ON PARTIALLY OBSERVED JUMP DIFFUSIONS III. REGULARITY OF
THE FILTERING DENSITY

FABIAN GERM AND ISTVÁN GYÖNGY

Abstract. The filtering equations associated to a partially observed jump diffusion model
$$(Z_t)_{t \in [0,T]} = (X_t, Y_t)_{t \in [0,T]},$$
driven by Wiener processes and Poisson martingale measures
are considered. Building on results from two preceding articles on the filtering equations,
the regularity of the conditional density of the signal
$X_t$, given observations
$Y_s$,
is investigated, when the conditional density of $X_0$ given $Y_0$ exists and belongs to a Sobolev
space, and the coefficients satisfy appropriate smoothness and growth conditions.

Contents

1. Introduction 1
2. Formulation of the main results 4
3. Preliminaries 6
4. Sobolev estimates 11
5. Solvability of the filtering equations in Sobolev spaces 25
6. Proof of Theorem 2.1 37
References 42

1. Introduction

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a complete filtered probability space, carrying a $d_1+d'$-dimensional
$\mathcal{F}_t$-Wiener process $(W_t, V_t)_{t \geq 0}$ and independent $\mathcal{F}_t$-Poisson martingale measures
$\tilde{N}_i(d\xi, dt) = N_i(d\xi, dt) - \nu_i(d\xi)dt$ on $\mathbb{R}_+ \times \mathcal{S}_i$, for $i = 0, 1$, with $\sigma$-finite characteristic measures $\nu_0$ and $\nu_1$ on
a separable measurable space $(\mathcal{S}_0, Z_0)$ and on $(\mathcal{S}_1, Z_1) = (\mathbb{R}^{d'} \setminus \{0\}, \mathcal{B}(\mathbb{R}^{d'} \setminus \{0\}))$, respectively,
where $\mathcal{B}(V)$ denotes the Borel $\sigma$-algebra on $V$ for topological spaces $V$.

We consider the signal and observation model

$$dX_t = b(t, X_t, Y_t) \, dt + \sigma(t, X_t, Y_t) \, dW_t + \rho(t, X_t, Y_t) \, dV_t$$
$$+ \int_{\mathcal{S}_0} \eta(t, X_{t-}, Y_{t-}, z) \tilde{N}_0(dz, dt) + \int_{\mathcal{S}_1} \xi(t, X_{t-}, Y_{t-}, z) \tilde{N}_1(dz, dt)$$
$$dY_t = B(t, X_t, Y_t) \, dt + dV_t + \int_{\mathcal{S}_1} \tilde{N}_1(dz, dt),$$

(1.1)
where \( b = (b^i) \), \( B = (B^i) \), \( \sigma = (\sigma^{ij}) \) and \( \rho = (\rho^i) \) are Borel functions on \( \mathbb{R}_+ \times \mathbb{R}^{d+d'} \), with values in \( \mathbb{R}^d \), \( \mathbb{R}^d \), \( \mathbb{R}^{d \times d_1} \) and \( \mathbb{R}^{d \times d'} \), respectively, \( \eta = (\eta^i) \) and \( \xi = (\xi^i) \) are \( \mathbb{R}^d \)-valued \( \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^{d+d'}) \otimes \mathcal{Z}_0 \)-measurable and \( \mathbb{R}^d \)-valued \( \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^{d+d'}) \otimes \mathcal{Z}_1 \)-measurable functions on \( \mathbb{R}_+ \times \mathbb{R}^{d+d'} \times \mathcal{Z}_0 \) and \( \mathbb{R}_+ \times \mathbb{R}^{d+d'} \times \mathcal{Z}_1 \), respectively.

This paper is a continuation of [5] and [6]. In [5] we derive the filtering equations, describing the time evolution of the conditional distribution \( P_t(dx) = P(X_t \in dx | \mathcal{F}_t^Y) \) and the unnormalised conditional distribution \( \mu_t(dx) = P_t(dx)\lambda_t \) of the unobserved component \( X_t \) given \( \mathcal{F}_t^Y \), the \( \sigma \)-algebra generated by the observations \( (Y_s)_{s \in [0,t]} \), where \( (\lambda_t)_{t \in [0,T]} \) is a normalising positive process. The equation for \( (\mu_t(dx))_{t \in [0,T]} \), referred to as Zakai equation, is given in Theorem 3.1 below and has the advantage of being linear in \( \mu \), making it easier to analyse in some situations. For more details on the filtering equations for partially observed (jump) diffusions as well as for a historical account we refer to [5] and the references therein.

In [6] it is shown that the conditional density \( \pi_t = dP_t/d\lambda_t \) exists for \( t > 0 \) and belongs to \( L_q \) for \( q \in [1,p] \), if \( \pi_0 = dP_0/d\lambda \in L_p \) for \( p \geq 2 \), the coefficients of (1.1) satisfy appropriate Lipschitz and growth conditions, and the derivatives of \( \xi \) and \( \eta \) in \( x \) are equicontinuous in \( x \), uniformly in their other variables. The aim of the present paper is to show that if in addition to the Lipschitz and growth conditions in [5], we assume that the coefficients have continuous and bounded derivatives in \( x \) up to order \( m+1 \), for some integer \( m \geq 0 \), and \( \pi_0 \in W_p^m \) for some \( p \geq 2 \), then \( (\pi_t)_{t \geq 0} \) is a \( W_p^m \)-valued weakly cadlag process.

For partially observed diffusion processes, i.e., when \( \xi = \eta = 0 \) and the observation process \( Y \) does not have jumps, the existence and the regularity properties of the conditional density \( \pi_t \) have been extensively studied in the literature. In [13], an early work on the regularity of the filtering density for continuous diffusions, it was shown that if the coefficients are bounded, \( \sigma \), \( \rho \) admit bounded derivatives in \( x \) up to order \( m+1 \), \( b,B \) admit bounded derivatives in \( x \) up to order \( m \), \( \sigma \sigma^* \) is uniformly non-degenerate and \( \pi_0 \in W_p^m \cap W_2^m \), then the filtering density \( (\pi_t)_{t \in [0,T]} \) is weakly continuous as an \( W_p^m \)-valued process, where \( p \geq 2 \) and \( m \geq 0 \). Later, under the uniform non-degeneracy condition on \( \sigma \sigma^* \), stronger results are obtained in [8] and [12] by the help of the \( L_p \)-theory of SPDEs developed there. Generalisations of the linear filtering theory are presented in [11] by the help of the theory of SPDEs with VMO leading coefficients and growing lower order coefficients, see [9], [10].

In [17] it was proven that the nondegeneracy condition can be dropped if one imposes \( m+2 \) bounded derivatives on \( \sigma \), \( \rho \) in \( x \), as well as \( m+1 \) derivatives on \( b,B \) in \( x \), to get that \( \pi \) is a \( W_p^m \)-valued weakly continuous process if \( \pi_0 \in W_p^m \cap W_2^m \) for \( p \geq 2 \) and \( m \geq 1 \). In [14] it was shown that the conditional density \( dP_t/d\lambda_t \) exists for any \( t > 0 \) and it is in \( L_2 \) if \( \pi_0 = dP_0/d\lambda \) exists, it belongs to \( L_2 \), and the coefficients are bounded and Lipschitz continuous. To achieve this, a nice calculation is presented in [14] to show that the \( L_2(\Omega \times \mathbb{R}_+,\mathbb{R}) \)-norm of \( P_t(\cdot) \), the conditional distribution \( P_t \) mollified by Gaussian kernels, can be estimated by the \( L_2(\Omega \times \mathbb{R}_+,\mathbb{R}) \)-norm of \( \pi_0 \), independently of \( \varepsilon > 0 \).

More recently filtering densities associated to signal and observation models \( (X_t,Y_t)_{t \geq 0} \) with jumps have been investigated in a growing number of publications. The above mentioned method from [14] was used in [1], [2], [16] and [19], under various conditions on the filtering models, to prove that the conditional density \( \pi_t = P(X_t \in dx | (Y_s)_{s \in [0,t]})/dx \) exists for \( t > 0 \) and it is in \( L_2 \), when the initial density exists and belongs to \( L_2 \). In [1] only the observation process has jumps. In [2] only the signal process has jumps due to an additive
noise component which is a cadlag process of bounded variation, adapted to the observation process. In [16] a fairly general jump diffusion model is considered, but, as in [1] and [2], the driving noises in the signal are independent of those in the observation. In [19] a general jump diffusion model with correlated signal and observation noises is considered, but as in the articles [1], [2] and [16], the coefficients in the signal do not depend on the observation process. In the above publications, [1], [2], [16] and [19], the coefficients of the SDE describing the signal and observation models are bounded and satisfy appropriate Lipschitz conditions.

An approach from [15] is adapted to study the uniqueness of measure-valued solutions to the filtering equations for a model with jumps in [18], and the existence of the filtering density in $L_2$ is obtained when the initial filtering density is in $L_2$ and the jump component in the signal is a symmetric $\alpha$-stable Lévy process. In this publication the signal and observation noises are independent of each other, the drift and diffusion coefficients in the signal process depend only on $x$, the variable for the signal, they are bounded and their derivatives up to first order and up second order, respectively, exist and are bounded functions.

As the present paper is a direct continuation of [6], it builds on the results of the latter. In [6] the method from [14], combined with methods from the theory of SPDEs, is applied to partially observed jump diffusions to show that the filtering density $\pi_t$ exists for $t > 0$ and belongs to $L_q$ for a $p \in [1, p]$ provided $\pi_0 \in L_p$ for a $p \geq 2$, the coefficients in the SDE satisfy appropriate Lipschitz conditions, the drift coefficient in the observation process is bounded, and the other coefficients satisfy a linear growth condition. In the present paper we investigate the regularity of the filtering density for the same filtering model as in [6]. In addition to these assumptions from [6], in the present paper we assume that for an integer $m \geq 0$ the initial conditional density $\pi_0$ is in $W^m_p$ for some $p \geq 2$ and the coefficients of the SDE admit bounded derivatives in $x$ up to order $m + 1$. Under these conditions we prove that $(\pi_t)_{t \geq 0}$ is a $W^m_p$-valued weakly cadlag process. Moreover, we show that if the coefficients are also bounded, then $(\pi_t)_{t \geq 0}$ is a $W^{s}_{p}$-valued strongly cadlag process for any $s < m$.

This article is structured as follows. Section 2 contains the main results along with the required assumptions. In section 3 we state some important results from [5] and [6] which we build on. Section 4 contains Sobolev estimates necessary to obtain a priori estimates for the smoothed filtering measures. In section 5 we investigate some solvability properties of the Zakai equation. Section 6 finally contains the proof of our main theorem, as well as some auxiliary results.

In conclusion we present important notions and notations used in this paper. For an integer $n \geq 0$ the notation $C^0_b(\mathbb{R}^d)$ means the space of real-valued bounded continuous functions on $\mathbb{R}^d$, which have bounded and continuous derivatives up to order $n$. (If $n = 0$, then $C^0_b(\mathbb{R}^d) = C_b(\mathbb{R}^d)$ denotes the space of real-valued bounded continuous functions on $\mathbb{R}^d$.) We denote by $\mathcal{M} = \mathcal{M}(\mathbb{R}^d)$ the set of finite Borel measures on $\mathbb{R}^d$ and by $\mathcal{M}^s = \mathcal{M}^s(\mathbb{R}^d)$ the set of finite signed Borel measures on $\mathbb{R}^d$. For $\mu \in \mathcal{M}$ we use the notation

$$\mu(\varphi) = \int_{\mathbb{R}^d} \varphi(x) \mu(dx)$$

for Borel functions $\varphi$ on $\mathbb{R}^d$. We say that a function $\nu : \Omega \to \mathcal{M}$ is $\mathcal{G}$-measurable for a $\sigma$-algebra $\mathcal{G} \subset \mathcal{F}$, if $\nu(\varphi)$ is a $\mathcal{G}$-measurable random variable for every bounded Borel function
\( \varphi \) on \( \mathbb{R}^d \). An \( \mathcal{M} \)-valued stochastic process \( \nu = (\nu_t)_{t \in [0,T]} \) is said to be weakly cadlag if almost surely \( \nu_t(\varphi) \) is a cadlag function of \( t \) for all \( \varphi \in C_b(\mathbb{R}^d) \). An \( \mathfrak{M} \)-valued process \( (\nu_t)_{t \in [0,T]} \) is weakly cadlag, if it is the difference of two \( \mathcal{M} \)-valued weakly cadlag processes. For processes \( U = (U_t)_{t \in [0,T]} \) we use the notation \( \mathcal{F}^U_t \) for the \( P \)-completion of the \( \sigma \)-algebra generated by \( \{U_s : s \leq t\} \). By an abuse of notation, we often write \( \mathcal{F}^U_t \) when referring to the filtration \( (\mathcal{F}^U_t)_{t \in [0,T]} \), whenever this is clear from the context. For a measure space \((\mathcal{Z}, \mathcal{F}, \nu)\) and \( p \geq 1 \) we use the notation \( \mathcal{L}^p(\mathcal{Z}) \) for the \( L^p \)-space of \( \mathcal{Z} \)-measurable processes defined on \( \mathcal{Z} \). However, if not otherwise specified, the function spaces are considered to be over \( \mathbb{R}^d \). We always use without mention the summation convention, by which repeated integer valued indices imply a summation. For a multi-index \( \alpha = (\alpha_1, \ldots, \alpha_d) \) of nonnegative integers \( \alpha_i, i = 1, \ldots, d \), a function \( \varphi \) of \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \) and a nonnegative integer \( k \) we use the notation

$$D^\alpha \varphi(x) = D_1^{\alpha_1} D_2^{\alpha_2} \ldots D_d^{\alpha_d} \varphi(x),$$

as well as

$$|D^k \varphi|^2 = \sum_{|\gamma| = k} |D^\gamma \varphi|^2,$$

where \( D_i = \frac{\partial}{\partial x^i} \) and \(| \cdot |\) denotes an appropriate norm. We also use the notation \( D_{ij} = D_i D_j \).

If we want to stress that the derivative is taken in a variable \( x \), we write \( D^\alpha_x \). If the norm \(| \cdot |\) is not clear from the context, we sometimes use appropriate subscripts, as in \(|\varphi|_{L^p} \) for the \( L^p \)-norm of \( \varphi \). For \( p \geq 1 \) and integers \( m \geq 0 \) we use the notation \( W^m_p \) for Borel functions \( f = f(x) \) on \( \mathbb{R}^d \) such that

$$|f|_{W^m_p}^p := \sum_{k=0}^m \int_{\mathbb{R}^d} |D^k f(x)|^p \, dx < \infty.$$ 

Throughout the paper we work on the finite time interval \([0,T]\), where \( T > 0 \) is fixed but arbitrary, as well as on a given complete probability space \((\Omega, \mathcal{F}, P)\) equipped with a filtration \((\mathcal{F}_t)_{t \geq 0}\) such that \( \mathcal{F}_0 \) contains all the \( P \)-null sets. For \( p, q \geq 1 \) and integers \( m \geq 1 \) we denote by \( \mathbb{W}^m_p = L_p(\Omega, \mathcal{F}_0, \mathcal{P}), W^m_p(\mathbb{R}^d) \) and \( \mathbb{W}^m_{p,q} \) the set of \( \mathcal{F}_0 \)-\( \mathcal{B}(\mathbb{R}^d) \)-measurable real-valued functions \( f = f(\omega, x) \) and \( \mathcal{F}_t \)-optional \( W^m_p \)-valued processes \( g = g_t(\omega, x) \) such that

$$|f|_{\mathbb{W}^m_p}^p := \mathbb{E}|f|_{W^m_p}^p < \infty \quad \text{and} \quad |g|_{\mathbb{W}^m_{p,q}}^p := \mathbb{E}\left( \int_0^T |g_t|^q_{W^m_p} \, dt \right)^{p/q} < \infty$$

respectively. If \( m = 0 \) we set \( \mathbb{L}_p = \mathbb{W}^0_p \) and \( \mathbb{L}_{p,q} = \mathbb{W}^0_{p,q} \). In case a different \( \sigma \)-algebra \( \mathcal{G} \) than \( \mathcal{F}_0 \) is considered above, we denote this explicitly by \( \mathbb{L}_p(\mathcal{G}) \) and \( \mathbb{W}^m_p(\mathcal{G}) \). If \( m \geq 0 \) is not an integer and \( p > 1 \) then \( W^m_p \) denotes the space of real-valued generalised functions \( h \) on \( \mathbb{R}^d \) such that

$$|h|_{W^m_p} := |(1 - \Delta)^{m/2} h|_{L_p} < \infty.$$ 

Finally, for real-valued functions \( f \) and \( g \) on \( \mathbb{R}^d \), we often denote by \( (f, g) \) the integral of \( f \cdot g \) over \( \mathbb{R}^d \).

### 2. Formulation of the main results

We fix nonnegative constants \( K_0, K_1, L, K \) and functions \( \xi \in L_2(\mathcal{Z}_1) = L_2(\mathcal{Z}_1, \mathcal{F}_1, \nu_1) \), \( \eta \in L_2(\mathcal{Z}_0) = L_2(\mathcal{Z}_0, \mathcal{F}_0, \nu_0) \), used throughout the paper, and make the following assumptions.
Assumption 2.1. 
(i) For $z_j = (x_j, y_j) \in \mathbb{R}^{d+i} \ (j = 1, 2)$, $t \geq 0$ and $3_i \in 3_i$ for $i = 0, 1$, 
\[|b(t, z_1) - b(t, z_2)| + |B(t, z_1) - B(t, z_2)| + |\sigma(t, z_1) - \sigma(t, z_2)| + |\rho(t, z_1) - \rho(t, z_2)| \leq L|z_1 - z_2|,\]
\[|\eta(t, z_1, 3_0) - \eta(t, z_2, 3_0)| \leq \tilde{\eta}(3_0)|z_1 - z_2|,\]
\[|\xi(t, z_1, 3_1) - \xi(t, z_2, 3_1)| \leq \tilde{\xi}(3_1)|z_1 - z_2|.\]

(ii) For all $z = (x, y) \in \mathbb{R}^{d+i}$, $t \geq 0$ and $3_i \in 3_i$ for $i = 0, 1$ we have 
\[|b(t, z)| + |\sigma(t, z)| + |\rho(t, z)| \leq K_0 + K_1|z|, \quad |B(t, z)| \leq K, \quad \int_{3_i} |3|^2 \nu_1(d3) \leq K_0^2,\]
\[|\eta(t, z, 3_0) + \eta(3_0)(K_0 + K_1|z|), |\xi(t, z, 3_1)| \leq \tilde{\xi}(3_1)(K_0 + K_1|z|).\]

(iii) The initial condition $Z_0 = (X_0, Y_0)$ is an $F_0$-measurable random variable with values in $\mathbb{R}^{d+i}$.

Assumption 2.2. The functions $\tilde{\eta} \in L_2(3_0, \mathbb{Z}, 0, \nu_0)$ and $\tilde{\xi} \in L_2(3_1, \mathbb{Z}, 1, \nu_1)$ are such that for constants $K_\eta$ and $K_\xi$ we have $\tilde{\eta}(3_0) \leq K_\eta$ and $\tilde{\xi}(3_1) \leq K_\xi$ for all $3_0 \in 3_0$, $3_1 \in 3_1$.

Assumption 2.3. For some $r > 2$ let $\mathbb{E}|X_0|^r < \infty$ and the measure $\nu_1$ satisfy 
\[K_r := \int_{3_i} |3|^r \nu_1(d3) < \infty.\]

By a well-known theorem of Itô one knows that Assumption 2.1 ensures the existence and uniqueness of a solution $(X_t, Y_t)_{t \geq 0}$ to (1.1) for any given $F_0$-measurable initial value $Z_0 = (X_0, Y_0)$, and for every $T > 0$, 
\[\mathbb{E}\sup_{t \leq T}(|X_t|^q + |Y_t|^q) \leq N(1 + \mathbb{E}|X_0|^q + \mathbb{E}|Y_0|^q)\] 
(2.1) holds for $q = 2$ with a constant $N$ depending only on $T, K_0, K_1, K_2, \tilde{\xi}|L_2, |\eta|L_2$ and $d + d'$. If in addition to Assumption 2.1 we assume Assumptions 2.2 and 2.3, then it is known, see e.g. [3], that the moment estimate (2.1) holds with $q := r$ for every $T > 0$, where now the constant $N$ depends also on $r, K_\xi, K_\eta$ and $K_\eta$.

Assumption 2.4. (i) For a constant $\lambda > 0$ we have 
\[\lambda|x - \bar{x}| \leq |x - \bar{x} + \theta(f_1(t, x, y, 3_1) - f_1(t, \bar{x}, y, 3_1))|\]
for all $\theta \in [0, 1], \ t \in [0, T], \ y \in \mathbb{R}^d, \ x, \bar{x} \in \mathbb{R}^d, \ 3_i \in 3_i, \ i = 0, 1$, and $f_0(t, x, y, 3_0) = \eta(t, x, y, 3_0), f_1(t, x, y, 3_1) = \xi(t, x, y, 3_1)$.

(ii) For all $(t, y) \in \mathbb{R}_+ \times \mathbb{R}^d$ and all $x_1, x_2 \in \mathbb{R}^d$, 
\[|(\rho B)(t, x_1, y) - (\rho B)(t, x_2, y)| \leq L|x_1 - x_2|,\]

(iii) The functions $f_0(t, x, y, 3) := \xi(t, x, y, 3)$ and $f_1(t, x, y, 3) := \eta(t, x, y, 3)$ are continuously differentiable in $x \in \mathbb{R}^d$ for each $(t, y, 3) \in \mathbb{R}_+ \times \mathbb{R}^d \times 3_i$, for $i = 0$ and $i = 1$, respectively, such that 
\[\lim_{\varepsilon \downarrow 0} \limsup_{t \in [0, T]} \sup_{3 \in 3_i} \sup_{|y| \leq R} \sup_{|x| \leq R} |D_x f_i(t, x, y, 3) - D_x f_i(t, \bar{x}, y, 3)| = 0\]
for every $R > 0$.

Assumption 2.5. Let $m \geq 0$ be an integer.
The following result was proven in [5]. In order to formulate it, we define $P$ and for each $\omega$ there is a set $\Omega$ of full probability and there is uniquely defined (up to indistinguishability) $\mathbb{M}$-valued processes $(\nu_t)_{t \geq 0}$ such that for every $\omega \in \Omega'$

\[
\nu_t^- (\varphi) = \lim_{s \uparrow t} \nu_s (\varphi) \quad \text{for all } \varphi \in C_b (\mathbb{R}^d) \text{ and } t > 0,
\]

and for each $\omega \in \Omega'$ we have $\nu_t^- = \nu_t$, for all but at most countably many $t \in (0, \infty)$. The following result was proven in [5]. In order to formulate it, we define

\[
\gamma_t = \exp \left( - \int_0^t B(s, X_s, Y_s) \, dV_s - \frac{1}{2} \int_0^t |B(s, X_s, Y_s)|^2 \, ds \right), \quad t \in [0, T].
\]
and note that since $B$ is bounded in magnitude, we have $E\gamma_T = 1$ and it is an $\mathcal{F}_t$-martingale under $P$. Thus we can define the equivalent probability measure $Q$ by $Q := \gamma_T P$.

**Theorem 3.1.** Let Assumption 2.1 hold. If $K_1 \neq 0$, then assume also $E|X_0|^2 < \infty$. Then there exist measure-valued $\mathcal{F}_t^Y$-adapted weakly cadlag processes $(P_t)_{t \in [0,T]}$ and $(\mu_t)_{t \in [0,T]}$ such that

$$P_t(\varphi) = \mu_t(\varphi)/\mu_t(1), \quad \text{for } \varphi \in \Omega, \ t \in [0,T],$$

$$P_t(\varphi) = E(\varphi(X_t)|\mathcal{F}_t^Y), \quad \mu_t(\varphi) = E_Q(\gamma_t^{-1}\varphi(X_t)|\mathcal{F}_t^Y) \ (a.s.) \ \text{for each } t \in [0,T].$$

We refer to $\mu_t$ (resp. $P_t$) as the unnormalised (resp. normalised) conditional distribution of $X_t$ given $\mathcal{F}_t^Y$, $t \in [0,T]$.

