Galois-theoretic features for 1-smooth pro-$p$ groups

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Abstract. Let $p$ be a prime. A pro-$p$ group $G$ is said to be 1-smooth if it can be endowed with a continuous representation $\theta : G \to \text{GL}_1(\mathbb{Z}_p)$ such that every open subgroup $H$ of $G$, together with the restriction $\theta|_H$, satisfies a formal version of Hilbert 90. We prove that every 1-smooth pro-$p$ group contains a unique maximal closed abelian normal subgroup, in analogy with a result by Engler and Koenigsmann on maximal pro-$p$ Galois groups of fields, and that if a 1-smooth pro-$p$ group is solvable, then it is locally uniformly powerful, in analogy with a result by Ware on maximal pro-$p$ Galois groups of fields. Finally we ask whether 1-smooth pro-$p$ groups satisfy a "Tits' alternative".

1 Introduction

Throughout the paper $p$ will denote a prime number, and $\mathbb{K}$ a field containing a root of unity of order $p$. Let $\mathbb{K}(p)$ denote the compositum of all finite Galois $p$-extensions of $\mathbb{K}$. The maximal pro-$p$ Galois group of $\mathbb{K}$, denoted by $G_{\mathbb{K}}(p)$, is the Galois group $\text{Gal}(\mathbb{K}(p)/\mathbb{K})$, and it coincides with the maximal pro-$p$ quotient of the absolute Galois group of $\mathbb{K}$. Characterising maximal pro-$p$ Galois groups of fields among pro-$p$ groups is one of the most important — and challenging — problems in Galois theory. One of the obstructions for the realization of a pro-$p$ group as maximal pro-$p$ Galois group for some field $\mathbb{K}$ is given by the Artin-Schreier theorem: the only finite group realizable as $G_{\mathbb{K}}(p)$ is the cyclic group of order 2 (cf. [1]).

The proof of the celebrated Bloch-Kato conjecture, completed by M. Rost and V. Voevodsky with Ch. Weibel’s “patch” (cf. [12, 27, 29]) provided new tools to study absolute Galois groups of field and their maximal pro-$p$ quotients (see, e.g., [2, 3, 17, 21]). In particular, the now-called Norm Residue Theorem implies that the $\mathbb{Z}/p$-cohomology algebra of a maximal pro-$p$ Galois group $G_{\mathbb{K}}(p)$

$$H^*(G_{\mathbb{K}}(p), \mathbb{Z}/p) := \bigoplus_{n \geq 0} H^n(G_{\mathbb{K}}(p), \mathbb{Z}/p),$$

with $\mathbb{Z}/p$ a trivial $G_{\mathbb{K}}(p)$-module and endowed with the cup-product, is a quadratic algebra: i.e., all its elements of positive degree are combinations of products of elements of degree 1, and its defining relations are homogeneous relations of degree 2 (see § 2.3 below). For instance, from this property one may recover the Artin-Schreier obstruction (see, e.g., [17, § 2]).

2020 Mathematics Subject Classification: Primary 12G05; Secondary 20E18, 20J06, 12F10.

Keywords: Galois cohomology, Maximal pro-$p$ Galois groups, Bloch-Kato conjecture, Kummerian pro-$p$ pairs, Tits' alternative.
More recently, a formal version of Hilbert 90 for pro-$p$ groups was employed to find further results on the structure of maximal pro-$p$ Galois groups (see [9, 19, 21]). A pair $\mathcal{G} = (G, \theta)$ consisting of a pro-$p$ group $G$ endowed with a continuous representation $\theta: G \rightarrow \text{GL}_1(\mathbb{Z}_p)$ is called a pro-$p$ pair. For a pro-$p$ pair $\mathcal{G} = (G, \theta)$ let $\mathbb{Z}_p(1)$ denote the continuous left $G$-module isomorphic to $\mathbb{Z}_p$, as an abelian pro-$p$ group, with $G$-action induced by $\theta$ (namely, $g.v = \theta(g) \cdot v$ for every $v \in \mathbb{Z}_p(1)$). The pair $\mathcal{G}$ is called a Kummerian pro-$p$ pair if the canonical map

$$H^1(G, \mathbb{Z}_p(1)/p^n) \rightarrow H^1(G, \mathbb{Z}_p(1)/p),$$

is surjective for every $n \geq 1$. Moreover the pair $\mathcal{G}$ is said to be a 1-smooth pro-$p$ pair if every closed subgroup $H$, endowed with the restriction $\theta|_H$, gives rise to a Kummerian pro-$p$ pair (see Definition 2.1). By Kummer theory, the maximal pro-$p$ Galois group $G_{\mathbb{K}}(p)$ of a field $\mathbb{K}$, together with the pro-$p$ cyclotomic character $\theta_{\mathbb{K}}: G_{\mathbb{K}}(p) \rightarrow \text{GL}_1(\mathbb{Z}_p)$ (induced by the action of $G_{\mathbb{K}}(p)$ on the roots of unity of order a $p$-power lying in $\mathbb{K}(p)$) gives rise to a 1-smooth pro-$p$ pair $\mathcal{G}_{\mathbb{K}}$ (see Theorem 2.8).

In [5] — driven by the pursuit of an "explicit" proof of the Bloch-Kato conjecture as an alternative to the proof by Voevodsky — C. De Clerq and M. Florence introduced the 1-smoothness property, and formulated the so-called "Smoothness Conjecture": namely, that it is possible to deduce the surjectivity of the norm residue homomorphism (which is acknowledged to be the "hard part" of the Bloch-Kato conjecture) from the fact that $G_{\mathbb{K}}(p)$ together with the pro-$p$ cyclotomic character is a 1-smooth pro-$p$ pair (see [5, Conj. 14.25] and [15, § 3.1.6], and Question 2.10 below).

In view of the Smoothness Conjecture, it is natural to ask which properties of maximal pro-$p$ Galois groups of fields arise also for 1-smooth pro-$p$ pairs. For example, the Artin-Scherier obstruction does: the only finite $p$-group which may complete into a 1-smooth pro-$p$ pair is the cyclic group $C_2$ of order 2, together with the non-trivial representation $\theta: C_2 \rightarrow \{\pm 1\} \subseteq \text{GL}_1(\mathbb{Z}_2)$ (see Example 2.9 below).

A pro-$p$ pair $\mathcal{G} = (G, \theta)$ comes endowed with a distinguished closed subgroup: the $\theta$-center $Z(\mathcal{G})$ of $\mathcal{G}$, defined by

$$Z(\mathcal{G}) = \left\{ h \in \text{Ker}(\theta) \mid ghg^{-1} = h^{\theta(g)} \forall g \in G \right\}.$$

This subgroup is abelian, and normal in $G$. In [11], A. Engler and J. Koenigsmann showed that if the maximal pro-$p$ Galois group $G_{\mathbb{K}}(p)$ of a field $\mathbb{K}$ is not cyclic then it has a unique maximal normal abelian closed subgroup (i.e., one containing all normal abelian closed subgroups of $G_{\mathbb{K}}(p)$), which coincides with the $\theta_{\mathbb{K}}$-center $Z(\mathcal{G}_{\mathbb{K}})$, and the short exact sequence of pro-$p$ groups

$$\{1\} \rightarrow Z(G_{\mathbb{K}}) \rightarrow G_{\mathbb{K}}(p) \rightarrow G_{\mathbb{K}}(p)/Z(G_{\mathbb{K}}) \rightarrow \{1\}$$

splits. We prove a group-theoretic analogue of Engler-Koenigsmann’s result for 1-smooth pro-$p$ groups.

