G₂-INSTANTONS ON THE 7-SPHERE

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Abstract. We study the deformation theory of G₂-instantons on the round 7-sphere, specifically those obtained from instantons on the 4-sphere via the quaternionic Hopf fibration. We find that the pullback of the standard ASD instanton lies in a smooth, complete, 15-dimensional family of G₂-instantons. In general, the space of infinitesimal G₂-instanton deformations on S₇ is identified with three copies of the space of ASD deformations on S⁴.

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1. Introduction

1.1. Background. Let M be an oriented 7-manifold with a G₂-structure, defined by a positive 3-form φ. Endow M with the corresponding Riemannian metric g = gφ and Hodge star operator * = *g (see §3.1 below).

A connection A on a vector bundle E → M is said to be a G₂-instanton if its curvature FA satisfies

$$F_A + * (\phi \wedge F_A) = 0.$$  (1.1)

This equation appeared in the physics literature during the early 1980s [12]. In the mid-1990s, not long after Joyce’s construction [24] of compact Riemannian manifolds with Hol(g) = G₂, Donaldson and Thomas [16] proposed to define an invariant of (torsion-free) G₂-structures by “counting” G₂-instantons in a manner analogous to the Casson invariant in dimension three.

Instantons on compact G₂-holonomy manifolds, with structure group SU(2) or SO(3), were first constructed by Walpuski [31] in the context of Joyce’s Kummer construction [24].

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Sá Earp and Walpuski [29], Walpuski [32], and Menet, Nordström, and Sá Earp [28] have succeeded in constructing instantons on several of the twisted connected sum $G_2$-holonomy manifolds due to Kovalev [27] and Corti-Haskins-Nordström-Pacini [13]. Despite this progress, instantons on compact manifolds with full $G_2$ holonomy remain extremely difficult to construct. The counting program has also encountered a host of fascinating difficulties; for extensive discussion, see Donaldson-Segal [17], Haydys-Walpuski [22], Joyce [23], and Doan-Walpuski [15].

A fruitful alternative is to consider metrics with less-than-full holonomy or $G_2$-structures with certain types of torsion. The instantons that appear in these contexts can demonstrate geometric phenomena that we ultimately hope to observe in the torsion-free case.

One such class is the nearly parallel $G_2$-structures, which satisfy

\[ d\phi = \tau_0 \psi. \]

Here $\psi = \ast_{\phi} \phi$ is the dual 4-form and $\tau_0$ is a constant. Nearly parallel $G_2$-structures have much in common with the torsion-free case: their associated metrics are Einstein and they enjoy a natural spinorial description [18]. More importantly for this work, instantons on nearly parallel $G_2$-manifolds are critical points of the Yang-Mills functional (see [21] or (3.12) below). This stands in contrast with general co-closed $G_2$-structures.

Most famously, the 7-sphere carries a “standard” $G_2$-structure induced by the octonionic structure of $\mathbb{R}^8$; for elementary reasons, this turns out to be nearly parallel. (In fact, $S^7$ also carries a second, “squashed,” nearly parallel $G_2$-structure—see Example 3.3 below and Appendix A—but we shall be concerned primarily with the standard structure.) There are three main reasons why the standard $S^7$ is an especially appealing arena for gauge theory.

First, determining the complete moduli space of instantons on $S^7$ presents a clear challenge for higher-dimensional gauge theory. This problem is at least as difficult as the corresponding problem on $S^4$, solved by the spectacular theorem of Atiyah-Drinfeld-Hitchin-Manin [2]. At present, we are limited to studying the deformations of a given $G_2$-instanton on $S^7$, in the spirit of Atiyah-Hitchin-Singer’s classic work on deformations of ASD instantons on $S^4$ [4].

The second motivation comes from another well-known difficulty in higher-dimensional gauge theory: the appearance of instantons with essential singularities. Given a Spin(7)-instanton with an isolated singularity on a Spin(7)-manifold, the cross-section of the tangent cone is a $G_2$-instanton on the standard $S^7$. Hence, these are the “building blocks” for the simplest non-removable singularities in dimension eight.

The third motivation is the relationship with gauge theory in fewer than seven dimensions. Since $G_2$-holonomy metrics typically collapse at the boundary of the moduli space, it is important to understand instantons on model $G_2$-manifolds coming from lower-dimensional geometries. Recently, Y. Wang [36] has shown that any $G_2$-instanton on $S^1 \times X$, for $X$ a Calabi-Yau 3-fold, is equivalent by a broken gauge transformation to the pullback of a Hermitian-Yang-Mills connection. This is the first case where the full moduli space of $G_2$-instantons on a given 7-manifold with holonomy contained in $G_2$ has been identified.

In the present case, a convenient link with 4-dimensional gauge theory is provided by the quaternionic Hopf fibration

\[ S^3 \to S^7 \to S^4. \]
1.2. **Summary.** In §2-3, we set our conventions and derive the basic results concerning instantons on nearly parallel $G_2$-manifolds, while reviewing the literature in this area.

In §4, we take a concrete approach to the special case of the standard instanton. In quaternionic notation on $\mathbb{R}^8 \cong \mathbb{H}^2$, the pullback by the (right) Hopf fibration of the standard ASD instanton may be given by

$$A_0(x, y) = \frac{1}{|x|^2 + |y|^2} \text{Im} \left[ |y|^2 x^{-1} dx + |x|^2 y^{-1} dy \right].$$

The restriction of $A_0$ to $S^7$ defines a smooth connection, dubbed the *standard $G_2$-instanton*. Proposition 4.4 describes a 15-dimensional space of infinitesimal deformations of $A_0$, generated by the deformations coming from $S^4$ together with the Spin$(7)$ rotations of $S^7$, modulo gauge. This will turn out to be the full space of deformations of $A_0$ as a $G_2$-instanton.

In §5, we prove our main result, Theorem 5.11, which determines the space of infinitesimal deformations of the pullback to $S^7$ of an arbitrary irreducible ASD instanton on $S^4$. The argument is elementary, but far from trivial. The difficulty is caused by two factors: first, according to the theorem of Bourguignon-Lawson-Simons [7, 8], the Yang-Mills stability operator necessarily has negative eigenvalues. Second, the Sp$(1)$-action giving the quaternionic Hopf fibration does not commute with the deformation operator (see Remark 2.1 below).

To obtain the result, we first prove a vanishing theorem for the vertical component of an infinitesimal deformation in Coulomb gauge, Theorem 5.8. This follows from a delicate analysis of the Weitzenbock formula for the stability operator, in which the first-order deformation equation is used crucially. Having established that the vertical component vanishes, we use another squaring trick, Lemma 5.9, to calculate the horizontal component of an infinitesimal deformation. This leads directly to Theorem 5.11, which equates the full space of infinitesimal deformations of the pullback, as a $G_2$-instanton, with three copies of the ASD deformations on $S^4$.

In §6, we briefly discuss the global structure of these families of $G_2$-instantons. In the case of charge $\kappa = 1$ and structure group SU$(2)$, which includes the standard instanton, we have the following result.

**Theorem 1.1.** The connected component of $A_0$ in the moduli space of $G_2$-instantons on $S^7$ is diffeomorphic to the tautological 5-plane bundle over the oriented real Grassmannian $G^{\omega}(5, 7)$.

In this description, the base space corresponds to the orbit of the Hopf fibration (1.3) under conjugation by Spin$(7)$, and the fiber corresponds to the pullback of the unit-charge ASD moduli space. The total space is 15-dimensional, agreeing with the dimension formula of Theorem 5.11. By contrast, in the case of higher charge, we do not expect all of the infinitesimal deformations identified by Theorem 5.11 to be integrable (see §6.3 below).
Lastly, we state Conjecture 6.3, due to Donaldson, which asserts that every $G_2$-instanton on $S^7$ having integral Chern-Simons value should arise from the pullback construction.

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2. Conventions

2.1. Quaternions, 2-forms, and Fueter maps. Let $\mathbb{H}$ denote the 4-dimensional algebra of quaternions, generated by $1, i, j, k$, and subject to the relations

\[ i^2 = j^2 = k^2 = ijk = -1. \]

The 3-dimensional space of imaginary quaternions $\text{Im} \mathbb{H}$, with commutator bracket, is isomorphic to the Lie algebra $\mathfrak{su}(2)$.

We shall identify $\mathbb{H}$ with $\mathbb{R}^4$ as follows:

\[ \{e_0, e_1, e_2, e_3\} = \{1, -i, -j, -k\}. \]

Denote by $\mathbb{H}_l$ and $\mathbb{H}_r$ the commuting subalgebras of $\text{End} \mathbb{R}^4$ corresponding to left- and right-multiplication by $\mathbb{H}$, respectively, and let $\text{Sp}(1)_l$ and $\text{Sp}(1)_r$ be the corresponding subgroups of $\text{SO}(4)$ generated by unit quaternions. With this convention, we have

\[ \text{Lie}(\text{Sp}(1)_l) = \Lambda^2^-, \quad \text{Lie}(\text{Sp}(1)_r) = \Lambda^2^+ \]

where $\Lambda^2^- \subset \Lambda^2 \mathbb{R}^4$ denotes the space of (anti-)self-dual 2-forms with respect to the Euclidean metric.

Choose the following standard basis for the self-dual 2-forms on $\mathbb{R}^4$:

\[ \omega_1 = dx^{01} + dx^{23}, \quad \omega_2 = dx^{02} - dx^{13}, \quad \omega_3 = dx^{03} + dx^{12}. \]

Here we abbreviate $dx^{01} = dx^0 \wedge dx^1$, etc. For $i = 1, 2, 3$, define the complex structure $I_i$ on $\mathbb{R}^4$ by

\[ \{I_i(v), w\} = \omega_i(v, w) \quad \forall v, w \in T\mathbb{R}^4. \]

Under the identification (2.1), $I_i$ corresponds to right-multiplication by the element $e_i$.

We define a Fueter map $L : \mathbb{R}^4 \to \mathbb{R}^4$ to be an endomorphism satisfying

\[ L + I_1 LI_1 + I_2 LI_2 - I_3 LI_3 = 0. \]

The 12-dimensional subspace of Fueter maps $\mathfrak{F} \subset \text{End} \mathbb{R}^4$ is the direct sum:

\[ \mathfrak{F} = \mathbb{H}_l \oplus \mathbb{H}_l I_1 \oplus \mathbb{H}_l I_2. \]

In particular, $\mathfrak{F}$ contains the space of linear maps for the standard complex structure, $I_1$. 
2.2. Spin(7) and its subgroups. The standard 4-form on $\mathbb{R}^8 = \mathbb{R}^4_x \oplus \mathbb{R}^4_y$ is defined by

$\Psi_0 = dx^{0123} + dy^{0123} + \omega^x_1 \wedge \omega^y_1 + \omega^x_2 \wedge \omega^y_2 - \omega^x_3 \wedge \omega^y_3$.}

The group Spin(7) consists of all linear transformations of $\mathbb{R}^8$ that preserve $\Psi_0$ under pullback. It is a simply-connected, simple, Lie subgroup of SO(8) of dimension 21 (see e.g. Walpuski and Salamon [34, §9]).

