Lax pair tensors and integrable spacetimes

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Abstract

The use of Lax pair tensors as a unifying framework for Killing tensors of arbitrary rank is discussed. Some properties of the tensorial Lax pair formulation are stated. A mechanical system with a well-known Lax representation – the three-particle open Toda lattice – is geometrized by a suitable canonical transformation. In this way the Toda lattice is realized as the geodesic system of a certain Riemannian geometry. By using different canonical transformations we obtain two inequivalent geometries which both represent the original system. Adding a timelike dimension gives four-dimensional spacetimes which admit two Killing vector fields and are completely integrable.

1 Introduction

Many problems in general relativity require an understanding of the global structure of the spacetime. Currently discussed global problems include the occurrence of naked singularities [1] and universality in gravitational collapse situations [2]. The study of global properties of spacetimes relies to a large extent on the ability to integrate the geodesic equations. In the absence of exact solutions numerical integration is often used to obtain a quantitative picture. However, in the quest for a deeper understanding the exact and numerical approaches should be viewed as complementary tools. To perform an exact investigation of the global properties of a given spacetime, not only must the spacetime itself be an exact solution of the Einstein equations, but in addition the geodesic equations must be integrable. Usually, in a $d$-dimensional space, integrability of the geodesic equations is connected with the existence of at least $d-1$ mutually commuting Killing vector fields which span a hypersurface in the spacetime. There are exceptions however. The most well-known example is the Kerr spacetime which has only two commuting Killing vectors. In that case it is the existence of an irreducible second rank Killing tensor which makes integration possible [3]. Another example is given by Ozsvath’s class III cosmologies [4]. In that case the geodesic system was integrated using the existence of a non-abelian Lie algebra of four Killing vectors [5]. In general integrability can only be guaranteed if there is a set of $d$ constants of the motion in involution (i.e. mutually Poisson commuting). Since the metric itself always provides one constant of the motion corresponding to the squared length of the geodesic tangent vector, the geodesic system will be integrable by Liouville’s theorem if there are $d-1$ additional Poisson commuting invariants.

Exact solutions of Einstein’s equations typically admit a number of Killing vector fields. Some of these Killing vector fields may be motivated by physical considerations. For example if one is interested in static stars the spacetime must have a timelike Killing vector. For such systems it is also very reasonable to assume spherical spatial symmetry leading to a total of four (noncommuting) Killing vectors. In most cases, the number of Killing vectors is limited by the physics of the problem. In a spherically symmetric collapse

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situations, for example, the spacetime admits exactly three noncommuting Killing vector fields which form an isometry group with 2-dimensional orbits. That structure is not sufficient for an exact integration of the geodesic equations. However, the physics of the problem does not impose any a priori restrictions on higher rank \((\geq 2)\) Killing tensors. A Killing vector field, \(\xi\), plays a double role; it is both an isometry for the metric \((\mathcal{L}_{\xi} g = 0)\) and a geodesic symmetry. This last property means that it can be interpreted as a symmetry transformation for the geodesic equations. By contrast higher rank Killing tensors are only geodesic symmetries. They have no obvious geometric interpretation (but cf. \[6\]). Because of the isometry property of the Killing vector fields, such symmetries can be incorporated right from the start by assuming a particular form the metric. In this way the field equations are actually simplified by the assumption of Killing vector symmetries. On the other hand, the higher rank Killing symmetries can at present not be used to simplify the form of the field equations. Instead the Killing tensor equations must be imposed as extra conditions thereby increasing both the number of dependent variables and the number of equations.

