A ROBUST COMPUTATIONAL FRAMEWORK FOR ANALYZING FRACTIONAL DYNAMICAL SYSTEMS

Khosro Sayevand* and Valeyollah Moradi
Faculty of Mathematical Sciences, Malayer University, Malayer, Iran

Abstract. This study outlines a modified implicit finite difference method for approximating the local stable manifold near a hyperbolic equilibrium point for a nonlinear systems of fractional differential equations. The fractional derivative is described in the Caputo sense of order $\alpha$ ($0 < \alpha \leq 1$) which is approximated based on the modified trapezoidal quadrature rule of order $O(\Delta t^{2-\alpha})$. The solution existence, uniqueness and stability of the proposed method is discussed. Three numerical examples are presented and comparisons are made to confirm the reliability and effectiveness of the proposed method.

1. Introduction. Fractional calculus is a subject between probability, differential equation and mathematical physics and deals with the study of real or complex order integral and derivative operators and their applications [2, 4, 7, 12–15, 22, 23, 32, 34–36, 38–41, 47, 49]. However, during the last 10 years, with the rapid development of nonlinear science, this theory has developed progressively and researchers have found that derivatives and integrals of non-integer order are suitable for the description of various physical phenomena. The non-local property and memory effect of fractional derivatives and integrals, is the main reason to use them in various fields of sciences. During the past decades especially the last years, fractional differential equations (FDEs) have played an important role in many fields such as physics, biology, mechanics and chemistry, etc. [19]. More recently, the applications of FDEs have been extended to quantum mechanics such that fractional quantum mechanics came into being [46, 48]. As we know, obtaining analytical solutions for problems based on fractional derivative and integral is very difficult. Therefore, approximate methods for finding the approximate solutions of these equations are very necessary and useful. In fact, finite difference methods, finite element methods, spectral methods, etc. have been presented to solve the various fractional equations. The review article [28] contains an up-to-date bibliography on finite difference methods for solving FDEs until 2012. At present, this field of study is still developing for its many applications in such varied fields as viscoelasticity, fluid flow and hydrology. The development of efficient numerical techniques for approximating the solutions of FDEs has been an important issue in the last decades. Therefore, some of the strategies such as the fractional finite volume method [30], Haar wavelet method [6], radial basis functions [16], operational matrix methods [24], Fourier spectral methods [5, 37] and other approaches [1, 11, 25, 26, 45, 50] were proposed.

2020 Mathematics Subject Classification. Primary: 41A58; 39A10 Secondary: 34K28; 41A10.
Key words and phrases. Caputo fractional derivatives, finite difference method, fractional dynamical systems, Hartman-Grobman theorem, stable manifold.
* Corresponding author: Khosro Sayevand.
For an interesting history and more scientific applications of fractional calculus, see [3, 27].

Finding accurate and efficient methods for solving fractional differential equations has become an important task. Solving the coupled nonlinear fractional dynamical systems is one of the most important issues in this field. Many researchers have done some work in this regard. Here are some examples of the latest works. Malik and Kumar in [33] established existence, uniqueness, Hyer-Ulam stability and controllability results for a coupled fractional dynamical system on time scales. Heydari et al. in [17] proposed a computational approach based on the shifted second-kind Chebyshev cardinal functions for obtaining an approximate solution of coupled variable order time-fractional sine-Gordon equations where the variable-order fractional operators are defined in the Caputo sense. Li and Ma in [31] established the similar relationship between a fractional differential equation and the corresponding fractional flow under a reasonable condition. They firstly presented some results on fractional dynamical system defined by the fractional differential equation with Caputo derivative. Furthermore, they showed the linearization and stability theorems of the nonlinear fractional system. Also, they proved Audounet-Matignon - Montseny conjecture. Jhinga and Daftardar in [20] presented a new numerical method for solving fractional delay differential equations along with its error analysis. Khader et al. in [21] introduced an implementation of an efficient numerical method for solving the system of coupled non-linear fractional dynamical model of marriage. Their method was based on the spectral collocation method using Legendre polynomials. They proved existence of local stable manifold around a hyperbolic equilibrium point of a fractional system. Deshpande and Daftardar in [8] proved existence of local stable manifold around a hyperbolic equilibrium point of a fractional system. As the latest sample of such works, Zhang et al. in [51] introduced some Banach spaces and then, based on these spaces and the coincidence degree theory, they considered a 2m-point boundary value problem for a coupled system of impulsive fractional differential equations at resonance, and obtained a new criterion on existence. In this paper, we are going to use of an so important theorem, named as the stable manifold theorem [18]. This theorem, is one of the most important results in the local qualitative theory of differential equations. In the Caputo fractional derivative of order $\alpha$ ($^{c}D^{\alpha}$), which is defined in the next section, this theorem shows that near a hyperbolic equilibrium point $x_{0}$, the nonlinear system

$$^{c}D^{\alpha}x = f(x), \quad 0 < \alpha \leq 1,$$

has stable and unstable manifolds $S$ and $U$ tangent at $x_{0}$ to the stable and unstable subspaces $E^{s}$ and $E^{u}$ of the linearized system

$$^{c}D^{\alpha}x = Ax, \quad 0 < \alpha \leq 1.$$  \hspace{1cm} (2)

Here $^{c}D^{\alpha}x = (^{c}D^{\alpha}_{t}x_{1},^{c}D^{\alpha}_{t}x_{2}, \ldots ,^{c}D^{\alpha}_{t}x_{m})^{T}$, $x_{j} := x_{j}(t)$; $j = 1, 2, \ldots , m$, and $A \in \mathbb{R}^{m \times m}$. Furthermore, $S$ and $U$ are of the same dimensions as $E^{s}$ and $E^{u}$, and if $\phi_{t}$ is the flow of the nonlinear system (1), then $S$ and $U$ are positively and negatively invariant under $\phi_{t}$ respectively, and satisfy

$$\lim_{t \to -\infty} \phi_{t}(c) = x_{0}, \quad \text{for all } c \in S,$$

and

$$\lim_{t \to \infty} \phi_{t}(c) = x_{0}, \quad \text{for all } c \in U.$$
We know that any linear system \( (2) \) has a unique solution in each point \( x_0 \) in \( \mathbb{R}^m \), the solution is given by \( x(t) = E_\alpha(At^\alpha)x_0 \), where \( E_\alpha(z) \) is the Mittag-Leffler function defined in (9).

In this paper, using an implicit finite difference method based on the modified trapezoidal quadrature rule, backward Euler differences and non-standard central approximations we would study nonlinear systems of fractional differential equations (1) near a hyperbolic equilibrium point \( x_0 \), where \( f : E \rightarrow \mathbb{R}^m \) and \( E \) is an open subset of \( \mathbb{R}^m \). We will show that under certain conditions on the function \( f \), the nonlinear system (1) has a unique solution in each point \( x_0 \in E \). We represent the stable manifold theorem and the Hartman-Grobman theorem [39] for nonlinear systems of fractional order which show that topologically the local behavior of the nonlinear system (1) near an equilibrium point \( x_0 \) where \( f(x_0) = 0 \) is typically determined by the behavior of the linear system (2) near the origin when the matrix \( A = Df(x_0) \). Furthermore, we establish the fundamental existence-uniqueness theorem for a nonlinear system of fractional differential equations (1) under the hypothesis that \( f \in C^1(E) \) where \( E \) is an open subset of \( \mathbb{R}^m \).

This paper has the following organisation. Section 2 introduces the fundamental definitions and properties of the fractional calculus. Section 3 is dedicated to the numerical approximation of the Caputo fractional derivative using a modified implicit finite difference method. In Section 4 the concept of fractional stable manifold and the Hartman-Grobman theorem is presented. In Section 5 the stability results for system (1) are proved in Theorems 3 and 4. It should also be noted that the main results of the paper are Lemma 5, Theorems 3 and 4, and the algorithm of Section 5.1. Section 6 illustrates the performance of the method for various examples. Finally, Section 7 presents the main conclusions.

