Asymptotic expansions of Laplace integrals for quantum state tomography

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Abstract

Bayesian estimation of a mixed quantum state can be approximated via maximum likelihood (MaxLike) estimation when the likelihood function is sharp around its maximum. Such approximations rely on asymptotic expansions of multi-dimensional Laplace integrals. When this maximum is on the boundary of the integration domain, as it is the case when the MaxLike quantum state is not full rank, such expansions are not standard. We provide here such expansions, even when this maximum does not belong to the smooth part of the boundary, as it is the case when the rank deficiency exceeds two. These expansions provide, aside the MaxLike estimate of the quantum state, confidence intervals for any observable. They confirm the formula proposed and used without precise mathematical justifications by the authors in an article recently published in Physical Review A.

1 Introduction

When the probability laws of the measurement data $Y$ with respect to the continuous parameter $p$ to estimate is given by an analytic model, a widely used way to fulfill this estimation is Maximum Likelihood (MaxLike) reconstruction (see, e.g., [5]). It consists in choosing as estimate $p_{\text{ML}}$, the value of $p$ that maximizes the conditional probability $\mathbb{P}(Y \mid p)$ of the data $Y$. Indeed, when the amount of independent measurements forming the data $Y$ is large, the function $p \mapsto \mathbb{P}(Y \mid p)$ becomes extremely sharp around its maximal value, and the MaxLike estimate $p_{\text{ML}}$ is a good approximation of the Bayesian mean estimate $p_{\text{BM}}$:

$$p_{\text{BM}} = \int_D p \mathbb{P}(p \mid Y) \, dp = \frac{\int_D p \mathbb{P}(Y \mid p) \, dp}{\int_D \mathbb{P}(Y \mid p) \, dp}$$

with $D \subset \mathbb{R}^{\dim p}$ being the set of physically acceptable values for $p$, $\mathbb{P}(Y \mid p)$ the probability density of $p$ knowing $Y$ and $\mathbb{P}_0(p)$ any a priori probability density for $p$.

Relying only on MaxLike estimation has the advantage of providing easy-to-compute algorithms. The first and second derivatives of $\mathbb{P}(Y \mid p)$ versus $p$ can be derived with finite difference method, gradient-like optimization methods can be used and one can extract the Cramér-Rao bound from the Hessian of the log-likelihood function to get a lower bound of the mean estimation error when this Hessian matrix is not degenerate. Nevertheless, some technicalities can arise, in particular for quantum state tomography [10], where the

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parameter \( p \) to estimate corresponds to a quantum state \( \rho \) an element of the compact convex domain \( D \) formed by the set of non negative Hermitian matrix of trace one. In practice, MaxLike estimates \( \rho_{\text{ML}} \) could be of low-rank, i.e. on the boundary of \( D \) as noticed in [3] and observed in [11].

All these reasons lead us to consider Bayesian Mean Estimations (BME) in the general setting when the parameter \( p \) lives in a finite dimensional and compact domain \( D \) with piece-wise smooth boundary. As the magnitude of \( \mathbb{P}(Y \mid p) \) grows (or decreases) exponentially fast compared to the number \( N \) of independent measurements generating the measurement set \( Y \), we consider the scaled log-likelihood function \( f(p) = \frac{1}{N} \log \left( \mathbb{P}(Y \mid p) \right) \). We then address the problem of computing, for any smooth scalar functions \( f \) and \( g \) and under various conditions, the asymptotic development when \( N \) tends towards infinity of the Laplace’s integral:

\[
\mathcal{I}_g(N) = \int_D g(p) \exp(N f(p)) \, dp. \tag{1}
\]

Such asymptotic expansions have been investigated since a long time. They involve integration by parts, Watson’s lemma, Laplace’s method, stationary phase, steepest descents and Hironaka’s resolution of singularities: see [2] for \( \dim p = 1 \) and the regular case when \( \dim p \geq 1 \); see [1] for the singular case in arbitrary dimension and its much more elaborate analysis. In the analytic case and around the maximum of \( f \) at \( \rho_{\text{ML}} \) inside the domain \( D \), these expansions rely on terms like \( e^{N f(\rho_{\text{ML}})} \log(N)^k \) where \( k \) is a non negative integer less than \( \dim p - 1 \) and where \( \alpha \) is rational and strictly positive [1, page 231]. From such series expansions, stem fundamental connections between algebraic geometry and statistical learning theory in the singular case, i.e. when the Hessian of \( f \) at \( \rho_{\text{ML}} \) is not negative definite. This is the object of singular learning theory developed in [12] and in [7].

It is interesting to notice that, as far as we know, very few results can be found when \( \rho_{\text{ML}} \) lies on the boundary of \( D \), excepted the case when \( \rho_{\text{ML}} \) is on a smooth part of the boundary. In [2, section 8.3], the derivation of the leading term is explained when \( \rho_{\text{ML}} \) is on the smooth part of the boundary and when the Hessian of the restriction of \( f \) to this smooth part is negative definite; sub-section 8.3.4 of [1] provides precise indications showing, when the Hessian of the restriction of \( f \) is degenerate, that an asymptotic expansion exists and is similar to the one obtained for \( \rho_{\text{ML}} \) in the interior of \( D \).

For quantum state estimation, this ensures the existence of asymptotic expansion in any case when \( \rho_{\text{ML}} \) has either a full rank (interior of \( D \)) or rank deficiency of one (smooth part of the boundary of \( D \)). For rank deficiency exceeding strictly one, \( \rho_{\text{ML}} \) does not belong to the smooth part of the boundary. As far as we know, the derivations of asymptotic expansions in these singular cases when the rank deficiency of \( \rho_{\text{ML}} \) exceeds two have not been precisely addressed up to now. This paper is a fist attempt to derive such asymptotic expansion of the Bayesian mean and variance when the log-likelihood function reaches its maximum on the boundary of \( D \), i.e. when \( \rho_{\text{ML}} \) is of low rank.

The goal of this paper is twofold. Firstly, we provide the leading terms of specific asymptotic expansions when \( \rho_{\text{ML}} \) lies in a half space. This is the object of section 2 where we assume that the restriction of \( f \) to the boundary admits a non-degenerate maximum at \( \rho_{\text{ML}} \) (see theorem 2). Secondly, we consider quantum state estimation and reformulate these leading terms intrinsically in terms of matrix product and trace. This is object of section 3, where we recall the precise structures of \( f \) and \( g \) in this case and exploit convexity and unitary invariance. We provide in this section precise mathematical justifications of the necessary and sufficient optimality conditions given without details in [11, eq. (8)] (see lemma 2 below) and of the Bayesian variance approximation corresponding to equation (10) in [11] (see theorem 3).
2 Asymptotic expansion of Laplace’s integral

Here, we assume that $p$ is of dimension $n$ and that $D = (-1,1)^n$. Set $p = z$ with $z \in \mathbb{R}^n$. Then (1) reads with:

$$I_p(N) = \int_{z \in (-1,1)^n} g(z) \exp (N f(z)) \, dz. \quad (2)$$

**Theorem 1.** Consider (2) where $f$ and $g$ are analytic functions of $z$ on a compact neighbourhood of $\overline{D}$, the closure of $D$. Assume that $f$ admits a unique maximum on $\overline{D}$ at $z = 0$ with $\frac{\partial^2 f}{\partial z^2} \bigg|_0$ negative definite.

If $g(0) \neq 0$, we have the following dominant term in the asymptotic expansion of $I_p(N)$ for large $N$:

$$I_p(N) = \left( \frac{g(0) \, (2\pi)^{n/2} \, e^{N f(0)} \, N^{-n/2}}{\sqrt{\det \left( \frac{\partial^2 f}{\partial z^2} \bigg|_0 \right)}} \right) + O\left( e^{N f(0)} N^{-n/2-1} \right). \quad (3)$$

If $g(0) = 0$, with $\frac{\partial g}{\partial z} \bigg|_0 = 0$ and $\frac{\partial h}{\partial z} \bigg|_0 = 0$, then we have:

$$I_p(N) = \left( \frac{\text{Tr} \left( -\frac{\partial^2 g}{\partial z^2} \bigg|_0 \left( \frac{\partial^2 f}{\partial z^2} \bigg|_0 \right)^{-1} \right) (2\pi)^{n/2}}{2 \sqrt{\det \left( \frac{\partial^2 f}{\partial z^2} \bigg|_0 \right)}} \right) e^{N f(0)} N^{-n/2-1} + O \left( e^{N f(0)} N^{-n/2-2} \right). \quad (4)$$

**Proof.** Since $f$ is analytic, $f(z) = f(0) - h(z)$ where $h$ is an analytic function of $z$ only with $h(0) = 0$, $\frac{\partial h}{\partial z} \bigg|_0 = 0$ and $\frac{\partial^2 h}{\partial z^2} \bigg|_0 = -\frac{\partial^2 f}{\partial z^2} \bigg|_0$ positive definite.

Via the Morse lemma (see, e.g., [9]), there exists a local diffeomorphism on $z$ around $0$, written $\tilde{z} = \psi(z)$, such that $\psi(0) = 0$ and $h(\tilde{z}) = \frac{1}{2} \sum_{k=1}^{n} (\psi_k(z))^2$. Moreover, we can chose $\psi$ such that $\frac{\partial \psi}{\partial z} \bigg|_0 = -\frac{\partial f}{\partial z} \bigg|_0$ is a positive definite symmetric matrix.

