Numerical solution of a time-fractional nonlinear Rayleigh-Stokes problem

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Abstract

We study a semilinear fractional-in-time Rayleigh-Stokes problem for a generalized second-grade fluid with a Lipschitz continuous nonlinear source term and initial data $u_0 \in \dot{H}^\nu(\Omega)$, $\nu \in [0, 2]$. We discuss stability of solutions and provide regularity results. Two spatially semidiscrete schemes are analyzed based on standard Galerkin and lumped mass finite element methods, respectively. Further, a fully discrete scheme is obtained by applying a convolution quadrature in time generated by the backward Euler method, and optimal error estimates are derived for smooth and nonsmooth initial data. Finally, numerical examples are provided to illustrate the theoretical results.

Key words. semilinear fractional Rayleigh-Stokes equation, lumped mass method, convolution quadrature, optimal error estimate, nonsmooth initial data.

AMS subject classifications. 65M60, 65M12, 65M15

1 Introduction

We consider a semilinear fractional-order Rayleigh-Stokes problem for a generalized second-grade fluid. Let $\Omega \subset \mathbb{R}^d (d = 1, 2, 3)$ be a bounded convex polygonal domain with its boundary $\partial \Omega$, and $T > 0$. The mathematical model is given by

$$
\begin{align*}
\partial_t u(x,t) - (1 + \gamma \partial_t^\alpha) \Delta u(x,t) &= f(u) \quad \text{in } \Omega \times (0,T], \\
u(x,t) &= 0 \quad \text{on } \partial \Omega \times (0,T], \\
u(x,0) &= u_0(x) \quad \text{in } \Omega,
\end{align*}
$$

where $\gamma > 0$ is a fixed constant, $u_0$ is a given initial data, $\partial_t = \partial/\partial t$ and $\partial_t^\alpha$ is the Riemann-Liouville fractional derivative in time with $\alpha \in (0, 1)$ defined by

$$
\partial_t^\alpha \varphi(t) = \frac{d}{dt} \int_0^t \omega_{1-\alpha}(t-s) \varphi(s) \, ds \quad \text{with} \quad \omega_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}.
$$

In (1.1a), $f : \mathbb{R} \to \mathbb{R}$ is a smooth function satisfying the Lipschitz condition

$$
|f(t) - f(s)| \leq L|t - s| \quad \forall t, s \in \mathbb{R},
$$

for some constant $L > 0$.

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The aim of this work is to study some aspects of the numerical solution of the semilinear problem (1.1). The linear case has been considered by several authors. For instance, in [9] and [10], implicit and explicit finite difference schemes have been proposed. A Fourier analysis was employed to investigate stability and convergence. In [22], a numerical scheme was derived and analyzed by transforming the problem into an integral equation. In [17], a numerical scheme was investigated using the reproducing kernel technique. In [23], Zaky applied the Legendre-tau method to problem (1.1) and discussed related convergence rates. The convergence analysis in all these studies assumes that the exact solution is sufficiently regular, including at $t = 0$, which is not practically the case. In [6], Jin et al. investigated a piecewise linear finite element method (FEM) in space and a convolution quadrature in time, and obtained optimal error estimates with respect to the solution smoothness, expressed through the initial data $u_0$. Most recently, a similar analysis was presented in [1] for a time-fractional Oldroyd-B fluid problem.

The numerical approximation of nonlinear time-fractional models has recently attracted the attention of many researchers. In particular, the time-fractional subdiffusion model

$$C^\alpha \partial_t^\alpha u(x,t) - \Delta u(x,t) = f(u)$$  \hfill (1.4)

has been given a special attention. Here, $C^\alpha \partial_t^\alpha$ denotes the Caputo fractional derivative in time of order $\alpha$. In [10], for instance, a linearized $L^1$-Galerkin FEM was proposed for solving a nonlinear time-fractional Schrödinger equation. Based on a temporal-spatial error splitting argument and a new discrete fractional Gronwall-type inequality, optimal error estimates of the numerical schemes are obtained without restrictions on the time step size. In [15], $L^1$-type schemes have been analyzed for approximating the solution of (1.4), and related error estimates have been derived. The estimates are obtained under high regularity assumptions on the exact solution. In [13], the numerical solution of (1.4) was investigated under the assumption that the nonlinear function $f$ is globally Lipschitz continuous and the initial data $u_0 \in H^2(\Omega) \cap H^1_0(\Omega)$. These results have been extended in [2] to problems with nonsmooth initial data. Recently, a numerical study with a more general condition on nonlinearity was presented in [3].

In this paper, we first investigate a lumped mass FE semidiscrete scheme in space for solving (1.1). Compared with the standard piecewise linear FEM [2, 6], the lumped mass FEM has the advantage that when representing the discrete solution in the nodal basis functions, it produces a diagonal mass matrix which enhances the computation procedure. Our aim is to derive optimal error estimates for solutions with smooth and nonsmooth initial data. The analysis will be based on a semi-group type approach. The FE solution will serve as an intermediate solution to establish error estimates for the lumped mass FEM. This technique was used for instance in [7, 8] and [4]. Our second objective is to investigate a time-stepping scheme using a first-order convolution quadrature in time. Pointwise-in-time optimal error estimates are then derived. The main technical tool relies on the use of the discrete propagator (discrete evolution operator) associated with the numerical method, see [11].

The paper is organized as follows. In Section 2, we represent the solution of (1.1) in an integral form and obtain regularity results. In Section 3, we derive error estimates for the standard Galerkin FEM. A convolution quadrature time discretization method is analyzed in Section 4, and related error estimates are established. In Section 5, we investigate a fully discrete scheme obtained by the lumped mass FEM combined with the convolution quadrature in time. Finally, we provide some numerical examples to confirm our theoretical results.

Throughout the paper, $c$ denotes a generic constant which may change at each occurrence but it is always independent of discretization parameters; mesh size $h$ and time step size $\tau$. We shall also use the notation $u'$ denoting $\partial u/\partial t$. 

