FPT and kernelization algorithms for the k-in-a-tree problem

Guilherme C. M. Gomes¹, Vinicius F. dos Santos¹, Murilo V. G. da Silva², and Jayme L. Szwarcfiter³,⁴

¹Departamento de Ciência da Computação, Universidade Federal de Minas Gerais – Belo Horizonte, Brazil
²Departamento de Informática, Universidade Federal do Paraná – Curitiba, Brazil
³Universidade Federal do Rio de Janeiro – Rio de Janeiro, Brazil
⁴Universidade do Estado do Rio de Janeiro – Rio de Janeiro, Brazil

Abstract

The three-in-a-tree problem asks for an induced tree of the input graph containing three mandatory vertices. In 2006, Chudnovsky and Seymour [Combinatorica, 2010] presented the first polynomial time algorithm for this problem, which has become a critical subroutine in many algorithms for detecting induced subgraphs, such as beetles, pyramids, thetas, and even and odd-holes. In 2007, Derhy and Picouleau [Discrete Applied Mathematics, 2009] considered the natural generalization to k mandatory vertices, proving that, when k is part of the input, the problem is NP-complete, and ask what is the complexity of four-in-a-tree. Motivated by this question and the relevance of the original problem, we study the parameterized complexity of k-in-a-tree. We begin by showing that the problem is W[1]-hard when jointly parameterized by the size of the solution and minimum clique cover and, under the Exponential Time Hypothesis, does not admit an n^{o(k)} time algorithm. Afterwards, we use Courcelle’s Theorem to prove fixed-parameter tractability under clique-width, which prompts our investigation into which parameterizations admit single exponential algorithms; we show that such algorithms exist for the unrelated parameterizations treewidth, distance to cluster, and distance to co-cluster. In terms of kernelization, we present a linear kernel under feedback edge set, and show that no polynomial kernel exists under vertex cover nor distance to clique unless NP ⊆ coNP/poly. Along with other remarks and previous work, our tractability and kernelization results cover many of the most commonly employed parameters in the graph parameter hierarchy.

1 Introduction

Given a graph G = (V, E) and a subset K ⊆ V(G) of size three – here called the set of terminal vertices – the three-in-a-tree problem consists of finding an induced tree of G that connects K. Despite the novelty of this problem, it has become an important tool in many detection algorithms, where it usually accounts for a significant part of the work performed during their executions.
It was first studied by Chudnovsky and Seymour \cite{1} in the context of theta and pyramid detection, the latter of which is a crucial part of perfect graph recognition algorithms \cite{2} and the former was an open question of interest \cite{3}. Across more than twenty pages, Chudnovsky and Seymour characterized all pairs $\langle G, K \rangle$ that do not admit a solution, which resulted in an $O(mn^2)$ time algorithm for the problem on $n$-vertex, $m$-edge graphs. Since then, three-in-a-tree has shown itself as a powerful tool, becoming a crucial subroutine for the fastest known even-hole \cite{4}, beetle \cite{4}, and odd-hole \cite{5} detection algorithms; to the best of our knowledge, these algorithms often rely on reductions to multiple instances of three-in-a-tree, e.g. the theta detection algorithm has to solve $O(mn^2)$ three-in-a-tree instances to produce its output \cite{6}. Despite its versatility, three-in-a-tree is not a silver bullet, and some authors discuss quite extensively why they think three-in-a-tree cannot be used in some cases \cite{5, 8}. Nevertheless, Lai et al. \cite{6} very recently made a significant breakthrough and managed to reduce the complexity of Chudnovsky and Seymour’s algorithm for three-in-a-tree to $O(m \log^2 n)$, effectively speeding up many major detection algorithms, among other improvements to the number of three-in-a-tree instances required to solve some other detection problems.

As pondered by Lai et al. \cite{6}, the usage of three-in-a-tree as a go-to solution for detection problems may, at times, seem quite unnatural. In the aforementioned cases, one could try to tackle the problem by looking for constant sized minors or disjoint paths between terminal pairs and then resort to Kawarabayashi et al.’s \cite{7} quadratic algorithm to finalize the detection procedure. The problem is that neither the minors nor the disjoint paths are guaranteed to be induced; to make the situation truly dire, this constraint makes even the most basic problems \textit{NP}-hard. For instance, Bienstock \cite{1, 2} proved that two-in-a-hole and three-in-a-path are \textit{NP}-complete. As such, it is quite surprising that three-in-a-tree can be solved in polynomial time and be of widespread importance. It is worth to note that the induced subgraph constraint is also troublesome from the parameterized point of view. Maximum Matching, for instance, can be solved in polynomial time \cite{22}, but if we impose that the matching must be induced subgraph, the problem becomes $W[1]$-hard when parameterized by the minimum number of edges in the matching \cite{30}.

Naturally, we may wonder how far we may push for polynomial time algorithms when considering larger numbers of terminal vertices, i.e. we are interested in the complexity of $k$-in-a-tree for $k \geq 4$. The first authors to examine this problem were Derhy and Picouleau, who proved in \cite{18} that $k$-in-a-tree is \textit{NP}-complete when the number of terminals is part of the input even on planar bipartite cubic graphs of girth four, but solvable in polynomial time if the girth of the graph is larger than the number terminals. A few years later, Derhy et al. \cite{19} showed that four-in-a-tree is solvable in triangle-free graphs, while Liu and Trotignon \cite{30} proved that so is $k$-in-a-tree on graphs of girth at least $k \geq 5$; their combined results imply that $k$-in-a-tree on graphs of girth at least $k$ is solvable in polynomial time. In terms of the $k$-in-a-path problem, Derhy and Picouleau \cite{18} argued that their hardness reduction also applies to this problem and showed that three-in-a-path is \textit{NP}-complete even on graphs of maximum degree three. Fiala et al. \cite{23} proved that $k$-in-a-path, $k$-induced disjoint paths, and $k$-in-a-cycle can be solved in polynomial time on claw-free graphs for every fixed $k$, but all of them are \textit{NP}-complete when $k$ is part of the input even on line graphs; in fact, they proved that the previous problems
are in XP when parameterized by the number of terminals on claw-free graphs.

Another related problem to \( k \)-in-a-tree is the well known Steiner Tree, where we want to find a subtree of the input with cost at most \( w \) connecting all terminals. Being one of Karp’s 21 NP-hard problems \([21]\), Steiner Tree has received a lot of attention over the decades. Relevant to our discussion, however, is its parameterized complexity. When parameterized by the number of terminals, it admits a single exponential time algorithm \([21]\); the same was proven to be true when treewidth \([32]\) is the parameter \([5]\). On the other hand, when parameterized by cliquewidth \([16]\), it is paraNP-hard since it is NP-hard even on cliques: we may reduce from Steiner Tree itself and add, for each non-edge of the input, an edge of cost \( w + 1 \). As we see below, our first two results are in complete contrast with the parameterized complexity of Steiner Tree.

**Our results.** We concern ourselves with the parameterized complexity of \( k \)-in-a-tree. We begin by presenting some algorithmic results for \( k \)-in-a-tree in Section 3, showing that the latter is \( \text{W}[1] \)-hard when simultaneously parameterized by the number of vertices in the solution and size of a minimum clique cover and, moreover, does not admit an \( n^{o(k)} \) time algorithm unless the Exponential Time Hypothesis \([26]\) (ETH) fails. This partially answers a (generalization) of Derhy and Picouleau’s question about the complexity of \( k \)-in-a-tree, in the sense that there is very little hope of obtaining an algorithm that runs in polynomial time only on the size of the input graph. On the positive side, we prove tractability under cliquewidth using Courcelle’s Theorem \([15]\), which prompts us, in Section 4, to turn our attention to which parameters allow us to devise single exponential time algorithms for \( k \)-in-a-tree. Using Bodlaender et al.’s dynamic programming optimization machinery \([5]\), we show that such algorithms exist under treewidth, distance to cluster, and distance to co-cluster. In Section 5, we present a kernel with \( 16q \) vertices and \( 17q \) edges when we parameterize \( k \)-in-a-tree by the size \( q \) of a minimum feedback edge set. In Section 6, we prove that the problem does not admit a polynomial kernel when parameterized by bandwidth, nor when simultaneously parameterized by the size of the solution, diameter, and distance to any graph class of your choosing. In particular, the latter shows that \( k \)-in-a-tree does not admit a polynomial kernel when parameterized by vertex cover nor when parameterized by distance to clique. All our negative kernelization results are obtained assuming \( \text{NP} \not\subseteq \text{coNP/poly} \).

In terms of tractability and kernelization, our results encompass most of the commonly employed parameters of Sorge and Weller’s graph parameter hierarchy \([33]\); we present a summary of our results in Figure 1. To see why the distance to solution parameter sits between vertex cover and feedback vertex set, we refer to the end of Section 3.

### 2 Preliminaries

We refer the reader to \([17]\) for basic background on parameterized complexity, and recall here only some basic definitions. A **parameterized problem** is a language \( L \subseteq \Sigma^* \times \mathbb{N} \). For an instance \( I = (x, q) \in \Sigma^* \times \mathbb{N} \), \( q \) is called the **parameter**. A parameterized problem is **fixed-parameter tractable** (FPT) if there exists an algorithm \( A \), a computable function \( f \), and a constant \( c \) such that given an instance \((x, q)\), \( A \) correctly decides whether \( I \in L \) in time bounded by \( f(q) \cdot |I|^c \); in this case, \( A \) is called an FPT algorithm. A fundamental concept
in parameterized complexity is that of kernelization; see [24] for a recent book on the topic. A kernelization algorithm, or just kernel, for a parameterized problem \(\Pi\) takes an instance \((x, q)\) of the problem and, in time polynomial in \(|x| + q\), outputs an instance \((x', q')\) such that \(|x'|, q' \leq g(q)\) for some function \(g\), and \((x, q) \in \Pi\) if and only if \((x', q') \in \Pi\). Function \(g\) is called the size of the kernel and may be viewed as a measure of the “compressibility” of a problem using polynomial-time pre-processing rules. A kernel is called polynomial (resp. quadratic, linear) if \(g(q)\) is a polynomial (resp. quadratic, linear) function in \(q\). A breakthrough result of Bodlaender et al. [3] gave the first framework for proving that some parameterized problems do not admit polynomial kernels, by establishing so-called composition algorithms. Together with a result of Fortnow and Santhanam [25], this allows to exclude polynomial kernels under the assumption that \(\text{NP} \not\subseteq \text{coNP}/\text{poly}\), otherwise implying a collapse of the polynomial hierarchy to its third level [35].

