Loops and legs beyond perturbation theory

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Within the non-perturbative \(1/N\) expansion, we discuss numerical methods for calculating multi-loop Feynman graph needed to derive physical scattering amplitudes. We apply higher order \(1/N\) methods to the scalar sector of the standard model, and show the existence of a mass saturation effect. The mass saturation has direct implications for future searches at the LHC and at possible muon colliders.

1. INTRODUCTION

There has been a lot of work lately in extending the calculation of radiative corrections to higher loop-orders and to multi-leg processes. The proceedings of this Workshop are suggestive for the complexity this field has reached since the pioneering work of Martinus Veltman and Gerardus ’t Hooft. Higher order QCD radiative corrections are necessary for interpreting the experimental data which the LHC experiments will provide. Higher order electroweak corrections are needed for interpreting precision electroweak measurements. Due to the time frame for the construction of future colliders, precision measurements will continue to be a main tool for new physics searches.

In this contribution we would like to discuss the use of multi-loop Feynman graphs for recovering results which are normally not within the scope of perturbation theory, namely deriving scattering amplitudes in the case of strongly coupled field theories.

A central question in particle physics is how is the electroweak symmetry broken. The experimental data currently available is compatible with a standard model scalar sector, and favors a light Higgs boson. However, additional degrees of freedom beyond the standard model have the potential to shift these fits. It is not at all excluded that the LHC experiments will see a scalar resonance at a considerably higher energy.

At strong coupling, the scalar sector was insufficiently explored. Qualitatively, one may expect new phenomena to appear: a Higgs particle strongly coupled to the vector bosons and to itself, anomalous vector boson self-interactions, and possibly the appearance of a spectrum of additional resonances in the scalar sector. Perturbative calculations loose their predictive power when radiative corrections blow up in higher loop orders, and the dependency on the renormalization scheme becomes substantial [1].

Ideally, one needs a solution for the scalar sector which is valid at strong coupling as well as at weak, and which is free of renormalization scheme uncertainty. This can be accomplished by explicitly summing up all loop orders within a non-perturbative \(1/N\) expansion. If a solution of sufficient accuracy is desired, such that it can be used in phenomenological calculations to compete with ordinary perturbation theory, then the \(1/N\) expansion must be carried out at higher order.

2. THE \(1/N\) SOLUTION

We are interested in a solution of the scalar sector of the standard model when the self-coupling of the Higgs field becomes strong. The non-perturbative effects are thus given by the scalar
field self-interaction, while the gauge coupling remains perturbative. It is then natural to resort to the equivalence theorem to relate amplitudes involving longitudinal electroweak gauge bosons to the corresponding amplitudes involving would-be Goldstone bosons. The problem is being reduced to solving a linear sigma model non-perturbatively.

2.1. The auxiliary field formalism

It is sufficient to consider the scalar sector alone, which is an SU(2)-symmetric linear sigma model. In the following, the scalar sector is extended to an $O(N)$-symmetric sigma model — with the standard model case recovered for $N = 4$ — such that an expansion in powers of $1/N$ can be performed:

$$
\mathcal{L}_1 = \frac{1}{2} \partial_{\mu} \Phi_0 \partial^{\mu} \Phi_0 - \frac{\mu_0^2}{2} \Phi_0^2 - \frac{\lambda_0}{4N} \Phi_0^4 , \\
\Phi_0 \equiv (\phi_0^1, \phi_0^2, \ldots, \phi_0^N)
$$

From the Lagrangian above, it is easy to derive scattering amplitudes at leading order in $1/N$. Typically, this involves summing up a geometric series of one-loop bubble diagrams which have Goldstone bosons in the loop.