The Lie algebra of Spin(7) corresponds to the subspace of 2-forms $\eta \in \Lambda^2 \mathbb{R}^8 \cong so(8)$ satisfying

$(2.7) \quad \eta + *(\Psi_0 \wedge \eta) = 0.$

Let $\Lambda^2 \subset \Lambda^2 \mathbb{R}^8$ be the subalgebra spanned by the three elements

$(2.8) \quad \omega^x_1 - \omega^y_1, \quad \omega^x_2 - \omega^y_2, \quad \omega^x_3 + \omega^y_3.$

Also denote the subspace

$(2.9) \quad \mathfrak{g}_{x,y} = \{L_{ij} dy^i \wedge dx^j \mid L \in \mathfrak{g}\} \subset \Lambda^2 \mathbb{R}^8.$

Then we have the following decomposition:

$(2.10) \quad \text{Lie(Spin(7))} = \Lambda^2_x \oplus \Lambda^2_y \oplus \Lambda^2_d \oplus \mathfrak{g}_{x,y}.$

One readily checks that each factor of (2.10) satisfies (2.7).

Take complex coordinates

$z^1 = x^0 + ix^1, \quad z^2 = x^2 + ix^3, \quad z^3 = y^0 + iy^1, \quad z^4 = y^2 + iy^3$

for $\mathbb{R}^8 \cong \mathbb{C}^4$. We have the standard holomorphic volume form and Kähler form

$(2.11) \quad \Omega = dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4, \quad \omega = \frac{i}{2} \sum_{i=1}^{4} dz^i \wedge d\bar{z}^i = \omega^x_1 + \omega^y_1.$

One may verify that

$(2.12) \quad \Psi_0 = \frac{1}{2} \omega \wedge \omega + \text{Re} \Omega.$

The 15-dimensional group SU(4) is therefore a subgroup of Spin(7). In particular, the 10-dimensional group Sp(2) of orthogonal quaternionic matrices, linear over $I_1, I_2,$ and $I_3$, is also a subgroup.

The 14-dimensional Lie group G$_2$ is the subgroup of Spin(7) stabilizing a point on $S^7$. Equivalently, G$_2$ is the subgroup of GL(7) that preserves the model 3-form

$(2.13) \quad \phi_0 = \frac{\partial}{\partial x^0} \wedge \Psi_0 = dx^{123} + dx^1 \wedge \omega^y_1 + dx^2 \wedge \omega^y_2 - dx^3 \wedge \omega^y_3.$

Denote the dual 4-form by

$(2.14) \quad \psi_0 = *_{\text{R}^7} \phi_0 = dy^{1234} + dx^{23} \wedge \omega^y_1 - dx^{13} \wedge \omega^y_2 - dx^{12} \wedge \omega^y_3.$

The Lie algebra Lie(G$_2$) corresponds to the subspace of 2-forms $\xi \in \Lambda^2 \mathbb{R}^7$ satisfying

$\xi + *(\phi_0 \wedge \xi) = 0,$

or equivalently

$\psi_0 \wedge \xi = 0.$
**Remark 2.1.** The fact that $\text{Sp}(2)\text{Sp}(1)$ is not a subgroup of $\text{Spin}(7)$ causes an essential difficulty. With our convention, $\text{Sp}(2)$ is a subgroup, but the block-diagonal $\text{Sp}(1)$, giving the Hopf fibration, is not. On the other hand, with the convention used for instance by Walpuski [33], $\text{Sp}(1)$ is a subgroup, but the commuting $\text{Sp}(2)$ is not. Our convention is necessary for Lemma 4.1 below.

### 3. Instantons on nearly parallel $G_2$-manifolds

In this section, we establish the basic facts about instantons on nearly parallel $G_2$-manifolds. Most of the results are known to researchers informally or by analogy with the nearly Kähler case (see Xu [37]), but some (in particular Proposition 3.8) have not appeared in their present form. For the spinorial formulation, see Harland-Nölle [21].

#### 3.1. Nearly parallel $G_2$-structures.

Let $M$ be an oriented 7-manifold. Recall that a $G_2$-structure on $M$ is defined by a global 3-form $\phi$ that is positive, in the sense that

$$G_\phi(v) = (v \cdot \phi) \wedge (v \cdot \phi) \wedge \phi > 0$$

for all $x \in M$ and $v \neq 0 \in T_x M$. Any such $\phi$ is pointwise equivalent to the model 3-form $\phi_0$, given by (2.13) above (see [34, Theorem 3.2]).

A positive 3-form defines a unique Riemannian metric $g_\phi$ on $M$ by the requirement

$$6g_\phi(v, v) \text{Vol}_{g_\phi} = G_\phi(v) \quad \forall \, v \in TM.$$  

We also associate to $\phi$ the dual 4-form

$$\psi = *_{g_\phi} \phi.$$  

Recall that a $G_2$-structure is said to be closed if $d\phi = 0$, and coclosed if $d\psi = 0$.

We are concerned with $G_2$-structures satisfying (1.2), where we assume

$$\tau_0 = \pm 4.$$  

The basic reference for nearly parallel structures is Friedrich-Kaehler-Moroianu-Semmelmann [18]. With the normalization (3.4), nearly parallel $G_2$-manifolds are Einstein, with

$$\text{Ric}_g = 6g.$$  

There are three further equivalent formulations of the nearly parallel condition (1.2). The first is that the induced $\text{Spin}(7)$-structure on the cone over $M$ be torsion-free. The second (and most frequently used) condition is that $M$ possess a nonzero Killing spinor. The third is as follows:

**Lemma 3.1.** A $G_2$-structure $\phi$ is nearly parallel if and only if

$$\nabla \phi = \frac{\tau_0}{4} \psi.$$  

Here $\nabla$ is the Levi-Civita connection associated to the metric $g_\phi$ defined by (3.2).

**Proof.** See Karigiannis [26, Theorem 2.27].
Example 3.2. Define the \textit{standard G}_2-\textit{structure on S}^7 by
\begin{equation}
\phi_{\text{std}} = \bar{r} \cdot \Psi_0|_{S^7},
\end{equation}
where
\[
\bar{r} = x^i \frac{\partial}{\partial x^i} + y^i \frac{\partial}{\partial y^i}
\]
is the coordinate vector field on \(\mathbb{R}^8\). The group of global automorphisms of \(\phi_{\text{std}}\) is \(\text{Spin}(7)\).

To obtain a more explicit expression for \(\phi_{\text{std}}\), we define the 3-form
\[
\nu^x = \bar{r} \cdot \text{Vol}_{S^7} = x^0dx^{-1}d^3y - x^1dx^0d^3y + x^2dx^0d^3y - x^3dx^0d^3y
\]
on \(\mathbb{R}^4\), and the 1-forms
\begin{align}
\zeta^1_x &= \bar{r} \cdot \omega^1_x = x^0dx^1 - x^1dx^0 + x^2dx^3 - x^3dx^2 \\
\zeta^2_x &= \bar{r} \cdot \omega^2_x = x^0dx^2 - x^2dx^0 + x^3dx^1 \\
\zeta^3_x &= \bar{r} \cdot \omega^3_x = x^0dx^3 - x^3dx^0 + x^1dx^2 - x^2dx^1.
\end{align}
Define \(\nu^y\) and \(\zeta^1_y\) similarly. We then have
\begin{equation}
\phi_{\text{std}} = \nu^x + \nu^y + \zeta^1_x \wedge \omega^y_x + \zeta^1_y \wedge \omega^x_x + \zeta^2_x \wedge \omega^y_x + \zeta^2_y \wedge \omega^x_x - \zeta^3_x \wedge \omega^y_x - \zeta^3_y \wedge \omega^x_x.
\end{equation}

It is easy to check, using the \(\text{Spin}(7)\)-invariance, that \(\phi_{\text{std}}\) defines the round metric on \(S^7\).

Notice that \(d\nu^x = 4\text{Vol}_{S^7}\) and \(d\zeta^x = 2\omega^x_x\), and similarly for \(y\), so
\begin{equation}
d\phi_{\text{std}} = 4\Psi_0|_{S^7}.
\end{equation}

Also note that
\begin{equation}
\psi_{\text{std}} = *_{\text{std}}\phi_{\text{std}} = *_{\text{std}}\Psi_0|_{S^7} = \Psi_0|_{S^7}.
\end{equation}

It follows from (3.9-10) that \(\phi_{\text{std}}\) is a nearly parallel \(G_2\)-structure.

Example 3.3. Define the \textit{squashed G}_2-\textit{structure}
\[
\phi_{\text{sq}} = \frac{27}{25} \left( \frac{1}{5} (\nu_x + \nu_y) + \frac{16}{5} \left( \zeta^1_x \wedge \zeta^2_x \wedge \zeta^3_x + \zeta^1_y \wedge \zeta^2_y \wedge \zeta^3_y + \zeta^3_x \wedge \zeta^2_x \wedge \zeta^1_x \right) \right).
\]
The squashed \(G_2\)-structure was discovered by Awada, Duff, and Pope [5], and has automorphism group \(\text{Sp}(2)\text{Sp}(1)\). Appendix A includes a proof that \(\phi_{\text{sq}}\) is nearly parallel.

Remark 3.4. Alexandrov and Semmelmann [1] have shown that both the standard and the squashed \(G_2\)-structures are rigid among nearly parallel \(G_2\)-structures. These remain the only known nearly parallel structures on the 7-sphere. We also note that the (non-)existence of a \textit{closed} \(G_2\)-structure on the 7-sphere is a well-known open problem.

Remark 3.5. The presence of the two distinct nearly parallel \(G_2\)-structures on \(S^7\) can be attributed to its 3-Sasakian structure (generated by \(I_1, I_2,\) and \(I_3\)). Any 3-Sasakian 7-manifold carries two non-isomorphic \(G_2\)-structures, one with standard fibers and one with “squashed” fibers; see Friedrich \textit{et al.} [18, Theorem 5.4] or Galicki-Salamon [20, Proposition 2.4]. Examples of 3-Sasakian 7-manifolds were constructed in abundance by Boyer, Galicki, Mann, and Rees [9], giving many compact inhomogeneous nearly parallel \(G_2\)-manifolds as a byproduct.
3.2. \(G_2\)-instantons. Recall that a connection \(A\) is called a \(G_2\)-instanton if its curvature satisfies (1.1), or equivalently

\[
\psi \wedge F_A = 0.
\]

If the \(G_2\)-structure \(\phi\) is nearly parallel, then from (1.1) and (1.2), we have

\[
0 = D^*_A F_A - * D_A (\phi \wedge F_A)
\]

\[
= D^*_A F_A - * (\tau_0 \psi \wedge F_A - \phi \wedge D_A F_A)
\]

\[
= D^*_A F_A.
\]

We have used (3.11) and the Bianchi identity in the last line. Hence, in the nearly-parallel case, any \(G_2\)-instanton is Yang-Mills. This observation goes back to Harland and Nölle [21].

The linearization of (3.11) is

\[
\psi \wedge D_A \alpha = 0,
\]

for \(\alpha \in \Omega^1 (\mathfrak{g}_E)\). Meanwhile, an infinitesimal gauge transformation \(u \in \Omega^0 (\mathfrak{g}_E)\) acts by

\[
u \mapsto D_A u.
\]

The infinitesimal deformations of a \(G_2\)-instanton \(A\), modulo gauge, therefore correspond to the first cohomology group of the following self-dual elliptic complex:

\[
\Omega^0 (\mathfrak{g}_E) \xrightarrow{D_A} \Omega^1 (\mathfrak{g}_E) \xrightarrow{\psi \wedge D_A} \Omega^6 (\mathfrak{g}_E) \xrightarrow{D_A} \Omega^7 (\mathfrak{g}_E).
\]

Folding (3.13) and writing \(d = D_A\), we obtain the deformation operator

\[
\mathcal{L}_A : \Omega^0 (\mathfrak{g}_E) \oplus \Omega^1 (\mathfrak{g}_E) \to \Omega^0 (\mathfrak{g}_E) \oplus \Omega^1 (\mathfrak{g}_E)
\]

\[
\begin{pmatrix} u \\ \alpha \end{pmatrix} \mapsto \begin{pmatrix} d^* \alpha \\ du + * (\psi \wedge d \alpha) \end{pmatrix}.
\]

This is a first-order, self-adjoint, elliptic operator.