All graphs in this work are finite and simple. We use standard graph theory notation and nomenclature for our parameters; for any undefined terminology in graph theory we refer to [6]. We denote the degree of vertex \(v\) on graph \(G\) by \(\text{deg}_G(v)\), and the set of natural numbers \(\{1, 2, \ldots, t\}\) by \([t]\). A graph is a cluster graph if each of its connected components is a clique, while a co-cluster graph is the complement of a cluster graph. The distance to cluster (co-cluster) of a graph \(G\), is the size of the smallest set \(U \subseteq V(G)\) such that \(G \setminus U\) is a cluster (co-cluster) graph. As defined in [8], a set \(U \subseteq V(G)\) is an \(\mathcal{F}\)-modulator of \(G\) if \(G \setminus U\) belongs to the graph class \(\mathcal{F}\). When the context is clear, we omit the qualifier \(\mathcal{F}\). For cluster and co-cluster graphs, one can decide if \(G\) admits a
3 Fixed-parameter tractability and intractability

While it has been known for some time that \( k\text{-IN-A-TREE} \) is \( \mathsf{NP} \)-complete even on planar bipartite cubic graphs, it is not known to be even in \( \mathsf{XP} \) when parameterized by the natural parameter, the number of terminals. We take a first step with a negative result about this parameterization, ruling out the existence of an \( \mathsf{FPT} \) algorithm unless \( \mathsf{FPT} = \mathsf{W[1]} \); in fact, we show for stronger parameterization: the maximum size of the induced tree that should contain the set of \( k \) terminal vertices \( K \) and the size of a minimum clique cover.

**Theorem 1.** \( k\text{-IN-A-TREE} \) is \( \mathsf{W[1]} \)-hard when simultaneously parameterized by the number of vertices of the induced tree and size of a minimum clique cover. Moreover, unless \( \mathsf{ETH} \) fails, there is no \( n^{o(k)} \) time algorithm for \( k\text{-IN-A-TREE} \).

**Proof.** Our reduction is from \( \text{Multicolored Independent Set} \) parameterized by the number of color classes \( \ell \). Formally, let \( H \) be the input to our source problem such that \( V(H) \) is partitioned into \( \ell \) color classes \( \{V_1, \ldots, V_\ell\} \) and each \( V_i \) induces a clique on \( H \). Our instance of \( k\text{-IN-A-TREE} \) \( (G, K) \) is such that \( K = \{v_0, \ldots, v_k\} \), \( V(G) = K \cup V(H) \), every edge of \( H \) is also in \( G \), each \( v_i \in K \) has \( N_G(v_i) = V_i \), and \( N(v_0) = V(H) \), so \( k = \ell + 1 \).

If \( I \) is a solution to \( \text{Multicolored Independent Set} \), \( I \cup \{v_0\} \) is a solution to \( (G, K) \): there are no cycles since \( I \) is also an independent set of \( G \), \( v_0 \) connects all vertices of \( I \), and each other terminal is connected exactly one vertex of \( I \). For the converse, if \( T \) is a solution to \( (G, K) \), we claim that \( I = T \setminus K \) is an independent set of size \( \ell \). To see that this is the case, note that: (i) \( T \cap V(H) \) must be independent, otherwise they would form a triangle with \( v_0 \); and (ii) \( |T \cap V_i| = 1 \) because each \( V_i \) is a clique and the only way to connect \( v_i \) to \( v_0 \) is by picking at least one vertex of \( V_i \). Since \( |K| = \ell + 1 \) we have that the solution size is at most \( 2k + 1 \) and, since \( Q_i = V_i \cup \{v_i\} \) is a clique, we can cover \( G \) with the \( k + 1 \) cliques \( \{\{v_0\}\} \cup_{i \in [k]} \{Q_i\} \). The second statement follows directly from the fact that \( \text{Multicolored Independent Set} \) has no \( n^{o(k)} \) time algorithm under \( \mathsf{ETH} \) and that our reduction exhibits a linear relation between the parameters of the source and destination problems.

Since the natural parameters offer little to no hope of fixed-parameter tractability, to obtain parameterized algorithms we turn our attention to the broad class of structural parameters. Our first positive result is a direct application of textbook MSO₁ formulae.

**Theorem 2.** \( k\text{-IN-A-TREE} \) parameterized by cliquewidth is in \( \mathsf{FPT} \).

**Proof.** Let \( (G, K) \) be the input to \( k\text{-IN-A-TREE} \), \( R \) be the binary relation represented by the formula \( \varphi_G(u, v, Y) = u \in Y \land v \in Y \land (e(u, v) \lor e(v, u)) \), and \( \text{TC}[R; x, y] \) be the reflexive and transitive closure of \( R \). As such, \( \text{conn}(Y) = \forall x, y, z \in \text{Y} \Rightarrow \text{TC}[R; x, y] \) is true if and only if \( G[Y] \) is connected \([15]\). Similarly, formula \( \text{cycle}(X) = \exists x, y, z \in X (x \neq y \land y \neq z \land x \neq z \land e(x, z) \land e(y, z) \land \exists Y' \subseteq \text{Y} \setminus \{x, y, z\} \cup Y \land y \in Y \land \text{conn}(Y')) \) is true if and only if \( G[X] \) has a cycle \([15]\). Putting the previous two formulae together, formula \( \text{indtree}(K) = \exists S (K \subseteq S \land \text{conn}(S) \land \neg \text{cycle}(S)) \) is true if and only if there is
some superset of $K$ that is connected and acyclic. By Courcelle’s Theorem \cite{Courcelle}, if $G$ has cliquewidth at most $q$, then one can determine in $f(q)n^{O(1)}$ time if $G$ satisfies indtree($K$) for some set $K$ of terminals.

Towards showing that the minimum number of vertices we must delete to obtain a solution sits between feedback vertex set and vertex cover in Figure 1, let $S \subset V(G)$ be such that $G \setminus S$ is a solution. First, note that $S$ is a feedback vertex set of $G$; for the other inequality, take a vertex cover $C$ of $G$ and note that placing two vertices of $G \setminus C$ with the same neighborhood in $C$ either generates a cycle in the solution or only one of them suffices — even if we have many terminals, we need to keep only two of them — so $|S| \leq |C| + 2^{|C|} + 1$. In terms of paraNP-hardness results, we can easily show that $k$-in-a-tree is paraNP-hard when parameterized by bisection width\cite{Bodlaender} to reduce from the problem to itself, we pick any terminal of the input and append to it a path with as many vertices as the original graph to obtain a graph with bisection width one. Similarly, when parameterizing by the size of a minimum dominating set and again reducing from $k$-in-a-tree to itself, we add a new terminal adjacent to any vertex of $K$ and a universal vertex, which can never be part of the solution since it forms a triangle with the new terminal and its neighbor.

4 Single exponential time algorithms

All results in this section rely on the rank based approach of Bodlaender et al.\cite{Bodlaender}, which requires the additional definitions we give below. Let $U$ be a finite set, $\Pi(U)$ denote the set of all partitions of $U$, and $\sqsubseteq$ be the coarsest relation defined on $\Pi(U)$, i.e. given two partitions $p,q \in \Pi(U)$, $p \sqsubseteq q$ if and only if each block of $q$ is contained in some block of $p$. It is known that $\Pi(U)$ together with $\sqsubseteq$ form a lattice, upon which we can define the usual join operator $\sqcup$ and meet operator $\sqcap$\cite{Bodlaender}. The join operation $p \sqcup q$ outputs the unique partition $z$ where two elements are in the same block of $z$ if and only if they are in the same block of $p$ or $q$. The result of the meet operation $p \sqcap q$ is the unique partition such that each block is formed by the non-empty intersection between a block of $p$ and a block of $q$. Given a subset $X \subseteq U$ and $p \in \Pi(U)$, $p_{|X} \in \Pi(X)$ is the partition obtained by removing all elements of $U \setminus X$ from $p$, while, for $Y \supseteq U$, $p_{|Y} \in \Pi(Y)$ is the partition obtained by adding to $p$ a singleton block for each element in $Y \setminus U$. For $X \subseteq U$, we shorthand by $U[X]$ the partition where one block is precisely $\{X\}$ and all other are the singletons of $U \setminus X$; if $X = \{a,b\}$, we use $U[ab]$.

A set of weighted partitions of a ground set $U$ is defined as $\mathcal{A} \subseteq \Pi(U) \times \mathbb{N}$. To speed up dynamic programming algorithms for connectivity problems, the idea is to only store a subset $\mathcal{A}' \subseteq \mathcal{A}$ that preserves the existence of at least one optimal solution. Formally, for each possible extension $q \in \Pi(U)$ of the current partitions of $\mathcal{A}$ to a valid solution, the optimum of $\mathcal{A}$ relative to $q$ is denoted by $\text{opt}(q,\mathcal{A}) = \min\{w : (p,w) \in \mathcal{A}, p \sqcup q = \{U\}\}$. $\mathcal{A}'$ represents $\mathcal{A}$ if $\text{opt}(q,\mathcal{A}') = \text{opt}(q,\mathcal{A})$ for all $q \in \Pi(U)$. The key result of Bodlaender et al.\cite{Bodlaender} is given by Theorem 3.

\footnote{The width of a bipartition $(A,B)$ of $V(G)$ is the number of edges between the parts. The bisection width is the minimum width of all bipartitions of $V(G)$ such that $|A| \leq |B| \leq |A| + 1$.}
Theorem 3 (3.7 of [5]). There exists an algorithm that, given \( A \) and \( U \), computes \( A' \) in time \(|A|2^{(\omega-1)|U|}|U|^{|\Omega(1)|}\) and \(|A'| \leq 2^{|U|-1}\), where \( \omega \) is the matrix multiplication constant.

