Abstract

We expand the family of tensors that can be efficiently decomposed and apply our new algorithmic ideas to blind deconvolution and Gaussian mixture models. Our basic contribution is an efficient algorithm to decompose certain symmetric overcomplete order-3 tensors, that is, three dimensional arrays of the form $T = \sum_{i=1}^{n} a_i \otimes a_i \otimes a_i$ where the $a_i$s are not linearly independent. Our second contribution builds on top of our tensor decomposition algorithm to expand the family of Gaussian mixture models whose parameters can be estimated efficiently. These ideas are also presented in a much more general framework of blind deconvolution that makes them applicable to mixture models of identical but very general distributions, including all centrally symmetric distributions with finite 6th moment.

1 Introduction

In this work we expand the family of tensors that can be efficiently decomposed and apply our new algorithmic ideas to blind deconvolution and Gaussian mixture models. We start by describing these tools and models.

Tensor decomposition is a basic tool in data analysis. The order-3 tensor decomposition problem can be stated as follows: Given an order-3 tensor $T = \sum_{i=1}^{n} a_i \otimes a_i \otimes a_i$, recover the vectors $a_i \in \mathbb{R}^d$. The problem is undercomplete if the $a_i$s are linearly independent, otherwise it is overcomplete. Two problems in data analysis motivate us here to study tensor decomposition: blind deconvolution and Gaussian mixture models (GMM).

A deconvolution problem can be formulated as follows: We have a $d$-dimensional random vector $Y = Z + \eta$ (1.1) where $Z$ and $\eta$ are independent random vectors. Given samples from $Y$, the goal is to determine the distribution of $Z$. We call the problem blind deconvolution when the distribution of $\eta$ is unknown, otherwise the problem is non-blind. It is called deconvolution because the distribution of $Y$ is the convolution of the probability distributions of $Z$ and $\eta$.

The following mixture model parameter estimation problem can be recast as a blind deconvolution problem: Let $X$ be a $d$-dimensional random vector distributed as the following mixture model: First sample $i$ from $[n]$, each value with probability $w_i$ ($w_i > 0$, $\sum_i w_i = 1$), then let $X = \mu_i + \eta$, where $\eta$ is a given $d$-dimensional random vector and $\mu_i \in \mathbb{R}^d$. The estimation problem is to estimate $\mu_i$s and $w_i$s from samples of $X$. It is a deconvolution problem $X = Z + \eta$ when $Z$ follows
the discrete distribution equal to $\mu_i$ with probability $w_i$. It is blind when the distribution of $\eta$ is unknown.

The **GMM parameter estimation problem** can be described as follows: Let $X \in \mathbb{R}^n$ be a random vector with density function $x \mapsto \sum_{i=1}^{k} w_i f_i(x)$ where $w_i > 0$, $\sum_i w_i = 1$ and $f_i$ is the Gaussian density function with mean $\mu_i \in \mathbb{R}^n$ and covariance matrix $\Sigma_i \in \mathbb{R}^{n \times n}$. GMM parameter estimation is the following algorithmic question: Given iid. samples from $X$, estimate $w_i$s, $\mu_i$s, $\Sigma_i$s.

The GMM parameter estimation problem is a deconvolution problem when the covariance matrices of the components are the same, namely $\Sigma_i = \Sigma$. Specifically, $X = Z + \eta$ where $Z$ follows a discrete distribution taking value $\mu_i$ with probability $w_i$, $i = 1, \ldots, k$ and $\eta$ is Gaussian with mean 0 and covariance $\Sigma$. It is a blind deconvolution if $\Sigma$ is unknown.

While the undercomplete tensor decomposition problem is well-understood (based on algorithmic techniques such as the tensor power method and Jennrich’s algorithm [16]), the overcomplete regime is much more challenging [20, Chapter 7]. Within the overcomplete case, there are fewer techniques available for the order-3 case than there are for higher order [20, Section 7.3]. We discuss some of these techniques and challenges below.

### 1.1 Related work

Among basic tensor decomposition techniques for the undercomplete case we have tensor power iteration (see [20] for example) and Jennrich’s algorithm ([16], also known as simultaneous diagonalization and rediscovered several times, with variations credited to [23]). Tensor power iteration is more robust than Jennrich’s algorithm, while Jennrich’s algorithm can be applied more generally: Tensor power iteration is mainly an algorithm for orthogonal tensors (orthogonal $a_i$s) and the general case with additional information, while Jennrich’s algorithm can decompose the general case without additional information. Our contributions below are based on Jennrich’s algorithm because of this additional power. The robustness of Jennrich’s algorithm is studied in several papers, our analysis builds on top of [14, 4].

For the overcomplete regime we have algorithms such as FOOBi [22] and the work of [4, 1, 2, 12, 25, 13, 17].

Many techniques for the overcomplete case only make sense for orders 4 and higher or have weaker guarantees in the order-3 case. For example, some techniques use the fact that an $d$-by-$d$-by-$d$-by-$d$ tensor can be seen as an $d^2$-by-$d^2$ matrix (and similarly for order higher than 4), while no equally useful operation is available for order-3 tensors. In particular, we are not aware of any efficient algorithm for arbitrary overcomplete order-3 tensors, even if only slightly overcomplete. Nevertheless, there are several results about decomposition in the order-3 case that are particularly relevant for our work: Kruskal’s uniqueness of decomposition [21], a robust version of Kruskal’s uniqueness and an algorithm running in time exponential in the number of components [5], an algorithm for tensors with incoherent components [1, 2], an algebraic algorithm [9, 10], a quasi-polynomial time algorithm (based on sum of squares) for tensors with random components [12] and polynomial time algorithms (also based on sum of squares) for tensors with random components [18, 26].

Blind deconvolution-type problems have a long history in signal processing and specifically in image processing as a deblurring technique (see, e.g., [24]). The idea of using higher order moments in blind identification problems is standard too in signal processing, specifically in Independent Component Analysis (see e.g. [7, 6]). Our model (1.1) is somewhat different but very natural and inspired by mixture models.
With respect to GMMs, we are interested in parameter estimation in high dimension with no separation assumption (the means $\mu_i$ can be arbitrarily close). Among the most relevant results in this context we have the following polynomial time algorithms: [19], for linearly independent means and spherical components (each $\Sigma_i$ is a multiple of the identity); [3], for $O(d^c)$ components with identical and known covariance $\Sigma$; [15, Section 7], [14], for linearly independent means and spherical components in the presence of Gaussian noise; [11], for a general GMM with $O(\sqrt{d})$ components in the sense of smoothed analysis.

1.2 Our results

Overcomplete tensor decomposition. We provide an algorithm that can recover the components $a_i \in \mathbb{R}^d$ to within error $\epsilon$ given a symmetric order-3 tensor $T = \sum_{i=1}^{d+k} a_i \otimes a_i \otimes a_i$, when any $d$-subset of the $a_i$s is linearly independent, in time polynomial in $d^k, 1/\epsilon^k$ and natural conditioning parameters. Even though the algorithm is exponential in the sense that it approximates the $a_i$s even when the input is a tensor that is $\epsilon'$-close to $T$. Also, it turns out that parameter $k$ above, the number of $a_i$s beyond the dimension $d$, is not the best notion of overcompleteness. In our result the tensor is of the form $T = \sum_{i=1}^{r+k} a_i \otimes a_i \otimes a_i$, where $r$ is the robust Kruskal rank of the $a_i$s (informally the maximum $r$ such that any $r$-subset is well-conditioned, Definition 2.1), so that $k$ is the number of components above the robust Kruskal rank. Thus, our analysis also applies when the Kruskal rank is less than $d$.

Blind deconvolution. We provide an efficient algorithm for the following blind deconvolution problem: Approximate the distribution of $Z$ (from (1.1)) when it is a $d$-dimensional discrete distribution supported on $d$ points satisfying a natural non-degeneracy condition (Assumption 4.1), the distribution of $\eta$ is unknown and the first and third moments of $\eta$ are 0 with finite 6th moment (this includes the natural case where $\eta$ has a centrally symmetric distribution with finite 6th moment). Equivalently, it can solve the mixture model parameter estimation problem above under the same conditions (alg. 4 and Theorem 4.2).

GMM. We show an efficient algorithm for the following GMM parameter estimation problem: Given samples from a $d$-dimensional mixture of $d$ identical and not necessarily spherical Gaussians with unknown parameters $w_i, \mu_i, \Sigma$, estimate all parameters (alg. 5 and Theorem 5.1).

It may seem as if the last two contributions (blind deconvolution and GMM) could be attacked with standard undercomplete tensor decomposition techniques given that the number of components is equal to the ambient dimension and therefore they could be linearly independent. It is not clear how that could actually happen, as the non-spherical unknown covariance seems to make standard approaches inapplicable and our contribution is a formulation that involves an overcomplete tensor decomposition and uses our overcomplete tensor decomposition algorithm in an essential way.

2 Notation and preliminaries

For $n \in \mathbb{N}$, the set $\{1, \ldots n\}$ is denoted by $[n]$. The unit sphere in $\mathbb{R}^d$ is denoted by $S^{d-1}$. 3
Matrices and vectors. For a matrix $A \in \mathbb{R}^{m \times n}$, we denote by $\sigma_i(A)$ its $i$-th largest singular value, by $A^\dagger$ its Moore-Penrose pseudoinverse, and by $\kappa(A) = \sigma_1(A)/\sigma_{\min(m,n)}(A)$ its condition number. Denote by $\text{vec}(A) \in \mathbb{R}^{mn}$ the vectorization of $A$, which is padding all of its entries into a vector successively. Denote by $\text{diag}(a)$ the diagonal matrix with diagonal entries in $a$, where $a$ is an indexed array of numbers. We use $\|\cdot\|_2$ to denote the spectral norm of matrix, and $\|\cdot\|_F$ to denote the Frobenius norm of a matrix.

In $\mathbb{R}^d$, we denote by $\langle a, b \rangle$ the inner product of two vectors $a, b$. Let $\hat{a} = a/\|a\|_2$. For a set of vectors $\{a_1, a_2, \ldots, a_n\}$, we denote their linear span by $\text{span}\{a_1, \ldots, a_n\}$. We use $[a_1, a_2, \ldots, a_n]$ to denote the matrix containing $a_i$s as columns. Particularly, if $A = [a_1, a_2, \ldots, a_n]$, we have $\hat{A} := [\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n]$ and $\hat{A}$ follows a similar definition. Besides, we denote by $A_m \in \mathbb{R}^{dxm}$ the matrix $[a_1, a_2, \ldots, a_m]$ for some $m < n$ and by $A_{\geq m} \in \mathbb{R}^{dx(n-m)}$ the matrix $[a_{m+1}, \ldots, a_n]$. We say the matrix $A$ is $\rho$-bounded if $\max_{i \in [n]} \|a_i\|_2 \leq \rho$.

Given a vector $a \in \mathbb{R}^d$ or a diagonal matrix $D \in \mathbb{R}^{d \times d}$, for $r \in \mathbb{R}$, notations $a^r$ and $D^r$ are used for entry-wise power.

**Definition 2.1** ([21, 5]). Let $A \in \mathbb{R}^{m \times n}$. The Kruskal rank of $A$, denoted $\text{K-rank}(A)$, is the maximum $k \in [n]$ such that any $k$ columns of $A$ are linearly independent. Let $\tau > 0$. The robust Kruskal rank (with threshold $\tau$) of $A$, denoted $\text{K-rank}_\tau(A)$, is the maximum $k \in [n]$ such that for any subset $S \subseteq [n]$ of size $k$ we have $\sigma_k(A_{S}) \geq 1/\tau$.

**Tensors.** For a symmetric order-3 tensor $T \in \mathbb{R}^{d \times d \times d}$ and a vector $x \in \mathbb{R}^d$, let $T_x$ denote the matrix $\sum_{i,j,k=1}^d T_{ijk} x_i e_j e_k \in \mathbb{R}^{d \times d}$. Let $a^{\otimes 3}$ be a shorthand for $a \otimes a \otimes a$. For a rank $n$ symmetric order-3 tensor $T = \sum_{i=1}^n a_i^{\otimes 3}$, we say the tensor $T$ is $\rho$-bounded if $\max_{i \in [n]} \|a_i\|_2 \leq \rho$.

**Cumulants.** The cumulants of a random vector $X$ are a sequence of tensors $K_1(X), K_2(X), \ldots$ related to the moment tensors of $X$. We only state the properties we need, see [27] for an introduction. We have for example: $K_1(X) = \mathbb{E}[X], K_2(X) = \text{cov}(X), K_3(X) = \mathbb{E}[(X - \mathbb{E}[X])^{\otimes 3}]$. Note that the first three cumulants coincide with the expectation, the covariance matrix and the third central moment of $X$. Cumulants have the property that for two independent random variables $X, Y$ we have $K_m(X + Y) = K_m(X) + K_m(Y)$. The first two cumulants of a standard Gaussian random variable are the mean and the covariance matrix, all subsequent cumulants are zero.

**Jennrich’s algorithm** [16]. The basic idea of Jennrich’s algorithm to decompose a symmetric order-3 tensor with linearly independent components $a_1, \ldots, a_d \in \mathbb{R}^d$ is the following: for random unit vectors $x, y \in \mathbb{R}^d$, compute the (right) eigenvectors of $T_x T_y^{-1}$. With probability 1, the set of eigenvectors is equal to the set of directions of $a_i$ (the eigenvectors recover the $a_i$s up to sign and norm). We use a version that allows for the number of $a_i$s to be less than $d$ and that includes an error analysis [15, 14].

**Error propagation.** We will frequently encounter the situation where we have “If $X \leq \epsilon$, then $Y$ is less than some polynomial in terms of $\epsilon, d$ and other related parameters”. To track the error for underlying $Y$ within each step of our algorithm, we will use subscripted symbols to indicate which step or theorem the underlying symbol is referring to. For example, $\epsilon_{A.4}$ refers to the error term used in Theorem A.4.
3 Overcomplete order-3 tensor decomposition

Throughout this section, we consider the problem of decomposing a symmetric order-3 tensor $T \in \mathbb{R}^{d \times d \times d}$ of rank $n$:

$$T = \sum_{i \in [n]} a_i^{\otimes 3}. \quad (3.1)$$

Given a symmetric order-3 tensor $T$, Jennrich’s algorithm is an efficient approach to recover its rank-1 components ($a_i$s). But it has no guarantees if the tensor rank $n$ is greater than $d$, or when its components are linearly dependent. Our intuition to deal with such cases is: it is still possible that a subset of $a_i$s is linearly independent, hence a tensor with such a subset of the components of $T$ can be efficiently decomposed. Our ideas work whenever a robust version of Kruskal’s uniqueness of decomposition holds and so we start with the same assumption as in that result, specialized to the symmetric tensor case (see Theorem A.1 for the version we use):

**Assumption 3.1.** Any $r$-subset among $a_i$s is linearly independent and overcompleteness parameter $k := n - r \leq (r - 2)/2$.

Given Assumption 3.1, we see that any rank-$r$ subset of components in the decomposition (3.1) contains linearly independent $a_i$s, which can be decomposed efficiently. The first goal of our algorithm is to find 2 vectors $x, y$, orthogonal to a $k$-subset of the $a_i$s. Without loss of generality the last $k$, namely

$$x, y \in S := \text{span}\{a_{r+1}, a_{r+2}, \ldots, a_{r+k}\}^\perp. \quad (3.2)$$

In this way, $T_x = T(x, \cdot, \cdot)$ and $T_y = T(y, \cdot, \cdot)$ are of rank $r$ with column spaces spanned by $a_1, \ldots, a_r$, which implies the following idealized steps to decompose $T$:

1. find $x, y \in S$;
2. apply Jennrich’s algorithm to $T_x, T_y$ to recover directions of $a_i$ for $i \in [r]$;
3. recover lengths of $a_i$s via a linear system;
4. deflate $T$ with the recovered components, then decompose the remaining tensor to recover $a_{r+i}$s.

