ON PSEUDO-AMENABILITY OF BEURLING ALGEBRAS

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Abstract. Amenability and pseudo-amenability of $\ell^1(S,\omega)$ is characterized, where $S$ is a left (right) zero semigroup or it is a rectangular band semigroup. The equivalence conditions to amenability of $\ell^1(S,\omega)$ are provided, where $S$ is a band semigroup. For a locally compact group $G$, pseudo-amenability of $\ell^1(G,\omega)$ is also discussed.

1. Introduction and Preliminaries

For a Banach algebra $A$ the projective tensor product $A\hat{\otimes}A$ is a Banach $A$-bimodule in a natural manner and the multiplication map $\pi : A\hat{\otimes}A \rightarrow A$ defined by $\pi(a \otimes b) = ab$ for $a, b \in A$ is a Banach $A$-bimodule homomorphism.

Amenability for Banach algebras introduced by B. E. Johnson [9]. Let $A$ be a Banach algebra and $E$ be a Banach $A$-bimodule. A continuous linear operator $D : A \rightarrow E$ is a derivation if it satisfies $D(ab) = D(a) \cdot b + a \cdot D(b)$ for all $a, b \in A$. Given $x \in E$, the inner derivation $ad_x : A \rightarrow E$ is defined by $ad_x(a) = a \cdot x - x \cdot a$. A Banach algebra $A$ is amenable if for every Banach $A$-bimodule $E$, every derivation from $A$ into $E^*$, the dual of $E$, is inner.

An approximate diagonal for a Banach algebra $A$ is a net $(m_i)$ in $A\hat{\otimes}A$ such that $a \cdot m_i - m_i \cdot a \rightarrow 0$ and $a\pi(m_i) \rightarrow a$, for each $a \in A$. The concept of pseudo-amenability introduced by F. Ghahramani and Y. Zhang in [5]. A Banach algebra $A$ is pseudo-amenable if it has an approximate diagonal. It is well-known that amenability of $A$ is equivalent to the existence of a bounded approximate diagonal.

The notions of biprojectivity and biflatness of Banach algebras introduced by Helemski˘ in [7]. A Banach algebra $A$ is biprojective if there is a bounded $A$-bimodule homomorphism $\rho : A \rightarrow A\hat{\otimes}A$ such that $\pi o \rho = I_A$, where $I_A$ is the identity map on $A$. We say that $A$ is biflat if there is a bounded $A$-bimodule homomorphism $\rho : A \rightarrow (A\hat{\otimes}A)^{**}$ such that $\pi^{**} o \rho = k_A$, where $k_A : A \rightarrow A^{**}$ is the natural embedding of $A$ into its second dual.

Let $S$ be a semigroup. A continuous function $\omega : S \rightarrow (0, \infty)$ is a weight on $S$ if $\omega(st) \leq \omega(s)\omega(t)$, for all $s, t \in S$. Then it is standard that

$$\ell^1(S,\omega) = \left\{ f = \sum_{s \in S} f(s)\delta_s : \|f\|_\omega = \sum_{s \in S} |f(s)|\omega(s) < \infty \right\}$$

is a Banach algebra with the convolution product $\delta_s * \delta_t = \delta_{st}$. These algebras are called Beurling algebras.

In this note, we study the earlier mentioned properties of Banach algebras for Beurling algebras. Firstly in section 2, we characterize amenability and pseudo-amenability of $\ell^1(S,\omega)$, for some certain class of semigroups. Let $S$ be a left or right zero semigroup. We prove that pseudo-amenability of $\ell^1(S,\omega)$ is equivalent to it’s amenability and these equivalent conditions imply that

2010 Mathematics Subject Classification. Primary: 22D15, 43A10; Secondary: 43A20, 46H25.

Key words and phrases. amenability, pseudo-amenability, Beurling algebra.
Proposition 2.1. Suppose that \( \rho \) and similarly \( \ell \) semigroup, and Remark 2.2. for each \( \mathbf{1} \) is singleton. We show that the same result holds for \( \ell^1(S, \omega) \), whenever \( S \) is a rectangular band semigroup and \( \omega \) is separable. Further, we investigate biprojectivity of \( \ell^1(S, \omega) \) whenever \( S \) is either left (right) zero semigroup or a rectangular band semigroup. For a band semigroup \( S \), we show that amenability of \( \ell^1(S, \omega) \) is equivalent to that of \( \ell^1(S) \) and these are equivalent to \( S \) being a finite semilattice.

