Exponential Bases for Partitions of Intervals.

David Walnut
Department of Mathematical Sciences
George Mason University
Fairfax, VA USA

Joint work with: S. Revay (Novetta and GMU) and G. Pfander (Katholische Universität Eichstätt-Ingolstadt)

FFT Online, 27 September 2021
Motivation: Riesz Bases of Exponentials

Basis Extraction/Complementation

Main Results

One Interval: Extracting a Basis

Two Intervals: Beatty-Fraenkel Sequences.

Three or more intervals: Calculus of Avdonin Maps
Orthogonal Bases of Exponentials

Definition

Given a countable set $\Lambda \subseteq \mathbb{R}^d$, define $\mathcal{E}(\Lambda)$ to be the exponential system

$$\mathcal{E}(\Lambda) = \{ e_\lambda(t) : \lambda \in \Lambda \} = \{ e^{2\pi i \langle \lambda, t \rangle} : \lambda \in \Lambda \}.$$ 

Theorem

$\mathcal{E}(\mathbb{Z}^d)$ is an orthonormal basis for $L^2[0, 1]^d$. $f \in L^2[0, 1]^d$ can be written

$$f(t) = \sum_{n \in \mathbb{Z}^d} \langle f, e_n \rangle e_n(t).$$
We will be working in $d = 1$ with the following easy variant.

**Theorem**

*Given $\alpha \in \mathbb{R}$ and $a > 0$. Then with $\Lambda = \frac{\mathbb{Z} + \alpha}{a}$, $\mathcal{E}(\Lambda)$ is an orthogonal basis for $L^2(I)$ where $I$ is any interval with $|I| = a$.***

What is most important here is the *length* of the interval, and less so the interval itself.
Fundamental Question: Given a domain $\Omega \subseteq \mathbb{R}^d$, does there exist a countable set $\Lambda$ such that $\mathcal{E}(\Lambda)$ is an orthogonal basis for $L^2(\Omega)$?

Fuglede (1974) conjectured the following: A domain $\Omega \subseteq \mathbb{R}^d$ admits an orthogonal basis of the form $\mathcal{E}(\Lambda)$ if and only if $\Omega$ tiles $\mathbb{R}^d$ by $\Lambda$, that is,

- $(\Omega + \lambda) \cap (\Omega + \lambda') = \emptyset$, a.e. if $\lambda$, $\lambda'$ are distinct elements of $\Lambda$,
- $\mathbb{R}^d = \bigcup_{\lambda \in \Lambda} (\Omega + \lambda)$. 

Exponential Bases for Partitions of Intervals.
Fuglede proved that the conjecture held for $\Omega$ a fundamental domain for a lattice $\Lambda$.

The conjecture is false for $d \geq 5$ (Tao, 2004), for $d = 4$ (Matolcsi, 2005), and for $d = 3$ (Matolcsi, Koulountzakis, Balint, Mora, 2005). However, the conjecture remains unsolved in full generality for $d = 1, 2$.

If $\Omega$ is a convex body in $\mathbb{R}^d$, then the conjecture holds in all dimensions. (Lev, Matolcsi, 2019).
By passing from an orthogonal to a non-orthogonal basis, we open up new possibilities.

**Definition**
A Riesz basis of a Hilbert space $\mathcal{H}$ is the image of an orthonormal basis under a bounded, invertible operator on $\mathcal{H}$.

**Theorem**
Given $\Omega \subseteq \mathbb{R}^d$, $\mathcal{E}(\Lambda)$ is a Riesz basis of $L^2(\Omega)$ if and only if

1. $\text{span} \mathcal{E}(\Lambda) = L^2(\Omega)$ and
2. for some $0 < A, B < \infty$ and every $\{c_\lambda\} \in \ell^2(\Lambda)$,

$$A \sum_\lambda |c_\lambda|^2 \leq \int_\Omega \left| \sum_{\lambda \in \Lambda} c_\lambda e^{2\pi i \langle \lambda, x \rangle} \right|^2 dx \leq B \sum_\lambda |c_\lambda|^2$$
**Theorem**

For a Riesz basis $\mathcal{E}(\Lambda)$ of $L^2(\Omega)$ exists $\{g_\lambda\}_{\lambda \in \Lambda}$ so that for all $f \in L^2(\Omega)$ we have

$$f(x) \equiv \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle e^{2\pi i \lambda x}$$

**Theorem (Kadec 1/4-theorem)**

For $\varphi : \frac{\mathbb{Z} + \alpha}{a} \to \mathbb{R}$, $\mathcal{E}(\text{Range}(\varphi))$ is a Riesz basis for $L^2(I)$ for any interval $I$ with $|I| = a$ if

$$\sup_{k \in \mathbb{Z}} \left| \varphi\left(\frac{k+\alpha}{a}\right) - \frac{k+\alpha}{a} \right| < \frac{1}{4a}.$$
Given a domain $\Omega \subseteq \mathbb{R}^d$, does there exist a countable set $\Lambda$ such that $E(\Lambda)$ is a Riesz basis for $L^2(\Omega)$?

There is no $\Omega$ for which such a Riesz basis is known not to exist.

In relatively few cases is it known how to construct such a basis.
Theorem

Let \{I_1, I_2, \ldots, I_n\} be a collection of disjoint subintervals of \([0, 1]\). Then there exists \(\Lambda \subseteq \mathbb{Z}\) such that \(\mathcal{E}(\Lambda)\) is a Riesz basis for \(L^2(I_1 \cup I_2 \cup \cdots \cup I_n)\).

In the paper the authors recount an imaginary conversation with a graduate student who asks: Why not just find sets \(\Lambda_k\) such that \(\mathcal{E}(\Lambda_k)\) is a Riesz basis for \(L^2(I_k)\), and let

\[\Lambda = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_n?\]
Are unions of Riesz bases Riesz bases of unions?

Sometimes they are.

Exponential Bases for Partitions of Intervals.
Are unions of Riesz bases Riesz bases of unions?

Sometimes they are.