Squaring (3.14), we obtain

\[
\mathcal{L}_A^2 \begin{pmatrix} u \\ \alpha \end{pmatrix} = \begin{pmatrix} d^* du - * (d \psi \wedge d \alpha + \psi \wedge F_A \wedge \alpha) \\ dd^* \alpha + * (\psi \wedge d (d^* \alpha) + F_A \wedge F_A) \end{pmatrix}
\]

\[
= \begin{pmatrix} d^* du \\ dd^* \alpha + * (\psi \wedge d (d^* \alpha)) \end{pmatrix}.
\]

Over a compact manifold, integration by parts implies

\[
\ker \mathcal{L}_A = \ker \mathcal{L}_A^2.
\]

Hence, \(du \equiv 0\) for any infinitesimal deformation on a compact nearly parallel \(G_2\)-manifold, and \(u \equiv 0\) if \(A\) is irreducible.

We shall use the following interior product notation:

\[
\begin{align*}
\frac{\partial}{\partial x^1} & = dx^0, \\
\frac{\partial}{\partial x^0} & = -dx^1. 
\end{align*}
\]

We also take interior products between differential forms, \(e.g.\)

\[
\begin{align*}
dx^{01} \lhd dx^1 & = dx^0, \\
dx^{01} \lhd dx^{01} & = 1,
\end{align*}
\]
and similarly in general using the metric. In particular, for a 2-form \( b \), we have

\[
* (\psi \wedge b) = \phi \wr b.
\]

For a \( g_\mathcal{E} \)-valued 1-form \( \alpha \), we shall write

\[
\mathcal{L}_A(\alpha) = \mathcal{L}_A \left( \begin{array}{c} 0 \\ \alpha \\ \phi \end{array} \right) = \left( \begin{array}{c} d^* \alpha \\ \phi \term \phi d \alpha \end{array} \right).
\]

Then (3.15) becomes

(3.17) \[
\mathcal{L}_A^2(\alpha) = dd^* \alpha + \phi \term \phi d (\phi \term \phi d \alpha).
\]

**Lemma 3.6** ([10, (2.7-2.8)]) The following identities hold between any positive 3-form \( \phi \), the associated metric \( g = g_\phi \), and the dual 4-form \( \psi = * \phi \):

\[
g^{pq} g_{ijk} \phi^{pq} = g_{ik} g_{jl} - g_{il} g_{jk} + \psi_{ijkl},
\]

\[
g^{pq} g^{\ell m} \phi_{pq} \psi_{ijkl} = 2 \phi_{ijkl}.
\]

**Proof.** Since these are zeroth-order identities, it suffices to check them for the standard 3- and 4-form, given by (2.13-2.14), and the standard metric. This is easily accomplished using the fact that \( G_2 \) acts transitively on orthonormal pairs of vectors. \( \square \)

**Lemma 3.7** (Cf. [35, Lemma 7.1]). For a nearly parallel \( G_2 \)-structure and any 2-form \( b \), there holds

(3.18) \[
d (b \term \phi) \term \phi = d^* b - db \term \psi + \frac{\tau_0}{2} b \term \phi.
\]

**Proof.** Write \( b = \frac{1}{2} b_{ij} dx^i \wedge dx^j \) in normal coordinates. We then have

\[
(d (b \term \phi) \term \phi)_{ik} = \frac{1}{2} \left( \frac{1}{2} (\nabla_m (b_{ij} \phi_{ij}) - \nabla_\ell (b_{ij} \phi_{ij})) \right) \phi_{mk}
\]

\[
= \frac{1}{4} \left( \nabla_m b_{ij} \phi_{ij} + b_{ij} \nabla_m \phi_{ij} - \nabla_\ell b_{ij} \phi_{ij} - b_{ij} \nabla_\ell \phi_{ij} \right) \phi_{mk}
\]

\[
= -\frac{1}{2} \nabla_m b_{ij} \phi_{ij} + \frac{\tau_0}{8} b_{ij} \psi_{mijk} \phi_{mk},
\]

where we have used Lemma 3.1 in the last line. By Lemma 3.6, this becomes

\[
(d (b \term \phi) \term \phi)_{ik} = -\nabla_i b_{ik} - \frac{1}{2} \nabla_m b_{ij} \psi_{mijk} + \frac{\tau_0}{4} b_{ij} \phi_{ijk},
\]

which agrees with the expression (3.18). \( \square \)

**Proposition 3.8.** For a \( G_2 \)-instanton with respect to a nearly parallel \( G_2 \)-structure \( \phi \), we have

(3.19) \[
\mathcal{L}_A^2(\alpha) = \frac{\tau_0}{2} \phi \term \phi d \alpha + \mathcal{J}_A(\alpha),
\]

where

\[
\mathcal{J}_A(\alpha) = \nabla_A^* \nabla_A \alpha + \text{Ric}(\alpha) - 2 [F_A \term \alpha]
\]
is the Yang-Mills stability operator (see Bourguignon-Lawson [7]). In the case of the round 7-sphere, we have

\[ \mathcal{L}_A^2(\alpha) = 2\phi \lrcorner d\alpha + \nabla_A^* \nabla_A \alpha + 6\alpha - 2 [F_A \lrcorner \alpha]. \]

**Proof.** From (3.17) and Lemma 3.7, we have

\[ \mathcal{L}_A^2(\alpha) = dd^* \alpha + \phi \lrcorner d(\phi \lrcorner d\alpha) \]

\[ = dd^* \alpha + d^* d\alpha - d^2 \alpha \lrcorner \psi + \frac{\tau_0}{2} \phi \lrcorner d\alpha \]

\[ = (dd^* + d^* d) \alpha - (F_A \wedge \alpha) \lrcorner \psi + \frac{\tau_0}{2} \phi \lrcorner d\alpha \]

\[ = \frac{\tau_0}{2} \phi \lrcorner d\alpha + (dd^* + d^* d) \alpha + (F_A \lrcorner \psi) \lrcorner \alpha. \]

But since \( A \) is an instanton, we have

\[ F_A \lrcorner \psi = *(F_A \wedge \phi) = -F_A. \]

Substituting into (3.21) and applying the Bochner formula yields (3.19). Then (3.20) is obtained by substituting \( \tau_0 = 4 \) and \( \text{Ric}_g = 6g \) on the round 7-sphere. \( \square \)

**Remark 3.9.** Ball and Oliveira [6] have studied instantons on the Aloff-Wallach spaces, which are nearly parallel. Singhal [30] also studies instantons on homogeneous nearly parallel \( G_2 \)-manifolds, using spinorial methods similar to those of Charbonneau and Harland [11] in the context of nearly Kähler manifolds.

4. **Hopf fibration and standard instantons**

In this section, we give an explicit description of the standard (A)SD instanton and its \( G_2 \) relative. We shall use a variant of Atiyah’s quaternionic notation [3] based on the convention (2.1):

\[ x = x^0 1 - x^1 i - x^2 j - x^3 k, \quad \bar{x} = x^0 1 + x^1 i + x^2 j + x^3 k \]

\[ dx = dx^0 1 - dx^1 i - dx^2 j - dx^3 k, \quad d\bar{x} = dx^0 1 + dx^1 i + dx^2 j + dx^3 k. \]

Here \( x \) and \( \bar{x} \) are \( \mathbb{H} \)-valued functions, and \( dx \) and \( d\bar{x} \) are \( \mathbb{H} \)-valued differential forms on \( \mathbb{R}^4 \).

We define \( y \) and \( \bar{y} \) similarly, and will identify

\[ \mathbb{R}^8 = \mathbb{H}_x \oplus \mathbb{H}_y \]

as above.

4.1. **Quaternionic Hopf fibration.** The (right) Hopf fibration is given by the quotient projection under right-multiplication by \( \mathbb{H}^x \):

\[ \pi : \mathbb{H}^2 \setminus \{(0,0)\} \to \mathbb{HP}^1. \]

The fibration (1.3) is obtained by restricting (4.1) to \( S^7 \), giving the quotient projection under right-multiplication by \( \text{Sp}(1) \cong S^3 \subset \mathbb{H}^x \).

To see the identification \( \mathbb{HP}^1 \cong S^4 \) explicitly, observe that \( \text{Sp}(2) \) acts on \( S^7 \) by isometries commuting with \( \text{Sp}(1)_x \). The stabilizer of an \( S^3 \) fiber of (1.3) is the subgroup \( \text{Sp}(1)_l \times \text{Sp}(1)_l \subset \text{Sp}(2) \).
G\textsubscript{2}-INFINITYONS ON THE 7-SPHERE

Meanwhile, Sp(2) acts by conjugation on the 5-dimensional space of 2 \times 2 traceless self-adjoint quaternionic matrices:

\[ W = \left\{ \begin{pmatrix} a & \bar{z} \\ z & -a \end{pmatrix} \mid a \in \mathbb{R}, z \in \mathbb{H} \right\}, \]

where the stabilizer of an axis is again Sp(1) \times Sp(1). We therefore have a map

\[ \mathbb{H}P^1 = \text{Sp}(2)/\text{Sp}(1) \times \text{Sp}(1) \to SO(5)/SO(4) = S^4 \]

which is an isometry, up to a factor of 1/2.

The basic link between instantons on \( S^7 \) and \( S^4 \) is as follows; a more general statement appears in Proposition 6.2 below.

**Lemma 4.1.** Let \( B \) be a connection on a principal bundle over \( S^4 \). Then \( B \) is an ASD instanton if and only if the pullback \( A = \pi^* B \) by the Hopf fibration is a \( G_2 \)-instanton, for either the standard or the squashed nearly parallel \( G_2 \)-structure on \( S^7 \).

**Proof.** By Sp(2)-invariance both of \( \phi_{\text{std}} \) and of the fibration, it suffices to consider the point \((1,0)\). From (3.8), we have

\[ \phi(x, 0) = \nu^x + \zeta^x_1 \wedge \omega^y_1 + \zeta^x_2 \wedge \omega^y_2 - \zeta^x_3 \wedge \omega^y_3. \]

and

\[ \phi(1, 0) = dx^{123} + dx^1 \wedge \omega^y_1 + dx^2 \wedge \omega^y_2 - dx^3 \wedge \omega^y_3. \]

The orthogonal complement of the fiber through \((1,0)\) is \( \mathbb{R}^4_y \), which is mapped conformally onto the tangent space of \( S^4 \) at \( p = \pi(1,0) \). Hence, \( F_A(1,0) = \pi^* F_B(p) \) is equal to a 2-form on \( \mathbb{R}^4_y \). It is clear from the expression (4.4) that \( \phi_\wedge F_A(1,0) \) vanishes if and only if this 2-form is ASD, which is equivalent to the same statement for \( F_B(p) \).