Take $\eta \in (0,1)$ small. There exists a $c < f(0)$ such that, $\forall z \in (-1,1)^n/(-\eta,\eta)^n$, $f(z) \leq c$. Since:

$$I_p(N) = \int_{z \in (-\eta,\eta)^n} g(z) e^{N f(z)} \, dz + \int_{z \not\in (-1,1)^n/(-\eta,\eta)^n} g(z) e^{N f(z)} \, dz$$

$$= e^{N f(0)} \left( \int_{z \in (-\eta,\eta)^n} g(z) e^{N (f(z) - f(0))} \, dz + O \left( e^{-N (f(0) - c)} \right) \right),$$

we only keep:

$$I_\eta(N) = \int_{z \in (-\eta,\eta)^n} g(z) e^{N (f(z) - f(0))} \, dz$$

Since $\eta$ is small, we can consider the change of variable $\tilde{z} = \psi(z)$ that yields:

$$I_\eta(N) = \int_{\tilde{z} \in \psi((-\eta,\eta)^n)} \tilde{g}(\tilde{z}) e^{-\frac{N}{2} \sum_{k=1}^{n} \tilde{z}_k^2} \, d\tilde{z}$$

where:

$$\tilde{g}(\tilde{z}) = \frac{g(\psi^{-1}(\tilde{z}))}{\sqrt{\det \left( \frac{\partial^2 \psi}{\partial \tilde{z}^2} \bigg|_0 \right)}} (1 + d(\tilde{z}))$$

(5)
and \( \tilde{d} \) is an analytic function with \( \tilde{d}(0) = 0 \). There exists \( \tilde{\eta} > 0 \) such that \((-\tilde{\eta}, \tilde{\eta})^n \subset \psi((-\eta, \eta)^n)\). Thus, similarly to the passage from \( I_\theta(N) \) to \( I_\theta(N) \), we can, up to exponentially small terms versus \( N \), just consider the asymptotic expansion of:

\[
\tilde{I}_\theta = \int_{\mathbb{R}^n} \tilde{g}(z)e^{-\tilde{\theta} \sum_k z_k^2} \, dz.
\]

When \( g(0) \neq 0 \), we have \( \tilde{g}(0) \neq 0 \). Set \( \tilde{g}(z) = \tilde{g}(0) + \sum_{k=1}^n \tilde{z}_k \tilde{h}_k(z) \) with \( \tilde{h}_k \) bounded analytic functions on \((-\tilde{\eta}, \tilde{\eta})^n\). We get:

\[
\tilde{I}_\theta = \tilde{g}(0) \int_{\mathbb{R}^n} e^{-\tilde{\theta} \sum_k z_k^2} \, dz + \int_{\mathbb{R}^n} \left( \sum_{k=1}^n \tilde{z}_k \tilde{h}_k(z) \right) e^{-\tilde{\theta} \sum_k z_k^2} \, dz.
\]

Up to exponentially small terms versus \( N \), the first integral in the right hand-side member can be replaced by:

\[
\int_{\mathbb{R}^n} e^{-\tilde{\theta} \sum_k z_k^2} \, dz = \left( \frac{2\pi}{N} \right)^{n/2}.
\]

A single integration by part versus \( z_k \) yields:

\[
\int_{\mathbb{R}^n} \tilde{z}_k \tilde{h}_k(z) e^{-\tilde{\theta} \sum_k z_k^2} \, dz = \frac{1}{N} \int_{\mathbb{R}^n} \frac{\partial \tilde{h}_k}{\partial z_k}(z) e^{-\tilde{\theta} \sum_k z_k^2} \, dz + O(\exp(-N^{2/3}/N)).
\]

This implies (3), via \( \tilde{I}_\theta = \tilde{g}(0) \left( \tilde{w} \right)^{n/2} (1 + O(1/N)) \) and \( \tilde{g}(0) = \frac{\sqrt{\det \left( \frac{\partial^2 f}{\partial z^2} \right)}_0}{\sqrt{\det \left( \frac{\partial^2 f}{\partial z^2} \right)}_0} \).

Assume now that \( g(0) = 0 \) and \( \frac{\partial g}{\partial z^2} \bigg|_0 = 0 \). Consider then the function \( \tilde{g} \) in (5). We have \( \tilde{g}(0) = 0 \) and \( \frac{\partial \tilde{g}}{\partial z^2} \bigg|_0 = 0 \). Moreover, writing:

\[
\kappa_0 = \sqrt{\left| \det \left( \frac{\partial^2 f}{\partial z^2} \right)_0 \right|},
\]

we have:

\[
\kappa_0 \tilde{g}(\psi(z)) = g(z) \left( 1 + e(z) \right)
\]

with \( e \) an analytic function with \( e(0) = 0 \). Thus, for any \( i, j \in \{1, \ldots, n\} \),

\[
\frac{\partial^2 g}{\partial z_i \partial z_j} \bigg|_0 = \kappa_0 \sum_{k, k'} \frac{\partial^2 \tilde{g}}{\partial z_k \partial z_{k'}} \bigg|_0 \frac{\partial \psi_k}{\partial z_i} \bigg|_0 \frac{\partial \psi_{k'}}{\partial z_j} \bigg|_0.
\]

Since \( \frac{\partial \psi}{\partial z} \bigg|_0 = \sqrt{-\frac{\partial^2 f}{\partial z^2} \bigg|_0} \), we have:

\[
\kappa_0 \frac{\partial^2 \tilde{g}}{\partial z^2} \bigg|_0 = \left( -\frac{\partial^2 f}{\partial z^2} \bigg|_0 \right)^{-1} \frac{\partial^2 g}{\partial z^2} \bigg|_0 \left( -\frac{\partial^2 f}{\partial z^2} \bigg|_0 \right)^{-1},
\]

and thus:

\[
\text{Tr} \left( \frac{\partial^2 \tilde{g}}{\partial z^2} \bigg|_0 \right) = \frac{\text{Tr} \left( \frac{\partial^2 g}{\partial z^2} \bigg|_0 \left( \frac{\partial^2 f}{\partial z^2} \bigg|_0 \right)^{-1} \right)}{\sqrt{\det \left( \frac{\partial^2 f}{\partial z^2} \bigg|_0 \right)}}.
\]

(7)
Since \( \tilde{g} \) and its first partial derivatives with respect to \( \tilde{z}_k \) vanish, we have:

\[
\tilde{g}(\tilde{z}) = \sum_{k,k'=1}^{n} \tilde{z}_k \tilde{z}_{k'} \tilde{b}_{k,k'}(\tilde{z}),
\]

where the function \( \tilde{b}_{k,k'} \) are analytic. To evaluate the integral in (6), we have to consider the dominant terms of the following integrals:

\[
B_{k,k'} = \int_{\tilde{z} \in (-\tilde{\eta},\tilde{\eta})^{n}} \tilde{z}_k \tilde{z}_{k'} \tilde{b}_{k,k'}(\tilde{z}) e^{-\frac{2\pi}{N} \sum_{i=1}^{n} \tilde{z}_i^2} d\tilde{z}.
\]

For \( k \neq k' \), one integration by part versus \( \tilde{z}_k \) followed by another one versus \( \tilde{z}_{k'} \), yield to \( B_{k,k'} = O \left( N^{-n/2-2} \right) \). For \( k = k' \), we can perform a single integration by part versus \( \tilde{z}_k \):

\[
\int_{\tilde{z} \in (-\tilde{\eta},\tilde{\eta})^{n}} \tilde{z}_k^2 \tilde{b}_{k,k}(\tilde{z}) e^{-\frac{2\pi}{N} \sum_{i=1}^{n} \tilde{z}_i^2} d\tilde{z}
\]

\[= \frac{1}{N} \int_{\tilde{z} \in (-\tilde{\eta},\tilde{\eta})^{n}} \left( \tilde{b}_{k,k}(\tilde{z}) + \tilde{z}_k \frac{\partial \tilde{b}_{k,k}}{\partial \tilde{z}_k}(\tilde{z}) \right) e^{-\frac{2\pi}{N} \sum_{i=1}^{n} \tilde{z}_i^2} d\tilde{z} + O(e^{-N\tilde{\eta}^2/2})
\]

\[= \frac{\tilde{b}_{k,k}(0)}{N} \left( \frac{2\pi}{N} \right)^{n/2} + O \left( N^{-n/2-2} \right).
\]

The sum \( \sum_{k,k'} B_{k,k'} \) corresponds to the integral \( \tilde{I}_\tilde{g} \) and reads:

\[
\tilde{I}_\tilde{g}(N) = \frac{\sum_{k=1}^{n} \tilde{b}_{k,k}(0)}{N} \left( \frac{2\pi}{N} \right)^{n/2} + O \left( N^{-n/2-2} \right).
\]

Since up to exponentially small terms, \( \tilde{I}_\tilde{g} \) and \( e^{-Nf(0)} I_g(N) \) coincide, we get (4) using (7) since \( \sum_{k=1}^{n} \tilde{b}_{k,k}(0) = \frac{1}{2} \mathrm{Tr} \left( \frac{\partial^2 f}{\partial z^2} \right) \).

We assume now that \( p \in \mathbb{R}^{n+1} \), \( n + 1 \) being the dimension of \( p \) (n non-negative integers), and that \( \mathcal{D} = (0,1) \times (-1)^n \). Set \( p = (x,z) \) with \( x \in \mathbb{R} \) and \( z \in \mathbb{R}^n \). Then (1) reads when \( g(x,z) \) is replaced by \( x^m g(x,z) \), with \( m \) a non negative integer:

\[
I_g(N) = \int_{x \in (0,1)} \int_{z \in (-1,1)^n} x^m g(x,z) \exp \left( N f(x,z) \right) dx dz. \tag{8}
\]

**Theorem 2.** Consider (8), where \( f \) and \( g \) are analytic functions of \( (x,z) \) on a compact neighbourhood of \( \overline{\mathcal{D}} \), the closure of \( \mathcal{D} \). Assume that \( f \) admits a unique maximum on \( \overline{\mathcal{D}} \) at \( (x,z) = (0,0) \), with \( \frac{\partial^2 f}{\partial z^2} \) negative definite and \( \frac{\partial f}{\partial x}(0,0) < 0 \).