\( \mathcal{E}(2\mathbb{Z}) \) is OB of \( L^2[0, \frac{1}{2}] \)
Sometimes they are.

\[ E(2\mathbb{Z}) \text{ is OB of } L^2[0, \frac{1}{2}] \]

\[ E(2\mathbb{Z} + 1) \text{ is OB of } L^2[\frac{1}{2}, 1] \]
Sometimes they are.

\[ \mathcal{E}(2\mathbb{Z}) \text{ is OB of } L^2[0, \frac{1}{2}] \]

\[ \mathcal{E}(2\mathbb{Z} + 1) \text{ is OB of } L^2[\frac{1}{2}, 1] \]

\[ \mathcal{E}(\mathbb{Z}) = \mathcal{E}(2\mathbb{Z} + 1) \cup \mathcal{E}(2\mathbb{Z}) \text{ is ONB of } L^2[0, 1] \]
Sometimes they’re not.

\[ \Lambda_1 = \{0\} \cup \{2n - \frac{1}{4}\}_{n>0} \cup \{2n + \frac{1}{4}\}_{n<0} \]

\[ \Lambda_2 = \{2n + 1 - \frac{1}{4}\}_{n>0} \cup \{2n - 1 + \frac{1}{4}\}_{n<0} \]
Sometimes they’re not.

\[ \Lambda_1 = \{0\} \cup \{2n - \frac{1}{4}\}_{n>0} \cup \{2n + \frac{1}{4}\}_{n<0} \]

\[ \Lambda_2 = \{2n + 1 - \frac{1}{4}\}_{n>0} \cup \{2n - 1 + \frac{1}{4}\}_{n<0} \]

\( \mathcal{E}(\Lambda_1) \) is RB of \( L^2[0, \frac{1}{2}] \)
Sometimes they’re not.

\[ \Lambda_1 = \{0\} \cup \{2n - \frac{1}{4}\}_{n>0} \cup \{2n + \frac{1}{4}\}_{n<0} \]
\[ \Lambda_2 = \{2n + 1 - \frac{1}{4}\}_{n>0} \cup \{2n - 1 + \frac{1}{4}\}_{n<0} \]

\( \mathcal{E}(\Lambda_1) \) is RB of \( L^2[0, \frac{1}{2}] \)
\( \mathcal{E}(\Lambda_2) \) is RB of \( L^2[\frac{1}{2}, 1] \)
Sometimes they’re not.

\[ \Lambda_1 = \{0\} \cup \{2n - \frac{1}{4}\}_{n>0} \cup \{2n + \frac{1}{4}\}_{n<0} \]

\[ \Lambda_2 = \{2n + 1 - \frac{1}{4}\}_{n>0} \cup \{2n - 1 + \frac{1}{4}\}_{n<0} \]

\[ \mathcal{E}(\Lambda_1) \text{ is RB of } L^2[0, \frac{1}{2}] \]

\[ \mathcal{E}(\Lambda_2) \text{ is RB of } L^2[\frac{1}{2}, 1] \]

\[ \mathcal{E}(\Lambda_1 \cup \Lambda_2) \text{ is not RB of } L^2[0, 1] \]
Theorem (Basis extraction)

Suppose that for $\Lambda \subseteq \mathbb{R}$, $\mathcal{E}(\Lambda)$ is a Riesz basis for $L^2[0,1]$. Then for every $0 < \alpha < 1$ there exists $\Lambda' \subseteq \Lambda$ such that $\mathcal{E}(\Lambda')$ is a Riesz basis for $L^2[0, \alpha]$.

**Question:** Is it necessarily true that $\mathcal{E}(\Lambda \setminus \Lambda')$ is a Riesz basis for $L^2[\alpha, 1]$?
Theorem (Basis complementation)

Let $0 < \alpha < 1$ and suppose that for $\Lambda \subseteq \mathbb{R}$, $\mathcal{E}(\Lambda)$ is a Riesz basis for $L^2[0, \alpha]$. Then there exists $\Lambda' \supseteq \Lambda$ such that $\mathcal{E}(\Lambda')$ is a Riesz basis for $L^2[0, 1]$.

**Question:** Is it necessarily true that $\mathcal{E}(\Lambda' \setminus \Lambda)$ is a Riesz basis for $L^2[\alpha, 1]$?
\[ \Lambda = \{2n\}_{n \leq 0} \cup \{2n - 1 + \frac{1}{8}\}_{n > 0} \]

\(\Lambda\) is a perturbation of \(\{2n - \frac{7}{16}\}_{n \in \mathbb{Z}}\)
\[ \Lambda = \{2n\}_{n \leq 0} \cup \{2n - 1 + \frac{1}{8}\}_{n > 0} \]

\(\Lambda\) is a perturbation of \(\{2n - \frac{7}{16}\}_{n \in \mathbb{Z}}\)

\(\mathcal{E}(\Lambda)\) is RB of \(L^2[0, \frac{1}{2}]\)
\[ \Lambda = \{2n\}_{n \leq 0} \cup \{2n - 1 + \frac{1}{8}\}_{n > 0} \]

\(\Lambda\) is a perturbation of \(\{2n - \frac{7}{16}\}_{n \in \mathbb{Z}}\)

\(\mathcal{E}(\Lambda)\) is RB of \(L^2[0, \frac{1}{2}]\)
\[ \Lambda = \{2n\}_{n \leq 0} \cup \{2n - 1 + \frac{1}{8}\}_{n > 0} \]

\[ \Lambda^\circ = \{2n\}_{n \in \mathbb{Z}} \cup \{2n - 1 + \frac{1}{8}\}_{n > 0} \cup \{2n + 1 - \frac{1}{8}\}_{n < 0} \]
Answer: No. (Dae Gwan Lee)

\[
\Lambda = \{2n\}_{n \leq 0} \cup \{2n - 1 + \frac{1}{8}\}_{n > 0}
\]

\[
\Lambda^\circ = \{2n\}_{n \in \mathbb{Z}} \cup \{2n - 1 + \frac{1}{8}\}_{n > 0} \cup \{2n + 1 - \frac{1}{8}\}_{n < 0}
\]

\(\mathcal{E}(\Lambda)\) is RB of \(L^2[0, \frac{1}{2}]\)
\[ \Lambda = \{ 2n \} \{ n \leq 0 \} \cup \{ 2n - 1 + \frac{1}{8} \} \{ n > 0 \} \]

\[ \Lambda^\circ = \{ 2n \} \{ n \in \mathbb{Z} \} \cup \{ 2n - 1 + \frac{1}{8} \} \{ n > 0 \} \cup \{ 2n + 1 - \frac{1}{8} \} \{ n < 0 \} \]

\[ \mathcal{E}(\Lambda) \text{ is RB of } L^2[0, \frac{1}{2}] \]
Answer: No. (Dae Gwan Lee)

\[ \Lambda = \{2n\}_{n \leq 0} \cup \{2n - 1 + \frac{1}{8}\}_{n > 0} \]

\[ \Lambda^\circ = \{2n\}_{n \in \mathbb{Z}} \cup \{2n - 1 + \frac{1}{8}\}_{n > 0} \cup \{2n + 1 - \frac{1}{8}\}_{n < 0} \]

\( \mathcal{E}(\Lambda) \) is RB of \( L^2[0, \frac{1}{2}] \)

\( \mathcal{E}(\Lambda^\circ) \) is RB of \( L^2[0, 1] \)
\[ \Lambda^0 \setminus \Lambda = \{2n\}_{n>0} \cup \{2n + 1 - \frac{1}{8}\}_{n<0} \]

