Transformations of coordinates and Hamiltonian formalism in deformed Special Relativity

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Abstract

We investigate the transformation laws of coordinates in generalizations of special relativity with two observer-independent scales. The request of covariance leads to simple formulas if one assumes noncanonical Poisson brackets, corresponding to noncommuting spacetime coordinates.

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1 Introduction

Recently, large interest has been devoted to a variety of deformations of special relativity admitting two invariant fundamental scales, the speed of light and an energy scale, to be identified with Planck energy $\kappa$ [1-3]. These theories aim to describe the dynamics of particles up to the Planck region, where the structure of spacetime may change due to quantum gravity effects. They were first introduced algebraically, investigating quantum deformations of the Lorentz group [2], and later rederived starting from a set of suitable physical postulates, as for example the request that special relativity is recovered in the low-energy limit [1].

All these models have in common the assumption that the momentum of a particle transforms nonlinearly under the Lorentz group, but differ in several respects. In fact, first of all, it is possible to construct many different nonlinear representations of the Lorentz group satisfying the postulates mentioned above; moreover, the theory can be defined either in standard spacetime, or in a spacetime with noncommuting coordinates [4]. In the latter case, when defining the classical dynamics through the Hamiltonian formalism, one is forced to use noncanonical Poisson brackets. Of course, depending on the choice of the previous basic assumptions, one obtains models that give rise to different physical predictions.

In order to compare the theory with experiment, one also needs a physical interpretation of the variables that enter in the theory in terms of measurable quantities. This is not always straightforward. We must consider in fact that the models under study, although classical, are supposed to be effective in the quantum domain, so that one cannot expect that all quantities (as for example velocity) can be defined in classical terms.

As we shall discuss in the following, for example, a difficulty in the interpretation of the theory arises from the fact that the transformation rules for the position of a particle depend on its momentum, and hence particles occupying the same position in a reference frame do not necessarily do so in another frame.

We believe that problems of this kind can be understood by recalling the quantum-mechanical origin of the models, which only in the limit of small momenta reduce to special relativity. This implies that the physical quantities should be interpreted accordingly. Clearly, we have no way to probe the sub-Planckian scales for which the theory is defined, and so we do not have to wonder if some of the usual assumption are not valid, provided that the
theory is logically consistent.

In this paper we try to clarify some of these questions by discussing the issue of the transformation laws of position and momentum in a Hamiltonian setting, assuming either commuting spacetime coordinates and standard Poisson brackets or noncommuting coordinates and deformed symplectic structure. We treat explicitly the Magueijo-Smolin (MS) model [3], since this is the simplest algebraically, but our considerations are valid also for different models. In the appendix, we shortly discuss the Lukierski-Nowicki-Ruegg (LNR) model [2]. For simplicity, we consider the case of 1+1 dimensions, but our results can be easily extended to four dimensions.

2 The transformation law for commuting coordinates

Let us consider the Hamiltonian formulation for a free particle in 1 + 1 dimensions. We introduce commuting spacetime coordinates \( q^a \) and momenta \( p_a \) (\( a = 0, 1 \)) for a particle, obeying canonical Poisson brackets, so that

\[
\{q^a, q^b\} = 0, \quad \{p_a, p_b\} = 0, \quad \{q^a, p_b\} = \delta^a_b. \tag{1}
\]

Deformed special relativity models [1-3] postulate a nonlinear Lorentz transformation law for the momenta, but do not specify the transformation law of the coordinates. More precisely, it is assumed that under an infinitesimal boost of generator \( J \),

\[
\delta p_a = \{J, p_a\} = w_a(p), \tag{2}
\]

where \( w_a(p) \) is a nonlinear function of the momentum \( p_a \). For the MS model [3], the functions \( w_a \) are

\[
w_0 = p_1 - \frac{p_0 p_1}{\kappa}, \quad w_1 = p_0 - \frac{p_1^2}{\kappa}. \tag{3}
\]

Now, it is easy to see that the transformation law (2) implies that also the coordinates must transform in a nontrivial way under Lorentz transformations. This is a simple consequence of the Jacobi identities. In fact, under a Lorentz transformations \( \delta q^a = \{J, q^a\} \), but

\[
\{\{J, q^a\}, p_b\} + \{\{p_b, J\}, q^a\} + \{\{q^a, p_b\}, J\} = 0 \tag{4}
\]
and hence
\[ \frac{\partial \{ J, q^a \}}{\partial q^b} = - \{ \{ J, q^a \}, p_b \} = - \frac{\partial w_b}{\partial p_a} \]  
(5)

Integrating these relations, one finds that
\[ \{ J, q^a \} = - \frac{\partial w_b}{\partial p_a} q^b + f^a(p) \]  
(6)

where \( f^a(p) \) are arbitrary functions of \( p^a \), satisfying \( \partial f^a/\partial p_b = \partial f^b/\partial p_a \), that can be set to zero.

