Decision Problems in Information Theory

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Abstract

Constraints on entropies are considered to be the laws of information theory. Even though the pursuit of their discovery has been a central theme of research in information theory, the algorithmic aspects of constraints on entropies remain largely unexplored. Here, we initiate an investigation of decision problems about constraints on entropies by placing several different such problems into levels of the arithmetical hierarchy. We establish the following results on checking the validity over all almost-entropic functions: first, validity of a Boolean information constraint arising from a monotone Boolean formula is co-recursively enumerable; second, validity of “tight” conditional information constraints is in $\Pi^0_3$. Furthermore, under some restrictions, validity of conditional information constraints “with slack” is in $\Sigma^0_2$, and validity of information inequality constraints involving max is Turing equivalent to validity of information inequality constraints (with no max involved). We also prove that the classical implication problem for conditional independence statements is co-recursively enumerable.

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1 Introduction

The study of constraints on entropies is a central topic of research in information theory. In fact, more than 30 years ago, Pippenger [40] asserted that constraints on entropies are the “laws of information theory” and asked whether the polymatroidal axioms form the complete laws of information theory, i.e., whether every constraint on entropies can be derived from the polymatroidal axioms. These axioms consist of the following three types of constraints: (1) $H(\emptyset) = 0$, (2) $H(X) \leq H(X \cup Y)$ (monotonicity), and (3) $H(X) + H(Y) \geq H(X \cap Y) + H(X \cup Y)$ (submodularity). It is known that the polymatroidal axioms are equivalent to Shannon’s basic inequalities, that is, to the non-negativity of the entropy, conditional entropy, mutual information, and conditional mutual information [17]. In a celebrated result published in 1998, Zhang and Yeung [52] answered Pippenger’s question negatively by finding a linear inequality that is satisfied by all entropic functions, but cannot be derived from the polymatroidal axioms.
Zhang and Yeung’s result became the catalyst for the discovery of other information laws that are not captured by the polymatroidal axioms (e.g., \cite{25, 34}). In particular, we now know that there are more elaborate laws, such as conditional inequalities, or inequalities expressed using max, which find equally important applications in a variety of areas. For example, implications between conditional independence statements of discrete random variables can be expressed as conditional information inequalities. In another example, we have recently shown that conjunctive query containment under bag semantics is at least as hard as checking information inequalities using max \cite{1}. Despite the extensive research on various kinds of information inequalities, to the best of our knowledge nothing is known about the algorithmic aspects of the associated decision problem: check whether a given information law is valid.

In this paper, we initiate a study of algorithmic problems that arise naturally in information theory, and establish several results. To this effect, we introduce a generalized form of information inequalities, which we call Boolean information constraints, consisting of Boolean combinations of linear information inequalities, and define their associated decision problems. Since it is still an open problem whether linear information inequalities, which are the simplest kind of information laws, are decidable, we focus on placing these decision problems in the arithmetical hierarchy, also known as the Kleene-Mostowski hierarchy \cite{41}. The arithmetical hierarchy has been studied by mathematical logicians since the late 1940s; moreover, it directly influenced the introduction and study of the polynomial-time hierarchy by Stockmeyer \cite{43}. The first level of the arithmetical hierarchy consists of the collection \( \Sigma^0_1 \) of all recursively enumerable sets and the collection \( \Pi^0_1 \) of the complements of all recursively enumerable sets. The higher levels \( \Sigma^0_n \) and \( \Pi^0_n \), \( n \geq 2 \), are defined using existential and universal quantification over lower levels. We prove a number of results, including the following.

1. Checking the validity of a Boolean information constraint arising from a monotone Boolean formula (in particular, a max information inequality) is in \( \Pi^0_1 \) (Theorem 7).
2. Checking the validity of a conditional information inequality whose antecedents are “tight” is in \( \Pi^0_3 \) (Corollary 11). “Tight” inequalities are defined in Section 4.2.2 and include conditional independence assertions between random variables.
3. Checking the validity of a conditional information inequality whose antecedents have “slack” and are group-balanced is in \( \Sigma^0_2 \) (Corollary 14).
4. Checking the validity of a group-balanced, max information inequality is Turing equivalent to checking the validity of an information inequality (Corollary 17).

While the decidability of linear information inequalities (the simplest kind considered in this paper) remains open, a separate important question is whether more complex Boolean information constraints are any harder. For example, some conditional inequalities, or some max-inequalities can be proven from a simple linear inequality, hence they do not appear to be any harder. However, Kaced and Romashchenko \cite{25} proved that there exist conditional inequalities that are essentially conditional, which means that they do not follow from a linear inequality. (We give an example in Equation (9).) We prove here that any conditional information inequality with slack is essentially unconditioned (Corollary 10; see also Equation (19)), and that any max-inequality also follows from a single linear inequality (Theorem 16).

A subtle complication involving these results is whether by “validity” it is meant that the given Boolean information constraint holds for the set of all entropic vectors over \( n \) variables, denoted by \( \Gamma^n \), or for its topological closure, denoted by \( \overline{\Gamma^n} \). It is well known that these two spaces differ for all \( n \geq 3 \). With the exception of (1) above, which holds for
Throughout this paper, vectors and tuples are denoted by bold-faced letters, and random variables are capital letters. For a random variable \( X \in \mathbb{R}^m \), its (binary) entropy is defined by

\[
H(X) \overset{\text{def}}{=} - \sum_{x \in D} p(x) \cdot \log p(x)
\]  

In this paper all logarithms are in base 2.

Fix a joint distribution over \( n \) finite random variables \( \mathbf{V} \overset{\text{def}}{=} \{ X_1, \ldots, X_n \} \). For each \( \alpha \subseteq \{1, \ldots, n\} \), let \( X_\alpha \) denote the random (vector-valued) variable \( (X_i : i \in \alpha) \). Define the set function \( h : 2^{\{n\}} \rightarrow \mathbb{R} \), by setting \( h(\alpha) \overset{\text{def}}{=} H(X_\alpha) \), for all \( \alpha \subseteq \{1, \ldots, n\} \). With some abuse, we blur the distinction between the set \( \{\alpha\} \) and the set of variables \( \mathbf{V} = \{X_1, \ldots, X_n\} \), and write \( H(X_\alpha), h(X_\alpha), \) or \( h(\alpha) \) interchangeably. We call the function \( h \) an entropic function, and also identify it with a vector \( h \overset{\text{def}}{=} (h(\alpha))_{\alpha \subseteq \{n\}} \in \mathbb{R}_+^{2^n} \), which is called an entropic vector. Note that most texts and papers on this topic drop the component \( h(\emptyset) \), which is always 0, leading to entropic vectors in \( \mathbb{R}_{+}^{2^n-1} \). We prefer to keep the \( \emptyset \)-coordinate to simplify notations. The implicit assumption \( h(\emptyset) = 0 \) is used through the rest of the paper.

The set of entropic functions/vectors is denoted by \( \Gamma_n^* \subset \mathbb{R}_+^{2^n} \). Its topological closure, denoted by \( \bar{\Gamma}_n^* \), is the set of almost entropic vectors (or functions). It is known [47] that \( \bar{\Gamma}_n^* \subseteq \Gamma_n^* \) for \( n \geq 3 \). In general, \( \Gamma_n^* \) is neither a cone nor convex, but its topological closure \( \bar{\Gamma}_n^* \) is a closed convex cone [47].

Every entropic function \( h \) satisfies the following basic Shannon inequalities:

\[
h(Y \cup X) \geq h(X) \quad \quad h(X) + h(Y) \geq h(X \cup Y) + h(X \cap Y)
\]
called monotonicity and submodularity respectively. Any inequality obtained by taking a positive linear combination of Shannon inequalities is called a Shannon-type inequality.

Throughout this paper we will abbreviate the union \( X \cup Y \) of two sets of variables as \( XY \). The quantities \( h(Y | X) \overset{\text{def}}{=} h(XY) - h(X) \) and \( I_h(Y; Z | X) \overset{\text{def}}{=} h(XY) + h(XZ) - h(XYZ) - h(X) \) are called the conditional entropy and the conditional mutual information respectively. It can be easily checked that \( h(Y | X) \geq 0 \) and \( I_h(Y; Z | X) \geq 0 \) are Shannon-type inequalities.

Remark 1. The established notation \( \Gamma_n^* \) [38, 51, 11] for the set of entropic vectors is unfortunate, because the star in this context does not represent the dual cone. We will continue to denote by \( \Gamma_n^* \) the set of entropic vectors (which is not a cone!), and use explicit parentheses, as in \( (\Gamma_n^*)^* \), to represent the dual cone.
3 Boolean information Constraints

Most of this paper considers the following problem: given a Boolean combination of information inequalities, check whether it is valid. However in Section 3, we briefly discuss the dual problem, namely, recognizing whether a given vector $\mathbf{h}$ is an entropic vector (or an almost entropic vector).

A Boolean function is a function $F : \{0, 1\}^m \rightarrow \{0, 1\}$. We often denote its inputs with variables $Z_1, \ldots, Z_m \in \{0, 1\}$, and write $F(Z_1, \ldots, Z_m)$ for the value of the Boolean function.

3.1 Problem Definition

A vector $\mathbf{c} \in \mathbb{R}^n$ defines the following (linear) information inequality:

$$\mathbf{c} \cdot \mathbf{h} = \sum_{\alpha \subseteq [n]} c_\alpha h(\mathbf{X}_\alpha) \geq 0. \quad (2)$$

The information inequality is said to be valid if it holds for all vectors $\mathbf{h} \in \Gamma_n^*; \text{ equivalently, } \mathbf{c} \text{ is in the dual cone, } \mathbf{c} \in \left(\Gamma_n^*\right)^\ast. \text{ By continuity, an information inequality holds } \forall \mathbf{h} \in \Gamma_n^* \iff \text{ it holds } \forall \mathbf{h} \in \Gamma_n^\ast. \text{ In 1986, Pippenger }[40] \text{ defined the "laws of information theory" as the set of all information inequalities, and asked whether all of them are Shannon-type inequalities. This was answered negatively by Zhang and Yeung in 1998 }[52]. \text{ We know today that several applications require more elaborate laws, such as max-inequalities and conditional inequalities. Inspired by these new laws, we define the following generalization.}$

Definition 2. To each Boolean function $F$ with $m$ inputs, and every $m$ vectors $\mathbf{c}_j \in \mathbb{R}^n, j \in [m]$, we associate the following Boolean information constraint:

$$F(\mathbf{c}_1 \cdot \mathbf{h} \geq 0, \ldots, \mathbf{c}_m \cdot \mathbf{h} \geq 0). \quad (3)$$

For a set $S \subseteq \mathbb{R}^n$, a Boolean information constraint is said to be $S$-valid if it holds for all $\mathbf{h} \in S$. Thus, we will distinguish between $\Gamma_n^*$-validity and $\Gamma_n^\ast$-validity. Unlike in the case of information inequalities, these two notions of validity no longer coincide for arbitrary Boolean information constraints in general, as we explain in what follows.

