A note on the relation between the Contextual Fraction and CNT$_2$

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Abstract

Contextuality (or lack thereof) is a property of systems of random variables. Among the measures of the degree of contextuality, two have played important roles. One of them, Contextual Fraction (CNTF) was proposed within the framework of the sheaf-theoretic approach to contextuality, and extended to arbitrary systems in the Contextuality-by-Default approach. The other, denoted CNT$_2$, was proposed as one of the measures within the Contextuality-by-Default approach. In this note, I prove that CNTF = 2CNT$_2$ within a class of systems, called cyclic, that have played a prominent role in contextuality research.

Contextuality (or lack thereof) is a property of systems of random variables. The theory of contextuality, which in psychology has developed from the extension of the theory of selective influences, provides the means to detect this property and, more recently, to measure the degree of (non)contextuality of a system. Among several measures of the degree of contextuality that have been proposed in the literature, I consider here two: the Contextual Fraction (CNTF), proposed by Abramsky, Barbosa, and Mansfeld [1], and CNT$_2$, proposed by Kujala and Dzhafarov [2]. Both have played prominent roles. CNTF has been associated with the quantification of the computational advantages of some quantum computation processes [1]. On the other hand, CNT$_2$ has been shown to be the only measure of contextuality proposed so far that can be extended to a measure of noncontextuality, NCNT$_2$ [2, 3]. Moreover, NCNT$_2$ is the only measure of noncontextuality currently available.

In psychology, several applications of contextuality analysis are due to the emergence of the quantum cognition field (see e.g., [4, 5, 6, 7, 8, 9]), whereas other applications come from the selective influences literature. In quantum cognition, the search for contextuality has mainly served as means to advocate for the appropriateness of the quantum formalism. For example, in the study of language, the presence of contextuality may witness the lack compositionality of concepts [10]. As an extension of the theory of selective influences, the aim has been to develop a principled approach to the ubiquitous violations of marginal selectivity in psychological research (e.g., [11, 12, 13, 14, 15, 16, 17, 18, 19]). Almost all of these applications include experimental data or designs whose

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representation is in a prominent class of systems of random variables, called cyclic systems (defined below). Dzhafarov, Kujala, and Cervantes [3, 20] have completely characterized the measure \( \text{CNT}_2 \) for cyclic systems. Using their results, I present a proof that, in the class of cyclic systems, \( \text{CNTF} = 2\text{CNT}_2 \).

1 Contextuality-by-Default

This section introduces the definitions and results from the Contextuality-by-Default theory (CbD) [21, 22, 23, 2, 24, 3] that are needed for the proof. Throughout the text, I shall use the data from Cervantes and Dzhafarov [13] to illustrate the defined objects and computations. Their data come from an online experiment in which participants were asked to make two conjoint choices, one of a character and one of a characteristic to describe the character, in accordance with a storyline sketching Hans Christian Andersen’s “The Snow Queen”. The manipulations determined the pair of characters, and the pair of characteristics from which a participant made their choices.

A system of random variables is a set of double-indexed random variables \( R_{c q} \), where \( q \in Q \) denotes their content, which can be interpreted as a question to which the random variable responds, and \( c \in C \) is their context, the conditions under which it is recorded. A system can be presented as

\[
\mathcal{R} = \{ R_{c q} : c \in C, q \in Q, q \prec c \},
\]

where \( q \prec c \) denotes that content \( q \) is recorded in context \( c \). The sets of contents and contexts for the “Snow Queen” experiment, \( Q_{SQ} = \{ q_1, q_2, q_3, q_4 \} \) and \( C_{SQ} = \{ c_1, c_2, c_3, c_4 \} \), are presented in Tables 1 and 2, and \( \mathcal{R}_{SQ} \) will denote the corresponding system of random variables.

For each \( q \in Q \), the subset of the system

\[
\mathcal{R}_q = \{ R_{c q} : c \in C, q \prec c \}
\]

(2)

is referred to as the connection corresponding to content \( q \). No two random variables within a connection \( \mathcal{R}_q \) are jointly distributed; thus, they are said to be stochastically unrelated.\(^1\) Analogously, for each \( c \in C \), the subset

\[
\mathcal{R}^c = \{ R_{c q} : q \in Q, q \prec c \}
\]

(3)

is referred to as the bunch for context \( c \). The variables within a bunch are jointly distributed. That is, bunches are random vectors with a given probability distribution. Table 3 presents the joint distributions of the four bunches of \( \mathcal{R}_{SQ} \).

The class of systems known as cyclic has had a prominent role in contextuality research. Cyclic systems are the object of Bell’s theorem [25, 26], the Leggett–Garg theorem [27], and Suppes and Zanotti’s theorem [28], among many other theoretical results (e.g., [29, 30]). It is also encountered in most applications which empirically probe for contextuality (e.g., [31, 32, 13, 16, 33, 34]).