We introduce the random differential operators

$$\tilde{L}_t = a_t^{ij}(x)D_{ij} + b_t^i(x)D_i + \beta_t^k \mathcal{M}_t^k, \quad \mathcal{M}_t^k = \rho_t^{ik}(x)D_i + B_t^k(x), \quad k = 1, 2, ..., d', \quad (3.1)$$

where $\beta_t := B_t(X_t)$ and

$$a_t^{ij}(x) := \frac{1}{2} \sum_{k=1}^{d} (\sigma_t^{ik} \sigma_t^{jk})^2(x) + \frac{1}{2} \sum_{l=1}^{d'} (\rho_t^{il} \rho_t^{jl})^2(x), \quad \sigma_t^{ik}(x) := \sigma^{ik}(t,x,Y_t), \quad \rho_t^{il}(x) := \rho^{il}(t,x,Y_t),$$

$$b_t^i(x) := b^i(t,x,Y_t), \quad B_t^k(x) := B^k(t,x,Y_t)$$

for $\varphi \in \Omega$, $t \geq 0$, $x = (x^1, ..., x^d) \in \mathbb{R}^d$, and $D_i = \partial/\partial x^i$, $D_{ij} = \partial^2/\partial x^i \partial x^j$ for $i,j = 1, 2, ..., d$.

Moreover for every $t \geq 0$ and $\xi \in \mathcal{F}_1$ we introduce the random operators $I_t^\xi$ and $J_t^\xi$ defined by

$$I_t^\xi \varphi(x,\xi) = \varphi(x + \xi_t(x,\xi),\xi)$$

$$J_t^\xi \phi(x,\xi) = I_t^\xi \phi(x,\xi) - \sum_{i=1}^{d} \xi_t^i(x,\xi) D_i \phi(x,\xi)$$

for functions $\varphi = \varphi(x,\xi)$ and $\phi = \phi(x,\xi)$ of $x \in \mathbb{R}^d$ and $\xi \in \mathcal{F}_1$, and furthermore the random operators $I_t^n$ and $J_t^n$, defined as $I_t^\xi$ and $J_t^\xi$, respectively, with $\eta_t(x,\xi)$ in place of $\xi_t(x,\xi)$, where

$$\xi_t(x,\xi_t) := \xi(t,x,Y_{t-},\xi_t), \quad \eta_t(x,\xi_t) := \eta(t,x,Y_{t-},\xi_t)$$

for $\varphi \in \Omega$, $t \geq 0$, $x \in \mathbb{R}^d$ and $\xi_t \in \mathcal{F}_t$ for $i = 0, 1$.

From [6] we know that if the unnormalised conditional distribution $\mu_t$ has a density such that $u_t = d\mu_t/dx$ (a.s.) for each $t \in [0,T]$ for an $L_\mu$-valued weakly cadlag process $(u_t)_{t \in [0,T]}$ for some $p \geq 2$, then it satisfies for each $\varphi \in C_{0}^\infty$ almost surely

$$(u_t, \varphi) = (\psi, \varphi) + \int_0^t (u_s, \tilde{L}_s \varphi) ds + \int_0^t (u_s, \mathcal{M}_s^k \varphi) dV_s^k + \int_0^t \int_{\mathcal{F}_s} (u_s, J_s^\eta \varphi) \nu_0(d\tilde{\eta}) ds$$

$$+ \int_0^t \int_{\mathcal{F}_s} (u_s, J_s^\xi \varphi) \nu_1(d\xi) ds + \int_0^t \int_{\mathcal{F}_s} (u_{s-}, I_s^\xi \varphi) \tilde{N}_1(ds, \xi) ds, \quad t \in [0,T]. \quad (3.2)$$

for all $t \in [0,T]$. Formally we may write (3.2) as the Cauchy problem

$$du_t = \tilde{L}_t^* u_t dt + \mathcal{M}_t^k u_t dV_t^k + \int_{\mathcal{F}_t} J_t^\eta u_t \nu_0(d\tilde{\eta}) dt$$

$$+ \int_{\mathcal{F}_t} J_t^\xi u_t \nu_1(d\xi) dt + \int_{\mathcal{F}_t} I_t^\xi u_{t-} \tilde{N}_1(ds, dt), \quad (3.3)$$
for a given \( \psi \).

**Definition 3.1.** Let integers \( m \geq 0 \) and \( p \geq 2 \). Let \( \psi \) be an \( W^m_p \)-valued \( \mathcal{F}_0 \)-measurable random variable. Then we say that a \( W^m_p \)-valued \( \mathcal{F}_t \)-adapted weakly cadlag process \((u_t)_{t \in [0,T]}\) is a \( W^m_p \)-solution of (3.3) with initial condition \( \psi \), if for each \( \varphi \in C_c^\infty \) almost surely (3.2) holds for every \( t \in [0,T] \).

If \( m = 0 \), then we call \( u \) an \( L_p \)-solution instead of a \( W^0_p \)-solution.

As in [6] we are interested in solutions that satisfy

\[
\text{ess sup}_{t \in [0,T]} |u_t|_{L^1} < \infty \quad \text{and} \quad \text{sup}_{t \in [0,T]} \int_{\mathbb{R}^d} |y|^2|u_t(y)| \, dy < \infty \quad \text{(a.s.)}.
\]  

(3.4)

To formulate the following results from [6], Lemma 5.7 and Theorem 2.1 therein, we recall that there exists a cadlag \( \mathcal{F}_t^\gamma \)-adapted process \((\gamma_t)_{t \in [0,T]}\), called the optional projection of \((\gamma_t)_{t \in [0,T]}\) under \( P \) with respect to \((\mathcal{F}_t^\gamma)_{t \in [0,T]}\), such that for every \( \mathcal{F}_t^\gamma \)-stopping time \( \tau \leq T \) we have

\[
\mathbb{E}(\gamma_\tau | \mathcal{F}_\tau^\gamma) = \gamma_\tau \quad \text{almost surely.}
\]  

(3.5)

Since for each \( t \), by known properties of conditional expectations, almost surely

\[
\mu_t(1) = \mathbb{E}Q(\gamma_t^{-1} | \mathcal{F}_t^\gamma) = 1/\mathbb{E}(\gamma_t | \mathcal{F}_t^\gamma) = 1/\gamma_t
\]

and \( P, \mu \) are weakly cadlag in the sense described above, we also have that almost surely \( P_t(\varphi) = \mu_t(\varphi) \gamma_t \) for each \( t \in [0,T] \) and \( \varphi \in C_b \).

**Theorem 3.2.** Let Assumptions 2.1, 2.2 and 2.4 hold. If \( K_1 \neq 0 \), then let additionally Assumption 2.3 hold for some \( r > 2 \). Assume the conditional density \( \pi_0 = P(X_0 = dx | Y_0) / dx \) exists almost surely and \( \mathbb{E}|\pi_0|^p_{L_p} < \infty \) for some \( p \geq 2 \).

(i) The unnormalized conditional density \((u_t)_{t \in [0,T]}\) exists almost surely and is an \( L_p \)-valued weakly cadlag process such that for each \( t \in [0,T] \) almost surely \( u_t = d \mu_t / dx \) and

\[
\mathbb{E} \sup_{t \in [0,T]} |u_t|^p_{L_p} \leq N \mathbb{E}|\pi_0|^p_{L_p}.
\]

for a constant \( N = N(d, d', p, K, K_\xi, K_\eta, L, T, \lambda, |\xi|_{L_2}, |\eta|_{L_2}) \). Moreover, \( u \) is the unique \( L_2 \)-solution to (3.3) satisfying the conditions in (3.4).

(ii) Almost surely the conditional density \( P(X_t = dx | \mathcal{F}_t^\gamma) / dx \) exists and belongs to \( L_p \) for all \( t \in [0,T] \). Moreover, there is an \( L_p \)-valued weakly cadlag process \((\pi_t)_{t \in [0,T]}\), such that for each \( t \in [0,T] \) almost surely \( \pi_t = P(X_t = dx | \mathcal{F}_t^\gamma) / dx \), as well as almost surely \( \pi_t = u_t \gamma_t \) for all \( t \in [0,T] \).

**Proof.** See Lemma 5.5 and Theorem 2.1 in [6]. \( \square \)

**Lemma 3.3.** Let \( 1 < p < \infty \) and let \((v_t)_{t \in [0,T]}\) be a weakly cadlag \( L_p \)-valued process. Assume moreover that for an \( m \geq 0 \) almost surely \( \text{ess sup}_{t \in [0,T]} |v_t|_{W^m_p} < \infty \) and \( v_T \in W^m_p \). Then \( v \) is weakly cadlag as a \( W^m_p \)-valued process.

**Proof.** Let \( \Omega' \) be the set of those \( \omega \in \Omega \) such that \((v_t(\omega))_{t \in [0,T]}\) is weakly cadlag as an \( L_p \)-valued function, \( v_T(\omega) \in W^m_p \) and \( \text{ess sup}_{t \in [0,T]} |v_t(\omega)|_{W^m_p} < \infty \). Then \( P(\Omega') = 1 \), and for
each $\omega \in \Omega'$ there exists a dense subset $T_\omega$ in $[0,T]$ such that $\sup_{t \in T_\omega} |v_t(\omega)|_{W_p^m} < \infty$. If $\omega \in \Omega'$ and $t \notin T_\omega$, $t \neq T$, then there exists a sequence $(t_n)_{n=1}^\infty \subset T_\omega$ such that $t_n \downarrow t$. Since $\sup_{t \in T_\omega} |v_t(\omega)|_{W_p^m} < \infty$ there exists a subsequence, also denoted by $(t_n)_{n=1}^\infty$, such that $v_{t_n}(\omega)$ converges weakly in $W_p^m$ to some element $\tilde{v} \in W_p^m$. However, as $v$ is weakly cadlag as an $L_p$-valued process, we know that $v_{t_n} \to v_t$ weakly in $L_p$ as $n \to \infty$ and hence $\tilde{v} = v_t \in W_p^m$. Thus clearly also $\sup_{t \in [0,T]} |v_t(\omega)|_{W_p^m} < \infty$ if $\omega \in \Omega'$. To see that $v$ is weakly cadlag as a $W_p^m$-valued process, note first that since $W_p^m$ is a reflexive space, which is embedded continuously and densely into $L_p$, we have that the dual $(L_p)^* = L_q$, $q = p/(p-1)$, is embedded continuously and densely into $(W_p^m)^*$. Therefore, for each $\varepsilon > 0$ and $\phi \in (W_p^m)^*$ there is an $\phi_\varepsilon \in L_q$ such that $|\phi - \phi_\varepsilon|_{(W_p^m)^*} < \varepsilon$. Fix a $t \in [0,T)$ and a sequence $t_n \downarrow t$. Then

$$\left| (v_n, \phi) - (v_t, \phi) \right| \leq \left| (v_n, \phi - \phi_\varepsilon) \right| + \left| (v_n, \phi_\varepsilon) - (v_t, \phi_\varepsilon) \right| + \left| (v_t, \phi_\varepsilon - \phi) \right|$$

$$\leq 2 \varepsilon \sup_{t \in [0,T]} |v_t|_{W_p^m} + \left| (v_n, \phi_\varepsilon) - (v_t, \phi_\varepsilon) \right|.$$

Recalling that $v$ is weakly cadlag as an $L_p$-valued process finishes the proof. \hfill \Box

The corollary of the following lemma will play an essential role in the proof of the statement on the strong cadlagness of $L_p$-solutions to the filtering equations, see Proposition 6.4.

**Lemma 3.4.** Let $\zeta$ be an $\mathbb{R}^d$-valued function on $\mathbb{R}^d$ such that for an integer $m \geq 1$ it is continuously differentiable up to order $m$, and

$$\inf_{\theta \in [0,1]} \inf_{x \in \mathbb{R}^d} |\det(I + \theta D\zeta(x))| =: \lambda > 0, \quad \max_{0 \leq k \leq m} \sup_{x \in \mathbb{R}^d} |D^k \zeta(x)| =: M_m < \infty. \quad (3.6)$$

Then the following statements hold.

(i) The function $\tau = x + \theta \zeta(x)$, $x \in \mathbb{R}^d$, is a $C^m$-diffeomorphism for each $\theta$, such that for all $x \in \mathbb{R}^d$, $\theta \in [0,1],$

$$\lambda' \leq \left| \det D\tau^{-1}(x) \right| \leq \lambda'', \quad \text{and} \quad \max_{1 \leq k \leq m} \sup_{x \in \mathbb{R}^d} |D^k \tau^{-1}| \leq M'_m < \infty, \quad (3.7)$$

with constants $\lambda' = \lambda'(d, M_1) > 0$, $\lambda'' = \lambda''(d, M_1)$ and $M'_m = M'_m(d, \lambda, M_m)$.

(ii) The function $\zeta^*(x) = -x + \tau^{-1}(x)$, $x \in \mathbb{R}^d$, is continuously differentiable up to order $m$, such that

$$\sup_{\mathbb{R}^d} |\zeta^*| = \sup_{\mathbb{R}^d} |\zeta|, \quad (3.8)$$

$$\sup_{\mathbb{R}^d} |D^k \zeta^*| \leq M^*_m \max_{1 \leq j \leq k} \sup_{\mathbb{R}^d} |D^j \zeta|, \quad \text{for } k = 1, 2, ..., m, \quad (3.9)$$

$$\inf_{\theta \in [0,1]} \inf_{x \in \mathbb{R}^d} |\det(I + \theta D\zeta^*)| \geq \lambda' \inf_{\theta \in [0,1]} \inf_{x \in \mathbb{R}^d} |\det(I + \theta D\zeta)|, \quad (3.10)$$

with a constant $M^*_m = M^*_m(d, \lambda, M_m)$ and with $\lambda'$ from (3.7).

(iii) For the function $c = D\zeta^* - 1$ we have

$$\sup_{x \in \mathbb{R}^d} |D^k c(x)| \leq N \max_{1 \leq j \leq k+1} \sup_{\mathbb{R}^d} |D^j \zeta|, \quad (3.11)$$

for $0 \leq k \leq m - 1$ with a constant $N = N(d, \lambda, m, M_m)$. 

Proof. Claims (i) and (ii) are Lemma 6.1 in [6]. To prove (iii) notice that for the function
\( F(A) = \det A \), considered as the function of the entries \( A^{ij} \) of \( d \times d \) real matrices \( A \), by Taylor’s formula

\[
\epsilon = \det(I + D\zeta^*) - \det I = \int_0^1 \frac{\partial}{\partial A^i} F(I + \theta D\zeta^*) \, d\theta D_i\zeta^*. 
\]

\( \square \)

Corollary 3.5. For \( \mathbb{R}^d \)-valued functions \( \zeta \) on \( \mathbb{R}^d \) we define the operators \( T^\zeta, I^\zeta \) and \( J^\zeta \) by

\( T^\zeta \varphi(x) = \phi(x + \zeta(x)) \) \quad for \( x \in \mathbb{R}^d \), and \quad \( I^\zeta \varphi = T^\zeta \varphi - \varphi \), \quad \( J^\zeta \varphi = I^\zeta - \zeta^i D_i \varphi \)

for differentiable functions \( \varphi \) on \( \mathbb{R}^d \). Assume that \( \zeta \) satisfies the conditions of Lemma 3.4 with \( m = 2 \). Then \( \tau_{\theta\zeta}(x) = x + \theta \zeta(x), \, x \in \mathbb{R}^d \) are \( C^2 \)-diffeomorphisms for each \( \theta \in [0,1] \), and for every \( v, \varphi \in C_0^\infty \) we have

\[
(v, I^\zeta \varphi) = (I^\zeta^i v, \varphi), \quad (v, J^\zeta \varphi) = (K^\zeta_i v, D_i \varphi),
\]

(3.12)

with

\[
I^\zeta^i v(x) = -\int_0^1 D_i \left( v(\tau_{\theta\zeta}^{-1}(x)) \zeta^i(\tau_{\theta\zeta}^{-1}(x)) | \det D\tau_{\theta\zeta}^{-1}(x) | \right) \, d\theta,
\]

\[
K^\zeta_i v(x) = \int_0^1 (\theta - 1) D_j \left( v(\tau_{\theta\zeta}^{-1}(x)) \zeta^i(\tau_{\theta\zeta}^{-1}(x)) \zeta^j(\tau_{\theta\zeta}^{-1}(x)) | \det D\tau_{\theta\zeta}^{-1}(x) | \right) \, d\theta,
\]

for \( i = 1,2,\ldots,d \). Moreover, for every \( x \in \mathbb{R}^d \) we have

\[
|I^\zeta^i v(x)| \leq N \sup_{x \in \mathbb{R}^d} (|\zeta(x)| + |D\zeta(x)|) \int_0^1 |v(\tau_{\theta\zeta}^{-1}(x))| + |(Dv)(\tau_{\theta\zeta}^{-1}(x))| \, d\theta \quad (3.13)
\]

\[
|K^\zeta_i v(x)| \leq N \sup_{x \in \mathbb{R}^d} (|\zeta(x)|^2 + |D\zeta(x)|^2) \int_0^1 |v(\tau_{\theta\zeta}^{-1}(x))| + |(Dv)(\tau_{\theta\zeta}^{-1}(x))| \, d\theta \quad (3.14)
\]

with a constant \( N = N(d,\lambda,M_1,M_2) \).

Proof. Using Taylor’s formula we have

\[
(v, I^\zeta \varphi) = \int_{\mathbb{R}^d} \int_0^1 v(x)(D_i \varphi)(\tau_{\theta\zeta}(x)) \zeta^i(x) \, d\theta \, dx,
\]

\[
(v, J^\zeta \varphi) = \int_{\mathbb{R}^d} \int_0^1 (1 - \theta)v(x)(D_{ij} \varphi)(\tau_{\theta\zeta}(x)) \zeta^i(x)\zeta^j(x) \, d\theta \, dx,
\]

and by a change of variables in the calculation of the integrals over \( \mathbb{R}^d \) and then integrating by parts we get the equations in (3.12). Hence we get the estimates (3.13) and (3.14) by applying standard derivative rules and using the estimates in (3.7). \( \square \)
4. Sobolev estimates

In this section we present some estimates which are needed in the subsequent sections. In the following lemmas note that by lower indices $i$ we mean the derivative with respect to $x^i$, i.e. $u_i = \frac{\partial}{\partial x^i} u(x)$. For $\varepsilon > 0$ we use the notation $k_\varepsilon$ for the Gaussian density function on $\mathbb{R}^d$ with mean 0 and variance $\varepsilon$. For linear functionals $\Phi$, acting on a real vector space $V$ containing $S = S(\mathbb{R}^d)$, the rapidly decreasing functions on $\mathbb{R}^d$, the mollification $\Phi^{(\varepsilon)}$ is defined by

$$\Phi^{(\varepsilon)}(x) = \Phi(k_\varepsilon(x - \cdot)), \quad x \in \mathbb{R}^d.$$  

In particular, when $\Phi = \mu$ is a (signed) measure from $S^*$, the dual of $S$, or $\Phi = f$ is a function from $S^*$, then

$$\mu^{(\varepsilon)}(x) = \int_{\mathbb{R}^d} k_\varepsilon(x - y) \mu(dy), \quad f^{(\varepsilon)}(x) = \int_{\mathbb{R}^d} k_\varepsilon(x - y) f(y) \, dy, \quad x \in \mathbb{R}^d,$$

and, using the formal adjoint $L^*$, we write

$$(L^* \mu)^{(\varepsilon)}(x) := \int_{\mathbb{R}^d} L_y k_\varepsilon(x - y) \mu(dy), \quad x \in \mathbb{R}^d$$

when $L$ is a linear operator on $V$ such that the integral is well-defined for every $x \in \mathbb{R}^d$. Here the subscript $y$ in $L_y$ indicates that the operator $L$ acts in the $y$-variable of the function $k_\varepsilon(x, y) := k_\varepsilon(x - y)$. For example, if $L$ is a differential operator of the form $a^{ij}D_{ij} + b^i D_i + c$, where $a^{ij}$, $b^i$ and $c$ are functions defined on $\mathbb{R}^d$, then

$$(L^* \mu)^{(\varepsilon)}(x) = \int_{\mathbb{R}^d} (a^{ij}(y) \frac{\partial^2}{\partial y^i \partial y^j} + b^i(y) \frac{\partial}{\partial y^i} + c(y)) k_\varepsilon(x - y) \mu(dy).$$

We will often use the following well-known properties of mollifications with $k_\varepsilon$:

(i) $|\varphi^{(\varepsilon)}|_{L_p} \leq |\varphi|_{L_p}$ for $\varphi \in L_p(\mathbb{R}^d)$, $p \in [1, \infty]$;

(ii) $\mu^{(\varepsilon)}(x) := \int_{\mathbb{R}^d} \mu^{(\varepsilon)}(x) \varphi(x) \, dx = \int_{\mathbb{R}^d} \varphi(x) \mu(dx) =: \mu(\varphi^{(\varepsilon)})$ for finite (signed) Borel measures $\mu$ on $B(\mathbb{R}^d)$ and $\varphi \in L_p(\mathbb{R}^d)$, $p \geq 1$;

(iii) $|\mu^{(\varepsilon)}|_{L_p} \leq |\mu^{(\varepsilon)}|_{L_p}$ for $0 \leq \varepsilon \leq \delta$, finite Borel measures $\mu$ on $\mathbb{R}^d$ and $p \geq 1$. This property follows immediately from (i) and the “semigroup property” of the Gaussian kernel,

$$k_{r+s}(y - z) = \int_{\mathbb{R}^d} k_r(y - x)k_s(x - z) \, dx, \quad y, z \in \mathbb{R}^d \text{ and } r, s \in (0, \infty).$$  

(4.1)

The following generalization of (iii) is also useful: for integers $p \geq 2$ we have

$$\rho^{(\varepsilon)}(y) := \int_{\mathbb{R}^d} \prod_{r=1}^p k_\varepsilon(x - y_r) \, dx = c_{p,\varepsilon} e^{-\sum_{1 \leq r < s \leq p} |y_r - y_s|^2/(2\varepsilon p)}, \quad y = (y_1, ..., y_p) \in \mathbb{R}^{pd},$$  

(4.2)

for $\varepsilon > 0$, with a constant $c_{p,\varepsilon} = c_{p,\varepsilon}(d) = p^{-d/2}(2\pi\varepsilon)^{(1-p)d/2}$. This calculation can be found in [6, Sec. 4]. Clearly, for every $r = 1, 2, ..., p$ and $i = 1, 2, ..., d$

$$\partial_{y^i} \rho^{(\varepsilon)}(y) = \frac{1}{\varepsilon^p} \sum_{s=1}^p (y_s^i - y_r^i) \rho^{(\varepsilon)}(y), \quad y = (y_1, ..., y_p) \in \mathbb{R}^d, \quad y_r = (y_1^r, ..., y_p^r) \in \mathbb{R}^d.$$

(4.3)
It is easy to see that
\[ \sum_{r=1}^{p} \partial_{y^r} \rho_\varepsilon(y) = 0 \quad \text{for } y \in \mathbb{R}^d, \quad j = 1, 2, \ldots, d, \]
which we will often use in the form
\[ \partial_{y^j} \rho_\varepsilon(y) = - \sum_{s \neq r}^{p} \partial_{y^s} \rho_\varepsilon(y) \quad \text{for } r = 1, \ldots, p \text{ and } j = 1, 2, \ldots, d. \] (4.4)
Moreover, we will use that for \( q = 1, 2 \), with a constant \( N = N(d, p, q) \),
\[ \varepsilon^{-q} \sum_{s \neq r} |y_s - y_r|^2 \rho_\varepsilon(y) \leq N \rho_\varepsilon(y), \quad y \in \mathbb{R}^d. \] (4.5)

The case of \( \alpha = 0 \) in the following Lemmas in this section is proven in [6] and hence this case will be omitted in the proofs.

The following estimates for \( \mu \in \mathcal{M} \) with density \( u = d\mu/dx \in W^m_p \), for \( m \geq 0 \) and \( p \geq 2 \) even, will be useful in later sections. In order for the left-hand side of these estimates to be well-defined, we require that
\[ K_1 \int_{\mathbb{R}^d} |x|^2 |u(x)| \, dx < \infty, \] (4.6)
where we use the formal convention that \( 0 \cdot \infty = 0 \), i.e. if \( K_1 = 0 \), then the second moment of \( |\mu(dx)| = |u(x)| dx \) is not required to be finite.