If \( g(0,0) \neq 0 \), we have the following dominant term in the asymptotic expansion of \( I_g(N) \) for large \( N \):

\[
I_g(N) = \left( \frac{g(0,0) m! (2\pi)^{n/2} e^{Nf(0,0)} N^{-m-n/2-1}}{\left| \det \left( \frac{\partial^2 f}{\partial z^2} \right) \right| (0,0)} \right)^{m+1} + O \left( e^{Nf(0,0)} N^{-n-m/2-2} \right). \tag{9}
\]
If \( g(0, 0) = 0 \), with \( \frac{\partial g}{\partial z} \big|_{(0,0)} = 0 \) and \( \frac{\partial g}{\partial x} \big|_{(0,0)} = 0 \), then we have:

\[
\mathcal{I}_g(N) = \left( \frac{\text{Tr} \left( -\frac{\partial^2 g}{\partial z^2} \big|_{(0,0)} \left( \frac{\partial^2 f}{\partial z^2} \big|_{(0,0)} \right)^{-1} \right) m! \left( 2\pi \right)^{n/2}}{2 \left| \det \left( \frac{\partial^2 f}{\partial z^2} \big|_{(0,0)} \right) \right|^{-m+1}} \right) e^{Nf(0,0)} N^{-m-n/2-2} + O \left( e^{Nf(0,0)} N^{-m-n/2-3} \right). \tag{10}
\]

For clarity’s sake, we consider here the analytic situation, despite the fact that the above asymptotics are also valid in the \( C^{m+3} \) case.

**Proof.** We adapt here the method sketched in section 8.3.4 of [1] for oscillatory integrals in a halfspace. Since \( f \) is analytic, we have

\[ f(x, z) = f(0, 0) - xf_1(x, z) - h(z) \]

where \( f_1 \) is analytic with \( f_1(0, 0) = -\frac{\partial f}{\partial z} \big|_{(0,0)} > 0 \), where \( h \) is an analytic function of \( z \) only, with \( h(0) = 0 \), \( \frac{\partial h}{\partial x} \big|_{x=0} = 0 \) and \( \frac{\partial^2 h}{\partial z^2} \big|_{z=0} = -\frac{\partial^2 f}{\partial z^2} \big|_{z=0} \) positive definite.

Set \( \phi(x, z) = xf_2(x, z) \). Consider the following map \((x, z) \mapsto (\tilde{x} = \phi(x, z), z)\). It is a local diffeomorphism around \((0, 0)\) that preserves the sign of \( x \), i.e. \( x \phi(x, z) \geq 0 \). Moreover, using the Morse lemma (see, e.g., [9]), there exists a local diffeomorphism on \( z \) around \( 0 \), \( \tilde{z} = \psi(z) \), such that \( \psi(0) = 0 \) and \( h(z) = \frac{1}{2} \sum_{k=1}^{n} (\psi(k))^2 \) (see, e.g., [9]). Moreover, we can chose \( \psi \) such that \( \frac{\partial \psi}{\partial z} \big|_{0} = \sqrt{-\frac{\partial^2 f}{\partial z^2} \big|_{0}} \) is a positive definite symmetric matrix.

To summarize, there is a local analytic diffeomorphism \( \Xi : V \ni (x, z) \mapsto (\tilde{x}, \tilde{z}) \in \tilde{V} \) from an open connected neighbourhood \( V \) of \( 0 \) to another open connected neighbourhood of \( 0 \) such that

- for all \((x, z) \in V\), we have \( \phi(x, z) > 0 \) (resp. \( < 0 \), \( = 0 \)) when \( x > 0 \) (resp. \( < 0 \), \( = 0 \))
- \( \forall (x, z) \in V \), \( f(x, z) = -\phi(x, z) - \frac{1}{2} \sum_{k=1}^{n} (\psi(k))^2 \).
- \( \det \left( \frac{\partial \phi}{\partial x} \big|_{(x, z)} \frac{\partial \phi}{\partial z} \big|_{(x, z)} \right) \big|_{(x, z)} = \left| \frac{\partial f}{\partial x} \big|_{(0, 0)} \right| \sqrt{\left| \det \left( \frac{\partial^2 f}{\partial z^2} \big|_{0} \right) \right|} \left( 1 + d(x, z) \right) \) where \( d \) is analytic on \( V \) with \( d(0, 0) = 0 \).

Since \( V \) is a neighbourhood of \( 0 \), there exists a \( \eta \in (0, 1) \) such that \( C_{\eta} = (0, \eta) \times (-\eta, \eta)^n \subset V \). Moreover, there exists \( c < f(0, 0) \) such that, \( \forall (x, z) \in \mathcal{D}/C_{\eta}, f(x, z) \leq c \). Since:

\[
\mathcal{I}_g(N) = \int_{(x, z) \in C_{\eta}} x^m g(x, z) e^{Nf(x, z)} \, dx \, dz + \int_{(x, z) \in \mathcal{D}/C_{\eta}} x^m g(x, z) e^{Nf(x, z)} \, dx \, dz = \int_{(x, z) \in C_{\eta}} x^m g(x, z) e^{Nf(x, z) - f(0, 0)} \, dx \, dz + e^{-N(f(0, 0) - c)} \int_{(x, z) \in \mathcal{D}/C_{\eta}} x^m g(x, z) e^{Nf(x, z) - c} \, dx \, dz
\]

\[
= e^{Nf(0,0)} \left( \int_{(x, z) \in C_{\eta}} x^m g(x, z) e^{Nf(x, z) - f(0, 0)} \, dx \, dz + e^{-N(f(0, 0) - c)} \int_{(x, z) \in \mathcal{D}/C_{\eta}} x^m g(x, z) e^{Nf(x, z) - c} \, dx \, dz \right)
\]

\[
= e^{Nf(0,0)} \left( \int_{(x, z) \in C_{\eta}} x^m g(x, z) e^{Nf(x, z) - f(0, 0)} \, dx \, dz + O \left( e^{-N(f(0, 0) - c)} \right) \right)
\]
we just have to consider the asymptotic expansion of:

$$I_\eta(N) = \int_{(x,z) \in C_\eta} x^m g(x,z) e^{N(f(x,z)-f(0,0))} \, dx \, dz$$

Since $C_\eta \subset V$, we can consider the change of variable $(\tilde{x}, \tilde{z}) = \Xi(x,z)$ that yields:

$$I_\eta(N) = \int_{(\tilde{x},\tilde{z}) \in \Xi(C_\eta)} \tilde{x}^m \tilde{g}(\tilde{x}, \tilde{z}) e^{-N(\tilde{x} + \frac{1}{2} \sum_{k=1}^{n} \tilde{z}_k^2)} \, d\tilde{x} \, d\tilde{z}$$

where:

$$\tilde{g}(\tilde{x}, \tilde{z}) = \frac{g(\Xi^{-1}(\tilde{x}, \tilde{z}))}{\left( f_1(\Xi^{-1}(\tilde{x}, \tilde{z})) \right)^m \left| \frac{\partial f}{\partial z} \right|_{(0,0)} \left| \det \left( \frac{\partial^2 f}{\partial z^2} \right|_{(0,0)} \right|} (1 + \tilde{d}(\tilde{x}, \tilde{z})),$$

and $\tilde{d}$ is an analytic function with $\tilde{d}(0,0) = 0$. Since, for all $(\tilde{x}, \tilde{z}) \in \Xi(C_\eta)$ we have $\tilde{x} \geq 0$, there exists a $\tilde{\eta} > 0$ such that $\tilde{C}_\eta = (0, \tilde{\eta}) \times (-\tilde{\eta}, \tilde{\eta})^n \subset \Xi(C_\eta)$. Thus, similarly to the passage from $I_\eta(N)$ to $I_\eta(N)$, we can just consider, up to exponentially small terms versus $N$, the asymptotic expansion of:

$$\tilde{I}_\eta = \int_{(\tilde{x},\tilde{z}) \in \tilde{C}_\eta} \tilde{x}^m \tilde{g}(\tilde{x}, \tilde{z}) e^{-N(\tilde{x} + \frac{1}{2} \sum_{k=1}^{n} \tilde{z}_k^2)} \, d\tilde{x} \, d\tilde{z}.$$

When $g(0,0) \neq 0$, we have $\tilde{g}(0,0) \neq 0$. Set $\tilde{g}(\tilde{x}, \tilde{z}) = \tilde{g}(0,0) + \tilde{x} \tilde{g}_1(\tilde{x}, \tilde{z}) + \sum_{k=1}^{n} \tilde{z}_k \tilde{h}_k(\tilde{x}, \tilde{z})$ with $\tilde{g}_1$ and $\tilde{h}_k$ bounded analytic functions on $\tilde{C}_\eta$. We get:

$$\tilde{I}_\eta = \tilde{g}(0,0) \int_{(\tilde{x},\tilde{z}) \in \tilde{C}_\eta} \tilde{x}^m e^{-N(\tilde{x} + \frac{1}{2} \sum_{k=1}^{n} \tilde{z}_k^2)} \, d\tilde{x} \, d\tilde{z} + \int_{(\tilde{x},\tilde{z}) \in \tilde{C}_\eta} \tilde{x}^{m+1} \tilde{g}_1(\tilde{x}, \tilde{z}) e^{-N(\tilde{x} + \frac{1}{2} \sum_{k=1}^{n} \tilde{z}_k^2)} \, d\tilde{x} \, d\tilde{z} + \int_{(\tilde{x},\tilde{z}) \in \tilde{C}_\eta} \tilde{x}^m \left( \sum_{k=1}^{n} \tilde{z}_k \tilde{h}_k(\tilde{x}, \tilde{z}) \right) e^{-N(\tilde{x} + \frac{1}{2} \sum_{k=1}^{n} \tilde{z}_k^2)} \, d\tilde{x} \, d\tilde{z}.$$

