PARABOLIC EQUATIONS INVOLVING LAGUERRE OPERATORS AND WEIGHTED MIXED-NORM ESTIMATES

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ABSTRACT. In this paper, we study evolution equation \( \partial_t u = -L_\alpha u + f \) and the corresponding Cauchy problem, where \( L_\alpha \) represents the Laguerre operator \( L_\alpha = \frac{1}{2} \left( -\frac{d^2}{dx^2} + x^2 + \frac{1}{x^2} (\alpha^2 - \frac{1}{4}) \right) \), for every \( \alpha \geq -\frac{1}{2} \). We get explicit pointwise formulas for the classical solution and its derivatives by virtue of the parabolic heat-diffusion semigroup \( \{e^{-\tau(\partial_t + L_\alpha)}\}_{\tau > 0} \). In addition, we define the Poisson operator related to the fractional power \( (\partial_t + L_\alpha)^s \) and reveal weighted mixed-norm estimates for relevant maximal operators.

1. Introduction. The generalized theory of singular integrals was initiated by Calderón and Zygmund \([7, 8]\), which generalize many results concerning the classical Hilbert transform to \( n \)-dimensional space. It is well known that the results have been widely applied to partial differential equations, due to the fact that certain derivatives of fundamental solutions of certain equations satisfy the conditions required by Calderón and Zygmund for kernels.

Sequently, Jones \([13]\) applied the theory of Calderón and Zygmund to the parabolic equation \( \partial_t u = \Delta u + f \) on \((0, \infty) \times \mathbb{R}^n\), obtained explicit pointwise formulas for the classical solution and its derivatives, and proved Sobolev estimates. In particular, Krylov \([14, 15]\) presented mixed norm estimates \( L^p_t(W^{2,p}_x) \) and \( L^p_t(C^{2,\alpha}_x) \) for parabolic equations by using parabolic Calderón-Zygmund theory. Recently, Li, Stinga and Torrea \([19]\) considered the heat equation \( \partial_t u = \Delta u + f \) and the harmonic oscillator evolution equation \( \partial_t u = \Delta u - |x|^2 u + f \) on the whole space \( \mathbb{R}^{n+1} \). By using the method of the parabolic semigroups, they obtained explicit pointwise formulas for the classical solution and its derivatives, and proved weighted mixed-norm Sobolev estimates, weighted mixed weak \( L^1 \)–estimates and a.e. convergence results of singular integrals associated to the parabolic operator \( \partial_t - \Delta \) and \( \partial_t - \Delta + |x|^2 \) in the Hermite setting. There are other numerous partial results and references therein studying parabolic equations by using harmonic analysis techniques \([9, 11, 18]\).

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On the other hand, there exists a close connection between the Hermite and Laguerre expansions. The harmonic analysis relating Hermite and Laguerre expansions first appeared in Muckenhoupt’s celebrated works [16, 17]. In [1, 12], the relation between Hermite and Laguerre was systematically used as a tool for studying operators associated to Laguerre polynomial expansions. Harmonic analysis associated with Laguerre operator (Riesz transform, maximal operators, oscillation and variation, Littlewood-Paley functions, square functions, multipliers,...) has been developed in the last years [1, 2, 12, 21].

Inspired by above, we will study the Laguerre parabolic equations

\[ \partial_t u(t, x) = -L_\alpha u(t, x) + f(t, x), \]

where \( L_\alpha \) is the Laguerre operator,

\[ L_\alpha = \frac{1}{2} \left( -\frac{d^2}{dx^2} + x^2 + \frac{1}{x^2}(\alpha^2 - \frac{1}{4}) \right), \]

for every \( \alpha \geq \frac{1}{2} \),

with \((t, x) \in (0, \infty) \times (0, \infty)\).

The first purpose of this paper is to study the classical solution of the evolution equation \( \partial_t u = -L_\alpha u + f \) and the corresponding Cauchy problem. As in other cases, we obtain explicit formulas for the classical solution and its derivatives. Because the heat-diffusion kernel relevant to the Laguerre operator contains Bessel function, the formula is more complicated, which causes considerable difficulties.

Our results concerning to the solution of (1.1) in the whole space \( \mathbb{R} \times (0, \infty) \) are the following.

**Theorem 1.1.** Suppose that \( f \in L^\infty(\mathbb{R} \times (0, \infty)) \) has compact support on \( \mathbb{R} \times (0, \infty) \). Then, the function \( u(t, x), (t, x) \in (0, \infty) \times (0, \infty) \), given by

\[ u(t, x) = \int_0^\infty \int_0^\infty W_\tau^{L_\alpha}(x, y) f(t - \tau, y) dy d\tau, \quad (t, x) \in (0, \infty) \times (0, \infty), \]

is well defined by an absolutely convergent integral, for every \((t, x) \in (0, \infty) \times (0, \infty)\). Moreover, if \( f \) is also \( C^2(\mathbb{R} \times (0, \infty)) \), then for every \((t, x) \in \mathbb{R} \times (0, \infty)\), \( u \) satisfies \( \partial_t u(t, x) = -L_\alpha u(t, x) + f(t, x) \). In this case, the following pointwise limits hold:

\[ \partial_t u(t, x) = \lim_{\epsilon \to 0^+} \int_0^\infty \int_0^\infty \partial_\tau W_\tau^{L_\alpha}(x, y) f(t - \tau, y) dy d\tau + f(t, x) \]

\[ = \lim_{\epsilon \to 0^+} \int_{\Omega_\epsilon(x)} \partial_\tau W_\tau^{L_\alpha}(x, y) f(t - \tau, y) dy d\tau + Af(t, x), \]

and

\[ \partial^2_t u(t, x) = \lim_{\epsilon \to 0^+} \int_0^\infty \int_0^\infty \partial^2_\tau W_\tau^{L_\alpha}(x, y) f(t - \tau, y) dy d\tau \]

\[ = \lim_{\epsilon \to 0^+} \int_{\Omega_\epsilon(x)} \partial^2_\tau W_\tau^{L_\alpha}(x, y) f(t - \tau, y) dy d\tau - (1 - A)f(t, x), \]

where, for every \( \epsilon, \tau \in (0, \infty) \), \( \Omega_\epsilon(x) = \{ (\tau, y) \in (0, \infty)^2 : \max(\tau^{\frac{1}{2}}, |x - y|) > \epsilon \} \), and \( A = \frac{1}{\sqrt{2\pi}} \int_0^1 e^{-\frac{x^2}{2w}} dw \), \( W_\tau^{L_\alpha}(x, y) \) are the heat kernel that will be defined precisely in Section 2.

We now consider the following Cauchy problem associated with (1.1):

\[ \begin{cases}
\partial_t u(t, x) = -L_\alpha u(t, x) + f(t, x), & t, x \in (0, \infty) \times (0, \infty) \\
 u(0, x) = g(x), & x \in (0, \infty)
\end{cases} \]

(1.3)
Theorem 1.2. Let $\alpha \geq -\frac{1}{2}$. Assume that $f \in L^\infty((0, \infty) \times (0, \infty))$ has compact support and $g \in L^\infty((0, \infty))$ with compact support. We define
\[
u(t, x) = \int_0^t \int_0^\infty W_t^{\alpha}(x, y)f(t - \tau, y)dyd\tau + \int_0^\infty W_t^{\alpha}(x, y)g(y)dy
\]where $t, x \in (0, \infty)$. Then, the last integral are absolutely convergent for every $(t, x) \in (0, \infty) \times (0, \infty)$. Moreover, if $f$ is also $C^2((0, \infty) \times (0, \infty))$, then the function $u$ defined by (1.4) is a classical solution of (1.3). And in this case, the following limits hold:
\[
\partial_t u(t, x) = \lim_{\epsilon \to 0^+} \int_0^{t - \epsilon} \int_0^\infty \partial_t W_t^{\alpha}(x, y)f(t - \tau, y)dyd\tau + \int_0^\infty \partial_t W_t^{\alpha}(x, y)g(y)dy
\]
and
\[
\partial^2_{tt} u(t, x) = \lim_{\epsilon \to 0^+} \int_0^{t - \epsilon} \int_0^\infty \partial^2_{tt} W_t^{\alpha}(x, y)f(t - \tau, y)dyd\tau + \int_0^\infty \partial^2_{tt} W_t^{\alpha}(x, y)g(y)dy.
\]

The second main result we obtain is weighted mixed norm $L^q(\mathbb{R}, v; L^p((0, \infty), w))$ estimates for maximal operator associated to parabolic Poisson operator, which is one kind of parabolic singular integrals. The operator we consider arises in applications to fractional nonlocal space-time equation and the master equation. The work of Stinga and Torrea [24] made a pretty deep analysis of solutions $u$ to fractional nonlocal equation $(\partial_t - \Delta)^s u(t, x) = f(t, x)$ on $\mathbb{R}^{n+1}$, for $0 < s < 1$. The fractional powers of the heat operator $(\partial_t - \Delta)^s$, $0 < s < 1$, can be characterized by a degenerate parabolic extension problem in one more dimension. The authors in [4] made a similar analysis. Indeed, the above problems are inspired by the fundamental works of Caffarelli and Silvestre (especially, see [5]), which is developed in [6, 23]. In our case, let $H = \partial_t + L_\alpha$, $u = u(t, x)$ be a (smooth) bounded function on $\mathbb{R}^{1+1}$. For $(t, x) \in \mathbb{R} \times (0, \infty)$, $y > 0$, and $0 < s < 1$, we define the Poisson operator related to the fractional power $(\partial_t + L_\alpha)^s$ as
\[
U(t, x, y) = \int_0^\infty \frac{y^{2s}}{4s^2} \int_0^\infty \int_0^\infty e^{-\frac{y^2}{4} e^{-\tau H} u(t, x)} d\tau \frac{d\tau}{\tau^{1+s}}.
\]
Moreover, $U$ is a solution to
\[
\begin{cases}
(HU, v) = (1 - 2s) \partial_y v + \partial_y y u, & \text{for each } v \in \text{Dom}(H) \text{ and } y > 0,
\lim_{y \to 0^+} U(t, x, y) = u(t, x), & \text{if } u(t, x) \in L^2(\mathbb{R}^{1+1})
\end{cases}
\]
which can be written in divergence form as
\[
\begin{cases}
\partial_t U = y^{1-2s} \text{div}_{x,y}(y^{1-2s} B_{x,y} \nabla_{x,y} U) - \frac{1}{2}(\alpha^2 \frac{4}{x^2} + x^2) U
\lim_{y \to 0^+} U(t, x, y) = u(t, x)
\end{cases}
\]
where $B(x) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$. Then, for every $(t, x) \in (0, \infty) \times (0, \infty)$,
\[- \lim_{y \to 0^+} y^{1-2s} U(t, x, y) = c_\alpha \left( \partial_t + \frac{1}{2} \left( - \frac{d^2}{dx^2} + x^2 + \frac{1}{x^2} (\alpha^2 - \frac{1}{4}) \right) \right)^s u(t, x).
\]
For all these details see [4]. Define the maximal operator as
\[
P_{L_\alpha}^s u(t, x) = \sup_{y > 0} \left| P_{L_\alpha}^y u(t, x) \right|, \quad \text{for } (t, x) \in \mathbb{R}^{1+1}.
\]

As it is well known, these maximal operators play an important role in understanding convergence of the solutions $U$ to the initial data $u$ or, in an equivalent way
as evidenced by the extension problem (1.6), in solving the fractional space-time nonlocal equations above.

