Tame hereditary path algebras and amenability

Sebastian Eckert

Fakultät für Mathematik, Universität Bielefeld, Bielefeld, Germany

Correspondence
Sebastian Eckert, Fakultät für Mathematik, Universität Bielefeld, Postfach 100 131, 33501 Bielefeld, Germany.
Email: seckert@math.uni-bielefeld.de

Funding information
Alexander von Humboldt Foundation; Max Planck Institute for Mathematics

Abstract
In this note we revisit the notion of amenable representation type introduced by Gábor Elek. We show that tame hereditary path algebras of quivers of extended Dynkin type over any field \( k \) are of amenable type. This verifies a conjecture of Elek, which draws similarities to the tame-wild dichotomy, for another class of tame algebras. We also show that path algebras of wild acyclic quivers over finite fields are not amenable.

MSC 2020
16G20, 16G60 (primary)

1 | INTRODUCTION

In the representation theory of algebras, an important objective is the study of categories of finitely generated modules over associative, unital algebras, finite-dimensional over a field \( k \). Given an algebra, one is interested in describing the structure of the module category, the indecomposable objects and the morphisms between them. Drozd [9] showed for algebraically closed fields that one can distinguish disjointly algebras of finite (there are only finitely many non-isomorphic finite-dimensional indecomposables), tame (in every dimension, the isomorphism classes of indecomposables — apart from finitely many exceptions — are parametrised by a finite number of one-parameter families) and wild (the classification problem roughly speaking already gives a classification for all finite-dimensional algebras, and is considered hopeless) representation type.

For arbitrary base fields, this trichotomy is not known. In addition, the precise notion of a one-parameter family depends on the base field, introducing additional complexity. Therefore and also inspired by a question of Ringel, one is interested in different characterisations of tameness that rely only on the module category. Crawley-Boevey [6] has studied generic modules, which are infinite-dimensional representations, and introduced the concept of generic tameness (for every natural number, there are only finitely many generic modules with this endolength), while
Krause [15] used fp-idempotent ideals to give a characterisation of tame algebras (for every natural number, there are only finitely many non-zero fp-idempotent and nilpotent ideals contained in the ideal of maps factoring through a finite coproduct of indecomposables of this endlength).

In [10], Elek has introduced the notions of amenable representation type and hyperfiniteness (HF) for countable sets of (finite-dimensional) modules. We deviate slightly and give the following definition, as we extend the notion of HF to arbitrary families and that of amenability to representation finite algebras.

**Definition 1.** Let $k$ be a field, $A$ a finite-dimensional $k$-algebra and let $\mathcal{M} \subseteq \text{mod } A$ be a family of finite-dimensional $A$-modules. We say that $\mathcal{M}$ is hyperfinite provided for every $\varepsilon > 0$ there exists $L_\varepsilon > 0$ such that for every $M \in \mathcal{M}$ there exists a submodule $N \subseteq M$ such that

$$\dim_k N \geq (1 - \varepsilon) \dim_k M,$$

(1)

and there are modules

$$N_1, N_2, \ldots, N_t \in \text{mod } A, \text{ with } \dim_k N_i \leq L_\varepsilon,$$

(2)

such that

$$N \cong \bigoplus_{i=1}^t N_i.$$  

(3)

The $k$-algebra $A$ is said to be of amenable representation type provided the set of all finite-dimensional $A$-modules (or more precisely, a set which meets every isomorphism class of finite-dimensional $A$-modules) is hyperfinite.

This definition characterises families of modules by the existence of submodules of nearly the same size that are direct sums of modules of bounded dimension. In this way, one can differentiate the $k$-algebras of amenable and non-amenable representation type. The notions are based on a novel quantitative approach to the representation theory of finite-dimensional algebras motivated by the emerging theory of graph limits and are inspired by the classical concepts of amenability and hyperfiniteness, and similar notions for skew fields and hyperfinite collections of graphs.

Elek further conjectures [10, Conjecture 1] that finite-dimensional algebras are of tame representation type if and only if they are of amenable representation type, thus yielding a dichotomy similar to Drozd’s trichotomy. While string algebras (over countable fields) are shown to be of amenable type in [10, Proposition 10.1], further examples are still lacking.

Recall that a quiver $Q$ is a directed graph on a finite set of vertices $Q_0$ having a finite set $Q_1$ of arrows. Denote the starting and terminating vertex of an arrow $\alpha : i \to j$ by $s(\alpha) = i$ and $t(\alpha) = j$, respectively. To a quiver $Q$, we associate the *path algebra* $kQ$ with basis the set of all paths in the quiver, including a path of length zero corresponding to each vertex. The multiplication is induced by concatenation of composable paths. Gabriel’s Theorem [11, Satz 1.2] then says that for a finite, connected quiver $Q$, the algebra $kQ$ is of finite representation type if and only if $Q$ is a Dynkin quiver, while a result by Donovan–Fraislish and Nazarova [8, 16] shows that a quiver $Q$ not of finite type is of tame type if and only if $Q$ is an extended Dynkin quiver.

In this note we prove the conjecture mentioned above for path algebras of acyclic quivers, exhibiting a new class of algebras of tame and amenable representation type.
Main Theorem. Let $Q$ be an acyclic quiver of extended Dynkin type $\widetilde{A}_n$, $\widetilde{D}_n$, $\widetilde{E}_6$, $\widetilde{E}_7$, or $\widetilde{E}_8$. Let $k$ be any field. Then the path algebra $kQ$ of $Q$ is of amenable representation type.

Moreover, if $k$ is a finite field and $Q$ is an acyclic quiver of wild type, then the path algebra $kQ$ of $Q$ is not of amenable type.

In order to prove our Main Theorem, we will first deduce some general results for hyperfinite families in Section 2. In Section 3, we will review some background on tame hereditary path algebras and discuss several technical lemmas, before studying the case of the 2-Kronecker quiver in Section 4. This result will then be used to prove the first half of our Main Theorem in Section 5, using the perpendicular calculus of [12] to reduce the number of vertices of the quivers, leading to a descent argument. The remaining statement will be shown in Section 6, using a similar simplification and suitable functors.

2 | HYPERFINITENESS AND AMENABILITY

We shall start the discussion by a further inspection of Definition 1.

*Remark.* Since $N$ is a submodule of $M$, the condition in (1) in the above definition is equivalent to

$$\dim_k(M/N) \leq \varepsilon \dim_k M$$

(4) because $\dim_k(M/N) = \dim_k M - \dim_k N$.

*Remark.* Finite sets (of finite-dimensional modules) are hyperfinite, since we can take $L_\varepsilon$ to be the maximum of the dimensions and put $N = M$. For the same reason, families of modules of bounded dimension are hyperfinite. Moreover, if $\mathcal{M}, \mathcal{M}'$ are hyperfinite families, so is $\mathcal{M} \cup \mathcal{M}'$: We can choose $L_{\varepsilon}$ to be the maximum of $L_{\varepsilon}^\mathcal{M}$ and $L_{\varepsilon}^\mathcal{M}'$, corresponding to $\mathcal{M}$ and to $\mathcal{M}'$, respectively. Similarly, any finite union of hyperfinite families is hyperfinite.

*Proposition 2.* Let $\mathcal{M}$ be a family of $A$-modules. If $\mathcal{M}$ is hyperfinite, so is the family of all finite direct sums of modules in $\mathcal{M}$.

*Proof.* Let $\mathcal{M}$ be hyperfinite and let $\varepsilon > 0$. Then there exists $L_{\varepsilon}$ satisfying the conditions in the definition. Now assume $X = \bigoplus_{i=1}^n M_i$ with $M_i \in \mathcal{M}$. For each $1 \leq i \leq n$, choose

$$\bigoplus_{j=1}^{t_i} N_{i,j} = N_i \subseteq M_i,$$

as for the HF of $\mathcal{M}$. Then

$$N := \bigoplus_{i=1}^n N_i \subseteq \bigoplus_{i=1}^n M_i = X,$$
as direct sums respect submodule inclusions. Also

\[
\dim_k N = \sum_{i=1}^{n} \dim_k N_i \geq (1 - \varepsilon) \sum_{i=1}^{n} \dim_k M_i = (1 - \varepsilon) \sum_{i=1}^{n} \dim_k M_i = (1 - \varepsilon) \dim_k X.
\]

Moreover, \(\dim_k N_{i,j} \leq L_\varepsilon\).

This shows that to check amenability, it is enough to check the criterion on the family of all indecomposable modules.