A function \( f : 2^{\Pi(U) \times N} \times Z \rightarrow 2^{\Pi(U) \times N} \) is said to preserve representation if \( f(A', z) = f(A, z) \) for every \( A, A' \in \Pi(U) \times N \) and \( z \in Z \); thus, if one can describe a dynamic programming algorithm that uses only transition functions that preserve representation, it is possible to obtain \( A' \). In the following lemma, let \( \text{rmc}(A) = \{(p, w) \in A \mid \exists \hat{(p, w')} \in A, w' < w\} \).

Lemma 4 (Proposition 3.3 and Lemma 3.6 of [5]). Let \( U \) be a finite set and \( A \subseteq \Pi(U) \times N \). The following functions preserve representation and can be computed in \(|A| \cdot |B| \cdot |U|^{|\Omega(1)|}\) time.

**Union.** For \( B \in \Pi(U) \times N \), \( A \uplus B = \text{rmc}(A \cup B) \).

**Insert.** For \( X \cap U = \emptyset \), \( \text{ins}(X, A) = \{(p_{x,U}, w) \mid (p, w) \in A\} \).

**Shift.** For any integer \( w' \), \( \text{shift}(w', A) = \{(p, w + w') \mid (p, w) \in A\} \).

**Glue.** Let \( \hat{U} = U \cup X \), then \( \text{glue}(X, A) = \text{rmc}\left(\{(\hat{U}[X] \uplus p_{i,G}, w) \mid (p, w) \in A\}\right) \).

**Project.** \( \text{proj}(X, A) = \text{rmc}\left(\{(p_{x,X}, w) \mid (p, w) \in A, \forall u \in X : \exists v \in X : p \subseteq U[wv]\}\right) \).

**Join.** If \( \hat{U} = U \cup U', A \subseteq \Pi(U) \times N \) and \( B \in \Pi(U') \times N \), then \( \text{join}(A, B) = \text{rmc}\left(\{(p_{U \uplus q_{U'}, w + w'} \mid (p, w) \in A, (q, w') \in B\}\right) \).

Even though our problem is unweighted, we found it convenient to solve a weighted version and of \( k\text{-in-a-tree} \). We state this slightly more general problem below.

**Light Connecting Induced Subgraph**

**Instance:** A graph \( G \), a set of \( k \) terminals \( K \subseteq V(G) \), and two integers \( \ell, f \).

**Question:** Is there a connected induced subgraph of \( G \) on \( \ell + k \) vertices and at most \( f \) edges that contains \( K \)?

Note that an instance \((G, K)\) of \( k\text{-in-a-tree} \) is positive if and only if there is some integer \( \ell \) where the Light Connecting Induced Subgraph instance \((G, K, \ell, \ell + k - 1)\) is positive. Our goal is to use the number of edges in the solution to Light Connecting Induced Subgraph as the cost of a partial solution in a dynamic programming algorithm. This shall be particularly useful for join nodes during our treewidth algorithm, as we may resort to the optimality of the solution to guarantee that the resulting induced subgraph of a \( \uplus \) operation is acyclic.

### 4.1 Treewidth

A tree decomposition of a graph \( G \) is a pair \( T = (T, \mathcal{B} = \{B_j \mid j \in V(T)\})\), where \( T \) is a tree and \( \mathcal{B} \subseteq 2^{V(G)} \) is a family where: \( \bigcup_{j \in \mathcal{B}} B_j = V(G) \); for every edge \( w \in E(G) \) there is some \( B_j \) such that \( \{u, v\} \subseteq B_j \); for every \( i, j, q \in V(T) \), if \( q \) is in the path between \( i \) and \( j \) in \( T \), then \( B_i \cap B_j \subseteq B_q \). Each \( B_j \in \mathcal{B} \) is called a bag of the tree decomposition. \( G \) has treewidth has most \( t \) if it admits a tree.
decomposition such that no bag has more than \( t \) vertices. For further properties of treewidth, we refer to [32]. After rooting \( T, G_t \) denotes the subgraph of \( G \) induced by the vertices contained in any bag that belongs to the subtree of \( T \) rooted at bag \( x \). An algorithmically useful property of tree decompositions is the existence of a nice tree decomposition that does not increase the treewidth of \( G \).

**Definition 5** (Nice tree decomposition). A tree decomposition \( \mathcal{T} \) of \( G \) is said to be nice if its tree is rooted at, say, the empty bag \( r(T) \) and each of its bags is from one of the following four types:

1. **Leaf node**: a leaf \( x \) of \( \mathcal{T} \) with \( B_x = \emptyset \).
2. **Introduce vertex node**: an inner bag \( x \) of \( \mathcal{T} \) with one child \( y \) such that \( B_x \setminus B_y = \{u\} \).
3. **Forget node**: an inner bag \( x \) of \( \mathcal{T} \) with one child \( y \) such that \( B_y \setminus B_x = \{u\} \).
4. **Join node**: an inner bag \( x \) of \( \mathcal{T} \) with two children \( y, z \) such that \( B_x = B_y = B_z \).

**Theorem 6.** There is an algorithm for Light Connecting Induced Subgraph that, given a nice tree decomposition of width \( t \) of the \( n \)-vertex input graph \( G \) rooted at the forget node for some terminal \( r \in K \), runs in time \( 2^{O(t)} n^{O(1)} \).

**Proof.** Let \( (G, K, t, f) \) be an instance of Light Connecting Induced Subgraph. For each bag \( x \), we compute the table \( g_x(S, \ell') \subseteq \Pi(S) \times \mathbb{N} \), where \( S \subseteq B_x \) contains the vertices of \( B_x \) that must be present in a solution and \( \ell' \) is the number of vertices we allow in the induced subgraphs of \( G_x \); each weighted partition \((p, w) \in g_x(S, \ell') \) corresponds to a choice of an induced subgraph of \( G_x \) with \( \ell' \) vertices, \( w + |E(G[S])| \) edges, and connected components given by the blocks of \( p \). If \( |S| > \ell' \), we define \( g_x(S, \ell') = \emptyset \). After every operation, we apply the algorithm of Theorem [3]

**Leaf node.** Since \( B_x = \emptyset \), the only possible connecting induced subgraph is precisely the empty graph, so we define:

\[
g_x(\emptyset, \ell') = \begin{cases} \{(\emptyset, 0)\}, & \text{if } \ell' = 0; \\ \emptyset, & \text{otherwise}. \end{cases}
\]

**Introduce node.** Let \( y \) be the child bag of \( x \) and \( B_x \setminus B_y = \{v\} \). We compute \( g_x(S, \ell') \) as follows, where \( A_y(S, \ell', v) = \text{ins}(\{v\}, g_y(S \setminus \{v\}, \ell' - 1)) \):

\[
g_x(S, \ell') = \begin{cases} \text{glue}(N[v] \cap S, A_y(S, \ell', v)), & \text{if } v \in S; \\ g_y(S, \ell'), & \text{if } v \not\in S \cup K; \\ \emptyset, & \text{otherwise}. \end{cases}
\]

On the third case above, if \( v \in K \) but \( v \not\in S \), we are not including a terminal into the induced subgraph, thus we cannot accept any partition with support \( S \) as valid. If \( v \not\in K \cup S \), then no changes are necessary, since the only vertex of \( G_x \) not in \( G_y \) is not considered for the solution. Finally, for the first case, since \( v \in S \), we must extend each partition of \( g_y(S \setminus \{v\}, \ell' - 1) \) to the ground set \( S \) (which we achieve by the insert operation); however, since we are looking
for an induced subgraph, we must use every edge between \( v \) and the neighbors of \( v \) in \( S \), merging the connected components containing them. Unlike in some connectivity problems such as Steiner Tree, we only count the edges within \( S \) while processing forget nodes; this shall simplify the join operation considerably, as we do not need to worry about repeatedly counting edges inside the current bag.

**Forget node.** Let \( y \) be the child bag of \( x \) and \( v \) be the forgotten vertex. The transition is directly computed by:

\[
g_x(S, \ell') = g_y(S, \ell') \oplus \text{shift}(|N(v) \cap S|, \text{proj}([v], g_y(S \cup \{v\}, \ell')))
\]

If \( v \) is not used in a partial solution, \( g_y(S, \ell') \) already correctly contains all the partial solutions where \( v \) is not used; on the other hand, if \( v \) was used in some solution, we must eliminate \( v \) from the partitions where it appears; however, we only keep the partitions that do not lose a block, otherwise we would have a connected component (represented by \( v \)) that shall never be connected to the remainder of the subgraph and, thus, cannot be extended to a valid solution.

**Join node.** If \( y, z \) are the children of bag \( x \), we compose its table by the equation:

\[
g_x(S, \ell') = \bigoplus_{\ell_1 + \ell_2 = \ell' + |S|, |S| \leq \ell_1, \ell_2} \text{join}(g_y(S, \ell_1), g_z(S, \ell_2))
\]

Where the union operator runs over all integer values satisfying the system \( \ell_1 + \ell_2 = \ell' + |S|, |S| \leq \ell_1, \ell_2 \); since we do not know how many vertices were used on the partial solutions of each subtree, we must try every combination to obtain all the partial solutions for the subtree rooted at \( x \). Since we force the vertices in \( S \) to be present in the solutions to the subtree rooted at bag \( x \), combining two partial solutions, represented by \((p, w) \in g_y(S, \ell_1)\) and \((q, w') \in g_z(S, \ell_2)\), corresponds to uniting the set of edges of the respective partial solutions \( G(p), G(q) \), which results in a merger of connected components.

Since the edges of \( S \) have not been counted towards the weights \( w, w' \), \( G(p \sqcup q) \) has exactly \( w + w' + |E(G[S])| \) edges. This is precisely the definition of the join operation.

In order to obtain the answer to the problem, we look at the child \( x \) of the forget node for terminal \( r \), that is, the child of the root of the tree, and check if \( g_x(\{r\}, \ell + |K|) \neq \emptyset \). In the affirmative, note that there is only one entry \( ([\{r\}], w) \in \text{proj}([\{r\}], \ell + |K|) \) and that the graph that connects all the terminals using \( \ell + |K| \) vertices has exactly \( w \) edges, since \( E(G[\{r\}]) \cap 0 \).