Up to exponentially small terms versus $N$, the first integral in the right-hand-side member can be replaced by:

$$\int_{(\tilde{x},\tilde{z}) \in (0, +\infty) \times (-\infty, +\infty)^n} \tilde{x}^m e^{-N(\tilde{x} + \frac{1}{2} \sum_{k=1}^{n} \tilde{z}_k^2)} \, d\tilde{x} \, d\tilde{z} = \frac{m!}{N^{m+1}} \left( \frac{2\pi}{N} \right)^{n/2}.$$

For the second integral, $m+1$ integrations by part versus $\tilde{x}$ are necessary:

$$\int_{(\tilde{x},\tilde{z}) \in \tilde{C}_\eta} \tilde{x}^{m+1} \tilde{g}_1(\tilde{x}, \tilde{z}) e^{-N(\tilde{x} + \frac{1}{2} \sum_{k=1}^{n} \tilde{z}_k^2)} \, d\tilde{x} \, d\tilde{z}$$

$$= \int_{\tilde{x} \in (-\tilde{\eta}, \tilde{\eta})} \left( \int_{0}^{\tilde{\eta}} \tilde{x}^{m+1} \tilde{g}_1(\tilde{x}, \tilde{z}) e^{-N\tilde{x}} \, d\tilde{x} \right) e^{-\frac{1}{2} \sum_{k=1}^{n} \tilde{z}_k^2} \, d\tilde{z},$$

where, via $m+1$ integrations by part, we get:

$$\int_{0}^{\tilde{\eta}} \tilde{x}^{m+1} \tilde{g}_1(\tilde{x}, \tilde{z}) e^{-N\tilde{x}} \, d\tilde{x} = \frac{1}{N^{m+1}} \int_{0}^{\tilde{\eta}} \tilde{g}_{m+2}(\tilde{x}, \tilde{z}) e^{-N\tilde{x}} \, d\tilde{x} + O(e^{-\tilde{\eta}N/N})$$
with \( \tilde{g}_{m+2} = \frac{g^{m+1}}{\lambda_0} (\tilde{x}^{m+1} \tilde{g}_1(\tilde{x}, \tilde{z})) \). We get:

\[
\int_{(\tilde{x}, \tilde{z}) \in \mathbb{C}_0} \tilde{x}^{m+1} \tilde{g}_1(\tilde{x}, \tilde{z}) e^{-N(\tilde{x}^2 + \frac{1}{2} \sum_{k=1}^n \tilde{z}_k^2)} \tilde{x} \tilde{z} \, d\tilde{x} \, d\tilde{z} = \frac{1}{N^{m+1}} \int_{(\tilde{x}, \tilde{z}) \in \mathbb{C}_0} \tilde{g}_{m+2}(\tilde{x}, \tilde{z}) e^{-N(\tilde{x}^2 + \frac{1}{2} \sum_{k=1}^n \tilde{z}_k^2)} d\tilde{x} \, d\tilde{z} + O(e^{-\tilde{\eta}_N / N}) = O \left( \frac{1}{N^{m+n/2+2}} \right),
\]

since \( \int_0 \tilde{g}_{m+2}(\tilde{x}, \tilde{z}) e^{-N\tilde{z}^2} \, d\tilde{x} \) is of order \( 1/N \).

Similarly, we get, with \( m \) integration by part versus \( \tilde{x} \),

\[
\int_{(\tilde{x}, \tilde{z}) \in \mathbb{C}_0} \tilde{x}^m \tilde{z}_k \tilde{h}_k(\tilde{x}, \tilde{z}) e^{-N(\tilde{x}^2 + \frac{1}{2} \sum_{k=1}^n \tilde{z}_k^2)} \tilde{x} \tilde{z} \, d\tilde{x} \, d\tilde{z} = \frac{1}{N^m} \int_{(\tilde{x}, \tilde{z}) \in \mathbb{C}_0} \tilde{z}_k \tilde{q}_{k,m}(\tilde{x}, \tilde{z}) e^{-N(\tilde{x}^2 + \frac{1}{2} \sum_{k=1}^n \tilde{z}_k^2)} d\tilde{x} \, d\tilde{z} + O(e^{-\tilde{\eta}_N / N}).
\]

where \( \tilde{q}_{k,m}(\tilde{x}, \tilde{z}) = \frac{\partial^{m+1}}{\partial \tilde{x}^{m+1}}(\tilde{x}^m \tilde{h}_k(\tilde{x}, \tilde{z})) \). A single integration by part versus \( \tilde{z}_k \) yields:

\[
\int_{(\tilde{x}, \tilde{z}) \in \mathbb{C}_0} \tilde{z}_k \tilde{q}_{k,m}(\tilde{x}, \tilde{z}) e^{-N(\tilde{x}^2 + \frac{1}{2} \sum_{k=1}^n \tilde{z}_k^2)} \tilde{x} \tilde{z} \, d\tilde{x} \, d\tilde{z} = \frac{1}{N} \int_{(\tilde{x}, \tilde{z}) \in \mathbb{C}_0} \frac{\partial \tilde{h}_{k,m}}{\partial \tilde{z}_k}(\tilde{x}, \tilde{z}) e^{-N(\tilde{x}^2 + \frac{1}{2} \sum_{k=1}^n \tilde{z}_k^2)} d\tilde{x} \, d\tilde{z} + O(e^{-\tilde{\eta}^2 N/2}).
\]

This implies that:

\[
\int_{(\tilde{x}, \tilde{z}) \in \mathbb{C}_0} \tilde{x}^m \tilde{z}_k \tilde{h}_k(\tilde{x}, \tilde{z}) e^{-N(\tilde{x}^2 + \frac{1}{2} \sum_{k=1}^n \tilde{z}_k^2)} \tilde{x} \tilde{z} \, d\tilde{x} \, d\tilde{z} = \frac{1}{N^{m+1}} \int_{(\tilde{x}, \tilde{z}) \in \mathbb{C}_0} \frac{\partial \tilde{h}_{k,m}}{\partial \tilde{z}_k}(\tilde{x}, \tilde{z}) e^{-N(\tilde{x}^2 + \frac{1}{2} \sum_{k=1}^n \tilde{z}_k^2)} d\tilde{x} \, d\tilde{z} + O(e^{-\tilde{\eta}^2 N/2}) = O \left( \frac{1}{N^{m+n/2+2}} \right).
\]

Thus, we get (9), thanks to \( \tilde{I}_g \equiv \frac{\tilde{g}(0,0)^m!}{N^{m+n/2}} (2 \pi / N)^{n/2} (1 + O(1/N)) \) and \( \tilde{g}(0,0) = \frac{g(0,0)}{\left| \left. \frac{\partial f}{\partial x} \right|_{(0,0)} \right|^{m+1} \sqrt{\left| \left. \frac{\det \left( \frac{\partial^2 f}{\partial z^2} \right) \right|_{(0,0)} \right|} \left( \left. \frac{\partial^2 f}{\partial x \partial z} \right|_{(0,0)} \right)^{m+1}} \). Assume now that \( g(0,0) = 0 \), \( \left. \frac{\partial f}{\partial x} \right|_{(0,0)} = 0 \) and \( \left. \frac{\partial f}{\partial z} \right|_{(0,0)} = 0 \). Consider then the function \( \tilde{g} \) in (11). We have \( \tilde{g}(0,0) = 0 \), \( \left. \frac{\partial^2 g}{\partial x \partial z} \right|_{(0,0)} = 0 \) and \( \left. \frac{\partial^2 g}{\partial z^2} \right|_{(0,0)} = 0 \). Moreover, denoting:

\[
\lambda_0 = \left| \left. \frac{\partial^2 f}{\partial z^2} \right|_{(0,0)} \right|^{m+1},
\]

we have:

\[
\lambda_0 \tilde{g}(\phi(x, z), \psi(z)) = g(x, z) \left( 1 + \epsilon(x, z) \right),
\]

with \( \epsilon \) an analytic function with \( \epsilon(0,0) = 0 \). Similarly to (7), we get:

\[
\text{Tr} \left( \frac{\partial^2 g}{\partial z^2} \right)_{(0,0)} = \text{Tr} \left( \frac{\partial^2 g}{\partial x \partial z} \right)_{(0,0)} \left( \frac{\partial^2 g}{\partial x \partial z} \right)_{(0,0)}^{-1} \left( \frac{\partial^2 g}{\partial z^2} \right)_{(0,0)}^{-1} \left( \frac{\partial^2 g}{\partial z^2} \right)_{(0,0)}^{m+1} \right). \tag{12}
\]
Since $\tilde{g}$ and its first partial derivatives versus $\tilde{x}$ and $\tilde{z}_k$ vanish, we have:

$$\tilde{g}(\tilde{x}, \tilde{z}) = \tilde{x}^2 \tilde{a}(\tilde{x}, \tilde{z}) + \sum_{k, k' = 1}^{n} \tilde{z}_k \tilde{z}_{k'} \hat{b}_{k, k'}(\tilde{x}, \tilde{z}) + \sum_{k = 1}^{n} \tilde{x} \tilde{z}_k \hat{c}_k(\tilde{x}, \tilde{z})$$

