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Non-Inertial Frames in Special and General Relativity

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Abstract

There is a review of what is known about global non-inertial frames in special and general relativity.

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I. INTRODUCTION

The aim of this contribution is to clarify what is known about non-inertial frames in special (SR) and general (GR) relativity. This topic is rarely discussed and till recently there was no attempt to develop a consistent general theory. All the results of the standard model of elementary particles are defined in the inertial frames of Minkowski space-time. Only at the level of neutron, atomic and molecular physics one needs a local study of non-inertial frames in SR, for instance the rotating ones for the Sagnac effect.

Moreover, relativistic metrology [1] and space physics around the Earth and in the Solar System [2] must take into account the gravitational field and the Post-Newtonian (PN) limit of GR, a theory in which global inertial frames are forbidden by the equivalence principle. In Einstein GR the gauge group of its Lagrangian formulation, the diffeomorphism group, implies that the 4-coordinates of the space-time (and therefore the local non-inertial frames) are \textit{gauge variables}. As a consequence, one would like to describe the effects of the physical degrees of freedom of the gravitational field by means of 4-scalars. This is an open theoretical problem. The praxis of experimentalists, who do not know which is the correct formulation of GR among the existing ones, is completely different.

Inside the Solar System the experimental localization of macroscopic classical objects is unavoidably done by choosing some \textit{convention} for the local 4-coordinates of space-time. Atomic physicists, NASA engineers and astronomers have chosen a series of reference frames and standards of time and length suitable for the existing technology [1, 2]. These conventions determine certain Post-Minkowskian (PM) 4-coordinate systems of an asymptotically Minkowskian space-time, in which the instantaneous 3-spaces are not strictly Euclidean. Then these reference frames are seen as a local approximation of a Celestial Reference Frame (ICRS), where however the space-time has become a cosmological Friedman-Robertson-Walker (FRW) one, which is only conformally asymptotically Minkowskian at spatial infinity. A search of a consistent patching of the 4-coordinates from inside the Solar System to the rest of the universe will start when the data from the future GAIA mission [3] for the cartography of the Milky Way will be available. This will allow a PM definition of a Galactic Reference System containing at least our galaxy. Let us remark that notwithstanding the FRW instantaneous 3-spaces are not strictly Euclidean, all the books on galaxy dynamics describe the galaxies by means of Kepler theory in Galilei space-time.

A well posed formulation of a PM ICRS (a global non-inertial frame for the 3-universe) would also be needed to face the main open problem of astrophysics, namely the dominance of \textit{dark} entities, the dark matter and the dark energy, in the existing description of the universe given by the standard $\Lambda$CDM cosmological model [4] based on the cosmological principle (homogeneity and isotropy of the space-time), which selects the class of Friedmann-Robertson-Walker (FWR) space-times. After the transition from quantum cosmology to classical astrophysics, with the Heisenberg cut roughly located at a suitable cosmic time ($\approx 10^5$ years after the big bang) and at the recombination surface identified by the cosmic microwave background (CMB), one has a description of the universe in which the known forms of baryonic matter and radiation contribute only with a few percents of the global budget. One has a great variety of models trying to explain the composition of the universe in accelerated expansion (based on data on high red-shift supernovae, galaxy clusters and CMB): WIMPS (mainly super-symmetric particles), $f(R)$ modifications of Einstein gravity (with a modified Newton potential), MOND (with a modification of Newton law),... for dark
matter; cosmological constant, string theory, back-reaction (spatial averages, non-linearity of Einstein equations), inhomogeneous space-times (Lemaître-Tolman-Bondi, Szekeres), scalar fields (quintessence, k-essence, phantom), fluids (Chaplygin fluid), ... for dark energy.

A PM ICRS would allow to interpret the astronomical data (luminosity, light spectrum, angles) on the 2-dimensional sky vault in a more realistic way (taking into account the inhomogeneities in the 3-universe) than in the nearly flat 3-spaces (as required by CMB data) of FWR space-times. In particular one needs new standards of time and length like the cosmic time and the luminosity distance extending the standard relativistic metrology inside the Solar System.

All these open problems justify the following description of what is known about non-inertial frames in SR and GR.

II. NON-INERTIAL FRAMES IN SPECIAL RELATIVITY

In non-relativistic (NR) Newtonian physics isolated systems are described in Galilei space-time, where both time and the instantaneous Euclidean 3-spaces are absolute quantities. As a consequence, the transition from the description of the system in NR inertial frames to its description in rigid non-inertial frames can be done by defining the following 3-coordinate transformation

\[ x^i = y^i(t) + \sigma^r R_{ri}(t). \]  

Here \( x^i \)'s are inertial Cartesian 3-coordinates centered on an inertial observer, while \( \sigma^r \)'s are rigid non-inertial 3-coordinates centered on an arbitrary observer whose trajectory is described by the Cartesian 3-coordinates \( y^i(t) \) in the inertial frame. This accelerated observer has a 3-velocity, which can be conveniently written in the form \( v^i(t) = R_{ij}(t) \frac{dy^j(t)}{dt} \). \( R(t) \) is a time-dependent rotation matrix \( (R^{-1} = R^T) \), which can be parametrized with three time-dependent Euler angles. The angular velocity of the rotating frame is \( \omega^i(t) = \frac{1}{2} \epsilon^{ijk} \Omega_{jk}(t) \) with \( \Omega_{jk}(t) = -\Omega_{kj}(t) = \left( \frac{dR(t)}{dt} R^T(t) \right)_{jk} \).

A particle of mass \( m_o \) with inertial Cartesian 3-coordinates \( x_o^i(t) \) is described in the non-inertial frame by 3-coordinates \( \eta^r(t) \) such that

\[ x_o^i(t) = y^i(t) + \eta^r(t) R_{ri}(t). \]

As shown in every book on Newtonian mechanics a particle satisfying the equation of motion \( m_o \frac{d^2 \bar{x}_o(t)}{dt^2} = -\frac{\partial V(t, x_o^i(t))}{\partial x_o^i} \), if an external potential \( V(t, x_o^i(t)) = \bar{V}(t, \eta^r(t)) \) is present, will satisfy the following equation of motion in the rigid non-inertial frame

\[ m_o \frac{d^2 \bar{\eta}(t)}{dt^2} = -\frac{\partial \bar{V}(t, \eta^r(t))}{\partial \bar{\eta}_o} - m_o \left[ \frac{d\bar{\omega}(t)}{dt} + \bar{\omega}(t) \times \bar{v}(t) \right. \]

\[ + 2 \bar{\omega}(t) \times \frac{d\bar{\eta}(t)}{dt} + \bar{\omega}(t) \times \left[ \bar{\omega}(t) \times \bar{\eta}(t) \right]. \]
where the standard Euler, Jacobi, Coriolis and centrifugal \textit{inertial forces} associated with the linear acceleration of the non-inertial observer and with the angular velocity of the rotating frame are present.

In Ref.[5] there is the extension to non-rigid non-inertial frames in which Eq.(2.1) is replaced by $x^i = A^i(t, \sigma^r)$ with $A^i$ arbitrary functions well behaved at spatial infinity. For instance, a differentially rotating non-inertial frame is described by Eq.(2.1) with a point-dependent rotation matrix $R(t, \sigma^r)$.

To go from NR inertial frames to the non-rigid non-inertial ones one has to replace the group of Galilei transformations, connecting the NR inertial frames, with some subgroup of the group of 3-diffeomorphisms of the Euclidean 3-space.