For an introduce bag \( x \) with child \( y \), the time taken to compute all entries of \( g_x \) is of the order of \( \ell \sum_{i=0}^{l(B_x)} \left( \binom{|B_x|}{i} \right) 2^{-2i}1^{\Omega(1)} \leq n(1 + 2\omega)1^{\Omega(1)} \); the term \( 2^{\omega i} \) comes from the need to execute the algorithm of Theorem \( \# \) upon a initial set of size \( 2^\ell \). For join nodes, the intermediate \( \sqcup \) operation may yield a set of size \( 2^{\omega 2^\ell} \), so we have that the tables can be computed in time \( \ell \sum_{i=0}^{l(B_x)} \left( \binom{|B_x|}{i} \right) 2^{(\omega + 1)\ell}1^{\Omega(1)} \leq (1 + 2^{(\omega + 1)\ell})1^{\Omega(1)}. \)

**Corollary 7.** There is an algorithm for \( k \text{-IN-A-TREE} \) that, given a nice tree decomposition of width \( t \) of the \( n \)-vertex input graph \( G \) rooted at the forget node for some terminal \( r \in K \), runs in time \( 2^{O(1)n1^{\Omega(1)}} \).
4.2 Distance to cluster

We now show that parameterizing by the distance to cluster \( q \) also yields an FPT algorithm. Throughout this section, \( G \) is the input graph, \( U \) is the cluster modulator, and \( C = \{C_1, \ldots, C_r\} \) are the maximal cliques of \( G \setminus U \). We also use the framework developed by Bodlaender et al.\cite{bodlaender2009} to optimize our dynamic programming algorithm.

**Theorem 8.** There is an algorithm for \( k \text{-IN-A-TREE} \) that runs in time \( 2^{O(q)} n^{O(1)} \) on graphs with distance to cluster at most \( q \) graphs.

**Proof.** Suppose we are given the instance \( (G, K) \) and the \( q \)-vertex cluster modulator \( U \subseteq V(G) \). We begin by guessing a subset \( K \cap U \subseteq S \subseteq U \) of vertices that will be present in a solution for the problem. Now, given \( S \), we execute the following pre-processing step: for each clique \( C_i \in C \), we discard all but one vertex of each maximal set of true twins; this way, we limit the size of \( C_i^* = C_i \setminus K \) to \( 2^q \).

An entry of our dynamic programming table \( f_S(i, c, \ell) \subseteq \Pi(S) \times \mathbb{N} \) is a set of partial solutions of \( G_i = G[K \cup \bigcup_{j=1}^i C_j] \), each of which uses exactly \( c \) vertices of \( C_i^* \) and induces a subgraph of \( G_i \) on \( \ell \) vertices. Note that we cannot use more than two vertices of each clique, so we only consider \( c \in \{0, 1, 2\} \). In each \( (p, w) \in f_S(i, c, \ell) \), each block of \( p \) corresponds to the vertices of \( S \) that lie in the same connected component of \( G_i \), and \( w \) is the number of edges used in the respective induced subgraph. Our transition is given by the following equation, where \( W(S, X) = |N(X) \cap (S \cup K)| + |E(G[X])| \); if either \( c + |K \cap C_i| > 2 \), \( c > \ell \), or there is some vertex in \( K \cap C_i \) that has no neighbor in \( S \) and \( c = 0 \), we define \( f_S(i, c, \ell) = \emptyset \).

\[
f_S(i, c, \ell) = 2 \bigcup_{j=0}^{c} \bigcup_{X \in \binom{C_i^*}{j}} \text{glue}_{W(S, X)}(N_S(X), f_S(i - 1, j, \ell - c))
\]

The above defines \( f_S \) for all \( (i, c, \ell) \in [r] \times \{0, 1, 2\} \times [n] \); we extend \( f_S \) to include the base case \( f_S(0, 0, |S \cup K|) = \{(p(S, K), E(G[S \cup K])) \} \), where \( p(S, K) \in \Pi(S) \) is the partition obtained by gluing together the connected components of \( G[S] \) which have a common neighbor in \( K \); for all other entries, \( f_S(r + 1, c, \ell) = \emptyset \). Our goal now is to show that there is a solution to our problem using the vertices of \( S \) if and only if \( (\{S\}, w) \in f_S(r, c, \ell) \), for some pair \( \ell \geq |S \cup K| \), \( c \geq |K \cap C_i| \), such that \( w = \ell - 1 \). To do so, we first prove that \( f(i, c, \ell) \) contains all partitions of \( S \) that represent all possible induced subgraphs of \( G_i \) on \( \ell \) vertices that use \( c \) vertices of \( C_i^* \), and that use as few edges as possible.

By induction, suppose that this holds for every entry \( f_S(i - 1, a, b) \). If \( c + |K \cap C_i| > 2 \) or \( c > \ell \), either we want to use more than two vertices of \( C_i \), which certainly implies that there is a copy of \( K_3 \) in the solution, or we want to use more than \( \ell \) vertices of \( C_i^* \), which is equally impossible, so \( f_S(i, c, \ell) = \emptyset \). If \( c = 0 \), we have that a weighted partition \((p, w)\) is valid in \( G_i \) if and only if it is valid in \( G_{i-1} \) since we use no vertices in \( V(G_i) \setminus V(G_{i-1}) \); thus, \((p, w) \in f_S(i, 0, \ell)\) if and only if \((p, w) \in f_S(i - 1, j, \ell)\) for some \( j \in \{0, 1, 2\} \). Otherwise, suppose we want to add a subset of vertices \( X \subseteq C_i^* \) to a partial solution \( H \), which is represented by \((p, w) \in f_S(i - 1, a, \ell - |X|)\). In this case, since \( N(X) \setminus C_i \subseteq U \) and \( U \cap V(H) \subseteq S \), we have that \( X \) can only reduce the
number of connected components of $H$ if $N(X)$ intersects two distinct blocks of $p$. Thus, the connected components of $G[V(H) \cup X]$ are represented, precisely, by glue$(N_S(X), p)$, and the only new edges used are those between $X$ and $S \cup K$, and the ones internal to $X$; this is precisely the shift accounted by $W(S, X)$. The minimality of $w$ for an entry $(p, w) \in f(i, c, \ell)$ is guaranteed by the rmc operation imbeded in both glue and \(\oplus\). Note that if there is some entry $(p, w) \in f_S(r, c, \ell)$ for some $c$ such that $p = \{S\}$ and $w = \ell - 1$, this means that there is an induced subgraph of graph that has $S$ contained in a connected component, uses $\ell$ vertices and $\ell - 1$ edges, and thus, must be an induced tree of $G$. That it connects all vertices of $K$ follows from the fact that $f_S(0, 0, |S \cup K|) contains p(S, K)$ and, in every clique $C_i$ that contains a vertex of $K$ with no neighbor in $S$, we force that at least one vertex of $C_i$ must be picked in a solution.

In terms of complexity, after every glue or \(\oplus\) operation, we apply the algorithm of Theorem 8. For each tuple $(S, i, c, \ell, j)$, we do so up to $|C_i^*| \leq 2^q$ times per tuple, which implies in a time requirement time of the order of $2^{q \cdot 2^{q-1}2^{(\omega-1)q\mathfrak{C}(1)}} \leq 2^{(\omega+3)q\mathfrak{C}(1)}$. Since we have $2^q\mathfrak{C}(1)$ tuples, our algorithm runs in $2^{(\omega+3)q\mathfrak{C}(1)}$ time.

4.3 Distance to co-cluster

We can use the result on distance to cluster to solve the problem on graphs with distance to co-cluster at most $q$ without much effort, as we see in the following proposition.

**Theorem 9.** There is an algorithm for $k$-in-a-tree that runs in time $2^{O(q^2)}n^{O(1)}$ where $q$ is the distance to co-cluster.

**Proof.** Suppose we are given a co-cluster modulator $U$ of the input graph $G$ and let $\mathcal{I} = \{I_1, \ldots, I_r\}$ be the family of independent sets of $G - U$. Since $G - U$ is a complete multipartite graph, if we pick vertices of three distinct elements of $\mathcal{I}$, we will form a $K_3$ in the induced subgraph. Moreover, for each pair $I_i, I_j \in \mathcal{I}$, at most one of them may have more than one vertex in any solution, the induced subgraph would contain a $C_4$. This implies that $K$ can intersect $V(G - U)$ in at most three vertices and at most two independent sets. If this intersection has size three, for each $K \cap U \subseteq S \subseteq U$, we can easily verify in polynomial time if $G[S \cup K]$ is a tree. Otherwise, for each pair $I_i, I_j \in \mathcal{I}$ such that $K \subseteq U \cup I_i \cup I_j$, we guess which one of them will have more than one vertex in the solution, say $I_i$, and which vertex $v \in I_i$ will be in the solution, with the restriction that $K \subseteq U \cup I_j \cup \{v\}$. Now, the graph $G' = G[U \cup I_j \cup \{v\}]$ has a cluster modulator $U \cup \{v\}$, and we can apply the algorithm of Theorem 8 on it to decide if there is an induced tree of $G'$ connecting $K$. It follows from the observations that there is a valid induced subtree of $G$ if and only if for some choice $I_i, I_j$ and $v \in I_i$, $G'$ has one such induced subgraph.

5 A linear kernel for Feedback Edge Set

In this section, we prove that $k$-in-a-tree admits a linear kernel when parameterized by the size $q$ of a minimum feedback edge set. Throughout this section, we denote our input graph by $G$, the set of terminals by $K$, and the tree obtained by removing the edges of a minimum size feedback edge set $F$ by $T(F)$.
Note that, if $G$ is connected and $F$ is of minimum size, $G \setminus F$ is a tree; we may safely assume the first, otherwise we either have that $(G, K)$ is a negative instance if $K$ is spread across multiple connected components of $G$, or there must be some edge of $F$ that merges two connected components of $G \setminus F$ and does not create a cycle, contradicting the minimality of $F$. The kernelization algorithm we describe works in two steps: it first finds a feedback edge set $F$ that minimizes the number of edges incident to vertices of degree two in $T(F)$, then compresses long induced paths of $G$. We denote the set of leaves of a tree $H$ by leaves $(H)$.