\((\Lambda^0 \setminus \Lambda) \cup \{0\}\) is a perturbation of \(\{2n + \frac{7}{16}\}_{n \in \mathbb{Z}}\)
\( \Lambda^o \setminus \Lambda = \{2n\}_{n>0} \cup \{2n + 1 - \frac{1}{8}\}_{n<0} \)

\((\Lambda^o \setminus \Lambda) \cup \{0\}\) is a perturbation of \(\{2n + \frac{7}{16}\}_{n \in \mathbb{Z}}\)

\(E(\Lambda)\) is RB of \(L^2[0, \frac{1}{2}]\)

\(E(\Lambda^o)\) is RB of \(L^2[0, 1]\)
\[ \Lambda^\circ \setminus \Lambda = \{2n\}_{n>0} \cup \{2n + 1 - \frac{1}{8}\}_{n<0} \]

\[(\Lambda^\circ \setminus \Lambda) \cup \{0\} \text{ is a perturbation of } \{2n + \frac{7}{16}\}_{n \in \mathbb{Z}}\]

\[\mathcal{E}(\Lambda) \text{ is RB of } L^2[0, \frac{1}{2}] \]
\[\mathcal{E}(\Lambda^\circ \setminus \Lambda) \text{ is not RB of } L^2[\frac{1}{2}, 1]\]

\[\mathcal{E}(\Lambda^\circ) \text{ is RB of } L^2[0, 1]\]
The following result of Lyubarski and Seip (2001) shows that extraction is always possible.

**Theorem**

Let $\mathcal{E}(\Lambda)$ be a Riesz basis of exponentials for $L^2[0, 1]$. For each $0 < a < 1$, there is a splitting

$$\Lambda = \Lambda' \cup \Lambda'', \quad \Lambda' \cap \Lambda'' = \emptyset$$

such that $\mathcal{E}(\Lambda')$ and $\mathcal{E}(\Lambda'')$ are Riesz bases for $L^2[0, a]$ and $L^2[a, 1]$ respectively.
If $\mathcal{E}(\Lambda)$ is an orthogonal basis, then extraction and complementation always go together.

**Theorem (Meyer, Matei (2009), Bownik, Casazza, Marcus, Speegle (2016))**

*Let $S \subseteq [0, 1]$ and suppose that for some $\Lambda \subseteq \mathbb{Z}$, $\mathcal{E}(\Lambda)$ is a Riesz basis for $L^2(S)$. Then $\mathcal{E}(\mathbb{Z} \setminus \Lambda)$ is a Riesz basis for $L^2([0, 1] \setminus S)$.***

Interestingly, Lee’s example seems to show that the assumption of orthogonality cannot be weakened.
Main Results

Theorem (Pfander, Revay, DW 2018)

Given a partition \(0 = a_0 < a_1 < a_2 < \cdots < a_n = 1\) of \([0, 1]\), there exists a partition of \(\mathbb{Z}\) into \(\Lambda_1, \ldots, \Lambda_n\) such that for each \(k\), \(\mathcal{E}(\Lambda_k)\) is a Riesz basis of \(L^2[a_{k-1}, a_k]\). In addition, \(\bigcup_{r=k}^{\ell} \mathcal{E}(\Lambda_r)\) is a Riesz basis of \(L^2[a_{k-1}, a_\ell]\).

\[
\begin{align*}
a_0 &= 0 & a_1 & a_2 & a_3 & a_4 & 1 = a_5 \\
\mathcal{E}(\Lambda_1) & & \mathcal{E}(\Lambda_2) & \mathcal{E}(\Lambda_3) & \mathcal{E}(\Lambda_4) & \mathcal{E}(\Lambda_5) \\
\end{align*}
\]

\[
\begin{align*}
a_0 &= 0 & a_1 & a_2 & a_3 & a_4 & 1 = a_5 \\
\mathcal{E}(\Lambda_2) \cup \mathcal{E}(\Lambda_3) \cup \mathcal{E}(\Lambda_4) \\
\end{align*}
\]
Main Results

Theorem (Pfander, Revay, DW 2018)

Let $b_1, \ldots, b_n > 0$ with $\sum_{j=1}^{n} b_j = 1$. Then there exist pairwise disjoint sets $\Lambda_1, \ldots, \Lambda_n \subseteq \mathbb{Z}$ such that $\bigcup_{j=1}^{n} \Lambda_j = \mathbb{Z}$ and for any $J \subseteq \{1, \ldots, n\}$, $\bigcup_{j \in J} E(\Lambda_j)$ is a Riesz basis for any interval of length $\sum_{j \in J} b_j$.

$$
\begin{array}{cccccc}
& b_1 & & b_2 & & b_3 & & b_4 & & b_5 \\
\hline
\mathcal{E}(\Lambda_1) & & \mathcal{E}(\Lambda_2) & & \mathcal{E}(\Lambda_3) & & \mathcal{E}(\Lambda_4) & & \mathcal{E}(\Lambda_5) \\
& b_2 + b_4 + b_5 & & & & & & & \\
\hline
\mathcal{E}(\Lambda_2) \cup \mathcal{E}(\Lambda_4) \cup \mathcal{E}(\Lambda_5) & & & & & & & & \\
\end{array}
$$
Theorem (Pfander, Revay, DW 2018)

Let $b_1, b_2, \ldots > 0$ with $\sum_{j=1}^{\infty} b_j = 1$ and $K \in \mathbb{N}$. Then there exist pairwise disjoint sets $\Lambda_1, \Lambda_2, \ldots \subseteq \mathbb{Z}$ such that for any $J \subseteq \mathbb{N}$ with $|J| \leq K$ or $|\mathbb{N} \setminus J| \leq K$, $\bigcup_{j \in J} \mathcal{E}(\Lambda_j)$ is a Riesz basis for any interval of length $\sum_{j \in J} b_j$. 