The Lax tensors introduced in \[7\] provide a unifying framework for Killing tensors of any rank and may lead to possibilities to incorporate the higher Killing symmetries in the field equations themselves. We will comment briefly on this issue below. A single Lax tensor may generate Killing tensors of varying ranks. Lax tensors arise from a covariant formulation of the Lax pair equation \[8\] for Riemannian and pseudo-Riemannian geometries. The standard Lax pair formulation involves a pair of matrices. In the covariant formulation on the other hand, the Lax pair is represented by two third rank tensors. The first Lax matrix corresponds exactly to the first Lax tensor while the second Lax matrix and the second Lax tensor differ by a term which coincides with the Levi-Civita connection. The derivative part of the tensorial Lax pair equation is identical to the Killing-Yano equation. Therefore Killing-Yano tensors are special cases of Lax tensors for which the second Lax tensor vanishes (the second Lax matrix however does not). However, whereas Killing-Yano tensors are by definition totally antisymmetric the Lax tensors have no a priori symmetry restrictions. In this paper we discuss methods for constructing spacetimes which admit a nontrivial pair of Lax tensors. We also give two examples of such spacetimes.

2 Lax pair tensors

In this section we outline the approach to integrable geometries as given in \[7\]. We consider a Riemannian or pseudo-Riemannian geometry with metric

\[
\mathrm{ds}^2 = g_{\mu\nu} \, dq^\mu \, dq^\nu .
\]

The geodesic equations can be represented by the Hamiltonian

\[
H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu ,
\]

(2)

together with the natural Poisson bracket (denoted by \{\, , \} ) on the cotangent bundle. The geodesic system is given by

\[
\dot{q}^\alpha = \{ q^\alpha , H \} = g^{\alpha\mu} p_\mu , \quad \dot{p}_\alpha = \{ p_\alpha , H \} = \Gamma^{\mu\nu}_\alpha p_\mu p_\nu .
\]

(3)

The complete integrability of this system can be shown with the help of a pair of matrices \(L\) and \(A\) with entries defined on the phase space (the cotangent bundle) and satisfying the Lax pair equation \[8\]

\[
\dot{L} = \{ L , H \} = [L,A] .
\]

(4)

It follows from \[4\] that the quantities \(I_k := \frac{1}{2} \, \text{Tr} \, L^k\) are all constants of the motion. If in addition they commute with each other \(\{ I_k , I_j \} = 0\) (Liouville integrability) then it is possible to integrate the system completely at least in principle (see e.g. \[6\]). The Lax representation \[4\] is not unique. In fact, the Lax pair equation is invariant under a transformation of the form

\[
\dot{L} = U L U^{-1} , \quad \dot{A} = U A U^{-1} - \dot{U} U^{-1} .
\]

(5)

We see that \(L\) transforms as a tensor while \(A\) transforms as a connection. As we will see, these statements acquire a more precise meaning in the geometric formulation which we will now describe.
Typically, the Lax matrices are linear in the momenta and in the geometric setting they may also be assumed to be homogeneous. This motivates the introduction of two third rank geometrical objects, \( L^\alpha_{\beta\gamma} \) and \( A^\alpha_{\beta\gamma} \), such that the Lax matrices can be written in the form

\[
L = (L^\alpha_{\beta}) = (L^\alpha_{\beta\gamma} p_\mu), \quad A = (A^\alpha_{\beta}) = (A^\alpha_{\beta\gamma} p_\mu).
\]

(6)

We will refer to \( L^\alpha_{\beta\gamma} \) and \( A^\alpha_{\beta\gamma} \) as the Lax tensor and the Lax connection, respectively. Defining

\[
B = (B^\alpha_{\beta}) = (B^\alpha_{\beta\gamma} p_\mu) = A - \Gamma,
\]

(7)

where

\[
\Gamma = (\Gamma^\alpha_{\beta}) = (\Gamma^\alpha_{\beta\gamma} p_\mu)
\]

(8)
is the Levi-Civita connection with respect to \( g_{\alpha\beta} \), it then follows that the Lax pair equation takes the covariant form (see [1] for details)

\[
L^{\alpha}_{\beta}(\gamma;\delta) = L^{\alpha}_{\mu}(\gamma B|\mu\beta\delta) - B^{\alpha}_{\mu}(\gamma L|\mu\beta\delta),
\]

