THREE DIMENSIONAL CASIMIR PISTON FOR MASSIVE SCALAR FIELDS

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ABSTRACT. We consider Casimir force acting on a three dimensional rectangular piston due to a massive scalar field subject to periodic, Dirichlet and Neumann boundary conditions. Exponential cut-off method is used to derive the Casimir energy in the interior region and the exterior region separated by the piston. It is shown that the divergent term of the Casimir force acting on the piston due to the interior region cancels with that due to the exterior region, thus render a finite well-defined Casimir force acting on the piston. Explicit expressions for the total Casimir force acting on the piston is derived, which show that the Casimir force is always attractive for all the different boundary conditions considered. As a function of $a$ — the distance from the piston to the opposite wall, it is found that the magnitude of the Casimir force behaves like $1/a^4$ when $a \to 0^+$ and decays exponentially when $a \to \infty$. Moreover, the magnitude of the Casimir force is always a decreasing function of $a$. On the other hand, passing from massless to massive, we find that the effect of the mass is insignificant when $a$ is small, but the magnitude of the force is decreased for large $a$ in the massive case.

1. INTRODUCTION

Casimir effect associated with piston geometry has attracted considerable interest since its introduction by Cavalcanti [1] a few years ago. The main attraction of Casimir piston is that such a geometric setup can resolve a serious divergent problem that plagues the Casimir calculations. In the conventional calculations of Casimir force inside a confined region such as a rectangular box, the nontrivial contribution of vacuum energy outside the box has been ignored, and the surface divergent terms which depends on the dimensions and geometry are discarded to obtain finite results under the pretext of some regularization schemes. It has been shown that such surface divergence cannot be removed by renormalization of physical parameters of the theory [2, 3]. Calvacanti [1] considered a new geometric configuration – a two dimensional rectangular piston, and showed that for a massless scalar field subject to Dirichlet boundary conditions, the divergent part of the Casimir force acting on the piston due to the interior and exterior regions cancel with each other and the net result is a finite attractive Casimir force without the surface divergence. This result has stimulated an interest in studying the Casimir force on pistons with different geometric setups and boundary conditions. In particular, Hertzberg et al [4] have studied the Casimir effect for electromagnetic fields in three dimensional rectangular pistons with perfectly conducting walls. It was found that the Casimir force is always attractive. This work was generalized in [5] where finite temperature effect was taken into account and pistons with general

Key words and phrases. Casimir force, rectangular piston, massive scalar field, divergence cancelation.
cross sections were considered. Around the same time, Marachevsky also carried out a similar investigation using a different approach \[6,7,8\]. In \[9\], Edery studied the case of a massless scalar field with Dirichlet boundary conditions for three dimensional rectangular pistons. He also found that the force on the piston is always attractive, in contrast to the fact that the regularized Casimir force for a three dimensional rectangular cavity for massless scalar field with Dirichlet boundary conditions can be attractive or repulsive depending on the relative size of the cavity. This work was later generalized in \[10\] to massless scalar fields with Dirichlet and Neumann boundary conditions in any dimensions. In all the above scenarios, the Casimir force was found to be attractive. In \[11\], Barton found out that when weakly reflecting dielectric materials are used, the Casimir force on the three dimensional piston can become repulsive when the plate separation is sufficiently large. On the other hand, the work of Zhai and Li \[12\] showed that in the case of mixed boundary conditions (one plate with Dirichlet boundary conditions and one with Neumann boundary conditions), the Casimir force on a rectangular piston in one, two and three dimensions is always repulsive. More recently, Casimir effect has been investigated for electromagnetic fields with perfect magnetic conditions in rectangular pistons of arbitrary dimensions \[13\] and for massless scalar fields with Dirichlet boundary conditions in pistons inside space-time with extra compactified dimensions \[14\].

To the best of our knowledge, no work has been carried out on the Casimir effect for massive scalar fields in the piston setting. In this work, we consider a massive scalar field in a three dimensional rectangular piston subject to periodic boundary conditions, Dirichlet boundary conditions as well as Neumann boundary conditions. We use exponential cut-off method to compute the cut-off dependent Casimir energy inside and outside the piston. We show that the sum of the Casimir energies has a divergent part that does not depend on the position of the piston, and therefore, the force acting on the piston is finite, without any regularization. Explicit formulas for the Casimir force are derived, which can be written as an infinite convergent sums of Bessel functions. From the formulas, it is easy to deduce that the Casimir force is always attractive under all the different boundary conditions. We also proved a stronger result: as a function of \(a\) — the separation between the piston and the opposite wall, the magnitude of the Casimir force behaves like \(1/a^4\) when \(a \to 0^+\) and it decays exponentially when \(a \to 0\). Moreover, the magnitude of the force is always decreasing from infinity to 0. The mass effect to the Casimir force is considered and it is found that the mass effect is significant only when \(a\) is large. Some numerical simulations have been carried out.

2. Cancellation of divergence in three dimensional Casimir piston

Consider a 3-dimensional rectangular piston in the form of a rectangular cavity \([0, L_1] \times [0, L_2] \times [0, L_3]\) separated by a plane \(x_1 = a\) (the piston) into two regions: the interior region \([0, a] \times [0, L_2] \times [0, L_3]\) and the exterior region \([a, L_1] \times [0, L_2] \times [0, L_3]\) (see Figure 1). At the end we are going to let \(L_1 \to \infty\) so that the exterior region becomes an open region. We want to compute the Casimir force acting on the piston due to a massive \((m > 0)\) scalar field \(\phi(x, t)\) satisfying the Klein–Gordon equation in Minkowski space-time:

\[
(\Box + m^2) \phi(x, t) = 0, \quad (x, t) \in \mathbb{R}^{3+1},
\]
Figure 1: The three dimensional rectangular pistons

where

\[ \Box = \frac{\partial^2}{\partial t^2} - \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}. \]

The periodic boundary condition (bc), Dirichlet bc and Neumann bc will be considered.

The Casimir energy of the piston system is the sum of the Casimir energy of the internal region \( E_{\text{cas}}(a, L_2, L_3; m) \) and the Casimir energy of the external region \( E_{\text{cas}}(L_1 - a, L_2, L_3; m) \). The Casimir force acting on the piston is then obtained by

\[ (2.2) \quad F_{\text{Cas}} = \lim_{L_1 \to \infty} -\frac{\partial}{\partial a} \left( E_{\text{Cas}}(a, L_2, L_3; m) + E_{\text{Cas}}(L_1 - a, L_2, L_3; m) \right). \]

In the periodic bc case, the eigenmodes of the \( (d+1) \) - dimensional field \( \phi(x, t) \) in a \( d \)-dimensional rectangular cavity \([0, L_1] \times \cdots \times [0, L_d]\) satisfying (2.1) (with \( 3 \) replaced by \( d \)) are given by

\[ \omega^P_k = \sqrt{\sum_{i=1}^{d} \left[ \frac{2\pi}{L_i} \right]^2 + m^2}, \quad k \in \mathbb{Z}^d; \]

and the Casimir energy is defined as the divergent sum

\[ (2.3) \quad E_{\text{Cas}}^P(L_1, \ldots, L_d; m) = \frac{1}{2} \sum_{k \in \mathbb{Z}^d} \omega^P_k. \]

For the Dirichlet bc and Neumann bc cases, the eigenmodes are respectively

\[ \omega^D_k = \sqrt{\sum_{i=1}^{d} \left[ \frac{\pi}{L_i} \right]^2 + m^2}, \quad k \in \mathbb{N}^d; \]

and

\[ \omega^N_k = \sqrt{\sum_{i=1}^{d} \left[ \frac{\pi}{L_i} \right]^2 + m^2}, \quad k \in (\mathbb{N} \cup \{0\})^d. \]
Therefore when $d = 3$ \cite{15},

\begin{equation}
E_{\text{Cas}}^{D/N}(L_1, L_2, L_3; m) = \frac{1}{8} \left\{ E_{\text{Cas}}^P(2L_1, 2L_2, 2L_3; m) \right. \\
\left. + \left[ E_{\text{Cas}}^P(2L_1, 2L_2; m) + E_{\text{Cas}}^P(2L_1, 2L_3; m) + E_{\text{Cas}}^P(2L_2, 2L_3; m) \right] \\
+ E_{\text{Cas}}^P(2L_1; m) + E_{\text{Cas}}^P(2L_2; m) + E_{\text{Cas}}^P(2L_3; m) - \frac{1}{2}m \right\}.
\end{equation}