**Theorem 1.1** Let $G$ be a torsion-free pro-$p$ group, $G \neq \mathbb{Z}_p$, endowed with a representation $\theta: G \rightarrow \text{GL}_1(\mathbb{Z}_p)$ such that $\mathcal{G} = (G, \theta)$ is a 1-smooth pro-$p$ pair. Then $Z(\mathcal{G})$ is the unique maximal normal abelian closed subgroup of $G$, and the quotient $G/Z(\mathcal{G})$ is a torsion-free pro-$p$ group.
In [28], R. Ware proved the following result on maximal pro-$p$ Galois groups of fields: if $G_{\mathbb{K}}(p)$ is solvable, then it is locally uniformly powerful, i.e., $G_{\mathbb{K}}(p) \cong A \rtimes \mathbb{Z}_p$, where $A$ is a free abelian pro-$p$ group, and the right-side factor acts by scalar multiplication by a unit of $\mathbb{Z}_p$ (see § 3.1). We prove that the same property holds also for 1-smooth pro-$p$ groups.

**Theorem 1.2** Let $G$ be a solvable torsion-free pro-$p$ group, endowed with a representation $\theta: G \to \text{GL}_1(\mathbb{Z}_p)$ such that $\mathcal{G} = (G, \theta)$ is a 1-smooth pro-$p$ pair. Then $G$ is locally uniformly powerful.

This gives a complete description of solvable torsion-free pro-$p$ groups which may be completed into a 1-smooth pro-$p$ pair. Moreover, Theorem 1.2 settles the Smoothness Conjecture positively for the class of solvable pro-$p$ groups.

**Corollary 1.3** If $\mathcal{G} = (G, \theta)$ is a 1-smooth pro-$p$ pair with $G$ solvable, then $G$ is a Bloch-Kato pro-$p$ group, i.e., the $\mathbb{Z}/p$-cohomology algebra of every closed subgroup of $G$ is quadratic.

**Remark 1.4** After the submission of this paper, I. Snopce and S. Tanushevski showed in [24] that Theorems 1.2–1.1 hold for a wider class of pro-$p$ groups. A pro-$p$ group is said to be Frattini-injective if distinct finitely generated closed subgroups have distinct Frattini subgroups (cf. [24, Def. 1.1]). By [24, Thm. 1.11 and Cor. 4.3], a pro-$p$ group which may complete into a 1-smooth pro-$p$ pair is Frattini-injective. By [24, Thm. 1.4] a Frattini-injective pro-$p$ group has a unique maximal normal abelian closed subgroup, and by [24, Thm. 1.3] a Frattini-injective pro-$p$ group is solvable if, and only if, it is locally uniformly powerful.

A solvable pro-$p$ group does not contain a free non-abelian closed subgroup. For Bloch-Kato pro-$p$ groups — and thus in particular for maximal pro-$p$ Galois groups of fields containing a root of unity of order $p$ — Ware proved the following Tits’ alternative: either such a pro-$p$ group contains a free non-abelian closed subgroup; or it is locally uniformly powerful (see [28, Cor. 1] and [17, Thm. B]). We conjecture that the same phenomenon occurs for 1-smooth pro-$p$ groups.

**Conjecture 1.5** Let $G$ be a torsion-free pro-$p$ group which may be endowed with a representation $\theta: G \to \text{GL}_1(\mathbb{Z}_p)$ such that $\mathcal{G} = (G, \theta)$ is a 1-smooth pro-$p$ pair. Then either $G$ is locally uniformly powerful, or $G$ contains a closed non-abelian free pro-$p$ group.

2 **Cyclotomic pro-$p$ pairs**

Henceforth, every subgroup of a pro-$p$ group will be tacitly assumed to be closed, and the generators of a subgroup will be intended in the topological sense.

In particular, for a pro-$p$ group $G$ and a positive integer $n$, $G^{p^n}$ will denote the closed subgroup of $G$ generated by the $p^n$-th powers of all elements of $G$. Moreover, for two elements $g, h \in G$, we set

$$h^g = g^{-1}hg, \quad \text{and} \quad [h, g] = h^{-1} \cdot h^g,$$
and for two subgroups $H_1, H_2$ of $G$, $[H_1, H_2]$ will denote the closed subgroup of $G$

generated by all commutators $[h, g]$ with $h \in H_1$ and $g \in H_2$. In particular, $G'$ will
denote the commutator subgroup $[G, G]$ of $G$, and the Frattini subgroup $G^p \cdot G'$ of $G$
is denoted by $\Phi(G)$. Finally, $d(G)$ will denote the minimal number of generator of $G$,
i.e., $d(G) = \dim(G/\Phi(G))$ as a $\mathbb{Z}/p$-vector space.

2.1 Kummerian pro-$p$ pairs

Let $1 + p\mathbb{Z}_p = \{ 1 + p\lambda \mid \lambda \in \mathbb{Z}_p \} \subseteq \text{GL}_1(\mathbb{Z}_p)$ denote the pro-$p$ Sylow subgroup of the

group of units of the ring of $p$-adic integers $\mathbb{Z}_p$. A pair $\mathcal{G} = (G, \theta)$ consisting of a pro-$p$

group $G$ and a continuous homomorphism

$$\theta : G \longrightarrow 1 + p\mathbb{Z}_p$$

is called a cyclotomic pro-$p$ pair, and the morphism $\theta$ is called an orientation of $G$ (cf. [7, § 3] and [21]).

A cyclotomic pro-$p$ pair $\mathcal{G} = (G, \theta)$ is said to be torsion-free if $\text{Im}(\theta)$ is torsion-free:
this is the case if $p$ is odd; or if $p = 2$ and $\text{Im}(\theta) \subseteq 1 + 4\mathbb{Z}_2$. Observe that a cyclotomic
pro-$p$ pair $\mathcal{G} = (G, \theta)$ may be torsion-free even if $G$ has non-trivial torsion—e.g., if $G$
is the cyclic group of order $p$ and $\theta$ is constantly equal to 1. Given a cyclotomic pro-$p$
pair $\mathcal{G} = (G, \theta)$ one has the following constructions:

(a) if $H$ is a subgroup of $G$, $\text{Res}_H(\mathcal{G}) = (H, \theta|_H)$;

(b) if $N$ is a normal subgroup of $G$ contained in $\text{Ker}(\theta)$, then $\theta$ induces an orientation

$$\theta : G/N \longrightarrow 1 + p\mathbb{Z}_p,$$

and we set $\mathcal{G}/N = (G/N, \theta)$;

(c) if $A$ is an abelian pro-$p$ group, we set $A \rtimes \mathcal{G} = (A \rtimes G, \theta \circ \pi)$, with $a^g = a^{\theta(g)-1}$ for

all $a \in A$, $g \in G$, and $\pi$ the canonical projection $A \rtimes G \rightarrow G$.

Given a cyclotomic pro-$p$ pair $\mathcal{G} = (G, \theta)$, the pro-$p$ group $G$ has two distinguished
subgroups:

(a) the subgroup

$$K(\mathcal{G}) = \left\{ h^{-\theta(h)} \cdot h^{\theta^{-1}} \mid g \in G, h \in \text{Ker}(\theta) \right\}$$

introduced in [9, § 3];

(b) the $\theta$-center

$$Z(\mathcal{G}) = \left\{ h \in \text{Ker}(\theta) \mid ghg^{-1} = h^\theta g \forall g \in G \right\}$$

introduced in [17, § 1].

