FIRST AND SECOND ORDER APPROXIMATIONS FOR A NONLINEAR WAVE EQUATION

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ABSTRACT. We consider the following nonlinear wave equation:

\[(\text{NLW}) \quad i\partial_t v - |D|v = |v|^2 v,\]

where \(D = -i\partial_x\), both on \(\mathbb{R}\) and \(\mathbb{T}\).

In the case of \(\mathbb{R}\), we prove that if the initial condition is of order \(O(\varepsilon)\) and supported on positive frequencies only, then the corresponding solution can be approximated by the solution of the Szegő equation. The Szegő equation writes \(i\partial_t u = \Pi_+ (|u|^2 u)\), where \(\Pi_+\) is the Szegő projector onto non-negative frequencies, and is a completely integrable system.

The approximation holds for a long time \(0 \leq t \leq C \varepsilon^2 (\log(\frac{1}{\varepsilon}) )^{1-2\alpha}, 0 \leq \alpha \leq 1/2\). The proof is based on the renormalization group method, first introduced in the context of theoretical physics by Chen, Goldenfeld, and Oono.

As a corollary, we give an example of solution of (NLW) on \(\mathbb{R}\) whose high Sobolev norms inflate over the time, relatively to the norm of the initial condition.

An analogous result of approximation was proved by Gérard and Grellier [10], in the case of \(\mathbb{T}\), using the theory of Birkhoff normal forms. We improve their result by finding the second order approximation with the help of an averaging method introduced by Temam and Wirosoetisno in [24]. We show that the effective dynamics will no longer be given by the Szegő equation.

1. Introduction

One of the most important properties in the study of the nonlinear Schrödinger equations (NLS) is dispersion. It is often exhibited in the form of the Strichartz estimates of the corresponding linear flow. In case of the cubic NLS:

\[(1.1) \quad i\partial_t u + \Delta u = |u|^2 u, \quad (t,x) \in \mathbb{R} \times M,\]

Burq, Gérard, and Tzvetkov [3] observed that the dispersive properties are strongly influenced by the geometry of the underlying manifold \(M\). Taking this idea further, Gérard and Grellier [8] remarked that dispersion disappears completely when \(M\) is a sub-Riemannian manifold or when the Laplacian is replaced by the Grushin operator. In those cases, by conveniently decomposing the function \(u\), we obtain that at least in the radial case, the Schrödinger equation is equivalent to the following system of transport equations:

\[(1.2) \quad i(\partial_t \pm (2m + 1)\partial_x)u_m = \Pi_m (|u|^2 u),\]

where \(\Pi_m\) are pseudo-differential orthogonal projectors. Therefore, studying the Schrödinger equation in a non-dispersive situation comes down to studying a system of the above type.

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In this paper we consider the following nonlinear wave equation on $\mathbb{R}$ and $\mathbb{T}$:

\[
\begin{align*}
\text{(NLW)} \\
&\begin{cases}
  i\partial_t v - |D|v = |v|^2 v, \\
v(0) = v_0
\end{cases}
\end{align*}
\]

where $D = -i\partial_x$. It is indeed a nonlinear wave equation since by applying the operator $i\partial_t + |D|$ to both sides of the equation, we obtain:

\[-\partial_t v + \Delta v = |v|^4 v + 2|v|^2(|D|v) - v^2(|D|v) + |D|(|v|^2 v)\].

Equation (NLW) was studied on $\mathbb{T}$ by Gérard and Grellier in [10]. We also define $\Pi^+ = L^2(\mathbb{T}) \to L^2(\mathbb{T})$, defined by:

\[\Pi^+ f(x) = \sum_{k=0}^{\infty} \hat{f}(k)e^{ikx}.\]

In the case of $\mathbb{R}$, the Szegő projector $\Pi^+ : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ can be defined similarly by:

\[\widehat{\Pi^+} f(\xi) = \begin{cases} 
\hat{f}(\xi), & \text{if } \xi \geq 0, \\
0, & \text{if } \xi < 0
\end{cases}\]

We also define $\Pi^+ = I - \Pi_+$, where $I$ is the identity operator. Applying the projectors $\Pi^+$ and $\Pi_-$ and writing $v = v^+ + v^-$, where $v^+ = \Pi^+(v)$, and $v^- = \Pi^-(v)$, we obtain that equation (NLW) is equivalent to the following system:

\[
\begin{align*}
\text{(1.3)} \\
&\begin{cases}
  i(\partial_t v^+ + \partial_x v^+) = \Pi^+ (|v|^2 v) \\
  i(\partial_t v^- - \partial_x v^-) = \Pi^- (|v|^2 v).
\end{cases}
\end{align*}
\]

Notice that this is a system of transport equations similar to the one obtained from the Schrödinger equation (1.2). We expect that the study of this system and therefore the study of the (NLW) equation help us understand better NLS in the case of lack of dispersion.

The (NLW) equation is a Hamiltonian evolution associated to the Hamiltonian

\[E(v) = \frac{1}{2}(|D|v, v) + \frac{1}{4}\|v\|_{L^4}^4,\]

with respect to the symplectic form $\omega(u, v) = \text{Im } \int u\overline{v}dx$. From this structure, we obtain the formal conservation law of energy $E(v(t)) = E(v(0))$. The invariance under translations and under modulations provides two more conservation laws, $Q(v(t)) = Q(v(0))$ and $M(v(t)) = M(v(0))$, where

\[Q(v) = \|v\|_{L^2}^2 \quad \text{and} \quad M(v) = (Dv, v).\]

The conservation of the mass and energy yields a uniform bound on the $H^{1/2}$-norm of the solution of (NLW) in $H^{1/2}$. The following result from [10] states that indeed, the (NLW) equation on $\mathbb{T}$ is globally well-posed in $H^{1/2}(\mathbb{T})$. 
Proposition 1.1 ([10]). The nonlinear wave equation \((\text{NLW})\) is globally well-posed in \(H^s(\mathbb{T})\). Moreover, if \(v_0 \in H^s(\mathbb{T})\) for some \(s > \frac{1}{2}\), then \(v \in C(\mathbb{R}; H^s)\).

An analogous result holds for \((\text{NLW})\) equation on \(\mathbb{R}\).

In this paper we prove that the solution of the \((\text{NLW})\) equation on \(\mathbb{R}\) with an initial condition of order \(O(\varepsilon)\) and supported only on positive frequencies, can be approximated by the solution of a simpler equation with the same initial data. The approximation is of order \(O(\varepsilon^2)\) and holds for a long time. The approximate equation is the Szegö equation, recently introduced by Gérard and Grellier:

\[
(1.4) \quad i \partial_t u = \Pi_+ (|u|^2 u).
\]

This equation was studied in details on \(\mathbb{T}\) in [8, 9] and on \(\mathbb{R}\) in [22, 23]. It is globally well-posed in \(H^s_+(\mathbb{T})\) and \(H^s_+(\mathbb{R})\) for \(s \geq 1/2\). Its most remarkable property is that it is completely integrable, in the sense that it admits a Lax pair. In particular, it possesses an infinite sequence of conservation laws, the strongest one being the \(H^{1/2}_+\)-norm.

The approximation result for the \((\text{NLW})\) equation on \(\mathbb{R}\) was motivated by a similar one proved by Gérard, Grellier [10] in the case of \(\mathbb{T}\). The case of \(\mathbb{R}\) brings new difficulties related, as we see below, to low frequencies. Moreover, the method used in the case of \(\mathbb{T}\) is the theory of Birkhoff normal forms. It seems difficult to use normal forms on \(\mathbb{R}\) due to small divisors problems. Our result will be proved using the renormalization group method of Chen, Goldenfeld and Oono [4, 5] coming from theoretical physics.

The heuristic idea that motivated our result on \(\mathbb{R}\) and the previous result on \(\mathbb{T}\) in [10] is the following. Consider the \((\text{NLW})\) equation with an initial condition \(v_0\) such that \(v_0 = \varepsilon u_0\), where \(u_0 \in H^{1/2}_+\). Since we have conservation of the momentum and of the energy, it follows that \(2E(v(t)) - M(v(t)) = 2E(v_0) - D(v_0)\). This yields:

\[
2(\|D|v_-(t), v_-(t)\| + \frac{1}{2}\|v(t)\|_{L^4}^2) = \frac{1}{2}\|v_{0,+}\|_{L^4}^2 = O(\varepsilon^4).
\]

Thus, \(\|v_-(t)\|_{H^{1/2}_+} = O(\varepsilon^2)\) for all \(t \in \mathbb{R}\). Moreover, we have

\[
\|v_-(t)\|_{H^{1/2}_+}^2 = \sum_{k \leq -1} (1 + |k|^2)^{1/2} |\hat{v}(k)|^2 \leq 2 \sum_{k \leq -1} |k|^2 |\hat{v}(k)|^2 \leq 2 \|v_-(t)\|_{H^{1/2}_+}^2 = O(\varepsilon^4).
\]

Then, \(\|v_-(t)\|_{H^{1/2}_+} = O(\varepsilon^2)\). Therefore, \(v_-(t)\) is \(\varepsilon^2\)-small, while the solution \(v(t)\) is only \(\varepsilon\)-small. It seems thus that the dynamics of \((\text{NLW})\) is dominated by \(v_+(t)\). We omit then all the terms containing \(v_-\) in the nonlinearity of the first equation in [10, 3], since they are supposed to be small. We obtain that \(u(t, x) = v_+(t, x + t)\) almost satisfies the Szegö equation

\[
i \partial_t u = \Pi_+ (|u|^2 u).
\]

Hence, it is natural to expect that the Szegö equation provides us with an approximation of the \((\text{NLW})\) equation with a small initial condition supported on positive frequencies.

In the case of \(\mathbb{R}\), the conservation of energy and momentum still gives \(\|v_-(t)\|_{H^{1/2}_+} = O(\varepsilon^2)\), while we have that \(\|v(t)\|_{H^{1/2}_+} = O(\varepsilon)\) for all \(t \in \mathbb{R}\). However, we have no other information on the \(L^2\)-norm of \(v_-(t)\). This suggests that the low frequencies cause some new difficulty in proving that \(v_-\) is small, and thus in proving that the flow of \((\text{NLW})\) can be approximated by that of the Szegö equation.

In what follows we state a weaker version of the approximation result for the \((\text{NLW})\) equation on \(\mathbb{T}\) in [10]. The original result holds for a slightly longer time \(0 \leq t \leq \frac{1}{\varepsilon} \log \left( \frac{1}{\varepsilon^2} \right)\).
and without assuming any bound on the solution of the Szegö equation. However, in the proof the authors use the complete integrability of the Szegö equation, while in Section 3, we will prove this weaker version without using the complete integrability.

**Theorem 1.2 (Gérard-Grellier [10]).** Let \(0 < \varepsilon \ll 1\), \(0 \leq \alpha \leq 1/2\), and \(\delta > 0\) sufficiently small. Let \(s > \frac{1}{2}\) and \(W_0 \in H^s_+(\mathbb{T})\). Let \(v(t)\) be the solution of the \((NLW)\) equation on \(\mathbb{T}\)

\[
\begin{aligned}
\begin{cases}
i\partial_t v - |D|v = |v|^2v \\
v(0) = W_0 := \varepsilon W_0.
\end{cases}
\end{aligned}
\]

(1.5)

Denote by \(W \in C(\mathbb{R}, H^{1/2}_+(\mathbb{T}))\) the solution of the Szegö equation on \(\mathbb{T}\):

\[
\begin{aligned}
\begin{cases}
i\partial_t W = \Pi_\perp (|W|^2 W) \\
W(0) = W_0
\end{cases}
\end{aligned}
\]

(1.6)

with the same initial data. Suppose that \(\|W(t)\|_{H^s} \leq C \varepsilon \left(\log\left(\frac{1}{\varepsilon \delta}\right)\right)^\alpha\) for all \(t \in \mathbb{R}\).

Then, if \(0 \leq t \leq \frac{1}{\varepsilon^2} \left(\log\left(\frac{1}{\varepsilon \delta}\right)\right)^{1-2\alpha}\), we have

\(\|v(t) - e^{-i|D|t}W(t)\|_{H^s} \leq \varepsilon^{3-C_0 \delta}\),

where \(C_0 > 0\) is an absolute constant.

In the second half of this paper, we improve the above result on \(\mathbb{T}\). We find a second order approximate solution, given by an equation which is more complex than the Szegö equation, but which provides a smaller error of order \(\varepsilon^5\) instead of \(\varepsilon^3\), in the approximation. For this purpose, we use the averaging method introduced by Temam and Wirosoetisno in [24].

In what follows we state and briefly comment the main results of the paper.

### 1.1. Main results.

First, in the case of \(\mathbb{R}\), we consider an initial condition for \((NLW)\) which is supported on positive frequencies only, is of order \(O(\varepsilon)\), and such that the corresponding solution of the Szegö equation is bounded for all times by \(C \varepsilon \left(\log\left(\frac{1}{\varepsilon \delta}\right)\right)^\alpha\), \(0 \leq \alpha \leq 1/2\). Then the solution of the \((NLW)\) equation with this initial condition stays \(\varepsilon^2\)-close to the solution of the Szegö equation with the same initial condition, for times \(0 \leq t \leq \frac{C \varepsilon^2 \left(\log\left(\frac{1}{\varepsilon \delta}\right)\right)^{1-2\alpha}}{\varepsilon^2}\).