![Diagram showing the sets and their corresponding basis functions.]
Theorem (Avdonin 1974)

For $\varphi : \mathbb{Z} + \alpha a \to \mathbb{R}$ injective with separated range, $\mathcal{E}(\text{Range}(\varphi))$ is a Riesz basis for $L^2[0, a]$ if there exists $R > 0$ such that

$$\sup_{m \in \mathbb{Z}} \left| \frac{1}{R} \sum_{\frac{k+\alpha}{a} \in [mR, (m+1)R]} \varphi\left(\frac{k+\alpha}{a}\right) - \frac{k+\alpha}{a} \right| < \frac{1}{4a}.$$ 

- Says essentially that if a separated set $\Lambda$ is “on average” close to a set whose exponentials form a Riesz basis for $L^2(I)$ ($I$ an interval), then $\mathcal{E}(\Lambda)$ is also a Riesz basis for $L^2(I)$.
- The above is not the most general statement of the theorem, but is more than good enough for our purposes.
Definition (Avdonin map)

Let $\epsilon, a > 0$ and $\alpha \in \mathbb{R}$. An injective map $\varphi: \frac{\mathbb{Z} + \alpha}{a} \rightarrow \mathbb{R}$ with separated range is an $\epsilon$-Avdonin map for $\frac{\mathbb{Z} + \alpha}{a}$ if for all $R > 0$ sufficiently large,

$$\sup_{m \in \mathbb{Z}} \left| \frac{1}{R} \sum_{\frac{k+\alpha}{a} \in [mR,(m+1)R]} \varphi\left(\frac{k + \alpha}{a}\right) - \left(\frac{k + \alpha}{a}\right) \right| < \epsilon. \quad (1)$$

Theorem

If $\varphi$ is an $\epsilon$-Avdonin map for $\frac{\mathbb{Z} + \alpha}{a}$ with $\epsilon \leq 1/4$, then $\mathcal{E}\left(\text{Range}(\varphi)\right)$ is a Riesz basis of exponentials for $L^2(I)$ for any interval $I$ with $|I| = a$. 

Walnut (GMU) Exponential Bases for Partitions of Intervals.
Our first goal is to prove the following theorem.

**Theorem (Avdonin 1991, Seip 1995)**

Given $0 < a \leq 1$, there exists $\Lambda \subseteq \mathbb{Z}$ such that $\mathcal{E}(\Lambda)$ is a Riesz basis for $L^2[0, a]$.

- $a$ irrational is the interesting case.
- We know that if $\Gamma = \frac{\mathbb{Z} + \frac{1}{2}}{a}$, then $\mathcal{E}(\Gamma)$ is a Riesz basis for $L^2[0, a]$.
- Round each element of $\Gamma$ to the nearest element of $\mathbb{Z} + \frac{1}{2}$. For any $x \in \mathbb{R}$, this is just $\lfloor x \rfloor + \frac{1}{2}$. 
Look at \( a = \frac{\sqrt{2}}{2} \).

\[
\frac{\mathbb{Z} + \frac{1}{2}}{a} = \sqrt{2}\mathbb{Z} + \frac{1}{\sqrt{2}}
\]
Look at $a = \sqrt{2}/2$.

$$a = \frac{1}{\sqrt{2}}$$

$$\frac{\mathbb{Z} + \frac{1}{2}}{a} = \sqrt{2}\mathbb{Z} + \frac{1}{\sqrt{2}}$$
Look at $a = \sqrt{2}/2$.

\[
a = \frac{1}{\sqrt{2}}
\]

\[
\frac{\mathbb{Z} + \frac{1}{2}}{a} = \sqrt{2}\mathbb{Z} + \frac{1}{\sqrt{2}}
\]
Look at $a = \sqrt{2}/2$.

$$a = \frac{1}{\sqrt{2}}$$

$$\frac{\mathbb{Z} + \frac{1}{2}}{a} = \sqrt{2}\mathbb{Z} + \frac{1}{\sqrt{2}}$$
What is the “average perturbation”?

**Theorem (Weyl Equidistribution Theorem)**

*Given* $\alpha$ *irrational,*

$$\lim_{R \to \infty} \frac{1}{R} \sum_{k=1}^{R} k \alpha \mod 1 = \frac{1}{2}$$

This is not quite what we want.
Theorem (Weyl-Khinchin)

Let a irrational and $\epsilon > 0$. Then for all $R$ sufficiently large,

$$\sup_{m \in \mathbb{Z}} \left| \frac{1}{R} \sum_{k=mR}^{(m+1)R-1} \frac{k + \frac{1}{2}}{a} \mod 1 - \frac{1}{2} \right| < \epsilon.$$ 

Note that

$$\left| \frac{1}{R} \sum_{k=mR}^{(m+1)R-1} \frac{k + \frac{1}{2}}{a} \mod 1 - \frac{1}{2} \right| = \left| \frac{1}{R} \sum_{k=mR}^{(m+1)R-1} \frac{k + \frac{1}{2}}{a} - \left( \left\lfloor \frac{k + \frac{1}{2}}{a} \right\rfloor + \frac{1}{2} \right) \right| < \epsilon.$$
Consequently,\[\varphi: \frac{\mathbb{Z} + \frac{1}{2}}{a} \to \mathbb{R}\]
given by\[\varphi\left(\frac{k + \frac{1}{2}}{a}\right) = \left\lfloor \frac{k + \frac{1}{2}}{a} \right\rfloor_{\mathbb{Z}} + \frac{1}{2}\]
is an $\epsilon$-Avdonin map, \textit{for every} $\epsilon > 0$.
Taking $\epsilon \leq \frac{1}{4a}$ gives the result.
In our previous example, look what happens when we also consider the interval \([a, 1]\).

\[
\frac{\mathbb{Z} + \frac{1}{2}}{a} = \sqrt{2}\mathbb{Z} + \frac{1}{\sqrt{2}}
\]
In our previous example, look what happens when we also consider the interval $[a, 1]$.

\[ a = \frac{1}{\sqrt{2}} \]

\[ \frac{n + \frac{1}{2}}{a} = \sqrt{2}n + \frac{1}{\sqrt{2}} \]
In our previous example, look what happens when we also consider the interval \([a, 1]\).

\[
\begin{align*}
0 & \quad a = \frac{1}{\sqrt{2}} \\
\frac{\mathbb{Z} + \frac{1}{2}}{a} &= \sqrt{2}\mathbb{Z} + \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \quad b = 1 - \frac{1}{\sqrt{2}} \\
\frac{\mathbb{Z} + \frac{1}{2}}{b} & \approx 3.41\mathbb{Z} + 1.71
\end{align*}
\]
In our previous example, look what happens when we also consider the interval \([a, 1]\).

\[
\frac{1}{\sqrt{2}} \quad b = 1 - \frac{1}{\sqrt{2}} \\
\frac{\mathbb{Z} + \frac{1}{2}}{a} = \sqrt{2}\mathbb{Z} + \frac{1}{\sqrt{2}} \\
\frac{\mathbb{Z} + \frac{1}{2}}{b} \approx 3.41\mathbb{Z} + 1.71
\]
In our previous example, look what happens when we also consider the interval \([a, 1]\).

\[
\begin{align*}
a &= \frac{1}{\sqrt{2}} \\
\frac{\mathbb{Z} + \frac{1}{2}}{a} &= \sqrt{2}\mathbb{Z} + \frac{1}{\sqrt{2}} \\
b &= 1 - \frac{1}{\sqrt{2}} \\
\frac{\mathbb{Z} + \frac{1}{2}}{b} &\approx 3.41\mathbb{Z} + 1.71
\end{align*}
\]
Beatty sequences

**Theorem**

For $a, b$ irrational with $a + b = 1$, the sets $\mathcal{A} = \left\{ \left\lfloor \frac{k}{a} \right\rfloor \right\}_{k \in \mathbb{N}}$ and $\mathcal{B} = \left\{ \left\lfloor \frac{k}{b} \right\rfloor \right\}_{k \in \mathbb{N}}$ partition $\mathbb{N}$.

