Generalized Degrees of Freedom for Network-Coded Cognitive Interference Channel

Song-Nam Hong
Dep. of Electrical Eng.
University of Southern California
Los Angeles, USA
Email: songnamh@usc.edu

Giuseppe Caire
Dep. of Electrical Eng.
University of Southern California
Los Angeles, USA
Email: caire@usc.edu

Abstract—We study a two-user cognitive interference channel (CIC) where one of the transmitters (primary) has knowledge of a linear combination (over an appropriate finite field) of the two information messages. We refer to this channel model as Network-Coded CIC, since the linear combination may be the result of some linear network coding scheme implemented in the backbone wired network. In this paper, we characterize the generalized degrees of freedom (GDoF) for the Gaussian Network-Coded CIC. For achievability, we use the novel Precoded Compute-and-Forward (PCoF) and Dirty Paper Coding (DPC), based on nested lattice codes. As a consequence of the GDoF characterization, we show that knowing “mixed data” (linear combinations of the information messages) provides a multiplicative gain for the Gaussian CIC, if the power ratio of signal-to-noise (SNR) to interference-to-noise (INR) is larger than certain threshold. For example, when SNR = INR, the Network-Coded cognition yields a 100% gain over the classical Gaussian CIC.

I. INTRODUCTION

Transmitters or receivers, in many practical communication systems, are not isolated, and they can share certain amount of information (i.e., information messages, channel state information, and so on). For example, in a cloud base station architecture, small base stations (BSs) are spatially distributed over a certain area, and connected to the infrastructure networks via wired backhaul [1]. Cooperation among transmitters or receivers can mitigate interferences by forming distributed MIMO systems. One special case of particular interest is the two-user Cognitive Interference Channel (CIC), where one of the transmitters (referred to as “cognitive”) has knowledge of both information messages to the two users, while the other (referred to as “primary”) has knowledge of the message destined to its intended receiver only. This model is relevant under certain assumptions on the underlying wired backbone network connecting the two transmitters. For example, in the case of unidirectional cooperation, the primary transmitter sends its message to the cognitive transmitter via an a wired link of infinite capacity. Another example is the asymmetric situation shown in Fig. 1 where one transmitter (cognitive) has larger wired backhaul capacity, and therefore is able to observe both messages. The CIC has been extensively investigated in the literature. The capacity region of the strong interference regime was characterized in [2]. When the interference at the primary receiver is weak, the capacity region was characterized in [3]-[5]. Recently, the capacity region for Gaussian CIC was approximately characterized within 1.87 bits, regardless of channel parameters [5].

For wired networks, routing is generally optimal only for a single source, multiple intermediate nodes, and a single destination [6]. Yet, it cannot achieve the maximum throughput in the more general case of multiple sources and multiple destinations (multi-source multicasting). In this case, it is well-known that by allowing intermediate nodes to forward functions of the incoming messages (Network Coding), the capacity of multi-source multicasting relay networks can be achieved and coincides with the min-cut max-flow bound [7]. Random linear network coding (RLNC) is of particular interest for its practical simplicity. In this case, intermediate nodes forward liner combinations of the incoming messages by randomly and independently choosing the coefficients from an appropriate finite-field [8]. Assuming that RLNC is used over the wired network, in this paper we introduce the Network-Coded CIC as a generalization of the classical CIC, where the primary transmitter knows a linear combination of the information messages (referred to as “mixed data”). This is motivated in Fig. 1 by introducing RLNC instead of just routing in the backbone network. Since delivering mixed data at the primary transmitter has the same cost (in terms of backhaul capacity) than delivering a single message, a natural question arises: Does mixed data at the primary transmitter provide capacity increase “for free” for cognitive interference channel?
Our main contribution is to approximately characterize the sum capacity of Gaussian Network-Coded CIC in terms of the sum Generalized Degrees of Freedom (GDoF) \cite{Avestimehr2010} of the Gaussian Network Coded CIC. This is enabled by properly using a novel Aligned Precoded Compute-and-Forward (PCoF) and Dirty Paper Coding (DPC). As a consequence of the GDoF analysis, we show that Network-Coded cognition can provide multiplicative gain in cognitive interference channels. As shown in Fig. 2, the gain of Network-Coded cognition becomes arbitrary large as SNR and INR go to infinity as long as \( \rho \geq 1/2 \) (i.e., except the weak interference regime). Namely, if \( \rho \geq 1/2 \), the performance gap between the Network-Coded CIC and the classical CIC becomes unbounded. For example, when \( \rho = 1 \) (i.e., \( \text{SNR} = 1\text{INR} \)), Network-Coded cognition provides 100\% gain over classical cognition.