The same argument applies on \( \phi_{\text{sq}} \). \( \square \)

4.2. **Standard (A)SD instanton.** Let \( P^+ \) denote the principal Sp(1)-bundle associated to the right Hopf fibration. We define the standard self-dual instanton to be the connection on \( P^+ \) induced by the round metric on the total space. The corresponding connection form on \( P^+ \) is the \( \mathfrak{su}(2) \)-valued 1-form

\[ \text{Im} \left[ \bar{x} \, dx + \bar{y} \, dy \right] \left/ \left| x \right|^2 + \left| y \right|^2 \right.. \]

Pulling back to \( \mathbb{R}^4 \) by the map \( x \mapsto (x, 1) \), we obtain the well-known connection matrix

\[ \text{Im} \bar{x} \, dx \left/ 1 + \left| x \right|^2 \right.. \]

whose curvature is the \( \mathfrak{su}(2) \)-valued self-dual 2-form

\[ d\bar{x} \wedge dx \left/ (1 + \left| x \right|^2)^2 \right.. \]

See Atiyah [3, §1] for these formulae, as well as a generalization giving the complete ADHM construction.
Similarly, we define the standard anti-self-dual (ASD) instanton \( P^- \) to be the principal bundle associated to the left Hopf fibration, with connection form

\[
(4.5) \quad \frac{\text{Im} \left[ w \, d\bar{w} + z \, d\bar{z} \right]}{|w|^2 + |z|^2}.
\]

In the stereographic chart on \( \mathbb{R}^4 \), this has a connection matrix

\[
(4.6) \quad B_0(x) = \frac{\text{Im} \, x \, d\bar{x}}{1 + |x|^2}
\]

and curvature the \( \text{Im} \, \mathbb{H} \)-valued self-dual 2-form

\[
(4.7) \quad F_{B_0}(x) = \frac{dx \wedge d\bar{x}}{(1 + |x|^2)^2}.
\]

4.3. **Standard \( G_2 \)-instanton.** Let

\[
(4.8) \quad P = P^- \times_{S^1} P^+ \xrightarrow{\pi_2} S^7
\]

be the fiber product over \( S^1 \) of \( P^- \) with \( P^+ \), considered as a principal \( \text{Sp}(1) \)-bundle via the projection to \( P^+ = S^7 \). According to §4.1 and Lemma 4.1, the pullback of the standard ASD instanton on \( P^- \) is a \( G_2 \)-instanton on \( P \to S^7 \), which we call the standard \( G_2 \)-instanton, \( A_0 \).

We may obtain a connection matrix for \( A_0 \) by pulling back the connection form of the standard ASD instanton (4.5) by the fiber-preserving map \( \mathbb{H}_r^2 \to \mathbb{H}_t^2 \) given by

\[
\begin{align*}
w & = y^{-1} \\
z & = x^{-1}.
\end{align*}
\]

This gives

\[
A_0(x, y) = \frac{1}{|x|^2 + |y|^2} \text{Im} \left[ x^{-1} d(x^{-1}) + y^{-1} d(y^{-1}) \right]
= \frac{|x|^2 |y|^2}{|x|^2 + |y|^2} \text{Im} \left[ -x^{-1} \bar{x} \, dx^{-1} - y^{-1} \bar{y} \, dy^{-1} \right]
= \frac{1}{|x|^2 + |y|^2} \text{Im} \left[ |y|^2 x^{-1} \, dx + |x|^2 y^{-1} \, dy \right],
\]

which is (1.4) above. The singularity along the \( x \)-axis can be removed by applying the gauge transformation \( g(x, y) = y/|y| \) : noting that \( dgg^{-1} = -\text{Im} \, y \, d\bar{y} / |y|^2 \), we obtain

\[
g(A_0) = gA_0 \, \bar{g} - dg \bar{g} = \frac{1}{|x|^2 + |y|^2} \text{Im} \left[ yx^{-1} \, dx \bar{y} + \frac{|x|^2}{|y|^2} \, dy \bar{y} + \frac{|x|^2 + |y|^2}{|y|^2} \, y \, d\bar{y} \right]
= \frac{1}{|x|^2 + |y|^2} \text{Im} \left[ y \, x^{-1} \, dx \bar{y} + y \, d\bar{y} \right].
\]

The singularity along the \( y \)-axis can be removed similarly. However, they cannot be removed simultaneously, for according to the following proposition, the bundle \( P \to S^7 \) is nontrivial. Recall that by the clutching construction, topological SU(2)-bundles on \( S^7 \) are classified by \( \pi_6(S^3) = \mathbb{Z}_{12} \).
Proposition 4.2 (Crowley-Goette [14, (1.18)]). For an SU(2)-bundle on $S^4$ with $c_2(E) = \kappa$, the pullback bundle on $S^7$ has homotopy class

$$\frac{\kappa (\kappa + 1)}{2} \in \mathbb{Z}_{12}.$$

4.4. Curvature calculation. We check directly that $A_0$ is a $G_2$-instanton on $S^7$. By (3.6), this is equivalent to showing that (1.4) is a Spin(7)-instanton on $\mathbb{R}^8$. We calculate

$$d(x^{-1}dx) = -x^{-1}dx \wedge x^{-1}dx$$

and

$$d \left( \frac{|y|^2}{|x|^2 + |y|^2} \right) = \frac{|x|^2|y|^2}{(|x|^2 + |y|^2)^2} 2 \Re \left[ y^{-1}dy - x^{-1}dx \right] = -d \left( \frac{|x|^2}{|x|^2 + |y|^2} \right).$$

This gives

(4.9)

$$dA_0 = \frac{1}{(|x|^2 + |y|^2)^2} \Im \left[ x^{-1}dx \wedge (-2|x|^2|y|^2 \Re \left[ y^{-1}dy - x^{-1}dx \right] - (|y|^4 + |x|^2|y|^2)x^{-1}dx \right]$$

$$+ y^{-1}dy \wedge (-2|x|^2|y|^2 \Re \left[ x^{-1}dx - y^{-1}dy \right] - (|x|^4 + |x|^2|y|^2)y^{-1}dy \right]$$

$$= \frac{1}{(|x|^2 + |y|^2)^2} \Im \left[ x^{-1}dx \wedge (-2|x|^2|y|^2 \Re y^{-1}dy + |y|^2\bar{dx}x - |y|^4x^{-1}dx \right]$$

$$+ y^{-1}dy \wedge (-2|x|^2|y|^2 \Re x^{-1}dx + |x|^2d\bar{y}y - |x|^4y^{-1}dy \right]$$

$$= \frac{1}{(|x|^2 + |y|^2)^2} \Im \left[ |y|^2x^{-1}dx \wedge \bar{dx}x + |x|^2y^{-1}dy \wedge d\bar{y}y - \bar{dx}x \wedge d\bar{y}y - ydy \wedge \bar{dx}x \right]$$

$$- 2|x|^2|y|^2y^{-1}dy \wedge x^{-1}dx - |x|^4y^{-1}dy \wedge y^{-1}dy - |y|^4x^{-1}dx \wedge x^{-1}dx \right].$$

On the other hand, we have

(4.10)

$$A_0 \wedge A_0 = \frac{1}{(|x|^2 + |y|^2)^2} \Im \left[ |y|^4x^{-1}dx \wedge x^{-1}dx + |x|^4y^{-1}dy \wedge y^{-1}dy + 2|x|^2|y|^2x^{-1}dx \wedge y^{-1}dy \right].$$

Adding (4.9) and (4.10), we obtain the curvature form

$$F_{A_0}(x, y) = dA_0 + A_0 \wedge A_0$$

(4.11)

$$= \frac{1}{(|x|^2 + |y|^2)^2} \Im \left[ |y|^2x^{-1}dx \wedge \bar{dx}x + |x|^2y^{-1}dy \wedge d\bar{y}y - 2\bar{dx}x \wedge d\bar{y}y \right]$$

$$= \frac{1}{(|x|^2 + |y|^2)^2} \Im \left[ (d\bar{dx}y|x|^2 - d\bar{dy}y|x|^2) \wedge (\bar{dx}x|y|^2 - d\bar{dy}y|x|^2) \right].$$

The 2-forms $dx \wedge d\bar{x}$, $dy \wedge d\bar{y}$, and $dx \wedge d\bar{y}$ are each invariant under Sp(1),. Therefore, the 2-form part of $F_{A_0}$ lies in Lie(Sp(2)) ⊂ Lie(Spin(7)), as claimed.

4.5. Linear deformations. Let $A$ be a conical instanton on $\mathbb{R}^8 \setminus \{0\}$ whose curvature $F_A$ takes values in Lie(Sp(2)) ⊗ $g_E$.

Given any $8 \times 8$ matrix $M$, we associate the vector field

$$X_M = M \cdot \bar{r} = M^i_j x^j \frac{\partial}{\partial x^i}$$
on $\mathbb{R}^8$, as well as the $\mathfrak{g}_E$-valued 1-form

\[(4.12) \quad \alpha_M = X_M - F_A \in \Omega^1(\mathfrak{g}_E).\]

Notice that (4.12) corresponds to pushforward by the diffeomorphism generated by $X_M$, with its horizontal lift to the bundle. For, working in a local gauge where $A_{X_M} = 0$, we have

\[\frac{d}{dt}\exp(-tX_M)^*A = X_M(A) = F(X_M, -) = \alpha_M.\]

The action on the curvature is given by

\[(4.13) \quad \frac{d}{dt}\exp(-tX_M)^*F_A = \frac{d}{dt}F_{\exp(-tX_M)A} = DA\alpha_M.\]

**Lemma 4.3.** For $M \in \text{Lie}(\text{Sp}(2))^\perp \subset \mathbb{R}^{8 \times 8}$, we have

\[(4.14) \quad (D^{S^7}_A)^*\alpha_M = 0.\]

**Proof.** Let $\alpha = \alpha_M$ and $F = F_A$. In coordinates on $\mathbb{R}^8$, we have

\[(4.15) \quad (D^{\mathfrak{g}_8}_A)^*\alpha = \nabla^i X^j F_{ij} + X^j \nabla^i F_{ij} = M_{ij} F_{ij}.\]

Since the curvature takes values in $\text{Lie}(\text{Sp}(2))$ and $M$ belongs to the orthogonal complement, the expression (4.15) vanishes identically. The result then follows from the formula

\[(D^{\mathfrak{g}_8}_A)^*\alpha = (D^{S^7}_A)^*\alpha - \langle \hat{r}, \nabla_{\hat{r}}(\alpha(\hat{r})) \rangle\]

and the fact that $\alpha(\hat{r}) \equiv 0$ for a conical instanton. \qed

**Proposition 4.4.** Let $W$ be the 5-dimensional space (4.2). Then $\ker L_A$ contains the space

\[(4.16) \quad \mathcal{V}_A = \{\alpha_M \mid M \in W \oplus WI_1 \oplus WI_2\},\]

where $\alpha_M$ is defined by (4.12).

If $F_A(x, y)$ spans $\text{Lie}(\text{Sp}(2))$ as $(x, y)$ varies over $\mathbb{R}^8$, then $\dim(\mathcal{V}_A) = 15$.

**Proof.** According to (4.13), the subspace $\{\alpha_M \mid M \in W\}$ corresponds to pushforward by elements of $\text{SL}(2, \mathbb{H})$, which preserve the algebra $\text{Lie}(\text{Sp}(2))$; hence, these correspond to infinitesimal deformations. By Lemma 4.3, the space (4.16) is in Coulomb gauge, so the first factor lies in the kernel of $L_A$.

