THREE WAYS TO LOOK
AT
MUTUALLY UNBIASED BASES

Ingemar Bengtsson

Stockholm University, Alba Nova
Fysikum
S-106 91 Stockholm, Sweden

Abstract

This is a review of the problem of Mutually Unbiased Bases in finite dimensional Hilbert spaces, real and complex. Also a geometric measure of "mubness" is introduced, and applied to some explicit calculations in six dimensions (partly done by Björck and by Grassl). Although this does not yet solve any problem, some appealing structures emerge.

---

1Talk at the Växjö conference on Foundations of Probability and Physics, June 2006.
2Email address: ingemar@physto.se. Supported by VR.
1. The problem

A formula that has been of central importance in many discussions about the foundations of quantum mechanics is

\[ |\langle x|p\rangle|^2 = \text{constant} \, . \]  

(1)

It expresses the complementarity of position and momentum. If we know everything about position, we know nothing of momentum. In Hilbert spaces of finite dimension \( N \), the analogous equation would concern two orthonormal bases \( |e_a\rangle \) and \( |f_a\rangle \) such that

\[ |\langle e_a|f_b\rangle|^2 = \text{constant} = \frac{1}{N} \, , \quad 0 \leq a, b \leq N - 1 \, . \]  

(2)

The important thing is that the right hand side is independent of \( a \) and \( b \). If such bases can be found, they are said to be mutually unbiased bases, or MUBs for short. The name emphasises that the information obtained in a projective measurement associated to one of the bases is completely unrelated to the information obtained from a projective measurement associated to the other. The question is: how many MUBs can one introduce in a given Hilbert space? If \( N \) is a power of a prime number then one can find \( N + 1 \) MUBs \([1] [2]\), but for other values of \( N \) the number might be smaller. On the other hand it might not: I simply do not know.

Why should you care about this problem? Apart from the fact that it is easy to state, several answers can be given. One answer is that MUBs are useful in quantum state tomography \([3, 4]\). Another has to do with various cryptographic protocols \([5]\). Thus, whether one wants to find or hide information, unbiasedness is a useful property. A third answer is that when one begins to look into it, the MUB problem leads one into many corners of mathematics that have been explored in communication theory, computer science, and so on, but which are relatively unknown to quantum physicists. If the essence of quantum mechanics is that it permits one to do things that cannot be done in a classical world, then many surprises may be lurking in those corners.

As a matter of fact, over the past three years or so, papers about the MUB problem have appeared at a rate larger than once a month. I found about forty of those papers interesting, but I decided to quote only a small
fraction here. On the other hand I will borrow results freely from people that I mention in the acknowledgements; we recently wrote a paper containing much more detail than did my talk (details that this written version will occasionally hint at) [6].

2. Existing constructions

Let us begin by taking a look at existing constructions. Recall that a discussion of position and momentum usually begins with the equation

\[ [x, p] = i , \]  

or in terms of unitary operators

\[ e^{iux} e^{ivp} = e^{-iuv} e^{ivp} e^{iux} . \]  

In his 1928 book, Hermann Weyl [7] considered a finite dimensional analogue of this. We look for two unitary operators \( X \) and \( Z \)—the notation is supposed to suggest an analogy to the Pauli matrices—such that

\[ XZ = q ZX , \]  

where \( q \) is a phase factor. Weyl found that if \( q \) is a primitive root of unity, say if

\[ q = e^{\frac{2\pi i}{N}} , \]  

then eq. (5) admits a representation that is unique up to unitary equivalence, and the eigenbases of \( X \) and \( Z \) are indeed mutually unbiased (although this piece of terminology came later!). If the eigenbasis of \( Z \) is chosen to be the standard basis, then the eigenbasis of \( X \) consists of the columns of the Fourier matrix \( F \), whose matrix elements are

\[ F_{ab} = q^{ab} , \quad 0 \leq a, b \leq N - 1 . \]  

It is called the Fourier matrix because it appears in the discrete Fourier transform. So far, all statements are independent of the dimension \( N \). Closer examination of the group that is generated by \( X \) and \( Z \) reveals some dimension dependent things; notably if \( N = p \) is a prime number, then one can
use the Weyl group to generate a set of $N + 1$ MUBs. But sometimes only three MUBs turn up in this way. The special status of prime numbers has to do with the fact that Weyl’s representation theorem requires a phase factor that is a primitive root of unity, that is a root of unity such that $q^k \neq 1$ for $k < N$. This is true for all the roots of unity only if $N$ is prime.

