New exact solutions of the discrete fourth Painlevé equation

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Abstract

In this paper we derive a number of exact solutions of the discrete equation

\[ x_{n+1}x_{n-1} + x_n(x_{n+1} + x_{n-1}) = \frac{-2z_n x_n^3 + (\eta - 3\delta^2 - z_n^2)x_n^2 + \mu^2}{(x_n + z_n + \gamma)(x_n + z_n - \gamma)}, \tag{1} \]

where \( z_n = n\delta \) and \( \eta, \delta, \mu \) and \( \gamma \) are constants. In an appropriate limit (1) reduces to the fourth Painlevé (PIV) equation

\[ \frac{d^2w}{dz^2} = \frac{1}{2w} \left( \frac{dw}{dz} \right)^2 + \frac{3}{2} w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w}, \tag{2} \]

where \( \alpha \) and \( \beta \) are constants and (1) is commonly referred to as the discretised fourth Painlevé equation. A suitable factorisation of (1) facilitates the identification of a number of solutions which take the form of ratios of two polynomials in the variable \( z_n \). Limits of these solutions yield rational solutions of PIV (2). It is also known that there exist exact solutions of PIV (2) that are expressible in terms of the complementary error function and in this article we show that a discrete analogue of this function can be obtained by analysis of (1).

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1. Introduction

The discrete fourth Painlevé equation (d-PIV) was recently derived by Ramani, Grammaticos & Hietarinta [1] using the method of singularity confinement [2]. It is given by the three-point, non-autonomous mapping

\[
x_{n+1}x_{n-1} + x_n(x_{n+1} + x_{n-1}) = \frac{-2x_n^3 + (\eta - 3\delta - \gamma)x_n^2 + \mu^2}{(x_n + x_n + \gamma)(x_n + x_n - \gamma)},
\]

(1)

with the variable \(x_n\) to be found in terms of \(z_n \equiv n\delta + \zeta\). This mapping is identified as the discretised version of the continuous fourth Painlevé equation (PIV)

\[
d^2w/dz^2 = \frac{1}{2w} \left( \frac{dw}{dz} \right)^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w},
\]

(2)

where \(\alpha\) and \(\beta\) are arbitrary constants. This is observed by taking the limit of (1) as \(\delta \to 0\), with \(\gamma = 1/\delta\) and \(\eta\) and \(\mu\) finite. This process yields (2) with the parameters \(\alpha\) and \(\beta\) in that equation related to \(\eta\) and \(\mu\) according to

\[
\alpha = \frac{1}{4}\eta \quad \text{and} \quad \beta = -\frac{1}{2}\mu^2.
\]

(3a,b)

The six (continuous) Painlevé equations (PI–PVI) were first derived around the turn of the century in an investigation by Painlevé and his colleagues into which second-order ordinary differential equations have the property that the singularities other than poles of any of the solutions are independent of the particular solution and so are dependent only upon the equation (cf., [3]); this property is now known as the Painlevé property. There has been considerable interest in Painlevé equations over the last few years primarily due to the fact that they arise as reductions of soliton equations solvable by inverse scattering as first demonstrated by Ablowitz and Segur [4]. Although first discovered from strictly mathematical considerations, the Painlevé equations have appeared in various of physical applications (cf., [5] and the references therein). The Painlevé equations may also be thought of as nonlinear analogues of the classical special functions though they are known to be transcendental since their solution is not expressible in terms of elementary functions. However rational solutions and one-parameter families of solutions of the Painlevé equations are expressible in terms of special functions are known to exist for particular values of the parameters. For example, there exist solutions of PII, PIII and PIV that are expressed in terms of Airy, Bessel and parabolic cylinder functions, respectively (cf., [6]).

Recently there has been considerable interest in integrable mappings and discrete systems, including discrete analogues of the Painlevé equations. Some of these mapping and discrete equations arise in physical applications. For example, a discrete analogue of PI (d-PI) arose in the study of the partition function in a two-dimensional model of quantum gravity [7–10]. Subsequently a discrete analogue of PII (d-PII) was derived in [11,12] and later discrete analogues of PIII–PV (d-PIII–d-PV) were obtained [1]; for further details on the derivation of the discrete Painlevé equations see, for example, [13,14]. One important result of the investigations is that the form of the discrete Painlevé equations is not unique since there exist several possible discrete analogues of the Painlevé equations. Kajiwara et al. [15] and Grammaticos et al. [16] have derived exact solutions of d-PII and d-PIII in terms of discrete Airy and discrete Bessel functions, respectively, in analogue to the aforementioned results for the associated continuous Painlevé equations. We further remark that Lax pairs and isomonodromic deformation problems are known to exist for d-PI [10], d-PII [12]...
and d-PIII [17]. However, at present, there is no discrete analogue of PVI, nor are there Lax pairs for the versions of d-PIV and d-PV derived in [16].

In recent work [18,19], we have been concerned with the investigation of Bäcklund transformations and exact solutions for PIV (2) together with an examination of various applications of these solutions to several physically motivated nonlinear partial differential equations. The purpose of the present article is to discuss some new solutions of d-PIV (1). In [18] we demonstrated how all known exact solutions of (2) can be categorised into one of three families; in two of these solutions can be determined in terms of the complementary error and parabolic cylinder functions whilst the third family consists of solutions which can be expressed as the ratio of two polynomials in \( z \). Since d-PIV reduces to (2) in the appropriate limit then it can be expected that exact solutions of (1) exist which should tend to known continuous solutions in the same limit. In §2 we shall principally concern ourselves with rational solutions of d-PIV and in §3 we identify solutions of (1) which can be thought of as discrete analogues of the complementary error function hierarchy discussed in [18]. In §4 we make a few closing remarks.

