Phase Transition in Interacting Boson System at Finite Temperatures

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Abstract

Thermodynamical properties of an interacting boson system at finite temperatures and zero chemical potential are studied within the framework of the Skyrme-like mean-field model. It is assumed that the mean field contains both attractive and repulsive terms. Self-consistency relations between the mean field and thermodynamic functions are derived. It is shown that for sufficiently strong attractive interactions this system develops a first-order phase transition via formation of Bose condensate. An interesting prediction of this model is that the condensed phase is characterized by a constant total density of particles. The thermodynamical characteristics of the system are calculated for the liquid-gas and condensed phases. The energy density exhibits a jump at the critical temperature.

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I. INTRODUCTION

In recent years the properties of hot and dense hadronic matter have attracted considerable interest. Such matter can be produced in relativistic nucleus-nucleus collisions which are under investigations in many laboratories. QCD motivated effective models and lattice simulations indicate that the chiral symmetry restoration and the deconfinement phase transition should take place at high temperatures and particle densities. In hot medium, the properties of the hadrons are expected to be modified. In this paper we study specifically the properties of interacting boson systems within the framework of a thermodynamically consistent mean-field approach. In particular, we investigate the possibility of Bose-Einstein condensation of interacting bosonic particles. This problem has been studied previously, starting from the pioneer works of A.B. Migdal and coworkers [1–4] and later by many authors using different models and methods. A possible formation of classical pion fields in relativistic nucleus-nucleus collisions was discussed in refs. [5–8]. In more recent studies [9–13], pionic systems with a finite isospin chemical potential and low temperatures have been considered. Interesting new results concerning dense pionic systems have been obtained recently using lattice methods [14, 15].

In the present paper we consider interacting boson systems at zero chemical potential, but high temperatures, where thermally produced particles have rather high densities. Simple calculations for noninteracting hadron resonance gas show that the particle density may reach values $(0.1 - 0.2) \text{ fm}^{-3}$ at temperatures $100 - 150 \text{ MeV}$, which are below the deconfinement phase transition, see e.g. refs. [16, 17]. Under such conditions the interaction effects should become important. To account for the interaction between the bosons we introduce a phenomenological Skyrme-like mean field $U(n)$, which depends only on the particle density $n$. Then the thermo-dynamical consistency relations are used to calculate the particle density, energy density and pressure as functions of temperature. An important difference of the considered system is that, in contrast to e.g. the nucleonic matter, the boson number is not conserved, but is determined by the minimization of a thermodynamic potential.

The paper is organized as follows. Section II shortly describes the thermodynamic mean-field model which is used in the presented calculations. The self-consistency relations between the mean field and thermodynamic functions are derived. In Section III we introduce a Skyrme-like parametrization of the mean field and calculate corresponding thermodynamic functions. In Section IV we demonstrate the possibility of the Bose condensation when the attractive interaction is strong enough. Our conclusions are summarized in Section V.
II. THERMODYNAMICALLY CONSISTENT MEAN-FIELD MODEL

First we shortly remind the basics of the thermodynamical mean-field model which was introduced in ref. [18], see more details in [19, 20].

Let us consider the system of interacting particles from general thermodynamic point of view. One can describe such a system in terms of the free energy density $\phi(n, T)$, which depends on particle density $n$ and temperature $T$. The free energy density (FED) is related to other thermodynamical quantities as follows\(^1\)

\[ \phi(n, T) = \varepsilon(n, T) - T s(n, T), \]
\[ \phi(n, T) = n \mu(n, T) - p(n, T), \]

where $\varepsilon(n, T)$ is the energy density and $p(n, T)$ is the pressure. Two quantities $\mu(n, T)$ (the chemical potential) and $s(n, T)$ (the entropy density) are given as partial derivatives with respect to independent variables $(n, T)$

\[ \mu = \left( \frac{\partial \phi}{\partial n} \right)_T, \quad s = - \left( \frac{\partial \phi}{\partial T} \right)_n. \]

Very generally, for a system of interacting particles the FED can be written as a sum of free and interacting contributions

\[ \phi(n, T) = \phi_0(n, T) + \phi_{\text{int}}(n, T), \]

where $\phi_0$ is the FED of the noninteracting system. The chemical potential can also be splitted into “free” and “interacting” pieces. In accordance with eq. (3) we obtain

\[ \mu = \mu_0 + \left( \frac{\partial \phi_{\text{int}}}{\partial n} \right)_T, \quad \text{where} \quad \mu_0 \equiv \left( \frac{\partial \phi_0}{\partial n} \right)_T. \]