The transition to SR is highly non trivial because, due to the Lorentz signature of Minkowski space-time, time is no more absolute and there is no notion of instantaneous 3-space: the only intrinsic structure is the conformal one, i.e. the light-cone as the locus of incoming and outgoing radiation. A \textit{convention on the synchronization of clocks is needed to define an instantaneous 3-space}. For instance the 1-way velocity of light from one observer A to an observer B has a meaning only after a choice of a convention for synchronizing the clock in A with the one in B. Therefore the crucial quantity in SR is the 2-way (or round trip) velocity of light $c$ involving only one clock. It is this velocity which is isotropic and constant in SR and replaces the standard of length in relativistic metrology

Einstein convention for the synchronization of clocks in Minkowski space-time uses the 2-way velocity of light to identify the Euclidean 3-spaces of the inertial frames centered on an inertial observer A by means of only its clock. The inertial observer A sends a ray of light at $x_i^o$ towards the (in general accelerated) observer B; the ray is reflected towards A at a point P of B world-line and then reabsorbed by A at $x^o_j$; by convention P is synchronous with the mid-point between emission and absorption on A’s world-line, i.e. $x_P^o = x_i^o + \frac{1}{2} (x_j^o - x_i^o) = \frac{1}{2} (x_i^o + x_j^o)$. This convention selects the Euclidean instantaneous 3-spaces $x^o = ct = \text{const.}$ of the inertial frames centered on A. Only in this case the one-way velocity of light between A and B coincides with the two-way one, $c$. However if the observer A is accelerated, the convention can breaks down due the possible appearance of coordinate singularities.

The existing coordinatizations, like either Fermi or Riemann-normal coordinates, hold only locally They are based on the \textit{1+3 point of view}, in which only the world-line of a time-like observer is given. In each point of the world-line the observer 4-velocity determines an orthogonal 3-dimensional space-like tangent hyper-plane, which is identified with an instantaneous 3-space. However, these tangent planes intersect at a certain distance from the world-line (the so-called acceleration length depending upon the 4-acceleration of the observer [6]), where 4-coordinates of the Fermi type develop a coordinate singularity. Another type of coordinate singularity is developed in rigidly rotating coordinate systems at a distance $r$ from the rotation axis where $\omega r = c$ ($\omega$ is the angular velocity and $c$ the two-way velocity of light). This is the so-called “horizon problem of the rotating disk”: a time-like 4-velocity becomes a null vector at $\omega r = c$, like it happens on the horizon of a black-hole.

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1 See Ref.[1] for an updated review on relativistic metrology on Earth and in the Solar System.
See Ref.[7] for a classification of the possible pathologies of non-inertial frames and on how to avoid them.

As a consequence, a theory of global non-inertial frames in Minkowski space-time has to be developed in a metrology-oriented way to overcame the pathologies of the 1+3 point of view. This has been done in the papers of Ref.[7] by using the 3+1 point of view in which, besides the world-line of a time-like observer, one gives a global nice foliation of the space-time with instantaneous 3-spaces.

Assume that the world-line $x^\mu(\tau)$ of an arbitrary time-like observer carrying a standard atomic clock is given: $\tau$ is an arbitrary monotonically increasing function of the proper time of this clock. Then one gives an admissible 3+1 splitting of Minkowski space-time, namely a nice foliation with space-like instantaneous 3-spaces $\Sigma_\tau$. It is the mathematical idealization of a protocol for clock synchronization: all the clocks in the points of $\Sigma_\tau$ sign the same time of the atomic clock of the observer. On each 3-space $\Sigma_\tau$ one chooses curvilinear 3-coordinates $z^\nu$ having the observer as origin. These are the Lorentz-scalar and observer-dependent radar 4-coordinates $\sigma^A = (\tau; \sigma^r)$, first introduced by Bondi [8].

If $x^\mu \mapsto \sigma^A(x)$ is the coordinate transformation from the Cartesian 4-coordinates $x^\mu$ of a reference inertial observer to radar coordinates, its inverse $\sigma^A \mapsto x^\mu = z^\mu(\tau, \sigma^r)$ defines the embedding functions $z^\mu(\tau, \sigma^r)$ describing the 3-spaces $\Sigma_\tau$ as embedded 3-manifold into Minkowski space-time. The induced 4-metric on $\Sigma_\tau$ is the following functional of the embedding $g_{AB}(\tau, \sigma^r) = [z^A_\mu \eta_{\mu\nu} z^B_\nu](\tau, \sigma^r)$, where $z^A_\mu = \partial z^\mu / \partial \sigma^A$ and $\eta_{\mu\nu} = \epsilon (+ - - -)$ is the flat metric. While the 4-vectors $z^\mu(\tau, \sigma^u)$ are tangent to $\Sigma_\tau$, so that the unit normal $l^\mu(\tau, \sigma^u)$ is proportional to $\epsilon^\mu_{\alpha\beta\gamma} [z^\alpha_1 z^\beta_2 z^\gamma_3](\tau, \sigma^u)$, one has $z^\mu_\nu(\tau, \sigma^r) = [N l^\mu + N^r z^\mu_r](\tau, \sigma^r)$ with $N(\tau, \sigma^r) = \epsilon [z^\mu_\nu l^\nu](\tau, \sigma^r) = 1 + n(\tau, \sigma^r)$ and $N^r(\tau, \sigma^r) = -\epsilon g^r_r(\tau, \sigma^r)$ being the lapse and shift functions respectively.

As a consequence, the components of the 4-metric $g_{AB}(\tau, \sigma^r)$ have the following expression

$$
\epsilon^4 g_{\tau\tau} = N^2 - N_\tau N^\tau, \quad -\epsilon^4 g_{r\tau} = N_\tau = 3 g_{rs} N^s,
$$

$$
3 g_{rs} = -\epsilon^4 g_{rs} = \sum_{a=1}^3 3 e_{(a)r}^3 e_{(a)s} = \bar{\phi}^{2/3} \sum_{a=1}^3 \epsilon^2 \Sigma_{b=1}^3 \gamma_{ba} R_b V_{ra}(\theta^i) V_{sa}(\theta^i), \quad (2.4)
$$

where $e_{(a)r}(\tau, \sigma^u)$ are cotriads on $\Sigma_\tau$, $\bar{\phi}^2(\tau, \sigma^r) = det 3 g_{rs}(\tau, \sigma^r)$ is the 3-volume element on $\Sigma_\tau$, $\lambda_0(\tau, \sigma^r) = [\bar{\omega}^{1/3} e^2 \Sigma_{b=1}^3 \gamma_{ba} R_b](\tau, \sigma^r)$ are the positive eigenvalues of the 3-metric ($\gamma_{ba}$ are suitable numerical constants) and $V(\theta^i(\tau, \sigma^r))$ are diagonalizing rotation matrices depending on three Euler angles.

$^2$ It is the non-factual idealization required by the Cauchy problem generalizing the existing protocols for building coordinate system inside the future light-cone of a time-like observer.