The second factor may be described as follows:

\[WI_1 = \left\{ \begin{pmatrix} aI_1 & zI_1 \\ z\bar{I}_1 & -aI_1 \end{pmatrix} \mid a \in \mathbb{R}, z \in \mathbb{H} \right\}.\]

The matrix \( \begin{pmatrix} I_1 & 0 \\ 0 & -I_1 \end{pmatrix} \) is just the first element in (2.8), which belongs to the subspace $\Lambda_2^2 \subset \text{Lie}(\text{Spin}(7))$. Since $I_1$ commutes with $\mathbb{H}I_1$, we also have

\[\begin{pmatrix} 0 & zI_1 \\ z\bar{I}_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\bar{z}I_1 \\ zI_1 & 0 \end{pmatrix},\]

which belongs to the subspace $\mathfrak{f}_{x,y} \subset \text{Spin}(7)$ given by (2.9). Hence, the second factor corresponds to pushforward by elements of $\text{Spin}(7)$, which are in Coulomb gauge by Lemma
4.3. The same is true of the third factor. Therefore the space $\mathcal{Y}_A$ is contained in the kernel of the deformation operator $\mathcal{L}_A$.

For the second statement, if the curvature $F_A(x, y)$ spans Lie($\text{Sp}(2)$), then for any nonzero $M$, the element $\alpha_M$ must be nonzero. In particular, the map $M \mapsto \alpha_M$ has vanishing kernel, hence rank 15.

\[ \square \]

Remark 4.5. The master’s thesis of Jurke [25] studies Spin($7$)-instantons using a quaternionic approach similar to that of this section. The author thanks an anonymous referee for pointing out this reference.

5. Infinitesimal deformations

In this section, we calculate the space of infinitesimal deformations, as a $G_2$-instanton, of the pullback to $S^7$ of a general irreducible ASD instanton on $S^4$. We write

\[ \nabla = \nabla^{S_7} = \pi_{S^7} \circ \nabla^{\mathbb{R}^8} \]

for the Levi-Civita connection on $S^7_{\text{std}}$, which we shall couple to the connection on any auxiliary bundle. We shall also write

\[ (5.1) \quad \nabla^v = \nabla_{\pi_v}, \quad \nabla^h = \nabla_{\pi_h}. \]

Here, $\pi_v$ is the orthogonal projection to the the vertical tangent space of the Hopf fibration and $\pi_h$ is the complementary projection, with respect to the round metric.

5.1. **Vertical and horizontal components.** Let $\Omega^1_v$ be the annihilator of vertical vector fields along the Hopf fibration, and $\Omega^1_h$ its orthogonal complement. We have

\[ \Omega^1_{S^7} = \Omega^1_v \oplus \Omega^1_h. \]

Letting

\[ \Omega^{(p,q)} = \Lambda^p \Omega^1_v \otimes \Lambda^q \Omega^1_h \subset \Omega^{p+q}_{S^7}, \]

we have a decomposition

\[ \Omega^k_{S^7} = \bigoplus_{p+q=k} \Omega^{(p,q)}_{S^7}. \]

An element of $\Omega^{(p,q)}$ will be referred to as a $(p,q)$-form.

Let $\nu$ denote the $(3,0)$ volume form of the Hopf fibration, and let

\[ \bar{\nu} = \ast \nu. \]

The $(0,2)$-forms split as

\[ \Omega^{(0,2)} = \Omega^{2+}_{h} \oplus \Omega^{2-}_{h}, \]

where $\Omega^{2\pm}_{h}$ are the (anti-)self-dual components with respect to the $(0,4)$ volume form $\bar{\nu}$.

For a $(0,1)$-form $b$, we shall write $d^v b$ for the $(1,1)$ part of $db$ and $d^h b$ for the $(0,2)$ part. A similar notation will be used for $(1,0)$-forms (see Lemma 5.3 below).
**Definition 5.1.** Let $\omega^x_i, \omega^y_i, \xi^x_i,$ and $\xi^y_i$ be as in (2.2) and (3.7). For $i = 1, 2, 3,$ define the global Sp(2)-invariant forms on $S^7$:

\[
\begin{align*}
\zeta_i &= \tilde{f} \cdot (\omega^x_i + \omega^y_i) \\
\omega_i^0 &= \omega_i^x + \omega_i^y|_{S^7} \\
\tilde{\omega}_i &= \omega_i^0 - \frac{1}{2} \epsilon_{ijk} \xi_j \wedge \xi_k.
\end{align*}
\]

Notice that $\{\zeta_i\}$ and $\{\tilde{\omega}_i\}$ are global frames for $\Omega^1_v$ and $\Omega^2_h$, respectively. The vertical volume form is given by

\[\nu = \zeta_1 \wedge \zeta_2 \wedge \zeta_3.\]

From (3.8) and the Sp(2)-invariance, we may re-express $\phi_{\text{std}}$ as follows:

\[
(5.2) \quad \phi_{\text{std}} = \frac{1}{\nu} \overbrace{(3.0)} + \underbrace{(1.2)}_{2} \zeta_1 \wedge \zeta_2 \wedge \zeta_3 \wedge \tilde{\omega}_3.
\]

We now derive the basic properties of these frames, and use them to decompose the Laplace operator into horizontal and vertical parts.

**Lemma 5.2.** The frame $\{\zeta_i\}$ is coclosed, and satisfies

\[
(5.3) \quad \nabla^v \zeta_i = \frac{1}{2} \epsilon_{ijk} \xi_j \wedge \xi_k, \quad \nabla^h \zeta_i = \tilde{\omega}_i
\]

\[
(5.4) \quad \nabla^s \nabla^v \zeta_i = 2\zeta_i, \quad \nabla^s \nabla^h \zeta_i = 4\zeta_i.
\]

Here $\nabla^v, \nabla^h$ are defined by (5.1) above.

**Proof.** From the Definition 5.1 and (3.7), for $X,Y \in TS^7$, we have

\[
(5.5) \quad \left(\nabla_X \zeta_i\right)(Y) = \left(\nabla_X^h \zeta_i\right)(Y) = (\omega^x_i + \omega^y_i)(X,Y) = \omega_i^s(X,Y).
\]

Coclosedness of $\zeta_i$ and (5.3) follow directly from (5.5).

Since $\zeta_i$ is coclosed and equal to the restriction of a linear form, we have

\[
(5.6) \quad \nabla^s \nabla_X^h \zeta_i = 2\zeta_i, \quad \nabla^s \nabla_Y^h \zeta_i = 6\zeta_i
\]

as can be verified directly from (5.5). Then (5.4) follows from (5.6) and the fact that $\nabla = \nabla^v + \nabla^h$. \hfill \Box

**Lemma 5.3.** Let $\alpha = a + b$ be a 1-form on $S^7$, with $a = f_i \zeta_i \in \Omega^1_v$ and $b \in \Omega^1_h$. Then

\[
\begin{align*}
\,d\alpha &= d^v a + d^h f_i \wedge \zeta_i + 2 f_i \tilde{\omega}_i + d^h b
\end{align*}
\]

**Proof.** This is a standard decomposition result, which can be seen directly as follows. By Lemma 5.2, we have

\[
d\zeta_i = 2\omega_i^0, \quad d^v \zeta_i = \epsilon_{ijk} \zeta_j \wedge \zeta_k, \quad i = 1, 2, 3.
\]

Therefore

\[
(5.7) \quad d\zeta_i = \epsilon_{ijk} \zeta_j \wedge \zeta_k + 2\tilde{\omega}_i.
\]
The result follows directly from (5.7). \qed

**Lemma 5.4.** If \( a = f_i \zeta_i \) is (co)closed on each \( S^3 \) fiber, then \((\nabla^* \nabla^h f_i) \zeta_i \) is again fiberwise (co)closed.

**Proof.** Let \( U_j \) be the dual vector field of \( \zeta_j \), for \( j = 1, 2, 3 \). The Killing vector fields \( U_j \) commute with the operators \( \nabla^* \nabla \) and \( \nabla^* \nabla^v \), hence also with \( \nabla^* \nabla^h = \nabla^* \nabla - \nabla^* \nabla^v \). Since \( d^v a \) and \( (d^v)^* a = d^* a \) are determined by \( U_j(f_i) \), we conclude that \( \nabla^* \nabla^h \) preserves (co)closedness on the fibers. \( \square \)

**Proposition 5.5.** Let \( \alpha = a + b \) be a 1-form as above, where \( a = f_i \zeta_i \) and \( b \in \Omega^1_h \). The vertical component of the Laplacian on \( S^7 \) is given by

\[
(\nabla^* \nabla \alpha)^v = \nabla^* \nabla^v a + 4a + \left( \nabla^* \nabla^h f_i + 2(d^h b, \bar{\omega}_i) \right) \zeta_i.
\]

Here \( \nabla^* \nabla^v \) denotes the Laplacian on the \( S^3 \) fiber.

**Proof.** We have

\[
(\nabla^* \nabla \alpha)^v = (\nabla^* \nabla a)^v + (\nabla^* \nabla b)^v.
\]

For the first term, we write

\[
(\nabla^* \nabla a)^v = \nabla^* \nabla^v a + \left( \nabla^* \nabla^h a \right)^v
\]

and

\[
\left( \nabla^* \nabla^h a \right)^v = \left( \nabla^* \left( \nabla^h f_i \zeta_i + f_i \nabla^h \zeta_i \right) \right)^v = \left( \nabla^* \nabla^h f_i \right) \zeta_i + \left( \nabla^h f_i \nabla^* \zeta_i - \nabla f_i \omega_i \right)^v + f_i \nabla^* \nabla^h \zeta_i = \left( \nabla^* \nabla^h f_i \right) \zeta_i + 4f_i \zeta_i
\]

where we have used Lemma 5.2. Then (5.9) yields

\[
(\nabla^* \nabla a)^v = \nabla^* \nabla^v a + 4a + \nabla^* \nabla^h f_i \zeta_i
\]

which gives the case \( b = 0 \) of the formula (5.8).

Assuming now that \( Y \) is horizontal and \( b \) is of type \((0,1)\), we compute

\[
0 \equiv \nabla_Y (\zeta_i, b) = \langle \nabla_Y \zeta_i, b \rangle + \langle \zeta_i, \nabla_X b \rangle
\]

\[
\langle \zeta_i, \nabla_Y b \rangle = -\langle \bar{\omega}_i(Y, -), b \rangle.
\]

For \( U \) vertical, we have

\[
0 \equiv \nabla_U (\zeta_i, b) = \langle \nabla_U \zeta_i, b \rangle + \langle \zeta_i, \nabla_U b \rangle
\]

and

\[
\langle \zeta_i, \nabla_U b \rangle = 0.
\]

Hence, for \( X \in TS^7 \), we have

\[
(\nabla_X b)^v = -\zeta_i \langle \bar{\omega}_i(X, -), b \rangle.
\]
Next, let \( \{e_j\}_{j=1}^7 \) be an orthonormal basis of vector fields that satisfies \( \nabla_{e_j} e_k = 0 \) at a given point. We compute

\[
0 \equiv \nabla_{e_j} \nabla_{e_j} \langle \zeta_i, b \rangle \\
= \langle \nabla_{e_j} \nabla_{e_j} \zeta_i, b \rangle + 2\langle \omega^i (e_j, -) , \nabla_{e_j} b \rangle + \langle \zeta_i , \nabla_{e_j} \nabla_{e_j} b \rangle \\
= -6\langle \zeta_i, b \rangle + 2\langle \omega^i (e_j, -) , \nabla_{e_j} b \rangle + \langle \zeta_i , \nabla_{e_j} \nabla_{e_j} b \rangle.
\]

Since \( \langle \zeta_i, b \rangle \equiv 0 \), we are left with

\[
\langle \zeta_i , \nabla_{e_j} \nabla_{e_j} b \rangle = -2\langle \omega^i (e_j, -) , \nabla_{e_j} b \rangle
\]

which may be rewritten as

\[
(\nabla^* \nabla b)^v = 2\zeta_i (\bar{\omega}_i, db).
\]

Combining (5.10) and (5.11) yields (5.8). □

**Remark 5.6.** The previous results carry over when \( \nabla \) is coupled to a connection that is trivial along the fibers of the Hopf fibration.

