Swimming of a uniform deformable sphere in a viscous incompressible fluid with inertia

B. U. Felderhof

Institut für Theorie der Statistischen Physik
RWTH Aachen University
Templergraben 55
52056 Aachen
Germany

R. B. Jones

Queen Mary University of London,
The School of Physics and Astronomy,
Mile End Road, London E1 4NS, UK

(Dated: November 20, 2018)

Abstract

The swimming of a deformable uniform sphere is studied in second order perturbation theory in the amplitude of the stroke. The effect of the first order reaction force on the first order center of mass velocity is calculated in linear response theory by use of Newton’s equation of motion. The response is characterized by a dipolar admittance, which is shown to be proportional to the translational admittance. As a consequence the mean swimming velocity, calculated in second order perturbation theory, depends on the added mass of the sphere. The mean swimming velocity and the mean rate of dissipation are calculated for several selected strokes.

PACS numbers: 47.15.G-, 47.63.mf, 47.63.Gd, 87.17.Jj
I. INTRODUCTION

In recent work [1] on the dynamics of swimming of a deformable sphere in a viscous incompressible fluid with inertia we showed that if the distortion of the spherical surface has an oscillatory dipolar component, then a reaction force is generated which can cause an oscillatory motion of the sphere. To first order the motion is linear in the amplitude of the dipolar distortion. Its magnitude depends on the inner dynamics of the swimmer. In earlier work on swimming in a fluid with inertia [2] we put the first order velocity equal to zero. We view this now as a kinematic condition which is realized only if the reaction force is fully absorbed by the sphere and has no effect on its surface motion. More generally, the effect of the oscillatory reaction force must be taken into account.

For a uniformly deforming sphere the velocity of the surface is identical with that of its center of mass, and the linear velocity cannot be neglected. In the following we show that the linear oscillatory motion affects the first order flow pattern and hence also the mean swimming velocity, where the mean is defined as the time average over a period of the stroke. The effect is dependent on the mass density of the sphere.

The effect of fluid inertia on swimming is characterized by a scale number $s$, defined by $s = a \sqrt{\omega \rho / 2 \eta}$, where $a$ is the radius of the undistorted sphere, $\omega$ is the frequency of the stroke, $\rho$ is the mass density of the fluid, and $\eta$ is its shear viscosity. In the Stokes limit $s = 0$ inertia can be neglected. In the limit of large $s$ the swimming is dominated by fluid inertia.

We showed earlier [3] for situations with vanishing linear velocity that the mean swimming velocity and the mean rate of dissipation depend intricately on the scale number $s$. For certain strokes the mean swimming velocity can change sign as $s$ increases. Such a reversal of swimming velocity was found also in computer simulation of a two-sphere system by Jones et al. [4]. In their model the two spheres are rigid, but oscillate relative to each other. The model was introduced earlier by Klotsa et al. [5]. A mechanical model with simplified hydrodynamic interactions [6] does not show the reversal of mean swimming velocity. Apparently the reversal is related to the details of the oscillatory flow pattern and the combination of friction and inertia.

In Sec. III of this article we elucidate the mechanism which couples the dipolar surface distortion to the linear motion of a uniform sphere. The reaction force which generates the linear center of mass motion must be calculated self-consistently. The linear response to the dipolar distortion follows from Newton’s equation and is characterized by a transport coefficient, which we call the dipolar admittance. This quantity depends on the scale number $s$ and the ratio of mass densities $\rho_0 / \rho$, and turns out to be proportional to the translational admittance.

In Secs. IV and V we calculate elements of the two matrices which enter the calculation of the mean swimming velocity and the mean rate of dissipation in second order perturbation theory in the amplitude of the stroke. As before [3] it is useful to introduce a Stokes representation to facilitate the calculation. In Sec. VI we study the mean swimming velocity and the mean rate of dissipation as functions of scale number and the ratio of mass densities. The article is concluded with a discussion.
II. FLUID MOTION

We consider a flexible sphere of radius $a$ immersed in a viscous incompressible fluid of shear viscosity $\eta$ and mass density $\rho$. The fluid is set in motion by time-dependent distortions of the sphere. We shall study axisymmetric periodic distortions which lead to a translational swimming motion of the sphere in the $z$ direction in a Cartesian system of coordinates. The analysis is based on a perturbation expansion of the Navier-Stokes equations in powers of the amplitude of distortions \cite{2}. The no-slip boundary condition is applied on the surface of the distorted sphere.

The surface distortion is written as

$$\xi(\theta, t) = \text{Re}[\xi_{\omega}(\theta)e^{-i\omega t}],$$

with polar angle $\theta$ and complex amplitude $\xi_{\omega}(\theta)$. The distortion $\xi$ is defined in the co-moving frame of the body with the origin at rest. In our recent analysis of the dipole-quadrupole (DQ) model and the quadrupole-octupole (QO) model \cite{7} we showed that in the DQ model the generated flow gives rise to a first order oscillating force on the body which in general leads to a first order motion \cite{1}. By definition we put the first order velocity equal to zero, in accordance with our earlier considerations \cite{2}, and we argued that this can be justified on the basis of an assumption on the inner structure of the body \cite{1}.

More generally we must allow internal dynamics for which the first order velocity does not vanish. In this article we assume in particular that the distorting sphere remains uniform with mass density $\rho_0$. If the distortion has a dipolar component then the hydrodynamic reaction force on the body leads to an oscillating first order velocity $U_1(t) = U_1(t)e_z$ with vanishing mean.