The remaining of this section is outlined as follows: we will first illustrate the idea in Section 3.1 of our tensor decomposition algorithm assuming that we have what (at this point) we could call “magic” vectors $x, y \in S$, that is, when step 1 happens exactly. Then, in Section 3.2, we present our actual approximation algorithm (alg. 2, for the case where we do not have access to $S$), along with the main theorem on the correctness of our algorithm and error propagation (Theorem 3.3). The main idea here is to find $x, y$ close to $S$ by repeatedly trying random choices, which is efficient for small $k$ as $\dim S = d - k$. The proof of our main theorem proposed in Section 3.2 will be presented in Appendix A.

3.1 Exact case

Given two “magic” vectors $x, y \in S$, we will use the evaluations of $T$ at $x$ and $y$, namely $T_x$ and $T_y$ and their simultaneous diagonalization to recover $a_i$. The ideal tensor decomposition algorithm is alg. 1. Its correctness is shown in Theorem 3.2.
Algorithm 1: Order-3 tensor decomposition (ideal)

Input: tensor $T = \sum_{i \in [r+k]} a_i \otimes^3, x, y \in S$

1. (direction recovery) find the top $r$ eigenvectors of $T_x T_y^\dagger, \tilde{a}_i$ for $i \in [r]$;
2. (length recovery) solve the linear system: $\hat{A}_r \text{diag}(\langle x, \tilde{a}_i \rangle) \text{diag}(\xi_i) \hat{A}_r^\dagger = T_x$ for $\text{diag}(\xi_i)$;
3. deflate the tensor $T$ by known components: $R = T - \sum_{i \in [r]} \xi_i \tilde{a}_i \otimes^3$;
4. pick $x', y'$ uniformly at random in $S^{d-1}$;
5. find the top $k$ eigenvectors of $R_{x'} R_{y'}^\dagger$, $\tilde{a}_{r+i}$ for $i \in [k]$;
6. solve the linear system: $\hat{A}_{>r} \text{diag}(\langle x', \tilde{a}_{r+i} \rangle) \text{diag}(\xi_{r+i}) \hat{A}_{>r}^\dagger = R_{x'}$ for $\text{diag}(\xi_{r+i})$;

Output: $\{\xi_1^{1/3} \tilde{a}_1, \xi_2^{1/3} \tilde{a}_2, \ldots, \xi_{r+k}^{1/3} \tilde{a}_{r+k}\}$

Theorem 3.2 (Noiseless case). Let $T$ be as in Eq. (3.1) and satisfy Assumption 3.1. Let $S = \text{span}\{a_{r+1}, a_{r+2}, \ldots, a_{r+k}\}$. Let $x, y \in S$ be such that $\langle x, a_i \rangle$ is nonzero for $i \in [r]$ and $\langle y, a_i \rangle \neq 0$ for $i \neq j$. Then alg 1 with inputs $T$ and $x, y$ finds vectors $\xi_i^{1/3} \tilde{a}_i$ such that $a_\pi(i) = \xi_i^{1/3} \tilde{a}_i$ for some permutation $\pi : [r+k] \rightarrow [r+k]$.

Proof. The proof builds on top of the standard analysis of Jennrich’s algorithm but it is self-contained. Let $A_r$ be the matrix containing $a_1, \ldots, a_r$ as columns, $D_x = \text{diag}(\langle x, a_1 \rangle)$ and $D_y = \text{diag}(\langle y, a_i \rangle)$. Since $x, y \in S$, we can rewrite $T_x$ and $T_y$ as:

$$T_x = A_r D_x A_r^\dagger, \quad T_y = A_r D_y A_r^\dagger.$$ 

Furthermore, we have:

$$T_x T_y^\dagger = A_r D_x D_y^{-1} A_r^\dagger.$$ 

It follows by Assumption 3.1 that the set of eigenvectors of $T_x T_y^\dagger$ equals the set of $\tilde{a}_i$s up to sign, and $\tilde{a}_i$s are recovered up to some permutation of $[r]$ and signs $s_i \in \{1, -1\}$ for $i \in [r]$. Without loss of generality, assume the permutation is the identity. The linear system in step 2 can be rewritten as:

$$\hat{A}_r \text{diag}(\langle x, \tilde{a}_i \rangle) \text{diag}(\xi_i) \hat{A}_r^\dagger = \hat{A}_r \text{diag}(\|a_i\|_2^3 \langle x, \tilde{a}_i \rangle) \hat{A}_r^\dagger,$$

which implies the set of $\xi_i$s equals to the set of $\|a_i\|_2^3$ up to the same signs. Therefore we have $a_i = \xi_i^{1/3} \tilde{a}_i$ for $i \in [r]$. By deflation, we have the remainder

$$R = T - \sum_{i \in [r]} \xi_i \tilde{a}_i \otimes^3 = \sum_{i \in [k]} a_i \otimes^3.$$ 

Let $x', y'$ be two random vector chosen uniformly on $S^{d-1}$. With probability 1, we have that $\text{diag}(\langle x', a_{r+i} \rangle)/\text{diag}(\langle y', a_{r+i} \rangle)$ has distinct and nonzero diagonal entries. An argument on $R_{x'}, R_{y'}$ similar to the one on $T_x, T_y$ can be made to show that $a_{r+i}$ are recovered by $\xi_{r+i}^{1/3} \tilde{a}_{r+i}$ for $i \in [k]$ up to some permutation. To wrap up, we have $a_\pi(i) = \xi_i^{1/3} \tilde{a}_i$ for some permutation $\pi$ of $[r+k]$.

3.2 Approximation algorithm and main theorem

In the previous subsection we showed that given “magic” vectors orthogonal to $a_{r+1}, \ldots, a_{r+k}$, then $T$ can be still efficiently decomposed. However, we have neither access to these $k$ components nor
their span (subspace $S$). Besides, in applications, instead of the true tensor $T$, we usually have only an approximation $\tilde{T}$ of it. In this subsection, we show that repeatedly trying random choices can help us finding $x, y$ close to $S$ and an approximation algorithm is proposed.

We will first summarize our result in this section as alg. 3 and Theorem 3.3.

**Algorithm 2:** JENNRRICH [15, Algorithm “Diagonalize”]

- **Input:** $M_\mu, M_\lambda \in \mathbb{R}^{d \times d}$, number of vectors $r$
- **Output:** columns of $WP$.

**Algorithm 3:** Approximate tensor decomposition

- **Input:** Tensor $\tilde{T} \in \mathbb{R}^{d \times d \times d}$, error tolerance $\epsilon$, tensor rank $n$, overcompleteness $k$, upper bound $M$ on $\|a_i\|_2$ for $i \in [\epsilon]$
- **Output:** $\{\xi_i^{1/3}\tilde{a}_i : i \in [r + k]\}$

**Theorem 3.3** (Correctness of alg. 3). Let $1 \leq k \leq (r - 2)/2$, $\tau > 0$. Let $T = \sum_{i \in [r+k]} a_i^{\otimes 3}$, with $a_i \in \mathbb{R}^d$. Let $A = [a_1, \ldots, a_{r+k}]$ with K-rank$_r(A) \geq r$. Let $0 < \epsilon_{\text{out}} \leq \min\{1/2, \min_{i \in [r+k]} \|a_i\|_2^2/2\}$. Let $M \geq \max_{i \in [r+k]} \|a_i\|_2$. There exist polynomials $\text{poly}_{3.3}(d, r, k, \tau, M)$, $\text{poly'}_{3.3}(d, r, k, \tau, M)$ such that if $\epsilon_{\text{in}} \leq \epsilon_{\text{out}}/\text{poly}_{3.3}$ and $\tilde{T}$ is a tensor such that $\|T - \tilde{T}\|_F \leq \epsilon_{\text{in}}$, then alg. 3 on input $\tilde{T}$ and $\epsilon = \epsilon_{\text{out}}/\text{poly'}_{3.3}$, outputs unit vectors $\tilde{a}_1, \ldots, \tilde{a}_{r+k}$ and numbers $\xi_1, \ldots, \xi_{r+k}$ such that for some permutation $\pi$ of $[r+k]$, we have

$$\|a_{\pi(i)} - \xi_i^{1/3}\tilde{a}_i\|_2 \leq \epsilon_{\text{out}}, \quad \forall i \in [r+k].$$

Besides, the expected running time is at most $\text{poly}(d^k, r^k, k^k, 1/\epsilon_{\text{out}}, r^k, M^k)$.

**Proof idea.** The proof has three parts. In the first part we establish that if $T'$, with which the algorithm finishes, is close to $\tilde{T}$ and has bounded components, then the components of $T'$,
\( \{ \xi_i^{1/3} \bar{a}_i : i \in [r+k] \} \), are close to those of \( T \). In the second part we show that, when assuming the “magic” vectors are found, the algorithm indeed finishes with a tensor \( T' \) that is close to \( \bar{T} \) (and therefore, close to \( T \) via triangle inequality), and how the error will propagate. In the third part we will investigate the probability that random sampling on the sphere will give the vectors satisfying our requirements.

The first part follows from [5, Theorem 2.6] (the version we need stated as Theorem A.1 here).

To see the second part, we will first assume that we have found “magic” vectors \( x \) and \( y \) that are close to \( S = \text{span}\{a_{r+1}, a_{r+2}, \ldots, a_{r+k}\} \). If they are close enough, Theorem A.2 guarantees that we can simultaneously diagonalize matrices \( \bar{T}_x \) and \( \bar{T}_y \) using Jennrich’s algorithm (alg. 2), and the outputs are close to the directions of \( a_i \)'s. Next, we show in Theorem A.3 that we can recover approximately the length of \( a_i \)'s via a linear system once we have the directions. At this point we completed the recovery of \( r \) components. Thereafter, we show in Theorem A.4 that when the deflation error is small, the residual tensor \( R \) can be decomposed in the same way and the last \( k \) directions are recovered. At the end of the second part, we similarly show in Theorem A.5 that the lengths of the last \( k \) components are approximately recovered.

Finally, we show that repeatedly choosing vectors at random uniformly on the sphere will return the desired “magic” vectors with positive probability leading to the claimed running time. □

The proof of Theorem 3.3 is in Appendix A.

### 4 Blind deconvolution of discrete distribution

In this section we provide an application of alg. 3: perform blind deconvolution of an additive mixture model of the form

\[
X = Z + \eta
\]

in \( \mathbb{R}^d \), where \( Z \) follows a discrete distribution that takes value \( \mu_i \) with probability \( w_i \) for \( i \in [d] \), and \( \eta \) is an unknown random variable independent of \( Z \) with zero mean, zero 3rd moment and finite 6th moment.

Our objective is to recover the parameters of \( Z \) when given finite samples coming from \( X \). By estimating the overall mean and translating the samples we can without loss of generality assume that \( \bar{Z} = \sum_{i \in [d]} w_i \mu_i = 0 \) for the rest of this section.

First we see that the parameters of the hidden variable, \( Z \), are identifiable from the 3rd cumulant of \( X \) as the first and third moments of \( \eta \) are zero. Let \( K_m(X) \) be the \( m \)-th cumulant of \( X \). By properties of cumulants (see Section 2):

\[
K_3(X) = K_3(Z) + K_3(\eta) = \sum_{i \in [d]} w_i \mu_i^{\otimes 3}.
\] (4.2)

If one can decompose the cumulant tensor \( K_3(X) \), then the function \( w_i^{1/3} \mu_i \) of the centers \( \mu_i \) and the mixing weights \( w_i \) is recovered. However the component vectors satisfy \( \sum_i w_i \mu_i = 0 \) (they are always linearly dependent) and therefore applying Jennrich’s algorithm naively has no guarantee to succeed.\(^1\) We show that, under a natural non-degeneracy condition, our overcomplete tensor decomposition algorithm (alg. 3) works successfully.

\(^1\)Note that even when the overall mean is non-zero and the means are linearly independent, \( K_3 \) still has linearly dependent components as it is the central 3rd moment. If one gives up on using \( K_3 \), then one loses (4.2).
**Assumption 4.1.** K-rank$_r([\mu_1, \ldots, \mu_d]) = d - 1.$

Under Assumption 4.1, we can decompose (4.2) with alg. 3. For simplicity in the demonstration, we reformulate the problem: let $a_i = \tilde{\mu}_i$, and $\rho_i = \|\mu_i\|_2$, our objective becomes

$$\text{decompose } T = \sum_{i \in [d]} w_i \rho_i^3 a_i^3,$$

such that $\sum_{i \in [d]} w_i = 1$, $\sum_{i \in [d]} w_i \rho_i a_i = 0.$ \hfill (4.3)

At this point, we are ready to state our algorithm (alg. 4) for learning mixture models.

**Algorithm 4:** Blind deconvolution of discrete distribution

**Input:** iid. samples $x_1, \ldots, x_N$ from mixture $X$, error tolerance $\epsilon'$, upper bound $\rho_{\text{max}}$ on $\|\mu_i\|_2$ for $i \in [d]$, lower bound $w_{\text{min}}$ on $w_i$ for $i \in [d]$, robust Kruskal rank threshold $\tau$

1. compute the sample 3rd cumulant $\tilde{T}$ using Fact B.1;
2. invoke alg. 3 with error tolerance $\epsilon_{4.2} = \epsilon'/\text{poly}_{4.2}$, tensor rank $d$ and overcompleteness 1 to decompose $\tilde{T}$, thus obtain $\tilde{\alpha}_i$ and $\xi_i$ for $i \in [d]$;
3. set $\tilde{v}$ to the right singular vector associated with the minimum singular value of $\tilde{A}D_{\xi}^{1/3}$;
4. set $\tilde{\omega} = \tilde{v}^{3/2}/(\sum_{i \in [d]} v_i^{3/2})$, $\tilde{\rho}_i = (\xi_i/\tilde{\omega})^{1/3}$ for $i \in [d]$;
5. set $\tilde{\mu}_i = \tilde{\rho}_i \tilde{\omega}$ for $i \in [d]$;

**Output:** estimated mixing weights $\tilde{\omega}_1, \ldots, \tilde{\omega}_d$, and estimated means $\tilde{\mu}_1, \ldots, \tilde{\mu}_d$

The correctness of alg. 4 is summarized below:

**Theorem 4.2** (Correctness of alg. 4). Let $X = Z + \eta$ be a random $d$-dimensional vector as in model (4.1) satisfying Assumption 4.1 and $X_i$ be its $i$-th coordinate. Let $0 < w_{\text{min}} \leq \min_{i \in [d]} w_i$, and let $\rho_{\text{max}} \geq \max_{i \in [d]} \rho_i$. Let $0 < \epsilon' \leq \min\{1/2, \min_{i \in [d]} w_i \rho_i^3/2\}, \delta \in (0, 1)$. There exists a polynomial $\text{poly}_{4.2}(d, \tau, \rho_{\text{max}}, w_{\text{min}}^{-1})$ such that if $\epsilon_{4.2} \leq \epsilon'/\text{poly}_{4.2}$, then given $N$ iid. samples of $X$, with probability $1 - \delta$ over the randomness in the samples, alg. 4 outputs the estimated centers $\tilde{\mu}_1, \ldots, \tilde{\mu}_d$ and mixing weights $\tilde{\omega}_1, \ldots, \tilde{\omega}_d$ of $Z$ such that

$$\|\mu_{\pi(i)} - \tilde{\mu}_i\|_2 \leq \epsilon', \quad |w_{\pi(i)} - \tilde{\omega}_i| \leq \epsilon', \quad \forall i \in [d].$$

for some permutation $\pi$ of $[d]$. The expected running time over the randomness of alg. 3 is at most $\text{poly}(d, \epsilon'^{-1}, \delta^{-1}, \tau, \rho_{\text{max}}, w_{\text{min}}^{-1})$ and will use $N = O(\epsilon'^{-2} \delta^{-1} d^{3\tau} \max_{i \in [d]} \mathbb{E}[X_i^6](\text{poly}_{4.2})^2)$ samples.