Finally in section 3, we investigate pseudo-amenability of \( L^1(G, \omega) \) where \( G \) is a locally compact group and \( \omega \) is a weight on \( G \). We prove that pseudo-amenability of \( L^1(G, \omega) \) implies amenability of \( G \), and under a certain condition it implies diagonally boundedness of \( \omega \). Next, if \( L^1(G, \omega) \) is pseudo-amenable we may obtain a character \( \varphi \) on \( G \) for which \( \varphi \leq \omega \).

2. Amenability and pseudo-amenability of \( \ell^1(S, \omega) \)

A semigroup \( S \) is a **left zero semigroup** if \( st = s \), and it is a **right zero semigroup** if \( st = t \) for each \( s, t \in S \). Then for \( f, g \in \ell^1(S, \omega) \), it is obvious that \( f \ast g = \varphi_S(f)g \) if \( S \) is a right zero semigroup, and \( f \ast g = \varphi_S(g)f \) if \( S \) is a left zero semigroup, where \( \varphi_S \) is the **augmentation character** on \( \ell^1(S, \omega) \).

We extend somewhat the obtained results for \( \ell^1(S) \) in [2,3] to the weighted case \( \ell^1(S, \omega) \).

**Proposition 2.1.** Suppose that \( S \) is a right (left) zero semigroup and \( \omega \) be a weight on \( S \). Then \( \ell^1(S, \omega) \) is biprojective.

**Proof.** We only give the proof in the case \( S \) is a right zero semigroup. Define \( \rho : \ell^1(S, \omega) \to \ell^1(S, \omega) \otimes \ell^1(S, \omega) \) by \( \rho(f) = \delta_t \otimes f \), where \( t_0 \) is an arbitrary element \( S \). Then for each \( f, g \in \ell^1(S, \omega) \) we have

\[
\rho(f \ast g) = \delta_t \otimes (f \ast g) = \varphi_S(f)(\delta_t \otimes g) = (f \ast \delta_{t_0}) \otimes g = f \cdot (\delta_{t_0} \otimes g) = f \cdot \rho(g)
\]

and similarly \( \rho(f \ast g) = \rho(f) \cdot g \). Further, \( \pi \rho \) is the identity map on \( \ell^1(S, \omega) \), as required. \( \square \)

**Remark 2.2.** It is known that every biprojective Banach algebra is biflat. Hence Proposition 2.1 shows that for every right or left zero semigroup \( S \), \( \ell^1(S, \omega) \) is biflat.

Given two semigroups \( S_1 \) and \( S_2 \), we say that a weight \( \omega \) on \( S := S_1 \times S_2 \) is **separable** if there exist two weights \( \omega_1 \) and \( \omega_2 \) on \( S_1 \) and \( S_2 \), respectively such that \( \omega = \omega_1 \otimes \omega_2 \). It is easy to verify that \( \ell^1(S, \omega) \cong \ell^1(S_1, \omega_1) \otimes \ell^1(S_2, \omega_2) \).

Let \( S \) be a semigroup and let \( E(S) = \{ p \in S : p^2 = p \} \). We say that \( S \) is a **band semigroup** if \( S = E(S) \). A band semigroup \( S \) satisfying \( st = s \), for each \( s, t \in S \) is called a **rectangular band semigroup**. For a rectangular band semigroup \( S \), it is known that \( S \cong L \times R \), where \( L \) and \( R \) are left and right zero semigroups, respectively [8, Theorem 1.1.3].

**Proposition 2.3.** Let \( S \) be a rectangular band semigroup and \( \omega \) be a separable weight on \( S \). Then \( \ell^1(S, \omega) \) is biprojective, and so it is biflat.

