EXISTENCE THEOREM
FOR WEAK QUASIPERIODIC SOLUTIONS
OF LAGRANGIAN SYSTEMS
ON RIEMANNIAN MANIFOLDS

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Abstract. We establish new sufficient conditions for the existence of weak Besicovitch quasiperiodic solutions for natural Lagrangian system on Riemannian manifold with time-quasiperiodic force function.

1. Introduction. Let $\mathcal{M}$ be a smooth complete connected $m$-dimensional Riemannian manifold equipped with an inner product $\langle \cdot, \cdot \rangle$ on fibers $T_x \mathcal{M}$ of tangent bundle $T \mathcal{M}$. Consider a natural system on $\mathcal{M}$ with Lagrangian function of the form $L|_{T_x \mathcal{M}} = \frac{1}{2}\langle \dot{x}, \dot{x} \rangle - \Pi(t, x)$ where $\frac{1}{2}\langle \dot{x}, \dot{x} \rangle$ and $\Pi(t, x)$ stand for kinetic and potential energy respectively. We suppose that the potential energy is represented as $\Pi := -W(\omega t, x)$ where $W(\omega t, x)$ is $\omega$-quasiperiodic force function generated by a function $W(\cdot, \cdot) \in C^{0,2}(\mathbb{T}^k \times \mathcal{M} \rightarrow \mathbb{R})$ ($W(\cdot, \cdot)$ is continuous together with $W''_{xx}(\cdot, \cdot)$); here $\mathbb{T}^k = \mathbb{R}^k/2\pi\mathbb{Z}^k$ is $k$-dimensional torus and $\omega = (\omega_1, ..., \omega_k) \in \mathbb{R}^k$ is a frequencies vector with rationally independent components. The problem is to detect in such a system $\omega$-quasiperiodic oscillations.

J. Blot in his series of papers [1–4] applied variational method to establish the existence of weak almost periodic solutions for systems in $\mathbb{E}^m$. Later, this method was used in [5–8] to prove the existence of weak and classical almost periodic solutions for systems of variational type. In [9, 10], weak and classical quasiperiodic solutions were found for natural mechanical systems in convex compact subsets of Riemannian manifolds with non-positive sectional curvature. The goal of the present paper is to extend these results to natural systems on arbitrary Riemannian manifolds.

2. Variational method. One can interpret a natural system on $\mathcal{M}$ as a natural system in Euclidean space $\mathbb{E}^n$ (of appropriate dimension $n$) with holonomic constraint. Namely, in view of the Nash embedding theorem [11] we consider $\mathcal{M}$ as a submanifold of $\mathbb{E}^n$ for some natural $n > m$. The set $\mathcal{M} \subset \mathbb{E}^n$ play the role of holonomic constraint for natural system in $\mathbb{E}^n$ with kinetic energy $K = \frac{1}{2}\langle \dot{y}, \dot{y} \rangle_{\mathbb{E}^n}$ and potential energy $-W(\omega t, y)$, if we suppose that $W(\cdot, \cdot)$ is defined in $\mathbb{T}^k \times \mathbb{E}^n$.

In what follows we shall use identical notations for inner product $\langle \cdot, \cdot \rangle_{\mathbb{E}^n}$ of $\mathbb{E}^n$ and the induced inner product $\langle \cdot, \cdot \rangle$ on $T \mathcal{M}$. Let $\nabla_\xi$ stands for the covariant differentiation of Levi-Civita connection in the direction of vector
\( \xi \in TM \), and let \( \nabla f \) stands for gradient vector field of a scalar function \( f(\cdot) : M \to \mathbb{R} \), i.e. \( \langle \nabla f(x), \xi \rangle = df(x)(\xi) \) for any \( \xi \in T_xM \).

Denote by \( H(\mathbb{T}^k \to \mathbb{E}^n) \) the space of \( \mathbb{E}^n \)-valued functions on \( k \)-torus which are integrable with the square of Euclidean norm \( \| \cdot \| := \sqrt{\langle \cdot, \cdot \rangle} \). Define on \( H(\mathbb{T}^k \to \mathbb{E}^n) \) the standard scalar product \( \langle \cdot, \cdot \rangle_0 = (2\pi)^{-k} \int_{\mathbb{T}^k} \langle \cdot, \cdot \rangle_0 d\varphi \) and the corresponding semi-norm \( \| \cdot \|_0 := \sqrt{\langle \cdot, \cdot \rangle_0} \). By \( H^1_\omega(\mathbb{T}^k \to \mathbb{E}^n) \) denote the space of functions \( f(\cdot) \in H(\mathbb{T}^k \to \mathbb{E}^n) \) each of which has weak (Sobolev) derivative \( D_\omega f(\cdot) \in H(\mathbb{T}^k \to \mathbb{E}^n) \) in the direction of vector \( \omega \). Recall that a function \( u(\cdot) \in H(\mathbb{T}^k \to \mathbb{E}^n) \) with Fourier series \( \sum_{n \in \mathbb{Z}^k} u_n e^{i \omega \cdot \xi} \) has a weak derivative iff the series \( \sum_{n \in \mathbb{Z}^k} |n \cdot \omega|^2 \|u_n\|^2 \) converges and then the Fourier series of \( D_\omega u(\cdot) \) is \( \sum_{n \in \mathbb{Z}^k} i(n \cdot \omega) u_n e^{i \omega \cdot \xi} \).

The space \( H^1_\omega(\mathbb{T}^k \to \mathbb{E}^n) \) is equipped with the semi-norm \( \| \cdot \|_1 \) generated by the scalar product \( \langle D_\omega \cdot, D_\omega \cdot \rangle_0 + \langle \cdot, \cdot \rangle_0 \). After identification of functions coinciding a.e., both spaces becomes Hilbert spaces with norms \( \| \cdot \|_0 \) and \( \| \cdot \|_1 \) respectively.

To any function \( u(\cdot) \in H(\mathbb{T}^k \to \mathbb{E}^n) \) with Fourier series \( \sum_{n \in \mathbb{Z}^k} u_n e^{i \omega \cdot \xi} \), one can put into correspondence a Besicovitch quasiperiodic function \( x(t) = u(\omega t) \) defined by its Fourier series \( \sum_{n \in \mathbb{Z}^k} u_n e^{i(n \omega) t} \). If \( u(\cdot) \in H^1_\omega(\mathbb{T}^k \to \mathbb{E}^n) \) then \( \dot{x}(t) \) denotes a Besicovitch quasiperiodic function \( D_\omega u(\omega t) \).

