Jacobi Metric and Morse Theory of Dynamical Systems

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Abstract

The generalization of the Maupertuis principle to second-order Variational Calculus is performed. The stability of the solutions of a natural dynamical system is thus analyzed via the extension of the Theorem of Jacobi. It is shown that the Morse Theory of the trajectories in the dynamical system is identical to the Morse Theory of geodesics in the Jacobi metric, even though the second-variation functionals around the action and the Jacobi length do not coincide. As a representative example, we apply this result to the study of the separatrix solutions of the Garnier System.

Key words: Maupertuis Principle, Morse Theory.

1 Introduction

The Jacobi version of the Maupertuis principle establishes that the dynamics in a natural system can be viewed as geodesic motion in an associated Riemannian manifold: If \( S = \int (T - U)dt \) is the (natural) action functional of a system defined in a Riemannian manifold, \((M, g)\), the critical trajectories of \( S \) with energy \( E = T + U \) coincide with the extremals (geodesics) of the length functional \( L_J = \int ds \) defined in \((M, h)\), where \( h \) is the Jacobi metric \( h = 2(E - U)g \).

In this work we analyze whether or not this idea works at the level of second-order variational calculus. Namely, can the Hessian operator of a natural dynamical system be viewed as the geodesic deviation operator of the
associated Riemannian manifold? Early on, it was pointed out that the answer is negative [9] and consequently the stability criterion provided by the geodesic deviation operator (“geometric stability criterion”) does not coincide with the usual criterion based on analysis of the spectrum of the Hessian operator (“dynamical stability criterion”). We shall show that even in this situation the Morse theory associated with the dynamical problem is the same as the Morse theory built on the geodesics of the Jacobi metric. The crux of the matter is that the Jacobi weak stability criterion provided by the number of conjugate points crossed by a trajectory is the same from both points of view.

There are several topics where the geometric criterion has been used, e.g., study of the chaotic behaviour of Hamiltonian systems [4] or the non-integrability of dynamical systems [8]. The Jacobi-Maupertuis principle has been extended to the case of Lorentzian manifolds in [12].

2 The Maupertuis-Jacobi Principle

2.1 Preliminaries and Notation

Let \( M \equiv (M, g) \) be a Riemannian manifold. Geodesics in \( M \) are extremals of the free-action \( S_0 \) or length functionals \( L \) for any differentiable curve \( \gamma : [t_1, t_2] \rightarrow M \):

\[
S_0[\gamma] = \int_{t_1}^{t_2} \frac{1}{2} \| \dot{\gamma}(t) \|^2 dt; \quad L[\gamma] = \int_{t_1}^{t_2} \| \dot{\gamma}(t) \| dt .
\]

(1)

The critical paths are the solutions of the Euler-Lagrange equations:

\[
\delta S_0 = 0 \Rightarrow \nabla_\gamma \dot{\gamma} = 0; \quad \delta L = 0 \Rightarrow \nabla_\gamma \dot{\gamma} = \dot{\lambda}(t) \dot{\gamma}, \quad \lambda(t) = -\frac{d^2t}{ds^2} \left( \frac{ds}{dt} \right)^2 .
\]

(2)

As a consequence of the invariance of the length functional under reparametrizations, variational calculus on \( L \) leads to equations where the geodesics are parametrized by an arbitrary parameter \( t \) (often called pre-geodesics).

To decide whether a critical path is a local minimum we focus on the second variation functionals:

\[
\delta^2 S_0 = \int_{s_1}^{s_2} \langle \Delta V, V \rangle ds, \quad \delta^2 L = \int_{s_1}^{s_2} \left\langle \Delta V^\perp, V^\perp \right\rangle ds
\]

\[
\Delta V = -\nabla_\gamma \nabla_\gamma' V - R(\gamma', V)\gamma' = -\frac{D^2V}{ds^2} - K_{\gamma'}(V) .
\]
Here, $V$ denotes a proper variation ($V(s_1) = V(s_2) = 0$, see for instance [7] for details) and $V^\perp$ is the component of $V$ orthogonal to the geodesic. $R$ and $K$ are respectively the curvature and sectional curvature tensors. Thus, $\Delta$ is the geodesic deviation operator.