\(^1\)More generally, any two \( R_{q}^c, R_{q'}^{c'} \in \mathcal{R} \) with \( c \neq c' \) are stochastically unrelated.
Table 1: Contents of the “Snow Queen” experiment.

| Content | Choice between |
|---------|----------------|
| q1      | Gerda          |
| q2      | Beautiful      |
| q3      | The Snow Queen |
| q4      | Kind           |

Table 2: Contexts of the “Snow Queen” experiment.

| Context | Conjoint choices made |
|---------|------------------------|
| c1      | q1 and q2              |
| c2      | q2 and q3              |
| c3      | q3 and q4              |
| c4      | q4 and q1              |

Table 3: Joint distributions of each of the bunches of the “Snow Queen” experiment.

| R_{ij}^{R} | R_{ij}^{R_1} | Mar. character | R_{ij}^{R_1} | R_{ij}^{R_1} | Mar. character |
|------------|--------------|----------------|--------------|--------------|----------------|
| Beautiful (0) Unattractive (1) character | .843 .020 .864 Gerda (0) | .797 .035 .832 Troll (1) |
| Troll (1) | .029 .107 .136 Troll (1) | .018 .150 .168 Troll (1) |
| Mar. characteristic | .872 .128 Mar. characteristic | .815 .185 Mar. characteristic |

| R_{ij}^{R_2} | R_{ij}^{R_1} | Mar. character | R_{ij}^{R_1} | R_{ij}^{R_1} | Mar. character |
|--------------|--------------|----------------|--------------|--------------|----------------|
| Beautiful (0) Unattractive (1) character | .760 .111 .780 Snow Queen (0) | .135 .536 .671 Finn woman (1) |
| Finn woman (1) | .070 .150 .220 Finn Woman (1) | .320 .009 .329 Finn Woman (1) |
| Mar. characteristic | .839 .161 Mar. characteristic | .455 .545 Mar. characteristic |

Note 1. Mar. = marginal observed proportion
Note 2. Adapted from Table 3 of [13]. The value encoding each choice for the variables R_{ij} is shown in parentheses.
Moreover, it has been shown that a system without cyclic subsystems is necessarily noncontextual [3, 35]. A system $R$ is cyclic if

1. each of its contexts contains two jointly distributed binary random variables,
2. each content is measured in two contexts, and
3. there is no proper subsystem of $R$ that satisfies 1 and 2.

A cyclic system with $n$ contexts (and $n$ contents) is said to be a cyclic system of rank $n$. For any cyclic system, the contexts and contents can be rearranged and numbered so that the system can be written as

$$R_n = \{ \{ R^i_i, R^i_{i \oplus 1} \} : i = 1, \ldots, n \},$$

where $R^i_i$ stands for $R^c_i$, and $\oplus 1$ denotes cyclic shift $1 \mapsto 2, \ldots, n - 1 \mapsto n, n \mapsto 1$. Thus, the variables $\{ R^i_i, R^i_{i \oplus 1} \}$ constitute the bunch corresponding to context $c_i$. The following matrices depict three cyclic systems: a cyclic system of rank 2 (where rank refers to the number of contexts, or, equivalently, the number of contexts), a cyclic system of rank 3, and a cyclic system of rank 4. The latter gives the structure of the system $R_{SQ}$ that represents the “Snow Queen” experiment.

$$
\begin{array}{ccc}
R^1_1 & R^1_2 & c_1 \\
R^2_1 & R^2_2 & c_2 \\
q_1 & q_2 & R_2
\end{array}

\begin{array}{ccc}
R^1_1 & R^1_2 & R^1_3 & c_1 \\
R^2_1 & R^2_2 & R^2_3 & c_2 \\
R^3_1 & R^3_2 & R^3_3 & c_3 \\
q_1 & q_2 & q_3 & R_3
\end{array}

\begin{array}{ccc}
R^1_1 & R^1_2 & R^1_3 & R^1_4 & c_1 \\
R^2_1 & R^2_2 & R^2_3 & R^2_4 & c_2 \\
R^3_1 & R^3_2 & R^3_3 & R^3_4 & c_3 \\
R^4_1 & R^4_2 & R^4_3 & R^4_4 & c_4 \\
q_1 & q_2 & q_3 & q_4 & R_4
\end{array}
$$

Without loss of generality, we assume that the binary random variables in our systems take the values 0 and 1. Table 3 shows which choice of character and of characteristic is encoded as 0 and which as 1 for the description and contextuality computations for the “Snow Queen” experiment. Kujala and Dzhafarov [2] introduce some notation to produce a vectorial description of such a system. This representation is obtained by taking the probabilities of events for each random variable, for each bunch, and for each connection in the system. Let $l(\cdot)$, $b(\cdot)$, and $c(\cdot)$ denote these vectors, respectively. For describing a cyclic system, the first two vectors are written:

$$l(\cdot) = \left( \frac{\Pr(R^i_i = r^i_i)}{\Pr(R^i_{i \oplus 1} = r^i_{i \oplus 1})} \right)_{i=1,\ldots,n},$$

$$b(\cdot) = \left( \Pr(R^i_i = r^i_i, R^i_{i \oplus 1} = r^i_{i \oplus 1}) \right)_{i=1,\ldots,n},$$

with $r^i_i, r^i_{i \oplus 1} = 0, 1$.  