2. Fundamental concepts and notation. This section is devoted to a description of the operational properties in order to be acquainted with sufficient FC theory and enable us to follow the solutions of the problems given in this paper.

**Definition 1.** ( [36, 39]) A real function \( f(x), x > 0 \), is said to be in the space \( C_\alpha, \alpha \in \mathbb{R} \), if there exists a real number \( p(\alpha > \alpha) \), such that \( f(x) = x^p f_1(x) \), where \( f_1(x) \in C[0, \infty) \), and it is said to be in the space \( C_\alpha^n, n \in \mathbb{N} \cup \{0\} \), if and only if \( f^{(n)}(x) \in C_\alpha \).

**Definition 2.** ( [36]) The Riemann-Liouville fractional integral of order \( \alpha > 0 \) of a function \( f(x) \in C_\alpha, \alpha \geq -1 \), is defined as

\[
I_\alpha^x f(x) = \begin{cases} 
\frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(\tau)}{(x-\tau)^{1-\alpha}} d\tau, & \alpha > 0, \quad x > 0, \\
f(x), & \alpha = 0
\end{cases}
\]

(5)

\[
I_\alpha^x f(x, t) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(s, t)}{(x-s)^{1-\alpha}} ds, \quad \alpha > 0, \quad x > 0,
\]

(6)

where \( \Gamma(\alpha) \) is the well-known Gamma function.

**Definition 3.** ( [36]) The Caputo fractional derivative of order \( \alpha > 0 \) of a function \( f(x), f(x) \in C_{\alpha-1}^n, n \in \mathbb{N} \cup \{0\} \), is defined as
\[ cD^\alpha f(x) = \begin{cases} 
\left[ I^{n-\alpha} f^{(n)}(x) \right] & n-1 < \alpha < n, \ n \in \mathbb{N}, \\
n \frac{d^n}{dx^n} f(x) & \alpha = n, 
\end{cases} \quad (7) \]

\[ cD_x^\alpha f(x, t) = I_x^{-\alpha} \frac{\partial^n f(x, t)}{\partial x^n}, \ n-1 < \alpha < n. \quad (8) \]

**Definition 4.** ([36]) The Mittag-Leffler function \( E_{\alpha, \beta}(z) \) with \( \alpha > 0, \ \beta > 0 \) is defined by the following series representation, valid in the whole complex plane

\[ E_{\alpha, \beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + \beta)}, \ z \in \mathbb{C}. \quad (9) \]

For \( \beta = 1 \), we obtain the Mittag-Leffler function in one parameter:

\[ E_{\alpha, 1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + 1)} \equiv E_{\alpha}(z). \quad (10) \]

**Definition 5.** ([9, 30]) Two-parametric Mittag-Leffler function having matrix argument for \( A \in \mathbb{R}^{n \times n} \) is defined as:

\[ E_{p, \beta}(A) = \sum_{k=0}^{\infty} \frac{A^k}{\Gamma(kp + \beta)}, \ \beta > 0. \]

3. A modified implicit finite difference method in the Caputo sense. In this section, we consider a new evaluation of the Caputo fractional derivative for \( 0 < \alpha < 1 \). First, we examine the following issues:

**Theorem 1.** If \( f \in C^4[a, b] \) then the approximation error of the composite modified trapezoidal rule can be expressed as follows:

\[ \int_a^b f(x)dx - [T(h) + \frac{h^2}{12} (f'(b) - f'(a))] = \frac{(b - a)}{720} h^4 f^{(4)}(\eta), \ a < \eta < b, \]

where \( T(h) \) is the approximation value achieved from the trapezoidal rule.

**Proof.** See [42].

**Lemma 1.**

\[ f(t_n, x_{k+\frac{1}{2}}) = -\frac{3}{8} f(t_n, x_k) + \frac{3}{4} f(t_n, x_{k+1}) - \frac{1}{8} f(t_n, x_{k+2}) = O(\Delta x^3). \]

**Proof.** See [43].

**Lemma 2.**

\[ f'(x_k) - \frac{1}{12\Delta x} \left( f(x_{k-2}) - 8 f(x_{k-1}) + 8 f(x_{k+1}) - f(x_{k+2}) \right) = O(\Delta x^4). \]

**Proof.** See [43].

**Lemma 3.** Suppose that \( f(x) \in C^2[0, x_n] \) and let

\[ R^\alpha f := \frac{\partial^{\alpha} f(x)}{\partial x^{\alpha}} \bigg|_{x=x_n} - \frac{\partial^{\alpha} f(x_n)}{\partial x^{\alpha}}, \ 0 < \alpha < 1, \]

then for \( x \in [0, x_n] \) we have

\[ | R^\alpha f | \leq \frac{h^{2-\alpha}}{\Gamma(2-\alpha)} \left[ \frac{1 - \alpha}{12} + \frac{2^{2-\alpha}}{2 - \alpha} - (1 + 2^{2-\alpha}) \right] \max | f''(x) |, \ h = x_k - x_{k-1}. \]
\begin{proof}

By assuming the \(0 = x_0 < x_1 < x_2 < \cdots < x_m = 1\) and \(\Delta x = x_j - x_{j-1}\), and splitting the convolution integral
\[
\frac{e}{0+1}D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_{x_0}^{x} (x - \zeta)^{n-1-\alpha} f^{(\alpha)}(\zeta) d\zeta,
\]
into a sum of two parts, a local part and a history part, whereas \(0 < \alpha < 1\), and \(x = x_j\) we get
\[
\frac{e}{0+1}D^\alpha f(x_j) = \frac{1}{\Gamma(1-\alpha)} \int_{x_0}^{x_j} (x_j - \zeta)^{-\alpha} f'(\zeta) d\zeta
\]
\[
= \frac{1}{\Gamma(1-\alpha)} \int_{x_0}^{x_j} (x_j - \zeta)^{-\alpha} f'(\zeta) d\zeta + \frac{1}{\Gamma(1-\alpha)} \int_{x_{j-1}}^{x_j} (x_j - \zeta)^{-\alpha} f'(\zeta) d\zeta
\]
\[
= C_l(x_j) + C_h(x_j).
\]

For approximate the local part, we apply the standard \(L_1\) approximation, thus we have
\[
C_l(x_j) = \frac{f(x_j) - f(x_{j-1})}{h\Gamma(1-\alpha)} \int_{x_{j-1}}^{x_j} (x_j - \zeta)^{-\alpha} d\zeta = \frac{f(x_j) - f(x_{j-1})}{h\Gamma(2-\alpha)}.
\]

For the history part, by applying the integration by part, we can eliminate \(f'(\zeta)\) and therefore we have
\[
C_h(x_j) = \frac{1}{\Gamma(1-\alpha)} \int_{x_0}^{x_j-1} (x_j - \zeta)^{-\alpha} f'(\zeta) d\zeta
\]
\[
= \frac{1}{\Gamma(1-\alpha)} \left[ f(x_{j-1}) h^{-\alpha} - f(x_0) j^{-\alpha} - \alpha \int_{x_0}^{x_{j-1}} (x_j - \zeta)^{-1-\alpha} f(\zeta) d\zeta \right]
\]
\[
= \frac{h^{-\alpha}}{\Gamma(1-\alpha)} \left[ f(x_{j-1}) - f(x_0) j^{-\alpha} \right] - \frac{\alpha}{\Gamma(1-\alpha)} \int_{x_0}^{x_{j-1}} (x_j - \zeta)^{-1-\alpha} f(\zeta) d\zeta.
\]