**Proof.** In view of earlier argument, it follows from Proposition 2.1, and then from [10, Proposition 2.4].

**Theorem 2.4.** Let \( S \) be a rectangular band semigroup and \( \omega \) be a weight on \( S \). Then \( \ell^1(S, \omega) \) is amenable if and only if \( S \) singleton.
Theorem 2.7. Let \( \ell^1(S) \) be amenable. Then it is immediate by [2, Theorem 3.3].

For a semigroup \( S \), we denote by \( S^{op} \) the semigroup whose underlying space is \( S \) but whose multiplication is the multiplication in \( S \) reversed.

**Proposition 2.5.** Let \( S \) be a right (left) zero semigroup and \( \omega \) be a weight on \( S \). Then \( \ell^1(S, \omega) \) is amenable if and only if \( S \) is singleton.

**Proof.** Suppose that \( S \) is a left zero semigroup, and that \( \ell^1(S, \omega) \) is amenable. Then \( S^{op} \) is a right zero semigroup. It is readily seen that \( S \times S^{op} \) is a rectangular band semigroup, and \( \ell^1(S^{op}, \omega) \) is amenable. Hence \( \ell^1(S, \omega) \otimes \ell^1(S^{op}, \omega) \cong \ell^1(S \times S^{op}, \omega \otimes \omega) \) is amenable. Now, by Theorem 2.4, \( S \) is singleton.

Let \( \mathcal{A} \) be Banach algebra, \( \mathcal{I} \) be a semilattice (i.e., \( \mathcal{I} \) is a commutative band semigroup) and \( \{ \mathcal{A}_\alpha : \alpha \in \mathcal{I} \} \) be a collection of closed subalgebras of \( \mathcal{A} \). Then \( \mathcal{A} \) is \( \ell^1 \)-graded of \( \mathcal{A}_\alpha \)'s over the semilattice \( \mathcal{I} \), denoted by \( \mathcal{A} = \bigoplus_{\alpha \in \mathcal{I}} \mathcal{A}_\alpha \), if it is \( \ell^1 \)-directsum of \( \mathcal{A}_\alpha \)'s as Banach space such that \( \mathcal{A}_\alpha \mathcal{A}_\beta \subseteq \mathcal{A}_{\alpha \beta} \), for each \( \alpha, \beta \in \mathcal{I} \).

Suppose that \( S^1 \) is the unitization of a semigroup \( S \). An equivalence relation \( \tau \) on \( S \) is defined by \( s \tau t \iff S^1 s S^1 = S^1 t S^1 \), for all \( s, t \in S \). If \( S \) is a band semigroup, then by [8, Theorem 4.4.1], \( S = \bigcup_{\alpha \in \mathcal{I}} S_{\alpha} \) is a semilattice of rectangular band semigroups, where \( \mathcal{I} = \widehat{S}^\tau \) and for each \( \alpha = [s] \in \mathcal{I} \), \( S_{\alpha} = [s] \).

**Theorem 2.6.** Let \( S \) be a band semigroup and \( \omega \) be a weight on \( S \). Then the following are equivalent:

(i) \( \ell^1(S, \omega) \) is amenable.

(ii) \( S \) is finite and each \( \tau \)-class is singleton.

(iii) \( \ell^1(S) \) is amenable.

(iv) \( S \) is a finite semilattice.

**Proof:** The implications (ii) to (iv) are equivalent [2, Theorem 3.5]. We establish (i) \( \implies \) (ii) and (iv) \( \implies \) (i).

(i) \( \implies \) (ii) If \( \ell^1(S, \omega) \) is amenable, then \( E(S) = S \) is finite and so \( \mathcal{I} = \widehat{S}^\tau \) is a finite semilattice.

Hence \( \ell^1(S, \omega) \cong \bigoplus_{\alpha \in \mathcal{I}} \ell^1(S_{\alpha}, \omega_{\alpha}) \), where \( \omega_{\alpha} = \omega|_{S_{\alpha}} \). Then by [6, Proposition 3.1], each \( \ell^1(S_{\alpha}, \omega_{\alpha}) \) is amenable. Now by Theorem 2.4, \( S_{\alpha} \) is singleton for each \( \alpha \in \mathcal{I} \), as required.