In general the following result has been established. Let $N = p_1^{n_1} \cdot p_2^{n_2} \cdot \ldots \cdot p_k^{n_k}$ be the prime number decomposition of $N$, with $p_1^{n_1} < p_2^{n_2} < \ldots < p_k^{n_k}$. Then the number of constructed MUBs obeys

$$p_1^{n_1} + 1 \leq \# \text{ MUBs} \leq N + 1 .$$

A complete set of $N + 1$ MUBs has been constructed for all prime power dimensions. Without going into any details (see Bandyopadhyay et al. [8] for a useful account), let me observe that the constructions differ somewhat depending on whether $N$ is a power of an even or an odd prime. In particular the behaviour of the MUBs under complex conjugation (relative to the standard basis) is strikingly different. I do not know what this means, if anything.

When $N$ is not a prime power, the known bounds are not very sharp. For some very special choices of $N$ a somewhat higher lower bound is known [9], but in general the problem is wide open. For $N = 6$ several attempts to construct more than 3 MUBs have been made, using either group theoretical tricks or computer searches. There seems to be a growing consensus that 3 is the best one can do. Perhaps this is related [10, 11] to the fact that no Graeco-Latin square of order 6 exists [12]. A set of $N - 1$ Latin squares that are mutually Graeco-Latin can be used to construct a finite affine plane. I leave this obscure remark as a hint that there are interesting connections between MUBs and combinatorics; for now let me just say that the available evidence concerning MUBs for $N = 6$ strikes me as rather weak.

The MUB problem appears also in other branches of learning. Indeed the same, or nearly the same, problem has occurred in the theory of radar signals [13], to operator algebra theorists [14], in Lie algebra theory [15], and in coding theory [16]. Unfortunately all technical terms differ between these groups of authors, so it is quite difficult for a quantum theorist to extract the information gathered in any of the other fields. Suffice it to say that the Lie algebra theorists have the best name for the problem. They call it the “Winnie-the-Pooh problem”, for reasons that would take us too far afield.
(It has to do with a free Russian translation of a verse hummed by Pooh.) The actual results achieved in the various fields appear to be roughly the same, as far as I can tell.

3. A packing problem

How should we look at the MUB problem? A first way is to view it as a packing problem in Hilbert space, or more accurately in complex projective space.

In Hilbert space a basis can be represented as the columns of a unitary matrix. Assuming one basis is represented by the unit matrix, all bases that are MUB with respect to it must be represented by unitary matrices of the form (for \( N = 3 \), say)

\[
U = \frac{1}{\sqrt{N}} \begin{bmatrix}
1 & 1 & 1 \\
\mathrm{e}^{i\phi_{10}} & \mathrm{e}^{i\phi_{11}} & \mathrm{e}^{i\phi_{12}} \\
\mathrm{e}^{i\phi_{20}} & \mathrm{e}^{i\phi_{21}} & \mathrm{e}^{i\phi_{22}}
\end{bmatrix}.
\]  

(9)

This is (except for the normalizing factor) a complex Hadamard matrix. The first row has been chosen to contain only ones by convention. All the vectors in all bases that are MUB with respect to the standard basis can therefore be found on a torus parametrized by \( N - 1 \) phases. This torus has a natural interpretation as the maximal flat torus in complex projective space (equipped with the Fubini–Study metric). Now I seem to be saying that finding the MUBs is equivalent to a packing problem on a flat torus, but unfortunately this is not quite true, because the tori in complex projective space are not totally geodesic. What this means is that intrinsic distance on the torus does not directly reflect the Fubini-Study distance [17]. We do have a packing problem in complex projective space, but packing problems are difficult, and moreover their solutions tend to depend on dimension in peculiar ways.