\footnote{Similarly, $\ominus 1$ will denote the inverse shift of $\oplus 1$.}
of the multimaximal coupling of connection \( R \), maximal possible given their individual distributions. If we denote the variables \( \{X_i\}_i \) a new set of jointly distributed random variables, one obtains the vector \( \{Y_i\}_i \) that describes a coupling of the distributions of \( X_i \) and \( Y_i \) coincide. In CbD, the couplings chosen for each connection are their multimaximal couplings. In a multimaximal coupling of a set of random variables, for any two \( Y_i, Y_j \), the probability \( \Pr(Y_i = Y_j) \) is the maximal possible given their individual distributions. If we denote the variables of the multimaximal coupling of connection \( R_{q_j} \) by

\[
T_{q_j} = \{T_i^j : c_i \in C, q_j < c_i\},
\]

one obtains the vector

\[
c_{(\cdot)} = \left( \Pr(T_i^j = r_i^j, T_i^j \ominus 1 = r_i^j \ominus 1) \right)_{i=1,\ldots,n},
\]

with \( r_i^j, r_i^j \ominus 1 = 0, 1 \), and where

\[
\begin{align*}
\Pr(T_i^j = 0, T_i^j \ominus 1 = 0) &= \min(\Pr(R_i^j = 0), \Pr(R_i^j \ominus 1 = 0)), \\
\Pr(T_i^j = 1, T_i^j \ominus 1 = 1) &= \min(\Pr(R_i^j = 1), \Pr(R_i^j \ominus 1 = 1)), \\
\Pr(T_i^j = 0, T_i^j \ominus 1 = 1) &= \Pr(R_i^j \ominus 1 = 1) - \Pr(T_i^j = 1, T_i^j \ominus 1 = 1), \\
\text{and} \quad \Pr(T_i^j = 1, T_i^j \ominus 1 = 0) &= \Pr(R_i^j = 1) - \Pr(T_i^j = 1, T_i^j \ominus 1 = 1).
\end{align*}
\]

Using this notation, Table 4 presents the multimaximal couplings for each of the four connections of system \( \mathcal{R}_{SQ} \).

The system \( \mathcal{R}_{SQ} \) is then described by vectors \( \mathbf{b}_{q_j} \) for each connection.

However, the third probability in vector \( \mathbf{b}_{q_j} \) has been modified to .021 to adjust for rounding error, so that the four shown values of the first context precisely sum to 1.

| Table 4: Distributions of the multimaximal couplings of each of the connections of the “Snow Queen” experiment. |
|---|---|---|---|---|---|---|
| $T_{q_1}$ | $T_{q_2}$ | $T_{q_3}$ | $T_{q_4}$ |
| $T_{q_1}$ | $T_{q_2}$ | $T_{q_3}$ | $T_{q_4}$ |
| Troll (0) | .832 | .032 | .964 | Beautiful (0) | .839 | .033 | .872 |
| Troll (1) | .000 | .136 | .136 | Unattractive (1) | .000 | .128 | .128 |
| Mar. characteristic | .832 | .168 | Mar. characteristic | .839 | .161 |
| $T_{q_2}$ | $T_{q_3}$ | $T_{q_4}$ | $T_{q_1}$ |
| $T_{q_2}$ | $T_{q_3}$ | $T_{q_4}$ | $T_{q_1}$ |
| Kind (0) | .475 | .000 | .475 | Snow Queen (0) | .671 | .000 | .671 |
| Evil (1) | .360 | .185 | .545 | Finn Woman (1) | .169 | .220 | .329 |
| Mar. characteristic | .815 | .185 | Mar. characteristic | .790 | .220 |

Note 1: Mar. = marginal observed proportion.

Note 2: The variable $T_i^j$ denotes the variable in the coupling corresponding to $R_{q_j}$. 

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\( T_{q_j} \) denotes the variable in the coupling corresponding to $R_{q_j}$. 

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3 The third probability in vector $\mathbf{b}_{q_j}$ has been modified to .021 to adjust for rounding error, so that the four shown values of the first context precisely sum to 1.
A system is noncontextual if it is possible to construct a coupling of \( \mathcal{R} \) that agrees simultaneously with the bunches and with the multimaximal couplings of the connections \([21, 22]\). If such a coupling exists, it can be found as the solution to a system of linear equations defined using vectors \( l_{(i)} \), \( b_{(i)} \), and \( c_{(i)} \).

First, let \( p_{(i)} \) be the concatenation of the three vectors \( l_{(i)} \), \( b_{(i)} \), and \( c_{(i)} \). Next, consider a vector \( s \) of length \( 2^m \) that is noncontextual if and only if there exists a solution to a system of linear equations defined using vectors \( l_{(i)}, b_{(i)}, c_{(i)} \). If such a coupling exists, it can be found as the solution \( h \) is a vector \( (\geq 0) \) (component-wise) such that

\[
\{ S_j^i = r_j^i : i, j = 1, \ldots, n, q_j < c_i \},
\]

with \( r_j^i = 0, 1 \). These values are the components of \( s \). Let \( M_{(i)} \) be an incidence \((0/1)\) matrix with \( 2^m \) columns labeled by the elements of \( s \) and \( 12n \) rows labeled by the events whose probabilities are the components of \( p_{(i)} \). This matrix is constructed as follows:

- If the \( u \)th component of \( p_{(i)} \) is \( \Pr(R_j^i = r_j^i) \) and the \( v \)th component of \( s \) includes the event \( \{ S_j^i = r_j^i \} \), then the cell \((u, v)\) of \( M_{(i)} \) is a 1;
- if the \( u \)th component of \( p_{(i)} \) is \( \Pr(R_j^i = r_j^i, R_j^i \oplus = r_j^i \oplus 1) \) and the \( v \)th component of \( s \) includes the event \( \{ S_j^i = r_j^i, S_j^i \oplus = r_j^i \oplus 1 \} \), then the cell \((u, v)\) is a 1;
- if the \( u \)th component of \( p_{(i)} \) is \( \Pr(T_j^i = r_j^i, T_j^i \ominus = r_j^i \ominus 1) \) and the \( v \)th component of \( s \) includes the event \( \{ S_j^i = r_j^i, S_j^i \ominus = r_j^i \ominus 1 \} \), then the cell \((u, v)\) is a 1;
- all other cells have zeroes.