The flow velocity $u(r, t)$ and pressure $p(r, t)$ in the laboratory frame are assumed to satisfy the Navier-Stokes equations

$$\rho \left[ \frac{\partial u}{\partial t} + u \cdot \nabla u \right] = \eta \nabla^2 u - \nabla p, \quad \nabla \cdot u = 0. \quad (2.2)$$

We consider a solution of these equations which varies periodically, as caused by the periodically varying shape of the body. In the laboratory frame the flow velocity tends to zero at infinity and the pressure tends to the ambient value $p_0$. The periodicity of the solution implies

$$u(r - \bar{U}Te_z, t + T) = u(r, t), \quad p(r - \bar{U}Te_z, t + T) = p(r, t), \quad (2.3)$$

where $\bar{U}$ is the mean swimming velocity, $T = 2\pi/\omega$ is the period, and $e_z$ is the unit vector in the $z$ direction. The mean swimming velocity $\bar{U}$ is of second order in the surface distortion $\xi$.

We can assume that at time $t = 0$ the centroid of the body is at the origin. To first order after period $T$ the centroid is again at the origin. The first order velocity and pressure take the form

$$u^{(1)}(r, t) = \text{Re}[u_{\omega}(r)e^{-i\omega t}], \quad p^{(1)}(r, t) = \text{Re}[p_{\omega}(r)e^{-i\omega t}] \quad (2.4)$$

with amplitudes $u_{\omega}(r), p_{\omega}(r)$ which satisfy the linearized Navier-Stokes equations

$$\eta[\nabla^2 u_{\omega} - \alpha^2 u_{\omega}] - \nabla p_{\omega} = 0, \quad \nabla \cdot u_{\omega} = 0, \quad (2.5)$$
with the variable
\[ \alpha = (-i\omega\rho/\eta)^{1/2} = (1 - i)(\omega\rho/2\eta)^{1/2}. \] (2.6)

The solution of Eq. (2.5) can be expressed as a linear superposition of modes [8]
\[ v_l(r, \alpha) = \frac{2}{\pi} e^{ia\alpha[0(l+1)k_{l-1}(\alpha r)A_l(\hat{r}) + l\kappa_{l+1}(\alpha r)B_l(\hat{r})]}, \]
\[ u_l(r) = -\left(\frac{a}{r}\right)^{l+2} B_l(\hat{r}), \quad p_l(r, \alpha) = \eta a^2 a\left(\frac{a}{r}\right)^{l+1} P_l(\cos \theta), \] (2.7)
with modified spherical Bessel functions \( k_l(z) \), radial unit vector \( \hat{r} = r/r = e_r \), and vector spherical harmonics \( \{A_l, B_l\} \) defined by [8]
\[ A_l = \hat{A}_l = lP_l(\cos \theta)e_r - P_l'(\cos \theta)e_\theta, \]
\[ B_l = \hat{B}_l = -(l + 1)P_l(\cos \theta)e_r - P_l'(\cos \theta)e_\theta, \] (2.8)
with Legendre polynomials \( P_l \) and associated Legendre functions \( P_l' \) in the notation of Edmonds [10].

The surface distortion function \( \xi_\omega(s) \) in the co-moving frame is prescribed and expanded as
\[ \xi_\omega(s) = -ia \sum_{l=1}^{\infty} [\kappa_l v_l(s, \alpha) + \mu_l u_l(s)], \] (2.9)
with \( s = a\hat{r} \) and complex coefficients \( \{\kappa_l, \mu_l\} \). The mode \( v_1(s, \alpha) \) involves the vector spherical harmonic \( A_1 = e_z \) corresponding to uniform displacement. The absence of uniform displacement in the co-moving frame implies the constraint \( \kappa_1 = 0 \). The first order fluid velocity at the surface in the co-moving frame is given by \( v^{(1)}(s, t) = \partial \xi(s, t)/\partial t \) according to the no-slip boundary condition. In the laboratory frame the surface is assumed to move with first order velocity \( U_1(t) = \Re[U_{1\omega} \exp(-i\omega t)] \). The first order flow velocity in the laboratory frame is related to that in the co-moving frame by
\[ u^{(1)}(r, t) = v^{(1)}(r, t) + U_1(t). \] (2.10)

The displacement-distortion at the surface \( r = a \) in the laboratory frame \( \xi_\omega'(s) \) is therefore given by
\[ -i\omega \xi_\omega'(s) = -i\omega \xi_\omega(s) + U_{1\omega}. \] (2.11)
Here \( s \) denotes a labeled point on the surface of the undistorted sphere. We show in the next section that for a uniform sphere the velocity \( U_{1\omega} \) can be calculated from the distortion \( \xi_\omega(s) \) by use of Newton’s equation.

III. DIPOLAR DISTORTION AND FIRST ORDER MOTION

The surface displacement-distortion function \( \xi_\omega'(s) \) can be expanded as
\[ \xi_\omega'(s) = -ia [\kappa_1 v_1(s, \alpha) + \mu_1 u_1(s) + \sum_{l=2}^{\infty} [\kappa_l v_l(s, \alpha) + \mu_l u_l(s)]], \] (3.1)
with the same coefficients for \( l \geq 2 \) as in Eq. (2.9). Here we derive expressions for the two coefficients \( \kappa'_1, \mu'_1 \). We write the \( l = 1 \) contribution to the velocity at the surface \( r = a \) in the laboratory frame in terms of vector spherical harmonics as

\[
\mathbf{u}^{(1)}_{\omega}(s) = c_{A1} \mathbf{A}_1 + c_{B1} \mathbf{B}_1.
\]  

Here \( \mathbf{A}_1 = e_z \) corresponds to displacement, and the vector spherical harmonic \( \mathbf{B}_1 = e_z - 3 \cos \theta \ e_r \) corresponds to a dipolar distortion. The corresponding \( l = 1 \) contribution to the flow velocity takes the form

\[
\mathbf{u}^{(1)}_{\omega}(r) = \frac{\pi c_{A1}}{4 e^{\alpha a} k_0(\alpha a)} \mathbf{v}_1(r, \alpha) + \left( \frac{k_2(\alpha a)}{2 k_0(\alpha a)} c_{A1} - c_{B1} \right) \mathbf{u}_1(r).
\]  

