) and the definition of the approximating sequence \( \{x_\varepsilon \} \) implies that, for \( u_0 \in \text{Dom}(\varphi) \), we have
\[
|x_\varepsilon(t) - u_0|^2 + \varphi_\varepsilon(x_\varepsilon(t)) + \int_0^t \langle H(x_\varepsilon(s)) \nabla \varphi_\varepsilon(x_\varepsilon(s)), 2[x_\varepsilon(s) - u_0] + \nabla \varphi_\varepsilon(x_\varepsilon(s)) \rangle ds \leq |x_\varepsilon(t) - u_0|^2 + \varphi_\varepsilon(x_\varepsilon(t)) + \int_0^t \langle 2[x_\varepsilon(s) - u_0] + \nabla \varphi_\varepsilon(x_\varepsilon(s)), f(s - \varepsilon, \pi_D(x_\varepsilon(s - \varepsilon))) + m'(s - \varepsilon) ds \rangle.
\]

Let consider an arbitrary fixed pair \((u_0, \hat{u}_0) \in \partial \varphi\). Since \( \nabla \varphi_\varepsilon(u_0) = \partial \varphi_\varepsilon(u_0) \), then it is easy to verify, from the definition of the subdifferential operator, that
\[
|x_\varepsilon(x_\varepsilon(u_0)) - \varphi_\varepsilon(u_0)| + \varphi_\varepsilon(u_0) - 2|\nabla \varphi_\varepsilon(u_0)||x_\varepsilon - u_0| \leq \varphi_\varepsilon(x_\varepsilon).
\]
Also, since \( \nabla \varphi_\varepsilon(u_0) \in \partial \varphi(J_\varepsilon(u_0)) \), where \( J_\varepsilon(x) = x - \varepsilon \nabla \varphi_\varepsilon(x) \), then
\[
\langle \hat{u}_0 - \nabla \varphi_\varepsilon(u_0), u_0 - (u_0 - \varepsilon \nabla \varphi_\varepsilon(u_0)) \rangle \geq 0,
\]
which yields, after short computations, \( |\nabla \varphi_\varepsilon(u_0)| \leq |\hat{u}_0| \). Moreover,
\[
-\varepsilon|\hat{u}_0|^2 \leq -\varepsilon \langle \hat{u}_0, \nabla \varphi_\varepsilon(u_0) \rangle = \langle \hat{u}_0, J_\varepsilon(u_0) - u_0 \rangle \leq \varphi(J_\varepsilon(u_0)) - \varphi(u_0) \leq \varphi(u_0) - \varphi(u_0) \leq 0.
\]
Due to
\[
\varphi_\varepsilon(x_\varepsilon(t)) \geq |\varphi_\varepsilon(x_\varepsilon(t)) - \varphi_\varepsilon(u_0)| + \varphi(u_0) - |\hat{u}_0|^2 - 2|\hat{u}_0||x_\varepsilon(t) - u_0|,
\]
Denoting by $C$ a generic constant independent of $\varepsilon$ ($C$ depends only of $c$ and $u_0$), the following estimates hold (to be shortened we omit the argument $s$, writing $x_\varepsilon$ in the place of $x_\varepsilon(s)$):

- $\frac{1}{c} |\nabla \varphi_\varepsilon (x_\varepsilon)|^2 \leq \langle H (x_\varepsilon) \nabla \varphi_\varepsilon (x_\varepsilon), \nabla \varphi_\varepsilon (x_\varepsilon) \rangle$,

- $\langle H (x_\varepsilon) \nabla \varphi_\varepsilon (x_\varepsilon), 2 (x_\varepsilon - u_0) \rangle \geq -2 |x_\varepsilon - u_0| |H (x_\varepsilon) \nabla \varphi_\varepsilon (x_\varepsilon)| \geq -2c |x_\varepsilon - u_0| |\nabla \varphi_\varepsilon (x_\varepsilon)| \geq -C \sup_{r \leq s} |x_\varepsilon(r) - u_0|^2 - \frac{1}{4c} |\nabla \varphi_\varepsilon (x_\varepsilon)|^2$,

- $2 |\dot{u}_0| |x_\varepsilon(t) - u_0| \leq \frac{1}{2} \sup_{r \leq t} |x_\varepsilon(r) - u_0|^2 + 2 |\dot{u}_0|^2$,

- $\langle 2 (x_\varepsilon(s) - u_0) + \nabla \varphi_\varepsilon (x_\varepsilon(s)), f (s - \varepsilon, \pi_D (x_\varepsilon(s - \varepsilon))) + m'(s - \varepsilon) \rangle \leq \frac{1}{8c} |2 (x_\varepsilon(s) - u_0) + \nabla \varphi_\varepsilon (x_\varepsilon(s))|^2 + 2c |f (s - \varepsilon, \pi_D (x_\varepsilon(s - \varepsilon))) + m'(s - \varepsilon)|^2 \leq \frac{1}{4c} |\nabla \varphi_\varepsilon (x_\varepsilon(s))|^2 + \frac{1}{c} |x_\varepsilon(s) - u_0|^2 + 4c \left[ (f^#(s - \varepsilon))^2 + |m'(s - \varepsilon)|^2 \right]$. 

Using the above estimates in (17) we infer

$$\begin{align*}
|x_\varepsilon(t) - u_0|^2 + |\varphi_\varepsilon (x_\varepsilon(t)) - \varphi_\varepsilon (u_0)| + \frac{1}{2c} \int_0^t |\nabla \varphi_\varepsilon (x_\varepsilon(r))|^2 dr & \\
\leq |x_0 - u_0|^2 + \varphi(x_0) - \varphi(u_0) + 3 |\dot{u}_0|^2 & + \frac{1}{2} \sup_{\theta \leq t} |x_\varepsilon(\theta) - u_0|^2 + 4c \int_0^t \left[ (f^#(r - \varepsilon))^2 + |m'(r - \varepsilon)|^2 \right] dr & + C \int_0^t \sup_{\theta \leq r} |x_\varepsilon(\theta) - u_0|^2 dr.
\end{align*}$$
We write the inequality for $s \in [0, t]$ and then we take the $\sup_{s \leq t}$. Hence
\[
\|x_\varepsilon - u_0\|^2_t + \sup_{s \leq t} |\varphi_\varepsilon (x_\varepsilon (s)) - \varphi_\varepsilon (u_0)| + \int_0^t |\nabla \varphi_\varepsilon (x_\varepsilon (r))|^2 \, dr \\
\leq 2 \left[ \|x_\varepsilon - u_0\|^2 + \varphi (x_0) - \varphi (u_0) + |\hat{u}_0|^2 \right] \\
+ 8c \int_{-1}^1 \left[ (f^\# (r))^2 + |m' (r)|^2 \right] \, dr + C \int_0^t \|x_\varepsilon - u_0\|^2 \, dr.
\]
By the Gronwall inequality we have
\[
\|x_\varepsilon - u_0\|^2_t \leq Ce^{Ct} \left[ \|x_0 - u_0\|^2 + \varphi (x_0) - \varphi (u_0) + |\hat{u}_0|^2 \right] + \int_{-1}^1 \left[ (f^\# (r))^2 + |m' (r)|^2 \right] \, dr.
\]
Hence, there exists a constant $C_T$, independent of $\varepsilon$, such that
\[
\sup_{t \in [0, T]} |x_\varepsilon (t)|^2 + \sup_{t \in [0, T]} |\varphi_\varepsilon (x_\varepsilon (t))| + \int_0^T |\nabla \varphi_\varepsilon (x_\varepsilon (s))|^2 \, ds \leq C_T.
\] (18)

Since $\nabla \varphi_\varepsilon (x) = \frac{1}{\varepsilon} (x - J_\varepsilon x)$, then, we also obtain
\[
\int_0^T |x_\varepsilon (s) - J_\varepsilon (x_\varepsilon (s))|^2 \, ds \leq \varepsilon C_T.
\] (19)

From the approximating equation, for all $0 \leq s \leq t \leq T$, we have
\[
\left| x_\varepsilon (t) - x_\varepsilon (s) \right| \\
\leq \left| \int_s^t H (x_\varepsilon (r)) \nabla \varphi_\varepsilon (x_\varepsilon (r)) \, dr \right| + \left| \int_{s-\varepsilon}^{t-\varepsilon} f (r, \pi_D (x_\varepsilon (r))) \, dr \right| \\
+ |m (t - \varepsilon) - m (s - \varepsilon)| \\
\leq c \int_s^t |\nabla \varphi_\varepsilon (x_\varepsilon (r))| \, dr + \int_{s-\varepsilon}^{t-\varepsilon} f^\# (r) \, dr + m_m (t - s) \\
\leq c \sqrt{t - s} \left( \int_s^t |\nabla \varphi_\varepsilon (x_\varepsilon (r))|^2 \, dr \right)^{1/2} + \sqrt{t - s} \left( \int_{s-\varepsilon}^{t-\varepsilon} (f^\# (r))^2 \, dr \right)^{1/2} + m_m (t - s) \\
\leq C_T \left[ \sqrt{t - s} + m_m (t - s) \right].
\]

In fact, moreover we have
\[
\| x_\varepsilon \|^2_{[s, t]} \leq \int_s^t |H (x_\varepsilon (r)) \nabla \varphi_\varepsilon (x_\varepsilon (r))| \, dr + \int_{s-\varepsilon}^{t-\varepsilon} |f (r, \pi_D (x_\varepsilon (r)))| \, dr + \int_{s-\varepsilon}^{t-\varepsilon} |m' (r)| \, dr \\
\leq C_T \sqrt{t - s}.
\]

Hence $\{x_\varepsilon : \varepsilon \in (0, 1]\}$ is a bounded and uniformly equicontinuous subset of $C ([0, T]; \mathbb{R}^d)$. From Ascoli-Arzela’s theorem it follows that there exists $\varepsilon_n \to 0$ and $x \in C ([0, T]; \mathbb{R}^d)$ such that
\[
\lim_{n \to \infty} \sup_{t \in [0, T]} \left| x_\varepsilon_n (t) - x (t) \right| = 0.
\]
By (19), there exists \( h \in L^2(0, T; \mathbb{R}^d) \) such that, on a subsequence, denoted also \( \varepsilon_n \), we have
\[
J_{\varepsilon_n}(x_{\varepsilon_n}) \to x \quad \text{in } L^2(0, T; \mathbb{R}^d) \quad \text{and a.e. in } [0, T], \quad \text{as } \varepsilon_n \to 0
\]
and
\[
\nabla \varphi_{\varepsilon_n}(x_{\varepsilon_n}) \rightharpoonup h, \quad \text{weakly in } L^2(0, T; \mathbb{R}^d).
\]
Therefore, for all \( t \in [0, T] \),
\[
\lim_{n \to \infty} \int_0^t H(x_{\varepsilon_n}(s)) \nabla \varphi_{\varepsilon_n}(x_{\varepsilon_n}(s)) \, ds = \int_0^t H(x(s)) h(s) \, ds. \tag{20}
\]
The lower semicontinuity property of \( \varphi \) yields, a.e. \( t \in [0, T] \),
\[
\varphi(x(t)) \leq \liminf_{n \to +\infty} \varphi(J_{\varepsilon_n}(x_{\varepsilon_n}(t))) \leq \liminf_{n \to +\infty} \varphi_{\varepsilon_n}(x_{\varepsilon_n}(t)) \leq C_T.
\]
Since \( \nabla \varphi_{\varepsilon}(x_{\varepsilon}) \in \partial \varphi(J_{\varepsilon}(x_{\varepsilon})) \), then for all \( y \in C([0, T] ; \mathbb{R}^d) \),
\[
\int_s^t \langle \nabla \varphi_{\varepsilon}(x_{\varepsilon}(r)), y(r) - J_{\varepsilon}(x_{\varepsilon}(r)) \rangle \, dr + \int_s^t \varphi(J_{\varepsilon}(x_{\varepsilon}(r))) \, dr \leq \int_s^t \varphi(y(r)) \, dr;
\]
passing to \( \liminf_{\varepsilon_n \to 0} \) we obtain
\[
\int_s^t \langle h(r), y(r) - x(r) \rangle \, dr + \int_s^t \varphi(x(r)) \, dr \leq \int_s^t \varphi(y(r)) \, dr,
\]
for all \( 0 \leq s \leq t \leq T \) and \( y \in C([0, T] ; \mathbb{R}^d) \), that is \( h(r) \in \partial \varphi(x(r)) \) a.e. \( t \in [0, T] \).

