The turnpike property in nonlinear optimal control — A geometric approach

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Abstract—This paper presents, using dynamical system theory, a framework for investigating the turnpike property in nonlinear optimal control. First, it is shown that a turnpike-like property appears in general dynamical systems with hyperbolic equilibrium and then, apply it to optimal control problems to obtain sufficient conditions for the turnpike occurs. The approach taken is geometric and gives insights for the behaviors of controlled trajectories, allowing us to find simpler proofs for existing results on the turnpike properties.

I. INTRODUCTION

The turnpike property was first recognized in the context of optimal growth by economists (see, e.g., [1]). The turnpike theorems say that for a long-run growth, regardless of starting and ending points, it will pay to get into a growth phase, called von Neumann path, in the most of intermediate stages. It is exactly like a turnpike and a network of minor roads; “if origin and destination are far enough apart, it will always pay to get on the turnpike and cover distance at the best rate of travel ...” (quoted from [2]).

In control theory, independently of the turnpike theorems in econometrics, this property was investigated as dichotomy in linear optimal control [3], [4] and later extended to nonlinear systems [5]. In optimal control, the turnpike property essentially means that the solution of an optimal control problem is determined by the system and cost function and independent of time intervals, initial and terminal conditions except in the thin layers at the both ends of the time interval (see, e.g., [6], [7], [8]). In the last decades, much progress has been made in the theory of turnpike for finite or infinite dimensional and linear or nonlinear control systems. In [9], the authors study the turnpike for linear finite and infinite dimensional systems and derive a simple but meaningful inequality, for which we term turnpike inequality. Their works are extended to finite-dimensional nonlinear systems [10], the semi-linear heat equation [11], the wave equation [12], [13] and periodic turnpike for systems in Hilbert spaces[14]. The authors in [15], [16], [17], [18], [19] investigate the turnpike property from the viewpoint of dissipative systems (see, e.g., [20], [21]).

In this paper, we first show that turnpike-like behaviors naturally appear in general dynamical systems with hyperbolic equilibrium. The main technique we use is the \( \lambda \)-Lemma which describes trajectory behaviors near invariant manifolds such as stable and unstable manifolds. That the turnpike inequality holds implies that if one fixes two ends of a trajectory close to stable and unstable manifolds and designates the time duration sufficiently large, then the trajectory necessarily converges to these manifolds to spend the most of time near the equilibrium. This is exactly the turnpike property. It should be noted that the two ends, as long as they are close to the manifold, do not need to lie in the vicinity of the equilibrium.

We, then, apply this inequality to an optimal control problems in which terminal states are not specified and the steady optimal solutions are not the origin as in [9], [11], [13], [14] and to an optimal control problem in which two terminal states are specified and the steady optimal point is the origin as in [3], [5], [10]. For both classes of problems, we employ a Dynamic Programming approach with Hamilton-Jacobi equations (HJE). The characteristic equations for HJE are Hamiltonian systems and the stabilizability (controllability for the second class) and detectability conditions assure that the equilibriums of the Hamiltonian systems are hyperbolic. The controlled trajectories appear in these Hamiltonian systems and we apply the turnpike result for dynamical system. Then, the existence of the trajectory satisfying initial and boundary conditions is guaranteed. In this paper, we derive sufficient conditions for optimality by using the Dynamic Programming and HJE and by imposing a condition that guarantees the existence of the solution to the HJE (Lagrangian submanifold property, see, e.g., [22]).

The organization of the paper is as follows. In Section II we review some of key tools from dynamical...
system theory and derive the turnpike inequality. In Section III, we apply it to optimal control problems. Section III-A handles the problem where boundary state is free and Section III-B handles the problem where initial and boundary states are fixed.

II. TURNPIKE IN DYNAMICAL SYSTEMS

Let us consider a nonlinear dynamical system of the form

$$\dot{x} = f(x),$$

where \( f : \mathbb{R}^n \to \mathbb{R}^n \) is of \( C^r \) class \((r \geq 1)\). We assume that \( f(0) = 0 \) and the hyperbolicity of \( f \) at 0, namely, assume that \( Df(0) \in \mathbb{R}^{n \times n} \) has \( k \) eigenvalues with strictly negative real parts and \( n - k \) eigenvalues with strictly positive real parts.