**Theorem 1.3. (Mixed-norm estimates with weights for extensions).** Suppose that $u \in L^q(\mathbb{R}, v; L^p((0, \infty), w))$ for $1 \leq p, q < \infty$, where $v \in A_p(\mathbb{R})$ and $w \in A_p((0, \infty))$. If $1 < p, q < \infty$ then

$$\|P_{L_\alpha}^{s,t}u\|_{L^q(\mathbb{R}, v; L^p((0, \infty), w))} \leq C_{p,q,v,w}\|u\|_{L^s(\mathbb{R}, v; L^p((0, \infty), w))}.$$  

If $q = 1$ and $1 < p < \infty$ then a weak-type estimate holds: for any $\lambda > 0$,

$$v(\{t \in \mathbb{R} : \|P_{L_\alpha}^{s,t}u(t, \cdot)\|_{L^p((0, \infty), w)} > \lambda\}) \leq \frac{C_{p,v,w}}{\lambda}\|u\|_{L^1(\mathbb{R}, v; L^p((0, \infty), w))}.$$  

**Remark 1.** A weight $v = v(t) \in A_p(\mathbb{R})$, $1 \leq p < \infty$, also satisfies the definition of Muckenhoupt condition by changing the distance $|t - s|$ by the distance $|t - s|^{\frac{1}{2}}$. Consequently, if $v = v(t) \in A_p(\mathbb{R})$ and $w = w(x) \in A_p(\mathbb{R})$, then the tensor product weight $w(t, x) := v(t)w(x) \in A_p^2(\mathbb{R}^{1+1})$, for any $1 \leq p < \infty$.

**Remark 2.** Along this paper we shall use the vector-valued Calderón-Zygmund theory in the spaces of homogeneous type. If $m$ denotes the Lebesgue measure on $\mathbb{R} \times (0, \infty)$ and $d$ represents the parabolic metric defined by $d((t, x), (\tau, y)) = \max(|t - \tau|^2, |x - y|)$, $t, \tau \in \mathbb{R}$ and $x, y \in (0, \infty)$, the triple $(\mathbb{R} \times (0, \infty), m, d)$ is a space of homogeneous type in the sense of Coifman and Weiss [10]. We represent, for every $1 \leq p < \infty$, by $A_p^2(\mathbb{R} \times (0, \infty))$ the class of Muckenhoupt weights in the space of homogeneous type $(\mathbb{R} \times (0, \infty), m, d)$.

In the next sections we will prove our theorems. In order to show our results, we use two different ways. Apart from vector-valued Calderón-Zygmund theory in the parabolic context, we also use a comparative approach, where we take advantage of the relation between the Hermite and Laguerre expansions mentioned above. Consequently, it is possible to deduce the properties of our integral operators from the corresponding ones associated to the Hermite operator established in [19].

We will also make a frequent use of the fact that, for every positive $A$ and every nonnegative $a$,

$$\sup_{t > 0} t^a e^{-tA} < \infty.$$  

(1.7)

We employ the letter $C$ to denote by any constant that can be explicitly computed in terms of known quantities. The exact value denoted by $C$ may therefore change from line to line in a given computation.

2. Preliminaries. We consider the Laguerre differential operator $L_\alpha$ defined by

$$L_\alpha = \frac{1}{2} \left( -\frac{d^2}{dx^2} + x^2 + \frac{1}{x^2}(\alpha^2 - \frac{1}{4}) \right), \quad x \in (0, \infty),$$

(2.1)

where $\alpha \geq -\frac{1}{2}$. This operator is self-adjoint in $L^2((0, \infty), dx) \cap C^2_0(0, \infty)$ and the Laguerre functions $\{\varphi_n^\alpha\}_{n \in \mathbb{N}}$ are eigenfunctions of this operator in the following sense: for every $n \in \mathbb{N}$,

$$L_\alpha \varphi_n^\alpha = (2n + \alpha + 1)\varphi_n^\alpha,$$

where

$$\varphi_n^\alpha(x) = \frac{\Gamma(n+1)}{\Gamma(n+1+\alpha)} \left( \frac{\Gamma(n+1+\alpha)}{\Gamma(n+1+\alpha)} \right)^{\frac{1}{2}} e^{-\frac{x}{2}} x^\alpha L_n^\alpha(x^2)(2x)^{\frac{1}{2}}; \quad x \in (0, \infty).$$
Here, for every \( n \in \mathbb{N} \), \( L_n^\alpha \) denotes the Laguerre polynomial of type \( \alpha \). Note that \( \{\varphi_n^\alpha\}_{n \in \mathbb{N}} \) is an orthonormal basis of \( L^2((0, \infty), dx) \). More details see [26].

The heat semigroup associated with \( \{\varphi_n^\alpha\}_{n \in \mathbb{N}}, \{e^{-tL_n^\alpha}\}_{t \geq 0} \), with infinitesimal generator \(-L_\alpha\), that we will denote by \( \{W_t^{L_\alpha}\}_{t \geq 0} \) is given in the spectral sense as

\[
W_t^{L_\alpha}(f) = \sum_{n=0}^{\infty} e^{-(2n+\alpha+1)t} c_n^\alpha \varphi_n^\alpha, \quad t > 0
\]

where

\[
c_n^\alpha(f) = \int_0^\infty \varphi_n^\alpha(x)f(x)dx, \quad n \in \mathbb{N}
\]

for \( f \in L^2((0, \infty), dx) \). Using the Mehler formula for Laguerre functions \( \{\varphi_n^\alpha\}_{n \in \mathbb{N}} \) we obtain, for every \( t > 0 \), the following integral representation of \( W_t^{L_\alpha}(f) \) against a kernel when \( f \in L^2((0, \infty), dx) \):

\[
W_t^{L_\alpha}(f)(x) = \int_0^\infty W_t^{L_\alpha}(x,y)f(y)dy, \quad x \in (0, \infty),
\]

in which, for \( x, y, t \in (0, \infty) \),

\[
W_t^{L_\alpha}(x,y) = 2(xy)^{\frac{\alpha}{2}} \frac{e^{-t}}{1-e^{-2t}} I_\alpha \left( \frac{2xye^{-t}}{1-e^{-2t}} \right) e^{-\frac{t}{2}(x^2+y^2)\frac{1+e^{-2t}}{1-e^{-2t}}} \tag{2.3}
\]

where \( I_\alpha \) denotes the modified Bessel function of the first kind and of order \( \alpha \) [25].

A direct calculations, the details of which we omit, verifies

\[
0 \leq W_t^{L_\alpha}(x,y) \leq C \frac{e^{-\frac{(x-y)^2}{t}}}{\sqrt{t}}, \quad t, x, y \in (0, \infty). \tag{2.4}
\]

And as the operator \( \partial_t \) and \( L_\alpha \) commute, the semigroup \( \{e^{-\tau(\partial_t+L_\alpha)}\}_{\tau \geq 0} \) is given by the composition \( e^{-\tau(\partial_t+L_\alpha)} = e^{-\tau \partial_t} \circ e^{-\tau L_\alpha} \). In particular, for smooth functions \( f(t, x) \) with rapid decay at infinity we have

\[
e^{-\tau(\partial_t+L_\alpha)}f(t, x) = e^{-\tau L_\alpha}f(t - \tau, x) = \int_0^\infty W_{\tau}^{L_\alpha}(x,y)f(t - \tau, y)dy.
\]

In this paper we are going to quote some properties of modified Bessel function \( I_\alpha(z) \) we need, which can be found in [16]. Let us present some basic facts about them now. For \( \alpha \geq -\frac{1}{2} \), the modified Bessel function \( I_\alpha \) is defined by

\[
I_\alpha(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{\alpha+2k}}{\Gamma(k+1)\Gamma(k+\alpha+1)}, \quad |z| < \infty, \quad |\arg z| < \pi.
\]

The following properties hold

\[
I_\alpha(z) \sim z^\alpha, \quad z \to 0^+;
\]

for \( \alpha \geq -\frac{1}{2} \) and every \( z \in (0, \infty) \),

\[
\sqrt{z}I_\alpha(z) = \frac{e^z}{\sqrt{2\pi}} \left( 1 + O\left( \frac{1}{z} \right) \right), \tag{2.6}
\]

and

\[
\frac{d}{dz}(z^{-\alpha}I_\alpha(z)) = z^{-\alpha}I_{\alpha+1}(z), \quad z \in (0, \infty). \tag{2.7}
\]
3. **Proof of Theorem 1.1.** Let us start with a technical lemma which will be used later.