where the function $\tilde{a}$, $\hat{b}_{k, k'}$, and $\hat{c}_k$ are analytic. To evaluate the integrals in (11), we have to consider the dominant terms of three kinds of integrals:

$$A = \int_{(\tilde{x}, \tilde{z}) \in \mathcal{C}_0} \tilde{x}^{m+2} \tilde{a}(\tilde{x}, \tilde{z}) e^{-N(\tilde{x} + \frac{1}{2} \sum_{i=1}^{n} \tilde{z}_i^2)} \, d\tilde{x} \, d\tilde{z},$$

$$B_{k, k'} = \int_{(\tilde{x}, \tilde{z}) \in \mathcal{C}_0} \tilde{x}^{m} \tilde{z}_k \tilde{z}_{k'} \hat{b}_{k, k'}(\tilde{x}, \tilde{z}) e^{-N(\tilde{x} + \frac{1}{2} \sum_{i=1}^{n} \tilde{z}_i^2)} \, d\tilde{x} \, d\tilde{z},$$

$$C_k = \int_{(\tilde{x}, \tilde{z}) \in \mathcal{C}_0} \tilde{x}^{m+1} \tilde{z}_k \hat{c}_k(\tilde{x}, \tilde{z}) e^{-N(\tilde{x} + \frac{1}{2} \sum_{i=1}^{n} \tilde{z}_i^2)} \, d\tilde{x} \, d\tilde{z}.$$

As done previously, $m + 2$ integrations by part on $\tilde{x}$ yield $A = O\left(N^{-m-n/2-3}\right)$. As done previously, $m + 1$ integrations by part versus $\tilde{x}$ and a single integration by part versus $\tilde{z}_k$ provide $C_k = O\left(N^{-m-n/2-3}\right)$. For $k \neq k'$, $m$ integrations by part versus $\tilde{x}$, one integration by part versus $\tilde{z}_k$ followed by another one versus $\tilde{z}_{k'}$, yield similarly to $B_{k, k'} = O\left(N^{-m-n/2-1}\right)$. For $k = k'$, we start with $m$ integrations by part versus $\tilde{x}$:

$$B_{k, k} = \int_{(\tilde{x}, \tilde{z}) \in \mathcal{C}_0} \tilde{x}^{m} \tilde{z}_k^2 \hat{b}_{k, k}(\tilde{x}, \tilde{z}) e^{-N(\tilde{x} + \frac{1}{2} \sum_{i=1}^{n} \tilde{z}_i^2)} \, d\tilde{x} \, d\tilde{z}$$

$$= \frac{1}{Nm} \int_{(\tilde{x}, \tilde{z}) \in \mathcal{C}_0} \tilde{z}_k^2 \tilde{q}_{k, m}(\tilde{x}, \tilde{z}) e^{-N(\tilde{x} + \frac{1}{2} \sum_{i=1}^{n} \tilde{z}_i^2)} \, d\tilde{x} \, d\tilde{z} + O(e^{-\tilde{q}N/N}).$$

where $\tilde{q}_{k, m}(\tilde{x}, \tilde{z}) = \frac{\partial}{\partial \tilde{z}_k} (\tilde{x}^m \tilde{b}_{k, k}(\tilde{x}, \tilde{z}))$. We can notice that $\tilde{q}_{k, m}(0) = m! \tilde{b}_{k, k}(0)$. A single integration by part versus $\tilde{z}_k$ yields to:

$$\int_{(\tilde{x}, \tilde{z}) \in \mathcal{C}_0} \tilde{z}_k \tilde{q}_{k, m}(\tilde{x}, \tilde{z}) e^{-N(\tilde{x} + \frac{1}{2} \sum_{i=1}^{n} \tilde{z}_i^2)} \, d\tilde{x} \, d\tilde{z}$$

$$= \frac{1}{N} \int_{(\tilde{x}, \tilde{z}) \in \mathcal{C}_0} \left( \tilde{q}_{k, m}(\tilde{x}, \tilde{z}) + \tilde{z}_k \frac{\partial \tilde{q}_{k, m}(\tilde{x}, \tilde{z})}{\partial \tilde{z}_k} \right) e^{-N(\tilde{x} + \frac{1}{2} \sum_{i=1}^{n} \tilde{z}_i^2)} \, d\tilde{x} \, d\tilde{z} + O(e^{-N\tilde{q}^2/N^2})$$

$$= \tilde{q}_{k, m}(0) \frac{1}{N^2} \left( \frac{2\pi}{N} \right)^{n/2} + O\left(N^{-n/2-3}\right).$$

With $\tilde{q}_{k, m}(0) = m! \tilde{b}_{k, k}(0)$, the sum $A + \sum_k C_k + \sum_{k, k'} B_{k, k'}$ corresponding the integral in (11) reads:

$$\tilde{I}_g(N) = \frac{\sum_{k=1}^{n} m! \tilde{b}_{k, k}(0)}{Nm} \left( \frac{2\pi}{N} \right)^{n/2} + O\left(N^{-m-n/2-3}\right).$$

Since up to exponentially small terms, $\tilde{I}_g$ and $e^{-Nf(0)}\tilde{I}_q(N)$ coincide, we get (10) using (12), since $\sum_{k=1}^{n} \tilde{b}_{k, k}(0) = \frac{1}{2} \text{Tr} \left( \frac{\partial^2 f}{\partial z^2} |_{0} \right)$.

The asymptotic expansions of theorems 1 and 2 yield directly the following approximations of the Bayesian mean and variance.

**Corollary 1.** Consider the analytic function $f(z)$ of theorem 1. Then we have the following asymptotic for any analytic function $g(z)$:

$$\mathcal{M}_g(N) \triangleq \frac{\int_{z \in (-1, 1)^n} g(z) \exp\left( N f(z) \right) \, dz}{\int_{z \in (-1, 1)^n} \exp\left( N f(z) \right) \, dz} = g(0) + O(N^{-1})$$

(13)
Lemma 1. Take two $C^2$ real-value functions $f$ and $g$ of $z \in \mathbb{R}^n$. Assume that $0$ is a regular critical point of $f$ and just a critical point of $g$. Take any $C^2$ diffeomorphism $\phi$ defined locally around $0$: $\tilde{z} = \phi(z)$. Then:

$$\text{Tr} \left( - \frac{\partial^2 g}{\partial z^2} \right) \left( \frac{\partial^2 f}{\partial z^2} \right)^{-1} = \text{Tr} \left( - \frac{\partial^2 \tilde{g}}{\partial \tilde{z}^2} \right) \left( \frac{\partial^2 \tilde{f}}{\partial \tilde{z}^2} \right)^{-1},$$

where $\tilde{f}(\phi(z)) = f(z)$ and $\tilde{g}(\phi(z)) = g(z)$.

This lemma just says that the above trace formula is coordinate-free, i.e., independent of the local coordinates chosen to compute the Hessian of $f$ and $g$ at their common critical point.

3 Application to quantum state tomography

As explained in [11], the parameter $p$ to estimate corresponds to a density operator $\rho$ (quantum state), a square matrix with complex entries and belonging to the convex compact set $\mathcal{D}$ formed by Hermitian $d \times d$ non-negative matrices of trace one. Then, the log-likelihood function admits the following structure:

$$f(\rho) = \sum_{\mu \in \mathcal{M}} \log \left( \text{Tr}(\rho Y_\mu) \right)$$

where the set $\mathcal{M}$ is finite and each measurement data $Y_\mu$ belongs also to $\mathcal{D}$. For any Hermitian $d \times d$ matrix $A$, (a quantum observable) we are interested to provide an approximation of Bayesian estimate of $\text{Tr}(\rho A)$,

$$I_A(N) = \frac{\int_{\mathcal{D}} \text{Tr}(\rho A) e^{N f(\rho)} \mathbb{P}_0(\rho) \, d\rho}{\int_{\mathcal{D}} e^{N f(\rho)} \mathbb{P}_0(\rho) \, d\rho},$$

(18)
and of the Bayesian variance:

\[ V_A(N) = \frac{\int_D \left( \text{Tr}(\rho A) - I_A(N) \right)^2 e^{Nf(\rho)} \, \mathbb{P}_0(\rho) \, d\rho}{\int_D e^{Nf(\rho)} \, \mathbb{P}_0(\rho) \, d\rho}. \]

Here above \( d\rho \) stands for the standard Euclidean volume element on \( D \), derived from the Frobenius product between \( n \times n \) Hermitian matrices, and \( \mathbb{P}_0 > 0 \) is a probability density on \( \rho \) prior to the measurement data \((Y_\mu)\). Since the number of real parameters to describe \( \rho \) is large in general, it is difficult to compute these integrals even numerically via Monte-Carlo method.

The following lemma provides a unitary invariance characterization of any \( \rho \) argument of the maximum of \( f \) on \( D \).

