A Variable Gain Sliding Mode Tracking Differentiator for Derivative Estimation of Noisy Signals

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ABSTRACT Estimating reliable signal component and its derivatives from noisy feedback signal is important in control systems. Toward this problem, this article presents a new model-free variable gain sliding mode tracking differentiator for derivative estimation of noisy signals by modifying a Levant and Yu’s sliding mode tracking differentiator. Specifically, different from Levant and Yu’s TD, the new TD employs an additional variable that contributes to overshoot reduction. In addition, the new TD adaptively changes its gains for improving the tracking and filtering performances. Moreover, the new TD only uses previous output values and it does not require input signal model in advance. The advantages of the new TD over previous TDs are confirmed through numerical examples.

INDEX TERMS Sliding mode, tracking differentiator, high order, noise attenuation, overshoot, variable gain.

I. INTRODUCTION

Estimating reliable signal component and its derivatives from noisy feedback signal is important in control systems. Traditional linear filters [1] are simple in structure and low in computational cost, and thus they are often the first option. However, in the case of strong noise reduction, a linear filter introduces a significant phase lag in the output. In addition, a linear filter proportionally transfers any signal component into the output. These problems of linear filters may cause instability of the controlled system in the case where system hardware generates significant phase lag and noise component has high amplitude.

Sliding mode tracking differentiators are studied for avoiding the disadvantages of linear filters. For example, Han and Wang [2] proposed a sliding mode tracking differentiator (TD [2]), of which the sliding surface takes a parabolic curve-like shape. One major advantage of TD [2] is that it realizes minimum-time convergence. Because of its effectiveness, TD [2] has been applied in many applications, such as vehicle suspension system [3], induction motor system [4] and wind turbine system [5]. However, TD [2] is prone to overshoot during convergence. Jin et al. [6] presented a modification of TD [2] by introducing a multi-level set-valued mapping (TD [6]). It is reported that, compared with TD [2], TD [6] is less prone to overshoot, and it produces smaller phase lag. After that, extensions of TD [6] are reported in the literature [7]–[10]. However, this class of tracking differentiators is limited to estimations of the input and its 1st-order derivative.

Toward the problem of high-order derivative estimation, in a subsequent study [15], Levant presented a TD [11]’s extension (TD [15]) that realizes the estimation of arbitrary
order derivative of the input. In addition, TD [15] can effectively remove small magnitude noise contained in the input, and it can reduce the chattering phenomenon by increasing the system order. After that, in [16], Levant and Yu proposed a TD [15]’s modification (TD [16]) that provides better tracking accuracy and filtering performance than TD [15] does. However, the performance of TD [16] is still not satisfactory enough in the cases where the magnitude of noise is high and the frequency range of the input is board. Moreover, TD [16] and its predecessors produce significant overshoot during the convergence.

Some variable gain techniques can be applied for improving the performance of Levant and his colleagues’ tracking differentiators. For example, in [17], a method of constructing adaptive variable gains is presented. Although the method improves the tracking performance, but the variable gains are constructed based on the input signal model, which cannot be always obtained in advance. As another example, in [18]–[21], state-norm observer based adaptive variable gain techniques are introduced for improving tracking performance. However, the construction of state-norm observer requires lower and upper bounds for the parameters of the system dynamic model as well as the noise source.

This article presents a new model-free variable gain sliding mode tracking differentiator for derivative estimation of noisy signals (new TD), which is an improvement of TD [16]. Concretely, different from TD [16], the new TD employs an additional variable that contributes overshoot reduction. In addition, the new TD adaptively changes its gains for improving the tracking and filtering performances. Moreover, the new TD only uses previous output values, and it does not require input signal model in advance. The effectiveness of the new TD is validated through numerical simulations.

The rest of this article is organized as follows: Section II demonstrates the existing problem of TD [16] by using a numerical example. Section III presents a new high-order adaptive sliding mode tracking differentiator for solving the problem of TD [16]. Section IV evaluates the advantages of the new TD through numerical simulations. Finally, Section V concludes the paper.