**Lemma 3.1.** [22, Lemma 1.1] Let $B > 0$. If $b \geq \max\{2(a-1),0\}$, excluding the case when $b = 2(a-1) = 0$, then

$$
\int_0^1 s^{-\alpha} \exp \left( -\frac{B T^2}{s} \right) ds \leq C_{\alpha,b,B} T^{-b}, \quad T > 0.
$$

**Lemma 3.2.** Let $\alpha \geq -\frac{1}{2}$. Then,

$$
\int_0^\infty |\partial_x W_t^{L_\alpha}(x,y)| dt \leq C y^{\alpha+\frac{1}{2}} x^{\alpha+\frac{1}{2}}, \quad 0 < y < \frac{x}{2}, \tag{3.1}
$$
$$
\int_0^\infty |\partial_x W_t^{L_\alpha}(x,y)| dt \leq C x^{\alpha+\frac{1}{2}} y^{\alpha+\frac{1}{2}}, \quad 2x < y < \infty, \tag{3.2}
$$
$$
\int_0^\infty |\partial_x (W_t^{L_\alpha}(x,y) - W_t(x,y))| dt \leq \frac{C}{x} (1 + \sqrt{\frac{x}{|x-y|}}) \frac{x}{2} < y < 2x, \tag{3.3}
$$

where

$$
W_t(x,y) = \left( \frac{e^{-t}}{\pi(1 - e^{-2t})} \right)^{\frac{1}{2}} e^{-\frac{1}{2}(x^2 + y^2) \frac{1 + e^{-2t}}{1 - e^{-2t}} + \frac{2e^{-t}}{1 - e^{-2t}}}, \quad t \in (0, \infty), \quad x, y \in \mathbb{R}, 
$$

is the Mehler kernel of the heat semigroup generated by the harmonic operator $H = \frac{1}{2} \left( -\frac{d^2}{dx^2} + x^2 \right)$ on $\mathbb{R}$.

**Proof.** By virtue of (2.3) and the property of modified Bessel function (2.7), for $t, x, y \in (0, \infty)$, we obtain

$$
\partial_x W_t^{L_\alpha}(x,y) = \partial_x \left[ \left( \frac{2e^{-t}}{1 - e^{-2t}} \right)^{\frac{1}{2}} \left( \frac{2xye^{-t}}{1 - e^{-2t}} \right)^{\alpha+\frac{1}{2}} e^{-\frac{1}{2}(x^2 + y^2) \frac{1 + e^{-2t}}{1 - e^{-2t}}} \times \right.
$$
$$
\times \left[ (\alpha + \frac{1}{2}) \left( \frac{2xye^{-t}}{1 - e^{-2t}} \right)^{-\alpha} I_\alpha \left( \frac{2xye^{-t}}{1 - e^{-2t}} \right) \right]
$$
$$
= \left[ \left( \frac{2e^{-t}}{1 - e^{-2t}} \right)^{\frac{1}{2}} \left( \alpha + \frac{1}{2} \right) \left( \frac{2xye^{-t}}{1 - e^{-2t}} \right)^{\alpha-\frac{1}{2}} \frac{2ye^{-t}}{1 - e^{-2t}} \times \right.
$$
$$
\times e^{-\frac{1}{2}(x^2 + y^2) \frac{1 + e^{-2t}}{1 - e^{-2t}}}
$$
$$
+ \left( \frac{2xye^{-t}}{1 - e^{-2t}} \right)^{\alpha+\frac{1}{2}} e^{-\frac{1}{2}(x^2 + y^2) \frac{1 + e^{-2t}}{1 - e^{-2t}}} \left( \frac{2e^{-t}}{1 - e^{-2t}} \right)^{\frac{1}{2}} \left( \frac{2xye^{-t}}{1 - e^{-2t}} \right)^{\alpha+\frac{1}{2}} e^{-\frac{1}{2}(x^2 + y^2) \frac{1 + e^{-2t}}{1 - e^{-2t}}}
$$
$$
\times I_\alpha \left( \frac{2xye^{-t}}{1 - e^{-2t}} \right) \left( \frac{2xye^{-t}}{1 - e^{-2t}} \right)^{-\alpha} \frac{2ye^{-t}}{1 - e^{-2t}}
$$
$$
\left. \times \left[ (\alpha + \frac{1}{2}) \left( \frac{1}{x} + \frac{e^{-2t}}{1 - e^{-2t}} (-x) \right) \left( \frac{2xye^{-t}}{1 - e^{-2t}} \right)^{-\alpha} I_\alpha \left( \frac{2xye^{-t}}{1 - e^{-2t}} \right) \right] \right]
$$

is the Mehler kernel of the heat semigroup generated by the harmonic operator $H = \frac{1}{2} \left( -\frac{d^2}{dx^2} + x^2 \right)$ on $\mathbb{R}$.
We conclude from Lemma (3.1) with a \((1.7)\) yields
\[
\int_1 \partial_x \frac{\alpha}{1-e^{-2t}} \left( \int_0^\infty \sum_{k=0}^{\alpha} \frac{2xye^{-t}}{1-e^{-2t}} \right)^{-\alpha+1} \left( \frac{2xye^{-t}}{1-e^{-2t}} \right) I_{\alpha+1} \left( \frac{2xye^{-t}}{1-e^{-2t}} \right) dt.
\]
To estimate \(\partial_x W_t^{L_\alpha}(x, y, t)\), \(x, y, t \in (0, \infty)\), according to (2.5) and (2.6), we have following arguments.

If \(x, y, t \in (0, \infty)\) and \(\frac{xye^{-t}}{1-e^{-2t}} \leq 1\), (2.5), (3.5) along with (1.7) verifies
\[
| \partial_x W_t^{L_\alpha}(x, y) | \leq 2^{\alpha+1} x^{\alpha-\frac{1}{2}} y^{\alpha+\frac{1}{2}} e^{-t(\alpha+1)} \left( \frac{2xye^{-t}}{1-e^{-2t}} \right)^{-\alpha} \times I_{\alpha+1} \left( \frac{2xye^{-t}}{1-e^{-2t}} \right) \left( \frac{2xye^{-t}}{1-e^{-2t}} \right)^{-\alpha} \leq 2^{\alpha+1} x^{\alpha-\frac{1}{2}} y^{\alpha+\frac{1}{2}} e^{-t(\alpha+1)} \left( \frac{2xye^{-t}}{1-e^{-2t}} \right)^{-\alpha} \times I_{\alpha+1} \left( \frac{2xye^{-t}}{1-e^{-2t}} \right) \left( \frac{2xye^{-t}}{1-e^{-2t}} \right)^{-\alpha} \leq C x^{\alpha-\frac{1}{2}} y^{\alpha+\frac{1}{2}} e^{-\frac{1}{2}(x^2+y^2) \frac{1+\alpha-2i}{1-e^{-2t}}} e^{-\frac{1}{2}(x^2+y^2) \frac{1+\alpha-2i}{1-e^{-2t}}}.
\]

On other hand, for every \(x, y, t \in (0, \infty)\) such that \(\frac{xye^{-t}}{1-e^{-2t}} \geq 1\), (2.6) combined with (1.7) yields
\[
| \partial_x W_t^{L_\alpha}(x, y) | \leq C e^{-\frac{1}{2}(x^2+y^2) \frac{1+\alpha-2i}{1-e^{-2t}}} e^{-\frac{1}{2}(x^2+y^2) \frac{1+\alpha-2i}{1-e^{-2t}}} e^{-t/2} \left( \frac{1}{1-e^{-2t}} \right)^{3/2} \leq C e^{-\frac{1}{2}(x^2+y^2) \frac{1+\alpha-2i}{1-e^{-2t}}} e^{-t/2} \left( \frac{1}{1-e^{-2t}} \right)^{3/2} \left( \frac{1}{1+2x+y} \right).
\]

Assume \(x, y, t \in (0, \infty)\) and \(0 < y < \frac{x}{2}\), to show (3.1), we split the integral into two parts as follows:
\[
\int_0^\infty | \partial_x W_t^{L_\alpha}(x, y) | \ dt
= \int_{\{t > 0, \frac{xye^{-t}}{1-e^{-2t}} \leq 1\}} | \partial_x W_t^{L_\alpha}(x, y) | \ dt + \int_{\{t > 0, \frac{xye^{-t}}{1-e^{-2t}} > 1\}} | \partial_x W_t^{L_\alpha}(x, y) | \ dt
= I_1(x, y) + I_2(x, y).
\]

We conclude from Lemma (3.1) with \(a = \alpha + 1, b = 2\alpha + 1\), (2.5) and (1.7) that
\[
I_1(x, y) \leq C x^{\alpha-\frac{1}{2}} y^{\alpha+\frac{1}{2}} \int_{\{t > 0, \frac{xye^{-t}}{1-e^{-2t}} \leq 1\}} e^{-\frac{1}{2}(x^2+y^2) \frac{1+\alpha-2i}{1-e^{-2t}}} e^{-t(\alpha+1)} \left( \frac{2xye^{-t}}{1-e^{-2t}} \right)^{\alpha+2} dt
\leq C x^{\alpha-\frac{1}{2}} y^{\alpha+\frac{1}{2}} \left( \int_0^1 \frac{1}{t^{\alpha+1}} e^{-\frac{1}{2}(x^2+y^2) \frac{1+\alpha-2i}{1-e^{-2t}}} dt \right) + e^{-c(x^2+y^2)} \int_1^\infty e^{-t(\alpha+1)} dt.
\]
\begin{equation}
\leq C x^{\alpha - \frac{1}{2}} y^{\alpha + \frac{1}{2}} \left( \frac{1}{(x^2 + y^2)^{\alpha + \frac{1}{2}}} + e^{-c(x^2 + y^2)} \right) \leq C y^{\alpha + \frac{1}{2}} x^{\alpha + \frac{1}{2}}.
\end{equation}

Moreover, by inspection of $0 < y < \frac{x}{2}$, (3.7) and Lemma 3.1 with $a = \alpha + 2, b = 2(\alpha + 1)$, we are led to

\begin{align*}
I_2(x, y) & \leq C \left( \frac{1}{x} + x + y \right) \int_{\{t > 0, \frac{xye^{-t}}{1 - e^{-2t}} > 1\}} e^{-\frac{1}{2} \left( x^2 + y^2 \right) \frac{1 + \frac{2}{a}}{1 - e^{-2t}}} e^{-\frac{t}{2}} (1 - e^{-2t})^{3/2} dt \\
& \leq C \left( \frac{1}{x} + x \right) \int_{\{t > 0, \frac{xye^{-t}}{1 - e^{-2t}} > 1\}} e^{-\frac{t}{2}} (1 - e^{-2t})^{3/2} \left( \frac{xye^{-t}}{1 - e^{-2t}} \right)^{\alpha + \frac{1}{2}} dt \\
& \leq C (xy)^{\alpha + \frac{1}{2}} \int_{0}^{\infty} e^{-\frac{t}{2}} (1 - e^{-2t})^{3/2} dt \\
& \leq C (xy)^{\alpha + \frac{1}{2}} \left( \int_{0}^{1} e^{-\frac{ct}{2}} dt + e^{-ct} \int_{1}^{\infty} e^{-t(\alpha + 2)} dt \right) \\
& \leq C (xy)^{\alpha + \frac{1}{2}} \left( \frac{1}{x^{2a + 2}} + e^{-ct} \right) \leq C y^{\alpha + \frac{1}{2}} x^{\alpha + \frac{1}{2}}.
\end{align*}

Similarly, we obtain (3.2).