Both $Z(\mathcal{G})$ and $K(\mathcal{G})$ are normal subgroups of $G$, and they are contained in $\text{Ker}(\theta)$.
Moreover, $Z(\mathcal{G})$ is abelian, while

$$K(\mathcal{G}) \supseteq \text{Ker}(\theta)' \quad \text{and} \quad K(\mathcal{G}) \subseteq \Phi(G).$$

Thus, the quotient $\text{Ker}(\theta)/K(\mathcal{G})$ is abelian, and if $\mathcal{G}$ is torsion-free one has an isomor-
phism of pro-$p$ pairs

$$\mathcal{G}/K(\mathcal{G}) \cong (\text{Ker}(\theta)/K(\mathcal{G})) \rtimes (\mathcal{G}/\text{Ker}(\theta)),$$
namely, $G/K(G) \simeq (\text{Ker}(\theta)/K(G)) \times (G/\text{Ker}(\theta))$ (where the action is induced by $\theta$, in the latter), and both pro-$p$ groups are endowed with the orientation induced by $\theta$ (cf. [18, Prop. 3.1]).

**Definition 2.1** Given a cyclotomic pro-$p$ pair $G = (G, \theta)$, let $\mathbb{Z}_p(1)$ denote the continuous $G$-module of rank 1 induced by $\theta$, i.e., $\mathbb{Z}_p(1) \simeq \mathbb{Z}_p$ as abelian pro-$p$ groups, and $g \cdot \lambda = \theta(g) \cdot \lambda$ for every $\lambda \in \mathbb{Z}_p(1)$. The pair $G$ is said to be Kummerian if for every $n \geq 1$ the map

$$H^1(G, \mathbb{Z}_p(1)/p^n) \to H^1(G, \mathbb{Z}_p(1)/p),$$

induced by the epimorphism of $G$-modules $\mathbb{Z}_p(1)/p^n \to \mathbb{Z}_p(1)/p$, is surjective. Moreover, $G$ is 1-smooth if $\text{Res}_H(G)$ is Kummerian for every subgroup $H \subseteq G$.

Observe that the action of $G$ on $\mathbb{Z}_p(1)/p$ is trivial, as $\text{Im}(\theta) \subseteq 1 + p\mathbb{Z}_p$. We say that a pro-$p$ group $G$ may complete into a Kummerian, or 1-smooth, pro-$p$ pair if there exists an orientation $\theta : G \to 1 + p\mathbb{Z}_p$ such that the pair $(G, \theta)$ is Kummerian, or 1-smooth.

Kummerian pro-$p$ pairs and 1-smooth pro-$p$ pairs were introduced in [9] and in [5, § 14] respectively. In [21], if $G = (G, \theta)$ is a 1-smooth pro-$p$ pair, the orientation $\theta$ is said to be 1-cyclotomic. Note that in [5, § 14.1], a pro-$p$ pair is defined to be 1-smooth if the maps (2.4) are surjective for every open subgroup of $G$, yet by a limit argument this implies also that the maps (2.4) are surjective also for every closed subgroup of $G$ (cf. [21, Cor. 3.2]).

**Remark 2.1** Let $G = (G, \theta)$ be a cyclotomic pro-$p$ pair. Then $G$ is Kummerian if, and only if, the map

$$H^1_{\text{cts}}(G, \mathbb{Z}_p(1)) \to H^1(G, \mathbb{Z}_p(1)/p),$$

induced by the epimorphism of continuous left $G$-modules $\mathbb{Z}_p(1) \to \mathbb{Z}_p(1)/p$, is surjective (cf. [21, Prop. 2.1]) — here $H^1_{\text{cts}}$ denotes continuous cochain cohomology as introduced by J. Tate in [26].

One has the following group-theoretic characterization of Kummerian torsion-free pro-$p$ pairs (cf. [9, Thm. 5.6 and Thm. 7.1] and [20, Thm. 1.2]).

**Proposition 2.2** A torsion-free cyclotomic pro-$p$ pair $G = (G, \theta)$ is Kummerian if and only if $\text{Ker}(\theta)/K(G)$ is a free abelian pro-$p$ group.

**Remark 2.3** Let $G = (G, \theta)$ be a cyclotomic pro-$p$ pair with $\theta \equiv 1$, i.e., $\theta$ is constantly equal to 1. Since $K(G) = G'$ in this case, $G$ is Kummerian if and only if the quotient $G/G'$ is torsion-free. Hence, by Proposition 2.2, $G$ is 1-smooth if and only if $H/H'$ is torsion-free for every subgroup $H \subseteq G$. Pro-$p$ groups with such property are called absolutely torsion-free, and they were introduced by T. Würfel in [30]. In particular, if $G = (G, \theta)$ is a 1-smooth pro-$p$ pair (with $\theta$ non-trivial), then $\text{Res}_{\text{Ker}(\theta)}(G) = (\text{Ker}(\theta), 1)$ is again 1-smooth, and thus $\text{Ker}(\theta)$ is absolutely torsion-free. Hence, a pro-$p$ group which may complete into a 1-smooth pro-$p$ pair is an absolutely torsion-free-by-cyclic pro-$p$ group.
Example 2.4
(a) A cyclotomic pro-$p$ pair $(G, \theta)$ with $G$ a free pro-$p$ group is 1-smooth for any orientation $\theta$: $G \to 1 + p\mathbb{Z}_p$ (cf. [21, § 2.2]).

(b) A cyclotomic pro-$p$ pair $(G, \theta)$ with $G$ an infinite Demushkin pro-$p$ group is 1-smooth if and only if $\theta: G \to 1 + p\mathbb{Z}_p$ is defined as in [14, Thm. 4] (cf. [9, Thm. 7.6]). E.g., if $G$ has a minimal presentation

$$G = \left\{ x_1, \ldots, x_d \mid x_1^{p^d}, [x_1, x_2] \cdots [x_{d-1}, x_d] = 1 \right\}$$

with $f \geq 1$ (and $f \geq 2$ if $p = 2$), then $\theta(x_2) = (1 - p^f)^{-1}$, while $\theta(x_i) = 1$ for $i \neq 2$.

(c) For $p \neq 2$ let $G$ be the pro-$p$ group with minimal presentation

$$G = \langle x, y, z \mid [x, y] = z^p \rangle.$$

Then the pro-$p$ pair $(G, \theta)$ is not Kummerian for any orientation $\theta$: $G \to 1 + p\mathbb{Z}_p$ (cf. [9, Thm. 8.1]).

(d) Let

$$H = \left\{ \begin{pmatrix} a & b & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{Z}_p \right\}$$

be the Heisenberg pro-$p$ group. The pair $(H, 1)$ is Kummerian, as $H/H' \cong \mathbb{Z}_p^2$, but $H$ is not absolutely torsion-free. In particular, $H$ can not complete into a 1-smooth pro-$p$ pair (cf. [18, Ex. 5.4]).

(e) The only 1-smooth pro-$p$ pair $(G, \theta)$ with $G$ a finite $p$-group is the cyclic group of order $2 G \cong \mathbb{Z}/2$, endowed with the only non-trivial orientation $\theta: G \to \{\pm 1\} \subseteq 1 + 2\mathbb{Z}_2$ (cf. [9, Ex. 3.5]).

Remark 2.5
By Example 2.4–(e), if $G = (G, \theta)$ is a torsion-free 1-smooth pro-$p$ pair, then $G$ is torsion-free.

