Monotone Iterative and Upper–Lower Solution Techniques for Solving the Nonlinear ψ—Caputo Fractional Boundary Value Problem

Abdelatif Boutiara, Maamar Benbachir, Jehad Alzabut, and Mohammad Esmael Samei

Abstract: The objective of this paper is to study the existence of extremal solutions for nonlinear boundary value problems of fractional differential equations involving the ψ—Caputo derivative with integral boundary conditions. Our main results are obtained by applying the monotone iterative technique combined with the method of upper and lower solutions. Further, we consider three cases for ψ* (t) as t, Caputo, and Katugampola (for ρ = 0.5) derivatives and examine the validity of the acquired outcomes with the help of two particular examples.

Keywords: extremal solutions; monotone iterative technique; ψ—Caputo fractional derivative; upper and lower solutions

MSC: 26A33; 34A08; 34B18

1. Introduction

The notion of fractional calculus refers to the last three centuries and it can be described as the generalization of classical calculus to orders of integration and differentiation that are not necessarily integers. Many researchers have used fractional calculus in different scientific areas [1–4].

In the literature, various definitions of the fractional-order derivative have been suggested. The oldest and the most famous ones advocate for the use of the Riemann–Liouville and Caputo settings. One of the most recent definitions of a fractional derivative was delivered by Kilbas et al., where the fractional differentiation of a function with respect to another function in the sense of Riemann–Liouville was introduced [5]. They further defined appropriate weighted spaces and studied some of their properties by using the corresponding fractional integral. In [6], Almada defined the following new fractional derivative and integrals of a function with respect to some other function:

\[ [D]^{\rho}_{a^+} \psi^\ast (t) := \left( \frac{1}{\psi^\ast (\xi)} \frac{d}{d\xi} \right)^n [I]^{n-\sigma}_{a^+} \psi^\ast (t) \]

\[ = \left( \frac{1}{\psi^\ast (\xi)} \frac{d}{d\xi} \right)^n \int_a^t \psi^\ast (\xi) \Gamma (n-\sigma) (\psi^\ast (t) - \psi^\ast (\xi))^{n-\sigma-1} \psi (\xi) d\xi, \]  

(1)
where \( n = [\sigma] + 1 \) and
\[
\mathbb{I}_{a}^{n-\sigma} \phi^* e(x) := \int_{a}^{x} \frac{\phi'(\xi)}{\Gamma(n-\sigma)} (\psi^*(x) - \psi^*(\xi))^{n-1} \phi(\xi) \, d\xi,
\]
respectively. He called the fractional derivative the \( \psi \)-Caputo fractional operator. In the above definitions, we get the Riemann–Liouville and Hadamard fractional operators whenever we consider \( \psi^*(y) = y \) or \( \psi^*(y) = \ln y \), respectively. Many researchers used this \( \psi \)-Caputo fractional derivative (see [7–13] and the references therein). Abdo et al., in [14], investigated the BVP for a fractional differential equation (FDE) involving \( \psi^* \) and lower solutions to prove the existence of extremal solutions for the following BVP of an FDE involving the operator remains rare.

In the present paper, we are interested in the MIT blended with the method of upper and lower solutions to prove the existence of extremal solutions for the following BVP of an FDE involving the operator
\[
\begin{cases}
\mathbb{C}D_{t}^{\psi^*} e(t) = \gamma(t, e(t)), & t \in I, \\
e(t_1) = \lambda \mathbb{I}^{\psi^*} e(\eta) + \delta,
\end{cases}
\]
where \( \mathbb{C}D_{t}^{\psi^*} \) is the operator (1) of order \( 0 < \sigma, \nu \leq 1 \), \( \mathbb{I}^{\psi^*} \) is the operator (2), the function \( \gamma' : [t_1, t_2] \times \mathbb{R} \rightarrow \mathbb{R} \) is continuous, \( \lambda \) and \( \delta \) are real constants, and \( \eta \in (t_1, t_2) \). It is worth mentioning that the MIT is efficiently used in the literature to investigate the existence of extremal solutions to many applied problems of nonlinear equations [25–38].

The rest of this paper is organized as follows. In Section 2, we recall some preliminary concepts, definitions, and lemmas that will act as prerequisites to proving the main results. The main results are stated and proved in Section 3. Finally, we give numerical examples to illustrate the correctness of the outcome.

2. Preliminaries

Let \( \sigma > 0 \). The left-sided \( \psi \)-Riemann–Liouville fractional integral (l-s-\( \psi \)-RLfi) of order \( \sigma \) for an integrable function \( e : I \rightarrow \mathbb{R} \) with respect to another function \( \psi^* : I \rightarrow \mathbb{R} \), which is an increasing differentiable function such that \( \psi^*(t) \neq 0, (\forall t \in I) \), is defined as follows:
\[
\mathbb{I}_{t_1}^{\psi^*} e(t) = \frac{1}{\Gamma(\sigma)} \int_{t_1}^{t} \psi^*(\xi)(\psi^*(t) - \psi^*(\xi))^{\sigma-1} e(\xi) \, d\xi,
\]
where \( \Gamma \) is the classical Euler Gamma function [5,6]. Algorithm A1 shows the MATLAB lines for the calculation of the l-s-\( \psi \)-RLfi. Let \( n \in \mathbb{N} \) and \( \psi^*, e \in C^n(I, \mathbb{R}) \) be two functions such that \( \psi^* \) is increasing and \( \psi^*(t) \neq 0, (\forall t \in I) \). The left-sided \( \psi \)-Riemann–Liouville fractional derivative (l-s-\( \psi \)-RLfd) of a function \( e \) of order \( \sigma \) is defined by
\[
\mathbb{D}_{t_1}^{n-\sigma} \psi^* e(t) = \left( \frac{1}{\psi^*(t)} \frac{d}{dt} \right) \mathbb{I}_{t_1}^{n-\sigma} \psi^* e(t)
\]
\[
= \frac{1}{\Gamma(n-\sigma)} \left( \frac{1}{\psi^*(t)} \frac{d}{dt} \right)^n \int_{t_1}^{t} \psi^*(\xi)(\psi^*(t) - \psi^*(\xi))^{n-\sigma-1} e(\xi) \, d\xi,
\]
where \( n = \lfloor \sigma \rfloor + 1 \) [6]. Algorithm A2 shows the MATLAB lines for the calculation of the \( l\)-\( \psi \)-RLfD. In addition, the left-sided \( \psi \)-Caputo fractional derivative \((l\)-\( \psi \)-Cfd\) of a function \( q \) of order \( \sigma \) is determined by

\[
C_D^\alpha_{t_1} q(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_1}^{t} \psi^\alpha(\xi)(\psi^\alpha(t) - \psi^\alpha(\xi))^{n-\alpha-1} q^{[n]}(\xi) d\xi, \quad \sigma \notin \mathbb{N},
\]

\[
C_D^\alpha_{t_1} q(t) = \begin{cases} 
\frac{1}{\Gamma(n-\sigma)} \int_{t_1}^{t} \psi^\sigma(\xi)(\psi^\sigma(t) - \psi^\sigma(\xi))^{n-\sigma-1} q^{[n]}(\xi) d\xi, & \sigma \notin \mathbb{N}, \\
q^{[n]}(t), & \sigma \in \mathbb{N}.
\end{cases}
\]