Definition 3. Let $F$ be a Boolean function. The entropic Boolean information constraint problem parameterized by $F$, denoted by $\text{EBIC}(F)$, is the following: given $m$ integer vectors $\mathbf{c}_j \in \mathbb{Z}^n, j \in [m]$, check whether the constraint \((3)\) holds for all entropic functions $\mathbf{h} \in \Gamma_n^*$. In the almost-entropic version, denoted by $\text{AEBIC}(F)$, we replace $\Gamma_n^*$ by $\Gamma_n^\ast$.

The inputs $\mathbf{c}_j, j \in [m]$, to these problems are required to be integer vectors in order for $\text{EBIC}(F)$ and $\text{AEBIC}(F)$ to be meaningful computational problems. Equivalently, one can require the inputs to be rational vectors $\mathbf{c}_j \in \mathbb{Q}^n, j \in [m]$.

Let $F$ be a Boolean function. $F$ can be written as a conjunction of clauses $F = C_1 \land C_2 \land \cdots$, where each clause is a disjunction of literals. Equivalently, a clause $C$ has this form:

$$(Z'_1 \land \cdots \land Z'_\ell) \Rightarrow (Z_1 \lor \cdots \lor Z_\ell) \quad (4)$$

where $Z'_1, \ldots, Z'_\ell, Z_1, \ldots, Z_\ell$ are distinct Boolean variables. Checking $\text{EBIC}(F)$ is equivalent to checking $\text{EBIC}(C)$, for each clause of $F$ (and similarly for $\text{AEBIC}(F)$); therefore and without loss of generality, we will assume in the rest of the paper that $F$ consists of a single clause \((4)\) and study the problem along these dimensions:

Conditional and Unconditional Constraints When $k = 0$ (i.e., when the antecedent is empty), the formula $F$ is monotone, and we call the corresponding Boolean information
constraint unconditional. When $k > 0$, the formula $F$ is non-monotone, and we call the corresponding constraint conditional.

**Simple and Max Constraints** When $k = 0$ and $\ell = 1$, then we say that $F$ defines a simple inequality: when $k = 0$ and $\ell > 1$, then we say that $F$ defines a max-inequality. The case when $\ell = 0$ and $k > 0$ is not interesting because $F$ is not valid, since the zero-vector $h = 0$ violates the constraint.

### 3.2 Examples and Applications

This section presents examples and applications of Boolean Function Information Constraints and their associated decision problems. A summary of the notations is in Fig. 1.

#### 3.2.1 Information Inequalities

We start with the simplest form of a Boolean information constraint, namely, the linear information inequality in Eq. (2), which arises from the single-variable Boolean formula $Z_1$. We will call the corresponding decision problem the information-inequality problem, denoted by $\text{IIP}$: given a vector of integers $c$, check whether Eq. (2) is $\Gamma_{c}^{\ast}$-valid or, equivalently, $\Gamma_{c}$-valid. Pippenger’s question from 1986 was essentially a question about decidability. Shannon-type inequalities are decidable in exponential time using linear programming methods, and software packages have been developed for this purpose [47, Chapter 13] (it is not known, however, if there is a matching lower bound in the complexity of this problem). Thus, if every information inequality were a Shannon-type inequality, then information inequalities would be decidable. However, Zhang and Yeung’s gave the first example of a non-Shannon-type information inequality [52]. Later, Matúš [34] proved that, when $n \geq 4$ variables, there exists infinitely many inequivalent non-Shannon entropic inequalities. More precisely, he proved that the following is a non-Shannon inequality, for every $k \geq 1$:

$$I_h(C; D|A) + \frac{k + 3}{2} I_h(C; D|B) + I_h(A; B) + \frac{k - 1}{2} I_h(B; C|D) + \frac{1}{k} I_h(B; D|C) \geq I_h(C; D)$$

(5)

This ruined any hope of proving decidability of information inequalities by listing a finite set of axioms. To date, the study of non-Shannon-type inequalities is an active area of research [30, 41, 49], and the question whether $\text{IIP}$ is decidable remains open.

Hammer et al. [20], showed that, up to logarithmic precision, information inequalities are equivalent to linear inequalities in Kolmogorov complexity (see also [20, Theorem 3.5]).
3.2.2 Max Information Inequalities

Next, we consider constraints defined by a disjunction of linear inequalities, in other words $(c_1 \cdot h \geq 0) \lor \cdots \lor (c_m \cdot h \geq 0)$, where $c_j \in \mathbb{R}^+$. This is equivalent to:

$$\max(c_1 \cdot h, c_2 \cdot h, \ldots, c_m \cdot h) \geq 0 \quad (6)$$

and, for that reason, we call them Max information inequalities and denote the corresponding decision problem by MaxIIP. As before, $\Gamma_n^*$-validity and $\Gamma_n^+$-validity coincide.

**Application to Constraint Satisfaction and Database Theory**

Given two finite structures $A$ and $B$, we write $\text{HOM}(A, B)$ for the set of homomorphisms from $A$ to $B$. We say that $B$ dominates structure $A$, denote by $A \preceq B$, if for every finite structure $C$, we have that $|\text{HOM}(A, C)| \leq |\text{HOM}(B, C)|$. The homomorphism domination problem asks whether $A \preceq B$, given $A$ and $B$. In database theory this problem is known as the query containment problem under bag semantics [13]. In that setting we are given two Boolean conjunctive queries $Q_1, Q_2$, which we interpret using bag semantics, i.e., given a database $D$, the answer $Q_1(D)$ is the number of homomorphisms $Q_1 \rightarrow D$ [28]. $Q_1$ is contained in $Q_2$ under bag semantics if $Q_1(D) \leq Q_2(D)$ for every database $D$. It is open whether the homomorphism domination problem is decidable.

Kopparty and Rossman [29] described a MaxIIP problem that yields a sufficient condition for homomorphism domination. In recent work [1] we proved that, when $B$ is acyclic, then that condition is also necessary, and, moreover, the domination problem for acyclic $B$ is Turing-equivalent to MaxIIP. Hence, any result on the complexity of MaxIIP immediately carries over to the homomorphism domination problem for acyclic $B$, and vice versa.

We illustrate here Kopparty and Rossman’s MaxIIP condition on a simple example. Consider the following two Boolean conjunctive queries: $Q_1() = R(u, v) \land R(v, w) \land R(w, u)$, $Q_2() = R(x, y) \land R(x, z)$; interpreted using bag semantics, $Q_1$ returns the number of triangles and $Q_2$ the number of V-shaped subgraphs. Kopparty and Rossman proved that $Q_1 \preceq Q_2$ follows from the following max-inequality:

$$\max\{2h(XY) - h(X) - h(XYZ), 2h(YZ) - h(Y) - h(XYZ), 2h(XZ) - h(Z) - h(XYZ)\} \geq 0 \quad (7)$$

3.2.3 Conditional Information Inequalities

A conditional information inequality has the form:

$$(c_1 \cdot h \geq 0 \land \cdots \land c_k \cdot h \geq 0) \Rightarrow c_0 \cdot h \geq 0 \quad (8)$$

Here we need to distinguish between $\Gamma_n^*$-validity and $\Gamma_n^+$-validity, and denote by ECII and AEII the corresponding decision problems. Notice that, without loss of generality, we can allow equality in the antecedent, because $c_j \cdot h = 0$ is equivalent to $c_j \cdot h \geq 0$. Suppose that there exist $\lambda_1 \geq 0, \ldots, \lambda_m \geq 0$ such that the inequality $c_0 \cdot h - (\sum \lambda_1 c_1) \cdot h \geq 0$ is valid; then Eq. (8) is, obviously, also valid. Kaced and Romashchenko [25] called Eq. (8) an essentially conditioned inequality if no such $\lambda_i$’s exist, and discovered several valid conditional inequalities that are essentially conditioned.

**Application to Conditional Independence**

Fix three set of random variables $X, Y, Z$. A conditional independence (CI) statement is a statement of the form $\phi = (Y \perp Z \mid X)$, and it asserts that $Y$ and $Z$ are independent conditioned on $X$. A CI implication is a statement $\phi_1 \land \cdots \land \phi_k \Rightarrow \phi_0$, where $\phi_i, i \in \{0, \ldots, k\}$ are CI statements. The CI implication
**problem** is: given an implication, check if it is valid for all discrete probability distributions. Since \((Y \perp\!\!\!\perp Z \mid X) \iff I_h(Y; Z \mid X) = 0 \iff -I_h(Y; Z \mid X) \geq 0\), the CI implication problem is a special case of ECIIP.

The CI implication problem has been studied extensively in the literature [30, 44, 18, 27]. Pearl and Paz [39] gave a sound, but incomplete, set of *graphoid axioms*. Studený [44] proved that no finite axiomatization exists, while Geiger and Pearl [18] gave a complete axiomatization for two restricted classes, called saturated, and marginal CIs. See [16, 21, 38] for some recent work on the CI implication problem. The decidability of the CI implication problem remains open to date.

Results in [25] imply that the following CI implication is essentially conditioned (see [27]):

\[
I_h(C; D|A) = I_h(C; D|B) = I_h(A; B) = I_h(B; C|D) = 0 \implies I_h(C; D) = 0
\]  

While a CI implication problem is an instance of an *entropic* conditional inequality, one can also consider the question whether a CI implication statement holds for all *almost entropic* functions; for example the implication (9) holds for all almost entropic functions. Kaced and Romashchenko [25] proved that these two problems differ, by giving examples of CI implications that hold for all entropic functions but fail for almost entropic functions.

3.2.4 Group-Theoretic Inequalities

There turns out to be a way to “rephrase” IIP as a decision problem in group theory; This was a wonderful result by Chan and Yeung [12] (see also [11]). A tuple \((G; G_1, \ldots, G_n)\) is called a *group system* if \(G\) is a finite group and \(G_1, \ldots, G_n \subseteq G\) are \(n\) subgroups. For any \(\alpha \subseteq \{n\}\), define \(G_\alpha := \bigcap_{i \in \alpha} G_i\); implicitly, we set \(G_\emptyset := G\). A vector \(c \subseteq \mathbb{R}^{2^n}\) defines the following *group-theoretic inequality*:

\[
\sum_{\alpha \subseteq \{n\}} c_{\alpha} \log \frac{|G|}{|G_\alpha|} \geq 0
\]  

\[\textbf{Theorem 4 (12).} \ An \ information \ inequality \ (2) \ is \ \Gamma^*_n\text{-valid if and only if the corresponding group-theoretic inequality (10) holds for all group systems } (G; G_1, \ldots, G_n).\]

In particular, a positive or negative answer to the decidability problem for IIP immediately carries over to the validity problem of group-theoretic inequalities of the form (10). We note that the group-theoretic inequalities considered here are different from the word problems in group, see e.g. the survey [35]; the undecidability results for word problems in groups do not carry over to the group-theoretic inequalities and, thus, to information inequalities.