The contribution to the first order pressure is

\[
p^{(1)}_{\omega}(r) = \eta \alpha^2 a \left( \frac{k_2(\alpha a)}{2 k_0(\alpha a)} c_{A1} - c_{B1} \right) \left( \frac{a}{r} \right)^2 \cos \theta.
\]  

From the stress tensor we find for the first order force exerted by the fluid on the sphere \( \mathbf{K}^{(1)}_{\omega} = K^{(1)}_{\omega} e_z \) with

\[
K^{(1)}_{\omega} = \frac{\pi c_{A1}}{4 e^{\alpha a} k_0(\alpha a)} k_{v1}(\alpha) + \left( \frac{k_2(\alpha a)}{2 k_0(\alpha a)} c_{A1} - c_{B1} \right) k_{u1}(\alpha),
\]  

with functions \( k_{v1}(\alpha), k_{u1}(\alpha) \) given by

\[
k_{v1}(\alpha) = -8\pi \frac{1 + s - is}{s - is} \eta a \quad k_{u1}(\alpha) = \frac{8\pi i}{3} s^2 \eta a,
\]  

with \( \alpha = (s - is)/a \) and scale number \( s = a \sqrt{\omega \rho / 2\eta} \). The contribution \( k_{v1}(\alpha) \) is purely viscous, and \( k_{u1}(\alpha) \) arises from the pressure.

The mean surface velocity is \( \mathbf{U}_{1\omega} = c_{A1} e_z \). For a uniform sphere this must equal the center of mass velocity, so that we obtain by Newton’s equation

\[
c_{A1} = -\frac{1}{i \omega m_0} K^{(1)}_{\omega},
\]  

where \( m_0 = 4\pi a^3 \rho_0 / 3 \) is the mass of the sphere. By substitution of Eq. (3.5) this yields a relation between the coefficients \( c_{A1}, c_{B1} \) of the form

\[
c_{A1} = \gamma_D(\omega) c_{B1}.
\]  

We find the explicit expression

\[
\gamma_D(\omega) = -i \omega m_f \mathcal{Y}_t(\omega),
\]  

where \( m_f \) is the mass of fluid displaced by the sphere, and \( \mathcal{Y}_t(\omega) \) is the translational admittance of the sphere, given by [11]

\[
m_f = \frac{4\pi}{3} a^3 \rho, \quad \mathcal{Y}_t(\omega) = \left[ -i \omega (m_0 + \frac{1}{2} m_f) + \zeta(\omega) \right]^{-1},
\]  

5
with frequency-dependent friction coefficient
\[ \zeta(\omega) = 6\pi \eta a (1 + \alpha a). \]

Here the contribution \( \frac{1}{2} m_f \) is the so-called added mass [12]. We call \( \gamma_D(\omega) \) the dipolar admittance, since it relates the convective motion of the sphere to the coefficient of dipolar distortion. The admittance may be expressed as
\[ \gamma_D(\omega) = \frac{4s^2 \rho}{4s^2 \rho_0 + [9i + (9 + 9i)s + 2s^2] \rho}. \]

For known amplitude \( c_{B1} \) of dipolar distortion the coefficient \( c_{A1} \) follows from Eq. (3.8). This determines the amplitude of first order convective motion. In turn this determines the modification of the flow pattern from dipolar form, as given by Eq. (3.3). From a comparison of Eqs. (3.1) and (3.3) we find for the coefficients \( \kappa'_1, \mu'_1 \) by use of the no-slip boundary condition
\[ \kappa'_1 = -\frac{\alpha}{2\omega} c_{A1}, \quad \mu'_1 = -\frac{1}{\omega a} \left( \frac{k_2(aa)}{2k_0(aa)} c_{A1} - c_{B1} \right). \]

Using the relation
\[ e_z = \frac{k_2(aa)}{2k_0(aa)} u_1(s) + \frac{1}{2} \alpha a v_1(s, \alpha), \]
and comparing with Eq. (2.11) we find
\[ \kappa_1 = 0, \quad \mu_1 = \frac{1}{\omega a} c_{B1}. \]

Given \( \mu_1 \) this yields \( c_{B1} \). Hence we find the velocity \( U_{1\omega} = c_{A1} \) by use of \( c_{A1} = \gamma_D(\omega) c_{B1} \). The coefficients \( \kappa'_1, \mu'_1 \) are used in the calculation of the mean swimming velocity, as discussed below.

IV. MEAN SWIMMING VELOCITY AND MEAN DISSIPATION

The first order velocity \( U_1(t) \) as calculated above affects also the first order flow pattern. The flow during the first period can be expressed in complex notation in terms of the amplitude functions as
\[ u_1^{(1)}(r) = -\omega a \left[ \kappa'_1 v_1(r, \alpha) + \mu'_1 u_1(r) + \sum_{l=2}^{\infty} \left[ \kappa_l v_l(r, \alpha) + \mu_l u_l(r) \right] \right], \]
\[ p_1^{(1)}(r) = -\eta \omega \alpha^2 a^2 \left[ \mu'_1 \left( \frac{a}{r} \right)^2 \cos \theta + \sum_{l=2}^{\infty} \mu_l \left( \frac{a}{r} \right)^{l+1} P_l(\cos \theta) \right]. \]

The mean second order flow velocity \( \bar{v}^{(2)} \) and pressure \( \bar{p}^{(2)} \) in the rest frame, moving with mean swimming velocity \( \bar{U}^{(2)} e_z \) with respect to the laboratory frame, satisfy the inhomogeneous Stokes equations [2]
\[ \eta \nabla^2 \bar{v}^{(2)} - \nabla \bar{p}^{(2)} = \frac{1}{2} \rho \text{Re} \left[ u_1^{(1)*} \cdot \nabla u_1^{(1)} \right], \quad \nabla \cdot \bar{v}^{(2)} = 0. \]
with boundary condition

\[ \mathbf{v}^{(2)}|_{r=a} = \mathbf{u}_S(\theta) = -\frac{1}{2} \text{Re} \left[ \xi^{*} \omega \cdot \nabla u^{(1)} \right] \bigg|_{r=a}. \]  