A detailed description of the construction of \( M_{(i)} \) for general systems of binary random variables can be found in Dzhafarov and Kujala \([23]\).

The system \( \mathcal{R}_n \) described by \( p_{(i)}^* \), is noncontextual \([23]\) if and only if there is a vector \( h \geq 0 \) (component-wise) such that

\[
M_{(i)} h = p_{(i)}^*.
\]  

(10)

Any solution \( h^* \) provides the probability distribution of a coupling \( S_n \) of \( \mathcal{R}_n \) that contains as its marginals both the bunch distributions and the multimaximal couplings of the connections of \( \mathcal{R}_n \).

It is evident that the rows of \( M_{(i)} \) are not linearly independent. For example, one can compute

\[
\Pr(R_j^i = 0) + \Pr(R_j^i = 1) = 1,
\]
and
\[
\Pr(R_i^j = 0, R_i^{j1} = 0) + \Pr(R_i^j = 0, R_i^{j1} = 1) + \Pr(R_i^j = 1, R_i^{j1} = 0) + \Pr(R_i^j = 1, R_i^{j1} = 1) = 1.
\]
Hence, the row of \( M_{(i)} \) corresponding to \( \Pr(R_i^j = 0) \) is a linear combination of the rows corresponding to \( \Pr(R_i^j = 0, R_i^{j1} = 0) \), \( \Pr(R_i^j = 0, R_i^{j1} = 1) \), \( \Pr(R_i^j = 1, R_i^{j1} = 0) \), \( \Pr(R_i^j = 1, R_i^{j1} = 1) \), and \( \Pr(R_i^j = 1) \).

The vectorial description can be reduced by taking only a subset of the components of \( p_{(i)} \), such that the corresponding rows of \( M_{(i)} \) are linearly independent. Let us choose the following reductions of the vectors \( l_{(i)}, b_{(i)}, \) and \( c_{(i)} \) and denote them \( l, b, \) and \( c: \)
\[
\begin{align*}
  l &= (p_{(i)})_{i = 1, \ldots, n, q_j < c_i} = (\Pr(R_i^j = 1))_{i = 1, \ldots, n, q_j < c_i}, \\
  b &= (p_{(i)})_{i = 1, \ldots, n} = (\Pr(R_i^j = 1, R_i^{j1} = 1))_{i = 1, \ldots, n}, \quad \text{and} \\
  c &= (p_{(i)})_{i = 1, \ldots, n} = (\Pr(T_i^j = 1, T_i^{j1} = 1))_{i = 1, \ldots, n}.
\end{align*}
\]
The system \( R_{SQ} \) can then be described by the reduced vectors
\[
\begin{align*}
  l_{SQ}^* &= (.136, .128, .161, .220, .329, .545, .185, .168), \\
  b_{SQ}^* &= (.107, .150, .009, .150), \quad \text{and} \\
  c_{SQ}^* &= (.136, .128, .220, .185).
\end{align*}
\]
Using the reduced vectors \( l_{SQ}^*, b_{SQ}^*, c_{SQ}^* \), Kujala and Dzhafarov \cite{2} define the linear programming tasks used to compute the CbD-based measures of contextuality.

Let a system \( R_n \) be described by
\[
p^* = \begin{pmatrix} l^* \\ b^* \\ c^* \end{pmatrix},
\]
where \( l^* \) and \( b^* \) are the empirical probabilities of the system, and \( c^* \) are the probabilities found from the multimaximal couplings of each of its connections. Let \( M \) be the incidence matrix found by taking the rows of \( M_{(i)} \) corresponding to the elements of \( p^* \). Note that the system is noncontextual if and only if there is a vector \( h \geq 0 \) (component-wise) such that
\[
Mh = p^*,
\]
subject to \( h^T h = 1 \) \cite{23}. Denoting the rows of \( M \) that correspond to \( l^*, c^*, b^* \) by, respectively, \( M_l, M_b, M_c \), we can rewrite \( 13 \) in extenso:
\[
\begin{pmatrix} M_l \\ M_b \\ M_c \end{pmatrix} h = \begin{pmatrix} l^* \\ b^* \\ c^* \end{pmatrix}.
\]
For the system \( R_{SQ} \) that describes the data from the “Snow Queen” experiment, the matrix \( M \) and the corresponding submatrices \( M_l, M_b, \) and \( M_c \) are depicted in Table \( \text{II} \) in the Appendix.
The contextuality measure $\text{CNT}_2(\mathcal{R}_n)$ can then be computed solving the linear programming task

$$
\begin{array}{l}
\text{find minimize} & \text{subject to} \\
x & 1^T d & -d \leq b^* - M_0 x \leq d \\
x, d \geq 0 \\
1^T x = 1 \\
M_l x = l^* \\
M_c x = c^*
\end{array}
$$

For any solution $x^*$, $\text{CNT}_2(\mathcal{R}_n) = \|b^* - M_0 x^*\|_1 = \|p^* - M x^*\|_1$. \cite{3} showed that for a cyclic system, the value of $\text{CNT}_2(\mathcal{R})$ is a one-coordinate distance; that is, if $\mathcal{R}_n$ is contextual, there exists a solution $x^*$ such that $d^*$ has zeroes for all but one of its components. They further showed that for each coordinate $i = 1, \ldots, n$ of $d$, there is a solution $x^*$ such that $d_i$ is the nonzero component. Note that there are infinitely many solutions to the linear programming task, since there are at least two solutions. Table 2 in the Appendix presents one such solution to this task for system $\mathcal{R}_{SQ}$ representing the “Snow Queen” experiment. The contextuality measure $\text{CNT}_2$ of the system is .069.

