Speed of Social Learning from Reviews in Non-Stationary Environments

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Abstract. Potential buyers of a product or service tend to first browse feedback from previous consumers through review platforms. This behavior is modeled by a market of Bayesian consumers with heterogeneous preferences, who sequentially decide whether to buy an item based on reviews of previous buyers. While the belief of the item quality in simple settings is known to converge to its true value, this paper extends this result to more general cases, besides providing convergence rates. In practice, the quality of an item may change over time as new competitors can appear in the market or the product/service can undergo modifications. This paper studies such dynamics with changing points model and shows that the cost of learning remains low, when expressed in total utility earned by consumers.

Keywords: Social Learning, Bayesian Estimation, Non-Stationary Environment.

1 Introduction

In our society many forms of learning are not from direct experience, but rather from observing the behavior of other people who themselves are trying to learn. In other words, people engage in social learning. For instance, before deciding whether to buy a product or service, consumers observe the past behavior of previous consumers and use this observation to make their own decision. Once their decision is made, this becomes a piece of information for the future consumers. In the old days, it was common to consider a crowd in a restaurant as a sign that the food was likely good. Nowadays, there are more sophisticated ways to learn from previous consumers. After buying a product and experiencing its features, people often leave reviews on sites such as Amazon, Tripadvisor, Yelp, etc. When consumers observe only the purchasing behavior of previous consumers, there is a risk of a cascade of bad decisions: if the first agents make the wrong decision,
the following agents may follow them thinking that what they did was optimal and herding happens. Interestingly enough, this is not necessarily the effect of bounded rationality. It can actually be the outcome of a Bayesian equilibrium in a game with fully rational players. It seems reasonable to conjecture that, if consumers write reviews about the product that they bought, then social learning will be achieved. This is not always the case, when consumers are heterogeneous and the reviews that they write depends on the quality of the object but also on their idiosyncratic attitude toward the product they bought. Also, consumers might value more recent reviews as they describe more accurately the current state of the product, whose quality indeed changes over time. Considering a stationary learning environment is thus a simplification that does not reflect the true behavior of consumers on review platforms. Withdrawing this assumption does not affect substantially the outcome of the learning process.

1.1 Main contributions

We consider a model where heterogeneous consumers arrive sequentially to a monopolistic market and before deciding whether to buy a product of unknown quality, they observe the binary reviews (like/dislike) of the previous buyers. Each buyer posts a sincere review that summarizes the experienced quality of the product and an idiosyncratic attitude to it. [Ifrach et al. (2019)] introduced this model in the case where the intrinsic quality of the product is fixed over time and can assume just two values, and studied conditions for social learning to be achieved. Building on their results, here we consider a larger set of possible values for the unknown quality and provide rates of convergence of the posterior distribution of the quality. For continuous quality spaces, we also provide an efficient non-Bayesian estimator of the quality.

We then consider a more challenging model where the unknown quality may vary over time. The criterion that we use in this dynamical setting is the utility loss that a non-informed consumer incurs with respect to a fully informed consumer, who at every time knows the true quality of the product. We show that, when the quality can assume only two values, the learning cost is a logarithmic factor of the changing rate of the quality. The bound for the case of continuous quality is more intricate. Here, too, we consider a non-Bayesian estimator.

Table 1 below summarizes the proved bounds for all considered settings. In the analysis we also consider the case of imperfect learners, who are not aware of the dynamical nature of the quality, and we quantify the loss they incur.

1.2 Related literature

The problem of social learning goes back to [Banerjee (1992)] and [Bikhchandani et al. (1992)] who considered models where Bayesian rational agents, who arrive at a market sequentially, observe the actions of the previous agents, and decide based on their private signals and the public observations, in equilibrium may herd into a sequence of bad decisions. In other words, social learning may fail with positive probability. [Smith and Sørensen (2000)] showed that this learning failure is due
Quality space & Stationary & Utility Loss & Tight Upper Bound \\
\{q_0, \ldots, q_K\} & ✓ & \mathcal{O}(1) & ✓ \\
[q, \bar{q}] & ✓ & \mathcal{O}(\sqrt{T}) & ✓ \\
\{L, H\} & ✗ & \mathcal{O}(\ln(1/\eta)\eta T) & ✓ \\
[q, \bar{q}] & ✗ & \mathcal{O}(\eta^{1/4}T) & ? \\

Table 1: Bounds summary. In a non-stationary environment, the quality changes with probability $\eta$ at each round, while the utility loss is summed over $T$ rounds.

to the fact that signals are bounded. In the presence of unbounded signals that can overcome any observed behavior, herding cannot happen.

Different variations of the above model have been considered, where either agents observe only a subset of the previous agents (see, e.g., [Celen and Kariv, 2004; Acemoglu et al., 2011; Lobel and Sadler, 2015], or the order in which actions are taken is not determined by a line, but rather by a lattice (Arieli and Mueller-Frank, 2019b). A general analysis of social learning models can be found in Arieli and Mueller-Frank (2019a).

A more recent stream of literature deals with models where agents observe not just the actions of the previous agents, but also their ex-post reaction to the actions they took. For instance, before buying a product of unknown quality, consumers read the reviews written by the previous consumers. In particular, Besbes and Scarsini (2018) dealt with some variation of a model of social learning in the presence of reviews with heterogeneous consumers. In one case agents observe the whole history of reviews and can use Bayes rule to compute the conditional expectation of the unknown quality and learning is achieved. In the other case they only observe the mean of past reviews. Interestingly, even in this case, learning is achieved and the speed of convergence is of the same order as in the previous case. Ifrach et al. (2018) studied a model where the unknown quality is binary and the reviews are also binary (like or dislike). They considered the optimal pricing policy and looked at conditions that guarantee social learning. A non-Bayesian version of the model was considered in Crapis et al. (2017), where mean-field techniques were adopted to study the learning trajectory.

The speed of convergence in social learning has been considered by Rosenberg and Vieille (2019) in models where only the actions of the previous agents are observed and by Acemoglu et al. (2017) when reviews are present. This last paper is the closest to the spirit of our own paper.

Learning problems in non-stationary environment have been considered, for instance, by Besbes et al. (2018, 2019) in a context where the function that is being learned changes smoothly, rather than abruptly as in our model in Section II.
1.3 Organization of the paper

Section 2 introduces the model of social learning from consumer reviews. Section 3 studies the stationary setting where the quality is fixed. Sections 3.1 and 3.2 respectively consider discrete and continuous quality spaces, while Section 3.3 bridges these two cases. Section 4 introduces the dynamical setting, where the quality changes over time. Section 4.1 and 4.3 respectively consider binary and continuous quality spaces. Section 4.2 on the other hand shows that the knowledge of the dynamical structure is crucial to the consumer utility.

2 Model

A monopolist sells at a price \( p \) some product of quality \( Q \in \mathbb{Q} \subseteq \mathbb{R} \) to a market of heterogeneous consumers. In the following, the quality space is assumed to be compact: \( \mathbb{Q} \subseteq [q, \bar{q}] \). At time \( t = 0 \), \( Q \) is drawn at random according to some prior \( \pi_0 \) and remains unknown to the consumers. We assume for simplicity that \( \pi_0 \) is uniform over \( Q \), but our results can naturally be generalized to any prior.

A new consumer arrives at each time \( t \in \mathbb{N}^* \) with individual random preference \( \theta_t \in \mathbb{R} \). This preference is private information of consumer \( t \). The random variables \( \theta_t \) are i.i.d. with cumulative distribution function \( F_{\theta} \).

Based on the history \( H_t \) of past observations and her preference \( \theta_t \), consumer \( t \) decides whether to buy the product. The history \( H_t \) is rigorously formalized below. In case of purchase, she receives the utility \( u_t := Q + \theta_t - p + \varepsilon_t \) where \( p \) is an exogenously determined price and \( \varepsilon_t \) are i.i.d. variations caused by different factors, e.g., fluctuations in the product quality or imperfect perception of the quality by the consumer. If the consumer does not buy, she gets \( u_t = 0 \).

Bayesian myopic rationality is assumed, so consumer \( t \) buys the product if and only if
\[
\mathbb{E}[u_t | H_t] \geq 0,
\]
that is, if and only if \( \theta_t \geq p - \mathbb{E}[Q | H_t] \). The consumer then reviews the product by giving the feedback \( Z_t = \text{sign}(u_t) \). In words, \( Z_t = 0 \) without purchase, \( Z_t = -1 \) if the product is bought but disliked (\( u_t < 0 \)) and \( Z_t = 1 \) if it is bought and liked. The history is now defined as \( H_t = (Z_s)_{s<t} \).