Next we will show (3.3). To this end, we split the integral into two parts.

\begin{equation}
\int_{0}^{\infty} \left| \partial_x (W_{t,x}^L (x, y) - W_t (x, y)) \right| dt
\end{equation}

\begin{align*}
&= \int_{\{t > 0, \frac{xye^{-t}}{1 - e^{-2t}} \leq 1\}} \left| \partial_x (W_{t,x}^L (x, y) - W_t (x, y)) \right| dt \\
& \quad + \int_{\{t > 0, \frac{xye^{-t}}{1 - e^{-2t}} > 1\}} \left| \partial_x (W_{t,x}^L (x, y) - W_t (x, y)) \right| dt \\
& = J_1(x, y) + J_2(x, y).
\end{align*}

For $J_1(x, y)$, we have

\begin{align*}
J_1(x, y) & \leq \int_{\{t > 0, \frac{xye^{-t}}{1 - e^{-2t}} \leq 1\}} \left| \partial_x W_{t,x}^L (x, y) \right| dt + \int_{\{t > 0, \frac{xye^{-t}}{1 - e^{-2t}} \leq 1\}} \left| \partial_x W_t (x, y) \right| dt \\
& = J_{1,1}(x, y) + J_{1,2}(x, y).
\end{align*}

Suppose that $t, x, y \in (0, \infty)$ and $\frac{xye^{-t}}{1 - e^{-2t}} \leq 1, \frac{x}{2} < y < 2x$, owing to Lemma 3.1 with $\alpha = \alpha + 1, b = 2\alpha + 1$ and (3.6) we deduce that

\begin{align*}
J_{1,1}(x, y) & \leq C x^{\alpha - \frac{1}{2}} y^{\alpha + \frac{1}{2}} \int_{0}^{\infty} e^{-\frac{1}{2} \left( x^2 + y^2 \right) \frac{1 + \frac{2}{a}}{1 - e^{-2t}}} e^{-t(\alpha + 1)} (1 - e^{-2t})^{a + 2} dt \\
& \leq C x^{2\alpha} \left( \int_{0}^{1} \frac{1}{t^{\alpha + 1}} e^{-\frac{ct}{2}} dt + e^{-c(x^2 + y^2)} \int_{1}^{\infty} e^{-t(\alpha + 1)} dt \right) \\
& \leq C x^{2\alpha} \left( \frac{1}{(x^2 + y^2)^{\alpha + \frac{1}{2}}} + e^{-c(x^2 + y^2)} \right) \leq C x.
\end{align*}

And it is easy to compute that

\begin{align*}
\partial_x W_t (x, y) & = \partial_x \left[ \left( \frac{e^{-t}}{\pi (1 - e^{-2t})} \right)^{\frac{1}{2}} e^{-\frac{1}{2} \left( x^2 + y^2 \right) \frac{1 + \frac{2}{a}}{1 - e^{-2t}}} \right] \\
& = \partial_x \left[ \left( \frac{e^{-t}}{\pi (1 - e^{-2t})} \right)^{\frac{1}{2}} e^{-\frac{1}{2} \left( x^2 + y^2 \right) \frac{1 + \frac{2}{a}}{1 - e^{-2t}}} \right].
\end{align*}
To estimate (2.6), (3.5) and (3.11), it is clear that, for $xye \frac{x^2+y^2}{1-x^2}$ we conclude from (3.10) with (3.12) that

$$By means of (3.11), if \frac{a}{b} < b \leq 1, we arrive at$$

$$| \partial_x W_t(x,y) | \leq C \frac{e^{-\frac{1}{2}}}{1-e^{-2t}} e^{-\frac{1}{2}(x^2+y^2) \frac{1+e^{-2t}}{1-x^2}},$$

Hence, by aid of Lemma 3.1 with $a = \frac{1}{2}, b = 1$ and (1.7) again, it follows that

$$J_{1,2}(x,y) \leq \int_0^\infty \frac{e^{-t}}{(1-e^{-2t})^2} e^{-\frac{1}{2}(x^2+y^2) \frac{1+e^{-2t}}{1-x^2}} dt$$

$$\leq C \left( \int_0^1 t^{-\frac{1}{2}} e^{-c(x^2+y^2)} dt + e^{-c(x^2+y^2)} \int_1^\infty e^{-\frac{1}{2} \frac{1}{2} t} dt \right)$$

$$\leq C \left( \frac{1}{(x^2+y^2)^{\frac{1}{2}}} + e^{-c(x^2+y^2)} \right) \leq C \frac{e^{-\frac{1}{2}}}{x}. \quad (3.12)$$

We conclude from (3.10) with (3.12) that

$$J_1(x,y) \leq C \frac{e^{-\frac{1}{2}}}{x}. \quad (3.13)$$

In inspection of (2.6), (3.5) and (3.11), it is clear that, for $\frac{xye^{-t}}{1-e^{-2t}} > 1,$

$$\partial_x(W_t^{\alpha}(x,y) - W_t(x,y))$$

$$= \left( \frac{e^{-t}}{\pi(1-e^{-2t})} \right)^{\frac{1}{2}} e^{-\frac{1}{2}(x^2+y^2) \frac{1+e^{-2t}}{1-x^2}} \left[ \left( 1 + O \left( \frac{1}{2x ye^{-t}} \right) \right) \left( (\alpha + \frac{1}{2}) \frac{1}{x} + \frac{1+e^{-2t}}{1-e^{-2t}} (-x) \right) \right.$$

$$\left. + \frac{2ye^{-t}}{1-e^{-2t}} \right] - \left( \frac{1+e^{-2t}}{1-e^{-2t}} (-x) + \frac{2ye^{-t}}{1-e^{-2t}} \right)$$

$$= W_t(x,y) \left[ \left( 1 + O \left( \frac{1}{2x ye^{-t}} \right) \right) (\alpha + \frac{1}{2}) \frac{1}{x} \right.$$ $$+ O \left( \frac{1}{2x ye^{-t}} \right) \left( \frac{1+e^{-2t}}{1-e^{-2t}} (-x) + \frac{2ye^{-t}}{1-e^{-2t}} \right). \quad (3.14)$$

To estimate $J_2,$ we then study the following integrals:

$$K_1(x,y) = \int_{\{ \frac{xye^{-t}}{1-e^{-2t}} > 1 \}} W_t(x,y) \frac{2ye^{-t}}{1-e^{-2t}} dt,$$

$$K_2(x,y) = \int_{\{ \frac{xye^{-t}}{1-e^{-2t}} > 1 \}} W_t(x,y) \frac{1+e^{-2t}}{1-e^{-2t}} x dt,$$

and

$$K_3(x,y) = \int_{\{ \frac{xye^{-t}}{1-e^{-2t}} > 1 \}} W_t(x,y) \frac{1}{x} dt.$$

Note previously, if $0 < a < b,$ we have

$$|b - e^{-t}a|^2 = |(b-a) + a(1-e^{-t})|^2 \geq (b-a)^2 + a^2(1-e^{-t})^2, \quad t \in (0, \infty).$$
Then as $\frac{x}{2} < y < 2x$, for a certain $c > 0$, it has
\[
|x - e^{-t}y|^2 + |y - e^{-t}x|^2 \geq (x - y)^2 + cx^2(1 - e^{-t})^2, \quad t \in (0, \infty).
\] (3.15)

Once more using Lemma 3.1 along with (1.7), we could derive
\[
K_1(x, y) \leq \int_0^\infty \left( \frac{e^{-t}}{\pi(1 - e^{-2t})} \right)^{\frac{1}{2}} e^{-\frac{|x - e^{-t}y|^2 + |y - e^{-t}x|^2}{2(1 - e^{-2t})}} \frac{2ye^{-t}}{1 - e^{-2t}} dt
\]
\[
\leq C \int_0^\infty \frac{ye^{-3t/2}}{(1 - e^{-2t})^{3/2}} e^{-\frac{(x-y)^2}{2(1-e^{-2t})}} dt
\]
\[
\leq C \int_0^\infty \frac{ye^{-3t/2}}{(1 - e^{-2t})^{3/2}} e^{-\frac{c x^2}{2(1-e^{-2t})}} dt
\]
\[
\leq C \int_0^1 \frac{ye^{-\frac{t(x-y)^2}{2}}}{e^{ct^2}} dt + ye^{-x^2} \int_1^\infty e^{-\frac{2}{3}t} dt
\]
\[
\leq C \frac{1}{\sqrt{x}} \int_0^1 t^{-\frac{1}{2}} e^{-\frac{t(x-y)^2}{2}} dt + \frac{1}{x} \int_1^\infty e^{-\frac{2}{3}t} dt
\]
\[
\leq C \frac{1}{\sqrt{x}} \left( \frac{1}{\sqrt{|x - y|}} + \frac{1}{\sqrt{x}} \right).
\]

For $K_2$, we infer
\[
K_2(x, y) \leq \int_0^\infty \left( \frac{e^{-t}}{\pi(1 - e^{-2t})} \right)^{\frac{1}{2}} e^{-\frac{|x - e^{-t}y|^2 + |y - e^{-t}x|^2}{2(1 - e^{-2t})}} \frac{1 + e^{-2t}}{1 - e^{-2t}} dt
\]
\[
\leq C \int_0^\infty \frac{e^{-t/2}}{(1 - e^{-2t})^{3/2}} e^{-\frac{(x-y)^2}{2(1-e^{-2t})}} dt
\]
\[
\leq C \int_0^1 t^{-1} e^{-\frac{(x-y)^2}{2e^{ct^2}}} dt + e^{-cx^2} \int_1^\infty e^{-\frac{2}{3}t} dt
\]
\[
\leq C \frac{1}{\sqrt{x}} \int_0^1 t^{-\frac{1}{2}} e^{-\frac{t(x-y)^2}{2}} dt + \frac{1}{x} \int_1^\infty e^{-\frac{2}{3}t} dt
\]
\[
\leq C \frac{1}{\sqrt{x}} \left( \frac{1}{\sqrt{|x - y|}} + \frac{1}{\sqrt{x}} \right).
\]