To calculate the second term in (14), by partitioning the interval \([x_0, x_{j-1}]\) such that \(x_0 < x_1 < x_2 < \cdots < x_{j-1}\) and also defining \(h = x_k - x_{k-1}\) as stepsize, we approximate \(f(\zeta)\) by the following quadratic interpolation:
\[
f(\zeta) \approx \left( \frac{\zeta - x_{k+\frac{1}{2}}}{x_{k+\frac{1}{2}} - x_k} \right) f(x_k) + \left( \frac{\zeta - x_k}{x_{k+\frac{1}{2}} - x_k} \right) f(x_{k+\frac{1}{2}})
\]
\[
+ \left( \frac{\zeta - x_k}{x_k - x_{k+\frac{1}{2}}} \right) f(x_{k+1}),
\]
and \(f(x_{k+\frac{1}{2}})\) as
\[
f(x_{k+\frac{1}{2}}) = \frac{3}{8} f(x_k) + \frac{3}{4} f(x_{k+1}) - \frac{1}{8} f(x_{k+2}).
\]

Consequently, we obtain
\[
f(\zeta) \approx f(x_k) \left\{ \frac{(\zeta - x_{k+\frac{1}{2}})(\zeta - x_k)}{(x_{k+\frac{1}{2}} - x_k)(x_k - x_{k+1})} + \frac{3}{8} \frac{(\zeta - x_k)(\zeta - x_{k+1})}{(x_{k+\frac{1}{2}} - x_k)(x_k - x_{k+1})} \right\}
\]
\[
f(x_{k+1}) \left\{ \frac{(\zeta - x_k)(\zeta - x_{k+\frac{1}{2}})}{x_{k+1} - x_k} + \frac{3}{4} \frac{(\zeta - x_k)(\zeta - x_{k+1})}{(x_{k+\frac{1}{2}} - x_k)(x_k - x_{k+1})} \right\}
\]
\[
+ f(x_{k+2}) - \frac{1}{8} \frac{(\zeta - x_k)(\zeta - x_{k+1})}{(x_{k+\frac{1}{2}} - x_k)(x_k - x_{k+1})},
\]
\end{proof}
where
\[ x_k = k \triangle x = kh, \quad x_{k+\frac{1}{2}} = (k + \frac{1}{2})h. \]  
\hfill (18)

Finally,
\[
f(\zeta) = \frac{1}{2h^2} \left( f(x_k) \left\{ \zeta^2 - (2k + 3)\zeta h + (k^2 + 3k + 2)h^2 \right\} + f(x_{k+1}) \left\{ -2\zeta^2 + (4k + 4)\zeta h - (2k^2 + 4k)h^2 \right\} + f(x_{k+2}) \left\{ \zeta^2(2k + 1)\zeta h + (k^2 + k)h^2 \right\} \right). \hfill (19)
\]

Now, we use \( f(\zeta) \) to calculate the general integral \( \int_{x_k}^{x_{k+1}} (x - \zeta)^{-1-\alpha} f(\zeta) d\zeta, \) \( k \neq j - 1. \) Denoting \( t = x - \zeta, \) we will have
\[
\int_{x_k}^{x_{k+1}} (x - \zeta)^{-1-\alpha} f(\zeta) d\zeta = \hat{D} \left( f(x_k) \left\{ (\alpha^2 - \alpha)(j - k)^{2-\alpha} - (j - k - 1)^{2-\alpha} \right\} + (\alpha^2 - 2\alpha)(2k - 2j + 3) \left\{ (\alpha^1 - \alpha)(j - k)^{1-\alpha} - (j - k - 1)^{1-\alpha} \right\} \\
+ (\alpha^2 - 3\alpha + 2) \left\{ j^2 - (2k + 3)j + k^2 + 3k + 2 \right\} (j - k)^{-\alpha} - (j - k - 1)^{-\alpha} \right\} + f(x_{k+1}) \left\{ -2(\alpha^2 - \alpha)(j - k)^{2-\alpha} - (j - k - 1)^{2-\alpha} \right\} \\
+ 4(\alpha^2 - 2\alpha)(j - k - 1) (j - k)^{1-\alpha} - (j - k - 1)^{1-\alpha} \right\} + f(x_{k+2}) \left\{ (\alpha^2 - \alpha)(j - k)^{2-\alpha} - (j - k - 1)^{2-\alpha} \right\} \\
+ (\alpha^2 - 2\alpha)(2k - 2j + 1) (j - k)^{1-\alpha} - (j - k - 1)^{1-\alpha} \right\} + (\alpha^2 - 3\alpha + 2) \left\{ j^2 - (2k + 1)j + k(k + 1) \right\} (j - k)^{-\alpha} - (j - k - 1)^{-\alpha} \right\} \right),
\]

and we can conclude
\[
\int_{x_{j-2}}^{x_j} (x - \zeta)^{-1-\alpha} f(\zeta) d\zeta = \hat{D} \left( f(x_{j-2}) \left\{ (\alpha^2 - \alpha)(2^{2-\alpha} - 1) \\
- (\alpha^2 - 2\alpha)(2^{1-\alpha} - 1) \right\} + f(x_{j-1}) \left\{ -2(\alpha^2 - \alpha)(2^{2-\alpha} - 1) + 4(\alpha^2 - 2\alpha)(2^{1-\alpha} - 1) \right\} \\
+ f(x_j) \left\{ (\alpha^2 - \alpha)(2^{2-\alpha} - 1) - 3(\alpha^2 - 2\alpha)(2^{1-\alpha} - 1) \right\} \\
+ 2(\alpha^2 - 3\alpha + 2)(2^{2-\alpha} - 1) \right\} \right), \hfill (20)
\]

where \( \hat{D} = \frac{h^{-\alpha}}{2(1-\alpha)(1-\alpha)^{2-\alpha}}. \) Now, we calculate the integral \( \int_{x_2}^{x_{j-2}} (x_j - \zeta)^{-1-\alpha} f(\zeta) d\zeta \) by means of the modified trapezoidal rule. Recall the modified trapezoidal quadrature:
\[
\int_a^b F(\zeta) d\zeta = T(b) + \frac{h^2}{12} \left[ F'(b) - F'(a) \right] + \frac{b - a}{720} h^4 F^{(4)}(\eta), \quad a < \eta < b, \hfill (21)
\]
where $T(h)$ denotes the compound trapezoidal rule. Since, we do not have the derivative of $F(\zeta)$ we need to adopt an approximation, namely be means of the central formula

$$F'(x_j) = \frac{-F(x_{j+2}) + 8F(x_{j+1}) - 8F(x_{j-1}) + F(x_{j-2})}{12\Delta x}.$$  

Therefore, we have

$$\int_{x_2}^{x_{j-2}} F(\zeta) \, d\zeta \approx T(h) + \frac{h^2}{12} \left[ -F(x_4) + 8F(x_3) - 8F(x_1) + F(x_0) \right]$$

where the coefficients $w_j$ are defined as follows:

$$w_j = \left( \alpha^2 - \alpha \right) \left( j^2 - \alpha \right) + \left( \alpha^2 - 2\alpha \right) (3 - 2j) \left( j^{1-\alpha} - (j - 1)^{1-\alpha} \right)$$

and

$$w_j = -2\left( \alpha^2 - \alpha \right) \left( j^{2-\alpha} - (j - 1)^{2-\alpha} \right) + 4(\alpha^2 - 2\alpha)(j - 1) \left( j^{1-\alpha} - (j - 1)^{1-\alpha} \right).$$

In case of $k = j + 1, j, j - 1$, we have $w_0,j = 4 - \alpha$, $w_{1,j} = 2\alpha^2 - 6\alpha$, $w_{2,j} = \alpha$. For more details see [43].