A torsion-free pro-$p$ pair $G = (G, \theta)$ is said to be $\theta$-abelian if the following equivalent conditions hold:

(i) $\text{Ker}(\theta)$ is a free abelian pro-$p$ group, and $G \cong \text{Ker}(\theta) \rtimes (G/\text{Ker}(\theta))$;

(ii) $Z(G)$ is a free abelian pro-$p$ group, and $Z(G) = \text{Ker}(\theta)$;

(iii) $G$ is Kummerian and $K(G) = \{1\}$

(cf. [17, Prop. 3.4] and [20, § 2.3]). Explicitly, a torsion-free pro-$p$ pair $G = (G, \theta)$ is $\theta$-abelian if and only if $G$ has a minimal presentation

$$G = \langle x_0, x_i, i \in I \mid [x_0, x_i] = x_j^q, [x_i, x_j] = 1 \forall i, j \in I \rangle \cong \mathbb{Z}_p^I \rtimes \mathbb{Z}_p$$

for some set $I$ and some $p$-power $q$ (possibly $q = p^\infty = 0$), and in this case $\text{Im}(\theta) = 1 + q\mathbb{Z}_p$. In particular, a $\theta$-abelian pro-$p$ pair is also 1-smooth, as every open subgroup $U$ of $G$ is again isomorphic to $\mathbb{Z}_p^I \rtimes \mathbb{Z}_p$, with action induced by $\theta|_U$, and therefore $\text{Res}_U(G)$ is $\theta|_U$-abelian.

Remark 2.6
From [9, Thm. 5.6], one may deduce also the following group-theoretic characterization of Kummerian pro-$p$ pairs: a pro-$p$ group $G$ may complete into a Kummerian oriented pro-$p$ group if, and only if, there exists an epimorphism of pro-$p$ groups.
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2.2 The Galois case

Let $\varphi : G \to G$ such that $G$ has a minimal presentation (2.5), and $\operatorname{Ker}(\varphi)$ is contained in the Frattini subgroup of $G$ (cf., e.g., [22, Prop. 3.11]).

Remark 2.7 If $G \cong \mathbb{Z}_p$, then the pair $(G, \theta)$ is $\theta$-abelian, and thus also 1-smooth, for any orientation $\theta : G \to 1 + p\mathbb{Z}_p$.

On the other hand, if $G = (G, \theta)$ is a $\theta$-abelian pro-$p$ pair with $d(G) \geq 2$, then $\theta$ is the only orientation which may complete $G$ into a 1-smooth pro-$p$ pair. Indeed, let $G' = (G, \theta')$ be a cyclotomic pro-$p$ pair, with $\theta' : G \to 1 + p\mathbb{Z}_p$ different to $\theta$, and let $\{x_0, x_i, i \in I\}$ be a minimal generating set of $G$ as in the presentation (2.5) — thus, $\theta(x_i) = 1$ for all $i \in I$, and $\theta(x_0) \in 1 + q\mathbb{Z}_p$. Then for some $i \in I$ one has $\theta' \mid_H \neq \theta \mid_H$, with $H$ the subgroup of $G$ generated by the two elements $x_0$ and $x_i$. In particular, one has $\theta([x_0, x_i]) = \theta'([x_0, x_i]) = 1$.

Suppose that $G'$ is 1-smooth. If $\theta'(x_i) \neq 1$, then

$$x_i^q = x_i \cdot x_i^q \cdot x_i^{-1} = (x_i^q)^{\theta'(x_i)} = x_i^{q \theta'(x_i)},$$

hence $x_i^{q(1 - \theta'(x_i))} = 1$, a contradiction as $G$ is torsion-free by Remark 2.5. If $\theta'(x_i) = 1$ then necessarily $\theta'(x_0) \neq \theta(x_0)$, and thus

$$x_i^{\theta(x_0)} = x_0 \cdot x_i \cdot x_0^{-1} = x_i^{\theta(x_0)},$$

hence $x_i^{\theta(x_0) - \theta'(x_0)} = 1$, again a contradiction as $G$ is torsion-free. (See also [21, Cor. 3.4].)

2.2 The Galois case

Let $\mathbb{K}$ be a field containing a root of 1 of order $p$, and let $\mu_{p^\infty}$ denote the group of roots of 1 of order a $p$-power contained in the separable closure of $\mathbb{K}$. Then $\mu_{p^\infty} \subseteq \mathbb{K}(p)$, and the action of the maximal pro-$p$ Galois group $G_{\mathbb{K}}(p) = \operatorname{Gal}(\mathbb{K}(p)/\mathbb{K})$ on $\mu_{p^\infty}$ induces a continuous homomorphism

$$\theta_{\mathbb{K}} : G_{\mathbb{K}}(p) \to 1 + p\mathbb{Z}_p$$

— called the pro-$p$ cyclotomic character of $G_{\mathbb{K}}(p)$ —, as the group of the automorphisms of $\mu_{p^\infty}$ which fix the roots of order $p$ is isomorphic to $1 + p\mathbb{Z}_p$ (see, e.g., [8, p. 202] and [9, § 4]). In particular, if $\mathbb{K}$ contains a root of 1 of order $p^k$ for $k \geq 1$, then $\operatorname{Im}(\theta_{\mathbb{K}}) \subseteq 1 + p^k\mathbb{Z}_p$.

Set $G_{\mathbb{K}} = (G_{\mathbb{K}}(p), \theta_{\mathbb{K}})$. Then by Kummer theory one has the following (see, e.g., [9, Thm. 4.2]).

Theorem 2.8 Let $\mathbb{K}$ be a field containing a root of 1 of order $p$. Then $G_{\mathbb{K}} = (G_{\mathbb{K}}(p), \theta_{\mathbb{K}})$ is 1-smooth.

1-smooth pro-$p$ pairs share the following properties with maximal pro-$p$ Galois groups of fields.

Example 2.9 (a) The only finite $p$-group which occurs as maximal pro-$p$ Galois group for some field $\mathbb{K}$ is the cyclic group of order 2, and this follows from the pro-$p$ version of the Artin-Schreier Theorem (cf. [1]). Likewise, the only finite $p$-group
which may complete into a 1-smooth pro-$p$ pair, is the cyclic group of order 2 (endowed with the only non-trivial orientation onto \{$\pm 1$\}), as it follows from Example 2.4–(e) and Remark 2.5.

(b) If $x$ is an element of $G_\mathbb{K}(2)$ for some field $\mathbb{K}$ and $x$ has order 2, then $x$ self-centralizes (cf. [4, Prop. 2.3]). Likewise, if $x$ is an element of a pro-$2$ group $G$ which may complete into a 1-smooth pro-2 pair, then $x$ self-centralizes (cf. [21, § 6.1]).

2.3 Bloch-Kato and the Smoothness Conjecture

A non-negatively graded algebra $A_\bullet = \bigoplus_{n \geq 0} A_n$ over a field $F$, with $A_0 = F$, is called a quadratic algebra if it is 1-generated — i.e., every element is a combination of products of elements of degree 1 —, and its relations are generated by homogeneous relations of degree 2. One has the following definitions (cf. [5, Def. 14.21] and [17, § 1]).

**Definition 2.2** Let $G$ be a pro-$p$ group, and let $n \geq 1$. Cohomology classes in the image of the natural cup-product

$$H^1(G, \mathbb{Z}/p) \times \ldots \times H^1(G, \mathbb{Z}/p) \xrightarrow{\cup} H^n(G, \mathbb{Z}/p)$$

are called symbols (relative to $\mathbb{Z}/p$, viewed as trivial $G$-module).

