ORDINARY VARIETIES AND THE COMPARISON BETWEEN MULTIPLIER IDEALS AND TEST IDEALS II

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Abstract. We consider the following conjecture: if $X$ is a smooth $n$-dimensional projective variety in characteristic zero, then there is a dense set of reductions $X_s$ to positive characteristic such that the action of the Frobenius morphism on $H^n(X_s, \mathcal{O}_{X_s})$ is bijective. We also consider the conjecture relating the multiplier ideals of an ideal $a$ on a nonsingular variety in characteristic zero, and the test ideals of the reductions of $a$ to positive characteristic. We prove that the latter conjecture implies the former one.

1. Introduction

This note is motivated by the joint paper with V. Srinivas [MS], aimed at understanding the following conjecture relating invariants of singularities in characteristic zero with corresponding invariants in positive characteristic. For a discussion of the notions involved, see below.

Conjecture 1.1. Let $Y$ be a smooth, irreducible variety over an algebraically closed field $k$ of characteristic zero, and $a$ a nonzero ideal on $Y$. Given any model $Y_A$ and $a_A$ for $Y$ and $a$ over a subring $A$ of $k$, finitely generated over $\mathbb{Z}$, there is a dense set of closed points $S \subset \text{Spec } A$ such that

$$J(Y, a^\lambda)_s = \tau(Y_s, a_s^\lambda)$$

for every $\lambda \in \mathbb{R}_{\geq 0}$ and every $s \in S$.

In the conjecture, we denote by $Y_s$ the fiber of $Y_A$ over $s \in S$, and $a_s$ is the ideal on $Y_s$ induced by $a_A$. The ideals $J(Y, a^\lambda)$ are the multiplier ideals of $a$. These are fundamental invariants of the singularities of $a$, that have seen a lot of recent applications due to their appearance in vanishing theorems (see [Laz, Chapter 9]). The ideals $\tau(Y_s, a_s^\lambda)$ are the (generalized) test ideals of Hara and Yoshida [HY], defined in positive characteristic using the Frobenius morphism. The above conjecture asserts therefore that for a dense set of closed points, we have the equality between the test ideals of $a$ and the reductions of the multiplier ideals of $a$ for all exponents. We note that it is shown in [HY] that if $\lambda \in \mathbb{R}_{\geq 0}$ is fixed, then the equality in (1) holds for every $s$ in an open subset of the closed points in $\text{Spec } A$.

The following conjecture was proposed in [MS].

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Conjecture 1.2. Let $X$ be a smooth, irreducible $n$-dimensional projective variety defined over an algebraically closed field $k$ of characteristic zero. If $X_A$ is a model of $X$ defined over a subring $A$ of $k$, finitely generated over $\mathbb{Z}$, then there is a dense set of closed points $S \subseteq \text{Spec} A$ such that the Frobenius action on $H^n(X_s, \mathcal{O}_{X_s})$ is bijective for every $s \in S$.

It is expected, in fact, that there is a set $S$ as in Conjecture 1.2 such that $X_s$ is ordinary in the sense of Bloch and Kato [BK] for every $s \in S$. In particular, this would imply that the action of the Frobenius on each cohomology group $H^i(X_s, \mathcal{O}_{X_s})$ is bijective (see [MS, Remark 5.1]). The main result of [MS] is that Conjecture 1.2 implies Conjecture 1.1. In this note we show that the converse is true:

Theorem 1.3. If Conjecture 1.1 holds, then so does Conjecture 1.2.