We have two things to clarify so that we can prove Theorem 4.2: the first is the accuracy of our sample cumulant. We will show that it can be made to $\varepsilon$ accuracy with polynomially many samples with any fixed $\varepsilon$. The second is in tensor decomposition, note that Theorem 3.3 guarantees that we can recover $\tilde{\alpha}_i$ approximately in the direction of $a_{\pi(i)}$ and $\xi_i$ close to $w_{\pi(i)}\rho_{\pi(i)}^3$ for some permutation $\pi$. However we are not finished yet as our goal is to recover both the centers and the mixing weights. Therefore we need to decouple $w_i$ and $\rho_i$ from $w_i\rho_i^3$, which corresponds to step 3 and 4 in alg. 4.
The analysis on sample cumulant is provided in Appendix B. We will only give the main result here:

**Lemma 4.3** (Sample cumulant). Let $T, \tilde{T}$ be the population and sample third cumulant of $X$, respectively. Let $X_i$ be the $i$-th coordinate of $X$. Given any $\varepsilon, \delta \in (0, 1)$, with probability $1 - \delta$ we have $\|T - \tilde{T}\|_F \leq \varepsilon$, given $N \geq O(d^3 \varepsilon^{-2} \delta^{-1} \max_{i \in [d]} \mathbb{E}[X_i^6])$.

**Proof.** Applying Lemma B.3 with accuracy $\varepsilon/d^3$, fail probability $\delta/d^3$ and taking the union bound over $d^3$ entries, one can see that $N \geq O(d^3 \varepsilon^{-2} \delta^{-1} \max_{i \in [d]} \mathbb{E}[X_i^6])$ many samples are enough to estimate $\|\tilde{T} - T\|_F$ up to accuracy $\varepsilon$ with probability $1 - \delta$.

**Decoupling.** In this part we will decouple the mixing weights $w_i$s and the norm $\rho_i$s after we decompose the tensor $\tilde{T}$. The true parameters satisfies $\sum_{i \in [d]} w_i \rho_i a_i = 0$ as $\mathbb{E}[X] = 0$, which can be reformulate as a linear system:

$$AD_{w_i \rho_i}^{1/3} w^{2/3} = 0$$

(4.4)

where $D_{w_i \rho_i} = \text{diag}(w_i \rho_i)$ and $A$ contains $a_i$ as columns. To decouple these parameters in the noiseless setting, one only need to solve this system under the constraint that $w$ is a probability vector. As $\text{rank}(A) = d - 1$, $w$ will be uniquely determined. Or in other words, $w^{2/3}$ lies in the direction of the right singular vector associated with the only zero singular value. It is natural to pursue the right singular vector associated to the minimum singular value of $\tilde{A}D_{\xi_i}^{1/3}$, where $\tilde{A} = [\tilde{a}_1, \ldots, \tilde{a}_d]$ and $D_{\xi_i} = \text{diag}(\xi_i)$. The following theorem guarantees this will work:

**Theorem 4.4** (Decoupling). Let $0 < w_{\min} \leq \min_{i \in [d]} w_i$, and let $\rho_{\max} \geq \max_{i \in [d]} \|\mu_i\|_2$. Suppose the outputs of step 2 in alg. 4: $\xi_1, \ldots, \xi_d$ and $\tilde{A} = [\tilde{a}_1, \ldots, \tilde{a}_d]$ satisfy Theorem 3.3 with

$$\epsilon_{\text{out}} < \frac{w_{\min}}{6 \sqrt{2}(1 + \sqrt{d})} \frac{1}{\tau} < \frac{1}{2\tau}$$

Let $v, \tilde{v}$ be the right singular vector associated with the minimum singular value of $AD_{w_i \rho_i}^{1/3}, \tilde{A}D_{\xi_i}^{1/3}$, respectively. Define

$$\tilde{w} = \frac{\tilde{v}^{3/2}}{(\sum_{i \in [d]} \tilde{v}_i^{3/2})}, \quad \tilde{\rho}_i = (\xi_i/\tilde{v}_i)^{1/3}.$$  

Then for the permutation $\pi$ such that $\|a_{\pi(i)} - \xi_i^{1/3} \tilde{a}_i\| \leq \epsilon_{\text{out}}$ for $i \in [d]$, we have

$$|w_{\pi(i)} - \tilde{w}_i| \leq 3 \sqrt{2(1 + \sqrt{d})} \tau \epsilon_{\text{out}}, \quad |\rho_{\pi(i)} - \tilde{\rho}_i| \leq [(2w_{\min}^{-1})^{1/3} + 2 \sqrt{2} \rho_{\max} w_{\min}^{-1} (1 + \sqrt{d}) \tau] \epsilon_{\text{out}}.$$

**Proof.** Since the least singular value of $AD_{w_i \rho_i}^{1/3}$ is zero, $v$ will be the solution to (4.4). Without loss of generality, we can assume $v$ has positive entries. We begin with bounding the singular value and vector of $\tilde{A}D_{\xi_i}^{1/3}$. By Theorem C.1, we have for the perturbed singular value:

$$\sigma_d(\tilde{A}D_{\xi_i}^{1/3}) = \sigma_d(\tilde{A}D_{\xi_i}^{1/3}) - \sigma_d(AD_{w_i \rho_i}^{1/3}) \leq \|AD_{w_i \rho_i}^{1/3} - \tilde{A}D_{\xi_i}^{1/3}\|_2 \leq \epsilon_{\text{out}} < \frac{1}{2\tau}.$$  

(4.5)

Note that Theorem C.1 applies to other singular values as well and implies they are bounded away from $1/(2\tau)$. By Theorem C.2, we have for the singular vectors:

$$\|v - \tilde{v}\|_2 \leq 2 \sqrt{2} \tau \epsilon_{\text{out}},$$  

(4.6)
where the perturbation in $v$ follows from setting $\Sigma_1 = \text{diag}(\sigma_1(AD_1^{1/3}), \ldots, \sigma_{d-1}(AD_1^{1/3}))$, $\Sigma_2 = 0$ and $\delta = 1/(2\tau)$ in Theorem C.2. Note that though what Theorem C.2 gives is the perturbation bound for the subspace spanned by rest $d - 1$ right singular vectors, the same bound applies to $\text{span}\{v\}$ since these two subspace are complementary.

It follows immediately that
\[ \tilde{v}_i \geq v_i - 2\sqrt{2}\tau\epsilon_{out} \geq w_{\text{min}}^{2/3} - 2\sqrt{2}\tau\epsilon_{out} > w_{\text{min}}^{2/3} - \frac{w_{\text{min}}}{1 + \sqrt{d}} > 0. \]

Now without loss of generality, take $\pi$ as the identity. The distance in mixing weights is bounded by:
\[ \|\tilde{w} - w\|_2 = \frac{\|\tilde{v}^{3/2}_i - v^{3/2}_i\|_2}{\sum_{i \in [d]} v^{3/2}_i} \leq \frac{\|\tilde{v}^{3/2}_i - v^{3/2}_i\|_2}{\sum_{i \in [d]} v^{3/2}_i} + \frac{\|\tilde{v}^{3/2}_i\|_2}{\sum_{i \in [d]} v^{3/2}_i} \left| \sum_{i} (v^{3/2}_i - \tilde{v}^{3/2}_i) \right|. \]  
(4.7)

We bound each term in (4.7) below:
\[ \|\tilde{v}^{3/2}_i\|_2 = \left( \sum_{i \in [d]} \tilde{v}^{3/2}_i \right)^{1/2} \leq \|\tilde{v}\|_2 = 1, \quad \sum_{i \in [d]} \tilde{v}^{3/2}_i \geq \|v\|_2 = 1, \quad \sum_{i \in [d]} \tilde{v}^{3/2}_i \geq \|\tilde{v}\|_2 = 1 \]

\[ \|\tilde{v}^{3/2}_i - v^{3/2}_i\|_2 = \left( \sum_{i \in [d]} (\tilde{v}^{3/2}_i - v^{3/2}_i)^2 \right)^{1/2} \leq \frac{3}{2} \left( \sum_{i \in [d]} (\tilde{v}_i - v_i)^2 \right)^{1/2} = \frac{3}{2} \|\tilde{v} - v\|_2, \]  
(4.8)

\[ \left| \sum_{i} (v^{3/2}_i - \tilde{v}^{3/2}_i) \right| \leq \|v^{3/2} - \tilde{v}^{3/2}\|_1 \leq \frac{3}{2} \|\tilde{v} - v\|_1 \leq \frac{3\sqrt{d}}{2} \|\tilde{v} - v\|_2, \]

where the last two lines use $|x^{3/2} - y^{3/2}| \leq \frac{3}{2} |x - y|$ for $x, y \in (0, 1)$. Combining (4.6) to (4.8):
\[ \|\tilde{w} - w\|_2 \leq \frac{3}{2} (1 + \sqrt{d}) \|\tilde{v} - v\|_2 \leq 3\sqrt{2}(1 + \sqrt{d})\tau\epsilon_{out}. \]  
(4.9)

By (4.9) and the assumption of the theorem, for $i \in [d]$, we have:
\[ \tilde{w}_i \geq w_i - 3\sqrt{2}(1 + \sqrt{d})\tau\epsilon_{out} \geq \frac{w_{\text{min}}}{2}. \]  
(4.10)

And the error in the magnitude is bounded by:
\[ |\tilde{\rho}_i - \rho_i| = \left| (\xi_i/\tilde{w}_i)^{1/3} - \rho_i \right| \leq \tilde{w}_i^{-1/3} \left( |\xi_i^{1/3} - w_i^{1/3}| \rho_i + \rho_i \left| w_i^{1/3} - \tilde{w}_i^{1/3} \right| \right) \]
\[ \leq \tilde{w}_i^{-1/3} (\epsilon_{out} + \rho_{\text{max}} \left| w_i^{1/3} - \tilde{w}_i^{1/3} \right|) \]
\[ \leq (2w_{\text{min}}^{-1})^{1/3} (\epsilon_{out} + \frac{1}{3} \rho_{\text{max}} (2w_{\text{min}}^{-1})^{1/3} |w_i - \tilde{w}_i|) \]
\[ \leq [(2w_{\text{min}}^{-1})^{1/3} + 2\sqrt{2}\rho_{\text{max}} w_{\text{min}}^{-1} (1 + \sqrt{d})\tau] \epsilon_{out}, \]

where the second inequality comes from Theorem 3.3, the third inequality comes from the Lipschitz property of $f(x) = x^{1/3}$ on some interval bounded away from zero, and the last follows from (4.9). \(\square\)
At the end of this section, we provide the proof of Theorem 4.2.

**Proof of Theorem 4.2.** Let

\[
\text{poly}_{4.2}(d, \tau, \rho_{\text{max}}, w_{\text{min}}^{-1}) = \text{poly}_{3.3}'(d, d - 1, 1, \tau, \rho_{\text{max}})[(2w_{\text{min}}^{-1})^{1/3} + 6\sqrt{2}\rho_{\text{max}}w_{\text{min}}^{-1}(1 + \sqrt{d})\tau + \rho_{\text{max}}]
\]

By Theorem 3.3, our choice of \(\epsilon_{4.2}\) guarantees that the output error of step 2 in alg. 4 is

\[
\epsilon_{\text{out}} = \frac{\epsilon'}{(2w_{\text{min}}^{-1})^{1/3} + 6\sqrt{2}\rho_{\text{max}}w_{\text{min}}^{-1}(1 + \sqrt{d})\tau + \rho_{\text{max}} < \frac{\epsilon'w_{\text{min}}}{6\sqrt{2}\rho_{\text{max}}(1 + \sqrt{d})\tau < \frac{w_{\text{min}}}{6\sqrt{2}\rho_{\text{max}}(1 + \sqrt{d})\tau}}.
\]

Every component in \(T\) is then estimated with \(\epsilon_{\text{out}}\) accuracy. Now Theorem 4.4 guarantees that \(\|\mu_i\|_2\) and \(w_i\) are not deviating too much. Precisely, assuming the permutation is the identity, for \(i \in [d]\) we have:

\[
\|\mu_i - \bar{\mu}_i\|_2 \leq \|\rho_i - \bar{\rho}_i\|_2 + \rho_i\|a_i - \bar{a}_i\|_2 \\
\leq [(2w_{\text{min}}^{-1})^{1/3} + 2\sqrt{2}\rho_{\text{max}}w_{\text{min}}^{-1}(1 + \sqrt{d})\tau]\epsilon_{\text{out}} + \rho_{\text{max}}\epsilon_{A.4} \quad (4.13)
\]

and

\[
|w_i - \tilde{w}_i| \leq 3\sqrt{2}(1 + \sqrt{d})\tau\epsilon_{\text{out}} \leq \epsilon'. \quad (4.14)
\]

Note that \(\epsilon_{\text{out}} = O(\epsilon'd^{-1/2})\), which by (A.36), requires the \(\epsilon_{\text{in}}\) for alg. 3 to be \(O(\epsilon'/(d^{14}\text{poly}_{4.2}))\) to apply Theorems 3.3 and 4.4. By Lemma 4.3, \(N = O(\epsilon'^{-2}\delta^{-1}d^{37}\max_{i \in [d]} \mathbb{E}[X_i^6](\text{poly}_{4.2})^2)\) many samples are needed. The running time has an extra dependence on \(\delta^{-1}\) since \(N\) is polynomial in \(\delta^{-1}\).

\[\square\]

## 5 Parameter estimation of Gaussian mixture models

In this section we consider a specific family of mixture models, namely GMM with identical but unknown covariance matrices:

\[
X = \sum_{i \in [d]} w_i\mathcal{N}(\mu_i, \Sigma) = Z + \mathcal{N}(0, \Sigma).
\]

Our objective is to recover all parameters \(\{w_i, \mu_i : i \in [d], \Sigma\}\) of the mixture. Again, suppose Assumption 4.1 holds and the mixture is mean zero by translating the samples as in Section 4. Algorithm 4 guarantees that we can recover the mixing weights \(w_i\)'s and centers \(\mu_i\)'s of \(Z\). To recover the covariance matrix of the Gaussian, notice that since the mixture is mean zero,

\[
\text{cov}(X) = \mathbb{E}[XX^\top] = \sum_{i \in [d]} w_i\mu_i\mu_i^\top + \Sigma.
\]

The covariance matrix can be recovered by taking the difference of the sample second moment of \(X\), and the estimated second moment of \(Z\). We propose alg. 5 for parameter estimation for Gaussian mixtures. The correctness of alg. 5 is summarized in the theorem below:
Algorithm 5: Parameter estimation for Gaussian mixtures

**Input:** iid. samples $x_1, \ldots, x_N$ from the mixture $X$, error tolerance $\epsilon''$, upper bound $\rho_{\text{max}}$ on $||\mu_i||_2$ for $i \in [d]$, lower bound $w_{\text{min}}$ on $w_i$ for $i \in [d]$, robust Kruskal rank threshold $\tau$

1. invoke alg. 4 with samples from $X$ and parameters $\epsilon' = \epsilon'' / \text{poly}_{5.1}, \rho_{\text{max}}, w_{\text{min}}, \tau$ to get $\tilde{w}_i$ and $\tilde{\mu}_i$ for $i \in [d]$;
2. set $\tilde{\Sigma} = \frac{1}{N} \sum_{j \in [N]} x_j x_j^\top - \sum_{i \in [d]} \tilde{w}_i \tilde{\mu}_i \tilde{\mu}_i^\top$;

**Output:** estimated mixing weights and means $\tilde{w}_i, \tilde{\mu}_i : i \in [d]$, estimated covariance matrix $\tilde{\Sigma}$

**Theorem 5.1** (Correctness of alg. 5). Let $X$ be a Gaussian mixture model as in (5.1) satisfying Assumption 4.1. Let $0 < w_{\text{min}} \leq \min_{i \in [d]} w_i$, and let $\rho_{\text{max}} \geq \max_{i \in [d]} \rho_i$. Let $0 < \epsilon'' \leq \min \{1/2, \min_{i \in [d]} w_i \rho_i^3/2\}, \delta \in (0,1)$. There exist a polynomial $\text{poly}_{5.1}(d, \rho_{\text{max}})$ such that if $\epsilon' \leq \epsilon'' / \text{poly}_{5.1}$, given $N$ iid. samples of $X$, with probability $1 - \delta$ over the randomness in the samples, alg. 5 outputs the estimated parameters of $X$: $\tilde{\mu}_1, \ldots, \tilde{\mu}_d, \tilde{w}_1, \ldots, \tilde{w}_d$ and $\tilde{\Sigma}$ such that

$$
||\tilde{\Sigma} - \Sigma||_F \leq \epsilon'', \quad |w_{\pi(i)} - \tilde{w}_i| \leq \epsilon', \quad ||\mu_{\pi(i)} - \tilde{\mu}_i||_2 \leq \epsilon', \quad \forall i \in [d]
$$

for some permutation $\pi$ of $[d]$. The expected running time over the randomness in alg. 3 is at most $\text{poly}(d, \epsilon''^{-1}, \delta^{-1}, w_{\text{min}}^{-1}, \tau, \rho_{\text{max}})$ and will use $N = O(\epsilon''^3 \delta^{-1} d^{\Theta} \max_{i \in [d]} \Sigma_{ii}^3 \text{poly}_{4.2}^2)$ samples.