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2 Continuous problem

This section is devoted to the analysis of the continuous problem \([1.1]\). Based on an integral representation of its solution, we prove regularity results, which will play a key role in the error analysis. We begin by introducing some notations. For \(r \geq 0\), we denote by \(\dot{H}^r(\Omega) \subset L^2(\Omega)\) the Hilbert space induced by the norm \(\|v\|_{\dot{H}^r(\Omega)} = \sum_{j=1}^{\infty} \lambda_j(v, \phi_j)^{2}\), where \(\{\lambda_j, \phi_j\}_{j=1}^{\infty}\) are the Dirichlet eigenpairs of \(A := -\Delta\) on \(\Omega\) with \(\{\phi_j\}_{j=1}^{\infty}\) being an orthonormal basis in \(L^2(\Omega)\). Thus, \(\|v\|_{\dot{H}^0(\Omega)} = \|v\|\) is the norm in \(L^2(\Omega)\), \(\|v\|_{\dot{H}^1(\Omega)}\) is the norm in \(H^1_0(\Omega)\), and \(\|v\|_{\dot{H}^2(\Omega)} = \|Av\|\) is the equivalent norm in \(H^2(\Omega) \cap H^1_0(\Omega)\) \([21]\).

For a given \(\theta \in (\pi/2, \pi)\), we define the sector \(\Sigma_\theta = \{z \in \mathbb{C}, z \neq 0, |\arg(z)| < \theta\}\). Since \(A\) is selfadjoint and positive definite, the operator \((z^\alpha I + A)^{-1} : L^2(\Omega) \to L^2(\Omega)\) satisfies the bound

\[
\|(z^\alpha I + A)^{-1}\| \leq M|z|^{-\alpha} \quad \forall z \in \Sigma_\theta,
\]

where \(M\) depends on \(\theta\).

Let \(\hat{u}(x, z)\) denote the the Laplace transform of \(u(x, t)\). Set \(w(t) = f(u(t))\). Then, by taking Laplace transforms in \([1.1]\), we obtain

\[
z\hat{u} - u_0 + A\hat{u} + \gamma z^\alpha A\hat{u} = \hat{w}(z).
\]

Hence,

\[
\hat{u} = \frac{g(z)}{z} (g(z)I + A)^{-1} (u_0 + \hat{w}(z)),
\]

where \(g(z) = \frac{z}{1 + \gamma z^\alpha}\). By means of the inverse Laplace transform, we have

\[
u(t) = E(t)u_0 + \int_0^t E(t-s)f(u(s)) \, ds, \quad t > 0,
\]

with the operator \(E(t) : L^2(\Omega) \to L^2(\Omega)\) being defined by

\[
E(t) = \frac{1}{2\pi i} \int_{\Gamma_{\theta,\delta}} e^{zt} \frac{g(z)}{z} (g(z)I + A)^{-1} \, dz,
\]

where, for fixed \(\delta > 0\), \(\Gamma_{\theta,\delta} := \{\rho e^{\pm i\theta} : \rho \geq \delta\} \cup \{\delta e^{i\psi} : |\psi| \leq \theta\}\) is oriented with an increasing imaginary part.

The following estimates hold, see \([3\text{ Theorem 2.1}]\).

**Lemma 2.1.** The operator \(E(t)\) satisfies

\[
\|\partial_t^m E(t)v\|_{\dot{H}^r(\Omega)} \leq ct^{-m - (1-\alpha)(p-q)/2}\|v\|_{\dot{H}^r(\Omega)},
\]

where \(m = 0\) and \(0 \leq q < p \leq 2\) or \(m > 0\) and \(0 \leq p, q \leq 2\).

In the sequel, we shall use the following generalization of Grönwall’s inequality \([5]\).

**Lemma 2.2.** Let \(T > 0, 0 \leq \alpha, \beta < 1\) and \(A, B \geq 0\). Then there is a positive constant \(C = C(T, B, \alpha, \beta)\) such that

\[
y(t) \leq At^{-\alpha} + B \int_0^t (t-s)^{-\beta} y(s) \, ds, \quad 0 < t \leq T,
\]

implies

\[
y(t) \leq CAt^{-\alpha}, \quad 0 < t \leq T.
\]
Note that, by the Lipschitz continuity of $f$,
\[
\|f(u)\| \leq \|f(u) - f(0)\| + \|f(0)\| \leq L\|u\| + \|f(0)\|.
\]
Using (2.2) and Lemma 2.1 we then get
\[
\|u(t)\| \leq c\|u_0\| + c\int_0^t \|f(u(s))\| \, ds
\leq c\|u_0\| + ct\|f(0)\| + cL\int_0^t \|u(s)\| \, ds.
\]
By Lemma 2.2, we obtain the stability result
\[
\|u(t)\| \leq c(\|u_0\| + t\|f(0)\|).
\]
Further properties of the solution $u$ are given below.

**Theorem 2.1.** Assume $u_0 \in \dot{H}^\nu(\Omega)$, $\nu \in [0, 2]$. Then problem (1.1) has a unique solution $u$ satisfying
\[
u \in C([0, T]; \dot{H}^\nu(\Omega)) \cap C((0, T]; \dot{H}^2(\Omega)).
\]
Furthermore,
\[
\|u(t)\|_{\dot{H}^\nu(\Omega)} \leq ct^{(\alpha - 1)(p - \nu) / 2}, \quad 0 \leq \nu \leq p \leq 2,
\]
and
\[
\|u'(t)\|_{\dot{H}^\nu(\Omega)} \leq ct^{(\alpha - 1)(p - \nu) / 2 - 1}, \quad p \in [0, 1].
\]
The constant $c$ may depend on $T$.