$^3$ $\epsilon = \pm 1$ according to either the particle physics $\epsilon = 1$ or the general relativity $\epsilon = -1$ convention.
Therefore starting from the four independent embedding functions \( z^\mu(\tau, \sigma^r) \) one obtains the ten components \( 4g_{AB} \) of the 4-metric (or the quantities \( N, N_r, \phi, R\dot{a}, \theta^i \)), which play the role of the inertial potentials generating the relativistic apparent forces in the non-inertial frame. It can be shown [7] that the usual NR Newtonian inertial potentials are hidden in the functions \( N, N_r \) and \( \theta^i \). The extrinsic curvature tensor
\[
3K_{rs}(\tau, \sigma^u) = \left[ \frac{1}{2N} (N_r|s + N_s|r - \partial_r \tau, \sigma^u) \right]_{(\tau, \sigma^u)} ,
\]
describing the shape of the instantaneous 3-spaces of the non-inertial frame as embedded 3-sub-manifolds of Minkowski space-time, is a secondary inertial potential, functional of the ten inertial potentials \( 4g_{AB} \). Now a relativistic positive-energy scalar particle with world-line \( x^\mu_0(\tau) \) is described by 3-coordinates \( \eta^r(\tau) \) defined by \( x^\mu_0(\tau) = z^\mu(\tau, \eta^r(\tau)) \), satisfying equations of motion containing relativistic inertial forces whose non-relativistic limit reproduces Eq.(2.3) as shown in Ref.[5, 7].

The foliation is nice and admissible if it satisfies the conditions:

1) \( N(\tau, \sigma^r) > 0 \) in every point of \( \Sigma_\tau \) so that the 3-spaces never intersect, avoiding the coordinate singularity of Fermi coordinates;

2) \( \epsilon^4 g_{\tau\tau}(\tau, \sigma^r) > 0 \), so to avoid the coordinate singularity of the rotating disk, and with the positive-definite 3-metric \( 3g_{rs}(\tau, \sigma^u) = -\epsilon^4 g_{rs}(\tau, \sigma^u) \) having three positive eigenvalues (these are the Møller conditions [9]);

3) all the 3-spaces \( \Sigma_\tau \) must tend to the same space-like hyper-plane at spatial infinity with a unit normal \( \epsilon^\mu_r \), which is the time-like 4-vector of a set of asymptotic ortho-normal tetrads \( \epsilon^A_\mu \). These tetrads are carried by asymptotic inertial observers and the spatial axes \( \epsilon^\mu_r \) are identified by the fixed stars of star catalogues. At spatial infinity the lapse function tends to 1 and the shift functions vanish.

By using the asymptotic tetrads \( \epsilon^\mu_A \) one can give the following parametrization of the embedding functions
\[
\begin{align*}
  z^\mu(\tau, \sigma^r) &= x^\mu(\tau) + \epsilon^{\mu}_A F^A(\tau, \sigma^r), \quad F^A(\tau, 0) = 0, \\
  x^\mu(\tau) &= x^\mu_0 + \epsilon^{\mu}_A f^A(\tau),
\end{align*}
\]
where \( x^\mu(\tau) \) is the world-line of the observer. The functions \( f^A(\tau) \) determine the 4-velocity \( u^\mu(\tau) = \dot{x}^\mu(\tau)/\sqrt{\epsilon \dot{x}^2(\tau)} \) \( (\dot{x}^\mu(\tau) = \frac{dx^\mu(\tau)}{d\tau}) \) and the 4-acceleration \( a^\mu(\tau) = \frac{du^\mu(\tau)}{d\tau} \) of the observer.

The Møller conditions are non-linear differential conditions on the functions \( f^A(\tau) \) and \( F^A(\tau, \sigma^r) \), so that it is very difficult to construct explicit examples of admissible 3+1 splittings. When these conditions are satisfied Eqs.(2.5) describe a global non-inertial frame in Minkowski space-time.

Till now the solution of Møller conditions is known in the following two cases in which the instantaneous 3-spaces are parallel Euclidean space-like hyper-planes not equally spaced due to a linear acceleration.

A) **Rigid non-inertial reference frames with translational acceleration.** An example are the following embeddings
\[
  z^\mu(\tau, \sigma^u) = x^\mu_0 + \epsilon_\tau^\mu f(\tau) + \epsilon^\mu_\sigma r
\]

\[
g_{\tau\tau}(\tau, \sigma^u) = \epsilon \left( \frac{df(\tau)}{d\tau} \right)^2, \quad g_{\sigma\sigma}(\tau, \sigma^u) = 0, \quad g_{rs}(\tau, \sigma^u) = -\epsilon \delta_{rs}.
\]

(2.6)

This is a foliation with parallel hyper-planes with normal \( l^\mu = \epsilon_\tau^\mu = const. \) and with the time-like observer \( x^\mu(\tau) = x^\mu_0 + \epsilon_\tau^\mu f(\tau) \) as origin of the 3-coordinates. The hyper-planes have translational acceleration \( \ddot{x}^\mu(\tau) = \epsilon_\tau^\mu \dot{f}(\tau) \), so that they are not uniformly distributed like in the inertial case \( f(\tau) = \tau \).

B) **Differentially rotating non-inertial frames** without the coordinate singularity of the rotating disk. The embedding defining this frames is

\[
z^\mu(\tau, \sigma^u) = x^\mu(\tau) + \epsilon^\mu_\tau R^\tau_s(\tau, \sigma) \sigma^s \rightarrow_{\sigma \rightarrow \infty} x^\mu(\tau) + \epsilon^\mu_\sigma r,
\]

\[
R^\tau_s(\tau, \sigma) = R^\tau_s(\alpha_i(\tau, \sigma)) = R^\tau_s(F(\sigma) \tilde{\alpha}_i(\tau)),
\]

\[
0 < F(\sigma) < \frac{1}{A \sigma^2}, \quad \frac{dF(\sigma)}{d\sigma} \neq 0 (Moller conditions),
\]

\[
z^\mu(\tau, \sigma^u) = \dot{x}^\mu(\tau) - \epsilon^\mu_\tau R^\tau_s(\tau, \sigma) \delta^s_{\mu \nu} \epsilon_{\nu \mu \nu} \sigma^u \frac{\Omega^\nu(\tau, \sigma)}{c},
\]

\[
z^\mu(\tau, \sigma^u) = \epsilon^\mu_k R^k_v(\tau, \sigma) \left( \delta^v_{\nu} + \Omega^v_{(\nu)}(\tau, \sigma) \sigma^u \right),
\]

(2.7)

where \( \sigma = |\tilde{\sigma}| \) and \( R^\tau_s(\alpha_i(\tau, \sigma)) \) is a rotation matrix satisfying the asymptotic conditions \( R^\tau_s(\tau, \sigma) \rightarrow_{\sigma \rightarrow \infty} \delta^\tau_s, \quad \partial_\tau R^\tau_s(\tau, \sigma) \rightarrow_{\sigma \rightarrow \infty} 0 \), whose Euler angles have the expression \( \alpha_i(\tau, \tilde{\sigma}) = F(\sigma) \tilde{\alpha}_i(\tau), \quad i = 1, 2, 3 \). The unit normal is \( l^\mu = \epsilon_\tau^\mu = const. \) and the lapse function is \( 1 + n(\tau, \sigma^u) = \epsilon \left( z^\mu l_\mu \right)(\tau, \sigma^u) = \epsilon \epsilon^\mu_\tau \dot{x}_\mu(\tau) > 0 \). In Eq.(2.7) one uses the notations \( \Omega^{(v)}(\tau, \sigma) = R^{-1}(\tau, \tilde{\sigma}) \partial_\tau R(\tau, \sigma) \) and \( \left( R^{-1}(\tau, \sigma) \partial_\tau R(\tau, \sigma) \right)^{\nu} = \delta^{uw} \epsilon_{\mu \nu \nu} \frac{\Omega^\mu(\tau, \sigma)}{c} \), with \( \Omega^\nu(\tau, \sigma) = F(\sigma) \tilde{\Omega}(\tau, \sigma) \dot{\alpha}(\tau, \sigma) \) \( \hat{\alpha}(\tau, \sigma) \) being the angular velocity. The angular velocity vanishes at spatial infinity and has an upper bound proportional to the minimum of the linear velocity \( v_i(\tau) = \dot{x}_\mu l^\mu \) orthogonal to the space-like hyper-planes. When the rotation axis is fixed and \( \tilde{\Omega}(\tau, \sigma) = \omega = const. \), a simple choice for the function \( F(\sigma) \) is \( F(\sigma) = \frac{1}{1+\frac{\sigma^2}{c^2}} \).