3.2.5 Application to Relational Query Evaluation

The problem of bounding the number of copies of a graph inside of another graph has a long and interesting history [17, 4, 14, 36]. The subgraph homomorphism problem is a special case of the relational query evaluation problem, in which case we want to find an upper bound on the output size of a full conjunctive query. Using the entropy argument from [14], *Shearer’s lemma* in particular, Atserias, Grohe, and Marx [5] established a tight upper bound on the answer to a full conjunctive query over a database. Note that Shearer’s lemma is a Shannon-type inequality. Their result was extended to include functional dependencies and more generally degree constraints in a series of recent work in database theory [19, 2] [3]. All these results can be cast as applications of Shannon-type inequalities. For a simple example,
let $R(X,Y), S(Y,Z), T(Z,U)$ be three binary relations (tables), each with $N$ tuples, then their join $R(X,Y) \bowtie S(Y,Z) \bowtie T(Z,U)$ can be as large as $N^2$ tuples. However, if we further know that the functional dependencies $XZ \rightarrow U$ and $YU \rightarrow X$ hold in the output, then one can prove that the output size is $\leq N^{3/2}$, by using the following Shannon-type information inequality:

$$h(XY) + h(YZ) + h(ZU) + h(X|YU) + h(U|XZ) \geq 2h(XYZU)$$

(11)

While the tight upper bound of any conjunctive query can be proven using only Shannon-type inequalities, this no longer holds when the relations used in the query are constrained to satisfy functional dependencies. In that case, the tight upper bound can always be obtained from an information inequality, but Abo Khamis et al. [3] gave an example of a conjunctive query for which the tight upper bound requires a non-Shannon inequality.

### 3.2.6 Application to Secret Sharing

An interesting application of conditional information inequalities is secret sharing, which is a classic problem in cryptography, independently introduced by Shamir [42] and Blakley [7]. The setup is as follows. There is a set $P$ of participants, a dealer $d \notin P$, and an access structure $\mathcal{F} \subset 2^P$. The access structure is closed under taking superset: $A \in \mathcal{F}$ and $A \subseteq B$ implies $B \in \mathcal{F}$. The dealer has a secret $s$, from some finite set $K$, which she would like to share in such a way that every set $F \in \mathcal{F}$ of participants can recover the secret $s$, but every set $F \notin \mathcal{F}$ knows nothing about $s$. The dealer shares her secret by using a secret sharing scheme, in which she gives each participant $p \in P$ a share $s_p \in K_p$, where $K_p$ is some finite domain. The scheme is designed in such a way that from the tuple $(s_p)_{p \in P}$ one can recover $s$ if $F \in \mathcal{F}$, and conversely one cannot infer any information about $s$ if $F \notin \mathcal{F}$.

One way to formalize secret sharing uses information theory (for other formalisms, see [6]). We identify the participants $P$ with the set $[n-1]$, and the dealer with the number $n$. A secret sharing scheme on $P$ with access structure $\mathcal{F} \subset 2^P$ is a joint distribution on $n$ discrete random variables $(X_1, \ldots, X_n)$ satisfying:

(i) $H(X_n) > 0$

(ii) $H(X_n \mid X_F) = 0$ if $F \in \mathcal{F}$

(iii) $H(X_n \mid X_F) = H(X_n)$ if $F \notin \mathcal{F}$; equivalently, $I_H(X_n; X_F) = 0$.

Intuitively, $X_i$ denotes the share given to the $i$th participant, and $X_n$ is the unknown secret. It can be shown, without loss of generality, that (i) can be replaced by the assumption that the marginal distribution on $X_n$ is uniform [8], which encodes the fact that the scheme does not reveal any information about the secret $X_n$. Condition (ii) means one can recover the secret from the shares of qualified participants, while condition (iii) guarantees the complete opposite. A key challenge in designing a good secret sharing scheme is to reduce the total size of the shares. The only known [15] [10] [26] way to prove a lower bound on share sizes is to lower bound the information ratio $\max_{p \in P} \frac{H(X_p)}{H(X_n)}$. In order to prove that some number $\ell$ is a lower bound on the information ratio, we need to check that $\max_{i \in [n-1]} \{h(X_i) - \ell \cdot h(X_n)\} \geq 0$ holds for all entropic functions $h \in \Gamma_n^+$ satisfying the extra conditions (i), (ii), and (iii) above.

Equivalently, $\ell$ is a lower bound on the information ratio if and only if the following Boolean information constraint is $\Gamma_n^+$-valid:

$$\bigwedge_{F \in \mathcal{F}} (h(X_n \mid X_F) = 0) \wedge \bigwedge_{F \notin \mathcal{F}} (I_h(X_n; X_F) = 0) \implies (h(X_n) = 0) \vee \left[ \bigvee_{i \in [n-1]} \{h(X_i) \geq \ell \cdot h(X_n)\} \right]$$
4 Placing EBIC and AEBIC in the Arithmetical Hierarchy

What is the complexity of EBIC($F$) / AEBIC($F$)? Is it even decidable? As we have seen there are numerous applications of the Boolean Information Constraint problem, hence any positive or negative answer, even for special cases, would shed light on these applications. While their (un)decidability is currently open, in this paper we provide several upper bounds on their complexity, by placing them in the arithmetical hierarchy.

We briefly review some concepts from computability theory. In this setting it is standard to assume objects are encoded as natural numbers. A set $A \subseteq \mathbb{N}^k$, for $k \geq 1$, is *Turing computable*, or *decidable*, if there exists a Turing machine that, given $x \in \mathbb{N}^k$ decides whether $x \in A$. A set $A$ is *Turing reducible to $B$* if there exists a Turing machine with an oracle for $B$ that can decide membership in $A$. The *arithmetical hierarchy* consists of the classes of sets $\Sigma^0_n$ and $\Pi^0_n$ defined as follows. The class $\Sigma^0_n$ consists of all sets of the form $\{ x \mid \exists y_1 \forall y_2 \exists y_3 \cdots Q y_n R(x, y_1, \ldots, y_n) \}$, where $R$ is an $(n+1)$-ary decidable predicate, $Q = \exists$ if $n$ is odd, and $Q = \forall$ if $n$ is even. In a dual manner, the class $\Pi^0_n$ consists of sets of the form $\{ x \mid \forall y_1 \exists y_2 \forall y_3 \cdots Q y_n R(x, y_1, \ldots, y_n) \}$. Then $\Sigma^0_0 = \Pi^0_0$ are the decidable sets, while $\Sigma^0_1$ consists of the recursively enumerable sets, and $\Pi^0_1$ consists of the co-recursively enumerable sets. It is known that these classes are closed under union and intersection, but not under complements, and that they form a strict hierarchy, $\Sigma^0_n, \Pi^0_n \subsetneq (\Sigma^0_{n+1} \cap \Pi^0_{n+1})$. For more background, we refer to [11]. Our goal is to place the problems EBIC($F$), AEBIC($F$), and their variants in concrete levels of the arithmetical hierarchy.

4.1 Unconditional Boolean Information Constraints

We start by discussing unconditional Boolean information constraints, or, equivalently, a Boolean information constraint defined by a monotone Boolean formula $F$. The results here are rather simple; we include them only as a warmup for the less obvious results in later sections. Based on our discussion in Sections 3.2.1 and 3.2.2, we have the following result.

**Theorem 5.** If $F$ is monotone, then EBIC($F$) and AEBIC($F$) are equivalent problems.

Next, we prove that these problems are co-recursively enumerable, by using the following folklore fact. A *representable set of $n$ random variables* is a finite relation $\Omega$ with $N$ rows and $n+1$ columns $X_1, \ldots, X_n, p$, where column $p$ contains rational probabilities in $[0,1] \cap \mathbb{Q}$ that sum to 1. Thus, $\Omega$ defines $n$ random variables with finite domain and probability mass given by rational numbers. We denote $h^\Omega$ its entropic vector. By continuity of Eq.(1), we obtain:

**Proposition 6.** For every entropic vector $h \in \Gamma^*_n$ and every $\varepsilon > 0$, there exists a representable space $\Omega$ such that $\| h - h^\Omega \| < \varepsilon$.

The group-characterization proven by Chan and Yeung [12] implies a much stronger version of the proposition; we do not need that stronger version in this paper.

**Theorem 7.** Let $F$ be a monotone Boolean formula. Then EBIC($F$) (and, hence, AEBIC($F$)) is in $\Pi^0_1$, i.e., it is co-recursively enumerable.

**Proof.** Fix $F = Z_1 \lor \cdots \lor Z_m$ and $c_i \in \mathbb{Z}^{2^n}$, $i \in [m]$. We need to check:

$$\forall h \in \Gamma^*_n: \quad c_1 \cdot h \geq 0 \lor \cdots \lor c_m \cdot h \geq 0 \quad (12)$$

We claim that (12) is equivalent to:

$$\forall \Omega: \quad c_1 \cdot h^\Omega \geq 0 \lor \cdots \lor c_m \cdot h^\Omega \geq 0 \quad (13)$$
Obviously (12) implies (13), and the opposite follows from Prop. 6 if (12) fails on some entropic vector \( h \), then it also fails on some representable \( h^\Omega \) close enough to \( h \). Finally, (13) is in \( \Pi^0_1 \) because, the property after \( \forall \Omega \) is decidable, by expanding the definition of entropy in each condition \( c_i \cdot h^\Omega \geq 0 \), and writing the latter as \( \sum_j a_j \log b_j \geq 0 \), or, equivalently, \( \prod_j (b_j)^{a_j} \geq 1 \), where \( a_j, b_j \) are rational numbers, which is decidable.

### 4.2 Conditional Boolean Information Constraints

We now consider non-monotone Boolean functions, in other words, conditional information constraints [8]. Since \( \Gamma^*_n \) and \( \Gamma_n \)-validity no longer coincide, we study EBIC\( (F) \) and AEBIC\( (F) \) separately. The results here are non-trivial, and some proofs are included in the Appendix.

#### 4.2.1 The Entropic Case

Our result for EBIC\( (F) \) is restricted to the CI implication problem. Recall from Sec. 3.2.3 that this problem consists of checking whether an implication between statements of the form \( (Y \equiv Z | X) \) holds for all random variables with finite domain, and this is equivalent to checking whether a certain conditional inequality holds for all entropic functions. We prove that this problem is in \( \Pi^0_1 \) by using Tarski’s theorem of the decidability of the theory of reals with +, * [16].

\[ \text{Theorem 8.} \] The CI implication problem (Section 3.2.3) is in \( \Pi^0_1 \).

\[ \text{Proof.} \] Tarski has proven that the theory of reals with +, * is decidable. More precisely, given a formula \( \Phi \) in FO with symbols + and *, it is decidable whether that formula is true in the model of reals \( (\mathbb{R}, +, *) \); for example, it is decidable whether \( \exists x \exists y \exists z (x^2 + 3y \geq z \land (y^3 + yz \leq xy^2)) \) is true. We will write \( (\mathbb{R}, +, *) = \Phi \) to denote the fact that \( \Phi \) is true in the model of reals.