In a natural dynamical system where the configuration space is the Riemannian manifold $(M, g)$, the dynamics is governed by the action:

$$S[\gamma] = \int_{t_1}^{t_2} \left( \frac{1}{2} \|\dot{\gamma}\|^2 - U(\gamma(t)) \right) \, dt ,$$

and the Euler-Lagrange-Newton equations are:

$$\delta S = 0 \Rightarrow \frac{D\dot{\gamma}}{dt} = -\text{grad}U .$$

The local stability of a given extremal under proper variations is determined by the Hessian or second-variation functional:

$$\delta^2 S = \int_{t_1}^{t_2} dt \left( \langle \Delta V, V \rangle - H(U)(V, V) \right) = \int_{t_1}^{t_2} dt \langle \Delta V - \nabla_V \text{grad}U, V \rangle .$$

and we shall denote the differential operator in the quadratic form (5) as: $\Delta V = \Delta V - \nabla_V \text{grad}U$.

### 2.2 Geodesics in the Jacobi Metric

Let $x^i$ be a system of local coordinates in $M$, and let us set $ds_g^2 = \sum_{ij} g_{ij} dx^i \otimes dx^j \equiv \sum_{ij} g_{ij} dx^i dx^j$. The Jacobi metric $h$ is defined as a conformal transformation of $g$: $ds_h^2 = \sum_{ij} h_{ij} dx^i dx^j$, $h_{ij} = 2(i_1 - U)g_{ij}$, where $i_1$ is a constant and the Riemannian character of $h$ restricts the admissible values for $i_1$ by means of the inequality $U < i_1$.

Given any two vector fields $X, Y \in \Gamma(TM)$, we shall use the following notation: $h(X, Y) = \langle X, Y \rangle^J$, $\|X\|^J = \sqrt{\langle X, X \rangle^J}$, and we shall write $s$ for $s_h$ for simplicity: $ds \equiv ds_h$.

**Theorem of Jacobi.** The extremal trajectories of the variational problem associated with the functional (3) with energy $E = i_1$, are the geodesics of the manifold $(M, h)$, where $h$ is the Jacobi metric: $h = 2(i_1 - U) g$.

The proof of this theorem can be found in several References (see [9] for instance). From an analytic point of view, the theorem simply establishes that the Newton equations (4) arising from the action $S$ are tantamount to the geodesic equations in $(M, h)$: $\nabla^J_{\gamma'} \gamma' = 0$, $\gamma' = \frac{dx}{ds}$. The equivalence is shown by performing both a conformal transformation, $h = 2(i_1 - U) g$, and
a reparametrization from the dynamical time $t$ to the arc-length parameter $s$ in $(M, h)$,
\[
\frac{ds}{dt} = 2\sqrt{(i_1 - U(\gamma(s)))T} = (i_1 - U(\gamma(s))) ,
\] (6)
over the solutions (note that in the domain $U < i_1$ the reparametrization is well defined).

3 "Geometric Stability" versus "Dynamical Stability"

In the language of Variational Calculus, the Jacobi theorem states that $\delta S = 0 \iff \delta S_0^J = 0$, or, for geodesics parametrized with respect to the arc-length, $\delta S = 0 \iff \delta L^J = 0$. In [1] we showed the following theorems referring to the second-variation functionals:

**Theorem 1.** Let $\gamma(t)$ be an extremal of the $S[\gamma]$ functional, and let $S_0^J[\gamma]$ be the free-action functional for the Jacobi metric associated with $S[\gamma]$. The corresponding Hessian functionals satisfy:
\[
\delta^2 S_0^J[\gamma] = \delta^2 S[\gamma] + \int_{t_1}^{t_2} dt \frac{d}{dt} 2\left\langle \dot{\gamma}, \frac{D}{dt} V \right\rangle \langle F, V \rangle ,
\] (7)
where $F = \text{grad} \ln(2(i_1 - U))$.

