The information content of gravitational wave harmonics in compact binary inspiral

Ronald W. Hellings
Department of Physics, Montana State University, Bozeman MT 59715

Thomas A. Moore
Department of Physics and Astronomy, Pomona College, Claremont CA 91109

ABSTRACT: The nonlinear aspect of gravitational wave generation that produces power at harmonics of the orbital frequency, above the fundamental quadrupole frequency, is examined to see what information about the source is contained in these higher harmonics. We use an order (4/2) post-Newtonian expansion of the gravitational wave waveform of a binary system to model the signal seen in a spaceborne gravitational wave detector such as the proposed LISA detector. Covariance studies are then performed to determine the ultimate accuracy to be expected when the parameters of the source are fit to the received signal. We find three areas where the higher harmonics contribute crucial information that breaks degeneracies in the model and allows otherwise badly-correlated parameters to be separated and determined. First, we find that the position of a coalescing massive black hole binary in an ecliptic plane detector, such as OMEGA, is well-determined with the help of these harmonics. Second, we find that the individual masses of the stars in a chirping neutron star binary can be separated because of the mass dependence of the harmonic contributions to the wave. Finally, we note that supermassive black hole binaries, whose frequencies are too low to be seen in the detector sensitivity window for long, may still have their masses, distances, and positions determined since the information content of the higher harmonics compensates for the information lost when the orbit-induced modulation of the signal does not last long enough to be apparent in the data.
I. Introduction

As two compact stars in a tight binary system approach each other, the nonlinear effects of relativistic gravity produce two effects in the waveform of the gravitational wave emitted by the binary. First, the loss of energy to gravitational radiation tightens the binary, decreasing the gravitational wave period and increasing its amplitude in a “chirp”. Second, the scattering of the emitted gravitational waves off of the strong gravitational field around the binary converts some of the energy to higher harmonics, modifying the shape of the waves. Several papers have been written [1,2] exploring the sensitivity of the signals from these gravitational wave sources to the parameters of the systems, but only one of these (Moore and Hellings [2], hereafter referred to as MH) has taken the information content of the higher harmonics into account. It is the purpose of this paper to expand on MH by particularly investigating the information content of the higher harmonics of the gravitational wave waveform, identifying three places where these harmonics break degeneracies in the solutions and allow otherwise poorly-separated parameters to be determined.

The remainder of this paper is organized as follows. In Section II, we present the mathematical form of the gravitational waves from inspiraling compact binaries, as seen in the detectors. The detectors considered will be the same as those in MH, namely an ecliptic-plane case, similar to the proposed OMEGA mission [3], and a precessing-plane case, similar to the proposed LISA mission [4]. In discussing the information content of the various contributions to the received signal, we will need rather complete expressions, so we will reproduce here the exposition as it is given in MH. In sections III through V, we will discuss the three places where the higher harmonics contribute to the determination of the binary parameters, giving the results of covariance studies to predict the uncertainties in the realistic parameter-estimation process. For brevity, we will omit many of the details of the covariance study procedure, referring the reader instead to MH for more information. Finally, in section VI, we will draw a general conclusion from this study.

II. The received gravitational wave waveform

The gravitational wave from a binary system of non-spinning point masses in quasi-circular orbit has been worked out by Blanchet et al. [5]. The waveform, to order $(4/2)$ in $v/c$, can be written in the frame of the detector as

$$h_{+,\times}(t) = \frac{2\tau c\eta}{5R_L} (1 + z)\varepsilon^2 \left[ H_{+,\times}^{(0)} + \varepsilon H_{+,\times}^{(1/2)} + \varepsilon^2 H_{+,\times}^{(2/2)} + \varepsilon^3 H_{+,\times}^{(3/2)} + \varepsilon^4 H_{+,\times}^{(4/2)} \right],$$

where $R_L$ is the luminosity distance to the source in an assumed flat Friedman universe, $c$ is the speed of light, $\tau \equiv 5G(m_1 + m_2)/c^3$ is proportional to the total mass in units of time, $\eta \equiv m_1 m_2/(m_1 + m_2)^2$ is the ratio of reduced mass to total mass, $\varepsilon \equiv (\tau \omega_s/5)^{1/3}$ is a time-dependent expansion parameter that is of order $v/c$ for the system, and $z$ is the cosmological redshift. The $\omega_s$ that appears in the definition of $\varepsilon$ is the time-dependent angular frequency of the binary orbit in its own frame. The terms in the multipole expansion of the waveform are given by