(iv) \( \implies \) (i) In this case \( \ell^1(S, \omega) \cong \ell^1(S) \), and \( \ell^1(S) \) is amenable.

**Theorem 2.7.** Let \( S \) be a rectangular band semigroup, and let \( \omega \) be a separable weight on \( S \). Then \( \ell^1(S, \omega) \) is pseudo-amenable if and only if \( S \) is singleton.

**Proof.** There is a left zero semigroup \( L \) and a right zero semigroup \( R \), and there are weights \( \omega_L \) and \( \omega_R \) on \( L \) and \( R \), respectively such that \( S \cong L \times R \) and \( \omega = \omega_L \otimes \omega_R \). We have \( \ell^1(S, \omega) \cong \ell^1(L, \omega_L) \otimes \ell^1(R, \omega_R) \). Hence the map \( \theta : \ell^1(S, \omega) \rightarrow \ell^1(L, \omega_L) \) defined by \( \theta(f \otimes g) = \varphi_R(g)f \) for \( f \in \ell^1(L, \omega_L) \) and \( g \in \ell^1(R, \omega_R) \), is an epimorphism of Banach algebras, whereas \( \varphi_R \) is the augmentation character on \( \ell^1(R, \omega_R) \). Whence \( \ell^1(L, \omega_L) \) has left and right approximate identity. Therefore \( L \) is singleton, because it is left zero semigroup. Similarly \( R \) is singleton, so is \( S \).

**Corollary 2.8.** Let \( S \) be a right (left) zero semigroup and \( \omega \) be a weight on \( S \). Then the following are equivalent:

(i) \( \ell^1(S, \omega) \) is pseudo-amenable.
(ii) $S$ is singleton.
(iii) $\ell^1(S,\omega)$ is amenable.

**Proof.** The implication $(ii) \iff (iii)$ is Proposition 2.5. For $(i) \implies (ii)$, we apply Theorem 2.7 for the rectangular band semigroup $S \times S^{op}$ with $\omega_L = \omega_R = \omega$.

The following is a combination of Theorems 2.4 and 2.7. Notice that in Theorem 2.4, we need not $\omega$ to be separable.

**Corollary 2.9.** Let $S$ be a rectangular band semigroup, and let $\omega$ be a separable weight on $S$. Then the following are equivalent:

(i) $\ell^1(S,\omega)$ is pseudo-amenable.
(ii) $S$ is singleton.
(iii) $\ell^1(S,\omega)$ is amenable.

For the left cancellative semigroups we have the following.

**Theorem 2.10.** Suppose that $S$ is a left cancellative semigroup and $\omega$ is a weight on $S$. If $\ell^1(S,\omega)$ is pseudo-amenable, then $S$ is a group.

**proof:** This is a more or less verbatim of the proof of [3, Theorem 3.6 $(i) \implies (ii)$].

3. **Pseudo-amenability of $L^1(G,\omega)$**

Throughout $G$ is a locally compact group and $\omega$ is a weight on $G$. The weight $\omega$ is *diagonally bounded* if $\sup_{g \in G} \omega(g)\omega(g^{-1}) < \infty$. It seems to be a right conjecture that $L^1(G,\omega)$ will fail to be pseudo-amenable whenever $\omega$ is not diagonally bounded. Although we are not able to prove (or disprove) the conjecture, the following is a weaker result.

The proofs in this section owe much to those of [4, Section 8].

**Theorem 3.1.** Suppose that $L^1(G,\omega)$ is pseudo-amenable for which there is an approximate diagonal $(m_i)$, such that $m_i - \delta_g \cdot m_i \cdot \delta_{g^{-1}} \to 0$ uniformly on $G$. Then $\omega$ is diagonally bounded.