Finally, taking into account (20), by passing to limit for \( \varepsilon = \varepsilon_n \to 0 \) in the approximating equation (15), via the Lebesgue dominated convergence theorem for the integral from the right-hand side, we get
\[
x(t) + \int_0^t H(x(s)) \, dk(s) = x_0 + \int_0^t f(s, x(s)) \, ds + m(t),
\]
where
\[
k(t) = \int_0^t h(s) \, ds.
\]

**Step 2.** Case \( m \in C([0, T] ; \mathbb{R}^d) \).

Let extend again \( m(s) = 0 \) for \( s \leq 0 \) and define
\[
m_{\varepsilon}(t) = \frac{1}{\varepsilon} \int_{t-\varepsilon}^t m(s) \, ds = \frac{1}{\varepsilon} \int_0^\varepsilon m(t+r-\varepsilon) \, dr,
\]
We have
\[
m_{\varepsilon} \in C^1([0, T] ; \mathbb{R}^d), \quad \|m_{\varepsilon}\|_T \leq \|m\|_T \quad \text{and} \quad m_{m_{\varepsilon}}(\delta) \leq m_m(\delta).
\]
Let \((x_\varepsilon, k_\varepsilon)\) be a solution of the approximating equation
\[
\begin{aligned}
&x_\varepsilon(t) + \int_0^t H(x_\varepsilon(r)) \, dk_\varepsilon(r) = x_0 + \int_0^t f(r, x_\varepsilon(r)) \, dr + m_\varepsilon(t), \quad t \geq 0, \\
dk_\varepsilon(r) \in \partial \varphi(x_\varepsilon(r))(dr),
\end{aligned}
\]
solution which exists according to the first step of the proof. We have
\[
k_\varepsilon(t) = \int_0^t h_\varepsilon(s) \, ds, \quad h_\varepsilon \in L^2(0, T; \mathbb{R}^d),
\]
and
\[
\int_s^t \langle y(r) - x_\varepsilon(r), dk_\varepsilon(r) \rangle + \int_s^t \varphi(x_\varepsilon(r))(dr) \leq \int_s^t \varphi(y(r))(dr), \tag{21}
\]
for all \(0 \leq s \leq t \leq T\) and \(y \in C([0, T]; \mathbb{R}^d)\).

From Lemma \([12]\) with \(m\) replaced by
\[
M_\varepsilon(t) = \int_0^t f(r, x_\varepsilon(r)) \, dr + m_\varepsilon(t),
\]
we have
\[
\|x_\varepsilon\|_T + \|k_\varepsilon\|_T \leq C_T(\|M_\varepsilon\|_T) \quad \text{and} \quad |x_\varepsilon(t) - x_\varepsilon(s)| + \|k_\varepsilon(t) - k_\varepsilon(s)\| \leq C_T(\|M_\varepsilon\|_T) \times \sqrt{\|M_\varepsilon\|(t - s)},
\]
where, for \(\delta > 0\), \(\mu_g(\delta) \overset{\text{def}}{=} \delta + m_g(\delta)\) and \(m_g\) is the modulus of continuity of the continuous function \(g : [0, T] \to \mathbb{R}^d\) (for more details see Annex 4.1.). Since, for all \(0 \leq s \leq t \leq T\),
\[
\mu_{M_\varepsilon}(t - s) \leq t - s + \sqrt{t - s} \int_0^T (f^\#(r))^2 \, dr + m(t - s) \overset{\text{def}}{=} \gamma(t - s) \quad \text{and} \quad \|M_\varepsilon\|_T = m_{M_\varepsilon}(T) \leq \int_0^T f^\#(r) \, dr + \|m\|_T \overset{\text{def}}{=} \gamma_T,
\]
then there exist the positive constants \(C_T(\gamma_T)\) and \(\tilde{C}_T(\gamma_T)\) such that
\[
\|x_\varepsilon\|_T + \|k_\varepsilon\|_T \leq C_T(\gamma_T) \quad \text{and} \quad m_{x_\varepsilon}(t - s) + \|k_{x_\varepsilon}(t) - k_{x_\varepsilon}(s)\| \leq \tilde{C}_T(\gamma_T) \times \sqrt{\gamma(t - s)}.
\]
By Ascoli-Arzelà’s theorem it follows that there exists \(\varepsilon_n \to 0\) and \(x, k \in C([0, T]; \mathbb{R}^d)\) such that
\[
x_{\varepsilon_n} \to x \quad \text{and} \quad k_{\varepsilon_n} \to k \quad \text{in} \ C([0, T]; \mathbb{R}^d).
\]
Moreover, since \(\downarrow \uparrow : C([0, T]; \mathbb{R}^d) \to \mathbb{R}\) is a lower semicontinuous function, then
\[
\downarrow k_{\varepsilon_n} \uparrow T \leq \liminf_{n \to +\infty} \downarrow k_{\varepsilon_n} \uparrow T \leq C_{T,m}.
\]
By Helly-Bray theorem, we can pass to the limit and we have, for all $0 \leq s \leq t \leq T$,

$$\lim_{n \to \infty} \int_s^t \langle y(r) - x_{\varepsilon_n}(r), dk_{\varepsilon_n}(r) \rangle = \int_s^t \langle y(r) - x(r), dk(r) \rangle$$

Passing now to $\liminf_{n \to +\infty}$ in (21) we infer $dk(r) \in \partial \varphi(x(r)) (dr)$. Finally, taking $\lim_{n \to \infty}$ in the approximating equation we obtain that $(x, k)$ is a solution of the equation (14). The proof is now complete.

In the next step we will show in which additional conditions the equation (12) admits a unique solution.

**Proposition 3.** Let the assumptions (6), (5), (7), (9) and (13) be satisfied. Assume also that there exists $\mu \in L^1_{\text{loc}}(\mathbb{R}^+; \mathbb{R}^d)$ such, that for all $x, y \in \mathbb{R}^d$,

$$|f(t, x) - f(t, y)| \leq \mu(t)|x - y|, \quad a.e. \ t \geq 0. \quad (22)$$

If $m \in BV_{\text{loc}}(\mathbb{R}^+; \mathbb{R}^d)$, then the generalized convex Skorohod problem with oblique subgradients (4) admits a unique solution $(x, k)$ in the space $C(\mathbb{R}^+; \mathbb{R}^d)$ such, that $f(x, k)$ is a solution of the equation (14). The proof is now complete.

**Proof.** The existence was proved in Theorem 2. Let us prove the inequality (23) which clearly yields the uniqueness.

Consider the symmetric and strict positive matrix $Q(r) = [H(x(r))]^{-1} + [H(\dot{x}(r))]^{-1}$. Remark that

$$Q(r) \left[ H(\dot{x}(r)) dk(r) - H(x(r)) dk(r) \right] = \left( [H(x(r))]^{-1} - [H(\dot{x}(r))]^{-1} \right) \left[ H(\dot{x}(r)) dk(r) + H(x(r)) dk(r) \right] + 2 \left[ dk(r) - dk(r) \right].$$

Let $u(r) = Q^{1/2}(r) (x(r) - \dot{x}(r))$. Then

$$du(r) = \left[ dQ^{1/2}(r) \right] (x(r) - \dot{x}(r)) + Q^{1/2}(r) d[x(r) - \dot{x}(r)]$$

$$= [\alpha(r) dx(r) + \dot{\alpha}(r) d\dot{x}(r)] (x(r) - \dot{x}(r))$$

$$+ Q^{1/2}(r) [f(r, x(r)) - f(r, \dot{x}(r))] dr$$

$$+ Q^{1/2}(r) [dm(t) - d\dot{m}(t)]$$

$$+ Q^{1/2}(r) \left[ -H(x(r)) dk(r) + H(\dot{x}(r)) d\dot{k}(r) \right].$$
with $\alpha, \hat{\alpha} \in \mathcal{L}(\mathbb{R}_+; \mathbb{R}^{d \times d})$, where $\mathcal{L}(\mathbb{R}_+; \mathbb{R}^{d \times d})$ is the space of continuous linear operators from $\mathbb{R}_+$ into $\mathbb{R}^{d \times d}$.

Using (24) and the assumptions on the matrix-valued functions $x \mapsto H(x)$ and $x \mapsto [H(x)]^{-1}$, we have (as signed measures on $\mathbb{R}_+$), for some positive constants $C_1, C_2, C_3, C$ depending only on the constants $c$ and $b$,

\[
\langle u(r), du(r) \rangle \leq C_1 |u(r)|^2 (d \hat{\|} x \hat{\|}_r + d \hat{\|} \hat{x} \hat{\|}_r) + \frac{C_2\mu(r)}{u(r)} |u(r)|^2 dr
+ C_3 |u(r)| d \hat{\|}m - \hat{m} \hat{\|}_r
+ \left\langle x(r) - \hat{x}(r), Q(r) \left[ H(\hat{x}(r)) \hat{d}k(r) - H(x(r)) dk(r) \right] \right\rangle
\leq C |u(r)| d \hat{\|}m - \hat{m} \hat{\|}_r + C |u(r)|^2 dV(r),
\]

with $V(t) = \hat{\|}x\hat{\|}_t + \hat{\|} \hat{x} \hat{\|}_t + \hat{\|} \hat{k} \hat{\|}_t + \int_0^t \mu(r) dr$. Now, by (16), we infer, for all $t \geq 0$,

\[
|u(t)| \leq e^{CV(t)} |x_0 - \hat{x}_0| + \int_0^t C e^{C[V(t)-V(r)]} d \hat{\|}m - \hat{m} \hat{\|}_r.
\]

and the inequality (23) follows. \hfill \Box

**Proposition 4.** Under the assumptions of Proposition 3 and, for $m \in C^1(\mathbb{R}_+; \mathbb{R}^d)$, the solution $(x_\varepsilon)_0 < \varepsilon \leq 1$ of the approximating equation

\[
x_\varepsilon(t) + \int_0^t H(x_\varepsilon(s)) dk_\varepsilon(s) = x_0 + \int_0^t f(s, \pi_D(x_\varepsilon(s))) ds + m(t), \quad t \geq 0,
\]

\[
dk_\varepsilon(s) = \nabla \varphi_\varepsilon(x_\varepsilon(s)) ds,
\]

has the following properties:

- for all $T > 0$ there exists a constant $C_T$, independent of $\varepsilon, \delta \in [0, 1]$, such that

\[
(j) \quad \sup_{t \in [0, T]} |x_\varepsilon(t)|^2 + \sup_{t \in [0, T]} |\varphi_\varepsilon(x_\varepsilon(t))| + \int_0^T |\nabla \varphi_\varepsilon(x_\varepsilon(s))|^2 ds \leq C_T,
\]

\[
(jj) \quad \hat{\|}x_\varepsilon\hat{\|}[s, t] \leq C_T \sqrt{t - s}, \quad \text{for all } 0 \leq s \leq t \leq T,
\]

\[
(jjj) \quad \|x_\varepsilon - x_\delta\|_T \leq C_T \sqrt{\varepsilon + \delta}.
\]

- Moreover, there exist $x, k \in C([0, T]; \mathbb{R}^d)$ and $h \in L^2(0, T; \mathbb{R}^d)$, such that

\[
\lim_{\varepsilon \to 0} k_\varepsilon(t) = k(t) = \int_0^t h(s) ds, \quad \text{for all } t \in [0, T],
\]

\[
\lim_{\varepsilon \to 0} \|x_\varepsilon - x\|_T = 0
\]

and $(x, k)$ is the unique solution of the variational inequality with oblique subgradients [14].
Proof. The proof for the estimates \((j)\) and \((jj)\) are exactly as in the proof of Theorem 2.