(4.3)

The mean is defined as the time-average over the period. The boundary condition is the time average of the no-slip condition calculated to second order in the amplitude of distortion,

\[ \mathbf{v}^{(2)}|_{r=a} + \xi' \cdot \nabla u^{(1)} \bigg|_{r=a} = 0, \]  

(4.4)

applied at the undistorted spherical surface. The right hand side in Eq. (4.2) represents the mean Reynolds force density \( \mathbf{f}_R^{(2)} = -\rho \mathbf{u}^{(1)} \cdot \nabla \mathbf{u}^{(1)} \). We write \( \mathbf{v}^{(2)} = \mathbf{v}_V^{(2)} + \mathbf{v}_S^{(2)} \). The volume part of the second order flow \( \mathbf{v}_V^{(2)} \) satisfies Eq. (4.2) with the boundary condition \( \mathbf{v}_{r=a}^{(2)} = 0 \). The surface part \( \mathbf{v}_S^{(2)} \) satisfies Eq. (4.2) with right hand side equal to zero and with boundary condition Eq. (4.3).

We define the multipole moment vectors \( \mathbf{\psi} \) and \( \mathbf{\hat{\psi}} \) as the one-dimensional arrays of complex coefficients

\[ \mathbf{\psi} = (\kappa_1', \mu_1', \kappa_2, \mu_2, ...), \quad \mathbf{\hat{\psi}} = (\mu_1, \kappa_2, \mu_2, ...). \]  

(4.5)

The mean swimming velocity \( \bar{U}^{(2)} = \bar{U}_2 \) and the mean rate of dissipation \( \bar{D}^{(2)} = \bar{D}_2 \) are bilinear in the vector \( \mathbf{\psi} \). The bilinear dependence can be expressed with scalar products as

\[ \bar{U}_2 = \frac{1}{2} \omega a (\mathbf{\psi} | \mathbf{B} | \mathbf{\psi}), \quad \bar{D}_2 = 8\pi \eta \omega^2 a^3 (\mathbf{\psi} | \mathbf{A} | \mathbf{\psi}), \]  

(4.6)

with hermitian matrices \( \mathbf{B} \) and \( \mathbf{A} \). The matrix \( \mathbf{B} \) has non-vanishing elements only for indices corresponding to pairs of angular numbers \( l, l-1 \) and \( l, l+1 \), and the matrix \( \mathbf{A} \) has non-vanishing elements only for indices corresponding to the pair \( l, l \). In our earlier work [8] we calculated

\[ \bar{U}_2 = \frac{1}{2} \omega a (\mathbf{\hat{\psi}} | \hat{\mathbf{B}} | \mathbf{\hat{\psi}}), \quad \bar{D}_2 = 8\pi \eta \omega^2 a^3 (\mathbf{\hat{\psi}} | \hat{\mathbf{A}} | \mathbf{\hat{\psi}}). \]  

(4.7)

with truncated matrices \( \hat{\mathbf{A}} \) and \( \hat{\mathbf{B}} \) obtained from \( \mathbf{A} \) and \( \mathbf{B} \) by dropping the first row and column. We can write

\[ \bar{U}_2 = \bar{U}_2' + \bar{U}_2'', \quad \bar{D}_2 = \bar{D}_2' + \bar{D}_2'', \]  

(4.8)

with correction terms \( \bar{U}_2' \) and \( \bar{D}_2' \).

In Appendix B of Ref. 8 we provided explicit expressions for the matrix elements of \( \hat{\mathbf{A}} \) and \( \hat{\mathbf{B}} \) up to maximum angular number \( l_{\text{max}} = L = 3 \). For our present purpose we need to evaluate in addition the elements \( A_{00}, A_{01}, A_{10} \), where the subscript 0 refers to the first element of a row or column, and the subscript 1 refers to the second element, etc.. We also need elements \( B_{02}, B_{20}, B_{03}, B_{30} \).

The explicit expressions for the elements \( A_{00}, A_{01}, A_{10} \) are

\[ A_{00} = \frac{3}{8s^6} [9 + 18s + 18s^2 + 12s^3 + 6s^4 + 2s^5], \]

\[ A_{01} = A_{10}^* = \frac{3}{4s^3} [3 + 3i + 6is - (2 - 2i)s^2]. \]  

(4.9)

The non-vanishing values of the other elements are listed in Appendix B of Ref. 8.
The matrix $\mathbf{B}$ is conveniently written as a sum of two parts

$$
\mathbf{B} = \mathbf{B}_S + \mathbf{B}_B,
$$  \hspace{1cm} (4.10)

where $\mathbf{B}_S$ follows directly from the second order surface velocity $\mathbf{\pi}_S(\theta)$, and $\mathbf{B}_B$ follows from the Reynolds force density. The calculations are performed in the same manner as before [3]. The explicit expressions for the elements $B_{S02}, B_{S20}, B_{S03}, B_{S30}$ are

$$
B_{S02} = B_{S20}^* = \frac{3 - 3i}{8s^7} [45 + 90s + (90 - 6i)s^2 + (60 - 12i)s^3 + (28 - 12i)s^4 + (8 - 8i)s^5],
$$

$$
B_{S03} = B_{S30}^* = \frac{-3i}{20s^3} [15 + 15i + 30is - 12(1 - i)s^2 - 4s^3].
$$  \hspace{1cm} (4.11)

The explicit expressions for the elements $B_{B02}, B_{B20}, B_{B03}, B_{B30}$ are

$$
B_{B02} = B_{B20}^* = \frac{-1 - i}{16s^3} [9i + 18is + (6 + 18i)s^2 - (4 + 36i)s^3
+ 4s^4 - 8s^5 + 8s^4(9i + 2s^2)e^{2s}\Gamma(0, 2s)],
$$

$$
B_{B03} = B_{B30}^* = \frac{-s - is}{120} [3i - (3 - 3i)s + 6s^2 - (8 + 8i)s^3
- is^4 - (1 - i)s^5 + 2s^4(9i + s^2)e^{s+is}\Gamma(0, s + is)],
$$  \hspace{1cm} (4.12)

where $\Gamma(0, z)$ is an incomplete Gamma-function [9]. All other elements of the matrices vanish.

