Jawerth-Franke embeddings of Herz-type Besov and Triebel-Lizorkin spaces

Douadi Drihem *

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Abstract

In this paper we prove the Jawerth-Franke embeddings of Herz-type Besov and Triebel-Lizorkin spaces. Moreover, we obtain the Jawerth-Franke embeddings of Besov and Triebel-Lizorkin spaces equipped with power weights. An application we present new embeddings between Besov and Herz spaces.

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1 Introduction

The Herz-type Besov-Triebel-Lizorkin spaces initially appeared in the papers of J. Xu and D. Yang [21] and [22]. Several basic properties were established, such as the Fourier analytical characterisation and lifting properties. When $\alpha = 0$ and $p = q$ they coincide with the usual function spaces $F^s_{p,q}$.

The interest in these spaces comes not only from theoretical reasons but also from their applications to several classical problems in analysis. In [13], Lu and Yang introduced the Herz-type Sobolev and Bessel potential spaces. They gave some applications to partial differential equations. Also in [18], Y. Tsutsui, studied the Cauchy problem for Navier-Stokes equations on Herz spaces and weak Herz spaces.

The main aim of this paper is to prove the Jawerth-Franke embeddings in $\dot{K}^\alpha p F^s_\beta$ and $\dot{K}^\alpha p B^s_\beta$ spaces, where we use the so-called $\varphi$-transform characterization in the sense of Frazier and Jawerth. As a consequence, we present the Jawerth-Franke embeddings of Besov and Triebel-Lizorkin spaces equipped with power weights. Also, we present new embeddings between Besov and Herz spaces. All these results generalize the existing classical results on Besov and Triebel-Lizorkin spaces.

For any $u > 0, k \in \mathbb{Z}$ we set $C(u) = \{ x \in \mathbb{R}^n : u/2 \leq |x| < u \}$ and $C_k = C(2^k)$. For $x \in \mathbb{R}^n$ and $r > 0$ we denote by $B(x,r)$ the open ball in $\mathbb{R}^n$ with center $x$ and radius $r$. Let $\chi_k$, for $k \in \mathbb{Z}$, denote the characteristic function of the set $C_k$. The expression

* M’sila University, Department of Mathematics, Laboratory of Functional Analysis and Geometry of Spaces, P.O. Box 166, M’sila 28000, Algeria, e-mail: douadidr@yahoo.fr

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We denote by $|\Omega|$ the $n$-dimensional Lebesgue measure of $\Omega \subseteq \mathbb{R}^n$. For any measurable subset $\Omega \subseteq \mathbb{R}^n$ the Lebesgue space $L^p(\Omega), 0 < p \leq \infty$ consists of all measurable functions for which $\left\| f | L^p(\Omega) \right\| = \left( \int_{\Omega} |f(x)|^p \, dx \right)^{1/p} < \infty, 0 < p < \infty$ and $\left\| f \right\|_{L^\infty(\Omega)} = \text{ess-sup} |f(x)| < \infty$. If $\Omega = \mathbb{R}^n$ we put $L^p(\mathbb{R}^n) = L^p$ and $\left\| f \right\|_{L^p(\mathbb{R}^n)} = \left\| f \right\|_p$. The symbol $\mathcal{S}(\mathbb{R}^n)$ is used in place of the set of all Schwartz functions and we denote by $\mathcal{S}'(\mathbb{R}^n)$ the dual space of all tempered distributions on $\mathbb{R}^n$. We define the Fourier transform of a function $f \in \mathcal{S}(\mathbb{R}^n)$ by $\mathcal{F}(f)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) \, dx$. Its inverse is denoted by $\mathcal{F}^{-1} f$. Both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are extended to the dual Schwartz space $\mathcal{S}'(\mathbb{R}^n)$ in the usual way.

Let $\mathbb{Z}^n$ be the lattice of all points in $\mathbb{R}^n$ with integer-valued components. If $v \in \mathbb{N}_0$ and $m = (m_1, ..., m_n) \in \mathbb{Z}^n$ we denote $Q_{v,m}$ the dyadic cube in $\mathbb{R}^n$;

$$Q_{v,m} = \{(x_1, ..., x_n) : m_i \leq 2^v x_i < m_i + 1, i = 1, 2, ..., n\}.$$  

By $\chi_{v,m}$ we denote the characteristic function of the cube $Q_{v,m}$.

Given two quasi-Banach spaces $X$ and $Y$, we write $X \hookrightarrow Y$ if $X \subseteq Y$ and the natural embedding of $X$ in $Y$ is continuous. We use $c$ as a generic positive constant, i.e. a constant whose value may change from appearance to appearance.

## 2 Function spaces

We start by recalling the definition and some of the properties of the homogenous Herz spaces $\dot{K}^{\alpha,p}_q$.

**Definition 1** Let $\alpha \in \mathbb{R}, 0 < p, q \leq \infty$. The homogenous Herz space $\dot{K}^{\alpha,p}_q$ is defined by

$$\dot{K}^{\alpha,p}_q = \{ f \in L^q_{\text{loc}}(\mathbb{R}^n) \setminus \{0\} : \|f\|_{\dot{K}^{\alpha,p}_q} < \infty \},$$

where

$$\|f\|_{\dot{K}^{\alpha,p}_q} = \left( \sum_{k = -\infty}^{\infty} 2^{k\alpha p} \|f \chi_k\|_q^p \right)^{1/p},$$

with the usual modifications made when $p = \infty$ and/or $q = \infty$.

The spaces $\dot{K}^{\alpha,p}_q$ are quasi-Banach spaces and if $\min(p, q) \geq 1$ then $\dot{K}^{\alpha,p}_q$ are Banach spaces. When $\alpha = 0$ and $0 < p = q \leq \infty$ then $\dot{K}^{0,p}_q$ coincides with the Lebesgue spaces $L^p$. A detailed discussion of the properties of these spaces may be found in the papers [9], [11], [12], and references therein.

Now, we present the Fourier analytical definition of Herz-type Besov and Triebel-Lizorkin spaces and recall their basic properties. We first need the concept of a smooth dyadic resolution of unity. Let $\phi_0$ be a function in $\mathcal{S}(\mathbb{R}^n)$ satisfying $\phi_0(x) = 1$ for $|x| \leq 1$ and $\phi_0(x) = 0$ for $|x| \geq 2$. We put $\phi_j(x) = \phi_0(2^{-j}x) - \phi_0(2^{1-j}x)$ for $j = 1, 2, 3, ...$. Then $\{\phi_j\}_{j \in \mathbb{N}_0}$ is a resolution of unity, $\sum_{j=0}^{\infty} \phi_j(x) = 1$ for all $x \in \mathbb{R}^n$. Thus we obtain the
Littlewood-Paley decomposition \( f = \sum_{j=0}^{\infty} \mathcal{F}^{-1} \phi_j \ast f \) of all \( f \in S'(\mathbb{R}^n) \) (convergence in \( S'(\mathbb{R}^n) \)).

We are now in a position to state the definition of Herz-type Besov and Triebel-Lizorkin spaces.

**Definition 2** Let \( \alpha, s \in \mathbb{R}, 0 < p, q \leq \infty \) and \( 0 < \beta \leq \infty \).

(i) The Herz-type Besov space \( \dot{K}^{\alpha,p}_{q}B_{\beta}^s \) is the collection of all \( f \in S'(\mathbb{R}^n) \) such that

\[
\|f\|_{\dot{K}^{\alpha,p}_{q}B_{\beta}^s} = \left( \sum_{j=0}^{\infty} 2^{js\beta} \|\mathcal{F}^{-1} \phi_j \ast f\|_{K^{\alpha,p}_{q}}^{\beta} \right)^{1/\beta} < \infty,
\]

with the obvious modification if \( \beta = \infty \).

