A NOTE ON ZERO-SET OF FRACTIONAL SOBOLEV FUNCTIONS WITH NEGATIVE POWER OF INTEGRABILITY

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Abstract. We extend a Poincaré-type inequality for functions with large zero-sets by Jiang and Lin to fractional Sobolev spaces. As a consequence, we obtain a Hausdorff dimension estimate on the size of zero-sets for fractional Sobolev functions whose inverse is integrable. Also, for a suboptimal Hausdorff dimension estimate, we give a completely elementary proof based on a pointwise Poincaré-style inequality.

1. Introduction

Let \( \Omega \subset \mathbb{R}^n \) be an open set. For functions \( u : \Omega \rightarrow \mathbb{R}^n \) we are interested in the size of the zero set \( \Sigma \),

\[
\Sigma := \{ x \in \Omega : \lim_{r \to 0} \int_{B_r(x)} |f| = 0 \},
\]

under the condition that for some \( \alpha > 0 \),

\[
\int_{\Omega} |f|^{-\alpha} < \infty.
\]

Here and henceforth, for a measurable set \( A \subset \mathbb{R}^n \) we denote the mean value integral

\[
\int_{A} f \equiv (f)_{A} := |A|^{-1} \int_{A} f.
\]

In [8] Jiang and Lin showed that if \( f \in W^{1,p}(\Omega) \), then

\[
\mathcal{H}^s(\Sigma) = 0 \quad \text{where} \quad s = \max\{0, n - \frac{\alpha}{p}\}.
\]

They were motivated by the analysis of rupture sets of thin films, which is described by a singular elliptic equation. We do not go into the details of this; instead, for applications, we refer to, e.g., [2,3,6,7].

In this note, we extend Jiang and Lin’s result to fractional Sobolev spaces and obtain

**Theorem 1.1.** For \( \sigma \in (0,1] \) and for any \( f \in W^{\sigma,p}(\Omega) \) satisfying (1.1), \( \mathcal{H}^s(\Sigma) = 0 \), where \( s = \max\{0, n - \frac{\alpha}{p+\sigma}\} \).

Here, we use the following definitions for the (fractional) Sobolev space. For more on these we refer to, e.g., [1,4,10].
Definition 1.2. The homogeneous $W^{\sigma,p}$-norms are defined as follows:

$$[f]_{W^{\sigma,p}(\Omega)} := \left\| \nabla f \right\|_{L^p(\Omega)}.$$ 

For $\sigma \in (0,1)$ we define the Sobolev space $W^{\sigma,p}$, $\sigma \in (0,1]$, $p \in [1,\infty]$ is then the collection of functions $f : \Omega \to \mathbb{R}$ with finite Sobolev norms $\|f\|_{W^{\alpha,p}(\Omega)}$,

$$\|f\|_{W^{\alpha,p}(\Omega)} := \|f\|_{L^p(\Omega)} + [f]_{W^{\sigma,p}(\Omega)}.$$ 

To prove Theorem 1.1, the case $p \leq n/\sigma$ is the relevant one, since for the other cases we can use the embedding into the Hölder spaces; see [8]. We have the following extension to fractional Sobolev spaces of a Poincaré-type inequality from [8].

Theorem 1.3. For any $\theta > 0$, $\sigma \in (0,1]$, $p \in (1,n/\sigma]$, the there is a closed set $T \subset \{x \in B_R : \limsup_{r \to 0} \int_{B_r} |f| = 0\}$, that satisfies (1.2)

$$\mathcal{H}^s(T) > \frac{1}{\theta} R^s,$$

and for any ball $B_r$ with some radius $r > 0$,

(1.3) $$\mathcal{H}^s(T \cap B_r) \leq \theta r^s.$$ 

Then,

$$\|f\|_{L^p(B_R)} \leq C R^\sigma [f]_{W^{\sigma,p}(B_R)}.$$ 

In [8] this was proven for the classical Sobolev space $W^{1,p}$, using an argument based on the $p$-Laplace equation with measures and the Wolff potential. Our argument, on the other hand, is completely elementary and adapts the classical blow-up proof of the Poincaré inequality; see Section 2.