2. Rational solutions of d-PIV

Before we start our study of rational solutions of (1) it is convenient to summarise the state of knowledge of rational solutions for PIV (2). It is easily verified that \( w = 1/z \) satisfies (2) with the parameter choices \( \alpha = 2, \beta = -2 \) and this is a simple example of a rational solution of this equation. Two families of such solutions take the forms

\[
\begin{align*}
    w(z; \alpha, \beta) &= \frac{P_{n-1}(z)}{Q_n(z)}, \\
    w(z; \alpha, \beta) &= -2z + \frac{P_{n-1}(z)}{Q_n(z)},
\end{align*}
\]

(4a,b)

where \( P_m(z) \) and \( Q_m(z) \) denote some polynomials of degree \( m \) consisting of either entirely even or else entirely odd powers of \( z \). Murata [20] has shown that (2) admits unique rational solutions of type (4a) or (4b) if the parameters are of the form

\[
(\alpha, \beta) = (\pm k, -2(1 + 2n + k)^2); \quad k, n \in \mathbb{Z}, \quad n \leq -1, \quad k \geq -2n,
\]

(5a)

and

\[
(\alpha, \beta) = (k, -2(1 + 2n + k)^2); \quad k, n \in \mathbb{Z}, \quad n \geq 0, \quad k \geq -n,
\]

(5b)

respectively. A further family of rational solutions of PIV is characterised by

\[w(z; \alpha, \beta) = -\frac{2}{3}z + \frac{P_{n-1}(z)}{Q_n(z)},\]

(6)

where \( (\alpha, \beta) = (n_1, -\frac{2}{3}(1 + 3n_2)^2) \) with \( n_1 \) and \( n_2 \) either both even or both odd integers. An extended discussion of the important properties of these rational solution families together with tables containing the first few solutions in each of these three hierarchies are to be found in [18].

Tamizhmani, Grammaticos & Ramani [21] noted that if we set

\[
\eta = 3\delta^{-2} + \gamma^2 - 2(a^2 + b^2), \quad \mu = a^2 - b^2,
\]

then the d-PIV equation (1) can be factorised as

\[
(x_{n+1} + x_n)(x_{n-1} + x_n) = \frac{(x_n + a + b)(x_n + a - b)(x_n - a + b)(x_n - a - b)}{(x_n + z_n + \gamma)(x_n + z_n - \gamma)},
\]

(7)
and the parameters $\alpha$ and $\beta$, as defined by (3), are now given by
\[
\alpha = \frac{3}{4}\delta^2 + \frac{1}{4}\gamma^2 - \frac{1}{2}(a^2 + b^2), \quad \beta = -\frac{1}{2}(a^2 - b^2)^2.
\] (8)

Simple rational solutions of (7) can be found with $x_n$ proportional to $z_n$; these are
\[
x_n = -2z, \quad a = \frac{1}{2}\delta + \gamma, \quad b = \frac{1}{2}\delta - \gamma, \quad \alpha = \frac{3}{4}\delta^2 - \frac{1}{4}\gamma^2, \quad \beta = -2\gamma^2\delta^2,
\] (9)

and
\[
x_n = -\frac{2}{3}z, \quad a = \frac{1}{6}\delta + \gamma, \quad b = \frac{1}{6}\delta - \gamma, \quad \alpha = \frac{3}{4}\delta^2 - \frac{1}{4}\gamma^2, \quad \beta = -\frac{2}{3}\gamma^2\delta^2.
\] (10)

In the limit as $\delta \to 0$, with $\gamma = 1/\delta$, these discrete solutions tend to
\[
w(z; 0, -2) = -2z, \quad w(z; 0, -\frac{2}{3}) = -\frac{2}{3}z,
\]
respectively, which are the first members of the PIV hierarchies typified by (4b) and (6). If solutions of (7) proportional to $1/z_n$ are sought then it is found that
\[
x_n = -\frac{\delta(\delta \pm \gamma)}{z_n}, \quad a = \frac{3}{2}\delta \pm \gamma, \quad b = \frac{1}{2}\delta \pm \gamma, \quad \alpha = \delta^2 - \frac{3}{2}\gamma^2 \mp 2\gamma\delta - \frac{1}{4}\delta^2, \quad \beta = -2\delta^2(\delta \pm \gamma)^2,
\] (11)

and so, in the limit as $\delta \to 0$, with $\gamma\delta = 1$, we have $x_n \to w(z; \pm 2, -2) = \pm 1/z$ and these continuous solutions are members of the family described by (4a).

More complicated rational solutions of d-PIV (1) can be deduced by rewriting (7) as the pair
\[
x_n + x_{n-1} = \frac{(x_n + a + b)(x_n + a - b)}{(x_n + z_n \pm \gamma)},
\] (12a)
\[
x_n + x_{n+1} = \frac{(x_n - a + b)(x_n - a - b)}{(x_n + z_n \mp \gamma)}.
\] (12b)

Tamizhmani et al. [21] speculated that such a formalism ought to lead to discrete rational solutions though they did not present any such solutions. It is a routine calculation to verify that equations (12a) and (12b) are compatible if and only if
\[
a = \frac{1}{2}\delta \pm \gamma.
\] (13)

[We remark at this stage that (12) is not the only possibility for the splitting of (7). Easy generalisations of (12) include the multiplication of the right hand sides of (12a) and (12b) by constants $C$ and $1/C$ respectively or the taking of the factors in different pairings. However, it can be shown that in either of these cases we obtain an incompatible couple of equations so that the choice of factorisation (12) is not as specialised and restrictive as it might first appear.]

If the condition (13) is satisfied then we can seek solutions of (1) by finding solutions of the simpler form (12b) (or (12a) would do equally well). If we write $x_n = P_n/Q_n$ then (12b) can be recast as
\[
\kappa_{n+1}P_{n+1} = (z_n \pm \gamma + \delta)P_n - \mu Q_n,
\] (14a)
\[
\kappa_{n+1}Q_{n+1} = -(z_n \mp \gamma)Q_n - P_n,
\] (14b)
where the ‘separation’ parameter \( \kappa_{n+1} \) can depend both on \( n \) and \( z_n \) (recall that \( \mu = a^2 - b^2 \)). However, if for the moment we take \( \kappa_{n+1} = 1 \), then eliminating \( P_n \) between (14a) and (14b), setting \( \gamma = 1/\delta \) and taking the limit as \( \delta \to 0 \) shows that \( Q_n \to q(z) \) where \( q(z) \) satisfies where

\[
\frac{d^2q}{dz^2} = (z^2 + \mu - 1)q. \tag{15a}
\]