Different regularization schemes have been employed to define a regularized Casimir energy from (2.3). We adopt here the exponential cut-off method which allows us to retain the divergent terms. Define the $\lambda$-dependent Casimir energy by

\begin{equation}
E_{\text{Cas}}^P(\lambda; L_1, \ldots, L_d; m) = \frac{1}{2} \sum_{k \in \mathbb{Z}^d} \omega_k^P e^{-\lambda \omega_k^P}.
\end{equation}

In the Appendix A, we show that

\begin{equation}
E_{\text{Cas}}^P(\lambda; L_1, \ldots, L_d; m) = E_{\text{Cas}, \text{div}}^P(\lambda; L_1, \ldots, L_d; m) + E_{\text{Cas}, \text{reg}}^P(L_1, \ldots, L_d; m) + O(\lambda),
\end{equation}

where for $d = 1, 2$ and 3, the $\lambda \to 0^+$ divergent term $E_{\text{Cas}, \text{div}}^P(\lambda; L_1, \ldots, L_d; m)$ and the regularized Casimir energy $E_{\text{Cas}, \text{reg}}^P(L_1, \ldots, L_d; m)$ are given respectively by

\begin{equation}
E_{\text{Cas}, \text{div}}^P(\lambda; L_1; m) = \frac{L_1}{2\pi} \lambda^{-2} - \frac{m^2 L_1}{4\pi} \log \lambda;
\end{equation}

\begin{equation}
E_{\text{Cas}, \text{reg}}^P(\lambda; L_1; m) = -\frac{m^2 L_1}{4\pi} \left( \log m - \log 2 - \frac{1}{4} + \gamma \right) - \frac{m}{\pi} \sum_{k=1}^{\infty} k^{-1} K_1(mkL_1);
\end{equation}

\begin{equation}
E_{\text{Cas}, \text{div}}^P(\lambda; L_1, L_2; m) = \frac{L_1 L_2}{2\pi} \lambda^{-3};
\end{equation}

\begin{equation}
E_{\text{Cas}, \text{reg}}^P(\lambda; L_1, L_2; m) = -\frac{L_1 L_2}{12\pi} m^3 - \frac{L_1 L_2}{\sqrt{8\pi^3}} m^{3/2} \times \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} \left( \sum_{i=1}^{2} \left| k_i L_i \right|^2 \right)^{-\frac{3}{2}} K_{3/2} \left( m, \sqrt{\sum_{i=1}^{2} \left| k_i L_i \right|^2} \right);
\end{equation}

\begin{equation}
E_{\text{Cas}, \text{div}}^P(\lambda; L_1, L_2, L_3; m) = \frac{3L_1 L_2 L_3}{2\pi^2} \lambda^{-4} - \frac{L_1 L_2 L_3}{8\pi^2} m^2 \lambda^{-2} + \frac{L_1 L_2 L_3}{32\pi^2} m^4 \log \lambda;
\end{equation}

\begin{equation}
E_{\text{Cas}, \text{reg}}^P(\lambda; L_1, L_2, L_3; m) = -\frac{L_1 L_2 L_3}{12\pi} m^2 - \frac{L_1 L_2 L_3}{2\pi^2} m^{1/2} \times \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \left( \sum_{i=1}^{3} \left| k_i L_i \right|^2 \right)^{-\frac{3}{2}} K_{3/2} \left( m, \sqrt{\sum_{i=1}^{3} \left| k_i L_i \right|^2} \right).
\end{equation}
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\[ E_{\text{Cas, reg}}^P(L_1, L_2, L_3; m) = \frac{L_1 L_2 L_3}{32\pi^2} m^4 \left( -\frac{1}{2} + \gamma + \log m - \log 2 \right) - \frac{L_1 L_2 L_3}{4\pi^2} m^2 \]

\[ \times \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \left( \sum_{i=1}^3 |k_i L_i|^2 \right)^{-1} K_2 \left( m \sqrt{\sum_{i=1}^3 |k_i L_i|^2} \right). \]

Here \( \gamma \) is the Euler constant and \( K_\nu(z) \) is the modified Bessel function. From these formulas and \( (2.4) \), we can compute the cut-off dependent Casimir energies for massive scalar fields inside three dimensional rectangular cavities under various boundary conditions. Conventionally, a finite Casimir energy is defined by taking \( \lambda \to 0^+ \) limit in the regular term of the Casimir energy. We would like to comment that the result will agree with the result derived by zeta regularization method (see Appendix B) if one defines the normalization constant \( \mu \) appearing in the zeta regularization method as \( e^{-\gamma} \).

Now we analyze the divergent term of the Casimir energy. In the periodic bc case, we observe that the divergent term depends linearly on the volume \( V = L_1 L_2 L_3 \) of the rectangular cavity. However, for the cases of Dirichlet bc and Neumann bc, \( (2.4) \) shows that the divergent term can be written in the form

\[ E_{\text{Cas, div}}^{D/N}(\lambda; L_1, L_2, L_3; m) = \mathcal{D}_V(\lambda; m) L_1 L_2 L_3 \pm \mathcal{D}_S(\lambda; m)(L_1 L_2 + L_2 L_3 + L_1 L_3) + \mathcal{D}_L(\lambda; m)(L_1 + L_2 + L_3). \]

In other words, besides the bulk divergence \( \mathcal{D}_V(\lambda; m) L_1 L_2 L_3 \) which only depends on the volume, there are also surface divergence \( \mathcal{D}_S(\lambda; m)(L_1 L_2 + L_2 L_3 + L_1 L_3) \) and divergence due to corners \( \mathcal{D}_L(\lambda; m)(L_1 + L_2 + L_3) \).

For the piston geometry, it is easy to see from \( (2.10) \) that the divergent term for the sum of the Casimir energy of the interior region and the exterior region given by \( E_{\text{Cas}}^{P/D/N}(\lambda; a, L_2, L_3; m) + E_{\text{Cas}}^{P/D/N}(\lambda; L_1 - a, L_2, L_3; m) \) depends only on \( L_1, L_2, L_3 \), but not on the position of the piston \( x_1 = a \). In other words, the divergent parts of the Casimir energies in the interior and exterior regions contribute Casimir force of same magnitude but opposite signs to the piston, and thus cancel with each other. This renders a finite quantity for the Casimir force acting on the piston given by \( (2.2) \), which can be rewritten as

\[ F_{\text{Cas}}^{P/D/N} = \lim_{L_1 \to \infty} -\frac{\partial}{\partial a} \left( E_{\text{Cas, reg}}^{P/D/N}(\lambda = 0; a, L_2, L_3; m) + E_{\text{Cas, reg}}^{P/D/N}(\lambda = 0; L_1 - a, L_2, L_3; m) \right). \]

3. Analysis of the Casimir Force

From \( (2.7), (2.8) \) and \( (2.9) \), one observes that the regularized Casimir energy \( E_{\text{Cas, reg}}^P(\lambda = 0; L_1, \ldots, L_d; m), d = 1, 2, 3 \) inside a rectangular cavity can be written
as a term $\mathcal{B}_d(L_2, \ldots, L_d; m) L_1$ which depends linearly on $L_1$, plus a Bessel series

$$R_d(L_1, \ldots, L_d; m) = -\frac{\prod_{i=1}^d L_i}{(2\pi)^{d-1}} \sum_{k_1 \in \mathbb{Z} \setminus \{0\}} \left( \sum_{i=1}^d |k_i L_i|^2 \right)^{-\frac{d+1}{2}} \times K_{\frac{d+1}{2}} \left( m \sqrt{\sum_{i=1}^d |k_i L_i|^2} \right).$$