### 5.2. Vanishing of the vertical component

This subsection proves our vanishing theorem for the vertical component of an infinitesimal deformation in Coulomb gauge.

**Lemma 5.7.** Let \( B \) be an ASD instanton on \( S^4 \), and put \( A = \pi^* B \). Assume that \( f \) is a section of \( \pi^* g_E \to S^7 \) satisfying

\[
\nabla^h_A f = 0.
\]

Then \( f = \pi^* h \) is the pullback of a section of \( g_E \to S^4 \), with

\[
\nabla_B h = 0.
\]

**Proof.** We have

\[
0 = \nabla^v \nabla^h f = \nabla^h \nabla^v f
\]

since \( f \) is a 0-form and the curvature \( F_A \) is purely horizontal. By (5.12), we have

\[
D_A f = D^v_A f.
\]

Applying \( D_A \) to both sides, by Proposition 5.3, we obtain

\[
D^2_A f = [F_A, f] = D_A D^v_A f
\]

\[
= (D^v_A)^2 f + D^h_A D^v_A f + 2D^v_A f(U_i) \bar{\omega}_i.
\]

By (5.14) and the fact that \( (D^v_A)^2 f = 0 \) for a pullback connection, the first two terms vanish, yielding

\[
[F_A, f] = 2 (\nabla U_i f) \bar{\omega}_i.
\]

Each side of (5.15) is of type \((0, 2)\); however, the LHS is anti-self-dual and the RHS is self-dual. Therefore both sides of (5.15) must vanish, giving

\[
\nabla^v f = 0.
\]

Hence, \( f \) is constant on the fibers, so \( f = \pi^* h \) for a section \( h \) of \( g_E \). Then (5.12) implies (5.13), as desired. □
Theorem 5.8. Let $B$ be an irreducible ASD instanton on a bundle $E \to S^4$, and $A = \pi^* B$ its pullback under the Hopf fibration. If $\alpha \in \Omega^1 (\pi^* g_E)$ satisfies $\mathcal{L}_A \alpha = 0$, then the vertical part of $\alpha$ vanishes.

Proof. We write $\nabla = \nabla_A$ and $d = d_A$ throughout the proof.

Decompose $\alpha = a + b$ into vertical and horizontal parts as above. Let $a = f_i \zeta_i$, and write the self-dual part of $d^h b$ as

$$(d^h b)^+ = c_i \bar{\omega}_i.$$ 

From Proposition 5.3 and (5.2), our assumption $\mathcal{L}_A \alpha = 0$ implies

$$(5.16) \quad 0 = \phi \lrcorner \ d\alpha = \underbrace{\phi \lrcorner \ d^v a}_{(1,0)} + \underbrace{\phi_{1,2} \lrcorner \ (d^h a + d^v b)}_{(0,1)} + \phi_{1,2} \lrcorner \ (2 f_i \bar{\omega}_i + d^h b).$$

Noting that $\bar{\omega}_i \lrcorner \ \bar{\omega}_i = 2$, according to (5.2), the $(1,0)$-part of (5.16) comes out to

$$0 = \zeta_1 \ (\ast d^v a(U_1) - 4 f_1 + 2 c_1) + \zeta_2 \ (\ast d^v a(U_2) + 4 f_2 + 2 c_2) + \zeta_3 \ (\ast d^v a(U_3) - 4 f_3 - 2 c_3)$$

where $U_j$ is the dual vector field to $\zeta_j$, for $j = 1, 2, 3$, as above. We conclude that

$$c_1 = -2 f_1 - \frac{1}{2} \ast d^v a(U_1)$$

$$c_2 = -2 f_2 - \frac{1}{2} \ast d^v a(U_2)$$

$$c_3 = -2 f_3 + \frac{1}{2} \ast d^v a(U_3).$$

We rewrite this as

$$c_i \zeta_i = -2 a + \frac{1}{2} \mu(a)$$

where $\mu : \Omega^1_v \to \Omega^1_v$, defined by (5.17), obeys

$$|\mu(a)| = |\ast d^v a| = |d^v a|.$$ 

Now, the decomposition of the Laplacian (5.8) reads

$$(\nabla^* \nabla) v = \nabla^* \nabla^v a + 4 a + 4 c_i \zeta_i + \left( \nabla^* \nabla^h f_i \right) \zeta_i$$

$$= \nabla^* \nabla^v a - 4 a + 2 \mu(a) + \left( \nabla^* \nabla^h f_i \right) \zeta_i$$

where we have used (5.18). Returning to (3.20), we have

$$0 = \mathcal{L}_A^2 \alpha = 2 \phi_{std} \lrcorner \ \alpha + \nabla^* \nabla \alpha - 2 [F_A \lrcorner \ \alpha] + 6 \alpha$$

$$= \nabla^* \nabla \alpha - 2 [F_A \lrcorner \ b] + 6 \alpha.$$ 

Taking the vertical part and inserting (5.20), since $F_A$ is purely horizontal, we obtain

$$(5.21) \quad 0 = \left( \mathcal{L}_A^2 \alpha \right)^v = \nabla^* \nabla^v a + 2 a + 2 \mu(a) + \left( \nabla^* \nabla^h f_i \right) \zeta_i.$$
Applying the Bochner formula on $S^3$ yields
\[
0 = (d^* d^v + d^v d^*) a + 2 \mu(a) + \left( \nabla^* \nabla^h f_i \right) \zeta_i.
\]
We now split $a$ into fiberwise closed and coclosed parts
\[
a = a_{cl} + a_{cocl}
\]
and write
\[
a_{cocl} = g_i \zeta_i.
\]
Taking an inner product with $a_{cocl}$ in (5.22), and integrating over $S^7$, yields
\[
0 = \left\| d^v a \right\|^2 + 2 \left( a_{cocl}, \mu(a) \right) + \sum_{i=1}^3 \left\| \nabla^h g_i \right\|^2.
\]
Here we have used the orthogonality between fiberwise closed and coclosed vertical 1-forms, the fact that $d^* a_{cocl} = (d^v)^* a_{cocl} = 0$, and Lemma 5.4.

Note that the first eigenvalue of the Hodge Laplacian on coclosed 1-forms on $S^3$ is 4, so
\[
0 \geq 2 \left\| d^v a \right\|^2 - 2 \left( a_{cocl}, \mu(a) \right) + \sum_{i=1}^3 \left\| \nabla^h g_i \right\|^2.
\]
Inserting this into (5.23), using Cauchy-Schwarz and (5.19), we obtain
\[
0 \geq 2 \left\| d^v a \right\|^2 - 2 \left( a_{cocl}, \mu(a) \right) + \sum_{i=1}^3 \left\| \nabla^h g_i \right\|^2.
\]
This yields
\[
\nabla^h g_i \equiv 0
\]
for $i = 1, 2, 3$. Since $B$ is assumed irreducible, Lemma 5.7 implies that $g_i \equiv 0$, and hence $a_{cocl} \equiv 0$.

But then $a = a_{cl}$ is fiberwise closed, so $\mu(a) = 0$ by (5.19). Since the RHS of (5.21) is then a positive operator, we conclude that $a \equiv 0$, as desired. \qed

5.3. **Space of infinitesimal deformations.** This subsection proves our main theorem, which calculates the deformations of a pulled-back instanton on $S^7$ in terms of the ASD deformations on $S^4$.

Consider the operator
\[
\delta = (\phi \cdot d(\cdot))^h : \Omega^1_h \rightarrow \Omega^1_h
\]
given by the restriction of $\phi \cdot d(\cdot)$ to the space of horizontal 1-forms. For $b \in \Omega^1_h$, we have
\[
\delta(b) = (\phi \cdot db)^h = (\phi^{(1,2)} \cdot d^v b)^h.
\]
Let $S^3_0 \subset \{(x, 0)\} \subset \mathbb{H}^2$ be the fiber of the Hopf fibration contained in the $x$-axis, and let
\[
\delta_0 = \delta|_{S^3_0}.
\]
Notice that
\[
dy^j \cdot \zeta_i = (\zeta^y_i)^j = \omega_{ijk} y^k.
\]
We may therefore define a local frame for $\Omega^1_h$ near $S^3_0$ by
\[
\bar{e}^j = dy^j - \zeta_i \omega_{ijk}y^k, \quad j = 0, \ldots, 3.
\]
We calculate
\[
d\bar{e}^j |_{S^3_0} = \omega_{ijk} \zeta_i \wedge \bar{e}^k
= d^c \bar{e}^j |_{S^3_0}.
\]
Then
\[
\delta_0 \left( \bar{e}^j \right) = \left( \phi - d^c \bar{e}^j \right)^h = - \left( \omega_{1jk}\omega_{1kl} + \omega_{2jk}\omega_{2kl} - \omega_{3jk}\omega_{3kl} \right) \bar{e}^\ell
\]
and
\[
\delta_0 \left( \bar{e}^j \right) = \bar{e}^j.
\]

**Lemma 5.9.** The kernel of $\delta_0$ consists of sections of the form
\[
L_{ij} x^i \bar{e}^j
\]
where $L \in \mathfrak{F}$ is a Fueter map of $\mathbb{R}^4$, according to (2.4) above.

**Proof.** Let $\alpha = \alpha_j \bar{e}^j$ be an arbitrary section of $\Omega^1_h$ near $S^3_0$. According to (5.24) and (4.3), we have
\[
\delta_0 \left( \alpha \right) = \phi - \left( U_1 (\alpha_j) \zeta_i \wedge \bar{e}^i \right) + \alpha_j \bar{e}^j
= (U_1 (\alpha_j) \omega_{1j\ell} + U_2 (\alpha_j) \omega_{2j\ell} - U_3 (\alpha_j) \omega_{3j\ell} + \alpha_{\ell}) \bar{e}^\ell
= \beta_{\ell} \bar{e}^\ell.
\]
Further, we calculate
\[
\delta_0^2 \left( \alpha \right) = U_1 (\beta_{\ell}) \omega_{1\ell i} + U_2 (\beta_{\ell}) \omega_{2\ell i} - U_3 (\beta_{\ell}) \omega_{3\ell i} + \beta_i
= \omega_{1\ell i} U_1 (U_1 \alpha_j \omega_{1j\ell} + U_2 \alpha_j \omega_{2j\ell} - U_3 \alpha_j \omega_{3j\ell} + \alpha_{\ell})
+ \omega_{2\ell i} U_2 (U_1 \alpha_j \omega_{1j\ell} + U_2 \alpha_j \omega_{2j\ell} - U_3 \alpha_j \omega_{3j\ell} + \alpha_{\ell})
- \omega_{3\ell i} U_3 (U_1 \alpha_j \omega_{1j\ell} + U_2 \alpha_j \omega_{2j\ell} - U_3 \alpha_j \omega_{3j\ell} + \alpha_{\ell})
+ \beta_i
= - (U_1^2 + U_2^2 + U_3^2) \alpha_i
+ \omega_{2\ell i} \omega_{1\ell i} U_1 U_2 (\alpha_j) + \omega_{1\ell i} \omega_{2\ell i} U_2 U_1 (\alpha_j)
- \omega_{3\ell i} \omega_{1\ell i} U_1 U_3 (\alpha_j) - \omega_{1\ell i} \omega_{3\ell i} U_3 U_1 (\alpha_j)
- \omega_{3\ell i} \omega_{2\ell i} U_2 U_3 (\alpha_j) - \omega_{2\ell i} \omega_{3\ell i} U_3 U_2 (\alpha_j)
+ (\beta_{\ell} - \alpha_{\ell}) + \beta_i
= - (U_1^2 + U_2^2 + U_3^2) \alpha_i + \omega_{3\ell i} [U_1, U_2] \alpha_j + \omega_{2\ell i} [U_1, U_3] \alpha_j - \omega_{1\ell i} [U_2, U_3] \alpha_j
+ 2 \beta_{\ell} - \alpha_{\ell}
= - (U_1^2 + U_2^2 + U_3^2) \alpha_i + 2 (\omega_{3\ell i} U_3 + \omega_{2\ell i} U_2 + \omega_{1\ell i} U_1) \alpha_j
+ 2 \beta_{\ell} - \alpha_{\ell}.
Again applying (5.25), we obtain
\begin{equation}
\delta_0^3 (\alpha) - e^i = - (U_1^2 + U_2^2 + U_3^2) \alpha_i + 4 \beta_1 - 3 \alpha_i.
\end{equation}