II. PRELIMINARIES

In this section we provide some basic definitions and results which will be extensively used in the sequel.

A. System Model

A two-user Gaussian Network-Coded CIC consists of a Gaussian interference channel where transmitter 1 (the cognitive transmitter) knows both user 1 and user 2 information messages (or, equivalently, two linearly independent linear combinations thereof) and transmitter 2 (the primary transmitter) only knows only one linear combination of the messages. Without loss of generality, we assume that the cognitive transmitter knows \( (\mathbf{w}_1, \mathbf{w}_2) \), and the primary transmitter has \( \mathbf{w}_1 \oplus \mathbf{w}_2 \), where \( \mathbf{w}_i \in \mathbb{F}_q^m \) denotes the information message desired at receiver \( \ell \), at rate \( R_\ell \text{ bit/symbol} \), for \( \ell = 1, 2 \). We assume that if \( R_1 \neq R_2 \) then the lowest rate message is zero-padded such that both messages have a common length, given by \( r = \max\{nR_1, nR_2 \} \), where \( n \) denotes the coding block length. A block of \( n \) channel uses of the discrete-time complex baseband two-user IC is described by

\[
\mathbf{y}_1 = h_{11}\mathbf{x}_1 + h_{12}\mathbf{x}_2 + \mathbf{z}_1, \quad (1)
\]

\[
\mathbf{y}_2 = h_{21}\mathbf{x}_1 + h_{22}\mathbf{x}_2 + \mathbf{z}_2, \quad (2)
\]

where \( \mathbf{z}_\ell \in \mathbb{C}^{n \times 1} \) contains i.i.d. Gaussian noise samples \( \sim \mathcal{CN}(0,1) \) and \( h_{ij} \in \mathbb{C} \) denotes the channel coefficients, assumed to be constant over the whole block of length \( n \) and known to all nodes. Also, we have a power constraint, given by \( \frac{1}{n} \mathbb{E}[\|\mathbf{y}_\ell\|^2] \leq \text{SNR} \) for \( \ell = 1, 2 \), where \( \| \cdot \| \) denotes the \( \ell_2 \)-norm. Each receiver \( \ell \) observes the channel output \( \mathbf{y}_\ell \) and produces an estimate \( \hat{\mathbf{w}}_\ell \) of the desired message \( \mathbf{w}_\ell \). We say that receiver \( \ell \) is in error whenever \( \hat{\mathbf{w}}_\ell \neq \mathbf{w}_\ell \). A rate pair \( (R_1, R_2) \) is achievable if there exists a family of codebooks with codewords satisfying the power constraint, and corresponding decoding functions such that the average decoding error probability satisfies \( \lim_{n \to \infty} \mathbb{P}(\hat{\mathbf{w}}_\ell \neq \mathbf{w}_\ell) = 0 \), for \( \ell = 1, 2 \).

B. Nested Lattice Codes

Let \( \mathbb{Z}[j] \) be the ring of Gaussian integers and \( p \) be a prime. Let \( \oplus \) denote the addition over \( \mathbb{F}_q \) with \( q = p^2 \), and let \( g : \mathbb{F}_q \to \mathbb{C} \) be the natural mapping of \( \mathbb{F}_q \) onto \( \{a + jb : a, b \in \mathbb{Z}_p \} \subset \mathbb{C} \). We recall the nested lattice code construction given in \cite{Avestimehr2009}. Let \( \Lambda = \{ \lambda = \mathbf{z}\mathbf{T} : \mathbf{z} \in \mathbb{Z}[j] \} \) be a lattice in \( \mathbb{C}^n \), with full-rank generator matrix \( \mathbf{T} \in \mathbb{C}^{n \times r} \). Let \( \mathcal{C} = \{ \mathbf{c} = \mathbf{w}\mathbf{G} : \mathbf{w} \in \mathbb{F}_q^m \} \) denote a linear code over \( \mathbb{F}_q \) with block length \( n \) and dimension \( r \), with generator matrix \( \mathbf{G} \). The lattice \( \Lambda_1 \) is defined through “construction A” (see \cite{Huffman2003} and references therein) as

