Long Time Asymptotic Behavior for the Nonlocal mKdV Equation in Solitonic Space–Time Regions

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Abstract
We study the long time asymptotic behavior for the Cauchy problem of an integrable real nonlocal mKdV equation with nonzero initial data in the solitonic regions

\[ q_t(x, t) - 6\sigma q(x, t)q(-x, -t)q_x(x, t) + q_{xxx}(x, t) = 0, \]
\[ q(x, 0) = q_0(x), \quad \lim_{x \to \pm \infty} q_0(x) = q_{\pm}, \]

where \(|q_\pm| = 1\) and \(q_+ = \delta q_-, \sigma \delta = -1\). In our previous article, we have obtained long time asymptotics for the nonlocal mKdV equation in the solitonic region \(-6 < \xi < 6\) with \(\xi = \frac{x}{t}\). In this paper, we give the asymptotic expansion of the solution \(q(x, t)\) for other solitonic regions \(\xi < -6\) and \(\xi > 6\). Based on the Riemann–Hilbert formulation of the Cauchy problem, further using the \(\bar{\partial}\) steepest descent method, we derive different long time asymptotic expansions of the solution \(q(x, t)\) in above two different space-time solitonic regions. In the region \(\xi < -6\), phase function \(\theta(z)\) has four stationary phase points on the \(\mathbb{R}\). Correspondingly, \(q(x, t)\) can be characterized with an \(N(\Lambda)\)-soliton on discrete spectrum, the leading order term on continuous spectrum and an residual error term, which are affected by a function \(\text{Im}\nu(\xi_i)\). In the region \(\xi > 6\), phase function \(\theta(z)\) has four stationary phase points on \(i\mathbb{R}\), the corresponding asymptotic approximations can be characterized with an \(N(\Lambda)\)-soliton with diverse residual error order \(O(t^{-1})\).

Keywords Nonlocal mKdV equation · Riemann–Hilbert problem · \(\bar{\partial}\)-steepest descent method · Long time asymptotics

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1 Introduction

In this paper, we investigate the long-time asymptotic behavior for the Cauchy problem of an integrable real nonlocal mKdV equation under nonzero boundary conditions

\[
q_t(x, t) - 6\sigma q(x, t)q(-x, -t)q_x(x, t) + q_{xxx}(x, t) = 0, \quad (1.1)
\]

\[
q(x, 0) = q_0(x), \quad \lim_{x \to \pm \infty} q_0(x) = q_{\pm}, \quad (1.2)
\]

where \(|q_{\pm}| = 1\) and \(q_+ = \delta q_-\), \(\sigma \delta = -1\). The nonlocal equation (1.1) was introduced in [1, 2], where \(\sigma = \pm 1\) denote the defocusing and focusing cases, respectively, and \(q(x, t)\) is a real function. The nonlocal equation (1.1) can be regarded as the integrable nonlocal extension of the well-known classical mKdV equation

\[
q_t(x, t) - 6\sigma q^2(x, t)q_x(x, t) + q_{xxx}(x, t) = 0, \quad (1.3)
\]

which appears in various of physical fields [3]. From the perspective of their structure, (1.3) can be translated to (1.1) by replacing \(q^2(x, t)\) with the \(PT\)-symmetric term \(q(x, t)q(-x, -t)\) (In the real field, the definition of \(PT\) symmetry is given by \(P: x \rightarrow -x\) and \(T: t \rightarrow -t\) [4]. The general form of the nonlocal mKdV equation was shown to appear in the nonlinear oceanic and atmospheric dynamical system [5].

There is much work on the study of various mathematical properties for the nonlocal mKdV equation. Zhang and Yan rigorously analyzed dynamical behaviors of solitons and their interactions for four distinct cases of the reflectionless potentials for both focusing and defocusing nonlocal mKdV equations with NZBCs [6]. The Darboux transformation was used to seek for soliton solutions of the focusing nonlocal mKdV equation (1.1) [7], and the IST for the focusing nonlocal mKdV equation (1.1) with ZBC was presented [8]. In [9], the long-time asymptotics for the nonlocal defocusing mKdV equation with decaying initial data were investigated via Deift-Zhou steepest-descent method developed by Deift and Zhou [10]. Such method has been widely applied to integrable systems, such as KdV, NLS, sine-Gordon, Camassa-Holm, Degasperis-Procesi, the Fokas-Lenells etc [11–18]. Recently, Deift-Zhou steepest-descent method was further generalized to \(\tilde{\partial}\)-steepest-descent method by McLaughlin and Miller [19, 20]. This method was successfully adapted to obtain the long-time asymptotics for solutions to the integrable systems with weighted Sobolev initial data, such as NLS, derivative NLS, modified Camassa-Holm, Fokas-Lenells, mKdV etc [21–30].

In our previous work [31], during analysis on the long-time asymptotics of the Cauchy problem (1.1)-(1.2), we found that both unit circle and real axis are the jump contour, on which there are six stationary phase points, among them the phase points \(\pm 1\) are fixed. Therefore according to the distribution of other four phase points on the jump contour, we can divide the upper half-plane \(\{(x, t) : x \in \mathbb{R}, \ t > 0\}\) into three space-time regions:
Fig. 1 The different space-time cones for $\xi = \frac{x}{t}$. Except phase points $z = \pm 1$, the function $\theta(z)$ has other four phase points, whose distribution depends on different space-time cones: Region I: $\xi < -6$, four phase points are located on real axis $\mathbb{R}$; Region II: $-6 < \xi < 6$, four phase points are located on the unit circle; Region III: $\xi > 6$, four phase points are located on the imaginary axis $i\mathbb{R}$.

Region I: $\xi := x/t < -6$, four phase points are located on real axis $\mathbb{R}$;
Region II: $-6 < \xi < 6$, four phase points are located on the unit circle;
Region III: $\xi > 6$, four phase points are located on the imaginary axis $i\mathbb{R}$.

See Fig. 1. We have obtained the the long-time asymptotic behavior of solutions to the nonlocal mKdV equation (1.1) in the Region II in [31]. In this work, we further obtain the long-time asymptotic behavior of solutions to the nonlocal mKdV equation (1.1) in the Region I and Region III. Different from our results in the Region II [31], in the Cone I, the value of $\nu(z)$ at four phase points $\zeta_i$, $i = 1, 2, 5, 6$ is not real. Moreover, the second leading term and the error term in the asymptotic expansion of the solution $q(x, t)$ are closely related to $\text{Im}\nu(\zeta_i)$, $i = 1, 2, 5, 6$. In the Region III, there is no phase point on the deformed jump contour $\Sigma(2)$, which implies that it is not necessary to consider local solvable RH models near phase points as [31]. Here we outline our main results as follows.

1.1 Main Results

The central results of this work give the long-time asymptotic behavior of the solutions $q(x, t)$ of nonlocal mKdV equation (1.1) in space-time regions I and III.

**Theorem 1.1** Let $q(x, t)$ be the solution for the initial-value problem (1.1)–(1.2) with generic data $q_0(x) \mp q_{\pm} \in L^{1,2}(\mathbb{R})$, $q' \in W^{1,1}(\mathbb{R})$ and scattering data $\left\{\rho(z), \bar{\rho}(z), \{\eta_k, A[\eta_k]\}\right\}_{k=1}^{2N_1+N_2}$. And $q^\Lambda(x, t)$ denote $N(\Lambda)$-soliton solution corresponding to scattering data $\left\{0, 0, \{\eta_k, A[\eta_k]\}\right\}_{k\in\Lambda}$ shown in Proposition 4.5, where $\Lambda$ is defined in (4.8). Then for $t \to \infty$,

1. In region I: $\xi < -6$, we have asymptotic expansion:

$$q(x, t) = q^\Lambda(x, t) - iT(\infty)^{-2} \sum_{i=1,2,5,6} t^{-\frac{1}{2} + \text{Im}(i)} f_i + R(t; \xi).$$  

The above $R(t; \xi)$ in (1.4) can be written as:
This paper is organized as follows.

1.2 Outline of this Paper

1. In Section 2, we get down to the spectral analysis on the Lax pair. Based on the analyticity, symmetry, and asymptotics of the Jost solutions and scattering data, the RH problem for the Cauchy problem \((1.1)-(1.2)\) is established.

2. In Section 3, we analyze the distribution of saddle points for different \(\xi\) and depict the decay regions of \(|e^{\pm 2i\omega(z)}|\) by some figures.

3. In Section 4 shows key technical processing for deforming and decomposing the RH problem in the case of \(\xi < -6\). In Sect. 4.1, we introduce a matrix-valued function \(T(z)\) to define a new RH problem for \(m^{(1)}(z)\), which admits a regular discrete spectrum and two triangular decompositions of the jump matrix. In Sect. 4.2, we introduce \(R^{(2)}(z)\) to make continuous extension for the jump matrix and remove the jump from \(\Sigma\) in such away that the new problem takes advantage of the decay of \(|e^{\pm 2i\omega(z)}|\) for \(z \notin \Sigma\). Consequently, a mixed \(\tilde{\partial}\)-RH problem is set up in Sect. 4.2.2. We further decompose the mixed \(\tilde{\partial}\)-RH problem for \(m^{(2)}(z)\) into a pure RH problem \(m^{rhp}(z)\) and a pure \(\tilde{\partial}\)-problem for \(m^{(3)}(z)\) in Sect. 4.2.3. In Sect. 4.3, we obtain the solution of the pure RH problem \(m^{rhp}(z)\) via an outer model \(m^{sol}(z)\) for the soliton components to be solved in Sect. 4.3.1, and an inner model \(m^{lo}(z)\) which are approximated by solvable models for \(m^{pc}_{i}, i = 1, 2, 5, 6\) obtained in Sect. 4.3.2. The error function \(E(z)\) between \(m^{rhp}(z)\) and \(m^{sol}(z)\) satisfies a small norm RH problem which is shown in Sect. 4.3.3. As for the pure \(\tilde{\partial}\)-problem \(m^{(3)} = m^{(2)}(m^{rhp})^{-1},\) we will present details in the Sect. 4.4.

4. In Sect. 5, we discuss related properties of the RH problem in the case of \(\xi > 6\) similar to Sect. 4.

5. Finally, in Sect. 6, based on a series of transformations above, a decomposition formula for \(m(z)\) is found

\[
m(z) = T(\infty)^{-\sigma_3}m^{(3)}(z)E(z)m^{sol}(z)T(\infty)^{\sigma_3} \left[I + z^{-1}T_1^{\sigma_3} + \mathcal{O}(z^{-2})\right],
\]

from which we obtain the long-time asymptotic behavior for the solutions of the Cauchy problem \((1.1)-(1.2)\) of the nonlocal mKdV equation in regions \(\xi < -6\) and \(\xi > 6\).
2 The Spectral Analysis and the RH Problem

The nonlocal mKdV equation (1.1) possesses the nonlocal Lax pair

\[ \Phi_x = X \Phi, \quad \Phi_t = T \Phi, \]  

(2.1)

where

\[ X = ik\sigma_3 + Q, \]
\[ T = [4k^2 + 2\sigma q(x, t)q(-x, -t)]X - 2i k\sigma_3 Q_x + [Q_x, Q] - Q_{xx}, \]
\[ Q = \begin{bmatrix} 0 & q(x, t) \\ \sigma q(-x, -t) & 0 \end{bmatrix}, \quad \sigma = \pm 1, \]

(2.2)

and \( k \) is a spectral parameter.

Taking \( q(x, t) = q_\pm \) in the Lax pair (2.1), we get the spectral problems

\[ \phi_x = X_\pm \phi, \quad \phi_t = T_\pm \phi, \]

(2.3)

with

\[ X_\pm = ik\sigma_3 + Q_\pm, \quad T_\pm = (4k^2 - 2)X_\pm, \]

(2.4)

and

\[ Q_\pm = \begin{bmatrix} 0 & q_\pm \\ -q_\pm & 0 \end{bmatrix}. \]

(2.5)

The eigenvalues of \( X_\pm \) are \( \pm i\lambda \), which satisfy

\[ \lambda^2 - k^2 = 1. \]

(2.6)

To avoid multi-valued case of eigenvalue \( \lambda \), we introduce a uniformization variable

\[ z = k + \lambda, \]

(2.7)

and obtain two single-valued functions

\[ k = \frac{1}{2}(z - \frac{1}{z}), \quad \lambda = \frac{1}{2}(z + \frac{1}{z}). \]

(2.8)

We can define two domains \( D_+, D_- \) and their boundary \( \Sigma \) on \( z \)-plane by

\[ D_+ = \{ z \in \mathbb{C} : (|z| - 1)\text{Im}z > 0 \}, \]
\[ D_- = \{ z \in \mathbb{C} : (|z| - 1)\text{Im}z < 0 \}, \]
\[ \Sigma = \mathbb{R} \cup \{ z \in \mathbb{C} : |z| = 1 \}, \]
Fig. 2 The complex $z$-plane showing the discrete spectrums [zeros of scattering data $s_{11}(z)$ (red) in yellow region and those of scattering data $s_{22}(z)$ (blue) in white region]. The yellow and white regions stand for $D_+$ and $D_-$, respectively.

which are yellow region and white region respectively shown in the Fig. 2.

By calculation, we know that the Jost solutions $\Phi_{\pm}(x, t, z)$ satisfy

$$\Phi_{\pm}(x, t, z) \sim E_{\pm}(z)e^{i\theta(x, t; z)\sigma_3},$$

(2.9)

where

$$E_{\pm}(k) = \begin{bmatrix} 1 & \frac{iq_{\pm}}{k+i\lambda} \\ \frac{iq_{\pm}}{k+i\lambda} & 1 \end{bmatrix}, \quad \theta(x, t; k) = \lambda \left[ \frac{x}{t} + (4k^2 - 2) \right].$$

For convenience, we introduce the modified Jost solutions $\mu_{\pm}(x, t, z)$ by eliminating the exponential oscillations

$$\mu_{\pm}(x, t, z) = \Phi_{\pm}(x, t, z)e^{-i\theta(x, t; z)\sigma_3},$$

(2.10)

such that

$$\lim_{x \to \pm\infty} \mu_{\pm}(x, t, z) = E_{\pm}(z),$$

(2.11)

and $\mu_{\pm}$ admit the Volterra type integral equations

$$\mu_{\pm}(x, t; z) = E_{\pm}(z)$$

$$+ \begin{cases} \int_{\pm\infty}^{x} E_{\pm}(z)e^{i\lambda(x-y)\sigma_3}[E_{\pm}^{-1}(z)\Delta Q_{\pm}(y, t)\mu_{\pm}(y, t, z)]dy, & k \neq \pm i, \\
\int_{\pm\infty}^{x} [I + (x-y)X_{\pm}]\Delta Q_{\pm}(y, t)\mu_{\pm}(y, t, z)dy, & k = \pm i, 
\end{cases}$$

(2.12)
Proposition 2.1 Given $n \in \mathbb{N}_0$, let $q \in L^{1,n+1}(\mathbb{R})$, $q' \in W^{1,1}(\mathbb{R})$.

- $\mu_{+,1}$ and $\mu_{-,2}$ can be analytically extended to $D_+$ and continuously extended to $D_+ \cup \Sigma_0$, $\mu_{-,1}$ and $\mu_{+,2}$ can be analytically extended to $D_-$ and continuously extended to $D_- \cup \Sigma_0$;

- The map $q \to \frac{\partial^n}{\partial x^n} \mu_{\pm,i}(z)$ ($i = 1, 2, n \geq 0$) are Lipschitz continuous, specifically, for any $x_0 \in \mathbb{R}$, $\mu_{-,1}(z)$ and $\mu_{+,2}(z)$ are continuously differentiable mappings:

$$
\partial^n_x \mu_{+,1} : \tilde{D}_+ \setminus \{0, \pm i\} \to L^\infty_{\text{loc}}(\tilde{D}_+ \setminus \{0, \pm i\}, C^1([x_0, \infty), \mathbb{C}^2)) \\
\cap W^{1,\infty}(\{x_0, \infty\}, C^2)),
$$

$$
\partial^n_x \mu_{-,2} : \tilde{D}_+ \setminus \{0, \pm i\} \to L^\infty_{\text{loc}}(\tilde{D}_+ \setminus \{0, \pm i\}, C^1((-\infty, x_0], \mathbb{C}^2)) \\
\cap W^{1,\infty}((-\infty, x_0], C^2)),
$$

$\mu_{+,1}(z)$ and $\mu_{-,2}(z)$ are continuously differentiable mappings:

$$
\partial^n_x \mu_{-,1} : \tilde{D}_- \setminus \{0, \pm i\} \to L^\infty_{\text{loc}}(\tilde{D}_- \setminus \{0, \pm i\}, C^1((-\infty, x_0], \mathbb{C}^2)) \\
\cap W^{1,\infty}((-\infty, x_0], C^2)),
$$

$$
\partial^n_x \mu_{+,2} : \tilde{D}_- \setminus \{0, \pm i\} \to L^\infty_{\text{loc}}(\tilde{D}_- \setminus \{0, \pm i\}, C^1([x_0, \infty), \mathbb{C}^2)) \\
\cap W^{1,\infty}([x_0, \infty], C^2)).
$$

- Let $K$ be a compact neighborhood of $\{-i, i\}$ in $\tilde{D}_+ \setminus \{0\}$. Set $x^\pm = \max\{\pm x, 0\}$, then there exists a constant $C$ such that for $z \in K$ we have

$$
|\mu_{+,1}(z) - (1, iz^{-1})^T| \leq C(x^-)e^{C\int_x^\infty (y-x)q^{-1}dy}\|q - \tilde{q}\|_{L^{1,1}(x, \infty)}, \quad (2.13)
$$

i.e., the map $z \to \mu_{+,1}(z)$ extends as a continuous map to the points $\pm i$ with values in $C^1([x_0, \infty), \mathbb{C}) \cap W^{1,\infty}([x_0, \infty], \mathbb{C})$ for any preassigned $x_0 \in \mathbb{R}$. Moreover, the map $q \to \mu^1_1(z)$ is locally Lipschitz continuous from:

$$
L^{1,1}(\mathbb{R}) \to L^\infty(\tilde{D}_+ \setminus \{0\}, C^1([x_0, \infty), \mathbb{C}) \cap W^{1,\infty}([x_0, \infty], \mathbb{C}). \quad (2.14)
$$

Analogous statements hold for $\mu_{+,2}$ and for $\mu_{-,j}$ ($j = 1, 2$). Furthermore, the maps $z \to \partial^n_x \mu_{+,1}(z)$ and $q \to \partial^n_x \mu_{+,1}(z)$ also satisfy

$$
|\partial^n_x \mu_{+,1}(z)| \leq F_n \left[ (1 + |x|)^{n+1}\|q - 1\|_{L^{1,1}(x, \infty)} \right], \quad z \in K. \quad (2.15)
$$
The asymptotic behavior of $\mu_{\pm,j}$, $j = 1, 2$ could be described by following proposition.