A straightforward approach, while we remain in the \( N \) dimensional Hilbert space, is to begin by asking for a classification of all complex Hadamard matrices. But here the existing results are very incomplete [18, 19, 20]. For any \( N \), the Fourier matrix exists, hence one basis that is MUB relative to the standard basis always exists. Are there more? We can multiply the Fourier matrix from the left with a diagonal unitary matrix, that is to say we can
multiply the rows with suitable phase factors, and hope that with appropriate choices of the phase factors we can find further bases that are MUB relative to both the standard basis and to the Fourier basis. We can also ask if there are further Hadamard matrices, not related to the Fourier matrix in this way. One would naively expect that the MUB problem should become easier the more such Hadamards one finds. This expectation is not borne out however.

To classify (complex) Hadamard matrices, it is helpful to begin by declaring two Hadamard matrices to be equivalent if one can be reached from the other by permutations of rows and columns, and by multiplying rows and columns with arbitrary phase factors. Since we think of the columns of an Hadamard matrix as representing a basis in Hilbert space, these operations applied to columns mean nothing at all to us, while the same operations applied to rows are achieved by unitary transformations that leave the standard basis invariant. I will refer to the standard basis as the “zeroth MUB”. Then I do not loose any generality if I assume that the first vector in the “first MUB” has entries $1/\sqrt{N}$ only. Thus $(\text{MUB})_1$ is represented by an Hadamard matrix in dephased form (the first row and the first column has all entries real and positive), while any further MUBs are represented by enphased Hadamard matrices.

When $N = 2$ or 3 the Fourier matrix is unique up to equivalences. Hence, once $(\text{MUB})_0$ and the first vector of $(\text{MUB})_1$ are chosen, everything else is forced. But when $N = 4$ there exists a one parameter family of inequivalent Hadamard matrices, found (appropriately) by Hadamard himself. It is

$$H_4(\phi) = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\
1 & e^{i\phi} & 1 & -1 \\
1 & -1 & 1 & -1 \\
1 & -e^{i\phi} & -1 & e^{i\phi} \end{bmatrix}.$$ 

(10)

Hence there is some freedom in choosing $(\text{MUB})_1$. And the choice matters—we have to set $\phi$ equal to zero, or to $\pi$, if we want to have an additional three MUBs.

When $N = 5$ the Fourier matrix is again unique [18], but when $N = 6$ there are many choices. I will discuss them later. When $N$ is a prime number one cannot introduce any free parameters into the Fourier matrix, but at least when $N = 7$ another, unrelated, one parameter family is known. In brief, the situation is confusing, but based on the information available one would
be inclined to guess that finding MUBs should be particularly easy when \( N = 6 \). Which is definitely not the case!

### 4. The shape of the body of density matrices

A second way to look at MUBs is to look at density matrices. Each vector in an \( N \) dimensional Hilbert space can be thought of as a rank one projector in the set of Hermitian matrices of unit trace, which has real dimension \( N^2 - 1 \). It is convenient to think of this space as a vector space, with the origin placed at

\[
\rho_* = \frac{1}{N} 1 .
\]  

(11)

Then a unit vector \( |e\rangle \) in Hilbert space corresponds to a real unit vector \( e \) in an \( N^2 - 1 \) dimensional vector space, whose elements are traceless matrices. The explicit correspondence is

\[
|e\rangle \rightarrow e = \sqrt{\frac{2N}{N - 1}} (|e\rangle\langle e| - \rho_*) .
\]  

(12)

Our vector space is equipped with the scalar product

\[
e \cdot f = \frac{1}{2} \text{Tr} ef .
\]  

(13)

The catch is that only a small subset of all unit vectors in the large vector space arises in this way. Anyway, the body of density matrices is now obtained by taking the convex cover of all those points on the unit sphere where some vector \( e \) ends. Equivalently, the body of density matrices consists of all Hermitian matrices of unit trace and positive spectrum. Either way, it is a body with an intricate shape that is difficult to visualize. It touches its \( N^2 - 2 \) dimensional outsphere in a small \( 2(N - 1) \) dimensional subset, arising from vectors in Hilbert space through the above correspondence. These are the pure states. The whole space is naturally an Euclidean space, with a squared distance given by

\[
D^2(A, B) = \frac{1}{2} \text{Tr}(A - B)^2 .
\]  

(14)
This is the Hilbert-Schmidt distance. The geometry induced on the set of pure states is of course precisely the Fubini-Study geometry on complex projective \((N-1)\)-space. Alternatively, the Hilbert-Schmidt distance provides the chordal distance between two points on the outsphere of the body of density matrices [17].