We define weak solution of Lagrangian system on \( M \) with density \( L = \frac{1}{2} \langle \dot{x}, \dot{x} \rangle + W(\omega t, x) \) in a slightly different way then in [7]. First, for any bounded subset \( A \subseteq M \), put
\[
S_A := C^\infty(\mathbb{T}^k \to A).
\]

Observe that if \( u_j(\cdot) \in S_A \) is a sequence bounded in \( H^1_\omega(\mathbb{T}^k \to \mathbb{E}^n) \) and convergent to a function \( u(\cdot) \) by norm of the space \( H(\mathbb{T}^k \to \mathbb{E}^n) \) (recall that we consider the set \( A \subseteq M \) both as a subset of \( \mathbb{E}^n \)), then for any \( n \in \mathbb{Z}^k \), the sequence of Fourier series coefficients \( u_{n,j} \) converges to \( u_n \) and for some \( K > 0 \) we have
\[
\sum_{|n| \leq N} |n \cdot \omega|^2 \|u_n\|^2 = \lim_{j \to \infty} \sum_{|n| \leq N} |n \cdot \omega|^2 \|u_{j,n}\|^2 \leq \liminf_{j \to \infty} \sum_{n \in \mathbb{Z}^k} |n \cdot \omega|^2 \|u_{j,n}\|^2 \leq K \quad \forall N \in \mathbb{N}.
\]

Hence, \( u(\cdot) \in H^1_\omega(\mathbb{T}^k \to \mathbb{E}^n) \) and \( \|D_\omega u\|_0 \leq \liminf_{j \to \infty} \|D_\omega u_j\|_0 \). Moreover, \( u_j(\cdot) \) converges to \( u(\cdot) \) weakly in \( H^1_\omega(\mathbb{T}^k \to \mathbb{E}^n) \).

Next, for any bounded subset \( A \subseteq M \) define a functional space \( H_A \) in a following way: \( u(\cdot) \in H_A \) iff there exists a sequence \( u_j(\cdot) \in S_A \) bounded in \( H^1_\omega(\mathbb{T}^k \to \mathbb{E}^n) \) and convergent to \( u(\cdot) \) by norm of the space \( H(\mathbb{T}^k \to \mathbb{E}^n) \) (recall that we consider the set \( A \subseteq M \) both as a subset of \( \mathbb{E}^n \)). As it was noted above \( H_A \subseteq H^1_\omega(\mathbb{T}^k \to \mathbb{E}^n) \). We shall say that \( h(\cdot) \in H^1_\omega(\mathbb{T}^k \to \mathbb{E}^n) \) is a vector field along the map \( u(\cdot) \in H_A \) defined in the above sens by a sequence \( u_j(\cdot) \) if there exists a sequence \( h_j(\cdot) \in C^\infty(\mathbb{T}^k \to TM) \) such that
both on the set $A$ determine whether the functional of averaged kinetic energy, namely Riemannian manifold of non-positive sectional curvature, we are not able to formulated by means of geodesics. But in the case where $(\mathcal{M}, \langle \cdot, \cdot \rangle)$ equipped with corresponding Levi-Civita connection is complete.

Definition 1. A Besicovitch quasiperiodic function $u(\omega t)$ generated by a function $u(\cdot) \in \mathcal{H}_A$ is called a weak quasiperiodic solution of the natural system on $\mathcal{M}$ if it satisfies the equality

$$\langle D_\omega u(\varphi), D_\omega h(\varphi) \rangle_0 + \langle W'_x(\varphi, u(\varphi)), h(\varphi) \rangle_0 = 0$$

for any vector field $h(\cdot)$ along $u(\cdot)$.

This definition is natural since the equality holds true for any classical quasiperiodic solution $u(\omega t)$ and continuous vector field $h(\varphi)$ along $u(\cdot)$ with continuous derivative $D_\omega h(\cdot)$. It should be also noted the following fact.

The application of variational approach to the problem of detecting weak quasiperiodic solution consists in finding a function $u(\cdot) \in \mathcal{H}_A$ which takes values in appropriately chosen bounded subset $A \subset \mathcal{M}$ and which is a strong limit in $H(\mathbb{T}^k \rightarrow \mathbb{R}^n)$ of minimizing sequence for the functional (the averaged Lagrangian)

$$J[u] = \int_{\mathbb{T}^k} \left[ \frac{1}{2}||D_\omega u(\varphi)||^2 + W(\varphi, u(\varphi)) \right] d\varphi$$

restricted to $\mathcal{S}_A$. It is naturally to expect that the first variation of $J$ at $u*(\cdot)$ vanishes, i.e.

$$J'[u_*(h)] := \langle D_\omega u_*(\varphi), D_\omega h(\varphi) \rangle_0 + \langle W'_x(\varphi, u_*(\varphi)), h(\varphi) \rangle_0 = 0$$

for any vector field $h(\cdot)$ along $u_*(\cdot)$. In such a case $u_*(\omega t)$ is a weak quasiperiodic solution.

In order to guarantee the convergence of a minimizing sequence $u_j(\cdot) \in \mathcal{S}_A$ for $J|_{\mathcal{S}_A}$ by norm $\|\cdot\|_0$ it is naturally to impose some convexity conditions both on the set $A$ and on the functional $J$. Usually, such conditions are formulated by means of geodesics. But in the case where $(\mathcal{M}, \langle \cdot, \cdot \rangle)$ is not a Riemannian manifold of non-positive sectional curvature, we are not able to determine whether the functional of averaged kinetic energy, namely $J_1[u] := \frac{1}{2} \int_{\mathbb{T}^k} ||D_\omega u(\varphi)||^2 d\varphi$, is convex using geodesics of Levi-Civita connection $\nabla$, if $(\mathcal{M}, \langle \cdot, \cdot \rangle)$. (The case of Riemannian manifold of non-positive sectional curvature was considered in [9, 10].)

In order to overcome the above difficulty we introduce a conformally equivalent inner product of the form $(\cdot, \cdot)_V|_{T_x\mathcal{M}} := e^{V(x)} \langle \cdot, \cdot \rangle|_{T_x\mathcal{M}}$ with appropriately chosen smooth function $V(\cdot) : \mathcal{M} \mapsto \mathbb{R}$. With this approach we succeed in establishing a required convexity properties of averaged Lagrangian under certain convexity conditions imposed on functions $V(\cdot)$ and $W(\varphi, \cdot)$.