Algorithm A3 shows the MATLAB lines for the calculation of \( C_D^\alpha_{t_1} q(t) \). If \( q \in C^n(i, \mathbb{R}) \), then the \( \psi \)Cfd of order \( \sigma \) of \( q \) is determined as ([6], Theorem 3):

\[
C_D^\alpha_{t_1} q(t) = D_{t_1}^\alpha q(t) - \psi^\alpha(t_1) + \frac{1}{\Gamma(n-\alpha)} \int_{t_1}^{t} \psi^\alpha(\xi)(\psi^\alpha(t) - \psi^\alpha(\xi))^{n-\alpha-1} q^{[n]}(\xi) d\xi, \quad \sigma \notin \mathbb{N},
\]

\[
C_D^\alpha_{t_1} q(t) = \begin{cases} 
\frac{1}{\Gamma(n-\sigma)} \int_{t_1}^{t} \psi^\sigma(\xi)(\psi^\sigma(t) - \psi^\sigma(\xi))^{n-\sigma-1} q^{[n]}(\xi) d\xi, & \sigma \notin \mathbb{N}, \\
q^{[n]}(t), & \sigma \in \mathbb{N}.
\end{cases}
\]

Lemma 1 ([8]). Let \( \sigma, \nu > 0 \), and \( q \in L^1(i, \mathbb{R}) \). Then, \( C_D^\alpha_{t_1} \|D^\nu\psi^\alpha q(t) = \|D^\nu\psi^\alpha q(t), (t \in i) \). In particular, if \( q \in C(i, \mathbb{R}) \), then \( C_D^\alpha_{t_1} \|D^\nu\psi^\alpha q(t) = \|D^\nu\psi^\alpha q(t), (t \in i) \).

Lemma 2 ([8]). Let \( \sigma > 0 \). If \( q \in C(i, \mathbb{R}) \), then \( C_D^\alpha_{t_1} q(t) = q(t), (t \in i) \), and

\[
\|D^\nu\psi^\alpha \left( C_D^\alpha_{t_1} q(t) \right) \| = \|D^\nu\psi^\alpha q(t) - \psi^\alpha(t_1)\| = q(t) - \psi^\alpha(t_1).
\]

where \( \| \) denotes the \( L^1 \) norm.

Lemma 3 ([5,8]). Let \( t > t_1 \), \( \sigma \geq 0 \), and \( \nu > 0 \). Then,

\[
(1) \quad \frac{\|D^\nu\psi^\alpha q(t) - \psi^\alpha(t_1)\|}{\|D^\nu\psi^\alpha q(t) - \psi^\alpha(t_1)\|} = \frac{\|q(t) - \psi^\alpha(t_1)\|}{\|q(t) - \psi^\alpha(t_1)\|},
\]

\[
(2) \quad C_D^\alpha_{t_1} q(t) - \psi^\alpha(t_1)\| = \frac{\|q(t) - \psi^\alpha(t_1)\|}{\|q(t) - \psi^\alpha(t_1)\|},
\]

\[
(3) \quad C_D^\alpha_{t_1} q(t) - \psi^\alpha(t_1)\| = 0, (\forall k \in \{0, \ldots, n-1\} \text{ and } n \in \mathbb{N}).
\]

3. Main Results

First, we start the following key fixed-point theorem.

Theorem 1 ([16,17]). Consider \( i \subset \mathcal{O} \) of an ordered Banach space \( \mathcal{B} \) and a nondecreasing mapping \( u : i \rightarrow i \). If each sequence \( \{u^n\} \subset u(i) \) converges whenever \( \{u^n\} \) is a monotone sequence in \( i \), then the sequence of the \( \psi \)-iteration of \( t_1 \) converges to the least fixed point \( q_+ \) of \( u \), and the sequence of the \( \psi \)-iteration of \( t_2 \) converges to the greatest fixed point \( q_- \) of \( u \). Moreover, \( q_+ = \min \{ t \in i : t \geq ut \} \), and \( q_- = \max \{ t \in i : t \leq ut \} \).
In fact, a function $q \in C(t, \mathbb{R})$ is said to be a solution of Equation (3) if $q$ satisfies the equation $CD_{t_i}^{\sigma,\psi^*} q(t) = f(t, q(t)), \ (\forall t \in I)$ and the condition $q(0) = \lambda CD_{t_i}^{\sigma,\psi^*} q(\eta) + \delta$. Now, we prove the the next key lemma of a solution for problem (3).

**Lemma 4.** Let $y \in C(t, \mathbb{R})$ and $\sigma \in (n - 1, n)$, $(n \in \mathbb{N})$; the linear fractional initial value problem

$$
\begin{cases}
CD_{t_i}^{\sigma,\psi^*} q(t) = y(t), & t \in I, \\
q(t_1) = \lambda CD_{t_i}^{\sigma,\psi^*} q(\eta) + \delta,
\end{cases} 
$$

(9)

has the following unique solution:

$$
q(t) = I_{t_i}^{\sigma,\psi^*} y(t) + \frac{1}{\Lambda} \left( \lambda I_{t_i}^{\sigma,\psi^*} y(\eta) + \delta \right)
$$

where $\Lambda = 1 - \frac{\lambda}{\Gamma(\nu + 1)} (\psi^*(\eta) - \psi^*(t_1))^\nu$.

