A DUALITY THEOREM FOR CERTAIN FOCK SPACES

JOCELYN GONESSA

Abstract. We characterise functions for the dual spaces of entire functions $f$ such that $fe^{-\phi} \in L^p(C^n, \rho^{-2}dA)$, $0 < p \leq 1$, where $\phi$ is a subharmonic weight and $\rho^{-2}$ is a positive function called under certain conditions regularised version of Laplacian $\Delta \phi$, as described in [4].

1. Introduction and main result.

Let a subharmonic function $\phi$ be given. The spaces we deal with are follows:

\[ F_p^\phi = \left\{ f \in \mathbb{C}^n : \|f\|_{p,\phi}^p = \int_{\mathbb{C}^n} |f(z)e^{-\phi(z)}|^p \rho^{-2}(z)dA(z) < \infty \right\}, \quad 1 \leq p < \infty \]

\[ F_\infty^\phi = \left\{ f \in \mathbb{C}^n : \|f\|_{\infty,\phi} = \sup_{z \in \mathbb{C}^n} |f(z)e^{-\phi(z)}| < \infty \right\} \]

where $\rho^{-2}$ is a positive function given. Here $dA$ is the Lebesgue measure on $\mathbb{C}^n$ normalized so that the volume of the unit ball is equal to one.

Our objective in this work is to prove for certain $\phi$ and $\rho$ the dual of $F_p^\phi$ is $F_\infty^\phi$, $0 < p \leq 1$. More precisely we study the following particular case.

\[ \phi(z) = \frac{sN_\star(z) - \log|z \bullet z|}{2}, \quad s > 0 \]

\[ \rho(z) = \sqrt{|z \bullet z|} \]

where $N_\star(z) = \sqrt{|z|^2 + |z \bullet z|}$

with $|z|^2 = z \bullet \bar{z}$ and $z \bullet w = z_1w_1 + \cdots + z_nw_n$ for all $z = (z_1, \ldots, z_n)$, $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$.

Namely, $N_\star/\sqrt{2}$ is a norm introduced by Hahn and Pflug, see [10]. It was shown to be the smallest norm in $\mathbb{C}^n$ that extends the euclidean norm in $\mathbb{R}^n$ in the following sense. If $N$ is any complex norm in $\mathbb{C}^n$ such that $N^2(x) = \|x\|^2 = \sum_{j=1}^{n} x_j^2$ for $x \in \mathbb{R}^n$ and $N(z) = \|z\|$ for $z \in \mathbb{C}^n$, then $N_\star(z)/\sqrt{2} \leq N(z)$ for $z \in \mathbb{C}^n$. Moreover, this norm was shown to be of interest in the study of several problems related to proper holomorphic mappings and the Bergman kernel, see [7, 8, 13, 14, 15] for example. For any $s > 0$ and $0 < p \leq \infty$ we let $L^p_s$ denote the space of Lebesgue measurable functions $f$ on $\mathbb{C}^n$ such that $fe^{-\phi} \in L^p(\mathbb{C}^n, \rho^{-2}dA)$. In this paper we are going to call $F_p^\phi$ the Fock spaces $F_p^\phi$, for no particular reason than use notations in [13]. So we will do the following identifications. $\|\| L^p_s := \|\|_{L^p_s,\phi}$ and $\|\|_{F_p^\phi} := \|\|_{F_p^\phi,\phi}$. Let us remark that if $n = 1$ then the spaces $F_p^\phi$ consist of all entire functions $f$ such that $f(z)e^{-s|z|^2} \in L^p(\mathbb{C}, |z|^{-p}dA)$. These are the classical Fock spaces.

2000 Mathematics Subject Classification. Primary 47B35, 32A36, 30H25, 30H30, 46B70, 46M35.

Key words and phrases. Bergman projection, Bergman spaces, Bloch space.