However, the auxiliary field formalism which was proposed in ref. [2] proves to be most useful in organizing sub-leading orders in the $1/N$ expansion in a diagrammatically manageable way. It consists in introducing an additional unphysical field $\chi$ in the Lagrangian:

$$
\mathcal{L}_2 = \mathcal{L}_1 + \frac{3N}{2\lambda_0} (\lambda_0 - \frac{\lambda_0}{6N} \Phi_0^2 - \mu_0^2)^2 \\
= \frac{1}{2} \partial_{\mu} \Phi_0 \partial^{\mu} \Phi_0 - \frac{1}{2} \lambda_0 \Phi_0^2 + \frac{3N}{2\lambda_0} \lambda_0^2 \\
- \frac{3\mu_0^2}{\lambda_0} \lambda_0 + \text{const.}
$$

Because the equation of motion for the auxiliary field $\chi$ is a constant, this addition does not change the dynamics. Green’s functions having Higgs and Goldstone bosons on the legs are the same, whether calculated with the Feynman rules given by $\mathcal{L}_1$ or by $\mathcal{L}_2$.

The Feynman rules, on the other hand, are changed. $\mathcal{L}_1$ contains the trilinear and quartic vertices $\sigma \pi \pi$, $\sigma \sigma$, $\sigma \sigma \sigma$, and $\pi \pi \pi$. Here $\sigma$ and $\pi$ are the massive and massless modes stemming from Lagrangian $\mathcal{L}_1$ after spontaneous symmetry breaking, respectively. $\mathcal{L}_2$ contains only the trilinear couplings $\chi \sigma \sigma$ and $\chi \pi \pi$.

This simplifies enormously the topological classification of Feynman diagrams according to their power of $1/N$ beyond leading order. The reader can easily convince himself of the utility of the auxiliary field formalism by writing down the diagrams which contribute to Goldstone-Goldstone scattering at NLO in $1/N$ in both formalisms.

2.2. Tachyonic regularization

By summing up the chains of one-loop bubble self-energy insertions, one encounters an ultraviolet renormalon. This gives rise to an additional, tachyonic pole in the propagators, apart from the expected physical spectrum containing one Higgs boson and $N - 1$ Goldstone modes.

By direct evaluation of the leading order contribution in $1/N$ to the two-point functions, one obtains the following propagators [2, 3]:

$$
D_{\sigma \sigma}(s) = \frac{i}{s - m^2(s)} \\
D_{\chi \chi}(s) = \frac{1}{Nv^2} \frac{ism^2(s)}{s - m^2(s)} \\
D_{\chi \sigma}(s) = \frac{i}{\sqrt{Nv}} \frac{im^2(s)}{s - m^2(s)} \\
D_{\pi, \pi}(s) = \frac{i}{s} \delta_{ij} 
$$

where

$$
m^2(s) = \frac{v^2}{\lambda} + \hat{\alpha}^{(0)}(s) \\
\equiv \frac{v^2}{\lambda} - \frac{1}{32\pi^2} \log \left(-\frac{s + 10}{\mu^2} \right) .
$$

Here $\hat{\alpha}^{(0)}(s)$ is the ultraviolet finite part of the one-loop self-energy bubble diagram, with a Goldstone boson in the loop. $\mu$ is the ultraviolet subtraction scale.
The propagators contain an Euclidian pole in the ultraviolet region at an energy $s = -\Lambda_t^2$ given by the following transcendental equation:

$$\frac{v^2}{\Lambda_t^2} - \frac{1}{32\pi^2} \log \left( \frac{\Lambda_t^2}{\mu^2} \right) + \frac{3}{\lambda} = 0 .$$  \hspace{1cm} (5)

The tachyonic scale is in the ultraviolet region for low values of the coupling, and tends to move towards low energy when the coupling is increased.

In higher order $1/N$ corrections, the Euclidian pole appears in the loop momentum integration. This induces causality violating contributions, even though these effects are numerically small as long as the tachyonic scale is high enough. For this reason, it is necessary to investigate the origin of this singularity and find a way for dealing with it.