**Proof.** We follow the standard argument in [4, Proposition 8.7]. Choose $f \in L^1(G,\omega)$ such that $K := \text{supp}f$ is compact and $\int f \neq 0$. Putting $F := f \cdot \chi_K \in L^\infty(G,\omega^{-1})$, we see that $\pi^*(F) \in L^\infty(G \times G,\omega^{-1} \times \omega^{-1})$ with

$$\pi^*(F)(x,y) = F(xy) = \int \chi_K(xyt)f(t)dt.$$ 

Let $(m_i) \subseteq L^1(G \times G,\omega \times \omega)$ be an approximate diagonal for $L^1(G,\omega)$ such that $\delta_g \cdot m_i \cdot \delta_{g^{-1}} - m_i \to 0$ uniformly on $G$, and $\pi(m_i)f - f \to 0$. Then for each $i$

$$\langle \pi^*(F),m_i \rangle = \langle F,\pi(m_i) \rangle = \langle \chi_K,\pi(m_i)f \rangle \to \langle \chi_K,f \rangle = \int f.$$ 

Consequently

$$\lim_i \langle \pi^*(F),m_i \rangle \neq 0. \quad (1)$$

We define $E := K \cdot \overline{K}$, and $A := \{(x,y) \in G \times G : xy \in E\}$. For $r > 0$, we define $A_r := \{(x,y) \in A : \omega(x)\omega(y) < r\}$, and $B_r := \{(x,y) \in A : \omega(x)\omega(y) \geq r\}$. Obviously, $\pi^*(F)\chi_A$ and $\pi^*(F)\chi_B$ both are in $L^\infty(G \times G,\omega^{-1} \times \omega^{-1})$, and $\pi^*(F) = \pi^*(F)\chi_A = \pi^*(F)\chi_A + \pi^*(F)\chi_B$. For every $i$, it is easy to see that

$$|\langle \pi^*(F)\chi_{B_r},m_i \rangle| \leq \|m_i\| \|F\| r^{-1} c_1$$
where \( c_1 := \sup_{t \in E} \omega(t) \). Hence
\[
\lim_{r \to \infty} \langle \pi^*(F) \chi_{B_r}, m_i \rangle = 0. \tag{2}
\]
Next, for every \( g \in G \), \( r > 0 \), and \( i \), we obtain
\[
|\langle \pi^*(F) \chi_{A_r}, \delta_g \cdot m_i \cdot \delta_{g^{-1}} \rangle| \leq \|m_i\| \|F\| r \ c_2 \frac{1}{\omega(g)\omega(g^{-1})} \]
where \( c_2 := \sup_{t \in E^{-1}} \omega(t) \). Therefore
\[
|\langle \pi^*(F) \chi_{A_r}, m_i \rangle| \leq |\langle \pi^*(F) \chi_{A_r}, m_i - \delta_g \cdot m_i \cdot \delta_{g^{-1}} \rangle| + |\langle \pi^*(F) \chi_{A_r}, \delta_g \cdot m_i \cdot \delta_{g^{-1}} \rangle|
\]
\[
\leq \|\pi^*(F)\| \sup_{g \in G} \|m_i - \delta_g \cdot m_i \cdot \delta_{g^{-1}}\| + \|m_i\| \|F\| r \ c_1 \ c_2^2 \frac{1}{\omega(g)\omega(g^{-1})}. \tag{3}
\]
Towards a contradiction, we assume that \( \omega \) is not diagonally bounded. Then there is a sequence \((g_n)\) in \( G \) such that \( \lim_n \omega(g_n)\omega(g_n^{-1}) = \infty \). Whence, it follows from (3) that for each \( i \) and \( r > 0 \)
\[
|\langle \pi^*(F) \chi_{A_r}, m_i \rangle| \leq \|\pi^*(F)\| \sup_{g \in G} \|m_i - \delta_g \cdot m_i \cdot \delta_{g^{-1}}\|.
\]
Hence
\[
|\langle \pi^*(F), m_i \rangle| \leq \|\pi^*(F)\| \sup_{g \in G} \|m_i - \delta_g \cdot m_i \cdot \delta_{g^{-1}}\| + |\langle \pi^*(F) \chi_{B_r}, m_i \rangle|.
\]
Putting (2) and (4) together, we may see that
\[
\lim_i \langle \pi^*(F), m_i \rangle = 0
\]
contradicting (1). \( \square \)

**Theorem 3.2.** Suppose that \( L^1(G, \omega) \) is pseudo-amenable, and that \( \omega \) is bounded away from 0. Then \( G \) is amenable.