(ii) Let \( 0 < p, q < \infty \). The Herz-type Triebel-Lizorkin space \( \dot{K}^{\alpha,p}_{q}F_{\beta}^s \) is the collection of all \( f \in S'(\mathbb{R}^n) \) such that

\[
\|f\|_{\dot{K}^{\alpha,p}_{q}F_{\beta}^s} = \left( \sum_{j=0}^{\infty} 2^{js\beta} |\mathcal{F}^{-1} \phi_j \ast f|^{\beta} \right)^{1/\beta} < \infty,
\]

with the obvious modification if \( \beta = \infty \).

**Remark 1** Let \( s \in \mathbb{R}, 0 < p, q \leq \infty, 0 < \beta \leq \infty \) and \( \alpha > -n/q \). The spaces \( \dot{K}^{\alpha,p}_{q}B_{\beta}^s \) and \( \dot{K}^{\alpha,p}_{q}F_{\beta}^s \) are independent of the particular choice of the smooth dyadic resolution of unity \( \{\phi_j\}_{j \in \mathbb{N}_0} \) (in the sense of equivalent quasi-norms). In particular \( \dot{K}^{\alpha,p}_{q}B_{\beta}^s \) and \( \dot{K}^{\alpha,p}_{q}F_{\beta}^s \) are quasi-Banach spaces and if \( p, q, \beta \geq 1 \), then \( \dot{K}^{\alpha,p}_{q}B_{\beta}^s \) and \( \dot{K}^{\alpha,p}_{q}F_{\beta}^s \) are Banach spaces. Further results, concerning, for instance, lifting properties, Fourier multiplier and local means characterizations can be found in \([21],[22],[23]\), and \([25]\).

Now we give the definitions of the spaces \( B_{\beta}^s \) and \( F_{\beta}^s \).

**Definition 3** (i) Let \( s \in \mathbb{R} \) and \( 0 < p, \beta \leq \infty \). The Besov space \( B_{p,\beta}^s \) is the collection of all \( f \in S'(\mathbb{R}^n) \) such that

\[
\|f\|_{B_{p,\beta}^s} = \left( \sum_{j=0}^{\infty} 2^{js\beta} \|\mathcal{F}^{-1} \phi_j \ast f\|_{p}^{\beta} \right)^{1/\beta} < \infty.
\]

(ii) Let \( s \in \mathbb{R}, 0 < p < \infty \) and \( 0 < \beta \leq \infty \). The Triebel-Lizorkin space \( F_{p,\beta}^s \) is the collection of all \( f \in S'(\mathbb{R}^n) \) such that

\[
\|f\|_{F_{p,\beta}^s} = \left( \sum_{j=0}^{\infty} 2^{js\beta} |\mathcal{F}^{-1} \phi_j \ast f|^{\beta} \right)^{1/\beta} < \infty.
\]

The theory of the spaces \( B_{p,\beta}^s \) and \( F_{p,\beta}^s \) has been developed in detail in \([15],[16]\) and \([17]\) but has a longer history already including many contributors; we do not want to discuss this here. Clearly, for \( s \in \mathbb{R}, 0 < p < \infty \) and \( 0 < \beta \leq \infty \),

\[
\dot{K}^{0,p}_{p}F_{\beta}^s = F_{p,\beta}^s.
\]
We introduce the sequence spaces associated with the function spaces $K_{q}^\alpha A_{\beta}^s$. If

$$\lambda = \{ \lambda_{v,m} \in \mathbb{C} : v \in \mathbb{N}_0, m \in \mathbb{Z}^n \},$$

$\alpha, s \in \mathbb{R}, 0 < p, q \leq \infty$ and $0 < \beta \leq \infty$, we set

$$\| \lambda \|_{K_{q}^\alpha p_{\beta}^s} = \left( \sum_{r=0}^{\infty} 2^{rs\beta} \left| \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} \right|^2 \right)^{1/\beta}$$

and, with $0 < p, q < \infty$,

$$\| \lambda \|_{K_{q}^\alpha p_{f\beta}^s} = \left( \sum_{r=0}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{rs\beta} | \lambda_{v,m} |^2 \right)^{1/\beta}$$

Let $\Phi$, $\psi$, $\varphi$ and $\Psi$ satisfy

$$\Phi, \Psi, \varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$$

and

$$\text{supp} \mathcal{F} \Phi, \text{supp} \mathcal{F} \Psi \subset B(0,2), \text{supp} \mathcal{F} \varphi, \text{supp} \mathcal{F} \psi \subset B(0,2)$$

where $c > 0$. Recall that the $\varphi$-transform $S_{\varphi}$ is defined by setting $(S_{\varphi} f)_{0,m} = \langle f, \Psi_m \rangle$ where $\Psi_m(x) = \Psi(x-m)$ and $(S_{\varphi} f)_{v,m} = \langle f, \varphi_{v,m} \rangle$ where $\varphi_{v,m}(x) = 2^{m/2} \varphi(2^{m}x - m)$ and $v \in \mathbb{N}$. The inverse $\varphi$-transform $T_{\psi}$ is defined by

$$T_{\psi} \lambda = \sum_{m \in \mathbb{Z}^n} \lambda_{0,m} \Psi_m + \sum_{v=1}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} \varphi_{v,m},$$

where $\lambda = \{ \lambda_{v,m} \in \mathbb{C} : v \in \mathbb{N}_0, m \in \mathbb{Z}^n \}$, see [3].

For simplicity, in what follows, we use $\hat{K}_{p}^\alpha A_{\beta}^s$ to denote either $\hat{K}_{p}^\alpha A_{\beta}^s$ or $\hat{K}_{p}^\alpha B_{\beta}^s$. If $\hat{K}_{p}^\alpha A_{\beta}^s$ means $\hat{K}_{p}^\alpha A_{\beta}^s$ then the case $p = \infty$ is excluded. To prove the main results of this paper we need the following theorem, see [3].

**Theorem 1** Let $\alpha, s \in \mathbb{R}, 0 < p, q \leq \infty, 0 < \beta \leq \infty$ and $\alpha > -n/q$. Suppose that $\varphi$ and $\Phi$ satisfy (3)-(6). The operators $S_{\varphi} : \hat{K}_{q}^\alpha A_{\beta}^s \rightarrow \hat{K}_{q}^\alpha A_{\beta}^s$ and $T_{\psi} : \hat{K}_{p}^\alpha A_{\beta}^s \rightarrow \hat{K}_{q}^\alpha A_{\beta}^s$ are bounded. Furthermore, $T_{\psi} \circ S_{\varphi}$ is the identity on $\hat{K}_{q}^\alpha A_{\beta}^s$.

We end this section with one more lemma, which is basically a consequence of Hardy’s inequality in the sequence Lebesgue space $\ell_q$.

**Lemma 1** Let $0 < a < 1$ and $0 < q \leq \infty$. Let $\{ \varepsilon_k \}$ be a sequences of positive real numbers and denote $\delta_k = \sum_{j=0}^{k} a^{k-j} \varepsilon_j$ and $\eta_k = \sum_{j=k}^{\infty} a^{j-k} \varepsilon_j, k \in \mathbb{N}_0$. Then there exists constant $c > 0$ depending only on $a$ and $q$ such that

$$\left( \sum_{k=0}^{\infty} \delta_k^{q} \right)^{1/q} + \left( \sum_{k=0}^{\infty} \eta_k^{q} \right)^{1/q} \leq c \left( \sum_{k=0}^{\infty} \varepsilon_k^{q} \right)^{1/q}.$$
3 Jawerth embedding

The classical Jawerth embedding says that:

\[ F_{q,\infty}^{s_2} \hookrightarrow B_{s,q}^{s_1} \]

if \( s_1 - n/s = s_2 - n/q \) and \( 0 < q < s < \infty \), see e.g. [7]. We will extend this embeddings to Herz-type Besov Triebel-Lizorkin spaces. We follow some ideas of Vybíral, [19], where use the technique of non-increasing rearrangement.