(i) If for every open subgroup $U \subseteq G$ every element $\alpha \in H^n(U, \mathbb{Z}/p)$, for every $n \geq 1$, can be written as

$$\alpha = \text{cor}_{V_i, U}^n(\alpha_1) + \ldots + \text{cor}_{V_i, U}^n(\alpha_r),$$

with $r \geq 1$, where $\alpha_i \in H^n(V_i, \mathbb{Z}/p)$ is a symbol and

$$\text{cor}_{V_i, U}^n : H^n(V_i, \mathbb{Z}/p) \to H^n(U, \mathbb{Z}/p)$$

is the corestriction map (cf. [16, Ch. I, § 5]), for some open subgroups $V_i \subseteq U$, then $G$ is called a weakly Bloch-Kato pro-$p$ group.

(ii) If for every closed subgroup $H \subseteq G$ the $\mathbb{Z}/p$-cohomology algebra

$$H^\bullet(H, \mathbb{Z}/p) = \bigoplus_{n \geq 0} H^n(H, \mathbb{Z}/p),$$

endowed with the cup-product, is a quadratic algebra over $\mathbb{Z}/p$, then $G$ is called a Bloch-Kato pro-$p$ group. As the name suggests, a Bloch-Kato pro-$p$ group is also weakly Bloch-Kato.

By the Norm Residue Theorem, if $\mathbb{K}$ contains a root of unity of order $p$, then the maximal pro-$p$ Galois group $G_\mathbb{K}(p)$ is Bloch-Kato. The pro-$p$ version of the “Smoothness Conjecture”, formulated by De Clerq and Florence, states that being 1-smooth is a sufficient condition for a pro-$p$ group to be weakly Bloch-Kato (cf. [5, Conj. 14.25]).

**Conjecture 2.10** Let $\mathcal{G} = (G, \theta)$ be a 1-smooth pro-$p$ pair. Then $G$ is weakly Bloch-Kato.

In the case of $\mathcal{G} = G_\mathbb{K}$ for some field $\mathbb{K}$ containing a root of 1 of order $p$, using Milnor $K$-theory one may show that the weak Bloch-Kato condition implies that $H^\bullet(G, \mathbb{Z}/p)$ is 1-generated (cf. [5, Rem. 14.26]). In view of Theorem 2.8, a positive answer to the
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Smoothness Conjecture would provide a new proof of the surjectivity of the norm residue isomorphism, i.e., the "surjectivity" half of the Bloch-Kato conjecture (cf. [5, § 1.1]).

Conjecture 2.10 has been settled positively for the following classes of pro-$p$ groups.

(a) Finite $p$-groups: indeed, if $G = (G, \theta)$ is a 1-smooth pro-$p$ pair with $G$ a finite (non-trivial) $p$-group, then by Example 2.4–(e) $p = 2$, $G$ is a cyclic group of order two and $\theta: G \to \{\pm 1\}$, so that $G \cong (\text{Gal}(\mathbb{C}/\mathbb{R}), \theta_{\mathbb{R}})$, and $G$ is Bloch-Kato.

(b) Analytic pro-$p$ groups: indeed if $G = (G, \theta)$ is a 1-smooth pro-$p$ pair with $G$ a $p$-adic analytic pro-$p$ group, then by [18, Thm. 1.1] $G$ is locally uniformly powerful and thus Bloch-Kato (see § 3.1 below).

(c) Pro-$p$ completions of right-angled Artin groups: indeed, in [25] it is shown that if $G = (G, \theta)$ is a 1-smooth pro-$p$ pair with $G$ the pro-$p$ completion of a right-angled Artin group induced by a simplicial graph $\Gamma$, then necessarily $\theta$ is trivial and $\Gamma$ has the diagonal property — namely, $G$ may be constructed starting from free pro-$p$ groups by iterating the following two operations: free pro-$p$ products, and direct products with $\mathbb{Z}_p$ —, and thus $G$ is Bloch-Kato (cf. [25, Thm. 1.2]).

3 Normal abelian subgroups

3.1 Powerful pro-$p$ groups

Definition 3.1 A finitely generated pro-$p$ group $G$ is said to be powerful if one has $G' \subseteq G^p$, and also $G' \subseteq G^4$ if $p = 2$. A powerful pro-$p$ group which is also torsion-free and finitely generated is called a uniformly powerful pro-$p$ group.

For the properties of powerful and uniformly powerful pro-$p$ groups we refer to [6, Ch. 4].

A pro-$p$ group whose finitely generated subgroups are uniformly powerful, is said to be locally uniformly powerful. As mentioned in the Introduction, a pro-$p$ group $G$ is locally uniformly powerful if, and only if, $G$ has a minimal presentation (2.5) — i.e., $G$ is locally powerful if, and only if, there exists an orientation $\theta: G \to 1 + p\mathbb{Z}_p$ such that $(G, \theta)$ is a torsion-free $\theta$-abelian pro-$p$ pair (cf. [17, Thm. A] and [3, Prop. 3.5]).

Therefore, a locally uniformly powerful pro-$p$ group $G$ comes endowed authentically with an orientation $\theta: G \to 1 + p\mathbb{Z}_p$ such that $G = (G, \theta)$ is a 1-smooth pro-$p$ pair. In fact, finitely generated locally uniformly powerful pro-$p$ groups are precisely those uniformly powerful pro-$p$ groups which may complete into a 1-smooth pro-$p$ pair (cf. [18, Prop. 4.3]).

Proposition 3.1 Let $G = (G, \theta)$ be a 1-smooth torsion-free pro-$p$ pair. If $G$ is locally powerful, then $G$ is $\theta$-abelian, and thus $G$ is locally uniformly powerful.

It is well-known that the $\mathbb{Z}/p$-cohomology algebra of a pro-$p$ group $G$ with minimal presentation (2.5) is the exterior $\mathbb{Z}/p$-algebra $H^*(H, \mathbb{Z}/p) \cong \bigwedge_{n \geq 0} H^1(H, \mathbb{Z}/p)$.
— if \( p = 2 \) then \( \wedge_{n \geq 0} V \) is defined to be the quotient of the tensor algebra over \( \mathbb{Z}/p \) generated by \( V \) by the two-sided ideal generated by the elements \( v \otimes v, v \in V \), so that \( H^*(G, \mathbb{Z}/p) \) is quadratic. Moreover, every subgroup \( H \subseteq G \) is again locally uniformly powerful, and thus also \( H^*(H, \mathbb{Z}/p) \) is quadratic. Hence, a locally uniformly powerful pro-\( p \) group is Bloch-Kato.

3.2 Normal abelian subgroups of maximal pro-\( p \) Galois groups

Let \( K \) be a field containing a root of 1 of order \( p \) (and also \( \sqrt{-1} \) if \( p = 2 \)). In Galois theory one has the following result, due to A. Engler, J. Koenigsmann and J. Nogueira (cf. [10] and [11]).

**Theorem 3.2** Let \( K \) be a field containing a root of 1 of order \( p \) (and also \( \sqrt{-1} \) if \( p = 2 \)), and suppose that the maximal pro-\( p \) Galois group \( G_K(p) \) of \( K \) is not isomorphic to \( \mathbb{Z}_p \). Then \( G_K(p) \) contains a unique maximal abelian normal subgroup. By [21, Thm. 7.7], such a maximal abelian normal subgroup coincides with the \( \theta_K \)-center \( Z(G_K) \) of the pro-\( p \) pair \( (G_K(p), \theta_K) \) induced by the pro-\( p \) cyclotomic character \( \theta_K \) (cf. § 2.2). Moreover, the field \( K \) admits a \( p \)-Henselian valuation with residue characteristic not \( p \) and non-\( p \)-divisible value group, such that the residue field \( \kappa \) of such a valuation gives rise to the cyclotomic pro-\( p \) pair \( G\kappa \) isomorphic to \( G_K/Z(G_K) \), and the induced short exact sequence of pro-\( p \) groups

\[
\{1\} \rightarrow Z(G_K) \rightarrow G_K(p) \rightarrow G\kappa(p) \rightarrow \{1\} \tag{3.1}
\]
splits (cf. [11, § 1] and [8, Ex. 22.1.6] — for the definitions related to \( p \)-henselian valuations of fields we direct the reader to [8, § 15.3]). In particular, \( G_K(p)/Z(G_K) \) is torsion-free.