**Theorem 2.** Let $\gamma$ be an extremal of the $S[\gamma]$ functional and let $L^J[\gamma]$ be the length functional of the Jacobi metric associated with $S[\gamma]$. The corresponding Hessian functionals satisfy:
\[
\delta^2 L^J[\gamma] = \delta^2 S[\gamma] - \int_{t_1}^{t_2} dt \frac{d}{dt} \frac{1}{2(i_1 - U)} \left[ \langle \nabla_{\dot{\gamma}} \dot{\gamma}, V \rangle - \langle \dot{\gamma}, \nabla_{\dot{\gamma}} V \rangle \right]^2
\] (8)

Equation (8) shows that geodesics of minimal Jacobi length $L^J[\gamma]$ are equivalent to stable solutions of the dynamical system, although the converse is not necessarily true. Allowing only orthogonal variations, $V = V^\perp$, (8) can be re-written as:
\[
\delta^2 S\big|_{V = V^\perp} = \delta^2 L^J + \int_{s_1}^{s_2} ds \left( \langle F^J, V^\perp \rangle \right)^2
\]

The proofs of these two theorems are based on the behaviour of the covariant derivatives and the curvature tensor under reparametrizations and conformal transformations of the metric tensor.
4 Jacobi Fields and Morse Theory

A basic tool for the analysis of the stability properties of a given extremal of a functional is the study of the Jacobi fields: the elements of the Kernel of the second-variation or Hessian operator. Direct computation of the Jacobi fields in a given problem is in general a very complicated task: one needs to solve the Euler-Lagrange equations, calculate the second-variation operator and, finally, determine the Kernel. Fortunately, there exists a short-cut if one knows a parametric family of extremals (see, for instance, [6] for details):

**Proposition.** If $\gamma = \gamma(t, a)$ is a family of extremals of $S[\gamma]$ characterized by the value of the real parameter $a$, then the vector field $\frac{\partial \gamma}{\partial a}$ is a Jacobi field.

Knowledge of the Jacobi fields allows us to calculate the conjugate points along an extremal $\gamma$, and it is thus possible to apply the Jacobi criterion for weak minimizers of a functional which counts the number of conjugate points crossed by the extremal as a measure of the instability.

Morse Theory establishes a link between the topology of a manifold and the critical-point structure of the functions defined in the manifold [11]. We shall deal with Morse Theory in spaces of closed paths with a fixed point in Riemannian manifolds. The aim is the application to the separatrix trajectories in the Garnier System. A formulation of this theory à la Bott [3] is as follows:

Let $\Omega M = \{ \gamma : S^1 \to M / \gamma(0) = \gamma(1) = m_0 \}$ be the loop space in $M$ with a fixed base point. Then, some of the topological properties of $\Omega M$ are codified in the Poincaré series $P_t(\Omega M^n) = \sum_{k=0}^{\infty} b_k t^k$, where $b_k = \dim H_k(\Omega M, \mathbb{R})$ are the corresponding Betti numbers.

For any functional $S$ defined on $\Omega M$, we define the Morse series as $M_t(S) = \sum_{N_c} P_t(N_c) \mu(N_c)$, where the sum is over the critical manifolds $N_c$, formed either by a single isolated critical path or by a continuous set of critical paths. $\mu(\gamma_c)$, the Morse index, is the dimension of the subspace where the Hessian operator of the functional $S$, $\Delta$, along $\gamma_c$ is negative definite.

The Morse inequalities, $M_t(S) \geq P_t(\Omega M)$, tell us that the topology of $\Omega M$ forces the existence of most of the extremals of the functional $S$.

The Morse index can be computed using the **Morse index Theorem**: The Morse index of a critical path $\gamma_c$ is equal to the number of conjugate points to the base point crossed by $\gamma_c$ counted with multiplicity. Note that this theorem is equivalent to the Jacobi criterion for weak minimizers.