Definition 4 Let \( \mu \) be the Lebesgue measure in \( \mathbb{R}^n \). If \( f \) is a measurable function on \( \mathbb{R}^n \), we define the non-increasing rearrangement of \( f \) through

\[ f^*(t) = \sup \{ \lambda > 0 : m_f(\lambda) > t \} \]

where \( m_f \) is the distribution function of \( f \).

We shall use the following properties. If \( 0 < p < \infty \), then

\[ \| f \|_p = \| f^* \mid L^p(0, \infty) \| \]  \hspace{1cm} (7)

for every measurable function \( f \). Let \( f \) and \( g \) be two non-negative measurable functions on \( \mathbb{R}^n \). If \( 1 \leq p \leq \infty \), then

\[ \| f + g \|_p \leq \| f^* + g^* \mid L^p(0, \infty) \| \]. \hspace{1cm} (8)

The proof follows from Theorems 3.4 and 4.6 in [1]. First, we will prove the discrete version of Jawerth embedding.

Theorem 2 Let \( \alpha_1, \alpha_2, s_1, s_2 \in \mathbb{R} \), \( 0 < s, p \leq \infty, 0 < q, r < \infty, \alpha_1 > -n/s \) and \( \alpha_2 > -n/q \). We suppose that

\[ s_1 - n/s - \alpha_1 = s_2 - n/q - \alpha_2. \] \hspace{1cm} (9)

Under the following assumptions

\[ 0 < q < s \leq \infty, q \leq r \quad \text{and} \quad \alpha_2 > \alpha_1 \] \hspace{1cm} (10)

or

\[ 0 < q < \min(s, p), q \leq r \leq \min(s, p) \quad \text{and} \quad \alpha_2 = \alpha_1 \] \hspace{1cm} (11)

or

\[ 0 < s \leq q < \infty, \alpha_2 + n/q > \alpha_1 + n/s \] \hspace{1cm} (12)

or

\[ 0 < s \leq q < \infty, q \leq r \leq p \leq \infty \quad \text{and} \quad \alpha_2 + n/q = \alpha_1 + n/s, \]

we have

\[ \dot{K}_q^{\alpha_2, r} f_\theta^{s_2} \hookrightarrow \dot{K}_s^{\alpha_1, p} b_r^{s_1}, \] \hspace{1cm} (13)

where

\[ \theta = \begin{cases} \begin{align*} r & \text{if } 0 < s \leq q < \infty, q \leq r \leq p \leq \infty \quad \text{and} \quad \alpha_2 + n/q = \alpha_1 + n/s, \\ \infty & \text{otherwise}. \end{align*} \end{cases} \]
Proof. Let \( \lambda \in \dot{K}^{\alpha_2, r}_{q} f^s_{p_0} \). We have
\[
\left\| \lambda \right\|_{\dot{K}^{\alpha_1, p}_s b^s_1}^r = \sum_{v=0}^{\infty} \left( \sum_{k=-\infty}^{\infty} \left( 2^{(k \alpha_1 + v s_1)} \right) \right)^{r/p} = \sum_{v=0}^{\infty} \left( \sum_{k=-\infty}^{v} \cdots \right)^{r/p} + \sum_{v=0}^{\infty} \left( \sum_{k=-v}^{\infty} \cdots \right)^{r/p} = I + II.
\]

**Step 1.** We prove our embedding under the assumption (10) and we will estimate \( I \) and \( II \), respectively. We will treat the case only where \( 0 < s, p < \infty \). The case \( s = \infty \) follows by the embedding
\[
\dot{K}^{\alpha_1, p}_{p_0} b^s r \hookrightarrow \dot{K}^{\alpha_1, p}_s b^s r
\]
for some \( 0 < q < p_0 < s \leq \infty \), see [2, Theorem 5.9].

**Estimation of \( I \).** Let \( x \in C_k \cap Q_{e, m} \) and \( y \in Q_{v, m} \). We have \( |x - y| \leq 2\sqrt{n}2^{-v} < 2^{c_n - v} \) and from this it follows that \( |y| < 2^{c_n - v} + \frac{k}{2} \leq 2^{c_n - v + 2} \), which implies that \( y \) is located in some ball \( B(0, 2^{c_n - v + 2}) \) and
\[
|\lambda_{v, m}| \leq 2^{n v} \int_{B(0, 2^{c_n - v + 2})} |\lambda_{v, m}| |\chi_{v, m}(y)| dy,
\]
where \( t > 0 \). Then for any \( x \in C_k \) we obtain
\[
\sum_{m \in \mathbb{Z}^n} |\lambda_{v, m}|^t \chi_{v, m}(x) \leq 2^{n v} \int_{B(0, 2^{c_n - v + 2})} \left| \sum_{m \in \mathbb{Z}^n} \lambda_{v, m} \chi_{v, m}(y) \right|^t dy = 2^{n v} \left\| \sum_{m \in \mathbb{Z}^n} \lambda_{v, m} \chi_{v, m} \chi_{B(0, 2^{c_n - v + 2})} \right\|_t.
\]

Consequently,
\[
2^{k \alpha_1 + v s_1} \left\| \sum_{m \in \mathbb{Z}^n} \lambda_{v, m} \chi_{v, m} \chi_k \right\|_q \leq 2^{v(s_1 + \frac{r}{q} - \frac{q}{2} - 2 \sigma)} \left\| \sum_{m \in \mathbb{Z}^n} \lambda_{v, m} \chi_{v, m} \chi_{B(0, 2^{c_n - v + 2})} \right\|_t.
\]

We may choose \( t > 0 \) such that \( \frac{1}{t} > \max(\frac{1}{q}, \frac{1}{q} + \frac{\alpha_2}{n}) \). Therefore, since \( \alpha_1 + \frac{n}{s} > 0 \),
\[
I \leq \sum_{v=0}^{\infty} 2^{v(s_1 + \frac{r}{q} - \frac{q}{2} - 2 \sigma v r sup 2^{i s_2 r} \left\| \sum_{m \in \mathbb{Z}^n} \lambda_{j, m} \chi_{j, m} \chi_{B(0, 2^{c_n - v + 2})} \right\|_t},
\]
which can be estimated by, using (9),
\[
C \sum_{v=0}^{\infty} 2^{v \alpha_2} \left( \sum_{i = -\infty}^{-v} 2^{i \sigma \alpha_2} \sup_{j \geq 0} 2^{i s_2 \sigma} \left\| \sum_{m \in \mathbb{Z}^n} \lambda_{j, m} \chi_{j, m} \chi_{i + c_n + 2} \right\|_q \right)^{r/\sigma},
\]
by Hölder’s inequality, with \( \sigma = \min(1, t) \) and \( \frac{r}{\sigma} = \frac{n}{s} - \frac{\alpha_2}{n} \). Hence Lemma 1 implies that
\[
I \leq \sum_{i = 0}^{\infty} 2^{-\alpha_2 i r} \sup_{j \geq 0} 2^{j s_2 r} \left\| \sum_{m \in \mathbb{Z}^n} \lambda_{j, m} \chi_{j, m} \chi_{2 - i + c_n} \right\|_q \lesssim \| \lambda \|_{\dot{K}^{\alpha_2, r}_{q} f^s_{p_0}}.
\]
Estimation of $II$. Our estimate use partially some decomposition techniques already used in [19]. Set

$$h_k(x) = \sup_{v \geq 0} 2^{v s_2} \sum_{m \in \mathbb{Z}^n} |\lambda_{v,m}| \chi_{v,m}(x) \chi_k(x).$$