Let us prove \((jjj)\). Similarly to the proof of the uniqueness result (Proposition 4), we introduce \(Q_{\varepsilon,\delta}(s) = [H(x_\varepsilon(s))]^{-1} + [H(x_\delta(s))]^{-1}\). Once again, to simplify the reading, we omit \(s\) in the argument of \(x_\varepsilon(s)\) and \(x_\delta(s)\). Remark that

\[
Q_{\varepsilon,\delta}(s)[H(x_\delta) \nabla \varphi_\delta(x_\delta) - H(x_\varepsilon) \nabla \varphi_\varepsilon(x_\varepsilon)]
\]

\[
= \left([H(x_\varepsilon)]^{-1} - [H(x_\delta)]^{-1}\right)[H(x_\delta) \nabla \varphi_\delta(x_\delta) + H(x_\varepsilon) \nabla \varphi_\varepsilon(x_\varepsilon)]
\]

\[
+ 2[dk_\delta(s) - dk_\varepsilon(s)].
\]

Let \(u_{\varepsilon,\delta}(s) = Q_{\varepsilon,\delta}^{1/2}(s)(x_\varepsilon(s) - x_\delta(s))\). Then

\[
du_{\varepsilon,\delta}(s) = \left[dQ_{\varepsilon,\delta}^{1/2}(s)\right](x_\varepsilon - x_\delta) + Q_{\varepsilon,\delta}^{1/2}(s)d[x_\varepsilon - x_\delta]
\]

\[
= \left[\alpha_{\varepsilon,\delta}(s)dx_\varepsilon + \beta_{\varepsilon,\delta}(s)dx_\delta\right](x_\varepsilon - x_\delta)
\]

\[
+ Q_{\varepsilon,\delta}^{1/2}(s)[f(s,\pi_D(x_\varepsilon)) - f(s,\pi_D(x_\delta))]ds
\]

\[
+ Q_{\varepsilon,\delta}^{1/2}(s)[-H(x_\varepsilon) \nabla \varphi_\varepsilon(x_\varepsilon) + H(x_\delta) \nabla \varphi_\delta(x_\delta)]ds,
\]

where \(\alpha_{\varepsilon,\delta}, \beta_{\varepsilon,\delta} : \mathbb{R}_+ \to \mathbb{R}^{d \times d}\) are some continuous functions which are bounded uniformly in \(\varepsilon, \delta\).

Therefore, for \(s \in [0,T]\),

\[
\langle u_{\varepsilon,\delta}(s), du_{\varepsilon,\delta}(s) \rangle \leq C|u_{\varepsilon,\delta}(s)|^2(\langle d\, x_{\varepsilon}\rangle + d\, x_{\delta}\rangle) + C\mu(s)|u_{\varepsilon,\delta}(s)|^2ds
\]

\[
+ 2\langle x_\varepsilon - x_\delta, Q_{\varepsilon,\delta}(s)[H(x_\delta) \nabla \varphi_\delta(x_\delta) - H(x_\varepsilon) \nabla \varphi_\varepsilon(x_\varepsilon)]\rangle ds
\]

\[
\leq C|u_{\varepsilon,\delta}(s)|^2dV(s) + 4\langle x_\varepsilon - x_\delta, \nabla \varphi_\delta(x_\delta) - \nabla \varphi_\varepsilon(x_\varepsilon)\rangle ds,
\]

with \(V(s) = \langle x_{\varepsilon}\rangle + \langle x_{\delta}\rangle + \langle k_{\varepsilon}\rangle + \langle k_{\delta}\rangle + \int_0^s \mu(r)dr \leq CT\).

Since, according to Asiminoaei & Răşcanu [3],

\[
\langle \nabla \varphi_\varepsilon(x) - \nabla \varphi_\delta(y), x - y \rangle \geq - (\varepsilon + \delta) |\nabla \varphi_\varepsilon(x)||\nabla \varphi_\delta(y)|,
\]

we have

\[
\langle x_\varepsilon(r) - x_\delta(r), dk_\delta(r) - dk_\varepsilon(r) \rangle = \langle x_\varepsilon(r) - x_\delta(r), \nabla \varphi(x_\delta(r)) - \nabla \varphi(x_\varepsilon(r)) \rangle dr
\]

\[
\leq (\varepsilon + \delta)|\nabla \varphi(x_\delta(r))||\nabla \varphi(x_\varepsilon(r))| dr.
\]

Consequently,

\[
\langle u_{\varepsilon,\delta}(r), du_{\varepsilon,\delta}(r) \rangle \leq 4(\varepsilon + \delta)|\nabla \varphi(x_\delta(r))||\nabla \varphi(x_\varepsilon(r))|dr + C|u_{\varepsilon,\delta}(r)|^2dV(r),
\]

Using inequality (45) from Annex 4.3. we deduce that there exists some positive constants,
that will be denoted by a generic one $C$, such that

$$\|x_\varepsilon - x_\delta\|_T \leq C \|u_\varepsilon,\delta\|_T$$

\[
\leq C\sqrt{\varepsilon + \delta} \left( \int_0^T |\nabla \varphi(x_\delta(r))| |\nabla \varphi(x_\varepsilon(r))| \, dr \right)^{1/2}
\]

\[
\leq C\sqrt{\varepsilon + \delta} \left[ \left( \int_0^T |\nabla \varphi(x_\delta(r))|^2 \, dr \right)^{1/2} + \left( \int_0^T |\nabla \varphi(x_\varepsilon(r))|^2 \, dr \right)^{1/2} \right]
\]

\[
\leq C\sqrt{\varepsilon + \delta}.
\]

Now, the other assertions clearly follows and the proof is complete. □

**Corollary 5.** If $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$ is a stochastic basis and $M$ a $\mathcal{F}_t$—progressively measurable stochastic process such that $M(\omega) \in C^1(\mathbb{R}_+; \mathbb{R}^d)$, $\mathbb{P}$—a.s. $\omega \in \Omega$, then, under the assumptions of Proposition 3, $\mathbb{P}$—a.s. $\omega \in \Omega$, the random generalized Skorohod problem with oblique subgradients:

\[
\begin{cases}
X_t(\omega) + \int_0^t H(X_s(\omega)) \, dK_t(\omega) = x_0 + \int_0^t f(s, X_s(\omega)) \, ds + M_t(\omega), & t \geq 0, \\
dK_t(\omega) \in \partial \varphi(X_t(\omega)) \,(dt)
\end{cases}
\]

admits a unique solution $(X(\omega), K(\omega))$. Moreover $X$ and $K$ are $\mathcal{F}_t$—progressively measurable stochastic processes.

**Proof.** In this moment we have to prove that $X$ and $K$ are $\mathcal{F}_t$—progressively measurable stochastic processes. But this follows from Proposition 4, since the approximating equation (26) admits a unique solution $(X^\varepsilon, K^\varepsilon)$, which is a progressively measurable continuous stochastic process. □

### 3. SVI with oblique subgradients

#### 3.1. Notations. Hypotheses

In this section we will present the Stochastic Variational Inequalities (for short, SVI) with oblique subgradient and the definition of theirs strong and weak solutions. The proof of the existence and uniqueness results are given in the next subsection.

Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$ be a stochastic basis and $\{B_t : t \geq 0\}$ a $\mathbb{R}^k$—valued Brownian motion. Our objective is to solve the SVI with oblique reflection

\[
\begin{cases}
X_t + \int_0^t H(X_s) \, dK_t = x_0 + \int_0^t f(s, X_s) \, ds + \int_0^t g(s, X_s) \, dB_s, & t \geq 0, \\
dK_t \in \partial \varphi(X_t)(dt),
\end{cases}
\]  
(27)
where \( x_0 \in \mathbb{R}^d \) and

\[
\begin{align*}
(i) & \quad (t, x) \mapsto f (t, x) : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d \text{ and } (t, x) \mapsto g (t, x) : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d \times k} \text{ are } \\
\text{Carathéodory functions (i.e. measurable w.r. to } t \text{ and continuous w.r. to } x), \quad \\
(ii) & \quad \int_0^T (f^\# (t))^2 \, dt + \int_0^T (g^\# (t))^4 \, dt < \infty,
\end{align*}
\]

with

\[
f^\# (t) \overset{\text{def}}{=} \sup_{x \in \text{Dom}(\varphi)} \left| f (t, x) \right| \quad \text{and} \quad g^\# (t) \overset{\text{def}}{=} \sup_{x \in \text{Dom}(\varphi)} \left| g (t, x) \right|.
\]

We also add Lipschitz continuity conditions:

\[
\exists \mu \in L_1^1 (\mathbb{R}_+), \exists \ell \in L_2^2 (\mathbb{R}_+) \text{ s.t. } \forall x, y \in \mathbb{R}^d, \text{ a.e. } t \geq 0,
\]

\[
(i) \quad | f (t, x) - f (t, y) | \leq \mu (t) \left| x - y \right|,
\]

\[
(ii) \quad | g (t, x) - g (t, y) | \leq \ell (t) \left| x - y \right|.
\]

**Definition 3.** (I) Given a stochastic basis \( (\Omega, \mathcal{F}, \mathbb{P}, \{ \mathcal{F}_t \}_{t \geq 0}) \) and a \( \mathbb{R}^k \)-valued \( \mathcal{F}_t \)-Brownian motion \( \{ B_t : t \geq 0 \} \), a pair \( (X, K) : \Omega \times [0, \infty) \to \mathbb{R}^d \times \mathbb{R}^d \) of continuous \( \mathcal{F}_t \)-progressively measurable stochastic processes is a strong solution of the SDE \( \text{(27)} \) if, \( \mathbb{P} \) – a.s. \( \omega \in \Omega : \)

\[
\begin{align*}
i) & \quad X_t \in \text{Dom}(\varphi), \forall t \geq 0, \varphi (X_t) \in L_1^1 (\mathbb{R}_+), \\
ii) & \quad K_t \in BV_{\text{loc}} ([0, \infty[; \mathbb{R}^d), \quad K_0 = 0, \\
iii) & \quad X_t + \int_0^t H (X_s) \, dK_s = x_0 + \int_0^t f (s, X_s) \, ds + \int_0^t g (s, X_s) \, dB_s, \forall t \geq 0, \\
iv) & \quad \forall 0 \leq s \leq t, \forall y : \mathbb{R}_+ \to \mathbb{R}^d \text{ continuous:} \\
& \quad \int_s^t (y (r) - X_r, dK_r) + \int_s^t \varphi (X_r) \, dr \leq \int_s^t \varphi (y (r)) \, dr.
\end{align*}
\]

That is

\[
(X. (\omega), K. (\omega)) \in \mathcal{SP} (H \varphi; x_0, M. (\omega)), \quad \mathbb{P} \text{ – a.s. } \omega \in \Omega,
\]

with

\[
M_t = \int_0^t f (s, X_s) \, ds + \int_0^t g (s, X_s) \, dB_s .
\]

(II) If there exists a stochastic basis \( (\Omega, \mathcal{F}, \mathbb{P}, \{ \mathcal{F}_t \}_{t \geq 0}) \), a \( \mathbb{R}^k \)-valued \( \mathcal{F}_t \)-Brownian motion \( \{ B_t : t \geq 0 \} \) and a pair \( (X, K) : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d \times \mathbb{R}^d \) of \( \mathcal{F}_t \)-progressively measurable continuous stochastic processes such that

\[
(X. (\omega), K. (\omega)) \in \mathcal{SP} (H \varphi; x_0, M. (\omega)), \quad \mathbb{P} \text{ – a.s. } \omega \in \Omega,
\]

then the collection \( (\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t, B_t, X_t, K_t)_{t \geq 0} \) is called a weak solution of the SVI \( \text{(27)} \).