**Proof.**

Clearly $\mathcal{A}$ and $\mathcal{B}$ can never coincide.
Given $N \in \mathbb{N}$, $|\mathcal{A} \cap [0, N)| = \lfloor aN \rfloor$ and $|\mathcal{B} \cap [0, N)| = \lfloor bN \rfloor$

$$aN - 1 < \lfloor aN \rfloor < aN, \quad \text{and} \quad bN - 1 < \lfloor bN \rfloor < bN$$

and summing

$$N - 2 = aN - 1 + bN - 1 < \lfloor aN \rfloor + \lfloor bN \rfloor < aN + bN = N.$$ 

Hence $|(\mathcal{A} \cup \mathcal{B}) \cap [0, N)| = N - 1.$
Beatty sequences arose as a solution to a problem posed in the Mathematical Monthly in 1926, but were known and mentioned in the nineteenth century by Lord Rayleigh in relation to the study of sound waves.

In 1969, Fraenkel considered sequences of the form \( \lfloor n\alpha + \gamma \rfloor : n \in \mathbb{Z} \) which he referred to as inhomogeneous Beatty sequences. It is his results that we need here.

**Theorem (Beatty-Fraenkel)**

Let \( a, b \) irrational with \( a + b = 1 \). Then the sets \( \left\lfloor \frac{k + \frac{1}{2}}{a} \right\rfloor \}_{k \in \mathbb{Z}} \) and \( \left\lfloor \frac{\ell + \frac{1}{2}}{b} \right\rfloor \}_{\ell \in \mathbb{Z}} \) partition \( \mathbb{Z} \).
Combining Beatty-Fraenkel and Weyl-Khinchin:

**Theorem (Pfander, Revay, DW, 2018)**

Given $a, b > 0$, there exist injective maps

$$
\varphi: \frac{\mathbb{Z} + \frac{1}{2}}{a} \rightarrow \frac{\mathbb{Z} + \frac{1}{2}}{a + b}, \quad \psi: \frac{\mathbb{Z} + \frac{1}{2}}{b} \rightarrow \frac{\mathbb{Z} + \frac{1}{2}}{a + b}
$$

such that

(1) $\text{Range}(\varphi)$ and $\text{Range}(\psi)$ partition $\frac{\mathbb{Z} + \frac{1}{2}}{a + b}$,

(2) for every $\epsilon > 0$, $\varphi$ and $\psi$ are $\epsilon$-Avdonin maps for $\frac{\mathbb{Z} + \frac{1}{2}}{a}$ and $\frac{\mathbb{Z} + \frac{1}{2}}{b}$ (resp.).
By Beatty-Fraenkel, (1) is satisfied with

\[
\varphi\left(\frac{k + \frac{1}{2}}{a}\right) = \left\lfloor \frac{k + \frac{1}{2}}{a} \right\rfloor + \frac{1}{2}, \quad \psi\left(\frac{k + \frac{1}{2}}{b}\right) = \left\lfloor \frac{k + \frac{1}{2}}{b} \right\rfloor + \frac{1}{2}
\]

Since both Range (\(\varphi\)) and Range (\(\psi\)) come from rounding, Weyl-Khinchin implies that the average perturbation from the lattices \(\mathbb{Z} + \frac{1}{2a}\) and \(\mathbb{Z} + \frac{1}{2b}\) can be made as small as desired. This is (2).

Taking \(a + b = 1\) gives the partition result for two intervals.
Three intervals

\[ b_1 = \frac{1}{5} \quad \frac{\frac{Z+\frac{1}{2}}{b_1}}{b_1 + b_2} = \sqrt{2}Z + \frac{1}{\sqrt{2}} \]

\[ b_2 = \frac{1}{\sqrt{2}} - \frac{1}{5} \]

\[ b_3 = 1 - \frac{1}{\sqrt{2}} \]

\[ \frac{Z+\frac{1}{2}}{b_1} = 5Z + \frac{5}{2} \]

\[ \frac{Z+\frac{1}{2}}{b_2} \approx 1.97Z + 0.99 \]

\[ \frac{Z+\frac{1}{2}}{b_3} \approx 3.41Z + 1.71 \]
Three intervals

\[ \frac{Z + \frac{1}{2}}{b_1 + b_2} = \sqrt{2}Z + \frac{1}{\sqrt{2}} \]

\[ b_2 = \frac{1}{\sqrt{2}} - \frac{1}{5} \]

\[ b_3 = 1 - \frac{1}{\sqrt{2}} \]

\[ \frac{Z + \frac{1}{2}}{b_1} = 5Z + \frac{5}{2} \]

\[ \frac{Z + \frac{1}{2}}{b_2} \approx 1.97Z + 0.99 \]

\[ \frac{Z + \frac{1}{2}}{b_3} \approx 3.41Z + 1.71 \]
Three intervals

\[ b_1 = \frac{1}{5}, \quad b_2 = \frac{1}{\sqrt{2}} - \frac{1}{5}, \quad b_3 = 1 - \frac{1}{\sqrt{2}} \]

\[ \frac{z + \frac{1}{2}}{b_1} = 5z + \frac{5}{2}, \quad \frac{z + \frac{1}{2}}{b_2} \approx 1.97z + 0.99, \quad \frac{z + \frac{1}{2}}{b_3} \approx 3.41z + 1.71 \]
Three intervals

\[ \frac{Z + \frac{1}{2}}{b_1 + b_2} = \sqrt{2}Z + \frac{1}{\sqrt{2}} \]

\[ b_1 = \frac{1}{5} \]

\[ b_2 = \frac{1}{\sqrt{2}} - \frac{1}{5} \]

\[ \frac{1}{\sqrt{2}} \]

\[ b_3 = 1 - \frac{1}{\sqrt{2}} \]

\[ \frac{Z + \frac{1}{2}}{b_1} = 5Z + \frac{5}{2} \]

\[ \frac{Z + \frac{1}{2}}{b_2} \approx 1.97Z + 0.99 \]