$$H_{+,\times}^{(0)} = -(1 + \cos^2 \theta) \cos 2\phi,$$
\begin{align*}
H_x^{(0)} &= -2 \cos i \sin 2\phi_r \\
H_x^{(1/2)} &= -\frac{\delta}{8} \sin i \left[ (5 + \cos^2 i) \cos \phi_r - 9(1 + \cos^2 i) \cos 3\phi_r \right] \\
H_x^{(1/2)} &= -\frac{3\delta}{4} \cos i \sin i \left[ \sin \phi_r - 3 \sin 3\phi_r \right]
\end{align*}

where \( i \) is the angle of the source’s orbital angular momentum vector relative to a unit vector pointing from the source to the detector, \( \delta \equiv (m_1 - m_2)/(m_1 + m_2) \), and \( \phi_r \) is the received phase as a function of time. The complete expressions for the higher-order \( H_s \) are found in Blanchet et al. [5].

To the level of approximation required, the phase received at time \( t \) is the Doppler-shifted version of the phase of the signal at the source

\[ \phi_r(t) = \phi(t_s) - \frac{\Omega R}{c(1 + z)} \sin \Theta I_0(t) + \phi_0, \tag{3a} \]

where

\[ I_0(t) \equiv \int_0^t \omega_s \sin(\Omega t - \Phi) dt. \tag{3b} \]

In Eqs. 3, \( t_s = t/(1 + z) \) is the time at the source, \( \Omega \) and \( R \) are the angular frequency and radius of the detector’s orbit around the sun, \( \Theta \) and \( \Phi \) are the source’s ecliptic coordinates in the sky (oriented so that \( \Phi = 0 \) corresponds to the direction of the detector relative to the sun at \( t = 0 \)), and \( \phi_0 \) is the phase of the received signal at \( t = 0 \).

According to Blanchet et al. [5], the orbital phase is in turn given by

\[ \phi(t_s) = \frac{1}{\eta} [F(t_s) - F(0)] \tag{4a} \]

where

\[ F(t_s) = G^5 + \eta_1 G^3 - \frac{3\pi}{4} G^2 + \eta_2 G \tag{4b} \]

with

\[ G(t_s) \equiv \left[ \frac{\eta}{\tau}(t_c - t_s) \right]^{1/8} \tag{4c} \]

\[ \eta_1 \equiv \frac{3.715}{8,064} + \frac{55}{96\eta} \tag{4d} \]

\[ \eta_2 \equiv \frac{9,275,495}{14,450,688} + \frac{284,875}{258,048\eta} + \frac{1,855}{2,048\eta^2} \tag{4e} \]

where \( t_c \) is the time to coalescence from \( t_s = 0 \), as measured in the source frame. (Specifying \( t_c \) is equivalent to specifying the initial separation of the orbiting masses.) Taking the time-derivative of Eq. 4a gives the orbital angular frequency \( \omega_s \equiv d\phi/dt_s \) in the source frame. The orbital phase is thus a complicated function of \( t_s \), parameterized by the four parameters \{\( \phi_0, \eta, \tau, t_c \}\).
The gravitational wave detectors are assumed to be of two types: an ecliptic plane detector, like OMEGA [3], and a precessing-plane detector, like LISA [4]. Each of these detectors consists of three coplanar laser-beam arms that form an equilateral triangle with spacecraft at each vertex of the triangle. The three arms define the detector plane. An interferometer is created by combining the beams in adjacent arms, the three possible combinations representing only two independent information streams. The signal received in the detectors is a phase measurement of the laser interferometer and may be written

\[ \frac{\phi(t)}{\nu T} = h(t) = F_+(t)h_+(t) + F_\times(t)h_\times(t) \]  

(5)

where \( \nu \) is the laser frequency and \( T \) is the light-travel-time along the interferometer arm. The time-dependent functions \( F_+ (t) \) and \( F_\times(t) \) are the beam-pattern functions for the interferometer pair. For both the precessing-plane case and the ecliptic plane case, the arms in a given interferometer pair make a 60° angle with respect to each other, and the beam-pattern functions for the interferometer formed by one such pair are

\[ F_+ = \frac{\sqrt{3}}{2} \left[ \frac{1}{2} (1 + \cos^2 \theta_D) \cos 2\phi_D \cos 2\psi_D \right. \]  

\[ \left. - \cos \theta_D \sin 2\phi_D \sin 2\psi_D \right] \]  

