(PARA-)HERMITIAN AND (PARA-)KÄHLER SUBMANIFOLDS OF A PARA-QUATERNIONIC KÄHLER MANIFOLD

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Abstract: On a para-quaternionic Kähler manifold $(\tilde{M}^{4n}, Q, \tilde{g})$, which is first of all a pseudo-Riemannian manifold, a natural definition of (almost) Kähler and (almost) para-Kähler submanifold $(M^{2m}, J, g)$ can be given where $J = J_1|_M$ is a (para-)complex structure on $M$ which is the restriction of a section $J_1$ of the para-quaternionic bundle $Q$. In this paper, we extend to such a submanifold $M$ most of the results proved by Alekseevsky and Marchiafava, 2001, where Hermitian and Kähler submanifolds of a quaternionic Kähler manifold have been studied.

Conditions for the integrability of an almost (para-)Hermitian structure on $M$ are given. Assuming that the scalar curvature of $\tilde{M}$ is non zero, we show that any almost (para-)Kähler submanifold is (para-)Kähler and moreover that $M$ is (para-)Kähler iff it is totally (para-)complex. Considering totally (para-)complex submanifolds of maximal dimension $2n$, we identify the second fundamental form $h$ of $M$ with a tensor $C = J_2 \circ h \in T^*M \otimes S^2 T^*M$ where $J_2 \in Q$ is a compatible (para-)complex structure anticommuting with $J_1$. This tensor, at any point $x \in M$, belongs to the first prolongation $S^{(1)}_J$ of the space $S_J \subset \text{End} T_x M$ of symmetric endomorphisms anticommuting with $J$. When $\tilde{M}^{4n}$ is a symmetric manifold the condition for a (para-)Kähler submanifold $M^{2n}$ to be locally symmetric is given.

In the case when $\tilde{M}$ is a para-quaternionic space form, it is shown, by using Gauss and Ricci equations, that a (para-)Kähler submanifold $M^{2n}$ is curvature invariant. Moreover it is a locally symmetric Hermitian submanifold iff the $u(n)$-valued 2-form $[C,C]$ is parallel. Finally a characterization of parallel Kähler and para-Kähler submanifold of maximal dimension is given.

1. Introduction

A pseudo-Riemannian manifold $(M^{4n}, g)$ with the holonomy group contained in $Sp_1(\mathbb{R}) \cdot Sp_n(\mathbb{R})$ is called a para-quaternionic Kähler manifold. This means that there exists a 3-dimensional parallel subbundle $Q \subset \text{End}TM$ of the bundle of endomorphisms which is locally generated by three skew-symmetric anticommuting endomorphisms $I, J, K$ satisfying the following para-quaternionic relations

$$-I^2 = J^2 = K^2 = Id, \quad IJ = -JI = K.$$ 

The subbundle $Q \subset \text{End}(TM)$ is called a para-quaternionic structure. Any para-quaternionic Kähler manifold is an Einstein manifold \[3\].
Let $\epsilon = \pm 1$; a submanifold $(M^{2m}, J^\epsilon = J^\epsilon|_{TM}, g)$ of the para-quaternionic Kähler manifold $(\tilde{M}^{4n}, Q, \tilde{g})$, where $M \subset \tilde{M}$ is a submanifold, the induced metric $g = \tilde{g}|_{M}$ is non-degenerate, and $J^\epsilon$ is a section of the bundle $Q|M \to M$ such that $J^\epsilon TM = TM$, $(J^\epsilon)^2 = \epsilon Id$, is called an almost $\epsilon$-Hermitian submanifold.

An almost $\epsilon$-Hermitian submanifold $(M^{2m}, J^\epsilon, g)$ of a para-quaternionic Kähler manifold $(\tilde{M}^{4n}, Q, \tilde{g})$ is called $\epsilon$-Hermitian if the almost $\epsilon$-complex structure $J^\epsilon$ is integrable, almost $\epsilon$-Kähler if the Kähler form $F = g \circ J^\epsilon$ is closed and $\epsilon$-Kähler if $F$ is parallel. Note that $\epsilon$-Kähler submanifolds are minimal ([2]).

We will always assume that $\tilde{M}^{4n}$ has non zero reduced scalar curvature $\nu = \text{scat}/(4n(n + 2))$.

In section 3 we study an almost $\epsilon$-Hermitian submanifold $(M^{2m}, J^\epsilon, g)$ of the para-quaternionic Kähler manifold $\tilde{M}^{4n}$ and give the necessary and sufficient condition to be $\epsilon$-Hermitian. If furthermore $M$ is analytic, we show that a sufficient condition for integrability is that $	ext{codim}(\tilde{T}_xM) > 2$ at some point $x \in M$ where by $\tilde{T}_xM$ we denote the maximal $Q_x$-invariant subspace of $T_x M$. Then, as an application, we prove that, if the set $U$ of points $x \in M$ where the Nijenhuis tensor of $J^\epsilon$ of an almost $\epsilon$-Hermitian submanifold of dimension $4k$ is not zero is open and dense in $M$ and $\nabla \tilde{T}_xM$ is non degenerate, then $M$ is a para-quaternionic submanifold.

In fact, by extending a classical result of quaternionic geometry (see [1], [12]), we show that a non degenerate para-quaternionic submanifold of a para-quaternionic Kähler manifold is totally geodesic, hence a para-quaternionic Kähler submanifold.

In section 4, we give two equivalent necessary and sufficient conditions for an almost $\epsilon$-Hermitian manifold to be $\epsilon$-Kähler. We prove that an almost $\epsilon$-Kähler submanifold $M^{2m}$ of a para-quaternionic Kähler manifold $\tilde{M}^{4n}$ is $\epsilon$-Kähler and, hence, a minimal submanifold (see [2]) and give some local characterizations of such a submanifold (Theorem 4.2). In Theorem 4.3 we prove that the second fundamental form $h$ of a $\epsilon$-Kähler submanifold $M$ satisfies the fundamental identity

$$h(J^\epsilon X, Y) = J^\epsilon h(X, Y) \quad \forall X, Y \in TM$$

and that, conversely, if the above identity holds on an almost $\epsilon$-Hermitian submanifold $M^{2m}$ of $\tilde{M}^{4n}$ then $M^{2m}$ is either a $\epsilon$-Kähler submanifold or a para-quaternionic (Kähler) submanifold and these cases cannot happen simultaneously. In particular, we prove that an almost $\epsilon$-Hermitian submanifold $M$ is $\epsilon$-Kähler if and only if it is totally $\epsilon$-complex, i.e. it satisfies the condition $J_2 T_xM \perp T_xM \quad \forall x \in M$, where $J_2 \in Q$ is a compatible para-complex structure anticommuting with $J^\epsilon$.

In section 5, we study an $\epsilon$-Kähler submanifold $M$ of maximal dimension $2n$ in a para-quaternionic Kähler manifold $(\tilde{M}^{4n}, Q, \tilde{g})$ (still assuming $\nu \neq 0$). Using the field of isomorphisms $J_2 : TM \to T^+M$ between the tangent and the normal bundle, we identify, as in [5], the second fundamental form $h$ of $M$ with a tensor $C = J_2 \circ h \in TM \otimes S^2 T^*M$. This tensor, at any point $x \in M$, belongs to the first prolongation $S_{J^\epsilon}^{(1)}$ of the space $S_{J^\epsilon} \subset \operatorname{End} T_xM$ of symmetric endomorphisms anticommuting with $J^\epsilon$. Using the tensor $C$, we present the Gauss-Codazzi-Ricci equations in a simple form and derive from it the necessary and sufficient conditions for the $\epsilon$-Kähler submanifold $M$ to be parallel and to be curvature invariant (i.e. $\tilde{R}_{XYZ} Z \in TM, \forall X, Y, Z \in TM$). In subsection 5.4 we study a maximal $\epsilon$-Kähler submanifold $M$ of a (locally) symmetric para-quaternionic Kähler space $\tilde{M}^{4n}$ and get the necessary and sufficient conditions for $M$ to be a locally symmetric manifold.
in terms of the tensor $C$. In particular, if $\tilde{M}^{4n}$ is a quaternionic space form, then the $\epsilon$-Kähler submanifold $M$ is curvature invariant. In this case, $M$ is symmetric if and only if the 2-form

$$[C, C] : X \wedge Y \mapsto [C_X, C_Y] \quad X, Y \in TM,$$

with values in the unitary algebra of the $\epsilon$-Hermitian structure and that satisfies the first and the second Bianchi identity, is parallel. Moreover $M$ is a totally $\epsilon$-complex totally geodesic submanifold of the quaternionic space form $\tilde{M}^{4n}$ if and only if

$$\text{Ric}_M = \frac{\nu}{2}(n+1)g$$

(see Proposition 5.14).

In Section 6 we characterize a maximal $\epsilon$-Kähler submanifold $M$ of the para-quaternionic Kähler manifold $\tilde{M}^{4n}$ with parallel non zero second fundamental form $h$, or shortly, parallel $\epsilon$-Kähler submanifold. In terms of the tensor $C$, this means that

$$\nabla_X C = -\omega(X) J^c \circ C, \quad X \in TM$$

where $\omega = \omega_1|_{TM}$ and $\nabla$ is the Levi-Civita connection of $M$. When $(M^{2n}, J, g)$, where $J = J^c, \epsilon = -1$, is a parallel not totally geodesic Kähler submanifold, the covariant tensor $g \circ C$ has the form $q C = q + \overline{q}$ where $q \in S^3(T^{1,0}x^2)M$ (resp. $\overline{q} \in S^3(T^{0,1}x^2)M$) is a holomorphic (resp. antiholomorphic) cubic form. We prove that any parallel, not totally geodesic, Kähler submanifold $(M^{2n}, J, g)$ of a para-quaternionic Kähler manifold $(\tilde{M}^{4n}, Q, g)$ with $\nu \neq 0$ admits a pair of parallel holomorphic line subbundle $L = \text{span}_C(q)$ of the bundle $S^3T^{1,0}M$ and $\overline{L} = \text{span}_C(\overline{q})$ of the bundle $S^3T^{0,1}M$ such that the connection induced on $L$ (resp. $\overline{L}$) has the curvature $R^L = -i\nu g \circ J = -i\nu F$ (resp. $R^\overline{L} = i\nu g \circ J = i\nu F$). In case $(M^{2n}, J, g)$ where $J = J^c, \epsilon = +1$, is a parallel not totally geodesic para-Kähler submanifold of $(\tilde{M}^{4n}, Q, \overline{g})$ we have $g C = q^+ + q^- \in S^3(T^{++}x^2) + S^3(T^{--}x^2)$ where $TM = T^+ + T^-$ is the bi-Lagrangian decomposition of the tangent bundle. We prove that, in this case, the pair of real line subbundle $L^+ := \mathbb{R}q^+ \subset S^3(T^{++}x^2)$ and $L^- := \mathbb{R}q^- \subset S^3(T^{--}x^2)$ are globally defined on $M$ and parallel w.r.t the Levi-Civita connection which defines a connection $\nabla^{L^+}$ on $L^+$ (resp. $\nabla^{L^-}$ on $L^-$) whose curvature is

$$R^{L^+} = \nu F, \quad (\text{resp.} \quad R^{L^-} = -\nu F).$$

## 2. PARA-QUATERNIONIC KÄHLER MANIFOLDS

For a more detailed study of para-quaternionic Kähler manifolds see [15], [2], [9], [8], [14]. Moreover for a survey on para-complex geometry see [4], [7].

