Preparing remotely two instances of quantum state

Ya-fei Yu\textsuperscript{a*}, Jian Feng\textsuperscript{b} and Ming-sheng Zhan\textsuperscript{a}

\textsuperscript{a}State Key Laboratory of Magnetic Resonance and Atomic and Molecular Physics, Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences, Wuhan 430071, PR China

\textsuperscript{b}Institute of Optical Communication, Liaocheng Teachers University, Liaocheng 252059, Shandong, PR China

In this short note, we propose a scheme, in which two instances of an equatorial state (or a polar state) can be remotely prepared in one-shot operation to different receivers with prior entanglement and 1 bit of broadcasting. The trade-off curve between the amount of entanglement and the achievable fidelity is derived.

Key words: remote state preparation, entanglement

PACS number(s): 03.67.HK

I. INTRODUCTION

Quantum entanglement and classical communication are two elementary resources in quantum information field. The trade-off relation between entanglement and classical communication is inspected in many quantum information processing. For example, in quantum teleportation [1] the utilization of quantum entanglement reduces greatly the cost of classical communication. Compared with quantum teleportation, remote state preparation (RSP) [2–7] exhibits the stronger trade-off between the required entanglement and the classical communication cost [4,5]. It is found that in the presence of a large amount of prior entanglement, the asymptotic classical communication cost of RSP for general states is 1 bit per qubit, half that of teleportation. On the other hand, the prior knowledge about quantum state in the process of quantum information reduces the communication cost including entanglement and classical communication. For instance in quantum remote control [8], for saving the communication requirement the teleportation of angles is divided into two cases of rotations by $\pi$ around any axis lying within the equatorial plane and arbitrary rotations around the $z$ axis [9]. As a special case of quantum remote control, the RSP strongly displays the trade-off relation between the restriction and the communication requirement: if only 1 ebit of entanglement is supplied, for general states the classical communication cost in RSP is equivalent to that in standard teleportation [3]; however, for constrained ensemble of states, the one is less than that in quantum teleportation [2,3].

Here we devise a scheme to generalize the RSP to one receiver to the case of multiple receivers. And differently from the previous work [2–7], we investigate the connection between the entanglement and the achievable fidelity under the conditions of certain cost of classical communication and constrained state. For simplicity, we consider the case of two receivers, in which Alice, using prior entanglement pre-shared with Bob and Charlie and broadcasting 1bit of classical information to Bob and Charlie, helps Bob and Charlie to recreate simultaneously an equatorial state (or a polar state) in their respective lab. We derive the trade-off curve between the amount of entanglement and the achievable fidelity. The detail is elaborated in Section II, and the results are summarized in Section III.

II. PREPARING TWO INSTANCES OF QUANTUM STATE REMOTELY

The problem is that Alice, using prior entanglement shared with Bob and Charlie and broadcasting 1 bit classical information to Bob and Charlie, aims to prepare remotely in one-shot operation an instance of the equatorial state to Bob and Charlie in their respective lab. The state to be remotely prepared has the form of $|\varphi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\phi} |1\rangle)$, where $\phi \in (0, 2\pi]$ is known to Alice, but unknown to Bob and Charlie.

According to the quantum cloning for equatorial qubit [10], a deliberate entangled state

\[
|\psi\rangle = \frac{1}{\sqrt{2}} (|0\rangle_a |\phi_1\rangle_{ABC} - |1\rangle_a |\phi_0\rangle_{ABC})
\] (1)

*Corresponding author. E-mail address: yfyu@wipm.ac.cn
is distributed among Alice, Bob and Charlie to prepare remotely two instances of the equatorial state $|\varphi\rangle$. The tripartite states $|\phi_0\rangle$ and $|\phi_1\rangle$ are defined as

$$
|\phi_0\rangle_{ABC} = \frac{1}{\sqrt{2}} (|0\rangle_A|0\rangle_B|0\rangle_C + \frac{1}{2} |1\rangle_A|0\rangle_B|1\rangle_C + \frac{1}{2} |1\rangle_A|1\rangle_B|0\rangle_C),
$$

$$
|\phi_1\rangle_{ABC} = \frac{1}{\sqrt{2}} (|1\rangle_A|1\rangle_B|1\rangle_C + \frac{1}{2} |0\rangle_A|0\rangle_B|1\rangle_C + \frac{1}{2} |0\rangle_A|1\rangle_B|0\rangle_C).
$$