\[
\Lambda_1 = p^{-1}g(\mathcal{C})\mathbf{T} + \Lambda, \quad (3)
\]

where \( g(\mathcal{C}) \) is the image of \( \mathcal{C} \) under the mapping \( g \) (applied component-wise). It follows that \( \Lambda \subseteq \Lambda_1 \subseteq p^{-1}\Lambda \) is a chain of nested lattices, such that \( |\Lambda_1/\Lambda| = p^{-2r} \) and \( |p^{-1}\Lambda/\Lambda_1| = p^{2(n-r)} \).

For a lattice \( \Lambda \) and \( \mathbf{r} \in \mathbb{C}^n \), we define the lattice quantizer \( Q_\Lambda(\mathbf{r}) = \arg\min_{\mathbf{x} \in \Lambda} \|\mathbf{r} - \mathbf{x}\|^2 \), the Voronoi region \( \mathcal{V}_\Lambda = \{ \mathbf{z} \in \mathbb{C}^n : Q_\Lambda(\mathbf{z}) = 0 \} \) and \( \max \mathcal{V}_\Lambda = \mathbf{r} - Q_\Lambda(\mathbf{r}) \). For \( \Lambda \) and \( \Lambda_1 \) given above, we define the lattice code \( \mathcal{L} = \Lambda_1 \cap \mathcal{V}_\Lambda \) with rate \( R = \frac{1}{n} \log |\mathcal{C}| = \frac{1}{n} \log q \). Construction A provides a natural labeling of the codewords of \( \mathcal{L} \) by the information messages \( \mathbf{w} \in \mathbb{F}_q^m \). Notice that the set \( p^{-1}g(\mathcal{C})\mathbf{T} \) is a system of coset representatives of the cosets of \( \Lambda \) in \( \Lambda_1 \). Hence, the natural labeling function \( f : \mathbb{F}_q^m \to \mathcal{L} \) is defined by \( f(\mathbf{w}) = p^{-1}g(\mathbf{w}\mathbf{G})\mathbf{T} \mod \Lambda \).

C. Compute-and-Forward

We recall here the CoF scheme of \cite{Avestimehr2009}. Consider the two-user Gaussian multiple access channel defined by

\[
\mathbf{y} = \sum_{k=1}^{2} h_k \mathbf{x}_k + \mathbf{z}, \quad (4)
\]

where \( \mathbf{h} = [h_1, h_2]^T \) and the elements of \( \mathbf{z} \) are i.i.d. \( \sim \mathcal{CN}(0,1) \). All users make use of the same nested lattice codebook \( \mathcal{L} = \Lambda_1 \cap \mathcal{V}_\Lambda \), where \( \Lambda \) has second moment
where the *dithering sequences* $d_k$’s are mutually independent across the users, uniformly distributed over $\mathcal{V}_\lambda$, and known to the receiver. The decoder’s goal is to recover a linear combination $\hat{v} = [\sum_{k=1}^{2} a_k t_k] \mod \Lambda$ with integer coefficient vector $a = [a_1, a_2]^T \in \mathbb{Z}[j]$. Since $\Lambda_1$ is a $\mathbb{Z}[j]$-module (closed under linear combinations with Gaussian integer coefficients), then $\hat{v} \in \mathcal{L}$. Letting $\hat{v}$ be the decoded codeword (for some decoding function which in general depends on $h$ and $a$), we say that a computation rate $R$ is achievable for this setting if there exists sequences of lattice codes $\mathcal{L}$ of rate $R$ and increasing block length $n$, such that the decoding error probability satisfies $\lim_{n \to \infty} \mathbb{P}(\hat{v} \neq v) = 0$.