From this point of view, knowledge of a one-parametric family of solutions $\gamma(t; a)$ of the dynamical problem (3) informs us about the geodesics $\gamma(s; a)$ in $(M, h)$. There is a one-to-one correspondence between the trajectories of the dynamical system and the geodesics of the Jacobi metric, which only differ by
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proper reparametrizations. Therefore, the two kinds of extremals share all the conjugate points, and the structure of the critical points of \( S \) and \( L' \) is the same. The Morse series coincide, \( \mathcal{M}(S) = \mathcal{M}(L') \), and the Morse theories built from the trajectories in \((M, g)\) and the geodesics in \((M, h)\) are identical.

5 Separatrix trajectories in the Garnier System

As an application, we shall analyze the separatrix trajectories in the Garnier System \([5]\). The configuration space is \((\mathbb{R}^N, \delta_{ij})\) and the action functional reads:

\[
S = \int dt \left\{ \frac{1}{2} \|\dot{\vec{q}}\|^2 + \frac{1}{2} (\vec{q} \cdot \vec{q} - 1)^2 + \sum_{i=1}^{N} \frac{1}{2} \sigma_i^2 q_i^2 \right\}.
\]

Here \( \sigma_i \) are constants, such that \( \sigma_i \neq \sigma_j, \forall i \neq j \), and the Euler-Lagrange equations are:

\[
\frac{d^2 q_i}{dt^2} = 2q_i (\vec{q} \cdot \vec{q} - 1) + \sigma_i^2 q_i, \quad i = 1, 2, \ldots, N.
\]

The system is completely integrable and, in fact, Garnier found all the periodic solutions in terms of hyperelliptic functions. In \([3]\) we computed the separatrix solutions lying on the boundary between bounded and unbounded motion. We shall focus in the \( N = 2 \) case for simplicity (in \([1]\) the very rich \( N = 3 \) case is analyzed thoroughly).

For any path \( \vec{q}(t) = (q_1(t), q_2(t)) \) in \((\mathbb{R}^2, \delta_{ij})\), the Lagrangian \( \mathcal{L} = \mathcal{L}(q_1, q_2, \dot{q}_1, \dot{q}_2) \) is:

\[
\mathcal{L} = \frac{1}{2} (q_1^2 + q_2^2) + \frac{1}{2} (\dot{q}_1^2 + \dot{q}_2^2 - 1)^2 + \frac{\sigma^2}{2} q_2^2.
\]

With no loss of generality, we have set \( \sigma_1 = 0, \sigma_2 = \sigma \). We shall only deal with the case \( \sigma < 1 \), because the separatrix solutions in this regime have a richer structure. In fact \( \sigma > 1 \) generates a situation where the separatrices are reduced to a single singular solution. The system is not only completely integrable, but Hamilton-Jacobi separable in Jacobi elliptical coordinates: \( q_1^2 = \sigma^{-2}(1-\mu_1)(1-\mu_2), q_2^2 = \sigma^{-2}(\bar{\sigma}^2-\mu_1)(\bar{\sigma}^2-\mu_2) \), where \( \sigma^2 = 1 - \sigma^2 \).

With this change, \( \mathbb{R}^2 \) is mapped into the infinite parallelogram determined by the inequalities: \(-\infty < \mu_1 < \sigma^2 < \mu_2 < 1\).

The choice of the separatrix trajectories is achieved by setting the values of the two invariants in involution to zero: \( i_1 = i_2 = 0 \). \( i_1 \) denotes the energy first-integral and \( i_2 \) is the second first-integral of the system, functionally independent of \( i_1 \) (see for instance \([9]\) for details). This choice confines the dynamics to the finite parallelogram \( \mathcal{P}(0) \equiv 0 \leq \mu_1 < \sigma^2 < \mu_2 < 1 \) (in Cartesian coordinates it is the area bounded by the ellipse \( q_1^2 + \frac{q_2^2}{\sigma^2} = 1 \)).
In elliptical coordinates the Euclidean metric \( \delta_{ij} \) becomes: \( g_{12} = 0 = g_{21}, \ g_{11} = \frac{-(\mu_1 - \mu_2)}{4(\mu_1 - 1)(\mu_1 - \sigma^2)}, \ g_{22} = \frac{-(\mu_2 - \mu_1)}{4(\mu_2 - 1)(\mu_2 - \sigma^2)}. \) The Christoffel symbols are obtained from