Notice that the summation over $k \in \mathbb{Z}^d \setminus \{0\}$ in (2.7), (2.8) and (2.9) is decomposed into summation over $k_1 = 0, (k_2, \ldots, k_d) \in \mathbb{Z}^{d-1} \setminus \{0\}$ and summation over $k_1 \neq 0, (k_2, \ldots, k_d) \in \mathbb{Z}^{d-1}$. The term that involves summation over $k_1 = 0, (k_2, \ldots, k_d) \in \mathbb{Z}^{d-1} \setminus \{0\}$ depends on $L_1$ linearly and is therefore combined into the term $\mathcal{B}_d(L_2, \ldots, L_d; m) L_1$. As in the case of the divergent terms, for periodic bc as well as Dirichlet bc and Neumann bc, the sum of the terms $\mathcal{B}_d(L_2, \ldots, L_d; m) L_1$ of the interior region and exterior region does not depend on the position of the piston and thus the corresponding Casimir force cancels with each other. In other words, the Casimir force acting on the three dimensional piston is given by

$$F_{\text{Cas}}^P(a; L_2, L_3; m) = -\lim_{L_1 \to -\infty} \frac{\partial}{\partial a} \left\{ R_3(a, L_2, L_3; m) + R_3(L_1 - a, L_2, L_3; m) \right\}$$

in the case of periodic bc; whereas for Dirichlet and Neumann bc, (2.4) shows that

$$F_{\text{Cas}}^{D/N}(a; L_2, L_3; m)$$

is given by

$$= -\frac{1}{8} \lim_{L_1 \to -\infty} \frac{\partial}{\partial a} \left( R_3(2a, 2L_2, 2L_3; m) + R_3(2L_1 - 2a, 2L_2, 2L_3; m) \right)$$

$$+ R_2(2a, 2L_2; m) - R_2(2L_1 - 2a, 2L_2; m) + R_2(2a, 2L_3; m)$$

$$+ R_2(2L_1 - 2a, 2L_3; m) + R_1(2a; m) + R_1(2L_1 - 2a; m).$$

Using the formula (106), #3.478, no. 4),

$$\int_0^\infty t^{\pm \nu - 1} \exp \left( -\alpha t - \frac{\beta}{t} \right) dt = 2 \left( \frac{\beta}{\alpha} \right)^{\frac{\nu}{2}} K_\nu(2\sqrt{\alpha \beta}),$$

we have

$$R_d(L_1, \ldots, L_d; m) = -\prod_{i=1}^d \frac{L_i}{2^{d+2} \pi^{d+1}} \int_0^\infty t^{-\frac{d+3}{2}} \sum_{k_1 \in \mathbb{Z} \setminus \{0\}} \exp \left( -tm^2 - \frac{1}{4t} \sum_{i=1}^d |k_i L_i|^2 \right) dt.$$  

From here, we see that $R_d(L_1, \ldots, L_d; m)$ decays exponentially as $L_1 \to \infty$. Taking derivative with respect to $a$ and letting $L_1 \to \infty$, we have

$$\lim_{L_1 \to -\infty} -\frac{\partial}{\partial a} R_d(L_1 - a, \ldots, L_d; m) = 0.$$  

In other words, there is no contribution to the Casimir force on the piston from the terms $R_d(L_1 - a, L_2, \ldots, L_d; m)$ of the exterior region. Therefore, the net contribution to the Casimir force on the piston are due to the terms $R_d(a, L_2, \ldots, L_d; m)$.
of the interior region. More precisely,

$$F_{\text{Cas}}^P(a; L_2, L_3; m) = -\frac{\partial}{\partial a} R_3(a, L_2, L_3; m),$$

and

$$F_{\text{Cas}}^{D/N}(a; L_2, L_3; m) = \frac{1}{8} \left\{ -\frac{\partial}{\partial a} R_3(2a, 2L_2, 2L_3; m) \pm \frac{\partial}{\partial a} R_2(2a; 2L_2; m) \pm \frac{\partial}{\partial a} R_2(2a; 2L_3; m) - \frac{\partial}{\partial a} R_1(2a; m) \right\}.$$  

Now applying the Jacobi inversion formula

$$\sum_{k=-\infty}^{\infty} e^{-tk^2} = \sqrt{\frac{\pi}{t}} \sum_{k=-\infty}^{\infty} e^{-\frac{2\pi^2}{t}}$$

to the summation over \((k_2, \ldots, k_d) \in \mathbb{Z}^{d-1}\) in (3.4), we find that

$$R_d(a, L_2, \ldots, L_d; m)$$

$$= -\frac{a}{8\pi} \int_0^\infty t^{-2} \sum_{\substack{k_1 \in \mathbb{Z} \setminus \{0\} \\ (k_2, \ldots, k_d) \in \mathbb{Z}^{d-1}}} \exp \left( -tm^2 - 4t\pi^2 \sum_{i=2}^d \left[ \frac{k_i}{L_i} \right] \right) \sqrt{\frac{2\pi k_1}{L_1}} K_1 \left( \sqrt{\frac{2\pi k_1}{L_1}} \right).$$

Eq. (3.5) then shows that in the periodic bc case,

$$F_{\text{Cas}}^P(a; L_2, L_3; m)$$

$$= \frac{1}{2\pi} \sum_{k_1=1}^{\infty} \sum_{(k_2, k_3) \in \mathbb{Z}^2} \frac{1}{k_1} \left[ m^2 + \sum_{i=2}^{3} \left[ \frac{\pi k_i}{L_i} \right] ^2 \right] K_1 \left( \sqrt{\frac{\pi k_1}{L_1}} \right).$$

On the other hand, using the fact that if \(f(k_2, k_3)\) is a totally even function, i.e. \(f(\pm k_2, \pm k_3) = f(k_2, k_3)\), then

$$\sum_{(k_2, k_3) \in \mathbb{Z}^2} \sum_{k_2, k_3 \in \mathbb{Z}} f(k_2, k_3) = \sum_{(k_2, k_3) \in \mathbb{Z} \times \mathbb{Z}} f(k_2, k_3);$$

Eqs. (3.6) and (3.8) give the expression

$$F_{\text{Cas}}^{D/N}(a; L_2, L_3; m)$$

$$= \frac{1}{2\pi} \sum_{k_1=1}^{\infty} \sum_{(k_2, k_3) \in \mathbb{N}^2/(\mathbb{N}\cup\{0\})^2} \frac{1}{k_1} \left[ m^2 + \sum_{i=2}^{3} \left[ \frac{\pi k_i}{L_i} \right] ^2 \right] K_1 \left( \sqrt{\frac{2\pi k_1}{L_1}} \right).$$
for the Casimir force in the Dirichlet bc and Neumann bc cases. Now using the formula [16], #8.486, no. 12

\[
\frac{d}{dz} K_\nu(z) = -\frac{\nu}{z} K_\nu(z) - K_{\nu-1}(z),
\]

we obtain finally the explicit expression for the Casimir force in the periodic bc, Dirichlet bc and Neumann bc cases:

\[
F_{\text{Cas}}^P(a; L_2, L_3; m) = -\frac{1}{\pi a} \sum_{k_1=1}^{\infty} \sum_{(k_2,k_3) \in \mathbb{Z}^2} \frac{1}{k_1} \left[ m^2 + \sum_{i=2}^{3} \left( \frac{2\pi k_i}{L_i} \right)^2 \right] K_0 \left( a k_1 \sqrt{m^2 + \sum_{i=2}^{3} \left( \frac{2\pi k_i}{L_i} \right)^2} \right) - \frac{1}{\pi a} \sum_{k_1=1}^{\infty} \sum_{(k_2,k_3) \in \mathbb{Z}^2} \left( m^2 + \sum_{i=2}^{3} \left( \frac{2\pi k_i}{L_i} \right)^2 \right) K_0 \left( a k_1 \sqrt{m^2 + \sum_{i=2}^{3} \left( \frac{2\pi k_i}{L_i} \right)^2} \right);
\]

