On the origin of generalized uncertainty principle from compactified $M5$-brane

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In this paper, we demonstrate that compactification in M-theory can lead to a deformation of field theory consistent with the generalized uncertainty principle (GUP). To investigate this, we have to obtain a relevant scale, so it is expected to produce a deformation similar to the GUP deformed field theory. In this paper, we demonstrate that compactification can be used to motivate the deformation of matter fields consistent with the GUP. To do this, we have to obtain a relevant action for the $M3$-brane.

This form ensures, via the Jacobi identity, that $[x_i, x_j] = 0 = [p_i, p_j]$ [3, 4].

The GUP has opened up an interesting window for quantum gravity phenomenology because it implies a modification of the fundamental commutation relation between position and momentum and this in turn also produces correction terms for all quantum mechanical systems [7, 8]. It may be noted that the GUP deformation has been generalized to include the full spacetime deformation, and the quantum field theory corresponding to such a deformation has also been constructed [5, 6].

Even though the GUP has been motivated from various approaches to quantum gravity including string theory, it has not been studied within the framework of compactification. On the other hand, compactification seems to deform spacetime geometry by a fundamental length scale, so it is expected to produce a deformation similar to the GUP deformed field theory. In this paper, we demonstrate that compactification can be used to motivate the deformation of matter fields consistent with the GUP. To investigate this, we have to obtain a relevant action for the $M3$-brane.

The gauge field in the action for two $M2$-branes is known to be valued in a Lie $3$-algebra rather than a Lie algebra [9, 10]. The action for two $M2$-branes can also be constructed without using a Lie $3$-algebra as long as the matter fields transform in the bi-fundamental representation. However, the Lie $3$-algebra valued gauge fields are important in the construction of a $M5$-brane action. This proposal for the $M5$-brane action is based on an infinite dimensional Lie $3$-algebra and a Nambu-Poisson structure on three dimensional manifolds. This model is expected to contain self-dual 2-form gauge fields in six dimensions, and so the resulting action may be interpreted as the action for $M5$-brane world-volume [11].

I. INTRODUCTION

Despite the fact that we as yet do not have a full theory of quantum gravity (QG), various different approaches to quantum gravity have been proposed. These approaches to quantum gravity can be used to gain phenomenologically understanding of the nature of the actual quantum theory of gravity. It is interesting to note that various different approaches to quantum gravity have predicted the existence of a minimum measurable length scale. These approaches include string theory and the semi-classical physics of black holes. The existence of a minimum length deforms the usual uncertainty principle to a (GUP) [1, 2].

$$\Delta x_i \Delta p_i \geq \frac{\hbar}{2}[1 + \beta ((\Delta p)^2 + p^2)] + 2\beta (\Delta p_i^2 + p_i^2) \] , \quad (1)$$

where $p^2 = \sum_j p_j p_j$, $p = p_0/(M_p c) = \beta_0 \ell_p^2$, $M_p$ = Planck mass, and $M_p c^2$ = Planck energy. However, as the uncertainty principle bears close resemblance to the Heisenberg algebra, the deformation of the uncertainty principle also deforms the Heisenberg algebra to [3, 4].

$$[x_i, p_j] = i\hbar(\delta_{ij} + \beta \delta_{ij} p^2 + 2\beta p_i p_j) \] . \quad (2)$$

This proposal for the $M5$-brane action is based on an infinite dimensional Lie $3$-algebra and a Nambu-Poisson structure on three dimensional manifolds. This model is expected to contain self-dual 2-form gauge fields in six dimensions, and so the resulting action may be interpreted as the action for $M5$-brane world-volume [11].
Motivated by these results, we propose a new model for constructing M3-branes from M0-branes. This is done by replacing the Nambu-Poisson structure in two dimensions by the structure in three dimensions. Then, we demonstrate that N M0-branes join each other and form a M5-brane. This M5-brane action is then compactified with a brane on two circles. Thus, we obtain a DBI action for M3-branes. This action contains extra higher order derivative terms. We observe that these higher order derivative terms are similar to the terms formed from a GUP deformation of the field theory. In fact, we will demonstrate that the explicit form of the modified commutation relations between position and momenta in this model is the same as that produced by the GUP. It is worth mentioning that the foundations of quantum commutation relations can be motivated from string theory [12].