Gonessa was supported by African Institute for Mathematical Sciences (in South Africa) and Agence Universitaire de la Francophonie.
The purpose of this paper is to describe the bounded linear functionals on Fock spaces $F^p_s$ for every $0 < p \leq 1$. The answer is well known when $1 < p < \infty$ and the problem is solved by using a special pairing, see [7]. The arguments previously provided in [7] and [23] play the key role in the present work. However the case $\phi$ is a (nonharmonic) subharmonic function, whose Laplacian satisfies $0 < m \leq \Delta \phi(z) \leq M$ ($m$, $M$ positive constants) and $\rho(z) = 1$ can be proved (used classical method in [23] and Lemma 1 in [17]). That is the dual of $F^p_s$ is $F^\infty_s$ for any $0 < p \leq 1$.

Even better if $\phi$ is a (nonharmonic) subharmonic function, whose $\mu = \Delta \phi$ is a doubling measure and $\rho^{-2}$ is a regularised version of $\mu$, i.e. the positive radius such that $\mu(D(z, \rho(z))) = 1$, and $\rho \geq 1$ then the dual of $F^p_s$ is $F^{\infty_s}$ for any $0 < p \leq 1$ (used classical method in [23] and Lemma 19 (a) in [12]). Here $D(z, \rho(z))$ is a ball centered in $z$ of radius $\rho(z)$. Our main result is the following.

**Theorem A.** Suppose $s > 0$ and $0 < p \leq 1$. Then the dual space of $F^p_s$ can be identified with $F^{\infty_s}$. More precisely, there is a bounded bilinear complex form $L$ on $F^p_s \times F^{\infty_s}$ such that every bounded linear functional on $F^p_s$ has the following form

$$f \mapsto L(f, g):= L(f, g)$$

for some unique $g \in F^{\infty_s}$. Furthermore the norm of the linear functional on $F^p_s$ is equivalent to the norm of $g$ in $F^{\infty_s}$. Namely, there exits a constant $C$ such that

$$C^{-1}\|g\|_{\infty,s} \leq \|Lg\| \leq C\|g\|_{\infty,s}$$

for all $g \in F^{\infty_s}$.

The problem of describing the bounded linear functionals on $L^p$, $0 < p \leq 1$, has been studied in several papers. In the case of Hardy spaces, the problem started in [19] and developed in [18, 6, 9]. For Bergman spaces, the problem was studied in [3, 21, 24, 25]. Namely, it was shown that with the classical integral pairing all the classical Fock spaces have the same dual space when $0 < p \leq 1$. That is the space of the bounded holomorphic functionals. In this note we use new tools under a special pairing to prove the same result for Fock spaces $F^p_s$.

This work began when I was visiting the African Institute for Mathematics Sciences in Cameroon. I wish to thank the Classical Analysis group managed by Professor David Békollé, for the full discussion. We achieved the work when I was visiting the African Institute for Mathematics Sciences in South Africa. I wish to thank the mathematics group, and the Directors Professor Mama Foupoouagnigni and Professor Barry Green in particular, for very nice visits.

Our starting point are some preliminaries results that we will need in the proof of the main theorem.

## 2. Preliminaries

Let $n \geq 2$ and consider the nonsingular cone

$$\mathbb{H} := \{z \in \mathbb{C}^{n+1} : z_1^2 + \cdots + z_{n+1}^2 = 0, \ z \neq 0\}.$$ 

This is the orbit of the vector $(1, i, 0, \ldots, 0)$ under the $SO(n + 1, \mathbb{C})$-action on $\mathbb{C}^{n+1}$. It is well-known that $\mathbb{H}$ can be identified with the cotangent bundle of the unit sphere $S^n$ in the $n$–dimensional sphere in $\mathbb{R}^{n+1}$ minus its zero section. It was proved in [18] that there is a unique (up to a multiplicative constant) $SO(n + 1, \mathbb{C})$–invariant holomorphic form $\alpha$ on $\mathbb{H}$. The restriction of this form to $\mathbb{H} \cap (\mathbb{C}\{0\})^{n+1}$ is given by