**Example 3.** Given an algebra \(A\) of finite representation type, there are only finitely many isomorphism classes of (finitely generated) indecomposable modules. Denoting this class by \(\text{ind} A\), their \(k\)-dimensions are bounded by

\[
\max_{M \in \text{ind} A} \{\dim_k M\}.
\]

This shows that such an algebra \(A\) is of amenable representation type.

A non-example is given in [10, Theorem 6] by the wild, generalised Kronecker algebras over countable fields, while string algebras were shown to be of amenable representation type, including the 2-Kronecker algebra (see also Theorem 16).

**Proposition 4.** Let \(A\) be a finite-dimensional \(k\)-algebra and \(\mathcal{M}, \mathcal{N} \subseteq \text{mod} A\) where \(\mathcal{N}\) is hyperfinite. If there is some \(H \geq 0\) such that for all \(M \in \mathcal{M}\), there exists a submodule \(N \subseteq M\) with \(N \in \mathcal{N}\), of codimension less than or equal to \(H\), then \(\mathcal{M}\) is also hyperfinite.

**Proof.** Let \(H \geq 0\) and \(\varepsilon > 0\). If the dimension of the modules in \(\mathcal{M}\) was bounded, say by \(K\), we can set \(L_\varepsilon := K\) and choose \(N = M\) for all \(M \in \mathcal{M}\), and we are done. On the other hand, if the dimension is not bounded, let us consider \(M \in \mathcal{M}\) indecomposable with \(\dim_k M > \frac{2H}{\varepsilon}\). We choose a submodule \(N \in \mathcal{N}\) of codimension bounded by \(H\). Since \(\mathcal{N}\) is hyperfinite, there is some submodule \(Y \subseteq N\) such that \(\dim Y \geq (1 - \frac{\varepsilon}{2}) \dim N\), while \(Y\) decomposes into direct summands of dimension less than or equal to \(\frac{L_\varepsilon N}{2}\).

We thus have that

\[
\dim Y \geq (1 - \frac{\varepsilon}{2}) \dim N = \dim N - \frac{\varepsilon}{2} \dim N \geq (\dim M - H) - \frac{\varepsilon}{2} \dim M \geq \dim M - \frac{\varepsilon}{2} \dim M - \frac{\varepsilon}{2} \dim M = (1 - \varepsilon) \dim M,
\]

since \(H \leq \frac{\varepsilon}{2} \dim M\). Moreover, \(Y\) decomposes into direct summands of dimension less than or equal to \(\frac{L_\varepsilon N}{2}\). If we therefore choose \(L_\varepsilon\) to be the maximum of \(\frac{L_\varepsilon N}{2}\) and \(\frac{2H}{\varepsilon}\), we have shown that \(\mathcal{M}\) is hyperfinite. \(\square\)
Proposition 5. Let $k$ be a field and let $A, B$ be two finite-dimensional $k$-algebras. Let $\mathcal{N} \subseteq \text{mod } A$ be a hyperfinite family. Let $K_1, K_2 > 0$. If $F : \text{mod } A \to \text{mod } B$ is an additive, left-exact functor such that

\begin{align}
K_1 \dim X &\leq \dim F(X), \quad \text{for all } X \in \mathcal{N}, \\
\dim F(X) &\leq K_2 \dim X, \quad \text{for all } X \in \text{mod } A,
\end{align}

then the family

$$
\mathcal{M} := \{F(X) : X \in \mathcal{N}\} \subseteq \text{mod } B
$$

is also hyperfinite.

Proof. By the hypothesis, for any $\varepsilon$ we can find some $L_{\varepsilon}^N > 0$ to exhibit the HF of the family $\mathcal{N}$. Let $M \in \text{mod } B$ such that there exists some $N \in \mathcal{N}$ with $F(N) = M$. Then there is a submodule $P \subseteq N$ such that $P = \bigoplus_{i=1}^l P_i$ with $\dim P_i \leq L_{\varepsilon}^N$ and $\dim P \geq (1 - \varepsilon) \dim N$. Since $F$ is additive, we have that $F(P) = \bigoplus_{i=1}^l F(P_i)$, and by (6),

$$
\dim F(P_i) \leq K_2 \dim P_i \leq K_2 L_{\varepsilon}^N.
$$

Moreover, the sequence

$$
0 \to F(P) \to M \to F(N/P)
$$

is exact, so $F(P)$ is a submodule of $M$, and by the rank-nullity theorem,

$$
\dim F(P) \geq \dim M - \dim F(N/P) \geq \dim M - K_2 \dim N/P \geq \dim M - K_2 \varepsilon \dim N \geq \dim M - \frac{K_2}{K_1} \varepsilon \dim M = (1 - \varepsilon) \dim M,
$$

if we choose $\varepsilon = \frac{K_1}{K_2} \varepsilon$. We can therefore choose $L_{\varepsilon}$ to be $K_2 L_{\varepsilon}^N$ to show the HF of $\mathcal{M}$. \qed

Definition 6. A functor $F$ satisfying the conditions of Proposition 5 is called HF-preserving.

Example 7.

(a) Equivalences are HF-preserving functors: they are left exact and the fact that simple modules are mapped to simple modules ensures the existence of suitable constants $K_1$ and $K_2$ (see the proof of Proposition 17 for the details on achieving the constants).

(b) If $A$ is the path algebra of a quiver $Q$ of amenable representation type, and $i \in Q_0$ is a sink, then we can take the Bernstein–Gelfand–Ponomarev (BGP) reflection functor $F = S_i^+$ to show that $kQ'$, where $Q' = \sigma_i(Q)$ is the quiver obtained from $Q$ by reversing all arrows starting or ending in $i$, is also amenable (see Section 3 for details on these functors). To see this, let $C = C_i$ be the full subcategory of $\text{mod } kQ'$ of objects having no direct summand isomorphic to $S(i)$. 
Then every indecomposable object of \( \text{mod } kQ' \) is either contained in \( C \) or is isomorphic to \( S(i) \). As \( C \) is in the essential image of \( F \), the family of indecomposable modules of \( \text{mod } kQ' \) is hyperfinite, proving the claim.

As a consequence of Proposition 5, we can also propagate the existence of non-hyperfinite families.

**Proposition 8.** Let \( k \) be a field and \( A, B \) be two finite-dimensional \( k \)-algebras. Let \( \{M_i : i \in I\} \subseteq \text{mod } A \) be a non-hyperfinite family of modules. Let \( K_1, K_2 > 0 \). If there exist additive functors \( F : \text{mod } A \to \text{mod } B \) and \( G : \text{mod } B \to \text{mod } A \) such that

- \( GF(M_i) \cong M_i \) for all \( i \in I \);
- \( G \) is left exact;
- \( K_1 \dim F(M_i) \leq \dim GF(M_i) \) for all \( i \in I \);
- \( \dim G(X) \leq K_2 \dim X \) for all \( X \in \text{mod } B \);

then \( \{ F(M_i) : i \in I \} \subseteq \text{mod } B \) is a non-hyperfinite family.

**Proof.** Consider the family \( \{ F(M_i) : i \in I \} \) in \( \text{mod } B \). Assume that it is hyperfinite. By Proposition 5 we know that \( \{ GF(M_i) : i \in I \} = \{ M_i : i \in I \} \) is hyperfinite. This yields a contradiction, so \( \{ F(M_i) : i \in I \} \) cannot be hyperfinite. \( \square \)

### 3 \ | \ EXTENDED DYNKIN QUIVERS

#### 3.1 | Preliminaries

From now on, let \( k \) be any field and \( Q \) be an acyclic, extended Dynkin quiver, that is, a quiver of type \( \tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \) or \( \tilde{E}_8 \) which has no oriented cycles. For more details on the following notions, we refer to [7, 18, 19].

We shall consider the category \( \text{mod } kQ \) of finite-dimensional left modules over the path algebra \( kQ \), which is equivalent to the category \( \text{rep}_k(Q) \) of finite-dimensional \( k \)-linear representations of \( Q \). Here, a *representation* \( (M_i, M_\alpha) \) of \( Q \) is given by a collection \( \{M_i : i \in Q_0\} \) of finite-dimensional \( k \)-vector spaces along with a collection of \( k \)-linear maps \( \{M_\alpha : \alpha \in Q_1\} \). It is well known that these categories are tame and hereditary and have the Krull–Schmidt property. We shall write \( \dim M \) for the *dimension vector* of the representation \( M \), where \( (\dim M)_i = \dim M = \dim_k M_i \). A special role will be played by the projective indecomposable modules \( P(i) \) corresponding to the vertices \( i \) of the quiver. They are the direct summands of \( kQ \) when viewed as a module over itself.