Once Theorem 1.3 is established, one can follow the arguments in [8] to obtain Theorem 1.1. These rely heavily on the theory of Souslin sets, [9], to find the closed set $T \subset \Sigma$ with the condition (1.2) and (1.3) satisfied. Those arguments are by no means elementary, but we were unable to remove them in order to show that $\mathcal{H}^s(\Sigma) = 0$. However, if one is satisfied in showing that $\mathcal{H}^t(\Sigma) = 0$ for any $t > s$, then there is a completely elementary argument, the details of which we will present in Section 3. There, we prove the following “pointwise” Poincaré-style inequality, from which the suboptimal Hausdorff dimension estimate easily follows; see Corollary 3.1.
**Lemma 1.4.** For any $\varepsilon > 0$, $p \in [1, \infty)$, there exists $C > 0$, such that the following holds. Let $f \in L^p_{\text{loc}}$, and assume $x \in \mathbb{R}^n$, such that

\begin{equation}
\lim_{r \to 0} \frac{1}{B_r(x)} \int_{B_r(x)} |f| = 0.
\end{equation}

Then for any $R > 0$, there exists $\rho \in (0, R)$ such that

$$
\int_{B_\rho(x)} |f|^p \leq C \left( \frac{R}{\rho} \right)^\varepsilon \int_{B_\rho(x)} \|f| - (|f|)_{B_\rho}\|^p.
$$

2. Poincaré Inequality: Proof of Theorem 1.3

By a scaling argument, Theorem 1.3 follows from Lemma 2.1.

**Lemma 2.1.** For any $\theta > 0$, $\sigma \in (0, 1]$, $\sigma \in (1, \frac{n}{\sigma}]$, $s \in (n - \sigma p, n]$, there is a constant $C > 0$ such that the following holds:

Let $f \in W^{\sigma,p}(B_1, [0, \infty))$, and assume that there is a closed set $T \subset B_1$ such that

$$
\mathcal{H}^s(T) \leq \frac{1}{\theta},
$$

as well as

$$
\mathcal{H}^s(T \cap B_r) \leq \theta r^s \text{ for any ball } B_r \text{ with radius } r > 0.
$$

Then,

$$
\|f\|_{L^p(B_1)} \leq C \left[ f \right]_{W^{\sigma,p}(B_1)}.
$$

**Proof.** We proceed by the usual blow-up proof of the Poincaré inequality: Assume the claim is false, and that for fixed $\theta, p, s, \sigma$ for any $k \in \mathbb{N}$ there are $f_k \in W^{\sigma,p}(B_1, [0, \infty))$ such that

$$
T_k \subset \{x \in B_1 : \limsup_{r \to 0} \frac{1}{B_r} \int_{B_r} f_k = 0\},
$$

and

$$
\mathcal{H}^s(T_k) > \frac{1}{\theta}, \quad \mathcal{H}^s(T_k \cap B_r) \leq \theta r^s \forall B_r,
$$

and

$$
\|f_k\|_{L^p(B_1)} > k \left[ f_k \right]_{W^{\sigma,p}(B_1)}.
$$

Replacing $f_k$ by $\frac{f_k}{\|f_k\|_p}$ (note that this does not change the definition and size of $T_k$), we can assume w.l.o.g.

$$
\|f_k\|_{L^p} \equiv 1
$$

and

$$
\left[ f_k \right]_{W^{\sigma,p}(B_1)} \xrightarrow{k \to \infty} 0.
$$

In particular, $f_k$ is uniformly bounded in $W^{\sigma,p}$, and by the Rellich-Kondrachov theorem, up to taking a subsequence, $f_k$ converges strongly in $L^p$, and weakly in $W^{\sigma,p}$ to some $f \in W^{\sigma,p}$, with $\left[ f \right]_{W^{\sigma,p}(B_1)} \equiv 0$, $\|f\|_{L^p} = 1$. Thus,

$$
f \equiv |B_1|^{-\frac{1}{p}},
$$
and setting \( g_k := |B_1|^{\frac{1}{p}} f_k \), we have found a sequence such that
\[
g_k \to 1 \quad \text{in } W^{\sigma,p}(B_1),
\]
\[
\mathcal{H}^s(T_k) > \frac{1}{\delta}
\]
and
\[
\mathcal{H}^s(T_k \cap B_r) \leq \theta r^s \quad \text{for any ball } B_r.
\]
This is a contradiction to Lemma 2.2. \( \square \)