**Lemma 2.** Assume that the \( d \times d \) Hermitian matrix \( \overline{\rho} \) is an argument of the maximum of \( f : D \ni \rho \mapsto f(\rho) \in [-\infty,0] \) defined in (17) over \( D \) (the set of density operators). Then necessarily, \( \overline{\rho} \) satisfies the following condition:

1. \( \text{Tr}(\overline{\rho} Y_\mu) > 0 \) for each \( \mu \in \mathcal{M} \);
2. \( \left[ \overline{\rho}, \nabla f_{|\overline{\rho}} \right] = \overline{\rho} \cdot \nabla f_{|\overline{\rho}} - \nabla f_{|\overline{\rho}} \overline{\rho} = 0 \), where \( \nabla f_{|\overline{\rho}} = \sum_{\mu \in \mathcal{M}} \frac{\nabla \mu}{\text{Tr}(\rho Y_\mu)} \) is the gradient of \( f \) at \( \overline{\rho} \) for the Frobenius scalar product;
3. there exists \( \lambda > 0 \) such that \( \lambda \overline{\rho} = \overline{\rho} \nabla f_{|\overline{\rho}} \) and \( \nabla f_{|\overline{\rho}} \leq \lambda I \), where \( \overline{\rho} \) is the orthogonal projector on the range of \( \overline{\rho} \) and \( I \) is the identity operator.

These conditions are also sufficient and characterize the unique maximum when, additionally, the vector space spanned by the \( Y_\mu \)'s coincides with the set of Hermitian matrices.

**Proof.** Since \( f \) is a concave function of \( \rho \), we can use the standard optimality criterion for a convex optimization problem (see, e.g., [4, section 4.2.3]): \( \overline{\rho} \) maximizes \( f \) over the convex compact set \( D \), if and only if, \( \rho \in D \), \( \text{Tr} \left((\rho - \overline{\rho}) \nabla f_{|\overline{\rho}}\right) \leq 0 \).

Assume that \( f(\overline{\rho}) \) is maximum. Since \( f(I) > -\infty \), for each \( \mu \) we have \( \text{Tr}(\overline{\rho} Y_\mu) > 0 \). Take \( \rho = e^{-iH\overline{\rho}e^{iH}} \) where \( H \) is an arbitrary Hermitian operator. We have:

\[ \text{Tr} \left(e^{-iH\overline{\rho}e^{iH}} \nabla f_{|\overline{\rho}}\right) \leq \text{Tr} \left(\overline{\rho} \nabla f_{|\overline{\rho}}\right). \]

For \( H \) close to zero, we have via the Baker-Campbell-Hausdorff formula, \( e^{-iH\overline{\rho}e^{iH}} = \overline{\rho} - i[H,\overline{\rho}] + O(\text{Tr}(H^2)) \). The above inequality implies that for all \( H \) small enough, \( \text{Tr} \left(H, \overline{\rho} \nabla f_{|\overline{\rho}}\right) = \text{Tr} \left(H, \overline{\rho}, \nabla f_{|\overline{\rho}}\right) = 0 \) and thus \( \overline{\rho} \) and \( \nabla f_{|\overline{\rho}} \) commute.

Consider the spectral decomposition \( \overline{\rho} = U \overline{\Sigma} U^\dagger \) where \( U \) is unitary and \( \overline{\Sigma} \) diagonal with entries \( 0 \leq \overline{\Sigma}_1 \leq \overline{\Sigma}_2 \leq \cdots \leq \overline{\Sigma}_d \leq 1 \). Since \( \overline{\rho} \) and \( \nabla f_{|\overline{\rho}} \) commute, we have also \( \nabla f_{|\overline{\rho}} = U \overline{\Sigma} U^\dagger \) with \( \overline{\Sigma} \) diagonal with entries \( (\overline{\Sigma}_k) \) Since \( \nabla f \) is non negative, these entries are non-negative too. Take \( \rho = U \Delta U^\dagger \) where \( \Delta \) is any diagonal matrix with non negative entries and of trace one. We have:

\[ \text{Tr} \left((\rho - \overline{\rho}) \nabla f_{|\overline{\rho}}\right) = \text{Tr} \left((\Delta - \overline{\Delta}) \overline{\Sigma}\right) \leq 0. \]

This means that, for any \( (\Delta_1,\ldots,\Delta_d) \in [0,1]^d \) such that \( \sum_{k=1}^d \Delta_k = 1 \) we have:

\[ \sum_{k=1}^d (\Delta_k - \overline{\Delta}_k) \overline{\Sigma}_k \leq 0. \]

Take \( \epsilon > 0 \), \( (k_1,k_2) \in \{1,\ldots,d\}^2 \) such that \( \overline{\Sigma}_{k_1} > 0 \) and \( k_2 \neq k_1 \). For \( k \in \{1,\ldots,d-1\}/\{k_1,k_2\} \) set \( \Delta_k = \overline{\Sigma}_k \) and take \( \Delta_{k_1} = \overline{\Sigma}_{k_1} - \epsilon \) with \( \Delta_{k_2} = \overline{\Sigma}_{k_2} + \epsilon \). By construction
\[ \text{Tr}(\Delta) = 1 \text{ and, for } \epsilon > 0 \text{ small enough, } \Delta_k \geq 0 \text{ for all } k \in \{1, \ldots, d\}. \] The previous inequality implies that:

\[ \forall (k_1, k_2) \in \{(1, \ldots, d)^2 \text{ such that } \Sigma_{k_1} > 0 \text{ and } k_1 \neq k_2, \quad \lambda_{k_2} \leq \lambda_{k_1}. \]

Thus for all \( k_1, k_2 \) such that \( \Sigma_{k_1} > 0 \) and \( \Sigma_{k_2} > 0 \), \( \lambda_{k_1} = \lambda_{k_2} = \lambda > 0 \). For \( k_1, k_2 \) such that \( \Sigma_{k_1} > 0 \) and \( \Sigma_{k_2} = 0 \), we have also \( \lambda_{k_2} \leq \lambda_{k_1} = \lambda \). Thus we get \( \lambda \leq \lambda_{k} \). With \( \Theta \) the diagonal matrix of entries \( \Theta_k = 0 \) (resp. \( = 1 \)) when \( \Sigma_k = 0 \) (resp. \( > 0 \)), we have \( \mathcal{P} = U \Theta U^\dagger \) we get \( \mathcal{X} \mathcal{P} = \mathcal{P} \nabla f|_\mathcal{P} \). Since \( \nabla f|_\mathcal{P} \) is non negative and cannot be zero, we have \( \lambda > 0 \).

Take \( \mathcal{P} \) satisfying the conditions of lemma 2. Since they are unitary invariant, we can assume that \( \mathcal{P} \) and \( \nabla f|_\mathcal{P} \) are diagonal operators \( \Sigma \) and \( \lambda \). Since we are in the convex situation, it is enough to prove that \( \mathcal{P} \) is a local maximum. Any local variation of \( \rho \) around \( \mathcal{P} \) and remaining inside \( \mathcal{D} \) is parameterized via the following mapping:

\[ (H, D) \mapsto e^{-iH} (\Sigma + D) e^{iH} = \rho_{H,D} \]

where \( H \) is any Hermitian matrix and \( D \) is any diagonal matrix of zero trace such that \( \Sigma + D \geq 0 \). We have the following expansion for \( H \) and \( D \) around zero:

\[ \rho_{H,D} = \Sigma + D - i[H, \Sigma] - i[H, D] - \frac{1}{2} [H, [H, \Sigma]] + O(\text{Tr} (H^3 + D^3)). \]

This yields to the following second order expansion of \( (H, D) \mapsto f(\rho_{H,D}) \) around zero:

\[ f(\rho_{H,D}) = f(\mathcal{P}) + \text{Tr} (\Sigma (D - i[H, \Sigma] - i[H, D] - \frac{1}{2} [H, [H, \Sigma]])) \]

\[ - \sum_{\mu \in \mathcal{M}} \frac{\text{Tr}^2 ((\rho_{H,D} - \mathcal{P}) Y_{\mu})}{2\text{Tr}^2 (\Sigma Y_{\mu})} + O(\|\rho_{H,D} - \mathcal{P}\|^3). \]

By assumptions, \( \Sigma, \Sigma \) and \( D \) are diagonal. Thus \( \text{Tr} (\Sigma (D - i[H, \Sigma] - i[H, D]))) = 0 \). Some elementary arguments exploiting \( \lambda \Sigma \leq \Sigma \leq \lambda I \), show that \( \text{Tr} (\Sigma D) \leq 0 \) since \( D \) is such that \( \Sigma + D \) is nonnegative and of trace one. We also have:

\[ -\text{Tr} (\Sigma ([H, [H, \Sigma]])) = \text{Tr} ([H, \Sigma] [H, \Sigma]) = -2 \sum_{k_1 \in P, k_2 \in Q} \Sigma_{k_1} (\Sigma_{k_2} - \lambda_{k_2}) |H_{k_1 k_2}|^2 \leq 0 \]

where \( P = \{k \mid \Sigma_k > 0\} \) and \( Q = \{k \mid \Sigma_k = 0\} \).

