On Gaussian multiplicative chaos

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Abstract

We propose a new definition of the Gaussian multiplicative chaos and an approach based on the relation of subcritical Gaussian multiplicative chaos to randomized shifts of a Gaussian measure. Using this relation we prove general results on uniqueness and convergence for subcritical Gaussian multiplicative chaos that hold for Gaussian fields with arbitrary covariance kernels.

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1 Introduction

Let $\langle T, \mu \rangle$ be a finite or $\sigma$-finite measure space. A subcritical Gaussian multiplicative chaos (GMC) is, loosely speaking, a renormalized exponential of a generalized Gaussian field $X$ parametrized by $t \in T$ — that is, a random measure $M$ on $T$ given by the formal expression

$$M(dt) := \exp \left[ X(t) - \frac{1}{2} \mathbb{E} |X(t)|^2 \right] \mu(dt) \quad (1)$$

In the cases of interest $X$ is only defined in a distributional sense, and, in particular, $X(t)$ is not a well-defined random variable, so (1) does not make sense literally. Accordingly, $M$ is almost surely singular w.r.t. $\mu$. The known rigorous constructions of $M$ proceed by approximating the field $X$ by “smooth” fields $X_n$ and taking the limit of the random measures

$$M_n(dt) := \exp \left[ X_n(t) - \frac{1}{2} \mathbb{E} |X_n(t)|^2 \right] \mu(dt)$$

There are two commonly used approximation techniques:

**Martingale approximation** [5] The martingale approximation is employed in Kahane’s original definition of GMC in [5]. In his construction the increments $X_n - X_{n-1}$ are independent, which implies that $(M_n)$ is a positive measure-valued martingale. The martingale property guarantees that $M_n$ converges to a random measure $M$. Moreover, $\mathbb{E}M = \mu$ iff the martingales $M_n[A]$ are uniformly integrable for all sets $A \subset T$, such that $\mu(A) < \infty$.

**Mollifying operators** [11, 3, 10] This technique restricts the generality to the case where $T$ is a domain in $\mathbb{R}^d$, $\mu$ is the Lebesgue measure, and the covariance kernel $K(t, s) := \mathbb{E}X(t)X(s)$ has the special form

$$K(t, s) = \gamma^2 \log^+ \|t - s\|^{-1} + g(t, s), \quad (2)$$

where the function $g : T \times T \to \mathbb{R}$ is bounded and continuous, and $\gamma^2 \in (0, 2d)$. The fields $X_n$ are constructed by convolution:

$$X_n(t) := \int_T X(t') \psi_{1/n}(t - t') dt'$$

$$\psi_{1/n}(x) := n^d \psi(nx)$$
where \( \psi : \mathbb{R}^d \to \mathbb{R}_+ \), \( \int \psi(t)\,dt = 1 \), subject to appropriate smoothness conditions. This was used in [11] for stationary fields on \( \mathcal{T} = \mathbb{R}^d \), and according to [10], the same techniques apply to the non-stationary setting. Also in [3] a related construction with circle averages was used in the special case where \( X \) is the Gaussian free field in dimension 2.

Below we list some of the most basic issues that need to be addressed by a theory of Gaussian multiplicative chaos.

**Problem 1.** Define a GMC as a random measure that is measurable w.r.t. a generalized Gaussian field.

**Problem 2.** Find sufficient conditions for convergence of GMC over Gaussian fields approximating \( X \).

**Problem 3.** Prove uniqueness of GMC or its independence of the specific approximation procedure used to construct it.

As far as we know, these problem have only been partially solved under unnecessarily restrictive assumptions.

- In Kahane’s work [5] the Gaussian field \( X \) itself is not introduced at all. The initial data in his framework is not \( X \) but rather an approximating sequence of Gaussian fields \( X_n \) with continuous covariance kernels, such that the increments \( X_1, X_2 - X_1, \ldots, X_n - X_{n-1}, \ldots \) are independent. Denote by \( p_n \) the covariance kernel of \( X_n - X_{n-1} \). Kahane defines the “covariance kernel of \( X \)” as the pointwise sum

\[
K(t,s) := \sum_n p_n(t,s)
\]  

and assumes that
\[
\forall t,s \in \mathcal{T} : p_n(t,s) \geq 0
\]

(in the pointwise sense). In his terminology functions \( K : \mathcal{T} \times \mathcal{T} \to [0, +\infty] \) that can be represented as (3), with \( p_n \) both positive definite and nonnegative pointwise, are called “kernels of \( \sigma \)-positive type”. The \( \sigma \)-positivity assumption is both restrictive — e.g. it doesn’t cover all positive definite logarithmic kernels [2] — and hard to verify for specific kernels.

- Under the \( \sigma \)-positivity assumption, [5 Théorème 1] asserts that the distribution of \( M \) depends only on \( K \) and not on the representation (3). Kahane does not discuss uniqueness of the joint distribution of \( M \) and the Gaussian randomness used to construct it, even though both are naturally defined on the same probability space (since the martingale \( M_n \) converges almost surely, not just in distribution).

- In [11] the Gaussian field \( X \) is defined on \( \mathcal{T} \subset \mathbb{R}^d \) in a distributional sense (so a priori the covariance kernel \( K \) is a distribution rather than a function), but only the special case of logarithmic kernels [2] is discussed. In
this setting, with an additional assumption of stationarity, [11, Theorem 2.1] asserts convergence and uniqueness in distribution for the convolution approximations. See also [10, Theorem 3.5] for a similar statement without the stationarity assumption. X and M are not defined on the same probability space.

- [8] addresses the problem of almost sure approximation and constructs M as a function of X. However, this is only done in the special case of the Gaussian free field in dimension 2 and its boundary values.

1.1 Our approach

In this paper we present an approach to the Gaussian multiplicative chaos that is based on the study of the functional dependence \( M = M (X) \), and eventually allows us to address Problems [11] mentioned above in the general setting of subcritical GMC over generalized Gaussian fields on standard measure spaces with no additional assumptions on the covariance kernels.

Our starting point is the following basic observation: the “exponential” (1) can be characterized by the way it changes when \( X \) is shifted by Cameron-Martin vectors \( \xi \) (i.e. functions in the reproducing kernel Hilbert space associated to \( X \)). Namely, for all such \( \xi \) the following should hold:

\[
M (X + \xi, dt) = e^{\xi(t)} M (X, dt)
\] (4)

This is taken as our definition of GMC. We call a GMC subcritical if \( E M \) is a \( \sigma \)-finite measure, in which case we can assume, without limitation of generality, that \( E M = \mu \). In general we always assume that \( E M \ll \mu \).

Our use of the term “subcritical” is compatible with the known notions of GMC for logarithmic kernels [2] with the subcritical and critical parameters (resp. \( 0 < \gamma < \sqrt{2d} \) and \( \gamma = \sqrt{2d} \)).

Note that even though \( X \) is only defined as a random distribution, for (4) to make sense \( \xi \) has to be a function, which means in our setting an element of \( L^0 (\mu) \). This motivates our choice for the measure-theoretic formalism for generalized Gaussian fields in which a Gaussian field is, loosely speaking, a “standard Gaussian random vector” in a Hilbert space \( H \) equipped with a continuous linear operator \( Y : H \to L^0 (\mu) \) that identifies Cameron-Martin vectors of \( X \) with functions on \( T \). Technically, \( X \) is an object of the same type as \( Y \) — an operator \( X : H \to L^0 (\Omega) \) — that sends vectors \( \xi \in H \to (Gaussian) random variables. It turns out to be convenient to treat both \( X \) and \( Y \) on equal grounds as “generalized random vectors in \( H \)”. From this point of view it is natural to write our generalized Gaussian field parametrized by \( t \in T \) as \( "(X, Y (t))" \), even though the value “\( Y (t)" \)” need not belong to \( H \) for any particular \( t \).

The importance of \( Y \) as a generalized \( H \)-valued function stems from the generalization of the Cameron-Martin formula that relates subcritical GMC to randomized shifts (also called “random translations” in [12]) via the so called
Peyrière measure construction:

\[ E \int f(X, t) M(X, dt) = E \int f(X + Y(t), t) E M(dt) \]  

(5)

for all bounded \((X, t)\)-measurable functions \(f\). Note that, in particular, this implies that the total mass \(M[T]\) is the density \(\text{Law}_{\omega} \circ \mu [X + Y] / \text{Law} X\). It turns out that, in fact, there is a bijective correspondence between subcritical GMC and generalized random vectors \(Y\) in \(H\), independent of \(X\), such that \(\text{Law}[X + Y] \ll \text{Law} X\) (see Theorem 29). This observation itself is anything but new (see, e.g., [12]), yet we haven’t seen it worked out clearly in proper generality (for instance, [12] only focuses on shifts with positive independent coordinates). We find it immensely useful to exploit systematically this relation between subcritical GMC and randomized shifts, and arguably it is the existence of such a relation that makes subcritical GMC way more tractable than GMC in general.

An important corollary of Theorem 29 is automatic uniqueness of subcritical GMC. This follows from the fact that (5) allows to recover \(M\) from \(Y\) and \(E M\).

After setting up a language for GMC and randomized shifts we proceed to study the following specific questions:

**Regularity of the kernel.** A priori, the covariance kernels \(K\) allowed in our setup are quite singular objects — namely, formal kernels of bounded bilinear forms on \(L^2\) (see Definition 9). In particular, they are not even functions on \(\mathcal{T} \times \mathcal{T}\). It turns out, however, that the existence of a subcritical GMC for a Gaussian field with covariance \(K\) implies that \(K\) is indeed a function in the sense that its bilinear form is Hilbert-Schmidt on \(L^2(\mu')\) for some equivalent measure \(\mu' \sim \mu\). Moreover, \(\mu'\) can be chosen in such a way that

\[ \forall n \geq 1 : \int_{\mathcal{T} \times \mathcal{T}} |K(t, s)|^n \mu'(dt) \mu'(ds) < \infty \]  

(6)

This is the content of our Theorem 31 and Corollary 33. While this polynomial moment condition is very far from sufficient (i.e. it fails to prove non-existence of GMC for logarithmic kernels [2] with \(\gamma > \sqrt{2d}\)), we still believe that the result is of some interest.

In relation with (6) one can also mention Kahane’s exponential moment conjecture [6]: a subcritical GMC exists for a kernel \(K\) iff there is an equivalent measure \(\mu' \sim \mu\) such that

\[ \int_{\mathcal{T} \times \mathcal{T}} \exp \frac{1}{2} K(t, s) \mu'(dt) \mu'(ds) < \infty \]  

(7)

This conjecture, while supported by the logarithmic kernel examples, has been disproved in general by Sato and Tamashiro [12]. As far as we know, finding a correct replacement for (7) remains an open problem.
Convergence of GMC. Assume that $X_n(t) := \langle X_n, Y_n(t) \rangle$ is a sequence of jointly Gaussian fields, depending on a single Gaussian vector $X$, that approximates the Gaussian field $X(t) := \langle X, Y(t) \rangle$ in the sense that

$$\forall \xi \in H : \langle \xi, Y_n \rangle \overset{L^1(T,\mu)}{\rightarrow} \langle \xi, Y \rangle.$$ 

Assume that there are subcritical GMCs $M_n$ over the fields $X_n(t)$ with fixed expectation measure $\mu$. Our main result is a very general convergence theorem (Theorem 43) that works with the bare measure-theoretic structure and reduces the convergence problem to verifying uniform integrability of the family of $L^1$ random variables $\{M_n[T]\}$, which can be done in practice using Kahane’s comparison inequality [5, Lemme 1].

As a sample application of Theorem 43 we recover and improve on a convergence result for logarithmic kernels. Namely, for a Gaussian field $X$ on a bounded domain $T \subset \mathbb{R}^d$ with covariance kernel $K$, such that

$$K(t, t') \leq (2d - \delta) \log \|t - t'\|^{-1} + O(1)$$

for some $\delta > 0$, and a positive bounded function $\psi : \mathbb{R}^d \rightarrow \mathbb{R}$, $\psi_\varepsilon(x) := \varepsilon^{-d} \psi(\varepsilon^{-1}x)$, we show as a direct consequence of Theorem 43 that the random measures

$$\exp\left[ X \ast \psi_\varepsilon - \frac{1}{2} \mathbb{E}[|X \ast \psi_\varepsilon|^2] \right] dt$$

converge in probability as $\varepsilon \rightarrow 0$ (in the space of measures equipped with the weak topology) to a subcritical GMC.

1.2 Structure of the paper

- In Section 2 we introduce an abstract setting for generalized Gaussian fields on measure spaces.
- In Section 3 we define the Gaussian multiplicative chaos and prove a technical Lemma 22 that simplifies verifying our definition.
- In Section 4 we introduce randomized shifts and discuss their connection with subcritical GMC in its most general form (Theorem 29).
- In Section 5 we prove our main result on the regularity of the kernels of GMC (Theorem 31 and Corollary 33).
- In Section 6 we prove a technical result (Lemma 36) that we need for our approximation theorem.
- In Section 7 we prove our main approximation theorem (Theorem 43).
- In Section 8 we apply Theorem 43 to logarithmic kernels to improve on a known approximation result.
1.3 Notation and standard assumptions

We always denote by $H$ a separable infinite-dimensional real Hilbert space; vectors in $H$ are denoted by $\xi, \eta, \ldots$, generalized random vectors (Definition 4) — by uppercase $X, Y, Z, \ldots$. Among the latter, $X$ is reserved for a standard Gaussian in $H$ (Example 6), defined on a standard probability space $(\Omega, \mathcal{F}, P)$ (i.e. isomorphic to a Polish space equipped with a Borel probability measure). $Y, Z, \ldots$ are defined on independent standard measurable spaces $T, S, \ldots$ equipped with finite or $\sigma$-finite positive measures $\mu, \nu, \ldots$. These $T, S, \ldots$ serve as parameter spaces for generalized Gaussian fields.

The scalar product notation $\langle \cdot, \cdot \rangle$ refers to the following things, in the order of increasing generality:

- $\langle \xi, \eta \rangle$ — the scalar product of vectors $\xi, \eta \in H$.
- $\langle X, \xi \rangle$ or $\langle \xi, X \rangle$ — the value of a generalized random vector $X$ on a vector $\xi \in H$.
- $\langle Y, Z \rangle$ — the “scalar product” of two independent generalized random vectors (Definition 17). Another notation we use for it is $K_{YZ}(t, s) := \langle Y(t), Z(s) \rangle$; $K_{YY}$ is abbreviated to $K$ when there is no risk of ambiguity.

Assuming $Y$ is defined on $T$, a Gaussian multiplicative chaos with shift $Y$ (Definition 20) is denoted by $M(X, dt)$ or, in cases of ambiguity, $M_Y(X, dt)$.

2 Generalized Gaussian fields

2.1 Generalized random vectors

It is a well-known fact that there is no standard Gaussian random vector in an infinite-dimensional Hilbert space $H$. What does exist, however, is a linear operator $X : H \to L^0(\Omega, \mathcal{F}, P)$ that takes vectors in $\xi \in H$ to Gaussian random variables, such that $\mathbb{E}\langle X, \xi \rangle^2 = \|\xi\|^2$. This object is a prototypical example of the following notion:

Definition 4. A generalized random vector in Hilbert space $H$ is a continuous linear operator $X : H \to L^0(\Omega, \mathcal{F}, P)$ that takes vectors in $\xi \in H$ to Gaussian random variables, such that $\mathbb{E}\langle X, \xi \rangle^2 = \|\xi\|^2$. This object is a prototypical example of the following notion:

Remark 5. In this paper by a measure $\mu$ we always mean a measure equivalent to a probability measure. Note that if $\mu', \mu''$ are two equivalent probability measures then the topologies of convergence in measure are the same for $\mu'$ and $\mu''$, so $L^0(\mu')$ and $L^0(\mu'')$ can be identified in a canonical way. In other words, $L^0$ depends only on an equivalence class of measures, which we call a “measure type”. In the sequel we define $L^0(\mu)$ to be $L^0(\mu')$ for a probability measure $\mu' \sim \mu$. 