A system is said to be consistently connected if, for any two $\mathcal{R}^c_q$, $\mathcal{R}^c_q'$, the respective distributions of $\mathcal{R}^c_q$ and $\mathcal{R}^c_q'$ coincide. This property of a system corresponds to the marginal selectivity condition in the selective influences literature in psychology, as well as to the no-disturbance or the no-signaling requirements in quantum physics. If this property is not satisfied, the system is said to be inconsistently connected. In general, a system of random variables $\mathcal{R}$ is not consistently connected. Note that whenever a system $\mathcal{R}$ is consistently connected, then for any two $\mathcal{R}^c_q$, $\mathcal{R}^c_q'$, the corresponding variables of a multimaximal coupling of $\mathcal{R}_q$ are almost always equal (that is, $\Pr(T^c_q = T^c_q') = 1$).

The contextuality measure $\text{CNTF}$ proposed in \cite{1} is defined only for consistently connected systems. However, there is a procedure that can be applied to any system of binary random variables to generate a new system $\mathcal{R}^\updownarrow$ that is consistently connected and whose contextually status is the same as that of the original system $\mathcal{R}$. Moreover, if $\mathcal{R}$ is consistently connected, then $\text{CNTF}(\mathcal{R})$ and $\text{CNTF}(\mathcal{R}^\updownarrow)$ are equal \cite{24}. The consistification of system $\mathcal{R}$ is obtained by constructing a new system $\mathcal{R}^\updownarrow$ in the following manner. First, define the set of contents $Q^\updownarrow$ of the new system as

$$
Q^\updownarrow = \{q_{ij} : c_i \in C, q_j \in Q, q_j \prec c_i\}.
$$

That is, for each content $q_j$ and each of the contexts $c_i$ in which it is recorded, we define a content $q_{ij} = "q_j$ recorded in context $c_i"$. Next, define the new set of contexts $C^\updownarrow$ as

$$
C^\updownarrow = C \sqcup Q,
$$

the disjoint union of the contexts and the contents of the system $\mathcal{R}$. Then, define the new relation

$$
\prec^\updownarrow = \{(q_{ij}, c_i) : q_j \in Q, c_i \in C, q_j \prec c_i\} \sqcup \{(q_{ij}, q_j) : q_j \in Q, c_i \in C, q_j \prec c_i\}.
$$
That is, the new content $q_{ij}$ is recorded in precisely two of the new contexts, $c_i \in C^\cap$ and $q_j \in C^\cap$. In this manner, the bunch

$$R^{c_i} = \{ R^{c_i}_{q_{ij}} : q_{ij} \in Q^\cap, q_{ij} \prec c_i \}$$

coinsides with the bunch

$$R^{c_i} = \{ R^{c_i}_{q} : q \in Q, q \prec c_i \}$$
of the original system; while the bunch

$$R^q_j = \{ R^q_j_{q_{ij}} : q_{ij} \in Q^\cap, q_{ij} \prec q_j \}$$
is constructed by defining new jointly distributed random variables $\{ R^q_j_{q_{ij}} \}_{q_{ij} \prec q_j}$ such that $R^q_j$ is the multimaximal coupling of $R^q_j$. Note that if $R$ is a cyclic system of rank $n$, then its consistified system $R^\cap$ is a consistently connected cyclic system of rank $2n$. The following matrices show the consistification of system $R_2$ and how its bunches relate to the bunches of the original system and the multimaximal couplings of its connections.

| $R_{q_{11}}^{c_1}$ | $R_{q_{12}}^{c_1}$ | $c_1$ |
|---------------------|---------------------|------|
| $R_{q_{22}}^{c_2}$ | $R_{q_{22}}^{c_2}$ | $q_2$|
| $R_{q_{21}}^{c_2}$ | $R_{q_{21}}^{c_2}$ | $c_2$|
| $q_{11}$           | $q_{12}$           | $q_{22}$ |
| $q_{21}$           | $R^1_2$            | $R^1_2$ |

If a system $R$ is consistently connected, the contextual fraction proposed by Abramsky et al. [1] can be computed solving the following linear programming task:

$$\begin{align*}
\text{find} & \quad \mathbf{z}^{\top} \mathbf{z} \\
\text{maximizing} & \quad \mathbf{M}_{(\cdot,\cdot)} \mathbf{z} \leq \mathbf{p}_{(\cdot)} \\
\text{subject to} & \quad \mathbf{z} \geq 0 \\
& \quad \mathbf{1}^{\top} \mathbf{z} \leq 1 
\end{align*}$$

(17)