3. Convexity of averaged Lagrangian. It is easily seen that if $V(\cdot) \in C^\infty(\mathcal{M} \mapsto \mathbb{R})$ is a bounded function on $\mathcal{M}$ then the Riemannian manifold $(\mathcal{M}, \langle \cdot, \cdot \rangle_V)$ equipped with corresponding Levi-Civita connection is complete.
In fact, by definition, the standard distance between any two points \( x_1, x_2 \in (\mathcal{M}, \langle \cdot, \cdot \rangle) \) is defined as

\[
\rho(x_1, x_2) := \inf \{ l(c) : c \in \mathcal{C}_{x_1,x_2} \},
\]

where \( \mathcal{C}_{x_1,x_2} \) is the set of all piecewise differentiable paths \( c : [0,1] \to \mathcal{M} \) connecting \( x_1 \) with \( x_2 \), and \( l(c) \) is the length of \( c \) on \( (\mathcal{M}, \langle \cdot, \cdot \rangle) \). If we denote by \( l_V(c) \) the length of path \( c \) on \( (\mathcal{M}, \langle \cdot, \cdot \rangle_V) \), then

\[
\inf_{x \in \mathcal{M}} \sqrt{e^V(x) l(c)} \leq l_V(c) \leq \sup_{x \in \mathcal{M}} \sqrt{e^V(x) l(c)}.
\]

Hence, the metric \( \rho(\cdot, \cdot) \) and the metric \( \rho_V(\cdot, \cdot) \) of \( (\mathcal{M}, \langle \cdot, \cdot \rangle_V) \) are equivalent.

Now it remains only to apply the Hopf-Rinow theorem (see, e.g., \cite{13}, sect. 5.3).

In order to distinguish geodesics of metrics \( \rho \) and \( \rho_V \) we shall call them \( \rho \)-geodesic and \( \rho_V \)-geodesic respectively.

For \( x \in \mathcal{M} \), let \( \exp_x(\cdot) : T_x \mathcal{M} \to \mathcal{M} \) denotes the exponential mapping of Riemannian manifold \( (\mathcal{M}, \langle \cdot, \cdot \rangle) \) with Levi-Civita connection \( \nabla \) and let \( \exp^V_x(\cdot) : T_x \mathcal{M} \to \mathcal{M} \) be the analogous mapping of Riemannian manifold \( (\mathcal{M}, \langle \cdot, \cdot \rangle_V) \) with corresponding Levi-Civita connection \( \nabla^V \). Note that since both manifolds are complete the domains of both exponential mappings coincide with entire \( T_x \mathcal{M} \).

Recall that a set of a Riemannian manifold is called convex if together with any two points \( x_1, x_2 \) this set contains a (unique) minimal geodesic segment connecting \( x_1 \) with \( x_2 \) (see, e.g., \cite{12}, sect. 11.8) or \cite{13}, sect. 5.2). It is well known that for any point \( x_0 \) an open ball of sufficiently small radius centered at point \( x_0 \) is convex. A function \( f : D_f \to \mathbb{R} \) with convex domain \( D_f \subset \mathcal{M} \) is convex iff its superposition with any naturally parametrized geodesic containing in \( D_f \) is convex.

Recall also that for the function \( V(\cdot) \), the Hesse form \( H_V(x) \) at point \( x \) (see., e.g., \cite{13}) is defined by the equality

\[
\langle H_V(x) \xi, \eta \rangle := \langle \nabla_x \nabla V(x), \eta \rangle \quad \forall \xi, \eta \in T_x \mathcal{M}.
\]

In addition, let us introduce the following quadratic form

\[
\langle G_V(x) \xi, \xi \rangle := \langle H_V(x) \xi, \xi \rangle - \frac{1}{2} \langle \nabla V(x), \xi \rangle^2 \quad \forall \xi \in T_x \mathcal{M},
\]

and denote

\[
\lambda_V(x) := \min_{\xi \in T_x \mathcal{M} \setminus \{0\}} \frac{\langle H_V(x) \xi, \xi \rangle}{\| \xi \|^2},
\]

\[
\mu_V(x) := \min_{\xi \in T_x \mathcal{M} \setminus \{0\}} \frac{\langle G_V(x) \xi, \xi \rangle}{\| \xi \|^2}.
\]

We accept the following hypotheses concerning convexity properties of functions \( V(\cdot) \) and \( W(\cdot) \):
Theorem 1. Let the Hypotheses (H1)–(H3) hold true. Then there exist positive constants $C$ such that

\[ \lambda_V(x) + \frac{1}{2} \| \nabla V(x) \|^2 \geq 0, \quad \forall x \in D; \quad (4) \]

(H2): there exist a noncritical value $v \in V(D)$ and a connected component $\Omega$ of open sublevel set $V^{-1}(((-\infty, v))$ with the following properties: (a) for any $x, y \in \Omega$ the domain $D$ contains a unique minimal $\rho_V$-geodesic segment with endpoints $x, y$; (b) the second fundamental form of $\partial \Omega$ is positive at each point $x \in \partial \Omega$ (i.e. for any $x \in \partial \Omega$ the restriction of $H_V(x)$ to $T_x \partial \Omega$ is positive definite); (c) the function $V(\cdot)$ satisfies the inequality

\[ \mu_V(x) \geq 2K^*(x) \quad \forall x \in \Omega \quad (5) \]

where

\[ K^*(x) := \max_{\sigma(x, \eta)} \frac{(R(\eta, \xi, \eta)}{\| \sigma \|^2 - \| \eta \|^2} \]

is the maximum sectional curvature at point $x$, $R$ is the Riemann curvature tensor of $(\mathcal{M}, \langle \cdot, \cdot \rangle)$, $\sigma(x, \eta)$ is a plane defined by vectors $\xi, \eta \in T_x \mathcal{M}$, and $K(\sigma(x, \eta))$ is a sectional curvature in direction $\sigma(x, \eta)$ [3];

(H3): the function $W(\cdot, \cdot)$ satisfies the following inequalities

\[ \lambda_W(\varphi, x) + \frac{1}{2} \langle \nabla W(\varphi, x), \nabla V(x) \rangle > 0 \quad \forall (\varphi, x) \in \mathbb{T}^k \times \bar{\Omega} \quad (\Omega := \Omega \cup \partial \Omega), \]

\[ \langle \nabla W(\varphi, x), \nabla V(x) \rangle > 0 \quad \forall (\varphi, x) \in \mathbb{T}^k \times \partial \Omega \]

where $\lambda_W(\varphi, x)$ is minimal eigenvalue of Hesse form $H_W(\varphi, x)$ for the function $W(\varphi, \cdot) : \mathcal{M} \mapsto \mathbb{R}$.

