SPECTROSCOPY OF STRINGY BLACK HOLE

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Abstract. The Maggiore’s method, which evaluates the transition frequency that appears in the adiabatic invariant from the highly damped modes, is used to investigate the entropy/area spectra of the Garfinkle–Horowitz–Strominger black hole (GHS-BH). We compute the resonance modes of the GHS-BH by using the confined scalar waves having high azimuthal quantum number. Although the area and entropy are characterized by the GHS-BH parameters, their quantizations are shown to be independent of those parameters. However, both spectra are equally spaced.

INTRODUCTION

The effects of gravity are very strong near the black holes (BHs) and quantum effects could not be ignored near the event horizon. Quantum gravity theory (QGT) [1] seeks to describe gravity according to the principles of quantum mechanics. One of the difficulties of formulating the QGT is that quantum gravitational effects only appear at length scales near the Planck scale, around $10^{-35}$ meter, a scale far smaller, and equivalently far larger in energy, than those currently accessible by high energy particle accelerators. Therefore, we physicists lack experimental data. The onset of the QGT dates back to the seventies. Hawking [2, 3] and Bekenstein [4, 5, 6, 7, 8] showed that a BH can be considered as a quantum mechanical system rather than a classical object.

The quantization of BHs was first proposed in the seminal works of Bekenstein [5, 6] in which the quantization procedure is based on the surface area of a BH that acts as a classical adiabatic invariant. According to the Ehrenfest principle [9], any classical adiabatic invariant should have a quantum entity with a discrete spectrum. Thus, Bekenstein [5, 6] suggested that the area of a quantum BH should have a discrete and equally spaced spectrum:

$$\mathcal{A}_n = \epsilon n \hbar \quad (n = 0, 1, 2, 3, \ldots),$$

(1)

where $\epsilon$ is an unknown “fudge” factor [10] and $\xi$ is of the order of unity. One can immediately deduce from Eq. (1) that the minimum increase in the area should be $\Delta \mathcal{A}_{\min} = \epsilon \hbar$. Moreover, Bekenstein conjectured that for the family Schwarzschild BHs (including the Kerr-Newman BH) the value of $\epsilon$ is $8\pi$ [6]. In the sequel, various methods were suggested to study the area spectrum of the BHs and to determine the value of the coefficient $\epsilon$ (for the topical review, a reader may refer to [11] and references therein). Among the methods proposed, Maggiore’s method (MM) [12] is the one that its result perfectly fits with the Bekenstein’s conjecture. In the MM, the adiabatic invariant quantity ($I_{adb}$) is given by [13, 14, 15, 16]

$$I_{adb} = \hbar \int \frac{T \, dS}{\Delta \omega},$$

(2)

in which $\Delta \omega = \omega_{n+1} - \omega_n$ stands for the transition frequency between the subsequent levels of a BH with temperature ($T$) and entropy ($S$). On the other hand, according to the Bohr-Sommerfeld quantization rule, $I_{adb}$ is quantized as $I_{adb} \simeq n \hbar$ as $n \to \infty$. To obtain $\Delta \omega$, Maggiore [12] embraced the BH as a damped harmonic oscillator having a characteristic frequency in the form of $\omega = (\omega_R^2 + \omega_I^2)^{1/2}$: $\omega_R$ and $\omega_I$ are the real and imaginary parts of the
frequency, respectively. For the ultrahigh dampings \((n \to \infty)\), \(\omega_I \gg \omega_R\) and therefore \(\Delta \omega \approx \Delta \omega_I\). Hod [17, 18] was the first physicist who considered the quasinormal modes (QNMs) or the so-called ringing modes for computing \(\Delta \omega\) and hence obtaining \(I_{adh}\). Then, many studies have used the MM to test the Bekenstein’s conjecture for various BH solutions (see for instance [19, 20, 21, 22, 23, 24, 25, 26, 27, 28]). Today, there are several methods to compute the QNMs, such as the WKB method, the phase integral method, continued fractions and direct integrations of the wave equation in the frequency domain [29].