It is known, as the stable manifold theorem, that there exist \( C^r \) manifolds \( S \) and \( U \), called stable manifold and unstable manifold of \((1)\) at 0, respectively, defined by

$$S := \{ x \in \mathbb{R}^n \mid \varphi(t,x) \to 0 \text{ as } t \to \infty \},$$

$$U := \{ x \in \mathbb{R}^n \mid \varphi(t,x) \to 0 \text{ as } t \to -\infty \},$$

where \( \varphi(t,x) \) is the solution of \((1)\) starting \( x \) at \( t = 0 \). Let \( E^s, E^u \) be stable and unstable subspaces in \( \mathbb{R}^n \) of \( Df(0) \) with dimension \( k, n - k \), respectively. It is known that \( S, U \) are invariant under the flow of \( f \) and are tangent to \( E^s, E^u \), respectively, at \( x = 0 \). See, e.g., [23], [24] for more details on the theory of stable manifold.

We will consider limiting behavior of submanifolds under the flow of \( f \) and need to introduce topology for maps and manifolds. Let \( M \) be a compact manifold of dimension \( m \) and the space \( C^r(M, \mathbb{R}^1) \) of \( C^r \) maps, \( 0 \leq r \leq \infty \), defined on \( M \). There exists a natural vector space structure on \( C^r(M, \mathbb{R}^1) \). Since \( M \) is compact, we take a finite cover of \( M \) by open sets \( V_1, \ldots, V_k \) and take a local chart \((x_i, U_i)\) for \( M \) with \( x_i(U_i) = B(2) \) such that \( x_i(V_i) = B(1), i = 1, \ldots, k \), where \( B(1) \) and \( B(2) \) are the balls of radius 1 and 2 at the origin of \( \mathbb{R}^n \).

For a map \( g \in C^r(M, \mathbb{R}^1) \), we define a norm by

$$\|g\|_r := \max_{\{u \in B(1)\}} \|g(u)\|, \|Dg^i(u)\|, \ldots,$$

where \( g^i = g \circ x_i^{-1} \), local representation of \( g \), and \( \| \cdot \| \) is a norm for linear maps. It is known that \( \| \cdot \|_r \) does not depend on the choice of finite cover and we call it \( C^r \) norm. For maps in \( C^r(M, N) \) where \( N \) is a manifold, we embed \( N \) in a Euclidean space with sufficiently high dimension. Let \( L, L' \) be \( C^r \) submanifolds of \( M \) and let \( \varepsilon > 0 \). We say that \( L \) and \( L' \) are \( C^r \)-close if there exists a \( C^r \) diffeomorphism \( h : L \to L' \) such that \( \|i' \circ h - i\|_r < \varepsilon \), where \( i : L \to M \) and \( i' : L' \to M \) are inclusion maps.

The following lemma is known as the \( \lambda \)-lemma and plays a crucial role in the theory of dynamical systems (see [24], [25]).

Lemma 2.1 (The \( \lambda \)-Lemma): Suppose that \( x = 0 \) is a hyperbolic equilibrium for \((1)\). Suppose also that \( S \) and \( U \) are \( k, (n - k) \)-dimensional stable and unstable manifolds of \( f \) at 0, respectively. For any \( (n - k) \)-dimensional disc \( B \) in \( U \), any point \( x \in S \), any \( (n - k) \)-dimensional disc \( D \) transversal to \( S \) at \( x \) and any \( \varepsilon > 0 \), there exists a \( T > 0 \) such that if \( t > T, \varphi(t, D) \) contains an \( (n - k) \)-dimensional disc that is \( \varepsilon \)-close to \( B \).

Next, we show that the turnpike behavior appears in the transition of points near the stable manifold to points near the unstable manifold if the transition duration is designated large. Let \( x_0 \in S \) and \( x_1 \in U \) be given points. From the stable manifold theorem, it holds that

$$|\varphi(t, x_0)| \leq Ke^{-\mu t} \text{ for } t \geq 0$$

$$|\varphi(t, x_1)| \leq Ke^{\mu t} \text{ for } t \leq 0,$$

where \( K > 0 \) is a constant dependent on \( x_0 \) and \( x_1 \) and \( \mu > 0 \) is a constant independent of \( x_0 \) and \( x_1 \).