In the following, \( \pi_t \) denotes the posterior distribution of \( Q \), given \( H_t \), which determines the buyer decision. Similarly to Ifrach et al. (2019), we define \( b_k \) as the time of the \( k \)-th purchase:
\[
b_1 = \min \{ t \mid \theta_t \geq p - \mathbb{E}[Q | H_t] \},
b_{k+1} = \min \{ t > b_k \mid \theta_t \geq p - \mathbb{E}[Q | H_t] \}.
\]

We conversely define \( B(t) = \max \{ k \mid b_k < t \} \) the number of purchases before \( t \), and we introduce the function \( G \) intervening in the posterior update defined by
\[
G(z, \pi, q) = \mathbb{P}[Z_t = z \mid \pi_t = \pi, Q = q],
\]

\[
= \begin{cases} 
\int_{p-E[Q]}^{\infty} F_{\varepsilon}(p - q - x) \, dF_{\Theta}(x) & \text{if } z = 1, \\
\int_{p-E[Q]}^{\infty} F_{\varepsilon}(p - q - x) \, dF_{\Theta}(x) & \text{if } z = -1, \\
F_{\Theta}(p - E[Q]) & \text{if } z = 0.
\end{cases}
\]
In the following, we also use the notation $G(z, \pi) = \mathbb{E}_{q \sim \pi}[G(z, \pi, q)]$. Note that the purchase decision only depends on $H_t$ and $\theta_t$, so that it does not yield any additional information on the quality. This differentiates this model from the classical social learning models, where consumers have private signals that are correlated with the quality. As a consequence, the posterior $\pi_t$ only depends on $(Z_{b_k})_{b_k < t}$. We also consider the following two mild assumptions in the sequel.

**Assumption 1** $\bar{F}_\theta(p - q) > 0$, i.e., there is always a fraction of consumers who buy the product.

**Assumption 2** The distribution of $\varepsilon$ has a continuous positive density on $\mathbb{R}$.

Assumption 1 avoids situations where the consumers would stop buying when the quality belief becomes low. Without this assumption, learning might fail [Acemoglu et al., 2017; Ifrach et al., 2019]. Assumption 2 on the other hand is technical, and although not strictly necessary, it simplifies the exposition.

### 3 Stationary Environment

This section considers the case of a fixed quality across time. For a binary quality $Q = \{L, H\}$, [Frfach et al. 2019] showed that the posterior almost surely converges to the true quality, while [Acemoglu et al., 2017] showed an asymptotic exponential convergence rate. Besides extending these results to larger quality spaces, this section aims at showing convergence rates of the posterior. The study of convergence rates in social learning is a recent concern (Acemoglu et al., 2017; Rosenberg and Vieille, 2019) despite being central to online learning (Bottou, 1999) and Bayesian estimation (Ghosal et al., 2000). Moreover, convergence rates are of crucial interest when facing a dynamical quality in Section 4.

Section 3.1 shows an exponential convergence rate for a discrete quality space while Section 3.2 gives a $1/\sqrt{t}$ rate when it is continuous. Section 3.3 finally bridges these two results when discretizing $Q$.

#### 3.1 Discrete Quality

In this section, the space $Q$ is finite, its cardinality denoted by $K$. As a consequence, the posterior update is obtained using Bayes rule,

$$\pi_{t+1}(q) = \frac{G(Z_t, \pi_t, q)}{G(Z_t, \pi_t)} \pi_t(q).$$

(3)

Theorem 1 below gives a convergence rate of the posterior to the true quality. Similarly to [Acemoglu et al., 2017, Theorem 2], it shows an exponential convergence rate. While their result considers a binary quality space $\{L, H\}$ and an asymptotic convergence rate, we provide an anytime, but less tight, rate on a general discrete quality space with similar assumptions. We focus on anytime rates as they are highly relevant with a dynamical, evolving quality.

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1 Although our Assumption 2 is stronger than their Assumption 1, the proof of Theorem 1 only requires the conditions given by their Assumption 1. The stronger conditions of Assumption 2 are used in the continuous case in Section 3.2.
Theorem 1. For $q \neq q'$, it holds

$$
\mathbb{E}[\pi_{t+1}(q') \mid Q = q] \leq 2 \exp \left( -\frac{t\delta^2(q, q')}{6\gamma^2(q, q')} \right),
$$

where $\delta(q, q') := \min_{\pi \in \mathcal{P}(Q)} |G(-1, \pi, q) - G(-1, \pi, q')| + |G(1, \pi, q) - G(1, \pi, q')|$ and $\gamma(q, q') := \max_{\pi \in \mathcal{P}(Q)} \left| \ln \left( \frac{G(-1, \pi, q)}{G(-1, \pi, q')} \right) \right| + \left| \ln \left( \frac{G(1, \pi, q)}{G(1, \pi, q')} \right) \right|$. Notice that $\delta$ minimizes the total variation between $Z_t$ conditioned either on $(\pi, Q = q)$ or $(\pi, Q = q')$. Thanks to Assumption 2, both quantities are positive, guaranteeing an exponential convergence rate of the posterior.

In the proof below, we shall use the notation $\text{KL}(\mu, \nu)$ for the Kullback-Leibler divergence between the distributions $\mu$ and $\nu$, which is defined as

$$
\text{KL}(\mu, \nu) = \mathbb{E}_{x \sim \mu} \left[ \ln \left( \frac{\mu(x)}{\nu(x)} \right) \right].
$$

Proof. Recall the prior is assumed uniform on $Q$. Equation (3) then gives $\ln \left( \frac{\pi_{t+1}(q)}{\pi_{t+1}(q')} \right) = \sum_{s=1}^{t} \ln \left( \frac{G(Z_s, \pi_s, q)}{G(Z_s, \pi_s, q')} \right)$. Define now

$$
X_t = \ln \left( \frac{G(Z_t, \pi_t, q)}{G(Z_t, \pi_t, q')} \right) - \text{KL}(G(\cdot, \pi_t, q), G(\cdot, \pi_t, q')).
$$

Notice that $\mathbb{E}[X_t \mid \mathcal{H}_t, Q = q] = 0$ and $X_t \in [Y_t, Y_t + \gamma(q, q')]$ almost surely for some $\mathcal{H}_t$-measurable variable $Y_t$. Azuma-Hoeffding inequality then yields:

$$
\mathbb{P} \left[ \sum_{s=1}^{t} X_s \leq -\lambda \left| \mathcal{H}_t, Q = q \right. \right] \leq \exp \left( -\frac{2\lambda^2}{t\gamma^2(q, q')} \right),
$$

which is equivalent to

$$
\mathbb{P} \left[ \frac{\pi_{t+1}(q')}{\pi_{t+1}(q)} \geq \exp \left( \lambda - \sum_{s=1}^{t} \text{KL}(G(\cdot, \pi_s, q), G(\cdot, \pi_s, q')) \right) \left| \mathcal{H}_t, Q = q \right. \right] \leq \exp \left( -\frac{2\lambda^2}{t\gamma^2(q, q')} \right).
$$

By Pinsker inequality, $\text{KL}(G(\cdot, \pi_s, q_k), G(\cdot, \pi_s, q_{k'})) \geq \delta^2(q, q')/2$, which becomes

$$
\mathbb{P} \left[ \pi_{t+1}(q') \geq \exp \left( \lambda - t\delta^2(q, q')/2 \right) \left| Q = q \right. \right] \leq \exp \left( -\frac{2\lambda^2}{t\gamma^2(q, q')} \right).
$$

So this yields

$$
\mathbb{E}[\pi_{t+1}(q')|Q = q] \leq \exp \left( \lambda - \frac{t\delta^2(q, q')}{2} \right) + \mathbb{P} \left[ \pi_{t+1}(q') \geq \exp \left( \lambda - t\delta^2(q, q')/2 \right) \left| Q = q \right. \right] \leq \exp \left( \lambda - \frac{t\delta^2(q, q')}{2} \right) + \exp \left( -\frac{2\lambda^2}{t\gamma^2(q, q')} \right).
$$
Theorem 2. For 

Now consider the continuous case 

3.2 Continuous Quality 

As 

 cannot depend on other problem parameters such as the distributions of 

 and so 

\[ \gamma \]

 constant 

\[ c \]

\[ \lambda = -x + \sqrt{2xy + x^2} \]

equals the two terms:

\[ \mathbb{E} | \pi_{t+1}(q') | Q = q | \leq 2 \exp(-x - y + \sqrt{x^2 + 2xy}) \leq 2 \exp \left( -\frac{y^2}{2(x+y)} \right). \]

The second inequality is given by the convex inequality 

\[ \sqrt{a} - \sqrt{a+b} \leq -\frac{b}{2\sqrt{a+b}} \]

for \( a = x^2 + 2xy \) and \( b = y^2 \). From the definitions of \( x \) and \( y \), this yields:

\[ \mathbb{E} | \pi_{t+1}(q') | Q = q | \leq 2 \exp \left( -\frac{t \delta^2(q, q')}{2\gamma^2(q, q') + 4\delta^2(q, q')} \right). \]

As \( G \) has values in \([0, 1]\), 

\[ | \ln(G(z, \pi, q)) - \ln(G(z, \pi, q')) | \geq | G(z, \pi, q) - G(z, \pi, q') | \]

and so \( \gamma(q, q') \geq \delta(q, q') \), which finally yields Theorem 2. \( \square \)

3.2 Continuous Quality 

Now consider the continuous case \( Q = [q, \bar{q}] \). The argument from Section 3.1 cannot be adapted to this case. Instead, we first show the existence of a “good” non-Bayesian estimator. The posterior will also have similar, if not better, performances as it minimizes the Bayesian risk.

In this section, we use the notation \( g(t) = \mathcal{O}(f(t)) \) if there exists a positive constant \( c \), independent of \( t \), such that for all \( t \in \mathbb{N}^* \), \( g(t) \leq cf(t) \). Note that \( c \) can depend on other problem parameters such as the distributions of \( \varepsilon \) and \( \theta \).