**Remark 3.3** By [21, Thm. 1.2 and Thm. 7.7], Theorem 3.2 and the splitting of (3.1) generalize to 1-smooth pro-\( p \) pairs whose underlying pro-\( p \) group is Bloch-Kato. Namely, if \( G = (G, \theta) \) is a 1-smooth pro-\( p \) pair with \( G \) a Bloch-Kato pro-\( p \) group, then \( Z(G) \) is the unique maximal abelian normal subgroup of \( G \), and it has a complement in \( G \).

3.3 Proof of Theorem 1.1

In order to prove Theorem 1.1 (and also Theorem 1.2 later on), we need the following result.

**Proposition 3.4** Let \( G = (G, \theta) \) be a torsion-free 1-smooth pro-\( p \) pair, with \( d(G) = 2 \) and \( G = \langle x, y \rangle \). If \( [[x, y], y] = 1 \), then \( \text{Ker}(\theta) = \langle y \rangle \) and

\[ xyx^{-1} = y^{\theta(x)}. \]

**Proof** Let \( H \) be the subgroup of \( G \) generated by \( y \) and \( [x, y] \). Recall that by Remark 2.5, \( G \) (and hence also \( H \)) is torsion-free.
If \( d(H) = 1 \) then \( H \cong \mathbb{Z}_p \), as \( H \) is torsion-free. Moreover, \( H \) is generated by \( y \) and \( x^{-1}y, \) and thus \( xHx^{-1} \subseteq H. \) Therefore, \( x \) acts on \( H \cong \mathbb{Z}_p \) by multiplication by \( 1 + p\lambda \) for some \( \lambda \in \mathbb{Z}_p. \) If \( \lambda = 0 \) then \( G \) is abelian, and thus \( G \cong \mathbb{Z}_p^2 \) as it is absolutely torsion-free, and \( \theta \equiv 1 \) by Remark 2.7. If \( \lambda \neq 0 \) then \( x \) acts non-trivially on the elements of \( H, \) and thus \( \langle x \rangle \cap H = \{1\} \) and \( G = H \rtimes \langle x \rangle \) by (2.5), \( (G, \theta') \) is a \( \theta' \)-abelian pro-\( p \) pair, with \( \theta' : G \rightarrow 1 + p\mathbb{Z}_p \) defined by \( \theta'(x) = 1 + p\lambda \) and \( \theta'(y) = 1. \) By Remark 2.7, one has \( \theta' \equiv \theta, \) and thus \( \theta(x) = 1 + p\lambda \) and \( \theta(y) = 1. \)

If \( d(H) = 2, \) then \( H \) is abelian by hypothesis, and torsion-free, and thus \( (H, \theta') \) is \( \theta' \)-abelian, with \( \theta' \equiv 1 : H \rightarrow 1 + p\mathbb{Z}_p \) trivial. By Remark 2.7, one has \( \theta' = \theta|_H, \) and thus \( y, [x, y] \in \text{Ker}(\theta). \) Now put \( z = [x, y] \) and let \( U \) be the open subgroup of \( G \) generated by \( x, z, t. \) Clearly, \( \text{Res}_U(G) \) is again 1-smooth. By hypothesis one has \( z^{y} = z, \) and hence commutator calculus yields

\[
[x, t] = [x, y^p] = z\cdot z^{y} \cdots z^{y^{p-1}} = z^p. \tag{3.2}
\]

Put \( \lambda = 1 - \theta(x)^{-1} \in p\mathbb{Z}_p. \) Since \( t \in \text{Ker}(\theta), \) by (2.1) \( [x, t] \cdot t^{-\lambda} \) lies in \( K(\text{Res}_U(G)). \) Since \( t \) and \( z \) commute, from (3.2) one deduces

\[
[x, t]t^{-\lambda} = z^p\cdot t^{-\lambda} = z^p\cdot t^{-\lambda/p} = \left(z^p(t^{-\lambda/p})\right)^p \in K(\text{Res}_U(G)). \tag{3.3}
\]

Moreover, \( z^p(t^{-\lambda/p}) \in \text{Ker}(\theta|_U). \) Since \( \text{Res}_U(G) \) is 1-smooth, by Proposition 2.2 the quotient \( \text{Ker}(\theta|_U)/K(\text{Res}_U(G)) \) is a free abelian pro-\( p \) group, and therefore (3.3) implies that also \( z^p(t^{-\lambda/p}) \) is an element of \( K(\text{Res}_U(G)). \)

Since \( K(\text{Res}_U(G)) \subseteq \Phi(U), \) one has \( z \equiv t^{\lambda/p} \mod \Phi(U). \) Then by [6, Prop. 1.9] \( d(U) = 2 \) and \( U \) is generated by \( x \) and \( t. \) Since \( [x, t] \in U^p \) by (3.2), the pro-\( p \) group \( U \) is powerful. Therefore, \( \text{Res}_U(G) \) is \( \theta|_U \)-abelian by Proposition 3.1. In particular, the subgroup \( K(\text{Res}_U(G)) \) is trivial, and thus

\[
[x, y] = z = t^{\lambda/p} = y^{1-\theta(x)^{-1}},
\]

and the claim follows.

\[\square\]

Proposition 3.4 is a generalization of [18, Prop. 5.6].

**Theorem 3.5** Let \( G = (G, \theta) \) be a torsion-free 1-smooth pro-\( p \) pair, with \( d(G) \geq 2. \)

(i) The \( \theta \)-center \( Z(G) \) is the unique maximal abelian normal subgroup of \( G. \)

(ii) The quotient \( G/Z(G) \) is a torsion-free pro-\( p \) group.

\[\text{Proof} \quad \text{Recall that} \ G \text{ is torsion-free by Remark 2.5. Since} \ Z(G) \text{ is an abelian normal subgroup of} \ G, \text{ by definition, in order to prove (i) we need to show that if} \ A \subset Z(G). \text{ then} \ A \subseteq Z(G). \]

First, we show that \( A \subseteq \text{Ker}(\theta). \) If \( A \cong \mathbb{Z}_p, \) let \( y \) be a generator of \( A. \) For every \( x \in G \) one has \( xyyx^{-1} \in A, \) and thus \( xyyx^{-1} = y^\lambda, \) for some \( \lambda \in 1 + p\mathbb{Z}_p. \) Let \( H \) be the subgroup of \( G \) generated by \( x \) and \( y, \) for some \( x \in G \) such that \( d(H) = 2. \) Then the pair \( (H, \theta') \) is \( \theta' \)-abelian for some orientation \( \theta' : H \rightarrow 1 + p\mathbb{Z}_p \) such that \( y \in \text{Ker}(\theta'), \) as \( H \) has a presentation as in (2.5). Since both \( \text{Res}_H(G) \) and \( (H, \theta') \) are 1-smooth pro-\( p \) pairs, by Remark 2.7 one has \( \theta' = \theta|_H, \) and thus \( A \subseteq \text{Ker}(\theta). \)
If \( A \neq \mathbb{Z}_p \), then \( A \) is a free abelian pro-\( p \) group with \( d(A) \geq 2 \), as \( G \) is torsion-free. Therefore, by Remark 2.3 the pro-\( p \) pair \((A, 1)\) is 1-smooth. Since also \( \text{Res}_A(G) \) is 1-smooth, Remark 2.7 implies that \( \theta|_A = 1 \), and hence \( A \subseteq \text{Ker}(\theta) \).