\[
F_{\text{Cas}}^{D/N}(a; L_2, L_3; m) = -\frac{1}{2\pi a} \sum_{k_1=1}^{\infty} \sum_{(k_2,k_3) \in \mathbb{Z}^2 \cap \mathbb{N}^2} \frac{1}{k_1} \left[ m^2 + \sum_{i=2}^{3} \left( \frac{\pi k_i}{L_i} \right)^2 \right] K_1 \left( 2a k_1 \sqrt{m^2 + \sum_{i=2}^{3} \left( \frac{\pi k_i}{L_i} \right)^2} \right) - \frac{1}{\pi a} \sum_{k_1=1}^{\infty} \sum_{(k_2,k_3) \in \mathbb{Z}^2 \cap \mathbb{N}^2} \left( m^2 + \sum_{i=2}^{3} \left( \frac{\pi k_i}{L_i} \right)^2 \right) K_0 \left( 2a k_1 \sqrt{m^2 + \sum_{i=2}^{3} \left( \frac{\pi k_i}{L_i} \right)^2} \right).
\]

These formulas are useful for studying the large–\(m\) and large–\(a\) behavior of the Casimir force. More precisely, since the modified Bessel function \(K_\nu(z)\) decays exponentially as \(z \to \infty\), with leading term

\[
K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z},
\]

Eqs. (3.10) and (3.11) show that for fixed \(m, L_2\) and \(L_3\), in the large \(a\) \((a \gg 1)\) limit, the Casimir force \(F_{\text{Cas}}^{P/D/N}(a; L_2, L_3; m)\) decays exponentially with leading terms being

\[
F_{\text{Cas}}^P(a; L_2, L_3; m) \sim -m \sqrt{\frac{2m}{\pi a}} e^{-am},
\]

\[
F_{\text{Cas}}^{N}(a; L_2, L_3; m) \sim -\frac{m}{2} \sqrt{\frac{m}{\pi a}} e^{-2am},
\]

\[
F_{\text{Cas}}^{D}(a; L_2, L_3; m) \sim -\frac{1}{2} \sqrt{\frac{1}{\pi a} \left( m^2 + \frac{\pi^2}{L_2^2} + \frac{\pi^2}{L_3^2} \right) \frac{3}{4}} \exp \left( -2a \sqrt{m^2 + \frac{\pi^2}{L_2^2} + \frac{\pi^2}{L_3^2}} \right)
\]

respectively. In other words, for fixed \(L_2\) and \(L_3\), when the piston is very far away from the opposite wall, the Casimir force is an exponentially weak attractive force. In the massless \(m \to 0^+\) limit, the Casimir force \(F_{\text{Cas}}^{P}(a; L_2, L_3; 0)\) for Dirichlet bc still decays exponentially with respect to \(a\). However, the Casimir force
Figure 2: These figures show the dependence of the Casimir force $F_{\text{Cas}}^{P/D/N}(a; L_2, L_3; m)$ on $a$ when $L_2 = L_3 = 1$ and $m = 0.1, 1, 10, 30$ respectively.

$F_{\text{Cas}}^{P}(a; L_2, L_3; 0)$ and $F_{\text{Cas}}^{N}(a; L_2, L_3; 0)$ for periodic bc and Neumann bc becomes polynomially decay with order $1/a^2$.

Another advantage of the explicit formulas (3.10) and (3.11) is that since the Bessel function $K_\nu(z)$ is positive for all $z > 0$, Eqs. (3.10) and (3.11) show immediately that the sign of the Casimir force $F_{\text{Cas}}^{P/D/N}(a; L_2, L_3; m)$ for all the three boundary conditions is negative, and therefore they are all attractive forces. This extends the known results about the massless scalar fields [1, 9, 10]. Extension of this result to massive scalar fields in arbitrary dimensions will be considered in a future work. Here we would like to remark that as is shown in Appendix B, the Casimir force acting on the piston due to the regularized Casimir energy of the interior region alone can be either attractive or repulsive for any of the boundary conditions. It is the total effect of the Casimir force from both the interior and exterior regions that gives a net finite force which is attractive.

For the behavior of the Casimir force when the separation distance $a$ is small compared to $L_2$ and $L_3$, we apply again the formula (3.7) to the summation over $k_1$ term of (3.3). The detailed derivation is given in the Appendix C. Here we present...
the end results:

\[(3.12)\]

\[
F_{\text{Cas}}^P (a; L_2, L_3; m) = \frac{L_2 L_3}{2 \pi^2 a^2} m^2 \sum_{k_1 = 1}^{\infty} \frac{1}{k_1^2} K_2(2amk_1) - \frac{L_2 L_3}{2 \pi^2 a^3} m^3 \sum_{k_1 = 1}^{\infty} \frac{1}{k_1} K_3(2amk_1) - \frac{2 \sqrt{2} \pi L_2 L_3}{a^2} \sum_{k_1 = 1}^{\infty} \sum_{(k_2,k_3) \in \mathbb{Z}^2 \setminus \{0\}} \frac{k_2^2 (m^2 + \frac{4 \pi^2 k_2^2}{a^2})^{1/4}}{(m^2 + \frac{4 \pi^2 k_2^2}{a^2}) \left( \sum_{i=2}^{\infty} \frac{[k_i L_i]^2}{[k_i L_i]^2} \right)} K_{1/2}(K_2(2amk_1))
\]

for the periodic bc case; and

\[(3.13)\]

\[
F_{\text{Cas}}^{D/N} (a; L_2, L_3; m) = \frac{L_2 L_3}{8 \pi^2 a^2} m^2 \sum_{k_1 = 1}^{\infty} \frac{1}{k_1^2} K_2(2amk_1) - \frac{L_2 L_3}{4 \pi^2 a^3} m^3 \sum_{k_1 = 1}^{\infty} \frac{1}{k_1} K_3(2amk_1) + \frac{(L_2 + L_3) m^{3/2}}{8 \pi^{3/2} a^{3/2}} \sum_{k_1 = 1}^{\infty} k_1^{-3/2} K_{3/2}(2amk_1) \pm \frac{(L_2 + L_3) m^{5/2}}{4 \pi^{3/2} a^{1/2}} \sum_{k_1 = 1}^{\infty} k_1^{-1/2} K_{5/2}(2amk_1)
\]

\[
+ \frac{1}{2 \pi} \frac{m}{a} \sum_{k_1 = 1}^{\infty} \frac{1}{k_1} K_1(2amk_1) - \frac{m^2}{4 \pi} \sum_{k_1 = 1}^{\infty} K_2(2amk_1)
\]

\[
+ \frac{m^{3/2}}{8 \pi^{3/2} L_2^{1/2}} \sum_{k_1 = 1}^{\infty} \sum_{k_2 = 0}^{\infty} k_2^{-3/2} K_{3/2}(2mk_2 L_2) \pm \frac{m^{3/2}}{8 \pi^{3/2} L_3^{1/2}} \sum_{k_1 = 1}^{\infty} \sum_{k_3 = 0}^{\infty} k_3^{-3/2} K_{3/2}(2mk_3 L_3)
\]

\[
- \frac{L_2 L_3}{16 \pi^2 a^2} m^2 \sum_{(k_2,k_3) \in \mathbb{Z}^2 \setminus \{0\}} \left( \sum_{i=2}^{\infty} \frac{[k_i L_i]^2}{[k_i L_i]^2} \right)^{-1} K_2 \left( 2m \left( \sum_{i=2}^{\infty} \frac{[k_i L_i]^2}{[k_i L_i]^2} \right)^{1/4} \right) \left( m^2 + \frac{\pi^2 k_2^2}{a^2} \right) K_0 \left( 2k_2 L_2 \sqrt{m^2 + \frac{\pi^2 k_2^2}{a^2}} \right)
\]