(In both cases (I) and (II) we will say that \( (X_t, K_t) \) is a solution of the oblique reflected SVI \( \text{(27)} \).)
3.2. Existence and uniqueness

In this section we will give the result of existence and uniqueness of the solution for the stochastic variational inequality with oblique subgradients introduced before. Theorem 6 deals with the existence of a weak solution in the sense of Definition 3, while Theorem 7 proves the uniqueness of a strong solution.

**Theorem 6.** Let the assumptions (6), (7), (9) and (28) be satisfied. Then the SVI (27) has at least one weak solution \((Ω, F, P, F_t, B_t, X_t, K_t)_{t \geq 0}\).

**Proof.** The main ideas of the proof come from Rascanu [14]. We extend \(f(t, x) = 0\) and \(g(t, x) = 0\), for \(t < 0\).

**Step 1.** Approximating problem.

Let \(0 < \varepsilon \leq 1\) and consider the approximating equation

\[
\begin{aligned}
X^n_t &= x_0, \quad \text{if } t < 0, \\
X^n_t + \int_0^t H(X^n_t) dK^n_t &= x_0 + M^n_t, \quad t \geq 0, \\
dK^n_t &\in \partial \varphi(X^n_t) dt,
\end{aligned}
\]

where

\[
M^n_t = \int_0^t f(s, \pi_D(X^n_{s-1/n})) ds + n \int_{t-1/n}^t \left[ \int_0^s g(r, \pi_D(X^n_{r-1/n})) dB_r \right] ds
\]

\[
= \int_0^t f(s, \pi_D(X^n_{s-1/n})) ds + \int_0^1 \left[ \int_0^{t-1/n+1/n} g(r, \pi_D(X^n_{r-1/n})) dB_r \right] du
\]

and \(\pi_D(x)\) is the orthogonal projection of \(x\) on \(D = \overline{\text{Dom} (\varphi)}\). Since \(M^n\) is a \(C^1\)-continuous progressively measurable stochastic process, then by Corollary 1, the approximating equation (31) has a unique solution \((X^n, K^n)\) of continuous progressively measurable stochastic processes.

**Step 2.** Tightness.

Let \(T \geq 0\) be arbitrary fixed. We will point out the main reasonings of this step.
• Since, by standard arguments,

\[
\mathbb{E} \left[ \sup_{0 \leq \theta \leq \varepsilon} |M^n_{t+\theta} - M^n_t|^4 \right] \\
\leq 8 \left( \int_t^{t+\varepsilon} f^\#(r) \, dr \right)^4 + 8 \int_0^1 \mathbb{E} \left[ \sup_{0 \leq \theta \leq \varepsilon} \left( \int_{t-\frac{1}{n}+\frac{1}{n}u}^{t+\theta-\frac{1}{n}+\frac{1}{n}u} \sigma(r, \pi D(X^n_{r-1/n})) \, dB_r \right)^4 \right] \, du
\]

\[
\leq 8\varepsilon \left( \int_t^{t+\varepsilon} |f^\#(r)|^2 \, dr \right)^2 + C \varepsilon \int_0^1 \left( \int_{t-\frac{1}{n}+\frac{1}{n}u}^{t+\theta-\frac{1}{n}+\frac{1}{n}u} |g^\#(r)|^2 \, dr \right)^2 \, du
\]

\[
\leq 8\varepsilon \left( \int_t^{t+\varepsilon} |f^\#(r)|^2 \, dr \right)^2 + C\varepsilon \int_0^1 \left( \int_{t-\frac{1}{n}+\frac{1}{n}u}^{t+\theta-\frac{1}{n}+\frac{1}{n}u} |g^\#(r)|^4 \, dr \right) \, du
\]

\[
\leq C'\varepsilon \times \sup \left\{ \left( \int_s^\tau |f^\#(r)|^2 \, dr \right)^2 + \int_s^\tau |g^\#(r)|^4 \, dr ; 0 \leq s < \tau \leq T, \tau - s \leq \varepsilon \right\}
\]

in conformity with Proposition 16 the family of laws of \( \{ M^n : n \geq 1 \} \) is tight on \( C ([0, T] ; \mathbb{R}^d) \).

• We now show that the family of laws of the random variables \( U^n = (X^n, K^n, \uparrow K^n, \downarrow) \) is tight on \( C ([0, T] ; \mathbb{R}^d) \times C ([0, T] ; \mathbb{R}) \times C ([0, T] ; \mathbb{R}) \times C ([0, T] ; \mathbb{R}^{2d+1}) \). From Proposition 1 we deduce

\[
\| U^n \|_T \leq C_T (\| M^n \|_T),
\]

\[
m_{U^n} (\varepsilon) \leq C_T (\| M^n \|_T) \times \sqrt{\varepsilon + m_{M^n} (\varepsilon)},
\]

and, from Lemma 19 it follows that \( \{ U^n ; n \in \mathbb{N}^* \} \) is tight on \( C ([0, T] ; \mathbb{R}^{2d+1}) \).

• By the Prohorov theorem there exists a subsequence such that, as \( n \to \infty \),

\( (X^n, K^n, \uparrow K^n, \downarrow, B) \to (X, K, V, B) \), in law

on \( C ([0, T] ; \mathbb{R}^{2d+1+k}) \) and, by the Skorohod theorem, we can choose a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) and some random quadruples \( (\hat{X}^n, \hat{K}^n, \hat{V}^n, \hat{B}^n) \), \( (\bar{X}, \bar{K}, \bar{V}, \bar{B}) \) defined on \( (\Omega, \mathcal{F}, \mathbb{P}) \), having the same laws as resp. \( (X^n, K^n, \uparrow K^n, \downarrow, B) \) and \( (X, K, V, B) \), such that, in \( C ([0, T] ; \mathbb{R}^{2d+1+k}) \), as \( n \to \infty \),

\( (\hat{X}^n, \hat{K}^n, \hat{V}^n, \hat{B}^n) \overset{P}{\to} a.s. (\bar{X}, \bar{K}, \bar{V}, \bar{B}) \).

• Remark that, by Lemma 20 \( (\bar{B}^n, \{F_t^{\hat{X}^n, \hat{K}^n, \hat{V}^n, \hat{B}^n} \}, n \geq 1 \), and \( (\bar{B}, \{F_t^{\bar{X}, \bar{K}, \bar{V}, \bar{B}} \}) \) are \( \mathbb{R}^k \)-Brownian motion.

\textit{Step 3. Passing to the limit.}
Since we have \((X^n, K^n, \uparrow K^n \downarrow, B) \to (\bar{X}, \bar{K}, \bar{V}, \bar{B})\) in law, then by Proposition 17 we deduce that, for all \(0 \leq s \leq t, P - a.s.,\)

\[\begin{align*}
\bar{X}_0 &= x_0, \quad \bar{K}_0 = 0, \quad \bar{X}_t \in E, \\
\uparrow \bar{K}_t - \uparrow \bar{K}_s &\leq \bar{V}_t - \bar{V}_s \quad \text{and} \quad 0 = \bar{V}_0 \leq \bar{V}_s \leq \bar{V}_t.
\end{align*}\] (32)

Moreover, since for all \(0 \leq s < t, n \in \mathbb{N}^*\)

\[
\int_s^t \varphi (X^n_r) \, dr \leq \int_s^t \varphi (y (r)) \, dr - \int_s^t \langle y (r) - X^n_r, dK^n_r \rangle \quad a.s.,
\]

then, by Proposition 17, we infer

\[
\int_s^t \varphi (\bar{X}_r) \, dr \leq \int_s^t \varphi (y (r)) \, dr - \int_s^t \langle y (r) - \bar{X}_r, d\bar{K}_r \rangle.
\] (33)

Hence, based on (32) and (33), we have

\[d\bar{K}_r \in \partial \varphi (\bar{X}_r) (dr)\]

Using the Lebesgue theorem and, once again Lemma 20, we infer for \(n \to \infty,\)

\[
\bar{M}^n = x_0 + \int_0^t f(s, \pi_D (\bar{X}^{n}_{s-1/n})) ds + n \int_{-1/n}^0 \left[ \int_0^s g(r, \pi_D (\bar{X}^{n}_{r-1/n})) dB_r \right] ds \\
\to \bar{M} = x_0 + \int_0^t f(s, \bar{X}_s) ds + \int_0^t g(s, \bar{X}_s) dB_s, \quad \text{in} \quad S_0^d [0, T],
\]

where \(S_0^d [0, T]\) is the space of progressively measurable continuous stochastic processes defined in Annex, Section 4.3.

By Proposition 18 it follows that the probability laws equality holds

\[\mathcal{L} (\bar{X}^n, \bar{K}^n, \bar{B}^n, \bar{M}^n) = \mathcal{L} (X^n, K^n, B^n, M^n) \quad \text{on} \quad C (\mathbb{R}_+; \mathbb{R}^{d+d+k+d}),\]

where by \(\mathcal{L}(\cdot)\) we mean the probability law of the random variable.

Since, for every \(t \geq 0,\)

\[X^n_t + \int_0^t H (X^n_r) dK^n_r - M^n_t = 0, \quad a.s.,\]

then, by Proposition 17 we have

\[\bar{X}_t^n + \int_0^t H (\bar{X}_s^n) d\bar{K}_s^n - \bar{M}_t^n = 0, \quad a.s.\]

Letting \(n \to \infty,\)

\[\bar{X}_t + \int_0^t H (\bar{X}_s) d\bar{K}_s - \bar{M}_t = 0, \quad a.s.,\]
that is, \( \mathbb{P} - \text{a.s.} \),

\[
X_t + \int_0^t H \left( \bar{X}_s \right) d\bar{K}_s = x_0 + \int_0^t f \left( s, \bar{X}_s \right) ds + \int_0^t g \left( s, \bar{X}_s \right) d\bar{B}_s, \quad \forall \ t \in [0, T].
\]

Consequently \((\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}, \mathcal{F}_t, \bar{X}_t, \bar{K}_t, \bar{B}_t)_{t \geq 0}\) is a weak solution of the SVI (27). The proof is complete.