**Proof.** Assume that $q$ satisfies (9). Then, Lemma 2 implies that

$$
q(t) = I_{t_i}^{\sigma,\psi^*} y(t) + c_1.
$$

(11)

The condition of problem (9) implies that $q(0) = c_1$ and

$$
I_0^{\nu} q(0) = I_0^{\sigma + \nu} \sigma(\eta) + c_1 \left( \psi^*(\eta) - \psi^*(t_1) \right)^\nu.
$$

Thus,

$$
c_1 \left( 1 - \frac{\lambda}{\Gamma(\nu + 1)} (\psi^*(\eta) - \psi^*(t_1))^\nu \right) = \lambda I_{t_i}^{\sigma,\psi^*} y(\eta) + \delta.
$$

Consequently,

$$
c_1 = \frac{1}{\Lambda} \left( \lambda I_{t_i}^{\sigma,\psi^*} y(\eta) + \delta \right).
$$

Finally, we obtain the solution (10):

$$
q(t) = I_{t_i}^{\sigma,\psi^*} y(t) + \frac{1}{\Lambda} \left( \lambda I_{t_i}^{\sigma,\psi^*} y(\eta) + \delta \right),
$$

which completes the proof. \(\square\)

**Lemma 5** (Comparison result). Let $q \in C(t, \mathbb{R})$ satisfy the following inequalities:

$$
\begin{cases}
CD_{t_i}^{\sigma,\psi^*} q(t) \geq 0, & t \in I, \\
q(0) \geq \lambda CD_{t_i}^{\sigma,\psi^*} q(\eta).
\end{cases}
$$

(12)

Then, $q(t) \geq 0, \ \forall t \in I$, where $0 < \sigma \leq 1$ is fixed.

**Proof.** Lemma 4 implies that the problem (9) has the following unique solution:

$$
q(t) = I_{t_i}^{\sigma,\psi^*} y(t) + \frac{1}{\Lambda} \left( \lambda I_{t_i}^{\sigma,\psi^*} y(\eta) + \delta \right).
$$

(13)
Moreover, Lemma 5 follows from (13).

**Theorem 2.** Consider the function \( \varphi \in C(\mathbb{R}, \mathbb{R}) \) and the following assumptions:

(H1) \( \exists \underline{u}_0, \overline{u}_0 \in C(I, \mathbb{R}) \) such that \( \underline{u}_0 \) and \( \overline{u}_0 \) are the \( l \)-solution and \( u \)-solution of problem (3), respectively, with \( \underline{u}_0(t) \leq u \leq \overline{u}_0(t) \), \( \forall t \in I \);

(H2) \( \exists \) a function \( \varphi \in C(I, \mathbb{R}) \) such that \( \varphi(t, \underline{u}) - \varphi(t, \overline{u}) \geq 0 \), for \( \underline{u}_0 \leq u \leq \overline{u}_0 \);

(H3) \( \lambda > 0 \) and \( \lambda(\psi^*(\eta) - \psi^*(t_1))^\psi < \Gamma(v + 1) \).

Then, there exist monotone iterative sequences \( \{\underline{u}_n\} \) and \( \{\overline{u}_n\} \) that converge uniformly on the interval \( I \) to the extremal solutions \( \underline{u}_0 \leq \underline{u} \leq \overline{u} \leq \overline{u}_0 \), respectively, of BVP (3), where

\[
\underline{u}_0 \leq \underline{u}_1 \leq \cdots \leq \underline{u}_n \leq \cdots \leq \underline{u}, \; \leq \cdots \leq \overline{u}_n \leq \cdots \leq \overline{u}_1 \leq \overline{u}_0.
\]

**Proof.** First, for any \( \underline{u}_0(t), \overline{u}_0(t) \in C(I, \mathbb{R}) \), we consider the BVPs of fractional order

\[
\begin{align*}
\mathcal{D}^\sigma_{t_1}^{\psi^*} \underline{u}_{n+1}(t) & = \varphi(t, \underline{u}_n(t)), \\
\underline{u}_{n+1}(0) & = \lambda \Gamma(v)^\psi \underline{u}_0(\eta) + \delta,
\end{align*}
\]

for all \( t \in I \).

**Definition 1.** A function \( \underline{u}_0 \in C(I, \mathbb{R}) \) is said to be a lower solution (\( l \)-solution) and upper solution (\( u \)-solution) of problem (3) if it satisfies

\[
\begin{align*}
\mathcal{D}^\sigma_{t_1}^{\psi^*} \underline{u}_0(t) & \geq \varphi(t, \underline{u}_0(t)), \\
\underline{u}_0(0) & \geq \lambda \Gamma(v)^\psi \underline{u}_0(\eta) + \delta,
\end{align*}
\]

and

\[
\begin{align*}
\mathcal{D}^\sigma_{t_1}^{\psi^*} \overline{u}_0(t) & \leq \varphi(t, \overline{u}_0(t)), \\
\overline{u}_0(0) & \leq \lambda \Gamma(v)^\psi \overline{u}_0(\eta) + \delta,
\end{align*}
\]

for all \( t \in I \). Then, we structure the proof as follows. For any \( h \in [\underline{u}_0, \overline{u}_0] \), define an operator \( \mathcal{F} \) with \( \mathcal{F}(h) = \varphi(h) \). As a first step, we show that the operator \( \mathcal{F} : [\underline{u}_0, \overline{u}_0] \to [\underline{u}_0, \overline{u}_0] \).

Let \( \underline{u}_1 = \mathcal{F}\underline{u}_0, \overline{u}_1 = \mathcal{F}\overline{u}_0 \). Then, \( \underline{u}_1, \overline{u}_1 \) are well defined and satisfy

\[
\begin{align*}
\mathcal{D}^\sigma_{t_1}^{\psi^*} \underline{u}_1(t) & = \varphi(t, \underline{u}_0(t)), \quad t \in I, \\
\underline{u}_1(t_1) & = \lambda \Gamma(v)^\psi \underline{u}_0(\eta) + \delta,
\end{align*}
\]

and

\[
\begin{align*}
\mathcal{D}^\sigma_{t_1}^{\psi^*} \overline{u}_1(t) & = \varphi(t, \overline{u}_0(t)), \quad t \in I, \\
\overline{u}_1(t_1) & = \lambda \Gamma(v)^\psi \overline{u}_0(\eta) + \delta.
\end{align*}
\]
We set \( \mu(t) = \varrho_1(t) - \varrho_0(t) \). From (15) and Definition 1, we get

\[
\int_{\tilde{t}}^{t} \mathcal{D}_{\tilde{t}}^\sigma \psi^\prime \mu(t) \geq 0.
\]

Again, since \( \mu(t_1) = \lambda \mathcal{D}_{\tilde{t}}^\sigma \psi^\prime \mu(q) \), by Lemma 5, \( \mu(t) \geq 0 \), \( (\forall t \in i) \). That is,

\[
\varrho_0(t) \leq \mathcal{F} \varrho_0(t) = \varrho_1(t).
\]

