THE GEOMETRY OF DEFORMATION QUANTIZATION AND SELF-DUAL GRAVITY

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A geometric formulation of the Moyal deformation for the Self-dual Yang-Mills theory and the Chiral Model approach to Self-dual gravity is given. We find in Fedosov’s geometrical construction of deformation quantization the natural geometrical framework associated to the Moyal deformation of Self-dual gravity.

1 Introduction

The purpose of this survey is to describe some conjectures in the geometry of deformation quantization for Self-dual Yang-Mills (SDYM) theory and the Chiral Model approach to Self-dual Gravity (SDG) worked out in detail there. This relation was originally suggested by I.A.B. Strachan. He has developed a deformed differential commutative geometry and has applied it to describe, within this geometrical framework, the multidimensional integrable systems. Here we intend to consider the application of some non-commutative geometry (Fedosov’s geometry) to self-dual gravity. The relation, for instance, between SDYM theory, Conformal Field Theory and Principal Chiral Model, all them with gauge group SDiff (Σ) (area-preserving diffeomorphism group of two-dimensional simply connected and symplectic manifold Σ), has been quite studied only at the algebraic level. The standard approach consists in considering a classical field theory invariant under some symmetry group, for instance, SU(N) and then take its large-N limit. In the case of Yang-Mills theory (both full and SD) its large-N limit (N → ∞) is somewhat mysterious, however it is very necessary to understand it in the searching for new faces of integrability. Drastic simplifications in some classical equations seem to confirm these speculations. However, geometric and topological aspects of the correspondence SDYM and SDG remain to be clarified.

2 Fedosov’s Geometry and the Moyal Deformation of Self-dual Gravity

First of all the Principal Chiral equations can be normally written as

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\begin{equation}
F = dA + A \wedge A = 0, \quad d \star A = 0
\end{equation}
where $\star$ is the standard Hodge operator and $A \in \mathcal{E}(\mathcal{G} \otimes \Lambda^1)$ is the connection one form. The corresponding equations are

\begin{equation}
A = g^{-1} dg, \quad d \star (g^{-1}dg) = 0.
\end{equation}

The first equation is the condition of flat connection and the second one is the equation of motion. In coordinates $(x, y) \in \Omega$, $A = A_\mu dx + A_\mu dy$, with $A_\mu(x, y) = \sum_{a=0}^{dim \ G} A^a_\mu(x, y) \tau_a \in \mathcal{G} \otimes C^\infty(\Omega)$, $\mu = x, y$. Now we generalize this gauge connection from $\mathcal{G}$-valued connection one-form to the corresponding $\mathcal{E}(\mathcal{W}_D)$-valued connection one-form $\tilde{A} = A_\mu dx + A_\mu dy$, with $A_\mu(x, y, p, q; h)$. The mentioned correspondence also implies that Eqs. (1) have a counterpart in terms of Fedosov’s geometry.

\begin{equation}
\tilde{F} = d\tilde{A} + \tilde{A} \wedge \tilde{A} = 0, \quad d \star \tilde{A} = 0,
\end{equation}
where $\tilde{A} \in \mathcal{E}(\mathcal{E}(\mathcal{W}_D) \otimes \Lambda^1)$ and $\tilde{F} \in \mathcal{E}(\mathcal{E}(\mathcal{W}_D) \otimes \Lambda^2)$ and where the multiplication $\wedge$ is defined by $a \wedge b = a_{ij}...lq \bullet b_{ij}...pq dx^j \wedge ... \wedge dx^l \wedge dx^i \wedge ... \wedge dx^q$, for all $a = \sum_k h^k a_{ij}...lq(x, y) dx^j \wedge ... \wedge dx^l \in \mathcal{E}(\mathcal{W} \otimes \Lambda^p)$ and $b = \sum_k h^k b_{ij}...lq(x, y) dx^i \wedge ... \wedge dx^q \in \mathcal{E}(\mathcal{W} \otimes \Lambda^q)$. $a \wedge b$ is defined by the usual wedge product on $\mathcal{M}$ and the product $\bullet$ in the Weyl algebra, $a \bullet b \equiv \exp\left(\frac{i}{\hbar} \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + \frac{\partial}{\partial y^*} \wedge \frac{\partial}{\partial z^*}\right)a(y, h)b(z, h)|_{z=y} = \sum_{k=0}^{\infty} \frac{\hbar^k}{k!} \omega_{i_{1}j_{1}} \ldots \omega_{i_{k}j_{k}} \frac{\partial^{i_{1}} a}{\partial y^{i_{1}} \ldots \partial y^{i_{k}}} \frac{\partial^{j_{1}} b}{\partial y^{j_{1}} \ldots \partial y^{j_{k}}}$. The corresponding to Eqs. (2) are