And $K_3$ becomes
\[
K_3(x, y) \leq \int_0^\infty \left( \frac{e^{-t}}{\pi(1 - e^{-2t})} \right)^{\frac{1}{2}} e^{-\frac{|x - e^{-t}y|^2 + |y - e^{-t}x|^2}{2(1 - e^{-2t})}} \frac{x ye^{-t}}{x \int_1^\infty e^{-2t} dt}
\]
\[
\leq C \int_0^\infty \frac{ye^{-3t/2}}{(1 - e^{-2t})^{3/2}} e^{-\frac{c x^2}{2(1-e^{-2t})}} dt
\]
\[
\leq C \int_0^1 \frac{yt^{-\frac{1}{2}} e^{-\frac{(x-y)^2}{2e^{ct^2}}}}{e^{ct^2}} dt + ye^{-x^2} \int_1^\infty e^{-\frac{2}{3}t} dt
\]
\[
\leq C \frac{1}{\sqrt{x}} \int_0^1 t^{-\frac{1}{2}} e^{-\frac{t(x-y)^2}{2}} dt + \frac{1}{x} \int_1^\infty e^{-\frac{2}{3}t} dt
\]
\[
\leq C \frac{1}{\sqrt{x}} \left( \frac{1}{\sqrt{|x - y|}} + \frac{1}{\sqrt{x}} \right).
\]
Combining the above estimates, we conclude that

$$J_2(x, y) \leq \frac{C}{x} \left(1 + \frac{\sqrt{x}}{\sqrt{|x-y|}}\right).$$  \tag{3.16}

Inserting (3.13) and (3.16) into the right hand side of (3.8), (3.3) is established, from which our conclusion follows.

**Lemma 3.3.** [2, Lemma 2.1] Let $\alpha \geq -\frac{1}{2}$. Then,

$$\int_0^\infty |\partial_t W_{L^\alpha}^t (x, y)| dt \leq C \frac{y^{\alpha+\frac{1}{2}}}{x^{\alpha+\frac{1}{2}}}, \quad 0 < y < \frac{x}{2},$$  \tag{3.17}

and

$$\int_0^\infty |\partial_t (W_{L^\alpha}^t (x, y) - W_t (x, y))| dt \leq C \frac{x^{\alpha+\frac{1}{2}}}{y^{\alpha+\frac{1}{2}}}, \quad 2x < y < \infty,$$  \tag{3.18}

and

$$\int_0^\infty |\partial_t (W_{L^\alpha}^t (x, y) - W_t (x, y))| dt \leq C \left(1 + \frac{\sqrt{x}}{\sqrt{|x-y|}}\right), \quad \frac{x}{2} < y < 2x, \quad \tag{3.19}$$

Proof of Theorem 1.1. From (2.4), we get, for any $(t, x) \in (0, \infty) \times (0, \infty)$ there exists a constant $c_t$ depending on the support of $f$ such that

$$\int_0^\infty \int_0^\infty W_{L^\alpha}^t (x, y) f(t, \tau, y) dy \leq C \|f\|_\infty \int_0^{c_t} \int_0^\infty e^{-\frac{(x-y)^2}{\sqrt{t}}} dy d\tau = C_t.$$  

Hence, the integral defining $u(t, x)$ is absolutely convergent. Assume now that $f \in C^1(\mathbb{R} \times (0, \infty))$ with the compact support. By proceeding as above we prove that

$$\partial_t u(t, x) = \int_0^\infty \int_0^\infty W_{L^\alpha}^t (x, y) \partial_t f(t, \tau, y) dy d\tau$$

$$= -\int_0^\infty \int_0^\infty W_{L^\alpha}^t (x, y) \partial_\tau f(t, \tau, y) dy d\tau, \quad t, x \in (0, \infty),$$

where the integral are absolutely convergent. We write

$$\partial_t u(t, x) = \int_0^{2x} \int_0^\infty (W_{L^\alpha}^t (x, y) - W_{L^\alpha}^t (x, y)) \partial_\tau f(t, \tau, y) d\tau dy$$

$$- \int_0^{\frac{x}{2}} \int_0^\infty W_{L^\alpha}^t (x, y) \partial_\tau f(t, \tau, y) d\tau dy$$

$$- \int_0^{\frac{x}{2}} \int_0^\infty W_{L^\alpha}^t (x, y) \partial_\tau f(t, \tau, y) d\tau dy$$

$$- \int_0^{2x} \int_0^\infty W_{L^\alpha}^t (x, y) \partial_\tau f(t, \tau, y) d\tau dy.$$

By (2.4) and integration by parts, we get

$$\int_0^{\frac{x}{2}} \int_0^\infty W_{L^\alpha}^t (x, y) \partial_\tau f(t, \tau, y) d\tau dy$$

$$= -\int_0^{\frac{x}{2}} \int_0^\infty \partial_\tau W_{L^\alpha}^t (x, y) f(t, \tau, y) d\tau dy,$$  \tag{3.21}
with \( t, x \in (0, \infty) \). Also, we obtain that
\[
\int_{2x}^{\infty} \int_{0}^{\infty} W_{\tau}^{L_{n}}(x, y) \partial_{\tau} f(t - \tau, y) \, dy \, d\tau
= - \int_{2x}^{\infty} \int_{0}^{\infty} \partial_{\tau} W_{\tau}^{L_{n}}(x, y) f(t - \tau, y) \, dy \, d\tau,
\]
\[ t, x \in (0, \infty). \tag{3.22} \]

From (2.3) and (3.4), we deduce the following pointwise relation
\[
W_{\tau}^{L_{n}}(x, y) - W_{\tau}(x, y)
= \left\{ \sqrt{2\pi} \left( \frac{2x e^{-\tau}}{1 - e^{-2\tau}} \right)^{\frac{1}{2}} I_{0} \left( \frac{2x e^{-\tau}}{1 - e^{-2\tau}} e^{-2x e^{-\tau}} \right) - 1 \right\} W_{\tau}(x, y).
\]

By aid of (2.6) and [1, (28)] we have for \( \tau, x, y \in (0, \infty) \),
\[
\left| W_{\tau}^{L_{n}}(x, y) - W_{\tau}(x, y) \right| \leq C \frac{1 - e^{-2\tau}}{2xe^{-\tau}} W_{\tau}(x, y) \leq C \frac{1 - e^{-2\tau}}{2xe^{-\tau}} e^{-\frac{e(x-y)^{2}}{\tau}}.
\]

We can apply integration by parts to get, for \( t, x \in (0, \infty) \),
\[
\int_{\frac{t}{2}}^{\infty} \int_{0}^{\infty} (W_{\tau}^{L_{n}}(x, y) - W_{\tau}(x, y)) \partial_{\tau} f(t - \tau, y) \, dy \, d\tau
= - \int_{\frac{t}{2}}^{\infty} \int_{0}^{\infty} \partial_{\tau} (W_{\tau}^{L_{n}}(x, y) - W_{\tau}(x, y)) f(t - \tau, y) \, dy \, d\tau. \tag{3.23}
\]

We are going to see that the integral on the right hand of (3.21), (3.22) and (3.23) are absolutely convergent. Let \( x \in (0, \infty) \) and \( t \in \mathbb{R} \). Since \text{supp} f is compact, there exists \( 0 < a < \frac{x}{2}, 2x < b \) and \( c > 0 \), such that \( f(t - \tau, y) = 0, (\tau, y) \notin (\infty, c) \times (a, b) \). Using Lemma 3.3, we obtain
\[
\int_{0}^{\frac{t}{2}} \int_{0}^{\infty} |\partial_{\tau} W_{\tau}^{L_{n}}(x, y) f(t - \tau, y)| \, dy \, d\tau
\leq C \int_{0}^{\frac{t}{2}} \int_{0}^{\infty} |\partial_{\tau} W_{\tau}^{L_{n}}(x, y)| \, dy \, d\tau
\leq C \int_{a}^{b} \frac{y^{\alpha + \frac{1}{2}}}{x^{\alpha + \frac{1}{2}}} \, dy < \infty.
\]

In similar way, we infer that
\[
\int_{2x}^{\infty} \int_{0}^{\infty} |\partial_{\tau} W_{\tau}^{L_{n}}(x, y) f(t - \tau, y)| \, dy \, d\tau
\leq C \int_{2x}^{b} \int_{0}^{\infty} |\partial_{\tau} W_{\tau}^{L_{n}}(x, y)| \, dy \, d\tau
\leq C \int_{2x}^{b} \frac{y^{\alpha + \frac{1}{2}}}{y^{\alpha + \frac{1}{2}}} \, dy < \infty,
\]
and
\[
\int_{\frac{t}{2}}^{\infty} \int_{0}^{\infty} |\partial_{\tau} W_{\tau}^{L_{n}}(x, y) - W_{\tau}(x, y)) f(t - \tau, y)| \, dy \, d\tau
\leq C \int_{\frac{t}{2}}^{\infty} \int_{0}^{\infty} |\partial_{\tau} W_{\tau}^{L_{n}}(x, y) - W_{\tau}(x, y)| \, dy \, d\tau
\leq C \int_{\frac{t}{2}}^{\infty} \frac{1}{x} \left( 1 + \frac{\sqrt{x}}{\sqrt{|x - y|}} \right) \, dy < \infty.
\]
On the other hand,
\[
\int_{\frac{a}{2}}^{2x} \int_0^\infty W_\tau(x,y) \partial_\tau f(t-\tau,y)d\tau dy
\]
\[
= \lim_{\epsilon \to 0} \int_{\frac{a}{2}}^{2x} \int_\epsilon^\infty W_\tau(x,y) \partial_\tau f(t-\tau,y)d\tau dy
\]
\[
= - \lim_{\epsilon \to 0} \left( \int_{\frac{a}{2}}^{2x} \int_\epsilon^\infty \partial_\tau W_\tau(x,y) f(t-\tau,y)d\tau dy + \int_{\frac{a}{2}}^{2x} W_\tau(x,y) f(t-\epsilon,y)dy \right)
\]
\[
= - \lim_{\epsilon \to 0} \int_{\frac{a}{2}}^{2x} \int_\epsilon^\infty \partial_\tau W_\tau(x,y) f(t-\tau,y)d\tau dy - f(t,x), \quad t, x \in (0, \infty). \quad (3.24)
\]