Further, taking into account eqs. (2), (4) and (5) one can represent the pressure in the following form

\[ p = n \mu(n, T) - \phi(n, T) = p_0(n, T) + n \left( \frac{\partial \phi_{\text{int}}}{\partial n} \right)_T - \phi_{\text{int}}, \]

where

\[ p_0(n, T) = n \mu_0(n, T) - \phi_0(n, T). \]

One can put this expression in correspondence to the pressure of the ideal gas $\tilde{p}_0$ calculated in the grand canonical ensemble for the same values $T$ and $\mu_0$ as they are taken in (7)

\[ \tilde{p}_0(T, \mu_0) = \frac{g}{3} \int \frac{d^3 k}{(2\pi)^3} \frac{k^2}{\sqrt{m^2 + k^2}} f(k; T, \mu_0), \]

\(^1\) Here and below we adopt the system of units $\hbar = c = 1, k_n = 1$.\]
where \( g \) is the degeneracy factor, \( f(\mathbf{k}; T, \mu_0) \) is the ideal gas distribution function (the Boltzmann, Fermi-Dirac or Bose-Einstein one), which depends on temperature and ideal chemical potential \( \mu_0 \).

Now we introduce the following important notations:

\[
U(n, T) = \left[ \frac{\partial \phi_{\text{int}}(n, T)}{\partial n} \right]_T,
\]
(9)

\[
P^{\text{ex}}(n, T) = n \left[ \frac{\partial \phi_{\text{int}}(n, T)}{\partial n} \right]_T - \phi_{\text{int}}(n, T).
\]
(10)

One can immediately see that these quantities are related as

\[
n \frac{\partial U(n, T)}{\partial n} = \frac{\partial P^{\text{ex}}(n, T)}{\partial n}.
\]
(11)

By subtracting eq. (10) from eq. (6) one can rewrite the total pressure as

\[
p = p_0(n, T) + P^{\text{ex}}(n, T).
\]
(12)

Evidently, if one defines \( p_0(n, T) \) as the pressure of the ideal gas, then, the quantity \( P^{\text{ex}}(n, T) \) should be regarded as an excess pressure, which is due to the interaction between particles.

Next, in our evaluations of the thermodynamic quantities of the interacting system we would like to use formula (8) for the pressure of the ideal gas. In the canonical ensemble the independent variables are \( n \) and \( T \), whereas in the grand canonical ensemble they are \( \mu \) and \( T \). Hence, it is necessary to express the free chemical potential \( \mu_0 \) through these variables. One can do this by substituting eq. (9) into eq. (5), thus obtaining

\[
\mu = \mu_0 + U(n, T).
\]
(13)

It is convenient to introduce also the single-particle energy for interacting particles

\[
E(\mathbf{k}, n) = \sqrt{m^2 + \mathbf{k}^2} + U(n).
\]
(14)

In the grand canonical ensemble we treat the particle density \( n \) as \( n(\mu, T) \), and as a result, the pressure of interacting particles can be expressed as

\[
p(T, \mu) = \frac{g}{3} \int \frac{d^3k}{(2\pi)^3} \frac{k^2}{\sqrt{m^2 + k^2}} f(\mathbf{k}; T, \mu) + P^{\text{ex}}(n, T),
\]
(15)

where

\[
f(\mathbf{k}; T, \mu) = \left\{ \exp \left[ \frac{E(\mathbf{k}, n) - \mu}{T} \right] + a \right\}^{-1}
\]
(16)

with \( a = +1 \) for fermions, \( a = -1 \) for bosons and \( a = 0 \) for the Boltzmann statistics.
In a homogeneous system, where the thermodynamic potential can be expressed as 
\[ \Omega(T, \mu, V) = -p(T, \mu)V, \]
the particle density reads \( n(T, \mu) = \frac{\partial p(T, \mu)}{\partial \mu} \). Then, with the help of relations (11) and (15) we obtain the standard relation
\[
n = g \int \frac{d^3k}{(2\pi)^3} f(k; T, \mu). \quad (17)
\]
Since the distribution function \( f(k; T, \mu) \) is itself a function of \( n \), this expression is not a formula for particle density \( n \), but an equation to be solved in a selfconsistent way for every point of the \((T, \mu)\) plane. The solution will result in the explicit dependence \( n = n(T, \mu) \), which in general differs from the ideal gas expression, \( n_0(T, \mu_0) \). Using \( s = \frac{\partial p}{\partial T} \) and the Euler relation, \( \varepsilon + p = Ts + \mu n \), one obtains for the energy density
\[
\varepsilon(T, \mu) = g \int \frac{d^3k}{(2\pi)^3} \sqrt{m^2 + k^2} f(k; T, \mu) + n U(n, T) + n \mu - P^{\text{ex}}(n, T) + 
\]
\[
+ T \left\{ \left[ \frac{\partial P^{\text{ex}}(n, T)}{\partial T} \right]_n - n \left[ \frac{\partial U(n, T)}{\partial T} \right]_n \right\}. \quad (18)
\]

