A MONOTONICITY PROPERTY OF A NEW BERNSTEIN TYPE OPERATOR

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ABSTRACT. In the present paper we prove that the probabilities of the Pólya urn distribution (with negative replacement) satisfy a monotonicity property similar to that of the binomial distribution (Pólya urn distribution with no replacement).

As a consequence, we show that the random variables with Pólya urn distribution (with negative replacement) are stochastically ordered with respect to the parameter giving the initial distribution of the urn. An equivalent formulation of this result shows that the new Bernstein operator recently introduced in [3] is a monotone operator.

The proofs are probabilistic in spirit and rely on various inequalities, some of which are of independent interest (e.g. a refined version of the reversed Cauchy-Bunyakovsky-Schwarz inequality).

1. INTRODUCTION

Pólya urn model (also known as Plya-Eggenberger urn model) is an experiment in which one observes the number of white balls extracted from an urn containing initially $a$ white balls and $b$ black balls, when the extracted ball is replaced in the urn together with $c$ balls of the same color before the next extraction from the urn.

Denoting by $X_{n,a,b,c}$ the random variable representing the number of white balls obtained in $n \geq 1$ extractions from the urn, it can be shown (e.g. [2]) that the model is well defined (defines a distribution) for $a, b > 0$ and $c \in \mathbb{R}$ satisfying $c \geq -\min\{a, b\}/(n - 1)$, and the distribution is given by

$$p_{n,k}^{a,b,c} = P(X_{n,a,b,c} = k) = C_n^{k} \frac{a^{k} b^{n-k} c^{(n-1)(a+b)}}{1^{(n,c)}}, \quad k \in \{0, 1, \ldots, n\},$$

(1.1)

where $x^{(0,h)} = 1$, $x^{(k,h)} = x(x+h) \cdots (x+(k-1)h)$ for $k \geq 1$, denotes the rising factorial with increment $h$.

In the present paper we focus on the case $a = x \in [0,1]$, $b = 1 - x$, and the minimal choice of the replacement parameter, $c = -\min\{x, 1-x\}/(n - 1)$. The reason for this choice is two-fold. The first is that it is an interesting problem of study from the probabilistic point of view, and the second relates to the newly introduced operator $R_n$ ([3, 4]) defined by

$$R_n(f, x) = Ef \left( \frac{1}{n} X_{n,a,b,c}^{x,1-x,-\min\{x,1-x\}/(n-1)} \right).$$

(1.2)

We show that in this case the corresponding probabilities satisfy a monotonicity property with respect to the initial urn distribution, similar to that of the binomial distribution (the case of the replacement parameter $c = 0$), and as a consequence, we show that the random variables with Pólya urn distribution satisfy a natural stochastic ordering. In an equivalent formulation, this results shows that the operator $R_n$ has a shape preserving property (monotonicity property), similar to that of the classical Bernstein operator (the case $c = 0$, see e.g. [1]).

The structure of the paper is the following. Section 2 contains some auxiliary results of independent interest, needed in the sequel. In Lemma 2.1 we prove an interesting inequality, which may be seen as a refined version of a reversed Cauchy-Bunyakovsky-Schwarz inequality (see Remark 2.1). In Lemma 2.3 we give estimates for sums involving functions with three positive continuous derivatives. Lemma 2.4 is a technical result concerning the sign of a certain function, essential for proving our main results.

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In Section 3 we first prove that the Pólya urn probabilities satisfy a certain monotonicity property with respect to the initial urn distribution, similar to that of the binomial distribution (Theorem 3.1). Using this, in Theorem 3.3 we show that the Pólya random variables satisfy a natural stochastic ordering. In an equivalent formulation (Theorem 3.4), this results shows that the operator $R_n$ is, similarly to the classical Bernstein operator, a monotone operator.

2. Auxiliary results

We begin with the following auxiliary result of independent interest.