Then

$$\|\lambda\|_{K_q^{r_1} f_{\frac{s_2}{r}}} = \left( \sum_{k=-\infty}^{\infty} 2^{k r_2} \|h_k\|_{q}^{r_1} \right)^{1/r}$$

and

$$|\lambda_{v,m}| \leq 2^{-v s_2} \inf_{x \in Q_{v,m}} h_k(x), \quad v \in \mathbb{N}_0, m \in \mathbb{Z}^n.$$ 

Using the fact that $\alpha_2 > \alpha_1$ and the assumption (9) we estimate $II$ by

$$\sum_{v=0}^{\infty} 2^{v(\frac{\alpha_2}{2} - \frac{s_2}{r} + s_2)} \sup_{k \geq -v} 2^{k r_2} \left| \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} \chi_{v,m} \chi_k \right|_r \leq \sum_{k=-\infty}^{\infty} 2^{k r_2} \left( \sum_{v=0}^{\infty} 2^{v(\frac{\alpha_2}{2} - \frac{s_2}{r} + s_2)} \sup_{k \geq -v} \left| \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} \chi_{v,m} \chi_k \right|_r \right)^{1/t},$$

where

$$q < r < \min(1, \frac{s}{r})$$

if $q < r$ and $t = 1$ if $q = r$.

We can easily prove the estimate:

$$2^{v s_2} \left| \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} \chi_{v,m} \chi_k \right|_r \leq 2^{-v n t} \sum_{m \in \mathbb{Z}^n} \left( \inf_{x \in Q_{v,m}} h_k(x) \right)^t.$$

Therefore, the sum $\sum_{v=0}^{\infty} \cdots$ in (15) can be estimated by

$$\sum_{v=0}^{\infty} 2^{-v n t} \left( \sum_{m \in \mathbb{Z}^n} \left( \inf_{x \in Q_{v,m}} h_k(x) \right)^t \right)^{r/t} \leq \sum_{v=0}^{\infty} 2^{-v n t} \left( \sum_{i=1}^{\infty} \left( (h_k)^*(2^{-v n i}) \right)^t \right)^{r/t}.$$

Using the monotonicity of $h$ and the inequality $r t < s$, the last term is bounded by

$$c \sum_{v=0}^{\infty} 2^{-v n t} \left( \sum_{l=0}^{\infty} 2^{n l} \left( (h_k)^*(2^{l-v n i}) \right)^t \right)^{r/t} \leq \sum_{v=0}^{\infty} 2^{-v n t} \sum_{l=0}^{\infty} 2^{n l (\frac{r}{t} - \frac{r}{s})} \left( (h_k)^*(2^{l-v n i}) \right)^t = c \sum_{v=0}^{\infty} 2^{-v n t} \sum_{j=-v}^{\infty} 2^{n (j+v) \frac{r}{t}} \left( (h_k)^*(2^{n j}) \right)^t = \sum_{j=-\infty}^{\infty} 2^{\frac{n j t}{r}} \left( (h_k)^*(2^{n j}) \right)^t$$
Since \( q < rt \), using the embedding \( \ell_q \hookrightarrow \ell_{rt} \), we get
\[
\sum_{j=-\infty}^{\infty} 2^{\frac{srt}{q}} ((h_k)^* (2^{nj}))^{rt} \leq \left( \sum_{j=-\infty}^{\infty} 2^{nj} ((h_k)^* (2^{nj}))^q \right)^{rt/q} = \|h_k\|_{rt}^{rt}.
\]
Consequently, we obtain \( II \lesssim \|\lambda\|_{K_q^{2r^2} F_t^{2^2}} \).

**Step 2.** We prove our embedding under the assumption (14) and we need only to estimate \( II \). Since \( q \leq r \leq \min(s, p) \), we can estimate \( II \) by (15) with 1 in place of \( t \). Using similar arguments of Step 1, we get the desired estimate. Notice that the case \( s = \infty \) follows by the embedding (13) for some \( 0 < q < p_0 < \min(s, p) \leq \infty \).

**Step 3.** We prove our embedding under the assumption (12) and again we need only to estimate \( II \). By Hölder’s inequality we get
\[
2^{es_1} \left\| \sum_{m \in \mathbb{Z}^n} |\lambda_{v,m} x_{v,m} x_k| \right\|_s \leq 2^{(\frac{2-s}{q})k+s_1} \left\| \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} x_{v,m} x_k \right\|_q.
\]
Hence \( II \) can be estimated by
\[
c \sum_{v=0}^{\infty} 2^{es_2 r} \left( \sum_{k=-v}^{\infty} 2^{k} (\alpha_1 + \frac{n}{q} - \frac{n}{q}) \left\| \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} x_{v,m} x_k \right\|_q^{r/p} \right)^{r/p} \leq \sum_{v=0}^{\infty} 2^{es_2 r} \left( \sum_{k=-v}^{\infty} 2^{(k+r)(\alpha_1 - \alpha_2 + \frac{n}{q})} \left\| \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} x_{v,m} x_k \right\|_q^{r/p} \right)^{r/p} \leq \sum_{v=-\infty}^{\infty} \left( \sum_{k=-v}^{\infty} 2^{(k+r)(\alpha_1 - \alpha_2 + \frac{n}{q})} \left\| \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} x_{v,m} x_k \right\|_q^{r/p} \right)^{r/p} \leq \left\| \lambda \right\|_{K_q^{2r^2} F_t^{2^2}}^{r}\]
by Lemma 1. If \( \alpha_2 + n/q = \alpha_1 + n/s \) and \( r \leq p \), then
\[
II \lesssim \sum_{v=0}^{\infty} 2^{es_2 r} \sum_{k=-v}^{\infty} 2^{k\alpha_2 r} \left\| \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} x_{v,m} x_k \right\|_q^{r} \leq \sum_{k=-\infty}^{\infty} 2^{k\alpha_2 r} \left\| \left( \sum_{v=0}^{\infty} 2^{es_2 r} \sum_{m \in \mathbb{Z}^n} |\lambda_{v,m}| x_{v,m} x_k \right)^{1/r} \right\|_q^{r} \leq \left\| \lambda \right\|_{K_q^{2r^2} F_t^{2^2}}^{r}\]

The proof is complete. \( \Box \)

We would like to mention that \( r \) on the right hand side of (13) is optimal. Indeed, for \( v \in \mathbb{N}_0 \) and \( N \geq 1 \), we put
\[
\lambda_{v,m}^N = \begin{cases} 2^{-(s_1 + \frac{n}{q} - \alpha_1) v} \sum_{i=1}^{N} c_i (2^{v-1}) & \text{if } m = 1, \\
0 & \text{otherwise},
\end{cases}
\]
and \( \lambda^N = \{ \lambda_{v,m}^N : v \in \mathbb{N}_0, m \in \mathbb{Z} \} \). We have
\[
\left\| \lambda^N \right\|_{K_q^{2r^2} F_t^{2^2}}^{r} = \sum_{k=-\infty}^{\infty} 2^{k\alpha_2 r} \left\| \left( \sum_{v=0}^{\infty} 2^{es_2 r} |\lambda_{v,1}| x_{v,1} \right)^{1/r} \right\|_q^{r}.
\]
We can rewrite the last statement as follows:

\[
\sum_{k=1-N}^{0} 2^{skr} \left( \sum_{v=1}^{N} 2^{(s_2-s_1+\frac{1}{2}+\alpha_1)v\theta} \chi_{v,1} \right) \frac{1}{q} \chi_k \right)^{r/q} = 0 \sum_{k=1-N}^{0} 2^{skr} \left( 2^{(s_2-s_1+\frac{1}{2}+\alpha_1)(1-k)} \chi_{1-k,1} \right)^{r/q} = cN,
\]

where the constant \( c > 0 \) does not depend on \( N \). Now

\[
\|\chi_N\|^\sigma = \sum_{v=0}^{\infty} 2^{s_1\sigma} \left( \sum_{k=-\infty}^{\infty} \|\sum_{m\in\mathbb{Z}} A_{v,m} \chi_{v,m} \chi_k \|^{p} \right)^{\sigma/p} = cN,
\]