**Proof.** Since \( L^1(G, \omega) \) is unital, pseudo-amenity and approximate amenability are the same [5, Proposition 3.2]. Now, it is immediate by [4, Proposition 8.1]. \( \square \)

We conclude by the following which is an analogue of [4, Proposition 8.9].

**Proposition 3.3.** Let \( L^1(G, \omega) \) be pseudo-amenable. Then there is a continuous positive character \( \varphi \) on \( G \) such that \( \varphi \leq \omega \).

**Proof.** Suppose that \((m_i)_i \subseteq L^1(G \times G, \omega \times \omega)\) be an approximate diagonal for \( L^1(G, \omega) \). For each \( i \) and \( f \in L^\infty(G \times G, \omega^{-1} \times \omega^{-1})^+ \) we define
\[
\tilde{m}_i(f) := \sup \{ Re(\langle m_i, \psi \rangle) : 0 \leq \psi \leq f, \ \psi \in L^\infty(G \times G, \omega^{-1} \times \omega^{-1}) \}.
\]
Then \( \tilde{m}_i \neq 0 \) on \( L^\infty(G \times G, \omega^{-1} \times \omega^{-1})^+ \) and we may extend \( \tilde{m}_i \) to a bounded functional on \( L^\infty(G \times G, \omega^{-1} \times \omega^{-1}) \) in the obvious manner. It is readily seen that \( \tilde{m}_i \neq 0, \langle \tilde{m}_i, f \rangle \geq 0 \), and \( \delta_{g^{-1}} \cdot \tilde{m}_i \cdot \delta_g - \tilde{m}_i \to 0 \), for every \( f \in L^\infty(G \times G, \omega^{-1} \times \omega^{-1})^+ \) and \( g \in G \).

Putting \( \bar{\omega}(x) := \sup_{y \in G} \omega(g^{-1}xy), \ x \in G \). Then \( \bar{\omega} \in L^\infty(G, \omega^{-1}), \ \bar{\omega}(xy) = \bar{\omega}(yx), \ \pi^*(\bar{\omega}) \in L^\infty(G \times G, \omega^{-1} \times \omega^{-1}) \) and \( \delta_g \cdot \pi^*(\bar{\omega}) \cdot \delta_{g^{-1}} = \pi^*(\bar{\omega}) \).

Take \( f \in C_c(G)^+ \) with \( \int f = 1 \), and then \( h := f \cdot \chi_K \), where \( K := supp f \). One may see that \( h \) is continuous, and there is \( c > 0 \) such that \( \pi^*(h) \geq c \pi^*(h) \). Hence
\[
\lim_i \langle \pi^*(\bar{\omega}), m_i \rangle \geq c \lim_i \langle \tilde{m}_i, \pi^*(h) \rangle \geq c \lim_i Re(\langle m_i, \pi^*(h) \rangle) = c \lim_i Re(\pi(m_i), h)
\]
\[
= c \lim_i Re(\pi(m_i) \cdot f, \chi_K) = c Re(f, \chi_K) = c > 0.
\]
Therefore there is $i_0$ for which $\langle \tilde{m}_{i_0}, \pi^*(\tilde{\omega}) \rangle > 0$. Set $F := \langle \tilde{m}_{i_0}, \pi^*(\tilde{\omega}) \rangle^{-1} \pi^*(\tilde{\omega})$, and for $g \in G$ we put

$$A_g(x, y) := \frac{1}{2} \left( \log \frac{\omega(gx)\omega(gy^{-1})}{\omega(x)\omega(y^{-1})} \right) F(x, y), \quad (x, y \in G).$$

Finally, for each $g \in G$, we define $\varphi(g) := \exp(\langle \tilde{m}_{i_0}, A_g \rangle)$. A similar argument used in [4, Proposition 8.9], shows that $\varphi$ is the desired character on $G$. \hfill $\Box$

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