**Lemma 4.** ([43]) The coefficients $w_{k,j}$ satisfy the following conditions:

$$w_{1,j} < 0, \quad w_{k,j} > 0, \quad k \neq 1,$$

$$\sum_{k=0}^{j} w_{k,j} < 0,$$

$$\sum_{k=0}^{\infty} w_{k,j} = 0.$$

**Lemma 5.**

$$C_{\alpha} D^\alpha f(x) = \frac{h^{-\alpha}}{2\Gamma(3-\alpha)} \sum_{k=0}^{j+1} w_{j+1-k,j} f(x_k) + O(\Delta x^{2-\alpha}).$$
Proof. Making use of Lemmas 1-3 and Theorem 1 the proof is straightforward. □

4. Fractional stable manifold and the Hartman-Grobman theorem. Hereunder the stable manifold and the Hartman-Grobman theorem are presented in the fractional sense.

**Definition 6.** Let $E$ be an open subset of $\mathbb{R}^m$ and let $f \in C^1(E)$. For $x_0 \in E$, let $\phi(t, x_0)$ be the solution of the initial value problem

\[
^{c}D^\alpha x(t) = f(x(t)), \quad 0 < \alpha \leq 1, \\
x(0) = x_0,
\]

defined on its maximal interval of existence $I(x_0)$. Then, for $t \in I(x_0)$, the set of mappings $\phi_t$ defined by

\[
\phi_t(x_0) = \phi(t, x_0),
\]

is called the flow of the differential equation (1) or the flow defined by the differential equation (1), $\phi_t$ is also referred to as the flow of the vector field $f(x)$.

**Definition 7.** A point $x_0 \in \mathbb{R}^m$ is called an equilibrium point or critical point of (1) if $f(x_0) = 0$. An equilibrium point $x_0$ is called a hyperbolic equilibrium point of (1) if none of the eigenvalues of the matrix $Df(x_0)$ have zero real part. The linear system (2) with the matrix $A = Df(x_0)$ is called the linearization of (1) at $x_0$.

**Theorem 2.** (The stable manifold theorem for fractional differential systems) ([39]) Let $E$ be an open subset of $\mathbb{R}^m$ containing the origin, let $f \in C^1(E)$, and let $\phi_t$ be the flow of the nonlinear system (1). Suppose that $f(0) = 0$ and that $Df(0)$ has $k$ eigenvalues with negative real part and $m - k$ eigenvalues with positive real part. Then, there exists a $k$-dimensional differentiable manifold $S$ tangent to the stable subspace $E^s$ of the linear system (2) at 0 such that for all $t \geq 0$, $\phi_t(S) \subset S$ and for all $x_0 \in S$

\[
\lim_{t \to \infty} \phi_t(x_0) = 0,
\]

and there exists an $m - k$-dimensional differentiable manifold $U$ tangent to the unstable subspace $E^u$ of (33) at 0 such that for all $t \leq 0$, $\phi_t(U) \subset U$ and for all $x_0 \in U$,

\[
\lim_{t \to -\infty} \phi_t(x_0) = 0.
\]

By the stable manifold theorem, the nonlinear (1) has stable manifold $S$ at $x_0$ to the stable subspace $E^S$ of the linearized system

\[
^{c}D^\alpha x(t) = A(x(t)), \quad 0 < \alpha \leq 1,
\]

where $A = Df(x_0)$. We remark that if $f \in C^1(E)$ and $f(0) = 0$, then, the system (1) can be written as

\[
^{c}D^\alpha x(t) = Ax(t) + F(x(t)), \quad 0 < \alpha \leq 1,
\]

where $A = Df(0)$, $F(x(t)) = f(x(t)) - Ax(t)$, $F \in C^1(E)$, $F(0) = 0$ and $DF(0) = 0$.

**Corollary 1.** Initial value problem (34) is equivalent to the following integral equation

\[
\phi_t(x_0) := x(t) = E_\alpha(t^\alpha A)x_0 + \int_0^t (t - \tau)^{\alpha - 1} E_{\alpha, \alpha}(A(t - \tau)^\alpha) F(x(\tau)).
\]
Now, consider the initial value problem (1) as follows

\[ ^cD^\alpha x(t) = f(t, x_1(t), x_2(t), \ldots, x_m(t)), \quad t \in [0, +\infty), \]

\[ x(0) = x_0, \quad x_0 \in \mathbb{R}^m, \]

where \( x = (x_1, x_2, \ldots, x_m)^T \in \mathbb{R}^m, \ f = (f_1, f_2, \ldots, f_m)^T \) and \( f_i : [0, +\infty) \times \mathbb{R}^m \to \mathbb{R}^m; i = 1, 2, \ldots, m \) are continuous at \( t \) and locally Lipschitz in \( x \).

**Definition 8.** ([10, 29]) System (1) is said to be \( \alpha \)–exponentially stable if there exist two positive constants \( M \) and \( \lambda \) such that for any two solutions \( x(t) \) and \( y(t) \) of this system with different initial values denoted by \( x_0 \) and \( y_0 \), one has

\[ \| x(t) - y(t) \| \leq M \| x_0 - y_0 \| \exp(-\lambda t^\alpha), \quad t \geq 0, \]

where \( \| . \| \) denotes the Euclidian norm.

**Lemma 6.** ([44]) (Bellman-Gronwall Inequality) Assume that function \( y(t) \) satisfies in the following form

\[ y(t) \leq \int_0^t a(\tau)y(\tau)d\tau + b(t), \]

with \( a(t) \) and \( b(t) \) being known real functions. Then

\[ y(t) \leq \int_0^t a(\tau)b(\tau)\exp[\int_\tau^t a(r)dr]d\tau + b(t). \]

**Lemma 7.** ([49]) If \( ^cD^\alpha x(t) \leq ^cD^\alpha y(t) \) where \( 0 < \alpha < 1 \) and \( x(0) = y(0) \), then \( x(t) \leq y(t) \).

**Lemma 8.** Let \( V(t) \) be a continuous function on \([0, +\infty)\) and satisfies \( ^cD^\alpha V(t) \leq \theta V(t) \) where \( 0 < \alpha < 1 \) and \( \theta \) is a constant. Then

\[ V(t) \leq V(0)\exp[\frac{\theta}{1+\alpha} t^\alpha]. \]

**Proof.** See [49]. \( \square \)

5. \( \alpha \)–exponential stability of fractional-order system. Consider the fractional system (36) as follows

\[ ^cD^\alpha x_i(t) = c_i x_i(t) + \sum_{j=1}^{m} a_{ij} f_{ij}(x_j(t)), \quad i = 1, \ldots, m, \quad 0 < \alpha < 1, \]

with the following assumptions:

(I) Functions \( f_{ij} \) are Lipschitz-continuous on \( \mathbb{R} \) with Lipschitz constants \( L_{ij} > 0 \), i.e.,

\[ |f_{ij}(u) - f_{ij}(v)| \leq L_{ij}|u - v|, \quad u, v \in \mathbb{R}. \]

(II) There exist positive constant \( \mu_i, i = 1, \ldots, m, \) such that

\[ \sum_{j=1}^{m} \frac{\mu_j}{\mu_i |c_i|} |a_{ji}|L_{ji} < 1. \]

**Theorem 3.** Assume that assumptions (I),(II) hold. Then, system (1) is \( \alpha \)–exponentially stable.
Proof. Let \( x(t) = (x_1(t), \ldots, x_m(t))^T \) and \( y(t) = (y_1(t), \ldots, y_m(t))^T \) are two solutions for system (1) with different initial values. Denoting \( e_i(t) = y_i(t) - x_i(t), i = 1, \ldots, m \), we have \( e_i(0) \neq 0 \) and

\[
^cD^\alpha e_i(t) = c_i e_i(t) + \sum_{j=1}^{m} a_{ij} (f_{ij}(y_j(t) - f_{ij}(x_j(t)))).
\]