Proposition 2.2: (i) There exists a \( T_0 > 0 \) such that for every \( T > T_0 \) there exists a \( \rho > 0 \) such that

$$|\varphi(t, y)| \leq Ke^{-\mu t} \text{ for } t \in [0, T], \ y \in B(x_0, \rho),$$

where \( B(x_0, \rho) \) is the \( n \)-dimensional ball centered at \( x_0 \) with radius \( \rho \). Moreover, \( \rho \to 0 \) when \( T \to \infty \).

(ii) There exist a \( T_0 < 0 \) such that for every \( T < T_0 \) there exists a \( \rho > 0 \) such that

$$|\varphi(t, y)| \leq Ke^{\mu t} \text{ for } t \in (T, 0], \ y \in B(x_1, \rho).$$

Moreover, \( \rho \to 0 \) when \( T \to -\infty \).

(iii) There exists a \( T_0 \) such that for any \( T > T_0 \) there exist an \((n - k)\)-dimensional disc \( D \) transversal to \( S \) at \( x_0 \) and a \((k - 1)\)-dimensional disc \( E \) transversal to \( U \) at \( x_1 \) such that \( \varphi(T, D) \) intersects \( \varphi(-T, E) \) at a single point.

Proof: (iii) Let \( d(\cdot, \cdot) \) denote the \( C^1 \)-norm of two submanifolds. First we take an \((n-k)\)-dimensional disc \( U_0 \) in \( U \) passing through \( 0 \), a \( k \)-dimensional disc \( S_0 \) in \( S \) passing through \( 0 \) and an \( \varepsilon > 0 \) so that any \((n-k), k\) dimensional discs \( D \) and \( E \) satisfying \( d(U_0, D) < \varepsilon \) and \( d(S_0, E) < \varepsilon \) intersect at a single point. This is possible because \( D \cap E = \emptyset \). This \( \varepsilon \) determines, based on the \( \lambda \)-Lemma, a \( T_0 > 0 \) such that for any \( T > T_0 \), there exist an \((n-k)\)-dimensional disc \( D \) transversal to \( S \) at \( x_0 \) and a \((k-1)\)-dimensional disc \( E \) transversal to \( U \) at \( x_1 \) such that \( d(U_0, \varphi(T, D)) < \varepsilon \) and \( d(S_0, \varphi(-T, E)) < \varepsilon \). Thus, \( \varphi(T, D) \) and \( \varphi(-T, E) \) intersect at a single point.

Remark 2.1: It should be noted that the above statements, especially (i) and (ii), are only on finite interval \([0, T] \). This is the major difference from the trajectories on the stable and unstable manifolds.

Theorem 2.3: There exists a \( T_0 > 0 \) such that for every \( T > T_0 \) there exist a \( \rho > 0 \), \( y_0 \in B(x_0, \rho) \) and \( y_1 \in B(x_1, \rho) \) such that \( \varphi(T, y_0) = y_1 \) and

$$|\varphi(t, y_0)| \leq K \left[e^{-\mu t} + e^{\mu(T-t)} \right] \text{ for } t \in [0, T].$$
Moreover, $\rho \to 0$ when $T \to \infty$.

Proof: Take the largest $T_0$ and the smallest $\rho$ in Proposition 2.2. We rename this $T_0$ as $T_0/2$. Take arbitrary $T > T_0$ and use Proposition 2.2(iii) to get a disc $D$ which is $(n-k)$-dimensional and transversal to $S$ at $x_0$ and a disc $E$ which is $k$-dimensional and transversal to $U$ at $x_1$ satisfying $D \subset B(x_0, \rho)$ and $E \subset B(x_1, \rho)$. This is possible by taking smaller $S_0$ and $U_0$ in the proof of Proposition 2.2(iii). Then, there exists a single point $z$ such that $\varphi(T/2, D) \cap \varphi(-T/2, E) = \{ z \}$ (see Fig 1).