For any solution $\mathbf{z}^*$, $\text{CNTF}(R) = 1 - \mathbf{1}^{\top} \mathbf{z}^*$. As with the task to compute CNT$_2$, there are infinitely many solutions to this task. The previous task is equivalent to the one proposed in [1] which uses a simpler representation of the system [24]. Now, if we consider the consistified system $R^\cap_n$ of a cyclic system $R_n$, then equality (18) is satisfied by Th. 18 of [3] and, if $R_n$ is consistently connected, equality (19) is satisfied by Th. 7 of [24].

$$\begin{align*}
\text{CNT}_2(R_n) &= \text{CNT}_2(R^\cap_n), \\
\text{CNTF}(R_n) &= \text{CNTF}(R^\cap_n).
\end{align*}$$

(18), (19)

Moreover, Th. 7 of [24] also shows that, regardless of consistent connectedness, the linear programming task to compute CNTF($R^\cap$) is equivalent to the
task in expression (17) where $p_\ast$ describes system $R^\dagger$. Hence, we will abuse notation and simply write CNT\textsubscript{2} and CNT\textsubscript{F} for the measures of the systems $R_n$ and $R_n^\dagger$, and use equality (19) as the definition of the contextual fraction for inconsistently connected systems. Table 2 presents one solution to the task (Expression 17) for the system $R_{SO}$. The contextuality measure CNT\textsubscript{F} of the system is .138.

Lastly, we will need the definition of the noncontextuality polytope and the related objects defined by [3] to describe its geometry. The noncontextuality polytope is usually defined with respect to the expected values of the variables codified with outcomes $-1/1$ and their products, rather than with respect to the probabilities. These expected values can be obtained by transforming the components of vectors $l$, $b$, and $c$ as follows

$$e_j^i = 2p_j^i - 1, \quad q_j < c_i, \quad (20)$$

$$e_{i,i\oplus1}^i = 4p_{i,i\oplus1}^i - 2p_i^i - 2p_{i\oplus1}^i + 1, \quad i = 1, \ldots, n, \quad (21)$$

$$e_{i,i\ominus1}^i = 4p_{i,i\ominus1}^i - 2p_i^i - 2p_{i\ominus1}^i + 1, \quad i = 1, \ldots, n. \quad (22)$$

The resulting vectors will be denoted as

$$\phi(l) = (e_j^i)_{i=1,\ldots,n, \ q_j < c_i},$$

$$\phi(b) = (e_{i,i\oplus1}^i)_{i=1,\ldots,n},$$

and

$$\phi(c) = (e_{i,i\ominus1}^i)_{i=1,\ldots,n}. \quad (23)$$

Given vectors $\phi(l^\ast)$ and $\phi(c^\ast)$, the noncontextuality polytope is the set

$$E_b = \left\{ \phi(b) : \exists h \text{ s.t. } \begin{pmatrix} M_l & M_b & M_c \end{pmatrix} h = \begin{pmatrix} l^\ast & b & c^\ast \end{pmatrix} \right\}. \quad (24)$$

That is, the noncontextuality polytope contains the set of expected values of the products of the two variables in each bunch that are consistent with the marginal distributions and the implied multimaximal couplings of a given system.

The polytope $E_b$ is a subset of the $n$-cube

$$C_b = [-1,1]^n, \quad (25)$$

which consists of all the possible values of $\phi(b)$. Given an arbitrary $n$-box

$$B = \prod_{i=1}^n [\min x_i, \max x_i] \subseteq C_b, \quad (26)$$

a vertex $V$ of $B$ will be called odd if its coordinates contain an odd number of $\min x_i$ and even, otherwise. In particular, a vertex of $C_b$ is even when the product of its coordinates equals 1. [3] showed that the polytope $E_b$ is the intersection of two subsets of $C_b$. For a consistently connected cyclic system,
these sets are $D_b$, defined as the $n$-demicube on the even vertices of $C_b$—that is, the convex hull of the even vertices of the $n$-cube $C_b$—and the $n$-box

$$\mathbb{R}_b = \prod_{i=1}^{n} \left[ |e_i^1 + e_i^{1+1}| - 1, 1 - |e_i^1 - e_i^{1+1}| \right].$$  \hfill (27)

2 Main theorem

**Theorem 1.** If $R_n$ is a cyclic system of rank $n$, then $\text{CNTF} = 2\text{CNT}_2$.

**Proof.** Let $R_n$ be a cyclic system of rank $n$. Note that if $R_n$ is noncontextual, both measures $\text{CNT}_2$ and $\text{CNTF}$ equal zero and, trivially, $\text{CNTF} = 2\text{CNT}_2$. Thus, we will assume that $R_n$ is contextual.

Without loss of generality, assume that the system $R_n$ is consistently connected. If not, take the consistification $R_n^1$ of $R_n$. Consider the vector $\phi(b^*)$ and the corresponding sets $C_b$, $D_b$, $R_b$, and $E_b$. Theorem 15 of [3] states that $\text{CNT}_2 = \frac{1}{2} ||\phi(b^*), E_b||_1$ is a single coordinate distance. Since the system is contextual and consistently connected, $||\phi(b^*), E_b||_1 = ||\phi(b^*), D_b||_1 > 0$. The point $\phi(b^*)$ clearly lies within a corner formed at an odd vertex of $E_b$ and cut by a facet of $D_b$ (cf. Lemmas 10 and 11 of [3]. See Figure 1 for an illustration).

Let $x^*$ be any of the $n$ points of the facet of $D_b$ that differs from $b^*$ on a single coordinate, and let $\bar{A}$ denote the line segment between $\phi(b^*)$ and $x^*$. Clearly, the length $|\bar{A}|$ equals $4\text{CNT}_2$.