In the scheme of [10], the receiver computes

$$\hat{y} = \left[ a_1 y - \sum_{k=1}^{2} a_k d_k \right] \mod \Lambda
= [v + \hat{z}_2(h, a, \alpha)] \mod \Lambda,$$

(6)

where

$$\hat{z}_2(h, a, \alpha) = \sum_{k=1}^{2} (\alpha h_k - a_k) x_k + \alpha \hat{z}$$

(7)

denotes the effective noise, including the non-integer self-interference (due to the fact that $\alpha h_k \notin \mathbb{Z}[j]$ in general) and the additive Gaussian noise term. The scaling, dither removal and modulo-$\Lambda$ operation in (6) is referred to as the CoF receiver mapping in the following. From [10], we know that by applying lattice decoding to $\hat{y}$ given in (6) the following computation rate is achievable:

$$R(h, a, \alpha, \text{SNR}) = \log^+ \left( \frac{\text{SNR}}{\|a\|^2 + \|\alpha h - a\|^2 + \text{SNR}} \right),$$

(8)

where $\log^+ (x) \triangleq \max\{\log(x), 0\}$.

III. AN ACHIEVABLE RATE REGION FOR GAUSSIAN NETWORK-CODED CIC

Using the fact that the cognitive transmitter has non-causal information of the primary transmitter signal, it can totally eliminate the known interference at its own intended receiver by using DPC. Also, we can remove the interference at the receiver 2 using Aligned PCoF. Using CoF decoding, the receiver 2 can reliably decode an integer linear combination of the lattice codewords sent by transmitters. The “interference” in the finite-field domain can be completely eliminated by precoding over the finite-field at the cognitive transmitter. It is well known that the performance of CoF is deteriorated by the non-integer penalty (i.e., the residual “self-interference” due to the fact that the channel coefficients take on non-integer values in practice). In order to eliminate this penalty, the primary transmitter scales its signal by some constant $\beta \in \mathcal{P}$ to create more favorable channel for CoF receiver mapping where $\mathcal{P} = \{\beta \in \mathbb{C} : \|\beta\| \leq 1\}$.

We let $a = (a_1, a_2) \in \mathbb{Z}[j]^2$ denote the integer coefficients vector used at receiver 2 for the modulo-$\Lambda$ receiver mapping (6), and we let $q_0 = g^{-1}(a_1 \mod p\mathbb{Z}[j])$. For the time being, it is assumed that $q_1, q_2 \neq 0$ over $\mathbb{F}_q$. The proposed achievability scheme proceeds as follows

- The primary transmitter produces the lattice codeword $v_2 = f(w_1 \oplus w_2)$ and transmits the following channel inputs:
  $$x_2 = [v_2 + d_2] \mod \Lambda.$$

- The cognitive transmitter produces the precoded message $bw_1$ where $b \in \mathbb{F}_q$ is given by
  $$q_1 b \oplus q_2 = 0 \implies b = (q_1)^{-1}(-q_2)$$
  (10)
  where $(q_1)^{-1}$ denotes the multiplicative inverse of $q_1$ and $(-q_2)$ denotes the additive inverse of $q_2$.

- The cognitive transmitter performs DPC using the known interference signal $h_{12} x_2$ to get:
  $$x_1 = [v_1 - \alpha_1(h_{12}/h_{11}) x_2 + d_1] \mod \Lambda,$$
  (11)

where $v_1 = f(b w_1)$. The known interference signal is in fact generated by using the knowledge of the message $w_1 \oplus w_2$, the dense lattice codebooks, and the dithering sequence $d_2$. The $d_2$’s are mutually independent across the transmitters, uniformly distributed over $\mathcal{V}_\lambda$, and known to all nodes.

Because of linearity, the precoding and the encoding over the finite-field commute. Therefore, we can write

$$y_1 = g(b) t_1 \mod \Lambda$$

(12)

$$y_2 = t_2 + t_2 \mod \Lambda$$

(13)

where $t_1 = f(w_1)$ and $t_2 = f(w_2)$. Receiver 1 performs the inflated modulo-lattice mapping as $\hat{y}_1 = [a_1 y_1/h_{11} - d_1] \mod \Lambda$. Then the resulting channel is a modulo-$\Lambda$ additive noise channel as

$$\hat{y}_1 = [v_1 - (1 - \alpha_1 h_{11}) u_1 + \alpha_1 \hat{z}_1/h_{11}] \mod \Lambda$$

where $u_1$ is uniformly distributed on $\mathcal{V}_\lambda$ and is independent of $\hat{z}_1$ and $y_1$ by Crypto Lemma. From standard DPC results [12], choosing

$$\alpha_1 = \alpha_{1, \text{snr}} \triangleq \frac{\text{SNR} \|h_{11}\|^2}{1 + \text{SNR} \|h_{11}\|^2},$$