**Reduction Rule 1.** If $G$ has a vertex $v$ of degree one, remove $v$ and, if $v \in K$, add the unique neighbor of $v$ in $G$ to $K$.

**Proof of safeness of Rule 1.** Safeness follows from the fact that a degree one vertex is in the solution if and only if its unique neighbor also is.

**Observation 10.** After exhaustively applying Rule 1 for every minimum feedback edge set $F$ of $G$, $T(F)$ has at most $2q$ leaves. Moreover, $T(F)$ has at most as many vertices of degree at least three as leaves.

We begin with any minimum feedback edge set $F$ of $G$. We partition $T(F) \setminus \text{leaves}(T(F))$ into $(D_2, D_4)$ according to the degree of the vertices of $G$ in $T(F)$: $v \in D_2$ if and only if $\deg_{T(F)}(v) = 2$. For $u, v, f \in V(G)$, we say that $u F$-links $v$ to $f$ if $v = u$ or if $T(F) \setminus \{u\}$ has no $v - f$ path. We say that vertices $u, f$ are an $F$-pair if the set of internal vertices of the unique $u - f$ path $P_F(u, f)$ of $T(F)$ is entirely contained in $D_2$; we denote the set of internal vertices by $P^*_F(u, f)$.

**Reduction Rule 2.** Let $u, f, w_1, w_2 \in V(G)$ be such that $u, f$ form an $F$-pair, $w_1 \neq w_2 \neq u \neq w_1$, $w_2$ is the unique neighbor of $f$ that $F$-links it to $u$ and $w_1$ $F$-links $w_2$ to $u$. If $u w_1 F$-links $f$ to $w_1$; as such, edge $f w_2$ is in the unique cycle of $G \setminus F^*$, so $F'' = F^* \cup \{f w_2\}$ is a feedback edge set of $G$ of size $q$. Furthermore, $w_2$ is the only vertex that has fewer neighbors in $T(F'')$ than in $T(F)$; since $w_2$ had two neighbors in $T(F)$, and $F''$ is a minimum feedback edge set of $G$, $w_2$ is a leaf of $T(F'')$, so it holds that leaves $(T(F)) \subset \text{leaves}(T(F''))$.

Reduction Rule 2 guarantees that there are no edges in $F$ between vertices of the paths between $F$-pairs, otherwise we could increase the number of leaves of our tree.

**Reduction Rule 3.** Let $f, u, v \in V(G)$ be such that $v \notin \text{leaves}(T(F)) \cup P_F(u, f)$, $u, f$ form an $F$-pair, and $|P_F(u, f)| \geq 4$. If there are adjacent vertices $w_1, w_2 \in P^*_F(u, f)$ with $vw_1 \in F$ and $w_2 F$-linking $v$ and $w_1$, remove edge $vw_1$ from $F$ and add edge $w_1 w_2$ to $F$.

**Proof of safeness of Rule 3.** Let $F' = F \setminus \{w_1 w_2\}$. Since $G \setminus F'$ has one more edge than $T(F')$ and $w_2 F$-links $v$ and $w$ (see Figure 2), the unique cycle of $G \setminus F'$ contains edge $w_1 w_2$, so $F''' = F' \cup \{vw_1\}$ is a feedback edge set of $G$ of size $q$. Since neither $v$ nor $w$ are leaves of $T(F')$ and $w_2 \in D_2$, $\deg_{T(F'''}(w_2) = 1$, so it holds that $|\text{leaves}(T(F'))| < |\text{leaves}(T(F'''))|$. 


Reduction Rule 4. Let $f, u, v, z, w_2 \in V(G)$ be such that $v \in \text{leaves}(T(F)) \setminus \{f\}$, $w_2 \in D_2$ is the unique neighbor of $v$ in $T(F)$, $u, f$ and $z, v$ are $F$-pairs, and $u$ $F$-links $f$ to $z$. If there is some $w_1 \in P^*_{F}(u, f)$ with $w_1 \in F$, remove $vw_2$ from $F$ and add $vw_1$.

Proof of safeness of Rule 4. Let $F' = F \setminus \{vw_2\}$. Since $T(F)$ is a tree and $w_2$ $F$-links $w_1$ and $v$, edge $vw_2$ is contained in the unique cycle of $G \setminus F'$; consequently, $F'' = F' \cup \{vw_2\}$ is a feedback edge set of $G$ of size $q$, but it holds that the degrees of $v$ and $w_2$ in $T(F'')$ are equal to one. Since $w_2 \notin \text{leaves}(T(F))$, we have that $\text{leaves}(T(F)) \subset \text{leaves}(T(F''))$.

Our analysis for Rule 4 works even if $u = z$ or $z = w_2$: what is truly crucial is that $w_2 \in D_2$ and that $v \neq f$. We present an example of the general case in Figure 3.

Reduction Rule 5. Let $f, u, v, z, w_2, w_3 \in V(G)$ be such that $v \in \text{leaves}(T(F))$, $w_3 \in N_{T(F)}(u) \cap P_{F}(u, f)$, $w_2w_3, vz \in E(T(F))$, $u, f$ is an $F$-pair, $z \in D_z$, and $u$ $F$-links $f$ to $v$. If $v$ is adjacent to some $w_1 \in P_{F}(u, f) \setminus \{w_2\}$ that $F$-links $w_2$ to $f$, remove $vw_1$ from $F$ and add $w_2w_3$ to $F$.

Proof of safeness of Rule 5. Let $F' = F \setminus \{vw_3\}$. Since $w_1$ $F$-links $w_2$ to $f$, we have that $w_2$ $F$-links $w_1$ to $v$, so edge $w_2w_3$ belongs to the unique cycle of $G \setminus F'$ and, consequently, $F'' = F' \cup \{w_2w_3\}$ is a feedback edge set of $G$ of size $q$. Since $\deg_{T(F)}(w_3) = \deg_{T(F)}(w_2) = 2$, $\deg_{T(F')} (w_3) = \deg_{T(F')} (w_2) = 1$, however,
v is adjacent to both z and w_1 in T(F’’), so it holds that leaves(T(F’’)) = leaves(T(F)) \cup \{w_2, w_3\} \setminus \{v\}.

Figure 4: Example for Reduction Rule 5, where the dotted edge w_2w_3 is added to F and the thick edge vw_1 is removed from F.

Our next lemma guarantees that the exhaustive application of rules 2 through 5 finds a set of paths in T(F) that have many vertices of degree two in G; essentially, at this point, we are done minimizing the number of incident edges to vertices of D_2.

**Lemma 11.** Let a, b ∈ V(G) be an F-pair such that a, b ∉ D_2, |P_F^∗(a, b)| ≥ 5, and let w be one of its inner vertices at distance at least three from both a and b. If none of the rules between Rule 3 and Rule 5 are applicable, then deg_G(w) = deg_T(F)(w).

**Proof.** Suppose that this is not the case, and let v ∈ N_G(w) \ N_{T(F)}(w).

- If v ∈ P_F(a, b), suppose w.l.o.g. that w F-links v to a; moreover, let w_2 be the unique neighbor of v that F-links it to w. In this case, Rule 2 is applicable: a, v are an F-pair with the required properties, w_2 has the same role here as in the definition of the rule, and we may set w as w_1.

- If v ∉ leaves(T(F)), we may assume, w.l.o.g., that w F-links v to b. We can apply Rule 3; v is not adjacent to w in T(F), so w has one neighbor w_2 ∈ D_2 that F-links it to v.

- If v ∈ leaves(T(F)) and its unique neighbor is w_2 ∈ D_2 \ P_F(a, b), we again may assume w.l.o.g. that w_2 F-links a and v. In this case, Rule 4 is applicable: there is some z ∉ leaves(T(F)) (possibly z ∈ \{a, w_2\}) that forms an F-pair with v, where P_F(z, v) \ {z, v} may be empty if z = w_2.

- If v ∈ leaves(T(F)) and z ∈ D_4 is its unique neighbor in T(F), then, since there are at least two other vertices between w and each of the endpoints of P_F(a, b), Rule 5 is applicable; to see that this is the case, set w to w_3 in the definition of the rule and w_1, w_2 as appropriate to depending on which endpoint of P_F(a, b) F-links w to v.

Thus, we conclude that v cannot exist and that the statement holds.

At this point, paths between F-pairs are mostly the same as in G: only the to vertices closest to each endpoint may be adjacent to some leaves of T(F), while all others have degree two in G. We say that u, f are a strict F-pair if for every w ∈ P_F^∗(u, f), deg_G(w) = 2.
Reduction Rule 6. Let \( u, f \in V(G) \) be a strict F-pair. If there are adjacent vertices \( w_1, w_2 \in P_F^*(u, f) \) such that either \( w_1, w_2 \in K \) or \( w_1, w_2 \notin K \), add a new vertex \( w^* \) to \( G \) that is adjacent to \( N_G(w_1) \cup N_G(w_2) \) \( \setminus \{w_1, w_2\} \) and remove both \( w_1, w_2 \in G \). If \( w_1, w_2 \in K \), set \( w^* \) as a terminal vertex.

Proof of safeness of Rule 6. Correctness follows directly from the hypotheses that \( w_1 \in K \) if and only if \( w_2 \in K \) and that both are degree two vertices. So, in a minimal solution \( H \) to \((G, K)\), either both vertices are in \( H \) or neither is in \( H \). For the converse, any minimal solution \( H' \) to the reduced instance \((G', K')\) either has \( w^* \), in which \( H \) is obtained by replacing \( w^* \) with both \( w_1 \) and \( w_2 \), or \( w^* \notin V(H) \), in which case \( H' \) itself is a solution to \((G, K)\). \( \square \)

Reduction Rule 7. Let \( u, f \in V(G) \) be a strict F-pair such that \( P_F^*(u, f) \geq 4 \). If Rule 6 is not applicable, replace \( P_F^*(u, f) \) with three vertices \( a, t, b \) so that \( a \) is adjacent to \( u, b \) to \( f \), and \( t \) to both \( a \) and \( b \). Furthermore, \( t \) is a terminal of the new graph if and only if \( K \setminus P_F^*(u, f) \neq \emptyset \).