Let $y_0 := \varphi(-T/2, z)$. Then, $y_0 \in D \subset B(x_0, \rho)$ and by Proposition 2.2(i), we have

$$|\varphi(t, y_0)| \leq Ke^{-\mu t} \text{ for } 0 \leq t \leq T/2.$$ \hspace{1cm} (4)

Let $y_1 := \varphi(T/2, z)$. Then, $y_1 \in E \subset B(x_1, \rho)$ and

$$|\varphi(t, y_1)| \leq Ke^{\mu t} \text{ for } -T/2 \leq t \leq 0.$$ \hspace{1cm} (5)

Combining (4) and (5), we get the inequality in the theorem. The last assertion follows from Proposition 2.2(ii) and (ii).

![A schematic of the proof of Theorem 2.3](image)

Fig. 1. A schematic of the proof of Theorem 2.3

III. TURNPIKE IN NONLINEAR OPTIMAL CONTROL

Let us consider a nonlinear control system

$$\dot{x} = f(x) + g(x)u, \quad x(t_0) = x_0, \quad (6)$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $g : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ are of $C^2$ class, $x(t) \in \mathbb{R}^n$ is state variables and $u(t) \in \mathbb{R}^m$ is control input. The optimal control problem or OPC is to find a control input for (6) such that the cost functional

$$J = \int_0^T L(x(t), u(t)) \, dt$$

is minimized. There are several types in OCPs depending on whether or not the terminal time $T$ is specified and whether or not the state variables are specified at the terminal time. In this paper, we consider OCPs where the terminal time $T$ is specified and two types of OCPs; one in which the state variables are free at $t = T$ and another in which they are fixed at $t = T$.

A. The OCP with state variables unspecified at the terminal time

For system (6), we consider the following cost functional

$$J(u) = \frac{1}{2} \int_0^T |Cx(t) - z|^2 + |u(t)|^2 \, dt,$$

where $C \in \mathbb{R}^{r \times n}$ and $z \in \mathbb{R}^r$ is a given vector. We call this problem (OCP)$_T$:

(OCP)$_T$ : Find a control $u \in L^\infty(0, T; \mathbb{R}^m)$ such that $J(u)$ along (6) is minimized over all $u \in L^\infty(0, T; \mathbb{R}^m)$.

Associated with (OCP)$_T$, we consider a steady optimization problem

(SOP) : Minimize $J_s(x, u) = \frac{1}{2}(|Cx - z|^2 + |u|^2)$

over all $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$ such that $f(x) + g(x)u = 0$.

We assume the following.

Assumption 1: (SOP) has a solution $(\bar{x}, \bar{u})$.

Also, associated with (OCP)$_T$, we can derive a Hamilton-Jacobi equation

$$V_i(t, x) + V_z(t, x)f(x) + \frac{1}{2}V_z(t, x)g(x)g(x)^\top V_z(t, x) + \frac{1}{2}|Cx - z|^2 = 0,$$

$$V_z(T, x) = 0,$$

for $V(t, x)$, where $V_i = D_i V$, $V_z = D_z V$. Defining a Hamiltonian

$$H(x, p) = p^\top f(x) - \frac{1}{2}p^\top g(x)g(x)^\top p + \frac{1}{2}|Cx - z|^2,$$

we consider the corresponding characteristic equation for (7)-(8)

$$\dot{x}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x_i}, \quad i = 1, \ldots, n$$

with $p_i(T) = 0, i = 1, \ldots, n$. Note that since the system (6) is time-invariant, the equation corresponding to $V_i$ is not necessary. The following fact is readily verified.

Fact. A solution $(\bar{x}, \bar{u})$ of (SOP) corresponds to an equilibrium point $(\bar{x}, \bar{p})$ of (9) with $\bar{u} = -g(\bar{x})^\top \bar{p}$.

Assumption 2: Let $A = D_x D_p H(\bar{x}, \bar{p}), B = g(\bar{x})$. The triplet $(C, A, B)$ is stabilizable and detectable.