Consequently:

\[ f(\rho_{H,D}) \leq f(\mathcal{P}) - \sum_{\mu \in \mathcal{M}} \frac{\text{Tr}^2 ((\rho_{H,D} - \mathcal{P}) Y_{\mu})}{2\text{Tr}^2 (\Sigma Y_{\mu})} + O(\|\rho_{H,D} - \mathcal{P}\|^3). \]

Since the vector space spanned by the \( Y_{\mu} \)'s coincide with the set of Hermitian matrices, the quadratic form \( X \mapsto \sum_{\mu \in \mathcal{M}} \frac{\text{Tr}^2 (XY_{\mu})}{2\text{Tr}^2 (\Sigma Y_{\mu})} \) is non-degenerate (\( X \) is any Hermitian matrix) and \( f \) is strongly concave. Thus we have \( f(\rho) < f(\mathcal{P}) \) for \( \rho \neq \mathcal{P} \) close to \( \mathcal{P} \). Consequently, \( \mathcal{P} \) is a strict local maximum and this maximum is unique and global since \( f \) is concave.

\[ \square \]

**Theorem 3.** Consider the log-likelihood function \( f \) defined in (17). Assume that the \( Y_{\mu} \)'s span the set of Hermitian matrices. Denote by \( \mathcal{P} \) the unique maximum of \( f \) on \( \mathcal{D} \) and define a projector \( \mathcal{P} \) such that, in addition to the necessary and sufficient conditions of lemma 2, we have \( \text{ker} (\mathcal{X} \mathcal{I} - \nabla f|_\mathcal{P}) = \text{ker}(I - \mathcal{P}) \). Then, for any Hermitian operator \( A \), its Bayesian mean defined in (18) admits the following asymptotic expansion

\[ I_A(N) = \text{Tr}(A\mathcal{P}) + O(1/N) \]
and its Bayesian variance defined in (19) satisfies
\[ V_A(N) = \text{Tr} \left( A|| \left( F^{-1} A|| \right) \right) /N + O(1/N^2) \]

where

- for any Hermitian operator \( B \), \( B|| \) stands for is orthogonal projection on the tangent space at \( \mathfrak{M} \) to the submanifold of Hermitian matrices with a rank equal to the rank of \( \mathfrak{M} \) and of unit trace. It reads
\[ B|| = B - \frac{\text{Tr} (B F)}{\text{Tr} (F)} F - (I - F) B (I - F); \]

when \( \mathfrak{M} \) is full rank, \( B|| = B - \text{Tr} (B) I / d \) since \( F = I \);

- the linear super-operator \( F \) corresponds to the Hessian at \( \mathfrak{M} \) of the restriction of \( f \) to the manifold of Hermitian matrices of rank equal to the rank of \( \mathfrak{M} \) and with trace one. Its reads for any Hermitian operator \( X \),
\[ F(X) = \sum_{\mu} \frac{\text{Tr}(XY_{\mu})}{\text{Tr}^2(\mu Y_{\mu})} Y_{\mu||} + \left( \mathfrak{M} - \nabla f_{\mathfrak{M}} \right) \mathfrak{M}^+ X \left( \mathfrak{M} - \nabla f_{\mathfrak{M}} \right) \]

with \( \mathfrak{M}^+ \) the Moore-Penrose pseudo-inverse of \( \mathfrak{M} \); the restriction of \( X \mapsto \text{Tr} (X F(X)) \) to the tangent space at \( \mathfrak{M} \) is positive definite; thus the restriction of \( F \) to this tangent space is invertible and can be seen as the analogue of the Fisher information; its inverse at \( A|| \) is denoted here above by \( (F)^{-1} (A||) \).

Proof. The Hessian of \( f \) at \( \rho \in \mathcal{D} \) where \( f(\rho) > -\infty \) reads:
\[ \nabla^2 f_{\rho}(X, Z) = - \sum_{\mu} \frac{\text{Tr}(XY_{\mu}) \text{Tr}(ZY_{\mu})}{\text{Tr}^2(\mu Y_{\mu})} \]

where \( X \) and \( Z \) are any Hermitian matrices. Since it is positive definite, \( f \) is strongly concave. Consequently the argument of maximum of \( f \) on \( \mathcal{D} \) is unique, denoted \( \mathfrak{M} \) and satisfies the condition of lemma 2. Take a small neighbourhood \( V \) of \( \mathfrak{M} \) in \( \mathcal{D} \). Then there exists a \( \epsilon > 0 \) such that, for \( \rho \in \mathcal{D} \), \( f(\rho) \leq f(\mathfrak{M}) - \epsilon \). To investigate \( \int_V e^{N(f(\rho) - f(\mathfrak{M}))} \mathcal{P}_0(\rho) \ D\rho \), we consider the following local coordinates based on the spectral decomposition of \( \mathfrak{M} = U \Delta U^\dagger \) with \( U \) unitary and \( \Delta \) diagonal with entries \( 0 = \delta_1 \leq \delta_2 \leq \ldots \leq \delta_d \leq 1 \) with \( \sum_{k=1}^d \delta_k = 1 \). Denote by \( r \) the rank of \( \mathfrak{M} \) and assume that \( r < d \) (the case \( r = d \) is much simpler, relies on theorem 1 and is left to the reader). We have \( \delta_k = 0 \) for \( k \) between 1 and \( d - r \) and \( \delta_k > 0 \) for \( k \) between \( d - r + 1 \) and \( d \). Since the volume element \( D\rho \) used in (18) and (19) is unitary invariant [8, page 42], we can assume without lost of generality that \( \mathfrak{M} \) is diagonal (change \( \mathcal{P}_0(\bullet) \) to \( \mathcal{P}_0(U \bullet U^\dagger) \) and replace each \( Y_{\mu} \) by \( U^\dagger Y_{\mu} U \) in the definition (17) of \( f \)). Consider the following map
\[ (\xi, \zeta, \omega) \mapsto T = \exp \left( \begin{bmatrix} 0 & \omega^\dagger \\ -\omega & 0 \end{bmatrix} \right) \left( \begin{bmatrix} \xi & 0 \\ 0 & \Delta_r + \zeta - \frac{\text{Tr}(\xi)}{r} I_r \end{bmatrix} \right) \exp \left( \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \right) \]

where \( \xi \) is a \( (d - r) \times (d - r) \) Hermitian matrix , \( \omega \) is \( (d - r) \times r \) matrix with complex entries, \( \zeta \) is a \( r \times r \) Hermitian matrix of trace 0, \( I_r \) is the identity matrix of size \( r \) and \( \Delta = \begin{bmatrix} 0 & 0 \\ 0 & \Delta_r \end{bmatrix} \). This map is a local diffeomorphism from a neighbourhood of \( (0, 0, 0) \) to a neighbourhood of \( \mathfrak{M} \) in the set of Hermitian matrices of trace one since its tangent map at zero, given by:
\[ (\delta\xi, \delta\zeta, \delta\omega) \mapsto \begin{bmatrix} \delta\xi & 0 & \delta\omega \Delta_r \\ \Delta_r \delta\omega^\dagger & 0 & \delta\zeta - \frac{\text{Tr}(\delta\omega)}{r} I_r \end{bmatrix} = \delta\rho \]
is bijective (local inversion theorem). Thus, we have:

\[
\int_V e^{N(f(\rho)-f(\varphi))} \mathbb{P}_0(\rho) \, d\rho = \int_{\mathcal{Y}^{-1}(V)} e^{N(f(\xi,\xi,\omega)-f(0,0,0))} \mathbb{P}_0(\xi,\xi,\omega) J(\xi,\xi,\omega) \, d\xi \, d\zeta \, d\omega
\]

where \(f(\xi,\xi,\omega)\) and \(\mathbb{P}_0(\xi,\xi,\omega)\) stand for \(f(\mathcal{Y}(\xi,\xi,\omega))\) and \(\mathbb{P}_0(\mathcal{Y}(\xi,\xi,\omega))\) and where \(J(\xi,\xi,\omega)\) is the Jacobian of this change of coordinates.

Since the constraint \(\Upsilon(\xi,\xi,\omega) \geq 0\) reads \(\xi \geq 0\), we consider another change of variables to parameterize \(\xi \geq 0\) around 0: \(\Xi : (x,\sigma,\xi,\omega) \mapsto (x\sigma = \xi,\xi,\omega)\), where \(x \geq 0\) and \(\sigma\) is a \((d-r) \times (d-r)\) density matrix. Then:

\[
\int_V e^{N(f(\rho)-f(\varphi))} \mathbb{P}_0(\rho) \, d\rho = \int_{\Xi^{-1}(\mathcal{Y}^{-1}(V))} e^{N(f(x\sigma,\xi,\omega)-f(0,0,0))} \mathbb{P}_0(x\sigma,\xi,\omega) J(x\sigma,\xi,\omega) x^m \, dx \, d\sigma \, d\zeta \, d\omega
\]

with \(m = (d-r+1)(d-r-1)\). This change of variables is singular, since for \(x = 0\) it is not invertible. Nevertheless, the set of coordinates verifying \(x = 0\) is of zero measure, and then this has no impact on the integral. Take \(\eta > 0\) small enough and adjust the neighbourhood \(V\) of \(\varphi\) such that \(\Xi^{-1}(\mathcal{Y}^{-1}(V))\) coincides with the set where \(x \in (0,\eta)\), \(\sigma \in D_{d-r}\), and all the real and imaginary parts of \(\zeta\) and \(\omega\) entries belong to \((-\eta,\eta)\). Following the notations of theorem 1, set \(z = (\zeta,\omega)\). We have \(z \in (-\eta,\eta)^n\) with \(n = 2r(d-r) + (r+1)(r-1)\) and:

\[
\int_V e^{N(f(\rho)-f(\varphi))} \mathbb{P}_0(\rho) \, d\rho = \int_{\sigma \in D_{d-r}} \left( \int_{(x,z) \in (0,\eta)^n} e^{N(f(x\sigma,z)-f(0,0,0))} x^m f(x\sigma,z) \mathbb{P}_0(x\sigma,z) \, dx \, dz \right) \, d\sigma.
\]