Consider a conditional inequality over a set of \( n \) joint random variables:

\[ I_h(Y_1; Z_1 | X_1) = 0 \wedge \cdots \wedge I_h(Y_k; Z_k | X_k) = 0 \Rightarrow I_h(Y; Z | X) = 0 \]

The following algorithm returns \textit{false} if the inequality fails on some entropic function \( h \), and runs forever if the inequality holds for all \( h \), proving that the problem is in \( \Pi^0_1 \):

\[ \begin{align*}
\text{= } & \text{ Iterate over all } N \geq 0. \text{ For each } N, \text{ do the following steps.} \\
\text{= } & \text{ Consider } n \text{ joint random variables } X_1, \ldots, X_n \text{ where each has outcomes in the domain } [N]; \text{ thus there are } N^n \text{ possible outcomes. Let } p_1, \ldots, p_N \text{ be real variables representing the probabilities of these outcomes.} \\
\text{= } & \text{ Construct a formula } \Delta \text{ stating “there exist probabilities } p_1, \ldots, p_N \text{ for these outcomes, whose entropy fails the conditional inequality”. More precisely, the formula consists of the following:} \\
\text{= } & \text{ Convert each conditional independence statement in the antecedent } I_h(Y_i; Z_i | X_i) = 0 \text{ into its equivalent statement on probabilities: } p(X_i Y_i Z_i) p(X_i) = p(X_i Y_i) p(X_i Z_i). \\
\text{= } & \text{ Replace each such statement with a conjunction of statements of the form } p(X_i = x, Y_i = y, Z_i = z) \cdot p(Y_i = x) = p(X_i = x, Y_i = y) \cdot p(X_i = x, Z_i = z), \text{ for all combinations of values } x, y, z. \text{ If } X_i, Y_i, Z_i \text{ have in total } k \text{ random variables, then there are } N^k \text{ combinations of values } x, y, z, \text{ thus we create a conjunction of } N^k \text{ equality statements.} \\
\end{align*} \]

\[ 3y \text{ is a shorthand for } y + y + y \text{ and } x \geq y \text{ is a shorthand for } \exists u(x = y + u^2). \]
Each marginal probability is a sum of atomic probabilities, for example \( p(X_i = x, Y_i = y) = p_{kl} + p_{kj} + \cdots \) where \( p_{kl}, p_{kj}, \ldots \) are the probabilities of all outcomes that have \( X_i = x \) and \( Y_i = y \). Thus, the equality statement in the previous step becomes the following formula: \( (p_{i1} + p_{i2} + \cdots)(p_{j1} + p_{j2} + \cdots) = (p_{k1} + p_{k2} + \cdots)(p_{l1} + p_{l2} + \cdots) \). There is one such formula for every combination of values \( x, y, z \); denote \( \Phi_i \) the conjunction of all these formulas. Thus, \( \Phi_i \) assures \( I_b(Y; Z_i|X_i) = 0 \).

- Let \( \Phi = \Phi_1 \land \cdots \land \Phi_k \). Let \( \Psi \) be the similar formula for the consequent: thus, \( \Psi \) asserts \( I_b(Y; Z|X) = 0 \).

- Finally, construct the formula \( \Delta \overset{\text{def}}{=} \exists \phi_1, \ldots, \exists \phi_N, (\Phi \land \neg \Psi) \).

- Check whether \( (\mathbb{R}, +, \cdot) \models \Delta \). By Tarski’s theorem this step is decidable.

- If \( \Delta \) is true, then return \( \text{false} \); otherwise, continue with \( N + 1 \).

### Tarski’s exponential function problem

One may attempt to extend the proof above from the CI implication problem to arbitrary conditional inequalities \([\mathbf{8}]\). To check if a conditional inequality is valid for all entropic functions, we can repeat the argument above: iterate over all domain sizes \( N = 1, 2, 3, \ldots \), and check if there exists probabilities \( p_1, \ldots, p_N \) that falsify the implication \((c_1 \cdot h \geq 0 \land \cdots \land c_k \cdot h \geq 0) \Rightarrow c_0 \cdot h \geq 0 \). The problem is that in order to express \( c_1 \cdot h \geq 0 \) we need to express the vector \( h \) in terms of the probabilities \( p_1, \ldots, p_N \). To apply directly the definition of entropy in \([\mathbf{1}]\) we need to use the log function, or, alternatively, the exponential function, and this takes us outside the scope of Tarski’s theorem. A major open problem in model theory, originally formulated also by Tarski, is whether decidability continues to hold if we augment the structure of the real numbers with the exponential function (see, e.g., \([\mathbf{32}]\) for a discussion). Decidability of the first-order theory of the reals with exponentiation would easily imply that the entropic conditional information inequality problem ECICP (not just the entropic conditional independence (CI) implication problem) is in \( \Pi^1_1 \), because every condition \( c \cdot h \geq 0 \) can be expressed using \(+, \cdot, \cdot\) and the exponential function, by simply expanding the definition of entropy in Equation \([\mathbf{1}]\).

#### 4.2.2 The Almost-Entropic Case

Suppose the antecedent of \([\mathbf{4}]\) includes the condition \( c \cdot h \geq 0 \). Call \( c \in \mathbb{R}^2 \) tight if \( c \cdot h \leq 0 \) is \( \Gamma_n \)-valid. When \( c \) is tight, we can rewrite \( c \cdot h \geq 0 \) as \( c \cdot h = 0 \). If \( c \) is not tight, then there exists \( h \in \Gamma_n \) such that \( c \cdot h > 0 \); in that case we say that \( c \) has slack. For example, all conditions occurring in CI implications are tight, because they are of the form \(-I_b(Y; Z|X) \geq 0\), and more conveniently written \( I_b(Y; Z|X) = 0 \), while a condition like \( 3h(X) - 4h(YZ) \geq 0 \) has slack. We extend the definition of slack to a set. We say that the set \( \{c_1, \ldots, c_k\} \subset \mathbb{R}^2 \) has slack if there exists \( h \in \Gamma_n \) such that \( c_i \cdot h > 0 \) for all \( i = 1, k \); notice that this is more restricted than requiring each of \( c_i \) to have slack. We present below results on the complexity of \( \text{AEBIC}(F) \) in two special cases: when all antecedents are tight, and when the set of antecedents has slack. Both results use the following theorem, which allows us to move one condition \( c_k \cdot h \geq 0 \) from the antecedent to the consequent:
Theorem 9. The following statements are equivalent:

\[ \forall h \in \Gamma_n^\dagger: \quad \bigwedge_{i \in \{k\}} c_i \cdot h \geq 0 \Rightarrow c \cdot h \geq 0 \]  \hspace{1cm} (14)

\[ \forall \varepsilon > 0, \exists \lambda \geq 0, \forall h \in \Gamma_n^\dagger: \quad \bigwedge_{i \in \{k\}} c_i \cdot h \geq 0 \Rightarrow c \cdot h + \varepsilon h([n]) \geq \lambda c_k \cdot h \]  \hspace{1cm} (15)

Moreover, if the set \( \{c_1, \ldots, c_k\} \) has slack, then one can set \( \varepsilon = 0 \) in Eq. (15).

Proof. We prove here only the implication from (15) to (14); the other direction is non-trivial and is proven in Appendix B using only the properties of closed convex cones. Assume condition (15) holds, and consider any \( h \in \Gamma_n \) s.t. \( \wedge_{i \in \{k\}} c_i \cdot h \geq 0 \). We prove that \( c \cdot h \geq 0 \).

For any \( \varepsilon > 0 \), condition (15) states that there exists \( \lambda > 0 \) such that \( c \cdot h + \varepsilon h([n]) \geq \lambda c_k \cdot h \) and therefore \( c \cdot h \geq 0 \). Since \( \varepsilon > 0 \) is arbitrary, we conclude that \( c \cdot h \geq 0 \), as required.

By applying the theorem repeatedly, we can move all antecedents to the consequent:

Corollary 10. Condition (14) is equivalent to:

\[ \forall \varepsilon > 0, \exists \lambda_1 \geq 0, \ldots, \exists \lambda_k \geq 0, \forall h \in \Gamma_n^\dagger: \quad c \cdot h + \varepsilon h([n]) \geq \sum_{i \in \{k\}} \lambda_i c_i \cdot h \]  \hspace{1cm} (16)

Moreover, if the set \( \{c_1, \ldots, c_k\} \) has slack, then one can set \( \varepsilon = 0 \) in Eq. (16).

Antecedents Are Tight. We consider now the case when all antecedents are tight, a condition that can be verified in \( \Pi_3^0 \), by Th. 7. In that condition, Eq. (14) is equivalent to:

\[ \forall p \in \mathbb{N}, \exists q \in \mathbb{N}, \forall h \in \Gamma_n^\dagger: \quad c \cdot h + \frac{1}{p} h([n]) \geq q \sum_{i \in \{k\}} c_i \cdot h \]  \hspace{1cm} (17)

Indeed, the non-trivial direction (16) \( \Rightarrow \) (17) follows by setting \( q \overset{\text{def}}{=} \max(\lambda_1, \ldots, \lambda_k) \in \mathbb{N} \) and noting that \( c_i \) is tight, hence \( c_i \cdot h \leq 0 \) and therefore \( \lambda_i c_i \cdot h \geq q c_i \cdot h \).

Corollary 11. Consider a conditional inequality (8). If all antecedents are tight, then the corresponding decision problem AECIIP is in \( \Pi_3^0 \).

Proof. Based on our discussion, the inequality (8) is equivalent to condition (17), which is of the form \( \forall p \exists q \forall h \). Replace \( h \) with a representable entropic vector \( h^\Omega \), as in the proof of Theorem 7, and it becomes \( \forall p \exists q \forall h^\Omega \), placing it in \( \Pi_3^0 \).

Recall that the implication problem for CI is a special case of a conditional inequality with tight antecedents. We have seen in Theorem 8 that the entropic version of the CI implication problem is in \( \Pi_3^0 \); Corollary 11 proves that the almost entropic version is in \( \Pi_3^0 \).

Consider any conditional inequality (8) where the antecedents are tight. If this inequality holds for all almost entropic functions, then it can be proven by proving a family of (unconditional) inequalities (17). In fact, some conditional inequalities in the literature have been proven precisely in this way. For example, consider the CI implication (9) (Sec. 3.2.3), and replace each antecedent \( I_h(Y; Z|X) = 0 \) with \( -I_h(Y; Z|X) \geq 0 \). By Eq. (17), the following condition holds: \( \forall p \in \mathbb{N}, \exists q \in \mathbb{N} \) such that

\[ q(I_h(C; D | A) + I_h(C; D | B) + I_h(A; B) + I_h(B; C | D)) + \frac{1}{p} h(ABCD) \geq I_h(C; D) \]  \hspace{1cm} (18)
Thus, in order to prove (18), it suffices to prove (19). Matúš’s inequality [5] provides precisely the proof of (19) (by setting \( k \overset{\text{def}}{=} p, q \overset{\text{def}}{=} \max\left(\frac{k+2}{2}, 1\right) \), and observing that \( I_s(B; D \mid C) \leq h(ABC D) \).