**Proof.** Let $\text{poly}_{5.1}(d, \rho_{\text{max}}) = 1 + d \rho_{\text{max}}^2 + 2d(2\rho_{\text{max}} + 1)$. (5.2)

By Theorem 4.2, the mixing weights $w_i$ and means $\mu_i$ will be estimated to $\epsilon'$ accuracy. The sample complexity and running time follows from therein, where the differences are that we have $\epsilon'' = O(\epsilon d^{-1})$ and that we can write $\max_{i \in [d]} \mathbb{E}[X_i^6] = \max_{i \in [d]} 15\Sigma_{ii}^3$ explicitly in the sample complexity for our Gaussian mixture.

Note that when the sample complexity guarantees the $K_3(X)$ is estimated to $\epsilon_{in}$ accuracy with probability $1 - \delta$, it can also guarantee $\text{cov}(X)$ is estimated to $\epsilon_{in}$ accuracy with probability $1 - \delta$ since the latter takes $O(d^{3} \epsilon_{in}^{-2} \delta^{-1} \max_{i \in [d]} \Sigma_{ii}^2)$ many samples by a similar argument to Lemmas B.2 and B.3. Now we bound the error in the covariance matrix:

$$
||\tilde{\Sigma} - \Sigma||_F = \frac{1}{N} \sum_{j \in [N]} x_j x_j^\top - \sum_{i \in [n]} \tilde{w}_i \tilde{\mu}_i \tilde{\mu}_i^\top - \Sigma||_F \leq \epsilon_{in} + \sum_{i \in [d]} w_i \mu_i \mu_i^\top + \tilde{\mu}_i \tilde{\mu}_i^\top ||_F
$$

$$
\leq \epsilon_{in} + \sum_{i \in [d]} |w_i - \tilde{w}_i| ||\mu_i \mu_i^\top||_F + \sum_{i \in [d]} \tilde{w}_i ||\mu_i \mu_i^\top - \tilde{\mu}_i \tilde{\mu}_i^\top||_F
$$

$$
\leq \epsilon_{in} + d \rho_{\text{max}}^2 \epsilon' + \sum_{i \in [d]} \tilde{w}_i ||\mu_i + \tilde{\mu}_i||_2 \epsilon'
$$

$$
\leq \epsilon_{in} + d \rho_{\text{max}}^2 \epsilon' + \sum_{i \in [d]} (w_i + \epsilon') (2||\mu_i||_2 + \epsilon') \epsilon'
$$

$$
\leq (1 + d \rho_{\text{max}}^2 + 2d(2\rho_{\text{max}} + 1)) \epsilon' \leq \epsilon'',
$$

where the first inequality comes from estimation in the variance, and the last inequality follows from crudely bounding $\epsilon_{in}$ by $\epsilon'$ and $w_i, \epsilon'$ by $1$. 

\[\square\]
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A Proof of Theorem 3.3

For the rest of this section, we will take a walk-through of the three parts mentioned in the “proof idea”. The first part is stated in Appendix A.1, the second part is shown in Appendix A.2, and the third part is shown in Appendix A.3. We will wrap up everything in the proof in Appendix A.4.

A.1 Uniqueness of decomposition

We first provide a theorem which guarantees the uniqueness of tensor decomposition within an $\varepsilon$ reconstruction error.

**Theorem A.1** ([5, Theorem 2.6]). Suppose a rank $R$ tensor $T = \sum_{i \in [R]} a_i \otimes^3 a_i \in \mathbb{R}^{d \times d \times d}$ is $\rho$-bounded. Let $A = [a_1, \ldots, a_R]$ with K-rank$_r(A) \geq r$, satisfying $3r \geq 2R + 2$. Then for every $0 < \varepsilon' \leq 1/2$, there exists

$$\varepsilon = \varepsilon'/\text{poly}_{A,1}(R, \tau, \rho', d)$$

for a fixed polynomial poly$_{A,1}$ so that for any other $\rho'$-bounded decomposition $T' = \sum_{i \in [R]} (a'_i) \otimes^3 a'_i$ with $\|T' - T\|_F \leq \varepsilon$, there exists a permutation $\pi$ of $[R]$ such that

$$\|a_{\pi(i)} - a'_i\|_2 \leq \varepsilon', \quad \forall i \in [R].$$

The theorem applies immediately to our case when K-rank$_r(A) \geq r$ and $R = r + k$. Note that even when we have only a noisy tensor $\tilde{T}$, as long as the contamination is not too much, if the algorithm returns with a low reconstruction error with respect to $\tilde{T}$, the vectors we recover will be close to those of $T$, by triangle inequality.

A.2 Robust decomposition

In this subsection, we will take a walk-through of the forward error propagation of alg. 3. We will assume, throughout this subsection, that we already have 2 vectors $x, y$ that are close to $S$, that is,

$$|\langle x, \hat{a}_{r+i} \rangle|, |\langle y, \hat{a}_{r+i} \rangle| \leq \theta$$

for $i \in [k]$, where $\theta \in (0, 1)$ will be chosen later, and K-rank$_r(A) \geq r$. Let $E_{input} = T - \tilde{T}$ be the input error tensor.

Part 1: robust diagonalization. We first cite the robust analysis on alg. 2.

**Theorem A.2** ([14], Theorem 5.4, Lemma 5.1, Lemma 5.2). Let $T_\mu, T_\lambda \in \mathbb{R}^{d \times d}$ be of the form:

$$T_\mu = \sum_{i \in [r]} \mu_i a_i a_i^T = A \text{diag}(\mu) A^T, \quad T_\lambda = \sum_{i \in [r]} \lambda_i a_i a_i^T = A \text{diag}(\lambda) A^T$$

with $A = (a_1, \ldots, a_r)$, $a_i \in \mathbb{R}^d$, $\|a_i\| = 1$, $\lambda_i, \mu_i \in \mathbb{R}$ for $i \in [r]$. Suppose

1. $\sigma_r(A) > 0$,
2. $0 < k_l \leq |\mu_i|, |\lambda_i| \leq k_u$ for all $i$,
3. $\left|\frac{\mu_i}{\lambda_i} - \frac{\mu_j}{\lambda_j}\right| \geq \alpha > 0$ for all $i \neq j$. 

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Let \( \tilde{T}_\mu, \tilde{T}_\lambda \) be matrices such that
\[
\|T_\mu - \tilde{T}_\mu\|_F, \|T_\lambda - \tilde{T}_\lambda\|_F \leq \frac{\epsilon k^3 \sigma_r(A)^3 \min\{\alpha, 1\}}{2^{11} K(A) k u r^2}.
\]

Then algorithm JENNRIICH on input \( \tilde{T}_\mu, \tilde{T}_\lambda \) outputs unit vectors \( \tilde{a}_1, \ldots, \tilde{a}_r \) such that for some permutation \( \pi \) of \([r]\) and signs \( s_1, \ldots, s_r \in \{-1, 1\} \) for all \( i \in [r] \) we have
\[
\|a_{\pi(i)} - s_i \tilde{a}_i\| \leq \epsilon.
\]

The algorithm runs in time \( \text{poly}(d, 1/\alpha, 1/k, 1/\sigma_r(A), 1/\epsilon) \).

There is a notable difference from the original version of Theorem A.2: since all matrices and numbers are real here, the phase term in the original theorem becomes the sign \( s_i \).

Now we argue that in our case, Theorem A.2 can be applied to \( \tilde{T}_x, T_x \) and \( \tilde{T}_y, T_y \). \( \tilde{T}_x, \tilde{T}_y \) are given by:
\[
\tilde{T}_x = \tilde{T}(x, \cdot, \cdot) = \hat{A}_r D_x \hat{A}_r^\top + \hat{A}_{>r} D'_{>r} \hat{A}_{>r}^\top + E_{\text{input}}(x, \cdot, \cdot),
\]
\[
\tilde{T}_y = \tilde{T}(y, \cdot, \cdot) = \hat{A}_r D_y \hat{A}_r^\top + \hat{A}_{>r} D'_{>r} \hat{A}_{>r}^\top + E_{\text{input}}(y, \cdot, \cdot),
\]
(A.1)

where \( \hat{A}_r \) contains \( \tilde{a}_i \) as columns, \( D_x = \text{diag}(\|a_i\|^3 \langle x, \tilde{a}_i \rangle) \) for \( i \in [r] \) and \( \hat{A}_{>r} \) contains \( \tilde{a}_{r+i} \), \( D'_x = \text{diag}(\|a_{r+i}\|^3 \langle x, \tilde{a}_{r+i} \rangle) \) for \( i \in [k] \). Similar notations hold for \( D_y, D'_y \). The error matrix for \( \tilde{T}_x \) is given by \( E_x \):
\[
E_x = \hat{A}_{>r} D'_{>r} \hat{A}_{>r}^\top + E_{\text{input}}(x, \cdot, \cdot),
\]
and similarly for \( E_y \). Condition 1 holds since \( \text{K-rank}_r(A) \geq r \). Precisely, we have
\[
\sigma_r(\hat{A}_r) \geq \sigma_r(A)/\max_{i \in [r]}\|a_i\|_2 \geq 1/(\tau M).
\]
The diagonal terms \( \mu_i \) and \( \lambda_i \) become \( \|a_i\|^3 \langle x, \tilde{a}_i \rangle \) and \( \|a_i\|^3 \langle y, \tilde{a}_i \rangle \) for \( i \in [r] \). Since \( x, y \) are chosen at random, we will provide the probability for conditions 2 and 3 to hold, as well as the choice for \( k_u, k_1, \alpha \) in Appendix A.3. For the error matrices, we have:
\[
\|E_x\|_F, \|E_y\|_F \leq \epsilon_{in} + k M^3 \theta.
\]
(A.2)

Hence Theorem A.2 applies to our case provided that the error from the input and from cancelling the remaining \( k \) components is not too large. To avoid ambiguity with Theorem 3.3, we denote the error between \( \tilde{a}_i \) and \( \tilde{a}_i \) for \( i \in [r] \) by \( \epsilon_{A.2} \).

**Part 2: norm estimation.** Suppose after applying alg. 2 on \( \tilde{T}_x, \tilde{T}_y \), we have the estimated eigenvectors \( \tilde{a}_1, \ldots, \tilde{a}_r \) close to their counterparts up to signs and permutation within \( \epsilon \) distance, the next step is to recover the length \( \|a_i\|_2 \). This can be done by finding a diagonal matrix \( D_\xi = \text{diag}(\xi_i) \), such that
\[
\hat{A}_r \text{diag}(\langle x, \tilde{a}_i \rangle) D_\xi \hat{A}_r^\top = \tilde{T}_x.
\]
(A.3)

To see this, one can easily verify that \( D_\xi = \text{diag}(s_i \|a_i\|^2) \) where \( s_i \in \{-1, 1\} \) is the sign factor in the noiseless setting. The following theorem guarantee that we can recover the norm:
Theorem A.3 (Norm estimation). Let \( \vec{b}_i \)s be the columns of \((\vec{A}_r^†)^\top\). In the setting of Theorem A.2, suppose \( \epsilon_{A,2} \leq \min\{k_l/2, (2\sqrt{r}T)M^{-1}\} \). The solution to (A.3) is given by

\[
\xi_i = \frac{\bar{T}(x, \vec{b}_i, \vec{b}_i)}{\langle x, \vec{a}_i \rangle}, \quad \text{for } i \in [r].
\]

Define:

\[
\epsilon_{A,3} = \frac{2}{k_l} \left[ 3M^3\epsilon_{A,2} + (r - 1)M^3\epsilon_{A,2}^2 + 4M^2\tau^2(\epsilon_{in} + kM^2\theta) \right].
\]

Then for the permutation \( \pi \) and any \( i \in [r] \) we have

\[
\|a_{\pi(i)}\|_2^3 - |\xi_i| \leq \epsilon_{A,3}.
\]

Proof. By choosing the direction of \( \vec{a}_i \), we can without loss of generality assume \( \xi_i \) is positive. And for the simplicity we assume the permutation is the identity.

We start with showing \( \sigma_r(\vec{A}_r) > 0 \), which implies there is a unique solution to (A.3). By Theorem A.2, the column-wise error of \( \vec{A}_r \) and \( \vec{A}_r \) is at most \( \epsilon_{A,2} \), therefore we have:

\[
|\sigma_r(\vec{A}_r) - \sigma_r(\vec{A}_r)| \leq \|\vec{A}_r - \vec{A}_r\|_2 \leq \sqrt{r}\epsilon_{A,2},
\]

and

\[
\sigma_r(\vec{A}_r) \geq \sigma_r(\vec{A}_r) - \sqrt{r}\epsilon_{A,2} \geq \frac{1}{\tau M} - \sqrt{r}\epsilon_{A,2} > 0,
\]

given \( \epsilon_{A,2} \leq (2\sqrt{r}T)M^{-1} \).

Next, we show that \( \bar{T}(x, \vec{b}_i, \vec{b}_i) / \langle x, \vec{a}_i \rangle \) is the solution to (A.3). The linear system can be rewritten with vectorizing as:

\[
\vec{A}^\odot \mathbf{vec}(\langle x, \vec{a}_i \rangle)\xi_i = \mathbf{vec}(\vec{T}_x),
\]

where \( \vec{A}^\odot = [\mathbf{vec}(\vec{a}_1\vec{a}_1^\top), \ldots, \mathbf{vec}(\vec{a}_r\vec{a}_r^\top)] \). The solution to this linear system is given by:

\[
\mathbf{vec}(\langle x, \vec{a}_i \rangle)\xi_i = \vec{B}^\odot \mathbf{vec}(\vec{T}_x),
\]

where \( \vec{B}^\odot = [\mathbf{vec}(\vec{b}_1\vec{b}_1^\top), \ldots, \mathbf{vec}(\vec{b}_r\vec{b}_r^\top)]^\top \). Putting everything back in the matrix form gives:

\[
\mathbf{diag}(\langle x, \vec{a}_i \rangle)\xi_i = \mathbf{diag}(\vec{b}_i\vec{T}_x\vec{b}_i),
\]

which implies \( \xi_i = \bar{T}(x, \vec{b}_i, \vec{b}_i) / \langle x, \vec{a}_i \rangle \).