If in this equation we set \( q(z) = \eta(\xi) \) with \( \xi = \sqrt{2} z \) then we obtain

\[
\frac{d^2\eta}{d\xi^2} = \left(\frac{1}{4} \xi^2 + \frac{1}{2} \mu - \frac{1}{2}\right)\eta, \tag{15b}
\]

which is the parabolic cylinder equation (cf., [22]). It is well known that when \( \mu = -2n \) this equation admits the solution

\[
\eta(\xi) = \text{He}_n(\xi) \exp\left(-\frac{\xi^2}{4}\right), \tag{16a}
\]

where \( \text{He}_n(\xi) \) is the Hermite polynomial of degree \( n \) given by

\[
\text{He}_n(\xi) = (-1)^n \exp\left(\frac{\xi^2}{2}\right) \frac{d^n}{d\xi^n} \left[\exp\left(-\frac{\xi^2}{2}\right)\right]. \tag{16b}
\]

If the variable \( Q_n \) is eliminated from equations (14) and the usual limit taken then it follows in a manner similar to that outlined above that rational solutions \( x_n = P_n/Q_n \) of (1) exist which tend to a function of the form \( -\text{He}_n(\xi)/\text{He}_m(\xi) \) whenever \( \mu = -2m \), with \( m \in \mathbb{N} \). It has been long established that solutions of (continuous) PIV (2) can be related to Hermite functions whenever the parameters \( \alpha \) and \( \beta \) take certain values. In particular, Lukashevich [23] proved that

\[
w(z; -(\nu + 1), -2\nu^2) = -\frac{1}{\phi_\nu} \frac{d\phi_\nu}{dz}, \tag{17a}
\]

\[
w(z; -\nu, -2(\nu + 1)^2) = -2z - \frac{1}{\phi_\nu} \frac{d\phi_\nu}{dz}, \tag{17b}
\]

\[
w(z; \nu, -2(\nu + 1)^2) = -2z - \frac{1}{\psi_\nu} \frac{d\psi_\nu}{dz}, \tag{17c}
\]

\[
w(z; \nu + 1, -2\nu^2) = \frac{1}{\psi_\nu} \frac{d\psi_\nu}{dz}, \tag{17d}
\]

where \( \phi_\nu(z) \) and \( \psi_\nu(z) \) are any solutions of the Weber-Hermite equations

\[
\frac{d^2\phi_\nu}{dz^2} - 2z \frac{d\phi_\nu}{dz} + 2\nu \phi_\nu = 0, \tag{18a}
\]

\[
\frac{d^2\psi_\nu}{dz^2} + 2z \frac{d\psi_\nu}{dz} - 2\nu \psi_\nu = 0. \tag{18b}
\]

If \( \nu = n \), with \( n \) a positive integer, then \( \phi_n(z) \) and \( \psi_n(z) \) are polynomials of degree \( n \) given by

\[
\phi_n(z) = (-1)^n \exp\left(z^2\right) \frac{d^n}{dz^n} \left[\exp\left(-z^2\right)\right], \quad \psi_n(z) = \exp\left(-z^2\right) \frac{d^n}{dz^n} \left[\exp\left(z^2\right)\right].
\]

(These are also expressible in terms of the Hermite polynomial \( \text{He}_n(\xi) \) defined above.)

Motivated by these comments concerning the continuous PIV case, we return to equations (14) for the discrete situation. If we write

\[
(P_n, Q_n) = (A_n, B_n) \times (-\delta)^n \Gamma \left(\frac{z_n - m\delta + \gamma}{\delta}\right), \tag{19}
\]
respectively. The first few solutions in this hierarchy are

\[(z_n - m\delta \mp \gamma)A_{n+1} + (z_n + \delta \pm \gamma)A_n = \mu B_n, \quad \text{(20a)}\]
\[(z_n - m\delta \mp \gamma)B_{n+1} - (z_n \mp \gamma)B_n = A_n. \quad \text{(20b)}\]

For \(m \in \mathbb{N}\) we can find exact solutions of these equations with \(A_n\) and \(B_n\) taking the forms of polynomials in \(z_n\), consisting of either only even or only odd powers, and of degrees \(m - 1\) and \(m\) respectively. The first few solutions in this hierarchy are

\[m = 1, \quad x_n = -\frac{\delta(\delta \pm \gamma)}{z_n}, \quad \text{(21a)}\]
\[m = 2, \quad x_n = -\frac{2\delta(3\delta \pm 2\gamma)z_n}{2z_n^2 - \delta(2\delta \pm \gamma)}, \quad \text{(21b)}\]
\[m = 3, \quad x_n = -\frac{3\delta(2\delta \pm \gamma)[2z_n^2 - \delta(3\delta \pm \gamma)]}{z_n[2z_n^2 - \delta(8\delta \pm 3\gamma)]}, \quad \text{(21c)}\]
\[m = 4, \quad x_n = -\frac{4\delta(5\delta \mp 2\gamma)[2z_n^2 - \delta(11\delta \pm 3\gamma)]z_n}{4z_n^4 - 4\delta(10\delta \pm 3\gamma)z_n^2 + 3\delta^2(3\delta \pm \gamma)(4\delta \pm \gamma)}, \quad \text{(21d)}\]
\[m = 5, \quad x_n = -\frac{5\delta(3\delta \pm \gamma)[4z_n^4 - 4\delta(13\delta \pm 3\gamma)z_n^2 + 3\delta^2(3\delta \pm \gamma)(4\delta \pm \gamma)]}{z_n[4z_n^4 - 20\delta(4\delta \pm \gamma) + \delta^2(256\delta^2 \pm 125\delta^2 + 15\gamma^2)]}. \quad \text{(21e)}\]

In each case the corresponding value of \(\mu\) in (20a) is \(\mu = -\delta m [(m + 1)\delta \pm 2\gamma]\) and this leads to the respective values of parameters \(\alpha\) and \(\beta\), as defined by (8), given by

\[\alpha = \frac{4\delta^2 - \gamma^2}{4\delta} \mp (m + 1)\gamma \delta - \frac{4\delta^2}{4\delta} [2m(m + 1) + 1], \quad \beta = -\frac{1}{4} m^2 \delta^2 [(m + 1)\delta \pm 2\gamma]^2. \quad \text{(22)}\]