(6a)

\[ F_\times = \frac{\sqrt{3}}{2} \left[ \frac{1}{2} (1 + \cos^2 \theta_D) \cos 2\phi_D \sin 2\psi_D \right. \]  

\[ \left. + \cos \theta_D \sin 2\phi_D \cos 2\psi_D \right] \]  

(6b)

where \( \theta_D \) and \( \phi_D \) are the instantaneous angular coordinates of the source measured relative to the frame of the detector and \( \psi_D \) specifies the orientation of the binary’s principal polarization axes around the fixed line of sight. The latter variable can be understood as follows. If the orbital inclination \( i \) to the line of sight is not zero, the quasicircular binary orbit will look elliptical to a viewer at the detector. The angle \( \psi_D \) specifies the orientation of the major axis of the ellipse as viewed by this observer, measured in the plane perpendicular to the line of sight and from a reference direction that is perpendicular to both the line of sight and the normal to the plane of the detector array. If \( \hat{L}, \hat{n}, \) and \( \hat{z}_D \) are unit vectors parallel to the binary’s conserved angular momentum, the direction of the line of sight, and the normal to the detector array, respectively, then we may define \( \psi_D \) via

\[ \tan \psi_D = \frac{\hat{L} \cdot [\hat{n} \times (\hat{z}_D \times \hat{n})]}{\hat{L} \cdot (\hat{z}_D \times \hat{n})} \]  

(7)

In both the precessing-plane and ecliptic-plane case, the equilateral triangle arrangement means that the detector array possesses two independent pairs of interferometer arms, so we will have two independent signals \( h_k(t) \) (where \( k = 1, 2 \)). Because one pair of arms is rotated 60° relative to the other in the plane of the detector array, the value of \( \phi_D \) will depend on the choice of interferometer pair as well, and so is denoted \( \phi_{D,k} \).

For the ecliptic plane case, the zenith of the detector plane is the same as the zenith of the ecliptic, so \( \theta_D = \Theta \) and \( \psi_D = \psi \). If the satellites orbit the Earth with angular frequency \( \omega_d \), then the apparent azimuth of the source relative to the detector arm will be given by

\[ \phi_{D,k} = \alpha_k(t) \quad \text{where} \quad \alpha_k(t) \equiv \Phi - \omega_d t + \alpha_0k \]
Here the constant $\alpha_{0k}$ specifies the orientation of the interferometer pair at $t = 0$. While we must have $\alpha_{02} = \alpha_{01} + \pi/3$, the constant $\alpha_{01} \equiv \alpha_0$ is arbitrary. In terms of these variables, the beam-pattern functions become

$$F_{+,k} = \frac{\sqrt{3}}{2} \left[ \frac{1}{2}(1 + \cos^2 \Theta) \cos 2\alpha_k \cos 2\psi - \cos \Theta \sin 2\alpha_k \sin 2\psi \right]$$

$$F_{x,k} = \frac{\sqrt{3}}{2} \left[ \frac{1}{2}(1 + \cos^2 \Theta) \cos 2\alpha_k \cos 2\psi + \cos \Theta \sin 2\alpha_k \cos 2\psi \right]$$

In the precessing-plane case, the orientation of the detector plane changes with time, making the expressions for the beam pattern functions more complicated. In this case, we have

$$\cos \theta_D = \frac{1}{2} \cos \Theta - \frac{\sqrt{3}}{2} \sin \Theta \cos \beta$$

$$\phi_{D,k} = \alpha_k(t) + \tan^{-1} \left[ \frac{\sqrt{3} \cos \Theta + \sin \Theta \cos \beta}{2 \sin \Theta \sin \beta} \right]$$

where $\alpha_k(t) \equiv \Omega t + \alpha_{0k}$ and $\beta(t)$ specifies the angular position of the detector array’s center of mass in the plane of the ecliptic. Taking account of the way that we have defined $\Phi = 0$, the quantity $\beta(t)$ is given by:

$$\beta(t) \equiv \Omega t - \Phi.$$  

One can also show that $\psi_D$ in this case is given by

$$\tan \psi_D = \frac{-a \cos \psi + b \sin \psi}{a \sin \psi + b \cos \psi}$$

where $a \equiv \sqrt{3} \sin \beta$ and $b \equiv \sqrt{3} \cos \Theta \cos \beta + \sin \Theta$.