In the present study, we study the entropy/area quantization (spectroscopy) of the GHS-BH [30]. GHS-BH is a solution to the low-energy limit of the string theory whose its action is obtained when the Einstein–Maxwell theory is expanded to involve a dilaton field \(\phi\). That is why physical properties of the GHS-BH is significantly different from the Reissner-Nordström BH. The spectroscopy problem of the GHS-BH was first studied by Wei et al. [31]. In the adiabatic invariant quantity (2), they used the ordinary QNMs of Chen and Jing [32] who obtained the associated QNMs by using the monodromy method [29]. Thus, it was shown that GHS-BH has an equidistant area spectrum at the high frequency modes. Later on, Sakalli and Gülnihal [33] reconsidered the problem of GHS-BH spectroscopy. However, instead of the ordinary QNMs they computed the “boxed QNMs” which are the characteristic resonance spectra of the scalar clouds. For that purpose, it was assumed that there exists a mirror or confining cavity surrounding the GHS-BH which is placed at a constant radial coordinate with a radius \(r_m\), which is very close to the horizon. The scalar field is imposed to be vanished at the mirror’s location (\(r_m\)), which requires to use both the Dirichlet and Neumann conditions. In that scenario, the radial wave equation was studied near the horizon region. Now, we want to reconsider the same problem with another scenario. As seen in the following sections, the effective potential generated from the massless Klein-Gordon equation (KGE) performs a barrier peak (BP) for the propagating scalar waves when the azimuthal quantum number \(l\) gets high values. As a result, the scalar waves are confined between the horizon and the BP. This will yield characteristic resonance modes (RMs) of the confined scalar fields in the GHS-BH geometry. To this end, the scalar field is imposed to be terminated at the BP and to be purely ingoing wave at the horizon. In fact, our method is similar to researches [34, 35, 36, 37] that are mainly inspired from the studies of [38, 39] in which the QNMs are computed using the poles of the scattering amplitude in the Born approximation. After reducing the radial KGE to the one-dimensional Schrödinger like wave equation [40], we show that it becomes a confluent hypergeometric differential equation [41] near the horizon. Imposing the relevant boundary condition and then using the pole feature of the Gamma function, we obtain the RMs of the GHS-BH. Using the highly damping RMs in the MM, we get the entropy/area spectra of the GHS-BH.

The paper is organized as follows. In Sec. 2, we introduce the GHS-BH metric and study the massless KGE on this geometry. We also show how the radial equation reduces to a one-dimensional Schrödinger like wave equation. Sec. 3 is devoted to the computation of the RMs of the GHS-BH. To this end, we show how the Schrödinger like wave equation reduces to a confluent hypergeometric differential equation. Then, we apply the MM to obtain the entropy/area spectra of the GHS-BH. We present our conclusions in Sec. 4. (Throughout this paper, we use the unit of \(c = G = k_B = 1\)).

**KGE ON GHS-BH GEOMETRY**

GHS-BH is the solution to the four-dimensional Einstein-Maxwell-dilaton low-energy action [30]. It has the following static and spherically symmetric metric:

\[
ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + g(r)d\Omega^2 ,
\]

(3)

where the metric functions are given by

\[
f(r) = \frac{r - 2M}{r},
\]

(4)

\[
g(r) = r^2 - 2ar,
\]

(5)

and

\[
a = \frac{Q^2}{2M} e^{-2\phi_0}.
\]

(6)
Physical quantities $Q$, $M$, and $\phi_0$ denote the magnetic charge, mass, and the asymptotic value (constant) of the dilaton field, respectively. $r_h = 2M$ corresponds to the event horizon of the GHS-BH. It is worth noting that in the GHS-BH geometry the dilaton field reads

$$e^{-2\phi} = \left(1 - \frac{2a}{r}\right)e^{-2\phi_0}. \quad (7)$$

On the other hand, the Maxwell field is given by

$$F = Q \sin \theta d\theta \wedge d\phi. \quad (8)$$

One can also get the electric charge case via the following duality transformations:

$$\tilde{F}_{\mu\nu} = \frac{e^{-2\phi}}{2} \epsilon^{\alpha\beta\mu\nu} F_{\alpha\beta}, \quad \text{and} \quad \phi \to -\phi. \quad (9)$$