To evaluate the non-relativistic limit for \( c \rightarrow \infty \), where \( \tau = ct \) with \( t \) the absolute Newtonian time, one chooses the gauge function \( F(\sigma) = \frac{1}{1+\frac{\sigma^2}{c^2}} \rightarrow_{c \rightarrow \infty} 1 - \frac{\sigma^2}{c^2} + O(c^{-4}) \).

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\(^4\) \( \dot{\alpha}(\tau, \sigma) \) defines the instantaneous rotation axis and \( 0 < \tilde{\Omega}(\tau, \sigma) < 2 \max \left( \dot{\alpha}(\tau, \sigma), \dot{\beta}(\tau, \sigma), \dot{\gamma}(\tau, \sigma) \right) \).

\(^5\) Nearly rigid rotating systems, like a rotating disk of radius \( \sigma_o \), can be described by using a function \( F(\sigma) \) approximating the step function \( \theta(\sigma - \sigma_o) \).
This implies that the corrections to rigidly-rotating non-inertial frames coming from Møller conditions are of order $O(c^{-2})$ and become important at the distance from the rotation axis where the horizon problem for rigid rotations appears.

As shown in the first paper in Refs.[7], global rigid rotations are forbidden in relativistic theories, because, if one uses the embedding $z^a(\tau, \sigma^u) = x^a(\tau) + \epsilon^a_r \, R^r_s(\tau) \sigma^s$ describing a global rigid rotation with angular velocity $\Omega^r = \Omega^r(\tau)$, then the resulting $g_{\tau\tau}(\tau, \sigma^u)$ violates Møller conditions, because it vanishes at $\sigma = \sigma_R = \frac{1}{\Omega(\tau)} \left[ \sqrt{\dot{x}^2(\tau) + [\dot{x}_\mu(\tau) \epsilon^a_r \, R^r_s(\tau)(\hat{\sigma} \times \hat{\Omega}(\tau))]}^2 - \dot{x}_\mu(\tau) \epsilon^a_r \, R^r_s(\tau)(\hat{\sigma} \times \hat{\Omega}(\tau)) \right]$ ($\sigma^u = \sigma \hat{\sigma}^u$, $\Omega^r = \Omega \hat{\Omega}^r$, $\hat{\sigma}^2 = \hat{\Omega}^2 = 1$). At this distance from the rotation axis the tangential rotational velocity becomes equal to the velocity of light. This is the horizon problem of the rotating disk (the horizon is often named the light cylinder). Let us remark that even if in the existing theory of rotating relativistic stars [10] one uses differential rotations, notwithstanding that in the study of the magnetosphere of pulsars often the notion of light cylinder is still used.

The search of admissible 3+1 splittings with non-Euclidean 3-spaces is much more difficult. The simplest case is the following parametrization of the embeddings (2.4) in terms of Lorentz matrices $\Lambda_A^B(\tau, \sigma) \rightarrow_{\sigma \rightarrow \infty} \delta^A_B$ with $\Lambda^A_B(\tau, 0)$ finite. The Lorentz matrix is written in the form $\Lambda = \mathcal{B} \mathcal{R}$ as the product of a boost $\mathcal{B}(\tau, \sigma)$ and a rotation $\mathcal{R}(\tau, \sigma)$ like the one in Eq.(2.7) ($\mathcal{R}_r^r = 1$, $\mathcal{R}_r^s = 0$, $\mathcal{R}_s^s = \mathcal{R}_r^r$). The components of the boost are $\mathcal{B}_{\tau\tau}(\tau, \sigma) = \gamma(\tau, \sigma) = \frac{1}{\sqrt{1 - \hat{\beta}^2(\tau, \sigma)}}$, $\mathcal{B}_{r\tau}(\tau, \sigma) = \gamma(\tau, \sigma) \beta_r(\tau, \sigma)$, $\mathcal{B}_{s\tau}(\tau, \sigma) = \delta_s^r + \frac{\gamma^r_s \beta_s(\tau, \sigma)}{1 + \gamma}$, with $\beta_r(\tau, \sigma) = G(\sigma) \beta^r(\tau)$, where $\beta^r(\tau)$ is defined by the 4-velocity of the observer $u^\mu(\tau) = \epsilon^\mu_r \beta^A(\tau)/\sqrt{1 - \hat{\beta}^2(\tau)}$, $\beta^A(\tau) = (1; \beta^r(\tau))$. The Møller conditions are restrictions on $G(\sigma) \rightarrow_{\sigma \rightarrow \infty} 0$ with $G(0)$ finite, whose explicit form is still under investigation.

The embedding (2.7) has been studied in details in Ref.[11] for the development of quantum mechanics in non-inertial frames.

See the second paper of Ref.[7] for the description of the electro-magnetic field and of phenomena like the Sagnac effect and the Faraday rotation in this framework for non-inertial frames.

The previous approach based on the 3+1 point of view has allowed a complete reformulation of relativistic particle mechanics in SR [7, 12, 13]. By means of parametrized Minkowski theories [12, 7], one can get the description of arbitrary isolated systems (particles, strings, fluids, fields) admitting a Lagrangian formulation in arbitrary non-inertial frames. To get it the Lagrangian is coupled to an external gravitational field and then the gravitational 4-metric is replaced with the 4-metric $\gamma_{AB}(\tau, \sigma^r)$, a functional of the embedding $z^a(\tau, \sigma^r)$, induced by an admissible 3+1 splitting of Minkowski space-time. The new Lagrangian, a

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6 It corresponds to the locality hypothesis of Ref.[6], according to which at each instant of time the detectors of an accelerated observer give the same indications as the detectors of the instantaneously comoving inertial observer.

7 See Ref.[5] for the definition of parametrized Galilei theories in NR mechanics.
function of the matter and of the embedding, is invariant under the frame-preserving diffeomorphisms of Ref.[14] 8. This kind of general covariance implies that the embeddings are \textit{gauge variables}, so that the transition among non-inertial frames is described as a \textit{gauge transformation}: only the appearances change, not the physics.

This framework allows us to define the \textit{inertial and non-inertial rest frames} of the isolated systems, where to develop the rest-frame instant form of the dynamics and to build the explicit form of the Lorentz boosts for interacting systems. While the inertial rest frames have their Euclidean 3-spaces defined as space-like 3-manifolds of Minkowski space-time orthogonal to the conserved 4-momentum of the isolated system, the non-inertial rest frames are admissible non-inertial frames whose 3-spaces tend to those of some inertial rest frame at spatial infinity, where the 3-space becomes orthogonal to the conserved 4-momentum.