The following is an outline of the proof. Given a variety $X$ as in Conjecture 1.2, we embed it in a projective space $\mathbb{P}_k^n$ such that $r := n - n \geq n + 1$, and the ideal $a \subseteq k[x_0, \ldots, x_N]$ defining $X$ is generated by quadrics. In this case it is easy to compute the multiplier ideals $\mathcal{J}(A_k^{N+1}, a^\lambda)$ for $\lambda < r$, and in particular we see that $(x_0, \ldots, x_N)^{2r - N - 1} \subseteq \mathcal{J}(A_k^{N+1}, a^\lambda)$ for every $\lambda < r$. It follows from a general property of multiplier ideals that if $g_1, \ldots, g_r$ are general linear combinations of a system of generators of $a$, and if $h = g_1 \cdots g_r$, then $\mathcal{J}(A_k^{N+1}, a^\lambda) = \mathcal{J}(A_k^{N+1}, h^{\lambda/r})$ for every $\lambda < r$. In this case, Conjecture 1.1 implies that for a dense set of closed points $s \in \text{Spec} A$, the ideal $(x_0, \ldots, x_N)^{2r - N - 1}$ is contained in $\tau(A_k^{N+1}, h^\mu)$ for every $\mu < 1$. Using some basic properties of test ideals, we deduce that the Frobenius action on $H^{N-1}(D_s, \mathcal{O}_{D_s})$ is bijective, where $D_s \subset \mathbb{P}_k^n$ is the hypersurface defined by $h_s$. We show that this in turn implies the bijectivity of the Frobenius action on $H^n(X_s, \mathcal{O}_{X_s})$, hence proves the theorem.

2. Proof of the main result

We start by recalling the definition of multiplier ideals and test ideals. Suppose first that $Y$ is a smooth, irreducible variety over an algebraically closed field $k$ of characteristic zero, and $a$ is a nonzero ideal on $Y$. A log resolution of $a$ is a projective, birational morphism $\pi: W \to Y$, with $W$ smooth, such that $a \cdot \mathcal{O}_W$ is the ideal of a divisor $D$ on $W$, with $D + K_{W/Y}$ having simple normal crossings (as usual, $K_{W/Y}$ denotes the relative canonical divisor of $W$ over $Y$). With this notation, for every $\lambda \in \mathbb{R}_{\geq 0}$ we have

$$\mathcal{J}(Y, a^\lambda) = \pi_* \mathcal{O}_W(K_{W/Y} - [\lambda D]).$$

Recall that if $E = \sum_i a_i E_i$ is a divisor with $\mathbb{R}$-coefficients, then $[E] = \sum_i [a_i] E_i$, where $[t]$ is the largest integer $\leq t$. It is a well-known fact that the above definition is independent of the choice of log resolution. For this and other basic facts about multiplier ideals, see [Laz, Chapter 9].

Suppose now that $Y = \text{Spec} R$ is an affine smooth, irreducible scheme of finite type over a perfect field $L$ of positive characteristic $p$ (in the case of interest for us, $L$ will be a finite field). Under these assumptions, the test ideals admit the following simple description that we will use, see [BMS2]. Recall that for an ideal $J$ and for $e \geq 1$, one denotes by $J^{[e]}$ the ideal $(h^{[e]} | h \in J)$. One can show that given an ideal $b$ in $R$, there
is a unique smallest ideal \( J \) such that \( b \subseteq J^{[p^e]} \); this ideal is denoted by \( b^{[1/p^e]} \). Suppose now that \( a \) is an ideal in \( R \) and \( \lambda \in R_{\geq 0} \). One can show that for every \( e \geq 1 \) we have the inclusion

\[
(a^{[\lambda p^e]}{[1/p^e]}^e) \subseteq (a^{[\lambda p^e+1]}{[1/p^e+1]}),
\]

where \([t]\) denotes the smallest integer \( \geq t \). Since \( R \) is Noetherian, it follows that \((a^{[\lambda p^e]}{[1/p^e]}^e)\) is constant for \( e \gg 0 \). This is the test ideal \( \tau(Y, a^\lambda) \). For details and a discussion of basic properties of test ideals in this setting, we refer to [BMS2]. For a comparison of general properties of multiplier ideals and test ideals, see [HY] and [MY].

