ON THE DIOPHANTINE EQUATION $(\binom{n}{k}) = (\binom{m}{l}) + d$

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Abstract. By finding all integral points on certain elliptic and hyperelliptic curves we completely solve the Diophantine equation $(\binom{n}{k}) = (\binom{m}{l}) + d$ for $-3 \leq d \leq 3$ and $(k, l) \in \{(2, 3), (2, 4), (2, 5), (2, 6), (2, 8), (3, 4), (3, 6), (4, 6), (4, 8)\}$. Moreover, we present some other observations of computational and theoretical nature concerning the title equation.

1. Introduction

There are many nice results related to the equation

$$\binom{n}{k} = \binom{m}{l},$$

in unknowns $k, l, m, n$. This is usually considered with the restrictions $2 \leq k \leq n/2, 2 \leq l \leq m/2$ and $k < l$. The only known solutions (with the above mentioned restrictions) are the following

$$\binom{16}{2} = \binom{10}{3}, \binom{56}{2} = \binom{22}{3}, \binom{120}{2} = \binom{36}{3},$$

$$\binom{21}{2} = \binom{10}{4}, \binom{153}{2} = \binom{19}{5}, \binom{78}{2} = \binom{15}{5} = \binom{14}{6};$$

$$\binom{221}{2} = \binom{17}{8}, \binom{F_{2i+2}F_{2i+3}}{F_{2i}F_{2i+3}} = \binom{F_{2i+2}F_{2i+3} - 1}{F_{2i}F_{2i+3} + 1} \text{ for } i = 1, 2, \ldots,$$

where $F_n$ is the $n$th Fibonacci number. The infinite family of solutions involving Fibonacci numbers was found by Lind [17] and Singmaster [21].

Equation [11] has been completely solved for pairs

$$(k, l) = (2, 3), (2, 4), (2, 6), (2, 8), (3, 4), (3, 6), (4, 6), (4, 8).$$

In cases of these pairs one can easily reduce the equation to the determination of solutions of a number of Thue equations or elliptic Diophantine equations. In 1966, Avanesov [11] found all integral solutions of equation [11] with $(k, l) = (2, 3)$. De Weger [10] and independently Pintér [19] provided all the solutions of the equation with $(k, l) = (2, 4)$. The case $(k, l) = (3, 4)$ reduces to the equation $Y(Y + 1) = X(X + 1)(X + 2)$ which was solved by Mordell [18]. The remaining pairs $(2, 6), (2, 8), (3, 6), (4, 6), (4, 8)$ were handled by Stroeker and de Weger [27], using linear forms in elliptic logarithms. The case with
(k, l) = (2, 5) was completely solved by Bugeaud, Mignotte, Siksek, Stoll and Tengely \[8\], the integral solutions are as follows

\[(n, m) = (0, 0), (0, 1), (1, 0), (1, 1), (2, 0), (2, 1), (3, 0), (3, 1), (4, 0), (4, 1), (5, -1), (5, 2), (6, -3), (6, 4), (7, -6), (7, 7), (15, -77), (15, 78), (19, -152), (19, 153).\]

In a recent paper Blokhuis, Brouwer and de Weger \[4\] determined all non-trivial solutions with \(\left(\begin{smallmatrix} n \\ k \end{smallmatrix}\right) \leq 10^{60}\) or \(n \leq 10^6\). General finiteness results are also known. In 1988, Kiss \[15\] proved that if \(k = 2\) and \(l\) is a given odd prime, then the equation has only finitely many positive integral solutions. Using Baker’s method, Brindza \[6\] showed that equation (1) with \(k = 2\) and \(l \geq 3\) has only finitely many positive integral solutions.

In case of the more general equation

\[(n \atop k) \equiv (m \atop l) + d \pmod{p}, \tag{2}\]

Blokhuis, Brouwer and de Weger \[4\] determined all non-trivial solutions with \(d = 1\) and \((k, l), (l, k) = (2, 3), (2, 4), (2, 6), (3, 4), (4, 6), (4, 8)\) and \((k, l) = (2, 8)\). They provided a complete list of solutions for the above cases and if \(\left(\begin{smallmatrix} n \\ k \end{smallmatrix}\right) \leq 10^{30}\).

| n  | k  | m  | l  |
|----|----|----|----|
| 11 | 2  | 8  | 3  |
| 60 | 2  | 23 | 3  |
| 160403633 | 2 | 425779 | 3 |
| 6  | 3  | 7  | 2  |
| 7  | 3  | 9  | 2  |
| 16 | 3  | 34 | 2  |
| 27 | 3  | 77 | 2  |
| 29 | 3  | 86 | 2  |
| 34 | 3  | 21 | 4  |

Table 1. Known solutions of the Diophantine equation \(\left(\begin{smallmatrix} n \\ k \end{smallmatrix}\right) = \left(\begin{smallmatrix} m \\ l \end{smallmatrix}\right)\).

If \(d\) is not fixed they also obtained some interesting infinite families, an example is given by

\[\left(12x^2 - 12x + 3\right) + \binom{x}{2} = \left(24x^3 - 36x^2 + 15x - 1\right) \pmod{2}.\]

In 2019, Katsipis \[14\] completely resolved the case with \((k, l) = (8, 2)\) and he also determined the integral solutions if \((k, l), (l, k) = (3, 6)\) and \(d = 1\).

The aim of this paper is to extend results mentioned above and offer some general observations and computational results.

2. Main results

We start our discussion with some numerical observations. More precisely, we observed that for certain pairs \((k, l)\) and an integer \(d\), the congruence

\[(n \atop k) \equiv (m \atop l) + d \pmod{p}, \tag{3}\]
Theorem 1. If \((k,l) = (2,4), d \in \mathbb{Z}\) and 3 is a quadratic non-residue modulo \(p > 4\), where the \(p\)-adic valuation of \(12d+1\) is odd, then congruence \((2)\) has no solutions. In particular, equation \((2)\) has no solutions in integers.

Remark. Based on the previous theorem we may provide some explicit results, for example if \(d \equiv u \pmod{75}\), where \(u \in \{7,12,17,22,32,37,42,47,57,62,67,72\}\), then equation \((2)\) has no solutions in integers with \((k,l) = (2,4)\).

By using elementary number theory we compute all integral solutions of equation \((2)\) for some values of \(k\) and \(d\) with \(l = k\) and \(d \neq 0\). We note that the case \(k = 2\) is in some sense trivial. Indeed, in this case the solvability of equation \((2)\) is equivalent to the existence of integers \(u,v\) such that \(u^2 - v^2 = 8d\) and \(u \equiv v \equiv 1 \pmod{2}\). Equivalently, we need to determine integers \(d_1, d_2\) with \(d_1 \leq d_2\) and \(8d = d_1d_2\) satisfying the conditions

\[
d_1 + d_2 \equiv 2 \pmod{4}, \quad d_2 - d_1 \equiv 2 \pmod{4}.
\]

Thus, if \(d\) is odd, one can take \(d_1 = 4z_1, d_2 = 2z_2\), where \(d = z_1z_2\), i.e., the number of solutions of our equation is at least \(\sigma_0(d)\), where \(\sigma_0(n) = \sum_{k|n} 1\). If \(d\) is even one possible choice is \(d_1 = 2, d_2 = 4d\).