**Proof.** For $\nu \in (0, 2]$, the proof follows the same lines as that of Theorem 3.1 in [2]. The latter also covers the estimate (2.1) when $\nu = 0$, see Step 3 in that proof. Thus, we shall only prove (2.5) for $\nu = 0$. To do so, we differentiate both sides of (2.2) with respect to $t$ so that
\[
u'(t) = E'(t)u_0 + E(t)f(u_0) + \int_0^t E(t - s)f'(u(s))u'(s) \, ds.
\]
Multiplying by $t$, we have
\[
tu'(t) = tE'(t)u_0 + tE(t)f(u_0) + \int_0^t sE(t - s)f'(u(s))u'(s) \, ds + \int_0^t (t - s)E(t - s)f'(u(s))u'(s) \, ds.
\]
Following [20, Lemma 5.2] and integrating by parts the last term on the right hand side, we get
\[
\int_0^t (t - s)E(t - s)f'(u(s))u'(s) \, ds = -tE(t)f(u_0) + \int_0^t E(t - s)f(u(s)) \, ds + \int_0^t (t - s)E'(t - s)f(u(s)) \, ds.
\]
Hence,
\[
tu'(t) = tE'(t)u_0 + \int_0^t sE(t - s)f'(u(s))u'(s) \, ds + \int_0^t E(t - s)f(u(s)) \, ds + \int_0^t (t - s)E'(t - s)f(u(s)) \, ds.
\]
Using Lemma 2.1 we thus deduce that
\[
\|tu'(t)\| \leq c + c\int_0^t \|s u'(s)\| \, ds,
\]
which, by Lemma 2.2, implies that $\|tu'(t)\| \leq c$. The $H^1(\Omega)$-estimate $\|\nabla u'(t)\| \leq ct^{(\alpha - 1)(1 - \nu) / 2 - 1}$ is derived in a similar manner. The desired estimate (2.5) follows then by interpolation, which completes the proof. \(\square\)
3 Semidiscrete FE scheme

Let \( \mathcal{T}_h \) be a shape regular and quasi-uniform triangulation of the domain \( \Omega \) into triangles \( K \), and let \( h = \max_{K \in \mathcal{T}_h} h_K \), where \( h_K \) denotes the diameter of \( K \). The approximate solution \( u_h \) of the Galerkin FEM will be sought in the FE space \( V_h \) of continuous piecewise linear functions over the triangulation \( \mathcal{T}_h \)

\[
V_h = \{ v_h \in C^0(\overline{\Omega}) : v_h|_K \text{ is linear for all } K \in \mathcal{T}_h \text{ and } v_h|_{\partial \Omega} = 0 \}.
\]

The semidiscrete Galerkin FEM for problem (1.1) now reads: find \( u_h(t) \in V_h \) such that

\[
(\partial_t u_h, \chi) + a(u_h, \chi) + \gamma a(\partial_t^2 u_h, \chi) = (f(u_h), \chi) \quad \forall \chi \in V_h, \quad t \in (0, T], \quad u_h(0) = P_h u_0,
\]

(3.1)

where \( (\cdot, \cdot) \) is the inner product in \( L^2(\Omega) \), \( a(v, w) := (\nabla v, \nabla w) \) and \( P_h : L^2(\Omega) \to V_h \) is the orthogonal \( L^2(\Omega) \)-projection. Upon introducing the discrete operator \( A_h : V_h \to V_h \) defined by

\[
(A_h \psi, \chi) = (\nabla \psi, \nabla \chi) \quad \forall \psi, \chi \in V_h,
\]

the spatially discrete problem (3.1) is equivalent to

\[
\partial_t u_h(t) + (1 + \gamma \partial_t^2) A_h u_h = P_h f(u_h(t)), \quad t \in (0, T], \quad u_h(0) = P_h u_0.
\]

(3.2)

Following the analysis in the previous section, we represent the solution of (3.2) as

\[
u_h(t) = E_h(t)P_h u_0 + \int_0^t E_h(t - s)P_h f(u_h(s)) \, ds,
\]

(3.3)

where \( E_h(t) : V_h \to V_h \) is defined by

\[
E_h(t) = \frac{1}{2\pi i} \int_{\Gamma_{\theta,\delta}} e^{zt} \frac{g(z)}{z} (g(z)I + A_h)^{-1} \, dz.
\]

In order to bound the FE error \( e_h(t) := u_h(t) - u(t) \), we introduce the operator

\[
S_h(z) := (g(z)I + A_h)^{-1} P_h - (g(z)I + A)^{-1},
\]

which satisfies the following properties, see [19].

**Lemma 3.1.** The following estimate holds for all \( z \in \Sigma_\theta \),

\[
\|S_h(z)v\| + h\|\nabla S_h(z)v\| \leq c h^2 \|v\|
\]

where \( c \) is independent of \( h \).

Let \( F_h(t) = E_h(t)P_h - E(t) \). Then, by Lemma 3.1 \( F_h(t) \) satisfies

\[
\|F_h(t)v\| + h\|\nabla F_h(t)v\| \leq c t^{-(1-\alpha)(1-\nu/2)} h^2 \|v\|_{H^\nu(\Omega)}, \quad \nu \in [0, 2].
\]

(3.4)

Now we are ready to prove an error estimate for the semidiscrete problem (3.2).

**Theorem 3.1.** Let \( u_0 \in \dot{H}^\nu(\Omega), \nu \in [0, 2] \). Let \( u \) and \( u_h \) be the solutions of problems (1.1) and (3.2), respectively. Then

\[
\|e_h(t)\| + h\|\nabla e_h(t)\| \leq c h^2 t^{-(1-\alpha)(1-\nu/2)}, \quad t \in (0, T].
\]

(3.5)

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Proof. Let $\beta = (1 - \alpha)(1 - \nu/2)$. Then, from (2.2) and (3.3), we obtain after rearrangements
\[
e_h(t) = F_h(t)u_0 + \int_0^t E_h(t-s)P_h[f(u_h(s)) - f(u(s))]ds + \int_0^t F_h(t-s)f(u(s))ds.
\]
Using the properties of $F_h$ in (5.34) and the boundedness of $\|E_h(s)\|$ and $\|f(u(s))\|$, we deduce
\[
\|e_h(t)\| \leq \|F_h(t)u_0\| + cL \int_0^t \|e(s)\|ds + \int_0^t \|F_h(t-s)f(u(s))\|ds
\]
\[
\leq ch^{2t-\beta}\|u_0\| + cL \int_0^t \|e(s)\|ds + ch^{2(\alpha-1)}ds
\]
\[
\leq ch^{2t-\beta} + cL \int_0^t \|e(s)\|ds + ch^2.
\]
An application of Lemma 2.2 yields $\|e_h(t)\| \leq ch^{2t-\beta}$. The $H^1(\Omega)$-error estimate is derived analogously, which completes the proof. 