**Antecedents Have Slack** Next, we consider the case when the antecedents have slack, which is a recursively enumerable condition. In that case, condition (16) is equivalent to:

\[
\exists \lambda_1 \geq 0, \ldots, \exists \lambda_k \geq 0, \forall h \in \Gamma_n^*:\quad c \cdot h \geq \sum_{i\in[k]} \lambda_i c_i \cdot h
\]  

In other words, we have proven the following result of independent interest: any conditional implication with slack is essentially unconditioned. However, we cannot immediately use (19) to prove complexity bounds for \( \text{AEBIC}(F) \), because the \( \lambda_i \)'s in (19) are not necessarily rational numbers. When we derived Eq. (17) we used the fact that the antecedents are tight, hence \( c_i \cdot h \leq 0 \), hence we could replace the \( \lambda_i \)'s with some natural number \( q \) larger than all of them. But now, the sign of \( c_i \cdot h \) is unknown. We prove below that, under a restriction called group balance, the \( \lambda_i \)'s can be chosen in \( \mathbb{Q} \), placing the decision problem in \( \Sigma_2^p \). Group balance generalizes Chan’s notion of a balanced inequality, which we review below.

In Appendix C we give evidence that some restriction is necessary to ensure the \( \lambda_i \)'s are rationals (Example 28), and also show that every conditional inequality can be strengthened to be group balanced (Prop 29).

A vector \( h \in \mathbb{R}^{n_a} \) is called modular if \( h(X) + h(Y) = h(X \cup Y) + h(X \cap Y) \) for all sets of variables \( X, Y \subseteq V \). Every non-negative modular function is entropic [47], and is a non-negative linear combination of the basic modular functions \( h^{(1)}, \ldots, h^{(n)} \), where \( h^{(1)}(\alpha) \overset{\text{def}}{=} 1 \) when \( j \in \alpha \) and is \( h^{(j)}(\alpha) \overset{\text{def}}{=} 0 \) otherwise. Chan [22] called an inequality \( c \cdot h \geq 0 \) balanced if \( c \cdot h^{(j)} = 0 \) for every \( j \in [n] \). He proved that any valid inequality can be strengthened to a balanced one. More precisely: \( c \cdot h \geq 0 \) is valid if \( c \cdot h^{(j)} \geq 0 \) for all \( i \in [n] \) and \( c \cdot h - \sum_i (c \cdot h^{(j)}) h(X_i \mid X_{[n]} \setminus \{i\}) \geq 0 \) is valid; notice that the latter inequality is balanced. For example, \( h(XY) + h(XZ) - h(X) - h(XYZ) \geq 0 \) is balanced, while \( h(XY) - h(X) \geq 0 \) is not balanced, and can be strengthened to \( h(XY) - h(X) - h(Y) \geq 0 \). We generalize Chan’s definition:

**Definition 12.** Call a set \( \{d_1, \ldots, d_k\} \subseteq \mathbb{R}^{n_a} \) group balanced if (a) \( \text{rank}(A) = k - 1 \) where \( A \) is the \( k \times n \) matrix \( A_{ij} = d_i \cdot h^{(j)} \), and (b) there exists a non-negative modular function \( h^{(\alpha)} \neq 0 \) such that \( d_i \cdot h^{(\alpha)} = 0 \) for all \( i \).

If \( k = 1 \) then \( \{d_1\} \) is group balanced iff \( d_1 \) is balanced, because the matrix \( A \) has a single row \( (d \cdot h^{(1)} - d \cdot h^{(n)}) \), and its rank is 0 iff all entries are 0. We prove in Appendix C

**Theorem 13.** Consider a group balanced set of \( n \) vectors with rational coefficients, \( D = \{d_1, \ldots, d_n\} \subseteq \mathbb{Q}^{n_a} \). Suppose the following condition holds:

\[
\exists \lambda_1 \geq 0, \ldots, \exists \lambda_n \geq 0, \sum_{i\in[n]} \lambda_i = 1, \forall h \in \Gamma_n^*:\quad \sum_{i\in[n]} \lambda_i d_i \cdot h \geq 0
\]  

Then there exists rational \( \lambda_1, \ldots, \lambda_k \geq 0 \) with this property.

This implies that, if \( c_1, \ldots, c_k \) have slack and \( \{c, -c_1, \ldots, -c_k\} \) is group balanced, then there exist rational \( \lambda_i \)'s for inequality (19). In particular:

**Corollary 14.** Consider a conditional inequality (8). If the antecedents have slack and \( \{c, -c_1, \ldots, -c_k\} \) is group balanced, then the corresponding decision problem is in \( \Sigma_2^p \).

We end this section by illustrating with an example:
A proof of the decidability of
we verify that the matrix $2$
verify that

4.3 Discussion on the Decidability of MaxIIP

A proof of the decidability of MaxIIP would immediately imply that the domination problem $A \leq B$ for acyclic structures $B$ is also decidable $[1]$. It is currently open whether MaxIIP is decidable, or even if the special case IIP is decidable. But what can we say about the domination problem if IIP were decidable? Theorem $7$ only says that both problems are in $IIP^0$, and does not tell us anything about MaxIIP if IIP were decidable. We prove here that, the decidability of IIP implies the decidability of group-balanced MaxIIP. We start with a result of general interest, which holds even for conditional Max-Information constraints.

\begin{theorem}
The following two statements are equivalent:

\begin{align}
\forall h \in \Gamma_n^* : & \quad \bigwedge_{i \in [k]} c_i \cdot h \geq 0 \implies \bigvee_{j \in [m]} d_j \cdot h \geq 0 \quad (22) \\
3\lambda_1, \ldots, \lambda_m \geq 0, \sum_j \lambda_j = 1, \forall h \in \Gamma_n^* : & \quad \bigwedge_{i \in [k]} c_i \cdot h \geq 0 \implies \sum_{j \in [m]} \lambda_j d_j \cdot h \geq 0 \quad (23)
\end{align}

\end{theorem}

The theorem says that every max-inequality is essentially a linear inequality. The proof of $(23) \implies (22)$ is immediate; we prove the reverse in Appendix $D$. As before, we don’t know whether these coefficients $\lambda_i$ can be chosen to be rational numbers in general, but by Theorem $13$ this is the case when $\{c_1, \ldots, c_k\}$ is group-balanced, and this implies:

\begin{corollary}
The MaxIIP problem where the inequalities $c_1, \ldots, c_n$ are group balanced is Turing equivalent to the IIP problem.
\end{corollary}

\begin{proof}
We describe a Turing reduction from MaxIIP to IIP. Consider a MaxIIP problem, $\bigvee_{j \in [m]} (c_j \cdot h \geq 0)$. We run two computations in parallel. The first computation iterates over all representable spaces $\Omega$, and checks whether $\Lambda_j(c_j \cdot h^{\Omega} < 0)$; if we find such a space then

\begin{footnote}
Where $h(X)$ denotes the basic modular function at $X$, i.e. $h(X)(X) = 1, h(X)(Y) = h(X)(Z) = 0$.
\end{footnote}
we stop and we return false. If the inequality is invalid then this computation will eventually terminate because in that case there exists a representable counterexample \( \Omega \). The second computation iterates over all \( m \)-tuples of natural numbers \( (\lambda_1, \ldots, \lambda_m) \in \mathbb{N}^m \) and checks \( \forall h \in \Gamma_n, \sum \lambda_j c_j \cdot h \geq 0 \) by using the oracle for \( \Pi \Pi \Pi \): if it finds such \( \lambda_j \)'s, then it stops and returns true. If the inequality is valid then this computation will eventually terminate, by Theorems \[16\] and \[13\].

We illustrate with an example.

\[\textbf{Example 18.}\] Consider Kopparty and Rosman’s inequality \([7]\), which can be stated as \(\max(c_1, c_2, c_3) \geq 0\), where \(c_1, c_2, c_3\) define the three expressions in \([7]\). To prove that it is valid, it suffices to prove that their sum is \(\geq 0\); we show this briefly here\[^3\]

\[
(2h(XY) - h(X)) + (2h(YZ) - h(Y)) + (2h(XZ) - h(Z)) - 3h(XYZ) \\
= (h(XY) + h(YZ) + h(XZ)) + (h(XY) - h(X)) + (h(YZ) - h(Y)) + (h(XZ) - h(Z)) \\
- 3h(XYZ) \\
\geq (h(XY) + h(YZ) + h(XZ)) + (h(XYZ) - h(XZ)) + (h(XY Z) - h(XY)) \\
+ (h(XYZ) - h(YZ)) - 3h(XYZ) = 0
\]

Theorem \[16\] proves that any max-inequality necessarily follows from such a linear inequality; we just have to find the right \(\lambda_j\)'s. In this example, the set \(c_1, c_2, c_3\) is group balanced (as we showed in Example \[15\]), therefore there exists rational \(\lambda_j\)'s; indeed, our choice here is \(\lambda_1 = \lambda_2 = \lambda_3 = 1\).

5 \ The Recognizability Problems

We study here two problems that are the dual of the Boolean information constraint problem. The \textit{entropic-recognizability problem} takes as input a vector \(h\) and checks if \(h \in \Gamma_n\). The \textit{almost-entropic-recognizability problem} checks if \(h \in \Gamma_n\). We will prove that the latter is in \(\Pi_2^P\), and leave open the complexity of the former.

Before we define these problems formally, we must first address the question of how to represent the input \(h\). One possibility is to represent \(h\) as a vector of rational numbers, but this is unsatisfactory, because usually entropies are not rational numbers. Instead, we will allow a more general representation. To justify it, assume first that \(h\) were given by some representable space \(\Omega\) (Sec. \[4.1\]), where all probabilities are rational numbers. In that case, every term \(p_i \log p_i\) in the definition of the entropy can be written as \(\log(p_i^{n_i})\), hence the quantity \(h(X)\) has the form \(h(X) = \log \prod_i p_i^{n_i}\). In general, any product \(\prod_i m_i^{n_i}\) where \(m_i, n_i \in Q\), for \(i = 1, \ldots, n\), can be rewritten as \((\frac{a}{b})\frac{1}{c}\), where \(a, b, c \in \mathbb{N}\). Indeed, writing \(m_i = u_i/v_i\) and \(n_i = s_i/t_i\) where \(u_i, v_i, s_i, t_i \in \mathbb{N}\), we have:

\[
\prod_i \left(\frac{u_i}{v_i}\right)^{t_i} = \prod_i \left(\frac{u_i}{v_i}\right)^{s_i} \cdot \prod_i \left(\frac{u_i}{v_i}\right) = \prod_i \left(\frac{u_i^{s_i}}{v_i^{s_i}}\right)^{t_i} = \left(\frac{a}{b}\right)^{\frac{1}{c}}
\]

\[a, b, c \in \mathbb{N}\]

Justified by this observation, we assume that the input to our problem consists of three vectors \((aX)Xv\), \((bX)Xv\), and \((cX)Xv\) in \(\mathbb{N}^3\), with the convention that \(h(X) \stackrel{\text{def}}{=} \frac{1}{X} \log \frac{aX}{X}\). Thus, we do not assume that these vectors come from a representable space \(\Omega\), we only assume their entropies can be represented in this form.

\[^3\] We apply submodularity: \(h(XY) - h(X) \geq h(XYZ) - h(XZ)\) etc.
Definition 19 ((Almost-)Entropic Recognizability Problem). Given natural numbers \((a_X)_{X \subseteq V}, (b_X)_{X \subseteq V}\), and \((c_X)_{X \subseteq V}\), check whether the vector \(h(X) \overset{\text{def}}{=} \frac{1}{c_X} \log \frac{a_X}{b_X}, X \subseteq V\), represents an entropic vector, or an almost-entropic vector.