From (3) and (6) one has
\[ \delta q^0 = - q^1 + \frac{p_1}{\kappa} q^0, \quad \delta q^1 = - \left( 1 - \frac{p_0}{\kappa} \right) q^0 + \frac{2p_1}{\kappa} q^1. \]  
(7)

Hence, the transformation properties of the position of a particle depend on its momentum, and two particles occupying the same position in a reference frame can occupy different positions in another. The momentum dependence of the transformations of the coordinates of a particle has been first remarked by Kowalski-Glikman [5] in a special case, but it seems to have been disregarded by other authors.

Actually, one can even derive the finite transformation properties of coordinates from those of momenta, by requiring covariance. In momentum space \( \mathcal{P} \), the effect of a boost is given by a nonlinear transformation
\[ p_a \rightarrow p'_a = W_a(p). \]  
(8)

The space of coordinates can be identified with the tangent space to \( \mathcal{P} \) and hence for covariance,
\[ q^a \rightarrow q'^a = L^a_b q^b, \]  
(9)

where
\[ L^a_b = \left( \frac{\partial W_a}{\partial p_b} \right)^{-1}. \]  
(10)

For the MS model, one has [3]
\[ W_0 = \frac{p_0 \cosh \xi + p_1 \sinh \xi}{\Delta}, \quad W_1 = \frac{p_1 \cosh \xi + p_0 \sinh \xi}{\Delta}, \]  
(11)

where
\[ \Delta = 1 + \frac{p_0}{\kappa} (\cosh \xi - 1) + \frac{p_1}{\kappa} \sinh \xi. \]
and $\xi$ is the rapidity parameter. It follows that

$$L^a_b = \Delta \left( \begin{array}{cc} \cosh \xi + \frac{p_0}{\kappa} (1 - \cosh \xi) & -\sinh \xi + \frac{p_0}{\kappa} (1 - \cosh \xi) \\ -\sinh \xi + \frac{p_0}{\kappa} \sinh \xi & \cosh \xi + \frac{p_0}{\kappa} \sinh \xi \end{array} \right).$$  

(12)

It is easy to check that the infinitesimal version of these relations is (7).

Given the transformation law (9), one can also define a covariant Lagrangian,

$$L = p_a \dot{q}^a - H(p).$$

(13)

In fact, $p_a \dot{q}^a$ is clearly invariant up to a total derivative under the Lorentz transformations (8-9). For a free particle, the Hamiltonian $H$ is given by the Casimir invariant of the algebra [3],

$$H = \frac{1}{2} \frac{p_0^2 - p_1^2}{(1 - \frac{p_0}{\kappa})^2}.$$  

(14)

It may be interesting to write down the Hamilton equations for (14):

$$\dot{q}^0 = \frac{\partial H}{\partial p_0} = \frac{p_0 - \frac{p_1^2}{\kappa}}{(1 - \frac{p_0}{\kappa})^2}, \quad \dot{q}^1 = \frac{\partial H}{\partial p_1} = -\frac{p_1}{(1 - \frac{p_0}{\kappa})^2},$$

$$\dot{p}_0 = -\frac{\partial H}{\partial q^0} = 0, \quad \dot{p}_1 = -\frac{\partial H}{\partial q^1} = 0.$$

The velocity of a particle obeying these equations of motion is given by

$$v = \frac{\dot{q}^1}{\dot{q}^0} = \frac{p_1 - p_0 p_1 / \kappa}{p_0 - p_1^2 / \kappa}.$$  

(15)

As discussed in the literature [6, 7], this definition of velocity is not satisfactory, since it implies that the velocity of a particle depends on its mass, and it is difficult to reconcile this fact with the role of velocity as the parameter of the Lorentz transformations.