Theorem 1. Let the Hypotheses (H1)–(H3) hold true. Then there exist positive constants $C, C_1$ and $c$ such that for any $u_0(\cdot), u_1(\cdot) \in C^\infty(\mathbb{T}^k \mapsto \Omega)$ one can choose a vector field $h(\cdot) \in C^\infty(\mathbb{T}^k \mapsto TM)$ along $u_0(\cdot)$ (this implies that $h(\varphi) \in T_u(\varphi) \mathcal{M}$ for all $\varphi \in \mathbb{T}^k$) in such a way that the following inequalities hold true

\[ c \rho(u_0(\varphi), u_1(\varphi)) \leq \| h(\varphi) \| \leq C \rho(u_0(\varphi), u_1(\varphi)) \quad \forall \varphi \in \mathbb{T}^k, \]

\[ \| D_\omega h(\varphi) \| \leq C_1 \{ \| D_\omega u_0(\varphi) \| + \| D_\omega u_1(\varphi) \| \} \quad \forall \varphi \in \mathbb{T}^k, \]

\[ J[u_1] - J[u_0] - J'[u_0](h) \geq \frac{\kappa^2}{2} \int_{\mathbb{T}^k} \rho^2(u_0, u_1) d\varphi \]

where $\kappa := \min \{ \lambda_W(\varphi, x) + \frac{1}{2} \langle \nabla W(\varphi, x), \nabla V(x) \rangle : (\varphi, x) \in \mathbb{T}^k \times \bar{\Omega} \}$.

The proof of this theorem needs several auxiliary statements and will be given below at the end of present Section.
Proposition 1. The Euler-Lagrange equation for $\rho_V$-geodesic on Riemannian manifold $(\mathcal{M}, \langle \cdot, \cdot \rangle)$ has the form

$$\nabla_x \dot{x} = -\langle \nabla V(x), \dot{x} \rangle \dot{x} + \frac{||\dot{x}||^2}{2} \nabla V(x),$$

(6)

Proof. A $\rho_V$-geodesic segment with endpoints $x_0, x_1 \in \mathcal{M}$ is an extremal of functional $\Phi[x(\cdot)] = \int_0^1 e^{V\rho(x(t))} ||\dot{x}(t)||^2 \, dt$ defined on the space $C^2_{x_0,x_1}$ of twice continuous differentiable curves $x = x(t)$, $t \in [0,1]$, such that $x(0) = x_0$, $x(1) = x_1$. We are going to derive the Euler-Lagrange equation using the connection $\nabla$. Consider a variation of $x(\cdot)$ defined by a smooth mapping $y(\cdot, \lambda) : [0,1] \times (-\varepsilon, \varepsilon) \rightarrow \mathcal{M}$ such that $y(\cdot, \lambda) \in C^\infty_{x_0, x_1}$ for any fixed $\lambda \in (-\varepsilon, \varepsilon)$ and $y(t, 0) \equiv x(t)$. Put

$$y(t, \lambda) := \frac{\partial}{\partial \lambda} y(t, \lambda), \quad y'(t, \lambda) := \frac{\partial}{\partial \lambda} y(t, \lambda).$$

Obviously, $\dot{y}(t, 0) = \dot{x}(t)$, $y(i, \lambda) \equiv x_i$, and $y'(i, \lambda) = 0$, $i = 0, 1$. Then since $\nabla_{y'} \dot{y} = \nabla_{y} y'$, we have

$$\frac{d}{d\lambda} \bigg|_{\lambda=0} \int_0^1 e^{V\rho} ||\dot{y}||^2 \, ds =$$

$$= \int_0^1 \left[ e^{V\rho} \langle \nabla V \circ y', y' \rangle ||\dot{y}||^2 + 2e^{V\rho} \langle \nabla y' \dot{y}, \dot{y} \rangle \right]_{\lambda=0} \, dt =$$

$$= \int_0^1 \left[ e^{V\rho} \langle \nabla V \circ y', y' \rangle ||\dot{y}||^2 + 2e^{V\rho} \langle \nabla y' \dot{y}, \dot{y} \rangle \right]_{\lambda=0} \, dt.$$

Taking into account that

$$\frac{\partial}{\partial t} e^{V\rho} \langle y', \dot{y} \rangle = e^{V\rho} \langle \nabla V \circ y, \dot{y} \rangle \langle y', \dot{y} \rangle + e^{V\rho} \langle \nabla y' \dot{y}, \dot{y} \rangle + e^{V\rho} \langle y', \nabla \dot{y} \rangle$$

and $e^{V\rho} \langle y', \dot{y} \rangle \big|_{t=0,1} = 0$, we get

$$\int_0^1 e^{V\rho} \langle \nabla y' \dot{y}, \dot{y} \rangle \, dt = -\int_0^1 e^{V\rho} \left[ \langle \nabla V \circ y, \dot{y} \rangle \langle y', \dot{y} \rangle + \langle y', \nabla \dot{y} \rangle \right] \, dt.$$

From this it follows that the first variation on functional $\Phi$ is

$$\frac{d}{d\lambda} \bigg|_{\lambda=0} \Phi[y(\cdot, \lambda)] = \Phi'[x(\cdot)] (y'(\cdot, 0)) =$$

$$= \int_0^1 \left[ e^{V} \left( \langle \nabla V, y' \rangle ||\dot{x}||^2 - 2 \langle \nabla V, \dot{x} \rangle \langle \dot{x}, y' \rangle - 2 \langle \nabla \dot{x}, y' \rangle \right) \right]_{x=x(t), \lambda=0} \, dt,$$

and the Euler-Lagrange equation is exactly (6). \qed
Proposition 2. Let the Hypothesis (H1) holds true. If a $\rho_V$-geodesic segment connecting points $x_0, x_1$ of the set $\Omega$ belongs to $D$, then this segment belongs to $\Omega$.

Proof. Let $x(\cdot) \in C^2_{x_0, x_1}$ satisfies (6) and let $x(t) \in D$ for all $t \in [0, 1]$. Then

$$\frac{d^2}{dt^2}e^V|_{x=x(t)} = \left[ e^V \left( \langle \nabla_x \nabla V, \dot{x} \rangle + \langle \nabla V, \dot{x} \rangle + \frac{1}{2} \| \dot{x} \|^2 \right) \right]|_{x=x(t)} = \left[ e^V \left( \langle \nabla_x \nabla V, \dot{x} \rangle + \| \dot{x} \|^2 \| \nabla V \|^2 / 2 \right) \right]|_{x=x(t)} \geq 0.$$

Hence, $e^{V \circ x(\cdot)}$ is convex and this implies $V \circ x(t) < v$ for all $t \in [0, 1]$. \qed

Proposition 3. Under the Hypotheses (H1)-(H2), the minimal $\rho_V$-geodesic segment connecting any two points $x, y \in \Omega$ does not contain conjugate points.