Recall also the following method to move between different orientations of a quiver: Given a sink \( i \in Q_0 \), denote by \( \sigma_i Q \) the quiver obtained by reversing all arrows involving \( i \). The *BGP reflection functor*

\[
S_i^+ : \text{rep}_k(Q) \to \text{rep}_k(\sigma_i Q)
\]

induces a bijection between the indecomposable representations of \( Q \) and \( \sigma_i Q \), if we ignore the simple representation corresponding to vertex \( i \) (see, for example, [7, Chapter 2]). Now, for any two orientations of an acyclic quiver, there is a sequence of reflections to move between them.
Also note that the action of $S^+_i$ on the dimension vectors is given by a linear transformation $s_i : \mathbb{Z}\langle Q_0 \rangle \to \mathbb{Z}\langle \sigma_i Q_0 \rangle$.

The Euler bilinear form of a quiver is defined on $\mathbb{Z}\langle Q_0 \rangle$ by

$$\langle x, y \rangle := \sum_{i \in Q_0} x_i y_i - \sum_{\alpha \in Q_1} x_{s(\alpha)} y_{t(\alpha)},$$

and agrees with the homological bilinear form on the dimension vectors of the representations, that is,

$$\langle \dim X, \dim Y \rangle = \dim_k \Hom(X, Y) - \dim_k \Ext^1_k Q(X, Y). \quad (7)$$

Recall the Tits form $q : \mathbb{Z}\langle Q_0 \rangle \to \mathbb{Z}$, that is, the corresponding quadratic form, $q(x) = \langle x, x \rangle$, which is positive semidefinite in the tame case, and its radical

$$\text{rad } q = \{ x \in \mathbb{Z}\langle Q_0 \rangle : q(x) = 0 \}.$$ 

It is one-dimensional and generated by the minimal radical vector $h_Q$. Note that $h_Q$ is independent of the orientation of the arrows of $Q$ and that there is an indecomposable module $T$ with $\dim T = h_Q$. Furthermore, there is a linear form $\partial$, the defect, such that

$$\partial(X) = \partial(\dim X) = \langle h_Q, \dim X \rangle.$$ 

This forms allows us to distinguish between preprojective ($\partial < 0$), regular ($\partial = 0$) and preinjective ($\partial > 0$) indecomposables of $\text{mod } kQ$. For a fixed algebra, the subcategories of the additive closures of preprojective, preinjective and regular modules will be denoted by $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$, respectively. Non-zero homomorphism between these subcategories only exist from $\mathcal{P}$ to $\mathcal{R}$ and $\mathcal{Q}$ and from $\mathcal{R}$ to $\mathcal{Q}$. We say that a module $M$ is exceptional provided it is indecomposable and has no self-extensions, that is, $\Ext^1_k Q(M, M) = 0$.

Recall also that $\text{mod } kQ$ has Auslander–Reiten (AR) sequences, giving rise to the AR translate $\tau$ and its inverse $\tau^−$. The category mod $kQ$ may be described by its AR quiver $\Gamma_k Q$, a translation quiver having vertices for each isomorphism class of indecomposable modules and arrows for the irreducible morphisms, that is, for those morphisms $g$ which are neither split monomorphisms nor split epimorphisms and for which $\gamma = ts$ implies that $s$ is a split monomorphism or $t$ is a split epimorphism (for more on the AR theory, see, for example, [2]). The translate also gives the following AR formulae

$$D \Hom_k Q(Y, \tau X) \cong \Ext^1_k Q(X, Y) \cong D \Hom_k Q(\tau^− Y, X), \quad (8)$$

where $D = \Hom_k (-, k)$ is the $k$-dual. Moreover, there exists a transformation $c$, called the Coxeter transformation, such that for any module $X$ without projective direct summands, we have

$$\dim \tau(X) = c(\dim X), \quad (9)$$

and such that

$$\langle x, y \rangle = -\langle y, c(x) \rangle = \langle c(x), c(y) \rangle \text{ for all } x, y \in \mathbb{Z}\langle Q_0 \rangle. \quad (10)$$

Now, the defect number $d_Q$ is the smallest positive integer $d$ such that

$$c^d(x) = x + \partial(x) h_Q \text{ for each } x \in \mathbb{Z}\langle Q_0 \rangle.$$
We can also use \( \tau \) to describe the preprojective and preinjective indecomposables: The preprojective indecomposable modules are of the form \( \tau^{-r}P(i) \) for some non-negative integer \( r \) and some \( i \in Q_0 \). The preinjective indecomposable modules are of the form \( \tau^rI(i) \) for some non-negative integer \( r \) and some \( i \in Q_0 \). Here, the \( I(i) \) denote the indecomposable, injective modules.

Moreover, we recall that the subcategory \( R \) of the regular modules is exact and Abelian. It decomposes into a direct sum of serial categories. Their AR quivers are tubes of the form \( \mathbb{Z}A_{\infty}/m \). The simple regular modules of each tube form a cycle under \( \tau \), and we refer to the cycle length \( m \) as the rank of the tube. At most three of the tubes are inhomogeneous, that is, have more than one simple regular module. We may therefore use a triple \((p, q, r)\) of positive integers to list the ranks of the inhomogeneous tubes and call it the tubular type of \( Q \). Note that the module \( T \) mentioned above is a simple regular module in a tube of rank one. Hence \( h_Q \) is \( c \)-invariant.

We will further use the notion of a perpendicular category of some module \( X \in \mod kQ \), defined by

\[
X^\perp := \{ Y \in \mod kQ : \Hom(X, Y) = 0 = \Ext^1_{kQ}(X, Y) \}.
\]

3.2 Perpendicular categories of simple regular modules

To proceed with the proof of the main theorem, we gather some technical lemmas. For the remainder of this section, let us introduce some notation: We denote by \( T_1, \ldots, T_m \) the isomorphism classes of the simple regular modules on the mouth of a given tube \( \mathbb{T} \) of rank \( m \) such that \( \tau T_i = T_{i-1} \) for \( i = 2, \ldots, m \) and \( \tau T_1 = T_m \). Following [19, Chapter 3], we define the objects \( T_i[\ell] \).

First, let \( T_i[1] := T_i \) for each \( 1 \leq i \leq m \). Now, for \( \ell \geq 2 \), recursively define \( T_i[\ell] \) to be the indecomposable module in \( \mathbb{T} \) with \( T_i[1] \) as a submodule such that \( T_i[\ell]/T_i[1] \cong T_{i+1}[\ell - 1] \). Thus \( T_i[\ell] \) is the regular module of regular length \( \ell \) with regular socle \( T_i \). Now, any regular indecomposable in \( \mathbb{T} \) will occur as some \( T_i[\ell] \). We may define \( T_i[\ell] \) for all \( i \in \mathbb{Z} \) by letting \( T_i[\ell] \cong T_{i}[\ell] \) iff \( i \equiv j \mod m \). Note that

\[
\dim T_i[\ell] = \sum_{j=1}^{i+\ell-1} \dim T_j.
\]

Lemma 9. Let \( k \) be any field. Let \( \mathbb{T} \) be an inhomogeneous tube of rank \( m \) in \( \Gamma_{kQ} \). Then \( \sum_{i=1}^{m} \dim T_i = h_Q \).

Proof. For a fixed orientation of each extended Dynkin quiver, the relevant tables of [7, Chapter 6] list the dimension vectors of the simple regular modules. To pass to a different orientation, we use the BGP reflection functors. In particular, by [7, Proposition 2.3], they induce isomorphisms between extension groups. Now, if \( \sum_{i=1}^{m} \dim T_i = h_Q \) for some orientation, the indecomposable representations \( S^+_k(T_i) \) of \( \sigma_kQ \) will again lie in the same tube: None of them is the simple representation for vertex \( k \), which is not regular because \( k \) is a sink. Thus \( S^+_k(T_i) \neq 0 \). Moreover, by the orthogonality of the tubes and [19, 3.1.(3')] the tubes and the enumeration of their regular simples are preserved under \( S^+_k \). Hence

\[
s_k(h_Q) = \sum_{i=1}^{m} s_k(\dim T_i) = \sum_{i=1}^{m} \dim S^+_k(T_i).
\]
Since $h_Q$ is invariant under $s_k$ and the minimal radical vector is independent of the orientation, the claim follows. □

**Lemma 10.** The entries of the minimal radical vector $h_Q$ give the absolute values of the defect for the indecomposable preprojective and preinjective modules, in particular,

$$-\delta(P(i)) = (h_Q)_i = \delta(I(i)).$$

**Proof.** Let $X = \tau^{-r}P(i)$ be an indecomposable preprojective module. Put $d = -\delta(X)$. Since $c(h_Q) = h_Q$, we have that

$$d = -\langle h_Q, \dim X \rangle \overset{(10)}{=} \langle \dim X, h_Q \rangle.$$

Now use that the bilinear form is invariant under $c$ to see that

$$\langle \dim X, h_Q \rangle = \langle \dim P(i), h_Q \rangle = (h_Q)_i,$$

where the last equality follows from $\langle \dim P(i), e_j \rangle = \delta_{ij}$ (see [19, Proof of Lemma in Section 2.4]). Hence $h_Q$ has $i$th component equal to $d$. A similar argument can be made for an indecomposable preinjective module $Y = \tau^r I(i)$. □

**Lemma 11.** Let $\mathbb{T}$ be a tube of rank $m \geq 2$ in $\Gamma_{kQ}$. Let $X$ be an indecomposable regular module in $\mathbb{T}$. Then there exists a submodule $Y \subseteq X$ of codimension bounded by the sum of the entries of $h_Q$ and a simple regular module $T \in \mathbb{T}$ such that $Y \in T^\perp$.