**Lemma 2.1.** For integers $n \geq 2$ and $k \in \{1, \ldots, n-1\}$, and positive real numbers $a_1, \ldots, a_n, b_1, \ldots, b_{n-k}$ with $\max_{1 \leq i \leq n} a_i \leq \min_{1 \leq j \leq n-k} b_j$ and $\sum_{i=1}^{n} a_i = \sum_{j=1}^{n-k} b_j$, we have

$$\sum_{i=1}^{n} a_i^2 \leq \sum_{j=1}^{n-k} b_j^2.$$  

**Proof.** Without loss of generality we may assume assume $a_1 \leq \ldots \leq a_n \leq b_1 \leq \ldots \leq b_{n-k}$.

Note that $\frac{\sum_{i=1}^{n} a_i}{n-k} = \frac{\sum_{j=1}^{n-k} b_j}{n-k} \geq b_1 = a_n$, and therefore $a_i \leq a_n \leq \frac{\sum_{i=1}^{n} a_i}{n-k}$, for $i \in \{1, \ldots, n\}$.

We obtain

$$\sum_{i=1}^{n} a_i^2 = \sum_{i=1}^{n} (a_i \cdot a_i) \leq \frac{1}{n-k} \sum_{i=1}^{n} \left( a_i \sum_{j=1}^{n} a_j \right) = \frac{1}{n-k} \left( \sum_{i=1}^{n} a_i \right)^2,$$

and using Cauchy-Bunyakovsky-Schwarz inequality we conclude

$$\sum_{i=1}^{n} a_i^2 \leq \frac{1}{n-k} \left( \sum_{i=1}^{n} a_i \right)^2 \leq \frac{1}{n-k} \left( \sum_{j=1}^{n-k} b_j \right)^2 \leq \sum_{j=1}^{n-k} b_j^2.$$

We have left to show that the inequality is a strict inequality. If the inequality in the statement of the lemma were an equality, from the proof above we conclude that we must have $a_1 = \ldots = a_n = a$ and $b_1 = \ldots = b_{n-k} = b$. The hypothesis of the lemma becomes in this case $na = (n-k)b$, and in turn, this shows $\sum_{i=1}^{n} a_i^2 = na^2 = \frac{1}{n} (n-k)^2 b^2 = \frac{n-k}{n} \sum_{j=1}^{n-k} b_j^2 < \sum_{j=1}^{n-k} b_j^2$, thus the equality cannot hold, concluding the proof. □

**Remark 2.2.** Remark on reversed CBS inequality The inequality (2.1) in the proof above is a particular case of a reversed Cauchy-Bunyakovsky-Schwarz inequality. For example, the Pólya-Szegő’s inequality for sequences of positive numbers is

$$\frac{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2}{(\sum_{i=1}^{n} a_i b_i)^2} \leq \frac{1}{4} \left( \sqrt{M_1 M_2} + \sqrt{m_1 m_2} \right)^2,$$

where $0 < m_1 \leq a_i \leq M_1 < \infty$ and $0 < m_2 \leq b_i \leq M_2 < \infty$, $i \in \{1, \ldots, n\}$. Taking $b_1 = \ldots = b_n = 1$ (thus $m_2 = M_2 = 1$, $m_1 = a_1$, $M_2 = a_n$) we obtain

$$\sum_{i=1}^{n} a_i^2 \leq \frac{1}{4} \left( \sqrt{\frac{a_n}{a_1}} + \sqrt{\frac{a_1}{a_n}} \right)^2 \left( \sum_{i=1}^{n} a_i \right)^2,$$

for any positive sequence $a_1 \leq a_2 \leq \ldots \leq a_n$. Since $\frac{1}{4} \left( \sqrt{\frac{a_n}{a_1}} + \sqrt{\frac{a_1}{a_n}} \right)^2 \geq 1$, we see that the inequality (2.1) improves the Pólya-Szegő inequality (under the additional hypotheses in the statement of the lemma).

We will also need the following auxiliary result.

**Lemma 2.3.** Suppose $f \in C^3 ([0, 1])$ is such that $f, f', f'', f''' \geq 0$ on $[0, 1]$. Then for any integer $N \geq 1$ we have

$$0 \leq \sum_{i=0}^{N} f \left( \frac{i}{N} \right) - N \int_{0}^{1} f (t) \, dt - \frac{f (0) + f (1)}{2} \leq \frac{f' (1) - f' (0)}{4N}.$$  

(2.3)
Figure 1. Trapezoids $T_i$ and $T'_i$ in the proof of Lemma 2.3: \[ \text{Area}(T'_i) \leq \int_{x_i}^{x_{i+1}} f(x) \, dx \leq \text{Area}(T_i). \]

**Proof.** Since $f$ is convex, the sum of the areas of trapezoids $T_0, \ldots, T_{N-1}$ (see Figure 1) is larger than the area under the graph of $f$, thus
\[
\int_0^1 f(t) \, dt \leq \sum_{i=0}^{N-1} \frac{f(x_i) + f(x_{i+1})}{2} \cdot \frac{1}{N},
\]
where $x_i = \frac{i}{N}, \; i \in \{0, 1, \ldots, N\}$. Rearranging the terms of the sum we obtain
\[
N \int_0^1 f(t) \, dt \leq -\frac{f(0) + f(1)}{2} + \sum_{i=0}^{N} f\left(\frac{i}{N}\right),
\]
which proves the left inequality in (2.3).