7
We use the “scalar product” notation $\langle X, \xi \rangle$ for a generalized random vector (or vector-valued function) $X$ and a vector $\xi \in H$ to denote the corresponding random variable (resp. function). The value $(X, \xi)(\omega)$ is written as $\langle X(\omega), \xi \rangle$, thus identifying random vectors with a special case of generalized random vectors.

The precise meaning of the symbol “$X(\omega)$” and the choice of the space where $X$ is a random element is arbitrary and in fact irrelevant for our purposes. For the sake of concreteness we may take an arbitrary basis $(e_n)$ of $H$ and identity $X(\omega)$ with its sequence of coordinates $(\langle X(\omega), e_n \rangle, n \in \mathbb{N})$, which is a random vector in the space of sequences $\mathbb{R}^\mathbb{N}$.

**Example 6.** We call a generalized random vector $X$ standard Gaussian in $H$ if all $\langle X, \xi \rangle, \xi \in H$ are centered Gaussian with variance $\mathbb{E} \langle X, \xi \rangle^2 = \|\xi\|^2$.

The following notions will be used throughout the text:

- The distribution of a generalized random vector $Y$, denoted $\text{Law} Y$, is the joint distribution of $\{(Y, \xi) \mid \xi \in H\}$.
- The $\sigma$-algebra generated by $Y$ is the one generated by $\{(Y, \xi) \mid \xi \in H\}$.
- For a bounded operator $A : H \to H$ the image $AY$ is the generalized random vector defined by $\langle AY, \xi \rangle := \langle Y, A^* \xi \rangle, \xi \in H$.
- For a random variable $f$ the product $f \cdot Y$ is defined by $\langle f \cdot Y, \xi \rangle := f \cdot \langle Y, \xi \rangle, \xi \in H$.
- If for every $\xi \in H$ the random variable $\langle Y, \xi \rangle$ is in $L^1$ then it follows easily from the closed graph theorem that there is a vector $EY \in H$, defined by $\langle EY, \xi \rangle := \mathbb{E} \langle Y, \xi \rangle$.

In case $Y$ is defined over a measure space $(T, \mu)$ we use the notation $E_\mu Y$.

For a generalized $H$-valued function $Y$ defined on $(T, \mu)$ the image of the operator $Y : H \to L^0(\mu)$ is a Hilbert space of equivalence classes of functions. Generally speaking, Hilbert spaces of functions are known otherwise as reproducing kernel Hilbert spaces (RKHS) associated with positive definite kernels. It turns out to be possible to build the basic RKHS theory with functions replaced by equivalence classes of functions. Since RKHS theory is only tangentially related to our main topic, we are only going to give the relevant definitions.

### 2.2 A factorization theorem

Our treatment of generalized random vectors and the related notions defined later in this section is based on a factorization theorem due to Nikishin and Maurey [9, 7, 8]. We state it below in two versions, the second one being a refinement of the first one that we will only need in Section [6].
Theorem 7 (Nikishin, Maurey). Let $H$ be a Hilbert space, let $(T, \mu)$ be a standard measure space.

1. Let $Y : H \to L^0(\mu)$ be a continuous linear operator. Then there exists a measure $\mu'$ equivalent to $\mu$, and a bounded operator $Y' : H \to L^2(\mu')$, such that $Y$ factors through the tautological embedding $\text{id} : L^2(\mu') \to L^0(\mu)$ as follows:

$$
\begin{array}{ccc}
H & \xrightarrow{Y} & L^0(\mu) \\
\downarrow{Y'} & & \downarrow{\text{id}} \\
L^2(\mu') & & \\
\end{array}
$$

2. Let $\{Y_\alpha | \alpha \in I\}$ be a family of continuous linear operators $H \to L^0(\mu)$ that is equicontinuous on the unit ball of $H$. Then the measures $\mu_\alpha$ and operators $Y'_\alpha : H \to L^2(\mu'_\alpha)$ constructed in the previous statement can be required to satisfy the following:

(a) $\sup_\alpha \|Y'_\alpha\| < \infty$

(b) the densities $\mu_\alpha / \mu$ are bounded away from 0 in measure $\mu$, i.e.

$$
\sup_\alpha \mu \left\{ \frac{\mu_\alpha(dt)}{\mu(dt)} < \varepsilon \right\} \to 0
$$

as $\varepsilon \to 0$.

The proof below is a fairly standard reduction to [8, Theoréme 3 b)].

Proof. We prove the second statement, since it obviously includes the first one. We apply to every $Y_\alpha$ individually [8, Theoréme 3 b)] with $q := 2$ and our $H$ identified with their $L^q$, deducing that for every $x \in (0, \frac{1}{8})$ there exists a subset $T_{\alpha,x} \subset T, \mu \left[ T \setminus T_{\alpha,x} \right] \leq 8x$, such that

$$
\forall \xi \in H : \int_{T_{\alpha,x}} |\langle Y_\alpha(t), \xi \rangle|^2 \mu(dt) \leq \text{const} \cdot x^{-1/2} J_x(Y_\alpha) \|\xi\| ,
$$

where $x \mapsto J_x(Y_\alpha)$ is the uniform bound on the decreasing rearrangement of $\langle Y_\alpha, \xi \rangle, \|\xi\| \leq 1$, i.e.

$$
J_x(Y_\alpha) := \sup_{\|\xi\| \leq 1} \sup_{j \in \mathbb{R}_+} \{j : |\langle Y_\alpha(t), \xi \rangle| > j\} \geq x \}
$$

Now note that since $Y_\alpha$ were required to be equicontinuous, we have

$$
\forall x : \sup_\alpha J_x(Y_\alpha) < \infty \quad (8)
$$
Therefore, there exists a decreasing sequence of positive reals \( c_n, n \geq 1, \sum_n c_n < \infty \), decaying sufficiently fast, such that
\[
\sum_n c_n \sup_{\alpha} \int_{\mathcal{T}_{\alpha,1/n}} |\langle Y_{\alpha} (t), \xi \rangle|^2 \mu (dt) < \infty
\]
Now we take
\[
\mu_{\alpha} (dt) := \sum_n c_n 1_{[\mathcal{T}_{\alpha,1/n}]} \mu (dt)
\]
Note that by construction
\[
\forall \xi \in H : \int |\langle Y_{\alpha} (t), \xi \rangle|^2 \mu_{\alpha} (dt) < \infty,
\]
so the factorizations of \( Y_{\alpha} \) through \( \text{id} : L^2 (\mu_{\alpha}) \rightarrow L^0 (\mu) \) exist by the closed graph theorem. Moreover, (8) implies that the bound (9) is uniform in \( \alpha \), thus
\[
\sup_{\alpha} \| Y_{\alpha} \|^2 < \infty,
\]
so the factorizations of \( Y_{\alpha} \) through \( \text{id} : L^2 (\mu_{\alpha}) \rightarrow L^0 (\mu) \) exist by the closed graph theorem. Moreover, (8) implies that the bound (9) is uniform in \( \alpha \), thus
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\[
\sup_{\alpha} \| Y_{\alpha} \|^2 < \infty,
\]
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\[
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\]
so the factorizations of \( Y_{\alpha} \) through \( \text{id} : L^2 (\mu_{\alpha}) \rightarrow L^0 (\mu) \) exist by the closed graph theorem. Moreover, (8) implies that the bound (9) is uniform in \( \alpha \), thus
\[
\sup_{\alpha} \| Y_{\alpha} \|^2 < \infty,
\[
= \int K(t_3, t_2) \left( f_1 (t_1) \frac{\mu''_1 (dt_1)}{\mu'_1 (dt_1)} \right) \left( f_2 (t_2) \frac{\mu''_2 (dt_2)}{\mu'_2 (dt_2)} \right) \mu'_1 (dt_1) \mu'_2 (dt_2), 
\]

where \( \mu''_1 (dt_1) / \mu'_1 (dt_1) \) denote Radon-Nikodym derivatives (sometimes we suppress \( dt_i \) and write just \( \mu''_i / \mu'_i \)). Note also that if \( \mu''_i \leq \mu'_i \) (in the sense that \( \mu''_i [A] \leq \mu'_i [A] \) for all measurable sets \( A \)) then boundedness in \( L^2 (\mu'_i) \) of the left-hand side of (11) implies boundedness in \( L^2 (\mu''_i) \) of the right-hand side. Thus the kernel must act on all \( L^2 \)'s over all small enough measures.

**Definition 9.** Let \((\mathcal{T}_i, \mu_i), i = 1, 2\) be standard measure spaces. A kernel \( K \) on \((\mathcal{T}_1, \mu_1) \times (\mathcal{T}_2, \mu_2)\) is an equivalence class of tuples \((\mu'_1, \mu'_2, K_{\mu'_1, \mu'_2})\), where \( \mu_i \sim \mu_i \) are finite measures on \( \mathcal{T}_i \) and \( K_{\mu'_1, \mu'_2} \) is a bounded bilinear form on \( L^2 (\mu'_1) \times L^2 (\mu'_2) \). The equivalence relation is

\[(\mu'_1, \mu'_2, K_{\mu'_1, \mu'_2}) \sim (\mu''_1, \mu''_2, K_{\mu''_1, \mu''_2})\]

iff for all \( f_i \in L^2 (\mu''_i + (\mu''_i / \mu'_i)^2 \cdot \mu'_i)\):

\[K_{\mu''_1, \mu''_2} (f_1, f_2) = K_{\mu'_1, \mu'_2} \left( \frac{\mu''_1}{\mu'_1} f_1, \frac{\mu''_2}{\mu'_2} f_2 \right)\]

(compare with (11)).

Whenever this equivalence class has a representative \((\mu'_1, \mu'_2, K_{\mu'_1, \mu'_2})\) we say that \( K \) is bounded on \( L^2 (\mu'_1) \times L^2 (\mu'_2) \).

If \( \mu''_i \leq \mu'_i \) then every \((\mu'_1, \mu'_2, K_{\mu'_1, \mu'_2})\) is equivalent to a unique \((\mu''_1, \mu''_2, K_{\mu''_1, \mu''_2})\), since the multiplication operator \( f_i \mapsto \frac{\mu''_i}{\mu'_i} f_i \) is bounded from \( L^2 (\mu''_i) \) to \( L^2 (\mu'_i) \). This means that the space of kernels is in fact an inductive limit (in the category of vector spaces) of spaces of bilinear forms on \( L^2 (\mu'_1) \times L^2 (\mu'_2) \) over the set of pairs of measures \((\mu'_1, \mu'_2)\) directed by the partial order

\[(\mu''_1, \mu''_2) \leq (\mu'_1, \mu'_2) \Leftrightarrow \forall i : \mu''_i \leq \mu'_i\]

Note that if \( \mathcal{T}_1 = \mathcal{T}_2 =: \mathcal{T} \) with \( \mu_1 \sim \mu_2 \) then it is enough to consider pairs of equal measures \((\mu'_1, \mu'_1)\), in the sense that every equivalence class contains representatives with a pair of equal measures. Indeed, for every \( \mu'_1, \mu'_2 \) we can take \( \mu'_i := \mu'_1 \wedge \mu'_2 \). From now on whenever \( \mathcal{T}_1 = \mathcal{T}_2 = \mathcal{T} \) with \( \mu_1 \sim \mu_2 \), we only consider bilinear forms on \( L^2 (\mu') \times L^2 (\mu') \).

**Definition 10.** If \( \mathcal{T}_1 = \mathcal{T}_2 = \mathcal{T}, \mu_1 \sim \mu_2 \) as above, then a kernel is called symmetric if the corresponding bilinear forms on \( L^2 (\mu') \times L^2 (\mu') \) are symmetric, and positive definite if so are the bilinear forms (positivity is always non-strict, i.e. \( \forall f \in L^2 (\mu') : K_{\mu', \mu'} (f, f) \geq 0 \)).

The partial order on symmetric kernels is defined accordingly, i.e. \( K_1 \leq K_2 \) iff \( K_2 - K_1 \) is positive definite.

It is straightforward to see that Definition 10 does not depend on the choice of measure \( \mu' \) within its equivalence class.

Here are some important classes of kernels, in the order of increasing generality:
Example 11 (Rank one kernels). For any functions $\varphi_1 \in L^0 (\mu_i)$ there is a kernel $\varphi_1 \otimes \varphi_2$, defined by the bilinear form

$$K_{\mu_1', \mu_2'} (f_1, f_2) := \langle f_1, \varphi_1 \rangle_{L^2 (\mu_1')} \langle f_2, \varphi_2 \rangle_{L^2 (\mu_2')}$$

$f_i \in L^2 (\mu_i')$, for small enough $\mu_i'$ (e.g. such that $\varphi_i \in L^2 (\mu_i')$).

Example 12 (Trace class kernels). A trace class kernel is a kernel $K$, for which there are measures $\mu_i' \sim \mu_i$, such that $K_{\mu_1', \mu_2'}$ is a trace class bilinear form on $L^2 (\mu_1') \times L^2 (\mu_2')$ (i.e. it defines a trace class operator). Since any trace class operator is an integral operator with a continuous kernel w.r.t. some Polish space topology, it follows that for trace class kernels it makes sense to define them as functions almost everywhere w.r.t. any measure $\nu$ on $T_1 \times T_2$ that has $\mu_i$-absolutely continuous projections on $T_i$. In particular, for $T_1 = T_2$, $\mu_1 \sim \mu_2$ it makes sense to define the value of $K$ at almost all points on the diagonal. In the sequel the notation $K(t, t)$ will be used for these properly defined diagonal values.

Example 13 ($L^2$ functions). For any $\mu_i'$ on $T_i$ and any function $K \in L^2 (T_1 \times T_2, \mu_1' \otimes \mu_2')$ the bilinear form (10) is bounded (actually, Hilbert-Schmidt) on $L^2 (\mu_1') \times L^2 (\mu_2')$, and thus defines a kernel.

Example 14 (Measures). For any finite measure $\nu$ on $T_1 \times T_2$ with $\mu_i$-absolutely continuous projections onto $T_i$ there exist measures $\mu_i' \sim \mu_i$, such that the bilinear form

$$K_{\mu_1', \mu_2'} (f_1, f_2) := \int f_1 (t_1) f_2 (t_2) \nu (dt_1 dt_2)$$

is bounded on $L^2 (\mu_1') \times L^2 (\mu_2')$, and thus $(\mu_1', \mu_2', K_{\mu_1', \mu_2'})$ defines a kernel. This follows from the Grothendieck’s factorization theorem [1], Theorem V.2.

Remark 15. There are kernels that are not measures; the most general notion here is known as “bimeasures” or “2-dimensional Fréchet measures”; see [1], Chapters IV, VI] for the details.

With our definition of kernel we can finally define a reproducing kernel Hilbert space associated to a kernel:

**Definition 16.** Given a positive definite kernel $K$ on $(T, \mu) \times (T, \mu)$, the reproducing kernel Hilbert space $\text{RKHS} (K)$ is the space of functions $\varphi \in L^0 (T, \mu)$ such that

$$\|\varphi\|^2_{\text{RKHS}} := \inf \{c \geq 0 \mid \varphi \otimes \varphi \leq cK\} < \infty$$

It is straightforward to show that $\text{RKHS} (K)$ is a Hilbert space for the norm $\|\cdot\|_{\text{RKHS}}$, and that it is a subspace of $L^2 (\mu')$ (not closed, but rather continuously embedded) for any measure $\mu'$, such that $K$ is bounded on $L^2 (\mu') \times L^2 (\mu')$. For continuous (or more generally, trace class) kernels Definition [10] is equivalent to the commonly used one.
The tautological embedding $\text{RKHS}(K) \to L^0(\mu)$ can be thought of as a generalized RKHS $(K)$-valued function, and it is natural, by analogy with classical RKHS theory, to denote the “value” of this “function” at $t$ by $K(t, \cdot)$. The linear form corresponding to it is the evaluation functional: for every $\varphi \in \text{RKHS}(K)$ we have

$$\langle K(t, \cdot), \varphi \rangle_{\text{RKHS}} = \varphi(t)$$

for almost all $t$. Note that unlike in classical RKHS theory, it is not a “usual” RKHS $(K)$-valued function — that is, unless $K$ is trace class (Example 12), in which case

$$\forall t_1, t_2 : \langle K(t_1, \cdot), K(t_2, \cdot) \rangle_{\text{RKHS}} = K(t_1, t_2)$$

(12)

In this section we will make sense of (12) for general kernels without the trace class assumption.