If \( a \) is an ideal in the polynomial ring \( k[x_0, \ldots, x_N] \), where \( k \) is a field of characteristic zero, a model of \( a \) over a subring \( A \) of \( k \), finitely generated over \( \mathbb{Z} \), is an ideal \( a_A \) in \( A[x_0, \ldots, x_N] \) such that \( a_A \cdot k[x_0, \ldots, x_N] = a \). We can obtain such a model by simply taking \( A \) to contain all the coefficients of a finite system of generators of \( a \). Of course, we may always replace \( A \) by a larger ring with the same properties; in particular, we may replace \( A \) by a localization \( A_a \) at a nonzero element \( a \in A \). If \( s \in \text{Spec} \; A \) and if \( a_A \) is a model of \( a \), then we obtain a corresponding ideal \( a_s \) in \( k(s)[x_0, \ldots, x_N] \). Note that if \( s \) is a closed point, then the residue field \( k(s) \) is a finite field.

Suppose now that \( X \subseteq P_k^N \) is a projective subscheme defined by the homogeneous ideal \( a \subseteq k[x_0, \ldots, x_N] \). If \( a_A \subseteq A[x_0, \ldots, x_N] \) is a model of \( a \) over \( A \), which we may assume homogeneous, then the subscheme \( X_A \) of \( P_A^N \) defined by \( a_A \) is a model of \( X \) over \( A \). If \( s \in \text{Spec} \; A \), then the subscheme \( X_s \subseteq P_{k(s)}^N \) is defined by the ideal \( a_s \). We refer to [MS, §2.2] for some of the standard facts about reduction to positive characteristic. We note that given \( a \) as above, we may consider simultaneously all the reductions \( J(A_k^{N+1}, a^\lambda)_s \) for all \( \lambda \in R_{\geq 0} \). This is due to the fact that for bounded \( \lambda \) we only have to deal with finitely many ideals, while for \( \lambda \gg 0 \), the multiplier ideals are determined by the lower ones via a Skoda-type theorem (see [MS, §3.2] for details).

We can now give the proof of our main result stated in Introduction.

**Proof of Theorem 1.3.** Let \( X \) be a smooth, irreducible \( n \)-dimensional projective variety over an algebraically closed field \( k \) of characteristic zero, with \( n \geq 1 \). It is clear that the assertion we need is independent of the model \( X_A \) that we choose. Consider a closed embedding \( X \hookrightarrow P_k^N \). After replacing this by a composition with a \( d \)-uple Veronese embedding, for \( d \gg 0 \), we may assume that the saturated ideal \( a \subseteq R = k[x_0, \ldots, x_N] \) defining \( X \) is generated by homogeneous polynomials of degree two (see [ERT, Proposition 5]). Furthermore, we may clearly assume that \( r := N - n \geq n + 1 \). Under these assumptions, it is easy to determine the multiplier ideals of \( a \) of exponent \( < r \).

**Lemma 2.1.** With the above notation, if \( m = (x_0, \ldots, x_N) \), then

\[
J(A_k^{N+1}, a^\lambda) = \left\{ \begin{array}{ll}
R, & \text{if } 0 \leq \lambda < \frac{N+1}{2}; \\
0, & \text{if } \frac{N+1}{2} \leq \lambda < r.
\end{array} \right.
\]

**Proof.** Let us fix \( \lambda \in R_{\geq 0} \), with \( \lambda < r \). We denote by \( Z \) the subscheme of \( A_k^{N+1} \) defined by \( a \). Let \( \varphi : W \to A_k^{N+1} \) be the blow-up of the origin, with exceptional divisor \( E \). Since \( a \) is generated by homogeneous polynomials of degree two, it follows that \( a \cdot O_W = O_W(-2E) \tilde{a} \),
where $\mathfrak{a}$ is the ideal defining the strict transform $\widetilde{Z}$ of $Z$ on $W$. We have $K_{W/A_k^{N+1}} = N\mathfrak{e}$, hence the change of variable formula for multiplier ideals (see [Laz, Theorem 9.2.33]) implies
\[
\mathcal{J}(A_k^{N+1}, a^\lambda) = \varphi_*(\mathcal{J}(W, (a \cdot \mathcal{O}_W)^\lambda) \otimes \mathcal{O}_W(N\mathfrak{e})).
\]