\[ \frac{Z + \frac{1}{2}}{b_3} \approx 3.41Z + 1.71 \]
Three intervals

$\frac{Z + \frac{1}{2}}{b_1 + b_2} = \sqrt{2}Z + \frac{1}{\sqrt{2}}$

$\frac{Z + \frac{1}{2}}{b_2} = \frac{1}{\sqrt{2}} - \frac{1}{5}$

$\frac{Z + \frac{1}{2}}{b_3} = 1 - \frac{1}{\sqrt{2}}$

$\frac{Z + \frac{1}{2}}{b_1} = 5Z + \frac{5}{2}$

$\frac{Z + \frac{1}{2}}{b_2} \approx 1.97Z + 0.99$

$\frac{Z + \frac{1}{2}}{b_3} \approx 3.41Z + 1.71$
Three intervals

\[
\begin{align*}
0 & \quad b_1 = \frac{1}{5} \\
\frac{z + \frac{1}{2}}{b_1 + b_2} & = \sqrt{2}z + \frac{1}{\sqrt{2}} \\
1 & \quad b_2 = \frac{1}{\sqrt{2}} - \frac{1}{5} \\
\frac{1}{\sqrt{2}} & \quad b_3 = 1 - \frac{1}{\sqrt{2}} \\
\frac{z + \frac{1}{2}}{b_1} & = 5z + \frac{5}{2} \\
\frac{z + \frac{1}{2}}{b_2} & \approx 1.97z + 0.99 \\
\frac{z + \frac{1}{2}}{b_3} & \approx 3.41z + 1.71
\end{align*}
\]
Three intervals

\[ \frac{Z + \frac{1}{2}}{b_1 + b_2} = \sqrt{2}Z + \frac{1}{\sqrt{2}} \]

\[ b_1 = \frac{1}{5}, \quad b_2 = \frac{1}{\sqrt{2}} - \frac{1}{5}, \quad b_3 = 1 - \frac{1}{\sqrt{2}} \]

\[ \frac{Z + \frac{1}{2}}{b_1} = 5Z + \frac{5}{2} \]

\[ \frac{Z + \frac{1}{2}}{b_2} \approx 1.97Z + 0.99 \]

\[ \frac{Z + \frac{1}{2}}{b_3} \approx 3.41Z + 1.71 \]
Three intervals

\[ \frac{Z + \frac{1}{2}}{b_1 + b_2} = \sqrt{2}Z + \frac{1}{\sqrt{2}} \]

\[ b_1 = \frac{1}{5} \]

\[ \frac{Z + \frac{1}{2}}{b_2} = \frac{1}{\sqrt{2}} - \frac{1}{5} \approx 1.97Z + 0.99 \]

\[ b_2 = \frac{1}{\sqrt{2}} - \frac{1}{5} \]

\[ \frac{Z + \frac{1}{2}}{b_3} = 1 - \frac{1}{\sqrt{2}} \approx 3.41Z + 1.71 \]

\[ b_3 = 1 - \frac{1}{\sqrt{2}} \]
Three intervals

\[
\begin{align*}
0 & \quad b_1 = \frac{1}{5} \\
\frac{z + \frac{1}{2}}{b_1 + b_2} & = \sqrt{2}z + \frac{1}{\sqrt{2}} \\
\frac{z + \frac{1}{2}}{b_1} & = 5z + \frac{5}{2} \\
\frac{z + \frac{1}{2}}{b_2} & \approx 1.97z + 0.99 \\
\frac{z + \frac{1}{2}}{b_3} & \approx 3.41z + 1.71 \\
1 & \quad b_3 = 1 - \frac{1}{\sqrt{2}}
\end{align*}
\]
Three intervals

\[ \frac{Z + \frac{1}{2}}{b_1 + b_2} = \sqrt{2}Z + \frac{1}{\sqrt{2}} \]

\[ b_1 = \frac{1}{5} \]

\[ \frac{Z + \frac{1}{2}}{b_1} = 5Z + \frac{5}{2} \]

\[ b_2 = \frac{1}{\sqrt{2}} - \frac{1}{5} \]

\[ \frac{1}{\sqrt{2}} \approx 1.97Z + 0.99 \]

\[ b_3 = 1 - \frac{1}{\sqrt{2}} \]

\[ \frac{Z + \frac{1}{2}}{b_3} \approx 3.41Z + 1.71 \]
Three intervals

\[
\begin{align*}
0 & \quad b_1 = \frac{1}{5} \\
\frac{z + \frac{1}{2}}{b_1 + b_2} & = \sqrt{2}z + \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & = b_2 = \frac{1}{\sqrt{2}} - \frac{1}{5} \\
\frac{z + \frac{1}{2}}{b_2} & \approx 1.97z + 0.99 \\
\frac{z + \frac{1}{2}}{b_3} & \approx 3.41z + 1.71 \\
1 & \quad b_3 = 1 - \frac{1}{\sqrt{2}}
\end{align*}
\]
Three intervals

\[
\frac{Z + \frac{1}{2}}{b_1 + b_2} = \sqrt{2}Z + \frac{1}{\sqrt{2}}
\]

\[
b_1 = \frac{1}{5}
\]

\[
b_2 = \frac{1}{\sqrt{2}} - \frac{1}{5}
\]

\[
b_3 = 1 - \frac{1}{\sqrt{2}}
\]

\[
\frac{Z + \frac{1}{2}}{b_1} = 5Z + \frac{5}{2}
\]

\[
\frac{Z + \frac{1}{2}}{b_2} \approx 1.97Z + 0.99
\]

\[
\frac{Z + \frac{1}{2}}{b_3} \approx 3.41Z + 1.71
\]
Three intervals

\[ \frac{z + \frac{1}{2}}{b_1 + b_2} = \sqrt{2}z + \frac{1}{\sqrt{2}} \]

\[ b_1 = \frac{1}{5} \]

\[ b_2 = \frac{1}{\sqrt{2}} - \frac{1}{5} \]

\[ \frac{z + \frac{1}{2}}{b_3} \approx 1.97z + 0.99 \]

\[ b_3 = 1 - \frac{1}{\sqrt{2}} \]

\[ \frac{z + \frac{1}{2}}{b_3} \approx 3.41z + 1.71 \]
Three intervals

\[ \frac{Z + \frac{1}{2}}{b_1 + b_2} = \sqrt{2}Z + \frac{1}{\sqrt{2}} \]

\[ b_1 = \frac{1}{5} \]

\[ b_2 = \frac{1}{\sqrt{2}} - \frac{1}{5} \]

\[ b_3 = 1 - \frac{1}{\sqrt{2}} \]

\[ \frac{Z + \frac{1}{2}}{b_1} = 5Z + \frac{5}{2} \]

\[ \frac{Z + \frac{1}{2}}{b_2} \approx 1.97Z + 0.99 \]

\[ \frac{Z + \frac{1}{2}}{b_3} \approx 3.41Z + 1.71 \]

Walnut (GMU) Exponential Bases for Partitions of Intervals.
Three intervals

\[ \frac{z + \frac{1}{2}}{b_1 + b_2} = \sqrt{2}z + \frac{1}{\sqrt{2}} \]

\[ b_1 = \frac{1}{5} \]

\[ \frac{z + \frac{1}{2}}{b_2} = 5z + \frac{5}{2} \approx 1.97z + 0.99 \]

\[ \frac{z + \frac{1}{2}}{b_3} = \frac{1}{\sqrt{2}} \]

\[ b_2 = \frac{1}{\sqrt{2}} - \frac{1}{5} \]

\[ b_3 = 1 - \frac{1}{\sqrt{2}} \]

\[ \frac{z + \frac{1}{2}}{b_3} \approx 3.41z + 1.71 \]