Notice from (5.3) that $\nabla_{U_i} U_i = 0$. Hence the first term on the RHS of (5.26) is the Laplace-Beltrami operator on $S^3_0$, applied component-wise to $\alpha$. We conclude that
\begin{equation}
\delta_0^3 (\alpha) = \Delta_S \alpha + 4 \delta_0 (\alpha) - 3 \alpha.
\end{equation}

In particular, if $\delta_0 (\alpha) = 0$, we have
\begin{equation}
\Delta_S \alpha_j = 3 \alpha_j.
\end{equation}

Hence, $\alpha_j$ lies in the first nonzero eigenspace, and is the restriction of a linear function on $\mathbb{R}^4$:
\begin{equation}
\alpha_j = x^i L_{ij}.
\end{equation}

Substituting back into (5.25), we have
\begin{equation}
\delta_0 (\alpha) = x^i (L_{ij} + \omega_{1ik} L_{k\ell} \omega_{1\ell j} + \omega_{2ik} L_{k\ell} \omega_{2\ell j} - \omega_{3ik} L_{k\ell} \omega_{3\ell j}) \partial^j = 0
\end{equation}
which recovers (2.4), as desired. \hfill \square

**Proposition 5.10.** The kernel of $\delta$ is the subspace of $\Omega^1_h$ given by
\begin{equation}
\pi^* \Omega^1_{S^4} \oplus I_1 (\pi^* \Omega^1_{S^4}) \oplus I_2 (\pi^* \Omega^1_{S^4}).
\end{equation}
Here the complex structures $I_1$ and $I_2$ act pointwise on $\Omega^1_h \subset \Omega^1_{\mathbb{R}^8}$.

**Proof.** Over the fiber $S^3_0$, this follows directly from Lemma 5.9 and the description (2.5) of $\mathfrak{g}$. Since Sp(2) preserves $\pi^* \Omega^1_{S^4}$ and commutes with $I_1$ and $I_2$, the subspace (5.27) is also invariant under Sp(2). Hence, it agrees with $\ker \delta$ globally. \hfill \square

**Theorem 5.11.** Let $A = \pi^* B$ be the pullback of an irreducible ASD instanton on $S^4$, and let
\begin{equation}
V = \ker (D_B^+ \oplus D_B^-) \subset \Omega^1_{S^4} (\mathfrak{g}_E)
\end{equation}
denote the space of infinitesimal deformations of $B$. The space of infinitesimal deformations of $A$, as a $G_2$-instanton, is given by
\begin{equation}
\ker \mathcal{L}_A = \pi^* V \oplus I_1 (\pi^* V) \oplus I_2 (\pi^* V) \subset \Omega^1_{S^7} (\mathfrak{g}_{\pi^* E}).
\end{equation}

For structure group $SU(2)$, this has dimension $3 (8k - 3)$.

**Proof.** Let $\alpha \in \ker \mathcal{L}_A$. According to Theorem 5.8, the vertical component of $\alpha$ vanishes, so it is a global section of $\Omega^1_h (\mathfrak{g}_E)$. The image $\delta (\alpha) \in \Omega^1_h$ under the horizontal component of the deformation operator must therefore vanish; by Proposition 5.10, we have
\begin{equation}
\alpha = \alpha_0 + \alpha_1 + \alpha_2
\end{equation}
according to (5.27).

It remains to show that $\mathcal{L}_A (\alpha) = 0$ if and only if $d^* \alpha_1 = 0$ and $(d^h \alpha_i)^+ = 0$, for $i = 0, 1, 2$. Clearly $d^* \alpha = 0$ if and only if $d^* \alpha_i = 0 \forall i$. We have
\begin{equation}
(d^h \alpha)^+ = \sum_{i=1}^3 (d^h \alpha_i)^+
\end{equation}
where \((d^h \alpha_i)^+\) are linearly independent. But the map \(\phi_\prec\) is a bundle isomorphism from \(\Omega^2_h\) to \(\Omega^1\), so
\[(\phi_\prec d\alpha)^v = \phi_\prec (d^h \alpha)^+ = 0\]
if and only if \((d^h \alpha)^+ = 0\). By (5.29), this implies that \((d^h \alpha_i)^+ = 0\) for \(i = 0, 1, 2\), as desired.

The dimension formula for structure group SU(2) follows from the Atiyah-Hitchin-Singer Theorem [4]. □

6. Global picture

In this section, we discuss the global structure of the components of the moduli space obtained by pullback under the Hopf fibration. We first prove Theorem 1.1 on the structure of the \(\kappa = 1\) component. For higher charge, the picture necessarily involves Hermitian-Yang-Mills connections on the twistor space \(\mathbb{CP}^3 \to S^4\).

6.1. Proof of Theorem 1.1. Let \(W\) be given by (4.2). Taking \(A_0\) in the gauge (1.4), we may define the smooth 5-dimensional family of connections
\[V = \{\exp(w)^* A_0 \mid w \in W\} \subset \mathcal{A}_E.\]
By the construction of §4.1, this family is equal to the pullback by the Hopf fibration of the 5-dimension family of unit-charge ASD instantons on \(S^4\). Further define a smooth map
\[
(\text{Spin}(7) \times V) \to \mathcal{A}_E
\]
\[(\sigma, A) \mapsto \sigma^* A.\]
(6.1)
By construction, the image of (6.1) consists of \(G_2\)-instantons.

Denote the principal bundle
\[Q = \text{Spin}(7) \to \text{Spin}(7)/\text{Sp}(2) \times U(1).\]
The stabilizer of \(V\) is \(\text{Sp}(2) \times U(1) \subset \text{Spin}(7)\), which acts on \(V\), modulo gauge, by the 5-dimensional representation of \(\text{Sp}(2)\) and the trivial representation of \(U(1)\). Taking the quotient by the gauge group \(\mathcal{G}_E\), the map (6.1) descends to a smooth map from the associated bundle
\[X = Q \times_{\text{Sp}(2) \times U(1)} V\]
to the space of connections modulo gauge:
\[\Phi : X \to \mathcal{A}_E/\mathcal{G}_E.\]
Notice that
\[\text{Spin}(7)/\text{Sp}(2) \times U(1) \cong SO(7)/SO(5) \times SO(2) = G^{or}(5, 7)\]
Hence, \(X\) is equal to the vector bundle over \(G^{or}(5, 7)\) associated to the standard representation of \(SO(5)\), i.e., the tautological 5-plane bundle. This is a 15-dimensional manifold.

We claim that the above map \(\Phi\) is a proper embedding. For each \(A \in V\), the image of the differential of \(\Phi\) is equal (modulo gauge) to the space of linear deformations \(\mathcal{Y}_A\) given by (4.16). By examining (4.11), it is easy to see that \(F_{A_0}(x, y)\) spans \(\text{Lie}(\text{Sp}(2))\) as \((x, y)\) varies over \(S^7\); the same is true of each element of the 5-dimensional family \(V\). Hence, according to Proposition 4.4, \(\dim(\mathcal{Y}_A) = 15\), and the differential of \(\Phi\) has full rank at each point of
V. Since the construction is invariant under the action of Spin(7), the same is true at each point of X. Hence, the map Φ is an embedding.

Next, note that as \( x \in X \) tends to infinity in a fiber V, the curvature of \( \Phi(x) \) concentrates along an associative great sphere (the preimage under the Hopf fibration of a point in \( S^4 \)). Hence, \( \Phi(x) \) tends to infinity in \( \mathcal{A}_E/G_E \). Since the base space \( G^{or}(5, 7) \) is compact, this shows that \( \Phi \) is a proper map.

Now, since for each \( x \in X \), \( \Phi(x) \) is equivalent modulo Spin(7) to a pullback from \( S^4 \), Theorem 5.11 implies that the space of infinitesimal deformations at \( \Phi(x) \) has dimension 15. Since \( \dim(X) = 15 \) and \( \Phi \) is a proper embedding, we conclude that \( \Phi \) is in fact a diffeomorphism onto a connected component of the G_2-instanton moduli space. □

**Remark 6.1.** By the same construction, we may compactify \( X \) fiberwise to a \( \bar{B}^5 \)-bundle, where a boundary point records bubbling along an associative great sphere.

### 6.2. Chern-Simons functional and Hermitian-Yang-Mills connections

Given two connections \( A \) and \( A_1 \) on a bundle \( E \), let \( a = A - A_1 \) and define the relative Chern-Simons 3-form:

\[
cs(A, A_1) = -Tr \left( a \wedge \left( F_{A_1} + \frac{1}{2} d_{A_1} a + \frac{1}{3} a \wedge a \right) \right).
\]

This satisfies

\[
dcs(A, A_1) = Tr \left( F_A \wedge F_A - F_{A_1} \wedge F_{A_1} \right).
\]

On a 7-manifold \( M \) with G_2-structure, we may define the global relative Chern-Simons functional

\[
CS_\psi(A, A_1) = \frac{-1}{4\pi^4} \int_M \psi \wedge cs(A, A_1).
\]

This normalization is chosen so that the formula in the following proposition will be integer-valued. Let \( \Pi : S^7_{std} \to \mathbb{CP}^3 \) be the natural projection for the standard complex structure, \( I_1 \).

**Proposition 6.2.** Let \( B \) be a connection on an SU(\( n \))-bundle \( E \to \mathbb{CP}^3 \). Then \( A = \Pi^* B \) is a G_2-instanton on \( S^7_{std} \) if and only if \( B \) is Hermitian-Yang-Mills.