The surface gravities [42] of the GHS-BH and the Schwarzschild BH are the same:

$$\kappa = \lim_{r \to r_h} \sqrt{-\frac{1}{2} \nabla_\mu \chi_\nu \nabla^\mu \chi^\nu} = \frac{1}{2} f'(r) \bigg|_{r=r_h} = \frac{1}{4M}, \quad (10)$$

where $\chi^\nu = [1, 0, 0, 0]$ is the timelike Killing vector and the prime symbol denotes the differentiation with respect to $r$. So, one can easily obtain the Hawking temperature of the GHS-BH as

$$T_H = \frac{\hbar \kappa}{2\pi} = \frac{\hbar}{8\pi M}. \quad (11)$$

On the other hand, the areal radii of the GHS-BH and the Schwarzschild BH are different. Thus, the area and entropy of the GHS-BH are not same with the Schwarzschild BH’s ones:

$$S = \frac{\mathcal{A}}{4\hbar} = \frac{\pi (r_+ - 2a) r_+}{\hbar}. \quad (12)$$

When $a = M$ (the extremal charge case: $Q = \sqrt{2}Me^{\phi_0}$), the GHS-BH’s area (and hence its entropy) vanishes. In fact, this extremal GHS-BH is indeed a naked singularity. Moreover, for having physical entropy $S \geq 0$, it is necessary to have $a \leq M$. The singularity of GHS-BH is null (unlike to the timelike singularity of Reissner-Nordström BH) and therefore outward radial null geodesics do not hit it (a reader may refer to [43]). The first law of thermodynamics for the GHS-BH takes the following form: $T_H dS = dM - UdQ$. Here, $U$ is the electric potential defined on the horizon: $U = aQ^{-1}$.

The massless KGE for the scalar field $\Psi$ is given by

$$\left(\sqrt{|g|}\right)^{-1} \partial_\nu \left(\sqrt{|g|} g^{\mu\nu} \partial_\mu \Psi\right) = 0. \quad (13)$$

We shall use the following ansatz for the scalar field $\Psi$:

$$\Psi = g(r)^{-1/2} H(r) e^{i\omega t} Y^m_l(\theta, \varphi), \quad \text{Re}(\omega) > 0, \quad (14)$$

where $\omega$ is the frequency of the propagating scalar wave. $Y^m_l(\theta, \varphi)$ represents the spheroidal harmonics with the eigenvalue $-l(l + 1)$ and magnetic quantum number $m$, respectively. Here, $l$ is the azimuthal quantum number, which is a type of quantum number defined for an orbital which determines its orbital angular momentum and also describes the shape of the orbital of a particle within the associated geometry. It is worth noting that orbitals can take even more complex shapes according to the higher values of $l$. A spherical orbit ($l = 0$) can be oriented in space in only one way. However orbital that has polar or cloverleaf shapes can point in different directions. To describe the orientation in space of a particular orbital, that is why one always needs the magnetic quantum number $m$.

After some algebra, the radial equation becomes

$$\left(-\partial_r^2 + V\right) H(r) - \left(\frac{\omega}{\hbar}\right)^2 H(r) = 0, \quad (15)$$
which is a one-dimensional Schrödinger type differential equation [40]. \( r^* \) represents the tortoise coordinate, which is defined as

\[
r^* = \int f(r)^{-1} dr.
\]

(16)

Evaluating integral of Eq. (16), we get

\[
r^* = r + r_h \ln \left( \frac{r - r_h}{r_h} \right).
\]

(17)

On the other hand, one can express \( r \) in terms of \( r^* \) as follows

\[
r = r_h [1 + \Omega (z)],
\]

(18)

where \( z = \exp \left( \frac{r - r_h}{r_h} \right) \) and \( \Omega (z) \) is the Lambert-W or the so-called omega function [44]. The tortoise coordinate has the following limits:

\[
\lim_{r \to r_h} r^* = -\infty, \quad \text{and} \quad \lim_{r \to \infty} r^* = \infty.
\]

(19)

The effective potential \( V(r) \) seen in Eq. (15) is found to be

\[
V(r) = f(r) \left[ \frac{l(l + 1)}{r(r - 2a)} + \frac{r_h(r - a)}{r^2(r - 2a)} - \frac{a^2 f(r)}{[r(r - 2a)]^2} \right].
\]

(20)

Fig. (1) exhibits the plot of \( V(r) \) versus \( r^* \) for various values of the azimuthal quantum number \( l \) with \( M = 1 \) and \( a = 0.5 \). It can be deduced from Fig. (1) that when \( l \)-parameter takes higher values, the effective potential tends to make a "BP" at a specific point which is among the event horizon and spatial infinity. This means that the scalar waves having very high azimuthal quantum number (\( l \gg 1 \)) that are not sufficiently energetic could not pass that BP and would be confined in a small region. To obtain those RMs, as being discussed in the following section, we shall make the analysis around the near horizon region of the GHS-BH.