4 Time discretization

This section is devoted to the analysis of a convolution quadrature time discretization for (3.2) generated by the backward Euler (BE) method. Let $0 = t_0 < t_1 < ... < t_N = T$ be a uniform partition of the time interval $[0, T]$, with grid points $t_n = n\tau$ and step size $\tau = T/N$. Integrating both sides of (3.2) over $(0, t)$, we get
\[
u/h(t) - u_h(0) + (\partial_{\tau}^{-1} + \gamma\partial_{\tau}^{\alpha-1})A_hu_h(t) = \partial_{\tau}^{-1}P_hf(u_h(t)).
\]
The fully discrete problem is then obtained by approximating the continuous integral by the convolution quadratures $\partial_{\tau}^{-1}, \partial_{\tau}^{\alpha-1}$ and $\partial_{\tau}^{-1}$, respectively, generated by the BE method, see [18, 11]. The resulting time-stepping scheme reads: with $U^0_h = P_hu_0$, find $U^n_h \in V_h$, $n = 1, 2, \ldots, N$, such that
\[
U^n_h - U^0_h + (\partial_{\tau}^{-1} + \gamma\partial_{\tau}^{\alpha-1})A_hU^n_h = \partial_{\tau}^{-1}P_hf(U^{n-1}_h).
\] (4.1)
We shall investigate a linearized version of (4.1) defined by: with $U^0_h = P_hu_0$, find $U^n_h$, $n = 1, 2, \ldots, N$, such that
\[
U^n_h - U^0_h + (\partial_{\tau}^{-1} + \gamma\partial_{\tau}^{\alpha-1})A_hU^n_h = \partial_{\tau}^{-1}P_hf(U^{n-1}_h).
\] (4.2)
In an expanded form, we have
\[
U^n_h - U^0_h + \tau A_h \sum_{j=0}^n q^{(1)}_{n-j} U^n_h + \gamma(1-\alpha)A_h \sum_{j=0}^n q^{(1-\alpha)}_{n-j} U^n_h = \tau \sum_{j=1}^n q^{(1)}_{n-j} f_h(U^{j-1}_h),
\]
where $f_h = P_hf$ and $q^{(\alpha)}_j = (-1)^j \binom{-\alpha}{j}$, see [18, 11]. Rewriting (4.2) as
\[
U^n_h = (I + (\partial_{\tau}^{-1} + \gamma\partial_{\tau}^{\alpha-1})A_h)^{-1} U^0_h + \partial_{\tau}^{-1}f_h(U^{n-1}_h),
\] (4.3)
and noting that $U^n_h$ depends linearly and boundedly on $U^0_h$ and $f_h(U^{j-1}_h)$, $1 \leq j \leq n$, we deduce the existence of linear and bounded operators $P_n$ and $R_n : V_h \to V_h$, $n \geq 0$, such that $U^n_h$ is represented by
\[
U^n_h = P_nU^0_h + \tau \sum_{j=1}^n R_{n-j}f_h(U^{j-1}_h),
\] (4.4)
see [11] Section 4. The operators $\tau R_n$, $n \geq 0$, in [14] are the convolution quadrature weights corresponding to the Laplace transform $K(z) = z^{-1}(I + (z^{-1} + \gamma z^{\alpha - 1})A_h)^{-1}$. Since $\|K(z)\| \leq c|z|^{-1}$, an application of Lemma 3.1 in [11], with $\mu = 1$, shows that there is a constant $B > 0$, independent of $\tau$, such that

$$\|R_n\| \leq B, \quad n = 0, 1, 2, \ldots.$$  \hfill (4.5)

For the error analysis, we introduce the intermediate $v_h(t) \in V_h$ satisfying

$$\partial_t v_h + (1 + \gamma \partial_t^\alpha) A_h v_h = P_h f(u(t)), \quad v_h(0) = P_h u_0,$$  \hfill (4.6)

and the discrete solution $v^n_h \in V_h$ defined by

$$\partial_t v^n_h + (1 + \gamma \partial_t^\alpha) A_h v^n_h = P_h f(u(t_n)), \quad n \geq 1, \quad v^0_h = U^0_h.$$  \hfill (4.7)

Then an estimation of $u(t_n) - v^n_h$ is given in the following lemma.

**Lemma 4.1.** Let $v^n_h$ be the solution to problem (4.7) with $u_0 \in \dot{H}^\nu(\Omega)$, $\nu \in (0, 2]$. Then there holds

$$\|u(t_n) - v^n_h\| \leq ct_n^{(1-\alpha)\nu/2-1} \tau + ct_n^{-(1-\alpha)(2-\nu)/2} h^2.$$  \hfill (4.8)

**Proof.** Note that (4.3) and (4.7) can be seen as semidiscrete and full discretizations of (1.1) with a given right-hand side function $f(u(t))$, respectively. For the homogeneous case $f = 0$, the bound (4.8) can be found in [3] Remark 4.3]. For the inhomogeneous case with $u_0 = 0$, we consider the splitting

$$u(t_n) - v^n_h = (u(t_n) - v_h) + (v_h - v^n_h) =: I_1 + I_2.$$  

Then, from the proof of Theorem 2.1 it is easily seen that $\|I_1\| \leq ch^2$. To estimate $\|I_2\|$, we follow the arguments in the proof of [12] Theorem 3.6 with $G(z) = \frac{g(z)}{z}(g(z)I + A_h)^{-1}$. Using the bound $\|u'(s)\| \leq cs^{(1-\alpha)\nu/2-1}$ in Theorem 2.1, we then deduce that

$$\|I_2\| \leq c\tau \|f(u_h(0))\| + c\tau \int_0^{t_n} \|f'(u(s))u'(s)\| \, ds \leq ct_n^{(1-\alpha)\nu/2} \tau,$$

which completes the proof. \hfill \qed

**Remark 4.1.** The bound for $\|I_2\|$ does not hold when $\nu = 0$, i.e., $u_0 \in L^2(\Omega)$. This is due to the strong singularity in the bound of $\|u'(s)\|$.