Assume now that

\[
V(t) = \sum_{i=1}^{m} \mu_i |e_i(t)|,
\]

where \( \mu_i > 0 \). Calculating the derivative of \( V(t) \) of \( \alpha \)-order and considering (45), we get

\[
^cD^\alpha V(t) = \sum_{i=1}^{m} \mu_i ^cD^\alpha |e_i(t)| = \sum_{i=1}^{m} \mu_i sgn(e_i(t)) ^cD^\alpha(e_i(t))
\]

\[
= \sum_{i=1}^{m} \mu_i sgn(e_i(t)) \{ c_i e_i(t) + \sum_{j=1}^{m} a_{ij} (f_{ij}(y_j(t) - f_{ij}(x_j(t)))) \}
\]

\[
\leq \sum_{i=1}^{m} \mu_i \{ |c_i| |e_i(t)| + \sum_{j=1}^{m} |a_{ij}| |L_{ij}| |e_j(t)| \}
\]

\[
= \sum_{i=1}^{m} \mu_i \{ |c_i| + \sum_{j=1}^{m} \frac{\mu_j}{\mu_i} |a_{ji}| |L_{ji}| |e_i(t)| \}
\]

\[
\leq max(|c_i| + \sum_{j=1}^{m} \frac{\mu_j}{\mu_i} |a_{ji}| |L_{ji}|) V(t), \ 1 \leq i \leq m.
\]

Let

\[
\lambda = max(|c_i| + \sum_{j=1}^{m} \frac{\mu_j}{\mu_i} |a_{ji}| |L_{ji}|).
\]

Taking advantage of Lemma 3, we have

\[
V(t) \leq V(0) \exp \left[ \frac{\lambda}{\Gamma(1+\alpha)} t^\alpha \right],
\]

which means that, system (42) is \( \alpha \)-exponentially stable. \( \square \)

Theorem 4. Under the assumptions (I),(II) system (1) has a unique equilibrium point.

Proof. Let us suppose that \( \Phi(u) = (\Phi_1(u), \Phi_2(u), \ldots, \Phi_m(u))^T \) be a mapping, where \( u = (u_1, u_2, \ldots, u_m)^T \) and

\[
\Phi_i(u) = \mu_i \sum_{j=1}^{m} a_{ij} f_{ij}(\frac{u_j}{c_j \mu_j}), \ i = 1, \ldots, m.
\]

Assumption (I), for any two vectors \( u = (u_1, u_2, \ldots, u_m)^T \) and \( v = (v_1, v_2, \ldots, v_m)^T \) in \( \mathbb{R}^m \), gives

\[
|\Phi_i(u) - \Phi_i(v)| = |\mu_i \sum_{j=1}^{m} a_{ij} [f_{ij}(\frac{u_j}{c_j \mu_j}) - f_{ij}(\frac{v_j}{c_j \mu_j})]|.
\]
Finally, assumption (II) gives
\[
\sum_{i=1}^{m} |\Phi(u) - \Phi(v)| \leq \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{\mu_i}{|c_j|} |a_{ij}| |L_{ij}| |u_j - v_j| = \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{\mu_j}{\mu_i |c_i|} |a_{ji}| |L_{ji}| |u_i - v_i|< \sum_{i=1}^{m} |u_i - v_i|. \tag{49}
\]
In other words
\[
\| \Phi(u) - \Phi(v) \| < \| u - v \|. 
\]
Thus, the mapping \( \Phi : \mathbb{R}^m \to \mathbb{R}^m \) is a contraction mapping on \( \mathbb{R}^m \), and therefore there is a unique fixed point \( u^* \in \mathbb{R}^m \) such that \( \Phi(u^*) = u^* \).

**Corollary 2.** Let \( x_i^* = \frac{u_i^*}{|c_i| |\mu_i|} \); \( i = 1, \ldots, m \), \( |c_i| |\mu_i| \neq 0 \), then \( x^* \) is a unique \( \alpha \)-expansive stable equilibrium point of system (1).

**Proof.** From \( \Phi(u^*) = u^* \), we have
\[
u_i^* = \mu_i \sum_{j=1}^{m} a_{ij} f_j(x_j^*). 
\]
Consequently
\[
a_i x_i^* + \sum_{j=1}^{m} a_{ij} f_j(x_j^*) = 0, \quad i = 1, \ldots, m. 
\]
Thus, using uniqueness property of \( u^* \) it is easily verified that \( x^* \) is a unique equilibrium point of system (1). Furthermore, it follows from Theorem 3 that \( x^* \) is a unique \( \alpha \)-expansive stable equilibrium point of system (1).

Now consider the local stable set of neighborhood \( N_r(0) \) in the following form
\[
W_{loc}^s(N_r(0)) := \{ x_0 \in N_r(0) : \phi_t(x_0) \in N_r(0), \forall t \geq 0, \lim\|\phi_t(x_0)\| = 0, \text{ when } t \to \infty \}, \tag{50}
\]
where
\[
N_r(0) := x_0 \in \mathbb{R}^m : \|x_0\| < r.
\]
For the linear counterpart of the system (34),
\[
^cD^\infty x(t) = Ax(t), \quad x(0) = x_0, \tag{51}
\]
the solution is explicitly given as \( x(t) = E_\alpha(t^n A)x_0 \).

Let \( F \in C^1[\mathbb{R}^m, \mathbb{R}^m] \) and \( DF(0) = 0 \), then \( F \) is a Lipschitz continuous function with \( r > 0 \) and \( t \geq 0 \) i.e.
\[
\|F(x(t)) - F(y(t))\| \leq \epsilon_r \|x(t) - y(t)\|, \quad x, y \in C[I, \mathbb{R}^m], \tag{52}
\]
whenever \( x(t), y(t) \in N_r(0) \).
Theorem 5. For the fractional system
\[ ^cD^\alpha x(t) = Ax(t) + F(x(t)), \quad x(0) = x_0 \in \mathbb{R}^m, \quad A \in \mathbb{R}^{m \times m}, \quad 0 < \alpha < 1, \]
with origin as an hyperbolic equilibrium point, \( F \in C[U, \mathbb{R}^m] \) where \( U \) a neighborhood of origin, \( F(0) = 0, \ DF(0) = 0 \), then there exists \( r > 0 \) such that for \( N_r(0) \), origin belongs to \( W^s_{loc}(N_r(0)) \) and \( W^u_{loc}(N_r(0)) \) forms a Lipschitz continuous function over \( E^s \), where
\[
E^s = \{ x_0 \in \mathbb{R}^m : \| E(t^\alpha(t)A)x_0 \| < \epsilon, \text{ some } \epsilon > 0 \text{ and } t \geq 0 \}. \tag{54}
\]
Proof. The proof is similar to the proof of Theorem 3, in [8]. \qed

5.1. An efficient algorithm for analysing the system (42). In this section, we are purposed to solve the system (42) via a specific numerical method. For this purpose, suppose that \( x(t) \) is computed up to the stage \( n \). To calculate \( x(t) \) in stage \( n + 1 \), at the beginning, we write system (42) as follows
\[
\sum_{j=0}^{n+1} w_{j}x_{n+1-j}^{i} = c_i x_i^{n+1} + \sum_{j=1}^{m} f_{ij} x_j^{n}, \quad i = 1, 2, \cdots, m, \tag{55}
\]
or in matrix form:
\[
x^{n+1} = A^{-1}(Bx^n - C), \tag{56}
\]
where
\[
A = \text{diag}(w_{0,n+2} - c_1, \ldots, w_{0,n+2} - c_m),
B = \text{diag}(-w_{1,n+2} + f_{11}, \ldots, w_{1,n+2} + f_{mm}),
C = (\sum_{j=2}^{n+1} w_{j,n+2}x_{1}^{n+1-j}, \ldots, \sum_{j=2}^{n+1} w_{j,n+2}x_{m}^{n+1-j})^T.
\]
Therefore, a sequence of quantities of \( x \) can be computed. Consider that the coefficients \( w_{i,j} \) are already calculated.