Theorem 2. All integral solutions \((n,m)\) of equation \((2)\) with \(l = k, k \in \{3,4,5\}\) and \(d \neq 0, d \in \{1,2,\ldots,20\}\) are as follows

| \((k,d)\) solutions | \((k,d)\) solutions |
|---------------------|---------------------|
| \((3,3),(4,3)\)   | \((4,4),(5,4)\)   |
| \((3,6),(5,4)\)   | \((4,10),(6,5)\)  |
| \((3,9),(5,3)\)   | \((4,14),(6,6)\)  |
| \((3,10),(6,5)\)  | \((4,20),(7,6)\)  |
| \((3,15),(7,6)\)  | \((5,4),(6,5)\)   |
| \((3,16),(6,4)\)  | \((5,15),(7,6)\)  |
| \((3,19),(6,3)\)  | \((5,20),(7,5)\)  |

In the next result we deal with the cases that can be reduced to elliptic curves.

Theorem 3. All integral solutions \((m,n)\) of equation \((2)\) with \(d \in \{-3,\ldots,3\}\) and \(n \geq k, m \geq l\) are as follows.

| \(d\) | \((k,l) = (2,3)\) |
|-------|-----------------|
| 3     | \((75,368),(77,383),(421726,158118758)\) |
| 2     | \((3,3),(4,4),(104,604)\) |
| 1     | \((6,7),(7,9),(16,34),(27,77),(29,86),(260,2407),(665,9879),(19630,1587767)\) |
| 0     | \((3,2),(5,5),(10,16),(22,56),(36,120)\) |
| -1    | \((4,3),(8,11),(23,60),(425779,160403633)\) |
| -2    | |
| -3    | \((4,2)\) |
Among the solutions given by Blokhuis, Brouwer and de Weger [4] there are some with \((k, l) = (2, 5)\) e.g.:
\[
\binom{10}{5} + 1 = \binom{23}{2}, \quad \binom{22}{5} + 1 = \binom{230}{2}, \quad \binom{62}{5} + 1 = \binom{3598}{2}
\]
in these cases the problem can be reduced to genus 2 curves.

**Theorem 4.** All integral solutions \((n, m)\) of equation (2) with \(d \in \{-3, \ldots, 3\}, k = 2, l = 5\) are as follows.

\[
\begin{array}{c|c|c|c|c}
\hline
d & (k, l) = (2, 4) & d & (k, l) = (2, 6) & d & (k, l) = (2, 8) \\
\hline
3 & \{4, 3\} & 3 & \{(7, 5), (11, 31), (50, 5638)\} & 3 & \{4, 4\} \\
2 & \{5, 4, 7, 9, 12, 32, 93, 2417\} & 2 & \{6, 3\} & 2 & \{7, 4\} \\
1 & \{4, 2, 6, 6, 10, 21\} & 1 & \{6, 2, 8, 8, 10, 21, 14, 78\} & 1 & \{7, 4\} \\
0 & \{6, 2\} & 0 & \{6, 2, 8, 8, 10, 21, 14, 78\} & 0 & \{7, 4\} \\
-1 & \{5, 4\} & -1 & \{5, 4, 21, 34\} & -1 & \{7, 4\} \\
-2 & \{5, 4\} & -2 & \{7, 4\} & -2 & \{7, 4\} \\
-3 & \{7, 4\} & -3 & \{7, 4\} & -3 & \{7, 4\} \\
\hline
\end{array}
\]

\[
\begin{array}{c|c|c|c|c}
\hline
d & (k, l) = (3, 6) & d & (k, l) = (4, 6) & d & (k, l) = (4, 8) \\
\hline
3 & \{6, 4, 7, 5\} & 3 & \{4, 4\} & 3 & \{4, 4\} \\
2 & \{8, 3\} & 2 & \{4, 4\} & 2 & \{4, 4\} \\
1 & \{5, 9, 32, 12\} & 1 & \{4, 3, 7, 7\} & 1 & \{4, 3, 7, 7\} \\
0 & \{8, 2, 10, 10, 14, 78, 17, 221\} & 0 & \{4, 3, 7, 7\} & 0 & \{4, 3, 7, 7\} \\
-1 & \{6, 3, 9, 9\} & -1 & \{5, 4, 21, 34\} & -1 & \{5, 4, 21, 34\} \\
-2 & \{7, 5\} & -2 & \{7, 5\} & -2 & \{7, 5\} \\
-3 & \{7, 4\} & -3 & \{7, 4\} & -3 & \{7, 4\} \\
\hline
\end{array}
\]

Let \(k \in \mathbb{N}\) be odd. In the following theorem we consider the Diophantine equation
\[
\left( \frac{f_1(x)}{k} \right) + \left( \frac{x}{2} \right) = \left( \frac{f_2(x)}{2} \right)
\]
in polynomials $f_1, f_2 \in \mathbb{Q}[x]$ satisfying the condition $\deg f_1 = 2, \deg f_2 = k$. Note that if $f_1(x), f_2(x)$ is a solution of \((1)\), then due to the identity $\left(\frac{x}{2}\right) = \left(\frac{1-x}{2}\right)$, $f_1(1-x), f_2(1-x)$ is also a solution. In the sequel we count such pairs of solutions as one. We are motivated by findings presented in \([4]\).

**Theorem 5.** Let $x$ be a variable.

1. For $k = 3, 5$ equation \((1)\) has exactly three solutions.
2. For $k = 7$ equation \((1)\) has exactly one solution.
3. For $k \in \{9, 11, 13, 15, 17, 19\}$ equation \((1)\) has no solutions.

3. Proofs of the theorems

**Proof of Theorem 5.** In order to get the result it is enough to note that the equation \(\left(\frac{y}{2}\right) + d = \left(\frac{x}{2}\right)\) can be rewritten as

\[
X^2 - 3Y^2 = -2(12d + 1),
\]

where $X = x^2 - 3x + 1, Y = 2y - 1$. If $2(12d + 1) \equiv 0 \pmod{p}$, then $X^2 \equiv 3Y^2 \pmod{p}$. Under our assumption on $p$ we see that 3 is quadratic non-residue modulo $p$ and congruence \((3)\), and hence equation \((2)\), has no integer solutions.