Now we are ready to derive error estimates for the linearized time-stepping scheme (4.2).

**Theorem 4.1.** Let $u_0 \in \dot{H}^\nu(\Omega)$, $\nu \in (0, 2]$. Then the fully discrete scheme (4.2) has a unique solution $U^n_h \in V_h$, $0 < n \leq N$, satisfying

$$\|U^n_h - u(t_n)\| \leq ct_n^{(1-\alpha)\nu/2-1} \tau + ct_n^{-(1-\alpha)(2-\nu)/2} h^2, \quad 0 < t_n \leq T,$$  \hfill (4.9)

where the constant $c = c(\alpha, \nu, T)$ is independent of $\tau$.

**Proof.** Notice that (4.2) is essentially a linear system with a symmetric positive definite matrix. Thus, for given $U^0_h, \cdots, U^{n-1}_h$, (4.2) has a unique solution $U^n_h \in V_h$. Similar to (4.4), the solution of (4.7) may be written as

$$v^n_h = P_h U^n_h + \tau \sum_{j=0}^{n} R_{n-j} f_h(u(t_j)), \quad n \geq 1,$$  \hfill (4.10)

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and in view of (4.10) and (4.4), we have for $0 < t_n \leq T$,
\[
U^n_h - u(t_n) = U^n_h - v^n_h + v^n_h - u(t_n) = v^n_h - u(t_n) + \tau \sum_{j=1}^{n} R_{n-j}(f_h(U^n_{h,j-1}) - f_h(u(t_{j-1}))) + \tau \sum_{j=1}^{n} R_{n-j}(f_h(u(t_{j-1})) - f_h(u(t_j))) - \tau R_n f_h(u(t_0)) =: \sum_{i=1}^{4} I_i.
\]
Using (4.8), we readily get
\[
\|I_2\| \leq LB\tau \sum_{j=0}^{n-1} \|U^n_h - u(t_j)\|.
\]
To bound $I_3$, we use (4.5), the Lipschitz continuity of $f$ and the estimate (4.5) to obtain (after a shifting in the summation),
\[
\|I_2\| \leq LB\tau \sum_{j=0}^{n-1} \|U^n_h - u(t_j)\|.
\]
To bound $I_3$, we use (4.3), the Lipschitz continuity of $f$ and the estimate $\|u'(t)\| \leq ct^{(1-\alpha)\nu/2-1}$, so that
\[
\|I_3\| \leq \tau LB \sum_{j=1}^{n-1} \|u(t_{j+1}) - u(t_j)\| + \tau LB \|u(t_1) - u(t_0)\|
\leq \tau LB \sum_{j=1}^{n-1} \tau \sup_{t_j \leq s \leq t_{j+1}} \|u'(s)\| + c\tau LB
\leq \tau LB \sum_{j=1}^{n-1} t_j^{(1-\alpha)\nu/2-1} \tau + c\tau LB
\leq c\tau LB t^{(1-\alpha)\nu/2},
\]
where $\|u(t)\| \leq c$ is used. For the last term, (4.3) and the Lipschitz continuity of $f$ implies that $\|I_4\| \leq cB\tau$. Altogether, we obtain
\[
\|U^n_h - u(t_n)\| \leq ct^{(1-\alpha)\nu/2-1} + ct^{(1-\alpha)(2-\nu)/2} h^2 + \tau LB \sum_{j=0}^{n-1} \|U^n_h - u(t_j)\|.
\]
Finally, the desired estimate (4.9) follows by applying the discrete Grönwall inequality.

5 The lumped mass FEM

In this section, we consider the lumped mass piecewise linear FE method and derive related convergence rates for smooth and nonsmooth initial data. We begin by defining the quadrature approximation of the $L^2(\Omega)$-inner product on $V_h$ by
\[
(w, \chi)_h = \sum_{K \in T_h} Q_{K,h}(w\chi) \quad \text{with} \quad Q_{K,h}(f) = \frac{|K|}{3} \sum_{i=1}^{3} f(P_i) \approx \int_{K} f \, dx,
\]
where $P_i, i = 1, 2, 3$ are vertices of the triangle $K \in T_h$. Then the spatially lumped mass FE scheme for (1.1) reads: find $\bar{u}_h(t) \in V_h$ such that
\[
(\partial_t \bar{u}_h, \chi)_h + a(\bar{u}_h, \chi) + \gamma a(\partial^2_t \bar{u}_h, \chi) = (f(\bar{u}_h), \chi) \quad \forall \chi \in V_h, \quad t \in (0, T), \quad \bar{u}_h(0) = P_h u_0. \quad (5.1)
\]
Next we introduce the projection operator \( \tilde{P}_h : L^2(\Omega) \to V_h \) defined by \((\tilde{P}_hv, \chi)_h = (v, \chi)\) for all \( \chi \in V_h \), and the discrete operator \( \tilde{A}_h : V_h \to V_h \) corresponding to the inner product \((\cdot, \cdot)_h\) satisfying

\[
(\tilde{A}_h\psi, \chi)_h = (\nabla\psi, \nabla\chi) \quad \forall \psi, \chi \in V_h.
\]

Then (5.1) is equivalent to

\[
\partial_t \tilde{u}_h(t) + (1 + \gamma \partial_t) \tilde{A}_h \tilde{u}_h = \tilde{P}_h f(\tilde{u}_h(t)), \quad \tilde{u}_h(0) = P_h u_0. \quad (5.2)
\]