We emphasise at this stage that the discrete solutions (21) are exact and are valid for any \(\delta\) and \(\gamma\). We can recover continuous solutions by letting \(\gamma = 1/\delta\) and taking the limit as \(\delta \to 0\) which yields \(\alpha = \mp (m + 1)\) and \(\beta = -2m^2\). Then from (21) we obtain

\[w(z; \mp 2, -2) = \mp \frac{1}{z}, \quad \text{(23a)}\]
\[w(z; \mp 3, -8) = \mp \frac{4z}{2z^2 \mp 1}, \quad \text{(23b)}\]
\[w(z; \mp 4, -18) = \mp \frac{3(2z^2 \mp 1)}{z(2z^2 \mp 3)}, \quad \text{(23c)}\]
\[w(z; \mp 5, -32) = \mp \frac{8z(2z^2 \mp 3)}{4z^4 \mp 12z^2 + 3}, \quad \text{(23d)}\]
\[w(z; \mp 6, -50) = \mp \frac{5[4z^4 \mp 12z^2 + 3]}{z(4z^4 \mp 20z^2 + 15)}. \quad \text{(23e)}\]

It is then clear that solutions (21) can be thought of as discrete analogues of the (continuous) solutions of PIV taking forms given by (17a) and (17d), with \(\nu = m\).

If in (14) we let

\[(P_n, Q_n) = (A_n, B_n) \times \delta^n \Gamma \left(\frac{z_n - m\delta \pm \gamma}{\delta}\right), \quad \text{(24)}\]

then we obtain

\[(z_n - m\delta \pm \gamma)(A_{n+1} - A_n) = -\mu B_n, \quad \text{(25a)}\]
\[(z_n - m\delta \pm \gamma)B_{n+1} + (z_n \mp \gamma)B_n = -A_n. \quad \text{(25b)}\]
As previously, exact polynomial solutions of these can be derived for integer \( m \) except that now \( A_n \) and \( B_n \) are of degrees \( m + 1 \) and \( m \) respectively. Then the first few solutions of this type are

\[
\begin{align*}
m = 0, & \quad x_n = -2z_n, \quad \text{(26a)} \\
m = 1, & \quad x_n = -\frac{2z_n^2 - \delta(\delta + \gamma)}{2z_n^2 - \delta(2\delta + \gamma)}, \quad \text{(26b)} \\
m = 2, & \quad x_n = -\frac{2z_n^2 - \delta(5\delta \mp 3\gamma)}{2z_n^2 - \delta(2\delta + \gamma)}, \quad \text{(26c)} \\
m = 3, & \quad x_n = -\frac{4z_n^4 - 4\delta(7\delta \mp 3\gamma)z_n^2 + 3\delta^2(3\delta \mp \gamma)(2\delta + \gamma)}{4z_n^4 - 4\delta(8\delta \mp 3\gamma)z_n^2 - 3\delta^2(3\delta \pm \gamma)(3\delta \mp \gamma)}, \quad \text{(26d)} \\
m = 4, & \quad x_n = -\frac{4z_n^4 - 20\delta(3\delta \mp \gamma)z_n^2 + \delta^2(146\delta^2 \mp 95\gamma\delta + 15\gamma^2)}{4z_n^4 - 16\delta(10\delta \mp 3\gamma)z_n^2 - 3\delta^2(4\delta \mp \gamma)(3\delta \mp \gamma)}, \quad \text{(26e)}
\end{align*}
\]

The corresponding values of the parameters \( \alpha \) and \( \beta \) are given by

\[
\alpha = \frac{3}{4}(\delta^{-2} - \gamma^2) + m\gamma\delta - \frac{1}{4}\delta^2[2m(m + 1) + 1], \quad \beta = -\frac{1}{4}(m + 1)^2\delta^2(m\delta + 2\gamma)^2 \quad \text{(27)}
\]

and in the limit \( \delta \to 0 \) with \( \gamma = 1/\delta \), solutions (26) reduce to the functions

\[
\begin{align*}
w(z; 0, -2) & = -2z, \quad \text{(28a)} \\
w(z; \pm 1, -8) & = -2z \mp 1, \quad \text{(28b)} \\
w(z; \pm 2, -18) & = -2z \mp \frac{4z}{2z^2 \pm 1}, \quad \text{(28c)} \\
w(z; \pm 3, -32) & = -2z \mp \frac{3(2z^2 \pm 1)}{z(2z^2 \pm 3)}, \quad \text{(28d)} \\
w(z; \pm 4, -50) & = -2z \mp \frac{8z(2z^2 \pm 3)}{4z^4 \pm 12z^2 \pm 3}. \quad \text{(28e)}
\end{align*}
\]

These are all members of the so-called ‘\(-2z\)’ hierarchy of rational solutions for PIV (2) and are of the form (17b) or (17c), with \( \nu = m \). It is straightforward to obtain further exact rational solutions of d-PIV by solving the pairs (20) or (25) for higher values of \( m \).

We note here that although we have found some rational solutions of d-PIV (1) it can be anticipated that further such solutions exist which are not derivable using the procedure described above. This deduction follows from the observation that rational solutions of PIV (2) are possible for all parameter values as described by (5) and that so far we have discrete solutions which, in the appropriate limit, tend to the continuous solutions of type (17). Therefore, there should be discrete counterparts to the remaining continuous rational solutions. These can be constructed by appealing to some Bäcklund transformations for d-PIV which were given by Tamizhmani et al. [21]. These authors presented a sequence of transformations which, given a solution of (7) with parameters \( a, b \) and \( \gamma \), yields a further solution of the same equation but now with parameters \( a + \delta, b \) and \( \gamma \). In short, suppose that \( x_n \) is a solution of d-PIV with parameters \( a, b \) and \( \gamma \). Then