We see that \( \vec{b}_i \) will be orthogonal to \( \vec{a}_j \) for \( j \neq i \), and \( \langle \vec{b}_i, \vec{a}_i \rangle = 1 \). By expanding \( \xi_i \) with (A.1), the error in norm estimation is bounded by:

\[
\|a_i\|_2^3 - \xi_i \leq \left( \|a_i\|_2^3 - \frac{\bar{T}(x, \vec{b}_i, \vec{b}_i)}{\langle x, \vec{a}_i \rangle} \right) = \left( \|a_i\|_2^3 - \frac{1}{\langle x, \vec{a}_i \rangle} \left( \sum_{j \in [r]} \langle x, a_j \rangle \langle \vec{b}_i, a_j \rangle^2 + \vec{b}_i^\top E_x \vec{b}_j \right) \right)
\]

\[
\leq \left( \|a_i\|_2^3 - \frac{\langle x, a_i \rangle \langle \vec{b}_i, a_i \rangle^2}{\langle x, \vec{a}_i \rangle} \right) - \frac{1}{\langle x, \vec{a}_i \rangle} \left( \sum_{j \in [r], j \neq i} \langle x, a_j \rangle \langle \vec{b}_i, a_j \rangle^2 + \vec{b}_j^\top E_x \vec{b}_j \right)
\]

\[
\leq \left( \frac{\langle x, \vec{a}_i \rangle \langle \vec{b}_i, \vec{a}_i \rangle^2}{\langle x, \vec{a}_i \rangle} - 1 \right) \|a_i\|_2^3 + \sum_{j \neq i, j \in [r]} \|a_j\|_2^3 \left( \frac{\langle x, a_j \rangle \langle \vec{b}_i, a_j \rangle^2}{\langle x, \vec{a}_i \rangle} \right) + \left| \frac{\vec{b}_i^\top E_x \vec{b}_i}{\langle x, \vec{a}_i \rangle} \right|.
\]
Now we analyze the bound of inner products therein. By standard arguments using triangular and Cauchy-Schwarz inequalities, we have:

\[
\|E_x\|_F \leq \epsilon_{in} + kM^3\theta.
\]

\[
|x, \tilde{a}_i| \geq |x, \tilde{a}_i| - \epsilon_{A.2} \geq k_l - \epsilon_{A.2}, \quad \text{for } i \in [r]
\]

\[
|x, \tilde{a}_j \rangle - \langle x, \tilde{a}_j \rangle \leq \epsilon_{A.2}, \quad \left| \langle \tilde{b}_i, \tilde{a}_j \rangle - \langle \tilde{b}_i, \tilde{a}_j \rangle \right| \leq \epsilon_{A.2}, \quad \text{for } j \in [r]
\]

where the first inequality comes from (A.2), the second inequality is from the assumptions of Theorem A.2, and last two inequalities follow from the result of Theorem A.2.

Combining (A.4) to (A.6) gives:

\[
\|a_i\|^3 - \xi_i \leq (k_l - \epsilon_{A.2})^{-1} M^3 \left( \|x, \tilde{a}_i\rangle \langle \tilde{b}_i, \tilde{a}_i\rangle^2 - \langle x, \tilde{a}_i\rangle \right) \sum_{j \neq i, j \in [r]} \|x, \tilde{a}_j\rangle \langle \tilde{b}_i, \tilde{a}_j\rangle^2 \right) \\
+ (k_l - \epsilon_{A.2})^{-1} \|E_x\|_F \leq (k_l - \epsilon_{A.2})^{-1} M^3 \left[ 3\epsilon_{A.2} + (r - 1)\epsilon_{A.2}^2 \right] + (k_l - \epsilon_{A.2})^{-1} \|E_x\|_F \leq (k_l - \epsilon_{A.2})^{-1} M^3 \left[ 3\epsilon_{A.2} + (r - 1)\epsilon_{A.2}^2 \right] + (k_l - \epsilon_{A.2})^{-1} \|E_x\|_F \leq (k_l - \epsilon_{A.2})^{-1} M^3 \left[ 3\epsilon_{A.2} + (r - 1)\epsilon_{A.2}^2 \right] + (k_l - \epsilon_{A.2})^{-1} \|E_x\|_F
\]

Applying the upper bound on \( \epsilon_{A.2} \) gives the desired result.

**Part 3: deflation.** After we deflated \( T \) with \( r \) components recovered, the error tensor in the deflation is given by

\[
E' = E_{\text{input}} + \sum_{i \in [r]} (a_i^{\otimes 3} - \xi_i \tilde{a}_i^{\otimes 3}).
\]

Immediately we have:

\[
\|E'\|_F \leq \|E_{\text{input}}\|_F + \sum_{i \in [r]} \left( \|a_i\|_2^2 \|\tilde{a}_i^{\otimes 3} - \tilde{\tilde{a}}_i^{\otimes 3}\|_F + \|\tilde{\tilde{a}}_i^{\otimes 3}\|_F \|a_i\|_2^2 - \xi_i \right)
\]

For any unit vector \( x' \), the norm of \( E'_x = E'(x', \cdot, \cdot) \) is also bounded by (A.8).

Now we show when the deflation does not cause too much error, the remaining tensor can be decomposed with the same strategy. This is done by using Theorem A.2 for the second time.

**Theorem A.4 (Deflation).** Let \( x', y' \) be two unit vectors. Let \( \bar{a}_{r+1}, \ldots, \bar{a}_{r+k} \) be the outputs of alg. 2 on inputs \( R_{x'}, R_{y'}, k \). Suppose the following conditions are true:

1. For all \( i \neq j, i, j \in [k], \frac{|\langle x', \bar{a}_{r+i} \rangle|}{|\langle y', \bar{a}_{r+i} \rangle|} \geq a' > 0 \).

2. For \( i \in [k], 0 < k_i' \leq |\langle x', \bar{a}_{r+i} \rangle|, |\langle y', \bar{a}_{r+i} \rangle| \leq k_i' \).
Define:

\[
\epsilon_{A.4} = \frac{2^{11}k^{5/2}k_u'r^4M^4}{(k'_l)^2 \min\{\alpha', 1\} \|E'\|_F}.
\]

Then there exists a permutation \( \pi' : [k] \to [k] \) and signs \( s_{d+i} \in \{\pm 1\} \) such that

\[
\|\hat{a}_{d+i}(i) - s_{d+i}\hat{a}_{d+i}\|_2 \leq \epsilon_{A.4}.
\]

**Proof.** We only need to show that Theorem A.2 can be applied here. Take

\[
T_\mu = \hat{A}_{>r} \text{diag}(\|a_{r+i}\|_2^2 \langle x', \hat{a}_{d+i} \rangle) \hat{A}_{>r}^\top, \quad T_\lambda = \hat{A}_{>r} \text{diag}(\|a_{r+i}\|_2^2 \langle y', \hat{a}_{d+i} \rangle) \hat{A}_{>r}^\top,
\]

\[
\bar{T}_\mu = R_{x'}, \quad \bar{T}_\lambda = R_{y'}.
\]

The 3 conditions in Theorem A.2 follow immediately with parameters \( k'_l, k'_u, \alpha' \). Besides, the error tensor is bounded by:

\[
\|\bar{T}_\mu - T_\mu\|_F, \|\bar{T}_\lambda - T_\lambda\|_F \leq \|E'\|_F = \frac{(k'_l)^2 \min\{\alpha', 1\}}{2^{11}k^2k'_u} \frac{1}{\sqrt{k\tau^4M^4}} \epsilon_{A.4},
\]

where the robust Kruskal rank condition guarantees that:

\[
\frac{\sigma_k(\hat{A}_{>r})^3}{\kappa(\hat{A}_{>r})} = \frac{\sigma_k(\hat{A}_{>r})^4}{\sigma_1(\hat{A}_{>r})} \geq \frac{1}{\tau^4M^4\|\hat{A}_{>r}\|_F} = \frac{1}{\sqrt{k\tau^4M^4}}.
\]

Hence Theorem A.2 holds and alg. 2 will output \( k \) vectors \( \epsilon_{A.4} \)-close to \( \hat{a}_{r+1}, \ldots, \hat{a}_{r+k} \) up to some permutation \( \pi' \) and signs within \( \epsilon_{A.4} \) distance.

With estimates for \( \hat{a}_{r+1}, \ldots, \hat{a}_{r+k} \), we can estimate the norm of \( a_{r+1}, \ldots, a_{r+k} \) in the same way we did for the first \( r \) components. Specifically, we solve the linear system for some diagonal matrix \( D'_k = \text{diag}(\xi_{r+i}) \):

\[
\hat{A}_{>r} \text{diag}(\langle x', \hat{a}_{r+i} \rangle) D'_k \hat{A}_{>r}^\top = R_{x'}
\]

(A.9)

**Theorem A.5.** Let \( \bar{b}_{r+i} \) be the columns of \( (\hat{A}_{>r})^\top \). In the settings of Theorem A.4, suppose

\[
\epsilon_{A.4} \leq \min\{k'_l/2, (2\sqrt{k}\tau M)^{-1}\}.
\]

Then the solution to (A.9) is given by:

\[
\xi_{r+i} = \frac{R(x', \bar{b}_{r+i}, \bar{b}_{r+i})}{\langle x', \hat{a}_{r+i} \rangle}, \quad \text{for } i \in [k].
\]

Define:

\[
\epsilon_{A.5} = \frac{2}{k'_l} \left[ M^3(3\epsilon_{A.4} + (k - 1)\epsilon_{A.4}^2) + 4M^2\tau^2\|E'\|_F \right].
\]

Then for the permutation \( \pi' \) in Theorem A.4,

\[
\|a_{r+i}(i)\|_3^3 - \|\xi_{r+i}\|_3 \leq \epsilon_{A.5}, \quad \text{for } i \in [k].
\]
Proof. Similar to the proof of Theorem A.3, assuming the permutation is the identity and \( \xi_{r+i} \) is positive. We start with bounding \( \sigma_k(\tilde{A}_{>r}) \) from below. By Theorem A.4, the column-wise error of \( \tilde{A}_r \) and \( \tilde{A}_r \) is at most \( \epsilon_{A.4} \), which implies the deviation in the spectral norm is at most \( \sqrt{k} \epsilon_{A.4} \). We have:

\[
\sigma_k(\tilde{A}_{>r}) \geq \sigma_k(\tilde{A}_{>r}) - \sqrt{k} \epsilon_{A.4} \geq \frac{1}{\tau M} - \sqrt{k} \epsilon_{A.4} \geq 0
\]  

(A.10)

We can write \( \|a_{r+i}\|^3 - \xi_{r+i} \) in the following 3 terms:

\[
\|a_{r+i}\|^3 - \xi_{r+i} = \left| \frac{1}{\langle x', \tilde{a}_{r+i} \rangle} \sum_{j \in [k]} \left( \|a_{r+j}\|^3 \langle x', \tilde{a}_{r+j} \rangle \langle \tilde{b}_{r+i}, \tilde{a}_{r+j} \rangle^2 \right) \right| + \frac{E'(x', \tilde{b}_{r+i}, \tilde{a}_{r+i})}{\langle x', \tilde{a}_{r+i} \rangle} - \|a_{r+i}\|^3
\]

\[
\leq \|a_{r+i}\|^3 \left| \frac{\langle x', \tilde{a}_{r+i} \rangle \langle \tilde{b}_{r+i}, \tilde{a}_{r+i} \rangle^2}{\langle x', \tilde{a}_{r+i} \rangle} - 1 \right| + \sum_{j \neq i, j \in [k]} \left( \|a_{r+j}\|^3 \left| \frac{\langle x', \tilde{a}_{r+j} \rangle \langle \tilde{b}_{r+i}, \tilde{a}_{r+j} \rangle^2}{\langle x', \tilde{a}_{r+i} \rangle} \right| + \frac{E'_2 \|\tilde{b}_{r+i}\|^2}{\|\langle x', \tilde{a}_{r+i} \rangle\|^2} \right).
\]

(A.11)

Similar to (A.6), we have the following bounds for the terms in (A.11):

\[
\|E'_2\|_2 \leq \epsilon_{in} + 3 \left( 2M^2 \epsilon_{A.2} + \epsilon_{A.3} \right)
\]

\[
\|\langle x', \tilde{a}_{r+i} \rangle\| \geq k_i' - \epsilon_{A.4} \quad \text{for } i \in [k]
\]

\[
\|\langle x', \tilde{a}_{r+j} \rangle - \langle x', \tilde{a}_{r+j} \rangle\| \leq \epsilon_{A.4}, \quad \left| \langle b_{r+i}, \tilde{a}_{r+j} \rangle - \langle \tilde{b}_{r+i}, \tilde{a}_{r+j} \rangle \right| \leq \epsilon_{A.4} \quad \text{for } i \in [k],
\]

(A.12)

where the first inequality is from (A.8), the second is from assumptions in Theorem A.4, the third and the last are from the conclusion of Theorem A.4. Now we can bound (A.11) with (A.10) and (A.12):

\[
\|a_{r+i}\|^3 - \xi_{r+i} \leq (k_i' - \epsilon_{A.4})^{-1} M^3 (3 \epsilon_{A.4} + (k - 1) \epsilon_{A.4}^2)
\]

\[
+ (k_i' - \epsilon_{A.4})^{-1} \left( \frac{1}{\tau M} - \sqrt{k} \epsilon_{A.4} \right) \geq \epsilon_{in} + r (3M^2 \epsilon_{A.2} + \epsilon_{A.3}).
\]

Applying the upper bound on \( \epsilon_{A.4} \) gives the desired result. \( \square \)

### A.3 Probability bounds

At this point, we have not covered yet how random sampling on the unit sphere will find two vectors close to \( S \), or how these vectors can meet the conditions in Theorem A.2 or Theorem A.4. In this subsection, we provide the probability bounds for these events. Our objective is to show that alg. 3 succeeds with positive probability in a single loop. Throughout this subsection, let \( x, y \) be two independent random vectors distributed uniformly on \( S^{d-1} \), and let \( \{\tilde{a}_1, \ldots, \tilde{a}_{r+k}\} \) be a set of unit vectors that has robust Kruskal rank \( r \) with threshold \( \tau M \).

We first list the events that we would like to bound to apply Theorem A.2:

1. vanishing \( k \) terms: \( \mathcal{E}_{1,y} = \{ i \in [k], \| \langle y, \tilde{a}_{r+i} \rangle \| \leq \theta \} \).
2. lower bounds on first \( r \) terms: \( \mathcal{E}_{2,y} = \{ i \in [r], \| \langle y, \tilde{a}_i \rangle \| \geq k_i \} \);
3. the eigenvalue gap: \( \mathcal{E}_3 = \{ \text{for all } i \neq j, i, j \in [r], \left| \frac{\langle x, \tilde{a}_i \rangle}{\langle y, \tilde{a}_i \rangle} - \frac{\langle x, \tilde{a}_j \rangle}{\langle y, \tilde{a}_j \rangle} \right| \geq \alpha \geq 0 \} \).


4. upper bounds on first $r$ terms: $\mathcal{E}_{4,y} = \{ \text{for } i \in [r], |\langle y, \hat{a}_i \rangle| \leq k_u \}$; 

And we have similar events $\mathcal{E}_{1,x}, \mathcal{E}_{2,x}, \mathcal{E}_{4,x}$. Note that here $k_l, k_u, \theta$ and $\alpha$ are considered as placeholder coefficients whose actual values will be chosen carefully in the proof of Theorem 3.3.

The structure of this subsection is stated as follows: we first give the probabilities for $\mathcal{E}_{4,x}, \mathcal{E}_{4,y}$ as they can be controlled with concentration inequalities with a union bound. Next, we will demonstrate our proof idea of controlling the probability of the first 3 set of events as the union bound would be too weak to work for them. After presenting our idea, we will first analyze the probability of $\mathcal{E}_{1,y} \cap \mathcal{E}_{2,y}$, then the probability of $\mathcal{E}_{1,x} \cap \mathcal{E}_{2,x} \cap \mathcal{E}_{3}$ when conditioned on the events of $y$. Finally we will collect these sub-events and give the probability that all of them will hold.

We first bound the probability of $\mathcal{E}_{4,y}$:

**Lemma A.6.** For $\hat{a}_1, \ldots, \hat{a}_r$, and for positive numbers $k_u \in (0, 1)$, we have

$$P \left[ |\langle y, \hat{a}_i \rangle| \leq k_u, \text{ for } i \in [r] \right] \geq 1 - 2re^{-cdk_u^2},$$

where $c$ is an absolute constant.

