ON THE EQUILIBRIUM CONFIGURATION OF THE BC-TYPE RUIJSENAARS-SCHNEIDER SYSTEM

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ABSTRACT. It is shown that the ground-state equilibrium configurations of the trigonometric BC-type Ruijsenaars-Schneider systems are given by the zeros of Askey-Wilson polynomials.

1. INTRODUCTION

The Ruijsenaars-Schneider systems [12, 5, 6] are integrable deformations of the celebrated Calogero-Moser n-particle models [10]. It is well-known that the equilibrium configurations of the Calogero-Moser models are described by the zeros of classical orthogonal polynomials such as the Hermite, Laguerre, Chebyshev, and Jacobi polynomials [2, 3, 4, 10]. This connection between the equilibria of one-dimensional integrable particle models and the locations of zeros of the classical hypergeometric orthogonal polynomials, first observed by Calogero, is closely related to a beautiful electrostatic interpretation of the zeros of orthogonal polynomials due to Stieltjes [14]. Recently, it was noticed that the equilibrium configurations of the Ruijsenaars-Schneider systems can also be described in a similar way by means of the zeros of some orthogonal polynomials [13, 11, 9]; all polynomials that appear in this context turn out to be classical in the sense that they sit somewhere in Askey’s hierarchy of (basic) hypergeometric orthogonal polynomials [1, 8]. The top of this hierarchy is formed by the Askey-Wilson polynomials [1]. (All other (basic) hypergeometric families of classical orthogonal polynomials are special (limiting) cases of the Askey-Wilson polynomials [8].) In this note we show that the zeros of these Askey-Wilson polynomials correspond to the ground-state equilibrium configurations of the trigonometric BC-type Ruijsenaars-Schneider systems introduced in Refs. [5, 6]. In the rational limit, one recovers the characterization of the ground-state equilibrium configurations of the rational BC-type Ruijsenaars-Schneider systems in terms of Wilson polynomials [15] due to Odake and Sasaki [9].

We will need to employ the following standard conventions from the theory of (basic) hypergeometric orthogonal polynomials [1, 8]: q-shifted factorials are denoted by

\[(a; q)_k := \begin{cases} 1 & \text{for } k = 0, \\ (1 - a)(1 - aq) \cdots (1 - aq^{k-1}) & \text{for } k = 1, 2, 3, \ldots, \end{cases}\]

with the convention that \((a; q)_{\infty} := \prod_{k=0}^{\infty}(1 - aq^k)\) (for \(|q| < 1\)); products of q-shifted factorials are abbreviated in the usual way via

\[(a_1, \ldots, a_r; q)_k := (a_1; q)_k \cdots (a_r; q)_k.\]
The $r+1\Phi_r$ terminating basic hypergeometric series is defined as

$$r+1\Phi_r \left[ \frac{q^{-n}, a_1, \ldots, a_r}{b_1, \ldots, b_r} \mid q; z \right] := \sum_{k=0}^{n} \frac{(q^{-n}, a_1, \ldots, a_r; q)_k}{(q, b_1, \ldots, b_r; q)_k} z^k$$

(where it is assumed that the parameters are such that denominators do not vanish). The hypergeometric degeneration of this series is given by

$$r+1\Phi_r \left[ -n, a_1, \ldots, a_r \mid b_1, \ldots, b_r, z \right] := \sum_{k=0}^{n} \frac{(-n, a_1, \ldots, a_r)_k}{(1, b_1, \ldots, b_r; q)_k} z^k,$$

where $(a_1, \ldots, a_r)_k := (a_1)_k \cdots (a_r)_k$ with $(a)_k := a(a+1) \cdots (a+k-1)$ (and $(a)_0 := 1$ by convention).