Alice is in possession of the qubits A and B, the qubits B and C belong to Bob and Charlie respectively. For RSP, Alice measures the qubit $a$ in the basis \{ $|\varphi\rangle_a$, $|\varphi^+\rangle_a$ \} where $|\varphi^+\rangle_a$ denotes the state orthogonal to $|\varphi\rangle_a$. The basis \{ $|\varphi\rangle_a$, $|\varphi^+\rangle_a$ \} is related to the old basis \{ $|0\rangle_a$, $|1\rangle_a$ \} in the following manner

$$
|0\rangle_a = \frac{1}{\sqrt{2}} (|\varphi\rangle_a + e^{i\phi} |\varphi^+\rangle_a),
$$

$$
|1\rangle_a = \frac{1}{\sqrt{2}} (e^{-i\phi} |\varphi\rangle_a - |\varphi^+\rangle_a).
$$

Rewriting the entangled state $|\psi\rangle$ in the basis \{ $|\varphi\rangle_a$, $|\varphi^+\rangle_a$ \} gives

$$
|\psi\rangle = \frac{1}{\sqrt{2}} (|0\rangle_a |\phi_1\rangle_{ABC} - |1\rangle_a |\phi_0\rangle_{ABC})
$$

$$
= \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{2}} (|\varphi\rangle_a + e^{i\phi} |\varphi^+\rangle_a) |\phi_1\rangle_{ABC} - \frac{1}{\sqrt{2}} (e^{-i\phi} |\varphi\rangle_a - |\varphi^+\rangle_a) |\phi_0\rangle_{ABC} \right)
$$

$$
= \frac{1}{\sqrt{2}} (|\varphi^+\rangle_a \otimes \frac{1}{\sqrt{2}} (|\phi_0\rangle_{ABC} + e^{i\phi} |\phi_1\rangle_{ABC}) - e^{-i\phi} |\varphi\rangle_a \otimes \frac{1}{\sqrt{2}} (|\phi_0\rangle_{ABC} - e^{i\phi} |\phi_1\rangle_{ABC})).
$$

If the result of the measurement gives $|\varphi^+\rangle_a$, three parties Alice, Bob and Charlie share a tripartite state $|\xi\rangle_{ABC} = \frac{1}{\sqrt{2}} (|\phi_0\rangle_{ABC} + e^{i\phi} |\phi_1\rangle_{ABC})$ as expected. Thus the reduced density matrices on the qubits B and C are given as

$$
\rho_B = Tr_{A,C} (|\xi\rangle_{ABC} \langle \xi |) = \rho_C = Tr_{A,B} (|\xi\rangle_{ABC} \langle \xi |)
$$

$$
= \frac{1}{2} (|0\rangle \langle 0| + |1\rangle \langle 1|) + \frac{1}{\sqrt{2}} (e^{-i\phi} |0\rangle \langle 1| + e^{i\phi} |1\rangle \langle 0|)
$$

$$
= \left( \frac{1}{2} + \frac{1}{2\sqrt{2}} \right) (|\varphi\rangle \langle \varphi|) + \left( \frac{1}{2} - \frac{1}{2\sqrt{2}} \right) (|\varphi^+\rangle \langle \varphi^+|).
$$

In the view of the RSP of single-partite state, this means that Alice assists Bob and Charlie in preparing an approximation to the equatorial qubit state $|\varphi\rangle$, simultaneously and respectively. The fidelity of the approximation with respect to the equatorial state is $\frac{1}{2} + \frac{1}{2\sqrt{2}}$, which can be proved to be optimal in this scheme of preparing remotely two instances of the equatorial state. And the fidelity of $\frac{1}{2} + \frac{1}{2\sqrt{2}}$ is also the optimal one of 1 → 2 cloning of the equatorial state [10].