Proof. It is known (see [13 sect. 3.6]) that the sectional curvature in direction $\sigma_x(\xi_1, \xi_2)$ on Riemannian manifold $(M, e^V \langle \cdot, \cdot \rangle)$ is represented in the form

$$K_V(\sigma_x(\xi_1, \xi_2)) = e^{-V}K(\sigma_x(\xi_1, \xi_2)) - \sum_{i=1}^2 \left[ \langle H_V(x)\xi_i, \xi_i \rangle - \frac{1}{2} \langle \nabla V(x), \xi_i \rangle^2 \right] - \frac{e^{-V}}{4} \| \nabla V(x) \|^2,$$

where $\xi_1, \xi_2$ is an orthonormal basis of the plane $\sigma_x(\xi_1, \xi_2)$, and the inequality [4] yields that this curvature is non-positive for any $x \in \Omega$. By the Morse–Schoenberg theorem any $\rho_V$-geodesic segment containing in $\Omega$ does not contain conjugate points. \qed

Proposition 4. Under the Hypotheses (H1)-(H3) there exists a smooth mapping $\zeta(\cdot, \cdot) : \Omega \times \Omega \rightarrow TM$ such that $\zeta(x, y) \in T_x M$ and

$$\exp^V_x(\zeta(x, y)) = y, \quad e^{V(x)/2} \| \zeta(x, y) \| = \rho_V(x, y), \quad (7)$$

$$\exp^V_x(t\zeta(x, y)) \in \Omega \quad \forall t \in [0, 1]. \quad (8)$$

Proof. It is known that if for some $\xi \in T_x M$ a geodesic segment $\exp^V_x(t\xi), \ t \in [0, 1]$, does not contain conjugate points then the mapping $\exp^V_x(\cdot)$ is local diffeomorphism at any point $t\xi$, $t \in [0, 1]$. Under the Hypothesis (H2) for any $x, y \in \Omega$ there exists a unique $\zeta(x, y)$ which satisfies conditions [5]. It follows from the implicit function theorem that the mapping $\zeta(\cdot, \cdot) : \Omega \times \Omega \rightarrow TM$ is smooth. \qed

If we define the mapping

$$\gamma_V(\cdot, \cdot) : [0, 1] \times \Omega \times \Omega \rightarrow \Omega, \quad \gamma_V(t, x, y) := \exp^V_x(t\zeta(x, y)),$$
then for any \( x, y \in \mathcal{D} \) the mapping \( \gamma_V(\cdot, x, y) : [0, 1] \mapsto \mathcal{D} \) satisfies the equation (6) together with boundary conditions \( \gamma_V(0, x, y) = x, \gamma_V(1, x, y) = y \).

The following scalar differential equation
\[
\frac{d\tau}{ds} = \exp (V \circ \gamma_V(\tau, x, y)) \int_0^1 \exp (-V \circ \gamma_V(t, x, y)) dt.
\]
has a unique strictly monotonically increasing solution
\[
\tau(\cdot, x, y) : [0, 1] \mapsto [0, 1], \quad \tau(0, x, y) = 0, \quad \tau(1, x, y) = 1.
\]
By means of reparametrisation \( t = \tau(s), x, y \) we define a smooth mapping
\[
\chi(\cdot, \cdot, \cdot) : [0, 1] \times \Omega \times \Omega \mapsto \Omega, \quad \chi(s, x, y) := \gamma_V(\tau(s, x, y), x, y)
\]
which plays an important role in subsequent reasoning. In [7] \( \chi(\cdot, \cdot, \cdot) \) is called the connecting mapping.

**Proposition 5.** For any \( x, y \in \Omega \) the mapping \( \chi(\cdot, x, y) : [0, 1] \mapsto \Omega \) satisfies the equation
\[
\nabla_{x'}x' = \frac{\|x'\|^2}{2} \nabla V(x),
\]
where \( x' = \frac{dx}{ds} \) and the boundary conditions \( \chi(0, x, y) = x, \chi(1, x, y) = y \).

**Proof.** The boundary conditions follow from definition of \( \gamma_V \) and (9). Let us show that (10) is obtained from (6) after the change of independent variable \( t = \tau(s) \). In fact, let \( \chi(s) = x \circ \tau(s) \). Then (6) takes the form
\[
\frac{1}{\tau'} \nabla \chi' \left( \frac{1}{\tau} \chi' \right) = -\frac{1}{(\tau')^2} \left\langle \nabla V \circ \chi, \chi' \right\rangle \chi' + \frac{\|\chi'\|^2}{2(\tau')^2} \nabla V \circ \chi,
\]
or
\[
-\frac{\tau''}{\tau'} \chi' + \nabla \chi' \chi' = -\left[ \frac{d}{ds} \nabla V \circ \chi \right] \chi' + \frac{\|\chi'\|^2}{2} \nabla V \circ \chi.
\]
From this it follows (10) since \( \tau''/\tau' = (V \circ \chi)' \).

**Proposition 6.** Let \( u_i(\cdot) \in \mathcal{S}_\Omega, i = 0, 1 \). Then under the hypotheses (H1)-(H2) the following inequality is valid
\[
\frac{d^2}{ds^2} \| D_{\omega} \chi(s, u_0(\varphi), u_1(\varphi)) \|^2 \geq 0 \quad \forall s \in [0, 1], \forall \varphi \in T^k.
\]