Now, by Theorem 1(iii) of [1], it is possible to write $\phi(b^*) = \text{CNTF}x^{SC} + (1 - \text{CNTF})x^{NC}$, where $x^{NC}$ is a point on the surface of $D_b$, and $x^{SC}$ is a point that maximally violates the Bell inequality

$$s_1(x) = \max_{\prod_{\lambda_i = -1}^{\lambda_i = 1}} \sum_{i=1}^{n} \lambda_i x_{i,i\oplus 1} > n - 2, \hfill (28)$$

where $\lambda_i = \pm 1$ and the coordinates of $x$ are labeled following the convention to label the coordinates of $\phi(b^*)$. Note that the values $\lambda_i^*$ for which the maximum is attained define the coordinates of an odd vertex $V^*$ of $C_b$. Also, the points $x^{SC}$ which maximally violate (28) lie in a hyperplane parallel to the facet of $D_b$ facing $V^*$ and $V^*$ is one of such points (see, e.g., Lemma 11 of [3]). Thus, one can choose $x^{SC} = V^*$ and find $x^{NC}$ as the point where the line that passes through $V^*$ and $\phi(b^*)$ intersects the corresponding face of $D_b$. Let $\bar{B}$ denote the line segment between $\phi(b^*)$ and $x^{NC}$, and $\bar{C}$, the line segment between $V^*$ and $x^{NC}$. Clearly, $\text{CNTF} = |\bar{B}|/|\bar{C}|$.

\footnote{Every $n$-cube can generate two $n$-demicubes. One the $n$-demicube described in Text. The other is the covex hull of its odd vertices.}

\footnote{For a cyclic system, the odd vertices of $C_b$ are indeed the only points that maximally violate (28) and are possible points $\phi(b)$ of a system; although, they may be unfeasible due to not belonging to the $n$-box $\mathbb{R}_b$ associated to certain systems.}

11
Let $\mathbf{P}$ be the plane determined by any one edge $\bar{L}$ of $\mathbb{C}_b$ emanating from $V^*$ and segment $\bar{C}$. Consider the hyperplanes

$$\sum_{i=1}^{n} \lambda_i^* x_{i,i+1} = \sum_{i=1}^{n} \lambda_i^* e_{i,i+1}$$  \hspace{1cm} (29)

and

$$\sum_{i=1}^{n} \lambda_i^* x_{i,i+1} = n - 2, \hspace{1cm} (30)$$

Hyperplane (30) is the hyperplane that contains the facet of $\mathbb{D}_b$ facing vertex $V^*$, and hyperplane (29) is the hyperplane parallel to (30) passing through the point $\phi(b^*)$. The intersections of hyperplanes (29) and (30) with $\mathbf{P}$ determine two parallel lines. Let $y$ be the point at the intersection of (29), $\mathbf{P}$, and $\bar{L}$. By Lemma 11 of [3], the line segment $\bar{A}'$ between $y$ and the end of $\bar{L}$ has length $|\bar{A}|$. Hence, by Thales’ intercept Theorem,

$$\text{CNTF} = \frac{|\bar{B}|}{|\bar{C}|} = \frac{|\bar{A}|}{|\bar{L}|} = \frac{4\text{CNT}_2}{2}. \hspace{1cm} (31)$$

Therefore, $\text{CNTF} = 2\text{CNT}_2$.

Figure 1 illustrates the construction presented in the proof for an example consistently connected system of rank 2. Figure 2 provides an example for a system of rank 3.

3 Discussion

The proof presented in this paper improves the understanding of the relationship between the different approaches to studying contextuality. [3] showed that in cyclic systems $\text{CNT}_2$ is equal to another measure, $\text{CNT}_1$, and in [2], it was conjectured that these measures are proportional to a measure computed based on negative probabilities, $\text{CNT}_3$. If this conjecture is found to be true, the current result shows that all these different approaches to the quantification of contextuality capture essentially the same feature of cyclic systems.

It should be noted, however, that outside the class of cyclic systems, these measures are not generally the same. [3, 36] present some examples that show that $\text{CNT}_2$ and $\text{CNT}_1$ are not in general functions of each other. It is also easy to see that $\text{CNT}_2$ and $\text{CNT}_F$ do not need to be proportional for noncyclic systems.

Take for example any number $m$ of separate systems of random variables and consider the system obtained from their disjoint union. That is, take systems $R_k$, with corresponding sets of contexts and contents $C_k$ and $Q_k$, $k = 1, \ldots, m$, and construct the system

$$\mathcal{R} = \left\{ R_q^c : c \in \bigsqcup_{k=1}^{m} C_k, q \in \bigsqcup_{k=1}^{m} Q_k, q \prec c \right\}.$$
Figure 1: Illustration of the proof for a consistently connected system $\mathcal{R}_2$.

Note: Example of the noncontextuality polytope and the contextuality measures for a consistently connected cyclic system of rank 2 with marginal expectations $e_1^1 = e_2^1 = -2$ and $e_1^2 = e_2^2 = 0$. The 2-demicube $D_b$ on the even vertices of $C_b$ is the diagonal line between the points $(-1, -1)$ and $(1, 1)$. The noncontextuality polytope $E_b$ is the segment of that diagonal that intersects the box $\mathbb{R}_b$. See the text for the remaining elements of the figure.
Figure 2: Illustration of the proof for a consistently connected system $\mathcal{R}_3$.