\[
\Gamma^1_{11} = \frac{1}{2(\mu_1 - \mu_2)} - \frac{2\mu_1 - (1 + \sigma^2)}{2(\mu_1 - 1)(\mu_1 - \sigma^2)}, \quad \Gamma^1_{12} = \frac{-1}{2(\mu_1 - \mu_2)}, \quad \Gamma^1_{22} = \frac{(\mu_1 - 1)(\mu_1 - \sigma^2)}{2(\mu_2 - 1)(\mu_2 - \sigma^2)(\mu_1 - \mu_2)}
\]

Replacing 1 by 2, \( \Gamma^2_{22} \) is equivalent to \( \Gamma^1_{11} \), and the same happens between \( \Gamma^2_{12}, \Gamma^1_{11}, \) and \( \Gamma^1_{12}, \Gamma^1_{22} \). The rest of the symbols are trivially related to these by symmetries or are null.

The Jacobi metric associated with the system, with \( i_1 = 0 \), is \( h = \frac{1}{\mu_2 - \mu_1}(-\mu_1^3 - \sigma^2 \mu_2^3) + (\mu_3^3 - \sigma^2 \mu_2^3)g \), and the Christoffel symbols can be obtained by direct calculation (\( \square \)).

### 5.1 Singular Solutions

There are two singular solutions to the system, and these live on the border of the parallelogram \( \mathbf{P}(0) \).

Taking \( q_2 = 0 \), the equations of the motion can be easily integrated, even in Cartesian coordinates, to find the solutions: \( \bar{q}(t) = (\bar{q}_1(t), 0) = (\pm \tanh(t - t_0), 0) \). In elliptical coordinates, these solutions lie on two edges of the border of \( \mathbf{P}(0) \): I, \( \bar{\mu}_1(t) = (1 - \tanh^2(\pm t), \bar{\sigma}) \), for \( t \in (-\infty, \arctanh(\mp \sigma)) \cup [\arctanh(\pm \sigma), \infty) \), and II, \( \bar{\mu}_{11}(t) = (\bar{\sigma}^2, 1 - \tanh^2(\mp t)) \) for \( t \in [\arctanh(\pm \sigma), \arctanh(\pm \sigma)] \) (taking \( t_0 = 0 \)).

The geodesic equations of the Jacobi metric reduced to the \( q_2 = 0 \) orbit become a single non-trivial differential equation, which is solved by a cubic algebraic equation:

\[
q_1'' + \frac{2q_1}{q_1^2 - 1}q_1^2 = 0 \Rightarrow \bar{q}_1 - \frac{q_1^3}{3} = \pm s \quad . \tag{12}
\]

\((t_1, t_2) = (-\infty, \infty)\) has been reparametrized to \([s_1, s_2] = [-\frac{2}{3}, \frac{2}{3}]\). The explicit solution of the cubic (\( \square \)) is

\[
q_1(s) = \bar{q}_1(s) = -\cos \frac{\theta}{3} + \sqrt{3} \sin \frac{\theta}{3} \quad , \tag{13}
\]

where \( \theta = \arctan \frac{\sqrt{3} - q_1^2}{-3s} \).

There exists a second type of singular solutions living on the edge \( \mu_1 = 0 \), which also belongs to the border of \( \mathbf{P}(0) \). This edge becomes the ellipse \( \bar{q}_1^2 + \frac{q_2^2}{\sigma^2} = 1 \) in Cartesian coordinates. The solutions are either \( \bar{q}(t) = (\tanh(\pm \sigma t), \bar{\sigma} \sech(\pm \sigma t)) \) in \( \mathbb{R}^2 \), or \( \bar{\mu}(t) = (0, 1 - \sigma^2 \tanh^2 \sigma t) \) in the elliptic plane.
The geodesic equations of the Jacobi metric reduced to the \( q_1^2 + \frac{q_2^2}{\sigma^2} = 1 \) orbit also become a single non-trivial differential equation, which is solved by a cubic algebraic equation:

\[ q''_1 - \frac{2\sigma^2 q_1}{1 - \sigma^2 q_1^2} q_1^2 = 0 \Rightarrow \sigma \left( \frac{q_1 - \sigma^2 q_1^3}{3} \right) = s. \quad (14) \]