**Theorem 2.** For \( M_t = \mathbb{E}[Q | \mathcal{H}_t], \mathbb{E}\left[ | M_t - Q | \right] = \mathcal{O}(1/\sqrt{t}) \).

Note that the rate \( \mathcal{O}(1/\sqrt{t}) \) is the best rate possible even if the reviews report exactly \( Q + \varepsilon_t \).

**Proof.** We first show the existence of a good non-Bayesian estimator. Define for any \( m, q \in Q \):

\[ \psi_m(q) = \phi(m, q) := \int_{p-m}^\infty \frac{F_x(p - q - x) \, dF_x(x)}{F_x(p - m)}. \]

which is the probability for a buyer to like a product, given that the posterior quality is \( m \) and the true quality is \( q \). Recall that \( B(t) \) is the number of purchases before consumer \( t \) and define \( L(t) \) the number of likes before consumer \( t \), i.e., 

\( L(t) = \sum_{s<t} \mathbb{1}_{z_s = 1} \).

**Lemma 1.** For \( M_t = \mathbb{E}[Q | \mathcal{H}_t] \) and \( P_{[q, \bar{q}]} \) the projection on the interval \([q, \bar{q}]\),

\[ \mathbb{E} \left[ \left( P_{[q, \bar{q}]} \circ \psi_t^{-1} (L(t)/B(t)) - Q \right)^2 \right] = \mathcal{O}(1/t), \]

where \( \psi_t(\cdot) := \frac{1}{B(t)} \sum_{k=1}^{B(t)} \psi_{M_k}(\cdot) \).
The detailed proof of Lemma 1 is postponed to Appendix A.1. We first provide a quick sketch of the proof. Because of Azuma-Hoeffding inequality, the expectation of \( \left( \frac{L(t)}{B(t)} - \tilde{\psi}_t(Q) \right)^2 \), for a fixed \( B(t) \), scales as \( O(1/B(t)) \). Thanks to Assumption 2, \( \tilde{\psi}_t^{-1} \) is Lipschitz when composed with the projection on \( Q \). This gives a \( O(1/B(t)) \) bound. Leveraging Assumption \( I \) then yields Lemma 1 when taking the expectation over \( B(t) \).

A property/characterization of the estimated posterior is that it minimizes the Bayesian mean square error among all \( H_t \)-measurable functions. In particular, it has a smaller error than the estimator given by Lemma 1:

\[
E \left[ (M_t - Q)^2 \right] \leq E \left[ \left( P_{[\bar{q},q]} \circ \tilde{\psi}_t^{-1} \left( \frac{L(t)}{B(t)} \right) - Q \right)^2 \right].
\]

Thanks to Lemma \( I \) the first term is in \( O(1/t) \) and Theorem 2 then follows. □

Remark 1. Lemma 1 gives a non-Bayesian estimator that converges to \( Q \) at rate \( 1/t \) in quadratic loss. Using arguments similar to Besbes and Scarsini (2018), this implies that \( M_t \xrightarrow{a.s.} Q \), thanks to a result from Le Cam and Yang (2000). Theorem 2 yields a different result: \( M_t \) converges to \( Q \) at a rate \( 1/\sqrt{t} \) in average.

3.3 Bridging discrete and continuous cases

This section bridges the two previous cases by illustrating how the exponential convergence rate degrades to a \( 1/\sqrt{t} \) convergence rate when a continuum is considered as the limit of discretizations. Assumption 2 implies that for some positive \( \lambda_1 \) and \( \lambda_2 \) depending only on \( Q, F_\theta \) and \( F_\varepsilon \):

\[
\forall q, q' \in Q, \lambda_1 |q - q'| \leq \delta(q, q') \leq \gamma(q, q') \leq \lambda_2 |q - q'|,
\]

where \( \delta \) and \( \gamma \) here correspond to the definitions given in Theorem 1, which yields in this case:

\[
E[\pi_{t+1}(q) \mid Q = q] \geq 1 - \sum_{q' \neq q} 2 \exp \left( -\frac{t\lambda_1^2 (q - q')^2}{6\lambda_2^2} \right).
\]

Assume, w.l.o.g. and for simplicity, that \( q = 0 \): then, if we uniformly discretize the interval \([0, \bar{q}]\) into \( K \) points, the last equation becomes for \( q_k = \frac{k}{K-1} \bar{q} \):

\[
E[\pi_{t+1}(q_k) \mid Q = q_k] \geq 1 - \sum_{k' \neq k} 2 \exp \left( -\frac{t\lambda_1^2 (k - k')^2 \bar{q}^2}{6K^2\lambda_2^2} \right)
\]

\[
\geq 1 - 4 \sum_{n=1}^\infty \exp \left( -\frac{t\lambda_1^2 n\bar{q}^2}{6K^2\lambda_2^2} \right).
\]

The last inequality comes from noting that every \( n \in \mathbb{N}^* \) appears at most twice from \((k - k')^2\) in the previous sum. The limit case of convergence is for \( K \approx \sqrt{t} \).
In the continuous case, we can thus discretize $Q$ as a grid of size $K$. This result then implies that for $K \approx \sqrt{t}$, the posterior after $t$ steps concentrates on a ball of radius $O\left(\frac{1}{\sqrt{t}}\right)$, which is the distance between two points of the grid. This implies a result similar to Theorem 2.

4 Dynamical Environment

The quality is now considered dynamical, i.e., evolving with time. We consider a Markovian model, where at each time step, the quality is redrawn according to the prior with some probability $\eta$:

$$Q_{t+1} = \begin{cases} Q_t & \text{with probability } 1 - \eta, \\ X_{t+1} & \text{with probability } \eta, \end{cases}$$

where the sequence $(X_t)_{t \in \mathbb{N}}$ is i.i.d. with distribution $\pi_0$ and $Q_0 = X_0$.

Studying the convergence of the posterior is irrelevant, as the quality regularly changes. Instead, we measure the quality of the posterior variations in term of the total utility loss $\sum_{t=1}^{T} \mathbb{E}[(Q_t + \theta_t - p)_+ - u_t]$, also known as “regret”. The first term $(Q_t + \theta_t - p)_+$ indeed corresponds to the utility a consumer would get if she knew the quality $Q_t$, while $u_t$ is the utility she actually gets. In the following, the loss defined as

$$L_T := \sum_{t=1}^{T} \mathbb{E}\left[\left|\mathbb{E}[Q_t | H_t] - Q_t\right|\right]$$

is instead used as it is easier to bound and justified by Lemma 2.

**Lemma 2.** $\sum_{t=1}^{T} \mathbb{E}[(Q_t + \theta_t - p)_+ - u_t] \leq L_T$.

A proof of Lemma 2 can be found in Appendix A.2. Moreover, Theorem 1 and Theorem 2 respectively imply in the stationary setting that the cumulative loss is of order 1 for the discrete case and $\sqrt{T}$ for the continuous case.

In this whole section, we use the following notations:

- $g(T, \eta) = O(f(T, \eta))$ if there exists a constant $c$ such that for all $T \in \mathbb{N}^*$ and $\eta \in (0, 1)$, $g(T, \eta) \leq c f(T, \eta)$,
- $g(T, \eta) = \Omega(f(T, \eta))$ if $f(T, \eta) = O(g(T, \eta))$.

4.1 Binary Quality

In this section, we consider a binary quality space $Q = \{L, H\} = \{0, 1\}$ for simplicity. Recall that the prior is assumed uniform on $Q$, i.e., $\pi_0(H) = \frac{1}{2}$. The posterior update is then given with $G$ as defined in Equation 2 by:

$$\pi_{t+1}(H) = (1 - \eta) \frac{G(Z_t, \pi_t, H)}{G(Z_t, \pi_t)} \pi_t(H) + \frac{\eta}{2}.$$
By induction, this leads to the following expression, which showcases how consumers value more recent reviews

\[
\pi_{t+1}(H) = \frac{(1 - \eta)^t}{2} \prod_{s=1}^{t} \frac{G(Z_s, \pi_s, H)}{G(Z_s, \pi_s)} + \frac{\eta}{2} \sum_{s=0}^{t-1} (1 - \eta)^s \prod_{i=t-s+1}^{t} \frac{G(Z_i, \pi_i, H)}{G(Z_i, \pi_i)}. \tag{9}
\]

This expression is more complex than in the stationary case, leading to a more intricate proof of error bounds besides taking in consideration changes of quality.

Theorem 3 below claims that the cumulated loss is of order \(\ln(1/\eta)\eta T\). Perfect learners, who could directly observe \(Q_{t-1}\) before taking the decision at time \(t\), would still suffer a loss \(\eta T/2\) as there is a constant uncertainty \(\eta/2\) about the next step quality. Theorem 3 thus claims that the cost of learning is just a logarithmic factor in the dynamical setting.

**Theorem 3.** \(L_T = O(\ln(1/\eta)\eta T)\) and if \(\eta T = \Omega(1)\), then \(L_T = \Omega(\ln(1/\eta)\eta T)\).

The proof of Theorem 3 is divided into two parts: first, the upper bound \(L_T = O(\ln(1/\eta)\eta T)\) and, second, the lower bound \(L_T = \Omega(\ln(1/\eta)\eta T)\). They use several intermediate results that are proved in Appendix A.2.