V. STOKES REPRESENTATION

The matrices $\mathbf{A}(s)$ and $\mathbf{B}(s)$ are singular at $s = 0$ which causes difficulties in numerical calculations and in the discussion of the relation to swimming in the Stokes limit. As we showed earlier [3], we can choose a more convenient matrix representation by expanding the surface displacement-distortion $\mathbf{\xi}_n(\mathbf{s})$ in terms of a different set of vector functions defined on the surface of the sphere $r = a$. It is of particular interest to use the set of functions found as limiting values on the sphere surface of the modes defined in the Stokes limit [3]. The mode functions $\mathbf{u}_l(\mathbf{r})$ in the Stokes limit are the same as in Eq. (2.5), but the functions $\mathbf{v}_l(\mathbf{r}, \alpha)$ are changed to

$$
\mathbf{v}_l^0(\mathbf{r}) = \left(\frac{a}{r}\right)^l \left[ (l + 1)P_l(\cos \theta)\mathbf{e}_r + \frac{l - 2}{l}P_{l-1}(\cos \theta)\mathbf{e}_\theta \right]
= \left(\frac{a}{r}\right)^l \left[ \frac{2l + 2}{l(2l + 1)} \mathbf{A}_l - \frac{2l - 1}{2l + 1} \mathbf{B}_l \right].
$$  \hspace{1cm} (5.1)

We denote the corresponding set of superposition coefficients as $\mathbf{\psi}^0 = (\kappa^0_1, \mu^0_1, \kappa^0_2, \mu^0_2, \ldots)$ and the corresponding Stokes representation of the matrices as $\mathbf{A}^I(s)$ and $\mathbf{B}^I(s)$. The Stokes limit is denoted as $\mathbf{A}^0 = \mathbf{A}^I(0)$ and $\mathbf{B}^0 = \mathbf{B}^I(0)$.

The two sets of mode coefficients $\psi$ and $\psi^I$ in the two representations are related by

$$
\psi = \Gamma \cdot \psi^I,
$$  \hspace{1cm} (5.2)
with a transformation matrix \( T \). The matrix \( T \) is block-diagonal, as given by a factor \( \delta_{w} \), with a 2-dimensional \( T_{l} \) at order \( l \) given by the relations
\[
\kappa_{l} = \frac{\pi}{l(2l+1)e^{2}k_{l-1}(z)} \kappa_{l}', \quad \mu_{l} = \frac{1}{2l+1} \left[ 2l - 1 + 2\frac{k_{l+1}(z)}{k_{l-1}(z)} \right] \kappa_{l}' + \mu_{l}', \quad z = (1 - i)s. \quad (5.3)
\]
The relation between the two sets of matrices is
\[
A' = T^{\dagger} \cdot A \cdot T, \quad B' = T^{\dagger} \cdot B \cdot T, \quad (5.4)
\]
where \( T^{\dagger} \) is the hermitian conjugate of \( T \). The mean swimming velocity \( \overline{U}_{2} \) and the mean rate of dissipation \( \overline{D}_{2} \) can be expressed alternatively as
\[
\overline{U}_{2} = \frac{1}{2} \omega a(\psi^{I}|B^{I}|\psi^{I}), \quad \overline{D}_{2} = 8\pi \eta \omega^{2} a^{3}(\psi^{I}|A^{I}|\psi^{I}). \quad (5.5)
\]
Earlier \([3]\) we gave the expressions of the matrix elements of the truncated matrices \( \hat{A}' \) and \( \hat{B}' \) up to order \( l = 3 \). The missing elements of the matrix \( A^{I} \) read
\[
A'_{00} = 1 + \frac{2}{3} s, \quad A'_{01} = A'_{10} = 1. \quad (5.6)
\]
All other elements of the first row and column vanish. The matrix \( B^{I} \) is a sum of two terms \( B^{I} = B_{S}^{I} + B_{B}^{I} \). The missing elements of the matrix \( B_{S}^{I} \) read
\[
B_{S02}^{I} = B_{S20}^{I*} = \frac{-11 + (3 + i)s + (4 - 4i)s^{2}}{5i + s + is}, \quad B_{S03}^{I} = B_{S30}^{I*} = \frac{-i}{5}[3 - (2 + 2i)s]. \quad (5.7)
\]
The missing elements \( B_{B02}^{I} \) and \( B_{B20}^{I} \) are given by
\[
B_{B02}^{I} = B_{B20}^{I*} = \frac{-is^{2}}{90(1 + s - is)} \left[ -27i - 54s + (39 - 18i)s^{2} - (36 - 6i)s^{3} - (22 + 3i)s^{4} 
\right. 
- (4 - 2i)s^{5} + 2is^{6} - s^{2} \left( 54 + (54 - 54i)s - 60is^{2} - (24 + 24i)s^{3} - 6s^{4} - 2(1 - i)s^{5} \right) F_{+} 
\left. - 36s^{2}[1 + s + is + is^{2}]F_{-} + 24s^{2}(9 - 2is^{2})F_{2} \right], \quad (5.8)
\]
with the abbreviations
\[
F_{+} = F(s + is), \quad F_{-} = F(s - is), \quad F_{2} = F(2s), \quad (5.9)
\]
where the function \( F(z) \) with complex variable \( z \) is defined by
\[
F(z) = e^{z}E_{1}(z) = \int_{0}^{\infty} \frac{e^{-u}}{z + u} \, du. \quad (5.10)
\]
The missing elements \( B_{B03}^{I} \) and \( B_{B30}^{I} \) are given by
\[
B_{B03}^{I} = B_{B30}^{I*} = \frac{s^{2}}{90} \left[ 3 + (3 + 3i)s - 6is^{2} - (8 - 8i)s^{3} - s^{4} 
+ (1 + i)s^{5} + 2s^{4}(9 - is^{2})F_{+} \right]. \quad (5.11)
\]
The elements are found from the matrices listed in Sec. IV by use of the transformation Eq. (5.4). Elements which are not listed vanish.
VI. MEAN SWIMMING VELOCITY AND MEAN RATE OF DISSIPATION FOR SOME SIMPLE STROKES