An orthonormal basis in Hilbert space is now represented as a regular simplex with \(N\) corners, spanning an \((N-1)\) dimensional flat subspace through the center of the outsphere. Two bases will be MUB if and only if their respective \((N-1)\)-planes are totally orthogonal. Since the whole space has dimension

\[
N^2 - 1 = (N + 1)(N - 1) ,
\]

it is clear that at most \(N + 1\) MUBs can be found.

After a moment’s reflection one sees that a complete set of MUBs defines a rather interesting convex polytope, which is the convex cover of \(N + 1\) regular simplices placed in totally orthogonal \((N-1)\)-planes. It is called the Complementarity Polytope [21]. Evidently such a polytope will exist in every \(N^2 - 1\) dimensional flat space. This does not yet solve our problem though, because given such a polytope its corners will correspond to vectors in Hilbert space if and only if we can rotate it so that all its corners fit into a special \(2(N-1)\) dimensional subset of its \(N^2 - 2\) dimensional outsphere. And when \(N > 2\) this is hard!

Nevertheless this is an appealing picture of MUBs. In particular, this is the way to see why MUBs solve the problem of optimal state determination in non-adaptive quantum state tomography [1, 2].

5. Real Hilbert spaces

It is instructive to pause to think about MUBs in real Hilbert spaces, because in this case it is easy to derive some negative results. Indeed real Hadamard matrices can exist only if \(N\) is two or divisible by four. So we see at once that in a three dimensional real Hilbert space, it is impossible to find even a pair of MUBs. We can see why geometrically—in real Hilbert space, not among the density matrices. If a vector is represented by a pair of antipodal points on the unit sphere in \(\mathbb{R}^3\), then a basis is represented by the corners of
an octahedron. It is then geometrically evident that there are four vectors
that are unbiased with respect to a given basis, and they form the corners of
a cube that is dual to the octahedron. But they do not form a basis!

In $\mathbb{R}^4$ the situation is different. It is still true that a basis is represented
by the corners of a cross polytope (the generalisation to arbitrary dimension
of the octahedron), and there will be eight vectors that are MUB with respect
to a given basis, again forming a cube that is dual with respect to the cross
polytope. But specifically in four dimensions, a cube can be regarded as the
convex cover of two symmetrically placed cross polytopes [22]. In this way
we end up with three bases represented by three symmetrically placed cross
polytopes, and their convex cover is a famous Platonic body known as the
24-cell (having no analogue in three dimensions). This gives us three MUBs,
and since the dimension of the set of real four-by-four density matrices is
nine, this is a complete set in four dimensions. It is a set that has acquired
some fame in quantum foundations, because a pair of dual 24-cells correspond
to 24 vectors that are uncolourable in the Kochen–Specker sense [23].

If you want to know what happens in higher real dimensions, consult
Boykin et al. [24].

6. Mubness

Let us return to the complex case. I will describe an attempt to investi-
gate what happens in six dimensions, but in order to say something more
interesting than the obvious “I failed”, I need a measure of how much I fail.
Many such measures, of varying degrees of sophistication, can be imagined.
The one we use is based on the picture of MUBs that emerged from the den-
sity matrix point of view, namely that they correspond to totally orthogonal
($N - 1$)-planes in an $N^2 - 1$ dimensional space. Just as vectors in an $N$
dimensional space can be regarded as rank one projectors in a higher dimen-
sional space, so one can regard $n$-planes in an $m$-dimensional space as rank $n$
projectors in a vector space of sufficiently high dimension. In mathematical
terms, this provides an embedding of the Grassmannian of $n$-planes into a
flat vector space. The rank $n$ projectors will sit on a sphere in this flat space,
and its natural Euclidean distance provides us with a chordal distance be-
tween the projectors. This notion of distance has been used to study packing
problems for $n$-planes [25], and it is the one we use to measure the distance
between bases in Hilbert space. The chordal distance attains its maximum if the $n$-planes are totally orthogonal, that is to say, if the bases are MUB. In this way it does provide a measure of “mubness”.