Proof of safeness of Rule 7. Let \( G' \) and \( K' \) be, respectively, the graph and set of terminals obtained after the application of the rule. Suppose \( H \) is a minimal solution to the \( k \)-in-a-tree instance \((G, K)\), i.e., every vertex of \( H \) is contained in a path between two terminals. Note that, if \( P_F^*(u, f) \cap V(H) = \emptyset \), \( H \) is also a solution to the instance \((G', K')\); as such, for the remainder of this paragraph, we may assume w.l.o.g. that \( P_F^*(u, f) \cap V(H) \neq \emptyset \) and that \( u \in V(H) \). If \( P_F^*(u, f) \setminus \{u\} \not\subseteq V(H) \), \( H' = H \cup \{a, t\} \setminus P_F^*(u, f) \) is a solution to \((G', K')\); since at least one vertex of \( P_F^*(u, f) \) is not in \( V(H) \) and every \( w \in P_F^*(u, f) \) has degree two in \( G \), the subpaths of \( P_F^*(u, f) \) in \( H \) are used solely for the collection of terminal vertices of \( P_F^*(u, f) \); consequently, \( H' \) is an induced tree of \( G' \) that contains all elements of \( K' \). On the other hand, if \( P_F^*(u, f) \subseteq V(H) \), \( H' = H \cup \{a, t, b\} \setminus P_F^*(u, f) \) is a solution to \((G', K')\); to see that this is the case, note that \( H \setminus P_F^*(u, f) \) is a forest with exactly two trees where \( u \) and \( f \) are in different connected components since \( P_F^*(u, f) \) is the unique path between them in \( H \), and \( K' \subseteq V(H') \) since \( P_F^*(u, f) \subseteq V(H) \) and \( K \setminus P_F^*(u, f) \subseteq V(H) \setminus P_F^*(u, f) \).

For the converse, let \( H' \) be a minimal solution to \((G', K')\). If \( \{a, t, b\} \not\subseteq V(H') \), \( H = H' \cup \{P_F^*(u, f) \setminus \{a, t, b\}\} \) is a solution of \((G, K)\), as we are replacing one path consisting solely of degree two vertices with another that satisfies the same property. If \( a \in V(H') \) but \( b \notin V(H') \), then \( t \in K' \) (recall that \( H' \) is minimal) and \( u \in V(H) \), implying that there is at least one terminal vertex in \( P_F^*(u, f) \). We branch our analysis in the following subcases, where \( v \in P_F^*(u, f) \setminus N_G(f) \):

- If \( v \notin K \), then \( H = H' \cup P_F^*(u, f) \setminus \{v\} \setminus \{a, t\} \) is a solution to \((G, K)\): all terminals of \( P_F^*(u, f) \) are contained in \( P_F^*(u, v) \) and no cycle is generated since all vertices of the path \( P_F^*(u, v) \) have degree two.
- If \( v \in K \) but \( f \notin V(H') \), \( H = H' \cup P_F^*(u, f) \setminus \{a, t\} \) is a solution to \((G, K)\): we cannot create any new cycle since \( f \notin V(H) \) and \( u, f \) form a strict F-pair, moreover all terminals of \( P_F^*(u, f) \) are contained in \( H \).
- If \( v \in K \) and \( f \in V(H') \), there is at least one non-terminal vertex \( w \in P_F^*(u, v) \) since Rule 6 is not applicable to \( P_F^*(u, f) \). As such, we set \( H = H' \cup P_F^*(u, w) \cup P_F^*(w, f) \setminus \{a, t\} \) and obtain a solution to \((G, K)\).
Finally, if \( \{a, t, b\} \cap V(H') = \emptyset \), it follows immediately from the assumption that 
\( H' \) is a solution to \((G', K')\) that \( H' \) is also a solution to \((G, K)\).

We are now ready to state our kernelization theorem.

**Theorem 12.** When parameterized by the size \( q \) of a feedback edge set, \( k\text{-IN-A}\)-
tree admits a kernel with \( 16q \) vertices and \( 17q \) edges that can be computed in 
\( O(q^2 + qn) \) time.

**Proof.** Let \( G \) be our \( n \)-vertex input graph and \( K \) a set of terminals. We begin by applying Rule 1 until no degree one vertex remains in \( G \). Then, we take any feedback edge set \( F \) of \( G \) – we can obtain one in \( O(n) \) time by listing the set of back edges of a depth-first search tree of \( G \) – and construct \( \mathcal{P}_F \) in \( O(n) \) time.

Let \( \mathcal{P}_F \) be the set of paths between all \( F \)-pairs such that, for each \( P_F(u, f) \in \mathcal{P}_F \) it holds that \( u, f \notin D_2 \). By Observation 10 we have at most \( 2q \) leaves in \( T(F) \) and \( 2q \) vertices in \( D_2 \), so there are at most \( 4q \) paths in \( \mathcal{P}_F \).

Each iteration of the first part of the algorithm is described below. If there is some path \( P_F(u, f) \in \mathcal{P}_F \) with \( f \in \text{leaves}(T(F)) \) and \( P_F^*(u, f) \neq \emptyset \), check if there is an edge in \( F \) between \( f \) and one of the vertices of \( P_F^*(u, f) \cup \{ u \} \); if this is the case, apply Rule 2 in \( O(n) \) time and move on to the next iteration; by Observation 10 \( |\mathcal{P}_F| \leq 4q \), so we can inspect each path and perform this step in \( O(q + n) \) time. Otherwise, let \( uv \in F \) be such that \( u \notin \text{leaves}(T(F)) \), and \( w_1 \in P_F(u, f) \in \mathcal{P}_F \). If \( v \notin \text{leaves}(T(F)) \) and \( w_1 \) is adjacent to some \( w_2 \in P_F^*(u, f) \) that \( F \)-links to \( v \), apply Rule 3 we can check if these conditions are satisfied in \( O(n) \) time, in particular, \( F \)-linking is a matter of testing if \( w_1 \) and \( v \) are in the same connected component of \( T(F) \setminus \{ w_2 \} \). If, however, \( v \in \text{leaves}(T(F)) \setminus \{ f \} \), \( v \) forms an \( F \)-pair with \( z \in V(G) \), \( w_2 \in D_2 \cap N_{T(F)}(v) \), and \( u F \)-links \( f \) to \( z \), Rule 4 is applicable in \( O(n) \) time. Finally, if \( v \in \text{leaves}(T(F)) \setminus \{ f \} \) is adjacent to some \( z \in D_2 \) which \( F \)-links \( u \) and \( v \), \( w_3 \in P_F(u, f) \cap N_{T(F)}(u) \) is adjacent to \( w_2 \) which in turn is \( F \)-linked to \( f \) by \( w_1 \), Rule 5 is applicable is \( O(n) \) time.

If none of the conditions stated in the previous paragraph is satisfied, we stop the algorithm: the number of leaves of \( T(F) \) cannot be increased by single edge swaps. As to the number of iterations, rules 2 through 5 guarantee that, when applicable, the number of leaves increases by exactly one. Since each one of their applications can be performed in \( O(q + n) \) time and we have at most \( \max_{P_F} |\text{leaves}(T(F))| \leq 2q \) iterations, the above algorithm can be executed in \( O(q^2 + qn) \) time.

Let \( P_\alpha \) be the set of paths of \( \mathcal{P}_F \) whose endpoints are strict \( F \)-pairs. For each path \( P_F(u, f) \in \mathcal{P}_F \setminus P_\alpha \), let \( u', f' \in P_F(u, f) \) be the strict \( F \)-pair that maximizes \( |P_F(u', f')| \). Note that \( |P_F(u, f) \setminus P_F(u', f')| \leq 4 \) if there are leaves adjacent to each of the vertices at distance two from the endpoints; we refer to Figure 5 for an example. Finally, define \( P_\beta \) as \( P_F(u', f') \) for each \( P_F(u, f) \in \mathcal{P}_F \setminus P_\alpha \).

Now, for each path in \( P_\alpha \cup P_\beta \), we can apply Rule 6 since at each step we remove one vertex from \( G \), all applications of the rule amount to \( O(n) \)-time. Afterwards, we apply Rule 7 to compress the paths as much as possible; again, this entire process is feasibly done in \( O(n) \) steps. As such, each path in \( P_\alpha \cup P_\beta \) has size at most three; consequently each path in \( \mathcal{P}_F \) has size at most seven. At first glance, this would yield a kernel of size \( 4q + 7 \cdot 4q = 32q \); however, we can observe that, for each edge in \( F \) incident to a vertex in some path of \( P_\beta \), we are
essentially reducing the number of leaves of $T(F)$ and large degree vertices in one unit each: the bound of $2q$ leaves is only met with equality if every edge of $F$ is incident to two leaves of $T(F)$. Therefore, if $\beta = |\mathcal{P}_\beta|$, the kernel’s size is given by $|\text{leaves}(T(F))| + |D_s| + 3|\mathcal{P}_x| + 7|\mathcal{P}_\beta| \leq (2q - \beta) + (2q - \beta) + 3(4q - 2\beta) + 7\beta = 16q - \beta$, which is maximized when $\beta = 0$. Regarding the number of edges, the contracted graph has $16q$ vertices and a feedback edge set of size $q$, so it has at most $17q$ edges. Finally, since the first part of the algorithm runs in $O(q^2 + qn)$ time and the latter in $O(n)$ time, we have a total complexity of $O(q^2 + qn)$ time. \hfill \Box

6 Kernelization lower bounds

In this section, we apply the cross-composition framework of Bodlaender et al. [4] to show that, unless $\text{NP} \subseteq \text{coNP}/\text{poly}$, $k$-in-a-tree does not admit a polynomial kernel under bandwidth, nor when parameterized by the distance to any graph class with at least one member with $t$ vertices for each integer $t$, which we collectively call non-trivial classes. We say that an NP-hard problem $R$ OR-cross-composes into a parameterized problem $L$ if, given $t$ instances $\{y_1, \ldots, y_t\}$ of $R$, we can construct, in time polynomial in $\sum_{i \in [t]} |y_i|$, an instance $(x, k)$ of $L$ that satisfies $k \leq p(\max_{i \in [t]} |y_i| + \log t)$ and admits a solution if and only if at least one instance $y_i$ of $R$ admits a solution; we say that $R$ AND-cross-composes into $L$ if the first two conditions hold but all $(x, k)$ has a solution if and only if all $t$ instances of $R$ admit a solution.