$$\alpha(z) = \sum_{j=1}^{n+1} \frac{(-1)^{j-1}}{z_j} d z_1 \wedge \cdots \wedge \widehat{d z_j} \wedge \cdots \wedge d z_{n+1}.$$
The orthogonal group $O(n + 1, \mathbb{R})$ acts transitively on the boundary $\mathcal{M}$ of the unit ball in $\mathbb{H}$. Thus there is a unique $O(n + 1, \mathbb{R})$-invariant probability measure $\mu$ on $\mathcal{M}$. This measure is induced by the Haar probability measure of $O(n + 1, \mathbb{R})$. Following [13] (see Lemma 2.1 page 506), we have for any $C^n$ function $f$ on $\mathbb{H}$ that

$$\int_{\mathbb{H}} f(z)\alpha(z) \wedge \overline{\alpha(z)} = m_n \int_{0}^{\infty} r^{2n-3} \int_{\partial \mathbb{M}} f(r\xi) d\mu(\xi) dr$$

provided that the integrals makes sense. Here

$$m_n = 2(n-1) \int_{\{z \in \mathbb{H} : \|z\| < 1\}} \alpha(z) \wedge \overline{\alpha(z)}$$

For each $s > 0$ and $0 < p < \infty$, let $L^p_s(\mathbb{H})$ denote the Lebesgue spaces of all functions $f$ on $\mathbb{H}$ such as $f \in L^p(\mathbb{H}, w_{s,p})$. Here $w_{s,p}$ is the Gaussian volume form defined on $\mathbb{H}$ as

$$w_{s,p}(z) = \frac{(sp)^{n-1} e^{-sp\|z\|^2/2}}{2^{n-2}m_n(n-2)!}, \quad z \in \mathbb{H}.$$ 

In the following we adopt some notations. For any $f \in L^p_s(\mathbb{H})$ we write

$$\|f\|_{L^p_s(\mathbb{H})} = \left( \int_{\mathbb{H}} |f(z)|^p w_{s,p}(z) \right)^{\frac{1}{p}}, \quad 0 < p < \infty$$

and

$$\|f\|_{L^\infty_s(\mathbb{H})} = \sup_{z \in \mathbb{H}} |f(z)| e^{-s \|z\|^2/2}$$

The weighted Bergman space $A^p_s(\mathbb{H})$ is the closed subspace of $L^p_s(\mathbb{H})$ consisting of holomorphic functions. When $p = 2$, the orthogonal projection $P_s$ from $L^2_s(\mathbb{H})$ onto $A^2_s(\mathbb{H})$ is called the weighted Bergman projection. It is well-known that $P_s$ is the integral operator on $L^2_s(\mathbb{H})$ given by the formula

$$P_s f(z) = \int_{\mathbb{H}} K_{s,\mathbb{H}}(z, w) f(w) w_{s,2}(w)$$

where

$$K_{s,\mathbb{H}}(z, w) = (-1)^{n(n+1)/2} (2i)^n (1 + \frac{2s}{n-1} z \cdot \bar{w}) e^{z \cdot \bar{w}}$$

is the reproducing kernel on $A^2_s(\mathbb{H})$, see [3]. This is the weighted Bergman kernel. Let an operator $T_p$ be defined as follows

$$T_p f(z) = C(p)^{1/p} z_{n+1} f(z_1, \ldots, z_n), \quad z = (z_1, \ldots, z_{n+1}) \in \mathbb{H}$$

where $C(p) = \frac{2^{n-3} m_n(n-2)!}{(sp)^{n-1}(n+1)!}$. As in [13] (see page 163) the operator $T_p$ will play a key role in our proof.

**Lemma 2.1.** For any $s > 0$ and $0 < p < \infty$, the operator $T_p$ is an isometry from $L^p_s(\mathbb{C}^n)$ into $L^p_s(\mathbb{H})$. More precisely, we have

$$\|T_p f\|_{L^p_s(\mathbb{H})} = \|f\|_{p,s}$$

In addition, the image $E^p_s(\mathbb{H})$ of $A^p_s(\mathbb{H})$ under $T_p$ is a closed proper subspace of $A^p_s(\mathbb{H})$ and $T_p$ is a unitary operator from $A^p_s(\mathbb{H})$ onto $E^p_s(\mathbb{H})$.