**Proof.** By [19, 3.1.(3')], we have that

$$\langle \dim T_i, \dim T_j \rangle = \begin{cases} 1, & i \equiv j \mod m, \\ -1, & i \equiv j + 1 \mod m, \\ 0, & \text{else}. \end{cases}$$

Using (11), for any $j \in \mathbb{Z}$, we have $\langle \dim T_j, \dim T_i[\ell'] \rangle = 0$, provided $\ell' \equiv 0 \mod m$. Since $T_i[\ell]$ is uniserial, $\Hom(T_j, T_i[\ell]) = 0$ if and only if $j \not\equiv i \mod m$. Because of (7), this implies that for all $i \in \mathbb{Z}$ and $\ell' \equiv 0 \mod m$, $T_i[\ell]$ is contained in the perpendicular category of some simple regular, that is in $T^\perp_{i+1}$. Note that $i + 1 \not\equiv i \mod m$ since $m \geq 2$.

If $\ell' \equiv 0 \mod m$, then write $\ell' = n \cdot m + r$, where $0 < r < m$. Then there is a short exact sequence

$$0 \to T_i[nm] \to T_i[\ell] \to Z \to 0,$$

and we have $\dim Z \leq h_Q$, using (11) and Lemma 9. Thus, we have found a suitable submodule $T_i[nm] \in T^\perp_{i+1}$. □

**Lemma 12.** Let $X = \tau^{-r}P(i)$ be some indecomposable preprojective $kQ$-module of defect $\delta(X) = -d < 0$. Let $\mathbb{T}$ be a tube of rank $m > d$. Then there is a simple regular $S \in \mathbb{T}$ such that $X \in S^\perp$.

**Proof.** By Lemma 9, $\sum_{j=1}^m \dim T_j = h_Q$ and by Lemma 10, $(h_Q)_i = d$. Now, since $m > d$, only up to $d$ out of the $m$ modules can have a vector space at vertex $i$ which is non-zero. Let $j_0$ be such
that $(\dim T_j)_i = 0$. Given $1 \leq j \leq m$, using (10), we have
\[
\langle \dim T_j, \dim \tau^{-r} P(i) \rangle = -\langle \dim \tau^{-r} P(i), c(\dim T_j) \rangle = -(\dim P(i), c^{r+1}(\dim T_j)) = -(\dim P(i), \dim T_{j-r-1}) = -(\dim T_{j-r-1})_i.
\]
Now, choose $j$ such that $j - r - 1 \equiv j_0 \mod m$. Since $T_j$ is regular and $X$ is preprojective, we have $\Hom(T_j, X) = 0$, so by (7) we must have that $\Ext^1_{kQ}(T_j, X) = 0$. Thus, $X \in T_j^\perp$. □

Indeed, a slightly stronger result can be shown.

**Lemma 13.** Let $X = \tau^{-r} P(i)$ be some indecomposable preprojective $kQ$-module of defect $\delta(X) = -d < 0$ and $r > d_Q$. Let $\mathbb{T}$ be an inhomogeneous tube of rank $m > \frac{d}{2}$. Then one of the following holds:

(a) There exists a simple regular module $S \in \mathbb{T}$ such that $X \in S^\perp$.

(b) There exists a submodule $Y \subseteq X$ and simple regular modules $S, T \in \mathbb{T}$ such that $0 \to Y \to X \to T \to 0$ is exact and $Y \in S^\perp$.

**Proof.** By Lemma 9, $\sum_{j=1}^m \dim T_j = h_Q$ and by Lemma 10, $(h_Q)_i = d$. We will write $d_j = (\dim T_j)_i$ and have $\sum_{j=1}^m d_j = d$.

Now, for $1 \leq j \leq m$,
\[
\dim \Hom(X, T_j) - \dim \Ext^1_{kQ}(X, T_j) \stackrel{(7)}{=} \langle \dim X, \dim T_j \rangle = \langle \dim \tau^{-r} P(i), \dim T_j \rangle \quad \stackrel{(10)}{=} \langle \dim P(i), \dim T_{j-r} \rangle = (\dim T_{j-r})_i = d_{j-r}.
\]

Note that $\dim \Ext^1_{kQ}(X, T_j) = \dim D \Hom(T_{j+1}, X) = 0$ by (8) and the fact that there are no non-zero maps from the regular to the preprojective component. Thus, if there is some $j$ such that $(\dim T_{j-r})_i > 0$, we can choose some non-zero map $\vartheta \in \Hom(X, T_j)$. The image $\text{im} \vartheta \subseteq T_j$ must be regular or has a preprojective summand. If there was a non-zero preprojective summand $Z$, it must be to the right of $X$ in $\Gamma_{kQ}$. But for any preprojective module $M$ in the $r$th translate of the projectives or further to the right in the AR quiver, we know that $\dim M = \dim M - \delta(M) h_Q > h_Q$ by the definition of the defect number. On the other hand, $\dim \text{im} \vartheta \leq \dim T_j \leq h_Q$, a contradiction. Thus, $\text{im} \vartheta$ must be regular. Since $T_j$ is a simple regular module, this implies that $\vartheta$ is surjective. We therefore have an exact sequence
\[
0 \to Y \to X \to T_j \to 0,
\]
by letting $Y := \ker \vartheta$. Applying $\Hom(-, T_j)$, we get an exact sequence
\[
\xi : 0 \to \Hom(T_j, T_j) \to \Hom(X, T_j) \to \Hom(Y, T_j) \to \Ext^1(T_j, T_j) \to \Ext^1(X, T_j).
\]
Since $T_j$ is an inhomogeneous simple regular module, there are no self-extensions, and we have $\Ext^1_{kQ}(T_j, T_j) = 0$. Hence, $\xi$ becomes the short exact sequence
\[
\xi' : 0 \to \End(T_j) \to \Hom(X, T_j) \to \Hom(Y, T_j) \to 0.
\]
Now, assume $d_{j-r} > 1$ for all $j$. Using the hypothesis, we would have

$$2m \leq \sum_{j=1}^{m} d_j = d < 2m,$$

a contradiction. Therefore, there is some $j_0$ with $d_{j_0-r} < 1$.

If $d_{j_0-r} = 1$, we have that $\dim \text{Hom}(X, T_{j_0}) = 1$, so the exact sequence $\xi''$ implies that $\text{Hom}(Y, T_{j_0})$ must be zero by dimension arguments, for $\dim \text{End}(T_{j_0}) \geq 1$. Hence, using (8),

$$\text{Ext}^1_{kQ}(T_{j_0+1}, Y) = D \text{Hom}(Y, T_{j_0}) = 0.$$

Along with the fact that $\text{Hom}(T_{j_0+1}, Y) = 0$, since there are no non-zero maps from regular to preprojective modules, this implies that $Y \in T_{j_0+1}^\perp$ and we are in case (b).

If $d_{j_0-r} = 0$, we have that $\dim \text{Hom}(X, T_{j_0}) = 0$. So, similarly,

$$\text{Ext}^1_{kQ}(T_{j_0+1}, X) = 0, \quad \text{and} \quad \text{Hom}(T_{j_0+1}, X) = 0,$$

and $X \in T_{j_0+1}^\perp$, showing that we are in case (a). \qed

To prove HF results for preinjective indecomposables, we will descend on the defect using the following lemmas.

**Lemma 14.** Let $X = \tau^i I(i)$ be some indecomposable preinjective $kQ$-module of defect $\delta(X) = d$. Then there is an injective module $I(j)$ such that there exists a non-zero homomorphism $\theta : X \to I(j)$ and for any direct summand $Z$ of $\ker \theta$, we have that $\delta(Z) < d$.