At any finite order of perturbation theory, the tachyon pole does not exist. It is an artifact of the bubble diagram summation. In the process of summing up all loop orders, an ambiguity is present, however. The summation of the perturbative series determines the result only up to functions which vanish in perturbation theory, such as $e^{-1/\lambda}$. Because the residuum of the tachyonic pole is precisely such a function which vanishes in perturbation theory, its presence or absence is completely arbitrary and cannot be determined within perturbation theory. For this reason, it is justified to restore causality by minimally subtracting the tachyon at its pole since the original information stemming from Feynman diagrams remains unchanged [3]:

$$D_{\sigma\sigma}(s) = i \left[ \frac{1}{s - m^2(s)} - \frac{\kappa}{s + \Lambda_t^2} \right]$$

$$D_{\chi\chi}(s) = \frac{is}{N\nu^2} \left[ \frac{m^2(s)}{s - m^2(s)} + \frac{\kappa \Lambda_t^2}{s + \Lambda_t^2} \right]$$

$$D_{\chi\sigma}(s) = i \sqrt{N\nu} \left[ \frac{m^2(s)}{s - m^2(s)} + \frac{\kappa \Lambda_t^2}{s + \Lambda_t^2} \right] ,$$  \hspace{1cm} (6)

were $\kappa = [1 + \Lambda_t^2/(32\pi^2 v^2)]^{-1}$ is the residuum of the tachyonic pole.

The tachyonic regularization can be seen as a prescription for summing up the perturbative series in a way which preserves causality.

### 2.3. $1/N$ renormalization

Performing renormalization at NLO in the $1/N$ expansion involves the treatment of ultraviolet divergences of diagrams with various numbers of loops. One way of keeping track of various loop order counterterms is to group them into $1/N$ counterterms. Each of the $1/N$ order counterterms $\delta\lambda, \delta t, \delta t_{\chi}, \delta Z_{\sigma, \sigma, \chi}$ is a power series in the coupling constant $\lambda$ [4]:

$$\frac{3}{\lambda_0} = \frac{3}{\lambda} + \Delta \lambda$$

$$= \frac{3}{\lambda} + \delta \lambda^{(0)} + \frac{1}{N} \delta \lambda + O \left( \frac{1}{N^2} \right)$$

$$\frac{3\mu_0}{\lambda_0} = -\frac{v^2}{2} (1 + \Delta t)$$

$$= -\frac{v^2}{2} \left[ 1 + \frac{1}{N} \delta t + O \left( \frac{1}{N^2} \right) \right]$$

$$\phi_i^0 = \pi_i Z_{\pi} \quad , \quad i = 1, \ldots, N - 1$$

$$= \pi_i \left[ 1 + \frac{1}{N} \delta Z_{\pi} + O \left( \frac{1}{N^2} \right) \right]$$

$$\phi_0^N = \sigma Z_{\sigma} + \sqrt{Nv}$$

$$= \sigma \left[ 1 + \frac{1}{N} \delta Z_{\sigma} + O \left( \frac{1}{N^2} \right) \right] + \sqrt{Nv}$$

$$\chi_0 = \chi Z_{\chi} + \hat{\chi} + \Delta t_{\chi}$$

$$= \chi \left[ 1 + \frac{1}{N} \delta Z_{\chi} + \frac{v^2}{N} \delta t_{\chi} + O \left( \frac{1}{N^2} \right) \right] + \sqrt{Nv}$$

In principle, these counterterms are sufficient to absorb the UV divergences of all diagrams of NLO in $1/N$.

When calculating $1/N$ diagrams at NLO, we resort to numerical integration. The ultraviolet divergence of the diagram must be removed before the numerical integration can be carried out, but the explicit $\epsilon$ expansion and isolation of UV poles is cumbersome because of the complexity of the diagrams. For this reason we perform a BPHZ-type renormalization [4]. We subtract the divergences of the diagrams according to the forest formula, as shown in figure 1. The subtracted expressions, being finite in the ultraviolet, can be integrated numerically. For a given physical process, the renormalization program then means to combine all UV subtraction terms from various diagrams with each other. Most subtractions can-
Figure 1. Ultraviolet subtractions of all NLO $1/N$ two- and three-point diagrams of the sigma model. The blob on the propagators denotes the resummed chain of one-loop Goldstone bubble diagrams. The solid line denotes Goldstone fields, the wavy line is the Higgs field, and the dashed line is the auxiliary field.