**Proof.** This is true by taking union bounds of Corollary C.4. □

To bound the probability of $\mathcal{E}_{1,y} \cap \mathcal{E}_{2,y}$, note that intuitively $\theta$ should be close to zero and much smaller than $k_l$, and in fact our “good events”, namely $\mathcal{E}_{1,y}$ happens with small probability. Hence the union bound is too weak to work here. We need to carefully bound the probability for these events. We first give the idea of our analysis below:

**Bands argument.** We analyze the events geometrically: the set $\{ |\langle y, \hat{a}_{r+i} \rangle| \leq \theta \}$ generates a banded area, while the set $\{ |\langle y, \hat{a}_j \rangle| \geq k_l \}$ generates the “outside” of a banded area. We call them bands of type I and type II, denoted by $B_{1,i}$ and $B_{2,j}$, respectively. To better illustrate this, we give a demonstration of bands as the shaded areas in the figure below:

![Figure 1: Example of bands](image)

We see that $\mathcal{E}_{1,y}, \mathcal{E}_{2,y}$ would be the intersection of these bands and the probability would become the area of intersection in the probability measure. Specifically, we have:

$$\mathcal{E}_{1,y} = \bigcap_{i=1}^{k} B_{1,i}, \quad \mathcal{E}_{2,y} = \bigcap_{j=1}^{r} B_{2,j}.$$
Before getting into $\mathcal{E}_{1,y} \cap \mathcal{E}_{2,y}$, we first see what would $B_{1,i} \cap B_{2,j}$ be: when $\hat{a}_{r,i}$ is orthogonal to $\hat{a}_j$, the intersection becomes $B_{1,i}$ with a rectangle region excluded. One can see this by choosing two canonical basis vectors. While in the general case, the excluded region is a parallelogram depending on $\langle \hat{a}_{r,i}, \hat{a}_j \rangle$. See the figure below for illustration:

![Figure 2: Intersection of bands](image)

When $\langle \hat{a}_{r,i}, \hat{a}_j \rangle = 0$, two bands will be orthogonal and two events will be independent, and in the extreme case the other way, two bands are parallel and hence the probability will be zero. But we have the fortune that the inner product will not be too close to one due to the robust Kruskal rank condition, which implies that we can, when estimating the probability, replace the parallelogram by a slightly larger rectangle area without damaging the final area too much, which is shown by the white dashed lines in Fig. 2b. This is essentially done by the orthogonal decomposition of $\hat{a}_j$ onto $S = \text{span}\{a_{r+1}, \ldots, a_{r+k}\}^\perp$ and $S^\perp$. For the rest of this section, we let $\text{proj}_{S^\perp}$ be the orthogonal projection onto $S$ and $\text{proj}_{S^\perp} = I - \text{proj}_{S}$ the orthogonal projection onto $S^\perp$.

Now we can bound the probability of $\mathcal{E}_{1,y} \cap \mathcal{E}_{2,y}$:

**Lemma A.7.** For $\hat{a}_1, \ldots, \hat{a}_{r+k}$, and for positive numbers $k_l < 1$ and $\theta < 1/\sqrt{d}$, let

$$p_1 = [1 - r\tau M \sqrt{ed}(k_l + \sqrt{r}\tau M\theta)]((c_1\theta\sqrt{d})^k - e^{-c_2d}),$$

where $c_1, c_2$ are absolute constants. Then we have

$$\mathbb{P}[\mathcal{E}_{1,y} \cap \mathcal{E}_{2,y}] = \mathbb{P}[(|\langle y, \hat{a}_{r+i} \rangle| \leq \theta, \text{ for } j \in [k]), (|\langle y, \hat{a}_i \rangle| \geq k_l, \text{ for } i \in [r]) \geq p_1.$$  

**Proof.** Using our notation of bands, we have

$$\mathbb{P}[\mathcal{E}_{1,y} \cap \mathcal{E}_{2,y}] = \mathbb{P}[\bigcap_{i \in [k]} B_{1,i} \cap \bigcup_{j \in [r]} B_{2,j}]$$

$$= \mathbb{P}\left[\bigcap_{i \in [k]} B_{1,i} \cap \bigcup_{j \in [r]} B_{2,j}\right]$$

$$\geq \mathbb{P}\left[\bigcap_{i \in [k]} B_{1,i}\right] - r\mathbb{P}\left[\bigcup_{i \in [k]} B_{1,i} \cap B_{2,i}^c\right],$$

(A.13)

where the last step comes from the union bound. Next, we give an upper bound for the second probability. Note that the second term is the intersection of $k + r$ type I bands but our idea still works as one of them comes from the complement of a type II band. To do this, we write $\hat{a}_1 = \text{proj}_S \hat{a}_1 + \text{proj}_{S^\perp} \hat{a}_1$, and suppose for a moment, that $\|\text{proj}_S \hat{a}_1\|_2$ is rather large. We have:

$$\mathbb{P}\left[\bigcap_{i \in [k]} B_{1,i} \cap B_{2,i}^c\right] = \mathbb{P}[|\langle y, \hat{a}_1 \rangle| \leq k_l |\cap_{i \in [k]} B_{1,i}], \mathbb{P}[|\cap_{i \in [k]} B_{1,i}]$$

$$\leq \mathbb{P}[|\langle y, \text{proj}_S \hat{a}_1 \rangle| \leq k_l + |\langle y, \text{proj}_{S^\perp} \hat{a}_1 \rangle|, |\cap_{i \in [k]} B_{1,i}] \mathbb{P}[|\cap_{i \in [k]} B_{1,i}].$$

(A.14)

Since $B_{1,i}$ is the event that $|\langle y, \hat{a}_{r+i} \rangle|$ is smaller than $\theta$, we can bound the inner product on the right hand side after conditioning:

$$|\langle y, \text{proj}_{S^\perp} \hat{a}_1 \rangle| = \left|y^\top \hat{A}_{r,\tau} \hat{A}_{r,\tau}^\top \hat{a}_1\right| \leq \sqrt{k}\theta\|\hat{A}_{r,\tau}^\top \hat{a}_1\|_2 \leq \sqrt{k}\theta\sigma_1(\hat{A}_{r,\tau}^\top) \leq \sqrt{k}\tau M\theta,$$

(A.15)
where the first equality comes from the definition of the projection, the second inequality follows from the conditioning, and the last comes from the robust Kruskal rank condition. Furthermore, we notice that \( \text{proj}_S \hat{a}_1 \) is orthogonal to vectors \( \hat{a}_{r+1}, \ldots, \hat{a}_{r+k} \) and the conditioning can therefore be dropped. We have:

\[
\mathbb{P} \left[ (\cap_{i \in [k]} B_{1,i}) \cap B_{2,1}^c \right] \leq \mathbb{P} \left[ \left\langle y, \text{proj}_S \hat{a}_1 \right\rangle \leq k_l + \sqrt{k \tau M} \right] \mathbb{P}[\cap_{i \in [k]} B_{1,i}]. \tag{A.16}
\]

By Lemma C.5, we have:

\[
\mathbb{P} \left[ \left\langle y, \text{proj}_S \hat{a}_1 \right\rangle \leq k_l + \sqrt{k \tau M} \right] \leq \frac{\sqrt{ed(k_l + \sqrt{k \tau M})}}{\|\text{proj}_S \hat{a}_1\|_2} \leq \tau M \sqrt{ed(k_l + \sqrt{k \tau M})}, \tag{A.17}
\]

where the last inequality follows from that the set \( \{\hat{a}_1, \hat{a}_{r+1}, \ldots, \hat{a}_{r+k}\} \) also satisfies the robust Kruskal rank condition. By Lemma C.6, we have:

\[
\mathbb{P}[\cap_{i \in [k]} B_{1,i}] \geq (c_1 \theta \sqrt{d})^k - e^{-c_2 d}. \tag{A.18}
\]

Combining (A.13), (A.14) and (A.16) to (A.18), we have

\[
\mathbb{P}[\mathcal{E}_{1,y} \cap \mathcal{E}_{2,y}] \geq [1 - \tau \tau M \sqrt{ed(k_l + \sqrt{k \tau M})}][(c_1 \theta \sqrt{d})^k - e^{-c_2 d}] \tag{A.19}
\]

At this point, we are ready to analyze the probability that there is a gap between any two ratios \( \frac{\langle x, \hat{a}_i \rangle}{\langle y, \hat{a}_i \rangle} \) and \( \frac{\langle x, \hat{a}_j \rangle}{\langle y, \hat{a}_j \rangle} \). We notice that once we condition on \( y \), the gap \( \frac{\langle x, \hat{a}_i \rangle}{\langle y, \hat{a}_i \rangle} - \frac{\langle x, \hat{a}_j \rangle}{\langle y, \hat{a}_j \rangle} \) becomes \( \langle x, C_i \hat{a}_i - C_j \hat{a}_j \rangle \) for some constants \( C_i, C_j \) and thus \( \mathcal{E}_3 \) is the intersection of \( \binom{r}{2} \) banded areas. For this reason, we consider the probability of \( \mathcal{E}_3 \cap \mathcal{E}_{1,x} \cap \mathcal{E}_{2,x} \) given \( \mathcal{E}_{1,y}, \mathcal{E}_{2,y}, \mathcal{E}_{4,y} \) hold.

**Lemma A.8.** For \( \hat{a}_1, \ldots, \hat{a}_{r+k} \), and positive numbers \( k_l, \alpha < 1, \theta < 1/\sqrt{d} \), let:

\[
p_2 = \left[ 1 - \tau \tau M \sqrt{ed(k_l + \sqrt{k \tau M})} - \left( \frac{r}{2} \right) \sqrt{\frac{ed}{2} M k_u (\alpha + \frac{2 \sqrt{k \theta M}}{k_l})} \right] (c_1 \theta \sqrt{d})^k - e^{-c_2 d},
\]

where \( c_1, c_2 \) are absolute constants. Then we have:

\[
\mathbb{P} [\mathcal{E}_3 \cap \mathcal{E}_{1,x} \cap \mathcal{E}_{2,x} \mid \mathcal{E}_{1,y}, \mathcal{E}_{2,y}, \mathcal{E}_{4,y}] \geq p_2.
\]

**Proof.** Since we condition on the bounds of \( y \), we have:

\[
\frac{\langle x, \hat{a}_i \rangle}{\langle y, \hat{a}_i \rangle} - \frac{\langle x, \hat{a}_j \rangle}{\langle y, \hat{a}_j \rangle} = \langle x, C_i \hat{a}_i - C_j \hat{a}_j \rangle, \tag{A.20}
\]

where \( |C_i|, |C_j| \) are bounded by \( k_u^{-1}, k_l^{-1} \) from below and above, respectively. We notice that the set \( \{|\langle x, C_i \hat{a}_i - C_j \hat{a}_j \rangle| \geq \alpha\} \) generates a type II band. For clarity we denote it by \( B_{3,ij} \). Also note that we can drop the conditioning and what remains is the intersection of \( k \) type I bands and \( \binom{r+1}{2} \) type II bands. The target probability becomes:

\[
\mathbb{P} \left[ (\cap_{i \in [k]} B_{1,i}) \cap (\cap_{j \in [r]} B_{2,j}) \cap (\cap_{i \neq j \in [r]} B_{3,ij}) \right] \\
= \mathbb{P} \left[ \cap_{i \in [k]} B_{1,i} \right] - \mathbb{P} \left[ (\cap_{i \in [k]} B_{1,i}) \cap \left( \cup_{j \in [r]} B_{2,j} \cup (\cup_{i \neq j \in [r]} B_{3,ij}) \right) \right] \\
\geq \mathbb{P} \left[ \cap_{i \in [k]} B_{1,i} \right] - \tau \mathbb{P} \left[ (\cap_{i \in [k]} B_{1,i}) \cap B_{2,1}^c \right] - \left( \frac{r}{2} \right) \mathbb{P} \left[ (\cap_{i \in [k]} B_{1,i}) \cap B_{3,12}^c \right]. \tag{A.21}
\]
where the last inequality is from the union bound. Note that the first two terms are the same as (A.13). Now we consider the last probability term, which is the intersection a type II band \(\{\langle x, C_1\hat{a}_1 - C_2\hat{a}_2 \rangle \geq \alpha \} \) with \(k\) type I bands. Following the same idea as Lemma A.7, we write \(v = C_1\hat{a}_1 - C_2\hat{a}_2 = \text{proj}_S v + \text{proj}_{S^c} v\). Then we have:

\[
\mathbb{P}[(\cap_{i \in [k]} B_{1,i}) \cap B_{3,12}^c] = \mathbb{P}[|\langle x, v \rangle| \leq \alpha | \cap_{i \in [k]} B_{1,i}] \mathbb{P}[\cap_{i \in [k]} B_{1,i}] \\
\leq \mathbb{P}[|\langle x, \text{proj}_S v \rangle| \leq \alpha + |\langle x, \text{proj}_{S^c} v \rangle|| \cap_{i \in [k]} B_{1,i}] \mathbb{P}[\cap_{i \in [k]} B_{1,i}] \tag{A.22}
\]

Again we have:

\[
|\langle x, \text{proj}_{S^c} v \rangle| = |x^\top \hat{A}_{\tau} \hat{A}_{\tau}^\top (C_1\hat{a}_1 - C_2\hat{a}_2)| \leq \sqrt{k\theta\|\hat{A}_{\tau}^\top\|_2\|C_1\hat{a}_1 - C_2\hat{a}_2\|_2 \leq \frac{2\sqrt{k\theta \tau M}}{k_l}}. \tag{A.23}
\]

So (A.22) can be further bounded by:

\[
\mathbb{P}[(\cap_{i \in [k]} B_{1,i}) \cap B_{3,12}^c] \leq \mathbb{P}\left[|\langle x, \text{proj}_S v \rangle| \leq \alpha + \frac{2\sqrt{k\theta \tau M}}{k_l}\right] \mathbb{P}[\cap_{i \in [k]} B_{1,i}] \tag{A.24}
\]

By Lemma C.5, the first term on the right hand side is bounded by:

\[
\mathbb{P}\left[|\langle x, \text{proj}_S v \rangle| \leq \alpha + \frac{2\sqrt{k\theta \tau M}}{k_l}\right] \leq \frac{\sqrt{ed}}{\|\text{proj}_S v\|_2}(\alpha + \frac{2\sqrt{k\theta \tau M}}{k_l}) \tag{A.25}
\]

The last inequality holds because the set \(\{\hat{a}_1, \hat{a}_2, \hat{a}_{\tau+1}, \ldots, \hat{a}_{\tau+k}\}\) satisfies the robust Kruskal rank condition, and thus \(\|\text{proj}_S v\|_2 = \|C_1\hat{a}_1 - C_2\hat{a}_2 - \text{proj}_{S^c} v\|_2 \geq \sqrt{C_1^2 + C_2^2(\tau M)^{-1}}\).