Walnut (GMU) Exponential Bases for Partitions of Intervals.
Three intervals

\[
\begin{align*}
0 & \quad \frac{Z + \frac{1}{2}}{b_1} = 5Z + \frac{5}{2} \\
\frac{1}{b_1 + b_2} & = \sqrt{2}Z + \frac{1}{\sqrt{2}} \\
\frac{1}{b_2} & = \frac{1}{\sqrt{2}} - \frac{1}{5} \\
\frac{1}{b_3} & = 1 - \frac{1}{\sqrt{2}} \\
\frac{Z + \frac{1}{2}}{b_1} & \approx 1.97Z + 0.99 \\
\frac{Z + \frac{1}{2}}{b_2} & \approx 3.41Z + 1.71
\end{align*}
\]
Three intervals

\[
\begin{align*}
\frac{Z + \frac{1}{2}}{b_1 + b_2} &= \sqrt{2}Z + \frac{1}{\sqrt{2}} \\
\frac{Z + \frac{1}{2}}{b_2} &\approx 1.97Z + 0.99 \\
\frac{Z + \frac{1}{2}}{b_3} &\approx 3.41Z + 1.71
\end{align*}
\]
Three intervals

\[
\begin{align*}
    b_1 &= \frac{1}{5} \\
    b_2 &= \frac{1}{\sqrt{2}} - \frac{1}{5} \\
    b_3 &= 1 - \frac{1}{\sqrt{2}}
\end{align*}
\]

\[
\begin{align*}
    \frac{z + \frac{1}{2}}{b_1} &= 5z + \frac{5}{2} \\
    \frac{z + \frac{1}{2}}{b_2} &\approx 1.97z + 0.99 \\
    \frac{z + \frac{1}{2}}{b_3} &\approx 3.41z + 1.71
\end{align*}
\]
Three intervals

\[ \frac{Z + \frac{1}{2}}{b_1} = \sqrt{2}Z + \frac{1}{\sqrt{2}} \]
\[ \frac{Z + \frac{1}{2}}{b_2} \approx 1.97Z + 0.99 \]
\[ \frac{Z + \frac{1}{2}}{b_3} \approx 3.41Z + 1.71 \]
Three intervals

\[ \frac{\lfloor z + \frac{1}{2} \rfloor}{b_1 + b_2} = \sqrt{2}z + \frac{1}{\sqrt{2}} \]

\[ b_2 = \frac{1}{\sqrt{2}} - \frac{1}{5} \]

\[ \frac{\lfloor z + \frac{1}{2} \rfloor}{b_1} = 5z + \frac{5}{2} \]

\[ \frac{\lfloor z + \frac{1}{2} \rfloor}{b_2} \approx 1.97z + 0.99 \]

\[ \frac{\lfloor z + \frac{1}{2} \rfloor}{b_3} \approx 3.41z + 1.71 \]
By working inductively, we can obtain a partition of $\mathbb{Z}$ into three sets.

However, there is no guarantee that the mappings so defined satisfy Avdonin’s Theorem

To get around this, we deploy a calculus of Avdonin maps.
Lemma

Suppose that there exist injective maps

\[ \hat{\phi} : \frac{\mathbb{Z} + \frac{1}{2}}{b_1} \rightarrow \frac{\mathbb{Z} + \frac{1}{2}}{b_1 + b_2}, \quad \hat{\psi} : \frac{\mathbb{Z} + \frac{1}{2}}{b_2} \rightarrow \frac{\mathbb{Z} + \frac{1}{2}}{b_1 + b_2}, \quad \sigma : \frac{\mathbb{Z} + \frac{1}{2}}{b_1 + b_2} \rightarrow \mathbb{Z} + \frac{1}{2} \]

such that

- Range (\(\hat{\phi}\)) \(\cup\) Range (\(\hat{\psi}\)) = \(\frac{\mathbb{Z} + \frac{1}{2}}{b_1 + b_2}\)

- \(\varphi\) and \(\psi\) are \(\delta\)-Avdonin maps for \(\frac{\mathbb{Z} + \frac{1}{2}}{b_1}\) and \(\frac{\mathbb{Z} + \frac{1}{2}}{b_2}\) (resp.), and

- \(\sigma\) is an \(\epsilon\)-Avdonin map for \(\frac{\mathbb{Z} + \frac{1}{2}}{b_1 + b_2}\).

Then \(\hat{\phi}, \hat{\psi}\) can be locally modified to \(\varphi, \psi\) so that in addition \(\sigma \circ \varphi\) and \(\sigma \circ \psi\) are \((\epsilon + 3\delta)\)-Avdonin maps.
Suppose that we are given \( b_1, b_2, b_3 > 0 \) so that \( b_1 + b_2 + b_3 = 1 \).

We can define \( \epsilon \)-Avdonin maps

\[
\varphi_1 : \frac{\mathbb{Z} + \frac{1}{2}}{b_1} \rightarrow \mathbb{Z} + \frac{1}{2}, \quad \psi_1 : \frac{\mathbb{Z} + \frac{1}{2}}{b_2 + b_3} \rightarrow \mathbb{Z} + \frac{1}{2}
\]

thereby partitioning \( \mathbb{Z} + \frac{1}{2} \) into \( \Lambda_1 = \text{Range} (\varphi_1) \) and \( \Gamma_1 = \text{Range} (\psi_1) \).

With \( \epsilon \) small enough, we immediately have that

\[
\mathcal{E} (\Lambda_1) \text{ is a Riesz basis for } L^2(I) \text{ with } |I| = b_1
\]

and

\[
\mathcal{E} (\Gamma_1) \text{ is a Riesz basis for } L^2(I) \text{ with } |I| = 1 - b_1 = b_2 + b_3
\]
Next define $\delta$-Avdonin maps

\[
\varphi_2 : \frac{Z + \frac{1}{2}}{b_2} \rightarrow \frac{Z + \frac{1}{2}}{b_2 + b_3}, \quad \psi_2 : \frac{Z + \frac{1}{2}}{b_3} \rightarrow \frac{Z + \frac{1}{2}}{b_2 + b_3}
\]

thereby partitioning $\frac{Z + \frac{1}{2}}{b_2 + b_3}$ into $\text{Range} (\varphi_2)$ and $\text{Range} (\psi_2)$.