6.1 Bandwidth

**Theorem 13.** When parameterized by bandwidth, $k$-in-a-tree does not admit a polynomial kernel unless $\text{NP} \subseteq \text{coNP}/\text{poly}$.

**Proof.** We are going to show that $k$-in-a-tree AND-cross-composes into itself. Let $\mathcal{H} = \{(H_1, K_1), \ldots, (H_t, K_t)\}$ be a set of instances of $k$-in-a-tree where each graph has $n$ vertices, $t \geq 3$ of which are terminals. The input $(G, K)$ to $k$-in-a-tree parameterized by bandwidth is constructed as follows: $G$ is initially the disjoint union of the $t$ input graphs and $K = \bigcup_{i \in [t]} K_i$; now, for each $i \in [t]$, take two distinct terminals $v_1(i), v_2(i)$ and add edge $v_2(i)v_1(i + 1)$ for every $i \in [t - 1]$. Essentially, we are organizing the $H_i$'s in a path.

Suppose now that every $H_i$ has a solution $T_i$ and note that $T = \bigcup_{i \in [t]} V(T_i)$ is a solution to $(G, K)$: $T$ is a tree and every terminal in $K$ has a path to
another. For the converse, take a solution \( T \) to \((G, K)\) and let \( T_i = T \cap V(H_i) \).

To see that \( T_i \) is in fact a solution to \((H_i, K_i)\), it suffices to observe that there can be no path between two vertices of \( H_i \) that contains vertices that do not belong to \( H_i \). As to the bandwidth, we claim that it is at most \( n - 1 \): we can set each vertex of \( H_i \) in the interval \([n(i - 1), ni - 1]\) arbitrarily as long as \( v_1(i) \) is placed at \( n(i - 1) \) and \( v_2(i) \) at \( ni - 1 \), obtaining the mapping \( f \). Consequently, every edge \( ab \in E(H_i) \) satisfies \( |f(a) - f(b)| \leq n \) and each edge \( v_2(i)v_1(i + 1) \) satisfies \( f(v_1(i + 1)) - f(v_2(i)) = n(i + 1 - 1) - (ni - 1) = 1 \).

### 6.2 Vertex cover

In this section, we show that Hamiltonian Path on cubic graphs OR-cross-composes into \( k \)-IN-A-TREE parameterized by vertex cover and number of terminals. Our construction, however, can be trivially adapted to different parameterizations, such as distance to clique. In both cases, we heavily rely on the original gadget by Derhy and Picouleau [18], but make some modifications to suit our needs. Let \( H \) be an instance of Hamiltonian Path on cubic graphs.

The Derhy-Picouleau graph of \( H \), which we denote by \( DP(H) \), is constructed as follows: for each \( v_i \in V(H) \), add to \( DP(H) \) one copy \( T_i \) of the gadget depicted in Figure 6 and, for each edge \( v_i v_j \in E(H) \), connect one of the black vertices of \( T_i \) to one of the black vertices of \( T_j \) so that the degree of each black vertex of \( DP(H) \) is three. We say that \( T_i \) and \( T_j \) are adjacent if there is an edge between a black vertex \( \alpha_i \) of \( T_i \) and a black vertex \( \beta_j \) of \( T_j \), where \( \{\alpha, \beta\} \subset \{a, b, c\} \).

The set of mandatory vertices of \( DP(H) \) is the set of gray vertices.

![Figure 6: Vertex gadget \( T_i \) for vertex \( v_i \). Vertex \( s_i \) is the only terminal of this gadget; white vertices are part of an independent set of maximum size.](image)

Before presenting the composition itself, we need to make some slight modifications to \( DP(H) \), to obtain what we dubbed the representative graph of \( H \). Ultimately, our goal is to overlay the multiple instances of Hamiltonian Path and, by applying an instance selector gadget, force the graph representing the selected instance to emerge from the confounding structure.

#### 6.2.1 Representative Graph

Our key modification to \( DP(H) \) is to replace the edge between black vertices with edge gadgets. Suppose that \( v_i v_j \in E(H) \), \( i < j \), and that \( \alpha_i \beta_j \in DP(H) \). We replace the latter edge with the four vertex gadget \( e(i, j, \alpha, \beta) \) as in Figure 7.

Note that \( e(i, j, \alpha, \beta) \) and \( e(i, j, \beta, \alpha) \) are different gadgets whenever \( \alpha \neq \beta \). By
doing this for every edge of $H$, we obtain the representative graph of $H$, denoted by $\text{Rep}(H)$. Intuitively, if $c_i, b_j$ are in the solution of the $k$-in-a-tree instance given by $\text{DP}(H)$, then $g_{ij}^{cb}$ is not in the solution of the instance whose input is $\text{Rep}(H)$. If either $c_i$ or $b_j$ are not in the solution, $g_{ij}^{cb}$ acts as a garbage collector and is used to connect $s_j$ and $s_{ij}^{cb}$.

![Figure 7: Edge gadget $e(i, j, c, b)$ for edge $v_i v_j \in E(H)$ ($i < j$).](image)

**Lemma 14.** There is an induced tree connecting the terminal vertices of $\text{DP}(H)$ if and only if there is an induced tree connecting the terminal vertices of $\text{Rep}(H)$.

**Proof.** Let $S$ be a solution to $\text{DP}(H)$; we construct the solution $S'$ to $\text{Rep}(H)$ as follows. $S'$ contains every vertex in $S$. For every pair of adjacent black vertices $\alpha_i, \beta_j$ add $s_{ij}^{\alpha\beta}$ to $S'$; if both $\alpha_i$ and $\beta_j$ are in $S$, add $\{p_{ij}^{\alpha\beta}, q_{ij}^{\alpha\beta}\}$ to $S'$; otherwise add $g_{ij}^{\alpha\beta}$ to $S'$; this concludes the definition of $S'$. To see that $S'$ induces a tree of $\text{Rep}(H)$, note that each path $\langle s_i, \text{white vertex}, \alpha_i, \beta_j, \text{white vertex}, s_j \rangle$ effectively had edge $\alpha_i\beta_j$ replaced by an induced $P_5$; furthermore, $g_{ij}^{\alpha\beta}$ is in $S'$ if and only if $\{\alpha_i, \beta_j\} \not\subseteq S$, so no cycle can be formed in the edge gadget; since $S$ induces a tree of $\text{DP}(H)$, we conclude that $S'$ induces a tree of $\text{Rep}(H)$.

Finally, $S'$ contains all terminal (gray) vertices of $\text{Rep}(H)$: all such vertices also in $\text{DP}(H)$ were already connected, while the new ones are either included in the induced $P_5$'s with endpoints $\alpha_i, \beta_j$, or are connected by $g_{ij}^{\alpha\beta}$ to $s_j$ (assuming $i < j$).

For the converse, suppose $S' \subseteq V(\text{Rep}(H))$ induces a tree of $\text{Rep}(H)$. Note that we can assume that $S'$ is minimal; in particular, we may safely assume that every black $\alpha_i$ vertex in $S'$ is used to connect $s_i$ to some other $s_j$ or, at the very least, to some terminal of an edge gadget. With this restriction in mind, we obtain our solution $S$ to $\text{DP}(H)$ by setting $S := S' \cap V(\text{DP}(H))$; note that, if $S'$ is not minimal, there could be a pair of vertices $\alpha_i \in V(T_i), \beta_j \in v(T_j)$ with $\alpha_i, \beta_j \in E(\text{DP}(H))$, which would imply that $S$ was not acyclic. Towards showing that $S$ induces a tree of $\text{DP}(H)$, let $s_i, s_j$ be two terminals of $\text{DP}(H)$ and $P_{\text{Rep}(i, j)}$ be the unique path between them in the subgraph of $\text{Rep}(H)$ induced by $S'$. We claim that there is no vertex $g_{ij}^{\alpha\beta}$ in $P_{\text{Rep}(i, j)}$: $g_{ij}^{\alpha\beta} \in S'$ implies that it is the unique neighbor of $s_{ij}^{\alpha\beta}$ in $S'$, otherwise $S'$ would not induce a tree of $\text{Rep}(H)$. Consequently, $P_{\text{Rep}(i, j)} \cap V(\text{DP}(H))$ induces a path of
Lemma 15. The graph $G$ has a vertex cover of size $O(n^2)$ and at most $O(n^2)$ terminals.

Proof. Note that $G$ has $7n$ vertices in vertex gadgets, at most $18(n^2 - 3n)$ vertices in edge gadgets, and $t + 1$ vertices in the instance selector gadget (recall that one vertex of $X$ is identified with a terminal of $T_1$). Since $Y$ is an independent set, $V(G) \setminus Y$ is a vertex cover with $O(n^2)$ elements. For the last part of the statement, it suffices to observe that the set of terminals of $G$ is a subset of $V(G) \setminus Y$. □

Lemma 16. There is no induced tree of $G$ connecting all terminal vertices with zero or more than one vertex of $Y$. Moreover, if $y_t \in Y$ is fixed, the graph obtained after removing $X$, $Y$, and all vertices that are in a triangle with $y_t$ is precisely $\text{Rep}(H_t)$.
Proof. If no vertex of \( Y \) is picked, then there is no path between the two terminals \( s_1, x \in X \) precisely because \( N(x) = Y \). If two vertices \( y_\ell, y_p \in Y \) are picked, then \( \{x, y_\ell, s_1, y_p\} \) is an induced \( C_4 \).

Now, let \( e(i, j, \alpha, \beta) \) be an edge gadget of \( G \). For the second part of the statement, recall that \( V(e(i, j, \alpha, \beta)) \subseteq N(y_\ell) \setminus X \) if and only if \( e(i, j, \alpha, \beta) \notin \mathcal{E}_\ell \), i.e., the vertices \( W_\ell \) of \( G \) that form a triangle with \( y_\ell \) are precisely those that belong to the edge gadgets \( e(i, j, \alpha, \beta) \) not present in \( \text{Rep}(H_\ell) \). As such, the induced subgraph of \( G \) that remains after the removal of \( W_\ell, X \) and \( Y \) is composed precisely of the vertex gadgets and \( \mathcal{E}_\ell \); since no extra edges were added within either group of gadgets or between them, we have that \( G \setminus (W_\ell \cup X \cup Y) = \text{Rep}(H_\ell) \).