**Proof.** Let $E(X)$ be the injective envelope of $X$, and take $I(j)$ to be some indecomposable direct summand of $E(X)$. This yields a non-zero homomorphism $\theta : X \to E(X) \to I(j)$. Consider the exact sequence

$$0 \to \ker \theta \to X \to \text{im} \theta \to 0.$$

Since there is a non-zero map from a preinjective module to $\text{im} \theta$, the latter must be preinjective or zero. Yet, $\text{im} \theta \neq 0$, since $\theta$ is non-zero. Thus, $\text{im} \theta$ has positive defect, implying that $\delta(\ker \theta) < \delta(X)$. If $\ker \theta$ only had preprojective or regular summands $Z$, we are done, for then $\delta(Z) \leq 0$. Thus, we may assume that there is some preinjective direct summand $Z$. Since $Z$ embeds into $\ker \theta$ and the kernel embeds into $X$, we get a short exact sequence

$$0 \to Z \to X \to X/Z \to 0.$$

Since $X$ is preinjective, again $X/Z$ must be preinjective or zero. If it was zero, then $Z \cong \ker \theta \cong X$, a contradiction, since $\delta(\ker \theta) \neq \delta(X)$. Thus, $\delta(X/Z) > 0$, and hence we may conclude that the defect $\delta(Z) < \delta(X) = d$. \qed

**Lemma 15.** Let $A$ be a tame hereditary algebra. Assume that the preprojective indecomposable modules $\text{ind} \ P$ and the regular indecomposable modules $\text{ind} \ R$ form hyperfinite families. Then $\text{ind} \ Q$ is also hyperfinite.
Proof. By Lemma 14, for each indecomposable preinjective module $X$, we can find a submodule $Y := \ker \theta$ of strictly smaller defect. Moreover, if $Y$ had a preinjective summand $Z$, it must have defect $\delta(Z) < \delta(X)$. We do an induction on the defect $d$. If $d = 1$, then we can choose the hyperfinite family $\mathcal{N}_0 = \mathcal{P} \cup \mathcal{R}$ of all preprojective and regular modules. For all preinjective indecomposables of defect $d = 1$, the submodule $Y$ must be in $\text{add } \mathcal{N}_0$, since there are no non-zero preinjective modules $Z$ with defect $\delta(Z) < 1$. The family $\mathcal{N}_0$ is hyperfinite by the hypothesis and Proposition 2. Moreover, the codimension of $Y$ is bounded by the dimension of the indecomposable injectives, of which there are only finitely many. Hence, we can use Proposition 4 to prove the HF of the indecomposable preinjective modules of defect one. We recursively define

$$\mathcal{N}_d := \mathcal{N}_{d-1} \cup \{\text{indecomposable preinjectives of defect } d\}.$$ 

Note that the base case implies that $\mathcal{N}_1$ is hyperfinite. For the induction, note that Lemma 14 also yields a submodule in $\text{add } \mathcal{N}_d$ for every indecomposable preinjective of defect $d + 1$ of bounded codimension. Assuming the HF of $\mathcal{N}_d$, Proposition 4 yields that $\mathcal{N}_{d+1}$ is hyperfinite. This concludes the induction, as the defect of the indecomposable modules is bounded. \qed

4 | THE 2-KRONECKER QUIVER CASE

The theorem of this section will be used as the base case in the proof of our main result. In the setting of [10, Proposition 10.1], it is a corollary to Elek’s result for countable, algebraically closed fields, as the path algebra of the 2-Kronecker quiver is a string algebra. In the following, we give a direct proof of the result for the Kronecker quiver different from Elek’s in our slightly more general setting.

**Theorem 16.** Let $k$ be any field. Then the path algebra of the 2-Kronecker quiver $\Theta(2)$ is of amenable representation type.

**Proof.** We fix notation for the vertices and arrows as follows.

![Diagram](image)

It is well known (see, for example, [4, Theorem 4.3.2] or [5]) that the indecomposable preprojective and preinjective $k$-representations of this quiver $Q$ are given by

$$P_n : \begin{cases} k^n & \xrightarrow{\text{id}} k^{n+1}, \\
0 & \xrightarrow{\text{id}} k^n \end{cases}, \quad Q_n : \begin{cases} k^{n+1} & \xrightarrow{\text{id} 0} k^n, \\
0 & \xrightarrow{0 \text{id}} k^{n+1} \end{cases}.$$
both for \( n \geq 0 \), while the indecomposable regular representations can be parametrised by

\[
R_n(\phi, \psi) : \quad k^n \xrightarrow{\phi} k^n, \quad \psi
\]

where either \( \phi \) is the identity and \( \psi \) is given by the companion matrix of a power of a monic irreducible polynomial over \( k \), or \( \psi \) is the identity and \( \phi \) is given by the companion matrix of a monomial.

We will show that each, the preprojective component \( \mathcal{P} \), the regular component \( \mathcal{R} \) and the preinjective component \( \mathcal{Q} \), are hyperfinite families to conclude the amenability of \( \text{mod } k\mathcal{Q} \). We will give an argument for the indecomposable objects in each component and then apply Proposition 2 to extend the result.

We start with the preprojectives, and let \( \varepsilon > 0 \). Set \( K_\varepsilon := \lceil \frac{1}{2\varepsilon} \rceil + 1 \) and \( L_\varepsilon = \frac{1}{\varepsilon} + 3 \). Let \( X = P_n \) be some indecomposable preprojective. If \( \dim X \leq L_\varepsilon \), there is nothing to show. We may thus assume that \( \dim X > L_\varepsilon \), implying \( n \geq K_\varepsilon \), and write \( n = j \cdot K_\varepsilon + r \), where \( 0 \leq r < K_\varepsilon \). Now consider the standard basis \( \{e_1, e_2, \ldots, e_n\} \) of \( k^n \). Let \( U \) be the submodule of \( X \) generated by the subset

\[
\{e_1, \ldots, e_{K_\varepsilon - 1}\} \cup \{e_{K_\varepsilon + 1}, \ldots, e_{2K_\varepsilon - 1}\} \cup \ldots
\]

\[
\cup \{e_{(j-1)K_\varepsilon + 1}, \ldots, e_{jK_\varepsilon - 1}\} \cup \{e_{jK_\varepsilon + 1}, \ldots, e_n\},
\]

dropping every \( K_\varepsilon \) th basis vector at the source. Then \( U \) decomposes into \( j \) direct summands of type \( P_{K_\varepsilon - 1} \) and a smaller remaining summand in case \( r \neq 0 \). All summands will thus be of \( k \)-dimension smaller than \( 2(K_\varepsilon - 1) + 1 < L_\varepsilon \). Moreover,

\[
\dim U = \dim X - j = \dim X - \frac{n - r}{K_\varepsilon} = \dim X - \frac{\dim X - 1}{2K_\varepsilon} + \frac{r}{K_\varepsilon} \\
\geq \dim X - \varepsilon(\dim X - 1) > (1 - \varepsilon) \dim X.
\]

This shows that the family of indecomposable preprojective modules \( \text{ind } \mathcal{P} \) is hyperfinite. We exemplify this process in Figure 1.

If \( X = R_n(\phi, \psi) \) is an indecomposable regular module, we may consider the submodule \( Y \) generated by the basis vectors \( \{e_1, \ldots, e_{n-1}\} \) of the vector space at vertex 1. Note that we assume that \( \psi \) corresponds to the Frobenius companion matrix of a power of a monic polynomial. Then \( Y \cong P_{n-1} \), so by the above it belongs to the hyperfinite family \( \mathcal{P} \). We have that \( \dim Y = \dim X - 1 \). Thus, an application of Proposition 4 with \( H = 1 \) gives the HF of the indecomposable regular modules.

To deal with the preinjective modules, we apply Lemma 15. Now apply Proposition 2 to \( \mathcal{P} \cup \mathcal{R} \cup \mathcal{Q} \) to see that \( \text{mod } k\mathcal{Q} \) is hyperfinite, and thus \( k\mathcal{Q} \) is amenable.
5 AMENABILITY OF EXTENDED DYNKIN QUIVERS

In order to prove our main result, we study the hyperfiniteness of perpendicular categories.

**Proposition 17.** Let $Q$ be a finite acyclic quiver.

(a) If $T \in \text{mod } kQ$ is an exceptional module without preprojective summands, $T^\perp$ is equivalent to $\text{mod } kQ'$ for some quiver $Q'$.