**Definition 2.1.** ([2]) Let $(\epsilon_1, \epsilon_2, \epsilon_3) = (-1, 1, 1)$ or a permutation thereof. An almost para-quaternionic structure on a differentiable manifold $\tilde{M}$ (of dimension $2m$) is a rank 3 subbundle $Q \subset \text{End} TM$, which is locally generated by three anticommuting fields of endomorphism $J_1, J_2, J_3 = J_1J_2$, such that $J^2_\alpha = \epsilon_\alpha \text{Id}$. Such a triple will be called a standard basis of $Q$. A linear connection $\nabla$ which preserves $Q$ is called an almost para-quaternionic connection. An almost para-quaternionic structure $Q$ is called a para-quaternionic structure if $\tilde{M}$ admits a para-quaternionic connection i.e. a torsion-free connection which preserves $Q$. An
(almost) para-quaternionic manifold is a manifold endowed with an (almost) para-quaternionic structure.

Observe that \( J_\alpha J_\beta = \epsilon_3 \epsilon_\gamma J_\gamma \) where \((\alpha, \beta, \gamma)\) is a cyclic permutation of \((1,2,3)\).

**Definition 2.2.** (2) An (almost) para-quaternionic Hermitian manifold \((\tilde{M}, Q, \tilde{g})\) is a pseudo-Riemannian manifold \((\tilde{M}, \tilde{g})\) endowed with an (almost) para-quaternionic structure \(Q\) consisting of skew-symmetric endomorphisms. The non degeneracy of the metric implies that \(\dim \tilde{M} = 4n\) and the signature of \(\tilde{g}\) is neutral. \((\tilde{M}^{4n}, Q, \tilde{g}), n > 1\), is called a para-quaternionic Kähler manifold if the Levi-Civita connection preserves \(Q\).

**Proposition 2.3.** (3) The curvature tensor \(\tilde{R}\) of a para-quaternionic Kähler manifold \((\tilde{M}, Q, \tilde{g})\), of dimension \(4n > 4\), at any point admits a decomposition

\[
\tilde{R} = \nu R_0 + W,
\]

where \(\nu = \frac{\text{scal}}{4n(n+2)}\) is the reduced scalar curvature,

\[
R_0(X,Y) := \frac{1}{2} \sum_\alpha \epsilon_\alpha \tilde{g}(J_\alpha X, Y) J_\alpha + \frac{1}{4}(X \wedge Y - \sum_\alpha \epsilon_\alpha J_\alpha X \wedge J_\alpha Y), \quad X, Y \in T_p\tilde{M},
\]

is the curvature tensor of the para-quaternionic projective space of the same dimension as \(\tilde{M}\) and \(W\) is a trace-free \(Q\)-invariant algebraic curvature tensor, where \(Q\) acts by derivations. In particular, \(\tilde{R}\) is \(Q\)-invariant.

We define a para-quaternionic Kähler manifold of dimension 4 as a pseudo-Riemannian manifold endowed with a parallel skew-symmetric para-quaternionic Kähler structure whose curvature tensor admits the decomposition (1).

Since the Levi-Civita connections \(\nabla\) of a para-quaternionic Kähler manifold preserves the para-quaternionic Kähler structure \(Q\), one can write

\[
\nabla J_\alpha = -\epsilon_\beta \omega_\gamma \otimes J_\beta + \epsilon_\gamma \omega_\beta \otimes J_\gamma,
\]

where the \(\omega_\alpha, \alpha = 1, 2, 3\) are locally defined 1-forms and \((\alpha, \beta, \gamma)\) is a cyclic permutation of \((1,2,3)\). We shall denote by \(F_\alpha := \tilde{g}(J_\alpha, \cdot)\) the Kähler form associated with \(J_\alpha\) and put \(F'_\alpha := -\epsilon_\alpha F_\alpha\).

We recall the expression for the action of the curvature operator \(\tilde{R}(X,Y), X,Y \in T\tilde{M}\) of \(\tilde{M}\), on \(J_\alpha\):

\[
[\tilde{R}(X,Y), J_\alpha] = \epsilon_3 \nu (-\epsilon_\beta F'_\gamma(X,Y) J_\beta + \epsilon_\gamma F'_\beta(X,Y) J_\gamma)
\]

where \((\alpha, \beta, \gamma)\) is a cyclic permutation of \((1,2,3)\).

**Proposition 2.4.** (2) The locally defined Kähler forms satisfy the following structure equations

\[
\nu F'_\alpha := -\epsilon_\alpha \nu F_\alpha = \epsilon_3 (d\omega_\alpha - \epsilon_\alpha \omega_\beta \wedge \omega_\gamma),
\]

where \((\alpha, \beta, \gamma)\) is a cyclic permutation of \((1,2,3)\).

By taking the exterior derivative of (5) we get

\[
\nu d F'_\alpha = \epsilon_3 d (d\omega_\alpha - \epsilon_\alpha \omega_\beta \wedge \omega_\gamma) = -\epsilon_3 (\epsilon_\alpha d\omega_\beta \wedge \omega_\gamma - \epsilon_\alpha \omega_\beta \wedge d\omega_\gamma).
\]

Since \(d\omega_\beta = \epsilon_3 \nu F'_\beta + \epsilon_\beta \omega_\gamma \wedge \omega_\alpha\) and \(d\omega_\gamma = \epsilon_3 \nu F'_\gamma + \epsilon_\gamma \omega_\alpha \wedge \omega_\beta\), we get

\[
\nu d F'_\alpha = -\epsilon_3 (\epsilon_\alpha \epsilon_3 \nu F'_\beta \wedge \omega_\gamma) - (\epsilon_\alpha \omega_\beta \wedge \epsilon_3 \nu F'_\gamma).
\]
that is $\nu[dF'_\alpha - \epsilon_\alpha(-F'_\beta \wedge \omega_\gamma + \omega_\beta \wedge F'_\gamma)] = 0$. Hence we have the following result.

**Proposition 2.5.** On a para-quaternionic Kähler manifold the following integrability conditions hold

$$\nu[dF'_\alpha - \epsilon_\alpha(-F'_\beta \wedge \omega_\gamma + \omega_\beta \wedge F'_\gamma)] = 0, \quad (\alpha, \beta, \gamma) = \text{cycl}(1, 2, 3).$$

3. **Almost $\epsilon$-Hermitian submanifolds of $\tilde{M}^{4n}$**

The definition of an (almost) complex structure on a differentiable manifold and the condition for its integrability are well known. We just recall the following other definitions (see [2]).

**Definition 3.1.** An (almost)para-complex structure on a differentiable manifold $M$ is a field of endomorphisms $J \in \text{End}TM$ such that $J^2 = \text{Id}$ and the $\pm 1$-eigenspace distributions $T^\pm M$ of $J$ have the same rank. An almost para-complex structure is called integrable, or para-complex structure, if the distributions $T^\pm M$ are integrable or, equivalently, the Nijenhuis tensor $N_J$, defined by

$$N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] + [X, Y], \quad X, Y \in TM$$

vanishes. An (almost)para-complex manifold $(M, J)$ is a manifold $M$ endowed with an (almost) para-complex structure.

**Definition 3.2.** An (almost) $\epsilon$-complex structure $\epsilon \in \{-1, 1\}$ on a differentiable manifold $M$ of dimension $2n$ is a field of endomorphisms $J \in \text{End}TM$ such that $J^2 = \epsilon \text{Id}$ and moreover, for $\epsilon = +1$ the eigendistributions $T^\pm M$ are of rank $n$. An $\epsilon$-complex manifold is a differentiable manifold endowed with an integrable (i.e. $N_J = 0$) $\epsilon$-complex structure.

Consequently, the notation (almost) $\epsilon$-Hermitian structure, (almost) $\epsilon$-Kähler structure, etc., will be used with the same convention.

Let recall that a submanifold of a pseudo-Riemannian manifold is non degenerate if it has non degenerate tangent spaces.

**Definition 3.3.** Let $(\tilde{M}^{4n}, Q, \tilde{g})$ be a para-quaternionic Kähler manifold. A $\tilde{g}$-non degenerate submanifold $M^{2m}$ of $\tilde{M}$ is called an almost $\epsilon$-Hermitian submanifold of $\tilde{M}$ if there exists a section $J^* : M \rightarrow Q|_M$ such that

$$J^*TM = TM \quad (J^*)^2 = \epsilon \text{Id}.$$

We will denote such submanifold $(M^{2m}, J^*, g)$ where $(g = \tilde{g}|_M, J^* = J^*|_M)$.

For a classification of almost (resp. para-)Hermitian manifolds see [13], (resp. [6], [11]).

Notice (see [21], [24], [22]) that in any point $x \in M$ the induced metric $g_x = <,>_x$ of an (almost) Hermitian submanifold has signature $2p, 2q$ with $p + q = m$ whereas the signature of the metric of an (almost) para-Hermitian submanifold is always neutral $(m, m)$. In both cases then the induced metric is pseudo-Riemannian (and Hermitian). Keeping in mind this fact, we will not use the suffix "pseudo" in the following.

For any point $x \in M^{2m}$, we can always include $J^*$ into a local frame $(J_1 = J^*, J_2, J_3 = J_1J_2 = -J_2J_1)$ of $Q$ defined in a neighbourhood $\tilde{U}$ of $x$ in $\tilde{M}$ such that $J_2^2 = \text{Id}$. Such frame will be called adapted to the submanifold $M$ and in fact,
where is even (not necessarily a multiple of 4) and the signature of \( g \) is:\n\[ \tilde{\nabla} J^c = -\omega_3 \otimes J_2 - \epsilon \omega_2 \otimes J_3 \]
where \( \tilde{\nabla} \) indicates the Levi-Civita connection on \( \tilde{M} \), and in complex case (\( \epsilon = -1 \)), from \((\epsilon_1, \epsilon_2, \epsilon_3) = (-1, 1, 1)\), we have \( J_2J_3 = -J_1, \) \( J_3J_1 = J_2 \) whereas in para-complex case, where \((\epsilon_1, \epsilon_2, \epsilon_3) = (1, 1, -1)\), we have \( J_2J_3 = -J_1, \) \( J_3J_1 = -J_2 \).

For any \( x \in M \) we denote \( T_x\tilde{M} \) the maximal para-quaternionic (Q-invariant) subspace of the tangent space \( T_xM \). Note that if \((J_1, J_2, J_3)\) is an adapted basis in a point \( x \in M \) then \( T_x\tilde{M} = T_xM \cap J_x\tilde{M} \).

We allow \( T_x\tilde{M} \) to be degenerate (even totally isotropic), hence its dimension is even (not necessarily a multiple of 4) and the signature of \( g|_{T_x\tilde{M}} \) is \((2k, 2s, 2k)\) where \( 2s = \text{dim} \ker g \) (see [20]). We recall that a subspace of a para-quaternionic vector space \((V, Q)\) is pure if it contains no non zero Q-invariant subspace. We write then
\[ T_xM = T_x\tilde{M} \oplus D_x \]
where \( D_x \) is any \( J^c \)-invariant pure supplement (the existence of such supplement is proved in [20]).

Recall that if \( M \) is a non degenerate submanifold of a pseudo-Riemannian manifold \((\tilde{M}, \tilde{g})\) and \( T_xM = T_x\tilde{M} \oplus T_x^\perp M \) is the orthogonal decomposition of the tangent space \( T_x\tilde{M} \) at point \( x \in M \) then the Levi-Civita covariant derivative \( \tilde{\nabla}_X \) of the metric \( \tilde{g} \) in the direction of a vector \( X \in T_xM \) can be written as:
\[ \tilde{\nabla}_X \equiv \begin{pmatrix} \nabla_X & -A_X^c \\ A_X^c & \tilde{\nabla}_X^c \end{pmatrix}. \]
that is
\[ \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \tilde{\nabla}_X \xi = -A_X^c X + \tilde{\nabla}_X^c \xi \]
for any tangent (resp. normal) vector field \( Y \) (resp. \( \xi \)) on \( M \). Here \( \nabla_X \) is the covariant derivative of the induced metric \( g \) on \( M \), \( \nabla_X^c \) is the normal covariant derivative in the normal bundle \( T^\perp M \) which preserves the normal metric \( g^\perp = g|_{T^\perp M} \), \( A_X^c Y = h(X, Y) \) in \( T^\perp M \) where \( h \) is the second fundamental form and \( A_X \xi = A^\xi X \), where \( A^\xi \) \( \in \text{End} \) \( TM \) is the shape operator associated with a normal vector \( \xi \).