\begin{equation}
\tilde{A} = g^{-1} \bullet dg, \quad d \star (g^{-1} \bullet dg) = 0.
\end{equation}

Now we will show that Eqs. (5) can be obtained from a variational principle from a Lagrangian of the standard $\mathcal{G}$ principal chiral model. First we recall that the action which gives Eqs. (2) reads $S = \int \Omega \mathcal{L}$ where $\mathcal{L} = \frac{1}{2} \text{Tr}(g^{-1}dg \wedge *g^{-1}dg)$, where $g : \Omega \to \mathcal{G}$ and $d$ is the exterior differential on $\Omega$ i.e. $d = dx \partial_x + dy \partial_y$ and $\text{Tr}$ is an invariant form on the Lie algebra of $\mathcal{G}$, $\text{Lie}(\mathcal{G}) = \mathcal{G}$. Here we have assumed that $\mathcal{G}$ is semisimple. The above action can be generalized to Fedosov’s geometry as follows $S^* = \int \Omega \mathcal{L}^*$ where $\mathcal{L}^* = -\frac{\omega}{\hbar^2} \wedge \text{Tr}(\tilde{A} \wedge \tilde{A}) = -\frac{\omega}{\hbar^2} \text{Tr}(g^{-1} \bullet dg \wedge \bullet g^{-1} \bullet dg)$, where ‘$\text{Tr}$’ is the Fedosov’s trace \footnote{Fedosov’s trace ‘$\text{Tr}$’ is defined for flat phase space as $\text{Tr}(a) := \int_{\mathbb{R}^{2n}} \sigma(a) \frac{n}{2\pi} \text{ with } a \in \mathcal{E}(\mathcal{W}_D)$ and $\sigma : \mathcal{E}(\mathcal{W}_D) \to \mathbb{Z}$ is a bijection.} and $g : \Omega \to \mathcal{G}_*$ is the generalized gravitational unitor. In the case of flat phase-space $R_{ijkl} = 0$, the trace can be expressed by
\begin{align*}
\mathcal{L}^* &= -\frac{\hbar^2}{2} \text{tr}(\tilde{A} \wedge \tilde{A}) = -\frac{\hbar^2}{2} \int_{\mathbb{R}^2} \sigma(\tilde{A} \wedge \star \tilde{A}) \, dp \wedge dq, \\
&= -\frac{\hbar^2}{2} \int_{\mathbb{R}^2} \sigma(g^{-\frac{1}{2}} \bullet dg \bullet \star g^{-\frac{1}{2}} \bullet dg) \, dp \wedge dq.
\end{align*}

This Lagrangian has precisely equation of motion (5). One can apply also the above procedure to the well known Yang and Donaldson-Nair-Schiff equations. In Ref. 1 we found that these equations admit suitable Moyal deformation via Fedosov’s geometry. One can follow the same procedure in order to reveal the underlying geometry of deformation quantization of the Moyal deformed WZW-like action of SDG obtained in Ref. 2.

3 Final Remarks

Some further questions remain to be overcome. For instance, it would be very interesting to investigate the behavior of heavenly hierarchies of conserved quantities within Fedosov’s geometry and some other alternative geometries of deformation quantization. There exists a strong relation between SDYM and SDG with the M(atrix) theory approach to $M$-theory. A similar relation exists between the later and $N = (2,1)$ strings. $N = (2,1)$ strings are also related to SDG through $N = 2$ heterotic strings. Since in both descriptions of $M$-theory is involved SDYM and SDG one would hope that some new geometrical descriptions of SDYM theory and SDG will be of some relevance to give more insight into $M$-theory.

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