### 2.4 Scalar product of independent generalized random vectors

**Definition 17.** Let $Y_i, i = 1, 2$ be generalized random vectors in a Hilbert space $H$, defined on measure spaces $(T_i, \mu_i)$. By Corollary 8 there are measures $\mu'_i \sim \mu_i$, for which all $\langle Y_i, \xi \rangle, \xi \in H$ have second moments. For these $\mu'_i$ define a kernel $K := \langle Y_1(\cdot), Y_2(\cdot) \rangle$ on $(T_1, \mu_1) \times (T_2, \mu_2)$ by its bilinear form on $L^2(\mu'_1) \times L^2(\mu'_2)$:

$$K_{\mu_1, \mu_2}(f_1, f_2) := \langle E_{\mu'_1} f_1 Y_1, E_{\mu'_2} f_2 Y_2 \rangle_H, f_i \in L^2(\mu'_i)$$

If $Y_i$ are functions then the $K$ defined above is a trace class kernel in the sense of Example 12 and $\langle Y(t_1), Y(t_2) \rangle = K(t_1, t_2)$ for $\nu$-almost all $(t_1, t_2)$ for every measure $\nu$ with $\mu_i$-absolutely continuous projections.

In the general case we still write for convenience $K(t_1, t_2) = \langle Y_1(t_1), Y_2(t_2) \rangle$, bearing in mind that neither are $Y_i$ “ordinary” $H$-valued functions, nor is $K$ a function on $T_1 \times T_2$. However, for those kernels $K$ that are relevant to subcritical GMC theory this abuse of notation is justified in retrospect by Theorem 31 and its Corollary 33 that asserts that these $K$ are actually functions.

We close our informal discussion of measurable RKHS theory by saying that it’s straightforward to prove, using Theorem 7 that the image of every embedding $Y : H \to L^0(T, \mu)$, equipped with the Hilbert space norm coming from its isomorphism with $H/\ker Y$, is exactly RKHS $(K)$ with $K(t_1, t_2) := \langle Y(t_1), Y(t_2) \rangle$. In other words, every Hilbert space that consists of $L^0$ functions is an RKHS.

### 2.5 Generalized Gaussian fields

A function of two variables $f : \Omega \times T \to \mathbb{R}$ can be treated equivalently as a “random field”, i.e. a function from $\Omega$ into a space of functions on $T$. Similarly, a kernel $F$ on $(\Omega, \mathcal{F}) \times (T, \mu)$ can be treated as a generalized random field — more precisely, a generalized random vector in $L^2(T, \mu')$, such that the random variables $\int F(\omega, t) \varphi(t) \mu'(dt), \varphi \in L^2(\mu')$ are described by the kernel $F$ via
their scalar products with random variables in $L^2(P'), P' \sim P$. Conversely, it follows from Theorem 7 that any generalized random vector in $L^2(\mu')$ for $\mu' \sim \mu$ is obtained this way from a kernel.

Among generalized random fields the Gaussian ones can be described as follows:

**Definition 18.** A generalized Gaussian field parametrized by a measure space $(T, \mu)$ is the kernel $\langle X(\omega), Y(t) \rangle$ on $(\Omega, P) \times (T, \mu)$, where $X$ is a standard Gaussian in a Hilbert space $H$ and $Y$ is a generalized $H$-valued function defined on $(T, \mu)$.

The covariance kernel of our Gaussian field is $K(t, t') := \langle Y(t), Y(t') \rangle$.

**Remark 19.** We say that a Gaussian field "has values at points" if its covariance kernel $K$ is trace class, or, equivalently, $t \mapsto Y(t)$ is an $H$-valued function. In this case for almost all $t \in T \langle X, Y(t) \rangle$ is a proper Gaussian random variable, which is, naturally, viewed as the value of the Gaussian field at $t$.

### 3 The definition of GMC

By a random measure on a measure space $(T, \mu)$ we always mean a random positive finite measure $M$ with $EM \ll \mu$. The measure $EM$ is defined by $(EM)[A] := E(M[A])$ for all measurable subsets $A \subset T$. Note that $EM$ need not be a $\sigma$-finite measure; however, it is equivalent to a finite one — namely, $E\left[(M[T] \vee 1)^{-1} M\right]$ — so it makes sense to require it to be $\mu$-absolutely continuous. Another equivalent way of stating the absolute continuity assumption is as follows: for every subset $A \subset T$, such that $\mu[A] = 0$, we have $M[A] = 0$ a.s. Clearly, this does not prevent $M$ from being almost surely singular w.r.t. $\mu$.

We adopt an approach to Gaussian multiplicative chaos based on its behavior w.r.t. the action of Cameron-Martin shifts on $X$ (cf. Theorem 57 for a definition).

**Definition 20.** A random measure $M$ on $T$ is called a Gaussian multiplicative chaos (GMC) over the Gaussian field $\langle X, Y(\cdot) \rangle$ if

1. $EM \ll \mu$
2. $M$ is measurable w.r.t. $X$ (which allows us to write $M = M(X)$
3. For all vectors $\xi \in H$
   \[
   M(X + \xi, dt) = e^{\langle Y(t), \xi \rangle} M(X, dt) \text{ a.s.} \tag{13}
   \]

The GMC is called subcritical if $EM$ is $\sigma$-finite.

Instead of "GMC over the Gaussian field $\langle X, Y(\cdot) \rangle$" we also say "GMC over $X$ with shift $Y$", especially in the subcritical case. The reason for this terminology is explained by Theorem 20.
Formula (13) requires a couple of comments. First, $X + \xi$ is a shifted Gaussian, so by the Cameron-Martin theorem, its distribution is equivalent to that of $X$, which makes $M (X + \xi)$ a well-defined random measure. Second, even though $\langle Y, \xi \rangle$ is only defined almost everywhere w.r.t. $\mu$, $e^{\langle Y, \xi \rangle} M$ is still a well-defined random measure, precisely because $E M$ is $\mu$-absolutely continuous. Indeed, if $\varphi, \tilde{\varphi}$ are two measurable functions on $T$ that are equal $\mu$-almost everywhere to $e^{\langle Y, \xi \rangle} M$ then for $P$-almost all $\omega$ and $M (X (\omega), dt)$-almost all $t$ we have $\varphi (t) = \tilde{\varphi} (t)$, thus almost surely $\varphi M = \tilde{\varphi} M$.

Obviously, for every GMC and every deterministic nonnegative function $f \in L^0 (T)$ the random measure $f (t) M (X, dt)$ is also a GMC. The transformation $M \mapsto f M$ preserves subcriticality, and in the subcritical case by choosing an appropriate $f > 0$ we can make $E M$ into a probability measure or assume that $E M = \mu$. In the sequel we are going to use this transformation to enforce even further nice properties of $E M$ (cf. Corollary 33).

Condition (20) is intended to make precise the notion of renormalized “exponential of a Gaussian field”. Indeed, if the Gaussian field has values at points, in the sense of Remark 19, the exponential in the naive pointwise sense is a GMC:

**Example 21** (Trivial GMC). If $K$ be a trace class kernel (Example 12) then $Y$ is a vector-valued function, and

$$M (X, dt) := \exp \left[ \langle X, Y (t) \rangle - \frac{1}{2} K (t, t) \right] \mu (dt) \quad (14)$$

is a subcritical GMC over the Gaussian field $\langle X, Y (\cdot) \rangle$ with $E M = \mu$.

In general verifying the assumption (13) in the definition of GMC can be nontrivial. The following lemma simplifies this problem considerably by allowing one to check (13) for $\xi$ in a dense subset $H' \subset H$. Moreover, in situations when we are only given a densely defined operator $Y : H' \to L^0 (T, \mu)$, the lemma effectively asserts that $L^0$ is the right target space for the extension of $Y$ to the whole $H$, thus justifying the choice of $L^0$ as the function space in our definition of Gaussian field.

**Lemma 22.** Let $X$ be a standard Gaussian in a Hilbert space $H$, and let $M$ be a random measure on a probability space $(T, \mu)$. Assume that

1. $E M$ is equivalent to $\mu$
2. $M$ is measurable w.r.t. $X$
3. There is a dense subset $H' \subset H$ and a map $Y : H' \to L^0 (T, \mu)$ (not assumed to be continuous or linear), such that for all vectors $\xi \in H'$

$$M (X + \xi, dt) = e^{\langle Y (\cdot), \xi \rangle} M (X, dt) \quad P\text{-a.s.} \quad (15)$$

Then $Y$ extends to a continuous linear operator $Y : H \to L^0 (T, \mu)$, and $M$ is a GMC over $\langle X, Y (\cdot) \rangle$.
In the proof we use the following simple fact:

**Lemma 23.** Let \((R_n)\) be sequence of random variables.

- \((\mathbb{R}^\infty\text{ case})\): there exists a deterministic sequence \((r_n)\), \(r_n \geq 0\), such that almost surely \(R_n = O(r_n), n \to \infty\).

- \((c_0 \text{ case})\): If, furthermore, \(R_n \to 0\) then in addition to \(R_n = O(r_n)\) we can also require \(r_n \to 0\).

**Proof.** We treat both the \(\mathbb{R}^\infty\) case and \(c_0\) case in parallel.

The measure \(\text{Law}(R_n)\) in \(\mathbb{R}^\infty\) (resp. \(c_0\)) is supported on a set \(\bigcup_m K_m\), where \(K_m\) are compact for the product topology (resp, \(c_0\) norm topology). Since \(K_m\) are compact, there are finite constants \(c_{m,n} := \sup_{x \in K_m} |x_n|\).

In the \(c_0\) case compactness implies \(c_{m,n} \to 0, n \to 0\) for every \(m\).

Since \((R_n)\) almost surely belongs to one of the \(K_m\)’s, to prove the lemma it’s enough to construct a deterministic sequence \(r_n\), such that

\[
\forall m : c_{m,n} = O(r_n), n \to \infty
\]  

and, in the \(c_0\) case, \(r_n \to 0\).

In the \(\mathbb{R}^\infty\) case we take

\[
r_n := \max_{m \leq n} c_{m,n}
\]

In the \(c_0\) case we consider indices \(\nu_t \in \mathbb{N}\), defined recursively by

\[
\nu_1 := 1
\]

\[
\nu_{t+1} := \min \{ k \geq \nu_t \mid \forall m \leq t, \forall n \geq k : c_{m,n} \leq 2^{-t} \}
\]

and take

\[
r_n := 2^{-t}, \nu_t \leq n < \nu_{t+1}
\]

Verification of (16) is left to the reader. \(\square\)

**Proof of Lemma 22.** For every two functions \(f, f' \in L^0(\mu)\) \(f M = f'M\) a.s. implies \(f = f'\) \(\mu\)-almost everywhere, since if, for example, \(\mu \{ f > f' \} > 0\), we would have with positive probability \(f M > f'M\) on the set \(\{ f > f' \}\). This shows that (15) determines \(\langle Y, \xi \rangle, \xi \in H'\) uniquely. Moreover, there is a unique maximal extension of \(Y\) for which (15) is true. Namely, take

\[
H_{\text{max}} := \{ \xi \in H \mid \exists f \in L^0(\mu), f > 0 : M (X + \xi, dt) = f (t) M (X, dt) \text{ P-a.s.} \}
\]

and define \(\langle Y, \xi \rangle\) for \(\xi \in H_{\text{max}}\) to be \(\log f\), where \(f\) is the unique function for which \(M (X + \xi) = f \cdot M (X)\). \(H_{\text{max}}\) is necessarily an additive subgroup of \(H\).
and $Y : H_{\text{max}} \to L^0(\mu)$ is a group homomorphism, since for every $\xi, \eta \in H_{\text{max}}$ we have

$$M((X + \eta) + \xi) = e^{(Y, \xi)}M(X + \eta) = e^{(Y, \xi)}e^{(Y, \eta)}M(X),$$

$$M(X) = e^{(Y, \xi)}M(X - \xi)$$

(these equalities are obtained by applying shifts $X \mapsto X + \eta$ and $X \mapsto X - \xi$, respectively, to both sides of (15)). From now on we denote by $Y$ this extension.

Since Cameron-Martin shifts act continuously, the operator $Y$ must be closed. This means that whenever we have $\xi, \xi_n \in H_{\text{max}}, \xi_n \to \xi, \langle Y, \xi_n \rangle _{L^0} \to \varphi$, this implies $\xi \in H_{\text{max}}, \langle Y, \xi \rangle = \varphi$. Indeed, by Theorem 59 we have for every function $g \in L^\infty(\mu)$

$$\int g(t) e^{(Y(t), \xi_n)} M(X, dt) = \int g(t) M(X + \xi_n, dt) \xrightarrow{L^0} \int g(t) M(X + \xi, dt)$$

Since $\xi_n \in H_{\text{max}}$ and $H_{\text{max}}$ is a subgroup, $2\xi_n \in H_{\text{max}}$. Using continuity of shifts once again and passing to a subsequence if necessary, we can ensure that almost surely

$$\sup_n \int e^{2(Y(t), \xi_n)} M(X, dt) < \infty,$$

so that, in particular, the functions $ge^{(Y, \xi_n)}$ are almost surely uniformly integrable w.r.t. $M$. This implies, together with convergence $e^{(Y, \xi_n)} \xrightarrow{L^0} e^\varphi$, that

$$\int g(t) e^{(Y(t), \xi_n)} M(X, dt) \to \int g(t) e^\varphi M(X, dt),$$

therefore, $M(X + \xi) = e^\varphi M(X)$, so $\xi \in H_{\text{max}}$ and $\langle Y, \xi \rangle = \varphi$.

We finally show that $Y$ is a bounded (and therefore continuous) operator. Let $\xi_n \in H_{\text{max}}$ be a bounded sequence of vectors. We show that the set $\{\langle Y, \xi_n \rangle \} \subset L^0$ is bounded (equivalently, tight).

Suppose, towards a contradiction, that $\{\langle Y, \xi_n \rangle \}$ is not bounded. Then by passing to a subsequence and changing signs ($\xi_n \mapsto -\xi_n$) if necessary, we may assume that there are constants $C_n \to +\infty$, such that

$$\liminf_{n \to \infty} \mu \{t : \langle Y(t), \xi_n \rangle \geq C_n \} > 0 \quad (17)$$

By construction, we have

$$\int 1\{\langle Y(t), \xi_n \rangle \geq C_n \} M(X, dt) \leq e^{-C_n} \int e^{(Y(t), \xi_n)} M(X, dt) = e^{-C_n} \int M(X + \xi_n, dt)$$

On the other hand, Theorem 59 together with the boundedness of $\xi_n$, implies that

$$\left\{ \int M(X + \xi_n, dt) \right\} \text{ is bounded in } L^0$$

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Thus
\[
R_n := \int \mathbf{1}\{\langle Y(t), \xi_n \rangle \geq C_n \} M(X, dt) \xrightarrow{L^0} 0 \quad (18)
\]

We will show that (18) contradicts (17). By passing to a subsequence, we may refine the convergence in probability in (18) to almost sure convergence and retain \( \liminf > 0 \) in (17) for that subsequence. By the \( c_0 \) case of Lemma 3, there is a deterministic sequence \( r_n \rightarrow 0 \), such that a.s. \( R_n = O(r_n) \). By passing to a subsequence for the last time, we may assume that \( \sum_n r_n < \infty \), and, therefore,
\[
\sum_n R_n < \infty \text{ a.s.}
\]

This implies
\[
\int \sum_n \mathbf{1}\{\langle Y(t), \xi_n \rangle \geq C_n \} M(X, dt) < \infty,
\]
so that, in particular, \( M = 0 \text{ a.s. on the set } \)
\[
A := \{\langle Y(t), \xi_n \rangle \geq C_n \text{ for infinitely many } n\}
\]

On the other hand, (17) implies that \( \mu[A] > 0 \), which is in contradiction with the equivalence \( E M \sim \mu \).