**Theorem 1.3.** Let \(0 < \varepsilon \ll 1\), \(s > \frac{1}{2}\), and \(W_0 \in H^s_+(\mathbb{R})\). Let \(v(t)\) be the solution of the \((NLW)\) on \(\mathbb{R}\)

\[
\begin{aligned}
\begin{cases}
i\partial_t v - |D|v = |v|^2v \\
v(0) = W_0 = \varepsilon W_0.
\end{cases}
\end{aligned}
\]

(1.7)

Denote by \(W \in C(\mathbb{R}, H^s_+(\mathbb{R}))\) the solution of the Szegö equation on \(\mathbb{R}\)

\[
\begin{aligned}
\begin{cases}
i\partial_t W = \Pi_\perp (|W|^2 W) \\
W(0) = W_0
\end{cases}
\end{aligned}
\]

(1.8)

with the same initial data. Assume that there exist \(0 \leq \alpha \leq \frac{1}{2}\) and \(\delta > 0\) small enough such that \(\|W(t)\|_{H^s} \leq C \varepsilon \left(\log\left(\frac{1}{\varepsilon \delta}\right)\right)^\alpha\) for all \(t \in \mathbb{R}\).
Remark 1.4. Theorem 1.2 was proved in [10] using the theory of Birkhoff normal forms. Rational functions, for example interesting solutions of the Szegö equation are those whose initial conditions are non-generic (NLW) equation from the known results one has for the Szegö equation. Some particularly by inspection of a naive perturbation expansion, than by inspection of the vector field. The advantage of the RG method over the normal form theory is that the secular terms are more readily identified with non-generic initial condition. This method seems to be difficult to adapt to the case of the one in Theorem 1.3 for the case of R. Notice that the approximation in Theorem 1.2 in [10] for the case of εW 0 > 0 is an absolute constant and C* is a constant depending only on the H^1_+ (R)-norm of W_0.

Notice that the approximation in Theorem 1.2 in [10] for the case of εW_0 > 0 is better than the one in Theorem 1.3 for the case of R (ε^3 instead of ε^2). This is what we expected even from our heuristic argument above. We will see in the proof that the estimates we have in the case of R are worse than those for the case of T, due to low frequencies.

Remark 1.4. Theorem 1.2 was proved in [10] using the theory of Birkhoff normal forms. This method seems to be difficult to adapt to the case of R. The method we use in this paper is the renormalization group method, coming from theoretical physics. The two methods are intimately related. In [25], it was noticed that, for a large class of autonomous ODEs, the nonlinearity which appears in the RG equation of order one is actually the Birkhoff normal form. This result was extended in [7] to order two, for the same class of autonomous ODEs, and to first order, for a class of non-autonomous ODEs. The advantage of the RG method over the normal form theory is that the secular terms are more readily identified by inspection of a naive perturbation expansion, than by inspection of the vector field.

The purpose of the approximation Theorem 1.3 is to deduce some information on the (NLW) equation from the known results one has for the Szegö equation. Some particularly interesting solutions of the Szegö equation are those whose initial conditions are non-generic rational functions, for example W_0 = 1/(x+i) - 2/(x+2i). For such solutions, we proved in [23] the following result:

Proposition 1.5 ([23]). Let s > 1/2. Let \( W \in C(\mathbb{R}, H^s_+ (\mathbb{R})) \) be the solution of the Szegö equation

\[
i \partial_t W = \Pi_+( |W|^2 W)
\]

with non-generic initial condition \( W_0 = 1/(x+i) - 2/(x+2i) \in H^s_+ (\mathbb{R}) \). Then, for \( t \) large enough, there exist \( C, c > 0 \) such that

\[
c^t^{2s-1} \leq \|W(t)\|_{H^s} \leq Ct^{2s-1}.
\]

In particular, \( \|W(t)\|_{H^s} \to \infty \) as \( t \to \infty \).

The following corollary proves that the high Sobolev norms of the (NLW) equation with initial condition \( \varepsilon W_0 = 1/(x+i) - 2/(x+2i) \) grow relatively to the norm of the initial condition.

Corollary 1.6. Let \( 0 < \varepsilon \ll 1, s > \frac{1}{2} \), and \( \delta > 0 \) sufficiently small. Let \( W_0 \in H^s_+ (\mathbb{R}) \) be the non-generic rational function \( W_0 = 1/(x+i) - 2/(x+2i) \). Denote by \( v(t) \) be the solution of the (NLW) equation on \( \mathbb{R} \)

\[
\begin{cases}
i \partial_t v - |D|v = |v|^2 v \\
v(0) = \varepsilon W_0.
\end{cases}
\]

Then, for \( \frac{1}{2s} \left( \log \left( \frac{1}{\varepsilon s} \right) \right)^{\frac{1}{2s-1}} \leq t \leq \frac{1}{2s} \left( \log \left( \frac{1}{\varepsilon s} \right) \right)^{\frac{1}{2s-1}} \), we have that

\[
\frac{\|v(t)\|_{H^s(\mathbb{R})}}{\|v(0)\|_{H^s(\mathbb{R})}} \geq C \left( \log \left( \frac{1}{\varepsilon s} \right) \right)^{\frac{4s-2}{4s-1}}.
\]
A similar result is available for the case of $\mathbb{T}$ [10].

The time on which the approximation in Theorem 1.3 is available, $t \leq \frac{1}{\varepsilon^2} \left( \log \left( \frac{1}{\varepsilon^2} \right) \right)^{1-2\alpha}$, does not allow us to prove the existence of a time $t^\varepsilon$ such that $\|v(t^\varepsilon)\|_{H^s(\mathbb{R})} \to \infty$ as $\varepsilon \to 0$. For that to happen, we would need an approximation at least up to a time of order $\frac{1}{\varepsilon^{2+\beta}}$ where $\beta > 0$.

In the case of $\mathbb{T}$, we find the second order approximation, that is an approximation with an error of order $\varepsilon^5$ instead of $\varepsilon^3$. We notice that the effective dynamics are no longer given by the Szegő equation.

**Theorem 1.7.** Let $0 < \varepsilon \ll 1$, $s > \frac{1}{2}$, $0 \leq \alpha \leq \frac{1}{2}$, and $\delta > 0$ small enough. Let $W_0 \in H^s_+ (\mathbb{T})$ be such that the solution of the Szegő equation (1.4) with initial condition $\varepsilon W_0$ is uniformly bounded by $\varepsilon \left( \log \left( \frac{1}{\varepsilon^2} \right) \right)^{\alpha}$ for all $t \in \mathbb{R}$. Denote by $v(t)$ the solution of the (NLW) equation on $\mathbb{T}$

\[
\begin{cases}
    i \partial_t v - |D|v = |v|^2 v \\
    v(0) = W_0 = \varepsilon W_0.
\end{cases}
\]

Consider $W \in C(\mathbb{R}, H^s_+(\mathbb{T}))$ to be the solution of the following equation on $\mathbb{T}$:

(1.9) \[
\begin{cases}
    i \partial_t W = \Pi(|W|^2 W) - \Pi_+ (|W|^2 \frac{1}{2} \Pi_- (|W|^2 W)) - \frac{1}{2} \Pi_+ (|W|^2 \frac{1}{2} \Pi_- (|W|^2 W)) \\
    W(0) = W_0.
\end{cases}
\]

with the same initial condition.

For a function $h \in H^s (\mathbb{T})$, set

\[
f_{osc}(h, t) = e^{i|D|^\varepsilon (|e^{-i|D|^\varepsilon h|^2} e^{-i|D|^\varepsilon h) - \frac{1}{2\pi} \int_0^{2\pi} e^{i|D|^\varepsilon (|e^{-i|D|^\varepsilon h|^2} e^{-i|D|^\varepsilon h) d\tau}.
\]

Denote by $F_{osc}(h, t)$ the unique function of mean zero in $t$ such that $\frac{\partial F_{osc}(h, t)}{\partial t} = f_{osc}(h, t)$.

Consider

\[
v_{app}(t) = e^{-i|D|^\varepsilon (W(t) + F_{osc}(W(t), t)).
\]

Then, if $0 \leq t \leq \frac{1}{\varepsilon^2} \left( \log \left( \frac{1}{\varepsilon^2} \right) \right)^{1-2\alpha}$, we have

\[
\|v(t) - v_{app}(t)\|_{H^s} \leq \varepsilon^{5-C_0 \delta},
\]

where $C_0 > 0$ is an absolute constant.

The above result cannot be directly extended to the case of $\mathbb{R}$. The main reason is that in equation (1.3) we see appear the operator $\frac{1}{2} \Pi_-$. In the case of $\mathbb{T}$, we have that $\frac{1}{2} \Pi_\varepsilon e^{ikx} = \frac{1}{\varepsilon} \Pi_{\varepsilon k < -1}$ and thus there is no problem related to small divisors. However, in the case of $\mathbb{R}$, if we pass into the Fourier space, we have $\frac{1}{\varepsilon} \Pi_{\varepsilon k < 0}$ and when $\varepsilon$ approaches zero, this gives a singularity. A way to get around this singularity would be to consider instead of resonances, i.e. frequencies for which a certain phase is null of resonances, i.e. resonances $|\phi| \leq \gamma$, for an optimal $\gamma > 0$. However, it seems that this would complicate significantly the dynamics (1.9).

In order to prove Theorem 1.7, we use an averaging method introduced by Temam and Wirosoetisno in [24].

We briefly describe in what follows the renormalization method, the averaging method, the concept of resonance, and their usage in the literature.
1.2. The renormalization group method, the averaging method, and the concept of resonance. The renormalization group (RG) method was introduced by Chen, Goldenfeld, and Oono [4, 5] in the context of theoretical physics, as a unified tool for asymptotic analysis. Its origin goes back to perturbative quantum field theory.

The method is most often used to find a long-time approximate solution to a perturbed equation. The main advantage of the RG method is that it provides an algorithm that can be easily applied to many equations. The starting point is a naive perturbation expansion, so that one does not need to guess or to make ad hoc assumptions about the structure of the perturbation series. Then, the divergent terms in the expansion (unbounded in time), are removed by renormalization. This leads to introducing the renormalization group equation. The solution of the RG equation is the main part of an approximate solution.

The effectiveness of the RG method was illustrated in a variety of examples of ordinary differential equations traditionally analyzed using disparate methods, including the method of multiple scales, boundary layer theory, the WKBJ method, the Poincaré-Lindstedt method, and the method of averaging.

The method was justified mathematically for a large class of ODEs in [25, 7]. It was also rigorously applied to some PDEs on bounded intervals, namely the Navier-Stokes equations [18], a slightly compressible fluid equation and the Swift-Hohenberg equation [19], and the primitive equations of the atmosphere and the ocean [21]. In [1] it was applied to the quadratic nonlinear Schrödinger equation on $\mathbb{R}^3$.

The idea behind the RG method is that the dynamics of an equation is dominated by its resonant part. This idea is also used by Colliander, Keel, Staffilani, Takaoka, and Tao in [6] to prove the existence of solutions for the cubic non-linear Schrödinger equation on $\mathbb{T}^2$ with arbitrarily large high Sobolev norms. They consider a reduced resonant equation for which they prove growth of high Sobolev norms, and then show that this resonant equation provides a good approximation for the initial one.

The averaging method we use in this paper was introduced by Temam and Wirosoetisno in [24] in the context of a class of differential equations. At first order it is related to the RG method, while at higher orders it is related to the asymptotic expansions of Bogolyubov and Mitropol’skii [2].

The RG method can also be applied at higher orders, as it was done for ODEs in [7]. In the case of the (NLW) equation on $\mathbb{T}$, we could prove that at second order the RG equation is exactly the averaged equation (1.9) in Theorem 1.7. However, the computations one needs to do when applying the RG method at second order are much more tedious than when applying the averaging method. Another reason why we preferred to present the averaging method for the second order approximation, is that this method does not only give the effective dynamics (1.9), but also gives an algorithm of how to build an approximate solution and how to estimate the error, which is not clear when one applies the RG method at higher orders.

Both the RG and the averaging methods are based on the concept of decomposing the nonlinearity into its resonant and non-resonant parts. Such a decomposition was very effective in proving global existence of small solutions of dispersive equations and scattering. This was done in several works of Germain, Masmoudi, and Shatah [11, 12, 13, 14, 15, 20], who treated the case of the gravity water waves equation in dimension 3, the coupled Klein-Gordon equations with different speeds, and the quadratic nonlinear Schrödinger equation in dimension 2 and 3. Gustafson, Nakanishi, and Tsai treated the case of the
Gross-Pitaevskii equation in dimension 3 in [16]. They use time, space, and space-time resonances, whereas in this paper we only consider time resonances.

The specificity of the (NLW) equation is that the resonant set does not have measure zero, as it was the case in the above cited papers. For this reason it is natural not to expect scattering, but a long-time approximation of the solution by some effective dynamics governed by the effect of the resonant part of the non-linearity. The decomposition in resonant and non-resonant part, was used in [1], precisely in this purpose in the case of the quadratic Schrödinger equation in dimension 3.

The structure of the paper is as follows. In the rest of the introduction, we heuristically explain the need of splitting the nonlinearity into its resonant and oscillatory part, which is at the basis of both the RG and averaging method. In Section 2, we present the RG method and use it to prove Theorem 1.3 dealing with the first order approximation in the case of $\mathbb{R}$. We also prove Corollary 1.6 which refers to high Sobolev norm inflation in the case of non-generic initial data. To have a good comparison between the case of $\mathbb{R}$ and that of $\mathbb{T}$, and for a better understanding of the second order approximation in the case of $\mathbb{T}$, in Section 3 we re-prove Theorem 1.2 from [10] using the RG method. In Section 4, we present the averaging method at second order and use it to prove Theorem 1.7 treating the second order approximation in the case of $\mathbb{T}$.