The details, adapted to our case, are as follows. Starting from a basis in Hilbert space, form the $N$ vectors $e_a$. Then form the $(N^2 - 1) \times N$ matrix

$$B = \sqrt{\frac{N - 1}{N}} [e_1 \ e_2 \ldots \ e_N].$$

(16)

It has rank $N - 1$. Next we introduce an $(N^2 - 1) \times (N^2 - 1)$ matrix of fixed trace, which is a projector onto the $(N - 1)$ dimensional plane spanned by the $e_a$:

$$P = BB^T = \frac{N - 1}{N} \begin{bmatrix} e_1^T \\ \vdots \\ e_N^T \end{bmatrix} \begin{bmatrix} e_1 & \ldots & e_N \end{bmatrix}. $$

(17)

It is easy to check (through acting on $e_a$ say) that this really is a projector. The chordal Grassmannian distance between two $N - 1$ planes spanned by two different bases then becomes

$$D^2_c(P_1, P_2) = \frac{1}{2} \text{Tr}(P_1 - P_2)^2 = N - 1 - \text{Tr}P_1P_2 .$$

(18)

It should be obvious that there is an analogy to how the density matrices were defined in the first place, and to the Hilbert-Schmidt distance between them. The difference is that now the projectors represent entire bases in Hilbert space, not single vectors.

Working through the details, one finds that

$$D^2_c(P_1, P_2) = N - 1 - \sum_a \sum_b \left( |\langle e_a | f_b \rangle|^2 - \frac{1}{N} \right)^2 .$$

(19)

Thus

$$0 \leq D^2_c \leq N - 1 ,$$

(20)

and the distance attains its maximum value if and only if the bases are MUB. As a measure of mubness, the chordal Grassmannian distance has the advantages that it is geometrically natural and simple to compute. It is
also natural from the tomographic point of view, although I certainly cannot claim any precise operational meaning for it.

7. $N = 6$

My third and final way to look at MUBs is to simply perform calculations to see what happens, without thinking very much. We tried the first open case: $N = 6$. Maybe this was a mistake, because 6 clearly sits astride the even and the odd prime numbers. Perhaps we should concentrate on $15 = 3 \cdot 5$? (The question how many mutually Graeco-Latin squares exist is actually open in this case.) On the other hand, a six dimensional Hilbert space is already a very large space to search in—and fifteen is larger. So we stick to six.

Let $(\text{MUB})_0$ be the standard basis, and $(\text{MUB})_1$ be given by the columns of some dephased Hadamard matrix (that is one whose first row and first column are real). We make a choice for $(\text{MUB})_1$, find all enphased Hadamard matrices that represent bases that are MUB with respect to $(\text{MUB})_1$, and then check how far apart the latter are, in the sense of the chordal distance.

When $N = 6$ there are several choices for $(\text{MUB})_1$. The following dephased Hadamard matrices are known [20]:

i) The Fourier matrix, augmented with two free parameters:

$$F_6(\phi_1, \phi_2) = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & q^{i\phi_1} & q^{2i\phi_2} & q^3 & 1^4q^{i\phi_1} & 1 \\
1 & q^2 & q^4 & 1 & q^2 & q^4 \\
1 & q^2e^{i\phi_1} & e^{i\phi_2} & q^3 & e^{i\phi_1} & q^3e^{i\phi_2} \\
1 & q^4 & q^2 & 1 & q^4 & q^2 \\
1 & q^5e^{i\phi_1} & q^4e^{i\phi_2} & q^3 & q^2e^{i\phi_1} & qe^{i\phi_2}
\end{bmatrix}, \quad q \equiv e^{\frac{2\pi i}{6}}. \quad (21)$$

There are various equivalences of the form $F(\phi_1, \phi_2) \approx F(\phi_3, \phi_4)$; this has been sorted out, but this is not the place to give all the details.

ii) The transpose of the above, again with two free parameters.

iii) A one-parameter family of matrices $D(\phi)$ whose entries, when the free phase is set to zero, are fourth roots of unity. It is known as the Diţă family.

iv) An isolated matrix $S$ whose entries are third roots of unity.
v) A matrix found by Björck, which is most conveniently given as the circulant matrix

\[
C = \begin{bmatrix}
1 & id & -d & -i & -\bar{d} & i\bar{d} \\
id & 1 & id & -d & -i & -\bar{d} \\
-\bar{d} & id & 1 & id & -d & i \\
-i & -\bar{d} & id & 1 & id & -d \\
-d & -i & -\bar{d} & id & 1 & id \\
id & -d & -i & -\bar{d} & i\bar{d} & 1
\end{bmatrix}.
\]