**Theorem 3.4.** Let \((M^{2m}, J^c, g), m > 1\), be an almost \( \epsilon \)-Hermitian submanifold of the para-quaternionic Kähler manifold \((\tilde{M}^{4n}, Q, \tilde{g})\). Then
(1) the almost \( \epsilon \)-complex structure \( J^c \) is integrable if and only if the local 1-form \( \psi = \omega_3 \circ J^c - \omega_2 \) on \( M^{2m} \) associated with an adapted basis \( H = (J_\alpha) \) vanishes.

(2) \( J^c \) is integrable if one of the following conditions holds:
(a) \( \dim(D_x) > 2 \) on an open dense set \( U \subset M \);
(b) \((M, J^c)\) is analytic and \( \dim(D_x) > 2 \) at some point \( x \in M \);

**Proof.** (1) Let proceed as in [5], Theorem 1.1. Remark that if \((M, J^c)\) is an almost \( \epsilon \)-complex submanifold of an almost \( \epsilon \)-complex manifold \((\tilde{M}, J^c)\) then the restriction of the Nijenhuis tensor \( N_{J^c} \) to the submanifold \( M \) coincides with the Nijenhuis
tensor $N_{\mathcal{J}^c}$ of the almost complex structure $\mathcal{J}^c = J'|_{TM}$. Then for any $X, Y \in TM$, we can write

$$\frac{1}{2} N_{\mathcal{J}^c}(X, Y) = \frac{1}{2} \left( [\mathcal{J}^c X, \mathcal{J}^c Y] - [\mathcal{J}^c \mathcal{J}^c X, \mathcal{J}^c Y] - \mathcal{J}^c [X, \mathcal{J}^c Y] + \mathcal{J}^c [X, \mathcal{J}^c Y] + \epsilon [X, Y] = \right.$$

$$\left. \frac{1}{2} \left( \nabla_\mathcal{J} X(\mathcal{J}^c Y) - \nabla_\mathcal{J} Y(\mathcal{J}^c X) - \mathcal{J}^c \nabla_\mathcal{J} X Y - \nabla_\mathcal{J} Y \mathcal{J}^c X \right) \right) - J^c (\nabla_\mathcal{J} X(\mathcal{J}^c Y) - \nabla_\mathcal{J} Y(\mathcal{J}^c X) - \nabla_\mathcal{J} X Y - \nabla_\mathcal{J} Y \mathcal{J}^c X)$$

and hence, from (3)

$$\frac{1}{2} N_{\mathcal{J}^c}(X, Y) = -[\omega_3(\mathcal{J}^c X) - \omega_3(X)] J_2 Y + \epsilon [\omega_2(\mathcal{J}^c X) + \omega_3(X)] J_2 Y$$

$$\left. + [-\epsilon \omega_2(\mathcal{J}^c Y) + \omega_3(Y)] J_2 X - [-\epsilon \omega_2(\mathcal{J}^c Y) + \omega_3(Y)] J_2 X \right)$$

where $(J_1, J_2, J_3)$ is an adapted local basis. This implies (1) in one direction.

Viceversa, let $N_{\mathcal{J}^c}(X, Y) = 0, \forall X, Y \in T_xM$. By applying $J_2$ to both members of the above equality, this is equivalent to the identity

$$\psi(X) Y + \epsilon \psi(\mathcal{J}^c X) \mathcal{J}^c Y = \psi(Y) X + \epsilon \psi(\mathcal{J}^c Y) \mathcal{J}^c X, \forall X, Y \in T_xM.$$  

Let assume that there exists a non zero vector $X \in T_xM$ such that $\psi(X) \neq 0$. We show that this leads to a contradiction. Let consider a vector $0 \neq Y \in T_xM$ which is not an eigenvector of $\mathcal{J}^c$ and such that $\text{span}(X, \mathcal{J}^c X) \cap \text{span}(Y, \mathcal{J}^c Y) = 0$. It is easy to check that such a vector $Y$ always exists. Then the vectors in both sides of (2) must be zero which implies in particular that $\psi(X) = 0$. Contradiction.

(2) We assume that $\mathcal{J}^c$ is not integrable. Then the 1-form $\psi = (\omega_3 \mathcal{J}^c - \omega_2)|_{TM}$ is not identically zero, by (1). Denote by $a = g^{-1}\psi$ the local vector field on $M$ associated with the 1-form $\psi$ and let $a = \bar{a} + a'$ with $\bar{a} \in \overline{TM}$ and $a' \in D$. Now we need the following

**Lemma 3.5.** Let $(M^{2m}, \mathcal{J}^c, g)$, $m > 1$, be an almost $\epsilon$-Hermitian submanifold of a para-quaternionic Kähler manifold $(\overline{M}^{4n}, Q, \overline{g})$. Then in any point $x \in M^{2m}$ where the Nijenhuis tensor $N(\mathcal{J}^c)_x \neq 0$, or equivalently the vector $a_x \neq 0$, any $\mathcal{J}^c$-invariant supplementary subspace $D_x$ is spanned by $a'_x$ and $\mathcal{J}^c a'_x$:

$$D_x = \text{span}(a'_x, \mathcal{J}^c a'_x).$$

Moreover if $T_xM$ is not para-quaternionic (i.e. $\dim D_x \neq 0$) then $\psi(T_xM) \equiv 0$.

**Proof.** Remark that

$$\frac{1}{2} N_{\mathcal{J}^c}(X, Y) = -\psi(X) J_2 Y + \epsilon \psi(\mathcal{J}^c X) J_2 Y + \psi(Y) J_2 X - \epsilon \psi(\mathcal{J}^c Y) J_2 X$$

$$= -J_2 [\psi(X) Y + \epsilon \psi(\mathcal{J}^c X) \mathcal{J}^c Y - \psi(Y) X - \epsilon \psi(\mathcal{J}^c Y) \mathcal{J}^c X],$$

that is $N_{\mathcal{J}^c}(X, Y) \in J_2 TM \cap TM = \overline{TM}$ for any $X, Y \in TM$. Hence

$$\psi(X) Y + \epsilon \psi(\mathcal{J}^c X) \mathcal{J}^c Y - \psi(Y) X - \epsilon \psi(\mathcal{J}^c Y) \mathcal{J}^c X \in \overline{TM} \quad \forall X, Y \in TM.$$  

Taking $X \in \overline{T_xM}$ and $0 \neq Y \in D_x$ the first two terms of (11) are in $D_x$ and the last two in $T_xM$. We conclude that $\psi(T_xM) \equiv 0$ if $\dim D_x \neq 0$. For $X = a = g^{-1}\psi$, since $g(a, \mathcal{J}^c a) = 0$, the last condition says that

$$b_Y := |a|^2 Y - \psi(Y) a - \epsilon \psi(\mathcal{J}^c Y) \mathcal{J}^c a \in \overline{TM} \quad \forall Y \in TM.$$  

Considering the $D$-component of the vector $b_Y$ for $Y = \overline{Y} \in \overline{TM}$ and $Y = Y' \in D$ respectively, we get the equations:

$$-\psi(\overline{Y}) a' - \epsilon \psi(\mathcal{J}^c \overline{Y}) \mathcal{J}^c a' = 0, \quad \forall \overline{Y} \in \overline{TM}$$
\(|a|^2 Y' - \psi(Y) a' - e\psi(J^* Y') J^* a' = 0 \quad \forall Y' \in \mathcal{D}.

The last equation shows that \(\mathcal{D}_x = \{a', J^* a'\}\) when \(a \neq 0\) (whereas (12) confirms that \(\psi(T_x M) = 0\) when \(\dim \mathcal{D} \neq 0\)). Observe that \(a'\) is never an eigenvector of the para-complex structure \(J\).

\(\square\)

**Continuing the proof of Theorem (3.4):** The Lemma implies statements (2a) and (2b) since in the analytic case the set \(U\) of points where the analytic vector field \(a \neq 0\) is open (complementary of the close set where \(a = 0\)) and dense (since otherwise it would exist an open set \(\tilde{U}\) with \(a(\tilde{U}) = 0\) which, by the analyticity of \(a\) it would imply \(a = 0\) everywhere) and \(\dim \mathcal{D}_x \leq 2\) on \(U\).

From (10) it follows the

**Corollary 3.6.** In case \(T_2 M\) is pure \(\epsilon\)-complex i.e. \(\overline{T_2 M} = 0\) in an open dense set in \(M\) than the almost Hermitian submanifold is Hermitian.

This is a generalization of the 2-dimensional case where clearly, by the non degeneracy hypotheses, \(T_2 M\) is pure for any \(x \in M\).

**Definition 3.7.** A submanifold \(M\) of an almost para-quaternionic manifold \((\widetilde{M}, Q)\) is an **almost para-quaternionic submanifold** if its tangent bundle is \(Q\)-invariant. Then \((M, Q|_{TM})\) is an almost para-quaternionic manifold.

The following proposition is the extension to the para-quaternionic case of a basic result in quaternionic case.

**Proposition 3.8.** A non degenerate almost para-quaternionic submanifold \(M^{4m}\) of a para-quaternionic Kähler manifold \((\widetilde{M}^{4n}, Q, \widetilde{g})\) is a totally geodesic para-quaternionic Kähler submanifold.

**Proof.** Let \(A\) be the shape operator of the para-quaternionic submanifold. Then, for any \(X, Y \in \Gamma(TM)\), \(\xi \in \Gamma(T^{-1}M)\),

\[
\tilde{g}(A^\epsilon(J_\alpha X), Y) = -\tilde{g}(\widetilde{\nabla}_Y \xi, J_\alpha X) = -\tilde{g}(\widetilde{\nabla}_Y \xi, J_\alpha X) = \tilde{g}(\xi, \widetilde{\nabla}_Y (J_\alpha X)) = \tilde{g}(\xi, \widetilde{\nabla}_Y (J_\alpha X) + J_\alpha \widetilde{\nabla}_Y X).
\]

Moreover

\[
\tilde{g}(\xi, \widetilde{\nabla}_Y X) = -\tilde{g}(J_\alpha \xi, \widetilde{\nabla}_Y X) = -\tilde{g}(J_\alpha \xi, \widetilde{\nabla}_Y X) - [X, Y])
= -\tilde{g}(J_\alpha \xi, \widetilde{\nabla}_Y X) = \tilde{g}(\xi, \widetilde{\nabla}_Y (J_\alpha Y) - (\widetilde{\nabla}_Y J_\alpha) Y) = \tilde{g}(\xi, \widetilde{\nabla}_X (J_\alpha Y)) = -\tilde{g}(\widetilde{\nabla}_X \xi, J_\alpha Y) = -\tilde{g}(J_\alpha A^\epsilon X, Y)
\]

and

\[
\tilde{g}(\xi, \widetilde{\nabla}_Y J_\alpha X) = \tilde{g}(\xi, -\epsilon_\beta \omega_\gamma (Y) J_\beta X + \epsilon_\gamma \omega_\beta (Y) J_\gamma) = 0
\]

since \(J_\beta X, J_\gamma X \in \Gamma(TM)\). It follows that \(AJ_\alpha = -J_\alpha A\), \(\alpha = 1, 2, 3\). Computing \(AJ_\alpha = -J_\alpha A\)

\[
AJ_\alpha = -J_\alpha A = -\epsilon_\beta \epsilon_\gamma (J_\beta J_\gamma A = -\epsilon_\beta \epsilon_\gamma A J_\beta J_\gamma = -\epsilon_\beta \epsilon_\gamma A J_\alpha = -\epsilon_\beta \epsilon_\gamma A J_\alpha\text{ for }A = 0 \text{ i.e. } h = 0. \text{ Now it is immediate to deduce that } (M^{4m}, Q|_{TM}, \tilde{g})\text{ is also para-quaternionic Kähler.}
\]

\(\square\)

**Corollary 3.9.** Let \((M^{4k}, J^*, g)\) be an almost \(\epsilon\)-Hermitian submanifold of dimension \(4k\) of a para-quaternionic Kähler manifold \(\widetilde{M}^{4n}\). Assume that the set \(U\) of points \(x \in M\) where the Nijenhuis tensor of \(J^*\) is not zero is open and dense in \(M\) and that, \(\forall x \in U, \overline{T_2 M}\) is non degenerate. Then \(M\) is a totally geodesic para-quaternionic Kähler submanifold.
M is called Kähler as in [5] by taking into account that, by the non degeneracy hypotheses of $T\omega M$, it is necessarily $\dim D_x = 0$.