SPECTROSCOPY ANALYSIS

In principle, Eq. (15) is solved for the QNMs with a particular set of boundary conditions: purely ingoing wave at the horizon and purely outgoing waves at the infinity. But unfortunately, Eq. (15) cannot be solved, analytically. Therefore to overcome this difficulty, one should take the help of some approximate method. In this section, we shall follow a particular method prescribed in [34, 35, 36, 37], which considers the wave dynamics in the vicinity of the event horizon. Since the effective potential (19) vanishes at the horizon \( (r^* \to -\infty) \) and makes a barrier for the scalar waves with high azimuthal quantum numbers \( (l \gg 1) \) at the intermediate region, therefore the RMs are defined to be those for which one has purely ingoing plane wave at the horizon and no wave at BP’s location: the latter condition is trivially satisfied. Namely, the relevant RMs should satisfy

\[
H(r)_{\text{RM}} \sim \begin{cases} e^{i \omega^*} & \text{at } r^* \to -\infty \\ 0 & \text{at } \text{BP} \end{cases}.
\]

(21)

The metric function \( f(r) \) can be expanded to series around the event horizon as follows

\[
f(r) \approx f'(r_+)(r-r_+) + \frac{f''(r_+)}{2}(r-r_+)^2 + O[(r-r_+)^3],
\]

\[
= 2\kappa y + \frac{f''(r_+)}{2} y^2 + O(y^3),
\]

(22)

where \( y = r - r_+ \). With this new variable, we apply the Taylor expansion around \( y = 0 \) to Eq. (20) and obtain the near horizon form of the effective potential as
Figure 1. Plot of $V$ versus $r^*$. The physical parameters are chosen to be $M = 1$ and $a = 0.5$. As $l$ gets bigger values, the potential barrier between the horizon and spatial infinity exhaustively increases.

$$V(y) \approx 2\kappa y \left[ l(l+1)(C + D y) + 2\kappa(G + H y) - 2\kappa y N \right],$$  \hspace{1cm} (23)

where the parameters are given by

$$C = \frac{1}{\varepsilon}, \quad D = \frac{-2x}{\varepsilon^2}, \quad G = \frac{x}{\varepsilon}, \quad H = \frac{-x^2 + a^2}{\varepsilon^2}, \quad N = \frac{a^2}{\varepsilon^2},$$  \hspace{1cm} (24)

with

$$x = r_+ - a, \quad \varepsilon = r_+(r_+ - 2a).$$  \hspace{1cm} (25)

Furthermore, the tortoise coordinate around the event horizon can be expressed as

$$r^* \approx \frac{1}{2\kappa} \ln y.$$  \hspace{1cm} (26)

Thus, near the horizon, the one-dimensional Schrödinger equation (15) behaves as follows

$$-4\kappa^2 y^2 \frac{d^2 H(y)}{dy^2} - 4\kappa^2 y \frac{dH(y)}{dy} + \left[ V(y) - \left( \frac{\omega^2}{\hbar} \right) \right] H(y) = 0.$$  \hspace{1cm} (27)

The above differential equation has two separate solutions, which can be expressed in terms of the Whittaker functions [41]. The solution can be transformed to the CH functions [41] and thus we have

$$H(y) \approx C_1 y^{\tilde{b}y} M(\tilde{a}, \tilde{b}, \tilde{c}y) + C_2 y^{\tilde{b}y} U(\tilde{a}, \tilde{b}, \tilde{c}y).$$  \hspace{1cm} (28)
The parameters of the above function are given by

\[ \tilde{a} = -\frac{i \gamma}{\sqrt{\kappa}} + \frac{\tilde{b}}{2}, \]
\[ \tilde{b} = 1 + i \frac{\omega}{\hbar \kappa}, \]
\[ \tilde{c} = i \frac{\lambda}{2 \sqrt{\kappa}}, \]  