The assumption \(\eta T = \Omega(1)\) guarantees that changes of quality actually have a non-negligible chance to happen in the considered time window. Without it, we would be back to the stationary case. In the extreme case \(\eta T \sim 1\), the error is thus of order \(\ln(T)\) against 1 in the stationary setting. This larger loss is actually the time needed to reverse the posterior belief after a change of quality. Indeed, assume that the posterior is very close to the true quality \(H\), i.e., \(\pi_t(L) \approx 0\); if the quality suddenly changes to \(L\), it will take a while to have a correct estimation again, i.e., to get again \(\pi_t(L) \approx 1\).

**Proof of Upper Bound.** In order to prove that \(L_T = O(\ln(1/\eta)\eta T)\), we will partition \(\mathbb{N}^*\) into blocks \([t_k + 1, t_{k+1}]\) of fixed quality and we show that the error on each block individually is \(O(\ln(1/\eta))\):

\[
t_1 = 0 \quad \text{and} \quad t_{k+1} = \max \{t > t_k | \forall s \in [t_k + 1, t], Q_s = Q_{t_{k+1}}\}. \tag{10}
\]

Define the stopping time

\[
\tau_k := \min \{\{t \in [t_k + 1, t_{k+1}] | \pi_t(Q_{t_{k+1}}) \geq 1/2\} \cup \{t_{k+1}\}\}. \tag{11}
\]

This is the first time\(^2\) in block \(k\) where the posterior of the true quality overcomes 1/2. The loss of a block is then decomposed as the terms before \(\tau_k\), which contribute to at most 1 per timestep, and the terms after \(\tau_k\). Lemma 3 bounds the first part and is proved in Appendix A.2.

**Lemma 3.** For any \(k\), \(\mathbb{P}[\tau_k - t_k \geq 2 + \frac{6 \gamma^2}{\delta^2} \ln(1/\eta)] \leq \eta\), where \(\delta := \delta(L, H)\) and \(\gamma := \gamma(L, H)\) as defined in Theorem 3.

\(^2\) It is set as the largest element of the block if such a criterion is never satisfied.
It can then be shown that past this stopping time, \( \frac{1}{\pi_t(Q_t)} \) cannot go above 2 in expectation as claimed by Lemma \ref{lemma:bound_p}

**Lemma 4.** For any \( k \in \mathbb{N}^* \) and \( t \in [\tau_k, t_{k+1}] \),

\[
E \left[ \frac{1}{\pi_t(Q_t)} \left| \tau_k, (t_n)_n \right. \right] \leq 2.
\]

We now bound the error after \( \tau_k \) on block \( k \). Assume w.l.o.g. that the quality is \( H \) on this block. Similarly to the proof of Theorem \ref{theorem:main}, Azuma-Hoeffding inequality on a single block leads to

\[
E \left[ \prod_{s=\tau_k}^{t-1} \frac{G(Z_s, \pi_s, L)}{G(Z_s, \pi_s)} \left| \pi_n, \forall s \in [n_0, t-1], Q_s = H \right. \right] \leq \frac{2}{\pi_n(H)} \exp \left( -\frac{(t-n)\delta^2}{6\gamma^2} \right).
\]

Moreover, Equation \ref{eq:bound_p} can be rewritten starting from \( n_0 \geq 1 \):

\[
\pi_{t+1}(q) = (1-\eta)^{t-n_0+1} \pi_{n_0}(q) \prod_{s=n_0}^{t} \frac{G(Z_s, \pi_s, q)}{G(Z_s, \pi_s)} + \frac{\eta}{2} \sum_{s=0}^{t-n_0} (1-\eta)^s \prod_{i=t-s+1}^{t} \frac{G(Z_i, \pi_s, q)}{G(Z_i, \pi)}.
\]

Define \( A^t_{\tau_k} = \{ s \in [\tau_k, \tau_k+t], Q_s = H \} \). Combining these last two equations, we obtain

\[
E \left[ \pi_{\tau_k+t}(L) \left| \pi_{\tau_k}, \tau_k, A^t_{\tau_k} \right. \right] \leq (1-\eta)^t \frac{2\pi_{\tau_k}(L)}{\pi_{\tau_k}(H)} \exp \left( -\frac{t\delta^2}{6\gamma^2} \right) + \frac{\eta}{2} \sum_{s=0}^{t-1} (1-\eta)^s E \left[ \frac{2}{\pi_{\tau_k+t-s}(H)} \left| \pi_{\tau_k}, \tau_k, A^t_{\tau_k} \right. \right] \exp \left( -\frac{s\delta^4}{6\gamma^2} \right).
\]

Thanks to Lemma \ref{lemma:bound_p} \( E \left[ \frac{2}{\pi_{\tau_k+t-s}(H)} \right| \pi_{\tau_k}, \tau_k, A^t_{\tau_k} \leq 4 \) and \( \frac{2\pi_{\tau_k}(L)}{\pi_{\tau_k}(H)} \leq 2 \), so that

\[
E \left[ \pi_{\tau_k+t}(L) \left| \pi_{\tau_k}, \tau_k, A^t_{\tau_k} \right. \right] \leq 2 \exp \left( -\frac{t\delta^4}{6\gamma^2} \right) + 2\eta \sum_{s=0}^{t-1} \exp \left( -\frac{s\delta^4}{6\gamma^2} \right)
\]

\[
\leq 2 \exp \left( -\frac{t\delta^4}{6\gamma^2} \right) + \frac{2\eta}{1 - \exp \left( -\frac{\delta^4}{6\gamma^2} \right)}. \tag{12}
\]

Finally, the loss incurred during the block \( k \) is at most \( \tau_k - t_k + \sum_{t=0}^{t_{k+1} - t_k - 1} \left( 2 \exp \left( -\frac{t\delta^4}{6\gamma^2} \right) + \frac{2\eta}{1 - \exp \left( -\frac{\delta^4}{6\gamma^2} \right)} \right) \), i.e.,

\[
\tau_k - t_k + \frac{2 + 2\eta(t_{k+1} - t_k)}{1 - \exp \left( -\frac{\delta^4}{6\gamma^2} \right)}.
\]

Lemma \ref{lemma:bound_p} yields that \( E[|\tau_k - t_k| (t_n)_n] \leq 2 + \frac{6\gamma^2}{\delta^4} \ln(1/\eta) + \eta(t_{k+1} - t_k) \). Thus in expectation, given \( (t_n)_n \), the loss over the block \( k \) is bounded by

\[
2 + \frac{6\gamma^2}{\delta^4} \ln(1/\eta) + \eta(t_{k+1} - t_k) + \frac{2 + 2\eta(t_{k+1} - t_k)}{1 - \exp \left( -\frac{\delta^4}{6\gamma^2} \right)}. \tag{13}
\]
Note then that \( t_{k+1} - t_k \) follows a geometric distribution of parameter \( \eta/2 \). In expectation the number of blocks counted before \( T \) is thus \( \mathcal{O}(\eta T) \) and summing Equation (13) over all these blocks yields \( \mathcal{L}_T = \mathcal{O}(\ln(1/\eta)\eta T) \).

**Proof of Lower Bound.** The proof of the lower bound is postponed to Appendix A.3. The idea is that the posterior cannot converge faster than exponentially on a single block. Thus, if the posterior converged in the last block, e.g., \( \pi_t(L) \approx \eta \) in a block of quality \( H \), then it would require a time \( \ln(1/\eta) \) before \( \pi_t(L) \geq 1/2 \) in the new block of quality \( L \), leading to a loss at least \( \ln(1/\eta) \).

### 4.2 Imperfect Learners

This section considers the same setting as Section 4.1 and shows that agnosticism to the dynamical structure of the problem leads to a considerable utility loss. In the following, we consider imperfect learners, i.e., consumers that are unaware of the dynamical structure of the problem. As a consequence, their posterior distribution \( \pi_{imp}^t \) follows the exact same update rule as in the stationary case given by Equation (3): \( \pi_{imp}^{t+1}(q) = \frac{G(Z_t, \sigma_{imp}^t, q)}{G(Z_t, \pi_{imp}^t)} \pi_{imp}^t(q) \). The error measure is then for imperfect learners,

\[
\mathcal{L}_{imp}^T := \sum_{t=1}^T \mathbb{E}\left[ 1 - \pi_{imp}^t(Q_t) \right].
\]

Theorem 4 below claims that the utility loss for imperfect learners is considerable, i.e., of order \( T \). It thus supports the significance of taking into account the dynamical structure of the problem in the learning process.

**Theorem 4.** If \( \eta T = \Omega(1) \), then \( \mathcal{L}_{imp}^T = \Omega(T) \).

The proof of Theorem 4 can be found in Appendix A.4 and shares similarities with the proof of the lower bound in Theorem 3. The posterior of imperfect learners converges quickly to the true quality. Because of this, after a change of quality, it takes a long time to reverse this posterior belief.

### 4.3 Continuous Quality

Now, we consider the continuous quality space \( Q = [q, \bar{q}] \) with the dynamical setting given by Equation (6). As in the stationary case, we first expose a satisfying non-Bayesian estimator, implying similar bounds on the posterior distribution.

**Theorem 5.** The loss of Bayesian consumers in the dynamical continuous case is bounded as \( \mathcal{L}_T = \mathcal{O}(\eta^{1/4}T) \).