(14)

we obtain

$$R_1 \leq \log(1 + \|h_{11}\|^2 \text{SNR}).$$

(15)

Letting $h = [h_{21}, h_{22}]$ with $h_{22} = h_{22} - \alpha_{1, \text{snr}} h_{12} h_{21}/h_{11}$, receiver 2 applies the CoF receiver mapping in (6) with integer coefficients vector $a = (a_1, a_2) \in \mathbb{Z}[j]^2$ and scaling factor

$$\beta \in \mathcal{P}.$$
\[ \alpha_2 = a_1/h_{21}, \text{ yielding} \]

\[
\tilde{\mathbf{y}}_2 = \begin{pmatrix} a_1 \mathbf{y}_1 + a_2 \mathbf{y}_2 + a_2 (h_{21} \mathbf{x}_1 + h_{22} \mathbf{x}_2 + \mathbf{z}_2) \\
- a_1 \mathbf{y}_1 + d_1 - a_2 (\mathbf{y}_2 + d_2) \end{pmatrix} \mod \Lambda
\]

\[
= \begin{pmatrix} a_1 \mathbf{y}_1 + a_2 \mathbf{y}_2 + (a_2 h_{21} - a_1) \mathbf{y}_1 + d_1 + (a_2 h_{22} - a_2) \mathbf{x}_2 + a_2 h_{21} \mathbf{A} + a_2 \mathbf{z}_2 \end{pmatrix} \mod \Lambda
\]

\[
= \begin{pmatrix} a \mathbf{T} \cdot \mathbf{y}_1 + a \mathbf{T} \cdot \mathbf{y}_2 \mod p \mathbf{Z}[j] \\
+ \mathbf{z}_g(\mathbf{h}, \mathbf{a}) \end{pmatrix} \mod \Lambda
\]

\[ (a) \]

\[
\tilde{\mathbf{y}}_2 = \begin{pmatrix} a \mathbf{T} \cdot \mathbf{y}_1 + a \mathbf{T} \cdot \mathbf{y}_2 \mod p \mathbf{Z}[j] \\
+ \mathbf{z}_g(\mathbf{h}, \mathbf{a}) \end{pmatrix} \mod \Lambda
\]

where \( \mathbf{A} = Q_\Lambda(\mathbf{y}_1 - \alpha_1 \mathbf{y}_2 + \mathbf{h}_{12} \mathbf{x}_1 + d_1) \). \( (a) \) is due to the fact that \( \alpha_2 h_{21} \mathbf{A} = a_1 \mathbf{A} \in \Lambda \), and \( (b) \) follows from the fact that the \( b \) is chosen to satisfy the \( [10] \), i.e., \( a_1 g(b) + a_2 \mod p \mathbf{Z}[j] = 0 \), and

\[
\tilde{\mathbf{z}}_g(\mathbf{h}, \mathbf{a}) = (\mathbf{a}_1 \tilde{\mathbf{h}}_{22} - h_{21} - a_2) \mathbf{u}_2 + (a_1/h_{21}) \mathbf{z}_2.
\]

By applying the lattice coding to \( \tilde{\mathbf{y}}_2 \), the receiver 2 can decode its message if

\[ R_2 \leq R(\mathbf{h}, a_1/h_{21}, a, \text{SNR}). \]

In order to mitigate the non-integer penalty at receiver 2, the primary transmitter only scales its signal by some constant \( \beta \in \mathcal{P} \). In this way, the rate \( R_1 \) in (20) is preserved, and the rate \( R_2 \) can be rewritten as a function of \( \beta \in \mathcal{P} \) as:

\[ R_2(\beta) \leq R(\mathbf{h}, a_1/h_{21}, a, \text{SNR}), \]

where now we have

\[ \tilde{h}(\beta) = \begin{pmatrix} h_{21}, \beta \left( h_{22} - \frac{\text{SNR}|h_{11}|^2}{1 + |h_{11}|^2} \right) \end{pmatrix}, \]

for some \( \beta \in \mathcal{P} \). Hence, we have proved the following:

**Theorem 1:** Aligned PCoF and DPC applied to Gaussian Network-Coded CIC with \( \mathbf{H} = [h_{ij}] \in \mathbb{C}^{2 \times 2} \) achieves the rate pairs \((R_1, R_2)\) such that

\[ R_1 \leq \log(1 + |h_{11}|^2 |\text{SNR}) \]

\[ R_2 \leq R(\mathbf{h}, a_1/h_{21}, a, \text{SNR}) \]

for any \( a \in \mathbb{Z}[j]^2 \) with \( a_1, a_2 \neq 0 \) and any \( \beta \in \mathcal{P} \), where \( \tilde{h} \) is given in (19). \( \nabla \)

**IV. GENERALIZED DEGREES OF FREEDOM**

In the high SNR regime, a useful proxy for the performance of wireless networks is provided by the sum Degree-of-Freedom (DoF), which is the pre-log factor (multiplexing gain) in the expression of the sum capacity in terms of SNR. In this section we study the symmetric Generalized DoF (GDoF) as introduced in [9], which is a more refined proxy for the high-SNR performance, capturing the relative strength of direct and interference links. We consider the following channel model:

\[ \mathbf{y}_1 = h_{11} \sqrt{\text{SNR}} \mathbf{x}_1 + h_{12} \sqrt{\text{INR}} \mathbf{x}_2 + \mathbf{z}_1 \]

\[ \mathbf{y}_2 = h_{21} \sqrt{\text{INR}} \mathbf{x}_1 + h_{22} \sqrt{\text{SNR}} \mathbf{x}_2 + \mathbf{z}_2 \]

where \( h_{ij} \in \mathbb{C} \) are bounded non-zero constants independent of SNR, INR, \( \mathbf{z} \) is the i.i.d. Gaussian noise \( \sim \mathcal{N}(0, 1) \), and \( \frac{1}{2} \mathbb{E}||\mathbf{x}||^2 \) \( \leq 1 \) for \( \ell = 1, 2 \). The channel is parameterized by SNR and INR, both growing to infinity. The way these parameters grow to infinity if defined by \( \rho > 0 \), given by

\[ \rho = \log \text{INR} / \log \text{SNR}, \]

i.e., by letting INR \( \sim \text{SNR}^\rho \) as SNR \( \to \infty \). The sum GDoF is defined by

\[ d_{\text{sum}}(\rho) = \lim_{\text{SNR} \to \infty} \frac{C_{\text{sum}} \log \text{SNR}}{\log \text{SNR}}. \]

The main result of this section is given by:

**Theorem 2:** For the Gaussian Network-Coded CIC, the sum symmetric GDoF is given by

\[ d_{\text{sum}}(\rho) = 1 + \rho. \]

**Proof:** See Appendix A.

**V. CONCLUDING REMARKS**

We investigated a two-user cognitive interference channel (CIC), in the case where the “primary” transmitter knows a linear combination of the information messages. The proposed combination of Aligned PCoF and Dirty Paper Coding, based on nested lattice codes, allowed us to characterize the sum generalized degrees-of-freedom of the Gaussian Network-Coded CIC. In particular, our result shows the surprising fact that, in certain regimes of the SNR/INR scaling region, network-coded cognition yields an unbounded gain (i.e., multiplicative gain) in the Gaussian CIC, with respect to the classical cognitive transmitter model.

**APPENDIX A**

**PROOF OF THEOREM 2**

**A. Achievable scheme**

We use the achievable rates given in Theorem 1. It is immediately shown that the achievable GDoF of the cognitive transmitter is 1, obtained by

\[ d_1(\rho) = \lim_{\text{SNR} \to \infty} \frac{\log(1 + |h_{11}|^2 |\text{SNR})}{\log \text{SNR}} = 1. \]