This makes possible to study the problem of the relativistic center of mass with the associated external and internal (i.e. inside the 3-space) realizations of the Poincaré algebra[15], relativistic bound states [16–18], relativistic kinetic theory and relativistic micro-canonical ensemble [19] and various other systems [20, 21]. Moreover a Wigner-covariant relativistic quantum mechanics [22], with a solution of all the known problems introduced by SR, has been developed after some preliminary work done in Ref.[11]. This allows the beginning of the study of relativistic entanglement taking into account all the consequences of the Lorentz signature of Minkowski space-time. As shown in Ref.[22] in SR the relativistic center of mass is a non-local non-measurable quantity: only relative variables have an operational meaning and this implies a spatial non-separability, i.e. some form of weak relationism in which all the objects know each other differently from the non-relativistic case where the center of a mass is a measurable quantity.

See Ref.[23] for an extended review of this approach both in SR and in GR. In the next Section there will be a sketch of the known results in GR.

III. NON-INERTIAL FRAMES IN GENERAL RELATIVITY

In GR global inertial frames are forbidden by the equivalence principle. Therefore gravitational physics has to be described in non-inertial frames.

While in SR Minkowski space-time is an absolute notion, unifying the absolute notions of time and 3-space of the NR Galilei space-time, in Einstein GR also the space-time is a dynamical object [24] and the gravitational field is described by the metric structure of the space-time, namely by the ten dynamical fields \(4g_{\mu\nu}(x)\) (\(x^\mu\) are world 4-coordinates) satisfying Einstein equations.

The ten dynamical fields \(4g_{\mu\nu}(x)\) are not only a (pre)potential for the gravitational field (like the electro-magnetic and Yang-Mills fields are the potentials for electro-magnetic and non-Abelian forces) but also determines the \textit{chrono-geometrical structure of space-time} through the line element \(ds^2 = 4g_{\mu\nu}dx^\mu dx^\nu\). Therefore the 4-metric teaches relativistic causality to the other fields: it says to massless particles like photons and gluons which are the allowed world-lines in each point of space-time. The ACES mission of ESA [25] will give

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8 This is the only paper known to us where here is an attempt to formulate a theory of non-inertial frames in SR.
the first precision measurement of the gravitational red-shift of the geoid, namely of the $1/c^2$ deformation of Minkowski light-cone caused by the geo-potential.

The metrology-oriented solution of the problem of clock synchronization used in SR can be extended to GR if Einstein space-times are restricted to the class of globally hyperbolic, topologically trivial, asymptotically Minkowskian space-times without super-translations.

As shown in the first paper of Ref. [26], in the chosen class of space-times the 4-metric $g_{\mu\nu}(x)$ tends in a suitable way to the flat Minkowski 4-metric $\eta_{\mu\nu}$ at spatial infinity and the ten strong asymptotic ADM Poincaré generators $P_{ADM}^A$, $J_{ADM}^{AB}$ (they are fluxes through a 2-surface at spatial infinity) are well defined functionals of the 4-metric fixed by the boundary conditions at spatial infinity.

These properties do not hold in generic asymptotically flat space-times, because they have the SPI group of asymptotic symmetries (direction-dependent asymptotic Killing symmetries) [27] and this is an obstruction to the existence of asymptotic Lorentz generators for the gravitational field [28]. However if one restricts the class of space-times to those not containing super-translations [29], then the SPI group reduces to the asymptotic ADM Poincaré group [30]: these space-times are asymptotically Minkowskian, they contain an asymptotic Minkowski 4-metric (to be used as an asymptotic background at spatial infinity in the linearization of the theory) and they have asymptotic inertial observers at spatial infinity whose spatial axes may be identified by means of the fixed stars of star catalogues [10]. Moreover, in the limit of vanishing Newton constant ($G = 0$) the asymptotic ADM Poincaré generators become the generators of the special relativistic Poincaré group describing the matter present in the space-time. This is an important condition for the inclusion into GR of the classical version of the standard model of particle physics, whose properties are all connected with the representations of this group in the inertial frames of Minkowski space-time. In absence of matter a sub-class of these space-times is the (singularity-free) family of Christodoulou-Klainermann solutions of Einstein equations [31] (they are near to Minkowski space-time in a norm sense and contain gravitational waves).

In the first paper of Ref. [26] it is also shown that the boundary conditions on the 4-metric required by the absence of super-translations imply that the only admissible 3+1 splittings of space-time (i.e. the allowed global non-inertial frames) are the non-inertial rest frames: their 3-spaces are asymptotically orthogonal to the weak ADM 4-momentum. Therefore one gets $\hat{P}_{ADM}^r \approx 0$ as the rest-frame condition of the 3-universe with a mass and a rest spin fixed by the boundary conditions. Like in SR the 3-universe can be visualized as a decoupled non-covariant (non-measurable) external relativistic center of mass plus an internal non-inertial rest-frame 3-space containing only relative variables (see the first paper in Ref. [32]).

In these space-times one can define global non-inertial frames by using the same admissible 3+1 splittings, centered on a time-like observer, and the observer-dependent radar 4-coordinates $\sigma^A = (\tau; \sigma^r)$ employed in SR. This will allow to separate the inertial (gauge)

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9 At this preliminary level these space-times must also be without Killing symmetries, because, otherwise, at the Hamiltonian level one should introduce complicated sets of extra Dirac constraints for each existing Killing vector.

10 The fixed stars can be considered as an empirical definition of spatial infinity of the observable universe.
degrees of freedom of the gravitational field (playing the role of inertial potentials) from the dynamical \textit{tidal} ones at the Hamiltonian level.

In GR the dynamical fields are the components $^{4}g_{\mu\nu}(x)$ of the 4-metric and not the embeddings $x^\mu = z^\mu(\tau, \sigma^r)$ defining the admissible 3+1 splittings of space-time like in the parametrized Minkowski theories of SR. Now the gradients $z^\mu_A(\tau, \sigma^r)$ of the embeddings give the transition coefficients from radar to world 4-coordinates, so that the components $^{4}g_{AB}(\tau, \sigma^r) = z^\mu_A(\tau, \sigma^r) z^\nu_B(\tau, \sigma^r) ^{4}g_{\mu\nu}(z(\tau, \sigma^r))$ of the 4-metric will be the dynamical fields in the ADM action. Like in SR the 4-vectors $z^\mu_A(\tau, \sigma^r)$, tangent to the 3-spaces $\Sigma_\tau$, are used to define the unit normal $l^\mu(\tau, \sigma^r) = z^\mu_A(\tau, \sigma^r) l_A^A(\tau, \sigma^r)$ to $\Sigma_\tau$, while the 4-vector $z^\mu_{\tau}(\tau, \sigma^r)$ has the lapse function as component along the unit normal and the shift functions as components along the tangent vectors.

Since the world-line of the time-like observer can be chosen as the origin of a set of the spatial world coordinates, i.e. $x^\mu(\tau) = (x^\sigma(\tau); 0)$, it turns out that with this choice the space-like surfaces of constant coordinate time $x^\sigma(\tau) = \text{const.}$ coincide with the dynamical instantaneous 3-spaces $\Sigma_\tau$ with $\tau = \text{const.}$. By using asymptotic flat tetrads $e^A_\mu = \delta^A_\mu \delta^r_A + \delta_i^A \delta^r_A$ (with $e^A_\mu$ denoting the inverse flat cotetrads) and by choosing a coordinate world time $x^\sigma(\tau) = x^\sigma_0 + \epsilon^\sigma_\tau = x^\sigma_0 + \tau$, one gets the following preferred embedding corresponding to these given world 4-coordinates

$$x^\mu = z^\mu(\tau, \sigma^r) = x^\mu(\tau) + \delta^\mu_\sigma x^\sigma_0 + \epsilon^\mu_\tau e^A_\mu.$$  \hspace{1cm} \text{(3.1)}

This choice implies $z^\mu_A(\tau, \sigma^r) = e^\mu_A$ and $^{4}g_{\mu\nu}(x = z(\tau, \sigma^r)) = \epsilon^A_\mu \epsilon^B_\nu g_{AB}(\tau, \sigma^r)$.