Since $f$ is convex, the tangent line to the graph of $f$ at $x_i$ lies below the graph of $f$ (see Figure 1). Summing over $i \in \{0, 1, \ldots, N - 1\}$ we obtain
\[
\sum_{i=0}^{N-1} \frac{f(x_i) + y_i}{2} \cdot \frac{1}{N} \leq \int_0^1 f(t) \, dt,
\]
where $y_i = f(x_i) + f'(x_i) \cdot \frac{1}{N}$, or equivalent
\[
N \int_0^1 f(t) \, dt \geq \sum_{i=0}^{N-1} \frac{f(x_i) + f(x_i) + f'(x_i)}{2N} = \sum_{i=0}^{N-1} f(x_i) + \frac{1}{2N} \sum_{i=0}^{N-1} f'(x_i),
\]
and therefore
\[
\sum_{i=0}^{N} f(x_i) \leq f(1) + \frac{1}{2N} f'(1) + N \int_0^1 f(t) \, dt - \frac{1}{2N} \sum_{i=0}^{N} f'(x_i).
\]

Using the inequality (2.4) with $f$ replaced by $f'$, we obtain
\[
\sum_{i=0}^{N} f(x_i) \leq f(1) + \frac{1}{2N} f'(1) + N \int_0^1 f(t) \, dt - \frac{1}{2N} \sum_{i=0}^{N} f'(x_i)
\]
\[
\leq f(1) + \frac{1}{2N} f'(1) + N \int_0^1 f(t) \, dt - \frac{1}{2N} \left( N \int_0^1 f'(t) \, dt + \frac{f'(0) + f'(1)}{2} \right)
\]
\[
= N \int_0^1 f(t) \, dt + \frac{f(0) + f(1)}{2} + \frac{f'(1) - f'(0)}{4N},
\]
concluding the proof. 

The following technical result is essential for the proof of our main results in the following section.
Lemma 2.4. For any integers \( n \geq 2 \) and \( k \in \{0, \ldots, n-1\} \), there exists \( x_{n,k} \in \left[ \frac{k-1}{n-1}, \frac{k}{n-1} \right] \) such that the function

\[
\varphi_{n,k}(x) = \sum_{i=0}^{n-1} \frac{1}{1 - \frac{i}{n-1} x} - \sum_{i=0}^{n-k-1} \frac{1}{1 - x - \frac{i}{n-1} x}, \quad x \in \left[ 0, \frac{n-1}{2n-k-2} \right],
\]

is positive on \([0, x_{n,k})\) and negative on \((x_{n,k}, \frac{n-1}{2n-k-2})\).

Proof. Under the above hypotheses on \( n \) and \( k \), it is easy to verify that \( \frac{k-1}{n-1} < \frac{k}{n} < \frac{n-1}{2n-k-2} \leq 1 \) (the last inequality is a strict inequality if \( k < n - 1 \)).

If \( k = 0 \), since \( \frac{1}{1-x-\frac{i}{n-1} x} > \frac{1}{1-x} \) for \( i \in \{0, \ldots, n-1\} \), we have \( \varphi_{n,0}(x) < 0 \) for \( x \in (0, \frac{n-1}{2n-1}) \), and the claim holds true with \( x_{n,0} = 0 \in \left( -\frac{1}{n-1}, 0 \right) \).

If \( k = n-1 \), \( \varphi_{n,k}(x) = \sum_{i=0}^{n-2} \frac{1}{1-x-\frac{i}{n-1} x} > 0 \) for \( x \in \left[ 0, \frac{n-1}{2n-(n-1)-2} \right] = [0, 1) \), and the claim holds true with \( x_{n,n-1} = 1 \in \left[ \frac{n-2}{n-1}, 1 \right] \).