The first order center of mass motion has an effect on the mean swimming velocity and the mean rate of dissipation given by the correction terms in Eq. (4.8). For a sphere only the dipolar surface distortion leads to oscillatory motion. Higher order multipole moments of the distortion do not couple by symmetry.

We consider the effect of first order motion on the mean swimming velocity for the simple swimmers studied in Ref. 3. As before we define the dimensionless reduced mean swimming velocity as

$$U_{\text{red}}(s) = \frac{(\psi^I|B^I(s)|\psi^I)}{(\hat{\psi}^I|A^0|\hat{\psi}^I)}.$$  \hspace{1cm} (6.1)

The denominator provides a measure of the intensity of surface agitation. Here the vector $\hat{\psi}^I$ is given by

$$\hat{\psi}^I = (\mu^I_1, \kappa^I_2, \mu^I_2, \ldots),$$  \hspace{1cm} (6.2)

with a chosen set of coefficients which are independent of $s$. Hence the vector $\psi^I = (\kappa^I_1(s), \mu^I_1, \kappa^I_2, \mu^I_2, \ldots)$, determined from Eqs. (3.13) and (5.3) by use of $c_{A1} = \gamma_D(\omega)c_{B1}$. Here Eq. (5.3) is used for $l = 1$ with left-hand side given by $\kappa^I_1, \mu^I_1$. This yields

$$\kappa^I_1(s) = \frac{-3s^2\rho}{4s^2\rho_0 + [9i + (9 + 9i)s + 3s^2]\rho} \mu^I_1.$$  \hspace{1cm} (6.4)

The moment $\kappa^I_1(s)$ may be regarded as being induced by $\mu^I_1$. We compare $U_{\text{red}}(s)$ with the previously defined quantity $[3]$

$$\hat{U}_{\text{red}}(s) = \frac{(\hat{\psi}^I|\hat{B}^I(s)|\psi^I)}{(\hat{\psi}^I|\hat{A}^0|\hat{\psi}^I)},$$  \hspace{1cm} (6.5)

which has a different numerator.

We consider first the Stokes limit $s = 0$ and the inertia limit $s \to \infty$. The upper left-hand corner of the matrices $A^0 = A^I(0)$ and $B^0 = B^I(0)$, truncated at $l = 4$, is given by Eqs. (7.17) and (7.11) of Ref. 13. We have $U_{\text{red}}(0) = \hat{U}_{\text{red}}(0)$, since $\kappa^I_1(0) = 0$. The value $U_{\text{red}}(0)$ can be evaluated for chosen stroke from the explicit matrix form. The upper left-hand corner of the matrix $B^I(\infty)$, truncated at $l = 3$ is given by

$$B^I_{13}(\infty) = \begin{pmatrix}
0 & 0 & \frac{2+23i}{5} & \frac{17i}{5} & 0 & 0 \\
0 & 0 & \frac{2-23i}{5} & \frac{3i}{35} & 0 & 0 \\
\frac{2-23i}{5} & \frac{3i}{35} & 0 & 0 & \frac{8+110i}{35} & \frac{6i}{7} \\
\frac{2+23i}{5} & \frac{3i}{35} & 0 & 0 & \frac{8-110i}{35} & \frac{6i}{7} \\
0 & 0 & \frac{8-110i}{35} & \frac{58i}{35} & 0 & 0 \\
0 & 0 & \frac{8+110i}{35} & \frac{58i}{35} & 0 & 0 \\
0 & 0 & \frac{6i}{7} & \frac{6i}{7} & 0 & 0 \\
0 & 0 & \frac{6i}{7} & \frac{6i}{7} & 0 & 0
\end{pmatrix}. \hspace{1cm} (6.6)
The subscripts 13 indicate that angular numbers \( l = 1, 2, 3 \) are involved. The matrix \( B_{13}(\infty) \) may be used to calculate the mean swimming velocity in the limit of large \( s \) for general stroke \( \hat{\boldsymbol{\psi}}^I = (\mu_1^I, \kappa_2^I, \mu_2^I, \kappa_3^I, \mu_3^I) \), together with

\[
\kappa_1^I(\infty) = \frac{-3\rho}{4\rho_0 + 3\rho} \mu_1^I. \tag{6.7}
\]

The corresponding surface agitation \( \langle \hat{\boldsymbol{\psi}}^I | \hat{\boldsymbol{A}}^0 | \hat{\boldsymbol{\psi}}^I \rangle \) follows from the matrix \( \hat{\boldsymbol{A}}^0_{13} \). From these expressions we can evaluate the limiting values \( \hat{U}_{\text{red}}(\infty) \) and \( \hat{U}_{\text{red}}(\infty) \) in the examples given below.

In our first example we consider the swimmer with a dipolar and a quadrupolar flow field, corresponding to moments \( \mu_1^I = 1, \mu_2^I = i/\sqrt{2} \), and all other moments vanishing. The dipolar and quadrupolar flow fields vary harmonically in time, and out of phase. For this swimmer the primary reduced swimming velocity \( \hat{U}_{\text{red}} = 1/\sqrt{2} = 0.701 \), independent of \( s \).