Applying the Lemma, we can adjust $\varphi_2$ and $\psi_2$ in such a way that

\[
\psi_1 \circ \varphi_2 : \frac{Z + \frac{1}{2}}{b_2} \rightarrow Z + \frac{1}{2}, \quad \psi_1 \circ \psi_2 : \frac{Z + \frac{1}{2}}{b_3} \rightarrow Z + \frac{1}{2}
\]

are $\epsilon + 3\delta$-Avdonin maps, and $\Gamma_1$ is partitioned into $\Lambda_2 = \text{Range} (\psi_1 \circ \varphi_2)$ and $\Lambda_3 = \text{Range} (\psi_1 \circ \psi_2)$.

With $\delta$ small enough, we immediately have $\mathcal{E}(\Lambda_2)$ RB for $L^2(I)$, $|I| = b_2$ and $\mathcal{E}(\Lambda_3)$ RB for $L^2(I)$, $|I| = b_3$.

$\Lambda_1$, $\Lambda_2$, $\Lambda_3$ is our desired partition.
Theorem (Pfander, Revay, DW, 2018)

Let $b_1, b_2, \ldots > 0$ with $\sum_{k=1}^{\infty} b_k = 1$, and $c_j = \sum_{k=j+1}^{\infty} b_k$ for $j \in \mathbb{N}$ so that $c_j + b_j = c_{j-1}$. Let $\delta > 0$ be given. Then there exist injective maps

$$\Phi_j : \frac{\mathbb{Z} + \frac{1}{2}}{b_j} \to \mathbb{Z} + \frac{1}{2}, \quad \Psi_j : \frac{\mathbb{Z} + \frac{1}{2}}{c_j} \to \mathbb{Z} + \frac{1}{2}$$

such that

(a) $\{\text{Range } (\Phi_k), \text{Range } (\Psi_j)\}_{k=1}^{j}$ is a partition of $\mathbb{Z} + \frac{1}{2}$,

(b) $\{\text{Range } (\Phi_{j+1}), \text{Range } (\Psi_{j+1})\}$ is a partition of $\text{Range } (\Psi_j)$, and

(c) $\Phi_j$ and $\Psi_j$ are $(1 - 2^{-j})\delta$-Avdonin maps for $\frac{\mathbb{Z} + \frac{1}{2}}{b_j}$ and $\frac{\mathbb{Z} + \frac{1}{2}}{c_j}$ (resp.)
For each $j$, define maps

$$\hat{\varphi}_j : \frac{\mathbb{Z} + \frac{1}{2}}{b_j} \rightarrow \frac{\mathbb{Z} + \frac{1}{2}}{b_j + c_j} = \frac{\mathbb{Z} + \frac{1}{2}}{c_{j-1}}$$

$$\hat{\psi}_j : \frac{\mathbb{Z} + \frac{1}{2}}{c_j} \rightarrow \frac{\mathbb{Z} + \frac{1}{2}}{b_j + c_j} = \frac{\mathbb{Z} + \frac{1}{2}}{c_{j-1}}$$

These can be simple rounding maps that we can take to be $\epsilon_j$-Avdonin maps with $\epsilon_1 = \frac{\delta}{2}$ and $\epsilon_j = \frac{\delta}{3.2^j}$ if $j \geq 2$. 
If $j = 1$ then $\Phi_1 = \hat{\varphi}_1$ and $\Psi_1 = \hat{\psi}_1$.

For $j = 2$, adjust the maps $\hat{\varphi}_2$ and $\hat{\psi}_2$ to $\varphi_2$ and $\psi_2$ so that

$$\Phi_2 = \Psi_1 \circ \varphi_2 : \frac{\mathbb{Z} + \frac{1}{2}}{b_2} \to \mathbb{Z} + \frac{1}{2},$$

$$\Psi_2 = \Psi_1 \circ \psi_2 : \frac{\mathbb{Z} + \frac{1}{2}}{c_2} \to \mathbb{Z} + \frac{1}{2}$$

are $\epsilon_1 + 3\epsilon_2 = (1 - \frac{1}{4})\delta$-Avdonin maps.

Proceed inductively.
Combining Intervals from Partition

Lemma

Let $a, b > 0$ and suppose that

\[\tau: \frac{\mathbb{Z} + \frac{1}{2}}{a} \rightarrow \mathbb{Z} + \frac{1}{2}, \quad \text{and} \quad \eta: \frac{\mathbb{Z} + \frac{1}{2}}{b} \rightarrow \mathbb{Z} + \frac{1}{2}\]

are $\epsilon$-Avdonin maps. Then there exists a $4\epsilon$-Avdonin map

\[\rho: \frac{\mathbb{Z} + \frac{1}{2}}{a + b} \rightarrow \mathbb{Z} + \frac{1}{2}\]

such that

\[\rho\left(\frac{\mathbb{Z} + \frac{1}{2}}{a + b}\right) = \tau\left(\frac{\mathbb{Z} + \frac{1}{2}}{a}\right) \cup \eta\left(\frac{\mathbb{Z} + \frac{1}{2}}{b}\right).\]
Theorem (Pfander, Revay, DW 2018)

Let $b_1, b_2, \ldots > 0$ with $\sum_{k=1}^{\infty} b_k = 1$ and $K \in \mathbb{N}$. Then there exist pairwise disjoint sets $\Lambda_1, \Lambda_2, \ldots \subseteq \mathbb{Z}$ such that for any $J \subseteq \mathbb{N}$ with $|J| \leq K$, $\bigcup_{j \in J} \mathcal{E}(\Lambda_j)$ is a Riesz basis for $L^2(I)$, $I$ an interval with $|I| = \sum_{j \in J} b_j$.

- Let $\delta = 4^{-K}$.
- Previous theorem allows us to obtain

$$\Phi_j : \frac{\mathbb{Z} + \frac{1}{2}}{b_j} \to \mathbb{Z} + \frac{1}{2}$$

a $4^{-K}$-Avdonin map.

- Letting

$$\{\Lambda_j = \text{Range } (\Phi_j)\}_{j=1}^{\infty},$$

gives pairwise disjoint subsets of $\mathbb{Z}$ such that $\mathcal{E}(\Lambda_j)$ is a Riesz basis for $L^2(I)$, $I$ an interval with $|I| = b_j$.
Given $J \subseteq \mathbb{N}$ with $|J| \leq K$, use the Lemma to combine bases pairwise to obtain a $4^{K-1}4^K = \frac{1}{4}$-Avdonin map from

$$P_J : \frac{\mathbb{Z} + \frac{1}{2}}{\sum_{j \in J} b_j} \to \mathbb{Z} + \frac{1}{2}$$

Hence

$$\bigcup_{j \in J} \mathcal{E}(\Lambda_j)$$

is a Riesz basis for $L^2(I)$, $I$ an interval with $|I| = \sum_{j \in J} b_j$.

The rapid growth of the Avdonin constants makes the a priori choice of $K \in \mathbb{N}$ necessary.