The following result is a crucial ingredient to prove the main lemma of this paper.

**Lemma 2.2** (See [12]). Let an integer $m$. Then for every $R > 0$ there exists $A = A(R)$ such that for all $z \in \mathbb{C}^n$

$$\sup_{\zeta \in D_n(z)} \left| \frac{1}{2} - |z|^2 - h_z(\zeta) \right| \leq A$$

where $h_z$ is a harmonic function in $D_n(z) = \{ \zeta \in \mathbb{C}^n : |\zeta - z| < R \}$ with $h_z(z) = 0$. 

3. Intermediate results

In this section we set out an important result of the paper.

Theorem B. Suppose $s > 0$ and $0 < p \leq 1$. Then the dual of $A^p_s(H)$ is $A^\infty_s(H)$ under the duality pairing

$$< f, g >_s = \int_H f(z)g(z)\overline{e^{-s|z|^2}}w_{s,2}(z)$$

(3.1)

The starting point of the proof of theorem B is the estimate of the reproducing kernel $\tilde{K}_{s,H}$ of $E^2_s(H)$.

3.1. Estimate reproducing kernel.

Lemma 3.1. Suppose $s > 0$ and $0 < p \leq 1$. We have

$$\int_H |\tilde{K}_{s,H}(z,w)|^p w_{s,p}(w) \leq Ce^{s\|z^2\|^2/2}$$

(3.2)

where $C$ is a constant.

Proof. If $P_r : \mathbb{C}^{n+1} \to \mathbb{C}^n$ is defined by $P_r(z_1, \ldots, z_n, z_{n+1}) = (z_1, \ldots, z_n)$, and $F = Pr|_{H}$, then $F : H \to \mathbb{C}^n \setminus \{0\}$ is a proper holomorphic mapping of degree 2. We denote by $W$ the branching locus of $F$. The image $F(W)$ of $W$ under $F$ is an analytic subset of $\mathbb{C}^n \setminus \{0\}$. We set $V = F(W) \cup \{0\}$. The local inverse $\phi$ and $\varphi$ of $F$ are given for $z \in \mathbb{C}^n \setminus V$ by

$$\phi(z) = (z, i\sqrt{z \cdot z})$$
$$\varphi(z) = (z, -i\sqrt{z \cdot z})$$

so that

$$\tilde{K}_{s,H}(z,w) = \frac{K^1_{s,H}(z,w) - K^2_{s,H}(z,w)}{2}$$

where

$$K^1_{s,H}(z,w) = \frac{\overline{w_{n+1}}}{\phi_{n+1}(F(w))} K_{s,H}(z, \phi(F(w)))$$
$$K^2_{s,H}(z,w) = \frac{\overline{w_{n+1}}}{\phi_{n+1}(F(w))} K_{s,H}(z, A(\phi(F(w))))$$

and $A$ is the transformation defined on $\mathbb{C}^{n+1}$ by

$$A(z_1, \ldots, z_{n+1}) = (z_1, \ldots, z_n, -z_{n+1})$$

When $0 < p \leq 1$, note that

$$\int_H |\tilde{K}_{s,H}(z,w)|^p w_{s,p}(w) \leq C \int_H |K_{s,H}(z,w)|^p w_{s,p}(w).$$