A matrix is said to be circulant if its columns are cyclic permutations of its first column. The complex number \(d\) has modulus unity and is

\[
d = \frac{1 - \sqrt{3}}{2} + i\sqrt{\frac{3}{2}} \quad \Rightarrow \quad d^2 - (1 - \sqrt{3})d + 1 = 0.
\]

Several people have invested some effort in making this list as long as it can be, and up to two weeks before my talk, I thought that it might well be a complete list. Still, we do have considerable latitude in how we choose the first MUB.

Let us begin with

\[(\text{MUB})_1 = F(0, 0) .\]

It happens that all vectors that are unbiased with respect to this choice of the zeroth and first MUB have been computed, first by Björck and Fröberg [26], and independently by Grassl [27]. Björck did not express the problem in these terms however. He was interested in biunimodular sequences, that is to say sequences of unimodular complex numbers \(x_a\) whose discrete Fourier transform

\[
\tilde{x}_a = \frac{1}{\sqrt{N}} \sum_{b=0}^{N-1} x_b q^{ba}
\]

also consists of unimodular complex numbers. On reflection, one sees that the two problems are equivalent. Björck and coauthors eventually solved this problem for all \(N \leq 8\). When \(N = 6\) there are altogether 48 such sequences; 12 Gaussian ones—they were known to Gauss—and an additional 36. The Gaussian ones have entries that are 12th roots of unity, while the additional
ones involve Björck’s magical number $d$. See also Haagerup [18], who seems to have been the first to get this quite right.

More is true. For a biunimodular sequence the autocorrelation function is

$$\gamma_b \equiv \frac{1}{N} \sum_{a=0}^{N-1} \bar{x}_a x_{a+b} = \frac{1}{N} \sum_{a=0}^{N-1} |\bar{x}_a|^2 q^{-ab} = \delta_{b,0}.$$  \hspace{1cm} (26)

Therefore $x_a$ and $x_{a+b}$, with $b$ fixed and non-zero, are orthogonal vectors. Then it follows that the $12 + 36$ vectors found by Björck can be assembled into $2 + 6$ unitary circulant matrices that are MUB with respect to the standard basis and the Fourier basis. When Grassl redid this calculation (using the program MAGMA) he observed that each of the 48 vectors can be used to form a basis in exactly two ways. Thus we end up with exactly $2 + 2 + 6 + 6 = 16$ possible choices for (MUB)$_2$. In itself, this is more than we need for a complete set of 7 MUBs. Provided that the Fourier basis is included, the question whether one can find a fourth MUB boils down to the question whether the chordal distance squared between any pair among the 16 is equal to 5, the maximal distance squared attained by MUBs in six dimensions.

The answer is no. The 4 Gaussian MUBs, composed of 12th roots, form a perfect square with side lengths squared $D^2 = 2$. The two groups of “non-Gaussian” MUBs are isometric copies of each other. One group consists of circulant matrices, while the other group consists of Fourier matrices emphased using the magical number $d$. The distance squared between any Gaussian and any non-Gaussian MUB is $D^2 \approx 4.62$ — rather close to 5, if one takes an optimistic view of things. The distance between the two six-plets of non-Gaussian MUBs is always $D^2 \approx 3.71$. Inside each group, distances reach all the way up to $D^2 \approx 4.64$, which is even closer to 5.

Still, although the pattern is nice, the conclusion is negative: the Fourier matrix cannot be included in a set of more than three MUBs. We do not have complete results for other choices of the first MUB. We made a program that lists all MUB triplets where all the entries of the matrices are 24th roots of unity. Quite a few triplets, with quite interesting structures, did turn up in this way, but there were no MUB quartets. Note that in prime power dimensions the standard solution for complete sets of MUBs contain $N$th or $2N$th roots of unity only (depending on whether $N$ is odd or even [2]);
in other words the analogous calculation in arbitrary dimension would have found the known complete sets.