Motivated by the result above, we performed numerical search for pairs $(k, l), k \leq l \leq 10, d \in \mathbb{Z}$ and prime numbers $p > l$ such that the congruence \((3)\) has no solutions modulo $p$. Here are results of our computations.

| $(k, l)$ | $p$ | $d \pmod{p}$ | $(k, l)$ | $p$ | $d \pmod{p}$ |
|---------|-----|--------------|---------|-----|--------------|
| (2, 6)  | 7   | (6, 8)       | (4, 11) | 11  | 11           |
| (2, 8)  | 11  | 7            | (4, 11) | 13  | 13           |
| (2, 9)  | 11  | (6, 9)       | (4, 11) | 19  | 19           |
| (2, 10) | 11  | 7, 8         | (4, 11) | 19  | 19           |
| (3, 4)  | 5   | 2            | (4, 5)  | 5   | 19           |
| (3, 5)  | 11  | 5            | (4, 5)  | 11  | 11           |
| (3, 10) | 11  | 5            | (4, 6)  | 7   | 19           |
| (4, 4)  | 5   | 2, 3         | (4, 6)  | 11  | 11           |
| (4, 5)  | 7   | 3            | (4, 6)  | 11  | 11           |
| (4, 6)  | 7, 10| 2, 3         | (4, 6)  | 11  | 11           |
| (4, 8)  | 11  | 8, 9         | (4, 8)  | 11  | 11           |
| (4, 9)  | 11  | 7, 8         | (4, 8)  | 11  | 11           |
| (4, 10) | 11  | 6, 7, 8, 9   | (4, 10) | 11  | 11           |
| (4, 10) | 11  | 6, 10        | (4, 10) | 11  | 11           |
| (5, 5)  | 7   | 3            | (5, 5)  | 11  | 11           |
| (5, 5)  | 11  | 3, 8         | (5, 5)  | 11  | 11           |
| (5, 6)  | 7   | 2, 3, 4      | (5, 5)  | 11  | 11           |
| (5, 6)  | 11  | 2, 7, 8      | (5, 5)  | 11  | 11           |
| (5, 8)  | 11  | 5            | (5, 6)  | 11  | 11           |
| (5, 9)  | 11  | 3, 8         | (5, 6)  | 11  | 11           |
| (5, 9)  | 11  | 10, 10       | (5, 6)  | 11  | 11           |
| (5, 10) | 11  | 2, 3, 7, 8   | (5, 6)  | 11  | 11           |
| (5, 10) | 11  | 2, 3, 4, 5   | (5, 6)  | 11  | 11           |
| (6, 6)  | 7   | 2, 3, 4, 5   | (5, 6)  | 11  | 11           |
| (6, 6)  | 11  | 2, 3, 8, 9   | (5, 6)  | 11  | 11           |
| (6, 6)  | 13  | 3, 10        | (5, 6)  | 11  | 11           |
| (6, 6)  | 13  | 23           | (5, 6)  | 11  | 11           |
| (6, 6)  | 23  | 5            | (5, 6)  | 11  | 11           |
| (6, 6)  | 29  | 6            | (5, 6)  | 11  | 11           |

Table 2. Pairs $(k, l), k \leq l \leq 10$ such that there exist $p \in \mathbb{P}, p \geq \max\{k, l\}$ such that for some $d \in \{1, \ldots, p\}$ the congruence \((3)\) has no solutions.
Proof of Theorem 2. Here we obtain that
\[
\prod_{i=0}^{k-1} (n-i) - \prod_{i=0}^{k-1} (m-i) = d \cdot k!,
\]
and the polynomial is reducible. It follows that
\[
(n-m)F(n,m) = d \cdot k!.
\]
Hence \((n-m)\) divides \(d \cdot k!\). It remains to solve the one variable polynomial equation
\[
F(m + d_1, m) - \frac{d \cdot k!}{d_1}
\]
for \(d_1 \mid (d \cdot k!)\). □

Remark. Let us note that if \(k = l > 2\), then in the considered range, i.e, \(d \in \{-20, \ldots, 20\}\) we have found at most one integer solution. It is an interesting problem to look for values of \(d\) such that the equation
\[
\binom{n}{k} - \binom{m}{k} = d
\]
has more than one solution in positive integers \(m, n\) satisfying \(n > m\). In order to construct values of \(d\) such that equation (5) has “many” solutions we used the following strategy. First, we computed the set
\[
D_k := \left\{ \binom{n}{k} - \binom{m}{k} : k < m < n \leq 10^4 \right\},
\]
and then looked for duplications in \(D_k\). We considered \(k \in \{3, \ldots, 10\}\). As one could expect, in the case \(k = 3\) the number of duplicates is big. In fact, we found 488 values of \(d\) which appeared at least three times in \(D_3\). The smallest value correspond to \(d = 2180\) with the solutions \((n, m) = (25, 10), (33, 28), (36, 32)\). We found only three values of \(d\) such that equation (5) has four solutions. The values of \(d\) and the corresponding solutions are as follows:
\[
\begin{align*}
d &= 10053736 \quad \quad & (n, m) &= (398, 132), (628, 572), (968, 946), (990, 969), \\
d &= 209920964 \quad & (n, m) &= (1081, 58), (1144, 617), (1242, 868), (3532, 3498), \\
d &= 1928818640 \quad & (n, m) &= (2266, 362), (2268, 428), (3622, 3300), (4991, 4831).
\end{align*}
\]

We strongly believe that the following is true.

Conjecture. For each \(N \in \mathbb{N}\) there is \(d_N \in \mathbb{N}\) such that the equation \(\binom{n}{3} - \binom{m}{3} = d_N\) has at least \(N\) positive integer solutions.

For \(k = 4\) we found 1190 values of \(d\) which appeared at least two times in \(D_4\). The smallest value corresponds to \(d = 680\) with the solutions \((n, m) = (13, 7), (18, 17)\). We found only one value of \(d\) such that equation (5) has three solutions. More precisely, for \(d = 18896570\) equation (5) has three solutions \((n, m) = (185, 163), (258, 251), (486, 485)\).
For $k = 5$ we found 4 values of $d$ which appeared at least 2 times in $D_5$. The values of $d$ and the corresponding solutions are as follows:

$$d = 146438643 \quad (n, m) = (117, 78), (133, 118),$$
$$d = 153852348 \quad (n, m) = (118, 78), (133, 117),$$
$$d = 817514347 \quad (n, m) = (160, 53), (209, 197),$$
$$d = 2346409884 \quad (n, m) = (197, 53), (209, 160).$$

For $k = 6$ we also found 4 values of $d$ which appeared at least 2 times in $D_6$. The values of $d$ and the corresponding solutions are as follows:

$$d = 3819816 \quad (n, m) = (40, 18), (57, 56),$$
$$d = 32449872 \quad (n, m) = (56, 18), (57, 40),$$
$$d = 6627315776 \quad (n, m) = (193, 66), (252, 243),$$
$$d = 26862437556 \quad (n, m) = (243, 66), (252, 193).$$

For $k = 7$ we found only one value of $d \in D_7$ such that equation (5) has two solutions. For $d = 8008$ we have solutions $(n, m) = (16, 14), (17, 16).$

For $k = 8, 9, 10$ there are no duplicates in the set $D_k$.