In the last equality we have taken into account that
\[
\lim_{s \to 0} \int_{\frac{a}{2}}^{2x} W_s(x,y) f(t-s,y)dy = f(t,x), \quad t, x \in (0, \infty).
\]

Indeed, let \( t, x \in (0, \infty) \). Since \( f \in C^1(\mathbb{R} \times (0, \infty)) \) with compact support, by using mean value theorem we deduce that \(|f(t-s,y) - f(t,y)| \leq Cs, s, y \in (0, \infty)\). Then, we can write
\[
\left| \int_{\frac{a}{2}}^{2x} W_s(x,y) f(t-s,y)dy - \int_{\frac{a}{2}}^{2x} W_s(x,y) f(t,y)dy \right|
\]
\[
\leq C \int_{\frac{a}{2}}^{2x} W_s(x,y) |f(t-s,y) - f(t,y)|dy
\]
\[
\leq Cs \int_{\frac{a}{2}}^{2x} W_s(x,y) dy \leq Cs \int_{\frac{a}{2}}^{2x} e^{-c(\frac{x-y}{s})^2} \frac{dy}{\sqrt{s}} < Cs. \quad (3.25)
\]

On the other hand, for a certain \( a > 0 \) such that \( \frac{a}{2} < x < \frac{a}{2}, f(t,y) = 0, y / \in (\frac{a}{2}, a) \).

It follows, with the obvious extension of \( f \), that
\[
\int_{-\infty}^{\frac{a}{2}} W_s(x,y) f(t,y)dy \leq C \int_{\frac{a}{2}}^{2x} e^{-c(\frac{x-y}{s})^2} \frac{dy}{\sqrt{s}} < Cs^{-\frac{1}{2}} e^{-c\frac{x^2}{s}}, \quad (3.26)
\]
and
\[
\int_{2x}^{\infty} W_s(x,y) f(t,y)dy \leq C \int_{2x}^{a} e^{-c(\frac{x-y}{s})^2} \frac{dy}{\sqrt{s}} < Cs^{-\frac{1}{2}} e^{-c\frac{x^2}{s}}. \quad (3.27)
\]

Moreover, it is well known that
\[
\lim_{s \to 0^+} e^{-sH(f(t, \cdot))(x)} = \lim_{s \to 0^+} \int_{\mathbb{R}} W_s(x,y) f(t,y)dy = f(t,x). \quad (3.28)
\]

Putting together all the above estimates we obtain
\[
\left| \int_{\frac{a}{2}}^{2x} W_s(x,y) f(t-s,y)dy - f(t,x) \right|
\]
\[
\leq \int_{\frac{a}{2}}^{2x} W_s(x,y) f(t-s,y)dy - \int_{\frac{a}{2}}^{2x} W_s(x,y) f(t,y)dy
\]
\[
+ \int_{2x}^{\infty} W_s(x,y) f(t,y)dy - \int_{-\infty}^{+\infty} W_s(x,y) f(t,y)dy
\]
From (3.20)–(3.24) we deduce that
\[ 5500 \text{ HUIYING FAN AND TAO MA} \]
As a consequence, we get
\[ \text{We conclude that} \]
\[ \lim_{s \to 0} \int_{\frac{1}{2}}^{2} W_s(x, y) f(t-s, y) dy = f(t, x). \]
From (3.20)–(3.24) we deduce that
\[ \partial_t u(t, x) = \lim_{\epsilon \to 0^+} \left( \int_{\epsilon}^{\infty} \int_{\frac{1}{2}}^{2} \partial_\tau (W_\tau^L(x, y) - W_\tau^L(x, y)) f(t-\tau, y) dy d\tau \right) \]
\[ + \int_{\epsilon}^{\infty} \int_{0}^{\frac{1}{2}} \partial_\tau W_\tau^L(x, y) f(t-\tau, y) dy d\tau \]
\[ + \int_{\epsilon}^{\infty} \int_{2}^{\infty} \partial_\tau W_\tau^L(x, y) f(t-\tau, y) dy d\tau \]
\[ + \int_{\epsilon}^{\infty} \int_{\frac{1}{2}}^{2} \partial_\tau W_\tau^L(x, y) f(t-\tau, y) dy d\tau + f(t, x), \quad t, x > 0. \]
As a consequence, we get
\[ \partial_t u(t, x) = \lim_{\epsilon \to 0^+} \int_{\epsilon}^{\infty} \int_{0}^{\infty} \partial_\tau W_\tau^L(x, y) f(t-\tau, y) dy d\tau + f(t, x), \quad t, x > 0. \]
Assume now that \( f \in C^2(\mathbb{R} \times (0, \infty)) \) and it has compact support. We consider the function
\[ H(t, x) = \int_{0}^{\infty} \int_{0}^{\infty} (W_\tau^L(x, y) - W_\tau^L(x, y)) f(t-\tau, y) dy d\tau, \quad t, x \in (0, \infty). \]
Let \( t, x \in (0, \infty). \) Since \( \text{supp} f \) is compact, there exists \( 0 < a < b < \infty \) and \( \tau_0 \in (0, \infty) \) such that
\[ H(t, x) = \int_{0}^{\tau_0} \int_{a}^{b} (W_\tau^L(x, y) - W_\tau^L(x, y)) f(t-\tau, y) dy d\tau, \quad t, x \in (0, \infty). \]
Note that the integrability of \( \partial_x W_\tau^L(x, y) \) by trial calculation, we conclude from Lemma 3.2 that
\[ \int_{0}^{\tau_0} \int_{a}^{b} |\partial_\tau (W_\tau^L(x, y) - W_\tau^L(x, y)) f(t-\tau, y)| dy d\tau \]
\[ \leq C \int_{a}^{b} \int_{0}^{\infty} |\partial_\tau (W_\tau^L(x, y) - W_\tau^L(x, y))| dy d\tau < \infty, \]
for \( x \in (0, \infty). \) Hence,
\[ \partial_\tau H(t, x) = \int_{0}^{\tau_0} \int_{0}^{\infty} \partial_\tau (W_\tau^L(x, y) - W_\tau^L(x, y)) f(t-\tau, y) dy d\tau, \]
for \( t, x \in (0, \infty), \) and the last integral is absolutely convergent. On the other hand, for \( x, y, \tau \in (0, \infty), \)
\[ \frac{1}{2} \partial_x^2 (W_\tau^L(x, y) - W_\tau^L(x, y)) - x^2 (W_\tau^L(x, y) - W_\tau^L(x, y)) \]
\[ \frac{\alpha^2 - \frac{1}{2} W_{r}^{L_\alpha}(x, y) + \partial_x (W_{r}^{L_\alpha}(x, y) - W_r(x, y))}{2x^2} = (3.30) \]

By proceeding as above we get that
\[ \partial^2_x \mathcal{H}(t, x) = \int_0^\infty \int_0^\infty \partial^2_x (W_{r}^{L_\alpha}(x, y) - W_r(x, y)) f(t - \tau, y) dy d\tau, \quad x \in (0, \infty), \]
and the last integral is absolutely convergent. We now consider the function
\[ \mathcal{H}(t, x) = \int_0^\infty \int_0^\infty W_r(x, y) f(t - \tau, y) dy d\tau, \quad t, x \in (0, \infty). \]

By extending \( f \) in the obvious way, we have for \( t, x \in (0, \infty) \),
\[
\mathcal{H}(t, x) = \int_0^\infty \int_{-\infty}^{+\infty} W_r(x, y) f(t - \tau, y) dy d\tau
\]
\[ = \int_0^\infty \int_{-\infty}^{+\infty} e^{-\frac{1}{2} |x-y|^2 \coth \tau - xy \tanh \tau \frac{\tau}{2}} f(t - \tau, y) dy d\tau
\]
\[ = \int_0^\infty \int_{-\infty}^{+\infty} e^{-\frac{1}{2} |y|^2 \coth \frac{\tau}{2} |xy|^2 \tanh \frac{\tau}{2}} f(t - \tau, x-y) dy d\tau
\]
\[ = \int_0^\infty \int_{-\infty}^{+\infty} e^{-\frac{1}{2} |x-y|^2 \coth \frac{\tau}{2} + |xy|^2 \tanh \frac{\tau}{2}} f(t - \tau, y) dy d\tau. \quad (3.31) \]

We introduce the following notation
\[ S(\tau) := \frac{1}{\sqrt{2\pi \sinh \tau}}, \quad H(\tau, y) := e^{-\frac{1}{2} |y|^2 \coth \frac{\tau}{2}}, \quad G(\tau, x, y) := e^{-\frac{1}{2} |2x-y|^2 \tanh \frac{\tau}{2}}, \quad F(\tau, x, y) := f(t - \tau, x - y). \]

Then, we rewrite (3.31) as
\[ \mathcal{H}(t, x) = \int_0^\infty \int_{-\infty}^{+\infty} S(\tau) H(\tau, y) G(\tau, x, y) F(\tau, x, y) dy d\tau. \]

Since the product \( G(\tau, x, y) F(\tau, x, y) \) is smooth with compact support, then by dominated convergence, we obtain that
\[ \partial_x \mathcal{H}(t, x) = \int_0^\infty \int_{-\infty}^{+\infty} S(\tau) H(\tau, y) \partial_x (G(\tau, x, y) F(\tau, x, y)) dy d\tau
\]
\[ = \int_0^\infty \int_{-\infty}^{+\infty} S(\tau) H(\tau, y) \left( \partial_x G(\tau, x, y) F(\tau, x, y) \right. \\
\[ + \left. G(\tau, x, y) \partial_x F(\tau, x, y) \right) dy d\tau
\]
\[ = \int_0^\infty \int_{-\infty}^{+\infty} S(\tau) H(\tau, y) (-2 \partial_y G(\tau, x, y) F(\tau, x, y)
\]
\[ - G(\tau, x, y) \partial_y F(\tau, x, y) dy d\tau. \]