In contrast to the discrete case, determining a lower bound in the continuous case remains open for the dynamical setting. Note that the total error is of order at least \( \sqrt{\eta T} \). Indeed, in the stationary case, no estimator converges faster than a rate \( 1/\sqrt{T} \). As the length of a block is around \( 1/\eta \), the loss per block is thus \( \Omega \left( 1/\sqrt{\eta} \right) \). Thanks to this, a tight bound should be between \( \sqrt{\eta T} \) and \( \eta^{1/4}T \).
Proof. In the stationary case, our non-Bayesian estimator comes from the ratio of observed likes. As highlighted by Equation (9), with a dynamical quality, recent reviews have a larger weight in the posterior. This leads to the following adapted discounted estimator for $\eta_1 \in (0,1)$:

$$\alpha^{u_1}(t) := \eta_1 \sum_{k=1}^{B(t)} (1 - \eta_1)^B(t)-k \mathbb{1}_{Z_{b_k}=1}. \tag{14}$$

We recall that $B(t)$ is the number of purchases before $t$ and $b_k$ is the time corresponding to the $k$-th purchase as defined in Equation (1). Lemma 5 below bounds the mean error for the estimator $\alpha^{u_1}$.

**Lemma 5.** For $\eta_1 = \sqrt{\eta}$,

$$\sum_{t=1}^{T} \sqrt{\mathbb{E} \left[ (\alpha^{u_1}(t) - \bar{\psi}_{t,\eta_1}(Q_t))^2 \right]} = O \left( \eta^{1/4}T \right),$$

where $\bar{\psi}_{t,\eta_1}(\cdot) := \eta_1 \sum_{k=1}^{B(t)} (1 - \eta_1)^B(t)-k \psi_{M_{b_k}}(\cdot)$, with $\psi$ defined by Equation (5).

The proof of Lemma 5 is given in Appendix A.5. Its main point is to show the following two inequalities for $t \in [t_i + 1, t_{i+1}]$:

$$\mathbb{E} \left[ (\alpha^{u_1}(t) - \eta_1 \sum_{k=1}^{B(t)} (1 - \eta_1)^B(t)-k \psi_{M_{b_k}}(Q_{b_k}) - \bar{\psi}_{t,\eta_1}(Q_t))^2 \right] = O (\eta_1),$$

$$\mathbb{E} \left[ (\eta_1 \sum_{k=1}^{B(t)} (1 - \eta_1)^B(t)-k \psi_{M_{b_k}}(Q_{b_k}) - \psi_{M_{b_k}}(Q_t)) \right]^2 = O \left( \frac{\eta}{\eta + \eta_1} \right).$$

The first one comes from Azuma-Hoeffding inequality, while the second one holds because the current block is not counted in the sum on the left hand side.

Similarly to the stationary setting, the error of the Bayesian estimator can be bounded by the error of the non-Bayesian one since the former is the minimizer of the quadratic loss among all $\mathcal{H}_t$-measurable functions:

$$\mathbb{E} \left[ (M_t - Q_t)^2 \right] \leq \mathbb{E} \left[ (P_{q,\bar{q}} \circ \bar{\psi}_{t,\eta_1}^{-1}(\alpha^{u_1}(t)) - Q_t)^2 \right].$$

Thanks to Assumption 2, $P_{q,\bar{q}} \circ \bar{\psi}_{t,\eta_1}^{-1}$ is Lipschitz. Theorem 5 then follows using Lemma 5 and Jensen inequality as in the proof of Theorem 2.

Lemma 5 uses the non-Bayesian estimator $\alpha^{u_1}(t)$ with the parameter $\eta_1$. Quite surprisingly, $\sqrt{\eta}$ seems to be the best choice of the parameter $\eta_1$, despite $\eta$ being the natural choice. Figure 5 below confirms this point empirically on a toy example. The experiment considers a quality space $Q = [0,1]$, a changing probability $\eta = 10^{-4}$ and gaussian distributions for $\theta$ and $\varepsilon$. Computing the exact posterior $M_t$ is intractable, so we remedy this point by assuming $M_t = 1$. 


all the time. This simplification does not affect the experiments run here as \( \psi_{t, \eta_1} \) uses \( M_t \) only to normalize by the fraction of buying consumers.

A larger \( \eta_1 \) allows to forget faster past reviews and thus gives a better adaptation after a quality change. However, a larger \( \eta_1 \) also yields a less accurate estimator in stationary phases.

The choice \( \eta_2^2/3 \) seems to be the best trade-off in Figure 1. The optimal choice of \( \eta_1 \) does not only depend on \( \eta \) but also on other parameters \( F_{\theta} \) and \( F_{\varepsilon} \). In the considered experiments, \( \eta \) is thus not small enough to ignore these other dependencies. Figure 1 yet illustrates the trade-off between small variance and fast adaptivity when tuning \( \eta_1 \).

![Fig. 1: Behavior of \( \alpha^{\eta_1} \) for different \( \eta_1 \).](image)

| Value of \( \eta_1 \) | \( \eta_1^{1/3} \) | \( \eta_1^{1/2} \) | \( \eta_1^{2/3} \) | \( \eta \) |
|----------------------|----------------|----------------|----------------|-------|
| Error               | 6362 3056 2327 5408 |

(a) Estimation error of \( \alpha^{\eta_1} \). The error is \( \sum_{t=1}^{T} \sqrt{E \left[ \left( \alpha^{\eta_1}(t) - \psi_{t, \eta_1}(Q_t) \right)^2 \right]} \) for \( T = 10^5 \), where the expectation is estimated by averaging over 2000 instances.

(b) Tracking of \( \psi_{t, \eta_1}(Q_t) \) by \( \alpha^{\eta_1}(t) \) over a single instance.

5 Conclusions

This work proposes a changing point framework for dynamical qualities in review based markets. Leveraging convergence rates in the stationary setting, regret bounds are proved for both discrete and continuous quality spaces with a dynamical quality. While the bound is tight in the former case, determining a tight bound in the continuous setting remains an open problem.

Many other directions also remain open for review based markets with changing quality. Especially, a slowly drifting quality can instead be considered. We only focus on the consumer side in this work, but the seller can also adaptively set the price of the item. What is a good seller strategy in this case? On the other hand, considering perfect Bayesian consumers might be unrealistic. In reality, consumers have limited computation capacity or can be risk averse, leading to different behaviors.
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Bibliography

D. Acemoglu, M. A. Dahleh, I. Lobel, and A. Ozdaglar. Bayesian learning in social networks. *Rev. Econ. Stud.*, 78(4):1201–1236, 2011. ISSN 0034-6527. https://doi.org/10.1093/restud/rdr004 URL https://doi.org/10.1093/restud/rdr004.

D. Acemoglu, A. Makhdoumi, A. Malekian, and A. Ozdaglar. Fast and slow learning from reviews. Technical report, National Bureau of Economic Research, 2017.

I. Arieli and M. Mueller-Frank. A general analysis of sequential social learning. Technical report, IESE Business School Working Paper No. WP-1119-E, 2019a.

I. Arieli and M. Mueller-Frank. Multidimensional social learning. *Rev. Econ. Stud.*, 86(3):913–940, 2019b. ISSN 0034-6527. https://doi.org/10.1093/restud/rdy029 URL https://doi.org/10.1093/restud/rdy029.

A. V. Banerjee. A simple model of herd behavior. *Quart. J. Econ.*, 107(3):797–817, 1992.

O. Besbes and M. Scarsini. On information distortions in online ratings. *Oper. Res.*, 66(3):597–610, 2018. ISSN 0030-364X. https://doi.org/10.1287/opre.2017.1676 URL https://doi.org/10.1287/opre.2017.1676.

O. Besbes, Y. Gur, and A. Zeevi. Non-stationary stochastic optimization. *Oper. Res.*, 63(5):1227–1244, 2015. ISSN 0030-364X. https://doi.org/10.1287/opre.2015.1408 URL https://doi.org/10.1287/opre.2015.1408.

O. Besbes, Y. Gur, and A. Zeevi. Optimal exploration-exploitation in a multi-armed bandit problem with non-stationary rewards. *Stoch. Syst.*, 9(4):319–337, 2019. https://doi.org/10.1287/stsy.2019.0033 URL https://doi.org/10.1287/stsy.2019.0033.

S. Bikchandani, D. Hirshleifer, and I. Welch. A theory of fads, fashion, custom, and cultural change as informational cascades. *J. Polit. Econ.*, 100(5):992–1026, 1992.

L. Bottou. On-line learning and stochastic approximations. In D. Saad, editor, *On-Line Learning in Neural Networks*, Publications of the Newton Institute, pages 9–42. Cambridge University Press, 1999. https://doi.org/10.1017/CBO9780511569920.003.

B. Çelen and S. Kariv. Observational learning under imperfect information. *Games Econom. Behav.*, 47(1):72–86, 2004. ISSN 0899-8256. https://doi.org/10.1016/S0899-8256(03)00179-9 URL https://doi.org/10.1016/S0899-8256(03)00179-9.

D. Crapis, B. Ifrach, C. Maglaras, and M. Scarsini. Monopoly pricing in the presence of social learning. *Management Sci.*, 63
A Additional proofs

This section contains detailed proofs of lemmas or theorems postponed to the Appendix.