This type of approach is widely used in relativistic mean field models of nuclear matter [21–23], where the nucleons interact with a scalar field \( \varphi \) (attraction) and a vector field \( V_\mu \) (repulsion). In the homogeneous static system only the “time” component of the vector field \( V_0 \) survives. In our notations it is equal to the mean field \( U(n, T) \).

III. APPLICATION FOR INTERACTING BOSON SYSTEMS WITH \( \mu = 0 \)

A. Skyrme-like parametrization of the mean field

The formalism described in the previous section has been applied for several physically interesting systems including the hadron-resonance gas [18] and the pionic gas [24]. In the present study we extend this approach to the case of a bosonic system which potentially can undergo Bose condensation. We assume that the interaction between particles is described by the Skyrme-like mean field
\[
U(n) = -An + Bn^2, \quad (19)
\]
where \( A \) and \( B \) are the model parameters, which should be specified for each particle species. A similar approach was used recently in Ref. [25] for describing Bose condensation in the system of interacting alpha-particles.

As well known, see e.g. ref. [26], the pion-pion interaction at low energies is rather small, because of their special nature as the Goldstone bosons for spontaneously broken chiral
symmetry. For the isospin-symmetric pion system, the first coefficient can be expressed as
\[ A = -\frac{4\pi}{3}(a_0^0 + 5a_0^2), \]
where \( a_0^0 \) and \( a_0^2 \) are the \( \pi\pi \) s-wave scattering lengths for isospin 0 and 2. Their values are well established both theoretically and experimentally: \( a_0^0 = 0.220 \frac{1}{m_\pi} \), \( a_0^2 = 0.044 \frac{1}{m_\pi} \), see e.g. ref. [27]. With these values the coefficient \( A \) practically vanishes. Nevertheless, some additional contribution to the attractive mean field at high temperatures, \( (T \geq 150 \text{ MeV}) \), may be provided by other hadrons present in the system, like \( \rho \)-mesons [28] or baryon-antibaryon pairs [29].

By this reason, in our calculations we consider a general case of \( A > 0 \), to study a bosonic system with both attractive and repulsive contributions to the mean field (19). But the numerical calculations will be done for bosons with mass \( m_\pi = 140 \text{ MeV} \) and degeneracy factor \( g = 3 \), which we call ”pions”. For the repulsive coefficient \( B \) we use a fixed value, obtained from an estimate based on the virial expansion [30], \( B = 10m_\pi b^2 \) with \( b \) equal to four times the proper volume of a particle, i.e. \( b = 16\pi r_0^3/3 \). In our numerical calculations we take \( b = 0.45 \text{ fm}^3 \) that corresponds to a pion radius \( r_0 \approx 0.3 \text{ fm} \).

In this paper we consider pion systems with \( \mu = 0 \). Then a nonzero pion density is only possible at \( T > 0 \). For each given temperature the particle density is found by self-consistently solving eq. (17) which now has the following form
\[ n = \frac{g}{2\pi^2} \int_0^\infty dk k^2 \left\{ \exp \left[ \frac{E(k, n)}{T} \right] - 1 \right\}^{-1}, \] (20)
where \( E(k, n) \) is given in eq. (14). From eq. (20) it is evident that the single-particle energy at \( |k| \to 0 \), \( E(0, n) = m + U(n) \), must be positive or zero. Otherwise, the occupation numbers become negative. The potential \( U(n) \) is shown in Fig. 1 for several values of the parameter \( \kappa = A/A_c \), characterizing the strength of the attractive interaction. Here \( A_c = 2\sqrt{mB} \) is the critical value of \( A \) at which the minimum of the potential reaches the energy level \(-m\). Below we choose \( \kappa \) as a variational parameter to characterize the vicinity to the critical point.