\[
\bar{y}_n = \frac{x_n x_{n+1} + x_n(\bar{z}_n + a) + x_n(\bar{z}_n - a) + b^2 - a^2}{x_n + x_{n+1}}, \quad \text{(29)}
\]

with \( \bar{z}_n = z_n + \frac{1}{2}\delta \) also satisfies d-PIV but now with parameters \( \bar{a} = a + \frac{1}{2}\delta, \bar{b} = \gamma \) and \( \bar{\gamma} = b \). The subtlety with transformation (29) is that \( \bar{y}_n \) is defined on a lattice of points which is offset by \( \frac{1}{2}\delta \)
from the original. In order to obtain a solution valid at points coinciding with the initial lattice it is therefore necessary to reapply (29) which has the overall effect of raising the value of $a$ by $\delta$ whilst keeping $b$ and $\gamma$ invariant. Clearly $M$ applications of this sequence will increment $a$ to $a + M\delta$ and leaves the other two parameters unchanged so that from a starting solution with parameters

$$a = \frac{1}{2}\delta \pm \gamma, \quad b = (N + \frac{1}{2})\delta \pm \gamma, \quad N = 0, 1, 2, \ldots$$

(which are the parameter values in (1) appropriate to the family of solutions whose first few members are listed in (21)) we can obtain a solution with

$$a = (M + \frac{1}{2})\delta \pm \gamma, \quad b = (N + \frac{1}{2})\delta \pm \gamma,$$

or, in terms of the parameters $\alpha$ and $\beta$ as given by (8),

$$\alpha = \frac{3}{4}(\delta^{-2} - \gamma^2) \pm \gamma\delta(M + N + 1) + \left[\frac{1}{4}(M - N)(M + N + 1) - (M + \frac{1}{2})^2\right] \delta^2,$$

$$\beta = -\frac{1}{2}(M - N)^2\delta^2 \left[\left(M + N + 1\right)\delta \pm 2\gamma\right]^2.$$  

In the usual limit our discrete rational solutions will then tend to continuous ones with associated parameters $\alpha = \mp(M + N + 1)$, $\beta = -2(M - N)^2$. Now $M$ and $N$ can be chosen so as to force these parameters to coincide with the form (5a) for any $k$ and $n$ in the permissible ranges; in this way we have a mechanism for deducing discrete analogues of all those rational solutions of PIV (2) which take the form $P_{n-1}(z)/Q_n(z)$.

A similar argument can be applied to the discrete solutions (28) and this demonstrates that given these results then suitable application of the Bäcklund transformations contained in [21] will generate another set of exact solutions of (1). These comprise the discrete counterpart to the ‘$-2z$’ rational hierarchy of (2) which itself is characterised by the parameter values given in (5b).

We remark here that of the three hierarchies of rational solutions for PIV (2), the simple procedures outlined above have yielded discrete analogues of only two of these; no solutions corresponding to the $-\frac{3}{2}z$ family (6) have been found. The reason for this is that although the full discrete equation (7) admits the solution $x_n = -\frac{2}{3}z_n$ with $a = \frac{1}{6}\delta \pm \gamma$, the splitting of this equation into the two Ricatti-like forms (12) gives a pair of equations whose compatibility requires that $a = \frac{1}{2}\delta \pm \gamma$. Thus, as noted by Tamizhmani et al. [21], this solution is not linearizable through the splitting assumption and thus it is unsurprising that the discrete analogue of the $-\frac{3}{2}z$ hierarchy of solutions cannot be generated in this way. However, we shall now show how use of the Bäcklund transformation (29) discussed above can be used to increment the values of $a$ or $b$ by $\pm \delta$ and thus lead to some more exact solutions in the discrete $-\frac{3}{2}z$ hierarchy. (It is noted that d-PIV (7) is invariant by interchange of parameters $a$ and $b$ or by a sign change of either of these. Therefore, given that two applications of (29) increases $a$ by $\delta$ it is easy to see how a suitable changes in sign of $a$ and $b$ or the swopping of these parameters allows transformations to be found that increase or decrease the values of $a$ or $b$ by integral multiples of $\delta$.) A few examples of simpler solutions in this hierarchy are:

$$x_n = -\frac{2}{3}z_n,$$

$$a = \frac{1}{6}\delta \pm \gamma, \quad b = -\frac{1}{6}\delta \pm \gamma, \quad \alpha = \frac{3}{4}\delta^{-2} - \frac{3}{2}\gamma^2 - \frac{1}{36}\delta^2, \quad \beta = -\frac{2}{3}\gamma^2\delta^2,$$

$$x_n = -\frac{2}{3}z_n - \frac{\delta(\delta \pm 3\gamma)}{3z_n},$$

$$a = \frac{1}{6}\delta \pm \gamma, \quad b = -\frac{1}{6}\delta \pm \gamma, \quad \alpha = \frac{3}{4}\delta^{-2} - \frac{3}{2}\gamma^2 - \frac{1}{36}\delta^2, \quad \beta = -\frac{2}{3}\gamma^2\delta^2.$$
\[ a = \pm \gamma - \frac{5}{6} \delta, \quad b = \mp \gamma + \frac{1}{6} \delta, \quad \alpha = \frac{3}{4} \left( \delta^{-2} - \gamma^2 \right) \pm \gamma \delta - \frac{13}{36} \delta^2, \quad \beta = -\frac{\delta}{9} (2\gamma \pm \delta)^2, \]

\[ x_n = -\frac{2 z_n (2 z_n^2 + 3 \gamma \delta + \delta^2)}{3 (2 z_n^2 + 3 \gamma \delta - 2 \delta^2)}, \]

\[ a = \pm \gamma - \frac{7}{6} \delta, \quad b = \mp \gamma + \frac{5}{6} \delta, \quad \alpha = \frac{3}{4} (\delta^{-2} - \gamma^2) \pm 2 \gamma \delta - \frac{37}{36} \delta^2, \quad \beta = -\frac{\delta}{9} (\gamma \mp \delta)^2, \]

\[ x_n = -\frac{2 z_n (4 z_n^4 - 45 \gamma^2 \delta^2 + \delta^4)}{3 (2 z_n^2 - 3 \gamma \delta - \delta^2)(2 z_n^2 + 3 \gamma \delta - \delta^2)}, \]