1.3. Heuristics of the proof of Theorem 1.3. The first approach to proving Theorem 1.3 is the following one. Consider the change of variables $u(t) = \frac{1}{\varepsilon} e^{i|D|t} v(t)$. Then $u$ satisfies the equation:

$$\begin{align}\tag{1.10} 
\partial_t u &= -i\varepsilon^2 e^{i|D|t} (e^{-i|D|t}u |e^{-i|D|t}u|^2 e^{-i|D|t}u) \\
u(0) &= W_0. 
\end{align}$$

Let us now set $W(t) := \frac{W(t)}{\varepsilon}$. Then $W(t)$ satisfies

$$\begin{align}\tag{1.11} 
i\partial_t W &= \varepsilon^2 \Pi_+ (|W|^2 W) \\
W(0) &= W_0. 
\end{align}$$

Then, setting $w(t) = u(t) - W(t)$, we have

$$\|u(t) - e^{-i|D|t} W\|_{H^s} = \varepsilon \|e^{-i|D|t}(u(t) - W(t))\|_{H^s} = \varepsilon \|w(t)\|_{H^s}.$$ 

We have that $w$ satisfies the equation

$$\begin{align} 
\partial_t w &= -i\varepsilon^2 e^{i|D|t} (e^{-i|D|t}u |e^{-i|D|t}u|^2 e^{-i|D|t}u) + i\varepsilon^2 \Pi_+ (|W|^2 W) \\
w(0) &= 0. 
\end{align}$$

Therefore,

$$w(t) = -i\varepsilon^2 \int_0^t \left(e^{i|D|\tau} (e^{-i|D|\tau}u |e^{-i|D|\tau}u|^2 e^{-i|D|\tau}u) - \Pi_+ (|W(\tau)|^2 W(\tau))\right) d\tau$$

The classical technique of estimating $w(t)$ consists in writing the right-hand side in such a way that we see $w(\tau)$ appear under the integral, and then use Gronwall’s inequality. However, $w(\tau) = u(\tau) - W(\tau)$, and in the above relation the only term in which $u$ appears is $f(u, \tau) := -i e^{i|D|\tau} (e^{-i|D|\tau}u |e^{-i|D|\tau}u|^2 e^{-i|D|\tau}u)$. It is thus natural to decompose the term $f(u, \tau)$ into a part which does not explicitly depend on $\tau$ called the resonant part, $f_{\text{res}}(u)$, and a part which depends on $\tau$ called the oscillatory part, $f_{\text{osc}}(u, \tau)$. Then, $f_{\text{res}}(u) - \Pi_+ (|W|^2 W)$ provides us with a term $w = u - W$. 

Since we have more information on $W(\tau)$, which can be transformed with a simple change of variables into the solution of the Szegö equation (1.4), it may be more convenient to decompose $f(W, \tau) = -ie^{i|D|\tau}|e^{-i|D|\tau}W|^2e^{-i|D|\tau}W$. It turns out that its resonant part is exactly $-i\Pi_+ (|W|^2W)$ and thus

$$f(W(\tau), \tau) = -i\Pi_+ (|W(\tau)|^2W(\tau)) + f_{osc}(W(\tau), \tau).$$

Therefore,

$$w(t) = \varepsilon^2 \int_0^t \left(f(u(\tau), \tau) - f(W(\tau), \tau)\right) d\tau + \int_0^t f_{osc}(W(\tau), \tau) d\tau$$

The first term will indeed yield $w = u - W$, and we are left with estimating the integral of the oscillatory part $f_{osc}(W(\tau), \tau)$. Since it depends on $\tau$ both explicitly and implicitly, it turns out that it can be difficult to estimate its integral. For that reason we consider in the following $F_{osc}(W(t), t) = \int_0^t f_{osc}(W(t), \tau) d\tau$, where the integrand depends only explicitly on $\tau$. We construct an ansatz using $F_{osc}(W, t)$ and we prove that with this ansatz, the error is indeed small.

2. First order approximation for the (NLW) equation on $\mathbb{R}$

2.1. The renormalization group method at order one. In what follows we describe the RG method of first order in the case of the (NLW) equation on $\mathbb{R}$.

In the (NLW) equation, we make the change of variables $u(t) = \frac{1}{\varepsilon}e^{i|D|t}v(t)$ and set $\tilde{\varepsilon} := \varepsilon^2$. Then $u$ satisfies the equation:

$$\begin{cases}
\partial_t u = -i\tilde{\varepsilon}e^{i|D|t}(|e^{-i|D|t}u|^2e^{-i|D|t}u) =: \tilde{\varepsilon}f(u, t) \\
u(0) = \frac{1}{\varepsilon}v_0 =: u_0.
\end{cases} \tag{2.1}$$

The starting point of the RG method is the naive perturbation expansion

$$u(t) = u^{(0)}(t) + \tilde{\varepsilon}u^{(1)}(t) + \tilde{\varepsilon}^2u^{(2)}(t) + \ldots$$

Taylor-expanding $f(u, t)$ around $u^{(0)}$, we obtain

$$f(u, t) = f(u^{(0)}, t) + f'(u^{(0)}, t)(u(t) - u^{(0)}(t)) + \ldots = f(u^{(0)}, t) + \tilde{\varepsilon}f'(u^{(0)}, t)u^{(1)}(t) + \ldots$$

Plugging the last two expansions into the equation (2.1) and identifying the coefficients according to the powers of $\tilde{\varepsilon}$, we obtain:

$$\begin{cases}
\partial_t u^{(0)} = 0 \\
\partial_t u^{(1)} = f(u^{(0)}, t) \\
\partial_t u^{(2)} = f'(u^{(0)}, t) \cdot u^{(1)}(t) \\
\ldots
\end{cases} \tag{2.2}$$

Therefore, $u^{(0)}(t) = u_0$ for all $t \in \mathbb{R}$, and using Duhamel’s formula we have

$$u^{(1)}(t) = \int_0^t f(u_0, s) ds.$$  

Here we assumed that $u^{(1)}(0) = 0$. As it was shown in [25], this assumption does not cause a loss of generality for an approximation of order $\tilde{\varepsilon}$. Thus, if we look for an approximation of the solution up to order $O(\tilde{\varepsilon})$ and neglect any terms $O(\tilde{\varepsilon}^2)$, we have

$$u(t) = u_0 + \tilde{\varepsilon}u^{(1)}(t) + O(\tilde{\varepsilon}^2) = u_0 + \tilde{\varepsilon} \int_0^t f(u_0, s) ds + O(\tilde{\varepsilon}^2). \tag{2.3}$$
Now we decompose the nonlinearity $f(u, t)$ into its resonant and non-resonant part. In order to do that, we first write the nonlinearity in the Fourier space:

$$
\mathcal{F}(f(u, s))(\xi) = -ie^{i|\xi|s} \mathcal{F}(|e^{-i|D|s}u|^2 e^{-i|D|s}u)(\xi)
$$

$$
= -ie^{i|\xi|s} \int_{\mathbb{R}} \mathcal{F}((e^{-i|D|s}u)^2)(\eta) \mathcal{F}(e^{-i|D|s}u)(\xi - \eta)d\eta
$$

$$
= -ie^{i|\xi|s} \int_{\mathbb{R}} \mathcal{F}(e^{-i|D|s}u)(\eta - \zeta) \mathcal{F}(e^{-i|D|s}u)(\zeta) \mathcal{F}(e^{-i|D|s}u)(\xi)d\zeta d\eta
$$

$$
= -i \int_{\mathbb{R}} e^{is(|\xi - |\eta - \zeta| - |\eta - \zeta|)} \hat{u}(\eta - \zeta) \hat{u}(\zeta) d\zeta d\eta.
$$

Setting $\phi(\xi, \eta, \zeta) := |\xi| - |\xi| + |\eta - \xi| - |\eta - \zeta|$, we can write

(2.4) \quad f(u, s) = f_{\text{res}}(u) + f_{\text{osc}}(u, s),

where

(2.5) \quad f_{\text{res}}(u) = -i\mathcal{F}^{-1} \int_{\{\phi = 0\}} \hat{u}(\eta - \zeta) \hat{u}(\zeta) \hat{u}(\eta - \xi) d\zeta d\eta,

$$
f_{\text{osc}}(u, s) = -i\mathcal{F}^{-1} \int_{\{\phi \neq 0\}} e^{is(|\xi - |\eta - \zeta| - |\eta - \zeta|)} \hat{u}(\eta - \zeta) \hat{u}(\zeta) d\zeta d\eta.
$$

As it will be proved in Lemma 2.1 in the next section, for fixed $\xi$, the set $\{\phi(\xi, \eta, \zeta) = 0\} \subset \mathbb{R}^2$ has non-zero Lebesgue measure, and thus it makes sense to integrate on this set. More precisely, $\{\phi(\xi, \eta, \zeta) = 0\}$ is the set of $(\eta, \zeta) \in \mathbb{R}^2$ such that $\zeta, \eta - \xi, \eta - \zeta$ have the same sign as $\xi$, or $\zeta = \xi$, or $\eta - \xi = \xi$.

Plugging the decomposition (2.4) into the equation (2.3), we obtain

$$u(t) = u_0 + \tilde{e}tf_{\text{res}}(u_0) + \tilde{e} \int_0^t f_{\text{osc}}(u_0, s) ds + O(\tilde{e}^2).$$

We notice that the resonant part of the non-linearity, which is constant in time, causes the appearance of the secular term $\tilde{e}tf_{\text{res}}(u_0)$. This term will grow with time and will cause the approximation to break down as time approaches $\frac{1}{\tilde{e}}$. The purpose of the renormalization group method consists in re-normalizing the secular term. By doing that, its main contribution is taken into account in such a way that the approximation of $u$ stays valid at least up to a time of order $\frac{1}{\tilde{e}}$. The idea behind the renormalization group method is to regard the term $u_0 + \tilde{e}tf_{\text{res}}(u_0)$ as being the Taylor expansion of order one of a function $W(t)$ around $t = 0$. Then, one introduces the renormalization group equation:

(2.6) \quad \begin{cases}
\partial_t W = \tilde{e}f_{\text{res}}(W) \\
W(0) = u_0
\end{cases}

An approximation of order $O(\tilde{e})$ of $u(t)$ is then

$$u(t) = W(t) + \tilde{e}F_{\text{osc}}(W(t), t),$$

where we set $F_{\text{osc}}(h, t) := \int_0^t f_{\text{osc}}(h, s) ds$ for all $h \in H^\frac{1}{2}.$

2.2. Approximate solution for the (NLW) equation on $\mathbb{R}$. In this section we construct an approximate solution based on the solution of the RG equation. We first determine the resonant part of the non-linearity $f_{\text{res}}$. For that purpose we fix $\xi \in \mathbb{R}$, and determine the area in the $(\zeta, \eta)$-plane in which $\phi(\xi, \eta, \zeta)$ vanishes.
Let us first make the following notations:
\[ \xi_1 = \xi, \quad \xi_2 = \zeta, \quad \xi_3 = \eta - \xi, \quad \xi_4 = \eta - \zeta. \]

Notice that \( \xi_1 - \xi_2 + \xi_3 - \xi_4 = 0 \). Then, \( \phi(\xi, \eta, \zeta) = 0 \) is equivalent to \( |\xi_1| - |\xi_2| + |\xi_3| - |\xi_4| = 0 \). We have the following lemma, whose proof follows its analogue in the case of \( T \).

**Lemma 2.1.** The set of \( (\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4 \) such that \( \xi_1 - \xi_2 + \xi_3 - \xi_4 = 0 \) and \( |\xi_1| - |\xi_2| + |\xi_3| - |\xi_4| = 0 \) is
\[
\begin{align*}
M := & \{ (\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4 \mid \xi_1 \neq \xi_2, \xi_1 \neq \xi_3, \xi_2, \xi_3, \xi_4 \geq 0 \} \\
& \cup \{ (\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4 \mid \xi_1 \neq \xi_2, \xi_1 \neq \xi_4, \xi_2, \xi_3, \xi_4 \leq 0 \} \\
& \cup \{ (\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4 \} \cup \{ (\xi_1, \xi_2, \xi_2, \xi_1) \in \mathbb{R}^4 \}.
\end{align*}
\]

Coming back to the notations in \( \xi, \eta, \) and \( \zeta \), we have that \( \phi(\xi, \eta, \zeta) = 0 \) in the following cases:

- If \( \xi > 0 \) and \( (\eta, \zeta) \in \{ (\eta, \zeta) \in \mathbb{R}^2 \mid \eta \geq \xi, \eta \geq \zeta \geq 0 \} \cup \{ \zeta = \xi \} \cup \{ \eta - \zeta = \xi \} \)
- If \( \xi < 0 \) and \( (\eta, \zeta) \in \{ (\eta, \zeta) \in \mathbb{R}^2 \mid \eta \leq \xi, \eta \leq \zeta \leq 0 \} \cup \{ \zeta = \xi \} \cup \{ \eta - \zeta = \xi \} \)

Since, for fixed \( \xi \in \mathbb{R} \), the sets \( \{ \zeta = \xi \} \) and \( \{ \eta - \zeta = \xi \} \) are of measure zero in the \( (\xi, \eta) \)-plane, they do not interfere in the integration in equation (2.5), and thus we can neglect them. We are therefore left with the following two terms of \( F(f_{\text{res}}(u)) \):

1. The case \( \xi > 0, \zeta \geq 0, \eta - \zeta \geq 0, \eta - \xi \geq 0 \):
   \[
   - i 1_{\xi > 0} \int \hat{u}(\eta - \xi) \hat{u}(\xi) \overline{u}(\eta - \xi) 1_{\xi \geq 0} 1_{\eta - \zeta \geq 0} 1_{\eta - \xi \geq 0} d\zeta d\eta \\
   = - i 1_{\xi > 0} \int \hat{u}_+(\eta - \zeta) \hat{u}_+(\zeta) \overline{u}_+(\eta - \xi) d\zeta d\eta = - i F(\Pi_+ (|u_+|^2 u_+)) (\xi) 1_{\xi > 0}.
   \]

2. The case \( \xi < 0, \zeta < 0, \eta - \zeta < 0, \eta - \xi < 0 \):
   \[
   - i 1_{\xi < 0} \int \hat{u}(\eta - \xi) \hat{u}(\xi) \overline{u}(\eta - \xi) 1_{\xi < 0} 1_{\eta - \zeta < 0} 1_{\eta - \xi < 0} d\zeta d\eta \\
   = - i 1_{\xi < 0} \int \hat{u}_-(\eta - \zeta) \hat{u}_-(\zeta) \overline{u}_-(\eta - \xi) d\zeta d\eta = - i F(\Pi_- (|u_-|^2 u_-)) (\xi) 1_{\xi < 0}.
   \]

Thus, the resonant part of the nonlinearity is
\[
(2.7) \quad f_{\text{res}}(u) = - i \left( \Pi_+ (|u_+|^2 u_+) + \Pi_- (|u_-|^2 u_-) \right)
\]

Let \( W_0 \in H^s_+ (\mathbb{R}) \), \( s > 1/2 \). We consider the renormalization group equation:
\[
(2.8) \quad \begin{cases}
\partial_t W = e^2 f_{\text{res}}(W) \\
W(0) = W_0
\end{cases}
\]

Projecting onto non-negative and negative frequencies, we obtain two equations, one for \( W_+ := \Pi_+ (W) \) and one for \( W_- := \Pi_- (W) \). Notice first that, since \( W_0 \in H^s_+ (\mathbb{R}) \), we have
By the Cauchy-Lipschitz theorem, we have that
\[
\frac{d}{dt}W = \varepsilon^2 \Pi_+ (|W|^2 W)
\]
and
\[
W(0) = W_0.
\]

By the Cauchy-Lipschitz theorem, we have that \( W_-(t) = 0 \) for all \( t \in \mathbb{R} \), and thus \( W = W_+ \).