We have proved the boundedness of the operator \( Y \). Since \( Y \) is closed on \( H_{\text{max}} \) and \( H_{\text{max}} \) is dense in \( H \), this implies that in fact \( H_{\text{max}} = H \), and \( Y \) is continuous. Every continuous group homomorphism between topological vector spaces is \( \mathbb{R} \)-linear, so \( Y \) is a generalized random vector. Now since (15) holds for all \( \xi \in H_{\text{max}} = H \), \( M \) is a GMC over \( \langle X, Y(\cdot) \rangle \).

As an application of Lemma 22 we prove a useful multiplicativity property of GMC with respect to a decomposition of a Gaussian field into a sum of two independent Gaussian fields.

Let \( J \subset H \) be a closed subspace, and consider the decomposition of the Gaussian field \( \langle X, Y(t) \rangle = \langle X, \text{pr}_J Y(t) \rangle + \langle X, \text{pr}_{J^\perp} Y(t) \rangle \). We would like to formalize the intuition that a subcritical GMC \( M_Y \) with shift \( Y \) is, conditionally on \( \text{pr}_J X \), a GMC with shift \( \text{pr}_{J^\perp} Y \) and (conditional) expectation \( E[M_Y|\text{pr}_J X] = M_{\text{pr}_J Y} \), that is, in turn, a subcritical GMC with shift \( \text{pr}_{J^\perp} Y \) and expectation \( E M_{\text{pr}_J Y} \). The problem here is that it is not trivial that \( \text{pr}_{J^\perp} Y \) is well-defined, as a generalized \( H \)-valued function, on the random measure space \( (T, M_{\text{pr}_J Y}) \). In other words, the functions \( \langle \text{pr}_{J^\perp} Y, \xi \rangle \) are almost surely well-defined \( M_{\text{pr}_J Y} \)-almost everywhere for individual \( \xi \in H \), and Lemma 22 is used to show that the “random process” \( \xi \mapsto \langle \text{pr}_{J^\perp} Y, \xi \rangle \), parametrized by \( H \) with values in \( L^0(T, M_{\text{pr}_J Y}) \), has a continuous version.

**Lemma 24.** Let \( J \subset H \) and \( Y \) be as in the discussion above, and let \( M_Y \) be a GMC with shift \( Y \).

1. If \( M_Y \) is subcritical then \( M_{\text{pr}_J Y} (\text{pr}_J X) := E[M_Y(X)|\text{pr}_J X] \) is a subcritical GMC over the standard Gaussian \( \text{pr}_J X \) in \( J \), with expectation \( E M \) and shift \( \text{pr}_J Y \).
2. Fix a dense countable set \( \{ \xi_n \} \subset J^\perp \). The map

\[
\{ \xi_n \} \to L^0 (\mathcal{F}, M_{pr_J Y} (pr_J X)), \xi_n \mapsto \langle pr_J Y, \xi_n \rangle
\]

extends almost surely to a continuous linear operator with domain \( J^\perp \) — that is, a generalized random vector — denoted by \( pr_J Y \).

Conditionally on \( pr_J X \), \( M_Y (X) \) is a GMC over \( pr_J X \) on the random measure space \((\mathcal{F}, M_{pr_J Y} (pr_J X))\) with shift \( pr_J Y \). If \( M_Y \) is subcritical then it is also conditionally subcritical.

**Proof.** The first statement is obvious, since Cameron-Martin shift action on \( L^1 (\Omega, \sigma (X)) \), restricted to shifts in \( J \subset H \), commutes with the conditional expectation operator \( E [\cdot | pr_J X] \). In other words, for all \( \xi \in J \)

\[
E [M_Y (X) | pr_J X = x + \xi] = E [M_Y (X) + \xi | pr_J X = x] = e^{Y \cdot \xi} E [M_Y (X) | pr_J X = x]
\]

for \( \text{Law}_{pr_J X} \)-almost all \( x \).

To prove the second statement we check the definition of GMC on the dense set \( \{ \xi_n \} \subset J^\perp \) and use Lemma 22. Note that shifts by \( J^\perp \) do not change \( pr_J X \), and that the equality

\[
M_Y (X + \xi_n) = e^{(pr_J Y \cdot \xi_n)} M_Y (X)
\]

is valid conditionally — that is, for almost any fixed value of \( pr_J X \) this is an equality of measures with (conditional) expectation equivalent to \( M_{pr_J Y} (pr_J X) \).

By Lemma 22 the map

\[
\xi_n \mapsto \langle pr_J Y, \xi_n \rangle \in L^0 (\mathcal{F}, M_{pr_J Y} (pr_J X))
\]

extends by continuity to a generalized random vector, and \( M_Y (X) \) is conditionally a GMC with this generalized random vector as its shift.

## 4 Randomized shifts

**Definition 25.** Let \( X \) be a standard Gaussian in \( H \), defined on a probability space \((\Omega, P)\). A generalized random vector \( Y \) in \( H \) over an independent probability space \((\mathcal{F}, \mu)\) is called a **randomized shift** (of \( X \)) if

\[
\text{Law}_{\otimes \mu} [X + Y] \ll \text{Law}_{\mu} X
\]

Note that being a randomized shift only depends on the measure type of \( \text{Law} Y \), so one may use the same term for \( \text{Law} Y \) or its measure type, and also talk about randomized shifts defined on measure spaces or measure type spaces.

**Example 26** (Trivial shifts). By the Cameron-Martin theorem (Theorem 67), every vector-valued function \( Y : \mathcal{T} \to H \) is a randomized shift.
These Cameron-Martin shifts are what we view as “trivial”. There are less trivial ones:

**Example 27** (Gaussian shifts). By the Hajek-Feldman theorem [2], a Gaussian generalized random vector in $H$ (i.e. that for which all linear functionals are Gaussian) is a randomized shift iff its covariance is Hilbert-Schmidt. Note also that being “trivial” in the above sense is equivalent to the covariance being trace class.

**Example 28** (Log-concave shifts). The above example can be generalized verbatim to generalized random vectors with log-concave distribution (i.e. such that all finite-dimensional projections are log-concave): they are randomized shifts iff their covariance is Hilbert-Schmidt.

In Theorem 29 we describe the relation between subcritical GMC over Gaussian fields with parameter space $(T, \mu)$ and randomized shifts defined on $(T, \mu)$. Here we view $(T, \mu)$ as an additional source of randomness, so functions on $\Omega \times T$ are treated as random variables — in particular, the projection map $\Omega \times T \to T$ is treated as “the” random point $t$ in $T$.

**Theorem 29.** There exists a subcritical GMC $M$ over the Gaussian field $X(t) = \langle X, Y(t) \rangle$ iff $Y$ is a randomized shift, in which case under the Peyrière measure on $\Omega \times T$

$$Q(d\omega, dt) := P(d\omega) M(\omega, dt) \quad (19)$$

we have

$$\text{Law}_Q[X, t] = \text{Law}_{P \otimes E_M}[X + Y(t), t] \quad (20)$$

Note that (20) characterizes uniquely the measure $Q$ on the $\sigma$-algebra generated by $(X, t)$, so by disintegration w.r.t. $X$ it also characterizes $M(X)$. Therefore, we have the following

**Corollary 30.** A subcritical GMC with a given expectation $E_M$ and a given shift $Y$ is unique whenever it exists. More precisely, $M$ can be recovered from $E_M$ and $Y$ as follows:

$$M[T] = \text{Law}_{P \otimes E_M}[X + Y(t)] \big/ \text{Law} X$$

$$M/M[T] = \text{Law}_{P \otimes E_M}[t \mid X + Y(t)]$$

**Proof of Theorem 29.** Assume first that there exists a subcritical GMC $M$. Since $E_M$ is required to be $\sigma$-finite, by a transformation $M \mapsto fM$ we may assume that $\mu := E_M$ is a probability measure.

Define a measure $Q$ on $\Omega \times T$ by (19). We are going to prove (20) by computing the conditional Laplace transform of $X$ given $t$ under the measure $Q$.

Let $\xi \in H$ and let $\varphi$ be an arbitrary positive measurable function on $T$ (defined $\mu$-almost everywhere). Then
\[ E_Q \phi(t) \exp \langle \xi, X \rangle = E \int \varphi(t) \exp \langle \xi, X \rangle M(X, dt) = \]

\[ \exp \frac{1}{2} \| \xi \|^2 \cdot E \int \varphi(t) M(X + \xi, dt) = \]

\[ \exp \frac{1}{2} \| \xi \|^2 \cdot E \int \varphi(t) \exp \langle \xi, Y(t) \rangle M(X, dt) = \]

\[ E \exp \langle \xi, X \rangle \cdot E_{\mu} \varphi(t) \exp \langle \xi, Y(t) \rangle = \]

\[ \exp \langle \xi, X \rangle \cdot E_{\mu} \varphi(t) \exp \langle \xi, X + Y(t) \rangle = \]

\[ \exp \langle \xi, X \rangle \cdot E_{\mu} \varphi(t) \exp \langle \xi, X + Y(t) \rangle = \]

\[ \text{(1)} \]

\[ \text{(2)} \]

\[ \text{(3)} \]

\[ \text{(21)} \]

"\( = \)" follows from the definition of \( Q \), "\( = \)" is an application of the Cameron-Martin theorem to the shift \( \xi \), and "\( = \)" is the definition of GMC. The equality of the left-hand side and the right-hand side of \( \text{(21)} \) for all \( \xi \) and \( \varphi \) implies \( \text{(20)} \), since it amounts to equality of conditional Laplace transforms of \( X \) conditioned on \( t \), together with the trivial fact \( \text{Law}_Q t = \text{Law}_{\mu} t = \mu \).

Note that if \( M \) exists then \( \text{(20)} \) implies that \( Y \) is a randomized shift. Indeed, \( \text{Law}_{\mu} [X + Y] \ll \text{Law}_X \) with density \( M[T] \).

Conversely, assume that \( Y \) is a randomized shift. Then define a measure \( Q' \) on \( \Omega \times T \) equipped with \( \sigma(X, t) \) by

\[ \text{Law}_{Q'}[X, t] := \text{Law}_{\mu} [X + Y(t), t] \]

The absolute continuity property in the definition of randomized shift amounts to saying that the \( \Omega \)-projection of \( Q' \) is absolutely continuous w.r.t. \( \mu(X) \), so, in particular, one can define a random measure \( M'(X, dt) \) via disintegration:

\[ \text{P}(d\omega) M'(X, dt) := Q'(d\omega, dt) \]

(this all happens on \( \Omega \times T, \sigma(X, t) \)), so \( M' \) is automatically measurable w.r.t. \( X \). \( \mathbb{E} M' \) is the \( T \)-projection of \( Q' \), so \( \mathbb{E} M' = \mu \) is \( \sigma \)-finite.

To check that \( M' \) is a GMC, introduce a measure type preserving action \( S_{\xi}, \xi \in H \) of the additive group of \( H \) on \( \Omega \times T, \sigma(X, t), Q' \) by

\[ S_{\xi}(X, t) := (X + \xi, t) \]

Since it really acts only on \( X \), there is also an action \( X \mapsto X + \xi \) on \( \Omega, \sigma(X), \mu \), which we also denote by \( S_{\xi} \).

\( S_{\xi} \) is measure type preserving \( Q' \), since it preserves the measure type of almost all fibers in the disintegration of \( Q' \) w.r.t. \( t \). Indeed, this amounts to saying that \( \text{Law}_{\mu} [X + Y(t) + \xi | t] \ll \text{Law}_{\mu} [X + Y(t) | t] \) for \( \mu \)-almost all \( t \), which is obvious from the Cameron-Martin theorem. Moreover, the same argument gives an expression for the density:

\[ \frac{(S_{\xi}) Q'(d\omega, dt)}{Q'(d\omega, dt)} = \exp \left[ -\langle X(\omega) - Y(t), \xi \rangle - \frac{1}{2} \| \xi \|^2 \right] \quad \text{(22)} \]
\((S_\xi)_* Q'\) is the pushforward of \(Q'\) by the map \(S_\xi\), i.e. \((S_\xi)_* Q'[A] = Q'\left[ S_\xi^{-1}[A] \right]\) for measurable sets \(A \in \sigma(X,t)\).

Now we compute the very same density in a different way, by disintegrating \(Q'\) w.r.t. \(X\) instead of \(t\). This shows how \(M'\) behaves w.r.t. shifts:

\[
\frac{(S_\xi)_* Q'(d\omega, dt)}{Q'(d\omega, dt)} = \frac{(S_\xi)_* [P(d\omega) M'(X(\omega), dt)]}{P(d\omega) M'(X(\omega), dt)} = \frac{(S_\xi)_* P(d\omega)}{P(d\omega)} = \frac{M'(X(\omega) + \xi, dt)}{M'(X(\omega), dt)}
\]

By comparing (22) to (23) we see that \(M'\) is a GMC.

5 Regularity of the kernel

For a Hilbert space \(H\) we denote by \(H^{\otimes n}\) its Hilbert \(n\)-th tensor power, also called the space of Hilbert-Schmidt tensors. By definition, it is the completion of the algebraic tensor power w.r.t. the scalar product

\[
\langle \xi_1 \otimes \cdots \otimes \xi_n, \eta_1 \otimes \cdots \otimes \eta_n \rangle := \langle \xi_1, \eta_1 \rangle \cdots \langle \xi_n, \eta_n \rangle,
\]

extended from decomposable tensors to all tensors by multilinearity. We denote by \(\|\cdot\|_2\) the Hilbert space norm corresponding to this scalar product.

For a standard Gaussian \(X\) in \(H\) there is a well-known theory of random variables that are polynomial in \(X\), also known as the Wick calculus (see [4, Chapter III]). Its basic construction is the Wick product of jointly Gaussian random variables, denoted by

\[
\langle X, \xi_1 \rangle \cdots \langle X, \xi_n \rangle := \langle X, \xi_1 \rangle \cdots \langle X, \xi_n \rangle,
\]

and defined by the polarization of the identity

\[
\langle X, \xi \rangle \cdots \langle X, \xi \rangle := \|\xi\|^n h_n \left( \|\xi\|^{-1} \langle X, \xi \rangle \right),
\]

where \(h_n\) is the \(n\)-th Hermite polynomial

\[
h_n(x) := e^{-\frac{1}{2}x^2} = x^n - \frac{1}{2} n(n-1)x^{n-2} + \ldots
\]

The basic fact is that \(\langle X^{\otimes n}, \lambda \rangle\), defined initially for finite-rank (i.e. “algebraic”) tensors \(\lambda\) by

\[
\langle X^{\otimes n}, \xi_1 \otimes \cdots \otimes \xi_n \rangle := \langle X, \xi_1 \rangle \cdots \langle X, \xi_n \rangle;
\]

extends by \(L^2\)-continuity to all Hilbert-Schmidt tensors \(\lambda \in H^{\otimes n}\). In our language this is stated as follows:

\[
X^{\otimes n} : \text{is a generalized random vector in } H^{\otimes n}
\]

It turns out that this implies a corresponding property for a randomized shift \(Y\) — with the crucial difference that \(Y\), unlike \(X\), does not need the Wick renormalization:

\[
Y^{\otimes n} \text{ is a generalized random vector in } H^{\otimes n}
\]
This is the content of our Theorem\[31\] By translating this into a property of the kernel \(K(t, s) := \langle Y(t), Y(s) \rangle\) we deduce in Corollary\[33\] that \(K\) is a function, in the sense of Example\[13\] which, furthermore, has polynomial moments w.r.t. equivalent measures.