Assume now \( n > 2 \) and \( k \in \{1, \ldots, n-2\} \). We will first show that if \( \varphi_{n,k}(x) = 0 \), then \( \varphi'_{n,k}(x) < 0 \) (note that \( \varphi_{n,k}(0) = k > 0 \), thus \( x \neq 0 \)).

We have

\[
\varphi'_{n,k}(x) = \frac{n-1}{x} \sum_{i=0}^{n-1} \left( 1 - \frac{i}{n-1} x \right)^2 - \sum_{i=0}^{n-k-1} \frac{1}{1 - x - \frac{i}{n-1} x} \left( 1 - \frac{i}{n-1} x \right)^2
\]

\[
= \frac{1}{x} \left( \sum_{i=0}^{n-1} \left( 1 - \frac{i}{n-1} x \right)^2 - \sum_{i=0}^{n-k-1} \frac{1}{1 - x - \frac{i}{n-1} x} \left( 1 - \frac{i}{n-1} x \right)^2 \right)
\]

\[
= \frac{1}{x} \varphi_{n,k}(x) + \frac{1}{x} \left( \sum_{i=0}^{n-1} \frac{1}{1 - \frac{i}{n-1} x} - \sum_{i=0}^{n-k-1} \frac{1}{1 - x - \frac{i}{n-1} x} \right).
\]

If \( \varphi_{n,k}(x) = 0 \), we obtain \( \varphi'_{n,k}(x) = \frac{1}{x} \left( \sum_{i=0}^{n-1} \frac{1}{1 - \frac{i}{n-1} x} - \sum_{i=0}^{n-k-1} \frac{1}{1 - x - \frac{i}{n-1} x} \right) \), and we have left to prove the implication

\[
\sum_{i=0}^{n-1} \frac{1}{1 - \frac{i}{n-1} x} = \sum_{i=0}^{n-k-1} \frac{1}{1 - x - \frac{i}{n-1} x} \\Rightarrow \sum_{i=0}^{n-1} \frac{1}{\left( 1 - \frac{i}{n-1} x \right)^2} < \sum_{i=0}^{n-k-1} \frac{1}{\left( 1 - x - \frac{i}{n-1} x \right)^2}.
\]

Choosing \( a_{i+1} = \frac{1}{1-x-\frac{i}{n-1} x} \), \( i \in \{0, \ldots, n-1\} \) and \( b_{j+1} = \frac{1}{1-x} \), \( j \in \{0, \ldots, n-k-1\} \), we have \( \max_{1 \leq i \leq n} a_i = a_n = \frac{1}{1-x} = b_1 = \min_{1 \leq j \leq n-k} b_j \), and the above implication follows from Lemma 2.1 concluding the proof of the claim.

We showed that \( \varphi_{n,k}(x) = 0 \) implies \( \varphi'_{n,k}(x) < 0 \). Since \( \varphi_{n,k} \) is continuously differentiable, a moment’s thought shows that this condition implies that \( \varphi_{n,k} \) can change signs at most once on the interval \([0, \frac{n-1}{2n-k-2})\).

Since \( \varphi_{n,k}(0) = n - (n-k) = k > 0 \) and \( \lim_{x \to \frac{n-1}{2n-k-2}} \varphi_{n,k}(x) = -\infty \), the function \( \varphi_{n,k} \) changes sign on \([0, \frac{n-1}{2n-k-2})\), and let \( x_{n,k} \) denote its unique root. We have left to show that \( x_{n,k} \) belongs to the specified interval.
Using Lemma 2.3 with $N = n - 1$ and $f(t) = \frac{1}{1-tx}$, respectively with $N = n - k - 1$ and $f(t) = \frac{1}{1-x-tx^{n-k-1}}$, we obtain:

\[
\varphi_{n,k}(x) \leq \left((n-1) \int_0^1 \frac{1}{1-tx} dt + \frac{1 + \frac{1}{1-x}}{2} + \frac{x}{4(n-1)}\right) \\
- \left((n-k-1) \int_0^1 \frac{1}{1-x-tx^{n-k-1}} dt + \frac{1 + \frac{1}{1-x-\frac{n-k-1}{n-1}x}}{2}\right)
\]