**Proof of Theorem 3.** All the equations related to this part can be reduced to elliptic curves given some model.

| $(k, l)$ | equation | transformation |
|---------|----------|---------------|
| (2,3)   | $Y^2 = X^3 - 36X^2 + 288X + 10368d + 1296$ | $X = 12m, Y = 216n - 108$ |
| (2,4)   | $Y^2 = 3X(X - 1)(X - 2)(X - 3) + 72d + 9$ | $X = m, Y = 6n - 3$ |
| (2,6)   | $Y^2 = X(X + 40)(X + 60) + 10^7 \cdot (72d + 9)$ | $X = 10m^2 - 50mn, Y = 600n - 300$ |
| (2,8)   | $Y^2 = 35X(X + 6)(X + 10)(X + 12) + 420^2(16d + 1)$ | $X = m^2 - 7m, Y = 420(2n - 1)$ |
| (3,4)   | $Y^2 = X(X - 3)(X - 8) - 384d + 16$ | $X = 4m, Y = 4m^2 - 12m + 4$ |
| (3,6)   | $15X(X - 1)(X + 1) = Y(Y - 3)(Y + 4) + 96d$ | $X = n - 1, Y = (m - 2)(m - 3)/2$ |
| (4,6)   | $Y^2 = X(X + 120)(X + 180) + 30^2 \cdot (24d + 1)$ | $X = 30m^2 - 150mn, Y = 900(n^2 - 3n + 1)$ |
| (4,8)   | $Y^2 = 305X(X + 6)(X + 10)(X + 12) + 420^2(24d + 1)$ | $X = m^2 - 7m, Y = 420(n^2 - 3n + 1)$ |

Table 3. Elliptic models of certain Diophantine equations of the form $\binom{m}{k} = \binom{n}{l} + d$

There exists a number of software implementations for finding integral points on elliptic curves [21, 22]. These procedures are based on a method developed by Stroeker and Tzanakis [28] and independently by Gebel, Pethő and Zimmer [13]. One may follow the transformations provided in [27] to handle these cases. Here we used the Magma procedures `IntegralPoints()` and `IntegralQuarticPoints()`. In some cases there exist no solution and we used `IsLocallySolvable()` and `TwoCoverDescent()` [7]. In cases related to $(k, l) = (3, 6)$ we follow the above mentioned elliptic logarithm method, the cases with $d = -1, 0, 1$ were solved earlier as given in the introduction, so it remains to deal with the values $d \in \{-3, -2, 2, 3\}$.

The case $d = 2$ yields an elliptic curve with Mordell-Weil rank 3 while the remaining three values of $d$ yield elliptic curves with Mordell-Weil rank 2; we only provide details for the case $d = 2$.

For this case we set $u = X, Y = v$ and we have the equation

(6) $C : g(u, v) = 0$, where $g(u, v) = 15u^3 - v^3 + 4v^2 - 15u - 3v - 180,$
where \( u = n - 1 \) and \( v = \frac{1}{2} \left( \left( m - \frac{5}{2} \right)^2 - \frac{1}{4} \right) = (m - 2)(m - 3)/2 \) and the Weierstrass model which is birationally equivalent to \( C \) over \( \mathbb{Q} \) is
\[
E : y^2 = x^3 - 1575x - 48749850 =: f(x).
\]

A notation remark: We will use “exponents” \( C \) and \( E \) on a point to declare whether the point is viewed as one on \( C \) or \( E \), respectively. Also, we will use \((u, v)\) or \((x, y)\) for the \( C \)-coordinates or the \( E \)-coordinates, respectively.

As already mentioned, \( E(\mathbb{Q}) \) has rank 3; its free part is generated by the points
\[
P_1^E = \left( \frac{10905}{4}, -\frac{1137285}{8} \right), \quad P_2^E = \left( \frac{7465}{9}, \frac{616040}{27} \right), \quad P_3^E = \left( \frac{10246}{25}, -\frac{551206}{125} \right)
\]
and the torsion subgroup is trivial.

The birational transformation between the models \( C \) and \( E \) is
\[
C \ni P_C := (u, v) \rightarrow (x, y) = (X(u, v), Y(u, v)) := P_E \in E
\]
\[
C \ni P_C =: (U(x, y), V(x, y)) = (u, v) \leftarrow (x, y) =: P_E \in E
\]
with
\[
X(u, v) = \frac{3(620u^2 + 235uv + 106v^2 - 210u - 438v + 1960)}{(u + 4)^2},
\]
\[
Y(u, v) = \frac{Y_{num}(u, v)}{(u + 4)^3},
\]
where
\[
Y_{num}(u, v) = 3(45795u^3 + 19080u^2v + 7285uv^2 - 35895u^2 - 16795uv - 4568v^2 + 32940u + 65744v - 408000)
\]
and
\[
U(x, y) = \frac{4x^3 - 465x^2 + 318xy + 3903030x - 94455y + 257567175}{-x^3 + 5580x^2 - 290250x + 161614575},
\]
\[
V(x, y) = \frac{9x^3 + 7020x^2 + 705xy - 9215775x + 205560y + 1359589050}{-x^3 + 5580x^2 - 290250x + 161614575}.
\]

With the aid of Maple we find out that there is exactly one conjugacy class of Puiseux series \( v(u) \) solving \( g(u, v) = 0 \). This unique class contains exactly three series and only the following one has real coefficients:
\[
v_1(u) = \zeta u + \frac{4}{3} + \left( \frac{7}{135} \zeta^2 - \frac{1}{3} \zeta \right) u^{-1} + \frac{968}{443} \zeta u^{-2} + \left( \frac{7}{405} \zeta^2 - \frac{1}{9} \zeta \right) u^{-3}
\]
\[
+ \left( \frac{6776}{32805} \zeta^2 - \frac{1936}{729} \zeta \right) u^{-4} + \ldots
\]

Here \( \zeta \) is the cubic root of 15. For every real solution of \( g(u, v) = 0 \) with \( |u| \geq 3 \) it is true that \( v = v_1(u) \) (according to Lemma 8.3.1 in \cite{29}).
Then the point \( P_0^E \) that plays a crucial role in the resolution (see \cite{29}, Definition 8.3.3) is

\[
P_0^E = (318\zeta^2 + 705\zeta + 1860, 21855\zeta^2 + 57240\zeta + 137385).
\]

Referring to the discussion of Section 1 of \cite{14}, we consider the linear form

\[
L(P) = \left( m_0 + \frac{s}{t} \right) \omega_1 + m_1 l(P_1) + m_2 l(P_2) + m_3 l(P_3) \pm l(P_0).
\]

Since \( f(X) \) has only one real root, namely \( e_1 \approx 366.7439448002 \), we have \( E(R) = E_0(R) \), therefore \( l(P_i) \) coincides with the elliptic logarithm of \( P_i^E \) for \( i = 1, \ldots, 3 \) (see Chapter 3 of \cite{29}, especially, Theorem 3.5.2). On the other hand, \( P_0^E \) has irrational coordinates.

As Magma does not possess a routine for calculating elliptic logarithms of non-rational points, we wrote our own routine in Maple for computing \( l(P_i) \) values of points with algebraic coordinates. Thus we compute

\[
l(P_1) \approx 0.0191558345, \quad l(P_2) \approx -0.0349501519, \quad l(P_3) \approx 0.0532999952, \quad l(P_0) \approx -0.00763363355.
\]

Note that the four points \( P_i^E, i = 0, 1, \ldots, 3 \) are \( \mathbb{Z} \)-linearly independent because their regulator is non-zero (see \cite{20}, Theorem 8.1). Therefore our linear form \( L(P) \) falls under the scope of the second “bullet” in \cite{29}, page 99] and we have

\[
r_0 = 1, \quad s/t = s_0/t_0 = 0/1 = 0, \quad d = 1, \quad r = 4, \quad n_i = m_i \text{ for } i = 1, \ldots, 3, \quad n_4 = \pm 1, \quad n_0 = m_0, \quad k = r + 1 = 4, \quad \eta = 1 \quad \text{and} \quad N = \max_{0 \leq i \leq 4} |n_i| \leq \min \{ M, \frac{1}{2}rM + 1 \} + \frac{1}{2} \eta r_0 = \frac{3}{2}M + \frac{3}{2},
\]

so that, in the relation (9.6) of \cite{29} we can take

\[
\alpha = 3/2, \quad \beta = 3/2.
\]