It is clear that $\widetilde{Z}$ is nonsingular over $A_k^{N+1} \setminus \{0\}$. Since $\widetilde{Z} \cap E \subseteq E \simeq \mathbb{P}^N$ is isomorphic to the scheme $X$, hence it is nonsingular, it follows that $\widetilde{Z}$ is nonsingular, and $\widetilde{Z}$ and $E$ have simple normal crossings. Let $\psi: \widetilde{W} \to W$ be the blow-up of $W$ along $\widetilde{Z}$, with exceptional divisor $T$, and let $\widetilde{E}$ be the strict transform of $E$. Note that $\widetilde{W}$ is nonsingular, and $\widetilde{E} + T$ has simple normal crossings. We have $K_{\widetilde{W}/W} = (r - 1)T$ and $a \cdot \mathcal{O}_{\widetilde{W}} = \mathcal{O}_{\widetilde{W}}(-2\widetilde{E} - T)$. Therefore $\psi$ is a log resolution of $a \cdot \mathcal{O}_W$, and by definition we have
\[
\mathcal{J}(W, (a \cdot \mathcal{O}_W)^\lambda) = \psi_*(\mathcal{O}_{\widetilde{W}}(-\lfloor |\lambda| - r + 1 \rfloor T - \lfloor 2\lambda \rfloor \widetilde{E}) = \mathcal{O}_{\widetilde{W}}(-\lfloor 2\lambda \rfloor \mathfrak{e})
\]
(recall that $\lambda < r$). The formula in the lemma follows from (3), (4), and the fact that $\varphi_*(\mathcal{O}_W(-i\mathfrak{e})) = m^i$ for every $i \in \mathbb{Z}_{\geq 0}$. \hfill $\Box$

Let $f_1, \ldots, f_m$ be a system of generators of $a$, with each $f_i$ homogeneous of degree two. We fix $g_1, \ldots, g_r$ general linear combinations of the $f_i$ with coefficients in $k$, and put $h = g_1 \cdots g_r$. In this case, we have
\[
\mathcal{J}(A_k^{N+1}, a^\lambda) = \mathcal{J}(A_k^{N+1}, h^{\lambda/r})
\]
for every $\lambda < r$ (see [Laz, Proposition 9.2.28]).

Suppose now that $a_A$ and $h_A$ are homogeneous models of $a$, and respectively $h$, over $A$. Let $X_A, D_A \subset \mathbb{P}^N_A$ be the projective schemes defined by $a_A$ and $h_A$, respectively. Note that $g_1, \ldots, g_r$ being general linear combinations of the $f_i$, the subscheme $V((g_1, \ldots, g_r)_{\mathbb{P}^N_k}) \subset \mathbb{P}_k^N$ has pure codimension $r$. Therefore we may assume that for every $s \in \text{Spec} \ A$, the scheme $V((g_1, \ldots, g_r)_{\mathbb{P}^N_k(s)})$ has pure codimension $r$ in $\mathbb{P}^N_k(s)$. We need to show that given models as above, there is a dense set of closed points $S \subset \text{Spec} \ A$ such that the Frobenius action on $H^n(X_s, \mathcal{O}_{X_s})$ is bijective for every $s \in S$. The next lemma shows that in fact, it is enough to find $S$ as above such that the Frobenius action on $H^{N-1}(D_s, \mathcal{O}_{D_s})$ is bijective for all $s \in S$.

**Lemma 2.2.** Let $L$ be a finite field, and $D_1, \ldots, D_r$ hypersurfaces in $\mathbb{P}^N = \mathbb{P}^N_L$, with $r \leq N$, such that the intersection scheme $Y = D_1 \cap \ldots \cap D_r$ has pure codimension $r$ in $\mathbb{P}^N$. If the Frobenius action on $H^{N-1}(D, \mathcal{O}_D)$ is bijective, where $D = \sum_{i=1}^r D_i$, then for every closed subscheme $X$ of $Y$, the Frobenius action on $H^{N-r}(X, \mathcal{O}_X)$ is bijective.

**Proof.** If $r = N$, then $X$ is zero-dimensional, and the Frobenius action on $\Gamma(X, \mathcal{O}_X)$ is bijective since $L$ is perfect. Therefore from now on we may assume that $r \leq N - 1$.