In the sections that follow, these expressions will be used to model the detected signal from a coalescing binary. The ability of the signal to determine the astrophysical parameters of the source will be simulated via a linear least-squares estimation process in which partial derivatives of the received signal with respect to the parameters are accumulated over a year of time-domain data and an information matrix is formed, weighted by the inverse of the $\text{rms}$ noise in the detector. The inverse of the information matrix is the covariance matrix, whose diagonal terms give the uncertainty to be expected in each parameter. Details of the procedure we use are given in MH and will not be repeated here. There is, however, one important difference between the covariance study in MH and the method we use here. This is in the way that the higher harmonics of the waveform are handled in the analysis.

The $\text{rms}$ noise that was assumed in the covariance studies in MH was calculated from the published noise spectra for each mission (OMEGA and LISA) using

$$\sigma_y = \sqrt{S_n(f)\Delta f},$$

where $\sigma_y$ is the rms noise in the detector, $S_n(f)$ is the published noise spectral density at the frequency $f$, taken to be the fundamental frequency of the source, and $\Delta f = 1/(2\Delta t)$ is the bandwidth, equal to the Nyquist frequency with sample time $\Delta t$. The Appendix in
MH explains that the noise in the space detectors is expected to be red noise with a power spectrum that falls off roughly as $f^{-4}$, and then justifies the use of Eq. 10, even in the presence of red noise, for the case where the signal is dominated by a single fundamental frequency. However, when a signal contains higher harmonics of the fundamental frequency, the analysis of red noise becomes more complicated.

Let us consider a time series $y(t) = h(t) + n(t)$, where $h(t)$ is the signal and $n(t)$ is the red noise. To avoid bias in the parameter estimation, the time series of length $T$ is first “prewhitened” by passing it through a linear filter, giving

$$x(t) = F(t) * y(t) \equiv \int_0^T F(t-\tau)y(\tau)d\tau. \quad (11)$$

The effect of the filter on the signal gives $g(t) = F(t) * h(t)$ and the effect on the noise is $m(t) = F(t) * n(t)$. The filter $F(t)$ is chosen so that the power spectrum of $m(t)$ will be flat. The Fourier transform of the filtered signal and noise are simple products, $g(f) = F(f)h(f)$ and $m(f) = F(f)m(f)$, where $F(f)$ is the transfer function of the filter. When the noise spectrum $S_n(f) = n^2(f)$ has a $f^{-4}$ power-law behavior, the prewhitening filter is $F(f) = d^2/dt^2$, with transfer function $F(f) = 4\pi^2 f^2$.

The signal-to-noise ratio for the filtered data is given by $(\text{SNR})^2 = \langle g^2(t) \rangle/\langle m^2(t) \rangle$, where the angle brackets denote a time average. The mean squared signal and noise strengths may in turn be written in terms of their spectral densities as

$$(\text{SNR})^2 = \frac{\int_{f_L}^{f_U} S_g(f)df}{\int_{f_L}^{f_U} S_m(f)df} = \frac{\int_{f_L}^{f_U} F^2(f)S_h(f)df}{S_m(f_H-f_L)} \quad (12),$$

where, in the last step, the fact that $S_m(f) = \text{const}$ has been used to complete the integral in the denominator and the fact that $g(f) = F(f)h(f)$ has been used to expand the numerator.

In MH, the signal $h(t)$ was assumed to be monochromatic, at frequency $f_0$, so that the power spectrum of $h$ would be $S_h(f) = \delta(f - f_0)S_h(f_0)$. However, when many harmonics are present, the power spectrum of the signal will be a series of delta functions, one at each of the harmonics. Thus we will have $S_h(f) = \sum_i |\delta(f - f_i)S_h(f_i)|$, which will complicate the numerator of Eq. 12. In the denominator, because $S_m(f)$ is constant, its relation to $S_n(f)$ may be worked out at any frequency desired; we choose $f_0$. Eq. 12 then becomes

$$(\text{SNR})^2 = \frac{\sum_i F^2(f_i) \int_{f_0}^{f_u} \delta(f - f_i)S_h(f_i)df}{F^2(f_0)S_n(f_0)(f_H-f_0)} = [S_n(f_0)(f_H-f_0)]^{-1}\sum_i \frac{F^2(f_i)}{F^2(f_0)} \langle h_i^2(t) \rangle \quad (13)$$

where we have assumed that the data have been high-pass filtered with cutoff at $f_L = f_0$. In the the last step in Eq. 13, we have recognized the power in each harmonic $\langle h_i^2(t) \rangle$ as the integral over the appropriate spike of the spectral density. It should be remembered that the $h_i(t)$ are the basic signals in the detector, before prewhitening.
The weighted information matrix used in the covariance analysis in this paper is found from Eq. 13. To calculate the rms noise, we take the noise spectral density at the fundamental frequency and multiply by the bandwidth $f_H - f_0 = 1/(2\Delta t)$, the Nyquist frequency $f_H = 1/(2\Delta t)$ being assumed to be much higher than the fundamental frequency $f_0$. To calculate the effective signal strength, each frequency component of the $H_\alpha$ in Eq. 2 is boosted by the ratio $F(f_i)/F(f_0) = f_i^2/f_0^2$. The information content of the higher harmonics will thus be improved over what would be calculated using the simple waveform.