2. The trigonometric $BC$-type Ruijsenaars-Schneider system

The trigonometric $BC$-type Ruijsenaars-Schneider system is a one-dimensional $n$-particle model characterized by the Hamiltonian \[5, 6\]

$$H(p, x) = \sum_{j=1}^{n} \left( \cosh(p_j) \sqrt{V_j(x)V_j(-x)} - (V_j(x) + V_j(-x))/2 \right), \quad (2.1a)$$

where

$$V_j(x) = w(x_j) \prod_{1 \leq k \leq n, k \neq j} v(x_j + x_k) v(x_j - x_k), \quad (2.1b)$$

$$v(x) = \frac{\sin(x + ig)}{\sin(x)}, \quad w(x) = \frac{\sin(x + ig_1) \cos(x + ig_2) \sin(x + ig_3) \cos(x + ig_4)}{\sin^2(x) \cos^2(x)} \quad (2.1c)$$

(and $i := \sqrt{-1}$). Throughout this note we will assume that the coupling parameters $g$ and $g_r$ ($r = 1, 2, 3, 4$) are positive. This guarantees in particular that the Hamiltonian $H(p, x)$ constitutes a nonnegative (smooth) function on the phase space

$$\Omega = \{ (p, x) \in \mathbb{R}^{2n} \mid 0 < x_1 < x_2 < \cdots < x_{n-1} < x_n < \pi/2 \}. \quad (2.2)$$

Indeed, one has that $H(p, x) \geq H(0, x) \geq 0$ (since $V_j(-x) = \overline{V_j(x)}$ and $|V_j(x)| \geq \Re(V_j(x))$).

3. The ground-state equilibrium configuration

The equilibrium configurations correspond to the critical points of the Hamiltonian $H(p, x)$ and the ground-state equilibrium configurations correspond in turn to the global minima. It is clear that the only way in which the nonnegative $BC$-type Ruijsenaars-Schneider Hamiltonian $H(p, x)$ may vanish (thus actually reaching the lower bound zero) is when $p = 0$ and $x$ is such that $V_j(x)$ is positive for $j = 1, \ldots, n$. This requires in particular that $V_j(x)$ is real-valued, i.e. that

$$V_j(x) = V_j(-x), \quad j = 1, \ldots, n, \quad (3.1a)$$
Here it is assumed that all parameters are real-valued subject to the constraints or more explicitly

\[
\prod_{1 \leq k < n, k \neq j} \frac{\sin(x_j + x_k + ig) \sin(x_j - x_k + ig)}{\sin(x_j + x_k - ig) \sin(x_j - x_k - ig)} = \quad (3.1b)
\]

\[
\frac{\sin(x_j - ig_1) \cos(x_j - ig_2) \sin(x_j - ig_3) \cos(x_j - ig_4)}{\sin(x_j + ig_1) \cos(x_j + ig_2) \sin(x_j + ig_3) \cos(x_j + ig_4)}, \quad j = 1, \ldots, n.
\]

We will see below that the nonlinear system of algebraic equations in Eq. (3.1b) has a unique solution given by the zeros of the Askey-Wilson polynomial of degree \(n\). It is not difficult to see that for this solution in fact \(V_j(x) > 0\) for \(j = 1, \ldots, n\), whence \(H(0, x) = 0\). Indeed, \(V_j(x)\) is real-valued by Eq. (3.1a). Furthermore, for sufficiently small values of the coupling parameters \(V_j(x)\) must be positive as the function in question tends to 1 for \(g, g_r \to 0\). This positivity remains valid for general positive parameter values \(g, g_r\) by a continuity argument revealing that the sign cannot flip (as none of the factors in \(V_j(x)\) becomes zero or singular).

We thus arrive at the following theorem.

**Theorem 3.1.** The trigonometric BC-type Ruijsenaars-Schneider Hamiltonian \(H(p, x)\) assumes the global minimum \(H = 0\) only at the point in the phase space \(\Omega\) such that \(p_1 = p_2 = \cdots = p_n = 0\) and \(0 < x_1 < x_2 < \cdots < x_n < \pi/2\) form a solution of the nonlinear system of algebraic equations in Eq. (3.1b).