In this proof, we show that the primary transmitter achieves the \( \rho \) GDoF by carefully choosing the \( \beta \). The effective channel for Aligned PCoF is given in (19) as \( \tilde{h}(\beta) = [h_{21} \sqrt{\text{INR}}, \beta(h_{22} \sqrt{\text{SNR}} - \alpha_1 \sqrt{h_{12} h_{21}/h_{11}}) \text{SNR}^{\rho - \frac{1}{2}}] \) and can be rewritten as

\[ \tilde{h}(\beta) = \sqrt{\text{SNR}^\rho}[h_{21}, \beta h_{22}] \]

where \( h_{22} = h_{22} \sqrt{\text{SNR}} \rho/2 - h \text{SNR}^{\rho/2} \) and \( h = \alpha_1 \sqrt{h_{12} h_{21}/h_{11}} \). Here, we choose \( \beta = \beta^* = h_{21}/(h_{22}) \).
where $\gamma \geq 1$ is an integer with $\gamma = \lfloor |h_{21}/\bar{h}_{22}|| \rfloor \in \mathbb{Z}_+$. This produces a kind of “aligned” channel:

$$\bar{h} = \text{SNR}^{\rho/2}[h_{21}, h_{21}/\gamma].$$

(29)

Letting $a_1 = \gamma$, and $a_2 = 1$, the effective noise in (10) is obtained by

$$z_{\text{eff}}(\bar{h}, a) = (\gamma/(h_{21}\text{SNR}^{\rho/2}))z_2.$$

(30)

This shows that non-integer penalty is completely eliminated. Also, we can use the zero forcing precoding over $F_h$ since the integer coefficients $a_1$ and $a_2$ are non-zero. From this, we have the lower bound on the achievable rate of Aligned PCoF:

$$\max_{\beta} R_2(\beta) \geq R_2(\beta^*) = \rho \log(\|h_{21}\|^2\text{SNR}) - 2\log(\gamma).$$

(31)

The lower and upper bounds on $\gamma$ is given by

$$1 \leq \gamma \leq 1 \pm \left\| \frac{h_{21}}{\text{SNR}^{1-\rho}/2 - h\text{SNR}^{(\rho-1)/2}} \right\|$$

(32)

where $\gamma$ converges to a const as $\text{SNR} \to \infty$. Finally, the achievable GDoF of the primary transmitter is derived as

$$d_2(\rho) \leq \lim_{\text{SNR} \to \infty} \frac{R_2(\beta^*)}{\log \text{SNR}} = \rho.$$

(33)

From (27) and (33), the achievable sum GDoF is $1 + \rho$.

B. Converse

For given rates $R_1$ and $R_2$, we define $R_{\text{sym}} = \min\{R_1, R_2\}$ and $\tilde{R} = \max\{R_1, R_2\} - R_{\text{sym}}$. If $R_1 > R_2$ then $W_1 = (M_1, \tilde{M})$ and $W_2 = (M_2, 0)$. In the reverse case, we have that $W_1 = (M_1, 0)$ and $W_2 = (M_2, \tilde{M})$. In both cases, the primary transmitter knows the linear combination, $W_1 + W_2 = (M_1 + M_2, \tilde{M})$. From the well-known Crypto Lemma, the $M_1 + M_2$ is mutually statistically independent of $M_1$, as well as $M_1 + M_2$ is mutually statistically independent of $M_2$. In this proof, we derive the upper bounds on $R_{\text{sym}}$ and $R_{\text{sym}} + \tilde{R}$.

First, we derive the upper bound on the symmetric rate $R_{\text{sym}}$:

$$nR_{\text{sym}} = H(M_1) = H(M_1|M_1 + M_2, \tilde{M}) + H(M_1|Y^n, M_1 + M_2, \tilde{M}) + H(M_1|Y^n, M_1 + M_2, \tilde{M})$$

$$\leq I(M_1; Y^n|1| M_1 + M_2, \tilde{M}) + n\epsilon_n$$

$$= h(Y^n|X^n_1, X^n_2) - h(Y^n|X^n_1, X^n_2) + n\epsilon_n$$

$$= I(X^n_1; Y^n_2|X^n_2) + n\epsilon_n$$

$$\leq n \log(1 + \|h_{11}\|^2\text{SNR}) + n\epsilon_n.$$  

From the above, we have the upper bound on $R_{\text{sym}}$ as

$$R_{\text{sym}} \leq \min\{\log(1 + \|h_{11}\|^2\text{SNR}), \log(1 + \|h_{21}\|^2\text{INR})\}.$$  

(34)