**Lemma 4.1.** Consider integers \( m \geq 0 \) and \( p \geq 2 \) even. Let \( \sigma = (\sigma^{ij}) \) be a Borel function on \( \mathbb{R}^d \) with values in \( \mathbb{R}^{d \times k} \), such that for some nonnegative constants \( K_0 \) and \( L \)
\[ |\sigma(x)| \leq K_0, \quad \sum_{k=1}^{m+1} |D^k \sigma(x)| \leq L, \] (4.7)
for all \( x, y \in \mathbb{R}^d \). Set \( a^{ij} = \sigma^{ik}\sigma^{jk}/2 \) for \( i, j = 1, 2, \ldots, d \). Let \( \mu \in \mathcal{M} \) such that it admits a density \( u = d\mu/dx \in W^m_p \) which satisfies (4.6). Then for \( \varepsilon > 0 \) we have
\[ A^\alpha := p((D^\alpha \mu)^{(\varepsilon)^{p-1}}, D^\alpha((\sigma^{ij} D^1)^{* \mu})^{(\varepsilon)}(\varepsilon)) + \frac{p(p-1)}{2}((D^\alpha \mu)^{(\varepsilon)^{p-2}}, D^\alpha((\sigma^{ij} D^1)^{* \mu})^{(\varepsilon)}, D^\alpha((\sigma^{jk} D^j)^{* \mu})^{(\varepsilon)}) \leq NL^2 |u|^p_{W^m_p} \] (4.8)
for multi-indices \( \alpha = (\alpha_1, \ldots, \alpha_d) \) such that \( 0 \leq |\alpha| \leq m \), where \( N \) is a constant depending only on \( d, m \) and \( p \).

**Proof.** Note first that using
\[ \sup_{x \in \mathbb{R}^d} \sum_{k=0}^{m+2} |D^k k_\varepsilon(x)| < \infty, \quad \sup_{x \in \mathbb{R}^d} \sum_{k=0}^{m+2} |D^k \rho_\varepsilon(x)| < \infty, \quad \text{for all } \varepsilon > 0 \] (4.9)
and
\[ \int_{\mathbb{R}^d} (1 + |x| + |x|^2) |u(x)| \, dx < \infty, \] (4.10)
as well as the conditions on $\sigma$, it is easy to verify that the left-hand side of (4.8) is well-defined. Changing the order of taking derivatives and integrals, then writing integer powers of integrals as iterated integrals and using

$$D_\varepsilon^\alpha k_\varepsilon(x-y) = (-1)^{[\alpha]} D_\varepsilon^\alpha k_\varepsilon(x-y),$$

we have

$$(D_\varepsilon^\alpha \mu(x))^{p-1} = \int_{\mathbb{R}^{(p-1)d}} \prod_{r=1}^{p-1} D_\varepsilon^\alpha k_\varepsilon(x-y_r) \mu(dy_1) \ldots \mu(dy_{p-1})$$

$$= \int_{\mathbb{R}^{(p-1)d}} (-1)^{[\alpha]} D_\varepsilon^\alpha \ldots D_\varepsilon^\alpha \prod_{r=1}^{p-1} k_\varepsilon(x-y_r) \mu(dy_1) \ldots \mu(dy_{p-1}),$$

$$D_\varepsilon^\alpha ((a^{ij} D_{ij})^* \mu(x))^{(\varepsilon)} = \int_{\mathbb{R}^d} a^{ij}(y) \partial_{y_p} \partial_{y_p} D_\varepsilon^\alpha k_\varepsilon(x-y) \mu(dy_p)$$

$$= \int_{\mathbb{R}^d} (-1)^{[\alpha]} a^{ij}(y) \partial_{y_p} \partial_{y_p} D_\varepsilon^\alpha k_\varepsilon(x-y) \mu(dy_p),$$

and hence for their product we get

$$(D_\varepsilon^\alpha \mu^{(\varepsilon)})^{p-1} D_\varepsilon^\alpha ((a^{ij} D_{ij})^* \mu)^{(\varepsilon)} = \int_{\mathbb{R}^{pd}} a^{ij}(y) \partial_{y_p} \partial_{y_p} D_\varepsilon^\alpha \Pi_{r=1}^p k_\varepsilon(x-y_r) \mu_p(dy)$$

(4.11)

where $D_\varepsilon^\alpha := D_\varepsilon^\alpha \ldots D_\varepsilon^\alpha$ and $\mu(dy) := \mu(dy_1) \ldots \mu(dy_{p-1})$. Similarly,

$$(D_\varepsilon^\alpha \mu^{(\varepsilon)})^{p-2} D_\varepsilon^\alpha ((\sigma^ik D_i)^* \mu)^{(\varepsilon)} D_\varepsilon^\alpha ((\sigma^jk D_j)^* \mu)^{(\varepsilon)}(x)$$

$$= \int_{\mathbb{R}^{pd}} \sigma^ik(y_{p-1}) \sigma^jk(y_p) \partial_{y_p} \partial_{y_p} D_\varepsilon^\alpha \Pi_{r=1}^p k_\varepsilon(x-y_r) \mu_p(dy).$$

Adding this to (4.11), then integrating against $dx$ over $\mathbb{R}^d$ and using (4.2) we obtain

$$A = \int_{\mathbb{R}^{pd}} (p a^{ij}(y) \partial_{y_p} \partial_{y_p} + \frac{p(p-1)}{2} \sigma^ik(y_{p-1}) \sigma^jk(y_p) \partial_{y_p} \partial_{y_p}) D_\varepsilon^\alpha \rho_\varepsilon(y) \mu_p(dy).$$

Using here the symmetry of $D_\varepsilon^\alpha \rho_\varepsilon(y)$ and $\mu_p(dy)$ in $y \in \mathbb{R}^{dp}$ and then interchanging differential operators we get

$$A = \int_{\mathbb{R}^{pd}} \left( \sum_{r=1}^p a^{ij}(y_r) D_\varepsilon^\alpha \partial_{y_r} \partial_{y_r} \rho_\varepsilon(y) + \sum_{1 \leq r < s \leq p} \sigma^ik(y_r) \sigma^jk(y_s) D_\varepsilon^\alpha \partial_{y_r} \partial_{y_s} \rho_\varepsilon(y) \right) \mu_p(dy)$$

Using

$$\partial_{y_r} \rho_\varepsilon(y) = - \sum_{s \neq r} \partial_{y_s} \rho_\varepsilon(y),$$

see (4.4), we have

$$\sum_{r=1}^p a^{ij}(y_r) D_\varepsilon^\alpha \partial_{y_r} \partial_{y_r} \rho_\varepsilon(y) = - \sum_{1 \leq r < s \leq p} (a^{ij}(y_r) + a^{ij}(y_s)) D_\varepsilon^\alpha \partial_{y_r} \partial_{y_s} \rho_\varepsilon(y),$$

and due to $a^{ij} = \sigma^ik \sigma^jk / 2$ we have

$$-2a^{ij}(y_r, y_s) := -2(a^{ij}(y_r) + a^{ij}(y_s)) + \sigma^ik(y_r) \sigma^jk(y_s) + \sigma^ik(y_s) \sigma^jk(y_r)$$

$$= -(\sigma^ik(y_r) - \sigma^ik(y_s))(\sigma^jk(y_r) - \sigma^jk(y_s)).$$
Hence
\[
A = -\frac{1}{2} \sum_{r \neq s} \int_{\mathbb{R}^d} a^{ij}(y_r, y_s) D^{\alpha} \partial_y^j \partial_y^i \rho_\varepsilon(y) \mu_p(dy),
\]  
(4.12)
that by integration by parts gives
\[
= -\frac{1}{2} \sum_{\beta \leq \alpha} \sum_{\gamma \leq \alpha} c_\beta^\alpha \int_{\mathbb{R}^d} \sum_{r \neq s} a^{ij}_{\beta \gamma}(y_r, y_s) \partial_y^j \partial_y^i \rho_\varepsilon(y) u_\beta(y_r) u_\gamma(y_s) \Pi_q \mu_p(dy),
\]  
(4.13)
where \(a^{ij}_{\beta \gamma}(x, r) := \partial_x^\beta \partial_x^\gamma a^{ij}(x, r)\) and \(u_\delta(x) := \partial_\delta u(x)\) for \(x, r \in \mathbb{R}^d\), for multi-indices \(\beta, \gamma\) and \(\delta, \delta' := \alpha - \delta\) for multi-indices \(\delta \leq \alpha\) (i.e. \(\delta_i \leq \alpha_i\) for \(i = 1, 2, \ldots, d\)), \(c_\beta^\gamma = \Pi_{i=1}^d c_\delta^\alpha\) with binomial coefficients \(c_\delta^\alpha\) for integers \(0 \leq k \leq n\),

\[u(y) := u(y_1) \ldots u(y_p) \text{ for } y = (y_1, \ldots, y_p) \in \mathbb{R}^{dp},\]

and \(dy = dy_1 \ldots dy_p\) is the Lebesgue measure on \(\mathbb{R}^{dp}\). For each \(\beta \leq \alpha\) and \(\gamma \leq \alpha\) we are going to estimate the integrand
\[
f^{\beta \gamma}(y) := \sum_{r \neq s} a^{ij}_{\beta \gamma}(y_r, y_s) \partial_y^j \partial_y^i \rho_\varepsilon(y) u_\beta(y_r) u_\gamma(y_s) \Pi_q \mu_p(dy), \quad y \in \mathbb{R}^{dp}, \quad \beta \leq \alpha, \quad \gamma \leq \alpha
\]
in the integral in (4.13). Because of the symmetry in \(\beta\) and \(\gamma\), we need only consider the following cases: (i) \(|\beta| \geq 1\) and \(|\gamma| \geq 1\), (ii) \(|\beta| \geq 1\) and \(\gamma = 0\) and (iii) \(\beta = \gamma = 0\). To proceed with the calculations in each of these cases, for functions \(h = h(y)\) and \(g = (y)\) of \(y \in \mathbb{R}^{dp}\) we will use the notations \(h \sim g\) if the integral of \(g - h\) against \(dy\) over \(\mathbb{R}^{dp}\) is zero. In case (i) by integration by parts we have
\[
f^{\beta \gamma} \sim \sum_{j=1}^4 f_j^{\beta \gamma}
\]
with
\[
f_1^{\beta \gamma} := \sum_{r \neq s} \partial_y^j \partial_y^i a^{ij}_{\beta \gamma}(y_r, y_s) \rho_\varepsilon(y) u_\beta(y_r) u_\gamma(y_s) \Pi_q \mu_p(dy),
\]
\[
f_2^{\beta \gamma} := \sum_{r \neq s} \partial_y^j \partial_y^i a^{ij}_{\beta \gamma}(y_r, y_s) \rho_\varepsilon(y) \partial_y^j u_\beta(y_r) u_\gamma(y_s) \Pi_q \mu_p(dy),
\]
\[
f_3^{\beta \gamma} := \sum_{r \neq s} \partial_y^j \partial_y^i a^{ij}_{\beta \gamma}(y_r, y_s) \rho_\varepsilon(y) u_\beta(y_r) \partial_y^j u_\gamma(y_s) \Pi_q \mu_p(dy),
\]
\[
f_4^{\beta \gamma} := \sum_{r \neq s} a^{ij}_{\beta \gamma}(y_r, y_s) \rho_\varepsilon(y) \partial_y^j u_\beta(y_r) \partial_y^i u_\gamma(y_s) \Pi_q \mu_p(dy).
\]
(4.14)
It is easy to see that for \(j = 1, 2, 3, 4\)
\[
|f_j^{\beta \gamma}(y)| \leq NL^2 \rho_\varepsilon(y) \sum_{|\delta| \leq m} |u_\delta(y_1)| \ldots \sum_{|\delta| \leq m} |u_\delta(y_p)|, \quad (y_1, y_2, \ldots, y_p) \in \mathbb{R}^{dp}
\]
with a constant \(N = N(d, m, p)\). Hence in the case (i) we get
\[
\int_{\mathbb{R}^{dp}} f^{\beta \gamma}(y) \, dy \sim NL^2 \int_{\mathbb{R}^{dp}} \sum_{|\delta| \leq m} |u_\delta(y_1)| \ldots \sum_{|\delta| \leq m} |u_\delta(y_p)| \rho_\varepsilon(y) \, dy
\]
Clearly, for \( r \neq s \) we have
\[
\hat{c}_{g_{y_1}}\beta_{\beta_{y}}(y_{r},y_{s}) = g^{\beta_{y}}(y_{r},y_{s}) + h^{\beta_{y}}(y_{r}),
\]
with
\[
g^{\beta_{y}}(y_{r},y_{s}) = \hat{c}_{y_{r}}\beta_{y_{r}}\sigma^{ik}(y_{r})(\sigma^{jy}(y_{r}) - \sigma^{jy}(y_{s})) + \hat{c}_{y_{r}}\beta_{y_{r}}\sigma^{jk}(y_{r})(\sigma^{ik}(y_{r}) - \sigma^{ik}(y_{s})),
\]
\[
h^{\beta_{y}}(y_{r}) = \sum_{1 \leq |\delta| < |\beta_{y}|} c_{\delta}^{\beta(i)} \sigma^{ik}(y_{r}) \sigma^{yk}(y_{r}) - \sigma^{ik}(y_{s})).
\]
where the multi-index \( \beta(i) \) is defined by \( \hat{c}_{\beta(i)} = \hat{c}_{y_{r}}\beta_{y_{r}}. \) Thus
\[
f_{1}^{\beta_{0}} = f_{11}^{\beta_{0}} + f_{12}^{\beta_{0}}
\]
with
\[
f_{11}^{\beta_{0}} = \sum_{r=1}^{p} \sum_{s \neq r} g^{\beta_{y}}(y_{r},y_{s}) \hat{c}_{y_{r}}\beta_{y_{r}}\rho_{\varepsilon}(y_{r}) u^{\beta_{y}}(y_{r}) \Pi_{q \neq r} u_{\alpha}(y_{q}),
\]
\[
f_{12}^{\beta_{0}} = \sum_{r=1}^{p} \sum_{s \neq r} h^{\beta_{y}}(y_{r}) \hat{c}_{y_{r}}\beta_{y_{r}}\rho_{\varepsilon}(y_{r}) u^{\beta_{y}}(y_{r}) \Pi_{q \neq r} u_{\alpha}(y_{q}).
\]
(4.15)

Since
\[
|g^{\beta_{y}}(y_{r},y_{s})| \leq NL^{2}|y_{r} - y_{s}| \quad j = 1, 2, \ldots, p,
\]
for some \( N = N(d, m, p) \), taking into account (4.3) we have
\[
|g^{\beta_{y}}(y_{r},y_{s})\hat{c}_{y_{r}}\beta_{y_{r}}\rho_{\varepsilon}(y)| \leq \frac{N}{\rho_{\varepsilon}} \sum_{1 \leq k < l \leq p} |y_{k} - y_{l}|^{2}\rho_{\varepsilon}(y) \leq N' \rho_{2\varepsilon}(y)
\]
and hence
\[
|f_{11}^{\beta_{0}}| \leq N' \rho_{2\varepsilon}(y) \sum_{|\delta| \leq m} |u_{\delta}(y_{1})| \ldots \sum_{|\delta| \leq m} |u_{\delta}(y_{p})|.
\]
with a constant $N' = N'(d, m, p)$. Remembering (4.4) by integration by parts we obtain

$$f_{12}^{\beta 0} = - \sum_{r=1}^{p} \int h^{\beta,j}(y_r) \partial_{y_{rj}} \rho_{\varepsilon}(y_r) u_{\beta}(y_r) \Pi_{q \neq r} u_{\alpha}(y_q) \sim f_{121}^{\beta 0} + f_{122}^{\beta 0}$$

with

$$f_{121}^{\beta 0} = \sum_{r=1}^{p} h^{\beta,j}(y_r) \rho_{\varepsilon}(y) \partial_{y_{rj}} u_{\beta}(y_r) \Pi_{q \neq r} u_{\alpha}(y_q),$$

$$f_{122}^{\beta 0} = \sum_{r=1}^{p} \partial_{y_{rj}} h^{\beta,j}(y_r) \rho_{\varepsilon}(y) u_{\beta}(y_r) \Pi_{q \neq r} u_{\alpha}(y_q).$$

Hence noting that

$$|h^{\beta,j}(y_r)| + |\partial_{y_{rj}} h^{\beta,j}(y_r)| \leq NL^2$$

with a constant $N = N(d, m, p)$, we get

$$|f_{121}^{\beta 0} + f_{122}^{\beta 0}| \leq NL^2 \rho_{\varepsilon}(y) \sum_{|\delta| \leq m} |u_{\delta}(y_1)| \ldots \sum_{|\delta| \leq m} |u_{\delta}(y_p)|$$

Consequently, for a constant $N' = N'(d, m, p)$,

$$\int_{\mathbb{R}^d} f_1^{\beta 0}(y) dy \leq NL^2 \int_{\mathbb{R}^d} \sum_{|\delta| \leq m} |u_{\delta}(y_1)| \ldots \sum_{|\delta| \leq m} |u_{\delta}(y_p)| \rho_{2\varepsilon}(y) dy$$

$$\leq N'L^2 \sum_{|\delta| \leq m} ||D^\delta u^{(2\varepsilon)}||_{L_p} \leq N'L^2 \sum_{|\delta| \leq m} ||D^\delta u^{(\varepsilon)}||_{L_p}. \tag{4.16}$$

Now we are going to estimate the integral of $f_2^{\beta 0}$. If $|\beta| = 1$, then

$$|a_{\beta 0}^{ij}(y_r, y_s)| \leq NL^2 |y_r - y_s|,$$

and taking into account (4.3), we get

$$|f_2^{\beta 0}| \leq NL^2 \rho_{2\varepsilon}(y) \sum_{|\delta| \leq m} |u_{\delta}(y_1)| \ldots \sum_{|\delta| \leq m} |u_{\delta}(y_p)|$$

with $N = N(d, p, m)$ in the same way as $|f_{11}|$ is estimated. Hence, as above,

$$\int_{\mathbb{R}^d} f_2^{\beta 0}(y) dy \leq NL^2 \sum_{|\delta| \leq m} ||D^\delta u^{(2\varepsilon)}||_{L_p} \leq NL^2 \sum_{|\delta| \leq m} ||D^\delta u^{(\varepsilon)}||_{L_p}. \tag{4.17}$$

for $|\beta| = 1$. If $|\beta| \geq 2$, then

$$a_{\beta 0}^{ij}(y_r, y_s) = g^{\beta,ij}(y_r, y_s) + h^{\beta,ij}(y_r)$$

with

$$g^{\beta,ij}(y_r, y_s) = \partial_{y_{rj}} \sigma^{\beta k}(y_r)(\sigma^{jk}(y_s) - \sigma^{jk}(y_r)) + \partial_{y_{rj}} \sigma^{jk}(y_r)(\sigma^{\beta k}(y_s) - \sigma^{\beta k}(y_r))$$

$$h^{\beta,ij}(y_r) = \sum_{1 \leq |\delta|, \delta < \beta} c_\delta^{\beta} \partial_{y_{rj}} \sigma^{\delta k}(y_r) \partial_{y_{rj}} \sigma^{j \delta}(y_r).$$
Noticing that for a constant $N = N(d, m, p)$,
\[ \sum_{ij} |g^{\beta,ij}(y_r, y_s)| \leq NL^2 |y_r - y_s| \]
and
\[ \sum_{ij} |h^{\beta,ij}(y_r)| + \sum_{ij} |\partial_{y_i} h^{\beta,ij}(y_r)| \leq NL^2, \]
we obtain (4.17) for $|\beta| \geq 2$ in the same way as the integral of $f_1^{30}$ is estimated. It remains to consider the case (iii), i.e., to estimate the integral of $f_0^0$. Since
\[ |a_{00}^{ij}(y_r, y_s)| \leq NL^2 |y_r - y_s|^2 \]
with a constant $N = N(d, m, p)$ and
\[ \partial_{y_i} \partial_{y_j} \rho_{\epsilon}(y) = \frac{1}{p \epsilon^2} \sum_{k=1}^{p} \sum_{l=1}^{p} (y_k^l - y_i^j)(y_k^j - y_l^i) + \frac{1}{p \epsilon} \delta_{ij}, \]
we have for a constant $N' = N'(d, m, p)$,
\[ |a_{00}^{ij}(y_r, y_s) \partial_{y_i} \partial_{y_j} \rho_{\epsilon}(y)| \leq \frac{N}{\epsilon} L^2 \sum_{1 \leq k < l \leq p} |y_k - y_l|^4 \rho_{\epsilon}(y) + \frac{N}{\epsilon} L^2 \sum_{1 \leq k < l \leq p} |y_k - y_l|^2 \rho_{\epsilon}(y) \]
\[ \leq N'L^2 \rho_{2\epsilon}(y) \text{ for } y = (y_1, ..., y_p) \in \mathbb{R}^{pd}. \]
Hence
\[ |f_0^0(y)| \leq NL^2 \rho_{2\epsilon}(y) \prod_{v=1}^{p} \left| u_\alpha(y_r) \right|, \]
that gives
\[ \int_{\mathbb{R}^d} f_0^0(y) \, dy \leq NL^2 \|D^\alpha u(\cdot)\|_{L_p}^p \leq NL^2 \|D^\alpha u^{(\epsilon)}(\cdot)\|_{L_p}^p \]
with a constant $N = N(d, m, p)$, and we finish the proof of (4.8) by using $\|v^{(\epsilon)}\|_{L_p} \leq \|v\|_{L_p}$ for $v \in L_p(\mathbb{R}^d)$.

**Corollary 4.2.** Let the conditions of Lemma 4.1 hold for integers $m \geq 0$ and $p \geq 2$ even. Then for $\epsilon > 0$ we have
\[ ((D^\alpha \mu^{(\epsilon)})^p)^{p-1}, D^\alpha(\sigma^{ij} D_{ij})^{(\epsilon)} \mu \leq NL^2 \|u\|_{W^{p}}^p \]
for multi-indices $\alpha = (\alpha_1, ..., \alpha_d)$ such that $0 \leq |\alpha| \leq m$, where $N$ is a constant depending only on $d, m$ and $p$.

**Proof.** It suffices to note that
\[ ((D^\alpha \mu^{(\epsilon)})^{p-2}D^\alpha((\sigma^{ik} D_i)^{\epsilon})\mu^{(\epsilon)}, D^\alpha((\sigma^{ij} D_{ij})^{\epsilon}) \mu^{(\epsilon)}) \]
\[ = \int_{\mathbb{R}^d} (D^\alpha \mu^{(\epsilon)})^{p-2}(x) \sum_{k=1}^{d} \left| D^\alpha((\sigma^{ik} D_i)^{\epsilon}) \mu^{(\epsilon)}(x) \right|^2 \, dx \geq 0 \]
\[ \square \]
Lemma 4.3. Let \( p \geq 2 \) and \( m \geq 0 \) be integers, and let \( \sigma = (\sigma^i) \) and \( b \) be Borel functions on \( \mathbb{R}^d \) with values in \( \mathbb{R}^d \) and \( \mathbb{R} \) respectively. Assume the partial derivatives of \( \sigma \) and \( b\sigma \) up to order \( m \) are functions such that there exist constants \( K \geq L \geq 1 \) such that

\[
\sum_{k=0}^{m+1} |D^k b(x)| \leq K, \quad |\sigma(x)| \leq K_0,
\]

\[
\sum_{k=1}^{m+1} |D^k \sigma(x)| + \sum_{k=1}^{m+1} |D^k (b\sigma)(x)| \leq L
\]

for all \( x, y \in \mathbb{R}^d \). Then for finite signed Borel measures \( \mu \) on \( \mathbb{R} \) with density \( u := d\mu/dx \in W_p^m \), satisfying (4.6), we have

\[
((D^\alpha \mu^{(\varepsilon)})^{p-2} D^\alpha (b\mu)^{(\varepsilon)}, D^\alpha (b\mu)^{(\varepsilon)}) \leq NK^2 |u|_{W_p^m}^p,
\]

(4.18)

\[
((D^\alpha \mu^{(\varepsilon)})^{p-2} D^\alpha ((\sigma^i D_i^*) \mu)^{(\varepsilon)} D^\alpha (b\mu)^{(\varepsilon)}) \leq NKL |u|_{W_p^m}^p
\]

(4.19)

for \( \varepsilon > 0 \) and multi-indices \( \alpha \) such that \( |\alpha| \leq m \), where \( N \) is a constant depending only on \( d, p, m \).