4. Zeros of the Askey-Wilson Polynomials

The nonlinear system of algebraic equations in Eq. (3.1b) turns out to be a special case of the Bethe Ansatz equations associated to \(q\)-Sturm-Liouville problems studied recently by Ismail et al. It follows from the machinery in loc. cit. that this algebraic system has a unique solution given by the zeros of the Askey-Wilson polynomial of degree \(n\). Below we will provide an independent direct proof of this fact.

To this end, we first need to recall some basic properties of the Askey-Wilson polynomials taken from Ref. [7]. The (monic) Askey-Wilson polynomials are trigonometric polynomials of the form

\[
p_n(x) = \cos(2nx) + \sum_{k=0}^{n-1} a_k \cos(2kx), \quad n = 0, 1, 2, \ldots, \quad (4.1a)
\]

obtained by applying Gram-Schmidt orthogonalization of the standard Fourier cosine basis \(1, \cos(2x), \cos(4x), \ldots\) on the interval \((0, \pi/2)\) with respect to the inner product

\[
\langle f, g \rangle_\Delta = \int_0^{\pi/2} f(x)g(x)\Delta(x)dx, \quad (4.1b)
\]

associated to the weight function

\[
\Delta(x) = \frac{1}{c(x)c(-x)}, \quad c(x) = \frac{ae^{2ix}b^{2ix}ce^{2ix}de^{2ix}q_\infty}{(e^{4ix};q_\infty)} \quad (4.1c)
\]

Here it is assumed that all parameters are real-valued subject to the constraints \(0 < q < 1\) and \(0 < |a|, |b|, |c|, |d| < 1\). These parameter restrictions ensure in particular that the weight function \(\Delta(x)\) is positive in the interval \((0, \pi/2)\).
The Askey-Wilson polynomials admit an explicit representation in terms of the following terminating basic hypergeometric series

\[ p_n(x) = \frac{(ab, ac, ad; q)_n}{2^n (abcdq^n; q)_n} \Phi_3 \left( \begin{array}{c} q^{-n}, abcdq^{n-1}, ae^{2ix}, ae^{-2ix} \\ ab, ac, ad \end{array} \right| q; q \right). \tag{4.2} \]

The polynomials under consideration are the eigenfunctions of a second-order difference operator. Upon performing the parameter substitution

\[ q = e^{-2g}, \quad a = e^{-2g_1}, \quad b = e^{-2g_2}, \quad c = e^{-2g_3}, \quad d = -e^{-2g_4}, \tag{4.3} \]

the corresponding eigenvalue equation becomes of the form

\[ D p_n(x) = E_n p_n(x), \tag{4.4a} \]

where \( D \) denotes the difference operator

\[ D = W(x)(T_{ig} - 1) + W(-x)(T_{-ig} - 1) \quad (T_{ig} f)(x) := f(x + ig), \tag{4.4b} \]

with

\[ W(x) = \frac{\sin(x + ig_1) \sin(x + ig_2) \sin(x + ig_3) \sin(x + ig_4)}{\sin(2x) \sin(2x + ig)}, \tag{4.4c} \]

and the eigenvalue is given by

\[ E_n = \frac{\cosh(\tilde{g} + 2ng) - \cosh(\tilde{g})}{2}, \quad \tilde{g} = g_1 + g_2 + g_3 + g_4 - g. \tag{4.4d} \]

After these preliminaries, we are now in the position to prove the main result. It follows from the general fact that the polynomials form an orthogonal system on the interval \((0, \pi/2)\) with respect to a positive weight function that the Askey-Wilson polynomial \( p_n(x) \) has \( n \) simple zeros inside the interval \((0, \pi/2)\). If we denote these zeros by \( x_1, \ldots, x_n \), then it is clear that the Askey-Wilson polynomial factorizes as

\[ p_n(x) = 2^{2n-1} \prod_{k=1}^n \sin(x_k + x) \sin(x_k - x) \tag{4.5} \]

(since \( 2 \sin(x_k + x) \sin(x_k - x) = \cos(2x) - \cos(2x_k) \)). After plugging the factorization of the Askey-Wilson polynomial from Eq. (4.5) into the difference equation in Eqs. (4.4a)–(4.4d), and setting of \( x \) equal to the \( j \)th root \( x_j \), one arrives at the identity

\[ W(x_j) \prod_{k=1}^n \sin(x_k + x_j + ig) \sin(x_k - x_j - ig) + \]

\[ W(-x_j) \prod_{k=1}^n \sin(x_k + x_j - ig) \sin(x_k - x_j + ig) = 0, \tag{4.6} \]

which amounts to Eq. (4.11). This shows that the roots of the Askey-Wilson polynomial solve the nonlinear system of algebraic equations in Eq. (4.11).