**Theorem 7.** If the assumptions (6), (7), (9), (28) and (29) are satisfied, then the SVI (27) has a unique strong solution \((X, K) \in S^0_d \times S^0_d\).

**Proof.** It is sufficient to prove the pathwise uniqueness, since by Theorem 1.1, page 149, from Ikeda & Watanabe [9] the existence of a weak solution and the pathwise uniqueness implies the existence of a strong solution.

Let \((X, K), (\hat{X}, \hat{K}) \in S^0_d \times S^0_d\) two solutions of the SVI with oblique reflection (27). Consider the symmetric and strict positive matrix

\[
Q_r = H^{-1}(X_r) + H^{-1}(\hat{X}_r).
\]

We have that

\[
dQ_r^{1/2} = dN_r + \sum_{j=1}^k \beta_r^{(j)} dB_r^{(j)},
\]

where \(N\) is a \(\mathbb{R}^{d \times d}\)–valued \(\mathcal{F}\)–measurable bounded variation continuous stochastic process (for short, m.b.v.c.s.p.), \(N_0 = 0\) and, for each \(j \in \{1, \ldots, k\}\), \(\beta_r^{(j)}\) is a \(\mathbb{R}^{d \times d}\)–valued \(\mathcal{F}\)–measurable stochastic process (for short, m.s.p.) such that \(\int_0^T |\beta_r^{(j)}|^2 dr < \infty, \ \text{a.s.}, \ \forall T > 0\).

Letting

\[
U_r = Q_r^{1/2} (X_r - \hat{X}_r),
\]

then

\[
dU_r = \left[ dQ_r^{1/2} \right] (X_r - \hat{X}_r) + Q_r^{1/2} d(X_r - \hat{X}_r) + \sum_{j=1}^k \beta_r^{(j)} (g(r, X_r) - g(r, \hat{X}_r)) e_j
\]

\[
= dK_r + \mathcal{G}_r dB_r,
\]

where

\[
dK_r = (dN_r) Q_r^{-1/2} U_r + Q_r^{1/2} \left[ H(\hat{X}_r) d\hat{K}_r - H (X_r) dK_r \right]
\]

\[
+ Q_r^{1/2} \left[ f (r, X_r) - f (r, \hat{X}_r) \right] dr + \sum_{j=1}^k \beta_r^{(j)} (g(r, X_r) - g(r, \hat{X}_r)) e_j,
\]

\[
\mathcal{G}_r = \Gamma_r + Q_r^{1/2} \left[ g(r, X_r) - g(r, \hat{X}_r) \right],
\]

and \(\Gamma_r\) is a \(\mathbb{R}^{d \times k}\) matrix with the columns \(\beta_r^{(1)} (X_r - \hat{X}_r), \ldots \), \(\beta_r^{(k)} (X_r - \hat{X}_r)\).
Using (24) and the properties of $H$ and $H^{-1}$, we have
\[
\langle U_r, Q_r^{1/2} \left[ H(\hat{X}_r) d\hat{K}_r - H(X_r) dK_r \right] \rangle
= \langle X_r - \hat{X}_r, \left( [H(X_r)]^{-1} - [H(\hat{X}_r)]^{-1} \right) [H(\hat{X}_r) d\hat{K}_r + H(X_r) dK_r] \rangle \\
- 2 \langle X_r - \hat{X}_r, dK_r - d\hat{K}_r \rangle
\leq bc |X_r - \hat{X}_r|^2 (d\downarrow K_r + d\downarrow \hat{K}_r).
\]
Hence, there exists a positive constant $C = C(b, c, r_0)$ such that
\[
\langle U_r, dK_r \rangle + \frac{1}{2} |G_r|^2 dt \leq |U_r|^2 dV_r,
\]
where
\[
dV_r = C \times \left( \mu(r) dr + \ell^2(r) dr + d\downarrow N_r^\uparrow + d\uparrow K_r^\downarrow + d\uparrow \hat{K}_r^\downarrow \right) + C \sum_{j=1}^k |\beta_j(r)|^2 dr.
\]
By Proposition 15 we infer
\[
\mathbb{E} \frac{e^{-2V_s} |U_s|^2}{1 + e^{-2V_s} |U_0|^2} \leq \mathbb{E} \frac{e^{-2V_0} |U_0|^2}{1 + e^{-2V_0} |U_0|^2} = 0.
\]
Consequently,
\[
Q_s^{1/2}(X_s - \hat{X}_s) = U_s = 0, \ P - a.s., \ for \ all \ s \geq 0
\]
and, by the continuity of $X$ and $\hat{X}$, we conclude that, $\mathbb{P} - a.s.,$
\[
X_s = \hat{X}_s \ \ for \ all \ s \geq 0.
\]

4. Annex

For the clarity of the proofs from the main body of this article we will group in this section some useful results that are used along this paper.

4.1. A priori estimates

We give five lemmas with a priori estimates of the solutions $(x, k) \in SP(H\partial\varphi; x_0, m)$. These lemmas and also theirs proofs are similar with those from the monograph of Pardoux & Răşcanu [12], but for the convenience of the reader we give here the proofs of the results in this new framework.

**Lemma 8.** If $(x, k) \in SP(H\partial\varphi; x_0, m)$ and $(\hat{x}, \hat{k}) \in SP(H\partial\varphi; \hat{x}_0, \hat{m})$, then for all $0 \leq s \leq t$:
\[
\int_s^t \left\langle x(r) - \hat{x}(r), dk(r) - d\hat{k}(r) \right\rangle \geq 0. \tag{34}
\]
Lemma 9. Let the assumptions (5), (6), (7) and (9) be satisfied. If $0 < s \leq t \leq T$,

$$m_t(t - s) \leq [(t - s) + m_m(t - s) + \sqrt{m_m(t - s)(\int k_{rs}^t - \int k_{rs}^t)}] \times \exp \{C[1 + (t - s) + (\int k_{rs}^t - \int k_{rs}^t) + (\int k_{rs}^t - \int k_{rs}^t)]\},$$

where $C = C(b,c,L) > 0$.

Proof. Let $0 \leq s \leq t$ and

$$h(t) = \langle H^{-1}(x(t)) [x(t) - m(t) - x(s) + m(s)], x(t) - m(t) - x(s) + m(s) \rangle.$$

We have

$$h(t) = 2 \int_s^t \langle H^{-1}(x(t)) [x(r) - m(r) - x(s) + m(s)], d[x(r) - m(r) - x(s) + m(s)] \rangle$$

$$= -2 \int_s^t \langle H^{-1}(x(t)) [x(r) - m(r) - x(s) + m(s)], H(x(r)) dk(r) \rangle$$

$$= 2 \int_s^t \langle H^{-1}(x(t)) [m(r) - m(s)], H(x(r)) dk(r) \rangle + 2 \int_s^t \langle x(s) - x(r), dk(r) \rangle$$

$$+ 2 \int_s^t \langle [H^{-1}(x(r)) - H^{-1}(x(t))] [x(r) - x(s)], H(x(r)) dk(r) \rangle.$$

Since

$$\int_s^t \langle x(s) - x(r), dk(r) \rangle \leq \int_s^t [\varphi(x(s)) - \varphi(x(r))] dr$$

$$\leq L(t - s) + L \int_s^t |x(s) - x(r)| dr$$

$$\leq L(t - s) + \frac{L}{2} (t - s) + \frac{L}{2} \int_s^t |x(r) - x(s)|^2 dr$$

and

$$\frac{1}{2c} |x(t) - x(s)|^2 - \frac{1}{c} |m(t) - m(s)|^2 \leq h(t),$$

then

$$|x(t) - x(s)|^2 \leq 2 m_m^2(t - s) + 4c^3 m_m(t - s) (\int k_{rs}^t - \int k_{rs}^t) + 6cL(t - s)$$

$$+ \int_s^t [2cL |x(r) - x(s)|^2 dr + 4bc^2 |x(r) - x(t)||x(r) - x(s)| d\int k_{rs}^t].$$
Here we continue the estimates by

\[ 4bc^2 \int_s^t |x(r) - x(t)| |x(r) - x(s)| \, d\xi_r s \]
\[ \leq 4bc^2 |x(s) - x(t)| \int_s^t |x(r) - x(s)| \, d\xi_r s + 4bc^2 \int_s^t |x(r) - x(s)|^2 \, d\xi_r s \]
\[ \leq \frac{1}{2} |x(s) - x(t)|^2 + \frac{1}{2} (4bc^2)^2 \left( \int_s^t |x(r) - x(s)| \, d\xi_r s \right)^2 \]
\[ + 4bc^2 \int_s^t |x(r) - x(s)|^2 \, d\xi_r s \]

and we obtain

\[ |x(t) - x(s)|^2 \]
\[ \leq 4 m_m^2 (t - s) + 8c^3 m_m (t - s) (\xi_k - \xi_{ks}) + 12cL (t - s) \]
\[ + 4cL \int_s^t |x(r) - x(s)|^2 \, dr + \left[ 16b^2 c^4 (\xi_k - \xi_{ks}) + 8bc^2 \right] \int_s^t |x(r) - x(s)|^2 \, d\xi_r s . \]

By the Stieltjes-Gronwall inequality, from this last inequality, the estimate (35) follows. □

For the next result we first remark that, if \( E \subset \mathbb{R}^d \) is a closed convex set such that

\[ \exists r_0 > 0, \ E_{r_0} \neq \emptyset \quad \text{and} \quad h_0 = \sup_{z \in E} \text{dist}(z, E_{r_0}) < \infty \]

(in particular if \( E \) is bounded), then for every \( 0 < \delta \leq \frac{r_0}{2 (1 + h_0)} \), \( y \in E \), \( \hat{y} = \pi_{E_{r_0}}(y) \)

\[ v_y = \frac{1}{1 + h_0} (\hat{y} - y) \]

and for all \( x \in E \cap \bar{B}(y, \delta) \) we have

\[ \overline{B}(x + v_y, \delta) \subset \overline{B}(y + v_y, r_0) \subset \text{conv} \{ y, \overline{B}(\hat{y}, r_0) \} \subset E. \]

\[ (36) \]

**Lemma 10.** Let the assumptions \( (3), (9), (7) \) and \( (2) \) be satisfied. If \( (x, k) \in \mathcal{SP}(H \partial \varphi; x_0, m) \), \( 0 \leq s \leq t \leq T \) and

\[ \sup_{r \in [s, t]} |x(r) - x(s)| \leq 2\delta_0 = \frac{\rho_0}{2bc} \wedge \rho_0 , \]

with \( \rho_0 = \frac{r_0}{2 (1 + r_0 + h_0)} \),

then

\[ \xi_k - \xi_{ks} \leq \frac{1}{\rho_0} |k(t) - k(s)| + \frac{3L}{\rho_0} (t - s) \]

\[ (37) \]

and

\[ |x(t) - x(s)| + \xi_k - \xi_{ks} \leq \sqrt{t - s + m_m (t - s)} \times e^{C_T (1 + ||m||^2_T)} , \]

\[ (38) \]

where \( C_T = C(b,c,r_0,h_0,L,T) > 0. \)
Proof. Remark first that $D_{r_0} \subset D_{d_0}$. Let $\alpha \in C \left( [0, \infty[; \mathbb{R}^d \right)$, $\|\alpha\|_{[s, t]} \leq 1$, be arbitrary. Consider $y = x(s) \in D$, $\hat{y} = \pi_{D_{r_0}}(y)$ and

$$v_y = \frac{1}{1 + h_0} (\hat{y} - y).$$

Let $z(r) = x(r) + v_y + \rho_0 \alpha(r)$, $r \in [s, t]$. Since $|x(r) - y| \leq 2d_0 \leq \rho_0$, then

$$x(r) + v_y + \rho_0 \alpha(r) \in \overline{B}(x(r) + v_y, \rho_0) \subset \overline{B}(y + v_y, \frac{r_0}{1 + h_0}) \subset D.$$

Remark that

$$|z(r) - x(r)| \leq \frac{h_0}{1 + h_0} + \rho_0 \leq 2$$

and

$$|\varphi(z(r)) - \varphi(x(r))| \leq 3L.$$

Therefore

$$\rho_0 \int_s^t \varphi(z(r)) dr \leq - \int_s^t \varphi(x(r)) dr + \int_s^t |\varphi(z(r)) - \varphi(x(r))| dr \leq - \varphi(x(s)) - \varphi(x(t)) + 3L(t-s).$$

Taking the sup$_{\|\alpha\|_{[s, t]} \leq 1}$, we infer

$$\rho_0 (\hat{k}^t - \hat{k}^s) \leq |k(t) - k(s)| + 3L(t-s),$$

that is 37.