\[
= \left(-\frac{n-1}{x} \ln(1-x) + \frac{1 + \frac{1}{1-x}}{2} + \frac{x}{4(n-1)}\right) \\
- \left(-\frac{n-1}{x} \ln \left(1 - \frac{n-k-1}{n-1}x\right) + \frac{1 + \frac{1}{1-x-\frac{n-k-1}{n-1}x}}{2}\right)
\]

\[
= \frac{1}{2} \left(1 - \frac{1}{1-x-\frac{n-k-1}{n-1}x}\right) + \frac{x^2(2-x)}{4(n-1)(1-x)^2} + \frac{n-1}{x} \ln \left(1 - \frac{n-k-1}{n-1}x\right) \\
+ \frac{1}{2} \left(1 - \frac{1}{1-x-\frac{n-k-1}{n-1}x}\right) + \frac{x^2(2-x)}{4(n-1)(1-x)^2} + \frac{n-1}{x} \ln \left(1 - \frac{n-k-1}{n-1}x\right).
\]

In particular, for $x = \frac{k}{n-1}$ we obtain $\varphi_{n,k}\left(\frac{k}{n-1}\right) \leq \frac{k(2n-2k)(2(n-1)^2)}{4(n-1)^2(n-k-1)^2} < 0$, which shows that $x_{n,k} < \frac{k}{n-1}$.

In order to obtain the lower bound for $x_{n,k}$, first note that for $k = 1$ the claim is trivial ($\varphi_{n,1}(0) = 1 > 0$, thus $x_{n,1} > 0$), so we may assume $k \in \{2, \ldots, n-1\}$. Using again Lemma 2.3 with the same choices as above, we obtain:

\[
\varphi_{n,k}(x) \geq \left((n-1) \int_0^1 \frac{1}{1-tx} dt + \frac{1 + \frac{1}{1-x}}{2} \right) - \left((n-k-1) \int_0^1 \frac{1}{1-x-tx^{n-k-1}} dt + \frac{1 + \frac{1}{1-x-\frac{n-k-1}{n-1}x}}{2}\right)
\]

\[
= \left(-\frac{n-1}{x} \ln(1-x) + \frac{1 + \frac{1}{1-x}}{2} \right) - \left(-\frac{n-1}{x} \ln \left(1 - \frac{n-k-1}{n-1}x\right) + \frac{1 + \frac{1}{1-x-\frac{n-k-1}{n-1}x}}{2}\right)
\]

\[
= \frac{1}{2} \left(1 - \frac{1}{1-x-\frac{n-k-1}{n-1}x}\right) - \left(\frac{1}{1-x-\frac{n-k-1}{n-1}x} - \frac{1}{1-x^2}\right) \frac{x}{4(n-1)}
\]

\[
+ \frac{n-1}{x} \ln \left(1 - \frac{n-k-1}{n-1}x\right) \\
+ \frac{n-1}{x} \ln \left(1 - \frac{n-k-1}{n-1}x\right).
\]
To simplify the following computation, denote by $A = \left( \frac{n-k}{n-1} \right)^2$, $B = \frac{k-1}{(n-1)^2}$, and $C = \frac{(n-k)^2}{k-1}$. For $x = \frac{k-1}{n-1}$, the above becomes

$$\varphi_{n,k}(x) \left( \frac{k-1}{n-1} \right) \geq \frac{1}{2} \left( 1 - \frac{1}{A+B} \right) - \left( \frac{1}{(A+B)^2} - \frac{1}{A^2} \right) B + \frac{1}{B} \ln \left( \frac{A+B}{A} \right)$$

$$= \frac{1}{2} + \frac{1}{B} \left( -\frac{1}{2(1+C)} - \frac{1}{4(1+C)^2} + \frac{1}{4C^2} + \ln \left( 1 + \frac{1}{C} \right) \right)$$

Since

$$\frac{d}{dC} \left( -\frac{1}{2(1+C)} - \frac{1}{4(1+C)^2} + \frac{1}{4C^2} + \ln \left( 1 + \frac{1}{C} \right) \right) = \left( -\frac{1}{2(1+C)^2} - \frac{1}{C(1+C)} \right) + \left( -\frac{1}{2(1+C)^2} - \frac{1}{2C^3} \right) < 0,$$

and $\lim_{C \to \infty} \left( -\frac{1}{2(1+C)} - \frac{1}{4(1+C)^2} + \frac{1}{4C^2} + \ln \left( 1 + \frac{1}{C} \right) \right) = 0$, we conclude that

$$\left( -\frac{1}{2(1+C)} - \frac{1}{4(1+C)^2} + \frac{1}{4C^2} + \ln \left( 1 + \frac{1}{C} \right) \right) > 0, \quad \text{for all } C > 0.$$

Using this and the previous inequality we obtain $x_{n,k} > \frac{k-1}{n-1}$, concluding the proof. \qed

3. MAIN RESULTS

We can now prove the first main result, as follows.