For technical reasons we will also need Theorem\[31\] in its refined “uniform” version. We present both versions, just like we did with Theorem\[7\]

We use the notation \(E_\mu := \int_\mathcal{T} \ldots \mu(\,dt)\) for convenience.

**Theorem 31.** Let \(X\) be a standard Gaussian in \(H\), and let \((\mathcal{T}, \mu)\) be a standard probability space.

1. Let \(Y\) be a randomized shift, defined on \((\mathcal{T}, \mu)\). For every \(n \in \mathbb{N}\) there is an equivalent measure \(\mu'_n \sim \mu\) on \(\mathcal{T}\), such that under \(\mu'_n\) all \(\langle Y, \xi \rangle\) have \(n\)-th moment, and the symmetric tensor \(E_{\mu'_n} Y \otimes^n\), defined by the polarization of \(\xi \mapsto E_{\mu'_n} \langle Y, \xi \rangle^n\), is Hilbert-Schmidt.

2. Let \(\{Y_\alpha | \alpha \in I\}\) be a family of randomized shifts over \((\mathcal{T}, \mu)\), and let \(M_{Y_\alpha}\) be the corresponding subcritical GMCs with expectation \(\mu\). Assume that \(\{M_{Y_\alpha}[T]\}\) is uniformly integrable. Then for every \(n \in \mathbb{N}\) there are measures \(\mu'_{\alpha, n} \sim \mu\), such that \(\langle Y_\alpha, \xi \rangle\) have \(n\)-th moments w.r.t. \(\mu'_{\alpha, n}\), and, furthermore, for a fixed \(n\)

   (a) the family of densities \(\{\mu'_{\alpha, n}/\mu\}\) is bounded away from 0 in measure, i.e.
   \[
   \sup_{\alpha} \mu \left\{ \mu'_{\alpha, n}/\mu < \varepsilon \right\} \to 0, \varepsilon \to 0 \quad (24)
   \]

   (b) \(E_{\mu'_{\alpha, n}} Y_\alpha \otimes^n\) is Hilbert-Schmidt, and \(\sup_{\alpha} \left\| E_{\mu'_{\alpha, n}} Y_\alpha \otimes^n \right\|_2 < \infty\).

The following basic observation will be useful in the proof:

**Lemma 32.** Let \(Y\) be a randomized shift defined on a probability space \((\mathcal{T}, \mu)\). Then for every bounded linear operator \(C : H \to H\), satisfying \(\|C\| \leq 1\), the generalized random vector \(CY\) is also a randomized shift. Moreover, if \(M_Y\) and \(M_{CY}\) are the subcritical GMC associated to the shifts \(Y\) and \(CY\) respectively then for every convex nonnegative \(f : \mathbb{R}_+ \to \mathbb{R}_+\) we have

\[
E f (M_{CY} [T]) \leq E f (M_Y [T])
\]

**Proof.** By introducing additional randomness into \(\Omega\), we may decompose \(X\) into a sum of independent components:

\[
X = C^* X' + (1 - C^* C)^{1/2} X'',
\]

where \(X'\) and \(X''\) are independent standard Gaussians in \(H\) — or, equivalently, \(X' \oplus X''\) is a standard Gaussian in \(H \oplus H\).

\[
\langle Y(t), X(\omega) \rangle = \left\langle CY(t) \oplus (1 - C^* C)^{1/2} Y(t), X'(\omega) \oplus X''(\omega) \right\rangle,
\]

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so in particular the GMC $M$ with shift $Y$ over $X$ is a GMC with shift $CY \oplus (1 - C^*C)^{1/2} Y$ over $X' \oplus X''$. Now apply the first statement of Theorem 24 to $X' \oplus X''$, the Hilbert space $H \oplus H$ and the subspace $J := H \oplus 0 \subset H \oplus H$. By Theorem 24

$$\mathbb{E} [M_Y (X) \mid X'] = M_{CY} (X'),$$

so in particular, $CY$ is indeed a randomized shift. The inequality $\mathbb{E} f (M_{CY} [T]) \leq \mathbb{E} f (M_Y [T])$ follows by Jensen’s inequality. 

\textbf{Proof of Theorem 31.} We prove the second statement, since it obviously contains the first one.

Fix $n$, and consider a Hilbert-Schmidt symmetric tensor $\lambda \in H^\otimes n$. Denote the Wick polynomial corresponding to $\lambda$ by

$$P_\lambda (X) := \langle \lambda, X^\otimes n \rangle :$$

It is well-known (see, e.g., [4]) that $\mathbb{E} |P_\lambda (X)|^2 = n! \|\lambda\|_2^2$, so in particular the family of random variables

$$\{P_\lambda (X), \|\lambda\|_2 \leq 1\}$$

is bounded in probability. Now by Lemma 32 we know that for every $|c| \leq 1$ the measures $\text{Law} (X + cY_\alpha)$ are absolutely continuous w.r.t. $\text{Law} X$, and moreover, by de la Vallee Poussin’s theorem, the family of densities $\{\text{Law} (X + cY_\alpha) / \text{Law} X\}$ is uniformly integrable. Therefore for fixed $c$ the family of random variables

$$\{P_\lambda (X + cY_\alpha) \mid \|\lambda\|_2 \leq 1, \text{rank} \lambda < \infty, \alpha \in I\}$$

is bounded in probability. Since $c \mapsto P_\lambda (X + cY)$ is an $n$-th degree polynomial, we can extract its $n$-th degree coefficient in $c$ (denoted by $[c^n] P_\lambda (X + cY_\alpha)$) by taking an appropriate linear combination of its values with different $c$. For instance, we can use the formula

$$[c^n] P_\lambda (X + cY_\alpha) = \frac{n^n}{n!} \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} P_\lambda \left( X + \frac{k}{n} Y_\alpha \right)$$

Note that for finite rank $\lambda$ the random variable $\langle \lambda, Y_\alpha^\otimes n \rangle$ is well-defined. Also note that the Wick product and the ordinary product only differ by lower degree terms, therefore,

$$\langle \lambda, Y_\alpha^\otimes n \rangle = [c^n] P_\lambda (X + cY_\alpha)$$

Now for fixed $n$, by (25), this family of random variables is bounded in probability as $\|\lambda\|_2 \leq 1, \text{rank} \lambda < \infty, \alpha \in I$. This implies that the $L^1$-valued operators $Y_\alpha^\otimes n$, defined on finite rank tensors, are bounded, and in fact equicontinuous, so they can be extended to equicontinuous operators defined on all Hilbert-Schmidt tensors $\lambda$. Therefore, by Theorem 7 for some measures $\mu_{\alpha,n}'$ equivalent to $\mu$

all linear functionals of $Y_\alpha^\otimes n$ have first moments, and $\sup_{\alpha} \left\| \mathbb{E} \mu_{\alpha,n}' \right\|_2 < \infty$, and the densities $\{\mu_{\alpha,n}' / \mu \mid \alpha \in I\}$ are bounded in measure from below. 

\text{\hfill \Box}
Corollary 33. In the setting of Theorem 31 the bilinear forms on $L^2(\mu'_{\alpha,n}) \times L^2(\mu'_{\alpha,n})$ corresponding to the kernels $K_{Y_\alpha,Y_\alpha}(t,t') := \langle Y_\alpha(t), Y_\alpha(t') \rangle$ are Hilbert-Schmidt. Thus $K_{Y_\alpha,Y_\alpha}$ can be viewed as functions on $\mathcal{T} \times \mathcal{T}$, defined $\mu \otimes \mu$-almost everywhere. Moreover,

$$\sup_n \int (K_{Y_\alpha,Y_\alpha}(t,t'))^n \mu'_{\alpha,n}(dt) \mu'_{\alpha,n}(dt') < \infty$$

Proof. For the sake of simplicity we will treat only the “non-uniform” case, i.e. a single $Y$ instead of the family $\{Y_\alpha\}$. The proof carries over verbatim the the “uniform” case.

Informally, if $Y$ was a random vector in $H$, we would write something like

$$\int (K(t,t'))^n \mu'_{\alpha,n}(dt) \mu'_{\alpha,n}(dt') = \int (Y(t), Y(t'))^n \mu'_{\alpha,n}(dt) \mu'_{\alpha,n}(dt') =$$

$$= \int \langle (Y(t))^\otimes n, (Y(t'))^\otimes n \rangle \mu'_{\alpha,n}(dt) \mu'_{\alpha,n}(dt') = \|E \mu_n Y^\otimes n\|_2^2,$$

which is finite for the measure $\mu'$ constructed in Theorem 31. What follows is a way to formalize this in the case of generalized random vectors.

Fix an orthonormal basis $(e_i)$ in $H$, and denote by $Y_i$ the $i$-th coordinate of $Y$ in this basis, i.e. $Y_i = (Y, e_i)$. Fix also a basis $(\varepsilon_p)$ in $L^2(\mu'_{\alpha,n})$. Then the Hilbert-Schmidt norm of $K$ may be rewritten as

$$\|K_{\mu'_{\alpha,n}, \mu'_{\alpha,n}}\|_2^2 = \sum_{p,q} \langle E_{\mu'_{\alpha,n}} \varepsilon_p Y, E_{\mu'_{\alpha,n}} \varepsilon_q Y \rangle_H^2 = \sum_{p,q} \left( \sum_i \left( E_{\mu'_{\alpha,n}} \varepsilon_p Y_i \cdot E_{\mu'_{\alpha,n}} \varepsilon_q Y_i \right) \right)^2 =$$

$$= \sum_{i,j} \sum_{p,q} E_{\mu'_{\alpha,n}} \varepsilon_p Y_i \cdot E_{\mu'_{\alpha,n}} \varepsilon_q Y_i \cdot E_{\mu'_{\alpha,n}} \varepsilon_p Y_j \cdot E_{\mu'_{\alpha,n}} \varepsilon_q Y_j =$$

$$= \sum_{i,j} \left( \sum_p E_{\mu'_{\alpha,n}} \varepsilon_p Y_i \cdot E_{\mu'_{\alpha,n}} \varepsilon_p Y_j \right)^2 \leq \sum_{i,j} \left( E_{\mu'_{\alpha,n}} Y_i Y_j \right)^2 = \|E \mu'_n Y^\otimes 2\|_2^2$$

which is finite by Theorem 31.

Now that $K$ is actually a function in the sense of Example 13 we may express it as the $L^2(\mu'_{\alpha,n} \otimes \mu'_{\alpha,n})$-convergent sum

$$K(t,t') = \sum_i Y_i(t) Y_i(t')$$

W.l.o.g. we may assume that $n$ is even. By Fatou’s lemma,

$$\int (K(t,t'))^n \mu'_{\alpha,n}(dt) \mu'_{\alpha,n}(dt') \leq \liminf_{N \to \infty} \int \left( \sum_{i \leq N} Y_i(t) Y_i(t') \right)^n \mu'_{\alpha,n}(dt) \mu'_{\alpha,n}(dt') =$$

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\[
\liminf_{N \to \infty} \int \sum_{i_1 \leq N} \cdots \sum_{i_n \leq N} Y_{i_1}(t) Y_{i_1}(t') \cdots Y_{i_n}(t) Y_{i_n}(t') \mu'_n(dt) \mu'_n(dt') = \\
= \liminf_{N \to \infty} \sum_{i_1 \leq N} \cdots \sum_{i_n \leq N} (E \nu'_n Y_{i_1} \cdots Y_{i_n})^2 = \left\| E \nu'_n Y \right\|_2^2
\]
which is finite by Theorem 31.

**Remark 34.** Yet another interpretation of this result could be: the existence of a subcritical GMC, which is the Wick exponential of the Gaussian field, implies that there are all Wick powers of that field, due to the formal identity

\[E \left| \int :X(t)\nu' :\mu'_n (dt) \right|^2 = \int (K(t,s)) \nu'_n \mu'_n (dt) \mu'_n (dt') < \infty \quad (26)\]

**Remark 35.** For a randomized shift \( Y \) there is a single measure \( \mu' \sim \mu \), such that

\[\int (K(t,s)) \nu'_n \mu'_n (dt) \mu'_n (dt') < \infty\]

for all \( n \) simultaneously.

**Proof.** Let \( \mu'_n \) be measures constructed in Theorem 31 for different \( n \), and let

\[R_n(t) := \mu(dt) / \mu'_n(dt)\]

By Lemma 3 there is a sequence \( r_n \to \infty \), such that \( R_n(t) = O(r_n), n \to \infty \) for \( \mu \)-almost all \( t \). It is straightforward to verify that the measure

\[\mu'(dt) := \inf_n (r_n/R_n(t)) \cdot \mu(dt)\]

satisfies our requirements.

\[\square\]

### 6 The second moment of subcritical GMC

Let \( Y \) and \( Z \) be two randomized shifts, defined on probability spaces \((T, \mu)\) and \((S, \nu)\) respectively. Both in this section and in the next one we need subcritical GMCs associated to different randomized shifts, so we use the notation \( M_Y, M_Z \) for the GMC associated to the shifts \( Y \) and \( Z \) with fixed expectation \( \mu \) and \( \nu \) respectively. By the definition of GMC,

\[M_Y(X + \xi, dt) = e^{\langle Y(t), \xi \rangle} M_Y(X, dt)\]

for any \( \xi \in H \). Note, however, that if we replace the deterministic shift \( \xi \) by the randomized shift \( Z \), both sides of the equation above still make sense as random measures defined on the extended probability space \((\Omega \times \mathcal{S}, P \otimes \nu)\). Indeed, \( M_Y(X + Z(s)) \) is well-defined because \( \text{Law}_P[X + Z(s)] \ll \text{Law}_P X \), and the function \((t, s) \mapsto e^{\langle Y(t), Z(s) \rangle}\) is well-defined for \( \mu \otimes \nu \)-almost all \((t, s)\) because the kernel \( K_{YZ}(t, s) := \langle Y(t), Z(s) \rangle \) is Hilbert-Schmidt with respect
to some equivalent product measure \( \begin{bmatrix} K_{YY} & K_{YZ} \\ K_{ZY} & K_{ZZ} \end{bmatrix} \) is positive definite, and its
diagonal blocks are Hilbert-Schmidt by Corollary 33 so the off-diagonal blocks
are also Hilbert-Schmidt).

The purpose of this section is to prove the following Lemma 36, which is a
crucial component in the proof of our main theorem on convergence of subcritical
GMC (Theorem 43).

**Lemma 36.** Let \( Y \) and \( Z \) be randomized shifts, and let \( M_Y, M_Z \) be the corre-
sponding subcritical GMCs with expectation \( \mu \) and \( \nu \) respectively. Then
\[
M_Y (X + Z (s), dt) = \exp K_{YZ} (t, s) \cdot M_Y (X, dt) \quad (P \otimes \nu \text{-a.s.})
\]
and
\[
E [M_Y (X) \otimes M_Z (X)] = \exp K_{YZ} \cdot \mu \otimes \nu
\]

In the case of “trivial” GMC (in the sense of Example 21) both state-
ments of Lemma 36 are elementary, so in general they may look form-
ally “obvious”. However, we believe that the real content of Lemma 36 lies in the
implied absolute continuity:
\[
M_Y (X + Z (s)) \ll M_Y (X) \quad (P \otimes \nu \text{-a.s.})
\]
\[
E [M_Y \otimes M_Z] \ll \mu \otimes \nu
\]

Indeed, the central difficulty in the proof is to show that the random measure
\( M_Y \otimes M_Z \) is small in probability on the set where \( \exp K_{YZ} \) is large, which
has small measure \( \mu \otimes \nu \). This is done under the additional assumption that
both \( (1 + \delta^2)^{1/2} Y \) and \( (1 + \delta^2)^{1/2} Z \) are randomized shifts for some \( \delta > 0 \),
which allows to construct conditional GMCs with kernels \( \delta^2 K_{YY} \) and \( \delta^2 K_{ZZ} \)
and conditional expectations \( M_Y \otimes M_Y \) and \( M_Z \otimes M_Z \) respectively. Existence
of such conditional GMCs allows us to obtain bounds on the kernel \( K_{YY}, K_{YZ} \) in
measure \( M_Y \otimes M_Y \) (resp. \( M_Z \otimes M_Z, M_Y \otimes M_Z \)) using Corollary 33.
The general case is then reduced to the “strictly subcritical” one.