A.1 Proof of Lemma 1

Note that
\[
\frac{L(t)}{B(t)} = \sum_{k=1}^{\mathbb{1}} \left| Z_{y_k} = 1 \right| \mathbb{I}_{Z_{y_k} = 1}, \quad \text{where } Z_{y_k} = 1 \mathbb{I}_{\mathbb{H}}(Q).
\]

By Azuma-Hoeffding inequality, we have
\[
\mathbb{P} \left( \left( \frac{L(t)}{B(t)} - \mathbb{E}_{\mathbb{1}}(Q) \right)^2 \geq \lambda \mid (B(n))n, Q \right) \leq 2e^{-2\lambda B(t)}.
\]

From this, we deduce a convergence rate \(1/B(t)\):
\[
\mathbb{E} \left( \left( \frac{L(t)}{B(t)} - \mathbb{E}_{\mathbb{1}}(Q) \right)^2 \mid (B(n))n, Q \right) \leq \int_0^{\infty} 2e^{-2\lambda B(t)} d\lambda = \frac{1}{B(t)}.
\]

Now remark from Assumption 2 that

1. \(\psi_m\) is increasing for any \(m \in Q\),
2. there is some $C > 0$ such that for any $q, q', m \in \mathbb{Q}$, $|\psi_m(q) - \psi_m(q')| \geq C^{-1}|q - q'|$. This is a consequence of the fact that the density of $\varepsilon$ is larger than some positive constant on any bounded interval.

The first point implies that $\bar{\psi}_t$ is invertible, while the second implies its inverse is $C$-Lipschitz when composed with the projection on $[q, \bar{q}]$. It follows:

$$
\mathbb{E} \left[ \left( P_{q, \bar{q}} \circ \bar{\psi}_t^{-1} \left( \frac{L(t)}{B(t)} \right) - Q \right)^2 \right] \leq \frac{C}{B(t)}.
$$

We now remove the conditioning on the values $(B(n))_n$. Recall that $B(t)$ stochastically dominates a binomial distribution of parameter $(t, \delta)$ for some positive $\delta$ thanks to Assumption 1.

$$
\mathbb{E} \left[ \left( P_{q, \bar{q}} \circ \bar{\psi}_t^{-1} \left( \frac{L(t)}{B(t)} \right) - Q \right)^2 \right] \leq \frac{2C}{B(t)} + (\bar{q} - q) \mathbb{P}(B(t) \leq \delta t/2) \leq \frac{2C}{B(t)} + (\bar{q} - q)e^{-\delta t/8}.
$$

Chernoff bound is used for the last inequality, which finally yields Lemma 1. □

A.2 Proof of Lemma 2

Define $M_t = \mathbb{E}[Q_t | \mathcal{H}_t]$. The inequality actually holds for each term individually, i.e., $\mathbb{E}[(Q_t + \theta_t - p)_+ - u_t] \leq \mathbb{E}[|M_t - Q_t|]$. Indeed,

$$
\mathbb{E}[(Q_t + \theta_t - p)_+ - u_t] = \mathbb{E}[(Q_t + \theta_t - p)(\mathbb{1}_{Q_t+\theta_t-p\geq 0} - \mathbb{1}_{M_t+\theta_t-p\geq 0})]
= \mathbb{E}[(Q_t + \theta_t - p)(\mathbb{1}_{Q_t+\theta_t-p\geq 0} - \mathbb{1}_{M_t+\theta_t-p\geq 0} - \mathbb{1}_{Q_t+\theta_t-p})].
$$

Now by distinguishing the two cases, we have:

$$
(Q_t + \theta_t - p)(\mathbb{1}_{Q_t+\theta_t-p\geq 0} - \mathbb{1}_{M_t+\theta_t-p\geq 0} - \mathbb{1}_{Q_t+\theta_t-p}) \\
\leq \begin{cases} 
(Q_t - M_t) & \text{if } Q_t + \theta_t - p \geq 0 \geq M_t + \theta_t - p, \quad (15) \\
(M_t - Q_t) & \text{if } M_t + \theta_t - p \geq 0 \geq Q_t + \theta_t - p.
\end{cases}
$$

And so this term is always smaller than $|M_t - Q_t|$ leading to Lemma 2. □

A.3 Proof of Theorem 3

Upper Bound.

Proof (Lemma 3). Assume w.l.o.g. in the following that the considered block $k$ is of quality $H$, i.e., $Q_{k+1} = H$. Note that for $t \in [t_k, \tau_k - 1]$, the posterior update given by Equation (3) yields:

$$
\pi_{t+1}(H) \geq \frac{G(Z_t, \pi_t, H)}{G(Z_t, \pi_t)} \pi_t(H) \quad \text{and} \quad \pi_{t+1}(L) \leq \frac{G(Z_t, \pi_t, L)}{G(Z_t, \pi_t)} \pi_t(L). \quad (16)
$$
It thus comes for \( t \in [t_k, \tau_k - 1] \), that 
\[
\frac{\pi_{t+1}(L)}{\pi_{t+1}(H)} \leq 2\eta^{-1} \prod_{s=t_k+1}^{t} G(Z_s, \pi_s, L) G(Z_s, \pi_s, H)
\]
as \( \pi_{t_k+1}(H) \geq \frac{n}{2} \). It has been shown in the proof of Theorem 1 that
\[
P\left[ \prod_{s=t_k+1}^{t} \frac{G(Z_s, \pi_s, L)}{G(Z_s, \pi_s, H)} > \exp \left( -\frac{(t-t_k)\delta^4}{6\gamma^2} \right) | \pi_{t_k+1}, \forall s \in [t_k+1, t], Q_s = H \right] \leq \exp \left( -\frac{(t-t_k)\delta^4}{6\gamma^2} \right).
\]

For \( n = \left\lceil \frac{6\gamma^2}{\delta^4} \ln(1/\eta) \right\rceil \), the previous inequality rewrites:
\[
P\left[ \prod_{s=t_k+1}^{t_k+n} \frac{G(Z_s, \pi_s, L)}{G(Z_s, \pi_s, H)} > 1 \bigg| \pi_{t_k+1}, \forall s \in [t_k+1, t_k+n], Q_s = H \right] \leq \eta.
\]

Note that by definition of \( \tau_k \), \( \frac{\pi_{\tau_k}(L)}{\pi_{\tau_k}(H)} \leq 1 \). This previous concentration inequality and the direct consequence of Equation (16) imply that \( P[\tau_k - t_k \geq n + 1] \leq \eta. \)

**Proof (Lemma 4).** Consider the block \( k \) in the following and assume w.l.o.g. that the quality is \( H \) on it. By definition of \( G \) and the posterior update, respectively given by Equations (2) and (3),
\[
E\left[ \frac{1}{\pi_{t+1}(H)} \bigg| Q_t = H, \pi_t \right] = \sum_{z=1}^{1} G(z, \pi_t, H) f \left( \frac{G(z, \pi_t)}{G(z, \pi_t, H) \pi_t(H)} \right),
\]
with \( f(x) = \frac{1}{2 + \frac{1}{x^2}} \). Note that \( f \) is concave on \( \mathbb{R}_+ \), so by Jensen inequality:
\[
E\left[ \frac{1}{\pi_{t+1}(H)} \bigg| Q_t = H, \pi_t \right] \leq f \left( \frac{1}{\pi_t(H)} \right).
\]

Lemma 4 then follows by induction
\[
E\left[ \frac{1}{\pi_{t+\tau_k+1}} \bigg| \tau_k, \forall s \in [\tau_k, t + \tau_k], Q_s = H \right] \leq E \left[ f \left( \frac{1}{\pi_{t+\tau_k}} \right) \bigg| \tau_k, \forall s \in [\tau_k, t + \tau_k], Q_s = H \right] \leq f \left( E \left[ \frac{1}{\pi_{t+\tau_k}} \bigg| \tau_k, \forall s \in [\tau_k, t + \tau_k], Q_s = H \right] \right) \leq f(2) = 2.
\]
The first inequality is a direct consequence of Equation (17), the second is Jensen inequality, while the third one is by induction using that \( f \) is increasing. 

**Lower Bound.** Consider the block \( k \) and assume w.l.o.g. that the quality is \( H \) during this block. The loss incurred during blocks \( k \) and \( k + 1 \) is at least \( (\tau_k - t_k + \tau_{k+1} - t_{k+1})/2 \).
Given the posterior update, $\pi_{t+1}(H) \leq c\pi_t(H)$ where $c = \max_{\pi} \frac{G(1, \pi, H)}{G(1, \pi, L)} > 1$.
As a consequence, $\tau_{k+1} - t_{k+1} \geq \min\left(\frac{\ln(2\pi_{k+1}(L))}{\ln(c)}, t_{k+2} - t_{k+1}\right)$. Assume in the following that $t_{k+2} - t_{k+1} \geq -\frac{\ln(\eta)}{\ln(c)}$, so that we actually have $\tau_{k+1} - t_{k+1} \geq -\frac{\ln(2\pi_{k+1}(L))}{\ln(c)}$.