It is trial to check that the last two obvious integrals are absolutely convergent. Then we write
\[ \partial_x \mathcal{H}(t, x) = \lim_{\varepsilon \to 0^+} \int_0^\infty \int_{-\infty}^{+\infty} S(\tau) H(\tau, y) \left( -2 \partial_y G(\tau, x, y) F(\tau, x, y)
\]
\[ - G(\tau, x, y) \partial_y F(\tau, x, y) dy d\tau =: \lim_{\varepsilon \to 0^+} (I_1' + I_2'). \]
Integration by parts leads to
\[ I_2' = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S(\tau) \partial_y (HG) F dy d\tau. \]

Hence,
\[
\partial_x H(t, x) = \lim_{\varepsilon \to 0^+} \int_{-\varepsilon}^{\varepsilon} \int_{-\infty}^{\infty} S(\tau) (-2\partial_y G(\tau, x, y) H(\tau, y) \\
+ \partial_y (G(\tau, x, y) H(\tau, y))) F(\tau, x, y) dy d\tau
= \lim_{\varepsilon \to 0^+} \int_{-\varepsilon}^{\varepsilon} \int_{-\infty}^{\infty} S(\tau) (-\partial_y G(\tau, x, y) H(\tau, y) \\
+ G(\tau, x, y) \partial_y H(\tau, y)) F(\tau, x, y) dy d\tau
= \lim_{\varepsilon \to 0^+} \int_{-\varepsilon}^{\varepsilon} \int_{-\infty}^{\infty} \partial_x W_\tau(x, y) f(t - \tau, y) dy d\tau, \quad t, x \in (0, \infty).
\]

In a similar way, we have
\[
\partial_x^2 H(t, x) = \lim_{\varepsilon \to 0^+} \int_{-\varepsilon}^{\varepsilon} \int_{-\infty}^{\infty} \partial_x^2 W_\tau(x, y) f(t - \tau, y) dy d\tau, \quad t, x \in (0, \infty).
\]

We conclude that, for \( i = 1, 2 \)
\[
\partial^i_x H(t, x) = \lim_{\varepsilon \to 0^+} \int_{-\varepsilon}^{\varepsilon} \int_{-\infty}^{\infty} \partial_x^i W_\tau(x, y) f(t - \tau, y) dy d\tau, \quad t, x \in (0, \infty). \tag{3.32}
\]

By combining (3.29) with (3.32), we obtain
\[
\partial_t u(t, x) + \frac{1}{2} \left( -\partial_x^2 + \frac{\alpha^2}{x^2} + x^2 \right) u(t, x)
= \lim_{\varepsilon \to 0^+} \int_{-\varepsilon}^{\varepsilon} \int_0^{\infty} \left( \partial_\tau + \frac{1}{2} \left( -\partial_x^2 + \frac{\alpha^2}{x^2} + x^2 \right) \right) W_\tau(x, y) f(t - \tau, y) dy d\tau + f(t, x)
= f(t, x), \quad t, x \in (0, \infty).
\]

The other representation of the derivatives of \( u \) as principal values can be proved by proceeding as above and by taking account into [19, Theorem 1.3]. The details are left to the interested reader. \( \square \)

4. **Proof of Theorem 1.2.** Supposed initially that \( f \in L_c^\infty ((0, \infty) \times (0, \infty)) \) and \( g \in L_c^\infty ((0, \infty)) \). We define
\[
u(t, x) = \int_0^t \int_0^\infty W_\tau L(x, y) f(t - \tau, y) dy d\tau + \int_0^\infty W_t L(x, y) g(y) dy
= u_1(t, x) + u_2(t, x).
\]

As the argument needed in order to the existence of \( u \), it can be checked that the integrals defining \( u_1(t, x) \) and \( u_2(t, x) \) are absolutely convergent. Observe that by linearity it is enough to solve equations
\[
\begin{cases}
\quad \partial_t u_1(t, x) + L_\alpha u_1(t, x) = f(t, x), \quad \text{for } t, x \in (0, \infty), \\
u_1(0, x) = 0, \quad \text{for } x \in (0, \infty),
\end{cases}
\]
and
\[
\begin{cases}
\quad \partial_t u_2(t, x) + L_\alpha u_2(t, x) = 0, \quad \text{for } t, x \in (0, \infty), \\
u_2(0, x) = g(x), \quad \text{for } x \in (0, \infty).
\end{cases}
\]
We will deal with \( u_1 \) and \( u_2 \) separately. On one hand, the solution \( u_2 \) is given by the heat-diffusion semigroup generated by \( L_\alpha \), namely, \( u_2(t,x) = e^{-tL_\alpha}g(x) \). Consequently, all the terms and properties of the problems in the statement are related to the initial datum \( g \) and established.

On the other hand, by proceeding as in the third section and taking into account \([19, (11)]\) we obtain that, for \( i = 1, 2 \),

\[
\partial_t^i u_i(t,x) = \lim_{\epsilon \to 0^+} \int_0^t \int_0^\infty \partial_x^i W_{\tau}^L f(t - \tau, y) \, dy \, d\tau, \quad t, x \in (0, \infty).
\]

Using parametric derivation and integration by parts we deduce that

\[
\partial_t u_1(t,x) = \int_0^t \int_0^\infty W_{\tau}^L(x,y) \partial_t f(t - \tau, y) \, dy \, d\tau + \int_0^\infty W_{\tau}^L(x,y) f(0,y) \, dy
\]

\[
= -\int_0^t \int_0^\infty W_{\tau}^L(x,y) \partial_x f(t - \tau, y) \, dy \, d\tau
\]

\[
- \int_0^t \int_0^\infty (W_{\tau}^L(x,y) - (W_{\tau}(x,y))) \partial_x f(t - \tau, y) \, dy \, d\tau
\]

\[
- \int_0^t \int_0^\infty W_{\tau}(x,y) \partial_x f(t - \tau, y) \, dy \, d\tau
\]

\[
= \lim_{\epsilon \to 0^+} \left( \int_\epsilon^t \int_0^\infty \partial_x (W_{\tau}^L(x,y) - (W_{\tau}(x,y))) f(t - \tau, y) \, dy \, d\tau \right)
\]

\[
= \lim_{\epsilon \to 0^+} \left( \int_\epsilon^t \int_0^\infty \partial_x W_{\tau}^L(x,y) \, dy \, d\tau \right)
\]

\[
= \lim_{\epsilon \to 0^+} \int_\epsilon^t \int_0^\infty \partial_x W_{\tau}^L(x,y) f(t - \tau, y) \, dy \, d\tau + \int_\epsilon^t \int_0^\infty \partial_t W_{\tau}(x,y) f(t - \tau, y) \, dy \, d\tau
\]

Putting together the above equalities we get

\[
\partial_t u_1(t,x) + L_\alpha u_1(t,x) = f(t,x), \quad \text{for } t, x \in (0, \infty).
\]

Moreover, by \((2.4)\), since \( f \) has compact support, we find \( a > 0 \) such that, for every \( x \in (0, \infty) \), there exists \( C > 0 \) for which

\[
|u_1(t,x)| \leq C \int_0^t \int_0^\infty \frac{e^{-\epsilon(x-y)^2}}{\sqrt{\tau}} \, dy \, d\tau \leq Ct, \quad 0 < t < a.
\]

Then, \( \lim_{\epsilon \to 0^+} u_1(t,x) = 0, \quad x \in (0, \infty) \). Thus, we prove that the function \( u \) is a classical solution of \((1.3)\).
5. **Proof of Theorem 1.3.** From Remark 1 we notice that our next result is slightly more general than Theorem 1.3 when $p = q > 1$.

**Theorem 5.1.** Let $u \in L^p(\mathbb{R} \times (0, \infty), w)$, for $w \in A^*_p(\mathbb{R} \times (0, \infty))$, $1 \leq p \leq \infty$. Let $P^s_{L_\alpha} u(t, x)$ be as in Theorem 1.3. If $1 < p \leq \infty$, then

$$
\|P^s_{L_\alpha} u(t, x)\|_{L^p(\mathbb{R} \times (0, \infty), w)} \leq C_{p, w} \|u\|_{L^p(\mathbb{R} \times (0, \infty), w)};
$$

if $p = 1$, then, for every $\lambda > 0$,

$$
\lambda \left\{ (t, x) \in \mathbb{R} \times (0, \infty) : |P^s_{L_\alpha} u(t, x)| > \lambda \right\} \leq \frac{C_w}{\lambda} \|u\|_{L^1(\mathbb{R} \times (0, \infty), w)}.
$$

**Proof.** For any function $u \in L^p(\mathbb{R} \times (0, \infty))$, we have

$$
P^s_{L_\alpha, y} u(t, x) = \frac{y^{2s}}{4^\Gamma(s)} \int_0^\infty e^{-s^2 t} \int_0^\infty W^s_{\tau, x}(x, z) u(t - \tau, z) dz \frac{d\tau}{\tau^{1+s}},
$$

where

$$
W^s_{\tau, x}(x, z) = 2(xz)^{\frac{k}{2}} \frac{e^{-\tau}}{1 - e^{-2\tau}} I_\alpha \left( \frac{2xz e^{-\tau}}{1 - e^{-2\tau}} \right) e^{-\frac{1}{2}(x^2 + z^2) \frac{1+2\tau}{1+2\tau}}.
$$

The operator above can be represented as

$$
P^s_{L_\alpha, y} u(t, x) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P^s_{L_\alpha, y}(\tau, x, z) u(t - \tau, z) dz d\tau,
$$

here,

$$
P^s_{L_\alpha, y}(\tau, x, z) = \frac{y^{2s}}{4^\Gamma(s)} \frac{e^{-s^2 t}}{\tau^{1+s}} 2(xz)^{\frac{k}{2}} \frac{e^{-\tau}}{1 - e^{-2\tau}} I_\alpha \left( \frac{2xz e^{-\tau}}{1 - e^{-2\tau}} \right) e^{-\frac{1}{2}(x^2 + z^2) \frac{1+2\tau}{1+2\tau}} \chi_{\tau > 0} \chi_{z > 0}.
$$

We will use the theory of vector-valued Calderón-Zygmund operators (see [3, 20]) to get the boundness of the maximal operators. Firstly, we have

$$
|P^s_{L_\alpha, y} u(t, x)| \leq C \|u\|_{L^\infty(\mathbb{R}^{1+1})} \frac{y^{2s}}{4^\Gamma(s)} \int_0^\infty e^{-s^2 t} \int_0^\infty e^{-\frac{e^{-\tau}(x-z)^2}{\tau}} \frac{d\tau}{\tau^{1+s}} dz = C \|u\|_{L^\infty(\mathbb{R}^{1+1})},
$$

(5.2)

Considering the $L^\infty(0, \infty)$-valued operator given by

$$
P^s_{L_\alpha} u(t, x) = \{P^s_{L_\alpha, y} u(t, x)\}_{y > 0}.
$$

(5.3)

Then (5.2) implies that $P^s_{L_\alpha}$ is a bounded operator from $L^\infty(\mathbb{R}^{1+1})$ to $L^\infty(\mathbb{R}^{1+1} ; L^\infty(0, \infty))$.