4. Almost $\epsilon$-Kähler, $\epsilon$-Kähler and totally $\epsilon$-complex submanifolds

**Definition 4.1.** The almost $\epsilon$-Hermitian submanifold $(M^{2n}, J^\epsilon, g)$ of a para-quaternionic Kähler manifold $(M^{4n}, Q, \overline{g})$ is called almost $\epsilon$-Kähler (resp., $\epsilon$-Kähler) if the Kähler form $F = F_1|TM = g\circ J^\epsilon$ is closed (resp. parallel). Moreover $M$ is called totally $\epsilon$-complex if

$$J_2T_xM \perp T_xM \quad \forall x \in M$$

where $(J_1, J_2, J_3)$ is an adapted basis (note that $J_2T_xM \perp T_xM \iff J_3T_xM \perp T_xM$).

For a study of (almost)-Kähler and totally complex submanifolds of a quaternionic manifold see [3, 10, 17, 18].

In case $\widetilde{M}$ is the $n$-dimensional para-quaternionic numerical space $\mathbb{H}^n$, the prototype of flat para-quaternionic Kähler spaces (see [21]), typical examples of such submanifolds are the flat Kähler (resp. para-Kähler) submanifolds $M^{2k} = \mathbb{C}^k$ (resp. $\mathbb{C}^k$) obtained by choosing the first $k$ para-quaternionic coordinates as complex (resp. para-complex) numbers and the remaining $n - k$ equals to zero. In case $M^{4n} = \mathbb{H}P^n$ is the para-quaternionic projective space endowed with the standard para-quaternionic Kähler metric (see [3]), examples of non flat Kähler (resp. para-Kähler) submanifolds are given by the immersions of the projective complex (resp. para-complex) spaces $\mathbb{C}P^{k-1}$ (resp. $\mathbb{C}P^{k-1}$) induced by the immersions considered above in the flat case.

From [3] one has

$$(\nabla_X J^\epsilon)Y = [-\omega_3(X)Id - \epsilon\omega_2(X)J^\epsilon] [J_2Y]^T \quad X, Y \in TM.$$  

and then, by arguing as in [5], the following theorem is deduced.

**Theorem 4.2.** Let $(\widetilde{M}^{4n}, Q, \overline{g})$ be a para-quaternionic Kähler manifold.

1) A totally $\epsilon$-complex submanifolds of $\widetilde{M}$ is $\epsilon$-Kähler.

2) If $\nu \neq 0$, for an almost $\epsilon$-Hermitian submanifold $(M^{2m}, J^\epsilon, g)$, $m > 1$, of $\widetilde{M}$ the following conditions are equivalent:

$$(k_1) \ M \ is \ \epsilon\text{-Kähler},$$  

$$(k_2) \ \omega_2|T_xM = \omega_3|T_xM = 0 \quad \forall x \in M,$$

$$(k_3) \ M \ is \ totally \ \epsilon\text{-complex}.$$  

**Proof.** The first statement follows from [14]. The second statement is proved in [2] Proposition 20.

**Theorem 4.3.** Let $(\widetilde{M}^{4n}, Q, \overline{g})$ be a para-quaternionic Kähler manifold with non vanishing reduced scalar curvature $\nu$ and $(M^{2m}, J^\epsilon, g)$ an almost $\epsilon$-Hermitian submanifold of $\widetilde{M}^{4n}$.

a) If $(M^{2m}, J^\epsilon, g)$ is $\epsilon$-Kähler then the second fundamental form $h$ of $M$ satisfies the identity

$$h(X, J^\epsilon Y) = h(J^\epsilon X, Y) = J^\epsilon h(X, Y) \quad \forall X, Y \in TM.$$  

In particular $h(J^\epsilon X, J^\epsilon Y) = eh(X, Y)$.
b) Conversely, if the identity (17) holds on an almost $\epsilon$-Hermitian submanifold $M^{2m}$ of $M^{4n}$ then it is either a $\epsilon$-Kähler submanifold or a para-quaternionic (Kähler) submanifold and these cases cannot happen simultaneously.

**Proof.** (a) Let $(M^{2m}, J^\epsilon, g)$ be an almost $\epsilon$-Hermitian submanifold of $\tilde{M}$. By (3),

$$
(\nabla_X J^\epsilon)Y = (\nabla_X J^\epsilon)Y + h(X, J^\epsilon Y) - J^\epsilon h(X, Y) = -\omega_3(X)J_2Y - \epsilon\omega_2(X)J_3Y, \quad X, Y \in TM.
$$

From Theorem (4.2), we get

$$
0 = (\nabla_X J^\epsilon)Y + h(X, J^\epsilon Y) - J^\epsilon h(X, Y), \quad \forall X, Y \in TM
$$

and, from $(\nabla_X J^\epsilon)Y = 0$ it is clear that if $(M, J^\epsilon)$ is $\epsilon$-Kähler then (15) holds.

(b) Conversely, let assume that (15) holds on the almost $\epsilon$-Hermitian submanifold $(M, J^\epsilon, g)$. Then for any $X, Y \in T_x M$, from (16) we have

$$
(\nabla_X J^\epsilon)Y = (\tilde{\nabla}_X J^\epsilon)Y.
$$

Hence, $\forall X, Y \in T_x M$,

$$(\nabla_X J^\epsilon)Y = -\omega_3(X)J_2Y - \epsilon\omega_2(X)J_3Y = (\omega_3(X)Id - \epsilon\omega_2(X)J^\epsilon) J_2Y \in T_x M.
$$

Then, either $J_2 T_x M = T_x M$ i.e. $T_x M$ is a para-quaternionic vector space or $\omega_2|_x = \omega_3|_x = 0$ and by Theorem (1.2) the two conditions cannot happen simultaneously.

The set $M_1 = \{ x \in M \mid J_2 T_x M = T_x M \}$ is a closed subset and the complementary open subset $M_2 = \{ x \in M \mid \omega_2|_x = \omega_3|_x = 0 \}$ is a closed subset as well since, from Theorem (4.2), $M_2 = \{ x \in M \mid J_2 T_x M \perp T_x M \}$. Then, either $M_2 = 0$ and $M = M_1$ is a para-quaternionic Kähler submanifold or $M_1 = 0$ and $M = M_2$ is $\epsilon$-Kähler.

**Corollary 4.4.** A totally geodesic almost $\epsilon$-Hermitian submanifold $(M, J^\epsilon, g)$ of a para-quaternionic Kähler manifold $(M^{4n}, Q, \tilde{g})$ with $\nu \neq 0$ is either a $\epsilon$-Kähler submanifold or a para-quaternionic submanifold and these conditions cannot happen simultaneously.

**Proof.** The statement follows directly from Theorem (4.3) since (15) certainly holds for a totally geodesic submanifold $(h = 0)$.

The following results have been proved in [2].

**Proposition 4.5.** ([2]) The shape operator $A$ of an $\epsilon$-Kähler submanifold $(M^{2m}, J^\epsilon, g)$ of a para-quaternionic Kähler manifold $(\tilde{M}^{4n}, Q, \tilde{g})$ anticommutes with $J^\epsilon$, that is $AJ^\epsilon = -J^\epsilon A$.

**Corollary 4.6.** ([2]) Any $\epsilon$-Kähler submanifold of a para-quaternionic Kähler manifold is minimal.

We conclude this section with the following result concerning almost $\epsilon$-Kähler submanifolds.

**Theorem 4.7.** Let $(\tilde{M}^{4n}, Q, \tilde{g})$ be a para-quaternionic Kähler manifold with non vanishing reduced scalar curvature $\nu$. Then any almost $\epsilon$-Kähler submanifold $(M^{2m}, J^\epsilon, g)$ of $\tilde{M}$ is $\epsilon$-Kähler.
Proof. By identity (9), the condition that the Kähler form $F = F_1|_M$ is closed can be written as
\[
F_2^T \wedge \omega_3^T = \epsilon F_3^T \wedge \omega_2^T,
\]
where $F_3^T, \omega_3^T$ are the restriction of the forms $F_3, \omega_3$ to $M$. We will prove that (17) implies integrability.

Let suppose that there exists a point $x$ of the almost $\epsilon$-Kähler submanifold $M$ where $N_{\mathcal{F}^1_x} \neq 0$. From Lemma (3.5) then dim $D_x = 0$ or 2 and from (11) one has dim $T_xM = 0$.

Let first consider the case that $T_xM = 0$. Observe that by hypotheses $T_xM$ is non degenerate (that dim $T_xM = 4k$). By applying (17) to the triple $(X, J_2X, J_1X)$ for $X \in T_xM$ no eigenvector of any compatible para-complex structure in $Q$, we have
\[
(F_2^T \wedge \omega_3^T)(X, J_2X, J_1X) = -\|X\|^2 \omega_3^T(J_1X) = -\omega_3^T(J_1X) = \epsilon(F_3^T \wedge \omega_2^T)(X, J_2X, J_1X) = \epsilon F_3^T(J_2X, J_1X) \omega_2^T(X) = -\|X\|^2 \omega_2^T(X).
\]
Hence $\omega_3^T = \epsilon \omega_2^T \circ J^e$ and, from Theorem (3.4), it follows that $N_{\mathcal{F}^1_x} = 0$. Contradiction.

Let now suppose that codim $T_xM = 2$. From Lemma (3.5) it is $\psi(T_xM) = 0$. If $T_xM$ is non degenerate, calculating both sides of equation (17) on vectors $X, J_2X, Y$, where $X$ is a unit vector from $T_xM$ and $Y \in D_x$ is the $J^e$-invariant orthogonal complement to $T_xM$ in $T_xM$ we get
\[
(F_2^T \wedge \omega_3^T)(X, J_2X, Y) = \epsilon \omega_2^T(Y) = -(F_3^T \wedge \eta)(X, J_2X, Y) = 0.
\]
Hence, $\omega_3^T|_{D_x} = 0 = \omega_2^T|_{D_x}$ which implies that $N_{\mathcal{F}^1_x} = 0$. Contradiction.