In Fig. 1 we compare \( U_{\text{red}}(s) \) with \( \hat{U}_{\text{red}}(s) \) as a function of \( s \) for \( \rho_0 = \rho \). In the Stokes limit \( s \to 0 \) the two quantities become identical. The comparison shows the effect of the linear notion on the mean swimming velocity. In situations without the linear motion the mean swimming velocity is given by \( \hat{U}_{\text{red}}(s) \).

For the stroke considered here the limiting value of \( U_{\text{red}}(s) \) at large \( s \) for general \( \rho_0 \) is given by

\[
U_{\text{red}}(\infty) = \frac{2\sqrt{2}}{5} \frac{5\rho_0 + 8\rho}{4\rho_0 + 3\rho}. \tag{6.8}
\]

For \( \rho_0 = \rho \) this takes the value \( 26\sqrt{2}/35 = 1.051 \).

In Fig. 2 we compare \( U_{\text{red}}(s) \) with \( \hat{U}_{\text{red}}(s) \) as a function of \( s \) for \( \rho_0 = \rho \) for the swimmer with stroke characterized by

\[
\mu_1^I = 1, \quad \kappa_2^I = -\mu_2^I, \quad \mu_2^I = \frac{5}{3} i \tag{6.9}
\]

This swimmer is a so-called \( B_1B_2 \)-squirmer with \( B_2/B_1 = 5 \). Ishikawa et al. [14] considered an active particle characterized by these coefficients. In the Stokes limit \( U_{\text{red}} \) and \( \hat{U}_{\text{red}} \) both tend to 48/43. For this swimmer the limiting value of \( U_{\text{red}}(s) \) at large \( s \) for general \( \rho_0 \) is given by

\[
U_{\text{red}}(\infty) = \frac{144}{43} \frac{2\rho_0 + \rho}{4\rho_0 + 3\rho}. \tag{6.10}
\]

For \( \rho_0 = \rho \) this takes the value \( 432/301 = 1.435 \).

In Fig. 3 we compare \( U_{\text{red}}(s) \) with \( \hat{U}_{\text{red}}(s) \) as a function of \( s \) for \( \rho_0 = \rho \) for the swimmer with stroke characterized by

\[
\mu_1^I = 1, \quad \kappa_2^I = -\frac{4i\sqrt{2}}{3}, \quad \mu_2^I = \frac{11i}{5\sqrt{2}}. \tag{6.11}
\]

In the Stokes limit \( U_{\text{red}} \) and \( \hat{U}_{\text{red}} \) both tend to \( 5/(3\sqrt{2}) = 1.179 \). For this swimmer the limiting value of \( U_{\text{red}}(s) \) at large \( s \) for general \( \rho_0 \) is given by

\[
U_{\text{red}}(\infty) = \frac{2\sqrt{2}}{75} \frac{205\rho_0 + 64\rho}{4\rho_0 + 3\rho}. \tag{6.12}
\]
For $\rho_0 = \rho$ this takes the value $538\sqrt{2}/525 = 1.449$.

We define the reduced mean rate of dissipation as

$$D_{\text{red}}(s) = \frac{\langle \psi^I | A^I(s) | \psi^I \rangle}{\langle \dot{\psi}^I | \dot{A}^0 | \dot{\psi}^I \rangle}.$$  \hfill (6.13)

In Fig. 4 we show the behavior for $\rho_0 = \rho$. In each case the mean rate increases in proportion to $s$ for large $s$.

In Fig. 5 we compare $U_{\text{red}}(s)$ with $\hat{U}_{\text{red}}(s)$ as a function of $s$ for $\rho_0 = \rho$ for the swimmer characterized by

$$\mu_1^I = 1, \quad \kappa_2^I = \frac{5}{3} \sqrt{\frac{230}{413}} i, \quad \mu_2^I = 0, \quad \kappa_3^I = -\frac{27}{59}, \quad \mu_3^I = 0.$$  \hfill (6.14)

Both $U_{\text{red}}(s)$ and $\hat{U}_{\text{red}}(s)$ change sign at some value of $s$. A sign change of the mean swimming velocity as a function of scale number $s$ was also observed in computer simulations of a two-sphere model by Jones et al. \[4\]. These authors also found a center of mass velocity oscillating at frequency $\omega$. We presume that the mechanism is similar to that of the model considered here.

In Fig. 6 we compare $U_{\text{red}}(s)$ with $\hat{U}_{\text{red}}(s)$ as a function of $s$ for $\rho_0 = \rho$ for the swimmer characterized by

$$\mu_1^I = 1, \quad \kappa_2^I = -1.553i, \quad \mu_2^I = 1.824i, \quad \kappa_3^I = 1.373, \quad \mu_3^I = -1.440.$$  \hfill (6.15)

For this swimmer $U_{\text{red}}(s)$ and $\hat{U}_{\text{red}}(s)$ are both positive. As in the other cases $U_{\text{red}}(s)$ is significantly larger than $\hat{U}_{\text{red}}(s)$ for large $s$. In the limit $U_{\text{red}}(\infty) = 2.109$ for $\rho_0 = \rho$.

VII. DISCUSSION

As we showed above, for a uniform deformable sphere the first order reaction force, which is linear in the amplitude of the stroke, causes a linear motion of the sphere which affects the first order flow pattern, and hence also the mean swimming velocity. The conclusion is more general. If the inner dynamics of the swimmer is such that the first order reaction force has an effect on the surface motion, then this must be taken into account, and its effect on the mean swimming velocity must be calculated. A purely kinematic theory, as we developed earlier \[2\], is valid only if the first order reaction force is fully absorbed and has no effect on the surface motion, or if the first order motion is simply prescribed along with the surface distortion.

In situations where the first order reaction force does have an effect on the surface motion, then in a complete theory the inner dynamics of the body must be considered. For a uniform body this can be circumvented by the requirement that the net surface motion is identical with that of the center of mass. For a uniform deformable sphere this leads to a calculation of the dipolar admittance, as shown in Sec. III. The calculation demonstrates that added mass has an effect on the mean swimming velocity, as was suggested earlier \[16\],\[17\].