(b) Assume $Q$ is of tame tubular type $(p, q, r)$, where $p > 1$, and all extended Dynkin quivers of type $(p-1, q, r)$ are amenable. If $T$ is an inhomogeneous simple regular module belonging to a tube of rank $p$ in $\Gamma_{kQ}$, then $T^\perp$ is hyperfinite.

**Proof.** By [12, Proposition 1.1], in both cases, $T^\perp$ is an exact Abelian subcategory of $\text{mod } kQ$ closed under the formation of kernels, cokernels and extensions. Moreover, [12, Theorem 4.16] yields that $T^\perp = \text{mod } \Lambda$ for some finite-dimensional hereditary algebra $\Lambda$, along with a homological epimorphism $\varphi : kQ \to \Lambda$, which induces a functor $\varphi_* : \text{mod } \Lambda \to \text{mod } kQ$. By Morita equivalence, we may assume that $\Lambda$ is basic.

To see that $\Lambda$ is given as the path algebra of a quiver, we want to apply [2, Proposition III.1.13]. We must therefore check that $\Lambda/J(\Lambda)$ is a product of copies of $k$, where $J$ denotes the Jacobson radical. Since we assume that $\Lambda$ is basic, we know that $\Lambda/J(\Lambda) \cong \bigoplus D_i$ for some division rings $D_i$. Indeed, from the Wedderburn–Artin theorem, we see that $D_i \cong \text{End}(S(i))$ where the $S(i)$ are the simple $\Lambda/J(\Lambda)$-modules, which are just the simple $\Lambda$-modules.

Now, if $S$ is any simple $\Lambda$-module, then $S \cong P/\text{rad}(P)$, where $P$ is a principal indecomposable $\Lambda$-module. Note that by [1, Corollary 17.12],

$$\text{End}_\Lambda(S) \cong \text{End}_\Lambda(P)/J(\text{End}_\Lambda(P)).$$

By [12, Theorem 4.4], the natural maps

$$\text{End}_\Lambda(-) \to \text{End}_kQ(\varphi_*-) \text{ and } \text{Ext}_\Lambda^1(-,-) \to \text{Ext}_kQ^1(\varphi_*-,\varphi_*-),$$

induced by $\varphi_*$, are isomorphisms, thus $\varphi_*$ maps exceptional modules to exceptional modules.

It follows from [20, Corollary 1] that $\text{End}_\Lambda(P) \cong \text{End}_{kQ}(\varphi_*P) \cong \text{End}_{kQ}(E)$ for some simple $kQ$-module $E$. But the simple $kQ$-modules all have trivial endomorphism ring $k$. Hence $\text{End}_\Lambda(S) \cong k$ and $\Lambda$ is isomorphic to $kQ'$ for some quiver $Q'$.

It remains to prove the additional statements of (b). By the previous argument, in this situation [12, Theorem 10.1(3)] applies for any field. Indeed, $\text{mod } \Lambda$ — being contained in $\text{mod } kQ$ — is of tame type and the properties of $\varphi_*$ with respect to the defect and AR sequences yield the needed tube ranks. Hence, $\Lambda \cong kQ'$ where $Q'$ has tubular type $(p-1, q, r)$. By the hypothesis, it is amenable.

Now, if $F : \text{mod } kQ' \to \text{mod } \Lambda \to T^\perp$ is an equivalence, the simples $S(i)$ of $kQ'$ get mapped to certain modules $B_i$ in $\text{mod } kQ$. The $k$-dimension of any module $M$ over a path algebra is determined by the length of any composition series. Such a series for $M$ in $kQ'$ gets mapped to a composition series in the perpendicular category, and thus a series in $\text{mod } kQ$, such that the factor modules are isomorphic to some $B_i$. Letting $K_2 := \max\{\text{dim } B_i\}$, we thus know that

$$\text{dim}_k F(M)_{kQ} \leq K_2 \text{dim}_k M_{kQ'}.$$
TABLE 1 Tubular types and minimal radical vector of the acyclic, extended Dynkin diagrams (see, for example, [18, p. 335]).

| Q      | (m_i)  | h_Q |
|--------|--------|-----|
| \( \tilde{\mathcal{A}}_{p,q} \) | (p, q) | 1...1 |
|        |        | 1   |
|        |        | 1...1 |
|        |        | 1   |
| \( \tilde{\mathcal{D}}_n \)    | (2, 2, n - 2) | 2 ... 2 |
|        |        | 1   |
| \( \tilde{\mathcal{E}}_6 \)    | (2, 3, 3) | 2   |
|        |        | 1   |
| \( \tilde{\mathcal{E}}_7 \)    | (2, 3, 4) | 2   |
|        |        | 1   |
| \( \tilde{\mathcal{E}}_8 \)    | (2, 3, 5) | 2   |
|        |        | 1   |

On the other hand, if \( F(M) \in T^\perp \), any submodule of \( F(M) \) in \( T^\perp \) is also a submodule in mod \( kQ \), so a composition series of \( F(M) \) in mod \( kQ \) is at least as long as one in \( T^\perp \). Thus,

\[
\dim_k M_{kQ'} \leq \dim_k F(M)_{kQ},
\]

using the fact that the length of \( M \) in mod \( kQ' \) equals the length of \( F(M) \) considered as an object of \( T^\perp \). Hence by Proposition 5, we have that each \( T^\perp \) is a hyperfinite family. \( \square \)

**Theorem 18.** Let \( Q \) be an acyclic quiver of extended Dynkin type \( \tilde{\mathcal{A}}_n, \tilde{\mathcal{D}}_n, \tilde{\mathcal{E}}_6, \tilde{\mathcal{E}}_7 \) or \( \tilde{\mathcal{E}}_8 \). Let \( k \) be any field. Then the path algebra \( kQ \) of \( Q \) is of amenable representation type.

**Proof.** Recall the tubular types and minimal radical vectors \( h_Q \) of the extended Dynkin diagrams; see Table 1. Note that in the case of \( \tilde{\mathcal{A}}_n \), we need to distinguish further: We say a quiver is of type \( \tilde{\mathcal{A}}_{p,q} \), if it has \( p + q = n + 1 \) vertices, where there are \( p \) arrows in clockwise and \( q \) arrows in anti-clockwise orientation.

We will prove the claim by induction on \( n \) for the cases of the acyclic \( \tilde{\mathcal{A}}_n \) and for \( \tilde{\mathcal{D}}_n \), and use the case of \( \tilde{\mathcal{D}}_5 \) to prove it for the \( \tilde{\mathcal{E}} \)-family, stepping from 6 to 7 to 8. One may use [12, Figure 1] to visualise the steps of the induction.

**Case \( \tilde{\mathcal{A}}_n \):** Given a quiver of type \( \tilde{\mathcal{A}}_1 \), the only acyclic orientation is the 2-Kronecker quiver, for which its path algebra has been shown to be of amenable type in Theorem 16.

Now assume all acyclic quivers \( \tilde{\mathcal{A}}_n \), for some \( n \geq 1 \), are of amenable representation type. Let \( Q \) be any quiver of type \( \tilde{\mathcal{A}}_{p,q} \) with \( p + q = n + 2 \geq 3 \). We may thus assume that \( p \geq 2 \), and choose a tube \( T \) of rank \( m := p \), and denote the isoclasses of simple regular modules in this tube by \( T_1, ..., T_m \). Using Lemma 10, we see from the minimal radical vector, that all indecomposable preprojective \( kQ \)-modules \( X \) have defect \( \delta(X) = -1 \). Then Lemma 12 implies that every indecomposable preprojective is contained in the perpendicular category \( T_i^\perp \) for some \( 1 \leq i \leq m \). By Proposition 17, each \( T_i^\perp \) is hyperfinite. This shows that the preprojectives form a hyperfinite family, using Proposition 2.

Next, we consider the regular modules. Indecomposable regular modules in a tube other than \( T \) will be contained in \( T_i^\perp \) by [19, 3.1.(3)']. By Lemma 11, any regular indecomposable in \( T \) either is contained in the perpendicular category of some simple regular in \( T \) or has a submodule of
globally bounded codimension that is in the perpendicular category of some simple regular in $\mathbb{T}$. But by the above argument, the perpendicular categories are hyperfinite. In the latter case, we can apply Proposition 4 to show the HF of these indecomposable regular modules.

For the preinjective modules, we apply Lemma 15.

Case $\tilde{D}_n$: For the case of $\tilde{D}_4$, choose a tube $\mathbb{T}$ of rank 2, and denote the simple regular modules in $\mathbb{T}$ by $S$ and $T$. In this case, as an extended Dynkin quiver of tubular type $(1,2,2)$ is one of type $\tilde{A}_{2,2}$, which is known by the above to have an amenable path algebra, Proposition 17 implies that $S^\perp$ and $T^\perp$ are hyperfinite.