II. PROBLEM DEMONSTRATION

Levant and Yu [16] proposed a high-order sliding mode tracking differentiator (TD [16]), of which continuous-time expression is given as follows:

\[
\dot{x}_{k-1}(t) = -\lambda_{k+1}L \frac{1}{T^2} |x_{k-1}(t)|^{\frac{k+1}{2}} \text{sgn}(x_{k-1}(t)) + x_0(t) - u(t), \tag{1a}
\]
\[
\dot{x}_0(t) = -\lambda_1 L \frac{1}{T^2} |x_0(t)|^{\frac{k}{2}} \text{sgn}(x_0(t)) + x_1(t), \tag{1b}
\]
\[
\dot{x}_{k-1}(t) = -\lambda_1 R \frac{1}{T^2} |x_{k-1}(t)|^{\frac{k}{2}} \text{sgn}(x_{k-1}(t)) + x_k(t), \tag{1c}
\]
\[
\dot{x}_k(t) = -\lambda_0 L |x_{k-1}(t)|^{\frac{k-1}{2}} \text{sgn}(x_{k-1}(t)), \tag{1d}
\]

where \( u \in \mathbb{R} \) is the input, \( x_0 \in \mathbb{R} \) is the estimated output of \( u \), \( x_k \in \mathbb{R}, k = 1, 2, \ldots \) is the estimation of the \( k \)-th order derivative of \( u \), \( x_{k-1} \in \mathbb{R} \) is an auxiliary variable, \( L > 0 \) is a parameter to be tuned, and \( \lambda_k \in \mathbb{R}, k = 0, 1, 2, \ldots \) is the constant that can be obtained from the recursive sequence reported in [16]. In addition, \( \text{sgn}(\cdot) \) is the conventional signum function.

For implementing TD [16] in discrete time, the continuous-time expression (1) must be discretized. The following is the explicit-Euler method based discrete-time algorithm of TD [16]:

\[
x_{k-1}(i) = x_{k-1}(i-1) + (-\lambda_{k+1}L \frac{1}{T^2} |x_{k-1}(i-1)|^{\frac{k+1}{2}} \times \text{sgn}(x_{k-1}(i-1)) + x_0(i-1) - u(i))T, \tag{2a}
\]
\[
x_0(i) = x_0(i-1) + (-\lambda_1 L \frac{2}{T^2} |x_0(i-1)|^{\frac{k}{2}} \times \text{sgn}(x_0(i-1)) + x_1(i-1))T, \tag{2b}
\]
\[
\vdots
\]
\[
x_{k-1}(i) = x_{k-1}(i-1) + (-\lambda_1 R \frac{1}{T^2} |x_{k-1}(i-1)|^{\frac{k}{2}} \times \text{sgn}(x_{k-1}(i-1)) + x_k(i-1))T, \tag{2c}
\]
\[
x_k(i) = x_k(i-1) + (-\lambda_0 L |x_{k-1}(i-1)|^{\frac{k-1}{2}} \times \text{sgn}(x_{k-1}(i-1)))T, \tag{2d}
\]

where \( i \) is the discrete-time index and \( T \) is the sampling interval.

![FIGURE 1. Responses of 1st-order TD [16] (\( \lambda_0 = 1.1, \lambda_1 = 2.12, \lambda_2 = 2 \)) under the input (3) with \( \eta = 0.01 \).](image-url)

Fig. 1 shows responses of 1st-order TD [16] under the following input:

\[
u(t) = \begin{cases} 
\cos(t) + \eta \delta(t) & \text{if } t \leq 10 \\
\cos(2t) + \eta \delta(t) & \text{otherwise,}
\end{cases} \tag{3}
\]

where \( \delta(t) \sim N(0, 1) \) is the unit white Gaussian noise with zero mean, and \( \eta \in \mathbb{R} \) is a scaling parameter. It is shown that, in the
case of \( L = 1 \), the convergence of the outputs is slow for the slow motion \((t \leq 10 \text{ s})\), and the outputs cannot track the input for the fast motion \((t > 10 \text{ s})\). The tracking performance can be improved by using a larger \( L \), e.g., \( L = 43 \), but it produces a significant overshoot, which is undesirable for a control system. In addition, a larger \( L \) degrades noise attenuating performance.

Toward to these problems of TD [16], a new model-free variable gain sliding mode tracking differentiator for derivative estimation of noisy signals is presented in the next section by improving TD [16].