**Proposition 2.2** Suppose that $q \mp q_{\pm} \in L^{1,n+1}(\mathbb{R})$ and $q' \in W^{1,1}(\mathbb{R})$. Then as $z \to \infty$, we have

$$
\mu_{\pm,1}(z) = e_1 - \frac{i}{z} \left( \int_{-\infty}^{\infty} \left[ \sigma q(y, t)q(-y, -t) + 1 \right] dy \right) + O(z^{-2}),
$$

$$
\mu_{\pm,2}(z) = e_2 + \frac{i}{z} \left( \int_{-\infty}^{\infty} \left[ q(x, t)q(y, t)q(-y, -t) + 1 \right] dy \right) + O(z^{-2}),
$$

and as $z \to 0$, we have

$$
\mu_{\pm,1}(z) = \frac{q_{\pm}}{z} e_2 + O(1),
$$

$$
\mu_{\pm,2}(z) = \frac{q_{\pm}}{z} e_1 + O(1),
$$

where $e_1 = (1, 0)^T$, $e_2 = (0, 1)^T$.

It follows that $\Phi_{\pm}(x, t; z)$ are fundamental solutions of Lax pair (2.1) as $z \in \Sigma_0$, thus there exists a constant scattering matrix $S(z) = (s_{ij}(z))_{2 \times 2}$ (independence of $x$ and $t$) such that

$$
\Phi_+(x, t, z) = \Phi_-(x, t, z) S(z), \quad z \in \Sigma_0.
$$

Owing to (2.10) and (2.20), $\Phi_\pm$ and $S(z)$ admits the following symmetry.

**Proposition 2.3** The symmetries for Jost solutions $\Phi_{\pm}(x, t, z)$ and scattering matrix $S(z)$ in $z \in \Sigma$ are listed as follows:

- **The first symmetry**

  $$
  \Phi_{\pm}(x, t, z) = \sigma_4 \Phi_{\mp}(-x, -t, -\bar{z}) \sigma_4, \quad S(z) = \sigma_4 [S(-\bar{z})]^{-1} \sigma_4,
  $$

  where $\sigma_4$ is defined as

  $$
  \sigma_4 = \begin{cases} 
  \sigma_1, & \sigma = -1, \\
  \sigma_2, & \sigma = 1. 
  \end{cases}
  $$

- **The second symmetry**

  $$
  \Phi_{\pm}(x, t, z) = \Phi_{\mp}(-x, -t, -\bar{z}), \quad S(z) = \bar{S}(-\bar{z}).
  $$

- **The third symmetry**

  $$
  \Phi_{\pm}(x, t, z) = \frac{i}{z} \Phi_{\pm}(x, t, -z^{-1}) \sigma_3 Q_{\pm}, \quad S(z) = (\sigma_3 Q_-)^{-1} S(-z^{-1}) (\sigma_3 Q_+).
  $$

\[Springer\]
The reflection coefficients are defined as:

\[ \rho(z) = \frac{s_{21}(z)}{s_{11}(z)}, \quad \tilde{\rho}(z) = \frac{s_{12}(z)}{s_{22}(z)}, \quad z \in \Sigma. \tag{2.25} \]

The following proposition provides some essential properties for \( s_{ij}(z), i, j = 1, 2 \) and \( \rho(z), \tilde{\rho}(z) \).

**Proposition 2.4** Let \( q \neq q_\pm \in L^{1,n+1}(\mathbb{R}) \) and \( q' \in W^{1,1}(\mathbb{R}) \), then

1. For \( z \in \{ z \in \mathbb{C} : |z| = 1 \}/\{ \pm i \} \),
   \[ |\rho(z)\tilde{\rho}(z)| \leq 1 - |s_{11}(z)|^{-2} < 1. \tag{2.26} \]

2. \( s_{ij}(z), i, j = 1, 2 \) and the reflection coefficient \( \rho(z), \tilde{\rho}(z) \) satisfy the symmetries
   \[ s_{11}(z) = -\sigma s_{22}(-z^{-1}), \quad s_{12}(z) = -\sigma s_{21}(-z^{-1}), \quad \tilde{\rho}(z) = \rho(-z^{-1}). \tag{2.27} \]

3. The scattering data have the asymptotics
   \[ \lim_{z \to \infty} (s_{11}(z) - 1)z = i \int_{\mathbb{R}} [\sigma q(y, t)q(-y, -t) - 1] dy, \tag{2.28} \]
   \[ \lim_{z \to 0} s_{11}(z) = -\sigma, \tag{2.29} \]
   \[ |s_{21}(z)| = \mathcal{O}(|z|^{-2}), \quad \text{as} \quad |z| \to \infty, \tag{2.30} \]
   \[ |s_{21}(z)| = \mathcal{O}(|z|^2), \quad \text{as} \quad |z| \to 0. \tag{2.31} \]

So that

\[ \rho(z), \tilde{\rho}(z) \sim z^{-2}, \quad |z| \to \infty; \quad \rho(z), \tilde{\rho}(z) \sim 0, \quad |z| \to 0. \tag{2.32} \]

4. Although \( s_{11}(z) \) and \( s_{21}(z) \) have singularities at points \( \pm i \), we can claim that the reflection coefficient \( \rho(z), \tilde{\rho}(z) \) remain bounded at \( z = \pm i \) and \( |\rho(\pm i)| = 1, |\tilde{\rho}(\pm i)| = 1 \). In fact, by direct calculation, we obtain

\[ s_{11}(z) = \frac{\mp s_{z}^{\pm}}{z \mp 1} + \mathcal{O}(1), \quad s_{21}(z) = \frac{-\sigma s_{z}^{\pm}}{z \mp 1} + \mathcal{O}(1), \tag{2.33} \]

\[ \lim_{z \to \pm i} \rho(z) = \lim_{z \to \pm i} \tilde{\rho}(z) = \pm \sigma, \tag{2.34} \]

where \( s_{z}^{\pm} = \frac{1}{2\pi} \text{det}(\Phi_{+1}(\pm i), \Phi_{-2}(\pm i)) \).

Since one cannot exclude the possibilities of zeros for \( s_{11}(z) \) and \( s_{22}(z) \) along \( \Sigma \). To solve the Riemann-Hilbert problem in the inverse process, we only consider the potentials without spectral singularities, i.e., \( s_{11}(z) \neq 0, s_{22}(z) \neq 0 \) for \( z \in \Sigma \).

The next proposition shows that, given data \( q_0(x) \) with sufficient smoothness and decay properties, the reflection coefficients will also be smooth and decaying [31].
Proposition 2.5 For given \( q \mp q_\pm \in L^{1,2}(\mathbb{R}) \) and \( q' \in W^{1,1}(\mathbb{R}) \), we then have \( \rho(z), \tilde{\rho}(z) \in H^1(\Gamma) \), where \( \Gamma \) is defined in (4.1).

Corollary 2.1 For given \( q \mp q_\pm \in L^{1,2}(\mathbb{R}) \), \( q' \in W^{1,1}(\mathbb{R}) \), we then have \( \rho(z), \tilde{\rho}(z) \in H^1.1(\Gamma) \).

Suppose that \( s_{11}(z) \) has \( N_1 \) and \( N_2 \) simple zeros, respectively, in \( D_+ \cap \{ z \in \mathbb{C} : \text{Re}z > 0 \} \) denoted by \( z_k, k = 1, 2, \ldots, N_1 \) and in \( D_+ \cap \{ z \in \mathbb{C} : \text{Re}z = 0 \} \) denoted by \( i\omega_k, k = 1, 2, \ldots, N_2 \). It follows from the symmetry relations of the scattering coefficients that

\[
\begin{align*}
  s_{11}(z_k) &= s_{11}(\bar{z}_k) = s_{22}(-z_k^{-1}) = s_{22}(\bar{z}_k^{-1}), \quad k = 1, 2, \ldots, N_1, \\
  s_{11}(i\omega_k) &= s_{22}(i\omega_k^{-1}), \quad k = 1, 2, \ldots, N_2.
\end{align*}
\]

It is convenient to define that

\[
\eta_k = \begin{cases} 
  z_k, & k = 1, 2, \ldots, N_1, \\
  -\bar{z}_{k-N_1}, & k = N_1 + 1, N_1 + 2, \ldots, 2N_1, \\
  i\omega_{k-2N_1}, & k = 2N_1 + 1, 2N_1 + 2, \ldots, 2N_1 + N_2,
\end{cases}
\]

and

\[
\hat{\eta}_k = -\eta_k^{-1}, \quad k = 1, 2, \ldots, 2N_1 + N_2.
\]

Thus, the discrete spectrum is given by

\[
\mathcal{Z} \cup \hat{\mathcal{Z}} = \{ \eta_k, \hat{\eta}_k \}_{k=1}^{2N_1+N_2},
\]

and the distribution of \( \mathcal{Z} \cup \hat{\mathcal{Z}} \) on the \( z \)-plane is shown in Fig. 2.

Given \( z_0 \in \mathcal{Z} \), it follows from the Wronskian representations and \( s_{11}(z_0) = 0 \) that \( \Phi_{+,1}(x,t,z_0) \) and \( \Phi_{-,2}(x,t,z_0) \) are linearly dependent; Given \( z_0 \in \hat{\mathcal{Z}} \), it follows from the Wronskian representations \( s_{22}(z_0) = 0 \) and that \( \Phi_{+,2}(x,t,z_0) \) and \( \Phi_{-,1}(x,t,z_0) \) are linearly dependent. For convenience, we denote the proportional coefficient in the following definition of \( b[z_0] \) by \( \frac{\Phi_{+,1}(x,t,z_0)}{\Phi_{-,2}(x,t,z_0)} \) or \( \frac{\Phi_{+,2}(x,t,z_0)}{\Phi_{-,1}(x,t,z_0)} \) according to the region \( z_0 \) belongs to. Let

\[
b[z_0] = \begin{cases} 
  \frac{\Phi_{+,1}(x,t,z_0)}{\Phi_{-,2}(x,t,z_0)}, & z_0 \in \mathcal{Z}, \\
  \frac{\Phi_{+,2}(x,t,z_0)}{\Phi_{-,1}(x,t,z_0)}, & z_0 \in \hat{\mathcal{Z}},
\end{cases}
\]

\[
A[z_0] = \begin{cases} 
  \frac{b[z_0]}{s_{11}(z_0)}, & z_0 \in \mathcal{Z}, \\
  \frac{b[z_0]}{s_{22}(z_0)}, & z_0 \in \hat{\mathcal{Z}}.
\end{cases}
\]

Proposition 2.6 For the given \( z_0 \in \mathcal{Z} \cup \hat{\mathcal{Z}} \), there exist three relations for \( b[z_0], s'_{11}(z_0) \) and \( s_{22}^2(z_0) \):

\[\text{Springer}\]
• The first relation
\[ b[z_0] = -\frac{\sigma}{b[-\bar{z}_0]}, \quad s'_{11}(z_0) = -s'_{11}(-\bar{z}_0), \quad s'_{22}(z_0) = -s'_{22}(-\bar{z}_0). \quad (2.40) \]

• The second relation
\[ b[z_0] = -b[-\bar{z}_0], \quad s'_{11}(z_0) = -s'_{11}(-\bar{z}_0), \quad s'_{22}(z_0) = -s'_{22}(-\bar{z}_0). \quad (2.41) \]

• The third relation
\[ b[z_0] = -\sigma b[-z_0^{-1}], \quad s'_{11}(z_0) = -\sigma z_0^{-2} s'_{22}(-z_0^{-1}). \quad (2.42) \]

From the first relation, one concludes that imaginary discrete spectrum \( i\omega_k \) exists if and only if \( \sigma = -1 \). That is to say that as \( \sigma = 1 \), one has \( N_2 = 0 \). We give the following relation among discrete spectrum \( \mathbb{Z} \cup \hat{\mathbb{Z}} \).

**Proposition 2.7** The relations for \( b[\cdot] \) and \( A[\cdot] \) in \( \mathbb{Z} \cup \hat{\mathbb{Z}} \) are given by
\[ b[\eta_k] = b[-\bar{\eta}_k] = -\sigma b[\bar{\eta}_k] = -\sigma b[\bar{\eta}_k^{-1}], \quad b^2[\eta_k] = 1 \]
\[ s'_{11}(\eta_k) = -s'_{11}(\bar{\eta}_k) = -\sigma \bar{\eta}_k^2 s'_{22}(\bar{\eta}_k) = \sigma \bar{\eta}_k^2 s'_{22}(\bar{\eta}_k^{-1}). \quad (2.43) \]

As \( \sigma = -1 \), one has
\[ b[i\omega_k] = b[i\omega_k^{-1}], \quad s'_{11}(i\omega_k) = -\omega_k^2 s'_{22}(i\omega_k^{-1}). \quad (2.44) \]

Then,
\[ A[\hat{\eta}_k] = \hat{\eta}_k^2 A[\eta_k]. \quad (2.45) \]

Define a sectionally meromorphic matrix as follows
\[ m(z) := m(x, t; z) = \begin{cases} \left( \frac{\mu_{+,1}(x, t; z)}{s_{11}(z)}, \mu_{+,2}(x, t; z) \right), & z \in D_+, \\ \left( \frac{\mu_{-,1}(x, t; z)}{s_{22}(z)}, \mu_{-,2}(x, t; z) \right), & z \in D_-, \end{cases} \quad (2.46) \]

and
\[ m_\pm(x, t, z) = \lim_{z' \to z} m(x, t, z') \quad z \in \Sigma, \quad (2.47) \]

the multiplicative matrix Riemann-Hilbert problem can be proposed as follows.

**RHP 2.1** Find a \( 2 \times 2 \) matrix-valued function \( m(z) := m(x, t; z) \) such that
• Analyticity: $m(z)$ is analytical in $\mathbb{C} \setminus (\Sigma \cup \mathcal{Z} \cup \hat{\mathcal{Z}})$ and has simple poles in $\mathcal{Z} \cup \hat{\mathcal{Z}} = \{\eta_k, \hat{\eta}_k\}_{k=1}^{2N_1+N_2}$.

• Jump relation: $m_+(x, t, z) = m_-(x, t, z)v(z)$, where

$$v(z) = \begin{bmatrix} 1 - \rho(z)\tilde{\rho}(z) & -\rho(z)e^{2it\theta(x,t,z)} \\ \rho(z)e^{-2it\theta(x,t,z)} & 1 \end{bmatrix}, \quad z \in \Sigma. \quad (2.48)$$

• Asymptotic behavior:

$$m(x, t; z) = I + O(z^{-1}), \quad z \to \infty, \quad (2.49)$$

$$m(x, t; z) = \frac{i}{z}\sigma_3 Q_- + O(1), \quad z \to 0. \quad (2.50)$$

• Residue conditions

$$\text{Res} m(z) = \lim_{z \to \eta_k} m(z) \begin{bmatrix} 0 \\ A[\eta_k]e^{-2it\theta(x,t,\eta_k)} \end{bmatrix}, \quad (2.51)$$

$$\text{Res} m(z) = \lim_{z \to \hat{\eta}_k} m(z) \begin{bmatrix} 0 A[\hat{\eta}_k]e^{2it\theta(x,t,\hat{\eta}_k)} \\ 0 \end{bmatrix}, \quad (2.52)$$

where $\theta(x, t, z) = \lambda(z)[\frac{x}{t} + (4k^2(z) - 2)]$.

The potential $q(x, t)$ is found by the reconstruction formula

$$q(x, t) = -i (m_1)_{12} = -i \lim_{z \to \infty} (zm)_{12}, \quad (2.53)$$

where $m_1$ appears in the expansion of $m = I + z^{-1}m_1 + O(z^{-2})$ as $z \to \infty$.