**Proof.** For any fixed \( \varphi \in T^k \) put
\[
\eta(s, t) := \frac{\partial}{\partial \varphi} \chi(s, u_0(\varphi + \omega t), u_1(\varphi + \omega t)),
\]
\[
\xi(s, t) := \frac{\partial}{\partial s} \chi(s, u_0(\varphi + \omega t), u_1(\varphi + \omega t)).
\]
Then in view of the well known relations (see. e.g., [13], DNF84)
\[
\nabla_\eta \xi = \nabla_\xi \eta, \quad \nabla_\eta \nabla_\xi \xi - \nabla_\xi \nabla_\eta \xi = R(\eta, \xi) \xi
\]
and (10), we have
\[
\nabla^2_\xi \eta = \nabla_\eta \nabla_\xi \xi - R(\eta, \xi) \xi = \\
= (\nabla_\eta \xi, \xi) \nabla V \circ \chi + \frac{\|\xi\|^2}{2} \nabla_\eta \nabla V \circ \chi - R(\eta, \xi) \xi
\]
and hence,
\[
\frac{d^2}{ds^2} \|\eta\|^2 = 2 \left[ (\nabla^2_\xi \eta, \eta) + \|\nabla_\xi \eta\|^2 \right] = \\
= 2 \|\nabla_\xi \eta\|^2 + 2 (\nabla_\xi \eta, \xi) (\nabla V \circ \chi, \eta) + \\
+ \|\xi\|^2 (\nabla_\eta \nabla V \circ \chi, \eta) - 2 (R(\eta, \xi) \xi, \eta) \geq \\
\geq 2 \|\nabla_\xi \eta\|^2 - 2 \|\nabla_\xi \eta\| \|\xi\| \|\nabla V \circ \chi, \eta\| + \\
+ \|\xi\|^2 (\nabla_\eta \nabla V \circ \chi, \eta) - 2 K^* \circ \chi \|\xi\|^2 \|\eta\|^2.
\]
Once the Hypothesis (H2) holds true, we get
\[
\frac{d^2}{ds^2} \|\eta\|^2 \geq \\
\geq 2 \|\xi\|^2 \|\eta\|^2 \left[ r^2 - \|\nabla V \circ \chi, \eta\| + \frac{1}{2} (\nabla e \nabla V \circ \chi, \eta) - K^* \circ \chi \right] \geq 0
\]
where \( r := \frac{\|\nabla_\xi \eta\|}{\|\xi\|}. \) \( \square \)

Now we are in position to prove the Theorem \( \square \). Let \( u_i(\cdot) \in S_0, \ i = 0, 1. \) By means of connecting mapping we get the following representation
\[
J[\chi(s, u_0, u_1)] = J[u_0] + sJ'[u_0] (\chi'_s(0, u_0, u_1)) + \frac{s^2}{2} \frac{d^2}{ds^2} \bigg|_{s=\theta} J[\chi(s, u_0, u_1)]
\]
with some \( \theta \in (0, 1). \) To estimate from below the term with second derivative we make use of Proposition \( \square \) which together with the Hypothesis (H3) implies
\[
\frac{d^2}{ds^2} \left[ \frac{1}{2} \|D_\omega \chi(s, u_0(\varphi), u_1(\varphi))\|^2 + W(\varphi, \chi(s, u_0, u_1)) \right] \geq \\
\geq \frac{d}{ds} \langle \nabla W(\varphi, \chi), \chi'_s \rangle = \langle \nabla_{\chi'_s} \nabla W(\varphi, \chi), \chi'_s \rangle + \langle \nabla W(\varphi, \chi), \nabla_{\chi'_s} \chi'_s \rangle = \\
= \langle \nabla_{\chi'_s} \nabla W(\varphi, \chi), \chi'_s \rangle + \frac{\|\chi'_s\|^2}{2} \langle \nabla W(\varphi, \chi), \nabla V(\chi) \rangle \geq \kappa \|\chi'_s\|^2.
\]
By the definition of \( \chi \) we have
\[
\chi'_s(s, u_0, u_1) = \tau'(s) \gamma_V(\tau(s), u_0, u_1) = \\
= \exp(V \circ \gamma_V(\tau(s), u_0, u_1)) \int_0^1 \exp(-V \circ \gamma_V(t, u_0, u_1)) dt \gamma_V(\tau(s), u_0, u_1).
\]
Since $\gamma_V(t, x, y)$ is $\rho_V$-geodesic, then $\exp(V \circ \gamma_V)\|\dot{\gamma}_V\|^2$ does not depend on $t$ and
\[
e^{V(x)/2}\|\dot{\gamma}_V(0, x, y)\| = e^{V(x)/2}\|\xi(x, y)\| = \rho_V(x, y).
\]
Hence
\[
\|\chi_s'(s, u_0, u_1)\|^2 = \left[\int_0^1 \exp(-V \circ \gamma_V(t, u_0, u_1)) \, dt\right]^2 \times 
\times \exp(V \circ \gamma_V(\tau(s), u_0, u_1)) \rho_V^2(u_0, u_1),
\]
and (11) implies that there exist positive constants $C, c$ dependent only on $V(\cdot)$ and $\Omega$ such that
\[
ce \rho(u_0, u_1) \leq \|\chi_s'(s, u_0, u_1)\| \leq C \rho(u_0, u_1). \tag{12}
\]
Define $h(\varphi) := \chi_s'(0, u_0(\varphi), u_2(\varphi))$. Then (11) with $s = 1$ yields
\[
J[u_1] - J[u_0] - J'[u_0] (\chi'(0, u_0, u_1)) \geq \frac{c \rho^2(u_0, u_1)}{2} \int_{\mathbb{T}^k} \rho^2(u_0, u_1) \, d\varphi.
\]
Finally, since the set $\Omega$ is bounded and the mapping $\chi$ is smooth, there exists positive constant $C_1$ such that
\[
\|D_\omega h(\varphi)\| \leq C_1 [\|D_\omega u_0(\varphi)\| + \|D_\omega u_1(\varphi)\|] \quad \forall \varphi \in \mathbb{T}^k.
\]
The proof of Theorem 1 is complete.

4. Main existence theorem. Now we proceed to the main result of this paper.

**Theorem 2.** Let the Hypotheses (H1)–(H3) hold true. Then the natural system on Riemannian manifold $(\mathcal{M}, \langle \cdot, \cdot \rangle)$ with Lagrangian density $L = \frac{1}{2} \langle \dot{x}, \dot{x} \rangle + W(\omega t, x)$ has a weak quasiperiodic solution.