Under Assumption 2, the equilibrium $(\bar{x}, \bar{p})$ is hyperbolic equilibrium for the Hamiltonian system (9) and
there exist stable and unstable manifolds for (9) at \((\bar{x}, \bar{p})\) which are expressed as

\[
S = \tilde{S} + \{(\bar{x}, \bar{p})\}, \quad U = \tilde{U} + \{(\bar{x}, \bar{p})\}.
\]

(10)

Here, \(\tilde{S}, \tilde{U}\) are the stable and unstable manifold of (9) in the coordinates \((\bar{x}, \bar{p})\), where \(\bar{x} = x - \bar{x}, \bar{p} = p - \bar{p}\), which is re-written as

\[
\frac{d}{dt} \begin{bmatrix} \bar{x} \\ \bar{p} \end{bmatrix} = \begin{bmatrix} A & -BB^\top \\ -C^\top C & -A^\top \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{p} \end{bmatrix} + o(|\bar{x}| + |\bar{p}|).
\]

We can now state the main theorem of this subsection. Let \(\pi_1 : (x,p) \mapsto x, \pi_2 : (x,p) \mapsto p\) be canonical projections.

Theorem 3.1: Suppose that \(x_0 \in \text{Int}(\pi_1(S))\), where \(\text{Int}(\cdot)\) is the interior of a set, and that \(U\) intersects \(p = 0\) transversally. If \(T\) is taken sufficiently large, then there exists a solution \((x(t,x_0),p(t,x_0))\) to (9) satisfying \(x(0,x_0) = x_0\) and \(p(T,x_0) = 0\). If, moreover,

\[
\det D_{x,p}x(t,x_0) \neq 0 \text{ for } t \in [0,T],
\]

then

\[
u^*(t) = -g(x(t,x_0))^\top p(t,x_0)
\]
is the local optimal solution for (OCP\(_1\)) and it holds that

\[
|\nu^*(t) - \bar{u}| + |x(t,x_0) - \bar{x}| \leq K \left[ e^{-\mu t} + e^{-\mu(T-t)} \right]
\]

(12)

for some constants \(K > 0\) and \(\mu > 0\) independent of \(T\).

Proof: Take a sufficiently large \(T > 0\). Then, Theorem 2.3 implies that there exist suitable \(p_0 \in \mathbb{R}^n\) and \(x_1 \in \mathbb{R}^n\) such that there exist a solution to (9) connecting \((x_0,p_0)\) and \((x_1,0)\). Also, there exist \(K' > 0\) and \(\mu > 0\) such that

\[
|x(t,x_0) - \bar{x}| + |p(t,x_0) - \bar{p}| \leq K'|e^{-\mu t} + e^{-\mu(T-t)}|
\]

for \(0 \leq t \leq T\).

Since \(|\nu^*(t) - \bar{u}| \leq \sup \|g(x)\| |p(t,x_0) - \bar{p}|\)

(12) holds with \(K = 2K'(1 + \sup \|g(x)\|)\), where supremum is taken along the trajectory. The condition (11) guarantees that there exists a Lagrangian submanifold in a neighborhood of this trajectory and this implies the existence of solution \(V(t,x)\) to (7)-(8) in the neighborhood. Then, the verification theorem in Dynamic Programming (see, e.g., [26]) shows that the control \(u^*\) is locally optimal.

Remark 3.1: The condition (11) guarantees that the solution \(V\) to (7) exists in a neighborhood of the trajectory \((x(t,x_0),p(t,x_0))\), \(0 \leq t \leq T\). The optimality of \(u^*\) is valid only in the neighborhood. This existence theory is described using the notion of Lagrangian submanifold (see, e.g., [22]) and when one seeks for larger domain of existence, the non-uniqueness issue of solution arises. We refer to [27] for general analysis of non-unique solutions and [28], [29], [30] for non-unique optimal controls for mechanical systems.

Next Corollary has been proved in [9], [14] in the study of the turnpike property for infinite dimensional systems. Their proofs are based on the estimates on adjoint variables in the linear Hamiltonian system (9) which is derived as a necessary condition of optimality. Here, we give an alternative proof using the geometric picture in Theorem 3.1.