To see that this is the only solution (up to permutation), we now assume—reversely—that the points \( 0 < x_k < \pi/2, \; k = 1, \ldots, n \) are such that they constitute any solution to Eq. (4.11), and show that this implies that the corresponding factorized polynomial of the form \( p_n(x) \) \((4.5)\) must be equal to the Askey-Wilson polynomial. To this end it is sufficient to infer that the factorized polynomial in question solves the eigenvalue equation in Eqs. (4.4a)–(4.4d) (since the spectrum of \( D \) is nondegenerate as a continuous function of the parameters and thus determines the eigenpolynomials uniquely). It is clear from the fact that the Askey-Wilson polynomials form the corresponding eigenbasis that acting with the operator \( D \)
A monic polynomial of the form in Eq. (4.5) produces the eigenvalue $E_n$ times a certain monic polynomial $q_n(x)$ of degree $n$. Furthermore, we have that $E_nq_n(x_j) = (Dp_n)(x_j) = 0$ for $j = 1, \ldots, n$, because of Eq. (4.6) (which holds since the points $x_1, \ldots, x_n$ solve Eq. (3.1b) by assumption). Hence, the monic polynomial $q_n(x)$ has the same roots as $p_n(x)$ and thus coincides with it. In other words, the factorized polynomial $p_n(x)$ solves the Askey-Wilson difference equation, and is thus equal to the Askey-Wilson polynomial, whence the roots $x_1, \ldots, x_n$ correspond to the roots of the Askey-Wilson polynomial. This gives rise to the following theorem.

**Theorem 4.1.** The unique (up to permutation) solution $0 < x_k < \pi/2$, $k = 1, \ldots, n$ of the nonlinear system of algebraic equations in Eq. (3.1b) is given by the (simple) roots of the Askey-Wilson polynomial $p_n(x)$ with parameters of the form in Eq. (4.3).

By combining Theorem 3.1 and Theorem 4.1, we end up with the desired characterization of the ground-state equilibrium configuration in terms of zeros of the Askey-Wilson polynomial.

**Corollary 4.2.** The trigonometric BC-type Ruijsenaars-Schneider Hamiltonian $H(p, x)$ in Eqs. (2.1a)-(2.1c) assumes the global minimum $H = 0$ only at the point in the phase space $\Omega$ such that $p_1 = p_2 = \cdots = p_n = 0$ and $0 < x_1 < x_2 < \cdots < x_n < \pi/2$ are given by the (simple) roots of the Askey-Wilson polynomial $p_n(x)$ with parameters of the form in Eq. (4.3).

5. **Rational degeneration**

By working the way down the Askey hierarchy of (basic) hypergeometric orthogonal polynomials, starting from the Askey-Wilson polynomials corresponding to the trigonometric BC-type Ruijsenaars-Schneider systems, one arrives at the equilibrium configurations associated to the degenerate Ruijsenaars-Schneider systems considered by Sasaki et al [11, 9] and at the equilibrium configurations associated to the Calogero-Moser systems considered by Calogero et al [2, 3, 4, 10]. As an example, we will wrap up by detailing the important case of the rational BC-type Ruijsenaars-Schneider system [5]. In this case the ground-state equilibrium turns out to be given by the zeros of the Wilson polynomials [15, 8].