6. Numerical simulations. We illustrate the performance of the method by means of three examples.

Example 1. Consider the nonlinear systems of fractional differential equations [39]
\[
\Lambda_1 : \begin{cases} ^cD^\alpha x_1(t) = -x_1(t) - (x_2(t))^2, \\ ^cD^\alpha x_2(t) = (x_1(t))^2 + x_2(t). \end{cases} \tag{57}
\]
Example 2. Consider the nonlinear systems of fractional differential equations [39]
\[
\Lambda_2 : \begin{cases} ^cD^\alpha x_1(t) = -x_1(t), \\ ^cD^\alpha x_2(t) = (x_1(t))^2 - 2x_2(t), \\ ^cD^\alpha x_3(t) = (x_1(t))^3 + x_3(t). \end{cases} \tag{58}
\]
We solve the system (57) and (58) for \( \alpha = 0.3, \alpha = 0.7, h = 0.1 \) and \( \alpha = 0.2, \alpha = 0.9, h = 0.1 \), respectively. In addition, Tables 1-4 report the obtained results for our method (OM) and the absolute errors (AE) for different values of \( t \) and \( \alpha \). Also, Figures 1-10 show the approximations obtained for different values of \( x_1, x_2 \) and \( x_3 \).

Example 3. Consider the following initial value problem [8]
\[
\Lambda_3 : ^cD^\alpha x(t) = Ax(t) + F(x(t)), \tag{59}
\]
Figure 1. Comparison of $x_1$ for different values of $t$ in Example 1 and corresponding to results obtained by Ref. [39] for $\alpha = 0.3$.

Figure 2. Comparison of $x_2$ for different values of $t$ in Example 1 and corresponding to results obtained by Ref. [39] for $\alpha = 0.3$.

Figure 3. Comparison of $x_1$ for different values of $t$ in Example 1 and corresponding to results obtained by Ref. [39] for $\alpha = 0.7$.

Figure 4. Comparison of $x_2$ for different values of $t$ in Example 1 and corresponding to results obtained by Ref. [39] for $\alpha = 0.7$. 
Table 1. Numerical results of Example 1 for different values of $t$ and $\alpha = 0.3$ and $h = 0.1$

| $t$ | Ref. [39] | $\text{OM}$ | $\text{AE}$ | Ref. [39] | $\text{OM}$ | $\text{AE}$ |
|-----|-----------|-------------|-------------|-----------|-------------|-------------|
| 0.3 | 0.00824   | 0.00835     | 0.00010     | -0.00002  | -0.00003    | 0.00000     |
| 0.6 | 0.00744   | 0.00753     | 0.00009     | -0.00002  | -0.00005    | 0.00003     |
| 0.9 | 0.00697   | 0.00705     | 0.00008     | -0.00002  | -0.00008    | 0.00006     |
| 1.2 | 0.00663   | 0.00676     | 0.00012     | -0.00002  | -0.00012    | 0.00009     |
| 1.5 | 0.00638   | 0.00655     | 0.00017     | -0.00001  | -0.00017    | 0.00015     |
| 1.8 | 0.00617   | 0.00637     | 0.00020     | -0.00001  | -0.00024    | 0.00022     |
| 2   | 0.00605   | 0.00627     | 0.00021     | -0.00001  | -0.00030    | -0.00028    |

Table 2. Numerical results of Example 1 for different values of $t$ and $\alpha = 0.7$ and $h = 0.1$

| $t$ | Ref. [39] | $\text{OM}$ | $\text{AE}$ | Ref. [39] | $\text{OM}$ | $\text{AE}$ |
|-----|-----------|-------------|-------------|-----------|-------------|-------------|
| 0.3 | 0.03226   | 0.03276     | 0.00049     | -0.00037  | -0.00029    | 0.00007     |
| 0.6 | 0.02539   | 0.02596     | 0.00056     | -0.00025  | -0.00014    | 0.00011     |
| 0.9 | 0.02109   | 0.02170     | 0.00060     | -0.00019  | -0.00007    | 0.00011     |
| 1.2 | 0.01808   | 0.01884     | 0.00076     | -0.00015  | -0.00004    | 0.00011     |
| 1.5 | 0.01584   | 0.01683     | 0.00098     | -0.00013  | -0.00003    | 0.00010     |
| 1.8 | 0.01411   | 0.01523     | 0.00111     | -0.00011  | -0.00002    | 0.00008     |
| 2   | 0.01315   | 0.01432     | 0.00116     | -0.00010  | -0.00002    | 0.00007     |

Figure 5. Comparison of $x_1$ for different values of $t$ in Example 2 and corresponding to results obtained by Ref. [39] for $\alpha = 0.2$.

Table 3. Numerical results of Example 2 for different values of $t$ and $\alpha = 0.2$ and $h = 0.1$

| $t$ | Ref. [39] | $\text{OM}$ | $\text{AE}$ | Ref. [39] | $\text{OM}$ | $\text{AE}$ | Ref. [39] | $\text{OM}$ | $\text{AE}$ | Ref. [39] | $\text{OM}$ | $\text{AE}$ |
|-----|-----------|-------------|-------------|-----------|-------------|-------------|-----------|-------------|-------------|-----------|-------------|-------------|
| 0.3 | 0.01064   | 0.01075     | 0.00111     | 0.01092   | 0.01101     | 0.0009     | 0.00000   | 0.00000     | 0.0000     |
| 0.6 | 0.00994   | 0.01003     | 0.0009     | 0.00997   | 0.00998     | 0.00001    | 0.00000   | 0.00000     | 0.0000     |
| 0.9 | 0.00952   | 0.00960     | 0.0007     | 0.00943   | 0.00941     | 0.00002    | 0.00000   | 0.00000     | 0.0000     |
| 1.2 | 0.00923   | 0.00935     | 0.00012    | 0.00907   | 0.00907     | 0.00000    | 0.00000   | 0.00000     | 0.0000     |
| 1.5 | 0.00901   | 0.00917     | 0.00016    | 0.00879   | 0.00880     | 0.00001    | 0.00000   | 0.00000     | 0.0000     |
| 1.8 | 0.00882   | 0.00901     | 0.00018    | 0.00856   | 0.00859     | 0.00002    | 0.00000   | 0.00000     | 0.0000     |
| 2   | 0.00872   | 0.00891     | 0.00019    | 0.00843   | 0.00846     | 0.00002    | 0.00000   | -0.00001    | 0.00001    |
Figure 6. Comparison of $x_2$ for different values of $t$ in Example 2 and corresponding to results obtained by Ref. [39] for $\alpha = 0.2$.

Figure 7. Comparison of $x_3$ for different values of $t$ in Example 2 and corresponding to results obtained by Ref. [39] for $\alpha = 0.2$.

Figure 8. Comparison of $x_1$ for different values of $t$ in Example 2 and corresponding to results obtained by Ref. [39] for $\alpha = 0.9$.

Figure 9. Comparison of $x_2$ for different values of $t$ in Example 2 and corresponding to results obtained by Ref. [39] for $\alpha = 0.9$. 
Our method
Ref./[LBracket139/RBracket1

Figure 10. Comparison of $x_3$ for different values of $t$ in Example 2 and corresponding to results obtained by Ref. [39] for $\alpha = 0.9$.