\[
+ \frac{\sqrt{\pi} L_2 L_3}{4 \pi a^3} \sum_{k_1 = 1}^{\infty} \sum_{(k_2,k_3) \in \mathbb{Z}^2 \setminus \{0\}} k_1^2 \left( \frac{m^2}{\sum_{i=2}^{\infty} [k_i L_i]^2} \right)^{1/4} K_{1/2} \left( 2 \sqrt{m^2 + \frac{\pi^2 k_2^2}{a^2}} \left( \sum_{i=2}^{\infty} \frac{[k_i L_i]^2}{[k_i L_i]^2} \right) \right)
\]

for the Dirichlet bc and Neumann bc cases. The last term on the right hand side of \[(3.12)\] and the last three terms on the right hand side of \[(3.13)\] vanish to zero exponentially fast when \(a \to 0^+\). The third term on the right hand side of \[(3.12)\] and the seventh, eighth and ninth terms on the right hand side of \[(3.13)\] are independent of \(a\). For the remaining terms, their behaviors as \(a \to 0^+\) are not obvious. Using the power series expansion of the Bessel function \(K_\nu(z)\) naively do not give us the correct asymptotic behavior due to the summation over \(k_1\). In Appendix C, we derive the correct asymptotic behavior for these terms when \(a \to 0^+\). The results
Figure 3: These figures show the dependence of the Casimir force \( F_{\text{Cas}}^{P/D/N}(a; L_2, L_3; m) \) on \( m \) when \( L_2 = L_3 = 1 \) and \( a = 0.1, 0.5, 1, 2 \) respectively.

are: for fixed \( m, L_2, L_3 \), as \( a \to 0^+ \),

\[
F_{\text{Cas}}^{P}(a; L_2, L_3; m) \sim -\frac{\pi^2 L_2 L_3}{30a^4} + \frac{L_2 L_3 m^2}{24a^2} + \frac{\pi^2 m^4}{32} L_2 L_3 \log a + O(a^0)
\]

\[
F_{\text{Cas}}^{D/N}(a; L_2, L_3; m) \sim -\frac{\pi^2 L_2 L_3}{480a^4} + \frac{\zeta(3)(L_2 + L_3)}{16\pi a^3} - \frac{\pi}{96a^2} + \frac{L_2 L_3 m^2}{96a^2} + \frac{m^2(L_2 + L_3)}{16\pi a} + \frac{\pi^2 m^4}{32} L_2 L_3 \log a - \frac{\pi m^2}{16} \log a + O(a^0).
\]

In particular, as \( a \to 0^+ \), the Casimir force becomes very negative (attractive) with leading term of order \( 1/a^4 \). Notice that this leading term is independent of the mass \( m \). We also observe that the first three leading terms of the Casimir force \( F_{\text{Cas}}^{D}(a; L_2, L_3; m) \) for Dirichlet bc has exactly the same form as that of the massless case derived in [9]. In fact, by taking the limit \( m \to 0^+ \) for \( F_{\text{Cas}}^{D}(a; L_2, L_3; m) \) in (3.13), we obtain the same expression as that derived in [9] when \( a \) is small.

From the above discussion, we conclude that the effect of mass is insignificant when \( a \) is small, but with nonzero mass, the Casimir force will be drastically reduced.

We would also like to mention that the expressions (3.12) and (3.13) can also be used to analyze the behavior of the Casimir force when \( L_2 \) and \( L_3 \) are large (parallel plate geometry). In this case, we find that the Casimir pressure (force per...
In the massless limit $m \to 0^+$, these become $P^P_{\text{Cas}}(a; 0) = -\pi^2/(30a^4)$ and $P^{D/N}_{\text{Cas}}(a; 0) = -\pi^2/(480a^4)$ respectively.
In Figure 2, 3, 4, 5, 6, 7 we show graphically the behavior of the Casimir force \( F_{\text{Cas}}^{P/D/N}(a; L_2, L_3; m) \). From (3.10) and (3.11), it is easy to verify that
\[
F_{\text{Cas}}^{P/D/N} \left( \lambda a; \lambda L_2, \lambda L_3; \frac{m}{\lambda} \right) = \lambda^{-2} F_{\text{Cas}}^{P/D/N}(a; L_2, L_3; m).
\]

Therefore, we have the freedom of fixing one of the variables which does not affect the shape (up to scaling) of the graphs. When one is not concerned with the dependence of the Casimir force on mass, one can let \( m = 1 \). From Figs. 2, 4, 5, 6, we see that fixing \( m, L_2 \) and \( L_3 \), the Casimir force always increase (as a function of \( a \)) from \(-\infty\) (when \( a \to 0^+ \)) to \( 0 \) (when \( a \to -\infty \)). In fact, by differentiating (3.10) and (3.11) again with respect to \( a \) and using (3.9), we can conclude that the Casimir force \( F_{\text{Cas}}^{P/D/N}(a; L_2, L_3; m) \) is always an increasing function of \( a \), and therefore the small \( a \) and large \( a \) behaviors show that it always increases from \(-\infty\) (when \( a \to 0^+ \)) to \( 0 \) (when \( a \to -\infty \)).

Fig. 3 shows the dependence of the Casimir force on mass \( m \). We see that the mass effect is more significant when \( a \) is large. On the other hand, we will tend to make conclusion that the Casimir force is an increasing function of \( m \) from Fig. 3. However, Fig. 7 invalidates this conjecture.

In Figure 8, we demonstrate the Casimir force on the piston, the regularized Casimir force and reduced regularized Casimir force (see Appendix B) due to the interior and exterior regions on a single graph, for the periodic bc and Dirichlet bc cases. The graphs show that the Casimir force due to the interior region and exterior region can be either attractive or repulsive. But the net effect on the Casimir piston is attractive.

In the above discussion, we are considering the case where the limit \( L_1 \to \infty \) has been taken. Now we briefly discuss the case where the limit \( L_1 \to \infty \) is not taken, i.e. the piston is assumed to be placed in a fixed closed rectangular box. To distinguish with the cases above, we denote the resulting Casimir force on the piston as \( F_{\text{Cas}}^{P/D/N}(a; L_2, L_3; m) \). From the discussion in the beginning of this section, \( F_{\text{Cas}}^{P/D/N}(a; L_2, L_3; m) \) is given by the right hand side of (3.1) and (3.2) without the limit \( \lim_{L_1 \to \infty} \). Therefore, in terms of the Casimir force \( F_{\text{Cas}}^{P/D/N}(a; L_2, L_3; m) \)
Figure 7: These figures show that the Casimir force $F_{\text{Cas}}(a; L_2, L_3; m)$ are not monotonic function of mass.

Figure 8: These figures show that despite the fact that the Casimir force acting on the piston is always attractive, the regularized Casimir force due to the interior region and exterior regions can be either attractive or repulsive.

computed for the case $L_1 \to \infty$, we have

$$F_{\text{Cas}}^{P/D/N}(a; L_2, L_3; m) = F_{\text{Cas}}^{P/D/N}(a; L_2, L_3; m) - F_{\text{Cas}}^{P/D/N}(L_1 - a; L_2, L_3; m)$$

Obviously $F_{\text{Cas}}^{P/D/N}(a; L_2, L_3; m) = 0$ when $a = L_1/2$, i.e. the piston is in an equilibrium position when it is placed in the middle of the box. However, if it is placed closer to the left end, i.e. $a < L_1 - a$, then since $F_{\text{Cas}}^{P/D/N}(a; L_2, L_3; m)$ is more negative than $F_{\text{Cas}}^{P/D/N}(L_1 - a; L_2, L_3; m)$, we find a net attractive force pulling the piston to the left end. In other words, the Casimir force always tends to move the piston to the nearest wall. In nanotechnology, this is known as the stiction effect and may have undesirable consequences to the functionality of nano-devices.