All preprojective modules $X$ of defect $\partial(X) = -1$ are in $S^\perp$ or $T^\perp$ by Lemma 12. Using Lemma 13, we can find a submodule $Y$ for all but finitely many indecomposable preprojectives $X$ of defect $\partial(X) = -2$, which are not themselves in $S^\perp$ or $T^\perp$. Since the dimension vector of a simple regular modules in $\mathbb{T}$ is bounded, the conditions of Proposition 4 are satisfied for all but finitely many indecomposable preprojectives of defect $-2$. This shows that the preprojectives form a hyperfinite family.

Moreover, the regular modules are hyperfinite: If they are in a tube other than $\mathbb{T}$, they will be contained in $S^\perp$ by [19, 3.1.($3'$)]. Choosing a second inhomogeneous tube $\mathbb{T}'$ and a simple regular $U \in \mathbb{T}'$, we know that $\mathbb{T} \subset U^\perp$, which is also hyperfinite.

We are left to deal with the preinjective modules. Here, we again apply Lemma 15. This proves the claim for $\tilde{D}_4$, using Proposition 2.

Now assume the case of $\tilde{D}_n$ has been established for some $n \geq 4$. To prove the amenability of quivers of type $\tilde{D}_{n+1}$, choose $\mathbb{T}$ to be the unique tube of maximal rank $n - 1$. Similar to the base case $\tilde{D}_4$, $S^\perp$ is of amenable type for $S \in \mathbb{T}$, since the tubular type $(2,2,(n + 1) - 2 - 1)$ belongs to $\tilde{D}_n$. By inspection of Table 1 and using Lemma 10, we see that the indecomposable preprojective have defect $-1$ or $-2$. Hence, they are in a hyperfinite family by Lemma 12. The regular indecomposables are hyperfinite by an argument similar to that of the base case. To deal with the indecomposable preinjectives, we again apply Lemma 15.

Case $\tilde{E}_n$: We proceed with $\tilde{E}_n$ for $n = 6, 7, 8$. Assume the path algebras of tubular type $(2,3, n - 4)$ have already been shown to be of amenable type. By choosing $\mathbb{T}$ to be a tube of maximal rank $m = n - 3$, we find simple regular modules $S$ such that $S^\perp$ is hyperfinite, because Proposition 17 reduces it to tubular type $(2,3, m - 1)$. Inspection via Lemma 10 tells us that any indecomposable preprojective module will have defect in absolute value less than $2m$. Thus, we can use Lemma 13 — if needed in connection with Proposition 4 — to show that all but finitely many, and hence all preprojective indecomposables form a hyperfinite family. For the indecomposable regular and preinjective modules, use the same arguments as for $\tilde{D}_n$. 

6 | NON-AMENABILITY OF WILD PATH ALGEBRAS

We say that a quiver $Q$ is wild if it has some connected component that is neither Dynkin nor extended Dynkin, that is, $Q$ has indefinite quadratic form.

Examples of such quivers include the wild, generalised Kronecker quivers $\Theta(m)$, that is, the quivers with two vertices and $m \geq 3$ parallel arrows between them.

**Proposition 19.** Let $Q$ be a quiver. If $Q$ has a subquiver $Q'$ such that $\text{mod} \ kQ'$ is not of amenable representation type, then neither is $\text{mod} \ kQ$ of amenable representation type.
Proof. Let $F : \text{mod} kQ' \to \text{mod} kQ$ be the embedding mapping any representation $M' = ((M'(i)), (M'(\alpha)))$ of $Q'$ to the representation $M = ((M(i)), (M(\alpha)))$ of $Q$ given by

$$M(i) = \begin{cases} M'(i), & i \in Q'_0, \\ 0, & \text{else}, \end{cases} \quad M(\alpha) = \begin{cases} M'(\alpha), & \alpha \in Q'_1, \\ 0, & \text{else}. \end{cases}$$

Since $\text{mod} kQ'$ is not of amenable representation type, there exists a non-hyperfinite family of modules $\{M'_j : j \in J\}$. Put $M_j = F M'_j$ and assume that $\{M_j : j \in J\}$ is hyperfinite, for otherwise we have found a non-hyperfinite sequence exhibiting the non-amenability of $\text{mod} kQ$. Note that any submodule $N$ of some $M_j$ is given by subspaces $N(i) \subseteq M_j(i)$ for each $i \in Q_0$ and linear maps $N(\alpha)$ for each $\alpha \in Q_1$ such that $\text{im} N(\alpha) \subseteq N(t(\alpha))$. Hence,

$$N' := \left( (N(i))_{i \in Q'_0}, (N(\alpha))_{\alpha \in Q'_1} \right),$$

is a subrepresentation of $M'_j$. Moreover, $\dim_k M_j = \dim_k M'_j$ and $\dim_k N = \dim_k N'$. Let $S$ be a direct summand of $N$, then each $S(i)$ is a direct summand of $N(i)$. This along with $S(i) = 0$ for all $i \in Q_0 \setminus Q'_0$ implies that $S$ also yields a direct summand $S'$ of $N'$ of dimension $\dim_k S' = \dim_k S$. Altogether, this implies that $\{M'_j : j \in J\}$ is hyperfinite, a contradiction. $\square$

**Theorem 20.** Let $k$ be a finite field and $Q$ a wild acyclic quiver. Then $\text{mod} kQ$ is not of amenable representation type.

**Proof.** If $Q$ contains the wild 3-Kronecker quiver $\Theta(3)$ as a subquiver, Proposition 19 along with [10, Theorem 6] implies that $\text{mod} kQ$ is not of amenable representation type.

If $Q$ is a wild quiver but does not contain a wild Kronecker quiver, it must have $n \geq 3$ vertices, hence $\text{mod} kQ$ has at least three isoclasses of simple modules. By [3, Theorem 2.1], which also holds for arbitrary fields, there exists a regular indecomposable module $S$ without self-extensions. Thus we can apply Proposition 17 to see that $S^\perp$ is isomorphic to $\text{mod} kQ'$ for some quiver $Q'$. Note that there is a corresponding homological epimorphism $\varphi : kQ \to kQ'$ and the induced functor $F = \varphi_* : \text{mod} kQ' \to \text{mod} kQ$ is fully faithful and exact. Indeed, $Q'$ has $n - 1$ vertices and [3, Theorem 4.1] shows that $kQ'$ is a wild quiver algebra.

Now, if we have some equivalence $\text{mod} kQ' \sim S^\perp$, the simples $S(i)$ of $kQ'$ are mapped to certain objects $B_i$, considered as modules in $\text{mod} kQ$. The $k$-dimension of any module $M$ over a path algebra is determined by the length of any composition series. Such a series for some $M'$ in $kQ'$ is mapped to a composition series in the perpendicular category, and thus a series in $\text{mod} kQ$, such that the factor modules are isomorphic to some $B_i$. This shows that

$$\dim_k F(M')_{kQ} \leq \max_{i=1,\ldots, n-1} \{\dim_k B_i\} \dim_k M'_{kQ'}. $$

Moreover, by [12, Proposition 1.1], $S^\perp$ is closed under kernels, cokernels and extensions. There is a (relative) projective generator $Z$ of $\text{mod} kQ'$, that is, for all $M' \in \text{mod} kQ'$, we have $\text{Ext}^1_{kQ'}(Z, M') = 0$ and there is $Z^r \to M'$. Now [13, Proposition A.1] shows that there exists a right adjoint to the inclusion $F$, which we denote by $G : \text{mod} kQ \to \text{mod} kQ'$. Indeed, for $M \in \text{mod} kQ$, $G(M)$ is given as a factor module of a right add($Z$)-approximation $Z_M$ of $M$. Since we may assume that $Z_M \cong Z \otimes_{\text{End}(Z)} \text{Hom}(Z, M)$, this implies that

$$\dim_k G(M) \leq \dim_k Z_M \leq \dim_k Z^2 \dim_k M.$$
Moreover, $G$ is left exact. Since $F$ is fully faithful, we have that $X \sim GF(X)$ for all $X \in \mod kQ'$.