A solution to this problem is to use deformed Poisson brackets [4]. This is very natural from the point of view of $\kappa$-Poincaré models, since deformed Poisson brackets can be considered as the classical limit of noncommuting spacetime coordinates. In the following section we reconsider the MS model from this point of view.
3 The transformation law for noncommuting coordinates

In the MS model, a suitable definition of the velocity of a particle is given by \( v = p_1/p_0 \). This coincides with \( \dot{q}_1/\dot{q}_0 \) if one introduces the following symplectic structure:\(^8\):

\[
\{q^0, q^1\} = \frac{q^1}{\kappa}, \quad \{p_0, p_1\} = 0, \quad \{q^0, p_0\} = 1 - \frac{p_0}{\kappa}, \\
\{q^1, p_1\} = 1, \quad \{q^0, p_1\} = -\frac{p_1}{\kappa}, \quad \{q^1, p_0\} = 0.
\]

The infinitesimal transformations of coordinates can then be deduced from the Jacobi identities as above. They read

\[
\delta q^0 = -q^1 + \frac{p_1}{\kappa} q^0, \quad \delta q^1 = -q^0 + \frac{p_1}{\kappa} q^1.
\]

For a free particle, with Hamiltonian \( (14) \), the Hamilton equations derived from \( (16) \) are

\[
\dot{q}^0 = \left(1 - \frac{p_0}{\kappa}\right) \frac{\partial H}{\partial p_0} - \frac{p_1}{\kappa} \frac{\partial H}{\partial p_1} = \frac{p_0}{\left(1 - \frac{p_0}{\kappa}\right)^2},
\]
\[
\dot{q}^1 = \frac{\partial H}{\partial p_1} = -\frac{p_1}{\left(1 - \frac{p_0}{\kappa}\right)^2},
\]
\[
\dot{p}_0 = -\left(1 - \frac{p_0}{\kappa}\right) \frac{\partial H}{\partial q^0} = 0,
\]
\[
\dot{p}_1 = -\frac{\partial H}{\partial q^1} + \frac{p_1}{\kappa} \frac{\partial H}{\partial q^0} = 0,
\]
and hence \( v = p_1/p_0 \), as expected.

It is known that the Hamilton equations for systems with nonstandard symplectic structure can be derived from an action principle \(^9\). Given a phase space with symplectic structure \( \{Q^A, Q^B\} = \phi^{AB} \), where \( Q^A \) denote either the coordinates or the momenta, one defines the functions \( R_A(Q^A) \) such that

\[
\frac{\partial R_A}{\partial Q^B} - \frac{\partial R_B}{\partial Q^A} = \phi_{AB},
\]

where \( \phi_{AB} \) is the inverse of \( \phi^{AB} \). The Hamilton equations can then be obtained varying with respect to \( Q^A \) the action

\[
I = \int d\tau (R_A \dot{Q}^A - H).
\]
Note that in general the action so defined contains derivatives of the momenta.

In our case, this procedure yields

\[ I = \int d\tau \left[ -\kappa \log \left(1 - \frac{p_0}{\kappa}\right) q^0 + p_1 \dot{q}^1 - \frac{p_1 q^1}{\kappa \left(1 - \frac{p_0}{\kappa}\right)} \dot{p}_0 - H \right]. \]  

(20)

After integration by parts, (20) can also be written as

\[ I = -\int d\tau \left[ \frac{q^0 + \frac{p_0}{\kappa} q^1}{1 - \frac{p_0}{\kappa}} \dot{p}_0 + q^1 \dot{p}_1 + H \right]. \]  

(21)

The transformation laws for the coordinates can now be obtained by requiring that the variables \( r^a \) conjugate to the momenta \( p_a \) in (21) transform covariantly, i.e., according to \( r^a \rightarrow r'^a = L^a_b r^b \). From (21) one has

\[ q^0 = \left(1 - \frac{p_0}{\kappa}\right) r^0 - \frac{p_1}{\kappa} r^1, \quad q^1 = r^1, \]  

(22)

and hence

\[ q'^0 = \left(1 - \frac{W_0}{\kappa}\right) L^0_a r^a - \frac{W_1}{\kappa} L^1_a r^a, \quad q'^1 = L^1_a r^a. \]