4. Conclusion

We have carried out the study of the Casimir effect for massive scalar fields in three dimensions subject to different boundary conditions in the piston setting. The methods we used here can be easily generalized to show that for pistons with
arbitrary cross sections in any dimensions, the Casimir force acting on the piston is always divergence free. For the cases considered here, the Casimir force is found to be always attractive. It would be interesting to investigate the possible repulsive Casimir force due to massive fields in the piston setting. Another thing we have not addressed here is the thermal effect, which we plan to carry out in the future.

**Acknowledgement** The authors would like to thank Malaysian Academy of Sciences, Ministry of Science, Technology and Innovation for funding this project under the Scientific Advancement Fund Allocation (SAGA) Ref. No P96c.

**Appendix A. Cut-off dependent Casimir energy for massive scalar fields in rectangular cavities**

The cut-off dependent Casimir energy $E_{\text{Cas}}^P(\lambda; L_1, \ldots, L_d; m)$ \[A.1\] for a massive scalar field with periodic bc inside a $d$-dimensional rectangular cavity can be written as

\[
E_{\text{Cas}}^P(\lambda; L_1, \ldots, L_d; m) = -\frac{1}{2} \frac{\partial}{\partial \lambda} K(\lambda; L_1, \ldots, L_d; m),
\]

where

\[
K(\lambda; L_1, \ldots, L_d; m) = \sum_{k \in \mathbb{Z}^d} e^{-\lambda \omega_k^P}.
\]

For a massless ($m = 0$) scalar field, the sum \[A.1\] has been computed in \[17\] using Euler-Maclaurin summation formula. Here we want to compute this kind of sum by a different method, making full advantage of the analytic structure of Epstein zeta function. Using inverse Mellin transform of exponential function

\[
e^{-z} = \frac{1}{2\pi i} \int_{c_0-i\infty}^{c_0+i\infty} \Gamma(w) z^{-w} dw, \quad c_0 > 0,
\]

we find that

\[
(A.2) \quad K(\lambda; L_1, \ldots, L_d) = \frac{1}{2\pi i} \int_{c_0-i\infty}^{c_0+i\infty} \lambda^{-w} \Gamma(w) Z_d\left(\frac{w}{2}, \frac{2\pi}{L_1}, \ldots, \frac{2\pi}{L_d}; m\right) dw, \quad c_0 > d,
\]

where $Z_d(s; a_1, \ldots, a_d; m)$ is the inhomogeneous Epstein zeta function defined by the series

\[
Z_d(s; a_1, \ldots, a_d; m) = \sum_{k \in \mathbb{Z}^d} \left(\sum_{i=1}^d |a_ik_i|^2 + m^2\right)^{-s}.
\]
where $\gamma(A.4)$

\[
Z_d(s; a_1, \ldots, a_d; m) = \frac{\pi^{\frac{d}{2}}}{\Gamma(s)} \frac{\Gamma(s - \frac{d}{2})}{\Gamma(s)} m^{d-2s} \\
+ \frac{2\pi^s}{\prod_{i=1}^d a_i} \frac{1}{\Gamma(s)} \frac{1}{m^{s-\frac{d}{2}}} \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \left( \sum_{i=1}^d \left( \frac{k_i}{a_i} \right)^2 \right)^{\frac{2s-d}{4}} K_{s-\frac{d}{4}} \left( 2\pi m \sqrt{\sum_{i=1}^d \left( \frac{k_i}{a_i} \right)^2} \right),
\]

where $K_\nu(z)$ is the modified Bessel function. The second term is an analytic function of $s$. The first term shows that when $d$ is odd, $Z_d(s; a_1, \ldots, a_d; m)$ has simple poles at $s = \frac{d}{2} - j$, $j = 0, 1, 2, \ldots$ with residues

\[
\text{Res}_{s=\frac{d}{2}-j} Z_d(s; a_1, \ldots, a_d; m) = \frac{(-1)^j \pi^{\frac{d}{2}}}{j!} \frac{m^{2j}}{\prod_{i=1}^d a_i} \Gamma \left( \frac{d}{2} - j \right).
\]

When $d$ is even, $Z_d(s; a_1, \ldots, a_d; m)$ only has poles at $s = \frac{d}{2} - j$ where $j = 0, 1, \ldots, \frac{d-2}{2}$, with the same residue \((A.4)\). By moving the contour of integration from the line $\Re w = c_0, c_0 > d$ to the line $\Re w = -2 + \varepsilon$, we obtain for $d = 1, 2, 3$, $K(\lambda; L_1; m) = \frac{L_1}{\pi} \lambda^{-1} + \frac{\lambda m^2 L_1}{2\pi} \log \lambda + \frac{\lambda m^2 L_1}{2\pi} \left( \log m - \log 2 - \frac{5}{4} + \gamma \right)$

\[
+ \frac{2\lambda m}{\pi} \sum_{k=1}^\infty k^{-1} K_1(mkL_1) + O(\lambda^2);
\]

\[
K(\lambda; L_1, L_2; m) = \frac{L_1 L_2}{2\pi} \lambda^{-2} - \frac{L_1 L_2}{4\pi} m^2 + \frac{L_1 L_2}{6\pi} \lambda m^3 + \frac{L_1 L_2}{\sqrt{2\pi}^3} \lambda m^{3/2}
\]

\[
\times \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} \left( \sum_{i=1}^2 \left( \frac{k_i}{L_i} \right)^2 \right)^{-\frac{1}{4}} \lambda^{3/2} \left( m^2 \sum_{i=1}^2 \left( \frac{k_i}{L_i} \right)^2 \right) + O(\lambda^2);
\]

K(\lambda; L_1, L_2, L_3; m) = \frac{L_1 L_2 L_3}{\pi^2} \lambda^{-3} - \frac{L_1 L_2 L_3}{4\pi^2} m^2 \lambda^{-1} - \frac{L_1 L_2 L_3}{16\pi^2} m^4 \lambda \log \lambda

\]

\[
- \frac{L_1 L_2 L_3}{16\pi^2} m^4 \lambda \left( -\frac{3}{2} + \gamma + \log m - \log 2 \right) + \frac{L_1 L_2 L_3}{2\pi^3} \lambda m^2
\]

\[
\times \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \left( \sum_{i=1}^3 \left( \frac{k_i}{L_i} \right)^2 \right)^{-1} \lambda \left( m^3 \sum_{i=1}^3 \left( \frac{k_i}{L_i} \right)^2 \right) + O(\lambda^2),
\]

where $\gamma$ is the Euler constant. Using \((A.1)\), we find that the cut-off dependent Casimir energy $E_{\text{Cas}}^P(\lambda; L_1, \ldots, L_3; m)$ for $d = 1, 2$ and $3$ are given respectively by

\[
E_{\text{Cas}}^P(\lambda; L_1; m) = \frac{L_1}{2\pi} \lambda^{-2} - \frac{m^2 L_1}{4\pi} \log \lambda - \frac{m^2 L_1}{4\pi} \left( \log m - \log 2 - \frac{1}{4} + \gamma \right)
\]

\[
- \frac{m}{\pi} \sum_{k=1}^\infty k^{-1} K_1(mkL_1) + O(\lambda);
\]
depends on the normalization constant $\mu$.

One can show that in the massless limit ($\mu = 0$), these results coincide with the corresponding results obtained in \[17, 10\].