**Theorem 3.1.** For arbitrarily fixed integers $n \geq 2$ and $k \in \{0, 1, \ldots, n\}$, the probability

$$p_{n,k}(x) = p_{n,k}^{x,1-x,\min\{x,1-x\}/(n-1)}(x)$$

given by (1.1) increases for $x \in \left[ 0, x_{n,k}^* \right]$ and decreases for $x \in \left[ x_{n,k}^*, 1 \right]$, where

$$x_{n,k}^* = \begin{cases} x_{n,k}, & \text{if } k \leq \frac{n-1}{2} \\ \frac{n}{2}, & \text{if } \frac{n}{2} < k < \frac{n+1}{2} \\ 1 - x_{n,\overline{n-k}}, & \text{if } k \geq \frac{n+1}{2} \end{cases},$$

(3.1)

and $x_{n,k} \in \left[ \frac{k-1}{n-1}, \frac{k}{n-1} \right]$ are given by Lemma 2.4.

**Proof.** First note that $p_{n,0}(x) = \prod_{i=0}^{n-1} \frac{1-x-i\min\{x,1-x\}/(n-1)}{1-i\min\{x,1-x\}/(n-1)}$ is a decreasing function of $x \in [0,1]$ (each factor decreases in $x$), and similarly $p_{n,n}(x) = \prod_{i=0}^{n-1} \frac{x-i\min\{x,1-x\}/(n-1)}{1-i\min\{x,1-x\}/(n-1)}$ is an increasing function of $x \in [0,1]$ (each factor increases in $x$). The claim of the theorem holds therefore true in the cases $k = 0$ and $k = n$ ($x_{n,0}^* = x_{n,0} = 0$, respectively $x_{n,n}^* = 1 - x_{n,0} = 1$), and we can assume that $k \in \{1, \ldots, n-1\}$.
For \( x \in (0, 1/2) \), we have
\[
\frac{d}{dx} \ln p_{n,k} (x) = \frac{d}{dx} \left( \ln C_n^k + \sum_{i=0}^{k-1} \ln \left( x - \frac{i}{n-1} \right) \right) + \sum_{i=0}^{n-k-1} \ln \left( 1 - x - \frac{i}{n-1} \right) - \sum_{i=0}^{n-k-1} \ln \left( 1 - \frac{i}{n-1} \right) \tag{3.2}
\]
\[
= \frac{k}{x} + \sum_{i=0}^{n-k-1} \frac{1}{1 - x - \frac{i}{n-1}} - \sum_{i=0}^{n-k-1} \frac{1}{1 - \frac{i}{n-1}} \]
\[
= \frac{k}{x} + \sum_{i=0}^{n-k-1} \frac{1}{1 - x - \frac{i}{n-1}} - \sum_{i=0}^{n-k-1} \frac{1}{1 - \frac{i}{n-1}} \]
\[
= \frac{k}{x} + \sum_{i=0}^{n-k-1} \frac{1}{1 - x - \frac{i}{n-1}} - \sum_{i=0}^{n-k-1} \frac{1}{1 - \frac{i}{n-1}} \]
\[
= \frac{1}{x} \varphi_{n,k} (x),
\]
in the notation of Lemma 2.4, and a similar computation shows
\[
\frac{d}{dx} \ln p_{n,k} (x) = -\frac{1}{1 - x} \varphi_{n,n-k} (1 - x), \quad x \in (1/2, 1),
\] (alternatively, in order to derive the above one can use the relation \( p_{n,k} (x) = p_{n,n-k} (1 - x) \), valid for any \( x \in [0,1] \), \( n \geq 2 \), and \( k \in \{0, 1, \ldots, n\} \)).

It remains to show that the information about the sign of \( \varphi_{n,k} (x) \) given by Lemma 2.4 translates into the monotonicity of \( p_{n,k} (x) \) indicated in the statement of the theorem.