The outline of our paper is as follows. In section II, we construct the M0-brane from the D0-brane and consider the relation between their algebras. We also show that M0-branes can unite and form a M5-brane. In section III, we will compactify the M5-brane on two circles and obtain the relevant action for the M3-brane. We also demonstrate the relation between the GUP and compactification in this section. The last section is devoted to the summary and conclusion.

II. M0-BRANES GROWING INTO M5-BRANES

In this section, inspired by the Lagrangian for M2-branes and M5-branes, we will replace the two dimensional Nambu-Poisson bracket by three one in D0-branes and build the DBI action for M0-branes with a Lie 3-algebra as the internal symmetry. D0-branes are 0 + 1 dimensional branes in a ten dimensional space-time of string theory which has nine transverse scalars corresponding to the nine transverse directions in their action. However, M0-branes are 0 + 1 dimensional branes in the eleven dimensional space-time of M-theory which has ten transverse scalars corresponding to the ten transverse directions. We will show that these M0-branes can join each other, grow and then make a transition to a M5-brane. Let us begin with the Born-Infeld action for a Dp-brane [13–19]:

\[
S = -T_p \int d^{p+1} \sigma S Tr \left( - \det(P_{ab}) E_{mn} + E_{mi}(Q^{-1} + \delta)^{ij} E_{jn} \right| + \lambda F_{ab} \right) det(Q_j^i)^{1/2}
\]

where

\[
E_{mn} = G_{mn} + B_{mn}, \quad Q_j^i = \delta_j^i + i \lambda [X^j, X^k] E_{kj}, \quad \lambda = 2 \pi l_s^2
\]

Dp-branes, \(i, j, k = p + 1, \ldots, 9\) are indices of the transverse space, and \(m, n\) are the ten-dimensional space-time indices. Also, \(T_p = \frac{1}{g_s (2 \pi)^p}\) is the tension of Dp-brane, \(l_s\) is the string length and \(g_s\) is the string coupling. Using this equation and assuming \(G_{ab} = \eta_{ab} + \partial_a X^i \partial_b X^i\), we can approximate Infeld action for D0-branes [13, 14, 20] as

\[
S_{D0} = \frac{1}{2 g_s l_s} \int dt Tr(\Sigma^9_{i=1} (\partial_t X^i \partial_i X^i - \frac{1}{4 \pi^2 l_s^4} [X^i, X^j]^2))
\]

where \([X^i = 1, 3, \ldots, 9]\) are transverse scalars. Following rules for commutation relations, we can write [13–15, 21, 22]

\[
[X^0, X^i] = i \lambda \partial_t X^i \quad [X^m, X^m] = 0
\]

Replacing \(\partial_t X^i\) by \([X^0, X^i]\) in equation (6), we get [13–15]

\[
S_{D0} = \frac{1}{8 g_s \pi^2 l_s^4} \int dt Tr(\Sigma_{m=1}^{9} [X^m, X^m]^2)
\]

Now, from the D0-branes, we can construct other Dp-branes by substituting the following rules in the action (7) [13, 15, 21, 22]:

\[
\Sigma_m \rightarrow \frac{1}{(2 \pi)^p} \int d^{p+1} \sigma \Sigma_{m-p-1}
\]

\[
[X^a, X^i] = i \lambda \partial_a X^i \quad [X^a, X^b] = i \lambda^2 F_{ab}
\]

\(i, j = p + 1, \ldots, 9\) \(a, b = 0, 1, \ldots, p\) \(m, n = 0, 1, \ldots, 9\)

Doing some calculations, we obtain

\[
S_{Dp} = -T_p \int d^{p+1} \sigma Tr \left( \Sigma_{i=p}^{9-p} (\partial_a X^i \partial_b X^i - \frac{1}{4 \pi^2 l_s^4} [X^i, X^j]^2 + \frac{\lambda^2}{4} (F_{ab})^2) \right)
\]

which is in agreement with studies done for D1, D3 and D5-branes [13, 15, 17, 20–22]. Now, we can extend this mechanism to M-Theory and obtain the relevant action for Mp-branes by replacing the two dimensional Nambu-Poisson bracket by three one-dimensional ones in the action (7) [13–15].