In case that $T_xM$ is degenerate (even totally isotropic) and dim $D_x = 2$ with $D_x$ any $J^e$-invariant complement to $T_xM$ in $T_xM$, by evaluating (17) on the triple $(Y, J^eY, X)$ with $\{Y, J^eY\}$ any basis of $D_x$ and $X \in \ker \eta$, it is
\[
F_2^T \wedge \omega_3^T(X, Y, J^eY) = <J_2X, J_3Y \geq \omega_3^T(J^eY) = <J_2X, J_3Y \geq \omega_3^T(Y);
\]
\[
\epsilon(F_3^T \wedge \omega_2^T)(X, Y, J^eY) = \epsilon <J_3X, J_2Y \geq \omega_2^T(J^eY) = \epsilon <J_3X, J_2Y \geq \omega_2^T(Y))\]
i.e.
\[
<J_2X, J_3Y \geq \omega_3(J^eY - \omega_2(Y)) = <J_3Y, J_2X \geq \omega_3(Y) - \omega_2(J^eY) = 0.
\]
Then, considering the non degeneracy of $T_xM$, the only solution is given by
\[
[\omega_3^T \circ J^eY - \omega_2^T] = \omega_3^T - \omega_2^T \circ J^eY = 0, \quad \forall Y \in D_x
\]
i.e. $\psi(D_x) = 0$ which leads again to the contradiction that $N_{\mathcal{F}^1_x} = 0$. \hfill \Box

We state the following corresponding result regarding quaternionic geometry:

**Theorem 4.8.** Let $(\tilde{M}^{4n}, Q, \tilde{g})$ be a quaternionic Kähler manifold with non zero reduced scalar curvature $\nu$. Then any almost Kähler submanifold $(M^{2m}, \mathcal{F}, g)$, $n \neq 2$ of $\tilde{M}$ is Kähler.

**Proof.** Here the condition for a submanifold to be almost-Kähler is given by the equation
\[
F_2^T \wedge \omega_3^T = F_3^T \wedge \omega_2^T.
\]
The result for dimension greater that 6 has been given in [5].
By applying the proof of our Theorem (4.7) to the other cases and considering that in quaternionic case the metric in each subspace of $T_x M$ is positive definite, the conclusion follows. With respect to the para-quaternionic case the difference concerning the dimension 4 follows from the fact that, in a point $x \in M$ where the tangent space $T_x M$ is a 4 dimensional (Euclidean) quaternionic vector space, the equation (15) admits the non trivial solution $(\omega^2, \omega^2 = \omega^2 \circ J)$ which does not imply $N_{J^2} = 0$ that happens iff $\omega^2 \circ J = \omega^2 = 0$. □

5. MAXIMAL $\epsilon$-KÄHLER SUBMANIFOLDS OF A PARA-QUATERNIONIC KÄHLER MANIFOLD

5.1. The shape tensor $C$ of a $\epsilon$-Kähler submanifold. Let $(M^{2n}, J^*, g)$ be a $\epsilon$-Kähler submanifold of maximal possible dimension $2n$ of a para-quaternionic Kähler manifold $(M^{4n}, Q, g)$ with $\nu \neq 0$. We fix an adapted basis $(J_1, J_2, J_3 = J_1 J_2, J_2^2 = \epsilon \text{Id}, J_2^2 = \text{Id}, J^* = J_1 |_{TM})$ of $Q$ and assume that it is defined on a neighbourhood of $M^{2n}$ in $M^{4n}$. From Theorem (4.2), the submanifold $M$ is totally $\epsilon$-complex. We have the orthogonal decomposition

$$T_x \tilde{M} = T_x M \oplus J_2 T_x M \quad \forall x \in M.$$ (19)

Since $\omega^2 |_{T_x M} = \omega^2 |_{T_x M} = 0 \quad \forall x \in M$, then the following equations hold:

$$\nabla_X J_1 = 0 \quad , \quad \nabla_X J_2 = \omega(X) J_3 \quad , \quad \nabla_X J_3 = \omega(X) J_2 \quad \forall X \in TM$$ (20)

where $\omega = \omega |_{TM}$ is a 1-form. We identify the normal bundle $T^\perp M$ with the tangent bundle $TM$ using $J_2$ (note that $J_2^{-1} = J_2$):

$$\varphi = J_2 |_{T^\perp M} : \quad T^\perp_x M \rightarrow T_x M \quad \xi \mapsto J_2 \xi.$$ Then the second fundamental form $h$ of $M$ is identified with the tensor field

$$C = J_2 \circ h \in TM \otimes S^2 T^* M$$

and the normal connection $\nabla^\perp$ on $T^\perp M$ is identified with a linear connection $\nabla^N = J_2 \circ \nabla^\perp \circ J_2$ on $TM$. We will call $C$ the shape tensor of the $\epsilon$-Kähler submanifold $M$. Note that $C$ depends on the adapted basis $(J_\alpha)$ and it is defined only locally. We recall (see [21]) that the 3-dimensional vector space $Q_x \subset \text{End}(T_x \tilde{M})$ has a natural pseudo-Euclidean norm defined by $L^2 = ||L||^2 \text{Id}$, $L \in Q$. W.r.t. the adapted basis above, if $L = a J_1 + b J_2 + c J_3 \in Q$ then $||L||^2 = a^2 - b^2 - c^2$ if $\epsilon = -1$ and $||L||^2 = -a^2 - b^2 + c^2$ if $\epsilon = 1$. Then if $(J_\alpha)$ is another adapted basis obtained by the pseudo-orthogonal transformation, represented in the base $(J_1, J_2, J_3)$, by the following matrices $B_\epsilon \in SO(2,1)$

$$B_{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix}, \quad B_{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh \theta & \sinh \theta \\ 0 & \sinh \theta & \cosh \theta \end{pmatrix}$$ (21)

then the shape tensor transforms as

$$C \mapsto C' = J_2 \circ h = \cos \theta C + \sin \theta J^* \circ C \quad (\text{resp.} \quad C' = \cosh \theta C + \sinh \theta J^* \circ C).$$

In the following $(E_i)$, $i = 1, \ldots, 2n$ will be an orthonormal basis of $T_x M$ and we will use the notation $\mu_i = <E_i, E_i>$.

**Lemma 5.1.** One has
(1) For any $X \in TM$ the endomorphism $C_X$ of $TM$ is symmetric and $C_X = -A^{X^{-1}} = -A^{X^2}$ where $A^\xi$ is the shape operator, defined in (3).

(2) $\nabla_X = \nabla_X - \omega(X,J^e)$, $X \in TM$.

(3) The curvature of the connection $\nabla^N$ is given by

$$R^N_{XY} = R_{XY} - ed\omega(X,Y)J^e.$$ 

(4) $\{C_X,J^e\} = C_X \circ J^e + J^e \circ C_X = 0$ and hence $trC = \sum_{i} h_i C_{E_i} E_i = 0$.

(5) The tensors $gC$ and $gC \circ J^e$ defined by

$$gC(X,Y,Z) = g(C_XY,Z), \quad (gC \circ J^e)(X,Y,Z) = gC(J^eX,Y,Z)$$

are symmetric, i.e., both $gC$ and $gC \circ J^e \in S^3T^*M$.

Proof. (1) Using (19) and (20), for any $X,Y,Z \in TM$ one has

$$\langle C_XZ,Y \rangle = \langle J_2h(X,Z),Y \rangle = -\langle h(X,Z),J_2Y \rangle = -\langle \nabla_X(Z),J_2Y \rangle$$

$$= \langle \nabla_X(J_2Y),Z \rangle = \langle (\nabla_X J_2)Y + J_2 \nabla_X Y, Z \rangle$$

$$= \langle \omega(X)J_3Y + J_2 \nabla_XY, Z \rangle = \langle J_2 \nabla_XY, Z \rangle$$

$$= -\langle \nabla_XJ_2Z, J_2Z \rangle = -\langle h(X,Y),J_2Z \rangle = \langle C_XY, Z \rangle.$$ 

Moreover, for any $X,Y,Z \in TM$,

$$\langle -A^{X^2}Y,Z \rangle = -\langle h(Y,Z),J_2X \rangle = \langle J_2h(Y,Z),X \rangle = \langle C_YZ,X \rangle = \langle Z,C_XY \rangle.$$ 

This implies that $C_X = -A^{X^2}$.

(2) Denoting by $[\cdot]_1$ the projection on $T^1M$ of a vector in $\tilde{TM}$, we have

$$\nabla^N_XY = J_2[\nabla_XY]_1 = J_2[\nabla_X(J_2Y)]_1 = J_2(\nabla_XJ_2Y) = (\nabla_XJ_2Y)$$

$$= J_2[(\omega(X)J_3Y + J_2 \nabla_XY + h(X,Y))]_1 = -\omega(X)J_2Y.$$ 

(3) $R^N_{XY} = [\nabla_X - \omega(X)J^e, \nabla_Y - \omega(Y)J^e](Z) - \nabla_{[X,Y]}Z + \omega([X,Y])J^eZ$

$$= R_{XY}Z + \nabla_X[-\omega(Y)J^eZ] - \omega(X)J^e \nabla_Y Z + \omega(X) \omega(Y)J^eZ + \epsilon \nabla_Y \omega(X)J^eZ + \epsilon \omega(Y)J^e \nabla_X Z - \omega(X) \omega(Y)J^eZ + \epsilon \omega([X,Y])J^eZ$$

$$= R_{XY}Z - \epsilon \{X \cdot \omega(Y) - Y \cdot \omega(X) - \omega([X,Y])\}J^eZ$$

$$= R_{XY}Z - \epsilon \omega(X)J^eZ.$$ 

(4) By using (19) we get

$$C_X(J^eY) = J_2h(X,J^eY) = J_2J^e h(X,Y) = -J^e C_XY.$$ 

Since $C_X = -J^e \circ C_X \circ J^e^{-1}$, then $trC_X = 0$ $\forall X \in TM$, which implies $trC = 0$.

(5) The first statement follows from (1) and the symmetry of $h$. Using (4) we prove the second one:

$$(gC \circ J^e)(X,Y,Z) = gC(J^eX,Y,Z) = \langle C_{J^eX}Y,Z \rangle = \langle C_Y(J^eX),Z \rangle$$

$$= -\langle J^e C_XY,Z \rangle = \langle C_YJ^eX,Z \rangle = \langle C_Y(J^eZ),X \rangle$$

$$= \langle C_{J^eZ}Y,X \rangle = (gC \circ J^e)(Z,Y,X).$$

Moreover from (1) it is $(gC \circ J^e)(X,Y,Z) = (gC \circ J^e)(X,Z,Y)$. 

We denote by $\nabla^l$ the linear connection in a tensor bundle which is a tensor product of a tangent tensor bundle of $M$ and a normal tensor bundle whose connections are respectively $\nabla$ and $\nabla^\perp$. For example, if $k$ is a section of the bundle
$T^\perp M \otimes S^2 T^\ast M$ then $(\nabla_X^k)(Y,Z) = \nabla_X^k(k(Y,Z)) - k(\nabla_X Y, Z) - k(Y, \nabla_X Z)$. By using (2) of Lemma \ref{lem:5.1}, we get

$$J_2(\nabla_X^k)(Y,Z) = J_2[\nabla_X^k(h(Y,Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) =$$

$$= \nabla_X^k J_2 h(Y,Z) - C_{\nabla_X Y} Z - C_Y \nabla_X Z$$

$$= (\nabla_X^k C) Y Z + C_{\nabla_X Y} Z - C_Y \nabla_X Z - C_Y \nabla_X Z$$

hence for the covariant derivative of the second fundamental form we have:

$$J_2(\nabla_X^k)(Y,Z) = (\nabla_X^k C) Y Z + 2\omega(X) J^c C_Y Z = (\nabla_X^k C) Y Z + \omega(X) J^c C_Y Z.$$

Denote by $S_{J^c} = \{ A \in \text{End}(TM), [A, J^c] = 0, g(A X, Y) = g(X, A Y) \}$ the bundle of symmetric endomorphisms of $TM$, which anticommutes with $J$ and by $S_{J^c}^{(1)} = \{ A \in \text{Hom}(TM, S_{J^c}) = T^\ast M \otimes S_{J^c}, A X Y = A Y X \}$ its first prolongation. Then conditions (4), (5) can be reformulated as follows.