Note that by Lemma 2.4 we have
\[
x_{n,k} \in \left[ \frac{k-1}{n-1}, \frac{k}{n-1} \right], \quad \text{for all } n \geq 2 \text{ and } k \in \{1, \ldots, n-1\}, \tag{3.4}
\]
If \( \frac{k-1}{n-1} \leq \frac{1}{2} \), Lemma 2.4 and (3.2) show that \( p_{n,k} \) increases on \([0, x_{n,k}]\), and decreases on \([x_{n,k}, 1/2] \) (note that \( x_{n,k} \leq 1/2 \) by (3.4) in this case). Since \( x_{n,n-k} \geq 1 - k/n \geq 1/2 \), the function \( \varphi_{n,n-k} (x) \) is positive for \( x \in [0,1/2] \subset [0, x_{n,n-k}] \), and from (3.3) it follows that \( p_{n,k} \) decreases on \([1/2, 1] \). Since \( p_{n,k} \) is a continuous function on \([0,1]\), it follows that \( p_{n,k} \) increases on \([0, x_{n,k}] \) and decreases on \([x_{n,k}, 1/2] \), and therefore the claim of the theorem holds in this case with \( x^*_{n,k} = x_{n,k} \).

If \( \frac{k-1}{n-1} \geq \frac{1}{2} \), Lemma 2.4 and (3.2) show that \( p_{n,k} \) increases on \([0, 1/2] \subset [0, x_{n,k}] \) (note that \( x_{n,k} \geq 1/2 \) by (3.4) in this case). Since \( x_{n,n-k} \leq 1 - k/n \leq 1/2 \), the function \( \varphi_{n,n-k} (x) \) is positive for \( x \in [0,1/2] \subset [0, x_{n,n-k}] \) and negative for \( x \in [x_{n,n-k}, 1/2] \), and from (3.3) it follows that \( p_{n,k} \) increases on \([1/2, 1 - x_{n,n-k}] \), and decreases on \([1 - x_{n,n-k}, 1] \). Since \( f \) is continuous, it follows that \( p_{n,k} \) increases on \([0,1/2] \), and decreases on \([1 - x_{n,n-k}, 1] \), and therefore the claim of the theorem holds in this case with \( x^*_{n,k} = 1 - x_{n,n-k} \).

We have left to consider the case \( \frac{k-1}{n-1} < 1/2 < k/n \), or equivalent \( n-2 < k < n-1 \). If \( n \) is odd, the previous double inequality is not satisfied for any integer \( k \), so assume \( n = 2m \) is even. The previous double inequality gives \( m - 1 < k < m + 1/2 \), which is satisfied only for \( k = m \). We have
\[
\sum_{i=0}^{2m-1} \frac{1}{1 - \frac{i}{n-1} \cdot \frac{1}{2}} > \sum_{j=0}^{m-1} \left( \frac{1}{1 - \frac{j}{n-1} \cdot \frac{1}{2}} + \frac{1}{1 - \frac{j}{n-1} \cdot \frac{1}{2}} \right) = \sum_{j=0}^{m-1} \frac{1}{1 - \frac{j}{n-1} \cdot \frac{1}{2}},
\]
and therefore
\[
\varphi_{2m,m} \left( \frac{1}{2} \right) = \sum_{i=0}^{2m-1} \frac{1}{1 - \frac{i}{n-1} \cdot \frac{1}{2}} - \sum_{j=0}^{m-1} \frac{1}{1 - \frac{j}{n-1} \cdot \frac{1}{2}} > 0.
\]
Since $\varphi_{n,k}(x) = \varphi_{2m,m}(x) = \varphi_{n,n-k}(x)$ for any $x \in [0, 1]$, the previous inequality shows that $x_{n,k} = x_{2m,m} = x_{n,n-k} > \frac{1}{2}$ (thus $\varphi_{n,k} = \varphi_{n,n-k}$ are positive on $[0, \frac{1}{2}]$). Using (3.2) and (3.3) we conclude that $p_{n,k}$ increases on $[0, \frac{1}{2}]$ and decreases on $[\frac{1}{2}, 1]$, thus the claim of the theorem holds with $x_{n,k}^* = \frac{1}{2}$ in this case, concluding the proof.

\[ \square \]

**Remark 3.2.** Since $p_{n,k}(x) = p_{n,n-k}(1 - x)$ for $x \in [0, 1]$, from the previous theorem it follows that $x_{n,k}^* = 1 - x_{n,n-k}^*$, for all $n \geq 2$ and $k \in \{0, \ldots, n\}$.