(9)

where \( L^\alpha_{\beta\gamma} \) and \( B^\alpha_{\beta\gamma} \) are tensorial objects. Note that the right-hand side of this equation is traceless, so that upon contracting over \( \alpha \) and \( \beta \) we obtain the Killing vector equation \( L^\mu_{\mu(\alpha\beta)} = 0 \). Splitting the Lax tensors in symmetric and antisymmetric parts with respect to the first two indices, \( S_{\alpha\beta\gamma} = L_{(\alpha\beta)\gamma} \), \( R_{\alpha\beta\gamma} = L_{(\alpha(\beta\gamma)} \), and \( Q_{\alpha\beta\gamma} = B_{(\alpha(\beta\gamma)} \), the Lax pair equation can be written as the system

\[
S^{\alpha\beta}_{(\gamma;\delta)} = -2S^{(\alpha}_{\mu}(\gamma Q^{\beta)\mu\delta)} + 2R^{(\alpha}_{\mu}(\gamma P^{\beta)\mu\delta)},
\]

(10)

\[
R^{\alpha\beta}_{(\gamma;\delta)} = -2R^{(\alpha}_{\mu}(\gamma Q^{\beta)\mu\delta)} + 2S^{(\alpha}_{\mu}(\gamma P^{\beta)\mu\delta)}.
\]

It is evident that this system is coupled via \( P_{\alpha\beta\gamma} \). Setting \( P_{\alpha\beta\gamma} = 0 \) gives the two separate sets of equations

\[
S^{\alpha\beta}_{(\gamma;\delta)} = -2S^{(\alpha}_{\mu}(\gamma Q^{\beta)\mu\delta)},
\]

(11)

\[
R^{\alpha\beta}_{(\gamma;\delta)} = -2R^{(\alpha}_{\mu}(\gamma Q^{\beta)\mu\delta)}.
\]

(12)

We will see below that the Lax tensors \( L_{\alpha\beta\gamma} \) and \( B_{\alpha\beta\gamma} \) in a geometrized version of the open Toda lattice are symmetric and antisymmetric respectively and therefore satisfy (11). If \( R_{\alpha\beta\gamma} \) is totally antisymmetric (with respect to all three indices) and \( Q_{\alpha\beta\gamma} = 0 \), then the equations (12) are identical to the third rank Killing-Yano equations (11). Therefore third rank Killing-Yano tensors are special cases of Lax tensors.

It is possible but not necessary to identify the invariant \( I_2 \) with the geodesic Hamiltonian (3). If such an identification is done then the metric is given by the relation

\[
g^{\alpha\beta} = L^\mu_{\mu\alpha} L^\nu_{\nu\beta}.
\]

(13)

Defining matrices \( L^\mu \) with components \((L^\mu)^{\alpha}_{\beta} = L^\alpha_{\beta\mu}\), the metric components are given by the formula

\[
g^{\alpha\beta} = \text{Tr}(L^\alpha L^\beta).
\]

(14)

This formula suggests using the components of the \( L^\mu \) (or some internal variables from which the \( L^\mu \) are built) as the basic variables already in the formulation of the field equations much like in the Ashtekar variable formalism (11).

3 Geometrization and tensorial representations of the three-particle open Toda lattice

Integrable systems are usually discussed in the context of classical mechanics. Classical Hamiltonians typically consist of a flat positive-definite kinetic energy together with a potential energy term. They are thus
superficially quite different from geometric Hamiltonians of the form (2). However, any classical Hamiltonian with a quadratic kinetic energy can be transformed to a geometric representation. One such geometrization results in the Jacobi Hamiltonian [12]. Another closely related geometrization was used in [7]. Both methods involve a reparameterization of the independent variable. Usually we will refer to the independent variable as the time, although its physical interpretation may vary. As a consequence of this feature, the original Lax representation is not preserved. It is known how to transform the invariants themselves under the time reparameterization [13, 14]. Given that the geometrized invariants are also in involution, the existence of a Lax representation is guaranteed [14]. However, to actually find such a Lax representation is non-trivial. Another geometrization scheme which does preserve the original Lax representation is to apply a suitable reparameterization [13, 14]. Given that the geometrized invariants are also in involution, the existence of a Lax representation is guaranteed [14]. However, to actually find such a Lax representation is non-trivial. Another geometrization scheme which does preserve the original Lax representation is to apply a suitable canonical transformation. This is however only possible for Hamiltonians with a potential of a special form. One such system that we will consider in this paper is the three-particle open Toda lattice