Substituting (11) and (12), after tedious but elementary algebraic manipulations, one gets the surprisingly simple result

\[ q'^0 = \Delta (q^0 \cosh \xi - q^1 \sinh \xi), \quad q'^1 = \Delta (-q^0 \sinh \xi + q^1 \cosh \xi). \]  

(23)

Thus, the transformation laws of coordinates are identical to the usual Lorentz transformations, except for the momentum-dependent factor \( \Delta \). It is then easy to build a simple “invariant length”

\[ (p_a p^a) q^a q_a \sim \left(1 - \frac{p_0}{\kappa}\right)^2 q^a q_a, \]  

(24)

from which an invariant line element can be defined. One may also define new (commuting) coordinates \( \bar{q}^a = (1 - p_0/\kappa) q^a \), which transform according to standard Lorentz transformations, so that \( \bar{q}^a \bar{q}_a \) is invariant. These coordinates however do not lead to a correct definition of the velocity of a particle.
4 Conclusions

We have discussed the coordinate transformation laws that allow a covariant definition of the Hamiltonian formalism in models of deformed special relativity, both in the case of commuting coordinates with canonical symplectic structure, and of noncommuting coordinates with deformed Poisson brackets. The transformation laws for coordinates are momentum dependent and in general rather complicated, except in the case of the MS model with noncommuting coordinates. In this case the transformation laws assume a natural form, that allows the definition of a simple (momentum-dependent) line element.

Of course, the interpretation of the momentum dependence of the transformation laws for coordinates is highly nontrivial. The deformed Lorentz transformations act on the full phase space and not separately on coordinate and momentum space. This implies that the coincidence of two events becomes observer-dependent. At first, this prediction may seem unphysical, but it must be considered that the theory is assumed to be effective at sub-Planckian scales, of which we do not have any direct experience.

A more conservative interpretation of the results of this paper would be to consider the $q^a$ simply as labels of the position of a particle, which should be connected to the spacetime coordinates $x^\mu$ by a sort of vierbein field: $q^a = e^a_\mu x^\mu$. The consistency of this approach is currently being investigated.

These considerations are also interesting in view of the inclusion of gravity in the theory [13]. A consistent approach should presumably lead to some kind of phase space extension of general relativity.

Note. While completing this work, I became aware of a paper [14] where the same transformations of coordinates (23) are proposed, starting from a different point of view.

5 Appendix

We report here the calculations analogous to those presented above, for the case of the LNR model [2].

The infinitesimal Lorentz transformations for LNR are given by (2), with

$$w_0 = p_1, \quad w_1 = \frac{\kappa}{2} \left( 1 - e^{-2p_0/\kappa} - \frac{p_1^2}{\kappa^2} \right).$$

(25)
It follows from (6) that
\[ \delta q^0 = e^{-2p_0/\kappa} q^1, \quad \delta q^1 = q^0 - \frac{p_1}{\kappa} q^1. \] (26)

The finite transformations are given by (8), with (10)
\[ W_0 = p_0 + \kappa \log \Gamma, \quad W_1 = \frac{p_1 \cosh \xi + \frac{\xi}{2} \left( 1 - e^{-2p_0/\kappa} + \frac{p_1^2}{\kappa^2} \right) \sinh \xi}{\Gamma}, \] (27)
where
\[ \Gamma = \frac{1}{2} \left( 1 + e^{-2p_0/\kappa} - \frac{p_1^2}{\kappa^2} \right) + \frac{1}{2} \left( 1 - e^{-2p_0/\kappa} + \frac{p_1^2}{\kappa^2} \right) \cosh \xi + \frac{p_1}{\kappa} \sinh \xi. \]

From (11) follows that \( q'^a = L^a_{\ b} q^b \), with
\[ L_1^1 = \frac{1}{2} \left( 1 - e^{-2p_0/\kappa} - \frac{p_1^2}{\kappa^2} \right) + \frac{1}{2} \left( 1 + e^{-2p_0/\kappa} + \frac{p_1^2}{\kappa^2} \right) \cosh \xi + \frac{p_1}{\kappa} \sinh \xi, \]
\[ L_0^1 = e^{-2p_0/\kappa} \left[ \frac{p_1}{\kappa} \left( 1 - \cosh \xi \right) - \sinh \xi \right], \] (28)
and \( L_0^0 = L_1^1 / \Gamma, \ L_0^1 = L_1^0 / \Gamma. \)