The Hamiltonian $H(p, x)$ of the rational BC-type Ruijsenaars-Schneider system is given by Eqs. (2.1a), (2.1c) with potentials of the form [5]

$$v(x) = (x + ig)/x, \quad w(x) = (x + ig_1)(x + ig_2)(x + ig_3)(x + ig_4)/x^2. \quad (5.1)$$

The phase space becomes in this situation

$$\Omega = \{(p, x) \in \mathbb{R}^{2n} \mid 0 < x_1 < x_2 < \cdots < x_{n-1} < x_n\}. \quad (5.2)$$

The following theorem characterizes the ground-state equilibrium configuration of the rational BC-type Ruijsenaars-Schneider Hamiltonian in terms of the zeros of the Wilson polynomials [15, 8]

$$p_n(x) = \frac{(-1)^n(a + b, a + c, a + d)_n}{(n + a + b + c + d - 1)_n} \times 4F_3\left[\begin{matrix} -n, n + a + b + c + d - 1, a + ix, a - ix \\ a + b, a + c, a + d \end{matrix} \mid 1 \right]. \quad (5.3a)$$
which satisfy the orthogonality relations

\[ \int_0^\infty p_n(x)p_m(x)\Delta(x)dx = 0, \quad n \neq m, \] (5.3b)

associated to the positive weight function

\[ \Delta(x) = \frac{1}{c(x), c(-x)}, \quad c(x) = \frac{\Gamma(2ix)}{\Gamma(a + ix)\Gamma(b + ix)\Gamma(c + ix)\Gamma(d + ix)}, \] (5.3c)

where \( a, b, c, d > 0 \) (and \( \Gamma(\cdot) \) refers to the gamma function).

**Theorem 5.1.** The rational BC-type Ruijsenaar-Schneider Hamiltonian \( H(p,x) \) from Eqs. (2.1a), (2.1b), with potentials of the form in Eq. (5.1), has a unique global minimum \( H = 0 \) in the phase space \( \Omega \) at \( p_1 = p_2 = \cdots = p_n = 0 \) and \( 0 < x_1 < x_2 < \cdots < x_n \) given by the (simple) roots of the rescaled Wilson polynomial \( p_n(x/g) \) with rescaled parameters \( a = g_1/g, b = g_2/g, c = g_3/g, d = g_4/g \); these roots in turn constitute the unique positive solution to the nonlinear system of algebraic equations

\[ \prod_{1 \leq k, l \leq n, k \neq l} \frac{\delta_{j, l} (x_j + x_k + ig)(x_j - x_k + ig)}{(x_j + x_k - ig)(x_j - x_k - ig)} = \frac{(x_j - ig_1)(x_j - ig_2)(x_j - ig_3)(x_j - ig_4)}{(x_j + ig_1)(x_j + ig_2)(x_j + ig_3)(x_j + ig_4)}, \]

\( j = 1, \ldots, n \).

It is clear from the theorem that varying the value of the coupling parameter \( g \) gives rise to a linear rescaling of the equilibrium positions. More specifically, starting from the \( g = 1 \) configuration corresponding to the zeros \( x_1, \ldots, x_n \) of the Wilson polynomials \( p_n(x) \) with \( a = g_1, b = g_2, c = g_3, d = g_4 \), one passes to the equilibrium configuration for general positive \( g \) via the rescaling \( x_j \to gx_j, j = 1, \ldots, n \).

The proof of the above theorem runs along the same lines of Sections 3 and 4 and hinges on the second-order difference equation for the rescaled Wilson polynomials of the form in Eqs. (4.4a), (4.4b) with

\[ W(x) = \frac{(x + ig_1)(x + ig_2)(x + ig_3)(x + ig_4)}{2x(2x + ig)}, \quad E_n = -ng(ng + \hat{g}) \] (5.4)

(cf. e.g. [8]). Indeed, it is immediate that the minimization condition \( V_j(x) = V_j(-x) \) now gives rise to the algebraic equations for the equilibrium points stated in the theorem; furthermore, the solution of this system readily follows upon substitution of the factorization \( p_n(x) = \prod_{k=1}^n (x + x_k)(x - x_k) \) into the difference equation for the rescaled Wilson polynomials.

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