3 Distribution of Saddle Points and Signature Table

We notice that the long-time asymptotic behavior of RHP 2.1 is influenced by the growth and decay of the exponential function

$$e^{\pm 2it\theta}, \quad \theta(z) = \frac{1}{2} \left( z + \frac{1}{z} \right) \left[ \frac{x}{t} - 2 + \left( z - \frac{1}{z} \right)^2 \right], \quad (3.1)$$

which not only appear in jump matrix $v(z)$ but also in the residue condition. Based on this observation, we shall make analysis for the real part of $\pm 2it\theta(z)$ to ensure the exponential decaying property. Let $\xi = \frac{x}{t}$, we consider the stationary phase points and the real part of $2it\theta(z)$:

$$\theta'(z) = -\frac{(1-z^2)(3z^4+\xi^2z^2+3)}{2z^4}, \quad (3.2)$$
Fig. 3  Plots of the distributions for saddle points: a $\xi < -6$, b $-6 < \xi < 6$, c $\xi > 6$. The black dotted curve is an unit circle. The red curve shows the $\text{Re}\theta'(z) = 0$, and the green curve shows the $\text{Im}\theta'(z) = 0$. The intersection points are the saddle points which express $\theta'(z) = 0$

\[
\text{Re}[2i\text{t}\theta(z)] = -2i\text{t}\text{Im}\theta(z) = -\text{t}\text{Im}z(1 - |z|^{-2}) \\
\left[\xi - 3 + (1 + |z|^{-2} + |z|^{-4})(3\text{Re}^2z - \text{Im}^2z)\right].
\] (3.3)

From Eq. (3.2), we find six stationary phase points of $\theta(z)$:

\[
\pm 1, \quad \pm \sqrt{-\xi \pm \sqrt{-36 + \xi^2}} \quad \text{with} \quad \sqrt{6}.
\] (3.4)

Except to $z = \pm 1$, there are also four stationary phase points, whose distribution depends on different $\xi$ is as follows:

(i) For $\xi < -6$, the four phase points are located on real axis $\mathbb{R}$ corresponding to Fig. 3a. Among them, two are inside the unit circle and the other two are outside the unit circle;

(ii) For $-6 < \xi < 6$, the four phase points are all located on the unit circle and they are symmetrical to each other, which is corresponded to Fig. 3b;

(iii) For $\xi > 6$, the four phase points are located on the imaginary axis $i\mathbb{R}$ corresponding to Fig. 3c. The two of them are inside the unit circle and the other two are outside the unit circle.;

Further, the decaying regions of $\text{Re}[2i\text{t}\theta(z)]$ are shown in Fig. 4. We will mainly discuss this case (i) and (iii) in the present paper.

Remark 3.1  According to (3.1), $\theta(z)$ allows the following symmetry:

\[
\theta(-z^{-1}) = -\theta(z), \quad \theta(-\bar{z}) = \overline{-\theta(z)}, \quad \theta(-\bar{z}^{-1}) = -\overline{\theta(z)}.
\] (3.5)
Fig. 4  Signature table of $\text{Re}(2i\theta)$ with different $\xi$: a $\xi < -6$, b $-6 < \xi < 6$, c $\xi > 6$. $\text{Re}(2i\theta) < 0$ in the green region and $\text{Re}(2i\theta) > 0$ in the white region. In other words, $|e^{2i\theta}| \to 0$ as $t \to \infty$ in the green region and $|e^{-2i\theta}| \to 0$ as $t \to \infty$ in the white region. Moreover, $\text{Re}(2i\theta) = 0$ on the green curve.

4 Asymptotic Analysis on RH Problem in Region $\xi < -6$

4.1 Jump Matrix Factorizations

In order to perform the long time analysis using the $\tilde{\partial}$ steepest descent method, we need to perform two essential operations:

(i) decompose the jump matrix $v(z)$ into appropriate upper/lower triangular factorizations so that the oscillating factor $e^{\pm 2i\theta(z)}$ are decaying in corresponding region respectively;

(ii) interpolate the poles by trading them for jumps along small closed loops enclosing each pole [22].

The first step is aided by two well known factorizations of the jump matrix $v(z)$

$$v(z) = \begin{cases} 
\begin{bmatrix}
1 - \rho(z)e^{2i\theta(z)} & 0 \\
0 & 1
\end{bmatrix} 
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}, & z \in \tilde{\Gamma} \\
\begin{bmatrix}
\frac{1}{1 - \rho(z)} & 0 \\
\frac{1}{1 - \rho(z)}e^{-2i\theta(z)} & 1
\end{bmatrix} 
\begin{bmatrix}
\frac{1}{\rho(z)}(1 - \rho(z)) & 0 \\
0 & \frac{1}{\rho(z)}(1 - \rho(z))
\end{bmatrix} 
\begin{bmatrix}
1 - \frac{\rho(z)}{1 - \rho(z)}e^{2i\theta(z)} & 0 \\
0 & 1
\end{bmatrix}, & z \in \Gamma,
\end{cases}$$

where $\zeta_j$, $j = 1, 2, \ldots, 6$ are six stationary phase points and

$$\tilde{\Gamma} = \{z \in \mathbb{C} : |z| = 1\} \cup (\zeta_6, \zeta_2) \cup (\zeta_1, \zeta_5);$$

$$\Gamma = \Sigma/\tilde{\Gamma} = (-\infty, \zeta_6) \cup (\zeta_2, 0) \cup (0, \zeta_1) \cup (\zeta_5, +\infty).$$

(4.1)

To remove the diagonal matrix in the middle of the second factorization, we introduce a scalar RH problem.

**RHP 4.1** Find a scalar function $\delta(z) := \delta(z; \xi)$, which is defined by the following properties:
• **Analyticity:** $\delta(z)$ is analytical in $\mathbb{C}\setminus\Gamma$.

• **Jump relation:**

$$
\begin{align*}
\delta^+(z) &= \delta^-(z)(1 - \rho(z)\tilde{\rho}(z)), \quad z \in \Gamma; \\
\delta^+(z) &= \delta^-(z), \quad z \in \tilde{\Gamma}.
\end{align*}
$$

• **Asymptotic behavior:**

$$
\delta(z) \to 1, \ z \to \infty
$$

Utilizing the Plemelj’s formula, we are arriving

$$
\delta(z) = \exp \left[ -\frac{1}{2\pi i} \int_{\Gamma} \log (1 - \rho(s)\tilde{\rho}(s)) \frac{1}{s-z} ds \right].
$$

Taking $\nu(z) = -\frac{1}{2\pi} \log(1 - \rho(z)\tilde{\rho}(z))$, then we can express

$$
\delta(z) = \exp \left( i \int_{\Gamma} \frac{\nu(s)}{s-z} ds \right).
$$

**Remark 4.1** $\nu(\zeta_i)$ are complex-valued and

$$
\text{Im} \nu(\zeta_i) = -\frac{1}{2\pi} \arg(1 - \rho(\zeta_i)\tilde{\rho}(\zeta_i)), \quad i = 1, 2, 5, 6.
$$

Assuming that $|\text{Im} \nu(z)| < \frac{1}{2}$ for all $z \in \mathbb{R}$, it means that $\log(1 - \rho(z)\tilde{\rho}(z))$ is single-valued and the singularity of $\delta(z)$ at $z = \zeta_i$ are square integrable. In a word, $\delta(z)$ is bounded at $z = \zeta_i$ [32].

For brevity, we denote $\mathcal{N} = \{1, 2, \ldots, 2N_1 + N_2\}$. Moreover, we introduce a small positive constant $\varrho$:

$$
\varrho = \frac{1}{2} \min \left\{ \min_{k \in \mathcal{N}} \{|\text{Im}(\eta_k)|, |\text{Im}(\hat{\eta}_k)|\}, \min_{\lambda, \mu \in \mathbb{Z} \cup \mathbb{\hat{Z}}, \lambda \neq \mu} |\lambda - \mu|, \min_{\lambda \in \mathbb{Z} \cup \mathbb{\hat{Z}}, i=1,2,\ldots,6} |\lambda - \zeta_i| \right\}.
$$

Taking $\delta_0 < \varrho$, we define $\Delta, \nabla$ and $\Lambda$ of $\mathcal{N}$ as follows:

$$
\begin{align*}
\Delta &= \{ k \in \mathcal{N} : \text{Re}(2i\theta(\eta_k)) < 0 \}, \quad \nabla = \{ k \in \mathcal{N} : \text{Re}(2i\theta(\eta_k)) > 0 \}, \\
\Lambda &= \{ k \in \mathcal{N} : |\text{Re}(2i\theta(\eta_k))| < \delta_0 \}.
\end{align*}
$$
To distinguish different type of zeros, we further give

\[ \Delta_1 = \{ k \in \{1, 2, \ldots, N_1\} : \text{Re}(2i\theta(z_k)) < 0 \}, \]
\[ \nabla_1 = \{ k \in \{1, 2, \ldots, N_1\} : \text{Re}(2i\theta(z_k)) > 0 \}, \]
\[ \Delta_2 = \{ l \in \{1, 2, \ldots, N_2\} : \text{Re}(2i\theta(i\omega_l)) < 0 \}, \]
\[ \nabla_2 = \{ l \in \{1, 2, \ldots, N_2\} : \text{Re}(2i\theta(i\omega_l)) > 0 \}, \]
\[ \Lambda_1 = \{ k \in \{1, 2, \ldots, N_1\} : |\text{Re}(2i\theta(z_k))| < \delta_0 \}, \]
\[ \Lambda_2 = \{ l \in \{1, 2, \ldots, N_2\} : |\text{Re}(2i\theta(i\omega_l))| < \delta_0 \}. \]

Define the function

\[
T(z) := T(z; \xi) = \prod_{k \in \Delta_1} \prod_{l \in \Delta_2} \frac{(z + z_k^{-1})(z - \bar{z}_k^{-1})(z - i\omega_l^{-1})}{(zz_k^{-1} - 1)(z\bar{z}_k^{-1} + 1)i\omega_l^{-1}z + 1} \exp \left[ i \int_\Gamma \nu(s) \left( \frac{1}{s - z} - \frac{1}{2s} \right) ds \right].
\]

**Proposition 4.1** The function defined by Eq. (4.9) has following properties:

(a) \( T(z) \) is meromorphic in \( \mathbb{C} \setminus \Gamma \), and for each \( k \in \Delta_1 \), \( l \in \Delta_2 \), \( z_k, -\bar{z}_k, i\omega_l \) are simple poles and \( -z_k^{-1}, \bar{z}_k^{-1}, i\omega_l^{-1} \) are simple zeros of \( T(z) \).

(b) \( T(z) = -[T(-z^{-1})]^{-1} \).

(c) For \( z \in \Gamma \),

\[
\frac{T_+(z)}{T_-(z)} = 1 - \rho(z)\tilde{\rho}(z). \tag{4.9}
\]

(d) For \( i = 1, 2, 5, 6 \), as \( z \to \zeta_i \) along any ray \( \zeta_i + re^{i\phi} \) with \( r > 0 \) and \( |\phi| < \pi \),

\[
|T(z) - T_i(z) - \text{i}^{iv(\zeta_i)}| \leq c \| \log(1 - \rho(z)\tilde{\rho}(z)) \|_{H^1(\Gamma)} |z - \zeta_i|^{1 - \text{Im}v(\zeta_i)}, \tag{4.10}
\]

\[
|T(z) - T(\pm 1)| \lesssim |z \mp 1|^\frac{1}{2}, \tag{4.11}
\]

where

\[
T_i(z) = \prod_{k \in \Delta_1} \prod_{l \in \Delta_2} \frac{(\zeta_i + z_k^{-1})(\zeta_i - \bar{z}_k^{-1})(\zeta_i - i\omega_l^{-1})}{(\zeta_i z_k^{-1} - 1)(\zeta_i \bar{z}_k^{-1} + 1)i\omega_l^{-1}\zeta_i + 1} \exp[i\beta_i(\zeta_i, \xi)],
\]

\[
\beta_1(z) = \left( \int_{-\infty}^{\xi} + \int_{0}^{\xi} + \int_{\xi}^{+\infty} \right) \frac{\nu(s)}{s - z} ds + \int_{0}^{\xi} \frac{\nu(s) + \nu(\xi)}{s - z} ds + \nu(\xi),
\]

\[
\log z - \int_{\Gamma} \frac{\nu(s)}{2s} ds.
\]
\[ \beta_2(z) = \left( \int_{-\infty}^{\xi_1} + \int_0^{\xi_2} + \int_{\xi_6}^{+\infty} \right) \frac{\nu(s)}{s-z} \, ds + \int_{\xi_2}^{0} \frac{\nu(s) - \nu(\xi_2)}{s-z} \, ds + \nu(\xi_2) \log (-z) \]
\[ - \int_{\Gamma} \frac{\nu(s)}{2s} \, ds, \]
\[ \beta_5(z) = \left( \int_{-\infty}^{\xi_6} + \int_0^{\xi_1} + \int_{\xi_5}^{+\infty} \right) \frac{\nu(s)}{s-z} \, ds + \int_{\xi_5}^{+\infty} \frac{\nu(s) - \chi_5(s) \nu(\xi_5)}{s-z} \, ds + \nu(\xi_5) \log (\xi_5 + 1 - z) \]
\[ - \int_{\Gamma} \frac{\nu(s)}{2s} \, ds, \]
\[ \beta_6(z) = \left( \int_{0}^{\xi_2} + \int_0^{\xi_1} + \int_{\xi_5}^{+\infty} \right) \frac{\nu(s)}{s-z} \, ds + \int_{\xi_5}^{+\infty} \frac{\nu(s) + \chi_6(s) \nu(\xi_6)}{s-z} \, ds + \nu(\xi_6) \log (z - \xi_6 + 1) \]
\[ - \int_{\Gamma} \frac{\nu(s)}{2s} \, ds. \] (4.12)

To trade the poles for jumps on small contours encircling each pole, we introduce a matrix-valued function \(G(z)\):

\[
G(z) = \begin{cases} 
& \begin{bmatrix} 1 & 0 \\ -A[\eta_k]e^{2i\theta(\eta_k)} & 1 \\ \frac{z-\eta_k}{A[\eta_k]e^{-2i\theta(\eta_k)}} & 1 \\ 0 & 1 \\ 1 - \frac{z-\eta_k}{A[\eta_k]e^{2i\theta(\eta_k)}} & 1 \\ -\frac{z-\hat{\eta}_k}{A[\hat{\eta}_k]e^{-2i\theta(\hat{\eta}_k)}} & 1 \\ 0 & 1 \\ 1 - \frac{z-\hat{\eta}_k}{A[\hat{\eta}_k]e^{2i\theta(\hat{\eta}_k)}} & 1 \\ 0 & 1 \\ 1 & \end{bmatrix}, \quad |z-\eta_k| < \rho, \quad k \in \nabla/\Lambda, \\
& |z-\hat{\eta}_k| < \rho, \quad k \in \nabla/\Lambda, \\
& \text{elsewhere.} 
\end{cases} \] (4.13)

Consider the following contour

\[ \Sigma^{(1)} = \Sigma \cup \Sigma_{cir}, \] (4.14)

where \( \Sigma_{cir} = \bigcup_{k \in \mathcal{N}/\Lambda} \{z \in \mathbb{C} : |z-\eta_k| = \rho \text{ or } |z-\hat{\eta}_k| = \rho \} \) depicted in Fig. 5.
Define the following transformation:

\[ m^{(1)}(z) = T(\infty)^{\sigma_3}m(z)G(z)T(z)^{-\sigma_3}, \quad (4.15) \]

which satisfies the following RH problem.

**RHP 4.2** Find a 2 × 2 matrix-valued function \( m^{(1)}(z) := m^{(1)}(x, t; z) \) such that

- \( m^{(1)}(z) \) is meromorphic in \( \mathbb{C} \setminus \Sigma^{(1)} \).
- The boundary value \( m^{(1)}(z) \) at \( \Sigma^{(1)} \) satisfies the jump condition \( m^{(1)}(z) = m^{(1)}_{\pm}(z)v^{(1)}(z) \), where

\[
\begin{cases}
\begin{bmatrix}
1 - \tilde{\rho}(z)T^2(z)e^{2i\theta(z)} \\
0 \\
1 - \tilde{\rho}(z)^2T^{-2}(z)e^{-2i\theta(z)} \\
1
\end{bmatrix}
\begin{bmatrix}
1 \\
0 \\
0 \\
1
\end{bmatrix}
, & \text{if } z \in \Gamma^t \\
\begin{bmatrix}
1 - \rho(z)T^{-2}(z)e^{-2i\theta(z)} \\
1 - \rho(z) \rho(z)T^{-2}(z)e^{-2i\theta(z)} \\
1 - \rho(z) \rho(z)T^{-2}(z)e^{-2i\theta(z)} \\
1
\end{bmatrix}
\begin{bmatrix}
1 \\
0 \\
0 \\
1
\end{bmatrix}
, & \text{if } z \in \Gamma
\end{cases}
\]

\[
v^{(1)}(z) = 
\begin{cases}
\begin{bmatrix}
1 - \frac{A[\eta_k]e^{2i\theta(\eta_k)}}{z-\eta_k} \\
0 \\
1 - \frac{A[\eta_k]e^{-2i\theta(\eta_k)}}{z-\eta_k} \\
0
\end{bmatrix}
, & |z - \eta_k| = \varrho, \quad k \in \nabla / \Lambda, \\
\begin{bmatrix}
1 - \frac{A[\hat{\eta}_k]e^{2i\theta(\hat{\eta}_k)}}{z-\hat{\eta}_k} \\
0 \\
1 - \frac{A[\hat{\eta}_k]e^{-2i\theta(\hat{\eta}_k)}}{z-\hat{\eta}_k} \\
0
\end{bmatrix}
, & |z - \hat{\eta}_k| = \varrho, \quad k \in \triangle / \Lambda, \\
\begin{bmatrix}
1 - \frac{A[\hat{\eta}_k]e^{2i\theta(\hat{\eta}_k)}}{z-\hat{\eta}_k} \\
0 \\
1 - \frac{A[\eta_k]e^{-2i\theta(\eta_k)}}{z-\eta_k} \\
0
\end{bmatrix}
, & |z - \eta_k| = \varrho, \quad k \in \nabla / \Lambda, \\
\begin{bmatrix}
1 \\
0 \\
0 \\
1
\end{bmatrix}
, & \text{elsewhere},
\end{cases}
\]

\[
(4.16)
\]
• Asymptotic behavior

\[ m^{(1)}(z) \to I, \quad z \to \infty; \quad m^{(1)}(z) \to \frac{i}{z} \sigma_1, \quad z \to 0. \] (4.17)

• Residue conditions

\[
\text{Res}_{z=\eta_k} m^{(1)}(z) = \begin{cases} 
\lim_{z \to \eta_k} m^{(1)}(z) & \left[ \begin{array}{cc}
0 & 0 \\
0 & 0 \\
\frac{1}{A[\eta_k]} & 0
\end{array} \right], \quad k \in \nabla \cap \Lambda,
\end{cases}
\]

\[
\lim_{z \to \eta_k} m^{(1)}(z) & \left[ \begin{array}{cc}
0 & 0 \\
0 & 0 \\
\frac{1}{A[\eta_k]} & 0
\end{array} \right], \quad k \in \nabla \cap \Lambda,
\]

(4.18)

\[
\text{Res}_{z=\hat{\eta}_k} m^{(1)}(z) = \begin{cases} 
\lim_{z \to \hat{\eta}_k} m^{(1)}(z) & \left[ \begin{array}{cc}
0 & 0 \\
0 & 0 \\
\frac{1}{A[\hat{\eta}_k]} & 0
\end{array} \right], \quad k \in \nabla \cap \Lambda,
\end{cases}
\]

\[
\lim_{z \to \hat{\eta}_k} m^{(1)}(z) & \left[ \begin{array}{cc}
0 & 0 \\
0 & 0 \\
\frac{1}{A[\hat{\eta}_k]} & 0
\end{array} \right], \quad k \in \nabla \cap \Lambda.
\]

(4.19)

4.2 Set Up and Decomposition of a Mixed $\bar{\partial}$-RH Problem

In this section, we want to remove the jump from the original jump contour $\Sigma$ in such a way that the new problem takes advantage of the decay of $e^{2i\theta(z)}$ or $e^{-2i\theta(z)}$ for $z \notin \Sigma$. Additionally, we hope to open the lenses in such a way that the lenses are bounded away from the disks containing poles introduced in Fig. 5.