### III. Position Sensitivity for OMEGA.

One of the important conclusions of MH was the necessity of including higher harmonics in evaluating the ability of an ecliptic-plane detector such as OMEGA to determine the sky position of a coalescing massive black hole binary. However, the improved information content of the higher harmonics, as given by Eq. 13, was not noted in that paper. Here we will review the conclusions of MH on this question, including the corrected treatment of the higher harmonic terms.

A covariance study was performed simulating one year of data in ecliptic-plane and precessing-plane detectors. The sources were coalescing massive black hole binaries with a range of masses and with ecliptic latitudes corresponding to a range in $\Theta$ from $0^\circ$ to $90^\circ$. The predicted uncertainties in the nine unknown parameters of the signal $\{\tau, t_c, z, \eta, i, \phi_0, \psi, \Theta, \Phi\}$ were determined from the covariance matrix of a linear least-squares parameter estimation process, as discussed in MH. The calculated uncertainties in $\Theta$ and $\Phi$ are combined to give a solid angle uncertainty via $\delta\Omega = \sin\Theta \delta\Theta \delta\Phi$. The results are shown in Figure 1. Two sets of curves are shown for each type of detector (precessing-plane and ecliptic-plane), one (with triangles) representing the values from MH in which the higher-order harmonics were not properly boosted and one (with squares) representing the new results with the correct treatment of higher harmonics included.

As may be seen in the figures, the boosted harmonics help in the determination of the angular position of the source in the sky for the ecliptic-plane configuration, especially at the middle ecliptic latitudes. By contrast, the precessing-plane configuration is little affected by the higher harmonics, except, surprisingly, in the case of two $10^5M_\odot$ coalescing black holes. The reason for this anomaly is that, at the lowest frequencies (largest black hole binaries) the position determination is supplied by the modulation of the signal created by the precessing detector plane, while, at the highest frequencies (smallest black hole binaries) the determination is dominated by the Doppler modulation provided by the motion of the detector around the sun. At the middle frequency, near $10^{-4}$ Hz, neither effect is able to provide strong position information independent of the binary orbit inclination $i$, and it is the higher harmonic of the gravitational wave waveform, even for the precessing-plane case, that allows the position to be found. In MH, we demonstrated the value of the higher-order harmonics for the ecliptic-plane case by plotting the $\Omega$ uncertainties with and without higher harmonics included (reference [2], Fig. 6). However, we did not investigate all black hole masses for the precessing-plane case and incorrectly concluded that, "artificially suppressing the higher-order terms in the waveform does not change the angular uncertainties very much." The true situation is shown in Fig. 2, where the uncertainty in $\Omega$ is plotted versus $\Theta$ for two $10^5M_\odot$ black holes and two $10^6M_\odot$ holes, with and without the higher-order harmonics included. The harmonics clearly contribute nothing for $10^6M_\odot$ holes, but make a substantial...
contribution in the case of $10^5 M_\odot$ holes.