So the desired inequality becomes

$$\int_H |K_{s,H}(z,w)|^p w_{s,p}(w) \leq Ce^{s\|z^2\|^2/2}.$$
In [13], Mengotti and Youssfi proved
\[ (3.3) \quad \int_{\mathbb{H}} p_k(z) (z \cdot \bar{\xi})^l d\mu(z) = \begin{cases} \frac{k!(n-1)!}{(k+n-2)!(2k+n-1)!} p_k(z) & \text{if } l = k \\ \text{else} & \end{cases} \]
where \( n, k \in \mathbb{N} \) and \( p_k \) is a homogeneous polynomial of degree \( k \) on \( \mathbb{H} \). Then binomial series expansion and (3.3) give that
\[
J_s(z) = \int_{\partial \mathbb{H}} \left( \sum_{k=0}^{+\infty} \frac{(ps)^k}{k!} (z \cdot \bar{w})^k \right)^2 w_{s,p}(w)
\]
\[
\approx \sum_{k=0}^{+\infty} (ps)^{2k} \int_0^{+\infty} e^{2k+2n-3} e^{-psr/2} dr \int_{\partial \mathbb{M}} |z \cdot \bar{\xi}|^{2k} d\mu(\xi)
\]
\[
\approx \sum_{k=0}^{+\infty} \frac{(ps)^{k-n+1} (n+k-2)!}{2k-n+2(k!)^2} \int_{\partial \mathbb{M}} |z \cdot \bar{\xi}|^{2k} d\mu(\xi)
\]
\[
\approx \sum_{k=0}^{+\infty} \frac{(ps)^{k-n} (n+k-2)!}{k!(2k+n-1)} \int_{\partial \mathbb{M}} |z \cdot \bar{\xi}|^{2k} d\mu(\xi)
\]
\[
\approx \frac{e^{psr/2}}{1 + 2ps \|z\|^2}.
\]
Similarly,
\[
I_s(z) \leq C \int_0^{+\infty} \left[ 1 + \left( \frac{2sr}{n-1} \right) \right] \left( \frac{\|z\|^2}{k!} \right)^2 \sum_{k=0}^{+\infty} \frac{(ps)^{k}}{k!} \left( z \cdot \bar{\xi} \right)^k \left( z \cdot \bar{w} \right)^k \left( \frac{\|z\|^2}{k!} \right)^2 \int_{\partial \mathbb{M}} \left( z \cdot \bar{\xi} \right)^{2k} d\mu(\xi)
\]
\[
\leq C \sum_{k=0}^{+\infty} \frac{(ps/2)^{2k} \|z\|^2}{k!(k+n-2)!(2k+n-1)} \left( 1 + \left( \frac{2sr}{n-1} \right) \right)^{2+\infty} \int_0^{+\infty} \left( \frac{2sr \|z\|^2}{n-1} \right) \left( 2sr \|z\|^2 \right)^{2k} e^{2k+2n-3} e^{-psr/2} dr
\]
\[
\leq C \sum_{k=0}^{+\infty} \frac{(2/s)^{k+n-2} (ps/2)^{2k} \|z\|^2}{k!(k+n-2)!(2k+n-1)} \left( k+n-2 \right)! \left( 2sr \|z\|^2 \right)^{2p} \Gamma(k+p+n-1)
\]
\[
\leq C(1 + 2ps \|z\|^2) e^{ps \|z\|^2/2}.
\]
Now using Hölder inequality we obtain that
\[
\int_{\mathbb{H}} |K_s(z, w)|^p w_{s,p}(w) \leq \sqrt{I_s(z)} J_s(z) \leq C e^{sp \|z\|^2/2}.
This completes the proof of lemma.

3.2. Pointwise estimates. In this section we give the natural growth of functions in $F^p_s$.

Lemma 3.2. For any holomorphic function $F$ on $\mathbb{H}$ and $z \in \mathbb{H}$ we have

$$|F(z)| \leq C \int_{D_{n+1}(z) \cap \mathbb{H}} |F(\zeta)|\alpha(\zeta) \wedge \overline{\alpha(\zeta)}$$

where $C$ is a constant independent on $z$.

Proof. $F$ being holomorphic then $\Delta F(\zeta) = 0$ where $\Delta = \sum_{i=1}^{n+1} \frac{\partial^2}{\partial \zeta_i \partial \overline{\zeta}_i}$. The divergence theorem implies that

$$0 = \int_{D_{n+1}(z) \cap \mathbb{H}} \Delta F(\zeta) dA(\zeta) = \int_{\partial(D_{n+1}(z) \cap \mathbb{H})} \frac{\partial F}{\partial \nu}(\zeta) d\mu(\zeta)$$
$$= \int_{\partial(D_{n+1}(0) \cap \mathbb{H})} \frac{\partial F}{\partial r}(z + r\xi) d\mu(\xi)$$
$$= \frac{\partial}{\partial r} \int_{\partial(D_{n+1}(0) \cap \mathbb{H})} F(z + r\xi) d\mu(\xi)$$

where $\frac{\partial}{\partial \nu}$ is the differentiation in the direction of the external normal. Since the mean value integral at $r = 0$ is equal to $F(z)$ then

$$\tag{3.4} F(z) = \int_{\partial(D_{n+1}(0) \cap \mathbb{H})} F(z + r\xi) d\mu(\xi)$$