Proof. First note that by (4.9) and (4.10), as well as the conditions on \( \sigma \) and \( b \), the left-hand sides of (4.18) and (4.19) are well-defined. Interchanging the order of integration and the differential operator \( D^\alpha \), rewriting the product of integrals as multiple integral, using Fubini’s theorem and the identity

\[
D^\alpha_k k_\varepsilon(x - z) = (-1)^{|\alpha|} D^\alpha_z k_\varepsilon(x - z), \quad x, y \in \mathbb{R}^d,
\]

as well as (4.2), for the left-hand side \( F_\alpha \) of (4.18) we compute

\[
F_\alpha = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} b(y_r)b(y_s) \Pi_{j=1}^p D^\alpha_x k_\varepsilon(x - y_j) \mu_p(dy) \, dx
\]

\[
= (-1)^{|\alpha|} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} b(y_r)b(y_s) \Pi_{j=1}^p D^\alpha_{y_j} k_\varepsilon(x - y_j) \mu_p(dy) \, dx
\]

\[
= (-1)^{|\alpha|} \int_{\mathbb{R}^d} b(y_r)b(y_s) \Pi_{j=1}^p D^\alpha_{y_j} k_\varepsilon(x - y_j) \, dx \, \mu_p(dy)
\]

\[
= (-1)^{|\alpha|} \int_{\mathbb{R}^d} b(y_r)b(y_s) \Pi_{j=1}^p \rho_\varepsilon(y) \Pi_{j=1}^p u(y_j) \, dy
\]

for any \( r, s \in \{1, 2, \ldots, p\} \) such that \( r \neq s \), where recall that \( dy = dy_1 \ldots dy_p \) and \( D^\alpha_{y_j} = \Pi_{j=1}^p D^\alpha_{y_j} \). Hence by integration by parts we obtain

\[
F_\alpha = \sum_{\beta \leq \alpha} \sum_{\gamma \leq \alpha} c^\beta_\gamma c^\alpha_\gamma \int_{\mathbb{R}^d} b(y_r)b(y_s) \Pi_{j=1}^p u(y_j) \Pi_{j \neq s,r}^p u_{\alpha}(y_j) \, dy,
\]

where \( v_\delta := D^\delta v \) and \( \tilde{\delta} := \alpha - \delta \) for functions \( v \) on \( \mathbb{R}^d \) and multi-indices \( \delta \leq \alpha \). Using here (4.2) and the boundedness condition on \( |b| \) and \( |D^\delta b| \) we have

\[
F_\alpha \leq NK^2 \sum_{\beta \leq \alpha} \sum_{\gamma \leq \alpha} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Pi_{j=1}^p k_\varepsilon(x - y_j) |u_\beta(y_r)||u_\gamma(y_s)||\Pi_{j \neq s,r}^p u_{\alpha}(y_j)| \, dx \, dy
\]
\[ R^\alpha = \int_{\mathbb{R}^p} f_{krr}(y) \, dy, \]

for any \( r, k \in \{1, 2, \ldots, p\} \) such that \( r \neq k \), where

\[ f_{krs}(y) := (-1)^{p+1}\|b(y_k)\sigma^j(y_r)\gamma \partial_y \rho_\varepsilon(y) u_\gamma(y_k)u_\beta(y_r)\Pi_{j \neq k,r} u_\alpha(y_j) \]

for \( k, r, s \in \{1, 2, \ldots, p\} \). As in the proof of Lemma 4.1, for real functions \( f \) and \( g \) we write \( f \sim g \) if they have the same (finite) Lebesgue integral against \( dy = dy_1 \ldots dy_p \) over \( \mathbb{R}^p \). We write \( f \leq g \) if the integrals of \( f \) and \( g \) against \( dy \) over \( \mathbb{R}^d \) are finite, and the integral of \( f - g \) can be estimated by \( NKL\|u\|_{W^{m}_p} \) for all \( u \in W^{m}_p \) with a constant \( N = N(d, m, p) \), independent of \( u \). By integration by parts we have

\[ f_{krr} \sim \sum_{\gamma \leq \alpha} \sum_{\beta \leq \alpha} f^{\gamma\beta}_{krr} \]

with

\[ f^{\gamma\beta}_{krs}(y) := c_\gamma^\beta c_\gamma^\beta b_\gamma(y_k)\sigma^j(y_r)\gamma \partial_y \rho_\varepsilon(y) u_\gamma(y_k)u_\beta(y_r)\Pi_{j \neq k,r} u_\alpha(y_j) \]

If \( \beta \neq 0 \) then by integration by parts (dropping \( \partial_y \) from \( \rho_\varepsilon \) to the other terms), and using the boundedness of \( b \), its derivatives up to order \( m + 1 \), and the boundedness of the derivatives of \( \sigma \) up to order \( m + 1 \), we see that \( f^{\gamma\beta}_{krr} \leq 0 \) for any \( k = 1, 2, \ldots, p, r \neq k \) and \( \gamma \leq \alpha \). If \( \beta = 0 \) and \( \gamma = 0 \), then \( f^{00}_{krr} \) can be estimated by an exact repetition of the proof of Lemma 4.2 in [6], by replacing \( \mu \) therein with \( u_\alpha dy \), to yield \( f^{00}_{krr} \leq 0 \). Consequently,

\[ f_{krr} \leq \sum_{0 \neq \gamma \leq \alpha} f^{00}_{krr} \quad \text{for every } k = 1, \ldots, p \text{ and } r \in \{1, 2, \ldots, p\} \setminus \{k\}. \]

Writing \( f^{00}_{krr}(y) = g^{\gamma}_{krr}(y)h^{\gamma}_{krr}(y) \), with

\[ g^{\gamma}_{krs}(y) := c_\gamma^\beta b_\gamma(y_k)\sigma^j(y_r)\gamma \partial_y \rho_\varepsilon(y), \quad h^{\gamma}_{krr}(y) := u_\gamma(y_k)\Pi_{j \neq k,r} u_\alpha(y_j), \]

we get

\[ p(p-1)(p-2)R^\alpha \leq \sum_{0 \neq \gamma \leq \alpha} \sum_{s=1}^{p} \sum_{r \neq s} \sum_{s \neq r} \int_{\mathbb{R}^p} g^{\gamma}_{kss}(y)h^{\gamma}_{krr}(y) \, dy + NKL\|u\|_{W^{m}_p}, \]

and by (4.4),

\[ p(p-1)R^\alpha \leq - \sum_{0 \neq \gamma \leq \alpha} \sum_{k=1}^{p} \sum_{r \neq k} \sum_{s \neq r} \int_{\mathbb{R}^p} g^{\gamma}_{krs}(y)h^{\gamma}_{krr}(y) \, dy + NKL\|u\|_{W^{m}_p} \]

\[ = - \sum_{0 \neq \gamma \leq \alpha} \sum_{s=1}^{p} \sum_{r \neq s} \sum_{k \neq r} \int_{\mathbb{R}^p} g^{\gamma}_{krs}(y)h^{\gamma}_{krr}(y) \, dy. \]
for all constant $R$. Summing up (4.21) and (4.22) we obtain

$$c_p \mathcal{R}^\alpha \leq \sum_{0 \neq \gamma \leq \alpha} \sum_{s=1}^p \sum_{r \neq s} \int_{\mathbb{R}^p} (g^\gamma_{kss}(y) - g^\gamma_{krs}(y))h^\gamma_k(y) \, dy + NKL|u|_{W^p}^p$$

(4.23)

where $c_p = p(p-1)^2$, and

$$(g^\gamma_{kss}(y) - g^\gamma_{krs}(y))h^\gamma_k(y) = c^\gamma_i b_\gamma(y_k)(\sigma^i(y_s) - \sigma^i(y_r))c_\gamma u_\gamma(y_k)\Pi_{j \neq k}u_\alpha(y).$$

By the boundedness of $|b_\gamma|$ and the Lipschitz condition on $\sigma$, using (4.5) we get

$$(g^\gamma_{kss} - g^\gamma_{krs})h^\gamma_k \leq 0, \quad \text{for all } 0 \neq \gamma \leq \alpha, \ s = 1, 2, \ldots, p \text{ and } r \neq s, \ k \neq r, s.$$

By integration by parts we have for the last term in (4.23),

$$g^\gamma_{krs}h^\gamma_s \leq 0, \quad \text{for } 0 \neq \gamma \leq \alpha \text{ and } s = 1, \ldots, p, \ r \neq s,$$

which finishes the proof of (4.19).

For vectors $\xi = \xi(x) \in \mathbb{R}^d$, depending on $x \in \mathbb{R}^d$ we consider the linear operators $I^\xi$ and $J^\xi$ defined by

$$T^\xi \varphi(x) = \varphi(x + \xi(x))$$

$$I^\xi \varphi(x) := T^\xi \varphi(x) - \varphi(x), \quad J^\xi \psi(x) := I^\xi \psi(x) - \xi(x)D_\xi \psi(x),$$

$x \in \mathbb{R}^d$, acting on functions $\varphi$ and differentiable functions $\psi$ on $\mathbb{R}^d$.

**Lemma 4.4.** Let $\xi = \xi(x, \xi)$ be an $\mathbb{R}^d$-valued function of $x \in \mathbb{R}^d$ for every $\xi \in \Xi$ for a set $\Xi$. Assume that for an integer $m \geq 1$ the partial derivatives of $\xi$ in $x \in \mathbb{R}^d$ up to order $m$ are functions on $\mathbb{R}^d$ for each $\xi \in \Xi$, such that for a constant $\lambda > 0$, a function $\xi$ on $\Xi$ and a constant $K_\xi \geq 0$ we have

$$|\xi(x, \xi)| \leq \bar{\xi}(\xi) \leq K_\xi,$$

$$\sum_{k=1}^{m+1} |D^k_\xi(x, \xi)| \leq \bar{\xi}(\xi), \quad |\det(\mathbb{I} + \theta D_x \xi(x, \xi))| \geq \lambda^{-1}$$

(4.25)

for all $x, y \in \mathbb{R}^d$, $\xi \in \Xi$ and $\theta \in [0, 1]$. Let $p \geq 2$ be an even integer. Then for every finite signed Borel measure $\mu$ with density $u = d\mu/dx \in W^m_p$, satisfying (4.6), we have

$$C := \int_{\mathbb{R}^d} p(D^\alpha_\mu(\xi))^{p-1} D^\alpha_\mu(J^\xi \mu)^{(\xi)} dx$$

$$+ \int_{\mathbb{R}^d} (D^\alpha_\mu + D^\alpha_\mu(I^\xi \mu)^{(\xi)})^{p} - (D^\alpha_\mu)^{p} - p(D^\alpha_\mu)^{p-1} D^\alpha_\mu(I^\xi \mu)^{(\xi)} dx$$

$$\leq N \xi^2(\xi)|u|_{W^p}^p \quad \text{for } \xi \in \Xi, \ v > 0$$

(4.26)

for multi-indices $\alpha$, $0 \leq |\alpha| \leq m$ with a constant $N = N(d, p, m, \lambda, K_\xi)$. 
Proof. Again we note that by (4.9) & (4.10), together with the conditions on $\xi$, it is easy to verify that $C$ is well-defined. Notice that

$$D_x^\alpha \mu^{(\varepsilon)} + D_x^\alpha (I^{\xi^*} \mu)^{(\varepsilon)} = D_x^\alpha (T^{\xi^*} \mu)^{(\varepsilon)}$$

and

$$p(D_x^\alpha \mu^{(\varepsilon)})^{p-1} D_x^\alpha (J^{\xi^*} \mu)^{(\varepsilon)} - p(D_x^\alpha \mu^{(\varepsilon)})^{p-1} D_x^\alpha (I^{\xi^*} \mu)^{(\varepsilon)}$$

$$+ p(D_x^\alpha \mu^{(\varepsilon)})^{p-1} D_x^\alpha ((\xi^i D_i)^* \mu)^{(\varepsilon)}$$

Hence

$$C = \int_{\mathbb{R}^d} (D_x^\alpha (T^{\xi^*} \mu)^{(\varepsilon)})^p - (D_x^\alpha \mu^{(\varepsilon)})^p + p(D_x^\alpha \mu^{(\varepsilon)})^{p-1} D_x^\alpha ((\xi^i D_i)^* \mu)^{(\varepsilon)} \, dx. \quad (4.27)$$

First we change the order of $D_x^\alpha$ and the integrals and operators $T_y^x$ and $I_y^x$ acting in the variable $y \in \mathbb{R}^d$, then we use

$$D_x^\alpha k_\varepsilon(x - y) = (-1)^{|\alpha|} D_y^\alpha k_\varepsilon(x - y)$$

to get

$$D_x^\alpha (T^{\xi^*} \mu)^{(\varepsilon)} = (-1)^{|\alpha|} \int_{\mathbb{R}^d} T_y^x D_y^\alpha k_\varepsilon(x - y) \, \mu(dy),$$

$$D_x^\alpha \mu^{(\varepsilon)} = (-1)^{|\alpha|} \int_{\mathbb{R}^d} D_y^\alpha k_\varepsilon(x - y) \, \mu(dy),$$

$$D_x^\alpha ((\xi^i D_i)^* \mu)^{(\varepsilon)} = (-1)^{|\alpha|} \int_{\mathbb{R}^d} \xi^i(y) \hat{c}_y^\varepsilon D_y^\alpha k_\varepsilon(x - y) \, \mu(dy).$$

Thus rewriting the product of integrals as multiple integrals, and using the product measure $\mu_p(dy) := \mu(dy_1) \ldots \mu(dy_p)$ on $\mathbb{R}^{dp}$ by Fubini’s theorem we get

$$(D_x^\alpha (T^{\xi^*} \mu)^{(\varepsilon)})^p(x) = \int_{\mathbb{R}^{dp}} \Pi_{r=1}^p (T_y^{x_r} D_{y_r}^\alpha k_\varepsilon(x - y_r)) \, \mu_p(dy)$$

$$= \int_{\mathbb{R}^{dp}} \Pi_{r=1}^p T_y^{x_r} D_{y_r}^\alpha \Pi_{r=1}^p k_\varepsilon(x - y_r) \, \mu_p(dy),$$

$$(D_x^\alpha \mu^{(\varepsilon)})^p = \int_{\mathbb{R}^{dp}} \Pi_{r=1}^p (D_{y_r}^\alpha k_\varepsilon(x - y_r)) \, \mu_p(dy)$$

$$= \int_{\mathbb{R}^{dp}} D_y^p \Pi_{r=1}^p k_\varepsilon(x - y_r) \, \mu_p(dy) \quad (4.28)$$

and

$$p(D_x^\alpha \mu^{(\varepsilon)})^{p-1} D_x^\alpha ((\xi^i D_i)^* \mu)^{(\varepsilon)} = p \int_{\mathbb{R}^{dp}} \Pi_{r=1}^p (D_{y_r}^\alpha k_\varepsilon(x - y_r)) \xi^i(y_p) \hat{c}_{y_p}^\varepsilon D_{y_p}^\alpha k_\varepsilon(x - y_p) \, \mu_p(dy)$$

$$= p \int_{\mathbb{R}^{dp}} \xi^i(y_p) \hat{c}_{y_p}^\varepsilon D_y^p \Pi_{r=1}^p k_\varepsilon(x - y_r) \, \mu_p(dy)$$

$$= \int_{\mathbb{R}^{dp}} \sum_{r=1}^p \xi^i(y_r) \hat{c}_{y_r}^\varepsilon D_y^p \Pi_{r=1}^p k_\varepsilon(x - y_r) \, \mu_p(dy), \quad (4.29)$$

where again

$$D_y^p := \Pi_{r=1}^p D_{y_r}^\alpha \quad \text{for } y = (y_1, \ldots, y_p) \in \mathbb{R}^{dp},$$
and the last equation is due to the symmetry of the function \( \Pi^{\rho}_{r=1}D^\rho_y k_\varepsilon(x - y_r) \) and the measure \( \mu_p(dy) \) in \( y = (y_1, \ldots, y_p) \in \mathbb{R}^p \). Thus from (4.27) we get

\[
C = \int_{\mathbb{R}^d} \int_{\mathbb{R}^p} L_y^\xi D^\rho_y \Pi^{\rho}_{r=1} k_\varepsilon(x - y_r) \mu_p(dy) \, dx
\]

with the operator

\[
L_y^\xi = \Pi^{\rho}_{r=1} T^\xi_{y_r} - I - \sum_{r=1}^p \xi^i(y_r) \partial_{y^i_r},
\]

defined by

\[
L_y^\xi \varphi(y) = \varphi(y_1 + \xi(y_1), \ldots, y_p + \xi(y_p)) - \varphi(y) - \sum_{r=1}^p \xi^i(y_r) \partial^i_y \varphi(y), \quad y = (y_1, \ldots, y_p) \in \mathbb{R}^p
\]

for differentiable functions \( \varphi \) of \( y = (y_1, \ldots, y_p) \in \mathbb{R}^p \). Using here Fubini’s theorem then changing the order of the operator \( L_y^\xi D^\rho_y \) and the integration against \( dx \), by virtue of (4.2) we have

\[
C = \int_{\mathbb{R}^d} \int_{\mathbb{R}^p} L_y^\xi D^\rho_y \Pi^{\rho}_{r=1} k_\varepsilon(x - y_r) \, dx \, \mu_p(dy) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^p} L_y^\xi D^\rho_y \rho_\varepsilon(y) \mu_p(dy), \quad (4.30)
\]

By Taylor’s formula

\[
L_y^\xi D^\rho_y \rho_\varepsilon(y) = \int_0^1 (1 - \vartheta) \xi^i(y_k) \xi^j(y_l) (\partial_{y^i_k} \partial_{y^j_l} D^\rho_y \rho_\varepsilon)(y + \vartheta \xi(y)) \, d\vartheta,
\]

where \( y = (y_1, \ldots, y_p) \in \mathbb{R}^d \), \( y_k \in \mathbb{R}^d \) for \( k = 1, 2, \ldots, p \), and \( \xi(y) := (\xi(y_1), \ldots, \xi(y_p)) \) for \( y = (y_1, \ldots, y_p) \in \mathbb{R}^d \). Thus by changing the order of integrals and then changing the variables \( y_k \) with \( y_k + \vartheta \xi(y_k) \) for \( k = 1, 2, \ldots, p \), from (4.30) we obtain

\[
C = \int_0^1 (1 - \vartheta) C(\vartheta) \, d\vartheta \quad (4.31)
\]

with

\[
C(\vartheta) = \int_{\mathbb{R}^d} \sum_{k=1}^p \sum_{i=1}^d \hat{\xi}^i(y_k) \hat{\xi}^j(y_l) \partial_{y^i_k} \partial_{y^j_l} D^\rho_y \rho_\varepsilon(y) \Pi^{\rho}_{r=1} \hat{u}(y_r) \, dy,
\]

where, with \( \tau_\vartheta(x) := x + \vartheta \xi(x) \),

\[
\hat{\xi}^i(x) := \xi^i(\tau_{\vartheta}^{-1}(x)), \quad \hat{u}(x) = u(\tau_{\vartheta}^{-1}(x)) \det D\tau_{\vartheta}^{-1}(x), \quad x \in \mathbb{R}^d, \quad i = 1, 2, \ldots, d, \quad (4.32)
\]

and \( dy := dy_1 dy_2 ... dy_p \) denotes the Lebesgue measure on \( \mathbb{R}^d \). Clearly,

\[
C(\vartheta) = C_1(\vartheta) + C_2(\vartheta)
\]

with

\[
C_1(\vartheta) = \int_{\mathbb{R}^d} \sum_{k=1}^p \hat{\xi}^i(y_k) \hat{\xi}^j(y_l) \partial_{y^i_k} \partial_{y^j_l} D^\rho_y \rho_\varepsilon(y) \Pi^{\rho}_{r=1} \hat{u}(y_r) \, dy,
\]

\[
C_2(\vartheta) = \int_{\mathbb{R}^d} \sum_{k=1}^p \sum_{l \neq k} \hat{\xi}^i(y_k) \hat{\xi}^j(y_l) \partial_{y^i_k} \partial_{y^j_l} D^\rho_y \rho_\varepsilon(y) \Pi^{\rho}_{r=1} \hat{u}(y_r) \, dy.
\]
Using (4.4) and the symmetry in $y_k$ and $y_l$, we have

$$C_1(\vartheta) = -\frac{1}{2} \int_{\mathbb{R}^d} \sum_{k=1}^{p} \sum_{l \neq k} (\tilde{e}_i(y_k)\tilde{e}_j(y_l) + \tilde{e}_i(y_l)\tilde{e}_j(y_k)) \partial_{y_k} \partial_{y_l} D^{\rho_{\vartheta}}(y) \Pi_{r=1}^{p} \tilde{u}(y) \, dy,$$

$$C_2(\vartheta) = \frac{1}{2} \int_{\mathbb{R}^d} \sum_{k=1}^{p} \sum_{l \neq k} (\tilde{e}_i(y_k)\tilde{e}_j(y_l) + \tilde{e}_i(y_l)\tilde{e}_j(y_k)) \partial_{y_k} \partial_{y_l} D^{\rho_{\vartheta}}(y) \Pi_{r=1}^{p} \tilde{u}(y) \, dy. \tag{4.33}$$

Hence

$$C(\vartheta) = \int_{\mathbb{R}^d} \sum_{r=1}^{p} \sum_{s \neq r} \tilde{a}^{rs}(y_r, y_s) \partial_{y_r} \partial_{y_s} D^{\rho_{\vartheta}}(y) \Pi_{r=1}^{p} \tilde{u}(y) \, dy \tag{4.34}$$

with

$$\tilde{a}^{rs}(y_r, y_s) = -\frac{1}{2} (\tilde{e}_i(y_r) - \tilde{e}_i(y_s)) (\tilde{e}_j(y_r) - \tilde{e}_j(y_s)).$$

Notice that the right-hand side of equation (4.34) is the same as the right-hand side of (4.12) with $\tilde{e}_i$ in place of $\sigma^i$ for each $i = 1, 2, \ldots, d$ and with $\tilde{u}$ in place of $u$. It is easy to verify, see Lemma 3.3 in [4], that for a constant $N = N(d, \lambda, m, K_{\xi})$ we have

$$\sum_{k=1}^{m+1} |D^k(\tau_{\vartheta}^{-1}(x))| \leq N, \quad \text{for each} \; \vartheta \in [0, 1], \; \xi \in \mathcal{F}, \; x \in \mathbb{R}^d.$$ 

Thus also for each $\vartheta \in [0, 1]$,

$$\sum_{k=1}^{m+1} |D^k \hat{\xi}(x, \xi)| \leq N \hat{\xi}(\xi) \quad \text{for} \; x \in \mathbb{R}^d, \; \xi \in \mathcal{F}, \tag{4.35}$$

with a constant $N = N(d, m, \lambda, K_{\xi})$, i.e., for each $\vartheta \in [0, 1]$ and $\xi \in \mathcal{F}$ the function $\hat{\xi}$ of $x \in \mathbb{R}^d$ satisfies the condition (4.7) on $\sigma$ in Lemma 4.1, with $N \hat{\xi}(\xi)$ in place of $L$. Consequently, copying the calculations which lead from equation (4.12) to the estimate (4.8) in the proof of Lemma 4.1, we obtain

$$C(\vartheta) \leq N \hat{\xi}^2(\xi) |\hat{u}|_{W^p_{\infty}} \quad \text{for each} \; \vartheta \in [0, 1], \; \xi \in \mathcal{F}$$

with a constant $N = N(d, m, p, \lambda, K_{\xi})$. Note that due to the condition (4.25) there is a constant $N = N(d, p, m, \lambda, K_{\xi})$ such that

$$|\hat{u}|_{W^p_{\infty}} \leq N |u|_{W^p_{\infty}} \quad \text{for all} \; \vartheta \in [0, 1]. \tag{4.36}$$

Hence by virtue of (4.31) the estimate (4.26) follows. \hfill \Box

**Corollary 4.5.** Let the conditions of Lemma 4.4 hold. Then for every finite signed Borel measure $\mu$ with density $u = d\mu/|dx| \in W^m_p$, satisfying (4.6), we have

$$\int_{\mathbb{R}^d} (D^\alpha \mu(\epsilon)) |u|_{W^p_{\infty}} \, dx \leq N \hat{\xi}^2(\xi) |u|_{W^p_{\infty}} \quad \text{for} \; \xi \in \mathcal{F}, \; \epsilon > 0 \tag{4.37}$$

for multi-indices $\alpha$, $0 \leq |\alpha| \leq m$ with a constant $N = N(d, p, m, \lambda, K_{\xi})$. 
Lemma 4.6. Let the conditions of Lemma 4.4 hold. Then for every finite signed Borel measure $\mu$ with density $u = d\mu/dx \in W^m_p$, satisfying (4.6), we have

$$\left| \int_{\mathbb{R}^d} (D^\alpha u(\varepsilon) + D^\alpha (I^\xi u(\varepsilon)))^p - (D^\alpha u(\varepsilon))^p \, dx \right| \leq N\xi(\beta) |u|^p_{W^m_p},$$

for a constant $N = N(d, p, m, \lambda, K_\xi)$ for $\beta \in \mathfrak{B}$, where the argument $x \in \mathbb{R}^d$ is suppressed in the integrand.