We have also

$$\hat{k}^t - \hat{k}^s = \frac{1}{\rho_0} |k(t) - k(s)| + \frac{3L}{\rho_0} (t-s)$$

$$= \frac{1}{\rho_0} \int_s^t [H^{-1}(x(r)) - H^{-1}(x(s))] H(x(r)) dk(r) + \frac{1}{\rho_0} H^{-1}(x(s)) \int_s^t H(x(r)) dk(r) + \frac{3L}{\rho_0} (t-s)$$

$$\leq \frac{bc}{\rho_0} \int_s^t |x(r) - x(s)| d\hat{k}^r + \frac{c}{\rho_0} \left| -x(t) + x(s) + m(t) - m(s) \right| + \frac{3L}{\rho_0} (t-s)$$

$$\leq \frac{bc}{\rho_0} 2d_0 (\hat{k}^t - \hat{k}^s) + \frac{c}{\rho_0} |x(t) - x(s)| + \frac{c}{\rho_0} m_m(t-s) + \frac{3L}{\rho_0} (t-s)$$

$$\leq \frac{1}{2} (\hat{k}^t - \hat{k}^s) + \frac{c}{\rho_0} |x(t) - x(s)| + \frac{c}{\rho_0} m_m(t-s) + \frac{3L}{\rho_0} (t-s)$$

24
and, consequently,
\[
\uparrow k_{t} - \downarrow k_{s} \leq \frac{2c}{\rho_0} \left| x(t) - x(s) \right| + \frac{2c}{\rho_0} m_m(t-s) + \frac{6L}{\rho_0} (t-s)
\]
\[
\leq \frac{1}{b} + \frac{2c}{\rho_0} m_m(t-s) + \frac{6L}{\rho_0} \quad \text{(39)}
\]
\[
\leq C_1 (1 + \|m\|_T),
\]
with \(C_1 = C_1 (T, b, c, \rho_0, L)\).

Now, plugging this estimate in (35), it clearly follows
\[
m_m(t-s) \leq \left[ (t-s) + m_m(t-s) + \sqrt{m_m(t-s)} \right] \exp \left[ C'(1 + \|m\|^2_T) \right],
\]
with \(C' = C'(b, c, L, r_0, h_0, T)\). Now, this last inequality, used in (39), yields the estimate (38).

**Lemma 11.** Let the assumptions (5), (6), (7) and (9) be satisfied. Let \((x, k) \in SP(H \partial \varphi; x_0, m)\), \(0 \leq s \leq t \leq T\) and \(x(r) \in D_{\delta_0}\) for all \(r \in [s,t]\). Then
\[
\uparrow k_{t} - \downarrow k_{t} \leq L \left( 1 + \frac{2}{\delta_0} \right) (t-s)
\]
and
\[
m_m(t-s) \leq C_T \times [(t-s) + m_m(t-s)],
\]
where \(C_T = C_T (b, c, r_0, h_0, L, T) > 0\).

**Proof.** Let \(y(r) = x(r) + \frac{\delta_0}{2} \alpha(r)\), with \(\alpha \in C \left( \mathbb{R}_+; \mathbb{R}^d \right)\), \(\|\alpha\|_{[s,t]} \leq 1\). Then \(y(r) \in D\) and
\[
\frac{\delta_0}{2} \int_s^t \langle \alpha(r), dk(r) \rangle = \int_s^t \langle y(r) - x(r), dk(r) \rangle
\]
\[
\leq \int_s^t |\varphi(y(r)) - \varphi(x(r))| \, dr
\]
\[
\leq L(t-s) + L \int_s^t |y(r) - x(r)| \, dr
\]
\[
\leq L(t-s) + L \frac{\delta_0}{2} (t-s).
\]
Taking the supremum over all \(\alpha\) such that \(\|\alpha\|_{[s,t]} \leq 1\), we have
\[
\uparrow k_{t} - \downarrow k_{t} \leq \left( \frac{2L}{\delta_0} + L \right) (t-s)
\]
and, by Lemma 9, the result follows.

Denote now \(\mu_m(\varepsilon) = \varepsilon + m_m(\varepsilon)\), \(\varepsilon \geq 0\).
Lemma 12. Let the assumptions (3), (7), (8) and (9) be satisfied and \((x, k) \in SP(H_\partial \phi; x_0, m)\). Then, there exists a positive constant \(C_T(\|m\|_T) = C (x_0, b, c, r_0, h_0, L, T, \|m\|_T)\), increasing function with respect to \(\|m\|_T\), such that, for all \(0 \leq s \leq t \leq T\):

\[
\begin{align*}
(\text{a}) \quad & \|x\|_T + \frac{1}{T} k_\downarrow T \leq C_T(\|m\|_T), \\
(\text{b}) \quad & |x(t) - x(s)| + \frac{1}{T} k_\downarrow t - \frac{1}{T} k_\downarrow s \leq C_T(\|m\|_T) \times \sqrt{\mu_m(t-s)}.
\end{align*}
\] (40)

Proof. We will follow the ideas of Lions and Sznitman from [11].

Step 1. Define the sequence

\[t_0 = T_0 = 0\]
\[T_1 = \inf \left\{ t \in [t_0, T] : \text{dist} (x(t), Bd(D)) \leq \frac{\delta_0}{2} \right\},\]
\[t_1 = \inf \left\{ t \in [T_1, T] : |x(t) - x(T_1)| > \delta_0 \right\},\]
\[T_2 = \inf \left\{ t \in [t_1, T] : \text{dist} (x(t), Bd(D)) \leq \frac{\delta_0}{2} \right\},\]
\[\ldots \ldots \ldots \ldots \ldots \]
\[t_i = \inf \left\{ t \in [T_i, T] : |x(t) - x(T_i)| > \delta_0 \right\},\]
\[T_{i+1} = \inf \left\{ t \in [t_i, T] : \text{dist} (x(t), Bd(D)) \leq \frac{\delta_0}{2} \right\},\]
\[\ldots \ldots \ldots \ldots \ldots \]

Clearly, we have

\[0 = T_0 = t_0 \leq t_1 \leq T_2 < \cdots \leq T_i < t_i \leq T_{i+1} < t_{i+1} \leq \cdots \leq T.\]

Denote

\[K(t) = \int_0^t H(x(r)) \, dk(r)\]

and it follows that there exists a positive constant \(\tilde{c}\) such that

\[\downarrow K_\downarrow t \leq c \downarrow k_\downarrow t \leq \tilde{c} \downarrow K_\downarrow t.\]

We have

- for \(t_i \leq s \leq t \leq T_{i+1}\):

\[|x(t) - x(s)| \leq \downarrow K_\downarrow t - \downarrow K_\downarrow s + |m(t) - m(s)|.\]

Since for \(t_i \leq r \leq T_{i+1}\), \(x(r) \in D_{\delta_0}\) then, by Lemma [11], for \(t_i \leq s \leq t \leq T_{i+1}\),

\[\downarrow k_\downarrow t - \downarrow k_\downarrow s \leq L \left( 1 + \frac{2}{\delta_0} \right) (t-s)\]

and

\[m_x(t-s) \leq [(t-s) + m_m(t-s)] \times C_T.\]
Hence, denoting in what follows by $C_T(\|m\|_T)$ a generic constant depending on the supremum norm of the continuous function $m$, we have

$$m_x(t-s) + \frac{1}{2}k^+_{t_i} - \frac{1}{2}k^+_{s_i} \leq \mu_m(t-s) \times C_T \leq \sqrt{\mu_m(t-s) \times C_T(\|m\|_T)}.$$

- for $T_i \leq s \leq t \leq T_i$, by Lemma [10] we have

  $$|x(t) - x(s)| + \frac{1}{2}k^+_{t_i} - \frac{1}{2}k^+_{s_i} \leq \sqrt{\mu_m(t-s) \times C_T(\|m\|_T)}.$$

- for $T_i \leq s \leq t_i \leq T_i + 1$,

  $$|x(t) - x(s)| + \frac{1}{2}k^+_{t_i} - \frac{1}{2}k^+_{s_i} \leq |x(t) - x(t_i)| + \frac{1}{2}k^+_{t_i} - \frac{1}{2}k^+_{s_i} + |x(t_i) - x(s)| + \frac{1}{2}k^+_{t_i} - \frac{1}{2}k^+_{s_i} \leq \sqrt{\mu_m(t-t_i) \times C_T + \mu_m(t_i-s) \times C_T(\|m\|_T)} \leq \sqrt{\mu_m(t-s) \times C_T(\|m\|_T)}.$$

Consequently, for all $i \in \mathbb{N}$ and $T_i \leq s \leq T_{i+1}$,

$$|x(t) - x(s)| + \frac{1}{2}k^+_{t_i} - \frac{1}{2}k^+_{s_i} \leq \sqrt{\mu_m(t-s) \times C_T(\|m\|_T)},$$

where $C_T(\|m\|_T) = C(b, c, r_0, h_0, L, \|m\|_T)$ is increasing with respect to $\|m\|_T$.

**Step 2.** Since $\mu_m : [0, T] \rightarrow [0, \mu_m(T)]$ is a strictly increasing continuous function, then the inverse function $\mu_m^{-1} : [0, \mu_m(T)] \rightarrow [0, T]$ is well defined and it is, also, a strictly increasing continuous function. We have

$$\delta \leq |x(t_i) - x(T_i)| \leq \sqrt{\mu_m(t_i-T_i) \times C_T(\|m\|_T)} \leq \sqrt{\mu_m(T_i+1-T_i) \times C_T(\|m\|_T)}$$

and, consequently,

$$T_{i+1} - T_i \geq \mu_m^{-1} \left[ \left( \frac{\delta}{C_T(\|m\|_T)} \right)^2 \right] \overset{\text{def}}{=} \frac{1}{\Delta_m} > 0.$$

Therefore, the bounded increasing sequence $(T_i)_{i \geq 0}$ is finite.