Although the evidence is incomplete, we do seem to be driven to admit that there can exist at most 3 MUBs when \( N = 6 \). A nagging doubt remains, because there is always the possibility that the above list of Hadamard matrices is incomplete. In particular, could the parameter spaces be incomplete? We do know that the number of free parameters in Fourier’s, Diţă’s, and Björck’s matrices is at most 4, while the matrix \( S \) cannot have any free parameters at all [28]. But we do not know if there are that many free parameters.

While I was thinking about what to say in my talk, Beauchamp and Nicoara [29] intervened. They found a new one-parameter family of Hadamard matrices, namely

\[
B = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & -\frac{1}{x} & -y & y & \frac{1}{x} \\
1 & -x & 1 & y & \frac{1}{x} & -\frac{1}{x} \\
1 & -\frac{1}{y} & \frac{1}{y} & y & \frac{1}{t} & \frac{1}{t} \\
1 & \frac{1}{y} & z & -t & 1 & -\frac{1}{x} \\
1 & x & -t & t & -x & -1
\end{pmatrix}
\]

(27)

where \((x, y, z, t)\) are complex numbers of modulus one, related by

\[
t = xyz
\]

(28)

\[
z = \frac{1 + 2y - y^2}{y(-1 + 2y + y^2)}
\]

(29)

\[
x = \frac{1 + 2y + y^2 \pm \sqrt{2\sqrt{1 + 2y + 2y^3 + y^4}}}{1 + 2y - y^2}
\]

(30)

with \(y\) remaining as a free phase factor. By construction this family contains all \( N = 6 \) Hermitian Hadamard matrices. The phase of \(y\) cannot be chosen quite arbitrarily; an interval around \(y = 1\) is excluded. On closer inspection one finds that this family starts from a matrix that is equivalent to Björck’s, passes through the Diţă family, comes back to Björck, repeats twice, and ends at Björck. The two branches of the square root lead to equivalent families.
What does this mean? I do not know. If it means that there are four dimensional families of Hadamard matrices, including the Fourier matrix and Björck’s matrix, then it also means that we have looked in a very small part of parameter space only. In fact we do have some reasons to believe that this is really so. Therefore it seems to me that the conclusion that we must draw about the MUB problem in six dimensions is: We have almost no evidence either way.

8. The real problem

The distance that we introduced can, in principle, be used to convert the MUB problem into that of maximising a function, such as

$$F = \sum_{i,j} D_{c}^{2}(P_i, P_j).$$

(31)

A similar procedure has been used [30] to find approximations to SIC-POVMs—this particular acronym stands for a kind of relative of the MUB problem—in dimensions up to $N = 45$. In our case the upper bound is attained if the $N + 1$ projectors represent totally orthogonal $(N - 1)$-planes. Whether we can reach the upper bound using $(N - 1)$-planes spanned by bases in the underlying Hilbert space is of course precisely the question.

I should add that I have not really done justice to the point of view that I tried to stress in the beginning, that the MUB problem leads one into many corners of useful mathematics that have not been very much explored by quantum physicists. But if you search your favourite eprint archive for some of the many papers, whose existence I hinted at, you will see what I mean. Meanwhile, the somewhat botanical spirit of my talk is perhaps appropriate in the town where Linnaeus was educated.

Anyway the real “MUB problem” is not how many MUBs we can find. The real MUB problem is to find out what we can do with those that exist.

Acknowledgments

I thank Åsa Ericsson, Jan-Åke Larsson, Wojciech Tadej, Wojciech Bruzda,
and Karol Życzkowski for collaborating with me. We thank Markus Grassl for sending his vectors for inspection, Bengt Nagel for insights, and Piotr Badziag for a good question. Three of us thank Andrei Khrennikov for inviting us to visit Småland.

References

[1] I. D. Ivanović, Geometrical description of quantal state determination, J. Phys. A14 (1981) 3241.

[2] W. K. Wootters and B. D. Fields, Optimal state-determination by mutually unbiased measurements, Ann. Phys. 191 (1989) 363.

[3] W. K. Wootters, A Wigner-function formulation of finite-state quantum mechanics, Ann. Phys. 176 (1987) 1.

[4] M. Appleby, these proceedings; also, at least one participant used MUBs for tomography in his lab.