With each other, and the small remaining set of subtractions is trivially absorbed into the $1/N$ local counterterms above. We have checked that all UV subtractions are polynomial, as they ought to be in order to be absorbed into local counterterms.

### 2.4. Numerical solution

The finite expressions depicted in figure 1 are calculated by numerical integration because so far no analytical approach exists for dealing with this type of diagrams.

For a numerical solution, it is advantageous to identify first all loop integrations which can be performed analytically. These are associated with closed Goldstone loops. For the diagrams shown in figure 1, it is possible to perform analytically all integrations except for one final loop integration. The final integration involves the resummed and tachyonically subtracted propagators of eqs. 6 and form factors from one-loop triangle or box diagrams involving massless Goldstone bosons in the loop. The final loop integration needs to be performed numerically. It can be reduced to a two-fold integral with the methods of refs. [5, 4], which do not resort to Feynman parameters.

### 2.5. Scheme independence of physical predictions

For a given order in the expansion parameter $1/N$, the Feynman diagrams of all loop orders are explicitly summed up. For this reason, the final result for a physical scattering amplitude is free of renormalization scheme dependence. In usual perturbative calculations, a renormalization scheme ambiguity is present because of the truncation of the perturbative expansion; this ambiguity is of higher order in the coupling constant.

We note that subtracted diagrams of the type shown in figure 1 are not individually free of renormalization scheme dependency. The internal subtractions which are performed for making the integrals finite in the ultraviolet, are calculated at a given subtraction point. This defines an intermediary renormalization scheme.

This renormalization scheme dependence cancels out only in physical results such as scatter-
ing amplitudes. In the following section we give the results for two scattering processes. We have checked explicitly that the two functions involved, $f_1$ and $f_2$ of eqns. 9, are independent of the subtraction point.

3. $\mu^+\mu^-$ COLLIDERS AND THE HIGGS MASS SATURATION EFFECT

Recently, feasibility studies for muon colliders attracted quite some attention. A muon collider would be an ideal $s$-channel Higgs factory. For a heavy Higgs boson, two processes dominate: $\mu^+\mu^- \rightarrow H \rightarrow t\bar{t}$ and $\mu^+\mu^- \rightarrow H \rightarrow W_L^+W_L^-$.

Within the $1/N$ expansion, the amplitudes for these processes at NLO are given by the following expressions:

\[ M_{f\bar{f}} = \frac{1}{s - m^2(s)} \left[ 1 - \frac{1}{N} f_1(s) \right] \]
\[ M_{ZZ} = \frac{m^2(s)}{\sqrt{Nv}} \frac{1 - \frac{1}{N} f_2(s)}{s - m^2(s)} \left[ 1 - \frac{1}{N} f_1(s) \right] . \]  

Here, the correction functions $f_1$ and $f_2$ are given by a combination of the two- and three-point functions defined in figure 1 (see [4]):

\[ f_1(s) = \frac{m^2(s)}{v^2} \hat{\alpha}(s) + 2\hat{\gamma}(s) + \frac{v^2}{m^2(s)} \left[ \hat{\beta}(s) - 2s - \frac{m^2(s)}{v^2} \left( \delta Z_\sigma - \delta Z_\pi \right) \right] \]
\[ f_2(s) = \frac{m^2(s)}{v^2} \hat{\alpha}(s) + \hat{\gamma}(s) - \hat{\phi}(s) - \frac{v^2}{m^2(s)} \hat{\eta}(s) . \]  

In figure 2 we give numerical results for these expressions. We plot the shape of the Higgs resonance for various strengths of the coupling. From the positions of the resonance maxima, it can be seen clearly that when the coupling is increased, a mass saturation effect appears. The position of the resonance's peak does not increase above a saturation value just under 1 TeV. The precise resonance shape and implicitly the saturation value are process dependent.