4.2.1 Characteristic Lines and Some Estimates for $\text{Re}[2i\theta(z)]$

We fix an angle $\theta_0 > 0$ sufficiently small such that the set \( \{ z \in \mathbb{C} : \left| \frac{\text{Re}z - \zeta_i}{z} \right| > \cos \theta_0, \left| \frac{\text{Re}z - \hat{\zeta}_i}{z} \right| > \cos \theta_0, i = 1, \ldots, 6 \} \) does not intersect the discrete spectrums set $\mathcal{Z} \cup \hat{\mathcal{Z}}$. For any $\xi < -6$, let

\[ \phi(\xi) = \min \left\{ \theta_0, \frac{\pi}{4} \right\}. \] (4.20)

Since the phase function (3.1) has six critical points at $\zeta_i, i = 1, 2, \ldots, 6$, our new contour is chosen to be

\[ \Sigma_{\text{jump}} = \bigcup_{j=1,2,3,4} \left( \Sigma_{0j} \cup \left( \bigcup_{i=1,2,\ldots,8} \Sigma_{ij} \right) \right) \] (4.21)

shown in Fig. 6, which consists of rays of the form $\zeta_i + re^{i\phi}$ with $r > 0$ and other line segments or arcs.
Lemma 4.1 Set $\xi = \frac{\pi}{4}$ and let $\xi < -6$. Then for $z = |z|e^{iw} \in \Omega_{0j}$, $j = 1, 2, 3, 4$, the phase $\theta(z)$ defined in (3.1) satisfies

$$\begin{align*}
\text{Re}[2i\theta(z)] &\geq c|\sin w|(\xi^{-1} - |z|), \quad z \in \Omega_{01}, \Omega_{02}, \\
\text{Re}[2i\theta(z)] &\leq -c|\sin w|(\xi^{-1} - |z|), \quad z \in \Omega_{03}, \Omega_{04},
\end{align*}$$

(4.22)

where $c = c(\xi) > 0$.

Proof We give a proof for $z = |z|e^{iw} \in \Omega_{01}$, the others are similar.

$$\text{Re}[2i\theta(z)] = -\sin w(|z| - |z|^{-1}) \left[ \xi - 3 + (F^2(|z|) - 1)(1 + 2\cos 2w) \right],$$

(4.23)

where $F(|z|) = |z| + |z|^{-1}$. Let

$$g(z) = \xi - 3 + (F^2(|z|) - 1)(1 + 2\cos 2w),$$

(4.24)

From $g(|z|) = 0$, we have

$$F^2(|z|) = 1 + \frac{3 - \xi}{1 + 2\cos 2w} := \alpha > 4.$$  

(4.25)

Further we have $F(|z|) = |z| + |z|^{-1} = \sqrt{\alpha}$, which leads to two solutions

$$|z|_1 = \frac{\sqrt{\alpha} - \sqrt{\alpha - 4}}{2}, \quad |z|_2 = \frac{\sqrt{\alpha} + \sqrt{\alpha - 4}}{2}.$$  

It’s easy to check that $h(|z|) = |z|^2 - \sqrt{\alpha}|z| + 1$ is decreasing in $(-\infty, |z|_1)$ and $F(|z|) > \sqrt{\alpha}$. Therefore,

$$\text{Re}[2i\theta(z)] > \sin w(|z|^{-1} - |z|) [\xi - 3 + (\alpha - 1)(1 + 2\cos 2w)] = 0.$$  

(4.26)

\[\Box\]

Corollary 4.1 For $z = |z|e^{iw} = u + iv \in \Omega_{0j}$, $j = 1, 2, 3, 4$,

$$\begin{align*}
\text{Re}[2i\theta(z)] &\geq c|v|, \quad z \in \Omega_{01}, \Omega_{02}, \\
\text{Re}[2i\theta(z)] &\leq -c|v|, \quad z \in \Omega_{03}, \Omega_{04},
\end{align*}$$

(4.27)

where $c = c(\xi) > 0$.

Lemma 4.2 For $z \in \Omega_{ij}$, $i = 1, 2, 5, 6$, $j = 1, 2, 3, 4$,

$$\begin{align*}
\text{Re}[2i\theta(z)] &\geq cv^2, \quad z \in \Omega_{i1}, \Omega_{i3}, i = 1, 2, 5, 6, \\
\text{Re}[2i\theta(z)] &\leq -cv^2, \quad z \in \Omega_{i2}, \Omega_{i4}, i = 1, 2, 5, 6,
\end{align*}$$

(4.28)

where $c = c(\xi) > 0$.  

\[\Box\]
Proof We take $z \in \Omega_{12}$ as an example. For $z = \zeta_1 + u + iv \in \Omega_{12}$, $0 < v < 1$, $\zeta_1 < |z| < \frac{1}{2} \sqrt{(\zeta_1 + \zeta_3)^2 + (\zeta_1 - \zeta_3)^2} \tan \phi$, we have

$$\text{Re}[2i\theta(z)] = (|z|^{-2} - 1)v \left[ \xi - 3 + 3(1 + |z|^2 + |z|^{-2}) - 4v^2(1 + |z|^{-2} + |z|^{-4}) \right]$$

$$\lesssim c \left[ \xi - 3 + 3(1 + |z|^2 + |z|^{-2}) - 4v^2(1 + |z|^{-2} + |z|^{-4}) \right]. \quad (4.29)$$

Let $\tau = |z|^2 \in \left( \xi_1^2, \frac{(\xi_1 + 1)^2}{4} \right)$ and define

$$h(\tau) = \xi - 3 + 3(1 + \tau + \tau^{-1}) - 4v^2(1 + \tau^{-1} + \tau^{-2}). \quad (4.30)$$

Due to $v \ll \tau$, then

$$h'(\tau) = 3 - 3\tau^{-2} + 4v^2\tau^{-2} + 8v^2\tau^{-3} \leq 0 \quad (4.31)$$

and $h(\tau)$ is decreasing in $\left( \xi_1^2, \frac{(\xi_1 + 1)^2}{4} \right)$. Thus,

$$h(\tau) \leq h\left( \xi_1^2 \right) = \xi - 3 + 3(1 + \xi_1^2 + \xi_1^{-2}) - 4v^2(1 + \xi_1^{-2} + \xi_1^{-4})$$

$$= \xi_1 = \frac{1}{6} \xi - 3 + 3(1 + \xi_1^2 + \xi_6^2) - 4v^2(1 + \xi_6^2 + \xi_6^4).$$
Recall $\theta'(\xi_6) = 0$, we obtain

$$3\xi_6^4 + \xi_6^2 + 3 = 0$$

and then

$$\xi - 3 = -3(\xi_6^{-2} + \xi_6^2 + 1) = -3(\xi_1^2 + \xi_6^2 + 1). \quad (4.32)$$

Substituting (4.32) into (4.32), it comes to

$$h(\tau) \leq h(\xi_1^2) = -4v^2(1 + \xi_6^2 + \xi_6^4). \quad (4.33)$$

As a consequence,

$$\text{Re}[2i\theta(z)] \leq -cv^2.$$

\[ \square \]

**Lemma 4.3** For $z \in \Omega_{ij}$, $i = 3, 4, 7, 8$, $j = 1, 2, 3, 4$,

\[
\begin{align*}
\text{Re}[2i\theta(z)] & \geq c \left| 1 - |z|^{-2} \right| v^2, \quad \text{if } z \in \Omega_{i1}, \Omega_{i3}, \Omega_{k1}, \Omega_{k2}, i = 3, 4, k = 7, 8, \\
\text{Re}[2i\theta(z)] & \leq -c \left| 1 - |z|^{-2} \right| v^2, \quad \text{if } z \in \Omega_{i2}, \Omega_{i4}, \Omega_{k3}, \Omega_{k4}, i = 3, 4, k = 7, 8,
\end{align*}
\] \hspace{1cm} \quad (4.34)

where $c = c(\xi_1) > 0$.

**Proof** We take $z \in \Omega_{74}$ as an example. For $z = u + iv \in \Omega_{74}$, we have $\xi_1 < |z| < 1$ and $0 < v < 1$, thus

$$\text{Re}[2i\theta(z)] \leq c(|z|^{-2} - 1) \left[ \xi - 3 + 3(1 + |z|^2 + |z|^{-2}) - 4v^2(1 + |z|^{-2} + |z|^{-4}) \right]. \quad (4.35)$$

Let $f(|z|) = |z|^2 + |z|^{-2}$, it is obvious that $f(|z|)$ is decreasing and $f(|z|) < f(\xi_1) = \xi_1^2 + \xi_1^{-2}$. Recall $\theta'(\xi_1) = 0$, we obtain

$$\xi - 3 = -3(\xi_1^2 + \xi_1^{-2} + 1). \quad (4.36)$$

Therefore,

$$\text{Re}[2i\theta(z)] \leq c(|z|^{-2} - 1) \left[ -3(1 + \xi_1^2 + \xi_1^{-2}) + 3(1 + \xi_1^2 + \xi_1^{-2}) - 4v^2(1 + |z|^{-2} + |z|^{-4}) \right]$$

$$\leq -c(\xi)(|z|^{-2} - 1)v^2. \quad \square$$
4.2.2 Opening Lenses

The estimates in Lemma 4.1, 4.2, 4.3 suggest that we should open lenses using (modified versions of) factorization for $z \in \Gamma$ and for $z \in \tilde{\Gamma}$ shown in (4.16). To do so, we need to define extensions of the off-diagonal entries of factorization matrices off $\Sigma$, which is the content of this subsection.

To be brief, we introduce the following notations and functions:

\[
\begin{align*}
    l_0^+ &= \left(0, \frac{\xi_1}{2}\right), \quad l_0^- = \left(\frac{\xi_2}{2}, 0\right), \\
    l_i^+ &= \left(\xi_i, \frac{\xi_i + \xi_{i+2}}{2}\right), \quad l_i^- = \left(\frac{\xi_{i-2} + \xi_i}{2}, \xi_i\right), \quad i = 1, 2, \\
    l_i^+ &= \left(\xi_i, \frac{\xi_i + \xi_{i+2}}{2}\right), \quad l_i^- = \left(\frac{\xi_{i-2} + \xi_i}{2}, \xi_i\right), \quad i = 3, 4, \\
    l_5^+ &= (\xi_5, +\infty), \quad l_6^+ = (-\infty, \xi_6), \quad l_i^- = \left(\frac{\xi_{i-2} + \xi_i}{2}, \xi_i\right), \quad i = 5, 6, \\
    \gamma_k &= \left\{z \in \mathbb{C} : z = e^{iw}, \frac{(k-1)\pi}{2} \leq w \leq \frac{k\pi}{2}\right\}, \quad k = 1, 2, 3, 4, \\
    r_1(z) &= \rho(z), \quad r_2(z) = \tilde{\rho}(z), \\
    r_3(z) &= \frac{\rho(z)}{1 - \rho(z)\tilde{\rho}(z)}, \quad r_4(z) = \frac{\tilde{\rho}(z)}{1 - \rho(z)\tilde{\rho}(z)}.
\end{align*}
\]

Further, we choose $\mathcal{R}^{(2)}(z) := \mathcal{R}^{(2)}(z; \xi)$ as:

\[
\mathcal{R}^{(2)}(z) = \begin{cases} 
    \begin{bmatrix} 1 & 0 \\ R_{ij}e^{-2it\theta} & 1 \end{bmatrix}, & z \in \Omega_{ij}, \quad i = 0, 7, 8, j = 1, 2, \\
    \begin{bmatrix} 1 & \overline{R_{ij}e^{2it\theta}} \\ 0 & 1 \end{bmatrix}^{-1}, & z \in \Omega_{ij}, \quad i = 0, 7, 8, j = 3, 4, \\
    \begin{bmatrix} 1 & 0 \\ R_{i1}e^{-2it\theta} & 1 \end{bmatrix}, & z \in \Omega_{i1}, \quad i = 1, 2, \ldots, 6, \\
    \begin{bmatrix} 1 & \overline{R_{i2}e^{2it\theta}} \\ 0 & 1 \end{bmatrix}^{-1}, & z \in \Omega_{i2}, \quad i = 1, 2, \ldots, 6, \\
    \begin{bmatrix} 1 & 0 \\ R_{i3}e^{-2it\theta} & 1 \end{bmatrix}, & z \in \Omega_{i3}, \quad i = 1, 2, \ldots, 6, \\
    \begin{bmatrix} 1 & \overline{R_{i4}e^{2it\theta}} \\ 0 & 1 \end{bmatrix}^{-1}, & z \in \Omega_{i4}, \quad i = 1, 2, \ldots, 6, \\
    1, & \text{elsewhere},
\end{cases}
\]

where the functions $R_{0j}, R_{ij}$ are defined as the following two propositions.
Proposition 4.2 \( R_{0j} : \overline{\Omega}_{0j} \to \mathbb{C} \) are continuous on \( \overline{\Omega}_{0j} \), \( j = 1, 2, 3, 4 \). Their boundary values are as follows:

\[
R_{0j}(z) = \begin{cases} 
  r_3(z)T^{-2}_{-}(z), & z \in l^+_0, l^-_0, \\
  r_3(\zeta_i)T^{-2}_{-}(\zeta_i)(z - \zeta_i)^{2i\nu(\zeta_i)}, & z \in \Sigma_{0j}, \quad j = 1, 2,
\end{cases}
\]

\[
R_{0j}(z) = \begin{cases} 
  r_4(z)T^{2}_{+}(z), & z \in l^+_0, l^-_0, \\
  r_4(\zeta_i)T^{2}_{+}(\zeta_i)(z - \zeta_i)^{-2i\nu(\zeta_i)}, & z \in \Sigma_{0j}, \quad j = 3, 4.
\end{cases}
\]

Moreover, \( R_{0j} \) have following property:

\[
|\partial_r R_{0j}(z)| \lesssim |z|^{-\frac{1}{2}} + |r_3'(\text{Re}z)|, \quad j = 1, 2, \tag{4.41}
\]

\[
|\partial_r R_{0j}(z)| \lesssim |z|^{-\frac{1}{2}} + |r_4'(\text{Re}z)|, \quad j = 3, 4.
\]

**Proof** The proof is the similar with [22, Lemma 6.5]. \(\square\)

Proposition 4.3 \( R_{ij} : \overline{\Omega}_{ij} \to \mathbb{C}, i = 1, 2, \ldots, 8, j = 1, 2, 3, 4 \) are continuous on \( \overline{\Omega}_{ij} \) with boundary values:

(i) For \( i = 1, 2, \)

\[
R_{i1}(z) = \begin{cases} 
  r_3(z)T^{-2}_{-}(z), & z \in l^-_i, \\
  r_3(\zeta_i)T^{-2}_{-}(\zeta_i)(z - \zeta_i)^{2i\nu(\zeta_i)}, & z \in \Sigma_{i1},
\end{cases}
\]

\[
R_{i2}(z) = \begin{cases} 
  r_2(z)T^{2}_{+}(z), & z \in l^+_i, \\
  r_2(\zeta_i)T^{2}_{+}(\zeta_i)(z - \zeta_i)^{-2i\nu(\zeta_i)}, & z \in \Sigma_{i2},
\end{cases}
\]

\[
R_{i3}(z) = \begin{cases} 
  r_1(z)T^{-2}_{-}(z), & z \in l^-_i, \\
  r_1(\zeta_i)T^{-2}_{-}(\zeta_i)(z - \zeta_i)^{2i\nu(\zeta_i)}, & z \in \Sigma_{i3},
\end{cases}
\]

\[
R_{i4}(z) = \begin{cases} 
  r_4(z)T^{2}_{+}(z), & z \in l^+_i, \\
  r_4(\zeta_i)T^{2}_{+}(\zeta_i)(z - \zeta_i)^{-2i\nu(\zeta_i)}, & z \in \Sigma_{i4}.
\end{cases}
\]