In the proof of Lemma 36 we will need the following general fact, which can
be seen as an analogue of Theorem 59 for randomized shifts.

**Lemma 37.** Let \( Z_n \) and \( Z \) be randomized shifts of \( X \) defined on the same proba-
bility space \( (S, \nu) \). Assume that the family of random variables \( \{ \int_T M_{Z_n} (X, dt), n \geq 1 \} \)
is uniformly integrable. Also assume that \( Z_n \rightarrow Z \) in the following sense:
\[
\forall \xi \in H : \langle Z_n, \xi \rangle \overset{L^0(T, \nu)}{\rightarrow} \langle Z, \xi \rangle
\]
Then for every \( f (X) \in L^0 (\Omega, \sigma (X), P) \) we have
\[
f (X + Z_n) \overset{L^0(\Omega, P) \otimes (T, \nu))}{\rightarrow} f (X + Z)
\]
Proof. For \( f(X) := \exp i \langle \xi, X \rangle, \xi \in H \), this follows from the assumptions. It is well-known that the linear combinations of exponentials \( \exp i \langle \xi, X \rangle \) are dense in \( L^0(\Omega, \sigma(X)) \). Let \( f \in L^0(\Omega) \) and let \( f_n(X) \to f(X) \) be a sequence of linear combinations of exponentials approximating \( f(X) \) in \( L^0(\Omega) \).

We claim that
\[
f_n(X + Z_m) \overset{L^0}{\to} f(X + Z_m) \quad \text{uniformly in } m
\]
The reason is uniform integrability of \( M_{Z_m} \). Indeed, for any \( \varepsilon \) there are large \( n \) such that \( \{|f_n - f| > \varepsilon\} \) has small measure w.r.t. \( P \). Thus this measure is also small for \( \int_\mathcal{T} M_{Z_m}(X, dt) \cdot P \), uniformly in \( m \), due to uniform integrability. This implies that \( f_n(X + Z) \overset{L^0}{\to} f(X + Z) \).

Proof of Lemma 36. The proof relies on approximating the shifts \( Y \) and \( Z \) by their projections \( P_n Y, P_n Z \), where \( (P_n) \) is an increasing sequence of finite-dimensional orthogonal projection operators in \( H \), converging strongly to 1. Note that for \( P_n Y, P_n Z \) both statements of Lemma 36 are satisfied, since \( P_n \) are finite-dimensional and therefore \( P_n Y, P_n Z \) are vector-valued functions.

We proceed via the following series of claims. Their proofs are given later in this section. In our notation we suppress the \( t, s, dt, ds \) for brevity.

Note that a sum of independent randomized shifts — and \( Y + Z \) in particular — is a randomized shift. Let \( M_{Y+Z} \) be the subcritical GMC on \( \mathcal{T} \times \mathcal{S} \) with expectation \( \mu \otimes \nu \), associated with the shift \( Y + Z \).

Claim 38. For any pair of randomized shifts \( Y, Z \) are equivalent. Furthermore, they are equivalent to
\[
M_Y \otimes M_Z = \exp K_{YZ} \cdot M_{Y+Z} \quad \text{a.s.}
\]

The statement of our lemma for a general pair of randomized shifts \( Y, Z \) can be reduced to a special case when \( Y \) and \( Z \) satisfy the following “strict subcriticality” assumption:

Assumption 39. There exists some \( \delta > 0 \), such that \( (1 + \delta^2)^{1/2} Y \) and \( (1 + \delta^2)^{1/2} Z \) are randomized shifts.

Note that for any randomized shifts \( Y, Z \) and any \( \delta_1, \delta_2 \in (0, 1) \) the shifts \( (1 - \delta_1) Y, (1 - \delta_2) Z \) satisfy Assumption 39. The following claim allows the reduction to this special case:

Claim 40. If the statements of Lemma 36 are true for \( (1 - \delta_1) Y, (1 - \delta_2) Z \) for every \( \delta_1, \delta_2 \in (0, 1) \) then they are true for \( Y, Z \).

In the remainder of the proof we assume that \( X \) and \( Y \) satisfy Assumption 39. We proceed by proving the lemma in the form (31) using an approximation argument.

Let \( \varepsilon > 0 \), and consider the following subsets of \( \mathcal{S} \times \mathcal{T} \):
\[
A_\varepsilon := \{\exp K_{YZ} \leq \varepsilon^{-1}\}
\]
\( A_\varepsilon^{(n)} := \{ \exp K_{P_n Y, P_n Z} \leq \varepsilon^{-1} \} \)

Assume that \( \varepsilon \) is such that for all \( n \) \((\mu \otimes \nu) \{ \exp K_{P_n Y, P_n Z} = \varepsilon^{-1} \} = 0 \) (this is true for all \( \varepsilon \) except at most countably many).

Claim 41. Fix \( \varepsilon > 0 \). For every \( f \in L^\infty (T, \mu) \), \( g \in L^\infty (S, \nu) \) the following sequences converge in probability as \( n \to \infty \):

\[
\int f \otimes g \cdot M_{P_n Y} \otimes M_{P_n Z} \to \int f \otimes g \cdot M_Y \otimes M_Z \quad (32)
\]

\[
\int f \otimes g \cdot \exp K_{P_n Y, P_n Z} \cdot M_{P_n Y + P_n Z} \to \int f \otimes g \cdot \exp K_{Y Z} \cdot M_{Y + Z} \quad (33)
\]

In addition to Claim 41 it remains to show that \( \int_{T \setminus (A_\varepsilon \cap A_\varepsilon^{(n)})} \int f \otimes g \cdot \exp K_{P_n Y, P_n Z} \cdot M_{P_n Y + P_n Z} \) converge to 0 in probability as \( \varepsilon \to 0 \) uniformly in \( n \).

Claim 42. Under Assumption 39 the random variables \( \int_{T \setminus (A_\varepsilon \cap A_\varepsilon^{(n)})} \int f \otimes g \cdot \exp K_{P_n Y, P_n Z} \cdot M_{P_n Y + P_n Z} \) converge to 0 in probability as \( \varepsilon \to 0 \) uniformly in \( n \).

Claims 41 and 42 together imply (31) under Assumption 39, and therefore also in the general case.

Proof of Claim 41. Theorem 29 applied to \( M_Z \) and its corresponding randomized shift \( Z \) implies that

\[
E_{\mathbb{P} \otimes \nu} [M_Y (X + Z (s)) \otimes \delta_s] = E [M_Y (X) \otimes M_Z (X)] \quad (34)
\]

The implication \((27) \Rightarrow (28)\) follows immediately from (34).

We prove now the converse implication \((28) \Rightarrow (27)\). \( (28) \) and (34) imply immediately that

\[
E_{\mathbb{P} \otimes \nu} [M_Y (X + Z (s), dt) \mid s] = \exp K_{Y Z} (t, s) \cdot \mu (dt),
\]

so both sides of (27) have the same conditional expectation w.r.t. \( s \). We show that both sides of (27) are GMCs conditionally on \( s \), with the same shift (namely, \( Y \)), and the claim follows from the uniqueness of subcritical GMC (Corollary 30). That the right-hand side of (27) is a conditional GMC with shift \( Y \) is obvious, since it is a product of a GMC \( M_Y \) and a function of \( t \) and \( s \). For the left-hand side we note that for any fixed \( \xi \in H \)

\[
M_Y (X + Z (s) + \xi, dt) = \exp \{ Y (t), \xi \} \cdot M_Y (X + Z (s), dt)
\]

for almost all \( s \), and apply Lemma 22 to any countable dense subset \( \{ \xi_n \} \subset H \).

Obviously, \( (31) \Rightarrow (28) \) by taking the expectation of both sides.

To prove the converse implication \((28) \Rightarrow (31)\) we notice that \( M_Y \otimes M_Z \) is a GMC with shift \( Y + Z \) by definition:

\[
\forall \xi \in H : M_Y (X + \xi) \otimes M_Z (X + \xi) = \exp ((Y, \xi) + (Z, \xi)) \cdot M_Y (X) \otimes M_Z (X)
\]

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implies that this GMC is subcritical. Therefore, both $M_Y \otimes M_Z$ and $\exp K_{YZ} \cdot M_{Y+Z}$ are subcritical GMC with the same shift and the same expectation, so (31) follows from uniqueness.

Proof of Claim 42. Among the equivalent statements of Lemma 36 it is convenient here to use (27). By our assumption we have for every $f \in L^\infty (\mathcal{T}, \mu)$

$$\int f(t) \cdot M_{(1-\delta_1)Y} (X + (1-\delta_2)Z; s) \, dt = \int f(t) \exp (1-\delta_1)(1-\delta_2) K_{YZ} (t,s) \cdot M_{(1-\delta_1)Y} (X, dt)$$

We fix $\delta_1$ and take $\delta_2 \to 0$. By Lemma 37 the left-hand side above converges in measure to $\int f \cdot M_{(1-\delta_1)Y} (X + Z)$. The right-hand side converges almost surely $\int f \exp (1-\delta_1) K_{YZ} \cdot M_{(1-\delta_1)Y} (X)$ by the monotone convergence theorem (applied separately to $\{K_{YZ} > 0\}$ and to $\{K_{YZ} \leq 0\}$). This shows that Lemma 36 holds for $(1-\delta_1)Y, Z$. By Claim 38 its statement is equivalent to one that is symmetric in $(Y, Z)$, so the same reasoning works with $Y$ and $Z$ interchanged, which allows to take $\delta_1 \to 0$ as well. \hspace{1cm} \square

Proof of Claim 41. It follows from the first part of Lemma 24 that for any $f \in L^\infty (\mathcal{T}, \mu)$, $g \in L^\infty (\mathcal{S}, \nu)$, $h \in L^\infty (\mathcal{T} \times \mathcal{S}, \mu \otimes \nu)$ the sequences $\int f \cdot M_{P_nY}$, $\int g \cdot M_{P_nZ}$ and $\int h \cdot M_{Y+P_n Z}$ are martingales that converge almost surely and in $L^1$ to $\int f \cdot M_Y$, $\int g \cdot M_Z$ and $\int h \cdot M_{Y+Z}$ respectively. This proves (32), as well as the following:

$$\int \mathbb{A}_\varepsilon \left[ f \otimes g \cdot \exp K_{YZ} \cdot M_{P_n Y + P_n Z} \right] \to \int \mathbb{A}_\varepsilon \left[ f \otimes g \cdot \exp K_{YZ} \cdot M_{Y+Z} \right]$$

It is enough to show that

$$\mathbb{E} \int \mathbb{A}_\varepsilon \left[ \exp K_{YZ} - \mathbb{I} \left[ A^{(n)}_\varepsilon \right] \exp K_{P_n Y, P_n Z} \right] M_{P_n Y + P_n Z} \to 0$$

However, this is clear. Indeed, $\mathbb{E} M_{P_n Y + P_n Z} = \mu \otimes \nu$, and it follows from the Hilbert-Schmidt property (Corollary 33) that $K_{P_n Y, P_n Z} \to K_{YZ}$ in measure $\mu \otimes \nu$. Thus by the bounded convergence theorem we have

$$\int \mathbb{A}_\varepsilon \left[ \exp K_{YZ} - \mathbb{I} \left[ A^{(n)}_\varepsilon \right] \exp K_{P_n Y, P_n Z} \right] \mu \otimes \nu \to 0$$

\hspace{1cm} \square

Proof of Claim 40. Let $X'$ be a standard Gaussian in $H$, independent of $X$. Consider the standard Gaussian $X \oplus X'$ in $H \oplus H$, and the generalized random vectors $Y \oplus \delta Y$ and $Z \oplus \delta Z$ in $H \oplus H$. They are both randomized shifts, since they are isometric images of $(1 + \delta^2)^{1/2} Y$ and $(1 + \delta^2)^{1/2} Z$. Let $M_{Y \oplus \delta Y} (X \oplus X')$
and $M_{Z\oplus\delta Z}(X \oplus X')$ denote their corresponding subcritical GMCs with expectation $\mu$ and $\nu$ respectively.

Let $f: \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing convex function, such that $f(x)/x \to +\infty, x \to +\infty$, and

$$E_f\left(\int M_{Y\oplus\delta Y}(X \oplus X', dt)\right), E_f\left(\int M_{Z\oplus\delta Z}(X \oplus X', ds)\right) < \infty$$

The operators $P_n \oplus 1$ and $P_n \oplus P_n$ from $H \oplus H$ to itself have norm $1$, so by Lemma 32 we have

$$\sup_n E_f\left(\int M_{P_n Y\oplus\delta Y}(X \oplus X', dt)\right) < \infty,$$

and similarly for $M_{P_n Y\oplus\delta P_n Y}$, $M_{P_n Z\oplus\delta Z}$ and $M_{P_n Z\oplus\delta P_n Z}$. Take $\varepsilon' > 0$, and consider the following set:

$$B_{\varepsilon', n} := \left\{X : \int M_{P_n Y}(X) > \varepsilon', E\left[f\left(\int M_{P_n Y\oplus\delta Y}(X \oplus X')\right) \left| X\right.\right] < \varepsilon'^{-1}\right\}$$

By Lemma 24, $M_{P_n Y\oplus\delta Y}(X \oplus X')/\int M_{P_n Y}(X)$ is, conditionally on $X$, a subcritical GMC over $X'$ with kernel $\delta^2 K_{YY}$ and conditional expectation $M_{P_n Y}(X)/\int M_{P_n Y}(X)$. Moreover, on $\{X \in B_{\varepsilon', n}\}$ the random variables $\int M_{P_n Y\oplus\delta Y}(X \oplus X')/\int M_{P_n Y}(X)$ are (conditionally on $X$) uniformly integrable. Therefore, Corollary 33 applies conditionally to the family of GMCs

$$\left\{M_{P_n Y\oplus\delta Y}(X \oplus X')/\int M_{P_n Y}(X) : X \in B_{\varepsilon', n}, n \geq 1\right\}$$

This yields existence of random measures $N_{Y,n}$ on $\mathcal{T}$, measurable w.r.t. $X$, such that the density $N_{Y,n}/M_{P_n Y}(X)$ is bounded in measure $(M_{P_n Y}(X))$ from below, such that

$$\text{ess sup}_{n \geq 1, X \in B_{\varepsilon', n}} \int K^2_{YY} \cdot N_{Y,n} \otimes N_{Y,n} < \infty$$

The same reasoning applies to $Z$, yielding the existence of random measures $N_{Z,n}$. Since $\begin{bmatrix} K_{YY} & K_{YZ} \\ K_{ZY} & K_{ZZ} \end{bmatrix}$ is positive definite, this gives a bound on $K_{YZ}$:

$$\text{ess sup}_{n \geq 1, X \in B_{\varepsilon', n}} \int K^2_{YZ} \cdot N_{Y,n} \otimes N_{Z,n} < \infty,$$

which implies that on $\{X \in B_{\varepsilon', n}\}$ the random variables $\int_{\mathcal{T} \setminus A_n} N_{Y,n} \otimes N_{Z,n}$ converge to $0$ as $\varepsilon \to 0$ uniformly in $n$. Since the densities $N_{Y,n}/M_{P_n Y}$ and $N_{Z,n}/M_{P_n Z}$ are bounded away from $0$ in measure, the inverse densities $M_{P_n Y}/N_{Y,n}$ and $M_{P_n Z}/N_{Z,n}$ are uniformly integrable w.r.t. $N_{Y,n}$ and $N_{Z,n}$ resp., which allows to conclude that on the event $\{X \in B_{\varepsilon', n}\}$ we have $\int_{\mathcal{T} \setminus A_n} M_{P_n Y} \otimes M_{P_n Z} \to 0$ uniformly in $n$ as $\varepsilon \to 0$.  