We used the following lemma, which essentially quantifies the intuition, that a function approximating 1 in \( W^{\sigma,p} \) cannot be zero on a large set.

**Lemma 2.2.** Let \( \sigma \in (0,1] \), \( s \in (n - \sigma p, n] \), \( f_k \in W^{\sigma,p}(B_1, [0, \infty)) \), and assume that
\[
\|f_k - 1\|_{W^{\sigma,p}(B_1)} \xrightarrow{k \to \infty} 0.
\]
Then, for any \( T_k \subset B_1 \) closed and
\[
T_k \subset \{ x \in B_1 : \limsup_{r \to 0} \int_{B_r} f_k = 0 \},
\]
as well as for some \( \theta > 0 \),
\[
(2.1) \quad \mathcal{H}^s(T_k \cap B_r) \leq \theta r^s \quad \text{for any } B_r, \text{ for all } k
\]
we have
\[
\lim_{k \to \infty} \mathcal{H}^s(T_k) = 0.
\]

**Proof.** By the subsequence principle, it suffices to show
\[
\liminf_{k \to \infty} \mathcal{H}^s(T_k) = 0.
\]
By extension, we also can assume that \( f_k - 1 \to 0 \) in \( W^{\sigma,p}(\mathbb{R}^n) \), and \( f_k \equiv 1 \) on \( \mathbb{R}^n \setminus B_2 \).

On the one hand, we have
\[
[f_k]_{W^{\sigma,p}(\mathbb{R}^n)} \xrightarrow{k \to \infty} 0.
\]
On the other hand, up to picking a subsequence, we can assume the existence of \( R_k \in (0,1) \), for \( k \in \mathbb{N} \), and \( \lim_{k \to \infty} R_k = 0 \), such that
\[
\inf_{r > R_k, x \in B_1} \int_{B_{r} \setminus x} f_k \geq \frac{9}{10}.
\]
Since for any point \( x \in T_k \) we have that \( \lim_{t \to 0} \int_{B_{r} \setminus x} f_k(x) = 0 \), we expect the average (fractional) gradient around \( x \) to be fairly large. More precisely, we have the following

**Claim.** There is a uniform constant \( c_{s,\sigma,p} > 0 \), such that the following holds: For any \( x \in T_k \), there exists \( \rho = \rho_{k,x} \in (0, R_k) \) such that
\[
(2.2) \quad c_{s,\sigma,p} \rho^s \leq \rho^{-\sigma p} \int_{B_{\rho}} |f_k - (f_k)_{B_{\rho}}|^p \leq C [f_k]_{W^{\sigma,p}(B_{\rho})}^p.
\]
Of course, we only have to show the first inequality; the second inequality is the classical Poincaré inequality.
For the proof let us write $f$ instead of $f_k$. Then, since for $x \in T$,

$$\lim_{l \to \infty} \int_{B_{2^{-l-1}R_k(x)}} f = 0,$$

we have that

$$\frac{9}{10} \leq \sum_{l=0}^{\infty} \left( \int_{B_{2^{-l}R_k(x)}} f - \int_{B_{2^{-l-1}R_k(x)}} f \right) \leq C \sum_{l=0}^{\infty} \left( (2^{-l}R_k)^{-n} \int_{B_{2^{-l}R_k}} |f - (f)_{B_{2^{-l}R_k}}| \right),$$

Consequently, for any $\varepsilon > 0$, there has to be some $c_\varepsilon > 0$ and some $l \in \mathbb{N}$ such that

$$\left( (2^{-l}R_k)^{-n} \int_{B_{2^{-l}R_k}} |f - (f)_{B_{2^{-l}R_k}}| \right) \geq c_\varepsilon (2^{-l}R_k)^{\varepsilon},$$

because if the opposite inequality was true for all $l \in \mathbb{N}$ we would have

$$\frac{9}{10} \leq C c_\varepsilon R_k \sum_{l \in \mathbb{N}} 2^{-l} \leq C c_\varepsilon \sum_{l \in \mathbb{N}} 2^{-\varepsilon l},$$

which is false for $c_\varepsilon$ small enough.