(ii) For \( i = 3, 4, \)

\[
R_{ij}(z) = \begin{cases} 
  r_1(z)T^{-2}_{-}(z), & z \in l^-_i, l^+_i, \\
  r_1(\zeta_i)T^{-2}_{-}(\zeta_i), & z \in \Sigma_{ij}, j = 1, 3,
\end{cases}
\]

\[
R_{ij}(z) = \begin{cases} 
  r_2(z)T^{2}_{+}(z), & z \in l^-_i, l^+_i, \\
  r_2(\zeta_i)T^{2}_{+}(\zeta_i), & z \in \Sigma_{ij}, j = 2, 4.
\end{cases}
\]

(iii) For \( i = 5, 6, \)

\[
R_{i1}(z) = \begin{cases} 
  r_1(z)T^{-2}_{-}(z), & z \in l^-_i, \\
  r_1(\zeta_i)T^{-2}_{-}(\zeta_i)(z - \zeta_i)^{2i\nu(\zeta_i)}, & z \in \Sigma_{i1},
\end{cases}
\]
\[ R_{i2}(z) = \begin{cases} r_4(z)T_+^2(z), & z \in l_i^+ \\ r_4(\zeta_i)T_i^2(\zeta_i)(z - \zeta_i)^{-2\nu(\zeta_i)}, & z \in \Sigma_{i2} \end{cases} \]

\[ R_{i3}(z) = \begin{cases} r_3(z)T_-^2(z), & z \in l_i^- \\ r_3(\zeta_i)T_i^2(\zeta_i)(z - \zeta_i)^{-2\nu(\zeta_i)}, & z \in \Sigma_{i3} \end{cases} \]

\[ R_{i4}(z) = \begin{cases} r_2(z)T_1^2(z), & z \in l_i^1 \\ r_2(\zeta_i)T_i^2(\zeta_i)(z - \zeta_i)^{-2\nu(\zeta_i)}, & z \in \Sigma_{i4} \end{cases} \]

(iv) For \( i = 7, 8, \)

\[ R_{i1}(z) = \begin{cases} r_1(z)T_-^2(z), & z \in \gamma_1, \gamma_4, \\ r_1(1)T_-^2(1)(1 - \chi(\zeta(z))), & z \in \Sigma_{i1}, \end{cases} \]

\[ R_{i2}(z) = \begin{cases} r_1(z)T_-^2(z), & z \in \gamma_2, \gamma_3, \\ r_1(-1)T_-^2(-1)(1 - \chi(\zeta(z))), & z \in \Sigma_{i2}, \end{cases} \]

\[ R_{i3}(z) = \begin{cases} r_2(z)T_1^2(z), & z \in \gamma_2, \gamma_3, \\ r_2(-1)T_1^2(-1)(1 - \chi(\zeta(z))), & z \in \Sigma_{i3}, \end{cases} \]

\[ R_{i4}(z) = \begin{cases} r_2(z)T_1^2(z), & z \in \gamma_1, \gamma_4, \\ r_2(1)T_1^2(1)(1 - \chi(\zeta(z))), & z \in \Sigma_{i4}, \end{cases} \]

where

\[ \chi(\zeta(z)) = \begin{cases} 1, & \text{dist}(z, \mathcal{Z} \cup \hat{\mathcal{Z}}) < \varrho / 3 \\ 0, & \text{dist}(z, \mathcal{Z} \cup \hat{\mathcal{Z}}) > 2\varrho / 3. \end{cases} \]

And \( R_{ij}(z), i = 1, 2, \ldots, 8, \ j = 1, 2, 3, 4 \) have following properties:

\[ |R_{ij}(z)| \leq c_1 + c_2|1 + z^2|^{-\frac{1}{4}}, \ z \in \Omega_{ij}, \]  
\[ |\bar{\partial}R_{ij}(z)| \leq c_1 + c_2|z - \zeta_i|^{1} + c_3|z - \zeta_i|^{-\alpha_i}, \ z \in \Omega_{ij}, \]

where

\[ \alpha_i = \begin{cases} \frac{1}{2} + \text{Im}(\zeta_i), & 0 < \text{Im}(\zeta_i) < \frac{1}{2}, \\ \frac{1}{2}, & -\frac{1}{2} < \text{Im}(\zeta_i) \leq 0. \end{cases} \]

Moreover, when \( z \to \pm i \)

\[ |\bar{\partial}R_{ij}(z)| \leq c|z - i|, \ z \in \Omega_{7j}, \ j = 1, 2, 3, 4, \]
\[ |\bar{\partial}R_{ij}(z)| \leq c|z + i|, \ z \in \Omega_{8j}, \ j = 1, 2, 3, 4. \]
Fig. 7 The jump contour $\Sigma_{jump}'$

**Proof** The proof is similar with [23, Lemma 4.1].

Define $\Sigma^{(2)} = \Sigma_{jum}' \cup \Sigma_{cir}$, where

$$\Sigma_{jum}' = \bigcup_{j=1,2,3,4} \left( \Sigma_{0j}' \cup \left( \bigcup_{i=3,4} \Sigma_{ij}' \right) \cup \left( \bigcup_{i=1,2,5,6,7,8} \Sigma_{ij} \right) \right)$$  \hspace{1cm} (4.60)

can be referred in Fig. 7. We now use $R^{(2)}(z)$ to define a new transformation

$$m^{(2)}(z) = m^{(1)}(z)R^{(2)}(z),$$  \hspace{1cm} (4.61)

which satisfies the following mixed $\bar{\partial}$-RH problem.

**RHP 4.3** Find a $2 \times 2$ matrix-valued function $m^{(2)}(z) := m^{(2)}(x, t; z)$ such that

- $m^{(2)}(z)$ is continuous in $\mathbb{C}\setminus(\Sigma^{(2)} \cup Z \cup \hat{Z})$.
- Jump relation: $m^{(2)}_+(z) = m^{(2)}_-(z)v^{(2)}(z)$, $z \in \Sigma^{(2)}$, where
For $z \in \mathbb{C} \setminus (\Sigma^{(2)} \cup \mathcal{Z} \cup \hat{\mathcal{Z}})$,

$$\tilde{\partial} m^{(2)} = m^{(2)} \tilde{\partial} R^{(2)}_+,$$

(4.62)
where

\[
\begin{align*}
\bar{\partial} \mathcal{R}^{(2)} &= \begin{cases}
\begin{bmatrix} 0 & 0 \\
\hat{\partial} R_{ij} e^{-2it\theta} & 0 
\end{bmatrix}, & z \in \Omega_{ij}, \ i = 0, 7, 8, \ j = 1, 2, \\
\begin{bmatrix} 0 & 0 \\
\hat{\partial} R_{1i} e^{-2it\theta} & 0 
\end{bmatrix}, & z \in \Omega_{i1}, \ i = 1, 2, \ldots, 6, \\
\begin{bmatrix} 0 & 0 \\
\hat{\partial} R_{2i} e^{-2it\theta} & 0 
\end{bmatrix}, & z \in \Omega_{i2}, \ i = 1, 2, \ldots, 6, \\
\begin{bmatrix} 0 & 0 \\
\hat{\partial} R_{3i} e^{-2it\theta} & 0 
\end{bmatrix}, & z \in \Omega_{i3}, \ i = 1, 2, \ldots, 6, \\
\begin{bmatrix} 0 & 0 \\
\hat{\partial} R_{4i} e^{-2it\theta} & 0 
\end{bmatrix}, & z \in \Omega_{i4}, \ i = 1, 2, \ldots, 6, \\
\begin{bmatrix} 0 & 0 \\
0 & 0 
\end{bmatrix}, & \text{elsewhere.}
\end{cases}
\end{align*}
\]

• Asymptotic behavior

\[
m^{(2)}(z) \rightarrow I, \ z \rightarrow \infty; \ m^{(2)}(z) \rightarrow \frac{i}{z} \sigma_1, \ z \rightarrow 0. \tag{4.64}
\]

• Residue conditions

\[
\begin{align*}
\text{Res}_{z = \eta_k} m^{(2)}(z) &= \begin{cases}
\lim_{z \rightarrow \eta_k} m^{(2)}(z) \begin{bmatrix} 0 & 0 \\
A[\eta_k] T^{-2}(\eta_k) e^{-2it\theta(\eta_k)} & 0 
\end{bmatrix}, & k \in \nabla \cap \Lambda, \\
\lim_{z \rightarrow \eta_k} m^{(2)}(z) \begin{bmatrix} 0 & 0 \\
\frac{1}{A[\eta_k]} \frac{1}{(T')^2(\eta_k)} & 0 
\end{bmatrix} e^{2it\theta(\eta_k)}, & k \in \triangle \cap \Lambda,
\end{cases} \\
\text{Res}_{z = \hat{\eta}_k} m^{(2)}(z) &= \begin{cases}
\lim_{z \rightarrow \hat{\eta}_k} m^{(2)}(z) \begin{bmatrix} 0 & 0 \\
A[\hat{\eta}_k] T^2(\hat{\eta}_k) e^{2it\theta(\hat{\eta}_k)} & 0 
\end{bmatrix}, & k \in \nabla \cap \Lambda, \\
\lim_{z \rightarrow \hat{\eta}_k} m^{(2)}(z) \begin{bmatrix} 0 & 0 \\
\frac{1}{A[\hat{\eta}_k]} \frac{T'(\hat{\eta}_k)}{(T'(\hat{\eta}_k))^2} e^{-2it\theta(\hat{\eta}_k)} & 0 
\end{bmatrix}, & k \in \triangle \cap \Lambda.
\end{cases}
\end{align*}
\tag{4.65}
\tag{4.66}
\]

4.2.3 Decomposition of the Mixed $\bar{\partial}$-RH Problem

To solve RH Problem 4.3, we decompose it into a model RH problem for $m^{hp}(z)$ with $\bar{\partial} \mathcal{R}^{(2)} = 0$ and a pure $\bar{\partial}$-problem for $m^{(3)}(z)$ with $\bar{\partial} \mathcal{R}^{(2)} \neq 0$, which can be shown as...
the following structure

\[ m^{(2)}(z) = \begin{cases} \tilde{\partial}R^{(2)} \equiv 0 \rightarrow m^{rh}(z), \\ \tilde{\partial}R^{(2)} \neq 0 \rightarrow m^{(3)}(z) = m^{(2)}(z)[m^{rh}(z)]^{-1} \end{cases} \quad (4.67) \]

with \( m^{rh}(z) \) satisfies the following RH problem.

**RHP 4.4** Find a \( 2 \times 2 \) matrix-valued function \( m^{rh}(z) := m^{rh}(x, t; z) \) such that

- \( m^{rh}(z) \) is analytic in \( \mathbb{C} \setminus (\Sigma(2) \cup \mathbb{Z} \cup \hat{\mathbb{Z}}) \).
- Jump relation: \( m^{rh}_{+}(z) = m^{rh}_{-}(z)v^{(2)}(z), z \in \Sigma(2). \)
- For \( z \in \mathbb{C} \setminus (\Sigma(2) \cup \mathbb{Z} \cup \hat{\mathbb{Z}}) \),

\[ \tilde{\partial}R^{(2)} \equiv 0. \quad (4.68) \]

- Asymptotic behavior

\[ m^{rh}(z) \rightarrow I, \quad z \rightarrow \infty; \quad m^{rh}(z) \rightarrow \frac{1}{z}\sigma_1, \quad z \rightarrow 0. \quad (4.69) \]

- \( m^{rh}(z) \) has same jump matrix and residue conditions as \( m^{(2)}(z) \).

We construct the solution \( m^{rh}(z) \) of the RH problem **4.4** in the following form

\[ m^{rh}(z) = \begin{cases} E(z)m^{sol}(z) = E(z)m^{err}(z)m^{A}(z), \quad z \in \mathbb{C} \setminus \bigcup_{i=1,2,5,6} U_{\zeta_i}, \\ E(z)m^{sol}(z)m^{lo}(z), \quad z \in \bigcup_{i=1,2,5,6} U_{\zeta_i}, \end{cases} \quad (4.70) \]

where \( U_{\zeta_i} \) is defined as

\[ U_{\zeta_i} = \{ z : |z - \zeta_i| < \varrho \}. \quad (4.71) \]

In the above, \( m^{rh}(z) \) decomposes into two parts: one part accounts for the solitons determined by discrete spectrums, and the other part accounts for the contribution of phase points. Specifically,

\[ m^{A}(z) = m^{sol}(z)|_{v^{(2)}=I, z \in \Sigma_{eir}} = m^{rh}(z)|_{v^{(2)}=I, z \in \Sigma_{jum}} \quad (4.72) \]

will be solved in next Sect. 4.3.1; \( m^{lo}(z) \) uses parabolic cylinder functions to build a matrix to match jumps of \( m^{rh}(z) \) in a neighborhood \( \zeta_i, i = 1, 2, 5, 6 \), which is shown in Sect. 4.3.2; \( E(z) \) is an error function and a solution of a small norm Riemann-Hilbert problem, which is shown in Sect. 4.3.3.
4.3 Analysis on the Pure RH Problem

4.3.1 Outer Model RH Problem

In this subsection, we build a reflectionless case of RHP 2.1 to show that its solution can approximated with \( \text{msol}(z) \). The following RH problem constructs the \( \text{msol} \).

RHP 4.5 Find a \( 2 \times 2 \) matrix-valued function \( \text{msol}(z) := \text{msol}(x, t; z) \) such that

- \( \text{msol}(z) \) is meromorphic in \( \mathbb{C} \setminus \Sigma_{\text{cir}} \).
- Jump relation: \( \text{msol}^+(z) = \text{msol}^-(z) v^{(2)}(z), z \in \Sigma_{\text{cir}} \).
- Asymptotic behavior:
  \[ \text{msol}(z) \to I, \quad z \to \infty; \quad \text{msol}(z) \to \frac{i}{z} \sigma_1, \quad z \to 0. \]
- \( \bar{\partial} R^{(2)} = 0 \).
- \( \text{msol}(z) \) has the same residue conditions as \( m^{(2)}(z) \).

Proposition 4.4 If \( \text{msol}(z) \) is the solution of RH problem 4.5 with scattering data

\[ \{ \rho(z), \tilde{\rho}(z), \{ \eta_k, C_k \}_{k=1}^{2N_1+N_2} \}, \quad (4.73) \]

\( \text{msol}(z) \) exists and unique.

Proof To transform \( \text{msol}(z) \) to the solution of RH problem 2.1, the jump and poles need to be restored. We reverse the triangularity effected in (4.15) and (4.61):

\[
\tilde{m}(z) = \left( -i \prod_{k \in \Delta_1} \prod_{k \in \Delta_2} |z_k|^2 \omega_l \right)^{-\sigma_3} m^{sol}(z) T(z)^{\hat{\sigma}_3} G(z)^{-1} \prod_{k \in \Delta_1} \prod_{k \in \Delta_2} \frac{(z + z_k^{-1})(z - \bar{z}_k^{-1})(z - i \omega_l^{-1})}{(zz_k^{-1} - 1)(zz_k^{-1} + 1)(i \omega_l^{-1} z + 1)}^{\sigma_3} \quad (4.74)
\]

with \( G(z) \) defined in (4.13). First we verify \( \tilde{m}(z) \) satisfying RH problem 2.1. The transformation to \( m^{sol}(z) \) preserves the normalization conditions at infinity obviously. And compare to (4.15), this transformation restores the jump on \( \Sigma_{\text{cir}} \) to residue for \( k \notin \Lambda \). As for \( k \in \Lambda \), take \( \eta_k \in \nabla \cap \Lambda \) as an example. Substituting (4.65) into this transformation:

\[
\text{Res}_{z=\eta_k} \tilde{m}(z) = (-i \prod_{k \in \Delta_1} \prod_{k \in \Delta_2} |z_k|^2 \omega_l)^{-\sigma_3} \text{Res}_{z=\eta_k} m^{sol}(z) T^{\hat{\sigma}_3} G^{-1}
\]

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Proposition 4.5
For given scattering data proposition. uniqueness comes from Liouville’s theorem. ⊓⊔

Proof is similar to Lemma 6.7 in [22].

The proof is similar to Lemma 6.7 in [22].

The main contribution to \( m^{\text{sol}}(z) \) comes from discrete spectrums \( \{ \eta_k, \check{\eta}_k, k \in \Lambda \} \), which can be described by the following PH problem 4.6.

RHP 4.6 Find a \( 2 \times 2 \) matrix-valued function \( m^\Lambda(z) := m^\Lambda(x, t; z) \) such that

- \( m^\Lambda(z) \) is analytic in \( \mathbb{C} \setminus (\mathbb{Z} \cup \mathbb{Z}) \).
- \( m^\Lambda(z) \rightarrow I, \ z \rightarrow \infty; \ m^\Lambda(z) \rightarrow i^\frac{1}{z} \sigma_1, \ z \rightarrow 0. \)
- \( v^{(2)} = I, \ \bar{\partial} R^{(2)} = 0. \)
- \( m^\Lambda(z) \) has same residue conditions as \( m^{(2)}(z) \).

The unique solution \( m^\Lambda(z) \) to the above RH problem 4.6 is given in the following proposition.

Proposition 4.5 For given scattering data \( \{ 0, 0, \eta_k, \check{\eta}_k \}_{k_0 \in \Lambda} \), the solution to RH problem 4.6 exists uniquely and can be constructed explicitly, where

\[
\check{C}_{k_0} = \begin{cases} 
A[\eta_{k_0}] \tilde{\delta}^{-2}(\eta_{k_0}), & k_0 \in \nabla \cap \Lambda, \\
- \frac{1}{A[\eta_{k_0}]} T'(\eta_{k_0}) \tilde{\delta}^{-2}(\eta_{k_0}), & k_0 \in \Delta \cap \Lambda.
\end{cases}
\] (4.75)

Proof The proof is similar to Lemma 6.7 in [22].