Similarly, using the definition of the upper solution, we can show that \( \varrho_1 = \mathcal{F} \varrho_0(t) \leq \varrho_0(t) \), \( (t \in i) \). Now, let \( \varrho(t) = \varrho_1(t) - \varrho_0(t) \). From (15), (16), and (H2), we have

\[
\int_{\tilde{t}}^{t} \mathcal{D}_{\tilde{t}}^\sigma \psi^\prime \varrho(t) = \varrho(t, \varrho_0(t)) - \varrho(t, \varrho_1(t)) \geq 0.
\]

Therefore,

\[
\varrho(t_1) = \varrho_1(t_1) - \varrho_0(t_1) = \lambda \mathcal{D}_{\tilde{t}}^\sigma \varrho(q).
\]

Moreover, \( \varrho(t) \geq 0 \) from Lemma 5. Thus, \( \mathcal{F} \varrho_0 \leq \mathcal{F} \varrho_0 \). This, together with \( \varrho_0 \leq \mathcal{F} \varrho_0 \) and \( \mathcal{F} \varrho_0 \leq \varrho_0 \), implies that \( \mathcal{F} \) is nondecreasing,

\[
\mathcal{F} : [\varrho_0, \varrho_0] \rightarrow [\varrho_0, \varrho_0],
\]

and \( \varrho_0 \leq \mathcal{F} \varrho \leq \varrho_0 \) for any \( \varrho \in [\varrho_0, \varrho_0] \). In consequence, \( \mathcal{F} \varrho_0 \subset \varrho_0 \) and

\[
\| \mathcal{F} \varrho \| \leq \max \{ \| \varrho_0 \|, \| \varrho_0 \| \} := d.
\]

Let \( \{ \varrho_n \} \) be an mis in \( [\varrho_0, \varrho_0] \). Then, \( \varrho_n \leq \mathcal{F} \varrho_n \leq \varrho_0 \) and \( \| \mathcal{F} \varrho_n \| \leq d \). For any \( (t, \varrho) \in i \times [-d, d] \), there exists a positive constant \( L_0 \) such that \( \| \varrho(t, \varrho) \| \leq L_0 \). Then, for any \( t_1, t_2 \in i \) with \( t_1 \leq t_2 \), we obtain

\[
| \mathcal{F} \varrho(t) - \mathcal{F} \varrho(\tilde{t}) | \leq \frac{1}{\Gamma(\sigma)} \int_{\tilde{t}}^{t} \psi^\prime(\xi) (\psi^*(\xi) - \psi^*(\xi))^{\sigma-1} \varrho(\xi, \varrho(\xi)) d\xi
\]

\[
- \frac{1}{\Gamma(\sigma)} \int_{\tilde{t}}^{t} \psi^\prime(\xi) (\psi^*(\xi) - \psi^*(\xi))^{\sigma-1} \varrho(\xi, \varrho(\xi)) d\xi
\]

\[
\leq \frac{1}{\Gamma(\sigma)} \left| \int_{\tilde{t}}^{t} \psi^\prime(\xi) \left[ (\psi^*(\xi) - \psi^*(\xi))^{\sigma-1}
\right.\right.
\]

\[
- (\psi^*(\xi) - \psi^*(\xi))^{\sigma-1}
\]

\[
\left. \varrho(\xi, \varrho(\xi)) d\xi \right| + \frac{1}{\Gamma(\sigma)} \int_{\tilde{t}}^{t} \psi^\prime(\xi) (\psi^*(\xi) - \psi^*(\xi))^{\sigma-1} \varrho(\xi, \varrho(\xi)) d\xi
\]

\[
\leq \frac{L_0}{\Gamma(\sigma + 1)} \left| (\psi^*(t) - \psi^*(t_1))^{\sigma} - (\psi^*(\tilde{t}) - \psi^*(t_1))^{\sigma}
\right.
\]

\[
- (\psi^*(t) - \psi^*(\tilde{t}))^{\sigma} + (\psi^*(t) - \psi^*(\tilde{t}))^{\sigma}
\]

which converges to zero as \( \tilde{t} \to t \). Let us observe that for \( t, \tilde{t} \in j \),

\[
| \mathcal{F} \varrho(t) - \mathcal{F} \varrho(\tilde{t}) | \to 0.
\]
when \( i \to t \). Thus, \( \{F \vartheta_n\} \) is equicontinuous on all \( j \). So, \( F \) is relatively compact on \([\vartheta_{\omega}, \overline{\vartheta}_0]\). Hence, the Arzelà–Ascoli theorem implies that \( F \) is compact on \([\vartheta_{\omega}, \overline{\vartheta}_0]\), and so,

\[
\{F \vartheta_n\} \subset \mathcal{F} \left( \left[ \vartheta_{\omega}, \overline{\vartheta}_0 \right] \right),
\]

converges. On the other hand, Theorem 1 implies that the sequence of the \( F \)-iteration of \( \vartheta_{\omega} \) and \( \overline{\vartheta}_0 \) converges to the least and the greatest fixed points \( \vartheta^{*} \) and \( \overline{\vartheta}^{*} \) of \( F \), respectively. This, in turn, implies that problem (3) has extremal solutions \( \vartheta \) derivatives in this example.

Three cases for of problem (21) for \( t \in [0, 1] \)

\[
V, \quad \text{and} \quad \text{respectively. Furthermore, we have}
\]

\[
\vartheta_{\omega} \leq \vartheta_{\omega i} \leq \cdots \leq \vartheta_{n} \leq \cdots \leq \vartheta_{s} \leq \cdots \leq \overline{\vartheta}_s \leq \cdots \leq \overline{\vartheta}_n \leq \cdots \leq \overline{\vartheta}_0.
\]

This completes the proof. \( \Box \)

4. Some Relevant Examples

Example 1. Consider the following problem:

\[
\begin{aligned}
\begin{cases}
\mathcal{C}D_{0}^{\frac{1}{4}} \Psi^{*}(t) = 2t - \varrho(t) + 1, & t \in [0, 1], \\
\Psi(0) = \frac{3}{4} \frac{2}{1 + 2}, & \varrho \in [0, 2).
\end{cases}
\end{aligned}
\]