Our result in this section is (see Appendix E for a proof):

Theorem 20. The almost entropic recognizability problem is in \(\Pi_0^0\).

We end with a brief comment on the complexity of the entropic-recognizability problem: given \(h\) (represented as in Def. 19) check if \(h \in \Gamma^*\). Consider the following restricted form of the problem: check if \(h\) is the entropic vector of a representable space \(\Omega\) (i.e. finite space with rational probabilities). This problem would remain in \(\Sigma_0^0\), because one can iterate over all representable spaces \(\Omega\) and check that their entropies are those required. However, in the general setting we ask whether any finite probability space has these entropies, not necessarily one with rational probabilities. This problem would remain in \(\Sigma_0^0\) if the theory of reals with exponentiation were decidable. Recall that Tarski’s theorem states that the theory of reals \(\text{FO}(\mathbb{R}, 0, 1, +, \ast)\) is decidable. A major open problem in model theory is whether the theory remains decidable if we add exponentiation. If that were decidable, then the entropic-recognizability problem would be in \(\Sigma_0^1\). To see this, consider the following semi-decision problem. Iterate over \(N = 1, 2, 3, \ldots\) and for each \(N\) check if there exists a probability space whose active domain has size \(N\) (thus, there are \(N^n\) outcomes, where \(n = |V|\) is the number of variables) and whose entropies are precisely those given. This statement that can be expressed using the exponential function (which we need in order to express the entropy as \(\sum_i p_i \log p_i\)). If there exists any finite probability space with the required entropies, then this procedure will find it; otherwise it will run forever, placing the problem in \(\Sigma_0^1\).

6 Discussion

CI Implication Problem The implication problem for Conditional Independence statements has been extensively studied in the literature, but its complexity remains an open problem. It is not even known whether this problem is decidable \([18, 37, 38]\). Our Theorem 8 appears to be the first upper bound on the complexity of the CI implication problem, placing it in \(\Pi_0^0\). Hannula et al. \([24]\) prove that, if all random variables are restricted to be binary random variables, then the CI implication problem is in EXPSPACE; the implication problem for binary random variables differs from that for general discrete random variables; see the discussion in \([18]\).

Finite, infinite, continuous random variables. In this paper, all random variables have a finite domain. There are two alternative choices: discrete random variables (possibly infinite), and continuous random variables. The literature on entropic functions has mostly alternated between defining entropic functions over finite random variables, or over discrete infinite random variables with finite entropy. For example discrete (possibly infinite) random variables are considered by Zhang and Yeung, \([51]\), by Chan and Yeung \([12]\), and by Chan \([22]\), while random variables with finite domains are considered by Matúš \([33, 34]\) and by Kaced and Romashchenko \([25]\). The reason for this inconsistency is that for information inequalities the distinction doesn’t matter: every entropy of a set of discrete random variables can be approximated arbitrarily well by the entropy of a set of random variables with finite domain, and Prop. 6 extends immediately to discrete random variables.\(^4\) However, the distinction

\(^4\) The idea of the proof relies on the fact that every entropy is required to converge, i.e. \(h(X_\alpha) =\)
is significant for conditional inequalities, and here the choice in the literature is always for finite domains. For example, the implication problem for conditional independence, i.e. the graphoid axioms, is stated for finite probability spaces by Geiger and Pearl [18], while Kaced and Romashchenko [25] also use finite distributions to prove the existence of conditional inequalities that hold over entropic but fail for almost-entropic functions. One could also consider continuous distributions, whose entropy is
\[
\int p(x) \log \left( \frac{1}{p(x)} \right) dx,
\]
where \( p \) is the probability density function. Chan [22] showed that an information inequality holds for all continuous distributions if it is balanced and it holds for all discrete distributions. For example, \( h(X) \geq 0 \) is not balanced, hence it fails in the continuous, because the entropy of the uniform distribution in the interval \([0, c]\) is \( \log c \), which is \(< 0 \) when \( c < 1 \).

**Strict vs. non-strict inequalities.** The literature on information inequalities always defines inequalities using \( \geq 0 \), in which case validity for entropic functions is the same as validity for almost entropic functions. One may wonder what happens if one examines strict inequalities \( c \cdot h > 0 \) instead. Obviously, each such inequality fails on the zero-entropic vector, but we can consider the conditional version \( h \neq 0 \Rightarrow c \cdot h > 0 \), which we can write formally as \( c \cdot h \leq 0 \Rightarrow h(V) \leq 0 \). This a special case of a conditional inequality as discussed in this paper. An interesting question is whether for this special case \( \Gamma_{n}^{*} \)-validity and \( \Gamma_{n}^{\dagger} \)-validity coincide; a negative answer would represent a significant extension of Kaced and Romashchenko’s result [25].

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\[ \sum p_i \log \frac{1}{p_i}, \text{ hence there exists a finite subspace of outcomes } \{1, 2, \ldots, N\} \text{ for which the sum is } \varepsilon \text{-close to } h(X_\alpha). \text{ The union of these spaces over all } \alpha \subseteq [n] \text{ suffices to approximate } h \text{ well enough.} \]
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We need the following non-obvious lemma due to Studený:

A Background on Cones

We will employ basic facts about closed, convex cones [9, 45], which we review briefly in this section.

Fix a set $S \subseteq \mathbb{R}^m$. We denote by $\bar{S}$ its topological closure. The set $S$ is convex if it is closed under taking convex combination, i.e. $x, y \in S$ and $\theta \in [0, 1]$ imply $\theta x + (1 - \theta) y \in S$. The set $S$ is a (Euclidean) cone if it is closed under taking non-negative multiple, i.e. $x \in S$ and $\theta \geq 0$ imply $\theta x \in S$. We use $\text{cone}(S)$ to denote its conic hull, i.e. the set of conic combinations $\sum \theta_i x_i$, where $\theta_i \geq 0$ and $x_i \in S$. It is easy to see that conic hulls are convex.

For any set $K \subseteq \mathbb{R}^m$, its dual cone $K^*$ is defined by

$$K^* := \{ y \mid x \cdot y \geq 0, \forall x \in K \} \tag{24}$$

If $x \in \mathbb{R}^m$ then we denote $x^* = \{ y \mid x \cdot y \geq 0 \}$ for short.

It is not hard to see that $K^*$ is always a closed, convex cone (regardless of whether $K$ is closed or convex or even a cone). For any two sets $K, L \subseteq \mathbb{R}^m$ it holds that $K \subseteq L^*$ iff $K^* \supseteq L$ (i.e. $(\cdot)^*$ forms an antitone Galois connection). It is also known that, taking duality twice, $K^{**}$ is the closure of the smallest convex cone containing $K$, i.e. $K^{**} = \text{cone}(K)$; in particular, $K^{**} = K$ iff $K$ is a closed convex cone.

A cone $K$ is called pointed if $x \in K$ and $-x \notin K$ imply $x = 0$; in other words, $K$ is pointed if it contains no line. If $K$ has a non-empty interior, then its dual $K^*$ is pointed. If $K$ is a cone and its closure is pointed, then $K^*$ has a non-empty interior. For any sets $K, L \subseteq \mathbb{R}^m$, it is easy to see that $(K \cup L)^* = K^* \cap L^*$. 

B Proof of Theorem 9

We need the following non-obvious lemma due to Studený:
Lemma 21. [45, Lemma 1] Let \( L \subseteq \mathbb{R}^m \) be a closed, convex cone, and \( y \in \mathbb{R}^m \) be a vector s.t. \( y \not\in L \). Then \( \text{cone}(L \cup \{y\}) \) is closed.

Example 22. The condition \( y \not\in L \) is necessary, as illustrated by the following example, also from [45]. Let

\[
L = \{(a, b, c) \mid a, b, c \in \mathbb{R}, a \geq 0, c \geq 0, ac \geq b^2\}
\]

One can check that this is a closed, convex cone. We start by proving that Condition (26) is equivalent to

\[
\forall x \in K : \bigwedge_{i \in [k]} (x \cdot y_i \geq 0) \Rightarrow (x \cdot y \geq 0)
\]

(26)

Fix a cone \( K \subseteq \mathbb{R}^m \). We say that a set of vectors \( \{y_1, \ldots, y_k\} \subseteq \mathbb{R}^m \) has slack w.r.t. \( K \) if there exists \( x \in K \) such that \( y_i \cdot x > 0 \) for \( i = 1, k \). We prove the following lemma, which is an extension of a result in [27]:

Lemma 23. Let \( K \subseteq \mathbb{R}^m \) be a closed convex cone. Then the following statements are equivalent:

\[
\forall x \in K : \bigwedge_{i \in [k]} (x \cdot y_i \geq 0) \Rightarrow (x \cdot y \geq 0)
\]

(26)

\[
\forall \varepsilon > 0, \exists \lambda \geq 0, \forall x \in K : \bigwedge_{i \in [k-1]} (x \cdot y_i \geq 0) \Rightarrow (x \cdot y + \varepsilon \|x\|_2 \geq \lambda x \cdot y_k)
\]

(27)

Moreover, if \( \{y_1, \ldots, y_k\} \) has slack w.r.t. \( K \) then we can set \( \varepsilon = 0 \) in Eq. (27).

Proof. The implication \( (27) \Rightarrow (26) \) is immediate, hence we prove \( (26) \Rightarrow (27) \). For every \( i \), the statement \( x \cdot y_i \geq 0 \) is equivalent to \( x \in y_i^\circ \). Denote \( L = K \cap \bigcap_{i \in [k-1]} y_i^\circ \) and notice that this is a closed, convex cone. We start by proving that Condition (26) is equivalent to \( y \in \text{cone}(L^* \cup \{y_k\}) \):

Condition (26) \( \iff \) \( L \cap y_i^\circ \subseteq y^\circ \)

\( \iff \) \( L^{**} \cap y_i^\circ \subseteq y^\circ \)

\( \iff \) \( (L^* \cup \{y_k\})^* \subseteq y^\circ \)

\( \iff \) \( y \in (L^* \cup \{y_k\})^{**} \)

\( \iff \) \( y \in \text{cone}(L^* \cup y_k) \)

Consider first the case when the set \( \{y_1, \ldots, y_k\} \) has slack in \( K \), and let \( x_0 \in K \) be such that \( x_0 \cdot y_i > 0 \) for all \( i = 1, k \); in particular, \( x_0 \in L \) and \( x_0 \cdot y_k > 0 \). This implies that \( y_k \not\in L^* \), hence, by Lemma 21 \( \text{cone}(L^* \cup \{y_k\}) \) is closed. Therefore we have that \( y \in \text{cone}(L^* \cup \{y_k\}) \), hence there exists \( z \in L^* \) and \( \lambda > 0 \) such that \( y = z + \lambda y_k \). To prove condition (27), it suffices to show that \( \forall x \in L, x \cdot y \geq \lambda x \cdot y_k \). This follows from the fact that \( x \cdot z \geq 0 \), which implies that \( x \cdot y = x \cdot z + \lambda x \cdot y_k \geq \lambda x \cdot y_k \) as required.