For comparison, we indicate in figure 2 the corresponding peak maxima which would be obtained by using usual perturbation theory at NNLO. A saturation is present qualitatively in this curve, too. However, it should be noticed that the perturbative curve is affected by large radiative corrections and large scheme uncertainties at such large coupling, and therefore is not reliable in the saturation region.

4. SATURATION EFFECT AT THE LHC

The main Higgs production mechanism at the LHC is gluon fusion which proceeds through a top loop [7]. The existing perturbative analyses indicate that the vector boson fusion becomes competitive at an energy of the order of 1 TeV.

In the presence of the saturation effect, the observation of a heavy Higgs resonance at the LHC will be different. In the saturation zone, the mass of the resonance remains more or less the same while the width increases with the coupling. The resonance becomes flatter and more difficult to detect.

In figure 3 we show a plot of the Higgs width...
The current knowledge of the Higgs width at strong coupling from perturbation theory and from the $1/N$ expansion. The parameters $M_{\text{PEAK}}$ and $\Gamma_{\text{PEAK}}$ are extracted from the position and the height of the Higgs resonance in fermion scattering as if the resonance was of Breit-Wigner type. For the perturbation theory curves we give the corresponding values of the on-shell mass parameter $m_H$.

as a function of the Higgs mass. The width and mass used in these plots, $M_{\text{PEAK}}$ and $\Gamma_{\text{PEAK}}$, are defined from the line shape of the resonance as seen in fermion-fermion scattering. They are derived from the position and height of the resonance as if it were of Breit-Wigner type. In this picture we show the calculations available so far in usual perturbation theory (LO, NLO, NNLO) [9], and in the $1/N$ expansion (LO and NLO) [4]. It can be seen that the two expansions converge nicely towards each other and display a mass saturation.

We studied the discovery potential of the LHC in the presence of the saturation effect. In ref. [8] we performed a Monte-Carlo simulation for the LHC. Only leptonic channels were considered, namely the “golden plated” channel ($l^+l^-$), and the $t\bar{t}H$ channel. When neutrinos in the final state are involved, the Higgs resonance appears as a Jacobian peak in the distribution of missing $p_T$. We included the gluon fusion process together with the relevant background. The strong interacting Higgs correction was included by using the NNLO perturbative calculation - it results into a faster Monte-Carlo while being a fair approximation of the non-perturbative $1/N$ result.

We used usual assumptions about the LHC energy and luminosity, and asked for a $5\sigma$ effect with respect to the background. The discovery potential estimated for the “golden plated” channel corresponds then to an on-shell Higgs mass of 830 GeV. The missing $p_T$ channel reaches up to an on-shell Higgs mass of 1030 GeV. Note that these values for the on-shell mass deviate considerably from the actual position of the resonance. The actual peak can be read from figure 3, where the on-shell mass is mapped onto the saturation curve.

5. CONCLUSIONS

The combinatorial structure of the sigma model makes possible an explicit calculation of the non-perturbative $1/N$ expansion at NLO.

The $1/N$ solution is valid at strong coupling as well as at weak coupling. It is also independent of the intermediate renormalization scheme which is being used. At NLO it also provides for ultraviolet finite wave function renormalization constants.

The non-perturbative solution of the sigma model is relevant for the standard model via the equivalence theorem. It implies that a mass saturation effect is present in the scalar sector. When the Higgs self-interaction is increased, the mass of the resonance increases only up to a maximum value under 1 TeV, while the width increases continuously.

An interesting question is how cutoff effects stemming from new physics at higher energies can modify the $1/N$ solution. This clearly deserves further investigation.

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