Table 4. Numerical results of Example 2 for different values of $t$ and $\alpha = 0.9$ and $h = 0.1$

| $t$ | $x_3$ Ref. [39] | OM | AE | $x_3$ Ref. [39] | OM | AE |
|-----|----------------|----|----|----------------|----|----|
| 0.3 | 0.01416        | 0.01414 | 0.00018 | 0.00515 | 0.00513 | 0.00020 | 0.00000 | 0.00000 | 0.00000 |
| 0.6 | 0.01062        | 0.01090 | 0.00028 | 0.00305 | 0.00327 | 0.00021 | 0.00000 | 0.00000 | 0.00000 |
| 0.9 | 0.00817        | 0.00849 | 0.00031 | 0.00211 | 0.00211 | 0.00016 | 0.00000 | 0.00000 | 0.00000 |
| 1.2 | 0.00640        | 0.00675 | 0.00034 | 0.00144 | 0.00144 | 0.00012 | 0.00000 | 0.00000 | 0.00000 |
| 1.5 | 0.00511        | 0.00547 | 0.00036 | 0.00104 | 0.00104 | 0.00008 | 0.00000 | 0.00000 | 0.00000 |
| 1.8 | 0.00443        | 0.00450 | 0.00037 | 0.00078 | 0.00078 | 0.00006 | 0.00000 | 0.00000 | 0.00000 |
| 2   | 0.00362        | 0.00398 | 0.00036 | 0.00066 | 0.00066 | 0.00004 | 0.00000 | 0.00000 | 0.00000 |

where

$$x(0) = x_0 = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix}, \quad x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$$ (60)

and

$$F(x(t)) = \begin{pmatrix} 0 \\ (x_1(t))^2 \\ 3(x_1(t))^2 \end{pmatrix}, \quad A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$ (61)

For sufficiently small neighborhood of origin and $t = 0$, $\sigma_1 = 0.001$ and by the same manipulation as Section 3, system (59) is analyzed. In addition, Table 5 reports the obtained results for our method and the absolute errors for different values of $t$ and $\alpha = 0.7$ and $h = 0.1$. Also, comparison of $x_i$, $i = 1, 2, 3$ for different values of $t$ and corresponding results obtained by [8] are presented in Figures 11-13. Furthermore, based on the assumptions of [8], the projection $\sigma_3$ versus $\sigma_1$ and $\sigma_2$ versus $\sigma_1$ of local stable manifold for $\alpha = 0.2, 0.5, 0.7, 1$ are presented in Figures 14-16. From the obtained results we make the following observations:

- The stable manifold is smooth in the neighborhood of the origin.
- Results are in a high agreement with those obtained in [8].
Table 5. Numerical results of Example 3 for different values of $t$ and $\alpha = 0.7$ and $h = 0.1$

| $t$  | $x_1$ Ref. [8] | $x_1$ OMT | AE   | $x_2$ Ref. [8] | $x_2$ OMT | AE   |
|------|----------------|-------------|------|----------------|-------------|------|
| 0.3  | 0.00064        | 0.00065     | 0.0000 | 0.00000        | 0.00000     | 0.0000 |
| 0.6  | 0.00051        | 0.00052     | 0.0000 | 0.00000        | 0.00000     | 0.0000 |
| 0.9  | 0.00042        | 0.00043     | 0.0000 | 0.00000        | 0.00000     | 0.0000 |
| 1.2  | 0.00036        | 0.00037     | 0.0000 | 0.00001        | 0.00001     | 0.0000 |
| 1.5  | 0.00031        | 0.00033     | 0.0000 | 0.00002        | 0.00003     | 0.0000 |
| 1.8  | 0.00028        | 0.00030     | 0.0000 | 0.00005        | 0.00009     | 0.0000 |
| 2    | 0.00026        | 0.00028     | 0.0000 | 0.00009        | 0.00016     | 0.0000 |

Figure 11. Comparison of $x_1$ for different values of $t$ in Example 3 and corresponding to results obtained by Ref. [8] for $\alpha = 0.7$ and $h = 0.1$.

Figure 12. Comparison of $x_2$ for different values of $t$ in Example 3 and corresponding to results obtained by Ref. [8] for $\alpha = 0.7$ and $h = 0.1$.

7. Concluding remarks. The fundamental goal of this study has been to approximate the local stable manifold near a hyperbolic equilibrium point for a nonlinear system of fractional differential equations. The goal has been achieved by using an implicit finite difference method based on the modified trapezoidal quadrature rule, backward Euler differences and non-standard central approximations. The solution existence, uniqueness and stability of our proposed method thoroughly investigated. Numerical examples has been presented to determine the efficiency and simplicity of our method for finding approximate solutions for the stable manifold of these systems. The appropriate tables and figures were presented and showed that our
modern numerical technique is very effective. Furthermore, the comparison of the results with other works, such as [30] show that in the new scheme procedure of construction the homeomorphism for fractional Hartman-Grobman theorem were not required and it has fewer and much forceful computation operations than the other methods. Notice that if a manifold is specified by a constraint equation $y \in \mathbb{R}^{n-k}$, and the dynamics given by

$$\begin{align*}
^cD^\alpha x &= f(x, y), \\
^cD^\alpha y &= h(x, y), \quad \alpha \in \mathbb{R},
\end{align*}$$

(62)

a new investigation can be appeared as invariant manifolds, which is still open. And finally, the present paper is only an introduction to the topic, and there remains further research to explore the topic. We point out that the corresponding analytical and numerical solutions were obtained using Mathematica.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure13.png}
\caption{Comparison of $x_3$ for different values of $t$ in Example 3 and corresponding to results obtained by Ref. [8] for $\alpha = 0.7$ and $h = 0.1$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure14.png}
\caption{Projection $\sigma_3$ versus $\sigma_1$ for different values of $\alpha$ in Example 3.}
\end{figure}
A ROBUST COMPUTATIONAL FRAMEWORK...

Figure 15. Projection $\sigma_2$ versus $\sigma_1$ for different values of $\alpha$ in Example 3.

Figure 16. Projection local stable manifold for different values of $\alpha$ in Example 3.

Acknowledgments. The authors wish to express their cordial thanks to the four anonymous referees for useful suggestions and comments.

REFERENCES

[1] M. M. Alsuyuti, E. Z. Doha, S. S. Ezz-Eldien, B. I. Bayoumi and D. Baleanu, Modified Galerkin algorithm for solving multitype fractional differential equations, Math. Methods Appl. Sci., 42 (2019), 1389–1412.

[2] D. Baleanu, R. Darzi and B. Agheli, Existence results for Langevin equation involving Atangana-Baleanu fractional operators, Mathematics, 8 (2020), 408.

[3] D. Baleanu, K. Diethelm, E. Scalas and J. J. Trujillo, Fractional Calculus Models and Numerical Models (Series on Complexity, Nonlinearity and Chaos), World Scientific, 2012.

[4] S. Bhatter, A. Mathur, D. Kumar and J. Singh, A new analysis of fractional Drinfeld-Sokolov-Wilson model with exponential memory, Physica A., 537 (2020), 122578, 13 pp.

[5] A. Bueno-Orovio, D. Kay and K. Burrage, Fourier spectral methods for fractional in space reaction-diffusion equations, BIT Numer. Math., 54 (2014), 937–954.

[6] Y. Chen, X. Han and L. Liu, Numerical solution for a class of linear system of fractional differential equations by the haar wavelet method and the convergence analysis, Comput. Model. Eng. Sci., 97 (2014), 391–405.

[7] M. Dehghan and M. Safarpour, Application of the dual reciprocity boundary integral equation approach to solve fourth-order time-fractional partial differential equations, Int. J. Comput. Math., 95 (2018), 2066–2081.
A. Deshpande and V. Daftardar-Gejji, Local stable manifold theorem for fractional systems, *Nonlinear Dynam.*, 83 (2016), 2435–2452.

V. Daftardar-Gejji and A. Babakhani, Analysis of a system of fractional differential equations, *J. Math. Anal. Appl.*, 293 (2004), 511–522.