Consider now the general case, when \( y \in \text{cone}(L^* \cup \{y_k\})\). Then, for every \( \varepsilon > 0 \) there exists \( y' \in \text{cone}(L^* \cup \{y_k\}) \) such that, denoting \( \delta = y' - y \), we have \( \|\delta\|_2 < \varepsilon \). Applying the argument above to \( y' \) instead of \( y \), we obtain \( x \cdot y' \geq \lambda x \cdot y_k \). On the other hand, \( x \cdot y' = x \cdot y + x \cdot \delta \leq x \cdot y + \|x\|_2 \|\delta\|_2 \leq x \cdot y + \varepsilon \|x\|_2 \).

\(^{5}\) \( L \) is isomorphic to the cone \( S^2 \) of positive semi-definite symmetric \( 2 \times 2 \) matrices.
Example 24. We show that the error term $\varepsilon\|x\|_2$ in Condition (27) is necessary in general. For that, consider the cone $L$ in Eq. (25). It satisfies the condition $\forall(a,b,c) \in L: a \leq 0 \Rightarrow b \leq 0$. Indeed, $a \leq 0$ is equivalent to $a = 0$, thus $b^2 \leq ac = 0$ implying $b = 0$, in particular $b \leq 0$. By writing the implication as $-a \geq 0 \Rightarrow -b \geq 0$, Lemma 23 says that $\forall \varepsilon > 0, \exists \lambda \geq 0$ such that $\forall(a,b,c) \in L: -b + \varepsilon \| (a,b,c) \|_2 \geq \lambda (-a)$ or, equivalently $b \leq \varepsilon \| (a,b,c) \|_2 + \lambda a$. When $\varepsilon = 0$ then this condition fails for any choice of $\lambda \geq 0$, for example it fails on the vector $(1,1+\lambda, (1+\lambda)^2) \in L$. On the other hand, if $\varepsilon > 0$, then the condition holds, for example we can choose $\lambda = 1/\varepsilon$ and we obtain $\varepsilon \| (a,b,c) \|_2 + \lambda a \geq \varepsilon c + \lambda a \geq 2\sqrt{ac} \geq 2|b| \geq b$.

We can now prove Theorem 9.

Proof. (of Theorem 9) It remains to prove that (14) implies (15). This follows from Lemma 23 applied to $K \overset{\text{def}}{=} \Gamma_n^*$, which is a closed, convex cone, see 27. By re-writing the vectors $x, y_i, y$ in Lemma 23 to $h, c_i, c$ in Theorem 9 and using $\varepsilon/2^n$ instead of $\varepsilon$, we use the Lemma to argue that (14) implies:

$$\forall \varepsilon > 0, \exists \lambda \geq 0, \forall h \in \Gamma_n^*: \bigwedge_{i \in [k-1]} c_i \cdot h \geq 0 \Rightarrow c \cdot h + \frac{1}{2^n} \varepsilon \| h \|_2 \geq \lambda c_h \cdot h$$

Condition (15) follows from the fact that $h([n]) \geq \frac{1}{2^n} \sum_{\alpha \in [n]} h(\alpha) = \frac{1}{2^n} \| h \|_1 \geq \frac{1}{2^n} \| h \|_2$. ▶

We end this section by stating an obvious consequence of Lemma 23 by applying it repeatedly, we can move all antecedents to the consequent, and obtain:

Corollary 25. Let $K \subseteq \mathbb{R}^m$ be a closed convex cone, and assume that $\{y_1, \ldots, y_k\}$ has slack w.r.t. $K$. Then the following are equivalent:

\[
\forall x \in K: \bigwedge_{i \in [k]} (x \cdot y_i \geq 0) \Rightarrow (x \cdot y \geq 0) \tag{28}
\]

\[
\exists \lambda_1, \ldots, \lambda_k \geq 0, \forall x \in K: (x \cdot y \geq \sum_{i \in [k]} \lambda_i x \cdot y_i) \tag{29}
\]

C Proof of Theorem 13

We prove Theorem 13 by generalizing it to arbitrary cones. First, we give the obvious generalization of the notion of group balanced. Fix a set of vectors $x_1, \ldots, x_n \in \mathbb{Q}^m$ with rational coordinates.

Definition 26. A set $D = \{y_1, \ldots, y_k\} \subseteq \mathbb{R}^m$ is called group balanced if (a) $\text{rank}(A) = k - 1$ where $A$ is the matrix $A_{ij} = x_j \cdot y_i$, and (b) there exists $x^{(\ast)} \in \text{conv}(x_1, \ldots, x_n)$ such that $x^{(\ast)} \cdot y_i = 0$ for all $i$.

Theorem 27. Let $K \subseteq \mathbb{R}^m$ be a cone and $D = \{y_1, \ldots, y_n\} \subseteq \mathbb{Q}^m$ be a group-balanced set of rational vectors. Suppose that the following condition holds:

\[
\exists \lambda_1 \geq 0, \ldots, \lambda_n \geq 0, \sum_i \lambda_i = 1, \forall x \in K: \sum_i \lambda_i x \cdot y_i \geq 0 \tag{30}
\]

Then there exist rational $\lambda_i$’s with this property.

Proof. Denote by $\Lambda \overset{\text{def}}{=} \{(\lambda_1, \ldots, \lambda_n) \mid \lambda_1 \geq 0, \ldots, \lambda_n \geq 0, \forall x \in K, \sum_i \lambda_i x \cdot y_i \geq 0\}$. Then $\Lambda$ is a convex cone, and is $\neq \{0\}$ by condition (30). To prove the theorem, we will show $\Lambda = \{t\lambda \mid t \geq 0\}$ for some rational vector $\lambda \in \mathbb{Q}^n$. Denote the matrix $A \overset{\text{def}}{=} (y_i \cdot x_j)_{ij}$;
by assumption its rank is \(n - 1\). Def. 26 (b) implies that there exist \(\mu_1 \geq 0, \ldots, \mu_n \geq 0\) such that, denoting \(x^{(s)} \defeq \sum_{i \in [n]} \mu_i x_i\), it holds that \(x^{(s)} \cdot y_i = 0\) for all \(i\). Since \(\text{rank}(A) = n - 1\), we have \(\mu_j > 0\) for all \(j\). Consider now any \((\lambda_1, \ldots, \lambda_n) \in \Lambda\), and let \(y^{(s)} = \sum_{i \in [n]} \lambda_i y_i\). We prove that, for every \(j \in [n]\), \(x_j \cdot y^{(s)} = 0\). To prove this, we note that \(x^{(s)} \cdot y_i = 0\) and \(x_j \cdot y^{(s)} \geq 0\) (condition 30 applied to \(x_j\)) imply:

\[
0 = \sum_{i \in [n]} \lambda_i x^{(s)} \cdot y_i = \sum_{i,j \in [n]} \lambda_i \mu_j x_j \cdot y_i = \sum_{j \in [n]} \mu_j (x_j \cdot y^{(s)}) \geq 0
\]

If \(x_j \cdot y^{(s)} > 0\) for some \(j\) then we obtain \(0 > 0\), a contradiction, hence \(x_j \cdot y^{(s)} = 0\) for all \(j\). Let \(v_1, \ldots, v_n\) be the column vectors of \(A\), and let \(V \defeq \text{span}(v_1, \ldots, v_n)\). We have proven that \(\Lambda \subseteq V^\perp\), where \(V^\perp\) denotes the orthogonal space. Since \(\dim(V) = n - 1\), we have that \(\dim(V^\perp) = 1\), in other words \(V^\perp = \{t\lambda \mid t \in \mathbb{R}\}\) for some non-zero vector \(\lambda\). Moreover, \(\lambda\) can be chosen to be a rational vector, because \(v_1, \ldots, v_n\) have rational coordinates. This proves the claim and the theorem.

Next we give an example showing that some additional condition on the vectors \(y_1, \ldots, y_k\) is necessary to ensure that the values \(\lambda_j\)'s can be chosen to be rational numbers.

**Example 28.** We show here an example where the values \(\lambda_j\) cannot be chosen to be rational numbers. We generalize Example 22 as follows. Fix two numbers \(\alpha, \gamma \in (0, 1)\) such that \(\alpha + \gamma = 1\) and \(\frac{\alpha}{\gamma} \notin \mathbb{Q}\). Consider the cone:

\[
K \defeq \{(a, b, c) \mid a \geq 0, c \geq 0, a^\alpha c^\gamma \geq b\}
\]

(The reader may verify that \(K\) is a closed, convex cone.) We claim that, for all \((a, b, c) \in K\), either \(a \geq b\) or \(c \geq b\), in other words \(K\) satisfies the max-inequality:

\[
\forall (a, b, c) \in K: \quad \max(a - b, c - b) \geq 0
\]

This follows from the inequality between the weighted arithmetic mean and weighted geometric mean: \(\alpha a + \gamma c \geq a^\alpha c^\gamma \geq b = (\alpha + \gamma)b\), hence, if both \(a < b\) and \(c < b\) then \(\alpha a + \gamma c < (\alpha + \gamma)b\); a contradiction. By Theorem 31 there exist \(\lambda_1, \lambda_2 \geq 0\), \(\lambda_1 + \lambda_2 = 1\) such that

\[
\forall (a, b, c) \in K: \quad \lambda_1(a - b) + \lambda_2(c - b) \geq 0
\]

Equivalently, \(\forall (a, b, c) \in K, \lambda_1 a + \lambda_2 c \geq b\). We prove that the only possible values are \(\lambda_1 = \alpha, \lambda_2 = \gamma\), and thus no rational values exist. For that we set \(b = a^\alpha c^{1-\alpha}\) in the inequality above and obtain:

\[
\forall a, c \geq 0: \quad \lambda_1 a + (1 - \lambda_1)c \geq a^\alpha c^{1-\alpha}
\]

\[
\forall a \geq 0, c > 0: \quad \lambda_1 \frac{a^\alpha}{c} + 1 - \lambda_1 \geq \left(\frac{a}{c}\right)^\alpha
\]

\[
\forall x \geq 0: \quad f(x) \defeq (1 - x^\alpha) - \lambda_1(1 - x) \geq 0
\]

Since \(f(1) = 0\) and \(f(x) \geq 0\) for \(x \in \mathbb{R}\), we must have \(f'(1) = 0\) by Lagrange’s theorem. Thus, \(-\alpha + \lambda_1 = 0\) or \(\lambda_1 = \alpha\), proving the claim.