Furthermore, the operator $P^s_{L_\alpha}$ can be written as

$$
P^s_{L_\alpha} u(t, x) = \int_{\mathbb{R}^{1+1}} \{P^s_{L_\alpha, y}(\tau, x, z)\}_{y > 0} u(t - \tau, z) dz d\tau,
$$

where $P^s_{L_\alpha, y} u(t, x)$ is given in (5.1). Using (1.7) and (2.4), it is easy to check that

$$
|P^s_{L_\alpha, y}(\tau, x, z)| \leq C \frac{e^{-\frac{c}{\tau}}}{\tau^{\frac{1}{2}+1}}.
$$
Thus (5.4) is established. In a similar way we can see that

\[ |P_{L_{n,y}}^{*}(\tau, x, z)| \leq C_{1} \left( \frac{\tau}{\tau^{2} + |x - z|} \right)^{1+2} e^{-\frac{(x-z)^{2}}{\tau^2}} \leq \frac{C_{1}}{\tau^{2} + |x - z|} \leq \frac{C_{1}}{\tau^{2} + |x - z|} \frac{1}{1+2}. \]

Next we estimate

\[ |\partial_{\tau} P_{L_{n,y}}^{*}(\tau, x, z)| \leq \frac{C}{(\tau^{2} + |x - z|)^{1+4}}. \]  

We have

\[ \partial_{\tau} P_{L_{n,y}}^{*}(\tau, x, z) = \left[ \frac{y^{2s}}{4 \Gamma(s)} \right] \left[ e^{-\frac{y^{2}}{\tau^2}} \left( \frac{y^{2}}{\tau} \right)^{s} \right] W_{L_{n}}^{\alpha}(x, z) \]

\[ = I_{1} + I_{2}. \]

For \( I_{1}, (1.7) \) and (2.4) imply that

\[ |I_{1}| = \left| \frac{y^{2s}}{4 \Gamma(s)} \left( \frac{1}{\tau^{1+s}} \right) e^{-\frac{y^2}{\tau^2}} \right| \leq \frac{C}{(\tau^{2} + |x - z|)^{1+4}}. \]

According to [2, (2.10)] and (1.7), we have

\[ |I_{2}| \leq C \frac{y^{2s}}{4 \Gamma(s)} \left( x \right)^{\alpha+1} \frac{1}{(1 - e^{-2\tau})^{\alpha+2}} \frac{1}{\left( \frac{1}{1 - e^{-2\tau}} \right)^{\frac{1}{2}}} \]

\[ \leq C \left( \frac{x}{1 - e^{-2\tau}} \right)^{\alpha+1} e^{-\frac{y^2}{\tau^2}} \leq \frac{C}{(\tau^{2} + |x - z|)^{1+4}}, \quad \frac{x}{1 - e^{-2\tau}} \leq 1, \]

and [2, (2.11)] and (1.7) imply that

\[ |I_{2}| \leq C \frac{y^{2s}}{4 \Gamma(s)} \left( x \right)^{\alpha+1} \frac{1}{(1 - e^{-2\tau})^{\alpha+2}} \frac{1}{\left( \frac{1}{1 - e^{-2\tau}} \right)^{\frac{1}{2}}} \]

\[ \leq \frac{C}{(\tau^{2} + |x - z|)^{1+4}}, \quad \frac{x}{1 - e^{-2\tau}} > 1. \]

Thus (5.4) is established. In a similar way we can see that

\[ |\nabla_{z} P_{L_{n,y}}^{*}(\tau, x, z)| \leq \frac{C}{(\tau^{2} + |x - z|)^{1+4}}. \]
Observe that if $2(|s|^{\frac{1}{2}} + |z - z_0|) \leq |	au - s|^{\frac{1}{2}} + |x - z_0|$, for some intermediate point $(\tau - \theta s, x, z_0)$ between $(\tau, x, z)$ and $(\tau - s, x, z_0)$, we have that

$$|P_{L_{\alpha,y}}^{\star}(\tau, x, z) - P_{L_{\alpha,y}}^{\star}(\tau - s, x, z_0)|$$

$$\leq |\partial_{\alpha}P_{L_{\alpha,y}}^{\star}(\tau - \theta s, x, z_0)||s| + |\nabla z P_{L_{\alpha,y}}^{\star}(\tau - \theta s, x, z_0)||z - z_0|$$

$$\leq \frac{|C||s|(|\tau - \theta s|^{\frac{1}{2}} + |x - z_0|)^{1+4}}{(|\tau - s|^{\frac{1}{2}} + |x - z_0|)^{1+3}} + \frac{C|z - z_0|}{(|\tau - s|^{\frac{1}{2}} + |x - z_0|)^{1+3}}$$

Therefore $\{P_{L_{\alpha,y}}^{\star}\}_{y>0}$ is a vector-valued Carleson-Zygmund kernel. As a consequence, $P_{L_{\alpha}}^{\star}$ is bounded from $L^p(\mathbb{R}^{1+1}, w)$ into $L^p(\mathbb{R}^{1+1}, w; L^\infty(0, \infty))$ for $w \in A^\infty_\alpha(\mathbb{R}^{1+1})$, $1 < p < \infty$, and it satisfies the corresponding weak-type estimate, namely, if $p = 1$, then, for any $\lambda > 0$,

$$w\left(\{(t, x) \in \mathbb{R}^{1+1} : \|P_{L_{\alpha}}^{\star} u(t, x)\|_{L^\infty(0, \infty)} > \lambda\}\right) \leq \frac{C_n}{\lambda} \|u\|_{L^1(\mathbb{R}^{1+1}, w)},$$

for $w \in A^\infty_\alpha(\mathbb{R}^{1+1})$. By observing that $P_{L_{\alpha}}^{\star} u(t, x) = \|P_{L_{\alpha}}^{\star} u(t, x)\|_{L^\infty(0, \infty)}$, the statement is established. \hfill \Box

Proof of Theorem 1.3. The kernel of $P_{L_{\alpha}}^{\star}$ is bounded by

$$K^{\star}_{L_{\alpha}}(t, x, z) = Cy^{2e}e^{-c\frac{z^2}{t}}\frac{e^{-c\frac{(x-z)^2}{t}}}{t^{1+\alpha}}\chi_{t>0}(\chi_{z>0}).$$

Moreover,

$$\int_{\mathbb{R}} K^{\star}_{L_{\alpha}}(t, x, z)dz \leq Cy^{2e}e^{-c\frac{z^2}{t}}\frac{1}{t^{1+\alpha}}. \quad (5.5)$$

Let us fix $1 < p < \infty$. By using Theorem 5.1 and Remark 1 we have

$$\|P_{L_{\alpha}}^{\star} u\|_{L^p(\mathbb{R}; L^p((0, \infty), w))} \leq C\|u\|_{L^p(\mathbb{R}; L^p((0, \infty), w))}.$$ 

This estimate with $v = 1$ implies in particular the boundedness

$$P_{L_{\alpha}}^{\star} : L^p(\mathbb{R}; L^p(\mathbb{R}, w)) \to L^p(\mathbb{R}; L^p(\mathbb{R}, w; L^\infty(0, \infty))),$$

in which $P_{L_{\alpha}}^{\star}$ is defined in the proof of Theorem 5.1 by (5.3). The kernel of $P_{L_{\alpha}}^{\star}$ is given by

$$\{P_{L_{\alpha,y}}^{\star}(t, \cdot)\}_{y} : L^p(\mathbb{R}, w) \to L^p(\mathbb{R}, w; L^\infty(0, \infty))$$

$$\varphi \mapsto \int_{\mathbb{R}} \{P_{L_{\alpha,y}}^{\star}(t, x, z)\}_{y} \varphi(z) w(z)dz. \quad (5.6)$$

In the case $w = 1$, Hölder inequality and (5.5) imply that the norm of the operator above is bounded by $Ct^{-1}$. Also in this case it is easy to check that the operator norm of $\{\partial_{\alpha}P_{L_{\alpha,y}}^{\star}(t, \cdot)\}_{y}$ is bounded by $Ct^{-2}$. On the other hand, for each fixed $t$, the $L^\infty(0, \infty)$-norm of the kernel (5.6) is bounded by $Ct^{-1}e^{-c\frac{(x-z)^2}{t}}\chi_{t>0}(\chi_{z>0}) \leq C^{-1}/(x-z)$, while its gradient with respect to $z$ is bounded $Ct^{-1}/|x-z|^{1+1}$. Then, the Calderón-Zygmund theory shows that the operator norm of (5.6) is bounded by $C_{1}t^{-1}$. The argument similar to the above can guarantee that the operator norm
CALDERÓN–ZYGMUND KERNEL ON THE REAL LINE. THEREFORE, WE OBTAIN
\[ \|P_{L_\alpha}^s u(t,x)\|_{L^\infty(0,\infty)} \]
AND THE CORRESPONDING WEAK TYPE ESTIMATE WHEN \( q = 1 \). THE RELATION \( P_{L_\alpha}^{s,+} u(t,x) = \|P_{L_\alpha}^s u(t,x)\|_{L^\infty(0,\infty)} \) COMPLETES THE PROOF.

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