TOPOLOGICAL EXPANSION OF THE $\beta$-ENSEMBLE MODEL AND QUANTUM ALGEBRAIC GEOMETRY IN THE SECTORWISE APPROACH

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We construct the solution of the loop equations of the $\beta$-ensemble model in a form analogous to the solution in the case of the Hermitian matrices $\beta = 1$. The solution for $\beta = 1$ is expressed in terms of the algebraic spectral curve given by $y^2 = U(x)$. The spectral curve for arbitrary $\beta$ converts into the Schrödinger equation $((\hbar\partial)^2 - U(x))\psi(x) = 0$, where $\hbar \propto (\sqrt{\beta} - 1/\sqrt{\beta})/N$. The basic ingredients of the method based on the algebraic solution retain their meaning, but we use an alternative approach to construct a solution of the loop equations in which the resolvents are given separately in each sector. Although this approach turns out to be more involved technically, it allows consistently defining the $B$-cycle structure for constructing the quantum algebraic curve (a D-module of the form $y^2 - U(x)$), where $[y, x] = \hbar$ and explicitly writing the correlation functions and the corresponding symplectic invariants $F_0$ or the terms of the free energy in an $1/N^2$-expansion at arbitrary $\hbar$. The set of “flat” coordinates includes the potential times $t_k$ and the occupation numbers $\tilde{\epsilon}_\alpha$. We define and investigate the properties of the $A$- and $B$-cycles, forms of the first, second, and third kinds, and the Riemann bilinear identities. These identities allow finding the singular part of $F_0$, which depends only on $\tilde{\epsilon}_\alpha$.

Keywords: Schrödinger equation, Bergman kernel, correlation function, Riemann identity, flat coordinates, Riccati equation

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1. Introduction

In contemporary mathematical physics, the notion of quantum surfaces is rather often encountered, appearing in many different guises. Having no intention to describe all problems in which quantization of the space–time coordinates themselves occurs (which pertains mainly to string or brane models), we nevertheless stress that the main feature of most, if not all, of these models is that the consideration is commonly restricted to the simple geometry of the sphere or torus. The observables in these theories are not the coordinates themselves, which cease to commute with each other and satisfy some postulated quantum