**Proof.** The proof will consist of three steps.

1. Construction of a projection mapping and its smooth approximation. Put $\Omega + \delta = (\bigcup_{x \in \Omega} B(x; \delta))$ where $B(x; \delta)$ stands for an open ball of radius $\delta$ centered at $x \in \mathcal{M}$ on Riemannian manifold $(\mathcal{M}, \langle \cdot, \cdot \rangle)$. Since by Hypothesis (H2) $\nu$ is a noncritical value, then $\partial \Omega = V^{-1}(\nu)$ is a regular hypersurface with unit normal field $\nu := \frac{\nabla V}{\|
abla V\|}$. As is well known (see, e.g., [12]), for sufficiently small $\delta > 0$, one can correctly define the projection mapping $P_\Omega : \Omega + \delta \to \Omega$ such that $P_\Omega x \in \Omega$ is the nearest point to $x \in \Omega + \delta$. If $x = X(q)$, $q \in \mathcal{Q} \subset \mathbb{R}^{m-1}$, is a smooth local parametric representation of $\partial \Omega$ in a neighborhood of a point $x_0 \in \partial \Omega$, then for sufficiently small $\delta_0 > 0$ the mapping
\[
\mathcal{Q} \times (-\delta_0, \delta_0) \ni (q, z) \mapsto \exp_{X(q)} (z\nu \circ X(q))
\]
introduces local coordinates with the following properties: local equation of $\partial \Omega$ is $z = 0$; each naturally parametrized $\rho$-geodesic $\gamma(s) = \exp_{X(q)} (s\nu \circ X(q))$.
exp_{X(q)}(s\nu \circ X(q)) is orthogonal to each hypersurface \( z = \text{const} \); the Riemannian metric takes the form \( \sum_{i,j=1}^{m-1} b_{ij}(q, z) dq_i dq_j + dz^2 \), where \( B(q, z) = \{ b_{ij}(q, z) \}_{i,j=1}^{m-1} \) is positive definite symmetric matrix; the function \( V(\cdot) \) is represented in the form \( V(q, z) = v + a(q)z + b(q, z)z^2; \) the mapping \( P_\Omega \) has the form

\[
P_\Omega(q, z) := \begin{cases} (q, 0) & \text{if } z \in (0, \delta_0), \\ (q, z) & \text{if } z \in (-\delta_0, 0). \end{cases}
\]

The projection mapping is continuous on \( \Omega + \delta \) and continuously differentiable on \((\Omega + \delta) \setminus \partial \Omega\). Moreover, it turns out that for sufficiently small \( \delta > 0 \) the derivative \( P_{\Omega_*} \) is contractive on \((\Omega + \delta) \setminus \partial \Omega\), i.e.

\[
\|P_{\Omega_*}\xi\| \leq \|\xi\| \quad \forall \xi \in T_x \mathcal{M}, \ x \in (\Omega + \delta) \setminus \partial \Omega. \quad (13)
\]

It is sufficiently to prove this inequality for any \( x \in (\Omega + \delta) \setminus \partial \Omega \). Let \( q = q(s), z = z(s) \) be natural equations of \( \rho \)-geodesic which starts at a point \( x_0 = (q_0, 0) \in \partial \Omega \) in direction of vector \( \eta = (q_0, 0) \in T_{x_0} \partial \Omega \). The hypothesis \((H2)\) implies that

\[
\langle \nabla_\eta \nabla V(x_0), \eta \rangle = \left. \frac{d^2}{ds^2} \right|_{s=0} V(q(s), z(s)) > 0 \quad \Leftrightarrow \quad a(q_0)\tilde{z}(0) > 0.
\]

Since \( a(q_0) > 0 \) (\( \nu \) is external normal to \( \partial \Omega \)) and z-component of geodesic equations is

\[
\ddot{z} = \frac{1}{2} \frac{\partial}{\partial z} \sum_{i,j=1}^{m-1} b_{ij}(q, z) \dot{q}_i \dot{q}_j,
\]

then the matrix \( B'(q_0, 0) \) is positive definite. From this it follows that \( B(q, z_1) > B(q, z_2) \) for all \( q \) from a neighborhood of \( q_0 \) and all \( z_1, z_2 \in (-\delta, \delta) \), \( z_1 > z_2 \) if \( \delta \in (0, \delta_0) \) is sufficiently small. Let \( \xi = (\dot{q}, \dot{z}) \) be a tangent vector at point \( (q, z) \) where \( z \in (0, \delta) \). Then

\[
\|\xi\|^2 = \sum_{i,j=1}^{m-1} b_{ij}(q, z) \dot{q}_i \dot{q}_j + \dot{z}^2 \geq
\]

\[
ge \sum_{i,j=1}^{m-1} b_{ij}(q, 0) \dot{q}_i \dot{q}_j = \|(\dot{q}, 0)\|^2 = \|P_{\Omega_*}\xi\|^2.
\]

Let us introduce a smooth approximation of projection mapping in a following way. For \( \varepsilon \in (0, \delta) \) define

\[
\varpi_\varepsilon(z) := \exp \left( \frac{1}{\varepsilon} - 1/(z + \varepsilon) \right), \quad z \in (-\varepsilon, 0), 0, \quad z \in \mathbb{R} \setminus (-\varepsilon, 0),
\]

\[
Z_\varepsilon(z) := \int_{-\varepsilon}^{\varepsilon} \frac{\varpi_\varepsilon(t)}{\int_{-\varepsilon}^{\varepsilon} \varpi_\varepsilon(t)dt} ds - \varepsilon, \quad z \in (-\delta_0, \delta_0)
\]
Obviously that the function $Z_{\varepsilon}(\cdot)$ is smooth, its derivative, $Z'_{\varepsilon}(z)$, equals 1 for $z \in (-\delta_{0}, -\varepsilon]$, monotonically decreases from 1 to 0 on $[-\varepsilon, 0]$, and equals 0 for $z \geq 0$. From this it follows that $Z_{\varepsilon}(z)$ equals $z$ for $z \in (-\delta_{0}, -\varepsilon]$ monotonically increases from $-\varepsilon$ to $Z_{\varepsilon}(0) \in (-\varepsilon, 0)$ on $[-\varepsilon, 0]$, and equals $Z_{\varepsilon}(0)$ for $z \in [0, \delta_{0})$. Now locally define

$$P_{\varepsilon, \Omega}(q, z) := \begin{cases} (q, Z_{\varepsilon}(0)) & \text{if } z \in (0, \delta_{0}), \\ (q, Z_{\varepsilon}(z)) & \text{if } z \in (-\delta_{0}, 0] \end{cases}$$

and for each point $x \in \Omega$ such that $B(x; \delta) \subset \Omega$ put $P_{\varepsilon, \Omega}(x) = x$. Since $Z_{\varepsilon}(0) < 0$, then

$$P_{\varepsilon, \Omega}(\Omega + \delta) \subset \Omega$$

and since $|Z'_{\varepsilon}(z)| \leq 1$, then for any $z \in (-\delta, \delta)$, and for any tangent vector $\xi = (\hat{q}, \hat{z})$ at point $(q, z)$ we have

$$\|\xi\|^2 = \sum_{i,j=1}^{m-1} b_{ij}(q, z) \hat{q}_i \hat{q}_j + z^2 \geq \sum_{i,j=1}^{m-1} b_{ij}(q, Z_{\varepsilon}(z)) \hat{q}_i \hat{q}_j + (Z'_{\varepsilon}(z) \hat{z})^2 = \||\hat{q}, Z'_{\varepsilon}(z) \hat{z}||^2 = \|P_{\varepsilon, \Omega} \xi\|.$$

From this it follows that

$$\|P_{\varepsilon, \Omega} \xi\| \leq \|\xi\| \quad \forall x \in \Omega + \delta, \forall \xi \in T_x M. \quad (14)$$

Besides, the Hypothesis (H3) implies

$$W(\varphi, P_{\varepsilon, \Omega} x) \leq W(\varphi, x) \quad \forall \varphi \in \mathbb{T}^m, \forall x \in \Omega + \delta \quad (15)$$

for sufficiently small $\delta$ and $\varepsilon \in (0, \delta)$.