As was pointed out in MH, the information content of the higher harmonics may be understood in the following way. We may consider the signals seen at each of the two detectors as depending on three effects. The first effect is the monotonic increase in the frequency of the source, as given by Eqs. 4. If this increase is written by expanding the frequency in a Taylor series, then the behavior can be expressed in terms of the derivatives $\omega_0$, $\dot{\omega}_0$, $\ddot{\omega}_0$, etc. As seen in Eqs. 4, these derivatives are linked to the basic variables $\eta$, $\tau$, and $t_c$. Observation of the time series will determine the frequency derivatives and will thus give $\eta$, $\tau$, and $t_c$ independently of any other features of the observed signal. The second effect is the variation of the signal with orientation of the detector, as given by Eqs. 6. However, the $F_{+\times}$, which depend explicitly on $\Theta$, $\Phi$ (through $\alpha_k$), and $\psi$, are not seen directly in the signal, but only in convolution with the third effect, the amplitudes of the two polarizations of the waves, as given by Eq. 1. The $h_{+\times}$ are determined by the already-known $\tau$, $\eta$, and $\omega$, and by the unknown parameters $R_L$, $i$, and $\phi_0$. There are thus six unknown parameters ($\Theta$, $\Phi$, $\psi$, $R_L$, $i$ and $\phi_0$) that must be determined from the waveform, without any help from the frequency derivatives. If only the fundamental frequency of the gravitational wave were present (Eqs. 2a–b), then each detector would see only a single harmonic, whose amplitude and phase would be the only observables. For two detectors, there would be two amplitude observables and two phase observables, but this would not be enough to determine the six unknown parameters, $\Theta$, $\Phi$, $\psi$, $R_L$, $i$, and $\phi_0$. However, if a second harmonic of the wave is included (the harmonics $H^{(1/2)}$ of Eqs. 2c–d), then a Fourier analysis of the detected signals will determine amplitudes and phases for both harmonics. Since the mix of phases and amplitudes between the two harmonics depends on $i$ and $\phi_0$, there will be nontrivial information in these additional terms, and all six gravitational wave parameters can be determined from the eight observed quantities. The ability of the higher harmonics to add information, however, is quenched as $\Theta \to 0$, because the form factors depend on $\Theta$ only through $\cos \Theta$ and also because $\Phi$ and $\psi$ become degenerate near the pole of the ecliptic.

IV. Masses of chirping binaries

It has long been known that, for a purely monochromatic signal from a binary star, the total mass of the system and the distance to the system are completely correlated in the parameterization of the signal, and so cannot be separately determined via the waveform. However, when a binary orbit is tight enough that a measurable change in the orbital frequency occurs over the time of observation, then a particular combination of the masses, the so-called “chirp mass”, may be determined independently of the distance. The formula for the chirp mass is derived from $G(t_s)$ in Eq. 4 and is written

$$\mathcal{M} = \left(1 - \frac{\delta^2}{4}\right)^{\frac{3}{4}} (m_1 + m_2) \quad (14)$$

Since the relative mass difference $\delta$ can vary between $-1$ and $1$, knowledge of $\mathcal{M}$ gives only a lower bound to the total mass of the system. It does not determine the total mass (which could in principle always be infinite, regardless of the value of $\mathcal{M}$), and is certainly not able to determine the two masses separately, at least not without additional information.

When the higher harmonics of the gravitational wave waveform are included in the analysis, there is additional information provided. The non-linearity that produces the chirp
also produces a non-zero higher harmonic of the gravitational wave, breaking the degeneracy between $\tau$ and $\delta$. The way the information enters is as follows. The gravitational wave phase, frequency, and frequency derivative (the chirp) provide information on the parameters $\{\phi_0, t_c, \tau, \delta\}$ that define the orbital phase, but it is not possible to determine the four parameters from the three measured quantities. (This is in contrast to the coalescing black hole case, where a frequency double-derivative is also large enough to be detected). However, the higher harmonics of Eqs. 2c and 2d depend on $\delta$ and $i$. By detecting these harmonics in the received signal, information on $\delta$ is provided that is independent of the phase derivatives.

We have performed a covariance study for a chirping neutron star binary located at the galactic center ($R_L = 8$ Mpc). Three different initial frequencies were considered, 200 s, 100 s, and 50 s, and three different mass ratios were taken, $\delta = 0$, 0.1, and 0.5. In each case a set of inclinations, from $0^\circ$ to $90^\circ$, were studied. Other parameters of the system were chosen arbitrarily. The results are shown in Figs. 3 and 4. In both figures, the increase of uncertainty at low inclination is due to the fact that the first harmonic (Eqs. 2c and 2d) is proportional to $\sin i$. As $i$ approaches zero, the first harmonic vanishes, leaving only the much weaker second harmonic as the next term in the gravitational wave waveform. The effect of the a priori value of $\delta$ on the final accuracy of the solution is seen in Fig. 3. In each case $\delta$ is determined to an uncertainty of about one part in 3. This knowledge breaks the degeneracy inherent in the chirp mass, allowing the total mass to be well determined. The increase in uncertainty of the total mass with increasing $\delta$ is a result of a greater correlation between $\tau$ and $\delta$ for larger values of $\delta$. Knowledge of both $\delta$ and $\tau$ allows the individual masses to be determined to roughly one part in 3, limited by the uncertainty in $\delta$. The effect of the choice of gravitational wave frequency on the uncertainties is shown in Fig. 4, where, at higher frequencies, both the frequency derivative and the value of $\varepsilon$ (Eq. 1) are larger, allowing both $\tau$ and $\delta$ to be determined with better accuracy.