B. Onset of Bose condensation

The values of \( \kappa \geq \kappa_c = 1 \) lead to a crossing of \(-m\) level and appearance of the density interval, where the function \( E(0, n) \) is negative. The end points of this interval are determined from equation
\[ U(n) + m = 0. \] (21)
Figure 1: The mean field potential $U$ versus particle density $n$ for an interacting pion system with $\mu = 0$. The blue curves are labeled by value of the parameter $\kappa$: $\kappa = 0$, $\kappa = 0.27$, $\kappa = 0.55$, $\kappa = 0.82$. In the case $\kappa = \kappa_c = 1.0$ (red solid curve), the corresponding curve touches the negative energy level $-m_\pi$. For the case $\kappa = 1.098$ (black solid curve) the curve contains a segment below $-m_\pi$ (black-dashed curve), which corresponds to a super-critical strength.

With $U(n)$ from (19) the solutions of this equation are

$$n_1 = \sqrt{\frac{m}{B}} \left( \kappa - \sqrt{\kappa^2 - 1} \right), \quad n_2 = \sqrt{\frac{m}{B}} \left( \kappa + \sqrt{\kappa^2 - 1} \right). \quad (22)$$

In the interval $n_1 < n < n_2$ the integral in eq. (17) is not positively definite, and therefore such densities can not be realized in an equilibrium system. At $\kappa > \kappa_c = 1$, the change of the pion density from $n = n_1$ to $n = n_2$ is only possible via the condensation of pions in zero-momentum mode, $|k| = 0$, so that their total density jumps from $n = n_1$ to $n = n_2$. As it is seen from eq. (22) the critical value of parameter $A$ is obtained when both roots coincide, i.e. when $\kappa = 1$ or $A = A_c = 2\sqrt{mB}$. For parameter $B$ specified above, $A_c = 2\sqrt{10m_\pi b} \approx 395 \text{ MeV-fm}^3$. The corresponding critical pion density, when the minimum of the potential reaches the level $-m_\pi$, is $n_0 = A_c/2B = \frac{1}{\sqrt{10b}} \approx 0.7 \text{ fm}^3$.

The limiting density of Bose particles, $n_{\text{lim}}(T)$, just before the formation of Bose condensate, i.e. at $U(n) = -m$, is the same as in the ideal gas at $\mu_0 = m$

$$n_{\text{lim}}(T) = \frac{g}{2\pi^2} \int_0^\infty dk k^2 \left[ \exp \left( \frac{\sqrt{m^2 + k^2} - m}{T} \right) - 1 \right]^{-1}. \quad (23)$$
Condensation in ideal gas

\[ n \left[ \text{1/fm}^3 \right] \]

\[ T \left[ \text{MeV} \right] \]

\[ n_{\text{lim}}(T) \]

\[ n_{\text{lg}} \]

\[ n_{\text{cond}} \]

\[ \mu_0 = m \]

\[ \mu_0 = 0 \]

Figure 2: Particle density versus temperature for the noninteracting pion gas with \( \mu_0 = m \) (dashed red curve labeled as \( n_{\text{lim}} \)). This curve separates the gas phase from the phase with the Bose condensate (dashed area). The same dependence corresponds to the interacting pion system with mean field \( U(n) = -m \) and \( \mu = 0 \). For comparison the blue line shows the density of ideal pion gas with \( \mu_0 = 0 \).

In Fig. 2 this dependence is depicted as a red-dashed line, which separates the normal phase from the phase with the Bose condensate. We come to the general conclusion: for mean-field potentials deeper than \(-m\), the equilibrated bosonic system will develop a Bose condensate.

C. Energy density and pressure in the liquid-gas phase

Let us consider first the interacting bosonic system without condensate, i.e. when \( U(n) + m > 0 \) for all \( n \). We call this state as a liquid-gas phase, to distinguish it from a weakly-interacting pion gas. In the case of \( \mu = 0 \) the pressure is expressed as (see [15])

\[
p(T) = \frac{g}{6\pi^2} \int_0^\infty dk \frac{k^4}{\sqrt{m^2 + k^2}} \left\{ \exp \left[ \frac{\sqrt{m^2 + k^2 + U(n)}}{T} \right] - 1 \right\}^{-1} + P^{\text{ex}}(n),
\]

(24)

where the particle density \( n(T) \) is obtained from eq. (20) assuming \( U(n) > -m \). Here the excess pressure is obtained using the condition of the thermodynamic consistency (11) with \( U(n) \) from eq. (19),

\[
P^{\text{ex}}(n) = \int_0^n dn' n' \frac{\partial U(n')}{\partial n'} = -\frac{1}{2} A n^2 + \frac{2}{3} B n^3.
\]

(25)

Fig. 3 shows the excess pressure as a function of the pion density for several values of \( \kappa \). One
Figure 3: Excess pressure versus pion density for interacting pion system with \( \mu = 0 \) for different values of parameter \( \kappa \) indicated in the figure. The lowest curve calculated for \( \kappa = 1.098 \) contains a segment \( n_1 < n < n_2 \) (black dashed line), where \( U(n) + m_\pi < 0 \).