2. Minimization of functional $J$ on $S_{\Omega + \delta}$. Obviously that the functional $J$ restricted to $S_{\Omega + \delta}$ is bounded from below. Let us show that

$$J_* := \inf J[S_{\Omega + \delta}] = \inf J[S_{\Omega}]. \quad (16)$$

In fact, if $v_j(\cdot) \in S_{\Omega + \delta}$ is such a sequence that $J[v_j]$ monotonically decreases to $J_*$, then (14) and (15) implies

$$J_* \leq J[P_{\varepsilon, \Omega + \delta} v_j] \leq J[v_j].$$

Hence, the sequence $u_j(\cdot) := P_{\varepsilon, \Omega + \delta} v_j(\cdot)$ is minimizing both for $J|_{S_{\Omega}}$ and for $J|_{S_{\Omega + \delta}}$.

3. Convergence of minimizing sequence to a weak solution. Let $u_j(\cdot) \in S_{\Omega}$ be a minimizing sequence for $J|_{S_{\Omega}}$. Without loss of generality, we may consider that

$$\|D\omega u_j\|_0^2 \leq M := \frac{2}{(2\pi)^k} \sup_{x \in \Omega} \int_{\mathbb{T}^k} W(\varphi, x) d\varphi - \frac{2}{(2\pi)^k} \int_{\mathbb{T}^k} \inf_{x \in \Omega} W(\varphi, x) d\varphi. \quad (17)$$
Let $h_j(\cdot) \in C^\infty(\mathbb{T}^k \mapsto TM)$ be a sequence of smooth mappings such that $h_j(\varphi) \in T_{u_j(\varphi)}M$ for any $\varphi \in \mathbb{T}^k$ and besides there exist positive constants $K, K_1$ such that

$$\|h_j\|_1 \leq K_1, \quad \|h_j(\varphi)\| \leq K \quad \forall \varphi \in \mathbb{T}^k, \quad \forall j = 1, 2, \ldots$$

(18)

Let us show that

$$\lim_{j \to \infty} J'[u_j](h_j) = 0.$$  \hfill (19)

On one hand, $J[u_j]$ decreases to $J_* := \inf J[\mathcal{S}_\Omega]$. On the other hand, for sufficiently small $s_0 \leq 1$ and for any $j \in \mathbb{N}$ there exists a number $\theta_j \in [-s_0, s_0]$ such that

$$J[\exp_{u_j}(sh_j)] = J[u_j] + sJ'[u_j](h_j) + \frac{s^2}{2} \frac{d^2}{ds^2} \bigg|_{s=\theta_j} J[\exp_{u_j}(sh_j)]$$

\forall s \in [-s_0, s_0], \quad \forall j \in \mathbb{N},

and, besides, there exists a constant $K_2 > 0$ such that

$$\frac{d^2}{ds^2} J[\exp_{u_j}(sh_j)] \leq K_2 \quad \forall s \in [-s_0, s_0], \quad \forall j \in \mathbb{N}.$$

If now we suppose that $\lim\sup_{j \to \infty} |J'[u_j](h_j)| > 0$ then one can choose $j$ and $s_j \in [-s_0, s_0]$ in such a way that

$$\exp_{u_j}(s_j h_j) \in \mathcal{S}_{\Omega+\delta}, \quad J[\exp_{u_j}(s_j h_j)] < J_*.$$

Thus, in view of (16), we arrive at contradiction with definition of $J_*$. Now by Theorem 4 for any pair $u_{i+j}(\cdot), u_j(\cdot)$ there exists a vector field $h_{ij}(\cdot)$ along $u_j(\cdot)$ such that

$$J[u_{i+j}] - J[u_j] - J'[u_j](h_{ij}) \geq \frac{\pi c^2}{2} \int_{\mathbb{T}^k} \rho^2((u_j, u_{i+j})d\varphi \geq \frac{(2\pi)^k \pi c^2}{2} \|u_{i+j} - u_j\|^2_0.$$

Since (19) implies $J'[u_j](h_{ij}) \to 0$ as $j \to \infty$, then the sequence $u_j(\cdot)$ is fundamental in $H(\mathbb{T}^k \mapsto \mathbb{E}^n)$ and in view of (17) converges to a function $u_\ast(\cdot)$ strongly in $H(\mathbb{T}^k \mapsto \mathbb{E}^n)$ and weakly in $H^1(\mathbb{T}^k \mapsto \mathbb{E}^n)$. Without loss of generality we may consider that $u_\ast(\cdot)$ is defined by a minimizing sequence which converges a.e.

Now it remains only to prove that $u_\ast(\cdot)$ is a weak solution, i.e. that there holds (3). Let $h(\cdot)$ be a vector field along $u_\ast(\cdot)$. By definition, there exists a sequence of smooth mappings $h_j(\varphi) \in T_{u_j(\varphi)}M$ which satisfies (18) and (19). Then, in view of (17), we get

$$\lim_{j \to \infty} \left| \langle D_\omega u_\ast, D_\omega h_j \rangle_0 - \langle D_\omega u_j, D_\omega h_j \rangle_0 \right| \leq \lim_{j \to \infty} \left| \langle D_\omega (u_\ast - u_j), D_\omega h_j \rangle_0 \right| + \sqrt{M} \lim_{j \to \infty} \|D_\omega (h - h_j)\|_0 = 0,$$
and by the Lebesgue theorem

$$\lim_{j \to \infty} \int_{\mathbb{T}^k} [W(\varphi, u_j(\varphi)) - W(\varphi, u_*(\varphi))] \, d\varphi = 0.$$ 

Hence,

$$J'[u_*](h) = \lim_{j \to \infty} J'[u_j](h_j) = 0. \quad \square$$

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