Proof. Define

$$F := \int_{\mathbb{R}^d} (D^\alpha u(\varepsilon) + D^\alpha (I^\xi u(\varepsilon)))^p - (D^\alpha u(\varepsilon))^p \, dx$$

where we use the operator $T$ defined in (4.24). As in the proof of Lemma 4.5 in [6] we define the operator

$$M^p_y = \Pi^p_{\beta=1} T^\xi_{y_\beta} - I$$

where $I$ is the identity operator. Observe that using Fubini’s theorem and the notation $D^\alpha_y = \Pi^p_{\beta=1} T^\xi_{y_\beta}$, $dy = dy_1 \cdots dy_p$, $y = (y_1, \ldots, y_p) \in \mathbb{R}^p$,

$$F = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (D^\alpha y_i \Pi^p_{j=1} (\xi_j \Pi^p_{k=1} u(y_k) - \Pi^p_{j=1} \xi_j \Pi^p_{k=1} u(y_k)) \, dy \, dx$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( M^p_y D^\alpha y_i \Pi^p_{j=1} \xi_j \Pi^p_{k=1} u(y_k) \right) \, dy \, dx = \int_{\mathbb{R}^d} \left( M^p_y D^\alpha y_i \rho_\varepsilon(y) \right) \Pi^p_{k=1} u(y_k) \, dy.$$ 

Next, note that by Taylor’s formula with $\xi(y) = (\xi(y_1), \ldots, \xi(y_p)) \in \mathbb{R}^p$,

$$M^p_y D^\alpha y_i \rho_\varepsilon(y) = \sum_{k=1}^p \int_0^1 (\partial_{y_k} D^\alpha y_i \rho_\varepsilon(y + \theta \xi(y))) \, d\theta \xi_i(y_k).$$

Thus, by a change of variables, Fubini’s theorem and the functions defined in (4.32),

$$F = \sum_{k=1}^p \int_0^1 \partial_{y_k} D^\alpha y_i \rho_\varepsilon(y) \xi_i(y_k) \Pi^p_{j=1} u(j) \, dy \, d\theta,$$

which by integration by parts gives, with multi-indices $\beta \leq \alpha, \beta := \alpha - \beta$ and constants $c_{\beta}$,

$$F = \sum_{\beta \leq \alpha} c_{\beta} \sum_{k=1}^p \int_0^1 \partial_{y_k} \rho_\varepsilon(y) \xi_i(y_k) \Pi^p_{j=1} u_j(y_k) \Pi^p_{j \neq k} \hat{u}_\alpha(y_j) \, dy \, d\theta,$$

$$= \sum_{\beta \leq \alpha} c_{\beta} \sum_{k=1}^p \int_0^1 f^\beta_k(\theta) \, d\theta,$$

where for $k = 1, \ldots, p, \theta \in [0, 1]$ and $\beta \leq \alpha$,

$$f^\beta_k(\theta) := \int_{\mathbb{R}^p} \partial_{y_k} \rho_\varepsilon(y) \xi_i(y_k) \Pi^p_{j=1} \hat{u}_\alpha(y_j) \, dy$$
and where \( \hat{u}_\gamma(y_k) = D^\gamma_k \hat{u}(y_k) \) for \( \gamma = \alpha, \beta \). We consider two cases. In the first case, let \( \beta < \alpha \) and hence \( |\beta| \geq 1 \). Then by integration by parts, for all \( k = 1, \ldots, p \) and a constant \( N = N(d, p, m, \lambda) \),

\[
f_k^\beta(\vartheta) = -\int_{\mathbb{R}^{dp}} \rho_\varepsilon(y)((\partial_{y_k}^\beta \hat{u}_\beta(y_k)) \hat{u}_\beta(y_k) + \hat{\xi}_k(y_k) (\partial_{y_k}^\beta \hat{u}_\beta(y_k))) \Pi_{j \neq k} \hat{u}_\alpha(y_j) dy d\vartheta \leq N \hat{\xi}(\vartheta) |u|^p_{W^p},
\]

where we used (4.35) and (4.36). In the second case \( \bar{\beta} = \alpha \) so that \( \beta = 0 \) and we have

\[
\sum_{k=1}^p f_k^0 = \sum_{k=1}^p \int_{\mathbb{R}^{dp}} \hat{\xi}_k(y) \Pi_{j=1}^p \hat{u}_\alpha(y_j) dy,
\]

as well as by using (4.4) and the symmetry in \( s \) and \( k \),

\[
\sum_{k=1}^p f_k^0 = -\sum_{k=1}^p \sum_{s \neq k} \int_{\mathbb{R}^{dp}} \hat{\xi}_k \rho_\varepsilon(y) (\hat{\xi}_s(y) - \hat{\xi}_s(y_s)) \Pi_{j=1}^p \hat{u}_\alpha(y_j) dy.
\]

Therefore also, with a constant \( N = N(d, p, m, \lambda, K_\xi) \),

\[
|\beta - (p - 1) \sum_{k=1}^p f_k^0 + \sum_{k=1}^p f_k^0| \leq N \hat{\xi}(\vartheta) |u|^p_{W^p},
\]

where we used (4.35) together with (4.5), as well as (4.36). This proves the lemma.

\[\square\]

5. Solvability of the Filtering Equations in Sobolev Spaces

The following two lemmas are essentially Lemma 5.2 in [6], where instead of \( D^\alpha_k \xi \) the kernel \( k_\varepsilon \) is considered. However, keeping this difference in mind, the arguments in the proofs of Lemma 5.2 in [6] can easily be adapted. Hence we only provide an outline and refer the reader to the preceding article [6] for full details.

**Lemma 5.1.** Let the Assumption 2.1 hold. Let \( u \) be an \( L_p \)-solution of (3.3), \( p \geq 2 \), such that (3.2) holds for each \( \varphi \in C_0^\infty(\mathbb{R}^d) \) almost surely for all \( t \in [0, T] \) and assume moreover that \( \text{ess sup}_{t \in [0, T]} |u_t|_{L_1} < \infty \). If \( K_1 \neq 0 \) in Assumption 2.1 (ii), then assume additionally

\[
\text{ess sup}_{t \in [0, T]} \int_{\mathbb{R}^d} |y|^2 |u_t(y)| dy < \infty, \quad \text{almost surely.} \tag{5.1}
\]

Then for each \( \varepsilon > 0 \) and integer \( m \geq 0 \), for any multi-index \( \alpha = (\alpha_1, \ldots, \alpha_d) \), \( |\alpha| \leq m \), for all \( x \in \mathbb{R}^d \) almost surely

\[
D^\alpha u^{(c)}_t(x) = D^\alpha u^{(c)}_0(x) + \int_0^t D^\alpha (L^*_{s} u^{(c)}_s)(x) ds + \int_0^t D^\alpha (M^*_{s} u^{(c)}_s)(x) dV^k_s + \int_0^t \int_0^{s_0} D^\alpha (J^{\nu_{s}^k}_{s} u^{(c)}_s)(x) d\nu_k + \int_0^t \int_0^{s_1} D^\alpha (J^{\nu_{s}^k}_{s} u^{(c)}_s)(x) d\nu_1 ds
\]

\[
+ \int_0^t \int_0^{s_3} D^\alpha (J^{\nu_{s}^k}_{s} u^{(c)}_s)(x) d\nu_3 ds + \int_0^t \int_0^{s_3} D^\alpha (I^{\nu_{s}^k}_{s} u^{(c)}_s)(x) dN_1 ds, \tag{5.2}
\]

where

\[\square\]
for all \( t \in [0, T] \).

**Proof.** The case of \( \alpha = 0 \) is Lemma 5.4 in [6]. The case of \( \alpha \neq 0 \) such that \( 0 < |\alpha| \leq m \) works exactly in the same way. We first define for a \( \psi \in C^2_0(\mathbb{R}) \) such that \( \psi(0) = 1, \psi(r) = 0 \) for \( |r| \geq 2 \), for \( n \geq 1 \), \( \psi_n(x) := \psi(|x|/n) \in C^2_0(\mathbb{R}^d) \). Setting \( \varphi_\varepsilon(y) := k_{\varepsilon, \alpha}(x - y) \psi_n(y) \in C^2_0(\mathbb{R}^d) \) in (3.3), where \( k_{\varepsilon, \alpha}(x - y) = D^2_\gamma k_{\varepsilon}(x - y), \) yields that for each \( x \in \mathbb{R}^d \) almost surely

\[
(u_t, k_{\varepsilon, \alpha}(x - \cdot)\psi_n) = (u_0, k_{\varepsilon, \alpha}(x - \cdot)\psi_n) + \int_0^t (u_s, \mathcal{L}_s(k_{\varepsilon, \alpha}(x - \cdot)\psi_n)) \, ds
\]

\[
+ \int_0^t (u_s, \mathcal{M}_s^k(k_{\varepsilon, \alpha}(x - \cdot)\psi_n)) \, dV^k_s + \int_0^t \int_{\mathbb{R}^d} (u_s, J^q_s(k_{\varepsilon, \alpha}(x - \cdot)\psi_n)) \, \nu_0(dq) \, ds
\]

\[
+ \int_0^t \int_{\mathbb{R}^d} (u_s, J^q_s(k_{\varepsilon, \alpha}(x - \cdot)\psi_n)) \, \nu_1(dq) \, ds + \int_0^t (u_{s-}, I^q_s(k_{\varepsilon, \alpha}(x - \cdot)\psi_n)) \, \tilde{N}_1(dq, ds)
\]

for all \( t \in [0, T] \). Then we notice that

\[
|k_{\varepsilon, \alpha}(x - y)| \leq \sum_{|\gamma| \leq m+2} |D^\gamma k_{\varepsilon}(x - y)| \leq N k_{2\varepsilon}(x - y),
\]

as well as that by Assumption for all \( x, y \in \mathbb{R}^d, s \in [0, T], \tilde{z}_i \in \tilde{z}_i, i = 0, 1 \) and \( n \geq 0 \) we have

\[
\sup_{x \in \mathbb{R}^d} |D^k \psi_n| = n^{-k} \sup_{\mathbb{R}^d} |D^k \psi| < \infty, \text{ for } k \in \mathbb{N}_.,
\]

\[
|\mathcal{L}_s(k_{\varepsilon, \alpha}(x - y)\psi_n(y))| + \sum_k |\mathcal{M}_s^k(k_{\varepsilon, \alpha}(x - y)\psi_n(y))|^2 \leq N(K^2_\varepsilon + K_1^2|y|^2 + K_1^2|Y_s|^2),
\]

\[
|J^q_s(k_{\varepsilon, \alpha}(x - y)\psi_n(y))| \leq \sup_{v \in \mathbb{R}^d} |D^2_v(k_{\varepsilon, \alpha}(x - v)\psi_n(v))||\eta_s(y, \tilde{z}_0)|^2 \leq N|\eta_s(y, \tilde{z}_0)|^2
\]

\[
\leq N\eta^2(\tilde{z}_0)(K^2_\varepsilon + K_1^2|y|^2 + K_1^2|Y_s|^2),
\]

and

\[
|J^q_s(k_{\varepsilon, \alpha}(x - y)\psi_n(y))| + |I^q_s(k_{\varepsilon, \alpha}(x - y)\psi_n(y))|^2
\]

\[
\leq \sup_{v \in \mathbb{R}^d} |D^2_v(k_{\varepsilon, \alpha}(x - v)\psi_n(v))||\xi_s(y, \tilde{z}_1)|^2 + \sup_{v \in \mathbb{R}^d} |D^2_v(k_{\varepsilon, \alpha}(x - v)\psi_n(v))|^2|\xi_s(y, \tilde{z}_1)|^2
\]

\[
\leq N|\xi_s(y, \tilde{z}_1)|^2 \leq N\xi^2(\tilde{z}_1)(K^2_\varepsilon + K_1^2|y|^2 + K_1^2|Y_s|^2),
\]

for a constant \( N = N(\varepsilon, m, d, K_0, K_1, K_\varepsilon, K_\eta) \). Using

\[
\text{ess sup}_{t \in [0, T]} \int_{\mathbb{R}^d} (1 + |y|^2 + |Y_t|^2)|u_t(y)| \, dy < \infty, \text{ (a.s.)}
\]

together with the estimates above, we can apply Lebesgue’s theorem on Dominated Convergence to get that for all \( x \in \mathbb{R}^d \),

\[
(u_t, k_{\varepsilon, \alpha}(x - \cdot)\psi_n) \to (u_t, k_{\varepsilon, \alpha}(x - \cdot)), \quad (u_0, k_{\varepsilon, \alpha}(x - \cdot)\psi_n) \to (u_0, k_{\varepsilon, \alpha}(x - \cdot)) \]

\[
\int_0^t (u_s, \mathcal{A}_s(k_{\varepsilon, \alpha}(x - \cdot)\psi_n)) \, ds \to \int_0^t (u_s, \mathcal{A}_s k_{\varepsilon, \alpha}(x - \cdot)) \, ds
\]
as \( n \to \infty \), almost surely uniformly in time, as well as that

\[
\lim_{n \to \infty} \int_0^t (u_s, \mathcal{M}_s^k(k_{\varepsilon, \alpha}(x - \cdot)\psi_n(\cdot))) \, dV^k_s = \int_0^t (u_s, \mathcal{M}_s^k k_{\varepsilon, \alpha}(x - \cdot)) \, dV^k_s,
\]
\[
\lim_{n \to \infty} \int_0^t \int_{\mathcal{F}} (u_{s-}, I^\xi_s(k_{\epsilon, \alpha}(x - \cdot) \psi_n)) \tilde{N}_1(d\zeta_3, ds) = \int_0^t \int_{\mathcal{F}} (u_{s-}, I^\xi_s k_{\epsilon, \alpha}(x - \cdot)) \tilde{N}_1(d\zeta_3, ds)
\]
in probability, uniformly in time. Thus, letting \( n \to \infty \) in (5.3) it remains to note that since \( \mathcal{A} \) acts in the \( y \) variable,

\[
(u_s, \mathcal{A} k_{\epsilon, \alpha}(x - \cdot)) = \int_{\mathbb{R}^d} u_s(y) A_s D^\alpha_y k_{\epsilon, \alpha}(x - y) dy
\]

\[
= D^\alpha_x \int_{\mathbb{R}^d} u_s(y) A_s k_{\epsilon, \alpha}(x - y) dy = D^\alpha(A^*_s u_s)(\epsilon)(x)
\]

does hold for all \((\omega, s, x) \in \Omega \times [0, T] \times \mathbb{R}^d\) if \( \mathcal{A} = \tilde{\mathcal{L}}, \mathcal{M}_k \) or the identity, as well as for all \((\omega, s, x, \tilde{\zeta}_i) \in \Omega \times [0, T] \times \mathbb{R}^d \times \tilde{\zeta}_i\) if \( \mathcal{A} = J^n \) or \( \mathcal{A} = \mathcal{I}^\xi, J^\xi \) with \( i = 0, 1 \) respectively.

\[\Box\]

**Lemma 5.2.** Let the Assumptions 2.1 and 2.2 hold. Let \( u \) be an \( L_p \)-solution of (3.3), \( p \geq 2 \) and assume moreover that \( \text{ess sup}_{t \in [0, T]} |u_t|_{L_p} < \infty \). If \( K_1 \neq 0 \) in Assumption 2.1 (ii), then assume additionally (5.1). Then for each \( \varepsilon > 0 \) and integer \( m \geq 0 \), for any multi-index \( \alpha = (\alpha_1, \ldots, \alpha_d) \), \( |\alpha| \leq m \), almost surely

\[
|D^\alpha u_t^{(\varepsilon)}|_{L_p}^p = |D^\alpha u_0^{(\varepsilon)}|_{L_p}^p + p \int_0^t \left( |D^\alpha u_s^{(\varepsilon)}|^{p-2} D^\alpha u_s^{(\varepsilon)}, D^\alpha (\tilde{\mathcal{L}}^*_s u_s^{(\varepsilon)}) \right) ds
\]

\[
+ p \int_0^t \left( |D^\alpha u_s^{(\varepsilon)}|^{p-2} D^\alpha u_s^{(\varepsilon)}, D^\alpha (\mathcal{M}_k^* u_s^{(\varepsilon)}) \right) ds
\]

\[
+ \frac{p(p-1)}{2} \sum_{k=1}^m \int_0^t \left( |D^\alpha u_s^{(\varepsilon)}|^{p-2}, |D^\alpha (\mathcal{M}_k^* u_s^{(\varepsilon)})|^2 \right) ds
\]

\[
+ p \int_0^t \int_{\mathcal{F}} (|D^\alpha u_s^{(\varepsilon)}|^{p-2} D^\alpha u_s^{(\varepsilon)}, D^\alpha (J^* \mu_s^{(\varepsilon)}) \nu_0(d\zeta_3)) ds
\]

\[
+ p \int_0^t \int_{\mathcal{F}} (|D^\alpha u_s^{(\varepsilon)}|^{p-2} D^\alpha u_s^{(\varepsilon)}, D^\alpha (J^* \mu_s^{(\varepsilon)}) \nu_1(d\zeta_3)) ds
\]

\[
+ p \int_0^t \int_{\mathcal{F}} (|D^\alpha u_s^{(\varepsilon)}|^{p-2} D^\alpha u_s^{(\varepsilon)}, D^\alpha (I^* \mu_s^{(\varepsilon)}) \tilde{N}_1(d\zeta_3)) ds
\]

\[
+ \int_0^t \int_{\mathbb{R}^d} \left\{ |D^\alpha u_s^{(\varepsilon)} + D^\alpha (I^* \mu_s^{(\varepsilon)})|^{p-2} |D^\alpha u_s^{(\varepsilon)}| \right\} dx N_1(d\zeta_3, ds)
\]

holds for all \( t \in [0, T] \).

**Proof.** We apply the Itô formula, Theorem 5.1 in [6], to \( |D^\alpha u_t^{(\varepsilon)}|_{L_p}^p \). In order to do that, we need to verify that almost surely for each \( x \in \mathbb{R}^d \) and \( \alpha \), such that \( 0 \leq |\alpha| \leq m \),

\[
\int_0^T |D^\alpha (\tilde{\mathcal{L}}^*_s u_s^{(\varepsilon)})(x)| ds < \infty, \quad \int_0^T \sum_{k} |D^\alpha (\mathcal{M}_k^* u_s^{(\varepsilon)})(x)|^2 ds < \infty,
\]
\[ \int_0^T \int_{\mathbb{R}^d} |D^\alpha(J^{\xi}_s u_s)(x)|^{\nu_0(d_3)} ds < \infty, \quad \int_0^T \int_{\mathbb{R}^d} |D^\alpha(I^{\xi}_s u_s)(x)|^{\nu_1(d_3)} ds < \infty, \]

that for every finite set \( \Gamma \in \mathcal{B}(\mathbb{R}^d) \), almost surely

\[ \int_\Gamma \int_0^T |D^\alpha(\bar{L}^*_s u_s)(x)|^{\nu_0(d_3)} ds < \infty, \quad \int_\Gamma \left( \int_0^T \left| \sum_k |D^\alpha(M^k_s u_s)(x)|^2 \right|^{\nu_1(d_3)} ds \right)^{1/2} dx < \infty, \]

as well as that almost surely

\[ A := \int_0^T \int_{\mathbb{R}^d} |D^\alpha(\bar{L}^*_s u_s)(x)|^p ds dx < \infty, \]

\[ A_\eta := \int_0^T \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} D^\alpha(J^\eta_s u_s)(x) \nu_0(d_3) \right|^p dx ds < \infty, \]

\[ A_\zeta := \int_0^T \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} D^\alpha(J^\zeta_s u_s)(x) \nu_1(d_3) \right|^p dx ds < \infty, \]

\[ B := \int_0^T \int_{\mathbb{R}^d} \left( \sum_k |D^\alpha(M^k_s u_s)(x)|^2 \right)^{p/2} ds dx < \infty, \]

\[ G := \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |D^\alpha(I^\xi_s u_s)(x, \bar{z})|^p |\nu_1(d_3)| dx ds < \infty, \]

\[ H := \int_0^T \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |D^\alpha(I^\xi_s u_s)(x, \bar{z})|^2 |\nu_1(d_3)| \right)^{p/2} dx ds < \infty. \]

For \( \alpha = 0 \) the claim is Lemma 5.4 in [6] and the estimates can be found in the proof of the preceding Lemma 5.2 therein. To prove the case where \( 0 < |\alpha| \leq m \), we note that for \( A = \bar{L}, \mathcal{M}, I^\xi, J^\xi, J^\eta \) we have

\[ D^\alpha(A^i u)^{(\xi)} = \int_{\mathbb{R}^d} D^\alpha_x (A_y k_\zeta(x - y)) u(y) dy = \int_{\mathbb{R}^d} (A_y k_{\zeta,\alpha}(x - y)) u(y) dy, \]

for \( k_{\zeta,\alpha}(x) = D^\alpha k_\zeta(x) \). Hence, a word for word repetition of the proof of Lemma 5.2 & 5.4 in [6], where we replace \( k_\zeta \) by \( k_{\zeta,\alpha} \) and recall (5.4), yields the desired result.

**Lemma 5.3.** Let Assumptions 2.1, 2.2, 2.5 and 2.4 hold with an integer \( m \geq 0 \) and let \( p \geq 2 \) be even. Let \( u \) be an \( W^m_p \)-solution to (3.3), such that \( \mathbb{E}|u_0|_{W^m_p} < \infty \) and almost surely \( \text{ess} \sup_{t \in [0, T]} |u_t|_{L_1} < \infty \). Then

\[ \mathbb{E} \sup_{t \in [0, T]} |u_t|_{W^m_p} \leq N \mathbb{E}|u_0|_{W^m_p} \]

for a constant \( N = N(m, d, p, K, \bar{K}, \bar{L}, T, \lambda, |\check{\xi}|_{L^2(3_1)}, |\check{\eta}|_{L^2(3_1)}) \).