We conclude that gravitational wave analysis of the several neutron star binaries that are expected in the galaxy, with frequencies around 0.01 Hz, will allow important population studies to be made. Knowledge of the three-dimensional position parameters will determine space densities and knowledge of the individual masses will provide the ground truth for evolutionary models.

V. Masses of coalescing supermassive black hole binaries

A recent paper by Hughes [7] has pointed out a potential problem in parameter estimation for massive black hole binaries. This is due to the fact that the more massive binaries ($> 10^6 M_\odot$) will spend very little time in the frequency band of the detector. As a result, the slow modulation of the signal will have little time to produce detectable effects in the waveform, and several parameters, notably the chirp mass, the reduced mass, and the luminosity distance to the source, will be very poorly determined. Hughes has performed covariance studies showing that, for a binary of two equal-mass $10^6 M_\odot$ black holes, the determination of these quantities is marginal at $z = 1$ and the solution matrix becomes singular at $z = 3$ (see Hughes [7], Tables 3 – 7). However, Hughes’s analysis used only the lowest gravitational wave harmonic, in a manner similar to that of Cutler and Vecchio. In conversations, and in his paper, Hughes acknowledges that the addition of the higher order harmonics could modify his conclusions.

We have performed covariance studies for parameter recovery for massive black hole
binaries, using the full harmonic analysis. We consider two cases in which the parameter estimation was very poor when only the fundamental quadrupole harmonic was included. First, we consider the case of two equal-mass $10^7 M_\odot$ black holes at a redshift $z = 3$. In this case we let the analysis run for a full year, allowing the high LISA noise at low frequencies to restrict the amount of significant signal available. Second, we consider two $10^6 M_\odot$ black holes, likewise at $z = 3$, and begin the analysis at a frequency of $10^{-4}$ Hz*. In this case, the binary system provides only about 2.7 days of data before the Post-Newtonian approximation breaks down ($\sim 10$ orbits before coalescence) and we terminate the simulation.

The output from these two cases is shown in Table 1. For each case, two runs are shown, one with the higher-order harmonics of the waveform included and one where they are set to zero. The formal uncertainties for each of the nine parameters are shown in the nine columns. In both cases without the higher-order harmonics, the information matrix is degenerate. When this occurs, the inversion program deletes one of the offending parameters and a solution of lower rank is found. The asterisks in the $\delta$ column in both cases without higher-order harmonics included indicate that the matrix was singular and that this parameter was dropped from the solution. The relative uncertainty in $\tau$, shown in the second column, would then be equivalent to the relative uncertainty in the chirp mass $M$.

| Case 1 | $R_L$ | $\tau$ | $\delta$ | $t_c$ | $i$ | $\psi$ | $\phi_0$ | $\Theta$ | $\Phi$ |
|--------|-------|--------|----------|-------|-----|--------|----------|----------|--------|
| with   | 0.081 | 0.00051| 0.029    | 0.0018| 0.15| 0.26   | 0.27     | 0.0083   | 0.0093 |
| without| 0.14  | 0.00049| ***      | 0.0018| 0.26| 0.49   | 0.49     | 0.0081   | 0.0094 |

| Case 2 | $R_L$ | $\tau$ | $\delta$ | $t_c$ | $i$ | $\psi$ | $\phi_0$ | $\Theta$ | $\Phi$ |
|--------|-------|--------|----------|-------|-----|--------|----------|----------|--------|
| with   | 0.26  | 0.012  | 0.058    | 0.0015| 0.43| 0.74   | 0.75     | 0.017    | 0.021  |
| without| 5.0   | 0.011  | ***      | 0.0015| 8.4 | 16     | 17       | 0.25     | 0.29   |

Table 1. Formal parameter uncertainties for runs with and without higher-order harmonics included in the simulation. Case 1 was for two $10^7 M_\odot$ black holes at $z = 3$, coalescing for one year. Case 2 was for two $10^6 M_\odot$ black holes, likewise at $z = 3$, beginning at an initial frequency of $10^{-4}$ Hz, with a resulting data span of only 2.7 days. The uncertainties for $R_L$ and $\tau$ are relative (i.e., $\sigma_\tau/\tau$, etc.), the uncertainty in $\delta$ is absolute and dimensionless, the uncertainty in $t_c$ is in years, and the uncertainties in angular quantities are in radians.