Note: Example of the noncontextuality polytope and the contextuality measures for a consistently connected cyclic system of rank 3 with marginal expectations $e_j^i = 0$. The 3-demicube $\mathcal{D}_b$ on the even vertices of $\mathcal{C}_b$ is the tetrahedron shaded in orange. Due to equal marginal expectations, the box $\mathcal{R}_b$ coincides with the ambient cube $\mathcal{C}_b$, and the noncontextuality polytope $\mathcal{E}_b$ coincides with $\mathcal{D}_b$. The sections of the (hyper-)planes (29) and (30) intersecting the ambient cube $\mathcal{C}_b$ are shaded in blue. The section of plane $P$ between segments $C$ and $L$, and the (hyper-)plane (30) is shaded in gray. All other elements as in Figure 1 and described in text.
The following matrix shows an example of such a system, one composed of two disjoint rank 2 cyclic subsystems.

\[
\begin{array}{ccc|c}
R_1^1 & R_1^2 & \cdots & c_1 \\
R_2^1 & R_2^2 & \cdots & c_2 \\
\hline
R_3^1 & R_3^2 & \cdots & c_3 \\
R_4^1 & R_4^2 & \cdots & c_4 \\
q_1 & q_2 & q_3 & q_4 & \mathcal{R}
\end{array}
\] (32)

As shown in Cervantes and Dzhafarov [37], for these type of systems, \( \text{CNT}_2(\mathcal{R}) = \sum_{k=1}^{m} \text{CNT}_2(\mathcal{R}_k) \), whereas \( \text{CNTF}(\mathcal{R}) = \max_{k=1,\ldots,m} \text{CNTF}(\mathcal{R}_k) \).

Lastly, in addition to the measures I have mentioned above, Cervantes and Dzhafarov [37] have recently proposed a hierarchical measure of contextuality and noncontextuality based on \( \text{CNT}_2 \). In the class of cyclic systems, this hierarchical measure reduces to the contextuality measure \( \text{CNT}_2 \) and the non-contextuality measure \( \text{NCNT}_2 \).

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A  Additional tables and computations for system $\mathcal{R}_{SQ}$

Table 1 presents the incidence matrix $\mathbf{M}$ for the contextuality analysis of system $\mathcal{R}_{SQ}$. Note that the same matrix may be used for the analysis of any cyclic system of rank 4. In Table 2, example solutions $\mathbf{x}^*$ and $\mathbf{z}^*$ to the linear programming tasks to compute $\text{CNT}_2$ (see Expression 15) and $\text{CNTF}$ (Expression 17), respectively, are presented together with the submatrix of $\mathbf{M}$ whose rows are labeled by the probabilities of bunch $R^2_{SQ}$. The solution $\mathbf{x}^*$ shown in Table 2 is such that $d^*_2$ is the only $d^*_i, i = 1, \ldots, 4$, greater than zero.
Table 1: Incidence matrix $M$ for the contextuality analysis of system $R_{SQ}$.

| $s_1 = 1$, $s_2 = 1$ | 11111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111
Table 2: Example solutions to the linear programming tasks for computing CNT\(_2\) and CNTF for system \(R_{SQ}\) describing the “Snow Queen” experiment.

| \(x^\top\) | \(m_{p_{\text{pr}}}^x\) | \(m_{p_{\text{gr}}}^z\) | \(p_{\text{f}}\) |
|---|---|---|---|
| \(S_2^2 = 1, S_2^3 = 1\) | 1 (5) 1 (17) 0 (95) 0 (14) 0 (63) 1 (31) 0 (15) 0 0 0 (1) 0 (3) 0 | 0.081 0.081 0.150 |
| \(S_2^2 = 0, S_2^3 = 1\) | 0 (5) 0 (17) 0 (95) 0 (14) 0 (63) 0 (31) 1 (15) 0 0 0 (1) 0 (3) 0 | 0.129 0.070 0.070 |
| \(S_2^2 = 1, S_2^3 = 0\) | 0 (5) 0 (17) 1 (95) 0 (14) 0 (63) 0 (31) 0 (15) 1 1 1 (1) 1 (3) 1 | 0.090 0.011 0.11 |
| \(S_2^2 = 0, S_2^3 = 0\) | 0 (5) 0 (17) 0 (95) 1 (14) 0 (63) 0 (31) 0 (15) 1 1 1 (1) 1 (3) 1 | 0.700 0.700 0.769 |

Note. The left panel shows vectors \(x^\top\) and \(z^\top\). Vector \(x^*\) solves the linear programming task for computing \(\text{CNT}_2\) (Expression 15) so that \(d_2 = \text{CNT}_2(R_{SQ}) = .069\). Vector \(z^*\) solves the task to compute \(\text{CNTF}\) (Expression 17). To help readability of the nonzero components, all zero components of the solutions are omitted and replaced by the number of omitted elements inside parentheses. The lower section shows the vectors of matrix \(M_{(.)}\) corresponding to the probabilities of bunch \(R_{SQ}^2\). The elements of these vectors corresponding to the zero components of the solutions have also been replaced by the number of omitted components. The right panel shows the results of the product of the vectors of matrix \(M_{(.)}\) and the solutions to each linear programming task, as well as the actual probabilities from bunch \(R_{SQ}^2\).