Set \( \tilde{e}_h = \tilde{u}_h(t) - u(t) \) and consider the splitting \( e_h = \tilde{u}_h(t) - u_h(t) + u_h(t) - u(t) =: \xi(t) + e_h(t) \)

where \( u_h \) is the solution of (5.2). Subtracting (5.1) from (5.2), we have \( \forall \chi \in V_h \)

\[
(\xi(t), \chi)_h + (\nabla \xi(t), \nabla \chi) + \gamma (\partial_t \nabla \xi(t), \nabla \chi) = (u'_h(t), \chi)_h - (u'_h, \chi)_h + (f(\tilde{u}_h), \chi) - (f(u_h), \chi).
\]

Hence, \( \xi(t) \) satisfies

\[
\xi'(t) + (1 + \gamma \partial_t) \tilde{A}_h \xi(t) = -\tilde{A}_h Q_h u'_h(t) + \tilde{P}_h (f(\tilde{u}_h(t)) - f(u_h(t))), \quad t \in (0, T], \quad \xi(0) = 0, (5.3)
\]

where \( Q_h : V_h \to V_h \) is the quadrature error defined by

\[
(\nabla Q_h \chi, \nabla \psi) = (\chi, \psi)_h - (\chi, \psi) \quad \forall \psi \in V_h.
\]

A key property of \( Q_h \) is given in the following lemma, see [7].

**Lemma 5.1.** Let \( Q_h \) be defined by (5.4). Then there holds

\[
\|\nabla Q_h \chi\| + h\|\tilde{A}_h Q_h \chi\| \leq c h^{p+1}\|\nabla^p \chi\| \quad \forall \chi \in V_h, \quad p = 0, 1,
\]

where the constant \( c \) is independent of \( h \).

Solving (5.3) for \( \xi \) using the Laplace transform, we have

\[
\xi(t) = e^{-1} \int^t_0 E_h(t-s) \left[ -\tilde{A}_h Q_h u'_h(s) + \tilde{P}_h (f(\tilde{u}_h(t)) - f(u_h(t))) \right] ds,
\]

where

\[
E_h(t) = \frac{1}{2\pi i} \int_{\Gamma_{\rho,\delta}} e^{zt} \frac{g(z)}{z} \left( g(z) I + \tilde{A}_h \right)^{-1} dz.
\]

Since the operator \( \tilde{A}_h \) is selfadjoint and positive definite, \( E_h(t) \) satisfies (see Lemma 2.1)

\[
\|\tilde{A}_h^{1/2} \partial_t \tilde{A}_h(t) v\| \leq c t^{-m-(1-\alpha)(p-q)/2} \|\tilde{A}^{q/2} v\|. \quad (5.6)
\]

Error estimates for smooth initial date are given in the following theorem.

**Theorem 5.1.** Let \( u \) be the solution of (1.1) with \( u_0 \in \dot{H}^\nu(\Omega), \nu \in [1, 2] \). Let \( \tilde{u}_h \) be the solution of (5.2). Then

\[
\|\tilde{e}_h(t)\| + h\|\nabla \tilde{e}_h(t)\| \leq c t^2 t^{-(1-\alpha)(1-\nu/2)}, \quad t \in (0, T]. \quad (5.7)
\]

**Proof.** Recall that \( \tilde{e}(t) = \xi(t) + e_h(t) \). In Theorem 3.1 a bound for \( e_h(t) \) is given. To estimate \( \xi(t) \), we modify the arguments presented in [7] for the parabolic case. We shall consider the cases \( \nu = 2 \) and \( \nu = 1 \) separately. For \( \nu = 2 \), we use (5.6) with \( p = 2, q = 1 \), the Lipschitz continuity of \( f \) and Lemma 5.1 to get

\[
\|\xi(t)\| \leq c \int^t_0 \left[ (t-s)^{(\alpha-1)/2}\|\nabla Q_h u'_h(s)\| + (t-s)^{(\alpha-1)/2}\|\xi(s)\| \right] ds
\]

\[
\leq c \int^t_0 \left[ t^{(\alpha-1)/2}\|\nabla Q_h u'_h(s)\| + \|\xi(s)\| \right] ds.
\]
Note that $\|\nabla u_h'(t)\| \leq ct^{-(\alpha+1)/2}$ by Theorem 2.2. Therefore

$$\|\xi(t)\| \leq c \int_0^t \left[ h^2(t-s)^{(\alpha-1)/2}s^{-(\alpha+1)/2} + \|\xi(s)\| \right] ds \leq ch^2,$$

where the last inequality follows by applying Lemma 2.2. Again, using (5.6) with $p = 1$, $q = 0$, the Lipschitz continuity of $f$ and Lemma 5.1 we find that

$$\|\nabla \xi(t)\| \leq \int_0^t \left[ \|\nabla \tilde{E}_h(t-s)\tilde{A}_h Q_h u_h'(s)\| + \|\nabla \tilde{E}_h(t-s)\tilde{P}_h(f(\tilde{u}_h(s)) - f(u_h(s)))\| \right] ds \leq c \int_0^t \left[ (t-s)^{-(\alpha-1)/2}\|\tilde{A}_h Q_h u_h'(s)\| + (t-s)^{(\alpha-1)/2}\|\xi(s)\| \right] ds \leq c \int_0^t \left[ h(t-s)^{-(\alpha-1)/2}\|\nabla u_h'(s)\| + (t-s)^{(\alpha-1)/2}\|\xi(s)\| \right] ds \leq c \int_0^t \left[ h(t-s)^{-(\alpha-1)/2}\|\nabla u_h(s)\| + (t-s)^{(\alpha-1)/2}\|\xi(s)\| \right] ds,$$

and therefore $\|\nabla \xi(t)\| \leq ch$ by Lemma 5.1. Hence, we have

$$\|\xi(t)\| + h\|\nabla \xi(t)\| \leq ch^2. \quad (5.8)$$

For $\nu = 1$, we split the integral in (5.5) as

$$\int_0^t \tilde{E}_h(t-s)\tilde{A}_h Q_h u_h'(s) ds = \left\{ \int_0^{t/2} + \int_{t/2}^t \right\} \tilde{E}_h(t-s)\tilde{A}_h Q_h u_h'(s) ds =: I_1 + I_2.$$