Now, for arbitrary elements \( x \in G \) and \( y \in A \), put \( z = [x, y] \). Since \( A \) is normal in \( G \), one has \( z \in A \), and since \( A \) is abelian, one has \([z, y] = 1\). Then Proposition 3.4 applied to the subgroup of \( G \) generated by \( \{x, y\} \) yields \( xyx^{-1} = x^{\theta(x)} \), and this completes the proof of statement (i).

In order to prove statement (ii), suppose that \( y^p \in Z(G) \) for some \( y \in G \). Then \( y^p \in \text{Ker}(\theta) \), and since \( \text{Im}(\theta) \) has no non-trivial torsion, also \( y \) lies in \( \text{Ker}(\theta) \). Since \( G \) is torsion-free by Remark 2.5, \( y^p \neq 1 \). Let \( H \) be the subgroup of \( G \) generated by \( y \) and \( x \), for some \( x \in G \) such that \( d(H) \geq 2 \). Since \( xy^p x^{-1} = (y^p)^{\theta(x)} \), commutator calculus yields
\[
y^p(x^{1-\theta(x)^{-1}}) = [x, y^p] = [x, y] \cdot [x, y]^p \cdot \cdots \cdot [x, y]^p = 1.
\]
(3.4)

Put \( z = [x, y] \), and let \( S \) be the subgroup of \( H \) generated by \( y, z \). Clearly, \( \text{Res}_S(G) \) is 1-smooth, and since \( y, z \in \text{Ker}(\theta) \), one has \( \theta|_S = 1 \), and thus \( S/S' \) is a free abelian pro-\( p \) group by Remark 2.3. From (3.4) one deduces
\[
y^{p(1-\theta(x)^{-1})} \cdot z^{-p} \equiv \left(y^{1-\theta(x)^{-1}} \cdot z^{-1}\right)^p \equiv 1 \mod S'. \tag{3.5}
\]
Since \( S/S' \) is torsion-free, (3.5) implies that \( z \equiv y^{1-\theta(x)^{-1}} \mod \Phi(S) \), so that \( S \) is generated by \( y \), and \( S \approx \mathbb{Z}_p \), as \( G \) is torsion-free. Therefore, \( S' = \{1\} \), and (3.5) yields \( [x, y] = y^{1-\theta(x)^{-1}} \), and this completes the proof of statement (ii).

\begin{remark}
Let \( G \) be a pro-\( p \) group isomorphic to \( \mathbb{Z}_p \), and let \( \theta \colon G \to 1 + p\mathbb{Z}_p \) be a non-trivial orientation. Then by Example 2.4–(a), \( G = (G, \theta) \) is 1-smooth. Since \( G \) is abelian and \( \theta(x) \neq 1 \) for every \( x \in G \), \( x \neq 1 \), \( Z(G) = \{1\} \), still every subgroup of \( G \) is normal and abelian.

In view of the splitting of (3.1) (and in view of Remark 3.3), it seems natural to ask the following question.
\end{remark}

\begin{question}
Let \( G = (G, \theta) \) be a torsion-free 1-smooth pro-\( p \) pair, with \( d(G) \geq 2 \). Is the pro-\( p \) pair \( G/Z(G) = (G/Z(G), \theta) \) 1-smooth? Does the short exact sequence of pro-\( p \) groups
\[
1 \longrightarrow Z(G) \longrightarrow G \longrightarrow G/Z(G) \longrightarrow 1
\]
split?
\end{question}

If \( G = (G, \theta) \) is a torsion-free pro-\( p \) pair, then either \( \text{Ker}(\theta) = G \), or \( \text{Im}(\theta) \approx \mathbb{Z}_p \), hence in the former case one has \( G \cong \text{Ker}(\theta) \rtimes (G/\text{Ker}(\theta)) \), as the right-hand factor is isomorphic to \( \mathbb{Z}_p \), and thus \( p \)-projective (cf. [16, Ch. III, § 5]). Since \( Z(G) \subseteq Z(\text{Ker}(\theta)) \) (and \( Z(G) = Z(G) \) if \( \text{Ker}(\theta) = G \)), and since \( \text{Ker}(\theta) \) is absolutely torsion-free if \( G \) is 1-smooth, Question 3.7 is equivalent to the following question (of its own group-theoretic interest): if \( G \) is an absolutely torsion-free pro-\( p \) group, does \( G \) split as direct product
\[
G \cong Z(G) \times (G/Z(G))?
\]
One has the following partial answer (cf. [30, Prop. 5]): if \( G \) is absolutely torsion-free, and \( Z(G) \) is finitely generated, then \( \Phi_n(G) = Z(\Phi_n(G)) \times H \), for some \( n \geq 1 \) and some subgroup \( H \subseteq \Phi_n(G) \) (here \( \Phi_n(G) \) denotes the iterated Frattini series of \( G \), i.e., \( \Phi_1(G) = G \) and \( \Phi_{n+1}(G) = \Phi(\Phi_n(G)) \) for \( n \geq 1 \)).

4 Solvable pro-\( p \) groups

4.1 Solvable pro-\( p \) groups and maximal pro-\( p \) Galois groups

Recall that a (pro-\( p \)) group \( G \) is said to be meta-abelian if there is a short exact sequence

\[
\{1\} \longrightarrow N \longrightarrow G \longrightarrow \overline{G} \longrightarrow \{1\}
\]

such that both \( N \) and \( \overline{G} \) are abelian; or, equivalently, if the commutator subgroup \( G' \) is abelian. Moreover, a pro-\( p \) group \( G \) is solvable if the derived series \( (G^n)_{n \geq 1} \) of \( G \) is finite, namely \( G^{(N+1)} = \{1\} \) for some finite \( N \).

Example 4.1 A non-abelian locally uniformly powerful pro-\( p \) group \( G \) is meta-abelian: if \( \theta : G \rightarrow 1 + p\mathbb{Z}_p \) is the associated orientation, then \( G' \subseteq \ker(\theta)^p \), and thus \( G' \) is abelian.

In Galois theory one has the following result by R. Ware (cf. [28, Thm. 3], see also [13] and [17, Thm. 4.6]).

Theorem 4.2 Let \( \mathbb{K} \) be a field containing a root of 1 of order \( p \) (and also \( \sqrt{-1} \) if \( p = 2 \)). If the maximal pro-\( p \) Galois group \( G_{\mathbb{K}}(p) \) is solvable, then \( G_{\mathbb{K}} \) is \( \theta_{\mathbb{K}} \)-abelian.

4.2 Proof of Theorem 1.2 and Corollary 1.3

In order to prove Theorem 1.2, we prove first the following intermediate results — a consequence of Würfel’s result [30, Prop. 2] and of [18, Prop. 6.11] —, which may be seen as the “1-smooth analogue” of [28, Thm. 2].

Proposition 4.3 Let \( G = (G, \theta) \) be a torsion-free 1-smooth pro-\( p \) pair. If \( G \) is meta-abelian, then \( G \) is \( \theta \)-abelian.

Proof Assume first that \( \theta \equiv 1 \) — i.e., \( G \) is absolutely torsion-free (cf. Remark 2.3). Then \( G \) is a free abelian pro-\( p \) group by [30, Prop. 2].