We compute the canonical heights of \( P_1^E, P_2^E, P_3^E \) using Magma\footnote{For the definition of the canonical height we follow J.H. Silverman; as a consequence the values displayed here for the canonical heights are the halves of those computed by Magma and the least eigenvalue \( \rho \) of the height-pairing matrix \( \mathcal{H} \) below, is half that computed by Magma; cf. “Warning” at bottom of p. 106 in \cite{29}.} and for the canonical height of \( P_0^E \) we confine ourselves to the upper bound by applying \cite{29}, Proposition 2.6.4]. Thus we have

\[
\hat{h}(P_1^E) \approx 3.6037959076, \quad \hat{h}(P_2^E) \approx 3.7072405585, \quad \hat{h}(P_3^E) \approx 4.8663287093, \quad \hat{h}(P_0^E) \leq 8.022765298 .
\]

The corresponding height-pairing matrix for the particular Mordell-Weil basis is

\[
\mathcal{H} \approx \begin{pmatrix}
3.6037959076 & -1.0424191872 & -1.2722619781 \\
-1.0424191872 & 3.7072405585 & 3.0174040388 \\
-1.2722619781 & 3.0174040388 & 4.8663287093
\end{pmatrix}
\]

with minimum eigenvalue

\[
\rho \approx 1.2142056695.
\]
Next we apply \cite{29} Proposition 2.6.3 in order to compute a positive constant \( \gamma \) with the property that \( \hat{h}(P^E) - \frac{1}{2}h(x(P)) \leq \gamma \) for every point \( P^E = (x(P), y(P)) \in E(\mathbb{Q}) \), where \( h \) denotes Weil height.\footnote{In the notation of \cite{29} Proposition 2.6.3, as a curve \( D \) we take the minimal model of \( E \) which is \( E \) itself.} It turns out that
\begin{equation}
\gamma \approx 4.8726444820.
\end{equation}

Finally, we have to specify the constants \( c_{12}, c_{13}, c_{14}, c_{15} \) defined in \cite{29} Theorem 9.1.2]. This can be carried out almost automatically with a Maple program. In this way we compute
\begin{equation}
c_{12} \approx 1.07690 \cdot 10^{27}, \quad c_{13} \approx 4.04702 \cdot 10^{162}, \quad c_{14} \approx 2.09861, \quad c_{15} \approx 24.99686.
\end{equation}

According to \cite{29} Theorem 9.1.3, applied to “case of Theorem 8.7.2”, if \( |u(P)| \geq \max\{B_2, B_3\} \), where \( B_2 \) and \( B_3 \) are explicit positive constants, then either \( M \leq c_{12} \), where \( c_{12} \) is an explicit constant, or
\begin{equation}
|u(P)| \geq 5, \quad B_2 = 4, \quad B_3 = 5, \quad \theta = 1, \quad c_9 = 0.17, \quad c_{10} = \log(11800), \quad c_{11} = 2.
\end{equation}

So, in view of (14) and (10), (11), (12), (13), we conclude that, if \( |u(P)| \geq 5 \), then either \( M \leq c_{12} \) or
\begin{equation}
M \leq \max\{c_{12}, 6.64 \cdot 10^{86}\} = 6.64 \cdot 10^{86} \quad \text{provided that } |u(P)| \geq 5.
\end{equation}

An easy straightforward computation shows that \( P^C = (-4, -9) \) is the only one integer point with \( |u(P)| \leq 4 \) (equivalently, the integer solution \((u, v)\) of (10) with \(|u| \leq 4\).

In order to find explicitly all points \( P^C \) with \( |u(P)| \geq 5 \) it is necessary to reduce the huge upper bound (15) to an upper bound of manageable size. This is accomplished with LLL-algorithm [10], in a similar way as in Appendix D in \cite{14}, and we obtain the reduced bound \( M \leq 10 \). Therefore, we have to check which points
\begin{equation}
P^E = m_1 P_1^E + m_2 P_2^E + m_3 P_3^E, \quad \text{with } \max_{1 \leq i \leq 3} |m_i| \leq 10,
\end{equation}
have the property that \( P^E = (x, y) \) maps via the transformation (5) to a point \( P^C = (u, v) \in C \) with integer coordinates. We remark here that every point \( P^C \) with \( u(P) \) integer and \( |u(P)| \geq 5 \) is obtained in this way, but the converse is not necessarily true; i.e. if \( \max_{1 \leq i \leq 3} |m_i| \leq 10 \) and the above \( P^E \) maps to \( P^C \) with integer coordinates, it is not necessarily true that \( |u(P)| \geq 5 \). After a computational search we find the only one point \( P^C = (-4, -9) \) which corresponds to the zero point \( O \in E \).

So no integral solution \((m, n)\) (with \( n \geq k \) and \( m \geq l \)) of equation (2) with \( d = 2 \) exists.
For the other three cases we provide some details in the tables below:

| $d$ | $a(d)$  | $r$ | Generators                                      | $\rho$ | $e_1$              |
|-----|---------|-----|-------------------------------------------------|--------|-------------------|
| -2  | -49559850 | 2   | $P_1 = (956289/4,935155287/8)$                   | 1.499191 | 368.748212        |
|     |         |     | $P_2 = (198006/169,-86888954/2197)$               |         |                   |
| -3  | -111271725 | 2   | $P_1 = (1230,-41805)$                            | 2.568215 | 482.072907        |
|     |         |     | $P_2 = (22159769795/91145209,103896688780607535/870163310323)$ |         |                   |
| 3   | -110056725 | 2   | $P_1 = (1230,-41805)$                            | 1.786872 | 480.319851        |
|     |         |     | $P_2 = (16866855/34969,7734674565/6539203)$       |         |                   |

Table 4. $C : 15u^3 - v^3 + 4v^2 - 15u - 3v - 90d$ and $E : y^2 = x^3 - 1575x + a(d)$

| $d$ | $B(M) :$ Initial bound | Reduced bound |
|-----|------------------------|---------------|
| -2  | $5.06 \cdot 10^{62}$   | 6             |
| -3  | $8.66 \cdot 10^{62}$   | 5             |
| 3   | $9.07 \cdot 10^{62}$   | 5             |

Table 5. Upper bounds of $M$.

| $d$ | $P^E = (x, y)$                        | $P^C = (u, v)$                      |
|-----|--------------------------------------|------------------------------------|
| -2  | $\mathcal{O}$                       | $(-2, 6)$                          |
| -3  | $\mathcal{O}$                       | $(3, 10)$                          |
| 3   | $\mathcal{O}$, $(-2053305)$          | $(3, 6)$, $(4, 10)$                |

Table 6. All points $P^E = \Sigma_i m_i P^E_i$ with $P^C = (u, v) \in \mathbb{Z} \times \mathbb{Z}$.

\[\square\]

**Proof of Theorem 4** We provide details only in case of $d = 3$, here the rank of the Jacobian is 6 (like in case of $d = 1$). Equation (\ref{eq:3}) with $d = 3$ defines the hyperelliptic curve

\[y^2 = 15x(x-1)(x-2)(x-3)(x-4) + 75^2.\]