While we have only investigated two cases and have neither determined the average sensitivity over all parameters nor examined the structure of the sensitivity as a function of the initial parameters, some conclusions can nevertheless be drawn from Table 1. First, although the gravitational wave in Case 1 remained buried in the noise throughout the entire year of data, the ability of the model to dig into the noise to recover the parameters of the coalescing massive black hole binary is still significant, even without the higher-order harmonics (though in this case one must give up the determination of the individual black hole masses). It therefore appears a scientifically undesirable thing to give up low-frequency sensitivity entirely by cutting off the position control system at too high a frequency. Second,

* It has been suggested, for engineering reasons, to cut off the accelerometer control law at $10^{-4}$ Hz, creating an effective sensitivity wall at this frequency.
the importance of including higher-order terms in the parameter estimation model is clear — it is the difference between determining and not determining the individual masses of the binary system and it significantly improves other parameters, notably the luminosity distance, $R_L$.

VI. Conclusions

The basic conclusion to be drawn from the last three sections is obvious — that it is important to include harmonics of the gravitational wave waveform when trying to recover source parameters from gravitational wave data. An accurate position determination for massive black hole binary coalescence requires higher-order harmonics for a detector with the ecliptic-plane configuration and also for a precessing-plane detector at intermediate frequencies near $10^{-4}$ Hz. Higher harmonics, in the case of a chirping neutron star binary, allow a determination to be made of the individual masses the stars in the binary and a very accurate determination to be made of the total mass. Finally, analysis of the signals from supermassive black hole binaries need the higher harmonics to provide the information that is lost due to the short time the systems are visible in the sensitivity window of the detector.

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FIGURE CAPTIONS

Figure 1. Angular uncertainty as a function of \( \Theta \) for equal-mass pairs of coalescing supermassive black holes at \( z = 1 \). In each case, there are two plots presented for each detector configuration — one (triangles) without the higher harmonics boosted according to Eq. 13 and one (squares) where the harmonics have been properly boosted. The total mass of the system for each case is displayed in the box on each plot.

Figure 2. Angular uncertainty as a function of \( \Theta \) for the precessing-plane configuration (LISA) for the \( 2 \times 10^5 \text{M}_\odot \) case of Fig. 1b and the \( 2 \times 10^6 \text{M}_\odot \) case of Fig. 1c. In each case, one plot is shown where the higher harmonics have been completely suppressed (without) and one where they are included, boosted in the proper way (with).

Figure 3. Uncertainties for \( \delta = (m_1 - m_2)/(m_1 + m_2) \) (solid line) and relative uncertainties for \( M_{\text{tot}} = m_1 + m_2 \) (dotted line), as functions of the source inclination \( i \) to the line of sight, for a neutron star binary with total mass \( M_{\text{tot}} = 2.8 \text{M}_\odot \) and initial frequency \( f = 0.01 \text{Hz} \). The three curves for each parameter are for three \( a \ priori \) values of \( \delta \).

Figure 4. Uncertainties for \( \delta = (m_1 - m_2)/(m_1 + m_2) \) (solid line) and relative uncertainties for \( M_{\text{tot}} = m_1 + m_2 \) (dotted line), as functions of the source inclination \( i \) to the line of sight, for a neutron star binary with total mass \( M_{\text{tot}} = 2.8 \text{M}_\odot \) and with \( \delta = 0 \). The three curves for each parameter are for three initial gravitational wave periods.
Figure 1: The logarithmic distribution of 4Q in the precessing-plane and ecliptic-plane for different $M_{\text{tot}}$.

(a) $M_{\text{tot}} = 2 \times 10^4 M_\odot$

(b) $M_{\text{tot}} = 2 \times 10^5 M_\odot$

(c) $M_{\text{tot}} = 2 \times 10^6 M_\odot$

(d) $M_{\text{tot}} = 2 \times 10^7 M_\odot$

The graphs show the distribution of 4Q as a function of $\Theta$ (degrees) for different masses.
Log $[\delta \Omega]$ (steradians) without with $10^5 M_\odot$ $10^6 M_\odot$ without with $10^5 M_\odot$ $10^6 M_\odot$ without with

$\Theta$ (degrees)
Log Uncertainty

\[ \delta = 0.5 \]

\[ \delta = 0.1 \]

\[ \delta = 0.0 \]

\[ i \text{ (degrees)} \]

\[ \log \text{Uncertainty} \]