One can see that it is negative at densities below \( 3A/4B = \frac{3}{2}\kappa n_0 \).

The energy density in the liquid-gas phase is obtained from eq. (18) with \( U(n) \) from eq. (19)

\[
\varepsilon(T) = g \int \frac{d^3k}{(2\pi)^3} \sqrt{m^2 + k^2} \left\{ \exp \left[ \frac{\sqrt{m^2 + k^2 + U(n)}}{T} \right] - 1 \right\}^{-1} + \varepsilon^{ex}(n). \tag{26}
\]

where, in accordance with (19) and (25) the excess-energy density is

\[
\varepsilon^{ex}(n) \equiv nU(n) - P^{ex}(n) = -\frac{1}{2}An^2 + \frac{1}{3}Bn^3. \tag{27}
\]

It is interesting to note that the critical condition (21) corresponds to the requirement that the total energy density at \( T = 0 \), i.e. \( m\cdot n + \varepsilon^{ex}(n) \), has its extremum. Indeed, the densities \( n_1 \) and \( n_2 \) given by eq. (22) correspond to the maximum and minimum of this function. Therefore, it is quite natural that the Bose condensate is formed at density \( n = n_2 \).

**IV. SELF-CONSISTENT SOLUTION FOR PION-CONDENSED PHASE**

**A. Particle density**

Let us return to the equation (20), which determines the particle density as a function of temperature. The solutions \( n(T) \) have been found iteratively for several values of \( \kappa \) as
Figure 4: Particle density versus temperature for interacting pion system with \( \mu = 0 \). The curves are labeled by the values of the interaction parameter \( \kappa = A/A_c \). The densities \( n_1, n_2 \) and \( n_0 \) are introduced in Fig. 1. The temperatures \( T_1, T_2 \) and \( T_0 \) indicate the points where these densities cross the limiting density line \( n_{\text{lim}}(T) \) (red dashed curve). For \( \kappa = 1.098 \) the lower piece of the blue dashed curve corresponds to the meta-stable states, while the upper piece is unstable. The phase transition from liquid-gas phase to mixed phase occurs at \( T_c = 116 \text{ MeV} \) and \( n_c = 0.065 \text{ fm}^{-3} \). For the non-interacting pion system the condensate appears above the line \( n_{\text{lim}} \) in the shaded area.

The limiting density \( n_{\text{lim}}(T) \), eq. (23), is also shown in this figure. One can see that the critical value \( \kappa_c = 1.0 \) separates two qualitatively different regimes. At \( \kappa < \kappa_c \) the curves \( n(T) \) are continuous, while at \( \kappa > \kappa_c \) they break down in two segments with a gap in between. One can check that this gap appears exactly between densities \( n_1 \) and \( n_2 \), where \( U(n) + m_\pi < 0 \), see Fig. 1. For parameter \( \kappa = 1.098 \) the lower branch exhibits a kind of backhanding while approaching the limiting density \( n_{\text{lim}}(T) \) from below (the blue-dashed segment)\(^2\). The appearance of such an unstable branch signals a first order phase transition in the system, i.e. the Bose condensation of pions. The critical temperature \( T_c \) is determined as a crossing point of pressure curves for the liquid-gas and condensate phases. This is graphically shown in Fig. 4. The appearance of such a bending point (cross in Fig. 4) was interpreted as a “limiting” temperature.

\(^2\) In ref. [24] the appearance of such a bending point (cross in Fig. 4) was interpreted as a “limiting” temperature.
Figure 5: The density of condensed pions versus temperature for the interacting pion system with the attraction parameter $\kappa = 1.098$. The important temperatures are: $T_c = 116.1$ MeV, where the condensate phase becomes stable, and $T_2 = 219$ MeV, where the condensate disappears shown in Fig. 6 which will be discussed in the next subsection. At $T > T_c$ the liquid-gas branch becomes metastable. The barrier between the two phases finally disappears at the end point marked by the cross (it corresponds to the banding point in Fig. 4). Obviously, if we reach the temperature $T_c$ and continue to pump energy into the multi-pion system, it will experience a phase transition leading to the formation of the Bose condensate, even in the system with $\mu = 0$. As a consequence the pion density will jump along the line $T = T_c$ from the value $n_c = 0.12$ fm$^{-3}$ to the value $n_2 = 1.09$ fm$^{-3}$. Because of the jump this is certainly a first order phase transition. It is rather obvious that with further increase of temperature the pion system will evolve along the horizontal line $n = n_2$ from $T_c = 116.1$ MeV till $T_2 = 219$ MeV, as shown in Fig. 4 (for $\kappa = 1.098$).