(21)

where

\[
\sigma = 1 \epsilon \nu = 1 \epsilon [0, 1], \lambda = 3 \epsilon 4 > 0, \eta = 2 \epsilon 3 \in (0, 1), \delta = 1 \epsilon 2, t_1 = 0, t_2 = 1,
\]

and \( \nu : t \times \mathbb{R} \to \mathbb{R} \) is given by

\[
\nu(t, \varrho) = 2t - \varrho^2 + 1,
\]

for \( t \in [0, 1] \). We take \( \vartheta_{\omega}(t) = 0 \) as the lower solution and \( \overline{\vartheta}_0(t) = 1 + 2t \) as the upper solution of problem (21), and we take \( \vartheta_{\omega} \leq \overline{\vartheta}_0 \) for \( t \in [0, 1] \). So, (H1) of Theorem 2 holds. Now, we consider three cases for \( \Psi^{*} \):

\[
\Psi_{1}^{*}(t) = t, \quad \Psi_{2}^{*}(t) = 2t, \quad \Psi_{3}^{*}(t) = \sqrt{t}.
\]

Note that \( \Psi_{1}^{*}(t) = t \) and \( \Psi_{3}^{*}(t) = \sqrt{t} \) give the Caputo and Katugampola (for \( \rho = 0.5 \)) derivatives in this example.

With the data provided in Tables 1 and 2, we can see from assumption (H3) that

\[
\frac{\lambda}{\Gamma(\nu + 1)} \left( \Psi^{*}(\eta) - \Psi^{*}(a) \right) = \frac{3}{4} \frac{2}{1 + 2} \left( \Psi^{*} \left( \frac{2}{3} \right) - \Psi^{*}(0) \right) < \frac{1}{2}
\]

\[
\begin{aligned}
\left\{ \begin{array}{l}
\frac{3}{4} \frac{2}{1 + 2} \left( \frac{2}{3} \right) \approx 0.690988 < 1, \quad \Psi^{*}(t) = t. \\
\frac{3}{4} \frac{2}{1 + 2} \left( 2^\frac{2}{3} - 1 \right) \approx 0.648610 < 1, \quad \Psi^{*}(t) = 2t. \\
\frac{3}{4} \frac{2}{1 + 2} \left( \sqrt{\frac{2}{3}} \right) \approx 0.764704 < 1, \quad \Psi^{*}(t) = \sqrt{t}.
\end{array} \right.
\end{aligned}
\]

Tables 1 and 2 show these results. One can see the 2D line plots of \( \vartheta_{\omega}(t) \) and \( \overline{\vartheta}_n(t) \) for the \( \Psi_{1}^{*}(t) = t \) Caputo derivative, \( \Psi_{2}^{*}(t) = 2t \), and the \( \Psi_{3}^{*}(t) = \sqrt{t} \) Katugampola derivative (for \( \rho = 0.5 \)) in Figure 1a,b. In addition, assumption (H2) is clearly satisfied.
Table 1. Numerical results of $\varrho_n$ for $\psi^*(t) = t, 2^t, \sqrt{t}$ in Example 1.

| $n$ | $t$  | $\psi^*(t) = t$ | $\psi^*(t) = 2^t$ | $\psi^*(t) = \sqrt{t}$ |
|-----|------|-----------------|-------------------|------------------------|
| 1   | 0.00000 | 6.16427         | 1.42292           | 9.07593                |
| 2   | 0.06250 | 4.28485         | 1.48618           | 6.19447                |
| 3   | 0.12500 | 6.94358         | 2.16988           | 10.07951               |
| 4   | 0.18750 | 3.08762         | 1.32280           | 4.40385                |
| 5   | 0.25000 | 7.27414         | 2.49829           | 10.38117               |
| 6   | 0.31250 | 1.44744         | 0.99029           | 2.03591                |
| 7   | 0.37500 | 6.79598         | 2.59341           | 9.54727                |
| 8   | 0.43750 | 1.96779         | 1.26042           | 2.73167                |
| 9   | 0.50000 | 7.91490         | 3.16083           | 10.95387               |
| 10  | 0.56250 | 3.00121         | 0.16361           | 4.18637                |
| 11  | 0.62500 | 0.40120         | 0.71785           | 0.62015                |
| 12  | 0.68750 | 6.91232         | 3.19814           | 9.34005                |
| 13  | 0.75000 | 1.12735         | 0.60722           | 1.66134                |
| 14  | 0.81250 | 6.93050         | 3.44249           | 9.21394                |
| 15  | 0.87500 | 2.52051         | 0.25975           | 3.59983                |
| 16  | 0.93750 | 5.59257         | 3.21189           | 7.26441                |
| 17  | 1.00000 | 1.08675         | 1.16584           |                       |

Table 2. Numerical results of $\varpi_n$ for $\psi^*(t) = t, 2^t, \sqrt{t}$ in Example 1.

| $n$ | $t$  | $\psi^*(t) = t$ | $\psi^*(t) = 2^t$ | $\psi^*(t) = \sqrt{t}$ |
|-----|------|-----------------|-------------------|------------------------|
| 1   | 0.00000 | 0.061643         | 0.014229           | 0.090759               |
| 2   | 0.06250 | 0.067849         | 0.019923           | 0.09536                |
| 3   | 0.12500 | 0.04538          | 0.014135           | 0.07156                |
| 4   | 0.18750 | 0.116960         | 0.023487           | 0.18527                |
| 5   | 0.25000 | 0.112735         | 0.060722           | 0.166134               |
| 6   | 0.31250 | 5.18945 × 10^2   | 0.25625            | 0.79212 × 10^2         |
| 7   | 0.37500 | 1.38157 × 10^3   | 0.01813            | 2.12337 × 10^6         |
| 8   | 0.43750 | 97.9661 × 10^12  | 0.027994           | 0.14979 × 10^14        |
| 9   | 0.50000 | 49.2583 × 10^26  | 0.034989 × 10^12   | 0.253138 × 10^26       |
| 10  | 0.56250 | 12.4534 × 10^54  | 0.84591 × 10^38    | 0.90406 × 10^54        |
| 11  | 0.62500 | 79.5978 × 10^108 | 0.05650 × 10^34    | 0.012170 × 10^110      |
| 12  | 0.68750 | 32.5184 × 10^218 | 0.023099 × 10^204  | 0.49793 × 10^218       |
| 13  | 0.75000 | NaN              | NaN                | NaN                    |
| 14  | 0.81250 | NaN              | NaN                | NaN                    |
| 15  | 0.87500 | NaN              | NaN                | NaN                    |
| 16  | 0.93750 | NaN              | NaN                | NaN                    |
| 17  | 1.00000 | NaN              | NaN                | NaN                    |