### III. A NEW VARIABLE GAIN SLIDING MODE TRACKING DIFFERENTIATOR

#### A. PROPOSAL OF NEW TD

Toward to the overshoot problem of TD [16], one possible remedy can be given as follows:

\[
\begin{align*}
v(t) &= -\lambda_k L \frac{1}{t_1^2} \left| x_{-1}(t) \right| \frac{1}{t_1^2} \text{sgn}(x_{-1}(t)) + x_0(t) - u(t), \\
\dot{x}_{-1}(t) &= -\lambda_k L \frac{1}{t_1^2} \left| x_{-1}(t) \right| \frac{1}{t_1^2} \text{sgn}(x_{-1}(t)) + x_0(t) - u(t) - \beta |v(t)| \text{sgn}(x_{-1}(t)), \\
\dot{x}_0(t) &= -\lambda_k L \frac{1}{t_1^2} \left| x_{-1}(t) \right| \frac{1}{t_1^2} \text{sgn}(x_{-1}(t)) + x_1(t), \\
\dot{\lambda}_k(t) &= -\lambda_2 |x_{-1}(t)|^\delta \text{sgn}(x_{-1}(t)),
\end{align*}
\]

where \( 0 \leq \beta < 1 \) is a constant, and \( v(t) \) is an intermediate variable that reflects some behavior of \( \dot{x}_{-1}(t) \).

Fig. 2 shows the responses of 1st-order TD (4) and 1st-order TD [16] with \( \beta = 0.5 \) under a step input. One can observe that, in the case of TD [16], \( x_{-1} \) cannot converge to 0 when the output \( x_0 \) reaches the desired value. Thus, \( x_{-1} \) continuously moves toward 0 after the reach of the desired value, and there occurs overshoot. On the other hand, it is shown that, similar to TD [16], the convergence of TD (4) can be divided into two parts. One is the part where \( \dot{x}_{-1} \) and \( x_{-1} \) have the same sign, indicated as \( S_+ \), and the other one is the part where \( \dot{x}_{-1} \) and \( x_{-1} \) have the opposite signs, indicated as \( S_- \). Specifically, in both TD [16] and TD (4), \( \dot{x}_- \) drives \( x_{-1} \) moves away from 0 during \( S_+ \), while \( \dot{x}_- \) forces \( x_{-1} \) moves toward 0 during \( S_- \). However, different from TD [16], the newly added term \( -\beta |v| \text{sgn}(x_{-1}) \) of TD (4) always has opposite sign with respect to \( x_{-1} \). This means that the deviation of TD (4)’s \( x_{-1} \) from 0 is smaller than that of TD [16] during \( S_+ \), while the convergence of TD (4)’s \( x_{-1} \) to 0 is faster than that of TD [16] during \( S_- \), resulting in shorter period of experiencing one circle of \( S_+ \) and \( S_- \) in TD (4). Thus, \( x_{-1} \) of TD (4) converges to 0 faster than that of TD [16] by producing smaller overshoot.

Unfortunately, when the input is corrupted by noise, TD (4) is still sensitive to noise, as shown in Fig. 3. It is clear that, in order to obtain a smoother response, parameter \( L \) of TD (4) should be set smaller, but it degrades convergence speed, as demonstrated in Section II. Toward this problem, this article presents the following new adaptive sliding mode tracking differentiator (new TD), which is a further modification of TD (4) by employing an adaptive gain:

\[
\begin{align*}
m(t) &= \alpha |x_{-1}(t)| + \sigma(t), \\
\dot{v}(t) &= -\lambda_k m(t) \frac{1}{t_1^2} \left| x_{-1}(t) \right| \frac{1}{t_1^2} \text{sgn}(x_{-1}(t)) + x_0(t) - u(t),
\end{align*}
\]
where $\alpha > 0$ is a constant. In addition, $\sigma(t)$ is a following defined measure that reflects the average change rate of the input:

$$\sigma(t) = \gamma \left( \max_{z \in [t-\tau, t]} (x_0(z)) - \min_{z \in [t-\tau, t]} (x_0(z)) \right)/\tau,$$

where $\gamma > 0$ is a constant. Here, it should be mentioned that the new TD only uses previous values of the output $x_0$, and it does not require input signal model in advance.