In a standard treatment of the Bose-Einstein condensation (see for instance [31]), above $T_c$ the particle density consists of two contributions: “gas-liquid” particles and “condensate” particles. Therefore, at $T > T_c$ we should represent the total particle density $n$ as

$$n = n_{\text{cond}}(T) + \frac{g}{2\pi^2} \int_0^\infty dk k^2 \left[ \exp \left( \frac{\sqrt{m^2+k^2-m}}{T} \right) - 1 \right]^{-1},$$

(28)

where $n_{\text{cond}}$ is the density of condensed pions. This equation is valid in our specific case too, where the evolution of the system goes along the constant density line $n = n_2$. Indeed, for every temperature $T$ from the interval $T_c < T \leq T_2$ (see Fig. 4) the density of particles is
Figure 6: Pressure versus temperature for an interacting pion system with $\mu = 0$ for super-critical interaction parameter $\kappa = 1.098$. Solid blue line labeled as $p_{lg}$ corresponds to the pressure in the liquid-gas phase, the dark blue line labeled as $p_{lg}^{(ms)}$ corresponds to meta-stable states in the liquid-gas phase, the “cross” marks the turning-point. Solid black lines labeled as $p_{mix}^{(1)}$ and $p_{mix}^{(2)}$ correspond to the pressures in the mixed phase (condensate plus liquid-gas) for the pion densities $n = n_1$ and $n = n_2$, respectively. C is the critical point. The branch $p_{mix}^{(1)}$ is unstable.

$n_2 = n_{lim}(T) + n_{cond}(T)$ and the value of the mean field is $U(n_2) = -m$. The behavior of $n_{cond}(T)$ is shown in Fig. 3. Hence, one should consider eq. (28) as a self-consistent description of the pion condensate in the framework of the mean-field approach.

**B. Pressure**

In the mixed phase with the Bose condensate the total particle density is fixed at $n = n_2$, and thus the pressure can be expressed as

$$p_{cond}(T) = g \int \frac{d^3k}{(2\pi)^3} \frac{k^2}{\sqrt{m^2 + k^2}} \left\{ \exp \left[ \frac{\sqrt{m^2 + k^2} - m}{T} \right] - 1 \right\}^{-1} + P^{ex}(n_2), \quad (29)$$

where $P^{ex}(n_2)$ is given by eq. (25) with $n = n_2$. One should bear in mind that the condensate particles with $k = 0$ give no contribution to the kinetic part of the pressure (first term), but contribute to the interaction pressure via $P^{ex}$ (second term). Note, that at $\kappa = 1.098$ $P^{ex}$ is negative for pion densities in the interval $0 < n < 1.16 \text{ fm}^{-3}$, see Fig. 3. Figure 6 shows several branches of pressure representing different phases. The critical temperature, $T_c = 116 \text{ MeV}$, is
obtained as the crossing point of two branches, representing the liquid-gas pressure for $\kappa = 1.098$ and pressure of the mixed phase at $n = n_2$. This explains why the Bose condensate appears only above $T_c$, when the additional thermal pressure compensates the negative contribution of $P^{ex}$. At $T < T_c$ the mixed-phase branch is metastable.

C. Energy density

In the mixed phase the energy density consists of the kinetic part, $\varepsilon_{\text{kin}}(T)$, which is produced by particles in the liquid-gas phase with density $n_{\text{lg}}(T) = n_{\text{lim}}(T)$, and the condensate particles with density $n_{\text{cond}}(T)$, due to the particles, with zero momentum. In accordance with self-consistent solution of eq. (28), in the mixed phase, a sum of these densities remains constant, $n_{\text{lg}}(T) + n_{\text{cond}}(T) = n_2$. This constant density $n_2$ determines the excess pressure $P^{ex}(n_2)$ and excess energy density $\varepsilon^{ex}(n_2)$ in the mixed phase. Therefore, the energy density in the mixed phase at $n = n_2$ reads

$$\varepsilon^{(2)}_{\text{mix}} = mn_{\text{cond}}(T) + g \int_{|k|\neq 0} \frac{d^3k}{(2\pi)^3} \sqrt{m^2 + k^2} \left[ \exp \left( \frac{\sqrt{m^2 + k^2} - m}{T} \right) - 1 \right]^{-1} + \varepsilon^{ex}(n_2), \quad (30)$$

where the condition $U(n_2) = -m$ has been used. Using eq. (27) and the expression for $n_{\text{cond}}(T)$ from eq. (28), one can rewrite this expression as

$$\varepsilon^{(2)}_{\text{mix}} = g \int_{|k|\neq 0} \frac{d^3k}{(2\pi)^3} \left( \sqrt{m^2 + k^2} - m \right) \left[ \exp \left( \frac{\sqrt{m^2 + k^2} - m}{T} \right) - 1 \right]^{-1} - P^{ex}(n_2). \quad (31)$$