---

6 For any set \(V \subseteq \mathbb{R}^k\), its orthogonal space is \(V^\perp \defeq \{u \mid \forall v \in V, u \cdot v = 0\}\).
Finally, we prove that the definition of strong balance is natural. Recall Condition (32) of Theorem 13, which we repeat here for readability:

\[ 3\lambda_1 \geq 0, \ldots, \lambda_n \geq 0, \sum_{i \epsilon[n]} \lambda_i = 1, \forall h \in \mathfrak{T}^T_n : \sum_i \lambda_i d_i \cdot h \geq 0 \] (32)

\[ \Box \text{Proposition 29.} \text{ For any set of vectors} D = \{d_1, \ldots, d_n\} \subseteq \mathbb{R}^n \text{ there exists a strongly balanced set} D' = \{d'_1, \ldots, d'_n\} \subseteq \mathbb{R}^n \text{ such that} D \text{ satisfies Condition (32) iff} D' \text{ satisfies it.} \]

\[ \text{Proof.} \text{ (Of Prop. 29) Fix any set} D = \{d_1, \ldots, d_n\} \in \mathbb{R}^n, \text{ and denote by} A \text{ the} n \times n \text{ matrix} \]

\[ A_{ij} = d_i \cdot h^{(j)}. \text{ We will modify} D \text{ to ensure that the matrix} A \text{ satisfies Def. 12} \]

First, we replace each \( d_i \) by \( d'_i \) obtained using Chan’s transformation:

\[ d'_i \cdot h = d_i \cdot h - \sum_j (d_j \cdot h^{(j)})h(X_j \mid X_{[n]-\{j\}}) \]

and denote by \( D' \) def \( \{d'_1, \ldots, d'_n\} \). First, we claim that \( D \) satisfies (32) iff \( D' \) does. This follows from Chan’s theorem, since, for any values \( \lambda_1, \ldots, \lambda_n \) that sum to 1, the expression \( d'' = \sum_i \lambda_i d'_i \) is precisely Chan’s transformation applied to \( d' = \sum_i \lambda_i d'_i \), hence \( d' \cdot h \geq 0 \) is valid iff \( d'' \cdot h \geq 0 \) is valid. After this transformation, every \( d'_i \) is balanced, and therefore the matrix \( A' \) associated to the new set \( D' \) is identically 0. Next, assume that \( D' \) satisfies condition (32), and let \( \lambda_1, \ldots, \lambda_n \) be the corresponding coefficients. We have \( \lambda_i > 0 \) for all \( i \) because the rank of \( A \) is \( n-1 \). Replace each \( d'_i \) by \( d''_i \) where:

\[ d''_i \cdot h = d'_i \cdot h + \frac{1}{\lambda_i} \left( nh(X_i \mid X_{[n]-\{i\}}) - \sum_{j \neq i} h(X_j \mid X_{[n]-\{j\}}) \right) \]

In other words, we add the term \( \frac{1}{\lambda_i} h(X_i \mid X_{[n]-\{i\}}) \) and subtract all terms \( \frac{1}{\lambda_i} h(X_j \mid X_{[n]-\{j\}}) \) for \( j \neq i \). This transformation does not affect the expression (32) because \( \sum_i \lambda_i d''_i = \sum_i \lambda_i d'_i \).

The new \( n \times n \) matrix \( A'' \) is as follows. The diagonal is \( A_{ii} = (n-1)/\lambda_i \), and all other entries are \( A_{ij} = -1/\lambda_i \). To compute its rank, multiply each row \( i \) with \( \lambda_i \), and the new matrix has \( n-1 \) on the diagonal and 1 everywhere else, hence the rank is \( n-1 \). Finally, we check that there exists \( h^{(*)} \) such that \( d''_i \cdot h^{(*)} = 0 \) for all \( i \), by setting \( h^{(*)} = \sum_j h^{(j)} \); then \( d''_i \cdot h^{(*)} \) is precisely the sum of the elements in row \( i \) of the matrix \( A'' \), hence it is \( 0 \). \( \Box \)

D Proof of Theorem 16

We prove the theorem by generalizing it to arbitrary cones, in Th. 31 below. Its proof, in turn, is based on the following lemma.

\[ \Box \text{Lemma 30.} \text{ If} K \subseteq \mathbb{R}^m \text{ is a convex cone such that} K \cap (-\infty, 0)^m = \emptyset, \text{ then} K^* \cap [0]^m \neq \{0\}. \]

In other words, if a convex cone \( K \) satisfies following property

\[ \forall x \in K, \quad \bigvee_{j \in [m]} x_j \geq 0 \] (33)

then there exists \( y \) s.t. \( y_j \geq 0 \) for all \( j \in [m] \), and \( x \cdot y \geq 0 \) for all \( x \in K \).

\[ \text{Proof.} \text{ Let} e_1, \ldots, e_m \text{ be the canonical basis of} \mathbb{R}^m, \text{ i.e.} \ (e_j)_i = \delta_{ij}, \text{ and let} L = \text{cone}(K \cup \{e_1, \ldots, e_m\}). \text{ We claim that} L \text{ also satisfies (33). Indeed, every} x' \in L \text{ has the form} \]

\[ x' = x + \sum_i \theta_i e_i \text{ with} x \in K \text{ and} \theta_i \geq 0, \text{ } i = 1, m. \text{ If} x'_j < 0 \text{ for all} j, \text{ then} x_j < 0 \text{ for all} j, \text{ which is a contradiction because} K \text{ satisfies property (33). Thus,} L \text{ is a convex cone, and} \]
disjoint from the strictly negative quadrant \((-\infty, 0)^m\); since the latter is an open set, it is also disjoint from \(L\). We claim that \(L^* \neq \{0\}\). Indeed, otherwise \(L^{**} = \{0\}^* = \mathbb{R}^m\), but \(L^{**} = \overline{L}\) is disjoint from \((-\infty, 0)^m\), which is a contradiction. Therefore there exists \(y \in L^*\) s.t. \(y \neq 0\). Since \(e_i \in L\), we have \(y_i = y \cdot e_i \geq 0\), i.e. \(y \in \mathbb{R}^m_+\). Since \(K \subseteq L\) we have \(L^* \subseteq K^*\), hence \(y \in K^*\) proving the lemma.

We prove Theorem 16 by generalizing it to arbitrary convex cones.

\[ \text{Theorem 31. Let } S \subseteq \mathbb{R}^m \text{ be any convex cone, and } y_1, \ldots, y_k \in \mathbb{R}^m. \text{ Then the following two properties are equivalent:} \]

\[ \forall x \in S : \max(x \cdot y_1, \ldots, x \cdot y_k) \geq 0 \]  \hspace{1cm} (34)

\[ \exists \lambda_1 \geq 0, \ldots, \lambda_k \geq 0, \sum_{i=1}^k \lambda_i = 1, \forall x \in S : \sum_{i=1}^k \lambda_i x \cdot y_i \geq 0 \]  \hspace{1cm} (35)

Theorem 16 is the special case when \(S \overset{\text{def}}{=} \Gamma^* \cap \cap_{i \in [k]} c_i^*\).

\[ \text{Proof. (of Theorem 31). We prove only the implication from (34) to (35); the other direction is immediate. Define:} \]

\[ K \overset{\text{def}}{=} \{ (x \cdot y_1, \ldots, x \cdot y_k) \mid x \in S \} \]

Since \(S\) is a convex cone, it follows that \(K\) is also a convex cone, hence we can apply Lemma 30 to \(K\). Condition (22) of the theorem states:

\[ \forall z \in K : \bigvee_{j \in [m]} z_j \geq 0 \]

Lemma 30 implies that there exists \(\lambda \in \mathbb{R}^k_+ \cap K^*\) s.t. \(\lambda \neq 0\):

\[ \forall z \in K : \lambda \cdot z \geq 0 \]

Expanding the definition of \(K\), this condition becomes:

\[ \forall x \in S : \sum_{i=1}^k \lambda_i x \cdot y_i \geq 0 \]  \hspace{1cm} (36)

We can assume w.l.o.g. that \(\sum \lambda_i = 1\) (otherwise we normalize it by dividing by \(\sum \lambda_i\)), and this proves (35).

\[ \Box \]

E Proof of Theorem 20

To prove the theorem, we need to establish the following (separating-hyperplane-type) lemma:

\[ \text{Lemma 32. Let } h \in \mathbb{R}^2^n \text{ and suppose } h \notin \Gamma^*_n. \text{ Then there exists an information theoretic inequality with integral coefficients that is not satisfied by } h. \text{ In other words, there exists } c \in \mathbb{Z}^2^n, \text{ such that } \forall h_0 \in \Gamma^*_n, c \cdot h_0 \geq 0, \text{ and } c \cdot h < 0. \]

To prove the lemma, we review some background, following Studený [45].

\[ \text{Definition 33. Given a cone } L, \text{ define its plane as } pl(L) \overset{\text{def}}{=} L \cap (-L). \]

\[ \text{Fact 34. [45, Fact 9] If } L \text{ is a closed, convex cone, then } pl(L) \text{ is a vector space.} \]
Definition 35. [45, Def. 6] A closed convex cone \( L \subseteq \mathbb{R}^m \) is called regular if \( Q \) is dense in \( pl(L) \), i.e. \( Q^m \cap pl(L) = pl(L) \).

Lemma 36. [45, Prop. 3] If \( L \) is regular, then \( Q^m \) is dense in \( L^* \).

These lemmas allow us to prove the separating-plane lemma:

Proof. (of Lemma 32) We use Lemma 36. Every pointed cone \( L \) is regular, because \( pl(L) = \{0\} \). The cone of almost entropic functions \( K = \Gamma_n^* \) is pointed, because for every \( h \in K \) and every \( X \subseteq [n] \), \( h(X) \geq 0 \); thus, if \( h_1, h_2 \in \Gamma_n^* \) and \( h_1 + h_2 = 0 \), then \( h_1 = h_2 = 0 \). Therefore \( K \) is regular, hence \( Q^{2^n} \) is dense in \( K^* \). Consider a vector that is not almost-entropic, \( h \notin K \). Then there exists an information inequality \( c_0 \in K^* \) s.t. \( c_0 \cdot h < 0 \). Since \( Q \) is dense in \( K^* \), there exists a sequence in \( Q^{2^n} \cap K^* \) that converges to \( c \), and therefore there exists some \( c_1 \in Q^{2^n} \cap K^* \) s.t. \( c_1 \cdot h < 0 \). Multiply \( c_1 \) with the product of all \( 2^n \) denominators of its components, and obtain a vector \( c \in \mathbb{Z}^{2^n} \cap K^* \) s.t. \( c \cdot h < 0 \). This completes the proof.

And, finally, we use it to place the almost-entropic-recognizability problem in \( \Pi_{0}^{\Pi} \)

Proof. (of Theorem 20) Given \( c \in \mathbb{Z}^{2^n} \), let \( P(c) \) be the following predicate:

\[
P(c) : \quad \forall h \in \mathbb{R}^{2^n} \ (h \in \Gamma_n^* \Rightarrow c \cdot h \geq 0)
\]

Thus, \( P(c) \) checks if \( c \) defines a valid information inequality, and this, by Theorem 7, is in \( \Pi_1^0 \). By Lemma 32, the almost-entropic recognizability problem is as follows. Given \( h \in \mathbb{R}^{2^n} \) (represented as in Def. 19):

\[
h \in \Gamma_n^* \iff \forall c(P(c) \rightarrow c \cdot h \geq 0) \iff \forall c(\neg P(c) \lor c \cdot h \geq 0)
\]

which places the problem in \( \Pi_1^0 \) because \( \neg P(c) \) is in \( \Sigma_1^0 \).