Thus, by Theorem 2, it follows that problem (21) has extremal solutions $\varrho^*, \varpi^* \in [0, 1 + 2t]$, which can be found by means of the iterative sequences $\{\varrho_n\}$ and $\{\varpi_n\}$ defined by (17) and (18), respectively, as follows:

$$\varrho_{n+1}(t) = \mathbb{I}_{t}^{\varrho^*} \psi^*(t, \varrho_n(t)) + \frac{1}{\lambda} \left( \lambda \mathbb{I}_{t}^{\varrho^*} \psi^*(\eta, \varrho_n(\eta)) + \delta \right)$$

$$= \mathbb{I}_{0}^{\varrho^*} \left( 2t - (\varrho_n(t))^2 + 1 \right) + \frac{1}{\lambda} \left[ \frac{3}{4} \mathbb{I}_{t}^{\varpi^*} \left( \frac{4}{3} - (\varrho_n(t))^2 + 1 \right) + \frac{1}{2} \right]$$
and

\[ \vartheta_{n+1}(t) = \mathbb{I}^\psi \Big( \psi(t, \vartheta_n(t)) + \frac{1}{\Lambda} \left( \lambda \mathbb{I}^\psi \psi'(\eta, \vartheta_n(\eta)) + \delta \right) \Big) \]

\[ = \mathbb{I}^\psi \left( 2t - (\vartheta_n(t))^2 + 1 \right) + \frac{1}{\Lambda} \left[ \frac{3}{4} 2t^{1/4} \left( \frac{4}{3} - (\vartheta_n(\eta))^2 + 1 \right) + \frac{1}{2} \right] \]

Figure 1. Graphical representation of \( \vartheta_n(t) \) and \( \vartheta_n(t) \) in (a) and (b) respectively for the \( \psi_1(t) = t \) Caputo derivative, \( \psi_2(t) = 2t \), and the \( \psi_3(t) = t^{1/2} \) Katugampola derivative (for \( \rho = 0.5 \)) in Figure 2a–c. Algorithm A4 shows how to calculate \( \vartheta_n(t) \) and \( \vartheta_n(t) \) for \( t \in \mathbb{I} \).
Example 2. Consider the following problem:

\[
\begin{cases}
\mathcal{C} D_{0^+}^{\sigma} \psi(t) = 2t - \varphi^2(t) + 1, & t \in \mathfrak{I} := [0.1, 1.5], \\
\varphi(0) = \varphi^0 + \varphi(\frac{3}{2}) + \frac{4}{5},
\end{cases}
\]

where

\[
\sigma = \frac{3}{5} \in (0, 1), \nu = \frac{2}{3}, \lambda = \frac{5}{6}, \eta = \frac{6}{7} \in (0, 1), \delta = \frac{9}{11}, t_1 = 0.1, t_2 = 1.5,
\]

and \(\mathcal{V} : \mathfrak{I} \times \mathbb{R} \to \mathbb{R}\) is given by

\[
\mathcal{V}(t, \varphi) = 0.6t - \varphi^2 + 8.5,
\]

for \(t \in \mathfrak{I}, \varphi \in \mathbb{R}\). We take \(\varphi^0(t) = 0.1\) as the \(l\)-solution and \(\varphi^0(t) = 1.1 + \sqrt{t}\) as the \(u\)-solution of problem (21), and we take \(\varphi^0(t) \leq \varphi(t)\) for \(t \in \mathfrak{I}\). So, (H1) of Theorem 2 holds. Now, we consider three cases for \(\psi^*\):

\[
\psi^*_1(t) = t, \quad \psi^*_2(t) = 2t^\nu, \quad \psi^*_3(t) = \sqrt{t}.
\]

Note that \(\psi^*_1(t) = t\) and \(\psi^*_3(t) = \sqrt{t}\) give the Caputo and Katugampola (for \(\rho = 0.5\)) derivatives in this example. These results are plotted in Figure 3a,b.
Figure 3. Graphical representation of $\varrho_n(t)$ and $\varpi_n(t)$ in (a) and (b) respectively for the $\psi_1(t) = t$ Caputo derivative, $\psi_2(t) = 2^t$, and the $\psi_3(t) = \sqrt{t}$ Katugampola derivative (for $\rho = 0.5$) in Example 2.
With the data provided, we can see from assumption (H3) that

\[
\lambda \Gamma(\nu + 1)(\Psi^*(\eta) - \Psi^*(t_1))^{\nu} = \frac{5^2}{\Gamma\left(\frac{5}{2} + 1\right)} \left(\Psi^*(\frac{6}{7}) - \Psi^*(0.1)\right)^{\frac{5}{2}}
\]

\[
= \begin{cases} 
\frac{5}{\Gamma\left(\frac{5}{2}\right)} \left(\frac{6}{7} - 0.1\right)^{\frac{5}{2}} \approx 0.766841 < 1, & \Psi^*(t) = t \ \\
\frac{5}{\Gamma\left(\frac{5}{2}\right)} \left(2^{\frac{5}{2}} - 1\right)^{\frac{5}{2}} \approx 0.755000 < 1, & \Psi^*(t) = 2^t \ \\
\frac{5}{\Gamma\left(\frac{5}{2}\right)} \left(\sqrt{\frac{5}{2}}\right)^{\frac{5}{2}} \approx 0.663661 < 1, & \Psi^*(t) = \sqrt{t}
\end{cases}
\]

Tables 3 and 4 show these results. One can see the 2D line plots of \(\bar{\varphi}_n(t)\) and \(\overline{n}(t)\) for the \(\Psi_1^*(t) = t\) Caputo derivative, \(\Psi_2^*(t) = 2^t\), and the \(\Psi_3^*(t) = \sqrt{t}\) Kattugamalpa derivative (for \(\rho = 0.5\)) in Figures 3a,b. Further, assumption (H2) is clearly satisfied. Thus, by Theorem 2, it follows that problem (22) has extremal solutions \(q_\ast, q_\ast^* \in [0,1,1+\sqrt{t}]\), which can be found by means of the iterative sequences \(\{\bar{\varphi}_n\}\) and \(\{\overline{n}_n\}\) defined by (17) and (18), respectively, as follows:

\[
\bar{\varphi}_{n+1}(t) = \frac{\bar{\varphi}_n(t) - \Psi^*(t)}{\bar{\varphi}_n(t)} + \frac{1}{\Lambda}\left(\lambda^{\frac{\nu}{\nu+\nu}} \Psi^*(\eta, \bar{\varphi}_n(t)) + \delta\right)
\]