Based on Stoll’s papers \cite{23, 24, 25} one can determine generators for the Mordell-Weil group by using Magma \cite{5}. We obtain that $J(\mathbb{Q})$ is free of rank 6 with Mordell-Weil basis given by (in Mumford representation)

\[
\begin{align*}
D_1 &= \langle x - 4, -75 \rangle, \\
D_2 &= \langle x - 3, 75 \rangle, \\
D_3 &= \langle x - 1, -75 \rangle, \\
D_4 &= \langle x, 75 \rangle, \\
D_5 &= \langle x^2 - 7x + 30, 195 \rangle, \\
D_6 &= \langle x^2 - 3x + 20, -30x - 45 \rangle
\end{align*}
\]
and the torsion subgroup is trivial. We apply Baker’s method \cite{2} to get a large upper bound for \( \log |x| \), here we use the improvements given in \cite{S} and \cite{12}. It follows that
\[
\log |x| \leq 1.028 \times 10^{612}.
\]
We have from Corollary 3.2 of \cite{12} that every integral point on the curve can be expressed in the form
\[
P - \infty = \sum_{i=1}^{6} n_i D_i
\]
with \(||(n_1, n_2, n_3, n_4, n_5, n_6)|| \leq 1.92 \times 10^{306} =: N\). Proposition 6.2 in \cite{12} gives an estimate for the precision we need to compute the appropriate matrices, this bound is as follows
\[
((1/5)(48\sqrt{r}Nt + 12\sqrt{r}N + 5N + 48))^{(r+4)/4} \approx 2.6 \times 10^{769},
\]
where in our case \( r = 6 \) and \( t = 1 \). We choose to compute the period matrix and the hyperelliptic logarithms with 1500 digits of precision. The hyperelliptic logarithms of the divisors \( D_i \) are given by
\[
\begin{align*}
\varphi(D_1) &= (0.087945\ldots + i0.112834\ldots, -0.473844\ldots - i0.741784\ldots) \in \mathbb{C}^2, \\
\varphi(D_2) &= (0.114612\ldots + i0.112834\ldots, -0.420527\ldots - i0.741784\ldots) \in \mathbb{C}^2, \\
\varphi(D_3) &= (-0.044486\ldots + i1.333456\ldots, -0.416321\ldots + i5.329970\ldots) \in \mathbb{C}^2, \\
\varphi(D_4) &= (0.127905\ldots + i0.112834\ldots, -0.413878\ldots - i0.741784\ldots) \in \mathbb{C}^2, \\
\varphi(D_5) &= (-0.118415\ldots + i0.037611\ldots, -0.857076\ldots - i0.247261\ldots) \in \mathbb{C}^2, \\
\varphi(D_6) &= (0.128537\ldots + i0.075223\ldots, -0.173077\ldots - i0.494522\ldots) \in \mathbb{C}^2.
\end{align*}
\]
We need now to choose an integer \( K \) that is larger than the constant given by Proposition 6.2 in \cite{12}. Setting \( K = 10^{1300} \) we get a new bound \( 126.98 \) for \(||(n_1, n_2, n_3, n_4, n_5, n_6)||\). We repeat the reduction process with \( K = 10^{16} \) that yields a better bound, namely \( 15.6 \). Two more steps with \( K = 6 \times 10^{11} \) and \( K = 2 \times 10^{11} \) provide the bounds \( 13.94 \) and \( 13.8 \). It remains to compute all possible expressions of the form
\[
n_1D_1 + \ldots + n_6D_6
\]
with \(||(n_1, n_2, n_3, n_4, n_5, n_6)|| \leq 13.8\). We performed a parallel computation to enumerate linear combinations coming from integral points on a machine having 12 cores. The computation took 3 hours and 23 minutes. We obtained the following non-trivial solutions
\[
\begin{align*}
\begin{pmatrix} 11 \\ 5 \end{pmatrix} + 3 &= \begin{pmatrix} 31 \\ 2 \end{pmatrix}, \\
\begin{pmatrix} 16 \\ 5 \end{pmatrix} + 3 &= \begin{pmatrix} 94 \\ 2 \end{pmatrix}, \\
\begin{pmatrix} 375 \\ 5 \end{pmatrix} + 3 &= \begin{pmatrix} 346888 \\ 2 \end{pmatrix}, \\
\begin{pmatrix} 379 \\ 5 \end{pmatrix} + 3 &= \begin{pmatrix} 356263 \\ 2 \end{pmatrix}.
\end{align*}
\]
If \( d = 1 \), then the rank of the Jacobian is 6 and the Baker bound is \( \log |x| \leq 1.225 \times 10^{532} \) and we have that \( ||(n_1, n_2, n_3, n_4, n_5, n_6)|| \leq 2.23 \times 10^{266} \). In three steps it is reduced to 14.97. In this case the non-trivial solutions are as follows

\[
\begin{align*}
\binom{10}{5} + 1 &= \binom{23}{2}, \\
\binom{22}{5} + 1 &= \binom{230}{2}, \\
\binom{62}{5} + 1 &= \binom{3598}{2}, \\
\binom{135}{5} + 1 &= \binom{26333}{2}, \\
\binom{139}{5} + 1 &= \binom{28358}{2}.
\end{align*}
\]

If \( d = -3, -1, 2 \), then the rank of the Jacobian is 3, we followed the arguments given in [8] and [11] to obtain a large bound for the size of possible integral solutions. We present them in the table below.

| \( d \) | bound for \( \log |x| \) |
|---|---|
| -3 | \( 2.91 \cdot 10^{488} \) |
| -1 | \( 1.21 \cdot 10^{552} \) |
| 2 | \( 3.25 \cdot 10^{590} \) |

Table 7. Upper bounds for \( \log |x| \).

In all three cases the rank of the Jacobians are equal to 3 and the torsion subgroup is trivial hence all points can be written as

\[ n_1 D_1 + n_2 D_2 + n_3 D_3, \]

where \( n_i \in \mathbb{Z} \). Using the previously applied hyperelliptic logarithm method the initial large upper bounds for \( \max\{|n_i|\} \) can be significantly reduced. If \( d = -3 \), then after one reduction step we get the bound 64 and other two steps make it 7. The only pair of integral points we obtain is given by \((6, \pm 75)\). Therefore we have

\[ \binom{3}{2} = \binom{6}{5} - 3. \]

If \( d = -1 \), then first we obtain a reduced bound 51 and finally it follows that \( \max\{|n_i|\} \leq 5 \). The complete list of integral points is given by \((5, \pm 15), (8, \pm 315)\). Thus we obtain

\[ \binom{11}{2} = \binom{8}{5} - 1. \]

Finally, in case of \( d = 2 \) the first reduction yields a bound 58 and the third one provides 6. The complete set of integral solutions is \( \{(-1, \pm 45), (5, \pm 75)\} \), so we do not get non-trivial solution of \( (2) \).