**Corollary 5.2.** The tensor $C = J_2 h$ belongs to the space $S_{J^c}^{(1)}$ and its covariant derivative is given by

$$\nabla_X C = J_2 \nabla_X h - \omega(X) J^c \circ C.$$

5.2. **Gauss-Codazzi-Ricci equations.** Let $M$ be a submanifold of a pseudo-Riemannian manifold $\tilde{M}$ and $\tilde{R}_{XY} = \tilde{R}_{XY}^X + \tilde{R}_{XY}^Y + \tilde{R}_{XY}^1 + \tilde{R}_{XY}^2$ the decomposition of the curvature operator $\tilde{R}_{XY}$, $X, Y \in T_x M$ of the manifold $\tilde{M}$ according to the decomposition $\text{End}(T_x \tilde{M}) = \text{End}(T_x M) + \text{Hom}(T_x M, T^\perp M) + \text{Hom}(T^\perp M, T_x M) + \text{End}(T^\perp M)$.

Using \cite{8} and calculating the curvature operator $\tilde{R}_{XY} = [\tilde{\nabla}_X, \tilde{\nabla}_Y] - [\nabla, [X, Y]]$ of the connection $\tilde{\nabla}$, we get the following **Gauss-Codazzi-Ricci equations**:

- **(Gauss)**
  $$R_{XY}^{TT} = R_{XY} - A_X A_Y^T + A_Y A_X^T = R_{XY} - \sum \kappa_i A_{\xi_i} X \wedge A_{\xi_i} Y \quad (TT)$$

- **(Codazzi 1)**
  $$R_{XY}^{XY} = h_X \nabla_Y h_Y - h_Y \nabla_X h_X + \tilde{h}_X \nabla_Y h_Y - \tilde{h}_Y \nabla_X h_X - h_{[X,Y]} \quad (\perp)$$

- **(Codazzi 2)**
  $$R_{XY}^{XX} = -h_X^T \nabla_Y h_Y + h_Y^T \nabla_X h_X + \tilde{h}_X^T \nabla_Y h_Y + \tilde{h}_Y^T \nabla_X h_X + h_{[X,Y]}^T \quad (\perp)$$

- **(Ricci)**
  $$R_{XY}^{YY} = R_{XY}^T - h_X \circ h_Y^T + h_Y \circ h_X^T = R^T(X,Y) - \sum \kappa_{a,b} \kappa_a \kappa_b < A^{\xi_a} X, A^{\xi_b} Y > \xi_a \wedge \xi_b \quad (\perp)$$
  $$R_{XY}^{XY} = R^T(X,Y) - \sum \kappa_{a,b} \kappa_a \kappa_b < [A^{\xi_a}, A^{\xi_b}] X, Y > \xi_a \wedge \xi_b$$
  $$R_{XY}^{XX} = R^T_{XX} \eta - \sum \kappa_{a} < A^{\xi} X, [A^{\xi}, A^{\eta}] Y > \xi_a \quad (\perp)$$

where $\xi_i$ is an orthonormal basis of $T^\perp M$ and $\kappa_i = < \xi_i, \xi_i >$, $X, Y \in TM, \eta \in T^\perp M,
\tilde{R}, R$ are the curvature tensors of the connections $\nabla$ and $\nabla^\perp$. We identify a bivector $X \wedge Y$ with the skew-symmetric operator $Z \mapsto (Y, Z) X - (X, Z) Y$ and denote by $h_X : T^\perp M \rightarrow TM$ the adjoint operator of $h_X = h(X, \cdot) : TM \rightarrow T^\perp M$. We recall that shape operator and second fundamental form satisfy $A^\eta X = h_X^T \eta$, $(AX = h_X^T \eta)$ or, equivalently, $< A^\eta X, Y > = h(X, Y, \eta)$.

**Definition 5.3.** Let $M$ be a submanifold $M$ of a Riemannian manifold $\tilde{M}$. Then

1. $M$ is called **curvature invariant** if $R_{XY} Z \in TM$, $\forall X, Y, Z \in TM$, or equivalently, $R^{TT} = R^{\perp T} = 0$. 


(2) $M$ is called **strongly curvature invariant** if it is curvature invariant and moreover \( \hat{R}_{\zeta \xi} \in T^1 M \), \( \forall \xi, \eta, \zeta \in T^1 M \).

(3) $M$ is called **parallel** if the second fundamental form is parallel: \( \nabla' h = 0 \).

Let us recall the following known result.

**Proposition 5.4.** A parallel submanifold $M$ of a locally symmetric manifold $\tilde{M}$ is curvature invariant and locally symmetric.

**Proof.** First statement follows from \((\perp T)\). For the second statement observe that \( \tilde{R}_{|T,M} = R^{TT} + R^{\perp T} = R^{TT} \). Then \( 0 = \tilde{\nabla} \tilde{R} = \nabla R^{TT} \). It follows

\[
0 = \nabla (R^{TT})(X,Y) = (\nabla R)_{XY} - \nabla (h_X \circ h_Y) + \nabla (h_Y \circ h_X) \quad \forall X, Y \in TM
\]

which implies \( \nabla R = 0 \). \( \square \)

5.3. **Gauss-Codazzi-Ricci equations for a \( \epsilon \)-Kähler submanifold.** By specifying the previous formulas to a totally \( \epsilon \)-complex submanifold and using Lemma \((5.1)\) and \((22)\) we get the following

**Proposition 5.5.** The Gauss-Codazzi-Ricci equations for a maximal totally \( \epsilon \)-complex submanifold \((M^{2n}, J^\epsilon, g)\) of a para-quaternionic Kähler manifold \((\tilde{M}^{4n}, Q, \tilde{g})\) can be written as

\[
\begin{align*}
(1) \quad & R_{XY}^{TT} = R_{XY} + [C_X, C_Y] \\
(2) \quad & J_2 R_{XY}^{\perp T} = R_{XY}^{TT} + [C_X, C_Y] = R_{XY} + [C_X, C_Y] - \epsilon d\omega(X, Y) J^\epsilon \\
(3) \quad & J_2 R_{XY}^{\perp} = P_{XY} - P_Y X
\end{align*}
\]

where \((J_n)\) is an adapted basis of \((M^{2n}, J^\epsilon)\), \( C = J_2 h \) is the shape tensor and \( P_{XY} := (\nabla_X C)_Y + \epsilon \omega(X) J^\epsilon \circ C_Y \in S_{J^\epsilon} \).

**Proof.** We prove the first two equations since the third comes directly from \((22)\)

\[
R_{XY}^{TT} = R_{XY} - A_X A_Y + A_Y A_X = R_{XY} + C_{J_2 A_X} X - C_{J_2 A_Y} Y
\]

\[
= R_{XY} + C_{J_2 h_X} X - C_{J_2 h_Y} Y = R_{XY} + C_{J_2 h_Y} X - C_{J_2 h_X} Y = R_{XY} + [C_X, C_Y]
\]

\[
J_2 R_{XY}^{\perp T} = J_2 (\nabla_X J_2 \nabla_Y + \nabla_Y J_2 \nabla_X) = J_2 (\nabla_X J_2 \nabla_Y + \nabla_Y J_2 \nabla_X) = J_2 (\nabla_X J_2 \nabla_Y + \nabla_Y J_2 \nabla_X) = R_{XY}^{TT} + J_2 h_X C_Y + J_2 h_Y C_X
\]

Now we prove that \( P_{XY} \in S_{J^\epsilon} \). Let \( X, Y, Z, T \in T_p M \). From \((22)\), \( P_{XY} = J_2 (\nabla_X h)_Y \) and computing

\[
\begin{align*}
& < P_{XY}, J^\epsilon Z, T > = < J_2 (\nabla_X h)_Y, J^\epsilon Z, T > \\
& = < J_2 \nabla_X [h(Y, J^\epsilon Z)], T > = < J_2 h(\nabla_X Y, J^\epsilon Z), T > \\
& = < J_2 J_1 [h(Y, Z)], T > = < J_2 J_1 [h(\nabla_X Y, Z)], T > = < J_2 J_1 [h(\nabla_X Y, Z)], T > \\
& = < P_{XY}, J^\epsilon Z, T >
\end{align*}
\]

Being \(- R_{XY}^{\perp T} \) the adjoint of \( R_{XY}^{\perp T} \), the operator \( R_{XY}^{\perp T} J_2 \) is the adjoint of \( J_2 R_{XY}^{\perp T} \). \( \square \)

**Corollary 5.6.** The Ricci tensor \( \text{Ric}_M \) of the \( \epsilon \)-Kähler submanifold \( M^{2n} \subset \tilde{M}^{4n} \) is given by

\[
\text{Ric}_M = \text{Ric}(R^{TT}) + \text{tr}_g (\langle C, C \rangle) = \text{Ric}(R^{TT}) + \langle \sum_i \mu_i C_{E_i}^2, \cdot \rangle
\]

or, more precisely,

\[
\text{Ric}_M(X, Y) = \text{Ric}(R^{TT})(X, Y) + \sum_{i=1}^{2n} \mu_i \langle C_{E_i} X, C_{E_i} Y \rangle \quad X, Y \in TM
\]

where \( \text{Ric}(R^{TT}) \) is the Ricci tensor of the tangential part \( R^{TT} \) of \( \tilde{R} \), that is \( \text{Ric}(R^{TT})(X, Y) = \text{tr}(Z \mapsto R_{ZX}^{TT} Y) \).
Proof.

\[
Ric(X, Y) = \sum_{i=1}^{2n} \mu_i \{g(R^{TT}(E_i, X)Y, E_i) - g([C_E, C_X]Y, E_i)\} = Ric(R^{TT})(X, Y) - \sum_{i=1}^{2n} \mu_i \{g(C_E, C_X Y, E_i) - g(C_X C_E, Y, E_i)\} = Ric(R^{TT})(X, Y) - \sum_{i=1}^{2n} \mu_i g(C_E, E_i, C_X Y) + \sum_{i=1}^{2n} \mu_i g(C_E, E_i, C_E, X) = Ric(R^{TT})(X, Y) + \sum_{i=1}^{2n} \mu_i g(C_E, E_i, C_E, X)
\]

\[\square\]

Proposition 5.7. Let \(M^{2n}\) be a \(c\)-Kähler submanifold of a para-quaternionic Kähler manifold \(\tilde{M}^{4n}\). Then

(i) \(M^{2n}\) is parallel if and only if \(P_{XY} := (\nabla_X C)_Y + cF(X, Y)J^c \circ C_Y = 0\);

(ii) \(M^{2n}\) is curvature invariant if and only if the tensor \(P_{XY}\) belongs to

\[
S_{J^c}^{(2)} = \{ A \in Hom(TM, S_{J^c}^{(1)}), A_{XY} = A_{YX}\}.
\]

Then \(M^{2n}\) is strongly curvature invariant.

Proof. 1) Follows from [22]. First statement of 2) follows from (3) of Proposition 5.7. To prove the last statement, we use the general identity for \(R\) of \(\tilde{M}^{4n}\) and \(R(J^c X, J^c Y)J^c T, J^c Z = R(X, Y)T, Z\) (it follows from repeated applications of [4]). By the curvature invariance and since \(J_2 T_x M = T^\perp_x M, \forall x \in M\), it is \(0 = R(X, Y)Z, \xi = R(J_2 X, J_2 Y)J_2 Z, J_2 \xi, X, Y, Z \in TM, \xi \in T^\perp M\). Then \(\tilde{R}_{\xi_1} \xi_2 \in T^\perp M, \forall \xi, \eta, \zeta \in T^\perp M\).