\[
= \bar{\varphi}_n(t) \left(2t - (\bar{\varphi}_n(t))^2 + 1\right) + \frac{1}{\Lambda} \left(\lambda^{\frac{\nu}{\nu+\nu}} \Psi^*(2\eta, (\bar{\varphi}_n(t))^2 + 1) + \delta\right)
\]

and

\[
\overline{n}_{n+1}(t) = \frac{\overline{n}_n(t) - \Psi^*(t)}{\overline{n}_n(t)} + \frac{1}{\Lambda}\left(\lambda^{\frac{\nu}{\nu+\nu}} \Psi^*(\eta, \overline{n}_n(t)) + \delta\right)
\]

\[
= \overline{n}_n(t) \left(2t - (\overline{n}_n(t))^2 + 1\right) + \frac{1}{\Lambda} \left(\lambda^{\frac{\nu}{\nu+\nu}} \Psi^*(2\eta, (\overline{n}_n(t))^2 + 1) + \delta\right)
\]

One can see the 2D line plots of \(\bar{\varphi}_n(t)\) and \(\overline{n}_n(t)\) for the \(\Psi_1^*(t) = t\) Caputo derivative, \(\Psi_2^*(t) = 2^t\), \(\Psi_3^*(t) = \sqrt{t}\), and the Kattugamalpa derivative (for \(\rho = 0.5\)) in Figures 3a,b. Algorithm A4 shows how to calculate \(\bar{\varphi}_n(t)\) and \(\overline{n}_n(t)\) for \(t \in I\).

Table 3. Numerical results of \(\bar{\varphi}_n\) for \(\Psi^*(t) = t, 2^t, \sqrt{t}\) in Example 2.

| \(n\) | \(t\) | \(\Psi^*(t) = t\) | \(\Psi^*(t) = 2^t\) | \(\Psi^*(t) = \sqrt{t}\) |
|------|------|-----------------|-----------------|-----------------|
| 1    | 0.10000 | 23.0887         | 21.4299         | 12.7468         |
| 2    | 0.22500 | 01.7027 \times 10^2 | 01.3640 \times 10^2 | 01.0319 \times 10^2 |
| 3    | 0.35000 | 01.3185 \times 10^4 | 99.6965 \times 10^2 | 82.2118 \times 10^4 |
| 4    | 0.47500 | 68.0783 \times 10^6 | 49.0307 \times 10^6 | 42.9176 \times 10^6 |
| 5    | 0.60000 | 17.1314 \times 10^{14} | 11.8461 \times 10^{14} | 10.8451 \times 10^{14} |
| 6    | 0.72500 | 01.0361 \times 10^{30} | 69.1949 \times 10^{28} | 65.6348 \times 10^{28} |
| 7    | 0.85000 | 36.4361 \times 10^{58} | 23.5622 \times 10^{58} | 23.0527 \times 10^{58} |
| 8    | 0.97500 | 04.3535 \times 10^{118} | 02.7357 \times 10^{118} | 02.7475 \times 10^{118} |
| 9    | 1.10000 | 06.0253 \times 10^{236} | 03.6879 \times 10^{236} | 03.7901 \times 10^{236} |
| 10   | 1.22500 | NaN             | NaN             | NaN             |
| 11   | 1.35000 | NaN             | NaN             | NaN             |
| 12   | 1.47500 | NaN             | NaN             | NaN             |
Table 4. Numerical results of $\varrho_n$ for $\psi^*(t) = t, 2^t, \sqrt{t}$ in Example 1.

| $n$ | $t$ | $\psi^*(t) = t$ | $\psi^*(t) = 2^t$ | $\psi^*(t) = \sqrt{t}$ |
|-----|-----|-----------------|-----------------|-----------------|
| 1   | 0.10000  | 23.0887         | 21.4299         | 12.7468         |
| 2   | 0.22500  | $0.11374 \times 10^2$ | 90.2639         | 69.2844         |
| 3   | 0.35000  | 58.6017 $\times 10^2$ | 43.0855 $\times 10^2$ | 37.2431 $\times 10^2$ |
| 4   | 0.47500  | 13.6661 $\times 10^6$ | 09.1617 $\times 10^6$ | 08.9616 $\times 10^6$ |
| 5   | 0.60000  | 71.6666 $\times 10^{12}$ | 41.3606 $\times 10^{12}$ | 49.1158 $\times 10^{12}$ |
| 6   | 0.72500  | 20.3287 $\times 10^{26}$ | 08.4297 $\times 10^{26}$ | 15.0160 $\times 10^{26}$ |
| 7   | 0.85000  | 02.0382 $\times 10^{54}$ | 35.0156 $\times 10^{52}$ | 01.6593 $\times 10^{54}$ |
| 8   | 0.97500  | 03.2721 $\times 10^{108}$ | 06.0417 $\times 10^{106}$ | 02.7695 $\times 10^{108}$ |
| 9   | 1.10000  | 10.7866 $\times 10^{216}$ | 17.9871 $\times 10^{212}$ | 08.9489 $\times 10^{216}$ |
| 10  | 1.22500  | $NaN$            | $NaN$           | $NaN$           |
| 11  | 1.35000  | $NaN$            | $NaN$           | $NaN$           |
| 12  | 1.47500  | $NaN$            | $NaN$           | $NaN$           |

5. Conclusions

In this study, we investigated the existence of solutions for a nonlinear FDE in the frame of the $\psi$—Caputo derivative with integral boundary conditions. To prove the main theorems, the monotone iterative and the upper–lower solution techniques in the sense of the $\psi$—Caputo fractional operator were used. Based on certain conditions, we constructed mis that uniformly converged to the extremal solutions of BVP. The results were tested by constructing two equations corresponding to BVP (3). Different values for $\psi$, such as the $\psi^*(t) = t$, Caputo, $\psi^*(t) = 2^t$, $\psi^*(t) = \sqrt{t}$, and Katugampola (for $\rho = 0.5$) derivatives and the upper and lower solutions, were examined and illustrated for the purpose of verification. We conclude that the results reported in this paper are of great significance for the relevant audience and can be applied to different types of fractional differential problems.