At last, collecting (A.16) to (A.18), (A.21), (A.22), (A.24) and (A.25) gives the desired probability. \(\Box\)

Finally, we are in a place to give the probability that all the events are true for \(x, y\):

**Lemma A.9.** For \(\hat{a}_1, \ldots, \hat{a}_{\tau+k}\), and for positive numbers \(k_u, k_l, \alpha < 1, k_l, \theta \leq 1/\sqrt{d}\). The probability that \(\mathcal{E}_{1,x}, \mathcal{E}_{1,y}, \mathcal{E}_{2,x}, \mathcal{E}_{2,y}, \mathcal{E}_{3}, \mathcal{E}_{4,x}, \mathcal{E}_{4,y}\) hold is at least:

\[
p_2(p_1 - 2re^{-c_1dk_1^2}) - 2re^{-c_1dk_1^2}
\]

where \(p_1, p_2\) are defined as:

\[
p_1 = [1 - r\tau M\sqrt{ed}(k_l + \sqrt{k\tau M\theta})](c_2\theta\sqrt{d})^k - e^{-c_3d})
\]

\[
p_2 = [1 - r\tau M\sqrt{ed}(k_l + \sqrt{k\tau M\theta}) - \left(\frac{r}{2}\right)\sqrt{\frac{ed}{2}}\tau M k_u(\alpha + \frac{2\sqrt{k\theta \tau M}}{k_l})](c_2\theta\sqrt{d})^k - e^{-c_3d}).
\]

Here \(c_1, c_2, c_3\) are absolute constants.
Proof. The proof is finished by combining Lemmas A.6 to A.8 and separating the intersection of all events:

\[
\begin{align*}
\Pr[\mathcal{E}_3 \cap \mathcal{E}_{1,y} \cap \mathcal{E}_{2,y} \cap \mathcal{E}_{4,y} \cap \mathcal{E}_{1,x} \cap \mathcal{E}_{2,x} \cap \mathcal{E}_{4,x}] & \geq \Pr[\mathcal{E}_3 \cap \mathcal{E}_{1,y} \cap \mathcal{E}_{2,y} \cap \mathcal{E}_{4,y} \cap \mathcal{E}_{1,x} \cap \mathcal{E}_{2,x}] - 2re^{-ck}\n
& = \Pr[\mathcal{E}_3 \cap \mathcal{E}_{1,x} \cap \mathcal{E}_{2,x} | \mathcal{E}_{1,y} \cap \mathcal{E}_{2,y} \cap \mathcal{E}_{4,y}] \Pr[\mathcal{E}_{1,y} \cap \mathcal{E}_{2,y} \cap \mathcal{E}_{4,y}] - 2re^{-ck}\n
& \geq \Pr[\mathcal{E}_3 \cap \mathcal{E}_{1,x} \cap \mathcal{E}_{2,x} | \mathcal{E}_{1,y} \cap \mathcal{E}_{2,y} \cap \mathcal{E}_{4,y}](\Pr[\mathcal{E}_{1,y} \cap \mathcal{E}_{2,y}] - 2re^{-ck}) - 2re^{-ck}\n
& \geq p_2(p_1 - 2re^{-ck}) - 2re^{-ck},
\end{align*}
\]

(A.26)

where the first and third inequalities follow from the union bound with \(\mathcal{E}_{4,x}, \mathcal{E}_{4,y}\).

At this point, we finished analyzing the randomness in the first tensor decomposition for the first \(r\) components. We give, in the following lemma, the probability that we could find some random vectors \(x', y'\) to perform the second tensor decomposition. The events will be denoted by \(\mathcal{E}_{4,x}', \mathcal{E}_{4,y}', \mathcal{E}_{4,x}'\), and \(\mathcal{E}_{4,y}'\) in analog to previous notations.

Lemma A.10. Let \(x', y'\) be two random vectors uniformly distributed on \(S^{d-1}\), for \(\hat{a}_{r+1}, \ldots, \hat{a}_{r+k}\), and for positive numbers \(k'\), \(k'\), \(\alpha' < 1\), the probability that \(\mathcal{E}_{4,x}', \mathcal{E}_{4,y}'\), \(\mathcal{E}_{4,x}', \mathcal{E}_{4,y}'\) hold is at least:

\[
\left(1 - \left(\frac{k}{2}\right)^\frac{ed}{2} \tau \sqrt{Mk'\alpha'} - \sqrt{e^{dk'}}\right)\left(1 - \sqrt{e^{dk'}} - 2re^{-ck}\right) - 2re^{-ck}.
\]

Proof. Again we separate the intersection of events and use previous results:

\[
\begin{align*}
\Pr[\mathcal{E}_{4,x} \cap \mathcal{E}_{4,y} \cap \mathcal{E}_{4,x} \cap \mathcal{E}_{4,y}] & \geq \Pr[\mathcal{E}_{4,x} \cap \mathcal{E}_{4,y} \cap \mathcal{E}_{4,x} \cap \mathcal{E}_{4,y}] - 2re^{-ck}\n
& = \Pr[\mathcal{E}_{4,x} \cap \mathcal{E}_{4,y} \cap \mathcal{E}_{4,x} \cap \mathcal{E}_{4,y}] \Pr[\mathcal{E}_{4,x} \cap \mathcal{E}_{4,y}] - 2re^{-ck}\n
& \geq (\Pr[\mathcal{E}_{4,x} \cap \mathcal{E}_{4,y}] - \Pr[\mathcal{E}_{4,x}')(\Pr[\mathcal{E}_{4,y}] - 2re^{-ck}) - 2re^{-ck}.
\end{align*}
\]

By Lemma C.5, we can lower bound \(\Pr[\mathcal{E}_{4,x}'], \Pr[\mathcal{E}_{4,y}']\) by \(1 - \sqrt{e^{dk'}}\). And for \(\Pr[\mathcal{E}_{4,x}'], \Pr[\mathcal{E}_{4,y}']\), we have:

\[
\Pr[\mathcal{E}_{4,x}'] \Pr[\mathcal{E}_{4,y}'] = \Pr\left[\min_{i \neq j, i,j \in [k]} \left| \frac{\langle x', \hat{a}_{r+i} \rangle}{\langle y', \hat{a}_{r+j} \rangle} - \frac{\langle x', \hat{a}_{r+j} \rangle}{\langle y', \hat{a}_{r+i} \rangle} \right| \leq \alpha' | \mathcal{E}_{4,x}', \mathcal{E}_{4,y}' \right]
\]

(A.28)

\[
\leq \Pr\left[\min_{i \neq j, i,j \in [k]} \left| \langle x, C_i \hat{a}_{r+i} - C_j \hat{a}_{r+j} \rangle \right| \leq \alpha' \right],
\]

where \(|C_i'|, |C_j'|\) are lower bounded by \(1/k'\). Therefore again using Lemma C.5, we have

\[
\Pr[\mathcal{E}_{4,x}'] \Pr[\mathcal{E}_{4,y}'] \leq \left(\frac{k}{2}\right)^\frac{ed}{2} \tau M k'\alpha'.
\]

(A.29)

Combining everything we have given the desired result.
A.4 Proof of Theorem 3.3

The proof idea of Theorem 3.3 can be restated as follows:

Proof idea. In Appendix A.3, we give the probability that the algorithm will succeed in a single loop. Though the specific choice of $k_u, k_u', \alpha, \theta, k_l, k_l', \alpha'$ is not given, we can sketch the scale of them: $k_u, k_u'$ can be chosen as $O(1)$ as they appear in exponentially decaying factors. $k_l, \alpha, k_l', \alpha'$ are chosen in some inverse polynomial in $r, d$, and $\theta$ will be chosen intuitively in some inverse polynomial smaller than the ones for $k_l$ and $\alpha$, to make the bounds in Lemmas A.9 and A.10 meaningful. But we would leave the choice for $\theta$ near the end of the proof as it appears at the beginning of error propagation.

The error propagation can be analyzed briefly as follows:

1. In the first tensor decomposition, we wish to find $r$ components of the input tensor, whose input error is polynomial in $\epsilon_{in}$ and $\theta$, as well as $r, d$ and the output error $\epsilon_{A.2}$ will be polynomial in them.

2. In norm estimation, the output error $\epsilon_{A.3}$ is also polynomial in $\epsilon_{in}, \theta, \epsilon_{A.2}$.

3. In deflation and the second tensor decomposition, the input deflation error will be polynomial in $\epsilon_{in}, \epsilon_{A.2}$ and $\epsilon_{A.4}$. Using Theorem A.2 again shows the output error is polynomial in the input error.

4. Finally, we will have output error $\epsilon_{A.5}$ polynomial in all previous terms.

Hence by backtracking the error propagation, we can find some polynomial $Q$ such that we can reach $\epsilon$ accuracy for reconstruction when $\epsilon_{in}, \theta \leq \epsilon/Q$. Note that this also guarantees that the success probability is at least inverse-polynomially bounded and the expected running time will be polynomial.

Finally we will argue that our algorithm can terminate with bounded components and therefore applying Theorem A.1 implies the component-wise $\epsilon_{out}$ accuracy can be attained. □

Proof. Without loss of generality, assume $\epsilon \leq \epsilon_{out}$. We trace the error propagation backwards and show how we can reach $\epsilon$ accuracy for the algorithm to terminate while maintaining positive success probability. For brevity in the discussion, in the following analysis, we will omit the constant terms involving $k, r, M$ and focus on the order of error propagation in terms of $r, d$ and other undetermined parameters.

The reconstruction error is bounded with Theorems A.2 to A.5:

\[
\|T' - \tilde{T}\|_F \leq \|T - \tilde{T}\|_F + \sum_{i \in [r+k]} \|a_i^{\otimes 3} - \xi_i \tilde{a}_i^{\otimes 3}\|_F \leq \epsilon_{in} + \sum_{i \in [r+k]} \|a_i^{\otimes 3} - \xi_i \tilde{a}_i^{\otimes 3}\|_F \\
\leq \epsilon_{in} + \sum_{i \in [r+k]} \|a_i\|_2^3 - \xi_i \|\tilde{a}_i^{\otimes 3}\|_F + \|\tilde{a}_i^{\otimes 3} - \tilde{a}_i^{\otimes 3}\|_F \|a_i\|_2^3 \leq \epsilon_{in} + 3rM^3\epsilon_{A.2} + r\epsilon_{A.3} + 3kM^3\epsilon_{A.4} + k\epsilon_{A.5} = O(\epsilon_{in} + r\epsilon_{A.2} + r\epsilon_{A.3} + \epsilon_{A.4} + \epsilon_{A.5}).
\]
Therefore we need:

\[ \epsilon_{A.t} = \frac{2^{11} r^{7/2} \kappa_{\rho}(\hat{A}_t)^3 k_t}{(k_t)^2 \kappa(\hat{A}_t) \min\{\alpha, 1\}} \| E_x \|^2_F \leq \frac{2^{11} r^{5/2} k_t \kappa^4 M^4}{(k_t)^2 \min\{\alpha, 1\}} \| E_x \|^2_F = O \left( \frac{k_t r^{5/2}}{k_t^2 \alpha} (\epsilon_{in} + \theta) \right) \]

\[ \epsilon_{A.3} = \frac{2}{k_t} \left[ 3M^3 \epsilon_{A.2} + (r - 1)M^3 \epsilon_{A.2}^2 + 4M^2 \tau^2 (\epsilon_{in} + kM^3 \theta) \right] = O \left( k_t^{-1}(\epsilon_{A.2} + r \epsilon_{A.2} + \epsilon_{in} + \theta) \right) = O \left( k_t^{-1}(r \epsilon_{A.2} + \epsilon_{in} + \theta) \right) \]

\[ \epsilon_{A.4} = \frac{2^{11} r^{5/2} k_t' k_u \kappa^4 M^4}{(k_t')^2 \min\{\alpha', 1\}} \| E' \|^2_F = O \left( \frac{k_t'}{(k_t')^2 \alpha'} \left( \epsilon_{in} + r \epsilon_{A.2} + r \epsilon_{A.3} \right) \right) \] (A.31)

\[ \epsilon_{A.5} = \frac{2}{k_t} \left[ M^3 (3 \epsilon_{A.4} + (k - 1) \epsilon_{A.4}^2) + 4M^2 \tau^2 \| E' \|^2_F \right] = O \left( \frac{(k_t')^{-1}(\epsilon_{A.4} + (k_t')^2 \alpha')}{k_t \epsilon_{A.4}} \right) = O \left( \frac{1}{k_t} + \frac{k_t' \alpha'}{k_t} \epsilon_{A.4} \right). \]

Before we move on, let us make the choice of \( k_u, k_l, \alpha, k_u', k_l', \alpha' \): to make Lemma A.9 meaningful, we need:

\[ r \tau M \sqrt{ed(k_l + \sqrt{k} \tau M \theta)} + \left( \frac{r}{2} \right) \sqrt{\frac{ed}{2} \tau M k_u (\alpha + \frac{2 \sqrt{k} \theta \tau M}{k_l})} \]

\[ = O(r \sqrt{d} k_l + r \theta + r^2 \sqrt{d} k_u \alpha + \frac{r^2 \sqrt{d} k_u \theta}{k_l}) \leq 1. \] (A.32)

Suppose we have:

\[ k_u = O(1), \quad k_l = O(r^{-1} d^{-1/2}), \quad \alpha = O(r^{-2} d^{-1/2}), \] (A.33)

then \( \theta \) would be at most \( O(r^{-3} d^{-1}) \). We will finalize the choice of \( \theta \) at the end of analysis. To make Lemma A.10 meaningful, we need:

\[ \left( \frac{k}{2} \right) \sqrt{\frac{ed}{2} \tau M k_u' \alpha' + \sqrt{ed} k l' \alpha'} = O\left( \sqrt{d} k_u' \alpha' + \sqrt{d} k l' \right) \leq 1. \] (A.34)

Therefore \( k_u' = O(1), k_l' = O(d^{-1/2}), \alpha' = O(d^{-1/2}) \) are sufficient. And now (A.31) can be further simplified as:

\[ \epsilon_{A.2} = O \left( \frac{k_u r^{5/2}}{k_t^2 \alpha} (\epsilon_{in} + \theta) \right) = O(r^{13/2} d^{3/2} (\epsilon_{in} + \theta)) \]

\[ \epsilon_{A.3} = O \left( k_t^{-1}(r \epsilon_{A.2} + \epsilon_{in} + \theta) \right) = O(r^{-1} d^{1/2} (r \epsilon_{A.2} + \epsilon_{in} + \theta)) \]

\[ = O(r^{17/2} d^2 (\epsilon_{in} + \theta)) \]

\[ \epsilon_{A.4} = O \left( \frac{k_t'}{(k_t')^2 \alpha'} (\epsilon_{in} + r \epsilon_{A.2} + r \epsilon_{A.3}) \right) = O(d^{3/2} (\epsilon_{in} + r \epsilon_{A.2} + r \epsilon_{A.3})) \]

\[ = O(r^{19/2} d^{7/2} (\epsilon_{in} + \theta)) \]

\[ \epsilon_{A.5} = O \left( \frac{1}{k_t} + \frac{k_t' \alpha'}{k_t} \epsilon_{A.4} \right) = O(r^{19/2} d^4 (\epsilon_{in} + \theta)), \] (A.35)
Therefore the $\epsilon$ reconstruction error is reached given $\epsilon_{in}$ and $\theta$ are $O(\epsilon r^{-19/2}d^{-4})$. In other words, there exists a polynomial $Q(d, r, k, \tau, M)$ such that if 

$$\epsilon_{in}, \theta \leq \frac{\epsilon}{Q(d, r, k, \tau, M)},$$

(A.37)

then with positive probability we have $\|T - \tilde{T}\|_F \leq \epsilon$. In the case when this happens, all intermediate error terms $\epsilon_{A.2}, \epsilon_{A.3}, \epsilon_{A.4}, \epsilon_{A.5}$ are small, and therefore $\epsilon_{A.2}, \epsilon_{A.4}$ meet the requirements of Theorems A.3 and A.5, respectively. Besides, we can in particular choose $Q$ such that $\epsilon_{A.5} \leq \epsilon$.

Now we argue that the termination condition:

$$\|T - \tilde{T}\|_F \leq \epsilon, \quad \max_{i \in [r+k]} |\xi_i|_1^{1/3} \leq 2M$$

is attainable. Without loss of generality, assume that $\max_{i \in [r+k]} |\xi_i|_1^{1/3} = |\xi_1|_1^{1/3}$.

When the algorithm finds the correct decomposition (note that this happens with positive probability), $|\xi_1|_1^{1/3}$ will not deviate from $\|a_i\|_2$ too much. For any $i \in [r+k]$, we must have:

$$\left|\|a_i\|_2 - |\xi_i|_1^{1/3}\right| \leq \frac{\|a_i\|_2^3 - |\xi_i|_1}{3(\|a_i\|_2^3 - |\xi_i|_1)} \leq \frac{\epsilon_{A.5}}{3(\|a_i\|_2^3 - |\xi_i|_1^{1/3})^{2/3}} \leq \frac{\epsilon}{3(\|a_i\|_2^3 - |\xi_i|_1^{1/3})^{2/3}},$$

(A.38)

which implies

$$|\xi_1|_1^{1/3} \leq M - M - |\xi_1|_1^{1/3} \leq M + \|a_i\|_2 - |\xi_i|_1^{1/3} \leq 2M + \|a_i\|_2 - |\xi_i|_1^{1/3} \leq 2M + \frac{\|a_i\|_2^3}{3(\|a_i\|_2^3/2)^{2/3}} - |\xi_i|_1^{1/3} \leq 2M - (1 - \frac{1}{3\sqrt{2}})\|a_i\|_2 < 2M,$$

(A.39)

where the fourth inequality comes from $\epsilon \leq \epsilon_{out} \leq \min_{i \in [r+k]} \|a_i\|_2^3/2 \leq \|a_1\|_2^3/2$. Therefore the algorithm is possible to terminate with a $2M$-bounded decomposition with $\epsilon$ reconstruction error.