As shown in Ref.[24], the dynamical nature of space-time implies that each solution (i.e. an Einstein 4-geometry) of Einstein’s equations (or of the associated ADM Hamilton equations) dynamically selects a preferred 3+1 splitting of the space-time, namely in GR the instantaneous 3-spaces are dynamically determined in the chosen world coordinate system, modulo the choice of the 3-coordinates in the 3-space and modulo the trace of the extrinsic curvature of the 3-space as a space-like sub-manifold of the space-time. Eq.(3.1) can be used to describe this 3+1 splitting and then by means of 4-diffeomorphisms the solution can be written in an arbitrary world 4-coordinate system in general not adapted to the dynamical 3+1 splitting. This gives rise to the 4-geometry corresponding to the given solution.

To define the canonical formalism the Einstein-Hilbert action for metric gravity (depending on the second derivative of the metric) must be replaced with the ADM action (the two actions differ for a surface term at spatial infinity). As shown in the first paper of Refs.[26], the Legendre transform and the definition of a consistent canonical Hamiltonian require the introduction of the DeWitt surface term at spatial infinity: the final canonical Hamiltonian turns out to be the \textit{strong} ADM energy (a flux through a 2-surface at spatial infinity), which is equal to the \textit{weak} ADM energy (expressed as a volume integral over the 3-space) plus constraints.

Therefore there is not a frozen picture like in the ”spatially compact space-times without boundaries” used in loop quantum gravity \cite{11}, but an evolution generated by a Dirac Hamiltonian equal to the weak ADM energy plus a linear combination of the first class constraints at the Hamiltonian level.

\textsuperscript{11} In these space-times the canonical Hamiltonian vanishes and the Dirac Hamiltonian is a combination of first-class constraints, so that it only generates Hamiltonian gauge transformations. In the reduced
constraints. Also the other strong ADM Poincaré generators are replaced by their weakly equivalent weak form $P^A_{\text{ADM}}, J^A_{\text{ADM}}$.

To take into account the fermion fields present in the standard particle model one must extend ADM gravity to ADM tetrad gravity. Since our class of space-times admits orthonormal tetrads and a spinor structure [34], the extension can be done by simply replacing the 4-metric in the ADM action with its expression in terms of tetrad fields, considered as the basic 16 configurational variables substituting the 10 metric fields. This can be achieved by decomposing the 4-metric on cotetrad fields (by convention a sum on repeated indices is assumed)

$$4g_{AB}(\tau, \sigma^r) = E^A_\alpha(\tau, \sigma^r) \eta^{(\alpha)(\beta)} E^B_\beta(\tau, \sigma^r), \quad (3.2)$$

by putting this expression into the ADM action and by considering the resulting action, a functional of the 16 fields $E^A_\alpha(\tau, \sigma^r)$, as the action for ADM tetrad gravity. In Eq.(3.2) $(\alpha)$ are flat indices and the cotetrad fields $E^A_\alpha$ are the inverse of the tetrad fields $E^A_\mu$, which are connected to the world tetrad fields by $E^A_\mu(x) = z^A_\mu(\tau, \sigma^r) E^A_\alpha(z(\tau, \sigma^r))$ by the embedding of Eq.(3.1).

This leads to an interpretation of gravity based on a congruence of time-like observers endowed with orthonormal tetrads: in each point of space-time the time-like axis is the unit 4-velocity of the observer, while the spatial axes are a (gauge) convention for observer’s gyroscopes. This framework was developed in the second and third paper of Refs.[26].

Even if the action of ADM tetrad gravity depends upon 16 fields, the counting of the physical degrees of freedom of the gravitational field does not change, because this action is invariant not only under the group of 4-diffeomorphisms but also under the O(3,1) gauge group of the Newman-Penrose approach [35] (the extra gauge freedom acting on the tetrads in the tangent space of each point of space-time).

The cotetrads $E^A_\alpha(\tau, \sigma^r)$ are the new configuration variables. They are connected to cotetrads $4^\phi_A(\tau, \sigma^r)$ adapted to the 3+1 splitting of space-time, namely such that the inverse adapted time-like tetrad $4^\phi_A(\tau, \sigma^r)$ is the unit normal to the 3-space $\Sigma_\tau$, by a standard Wigner boosts for time-like Poincaré orbits with parameters $\varphi(a)(\tau, \sigma^r)$, $a = 1, 2, 3$

$$E^A_\alpha = L^{(\alpha)}_{(\beta)}(\varphi(a)) \phi^{(\beta)}_A, \quad 4g_{AB} = 4E^A_\alpha \eta^{(\alpha)(\beta)} 4E^B_\beta,$$

$$L^{(\alpha)}_{(\beta)}(\varphi(a)) = \frac{\partial}{\partial z^\alpha} \left[ L^{(\beta)}_{(\gamma)}(V(z(\sigma))) ; \phi^\gamma \right] = \delta^{(\alpha)}_{(\beta)} + 2\epsilon V^{(\alpha)}(z(\sigma)) \phi^{(\beta)}_A - \epsilon \frac{(V^{(\alpha)}(z(\sigma)) + \phi^{(\alpha)}_A)(V^{(\beta)}(z(\sigma)) + \phi^{(\beta)}_A)}{1 + V^{(\alpha)}(z(\sigma))}. \quad (3.3)$$

phase space, quotient with respect the Hamiltonian gauge group, the reduced Hamiltonian is zero and one has a frozen picture of dynamics. This class of space-times fits well with Machian ideas (no boundary conditions) and with interpretations in which there is no physical time like the one in Ref.[33]. However, it is not clear how to include in this framework the standard model of particle physics.
In each tangent plane to a point of $\Sigma_r$ this point-dependent standard Wigner boost sends the unit future-pointing time-like vector $V^{(a)} = 4 E_{A}^{(a)} l^A = (1; 0)$ into the unit time-like vector $V^{(a)} = 4 E_{A}^{(a)} l^A = \left( \frac{1}{\sqrt{1 + \sum_a \varphi^2(a)}}, -\varphi(a) \right)$. As a consequence, the flat indices $(a)$ of the adapted tetrads and cotetrads and of the triads and cotriads on $\Sigma_r$ transform as Wigner spin-$1$ indices under point-dependent $\text{SO}(3)$ Wigner rotations $R_{(a)(b)}(V(z(\sigma)) ; \Lambda(z(\sigma)))$ associated with Lorentz transformations $\Lambda^{(a)}(\beta)(z)$ in the tangent plane to the space-time in the given point of $\Sigma_r$. Instead the index $(o)$ of the adapted tetrads and cotetrads is a local Lorentz scalar index.