To conclude the proof, just note that we have prepared a descent argument leading to a wild quiver with two vertices, which has to include a wild Kronecker quiver $\Theta(m)$ as a subquiver. Given any non-hyperfinite family $\{M_j\}_{j \in J}$ in $\mod k\Theta(m)$, we choose $Z$ as above and let $K_1 = \max\{\dim_k B_j\}^{-1}$ and $K_2 = (\dim_k Z)^2$. We have fulfilled the conditions of Proposition 8, which we apply to show the non-HF in each step, until we reach mod $kQ$. 

\[ \square \]

Remark. To prove the theorem, we have used [10, Theorem 6] showing that any wild Kronecker quiver is not of amenable representation type. While that result is stated in the context of finite fields, our reduction in the proof above will work over any field. 

\[ \square \]

Remark. The above theorem can also be proved more directly, by supplying a concrete embedding for each minimal wild quiver $Q$ (see [14, Section 4] for a list). In each case, we need to exhibit exceptional objects $X, Y \in \mod kQ$ such that $(X, Y)$ is an orthogonal exceptional pair, that is,

$$\text{Hom}(Y, X) = \text{Hom}(X, Y) = 0 = \text{Ext}^1_{kQ}(Y, X),$$

fulfilling the additional requirement that $m := \dim_k \text{Ext}^1_{kQ}(X, Y) \geq 3$.

Then, by [17, Lemma 1.5], there is a full exact embedding $F: \mod k\Theta(m) \to \mod kQ$. This will map the simple representations of $\Theta(m)$ to $X$ and $Y$, respectively. Now, if $M$ is a module for $k\Theta(m)$, any composition series will get mapped to a series in $\mod kQ$, such that the factor modules are isomorphic to either $X$ or $Y$. This shows that

$$\dim_k F(M) \leq \max\{\dim_k X, \dim_k Y\} \dim_k M.$$ 

Denoting the closure of the full subcategory of $\mod kQ$ containing $X$ and $Y$ under kernels, images, cokernels and extensions by $C(X, Y)$, $F$ induces an equivalence $\mod k\Theta(m) \sim C(X, Y)$, see, for example, [17, Section 1] in connection with [20, Corollary 1]. Assuming that $Z$ is a relative projective generator of $C(X, Y)$, [13, Proposition A.1] shows that there exists a right adjoint to the inclusion, which we denote by $G: \mod kQ \to C(X, Y)$. Moreover, if $M \in \mod kQ$, $G(M)$ is given as a factor module of a right $\text{add}(Z)$-approximation $Z_M$ of $M$. Since we may assume that $Z_M \cong Z \otimes_{\text{End}(Z)} \text{Hom}(Z, M)$, this implies that

$$\dim_k G(M) \leq \dim_k Z_M \leq (\dim_k Z)^2 \dim_k M.$$ 

Moreover, $G$ is left exact, and we have $GF(M) \cong M$ for all $M \in \mod k\Theta(m)$.

To conclude the proof in this case, we may chose a non-hyperfinite sequence $\{M_j\}_{j \in J}$ in $\mod k\Theta(m)$, guaranteed to exist by [10, Theorem 6]. Choose $X, Y, Z$ as above and let $K_1 = \max\{\dim_k X, \dim_k Y\}^{-1}$ and $K_2 = (\dim_k Z)^2$. We have then fulfilled the conditions of Proposition 8, which we apply to show the non-HF of $\mod kQ$. 


Examples 21.

(a) Let $Q$ be the five-subspace quiver $S(5)$. Choose the modules $X = S(5)$, the simple for vertex 5 and $Y = \tau^{-1}P(5)$. Then

$$\text{Hom}(X, Y) = 0 = \text{Hom}(Y, X),$$

since $X$ and $Y$ have disjoint support. Also, $\text{Ext}^1(Y, X) = 0$ since $X$ is injective. On the other hand, $\text{Ext}^1(X, Y) \cong k^3$ for dimension reasons.

(b) Let $Q$ be the one-point extension by a source $\infty$ at an extending vertex 0 of the subspace-oriented $\tilde{E}_6$. Choose $X = S(\infty)$ and $Y$ to be the representation induced by $\tau^{-6}P(1)$ of the underlying $\tilde{E}_6$.

(c) Let $Q$ be the one-point extension at an extending vertex 0 of a linearly oriented $D_5$ by a source $\infty$. Choose $X = S(\infty)$ and $Y$ as the representation induced by $\tau^{-3}P(3)$ of the underlying $D_5$.

(d) Let $Q$ be the one-point extension at an extending vertex 0 of a linearly oriented $D_5$ by a sink $\infty$. Choose $Y = S(\infty)$ and $X$ to be the representation induced by $\tau^{-3}P(3)$ of the underlying $D_5$.

**ACKNOWLEDGEMENTS**

This note is based on work done during the author’s doctorate studies at Bielefeld University. The author would like to thank his supervisor Professor W. Crawley-Boevey for his advice and guidance, A. Hubery for helpful discussions and two anonymous referees for valuable suggestions which improved the presentation. The author has been supported by the Alexander von Humboldt Foundation in the framework of an Alexander von Humboldt Professorship endowed by the German Federal Ministry of Education and Research. Moreover, he is grateful to the Max Planck Institute for Mathematics in Bonn for its hospitality and financial support.

Open access funding enabled and organized by Projekt DEAL.

**JOURNAL INFORMATION**

The *Bulletin of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission.
All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

REFERENCES
1. F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, 2nd ed., Graduate Texts in Mathematics, vol. 13, Springer, Berlin, 1992.
2. M. Auslander, I. Reiten, and S. O. Smalø, *Representation theory of Artin algebras*, Cambridge Studies in Advanced Mathematics, vol. 36, Cambridge University Press, Cambridge, 1995.
3. D. Baer, *A note on wild quiver algebras and tilting modules*, Commun. Algebra 17 (1989), no. 3, 751–757.
4. D. J. Benson, *Representations and Cohomology, I. Basic representation theory of finite groups and associative algebras*, vol. 30, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1998.
5. P.-F. Burgermeister, *Classification des représentations de la double flèche*, Enseign. Math. (2) 32 (1986), no. 3–4, 199–210.
6. W. Crawley-Boevey, *Tame algebras and generic modules*, Proc. Lond. Math. Soc. (3) 63 (1991), no. 2, 241–265.
7. V. Diab and C. M. Ringel, *Indecomposable representations of graphs and algebras*, Mem. Amer. Math. Soc. 6 (1976), no. 173.
8. P. Donovan and M. R. Freislich, *The Representation Theory of Finite Graphs and Associated Algebras*, vol. 5, Carleton Mathematical Lecture Notes, Carleton University, Ottawa, Ontario, 1973.
9. Y. A. Drozd, *Tame and wild matrix problems*, Representation Theory, II (Ottawa, Ontario, 1979), Lecture Notes in Mathematics, vol. 832, Springer, Berlin, 1980, pp. 242–258.
10. G. Elek, *Infinite dimensional representations of finite dimensional algebras and amenability*, Math. Ann. 369 (2017), no. 1, 397–439.
11. P. Gabriel, *Unzerlegbare Darstellungen. I*, Manuscr. Math. 6 (1972), 71–103.
12. W. Geigle and H. Lenzing, *Perpendicular categories with applications to representations and sheaves*, J. Algebra 144 (1991), no. 2, 273–343.
13. A. Hubery and H. Krause, *A categorification of non-crossing partitions*, J. Eur. Math. Soc. 18 (2016), no. 10, 2273–2313.
14. O. Kern, *Preprojective components of wild tilted algebras*, Manuscr. Math. 61 (1988), no. 4, 429–445.
15. H. Krause, *Finite versus infinite dimensional representations — a new definition of tameness*, Infinite Length Modules (Bielefeld, 1998), Trends in Mathematics, Birkhäuser, Basel, 2000, pp. 393–403.
16. L. A. Nazarova, *Representations of quivers of infinite type*, Izv. Akad. Nauk SSSR Ser. Mat. 37 (1973), no. 4, 752–791.
17. C. M. Ringel, *Representations of *K*-species and bimodules*, J. Algebra 41 (1976), no. 2, 269–302.
18. C. M. Ringel, *Infinite dimensional representations of finite dimensional hereditary algebras*, Abelian Groups and their Relationship to the Theory of Modules (Rome, 1977), vol. XXIII, Symposia Mathematica, Academic Press, London, 1979, pp. 321–412.
19. C. M. Ringel, *Tame algebras and integral quadratic forms*, vol. 1099, Lecture Notes in Mathematics, Springer, Berlin, 1984.
20. C. M. Ringel, *The braid group action on the set of exceptional sequences of a hereditary Artin algebra*, Abelian Group Theory and Related Topics (Oberwolfach, 1993), vol. 171, Contemporary Mathematics, American Mathematical Society, Providence, RI, 1994, pp. 339–352.