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It remains to show that
\[ \inf_n P \{ X \in B_{\varepsilon',n} \} \to 1, \varepsilon' \to 0 \]
This follows from uniform integrability of our definition of \( f \) and the fact that
\[ \sup_n P \left\{ \int M_{P_n Y} (X) < \varepsilon' \right\} \leq P \left\{ \inf_n \int M_{P_n Y} (X) < \varepsilon' \right\} \to 0 \]
(indeed, since \( \int M_{P_n Y} (X) \) is a nonnegative martingale, \( \inf_n \int M_{P_n Y} (X) = 0 \) with positive probability implies that \( M_Y (X) = 0 \) with positive probability, which is impossible, since the probability of the Cameron-Martin shift invariant event \( \{ M_Y (X) = 0 \} \) can be either 0 or 1).

The same reasoning applies with \( M_{P_n Y \oplus \delta Y} \) replaced by \( M_{P_n Y \oplus \delta P_n Y} \), yielding the same conclusion for \( \int_{T \setminus A_n} M_{P_n Y \otimes M_{P_n Z}} \).

\[ \square \]

7 Approximation

Let \( Y_n, n \geq 1 \) be randomized shifts defined on a probability space \((T, \mu)\). Let \( K_{Y_n Y_n} (t, s) := \langle Y_n (t), Y_n (s) \rangle \) be the corresponding kernel, which, by Corollary 33, we will always view as a function on \((T \times T, \mu \otimes \mu)\). Let \( M_{Y_n} \) be the subcritical GMC associated to \( Y_n \) with expectation \( \mu \).

In this section we prove our main result on approximation of subcritical GMC:

**Theorem 43.** Assume that:

- The family of random variables \( \{ M_{Y_n [T]} \} \) is uniformly integrable;
- There exists a generalized \( H \)-valued function \( Y \) defined on \((T, \mu)\) that is the limit of \( Y_n \) in the sense that
  \[ \forall \xi \in H : \langle Y_n, \xi \rangle \overset{L^0(\mu)}{\to} \langle Y, \xi \rangle \] (35)

Then \( Y \) is a randomized shift. If, furthermore,

- The kernels \( K_{Y_n Y_n} \) converge to \( K_{YY} \) in \( L^0 (\mu \otimes \mu) \)

Then the subcritical GMC \( M_Y \) (associated to \( Y \) with expectation \( \mu \)) is the limit of \( M_{Y_n} \) in the sense that
  \[ \forall f \in L^1 (\mu) : \int f(t) M_{Y_n} (X, dt) \overset{L^1}{\to} \int f(t) M_Y (X, dt) \] (36)

In the proof we will need a measure-theoretic tool (Lemma 10) for proving convergence of integrals of functions against random measures, which can be seen as a stochastic analogue of Lebesgue’s theorem. Just like Lebesgue’s theorem, it comes with a related notion of “uniform integrability”:

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Definition 44. A family of random measures \( \{M_\alpha\} \) on a measurable space \( T \) is called **uniformly stochastically absolutely continuous** (USAC) w.r.t. a deterministic probability measure \( \mu \) on \( T \) if

- \( \forall \alpha : E M_\alpha \ll \mu \)
- For every \( c > 0 \) we have
  \[ \sup_{A \subset T} \sup_{\rho[A] \leq \varepsilon} P \{M_\alpha[A] > c\} \rightarrow 0, \varepsilon \rightarrow 0 \]

Example 45. If \( EM_\alpha = \mu \) for all \( \alpha \) then \( \{M_\alpha\} \) is USAC. Indeed, if \( \mu[A] < \varepsilon \) then \( M_\alpha[A] \) are uniformly small in \( L^1 \), therefore uniformly small in probability.

Lemma 46. Let \( (M_n) \) be a sequence of random measures on \( T \), such that

- \( \forall \alpha : E M_\alpha \ll \mu \), and let \( F_n, F \in L^0(T, \mu) \). Assume that:
  - \( M_n \rightarrow M \) in the sense that
    \[ \forall f \in L^\infty(T, \mu) : \int f(t) M_n(dt) \stackrel{L^0}{\rightarrow} \int f(t) M(dt) \tag{37} \]
  - \( F_n \stackrel{L^0(\mu)}{\rightarrow} F \)
  - For all \( n \) we have
    \[ \int |F_n(t)| M_n(dt) < \infty \]
  - The family of random measures \( \{|F_n(t)| M_n(dt)\} \) is USAC w.r.t. \( \mu \).

Then

\[ \int |F(t)| M(dt) < \infty \]

and

\[ \int F_n(t) M_n(dt) \stackrel{L^0}{\rightarrow} \int F(t) M(dt) \tag{38} \]

The same is true if we replace convergence in \( L^0 \) by convergence in law both in (37) and in (38).

Proof. By passing to a subsequence, we may assume that \( F_n \rightarrow F \) almost everywhere. Fix \( \varepsilon > 0 \). By Egorov’s theorem, there is a set \( A_\varepsilon \subset T \), such that \( \mu[T \setminus A_\varepsilon] \leq \varepsilon \) and \( F_n \rightarrow F \) uniformly on \( A_\varepsilon \).

\[ \left| \int_{A_\varepsilon} F_n(t) M_n(dt) - \int F(t) M_n(dt) \right| \leq \]
The first term is small in probability uniformly in $n$ whenever $\varepsilon$ is small due to the USAC property. The second term is small in probability for fixed $\varepsilon$ and large $n$ because the family of random variables $\{M_n[T]\}$ is bounded in probability and $\text{ess sup}_{A_n} |F_n - F| \to 0, n \to \infty$.

On the other hand, $\int_{A_n} F(t) M_n(dt)$ is close in probability (resp. in law) to $\int F(t) M(dt)$ by assumption.

The next statement is based entirely on the main result of the previous section, Lemma 36.

**Lemma 47.** Let $\{Y_\alpha\}$ and $\{Z_\beta\}$ be families of randomized shifts on $(T, \mu)$ and $(S, \nu)$ respectively, and let $\{M_{Y_\alpha}[T]\}$ be uniformly integrable. Let $T$ be a random point in $T$ with law $\mu$, independent of $X$. Then the family of random measures $N_{\alpha\beta}$ on $T \times S$ given by

$$N_{\alpha\beta} := \exp \left\{ K_{Y_\alpha Z_\beta} (T, s) \cdot M_{Z_\beta} (X, ds) \right\}$$

is USAC w.r.t. $\mu \otimes \nu$.

**Proof.** Fix $\varepsilon > 0$. Consider a measurable subset $A \subset T \times S$, and denote by $A_t, t \in T$ its $t$-section, i.e.

$$A_t := \{ s \in S : (t, s) \in A \} \subset S$$

By Lemma 36

$$\text{P} \{ N_{\alpha\beta} [A] > c \} = \text{P} \left\{ \int_{A_T} \exp K_{Y_\alpha Z_\beta} (T, s) M_{Z_\beta} (X, ds) > c \right\} =$$

$$= \text{P} \left\{ \int_{A_T} M_{Z_\beta} (X + Y_\alpha (T), ds) > c \right\} =$$

$$= \text{E} \int_{A_t} \left\{ \int_{A_t} M_{Z_\beta} (X, ds) > c \right\} \leq$$

$$\leq \mu \left\{ t : \nu [A_t] > \varepsilon \right\} + \text{E} \int_{A_t} \left\{ \nu [A_t] \leq \varepsilon \right\} \left\{ \int M_{Z_\beta} (X, ds) > c \right\}$$

The first term is small whenever $(\mu \otimes \nu) [A]$ is small enough. The second term is small whenever $\varepsilon$ is small enough, since for $t$, such that $\nu [A_t] \leq \varepsilon$, we have

$$\text{P} \left\{ \int_{A_t} M_{Z_\beta} (X, ds) > c \right\} \leq c^{-1} \text{E} \int_{A_t} M_{Z_\beta} (X, ds) = c^{-1} \nu [A_t] \leq c^{-1} \varepsilon,$$

and $\{ \int M_{Y_\alpha} (X, dt) \}$ is assumed to be uniformly integrable. $\square$
Before we begin the proof of Theorem 43 we make some general observations. First, we note that families of random measures with fixed expectation have the following distributional precompactness property:

**Lemma 48.** Let \( \{ M_\alpha \} \) be a family of random measures on \( T \) with expectation \( \mu \). Then there exists a sequence \( (\alpha_n) \), and a random measure \( M \) on \( (T, \mu) \) with \( \mathbb{E} M \leq \mu \), possibly defined on an extended probability space, such that

\[
\forall \xi \in H, \forall f \in L^1 : \left( \langle X, \xi \rangle, \int f(t) M_{\alpha_n}(dt) \right) \overset{\text{Law}}{\rightarrow} \left( \langle X, \xi \rangle, \int f(t) M(dt) \right)
\]

(39)

In the sequel we abbreviate (39) to “\( (X, M_{\alpha_n}) \overset{\text{Law}}{\rightarrow} (X, M) \)”.

**Proof.** \( (T, \mu) \) is a standard measure space, so we may assume that its measurable structure comes from a Borel \( \sigma \)-algebra of a compact metrizable topology on \( T \). We identify \( X \) with a random element in the Polish space \( \mathbb{R}^\infty \).

The family \( \{ M_\alpha [T] \} \) is tight, therefore the family \( \{ (X, M_\alpha) \} \) of random elements of \( \mathbb{R}^\infty \times \text{Measures}(T) \) is tight when the space of measures is equipped with the weak topology. Thus for some sequence \( (\alpha_n) \) there is a distributional limit \( (X, M) \) of \( (X, M_{\alpha_n}) \), possibly on an extended probability space. This implies (39) for continuous \( f \) and \( \xi \in H \) such that \( X \mapsto \langle \xi, X \rangle \) are equivalent to continuous functions.

It’s easy to see that the family of maps

\[
H \times L^1(\mu) \to L^0(P) \times L^0(P)
\]

\[
(\xi, f) \mapsto \left( \langle X, \xi \rangle, \int f(t) M_{\alpha_n}(dt) \right)
\]

is equicontinuous, so (39) holds for all \( \xi \in H, f \in L^1 \).

The inequality \( \mathbb{E} M \leq \mu \) follows from \( \mathbb{E} M_\alpha = \mu \) together with Fatou’s lemma.

Lemma 48 applies to GMC in particular, so by passing to a subsequence, we may assume in Theorem 43 that for some random measure \( M \) with \( \mathbb{E} M \leq \mu \) we have

\[
(X, M_{Y_n}) \overset{\text{Law}}{\rightarrow} (X, M)
\]

(40)

in the sense of Lemma 48. Furthermore, \( \mathbb{E} M = \mu \) due to the uniform integrability assumption of Theorem 43.

Here is another general observation:

**Lemma 49.** Let \( (X, M_{Y_n}) \overset{\text{Law}}{\rightarrow} (X, M) \), and assume that \( M \) is measurable w.r.t. \( X \). Then \( M_{Y_n} \overset{L^0}{\rightarrow} M \) in the sense that

\[
\forall f \in L^1(\mu) : \int f(t) M_{Y_n}(dt) \to \int f(t) M(dt)
\]
Proof. Consider the sequence of triples \((X, M_{Y_n}, M)\). By the same reasoning as in Lemma 48, it has a subsequential distributional limit. Since \((X, M_{Y_n}) \xrightarrow{\text{Law}} (X, M)\), this distributional limit has the form \((X, M, M')\), where \((X, M)\) and \((X, M')\) have the same joint distribution. But since \(M\) (and therefore also \(M'\)) is a function of \(X\), we have \(M = M'\). This implies, in particular, that \((M_{Y_n}, M) \xrightarrow{\text{Law}} (M, M)\), so for every function \(f \in L^1(\mu)\)

\[
\int f(t) M_{Y_n}(dt) - \int f(t) M(dt) \xrightarrow{\text{Law}} \int f(t) M(dt) - \int f(t) M(dt) = 0
\]

Convergence in law to a constant is equivalent to convergence in probability, so \(\int f(t) M_{Y_n}(dt) \xrightarrow{\mu} \int f(t) M(dt)\).

By Lemmas 48 and 49, the main difficulty of the proof of Theorem 43 is to show that the measure \(M\) obtained in the distributional limit is measurable w.r.t. \(X\).

Proof of Theorem 43. By the earlier remarks we assume that there is a random measure \(M\) on \(\mathcal{T}\) that satisfies (40).

The proofs of the following Claims 50-53 are given in the end of this section.

Claim 50. \(Y\) is a randomized shift, with its subcritical GMC given by \(M_Y = E[M|X]\).

Claim 50 implies that it is enough to show that \(M\) is measurable w.r.t. \(X\), since, as noted above, this would imply convergence in \(L^0\) of \(M_{Y_n}\) to \(M\).

Fix any bounded continuous function \(G : \mathbb{R}_+ \rightarrow \mathbb{R}_+\), such that \(G\) is strictly concave and \(x \mapsto xG(x)\) is strictly convex — for instance, \(G(x) := \frac{x}{1+x}\). Our next goal will be to show that

\[
E M[\mathcal{T}] G(M[\mathcal{T}]) \leq E M_Y[\mathcal{T}] G(M_Y[\mathcal{T}]) \tag{41}
\]

(41) will immediately imply that \(M[\mathcal{T}]\) is measurable w.r.t. \(X\), due to the following chain of inequalities:

\[
E M[\mathcal{T}] G(M_Y[\mathcal{T}]) = E M_Y[\mathcal{T}] G(M_Y[\mathcal{T}]) \leq \tag{42}
\]

\[
\leq E M[\mathcal{T}] G(M[\mathcal{T}]) \leq E M_Y[\mathcal{T}] G(M_Y[\mathcal{T}]) = \tag{(i)}
\]

\[
E[M[\mathcal{T}]E[G(M[\mathcal{T}])|X]] \leq E M[\mathcal{T}] G(M_Y[\mathcal{T}])
\]

"\(\leq\)" is exactly (41), and the first and the last inequality are both instances of Jensen’s inequality, applied to \(xG(x)\) and \(G(x)\) respectively. Since these two functions are strictly convex and concave, respectively, equality in the Jensen’s inequality implies that \(M[\mathcal{T}] = E[M[\mathcal{T}]|X]\), i.e. that \(M[\mathcal{T}]\) is measurable w.r.t. \(X\). A similar argument works with \(M\) replaced by \(f \cdot M\) for any nonnegative bounded \(f\), which amounts to replacing \(M[\mathcal{T}]\) by \(\int f(t) M(dt)\), so in fact (41) implies that \(M\) is measurable w.r.t. \(X\).

Our proof of (41) consists of the following steps. First, we show how the distribution of \(M\) changes when multiplied by the density \(M[\mathcal{T}]\):

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Claim 51.
\[ E M \left(T\right) G \left(M \left[T\right]\right) = E_{P \otimes \mu} G \left(\int \exp \left(Y \left(t\right) , Y \left(s\right)\right) M \left(ds\right)\right) \] (43)

The next two claims concern a change of measure similar to the one above with \( M \left[T\right] \) replaced by a “true” exponential or GMC:

Claim 52. For every \( \xi \in H \)
\[ E \exp \left(\xi , X\right) - \frac{1}{2} \|\xi\|^2 \right) G \left(M \left[T\right]\right) = E G \left(\int \exp \left(\xi , Y \left(t\right)\right) M \left(dt\right)\right) \] (44)

Claim 53.
\[ E E \left[M \left[T\right] | X\right] G \left(M \left[T\right]\right) \geq E_{P \otimes \mu} G \left(\int \exp \left(Y \left(t\right) , Y \left(s\right)\right) M \left(ds\right)\right) \] (45)

Claims 51 and 53 together imply (41). As noted above, this is enough to prove the theorem.