For each \(\sigma \in D_{d-r}\), let us use (9), with \(J(x\sigma,z) \mathbb{P}_0(x\sigma,z)\) standing for \(g(x,z)\). We have \(g(0,0) = J(0,0) \mathbb{P}_0(0,0) > 0\). By construction, we have:

\[
(f(x\sigma,z) = x f_1(x,\sigma,z) + f(0,z)
\]

where \(f_1(x,\sigma,z)\) is analytic versus \((x,z)\) and \(f_1(0,\sigma,0) = (\text{Tr}(\Lambda_{d-r}\sigma) - \lambda)\). This is based on (22) and on the diagonal structure \(\nabla f_{\varphi} = \begin{bmatrix} \Lambda_{d-r} & 0 \\ 0 & \lambda I_{r} \end{bmatrix}\). By assumptions, \(\Lambda_{d-r} < \lambda I_{d-r}\). Thus, there exists \(\epsilon' > 0\) such that for all \(\sigma\), \(f_1(0,\sigma,0) < -\epsilon'\) and \(\frac{\partial f_1}{\partial \sigma} < -\epsilon'\) at \((x,z) = 0\), for any \(\sigma \in D_{d-r}\). Let us consider now the expansion of \(z \mapsto f(0,z)\) up to order 2 versus \(z\). Using \(\delta z = (\delta \zeta,\delta \omega)\) and (22), completed via second order terms derived form the Backer-Campbell-Hausdorff formula, we find:

\[
\delta \rho = \begin{bmatrix}
\delta \omega \Xi_r \delta \omega^\dagger \\
(\delta \zeta + \Xi_r \delta \omega) \delta \omega^\dagger \\
\delta \omega (\Xi_r + \delta \zeta) \\
\delta \omega \delta \omega^\dagger \Xi_r + \Xi_r \delta \omega^\dagger \delta \omega + O(||\delta z||^3)
\end{bmatrix}.
\]

Consequently,

\[
f(0,\delta z) = f(\varphi) + \text{Tr}\left(\nabla f_{\varphi} \delta \rho\right) + \frac{1}{2} \nabla^2 f_{\varphi}(\delta \rho,\delta \rho) + O(||\delta \rho||^3)
\]

\[
= f(\varphi) - \text{Tr}\left((\lambda I_{d-r} - \Lambda_{d-r})\delta \omega \Xi_r \delta \omega^\dagger\right) - \frac{1}{2} \sum_{\mu} \frac{\text{Tr}^2(\delta \rho Y_{\mu})}{\text{Tr}^2(\varphi Y_{\mu})}
\]

(23)
This shows that $\frac{\partial f}{\partial \rho}$ vanishes at $(0, z)$ and that $\frac{\partial^2 f}{\partial \sigma^2}$ is negative definite at $(0, z)$ ($\hat{\Lambda}_{d-r}$) and independent of $\sigma$. All the assumptions necessary for (9) are fulfilled and we can write:

$$\int_{\mathcal{D}} e^{Nf(\rho)} \mathbb{P}_0(\rho) \, d\rho = \kappa_0 \, e^{f(\mathcal{P})} N^{-m-n/2-1} \int_{\sigma \in \mathcal{D}_{d-r}} \frac{d\sigma}{(\lambda - \text{Tr} (\hat{\Lambda}_{d-r} \sigma))^{m+n}} + O\left(e^{f(\mathcal{P})} N^{-m-n/2-2}\right)$$

where $\kappa_0 = \frac{\mathbb{P}_0(\mathcal{P}, f(0,0))}{\sqrt{|\det \left( \frac{\partial^2 f}{\partial \sigma^2} (0,0) \right)|}}$.

Similarly we have:

$$\int_{\mathcal{D}} \text{Tr} (\rho A) \, e^{Nf(\rho)} \mathbb{P}_0(\rho) \, d\rho = \kappa_0 \, \text{Tr} (A\mathcal{P}) \, e^{f(\mathcal{P})} N^{-m-n/2-1} \int_{\sigma \in \mathcal{D}_{d-r}} \frac{d\sigma}{(\lambda - \text{Tr} (\hat{\Lambda}_{d-r} \sigma))^{m+n}} + O\left(e^{f(\mathcal{P})} N^{-m-n/2-2}\right).$$

Consequently, we have proved that: $I_A(\mathcal{N}) = \text{Tr} (\mathcal{P} A) + O(1/\mathcal{N})$.

Simple computations show that the expansion of $V_A(\mathcal{N})$ reduces to the expansion of the following integral $\int_{\mathcal{D}} \text{Tr}^2 ((\rho - \mathcal{P}) A) \, e^{Nf(\rho)} \, \mathbb{P}_0(\rho) \, d\rho$ based on (10) with $g(x, \sigma, z) = J(x\sigma, z) \mathcal{P}_0(x\sigma, z) h(x\sigma, z)$, $h(x\sigma, z) = \text{Tr}^2 (\hat{T} (x\sigma, z) - \mathcal{P} A)$ and $z = (\zeta, \omega)$. Since $\frac{\partial^2 f}{\partial \sigma^2} (0,0) = J(0,0) \mathcal{P}_0(\mathcal{P}) \frac{\partial^2 h}{\partial z^2} (0,0,0) \sigma$ is independent of $\sigma$, we have using (10):

$$\int_{\mathcal{D}} \text{Tr}^2 ((\rho - \mathcal{P}) A) \, e^{Nf(\rho)} \mathbb{P}_0(\rho) \, d\rho = \kappa_0 \, \text{Tr} \left( - \frac{\partial^2 f}{\partial \sigma^2} (0,0) \left( \frac{\partial^2 f}{\partial \sigma^2} (0,0) \right)^{-1} \right) e^{f(\mathcal{P})} N^{-m-n/2-2} \int_{\sigma \in \mathcal{D}_{d-r}} \frac{d\sigma}{(\lambda - \text{Tr} (\hat{\Lambda}_{d-r} \sigma))^{m+n}} + O\left(e^{f(\mathcal{P})} N^{-m-n/2-2}\right).$$

Consequently, we have $V_A(\mathcal{N}) = \frac{\text{Tr} \left( - \frac{\partial^2 f}{\partial \sigma^2} (0,0) \left( \frac{\partial^2 f}{\partial \sigma^2} (0,0) \right)^{-1} \right)}{2N} N^{-m-n/2-2} + O(1/\mathcal{N})$. The fact that the trace in the numerator coincides with $2\text{Tr} (A_{||} (\mathcal{P})^{-1} (A_{||}))$ results form the following computations.

- Formula (20) is unitary invariant. In the frame where $\mathcal{P} = \begin{bmatrix} 0 & 0 \\ 0 & \Delta_r \end{bmatrix}$ is diagonal, the tangent space to the manifold at $\mathcal{P}$ of rank $r$ Hermitian matrix is given by $\delta \rho$ satisfying (22) with $\delta \xi = 0$ and $(\delta \zeta, \delta \omega)$ arbitrary. One can check that (20) provides the following block decomposition $A = \begin{bmatrix} A_0 & A_{0,r} \\ A_{0,r}^* & A_r \end{bmatrix}$ for $A_{||}$ when $A = \begin{bmatrix} A_0 \\ A_{0,r}^* \end{bmatrix}$. One can also check that $A_{||}$ belongs to this tangent space and that $\text{Tr} (A \delta \rho) = \text{Tr} (A_{||} \delta \rho)$ for any tangent element $\delta \rho$.
- Since $h(0, z) = \text{Tr}^2 ((\hat{T}(0, z) - \mathcal{P}) A)$, we have:

$$\frac{\partial^2 h}{\partial z^2} (0,0) \sigma = 2\text{Tr}^2 (\delta \hat{T} A) = 2\text{Tr}^2 (\delta \hat{T} A_{||})$$
with $\delta Y = \begin{bmatrix} 0 \\ \delta \omega^t \delta \zeta \end{bmatrix}$ and $\delta z = (\delta \zeta, \delta \omega)$. This means that $\frac{\partial^2 h}{\partial z^2}|_{(0,0)}$ is colinear to the orthogonal projector on the direction given by $A_\parallel$ in the tangent space to $\mathcal{P}$. This implies that $\text{Tr} \left( \left( \frac{\partial^2 f}{\partial z^2} \right)|_{(0,0)} \left( \frac{\partial^2 f}{\partial z^2} \right)|_{(0,0)}^{-1} \right)$ corresponds to twice the value at $A_\parallel$ of the quadratic form attached to the inverse of the Hessian at $\mathcal{P}$ of the restriction of $f$ to the manifold of rank $r$ Hermitian matrices of trace one (we use here lemma 1).

- This Hessian is given by (21) since, for $X = \delta Y = \begin{bmatrix} 0 \\ \delta \omega^t \delta \zeta \end{bmatrix}$, we have:

$$\text{Tr} \left( X \left( \mathcal{X} - \nabla f|_{\mathcal{P}} \right) X \mathcal{P}^t + X \mathcal{P}^t X \left( \mathcal{X} - \nabla f|_{\mathcal{P}} \right) \right) = 2\text{Tr} \left( (\mathcal{X} - \nabla f|_{\mathcal{P}} - \lambda_{r-d}) \delta \omega \mathcal{X} \delta \omega^t \right)$$

because $\mathcal{P}^t = \begin{bmatrix} 0 & 0 \\ 0 & \Delta_r^{-1} \end{bmatrix}$. We recover from (23) that

$$f(0, z) = f(\mathcal{P}) - \frac{1}{2} \text{Tr} \left( X \mathcal{F}(X) \right),$$

i.e., that $\mathcal{F}$ is indeed the Hessian at $\mathcal{P}$ of the restriction of $f$ to rank-$r$ Hermitian matrices of trace one.

\[\square\]

## 4 Concluding remark

When maximum likelihood estimation provides a quantum state of reduced rank, we have provided, based on asymptotic expansions of specific multidimensional Laplace integrals, an estimate of the Bayesian mean and variance for any observable. We guess that similar asymptotic expansions could be of some interest for quantum compress sensing [6] when the dimension of underlying Hilbert space is large and the rank is small.

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