**Appendix B. Casimir Effect for Massive Scalar Fields in Rectangular Cavities by Zeta Regularization Method, and the Attractive/Repulsive Nature of the Casimir Force**

Using zeta regularization method, the Casimir energy $E_{\text{Cas}}^P(L_1, \ldots, L_d; m)$ \[2.3\] for a massive scalar field with periodic bc inside a $d$-dimensional rectangular cavity is defined as \[25\]

$$E_{\text{Cas}}^P(L_1, \ldots, L_d; m) = \frac{3L_1L_2L_3}{2\pi^2} \lambda^{-4} - \frac{L_1L_2L_3}{8\pi^2} m^2 \lambda^{-2} + \frac{L_1L_2L_3}{32\pi^2} m^4 \log \lambda$$

$$+ \frac{L_1L_2L_3}{32\pi^2} m^4 \left( -\frac{1}{2} + \gamma + \log m - \log 2 \right) - \frac{L_1L_2L_3}{4\pi^2} m^2$$

$$\times \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \left( \sum_{i=1}^{3} |k_i L_i|^2 \right)^{-1} K_2 \left( m\lambda \left( \sum_{i=1}^{3} |k_i L_i|^2 \right) \right) + O(\lambda).$$

Using zeta regularization method, the Casimir energy $\hat{E}_{\text{Cas}}^P(L_1, \ldots, L_d; m)$ \[2.3\] for a massive scalar field with periodic bc inside a $d$-dimensional rectangular cavity is defined as \[25\]

$$\hat{E}_{\text{Cas}}^P(L_1, \ldots, L_d; m) := \frac{1}{4} \lim_{\varepsilon \to 0} \left( \zeta_P \left( -\frac{1}{2} + \varepsilon \right) + \zeta_P \left( -\frac{1}{2} - \varepsilon \right) \right),$$

where $\zeta_P(s)$ is the zeta function

$$\zeta_P(s) = \mu^{1+2s} \sum_{k \in \mathbb{Z}^d} (\omega_k)^{-2s} = \mu^{1+2s} \zeta_P \left( s; \frac{2\pi}{L_1}, \ldots, \frac{2\pi}{L_d}; m \right),$$

and $\mu$ is a normalization constant. Using \[2.3\] we find that $\hat{E}_{\text{Cas}}^P(L_1, \ldots, L_d; m)$ depends on the normalization constant $\mu$ if and only if $d$ is odd. For $d = 1, 2, 3$, $E_{\text{Cas}}^P(L_1, \ldots, L_d; m)$ is given explicitly by

$$\hat{E}_{\text{Cas}}^P(L_1; m) = \frac{L_1}{4\pi} m^2 \log \mu - \frac{m^2 L_1}{4\pi} \left( \log m - \log 2 - \frac{1}{4} \right) - \frac{m}{\pi} \sum_{k=1}^{\infty} k^{-1} K_1(mkL_1),$$

$$\hat{E}_{\text{Cas}}^P(L_1, L_2; m) = -\frac{L_1L_2}{12\pi} m^3 - \frac{L_1L_2}{\sqrt{8\pi^3}} m^{3/2}$$

$$\times \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} \left( \sum_{i=1}^{2} |k_i L_i|^2 \right)^{-3} K_3 \left( m\lambda \left( \sum_{i=1}^{2} |k_i L_i|^2 \right) \right),$$

$$\hat{E}_{\text{Cas}}^P(L_1, L_2, L_3; m) = -\frac{3L_1L_2L_3}{2\pi^2} \lambda^{-4} - \frac{L_1L_2L_3}{8\pi^2} m^2 \lambda^{-2} + \frac{L_1L_2L_3}{32\pi^2} m^4 \log \lambda$$

$$+ \frac{L_1L_2L_3}{32\pi^2} m^4 \left( -\frac{1}{2} + \gamma + \log m - \log 2 \right) - \frac{L_1L_2L_3}{4\pi^2} m^2$$

$$\times \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \left( \sum_{i=1}^{3} |k_i L_i|^2 \right)^{-1} K_2 \left( m\lambda \left( \sum_{i=1}^{3} |k_i L_i|^2 \right) \right) + O(\lambda).$$
either attractive or repulsive depending on the values of $L$ to a massive scalar field with Dirichlet and Neumann boundary conditions can be force $\hat{E}_{\text{Cas}}^p(L_1, L_2, L_3; m) = \frac{L_1 L_2 L_3}{32 \pi^2} m^4 \log \mu + \frac{L_1 L_2 L_3}{32 \pi^2} m^4 \left( \log m - \log 2 - \frac{1}{2} \right)
- \frac{L_1 L_2 L_3}{4 \pi^2} m^2 \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \left( \sum_{i=1}^{3} |k_i L_i|^2 \right)^{-1} K_2 \left( m \sqrt{\sum_{i=1}^{3} |k_i L_i|^2} \right).

Compare to the result obtained by cut-off regularization method, we find that the zeta regularized Casimir energy $E_{\text{Cas}}^p(L_1, \ldots, L_d; m)$ will agree with the $\lambda$-independent part of the cut-off dependent Casimir energy $E_{\text{Cas}}^p(\lambda; L_1, \ldots, L_d; m)$ if and only if we take $\mu = e^{-\gamma}$, where $\gamma$ is the Euler constant. Using this prescription, we find that the zeta regularized Casimir force acting in the $x_1$ direction for $d = 3$ is

\begin{equation}
(B.1) \quad \hat{F}_{1,\text{Cas}}^p(L_1, L_2, L_3; m) = -\frac{L_2 L_3}{32 \pi^2} m^4 \left( \log m - \log 2 - \frac{1}{2} + \gamma \right)
+ \frac{L_2 L_3}{4 \pi^2} m^2 \sum_{(k_1, k_3) \in \mathbb{Z}^2 \setminus \{0\}} \left( \sum_{i=2}^{3} |k_i L_i|^2 \right)^{-1} K_2 \left( m \sqrt{\sum_{i=2}^{3} |k_i L_i|^2} \right)
+ \frac{\partial}{\partial L_1} \left( \frac{L_1 L_2 L_3}{4 \pi^2} m^2 \sum_{k_1 \in \mathbb{Z} \setminus \{0\}} \left( \sum_{i=1}^{3} |k_i L_i|^2 \right)^{-1} K_2 \left( m \sqrt{\sum_{i=1}^{3} |k_i L_i|^2} \right) \right) .
\end{equation}

As is shown in Section 3, the last term in (B.1) is the net contribution to the Casimir force acting on the Casimir piston. For any fixed $m, L_2, L_3$, it is always negative and increase from $-\infty$ (when $L_1 \to 0^+$) to 0 (when $L_1 \to \infty$). On the other hand, the first and second term in (B.1) are independent of $L_1$. Moreover, the second term is always positive and by taking $m$ sufficiently small, the first term is also positive. Therefore, for fixed $L_2, L_3$, when $m$ is sufficiently small and $L_1$ is sufficiently large, the zeta regularized Casimir force $\hat{F}_{1,\text{Cas}}^p(L_1, L_2, L_3; m)$ becomes repulsive. Similar reasonings can be used to show that the zeta regularized Casimir force $\hat{F}_{1,\text{Cas}}^p(L_1, L_2, L_3; m)$ acting in the $x_1$ direction of a rectangular cavity due to a massive scalar field with Dirichlet and Neumann boundary conditions can be either attractive or repulsive depending on the values of $L_1, L_2, L_3$ and $m$.