Denote \( m^{\text{err}}(z) \) as the error function generated by jumps on \( \Sigma_{\text{cir}} \), then \( m^{\text{err}}(z) = m^{\text{sol}}(z)[m^\Lambda(z)]^{-1} \) satisfies the following RH problem:

RHP 4.7 Find a \( 2 \times 2 \) matrix-valued function \( m^{\text{err}}(z) := m^{\text{err}}(x, t; z) \) such that
• \( m^{err}(z) \) is analytic in \( \mathbb{C}/\Sigma_{cir} \).
• Jump relation: 
  \[ v^{err}(z) = m^{err}(z)v^{err}(z), \quad z \in \Sigma_{cir}, \]
  where
  \[ v^{err}(z) = m^{\Lambda}(z)v^{(2)}(z)[m^{\Lambda}(z)]^{-1}. \]  
  (4.76)

• Asymptotic behavior
  \[ m^{err}(z) = I + O(z^{-1}), \quad z \to \infty. \]  
  (4.77)

For \( z \in \Sigma_{cir} \), the jump matrix \( v^{err}(z) \) satisfies
  \[ \|v^{err}(z) - I\|_{L^\infty(\Sigma_{cir})} \lesssim e^{-ct}. \]  
  (4.78)

From Beals-Coifman theorem [33], we obtain the solution of RH problem 4.7:
  \[ m^{err}(z) = I + \frac{1}{2\pi i} \int_{\Sigma_{cir}} \frac{\mu_e(s)(v^{err}(s) - I)}{s - z} ds, \]  
  (4.79)
where \( \mu_e(s) \in L^2(\Sigma_{cir}) \) satisfies
  \[ (1 - C_{we})\mu_e(z) = I. \]  
  (4.80)

Thus,
  \[ \|C_{we}\|_{L^2(\Sigma_{cir})} \leq \|C\|_{L^2(\Sigma_{cir})}\|v^{err}(z) - I\|_{L^\infty(\Sigma_{cir})} \lesssim e^{-ct}, \]  
  (4.81)
which implies \( 1 - C_{we} \) invertible for sufficiently large \( t \). Furthermore, we can prove
the existence and uniqueness for \( \mu_e, m^{err}(z) \) and we have following proposition.

**Proposition 4.6** For any \((x, t)\) such that \( \xi = \frac{x}{t} < -6 \) and \( t \gg 1 \), uniformly for \( z \in \mathbb{C} \) we have
  \[ m^{sol}(z) = m^{\Lambda}(z)\left[I + O(e^{-ct})\right], \]  
  (4.82)
and, in particular, for large \( z \) we have
  \[ m^{sol}(z) = m^{\Lambda}(z)\left[I + z^{-1}O(e^{-ct}) + O(z^{-2})\right]. \]  
  (4.83)

**Proof** The proof is similar to Lemma 6.7 in [22].

4.3.2 A Local Solvable RH Model Near Phase Points

In the neighborhood \( U_{\xi_i} \) of \( \xi_i, i = 1, 2, 5, 6 \), we establish a local model \( m^{lo}(z) \) which
exactly matches the jumps of \( m^{rhp}(z) \) on \( \Sigma^{lo} \) for function \( E(z) \) and then it has a
uniform estimate on the decay of the jump, where $\Sigma^{lo}$ is defined as:

$$\Sigma^{lo} = \bigcup_{i=1,2,5,6} (L_i \cap U_{\zeta_i}),$$

$$\Sigma^{lo}_i = L_i \cap U_{\zeta_i}, \quad i = 1, 2, 5, 6,$$

where

$$L_i = \bigcup_{j=1,2,3,4} \Sigma_{ij}, \quad i = 1, 2, 5, 6,$$

see in Fig. 8.

RHP 4.8 Find a $2 \times 2$ matrix-valued function $m^{lo}(z) := m^{lo}(x, t; z)$ such that

- Analyticity: $m^{lo}(z)$ is analytic in $\mathbb{C} \setminus \Sigma^{lo}$.
- Jump relation: $m^{lo}_+(z) = m^{lo}_-(z)v^{(2)}(z), \quad z \in \Sigma^{lo}$.
- Asymptotic behaviors: $m^{lo}(z) = I + O(z^{-1}), \quad z \to \infty$.

According to [31], solving for $m^{lo}(z)$ of RH problem 4.8 can be decomposed into RH problem of $m^{lo}_i(z), \quad i = 1, 2, 5, 6$, where $m^{lo}_i$ can be constructed by parabolic cylinder equation. Further, the solution $m^{lo}$ can be expressed as the following proposition as $t \to \infty$.

Proposition 4.7 As $t \to \infty$,

$$m^{lo}(z) = I + \frac{1}{2} t^{-\frac{1}{2}} \sum_{i=1,2,5,6} \frac{m^{pc}_{p i, 1}(\zeta_i)}{\sqrt{\theta''(\zeta_i)(z - \zeta_i)}} + O(t^{-1}),$$

where

$$m^{pc}_{p i, 1} = \begin{bmatrix} 0 & \beta_{12}^{\zeta_i} \\ -\beta_{21}^{\zeta_i} & 0 \end{bmatrix},$$

(4.87)
and

\[
\begin{align*}
\beta_{12}^i &= -\sqrt{\frac{2\pi e^{\frac{2i}{3}}e^{-\frac{2\nu}{3}}}{\rho_i \Gamma(-i\nu(\xi_i))}} \triangleq t^{1\text{Im}^*(\nu_i)} \tilde{\beta}_{12}^i, \\
\beta_{21}^i &= \sqrt{\frac{2\pi e^{-\frac{2i}{3}}e^{-\frac{2\nu}{3}}}{\rho_i \Gamma(i\nu(\xi_i))}} \triangleq t^{-1\text{Im}^*(\nu_i)} \tilde{\beta}_{21}^i.
\end{align*}
\]

### 4.3.3 The Small Norm RH Problem for Error Function

In this section, we consider the error matrix-function \( E(z) \) generated by the jumps on \( \Sigma'_{jup} \) outside \( U(\xi) \), where

\[
U(\xi) = \bigcup_{i=1,2,3,4} U_{\xi_i},
\]

and \( E(z) \) allows the following RHP:

**RHP 4.9** Find a \( 2 \times 2 \) matrix-valued function \( E(z) \) such that

- \( E(z) \) is analytical in \( \mathbb{C} \setminus \Sigma^E \), where
  \[
  \Sigma^E = \partial U(\xi) \cup \left( \Sigma'_{jum} \setminus U(\xi) \right);
  \]

- \( E(z) \) takes continuous boundary values \( E_{\pm}(z) \) on \( \Sigma^E \) and
  \[
  E_{+}(z) = E_{-}(z)v^E(z),
  \]
  where
  \[
  v^E(z) = \begin{cases} 
  m^{sol}(z)v^{(2)}(z) \left[ m^{sol}(z) \right]^{-1}, & z \in \Sigma'_{jum} \setminus U(\xi), \\
  m^{sol}(z)m^{lo}(z) \left[ m^{sol}(z) \right]^{-1}, & z \in \partial U(\xi).
  \end{cases}
  \]

- asymptotic behavior
  \[
  E(z) = I + \mathcal{O}(z^{-1}), \quad z \to \infty.
  \]

By simple calculation, we have the following estimates of \( v^E \):

\[
\|v^E - I\|_{L^\infty(\Sigma^E)} = \begin{cases} 
\mathcal{O}(e^{-ct}), & z \in \Sigma'_{jum} \setminus U(\xi), \\
\mathcal{O}\left( t^{-\frac{1}{2} + \max_{i=1,2,3,6} \text{Im}^*(\nu_i)} \right), & z \in \partial U(\xi).
\end{cases}
\]

**Proposition 4.8** RHP 4.9 has an unique solution \( E(z) \).
Fig. 9  The jump contours $\Sigma^E$ of $E(z)$

Proof  According to Beals–Coifman theory, the solution for RHP 4.9 can be written as:

$$E(z) = I + \frac{1}{2\pi i} \int_{\Sigma^E} \frac{\mu_E(v^E(s) - I)}{s - z} ds,$$

(4.92)

where $\mu_E \in L^2(\Sigma^E)$, and satisfies

$$(I - C_{w_E})\mu_E = I.$$

We can obtain the following estimate by the definition of $C_w$:

$$\|C_{w_E}\|_{L^2(\Sigma^E)} \leq \|C_{-}\|_{L^2(\Sigma^E)} \|v^E - I\|_{L^\infty(\Sigma^E)} \lesssim O\left(t^{-\frac{1}{2} + \max_{i=1,2,5,6} |\text{Im}(\zeta_i)|}\right),$$

which implies $I - C_{w_E}$ is invertible for sufficiently large $t$. Furthermore, the existence and uniqueness for $\mu$ and $E(z)$ can be proved. $\square$

In order to reconstruct the solution $q(x, t)$ of (1.1), we need the asymptotic behavior of $E(z)$ as $z \to \infty$.

Proposition 4.9  As $z \to \infty$, we have

$$E(z) = I + \frac{E_1}{z} + O(z^{-2}),$$

(4.93)
where

\[
E_1 = \sum_{i=1,2,5,6} \frac{t^{-1/2}}{2\sqrt{\theta^p(\zeta_i)}} m^{sol}(\zeta_i)m^{pc}_{i,1} m^{sol}(\zeta_i) + \left( \mathcal{O}(t^{-1 + \max_{i=1,2,5,6}|\text{Im}'(t_i)|}), \mathcal{O}(t^{-1 + \max_{i=1,2,5,6}|\text{Im}'(t_i)|}) \right).
\]

**Proof** Recall (4.92), we obtain

\[
E_1 = -\frac{1}{2\pi i} \int_{\Sigma^E} \mu_E(s)(v^E(s) - I) ds = I_1 + I_2 + I_3, \tag{4.94}
\]

where

\[
I_1 = -\frac{1}{2\pi i} \int_{\partial U(\xi)} (v^E(s) - I) ds, \tag{4.95}
\]

\[
I_2 = -\frac{1}{2\pi i} \int_{\Sigma^{E\setminus U(\xi)}} (v^E(s) - I) ds,
\]

\[
I_3 = -\frac{1}{2\pi i} \int_{\Sigma^E} (\mu_E(s) - I)(v^E(s) - I) ds. \tag{4.96}
\]

For \(I_2\) and \(I_3\), we have

\[
I_2 = \mathcal{O}(e^{-ct}),
\]

\[
I_3 = \mathcal{O}\left( t^{-\frac{1}{2} + \max_{i=1,2,5,6}|\text{Im}'(t_i)|} \right) \mathcal{O}\left( t^{-\frac{1}{2} + \min_{i=1,2,5,6}|\text{Im}'(t_i)|} \right) \mathcal{O}\left( t^{-\frac{1}{2} + \max_{i=1,2,5,6}|\text{Im}'(t_i)|} \right).
\]

As for \(I_1\), we deduce

\[
I_1 = -\frac{1}{2\pi i} \sum_{i=1,2,5,6} \int_{\partial U_{t_i}} t^{-1/2} \frac{m^{sol}(s)m^{pc}_{i,1} m^{sol}(s)^{-1} ds}{2\sqrt{\theta^p(\zeta_i)(z - \zeta_i)}}. \tag{4.97}
\]

Substitute (4.87) into the above formula to get the conclusion. \(\square\)

**4.4 Analysis on the Pure \(\bar{\partial}\)-Problem**

Now we define the function

\[
m^{(3)}(z) = m^{(2)}(z)(m^{rhp}(z))^{-1}. \tag{4.98}
\]

Then \(m^{(3)}\) satisfies the following pure \(\bar{\partial}\)-Problem.

\(\bar{\partial}\)-Problem 4.1 Find a \(2 \times 2\) matrix-valued function \(m^{(3)}(z) := m^{(3)}(x, t; z)\) such that
• $m^{(3)}(z)$ is continuous in $\mathbb{C}$ and analytic in $\mathbb{C} \setminus \bigcup_{i,j=1,2,\ldots,8} \Omega_{ij}$;

• asymptotic behavior

$$m^{(3)}(z) = I + \mathcal{O}(z^{-1}), \quad z \to \infty; \quad (4.99)$$

• For $z \in \mathbb{C}$, we have

$$\tilde{\partial}m^{(3)}(z) = m^{(3)}(z)W^{(3)}(z); \quad (4.100)$$

where $W^{(3)} = m^{rh}(z)\tilde{\partial}R^{(2)}(z)(m^{rh}(z))^{-1}$,

\[
W^{(3)}(z) = \begin{cases}
m^{rh}(z) \left[ \begin{array}{cc}
1 & 0 \\
0 & 1
\end{array} \right] (m^{rh}(z))^{-1}, & z \in \Omega_{ij}, \quad i = 0, 7, 8, \quad j = 1, 2,
m^{rh}(z) \left[ \begin{array}{cc}
1 & 0 \\
\tilde{\partial}R_{ij} & 1
\end{array} \right] (m^{rh}(z))^{-1}, & z \in \Omega_{0j}, \quad i = 0, 7, 8, \quad j = 3, 4,
m^{rh}(z) \left[ \begin{array}{cc}
0 & 0 \\
\tilde{\partial}R_{1i} & 0
\end{array} \right] (m^{rh}(z))^{-1}, & z \in \Omega_{i1}, \quad i = 1, 2, \ldots, 6,
m^{rh}(z) \left[ \begin{array}{cc}
0 & 0 \\
\tilde{\partial}R_{i2} & 0
\end{array} \right] (m^{rh}(z))^{-1}, & z \in \Omega_{i2}, \quad i = 1, 2, \ldots, 6,
m^{rh}(z) \left[ \begin{array}{cc}
0 & 0 \\
\tilde{\partial}R_{i3} & 0
\end{array} \right] (m^{rh}(z))^{-1}, & z \in \Omega_{i3}, \quad i = 1, 2, \ldots, 6,
m^{rh}(z) \left[ \begin{array}{cc}
0 & 0 \\
\tilde{\partial}R_{i4} & 0
\end{array} \right] (m^{rh}(z))^{-1}, & z \in \Omega_{i4}, \quad i = 1, 2, \ldots, 6,
\end{cases}
\]

Now we consider the long time asymptotic behavior of $m^{(3)}$. The solution of $\tilde{\partial}$-Problem 4.1 can be solved by the following integral equation

$$m^{(3)}(z) = I - \frac{1}{\pi} \iint_{\mathbb{C}} \frac{m^{(3)}(s)W^{(3)}(s)}{s-z} dA(s), \quad (4.101)$$

where $A(s)$ is the Lebesgue measure on $\mathbb{C}$. Denote $S$ as the Cauchy–Green integral operator

$$S[f](z) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{f(s)W^{(3)}(s)}{s-z} dA(s), \quad (4.102)$$

then (4.101) can be written as the following operator equation

$$(1 - S)m^{(3)}(z) = I. \quad (4.103)$$

To prove the existence of the operator at large time, we present the following lemma.
Lemma 4.4 The norm of the integral operator $S$ decay to zero as $t \to \infty$, and

$$
\|S\|_{L^\infty \to L^\infty} = O(t^{-\frac{1}{4}}) + O(t^{\frac{\alpha - 1}{2}}),
$$

(4.104)

where $\alpha = \max_{i=1,2,...,6} \alpha_i$.

Proof For any $f \in L^\infty$, we have

$$
\|Sf\|_{L^\infty} \leq \|f\|_{L^\infty} \frac{1}{\pi} \int \int_{C} \frac{|W^{(3)}(s)|}{|s - z|} dA(s),
$$

(4.105)

where

$$
|W^{(3)}(s)| \leq |m^{rhp}(s)|^2 |1 + s^{-2}|^{-1} |\bar{\partial}R^{(2)}(s)|.
$$

(4.106)

Recall $\bar{\partial}R^{(2)}(s) = 0$, $s \in \mathbb{C}/\Omega$, so we only need to consider this estimate in $\Omega$, where

$$
\Omega = \bigcup_{j=1,2,3,4} \left[ \Omega_{0j} \cup \left( \bigcup_{i=1,2,...,8} \Omega_{ij} \right) \right].
$$

(4.107)

Since $m^{rhp}(s) = m^{out}(I + s^{-1}E_1 + O(s^{-2}))$, we can bound $m^{rhp}$

$$
|m^{rhp}(s)| \lesssim 1 + |s|^{-1} = c\sqrt{(1 + |s|^{-1})^2} \lesssim \sqrt{1 + |s|^{-2}} = |s|^{-1/2} \sqrt{1 + |s|^2} = |s|^{-1/2}(s).
$$

(4.108)

Then we have

$$
|m^{rhp}(s)|^2 |1 + s^{-2}|^{-1} \lesssim \frac{|s|^{-2} \langle s \rangle^2}{|1 + s^{-2}|} = \begin{cases} 
O(1), & z \in \Omega_{ij}, \\
\frac{\langle s \rangle}{|s - i|}, & z \in \Omega_{7j}, \\
\frac{\langle s \rangle}{|s + i|}, & z \in \Omega_{8j},
\end{cases} \quad i = 0, 1, 2, \ldots, 6, j = 1, 2, 3, 4.
$$

Taking $z \in \Omega_{01}, \Omega_{11}, \Omega_{74}$ for examples, the other cases are similar.