For every subset $J \subseteq \{1, \ldots, r\}$, let $D_J = \bigcap_{j \in J} D_j$. By assumption, $Y$ is a complete intersection, hence there is an exact complex
\[
\mathcal{C}^* : \quad 0 \to \mathcal{C}^0 \xrightarrow{d^0} \mathcal{C}^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{r-1}} \mathcal{C}^r \to 0,
\]
where $C^0 = \mathcal{O}_D$, and $C^m = \bigoplus_{|J|=m} \mathcal{O}_{D_J}$ for $m \geq 1$. Note that we have a morphism of complexes $C^\bullet \to F_\ast(C^\bullet)$, where $F$ is the absolute Frobenius morphism on $X$. It follows that if we break up $C^\bullet$ into short exact sequences, the maps in the corresponding long exact sequences for cohomology are compatible with the Frobenius action.

Let $\mathcal{M}^i = \text{Im}(d^i)$, hence $\mathcal{M}^0 = C^0 = \mathcal{O}_D$ and $\mathcal{M}^r-1 = C^r = \mathcal{O}_Y$. Since each $D_J$ is a complete intersection in $\mathbb{P}^N$, it follows that $H^i(D_J,\mathcal{O}_{D_J}) = 0$ for every $i$ with $1 \leq i < \dim(D_J) = N - |J|$. We deduce that for every $i$ with $0 \leq i \leq r - 2$, the short exact sequence

$$0 \to \mathcal{M}^i \to C^{i+1} \to \mathcal{M}^{i+1} \to 0$$

gives an exact sequence

$$0 = H^{N-i-2}(\mathbb{P}^N,C^{i+1}) \to H^{N-i-2}(\mathbb{P}^N,\mathcal{M}^{i+1}) \to H^{N-i-1}(\mathbb{P}^N,\mathcal{M}^i).$$

Therefore we have a sequence of injective maps

$$H^{N-r}(Y,\mathcal{O}_Y) \hookrightarrow H^{N-r+1}(\mathbb{P}^N,\mathcal{M}^{r-2}) \hookrightarrow \ldots \hookrightarrow H^{N-2}(\mathbb{P}^N,\mathcal{M}^1) \hookrightarrow H^{N-1}(D,\mathcal{O}_D),$$

compatible with the Frobenius action. Since this action is bijective on $H^{N-1}(D,\mathcal{O}_D)$ by hypothesis, it follows that it is bijective also on $H^{N-r}(Y,\mathcal{O}_Y)$ (see, for example, [MS, Lemma 2.4]).

On the other hand, since $\dim(Y) = N - r$, the surjection $\mathcal{O}_Y \to \mathcal{O}_X$ induces a surjection $H^{N-r}(Y,\mathcal{O}_Y) \to H^{N-r}(X,\mathcal{O}_X)$, compatible with the Frobenius action. As we have seen, the Frobenius action is bijective on $H^{N-r}(Y,\mathcal{O}_Y)$, hence on every quotient (see [MS, Lemma 2.4]). This completes the proof of the lemma.

Returning to the proof of Theorem 1.3, we see that it is enough to show that there is a dense set of closed points $S \subset \text{Spec} \ A$ such that Frobenius acts bijectively on $H^{N-1}(D_s,\mathcal{O}_{D_s})$ for $s \in S$. We assume that Conjecture 1.1 holds, hence there is a dense set of closed points $S \subset \text{Spec} \ A$ such that $\tau(A_k^{N+1}, h^r_s) = J(A_k^{N+1}, h^r_\ast)$ for every $\lambda \in \mathbb{R}_{\geq 0}$ and every $s \in S$. In particular, it follows from Lemma 2.1 and (5) that $(x_0, \ldots, x_N)^{2rN-1} \subset \tau(A_k^{N+1}, h^r_s)$ for every $\lambda < 1$. Since $\deg(h_s) = 2r = (N + 1)$, Proposition 2.3 below implies that the Frobenius action on $H^{N-1}(D_s,\mathcal{O}_{D_s})$ is bijective for all $s \in S$. As we have seen, this completes the proof of Theorem 1.3.