\[ H = \frac{1}{2} \left( \tilde{p}_1^2 + \tilde{p}_2^2 + \tilde{p}_3^2 \right) + e^{2(q_1 - \bar{q})^2} + e^{2(q_2 - \bar{q})^2}. \]  

(15)

Below we will discuss two canonical transformations which correspond to inequivalent geometric representations of (13). For an explicit integration of the Toda lattice, see e.g. [15]. The standard symmetric Lax representation is [15]

\[ \mathbf{L} = \begin{pmatrix} \tilde{p}_1 & \tilde{a}_1 & 0 \\ \tilde{a}_1 & \tilde{p}_2 & \tilde{a}_2 \\ 0 & \tilde{a}_2 & \tilde{p}_3 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 0 & \tilde{a}_1 & 0 \\ -\tilde{a}_1 & 0 & \tilde{a}_2 \\ 0 & -\tilde{a}_2 & 0 \end{pmatrix}, \]  

(16)

where

\[ \tilde{a}_1 = \exp(q_1 - \bar{q}), \quad \tilde{a}_2 = \exp(q_2 - \bar{q}). \]  

(17)

Note that the definitions of \( \tilde{a}_1 \) and \( \tilde{a}_2 \) differ from the ones used in [7]. The Hamiltonian (14) admits the linear invariant \( I_1 = \text{Tr} \mathbf{L} = \tilde{p}_1 + \tilde{p}_2 + \tilde{p}_3 \), corresponding to translational invariance. The Lax representation also gives rise to the two invariants \( I_2 = \frac{1}{2} \text{Tr} \mathbf{L}^2 = H \) and \( I_3 = \frac{1}{4} \text{Tr} \mathbf{L}^3 \). As discussed above, we will assume that the tensorial Lax representation is linear and homogeneous in the momenta. A homogeneous Lax representation can be obtained from the standard representation by applying a canonical transformation of the phase space. This can be done in several ways. We will investigate two possibilities below.

### 3.1 Tensorial Lax representation I

In the first attempt we straightforwardly apply a simple canonical transformation that will give a linear and homogeneous Lax representation

\[ \tilde{q}_1 = q_1 + \ln p_1, \quad \tilde{p}_1 = p_1, \]
\[ \tilde{q}_2 = q_2, \quad \tilde{p}_2 = p_2, \]
\[ \tilde{q}_3 = q_3 - \ln p_3, \quad \tilde{p}_3 = p_3. \]  

(18)

The resulting Lax pair matrices are

\[ \mathbf{L} = \begin{pmatrix} p_1 & a_1 p_1 & 0 \\ a_1 p_1 & p_2 & a_2 p_3 \\ 0 & a_2 p_3 & p_3 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 0 & a_1 p_1 & 0 \\ -a_1 p_1 & 0 & a_2 p_3 \\ 0 & -a_2 p_3 & 0 \end{pmatrix}, \]  

(19)

where

\[ a_1 = \exp(q_1 - q_2^2), \]
\[ a_2 = \exp(q_2^2 - q_3^2). \]  

(20)

The Hamiltonian is now purely kinetic

\[ H = \frac{1}{2} \text{Tr} \mathbf{L}^2 = \frac{1}{2} \left[ (1 + 2a_1^2) p_1^2 + p_2^2 + (1 + 2a_2^2) p_3^2 \right]. \]  