Given any two such connections \( (E, B) \) and \( (E_1, B_1) \) for which \( \Pi^* E \cong \Pi^* E_1 \), we have

\[
CS_\psi(A, A_1) = \langle \omega_{FS} \cup (c_2(E) - c_2(E_1)), [\mathbb{CP}^3] \rangle.
\]

**Proof.** By SU(4)-invariance, it suffices to consider the point \( p = (1, 0, \ldots, 0) \in S^7 \). We shall write

\[
T_p S^7 = \mathbb{R} x_1 \oplus \mathbb{C}^3.
\]

Then, from the expression (2.12), we have

\[
\phi_{std}(p) = \frac{\partial}{\partial x_0} \cdot \Psi_0 = dx^1 \wedge \omega + \text{Re} dz_2 \wedge dz_3 \wedge dz_4
\]
where \( \omega \) is the standard Kähler form on \( \mathbb{C}^3 \), given by (2.11). Letting
\[
F = F_A(p) = \Pi^* F_B\big|_{T_pS^7} = F_B|_{\mathbb{C}^3}
\]
we have
\[
\phi_0 \sim F = dx^1 (\omega.F) + \text{Re } dz_2 \wedge dz_3 \wedge dz_4 \sim F^{2,0}.
\]
Therefore, \( A \) is a \( G_2 \)-instanton if and only if
\[
\omega.F = 0, \quad F^{2,0} = 0
\]
which is to say, \( B \) is Hermitian-Yang-Mills on \( \mathbb{CP}^3 \).

Next, we have
\[
-4\pi^4 CS_\psi(A, A_1) = \int_{S^7} \psi \wedge cs(A, A_1)
\]
\[
= \frac{1}{4} \int_{S^7} d\phi \wedge cs(A, A_1)
\]
\[
= -\frac{1}{4} \int_{S^7} \phi \wedge dcs(A, A_1)
\]
\[
= -\frac{1}{4} \int_{S^7} \phi \wedge Tr (F_A \wedge F_A - F_{A_1} \wedge F_{A_1}).
\]
(6.8)

Assume that both \( A \) and \( A_1 \) are pullbacks of Hermitian-Yang-Mills connections on \( \mathbb{CP}^3 \).

Computing at \( p \) as above, we have \( F = F_A^{1,1} \) and are left with
\[
\phi_0 \wedge Tr (F \wedge F) = dx^1 \wedge \omega \wedge Tr (F \wedge F).
\]
(6.9)

Since the Fubini-Study form on \( \mathbb{CP}^3 \) satisfies
\[
\Pi^* \omega_{FS} = \frac{\omega}{\pi}
\]
on \( S^7 \), the expression (6.9) agrees globally with
\[
\pi \zeta_1 \wedge \Pi^* (\omega_{FS} \wedge Tr (F_B \wedge F_B)).
\]

Integrating (6.8) over the fibers of \( \Pi \), we obtain
\[
-4\pi^4 CS_\psi(A, A_1) = -\frac{\pi}{4} \int_{S^7} \zeta_1 \wedge \Pi^* (\omega_{FS} \wedge Tr (F_B \wedge F_B - F_{B_1} \wedge F_{B_1}))
\]
\[
= -\frac{\pi^2}{2} \int_{\mathbb{CP}^3} \omega_{FS} \wedge Tr (F_B \wedge F_B - F_{B_1} \wedge F_{B_1})
\]
\[
= -\frac{\pi^2}{2} ([\omega_{FS}] \cup 8\pi^2 (c_2(E) - c_2(E_1)), [\mathbb{CP}^3])
\]
\[
= -4\pi^4 ([\omega_{FS}] \cup (c_2(E) - c_2(E_1)), [\mathbb{CP}^3])
\]
as claimed. \( \square \)
6.3. **Discussion.** Theorem 5.11 can be explained in light of Proposition 6.2, as follows. Owing to the Ward correspondence, the space of infinitesimal deformations of the pullback of an ASD instanton to the twistor space $\mathbb{CP}^3 \to S^4$, as a Hermitian-Yang-Mills connection, is the complexification of the space of ASD deformations. In fact, there is an $S^1$ family of complex structures

$$\cos(\theta)I_1 + \sin(\theta)I_2$$

such that the quotient map $\Pi_\theta$ satisfies Proposition 6.2 and factorizes the given Hopf fibration:

$$S^7 \xrightarrow{\Pi_\theta} \mathbb{CP}^3 \xrightarrow{\pi} S^4.$$  

The span of the pullbacks by $\Pi_\theta$ of the deformation space over $\mathbb{CP}^3$, for $\theta \in S^1$, gives the larger space (5.28).

According to this argument, the fibration to $\mathbb{CP}^3$ accounts for all of the infinitesimal deformations identified by Theorem 5.11. As such, we expect that only the deformations coming from (6.4), or its 6-dimensional family of Spin(7) conjugates, are integrable. So for $\kappa > 1$, the generic dimension of the pulled-back component of the $G_2$-instanton moduli space should be\(^1\)

$$2(8\kappa - 3) + 6 = 16\kappa. \tag{6.10}$$

These components appear to be singular along the subset of instantons coming from $S^4$ (i.e., instanton bundles on $\mathbb{CP}^3$ satisfying a reality condition), which are themselves manifolds of dimension

$$8\kappa - 3 + 10 = 8\kappa + 7.$$  

More generally, we would like to know where the instantons obtained via pullback fit within the full moduli space of $G_2$-instantons on $S^7$. While examples that are unrelated to any fibration may exist, the following guess is appropriate based on [36] and the present work. For the statement, fix a reference Hermitian-Yang-Mills connection $B_1$ on an $SU(n)$-bundle $E_1 \to \mathbb{CP}^3$, and let $E = \Pi^*E_1 \to S^7$ and $A_1 = \Pi^*B_1$.

**Conjecture 6.3 (Donaldson).** Let $A$ be a $G_2$-instanton on $E \to S^7_{std}$, for which

$$CS_{\psi}(A, A_1) \in \mathbb{Z}.$$  

Then $A$ is equivalent, modulo the action of Spin(7) and $\mathcal{G}_E$, to the pullback of a Hermitian-Yang-Mills connection on $\mathbb{CP}^3$.

\(^1\)The $\kappa = 1$ instantons all fail to be “generic” due to the extra U(1) stabilizer generated by $\begin{pmatrix} I_3 & 0 \\ 0 & I_3 \end{pmatrix}$.  

Appendix A. Squashed 7-sphere

As a check on our conventions, we include the following short proof that $\phi_{sq}$ is a nearly parallel $G_2$-structure.

Let $\zeta_i$ and $\bar{\omega}_i$ be the $\text{Sp}(2)$-invariant frames for $\Omega^1$ and $\Omega^2_h$ given by Definition 5.1, and $\nu$ the (3,0) volume form of the Hopf fibration. The squashed $G_2$-structure, defined in Example 3.3, may be rewritten

$$\phi_{sq} = \frac{27}{25} \left( \frac{1}{5} \nu - \zeta_1 \wedge \bar{\omega}_1 - \zeta_2 \wedge \bar{\omega}_2 - \zeta_3 \wedge \bar{\omega}_3 \right).$$

More generally, for $a, b > 0$, define the $\text{Sp}(2)\text{Sp}(1)$-invariant $G_2$-structure

$$\phi_{a, b} = a \nu - b \zeta_i \wedge \bar{\omega}_i.$$

One can check from (3.2) that the metric and volume form defined by $\phi_{a, b}$ are

$$g_{a, b} = a^{-1/3} \left( a g_{\text{std}}|_{T_v} + b g_{\text{std}}|_{T_h} \right), \quad \text{Vol}_{a, b} = a^{1/3} b^2 \text{Vol}_{\text{std}}.$$

In this metric, we have

$$|\nu|_{g_{a, b}} = a^{-1}, \quad |\zeta_i \wedge \bar{\omega}_j|_{g_{a, b}} = \sqrt{2b^{-1}}$$

$$*_{g_{a, b}} \nu = a^{-5/3} b^2 \bar{\nu}, \quad *_{g_{a, b}} \left( \zeta_i \wedge \bar{\omega}_j \right) = \frac{a^{1/3}}{2} \epsilon_{ik\ell} \zeta_k \wedge \zeta_\ell \wedge \bar{\omega}_j.$$

The dual 4-form is given by

$$\psi_{a, b} = *_{g_{a, b}} \phi_{a, b} = a^{-2/3} b \left( b \bar{\nu} - a \epsilon_{ijk} \zeta_i \wedge \zeta_j \wedge \bar{\omega}_k \right).$$

In this notation, we have

$$\phi_{sq} = \phi \frac{27}{125} \frac{47}{25} = \left( \frac{3}{5} \right)^3 \phi_{1,5}.$$

The associated metric and volume form are

$$g_{sq} = \frac{9}{5} \left( \frac{1}{5} g_{\text{std}}|_{T_v} + g_{\text{std}}|_{T_h} \right), \quad \text{Vol}_{sq} = \frac{3^7}{5^5} \text{Vol}_{\text{std}}.$$

To see that $\phi_{sq}$ is nearly parallel, we calculate as follows. According to (5.7), we have

$$d\zeta_i = \epsilon_{ijk} \zeta_j \wedge \zeta_k + 2 \bar{\omega}_i.$$

Applying $d$ to (5.7), we obtain

$$0 = 2 \epsilon_{ijk} d\zeta_j \wedge \zeta_k + 2 d\bar{\omega}_i$$

and

$$d\bar{\omega}_i = \epsilon_{ijk} \zeta_j \wedge d\zeta_k = 2 \epsilon_{ijk} \zeta_j \wedge \bar{\omega}_k + \epsilon_{ijk} \epsilon_{k\ell m} \zeta_j \wedge \zeta_\ell \wedge \zeta_m$$

$$= 2 \epsilon_{ijk} \zeta_j \wedge \bar{\omega}_k.$$

Next, we compute

$$d\nu = d\zeta_1 \wedge \zeta_2 \wedge \zeta_3 - \zeta_1 \wedge d\zeta_2 \wedge \zeta_3 + \zeta_1 \wedge \zeta_2 \wedge d\zeta_3$$

$$= 2 \left( \bar{\omega}_1 \wedge \zeta_2 \wedge \zeta_3 - \zeta_1 \wedge \bar{\omega}_2 \wedge \zeta_3 + \zeta_1 \wedge \zeta_2 \wedge \bar{\omega}_3 \right)$$

$$= \epsilon_{ijk} \bar{\omega}_i \wedge \zeta_j \wedge \zeta_k.$$
We then have
\[
d\phi_{a,b} = a\epsilon_{ijk}\tilde{\omega}_i \wedge \zeta_j \wedge \zeta_k - b(2\tilde{\omega}_i + \epsilon_{ijk}\zeta_j \wedge \zeta_k) \wedge \tilde{\omega}_i + 2b\zeta_i \wedge \epsilon_{ijk}\zeta_j \wedge \tilde{\omega}_k
\]
\[
= (a + b)\epsilon_{ijk}\zeta_i \wedge \zeta_j \wedge \tilde{\omega}_k - 2b\tilde{\omega}_i \wedge \tilde{\omega}_i
\]
\[
= (a + b)\epsilon_{ijk}\zeta_i \wedge \zeta_j \wedge \tilde{\omega}_k - 12b\nu.
\]
This gives
\[
d\phi_{1,5} = -\frac{12}{5}\psi_{1,5}.
\]
After normalizing by \(\frac{12}{5^4} = \frac{3}{5}\), we have
\[
d\phi_{sq} = -4\psi_{sq}
\]
as claimed.

**Remark A.1.** Notice that although \(\phi_{1,1}\) and \(\phi_{std}\) both correspond to the standard metric on \(S^7\), they are not isomorphic, since the former is not a nearly parallel \(G_2\)-structure. In fact, Friedrich [19] has shown that \(\phi_{std}\) is the unique nearly parallel \(G_2\)-structure (up to rotations) compatible with the round metric on \(S^7\).

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