H. Delavari, D. Baleanu and J. Sadati, Stability analysis of Caputo fractional order nonlinear systems revisited, *Nonlinear Dynam.*, 67 (2012), 2433–2439.

S. Esmaeili and R. Garrappa, A pseudo-spectral scheme for the approximate solution of a time-fractional diffusion equation, *Int. J. Comput. Math.*, 92 (2015), 980–994.

R. M. Ganji, H. Jafari and D. Baleanu, A new approach for solving multi variable orders differential equations with Mittag-Leffler kernel, *Chaos Soliton Fract.*, 130 (2020), 109405, 5 pp.

M. M. Ghalib, A. A. Zafar, M. B. Riaz, Z. Hammouch and K. Shabbir, Analytical approach for the steady MHD conjugate viscous fluid flow in a porous medium with nonsingular fractional derivative, *Physica A.*, 554 (2020), 123941, 15 pp.

M. M. Ghalib, A. A. Zafar, Z. Hammouch, M. B. Riaz and K. Shabbir, Analytical results on the unsteady rotational flow of fractional-order non-Newtonian fluids with shear stress on the boundary, *Disert Contin. Dyn. S.*, 13 (2020), 683–693.

R. M. Ganji, H. Jafari and D. Baleanu, A new approach for solving multi variable orders differential equations with Mittag-Leffler kernel, *Chaos Soliton Fract.*, 130 (2020), 109405, 4 pp.

V. R. Hosseini, W. Chen and Z. Avazzadeh, Numerical solution of fractional telegraph equation by using radial basis functions, *Eng. Anal. Boundary Elements*, 38 (2014), 31–39.

M. H. Heydari, Z. Avazzadeh, Y. Yang and C. A. Cattani, A cardinal method to solve coupled nonlinear variable-order time fractional sine-Gordon equations, *Comput. Appl. Math.*, 39 (2020), 2–23.

P. Hartman, *Ordinary Differential Equations*, John Wiley and Sons, New York, 1964.

D. Ingman and J. Suzdalnitsky, Control of damping oscillations by fractional differential operator with time-dependent order, *Comput. Methods Appl. Mech. Eng.*, 193 (2004), 5585–5595.

A. Jhinga and V. Daftardar-Gejji, A new numerical method for solving fractional delay differential equations, *Comput. Appl. Math.*, 38 (2019), 166–184.

M. M. Khader, A. Shloof and H. Ali, On the numerical simulation and convergence study for system of non-linear fractional dynamical model of marriage, *New Trends Math. Scie.*, 5 (2017), 130–141.

D. Kumar, J. Singh and D. Baleanu, On the analysis of vibration equation involving a fractional derivative with Mittag-Leffler law, *Math. Methods Appl. Sci.*, 43 (2019), 443–457.

S. Kazem and M. Dehghan, Semi-analytical solution for time-fractional diffusion equation based on finite difference method of lines (MOL), *Eng. Comput.*, 35 (2019), 229–241.

M. H. Kim, G. C. Ri and O. Hyong-Chol, Operational method for solving multi-term fractional differential equations with the generalized fractional derivatives, *Fract. Calculus Appl. Anal.*, 17 (2014), 79–95.

D. Kumar, R. P. Agarwal and J. Singh, A modified numerical scheme and convergence analysis for fractional model of lienard’s equation, *J. Comput. Appl. Math.*, 339 (2018), 405–413.

D. Kumar, F. Tchier, J. Singh and D. Baleanu, An efficient computational technique for fractal vehicular traffic flow, *Entropy*, 20 (2018), 259.

A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

C. Li and F. Zeng, Finite difference methods for fractional differential equations, *Int. J. Bifurcat Chaos*, 22 (2012), 1230014, 28 pp.

Y. Li, Y. Q. Chen and I. Podlubny, Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalized Mittag-Leffler stability, *Comput. Math. Appl.*, 59 (2010), 1810–1821.

F. Liu, P. Zhuang, I. Turner, K. Burrage and V. Anh, A new fractional finite volume method for solving the fractional diffusion equation, *Appl. Math. Model.*, 38 (2014), 3871–3878.

C. Li and Y. Ma, Fractional dynamical system and its linearization theorem, *Nonlinear Dynam.*, 71 (2013), 621–633.

K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Willey and Sons, Inc. New York, 1993.
[32] M. Malik and V. Kumar, Existence, stability and controllability results of coupled fractional dynamical system on time scales, Bull. Malays. Math. Sci. Soc., 43 (2020), 3369–3394.

[33] K. M. Owolabi and Z. Hammouch, Spatiotemporal patterns in the Belousov-Zhabotinskii reaction systems with Atangana-Baleanu fractional order derivative, Physica A., 523 (2019) 1072–1090.

[34] K. B. Oldham and J. Spanier, The Fractional Calculus, Theory and Applications of Differentiation and Integration to Arbitrary Order, Academic Press, New York, 1974.

[35] I. Podlubny, Fractional Differential Equations Calculus, Academic Press, New York, 1999.

[36] E. Pindza and K. M. Owolabi, Fourier spectral method for higher order space fractional reaction-diffusion equations, Commun. Nonlinear Sci. Numer. Simulat., 40 (2016), 112–128.

[37] J. Singh, D. Kumar, D. Baleanu and S. Rathore, On the local fractional wave equation in fractal strings, Math. Methods Appl. Sci., 42 (2019), 1588–1595.

[38] K. Sayevand and K. Pichaghchi, Successive approximation: A survey on stable manifold of fractional differential systems, Fract. Calc. Appl. Anal., 18 (2015), 621–641.

[39] K. Sayevand and M. Rostami, Fractional optimal control problems: optimality conditions and numerical solution, IMA J. Math. Control Info., 35 (2018), 123–148.

[40] K. Sayevand and M. Rostami, General fractional variational problem depending on indefinite integrals, Numer. Algor., 72 (2016), 959–987.

[41] J. Stoer, R. Bulirsch and R. Bartels, Introduction to Numerical Analysis, Springer, 2002.

[42] K. Sayevand, J. Tenreiro Machado and V. Moradi, A new non-standard finite difference method for analysing the fractional Navier-Stokes equations, Comput. Math. Appl., 78 (2019), 1681–1694.

[43] J. J. E. Slotine and W. Li, Applied Nonlinear Control, Prentice Hall, Englewood Cliffs, New Jersey, 1991.

[44] J. Singh, D. Kumar, D. Baleanu and S. Rathore, An efficient numerical algorithm for the fractional Drinfeld-Sokolov-Wilson equation, Appl. Math. Comput., 335 (2018), 12–24.

[45] V. E. Tarasov, Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media, Springer Science, Business Media, 2010.

[46] S. Ucar, E. Ucar, N. Ozdemir and Z. Hammouch, Mathematical analysis and numerical simulation for a smoking model with Atangana-Baleanu derivative, Chaos Solution Fract., 118 (2019), 300–306.

[47] V. V. Uchaikin, Fractional Derivatives for Physicists and Engineers , Vol. 2, Springer, Berlin, 2003.

[48] J. Yu, C. Hu and H. Jiang, \( \alpha \)-stability and \( \alpha \)-synchronization for fractional-order neural networks, Neural Netw., 35 (2012), 82–87.

[49] M. A. Zaky, A legendre spectral quadrature tau method for the multi-term time-fractional diffusion equations, Comput. Appl. Math., 37 (2018), 3525–3538.

[50] X. Zhang, C. Zhu and Z. Wu, Solvability for a coupled system of fractional differential equations with impulses at resonance, Bound. Value Probl., 2013 (2013), 80–103.

Received February 2020; 1st revision March 2020; final revision April 2020.

E-mail address: ksayehvand@malayeru.ac.ir
E-mail address: rmoradi1354@gmail.com