**Proposition 2.3.** Let $L$ be a perfect field of characteristic $p > 0$, and $h \in R = L[x_0, \ldots, x_N]$ a homogeneous polynomial of degree $d \geq N + 1$, with $N \geq 2$. If $(x_0, \ldots, x_N)^{d-N-1} \subset \tau(A_L^{N+1}, h^{1-\frac{1}{p}})$, then the Frobenius action on $H^{N-1}(D,\mathcal{O}_D)$ is bijective, where $D \subset \mathbb{P}^N_L$ is the hypersurface defined by $h$.

**Proof.** In the case $d = N+1$, this is a reformulation of a well-known fact due to Fedder [Fe]. We follow the argument from [MTW, Proposition 2.16], that extends to our more general setting. It is enough to show that the Frobenius action on $H^{N-1}(D,\mathcal{O}_D)$ is injective (see [MS, §2.1]).

Note first that $\tau(A_L^{N+1}, h^{1-\frac{1}{p}}) = (h^{p-1})^{1/p}$ (see [BMS1, Lemma 2.1]), hence by assumption $m^{d-N-1} \subset (h^{p-1})^{1/p}$, where $m = (x_0, \ldots, x_N)$. It is convenient to use the interpretation of the ideal $(h^{p-1})^{1/p}$ in terms of local cohomology. Let $E = H^{N+1}_m(R)$. 


Recall that this is a graded $R$-module, carrying a natural action of the Frobenius, that we denote by $F_E$. There is an isomorphism

$$E \simeq R_{x_0 \cdots x_N} / \sum_{i=0}^{N} R_{x_0 \cdots \hat{x}_i \cdots x_N}.$$ 

Via this isomorphism, $F_E$ is induced by the Frobenius morphism on $R_{x_0 \cdots x_N}$.

The annihilator of $(h^{p-1})^{[1/p]}$ in $E$ is equal to $\text{Ker}(h^{p-1} F_E)$ (see, for example, [BMS2, §2.3]). Therefore we have

$$(6) \quad \text{Ker}(h^{p-1} F_E) \subseteq \text{Ann}_E(m^{d-N-1}) = \bigoplus_{i \geq -d+1} E_i.$$ 

On the other hand, the exact sequence

$$0 \to R(-d) \xrightarrow{h} R \to R/(h) \to 0$$ 

induces an isomorphism

$$H^N_m(R/(h)) \simeq \{ u \in E \mid hu = 0 \}(-d),$$

such that the Frobenius action on $H^N_m(R/(h))$ is given by $h^{p-1} F_E$. Since $H^{N-1}(D, \mathcal{O}_D) \simeq H^N_m(R/(h))_0 \hookrightarrow E_{-d}$, (6) implies that the Frobenius action is injective on $H^{N-1}(D, \mathcal{O}_D)$. This completes the proof of the proposition. \qed

**Remark 2.4.** In the proof of Theorem 1.3 we only used the inclusion “$\subseteq$” in Conjecture 1.1. However, this is the interesting inclusion: the reverse one is known, see [HY] or [MS, Proposition 4.2]. It is more interesting that we only used Conjecture 1.1 when $Y = A^{N+1}_k$, $a$ is principal and homogeneous, and $\lambda = 1 - \frac{1}{p}$. By combining Theorem 1.3 with the main result in [MS], we see that in order to prove Conjecture 1.1 in general, it is enough to consider the case when $Y = A^n_k$, $a = (f)$ is principal and homogeneous, and show the following: if $b = J(Y, a^{1-\varepsilon})$ for $0 < \varepsilon \ll 1$, and if $f_A \in A[x_1, \ldots, x_n]$ is a model for $f$, then there is a dense set of closed points $S \subset \text{Spec} A$ such that

$$b_s \subseteq (f^{p-1})^{[1/p]}$$

for every $s \in S$, where $p = \text{char}(k(s))$.

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