(21)
Using (22) we identify a metric
\[
ds^2 = g_{11}(dq^1)^2 + (dq^2)^2 + g_{33}(dq^3)^2,
\]
where
\[
g_{11} = (1 + 2a_1^2)^{-1},
\]
\[
g_{33} = (1 + 2a_2^2)^{-1}.
\]
The non-zero Levi-Civita connection coefficients, \(\Gamma_{\beta\gamma}^\alpha = \Gamma_{(\beta\gamma)}^\alpha\), of this metric are
\[
\begin{align*}
\Gamma_{11}^1 &= -2a_1^2 g_{11}, & \Gamma_{23}^3 &= 2a_2^2 g_{33}, \\
\Gamma_{12}^2 &= 2a_1^2 g_{11}, & \Gamma_{32}^3 &= -2a_2^2 g_{33}, \\
\Gamma_{21}^3 &= -2a_1^2 (g_{11})^2,
\end{align*}
\]
(24)
Following the arguments above, the homogeneous Lax matrix should correspond to a tensor with mixed indices \(L_{\alpha\beta}\). It is a reasonable assumption that the covariant Lax formulation inherits the symmetries of the standard formulation we started with. We therefore expect \(L_{\alpha\beta}\) and \(B_{\alpha\beta}\) to have the symmetries \(L_{(\alpha\beta)} = L_{\alpha\beta}\) and \(B_{(\alpha\beta)} = B_{\alpha\beta}\). Note that the symmetry properties are not imposed on the Lax matrices, \(L_{\alpha\beta}\) and \(B_{\alpha\beta}\), themselves. In fact, the required symmetries are not consistent with the representation (23). We can however perform a similarity transformation \(\tilde{\Gamma}\) of the Lax matrix, \(L \rightarrow \tilde{L}\) in such a way that \(L_{\alpha\beta}\) will be symmetric. Using the transformation matrix
\[
U = \begin{pmatrix}
1/\sqrt{g_{11}} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1/\sqrt{g_{33}}
\end{pmatrix}
\]
(25)
will give a new Lax pair
\[
\begin{align*}
\tilde{L} &= \begin{pmatrix}
p_1 & a_1/\sqrt{g_{11}} p_1 & 0 \\
a_1/\sqrt{g_{11}} p_1 & p_2 & a_2/\sqrt{g_{33}} p_3 \\
0 & a_2/\sqrt{g_{33}} p_3 & p_3
\end{pmatrix}, \\
\tilde{A} &= \begin{pmatrix}
\Gamma_{11}^1 p_1 + \Gamma_{12}^2 p_2 & a_1/\sqrt{g_{11}} p_1 & 0 \\
-a_1/\sqrt{g_{11}} p_1 & a_2/\sqrt{g_{33}} p_3 & \Gamma_{32}^3 p_2 + \Gamma_{33}^3 p_3 \\
0 & -a_2/\sqrt{g_{33}} p_3 & \Gamma_{32}^3 p_2 + \Gamma_{33}^3 p_3
\end{pmatrix},
\end{align*}
\]
(26)
where \(\tilde{L}\) is such that \(L_{\alpha\beta}\) is symmetric. Defining \(\tilde{L} = (L_{\alpha\beta})\) and \(\tilde{A} = (A_{\alpha\beta})\) by
\[
\tilde{L} = gL, \quad \tilde{A} = gA,
\]
(28)
where \(g = (g_{\alpha\beta})\), we have
\[
\begin{align*}
\tilde{L} &= \begin{pmatrix}
g_{11} p_1 & a_1/\sqrt{g_{11}} p_1 & 0 \\
a_1/\sqrt{g_{11}} p_1 & p_2 & a_2/\sqrt{g_{33}} p_3 \\
0 & a_2/\sqrt{g_{33}} p_3 & g_{33} p_3
\end{pmatrix}, \\
\tilde{A} &= \begin{pmatrix}
\Gamma_{11}^1 p_1 + \Gamma_{12}^2 p_2 & a_1/\sqrt{g_{11}} p_1 & 0 \\
-a_1/\sqrt{g_{11}} p_1 & a_2/\sqrt{g_{33}} p_3 & \Gamma_{32}^3 p_2 + \Gamma_{33}^3 p_3 \\
0 & -a_2/\sqrt{g_{33}} p_3 & \Gamma_{32}^3 p_2 + \Gamma_{33}^3 p_3
\end{pmatrix}.
\end{align*}
\]
(29)
(30)
Note that the upper triangular parts of \(\tilde{L}\) and \(\tilde{A}\) coincide. This property is peculiar to the open Toda lattice. We also define the corresponding connection matrix \(\tilde{\Gamma} = g\Gamma\) given by
\[
\tilde{\Gamma} = \begin{pmatrix}
\Gamma_{11}^1 p_1 + \Gamma_{12}^2 p_2 & 2a_1^2 g_{11} p_1 & 0 \\
-2a_1^2 g_{11} p_1 & 0 & 2a_2^2 g_{33} p_3 \\
0 & -2a_2^2 g_{33} p_3 & \Gamma_{32}^3 p_2 + \Gamma_{33}^3 p_3
\end{pmatrix}.
\]
(31)
Finally expressing $\hat{I}$