We construct an approximate solution by
\[
(2.9) \quad u_{app}(t) = W(t) + \varepsilon^2 F_{osc}(W(t), t).
\]
Then, \( u_{app} \) satisfies the equation
\[
(2.10) \quad \begin{cases}
\partial_t u_{app} = \varepsilon^2 f(W(t), t) + \varepsilon^4 D_W F_{osc}(W(t), t) \cdot f_{res}(W(t)) \\
u_{app} = W_0.
\end{cases}
\]
By the Duhamel formula, we obtain that
\[
(2.11) \quad u_{app}(t) = W_0 + \varepsilon^2 \int_0^t f(u_{app}(s))ds + \int_0^t R_\varepsilon(s,W(s))ds,
\]
where
\[
R_\varepsilon(W(t), t) = \varepsilon^2 \left( f(W(t)) - f(u_{app}(t)) \right) + \varepsilon^4 D_W F_{osc}(W(t), t) \cdot f_{res}(W(t)).
\]

2.3. Estimates for the oscillatory part of the nonlinearity in the case of \( \mathbb{R} \).

**Lemma 2.2.** Let \( s \geq 1 \). Let \( W \in C(\mathbb{R}, H^s_+(\mathbb{R})) \) be such that \( W = \varepsilon W \) is the solution of the Szeg\ö equation \((1.3)\) with initial data \( W_0 = \varepsilon W_0 \). Then, we have that
\[
\|F_{osc}(W, t)\|_{H^s} \leq C_* t^{1/2} + C\|W\|_{H^s}^3,
\]
\[
\|D_W F_{osc}(W(t), t) \cdot f_{res}(W(t))\|_{H^s} \leq C_* t^{1/2} + C\|W\|_{H^s}^2,
\]
where \( C > 0 \) is an absolute constant and \( C_* > 0 \) is a constant depending only on the \( H^s_+(\mathbb{R}) \)-norm of \( W_0 \).

**Proof.** Since \( W \in L^2_+(\mathbb{R}) \) and using Lemma 2.1 we have that
\[
\hat{F}_{osc}(W(t), s, \xi) = -i \int \int_{\phi \neq 0} e^{i\phi(\xi, \eta, \zeta)} \hat{W}(t, \eta - \zeta) \hat{W}(t, \zeta) \overline{W}(t, \eta - \xi) 1_{\eta - \zeta \geq 0} 1_{\xi \geq 0} d\eta d\zeta
\]
\[
= -i 1_{\xi < 0} \int \int e^{i\phi(\xi, \eta, \zeta)} \hat{W}(t, \eta - \zeta) \hat{W}(t, \zeta) \overline{W}(t, \eta - \xi) 1_{\eta \geq 1} 1_{\xi \geq 0} d\eta d\zeta.
\]
Then,
\[
\hat{F}_{osc}(W(t), t, \xi) = i \int_0^t \hat{F}_{osc}(W(t), s, \xi) ds
\]
\[
= -i 1_{\xi < 0} \int \int e^{i\phi(\xi, \eta, \zeta)} \frac{1}{i\phi} \hat{W}(t, \eta - \zeta) \hat{W}(t, \zeta) \overline{W}(t, \eta - \xi) 1_{\eta \geq 1} 1_{\xi \geq 0} d\eta d\zeta.
\]
Notice that in the region \( \xi < 0 \) and \( \{ (\eta, \zeta) \in \mathbb{R}^2 | \eta \geq \zeta \geq 0 \} \), we have that
\[
\phi(\xi, \eta, \zeta) = |\xi| - |\zeta| + |\eta - \xi| - |\eta - \zeta| = -\xi - \zeta + \eta - \xi - \eta + \zeta = -2\xi.
\]
Then,

\begin{equation}
\tilde{F}_{osc}(W(t), t, \xi) = \frac{e^{-2it\xi} - 1}{2\xi} F(|W|^2 W)(\xi)1_{\xi<0}.
\end{equation}

We now compute the $L^2$-norm of $F_{osc}(W(t), t)$, using Parseval’s identity:

\[
2\pi \|F_{osc}(W(t), t)\|_{L^2(\mathbb{R})}^2 = \left\| \hat{F}_{osc}(W(t), t) \right\|_{L^2(\mathbb{R})}^2 = \int_{-\infty}^{0} \frac{\sin^2(t\xi)}{\xi^2} |F(|W|^2 W)(\xi)|^2 d\xi \\
\leq \|F(|W|^2 W)\|_{L^\infty(\mathbb{R})}^2 \int_{-\infty}^{0} \frac{\sin^2(t\xi)}{\xi^2} d\xi \\
\leq \||W|^2 W\|_{L^2(\mathbb{R})}^2 t \int_{0}^{\infty} \frac{\sin^2 x}{x^2} dx \\
\leq Ct\|W\|_{L^2(\mathbb{R})}^6 \leq Ct\|W(t)\|_{H^{1/2}_{+}(\mathbb{R})}^6 \leq C\|W_0\|_{H^{1/2}_{+}(\mathbb{R})}^6.
\]

The last inequality is due to the conservation of the $H^{1/2}_{+}$-norm by the flow of the Szegö equation. Therefore,

\[
\|F_{osc}(W(t), t)\|_{L^2(\mathbb{R})} \leq C_s t^{1/2} \text{ for all } t \in \mathbb{R}.
\]

Let us now estimate the $H^s$-norm of $F_{osc}(W(t), t)$ for $s \geq 1$.

\[
\|F_{osc}(W(t), t)\|_{H^s(\mathbb{R})}^2 = \int_{-\infty}^{0} \xi^{2s} \frac{\sin^2(t\xi)}{\xi^2} |F(|W|^2 W)(\xi)|^2 d\xi \\
\leq \int_{-\infty}^{0} \xi^{2(s-1)} |F(|W|^2 W)(\xi)|^2 d\xi \\
\leq \||W|^2 W\|_{H^{s-1}(\mathbb{R})}^2 \leq \||W|^2 W\|_{H^s(\mathbb{R})}^2 \leq \||W|^2 W\|_{H^s(\mathbb{R})}^6.
\]

Therefore,

\[
\|F_{osc}(W(t), t)\|_{H^s(\mathbb{R})}^2 \leq C_s t^{1/2} + C\|W\|_{H^s(\mathbb{R})}^3.
\]

We proceed similarly for $DWF_{osc}(W(t), t) \cdot f_{res}(W)$. First, we notice that

\[
\mathcal{F}(DWF_{osc}(W, t) \cdot f_{res}(W))(\xi) = 2\frac{e^{-2it\xi} - 1}{2\xi} \mathcal{F}(|W|^2 f_{res}(W))(\xi)1_{\xi<0} \\
+ \frac{e^{-2it\xi} - 1}{2\xi} \mathcal{F}(W^2 f_{res}(W))(\xi)1_{\xi<0}.
\]
We use in what follows the fact that $f_{\text{res}}(W) = \Pi_+ (|W|^2 W)$, which is a consequence of equation (2.7) and of $W \in L^2_t (\mathbb{R})$. We estimate the $L^2$-norm, using Parseval's identity:

$$2\pi \| D_W F_{\text{osc}} (W, t) \cdot f_{\text{res}}(W) \|_{L^2(\mathbb{R})}^2 = \| \mathcal{F} (D_W F_{\text{osc}} (W, t) \cdot f_{\text{res}}(W)) (\xi) \|_{L^\infty(\mathbb{R})}^2 \leq C \int_{-\infty}^0 \frac{\sin^2 (t \xi)}{\xi^2} |\mathcal{F}(|W|^2 f_{\text{res}}(W)) (\xi)|^2 d\xi + C \int_{-\infty}^0 \frac{\sin^2 (t \xi)}{\xi^2} |\mathcal{F}(W^2 f_{\text{res}}(W)) (\xi)|^2 d\xi \leq C \left( \| |W|^2 f_{\text{res}}(W) \|_{L^\infty(\mathbb{R})}^2 + \| \mathcal{F}(W^2 f_{\text{res}}(W)) \|_{L^\infty(\mathbb{R})}^2 \right) \int_{-\infty}^0 \frac{\sin^2 (t \xi)}{\xi^2} d\xi \leq C t \| W \|_{L^4(\mathbb{R})}^4 \| f_{\text{res}}(W) \|_{L^2(\mathbb{R})}^2 \leq C t \| W \|_{H^1(\mathbb{R})}^4 \| f_{\text{res}}(W) \|_{L^2(\mathbb{R})}^2 \leq C t \| W \|_{H^{1/2}(\mathbb{R})}^4 \leq C \| W \|_{H^{1/2}(\mathbb{R})}^{10} \leq C \| W \|_{H^{1/2}(\mathbb{R})}^{10} \leq C \| W \|_{H^{1/2}(\mathbb{R})}^{10}.$$

Therefore, for $s \geq 1$ we have

$$\| D_W F_{\text{osc}} (W, t) \cdot f_{\text{res}}(W) \|_{H^s(\mathbb{R})} \leq C_s t^{1/2} + C \| W \|_{H^s(\mathbb{R})}^5.$$

\[ \Box \]

2.4. Proof of Theorem 1.3

Proof of Theorem 1.3. Let $v$ be the solution of equation (1.7). With the change of variables $u(t) = \frac{1}{\varepsilon} e^{\varepsilon D} (v(t))$, we have that $u$ satisfies the equation (1.10). By the Duhamel formula, it follows that

$$u(t) = W_0 + \varepsilon^2 \int_0^t f(u(s), s) ds.$$  \hspace{1cm} (2.13)

Set $w(t) = u(t) - u_{\text{app}}(t)$, where $u_{\text{app}}$ is defined by (2.9). By equations (2.13) and (2.11), we have that

$$w(t) = \varepsilon^2 \int_0^t \left( f(u(s), s) - f(u_{\text{app}}(s), s) \right) ds + \int_0^t R_\varepsilon (W(s), s) ds = \varepsilon^2 \int_0^t \left( f(u(s), s) - f(u_{\text{app}}(s), s) \right) ds + \varepsilon^2 \int_0^t \left( f(W(s), s) - f(u_{\text{app}}(s), s) \right) ds - \varepsilon^4 \int_0^t D_W F_{\text{osc}} (W(s), s) \cdot f_{\text{res}}(W(s)) ds = I + II + III.$$

Here $W$ denotes the solution of the renormalization group equation (1.11). In what follows, we estimate each of the terms $I, II, III$ in the $H^s$-norm, $s > 1/2$. Using the definition of $u_{\text{app}}$
\[ \|u_{\text{app}}(t)\|_{H^s} \leq \|W\|_{H^s} + \varepsilon^2 \|F_{\text{osc}}(W,t)\|_{H^s} \leq \|W\|_{H^s} + \varepsilon^2 C \|W\|_{H^s}^3. \]

Then, we have
\[
\|I\|_{H^s} \leq \varepsilon^2 \int_0^t \|w(\tau)\|_{H^s}(\|u(\tau)\|_{H^s}^2 + \|u_{\text{app}}(\tau)\|_{H^s}^2) \, d\tau
\]
\[
\leq C \varepsilon^2 \int_0^t \|w(\tau)\|_{H^s}(\|w(\tau)\|_{H^s}^2 + \|u_{\text{app}}(\tau)\|_{H^s}^2) \, d\tau
\]
\[
\leq C \varepsilon^2 \int_0^t \|w(\tau)\|_{H^s}(\|w(\tau)\|_{H^s}^2 + \|W\|_{H^s}^2 + \varepsilon^4 C \|W\|_{H^s}^6) \, d\tau.
\]

Using \( W(s) - u_{\text{app}}(s) = -\varepsilon^2 F_{\text{osc}}(W(s),s) \), and proceeding as above, we obtain
\[
\|II\|_{H^s} \leq \varepsilon^4 t \|F_{\text{osc}}(t,W(t))\|_{L^\infty([0,t],H^s)}(\|W\|_{L^\infty([0,t],H^s)}^2 + \|u_{\text{app}}\|_{L^\infty([0,t],H^s)}^2)
\]
\[
\leq C \varepsilon^4 t(t^{1/2} + \|W\|_{L^\infty([0,t],H^s)}^3)(\|W\|_{L^\infty([0,t],H^s)}^2 + \|W\|_{H^s}^6) + \varepsilon^4 t + \varepsilon^4 \|W\|_{H^s}^5,
\]

and
\[
\|III\|_{H^s} \leq C \varepsilon^4 t(t^{1/2} + \|W\|_{L^\infty([0,t],H^s)}^5).
\]

In order to estimate \( w \) we will use a bootstrap argument. Let \( 0 \leq \alpha \leq \frac{1}{2}, \delta > 0 \) small enough, and set
\[
T := \sup \left\{ t \geq 0 : \|w(t)\|_{H^s} \leq 1 \right\}.
\]

We will prove that \( T > \frac{1}{\varepsilon^2} \left( \log \left( \frac{1}{\varepsilon^2} \right) \right)^{1-2\alpha} \). Suppose by contradiction that
\[
T \leq \frac{1}{\varepsilon^2} \left( \log \left( \frac{1}{\varepsilon^2} \right) \right)^{1-2\alpha}.
\]