Again we can rewrite the last statement as follows:

\[
\|\chi_N\|_{\mathcal{K}_s^{\alpha_1,p,b_s^1}} = \sum_{v=1}^{N} 2^{(s_2-s_1)+1/2} \left( \sum_{k=1-N}^{0} 2^{skr} \left( \sum_{m\in\mathbb{Z}} \|\chi_{v,m} \chi_k \|^{p} \right)^{\sigma/p} \right) = cN,
\]

where the constant \( c > 0 \) does not depend on \( N \). If the embeddings \( (13) \) holds then for any \( N \in \mathbb{N}, N^{s_2-s_1}/r \leq C \). Thus, we conclude that \( 0 < r \leq \sigma \leq \infty \) must necessarily hold by letting \( N \to +\infty \).

Using Theorems 1 and 2, we have the following Jawerth embedding.

**Theorem 3** Let \( \alpha_1, \alpha_2, s_1, s_2 \in \mathbb{R}, 0 < s, p < \infty, 0 < q, r \leq \infty, \alpha_1 > -n/s \) and \( \alpha_2 > -n/q \). Under the hypothesis of Theorem 2 we have

\[
\hat{K}_q^{\alpha_1,p} F_{s_1} \hookrightarrow \hat{K}_s^{\alpha_1,p} B_{s_1}^r, \quad (16)
\]

where

\[
\theta = \begin{cases} 
0 & \text{if } 0 < s \leq q < \infty, q \leq r \leq p \leq \infty \text{ and } \alpha_2 + n/q = \alpha_1 + n/s \\
\infty & \text{otherwise}.
\end{cases}
\]

From this theorem and the fact that \( \hat{K}_q^{\alpha_0,p} F_{\infty} = F_{q,\infty} \) and \( \hat{K}_s^{\alpha_0,p} B_{s_1}^r \hookrightarrow \hat{K}_s^{\alpha_0,p} B_{s_1}^r = B_{s_1}^r \), with \( 0 < p < s < \infty \), we obtain the following embeddings

\[
F_{s_2} \hookrightarrow \hat{K}_s^{\alpha_0,p} B_{s_1}^r \hookrightarrow B_{s_1}^r
\]

if

\[
0 < q < p < s < \infty \text{ and } s_1 - n/s = s_2 - n/q.
\]

From this theorem and the fact that \( \hat{K}_q^{\alpha_0,p} F_{s_2}^r = \hat{K}_q^{\alpha_0,p} \) for \( 1 < r, q < \infty \) and \( \alpha < n - \frac{n}{q} \), see [20] and again \( \hat{K}_s^{\alpha_0,p} B_{s_1}^r = B_{s_1}^r \) we immediately arrive at the following embedding between Herz and Besov spaces.

**Theorem 4** Let \( \alpha, s_1 \in \mathbb{R}, 0 < s \leq \infty, 1 < r, q < \infty \) and \( 0 \leq \alpha < n - \frac{n}{q} \). We suppose that \( \frac{\alpha}{s} - s_1 = \alpha + \frac{\alpha}{q} \). Let

\[
0 < q < s \leq \infty, q \leq r \text{ and } \alpha > 0
\]
or

\[ 0 < q < s \leq \infty, q \leq r \leq s \text{ and } \alpha = 0 \]

or

\[ 0 < s \leq q < \infty, \alpha > \frac{n}{s} - \frac{n}{q} \]

or

\[ 0 < s \leq q \leq \infty, q \leq r \leq s \leq \infty \text{ and } \alpha = \frac{n}{s} - \frac{n}{q}. \]

Then

\[ \dot{K}_{q}^{\alpha,r} \hookrightarrow B_{s,r}^{s_{1}}, \]

where

\[ r = 2 \text{ if } 0 < s \leq q < \infty, q \leq 2 \leq s \leq \infty \text{ and } \alpha = \frac{n}{s} - \frac{n}{q}. \]

Some embeddings between Herz spaces and homogenous Besov spaces can be found in [18].

### 4 Franke embedding

The classical Franke embedding may be rewritten as follows:

\[ B_{q,s}^{s_{2}} \hookrightarrow F_{s,\infty}^{s_{1}}, \]

if \( s_{1} - n/s = s_{2} - n/q \) and \( 0 < q < s < \infty \), see e.g. [4]. As in Section 3 we will extend this embeddings to Herz-type Besov-Triebel-Lizorkin spaces. Again, we follow some ideas of Vybíral, [19]. We will prove the discrete version of Franke embedding.

**Theorem 5** Let \( \alpha_{1}, \alpha_{2}, s_{1}, s_{2} \in \mathbb{R} \), \( 0 < s, p, q < \infty \), \( 0 < \theta \leq \infty \), \( \alpha_{1} > -\frac{n}{s} \) and \( \alpha_{2} > -\frac{n}{q} \). We suppose that

\[ s_{1} - \frac{n}{s} - \alpha_{1} = s_{2} - \frac{n}{q} - \alpha_{2}. \]

Let

\[ 0 < q < s < \infty, \alpha_{2} \geq \alpha_{1}, \quad (17) \]

or

\[ 0 < s \leq q < \infty \text{ and } \alpha_{2} + \frac{n}{q} > \alpha_{1} + \frac{n}{s}. \quad (18) \]

Then

\[ \dot{K}_{q}^{\alpha_{2},p\beta_{p}} \hookrightarrow \dot{K}_{s}^{\alpha_{1},p\beta_{s}}, \quad (19) \]

**Proof.** We prove our embedding under the conditions (17). Let \( \lambda \in \dot{K}_{q}^{\alpha_{2},p\beta_{p}} \). We have

\[
\| \lambda \|_{\dot{K}_{s}^{\alpha_{1},p\beta_{s}}}^{p} = \sum_{k=-\infty}^{\infty} 2^{k\alpha_{1}p} \left( \sum_{v} \sum_{m \in \mathbb{Z}^{n}} 2^{v\alpha_{1}p} |\lambda_{v,m}|^{p} \chi_{v,m} \chi_{k} \right)^{1/\theta} \| \lambda \|_{s}^{p}
\]

\[
= \sum_{k=-\infty}^{0} \cdots + \sum_{k=1}^{\infty} \cdots
\]

\[
= J_{1} + J_{2}.
\]
Estimation of $J_1$. Let $c_0 = 1 + \lfloor \log_2 (2\sqrt{n} + 1) \rfloor$. Obviously

$$J_1 \lesssim \sum_{k=-\infty}^{0} 2^{k\alpha_1 p} \left( \sum_{v=0}^{c_n-k+1} \cdots \right)^{1/\theta} \|s\|^p + \sum_{k=-\infty}^{0} 2^{k\alpha_1 p} \left( \sum_{v=c_n-k+2}^{\infty} \cdots \right)^{1/\theta} \|s\|^p$$

$$= T_1 + T_2.$$ 

The same analysis as in the proof of Theorem 2 shows that

$$\sum_{m \in \mathbb{Z}^n} |\lambda_{v,m}|^t \chi_{v,m}(x) \lesssim 2^{nu} \sum_{m \in \mathbb{Z}^n} |\lambda_{v,m}| \chi_{v,m}(B(0,2^{c_n-v-2})) \|t\|$$

for any $x \in C_k$. From Lemma 1, since $\alpha_1 + \frac{n}{q} > 0$, $T_1$ does not exceed

$$c \sum_{v=0}^{\infty} 2^{v(\alpha_1 - \frac{n}{q} + \frac{\alpha_2}{p})} \left\| \sum_{m \in \mathbb{Z}^n} |\lambda_{v,m}| \chi_{v,m}(B(0,2^{c_n-v-2}) \|t\|^p.$$