The upper bound on $R_\ell$ can be computed as

$$nR_\ell \leq H(W_\ell) = H(W_\ell) - H(W_\ell|Y^n_\ell) + H(W_\ell|Y^n_\ell)$$

$$\leq I(W_\ell; Y^n_\ell) + n\epsilon_n$$

$$= h(Y^n_\ell) - h(Y^n_\ell|W_1, W_2) + n\epsilon_n$$

$$= I(X^n_\ell; Y^n_\ell|X^n_\ell) + n\epsilon_n$$

$$\leq n \log(1 + \|h_{11}\|^2\text{SNR}) + \|h_{21}\|^2\text{INR} + n\epsilon_n.$$  

Since $R_{\text{sym}} + \tilde{R} = \max\{R_1, R_2\}$, we have:

$$R_{\text{sym}} + \tilde{R} \leq \min\{\log(1 + \|h_{11}\|^2\text{SNR}) + \|h_{21}\|^2\text{INR}\}.$$  

Using (34), and $\text{INR} = \text{SNR}^\rho$, we have the upper bounds in the asymptotic case:

$$\lim_{\text{SNR} \to \infty} \left( \frac{R_{\text{sym}}}{\log \text{SNR}} + \frac{R_{\text{sym}} + \tilde{R}}{\log \text{SNR}} \right) \leq \min\{1, \rho\} + \max\{1, \rho\}.$$  

Finally, we have the upper bound on the sum GDoF as

$$d_{\text{sym}} = \lim_{\text{SNR} \to \infty} \frac{2R_{\text{sym}} + \tilde{R}}{\log \text{SNR}} \leq 1 + \rho.$$  

(36)

This completes the proof.

REFERENCES

[1] S.-N. Hong and G. Caire, “Lattice Coding Strategies for Cooperative Distributed Antenna Systems,” *submitted to IEEE Transactions on Information Theory*, Oct. 2012. [Online] Available: http://arxiv.org/abs/1210.0760

[2] I. Maric, R. D. Yates, and G. Kramer, “Capacity of interference channels with partial transmitter cooperation,” *IEEE Transactions on Information Theory*, vol. 53, pp. 3536-3548, Oct. 2007.

[3] W. Wu, S. Vishwanath, and A. Arapostathis, “Capacity of a class of cognitive radio channels: Interference channels with degraded message sets,” *IEEE Transactions on Information Theory*, vol. 53, pp. 4391-4399, Nov. 2007.

[4] A. Jovicic and P. Viswanath, “Cognitive radio: An information-theoretic perspective,” *IEEE Transactions on Information Theory*, vol. 55, pp. 3945-3958, Sept. 2009

[5] S. Rini, D. Tinninetti, and N. Devroye, “The Capacity Region of Gaussian Cognitive Radio Channels to within 1.87 bits,” in *proceedings of IEEE Information Theory Workshop (ITW)*, Cairo, Egypt, Jan. 2010.

[6] L. R. Ford and D. R. Fulkerson, “Maximal flow through a network,” *Canadian Journal of Mathematics*, vol. 8, pp. 399-404, 1956.

[7] R. Ahlswede, N. Cai, S.-Y. R. Li, and R. W. Yeung, “Network information flow,” *IEEE Transactions on Information Theory*, vol. 49, pp. 371-381, Feb. 2003.

[8] T. Ho, M. Medard, R. Koetter, D. R. Karger, M. Effros, J. Shi, and B. Leong, “A Random Linear Network Coding Approach to Multicast,” *IEEE Transactions on Information Theory*, vol. 52, pp. 4413-4430, Oct. 2006.

[9] R. Etkin, D. N. C. Tse, and H. Wang, “Gaussian interference channel capacity to within one bit,” *IEEE Transactions on Information theory*, vol. 54, pp. 5534-5562, Dec. 2008.

[10] B. Nazer and M. Gastpar, “Compute-and-Forward: Harnessing Interference through Structured Codes,” *IEEE Transactions on Information Theory*, vol. 57, pp. 6463-6486, Oct. 2011.

[11] U. Erez and R. Zamir, “Achieving $\frac{1}{2} \log(1 + \text{SNR})$ on the AWGN channel with lattice encoding and decoding,” *IEEE Transactions on Information Theory*, vol. 50, pp. 2293-2314, Oct. 2004.

[12] R. Zamir, S. Shamai, and U. Erez, “Nested Linear/Lattice Codes for Structured Multiterminal Binning,” *IEEE Transactions on Information Theory*, vol. 48, pp. 1250-1276, June, 2002.