Thus, for $\rho := 2^{-l}R_k \in (0, R_k)$,

$$\rho^{n-\sigma+\varepsilon} \leq C_\varepsilon \rho^{-\sigma} \int_{B_\rho} |f - (f)_{B_\rho}| \leq C_\varepsilon \left( \rho^{-\sigma p} \int_{B_\rho} |f - (f)_{B_\rho}|^p \right)^{\frac{1}{p}} \rho^{n-\frac{n}{p}},$$

that is,

$$\rho^{n-\sigma p+\varepsilon p} \leq C_\varepsilon \rho^{-\sigma p} \int_{B_\rho} |f - (f)_{B_\rho}|^p.$$

Setting $\varepsilon = \frac{s-(n-\sigma p)}{p} > 0$, we have shown for any $x \in T$ the existence of some $\rho \in (0, R_k)$ satisfying (2.2), and the claim is proven.

For any $k$ we cover $T_k$ by the family

$$\mathcal{F}_k := \{B_\rho(x), \ x \in T, \ B_\rho(x) \text{satisfies (2.2)}\}.$$ 

Since $T \subset B_2$ is closed and bounded, i.e. compact, we can find a finite subfamily still covering all of $T_k$, and then using Vitali’s (finite) covering theorem, we find a subfamily $\tilde{\mathcal{F}}_k \subset \mathcal{F}_k$ of disjoint balls $B_\rho(x)$, so that the union of the $B_{5\rho}$ covers all of $T_k$. We use this $\tilde{\mathcal{F}}_k$ as a cover for an estimate of the Hausdorff measure:

$$\mathcal{H}^s(T_k) \leq \sum_{B_\rho \in \tilde{\mathcal{F}}_k} \mathcal{H}^s(B_{5\rho} \cap T_k) \leq \theta 5^s \sum_{B_\rho \in \tilde{\mathcal{F}}_k} \rho^s \leq C_{\theta, s} \sum_{B_\rho \in \tilde{\mathcal{F}}_k} [f_k]_{W^{s,p}(B_\rho)} \leq C_{\theta, s} [f_k]_{W^{s,p}(\mathbb{R}^n)} \xrightarrow{k \to \infty} 0.$$ 

$\square$
3. AN ELEMENTARY PROOF FOR THE SUBOPTIMAL CASE

We start with the proof of the pointwise inequality, Lemma 1.4.

Proof. First, let us show the claim for \( p = 1 \):

Fix \( R, \varepsilon > 0, f \in L^1_{loc} \) and assume \( x = 0 \). W.l.o.g., \( f \geq 0 \). Set

\[
(3.1) \quad \tau = 2^{-n-1} \left( \sum_{l = -\infty}^{0} 2^{\varepsilon l} \right)^{-1} R^{-\varepsilon},
\]

and \( C_\varepsilon := R^{-\varepsilon} \tau^{-1} \). Assume by contradiction that the claim was false, i.e. assume that for any \( \rho \in (0, R) \),

\[
(3.2) \quad \int_{B_\rho} |f - (f)_{B_\rho}| < \tau \rho^\varepsilon \int_{B_\rho} f.
\]

Then for any \( K \in \mathbb{N} \),

\[
\int_{B_\rho} |f - (f)_{B_\rho}| < \tau \rho^\varepsilon \sum_{k = -K}^{0} \int_{B_{2^{k+1} \rho}} f - \int_{B_{2^{k} \rho}} f + \tau \rho^\varepsilon \int_{B_{2-K-1} \rho} f.
\]

\[
\leq 2^n \tau \rho^\varepsilon \sum_{k = -K}^{0} \int_{B_{2^{k+1} \rho}} |f - (f)_{B_{2^{k+1} \rho}}| + \tau \rho^\varepsilon \int_{B_{2-K-1} \rho} f.
\]