The adapted tetrads and cotetrads have the expression

$$4 \overset{\circ}{E}_{A}^{(a)} = \frac{1}{1 + n} (1 - \sum_a n_a \varphi^2(a)) = l^A, \quad 4 \overset{\circ}{E}_{A}^{(o)} = (0; 3 \varphi(o)),$$

$$4 \overset{\circ}{E}_{A}^{(a)} = (1 + n) (1; \vec{0}) = \epsilon l_A, \quad 4 \overset{\circ}{E}_{A}^{(a)} = (n_a; 3 \varphi(o)_r), \quad 4 \overset{\circ}{E}_{A}^{(a)} = (n_a; 3 \varphi(o)_r), \quad (3.4)$$

where $3 \varphi(o)_r$ and $3 \varphi(o)_r$ are triads and cotriads on $\Sigma_r$ and $n_a = n_r 3 \varphi(o)_r = n_r 3 \varphi(o)_r$ $^{12}$ are adapted shift functions. In Eqs.(3.4) $N(\tau, \vec{\sigma}) = 1 + n(\tau, \vec{\sigma}) > 0$, with $n(\tau, \vec{\sigma})$ vanishing at spatial infinity (absence of super-translations), so that $N(\tau, \vec{\sigma}) d\tau$ is positive from $\Sigma_r$ to $\Sigma_{r+dr}$, is the lapse function; $N^r(\tau, \vec{\sigma}) = n^r(\tau, \vec{\sigma})$, vanishing at spatial infinity (absence of super-translations), are the shift functions.

The adapted tetrads $4 \overset{\circ}{E}_{A}^{(a)}$ are defined modulo $\text{SO}(3)$ rotations $4 \overset{\circ}{E}_{A}^{(a)} = \sum_b R_{(a)(b)}(\alpha(c)) 4 \overset{\circ}{E}_{A}^{(a)}$, $3 \varphi(o)_r = \sum_b R_{(a)(b)}(\alpha(c)) 3 \varphi(o)_r$, where $\alpha(a)(\tau, \vec{\sigma})$ are three point-dependent Euler angles. After having chosen an arbitrary point-dependent origin $\alpha(a)(\tau, \vec{\sigma}) = 0$, one arrives at the following adapted tetrads and cotetrads $[\bar{n}_a = \sum_b n_b R_{(b)(a)}(\alpha(c)), \sum_a n_a 3 \varphi(o)_r = \sum_a \bar{n}_a 3 \varphi(o)_r]$

$$4 \overset{\circ}{E}_{A}^{(a)} = 4 \overset{\circ}{E}_{(o)} = \frac{1}{1 + n} (1 - \sum_a \bar{n}_a 3 \varphi(o)_r) = l^A, \quad 4 \overset{\circ}{E}_{A}^{(a)} = (0; 3 \varphi(o)_r),$$

$$4 \overset{\circ}{E}_{A}^{(o)} = 4 \overset{\circ}{E}_{A}^{(o)} = (1 + n) (1; \vec{0}) = \epsilon l_A, \quad 4 \overset{\circ}{E}_{A}^{(a)} = (n_a; 3 \varphi(o)_r), \quad (3.5)$$

which one will use as a reference standard.

The expression for the general tetrad

$$4 E_{A}^{(a)} = 4 \overset{\circ}{E}_{A}^{(a)} L^{(\beta)}(\varphi(a)) = 4 \overset{\circ}{E}_{A}^{(a)} L^{(o)}(\varphi(c)) +$$

$$+ \sum_{ab} 4 E_{(b)} L^{(a)}(\varphi(c)), \quad (3.6)$$

$^{12}$ Since one uses the positive-definite 3-metric $\delta_{(a)(b)}$, one will use only lower flat spatial indices. Therefore for the cotriads one uses the notation $3 \varphi(o)_r = 3 \varphi(o)_r$ with $\delta_{(a)(b)} = 3 \varphi(o)_r 3 \varphi(o)_r$. 

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shows that every point-dependent Lorentz transformation $\Lambda$ in the tangent planes may be parametrized with the (Wigner) boost parameters $\varphi_{(a)}$ and the Euler angles $\alpha_{(a)}$, being the product $\Lambda = RL$ of a rotation and a boost.

The future-oriented unit normal to $\Sigma_\tau$ and the projector on $\Sigma_\tau$ are $l_A = \epsilon (1 + n) \left(1; 0\right)$,

$$4 g^{AB} l_A l_B = \epsilon, \quad l^A = \epsilon (1 + n) 4 g^{Ar} = \frac{1}{1 + n} \left(1; -n^r\right) = \frac{1}{1 + n} \left(1; -\sum_a \tilde{n}_{(a)} 3 \tilde{e}_{(a)r}\right), \quad 3 h^B_A = \delta^B_A - \epsilon l_A l^B.$$

The 4-metric has the following expression

$$4 g_{\tau\tau} = \epsilon [(1 + n)^2 - 3 g^{rs} n_r n_s] = \epsilon (1 + n)^2 - \sum_a \tilde{n}_{(a)}^2,$$

$$4 g_{\tau r} = -\epsilon n_r = -\epsilon \sum_a \tilde{n}_{(a)} 3 \tilde{e}_{(a)r},$$

$$4 g_{rs} = -\epsilon^2 g_{rs} = -\epsilon \sum_a 3 \tilde{e}_{(a)r} 3 \tilde{e}_{(a)s} = -\epsilon \sum_a 3 \tilde{e}_{(a)r} 3 \tilde{e}_{(a)s},$$

$$4 g^{\tau\tau} = \frac{\epsilon}{(1 + n)^2}, \quad 4 g^{\tau r} = -\epsilon \frac{n^r}{(1 + n)^2} = -\epsilon \sum_a 3 \tilde{e}_{(a)r} \tilde{n}_{(a)},$$

$$4 g^{rs} = -\epsilon \left(3 g^{rs} - \frac{n^r n^s}{(1 + n)^2}\right) = -\epsilon \sum_{ab} 3 \tilde{e}_{(a)r} 3 \tilde{e}_{(b)s} (\delta_{(a)(b)} - \frac{\tilde{n}_{(a)} \tilde{n}_{(b)}}{(1 + n)^2}),$$

$$\sqrt{-g} = \sqrt{|4 g|} = \frac{\sqrt{2 g}}{\sqrt{\epsilon 4 g^{\tau\tau}}} = \sqrt{\gamma} (1 + n) = 3 e (1 + n),$$

$$3 g = \gamma = (3 e)^2, \quad 3 e = det 3 \tilde{e}_{(a)r}. \quad (3.7)$$

The 3-metric $3 g_{rs}$ has signature $(++ +)$, so that one may put all the flat 3-indices down. One has $3 g^{ru} 3 g_{us} = \delta^r_s$.

After having introduced the kinematical framework for the description of non-inertial frames in GR, we must study the dynamical aspects of the gravitational field to understand which variables are dynamically determined and which are the inertial effects hidden in the general covariance of the theory. Since at the Lagrangian level it is not possible to identify which components of the 4-metric tensor are connected with the gauge freedom in the choice of the 4-coordinates and which ones describe the dynamical degrees of freedom of the gravitational field, one must restrict himself to the class of globally hyperbolic, asymptotically flat space-times allowing a Hamiltonian description starting from the description of Einstein GR in terms of the ADM action [36] instead than in terms of the Einstein-Hilbert one. In canonical ADM gravity one can use Dirac theory of constraints [37] to describe the Hamiltonian gauge group, whose generators are the first-class constraints of the model. The basic tool of this approach is the possibility to find so-called Shanmugadhasan canonical transformations [38], which identify special canonical bases adapted to the first-class constraints (and also to the second-class ones when present). In these special canonical bases the vanishing of
certain momenta (or of certain configurational coordinates) corresponds to the vanishing of well defined Abelianized combinations of the first-class constraints (Abelianized because the new constraints have exactly zero Poisson brackets even if the original constraints were not in strong involution). As a consequence, the variables conjugate to these Abelianized constraints are inertial Hamiltonian gauge variables describing the Hamiltonian gauge freedom.