Proof. For \( m = 0 \) the claim is Lemma 5.4 in [6]. We proceed similarly here. For the present case, fix a multi-index \( \alpha \) such that \( 0 \neq |\alpha| \leq m \), and define

\[
Q_p(\alpha, b, \sigma, \rho, \beta, u, k_z) = p((D^\alpha u^{(c)})^{p-1}, D^\alpha (\tilde{L} \ast u)^{(e)}) \\
+ \frac{p(p-1)}{2} \sum_k ((D^\alpha u^{(e)})^{p-2}, (D^\alpha (\mathcal{M} \ast k \ast u)^{(e)})^2),
\]

(5.7)

\[
Q_p^{(0)}(\alpha, \eta(\mathcal{S}_0), u, k_z) = p((D^\alpha u^{(c)})^{p-1}, D^\alpha (\mathcal{J} \eta(\mathcal{S}_0) \ast u)^{(e)}),
\]

(5.8)

\[
Q_p^{(1)}(\alpha, \xi(\mathcal{S}_1), u, k_z) = p((D^\alpha u^{(c)})^{p-1}, D^\alpha (\mathcal{J} \xi(\mathcal{S}_1) \ast u)^{(e)}),
\]

(5.9)

\[
R_p(\alpha, \xi(\mathcal{S}_1), u, k_z) = |D^\alpha u^{(c)} + D^\alpha (\mathcal{J} \xi(\mathcal{S}_1) \ast u)^{(e)}|_{L_p}^p - |D^\alpha u^{(c)}|_{L_p}^p - p((D^\alpha u^{(c)})^{p-1}, D^\alpha (\mathcal{J} \xi(\mathcal{S}_1) \ast u)^{(e)}),
\]

for \( u \in W^m_p \), \( \beta \in \mathbb{R}^d \), functions \( b, \sigma \) and \( \rho \) on \( \mathbb{R}^d \), with values in \( \mathbb{R}^d \), \( \mathbb{R}^{d \times d_1} \) and \( \mathbb{R}^{d \times d'} \), respectively, and \( \mathbb{R}^d \)-valued functions \( \eta(\mathcal{S}_0) \) and \( \xi(\mathcal{S}_1) \) for each \( \mathcal{S}_i \in \mathcal{S}_i \), \( i = 0, 1 \), where \( \beta_t = B_t(X_t) \),

\[\tilde{L} = \frac{1}{2}(\sigma^i \sigma^j + \rho^i \rho^j)D_{ij} + \beta^i \rho^i D_i + \beta^i B^i, \quad \mathcal{M}^k = \rho^i \rho^j D^k, \quad k = 1, 2, ..., d'.\]

By Lemma 5.2 almost surely

\[
d|D^\alpha u^{(c)}|^p_{L_p} = Q_p(\alpha, b_t, \sigma_t, \rho_t, \beta_t, u_t, k_z) dt + \int_{\mathcal{S}_0} Q_p^{(0)}(\alpha, \eta_t(\mathcal{S}_0), u_t, k_z) \nu_0(d\bar{d}) dt
\]

\[
+ \int_{\mathcal{S}_1} Q_p^{(1)}(\alpha, \xi_t(\mathcal{S}_1), u_t, k_z) \nu_1(d\bar{d}) dt + \int_{\mathcal{S}_1} R_p(\alpha, \xi_t(\mathcal{S}_1), u_t, k_z) N_1(d\bar{d}, dt) + d\zeta_1(\alpha, t) + d\zeta_2(\alpha, t),
\]

(5.10)

for all \( t \in [0, T] \) and

\[
\zeta_1(\alpha, t) = p \int_0^t ((D^\alpha u^{(c)})^{p-1}, D^\alpha (\mathcal{M} \ast k \ast u)^{(e)}) dV^k,
\]

(5.11)

\[
\zeta_2(\alpha, t) = p \int_0^t \int_{\mathcal{S}_1} ((D^\alpha u^{(c)})^{p-1}, D^\alpha (\mathcal{J} \ast u)^{(e)}) \tilde{N}_1(d\bar{d}, ds) \quad t \in [0, T]
\]

are local martingales under \( P \). We write

\[
\int_{\mathcal{S}_1} R_p(\alpha, \xi_t(\mathcal{S}_1), u_t, k_z) N_1(d\bar{d}, dt) = \int_{\mathcal{S}_1} R_p(\alpha, \xi_t(\mathcal{S}_1), u_{t-}, k_z) \nu_1(d\bar{d}) dt + d\zeta_3(\alpha, t)
\]

(5.12)

with

\[
\zeta_3(\alpha, t) = \int_0^t \int_{\mathcal{S}_1} R_p(\alpha, \xi_s(\mathcal{S}_1), u_{s-}, k_z) N_1(d\bar{d}, ds) - \int_0^t \int_{\mathcal{S}_1} R_p(\alpha, \xi_s(\mathcal{S}_1), u_s, k_z) \nu_1(d\bar{d}) ds,
\]

which we can justify if we show

\[
A := \int_0^T \int_{\mathcal{S}_1} |R_p(\alpha, \xi_s(\mathcal{S}_1), u_s, k_z)| \nu_1(d\bar{d}) ds < \infty \quad \text{(a.s.)}
\]

(5.13)

To this end observe that by Taylor's formula

\[
0 \leq R_p(\alpha, \xi_t(\mathcal{S}_1), u_t, k_z)) \leq N \int_{\mathbb{R}^d} (D^\alpha u^{(c)})^{p-2}(D^\alpha (\mathcal{J} \ast u)^{(e)})^2 + (D^\alpha (\mathcal{J} \ast u)^{(e)})^p dx
\]

(5.14)
with a constant \( N = N(d, p) \). Hence

\[
\int_{\mathbb{R}^d} R_p(\alpha, \xi_1(\t); u_t, k_\xi) \nu_1(d\delta) \leq N \int_{\mathbb{R}^d} (D^\alpha u_t(\cdot) |_{L^p})^p + A_1(t) + A_2(t) \]

with

\[
A_1(t) = \int_{\mathbb{R}^d} |(D^\alpha k_\xi(x - y)) u_t(x) dy|^p \,
dx,
\]

\[
A_2(t) = \int_{\mathbb{R}^d} |(D^\alpha k_\xi(x - y)) u_t(x) dy|^2 \,
dx,
\]

\[
A_1(t) = \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |(D^\alpha k_\xi(x - y)) u_t(x) dy|^p \,
dx \right) \nu_1(d\delta)^{p/2} \,
dx.
\]

and constants \( N \) and \( N' \) depending only on \( d \) and \( p \). By Minkowski's inequality and using again that \( D^\alpha \xi k_\xi(x - y) = I^\xi D^\alpha k_\xi(x - y) \),

\[
|D^\alpha u_t(\cdot) |_{L^p}^p = \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |(D^\alpha k_\xi(x - y)) u_t(x) dy|^p \,
dx \right) \nu_1(d\delta)^{p/2} \,
dx.
\]

where \( D^{\alpha+1} = D^\alpha \xi \) and similarly, using Assumption 2.2,

\[
A_2(t) = \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |(D^\alpha k_\xi(x - y)) u_t(x) dy|^p \,
dx \right) \nu_1(d\delta) \right) \,
dx.
\]

By (5.14)–(5.18) we have a constant \( N = N(p, d, \varepsilon, |\xi|_{L^2(\delta)}, K_\xi) \) such that

\[
A \leq N \int_0^T |D^\alpha u_t(\cdot) |_{L^p}^p + N \int_0^T \left( \int_{\mathbb{R}^d} (K_0 + K_1 |y|) u_t(x) dy \right) \,
dt < \infty \ (a.s.)
\]

Next we claim that, with the operator \( T^\xi \) defined in (4.24), we have

\[
\zeta_2(\alpha, t) + \zeta_3(\alpha, t) = \int_0^t \left( \int_{\mathbb{R}^d} (K_0 + K_1 |y|) u_t(x) dy \right) \,
dt =: \zeta(\alpha, t) \quad \text{for } t \in [0, T].
\]

(5.19)
For that purpose not first that \( D^\alpha u^{(c)} + D^\alpha (I^\xi (\xi_) u^{(c)}) = D^\alpha (T^\xi u^{(c)}) \). To see that the stochastic integral \( \zeta(\alpha, t) \) is well-defined as an Itô integral note that by Lemma 4.6,

\[
\int_0^T \int_{\mathcal{F}_1} |D^\alpha (T^\xi u_s^{(c)})^p_{L^p} - |D^\alpha u_s^{(c)}|_{L^p}|^2 \nu_1(d_3) ds \leq N |\hat{\xi}|^2_{L^2(\mathcal{F}_1)} \int_0^T |u_s^{2p}|_{L^p} ds < \infty \text{ (a.s.)}
\]

(5.20)

with a constant \( N = N(d, p, m, \lambda, K_\xi) \). Since \( \mathcal{F}_1 \) is \( \sigma \)-finite, there is an increasing sequence \( (\mathcal{F}_1)_{n=1}^\infty \), \( \mathcal{F}_1 \in \mathcal{F}_1 \), such that \( \nu_1(\mathcal{F}_1) \leq \infty \) for every \( n \) and \( \omega_{n=1}^\infty \mathcal{F}_1 = \mathcal{F}_1 \). Then it is easy to see that

\[
\tilde{\zeta}_n(\alpha, t) = p \int_0^t \int_{\mathcal{F}_1} \mathbf{1}_{\mathcal{F}_1}(\hat{\zeta}_n(\alpha, t) - \tilde{\zeta}_n(\alpha, t)) N(d_3, ds),
\]

\[
\hat{\zeta}_n(\alpha, t) = p \int_0^t \int_{\mathcal{F}_1} \mathbf{1}_{\mathcal{F}_1}(\hat{\zeta}_n(\alpha, t) - \tilde{\zeta}_n(\alpha, t)) \nu_1(d_3) ds,
\]

\[
\tilde{\zeta}_n(\alpha, t) = \int_0^t \int_{\mathcal{F}_1} \mathbf{1}_{\mathcal{F}_1}(\tau_1(\alpha, t), u_s, k_\xi) \nu_1(d_3, ds),
\]

\[
\hat{\zeta}_n(\alpha, t) = \int_0^t \int_{\mathcal{F}_1} \mathbf{1}_{\mathcal{F}_1}(\tau_1(\alpha, t), u_s, k_\xi) \nu_1(d_3) ds
\]

are well-defined, and

\[
\zeta_2(\alpha, t) = \lim_{n \to \infty} (\tilde{\zeta}_n(\alpha, t) - \hat{\zeta}_n(\alpha, t)), \quad \zeta_3(\alpha, t) = \lim_{n \to \infty} \tilde{\zeta}_n(\alpha, t) - \lim_{n \to \infty} \hat{\zeta}_n(\alpha, t),
\]

where the limits are understood in probability. Hence

\[
\zeta_2(\alpha, t) + \zeta_3(\alpha, t) = \lim_{n \to \infty} \left( \tilde{\zeta}_n(\alpha, t) + \tilde{\zeta}_n(\alpha, t) - \left( \hat{\zeta}_n(\alpha, t) + \hat{\zeta}_n(\alpha, t) \right) \right)
\]

\[
= \lim_{n \to \infty} \left( \int_0^t \int_{\mathcal{F}_1} \mathbf{1}_{\mathcal{F}_1}(\hat{\zeta}_n(\alpha, t) + \hat{\zeta}_n(\alpha, t) - \left( \hat{\zeta}_n(\alpha, t) + \hat{\zeta}_n(\alpha, t) \right)\right)
\]

\[
= \lim_{n \to \infty} \left( \int_0^t \int_{\mathcal{F}_1} \mathbf{1}_{\mathcal{F}_1}(\hat{\zeta}_n(\alpha, t) + \hat{\zeta}_n(\alpha, t) - \left( \hat{\zeta}_n(\alpha, t) + \hat{\zeta}_n(\alpha, t) \right)\right)
\]

which completes the proof of (5.19). Consequently, from (5.10)-(5.12) we have

\[
d|D^\alpha u^{(c)}_t|_{L^p} = Q_p(\alpha, b_1, \sigma_t, \rho_t, \beta_t, u_t, k_\xi) dt + \int_{\mathcal{F}_1} Q_p^{(i)}(\alpha, \eta_t(\alpha, t), u_t, k_\xi) \nu_0(d_3) dt
\]

\[
+ \int_{\mathcal{F}_1} Q_p^{(i)}(\alpha, \xi_t(\alpha, t), u_t, k_\xi) + R_p(\alpha, \xi_t(\alpha, t), u_t, k_\xi) \nu_1(d_3) dt + d\zeta_1(\alpha, t) + d\zeta_2(\alpha, t).
\]

(5.21)

By Lemma 4.1, Corollary 4.2 and Lemma 4.3 we have

\[
Q_p(\alpha, b_1, \sigma_t, \rho_t, \beta_t, u_t, k_\xi) \leq N (L^2 + K^2) |u_s|_{W^{2m}}^p
\]

(5.22)

with a constant \( N = N(d, p, m) \), and by Lemma 4.4 and Corollary 4.5, using that \( \tilde{\xi} \leq K_\xi \) and \( \tilde{\eta} \leq K_\eta \),

\[
Q_p^{(i)}(\alpha, \eta_t(\alpha, t), u_t, k_\xi) \leq N \tilde{\eta}^2(\tilde{z}) |u_s|_{W^{2m}}^p, \quad (Q_p^{(i)} + R_p)(\alpha, \xi_t(\alpha, t), u_t, k_\xi) \leq N \tilde{\xi}^2(\tilde{z}) |u_s|_{W^{2m}}^p
\]

(5.23)

with a constant \( N = N(K_\xi, K_\eta, d, p, \lambda, m) \). Thus from (5.21) we obtain that or all \( \alpha \) with \( |\alpha| \leq m \) almost surely

\[
|D^\alpha u^{(c)}_t|_{L^p} \leq |u_0|_{W^{2m}}^p + N \int_0^t |u_s|_{W^{2m}}^p ds + m(t) \quad \text{for all } t \in [0, T]
\]
with a constant $N=N(m, p, d, K, K, L, \lambda, \xi_{[L_2(\xi_1), [\eta_1[L_2(\eta_1)])}$ and the local martingale $m^\varepsilon(\alpha, t) = \zeta_1(\alpha, t) + \zeta(\alpha, t)$. Summing over all $|\alpha| \leq m$ gives

$$|u_t^\varepsilon|_{W_p}^p \leq |u_0^\varepsilon|_{W_p}^p + N \int_0^t |u_s|_{W_p}^p ds + m_t^\varepsilon \quad \text{for all } t \in [0, T]$$

(5.24)

with another constant $N=N(m, p, d, K, K, L, \lambda, \xi_{[L_2(\xi_1), [\eta_1[L_2(\eta_1)])}$ and a local martingale, denoted again by $m^\varepsilon$. For integers $n \geq 1$ set $\tau_n = \bar{\tau}_n \land \tilde{\tau}_n$, where $(\tilde{\tau}_n)_{\varepsilon \rightarrow 0}$ is a localising sequence of stopping times for $m^\varepsilon$ and

$$\tilde{\tau}_n = \inf \left\{ t \in [0, T] : \int_0^t |u_s|_{W_p}^p ds \geq n \right\}.$$ 

Then from (5.24), using also $|D^\alpha u^\varepsilon|_{L_p} = |(D^\alpha u)^\varepsilon|_{L_p} \leq |D^\alpha u|_{L_p}$ for multi-indices $\alpha \leq m$ and $\varepsilon > 0$ we get

$$\mathbb{E}|u_t^\varepsilon|_{W_p}^p \leq \mathbb{E}|u_0^\varepsilon|_{W_p}^p + N \int_0^t \mathbb{E}|u_s|_{W_p}^p ds < \infty \quad \text{for } t \in [0, T] \text{ and integers } n \geq 1.$$ 

Applying Fatou's lemma for the limit $\varepsilon \rightarrow 0$ followed by Grönwall's lemma gives

$$\mathbb{E}|u_t^\varepsilon|_{W_p}^p \leq \mathbb{E}|u_0^\varepsilon|_{W_p}^p \quad \text{for } t \in [0, T] \text{ and integers } n \geq 1$$

with a constant $N=N(m, p, d, K, K, L, \lambda, \xi_{[L_2(\xi_1), [\eta_1[L_2(\eta_1)])}$. Letting here $n \rightarrow \infty$, by Fatou's lemma we obtain

$$\sup_{t \in [0, T]} \mathbb{E}|u_t^\varepsilon|_{W_p}^p \leq N\mathbb{E}|u_0^\varepsilon|_{W_p}^p.$$ 

(5.25)

To prove (5.6) we define a localizing sequence of stopping times $(\tau_k^\varepsilon)_{k=1}^\infty$ for the local martingale $m^\varepsilon$, as well as

$$\tilde{\rho}_n = \inf \left\{ t \in [0, T] : \int_0^t |u_s|_{W_p}^{2p} ds \geq n \right\}, \quad \text{and} \quad \rho_{n,k}^\varepsilon = \tilde{\rho}_n \land \tilde{\rho}_k^\varepsilon.$$ 

Using the Davis inequality and Lemma 4.3 by standard calculations for every $n \geq 1$ we get for each $|\alpha| \leq m$ for the Doob-Meyer process of $\zeta_1$,

$$\mathbb{E}\sup_{t \in T} |\zeta_1(\alpha, t \land \rho_{n,k}^\varepsilon)| \leq 3\mathbb{E} \left( \sum_k \int_{\tilde{\rho}_n^\varepsilon}^{\tilde{\rho}_{n,k}^\varepsilon} \left( (D^\alpha u^\varepsilon)_{L_p}^{p-1}, D^\alpha (M^\varepsilon u^\varepsilon)_{L_p} \right)^2 ds \right)^{1/2}$$

(5.26)

$$\leq N\mathbb{E} \left( \int_{\tilde{\rho}_n^\varepsilon}^{\tilde{\rho}_{n,k}^\varepsilon} |u_s|_{W_p}^{2p} ds \right)^{1/2} < \infty,$$

and similarly, for each $|\alpha| \leq m$, the Doob-Meyer process of $\zeta(\alpha, \cdot)$ is

$$\langle \zeta(\alpha, \cdot) \rangle(t) = \int_0^t \int_{\tilde{\tau}_1} |D^\alpha (T^\varepsilon u^\varepsilon)_{L_p}^{(p)} - |D^\alpha u^\varepsilon|_{L_p}^{(p)} |^2 du_1 ds, \quad t \in [0, T].$$

Using the Davis inequality and Lemma 4.6,

$$\mathbb{E}\sup_{s \in T} |\zeta(\alpha, s \land \rho_{n,k}^\varepsilon)| \leq 3\mathbb{E} \langle \zeta(\alpha, \cdot) \rangle^{1/2} \langle T \land \rho_{n,k}^\varepsilon \rangle \leq N\mathbb{E} \left( \int_0^{\tilde{\rho}_n^\varepsilon} |u_s|_{W_p}^{2p} ds \right)^{1/2} < \infty,$$

(5.27)
with a constant $N = N(m, d, p, K, K, L, |\xi|_{L^2(\Omega)})$. Thus, due to (5.25) together with (5.26) and (5.27), we get from (5.24), with constant $N = N(m, p, d, T, K, K, L, |\xi|_{L^2(\Omega)}, |\eta|_{L^2(\Omega)})$,

$$
\mathbb{E} \sup_{t \in [0, T]} |u_{t, \rho_n}^{(\varepsilon)}|_{W^m_p}^p \leq \mathbb{N}E |u_0|_{W^m_p}^p + \sum_{|\alpha| \leq m} \mathbb{E} \sup_{t \leq T} |\zeta(\alpha, t \wedge \rho_n^\varepsilon)| + \sum_{|\alpha| \leq m} \mathbb{E} \sup_{t \leq T} |\zeta(\alpha, t \wedge \rho_n^\varepsilon)|
$$

$$
\leq \mathbb{N}E |u_0|_{W^m_p}^p + \mathbb{N}E \left( \int_0^T |u_{s, \rho_n}^{(\varepsilon)}|_{W^m_p} \, ds \right)^{1/2}.
$$

Letting here $k \to \infty$ and then $\varepsilon \to 0$, we obtain by Fatou's lemma with constants $N'$ and $N''$ only depending on $m, p, d, T, K, K, \xi, \eta, L, |\xi|_{L^2(\Omega)}$ and $|\eta|_{L^2(\Omega)}$,

$$
\mathbb{E} \sup_{t \in [0, T]} |u_{t, \rho_n}^{(\varepsilon)}|_{W^m_p} \leq \mathbb{N}E |u_0|_{W^m_p}^p + \mathbb{N}E \left( \sup_{t \in [0, T]} \int_0^T |u_{s, \rho_n}^{(\varepsilon)}|_{W^m_p} \, ds \right)^{1/2}
$$

$$
\leq \mathbb{N}E |u_0|_{W^m_p}^p + \mathbb{N}E \left( \sup_{t \in [0, T]} \int_0^T |u_{s, \rho_n}^{(\varepsilon)}|_{W^m_p} \, ds \right)^{1/2}
$$

$$
\leq \mathbb{N}E |u_0|_{W^m_p}^p + \frac{1}{2} \mathbb{E} \sup_{t \in [0, T]} |u_{t, \rho_n}^{(\varepsilon)}|_{W^m_p} + N' \int_0^T |u_{s, \rho_n}^{(\varepsilon)}|_{W^m_p} \, ds
$$

$$
\leq N'' \mathbb{E} |u_0|_{W^m_p}^p + \frac{1}{2} \mathbb{E} \sup_{t \in [0, T]} |u_{t, \rho_n}^{(\varepsilon)}|_{W^m_p},
$$

where we used Young's inequality. Thus also, we get for all $n$,

$$
\mathbb{E} \sup_{t \in [0, T]} |u_{t, \rho_n}^{(\varepsilon)}|_{W^m_p} \leq 2N'' \mathbb{E} |u_0|_{W^m_p}^p.
$$

Using Fatou's lemma we get the desired result. 

The following Lemma 5.5 is Lemma 6.4 in [6]. For integers $m \geq 0$ and real numbers $p \geq 1$ we define $W^m_p = W^m_p(\mathbb{R}^d)$ to be the space of $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}^d)$-measurable real valued random variables $\psi$ such that

$$
|\psi|^p_{W^m_p} := \mathbb{E} \sum_{k=0}^m \int_{\mathbb{R}^d} |D^k \psi(x)|^p \, dx < \infty.
$$

For $p, q \geq 1$ and integers $m \geq 0$ we denote by $W^m_{p,q}$ the space of $\mathcal{O} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable real valued functions $v = v_t(\omega, x)$ such that

$$
|v|^p_{W^m_{p,q}} := \mathbb{E} \left( \int_0^T |v_t|^q_{W^m_p} \, dt \right)^{p/q} < \infty.
$$

If $m = 0$ then we write $L_{p,q} := W^0_{p,q}$. Let $\mathbb{E}^m_0$ denote the space of those functions $\psi \in \cap_{p \geq 1} W^m_p$ such that

$$
\sum_{k=0}^m \sup_{\omega \in \Omega} \sup_{x \in \mathbb{R}^d} |D^k \psi(x)| < \infty \quad \text{and almost surely } \psi(x) = 0 \text{ for } |x| \geq R,
$$

for some constant $R$ depending on $\psi$. It is easy to see that $\mathbb{E}^m_0$ is a dense subspace of $W^m_p$ for every $p \in [1, \infty)$. For $\varepsilon > 0$ let in the following proposition $v^{(\varepsilon)}$ denote the convolution

$$
v^{(\varepsilon)}(x) = \int_{\mathbb{R}^d} \chi_\varepsilon(x-y)v(y) \, dy
$$
of a Borel function \( v \) on \( \mathbb{R}^d \), where \( \chi \) is a smooth, symmetric function of unit integral on \( \mathbb{R}^d \), such that \( \chi(x) = 0 \) for \( |x| \geq 1 \) and \( \chi(\cdot) := \varepsilon^{-d} \chi(\cdot/\varepsilon) \). Let

\[
\mathcal{M}_{t}^{\varepsilon k} = \rho_{t}^{(\varepsilon)ik} D_i + B_{t}^{(\varepsilon)k}, \quad k = 1, \ldots, d',
\]

\[
\mathcal{L}_{t}^{\varepsilon} = a_{t}^{\varepsilon,ij} D_{ij} + b_{t}^{(\varepsilon)j} D_i + \beta_{t}^{k} \mathcal{M}_{t}^{\varepsilon k}, \quad \beta_{t} = B(t, X_t, Y_t),
\]

\[
a_{t}^{\varepsilon,ij} := \frac{1}{2} \sum_{k} (\sigma_{t}^{(\varepsilon)(s)k} \sigma_{t}^{(\varepsilon)(s)j} + \rho_{t}^{(\varepsilon)ik} \rho_{t}^{(\varepsilon)jk}), \quad i, j = 1, 2, \ldots, d
\]

and let \( I^{\xi}, J^{\xi} \) and \( J^{\eta} \) be defined as \( I^{\xi}, J^{\xi} \) and \( J^{\eta} \), only with \( \xi^{(\varepsilon)} \) and \( \eta^{(\varepsilon)} \) instead of \( \xi \) and \( \eta \), respectively.