Setting now for \( l \in \mathbb{Z} \),

\[
a_l := \int_{B_{2^l R}} |f - (f)_{B_{2^l R}}|, \quad b_l := \int_{B_{2^l R}} f,
\]

the above equation applied to \( \rho = 2^l R \) reads as

\[
a_l \leq 2^n R^\varepsilon \tau 2^{\varepsilon l} \sum_{k = -K}^{0} a_{k+l} + \tau (2^l R)^\varepsilon b_{-K+l-1} \quad \text{for any } K \in \mathbb{N}, l \in -\mathbb{N}.
\]

In particular for any \( L \in \mathbb{N} \),

\[
\sum_{l = -L}^{0} a_l \leq 2^n R^\varepsilon \tau \sum_{l = -L}^{0} 2^{\varepsilon l} \sum_{k = -K}^{0} a_{k+l} + \tau R^\varepsilon \sum_{l = -L}^{0} 2^{\varepsilon l} b_{-K+l-1}
\leq 2^n R^\varepsilon \tau \sum_{l = -L}^{0} 2^{\varepsilon l} \sum_{k = -K+l}^{0} a_k + \tau R^\varepsilon \left( \sup_{j \leq -K} b_j \right) \sum_{l = -\infty}^{0} 2^{\varepsilon l}
\leq 2^n R^\varepsilon \tau \sum_{k = -L-K}^{0} a_k \sum_{l = -L}^{0} 2^{\varepsilon l} + \tau R^\varepsilon \left( \sup_{j \leq -K} b_j \right) \sum_{l = -\infty}^{0} 2^{\varepsilon l}
\leq \frac{1}{2} \sum_{k = -L-K}^{0} a_k + \frac{1}{2} \sup_{j \leq -K} b_j.
\]
Under the additional assumption that

\[(3.3) \quad \sum_{l=-\infty}^{0} a_l < \infty,\]

letting \(L, K \to \infty\), using that by (1.4) we have \(\lim_{l \to \infty} b_l = 0\), the above estimate implies that \(a_k = 0\) for all \(k \leq 0\). This means that \(f\) is a constant on \(B_R\), and in particular by (1.4), \(f\) is constantly zero in \(B_R\). This contradicts the strict inequality (3.2).

To see (3.3), fix \(K \in \mathbb{N}\) such that \(\sup_{j \leq -K} b_j \leq 2\). Then for \(c_L := \sum_{l=-L}^{0} a_l\), the above estimate becomes

\[c_L \leq \frac{1}{2} c_{L+K} + 1 \quad \text{for any } L \in \mathbb{N}.\]

In particular, for any \(i \in \mathbb{N}\),

\[c_{L+iK} \leq 2^{-i} c_L + \sum_{j=0}^{i} 2^{-j}.\]

Since \(c_i\) is monotonically increasing,

\[\sup_{i \geq L+K} c_i \leq c_L + \sum_{j=0}^{\infty} 2^{-j} < \infty.\]

This proves Lemma 1.4 for \(p = 1\).

If \(p > 1\), we apply this to \(f^p\), and obtain

\[(3.4) \quad \int_{B_\rho(x)} f^p \leq C \left(\frac{R}{\rho}\right)^{\varepsilon} \int_{B_\rho(x)} |f^p - (f^p)_{B_\rho}|.\]

We now need the following estimate, which holds for any \(p \in [1, \infty)\), and \(\delta \in (0, 1)\):

\[||a - b|^p - |a|^p - |b|^p| \leq \delta |a|^p + \frac{C_p}{\delta^p} |b|^p.\]

Since \(B_\rho\) is fixed, let us write \((f)\) for \((f)_{B_\rho}\). First, for any \(\delta \in (0, 1)\),

\[|f^p - (f^p)| \leq |f - (f)|^p + |(f)^p - (f^p)| + \frac{C}{\delta^p} |f - (f)|^p + \delta (f)^p.\]