Assume now that \( \theta \neq 1 \). Since \( G \) is 1-smooth, also \( \text{Res}_{\text{Ker}(\theta)}(G) \) and \( \text{Res}_{\text{Ker}(\theta)^p}(G) \) are 1-smooth pro-\( p \) pairs, and thus \( \text{Ker}(\theta) \) and \( \text{Ker}(\theta)^p \) are absolutely torsion-free. Moreover, \( \text{Ker}(\theta)^p \subseteq G' \), and since the latter is abelian, also \( \text{Ker}(\theta)^p \) is abelian, i.e., \( \text{Ker}(\theta) \) is meta-abelian. Thus \( \text{Ker}(\theta) \) is a free abelian pro-\( p \) group by [30, Prop. 2]. Consequently, for arbitrary \( y \in \text{Ker}(\theta) \) and \( x \in G \), the commutator \( [x, y] \) lies in \( \text{Ker}(\theta) \) and \( [[x, y], y] = 1 \). Therefore, Proposition 3.4 implies that \( x y x^{-1} = y^{\theta(y)} \) for every \( x \in G \) and \( y \in \text{Ker}(\theta) \), namely, \( G \) is \( \theta \)-abelian.

\[\square\]
Note that Proposition 4.3 generalizes [30, Prop. 2] from absolutely torsion-free pro-$p$ groups to 1-smooth pro-$p$ groups. From Proposition 4.3, we may deduce Theorem 1.2.

**Proposition 4.4** Let $G = (G, \theta)$ be a torsion-free 1-smooth pro-$p$ pair. If $G$ is solvable, then $G$ is locally uniformly powerful.

**Proof** Let $N$ be the positive integer such that $G^{(N)} \neq \{1\}$ and $G^{(N+1)} = \{1\}$. Then for every $1 \leq n \leq N$, the pro-$p$ pair Res$_{G^n}(G)$ is 1-smooth, and $G^{(n)}$ is solvable, and moreover $\theta|_{G^{(n)}} \equiv 1$ if $n \geq 2$.

Suppose that $N \geq 3$. Since $G^{(N-1)}$ is metabelian and $\theta|_{G^{(N-1)}} \equiv 1$, Proposition 4.3 implies that $G^{(N-1)}$ is a free abelian pro-$p$ group, and therefore $G^{(N)} = \{1\}$, a contradiction. Thus, $N \leq 2$, and $G$ is meta-abelian. Therefore, Proposition 4.3 implies that the pro-$p$ pair $G$ is $\theta$-abelian, and hence $G$ is locally uniformly powerful (cf. § 3.1).

Proposition 4.4 may be seen as the 1-smooth analogue of Ware’s Theorem 4.2. Corollary 1.3 follows from Proposition 4.4 and from the fact that a locally uniformly powerful pro-$p$ group is Bloch-Kato (cf. § 3.1).

**Corollary 4.5** Let $G = (G, \theta)$ be a torsion-free 1-smooth pro-$p$ pair. If $G$ is solvable, then $G$ is Bloch-Kato.

This settles the Smoothness Conjecture for the class of solvable pro-$p$ groups.

4.3 A Tits’ alternative for 1-smooth pro-$p$ groups

For maximal pro-$p$ Galois groups of fields one has the following Tits’ alternative (cf. [28, Cor. 1]).

**Theorem 4.6** Let $\mathbb{K}$ be a field containing a root of 1 of order $p$ (and also $\sqrt{-1}$ if $p = 2$). Then either $G_{\mathbb{K}}$ is $\theta_{\mathbb{K}}$-abelian, or $G_{\mathbb{K}}(p)$ contains a closed non-abelian free pro-$p$ group.

Actually, the above Tits’ alternative holds also for the class of Bloch-Kato pro-$p$ groups, with $p$ odd: if a Bloch-Kato pro-$p$ group $G$ does not contain any free non-abelian subgroups, then it can complete into a $\theta$-abelian pro-$p$ pair $G = (G, \theta)$ (cf. [17, Thm. B], this Tits’ alternative holds also for $p = 2$ under the further assumption that the Bockstein morphism $\beta: H^1(G, \mathbb{Z}/2) \to H^2(G, \mathbb{Z}/2)$ is trivial, see [17, Thm. 4.11]).

Clearly, a solvable pro-$p$ group contains no free non-abelian subgroups.

A pro-$p$ group is $p$-adic analytic if it is a $p$-adic analytic manifold and the map $(x, y) \mapsto x^{-1}y$ is analytic, or, equivalently, if it contains an open uniformly powerful subgroup (cf. [6, Thm. 8.32]) — e.g., the Heisenberg pro-$p$ group is analytic. Similarly to solvable pro-$p$ groups, a $p$-adic analytic pro-$p$ group does not contain a free non-abelian subgroup (cf. [6, Cor. 8.34]).

Even if there are several $p$-adic analytic pro-$p$ groups which are solvable (e.g., finitely generated locally uniformly powerful pro-$p$ groups), none of these two classes of pro-$p$ groups contains the other one: e.g.,
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(a) the wreath product $\mathbb{Z}_p \wr \mathbb{Z}_p \cong \mathbb{Z}_p \rtimes \mathbb{Z}_p$ is a meta-abelian pro-$p$ group, but it is not $p$-adic analytic (cf. [23]);

(b) if $G$ is a pro-$p$-Sylow subgroup of $\text{SL}_2(\mathbb{Z}_p)$, then $G$ is a $p$-adic analytic pro-$p$ group, but it is not solvable.

In addition, it is well-known that also for the class of pro-$p$ completions of right-angled Artin pro-$p$ groups one has a Tits’ alternative: the pro-$p$ completion of a right-angled Artin pro-$p$ group contains a free non-abelian subgroup unless it is a free abelian pro-$p$ group (i.e., unless the associated graph is complete) — and thus it is locally uniformly powerful.

In [18], it is shown that analytic pro-$p$ groups which may complete into a 1-smooth pro-$p$ pair are locally uniformly powerful. Therefore, after the results in [18] and [25], and Theorem 1.2, it is natural to ask whether a Tits’ alternative, analogous to Theorem 4.6 (and its generalization to Bloch-Kato pro-$p$ groups), holds also for all torsion-free 1-smooth pro-$p$ pairs.

**Question 4.7** Let $\mathcal{G} = (G, \theta)$ be a torsion-free 1-smooth pro-$p$ pair, and suppose that $\mathcal{G}$ is not $\theta$-abelian. Does $G$ contain a closed non-abelian free pro-$p$ group?

In other words, we are asking whether there exists torsion-free 1-smooth pro-$p$ pairs $\mathcal{G} = (G, \theta)$ such that $G$ is not analytic nor solvable, and yet it contains no free non-abelian subgroups. In view of Theorem 4.6 and of the Tits’ alternative for Bloch-Kato pro-$p$ groups [17, Thm. B], a positive answer to Question 4.7 would corroborate the Smoothness Conjecture.

Observe that — analogously to Quesion 3.7 — Question 4.7 is equivalent to asking whether an absolutely torsion-free 1-smooth pro-$p$ group which is not abelian contains a closed non-abelian free subgroup. Indeed, by Proposition 3.4 (in fact, just by [18, Prop. 5.6]), if $\mathcal{G} = (G, \theta)$ is a torsion-free 1-smooth pro-$p$ pair and $\text{Ker}(\theta)$ is abelian, then $\mathcal{G}$ is $\theta$-abelian.

Acknowledgement

The author thanks I. Efrat, J. Minac, N.D. Tǎn and Th. Weigel for working together on maximal pro-$p$ Galois groups and their cohomology; and P. Guillot and I. Snopce for the interesting discussions on 1-smooth pro-$p$ groups. Also, the author wishes to thank the editors of CMB-BMC, for their helpfulness, and the anonymous referee.

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