If \( d = -2 \), then the rank of the Jacobian is 1, therefore classical Chabauty’s method [9] can be applied, it is now implemented in Magma [5]. We obtain that the equation \( \binom{n}{2} = \binom{m}{5} - 2 \) has no non-trivial solution. \( \Box \)
Remark. Let

\[ C_d : y^2 = 15x(x-1)(x-2)(x-3)(x-4) + 15^2(8d + 1) \]

and write \( J_d := \text{Jac}(C_d) \). The curve \( C_d \) is isomorphic to the curve defined by the equation \( (y/2) = (x/3) + d \). We computed upper bounds for the numbers \( r_d = \text{rank} \, J_d(\mathbb{Q}) \) using the Magma procedure \texttt{RankBound}. We obtained the following data:

| \( i \) | the value of \( d \) such that \( r_d \leq i \) |
|-----|---------------------------------------------|
| 0   | \(-45, -40, -39, -37, -34, -10, -9, -4, 8, 25, 26, 40, 47\) |
| 1   | \(-47, -36, -33, -31, -28, -26, -25, -22, -14, -13, -8, -5, -2, 5, 11, 17, 20, 29, 32, 41, 50\) |
| 2   | \(-50, -46, -41, -38, -32, -30, -29, -24, -23, -19, -16, -7, 4, 13\) |
| 3   | \(-48, -44, -43, -42, -35, -21, -20, -15, -11, -3, -1, 2, 7, 16, 18\) |
| 4   | \(-49, -27, -18, -17, -12, -6, 9, 12, 22, 24, 34, 37, 46, 49\) |
| 5   | \(27, 36\) |
| 6   | \(0, 1, 3, 6, 10, 15, 45\) |
| 7   | \(21, 28\) |

Table 8. Upper bounds for the rank of Jacobian of the curve \( C_d \) for \( d \in \{-50, \ldots, 50\} \).

We checked that for \( i \in \{0, 4, 5, 6, 7\} \) the upper bounds computed by \texttt{RankBound} are actually equal to the ranks.

Let us note that \( 21 = \binom{3}{2} \) and \( 28 = \binom{8}{2} \). We checked that in both cases the rank is equal to 7. This follows from the existence of seven independent divisors in \( J_d(\mathbb{Q}) \). They are as follows:

\[ d = 21; \quad < x - 3, -345 >, < x - 1, -345 >, < x - 4, 345 >, < x, 345 >, \]
\[ < x + 3, 285 >, < x + 4, 135 >, < x - 11, 975 >, < x^2 + x + 30, -30x + 165 >, \]
\[ d = 28; \quad < x - 3, 225 >, < x - 1, -225 >, < x - 4, 225 >, < x - 12, 1215 >, \]
\[ < x - 17, -3345 >, < x, 225 >, < x^2 - x + 18, -135 >. \]

We also looked for high rank Jacobians for further values of \( d \) of the form \( \binom{w}{2} \). For \( d = 66 = \binom{12}{2} \) we obtained the equality \( r_{66} = 8 \) with the following independent divisors

\[ < x - 3, -345 >, < x - 1, -345 >, < x - 4, 345 >, < x, 345 >, \]
\[ < x + 3, 285 >, < x + 4, 135 >, < x - 11, 975 >, < x^2 + x + 30, -30x + 165 >. \]

The torsion part of \( J_{66}(\mathbb{Q}) \) is trivial. We conjecture that the only solutions in positive integers of the equation \( \binom{w}{2} = \binom{7}{2} + 66 \) are

\((x, y) = (1, 23), (2, 23), (3, 23), (4, 23), (11, 65), (28, 887), (7935, 1447264765), (7939, 1449089815)\).
The large points are explained by the fact that on the curve $C(\omega)$ we have the following solutions
\[
\begin{align*}
x &= 3 \cdot 5 \cdot (2w - 1)^2, \\
y &= 75(72w^4 - 1440w^3 + 1020w^2 - 300w + 31)(2w - 1) \text{ and} \\
x &= 3 \cdot 5 \cdot (2w - 1)^2 + 4, \\
y &= 75(72w^4 - 1440w^3 + 1140w^2 - 420w + 61)(2w - 1).
\end{align*}
\]
Hence we obtain the following divisors on $J(\omega)(\mathbb{Q})$
\[
\begin{align*}
(x, 30w - 15, 1), \\
(x - 1, 30w - 15, 1), \\
(x - 2, 30w - 15, 1), \\
(x - 3, 30w - 15, 1), \\
(x - 4, 30w - 15, 1), \\
(x - 60w^2 + 60w - 15, 108000w^5 - 2700000w^4 + 261000w^3 - 121500w^2 + 27150w - 2325, 1), \\
(x - 60w^2 + 60w - 19, 108000w^5 - 2700000w^4 + 279000w^3 - 148500w^2 + 40650w - 4575, 1).
\end{align*}
\]

**Remark.** In case of the equation
\[
\left(\begin{array}{c}
n \\
n
\end{array}\right) = \left(\begin{array}{c}
m \\
m
\end{array}\right) + d
\]
one obtains genus 3 curves. Stoll [26] proved that the rank of the Jacobian is 9 if $d = 0$. For other values of $d$ in the range $\{-3, \ldots, 3\}$ many of the genus 3 hyperelliptic curves have high ranks as well. Balakrishnan et. al. [3] developed an algorithm to deal with genus 3 hyperelliptic curves defined over $\mathbb{Q}$ whose Jacobians have Mordell-Weil rank 1. If $d = -2$, then the equation is isomorphic to the curve
\[
Y^2 = 70X^7 - 1470X^6 + 12250X^5 - 51450X^4 + 113680X^3 - 123480X^2 + 50400X - 66150
\]
and using Magma (with `SetClassGroupBounds("GRH")` to speed up computation) we get that the rank of the Jacobian is 1. Therefore we may try to use the Sage implementation described in [3] to compute the set of rational points on this curve. The affine points are $(8, \pm 1470)$, hence we have the solution
\[
\left(\begin{array}{c}
4 \\
2
\end{array}\right) = \left(\begin{array}{c}
8 \\
7
\end{array}\right) - 2.
\]

**Proof of Theorem 5** In each case we will be working in the same way. More precisely, for given $k$ we write $f_1(x) = a_2x^2 + a_1x + a_0$ and $f_2(x) = \sum_{i=0}^{k} b_i x^i$. The polynomial $\frac{f_1(x)}{k} + \frac{(x^2)}{2} - \frac{f_2(x)}{2} = \sum_{i=0}^{2k} A_i x^i$ needs to be zero. Thus the coefficient near $x^i$ in $F_k(x)$ need to be zero for $i = 0, \ldots, 2k$. In consequence, we are interested in solving the system of polynomial equations
\[
S_k : A_0 = A_1 = \ldots = A_{2k} = 0
\]
in $k + 4$ variables $a_0, a_1, a_2, b_0, \ldots, b_k$. We have $A_{2k} = \frac{a_k^2}{k} - \frac{b_k^2}{2}$ and thus $a_2 = \frac{k^2 t^2}{2}, b_k = (\frac{k^2 t}{2})^{\frac{k-1}{2}} t^k$ for some non-zero $t \in \mathbb{Q}$. We note that after the substitution of the computed
values of $a_2, b_k$ into the system $S_k$, the related system of equations

$$S'_k : A'_k = A'_{k+1} = \cdots = A'_{2k-1},$$

where $A'_i$ comes from $A_i$ after the substitution of the computed values of $a_2, b_k$, is triangular with respect to the variables $b_0, b_1, \ldots, b_{k-1}$. More precisely, we have $\deg_{b_i} A'_{k+i} = 1$ for $i = 0, \ldots, k - 1$. Moreover, the coefficient near $b_i$ is a power of $t$ times a rational number. Solving for $b_0, \ldots, b_{k-1}$ and substituting into $S'_k$ we are left with the system of equations

$$S''_k : A''_0 = A''_1 = \cdots = A''_{k-1},$$

in three variables $a_0, a_1, t$. The polynomial $A''_i$ is the numerator of the rational function $A'_i$ after substitution of the computed values $b_0, \ldots, b_{k-1}$. It seems that for each fixed odd $k \geq 3$, the system $S''_k$ can be solved using Gröbner bases techniques. More precisely, we compute $G_k$ - the Gröbner basis of the ideal generated by the polynomials $A''_i, i = 0, \ldots, k - 1$. For $k \geq 5$ we have more equations than variables we expect that the system $S''_k$ for all sufficiently large $k$ has no rational (and even complex) solutions. This can be confirmed with our approach for $k \in \{11, \ldots, 19\}$. However, we were unable to prove such a statement in full generality.