We may choose $t > 0$ such that $\frac{1}{t} > \max(\frac{1}{q}, \frac{1}{2} + \frac{\alpha_2}{n})$, $\sigma = \min(1, t)$ and $\frac{n}{q} = \frac{n}{t} - \frac{n}{q} - \alpha_2$. By Hölder’s inequality, this term is bounded by

$$c \sum_{v=0}^{\infty} 2^{v\frac{n}{q}p} \left( \sum_{i=-v}^{\infty} 2^{i\frac{n}{q} + \alpha_2 \sigma i} \sup_{j \geq 0} \left\| \sum_{m \in \mathbb{Z}^n} 2^{s_2 j} |\lambda_{j,m}| \chi_{j,m}(x_{1+c_n+2}) \|q \| \right)^{q/p \sigma}.$$

Using again Lemma 1, the last term is bounded by

$$c \sum_{i=0}^{\infty} 2^{-\alpha_2 ip} \sup_{j \geq 0} \left\| \sum_{m \in \mathbb{Z}^n} 2^{s_2 j} |\lambda_{j,m}| \chi_{j,m}(x_{1+c_n+2}) \|q \| \lesssim \|\lambda\|_{K_q^{\alpha_2, p, b_2}}.$$

Now we estimate $T_2$. First let us consider $\alpha_2 > \alpha_1$. We have

$$T_2 \lesssim \sum_{k=-\infty}^{0} 2^{k\alpha_2 p} \sup_{v \geq c_n+2-k} 2^{v(s_2 - \frac{n}{q} + \frac{\alpha_2}{p})p} \left\| \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} \chi_{v,m} \chi_k \right\|_s^p$$

$$= \sum_{k=-\infty}^{0} 2^{k\alpha_2 p} \sup_{v \geq c_n+2-k} 2^{v(s_2 - \frac{n}{q} + \frac{\alpha_2}{p})p} \|h_{v,k}\|_s^p.$$ 

Let us prove that

$$2^{v(s_2 - \frac{n}{q})} \|h_{v,k}\|_s \lesssim \left\| \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} \chi_{v,m} \chi_k \right\|_q = \delta,$$

where $\bar{C}_k = \cup_{i=-1}^{1} C_{k+i}$, which equivalent to

$$\int 2^{v(s_2 - \frac{n}{q})} (h_{v,k}(x)\delta^{-1})^s dx \lesssim 1.$$

Let $x \in C_k \cap Q_{v,m}$ and $y \in Q_{v,m}$ with $v \geq c_n - k + 2$. We have $|x-y| \leq 2\sqrt{n}2^{-v} < 2^{c_n-v}$ and from this it follows that $2^{k-2} < |y| < 2^{c_n-v} + 2^k < 2^{k+2}$, which implies that $y$ is located in $\bar{C}_k$. Therefore,

$$|\lambda_{v,m}|^q \chi_{v,m}(x) \lesssim 2^{nu} \int_{\bar{C}_k} |\lambda_{v,m}|^q \chi_{v,m}(y) dy.$$
if \( x \in C_k \cap Q_{v,m} \). We have

\[
\int 2^{\nu\alpha} (h_{v,k}(x) \delta^{-1})^s dx = \int (2^{-\nu \alpha} h_{v,k}(x) \delta^{-1})^s (h_{v,k}(x) \delta^{-1})^q dx \\
\leq \int (h_{v,k}(x) \delta^{-1})^q dx \\
\leq 1.
\]

Consequently,

\[
T_2 \lesssim \sum_{k=-\infty}^{0} 2^{k\alpha_1 p} \sup_{v \geq 0} 2^{v \alpha_2 p} \| h_{v,k} \|_q^p \leq \| \lambda \|_{\mathcal{L}^{\alpha_2 p} p}^p.
\]

Now let us consider \( \alpha_2 = \alpha_1 \). We can suppose that \( \theta \leq q \) and \( p \leq s \), since the opposite cases can be obtained by the fact that \( \ell_q \mapsto \ell_\theta \) and/or \( \ell_s \mapsto \ell_p \), respectively. Observe that

\[
|2^{-v} m| \leq |x - 2^{-v} m| + |x| \leq 2^k + \sqrt{n} 2^{-v} \leq 2^{k+1}
\]

and

\[
|2^{-v} m| \geq ||x - 2^{-v} m| - |x|| \geq 2^{k-1} - \sqrt{n} 2^{-v} \geq 2^{k-2}
\]

if \( x \in C_k \cap Q_{v,m} \) and \( v \geq c_n + 2 - k \). Hence \( m \) is located in

\[
\bar{C}_{k+v} = \{ m \in \mathbb{Z}^n : 2^{k+v-2} \leq |m| \leq 2^{k+v+1} \}.
\]

Therefore \( T_2 \) can be estimated by

\[
\sum_{k=-\infty}^{0} 2^{k\alpha_1 p} \left( \sum_{v=c_n+2-k}^{\infty} \sum_{m \in \bar{C}_{k+v}} 2^{v \alpha_2 p} |\lambda_{v,m}|^\theta \chi_{v,m} \right)^{1/\theta} \| p \|.
\]

Let

\[
\bar{\lambda}_{v,m_1}^{1,k} = \max_{m \in \bar{C}_{k+v}} |\lambda_{v,m}|, \quad m_1 \in \mathbb{Z}^n
\]

and (decreasing rearrangement of \( \{\lambda_{v,m} \}_{m \in \bar{C}_{k+v}} \))

\[
\bar{\lambda}_{v,m_j}^{j,k} = \max_{m \in \bar{C}_{k+v}} \sum_{i=1}^{j} |\lambda_{v,m_i}| - \sum_{i=1}^{j-1} \bar{\lambda}_{v,m_i}^{i,k}, \quad m_j \in \mathbb{Z}^n, j \geq 2.
\]