\[ a = \pm \gamma + \frac{5}{6} \delta, \quad b = \mp \gamma + \frac{5}{6} \delta, \quad \alpha = \frac{3}{4} (\delta^{-2} - \gamma^2) - \frac{25}{36} \delta^2, \quad \beta = -\frac{50}{9} \gamma^2 \delta^2, \]

\[ x_n = -\frac{8 z_n^6 - 4 \delta (\delta \pm 3 \gamma) z_n^4 - 26 \delta^2 (38 \delta^2 \mp 69 \gamma \delta + 27 \gamma^2) z_n^2 + 9 \delta^3 (4 \delta^3 \mp 11 \gamma \delta^2 + 10 \gamma^2 \delta \mp 3 \gamma^3)}{3 z_n \left( 4 z_n^2 - 4 \delta (4 \delta \mp 3 \gamma) z_n^2 \pm 9 \gamma \delta^2 (\delta \mp \gamma) \right)}, \]

\[ a = \pm \gamma - \frac{7}{6} \delta, \quad b = \mp \gamma + \frac{11}{6} \delta, \quad \alpha = \frac{3}{4} (\delta^{-2} - \gamma^2) \pm 3 \gamma \delta - \frac{85}{36} \delta^2, \quad \beta = \frac{2 \delta^2 (3 \delta \mp 2 \gamma)^2}{9} \]

In the limit as \( \delta \to 0 \), with \( \gamma \delta = 1 \), the solutions (33) reduce to

\[ w(z; 0, -\frac{2}{9}) = -\frac{2}{3} z, \]

\[ w(z; \pm 1, -\frac{8}{9}) = -\frac{2}{3} z \mp \frac{1}{z}, \]

\[ w(z; \pm 2, -\frac{2}{9}) = -\frac{2}{3} z \pm \frac{4z}{2z^2 \pm 3}, \]

\[ w(z; 0, -\frac{50}{9}) = -\frac{2}{3} z \pm \frac{24z}{(2z^2 - 3)(2z^2 + 3)}, \]

\[ w(z; \pm 3, -\frac{8}{9}) = -\frac{2}{3} z \pm \frac{3(4z^4 \pm 4z^2 + 3)}{z(4z^4 \pm 12z^2 - 9)}, \]

which belong to the \(-\frac{2}{3} z\) hierarchy of rational solutions of PIV (2); see Table 4.1.3 of [18] for a more extensive list of such solutions. The method of generating the \(-\frac{2}{3} z_n\) discrete rational solutions given in (33) is intrinsically less satisfying than that employed for the evaluation of solutions (21) and (26) in the other two families; this feature is a direct consequence of the fact that the \(-\frac{2}{3} z_n\) forms do not arise from a suitable factorisation of d-PIV into a pair of simple compatible equations akin to (12). Instead the present method is based on a direct implementation of the Bäcklund transformation (29), which, for the far more complicated solutions in the hierarchy, becomes an increasingly laborious task. However, we believe that the Bäcklund transformation (29) will inevitably be needed to make a complete evaluation of each of the hierarchies and provide a systematic and efficient procedure for doing this.

3. Discrete Complementary Error Function Solutions

In §2 we have deduced several new rational solutions of d-PIV (1). It has been known for a considerable time that PIV (2) admits a number of other exact solutions which can be categorised into separate families. In addition to the rational hierarchy, solutions can be found which are expressible in terms of the complementary error function and, for certain half-integer and integer values of the parameters \( \alpha \) and \( \beta \), in terms of parabolic cylinder functions. We note the close connection which exists between the complementary error function and rational solution hierarchies; solutions in the former can be found in terms of \( \Psi(z) \) where

\[ \Psi(z) = \frac{\psi'(z)}{\psi(z)}, \quad \psi(z) = 1 - C \text{ erfc}(z) \equiv 1 - \frac{2C}{\sqrt{\pi}} \int_z^\infty \exp(-t^2) \, dt, \]
with \( C \) an arbitrary constant. When we choose \( C = 0 \) then \( \Psi \equiv 0 \) and the complementary error solutions reduce to the rational forms whose discrete analogues have been described above. The formulation given in §2 enables us to deduce a discretised version of \( \Psi(z) \) and in order to do this we return to system (14). If we take \( \mu = 0 \) there and select \( \kappa_{n+1} = -(z_n + \gamma) \) then

\[
(z_n + \gamma)P_{n+1} + (z_n - \gamma + \delta)P_n = 0, \tag{36a}
\]

\[
(z_n + \gamma) [Q_{n+1} - Q_n] - P_n = 0. \tag{36b}
\]

If we choose \( z_n = n\delta \) then

\[
P_{n+1} = \left[ \frac{\gamma - (n + 1)\delta}{\gamma + n\delta} \right] P_n \quad \text{and} \quad Q_{n+1} - Q_n = \frac{P_n}{\gamma + n\delta}. \tag{37a,b}
\]

The first of these is easily solved to yield

\[
P_n = \frac{[\Gamma(\gamma/\delta)]^2}{\Gamma(-n + \gamma/\delta)\Gamma(n + \gamma/\delta)} P_0 \tag{38}
\]

and it is clear that this is an even function in \( n \). The continuous limit of (38) is best obtained by setting \( \gamma = 1/\delta \), \( \delta = z/n \) and noting that

\[
\ln(P_n/P_0) = \sum_{r=1}^{n} \ln \left(1 - \frac{r^2}{n^2}\right) - \sum_{r=1}^{n-1} \ln \left(1 + \frac{r^2}{n^2}\right). \tag{39}
\]