We now bound $\ln(\pi_{t+1}(L))$ in expectation. By concavity of the logarithm,
$$\mathbb{E}[\ln(\pi_{t+1}(L)) \mid (t_n)_n, \tau_k] \leq \ln\left(\mathbb{E}[\pi_{t+1}(L) \mid (t_n)_n, \tau_k]\right).$$
Equation (12) in the proof of the upper bound yields
$$\mathbb{E}[\pi_{t+1}(L) \mid (t_n)_n, \tau_k] \leq 2 \exp\left(-\frac{(t_{k+1} - \tau_k)\delta^4}{6\gamma^2}\right) + \frac{2\eta}{1 - \exp\left(-\frac{\delta^4}{6\gamma^2}\right)}.$$
And so, with $t_{k+2} - t_{k+1} \geq -\frac{\ln(\eta)}{\ln(c)}$, we have
$$\mathbb{E}[\tau_k - t_k + \tau_{k+1} - t_{k+1} \mid (t_n)_n, \tau_k] \geq \tau_k - t_k + \left(\begin{array}{c}
- \ln\left(2 \exp\left(-\frac{(t_{k+1} - \tau_k)\delta^4}{6\gamma^2}\right) + \frac{2\eta}{1 - \exp\left(-\frac{\delta^4}{6\gamma^2}\right)}\right) \\
\text{ln}(c)
\end{array}\right) +$$
$$+ \left(\begin{array}{c}
- \ln\left(\frac{2\eta}{1 - \exp\left(-\frac{\delta^4}{6\gamma^2}\right)}\right) - \frac{1 - \exp\left(-\frac{\delta^4}{6\gamma^2}\right)}{\text{ln}(c)} \exp\left(-\frac{(t_{k+1} - \tau_k)\delta^4}{6\gamma^2}\right) \\
\text{ln}(c)
\end{array}\right).$$
Where we used that $-\ln(x + y) \geq -\ln(x) - y/x$.
When looking at the variations of the right hand side with $\tau_k$, it is minimized either when $\tau_k = t_k$ or when the second term is equal to 0, i.e., $\tau_k = \frac{6\gamma^2}{\delta^4} \ln\left(-\eta \frac{2\eta}{1 - \exp\left(-\frac{\delta^4}{6\gamma^2}\right)}\right) + t_{k+1}$. Finally this yields when $t_{k+2} - t_{k+1} \geq -\frac{\ln(\eta)}{\ln(c)}$.

$$\mathbb{E}[\tau_k - t_k + \tau_{k+1} - t_{k+1} \mid (t_n)_n] \geq \min\left(\frac{- \ln\left(\frac{2\eta}{1 - \exp\left(-\frac{\delta^4}{6\gamma^2}\right)}\right) - \frac{1 - \exp\left(-\frac{\delta^4}{6\gamma^2}\right)}{\text{ln}(c)} \exp\left(-\frac{(t_{k+1} - t_k)\delta^4}{6\gamma^2}\right)}{\text{ln}(c)},
\frac{6\gamma^2}{\delta^4} \ln\left(-\eta \frac{2\eta}{1 - \exp\left(-\frac{\delta^4}{6\gamma^2}\right)}\right) + t_{k+1} - t_k\right).$$
Case $\eta T \geq 64$. Recall that $t_{k+1} - t_k$ are i.i.d. geometric variables of parameter $\eta/2$. Lemma 6 below provides some concentration bound for the sum of such variables. Its proof is given at the end of the section.

**Lemma 6.** Denote by $Y(n, p)$ the sum of $n$ i.i.d. geometric variables of parameter $p$. We have the following concentration bounds on $Y(n, p)$:

1. For $k \leq 1$ and $kn/p \in \mathbb{N}$, $\Pr[Y(n, p) < kn/p] \leq \exp\left(-\frac{(1-1/k)^2 kn}{1+1/k}\right)$.
2. For $k \geq 1$ and $kn/p \in \mathbb{N}$, $\Pr[Y(n, p) > kn/p] \leq \exp\left(-\frac{(1-1/k)^2 kn}{2}\right)$.

Let $\alpha \in \left[\frac{\eta}{72}, \frac{1}{4}\right]$ such that $\alpha \eta T \in 2\mathbb{N}$ and note that $\mathbb{1}_{t_{k+1} - t_k \geq x}$ follows a Bernoulli distribution of parameter $(1 - \eta/2)^x$. We then have the following concentration bounds:

$$
\Pr\left[\sum_{k=1}^{\alpha \eta T} \mathbb{1}_{t_{k+1} - t_k > T}\right] \leq \exp\left(-\frac{(1-2\alpha)^2 \eta T}{4}\right) \leq \exp\left(-\frac{\eta T}{16}\right) \leq e^{-4}. \quad (19)
$$

$$
\Pr\left[\sum_{k=1}^{\alpha \eta T/2} \mathbb{1}_{t_{2k+1} - t_{2k} \geq \frac{2}{\eta}} \mathbb{1}_{t_{2k+2} - t_{2k+1} \geq \frac{-\ln(\eta)}{\ln(c)}} \leq \frac{\alpha \eta T}{4} (1 - \eta/2)^{\frac{4}{7} - \frac{\ln(\eta)}{\ln(c)}}\right] \leq \exp\left(-\frac{\alpha \eta T (1 - \eta/2)^{\frac{4}{7} - \frac{\ln(\eta)}{\ln(c)}}}{16}\right). \quad (20)
$$

The first bound is a direct consequence of Lemma 3 while the second one is an application of Chernoff bound to Bernoulli variables of parameter $(1 - \eta/2)^{\frac{4}{7} - \frac{\ln(\eta)}{\ln(c)}}$. In the following, we only consider small $\eta$, since otherwise the dependence in $\eta$ in the lower bound does not matter. Thus assume that $\eta$ is small enough so that $\frac{\alpha}{\eta} \geq \frac{\ln(\eta)}{\ln(c)}$. The second bound then becomes:

$$
\Pr\left[\sum_{k=1}^{\alpha \eta T/2} \mathbb{1}_{t_{2k+1} - t_{2k} \geq \frac{2}{\eta}} \mathbb{1}_{t_{2k+2} - t_{2k+1} \geq \frac{-\ln(\eta)}{\ln(c)}} \leq \frac{\alpha \eta T}{4} (1 - \eta/2)^{\frac{4}{7} - \frac{\ln(\eta)}{\ln(c)}}\right] \leq \exp\left(-\frac{\alpha \eta T (1 - \eta/2)^{\frac{4}{7} - \frac{\ln(\eta)}{\ln(c)}}}{16}\right).
$$

Now note that for any $\eta \in (0, 1)$, $e^{-3} \leq (1 - \eta/2)^{\frac{4}{7} - \frac{\ln(\eta)}{\ln(c)}} \leq e^{-2}$, so that the last inequality implies

$$
\Pr\left[\sum_{k=1}^{\alpha \eta T/2} \mathbb{1}_{t_{2k+1} - t_{2k} \geq \frac{2}{\eta}} \mathbb{1}_{t_{2k+2} - t_{2k+1} \geq \frac{-\ln(\eta)}{\ln(c)}} \leq \frac{\alpha \eta T}{4} e^{-\frac{3}{2}}\right] \leq \exp\left(-\frac{\alpha \eta T e^{-\frac{3}{2}}}{16}\right)
$$

$$
\leq \exp\left(-\frac{\alpha \eta T e^{-\frac{3}{2}}}{8}\right).
$$

Now note that $e^{-4} + e^{-\frac{3}{2}} < 1$ so that neither the event in Equation (19) nor in Equation (20) hold with some constant probability. In that case, Equation (19) means that the $\alpha \eta T$ first blocks fully count in the regret. Equation (20) implies that Equation (18) holds for at least $\Omega(\eta T)$ pairs of blocks and for each of them, the incurred error is at least $\Omega((\ln(1/\eta)))$. This finally implies that $\mathcal{L}_T = \Omega(\ln(1/\eta) \eta T)$. 

Case $\eta T \leq 64$. Since $\eta T = \Omega(1)$, we can consider a constant $c_0 > 0$ such that $\eta T > c_0$. In that case, the desired bound can actually be obtained on the two first blocks only. Assume w.l.o.g. for simplicity that $T$ is a multiple of $4$ and that $\eta$ is small enough so that $T/4 \geq -\frac{\ln(\eta)}{\ln(c)}$.

$$\mathbb{P}\left(t_1 - t_0 \in [T/4, T/2] \text{ and } t_2 - t_1 \in [T/4, T/2]\right) = \left((1 - \eta/2)^{T/4} - (1 - \eta/2)^{T/2}\right)^2$$

$$= e^{\frac{\eta}{4}(1-\eta/2)(1 - e^{\frac{\eta}{4}(1-\eta/2)})^2}$$

$$\geq e^{-\frac{\eta}{4}(1 - e^{-\frac{\eta}{4}})^2}.$$

With a positive probability depending only on $c_0$, the two first blocks are completed before $T$, $t_1 - t_0 \geq T/4$ and $t_2 - t_1 \geq T/4$. Equation (18) then gives that the loss incurred during the two first blocks is $\Omega\left(\ln(1/\eta)\right)$. As $\eta T = O(1)$ in this specific case, this still leads to the lower bound $L_T = \Omega\left(\ln(1/\eta)\eta T\right)$. □

**Proof (Lemma 7).** Note that the probability that the sum of $n$ i.i.d. geometric variables of parameter $p$ are smaller than $kn/p$ is exactly the probability that the sum of $kn/p$ i.i.d. Bernoulli variables are larger than $n$. We can then use the Chernoff bound on these $kn/p$ Bernoulli variables. The same reasoning also leads to the second inequality. □

### A.4 Proof of Theorem 4

This proof relies on intermediate results given by Lemma 7. Its proof can be found below.