By applying the explicit-Euler discretization, the process of discrete-time implementation of the new TD is obtained as follows:

$$\begin{align}
\dot{x}_{n+1}(t) &= -\lambda_{k+1} m(t) \frac{1}{\tau^2} [x_{n}(t)]^{1+\frac{1}{\tau^2}} \text{sgn}(x_{n}(t)) + x_0(t) \\
\dot{x}_0(t) &= -\lambda_k m(t) \frac{1}{\tau^2} [x_{n}(t)]^{1+\frac{1}{\tau^2}} \text{sgn}(x_{n}(t)) + x_1(t), \quad (5c) \\
\dot{x}_{k-1}(t) &= -\lambda_{i+1} m(t) \frac{1}{\tau^2} [x_{n-1}(t)]^{1+\frac{1}{\tau^2}} \text{sgn}(x_{n-1}(t)) + x_n(t) \\
\dot{x}_k(t) &= -\lambda_0 m(t) [x_{n}(t)]^0 \text{sgn}(x_{n}(t)), \quad (5f)
\end{align}$$

where $n \in \mathbb{N}$.

Fig. 4 and Fig. 5 show the responses of 1st-order the new TD under the input $u(t) = 1$ and the input (3) with $\eta = 0$, respectively. In the two figures, the changes of the parameters with time are also included. It is shown that, in the case of large error between the input and the output, the first term of the adaptive gain $m$ will dominate for accelerating the convergence. On the other hand, in the case of small error, the second term of $m$ will dominate for balancing the trade-off between tracking speed and filtering performance, i.e., a small $\sigma$ for slow motion and a large $\sigma$ for fast motion.

The stability analysis of the new TD is provided in Appendix.

**B. PARAMETERS OF NEW TD**

Fig. 6 shows the responses of 1st-order new TD with different values of $\beta$ under the input $u(t) = 1$. It is shown that the overshoot magnitude decreases as $\beta$ increases. However, a large $\beta$ degrades the convergence performance, e.g., $\beta = 0.9$ v.s. $\beta = 0.5$ in Fig. 6.

Fig. 7 illustrates the influence of $\alpha$ on the tracking performance under the input (3) with $\eta = 0$. It is shown that,
in the case of $\alpha = 0$, the system fails to track the input signal because of $m(t) = 0$. On the other hand, by comparing $\alpha = 1000$ and $\alpha = 2500$, it is shown that increasing $\alpha$ contributes to the improvement of convergence in the case of large error between the input and the output, but it increases overshoot magnitude.

Fig. 8 demonstrates the influence of $\gamma$ on the tracking performance under the input (3) with $\eta = 0.01$. It is shown that the bias between the input and the output is largest when $\gamma = 0$. It is also shown that, although the increase of $\gamma$ decreases the bias and reduces overshoot during convergence, a large $\gamma$ results in the sensitivity of the output to the noise.

The above results indicates that the parameters of the new TD should be carefully chosen for achieving satisfactory performance.
IV. SIMULATION

This section evaluates the performance of the new TD by using the following two input signals:

\[ u(t) = 1 + 0.01\delta(t) \]  

(8)

and

\[ u(t) = \begin{cases} 
\cos(t) + 0.01\delta(t) & \text{if } t \leq 10 \text{ s} \\
\cos(5 + 0.15(t - 10)^2) + 0.01\delta(t) & \text{if } 10 < t \leq 25 \text{ s} \\
(1 + 0.15(t - 25))\sin(t) + 0.01\delta(t) & \text{otherwise,} 
\end{cases} \]

(9)

where the first term of the right-hand side of each input is the signal component, and the second term is the noise component. Here, it should be mentioned that, in (9), the inputs for \( t < 10 \text{ s} \), \( 10 < t \leq 25 \text{ s} \) and \( t > 25 \text{ s} \) are to estimate the performance under fixed frequency and magnitude, fixed magnitude with varying frequency, and fixed frequency with varying magnitude, respectively. In addition, the initial states of the new TD are set zeros at \( t = 0 \text{ s} \), and sampling time \( T = 0.001 \text{ s} \) is used for all simulations.
TABLE 2. Parameter $L$ of VGTD [16] for input (9).

| Order       | $t \leq 10$                        | $10 < t \leq 25$                      | $t > 25$                        |
|-------------|-----------------------------------|--------------------------------------|---------------------------------|
| 1st-order VGTD [16] | $1$                             | $0.09(t-10)^2 + 0.3$                | $0.15(t-25) + 1.3$             |
| 2nd-order VGTD [16] | $1$                             | $0.027(t-10)^3 + 0.27(t-10)$        | $0.15(t-25) + 1.45$            |
| 3rd-order VGTD [16] | $1$                             | $0.0081(t-10)^4 + 0.162(t-10)^2 + 0.27$ | $0.15(t-25) + 1.6$            |

**FIGURE 15.** Comparison among 1st-order TD [16], 1st-order VGTD [16] and 1st-order new TD under the input (9). For all TDs, $\lambda_0 = 1$, $\lambda_1 = 2.12$, $\lambda_2 = 2$.