The explicit solution of the cubic together with the elliptic orbit afford the geodesic:

\[ q_1(s) = -\frac{1}{\sigma} \cos \frac{\theta}{3} + \frac{\sqrt{3}}{\sigma} \sin \frac{\theta}{3}, \quad q_2(s) = \frac{2}{\sigma} \sqrt{-2 + \sigma^2 + \cos \frac{2\theta}{3} + \sqrt{3} \sin \frac{2\theta}{3}} \]

from \( s_1 = -\sigma(1 - \frac{\sigma^2}{3}) \) to \( s_2 = \sigma(1 - \frac{\sigma^2}{3}) \).

5.2 General Solution

The Hamiltonian associated with the Lagrangian (11), in elliptical coordinates is of the Stäckel type:

\[ H = \frac{1}{2(\mu_1 - \mu_2)} \left( -4(\mu_1 - 1)(\mu_1 - \bar{\sigma})^2 \pi_1^2 - 4(1 - \mu_2)(\mu_2 - \bar{\sigma})^2 \pi_2^2 \right. \]
\[ \left. - (\mu_1^3 - \sigma^2 \mu_1^2) + (\mu_2^3 - \sigma^2 \mu_2^2) \right) \]

\[ (15) \]
and the integration of the Hamilton-Jacobi equation, for separation constants equal to zero, leads to the equation of the orbits,

\[
\left( \frac{1}{\sqrt{1 - \mu_1 + \sigma}} \right) \cdot \left( \frac{1}{\sqrt{1 - \mu_1 - 1}} \right) \cdot \text{sign}(\pi_1) = e^{2 \sigma \bar{\sigma} a}, \quad (16)
\]

and the equation of the temporal dependence,

\[
\left( \frac{1}{\sqrt{1 - \mu_1 - \sigma}} \right) \cdot \left( \frac{1}{\sqrt{1 - \mu_1 + 1}} \right) \cdot \text{sign}(\pi_2) = e^{2(t + t_0)\sigma}. \quad (17)
\]

Here \(a\) is a constant that parametrizes the different orbits and \(t_0\) is another constant coming from the time-translation invariance.

At the limit \(a \to \pm \infty\), equation (16) reduces to the singular solution \(q_2 = 0\) together with the singular solution living on the "ellipse". Thus, identifying \(a = -\infty\) with \(a = \infty\), we see that all the orbits are parametrized by a periodic parameter \(a\): the space of all orbits is the \(S^1\) circle.

![Figure 3: Graphics of the orbits (16) in the Cartesian plane and in the elliptical one. The graphics were obtained by numerical calculations.](image)

A straightforward application of the Hamilton-Jacobi procedure to the free-Hamiltonian, \(H_J = \frac{1}{2} (h^{11} \pi_1^2 + h^{22} \pi_2^2)\), with \(\tilde{\pi}_j = \frac{d\mu_j}{ds}\), provides all the geodesics of the Jacobi metric with \(i_1 = 0\).

The equation giving the orbits is exactly (16), and the arc-length dependence is determined by the expression:

\[
s + s_0 = -(-1)^{\text{sign}(\pi_1)} \frac{\mu_1 + 2}{3} \sqrt{1 - \mu_1} - (-1)^{\text{sign}(\pi_2)} \frac{\mu_2 + 2}{3} \sqrt{1 - \mu_2} \quad (18)
\]