Proof of Claim 50. Since \( (X,M,Y_n) \) converges in law to \( (X,M) \) and \( \left\{M_{Y_n} \left[T\right]\right\} \) is uniformly integrable, we deduce that for every \( \xi \in H \) and every \( f \in L^1(\mu) \)
\[ E \left[\exp i \langle \xi , X\rangle \cdot \int f \left(t\right) M_{Y_n} \left(dt\right)\right] \rightarrow E \left[\exp i \langle \xi , X\rangle \cdot \int f \left(t\right) M \left(dt\right)\right] \] (46)

By Theorem 29, the left-hand side of (46) equals
\[ E_{P \otimes \mu} f \left(t\right) \exp i \langle \xi , X + Y_n \left(t\right)\rangle \]

Now since \( \langle \xi , Y_n \rangle \overset{L^1}{\to} \langle \xi , Y\rangle \) by assumption, we have
\[ E_{P \otimes \mu} f \left(t\right) \exp i \langle \xi , X + Y_n \left(t\right)\rangle \rightarrow E_{P \otimes \mu} f \left(t\right) \exp i \langle \xi , X + Y \left(t\right)\rangle \]

Therefore,
\[ E \left[\exp i \langle \xi , X\rangle \cdot \int f \left(t\right) M \left(dt\right)\right] = E_{P \otimes \mu} f \left(t\right) \exp i \langle \xi , X + Y \left(t\right)\rangle \] (47)

By taking \( f := 1 \) we deduce that Law \[X + Y\] is absolutely continuous w.r.t. Law \( X \) with density \( E \left[\int M \left(dt\right) | X\right] \), so that \( Y \) is a randomized shift. Moreover, (47) implies that the measures \( P \left(d\omega\right) M \left(\omega , dt\right) \) and \( P \left(d\omega\right) M_Y \left(\omega , dt\right) \) have the same restriction to \( \sigma \left(X,T\right) \), therefore \( E \left[M | X\right] = M_Y \).
Proof of Claim 51. By uniform integrability of \( \{M_{Y_n}[T]\} \) and convergence \( M_{Y_n}[T] \xrightarrow{\text{Law}} M[T] \) we have

\[
\mathbb{E} M[T] G(M[T]) = \lim_{n \to \infty} \mathbb{E} M_{Y_n}[T] G(M_{Y_n}[T])
\]

By Theorem 29 and Lemma 36 both applied to the randomized shift \( Y_n \), we have

\[
\mathbb{E} M_{Y_n}[T] G(M_{Y_n}[T]) = \mathbb{E}_{P \otimes \mu} G(M_{Y_n}(X + Y_n(t))[T]) = \mathbb{E}_{P \otimes \mu} G\left( \int \exp K_{Y_n,Y_n}(t,s) M_{Y_n}(X,ds) \right)
\]

Thus to prove the claim it’s enough to show that for a random point \( T \) in \( T \) with distribution \( \mu \), independent of \( X \), we have

\[
\int \exp K_{Y_n,Y_n}(T,s) M_{Y_n}(X,ds) \overset{\text{Law}}{\to} \int \exp K_{Y,Y}(T,s) M(ds) \quad (48)
\]

Somewhat tautologically, we rewrite both integrals in a way that involves random measures and deterministic (i.e. not dependent on \( T \)) functions:

\[
\int_T \exp K_{Y_n,Y_n}(T,s) M_{Y_n}(X,ds) = \int_{T \times T} \exp K_{Y_n,Y_n}(t,s) \cdot \delta_T \otimes M_{Y_n}(X,ds)
\]

\[
\int_T \exp K_{Y,Y}(T,s) M(ds) = \int_{T \times T} \exp K_{Y,Y}(t,s) \cdot \delta_T (dt) \otimes M(ds)
\]

Now we apply Lemma 17 to the randomized shifts \( \{Y_\alpha\} := \{Y_n\}, \{Z_\beta\} := \{Y_n\} \) and deduce that \( \{\exp K_{Y_n,Y_n} \cdot \delta_T \otimes M_{Y_n}\} \) is USAC. Then apply Theorem 40 to the random measures \( \delta_T \otimes M_{Y_n} \) and functions \( \exp K_{Y_n,Y_n} \). Note that \( M_{Y_n} \xrightarrow{\text{Law}} M \) trivially implies \( \delta_T \otimes M_{Y_n} \xrightarrow{\text{Law}} \delta_T \otimes M \), and that by our assumptions \( \exp K_{Y_n,Y_n} \xrightarrow{L^p(\mu \otimes \mu)} \exp K_{Y,Y} \), so indeed Theorem 40 is applicable in this case, yielding (48), and therefore also the claim. \( \square \)

Proof of Claim 52. By the definition of GMC and the Cameron-Martin theorem it’s easy to see that

\[
\mathbb{E} \exp \left[ \langle \xi, X \rangle - \frac{1}{2} \|\xi\|^2 \right] G(M_{Y_n}[T]) = \mathbb{E} G\left( \int \exp \langle \xi, Y_n(t) \rangle M_{Y_n}(X,dt) \right)
\]

The convergence \( (X, M_{Y_n}) \xrightarrow{\text{Law}} (X, M) \) implies

\[
\mathbb{E} \exp \left[ \langle \xi, X \rangle - \frac{1}{2} \|\xi\|^2 \right] G(M_{Y_n}[T]) \to \mathbb{E} \exp \left[ \langle \xi, X \rangle - \frac{1}{2} \|\xi\|^2 \right] G(M[T]), n \to \infty
\]

On the other hand,

\[
\mathbb{E} G\left( \int \exp \langle \xi, Y_n(t) \rangle M_{Y_n}(X,dt) \right) \to \mathbb{E} G\left( \int \exp \langle \xi, Y(t) \rangle M(dt) \right) \quad (49)
\]
Indeed, this follows from Lemma 47 applied to the families of randomized shifts \( \{Y_n\} := \{\xi\} \cup \{Z_n\} \), where \( \xi \) is identified with the vector-valued function \( * \mapsto \xi \) on the one-point set \( \{*\} \). Lemma 47 implies that \( \{\exp (\xi, Y_n) \cdot M_{Y_n}\} \) is USAC, so that we can apply Theorem 46 to the random measures \( M_{Y_n} \) and functions \( \exp (\xi, Y_n) \), implying (49). This proves the claim.

**Proof of Claim 53.** The strategy is to randomize the \( \xi \) in Claim 52 and thus approximate the randomized shift \( Y \). By applying (44) conditionally, we have for every measurable vector-valued function \( \xi : T \to H \)

\[
E_{P \otimes \mu} \exp \left[ \langle \xi (t), X \rangle - \frac{1}{2} \| \xi (t) \|^2 \right] G (M [T]) = E_{P \otimes \mu} G \left( \int \exp \langle \xi (t), Y (s) \rangle M (ds) \right)
\]

Take any increasing sequence \( (P_n) \) of finite-dimensional projections that converge strongly to 1. Since \( P_n \) has finite-dimensional range, \( P_n Y \) is in fact a vector-valued function. Therefore, we can take \( \xi (t) := P_n Y (t) \) above and obtain

\[
E_{P \otimes \mu} \exp \left[ \langle P_n Y (t), X \rangle - \frac{1}{2} \| P_n Y (t) \|^2 \right] G (M [T]) = E_{P \otimes \mu} G \left( \int \exp \langle P_n Y (t), Y (s) \rangle M (ds) \right)
\]

The function \( (t, s) \mapsto \langle P_n Y (t), Y (s) \rangle \) converges \( \mu \otimes \mu \)-almost everywhere to \( \langle Y (t), Y (s) \rangle \), so by Fatou’s lemma

\[
E_{P \otimes \mu} G \left( \int \exp \langle Y (t), Y (s) \rangle M (ds) \right) \leq \liminf_{n \to \infty} E_{P \otimes \mu} G \left( \int \exp \langle P_n Y (t), Y (s) \rangle M (ds) \right)
\]

On the other hand, \( t \) and \( M [T] \) are conditionally independent given \( X \), so

\[
E_{P \otimes \mu} \exp \left[ \langle P_n Y (t), X \rangle - \frac{1}{2} \| P_n Y (t) \|^2 \right] G (M [T]) =
\]

\[
= E \left[ \int \exp \left[ \langle P_n Y (t), X \rangle - \frac{1}{2} \| P_n Y (t) \|^2 \right] \mu (dt) \cdot G (M [T]) \right] =
\]

\[
= E M_{P_n Y} [T] G (M [T])
\]

On the other hand, by the martingale property (Theorem 24), \( M_{P_n Y} [T] = E [M_Y [T] | P_n X] \) is a uniformly integrable martingale that converges to \( M_Y [T] \), which, by Claim 50 coincides with \( E [M [T] | X] \). Thus

\[
\liminf_{n \to \infty} E_{P \otimes \mu} \exp \left[ \langle P_n Y (t), X \rangle - \frac{1}{2} \| P_n Y (t) \|^2 \right] G (M [T]) = E E [M [T] | X] G (M [T])
\]

This proves the claim. \( \square \)
8 An application to logarithmic kernels

Let \( T \subset \mathbb{R}^d \) be a bounded domain, let \( \mu \) be the Lebesgue measure on \( T \), and let \( K \) be a positive definite Hilbert-Schmidt kernel on \((T, \mu) \times (T, \mu)\), such that for some \( \delta > 0 \)

\[
K(t, s) \leq (2d - \delta) \log \|t - s\|^{-1} + O(1), \forall t, s \in T
\]  

(50)

Consider also a bounded function \( \psi \) on \( \mathbb{R}^d \) with compact support, such that \( \psi \geq 0 \), \( \int \psi(x) \, dx = 1 \), and denote \( \psi_\varepsilon(x) := \varepsilon^{-d} \psi(\varepsilon^{-1}x) \).

Theorem 54. Let \( X(t) := \langle X, Y(t) \rangle \) be a Gaussian field on \( T \) with covariance \( K \), and let

\[
X_\varepsilon(t) := \int_T X(t') \psi_\varepsilon(t - t') \, dt'
\]

\[
K_\varepsilon(t, s) := \mathbb{E} X_\varepsilon(t) X_\varepsilon(s)
\]

\[
M_\varepsilon(dt) := \exp \left[ X_\varepsilon(t) - \frac{1}{2} K_\varepsilon(t, t) \right]
\]

Then there exists a subcritical GMC \( M \) over the field \( X \) (with shift \( Y \)), and \( M_\varepsilon \to M \) in probability (the space of measures is equipped with the weak topology). This \( M \) does not depend on the function \( \psi \) used for approximation.

We will see that Theorem 54 follows from Theorem 43 once we have a way of verifying the uniform integrability assumption of the latter. This is done using classical results on existence of GMC for specific logarithmic kernels and the Kahane’s comparison inequality [5], stated below.

Theorem 55. Consider the following kernels on \( T \times T \):

\[
\tilde{K}_{C, \gamma}(t, s) := \gamma^2 \int_1^C e^{-u\|t - s\|} \frac{du}{u} = \gamma^2 \log \left( C \wedge \|t - s\|^{-1} \right) + O(1)
\]

Then for \( \gamma < \sqrt{2d} \) the family of GMCs with these kernels is uniformly integrable.

Theorem 56. Let \( X', X'' \) be Gaussian fields on \( T \) with continuous (or, more generally, trace class) covariance kernels \( K_1, K_2 \). Assume that

\[
\forall t, s : K_1(t, s) \leq K_2(t, s)
\]

Then for every convex function \( f : \mathbb{R} \to \mathbb{R} \)

\[
\mathbb{E} f \left( \int \exp \left[ X_1(t) - \frac{1}{2} K_1(t, t) \right] \, dt \right) \leq \mathbb{E} f \left( \int \exp \left[ X_2(t) - \frac{1}{2} K_2(t, t) \right] \, dt \right)
\]
Proof of Theorem 54. We may view the Gaussian field $X$ as the standard Gaussian in the Hilbert space $H := \text{RKHS}(K)$, with $Y(t) := K(t, \cdot)$. Denote by $Y_\varepsilon$ the approximation of $Y$:

$$Y_\varepsilon(t) := \int \! Y(t') \psi_\varepsilon(t - t') \, dt',$$

and note that indeed $X_\varepsilon(t) = \langle X, Y_\varepsilon(t) \rangle$.

All assumptions of Theorem 43 except uniform integrability are quite trivial to check. Indeed, for every $\xi \in H$ we can write

$$\langle \xi, Y_\varepsilon(t) \rangle = \xi_\varepsilon(t) := \int \! \xi(t') \psi_\varepsilon(t - t') \, dt',$$

so $\xi_\varepsilon$ is the restriction to $T$ of $\xi \ast \psi_\varepsilon$. Obviously, $\xi_\varepsilon \to \xi$ in $L^0(T)$. The same reasoning works for the kernels:

$$K_\varepsilon(t, s) = \int \! K(t', s') \psi_\varepsilon(t - t') \psi_\varepsilon(s - s') \, dt' ds',$$

so $K_\varepsilon \to K$ in $L^0(T \times T)$.

To verify uniform integrability we use Kahane’s inequality and a reference family of kernels for which uniform integrability is already known by other means.

It is straightforward to deduce from (50) that there exists a constant $C_0$, such that for every $\varepsilon > 0$ there exists $C(\varepsilon)$, such that

$$\forall t, s : K_\varepsilon(t, s) \leq \tilde{K}_{C(\varepsilon), \gamma}(t, s) + C_0,$$

where $\gamma = \sqrt{2d - \delta}$. Now by Theorem 55 GMCs with kernels $\tilde{K}_{C(\varepsilon), \gamma}(t, s)$, and thus also $\tilde{K}_{C(\varepsilon), \gamma}(t, s) + C_0$, are uniformly integrable. Therefore, by de la Vallee Poussin’s theorem and Kahane’s inequality, GMCs with kernels $K_\varepsilon(t, s)$ are also uniformly integrable. Therefore, all assumptions of Theorem 43 are verified, and we have convergence $M_\varepsilon \overset{L^0}{\to} M$. That $M$ does not depend on the approximation follows from our uniqueness argument (Corollary 50). \hfill \Box

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A Cameron-Martin shifts

Here we collect some standard facts concerning the deterministic (aka Cameron-Martin) shifts of Gaussian measures that we need throughout the paper. We refer to [4, Theorem 14.1].

Theorem 57. Let $X$ be a standard Gaussian vector in $H$. Then for every $\xi \in H$ the distribution of $X + \xi$ is equivalent to that of $X$ with density $\exp \left[ \langle X, \xi \rangle - \frac{1}{2} \|\xi\|^2 \right]$. That is, for every bounded random variable $f(X)$, measurable w.r.t. $X$, the random variable $f(X + \xi)$ is well-defined, and

$$
\mathbb{E} f(X + \xi) = \mathbb{E} f(X) \exp \left[ \langle X, \xi \rangle - \frac{1}{2} \|\xi\|^2 \right]
$$

The fact that $\text{Law}[X + \xi]$ is equivalent to $\text{Law} X$ can be stated in a different language: $H$ acts on $(\Omega, \sigma(X))$ by measure type preserving transformations. It is known that this shift action is ergodic, i.e. shift-invariant random variables are constant. More generally,

Theorem 58. Let $H' \subset H$ be a subspace (closed or not). A random variable $f(X)$ is invariant under shifts by $H'$:

$$
\forall \xi \in H': f(X) = f(X + \xi) \ a.s.
$$

if and only if $f(X)$ is measurable w.r.t. the orthogonal projection of $X$ onto $H'^\perp$.

A measure type preserving action of a group on a measure type space is the same thing as a representation of this group by automorphisms of the algebra $L^0$ over that space. It is useful to know that in our case this representation enjoys a strong continuity property:

Theorem 59. The map

$$
L^0(\Omega, \sigma(X)) \times H \to L^0(\Omega, \sigma(X))
$$

$$(f(X), \xi) \mapsto f(X + \xi)$$

is continuous.

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