Then

\[
\sum_{m \in \bar{C}_{k+v}} |\lambda_{v,m}| \chi_{v,m} = \sum_{i=1}^{L_{k+v}} \bar{\lambda}_{v,m_i}^{i,k} \chi_{v,m_i} = f
\]

for some \( L_{k+v} \in \mathbb{N} \). It is not difficult to see that

\[
f^*(t) = \sum_{i=1}^{L_{k+v}} \bar{\lambda}_{v,m_i}^{i,k} \chi_{[B_{i-1},B_i]}(t),
\]

with

\[
B_i = \sum_{j=1}^{i} |Q_{v,m_j}| = 2^{-vn} i, \quad i = 1, \ldots, L_{k+v}.
\]
Using the properties \(\text{(7)}\) and \(\text{(8)}\) we can estimate \(\| \cdots \|_s^p\) by

\[
c \left\| \sum_{v=c_0+2-k}^{L_{k+v}} \sum_{i=1}^{2^{v s_1}} \left| \tilde{\lambda}_{v,m_i}^{i,k} \right|^\theta \tilde{X}_{v,i} g_{v,i} \right\|^{p/\theta}_{L^{s/\theta}(0, \infty)}
\]

where \(\tilde{X}_{v,i}\) is a characteristic function of the interval \((2^{-v}(i-1), 2^{-v}i)\). By duality, the last norm may be rewritten as

\[
\sup \int_0^\infty \left\| \sum_{v=c_0+2-k}^{L_{k+v}} \sum_{i=1}^{2^{v s_1}} \left| \tilde{\lambda}_{v,m_i}^{i,k} \right|^\theta \tilde{X}_{v,i}(x) g(x) dx \right\|_{L^{s/\theta}(0, \infty)}
\]

where the supremum is taken over all non-increasing non-negative measurable functions \(g\) with \(\| g \|_{L^\beta(0, \infty)} \leq 1\) and \(\beta\) is the conjugated index to \(s/\theta\). Similarly, \(\tilde{g}\) stands for the conjugated index to \(q/\theta\). Let

\[
g_{v,i} = \int_0^\infty \tilde{X}_{v,i}(x) g(x) dx.
\]

Hölder’s inequality implies that

\[
\sum_{v=c_0+2-k}^{L_{k+v}} \sum_{i=1}^{2^{v s_1}} \left| \tilde{\lambda}_{v,m_i}^{i,k} \right|^\theta \tilde{X}_{v,i} g_{v,i} \leq \sum_{v=c_0+2-k}^{L_{k+v}} \left( \sum_{i=1}^{2^{v s_2}} \left| \tilde{\lambda}_{v,m_i}^{i,k} \right|^{\theta q} \right) \left( \sum_{h=1}^{2^{v s_1-s_2}} g_{v,h} \right)^{1/q} \]

\[
\leq \left( \sum_{v=c_0+2-k}^{L_{k+v}} \left( \sum_{i=1}^{2^{v s_2}} \left| \tilde{\lambda}_{v,m_i}^{i,k} \right|^{\theta q} \right)^{s/q} \right) \left( \sum_{h=1}^{2^{v s_1-s_2}} g_{v,h} \right)^{1/\beta} \left( \sum_{v=c_0+2-k}^{L_{k+v}} \left( \sum_{h=1}^{2^{v s_1-s_2}} g_{v,h} \right)^{\beta/q} \right)^{1/\beta}.
\]

As in \(\text{(19)}\) we can prove that the second term is bounded. Clearly the first term can be estimated by

\[
c \left( \sum_{v=c_0+2-k}^{L_{k+v}} \left( \sum_{m \in C_{k+i}} 2^{v s_2} |\lambda_{v,m}|^{q/\theta} \right)^{s/q} \right)^{\theta/s} \leq c \left( \sum_{v=c_0+2-k}^{L_{k+v}} \left( \sum_{m \in C_{k+i}} \lambda_{v,m} \lambda_{v,m} \chi_{v,m} \chi_{C_k} \right)^{s/q} \right)^{\theta/s},
\]

where \(\tilde{C}_{k+i} = \bigcup_{i=-2}^{i} C_{k+i}\). Using the well-known inequality

\[
\left( \sum_{j=0}^{\infty} |a_j|^\rho \right)^{1/\rho} \leq \sum_{j=0}^{\infty} |a_j|^\rho, \quad \{a_j\}_j \subset \mathbb{C}, \quad \rho \in (0, 1], \quad (21)
\]

we obtain that \(T_2\) can be estimated by \(c \| \lambda \|_{K_{\rho}^{s_2-q_1} b_{p_2}^{s_2}}^{p}\).

**Estimation of \(J_2\).** We use the same notations as in the estimation of \(J_1\). We have

\[
J_2 \leq \sum_{k=1}^{c_k+1} \cdots + \sum_{k=c_k+2}^{\infty} \cdots
\]
As in the estimation of $T_2$, the second term can be estimated by $c \|\lambda\|_{K_q^{\alpha_2,p}}^p$. Now the first term is bounded by

$$c \sum_{k=1}^{c_n+1} 2^{k\alpha_1 p} \left( \sum_{v=0}^{c_n-k+1} \cdots \right)^{1/\theta} \|s\|_p + \sum_{k=1}^{c_n+1} 2^{k\alpha_1 p} \left( \sum_{v=c_n-k+2}^{\infty} \cdots \right)^{1/\theta} \|s\|_p \leq \|\lambda\|_{K_q^{\alpha_2,p}}^p,$$

where again we used the same arguments as in the estimation of $T_1$ and $T_2$. Using a combination of the arguments used in Step 3 of the proof Theorem 2 we prove our embedding under the conditions (18). The proof is complete. □

Also as above $p$ on the right hand side of (19) is optimal.

Using Theorems 1 and 5, we have the following Franke embedding.

**Theorem 6** Let $\alpha_1, \alpha_2, s_1, s_2 \in \mathbb{R}$, $0 < s, q < \infty$, $0 < p < \infty$, $0 < \theta \leq \infty$, $\alpha_1 > -\frac{n}{s}$ and $\alpha_2 > -\frac{n}{q}$. Under the hypothesis of Theorem 5 we have

$$\dot{K}_q^{\alpha_2,p} B_{p}^{s_2} \hookrightarrow \dot{K}_s^{\alpha_1,p} F_{\theta}^{s_1}. \quad (22)$$

We would like to mention that from this theorem we have

$$B_{q,s}^{s_2} \hookrightarrow \dot{K}_q^{0,s} B_{q}^{s_2} \hookrightarrow F_{s,\theta}^{s_1},$$

if $0 < q < s < \infty$, $0 < \theta \leq \infty$ and

$$s_1 - \frac{n}{s} = s_2 - \frac{n}{q}.$$

Also we immediately arrive at the following embedding between Herz and Besov spaces.

**Theorem 7** Let $\alpha, s_2 \in \mathbb{R}$, $1 < s, q < \infty$ and $-\frac{n}{s} < \alpha \leq 0$. We suppose that $\frac{n}{q} - \frac{n}{s} = \frac{\alpha}{q} - \frac{\alpha}{s}$. Let

$$1 < q < s < \infty,$$

or

$$1 < s \leq q < \infty \text{ and } \alpha < \frac{n}{q} - \frac{n}{s}.$$

Then

$$B_{q,q}^{s_2} \hookrightarrow \dot{K}_s^{\alpha,q}.$$

By the same examples of [2], the assumptions $s_1 - \frac{n}{s} - \alpha_1 \leq s_2 - \frac{n}{q} - \alpha_2$ and $\alpha_2 + \frac{n}{q} \geq \alpha_1 + \frac{n}{s}$ are necessary. Indeed, let $\eta \in \mathcal{S} (\mathbb{R}^n)$ be a function such that $\text{supp} \mathcal{F} \eta \subseteq \{ \xi \in \mathbb{R}^n : 3/4 < |\xi| < 1 \}$. For $x \in \mathbb{R}^n$ and $N \in \mathbb{N}$ we put $f_N (x) = \eta (2^N x)$. First we have $\eta \in \dot{K}_s^{\alpha_1,p} \cap \dot{K}_q^{\alpha_2,r} \cap \dot{K}_q^{\alpha_2,p}$. Due to the support properties of the function $\eta$ we have for any $j \in \mathbb{N}_0$

$$\mathcal{F}^{-1} \phi_j * f_N = \begin{cases} f_N, & j = N \\ 0, & \text{otherwise.} \end{cases}$$
Hence
\[ \|f_N\|_{K_s^{a_1,p}B^b_p} = 2^{s_1N} \|f_N\|_{K_s^{a_1,p}} \]
\[ = 2^{s_1N} \left( \sum_{k=-\infty}^{\infty} 2^{k\alpha_1p} \|f_N\|_{s_k}^p \right)^{1/p} \]
\[ = 2^{(a_1-n/s)N} \left( \sum_{k=-\infty}^{\infty} 2^{k\alpha_1p} \|\eta\|_{s_k}^p \right)^{1/p} \]
\[ = 2^{(a_1-a_1-n/s)N} \|\eta\|_{K_s^{a_1,p}}. \]