Considering $j$ be such that $T = T_j$, we have

$$T = T_j = \sum_{i=1}^{j} (T_i - T_{i-1}) \geq \frac{j}{\Delta_m}.$$
Let $0 \leq s \leq t \leq T$ and we have

$$\uparrow k^t_T - \downarrow k^s_s = \sum_{i=1}^{j} \left( \uparrow k^p_{(t \wedge T_i) \vee s} - \downarrow k^p_{(T_i \wedge T_{i-1}) \vee s} \right)$$

$$\leq \sum_{i=1}^{j} \sqrt{\mu_m \left( (t \wedge T_i) \vee s - (t \wedge T_{i-1}) \vee s \right) \times C_T (\|m\|_T)}$$

$$\leq j \times \sqrt{\mu_m (t - s) \times C_T (\|m\|_T)}$$

$$\leq T \Delta_m \sqrt{\mu_m (t - s) \times C_T (\|m\|_T)}.$$ 

Consequently,

$$\uparrow k^t_T \leq T \Delta_m \sqrt{\mu_m (T) \times C_T (\|m\|_T)} \leq C'_T (\|m\|_T)$$

and

$$|x(t)| = \left| x_0 + m(t) - \int_0^t H(x(s)) \, ds \right|$$

$$\leq |x_0| + \|m\|_T + c \|k^t_T\|$$

$$\leq |x_0| + \|m\|_T + c \|k^t_T\|.$$

We conclude that there exists a positive constant $C_T (\|m\|_T) = C (b, c, r_0, h_0, L, \|m\|_T) > 0$ (increasing with respect to $\|m\|_T$) such that

$$\uparrow k^t_T \leq C_T (\|m\|_T) \quad \text{and} \quad \|x\|_T \leq |x_0| + C_T (\|m\|_T),$$

that is (4.10a).

By Lemma 9 for every $0 \leq s \leq t \leq T$:

$$m_x(t - s) \leq \left[ (t - s) + m_m(t - s) + \sqrt{m_m(t - s) C_T (\|m\|_T)} \right] \times C_T (\|m\|_T)$$

$$\leq C'_T (\|m\|_T) \times \sqrt{\mu_m (t - s)},$$

that means (4.10b) holds. The proof is now complete. 

4.2. Moreau-Yosida regularization of a convex function

By $\nabla \varphi_\varepsilon$ we denote the gradient of the Yosida’s regularization $\varphi_\varepsilon$ of the convex lower semicontinuous function $\varphi$, that is

$$\varphi_\varepsilon(x) = \inf \left\{ \frac{1}{2\varepsilon} |z - x|^2 + \varphi(z) : z \in \mathbb{R}^d \right\}$$

$$= \frac{1}{2\varepsilon} |x - J_\varepsilon x|^2 + \varphi(J_\varepsilon x),$$

28
where \( J_x = x - \varepsilon \nabla \varphi_\varepsilon(x) \). The function \( \varphi_\varepsilon : \mathbb{R}^d \to \mathbb{R} \) is convex and differentiable and, for all \( x, y \in \mathbb{R}^d, \varepsilon > 0 \):

\[
\begin{align*}
(a) & \quad \nabla \varphi_\varepsilon(x) = \partial \varphi_\varepsilon(x) \in \partial \varphi(J_x x) \text{ and } \varphi(J_x x) \leq \varphi_\varepsilon(x) \leq \varphi(x), \\
(b) & \quad |\nabla \varphi_\varepsilon(x) - \nabla \varphi_\varepsilon(y)| \leq \frac{1}{\varepsilon} |x - y|, \\
(c) & \quad \langle \nabla \varphi_\varepsilon(x) - \nabla \varphi_\varepsilon(y), x - y \rangle \geq 0, \\
(d) & \quad \langle \nabla \varphi_\varepsilon(x) - \nabla \varphi_\varepsilon(y), x - y \rangle \geq -(\varepsilon + \delta) (\nabla \varphi_\varepsilon(x), \nabla \varphi_\varepsilon(y)).
\end{align*}
\]

Moreover, in the case \( 0 = \varphi(0) \leq \varphi(x) \), for all \( x \in \mathbb{R}^d \), we have

\[
\begin{align*}
(a) & \quad 0 = \varphi_\varepsilon(0) \leq \varphi_\varepsilon(x) \quad \text{and} \quad J_\varepsilon(0) = \nabla \varphi_\varepsilon(0) = 0, \\
(b) & \quad \frac{\varepsilon}{2} |\nabla \varphi_\varepsilon(x)|^2 \leq \varphi_\varepsilon(x) \leq (\nabla \varphi_\varepsilon(x), x), \quad \forall x \in \mathbb{R}^d.
\end{align*}
\]

**Proposition 13.** Let \( \varphi : \mathbb{R}^d \to [-\infty, +\infty] \) be a proper convex l.s.c. function such that \( \text{int} (\text{Dom}(\varphi)) \neq \emptyset \). Let \((u_0, \hat{u}_0) \in \partial \varphi, r_0 \geq 0 \) and

\[
\varphi_{u_0,r_0}^\# \overset{\text{def}}{=} \sup \{ \varphi(u_0 + r_0 v) : |v| \leq 1 \}.
\]

Then, for all \( 0 \leq s \leq t \) and \( dk(t) \in \partial \varphi(x(t))(dt) \),

\[
r_0 (\text{\footnotesize \{} \text{\footnotesize $k^s_{\hat{u}_s} - k^s_{u_s}$} \text{\footnotesize \}}) + \int_s^t \varphi(x(r))dr \leq \int_s^t \langle x(r) - u_0, dk(r) \rangle + (t - s) \varphi_{u_0,r_0}^\#,
\]

and, moreover,

\[
r_0 (\text{\footnotesize \{} \text{\footnotesize $k^s_{\hat{u}_s} - k^s_{u_s}$} \text{\footnotesize \}}) + \int_s^t |\varphi(x(r)) - \varphi(u_0)| dr \leq \int_s^t \langle x(r) - u_0, dk(r) \rangle \\
+ \int_s^t (2 |\hat{u}_0||x(r) - u_0| + \varphi_{u_0,r_0}^\# - \varphi(u_0)) dr.
\]

### 4.3. Useful inequalities

Let now introduce the spaces that will appear in the next results.

Denote by \( S^p_d [0, T], p \geq 0 \), the space of progressively measurable continuous stochastic processes \( X : \Omega \times [0, T] \to \mathbb{R}^d \), such that

\[
\|X\|_{S^p_d} = \begin{cases} 
\left( \mathbb{E} \|X\|_T^p \right)^{\frac{1}{p}} & \text{if } p > 0, \\
\mathbb{E}[1 \wedge \|X\|_T] & \text{if } p = 0,
\end{cases}
\]

where \( \|X\|_T = \sup_{t \in [0, T]} |X_t| \). The space \( S^p_d [0, T], \|\cdot\|_{S^p_d} \), \( p \geq 1 \), is a Banach space and \( S^p_d [0, T], 0 \leq p < 1 \), is a complete metric space with the metric \( \rho(Z_1, Z_2) = \|Z_1 - Z_2\|_{S^p_d} \)

(when \( p = 0 \) the metric convergence coincides with the probability convergence).
Denote by \( \Lambda^p_{d\times k}(0, T) \), \( p \in [0, \infty[, \) the space of progressively measurable stochastic processes \( Z : \Omega \times [0, T) \rightarrow \mathbb{R}^{d \times k} \) such that

\[
\| Z \|_{\Lambda^p} = \begin{cases} 
\left[ \mathbb{E} \left( \int_0^T \| Z_s \|^2 \, ds \right) \right]^{\frac{p}{2}}_p, & \text{if } p > 0, \\
1 \land \left( \int_0^T \| Z_s \|^2 \, ds \right)^{\frac{1}{2}}, & \text{if } p = 0.
\end{cases}
\]

The space \( (\Lambda^p_{d\times k}(0, T), \| \cdot \|_{\Lambda^p}) \), \( p \geq 1 \), is a Banach space and \( \Lambda^p_{d\times k}(0, T), \) \( 0 \leq p < 1 \), is a complete metric space with the metric \( \rho(Z_1, Z_2) = \| Z_1 - Z_2 \|_{\Lambda^p} \).

**Proposition 14.** Let \( x \in BV_{\text{loc}} ([0, \infty[; \mathbb{R}^d) \) and \( V \in BV_{\text{loc}} ([0, \infty[; \mathbb{R}) \) be continuous functions. Let \( R, N : [0, \infty[ \rightarrow [0, \infty[ \) be two continuous increasing functions. If

\[
(x(t), dx(t)) \leq dR(t) + |x(t)| \, dN(t) + |x(t)|^2 \, dV(t)
\]

as signed measures on \([0, \infty[\), then for all \( 0 \leq t \leq T \),

\[
\| e^{-V} x \|_{[t, T]} \leq 2 \left[ \left( \int_t^T e^{-2V(s)} \, dR(s) \right)^{1/2} + \int_t^T e^{-V(s)} \, dN(s) \right]. \tag{45}
\]

If \( R = 0 \) then, for all \( 0 \leq t \leq s \),

\[
| x(s) | \leq e^{V(s) - V(t)} | x(t) | + \int_t^s e^{V(s) - V(r)} \, dN(r). \tag{46}
\]

**Proof.** Let \( u_\varepsilon(r) = | x(r) |^2 e^{-2V(r)} + \varepsilon, \) for \( \varepsilon > 0 \). We have as signed measures on \([0, \infty[\)

\[
du_\varepsilon(r) = -2e^{-2V(r)} | x(r) |^2 dV(r) + 2e^{-2V(r)} \langle x(r), dx(r) \rangle \leq 2e^{-2V(r)} dR(r) + 2e^{-2V(r)} | x(r) | dN(r) \leq 2e^{-2V(r)} dR(r) + 2e^{-V(r)} \sqrt{u_\varepsilon(r)} dN(r).
\]

If \( R = 0 \) then

\[
d \left( \sqrt{u_\varepsilon(r)} \right) = \frac{du_\varepsilon(r)}{2\sqrt{u_\varepsilon(r)}} \leq e^{-V(r)} dN(r),
\]

and, consequently, for \( 0 \leq t \leq s, \sqrt{u_\varepsilon(s)} \leq \sqrt{u_\varepsilon(t)} + \int_t^s e^{-V(r)} dN(r), \) that yields \(46\) by passing to limit as \( \varepsilon \to 0. \)

If \( R \neq 0 \) we have

\[
e^{-2V(s)} | x(s) |^2 \leq e^{-2V(t)} | x(t) |^2 + 2 \int_t^s e^{-2V(r)} dR(r) + 2 \int_t^s e^{-2V(r)} | x(r) | dN(r) \leq e^{-2V(t)} | x(t) |^2 + 2 \int_t^s e^{-2V(r)} dR(r) + 2 \| e^{-V} x \|_{[t, T]} \int_t^s e^{-V(r)} dN(r) \leq | e^{-V(t)} x(t) |^2 + 2 \int_t^T e^{-2V(r)} dR(r) + \frac{1}{2} \| e^{-V} x \|^2_{[t, T]} + 2 \left( \int_t^T e^{-V(r)} dN(r) \right)^2.
\]
Hence, for all \( t \leq \tau \leq T \),
\[
e^{-2V(\tau)}|x(\tau)|^2 \leq \|e^{-V}x\|^2_{[t,T]} \leq 2e^{-2V(t)}|x(t)|^2 + 4 \int_t^T e^{-2V(s)}dR(s) + 4 \left( \int_t^T e^{-V(s)}dN(s) \right)^2
\]
and the result follows. \( \square \)

Recall, from Pardoux & Răşcanu [12], an estimate on the local semimartingale \( X \in S_0^d \) of the form
\[
X_t = X_0 + K_t + \int_0^t G_s dB_s, \quad t \geq 0, \quad P - a.s.,
\]
where
\[
\diamond \quad K \in S_0^d, \quad K \in BV_{loc}([0,\infty[;\mathbb{R}^d), \quad K_0 = 0, \quad P - a.s., \\
\diamond \quad G \in \Lambda_0^{d \times k}.
\]
For \( p \geq 1 \) denote \( m_p \overset{def}{=} 1 \lor (p-1) \) and we have the following result.