According to the hypothesis on \( W \) and since \( W = \varepsilon W \), we have that \( \|W(t)\|_{H^s(\mathbb{R})} \leq C \left( \log \left( \frac{1}{\varepsilon^2} \right) \right)^\alpha \) for all \( t \in \mathbb{R} \). Using the estimates of I, II, III, we obtain for \( 0 \leq t \leq T \)
\[
\|w(t)\|_{H^s} \leq C \varepsilon^2 \int_0^t \|w(\tau)\|_{H^s}(1 + \|W\|_{H^s}^2 + \varepsilon^4 C \|W\|_{H^s}^6) \, d\tau
\]
\[
+ C \varepsilon^4 t(t^{1/2} + \|W\|_{H^s}^3)(\|W\|_{H^s}^2 + \varepsilon^4 t + \varepsilon^4 \|W\|_{H^s}^6) + C \varepsilon^4 t(t^{1/2} + \|W\|_{H^s}^5)
\]
\[
\leq C \varepsilon^2 \left( \log \left( \frac{1}{\varepsilon^2} \right) \right)^{2\alpha} \int_0^t \|w(\tau)\|_{H^s} \, d\tau + C \varepsilon \left( \log \left( \frac{1}{\varepsilon^2} \right) \right)^{2(1-2\alpha)}
\]
\[
+ C \varepsilon \left( \log \left( \frac{1}{\varepsilon^2} \right) \right)^{\frac{1}{2}(1-2\alpha)}.
\]

By Gronwall’s inequality it follows, for \( 0 \leq t \leq \frac{1}{\varepsilon^2} \left( \log \left( \frac{1}{\varepsilon^2} \right) \right)^{1-2\alpha} \), that
\[
\|w(t)\|_{H^s} \leq C \varepsilon \left( \log \left( \frac{1}{\varepsilon^2} \right) \right)^{\frac{2-\alpha}{2}} e^{C \log \left( \frac{1}{\varepsilon^2} \right)} \leq C \varepsilon \left( \log \left( \frac{1}{\varepsilon^2} \right) \right)^{\frac{2-\alpha}{2}} \frac{1}{\varepsilon^{C\delta}} \leq C \varepsilon^{1-C_0 \delta},
\]

If \( \delta \) is sufficiently small, this bound is much better than the one imposed in the definition of \( T \). Since \( w \) is continuous with respect to \( t \), it follows that there exists \( \gamma > 0 \) such that
\[
\|w(t)\|_{H^s} \leq 1,
\]
for $0 \leq t \leq \frac{1}{\varepsilon^2}(\log(\frac{1}{\varepsilon^2}))^{\frac{1-2\alpha}{2\alpha}} + \gamma$. This contradicts the assumption (2.15) we made on $T$. Therefore, $T > \frac{1}{\varepsilon^2}(\log(\frac{1}{\varepsilon^2}))^{\frac{1-2\alpha}{2\alpha}}$ and, moreover, $\|w(t)\|_{H^s} \leq \varepsilon^{1-C_0\delta}$ for all $0 \leq t \leq \frac{1}{\varepsilon^2}(\log(\frac{1}{\varepsilon^2}))^{\frac{1-2\alpha}{2\alpha}}$. This yields

$$\|u(t) - W(t) - \varepsilon^2 F_{osc}(W(t), t)\|_{H^s(\mathbb{R})} \leq C_s \varepsilon^{1-C_0\delta}$$

for all $0 \leq t \leq \frac{1}{\varepsilon^2}(\log(\frac{1}{\varepsilon^2}))^{\frac{1-2\alpha}{2\alpha}}$. Since by Lemma 2.2 we have that

$$\|\varepsilon^2 F_{osc}(W(t), t)\|_{H^s(\mathbb{R})} \leq \varepsilon^2(C_s t^{1/2} + C|W|^3_{H^s}) \leq C_s \varepsilon\left(\log(\frac{1}{\varepsilon^{2s}})\right)^{\frac{1}{4}(1-2\alpha)} \leq C_s \varepsilon^{1-C_0\delta}$$

for $0 \leq t \leq \frac{1}{\varepsilon^2}(\log(\frac{1}{\varepsilon^2}))^{\frac{1-2\alpha}{2\alpha}}$, we obtain

$$\|u(t) - W(t)\|_{H^s(\mathbb{R})} \leq C_s \varepsilon^{1-C_0\delta}.$$ 

Recalling that $u(t) = \frac{1}{\varepsilon^4}e^{iDt}v(t)$ and $W = \frac{1}{\varepsilon^2}W$, we obtain that

$$(2.16)\quad \|v(t) - e^{-i|D|t}W(t)\|_{H^s(\mathbb{R})} \leq C_s \varepsilon^{2-C_0\delta},$$

for $0 \leq t \leq \frac{1}{\varepsilon^2}(\log(\frac{1}{\varepsilon^2}))^{\frac{1-2\alpha}{2\alpha}}$. \hfill \qed

2.5. Proof of Corollary 1.6.

Proof of Corollary 1.6. Let $W$ be the solution of the equation

\[
\begin{align*}
    i\partial_t W &= \varepsilon^2 \Pi_+ (|W|^2 W) \\
    W(0) &= W_0.
\end{align*}
\]

With the change of variables $W(t, x) = y(\varepsilon^2 t, x)$, we have that $y$ satisfies the Szegö equation:

\[
\begin{align*}
    i\partial_t y &= \Pi_+ (|y|^2 y) \\
    y(0) &= W_0.
\end{align*}
\]

Then, according to Proposition 1.5, we have that $\|y(t)\|_{H^s(\mathbb{R})} \sim t^{2s-1}$, for all $s > \frac{1}{2}$ and for $t > 1$ sufficiently large. Consequently, we have

$$\|W(t)\|_{H^s(\mathbb{R})} \sim (\varepsilon^2 t)^{2s-1}$$

for $\varepsilon^2 t$ sufficiently large. Suppose $\frac{1}{\varepsilon^2}(\log(\frac{1}{\varepsilon^2}))^{\frac{1}{4\alpha-1}} \leq t \leq \frac{1}{\varepsilon^2}(\log(\frac{1}{\varepsilon^2}))^{\frac{1}{4\alpha-1}}$. Then, \begin{equation}
(2.17) \quad \frac{c}{2^{2s-1}} \left(\log\left(\frac{1}{\varepsilon^2}\right)\right)^{\frac{2s-1}{4\alpha-1}} \leq W(t), t \leq C\left(\log\left(\frac{1}{\varepsilon^2}\right)\right)^{\frac{2s-1}{4\alpha-1}}.
\end{equation}

Applying Theorem 1.3 with $\alpha = \frac{2s-1}{4\alpha-1} \in (0, \frac{1}{2})$, we obtain that

$$(2.18) \quad \|v(t) - e^{-i|D|t}\varepsilon W(t)\|_{H^s(\mathbb{R})} \leq C_s \varepsilon^{2-C_0\delta},$$

for $0 \leq t \leq \frac{1}{\varepsilon^2}(\log(\frac{1}{\varepsilon^2}))^{\frac{1}{4\alpha-1}}$. Then, equations (2.17) and (2.18) yield

$$\|v(t)\|_{H^s} \geq \|\varepsilon W(t)\|_{H^s} - \|v(t) - e^{-i|D|t}\varepsilon W(t)\|_{H^s} \geq \frac{c}{2^{2s-1}} \varepsilon\left(\log\left(\frac{1}{\varepsilon^2}\right)\right)^{\frac{2s-1}{4\alpha-1}} - C_s \varepsilon^{2-C_0\delta} \geq C\varepsilon\left(\log\left(\frac{1}{\varepsilon^2}\right)\right)^{\frac{2s-1}{4\alpha-1}}.$$
Since $v(0) = \varepsilon W_0$, it follows that, for $\frac{1}{2\varepsilon^2}\left(\log\left(\frac{1}{\varepsilon\delta}\right)\right)^{\frac{1}{4s-1}} \leq t \leq \frac{1}{\varepsilon^2}\left(\log\left(\frac{1}{\varepsilon\delta}\right)\right)^{\frac{2s-1}{4s-1}}$, we have

$$\frac{\|v(t)\|_{H^s}}{\|v(0)\|_{H^s}} \geq C\left(\log\left(\frac{1}{\varepsilon\delta}\right)\right)^{\frac{2s-1}{4s-1}}.$$

$\square$

3. **First order approximation for the (NLW) equation on $\mathbb{T}$**

3.1. **The renormalization group equation for the case of $\mathbb{T}$**. We decompose a $2\pi$-periodic function $a(t)$ in the following way:

$$(3.1) \quad a(t) = a_{res} + a_{osc}(t),$$

where

$$(3.2) \quad a_{res} = \frac{1}{2\pi} \int_0^{2\pi} a(\tau)d\tau$$

is the mean of the function $a(t)$ or equivalently, the Fourier coefficient at zero. The oscillatory part is then

$$(3.3) \quad a_{osc}(t) = \sum_{k \neq 0} \hat{a}(k)e^{ikx}.$$ 

With this decomposition, we notice that for the torus, the resonant and non-resonant part of the nonlinearity are the following:

$$f_{res}(u, x) = -i \sum_{k = -\infty}^{\infty} e^{ikx} \sum_{|k| - |l| + |m| - |j| = 0} \hat{u}(j)\hat{u}(l)\overline{\hat{u}(m)},$$

$$f_{osc}(u, s, x) = -i \sum_{k = -\infty}^{\infty} e^{ikx} \sum_{|k| - |l| + |m| - |j| \neq 0} e^{is(|k| - |l| + |m| - |j|)}\hat{u}(j)\hat{u}(l)\overline{\hat{u}(m)}.$$ 

A slight difference with the case of $\mathbb{R}$ is the definition of $F_{osc}(u, t)$:

$$F_{osc}(u, t, x) := -i \sum_{k = -\infty}^{\infty} e^{ikx} \sum_{|k| - |l| + |m| - |j| = 0} \frac{e^{it(|k| - |l| + |m| - |j|)}}{i(|k| - |l| + |m| - |j|)}\hat{u}(j)\hat{u}(l)\overline{\hat{u}(m)},$$

whereas for $\mathbb{R}$, we had $F_{osc}(u, t) = \int_0^t f_{osc}(u, s)ds$. Notice that in both cases we have that $\frac{\partial F_{osc}}{\partial t}(u, t, x) = f_{osc}(u, t, x)$.

As it was shown in [10], the following lemma holds:

**Lemma 3.1.** We have that $k - l + m - j = 0$ and $|k| - |l| + |m| - |j| = 0$ if and only if we are in one of the following cases:

(i) If $k > 0$ and $\{l, m, j \geq 0\} \cup \{k = l\} \cup \{k = j\}$

(ii) If $k = 0$ and $\{l, m, j \geq 0\} \cup \{l, m, j \leq 0\}$

(iii) If $k < 0$ and $\{l, m, j \geq 0\} \cup \{k = l\} \cup \{k = j\}$.
We decompose the region where $k - l + m - j = 0$ and $|k| - |l| + |m| - |j| = 0$ into disjoint sub-regions, and we compute the Fourier transform of the resonant part $f_{\text{res}}(u)$. We obtain the following ten terms:

1. The case $k, l, m, j \geq 0$:
   \[-i \sum_{k-l+m-j=0 \atop k,l,m,j \geq 0} \hat{u}(j)\hat{u}(l)\overline{u}(m) = -i\mathcal{F}(|u_+|^2u_+)(k)\mathbf{1}_{k \geq 0}.
   \]

2. The case $k \geq 0, k = l, m = j < 0$:
   \[-i \sum_{l=k \geq 0 \atop m=j<0} \hat{u}(j)\hat{u}(l)\overline{u}(m) = -i\hat{u}(k)\mathbf{1}_{k \geq 0} \sum_{j=-\infty}^{-1} |\hat{u}(j)|^2 = -i\|u_-\|_{L^2}^2\hat{u}_+(k)\mathbf{1}_{k \geq 0}.
   \]

3. The case $k \geq 0, k = j, m = l < 0$. We obtain as above $-i\|u_-\|_{L^2}^2\hat{u}_+(k)\mathbf{1}_{k \geq 0}$.

4. The case $k = 0$ and $l, m, j < 0$:
   \[-i \sum_{l+m-j=0 \atop l,m,j < 0} \hat{u}(j)\hat{u}(l)\overline{u}(m) = -i\mathcal{F}(|u_-|^2u_-)(0).
   \]

5. The case $k, l, m, j < 0$:
   \[-i \sum_{k-l+m-j=0 \atop k,l,m,j < 0} \hat{u}(j)\hat{u}(l)\overline{u}(m) = -i\mathcal{F}(|u_-|^2u_-)(k)\mathbf{1}_{k < 0}.
   \]

6. The case $k < 0, l = 0, j < 0, m < 0$:
   \[-i \sum_{k+m-j=0 \atop k,m,j < 0} \hat{u}(j)\hat{u}(l)\overline{u}(m) = -i\hat{u}(0)\mathbf{1}_{k < 0} \sum_{j=k-1}^{-1} \hat{u}(j)\overline{u}(j-k)
   = -i\hat{u}(0)\mathcal{F}(\Pi_-|u_-|^2)(k)\mathbf{1}_{k < 0}.
   \]

7. The case $k < 0, j = 0, l < 0, m < 0$. We obtain as above $-i\hat{u}(0)\mathcal{F}(\Pi_-|u_-|^2)(k)\mathbf{1}_{k < 0}$.