\[
S_{M0} = T_{M0} \int dt Tr(\Sigma_{M, N, L=0}^{10} [X^M, X^N, X^L]^2)
\]
method used in [23], and define $<X^8> = \frac{p}{l_p^4}$, where $l_p$ is the Planck length. Thus, we obtain [13–15]

$$S_{M0} = T_{M0} \int dt Tr(\Sigma_{M,N,L=0}^{10}[X^M, X^N, X^L]^2)$$

$$= 6T_{M0}(\frac{R^2}{l_p^4}) \int dt Tr(\Sigma_{M,N=0}^{9}[X^M, X^N]^2) + O(X^6)$$

$$= S_{D0} + O(X^6) \quad (11)$$

where we define $T_{D0} = 6T_{M0}(\frac{R^2}{l_p^4})$. The terms of order $O(X^6)$ have not been written explicitly because they will be ignorable at large radius ($R \to \infty$). Clearly, in this limit, the action (11) for compactified M-theory is in agreement with the action (7).

Different M$p$-branes can be built from M0-branes (similar to D$p$-branes) by using the following rules [11]

$$[X^a, X^b, X^c]^2 = \frac{1}{2}(\partial_a X^i)^2 \quad [X^a, X^b, X^c]^2 = (F_{abc})^2$$

$$\Sigma_m \to \frac{1}{(2\pi)^p} \int d^{p+1}\sigma \Sigma_{m-p-1}$$

$$i, j = p + 1, \ldots, 10 \quad a, b, c = 0, 1, \ldots, p \quad m, n = 0, \ldots, 10 \quad (12)$$

where

$$F_{abc} = \partial_a A_{bc} - \partial_b A_{ca} + \partial_c A_{ab} \quad (13)$$

and $A_{ab}$ is a 2-form gauge field. We can obtain the relevant action for the M5-brane by substituting (12) in the action (10):

$$S_{M5} = - T_{M5} \int d^6 \sigma Tr \left( \Sigma_{i=6}^{10} \left( \frac{1}{2}\partial_a X^i \partial_b X^i - \frac{1}{4}[X^i, X^j, X^k]^2 + \frac{1}{6}(F_{abc})^2 \right) \right) \quad (14)$$

Earlier studies done on the M5-branes [11] agree with the above. The derivation of the action for other M$p$-branes can be done by this mechanism. So, we can also use it to obtain a suitable action for D$p$-branes by compacting M-branes on a circle.

III. EMERGENCE OF GUP IN COMPACTIFIED M5-BRANES

In this section, we compactify the M5-brane on two circles to obtain M3-branes. We do this because our universe has 3 + 1 dimensions, and can be thought to be located on a 3 + 1 dimensional brane like a M3-brane. In order to do this, we will first write the 2-form gauge fields in M-branes in terms of 1-form gauge fields:

$$A_{ab} = \partial_a A_b - \partial_b A_a \quad (15)$$

For compacting the M5-brane to a M3-brane, we need to replace two gauge fields by two scalar fields [9–11, 13, 15]. We assume that $A^4 = X^4$ and $A^5 = X^5$. Thus, we obtain

$$A_{a4} = \partial_a X^4 - \partial_4 A_a \quad (16)$$

$$A_{a5} = \partial_a X^5 - \partial_5 A_a \quad (17)$$

where $a = 0, 1, 2, 3$. Substituting (16) and (17) in (13), we obtain

$$F_{aas} = \partial_a A_{as} - \partial_a A_{sa} + \partial_s A_{aa} \quad (18)$$

where $s = 4, 5$. We use the fact that $A_{as} = -A_{sa}$ and $A_{aa} = 0$ to obtain

$$F_{a44} = 2\partial_a^2 X^4 - 2\partial_4 \partial_a A_a$$

$$F_{a45} = 2\partial_a^2 X^5 - 2\partial_5 \partial_a A_a$$

and

$$F_{a55} = 2\partial_a^2 X^5$$

It may be noted that by compacting to a M3-brane, the gauge fields do not depend on the fourth and fifth dimensions. Thus, we can write

$$\partial_4 A_a = 0 \quad \partial_5 A_a = 0$$

$$\partial_4 \partial_a A_a = 0 \quad \partial_5 \partial_a A_a = 0$$

Using the above equations and assuming that the radius of the compactified dimensions are the same, we can calculate the integrals

$$x^4 = x^5 = 2\pi R \to \int dx^4 = \int dx^5 = 2\pi \int dr \to$$

$$\int dx^4 F_{a44} = -2 \int dx^4 \partial_4 \partial_a X^4 = -2 \int dx^5 \partial_5 \partial_a X^5 = -2 \int dx^4 \partial_4 \partial_a X^4 = 2\partial_a X^4$$

$$\int dx^5 F_{a55} = -2 \int dx^5 \partial_5 \partial_a X^5 = -2 \int dx^4 \partial_4 \partial_a X^5 = 2\partial_a X^5$$

$$\to \int dx^4 \int dx^5 (F_{a44})^2 = 4(\partial_a X^4)^2$$

and

$$\int dx^4 \int dx^5 (F_{a55})^2 = 4(\partial_a X^5)^2 \quad (22)$$
Now we can also obtain the rules for commutation as follows:

\[
[X^4, X^4, X^i]^2 = \frac{1}{2}(\partial_4 X^i)^2 \\
[X^5, X^5, X^i]^2 = \frac{1}{2}(\partial_4 X^i)^2
\]  

(23)

Substituting equations (21), (22) and (23) in equation (14), we can obtain the relevant action for the compactified M3-brane as

\[
S_{M3} = -T_{M3} \int d^4\sigma Tr \left( \sum_{\alpha=1}^{10} \left\{ \frac{1}{2} \partial_a X^i \partial_b X^i + 4(\partial_a^2 X^4)^2 \\
+ 4(\partial_a^2 X^5)^2 + 4(\partial_a \partial_b X^4)^2 + 4(\partial_a \partial_b X^5)^2 \\
- \frac{1}{4} [X^i, X^j, X^k]^2 + \frac{1}{6} (F_{abc})^2 \right\} \right)
\]

(24)

From this action, we can obtain the equations of motion for \(X^4\) and \(X^5\):

\[
\begin{align*}
\{\partial_a^2 + 4\partial_a \partial_b + \frac{\partial^2 V}{\partial (x^a)^2}\} X^4 &= 0 \\
\{\partial_a^2 + 4\partial_a \partial_b + \frac{\partial^2 V}{\partial (x^a)^2}\} X^5 &= 0 \\
V &= -\frac{1}{4} [X^i, X^j, X^k]^2
\end{align*}
\]

(25)

Replacing \(\partial_a = P_a\) and \(\frac{\partial^2 V}{\partial (x^a)^2} = m^2\), we obtain

\[
\begin{align*}
\{(P_a)^2 + 4(P_a)^2 + 4(P_a P_b)^2 + m^2\} X^4 &= 0 \\
\{(P_a)^2 + 4(P_a)^2 + 4(P_a P_b)^2 + m^2\} X^5 &= 0
\end{align*}
\]

(26)

We can compare these equations with the usual equations for scalar field theory

\[
\{(P_a)^2 + m^2\} X^4 = 0 \quad \{(P_a)^2 + m^2\} X^5 = 0
\]

(27)

As a result, we can redefine the momentum \(P_a\) as

\[
(P_a)^2 = (P_a)^2 + 4(P_a)^4 + 4(P_a P_b)^2 \rightarrow P_a \simeq P_a (1 + 2(P_a)^2 + 2(P_b)^2)
\]

(28)

Near the Planck scale, our model modifies the GUP as follows:

\[
[x_a, \tilde{P}_a] = i\hbar [1 + 6(P_a)^2 + 2(P_b)^2]
\]

(29)

This equation is the same as Eq. (2). Thus, the compactification of a M5-brane action to a M3-brane action produces the deformed GUP of Heisenberg. We can argue that compactification in M-theory can also be used as a motivation for GUP. Also, our calculations show that the GUP occurs only for momentum of scalar fields which are produced from compactification.

IV. SUMMARY AND DISCUSSION

In this paper, we have demonstrated that all \(Dp\)-branes can be formed from joining \(D0\)-branes. We generalized our model to M-theory and showed that M5-branes can also be obtained from M0-branes. Then, we compactified an M5-brane on two circles by replacing two gauge fields with two scalars. Thus, we were able to calculate the relevant action for M3-branes. We observed that extra derivatives terms are produced due to compactification. These higher derivative terms can be expressed in terms of higher order momentum terms in the equation of motion for new scalar fields. These terms can be obtained by modifying the usual uncertainty principle to the GUP. So, we have demonstrated that compactification in M-theory can be used as a motivation for the GUP.

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