To bound $I_1$, we integrate by parts so that

$$I_1 = \int_0^{t/2} \tilde{E}_h(t-s)\tilde{A}_h Q_h u_h'(s) ds = \tilde{E}_h(t/2)\tilde{A}_h Q_h u_h(t/2) - \tilde{E}_h(t)\tilde{A}_h Q_h u_h(0) + \int_0^{t/2} \tilde{E}_h(t-s)\tilde{A}_h Q_h u_h(s) ds.$$

By (5.8) and Lemma 5.1 it follows that

$$\|I_1\| \leq ct^{(\alpha-1)/2}\|\nabla Q_h u_h(t/2)\| + c \int_0^{t/2} (t-s)^{(\alpha-3)/2}\|\nabla Q_h u_h(s)\| ds \leq ch^2 t^{(\alpha-1)/2}\|\nabla u_h(t/2)\| + ch^2 \int_0^{t/2} (t-s)^{(\alpha-3)/2}\|\nabla u_h(s)\| ds \leq ch^2 t^{(\alpha-1)/2}.$$

For $I_2$, we apply (5.6) with $p = 2$, $q = 1$ and Lemma 5.1 to get

$$\|I_2\| = \| \int_{t/2}^t \tilde{E}_h(t-s)\tilde{A}_h Q_h u_h'(s) ds \| \leq ch^2 \int_{t/2}^t (t-s)^{(\alpha-1)/2}\|\nabla u_h'(s)\| ds \leq ch^2 \int_{t/2}^t (t-s)^{(\alpha-1)/2}s^{-1} ds \leq ch^2 t^{(\alpha-1)/2}.$$
From (5.3), we thus deduce that
\[ \|\xi(t)\| \leq ch^2 t^{(\alpha-1)/2} + c \int_0^t \|\xi(s)\| \, ds. \]

Then an application of Lemma 2.2 yields \( \|\xi(t)\| \leq ch^2 t^{(\alpha-1)/2} \). For the \( H^1(\Omega) \)-estimate of \( \xi \), we follow previous arguments, apply (5.6) with \( p = 1, q = 0 \) and use Lemma 5.1 to conclude that \( \|\nabla\xi(t)\| \leq ch t^{(\alpha-1)/2} \). Hence, for \( \nu = 1 \),
\[ \|\xi(t)\| + h\|\nabla\xi(t)\| \leq ch^2 t^{(\alpha-1)/2}. \] (5.9)

By interpolation of (5.8) and (5.9), we obtain
\[ \|\xi(t)\| + h\|\nabla\xi(t)\| \leq ch^2 t^{-(1-\alpha)(1-\nu/2)}, \quad \nu \in [1, 2]. \]

Together with the estimate (3.5), this completes the proof of (5.7). \( \square \)

In the next theorem, a nonsmooth data error estimate is derived. The proof is quite similar to the previous one and hence omitted.

**Theorem 5.2.** Let \( u \) be the solution of (1.1) with \( u_0 \in L^2(\Omega) \). Let \( \bar{\eta}_h \) be the solution of (5.2). Then
\[ \|\bar{\eta}_h(t)\| + h\|\nabla\bar{\eta}_h(t)\| \leq ch t^{(\alpha-1)} \quad t \in (0, T]. \] (5.10)

Furthermore, if the quadrature error operator \( Q_h \) satisfies
\[ \|Q_h \chi\| \leq ch^2 \|\chi\| \quad \forall \chi \in V_h, \] (5.11)
then the following optimal error estimate holds:
\[ \|\bar{\eta}_h(t)\| \leq ch^2 t^{(\alpha-1)}. \] (5.12)

**Remark 5.1.** For symmetric meshes, the operator \( Q_h \) satisfies (5.11), see [7, 8]. Thus, by interpolating (5.7) and (5.12), we get for \( u_0 \in H^\nu(\Omega) \) and \( \nu \in [0, 2] \),
\[ \|\bar{\eta}_h(t)\| \leq ch^2 t^{-(1-\alpha)(1-\nu/2)}. \]

Now we consider the lumped mass FE method combined with a time convolution quadrature generated by the backward Euler method. The resulting linearized time-stepping scheme is defined as follows: with \( \bar{u}^0_h = P_h u_0 \), find \( \bar{u}^n_h \in V_h \), \( n = 1, 2, \ldots, N \), such that
\[ \bar{u}^n_h - \bar{u}^0_h + (\partial_t^{-1} + \gamma \partial_t^{\alpha-1})\bar{A}_h \bar{u}^n_h = \partial_t^{-1} P_h f(\bar{u}^{n-1}_h). \] (5.13)

Following the analysis in Section 4, we obtain the following error estimate.

**Theorem 5.3.** Let \( u_0 \in H^\nu(\Omega) \), \( \nu \in (0, 2] \). Assume the mesh is symmetric. Then the fully discrete scheme (5.13) has a unique solution \( \bar{u}^n_h \in V_h \), \( 0 < n < N \), satisfying
\[ \|\bar{u}^n_h - u(t_n)\| \leq ch_n^{(1-\alpha)\nu/2-1} + ch_n^{-(1-\alpha)(2-\nu)/2} h^2, \quad 0 < t_n \leq T, \] (5.14)
where the constant \( c = c(\alpha, \nu, T) \) is independent of \( \tau \).
6 Numerical Experiments

In this section, numerical examples are provided to validate the theoretical results. We choose \( \Omega = (0,1)^2 \), fix \( T = 1 \) and consider problems with smooth and nonsmooth initial data. We let \( N \) denote the number of time steps and \( \tau = T/N \). Since exact solutions are difficult to obtain, we compute reference solutions on very refined meshes.