8. The case $k < 0, m = 0, l < 0, j < 0$:
   \[-i \sum_{k-l-j=0 \atop k,l,j < 0} \hat{u}(j)\hat{u}(l)\overline{u}(m) = -i\hat{u}(0)\mathbf{1}_{k < 0} \sum_{j=k-1}^{-1} \hat{u}(j)\overline{u}(k-j)
   = -i\hat{u}(0)\mathcal{F}(u_+^2)(k)\mathbf{1}_{k < 0}.
   \]

9. The case $k < 0, k = l, m = j \geq 0$:
   \[-i \sum_{k=l<0 \atop m=j \geq 0} \hat{u}(j)\hat{u}(l)\overline{u}(m) = -i\hat{u}(k)\mathbf{1}_{k < 0} \sum_{j=0}^{\infty} |\hat{u}(j)|^2 = -i\|u_+\|_{L^2}^2\hat{u}_-(k)\mathbf{1}_{k < 0}.
   \]

10. The case $k < 0, k = j, l = m \geq 0$. We obtain as above $-i\|u_+\|_{L^2}^2\hat{u}_-(k)\mathbf{1}_{k < 0}$.
    Thus, the resonant part of the nonlinearity is
    \[(3.4) \quad f_{\text{res}}(u, x) = -i\Pi_+(|u_+|^2u_+ - 2i\|u_-\|_{L^2}^2u_+ - i\mathcal{F}(|u_-|^2u_-)(0)
    - i\Pi_-(|u_-|^2u_- - 2i\hat{u}(0)\Pi_-|u_-|^2) - i\hat{u}(0)u_-^2 - 2i\|u_+\|_{L^2}^2u_-.)\]
Lemma 3.2. Let $s > \frac{1}{2}$ and $W_0 \in H^s_+(\mathbb{T})$. We consider the renormalization group equation:

\begin{align}
\frac{\partial u}{\partial t} &= \varepsilon^2 f_{\text{res}}(u) \\
 u(0) &= W_0
\end{align} \tag{3.5}

This equation has a unique global solution in $H^s(\mathbb{T})$ which coincides with $W \in C(\mathbb{R}, H^s(\mathbb{T}))$, the solution of the following equation:

\begin{align}
 i\frac{\partial W}{\partial t} &= \varepsilon^2 \Pi + (|W|^2 W) \\
 W(0) &= W_0 \tag{3.6}
\end{align}

In particular, $u_-(t) = 0$ for all $t \in \mathbb{R}$.

Proof. We first notice that $f_{\text{res}} : H^s(\mathbb{T}) \to H^s(\mathbb{T})$, $s > \frac{1}{2}$, defined in equation (3.4) is a locally Lipschitz mapping. Indeed, one can prove using the structure of algebra of $H^s(\mathbb{T})$, that

$$
\|f_{\text{res}}(u) - f_{\text{res}}(v)\|_{H^s} \leq \|u - v\|_{H^s} (\|u\|^2_{H^s} + \|v\|^2_{H^s}),
$$

for all $u, v \in H^s(\mathbb{T})$. Then, by the Cauchy-Lipschitz theorem it follows that equation (3.5) has an unique solution in $H^s(\mathbb{T})$.

With the change of variables $W(t, x) = y(\varepsilon^2 t, x)$, we obtain from equation (3.6) that $y$ satisfies the Szegő equation (1.4). The Szegő equation has a unique global solution supported on non-negative frequencies. Thus $W$ is unique and satisfies $W_-(t) = 0$ for all $t \in \mathbb{R}$. The only term in the expression of $f_{\text{res}}(u)$ (3.4), which does not contain $u_-$ is $-i\Pi(|u_+|^2 u_+)$. Therefore we immediately notice that the solution of the equation (3.6) is also the solution of the equation (3.5). $\square$

3.2. Estimates for the oscillatory part of the nonlinearity in the case of $\mathbb{T}$. To re-prove Theorem 1.2 we apply exactly the same method used in the proof of Theorem 1.3. The only changes that appear are in the estimate of $F_{\text{osc}}(W(t), t)$. We show that on $\mathbb{T}$ we obtain a better estimate than on $\mathbb{R}$.

Lemma 3.3. Let $s > \frac{1}{2}$. For all $W \in H^s_+(\mathbb{T})$, we have that

$$
\|F_{\text{osc}}(W, t)\|_{H^s(\mathbb{T})} \leq C_s \|W\|^3_{H^s(\mathbb{T})},
$$

$$
\|DWF_{\text{osc}}(W, t) \cdot f_{\text{res}}(W)\|_{H^s(\mathbb{T})} \leq C_s \|W\|^5_{H^s(\mathbb{T})},
$$

where $C_s$ is a constant depending only on $s$.

Proof. The Fourier coefficients of $F_{\text{osc}}(W, t)$ are:

$$
\mathcal{F}(F_{\text{osc}})(W, t, k) = -\sum_{k-l+m-j=0, \atop |k|-|l|+|m|-|j| \neq 0} \frac{\varepsilon^{|l|+|m|+|j|} e^{it(|k|-|l|+|m|-|j|)}}{i(|k|-|l|+|m|-|j|) \hat{W}(j) \hat{W}(l) \overline{\hat{W}(m)}}.
$$
Setting $\hat{W}_k := \hat{W}(k)$ for all $k \in \mathbb{Z}$, and using the convexity of the function $|x|^\alpha$ if $\alpha > 1$, we have that

$$\|F_{\text{osc}}(W, t)|_{H^s(T)}|^2 \leq \sum_{k \in \mathbb{Z}} (1 + |k|^2)^s \sum_{k = l - m + j = 0, \ |k| - |l| + |m| - |j| \neq 0} e^{it(|k| - |l| + |m| - |j|)} |\hat{W}_j \hat{W}_{l} \hat{W}_{m}|^2$$

$$\leq \sum_{k \in \mathbb{Z}} (1 + |k|^2)^s \left( \sum_{k = l - m + j = 0} |\hat{W}_j \hat{W}_{l} \hat{W}_{m}| \right)^2$$

$$\leq \sum_{k \in \mathbb{Z}} (1 + |k|^2)^s \sum_{k = l - m + j} |\hat{W}_j \hat{W}_{l} \hat{W}_{m}| \sum_{k = l - m + j} |\hat{W}_j \hat{W}_{l} \hat{W}_{m}|$$

$$\leq \sum_{k \in \mathbb{Z}} (1 + |k|^2)^s \sum_{k = l - m + j} |\hat{W}_j \hat{W}_{l} \hat{W}_{m}|$$

$$= \sum_{l - m + j = 0} \sum_{l - m + j = 0} (1 + |l|^2 + |m|^2 + |j|^2)^{s/2} |\hat{W}_j| |\hat{W}_{l}| |\hat{W}_{m}|$$

$$\leq \sum_{l - m + j = 0} (1 + |l|^2)^{s/2} (1 + |m|^2)^{s/2} (1 + |j|^2)^{s/2}$$

$$\leq C_s \sum_{l - m + j = 0} [(1 + |l|^2)^{s/2} + (1 + |m|^2)^{s/2} + (1 + |j|^2)^{s/2}]$$

$$\leq C_s \sum_{l - m + j = 0} \sum_{l - m + j = 0} (1 + |j|^2)^{s/2} |\hat{W}_j||\hat{W}_{l}| |\hat{W}_{m}|$$

We consider the functions $V^* = \sum_{j \in \mathbb{Z}} e^{i\pi j} \hat{V}_j^*$ and $U^* = \sum_{j \in \mathbb{Z}} e^{i\pi j} \hat{U}_j^*$, where

$$\hat{V}_j^* := |\hat{W}_j|$$

$$\hat{U}_j^* := (1 + |j|^2)^{s/2} |\hat{W}_j|.$$ 

Ignoring the other terms in the above sum, which can be treated in a similar manner as the term we keep, and using the Sobolev embedding $H^s(T) \subset L^\infty(T)$ if $s > 1/2$, we obtain

$$\|F_{\text{osc}}(W(t), t)|_{H^s(T)}|^2 \leq C_s \sum_{l - m + j = 0} \sum_{l - m + j = 0} \hat{U}_j^* \hat{V}_l^* \hat{U}_m^* \hat{V}_j^* \hat{W}_m \leq C_s \int_T U^* U^* (V^*)^2 (V^*)^2 dz$$

$$\leq C_s \int_T |U^*|^2 |V^*|^4 dz \leq C_s \|U^*\|_{L^2(T)} \|V^*\|_{L^\infty(T)}^4$$

$$\leq C_s \|U^*\|_{L^2(T)} \|V^*\|_{H^s(T)}^4 \leq C_s \|V^*\|_{H^s(T)}^2 \|V^*\|_{H^s(T)} \leq C_s \|W^*\|_{H^s(T)}^6,$$

where $C_s$ denotes a constant depending on $s$. 

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The second estimate in the statement, 
\[ \|D_W F_{osc}(W, t) \cdot f_{res}(W)\|_{H^s(\mathbb{T})} \leq C_s \|W\|_{H^s(\mathbb{T})}^5, \]
can be proved similarly.

3.3. **Proof of Theorem 1.2** By the hypothesis we have that \( \|W(t)\|_{H^s} \leq C \left( \log\left(\frac{1}{\varepsilon^2}\right) \right)^{\alpha} \).
Using the definition of \( u_{app} \) [24], and Lemma 3.3, we obtain that \( \|u_{app}(t)\|_{H^s} \leq C \left( \log\left(\frac{1}{\varepsilon^2}\right) \right)^{\alpha} \). Proceeding as in the proof of Theorem 1.3, we obtain for \( 0 \leq t \leq \frac{1}{\varepsilon^2} \log\left(\frac{1}{\varepsilon^2}\right)^{1-2\alpha} \)
\[ \|w(t)\|_{H^s} \leq C \left( \log\left(\frac{1}{\varepsilon^2}\right) \right)^{2\alpha} \varepsilon^2 \int_0^t \|w(\tau)\|_{H^s} d\tau + C\varepsilon^4 \left( \log\left(\frac{1}{\varepsilon^2}\right) \right)^{5\alpha} t. \]
This yields, by Gronwall’s inequality, that for \( 0 \leq t \leq \frac{1}{\varepsilon^2} \left( \log\left(\frac{1}{\varepsilon^2}\right) \right)^{1-2\alpha} \) we have
\[ \|w(t)\|_{H^s} \leq C\varepsilon^4 \left( \log\left(\frac{1}{\varepsilon^2}\right) \right)^{5\alpha} \varepsilon^2 \left( \log\left(\frac{1}{\varepsilon^2}\right) \right)^{2\alpha} t \leq \varepsilon^{2-C_0\delta}, \]
where \( C_0 > 0 \). Since \( w(t) = u(t) - W(t) - \varepsilon^2 F_{osc}(W(t), t) \) and
\[ \|F_{osc}(W(t), t)\|_{H^s} \leq C \left( \log\left(\frac{1}{\varepsilon^2}\right) \right)^{3\alpha}, \]
then, it follows that
\[ \|u(t) - W(t)\|_{H^s(\mathbb{T})} \leq C\varepsilon^{2-C_0\delta} \text{ if } 0 \leq t \leq \frac{1}{\varepsilon^2} \left( \log\left(\frac{1}{\varepsilon^2}\right) \right)^{1-2\alpha}. \]
Then, the changes of variables \( v(t) = \varepsilon e^{i|D|t} u(t) \) and \( W = \varepsilon W \) yield the conclusion
\[ \|v(t) - e^{-i|D|t} W(t)\|_{H^s(\mathbb{T})} \leq C\varepsilon^{3-C_0\delta} \text{ if } 0 \leq t \leq \frac{1}{\varepsilon^2} \left( \log\left(\frac{1}{\varepsilon^2}\right) \right)^{1-2\alpha}. \]

4. **Second order approximation for the (NLW) equation on \( \mathbb{T} \)**

4.1. **The averaging method at order two.** As before, in the (NLW) equation with initial condition \( v(0) = W_0 = \varepsilon W_0 \), we make the change of variables \( u(t) = \frac{1}{\varepsilon} e^{i|D|t} v(t) \). Then \( u \) satisfies the equation:
\[
\begin{align*}
\partial_t u &= -i\varepsilon^2 e^{i|D|t} (|e^{-i|D|t} u|^2 e^{-i|D|t} u) =: f(u, t) \\
u(0) &= W_0.
\end{align*}
\]
The averaging method at order two introduced by Temam and Wiwrosièno in [24], consists in considering the following averaging ansatz:
\[ (4.1) \quad u_{app}(t) = W(t) + \varepsilon^2 N_1(W, t) + \varepsilon^4 N_2(W, t) =: N(W, t, \varepsilon), \]
where \( W \) is a solution of the following averaged equation:
\[ (4.2) \quad \begin{cases}
\partial_t W = \varepsilon^2 R_1(W) + \varepsilon^4 R_2(W) =: R(W, \varepsilon) \\
W(0) = W_0.
\end{cases} \]
The use of these notations is explained by the fact that \( R_1, R_2 \) turn out to be resonant terms, while \( N_1, N_2 \) are non-resonant (oscillatory) terms.
A formal computation then shows that
\[ \partial_t u_{app}(t) = \frac{\partial N(W, t, \varepsilon)}{\partial t} = N'(W, t, \varepsilon) \cdot \frac{\partial W}{\partial t} + \frac{\partial N}{\partial t}(W, t, \varepsilon) \]
\[ = (\varepsilon^2 N'_1(W, t) + \varepsilon^4 N'_2(W, t)) \cdot (\varepsilon^2 R_1(W) + \varepsilon^4 R_2(W)) + \frac{\partial W}{\partial t} \]
\[ + \varepsilon^2 \frac{\partial N_1}{\partial t}(W, t) + \varepsilon^4 \frac{\partial N_2}{\partial t}(W, t) \]
\[ = \varepsilon^2 \left( R_1(W) + \frac{\partial N_1}{\partial t}(W, t) \right) \]
\[ + \varepsilon^4 \left( R_2(W) + N'_1(W, t) \cdot R_1(W) + \frac{\partial N_2}{\partial t}(W, t) \right) + O(\varepsilon^6). \]

We now formally Taylor-expand \( f(u_{app}(t), t) \) around \( W(t) \),
\[ f(u_{app}, t) = f(W, t) + f'(W, t) \cdot (u_{app} - W) + O(\varepsilon^4) \]
\[ = f(W, t) + \varepsilon^2 f'(W, t) \cdot N_1(W, t) + O(\varepsilon^4). \]

We replace the two expansions into the equation
\[ \partial_t u_{app} = \varepsilon^2 f(u_{app}, t) + O(\varepsilon^6) \]
in order to determine \( R_1, R_2, N_1, N_2 \) which yield an approximate solution. Identifying the coefficients according to the powers of \( \varepsilon \), we obtain the equations

\((4.3)\)  
\[ R_1(W) + \frac{\partial N_1}{\partial t}(W, t) = f(W, t) \]

\((4.4)\)  
\[ R_2(W) + N'_1(W, t) \cdot R_1(W) + \frac{\partial N_2}{\partial t}(W, t) = f'(W, t) \cdot N_1(W, t) \]