The linear invariant is

Using the relation $\hat{A} = \hat{\Gamma} + \hat{\mathcal{B}}$ where $\mathcal{B} = g \mathcal{B}$ we then find the following relation between the upper triangular components of $\hat{L}$ and $\hat{\mathcal{B}}$

Finally expressing $\hat{L}$ and $\hat{\mathcal{B}}$ in terms of $\hat{\Gamma}$ we now have for the upper triangular parts

Furthermore, the diagonal elements of $\hat{A}$ and $\hat{\Gamma}$ are identical. This implies that $\mathcal{B}$ is antisymmetric in agreement with our expectations.

### 3.2 Tensorial Lax representation II

The canonical transformation used in the previous section is not the only possible choice. The Toda lattice \([\mathbb{R}]\) has a Killing vector symmetry. In fact, by adapting the coordinates $(q^i, \bar{p}_i)$ to the linear symmetry, another representation is suggested. For this purpose, a suitable canonical transformation is

The Hamiltonian now becomes

The linear invariant is $I_1 = \sqrt{3} \bar{p}_3$. The form of the Hamiltonian now suggests applying a canonical transformation of the form

The resulting homogeneous Lax representation is

where

$$a_1 = \exp(\frac{1}{\sqrt{2}}q^1 + \frac{1}{\sqrt{2}}q^2),$$

$$a_2 = \exp(-\frac{1}{\sqrt{2}}q^1 - \frac{1}{\sqrt{2}}q^2).$$

$$a_1 = \exp(\frac{1}{\sqrt{2}}q^1 + \frac{1}{\sqrt{2}}q^2)$$

$$a_2 = \exp(-\frac{1}{\sqrt{2}}q^1 - \frac{1}{\sqrt{2}}q^2).$$
This gives the purely kinetic Hamiltonian

\[ H = \frac{1}{2} \text{Tr} L^2 = \frac{1}{2} \{ [1 + 2(a_1^2 + a_2^2)] p_1^2 + p_2^2 + p_3^2 \}, \quad (41) \]

and the corresponding metric becomes

\[ ds^2 = g_{11}(dq^1)^2 + (dq^2)^2 + (dq^3)^2, \quad (42) \]

where

\[ g_{11} = [1 + 2(a_1^2 + a_2^2)]^{-1}. \quad (43) \]

The non-zero Levi-Civita connection coefficients of this metric are

\[ \Gamma_{11}^1 = -\sqrt{2}(a_1^2 + a_2^2)g_{11}, \]
\[ \Gamma_{12}^1 = -\sqrt{6}(a_1^2 - a_2^2)g_{11}, \]
\[ \Gamma_{21}^1 = \sqrt{6}(a_1^2 - a_2^2)(g_{11})^2. \quad (44) \]