We would also like to remark that in some of the regularization schemes, the Casimir force on a wall in a rectangular cavity is defined by subtracting the Casimir force in the absence of boundary. In this case, the Casimir force is equal to the sum of the last two terms in (B.1), i.e. the first term is absent. We call the Casimir force defined in this way the reduced Casimir force. Notice that in the massless case, there is no difference between the Casimir force and reduced Casimir force in a rectangular cavity. On the other hand, it is easy to see that the argument given above shows that the reduced Casimir force in a rectangular cavity can be positive or negative, under different boundary conditions. In Figure 8, we demonstrate the Casimir force on the piston, the Casimir force (and reduced Casimir force) due to the interior and exterior regions on a single graph, for the periodic bc and Dirichlet bc cases.
Appendix C. An alternative expression for the Casimir force when \( a \) is small

To study the behavior of the Casimir force when \( a \) is small, we first split the summation over \( (k_2, \ldots, k_d) \in \mathbb{Z}^{d-1} \) into the term \( (k_2, \ldots, k_d) = 0 \) and the summation over \( (k_2, \ldots, k_d) \in \mathbb{Z}^{d-1} \setminus \{0\} \). Then applying the identity (3.7) to the summation over \( k_1 \in \mathbb{Z} \setminus \{0\} \), we find that

\[
R_d(a, L_2, \ldots, L_d; m) = -\frac{a \prod_{i=2}^d L_i}{2^{d+1} \pi^{d+2}} \int_0^\infty t^{-\frac{d+3}{2}} \sum_{k_1=1}^\infty \exp \left( -tm^2 - \frac{k_1^2 a^2}{4t} \right) dt

+ \frac{a \prod_{i=2}^d L_i}{2^{d+2} \pi^{d+2}} \int_0^\infty t^{-\frac{d+3}{2}} \sum_{(k_2, \ldots, k_d) \in \mathbb{Z}^{d-1} \setminus \{0\}} \exp \left( -tm^2 - \frac{1}{4t} \sum_{i=2}^d [k_i L_i]^2 \right) dt

- \frac{\prod_{i=2}^d L_i}{2^{d+1} \pi^{d+2}} \int_0^\infty t^{-\frac{d+3}{2}} \sum_{(k_2, \ldots, k_d) \in \mathbb{Z}^{d-1} \setminus \{0\}} \exp \left( -tm^2 - \frac{1}{4t} \sum_{i=2}^d [k_i L_i]^2 \right) dt.
\]

Taking derivative with respect to \( a \) gives

\[
\frac{\partial}{\partial a} R_d(a, L_2, \ldots, L_d; m) \]

The expressions for the Casimir force \( F_{\text{Cas}}^{P/D/N}(a; L_1, L_2, L_3; m) \) can then be obtained by using (3.3), (3.5) and (5.0). The results are given by (3.12) and (3.13).

The last term on the right hand side of (C.2) tends to zero exponentially fast when \( a \to 0^+ \). The third term is independent of \( a \). For the first two terms, their asymptotic behaviors when \( a \to 0^+ \) are not obvious. Let us call the sum of these two terms as \( \Xi(a; L_1, L_2, \ldots, L_d; m) \). Here we derive its asymptotic behavior when...
\( a \rightarrow 0^+ \). From the derivation above, we have

\[
\mathcal{I}(a; L_2, \ldots, L_d; m) = \frac{\partial}{\partial a} \left\{ \frac{a}{2^{d+1} \pi} \prod_{i=2}^{d} L_i \int_0^\infty t^{\frac{d+1}{2}} \sum_{k_1=1}^{\infty} \exp \left( -tm^2 - \frac{k_1^2 a^2}{4t} \right) dt \right\}
\]

\[
= \prod_{i=2}^{d} L_i \frac{\partial}{\partial a} \Psi \left( -\frac{d+1}{2}; a; m \right),
\]

where \( \Psi(s; a; m) \) is the analytic function (of \( s \)) defined by

\[
\Psi(s; a; m) = a \int_0^\infty t^{s-1} \sum_{k=1}^{\infty} \exp \left( -tm^2 - \frac{k_1^2 a^2}{4t} \right) dt.
\]

Making a change of variable \( t \mapsto a^2 t \), we have

\[
\Psi(s; a; m) = a^{1+2s} \int_0^\infty t^{s-1} \sum_{k=1}^{\infty} \exp \left( -ta^2 m^2 - \frac{k_1^2 a^2}{4t} \right) dt.
\]

For \( s > 1/2 \), Jacobi inversion formula (3.7) gives

\[
\Psi(s; a; m) = a^{1+2s} \int_0^\infty t^{s-1} \sum_{k=1}^{\infty} e^{-ta^2 m^2} \left\{ -\frac{1}{2} + \sqrt{\pi t} \sum_{k=-\infty}^{\infty} e^{-4\pi^2 k^2 t} \right\} dt.
\]

When \( a \) is small, we can use the Taylor expansion of \( e^{-ta^2 m^2} \) to get

\[
\Psi(s; a; m) = -\frac{am^{-2s}}{2} \Gamma(s) + \sqrt{\pi} m^{-1-2s} \Gamma \left( s + \frac{1}{2} \right) + 2 \sqrt{\pi} a^{1+2s} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} a^{2j} m^{2j} \int_0^\infty t^{s+j-1/2} \sum_{k=1}^{\infty} e^{-4\pi^2 k^2 t} dt
\]

\[
= -\frac{am^{-2s}}{2} \Gamma(s) + \sqrt{\pi} m^{-1-2s} \Gamma \left( s + \frac{1}{2} \right) + 2 \sqrt{\pi} a^{1+2s} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \frac{a^{2j} m^{2j}}{(2\pi)^{2j+2s+1}} \Gamma \left( s + j + \frac{1}{2} \right) \zeta(2s + 2j + 1),
\]

where \( \zeta(s) \) is the Riemann zeta function. One can check that the formula on the right hand side of the last equality define an analytic function of \( s \). Putting \( s = -(d+1)/2 \) and taking derivative with respect to \( a \), we find that if \( d \) is odd,

\[
\mathcal{I}(a; L_2, \ldots, L_d; m)
= 2\pi^{d/2} a^{-d-1} \prod_{i=2}^{d} L_i \sum_{j \in \mathbb{N} \cup \{0\}} \sum_{j \neq \frac{d+1}{2}} \frac{(-1)^j}{j!} (2\pi)^{-2j} m^{2j} a^{2j} \Gamma \left( j + 1 - \frac{d}{2} \right) \zeta(2j - d)
+ \frac{(-1)^{d+1}}{(2\pi)^{d+1}} m^{d+1} \frac{d}{2^{d+2} \pi^{d+2}} \prod_{i=2}^{d} L_i \left\{ 2 \log \frac{am}{4\pi} + 2 - \psi \left( \frac{d+3}{2} \right) + \gamma \right\},
\]
and when $d$ is even,
\[
\Xi(a; L_2, \ldots, L_d; m) = 2\pi^{d/2}a^{d-1} \left[ \prod_{i=2}^{d} L_i \right] \sum_{j\in\mathbb{N}\setminus\{0\}} \frac{(-1)^j}{j!} (2\pi)^{2j} m^{2j} \Gamma \left( j + 1 - \frac{d}{2} \right) \zeta(2j - d) 
\]
\[
- \frac{\Gamma \left( \frac{d+1}{2} \right)}{2^{d+2}a^{d+1}} \prod_{i=2}^{d} L_i \right] - \frac{(-1)^{d+1}}{(2!)^{d-1} 2^d \pi^d} \left[ \prod_{i=2}^{d} L_i \right] a^{-1}.
\]

Here $\psi(z)$ is the logarithm derivative of the gamma function, and for $s \leq 0$, $\Gamma(s/2 + 1)\zeta(s)$ is understood as
\[
\Gamma \left( \frac{s + 2}{2} \right) \zeta(s) = \frac{\pi^{s - \frac{1}{2}}}{2} \Gamma \left( 1 - \frac{s}{2} \right) \zeta(1 - s).
\]

When $d = 1, 2, 3$, using $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$, we have respectively
\[
\Xi(a; m) = -\frac{\pi}{6a^2} - \frac{\pi m^2}{4} \log a + O(a^0);
\]
\[
\Xi(a; L_2; m) = -\frac{L_2}{\pi a^3} \zeta(3) + \frac{m^2 L_2}{4 \pi a} + O(a^0);
\]
and
\[
\Xi(a; L_2, L_3; m) = -\frac{\pi^2 L_2 L_3}{30 a^4} + \frac{m^2 L_2 L_3}{24 a^2} + \frac{\pi^2 m^4}{32} L_2 L_3 \log a + O(a^0).
\]

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