Our theory of swimming is based on a perturbation theory to second order in the amplitude of the stroke. It is assumed that the flow remains laminar and that turbulence can be neglected in the full range of scale number. The assumption is well supported by the recent
computer simulations of a two-sphere system by Jones et al. [4]. In actual swimming the mean swimming velocity will be somewhat reduced by turbulent drag. For further confirmation of the theory it would be desirable to carry out computer simulations for a deformable sphere.
[1] B. U. Felderhof and R. B. Jones, "Dynamics of cruising swimming by a deformable sphere for two simple models". [arXiv:1810.02089][physics.flu-dyn].

[2] B. U. Felderhof and R. B. Jones, "Inertial effects in small-amplitude swimming of a finite body", Physica A 202, 94 (1994).

[3] B. U. Felderhof and R. B. Jones, "Effect of fluid inertia on swimming of a sphere in a viscous incompressible fluid", [arXiv:1803.02104][physics.flu-dyn].

[4] S. K. Jones, A. Pal Singh Bhatta, G. Katzikis, B. E. Griffith, and D. Klotsa, "Transition in motility mechanism due to inertia in a model self-propelled two-sphere swimmer", [arXiv:1801.03974][physics.flu-dyn].

[5] D. Klotsa, K. A. Baldwin, R. J. Hill, R. M. Bowley, and M. R. Swift, "Propulsion of a two-sphere swimmer", Phys. Rev. Lett. 115, 248102 (2015).

[6] B. U. Felderhof, "Effect of fluid inertia on the motion of a collinear swimmer", Phys. Rev. E 90, 063114 (2016).

[7] B. U. Felderhof and R. B. Jones, "Second harmonic generation and vortex shedding by a dipole-quadrupole and a quadrupole-octupole swimmer in a viscous incompressible fluid", [arXiv:1803.11037][physics.flu-dyn].

[8] B. U. Felderhof and R. B. Jones, "Swimming of a sphere in a viscous incompressible fluid with inertia", Fluid Dyn. Res. 49, 045510 (2017).

[9] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions (Dover, New York, 1965).

[10] A. R. Edmonds, Angular Momentum in Quantum Mechanics (Princeton University Press, Princeton (N.J.), 1974).

[11] B. U. Felderhof, "Linear response theory of the motion of a spherical particle in an incompressible fluid", Physica A 166, 492 (1990).

[12] J. Lighthill, An Informal Introduction to Theoretical Fluid Mechanics (Clarendon Press, Oxford, 1986).

[13] B. U. Felderhof and R. B. Jones, "Optimal translational swimming of a sphere at low Reynolds number", Phys. Rev. R 90, 023008 (2014).

[14] T. Ishikawa, M. P. Simmonds, and T. J. Pedley, "Hydrodynamic interaction of two swimming model microorganisms", J. Fluid Mech. 568, 119 (2006).

[15] B. U. Felderhof, "Stokesian spherical swimmers and active particles", Phys. Rev. E 91, 043018 (2015).

[16] T. L. Daniel, "Unsteady aspects of aquatic locomotion", Amer. Zool. 24,121 (1984).

[17] C. Caspersen, P. A. Berthelsen, M. Eik, C. Pakozi, and P.-L. Kjendlie, "Added mass in human swimmers: Age and gender differences", J. Biomech. 43, 2369 (2010).
Figure captions

Fig. 1
Plot of the reduced mean swimming velocity \( U_{\text{red}}(s) \) as a function of scale number \( s \) for the dipole-quadrupole swimmer with \( \rho_0 = \rho \) and stroke characterized by mode coefficients \( \mu_1^l = 1, \mu_2^l = i/\sqrt{2} \) (solid curve) compared with \( \hat{U}_{\text{red}}(s) \) for the same stroke of a swimmer without linear motion (dashed curve).

Fig. 2
Plot of the reduced mean swimming velocity \( U_{\text{red}}(s) \) as a function of scale number \( s \) for the swimmer with with \( \rho_0 = \rho \) and stroke characterized by mode coefficients specified in Eq. (6.7) (solid curve) compared with \( \hat{U}_{\text{red}}(s) \) for the same stroke of a swimmer without linear motion (dashed curve).

Fig. 3
Plot of the reduced mean swimming velocity \( U_{\text{red}}(s) \) as a function of scale number \( s \) for the swimmer with with \( \rho_0 = \rho \) and stroke characterized by mode coefficients specified in Eq. (6.9) (solid curve) compared with \( \hat{U}_{\text{red}}(s) \) for the same stroke of a swimmer without linear motion (dashed curve).

Fig. 4
Plot of the reduced mean rate of dissipation \( D_{\text{red}}(s) \) as a function of scale number \( s \) for the swimmer with with \( \rho_0 = \rho \) for the dipole-quadrupole swimmer of Fig. 1 (solid curve), for the swimmer with stroke Eq. (6.7) (long dashes), and for the swimmer with stroke Eq. (6.9) (short dashes).

Fig. 5
Plot of the reduced mean swimming velocity \( U_{\text{red}}(s) \) as a function of scale number \( s \) for the swimmer with with \( \rho_0 = \rho \) and stroke characterized by mode coefficients specified in Eq. (6.12) (solid curve) compared with \( \hat{U}_{\text{red}}(s) \) for the same stroke of a swimmer without linear motion (dashed curve).

Fig. 6
Plot of the reduced mean swimming velocity \( U_{\text{red}}(s) \) as a function of scale number \( s \) for the swimmer with with \( \rho_0 = \rho \) and stroke characterized by mode coefficients specified in Eq. (6.13) (solid curve) compared with \( \hat{U}_{\text{red}}(s) \) for the same stroke of a swimmer without linear motion (dashed curve).
FIG. 1:
FIG. 2:
FIG. 3:
FIG. 4:
FIG. 5:
FIG. 6: