TWISTED MODULI SPACES AND DUISTERMAAT-HECKMAN MEASURES

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Abstract. Following Boalch-Yamakawa and Meinrenken, we consider a certain class of moduli spaces on bordered surfaces from a quasi-Hamiltonian perspective. For a given Lie group $G$, these character varieties parametrize flat $G$-connections on “twisted” local systems, in the sense that the transition functions take values in $G \rtimes \text{Aut}(G)$. After reviewing the necessary tools to discuss twisted quasi-Hamiltonian manifolds, we construct a Duistermaat-Heckman (DH) measure on $G$ that is invariant under the twisted conjugation action $g \mapsto h g \kappa(h^{-1})$ for $\kappa \in \text{Aut}(G)$, and characterize it by giving a localization formula for its Fourier coefficients. We then illustrate our results by determining the DH measures of our twisted moduli spaces.

0. Introduction

The theory of quasi-Hamiltonian manifolds was initiated by Alekseev, Malkin and Meinrenken in [2], and one of its fundamental results is the bijective correspondence between such spaces and Banach manifolds with Hamiltonian actions of loop groups. On the one hand, this theory allows one to generalize several core aspects of symplectic geometry to the loop group setting (e.g. convexity properties, Duistermaat-Heckman distributions [3,5,26] and geometric quantization [20,28,38]), and on the other hand, it provides a finite-dimensional approach to the study of various types of moduli spaces of flat bundles over surfaces with boundary [7,22,27].

The present paper is concerned with Hamiltonian actions of twisted loop groups, for a twist given by a Lie group automorphism $\kappa \in \text{Aut}(G)$, and the corresponding twisted quasi-Hamiltonian $G$-spaces [9,19,25,29]. In the latter case, the moment map takes values in a Lie group $G$, and is equivariant with respect to the $\kappa$-twisted conjugation action of the group on itself:

$$\text{Ad}_g^\kappa(h) = gh \kappa(g^{-1}), \quad \forall g, h \in G,$$

Basic examples of twisted quasi-Hamiltonian $G$-spaces include orbits of the action $\text{Ad}_G^\kappa$, as well as twisted $G$-character varieties [9,29].

The first objective of this work is to construct and compute the Duistermaat-Heckman (DH) measure of a twisted quasi-Hamiltonian $G$-space $M$ (tq-Hamiltonian in short). As in the standard theory [3,5,26], the DH measure of $M$ encodes the volumes of its symplectic quotients. The main feature of the setup at hand is that the Fourier expansion is given in terms of twining characters [14,17,43], the class functions on $G$ with respect to twisted conjugation, which were introduced by Fuchs, Schellekens and Schweigert in [14]. The main result of this work is a localization formula for the Fourier coefficients of the DH measures.

The second objective of this article is to determine the DH measures explicitly for twisted $G$-character varieties [29]. Similar moduli spaces were studied from the algebro-geometric perspective [18,35], in view of obtaining a generalization of the Verlinde formula that incorporates...
the Dynkin diagram automorphisms of the group involved. Following the paradigm of Alekseev-
Meinrenken-Woodward [4, 28, 30], a symplectic approach to deriving such a Verlinde formula
would be to compute the quantization of the reduced spaces of twisted moduli spaces, which
requires a K-theoretic analogue of the localization formula obtained here. Another motivation
for the determination of the DH measures of twisted moduli spaces is the study of the cohomol-
ogy of their symplectic quotients, by generalizing Witten’s formulas for intersection pairings to
the present setting, and following [24, 26, 41, 42].

The contents of this paper are organized as follows. After reviewing the basics of tq-
Hamiltonian geometry, section 1 gives a detailed discussion of twisted moduli spaces. The
Duistermaat-Heckman measure of tq-Hamiltonian manifold and the localization theorem are
addressed in section 2, which ends with the computation of DH measures of twisted moduli
spaces. The first set of prerequisites are some properties of twisted conjugation, which are
summarized at the beginning of sections 1 and 2. The second set of prerequisites are the Dirac
geometry techniques used in Alekseev, Bursztyn and Meinrenken’s [1]. Section 3 gathers some
key results of that work in the context of twisted conjugation.

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CONTENTS

0. Introduction 1
Acknowledgements 2
1. Twisted quasi-Hamiltonian manifolds 2
1.1. Twisted conjugation 3
1.2. Quasi-Hamiltonian geometry 5
1.3. Moduli spaces 8
2. Duistermaat-Heckman Measures and Localization 17
2.1. Twining characters 17
2.2. Definitions and basic properties 20
2.3. Localization and Fourier coefficients 22
2.4. Application to moduli spaces 24
3. Dirac geometry and twisted conjugation 26
3.1. Reminders 26
3.2. Pure spinors on Lie groups 29
3.3. Dirac geometry of fusion 34
References 36

1. Twisted quasi-Hamiltonian manifolds

In this section, we establish the geometric setup of the present work. We start by recalling
some basics of the twisted conjugation action of a Lie group on itself in 1.1. Next, we review
the definition of twisted quasi-Hamiltonian manifolds in 1.2, the first examples thereof, and
Thus, to study the twisted conjugation action, it is sufficient to consider the case where

\[
\kappa : \text{Aut}(g) \to \text{Out}(G) \to \text{Out}(G) \to 1.
\]

We assume that the Lie algebra \( g \) is endowed with an \( \text{Aut}(G) \)-invariant, symmetric, and non-degenerate bilinear form \( B \in S^2 g^* \), and we often write \( \xi \cdot \zeta = B(\xi, \zeta) \) for the inner product of elements \( \xi, \zeta \in g \).

We denote the set of roots of \( G \) by \( \Phi \subseteq \mathfrak{t}^* \), and the Weyl group by \( W = N_G(T)/T \). We use real roots below: for \( \xi \in \mathfrak{t} \) and \( t = e^\xi \in T \), we write \( t^{\xi} = e^{2\pi i \langle \alpha, \xi \rangle} \) for the corresponding character. We fix a choice of positive roots \( \Phi_+ \subseteq \Phi \) and along with the identification \( t \cong t^\xi \) given by the invariant inner product \( B \) on \( g \), we denote the closed fundamental Weyl chamber by \( t_+^\xi \subseteq t^\xi \). For \( G \) simply connected, we denote the weight lattice by \( \Lambda^* = \text{Hom}(\Lambda, \mathbb{Z}) \), where \( \Lambda = \ker(\exp) \cap t \) is the integral lattice, and we denote the dominant weights by \( \Lambda^+_+ = \Lambda^* \cap t_+^\xi \).

1.1. Twisted conjugation.

1.1.1. Notation. For a Lie group \( G \) with (fixed) maximal torus \( T \), we denote the corresponding Lie algebras by \( \mathfrak{g} \) and \( \mathfrak{t} \). We denote the group of automorphisms of \( G \) by \( \text{Aut}(G) \), and its normal subgroup of inner automorphisms by \( \text{Inn}(G) \). The group of outer automorphisms \( \text{Out}(G) \) of \( G \) is defined by the exact sequence:

\[
1 \to \text{Inn}(G) \to \text{Aut}(G) \to \text{Out}(G) \to 1.
\]

1.1.2. The twisted conjugation action.

**Definition 1.1.** Let \( \kappa \in \text{Aut}(G) \) be an automorphism of \( G \). The \( \kappa \)-twisted conjugation action of \( G \) on itself is given by:

\[
\text{Ad}_g^\kappa(h) = gh\kappa(g^{-1}), \quad \forall g, h \in G.
\]

We denote by \( G\kappa \) the group \( G \) endowed with the action \( \text{Ad}_G^\kappa \). The orbits of this action are called the \( \kappa \)-twisted conjugacy classes of \( G \).

In terms of the semi-direct product \( G \rtimes \text{Aut}(G) \), twisted conjugation corresponds to the usual conjugation action of the subgroup \( G \kappa \subseteq G \rtimes \text{Aut}(G) \):

\[
(g, 1)(h, \kappa)(g, 1)^{-1} = (\text{Ad}_g^\kappa(h), \kappa).
\]

For any automorphism \( \tau = \text{Ad}_a \circ \kappa \) with \( a \in G \), the \( \kappa \)-twisted and \( \tau \)-twisted conjugacy classes are related by the right multiplication map \( R_a : x \mapsto xa^{-1} \), since:

\[
R_{a^{-1}} \circ \text{Ad}_g^{(\text{Ad}_a \circ \kappa)} = \text{Ad}_g^\kappa \circ R_{a^{-1}}, \quad \forall g \in G.
\]

Thus, to study the twisted conjugation action, it is sufficient to consider the case where \( \kappa \in \text{Aut}(G) \) is induced by a Dynkin diagram automorphism.

For the remainder of this subsection, we focus on the case of \( G \) compact 1-connected and simple. This setup will be of particular importance later (Proposition 1.21) and section 2), and several facts are worth mentioning. For more details, see [43, §§2-3] and references therein.

Any automorphism \( \kappa \in \text{Aut}(\Pi) \) of the Dynkin diagram of \( G \) permutes the simple roots \( \Pi \) of \( \Phi \) and preserves its Cartan matrix. If \( \{e_{\pm \alpha}\}_{\alpha \in \Phi_+} \) denote the Chevalley generators of \( \mathfrak{g}_\mathbb{C} \), there then exists a unique corresponding \( \kappa \in \text{Aut}(\mathfrak{g}_\mathbb{C}) \), such that \( \kappa(e_{\pm \alpha}) = e_{\pm \kappa(\alpha)} \) for all \( \alpha \in \Pi \). This \( \kappa \in \text{Aut}(\mathfrak{g}_\mathbb{C}) \) preserves \( \mathfrak{g} \subseteq \mathfrak{g}_\mathbb{C} \), \( \mathfrak{t} \subseteq \mathfrak{t}_\mathbb{C} \) and \( \mathfrak{t}_+^\xi \subseteq \mathfrak{t}_\mathbb{C}^* \), and exponentiates to an element \( \kappa \in \text{Aut}(G) \) preserving the maximal torus \( T \). We hence obtain a homomorphism
Let Proposition 1.2. references therein for the original authors of these results. For statement (5), we note that are contained in Lemma 3.5, Theorem 3.6, Proposition 3.9 and Proposition 2.4 in [43] (see references therein for the original authors of these results). For statement (5), we note that

| $\mathfrak{R}$ | $A_2$ | $A_{2n}$, $n \geq 2$ | $A_{2n-1}$, $n \geq 2$ | $D_{n+1}$, $n \geq 4$ | $D_4$ | $E_6$ |
|---------------|-------|-----------------|-----------------|-----------------|-------|-------|
| $\text{Out}(G)$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $S_3$ | $\mathbb{Z}_2$ |
| $\mathfrak{R}(\kappa)$ | $A_1$ | $C_n$ | $B_n$ | $B_n$ | $G_2$ if $|\kappa| = 3$ | $B_3$ if $|\kappa| = 2$ | $F_4$ |

Table 1. Non-trivial $\text{Out}(G)$ and orbit root systems

We now introduce the objects that control the $\kappa$-twisted conjugacy classes of $G$. Let $G^\kappa$ and its maximal torus $T^\kappa$ denote the $\kappa$-fixed subgroups of $G$ and $T$, and let $\mathfrak{g}^\kappa$ and $t^\kappa$ denote the corresponding Lie algebras. Let $T_\kappa \subseteq T$ denote the image of of the homomorphism $T \to T$, $t \mapsto t\kappa(t^{-1})$, and let $t_\kappa = \text{Lie}(T_\kappa)$. As for usual conjugation, we define the outer Weyl group $W^\kappa = N_G^\kappa(T^\kappa)/T^\kappa$, where $N_G^\kappa(T^\kappa) = \{ g \in G \mid \text{Ad}_g^\kappa(T^\kappa) = T^\kappa \}$. We denote by $W^\kappa \subseteq W$ the subgroup of elements commuting with $\kappa$.

For non-trivial $\text{Out}(G)$ (Table 1), one associates the orbit root system $\mathfrak{R}(\kappa)$ to the data $(\mathfrak{R}, \kappa)$. Let $p : t^* \to (t^*)^\kappa$ denote the orthogonal projection. For $\mathfrak{R} \neq A_{2n}$, the projection $p(\mathfrak{R})$ is an indecomposable root system, and its dual is equivalent to $\mathfrak{R}(\kappa)$. For $\mathfrak{R} = A_{2n}$ with $n > 1$, the orbit root system $\mathfrak{R}(\kappa)$ is equivalent to the dual of the $B_n$ subsystem of $p(\mathfrak{R}) = BC_n$. For $\mathfrak{R} = A_2$, $\mathfrak{R}(\kappa) = A_1$ is given by twice the highest root of $\mathfrak{R}$. We call the compact simply connected form of $\mathfrak{R}(\kappa)$ the orbit Lie group, and denote it by $G(\kappa)$. Note that the Weyl group of $\mathfrak{R}(\kappa)$ coincides with $W^\kappa$.

With the preliminaries of the last two paragraphs, we have:

**Proposition 1.2.** Let $G$ be compact 1-connected and simple with nontrivial $\kappa \in \text{Out}(G)$.

1. One has that $T = T^\kappa \cdot T_\kappa$, that $t = t^\kappa \oplus t_\kappa$, and that:
   
   $T^\kappa \cap T_\kappa \simeq \begin{cases} (\mathbb{Z}_2)^{\dim t_\kappa}, & |\kappa| = 2; \\ \mathbb{Z}_3, & |\kappa| = 3. \end{cases}$

2. One has that $W^\kappa \simeq (T^\kappa \cap T_\kappa) \rtimes W^\kappa$, and that any $\kappa$-twisted conjugacy class in $G$ intersects $T^\kappa$ in an orbit of $W^\kappa$.

3. The orbit Lie group is related to $G$ by the isomorphism $L^\kappa G(\kappa) \simeq (L^\kappa G)_0^\kappa$, and its maximal torus is isomorphic to $T^\kappa/(T^\kappa \cap T_\kappa)$ ($L^\kappa K$ denotes the Langlands dual of $K$, and $K_0 \subseteq K$ the identity component).

4. The weight lattice of $G(\kappa)$ coincides with $(\Lambda^*)^\kappa = \Lambda^* \cap (t^*)^\kappa$, the lattice of $\kappa$-fixed weights of $G$. The integral lattice of $G(\kappa)$ coincides with the lattice $\exp_{t^\kappa}^{-1}(T^\kappa \cap T_\kappa) \subset t^\kappa$.

5. The $\kappa$-twisted conjugacy classes of $G$ are parametrized by the fundamental alcove of the orbit group $G(\kappa)$.

In this proposition, statement (3) is discussed for $G$ complex in [21]. Parts (1), (2) and (4) are contained in Lemma 3.5, Theorem 3.6, Proposition 3.9 and Proposition 2.4 in [43] (see references therein for the original authors of these results). For statement (5), we note that
the affine Weyl group $\Lambda \times W$ is replaced by $\exp_\kappa^{-1}(T^\kappa \cap T_\kappa) \times W^\kappa$ when dealing with twisted conjugation. The orbit space $G/Ad^\kappa_G$ identifies with $T^\kappa/W(\kappa) \simeq \mathfrak{t}^\kappa/\left(\exp_\kappa^{-1}(T^\kappa \cap T_\kappa) \times W^\kappa\right)$, which is the fundamental alcove of $G(\kappa)$ by statements (2)-(4). An explicit description of this alcove is given in [34, §3] (c.f. [43, Prop.3.10]).

1.2. Quasi-Hamiltonian geometry.

1.2.1. Definition and first examples. Let $\theta^L, \theta^R \in \Omega^1(G, \mathfrak{g})$ denote the left- and right-invariant Maurer-Cartan 1-forms respectively, and let $\eta = \frac{1}{12}[\theta^k, \theta^l] \cdot \theta^k = \frac{1}{12}[\theta^R, \theta^R] \cdot \theta^R$ be the Cartan 3-form. We have the following definition:

**Definition 1.3.** Let $G$ be a Lie group, and let $\kappa \in \text{Aut}(G)$ be an automorphism. A **twisted quasi-Hamiltonian $G$-space** [9, 19, 23] (tq-Hamiltonian in brief) is a triple $(M, \omega, \Phi)$ where $M$ is a $G$-manifold, $\omega \in \Omega^2(M)^G$ an invariant 2-form, and $\Phi : M \to G\kappa$ is the **group-valued moment map** satisfying the following conditions:

(i) **Equivariance:** $\Phi(g \cdot x) = \text{Ad}^\kappa_g(\Phi(x))$, for all $g \in G$ and $x \in M$;

(ii) **Differential:** $d\omega = \Phi^* \eta$;

(iii) **Moment map condition:** $\iota_{\xi_M} \omega = \frac{1}{2} \Phi^* (\theta^L \cdot \kappa(\xi) + \theta^R \cdot \xi)$, for all $\xi \in \mathfrak{g}$;

(iv) **Minimal degeneracy:** $\ker \omega_x \cap \ker(\Phi_*|_x) = \{0\}$ for all $x \in M$.

Before addressing the basic examples, it will be useful to highlight a few observations.

**Remark 1.4.**

1. (Taking $\kappa = 1$ above, we recover the original definition [2] of a quasi-Hamiltonian manifold (up to a sign convention). Any tq-Hamiltonian $G$-space with moment map $\Phi : M \to G\kappa$ can be viewed as an untwisted q-Hamiltonian space with $G \times \langle \kappa \rangle$-valued moment map, and we only require that the subgroup $G \subseteq G \times \langle \kappa \rangle$ acts on $M$. Due to this observation, many results of q-Hamiltonian geometry carry over directly to the twisted case.

2. To be specific about the target of the moment map of $(M, \omega, \Phi)$, we will often refer to $(M, \omega, \Phi)$ as a $G\kappa$-valued tq-Hamiltonian space.

3. Definition 1.3 can be reformulated succinctly in the language of Dirac geometry. The triple $(M, \omega, \Phi)$ is a tq-Hamiltonian $G$-space if $(\Phi, \omega) : (M, TM, 0) \longrightarrow (G\kappa, E^\kappa_G, \eta)$ is a strong $G$-equivariant Dirac morphism [1, Def.2.4], where $E^\kappa_G \subseteq TG$ denotes the twisted Cartan-Dirac structure [23, Rk.3.5]. We review this viewpoint in section 3.1.

4. There exists an equivalence of categories between $G\kappa$-valued tq-Hamiltonian manifolds and weakly symplectic Banach manifolds equipped with a Hamiltonian action of the $\kappa$-twisted loop group $L(\kappa)G = \{ \gamma : \mathbb{R} \to G \mid \gamma(t+1) = \kappa(\gamma(t)) \}$. This is an extension of the equivalence theorem [2, Thm.8.3] to the twisted setup, and is explained in [19, Thm.2.4]. This equivalence of categories can be interpreted as a certain Morita equivalence of pre-symplectic groupoids, as explained in [25, Rk.3.3, §7].

Our first example is the quasi-Hamiltonian analogue of coadjoint orbits in symplectic geometry.

**Example 1.5.** (Twisted conjugacy classes) Let $a \in G$ be a given element, and denote its $\kappa$-twisted conjugacy class by $C = \text{Ad}^\kappa_G(a)$. This space has a natural structure of a $G\kappa$-valued tq-Hamiltonian space. The group $G$ acts on $C$ by $\kappa$-twisted conjugation, the moment
map $\Phi : C \to G\kappa$ is given by the inclusion $C \hookrightarrow G\kappa$, and the invariant 2-form is uniquely determined by the moment map condition:

\[
\omega(C)(\xi_C, \zeta_C)_x = -\frac{1}{2}((\text{Ad}_x \circ \kappa) - (\text{Ad}_x \circ \kappa)^{-1}) \xi \cdot \zeta, \quad \forall \xi, \zeta \in \mathfrak{g}.
\]

The identity $d\omega = \Phi^*\eta$ and the minimal degeneracy condition are verified by elementary computations.

For future reference, we give a special case of the example above that allows us to view the group $G\kappa$ itself as a tq-Hamiltonian $G \times G$-space. This example generalizes the discussion of [5, §2.6].

**Example 1.6. ($G\kappa$ as a twisted conjugacy class)** For a given Lie group $G$ and a fixed automorphism $\kappa \in \text{Aut}(G)$, let $K = G \times \langle \kappa \rangle$, and let $\nu \in \text{Aut}(K \times K)$ be the involution:

\[
\nu : K \times K \to K \times K, \quad (a, b) \mapsto (b, a).
\]

Letting $G \times G$ act on $K \times K$ via $\nu$-twisted conjugation, consider the orbit through $(\kappa, \kappa^{-1})$:

\[
C = \text{Ad}_G^\nu G \times G(\kappa, (g\kappa)^{-1}).
\]

We have a $G \times G$-equivariant diffeomorphism:

\[
G\kappa \to C, \quad g \mapsto (g\kappa, (g\kappa)^{-1}),
\]

where the $G \times G$ action on $G\kappa$ is given by:

\[
(g_1, g_2) \cdot g = g_1 g\kappa(g_2^{-1}), \quad \forall g, g_1, g_2 \in G.
\]

We can thus identify $G\kappa$ with the $\nu$-twisted conjugacy class $C$, with moment map:

\[
\Phi : G\kappa \to (K \times K)\nu, \quad \Phi(g) = (g\kappa, (g\kappa)^{-1}),
\]

and q-Hamiltonian form $\omega_{G\kappa} \equiv 0$ by equation (1.2), since for all $g \in G$:

\[
(\text{Ad}_{(g\kappa, (g\kappa)^{-1})}\nu)^{-1} = \text{Ad}_{(g\kappa, (g\kappa)^{-1})}\nu.
\]

In conclusion, $(G\kappa, 0, \Phi)$ is a $(K \times K)\nu$-valued tq-Hamiltonian $G \times G$-space.

As previously mentioned, the twisted moduli spaces of section 1.3 provide a large class of examples of tq-Hamiltonian manifolds. The construction of these spaces and their properties rely on fusion and reduction, which we now review.

1.2.2. **Fusion.** We start with internal fusion. Let $G$ and $H$ denote two Lie groups, and consider an untwisted q-Hamiltonian $G \times G \times H$-space $(M, \omega, \Phi)$ with moment map $\Phi : M \to G \times G \times H$. Internal fusion produces an alternative q-Hamiltonian structure $(M, \omega_{\text{fus}}, \Phi_{\text{fus}})$ on the same manifold $M$, with $\Phi_{\text{fus}}$ now taking values in $G \times H$. In the twisted setup, the precise definition is as follows:

**Proposition 1.7. (Internal fusion)** Let $G$ and $H$ be Lie groups, and $\kappa_1, \kappa_2 \in \text{Aut}(G)$, $\tau \in \text{Aut}(H)$. Let $(M, \omega, \Phi)$ be a tq-Hamiltonian $G \times G \times H$-manifold with moment map:

\[
\Phi : M \to G\kappa_1 \times G\kappa_2 \times H\tau, \quad \Phi(x) = (\Phi_1(x), \Phi_2(x), \Phi_3(x)).
\]
The triple \((M, \omega_{\text{fus}}, \Phi_{\text{fus}})\), where \(M\) is equipped with the \(G \times H\) action such that \(G\) acts diagonally, and where:

\[
\omega_{\text{fus}} = \omega - \frac{1}{2} \Phi_1^* \theta^L \cdot \Phi_2^* \kappa^R, \\
\Phi_{\text{fus}}(x) = (\Phi_1(x) \cdot \kappa_1(\Phi_2(x)), \Phi_3(x)), \quad \forall x \in M,
\]

is a \(G \kappa_1 \kappa_2 \times H \tau\)-valued tq-Hamiltonian \(G \times H\)-space, called the \textbf{internal fusion} of \((M, \omega, \Phi)\) with respect to the first two components of \(\Phi\).

Our next operation is a special case of the previous one. Given two tq-Hamiltonian \(G\)-spaces \((M_i, \omega_i, \Phi_i)\) with \(i = 1, 2\), it is easily seen from Definition 1.3 that \((M_1 \times M_2, \omega_1 + \omega_2, \Phi_1 \times \Phi_2)\) is a tq-Hamiltonian \(G \times G\)-space. Applying internal fusion to the direct product \(M_1 \times M_2\), we obtain the so-called \textbf{fusion product} of the \((M_i, \omega_i, \Phi_i)\).

\textbf{Proposition 1.8. (Fusion product)} Let \(G\) be a Lie group with automorphisms \(\kappa_1, \kappa_2 \in \text{Aut}(G)\). For \(i = 1, 2\), suppose that \((M_i, \omega_i, \Phi_i)\) are \(G \kappa_i\)-valued tq-Hamiltonian \(G\)-spaces. Let \(G\) act diagonally on \(M_1 \oplus M_2 := M_1 \times M_2\), and define:

\[
\omega_{\text{fus}} = \omega_1 + \omega_2 - \frac{1}{2} \Phi_1^* \theta^L \cdot \Phi_2^* \kappa^R, \\
\Phi_{\text{fus}}(x_1, x_2) = \Phi_1(x_1) \cdot \kappa_1(\Phi_2(x_2)), \quad \forall x_i \in M_i.
\]

The triple \((M_1 \oplus M_2, \omega_{\text{fus}}, \Phi_{\text{fus}})\) is a \(G \kappa_1 \kappa_2\)-valued tq-Hamiltonian space called the \textbf{fusion product} of \((M_1, \omega_1, \Phi_1)\) and \((M_2, \omega_2, \Phi_2)\).

\textbf{Remark 1.9.} We keep the notation of Propositions 1.7 and 1.8.

1. Fusion composes the twisting automorphisms assigned to the target spaces of moment maps. This is the first notable difference with the fusion of untwisted q-Hamiltonian manifolds.

2. From the standpoint of Dirac geometry, fusion is simply a composition of Dirac morphisms. As such, Propositions 1.7 and 1.8 are the twisted analogues of [1] Thm.5.6, and the main ingredient in the proof is the fact that the group multiplication map:

\[
\text{Mult} \circ (1 \times \kappa_1) : G \kappa_1 \times G \kappa_2 \rightarrow G \kappa_1 \kappa_2, \quad (a, b) \mapsto ak_1(b),
\]

extends to a \(G\)-equivariant Dirac morphism [1] Thm.3.9]. See Proposition 3.15 and Remark 3.16 for more details.

3. Continuing on the previous remark, composing \(\text{Mult} : G \times G \rightarrow G\) by elements of \(\text{Aut}(G)\) allows to define several fusion products on the category of tq-Hamiltonian manifolds. This is a second major difference in comparison with the untwisted theory.

\textbf{1.2.3. Reduction.} We will need the following generalization of [2] Prop.4.4] (see Remark 3.16):

\textbf{Proposition 1.10. (Inversion)} Let \(G\) be a Lie group and \(\kappa \in \text{Aut}(G)\) an automorphism. If \((M, \omega, \Phi)\) is a \(G\kappa\)-valued tq-Hamiltonian \(G\)-space, its \textbf{inverse} \((M^-, -\omega, \Phi^-)\) is a \(G\kappa^{-1}\)-valued tq-Hamiltonian manifold, where \(M^- = M\) as \(G\)-manifolds, and where the moment map is given by:

\[
\Phi^- : M \rightarrow G \kappa^{-1}, \quad \Phi^-(x) = \kappa^{-1}(\Phi(x)^{-1}).
\]

Reduction of tq-Hamiltonian \(G\)-spaces is defined in terms of the shifting trick [2] Rk.6.2], which brings the operation back to standard q-Hamiltonian reduction:
Definition 1.11. Given a $G_0$-valued tq-Hamiltonian space $(M, \omega, \Phi)$, let $C = \text{Ad}_{\kappa}^c(a)$ be the twisted conjugacy class of $a \in G$. The \textbf{q-Hamiltonian reduction} of $M$ at $a \in G$ is defined as the quotient:

$$M_a = (M \otimes C^-)/G.$$  

By Proposition 1.8, $M \otimes C^-$ is an untwisted q-Hamiltonian $G$-space, and by the discussion in \cite[§§5-6]{2} we can state:

**Proposition 1.12. (Reduction)** With the notation above, one has that:

1. A point $a \in G$ is a regular value of $\Phi : M \to G$ if and only if $e \in G$ is a regular value of the moment map $\Phi_a : M \otimes C^- \to G$.

2. If $Z_a^c \subseteq G$ denotes the stabilizer of $a \in G$ under $\text{Ad}^c_a$, one has that:

$$M_a = \Phi_a^{-1}(e)/G \simeq \Phi^{-1}(a)/Z_a^c.$$  

3. The 2-form on $\Phi_a^{-1}(e) \subset M \otimes C^-$ descends to a symplectic form $\omega_a \in \Omega^2(M_a)$. The space $M_a$ is then a (singular) symplectic space in the sense of Sjamaar-Lerman \cite{39}.

1.3. Moduli spaces. This section is divided into two parts. The first one sets up the notation, and reminds of several general constructions and facts pertaining to character varieties associated to bordered surfaces. Its purpose is to formulate a precise definition of our twisted moduli spaces, and to specify the type of flat connections they parametrize.

The second part concerns the tq-Hamiltonian geometry of our character varieties. After explaining the construction of the moment map and the invariant 2-form associated to a twisted moduli space, we illustrate the discussion of the previous section with concrete examples.

1.3.1. Character varieties. Let $X$ be a locally path-connected and locally simply connected topological space, and let $Y \subseteq X$ be a closed subspace. We employ the following conventions for the fundamental groupoid $\Pi_1(X,Y) \Rightarrow Y$. The source and target maps $s, t : \Pi_1(X,Y) \to Y$ are defined as $s[\gamma] = \gamma(0)$ and $t[\gamma] = \gamma(1)$ for $[\gamma] \in \Pi_1(X,Y)$. The groupoid product is given by $[\beta][\gamma] = [\beta \ast \gamma]$ if $[\beta], [\gamma] \in \Pi_1(X,Y)$ satisfy $t[\gamma] = s[\beta]$, where for representatives $\beta, \gamma : [0,1] \to X$:

$$(\beta \ast \gamma)(t) = \begin{cases} \gamma(2t), & t \in [0, \frac{1}{2}]; \\ \beta(2t - 1), & t \in [\frac{1}{2}, 1]. \end{cases}$$

For a pair $(X,Y)$ as above and a group $K$, the associated \textbf{$K$-character variety} is the space of groupoid homomorphisms:

$$\text{Hom}(\Pi_1(X,Y), K),$$

where $K$ is viewed as a groupoid with one object. We will use the following facts:

**Fact 1.13.** With the notation above:

(a) The $K$-character variety has a natural action of the gauge group $\text{Map}(Y,K)$, such that for all $\phi \in \text{Map}(Y,K)$ and $\rho \in \text{Hom}(\Pi_1(X,Y), K)$:

$$(\phi \cdot \rho)_a = \phi_{t(\alpha)} \rho_{s(\alpha)}^{-1} \in K, \quad \forall \alpha \in \Pi_1(X,Y).$$

(b) Any morphism of pairs of topological spaces $f : (X,Y) \to (X',Y')$ induces a map:

$$f^* : \text{Hom}(\Pi_1(X',Y'), K) \to \text{Hom}(\Pi_1(X,Y), K),$$

which intertwines the actions of the gauge groups $\text{Map}(Y', K)$ and $\text{Map}(Y, K)$. 

(c) Any morphism of groups $\varphi : K \to H$ induces a map:

$$\varphi_* : \text{Hom}(\Pi_1(X,Y),K) \to \text{Hom}(\Pi_1(X,Y),K),$$

which intertwines the actions of $\text{Map}(Y,K)$ and $\text{Map}(Y,H)$.

For the remainder of this section, $\Sigma$ denotes a compact oriented surface such that each connected component has a non-empty boundary, each boundary circle in $\partial \Sigma$ has precisely one basepoint $p_j$, and $S = \{p_j\}$ denotes the resulting finite collection of basepoints. When there is no risk of confusion, we will simply denote the fundamental groupoid of $\Sigma$ based at $S$ by $\Pi := \Pi_1(\Sigma,S)$. In view of the previous paragraph, we have the following definition:

**Definition 1.14.** Let $G$ be a Lie group, and let $p : G \rtimes \text{Aut}(G) \to \text{Aut}(G)$ denote the projection $(g,\kappa) \mapsto \kappa$. For an element $\sigma \in \text{Hom}(\Pi, \text{Aut}(G))$, called the twist, the $\sigma$-twisted moduli space associated to $(\Sigma,S,G)$ is the preimage of $\sigma$ under the induced map:

$$p_* : \text{Hom}(\Pi, G \rtimes \text{Aut}(G)) \to \text{Hom}(\Pi, \text{Aut}(G)).$$

We denote this space by:

$$M_\sigma(\Sigma,G) := \text{Hom}_\sigma(\Pi,G).$$

**Remark 1.15.** Let $\Pi$ and $\sigma \in \text{Hom}(\Pi, \text{Aut}(G))$ be as in the definition.

1. The notation $\text{Hom}_\sigma(\Pi,G)$ emphasizes the fact that we view its elements as $\sigma$-twisted groupoid morphisms $\Pi \to G$. That is, a map $\rho : \Pi \to G$ lies in $\text{Hom}_\sigma(\Pi,G)$ if and only if for all composable $\alpha, \beta \in \Pi$:

$$\rho_{\alpha \beta} = \rho_\alpha \sigma_\alpha(\rho_\beta), \quad \rho_{\alpha}^{-1} = \sigma_\alpha^{-1}(\rho_\alpha^{-1}).$$

2. With the notation of the definition, the subspace:

$$(p_*)^{-1}(\sigma) \subseteq \text{Hom}(\Pi, G \rtimes \text{Aut}(G))$$

is not invariant under the action of the full gauge group $\text{Map}(S, G \rtimes \text{Aut}(G))$. It is however invariant under the subgroup $\text{Map}(S,G) \simeq G^{|S|}$. The latter acts as follows on $\rho \in \text{Hom}_\sigma(\Pi,G)$:

$$(\phi \cdot \rho)_\alpha = \phi_{t(\alpha)} \rho_\alpha \sigma_\alpha \left(\phi_{t(\alpha)}^{-1}\right) \in G, \quad \forall \alpha \in \Pi, \phi \in \text{Map}(S,G).$$

We now turn to the type of objects parametrized by the spaces $M_\sigma(\Sigma,G)$. We recall the following (see also [9][10][32]):

**Definition 1.16.** Let $M$ be a manifold, let $K$ be a Lie group and let $G \to M$ be a Lie group bundle of typical fibre $K$.

(a) The bundle $G \to M$ is called a local system (of groups) on $M$ if it is equipped with a flat Ehresmann connection such that parallel transport between fibres is given by group isomorphisms.

(b) A bundle $\mathcal{P} \to M$ is called a $G$-torsor if it is equipped with a fibre-wise free and transitive action of the bundle $G \to M$.

(c) Let $\mathcal{P} \to M$ be a torsor. A framing of $\mathcal{P}$ at a point $x \in M$ is a choice of $G_x$-equivariant isomorphism $\psi_x : G_x \to P_x$.

In the case where $G \to M$ is a local system of groups and $\mathcal{P} \to M$ is a local system of simply transitive homogeneous spaces for $G$, we will say that $\mathcal{P}$ is a flat $G$-torsor.
Remark 1.17. For the sake of clarity, let us mention some facts that will be implicitly used later. More details can be found in [9, §2].

1. For any local system of groups \( G \rightarrow M \) with typical fibre \( K \), there exists a trivializing open cover \( \{U_i\}_{i \in I} \) of \( M \) such that the transition functions \( U_i \cap U_j \rightarrow \text{Aut}(K) \) are locally constant on \( U_i \cap U_j \neq \emptyset \).

2. If \( \mathcal{P} \rightarrow M \) is a \( G \)-torsor, its typical fibre is the manifold \( K \), without its group structure.

3. If we take \( G = M \times K \) to be the trivial \( K \)-bundle over \( M \), a \( G \)-torsor \( \mathcal{P} \rightarrow M \) is the same as a principal \( K \)-bundle over \( M \). As such, \( G \)-torsors give a natural generalization of principal bundles.

Let \( \Sigma \) be an oriented, compact and connected bordered surface. As above, let \( S \subseteq \partial \Sigma \) be the boundary basepoints, and let \( \Pi = \Pi_1(\Sigma, S) \) be the fundamental groupoid. Consider the universal cover of \( \Sigma \) based at \( S \):

\[
\tilde{\Sigma} := \{ \gamma : [0,1] \to \Sigma \mid \gamma(0) \in S \} / \{\text{homotopy rel. } \{0,1\}\},
\]

with projection \( \pi : \tilde{\Sigma} \to \Sigma, [\gamma] \mapsto t[\gamma] \). This projection coincides with the quotient map with respect to the action of \( \Pi \) on \( \tilde{\Sigma} \) by concatenation from the right:

\[
\Pi \times \tilde{\Sigma} \longrightarrow \tilde{\Sigma}, \quad (\alpha, x) \longmapsto x \cdot \alpha^{-1},
\]

for \( (\alpha, x) \in \Pi \times \tilde{\Sigma} \) such that \( s(x) = s(\alpha) \).

For a fixed Lie group \( G \) with automorphism group \( \text{Aut}(G) \), any twist \( \sigma \in \text{Hom}(\Pi, \text{Aut}(G)) \) gives rise to a local system of groups:

\[
(1.5) \quad \mathcal{E}_\sigma := \tilde{\Sigma} \times_{(\sigma, \Pi)} \text{Aut}(G) = \left( \tilde{\Sigma} \times \text{Aut}(G) \right) / \sim_{(\sigma, \Pi)},
\]

where the equivalence relation \( (\sigma, \Pi) \) is given by:

\[
(1.6) \quad (x, \kappa) \sim_{(\sigma, \Pi)} (y, \tau) \iff \exists \alpha \in \Pi : y = x \cdot \alpha^{-1}, \quad \tau = \sigma_\alpha \kappa,
\]

for \( (x, \kappa), (y, \tau) \in \tilde{\Sigma} \times \text{Aut}(G) \). We then have the following local system of groups:

\[
(1.7) \quad \mathcal{G}_\sigma := \mathcal{E}_\sigma \times_{\text{Aut}(G)} G,
\]

and for any \( \sigma \)-twisted homomorphism \( \rho \in \text{Hom}_\sigma(\Pi, G) \), we have a flat \( \mathcal{G}_\sigma \)-torsor:

\[
(1.8) \quad \mathcal{P}_\rho := \tilde{\Sigma} \times_{(\rho, \Pi)} G,
\]

where the equivalence relation \( \sim_{(\rho, \Pi)} \) on \( \tilde{\Sigma} \times G \) is given by:

\[
(1.9) \quad (x, g) \sim_{(\rho, \Pi)} (y, h) \iff \exists \alpha \in \Pi : y = x \cdot \alpha^{-1}, \quad h = \rho_\alpha \sigma_\alpha(g).
\]

Thus, the bundle \( \mathcal{P}_\rho \rightarrow \Sigma \) is naturally equipped with a flat Ehresmann connection, for which the holonomy along \( \alpha \in \Pi \) is given by \( \rho_{\alpha^{-1}} \in G \). Furthermore, a framing of \( \mathcal{P}_\rho \) at any basepoint \( p_j \in S \) induces a trivialization of \( \mathcal{P}_\rho|_S \).

The discussion above associates a flat \( \mathcal{G}_\sigma \)-torsor framed at \( S \subseteq \partial \Sigma \) to any \( \rho \in M_\sigma(\Sigma, G) \), and by modifying the proof of the Riemann-Hilbert correspondence [13, Cor.1.4] to account for boundary framings, we have:
Proposition 1.18. Let $\Sigma$ be a bordered surface with boundary basepoints $S \subseteq \partial \Sigma$, and let $G_\sigma \to \Sigma$ be the local system of groups \([1,6]\) obtained from a twist $\sigma \in \text{Hom}(\Pi, \text{Aut}(G))$. There is a bijective correspondence between elements of $M_\sigma(\Sigma, G)$ and flat $G_\sigma$-torsors on $\Sigma$ framed at the boundary basepoints $S \subseteq \partial \Sigma$.

Suppose now that $\Sigma = \Sigma^b_h$ is a connected surface of genus $h \geq 0$ with $b \geq 1$ boundary circles. Recall that the fundamental groupoid $\Pi = \Pi_1(\Sigma, S)$ admits a system of free generators $\mathcal{F}$. Such generators are obtained by considering a finite set of non-intersecting paths $\{P_i\}_{i \in I}$ in $\Sigma$ with endpoints in $S$, such that cutting $\Sigma$ along the $P_i$ results in a polygon. The system $\mathcal{F}$ is then given by the set $\{\{P_i\}_{i \in I} \subseteq \Pi$ and the homotopy classes of $(b - 1)$ boundary circles. Since any twisted homomorphism $\rho \in M_\sigma(\Sigma, G)$ is completely determined by its values on the elements of $\mathcal{F}$, we can state:

Proposition 1.19. Let $\Sigma = \Sigma^b_h$ be a connected bordered surface with boundary basepoints $S$, and consider a twist $\sigma \in \text{Hom}(\Pi, \text{Aut}(G))$. Any choice of a system of free generators of $\Pi$ gives rise to a diffeomorphism:

$$M_\sigma(\Sigma, G) \cong G^{2(h+b-1)}.$$

Remark 1.20. Let $\Sigma^b_h$ be as above, and let $S = \{p_j\}_{j=1}^b$ denote the boundary basepoints.

1) Given a system of free generators $\mathcal{F} \subseteq \Pi$, the twist $\sigma \in \text{Hom}(\Pi, \text{Aut}(G))$ is constructed in practice by the assigning values $\sigma_\alpha \in \text{Aut}(G)$ for $\alpha \in \mathcal{F}$, and then extending by the homomorphism property.

2) A concrete example of a system of free generators of $\Pi$ is described in detail in \cite[§9.2]{9}. There, the authors take the set:

$$\mathcal{F} = \{A_i, B_i, V_j, U_j \mid 1 \leq i \leq h, \ 2 \leq j \leq b\},$$

where the $\{A_i, B_i\}_{i=1}^h$ correspond to the handles of $\Sigma^b_h$, the class $V_j$ is that of the boundary circle based at $p_j \in S$, and $U_j$ is the class of a path from $p_j$ to $p_1$. These generators are then subject to the relation:

$$V_1 \prod_{j=2}^b U_j V_j U_j^{-1} \prod_{i=1}^h [A_i, B_i] = 1,$$

which is the word formed by the boundary segments of the polygon obtained by cutting $\Sigma^b_h$ along the $A_i$, $B_i$ and $U_j$.

To study twisted moduli spaces with $G$ compact, connected and simply connected, it is sufficient to consider twists taking values in the diagram automorphism group $\text{Out}(G)$, viewed as a subgroup of $\text{Aut}(G)$. This is explained by:

Proposition 1.21. Let $G$ be a compact $1$-connected Lie group, and let $\Sigma$ be a bordered surface with boundary basepoints $S \subseteq \partial \Sigma$. If the twists $\sigma, \tau \in \text{Hom}(\Pi, \text{Aut}(G))$ have the same image in $\text{Hom}(\Pi, \text{Out}(G))$, then:

(a) The local systems $G_\sigma$ and $G_\tau$ over $\Sigma$ are isomorphic.

(b) The moduli spaces $M_\sigma(\Sigma, G)$ and $M_\tau(\Sigma, G)$ are $G^{\text{dim} S}$-equivariantly isomorphic.

Outline of proof. Statement (b) follows from (a), equation \([1.7]\), and Proposition 1.18. Statement (a) is a consequence of the fact that if $\sigma, \tau \in \text{Hom}(\Pi, \text{Aut}(G))$ have the same image in
Hom($\Pi$, Out($G$)), then the local systems $E_\sigma$ and $E_\tau$ defined by equation (1.5) are isomorphic. Such an isomorphism can be obtained by constructing a map $\psi: \tilde{\Sigma} \to \text{Aut}(G)$ such that:

$$\psi(x \cdot \alpha^{-1}) = \tau_\alpha \circ \psi(x) \circ \sigma_\alpha^{-1}, \quad \forall x \in \tilde{\Sigma}, \ \alpha \in \Pi.$$

and use it to get an automorphism of the trivial principal Aut($G$)-bundle $\tilde{\Sigma} \times \text{Aut}(G)$ that maps the equivalence classes of $\sim_{(\sigma,\Pi)}$ to those of $\sim_{(\tau,\Pi)}$ (see notation after eq. (1.5)). An explicit construction of the map $\psi: \tilde{\Sigma} \to \text{Aut}(G)$ parallels the one given in [37, App.A]. □

Remark 1.22. Up to this point, we have only considered connected surfaces $\Sigma$, but this assumption is not really restrictive. For a surface $\Sigma = \Sigma_1 \cup \Sigma_2$ with two connected components $\Sigma_i = \Sigma_{i,h}$ ($i = 1, 2$), the fundamental groupoid decomposes as a product $\Pi = \Pi_1(\Sigma_1, S_1) \times \Pi_1(\Sigma_2, S_2)$, where $S_i \subseteq \partial \Sigma_i$ are the basepoints. The twist $\sigma$ decomposes accordingly, and the corresponding twisted moduli space is just a product:

$$M_\sigma(\Sigma_1 \cup \Sigma_2, G) = M_{\sigma_1}(\Sigma_1, G) \times M_{\sigma_2}(\Sigma_2, G).$$

The above discussion thus extends directly to compact surfaces with finitely many connected components.

1.3.2. The tq-Hamiltonian structure. The preceding discussion shows that the moduli spaces $M_\sigma(\Sigma, G)$ are naturally equipped with a group-valued moment map, which is given by evaluation along the boundary $\partial \Sigma$, and equivariant with respect to the natural gauge group action. To specify the q-Hamiltonian form $\omega_\sigma \in \Omega^2(M_\sigma(\Sigma, G))^G$, we follow Ševera’s formulation in [40, Thm.3.1].

Let $M = M_\sigma(\Sigma, G)$ with $\sigma \in \text{Hom}(\Pi, \text{Aut}(G))$ and $\Sigma = \Sigma_{i,h}$ connected. Let $\Gamma \subseteq \text{Aut}(G)$ be the subgroup generated by the image of $\sigma$, let $K = G \rtimes \Gamma$, and consider the group $K(M) := \mathcal{C}^\infty(M, K)$ with pointwise multiplication. The central extension $K(M) \times \Omega^2(M)$ then has multiplication and inversion given by:

$$\begin{align}
(1.8) \quad (q_1, \kappa_1, \omega_1) \cdot (q_2, \kappa_2, \omega_2) &= (q_1 \cdot \kappa_1 q_2, \kappa_1 \kappa_2, \omega_1 + \omega_2 - \frac{1}{2} q_1^* \theta^L \cdot \kappa_1(q_2^* \theta^R)), \\
(1.9) \quad (q_1, \kappa_1, \omega_1)^{-1} &= (\kappa_1^{-1}, q_1^{-1}, -\omega_1),
\end{align}$$

where $q_i : M \to G$, $\kappa_i \in \text{Aut}(G)$ and $\omega_i \in \Omega^2(M)$ for $i = 1, 2$. We are mainly interested in the elements $(\text{ev}_\gamma, 0) \in K(M) \times \Omega^2(M)$, where for $\gamma \in \Pi$:

$$\begin{align}
(1.10) \quad \text{ev}_\gamma : M_\sigma(\Sigma, G) \to K, \quad \rho \mapsto (\rho_\gamma, \sigma_\gamma).
\end{align}$$

Next, suppose $\Delta_F(\Sigma)$ is a polygon obtained from a system of free generators $F \subset \Pi$ (see parag. after Prop.1.18). If $\{E_i\}_{i=1}^{n_E}$ denote the edges of $\partial \Delta_F(\Sigma)$ ($n_E = 4h + 3b - 2$), then their homotopy classes in $\Pi$ satisfy the relation:

$$\prod_{i=1}^{n_E} [E_i] = [\partial \Delta_F(\Sigma)] = 1.$$

Using these objects, we state:

**Theorem 1.23.** Let $\Sigma = \Sigma_{i,h}$ be a bordered surface with boundary basepoints $S$, and fix a twist $\sigma \in \text{Hom}(\Pi, \text{Aut}(G))$. The triple $(M_\sigma(\Sigma, G), \omega_\sigma, \Phi_\sigma)$ is tq-Hamiltonian $G^b$-space, where:
(i) The moment map is given by the holonomies along the boundary circles $V_i$ of $\Sigma$:

$$\Phi_{\sigma} : M_\sigma(\Sigma, G) \rightarrow G\sigma_{V_1}^{-1} \times \cdots \times G\sigma_{V_b}^{-1},$$

$$\rho \mapsto (\rho_{V_1}^{-1}, \cdots, \rho_{V_b}^{-1}).$$

(ii) For any polygon presentation $\Delta_F(\Sigma)$ of $\Sigma$ with edges $\{E_i\}_{i=1}^{n_E}$, the invariant 2-form $\omega_\sigma \in \Omega^2(M_\sigma(\Sigma, G))^G$ is given by Ševera’s formula:

$$\omega_\sigma = \prod_{i=1}^{n_E} (ev_{E_i}, 0),$$

where the product is in the group $K(M_\sigma(\Sigma, G)) \times \Omega^2(M_\sigma(\Sigma, G)).$

Remark 1.24. Equation (1.12) can be obtained by combining the gauge theoretic construction in [2, §9] with the equivalence theorem for tq-Hamiltonian manifolds (Remark 1.4-(4)). For a $G_\sigma$-torsor $P \rightarrow \Sigma$, let $M$ denote the moduli space of flat connections on $P$, modulo gauge transformations that are trivial along $\partial \Sigma$. Then with $\kappa_i = \sigma_{V_i}^{-1}$, $M$ is the Hamiltonian $L(\kappa_1)G \times \cdots \times L(\kappa_b)G$-space equivalent to $M_\sigma(\Sigma, G)$. Modifying the proof of [2, Thm.9.3] to account for the twist $\sigma$, one obtains the invariant symplectic structure on $M$, which by [25, §7] gives rise to $\omega_\sigma$. The key observation with Ševera’s formulation is that the multiplication in the group $K(M_\sigma(\Sigma, G)) \times \Omega^2(M_\sigma(\Sigma, G))$ reproduces the equations of the fusion product of tq-Hamiltonian manifolds.

Example 1.25. (The double) We generalize the example $D(G) \simeq G \times G$ studied in [2, §3.2], by realizing it a twisted moduli space associated to an annulus (or a cylinder) $\Sigma^0_0$.

As generators of $\Pi = \Pi_1(\Sigma^2_0, \{p_1, p_2\})$, we take the paths $X$ and $Y$ depicted in Figure 1.1, and we define the twist $\sigma \in \text{Hom}(\Pi, \text{Aut}(G))$ by setting:

$$\sigma_X = \tau, \sigma_Y = \kappa \in \text{Aut}(G).$$

We use the parametrization $(x, y) = (\rho_X, \rho_Y)$ for the elements $\rho \in D_\sigma(G) = M_\sigma(\Sigma^2_0, G)$. The gauge action of an element $\phi = (g_1, g_2)$ of $\text{Map}(S, G) = G \times G$ is then expressed as:

$$(g_1, g_2) \cdot (x, y) = (g_1x\tau(g_2^{-1}), g_2x\kappa(g_1^{-1})).$$

![Figure 1.1. Generators of $\Pi_1(\Sigma^2_0, \{p_1, p_2\})$ and twist](image-url)
Since the boundary circles of $\Sigma^2$ are then given by $V_1 = Y^{-1}X^{-1}$ and $V_2 = YX$, the components of the moment map $\Phi_{D\sigma} : D\sigma(G) \to G\tau \times G^{-1}\kappa$ are given by:

\begin{equation}
\Phi_i(x, y) = \begin{cases}
x \tau(y), \\
\tau^{-1}(x^{-1}\kappa^{-1}(y^{-1}))
\end{cases},
\end{equation}

and satisfy:

$$\Phi_{D\sigma}((g_1, g_2) \cdot (x, y)) = \left(\text{Ad}_{g_1}^{\tau \kappa} \Phi_1(x, y), \text{Ad}_{g_2}^{\tau^{-1} \kappa^{-1}}(x, y)\right).$$

To construct the invariant 2-form $\omega_{D\sigma}$, we cut the annulus of Figure 1.1 along the path $Y$, and assigning the functions $(\text{ev}_{V_j}, 0), (\text{ev}_Y, 0) \in C^\infty(D\sigma(G), K) \times \Omega^2(D\sigma(G))$ to the edges of the obtained square, we have the situation depicted in Figure 1.2. By Ševera’s formula (1.12):

$$(e, 1, \omega_{D\sigma}) = (e, 1, -\frac{1}{2} (\tau^{-1} x^* \theta^L \cdot y^* \theta^R + x^* \theta^R \cdot \kappa^{-1} y^* \theta^L)), $$

so that:

$$\omega_{D\sigma} = -\frac{1}{2} (\tau^{-1} (x^* \theta^L \cdot y^* \theta^R + x^* \theta^R \cdot \kappa^{-1} y^* \theta^L)).$$

The Map($S, G$)-invariance of $\omega_\sigma$ easily follows from this equation. Modifying the identities used in the proof of [2, Prop.3.2] to incorporate the automorphisms $\tau, \kappa \in \text{Aut}(G)$, one checks the moment map condition, the equation $d\omega_{D\sigma} = \Phi^*_D \eta_{G \times G}$, as well as the minimal degeneracy condition.

1.3.3. Fusion and reduction. Let $\Sigma$ be a possibly disconnected bordered surface with boundary basepoints $S = \{p_j\}_{j=1}^b$, and fix a twist $\sigma \in \text{Hom}(\Pi, \text{Aut}(G))$. For simplicity, we denote the moment map of the associated moduli space by:

$$\Phi = (\Phi_1, \cdots, \Phi_b) : M_\sigma(\Sigma, G) \longrightarrow G\kappa_1 \times \cdots \times G\kappa_b,$$

where $\kappa_j = \sigma_{V_j^{-1}} \in \text{Aut}(G)$ for $j = 1, \cdots, b$.

For integers $1 \leq i < j \leq b$, the internal fusion $(M_\sigma(\Sigma, G))_{ij}$ corresponds to the moduli space of the surface $\Sigma_{(ij)} = \Sigma \cup_{V_i, V_j} \Sigma^3_0$, obtained by gluing a pair of pants $\Sigma^3_0$ to the boundary.
components $V_i$ and $V_j$ of $\Sigma$, as depicted in Figure 1.3. Proposition 1.7 gives the tq-Hamiltonian structure on $(M_\sigma(\Sigma,G))_{ij}$, for which the moment map is given by:

$$\Phi_{ij} : (M_\sigma(\Sigma,G))_{ij} \rightarrow G\kappa_i\kappa_j \times G\kappa_1 \times \cdots \times G\kappa_i \times \cdots \times G\kappa_b,$$

$$\rho \mapsto (\Phi_i \cdot (\kappa_i \circ \Phi_j), \Phi_1, \cdots, \Phi_i, \cdots, \Phi_j, \cdots, \Phi_b)(\rho).$$

We illustrate this with the generalization of [2, Ex.6.1] to our setup, the fused double $D_\phi(G)$.

**Example 1.26. (The fused double)** Keeping the notation and objects of Example 1.25, consider a pair of pants $\Sigma_3^3$ with paths $W_1$ and $W_2$ joining the basepoints as depicted in Figure 1.4, which we glue to the cylinder $\Sigma_0^2$ to obtain the surface $\Sigma_1^1$.

The paths $A = W_1XW_2^{-1}$ and $B = W_2YW_1^{-1}$ give a system of free generators for $\Pi = \Pi_1(\Sigma_1^1, \{p_1\})$, and we obtain the twist $\varphi \in \text{Hom}(\Pi, \text{Aut}(G))$ satisfying:

$$\varphi_A = \tau, \varphi_B = \kappa.$$
Parametrizing \( \rho \in \mathbb{D}_\varphi(G) = M_\varphi(\Sigma_1^1, G) \) using \((a, b) = (\rho_A, \rho_B)\), the action of the gauge group \( \text{Map}\{p_1\}, G = G \) is expressed as:

\[
g \cdot (a, b) = (\text{Ad}_g^{-\tau}(a), \text{Ad}_g^-\kappa(b)), \quad \forall g \in G,
\]

while the moment map \( \Phi_{\mathbb{D}_\varphi} : \mathbb{D}_\varphi(G) \to G[\tau, \kappa] \) is given by:

\[
\Phi_{\mathbb{D}_\varphi}(a, b) = \Phi_1(a, b) \tau \kappa (\Phi_2(a, b)),
\]

with \( \Phi_i \) as in equation (1.13) and \([\tau, \kappa] = \tau \kappa \tau^{-1} \kappa^{-1}\). The equivariance property in this case reads:

\[
\Phi_{\mathbb{D}_\varphi}(g \cdot (a, b)) = \text{Ad}_g^{-\tau,\kappa}(\Phi_{\mathbb{D}_\varphi}(a, b)), \quad \forall g \in G.
\]

Next, cutting \( \Sigma_1^1 \) along the paths \( A \) and \( B \), and assigning the appropriate elements of \( K(\mathbb{D}_\varphi(G)) \) to the edges of the resulting pentagon, equation (1.12) yields:

\[
(e, 1, \omega_{\mathbb{D}_\varphi}) = \left( \Phi_{\mathbb{D}_\varphi}, [\tau, \kappa], 0 \right)^{-1} (a, \tau, 0) (b, \kappa, 0) (a, \tau, 0)^{-1} (b, \kappa, 0)^{-1},
\]

\[
= \left( \Phi_{\mathbb{D}_\varphi}, [\tau, \kappa], 0 \right)^{-1} (\Phi_1, \tau \kappa, -\frac{1}{2} \tau^{-1} a^* \theta^L \cdot b^* \theta^R) (\Phi_2, \tau^{-1} \kappa^{-1} - \frac{1}{2} a^* \theta^R \cdot \kappa^{-1} b^* \theta^L),
\]

so that the resulting 2-form \( \omega_{\mathbb{D}_\varphi} \in \Omega^2(\mathbb{D}_\varphi(G))^G \) coincides with the one given by Proposition 1.7

\[
\omega_{\mathbb{D}_\varphi} = \omega_{D_\sigma} - \frac{1}{2} \Phi_1^* \theta^L \cdot \Phi_2^*(\tau \kappa)^* \theta^R,
\]

where as in Example 1.25

\[
\omega_{D_\sigma} = -\frac{1}{2} \left( \tau^{-1} (a^* \theta^L) \cdot b^* \theta^R + a^* \theta^R \cdot \kappa^{-1} (b^* \theta^L) \right).
\]

Thus, \( (\mathbb{D}_\varphi(G), \omega_{\mathbb{D}_\varphi}, \Phi_{\mathbb{D}_\varphi}) \) is a tq-Hamiltonian \( G \)-space with \( G[\tau, \kappa] \)-valued moment map.

More generally for \( b, h \geq 1 \), the connected bordered surface \( \Sigma_h^b \) is obtained by “fusing” \( h \) one-holed tori \( \Sigma_1^1 \) and \((b-1)\) cylinders \( \Sigma_0^2 \), and we can hence identify the corresponding moduli space with a product:

\[
M_\sigma(\Sigma_h^b, G) = \left( \bigoplus_{i=1}^h \mathbb{D}_{\varphi_i}(G) \right) \bigoplus \left( D_{\sigma_1}(G) \bigoplus \cdots \bigoplus D_{\sigma_{b-1}}(G) \right)
\]

for appropriate twists \( \sigma_i, \varphi_j \).

For the symplectic reduction of the spaces \( M_\sigma(\Sigma_h^b, G) \), let \( C_i \) be a \( \kappa_i \)-twisted conjugacy class in \( G \), and let:

\[
\bar{C} = (C_1, \cdots, C_b).
\]

By Proposition 1.12 the reduced space:

\[
\mathcal{M}_\sigma(\Sigma, \bar{C}) := \left( M_\sigma(\Sigma, G) \bigoplus \cdots \bigoplus C_i \bigoplus \cdots \bigoplus C_b \right) / G^b.
\]

is a singular symplectic space, and two of its main properties are that:

(i) As a moduli space of flat connections, \( \mathcal{M}_\sigma(\Sigma, \bar{C}) \) parametrizes the connections for which the \( \text{ith} \) boundary holonomy takes values in the conjugacy class \( C_i \subseteq G \kappa_i \).

(ii) If \( \Sigma \) is the surface obtained by capping-off the boundary components of \( \Sigma \), the representation variety:

\[
\text{Hom}_\sigma (\pi_1(\Sigma), G) / G \cong M_\sigma(\Sigma, G) / \text{Map}(S, G)
\]

carries a natural Poisson structure \( [\mathbb{S}], \mathbb{G} \), for which the symplectic leaves are precisely the quotients \( M_\sigma(\Sigma, \bar{C}) \).
Remark 1.27. We discussed tq-Hamiltonian manifolds \((M, \omega, \Phi)\) for which the moment map takes values in \(G \rtimes \text{Aut}(G)\), but where only the identity component \(G\) acts on \(M\). A natural question that arises is whether it is possible to develop a theory where the disconnected group \(G \rtimes \text{Aut}(G)\) acts on \(M\). Although this is an open question we hope to address in other work, we note that it is possible to construct such examples from character varieties. For instance, consider the surface \(\Sigma = \Sigma_2^2\) and \(\kappa \in \text{Aut}(G)\) with \(|\kappa| = 2\). One can implement an action of \(\langle \kappa \rangle\) on \(\Pi_1(\Sigma, \{p_1, p_2\})\), by letting \(\kappa\) permute the boundary circles and the generators \(\{A_i, B_i\}\) of the handles, which gives an action of \((G \rtimes \langle \kappa \rangle)^2\) on \(\text{Hom}(\Pi, G \rtimes \langle \kappa \rangle)\).

2. Duistermaat-Heckman Measures and Localization

This section studies the Duistermaat-Heckman measure associated to a twisted q-Hamiltonian manifold, by extending some of the main results of \cite{3,5} to our setup. To put things into context, let \((M, \omega, \Phi)\) be a Hamiltonian \(G\)-manifold with Liouville form \(\Lambda_M\) and DH measure \(\text{DH}_\Phi = \Phi_*|\Lambda_M| \in \mathcal{D}'(g^*)^G\). The Duistermaat-Heckman localization theorem states that at \(\xi \in g\), the Fourier transform of \(\text{DH}_\Phi\) localizes to integrals over the connected components of the vanishing set \((\xi_M)^{-1}(0)\). In equation form \cite{6,15,16,36}:

\[
(2.1) \quad \int_{g^*} e^{2\pi i \langle \mu, \xi \rangle} d\text{DH}_\Phi(\mu) = \sum_{Z \in (\xi_M)^{-1}(0)} \int_{Z} e^{\iota_Z(\omega + (\Phi, 2\pi i \xi))} Eul(N_Z, 2\pi i \xi),
\]

where \(\iota_Z : Z \hookrightarrow M\) is the inclusion and \(Eul(N_Z, \cdot)\) is the equivariant Euler form of the normal bundle \(N_Z = TM|_Z/TZ\).

The main result of this work, Theorem 2.9, generalizes equation (2.1) to the case of a \(G\)-valued tq-Hamiltonian manifold \((M, \omega, \Phi)\). Here, \(\text{DH}_\Phi\) is an \(\text{Ad}_{G}^\kappa\)-invariant measure on \(G\), and is given by a Fourier series in the \(\kappa\)-twining characters \(\{\tilde{\chi}_\lambda^\kappa\} \subset L^2(G^\kappa)^G\). The RHS of equation (2.1) is replaced by a Fourier coefficient \(\langle \text{DH}_\Phi, \tilde{\chi}_\lambda^\kappa \rangle\), while the localized integrals of the LHS now involve subgroups of the max torus \(T \subseteq G\) defined by \(\kappa \in \text{Aut}(G)\).

The upcoming subsections are organized as follows. Section 2.2 reviews twining characters and their properties. Section 2.3 discusses the basics of DH measures of tq-Hamiltonian manifold. One of the delicate points there is the construction of the corresponding Liouville form \(\Lambda_M\), which is carried out in section 3.2. Section 2.4 deals with the localization theorem, and also builds upon results in section 3.2. Finally, we return to twisted moduli spaces in section 2.4 where we compute their DH measures.

2.1. Twining characters. In the entirety of this section, \(G\) denotes a compact, connected, simply connected and simple Lie group. Unless otherwise stated, \(\kappa \in \text{Out}(G)\) is induced by a Dynkin diagram automorphism, and as explained in section 1.1 we view \(\text{Out}(G)\) as a subgroup of \(\text{Aut}(G)\). We continue with the notation introduced at the beginning of section 1.1.

2.1.1. Notation. For a dominant weight \(\lambda \in \Lambda^*_+\), we denote the corresponding irreducible representation by \((\rho_\lambda, V_\lambda)\), and the corresponding irreducible character by \(\chi_\lambda\). Letting \(v_\lambda \in V_\lambda\) denote the normalized highest weight vector, we will denote by:

\[
(2.2) \quad \Delta_\lambda(g) := \langle v_\lambda, \rho_\lambda(g) \cdot v_\lambda \rangle, \quad \forall g \in G,
\]
the “spherical harmonic” function corresponding to $\lambda \in \Lambda^*_+$. Letting $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$ denote the half-sum of positive roots of $G$, we note that $\rho \in (t^*)^\kappa$. Lastly, we denote the $\kappa$-fixed dominant weights by $(\Lambda^*_+)^\kappa = \Lambda^*_+ \cap (t^*)^\kappa$.

2.1.2. Definitions. Let $(\rho_V, V)$ be a representation of $G$. We say that such a representation is $\kappa$-admissible if it admits an implementation of $\kappa$ on $V$ that is compatible with the action of $G$, that is, if there exists a unitary operator $\tilde{\kappa}_V \in \text{Aut}(V)$ such that:

$$\rho_V (\kappa(g)) = \tilde{\kappa}_V \circ \rho_V (g) \circ \tilde{\kappa}_V^{-1}, \ \forall g \in G.$$ 

These representations are closed under direct sums and tensor products. In the particular case that $V = V_{\lambda}$ for $\lambda \in (\Lambda^*_+)^\kappa$, Schur’s lemma gives the existence of a unique unitary automorphism $\tilde{\kappa}_\lambda \in \text{Aut}(V_{\lambda})$ satisfying the equation above and preserving the highest weight vector $v_\lambda \in V_{\lambda}$.

**Definition 2.1.** Let $(\rho_V, V)$ be a $\kappa$-admissible representation of $G$, and let $\tilde{\kappa}_V \in \text{Aut}(V)$ be an implementation restricting to $\tilde{\kappa}_\lambda$ on each irreducible summand $V_{\lambda}$ with $\lambda \in (\Lambda^*_+)^\kappa$. The $\kappa$-twining character of $(\rho_V, V)$ is the function $G \rightarrow \mathbb{C}$ given for any $g \in G$ by:

$$\tilde{\chi}_V^\kappa (g) := \text{tr}_V (\tilde{\kappa}_V \circ \rho_V (g)).$$

For $V = V_{\lambda}$ irreducible, we denote by $\tilde{\chi}_V^\kappa$ the corresponding twining character.

For a $\kappa$-admissible representation $(\rho_V, V)$, the non-vanishing contributions to the function $\tilde{\chi}_V^\kappa$ only come from the irreducible factors $V_{\lambda} \subseteq V$ such that $\lambda \in (\Lambda^*_+)^\kappa$, since the implementation $\tilde{\kappa}_V \in \text{Aut}(V)$ necessarily acts a permutation on the remaining irreducible factors [43 §4.1]. For a second $\kappa$-admissible representation $(\rho_{V'}, V')$, one easily checks that:

$$\tilde{\chi}_{V \oplus V'}^\kappa = \tilde{\chi}_V^\kappa + \tilde{\chi}_{V'}^\kappa, \quad \tilde{\chi}_{V \otimes V'}^\kappa = \tilde{\chi}_V^\kappa \cdot \tilde{\chi}_{V'}^\kappa,$$

as it is the case with usual characters.

**Remark 2.2.** An alternative way of introducing $\kappa$-twining characters uses representations of the disconnected group $G \rtimes \langle \kappa \rangle$. A detailed reference for this material is Mohrdieck’s [33 §2.4]. By [33 Prop.2.7], an irrep $V$ of $G \rtimes \langle \kappa \rangle$ is either of the form $V = \bigoplus_{i=1}^{\lambda^\kappa} V_{\kappa(\lambda)}$ for $\lambda \in \Lambda^*_+$ such that $\kappa(\lambda) \neq \lambda$, or of the form $V = V_{\lambda}$ for $\lambda \in (\Lambda^*_+)^\kappa$, with $|\kappa|$ inequivalent homomorphisms $\rho_{\lambda,j} : G \rtimes \langle \kappa \rangle \rightarrow \text{Aut}(V_{\lambda})$, coming from the distinct implementations $\exp(\frac{2\pi i}{|\kappa|}j)\tilde{\kappa}_\lambda$ of $\kappa$ on $V_{\lambda}$.

Under the identification $G \kappa \equiv G$, the $\kappa$-admissible representations of $G$ are the restrictions to the component $G_\kappa$ of the representations of $G \rtimes \langle \kappa \rangle$. At the level of irreducible characters, the restriction $\chi_V|_{G_\kappa}$ vanishes for $\kappa(\lambda) \neq \lambda$ [33 Prop.2.8], while $\chi_V|_{G_\kappa} \equiv \exp(\frac{2\pi i}{|\kappa|}j)\hat{\chi}_\lambda^\kappa$ for $\lambda \in (\Lambda^*_+)^\kappa$, $j = 1, \ldots, |\kappa|$.

The reason for introducing twining characters is that we prefer to think in terms of class functions for $\text{Ad}_G^\kappa$ on the group $G$, as opposed to restrictions of functions defined on $G \rtimes \langle \kappa \rangle$. This is also the motivation for introducing $\kappa$-admissible representations.

2.1.3. Twining characters as $L^2$ functions. Let $L^2(G)$ be the space of $\mathbb{C}$-valued $L^2$-functions on $G$ with respect to the normalized Haar measure $dg$, and recall the convolution product on continuous functions:

$$(\psi \ast \varphi)(x) = \int_G \psi(xg^{-1}) \varphi(g) dg, \ \psi, \varphi \in C^0(G, \mathbb{C}).$$
Let $L^2(G_\kappa)^G \subset L^2(G)$ denote the subspace of $\text{Ad}_G^\kappa$-invariant functions. The irreducible twining characters $\{\tilde{\chi}_\lambda^\kappa\}_{\lambda \in (\Lambda_+^\kappa)}$ extend several properties of the usual characters:

**Proposition 2.3.** With the notations of this section:

1. The averaging map $\Lambda_{\text{v}}^\kappa : L^2(G) \to L^2(G_\kappa)^G$, $f \mapsto \int_G ((\text{Ad}_g^\kappa)f)dg$ is an orthogonal projection.

2. The twining characters $\{\tilde{\chi}_\lambda^\kappa\}_{\lambda \in (\Lambda_+^\kappa)}$ generate a dense subspace of $L^2(G_\kappa)^G$, and satisfy the orthogonality relations:

$$\langle \tilde{\chi}_\lambda^\kappa, \tilde{\chi}_\mu^\kappa \rangle_{L^2} = \delta_{\mu\lambda}, \quad \lambda, \mu \in (\Lambda_+^\kappa)^\kappa.$$

3. For a second Dynkin diagram automorphism $\tau \in \text{Out}(G)$, one has the identities:

$$\langle \tilde{\chi}_\lambda^\kappa \ast \tilde{\chi}_\mu^\kappa \rangle_{L^2} = (\dim V_\lambda)^{-1}\delta_{\mu\lambda}\tilde{\chi}_\lambda^\kappa(x), \quad \forall x \in G,$$

$$\langle \tilde{\chi}_\lambda^\kappa \otimes \tilde{\chi}_\mu^\kappa \rangle_{L^2} = \dim V_\lambda \int_G \tilde{\chi}_\lambda^{(\kappa\tau)}(gx_1\kappa_1(y)) dg, \quad \forall (x, y) \in G \times G,$$

for all $\lambda, \mu \in (\Lambda_+^\kappa) \cap (\Lambda_+^\kappa)^\tau$.

4. For $\lambda \in (\Lambda_+^\kappa)^\kappa$, the spherical harmonic $\Delta_\lambda : G \to \mathbb{C}$ of eq. (2.2) satisfies the identity:

$$\int_G \Delta_\lambda(\text{Ad}_g^\kappa x)dg = (\dim V_\lambda)^{-1}\tilde{\chi}_\lambda^\kappa(x).$$

We only prove equation (2.4) here, since the remaining statements of are examined in the proof of [43, Prop.4.4].

**Proof of eq.(2.4)** The function:

$$f(x, y) = \dim V_\lambda \int_G \tilde{\chi}_\lambda^{(\kappa\tau)}(gx_1\kappa_1(y)) dg,$$

is an $\text{Ad}_G^\kappa \times \text{Ad}_G^\kappa$-invariant $L^2$ function on $G \times G$. It suffices to show that $\langle \tilde{\chi}_\lambda^\kappa \otimes \tilde{\chi}_\mu^\kappa, f \rangle = 1$, which implies by the orthogonality and density of twining characters that $f \equiv \tilde{\chi}_\lambda^\kappa \otimes \tilde{\chi}_\mu^\kappa$. Using the invariance of the Haar measure under inversion and $\text{Out}(G)$, along with the convolution of twining characters and the identities:

$$\tilde{\chi}_\lambda^\kappa(y) = \tilde{\chi}_\lambda^{(\tau^{-1})}(y^{-1}) = \tilde{\chi}_\lambda^{(\tau^{-1})}(\tau^{-1}(y^{-1})),$$

one computes that for any $h \in G$:

$$\int_G \tilde{\chi}_\lambda^{(\kappa\tau)}(h\kappa_1(y)) \tilde{\chi}_\lambda(y)dy = \left(\tilde{\chi}_\lambda^{(\kappa\tau)} \ast \tilde{\chi}_\lambda^{(\tau^{-1})}\right)((\kappa\tau^{-1})(h)) = (\dim V_\lambda)^{-1}\tilde{\chi}_\lambda^\kappa(h).$$

Using this last equation with $h = gx_1\kappa_1(g^{-1})$, we now have:

$$\langle f, \tilde{\chi}_\lambda^\kappa \otimes \tilde{\chi}_\mu^\kappa \rangle = \dim V_\lambda \int_{G \times G} \left[ \int_G \tilde{\chi}_\lambda^{(\kappa\tau)}(gx_1\kappa_1(y)) \tilde{\chi}_\lambda(y)dy \right] \tilde{\chi}_\lambda^\kappa(x)dxdg = \int_{G \times G} \tilde{\chi}_\lambda^\kappa(gx_1\kappa_1(g^{-1})) \tilde{\chi}_\lambda^\kappa(x)dxdg = \left(\int_G dg \right) \langle \tilde{\chi}_\lambda^\kappa, \tilde{\chi}_\lambda^\kappa \rangle = 1,$$

where we used the $\text{Ad}_G^\kappa$-invariance of $\tilde{\chi}_\lambda^\kappa$ in the last equality. □
2.2. Definitions and basic properties. Let \((M, \omega, \Phi)\) denote a \(G\kappa\)-valued tq-Hamiltonian manifold. The data of \(G\) and \(\kappa \in \text{Out}(G)\) gives rise to a distinguished \(\text{Ad}^\kappa G\)-invariant form \(\psi_G^\kappa \in \Omega(G)\) (see eq. (3.11), and section 3.2 for the construction). The **Liouville form** associated to \((M, \omega, \Phi)\) is the \(G\)-invariant form defined by:

\[ \Lambda_M := (e^\omega \Phi^* \psi_G^\kappa)^{\text{top}} \in \Omega^{\text{top}}(M)^G. \]

In parallel to the symplectic context, we have:

**Definition 2.4.** The **Duistermaat-Heckman (DH) measure** associated to \((M, \omega, \Phi)\) is defined as the pushforward:

\[ \text{DH}_\Phi := \Phi_* |\Lambda_M| \in \mathcal{D}'(G\kappa)^G, \]

where \(|\Lambda_M| \in \mathcal{D}'(M)^G\) is the induced Liouville measure, and \(\mathcal{D}'(G\kappa)^G\) is the space of \(\text{Ad}^\kappa G\)-invariant distributions on \(G\).

By construction, the singular support of \(\text{DH}_\Phi\) coincides with the singular values of the map \(\Phi : M \to G\kappa\). Outside of that set, we think of the DH measure as (locally) given by a Fourier series:

\[ \text{DH}_\Phi = \frac{1}{\text{vol}_G} \left( \sum_{\lambda \in (\Lambda_+^\kappa)^G} \langle \text{DH}_\Phi, \chi_\lambda^\kappa \rangle \chi_\lambda^\kappa \right) d\text{vol}_G, \]

where \(d\text{vol}_G\) is the Riemannian measure on \(G\). (see Remark 2.8 for comments on this point).

For \(i = 1, 2\), let \(\kappa_i \in \text{Out}(G)\), and consider \(G\kappa_i\)-valued tq-Hamiltonian \(G\)-manifolds \((M_i, \omega_i, \Phi_i)\) with fusion product \((M_1 \otimes M_2, \omega_{\text{fus}}, \Phi_{\text{fus}})\), where:

\[ \Phi_{\text{fus}} : M_1 \otimes M_2 \to G\kappa_1 \kappa_2, (x,y) \mapsto \text{Mult} (\Phi_1(x) \times \kappa_1(\Phi_2(y))). \]

Under fusion, DH measures behave as follows:

**Proposition 2.5.** With the notation above, one has that:

\[ \text{DH}_{\Phi_{\text{fus}}} = \text{DH}_{\Phi_1} \ast (\kappa_1)_* \text{DH}_{\Phi_2} \in \mathcal{D}'(G\kappa_1 \kappa_2)^G. \]

For any \(\lambda \in (\Lambda_+^\kappa)^G \cap (\Lambda_+^\kappa)^G\), the Fourier coefficient of \(\text{DH}_{\Phi_{\text{fus}}}\) corresponding to \(\chi_\lambda^{(\kappa_1, \kappa_2)}\) is given by:

\[ \langle \text{DH}_{\Phi_{\text{fus}}}, \chi_\lambda^{(\kappa_1, \kappa_2)} \rangle = (\dim V_\lambda)^{-1} \langle \text{DH}_{\Phi_1}, \chi_\lambda^{\kappa_1} \rangle \langle \text{DH}_{\Phi_2}, \chi_\lambda^{\kappa_2} \rangle. \]

**Proof.** For the Liouville forms, we have by Corollary 3.18 that:

\[ \Lambda_{M_1 \otimes M_2} = \Lambda_{M_1} \otimes \Lambda_{M_2} \in \Omega^{\text{top}}(M_1 \times M_2)^G. \]

The definitions of \(\Phi_{\text{fus}}\) and \(\text{DH}_{\Phi_{\text{fus}}}\) yield:

\[ \text{DH}_{\Phi_{\text{fus}}} = \text{Mult}_* (\text{DH}_{\Phi_1} \otimes (\kappa_1)_* \text{DH}_{\Phi_2}), \]

and since convolution of measures on \(G\) is given by pushforward under the multiplication map \(\text{Mult} : G \times G \to G\), we obtain the first equation of the statement. The second equation follows...
The RHS of this equation is determined from the convolution of DH measures under fusion:

\[ \langle \text{DH}_{\Phi_{\text{flat}}}, \tilde{x}^{(k_1 \kappa_2)} \rangle = \int_{G \times G} \tilde{x}^{(k_1 \kappa_2)} (x \kappa_1(y)) \, dg \, d \text{DH}_{\Phi_1} (g^{-1} x \kappa_1(y)) \, d \text{DH}_{\Phi_2}(y) \]

\[ = \int_{G \times G} \left[ \int_G \tilde{x}^{(k_1 \kappa_2)} (g x \kappa_1(g^{-1}) \kappa_1(y)) \, dg \right] \, d \text{DH}_{\Phi_1}(x) \, d \text{DH}_{\Phi_2}(y) \]

\[ = \langle \dim V_\lambda \rangle^{-1} \langle \text{DH}_{\Phi_1}, \tilde{x}^{\kappa_1}_\lambda \rangle \langle \text{DH}_{\Phi_2}, \tilde{x}^{\kappa_2}_\lambda \rangle, \]

where the third equality comes from equation (2.4).

\[ \square \]

Remark 2.6. Suppose that we have a fusion product \( M_1 \oplus \cdots \oplus M_r \), where each \( M_i \) is a \( G_i \)-valued tq-Hamiltonian manifold. The above generalizes [5, Eq.(30)] to:

\[ \langle \text{DH}_{\Phi_{\text{flat}}}, \tilde{x}^{(k_1 \cdots \kappa_r)} \rangle = \langle \dim V_\lambda \rangle^{-1} \prod_{j=1}^r \langle \text{DH}_{\Phi_j}, \tilde{x}^{\kappa_j}_\lambda \rangle, \quad \lambda \in \bigcap_{i=1}^r (\Lambda_i^*)^{\kappa_i}. \]

The next point we address is the relation between the DH measure and the volumes of reduced spaces \( M_a = \Phi^{-1}(a)/Z_a^\circ \). Recall that the conjugacy classes of stabilizer subgroups \( H \subset G \) induce the orbit type stratification \( M = \cup_{(H)} M(H) \) [31, §A.1], and that there exists a unique open dense \( M_{\text{prin}} \subset M \) called the principal stratum. The latter corresponds to the smallest conjugacy class of stabilizers \( \Gamma \), where \( \Gamma \) is called a principal stabilizer.

**Proposition 2.7.** Let \( (M, \omega, \Phi) \) be a \( G_k \)-valued tq-Hamiltonian manifold with finite principal stabilizer \( \Gamma \). Let \( a \in G_k \) be such that \( \Phi^{-1}(a) \) intersects \( M_{\text{prin}} \), and let \( C = \text{Ad}^\circ G(a) \). At \( a \in G_k \), the Radon-Nikodym derivative of \( \text{DH}_\Phi \) with respect to the Riemannian measure on \( G \) is given by:

\[ \frac{d \text{DH}_\Phi}{d \text{vol}_G}(a) = \frac{1}{|\Gamma| \text{Vol}(C)} \text{Vol}(M_a). \]

**Proof.** We have the untwisted q-Hamiltonian manifold \( Y = M \otimes C^- \) with moment map \( \Psi(x, h) = \Phi(x) h^{-1} \). The symplectic quotient \( M_a = \Phi^{-1}(C)/G \) coincides with \( Y/G = \Psi^{-1}(e)/G \), and the principal stabilizers of both \( \Psi^{-1}(e) \) and \( \Phi^{-1}(a) \) are conjugate to \( \Gamma \). By [5, Thm.4.1] then:

\[ \frac{d \text{DH}_\Psi}{d \text{vol}_G}(e) = \frac{1}{|\Gamma| \text{Vol}_G \text{Vol}(M_a)}. \]

The RHS of this equation is determined from the convolution of DH measures under fusion:

\[ \frac{d \text{DH}_\Psi}{d \text{vol}_G}(e) = \langle \text{DH}_\Phi \ast \text{Inv}_a \text{DH}_G, \delta_e \rangle = \text{Vol}(C) \langle \text{DH}_\Phi, \delta_a \rangle = \text{Vol}(C) \frac{d \text{DH}_\Phi}{d \text{vol}_G}(a). \]

\[ \square \]

**Remark 2.8.** We conclude this subsection with some comments.

(a) The validity of the Fourier series in eq. (2.5) for \( \text{DH}_\Phi \) follows from the discussion in the last paragraphs of [5, §4.1], which is based on the approach in [23, §3]. Consider the Laplace-Beltrami operator \( \Delta \) on the disconnected group \( G \rtimes (\kappa) \). For \( t > 0 \), apply the smoothing operator to \( \text{DH}_\Phi \) and set:

\[ \frac{d \text{DH}_\Phi}{d \text{vol}_G} = \frac{1}{|\kappa| \text{Vol}_G} \lim_{t \to 0^+} \sum_{[V] \in \text{Inn}(G \rtimes (\kappa))} \langle \text{DH}_\Phi, e^{-t \Delta} \chi_V \rangle \chi_V. \]
Since $DH_\Phi$ is supported on the component $G\kappa$, we have by Remark 2.2 that $\langle DH_\Phi, \chi_V \rangle$ vanishes for $V$ corresponding to $\lambda \in \Lambda_+^* \setminus (\Lambda_+^*)^\kappa$, and $\langle DH_\Phi, \chi_V \rangle = \exp(2\pi i j)\langle DH_\Phi, \tilde{\chi}_\lambda \rangle$ for $V$ corresponding to $\lambda \in (\Lambda_+^*)^\kappa$, $j = 1, \ldots, |\kappa|$. The previous equation then becomes:

$$\frac{dDH_\Phi}{\text{vol}_G} = \frac{1}{\text{vol}_G} \lim_{t \to 0^+} \sum_{\lambda \in (\Lambda_+^*)^\kappa} e^{-tp(\lambda)}\langle DH_\Phi, \tilde{\chi}_\lambda \rangle \tilde{\chi}_\lambda,$$

where $p(\lambda) = ||\lambda + \rho||^2 - ||\rho||^2$ is the eigenvalue of $\Delta$ on the character $\chi_\lambda$. This regularization gives a rigorous meaning to equation (2.5), and is necessary when that sum diverges.

(b) The conditions on the principal stratum and stabilizer in Proposition 2.7 are there to guarantee that the formula holds for singular values of the moment map (see [31, §2A.1]). If we are merely interested in a regular value $a \in G\kappa$ of $\Phi$, then $\text{Im}(\Phi^* \theta^R)_x = g_x^\perp = g$ for all $x \in \Phi^{-1}(a)$ by [29, Prop.3.9], and the stabilizers $G_x \subset Z_\kappa^*$ are thus finite. The number $|\Gamma|$ in the formula of Proposition 2.7 is then replaced by the cardinality of the subgroup $\cap_{x \in \Phi^{-1}(a)} G_x \subset Z_\kappa^*$.

2.3. Localization and Fourier coefficients. The main result of this work is the following:

**Theorem 2.9.** Let $G$ be a compact 1-connected simple Lie group with $\kappa \in \text{Out}(G)$, and let $(M, \omega, \Phi)$ be a $G\kappa$-valued tq-Hamiltonian manifold. For a $\kappa$-fixed dominant weight $\lambda \in (\Lambda_+^*)^\kappa$, let $\xi_\lambda = 2\pi i B^\perp(\rho + \lambda) \in \mathfrak{t}^\perp$. The Fourier coefficient of $DH_\Phi$ corresponding to $\lambda$ is given by the localization formula:

$$\frac{\langle DH_\Phi, \tilde{\chi}_\lambda \rangle}{\dim V_\lambda} = \sum_{\substack{Z \subseteq (\xi_\lambda)^{-1}(0) \subset \Phi^{-1}(T)}} \int_{\text{Eul}(N_Z, \xi_\lambda)} \frac{(\Phi_Z)^{\lambda + \rho} e^{\omega_Z} \Phi_Z^* (\phi_\kappa)}{\text{Eul}(N_Z, \xi_\lambda)},$$

where:

- $Z$ is a connected component of the vanishing set $(\xi_\lambda)^{-1}(0) \subseteq \Phi^{-1}(T)$;
- $\omega_Z$ and $\Phi_Z$ are the pullbacks of $\omega$ and $\Phi$ to $Z \subset \Phi^{-1}(T)$;
- $\text{Eul}(N_Z, \cdot)$ is the $T$-equivariant Euler form of the normal bundle $N_Z = TM|_Z / T_Z$;
- The function $(\Phi_Z)^{\lambda + \rho}$ designates $(\Phi_Z(z))^{\lambda + \rho} = e^{2\pi i (\lambda + \rho, \zeta)}$, for $z \in Z$ and $\zeta \in \mathfrak{t}$ such that $\Phi_Z(z) = e^\zeta$.
- The differential form $\phi_\kappa \in \Omega(G)$ is given by:

$$\phi_\kappa = \begin{cases} 2^{-\frac{1}{2}} \dim T_\kappa \exp(\varpi) \wedge \text{dvol}_{T_\kappa}, & |\kappa| = 2; \\
2^{-1} \exp \left( \varpi + \sqrt{3} \text{dvol}_{T_\kappa} \right), & |\kappa| = 3; \end{cases}$$

with $\text{dvol}_{T_\kappa}$ denoting the Riemannian volume form of the subtorus $T_\kappa \subseteq T$, and $\varpi \in \Omega^2(G)_\mathbb{C}$ the form introduced in eq. (3.17).

The proof below is heavily based on the material of section 3.2 and can therefore be skipped on a first reading.

**Proof.** **Step 1:** The connected components $Z \subseteq (\xi_\lambda)^{-1}(0)$ are tq-Hamiltonian $T$-spaces. For any $z \in Z$, $\xi_\lambda \in (\mathfrak{t}^\perp)_\mathbb{C}$ implies that:

$$(\kappa(\xi_\lambda)^L - \xi_\lambda^R)_{\Phi(z)} = ((\text{Ad}_{\Phi(z)} - 1)\xi_\lambda)^R_{\Phi(z)} = 0,$$
and since $\xi_\lambda$ is a regular element of $t_\xi$ under $\text{Ad}_G$, we have that $\Phi(z) \in T$. Thus, $Z \subseteq \Phi^{-1}(T)$, and the restriction of the $G$ action to the action of $T$ preserves these components.

**Step 2:** Consider the equivariant form:

$$\hat{\Lambda}_M^\lambda := e^{\omega} \Phi^* (\Delta_\lambda \hat{\psi}_G^\kappa) \in \Omega_G(M),$$

where $\hat{\psi}_G^\kappa \in \Omega(G)$ is the restriction of the Gauss-Dirac spinor of eq. (3.16) to $G \subset G_C$, and $\Delta_\lambda$ is the spherical harmonic corresponding to $\lambda \in (\Lambda^*_+)$. The form $\hat{\Lambda}_M^\lambda$ is equivariantly closed at $\xi_\lambda \in (t^\kappa)_\xi$:

$$(d - t_{(\xi_\lambda)_M}) \hat{\Lambda}_M^\lambda = 0.$$

Proposition 3.13 (3) states that $\hat{\psi}_G^\kappa$ satisfies the differential equation:

$$(d + \eta) \left( \Delta_\lambda \hat{\psi}_G^\kappa \right) = \theta \left( e^{\omega}(\xi_\lambda) \right) \cdot \left( \Delta_\lambda \hat{\psi}_G^\kappa \right),$$

and since $(\Phi, \omega)$ is a Dirac morphism, taking the pullback under $\Phi : M \to G\kappa$ of this equation and then multiplying by $e^{\omega}$ yields $d\hat{\Lambda}_M^\lambda = t_{(\xi_\lambda)_M} \hat{\Lambda}_M^\lambda$.

Now, by the Berline-Vergne localization formula \[6\] Thm.7.13:

$$\int_M (\hat{\Lambda}_M^\lambda)[\text{top}] = \sum_{Z \subset (\xi_\lambda)_M} \int_{\text{Eul}(N_Z, \xi_\lambda)} \int Z \hat{\Lambda}_M^\lambda(\xi_\lambda).$$

**Step 3:** The LHS of equation (2.7) is given by:

$$\int_M (\hat{\Lambda}_M^\lambda)[\text{top}] = (\dim V_\lambda)^{-1} \langle DH_\Phi, \hat{\chi}_\lambda^\kappa \rangle.$$

We have $\hat{\Lambda}_M^\lambda = \Phi^* (\Delta_\lambda) e^{-t_{(\xi_\lambda)_M}} (e^{\omega} \Phi^* \hat{\psi}_G^\kappa)$ by 3.13 (2), and since $\exp(-t_{(\xi_\lambda)_M})$ does not modify top degree parts, the integrand in the RHS is fact $\Phi^* (\Delta_\lambda) \Lambda_M$. Next, by Proposition 2.3 (4):

$$\int_M (\hat{\Lambda}_M^\lambda)[\text{top}] = \int_M \Phi^* (\Delta_\lambda) \Lambda_M = \int_G \Delta_\lambda(x) dDH_\Phi(x)$$

$$= \int_G \left( \int_G \Delta_\lambda(gx\kappa_1(g^{-1})) dg \right) dDH_\Phi(x)$$

$$= (\dim V_\lambda)^{-1} \int_G \hat{\chi}_\lambda^\kappa(x) dDH_\Phi(x) = (\dim V_\lambda)^{-1} \langle DH_\Phi, \hat{\chi}_\lambda^\kappa \rangle.$$

**Step 4:** The integrals in the RHS of equation (2.7) reduce to:

$$\int_Z \frac{\hat{\Lambda}_M^\lambda(\xi_\lambda)}{\text{Eul}(N_Z, \xi_\lambda)} = \int_Z \frac{(\Phi_Z)^{\lambda+\rho} e^{\omega} \Phi_Z^* (\phi_\kappa)}{\text{Eul}(N_Z, \xi_\lambda)}.$$  

By proposition 3.14, we have that $(\hat{\psi}_G^\kappa)_t = t^\rho (\phi_\kappa)_t$ for all $t \in T$, where $\phi_\kappa$ is the form in the statement. Since $\Delta_\lambda(t) = t^\lambda$ for all $t \in T$, and since $\Phi_Z$ is $T$-valued, we have:

$$t^\rho \hat{\Lambda}_M^\lambda(\xi_\lambda) = e^{\omega x} \Phi_Z^* (\Delta_\lambda \hat{\psi}_G^\kappa) = (\Phi_Z)^{\lambda+\rho} e^{\omega} \Phi_Z^* (\phi_\kappa),$$

which completes the proof of the theorem. \qed
In comparison with the Alekseev-Meinrenken-Woodward localization formula [3, Thm.5.2], the main difference is the presence of the form $\phi_\kappa \in \Omega(G)$ in equation (2.6), which equals 1 for $\kappa = 1$. A second notable difference is the structure of the connected components of the vanishing set $(\xi_\lambda)^{-1}_M(0)$, as the next example shows.

**Example 2.10.** For $\kappa \in \text{Out}(G)$, let $a \in T^{\kappa}$ be an element with stabilizer $T^{\kappa}$ under $\text{Ad}^\kappa_G$, and consider the twisted conjugacy class $C = \text{Ad}^\kappa_G(a)$ with its tq-Hamiltonian structure of Example 1.2. Applying the localization formula (2.6) to $\lambda = 0$ gives the Liouville (or DH) volume of the twisted conjugacy class $C$. Here, the vanishing set of $\xi_0 = 2\pi B^2(\rho)$ is precisely $C \cap T$.

In the case where $\kappa = 1$, we have $C \cap T = \{ w \cdot a \}_{w \in W}$. For nontrivial $\kappa \in \text{Out}(G)$ with $|\kappa| = 2$, each connected component of $C \cap T$ is a $T_\kappa$-orbit $Z_w = T_\kappa \cdot (w \cdot a)$ for some $w \in W^{\kappa}$, with trivial normal bundle $N_{Z_w} \to Z_w$ such that [6, §7.1]:

$$\text{Eul}(N_{Z_w}, 2\pi \rho)^{-1} = (-1)^{\dim T^{-1}_\kappa} (-1)^{|w|} \frac{\text{vol}_G}{\text{vol}_T}.$$  

Using the properties of the orbit root system [43, §2.3] associated to $(\mathfrak{r}, \kappa)$ and [43, Lem.3.7]:

$$\text{Vol}(C) = \sum_{w \in W^{\kappa}} \int_{Z_w} \frac{(\Phi_{Z_w})^\rho e^{\omega_{Z_w} \Phi_{Z_w}^\kappa(\phi_\kappa)}}{\text{Eul}(N_{Z_w}, 2\pi \rho)} = \sum_{w \in W^{\kappa}} \frac{a^{w^{-1} \rho}}{\text{Eul}(N_{Z_w}, 2\pi \rho)} \int_{Z_w} e^{\omega_{Z_w} \Phi_{Z_w}^\kappa(\phi_\kappa)} \frac{\text{vol}_G}{\text{vol}_{T^{\kappa}}} = |T^{\kappa} \cap T_\kappa|^\frac{1}{2} \left( \sum_{w \in W^{\kappa}} (-1)^{|w|} a^{w^{-1} \rho} \right) \frac{\text{vol}_G}{\text{vol}_{T^{\kappa}}} = |\det(\kappa)^-1(\text{Ad}_a \kappa - 1)|^\frac{1}{2} \frac{\text{vol}_G}{\text{vol}_{T^{\kappa}}},$$

where the term $(\text{vol}_{T^{\kappa}})^{-1}$ comes from $T = T^{\kappa} \cdot T_\kappa$ and the fact that $\phi_\kappa$ involves $d\text{vol}_{T_\kappa}$. Thus, the Liouville volume of $C$ coincides with its Riemannian volume [5, Prop.4.2].

2.4 Application to moduli spaces. By decomposing a connected surface $\Sigma^h_0$ into $h$ 1-holed tori and $(b - 1)$ cylinders glued along pants $\Sigma^3_0$, the moduli space corresponding to a twist $\sigma \in \text{Hom}(\Pi, \text{Out}(G))$ can be realized as a fusion product:

$$M_\sigma(\Sigma^h_0, G) = (D_{\varphi_1}(G) \oplus \cdots \oplus D_{\varphi_h}(G)) \oplus (D_{\sigma_1}(G) \oplus \cdots \oplus D_{\sigma_{b-1}}(G)),$$

where the doubles $D_{\sigma_i}(G) = M_{\sigma_i}(\Sigma^2_0, G)$ and $D_{\varphi_i}(G) = M_{\varphi_i}(\Sigma^3_0, G)$ are as in Examples 1.25 and 1.26 respectively, for appropriate twists $\sigma_i \in \text{Hom}(\Pi_1(\Sigma^2_0, \{p_1, p_2\}), \text{Out}(G))$ and $\varphi_i \in \text{Hom}(\Pi_1(\Sigma^3_0, \{p\}), \text{Out}(G))$. From Proposition 2.5, the DH measure $\text{DH}_{\Phi_\sigma}$ associated to $M_\sigma(\Sigma^h_0, G)$ is a convolution product of measures $\text{DH}_{D_{\sigma_i}(G)}$ and $\text{DH}_{D_{\varphi_i}(G)}$, and it therefore suffices to determine the Fourier coefficients of the latter to obtain those of $\text{DH}_{\Phi_\sigma}$.

We first establish the following twisted counterparts to [5, Prop.4.4] and [5, §2.6]:

**Lemma 2.11.** For $G$ compact 1-connected and simple, let $\tau, \kappa \in \text{Out}(G)$.

(1) In the setup of Example 1.6, the Liouville measure of $G\kappa$ coincides with its Riemannian measure as a symmetric space, that is:

$$\text{Vol}(G\kappa) = \text{vol}_G.$$

(2) The twisted double can be realized as the fusion product:

$$D_\sigma(G) = G\tau \oplus G\kappa.$$
Proof. Let $K = G \times \langle \kappa, \tau \rangle$, and let $\nu$ denote the involution $K \times K \to K \times K$, $(a,b) \mapsto (b,a)$. Recall from Example 1.6 that $(G\tau,0,\Phi)$ is the $\nu$-twisted conjugacy class of $(\tau,\tau^{-1}) \in K \times K$ with moment map $\Phi(g) = (g\tau,(g\tau)^{-1})$.

(1) This follows from the proof of [5, Prop.4.4], with the next modifications. Here, $G\tau$ is identified with the symmetric space $(G \times G)/Z_{(\tau,\tau^{-1})}$, where the stabilizer of $(\tau,\tau^{-1})$ is given by:

$$Z_{(\tau,\tau^{-1})}^\nu = \{(\tau(g),g) \in G \times G \mid g \in G \}.$$

Letting $c = (\text{Lie}Z_{(\tau,\tau^{-1})}^\nu)^\perp = \ker(\nu + 1)$, we have by [5, Prop.4.2]:

$$\text{Vol}(G\tau) = 2\frac{\dim \theta}{\dim G} \frac{\text{vol}_G^2}{\text{vol}_{Z_{(\tau,\tau^{-1})}}^\nu} = \text{vol}_G = |\text{det}_c (\text{Ad}_{(\tau,\tau^{-1})} \nu - 1)| \frac{1}{2} \frac{\text{vol}_{G \times G}}{\text{vol}_{Z_{(\tau,\tau^{-1})}}^\nu},$$

since the induced Riemannian metric on $Z_{(\tau,\tau^{-1})}^\nu \subseteq G \times G$ is twice that of $G$.

(2) For this part, we endow the space $G\kappa$ with the $\kappa$-Hamiltonian structure $(G\kappa,0,\Psi)$, where $\Psi(g) = ((g\kappa)^{-1},g\kappa)$, and keep the structure above for $G\tau$. The moment map $\Phi_{\text{fus}}$ of the fusion product $G\tau \otimes G\kappa$ takes values in the component $G\tau \kappa \times G\tau^{-1}\kappa^{-1}$ of $K \times K$. Under our usual identification $G\tau \kappa \times G\tau^{-1}\kappa^{-1} \equiv G \times G$, the components of $\Phi_{\text{fus}}$ are the same as those of Example 1.25, and the resulting 2-form $\omega_{\text{fus}}$ is that of $D_\sigma(G)$. \hfill $\square$

**Proposition 2.12.** Let $\text{DH}_{D_\sigma(G)}$ and $\text{DH}_{\mathbb{D}_\varphi(G)}$ denote the DH measures of the double $D_\sigma(G)$ and its internal fusion $\mathbb{D}_\varphi(G)$ respectively. The Fourier coefficients are respectively given by:

$$\langle \text{DH}_{D_\sigma(G)}, \check{\chi}_\lambda^{\tau} \otimes \check{\chi}_\mu^{-1} \rangle = \delta_{\lambda\mu} \frac{\text{vol}_G^2}{\text{vol}_{G \times G}^2}, \quad \forall \lambda, \mu \in \Lambda_+^* \cap (\Lambda_+^*)^\tau.$$

$$\langle \text{DH}_{\mathbb{D}_\varphi(G)}, \check{\chi}_\lambda^{[\tau,\kappa]} \rangle = \dim \Gamma \text{det}_{c} (\text{Ad}_{(\tau,\kappa)} \nu - 1) \frac{1}{2} \frac{\text{vol}_{G \times G}^2}{\text{vol}_{Z_{(\tau,\kappa)}}^\nu} \text{vol}_{G \times G}^2,$$

$$\forall \lambda \in (\Lambda_+^*)^\kappa \cap (\Lambda_+^*)^\tau.$$

Proof. For the first equation, we have by the previous lemma that:

$$\langle \text{DH}_{D_\sigma(G)}, \check{\chi}_\lambda^{\tau} \otimes \check{\chi}_\mu^{-1} \rangle = \int_{G \times G} \check{\chi}_\lambda^{\tau} (a\tau(b)) \check{\chi}_\mu^{-1} (\tau^{-1}(a^{-1}\kappa^{-1}(b^{-1}))) \text{dvol}_{G \times G}(a,b).$$

Using the identities:

$$\check{\chi}_\mu^{(\kappa\tau)^{-1}}(g) = \check{\chi}_\mu^{(\kappa\tau)^{-1}}((\kappa\tau)^{-1}(g^{-1})),$$

$$\check{\chi}_\mu^{\tau} = \tau^* \check{\chi}_\mu^{\tau}, \quad (\text{Ad}_{(\kappa\tau)}^\mu)^* \check{\chi}_\mu^{\tau} = \check{\chi}_\mu^{\tau},$$

we have that:

$$\check{\chi}_\mu^{-1} (\tau^{-1}(a^{-1}\kappa^{-1}(b^{-1}))) = \check{\chi}_\mu^{-1} (\tau^{-1}(a^{-1}\kappa^{-1}(b^{-1}))) \text{dvol}_{G \times G}^2.$$
Corollary 2.13. Consider the surface \( \Sigma = \Sigma^1_h \) with fundamental groupoid \( \Pi \), let \( \tilde{\varphi}_i \in \text{Hom}(\Pi, \text{Out}(G)) \) be twists for \( 1 \leq i \leq h \) such that the moment map of \( \mathbb{D}_{\tilde{\varphi}_i}(G) \) is \( G_{\tau_i} \)-valued, and let \( \mathcal{C}_j \subseteq G \) be \( \kappa_j \)-twisted conjugacy classes for \( 1 \leq j \leq b \). Consider the space:

\[
M = \mathbb{D}_{\tilde{\varphi}_1}(G) \otimes \cdots \otimes \mathbb{D}_{\tilde{\varphi}_h}(G) \otimes \mathcal{C}_1 \otimes \cdots \otimes \mathcal{C}_b,
\]

with moment map:

\[
\Phi : M \to G_{\tau_1} \times \cdots G_{\tau_h} \times G_{\kappa_1} \times \cdots G_{\kappa_b}.
\]

For a dominant integral weight \( \lambda \in \Lambda^*_+ \) that is invariant under the \( \tau_i \)'s and \( \kappa_j \)'s, the corresponding Fourier coefficient of \( \text{DH}_{\Phi} \) is given by:

\[
\langle \chi_{\lambda}^{(\tau_1, \cdots, \tau_h, \kappa_1, \cdots, \kappa_b)}, \text{DH}_{\Phi} \rangle = \frac{(\text{vol}(G))^{2h}}{(\dim V_{\lambda})^{2h+b-t}} \prod_{j=1}^b \text{Vol}(\mathcal{C}_j) \chi^\kappa_j(\mathcal{C}_j).
\]

In conclusion, the results of [5] §4.3 extend to the case of twisted moduli spaces in the expected way.

3. DIRAC GEOMETRY AND TWISTED CONJUGATION

This section adapts certain key results from [1] to the setup of twisted conjugation, and gathers several technical facts used in the previous sections. Section 3.1 reviews the fundamental concepts of Dirac geometry, namely Dirac structures, Dirac morphisms and pure spinors. Section 3.2 summarizes the properties of certain Dirac structures on a Lie group, and discusses the pure spinors involved in the definition of Liouville forms and localization. Section 3.3 looks at the Dirac geometry of fusion. We continue with the notation introduced in sections 1.1 and 2.1

3.1. Reminders.

3.1.1. Dirac structures. The starting point of Dirac geometry is the following concept [1] §2.1:

**Definition 3.1.** Let \( M \) be a manifold with a closed 3-form \( \chi \in \Omega^3(M) \), and let \( \mathbb{T}M = TM \oplus T^*M \) be equipped with the split symmetric bilinear form:

\[
\langle X \oplus \alpha, Y \oplus \beta \rangle = \iota_X \beta + \iota_Y \alpha, \quad \forall X \oplus \alpha, Y \oplus \beta \in \Gamma(\mathbb{T}M).
\]

A (\( \chi \)-twisted) **Dirac structure** on \( M \) is a subbundle \( E \subset \mathbb{T}M \) which is Lagrangian (i.e. \( E^\perp = E \)), and whose sections are closed under the following Courant bracket on \( \Gamma(\mathbb{T}M) \):

\[
\llbracket X \oplus \alpha, Y \oplus \beta \rrbracket_\chi = [X, Y] + \mathcal{L}_X \beta - d\iota_Y \alpha + \iota_X \iota_Y \chi, \quad \forall X \oplus \alpha, Y \oplus \beta \in \Gamma(\mathbb{T}M).
\]

An obvious example of a Dirac structure on a manifold \( M \) is \( E = TM \) with \( \chi = 0 \). For \( \chi \neq 0 \), non-trivial examples of Dirac structures are given by graphs of 2-forms \( \omega \in \Omega^2(M) \) satisfying \( d\omega = \chi \), and by graphs of \( \chi \)-twisted Poisson structures, i.e. bivectors \( \pi \in \mathfrak{X}^2(M) \) satisfying \( \frac{1}{2} [\pi, \pi]_{\text{Sch}} + \pi^2 \chi = 0 \). Below, we focus on certain Dirac structures on a Lie group \( G \) with \( \chi \in \Omega^3(G) \) the Cartan 3-form \( \eta = \frac{1}{12} [\theta^L, \theta^L] \cdot \theta^L \). These are nicely described in terms of a distinguished trivialization of \( TG \) [1] §3 which we now remind of.

Let \( G \) be a Lie group, let \( B \) be an \( \text{Aut}(G) \)-invariant symmetric non-degenerate bilinear form on \( g = \text{Lie}(G) \). Equip \( G \) with the action of \( G \times G \) given by the map \( \mathcal{A} : G \times G \to \text{Diff}G \) given by:

\[
(3.1) \quad \mathcal{A}(a, b) \cdot g = bga^{-1}, \quad \forall a, b, g \in G.
\]
Letting \( \mathfrak{g} \) designate \( g \) equipped with the bilinear form \(-B\), let \( \mathfrak{d} := \mathfrak{g} \oplus \mathfrak{g} \) denote the Lie algebra of \( G \times G \) with the sum of Lie brackets and inner product \( B_0 = B \oplus -B \). The infinitesimal action of \( \mathfrak{d} \) on \( G \) lifts to the trivialization \( s : G \times \mathfrak{d} \to TG \) given by:

\[
(3.2) \quad s(\xi \oplus \zeta)_g = (\xi_g^L - \zeta^R_g) \oplus \frac{1}{2}(\theta_g^L \cdot \xi + \theta_g^R \cdot \zeta), \quad \forall (\xi \oplus \zeta) \in \mathfrak{d}, g \in G
\]

which is a \( G \times G \)-equivariant map such that [1, Prop.3.1]:

\[
B_0(x, y) = \langle s(x), s(y) \rangle, \quad s([x, y]_\mathfrak{d}) = [s(x), s(y)]_\mathfrak{y},
\]

for all \( x, y \in \mathfrak{d} \). A central property of this map is that it associates an \( \eta \)-twisted Dirac structure \( E^\eta = s(G \times s) \) to any Lagrangian subalgebra \( s \subset \mathfrak{d} [1, \S 3.2] \). The next examples look at two Dirac structures that are intimately related to \( \tau \)-Hamiltonian manifolds.

**Example 3.2. (Cartan-Dirac structure)** For a fixed automorphism \( \kappa \in \text{Aut}(G) \), consider the \( \kappa \)-twisted diagonal subalgebra \( \mathfrak{g}^\kappa_\Delta = \{ \kappa(\xi) \oplus \xi \mid \xi \in \mathfrak{g} \} \) of \( \mathfrak{d} \). Its image under the trivialization \( s \) is the \( \kappa \)-**twisted Cartan-Dirac structure** on \( G [29, \text{Rk.3.5}] \):

\[
E^\kappa_G = \left\{ (\kappa(\xi)^L - \xi^R) \oplus \frac{1}{2}(\kappa^{-1}\theta^L - \theta^R) \cdot \xi \mid \xi \in \mathfrak{g} \right\}.
\]

This Dirac structure is the twisted counterpart to the Cartan-Dirac structure studied in [1, \S 3.3]. For the upcoming discussion, we also introduce the following Lagrangian complement of \( E^\kappa_G \subset TG \):

\[
F^\kappa_G = \left\{ (\kappa(\xi)^L + \xi^R) \oplus \frac{1}{2}(\kappa^{-1}\theta^L - \theta^R) \cdot \xi \mid \xi \in \mathfrak{g} \right\}.
\]

We have that \( F^\kappa_G = s(G \times \mathfrak{g}^\kappa_\Delta^-) \), where \( \mathfrak{g}^\kappa_\Delta^- = \{ \kappa(\xi) \oplus -\xi \mid \xi \in \mathfrak{g} \} \) is the twisted anti-diagonal in \( \mathfrak{d} \). Since \( \mathfrak{g}^\kappa_\Delta^- \) is not a subalgebra, \( F^\kappa_G \) is not Courant integrable.

**Example 3.3. (Gauss-Dirac structure)** Suppose that \( G \) is a complex Lie group, and let \( \mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_- \) be the triangular decomposition of its Lie algebra. The \( \kappa \)-**twisted Gauss-Dirac structure** \( \bar{F}^\kappa_G \subset TG \) is the image under \( s : G \times \mathfrak{d} \to \Gamma(TG) \) of the Lagrangian subalgebra:

\[
(3.3) \quad \mathfrak{s}^\kappa = \left\{ (\xi_+ + \kappa(\xi_0)) \oplus (\xi_- - \xi_0) \mid \xi_\pm \in \mathfrak{n}_\pm, \xi_0 \in \mathfrak{h} \right\} \subset \mathfrak{d}.
\]

With this Dirac structure, we have an alternative Lagrangian splitting \( TG = E^\kappa_G \oplus \bar{F}^\kappa_G \). The link between \( F^\kappa_G \) and \( \bar{F}^\kappa_G \) is clarified in the next section.

### 3.1.2. Dirac morphisms.

**Definition 3.4.** For \( i = 1, 2 \), let \( M_i \) be manifolds with closed forms \( \chi_i \in \Omega^3(M_i) \), and let \( E_i \to M_i \) be Dirac structures. A (strong) **Dirac morphism** \((M_1, E_1, \chi_1) \to (M_2, E_2, \chi_2)\) is a pair \((\Phi, \omega)\), where \( \Phi : M_1 \to M_2 \) and \( \omega \) are \( \Omega^2(M_1) \) such that:

1. The forms \( \chi_i \) and \( \omega \) satisfy: \( \Phi^*\chi_2 = \chi_1 + d\omega \);
2. For all \( x \in M_1 \) and \( v_2 \oplus \alpha_2 \in E_2|_{\Phi(x)} \), there exists a unique \( v_1 \oplus \alpha_1 \in E_1|_x \) such that:

\[
v_2 = \Phi_*|_x v_1, \quad \Phi^*\alpha_2 = \alpha_1 + \iota_{v_1}\omega_x.
\]

Sections \( \zeta_i \in \Gamma(E_i) \) satisfying the equations in (2) for all \( x \in M_1 \) are called \((\Phi, \omega)\)-related, and are denoted by \( \zeta_1 \sim_{(\Phi, \omega)} \zeta_2 \).

Regarding the Dirac morphisms that we will encounter below, it is useful to recall the following points [1, \S§1.6, 2.2]:
3.1.3. Pure spinors. The composition of two Dirac morphisms \((\Phi_1, \omega_1)\) and \((\Phi_2, \omega_2)\) is given by:
\[
(\Phi_2, \omega_2) \circ (\Phi_1, \omega_1) = (\Phi_2 \circ \Phi_1, \omega_1 + \Phi_1^* \omega_2).
\]

(2) If the \(M_i\) are \(G\)-manifolds and the Dirac structures are \(G\)-equivariant, the Dirac morphism \((\Phi, \omega)\) is \(G\)-equivariant when \(\Phi : M_1 \to M_2\) is \(G\)-equivariant and \(\omega \in \Omega^2(M_1)^G\).

(3) For a (strong) Dirac morphism \((\Phi, \omega) : (M_1, E_1, \chi_1) \to (M_2, E_2, \chi_2)\), the uniqueness in condition (2) of the definition is equivalent to the transversality condition that:
\[
\ker(\Phi, \omega) \cap E_1 = 0,
\]
where \(\ker(\Phi, \omega) \subset TM\) is the subspace of elements \(X \oplus \alpha \sim_{(\Phi, \omega)} 0\) [1 §1.6].

Let \(G\) be a Lie group with an automorphism \(\kappa \in \text{Aut}(G)\), and let \(E_G^* \to G\) be the twisted Cartan-Dirac structure introduced previously. Suppose that \(M\) is a \(G\)-manifold, and let \((\Phi, \omega) : (M, TM, 0) \to (G\kappa, E_G^*, \eta)\) be a \(G\)-equivariant Dirac morphism. Unwinding the definition, this is equivalent to saying that the 2-form \(\omega \in \Omega^2(M)^G\) and the equivariant map \(\Phi : M \to G\kappa\) satisfy:

\[
\begin{align*}
(i) \ d\omega &= \Phi^* \eta; \\
(ii) \ \iota_{\xi_M} \omega &= \frac{1}{2} \Phi^* (\kappa^{-1} \theta^L + \theta^R) \cdot \xi, \text{ for all } \xi \in g; \\
(iii) \ \ker \omega_x \cap \ker(\Phi_{x|\mathfrak{x}}) &= 0, \text{ for all } x \in M;
\end{align*}
\]

which are precisely the axioms making \((\omega, \Phi)\) a tq-Hamiltonian \(G\)-space. The case of untwisted q-Hamiltonian manifolds \((\kappa = 1)\) is addressed in [1 §5.1], and this formulation in terms of Dirac morphisms was first discussed by Burzstyn and Crainic in [1].

3.1.3. Pure spinors. Recall that for a vector space \(V\) over \(\mathbb{R}\) or \(\mathbb{C}\) [1 §1], the pairing \(\langle \cdot, \cdot \rangle\) between vectors and covectors on \(V = V \oplus V^*\) is a non-degenerate symmetric bilinear form of split signature. Taking the Clifford algebra \(\text{Cl}(V)\) with product:

\[
xy + yx = \langle x, y \rangle 1, \quad \forall x, y \in V,
\]
the exterior algebra \(\wedge V^*\) gives a spinor module for \(\text{Cl}(V)\), where the isomorphism \(\varrho : \text{Cl}(V) \to \text{End}(\wedge V^*)\) is such that:

\[
\begin{equation}
\varrho(v \oplus \alpha) \phi = \iota_v \phi + \alpha \wedge \phi, \quad \forall v \oplus \alpha \in V, \phi \in \wedge V^*.
\end{equation}
\]

An element \(\phi \in \wedge V^* \setminus \{0\}\) is called a pure spinor if the subspace:

\[
N_\phi = \{ x \in V \mid \varrho(x) \phi = 0 \}
\]
is Lagrangian in \(V\). Furthermore, any Lagrangian subspace \(E \subset V\) arises as a subspace \(N_\phi\) for an appropriate \(\phi \in \wedge V^* \setminus \{0\}\) (see [2 III.1.9] or [1 Eq.(6)] for a precise expression).

The constructions of the previous paragraph extend to the smooth category. For a manifold \(M\), the bundle \(\wedge T^* M\) is a spinor module for \(\text{Cl}(TM)\), and any Lagrangian subbundle \(E \subset TM\) can be locally described by pure spinors in \(\Omega(M)\). If \(E \subset TM\) is \(\chi\)-twisted Dirac structure on \(M\) defined by a pure spinor \(\phi \in \Omega(M)\), then the latter satisfies a differential equation of the form:

\[
(d + \chi) \phi = \varrho(\sigma^E) \phi,
\]
where \(\sigma^E \in \Gamma(E^*)\) is a unique section depending on \(\phi\) (see [1 Prop.2.2] for more details).

Our main interest in pure spinors defining Dirac structures is their relation to volume forms. As a consequence of [1 Prop.1.15-(c)], we have the following fact:
Proposition 3.5. Let $M$ be a manifold, let $(N, E, \chi)$ be a Dirac structure on a manifold $N$, and let $F \subseteq \mathbb{T}N$ be a Lagrangian complement to $E$ defined by a pure spinor $\psi_F \in \Omega(N)$. If $(\Phi, \omega) : (M, TM, 0) \to (N, F, \chi)$ is a strong Dirac morphism, then the backward image:

$$(\Phi, \omega)^{-1}F = \{ x \in \mathbb{T}M \mid \exists y \in F \text{ such that } x \sim_{(\Phi, \omega)} y \}$$

is a Lagrangian subbundle transverse to $TM$ in $\mathbb{T}M$, and is defined by the pure spinor:

$$e^{\omega} \Phi^* \psi_F \in \Omega(M).$$

Furthermore, the top degree part $(e^{\omega} \Phi^* \psi_F)_{\text{top}} \in \Omega^{\text{top}}(M)$ is a volume form on $M$.

In the context of a tq-Hamiltonian $G$-manifold $(M, \omega, \Phi)$, this proposition states that the Dirac morphism $(\Phi, \omega) : (M, TM, 0) \to (G, E_G^\kappa, \eta)$ and the Lagrangian splitting $\mathbb{T}G = E_G^\kappa \oplus F_G^\kappa$ give rise to a volume form $(e^{\omega} \Phi^* \psi_G^\kappa)_{\text{top}}$ on $M$, with $\psi_G^\kappa \in \Omega(G)$ a pure spinor defining the Lagrangian subbundle $F_G^\kappa$. The next subsection addresses the construction and the properties of a distinguished $\psi_G^\kappa$, as well as its links to a certain pure spinor defining the Gauss-Dirac structure $\tilde{F}_G^\kappa$.

3.2. Pure spinors on Lie groups.

3.2.1. The trivialization $G \times \text{Cl}(\mathfrak{g}) \cong \wedge T^*G$. The bilinear form $B$ on $\mathfrak{g}$ induces an isometry between $(\mathfrak{g} \oplus \mathfrak{g}^*, \langle \cdot, \cdot \rangle)$ and $(\mathfrak{d}, B_\mathfrak{d})$ [1, §4.1]. This identification realizes $\text{Cl}(\mathfrak{g})$ as a spinor module for $\text{Cl}(\mathfrak{d})$, where the isomorphism $\varrho : \text{Cl}(\mathfrak{d}) \to \text{End}(\text{Cl}(\mathfrak{g}))$ is given by:

$$\varrho^\text{Cl}(\xi \oplus \zeta)x = \xi \cdot x - (-1)^{|\xi||\zeta|} x \cdot \zeta, \quad \forall \xi \oplus \zeta \in \mathfrak{d}, x \in \text{Cl}(\mathfrak{g}).$$

Next, by fixing a lift $\tau : G \to \text{Pin}(\mathfrak{g})$ of $\text{Ad} : G \to \text{O}(\mathfrak{g})$ such that for any orthonormal basis $\{v_i\} \subset \mathfrak{g}$:

$$\tau(\xi) = \frac{d}{dt} \left( \tau(e^{t\xi}) \right) |_{t=0} = -\sum_{i>j} B([\xi, v_i], v_j) \in \text{Cl}(\mathfrak{g}), \quad \forall \xi \in \mathfrak{g},$$

we obtain the following action of $G \times G$ on $\text{Cl}(\mathfrak{g})$:

$$(3.6) \quad \mathcal{A}^\text{Cl}(a, b)x = \tau(a)x \tau(b^{-1}), \quad \forall (a, b) \in G \times G, x \in \text{Cl}(\mathfrak{g}),$$

which extends the adjoint action of $G$ on $\text{Cl}(\mathfrak{g})$ [1, eq.(76)].

Now fix a generator $\text{vol}_\mathfrak{g} \in \wedge^{\text{top}} \mathfrak{g}$ with dual $\text{vol}_\mathfrak{g}^* \in \wedge^{\text{top}} \mathfrak{g}^*$ such that $\iota(\text{vol}_\mathfrak{g}^*) \text{vol}_\mathfrak{g} = 1$. In light of the previous paragraph, define the trivialization $\mathcal{R} : G \times \text{Cl}(\mathfrak{g}) \to \wedge T^*G$ by the prescription:

$$(3.7) \quad \mathcal{R}(x)_{|y} = (q \circ \iota)^{-1}(x \tau(g)), \quad \forall (g, x) \in G \times \text{Cl}(\mathfrak{g}),$$

where $q : \wedge \mathfrak{g} \to \text{Cl}(\mathfrak{g})$ denotes the quantization map, and where $\iota$ denotes the isomorphism:

$$\star : \wedge \mathfrak{g}^* \to \wedge \mathfrak{g}, \quad \phi \mapsto \iota(\phi) \text{vol}_\mathfrak{g}.$$

The upshot of these constructions is that $\mathcal{R}$ gives a transparent description of the pure spinors defining Lagrangian subbundles of $\mathbb{T}G$. The properties that we need are the following [1, Prop.4.2]:

Proposition 3.6. Consider the trivializations $s : G \times \mathfrak{d} \to \mathbb{T}G$ and $\mathcal{R} : G \times \text{Cl}(\mathfrak{g}) \to \wedge T^*G$ given by equations (3.2) and (3.7) respectively. One has that:
(1) $\mathcal{R}$ intertwines the Clifford actions (3.4) and (3.5):

$$\mathcal{R} \left( \rho^\text{Cl}(\xi \oplus \zeta) \cdot x \right) = \rho(\sigma(\xi \oplus \zeta)) \cdot \mathcal{R}(x), \quad \forall \xi \oplus \zeta \in \mathcal{O}, \ x \in \text{Cl}(g).$$

(2) $\mathcal{R}$ intertwines the actions (3.1) and (3.6) of $G \times G$:

$$\left( A^a(a^{-1}, b^{-1})^* \mathcal{R}(x) \right)_g = (-1)^{|a||g|+|x|} \mathcal{R} \left( A^\text{Cl}(a, b) \cdot x \right)_g, \quad \forall (a, b) \in G \times G, x \in \text{Cl}(g),$$

where $|g| := |\tau(g)|$ for all $g \in G$.

3.2.2. The Cartan-Dirac spinor and Liouville form. For the remainder of this section, we assume that we have a fixed generator $\text{vol}_g \in \wedge^\text{top} \mathfrak{g}$, as well as a fixed automorphism $\kappa \in \text{Aut}(G)$.

Using the same notation for the differential $\kappa \in \text{Aut}(\mathfrak{g})$, the latter can be viewed as an element of $\text{O}(\mathfrak{g})$, and we fix a lift $\tau(\kappa) \in \text{Pin}(\mathfrak{g})$, such that:

$$\tau(\kappa)x \tau(\kappa^{-1}) = (-1)^{|\tau(\kappa)||x|} \kappa(x), \quad \forall x \in \text{Cl}(\mathfrak{g}).$$

As seen in the previous section, we have a Lagrangian splitting $\mathcal{O} = \mathfrak{g}^\kappa_+ \oplus \mathfrak{g}^\kappa_-$, where the $\kappa$-twisted diagonal and anti-diagonal are given by:

$$\mathfrak{g}^\kappa_+ = \{ \kappa(\xi) \oplus \xi \mid \xi \in \mathfrak{g} \}, \quad \mathfrak{g}^\kappa_- = \{ \kappa(\xi) \oplus -\xi \mid \xi \in \mathfrak{g} \}.$$

By direct computation, it is easily verified that these Lagrangians are respectively defined by the following pure spinors in $\text{Cl}(\mathfrak{g})$:

$$x^\kappa_\Delta := \tau(\kappa), \quad x^\kappa_- := q(\text{vol}_g) \tau(\kappa).$$

By equation (3.8) in Prop. 3.6, the Lagrangian subbundles $E^\kappa_G = s(G \times \mathfrak{g}^\kappa_\Delta)$ and $F^\kappa_G = s(G \times \mathfrak{g}^\kappa_-)$ are thus defined by the pure spinors:

$$\left( \phi^\kappa_G \right)_g = \mathcal{R} \left( x^\kappa_\Delta \tau(\kappa^{-1}(g)) \right), \quad \left( \psi^\kappa_G \right)_g = \mathcal{R} \left( x^\kappa_- \tau(\kappa^{-1}(g)) \right),$$

in $\Omega(G)$. By equation (3.9), these forms behave as follows under $\kappa$-twisted conjugation:

$$((\text{Ad}_a^\kappa)^* \phi^\kappa_G)_g = (-1)^{|a||g|+|x^\kappa_\Delta|} (\phi^\kappa_G)_g, \quad ((\text{Ad}_a^\kappa)^* \psi^\kappa_G)_g = (-1)^{|a||g|+|x^\kappa_-|} (\psi^\kappa_G)_g, \quad \forall a, g \in G,$

in the particular case that $G$ is connected, the pure spinors $\phi^\kappa_G, \psi^\kappa_G \in \Omega(G)$ are $\text{Ad}_G^\kappa$-invariant.

Proposition 3.5 and the last paragraph justify the following definition:

**Definition 3.7.** Let $(M, \omega, \Phi)$ be a $G_k$-valued $\text{tq-Hamiltonian}$ manifold. The associated **Liouville form** on $M$ is defined to be the $G$-invariant volume form given by:

$$\Lambda_M := (e^{\omega(\Phi^\kappa)} \psi^\kappa_G)|_{\text{top}} \in \Omega^{\text{top}}(M)^G.$$

**Remark 3.8.** Although the trivialization $\mathcal{R} : G \times \text{Cl}(\mathfrak{g}) \to \wedge T^*G$ depends on the choice of the generator $\text{vol}_g \in \wedge^\text{top} \mathfrak{g}$, the Liouville form $\Lambda_M$ is unaffected by this choice [1 Rk.4.5-(a)].

Indeed, replacing $\text{vol}_g$ by $\lambda \text{vol}_g$ with $\lambda \neq 0$, the isomorphism $\star^{-1} : \wedge \mathfrak{g} \to \wedge^* \mathfrak{g}$ replaces $\mathcal{R}$ by $\lambda^{-1} \mathcal{R}$, but this does not affect $\psi^\kappa_G = \mathcal{R}(q(\text{vol}_g) \tau(\kappa)).$

For the sake of completeness, we give explicit formulas for the pure spinors $x^\kappa_\Delta, x^\kappa_- \in \text{Cl}(\mathfrak{g})$. Suppose for the remainder of this part that $G$ is a compact 1-connected simple Lie group, and that $\kappa \in \text{Out}(G)$ is induced by a non-trivial Dynkin diagram automorphism. To formulate the next proposition, we employ the following notation:
- For any subspace $a \subset g$, denote by $\text{vol}_a \in \wedge^{[\top]}a$ the generator induced by the orientation on $g$, with dual form $\text{vol}_a^* \in \wedge^{[\top]}a^*$ such that $i(\text{vol}_a^*)\text{vol}_a = 1$. For a subgroup $A \subseteq G$ with $a = \text{Lie}A$, denote by $d\text{vol}_A \in \Omega^{[\top]}(A)$ the Riemannian volume form such that $(d\text{vol}_A)|_e = \text{vol}_a^*$.

- Recall that $T_\kappa \subset T$ is the image of $T \rightarrow T, t \mapsto t\kappa(t^{-1})$, with Lie algebra $t_\kappa \subset t$. We have that $|T^\kappa \cap T_\kappa| = 2\dim t_\kappa$ for $|\kappa| = 2$, and that $|T^\kappa \cap T_\kappa| = 3$ for $|\kappa| = 3$.

- For $G$ of type $D_4$ and $|\kappa| = 3$, fix an orthonormal basis $\{a_i, b_i\}_{i=1}^7 \in (g^\kappa)^{\perp}$ such that:

$$\ker \left(\kappa - e^{\pm i\frac{7\pi}{2}}\right) = \text{Span}_C\{-a_i \pm b_i\}_{i=1}^7, \quad t_\kappa = \text{Span}_R\{a_1, b_1\}.$$ 

We can now state:

**Proposition 3.9.** Let $G$ be compact 1-connected and simple, and let $\kappa \in \text{Out}(G)$. A pure spinor $x^\kappa_\Delta \in \text{Cl}(g)$ defining the twisted diagonal $g^\kappa_\Delta \subset \mathfrak{g}$ is given by:

$$x^\kappa_\Delta = \begin{cases} 2^{\frac{1}{2}\dim(g^\kappa)}q(\text{vol}_g^\perp), & \text{for } |\kappa| = 2, \\ 2^{-\frac{1}{2}\dim(g^\kappa)}q\left(e^{-2\sqrt{3}\sum_j a_j \wedge b_j}\right), & \text{for } |\kappa| = 3. \end{cases}$$

A pure spinor $x^\kappa_{\Delta^-} \in \text{Cl}(g)$ defining the twisted anti-diagonal $g^\kappa_{\Delta^-} \subset \mathfrak{g}$ is given by:

$$x^\kappa_{\Delta^-} = \begin{cases} 2^{-\frac{1}{2}\dim(g^\kappa)}q(\text{vol}_g^\perp), & \text{for } |\kappa| = 2; \\ (\sqrt{3})^{\frac{1}{2}\dim(g^\kappa)}q\left(e^{\frac{2}{\sqrt{3}}\sum_j a_j \wedge b_j}\text{vol}_g^\perp\right), & \text{for } |\kappa| = 3. \end{cases}$$

**Outline of proof.** For $x^\kappa_\Delta = \tau(\kappa)$, it suffices to directly check that the given expression yields a lift of the differential $\kappa \in \text{O}(g)$ to $\text{Pin}(g)$, i.e. that:

$$\tau(\kappa)^\top\tau(\kappa) = 1; \quad \text{and } \tau(\kappa)\xi^\top(\kappa^{-1}) = (-1)^{\dim(g^\kappa)}\kappa(\xi), \quad \forall \xi \in g.$$

This is easily done with an orthonormal basis of $g$ when $|\kappa| = 2$, while for $|\kappa| = 3$, the exponential is re-written as a product:

$$x^\kappa_\Delta = 2^{-7}q(e^{-2\sqrt{3}\sum_j a_j \wedge b_j}) = 2^{-7} \prod_{i=1}^7 (1 - 2\sqrt{3}a_ib_i).$$

For the product $x^\kappa_{\Delta^-} = q(\text{vol}_g)^\tau(\kappa)$, one uses $q(\text{vol}_g) = q(\text{vol}_g^\perp)q(\text{vol}_g^{\perp})$, as well as $q(\text{vol}_g^{\perp}) = \prod_i a_ib_i$ when $|\kappa| = 3$. $\square$

**Remark 3.10.** This proposition allows one to determine $\Lambda_M$ and $\psi^\kappa_G$ explicitly in certain cases. For instance, if $|\kappa| = 2$, the approach in the proof of Thm.3.1 leads to:

$$(\psi^\kappa_G)_g = \pm |T^\kappa \cap T_\kappa|^{-\frac{1}{2}} \cdot \det_{L^2}^{\frac{1}{2}} \left(\frac{\text{Ad}_g\kappa + 1}{2}\right) |e^{-\frac{1}{2}B\left(\frac{\text{Ad}_g\kappa - 1}{2}L_g^L, L_g\right)} \wedge (d\text{vol}_T)_g|,$$

on the subset of $G$ where $(\text{Ad}_g\kappa + 1)$ is invertible. One obtains a similar but more cumbersome expression when $|\kappa| = 3$. With $\kappa = 1$, one recovers Thm.4.6.
3.2.3. The Gauss-Dirac spinor. Continuing with the notation above, let \( G_C \) be the complexification of \( G \), and let \( g_C = n_+ \oplus h \oplus n_- \) be the triangular decomposition of its Lie algebra. The twisted Gauss-Dirac structure is the image \( \hat{F}_G^\kappa = \mathfrak{s}(G \times \mathfrak{s}^\kappa) \), where:

\[
\mathfrak{s}^\kappa = \{(\xi_+ + \kappa(\xi_0)) \oplus (\xi_- - \xi_0) \mid \xi_\pm \in n_\pm, \xi_0 \in h\}.
\]

After clarifying the relation between the subalgebra \( \mathfrak{s}^\kappa \subset \mathfrak{d} \) and the twisted anti-diagonal \((\mathfrak{g}_C)^{\kappa}_{\Delta^-}\), we describe the Gauss-Dirac spinor \( \psi^\kappa_C \in \Omega(G_C) \) defining \( \hat{F}_G^\kappa \), and discuss its properties relevant to localization.

Let \( \tau = \frac{1}{2} \sum_{\alpha \in R_+} e_{-\alpha} \wedge e_\alpha \) denote the classical r-matrix \( \hat{I} \) \( \text{[3.6]} \), where \( \{e_\alpha\}_{\alpha \in R} \subset n_+ \oplus n_- \) is a basis of root vectors satisfying \( B(e_\alpha, e_\beta) = \delta_{\beta,-\alpha} \) and \( e_{-\alpha} = \overline{e_\alpha} \). Letting \( A^{-\tau} \in O(g_C \oplus g_C^\kappa) \) denote the map \( 0 \oplus \alpha \mapsto \lambda_\alpha \tau \oplus \alpha \) \( \hat{II} \) \( \text{[1.5]} \), the graph of \( \tau \in \wedge^2 g_C \) in \( g_C \oplus g_C^\kappa \) is given by \( Gr_\tau = A^{-\tau}(0 \oplus g_C^\kappa) \). Considering then the isometry:

\[
\i^\kappa : g_C \oplus g_C^\kappa \to \mathfrak{d}_C, \quad \i^\kappa(\xi \oplus \alpha) = \kappa \left( \xi + \frac{1}{2} B^\kappa \alpha \right) \oplus \left( -\xi - \frac{1}{2} B^\kappa \alpha \right),
\]

we have that \( \hat{II} \) \( \text{Lem.3.16] \): \( \mathfrak{s}^\kappa = \i^\kappa(Gr_\tau) \), \( (g_C)^{\kappa}_{\Delta^-} = \i^\kappa(0 \oplus g_C^\kappa) \), and \( (g_C)^{\kappa}_{\Delta} = \i^\kappa(g_C \oplus 0) \).

At the level of pure spinors defining Lagrangian subspaces, the present setup leads to:

**Proposition 3.11.** A pure spinor defining the Lagrangian subalgebra \( \mathfrak{s}^\kappa \subset \mathfrak{d}_C \) is given by:

\[
x^{\kappa}_{\Delta_-} := \varrho^{Cl} \left( e^{-\varphi(\i^\kappa_\kappa)(\tau)} \right) x^{\kappa}_{\Delta_-} \in \text{Cl}(g_C),
\]

where \( x^{\kappa}_{\Delta_-} \in \text{Cl}(g_C) \) is the pure spinor defining \( (g_C)^{\kappa}_{\Delta^-} \). For \( g_C \) simple and \( \kappa \in \text{Aut}(g_C) \) induced by a Dynkin diagram automorphism, one has that:

\[
x^{\kappa}_{\Delta^-} = \begin{cases} 2^{-\frac{1}{4}} \text{dim}_{t_\kappa} \left( \prod_{\alpha \in R_+} e_{-\alpha} \right) q(\text{vol}^\kappa), & \text{for } |\kappa| = 2; \\ \sqrt{\frac{2}{3}} \left( \prod_{\alpha \in R_+} e_{-\alpha} \right) q \left( e^{\frac{1}{2} \text{vol}_{\kappa}} \text{vol}^\kappa \right), & \text{for } |\kappa| = 3. \end{cases}
\]

**Outline of proof.** Since \( g_C^\kappa \subset g_C \oplus g_C^\kappa \) is defined by \( \text{vol}^\kappa \in \wedge^{[\text{top}]}g_C^\kappa \), and since \( Gr_\tau = A^{-\tau}(g_C^\kappa) \), the latter is defined by the pure spinor:

\[
\varrho(A^{-\tau})\text{vol}^\kappa = \exp(-\tau(\varphi))\text{vol}^\kappa \in \wedge^\kappa g_C^\kappa,
\]

where \( \widehat{A}^{-\tau} \in \text{Pin}(g \oplus g^\kappa)_C \) is the lift of \( A^{-\tau} \) in the representation \( \varrho : \text{Cl}(g \oplus g^\kappa)_C \to \text{End}(\wedge^\kappa g_C^\kappa) \). Next, using the isometry \( \i^\kappa \) and the quantization maps, one constructs the unique isomorphism of spinor modules \( R^\kappa : \wedge^\kappa g_C^\kappa \to \text{Cl}(g_C) \) mapping \( \text{vol}^\kappa \) to \( x^{\kappa}_{\Delta_-} \), and intertwining the Clifford actions of \( \text{Cl}(g \oplus g^\kappa)_C \cong \text{Cl}(\mathfrak{d}_C) \) via:

\[
R^\kappa \left( \varrho(\xi \oplus \alpha) \phi \right) = \varrho^{Cl} \left( \i^\kappa(\xi \oplus \alpha) \right) R^\kappa(\phi), \quad \forall \phi \in \wedge^\kappa g_C^\kappa, \xi \oplus \alpha \in (g \oplus g^\kappa)_C.
\]

Under the isomorphism \( R^\kappa \), the counterpart of the lift \( \widehat{A}^{-\tau} \) is the element \( e^{-\varphi(\i^\kappa)(\tau)} \in \text{Cl}(\mathfrak{d}_C) \), which yields equation \( \text{(3.13)} \).

In the case of \( g_C \) simple and \( \kappa \in \text{Aut}(g_C) \) induced by a diagram automorphism, equation \( \text{(3.14)} \) follows from a direct computation. One uses the expressions of \( x^{\kappa}_{\Delta_-} \) in Proposition \( \text{3.9} \) along with the identity:

\[
\varrho^{Cl} \left( \i^\kappa(\xi \oplus 0) \right) \circ q = q \circ \left( \i \left[ B^\kappa \left( \frac{\kappa - 1}{2} \xi \right) \right] + (\kappa - 1) \xi \right), \quad \forall \xi \in g_C,
\]
Remark 3.12. We omit the detailed construction of $R^\kappa : \wedge g^* \to \text{Cl}(g)$ above, since it would require a disgression that is not used elsewhere, with additional notation. The idea is as follows. Using $B_\kappa$ to identify $g_\kappa^\kappa$ with $(g_\kappa^\kappa)^*$ in the splitting $\mathfrak{g} = g_\Delta^\kappa \oplus g_\Delta^\kappa$, one constructs a first isomorphism of spinor modules $R^\kappa_1 : \wedge g_\Delta^\kappa \to \text{Cl}(g)$. On the other hand, the isometry of equation (3.12) extends to isomorphisms $i^\kappa : \wedge (g \oplus g^*) \to \wedge \mathfrak{g}$ and $i^\kappa : \wedge g^* \to \wedge g_\Delta^\kappa$, and using the quantization maps for $\text{Cl}(g \oplus g^*)$ and $\text{Cl}(\mathfrak{g})$, one constructs a second isomorphism of spinor modules $R^\kappa_2 : \wedge g^* \to \wedge g_\Delta^\kappa$ such that $R^\kappa = R^\kappa_1 \circ R^\kappa_2$ gives the desired properties.

Turning to Lagrangian subbundles of $\mathbb{T}G_C$, define the trivialization $e^\kappa : G \times g_C \to E^\kappa_G$ by:

$$e^\kappa(\xi)|_g = \mathfrak{s}(i^\kappa(\xi + 0))|_g, \ \forall (g, \xi) \in G \times g_C.$$ (3.15)

The twisted Gauss-Dirac structure $\tilde{F}^\kappa_{G_C}$ arises as the image of the Lagrangian complement $F^\kappa_{G_C}$ under the orthogonal transformation $A^{-e^\kappa(t)} \in \Gamma(O(\mathbb{T}G_C))$ [1, Cor.3.17]. By Proposition 3.6 and equation (3.13), the twisted Gauss-Dirac spinor is hence given by:

$$(\tilde{\psi}^\kappa_{G_C})_g := \mathcal{R}(x_{g^*} \tau (\kappa^{-1}(g))), \ \forall g \in G_C.$$ (3.16)

Its main properties are as follows:

**Proposition 3.13.** Suppose $G$ is compact 1-connected and simple with complexification $G_C$, and let $\kappa \in \text{Out}(G)$ be non-trivial. The twisted Gauss-Dirac spinor $\tilde{\psi}^\kappa_{G_C} \in \Omega(G_C)$ satisfies the following properties:

1. **Invariance properties:** For all $a_\pm \in \exp(n_\pm)$ and $t \in T_C$, one has that:

$$R^\kappa_{a_+} \tilde{\psi}^\kappa_{G_C} = L^\kappa_{a_-} \tilde{\psi}^\kappa_{G_C} = \tilde{\psi}^\kappa_{G_C},$$

$$R^\kappa_{t_+} \tilde{\psi}^\kappa_{G_C} = L^\kappa_{t_-} \tilde{\psi}^\kappa_{G_C} = t^\rho \tilde{\psi}^\kappa_{G_C}.$$ (2)

2. **Transformation by r-matrix:** The pure spinors $\tilde{\psi}^\kappa_{G_C}$ and $\psi^\kappa_{G_C}$ are related by the equation:

$$\tilde{\psi}^\kappa_{G_C} = g(\exp (-e^\kappa(t))) \cdot \psi^\kappa_{G_C}.$$ (3)

3. **Differential equations:** For any $\kappa$-invariant dominant weight $\lambda \in (\Lambda^*_+)^\kappa$, the spinor $\Delta^\lambda \tilde{\psi}^\kappa_{G_C}$ satisfies the differential equation:

$$(d + \eta)\Delta^\lambda \tilde{\psi}^\kappa_{G_C} = g(e^\kappa(2\pi i B^\kappa(\lambda + \rho))) \cdot \Delta^\lambda \tilde{\psi}^\kappa_{G_C},$$

where $\Delta^\lambda$ is the spherical harmonic of equation (2.2), and $\rho \in (t^*)^\kappa$ the half-sum of positive roots of $G$.

**Outline of proof.** The invariance properties follow from Proposition (3), while the equation in (2) follows from Proposition (1) and equation (3). The proof for the differential equation in (3) is the same as the proof of [1, Prop.4.18]. □

For the purposes of Duistermaat-Heckman localization, it is useful in practice to have explicit expressions for $(\tilde{\psi}^\kappa_{G_C})_t$ with $t \in T_C$. Using the basis $\{e_\alpha\}_{\alpha \in \mathbb{N}} \subset n_+ \oplus n_-$ introduced above, let
Let Inv : Gθ where we implicitly use below.

3.3. Dirac geometry of fusion. Given a Lie group G, let  and  denote the pullback of  (resp. ) to the ith component of  ×  × . Let  :  ×  ×  denote the multiplication map, and let  :  ×  →  denote inversion. Define the following forms on  ×  ×  ×  × 

\[ \zeta := -\frac{1}{2}B(\theta^{L,1}, \theta^{R,2}) \in \Omega^2(G \times G), \quad \eta_{G \times G} = \eta^1 + \eta^2 \in \Omega^3(G \times G), \]

where  denotes the Cartan 3-form with  and 

\[
(\psi^\kappa_{G\times G})_t = \begin{cases} 
(-1)^{\frac{1}{2}(\dim(T^\kappa) \cap T_\kappa)} \cdot t^\rho (\exp(\varpi) \wedge d\text{vol}_{T_\kappa})_t, & \text{for } |\kappa| = 2; \\
\frac{1}{2}t^\rho \exp(\varpi + \sqrt{3}d\text{vol}_{T_\kappa})_t, & \text{for } |\kappa| = 3.
\end{cases}
\]

\[ (\psi^\kappa_{G\times G})_t = \begin{cases} 
(\psi^\kappa_{G\times G})_t = \begin{cases} 
(-1)^{\frac{1}{2}(\dim(T^\kappa) \cap T_\kappa)} \cdot t^\rho (\exp(\varpi) \wedge d\text{vol}_{T_\kappa})_t, & \text{for } |\kappa| = 2; \\
\frac{1}{2}t^\rho \exp(\varpi + \sqrt{3}d\text{vol}_{T_\kappa})_t, & \text{for } |\kappa| = 3.
\end{cases}
\]

is a strong Dirac morphism. The purpose of the present section is to formulate the main results of [1] §3.4.3 in the context of twisted conjugation. Conceptually, the quickest way of doing so is to apply the theory developed in [1] §3.4.3 to the disconnected group  =  × , where  is a finitely generated subgroup. From this standpoint, the right multiplication map by  in  identifies the component  ×  with the group  equipped with the action  ×  ×  =  ×  |  ×  =  ×  |  ×  which we implicitly use below.

Given  = Aut(G), let  denote the group  equipped with the action  ×  ×  and consider the  -equivariant maps:

\[ \text{Mult}^\kappa_1 : G_{\kappa_1} \times G_{\kappa_2} \rightarrow G_{\kappa_1 \kappa_2}, \quad \text{Mult}^\kappa_1(a, b) = (\text{Mult} \circ (1 \times \kappa_1))(a, b); \]

\[ \text{Inv}^\kappa : G_{\kappa} \rightarrow G_{\kappa}^{-1}, \quad \text{Inv}^\kappa(a) = (\kappa^{-1} \circ \text{Inv})(a); \]

as well as the 2-form:

\[ \zeta^\kappa := (1 \times \kappa)^*\zeta \in \Omega^2(G \times G). \]

Modifying the proof [1] Thm.3.9 to take into account the effect of twisting automorphisms, it is easy to establish:

Proposition 3.14. With the assumptions of Proposition 3.13, we have for all  \( t \in T_G \) that:

\[ (\psi^\kappa_{G\times G})_t = \begin{cases} 
(-1)^{\frac{1}{2}(\dim(T^\kappa) \cap T_\kappa)} \cdot t^\rho (\exp(\varpi) \wedge d\text{vol}_{T_\kappa})_t, & \text{for } |\kappa| = 2; \\
\frac{1}{2}t^\rho \exp(\varpi + \sqrt{3}d\text{vol}_{T_\kappa})_t, & \text{for } |\kappa| = 3.
\end{cases} \]

3.3. Dirac geometry of fusion. Given a Lie group G, let  and  to the ith component of  ×  × , let Mult :  ×  →  denote the multiplication map, and let  :  →  denote inversion. Define the following forms on  ×  × 

\[ \zeta := -\frac{1}{2}B(\theta^{L,1}, \theta^{R,2}) \in \Omega^2(G \times G), \quad \eta_{G \times G} = \eta^1 + \eta^2 \in \Omega^3(G \times G), \]

Given  = Aut(G), let  denote the group  equipped with the action  ×  ×  and consider the  -equivariant maps:

\[ \text{Mult}^\kappa_1 : G_{\kappa_1} \times G_{\kappa_2} \rightarrow G_{\kappa_1 \kappa_2}, \quad \text{Mult}^\kappa_1(a, b) = (\text{Mult} \circ (1 \times \kappa_1))(a, b); \]

\[ \text{Inv}^\kappa : G_{\kappa} \rightarrow G_{\kappa}^{-1}, \quad \text{Inv}^\kappa(a) = (\kappa^{-1} \circ \text{Inv})(a); \]

as well as the 2-form:

\[ \zeta^\kappa := (1 \times \kappa)^*\zeta \in \Omega^2(G \times G). \]

Modifying the proof [1] Thm.3.9 to take into account the effect of twisting automorphisms, it is easy to establish:

Proposition 3.15. Let G be a Lie group with  \( \kappa_1, \kappa_2 \in \text{Aut}(G) \). With the notation of this section, the multiplication map extends to a Dirac morphism:

\[ (\text{Mult}^\kappa_1, \zeta^\kappa_1) : (G_{\kappa_1} \times G_{\kappa_2}, E^\kappa_1 \oplus E^\kappa_2, \eta_{G \times G}) \rightarrow (G_{\kappa_1 \kappa_2}, E^\kappa_1 \oplus E^\kappa_2, \eta), \]

where in terms of the map in equation (3.15):

\[ E^\kappa_1 \oplus E^\kappa_2 = \{ e^{\kappa_1,1}(\xi) + e^{\kappa_2,2}(\zeta) \mid \xi, \zeta \in g \}. \]
Similarly, the inversion map extends to a Dirac morphism:
\[(\text{Inv}^\kappa,0) : (G\kappa,E_G^\kappa,\eta) \rightarrow (G\kappa^{-1},(E_G^\kappa)^{-1},-\eta),\]
where \((E_G^\kappa)^{-1}\) is defined to be:
\[\{X \oplus -\alpha \mid X \oplus \alpha \in E_G^\kappa\}.\]

**Remark 3.16.** Recall that the composition of Dirac morphisms \((\Phi_i,\omega_i)\) \((i = 1,2)\) is given by:
\[(\Phi_2,\omega_2) \circ (\Phi_1,\omega_1) = (\Phi_2 \circ \Phi_1,\omega_1 + \Phi_1^*\omega_2).\]

1. With the notation of Proposition 1.7, let \((M,\omega,\Phi)\) be a \(G\kappa_1 \times G\kappa_2 \times H\tau\)-valued \(\text{tq-Hamiltonian}\) manifold. Proposition 1.7 follows from the fact that:
\[
(\Phi_{\text{tus}},\omega_{\text{tus}}) = (\text{Mult}^{\kappa_1}_1,\kappa_1) \circ (\Phi,\omega)
\]
is a strong Dirac morphism by the previous proposition. Proposition 1.8 is obtained as the special case where \(H = \{e\}\) and \((M,\omega,\Phi) = (M_1 \times M_2,\omega_1 + \omega_2,\Phi_1 \times \Phi_2)\), where \((M_1,\omega_1,\Phi_1)\) are \(G\kappa_1\)-valued \(\text{tq-Hamiltonian}\) manifolds.

2. Similarly, the last proposition implies Proposition 1.10 since the composition:
\[(\Phi^-, -\omega) = (\text{Inv}^\kappa,0) \circ (\Phi,\omega)\]
gives a strong Dirac morphism for any \(G\kappa\)-valued \(\text{tq-Hamiltonian}\) manifold \((M,\omega,\Phi)\).

The next topic we address is the Liouville form of a fusion product. Regarding the pure spinor \(\psi_G^\kappa \in \Omega(G)\) defining the Lagrangian complement \(F_G^\kappa \subset TG\), we have the following generalization of [1, Thm.4.9]:

**Proposition 3.17.** Let \(G\) be a Lie group, and \(\kappa_1,\kappa_2 \in \text{Aut}(G)\) two automorphisms. The pullback of \(\psi_G^{\kappa_1}\kappa_2 \in \Omega(G)\) under the map \(\text{Mult}^\kappa : G \times G \rightarrow G\) satisfies the equation:
\[
\exp(\kappa_1^*)\left(\psi_G^{\kappa_1}\kappa_2\right) = \varrho \left(\exp(-\hat{\gamma})\right) \cdot \left(\psi_G^{\kappa_1,1} \otimes \psi_G^{\kappa_2,2}\right),
\]
where for any orthonormal basis \(\{v_i\} \subset g\), the bi-vector \(\hat{\gamma} \in \Gamma\left(\Lambda^2(E_G^{\kappa_1,1} \oplus E_G^{\kappa_2,2})\right)\) is given by:
\[
\hat{\gamma} = \frac{1}{2} \sum_i e^{\kappa_1,1}(v_i) \wedge e^{\kappa_2,2}(v_i).
\]

**Outline of proof.** With the notation of the proposition, let \(K = G \times (\kappa_1,\kappa_2)\). Applying [1, Thm.4.9] to the disconnected group \(K\) yields:
\[
e^{\kappa_1}\text{Mult}^*_K \Psi_K = \varrho \left(\exp(-\frac{1}{2} \sum_i e_K^1(v_i) \wedge e_K^2(v_i))\right) \cdot (\psi_K^1 \otimes \psi_K^2),
\]
where \(\Psi_K = R(\varrho(\text{vol}_g))\) defines the Lagrangian complement of the untwisted Cartan-Dirac structure \(E_K \subset TK\), and \(e_K : g \rightarrow E_K\) is the untwisted analogue to the trivialization in equation (3.15). The equation in the statement is then obtained by first restricting equation (3.22) to the outer component \(G\kappa_1 \times G\kappa_2 \subset K \times K\), and then identifying \(G\kappa_1 \times G\kappa_2\) with \(G \times G\) using the right multiplication maps by \(\kappa_i^{-1}\).

We can now state:
Corollary 3.18. For \( i = 1, 2 \), let \( (M_i, \omega_i, \Phi_i) \) be \( G_{k_i} \)-valued \( tq \)-Hamiltonian \( G \)-spaces. The Liouville form of the fusion product \((M_1 \otimes M_2, \omega_{\text{fus}}, \Phi_{\text{fus}})\) is given by:

\[
\Lambda_{M_1 \otimes M_2} = \Lambda_{M_1} \otimes \Lambda_{M_2} \in \Omega^{\text{top}}(M_1 \times M_2)^G.
\]

Proof. Similarly to [1] Prop.5.15, the previous proposition gives the computation:

\[
e^{\omega_{\text{fus}}} \Phi_{\text{fus}}^* \eta_{\text{G}}^{k_1 k_2} = e^{\omega_1 + \omega_2} (\Phi_1 \times \Phi_2)^* \left( e^{\gamma^1} \left( \Phi_1^* \right)^* \left( \Phi_2^* \right)^{k_1 k_2} \right)
\]

\[
= e^{\omega_1 + \omega_2} (\Phi_1 \times \Phi_2)^* \left( \rho(e^{-\gamma}) \cdot (\psi_G^{k_1, 1} \otimes \psi_G^{k_2, 2}) \right)
\]

\[
= \exp \left( -t \left( \frac{1}{2} \sum_i (v_i)_{M_1} \wedge (v_i)_{M_2} \right) \right) \left( e^{\omega_1} \Phi_1^* \eta_{\text{G}}^{k_1, 1} \otimes e^{\omega_2} \Phi_2^* \eta_{\text{G}}^{k_2, 2} \right).
\]

The claim follows by taking the top degree part on both sides, since in the RHS, the term \( \Lambda_{M_1} \otimes \Lambda_{M_2} \) is unaffected by the exponential of the interior product. \( \square \)

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