The proof of the lemma arises from (3.4) and (2.1).

Lemma 3.3. Let $0 < p < \infty$. Then for any holomorphic function $f$ on $\mathbb{C}^n$ we have

$$\|f\|_{\infty,s} \leq C \|f\|_{p,s}$$

where $C$ is a constant.

Proof. Let $H_z$ a holomorphic function on $\mathbb{H}$ such that $h_z = \Re H_z$. Then

$$|T_p f(z)e^{-s|z|^2}|^p = |T_p f(z)e^{-sH_z(z)}| e^{-sp|z|^2}$$

From Lemma 3.2 we have that

$$|T_p f(z)e^{-s|z|^2}|^p \leq C \int_{D_{n+1}(z) \cap \mathbb{H}} |T_p f(w)e^{-sH_z(w)-s|z|^2}| \alpha(w) \wedge \overline{\alpha(w)}$$

Also, from Lemma 2.2 we obtain that.

$$\tag{3.5} |T_p f(z)e^{-s|z|^2}|^p \leq C \int_{\mathbb{H}} |T_p f(w)e^{-s|w|^2}| \alpha(w) \wedge \overline{\alpha(w)}$$

for all $z \in \mathbb{H}$. Finally the lemma arises from (3.5) combined with the estimate (2.2) at the point $(z,i\sqrt{z \cdot z}) \in \mathbb{H}$ where $z \in \mathbb{C}^n$. □
3.3. Inclusion.

Lemma 3.4. Suppose $0 < p \leq 1$. Then $\mathcal{E}_p^s(\mathbb{H}) \subset \mathcal{E}_s^1(\mathbb{H})$ and the inclusion is continuous.

Proof. The starting of the proof is the embedding $F_p^s \subset F_s^1$. Because the desired embedding follows by using the isometric $T_p$. So for any $f \in F_p^s$ the Lemma 3.3 yields

$$\|f\|_{s,1} \leq \int_{\mathbb{C}^n} |f(z)| e^{-sN^2(z)/2} |z \cdot z|^{-\frac{n}{2}} dA(z)$$

$$\leq \int_{\mathbb{C}^n} |f(z)|^p e^{-sN^2(z)/2} |z \cdot z|^{-\frac{n}{2}} dA(z)$$

$$\leq C \int_{\mathbb{C}^n} |f(z)|^p e^{-sN^2(z)/2} |z \cdot z|^{-\frac{n}{2}} \|g\|_{p,s}^{1-p} e^{-sN^2(z)/2} |z \cdot z|^{-\frac{n}{2}} dA(z)$$

$$\leq C \|f\|_{p,s}$$

This proves the desired embedding. \(\square\)

4. Proof of the theorem B

Proof of the theorem B. Consider the bilinear form $L$ defined on $\mathcal{E}_p^s(\mathbb{H}) \times \mathcal{E}_s^\infty(\mathbb{H})$ by

$$(f, g) \rightarrow Lg(f) := L(f, g) = \int_{\mathbb{H}} f(z) \overline{g(z)} w_{2,s}(z).$$