**Proposition 15.** Let \( X \in S_0^d \) be a local semimartingale of the form (47). Assume there exist \( p \geq 1 \) and \( V \in \mathcal{P}_{m.b.v.c.s.p.}, V_0 = 0, \quad P - a.s. \), such that as signed measures on \([0,\infty[:\)
\[
\langle X_t, dK_t \rangle + \frac{1}{2} m_p |G_t|^2 dt \leq |X_t|^2 dV_t, \quad P - a.s.. \quad (48)
\]
Then, for all \( \delta \geq 0, \quad 0 \leq t \leq s \), we have that
\[
\mathbb{E}^{\mathcal{F}_t} \left[ \frac{|e^{-V_s}X_s|^p}{\left( 1 + \delta |e^{-V_s}X_s|^2 \right)^{p/2}} \right] \leq \frac{|e^{-V_t}X_t|^p}{\left( 1 + \delta |e^{-V_t}X_t|^2 \right)^{p/2}}, \quad P - a.s.. \quad (49)
\]

4.4. Tightness results

The next five results are given without proofs; you can find them in the monograph [12].

**Proposition 16.** Let \( \{X^n_t : t \geq 0\}, \quad n \in \mathbb{N}^* \), be a family of \( \mathbb{R}^d \)-valued continuous stochastic processes defined on probability space \( \Omega, \mathcal{F}, P \). Suppose that, for every \( T \geq 0 \), there exist \( \alpha = \alpha_T > 0 \) and \( b = b_T \in C(\mathbb{R}_+) \) with \( b(0) = 0 \) (both independent of \( n \)), such that
\[
(i) \quad \lim_{N \to \infty} \left[ \sup_{n \in \mathbb{N}^*} P(\{|X^n_0| \geq N\}) \right] = 0,
\]
\[
(ii) \quad \mathbb{E} \left[ 1 \wedge \sup_{0 \leq s \leq \varepsilon} |X^n_{t+s} - X^n_t|^\alpha \right] \leq \varepsilon \cdot b(\varepsilon), \quad \forall \varepsilon > 0, n \geq 1, \quad t \in [0,T].
\]
Then \( \{X^n : n \in \mathbb{N}^*\} \) is tight in \( C(\mathbb{R}_+;\mathbb{R}^d) \).

31
Lemma 20. Let \( (X^n, K^n, V^n) \) be continuous stochastic processes such that

\[
\frac{(X^n, K^n, V^n)}{\text{law}}_{n \to \infty} (X, K, V)
\]

and, for all \( 0 \leq s < t \), and \( n \in \mathbb{N}^* \)

\[
\uparrow K^n \uparrow_t - \uparrow K^n \uparrow_s \leq V^n_t - V^n_s \quad \text{a.s.}
\]

and

\[
\int_s^t \varphi (X^n_r) \, dr \leq \int_s^t \langle X^n_r, dK^n_r \rangle, \quad \text{a.s.}
\]

Then \( \uparrow K \uparrow_t - \uparrow K \uparrow_s \leq V_t - V_s \), a.s. and

\[
\int_s^t \varphi (X_r) \, dr \leq \int_s^t \langle X_r, dK_r \rangle, \quad \text{a.s.}
\]

Proposition 17. Consider \( \varphi : \mathbb{R}^d \to [0, +\infty) \) a l.s.c. function. Let \( (X, K, V), (X^n, K^n, V^n), n \in \mathbb{N} \), be \( C \left( [0, T] ; \mathbb{R}^d \right) \times C \left( [0, T] ; \mathbb{R} \right) \) -valued random variables, such that

\[
(X^n, K^n, V^n) \xrightarrow{\text{law}}_{n \to \infty} (X, K, V)
\]

and, for all \( 0 \leq s < t \), and \( n \in \mathbb{N}^* \)

\[
\uparrow K^n \uparrow_t - \uparrow K^n \uparrow_s \leq V^n_t - V^n_s \quad \text{a.s.}
\]

and

\[
\int_s^t \varphi (X^n_r) \, dr \leq \int_s^t \langle X^n_r, dK^n_r \rangle, \quad \text{a.s.}
\]

Then \( \uparrow K \uparrow_t - \uparrow K \uparrow_s \leq V_t - V_s \), a.s. and

\[
\int_s^t \varphi (X_r) \, dr \leq \int_s^t \langle X_r, dK_r \rangle, \quad \text{a.s.}
\]

Proposition 18. Let \( X, \tilde{X} \in S^0 \mathbb{R} [0, T] \) and \( B, \tilde{B} \) be two \( \mathbb{R}^k \)-Brownian motions and \( g : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d \times k} \) be a function satisfying

\[
g (\cdot, y) \text{ is measurable } \forall \ y \in \mathbb{R}^d, \text{ and}
\]

\[
y \mapsto g (t, y) \text{ is continuous } dt - \text{a.e.}
\]

If

\[
\mathcal{L} (X, B) = \mathcal{L} (\tilde{X}, \tilde{B}), \quad \text{on } C (\mathbb{R}_+, \mathbb{R}^{d+k}),
\]

then

\[
\mathcal{L} \left( X, B, \int_0^t g (s, X_s) \, dB_s \right) = \mathcal{L} \left( \tilde{X}, \tilde{B}, \int_0^t g (s, \tilde{X}_s) \, d\tilde{B}_s \right), \quad \text{on } C (\mathbb{R}_+, \mathbb{R}^{d+k+d}).
\]

Lemma 19. Let \( g : \mathbb{R}_+ \to \mathbb{R}_+ \) be a continuous function satisfying \( g (0) = 0 \) and \( G : C \left( \mathbb{R}_+ ; \mathbb{R}^d \right) \to \mathbb{R}_+ \) be a mapping which is bounded on compact subsets of \( C \left( \mathbb{R}_+ ; \mathbb{R}^d \right) \). Let \( X^n, Y^n, n \in \mathbb{N}^*, \) be random variables with values in \( C \left( \mathbb{R}_+ ; \mathbb{R}^d \right) \). If \( \{ Y^n : n \in \mathbb{N}^* \} \) is tight and, for all \( n \in \mathbb{N}^* \),

\[
(i) \quad |X_0^n| \leq G (Y^n), \quad \text{a.s.},
\]

\[
(ii) \quad \mathbb{M}_{X^n} (\varepsilon; [0, T]) \leq G (Y^n) g (\mathbb{M}_{Y^n} (\varepsilon; [0, T])), \quad \text{a.s.}, \quad \forall \ \varepsilon, T > 0,
\]

then \( \{ X^n : n \in \mathbb{N}^* \} \) is tight.

Lemma 20. Let \( B^n, B^n : \Omega \times [0, \infty) \to \mathbb{R}^k \) and \( X, X^n : \Omega \times [0, \infty) \to \mathbb{R}^{d \times k} \) be continuous stochastic processes such that

\[
(i) \quad B^n \text{ is } \mathcal{F}_t^{B^n, X^n} \text{ Brownian motion, for all } n \geq 1,
\]

\[
(ii) \quad \mathcal{L} (X^n, B^n) = \mathcal{L} (\tilde{X}^n, B^n) \text{ on } C (\mathbb{R}_+, \mathbb{R}^{d \times k} \times \mathbb{R}^k), \quad \text{for all } n \geq 1,
\]
\[ (iii) \int_0^T |\bar{X}_n^s - \bar{X}_s|^2 \, ds + \sup_{t \in [0,T]} |\bar{B}_n^s - \bar{B}_t| \to 0 \text{ in probability, as } n \to \infty, \text{ for all } T > 0. \]

Then \((\bar{B}_n, \{\mathcal{F}_t^{B_n,X_n}\}), n \geq 1\), and \((\bar{B}, \{\mathcal{F}_t^{B,X}\})\) are Brownian motions and, as \(n \to \infty\),

\[ \sup_{t \in [0,T]} \left| \int_0^t \bar{X}_n^s d\bar{B}_n^s - \int_0^t \bar{X}_s d\bar{B}_s \right| \to 0 \quad \text{in probability.} \]

Acknowledgements

The authors are grateful to the referees for the attention in reading this paper and for theirs very useful suggestions.

References

[1] Asiminoaei, I.; Răşcanu, A., Approximation and simulation of stochastic variational inequalities-splitting up method, Numer. Funct. Anal. and Optim. 18, no. 3&4, pp. 251-282, 1997.

[2] Barbu, V.; Răşcanu, A., Parabolic variational inequalities with singular inputs, Differential Integral Equations 10, no. 1, pp. 67–83, 1997.

[3] Bensoussan, A; Răşcanu, A., Stochastic variational inequalities in infinite-dimensional spaces, Numer. Funct. Anal. Optim. 18, no. 1&2, pp. 19–54, 1997.

[4] Buckdahn, R.; Răşcanu, A., On the existence of stochastic optimal control of distributed state system, Nonlinear Anal. 52, no. 4, pp. 1153–1184, 2003.

[5] Cépa, E., Équations différentielles stochastiques multivoques. (French) [Multivalued stochastic differential equations], Séminaire de Probabilités, XXIX, Lecture Notes in Math., Springer, Berlin, pp. 86-107, 1995.

[6] Cépa, E., Problème de Skorohod multivoque. (French) [Multivalued Skorohod problem], Ann. Probab. 26, no 2, pp. 500-532, 1998.

[7] Dupuis, P.; Ishii, H., SDEs with oblique reflection on nonsmooth domains, Ann. Probab. 21, no. 1, pp. 554–580, 1993.

[8] Hamadène, S.; Hassani M., BSDEs with two reflecting barriers: the general result, Probab. Theory Relat. Fields 132, pp. 237-264, 2005.

[9] Ikeda, N.; Watanabe, S., Stochastic differential equations and diffusion processes, North-Holland/Kodansha, 1981.

[10] McKean, H., A. Skorohod’s stochastic integral equation for a reflecting barrier diffusion, J. Math. Kyoto Univ., Volume 3, Number 1, pp. 85-88, 1963.

[11] Lions, P.L.; Sznitman, A., Stochastic differential equations with reflecting boundary conditions, Comm. Pure Appl. Math. 37, no. 4, pp. 511–537, 1984.

[12] Pardoux, E.; Răşcanu, A., SDEs, BSDEs and PDEs, book, submitted to Springer.
[13] Răşcanu, A., *Deterministic and stochastic differential equations in Hilbert spaces involving multivalued maximal monotone operators*, Panamer. Math. J. 6, no. 3, pp. 83–119, 1996.

[14] Răşcanu, A., *Stochastic variational inequalities in non-convex domains*, submitted.

[15] Răşcanu, A.; Rotenstein. E., *The Fitzpatrick function - a bridge between convex analysis and multivalued stochastic differential equations*, J. Convex Anal., no. 18, no. 1, pp. 105-138, 2011.

[16] Skorohod, A., *Stochastic equations for diffusion processes in a bounded region*, Veroyatnost. i Primenen. no. 6, pp. 264-274, 1961; no. 7, pp. 3-23, 1962.

[17] Tanaka, H., *Stochastic Differential Equations with Reflecting Boundary Condition in Convex Regions*, Hiroshima Math. J., pp. 163-177, 1979.