In Fig. 7 we present the energy densities for different systems: interacting pions with $\kappa = 1.098$ (blue solid curve), interacting pions at critical value $\kappa = \kappa_c = 1.0$ (black solid curve) and the ideal pion gas (black dashed curve). One can see that for $\kappa = 1.098$ the model predicts upward jump of the energy density of about 30 MeV/fm$^3$ (latent heat), at critical temperature $T_c = 116.1$ MeV. This is another manifestation of the first order phase transition. One can see in Fig. 7 that in a wide interval of temperatures the energy density of the critical system with $\kappa = 1$ is larger than that in the condensed phase with $\kappa = 1.098$ represented by $\varepsilon^{(2)}_{\text{cond}}$.

V. CONCLUDING REMARKS

In this paper we have presented a thermodynamically consistent method to describe dense bosonic systems at high temperatures and zero chemical potential. A central step of this approach is to solve the self-consistent equations (20) or (28) for the pion density at a given
Figure 7: The energy density versus temperature for an interacting pion system with $\mu = 0$. The black solid curve labeled as $\kappa = 1.0$ corresponds to the energy density for the parameter $\kappa = \kappa_c$. The blue solid curve, which consists of several segments, labeled as $\varepsilon_{\text{lg}}$ and $\varepsilon_{\text{mix}}^{(2)}$, corresponds to $\kappa = 1.098$. The latter corresponds to the mixed phase with constant particle density $n = n_2$ in the temperature interval $T_c \leq T \leq T_2$. The energy density of meta-stable states in liquid-gas phase for $\kappa = 1.098$ is labeled as $\varepsilon_{\text{lg}}^{(\text{ms})}$ (blue dashed segment). The energy density in the mixed phase $\varepsilon_{\text{mix}}^{(1)}$ calculated for the constant density $n = n_1$ is depicted as black solid segment. The energy density of metastable states in the liquid-gas phase for $\kappa = 1.098$ is labeled as $\varepsilon_{\text{lg}}^{(\text{ms})}$ (blue dashed segment). The ideal gas energy density is depicted as black dashed curve.

temperature. The crucial point is calculation of the kinetic integral (23), which determines the pion density in the presence of the mean-field $U(n)$ by eq. (19). This integral is positive definite only if the condition $U(n) \geq -m$ is fulfilled. If the attractive mean field is so strong that this condition is violated, the multi-boson system develops a Bose condensate. Our analysis leads to the conclusion that in presence of the condensate, the allowed states of the system must satisfy the condition $U(n) + m = 0$, where $n$ is the total particle density including the condensate. This very unusual behavior is only possible if the attractive interaction between bosons is strong enough. However, as was already mentioned in Sec. III, the empirical data and theoretical calculations, see ref. [26], show that the pion-pion interaction is rather weak at energies $\leq 100$ MeV. Nevertheless, an additional contribution to the pion mean field can be
provided by the attractive pion-nucleon interaction in cold nuclear matter, as demonstrated in refs. [1]-[4], or by $\rho$-mesons and baryon-antibaryon pairs at high temperatures, as considered in refs. [28]-[29]. Another interesting possibility, studied in refs. [8]-[13], is Bose condensation in a pure pionic system with non-zero isospin chemical potential. We are planning to consider such interacting systems in the future.

Finally, we would like to point out that mesonic degrees of freedom may not be appropriate at high particle densities as considered in this paper. It is more likely that under such conditions the mesons will melt into quarks and antiquarks. Such multi-quark-antiquark systems can be studied by using either lattice methods, as in refs. [14]-[15], or QCD motivated effective models. For example, in ref. [32] the Nambu-Jona-Lasinio model has been used to describe baryon-free matter made of quarks and antiquarks, so called “meso-matter”. It was found that such systems are characterized by strong attractive and reduced repulsive interactions, leading to a strong liquid-gas phase transition. It would be also interesting to investigate the possibility of $q - \bar{q}$ pairing and Bose condensation in such systems.