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Appendix A. Supporting Informations

Algorithm A1 MATLAB lines for the calculation of $I^{\sigma}_{t_1} q(t)$ in Equation (4).

```matlab
1: LSfractionalintegral
Require: t_1, \sigma, \psi, q, t
2: syms v e;
3: E = int(subs(diff(\psi, v), v, e) * (eval(subs(\psi, v, t)) - subs(\psi, v, e))^(\sigma - 1) * subs(q, v, e), t_1, t);
4: mathbbI = round(1/\gamma(\sigma) * eval(E), 8);
5: return mathbbI
```

Algorithm A2 MATLAB lines for the calculation of $D^{\alpha}_{t_1} \phi(t)$ in Equation (5).

```matlab
1: LSfractionalderivative
Require: t_1, \sigma, \psi, \varrho, q, t
2: syms v e t;
3: n = floor(\alpha) + 1;
4: E = int(subs(diff(\psi, v), v, e) * (subs(\psi, v, t) - subs(\psi, v, e))^(n - \sigma - 1) * subs(\varrho, v, e), e, t_1, t);
5: F = diff(E, t, n);
6: F = (1/subs(diff(\psi, v), v, t))^n * F;
7: F = round(1/\gamma(n - \sigma), 8) * F;
8: mathbbD = F;
9: return mathbbD
```

Algorithm A3 MATLAB lines for the calculation of $C^{\sigma,\Psi^*}_{t_1} q(t)$ in Equation (6).

```matlab
1: LSCaputofractionalderivative
Require: t_1, \sigma, \psi, \varrho, q, t
2: syms v e;
3: n = floor(\sigma) + 1;
4: if fix(\alpha) == \sigma then
5: F = eval(subs(diff(wp, v, n), v, t));
6: E = F;
7: end
8: F = int(subs(diff(\psi, v), v, e) * (eval(subs(\psi, v, t)) - subs(\psi, v, e))^(n - \sigma - 1) * eval(subs(diff(\varrho, v, n), v, e)), t_1, t);
9: E = 1/\gamma(n - \sigma) * F;
10: end if
11: mathbbD = E;
12: return mathbbD
```
Algorithm A4 MATLAB lines for the calculation of $u_n(t)$ and $v_n(t)$ in Example 1.

Require: $t_1, \sigma, \psi, q, t$

1: sym $v$;
2: clear;
3: format long;
4: sym $v \in \mathbb{R}$;
5: $a = 1/4; \beta = 1/2; \lambda = 3/4; \eta = 2/3; \delta = 1/2$;
6: $a = 0; b = 1$;
7: $\psi = \varphi = v$;
8: $n = \text{floor}(a) + 1$;
9: $\Lambda = 1 - \lambda/\gamma(\beta + 1) * (\text{eval}(\text{subs}(\psi, v, \eta)) - \text{eval}(\text{subs}(\psi, v, a)))^\delta$;
10: $u_0 = 0; v_0 = 1 + 2 * v$;
11: $\psi = \text{sym}(v) \text{sym}(2^v) \text{sym}(v(1/2))$;
12: $h = 4; c = 1$;
13: for $i = 1$ to $3$
14: $t = a$;
15: $r = 1$;
16: MatrixResults($r, c = r$);
17: MatrixResults($r, c + 1 = t$);
18: MatrixResults($r, c + 2 = \text{round}(1 - \lambda/\gamma(\beta + 1) * (\text{eval}(\text{subs}(\psi(i), v, \eta)) - \text{eval}(\text{subs}(\psi(i), v, a)))^\delta, 6)$);
19: MatrixResults($r, c + 3 = 1 - 0 + 2 * t$);
20: MatrixResults($r, c + 4 = 1 - 0 + 2 * eta$);
21: MatrixResults($r, c + 5 = \text{round}(\text{abs}(\text{LSfractionalIntegral}(a, a, \psi(i)), \text{MatrixResults}(r, c + 3), t) + 1/\text{MatrixResults}(r, c + 2) * (\lambda * (\text{LSfractionalIntegral}(a, a + \beta, \psi(i)), \text{MatrixResults}(r, c + 4), \eta) + \delta)), 6)$);
22: MatrixResults($r, c + 6 = 1 - (1 + 2 * t) + 2 * t$);
23: MatrixResults($r, c + 7 = 1 - (1 + 2 * \eta) + 2 * \eta$);
24: MatrixResults($r, c + 8 = \text{round}(\text{abs}(\text{LSfractionalIntegral}(a, a, \psi(i)), \text{MatrixResults}(r, c + 3), t) + 1/\text{MatrixResults}(r, c + 2) * (\lambda * (\text{LSfractionalIntegral}(a, a + \beta, \psi(i)), \text{MatrixResults}(r, c + 4), \eta) + \delta)), 6)$);
25: $t = t + 1/2^b$;
26: $r = r + 1$;
27: while ($t \leq b$) do
28: MatrixResults($r, c = r$);
29: MatrixResults($r, c + 1 = t$);
30: MatrixResults($r, c + 2 = \text{round}(1 - \lambda/\gamma(\beta + 1) * (\text{eval}(\text{subs}(\psi(i), v, \eta)) - \text{eval}(\text{subs}(\psi(i), v, a)))^\delta, 6)$);
31: MatrixResults($r, c + 3 = 1 - (\text{MatrixResults}(r - 1, c + 3))^2 + 2 * t$);
32: MatrixResults($r, c + 4 = 1 - (\text{MatrixResults}(r - 1, c + 3))^2 + 2 * \eta)$;
33: MatrixResults($r, c + 5 = \text{round}(\text{abs}(\text{LSfractionalIntegral}(a, a, \psi(i)), \text{MatrixResults}(r, c + 3), t) + 1/\text{MatrixResults}(r, c + 2) * (\lambda * (\text{LSfractionalIntegral}(a, a + \beta, \psi(i)), \text{MatrixResults}(r, c + 4), \eta) + \delta)), 6)$);
34: MatrixResults($r, c + 6 = 1 - (\text{MatrixResults}(r - 1, c + 6))^2 + 2 * t$);
35: MatrixResults($r, c + 7 = 1 - (\text{MatrixResults}(r - 1, c + 7))^2 + 2 * \eta$);
36: MatrixResults($r, c + 8 = \text{round}(\text{abs}(\text{LSfractionalIntegral}(a, a, \psi(i)), \text{MatrixResults}(r, c + 6), t) + 1/\text{MatrixResults}(r, c + 2) * (\lambda * (\text{LSfractionalIntegral}(a, a + \beta, \psi(i)), \text{MatrixResults}(r, c + 7), \eta) + \delta)), 6)$);
37: $t = t + 1/2^b$;
38: $r = r + 1$;
39: end while
40: $c = c + 9$;
41: end for
42: return MatrixResults
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