Let

$$\text{poly}_{3,3}(d, r, k, \tau, M) = 2 \text{poly}_{A.1}(r + k, \tau, M, 2M, d)$$

poly_{3,3}(d, r, k, \tau, M) = Q(d, r, k, \tau, M) \text{poly}_{3,3}(d, r, k, \tau, M)) (A.40)

so that given $0 < \epsilon_{out} \leq \min\{1/2, \min_{i \in [r+k]} \|a_i\|_2^3/2\}$, $\epsilon = \epsilon_{out}/(\text{poly}_{3,3})$ and $\epsilon_{in} \leq \epsilon_{out}/(\text{poly}_{3,3})$, when the algorithm terminates, we have

$$\|T - T'\|_F \leq \epsilon + \epsilon_{in} \leq \frac{\epsilon_{out}}{2 \text{poly}_{3,3}} + \frac{\epsilon_{out}}{2Q \text{poly}_{3,3}} \leq \frac{\epsilon_{out}}{\text{poly}_{A.1}(r + k, \tau, M, 2M, d)}.$$

(A.41)

Now we can apply Theorem A.1 and obtain component-wise $\epsilon_{out}$ accuracy.

For the running time of the algorithm: as each step in our algorithm takes only polynomial time, a single loop of the algorithm will take only polynomial time. The success probability has leading term $O((\theta \sqrt{d})^{2k})$ as our choice of parameters guarantees that other terms in Lemmas A.9 and A.10 are $O(1)$, which implies that the algorithm expects to end in $O((\theta \sqrt{d})^{-2k})$ loops and the expectation of running time is $\text{poly}(d^k, r^k, k^k, 1/\epsilon^k, \tau^k, M^k)$. Since $\epsilon = \epsilon_{out}/(\text{poly}_{3,3})$, the running time is essentially $\text{poly}(d^k, r^k, k^k, 1/\epsilon_{out}^k, \tau^k, M^k).$
B Estimating cumulants

Previously, we have shown that one can identify the centers of a mixture model from its 3rd cumulant tensor. However what we have is only an unlabeled dataset drawn from the mixture distribution instead of the population statistics. In this section, we provide technical details about the unbiased estimator of cumulants, called k-statistics. They are the unbiased estimator for cumulants with the minimum variance, and are long studied in the statistics community. We provide the formula for the 3rd k-statistic given in chapter 4 of [27] here:

**Fact B.1.** Given $N$ iid. samples $x_1, \ldots, x_N$ of $X$, the k-statistic for the 3rd cumulant of $X$ is given by:

$$k_3(r, s, t) = \frac{1}{N} \sum_{i,j,k \in [N]} \phi^{(ijk)}(x_i)_r(x_j)_s(x_k)_t,$$

where $r, s, t$ are the position indices in the tensor, and $\phi^{(ijk)}$ is the coefficient such that $\phi$ is invariant when input indices are permuted, and for distinct $i, j, k \in [N]$:

$$\phi^{(iii)} = \frac{1}{N}, \quad \phi^{(iij)} = -\frac{1}{N - 1}, \quad \phi^{(ijk)} = \frac{2}{(N - 1)(N - 2)}.$$

To obtain the entry-wise concentration bound for $k_3$, we begin with bounding the variance of each entry in $k_3$:

**Lemma B.2.** Let $X$ follow a distribution in (4.1). The 3rd k-statistics $k_3$ of $X$ satisfies:

$$\text{Var}(k_3(r, s, t)) = O \left( \frac{\max_{t \in [d]} E[X_t^2]}{N} \right)$$

**Proof.** We start with defining some multi-indices to simply the computation: let $I = (r, s, t) \in [d]^3$ and $\alpha = (i, j, k) \in [N]^3$, then write

$$\phi^{(\alpha)}(X^{(I)}_{(\alpha)}) = \phi^{(ijk)}(x_i)_r(x_j)_s(x_k)_t.$$

Also, we define the intersection of 2 multi-indices $\alpha, \beta$ as:

$$\alpha \cap \beta = \{i \in \alpha : i \in \beta\}$$

Now we continue to bound the variance of $k_3$:

$$\text{Var}(k_3(I)) = \mathbb{E} \left[ \left( \frac{1}{N} \sum_{\alpha \in [N]^3} \phi^{(\alpha)}(X^{(I)}_{(\alpha)}) \right)^2 \right] - \mathbb{E} \left[ \frac{1}{N} \sum_{\alpha \in [N]^3} \phi^{(\alpha)}(X^{(I)}_{(\alpha)}) \right]^2$$

$$= \frac{1}{N^2} \sum_{\alpha, \beta \in [N]^3} \phi^{(\alpha)} \phi^{(\beta)} \mathbb{E}[X^{(I)}_{(\alpha)} X^{(I)}_{(\beta)}] - \frac{1}{N^2} \sum_{\alpha, \beta \in [N]^3} \phi^{(\alpha)} \phi^{(\beta)} \mathbb{E}[X^{(I)}_{(\alpha)}] \mathbb{E}[X^{(I)}_{(\beta)}]$$

$$= \frac{1}{N^2} \sum_{\alpha \cap \beta \neq \emptyset} \phi^{(\alpha)} \phi^{(\beta)} \mathbb{E}[X^{(I)}_{(\alpha)} X^{(I)}_{(\beta)}] - \frac{1}{N^2} \sum_{\alpha \cap \beta \neq \emptyset} \phi^{(\alpha)} \phi^{(\beta)} \mathbb{E}[X^{(I)}_{(\alpha)}] \mathbb{E}[X^{(I)}_{(\beta)}]$$

$$\leq \frac{1}{N^2} \sum_{\alpha \cap \beta \neq \emptyset} \phi^{(\alpha)} \phi^{(\beta)} \mathbb{E}[X^{(I)}_{(\alpha)} X^{(I)}_{(\beta)}].$$
Now let us consider how many terms there are indexed by $\beta$ that does not intersect with $\alpha$. Let $\text{dist}(\beta)$ be the number of distinct indices in $\beta$. Then there are \( \binom{N}{\text{dist}(\beta)} \) ways to generate $\beta$, out of which there are \( \binom{N-3}{\text{dist}(\beta)} \) that will absolutely have no intersection with $\alpha$ as $\alpha$ can have at most 3 different indices. That is, when fixing the number of distinct indices in $\beta$, at most a

\[
\left[ \left( \binom{N}{\text{dist}(\beta)} \right) - \binom{N-3}{\text{dist}(\beta)} \right] / \binom{N}{\text{dist}(\beta)}
\]

fraction of index sets in $\beta$ will intersect with $\alpha$. To proceed, we need some other estimations on the coefficients and the expectation. By (B.2) we have:

\[
\left| \phi^{(\alpha)} \right| = O(1/N^{\text{dist}(\beta)-1}), \quad \sum_{\alpha \in [N]^3} \left| \phi^{(\alpha)} \right| = O(N).
\]

For $\mathbb{E}[x_{(\alpha)}^{(I)} x_{(\beta)}^{(I)}]$, by Cauchy-Schwarz inequality we have:

\[
\mathbb{E}[x_{(\alpha)}^{(I)} x_{(\beta)}^{(I)}] \leq \max\{\mathbb{E}[(x_{(\alpha)}^{(I)})^2], \mathbb{E}[(x_{(\beta)}^{(I)})^2]\}.
\]

Take $\mathbb{E}[(x_{(\alpha)}^{(I)})^2] = \mathbb{E}[(x_i^2(x_j^2(x_k^2)_{\beta}))_I]$ for instance. We have:

\[
\mathbb{E}[(x_{(\alpha)}^{(I)})^2] = \begin{cases} 
\mathbb{E}[x_i^2 x_j^2 x_k^2] & i = j = k \\
\mathbb{E}[x_i^2 x_j^2] \mathbb{E}[x_k^2] & i = j \neq k \\
\mathbb{E}[x_i^2] \mathbb{E}[x_j^2] \mathbb{E}[x_k^2] & i \neq j \neq k.
\end{cases}
\]

Applying Cauchy-Schwarz inequality and Hölder’s inequality gives $\mathbb{E}[(x_{(\alpha)}^{(I)})^2] \leq \max_{t \in [d]} \mathbb{E}[X_t^6]$.

Then (B.3) can be further bounded by

\[
\text{Var}(k_3(I)) \leq \frac{1}{N^2} \sum_{\alpha \cap \beta \neq \emptyset} \left| \phi^{(\alpha)} \phi^{(\beta)} \mathbb{E}[x_{(\alpha)}^{(I)} x_{(\beta)}^{(I)}] \right|
\]

\[
\leq \frac{\max_{t \in [d]} \mathbb{E}[X_t^6]}{N^2} \sum_{\alpha \in [N]^3} \left| \phi^{(\alpha)} \right| \sum_{c=1}^{3} \sum_{\text{dist}(\beta) = c, \alpha \cap \beta \neq \emptyset} \left| \phi^{(\beta)} \right|
\]

\[
= O \left( \frac{\max_{t \in [d]} \mathbb{E}[X_t^6]}{N^2} \sum_{\alpha \in [N]^3} \left| \phi^{(\alpha)} \right| \sum_{c=1}^{3} \binom{N}{c} - \binom{N-3}{c} \frac{N^{1-c}}{N^{1-c} \sum_{\text{dist}(\beta) = c} 1} \right)
\]

\[
= O \left( \frac{\max_{t \in [d]} \mathbb{E}[X_t^6]}{N^2} \sum_{\alpha \in [N]^3} \left| \phi^{(\alpha)} \right| N^{c-1} N^{1-c} \right)
\]

\[
= O \left( \frac{\max_{t \in [d]} \mathbb{E}[X_t^6]}{N} \right),
\]

where the second last equality comes from the estimation on binomial coefficients. \( \square \)

Using Chebyshev’s inequality yields the follow sample bounds immediately:
Lemma B.3. Given \( \epsilon, \delta \in (0, 1) \), the entry-wise error between \( k_3 \) and \( K_3(X) \) is at most \( \epsilon \) with probability \( 1 - \delta \) using

\[
N \geq O \left( \frac{\max_{t \in [d]} \mathbb{E}[X_t^6]}{\epsilon^2 \delta} \right)
\]
samples.

C Technical lemmas

In this section we collect technical lemmas used throughout the paper.

C.1 Perturbed SVD bounds

Here we provide the Wedin’s theorem, a \( \sin(\theta) \) theorem for perturbed singular vectors as well as the Weyl’s inequality for SVD. The following results are from [28].

Theorem C.1 (Weyl’s inequality). Let \( A, A + E \in \mathbb{R}^{d_1 \times d_2} \) with \( d_1 \geq d_2 \). Denote the singular values of \( A, A + E \) by \( \sigma_i, \tilde{\sigma}_i \), respectively. Suppose all singular values are in descending order, i.e. \( \sigma_i \geq \sigma_{i+1} \) for \( i \leq \min\{d_1, d_2\} \). Then we have:

\[
|\sigma_i - \tilde{\sigma}_i| \leq \|E\|_2.
\]

Theorem C.2 (Wedin). In the same notation as Theorem C.1, let the singular value decomposition of \( A \) be:

\[
[U_1, U_2, U_3]^{\top} A [V_1, V_2] = \begin{bmatrix}
\Sigma_1 & 0 \\
0 & \Sigma_2 \\
0 & 0
\end{bmatrix},
\]

where the singular values need not be in descending order. And the perturbed version is:

\[
[\tilde{U}_1, \tilde{U}_2, \tilde{U}_3]^{\top} (A + E) [\tilde{V}_1, \tilde{V}_2] = \begin{bmatrix}
\tilde{\Sigma}_1 & 0 \\
0 & \tilde{\Sigma}_2 \\
0 & 0
\end{bmatrix}
\]

Let \( \Phi \) be the matrix of canonical angles between \( \text{span}\{U_1\}, \text{span}\{\tilde{U}_1\} \) and \( \Theta \) be that of \( \text{span}\{V_1\}, \text{span}\{\tilde{V}_1\} \). Let \( \delta = \min\{\min_i \tilde{\Sigma}_{1,ii}, \min_{i,j} |\tilde{\Sigma}_{1,ii} - \Sigma_{2,jj}|\} \). Then

\[
\sqrt{\|\sin \Phi\|_2^2 + \|\sin \Theta\|_2^2} \leq \frac{\sqrt{2}\|E\|_2}{\delta}.
\]

C.2 Probability tail bounds

We collect some useful probability tail bounds in this section:

Lemma C.3 (Concentration of Lipschitz function [29]). Let \( X \) be a random vector distributed uniformly over the unit sphere in \( \mathbb{R}^d \), \( f : \mathbb{S}^{d-1} \to \mathbb{R} \) be a Lipschitz function with Lipschitz constant \( L \). Then for every \( t \geq 0 \), we have:

\[
\mathbb{P}[|f(X) - \mathbb{E}[f(X)]| \geq t] \leq 2 \exp \left( -\frac{cdt^2}{L^2} \right)
\]
Corollary C.4. Let $X$ be a random vector distributed uniformly over the unit sphere in $\mathbb{R}^d$, $v \in \mathbb{R}^d$ be a unit vector. Then for any positive number $t \in (0, 1)$,

$$P[|\langle X, v \rangle| \geq t] \leq 2 \exp \left(-cdt^2\right)$$

Proof. This is a direct application of Lemma C.3. □

Lemma C.5 ([8, 19]). Pick any $\delta \in (0, 1)$, matrix $M \in \mathbb{R}_{d \times d}$ and finite subset $Q \subset \mathbb{R}^d$. If $X \in \mathbb{R}^d$ is a random vector distributed uniformly over the unit sphere in $\mathbb{R}^d$, then

$$P\left[\min_{q \in Q} |\langle X, Mq \rangle| \geq \frac{\delta \min_{q \in Q} \|Mq\|_2}{\sqrt{ed} |Q|}\right] \geq 1 - \delta$$

Lemma C.6. Let $X$ be a uniformly random unit vector in $\mathbb{R}^d$, $a_1, \ldots, a_k \in S^{d-1}$, and $t \in [0, 1/\sqrt{d}]$. Then

$$P[|\langle X, a_i \rangle| \leq t, \forall i \in [k]] \geq (c_1 t \sqrt{d})^k - e^{-c_2 d},$$

where $c_1, c_2$ are absolute constants.

Proof. Let $Y$ be a standard Gaussian random vector in $\mathbb{R}^d$ and write $X = Y/\|Y\|_2$. Note that for any $t \in [0, 1]$, by the Gaussian correlation inequality,

$$P[|\langle Y, a_i \rangle| \leq t, \forall i \in [k]] \geq \prod_{i=1}^k P[|\langle Y, a_i \rangle| \leq t] \geq \left(\frac{2t}{\sqrt{2\pi} e^{-t^2/2}}\right)^k$$

Thus,

$$P[|\langle X, a_i \rangle| \leq t, \forall i \in [k]] = P[|\langle Y, a_i \rangle| \leq t\|Y\|_2, \forall i \in [k]]$$

$$\geq P[|\langle Y, a_i \rangle| \leq t\sqrt{d}/2, \forall i \in [k]] - P[\|Y\|_2 \leq \sqrt{d}/2]$$

$$\geq \left(\frac{t\sqrt{d} e^{-t^2 d/8}}{\sqrt{2\pi} e^{-1/8}}\right)^k - e^{-c_2 d} = \left(c_1 t \sqrt{d}\right)^k - e^{-c_2 d},$$

where the last inequality comes from $t \leq 1/\sqrt{d}$ and the concentration of $\|Y\|_2$. □