### 5.3 Jacobi fields

In [2] the Jacobi fields are calculated on the trajectories. Here, we closely follow the same idea to compute the Jacobi fields on the geodesics, which
are the solutions of the equations (16)-(18). Thus, we have a two-parametric family $\vec{\mu}(s; a, s_0)$ of geodesics, and, accordingly, two Jacobi fields: $\frac{\partial \vec{\mu}}{\partial s_0}$ and $\frac{\partial \vec{\mu}}{\partial a}$; see Section §2.2. The first Jacobi field is tangent to the geodesic because it obeys the invariance under $s$ translations. We focus on the computation of the orthogonal Jacobi field: $J = \frac{\partial \vec{\mu}}{\partial a}$. An explicit formula for the geodesic $\vec{\mu}(s; a, s_0) = (\bar{\mu}_1, \bar{\mu}_2)$ in terms of known analytical functions is not available, but we implicitly derive the system (16)-(18) with respect to $a$ to obtain a linear system in the components of $J$:

\[
\begin{align*}
\frac{-(-1)^{\alpha_1}}{\bar{\mu}_1\sqrt{1 - \bar{\mu}_1(\bar{\sigma}^2 - \bar{\mu}_1)}} \frac{\partial \bar{\mu}_1}{\partial a} - \frac{(-1)^{\alpha_2}}{\bar{\mu}_2\sqrt{1 - \bar{\mu}_2(\bar{\sigma}^2 - \bar{\mu}_2)}} \frac{\partial \bar{\mu}_2}{\partial a} &= 2 \quad (19) \\
\frac{-(-1)^{\alpha_2}}{\sqrt{1 - \bar{\mu}_1}} \frac{\partial \bar{\mu}_1}{\partial a} + \frac{(-1)^{\alpha_2}}{\sqrt{1 - \bar{\mu}_2}} \frac{\partial \bar{\mu}_2}{\partial a} &= 0 \quad (20)
\end{align*}
\]

The Jacobi field that solves (19)-(20) is:

\[
J = j(\bar{\mu}_1, \bar{\mu}_2) \left( (-1)^{\alpha_1} \bar{\mu}_2 \sqrt{1 - \bar{\mu}_1} \frac{\partial}{\partial \bar{\mu}_1} + (-1)^{\alpha_2} \bar{\mu}_1 \sqrt{1 - \bar{\mu}_2} \frac{\partial}{\partial \bar{\mu}_2} \right) ,
\]

where

\[
j(\bar{\mu}_1, \bar{\mu}_2) = \frac{2\bar{\mu}_1\bar{\mu}_2(\bar{\mu}_1 - \bar{\sigma}^2)(\bar{\mu}_2 - \bar{\sigma}^2)}{(\bar{\mu}_1 - \bar{\mu}_2)(\bar{\mu}_1^2 + \bar{\mu}_2^2 + \bar{\mu}_1\bar{\mu}_2 - (\bar{\mu}_1 + \bar{\mu}_2)\bar{\sigma}^2)}, \quad \alpha_1, \alpha_2 = 0, 1
\]

$J$ is zero both at the starting point D and at the focus F of the ellipse, and, henceforth, the focus is a conjugate point of D of multiplicity 1.

5.4 The Morse series

Taking $M = P_2(0); \Omega M = C$, we have the following critical point structure for the geodesics that start and end at the point D. Firstly, the point D itself is a possible geodesic with a Morse index 0. Secondly, all members of the family determined by the general solution, $\vec{\mu}(s; a, s_0)$ cross the focus. Bearing in mind that $P_t(S^1) = (1 + t)$, the family contributes to the Morse series with $(1 + t)t$. Iterating two of the solutions, $\vec{\mu}^2\vec{\mu}$, D is a conjugate point to itself and the contribution to the Morse series is $(1 + t)t^3$. More iterations tell us that $M_t(L^J|C) = 1 + (1 + t)t + (1 + t)t^3 + \ldots = \frac{1}{1-t}$. The Morse series is equal to the Poincaré series of the space of closed geodesics in $S^2$: $M_t(S|C) = P_t(\Omega S^2)$.

In [2] we calculated the Morse series for the $N = 3$ case. The result is:

$M_t(L^J|C) = P_t(C) = \frac{1}{1-t} = P_t(\Omega S^3)$. 
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