Thus, \( R_1 \) is the part of \( f(W, t) \) which does not explicitly depend on \( t \). According to the decomposition given in equations \((3.1)\), \((3.2)\), and \((3.3)\), we have:

\[ R_1(W) = f_{res}(W) \quad \text{and} \quad N_1(W, t) = F_{osc}(W, t). \]

Then, from the second equation we have:

\((4.5)\)  
\[ R_2(W) = \{ f'(W, t) \cdot N_1(W, t) \}_\text{res} - \{ N'_1(W, t) \cdot R_1(W) \}_\text{res} \]
\[ \frac{\partial N_2}{\partial t}(W, t) = \{ f'(W, t) \cdot N_1(W, t) \}_\text{osc} - \{ N'_1(W, t) \cdot R_1(W) \}_\text{osc}. \]

Replacing \( R_1, N_1 \) and noticing that \( F'_{osc}(W, t) \cdot f_{res}(W) \) does not have a resonant part, we obtain:

\((4.6)\)  
\[ R_2(W) = \{ f'(W, t) \cdot F_{osc}(W, t) \}_\text{res} - \{ F'_{osc}(W, t) \cdot f_{res}(W) \}_\text{res} \]
\[ = \{ f'(W, t) \cdot F_{osc}(W, t) \}_\text{res} \]
\[ \frac{\partial N_2}{\partial t}(W, t) = \{ f'(W, t) \cdot F_{osc}(W, t) \}_\text{osc} - F'_{osc}(W, t) \cdot f_{res}(W). \]

We set \( w(t) := u(t) - u_{app}(t) \). In what follows, we determined a simplified version of the equation satisfied by \( w \). First, by the definition of \( u_{app} \), we have that \( w \) satisfies:

\[ \begin{cases} 
\frac{\partial w}{\partial t} = \varepsilon^2 f(u, t) - \frac{\partial W}{\partial t} - \varepsilon^2 \frac{\partial N_1}{\partial t}(W, t) - \varepsilon^2 N'_1(W, t) \cdot \frac{\partial W}{\partial t} - \varepsilon^4 \frac{\partial N_2}{\partial t}(W, t) - \varepsilon^4 N'_2(W, t) \cdot \frac{\partial W}{\partial t} \\
w(0) = 0 
\end{cases} \]
We consider the following Taylor expansion of \( f(u) \) around \( W \):
\[
\begin{align*}
f(u, t) &= f(w + u_{\text{app}}) = f(w + W + \varepsilon^2 N_1 + \varepsilon^4 N_2, t) \\
&= f(W, t) + f'(W, t) \cdot (w + \varepsilon^2 N_1 + \varepsilon^4 N_2) \\
&\quad + \int_0^1 f''(\alpha(w + \varepsilon^2 N_1 + \varepsilon^4 N_2) + W) \cdot (w + \varepsilon^2 N_1 + \varepsilon^4 N_2) \\
&\quad \otimes (w + \varepsilon^2 N_1 + \varepsilon^4 N_2)(1 - \alpha) d\alpha.
\end{align*}
\]
Replacing this into the equation of \( w \) and using the equation (4.2), we obtain that
\[
\frac{\partial w}{\partial t} = \varepsilon^2 f(W, t) + \varepsilon^2 f'(W, t) \cdot (w + \varepsilon^2 N_1 + \varepsilon^4 N_2)
\]
\[
\quad + \varepsilon^2 \int_0^1 f''(\alpha(w + \varepsilon^2 N_1 + \varepsilon^4 N_2) + W) \cdot (w + \varepsilon^2 N_1 + \varepsilon^4 N_2)
\]
\[
\quad \otimes (w + \varepsilon^2 N_1 + \varepsilon^4 N_2)(1 - \alpha) d\alpha
\]
\[
\quad - \varepsilon^2 R_1(W) - \varepsilon^4 R_2(W) - \varepsilon^2 \frac{\partial N_1}{\partial t}(W, t) - \varepsilon^2 N_1'(W, t) \cdot (\varepsilon^2 R_1(W) + \varepsilon^4 R_2(W))
\]
\[
\quad - \varepsilon^4 \frac{\partial N_2}{\partial t}(W, t) - \varepsilon^4 N_2'(W, t) \cdot (\varepsilon^2 R_1(W) + \varepsilon^4 R_2(W)).
\]
By the equations (4.3), it follows that
\[
\frac{\partial w}{\partial t} = \varepsilon^2 f'(W, t) \cdot (w + \varepsilon^4 N_2)
\]
\[
\quad + \varepsilon^2 \int_0^1 f''(\alpha(w + \varepsilon^2 N_1 + \varepsilon^4 N_2) + W) \cdot (w + \varepsilon^2 N_1 + \varepsilon^4 N_2)
\]
\[
\quad \otimes (w + \varepsilon^2 N_1 + \varepsilon^4 N_2)(1 - \alpha) d\alpha
\]
\[
\quad - \varepsilon^6 N_1'(W, t) \cdot R_2(W) - \varepsilon^4 N_2'(W, t) \cdot (\varepsilon^2 R_1(W) + \varepsilon^4 R_2(W)).
\]
Integrating from 0 to \( t \), we then obtain that
\[
(4.7) \quad w(t) = \varepsilon^2 \int_0^t f'(W, \tau) \cdot w(\tau) d\tau + \varepsilon^6 \int_0^t f''(W, \tau) \cdot N_2(W, \tau) d\tau
\]
\[
\quad - \varepsilon^6 \int_0^t N_1'(W, \tau) \cdot R_2(W) d\tau - \varepsilon^4 \int_0^t N_2'(W, \tau) \cdot (\varepsilon^2 R_1(W) + \varepsilon^4 R_2(W)) d\tau
\]
\[
\quad + \varepsilon^2 \int_0^t \int_0^1 f''(\alpha(w + \varepsilon^2 N_1 + \varepsilon^4 N_2) + W) \cdot (w + \varepsilon^2 N_1 + \varepsilon^4 N_2)
\]
\[
\quad \otimes (w + \varepsilon^2 N_1 + \varepsilon^4 N_2)(1 - \alpha) d\alpha d\tau.
\]
4.2. Study of the second order averaged equation in the case of \( T \). Let \( W_0 \in H^s_+(\mathbb{T}) \), \( s > 1/2 \). We consider the averaged equation
\[
\begin{cases}
\partial_t W = \varepsilon^2 R_1(W) + \varepsilon^4 R_2(W) \\
W(0) = W_0.
\end{cases}
\]
Since we already computed \( R_1 \) and \( R_2 \), we can rewrite this equation as:
\[
\begin{cases}
\partial_t W = \varepsilon^2 f_{\text{res}}(W) + \varepsilon^4 \{ f'(W, t) \cdot F_{\text{osc}}(W, t) \}_{\text{res}} \\
W(0) = W_0.
\end{cases}
\]
Setting \( \mathcal{W} = \varepsilon \mathcal{W} \), we have that \( \mathcal{W} \) satisfies the equation:

\[
\begin{aligned}
\frac{\partial \mathcal{W}}{\partial t} &= f_{\text{res}}(\mathcal{W}) + \{ f'(\mathcal{W}, t) \cdot F_{\text{osc}}(\mathcal{W}, t) \}_{\text{res}} \\
\mathcal{W}(0) &= \varepsilon \mathcal{W}_0 =: \mathcal{W}_0.
\end{aligned}
\]  

(4.8)

**Lemma 4.1.** Let \( s > \frac{1}{2} \). The problem \( \text{(4.8)} \) is locally well-posed in \( H^s(\mathbb{T}) \) at least on a time-interval \([0, \frac{C}{\varepsilon}]\), where \( C > 0 \).

**Proof.** We first estimate the two terms on the right-hand-side of equation \( \text{(4.8)} \). By equation \( \text{(3.4)} \), we have that

\[
\| R_1(\mathcal{W}) \|_{H^s(\mathbb{T})} = \| f_{\text{res}}(\mathcal{W}) \|_{H^s(\mathbb{T})} \leq C \| \mathcal{W} \|^2_{H^s(\mathbb{T})}.
\]

Then, we explicitly write the Fourier coefficients of \( \{ f'(\mathcal{W}, t) \cdot F_{\text{osc}}(\mathcal{W}, t) \}_{\text{res}} \). Since we have

\[
\mathcal{F}(f(\mathcal{W}))(k) = -i \sum_{k-l+m-j=0} e^{it(k+|-l|+|m|-|j|)} \hat{W}(j) \hat{W}(l) \hat{W}(m),
\]

it follows that

\[
\begin{aligned}
\mathcal{F}\left( f'(\mathcal{W}) \cdot F_{\text{osc}}(\mathcal{W}, t) \right)(k) &= -2i \sum_{k-l+m-j=0} e^{it(k+|-l|+|m|-|j|)} F_{\text{osc}}(\mathcal{W}, t)(j) \hat{W}(l) \hat{W}(m) \\
&\quad - i \sum_{k-l+m-j=0} e^{it(k+|-l|+|m|-|j|)} \hat{W}(j) \hat{W}(l) F_{\text{osc}}(\mathcal{W})(m) \\
&\quad + 2i \sum_{k-l+m-j=0} \sum_{j-n+p-q=0 \atop |j|-|n|+|p|-|q| \neq 0} e^{it(k+|-l|+|m|-|j|)} \hat{W}(j)(l) \hat{W}(m) \frac{e^{-it(|m|-|n|+|p|-|q|)}}{|m|-|n|+|p|-|q|} \hat{W}(n) \hat{W}(p) \hat{W}(l) \hat{W}(m).
\end{aligned}
\]

(4.9)

Then, \( R_2(\mathcal{W}) \), the resonant part of \( f'(\mathcal{W}, t) \cdot F_{\text{osc}}(\mathcal{W}, t) \), has the following Fourier coefficients:

\[
\mathcal{F}(R_2(\mathcal{W})) = \mathcal{F}\left( \{ f'(\mathcal{W}) \cdot F_{\text{osc}}(\mathcal{W}, t) \}_{\text{res}} \right)(k) = 2i \sum_{k-l+m-j=0 \atop j-n+p-q=0 \atop |k|-|l|+|m|-|n|+|p|-|q| \neq 0} \frac{1}{|j|-|n|+|p|-|q|} \hat{W}(j)(l) \hat{W}(m) \hat{W}(n) \hat{W}(p) \hat{W}(l) \hat{W}(m) \\
+ i \sum_{k-l+m-j=0 \atop m-n+p-q=0 \atop |m|-|n|+|p|-|q| \neq 0} \frac{1}{|m|-|n|+|p|-|q|} \hat{W}(j)(l) \hat{W}(n) \hat{W}(q) \hat{W}(p).
\]

(4.10)
Noticing that
\[
|\mathcal{F} \left( \{f'(W) \cdot F_{osc}(W, t)\}_{\text{res}} \right)(k)| \leq 2 \sum_{k-l+m-n+p-q=0} \left| \hat{W}(n) \hat{W}(q) \hat{W}(p) \hat{W}(l) \hat{W}(m) \right| + \sum_{k-l-j+n+p-q=0} \left| \hat{W}(j) \hat{W}(l) \hat{W}(n) \hat{W}(q) \hat{W}(p) \right|
\]
and proceeding as in the proof of Lemma 4.3, we obtain
\[
(4.11) \quad \|R_2(W)\|_{H^s(T)} = \|\{f'(W) \cdot F_{osc}(W, t)\}_{\text{res}}\|_{H^s} \leq C \|W\|_{H^s(T)}^5.
\]
We use a standard fixed point argument to prove that equation (4.8) is locally well-posed. Define
\[
AW(t) := W(0) + \int_0^t f_{\text{res}}(W(\tau)) d\tau + \int_0^t \{f'(W(\tau), \tau) \cdot F_{osc}(W(\tau), \tau)\}_{\text{res}} d\tau.
\]
We intend to show that there is \( T = \frac{C'}{\varepsilon^2} \) such that \( A \) is a contraction of the ball
\[
B(\varepsilon) = \left\{ W \in C([0, T], H^s) \left| \|W\|_{L^\infty([0, T], H^s(T))} \leq \varepsilon \right. \right\},
\]
where \( \varepsilon = 2 \|W_0\|_{H^s(T)} \). First we notice that \( A \) acts on the ball \( B(\varepsilon) \). Indeed, let \( W \in B(\varepsilon) \). Then,
\[
\|AW\|_{L^\infty([0, T], H^s(T))} \leq \|W(0)\|_{H^s(T)} + T \|f_{\text{res}}(W(\tau))\|_{L^\infty([0, T], H^s(T))} + T \|\{f'(W(\tau), \tau) \cdot F_{osc}(W(\tau), \tau)\}_{\text{res}}\|_{L^\infty([0, T], H^s(T))} \leq \|W(0)\|_{H^s(T)} + C T \|W\|_{H^s(T)}^3 (1 + \|W\|_{H^s(T)}^2) \leq \frac{\varepsilon}{2} + C T R^3 (1 + R^2).
\]
Choosing \( T = \frac{C'}{2 CR^3(1 + R^2)} \), we obtain \( \|AW\|_{L^\infty([0, T], H^s(T))} \leq R \) and thus \( AW \in B(\varepsilon) \). The fact that \( A \) is a contraction follows similarly. Therefore, there exists a unique solution of equation (4.8) in \( B(\varepsilon) \).}

Proposition 4.2. Let \( W_0 \in H^s_{L^2}(\mathbb{T}) \), \( s > 1/2 \). The solution of the Cauchy problem (4.8),
\[
\begin{cases}
\partial_t W = f_{\text{res}}(W) + \{f'(W(t), t) \cdot F_{osc}(W, t)\}_{\text{res}}, \\
W(0) = \varepsilon W_0 =: W_0,
\end{cases}
\]
coincides with the solution of the following Cauchy problem:
\[
(4.12) \quad \begin{cases}
\partial_t Y = -i \Pi_+ (|Y|^2 Y) - i \Pi_+ (|Y|^2 \frac{1}{2} \Pi_- (|Y|^2 Y)) - \frac{i}{2} \Pi_+ (Y^2 \frac{1}{2} \Pi_- (|Y|^2 Y)) \\
Y(0) = \varepsilon W_0 =: W_0
\end{cases}
\]
on its maximal interval of existence.