We shall apply the linearized time-stepping scheme (5.13) and perform the computation on symmetric and nonsymmetric triangular meshes. For the symmetric meshes, we divide the domain \( \Omega \) into regular right triangles with \( M \) equal subintervals of length \( h = 1/M \) on each side of the domain. The nonsymmetric meshes are constructed by choosing \( M \) subintervals of lengths \( 4/3 \) and \( 2/3 \) in the \( x \)-direction, distributed such that they form an alternating series, while the \( y \)-direction is divided into \( 3M/4 \) equally spaced subintervals with the assumption that \( M \) is divisible by 4.

| \( \alpha \) | case \( \\backslash \ M \) | 8 | 16 | 32 | 64 | 128 | rate |
|---|---|---|---|---|---|---|---|
| 0.25 | (a) | 1.03e-3 | 2.64e-4 | 6.63e-5 | 1.65e-5 | 4.06e-6 | 2.03 |
| | (b) | 1.03e-3 | 2.62e-4 | 6.57e-5 | 1.64e-5 | 4.03e-6 | 2.02 |
| 0.5 | (a) | 1.10e-3 | 2.81e-4 | 7.06e-5 | 1.76e-5 | 4.32e-6 | 2.03 |
| | (b) | 1.09e-3 | 2.77e-4 | 6.95e-5 | 1.73e-5 | 4.29e-6 | 2.02 |
| 0.75 | (a) | 1.16e-3 | 2.97e-4 | 7.47e-5 | 1.86e-5 | 4.57e-6 | 2.03 |
| | (b) | 1.16e-3 | 2.93e-4 | 7.32e-5 | 1.83e-5 | 4.54e-6 | 2.01 |

We consider the model (1.1) with the following data:

(a) \( u_0(x,y) = xy(1-x)(1-y) \in \dot{H}^2(\Omega) \) and \( f = \sqrt{1+u^2} \),

(b) \( u_0(x,y) = \chi_{(0,1/2) \times (0,1)}(x,y) \in \dot{H}^\epsilon(\Omega) \) for \( 0 \leq \epsilon < 1/2 \), and \( f = \sqrt{1+u^2} \),

where \( \chi_S \) denotes the characteristic function of the set \( S \).

The numerical results on symmetric meshes are presented in Tables 1-4. In Tables 1 and 2, we investigate the spatial and temporal convergence rates, respectively. From the tables, we observe an \( O(h^2) \) rate in space and \( O(\tau) \) rate in time which agrees well with our theoretical estimates.

Table 3 displays the space prefactor convergence rates with respect to \( t \). We notice that the spatial error essentially stays unchanged in the smooth case (a), whereas it behaves like \( O(t^{3(\alpha-1)/4}) \) in the nonsmooth case (b). These results confirm the estimates of Theorem 5.3.

By neglecting the spatial error, fixing the step size \( \tau = 10 \) and taking \( t_N \rightarrow 0 \), we examine the time prefactor. Theorem 5.3 indicates that the error behaves like \( O(t_N^{(1-\alpha)/2}) \) for \( u_0 \in \dot{H}^\epsilon(\Omega) \).
Table 3: $L^2$-error for cases (a) and (b) with $\alpha = 0.5$: $t \to 0$, $h = 1/64$, $N = 500$.  

| $t_N$ | 1e-3   | 1e-4   | 1e-5   | 1e-6   | 1e-7   | rate  |
|-------|--------|--------|--------|--------|--------|-------|
| (a)   | 8.04e-6| 1.25e-5| 1.52e-5| 1.63e-5| 1.68e-5| -0.01 (0) |
| (b)   | 1.89e-4| 4.68e-4| 1.12e-3| 2.65e-3| 6.15e-3| -0.36 (-0.375) |

The numerical results presented in Table 3 show a convergence rate of order $O(t_{N}^{0.5})$ for smooth data and $O(t_{N}^{1/8})$ for nonsmooth data, which confirms our convergence theory.

Table 4: $L^2$-error for cases (a) and (b) with $\alpha = 0.5$: $t \to 0$, $h = 1/512$, $N = 10$.  

| $t_N$ | 1e-3   | 1e-4   | 1e-5   | 1e-6   | 1e-7   | rate  |
|-------|--------|--------|--------|--------|--------|-------|
| (a)   | 2.01e-4| 8.63e-5| 2.92e-5| 9.43e-6| 3.01e-6| 0.49 (0.50) |
| (b)   | 4.16e-3| 3.21e-3| 2.30e-3| 1.73e-3| 1.30e-3| 0.12 (0.125) |

For the case of nonsymmetric meshes, we focus on spatial errors. Theorem 5.2 suggests convergence rates of order $O(h^2)$ for smooth initial data and, by interpolation, $O(h^{3/2})$ for $u_0 \in H^{1/2}$. In Table 5, the spatial discretization errors for cases (a) and (b) are presented. The results show convergence rates of order $O(h^2)$ in both cases, which may be seen unexpected. In our case, the particular choice of initial data could have a positive effect on the convergence rate. A similar fact was also observed in the case of the finite volume method [14].

Table 5: $L^2$-error for cases (a) and (b) on nonsymmetric meshes with $\alpha = 0.5$, $N = 500$.  

| case \ $M$ | 8    | 16   | 32   | 64   | 128  | rate  |
|------------|------|------|------|------|------|-------|
| (a)        | 1.70e-3| 4.40e-4| 1.11e-4| 2.76e-5| 6.64e-6| 2.05 (2.00) |
| (b)        | 1.65e-3| 4.20e-4| 1.05e-4| 2.61e-5| 6.29e-6| 2.05 (1.50) |

7 Conclusion

In this work, we have studied a semilinear time-fractional Rayleigh–Stokes problem involving a fractional derivative in time of Riemann-Liouville type. The nonlinear term satisfies a global Lipchitz condition. We discussed stability and provided regularity results for the exact solution. Two spatially semidiscrete schemes were investigated based on the standard Galerkin and lumped mass finite element methods, respectively. A fully discrete scheme was obtained via a convolution quadrature in time generated by the backward Euler method, and optimal error estimates were derived for smooth and nonsmooth initial data. Several numerical experiments were carried out on symmetric and nonsymmetric triangular meshes to validate the theoretical results.

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