We introduce an inequality which plays an vital role in our analysis. Make $s = z_0 + u + iv$, $z = \alpha + i\eta$, $u, v, \alpha, \eta > 0$ we have the inequality

$$
\left\| \frac{1}{s - z} \right\|_{L^q(v, \infty)} \lesssim \left( \int_{\mathbb{R}^+} \left[ 1 + \left( \frac{u + z_0 - \alpha}{v - \eta} \right)^2 \right]^{-\frac{q}{2}} (v - \eta)^{-q} du \right)^{\frac{1}{q}}
$$

$$
= |v - \eta|^{\frac{1}{q} - 1} \left( \int_{\mathbb{R}^+} \left[ 1 + \left( \frac{u + z_0 - \alpha}{v - \eta} \right)^2 \right]^{-\frac{q}{2}} d \left( \frac{u + z_0 - \alpha}{v - \eta} \right) \right)^{1/q}.
$$
For \( q > 1 \),
\[
0 < |v - \eta|^1/q - 1. 
\]

For \( z \in \Omega_{01} \), we make \( z = \alpha + i \eta, s = 0 + u + iv \). Thanks to (4.109), we have

\[
\begin{align*}
\frac{1}{\pi} \iint_{\Omega_{01}} \frac{|\tilde{\partial} R_{01}| |e^{-2it\theta}|}{|s - z|} dA(s) \\
\lesssim \int \int_{\Omega_{01}} \frac{|\tilde{\partial}'(\text{Res})| e^{-ctv}}{|s - z|} dA(s) + \int \int_{\Omega_{01}} \frac{|s|^{-\frac{1}{2}} e^{-ctv}}{|s - z|} dA(s) \\
\lesssim t^{-\frac{1}{2}}.
\end{align*}
\]

For \( z \in \Omega_{11} \), For \( s = \zeta_1 + u + iv \in \Omega_{11}, |s - \zeta_1|^{\frac{1}{2}} \) is bounded, then

\[
\begin{align*}
\frac{1}{\pi} \iint_{\Omega_{11}} \frac{|\tilde{\partial} R_{11}| |e^{-2it\theta}|}{|s - z|} dA(s) \\
\lesssim \int \int_{\Omega_{11}} \frac{e^{-ctv^2}}{|s - z|} dA(s) + \int \int_{\Omega_{11}} \frac{|s - \zeta_1|^{-\alpha_1} e^{-ctv^2}}{|s - z|} dA(s) \\
\lesssim I_1 + I_2.
\end{align*}
\]

First \( I_1 \) can be estimated as follows:

\[
\begin{align*}
I_1 &\lesssim \int_0^{\xi_1 \tan \phi} e^{-ctv^2} \left\| \frac{1}{s - z} \right\|_{L^2} dv \\
&\lesssim \int_0^{+\infty} e^{-ctv^2} \frac{1}{\sqrt{v - \eta}} dv \\
&= \int_0^{\eta} \frac{e^{-ctv^2}}{\sqrt{v - \eta}} dv + \int_{\eta}^{+\infty} \frac{e^{-ctv^2}}{\sqrt{\eta - v}} dv \\
&= I_{11} + I_{12}.
\end{align*}
\]

where

\[
I_{11} = \int_0^{1} \frac{\sqrt{\eta} e^{-ctw^2 \eta^2}}{\sqrt{1 - w}} dw \lesssim \int_0^{1} \frac{t^{-\frac{1}{4}} w^{-\frac{1}{2}}}{\sqrt{1 - w}} dw \lesssim t^{-\frac{1}{4}}
\]

and

\[
I_{12} = \int_0^{+\infty} \frac{e^{-ctw^2}}{\sqrt{w}} dw \lambda = t^{\frac{1}{2}} \int_0^{+\infty} \frac{e^{-c\lambda^2}}{\sqrt{\lambda}} d\lambda \lesssim t^{-\frac{1}{4}}.
\]
As for $I_2$, we have

$$I_2 = \int_0^{\xi_1} e^{-ctv^2} \left( \int_{v-\xi_1}^{\xi_1} \frac{|z-\xi_1|^{-\alpha_1}}{|s-z|} du \right) dv$$

where

$$I_{21} = \int_0^1 e^{-ct\eta^2w^2(\eta - w\eta)^{\frac{1}{q}} - 1} (w\eta)^{\frac{1}{p} - \alpha_1} dw$$

$$= \int_0^1 e^{-ct\eta^2w^2} \eta^{1-\alpha_1} (1 - w)^{\frac{1}{q} - 1} w^{\frac{1}{p} - \alpha_1} dw$$

$$\lesssim t^{\frac{q}{2} - \frac{1}{2}} \lesssim t^{\frac{q}{2} - \frac{1}{2}}.$$  \quad (4.112)

and

$$I_{22} = \int_0^{+\infty} e^{-ct(\eta + w)^2} w^{\frac{1}{q} - 1} (\eta + w)^{\frac{1}{p} - \alpha_1} dw \lesssim \int_0^{+\infty} e^{-ctw^2} w^{-\alpha_1} dw$$

$$\lesssim t^{\frac{q}{2} - \frac{1}{2}} \lesssim t^{\frac{q}{2} - \frac{1}{2}}.$$  \quad (4.113)

For $z = u + iv \in \Omega_{7^4}$, we have

$$\frac{1}{\pi} \int \int_{\Omega_{7^4}} \frac{\langle s \rangle |\bar{\partial} R_{7^4}| e^{2i\theta}}{|s - i||s - z|} dA(s)$$

$$\lesssim \int \int_{\Omega_{7^4}} \frac{\langle s \rangle |\bar{\partial} R_{7^4}| e^{2i\theta} |\chi_{U;\varepsilon}(|s|)}}{|s - i||s - z|} dA(s)$$

$$+ \int \int_{\Omega_{7^4}} \frac{|\langle s \rangle |\bar{\partial} R_{7^4}| e^{2i\theta} |\chi_{\Omega_{7^4}/U;\varepsilon}(|s|)}}{|s - i||s - z|} dA(s)$$

$$= I_3 + I_4,$$

where $\chi_{U;\varepsilon}(|s|) + \chi_{\Omega_{7^4}/U;\varepsilon}(|s|)$ is the partition of unity. Note $1 - v < u < \sqrt{1 - v^2}$, we have

$$e^{-ct|s|^{\alpha_1} - 1} v^2 = e^{-ct\frac{|1 - u^2 - v^2|^2}{u^2 + v^2}} v^2 \leq e^{-ct|1 - u - v^2|v^2} \leq e^{-ct|1 - \sqrt{1 - v^2} - v^2|v^2}. \quad (4.114)$$
For $I_3$, the singularity at $z = i$ can be balanced by (4.59) and $\langle s \rangle$ is bounded for $z \in \Omega_{74}$. Utilizing $e^{-z} \lesssim z^{-\frac{1}{4}}$, we have

$$I_3 \lesssim \int_0^1 \int_0^1 \frac{1}{|s-z|} e^{-ct|1-\sqrt{1-v^2}|v^2} dudv$$

$$\lesssim \int_0^1 \left\| \frac{1}{|s-z|} \right\|_{L^2} \left( \int_{1-v}^1 e^{-2ct|1-\sqrt{1-v^2}|v^2} du \right)^{\frac{1}{2}} dv$$

$$\lesssim \int_0^1 \frac{1}{\sqrt{|v-\eta|}} v^{\frac{1}{2}} e^{-ct|1-\sqrt{1-v^2}|v^2} dv$$

$$\lesssim t^{-\frac{1}{4}} \int_0^1 \frac{(\sqrt{1-v^2} - 1 + v^2)^{-\frac{1}{2}}}{\sqrt{|v-\eta|}} dv \lesssim t^{-\frac{1}{4}}.$$  (4.115)

Thanks to (4.109), we obtain

$$I_4 \lesssim \iint_{\Omega_{74}} \frac{c_1 + c_2 |z - 1|^{\frac{1}{2}}}{|s-z|} e^{2it\theta} |dA(s) + \iint_{\Omega_{74}} |z - 1|^{\frac{1}{2}} e^{2it\theta} |dA(s)$$

$$\lesssim I_{41} + I_{42},$$  (4.116)

where $I_{41}$ is similar to $I_3$,

$$I_{41} \lesssim \int_0^1 \int_0^1 \frac{1}{|s-z|} e^{-ct|1-\sqrt{1-v^2}|v^2} dudv \lesssim t^{-\frac{1}{4}}$$  (4.117)

and

$$I_{42} \lesssim \int_0^1 \int_0^1 \frac{|z-1|^{\frac{1}{2}}}{|s-z|} e^{-ct|\sqrt{1-v^2} - 1 + v^2|v^2} dudv$$

$$\lesssim \int_0^1 \left\| \frac{1}{|s-z|} \right\|_{L^p} \left\| \frac{1}{|s-z|} \right\|_{L^q} e^{-ct|\sqrt{1-v^2} - 1 + v^2|v^2} dv$$

$$\lesssim \int_0^1 v^{\frac{1}{p} - \frac{1}{2}} |v-\eta|^{\frac{1}{q} - 1} e^{-ct|\sqrt{1-v^2} - 1 + v^2|v^2} dv$$

$$\lesssim \int_{v=\eta w}^{+\infty} w^{\frac{1}{p} + \frac{1}{2}} |1-w|^{\frac{1}{q} - 1} e^{-ctw^2|\sqrt{1-\eta w^2} - 1 + \eta w^2|} dw$$

$$\lesssim t^{-\frac{1}{4}} \int_{0}^{+\infty} w^{\frac{1}{p}} |1-w|^{\frac{1}{q} - 1} \sqrt{|1-\eta w^2| - 1 + \eta w^2}^{-\frac{1}{4}} dw$$

$$\lesssim t^{-\frac{1}{4}}.$$  (4.118)
Based on the above discussion, we have the following proposition.

**Proposition 4.10** As \( t \to \infty \), \((I - S)^{-1}\) exists, which implies \( \tilde{\partial} \) Problem 4.1 has a unique solution.

Aim at the asymptotic behavior of \( m^{(3)} \), we make the asymptotic expansion

\[
m^{(3)} = I + z^{-1}m_1^{(3)}(x, t) + O(z^{-2}), \quad \text{as} \quad z \to \infty
\]

(4.119)

where

\[
m_1^{(3)}(x, t) = \frac{1}{\pi} \iint_{\mathbb{C}} m^{(3)}(s) W^{(3)}(s) dA(s).
\]

(4.120)

To recover the solution of (1.1), we shall discuss the asymptotic behavior of \( m_1^{(3)}(x, t) \), thus we have the following proposition.

**Proposition 4.11** As \( t \to \infty \),

\[
|m_1^{(3)}(x, t)| \lesssim t^{-\frac{3}{4}} + t^{\frac{a}{2} - 1}.
\]

(4.121)

**Proof** Take \( z \in \Omega_{01}, \Omega_{11}, \Omega_{74} \) as examples.

For \( z \in \Omega_{01} \), we make \( z = \alpha + i \eta, s = 0 + u + i v \). Thus,

\[
|m_1^{(3)}| \lesssim \iint_{\Omega_{01}} |\tilde{\rho}'(\text{Res})| e^{-ctv} dA(s) + \iint_{\Omega_{01}} |s|^{-1/2} e^{-ctv} dA(s)
\]

\[
= \int_{\mathbb{R}^+} \|\tilde{\rho}'(\text{Res})\|_{L^2(\mathbb{R}^+)} e^{-ctv} dv + \int_{\mathbb{R}^+} \|s|^{-1/2}\|_{L^p(\mathbb{R}^+)} \|e^{-ctv}\|_{L^q(\mathbb{R}^+)} dv,
\]

\[
\lesssim t^{-1}. \quad (4.122)
\]

For \( z \in \Omega_{11} \), we make \( z = \alpha + i \eta, s = \xi_1 + u + iv \). Then

\[
|m_1^{(3)}(z)| \lesssim \iint_{\Omega_{11}} e^{-ctv^2} dA(s) + \iint_{\Omega_{11}} |z - \xi_1|^{-\alpha_1} e^{-ctv^2} dA(s) = I_1 + I_2,
\]

(4.123)

where

\[
\iint_{\Omega_{11}} e^{-ctv^2} dA(s) = \int_0^{\frac{\xi_1}{2} \tan \phi} \int_{\frac{\xi_1}{2} + \frac{v}{\tan \phi}}^{\frac{\xi_1}{2} + \frac{v}{\tan \phi}} e^{-ctv^2} du dv
\]

\[
\lesssim \int_0^{\frac{\xi_1}{2} \tan \phi} \left( \int_{\frac{\xi_1}{2} + \frac{v}{\tan \phi}}^{\frac{\xi_1}{2} + \frac{v}{\tan \phi}} e^{-2ctv^2} du \right)^{1/2} dv
\]

\[
\lesssim \int_0^{+\infty} v^2 e^{-ctv^2} dv \lesssim t^{-\frac{3}{4}} \quad (4.124)
\]
and

\[
I_2 \lesssim \int_0^{+\infty} \| |z - \zeta|^{-\alpha_1} \|_{L^p} \| e^{-ctv^2} \|_{L^q} dv \\
\lesssim \int_0^{+\infty} v^{-\alpha_1 + \frac{1}{p} v^{\frac{1}{q}}} e^{-ctv^2} dv \\
\lesssim \int_0^{+\infty} v^{1-\alpha_1} e^{-ctv^2} dv \\
= \frac{1}{v^{\frac{3}{4}}} \int_0^{+\infty} w^{1-\alpha_1} e^{-w^2 - \frac{1}{2}t} dw \\
\lesssim t^{\frac{\alpha_1}{2} - 1} \lesssim t^{\frac{\alpha_1}{2} - 1}.
\]

(4.125)

For \( z \in \Omega_{74} \), for \( s = u + iv \), we have

\[
|m_1^{(3)}| \lesssim \frac{1}{\pi} \iint_{\Omega_{74}} \frac{\langle s \rangle |\tilde{\partial} R_{74}| |e^{2it\theta}|}{|s - i|} dA(s) \\
\lesssim \iint_{\Omega_{74}} \frac{\langle s \rangle |\tilde{\partial} R_{74}| |e^{2it\theta}| \chi_{U(1;\epsilon)}(|s|)}{|s - i|} dA(s) \\
+ \iint_{\Omega_{74}} \frac{\langle s \rangle |\tilde{\partial} R_{74}| |e^{2it\theta}| \chi_{\Omega_{74}/U(1;\epsilon)}(|s|)}{|s - i|} dA(s) \\
= \hat{I}_3 + \hat{I}_4,
\]

where \( \chi_{U(i;\epsilon)}(|s|) + \chi_{\Omega_{74}/U(i;\epsilon)}(|s|) \) is the partition of unity. With the help of (4.59), we can balance the singularity at \( z = i \). Utilizing \( e^{-z} \lesssim z^{-\frac{3}{4}} \), we derive

\[
\hat{I}_3 \lesssim \int_0^1 \int_{1-v}^1 e^{-ct\sqrt{1-v^2 - 1 + v^2}|v^2|} dudv \\
\lesssim \int_0^1 ve^{-ct\sqrt{1-v^2 - 1 + v^2}|v^2|} dv \\
\lesssim \int_0^{+\infty} ve^{-ct\sqrt{1-v^2 - 1 + v^2}|v^2|} dv \\
= t^{-1} \int_0^{+\infty} we^{-ctw^2\sqrt{1-t^{-1}w^2 - 1 + t^{-1}w^2}} dw \\
\lesssim t^{-1}.
\]

(4.126)

Notice \( \frac{\langle s \rangle}{|s - i|} = O(1) \) for \( s \in \Omega_{74}/U(i; \epsilon) \), we obtain

\[
\hat{I}_4 \lesssim \iint_{\Omega_{74}} (c_1 + c_2|z - 1|^\frac{1}{2})|e^{2i\theta}| dA(s) + \iint_{\Omega_{74}} |z - 1|^{-\frac{1}{2}}|e^{2i\theta}| dA(s) \\
= \hat{I}_{41} + \hat{I}_{42},
\]

(4.127)
where

\[
\hat{I}_{41} \lesssim \int_0^1 \int_{1-v}^1 e^{-ct(\sqrt{1-v^2-1+v^2})v^2} dudv \\
\lesssim t^{-1},
\]

which is similar to \(\hat{I}_3\).

\[
\hat{I}_{42} \lesssim \int_0^1 \int_{1-v}^1 |z - 1|^{-\frac{1}{2}} e^{-ct|\sqrt{1-v^2-1+v^2}|v^2} dudv
\]

\[
\lesssim \|z - 1\|_{L^p} \left( \int_{1-v}^1 e^{-ct|\sqrt{1-v^2-1+v^2}|v^2} du \right)^{\frac{1}{q}} dv
\]

\[
= \int_0^1 v^\frac{1}{2} e^{-ct|\sqrt{1-v^2-1+v^2}|v^2} dv
\]

\[
\lesssim \int_0^1 v^\frac{1}{2} e^{-ct|\sqrt{1-v^2-1+v^2}|v^2} dv
\]

\[
w = t^\frac{1}{4} v^\frac{1}{2} \int_0^\infty w^2 e^{-ctw^4|\sqrt{1-t^{-1}w^4}-1+t^{-1}w|dw}
\]

\[
\lesssim t^{-\frac{3}{4}}.
\]

5 Asymptotic Analysis on RH Problem in Region \(\xi > 6\)

In this section, we will discuss some results for the case \(\xi > 6\). Different from the case of \(\xi < -6\), the four phase points are not on the jump contours besides \(\pm 1\) in this case. It implies that the four phase points will not contribute to long-time behavior. Next, we will discuss below with the similar steps in Sect. 4.