Proof. First we make the observation that we can easily prove local well-posedness of equation (4.12), in \( H^s(\mathbb{T}) \), \( s > \frac{1}{2} \), on a time interval \( \left[ 0, \frac{\varepsilon}{C} \right] \), following the lines of the proof of Lemma 4.4. Notice that \( Y \in L^2_{L^2}(\mathbb{T}) \). Therefore \( Y_-(t) = 0 \) for all \( t \) in the maximal interval of existence of \( Y \).

In the following we prove that the only terms that do not contain \( W_- \) and thus, contain only \( W_+ \) in \( \{f'(W(t) \cdot F_{osc}(W, t)\}_{\text{res}} \) are \( -\Pi_+ (|W_+|^2 \frac{1}{2} \Pi_- (|W_+|^2 W_+)) \) and \( -\frac{1}{2} \Pi_+ (W_+^2 \frac{1}{2} \Pi_- (|W_+|^2 W_+)) \). Since all the other terms contain at least one factor \( W_- \), it results that the \( Y(t) = Y_+(t) \) is also solution for equation (4.8). By Lemma 4.4, we have
uniqueness of the solution of equation (4.8). Thus, $\mathcal{Y}$ is the unique solution of equation (4.8).

It is thus sufficient to determine the terms of $\{f'(\mathcal{W}, t) \cdot F_{osc}(\mathcal{W}, t)\}_{res}$ which do not contain $\mathcal{W}_-$. Let us consider the first term of the Fourier coefficient in equation (4.10):

$$2i \sum_{\begin{subarray}{l} k-l+m-j=0 \\ j-n+p-q=0 \\ |j|-|n|+|p|-|q|\neq 0 \\ |k|-|l|+|m|-|n|+|p|-|q|=0 \end{subarray}} \frac{1}{|j|-|n|+|p|-|q|} \mathcal{W}(n) \mathcal{W}(q) \mathcal{W}(p) \mathcal{W}(l) \mathcal{W}(m)$$

The first condition we have for the above sum is that $|j|-|n|+|p|-|q| \neq 0$. As we noticed in Lemma 3.1, it follows that if $j, n, p, q$ cannot be simultaneously non-positive or non-negative, $j \neq n$, and $j \neq q$. Since in the above expression we have the factor $\mathcal{W}(n) \mathcal{W}(q) \mathcal{W}(p)$, it follows that if we only want to have $\mathcal{W}_+$, then the only possibility is $p, n, q \geq 0$ and $j < 0$. In particular, this also satisfies $j \neq n$ and $j \neq q$.

The second condition we have for the above sum is $|k|-|l|+|m|-|n|+|p|-|q| = 0$. As a consequence, this yields $|k|-|l|+|m|-|j| = -(|j|-|n|+|p|-|q|) \neq 0$. Thus, $k, l, m, j$ cannot be simultaneously non-positive or non-negative, $k \neq l$, and $k \neq j$. Since in the above sum we see appear the product $\mathcal{W}(l) \mathcal{W}(m)$, if we only want to have $\mathcal{W}_+$, it follows that we have two choices:

(i) $k, l, m \geq 0$; $j < 0$ and $k \neq l$,
(ii) $k < 0$; $l, m \geq 0$; $j < 0$.

Note that if $k, l, m \geq 0$, $j < 0$ and if $k = l$, then $k-l+m-n = 0$ yields $m = j$, which contradicts the fact that $m$ and $j$ have different signs. Thus, the condition $k \neq l$ in (i) is redundant.

We compute $|k|-|l|+|m|-|n|+|p|-|q|$ for the second case (ii):

$$|k|-|l|+|m|-|n|+|p|-|q| = -k-l+m-n+p-q$$
$$= -2k + (k-l+m-j) + (j-n+p-q) = -2k < 0.$$  

This contradicts the condition $|k|-|l|+|m|-|n|+|p|-|q| = 0$, and thus the case (ii) does not take place.

In the case (i), we have

$$|k|-|l|+|m|-|n|+|p|-|q| = k-l+m-n+p-q$$
$$= (k-l+m-j) + (j-n+p-q) = 0.$$  

Moreover,

$$|j|-|n|+|p|-|q| = -j-n+p-q = -2j + (j-n+p-q) = -2j = -2(n-p+q).$$
Thus the only possible choice if we want to obtain terms that do not contain $W_-$, is the following:

$$2i \sum_{k-l+m-n+p+q=0 \atop n-p+q<0 \atop k,l,m,n,p,q \geq 0} \frac{1}{2(n-p+q)} \tilde{W}(n)\tilde{W}(q)\tilde{W}(p)\tilde{W}(l)\tilde{W}(m)$$

$$= -i \sum_{k-l+m-(n-p+q)=0 \atop n-p+q<0 \atop k,l,m,n,p,q \geq 0} \mathcal{F}\left( \frac{1}{D} \Pi_- (|W|^2 W) \right) (n-p+q)\tilde{W}(l)\tilde{W}(m)$$

$$= -i \mathcal{F}\left( \Pi (|W|^2 \frac{1}{D} \Pi_- (|W|^2 W)) \right) (k).$$

Proceeding similarly with the second resonant part in equation (4.10), which is equal to

$$i \sum_{k-l+m-j=0 \atop m-n+p-q=0 \atop |m|-|n|+|p|-|q| \neq 0 \atop |k|-|l|-|j|+|n|-|p|+|q|=0} \frac{1}{|m|-|n|+|p|-|q|} \tilde{W}(j)\tilde{W}(l)\tilde{W}(n)\tilde{W}(q)\tilde{W}(p),$$

we obtain that it contains only one term in which $W_-$ does not appear, which is

$$-\frac{i}{2} \mathcal{F}\left( \Pi_+ (W^2 \frac{1}{D} \Pi_- (|W|^2 W)) \right) (k).$$

Therefore, the conclusion of the proposition follows. \qed

**Proposition 4.3.** Let $s > \frac{1}{2}$, $0 \leq \alpha \leq \frac{1}{2}$, and $\delta > 0$ small enough. Consider the equations

$$\left\{ \begin{array}{l}
\partial_t Y = -i\varepsilon^2 \Pi_+ (|Y|^2 Y) - i\varepsilon^4 \Pi_+ (|Y|^2 \frac{1}{D} \Pi_- (|Y|^2 Y)) - \frac{i\varepsilon^4}{2} \Pi_+ (Y^2 \frac{1}{D} \Pi_- (|Y|^2 Y)) \\
Y(0) = W_0
\end{array} \right.$$

and

(4.13)

$$\left\{ \begin{array}{l}
\partial_t U = -i\varepsilon^2 \Pi_+ (|U|^2 U) \\
U(0) = W_0.
\end{array} \right.$$

Assume that $\|U(t)\|_{H^s} \leq C \left( \log \left( \frac{1}{\varepsilon^4} \right) \right)^\alpha$ for all $t \in \mathbb{R}$. Then, for $0 \leq t \leq \frac{1}{\varepsilon^4} \left( \log \left( \frac{1}{\varepsilon^4} \right) \right)^{1-2\alpha}$, we have that

$$\|Y(t) - U(t)\|_{H^s} \leq \varepsilon^{2-\alpha},$$

where $C_0 > 0$ is a constant and $\delta$ is chosen small enough such that $C_0 \delta < 1$. In particular, $\|Y(t)\|_{H^s} \leq C \left( \log \left( \frac{1}{\varepsilon^4} \right) \right)^\alpha$.

**Proof.** Set $Z := Y - U$. Then $Z$ satisfies the equation

$$\left\{ \begin{array}{l}
\partial_t Z = -i\varepsilon^2 \left( \Pi_+ (|Y|^2 Y) - \Pi_+ (|U|^2 U) \right) - i\varepsilon^4 \Pi_+ (|Y|^2 \frac{1}{D} \Pi_- (|Y|^2 Y)) \\
\quad - \frac{i\varepsilon^4}{2} \Pi_+ (Y^2 \frac{1}{D} \Pi_- (|Y|^2 Y)) \\
Z(0) = 0.
\end{array} \right.$$

...
We set also
\[ h(U) := -i\Pi_+([U]^2U) \]
\[ g(U) := -i\Pi_+([U]^2\frac{1}{D}\Pi_-([U]^2U)) - \frac{i}{2}\Pi_+(U^2\frac{1}{D}\Pi_-([U]^2U)). \]

Then, we have
\[ Z(t) = \varepsilon^2 \int_0^t \left( h(Y(\tau)) - h(U(\tau)) \right) d\tau + \varepsilon^4 \int_0^t g(Y(\tau)) d\tau. \]

Using the boundedness of the operators \( \Pi_+ \) and \( \frac{1}{D}\Pi_- \) on \( H^s(\mathbb{T}) \), and proceeding as in the proof of Theorem 4.3, we obtain that
\[ \|Z(t)\|_{H^s(\mathbb{T})} \leq \varepsilon^2 \left( \log\left( \frac{1}{\varepsilon^5} \right) \right)^{2\alpha} C \int_0^t \|Z(\tau)\| d\tau + \varepsilon^4 \left( \log\left( \frac{1}{\varepsilon^5} \right) \right)^{5\alpha} C t. \]

By Gronwall’s inequality, it follows that for \( 0 \leq t \leq \frac{1}{\varepsilon^5} \left( \log\left( \frac{1}{\varepsilon^5} \right) \right)^{1-2\alpha} \) we have
\[ \|Z(t)\|_{H^s(\mathbb{T})} \leq \epsilon^2 - C_0 \delta, \]
where \( C_0 > 0 \) is a constant.

4.3. **Proof of Theorem 1.7**

**Lemma 4.4.** For \( W \in H^s(\mathbb{T}) \) we have that
\[ \|f'(W,t)\| \leq C\|W\|_{H^s}, \]
\[ \|f''(W,t)\| \leq C\|W\|_{H^s}. \]

where \( \| \cdot \| \) denotes the operator norm of a bounded linear operator acting on \( H^s(\mathbb{T}) \). In addition, the following applications are continuous and \( N \)-linear on \( H^s(\mathbb{T}) \):

1. \( W \mapsto N_2(W,t) \) with \( N = 5 \),
2. \( W \mapsto f'(W,t) \cdot R_2(W), W \mapsto N'_1(W,t) \cdot R_2(W), W \mapsto N'_2(W,t) \cdot R_1(W) \) with \( N = 7 \),
3. \( W \mapsto N'_3(W,t) \cdot R_2(W) \) with \( N = 9 \).

In particular, if \( \|W\|_{H^s(\mathbb{T})} \leq \left( \log\left( \frac{1}{\varepsilon^5} \right) \right)^{\alpha} \), then their \( H^s(\mathbb{T}) \)-norms are all bounded by
\[ \left( \log\left( \frac{1}{\varepsilon^5} \right) \right)^{9\alpha}. \]

**Proof.** The proof follows the same lines as that of Lemma 4.3.

**Proof of Theorem 1.7**. By Lemma 4.2, we have that the solution of the averaged equation (4.8) is \( W(t) = Y(t) \). By hypothesis, we have that the solution of equation (4.13) satisfies
\[ \|U(t)\|_{H^s(\mathbb{T})} \leq C \left( \log\left( \frac{1}{\varepsilon^5} \right) \right)^{\alpha}. \]
Then, by Lemma 4.3, it follows that \( \|W(t)\|_{H^s(\mathbb{T})} = \|Y(t)\|_{H^s(\mathbb{T})} \leq C \left( \log\left( \frac{1}{\varepsilon^5} \right) \right)^{\alpha}. \) Using the estimates of Lemma 4.4, it follows from equation (4.17), that
\[ \|w(t)\|_{H^s(\mathbb{T})} \leq \varepsilon^2 \left( \log\left( \frac{1}{\varepsilon^5} \right) \right)^{2\alpha} C \int_0^t \|w(\tau)\|_{H^s(\mathbb{T})} d\tau + \varepsilon^6 \left( \log\left( \frac{1}{\varepsilon^5} \right) \right)^{9\alpha} C t, \]
for $0 \leq t \leq \frac{1}{\varepsilon^2} \left( \log \left( \frac{1}{\varepsilon^2} \right) \right)^{1-2\alpha}$. Then, by Gronwall’s inequality we obtain

$$
\|w(t)\|_{H^s(T)} \leq \varepsilon^6 \left( \log \left( \frac{1}{\varepsilon^2} \right) \right)^{9\alpha} C t \varepsilon^2 \left( \log \left( \frac{1}{\varepsilon^2} \right) \right)^{2\alpha} C t
\leq \varepsilon^4 \left( \log \left( \frac{1}{\varepsilon^2} \right) \right)^{1+7\alpha} C \varepsilon \log \left( \frac{1}{\varepsilon^2} \right) \leq \varepsilon^{4-C_0\delta},
$$

where $C_0 > 0$. Thus, $\|u(t) - W(t) - \varepsilon^2 N_1(W(t)) - \varepsilon^4 N_2(W(t))\|_{H^s(T)} \leq \varepsilon^{4-C_0\delta}$ for $0 \leq t \leq \frac{1}{\varepsilon^2} \left( \log \left( \frac{1}{\varepsilon^2} \right) \right)^{1-2\alpha}$. Since $\|N_2(W(t))\|_{H^s(T)} \leq C \left( \log \left( \frac{1}{\varepsilon^2} \right) \right)^{5\alpha}$, this yields

$$
\|u(t) - W(t) - \varepsilon^2 F_{osc}(W,t)\|_{H^s(T)} = \|u(t) - W(t) - \varepsilon^2 N_1(W,t)\|_{H^s(T)} \leq \varepsilon^{4-C_0\delta}.
$$

Changing back to the variables $v = \varepsilon e^{-|D|} u$ and $W = \varepsilon W$, the conclusion of the theorem follows:

$$
\|v(t) - e^{-|D|t}(W(t) + F_{osc}(W,t))\|_{H^s(T)} = \|u(t) - W(t) - \varepsilon^2 N_1(W,t)\|_{H^s(T)} \leq \varepsilon^{5-C_0\delta}.
$$

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