Also note that since $x_{n,k} \in \left[\frac{k-1}{n-1}, \frac{k}{n-1}\right]$, from (3.1) it follows that we also have $x_{n,k}^* \in \left[\frac{k-1}{n-1}, \frac{k}{n-1}\right]$ for all $n \geq 2$ and $k \in \{0, 1, \ldots, n\}$.

We are now ready to prove the main result. Recall that a random variables variable $X$ is smaller than the random variable $Y$ in the usual stochastic order (in symbols $X \leq_{st} Y$) if the corresponding distribution functions $F_X$ and $F_Y$ satisfy $F_X(x) \geq F_Y(x)$ for all $x \in \mathbb{R}$.

**Theorem 3.3.** For any $n \geq 2$, the random variables $X_n^{x,1-x,\min\{x,1-x\}/(n-1)}$ with Polya urn distribution given by (1.1) satisfy the following stochastic ordering

\[
X_n^{x,1-x,\min\{x,1-x\}/(n-1)} \leq_{st} X_n^{y,1-y,\min\{x,1-y\}/(n-1)}, \quad 0 \leq x \leq y \leq 1.
\]

**Proof.** Fix $n \geq 2$ and denote by $F_x(\cdot)$ the distribution function of the random variable $X_n^{x,1-x,\min\{x,1-x\}/(n-1)}$, $x \in [0, 1]$. In order to prove the claim, it suffices to show that for any $k \in \{0, 1, \ldots, n\}$, $F_x(k)$ is a decreasing function of $x \in [0, 1]$, and we will prove this inductively on $k$.

Since $F_x(0) = P(X_n^{x,1-x,\min\{x,1-x\}/(n-1)} = 0) = p_{n,0}(x)$ is a decreasing function of $x \in [0, 1]$ (by Theorem 3.1), the claim holds true for $k = 0$.

Assume now that the claim is true for $k - 1$, i.e. $F_x(k - 1)$ is decreasing in $x \in [0, 1]$.

**Theorem 3.1** shows that $p_{n,k}(x)$ is a decreasing function of $x \in [x_{n,k}^*, 1]$, and therefore $F_x(k - 1) + p_{n,k}(x)$ is decreasing for $x \in [x_{n,k}^*, 1]$.

Considering now $x \in \left[0, x_{n,k}^*\right]$, we observe that

\[
F_x(k) = \sum_{i=0}^{k} p_{n,i}(x) = 1 - \sum_{i=k+1}^{n} p_{n,i}(1-x) = 1 - \sum_{i=0}^{n-k-1} p_{n,i}(1-x).
\] (3.5)

Using Remark 3.2 we obtain $x_{n,k}^* + x_{n,k,n-k-1}^* \leq \frac{k}{n-1} + \frac{n-k-1}{n-1} = 1$, and it follows that for $x \in \left[0, x_{n,k}^*\right]$ we have

\[
1 - x \geq 1 - x_{n,k}^* \geq x_{n,n-k-1}^* \geq x_{n,i}^*, \quad i \in \{0, 1, \ldots, n - k - 1\},
\] (3.6)

since by Remark 3.2 we have $x_{n,i}^* \leq \frac{i}{n-1} \leq x_{n,i+1}^*$ for all $i \in \{0, 1, \ldots, n - 1\}$.

Using (3.5) and (3.6), together with the monotonicity of $p_{n,i}$ given by Theorem 3.1, it follows that $F_x(k)$ is also decreasing for $x \in \left[0, x_{n,k}^*\right]$, concluding the proof of the theorem.

\[ \square \]

It is known (e.g. [5], p. 4) that the stochastic comparison $X \leq_{st} Y$ is equivalent to $Ef(X) \leq Ef(Y)$ for all increasing functions $f$ for which the expectations exist.

Using this and the definition (1.2) of the operator $R_n$, we can restate the above theorem as follows.

**Theorem 3.4.** The operator $R_n$ defined by (1.2) is a monotone operator, that is, if $f : [0, 1] \to \mathbb{R}$ is monotone increasing (decreasing), then $R_n(f, \cdot) : [0, 1] \to \mathbb{R}$ is also monotone increasing (decreasing).

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