**Lemma 7.** For any $t \in [t_k + 1, t_{k+1}]$,

1. $\pi_t^{imp}(Q_t) \leq c^{t-t_k}\pi_{t_k}^{imp}(Q_t)$;
2. $\mathbb{E}\left[\ln(1 - \pi_t(Q_t)) \mid (t_n)_n, \pi_{t_k}\right] \leq \ln(2) - (t - t_k) \frac{\delta^4}{c^{t,t_k}}$;
3. $\mathbb{P}\left[\ln(1 - \pi_t^{imp}(Q_t)) - \mathbb{E}\left[\ln(1 - \pi_t^{imp}(Q_t)) \mid (t_n)_n, \pi_{t_k}\right] \geq \lambda \gamma \sqrt{t - t_k} \mid (t_n)_n, \pi_{t_k}\right] \leq \exp\left(-2\lambda^2\right)$;

where $c = \max_{\pi, z, q, q'} \frac{G(\pi, z, q) G(\pi, z, q)}{G(z, \pi, q)}$, $\delta = \delta(L, H)$ and $\gamma = \gamma(L, H)$ as defined in Theorem 1.

Consider two successive blocks $k$ and $k + 1$. We can assume w.l.o.g. that the quality is $H$ on the block $k$ and $L$ on the block $k + 1$. Similarly to the proof of Theorem 3 define $\tau_{k+1} = \min\left\{t \in [t_{k+1} + 1, t_{k+2}] \mid \pi_t(L) \geq 1/2\right\} \cup \{t_{k+2}\}$.

The first point of Lemma 7 implies that $\tau_{k+1} - t_{k+1} \geq \min\left(t_{k+2} - t_{k+1}, \frac{-\ln(2) - \ln(\pi_{t_{k+1}}(L))}{\ln(c)}\right)$. Moreover, thanks to the second and third points of Lemma 7 with probability at least $1 - e^{-2\lambda^2}$ for some $\lambda > 0$, $-\ln(\pi_{t_{k+1}}(L)) \geq (t_{k+1} - t_k) \frac{\delta^4}{c^{t_k,t_k}} - \ln(2) - \lambda \gamma \sqrt{t_{k+1} - t_k}$. So we can actually bound $\tau_{k+1} - t_{k+1}$ in expectation:
\[ \mathbb{E}[t_{k+1} - t_{k+1} | (t_n)_n] \geq (1 - e^{-2\lambda^2}) \frac{\delta^4}{6\gamma^2 \ln(c)} \min(t_{k+2} - t_{k+1}, t_{k+1} - t_k) - 2 \ln(2) + \lambda \gamma \sqrt{\frac{t_{k+1} - t_k}{\ln(c)}}. \tag{21} \]

**Case \( \eta T \geq 64 \).** Consider \( \alpha \in \left[ \frac{\ln(2)}{2}, \frac{1}{4} \right] \) such that \( \alpha \eta T \in 2\mathbb{N}^+ \). Recall that \( t_{k+1} - t_k \) are geometric variables of parameter \( \eta/2 \). Similarly to Equations (19) and (20) in the proof of Theorem 3, we can show

1. \( \mathbb{P} \left[ \sum_{k=1}^{\alpha \eta T} (t_{k+1} - t_k) > T \right] \leq \exp \left( -\left(1 - 2\alpha \right)^2 T \right) \); 
2. \( \mathbb{P} \left[ \sum_{k=1}^{\alpha \eta T/2} \mathbb{1}_{t_{k+1} - t_{k+2} \geq \frac{T}{2}} \mathbb{1}_{t_{k+2} - t_{k+3} \geq \frac{T}{2}} \leq \frac{\alpha \eta T}{4} \left(1 - \eta/2 \right)^{\frac{T}{16}} \right] \leq \exp \left( -\frac{\alpha \eta T(1-\eta/2)}{16} \right). \)

Similarly to the proof of Theorem 3, the sum of these two probabilities is below 1, so that none of these two events can happen with probability \( \Omega(1) \). When it is the case, the first point yields that the \( \alpha \eta T \) first blocks totally count in the loss before \( T \). The second point implies, thanks to Equation (21), that the total loss is \( \Omega(T) \) in this case.

**Case \( \eta T < 64 \).** Since \( \eta T = \Omega(1) \), we can consider a constant \( c_0 > 0 \) such that \( \eta T > c_0 \). Similarly to the case \( \eta T < 64 \) in the proof of Theorem 3, we can show that with a positive probability depending only on \( c_0 \), the two first blocks are completed before \( T \) and \( \min(t_1 - t_0, t_2 - t_1) \geq T/4 \). In that case, Equation (21) yields that the loss incurred during the two first blocks is \( \Omega(T) \).

**Proof (Lemma 7).**

1) This is a direct consequence of the posterior update given by Equation (3).

2) Jensen inequality gives that

\[ \mathbb{E}[\ln(1 - \pi_t(Q_t)) | (t_n)_n, \pi_{t_k}] \leq \ln(\mathbb{E}[1 - \pi_t(Q_t) | (t_n)_n, \pi_{t_k}]). \]

Theorem 11 claims that \( \mathbb{E}[1 - \pi_t(Q_t)](t_n)_n, \pi_{t_k}] \leq 2 \exp \left( -(t - t_k) \frac{\delta^4}{6\gamma^2 \ln(c)} \right) (1 - \pi_{t_k}(Q_t)), \) leading to the second point.

3) Recall that for \( q \neq Q_t \) \( \ln(\pi_{t_k}^{\text{imp}}(q)) = \ln(\pi_{t_k}^{\text{imp}}(q)) + \sum_{s=t_k}^{t-1} \ln(G(Z_s, \pi_s, q)) \) and \( \ln(G(Z_s, \pi_s, q)) \in [Y_s, Y_s + \gamma] \) for some variable \( Y_s \). The third point is then a direct application of Azuma Hoeffding inequality.

**A.5 Proof of Lemma 5**

First fix the quality blocks \((t_n)_n\) defined by Equation (10). Note that \( \eta_t \sum_{k=1}^{B(t)} (1 - \eta_t)^{B(t) - k} \psi_M t_k(Q_{t_k}) \) is exactly the expectation of \( a^n(t) \) given \( H_t \) and \((t_n)_n\).
positive variables then gives

\[ \alpha \in \mathbb{R}_+ \]

number of reviews between consumer \( t \) and consumer \( t+1 \). It thus dominates a binomial distribution of parameters \( (t-t_i-1) \), with \( c := F_\theta(p-q) > 0 \), thanks to Assumption \( \text{I} \). As a consequence, with \( \text{Bin}(t-t_i-1, c) \) being a binomial distribution of parameters \( (t-t_i-1, c) \):

\[ \mathbb{E} \left[ (1-\eta_1)^{2B(t)-2B(t_i+1)} \right]_{(t_n)} \leq (1-\eta_1)^{c(t-t_i-1)} + \mathbb{P} \left[ \text{Bin}(t-t_i+1, c) \leq \frac{c(t-t_i-1)}{2} \right] \]

\[ \leq (1-\eta_1)^{c(t-t_i-1)} + \exp \left( -\frac{c(t-t_i-1)}{8} \right). \]
The last inequality is obtained by Chernoff bound. Combining Equations (23) and (24) then gives for 
\( h(t) := \max\{t_i | t_i < t\} \),

\[
\mathbb{E} \left[ \left( \eta \sum_{k=1}^{B(t)} (1 - \eta)^{B(t)-k} \psi_{M_k} (Q_{b_k}) - \psi_{t,\eta_1} (Q_t) \right)^2 \right] \leq (1 - \eta_1)^c (t - h(t) - 1) + \exp \left( -\frac{c(t - h(t) - 1)}{8} \right).
\]

Note that 
\( h(t) = \max\{t' | t' < t, Q_{t'} \neq Q_t \} \cup \{0\} \). By reversing the time, 
\( t - h(t) - 1 \) is thus the minimum between a geometric variable of parameter \( \eta \) and \( t - 1 \):

\[
\mathbb{E} \left[ \left( \eta \sum_{k=1}^{B(t)} (1 - \eta)^{B(t)-k} \psi_{M_k} (Q_{b_k}) - \psi_{t,\eta_1} (Q_t) \right)^2 \right] \leq \eta \sum_{k=0}^{\infty} (1 - \eta)^k \left( (1 - \eta_1)^c + \exp \left( -\frac{ck}{8} \right) \right)
\]

As \( c \in [0, 1] \), the last inequality uses that \((1 - \eta_1)^c \leq 1 - c\eta_1 \).

Noting that \( 2x^2 + 2y^2 \geq (x + y)^2 \), we can now use Equations (22) and (26) to bound the total error on a round:

\[
\mathbb{E} \left[ \left( \alpha^{\eta_1} (t) - \psi_{t,\eta_1} (Q_t) \right)^2 \right] \leq 2\eta_1 + \frac{2\eta}{\eta + c\eta_1 - c\eta_1} + \frac{2\eta}{1 - \exp (-c/8)}.
\]

The error on a single round is of order \( O \left( \eta_1 + \frac{\eta}{\eta + c\eta_1} \right) \) in average; and for \( \eta_1 = \sqrt{\eta} \), it is then \( O \left( \sqrt{\eta} \right) \) in average. Summing the square root of this term over all rounds finally yields to Lemma 3.

---

3 Taking \( \eta_1 = \sqrt{\eta/c} \) actually is a better choice, but we just focus on the dependence with \( \eta \) here.