Theorem 17 is a direct consequence of Lemmas 14, 15, and 16.

Theorem 17. \( k\)-in-a-tree does not admit a polynomial kernel when parameterized by the number of vertices of the induced tree, and size of a minimum vertex cover unless \( \text{NP} \subseteq \text{coNP}/\text{poly} \).

As for Corollary 18, we observe that there is nothing special about set \( Y \) of our instance selector gadget being an independent set; the key feature is that only one of them may be used in a solution; as such, we may freely encode a member of whichever graph classes we are interested in \( G[Y] \).

Corollary 18. For every non-trivial graph class \( \mathcal{G} \), \( k\)-in-a-tree does not admit a polynomial kernel when parameterized by the number of vertices of the induced tree and size of a minimum \( \mathcal{G} \)-modulator unless \( \text{NP} \subseteq \text{coNP}/\text{poly} \).

7 Concluding Remarks

In this work, we performed an extensive study of the parameterized complexity of \( k\)-in-a-tree and the existence of polynomial kernels for the problem, motivated by the relevance of three-in-a-tree in subgraph detection algorithms and a question of Derhy and Picouleau\[18\] about the complexity of four-in-a-tree. We presented multiple positive and negative results on the problem, of which we highlight its \( \text{W}[1] \)-hardness under the natural parameter, the linear kernel under feedback edge set, and the nonexistence of a polynomial kernel under vertex cover/distance to clique. The main question about the complexity of \( k\)-in-a-tree for fixed \( k \), however, remains open; our hardness result showed that there is no \( n^{o(k)} \) time algorithm under ETH, but no XP algorithm is known to exist. It is worthwhile to revisit some cases where three-in-a-tree has not been successful to identify possible applications for \( k\)-in-a-tree. There are also no known running time lower bounds for the parameters we study, and determining whether or not we can obtain \( 2^{o(q)} n^{\Omega(1)} \) time algorithms seems a feasible research direction; still in terms of algorithmic results, it would be quite interesting to see how we can avoid Courcelle’s Theorem to get an algorithm when parameterizing by cliquewidth. The natural investigation of \( k\)-in-a-tree on different graph classes may provide some insights on how to tackle particular cases, such as four-in-a-tree; this study has already been started in[20] and in others — such as cographs — may even be trivial, but many other cases may be quite challenging and much still remains to be done.
References

[1] Dan Bienstock. On the complexity of testing for odd holes and induced odd paths. *Discrete Mathematics* 90(1):85 – 92, 1991. ISSN 0012-365X. doi: https://doi.org/10.1016/0012-365X(91)90098-M. URL http://www.sciencedirect.com/science/article/pii/0012365X9190098M

[2] Dan Bienstock. Corrigendum: To: D. bienstock, “On the complexity of testing for odd holes and induced odd paths” *Discrete Mathematics* 90 (1991) 85–92. *Discrete Mathematics*, 102(1):109, 1992. ISSN 0012-365X. doi: https://doi.org/10.1016/0012-365X(92)90357-L. URL http://www.sciencedirect.com/science/article/pii/0012365X9290357L

[3] Hans L. Bodlaender, Rodney G. Downey, Michael R. Fellows, and Danny Hermelin. On problems without polynomial kernels. *Journal of Computer and System Sciences*, 75(8):423 – 434, 2009. ISSN 0022-0000. doi: https://doi.org/10.1016/j.jcss.2009.04.001. URL http://www.sciencedirect.com/science/article/pii/S0022000009000282

[4] Hans L. Bodlaender, Bart M. P. Jansen, and Stefan Kratsch. Cross-composition: A new technique for kernelization lower bounds. In Proc. of the 28th International Symposium on Theoretical Aspects of Computer Science (STACS), volume 9 of LIPIcs, pages 165–176, 2011.

[5] Hans L. Bodlaender, Marek Cygan, Stefan Kratsch, and Jesper Nederlof. Deterministic single exponential time algorithms for connectivity problems parameterized by treewidth. *Information and Computation*, 243: 86 – 111, 2015. ISSN 0890-5401. doi: https://doi.org/10.1016/j.ic.2014.12.008. URL http://www.sciencedirect.com/science/article/pii/S0890540114001606

[6] J.A. Bondy and U.S.R Murty. *Graph Theory*. Springer Publishing Company, Incorporated, 1st edition, 2008. ISBN 1846289696.

[7] Anudhyan Boral, Marek Cygan, Tomasz Kociumaka, and Marcin Pilipczuk. A fast branching algorithm for cluster vertex deletion. *Theory of Computing Systems*, 58(2):357–376, 2016. doi: 10.1007/s00224-015-9631-7. URL https://doi.org/10.1007/s00224-015-9631-7

[8] Leizhen Cai. Parameterized complexity of vertex colouring. *Discrete Applied Mathematics*, 127(3):415 – 429, 2003. ISSN 0166-218X. doi: https://doi.org/10.1016/S0166-218X(02)00242-1. URL http://www.sciencedirect.com/science/article/pii/S0166218X02002421

[9] Hsien-Chih Chang and Hsueh-I Lu. A faster algorithm to recognize even-hole-free graphs. *Journal of Combinatorial Theory, Series B*, 113:141 – 161, 2015. ISSN 0095-8956. doi: https://doi.org/10.1016/j.jctb.2015.02.001. URL http://www.sciencedirect.com/science/article/pii/S0095895615000155

[10] Maria Chudnovsky and Rohan Kapadia. Detecting a theta or a prism. *SIAM Journal on Discrete Mathematics*, 22(3):1164–1186, 2008. doi: 10.1137/060672613. URL https://doi.org/10.1137/060672613
[23] Jiří Fiala, Marcin Kamiński, Bernard Lidický, and Daniël Paulusma. The k-in-a-path problem for claw-free graphs. Algorithmica, 62(1):499–519, 2012. doi: 10.1007/s00453-010-9468-z. URL https://doi.org/10.1007/s00453-010-9468-z.

[24] Fedor V. Fomin, Daniel Lokshtanov, Saket Saurabh, and Meirav Zehavi. Kernelization: Theory of Parameterized Preprocessing. Cambridge University Press, 2019. doi: 10.1017/9781107415157.

[25] Lance Fortnow and Rahul Santhanam. Infeasibility of instance compression and succinct pcps for np. Journal of Computer and System Sciences, 77(1):91 – 106, 2011. ISSN 0022-0000. doi: https://doi.org/10.1016/j.jcss.2010.06.007. URL http://www.sciencedirect.com/science/article/pii/S0022000010000917. Celebrating Karp’s Kyoto Prize.

[26] Russell Impagliazzo and Ramamohan Paturi. On the complexity of k-sat. Journal of Computer and System Sciences, 62(2):367 – 375, 2001. ISSN 0022-0000. doi: https://doi.org/10.1006/jcss.2000.1727. URL http://www.sciencedirect.com/science/article/pii/S0022000000917276.

[27] Richard M. Karp. Reducibility among Combinatorial Problems, pages 85–103. Springer US, Boston, MA, 1972. ISBN 978-1-4684-2001-2. doi: 10.1007/978-1-4684-2001-2_9. URL https://doi.org/10.1007/978-1-4684-2001-2_9.

[28] Ken-ichi Kawarabayashi, Yusuke Kobayashi, and Bruce Reed. The disjoint paths problem in quadratic time. Journal of Combinatorial Theory, Series B, 102(2):424 – 435, 2012. ISSN 0095-8956. doi: https://doi.org/10.1016/j.jctb.2011.07.004. URL http://www.sciencedirect.com/science/article/pii/S0095895611000712.

[29] Kai-Yuan Lai, Hsueh-I Lu, and Mikkel Thorup. Three-in-a-tree in near linear time. In Proceedings of the 52nd Annual ACM SIGACT Symposium on Theory of Computing, STOC 2020, page 1279–1292, New York, NY, USA, 2020. Association for Computing Machinery. ISBN 9781450369794. doi: 10.1145/3357713.3384235. URL https://doi.org/10.1145/3357713.3384235.

[30] W. Liu and N. Trotignon. The k-in-a-tree problem for graphs of girth at least k. Discrete Applied Mathematics, 158(15):1644 – 1649, 2010. ISSN 0166-218X. doi: https://doi.org/10.1016/j.dam.2010.06.005. URL http://www.sciencedirect.com/science/article/pii/S0166218X10002131.

[31] Hannes Moser and Somnath Sikdar. The parameterized complexity of the induced matching problem. Discrete Applied Mathematics, 157(4): 715 – 727, 2009. ISSN 0166-218X. doi: https://doi.org/10.1016/j.dam.2008.07.011. URL http://www.sciencedirect.com/science/article/pii/S0166218X08003211.

[32] Neil Robertson and Paul D. Seymour. Graph minors. ii. algorithmic aspects of tree-width. Journal of Algorithms, 7(3):309 – 322, 1986. ISSN 0196-6774. doi: https://doi.org/10.1016/0196-6774(86)90023-4. URL http://www.sciencedirect.com/science/article/pii/0196677486900234.
[33] Manuel Sorge and Mathias Weller. The graph parameter hierarchy. *unpublished manuscript*, 2019.

[34] Nicolas Trotignon and Kristina Vušković. A structure theorem for graphs with no cycle with a unique chord and its consequences. *Journal of Graph Theory*, 63(1):31–67, 2010. doi: 10.1002/jgt.20405. URL [https://onlinelibrary.wiley.com/doi/abs/10.1002/jgt.20405](https://onlinelibrary.wiley.com/doi/abs/10.1002/jgt.20405)

[35] Chee K. Yap. Some consequences of non-uniform conditions on uniform classes. *Theoretical Computer Science*, 26(3):287 – 300, 1983. ISSN 0304-3975. doi: [https://doi.org/10.1016/0304-3975(83)90020-8](https://doi.org/10.1016/0304-3975(83)90020-8). URL [http://www.sciencedirect.com/science/article/pii/0304397583900208](http://www.sciencedirect.com/science/article/pii/0304397583900208).