Hence it follows that

\[
\lim_{n \to \infty} \{\ln(P_n/P_0)\} = -z^2 + O(1/n),
\]

and so

\[
P_n \to P_0 \exp(-z^2) \equiv P_*(z),
\]

say. Consequently (38) can be thought of as a discrete version of \( \psi'(z) \propto \exp(-z^2) \) and so the solution of (36b) for \( Q \) should yield an appropriate discretised form for \( \psi(z) \) which by (35b) is closely related to the error function \( \text{erfc}(z) \). An explicit solution of equation (36b) is not immediately apparent, though it can readily be seen that

\[
Q_{n+1} = \sum_{j=0}^{n} \frac{P_j}{\gamma + j\delta} + Q_0, \tag{40}
\]

where \( Q_0 \) is a constant and the \( P_j \) are as in (38). This sum is not straightforward to evaluate but for the present purposes it is adequate to merely note that as \( \delta \to 0 \), with \( \gamma = 1/\delta \), then

\[
Q_{n+1} \longrightarrow \delta \sum_{j=0}^{n} P_j + Q_0,
\]

which in turn asymptotes to the integral of \( P_*(z) \); i.e. to a multiple of the complementary error function plus a constant. Thus in forming the solution \( x_n = P_n/Q_n \) we have a discrete equivalent to the continuous PIV solution \( w(z; 1, 0) = \Psi(z) \) defined above. Solutions in the (continuous) complementary error function hierarchy are of the form

\[
w(z; \alpha, \beta) = \frac{P(z, \Psi(z))}{Q(z, \Psi(z))},
\]
where $P$ and $Q$ are polynomials in $z$ and $\Psi(z)$, and exist for precisely those parameter values given in (5). What we have developed here is the discrete analogue of the simplest solution in the complementary error function hierarchy; furthermore, this is the discrete analogue of the most elementary of the “bound-state” solutions of PIV derived in [24] (see also [25]), which themselves form a special case complementary error function hierarchy.

This technique can be adapted to find discrete counterparts to further exact solutions within the complementary error function hierarchy. If we take

$$\alpha^{(m)} = \frac{3}{4} (\delta^{-2} - \gamma^2) + (m + 1)\gamma\delta - \frac{1}{4} [2m(m + 1) + 1] \delta^2,$$

$$\beta^{(m)} = -\frac{1}{2}m^2 \delta^2 [2\gamma - (m + 1)\delta]^2,$$

for $m = 0, 1, \ldots$, and also write the corresponding discretised solution $x^{(m)}_n = P^{(m)}_n/Q^{(m)}_n$, then equations (14), with the lower sign taken, may be written as

$$\kappa^{(m)}_{n+1} P^{(m)}_{n+1} = (z_n - \gamma + \delta) P^{(m)}_n - \mu^{(m)} Q^{(m)}_n,$$

$$\kappa^{(m)}_{n+1} Q^{(m)}_{n+1} = -(z_n + \gamma) Q^{(m)}_n - P^{(m)}_n,$$

where $\mu^{(m)} = m\delta (2\gamma - (m + 1)\delta)$. In the limit as $\delta \to 0$, with $\gamma\delta = 1$, it is clear from (41) that we will obtain solutions with $\alpha = m + 1$, $\beta = -2m^2\delta$ and it is the discrete complementary error function solution with index $m = 0$ that we have derived in (36)-(40) above. If we choose the separation variable $\kappa^{(m)}_{n+1} = -(z_n + \gamma - m\delta)$ then by cross-elimination we can show from (42) that the variables $P^{(m)}_n$ and $Q^{(m)}_n$ satisfy

$$[z_n + (\gamma - \delta) - m\delta] P^{(m)}_{n+2} - 2(\gamma - \delta) P^{(m)}_{n+1} - [z_n - (\gamma - \delta) + m\delta] P^{(m)}_n = 0,$$

$$[z_n + \gamma - (m + 1)\delta] Q^{(m)}_{n+2} - 2\gamma Q^{(m)}_{n+1} - [z_n - \gamma + (m + 1)\delta] Q^{(m)}_n = 0.$$

Now it can be seen that if for some particular choice $m = M$ we write $\hat{\gamma} = \gamma - \delta$ in (43a) then the governing equation for $P^{(M)}_n$ is identical to that for $Q^{(M-1)}_n$ with $\gamma$ in (43b) written as $\hat{\gamma}$. The result of this observation is that given the solution $Q^{(M-1)}_n$, we can immediately deduce the form of $P^{(M)}_n$ simply by replacing all occurrences of $\gamma$ in the former solution with $\gamma - \delta$. Furthermore, since the continuous limit of our discretised solutions is obtained by letting $\delta \to 0$ and $\gamma\delta \to 1$, it is clear that the appropriate limit for $P^{(M)}_n$ is precisely the same as that for $Q^{(M-1)}_n$. Thus, in practice, if we have solutions $P^{(M-1)}_n$ and $Q^{(M-1)}_n$ then $P^{(M)}_n$ is directly derived from $Q^{(M-1)}_n$ and $Q^{(M)}_n$ can be obtained by solving (42b).

The discretised complementary function solutions described here are exact in the sense that they are precise solutions of the d-PIV equation (7) but the practical difficulty with them is that the derivation of explicit formulae for the solutions is distinctly awkward. The main reason for this stems from the fact that a closed form for the $Q^{(0)}_n$ solution given by (40) is unknown and then, because of the nature of systematic manner for deriving further solutions that we have just described, this difficulty with the $Q^{(0)}_n$ form propagates itself through the rest of the hierarchy. However, we shall demonstrate how one can obtain the next complementary error function solution and this shows how the procedure operates in practice. Using the notation suggested by (41) we have the results that

$$P^{(0)}_n = \frac{[\Gamma(\gamma/\delta)]^2}{\Gamma(-n + \gamma/\delta)\Gamma(n + \gamma/\delta)} P^{(0)}_0,$$

$$Q^{(0)}_{n+1} = Q^{(0)}_0 + \sum_{j=0}^n \frac{P^{(0)}_j}{\gamma + j\delta},$$
which we have noted tend to multiples of the functions \( \psi'(z) \) and \( \psi(z) \) respectively (see (35)) in the appropriate limit. We obtain \( P_n^{(1)} \) immediately from (44b) by replacing \( \gamma \) by \( \gamma - \delta \), since \( P_n^{(1)} = Q_n^{(0)} \), and so

\[
P_n^{(1)} = Q_0^{(0)} + \sum_{j=0}^{n-1} \frac{P_j^{(0)}}{\gamma + (j-1)\delta}.
\]