Corollary 3.2: Suppose that the system (6) is linear, that is, \(f(x) = Ax\) and \(g(x) = B\) with real constant matrices \(A \in \mathbb{R}^{n \times n}\) and \(B \in \mathbb{R}^{n \times m}\). Under Assumption 2, (OCP\(_1\)) has the global solution \(u^*(t), 0 \leq t \leq T\) for any \(z \in \mathbb{R}^r\). Moreover, the Turnpike inequality (12) holds.

Proof: Let us first denote \(\text{Ham} = \begin{bmatrix} A & -BB^\top \\ -C^\top C & -A^\top \end{bmatrix}\) and employ the eigen structure analysis in [31] using a symplectic transform in [32], [31].

\[
\begin{bmatrix} I & L \\ P & PL + I \end{bmatrix} = \begin{bmatrix} A_c & 0 \\ 0 & -A_c^\top \end{bmatrix},
\]

(13)

where \(A_c := A - BB^\top P\) and \(P \geq 0\), \(L \leq 0\) are the solutions for the following Riccati and Lyapunov equations:

\[
PA + A^\top P - PBB^\top P + C^\top C = 0,
\]

\[
LA_c + A_cL = BB^\top.
\]

It should be noted that \(A_c\) is an asymptotically stable matrix and that \(\text{Ham}\) has no eigenvalues on the imaginary axis. The unique solution \((\bar{x}, \bar{p})\) to (SOC) is expressed as \(\bar{x} = -\text{Ham}^{-1} \begin{bmatrix} 0 \\ c \end{bmatrix}\). Then, \(U\) and \(S\) in (10) can be written as

\[
S = \{(u, P u) \mid u \in \mathbb{R}^n\} \cup \{(\bar{x}, \bar{p})\},
\]

\[
U = \{(Lu, (PL + I)u) \mid u \in \mathbb{R}^n\} \cup \{(\bar{x}, \bar{p})\}.
\]

It is readily seen that \(x_0 \in \text{Int}(\pi_1(S))\) for any \(x_0 \in \mathbb{R}^n\) since \(\pi_1(S) = \mathbb{R}^n\). Moreover, under Assumption 2, it can be shown that \(PL + I\) is nonsingular and therefore, \(U\) intersects \(p = 0\) transversally for any \(z \in \mathbb{R}^r\). As for the condition (11), we study the property of \(\Phi_{11}(t)\) in the transition matrix of \(\text{Ham}\):

\[
\exp[t\text{Ham}] = \begin{bmatrix} \Phi_{11}(t) & \Phi_{12}(t) \\ \Phi_{21}(t) & \Phi_{22}(t) \end{bmatrix}.
\]

It can be readily seen, using (13) that

\[
\Phi_{11}(t) = \exp[tA_c] x \left\{ I + \left( L - \exp[-tA_c]L \exp[-tA_c^\top]\right)P \right\}.
\]

It is possible to prove that \(\Phi_{11}(t)\) is nonsingular for \(t \geq 0\).

Remark 3.2: Although the problem in Corollary 3.2 is linear, it is not an easy task to explicitly write down the solution for (7)-(8) except for \(z = 0\). This corollary, however, says that the solution globally exists.
B. The OCP with state variables specified at the terminal time

In this subsection, we consider an OCP for \((x, u)\) with arbitrarily specified terminal states. Let \(x_1 \in \mathbb{R}^n\) be given. Let us define cost functional
\[
J(u) = \frac{1}{2} \int_0^T x(t)^T C^T C x(t) + |u|^2 dt,
\]
and consider
\[
\text{(OCP)}: \quad \text{Find a control } u \in L^\infty(0, T; \mathbb{R}^m) \text{ such that } J(u) \text{ along } u \text{ is minimized over all } u \in L^\infty(0, T; \mathbb{R}^m) \text{ such that } x(T) = x_1.
\]

We impose an additional assumption on \(f\).

Assumption 3: \(f(0) = 0\).

Then, the corresponding steady optimization problem has a unique solution \((x, u) = (0, 0)\). The Hamilton-Jacobi equation associated with \((\text{OCP})_T\)
\[
V_t(x, t) + V_x(x, t) f(x)
- \frac{1}{2} V_x(x, t) g(x) g(x)^T V_x(x, t)^T + \frac{1}{2} x^T C^T C x = 0.
\]
The Hamiltonian in this case is
\[
H(x, p) = p^T f(x) - \frac{1}{2} p^T g(x) g(x)^T p + \frac{1}{2} x^T C^T C x,
\]
and the corresponding characteristic equation for \((14)\) is
\[
\dot{x}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x_i}, \quad i = 1, \ldots, n
\]
with \(x(0) = x_0\) and \(x(T) = 0\).