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[1] A.B. Migdal, Sov. Phys. JETP 34(6), 1184 (1972).
[2] A.B. Migdal, O.A. Markin, I.N. Mishustin, Sov. Phys. JETP 39(2), 212 (1974).
[3] A.B. Migdal, Rev. Mod. Phys. 50, 107 (1978).
[4] A.B. Migdal, E. Saperstein, M. Troitsky, and D. Voskresensky, Phys. Rept. 192, 179 (1990).
[5] A. Anselm and M. Ryskin, Phys. Lett B226, 482 (1991).
[6] J.-P. Blaizot and Krzewiński, Phys. Rev. D46, 246 (1992).
[7] J.D. Bjorken, Intern. J. Mod. Phys. A7, 4189 (1992).
[8] I.N. Mishustin and W. Greiner, J. Phys. G: Nucl. Part. Phys. 19, L101 (1993).
[9] D.T. Son and M.A. Stephanov, Phys. Rev. Lett. 86, 592 (2001); arXiv:hep-ph/0005225; Phys.
Atom. Nucl. 64, 834 (2001).
[10] J. Kogut, and D. Toublan, Phys. Rev. D64, 034007 (2001); arXiv:hep-ph/0103271.
[11] D. Toublan, and J. Kogut, Phys. Lett. B564, 212 (2001); arXiv:hep-ph/0301183.
[12] A. Mammarella, and M. Mammarelli, Phys. Rev. D92, 085025 (2015); [1507.02934 [hep-ph]],
2015.
[13] S. Carignano, L. Lepori, A. Mammarelli, M. Mannarelli, G. Pagliaroli, Eur. Phys. J. A53, 35
(2017); arXiv:1610.06097 [hep-ph].
[14] B.B. Brandt, G. Endrodi, PoS LATTICE2016 039 (2016); arXiv:1611.06758 [hep-lat].
[15] B.B. Brandt, G. Endrodi, S. Schmalzbauer, arXiv:1709.10487 [hep-lat].
[16] L.M. Satarov, M.N. Dmitriev, I.N. Mishustin, Phys. Atom. Nucl. 72, 1390-1415 (2009).
[17] Volodymyr Vovchenko, Anton Motornenko, Paolo Alba, Mark I. Gorenstein, Leonid M. Satarov,
and Horst Stoecker, Phys. Rev. C 96, 045202 (2017).
[18] D. Anchishkin, V. Vovchenko, J. Phys. G: Nucl. Part. Phys. 42, 105102: 1-27 (2015);
arXiv:1411.1444 [nucl-th].
[19] D.V. Anchishkin, Sov. Phys. JETP 75, 195 (1992) [Zh. Eksp. Teor. Fiz. 102, 369 (1992)].
[20] D. Anchishkin, E. Suhonen, Nucl. Phys. A 586, 734-754 (1995).
[21] J.D. Walecka, Ann. Phys. (N.Y.) 83 (1974) 491.
[22] B.D. Serot and J.D. Walecka, Adv. Nucl. Phys. 16 (1986).
[23] B.M. Waldhauser, J.A. Maruhn, H. Stöcker, W. Greiner, Phys. Rev. C 38 (1988) No.2 1003-1009.
[24] R.V. Poberezhnyuk, V. Yu. Vovchenko, D.V. Anchishkin, M.I. Gorenstein, J. Phys. G: Nucl.
Part. Phys. 43, 095105 (2016); arXiv:1508.04585 [nucl-th].
[25] L.M. Satarov, M.I. Gorenstein, A. Motornenko, V. Vovchenko, I.N. Mishustin, H. Stoecker, J.
Phys. G44, 12 (2017).
[26] J. Gasser and H. Leutwyler, Nucl. Phys. A321, 387 (1989).
[27] G. Conangelo, J. Gasser and H. Leutwyler, Nucl. Phys. B603, 125 (2001).
[28] E.V. Shuryak, Nucl. Phys. A533, 761 (1991).
[29] J. Theis, G. Graebner, G. Buchwald, J. Maruhn, W. Greiner, H. Stoecker, J. Polonyi, Phys.
Rev. D 28, 2286 (1983).
[30] J.P. Hansen, I.R. McDonald, Theory of Simple Liquids, Academic Press, 2006.
[31] A.J. Leggett, Quantum Liquids, Oxford University Press Inc., New York, 2006.
[32] I.N. Mishustin, L.M. Satarov, H. Stoecker, and W. Greiner, Phys. Rev. C59, 3343 (1999).