Consider for \( \varepsilon \in (0, 1) \) the equation

\[
du_{t}^{\varepsilon} = \mathcal{L}_{t}^{\varepsilon} u_{t}^{\varepsilon} \, dt + \mathcal{M}_{t}^{\varepsilon k} u_{t}^{\varepsilon} \, dV_{t}^{k} + \int_{\mathcal{E}_{0}} I^{\xi}_{t} u_{t}^{\varepsilon} \, \nu_{0}(d\xi) \, dt
\]

\[
+ \int_{\mathcal{E}_{1}} J^{\xi}_{t} u_{t}^{\varepsilon} \, \nu_{1}(d\xi) \, dt + \int_{\mathcal{E}_{2}} J^{\eta}_{t} u_{t}^{\varepsilon} \, \bar{N}_{1}(d\eta) \, dt, \quad \text{with } u_{0}^{\varepsilon} = \psi^{(\varepsilon)}.
\]

**Proposition 5.4.** Let Assumptions 2.1, 2.2 and 2.4 hold with \( K_1 = 0 \) and let \( p \geq 2 \) be even. Assume that the following “support condition” holds: There is some \( R > 0 \) such that

\[
(b_{t}(x), B_{t}(x), \sigma_{t}(x), \rho_{t}(x), \eta_{t}(x, \xi_{t}(x, \xi_{t})), \xi_{t}(x, \xi_{t})) = 0
\]

for \( \omega \in \Omega, t \geq 0, \xi_{t} \in \mathcal{E}_{0}, \xi_{t} \in \mathcal{E}_{1} \) and \( x \in \mathbb{R}^d \) such that \( |x| \geq R \). Let \( \psi \in \mathbb{B}^{m}_{0} \) such that almost surely \( \psi(x) = 0 \) for \( |x| \geq R \). Then there exist \( \varepsilon_{0} > 0 \) and a \( \bar{R} = \bar{R}(R, K_{1}, K_{\xi}, K_{\eta}) \) such that the following statements hold.

(i) For each \( \varepsilon \in (0, \varepsilon_{0}) \) there exists a \( \mathcal{W}_{p}^{\varepsilon} \)-solution \( u^{\varepsilon} \) to (5.28), for every \( r \geq 1 \), with initial condition \( u_{0}^{\varepsilon} = \psi^{(\varepsilon)} \) and such that

\[
\mathbb{E} \sup_{t \in [0,T]} |u_{t}^{\varepsilon}|_{\mathcal{W}_{p}^{\varepsilon}} < \infty \quad \text{and} \quad u_{t}^{\varepsilon}(x) = 0 \quad \text{almost surely for } |x| \geq \bar{R} \text{ and } t \in [0,T].
\]

(ii) There exists a unique \( \mathcal{L}_{p}^{\varepsilon} \)-solution \( u \) to (3.3) (with non-smoothed coefficients) such that almost surely \( u_{t}(x) = 0 \) for \( dx \)-almost every \( x \in \{ x \in \mathbb{R}^d : |x| \geq \bar{R} \} \) for every \( t \in [0,T] \) and

\[
\mathbb{E} \sup_{t \in [0,T]} |u_{t}|_{\mathcal{L}_{p}} \leq \mathbb{E} |\psi|_{\mathcal{L}_{p}}^{\varepsilon}
\]

with a constant \( N = N(d, p, T, K_{1}, K_{\xi}, K_{\eta}, L, \mathbb{E} |\xi|_{L_{2}}, |\eta|_{L_{2}}) \).

(iii) There exists a sequence \( (\varepsilon_{n})_{n=1}^{\infty} \), \( \varepsilon \to 0 \), such that

\[
u^{\varepsilon_{n}} \to u \quad \text{weakly in } \mathcal{L}_{p,q} \quad \text{as } \varepsilon \to 0, \quad \text{for every integer } q \geq 2.
\]

**Proof.** See Lemma 6.4 in [6].

**Lemma 5.5.** Let Assumptions 2.1, 2.2, 2.5 and 2.4 hold with \( K_1 = 0 \). Consider integers \( m \geq 0 \) and \( p \geq 2 \) even. Let moreover the support condition (5.29) of Proposition 5.4 hold for some \( R > 0 \). Then there exists a unique \( \mathcal{W}_{p} \)-solution \( (u_{t})_{t \in [0,T]} \) to equation (3.3) with initial condition \( u_{0} = \psi \). Moreover, almost surely \( u_{t}(x) = 0 \) for \( dx \)-almost every \( x \in \{ x \in \mathbb{R}^d : |x| \geq \bar{R} \} \) for every \( t \in [0,T] \) for a constant \( \bar{R} = \bar{R}(R, K_{1}, K_{\xi}, K_{\eta}) \), and

\[
\mathbb{E} \sup_{t \in [0,T]} |u_{t}|_{\mathcal{W}_{p}} \leq \mathbb{E} |\psi|_{\mathcal{W}_{p}}^{\varepsilon}
\]

with a constant \( N = N(m, d, p, T, K_{1}, K_{\xi}, K_{\eta}, L, \mathbb{E} |\xi|_{L_{2}}, |\eta|_{L_{2}}) \).
Proof. By Proposition 5.4 (i) for \( \varepsilon > 0 \) sufficiently small there exists a \( W^m \)-valued weakly cadlag \( \mathcal{F}_t \)-adapted process \( (u^\varepsilon_t)_{t \in [0,T]} \), such that for each \( \varphi \in C^0_c \) almost surely

\[
(u^\varepsilon_t, \varphi) = (u^\varepsilon_t, \varphi) + \int_0^t (u^\varepsilon_s, \zeta_s^\varepsilon \varphi) \, ds + \int_0^t (u^\varepsilon_s, \mathcal{M}^\varepsilon_s \varphi) \, dV^s_k + \int_0^t \int_{3_1} (u^\varepsilon_s, J^\varepsilon_s \varphi) \, \nu_1(d3) \, ds
\]

\[
+ \int_0^t \int_{3_1} (u^\varepsilon_s, J^\varepsilon_s \varphi) \, \nu_1(d3) \, ds + \int_0^t \int_{3_1} (u^\varepsilon_s, I^\varepsilon_s \varphi) \, N_1(d3, d3),
\]

(5.31)

holds for all \( t \in [0,T] \). By Proposition 5.4 (ii), since almost surely \( u^\varepsilon_t = 0 \) for \( |x| \geq \bar{R} \) for all \( t \in [0,T] \) for a constant \( \bar{R} = \bar{R}(R, K, K_0, K_\lambda, K_\eta) \), we also have

\[
\mathbb{E} \sup_{\varepsilon \in [0,T]} |u^\varepsilon_t|_{L^p} \leq \bar{R}^{d/q} \mathbb{E} \sup_{\varepsilon \in [0,T]} |u^\varepsilon_t|^p_{L^p}
\]

for \( q = p/(p - 1) \). Next, note that the smoothed coefficients \( b^{(\varepsilon)}, B^{(\varepsilon)}, \sigma^{(\varepsilon)}, \rho^{(\varepsilon)}, \xi^{(\varepsilon)} \) and \( \eta^{(\varepsilon)} \) satisfy Assumptions 2.1, 2.2, 2.5 and Assumption 2.4 (ii) & (iii) with the same constants \( K_0, L, K_\lambda \) and \( K_\eta \), independent of \( \varepsilon \). By Remark 2.1 (i) we have that for all \( t \in [0,T], \theta \in [0,1], y \in \mathbb{R}^d \) and \( z \in \mathbb{R}^d \),

\[
\tau^{\varepsilon}_{t,\theta,z_0}(x) = x + \theta \eta^{(\varepsilon)}(x,z_0), \quad \text{and} \quad \tau^{\varepsilon}_{t,\theta,z_1}(x) = x + \theta \xi^{(\varepsilon)}(x,z_1)
\]

are \( C^1 \)-diffeomorphisms. Moreover, by Lemma 6.2 in [6], we know that for \( \varepsilon \) sufficiently small we have that for all \( t \in [0,T], \theta \in [0,1] \) and \( z \in \mathbb{R}^d \), \( |x| \geq \bar{R} \), the mappings

\[
(\tau^{\varepsilon}_{t,\theta,z_0})^{(\varepsilon)} = \tau^{\varepsilon}_{t,\theta,z_0}(x) = x + \theta \eta^{(\varepsilon)}(x,z_0) \quad \text{and} \quad (\xi^{\varepsilon}_{t,\theta,z_1})^{(\varepsilon)} = \xi^{\varepsilon}_{t,\theta,z_1}(x) = x + \theta \xi^{(\varepsilon)}(x,z_1)
\]

remain \( C^1 \)-diffeomorphisms such that

\[
|\det D\tau^{\varepsilon}_{t,\theta,z_0}(x)| \geq \lambda' \quad \text{and} \quad |\det D\tau^{\varepsilon}_{t,\theta,z_1}(x)| \geq \lambda',
\]

with a \( \lambda' = \lambda'(\lambda, K_\lambda, K_\eta, K_0) \) independent of \( \varepsilon \). By Remark 2.1 (ii) we then know that Assumption 2.4 (i) is satisfied with (another) \( \lambda'' = \lambda''(\lambda, K_\lambda, K_\eta, K_0) \) independent of \( \varepsilon \). Hence by Lemma 5.3 for each \( \varepsilon > 0 \) also

\[
\mathbb{E}|u^\varepsilon_t|^p_{W^m_p} + \mathbb{E} \left( \int_0^T |u^\varepsilon_t|^p_{W^m_p} \, dt \right)^{p/r} \leq \mathbb{E}|u^\varepsilon_t|^p_{W^m_p} + T^{p/r} \mathbb{E} \sup_{\varepsilon \in [0,T]} |u^\varepsilon_t|^p_{L^p} \leq N\mathbb{E}|\psi|^p_{W^m_p}
\]

(5.32)

for a constant \( N = N(m,d,p,K_\eta,K_\lambda,K_\xi,K,L,T,\lambda,|\tilde{\xi}|_{L^2(3_1)},|\eta|_{L^2(3_1)}) \) independent of \( \varepsilon \) for all integers \( r \geq 1 \). Letting \( (\varepsilon_n)_{n=1}^\infty \) be the sequence from Proposition 5.4 (iii), we know that

\[
u^{\varepsilon_n}_t \rightarrow u_T \quad \text{weakly in } \mathbb{L}^p(\mathcal{F}_T) \quad \text{and} \quad u^{\varepsilon_n} \rightarrow u \quad \text{weakly in } \mathbb{L}^p_{p,r} \quad \text{for integers } r \geq 1 \quad \text{as } n \rightarrow \infty
\]

where \( u \) is the unique \( L^p \)-solution to (3.3) and, if necessary by passing to a subsequence,

\[
u^{\varepsilon_n}_T \rightarrow u_T \quad \text{weakly in } \mathbb{W}^m_p(\mathcal{F}_T) \quad \text{and} \quad u^{\varepsilon_n} \rightarrow u \quad \text{weakly in } \mathbb{W}^m_p \quad \text{for integers } r \geq 1.
\]

Letting \( r \rightarrow \infty \) in (5.32) yields

\[
\mathbb{E}|u_T|^p_{W^m_p} + \mathbb{E} \sup_{\varepsilon \in [0,T]} |u^\varepsilon_t|^p_{W^m_p} < N\mathbb{E}|\psi|^p_{W^m_p}.
\]

By Lemma 3.3 \( u \) is weakly cadlag as \( W^m_p \)-valued process. Thus we can replace the essential supremum above by the supremum to obtain (5.30). By Proposition 5.4 (ii) we also have
that almost surely \( u_t(x) = 0 \) for \( dx \)-almost every \( x \in \{ x \in \mathbb{R}^d : |x| \geq R \} \) for every \( t \in [0, T] \) for a constant \( R = \bar{R}(R, K, K_0, K_\xi, K_\eta) \). This finishes the proof.

\[ \square \]

**Corollary 5.6.** Let Assumptions 2.1, 2.2, 2.4 and 2.5 hold with an integer \( m \geq 0 \). Assume, moreover that the support condition (5.29) holds for some \( R > 0 \). Then for every \( p \geq 2 \) there is a linear operator \( \mathbb{S} \) defined on \( \mathbb{W}_p^m \) such that \( \mathbb{S}\psi \) admits a \( P \otimes dt \)-modification \( u = (u_t)_{t \in [0, T]} \) which is a \( W_p^m \)-solution to equation (3.3) for every \( \psi \in \mathbb{W}_p^m \), with initial condition \( u_0 = \psi \), and

\[
\mathbb{E} \sup_{t \in [0, T]} |u_t|_{W_p^m}^p \leq N\mathbb{E}|\psi|_{W_p^m}^p
\]

with a constant \( N = N(m, d, p, T, K, K_\xi, K_\eta, L, \lambda, |\xi|_{L_2}, |\eta|_{L_2}) \). Moreover, if \( \psi \in \mathbb{W}_p^m \) such that almost surely \( \psi(x) = 0 \) for \( |x| \geq R \), then almost surely \( u_t(x) = 0 \) for \( |x| \geq \bar{R} \) for \( t \in [0, T] \) for a constant \( \bar{R} = \bar{R}(R, K, K_0, K_\xi, K_\eta) \).

**Proof.** By Corollary 6.5 in [6] we know that there exist linear operators \( \mathbb{S} \) and \( \mathbb{S}_T \) on \( L_p \) such that \( \mathbb{S}\psi \) admits a \( P \otimes dt \)-modification \( u = (u_t)_{t \in [0, T]} \) that is an \( L_p \)-solution to (3.3) such that \( u_T = \mathbb{S}_T \psi \) satisfies equation (3.3) for each \( \varphi \in C_0^\infty \) almost surely with \( u_T \) in place of \( u_t \) and \( t \coloneqq T \). By an abuse of notation we refer to this stochastic modification \( u \) whenever we write \( \mathbb{S}\psi \) in the following. It remains to show that if \( \psi \in \mathbb{W}_p^m \), then \( u \) is in particular a \( W_p^m \)-solution to (3.3), i.e. it is weakly cadlag as \( W_p^m \)-valued process.

If \( p \) is an even integer, then this follows from Lemma 5.5. Assume \( p \) is not an even integer. Then let \( p_0 \) be the greatest even integer such that \( p_0 \leq p \) and let \( p_1 \) be the smallest even integer such that \( p \leq p_1 \). By Lemma 5.5, in particular (5.30), we get that

\[
|\mathbb{S}_T \psi|_{W_p^{m}} + |\mathbb{S}\psi|_{W_p^{m}} \leq N_i|\psi|_{W_p^{m}} \quad \text{for } i = 0, 1 \quad (5.34)
\]

for every \( r \in [1, \infty) \) and constants \( N_i = N_i(m, d, p, T, K, K_\xi, K_\eta, L, \lambda, |\xi|_{L_2}, |\eta|_{L_2}) \), \( i = 0, 1 \), independent of \( r \). Hence, by a well-known generalization of the Riesz-Thorin interpolation theorem we also get for all \( r \geq 1 \),

\[
|\mathbb{S}_T \psi|_{W_p^{m}} + |\mathbb{S}\psi|_{W_p^{m}} \leq N|\psi|_{W_p^{m}} \quad \text{for } i = 0, 1 \quad (5.35)
\]

for (another) constant \( N = N(m, d, p, T, K, K_\xi, K_\eta, L, \lambda, |\xi|_{L_2}, |\eta|_{L_2}) \). Consider a sequence \( (\psi^n)_{n=1}^{\infty} \subset \mathbb{B}_m^m \) such that \( \psi^n \to \psi \) in \( \mathbb{W}_p^m \). For each \( n \), \( u^n = \mathbb{S}_T \psi^n \) is the unique \( W_p^m \)-solution to (3.3), \( i = 0, 1 \), with initial condition \( \psi^n \). By virtue of (5.35), using that \( |\psi^n - \psi|_{W_p^m} \to 0 \), as \( n \to \infty \) we know that also

\[ u^n \to u \quad \text{weakly in } \mathbb{W}_{p_r}^m \text{ for every integer } r \geq 1 \quad \text{and } u^n \to u_T \quad \text{weakly in } \mathbb{W}_p^m(\mathcal{F}_T), \]

where \( u = \mathbb{S}\psi \) is the unique \( L_p \)-solution introduced in the beginning of the proof, satisfying (5.35). To see that \( u \) is weakly cadlag as \( W_p^m \)-valued process, note that by letting \( r \to \infty \) in (5.35) or \( \mathbb{S}\psi = u \) and \( \mathbb{S}_T \psi = u_T \) yields

\[
\mathbb{E}|u_T|_{W_p^m}^p + \mathbb{E}\operatorname{ess} \sup_{t \in [0, T]} |u_t|_{W_p^m}^p \leq N\mathbb{E}|\psi|_{W_p^m}^p,
\]

for (another) constant \( N = N(m, d, p, T, K, K_\xi, K_\eta, L, \lambda, |\xi|_{L_2}, |\eta|_{L_2}) \). By Lemma 3.3 we then know that \( u \) is weakly cadlag as \( W_p^m \)-valued process. Thus we can replace the essential supremum above with the supremum, to obtain (5.33). To prove the claim about the support of \( u \), note that if \( \psi(x) = 0 \) for \( |x| \geq R \), for a constant \( R \), and \( \psi^n \to \psi \) in \( \mathbb{W}_p^m \), then
for sufficiently large $n$ we have $\psi^n(x) = 0$ for $|x| \geq 2R$. By Proposition 5.4 (ii) thus also $u^n_t(x) = 0$ for $dx$-almost every $x \in \{x \in \mathbb{R}^d : |x| \geq R\}$ for every $t \in [0,T]$ and $n$ sufficiently large, for a constant $\bar{R} = \bar{R}(R,K,K_0,K_{\xi},K_n)$. This is clearly preserved in the limit as $n \to \infty$. This finishes the proof.

\[\]

6. Proof of Theorem 2.1

Let $\chi$ be a smooth function on $\mathbb{R}$ such that $\chi(r) = 1$ for $r \in [-1, 1]$, $\chi(r) = 0$ for $|r| \geq 2$, $\chi(r) \in [0, 1]$ and $\sum_{k=1}^{m+2} |d^k/ (dr^k) \chi(r)| \leq C$ for all $r \in \mathbb{R}$ and a real nonnegative constant $C$. For integers $n \geq 1$ we define the function $\chi_n(x) = \chi(|x|/n), x \in \mathbb{R}^d$.

**Lemma 6.1.** (i) Let $b = (b^i)$ be an $\mathbb{R}^d$-valued function on $\mathbb{R}^m$ such that for a constant $L$

\[|b(v) - b(z)| \leq L|v - z| \quad \text{for all } v, z \in \mathbb{R}^m. \tag{6.1}\]

Then for $b_n(z) = \chi(|z|/n)b(z), z \in \mathbb{R}^m$, for integers $n \geq 1$ we have

\[|b_n(z)| \leq 2nL + |b(0)|, \quad |b_n(v) - b_n(z)| \leq (5L + 2|b(0)|)|v - z| \quad \text{for all } v, z \in \mathbb{R}^m. \tag{6.2}\]

(ii) Let additionally to (i) the function $b$ satisfy

\[\sum_{k=1}^m |D^k b| \leq M, \tag{6.3}\]

for a constant $M > 0$. Then $b_n$ satisfies (6.3) in place of $b$ with $M' = M'(M, C, m, |b(0)|)$ in place of $M$.

**Proof.** The proof of (i) is Lemma 7.2 in [6]. The proof of (ii) is an easy exercise. \[\]

To preserve the diffeomorphic property of the mappings

\[\tau^\eta_{t, \Omega, \theta}(x) = x + \theta \eta_t(x, \Omega) \quad \text{and} \quad \tau^{\xi}_{t, \Omega, \theta}(x) = x + \theta \xi_t(x, \Omega) \tag{6.4}\]

(for all $\omega \in \Omega, t \in [0, T], \theta \in [0, 1]$ and $\Omega_t \in \mathfrak{F}_t, i = 0, 1$) as a function of $x \in \mathbb{R}^d$, when the functions $\xi$ and $\eta$ are truncated, we introduce, for each fixed $R > 0$ and $\epsilon > 0$, the function $\kappa^R_\epsilon$ defined on $\mathbb{R}^d$ by

\[\kappa^R_\epsilon(x) = \int_{\mathbb{R}^d} \phi^R_{\epsilon}(x-y)k(y) dy, \quad \phi^R_{\epsilon}(x) \begin{cases} 1, & |x| \leq R + 1, \\ 1 + \epsilon \log \left( \frac{R+1}{|x|} \right), & R + 1 < |x| < (R + 1)e^{1/\epsilon}, \\ 0, & |x| \geq (R + 1)e^{1/\epsilon}, \end{cases} \tag{6.5}\]

where $k$ is a nonnegative $C^\infty$ mapping on $\mathbb{R}^d$ with support in $\{x \in \mathbb{R}^d : |x| \leq 1\}$.

**Lemma 6.2.** Let $\xi : \mathbb{R}^d \to \mathbb{R}^d$ be such that for a constant $L \geq 1$ and for every $\theta \in [0, 1]$ the function $\tau_\theta(x) = x + \theta \xi(x)$ is $L$-biLipschitz, i.e.

\[L^{-1}|x - y| \leq |\tau_\theta(x) - \tau_\theta(y)| \leq L|x - y| \tag{6.6}\]

for all $x, y \in \mathbb{R}^d$. Then for any $M > L$ and any $R > 0$ there is an $\epsilon = \epsilon(L, M, R, |\xi(0)|) > 0$ such that with $\kappa^R_\epsilon := \kappa^R_\epsilon \xi$ the function $\xi^R := \kappa^R_\epsilon \xi$ vanishes for $|x| \geq R$ for a constant $\bar{R} = \bar{R}(L, M, R, |\xi(0)|) > R$, $|\xi^R|$ is bounded by a constant $N = N(L, M, R, |\xi(0)|)$, and for every $\theta \in [0, 1]$ the mapping

\[\tau^R_\theta(x) = x + \theta \xi^R(x), \quad x \in \mathbb{R}^d\]

is $M$-biLipschitz.
Proof. This is Lemma 7.3 in [6].

We summarize the results of Lemmas 7.1, 7.2 and Remark 7.1 in [6] in the following lemma. For that purpose, define the functions $b^n = (b^m(t,z))$, $B^n = (B^m(t,z))$, $\sigma^n = (\sigma^{mij}(t,z))$, $\eta^n = (\eta^{mi}(t,z,\xi))$ and $\xi^n = (\xi^{mij}(t,z,\xi))$ by

\[
    (b^n, B^n, \sigma^n, \rho^n) = (b, B, \sigma, \rho)\chi_n, \quad (\eta^n, \xi^n) = (\eta, \xi)\chi_n
\]

(6.7)

for every integer $n \geq 1$, where $\chi_n$ and $\tilde{\chi}_n$ are functions on $\mathbb{R}^{d+d'}$ defined by $\chi_n(z) = \chi(|z|/n)$ and $\tilde{\chi}_n(x, y) = \kappa_R(|x|/n)|y|/n$ for $z = (x, y) \in \mathbb{R}^{d+d'}$, with $\chi$ used in Lemma 6.1 and with $\kappa^R = \kappa^R_\xi$ from Lemma 6.2, such that, by the $L$-biLipschitzness of the mappings in (6.4), the mappings

\[
    \tau_{t,\xi,\theta}(x) = x + \theta \eta^n(x, \xi) \quad \text{and} \quad \tau_{t,\xi,\theta}(x) = x + \theta \xi^n(x, \xi)
\]

are biLipschitz (for all $\omega \in \Omega$, $t \in [0, T]$, $\theta \in [0, 1]$ and $\xi \in \tilde{\xi}$, $i = 0, 1$).