Under Assumptions 2 and 3, the Hamiltonian system \((15)\) can be written as
\[
\frac{d}{dt} \begin{bmatrix} x \\ p \end{bmatrix} = \text{Ham} \begin{bmatrix} x \\ p \end{bmatrix} + \text{higher order terms},
\]
where \(\text{Ham} = \begin{bmatrix} A & -B^T \\ -C^T & -A^T \end{bmatrix}\) with \(A = Df(0), B = g(0)\) and the origin is a hyperbolic equilibrium with \(n\) stable and \(n\) unstable eigenvalues. Let \(S\) and \(U\) be the stable and unstable manifolds of \((15)\) at the origin.

Theorem 3.3: Suppose that \(x_0 \in \text{Int}(\pi_1(S))\) and \(x_1 \in \text{Int}(\pi_1(U))\). If \(T > 0\) is taken sufficiently large, there exists a solution \((x(t, x_0), p(t, x_0))\) to \((15)\) satisfying \(x(0) = x_0\) and \(x(T) = x_1\). If, moreover,
\[
\det D_{x_0} x(t, x_0) \neq 0 \text{ for } t \in [0, T),
\]
then
\[
u^*(t) = -g(x(t, x_0))^T p(t, x_0)
\]
is the local optimal solution for \((\text{OCP})_T\) and the same turnpike inequality as \((12)\).

Proof: For \(T > 0\) sufficiently large, from Theorem 2.3, there exist \(p_0, p_1 \in \mathbb{R}^m\) such that a solution to \((15)\) connecting \((x_0, p_0)\) and \((x_1, p_1)\) exists. The rest of the proof is almost the same as Theorem 3.3.

Corollary 3.4: Let us impose the controllability of \((A, B)\) in Assumption 2 instead of the stabilizability. Then, for sufficiently small \(|x_0|\) and \(|x_1|\) and sufficiently large \(T\), the local optimal control exists and the turnpike inequality \((12)\) holds.

Proof: We again employ the eigen structure analysis \((13)\). The tangent spaces of \(S\) and \(U\) at the origin are written as
\[
T_0 S = \{(u, Pu) \mid u \in \mathbb{R}^n\},
T_0 U = \{(u, (PL + I)L^{-1})u) \mid u \in \mathbb{R}^n\}.
\]
The latter is obtained by showing, using the controllability of \((A, B)\), that \(L\) is strictly negative definite. Therefore, \(x_0 \in \text{Int}(\pi_1(S))\) and \(x_1 \in \text{Int}(\pi_1(U))\) for sufficiently small \(|x_0|, |x_1|\). It is seen that the condition \((16)\) holds for these \(|x_0|, |x_1|\) (making them smaller if necessary) from the analysis on \(\Phi_{11}(t)\) in the proof of Theorem 3.1.

Remark 3.3: The linear counterpart of Corollary 3.4 is in [3] where they use anti-stabilizing solution \(P_u\) for the Riccati equation. In this case, the turnpike holds for all \(x_0\) and \(x_1\). It can be shown that \(P_u = (PL + I)L^{-1}\).

Note that we do not need the observability condition. Corollary 3.4 is obtained in [5] using the Hamilton-Jacobi theory. Compared with their conditions, we use only the linear controllability and detectability which can be easily checked. The authors of [10] obtain similar results to Corollary 3.4 with more generalized terminal conditions.

IV. CONCLUSIONS

In this paper, using techniques from dynamical system theory such as invariant manifolds and the \(\lambda\)-Lemma, we showed that turnpike-like behavior naturally appears in hyperbolic dynamical systems. This is then applied to analyze Hamiltonian systems describing controlled trajectories to obtain sufficient conditions for optimal controls yielding the turnpike to exist. The framework proposed in the paper is geometric and an alternative to existing ones. Application of this approach to specific problems is the future work.

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