We prove the first part of our theorem. However, we present details of the reasoning only for $k = 3$. The case $k = 5$ is proved in exactly the same way. We are interested in rational solutions of the system

$$S_3 : A_0 = \ldots = A_6 = 0.$$

We have $a_2 = 3t^2, b_3 = 3t^3$ for some $t \neq 0$. We put the values of $a_2, b_3$ into the system $S_3$ and solve corresponding system of equations

$$S'_3 : A'_3 = A'_4 = A'_5 = 0,$$

with respect to $b_0, b_1, b_2$. We get

$$b_0 = \frac{36a_0a_1t^2 - 36a_1t^2 - a_1^3 + 72t^3}{144t^3}, \quad b_1 = \frac{12a_0t^2 + a_1^2 - 12t^2}{8t}, \quad b_0 = \frac{3a_1t}{2}.$$

In consequence, after the substitution of the values of $a_2, b_0, b_1, b_2, b_3$ into the system $S_3$ we obtain the system

$$S''_3 : A''_0 = A''_1 = A''_2 = 0,$$

where $A''_i = t^{2(i-1)}A'_i \in \mathbb{Q}[t, a_0, a_1]$. It is an easy task to solve the system $S''_3$. Indeed, we compute Gröbner basis $G_3$, of the ideal generated by $A''_0, A''_1, A''_2$. The basis $G_3$ contains four polynomials. Two of them are the following

$$a_1^5(a_1 + 3)(a_1 + 12), \quad (4a_0 - 7)a_1^5(a_1 + 12)$$

and we easily obtain the following solutions

$$f_1(x) = 3(-1 + 2x)^2, \quad f_2(x) = 2 - 15x + 36x^2 - 24x^3,$$

$$f_1(x) = 5 - 12x + 12x^2, \quad f_2(x) = 5 - 21x + 36x^2 - 24x^3,$$

$$f_1(x) = \frac{1}{3}(12x^2 - 12x + 7), \quad f_2(x) = \frac{1}{3}(-24x^3 + 36x^2 - 18x + 7).$$

Note that the first two solutions were presented in [4]. Unfortunately, the polynomials from the third solution take only non-integer values.
For \( k = 5 \) we proceed in the same way and omit details. However, let us note that the Gröbner basis \( G_5 \) contains 7 polynomials. Two of them are the following
\[
a_9^1(a_1 + 60)(3a_1 + 80), \quad a_9^0(3a_0 - 26)(a_1 + 60)
\]
and we obtain two solutions with integer coefficients and the solution (corresponding to the triple \( t = 2/3, a_0 = 26/3, a_1 = -80/3 \))
\[
f_1(x) = \frac{2}{3}(40x^2 - 40x + 13), \quad f_2(x) = \frac{1}{27}(12800x^5 - 32000x^4 + 32000x^3 - 16000x^2 + 3955x - 364).
\]
By replacing \( x \) by \( 3x - 1 \) we obtain polynomial with integer coefficients, which is exactly the third solution from the paper [4].

For \( k = 7 \) the Gröbner basis \( G_7 \) contains 11 elements. In particular, the following three polynomials are in \( G_7 \):
\[
a_{12}^1(a_1 + 70), \quad a_{12}^1(2a_0 - 41), \quad a_{10}^1(420t - a_1)(a_1 + 420t).
\]
We found that the only solution (corresponding to \( t = 1/6, a_0 = 41/2, a_1 = -70 \)) is the following
\[
f_1(x) = \frac{1}{2}(140x^2 - 140x + 41),
\]
\[
f_2(x) = \frac{1}{96}(5488000x^7 - 19208000x^6 + 28812000x^5 - 24010000x^4 + 11997160x^3 - 3589740x^2 + 594370x - 41847).
\]

The last part of our theorem follows from certain Gröbner basis computations. For \( k \in \{9, 11, 13, 15, 17, 19\} \) we found that the \( G_k \) contains polynomial of the form \( tu^k \) for some \( u_k \in \mathbb{N}_+ \), i.e., \( t \) need to be zero which leads to contradiction. \( \square \)

Remark. Using the same approach as in the proof of the above theorem one can prove that the Diophantine equation \( (f_k^1(x))^2 - (f_k^2(x))^k = (f_k^0(x))^2) \) has no polynomial solutions \( f_1, f_2 \in \mathbb{Q}[x] \) satisfying \( \deg f_1 = 2, \deg f_2 = k \) for \( k \in \{3, 5, \ldots, 19\} \).

We also looked for solutions of the more general Diophantine equation

\[
\left( \frac{f_1(x)}{k} \right) + \left( \frac{f_0(x)}{2} \right) = \left( \frac{f_2(x)}{2} \right),
\]
where \( f_0 \) is of degree 2. By using the same approach as in the proof of Theorem [5] one can prove that for \( k \in \{5, 7, \ldots, 19\} \) there are no solutions \( f_0, f_1, f_2 \in \mathbb{Q}[x] \) of (16) satisfying \( \deg f_0 = \deg f_1 = 2 \) and \( \deg f_2 = k \).

However, if we allow \( f_0 \) to be of degree 3 we found the following solutions. For \( k = 5 \) we have the solution
\[
f_1(x) = 15x^2,
\]
\[
f_0(x) = \frac{1}{2}(30x^3 - 5x + 1),
\]
\[
f_2(x) = \frac{1}{2}(225x^5 - 75x^3 + 7x + 1).
\]
For \( k = 7 \) we have the solution
\[
\begin{align*}
f_1(x) &= 2520x^2 + 1, \\
f_0(x) &= \frac{1}{2} \left(17640x^3 - 23x + 1\right), \\
f_2(x) &= \frac{1}{2} \left(32006016000x^7 - 88905600x^5 + 52920x^3 + 7x + 1\right).
\end{align*}
\]
Note that in both cases by replacing \( x \) by \( 2x - 1 \) we get polynomials with integer coefficients.

Playing around with the Diophantine equation \( \left(f_0(x)^3\right) + \left(f_1(x)^3\right) = \left(f_2(x)^2\right) \) we also found the polynomial solution
\[
\begin{align*}
f_0(x) &= x(3x + 2), \\
f_1(x) &= (2x + 1)(3x + 2), \\
f_2(x) &= 9x^3 + 15x^2 + 6x + 1.
\end{align*}
\]

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