Proposition 5.8. For a \(c\)-Kähler submanifold \((M^{2n}, J^c, g)\) of a para-quaternionic Kähler manifold \(\tilde{M}^{4n}\), we have:

\[
R_{XY}^{\perp} = J_2 R_{XY}^{TT} J_2 + \nu F(X, Y)J^c
\]

i.e. Ricci equation follows from Gauss one. Moreover

\[
d_\omega(X, Y) = \nu F(X, Y).
\]

Proof. By proposition 2.23, the fact that \([J_\alpha, J_\beta] = 2\epsilon_3 \epsilon_7 J_7\), and from 4 one has

\[
\langle J_2 R_{XY}^{\perp} J_2 U, V \rangle = \langle J_2 \tilde{R}_{XY} J_2 U, V \rangle = \langle J_2 [\tilde{R}_{XY}, J_2] U + J_2 \tilde{R}_{XY} U, V \rangle = \langle \tilde{R}_{XY} U, V \rangle + \epsilon_3 \nu \langle J_2 (-F_1(X, Y)J_3 + F_3(X, Y)J_1) U, V \rangle = \langle R_{XY}^{TT} U, V \rangle - \epsilon_\omega \langle F(X, Y)J_1 U, V \rangle, \quad X, Y, U, V \in TM,
\]

that is (23). Since \(J_2 R_{XY}^{\perp} J_2 = R_{XY}^{TT} - \epsilon d\omega(X, Y)J^c\), the last identity follows.

5.4. Maximal \(c\)-Kähler submanifolds of a para-quaternionic symmetric space. Now we assume that the manifold \((\tilde{M}^{4n}, \tilde{g})\) is a (locally) symmetric manifold, i.e. \(\tilde{\nabla} \tilde{R} = 0\). By adapting the proof of Proposition 2.10 in [5] to the para-quaternionic case, we can state the following

Proposition 5.9. Let \((M^{2n}, J^c, g)\) be an \(c\)-Kähler submanifold of a para-quaternionic locally symmetric space \((\tilde{M}^{4n}, \tilde{g})\). Then the covariant derivatives of the tangential part \(R^{TT}\), the normal part \(R^{\perp\perp}\) and mixed part \(R^{\perp T}\) of the curvature tensor \(\tilde{R}|_M\) can be expressed in terms of these tensors and the shape operator \(C = J_2 \circ h\) as follows:

\[
\langle (\nabla_X R^{TT})(Y, Z)U, V \rangle = + \langle R^{T\perp T}(Y, Z)U, J_2 C_X V \rangle - \langle R^{\perp T T}(Y, Z)V, J_2 C_X U \rangle + \langle J_2 R^{\perp T T}(U, V)C_X Y, Z \rangle + \langle R^{\perp T T}(U, V)Y, J_2 C_X Z \rangle.
\]
(25) \( (\nabla_X R^\perp)(Y, Z)U = -J_2C_X R^T(Y, Z)U - R^\perp(Y, Z)J_2C_X U \)
\[ + [\tilde{R}(J_2C_X Y, Z)U + \tilde{R}(Y, J_2C_X Z)U]^\perp \]
\[ = J_2C_X R^T(Y, Z)U - J_2R^T(Y, Z)C_X U \]
\[ + \nu F(Y, Z)J_3C_X U + [\tilde{R}(J_2C_X Y, Z)U + \tilde{R}(Y, J_2C_X Z)U]^\perp \]

(26) \( (\nabla_X R^\perp)(Y, Z)\xi = +[\tilde{R}(J_2C_X Y, Z)\xi + \tilde{R}(Y, J_2C_X Z)\xi]^T \)
\[ + R^T(Y, Z)C_{J_3}\xi - C_X J_2 R^\perp(Y, Z)\xi \]

(27) \( (\nabla_X R^\perp)(Y, Z)j_2U, j_2V = \langle R^T U, V \rangle Z, J_2C_X Y - (R^T(U, V)Y, J_2C_X Z) \)
\[ + (R^T(Y, Z)C_X U, J_2U) + \langle C_X R^T(Y, Z)J_2U, V \rangle \]

for any \( X, Y, Z, U, V \in TM \), \( \xi \in T^\perp M \).

By (23) and (25) we get immediately the following result.

**Proposition 5.10.** If the \( \epsilon \)-Kähler submanifold \( M^{2n} \subset \tilde{M}^{2n} \) is curvature invariant then the tensor field \( R^{TT} \) is parallel i.e. \( \nabla R^{TT} = 0 \) and satisfies the identity

\[ C_X R^T(Y, Z) + R^T(Y, Z)C_X + \nu F(Y, Z)J^2C_X \]
\[ = [J_2(\tilde{R}(J_2C_X Y, Z) + \tilde{R}(Y, J_2C_X Z))]^{TT} \]

where \( (A)^{TT} \) denotes the \( \mathrm{End}(T_2M) \) component of an endomorphism \( A \) of \( T_2\tilde{M} \).

Denote by \( [C, C] \) the \( \mathrm{End}(T_2M) \) valued \( 2 \)-form, given by

\[ [C, C](X, Y) = [C_X, C_Y] \quad \forall X, Y \in TM. \]

One can easily check that it is globally defined on \( M \).

For a subspace \( \mathcal{G} \subset \mathrm{End}(T_2M) \) we define the space \( \mathcal{R}(\mathcal{G}) \) of the curvature tensors of type \( \mathcal{G} \) by

\[ \mathcal{R}(\mathcal{G}) = \{ R \in \mathcal{G} \otimes \Lambda^2 T^*_2M \mid \text{cycl } R(X, Y)Z = 0, \forall X, Y, Z \in T_2M \} \]

where \( \text{cycl} \) is the sum of cyclic permutations of \( X, Y, Z \).

Let denote by \( u_{p,q}^\epsilon \), the Lie algebra of the unitary Lie group of automorphisms of the Hermitian (para)-complex structure \( (J^\epsilon, g) \) where \( (p, q) \) corresponds to the signature of \( g \). As a Corollary of Propositions 5.9 and 5.5 (1) we have the following

**Proposition 5.11.** Under the assumptions of Proposition 5.10 the tensor field \( [C, C] = R^{TT} - R \) belongs to the space \( \mathcal{R}(u_{p,q}^\epsilon) \) and satisfies the second Bianchi identities:

\[ \text{cycl } \nabla_Z [C_X, C_Y] = 0. \]

**Proof.** The tensor \( [C, C] \) satisfies the first Bianchi identity since \( R \) and \( R^{TT} \) do it. Moreover \( J^\epsilon \circ [C_X, C_Y] = -C_X \circ J^\epsilon C_Y + C_Y \circ J^\epsilon C_X = C_X C_Y \circ J^\epsilon - C_Y C_X \circ J^\epsilon = [C_X, C_Y] \circ J^\epsilon \) i.e. \( [C_X, C_Y] \) commutes with \( J^\epsilon \). Furthermore, by the symmetry of \( C_X, <[C_X, C_Y] Z, T > = < C_Y Z, C_X T > = -< C_X Z, C_Y T > = < Z, [C_Y, C_X] T > \)

that is \( [C_X, C_Y] \) is skew-symmetric with respect to the metric \( g = < , , > \). Then the tensor \( [C, C] \) belongs to the space \( \mathcal{R}(u_{p,q}^\epsilon) \) of the \( u_{p,q}^\epsilon \)-curvature tensors. The last statement follows from remark that \( \text{cycl } \nabla_Z [C_X, C_Y] = \text{cycl } \nabla_Z (R^{TT} + R) \).

\[ \nabla R^{TT} = 0 \] and \( R \) satisfies the second Bianchi identity. \( \square \)
As another Corollary of Proposition (5.9) we get the following result.

**Proposition 5.12.** A maximal $\epsilon$-Kähler submanifold $M^{2n}$ of a locally symmetric para-quaternionic Kähler manifold $\tilde{M}^{4n}$ is locally symmetric (that is $\nabla R = 0$) if and only if the following identity holds:

\[
\langle \nabla_X [C,C]Y, Z U, V \rangle = \langle R^{1\perp T}(Y,Z)U, J_2 C_X V \rangle - \langle R^{1\perp T}(Y,Z)V, J_2 C_X U \rangle + \langle J_2 R^{1\perp T}(U,V)C_X Y, Z \rangle + \langle R^{1\perp T}(U,V)Y, J_2 C_X Z \rangle.
\]

If $M$ is curvature invariant then (29) reduces to the condition that the tensor field $\nabla [C,C]$ is parallel ($\nabla [C,C] = 0$).

**Proof.** The proof follows directly from the Gauss equation and (24). \qed

### 5.5. Maximal totally complex submanifolds of para-quaternionic space forms

Now we assume that $\tilde{(M^{4n}, Q, \tilde{g})}$ is a non flat para-quaternionic space form, i.e., a para-quaternionic Kähler manifold which is locally isometric to the para-quaternionic projective space $\mathbb{HP}^n$ or the dual para-quaternionic hyperbolic space $\tilde{H}^n$ with standard metric of reduced scalar curvature $\nu$. Recall that the curvature tensor of $\tilde{(M^{4n}, Q, \tilde{g})}$ is given by $\tilde{R} = \nu R_0$ (see (2)). We denote by $R_{C^p n}$ the curvature tensor of the $\epsilon$-complex projective space (normalized such that the holomorphic curvature equal to 1):

\[
R_{C^p n}(X, Y) = \frac{1}{4} \left( - \epsilon X \wedge Y + JX \wedge JY - 2(JX, Y)J \right).
\]

It is a straightforward to verify the

**Proposition 5.13.** Let $(M^{2n}, J^c, g)$ be a totally $\epsilon$-complex submanifold of the para-quaternionic space form $\tilde{M}^{4n}$. We have:

1. $R^{TT}_{XX} = -\epsilon (R_{C^p n})_{XY} = \frac{\epsilon}{4} \left( \epsilon X \wedge Y - J_1 X \wedge J_1 Y + 2(J_1 X, Y)J_1 \right)$.
2. $\text{Ric}(R^{TT}) = \frac{\epsilon}{4} (n+1) g$, $g = g_{\tilde{M}}$.
3. $R^{1\perp T} = R^{T\perp} = 0$.
4. $R_{XX}^{1\perp} = \frac{\epsilon}{4} \left( - J_2 X \wedge J_2 Y + \epsilon J_3 X \wedge J_3 Y + 2\epsilon (J_1 X, Y)J_1 \right)$.

As a consequence of Corollary (5.6) and Proposition (5.13) we get

**Proposition 5.14.** Let $M^{2n}$ be a $\epsilon$-Kähler submanifold of a para-quaternionic space form $\tilde{M}^{4n}$ with reduced scalar curvature $\nu$.

\[
\text{Ric}_M(X, X) = \frac{\nu}{2}(n+1) g(X, X) + \text{tr} C^2_X = \frac{\nu}{2} (n+1)||X||^2 - \sum_{i=1}^{2n} \mu_i ||h(E_i, X)||^2, \quad X \in T_x M
\]

Moreover the second fundamental form $h_x$ of $M$ at point $x \in M$ vanishes if and only if $(\text{Ric}_M)_x = \frac{\nu}{2}(n+1) g$. In particular $M$ is a totally $\epsilon$-complex totally geodesic submanifold if and only if

\[
\text{Ric}_M = \frac{\nu}{2}(n+1) g.
\]

From Proposition (5.12) we get

**Proposition 5.15.** A maximal $\epsilon$-Kähler submanifold $(M^{2n}, J^c, g)$ of a non flat para-quaternionic space form is locally symmetric if and only if the tensor field $[C,C]$ is parallel. In particular, any maximal $\epsilon$-Kähler submanifold with parallel second fundamental form is (locally ) symmetric.
Proof. It is sufficient to prove only the last statement. Assume that \( \nabla' h = 0 \). Then 
\[
\nabla_X C = \omega(X) J^T C \quad \text{and} \quad \nabla_X [C, C](Y, Z) = [\nabla_X C_Y, C_Z] + [C_Y, \nabla_X C_Z] = \omega(X) \left( [J^T C_Y, C_Z] + [C_Y, J^T C_Z] \right) = 0 \quad \text{since} \ C_Y \ \text{anticommutes with} \ J^T.
\]

\[\Box\]

6. The Parallel Cubic Line Bundles of a Maximal Parallel \( \epsilon \)-Kähler Submanifold

For a deep analysis of parallel submanifolds of a quaternionic manifold refer to \[\text{[5, 12]}\]. We will assume that \( \tilde{M}^{4n} \) is a para-quaternionic Kähler manifold with the reduced scalar curvature \( \nu \neq 0 \). We consider first the case that \( (M^{2n}, J, g) \) is a parallel totally complex submanifold of \( \tilde{M} \). From Proposition \( \text{(5.7)} \)

\[
P_{XY} := (\nabla_X C)_Y - \omega(X) J \circ C_Y = 0 \quad X, Y \in TM.
\]

We will assume moreover that \( M \) is not a totally geodesic submanifold, i.e. \( h \neq 0 \). By Proposition \( \text{(5.7)} \) \( M \) is a curvature invariant submanifold \( (R^{1T} = 0) \). We denote by \( T^C M = T^{1,0} M + T^{0,1} M \) the decomposition of the complexified tangent bundle into holomorphic and antiholomorphic parts and by \( T^{*C} M = T^{*1,0} M + T^{*0,1} M \) the dual decomposition of the cotangent bundle.

Denote by \( S_j^{(1)} C \) the complexification of the bundle \( S_j^{(1)} \) \( (\text{see Corollary \[\text{(5.2)}\]} \) and by \( g \circ S_j^{(1)} C \) the associated subbundle of the bundle \( S^3(T^* M)^C \). We will call 

\[S^3(T^* M)^C \]

the bundle of complex cubic forms.

Proposition 6.1. Let \( (M^{2n}, J, g) \) be a parallel Kähler submanifold of a para-quaternionic Kähler manifold \( \tilde{M}^{4n} \) with \( \nu \neq 0 \). If it is not totally geodesic then on \( M \) there is a pair of canonically defined parallel complex line subbundles \( L \) \( (\text{resp.} \ L') \) of the bundle \( S^3(T^{1,0} M) \) \( (\text{resp.} \ S^3(T^{0,1} M)) \) of holomorphic \( (\text{resp. antiholomorphic}) \) cubic forms such that the curvature induced by the Levi-Civita connection has the curvature form

\[
R^L = -i \nu F, \quad (\text{resp.} \ R^{L'} = i \nu F).
\]

Proof. We first prove the following

Lemma 6.2. \( g \circ S_j^{(1)} C = S^3(T^{1,0} M) + S^3(T^{0,1} M) \).

Proof. Since \( J|_{T^{1,0} M} = i \ Id \), \( J|_{T^{0,1} M} = -i \ Id \), one has 

\[
S_j^C = \text{Hom}(T^{1,0} M, T^{0,1} M) + \text{Hom}(T^{0,1} M, T^{1,0} M)
\]

where \( S_j^C \) is the space \( S_j^C \) of complex endomorphisms of \( T_j^C M \) which anticommute with \( J \). In fact, let \( X \in T^{1,0} M \), \( A \in S_j^C \), and denote by \( AX = Y = Y^{1,0} + Y^{0,1} \), \( Y^{1,0} \in T^{1,0} M \), \( Y^{0,1} \in T^{0,1} M \). Then \( AJX = iAX = iY^{1,0} + iY^{0,1} \) whereas 

\[
-JAX = -JY^{1,0} - JY^{0,1} = -iY^{1,0} + iY^{0,1} \text{ which implies} \ Y^{1,0} = 0. \quad \text{Analogously,} \text{ if} \ X \in T^{0,1} M, \text{ we get} \ Y^{0,1} = 0.
\]

Hence the space \( g \circ S_j^C \) of symmetric bilinear forms, associated with \( S_j^C \) is 

\[
g \circ S_j^C = S^2(T^{1,0} M) + S^2(T^{0,1} M).
\]

In fact, for \( X \in T^{1,0} M \) and \( A \in S_j^C \), 

\[
<AX, Y> = - <J^2AX, Y> = <JAX, JY> = - <AJX, JY> = -i <AX, JY>.
\]

This implies that \( JY = iY \) i.e. \( Y \in T^{1,0} M \). Analogously, if \( X \in T^{0,1} M \) then \( Y \in T^{0,1} M \). This proves the lemma. \[\Box\]
Using this lemma we can decompose the cubic form $gC \in g \circ S^1_3$ associated with the shape operator $C = J_3 h$ into holomorphic and antiholomorphic parts:

$$gC = q + \overline{q} \in S^3(\mathbb{T}^{1,0} M) + S^3(\mathbb{T}^{0,1} M).$$

Since, by assumption, $\nabla C = \omega(X)J \circ C$ we have

$$g \nabla C = \nabla C = \nabla \omega + q = \omega(X)g(J \circ C).$$

For $Y, Z \in \mathbb{T}^{1,0} M$, we get

$$\nabla C(Y, Z) = \omega(X)g(JC(Y, Z)) = -i\omega(X)gC(Y, Z)$$

since $C(Y, Z) \in \mathbb{T}^{0,1} M$ and $JC(Y, Z) = -iC(Y, Z)$. This shows that

$$\nabla C = -i\omega(X)q.$$

Under the changing of adapted basis $(J_\alpha) \to (J'_\alpha)$ represented in basis $(J_1, J_2, J_3)$ by the first matrix in (21), we have $J'_2 = \cos \theta J_2 + \sin \theta J_3$ and the cubic form $q$ changes by $q \to q' = (\cos \theta - i \sin \theta)q$. In fact, for $Y, Z \in \mathbb{T}^{1,0} M$,

$$q'(Y, Z) = \cos \theta J_2 \circ h(Y, Z) + \sin \theta J_1 (J_2 \circ h(Y, Z)) = \cos \theta q - i \sin \theta q$$

since $J_2 \circ h(Y, Z) \in \mathbb{T}^{0,1} M$. Analogously $\overline{q} \to \overline{q}' = (\cos \theta + i \sin \theta)\overline{q}$.

Note also that the cubic forms $q$ and $\overline{q}$ are not 0 at any point, since by assumption the second fundamental form $h$ is parallel and not zero. These show that the complex line bundle $L = \text{span}_C(q) \subset S^3(\mathbb{T}^{1,0} M)$ (resp. $\overline{L} = \text{span}_C(\overline{q}) \subset S^3(\mathbb{T}^{0,1} M)$) is globally defined and parallel, i.e. the Levi-Civita connection $\nabla$ preserves $L$ (resp. $\overline{L}$) and defines a connection $\nabla^L$ in $L$ (resp. $\nabla^{\overline{L}}$ in $\overline{L}$). Using (31), we calculate the curvature of $\nabla^L$ as follows:

$$R^L(X, Y)q = \left( [\nabla^L_X, \nabla^L_Y] - \nabla^L_{[X, Y]} \right) q = \left( [\nabla_X, \nabla_Y] - \nabla_{[X, Y]} \right) q$$

$$= -\nabla_X \left( \omega(Y)iq \right) + \nabla_Y \left( \omega(X)iq \right) + \omega([X, Y])iq$$

$$= -d\omega(X, Y)iq - \omega(Y)\omega(X)q + \omega(X)\omega(Y)q$$

$$= -d\omega(X, Y)iq = -\nu F(X, Y)iq$$

Analogously it is $\nabla_X \overline{q} = i\omega(X)\overline{q}$ and $R^{\overline{L}}(X, Y)\overline{q} = \nu F(X, Y)i\overline{q}$. \hfill \Box

**Definition 6.3.** A parallel subbundle $L \subset S^3(\mathbb{T}^{1,0} M)$ with the curvature form (30) on a Kähler manifold $M$ is called a parallel cubic line bundle of type $-\nu$.

**Corollary 6.4.** A parallel maximal Kähler not totally geodesic submanifold $M$ of a para-quaternionic Kähler manifold $\tilde{M}$ with $\nu \neq 0$ has a pair of parallel cubic line bundles of type $\pm \nu$.

Let consider now the case that $(M^{2n}, K, g)$ is a parallel, totally para-complex, not totally geodesic submanifold of $\tilde{M}$. Then

$$P_{XY} = (\nabla_X C)_Y + \omega(X)K \circ C_Y = 0 \quad X, Y \in TM \quad \text{and} \quad h \neq 0.$$

By Proposition (5.7) $M$ is a curvature invariant submanifold. Let $TM = T^+ M + T^- M$ be the bi-Lagrangean decomposition of the tangent bundle into the $(+1)$ and $(-1)$ eigenspaces of $K$ and by $T^+ M = (T^+ M) + (T^- M)$ the dual decomposition of the cotangent bundle. We will call $S^3(T^+ M)$ the bundle of real cubic forms.

We recall that $C \in S^1_3(K)$ (see Corollary (5.2)) and we denote by $g \circ S^1_3\overline{K}$ the associated subbundle of the bundle $S^3(T^* M)$. Following the same line of proof of lemma (6.2) we can affirm that
Lemma 6.5. $g \circ S^{(1)}_K = S^3(T^{*+}M) + S^3(T^{*-}M)$.

We can then decompose the cubic form $gC \in g \circ S^{(1)}_K$ associated with the shape operator $C = J_2 h$ according to:

$$gC = q^+ + q^- \in S^3(T^{*+}M) + S^3(T^{*-}M).$$

Proposition 6.6. Let $(M^{2n}, K)$ be a parallel para-Kähler submanifold of a para-quaternionic Kähler manifold $M^{4n}$ with $\nu \neq 0$. If it is not totally geodesic then on $M$ the pair of real line subbundle $L^+ := \mathbb{R}q^+ \subset S^3(T^{*+}M)$ and $L^- := \mathbb{R}q^- \subset S^3(T^{*-}M)$ are globally defined and parallel, i.e the Levi-Civita connection $\nabla$ preserves $L^+$ (resp. $L^-$) and defines a connection $\nabla_{L^+}$ on $L^+$ (resp. $\nabla_{L^-}$ on $L^-$) whose curvature is

$$R_{L^+} = \nu F,$$

where $F = g \circ K$ is the Kähler form of $M$.

Proof. Following the same line of proof of the previous Kähler case, we have

$$\nabla_X q^+ = \omega(X) q^+; \quad \nabla_X q^- = -\omega(X) q^-.$$

Under a changing of the adapted basis $(J_a) \rightarrow (J'_a)$, represented in basis $(J_1, J_2, J_3)$ by the second matrix in (21), we have that

$$q^+ \rightarrow q'^+ = (\cosh \theta - \sinh \theta) q^+ \quad \text{and} \quad q^- \rightarrow q'^- = (\cosh \theta + \sinh \theta) q^-.$$

Note that $q^+ \neq 0$ and $q^- \neq 0$ at any point, since by assumption the second fundamental form $h$ is parallel and not zero and the metric $g$ is non degenerate on $M$. Then the real line bundles $L^+ := \mathbb{R}q^+ \subset S^3(T^{*+}M)$ and $L^- := \mathbb{R}q^- \subset S^3(T^{*-}M)$ are globally defined and parallel, i.e the Levi-Civita connection $\nabla$ preserves $L^+$ (resp. $L^-$) and defines a connection $\nabla_{L^+}$ on $L^+$ (resp. $\nabla_{L^-}$ on $L^-$. Moreover

$$R_{L^+}(X, Y) q^+ = \left( [\nabla^+_X, \nabla^+_Y] - \nabla^+_{[X, Y]} \right) q^+ = \left( [\nabla_X, \nabla_Y] - \nabla_{[X, Y]} \right) q^+$$

$$= \nabla_X \left( \omega(Y)(q^+) \right) - \nabla_Y \left( \omega(X)(q^+) \right) - \omega([X, Y])(q^+)$$

$$= \omega(X)(Y)(q^+) = \nu F(X, Y) q^+.$$

Analogously $R_{L^-}(X, Y) q^- = -\nu F(X, Y) q^-$. 

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