Making a similarity transformation (5) with

\[ U = \begin{pmatrix} 1/\sqrt{g_{11}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (45) \]

we obtain a Lax pair with the desired symmetry properties

\[ L = \begin{pmatrix} \frac{1}{\sqrt{2}}p_1 + \frac{1}{\sqrt{6}}p_2 & a_1/\sqrt{g_{11}}p_1 & 0 \\ a_1/\sqrt{g_{11}}p_1 & -\frac{1}{\sqrt{2}}p_2 & a_2p_1 \\ 0 & a_2p_1 & -\frac{1}{\sqrt{2}}p_1 + \frac{1}{\sqrt{6}}p_2 \end{pmatrix} + \frac{1}{\sqrt{3}}p_3 1, \quad (46) \]

\[ A = \begin{pmatrix} \Gamma_{11}^1 p_1 + \Gamma_{12}^1 p_2 & a_1/\sqrt{g_{11}}p_1 & 0 \\ -a_1/\sqrt{g_{11}}p_1 & a_2p_1 & 0 \\ 0 & -a_2p_1 & 0 \end{pmatrix}. \quad (47) \]

The matrices \( \hat{L} \) and \( \hat{A} \) have the form

\[ \hat{L} = \begin{pmatrix} \frac{1}{\sqrt{2}}p_1 + \frac{1}{\sqrt{6}}p_2 & a_1/\sqrt{g_{11}}p_1 & 0 \\ a_1/\sqrt{g_{11}}p_1 & -\frac{1}{\sqrt{2}}p_2 & a_2p_1 \\ 0 & a_2p_1 & -\frac{1}{\sqrt{2}}p_1 + \frac{1}{\sqrt{6}}p_2 \end{pmatrix} + \frac{1}{\sqrt{3}}p_3 \text{g}, \quad (48) \]

\[ \hat{A} = \begin{pmatrix} \Gamma_{11}^1 p_1 + \Gamma_{12}^1 p_2 & a_1/\sqrt{g_{11}}p_1 & 0 \\ -a_1/\sqrt{g_{11}}p_1 & a_2p_1 & 0 \\ 0 & -a_2p_1 & 0 \end{pmatrix}. \quad (49) \]

As in the first case, the upper triangular parts of \( \hat{L} \) and \( \hat{A} \) coincide. There is however no simple relation like \( (32) \) between the components of \( \hat{L} \) and the corresponding components of the connection matrix \( \hat{\Gamma} \). The form of \( \hat{B} = \hat{A} - \hat{\Gamma} \) is

\[ \hat{B} = \begin{pmatrix} 0 & \hat{B}_{12} & 0 \\ -\hat{B}_{12} & 0 & a_2p_1 \\ 0 & -a_2p_1 & 0 \end{pmatrix}, \quad (50) \]

where

\[ \hat{B}_{12} = [a_1/\sqrt{g_{11}} + \sqrt{6}(a_1^2 - a_2^2)g_{11}]p_1. \quad (51) \]
4 Four-dimensional generalizations

We can obtain a four-dimensional spacetime simply by adding a time coordinate according to the prescription

\[ (4) ds^2 = -(dq^0)^2 + ds^2, \]  

where \( ds^2 \) is a three-dimensional positive-definite metric. It follows that

\[ (4) \Gamma = \begin{pmatrix} 0 & 0 \\ 0 & \Gamma \end{pmatrix}. \]  

For the cases obtained above this will lead to inequivalent spacetimes. One way to generalize the three-dimensional Lax pair is

\[ (4) L = \begin{pmatrix} i p_0 & 0 \\ 0 & L \end{pmatrix}, \quad (4) A = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}, \]  

for which

\[ (4) B = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}. \]  

This Lax pair gives the geodesic Hamiltonian of the corresponding spacetime metric as quadratic invariant.