Then we obtain \( Q_n^{(1)} \) by solving (42b) which yields

\[
Q_n^{(1)} = \left[ \gamma - (n-1)\delta \right] \sum_{j=0}^{n-2} \frac{P_j^{(1)}}{(\gamma + j\delta)[\gamma + (j+1)\delta]} + \frac{P_{n-1}^{(1)}}{\gamma + (n-2)\delta} + Q_0^{(1)}.
\]

We can readily see the difficulty in simplifying these solutions further but we can show from (45b) that as \( \delta \to 0 \) with \( \gamma \delta = 1 \) so

\[
Q_n^{(1)} \to \delta \sum_{j=0}^{n-1} P_j^{(1)} + Q_0^{(1)}.
\]

Since \( P_n^{(1)} \to \psi(z) \) in this limit, so (46) indicates that \( Q_n^{(1)} \) asymptotes the integral of \( \psi(z) \). Moreover, since \( \psi(z) \) satisfies

\[
\frac{d^2\psi}{dz^2} + 2z \frac{d\psi}{dz} = 0,
\]

this then means that

\[
Q_n^{(1)} \to z\psi(z) + \frac{1}{2} \frac{d\psi}{dz},
\]

and hence

\[
x_n^{(1)} = \frac{P_n^{(1)}}{Q_n^{(1)}} \to \frac{2\psi(z)}{2z\psi(z) + \psi'(z)} = \frac{2}{\Psi(z) + 2z},
\]

where \( \Psi(z) \) is as defined in (35a). Thus we conclude that \( x_n^{(0)} \) and \( x_n^{(1)} \) provide discretised forms of the first two complementary error function solutions of the continuous PIV equation (2) with parameters as given by (41), i.e. the continuous solutions \( \Psi(z) \) and \( 2/|\Psi(z) + 2z| \) (see Table 3.2.2 in [18]). Further discretised solutions could in principle by generated by proceeding up the chain of equations characterised by (42) and (43) although we shall not do this here. Nevertheless, what we have shown is how discrete analogues of complementary error function solutions of (2) may be obtained.

The solutions arising from equations (43) are the complementary error function-like generalisations of the discrete rational solutions whose first few members are listed in (21). These rational forms vanish as \( z \to \pm \infty \) as do the solutions which are obtained by following the method characterised by (41)-(48). However, we demonstrated in §2 how solutions within the so-called ‘\(-2z\)’ rational hierarchy can be obtained by appropriate choice of the separation constant in (14); now we shall again take \( \mu^{(m)} = -(m+1)\delta(2\gamma + m\delta) \), \( m = 0, 1, \ldots \) (with the corresponding values of \( \alpha \) and \( \beta \) then as defined in (27)) and the separation constant \( k^{(m)}_{n+1} = z_n + \gamma - m\delta \). Then, with the upper choice of signs, equations (14) show that in this case \( P_n^{(m)} \) and \( Q_n^{(m)} \) satisfy

\[
[z_n - (\gamma + \delta) - m\delta] P_{n+2}^{(m)} + 2(\gamma + \delta) P_{n+1}^{(m)} - [z_n + (\gamma + \delta) + m\delta] P_n^{(m)} = 0,
\]

\[
[z_n - \gamma - (m + 1)\delta] Q_{n+2}^{(m)} + 2\gamma Q_{n+1}^{(m)} - [z_n + \gamma + (m + 1)\delta] Q_n^{(m)} = 0,
\]

which are identical to (43) under the transformation \( \gamma \to -\gamma \). This then provides the starting point for the derivation of further complementary error function-like solutions which are the
natural extensions of the ‘$-2z_n$’ rational forms whose first few members are as in (26). The most straightforward example is provided by $m = 0$ whence, from (44), it can shown that

$$Q_n^{(0)} = \text{constant} + \left[ \frac{\Gamma (\gamma/\delta)}{\Gamma (-j + \gamma/\delta) \Gamma (j + \gamma/\delta)} \right]^2 \sum_{j=0}^{n-1} \frac{1}{(\gamma + j\delta)\Gamma (-j + \gamma/\delta) \Gamma (j + \gamma/\delta)}$$

and we have already shown how this function asymptotes to a multiple of $\psi(z)$ in the usual limit. It then follows that

$$(z_n + \gamma)P_{n+1}^{(0)} - (z_n + \gamma - \delta)P_{n}^{(0)} = 2\gamma\delta Q_n^{(0)}$$

and solution of this equation in the limit as $\delta \to 0$, with $\gamma\delta = 1$, leads to the result that

$$P_n^{(0)} \to P_0^{(0)} - 2\delta \sum_{j=0}^{n} Q_j^{(0)}.$$ 

Thus $P_n^{(0)}$ asymptotes to the function

$$P_s^{(0)} = -2 \int_{z}^{s} \psi(s) \, ds = -2z\psi(z) - \frac{d\psi}{dz},$$

and so the solution

$$x_n^{(0)} = \frac{P_n^{(0)}}{Q_n^{(0)}},$$

$$-2z\psi(z) + \psi'(z) = -2z - \Psi(z)$$

and thus we have the first in the sequence of ‘$-2z$’ complementary error function solutions of PIV (2). A list of the simpler solutions in this hierarchy may be found in Table 3.2.3 of [18] and discrete equivalents of these solutions may be derived using the ideas here; the essential ingredient of this process is the requirement to solve equations (49) for higher values of $m$.

4. Discussion

The main result of this paper has been the evaluation of a number of exact solutions of the d-PIV (1) and consequently answer a number of the questions raised by Tamizhmani et al. [21]. These solutions are analogues of known solutions of the continuous PIV (2). This provides further evidence of the close relationship and analogy between properties of the continuous and discrete Painlevé equations.

We believe that further exact solutions of d-PIV (1) will arise by studying the discrete Bäcklund transformations given in [21]. A detailed study is also required in order to find discrete analogues of these solutions of PIV (2) that are expressible in terms of the parabolic cylinder function $D_\nu(z)$, for non-integer $\nu$. An important open question for d-PIV (1) remains that of the derivation of an associated Lax pair and the form of the associated isomonodromic deformation problem.

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