Fig. 9 - Fig. 11 show the step responses of the 1st-order, 2nd-order and 3rd-order new TD and TD [16]. It should be mentioned here that the parameters of both TDs for each order system are set different because the system dynamics (e.g., $\lambda_k$) are different for each order. It is shown that the filtering performance of the new TD is slightly better than that of TD [16] with $L = 1$, and the convergence speed of the new TD is faster than that of TD [16] with $L = 1$. The figures also show that the convergence speed of TD [16] can be increased with a larger $L$, but it sacrifices the filtering performance. Fig. 12 - Fig. 14 show the results of two measures. One is the rising time (RT), which is defined as the time spent for the output $x_0$ first reaches the magnitude of the noiseless signal component of the input. The other one is the average error $|x_0 - u|$ (AE) during the time interval between 5 s and 10 s. It is known that, for a better tracking and filtering performance, both RT and AE should be maintained small. From the results, one can observe that TD [16] cannot reduce both RT and AE simultaneously to the levels of the new TD by adjusting its parameter.

**FIGURE 16.** Comparison among 2nd-order TD [16], 2nd-order VGTD [16] and 2nd-order new TD under the input (9). For all TDs, $\lambda_0 = 1.1$, $\lambda_1 = 3.06$, $\lambda_2 = 4.16$, $\lambda_3 = 3$. 

FIG. 12 - Fig. 14 show the results of two measures. One is the rising time (RT), which is defined as the time spent for the output $x_0$ first reaches the magnitude of the noiseless signal component of the input. The other one is the average error $|x_0 - u|$ (AE) during the time interval between 5 s and 10 s. It is known that, for a better tracking and filtering performance, both RT and AE should be maintained small. From the results, one can observe that TD [16] cannot reduce both RT and AE simultaneously to the levels of the new TD by adjusting its parameter.
of TD [16] and TD [16] that employs the variable gain scheme presented in [17] (VGT [16]) are also included in the figures. Here, the parameters of the new TD and TD [16] are set so that the times for reaching the noiseless signal component of the input are the same. In addition, the parameter of VGT [16] is obtained by applying the parameter tuning method presented in [17]. Table 1 shows the parameters of the new TD and TD [16], and Table 2 shows the parameter of VGT [16]. It should be mentioned that the variable gain scheme [17] leads to a fixed value of parameter $L$ when both frequency and magnitude of the input are fixed constant, i.e., VGT [16] reduces to TD [16] when an input with fixed frequency and magnitude is provided. It is shown that, compared with TD [16], the new TD produces smaller overshoot and converges faster. In addition, the new TD provides smoother estimations of the input and its derivatives. It is clear that the filtering performance of TD [16] can be improved by using a smaller value of $L$, but it sacrifices the tracking speed, as also demonstrated in Fig. 1 and Fig. 9 - Fig. 11. It is also shown that the new TD has similar noise reduction ability compared with VGT [16] after convergence. However, in the case of sudden change of the input, VGT [16] produces large overshoot and exhibits slow convergence. These tendencies of VGT [16] become more serious as the system order increases. Such behavior of VGT [16] may cause instability of the controlled system. Moreover, the results also show that, for all three TDs, the phase lag between the estimated signal and the noiseless signal component becomes larger as the order of derivative increases. This implies that the trade-off between the tracking and filtering performance and phase lag should be considered in applications.

As a whole, it can be confirmed that the new TD performs the best among the three TDs by considering overshoot, tracking accuracy and filtering performance.

V. CONCLUSION

This article has presented a new model-free variable gain sliding mode tracking differentiator for derivative estimation of noisy signals (new TD), which is an improvement of Levant and Yu’s TD [16] by employing an additional variable and an adaptive gain. Simulation results show that the new TD produces smaller overshoot and removes noise more effectively than TD [16] does without sacrificing tracking performance. The results also show that the new TD has similar noise reduction ability compared with VGT [16] after convergence, but it has faster convergence performance.

One limitation of this work is that the parameters of the new TD are selected in a try and error manner. Thus, in future research, theoretical and quantitative tuning method of parameters ($\alpha$, $\beta$ and $\gamma$) should be developed for better application of the new TD.