Also in the LNR case the standard hamiltonian formalism does not give a consistent definition of the particle velocity. A suitable definition of velocity is given instead by the right-invariant velocity (11),
\[ v = \frac{e^{p_0/\kappa} p_1}{\kappa \sinh \frac{p_0}{\kappa} + e^{p_0/\kappa} \frac{p_1}{\kappa}}, \] (29)
and can be obtained introducing the following nonstandard symplectic structure (12):
\[ \{ q^0, q^1 \} = \frac{q^1}{\kappa}, \quad \{ p_0, p_1 \} = 0, \quad \{ q^0, p_0 \} = 1, \]
\[ \{ q^1, p_1 \} = 1, \quad \{ q^0, p_1 \} = -\frac{p_1}{\kappa}, \quad \{ q^1, p_0 \} = 0. \] (30)

The infinitesimal transformations of coordinates can then be obtained as before from the Jacobi identities. They read
\[ \delta q^0 = -\frac{\kappa}{2} \left( 1 - e^{-2p_0/\kappa} - \frac{p_1^2}{\kappa^2} \right) q^1 + \frac{p_1}{\kappa} q^0, \quad \delta q^1 = -q^0. \] (31)
The Hamiltonian for a free particle is given by the Casimir invariant

\[ H = \frac{1}{2} \left[ \left( 2\kappa \sinh \frac{p_0}{2\kappa} \right)^2 - e^{p_0/\kappa} \frac{p_1^2}{\kappa^2} \right], \quad (32) \]

and the Hamilton equations are

\[
\begin{align*}
\dot{q}^0 &= \frac{\partial H}{\partial p_0} - \frac{p_1}{\kappa} \frac{\partial H}{\partial p_1} = \kappa \sinh \frac{p_0}{\kappa} + e^{p_0/\kappa} \frac{p_1^2}{\kappa^2}, \\
\dot{q}^1 &= \frac{\partial H}{\partial p_1} = -e^{p_0/\kappa} p_1, \\
\dot{p}_0 &= -\frac{\partial H}{\partial q_0} = 0, \\
\dot{p}_1 &= -\frac{\partial H}{\partial q_1} + p_1 \frac{\partial H}{\kappa \partial q_0} = 0, \quad (33)
\end{align*}
\]

which lead to the definition (29) of velocity.

As explained above, eqs. (33) can be obtained from an action principle. The action reads in this case,

\[ I = \int d\tau \left[ p_0 \dot{q}^0 + p_1 \dot{q}^1 - \frac{q_1 p_1}{\kappa} \dot{p}_0 - H \right], \quad (34) \]

or, after integration by parts,

\[ I = -\int d\tau \left[ \left( q_0 + \frac{p_1}{\kappa} q_1 \right) \dot{p}_0 + q_1 \dot{p}_1 + H \right]. \quad (35) \]

The transformation laws for the coordinates can now be obtained by requiring that the variables \( r^a \) conjugate to the momenta \( p_a \) in (35) transform covariantly. One has

\[ q^0 = r^0 - \frac{p_1}{\kappa} r^1, \quad q^1 = r^1, \quad (36) \]

and hence

\[ q^0 = L^0_a r^a - \frac{1}{\kappa} W_1 L^1_a r^a, \quad q^1 = L^1_a r^a. \]

After some algebra, one gets the result

\[ q^0 = q^0 (\cosh \xi + \frac{p_1}{\kappa} \sinh \xi) - \frac{q_1}{2} \left( 1 + e^{-2p_0/\kappa} - \frac{p_1^2}{\kappa^2} \right) \sinh \xi, \]
\[ q'^1 = -q^0 \left[ \sinh \xi + \frac{p_1}{\kappa} (1 - \cosh \xi) \right] + \frac{q^1}{2} \left[ \left( 1 - e^{-2p_0/\kappa} + \frac{p_1^2}{\kappa^2} \right) + \left( 1 + e^{-2p_0/\kappa} - \frac{p_1^2}{\kappa^2} \right) \cosh \xi \right]. \]

In this case, the transformation laws for coordinates are not especially simple and do not seem to lead to the same interesting developments as in the MS case.

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