Plugging this into (3.4), for \(\delta = \tilde{\delta}(R/\rho)^{-\varepsilon}\) small enough, we arrive at

\[(3.5) \quad \int_{B_\rho(x)} f^p \leq C \left(\frac{R}{\rho}\right)^{(1+p)\varepsilon} \int_{B_\rho(x)} |f - (f)|^p + C \rho^n \left(\frac{R}{\rho}\right)^{(1+p)\varepsilon} |(f)^p - (f^p)|.\]

Next,

\[|(f)^p - (f^p)| \leq ((|f|^p - f^p)|) \leq (|f - (f)|^p) + \delta f^p + \frac{C}{\delta^p} (|f - (f)|^p).\]
Plugging this now for $\delta = \tilde{\delta}(R/\rho)^{-1}(1+p)\varepsilon$ into (3.5), by absorbing we arrive at

$$\int_{B_\rho(x)} f^p \leq C \left( \frac{R}{\rho} \right)^{\varepsilon p} \int_{B_\rho(x)} |f - (f)|^p.$$ 

Since this holds for $\varepsilon > 0$ is arbitrarily small, this proves Lemma 1.4.

**Corollary 3.1.** For $\sigma \in (0, 1]$ and for any $f \in W^{\sigma,p}(\Omega)$ satisfying (1.1), $H^t(\Sigma) = 0$, whenever $t > s = \max\{0, n - \sigma \rho \alpha/\beta + \alpha\}$.

**Proof.** Let $\varepsilon > 0$, $R > 0$, and $x \in \Sigma$. Pick $\rho < R$ from Lemma 1.4 so that

$$\int_{B_\rho(x)} |f|^p \leq C R^{\varepsilon} \rho^{\sigma p - \varepsilon} [f]^p_{W^{\sigma,p}(B_\rho)}.$$ 

By Hölder and Young inequality, as in [8, Corollary 2.1],

$$\rho^{n + (2\varepsilon - \sigma p)\alpha/\beta} \leq C \int_{B_\rho(x)} |f|^p + C \rho^{\varepsilon} \int_{B_\rho(x)} |f|^{-\alpha} \leq C R^{2\varepsilon} [f]^p_{W^{\sigma,p}(B_\rho)} + C \int_{B_\rho(x)} |f|^{-\alpha}.$$ 

Now let $\varepsilon > 0$ such that $t > n + (2\varepsilon - \sigma p)\alpha/\beta$. Then what we have shown is that for any $R > 0$ and any $x \in \Sigma$ there exists $\rho \in (0, R)$ such that

$$\rho^t \leq C R^{2\varepsilon} [f]^p_{W^{\sigma,p}(B_\rho)} + C \int_{B_\rho(x)} |f|^{-\alpha}.$$ 

Now let

$$\mathcal{V}_R := \{B_\rho(x) : x \in \Sigma, \rho < R, (3.6) \text{ holds}\}.$$ 

Any countable disjoint subclass $\mathcal{U}_R \subset \mathcal{V}_R$ satisfies

$$\sum_{B_\rho \subset \mathcal{U}_R} \rho^t \leq C R^{2\varepsilon} [f]^p_{W^{\sigma,p}(\Omega)} + CR^{\varepsilon} \int_\Omega |f|^{-\alpha}.$$ 

By the Besicovitch covering theorem, as in, e.g., [5, Theorem 18.1], we find for any $R$ a countable subclass $\mathcal{U}_R \subset \mathcal{V}_R$, such that any point of $\Sigma$ is covered at least once, and at most a fixed number of times. Thus,

$$\mathcal{H}^t(\Sigma) = \lim_{R \to 0} \mathcal{H}^t_R(\Sigma) \leq C \lim_{R \to 0} \sum_{B_\rho \subset \mathcal{U}_R} \rho^t \leq C [f] \lim_{R \to 0} R^{\varepsilon} = 0.$$ 

$\square$
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