(29)

where

\[ \gamma = 2 \kappa x + l(l + 1), \]
\[ \lambda = 4 \sqrt{x(l + 1) + \kappa (z + 3a^2)}. \]  

(30)

By using one of the transformations of the confluent hypergeometric functions [41], we obtain the near horizon \((y \ll 1)\) behavior of the solution (28) as

\[ H(y) \sim \left[ C_1 + C_2 \frac{\Gamma(1 - \tilde{b})}{\Gamma(1 + \tilde{a} - \tilde{b})} \right] y^{\frac{\tilde{a}}{2}} + C_2 \frac{\Gamma(\tilde{b} - 1)}{\Gamma(\tilde{a})} y^{-\frac{\tilde{a}}{2}}, \]

(31)

Since the RMs impose that the outgoing waves must spontaneously terminate at the horizon, the second term must be vanished. This is possible with the poles of the Gamma function of the denominator seen in the second term. In short, if we set \(\tilde{a} = -n\) \((n = 0, 1, 2, ...\), the outgoing waves vanish and hence we read the frequencies of the RMs of the GHS-BH. The result is given by

\[ \omega_n = \frac{\hbar \sqrt{\kappa [2\kappa x + l(l + 1)]}}{2 \sqrt{x(l + 1) + \kappa (z + 3a^2)}} + i(2n + 1)\hbar \kappa, \]

\[ \approx \frac{\hbar \sqrt{\kappa}}{2} l + i(2n + 1)\hbar \kappa, \quad l \gg 1, \]  

(32)

where \(n\) is called the overtone quantum (resonance) number [45]. For the highly excited states \((n \to \infty\) and therefore \(\omega_I \gg \omega_R\)), we have

\[ \Delta \omega \approx \Delta \omega_I = 2\kappa \hbar = 4\pi T_H. \]

(33)

Substituting this into Eq. (2), we obtain

\[ I_{adh} = S \frac{\hbar}{4\kappa}. \]  

(34)

Recalling the Bohr-Sommerfeld quantization rule \((I_{adh} = \hbar n)\), we find the entropy spectrum as

\[ S_n = 4\pi n. \]  

(35)

Furthermore, since \(S = \frac{A}{4\pi}\), we can also read the area spectrum:

\[ A_n = 16\pi n. \]  

(36)

Thus, the minimum area spacing becomes

\[ \Delta A_{\text{min}} = 16\pi \hbar. \]  

(37)

which represents that the entropy/area spectra of the GHS-BH are evenly spaced. Same conclusion was obtained in the studies of [34, 35, 36, 37]. Moreover, our results support the Kothawala et al.’s conjecture [46] which claims that the BHs in Einstein’s theories should have equidistant area spectrum.
CONCLUSION

In this study, we have first studied the massless KGE on the GHSBH geometry. Next, we have shown that the one-dimensional Schrödinger type differential equation (15) can be obtained from the radial equation. From figure (1), it is seen that for the high azimuthal quantum numbers \( l \gg 1 \), the effective potential can form a BP just beyond the event horizon. In such a case, the scalar waves are confined between the horizon and BP. Then, we have applied the particular approximation method [34, 35, 36, 37] for finding the characteristic frequencies of the RMs at the near horizon region. We have shown that the one-dimensional Schrödinger type differential equation can be approximated to a confluent hypergeometric differential equation. After some straightforward computations, we have derived the RMs of the GHS-BH and in sequel applied the MM for the highly damped RMs to find out the entropy/area spectra of the GHS-BH. The obtained spectra are equally spaced and independent of the physical parameters of the GHS-BH. As a final remark, our calculations have revealed that the value of the dimensionless constant \( \epsilon = 16\pi \). This result may be questioned since it is different from the original result of Bekenstein: \( \epsilon = 8\pi \). However, as being emphasized by Hod [18], rather than the value of \( \epsilon \), the uniform quantization of the area/entropy spectra has the utmost importance in the subject of BH spectroscopy.

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