Otherwise, if the result of the single-partite measurement is $|\varphi\rangle_a$, we get an incorrect tripartite state, from which the correct state $|\xi\rangle_{ABC}$ can be recovered by performing Pauli operator $\sigma_z$ on each of the qubits A, B and C.

On the completion of the single-partite measurement, Alice broadcasts her result of the measurement through public channel to Bob and Charlie. Then depending on the result, the parties determine to rotate respective qubits or do nothing. Consequently, the scheme can be applied to prepare remotely and simultaneously two instances of an equatorial state to different receivers.

It is observed that in the cut of $a : B$ (or $a : C$) of the entangled state $|\psi\rangle$, the relative entropy of entanglement \cite{11,12} $E_r = \min_{\sigma \in D} S(\rho/\sigma) \approx 0.6095 \text{ ebit} < 1 \text{ ebit}$, where $S(\rho/\sigma) = Tr\{\rho (\log \rho - \log \sigma)\}$ is the quantum relative entropy, and the minimum is taken over D, the set of separable states. So the fidelity obtained in our scheme is less than 1 because of the limit of the lower bound to the necessary resources in the perfect RSP.

The scheme also succeeds in preparing remotely a polar state with the form of $|\varphi'\rangle = \cos(\theta) |0\rangle + \sin(\theta) |1\rangle$, where $\theta \in (0, \pi]$ is known only to Alice. Now the deliberate entangled state shared among Alice, Bob and Charlie is previously proposed in [13]. That is, the states $|\phi_0\rangle$ and $|\phi_1\rangle$ in Eq. (1) are defined as

$$
|\Phi_0\rangle_{ABC} = \sqrt{\frac{2}{3}} (|0\rangle_A |0\rangle_B |0\rangle_C + \sqrt{\frac{1}{6}} (|0\rangle_A |1\rangle_B |1\rangle_C + |1\rangle_B |0\rangle_C)
$$

(6)
\[ |\Phi_1\rangle_{ABC} = \sqrt{\frac{2}{3}} |1\rangle_A |1\rangle_B |1\rangle_C + \sqrt{\frac{1}{6}} |0\rangle_A (|0\rangle_B |1\rangle_C + |1\rangle_B |0\rangle_C). \]  

Similarly, when the single-particle measurement gives $|\varphi^+\rangle$, three parties do nothing to their qubit; or they apply a rotation operator $-i\sigma_y$ on their respective qubit to retrieve the desired tripartite state. Resultantly, Bob and Charlie obtain an approximation of $|\varphi'\rangle$ with the maximal fidelity $\frac{3}{\sqrt{2}}$, respectively. In this case, the relative entropy of entanglement in the cut of $a : B$ (or $a : C$) of the entangled state $|\psi\rangle$, $E_r \approx 0.4425$.

We examine further in the RSP of a polar state the relation between the entanglement and the fidelity under 1 bit classical communication, and rewrite the Equations (6) and (7) in general form:

\[ |\Phi_0\rangle_{ABC} = \alpha |0\rangle_A |0\rangle_B |0\rangle_C + \beta |1\rangle_A (|0\rangle_B |1\rangle_C + |1\rangle_B |0\rangle_C), \]

\[ |\Phi_1\rangle_{ABC} = \alpha |1\rangle_A |1\rangle_B |1\rangle_C + \beta |0\rangle_A (|0\rangle_B |1\rangle_C + |1\rangle_B |0\rangle_C), \]

where $\alpha, \beta$ are real numbers, and satisfy $\alpha^2 + 2\beta^2 = 1$. It can be deduced from the isotropy of the procedure that the fidelity $F = \frac{1+\alpha^2}{2}$ and $F_{\text{max}} = \frac{3}{2}$. A simple calculation gives

\[ E_r = \frac{3F-1}{2} \log\left(\frac{3F-1}{2}\right) + \frac{1-F}{2} \log\left(\frac{1-F}{2}\right) - F \log\left(\frac{F}{2}\right). \]

The analytic relation can be plotted in Fig.1. It is obvious that the required amount of entanglement increases with the fidelity in the limit of certain classical communication cost. And it is noticed that the trade-off relation ties in nicely with ideas concerning the relation between the amount of classical information on a quantum state and the available entanglement [14].