Starting from the ADM action for tetrad gravity one defines the Hamiltonian formalism in a phase space containing 16 configurational field variables and 16 conjugate moments. One identifies the 14 first-class constraints of the system and one finds that the canonical Hamiltonian is the weak ADM energy (it is given as a volume integral over 3-space). The existence of these 14 first-class constraints implies that 14 components of the tetrads (or of the conjugate momenta) are Hamiltonian gauge variables describing the inertial aspects of the gravitational field (6 of these inertial variables describe the extra gauge freedom in the choice of the tetrads and in their transport along world-lines). Therefore there are only 2+2 degrees of freedom for the description of the tidal dynamical aspects of the gravitational field. The asymptotic ADM Poincaré generators can be evaluated explicitly. Till now the type of matter studied in this framework [32] consists of the electro-magnetic field and of N charged scalar particles, whose signs of the energy and electric charges are Grassmann-valued to regularize both the gravitational and electro-magnetic self-energies (it is both a ultraviolet and an infrared regularization).

The remaining 2+2 conjugate variables describe the dynamical tidal degrees of freedom of the gravitational field (the two polarizations of gravitational waves in the linearized theory). If one would be able to include all the constraints in the Shanmugadhasan canonical basis, these 2+2 variables would be the Dirac observables of the gravitational field, invariant under the Hamiltonian gauge transformations. However such Dirac observables are not known: one only has statements about their existence [39]. Moreover, in general they are not 4-scalar observables. The problem of the connection between the 4-diffeomorphism group and the Hamiltonian gauge group was studied in Ref.[40] by means of the inverse Legendre transformation and of the notion of dynamical symmetry. The conclusion is that on the space of solutions of Einstein equations there is an overlap of the two types of observables: there should exists special Shanmugadhasan canonical bases in which the 2+2 Dirac observables become 4-scalars when restricted to the space of solutions of the Einstein equations. In any case the identification of the inertial gauge components of the 4-metric is what is needed to make a fixation of 4-coordinates as required by relativistic metrology.

It can be shown that there is a Shanmugadhasan canonical transformation [41] (implementing the so-called York map [42] and diagonalizing the York-Lichnerowics approach [43]) to a so-called York canonical basis adapted to 10 of the 14 first-class constraints. Only the super-Hamiltonian and super-momentum constraints, whose general solution is not known, are not included in the basis, but it is clarified which variables are to be determined by their solution, namely the 3-volume element (the determinant of the 3-metric) of the 3-space \( \Sigma_T \) and the three momenta conjugated to the 3-coordinates on \( \Sigma_T \). The 14 inertial gauge variables turn out to be: a) the six configurational variables \( \varphi_{(a)} \) and \( \alpha_{(a)} \) of the tetrads describing their \( \text{O}(3,1) \) gauge freedom; b) the lapse and shift functions; c) the 3-coordinates on the 3-space (their fixation implies the determination of the shift functions); d) the York time [44] \( \text{K} \), i.e. the trace of the extrinsic curvature of the 3-spaces as 3-manifolds embedded into the space-time (its fixation implies the determination of the lapse function).
It is the only gauge variable which is a momentum in the York canonical basis: this is due to the Lorentz signature of space-time, because the York time and three other inertial gauge variables can be used as 4-coordinates of the space-time (see Ref.[24] for this topic and for its relevance in the solution of the hole argument). In this way an identification of the inertial gauge variables to be fixed to get a 4-coordinate system in relativistic metrology was found. While in SR all the components of the tetrads and their conjugate momenta are inertial gauge variables, in GR the two eigenvalues of the 3-metric with determinant one and their conjugate momenta describe the physical tidal degrees of freedom of the gravitational field. In the first paper of Ref.[32] there is the expression of the Hamilton equations for all the variables of the York canonical basis.

An important remark is that in the framework of the York canonical basis the natural family of gauges is not the harmonic one, but the family of 3-orthogonal Schwinger time gauges in which the 3-metric in the 3-spaces is diagonal.

Both in SR and GR an admissible 3+1 splitting of space-time has two associated congruences of time-like observers [7], geometrically defined and not to be confused with the congruence of the world-lines of fluid elements, when relativistic fluids are added as matter in GR [45–47]. One of the two congruences, with zero vorticity, is the congruence of the Eulerian observers, whose 4-velocity field is the field of unit normals to the 3-spaces. This congruence allows us to re-express the non-vanishing momenta of the York canonical basis in terms of the expansion \( \theta = -\frac{3}{K} \) and of the shear of the Eulerian observers. This allows us to compare the Hamilton equations of ADM canonical gravity with the usual first-order non-Hamiltonian ADM equations deducible from Einstein equations given a 3+1 splitting of space-time but without using the Hamiltonian formalism. As a consequence, one can extend our Hamiltonian identification of the inertial and tidal variables of the gravitational field to the Lagrangian framework and use it in the cosmological (conformally asymptotically flat) space-times: in them it is not possible to formulate the Hamiltonian formalism but the standard ADM equations are well defined. The time inertial gauge variable needed for relativistic metrology is now the expansion of the Eulerian observers of the given 3+1 splitting of the globally hyperbolic cosmological space-time.

IV. CONCLUSION

In conclusion we now have a framework for non-inertial frames in GR and an identification of the inertial gauge variables in asymptotically Minkowskian and also cosmological space-times.

See Refs.[23, 32] for the possibility that dark matter is only a relativistic inertial effect induced by the inertial gauge variable \( ^3K \) (the York time): a suitable choice of the 3-space in PM Celestial Reference Frame could simulate the effects explained with dark matter.

Moreover in Ref.[23], at a preliminary level, it is also shown that the York time is connected also with dark energy in cosmological space-times [4]. In the standard FWR space-times the Killing symmetries connected with homogeneity and isotropy imply \( \tau \) is the

\(^{13}\) Instead in Yang-Mills theory all the gauge variables are configurational.
cosmic time, $a(\tau)$ the scale factor) that the York time is no more a gauge variable but coincides with the Hubble constant: $3K(\tau) = -\frac{\dot{a}(\tau)}{a(\tau)} = -H(\tau)$. However at the first order in cosmological perturbations (see Ref.[48] for a review) one has $3K = -H + 3K_{(1)}$ with $3K_{(1)}$ being again an inertial gauge variable to be fixed with a metrological convention. Therefore the York time has a central role also in cosmology and one needs to know the dependence on it of the main quantities, like the red-shift and the luminosity distance from supernovae [49], which require the introduction of the notion of dark energy to explain the 3-universe and its accelerated expansion in the framework of the standard $\Lambda$CDM cosmological model.

In particular it will be important to study inhomogeneous space-times without Killing symmetries like the Szekeres ones [50], where the York time remains an arbitrary inertial gauge variable, to see whether it is possible to find a 3-orthogonal gauge in them (at least in a PM approximation) in which the convention on the inertial gauge variable York time allows one to eliminate both dark matter and dark energy through the choice of a 4-coordinate system in a consistent PM reformulation of ICRS and simultaneously to save the main good properties of the standard $\Lambda$CDM cosmological model due to the inertial and dynamical properties of the space-time.
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