APPENDIX

**Proposition 1**: The new TD converges in finite time when $m(t)$ is fixed constant and $0 \leq \beta < 1$. 

Fig. 15 - Fig. 17 show the responses of the 1st-order, 2nd-order and 3rd-order new TD. For comparisons, the responses
Proof: Let us consider the input $u(t) = u_0(t) + \mu(t) + \mu_e(t)$, where $u_0(t)$ is a unknown noise-free component, and $\mu(t)$ and $\mu_e(t)$ are noise-related components that satisfy $|\mu(t)| \leq \rho$ for some unknown $\mu \geq 0$ and $\int_0^t \mu_e(t) \, ds \leq \varepsilon$ for some unknown $\varepsilon \geq 0$.

If $m(t)$ is fixed constant and $\beta = 0$, the following relations are held [16]:

\begin{align}
\omega &= \max((\frac{\varepsilon}{m(t)} )^{\frac{1}{\gamma}}, (\frac{\delta}{m(t)} )^{\frac{1}{\gamma}}) \quad (10a) \\
|x_1(t)|_{\beta=0} &\leq \lambda_{k+1} m(t) \omega^{k+2} = \varrho \quad (10b) \\
|x_k(t) - u(t)^{(k)}| &\leq \lambda_k m(t) \omega^{k+1} = \Omega. \quad (10c)
\end{align}

On the other hand, the convergence of the new TD can be divided into two parts. One is the part where $\dot{x}_{-1}(t)$ and $x_{-1}(t)$ have the same sign, denoted as $S_+$, and the other one is the part where $\dot{x}_{-1}(t)$ and $x_{-1}(t)$ have opposite signs, denoted as $S_-$. In the case of $S_+$, because:

\begin{equation}
(-\beta \dot{x}_{-1} |\text{sgn}(x_{-1})|) / (\dot{x}_{-1}) < 0, \quad (11)
\end{equation}

$\dot{x}(t)$ drives $x_{-1}(t)$ moves away from 0. Moreover, in the case of $S_-$, because:

\begin{equation}
(-\beta |x_{-1}| |\text{sgn}(x_{-1})|) / (\dot{x}_{-1}) > 0, \quad (12)
\end{equation}

$\dot{x}_{-1}(t)$ forces $x_{-1}(t)$ moves toward 0. Thus, when $0 \leq \beta < 1$, the term $(-\beta \dot{x}_{-1} |\text{sgn}(x_{-1})|)$ of the new TD decreases the value of $\dot{x}_{-1}(t)$ during $S_+$, while it increases the value of $\dot{x}_{-1}(t)$ during $S_-$, so that the maximum of $|x_{-1}(t)|_{0 \leq \beta < 1}$ does not exceed the maximum of $|x_{-1}(t)|_{\beta=0}$:

\begin{equation}
\max(|x_{-1}(t)|_{0 \leq \beta < 1}) \leq \max(|x_{-1}(t)|_{\beta=0}), \quad (13)
\end{equation}

which further leads to the followings:

\begin{align}
|x_{-1}(t)|_{0 \leq \beta < 1} &\leq \varrho \quad (14a) \\
|x_k(t) - u(t)^{(k)}| &\leq \Omega. \quad (14b)
\end{align}

Thus, the new TD converges in finite time when $m(t)$ is fixed constant and $0 \leq \beta < 1$. QED.

Proposition 2: The new TD converges in finite time when $m(t)$ is adaptively changed and $0 \leq \beta < 1$.

Proof: Let us set the maximum value of the adaptive gain $m(t)$ as $\max(m(t))$. Then, the following maximums are achieved:

\begin{align}
\max(\varrho) &= \lambda_{k+1} \max(m(t)) \omega^{k+2} \quad (15a) \\
\max(\Omega) &= \lambda_k \max(m(t)) \omega^{k+1}. \quad (15b)
\end{align}

Then, from (14), one can obtain the followings:

\begin{align}
|x_{-1}(t)|_{0 \leq \beta < 1} &\leq \max(\varrho), \quad (16a) \\
|x_k(t) - u(t)^{(k)}| &\leq \max(\Omega), \quad (16b)
\end{align}

which indicates the new TD converges in finite time when $m(t)$ is adaptively changed and $0 \leq \beta < 1$. QED.
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