We consider another situation of preparing remotely two instances of quantum state at the same location, where Alice holds only qubit $a$ and the qubits $A$, $B$ and $C$ belong to Bob. It requires 1 bit classical communication from Alice to Bob and 1 ebit in the cut of $a : ABC$. Instead, the task can be achieved by the combination of the RSP of one instance of quantum state and the optimal cloning procedure. In the two schemes the required amount of entanglement is equal for the same optimal fidelity and the same 1 bit classical communication.

In addition, there is something interesting in the scheme if we continue to inspect the scheme in view of the RSP of a tripartite state. It is noticed that after the RSP, a one-parameter tripartite entangled state $|\xi\rangle_{ABC}$ is constructed among the three parties Alice, Bob and Charlie. The parameter $\phi \in (0, 2\pi]$ (or $\theta \in (0, \pi]$) is dependent on Alice’s choice. So the scheme can also be used to prepare remotely a one-parameter tripartite state by Alice broadcasting 1 bit to and pre-sharing entanglement with Bob and Charlie.

### III. CONCLUSION

In this short note, we give a scheme in which two instances of an equatorial state (or a polar state) can be remotely prepared to Bob and Charlie by Alice broadcasting 1 bit to Bob and Charlie through public channel and sharing prior entanglement among three parties. The trade-off curve between the amount of entanglement and the achievable fidelity is derived under the conditions of certain classical communication and constrained state. And in view of the RSP of the tripartite state, the scheme can be used to prepare remotely a one-parameter tripartite entangled state by prior entanglement and 1 bit of broadcasting. It gives an example for the statement about the preparation of the tripartite state in [4]. The scheme can be easily generalized to the case of multiple (more than two) receivers by employing more parties entangled state. More generally, it inspires us to study quantum remote control on multiple separate objects, for example, teleportation of unitary operations [8,9] to several different receivers, and gives us an insight into the trade-offs between the communication requirement and the restrictions about operations and quantum states. We hope that it will shed some light on the understanding of the fundamental laws of quantum information processing and the research of quantum communication complexity.

### Acknowledgments

This work has been financially supported by the National Natural Science Foundation of China under the Grant No.10074072.
[1] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. 70, 1895 (1993).
[2] A. K. Pati, Phys. Rev. A 63, 014302 (2001).
[3] H. K. Lo, Phys. Rev. A 62, 012313 (2001).
[4] C. H. Bennett, D. P. Divincenzo, P. W. Shor, J. A. Smolin, B. M. Terhal, and W. K. Wootters, Phys. Rev. Lett. 87 (7), 077902 (2001).
[5] Igor Devetak and Toby Berger, Phys. Rev. Lett. 87 (19), 197901 (2001).
[6] Zheng Yi-Zhuang, Gu Yong-Jian, and Guo Guang-Can, Chin. Phys. Lett 19 (1), 14–19 (2002).
[7] B. S. Shi and A. Tomita, J. Opt. B: Quantum Semiclass. Opt. 4 (6), 380–382 (2002).
[8] S. F. Huelga, J. A. Vaccaro and A. Chefles, Phys. Rev. A 63, 042303 (2001).
[9] S. F. Huelga, M. B. Plenio and J. A. Vaccaro, Phys. Rev. A 65, 042316 (2002).
[10] H. Fan, K. Matsumoto, X. B. Wang, and M. Wadati, Phys. Rev. A 65, 012304 (2001).
[11] V. Vedral and M. B. Plenio, Phys. Rev. A 57 (3), 1619 (1998).
[12] V. Vedral and M. B. Plenio, M. A. Rippin, and P. L. Knight, Phys. Rev. Lett. 78 (12), 2275 (1997).
[13] M. Murao, D. Jonathan, M. B. Plenio, and V. Vedral, Phys. Rev. A 59, 156 (1999).
[14] J. Eisert, T. Felbinger, P. Papadopoulos, M. B. Plenio, and M. Wilkens, Phys. Rev. Lett. 84 (7), 1611 (2000).
Fig. 1 The trade-off curve between the amount of entanglement and the achievable fidelity for the RSP of a polar state is given.