The same arguments give
\[ \|f_N\|_{K_s^{a_1,p}F^{a_1}_s} = 2^{(a_1-a_1-n/s)N} \|\beta\|_{K_s^{a_1,p}}, \quad \|f_N\|_{K_q^{a_2,r}F^{a_2}_q} = 2^{(s_2-a_2-n/q)N} \|\eta\|_{K_q^{a_2,r}}, \]

and
\[ \|f_N\|_{K_q^{a_2,p}B^{a_2}_p} = 2^{(s_2-a_2-n/q)N} \|\eta\|_{K_q^{a_2,p}}. \]

If the embeddings (16) and (22) hold then for any \( N \in \mathbb{N} \)
\[ 2^{(s_1-s_2-a_2-n/s+n/q)N} \leq c. \]

Thus, we conclude that \( s_1 - \frac{n}{s} - \alpha_1 \leq s_2 - \frac{n}{q} - \alpha_2 \) must necessarily hold by letting \( N \to +\infty \).

Let now \( \omega \in S(\mathbb{R}^n) \) be a function such that \( \text{supp}\mathcal{F}\omega \subset \{ \xi \in \mathbb{R}^n : |\xi| < 1 \} \). For \( x \in \mathbb{R}^n \) and \( N \in \mathbb{Z}\setminus\mathbb{N} \) we put \( f_N(x) = \omega(2^Nx) \). We have \( \omega \in \mathcal{K}_s^{\alpha_1,p} \cap \mathcal{K}_q^{\alpha_2,r} \cap \mathcal{K}_q^{a_2,p} \). It easy to see that
\[ \mathcal{F}^{-1}\phi_j \ast f_N = \begin{cases} f_N, & j = 0 \\ 0, & \text{otherwise}. \end{cases} \]

Hence
\[ \|f_N\|_{\mathcal{K}_s^{\alpha_1,p}B^b_p} = \|f_N\|_{\mathcal{K}_s^{\alpha_1,p}} = 2^{-(a_1+n/s)N} \|\omega\|_{\mathcal{K}_s^{\alpha_1,p}}. \]

The same arguments give
\[ \|f_N\|_{\mathcal{K}_s^{\alpha_1,p}F^{a_1}_s} = 2^{-(a_1+n/s)N} \|\omega\|_{\mathcal{K}_s^{\alpha_1,p}} \]
\[ \|f_N\|_{\mathcal{K}_q^{a_2,r}F^{a_2}_q} = 2^{-(a_2+n/q)N} \|\omega\|_{\mathcal{K}_q^{a_2,r}} \]

and
\[ \|f_N\|_{\mathcal{K}_q^{a_2,p}B^{a_2}_p} = 2^{-(a_2+n/q)N} \|\omega\|_{\mathcal{K}_q^{a_2,p}}. \]

If the embeddings (16) and (22) hold then for any \( N \in \mathbb{Z}\setminus\mathbb{N} \)
\[ 2^{-(a_1-a_2+n/s-n/q)N} \leq c. \]

Thus, we conclude that \( \alpha_2 + \frac{n}{q} \geq \alpha_1 + \frac{n}{s} \) must necessarily hold by letting \( N \to -\infty \).
5 Applications

In this section, we give a simple application of Theorems 3 and 6. Let \( w \) denote a positive, locally integrable function and \( 0 < p < \infty \). Then the weighted Lebesgue space \( L^p(\mathbb{R}^n, w) \) contains all measurable functions such that

\[
\left\| f \right\|_{L^p(\mathbb{R}^n, w)} = \left( \int_{\mathbb{R}^n} |f(x)|^p w(x) \, dx \right)^{1/p} < \infty.
\]

For \( q \in [1, \infty) \) we denote by \( A_q \) the Muckenhoupt class of weights, and \( A_{\infty} = \bigcup_{q \geq 1} A_q \). We refer to [6] for the general properties of these classes. Let \( w < \beta \leq \infty \) and \( 0 < p < \infty \). We define weighted Triebel-Lizorkin spaces \( F^{s}_{p,q}(w) \) to be the set of all distributions \( f \in \mathcal{S}'(\mathbb{R}^n) \) such that

\[
\left\| f \right\|_{F^{s}_{p,q}(w)} = \left\| \left( \sum_{j=0}^{\infty} 2^{js\beta} |\mathcal{F}^{-1} \varphi_{j} * f|^\beta \right) \right\|_{L^p(\mathbb{R}^n, w)}^{1/\beta}
\]

is finite. In the limiting case \( q = \infty \) the usual modification is required. Also we define weighted Besov spaces \( B^{s}_{p,q}(w) \) to be the set of all distributions \( f \in \mathcal{S}'(\mathbb{R}^n) \) such that

\[
\left\| f \right\|_{B^{s}_{p,q}(w)} = \left( \sum_{j=0}^{\infty} 2^{j\beta} \left\| \mathcal{F}^{-1} \varphi_{j} * f \right\|_{L^p(\mathbb{R}^n, w)}^\beta \right)^{1/\beta}
\]

is finite. In the limiting case \( q = \infty \) the usual modification is required. The spaces \( F^{s}_{p,q}(w) \) and \( B^{s}_{p,q}(w) \) are independent of the particular choice of the smooth dyadic resolution of unity \( \{ \varphi_{j} \}_{j \in \mathbb{N}_0} \) appearing in their definitions. They are quasi-Banach spaces, Banach spaces for \( p, q \geq 1 \), moreover for \( w \equiv 1 \in A_{\infty} \) we obtain the usual (unweighted) Besov and Triebel-Lizorkin spaces. Let \( w_\gamma \) be a power weight, i.e., \( w_\gamma(x) = |x|^\gamma \) with \( \gamma > -n \). Then in view of the fact that \( L^p = \dot{K}^{0,p}_p \), we have

\[
\left\| f \right\|_{A^{s}_{p,\beta}(w_\gamma)} \approx \left\| f \right\|_{\dot{K}^{s_{2}}_{p,\beta} A^{s}_{p,\beta}}.
\]

Applying Theorems 3 and 6 in some particular cases yields the following embeddings.

**Corollary 1** Let \( s_1, s_2 \in \mathbb{R}, 0 < q < s < \infty, 0 < \beta \leq \infty \) and \( w_{\gamma_1}(x) = |x|^{\gamma_1}, w_{\gamma_2}(x) = |x|^{\gamma_2} \), with \( \gamma_1 > -n \) and \( \gamma_2 > -n \). We suppose that

\[
s_1 - \frac{n + \gamma_1}{s} = s_2 - \frac{n + \gamma_2}{q}
\]

and

\[
\frac{\gamma_2}{q} \geq \frac{\gamma_1}{s}.
\]

Then

\[
F^{s_2}_{q,\beta}(w_{\gamma_2}) \hookrightarrow B^{s_1}_{q,\beta}(w_{\gamma_1}) \quad \text{and} \quad B^{s_2}_{q,\beta}(w_{\gamma_2}) \hookrightarrow F^{s_1}_{q,\beta}(w_{\gamma_1}).
\]
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