This mapping is well-defined. Namely, the Lemma 3.4 gives that

$$|Lg(f)| \leq \|f\|_{\mathcal{E}_p^s(\mathbb{H})} \|g\|_{\mathcal{E}_s^\infty(\mathbb{H})}$$

$$\leq \|f\|_{\mathcal{E}_p^s(\mathbb{H})} \|g\|_{\mathcal{E}_s^\infty(\mathbb{H})}$$

for all $(f, g) \in \mathcal{E}_p^s(\mathbb{H}) \times \mathcal{E}_s^\infty(\mathbb{H})$. Conversely, if $G$ is a bounded linear functional on $\mathcal{E}_p^s(\mathbb{H})$ we must find $g \in \mathcal{E}_s^\infty(\mathbb{H})$ verifying

$$G(f) = \int_{\mathbb{H}} f(z) \overline{g(z)} w_{2,s}(z)$$

for all $f \in \mathcal{E}_p^s(\mathbb{H})$. For this goal we choose

$$g(w) = G(\tilde{K}_s(w, z)), \ w \in \mathbb{H}$$

where $\tilde{K}_s$ is the Bergman kernel of $\mathcal{E}_s^2(\mathbb{H})$. Let us prove that $g$ is the desired function. First, we observe that

$$|g(w)| \leq \|G\| \|\tilde{K}_s(\cdot, w)\|_{\mathcal{E}_p^s(\mathbb{H})}$$

$$\leq C \|g\| e^{\|w\|^2/2}$$

and thus $\|g\|_{\mathcal{E}_s^\infty(\mathbb{H})} \leq C \|G\|$. Second, let us show the following.

$$G(f) = \int_{\mathbb{H}} f(z) \overline{g(z)} w_{2,s}(z)$$

for all $f \in \mathcal{E}_p^s(\mathbb{H})$. To do that we can observe by reproducing property that (4.1) is true for $f(z) = \tilde{K}_s(z, w)$. Moreover the set of all finite linear combinations of reproducing kernel functions being dense in $\mathcal{E}_p^s(\mathbb{H})$ and $\mathcal{E}_p^s(\mathbb{H}) \subset \mathcal{E}_s^1(\mathbb{H}) \subset \mathcal{E}_s^2(\mathbb{H})$ then (4.1) is true. This completes the proof of the theorem B. \(\square\)
5. Proof of the main theorem

Proof. Consider the bilinear form \( L \) defined on \( F^p \times F^\infty_s \) by

\[
(f, g) \rightarrow Lg(f) := L(f, g) = \int_{\mathcal{H}} T_p f(z) \overline{T_g(z)} w_{2,s}(z)
\]

where \( T_g(z_1, \ldots, z_{n+1}) = z_{n+1} g(z_1, \ldots, z_n) \). The functional \( L \) is well-defined. Indeed Lemma 3.7 yields

\[
|L_g(f)| \leq C \|T_p f\|_{\mathcal{E}^s_{s+1}(\mathcal{H})} \|T_g\|_{\mathcal{E}^\infty_{s+1}(\mathcal{H})} \\
\leq C \|T_p f\|_{\mathcal{E}^s_{s+1}(\mathcal{H})} \|T_g\|_{\mathcal{E}^\infty_{s+1}(\mathcal{H})} \\
\leq C \|f\|_{p,s} \|g\|_{p,s}
\]

for all \((f, g) \in F^p \times F^\infty_s\). Conversely, if \( G \) is a bounded linear functional on \( F^p \), then \( G \circ T_p^{-1} \) is in the dual space of \( \mathcal{E}^s_{p}(\mathcal{H}) \). Hence, from Theorem 3.3 there exits \( \tilde{h} \in \mathcal{E}^\infty_{s+1}(\mathcal{H}) \) such that

\[
G \circ T_p^{-1}(\tilde{h}) = \int_{\mathcal{H}} \tilde{h}(z) \overline{T_g(z)} w_{2,s}(z)
\]

for all \( \tilde{h} \in \mathcal{E}^p_{s}(\mathcal{H}) \). Finally for \( g = T^{-1} h \) we get that,

\[
G(f) = G \circ T_p^{-1}(T_p f) = \int_{\mathcal{H}} T_p f(z) \overline{T_g(z)} w_{2,s}(z)
\]

for all \( f \in F^p \), This completes the proof of the main theorem. \( \square \)