4.1 Case I

Adding a time dimension to (22) we obtain the metric

\[ ds^2 = -(dq^0)^2 + g_{11}(dq^1)^2 + (dq^2)^2 + g_{33}(dq^3)^2, \]  

where

\[ g_{11} = \left(1 + 2a_1^2\right)^{-1}, \]
\[ g_{33} = \left(1 + 2a_2^2\right)^{-1}, \]
\[ a_1 = \exp(q^1 - q^2), \]
\[ a_2 = \exp(q^2 - q^3). \]  

This spacetime is of Petrov type I. The nonzero components of the energy-momentum tensor calculated in a Lorentz frame are \((\kappa = 1)\)

\[ T^{00} = -((g_{11})^2 T^{11} + T^{22} + (g_{33})^2 T^{33}) \]
\[ = -4a_1^2(g_{11})^2(a_1^2 - 1) + 4a_1^2a_2^2 g_{11}g_{33} - 4a_2^2(g_{33})^2(a_2^2 - 1), \]
\[ T^{11} = 4a_1^2(a_1^2 - 1), \]
\[ T^{22} = -4a_1^2a_2^2 g_{11}g_{33}, \]
\[ T^{33} = 4a_2^2(a_2^2 - 1). \]  

4.2 Case II

Adding a time dimension to (22) we obtain the metric

\[ ds^2 = -(dq^0)^2 + g_{11}(dq^1)^2 + (dq^2)^2 + (dq^3)^2, \]  

where

\[ g_{11} = \left[1 + 2(a_1^2 + a_2^2)\right]^{-1}, \]
\[ a_1 = \exp\left(\frac{1}{\sqrt{2}} q^1 + \sqrt{\frac{3}{2}} q^3\right), \]
\[ a_2 = \exp\left(\frac{1}{\sqrt{2}} q^1 - \sqrt{\frac{3}{2}} q^3\right). \]  

8
This spacetime is of Petrov type D. The nonzero components of the energy-momentum tensor calculated in a Lorentz frame are \((\kappa = 1)\)

\[
T^{00} = 12e^{2\sqrt{2}q^1}g_{11}^{-2}\left(4 - 2\sinh^2(\sqrt{2}q^2) + e^{-\sqrt{2}q^1}\cosh(\sqrt{2}q^2)\right),
\]

\[
T^{11} = -(g^{11})^2 T^{00}. \tag{61}
\]

4.3 Comment on the energy-momentum tensors

In both of the above cases, the energy-momentum tensor takes the form

\[
T^{\alpha\beta} = \begin{pmatrix}
\mu & 0 & 0 & 0 \\
0 & p_1 & 0 & 0 \\
0 & 0 & p_2 & 0 \\
0 & 0 & 0 & p_3
\end{pmatrix}, \tag{62}
\]

where \(\mu := T^{00}\) is the energy density, and \(p_i := T^{ii}, (i = 1, 2, 3)\) are anisotropic pressures. Such an energy-momentum tensor is physically meaningful if the weak energy condition \([16]\)

\[
\mu \geq 0,
\]

\[
\mu + p_i \geq 0, \quad i = 1, 2, 3, \tag{63}
\]

is satisfied. For case I, there is an unbounded subdomain of the space coordinates \((q^1, q^2, q^3)\) for which the weak energy condition holds. For case II, it is easily seen that the restrictions on the energy-momentum tensor are inconsistent, so that the weak energy condition never holds.

5 Discussion

In this paper we have presented the first application of the tensorial Lax pair approach to integrable geometries. Two inequivalent geometries representing the three-particle open Toda lattice were found. This reflects the fact that the same underlying mathematical structure may correspond to inequivalent physical systems.

The geometrization procedure used in this work relies on canonical transformations which are peculiar to the particular problem considered. Other more general geometrization schemes involving reparameterizations of the independent variable may also be used to construct integrable geometries. However it is not known at present how to transform the Lax representation under time reparameterizations. This is despite the fact that it is known how to transform the invariants themselves \([13, 7]\). The ansätze for the metrics \([56] \) and \([59] \) are not the most general one can think of. One possible generalization is to include time-dependence in the metric coefficients. Furthermore, by starting with other integrable systems, one would expect to obtain new examples of integrable spacetimes. The possibility to find physically interesting integrable geometries with a Lax pair is thus not exhausted by the present work.

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