[5] N. J. Cerf, M. Bourennane, A. Karlsson, and N. Gisin, Security of quantum key distribution using $d$-level systems, Phys. Rev. Lett. 88 (2002) 127902.

[6] I. Bengtsson, W. Bruzda, Å. Ericsson, J.-Å. Larsson, W. Tadej and K. Życzkowski, Mubs and Hadamards of order six, e-print quant-ph/0610161.

[7] H. Weyl: Gruppentheorie und Quantenmechanik, Hirzel, Leipzig 1928.

[8] S. Bandyopadhyay, P. O. Boykin, V. Roychowdhury, and F. Vatan, A new proof for the existence of mutually unbiased bases, Algorithmica 34 (2002) 512.

[9] P. Wocjan and T. Beth, New construction of mutually unbiased bases in square dimensions, Quant. Inf. and Comp. 5 (2005) 93.

[10] G. Zauner: Quantendesigns. Grundzüge einer nichtkommutativen Designtheorie, Ph D thesis, Univ. Wien 1999.
[11] M. Saniga, M. Planat and H. Rosu, Mutually unbiased bases and finite projective planes, J. Opt. B6 (2004) L19.

[12] R. A. Fisher and F. Yates, The 6 x 6 Latin squares, Proc. Camb. Phil. Soc. 30 (1934) 492.

[13] W. O. Alltop, Complex sequences with low periodic correlations, IEEE Trans. Inform. Theory 26 (1980) 350.

[14] S. Popa, Orthogonal pairs of *subalgebras in finite von Neumann algebras, J. Operator Theory 9 (1983) 253.

[15] A. I. Kostrikin, I. A. Kostrikin, and V. A. Ufnarovski˘i, Orthogonal decompositions of simple Lie algebras (type $A_n$), Proc. Steklov Inst. Math. 1983 (4), 113.

[16] A. R. Calderbank, P. J. Cameron, W. M. Kantor, and J. J. Seidel, $Z_4$-Kerdock codes, orthogonal spreads, and extremal Euclidean line-sets, Proc. London Math. Soc. 75 (1997) 436.

[17] I. Bengtsson and K. Życzkowski: Geometry of Quantum States, Cambridge UP 2006.

[18] U. Haagerup, Orthogonal maximal abelian *-subalgebras of the $n \times n$ matrices and cyclic $n$-roots, in Operator Algebras and Quantum Field Theory, Rome (1996), Internat. Press, Cambridge, MA 1997.

[19] P. Dita, Some results on the parametrization of complex Hadamard matrices, J. Phys. A37 (2004) 5355.

[20] W. Tadej and K. Życzkowski, A concise guide to complex Hadamard matrices, Open Sys. Information Dyn. 13 (2006) 133.

[21] I. Bengtsson and Å. Ericsson, Mutually unbiased bases and the Complementarity Polytope, Open. Sys. Information Dyn. 12 (2005) 107.

[22] H. S. M. Coxeter: Regular Polytopes, Dover, New York 1973.

[23] A. Peres, Two simple proofs of the Kochen–Specker theorem, J. Phys. A24 (1991) 175.
[24] P. O. Boykin, M. Sitharam, M. Tarifi, and P. Wocjan, Real mutually unbiased bases, arxiv eprint quant-ph/0502024.

[25] J. H. Conway, R. H. Hardin and N. J. A. Sloane, Packing lines, planes, etc: Packings in Grassmannian spaces, Exp. Math. 5 (1996) 139.

[26] G. Björck and R. Fröberg, A faster way to count the solutions of inhomogeneous systems of algebraic equations, with applications to cyclic $n$-roots, J. Symbolic Computation 12 (1991) 329.

[27] M. Grassl, On SIC-POVMs and MUBs in dimension 6, e-print quant-ph/0406175.

[28] W. Tadej, W. Słomczynski, and K. Życzkowski, Defect of a unitary matrix, to appear; see also the Concise Guide, op. cit..

[29] K. Beauchamp and R. Nicoara, Orthogonal maximal abelian $*$-subalgebras of the 6x6 matrices, arxiv eprint math.OA/0609076.

[30] J. M. Renes, R. Blume-Kohout, A. J. Scott, and C. M. Caves, Symmetric informationally complete quantum measurements, J. Math. Phys. 45 (2004) 2171.