References

[1] V. Azarin, Growth theory of subharmonic, Springer, Basel, Boston, Berlin, 2009.
[2] M. Christ, On the \( \partial \) equation in weighted \( L^2 \) norms in \( C^1 \), The Journal of Geometric Analysis. 1(3) (1991), 193-230.
[3] R. Coifman and R. Rochberg, Representation theorems for holomorphic and harmonic functions in \( L^p \), Astérisque. 77 (1980), 11-66.
[4] H. R. Cho and K. Zhu, Fock Sobolev spaces and their Carleson measures, Journal of Functional Analysis. 263 (2012) 24832506.
[5] P. Duren, B. Romberg, and A. Shields, Linear functionals on \( H^p \) spaces with \( 0 < p < 1 \), J. Reine Angew. Math. 238 (1969), 52-60.
[6] A. Frazier, The dual of \( H^p \) of the polydisk for \( 0 < p < 1 \), Duke Math. J 39 (1972), 369-379.
[7] J. Gonessa, Duality of Fock spaces with respect to the minimal norm, Archiv der Mathematik. 100 (2013), 439-447.
[8] J. Gonessa and E. H. Youssfi, The Bergman projection in spaces of entire functions, Ann. Polon. Math. 104 (2012), 161-174.
[9] K. Hahn and J. Mitchell, Representation of linear functionals in \( H^p \) spaces of bounded symmetric domains, J. Math. Anal. Appl. 56 (1976), 379-396.
[10] K. T. Hahn and P. Pflug, On a minimal complex norm that extends the real Euclidean norm, Monatsch. Math. 108 (1998), 107-112.
[11] S. Jason, J. Peret, and R. Rochberg, Hankel forms and the Fock space, Rev. Math. Iber. 3 (1987), 161-174.
[12] N. Marco, X. Massaneda and J. Ortega-Cerdà, Interpolating and sampling sequences for entire functions, Geometric and Functional Analysis. 13 (2003), 862-914.
[13] G. Mengotti and E. H. Youssfi, The weighted Bergman projection and related theory on the minimal ball and applications, Bull. sci. Math. 123 (1999), 501-525.
[14] G. Mengotti, Duality theorems for certain analytic spaces on the minimal ball, Arch. Math. (Bassel) 75 (2000), 389-394.
[15] G. Mengotti, The Bloch space for minimal ball, Studia. Math. 148 (2) (2001), 131-142.
[16] K. Oelijkaus, P. Pflug and E. H. Youssfi, proper holomorphic mappings and related automorphism groups, J. geom. Anal. 7(4) (1997), 623-636.
[17] J. Ortega-Cerdà and K. Seip, Beurling-type density theorems for weighted $L^p$ spaces of entire functions, J. Anal. Math. 75 (1998), 247-266.
[18] P. Pflug and E. H. Youssfi, The Bergman kernel of the minimal ball and applications, Ann. Inst Fourier (Grenoble) 47 (1997), 915-928.
[19] B. Romberg, The $H^p$ spaces with $0 < p < 1$, Doctoral dissertation, University of Rochester, 1960.
[20] J. Siciak, Extremal plurisubharmonic functions in $\mathbb{C}^n$, Ann. Pol. Math. 39 (1981), 175-211.
[21] J. Shapiro, reproducing kernels, and diagonal maps on the Hardy and Bergman spaces, Duke Math. J 43 (1976), 187-202.
[22] Y. Tung, Fock spaces, Ph.D. dissertation, University of Michigan (2005).
[23] K. Zhu, Analysis on Fock Spaces, Springer, GTM 263, New York, (2012).
[24] K. Zhu, Bergman and Hardy spaces with small exponents, Pacific J. Math. 162 (1994), 189-199.
[25] K. Zhu, Spaces of Holomorphic Functions in the Unit Ball, Springer, New York, 2005.
[26] K. Zhu, Operator Theory in Function Spaces, American Mathematical Society, 2007.

Gonessa: Université de Bangui, Département de mathématiques et Informatique, BP.908 Bangui-
République Centrafricaine
E-mail address: gonessa.jocelyn@gmail.com
E-mail address: jocelyn@aims.ac.za