**Lemma 6.3.** Let Assumptions 2.1, 2.2 and 2.5 hold. If $K_1 \neq 0$ in Assumption 2.1 (ii), then let additionally Assumption 2.3 for some $r > 2$ hold. Assume the initial conditional density $\pi_0 = P(X_0 \in d\pi | \mathcal{F}_0 Y)$ / $dx$ exists (a.s.) and satisfies $E|\pi_0|^p < \infty$ for some $p \geq 2$ and integer $m \geq 0$. Then there exist sequences

\[
    (X^n_0)_{n=1}^{\infty}, ((X^n_t, Y^n_t)_{t \in [0, T]}_{n=1}^{\infty}, \quad \text{as well as} \quad (\pi^n_0)_{n=1}^{\infty} \quad \text{and} \quad ((\pi^n_t)_{t \in [0, T]}_{n=1}^{\infty})
\]

such that the following are satisfied:

(i) For each $n \geq 1$ the coefficients $b^n, B^n, \sigma^n, \rho^n, \xi^n$ and $\eta^n$, defined in (6.7), satisfy Assumptions 2.1 and 2.2 with $K_1 = K_2 = 0$ and constants $K'_0 = K_0(n, K, K_0, K_1, K_\xi, K_\eta)$ and $L' = L'(K, K_0, K_1, L, K_\xi, K_\eta)$ in place of $K_0$ and $L$, Assumption 2.5 with a constant $K' = K'(K_0, K_1)$ in place of $L$, as well as Assumption 2.4 with $\lambda' = \lambda'(K_0, K_1, K_\xi, K_\eta, \lambda)$ in place of $\lambda$. Moreover, for each $n \geq 1$ they satisfy the support condition (5.29) of Lemma 5.5 for some $R = R(n)$.

(ii) For each $n \geq 1$ the random variable $X^n_0$ is $\mathcal{F}_0$-measurable and satisfies

\[
    \lim_{n \to \infty} X^n_0 = X_0, \quad \omega \in \Omega, \quad \text{and} \quad E|X^n_0|^r \leq N(1 + E|X_0|^r)
\]

for $r \geq 1$ with a constant $N$ independent of $n$.

(iii) $Z^n_t = (X^n_t, Y^n_t)$ is the solution to (1.1) with the coefficients $b^n, B^n, \sigma^n, \rho^n, \xi^n$ and $\eta^n$ in place of $b, B, \sigma, \rho, \xi$ and $\eta$, respectively, and with initial condition $Z_0^n = (X^n_0, Y^n_0)$.

(iv) For each $n \geq 1$ we have $\pi^n_0 = P(X^n_0 \in d\pi | \mathcal{F}_0 Y^n)/dx$, $\pi^n_0(x) = 0$ for $|x| \geq n + 1$ and

\[
    \lim_{n \to \infty} |\pi^n_0 - \pi_0|_{W^m} = 0,
\]

where $\pi_0 = P(X_0 \in d\pi | \mathcal{F}_0 Y)/dx$.

(v) For each $n \geq 1$ there exists an $L_r$-solution $u^n$ to (3.3), $r = 2, p$, such that $u^n$ is the unnormalised conditional density of $X^n$ given $Y^n$, almost surely

\[
    u^n_t(x) = 0 \quad \text{for } dx \text{-a.e. } x \in \{x \in \mathbb{R}^d : |x| \geq \tilde{R}\} \text{ for all } t \in [0, T]
\]

with a constant $\tilde{R} = \tilde{R}(n, K, K_0, K_\xi, K_\eta)$ and

\[
    E \sup_{t \in [0, T]} |u^n_t|_{L^p} \leq NE|\pi^n_0|_{L^p}
\]

(6.8)
with a constant $N = N(d, d', K, L, K_\xi, K_\eta, T, p, \lambda, \xi_{L_2}, \eta_{L_2})$. Moreover,
$$u^n \rightarrow u \quad \text{weakly in } L_{r,q} \text{ for } r = p, 2 \text{ and all integers } q > 1,$$
where $u$ is the unnormalised conditional density of $X$ given $Y$, satisfying (6.8) with the same constant $N$ and $(u, \pi_0)$ in place of $(u^n, \pi^n_0)$.

(vi) Consequently, for each $n \geq 1$ and $t \in [0, T]$ we have
$$\pi^n_t(x) = P(X^n_t \in dx|\mathcal{F}^Y_t)/dx = u^n_t(x)^{\alpha^n_t}, \quad \text{almost surely,}$$
as well as
$$\pi_t(x) = P(X_t \in dx|\mathcal{F}^Y_t)/dx = u_t(x)^{\alpha_t}, \quad \text{almost surely,}$$
where $\alpha^n_t$ and $\alpha_t$ are cadlag positive normalising process, adapted to $\mathcal{F}^Y_t$ and $\mathcal{F}^Y_t$ respectively.

Proof. This is Corollary 7.4 in [6]. \qed

Now we are in the position to prove our main result.

Proof of Theorem 2.1. Step I. Assume first that the support condition (5.29) holds with some $R > 0$ and that the initial conditional density $\pi_0$ is such that $\pi_0(x) = 0$ for $|x| \geq R$. By Corollary 5.6 we know that there exists a $W^m_p$-solution $(u_t)_{t \in [0, T]}$ to (3.3) with initial condition $\pi_0$, satisfying
$$\mathbb{E} \sup_{t \in [0, T]} |u_t|_{W^m_p}^p \leq N\mathbb{E}|\pi_0|_{W^m_p}^p \tag{6.9}$$

with a constant $N = N(m, d, p, T, K, K_\xi, K_\eta, L, \lambda, \xi_{L_2}, \eta_{L_2})$. Moreover, we have $u_t = 0$ for $|x| \geq R$, for a constant $\tilde{R} = \tilde{R}(R, K, K_0, K_1, K_\xi, K_\eta)$, and hence clearly
$$\sup_{t \in [0, T]} |u_t|_{L^1} \leq \tilde{R}^{d/\varrho} \sup_{t \in [0, T]} |u_t|_{L^p} \text{ and } \sup_{t \in [0, T]} \int_{\mathbb{R}^d} |y|^2 |u_t(y)| dy < \infty \text{ (a.s.),}$$
with $\varrho = p/(p - 1)$. Since also $\pi_0 = P(X_0 \in dx|\mathcal{F}^Y_0)/dx \in \mathbb{L}_1$, then in particular $\pi_0 \in \mathbb{L}_2$ and hence
$$\mathbb{E} \sup_{t \in [0, T]} |u_t|_{L^2}^2 \leq N\mathbb{E}|\pi_0|_{L^2}^2 \tag{6.10}$$

with a constant $N = N(d, p, T, K, K_\xi, K_\eta, L, \lambda, \xi_{L_2}, \eta_{L_2})$ By Lemma 5.5 $u$ is the unique $L^2$-solution and therefore by Theorem 3.2, $u$ is in particular the unnormalised conditional density, i.e., $u_t = d\mu_t/dx$ for all $t \in [0, T]$, almost surely, with $\mu$ the unnormalised conditional distribution from Theorem 3.1. Thus also for each $t \in [0, T]$,
$$\pi_t = P(X_t \in dx|\mathcal{F}^Y_t)/dx = u_t^{\alpha_t}, \quad \text{almost surely,}$$
where $\alpha_t$ is the $\mathcal{F}_t^Y$-optional projection of the normalising process $\gamma$ under $P$ introduced in (3.5).

Step II. Finally, we dispense with the assumption that the coefficients and the initial condition are compactly supported. Define the functions $b_n, B_n, \sigma_n, \rho_n, \xi_n$ and $\eta_n$ as in (6.7). Note that by Lemma 6.3 the truncated coefficients satisfy Assumptions 2.1 and 2.2 with $K_1 = K_2 = 0$ and constants $K'_0 = K'_0(n, K, K_0, K_1, K_\xi, K_\eta)$ and $L' = L'(K, K_0, K_1, L, K_\xi, K_\eta)$ in place of $K_0$ and $L$, the coefficients $b_n, B_n, \sigma_n, \rho_n$ satisfy Assumption 2.5 with a constant $K' = K'(m, K_0, K_1)$ in place of $L$, and moreover that the coefficients $\eta_n$ and $\xi_n$ satisfy Assumption 2.5 with $K'\bar{\eta}$ and $K'\bar{\xi}$ instead of $\bar{\eta}$ and $\bar{\xi}$ respectively. Furthermore,
by Lemma 6.2, for each \( n \geq 1 \) the coefficients \( \eta_n \) and \( \xi_n \) satisfy Assumption 2.4 with a constant \( \lambda' = \lambda'(\lambda, K_0, K_1, K_\eta, K_\xi) \) in place of \( \lambda \). Note that \( K', L' \) and \( \lambda' \) do not depend on \( n \). Moreover, for each \( n \geq 1 \) they satisfy the support condition (5.29) of Lemma 5.5 for some \( R = R(n) > 0 \). By assumption, \( \pi_0 = P(X_0 \in dx|F_0^Y)/dx \) exists almost surely and \( \mathbb{E}|\pi_0|^p_{W_p} < \infty \). Then let \( (X_0^n)_{n=1}^{\infty} \) and \( (\pi_0^n)_{n=1}^{\infty} \in \mathcal{W}_{p}^m \) be the sequences from Lemma 6.3 such that

\[
\lim_{n \to \infty} |\pi_0^n - \pi_0|_{\mathcal{W}_{p}}^p = 0, \tag{6.11}
\]

\( \pi_0^n(x) = 0 \) for \( |x| \geq R(n) \) and \( \pi_0^n = P(X_0^n \in dx|F_0^Y)/dx \) (a.s.), where \( (X_0^n,Y_0) \) is the initial condition to the system (1.1), and \( (R(n))_{n=1}^{\infty} \) is the sequence of positive numbers from the support condition for the coefficients \( (\sigma^n, \ldots, \xi^n) \). By Step I we know that there exists a \( W^m \)-solution \((u_t)_{t \in [0,T]}\) to (3.3) with initial condition \( \pi_0^n \), which is the unnormalized conditional density of \( X^n = (X^n_t)_{t \in [0,T]} \) given \( Y^n = (Y^n_t)_{t \in [0,T]} \), where \( Z^n = (X^n, Y^n) \) is the solution to (1.1) with initial condition \((X_0^n,Y_0)\). By Lemma 6.3 (v) we know moreover that

\[
u^n \to u \quad \text{weakly in } \mathbb{L}_{r,q} \quad \text{as } q \to r \quad \text{for } r = p,2 \quad \text{and all integers } q > 1,
\]

where \( u \) is the unnormalized conditional density of \( X \) given \( Y \) from Theorem 3.2, satisfying

\[
\mathbb{E} \sup_{t \in [0,T]} |u_t|^2_{L_2} \leq N \mathbb{E}|\pi_0|^2_{L_2},
\]

with a constant \( N = N(d,p,T,K,K_\xi,K_\eta,L,\lambda,|\xi|_{L_2},|\eta|_{L_2}) \) independent of \( n \). Moreover, \( u \) is an \( L_p \)-solution to (3.3) and by Theorem 3.2 (ii), it is the unique \( L_2 \)-solution to (3.3). It remains to show that \( u \) is also a \( W^m \)-solution to (3.3), as well as that it is strongly cadlag as \( W^s \)-valued process, for \( s \in [0,m] \). To prove the former, by (6.9) together with (6.11) we get that for \( n \) sufficiently large,

\[
\mathbb{E}|u^n_T|^p_{W^m} + \mathbb{E} \left( \int_0^T |u^n_r|^p_{W^m} \, dt \right)^{p/r} \leq \mathbb{E}|u_T|^p_{W^m} + T^{p/r} \mathbb{E} \sup_{t \in [0,T]} |u_t|^p_{W^m} \leq 2 N \mathbb{E}|\pi_0|^p_{W^m}. \tag{6.12}
\]

Hence we know that

\[
u^n \to u_T, \quad \text{weakly in } \mathcal{W}_{p}^m \quad \text{and} \quad \nu^n \to u, \quad \text{weakly in } \mathcal{W}_{p,r}^m \quad \text{for any } r > 1,
\]

where \( u \) satisfies for all \( r \geq 1,

\[
\mathbb{E}|u_T|^p_{W^m} + \mathbb{E} \left( \int_0^T |u_r|^p_{W^m} \, dt \right)^{p/r} \leq 2 N \mathbb{E}|\pi_0|^p_{W^m}.
\]

Letting \( r \to \infty \) above yields

\[
\mathbb{E}|u_T|^p_{W^m} + \mathbb{E} \operatorname{ess} \sup_{t \in [0,T]} |u_t|^p_{W^m} \leq 2 N \mathbb{E}|\pi_0|^p_{W^m}.
\]

By Lemma 3.3 we then know that \( u \) is weakly cadlag as a \( W^m \)-valued process, i.e. it is a \( W^m \)-solution to (3.3). Clearly, by Lemma 6.3, also for each \( t \in [0,T] \)

\[
\pi_t(x) = P(X_t \in dx|F_t^Y)/dx = u_t(x)^\gamma_t, \quad \text{almost surely},
\]

with \( \gamma_t \) from Theorem 3.2. We now show that if \( m \geq 1 \) and \( K_1 = 0 \), then \( u \) is strongly cadlag as \( W^s \)-valued process for \( s \in [0,m] \). To this and first we state \( u \) is strongly cadlag as an \( L_p \)-valued process.
Proposition 6.4. Let Assumptions 2.1 through 2.5 hold with $K_1 = 0$ and with $m = 1$. Let $p \geq 2$ and let $u = (u_t)_{t \in [0, T]}$ be a $W^1_p$-solution to (3.3). Then $u$ is strongly cadlag as an $L_p$-valued process.

Proof. We apply Theorem 2.2 in [7]. In order to do so, we rewrite equation (3.2) into the form used therein. Clearly, for $v \in W^1_p$ and $\varphi \in C^{\infty}_0$ we have

$$(v, \mathcal{M}^k \varphi) = (\mathcal{M}^k u, \varphi)$$

with $\mathcal{M}^k u = -D_i(p^k u) + B^k u$, $k = 1, 2, \ldots, d_1$

and

$$(v, \tilde{\mathcal{L}}^k \varphi) = - (D_j(a^j u), D_i \varphi) - (D_i(b^i u), \varphi) + \beta^k_s(\mathcal{M}^k u, \varphi),$$

Using Corollary 3.5 with $\eta(\cdot, \cdot)$ and $\xi(\cdot, \cdot)$ in place of $\zeta$ we can see that

$$(v, J^1 \varphi) = (K^1 u, D_i \varphi), \quad (v, J^2 \varphi) = (K^2 u, D_i \varphi) \quad \text{and} \quad (v, I^1 \varphi) = (I^1 u, \varphi),$$

for $v \in W^1_p$ and $\varphi \in C^\infty_0$, where $K^1 u$ and $K^2 u$ are defined as $K^k u$ in Corollary 3.5 with $\zeta$ replaced by $\eta(\cdot, \cdot)$ and $\xi(\cdot, \cdot)$, respectively, for $t \in [0, T], \omega \in \Omega, \eta_0 \in \zeta_0, \zeta_1 \in \zeta_1, i = 1, \ldots, d$, and $I^k u$ is defined as $I^k u$, with $\zeta$ replaced by $\xi(\cdot, \cdot)$. Thus for every $\varphi \in C^\infty_0$ almost surely

$$(u_t, \varphi) = (\psi, \varphi) - \int_0^t (D_j(a^j u), D_i \varphi) ds - \int_0^t (D_i(b^i u) + \beta^k_s \mathcal{M}^k u, \varphi) ds$$

$$+ \int_0^t (\mathcal{M}^k u, \varphi) dV^k_s + \int_0^t \int_{\zeta_0} (K^k u, D_i \varphi) \nu_0(\zeta_3) ds$$

$$+ \int_0^t \int_{\zeta_1} (K^k u, D_i \varphi) \nu_1(\zeta_3) ds + \int_0^t \int_{\zeta_1} (I^k u, \varphi) \tilde{N}_1(\zeta_3, ds)$$

for all $t \in [0, T]$. It is easy to see that almost surely

$$\int_0^T |D_i(a^j u)|^p ds < \infty, \quad \int_0^T |D_i(b^i u) + \beta^k_s \mathcal{M}^k u|^p ds < \infty,$$

$$\int_0^T \int_{\mathbb{R}^d} (\sum_k |(\mathcal{M}^k u)(x)|^2)^{p/2} dx ds < \infty. \quad (6.14)$$

By estimates (3.13) and (3.14), for all $x \in \mathbb{R}^d$ we have

$$|I^k u(x)| \leq N_\zeta \int_0^1 |u(\tau_{\theta^1}(x))| + |(Du)(\tau_{\theta^1}(x))| d\theta,$$

$$|K^k u(x)| \leq N_\zeta \int_0^1 |u(\tau_{\theta^1}(x))| + |(Du)(\tau_{\theta^1}(x))| d\theta,$$

$$|K^k u(x)| \leq N_\zeta \int_0^1 |u(\tau_{\theta^1}(x))| + |(Du)(\tau_{\theta^1}(x))| d\theta$$

for every $\omega \in \Omega, s \in [0, T], \zeta_i \in \zeta_i (i=0,1)$, suppressed in these estimates, with a constant $N = N(d, \lambda, L, K_\eta, K_\xi)$ and with the $C^2$-diffeomorphisms

$$\tau_{\theta^1}(x) = x + \theta^1(x) \quad \text{and} \quad \tau_{\theta^1}(x) = x + \theta^1(x).$$
Hence by Jensen’s inequality, Fubini’s theorem and Minkovski’s inequality we get
\[
\int_0^T \int_{\mathbb{R}^d} \left( \int_{\mathcal{F}_t} |\xi(x)|^2 \nu_1(dx) \right)^{p/2} dx ds \leq N |\tilde{\xi}|_{L^2(\mathcal{F}_t)}^p \int_0^T |u_s|_{L^p}^p ds < \infty \text{ (a.s.)} \tag{6.15}
\]
with a constant \( N = N(p, d, \lambda, L, K_\eta, K_\xi) \). By Jensen’s inequality and Fubini’s theorem we obtain
\[
\int_0^T \int_{\mathbb{R}^d} \int_{\mathcal{F}_t} |T_s u_s(x)|^p \nu_1(dx) dx ds \leq N |\tilde{\xi}|_{L^2(\mathcal{F}_t)}^p \int_0^T |u_s|_{L^p}^p ds < \infty \text{ (a.s.)}, \tag{6.16}
\]
and for every \( i = 1, 2, \ldots, d \)
\[
\int_0^T \int_{\mathbb{R}^d} \int_{\mathcal{F}_t} |K_s^{\xi} u_s(x)|^p \nu_1(dx) dx ds \leq N |\tilde{\xi}|_{L^2(\mathcal{F}_t)}^p \int_0^T |u_s|_{L^p}^p ds < \infty \text{ (a.s.)}, \tag{6.17}
\]
\[
\int_0^T \int_{\mathbb{R}^d} \int_{\mathcal{F}_t} |K_s^{\nu} u_s(x)|^p \nu_1(dx) dx ds \leq N |\tilde{\xi}|_{L^2(\mathcal{F}_t)}^p \int_0^T |u_s|_{L^p}^p ds < \infty \text{ (a.s.)} \tag{6.18}
\]
with a constant \( N = N(p, d, \lambda, L, K_\eta, K_\xi) \). Hence, by virtue of Theorem 2.2 in [7] we get from equation (6.13), taking into account (6.14) through (6.18), that \((u_t)_{t \in [0,T]}\) is strongly c\-adl\-ag as an \( L_p \)-valued process.

By the above proposition \( u \) is a strongly c\-adl\-ag \( L_p \)-valued process, as well as weakly c\-adl\-ag as an \( W^m_p \)-valued process. By interpolation we then have a constant \( N = N(d, m, s, p) \) such that
\[
|u_t - u_{t_n}|_{W^s_p} \leq N |u_t - u_{t_n}|_{W^m_p} |u_t - u_{t_n}|_{L^p} \leq 2N\zeta |u_t - u_{t_n}|_{L^p},
\]
\[
|u_{r_n} - u_{r-}|_{W^s_p} \leq N |u_{r_n} - u_{r-}|_{W^m_p} |u_{r_n} - u_{r-}|_{L^p} \leq 2N\zeta |u_{r_n} - u_{r-}|_{L^p}
\]
for any \( t \in [0,T) \), \( r \in (0,T] \), any strictly decreasing sequences \( t_n \to t \) and strictly increasing sequences \( r_n \to r \) with \( r_n, t_n \in (0,T) \), where \( u_{r-} \) denotes the weak limit in \( W^m_p \) of \( u \) at \( r \) from the left, and \( \zeta := \sup_{t \in [0,T]} |u_t|_{W^s_p} < \infty \text{ (a.s.)} \). Letting here \( n \to \infty \) we finish the proof. \( \square \)

Acknowledgements. The authors are very grateful to Nicolai Krylov, whose comments and suggestions greatly improved the presentation of the present article.

References

[1] S. Blackwood, Lévy processes and filtering theory, Dissertation, University of Sheffield, 2014.
[2] A. Calvia and G. Ferrari, Nonlinear Filtering of Partially Observed Systems Arising in Singular Stochastic Optimal Control, Applied Mathematics & Optimization 85.2 (2022), 1-43.
[3] K. A. Dareiotis, C. Kumar and S. Sabanis, On tamed Euler approximations of SDEs driven by Lévy noise with applications to delay equations, SIAM Journal on Numerical Analysis (2016)
[4] M. De-Léon Contreras, I. Gyöngy and S. Wu, On solvability of integro-differential equations, Potential Anal. 55 (2021), no. 3, 443-475.
[5] F. Germ and I. Gyöngy, On partially observed jump diffusions I. The filtering equations, arXiv:2205.08286, 2022
[6] F. Germ and I. Gyöngy, On partially observed jump diffusions II. The filtering density, arXiv:2205.14534, 2022
[7] I. Gyöngy and S. Wu, Itô’s formula for jump processes in \( L^p \)-spaces, Stochastic processes and their applications, 2021.
[8] N.V. Krylov, An analytic approach to SPDEs, Stochastic Partial Differential Equations: Six Perspectives, Mathematical Surveys and Monographs 64 (1999), 185-242.
REGULARITY OF THE FILTERING DENSITY

[9] N. V. Krylov, On divergence form SPDEs with VMO coefficients, SIAM J. Math. Anal. 40 (2009), no. 6, 2262-2285.

[10] N. V. Krylov, On divergence form SPDEs with growing coefficients in $W^2_2$ spaces without weights, SIAM J. Math. Anal. 42 (2010), 609-633.

[11] N. V. Krylov, Kalman-Bucy filter and SPDEs with growing lower-order coefficients in $W^1_2$ spaces without weights, Illinois Journal of Mathematics 54.3 (2010), 1069-1114.

[12] N. V. Krylov, Filtering equations for partially observable diffusion processes with Lipschitz continuous coefficients. The Oxford handbook of nonlinear filtering, 169-194. Oxford Univ. Press, Oxford, 2011.

[13] N.V. Krylov and B.L. Rozovskii, On conditional distributions of diffusion processes, Math. USSR Izv. 12 (1978), 336-356.

[14] T.G. Kurtz and J. Xiong, Particle representations for a class of nonlinear SPDEs, Stochastic Processes and their Applications 83 (1999).

[15] T.G. Kurtz and D.L. Ocone, Unique characterization of conditional distributions in nonlinear filtering, Annals of Probability (1988).

[16] V. Maroulas, X. Pan and J. Xiong, Large deviations for the optimal filter of nonlinear dynamical systems driven by Lévy noise, Stochastic Processes and their Applications 130 (2020), 203–231.

[17] B.L. Rozovskii, On conditional distributions of degenerate diffusion processes, Theory of Probability & its Applications 25.1 (1980), 147-151.

[18] H. Qiao and J. Duan, Nonlinear filtering of stochastic dynamical systems with Lévy noises, Advances in Applied Probability 47-3 (2015).

[19] H. Qiao, Nonlinear filtering of stochastic differential equations driven by correlated Lévy noises, Stochastics (2021).

School of Mathematics, University of Edinburgh, King’s Buildings, Edinburgh, EH9 3JZ, United Kingdom
Email address: fgerm@ed.ac.uk

School of Mathematics and Maxwell Institute, University of Edinburgh, Scotland, United Kingdom.
Email address: i.gyongy@ed.ac.uk