The jump matrix \(v(z)\) allows the following factorization:

\[
v(x, t; z) = \begin{bmatrix}
\frac{1}{1-\rho}\rho e^{-2it\theta} & 0 \\
1 - \rho & 0
\end{bmatrix}
\begin{bmatrix}
1 - \rho \bar{\rho} e^{2it\theta} & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
1 - \frac{\bar{\rho}}{1-\rho}\rho e^{2it\theta} & 0 \\
0 & 1
\end{bmatrix}, \quad z \in \Sigma. \quad (5.1)
\]

Then we choose

\[
T(z) := T(z; \xi) = \prod_{k \in \Delta_1} \prod_{l \in \Delta_2} \frac{(z + z_k^{-1})(z - z_k^{-1})(z - i\omega_l^{-1})}{(z\bar{z}_k^{-1} - 1)(z\bar{z}_k^{-1} + 1)(i\omega_l^{-1}z + 1)} \exp \left[i \int_{\Sigma} v(s) \left(\frac{1}{s - z} - \frac{1}{2s}\right) ds\right],
\]

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which allows the following estimate as $z \to \pm 1$:

$$\left| T(z) - T_{\pm 1}(\pm 1)|z \mp 1|^{-2i\nu(\pm 1)} \right| \leq c|z \mp 1|^\frac{1}{2}, \quad (5.2)$$

where

$$T_{\pm 1}(\pm 1) = \prod_{k \in \Delta_1} \prod_{l \in \Delta_2} \frac{(\pm 1 + z^{-1}_k)(\pm 1 - z^{-1}_k)(\pm 1 - i\omega_l^{-1})}{(\pm z^{-1}_k - 1)(\pm z^{-1}_k + 1)(\pm 1 - i\omega_l + 1)} \exp[i\beta_{\pm 1}(\pm 1)],$$

$$\beta_{-1}(z, \xi) = \nu(-1) \log |z + 2| + \int_{-\infty}^{-1} \frac{\nu(s) + \chi_1(s)\nu(-1)}{s - z} ds$$

$$+ \int_{-1}^{+\infty} \frac{\nu(s) - \chi_2(s)\nu(-1)}{s - z} ds,$$

$$\beta_{+1}(z, \xi) = \nu(1) \log |z - 2| + \int_{-\infty}^{-1} \frac{\nu(s) - \chi_3(s)\nu(1)}{s - z} ds$$

$$+ \int_{1}^{+\infty} \frac{\nu(s) - \chi_4(s)\nu(1)}{s - z} ds. \quad (5.3)$$

Utilizing transformation

$$m^{(1)}(z) = T(\infty)^{\sigma_3} m(z) G(z) T(z)^{-\sigma_3}, \quad (5.4)$$

it’s easy to verify $m^{(1)}(z)$ still satisfies the RHP 4.2 formally.

### 5.1 Opening Lenses

We fix an angle $\theta_0 > 0$ sufficiently small such that the set $\{z \in \mathbb{C} : |\text{Re}z|, |\text{Re}z^{-1}| > \cos \theta_0\}$, does not intersect the discrete spectrums set $Z \cup \hat{Z}$. For any $-\infty < \xi < 6$, let

$$\phi(\xi) = \min \left\{ \theta_0, \frac{\pi}{4} \right\} \quad (5.5)$$

and define

$$\Sigma_{jum} = \bigcup_{i, j = 1, 2, 3, 4} \left( \Sigma_{0j} \cup \Sigma_{ij} \right), \quad (5.6)$$

which is shown in Fig. 10 and consists of rays and other line segments or arcs.

**Lemma 5.1** Set $\xi = \frac{\xi}{7}$ and let $6 < \xi < +\infty$. Then for $z = u + iv \in \Omega_{0j}, \Omega_{ij}, i, j = 1, 2, 3, 4$, the phase $\theta(z)$ defined in (3.1) satisfies

$$\text{Re}[2i\theta(z)] \geq cv, \quad z \in \Omega_{k1}, \Omega_{k2}, \Omega_{i1}, \Omega_{i3}, k = 0, 3, i = 1, 2,$$

$$\text{Re}[2i\theta(z)] \leq -cv, \quad z \in \Omega_{k3}, \Omega_{k4}, \Omega_{i2}, \Omega_{i4}, k = 0, 3, i = 1, 2, \quad (5.7)$$
Fig. 10  The blue curves are the opening contours in region \( \{ z \in \mathbb{C} : |e^{-2i\theta(z)}| \to 0 \} \) while the red curves are the opening contours in region \( \{ z \in \mathbb{C} : |e^{2i\theta(z)}| \to 0 \} \). These arrows represent directions of jump contours

where \( c = c(\xi) > 0. \)

**Proof**  Take \( z \in \Omega_{01} \) as an example. For \( z = u + iv \in \Omega_{01} \), we have

\[
\text{Re}[2i\theta(z)] = -v(1 - |z|^{-2})[\xi - 3 + (1 + |z|^{-2} + |z|^{-4})(3u^2 - v^2)],
\]

(5.8)

where

\[
\xi - 3 + (1 + |z|^{-2} + |z|^{-4})(3u^2 - v^2) \geq 3 + 3(3u^2 - v^2) \geq 3.
\]

(5.9)

Thus,

\[
\text{Re}[2i\theta(z)] \geq cv,
\]

(5.10)

with \( c = c(\xi) > 0. \)  \( \Box \)
Redefine

\[ l_0^+ = (0, z_1), \quad l_0^- = (-z_1, 0), \]
\[ l_1^+ = (z_1, 1), \quad l_1^- = (1, +\infty), \quad l_2^+ = (-1, -z_1), \quad l_2^- = (-\infty, -1), \]
\[ \gamma_k = \left\{ z \in \mathbb{C} : z = e^{iw}, \frac{(k - 1)\pi}{2} \leq w \leq \frac{k\pi}{2} \right\}, \quad k = 1, 2, 3, 4, \]

and

\[
\mathcal{R}^{(2)}(z) = \begin{cases} 
1 - R_{ij}e^{2i\theta} & , \quad z \in \Omega_{ij}, \quad i = 0, 3, j = 1, 2, \\
0 & , \quad z \in \Omega_{ij}, \quad i = 0, 3, j = 3, 4, \\
1 - R_{i1}e^{2i\theta} & , \quad z \in \Omega_{i1}, \quad i = 1, 2, \\
0 & , \quad z \in \Omega_{i2}, \quad i = 1, 2, \\
1 - R_{i3}e^{2i\theta} & , \quad z \in \Omega_{i3}, \quad i = 1, 2, \\
0 & , \quad z \in \Omega_{i4}, \quad i = 1, 2, \\
I & , \text{ else where,}
\end{cases}
\]

(5.12)

where the functions \( R_{0j} \) is the same as defined in Proposition 4.2 and \( R_{ij}, i, j = 1, 2, 3, 4 \) are defined in the following proposition.

Proposition 5.1 \( R_{ij} : \overline{\Omega}_{ij} \rightarrow \mathbb{C}, i = 1, 2, \ldots, 8, j = 1, 2, 3, 4 \) are continuous on \( \overline{\Omega}_{ij} \) with boundary values:

(i) For \( i = 1, 2, \)

\[
R_{ij}(z) = \begin{cases} 
r_3(z)T_{-2}^{-2}(z), & z \in l_1^+, \quad l_1^-, \\
r_3(\pm 1)T_{\pm 1}^{-2}(\pm 1)|z + 1|^{4iv(\pm 1)}, & z \in \Sigma_{ij}, \quad j = 1, 3, \\
r_4(z)T_2^{2}(z), & z \in l_i^+, \\
r_4(\pm 1)T_{\pm 1}^{2}(\pm 1)|z + 1|^{-4iv(\pm 1)}, & z \in \Sigma_{ij}, \quad j = 2, 4,
\end{cases}
\]

(5.13)

in which take \(+1\) when \( i = 1 \) and take \(-1\) when \( i = 2.\)
(ii) For $i = 3,$

\[
R_{ij}(z) = \begin{cases} 
  r_3(z)T_{-1}^{-2}(z), & z \in \gamma_1, \gamma_4, \\
  r_3(\pm 1)T_{\pm 1}^{-2}(\pm 1)|z| \mp 1^{4i\nu(\pm 1)}(1 - \chi_2(z)), & z \in \Sigma_3, j = 1, 2,
\end{cases}
\]

\[
R_{ij}(z) = \begin{cases} 
  r_4(z)T_{+1}^2(z), & z \in \gamma_2, \gamma_3, \\
  r_4(\pm 1)T_{\pm 1}^2(\pm 1)|z| \mp 1^{-4i\nu(\pm 1)}(1 - \chi_2(z)), & z \in \Sigma_3, j = 3, 4,
\end{cases}
\]

(5.15)

\[
in which take $+1$ when $j = 1, 4$ and take $-1$ when $j = 2, 3.$
\]

Moreover, $R_{ij}(z), i, j = 1, 2, 3, 4$ have following properties:

\[
|R_{ij}(z)| \leq c_1 + c_2|1 + z^2|^{-\frac{1}{2}}, \quad z \in \Omega_{ij},
\]

\[
|\bar{\partial}R_{ij}(z)| \leq c_1 + c_2|z| \mp 1|^{\frac{1}{2}} + c_3|z| \mp 1|^{-\frac{1}{2}}, \quad z \in \Omega_{ij},
\]

(5.17)

and when $z \to \pm i$,

\[
|\bar{\partial}R_{ij}(z)| \leq c|z - i|, \quad z \in \Omega_3, \quad j = 1, 4,
\]

\[
|\bar{\partial}R_{ij}(z)| \leq c|z + i|, \quad z \in \Omega_3, \quad j = 2, 3.
\]

(5.18)

**Proof** The proof is similar to Proposition 4.3.

Define $\Sigma^{(2)} = \Sigma'_{jum} \cup \Sigma_{cir},$ where

\[
\Sigma'_{jum} = \bigcup_{j=1,2,3,4} \left( \Sigma_3 \cup \bigcup_{i=1,2} \Sigma'_{ij} \right),
\]

(5.19)

can be referred in Fig. 11. We get a RH problem of $m^{(2)}(z)$ by transformation

\[
m^{(2)}(z) = m^{(1)}(z)R^{(2)}(z),
\]

(5.20)

which has the same form as RH problem 4.3 with different jump matrix as:
$$v(2) = [\mathcal{R}^{(2)}_-]^{-1} v(1) \mathcal{R}^{(2)}_+ = \begin{cases} \begin{pmatrix} 1 & 0 \\ (R_{11} - R_{01})e^{-2i\theta} & 1 \end{pmatrix}, & z \in \Sigma'_{11}, \\
\begin{pmatrix} 1 & 0 \\ (R_{14} - R_{04})e^{2i\theta} & 1 \end{pmatrix}, & z \in \Sigma'_{14}, \\
\begin{pmatrix} 1 & 0 \\ (R_{12} - R_{02})e^{-2i\theta} & 1 \end{pmatrix}, & z \in \Sigma'_{21}, \\
\begin{pmatrix} 1 & 0 \\ (R_{24} - R_{03})e^{2i\theta} & 1 \end{pmatrix}, & z \in \Sigma'_{24}, \\
\begin{pmatrix} 1 & 0 \\ R_{i2}e^{2i\theta} & 1 \end{pmatrix}, & z \in \Sigma'_{i2}, \quad i = 1, 2, \\
\begin{pmatrix} 1 & 0 \\ R_{i3}e^{-2i\theta} & 1 \end{pmatrix}, & z \in \Sigma'_{i3}, \quad i = 1, 2, \\
\begin{pmatrix} 1 & 0 \\ -R_{ij}e^{-2i\theta} & 1 \end{pmatrix}, & z \in \Sigma_{3j}, \quad j = 1, 2, \\
\begin{pmatrix} 1 & 0 \\ R_{ij}e^{2i\theta} & 1 \end{pmatrix}, & z \in \Sigma_{3j}, \quad j = 3, 4, \\
\begin{pmatrix} 1 & 0 \\ -A[\eta_k] T^{-2}(z)e^{-2i\theta(\eta_k)} & 1 \end{pmatrix}, & |z - \eta_k| = \rho, \quad k \in \nabla / \Lambda, \\
\begin{pmatrix} 1 & 0 \\ z^{-\eta_k} T^2(z)e^{2i\theta(\eta_k)} & 1 \end{pmatrix}, & |z - \eta_k| = \rho, \quad k \in \Delta / \Lambda, \\
\begin{pmatrix} 1 & 0 \\ -\hat{A}[\hat{\eta}_k] T^{-2}(z)e^{-2i\theta(\hat{\eta}_k)} & 1 \end{pmatrix}, & |z - \hat{\eta}_k| = \rho, \quad k \in \nabla / \Lambda, \\
\begin{pmatrix} 1 & 0 \\ z^{-\hat{\eta}_k} T^2(z)e^{2i\theta(\hat{\eta}_k)} & 1 \end{pmatrix}, & |z - \hat{\eta}_k| = \rho, \quad k \in \Delta / \Lambda. \end{cases}$$

5.2 Decomposition of Mixed $\bar{\partial}$-RH Problem

Like the case $-\infty < \xi < -6$, we decompose $m^{(2)}(z)$ into the following structure

$$m^{(2)}(z) = \begin{cases} \bar{\partial} \mathcal{R}^{(2)} \equiv 0 \longrightarrow m^{rh}(z), \\
\bar{\partial} \mathcal{R}^{(2)} \neq 0 \longrightarrow m^{(3)}(z) = m^{(2)}(z)[m^{rh}(z)]^{-1}. \end{cases} \quad (5.21)$$

Remark 5.1 In this case, there is no phase point on the jump contour $\Sigma_{jump}'$. Thus, we construct the solution $m^{rh}(z)$ of the RH problem as

$$m^{rh}(z) = E(z)m^{sol}(z) = E(z)m^{err}(z)m^{\Lambda}(z). \quad (5.22)$$
In which, $m^{sol}(z)$ is given in (4.83). As for $E(z)$, it satisfies RH problem 4.9 with jump matrix

$$
v^{E}(z) = m^{sol}(z)v^{(2)}(z)[m^{sol}(z)]^{-1}, \quad z \in \Sigma_{jump}^\prime,
$$

and

$$
E(z) = I + z^{-1}O(e^{-ct}) + O(z^{-2}).
$$

5.3 Analysis on the Pure $\bar{\partial}$-Problem

Similar to the case of $-\infty < \xi < -6$, we focus our insights on the estimates for the Cauchy–Green operator $S$ defined by (4.102) and $m^{(3)}_{1}$ defined by (4.120). Then we have the following two estimations.

**Lemma 5.2** *The norm of the integral operator $S$ decay to zero as $t \to \infty$, and*

$$
\|S\|_{L^\infty \to L^\infty} = O(t^{-\frac{1}{2}}).
$$

**Proof** The proof is the analogue of Lemma 5.3. in [31].
Proposition 5.2 As $t \to \infty$, 

$$|m^{(3)}_1(x, t)| \lesssim t^{-1}. \quad (5.26)$$

Proof The proof is similar to Proposition 5.10. in [31].

6 Long-Time Asymptotics for the Nonlocal mKdV Equation

Now we start to construct the asymptotic solution for the nonlocal mKdV equation (1.1) in the case of $\xi < -6$ and $\xi > 6$ by proving Theorem 1.1. 

Proof Take $\xi < -6$ as an example, the case of $\xi > 6$ can be proved similarly. Denote

$$T_1 = 2i \sum_{k \in \Delta_1} (\text{Im} z_k + \text{Im} z_k^{-1}) + \sum_{l \in \Delta_1} (\text{Im} \omega_l + \text{Im} \omega_l^{-1}), \quad (6.1)$$

then $T(z)$ can be written as

$$T(z) = T(\infty) \left[ 1 + z^{-1} T_1 + \mathcal{O}(z^{-2}) \right].$$

Recall all the transformations for $m(x, t; z)$, we obtain

$$m(z) = T(\infty)^{-\sigma_3} m^{(3)}(z) E(z) m^{sol}(z) T(\infty)^{\sigma_3} \left[ I + z^{-1} T_1^{\sigma_3} + \mathcal{O}(z^{-2}) \right]. \quad (6.2)$$

From (4.83), the asymptotic behaviors can be written as follows:

$$m^{sol}(z) = I + z^{-1} \left( m^A_1 + \mathcal{O}(e^{-ct}) \right) + \mathcal{O}(z^{-2}). \quad (6.3)$$

Thus,

$$m(z) = I + z^{-1} T(\infty)^{-\sigma_3} \left[ m^{(3)}_1 + m^A_1 + E_1 + \mathcal{O}(e^{-ct} + T_1^{\sigma_3}) \right] T(\infty)^{\sigma_3} + \mathcal{O}(z^{-2}). \quad (6.4)$$

According to the potential recovering formulae (2.53), $q(x, t)$ is obtained:

$$q(x, t) = q^A(x, t) - iT(\infty)^{-2} \sum_{i=1, 2, 5, 6} i^{-\frac{1}{2} + \text{Im}^*(u_i)} f_i + \mathcal{O}\left( t^{-1+ \max_{i=1, 2, \ldots, 6} \text{Im}^*(u_i) + \max_{i=1, 2, \ldots, 6} \text{Im}^*(u_i)} \right),$$

where

$$q^A = -iT(\infty)^{-2} \lim_{z \to \infty} (zm^A)_12$$
and

\[ f_i = \frac{m_{11}(\zeta_i)^2 \beta_{12}(\zeta_i) \beta_{21}(\zeta_i)}{2 \sqrt{\theta'(\zeta_i)}} m^{sol}(\zeta_i) \det m^{sol}(\zeta_i), \quad m^{sol}(\zeta_i) = \begin{bmatrix} m_{11}(\zeta_i) & m_{12}(\zeta_i) \\ m_{21}(\zeta_i) & m_{22}(\zeta_i) \end{bmatrix}. \quad (6.5) \]

Denote

\[ R(t; \xi) = O(t^{-1+ \max_{i=1,2,\ldots,6} \text{Im} \hat{\omega}(\xi)_i + \max_{i=1,2,\ldots,6} |\text{Im} \hat{\omega}(\xi)_i|}). \]

Compare the powers of \( t \) in the secondary main term and the error term, (1.5) is derived.

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**Data Availability**  The data that supports the findings of this study are available within the article.

**Declarations**

**Conflict of interest**  The authors have no conflicts to disclose.

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