ON THE VOLUME OF DOUBLE TWIST LINK CONE-MANIFOLDS

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ABSTRACT. We consider the double twist link \( J(2m + 1, 2n + 1) \) which is the two-bridge link corresponding to the continued fraction \( (2m + 1) - 1/(2n + 1) \). It is known that \( J(2m + 1, 2n + 1) \) has reducible nonabelian \( SL_2(\mathbb{C}) \)-character variety if and only if \( m = n \). In this paper we give a formula for the volume of hyperbolic cone-manifolds of \( J(2m + 1, 2m + 1) \). We also give a formula for the A-polynomial 2-tuple corresponding to the canonical component of the character variety of \( J(2m + 1, 2m + 1) \).

1. Introduction

For a hyperbolic link \( \mathcal{L} \) in \( S^3 \), let \( E_\mathcal{L} = S^3 \setminus \mathcal{L} \) be the link exterior and let \( \rho_{\text{hol}} \) be a holonomy representation of \( \pi_1(\mathcal{E}_\mathcal{L}) \) into \( \text{PSL}_2(\mathbb{C}) \). Thurston [Th] showed that \( \rho_{\text{hol}} \) can be deformed into an \( \ell \)-parameter family \( \{ \rho_{\alpha_1, \ldots, \alpha_\ell} \} \) of representations to give a corresponding family \( \{ E_\mathcal{L}(\alpha_1, \ldots, \alpha_\ell) \} \) of singular complete hyperbolic manifolds, where \( \ell \) is the number of components of \( \mathcal{L} \). In this paper we consider only the case where all of \( \alpha_j \)'s are equal to a single parameter \( \alpha \). In which case we also denote \( E_\mathcal{L}(\alpha_1, \ldots, \alpha_\ell) \) by \( E_\mathcal{L}(\alpha) \). These \( \alpha \)'s and \( E_\mathcal{L}(\alpha) \)'s are called the cone-angles and hyperbolic cone-manifolds of \( \mathcal{L} \), respectively.

We consider the complete hyperbolic structure on a link complement as the cone-manifold structure with cone-angle zero. It is known that for a two-bridge link \( \mathcal{L} \) there exists an angle \( \alpha_\mathcal{L} \in \left[ \frac{2\pi}{3}, \pi \right) \) such that \( E_\mathcal{L}(\alpha) \) is hyperbolic for \( \alpha \in (0, \alpha_\mathcal{L}) \), Euclidean for \( \alpha = \alpha_\mathcal{L} \), and spherical for \( \alpha \in (\alpha_\mathcal{L}, \pi) \) [HLM, Ko1, Po, PW]. A method for computing the volume of hyperbolic cone-manifolds of links was outlined in [HLM], and explicit volume formulas have been known for hyperbolic cone-manifolds of the links \( 5_2, 6_2, 6_3, 7_2 \) (see [HLMR] and references therein) and of twisted Whitehead links [Tr].

For integers \( m \) and \( n \), consider the double twist link \( J(2m + 1, 2n + 1) \) which is the two-bridge link corresponding to the continued fraction \( (2m + 1) - 1/(2n + 1) \) (see Figure 1). It was shown by Petersen and the author [PT] that \( J(2m + 1, 2n + 1) \) has reducible nonabelian \( SL_2(\mathbb{C}) \)-character variety if and only if \( m = n \). In this paper we are interested in the double twist link \( \mathcal{L}_m = J(2m + 1, 2m + 1) \), since the canonical component of the character variety of \( \mathcal{L}_m \) has a rather nice form (see Remark 3.4). Here a canonical component of the character variety of a hyperbolic link \( \mathcal{L} \) is a component containing the character of a lift of a holonomy representation of \( \pi_1(\mathcal{E}_\mathcal{L}) \) to \( SL_2(\mathbb{C}) \).

Let \( \{ S_j(v) \}_{j \in \mathbb{Z}} \) be the sequence of Chebyshev polynomials of the second kind defined by \( S_0(v) = 1 \), \( S_1(v) = v \) and \( S_j(v) = vS_{j-1}(v) - S_{j-2}(v) \) for all integers \( j \). Let

\[
R_{\mathcal{L}_m}(s, z) = (s^2 + s^{-2} + 2 - z)(S_m^2(z) + S_{m-1}^2(z)) - 2(s^2 + s^{-2})S_m(z)S_{m-1}(z) - z.
\]

The volume of the hyperbolic cone-manifold of \( \mathcal{L}_m \) is computed as follows.

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Figure 1. The double twist link $J(2m+1, 2n+1)$. Here $2m+1$ and $2n+1$ denote the numbers of half twists in the boxes. Positive (resp. negative) numbers correspond to right-handed (resp. left handed) twists.

**Theorem 1.1.** For $\alpha \in (0, \alpha_L^m)$ we have

$$\text{Vol} E_{\mathcal{L}_m}(\alpha) = \int_{\alpha}^{\pi} \log \left| \frac{S_m(z) - e^{-i\omega}S_{m-1}(z)}{S_m(z) - e^{i\omega}S_{m-1}(z)} \right| d\omega$$

where $z$, with $\text{Im}(S_{m-1}(z)\overline{S_m(z)}) \geq 0$, is a certain root of $R_{\mathcal{L}_m}(e^{i\omega/2}, z) = 0$.

Note that the above volume formula for the hyperbolic cone-manifold $E_{\mathcal{L}_m}(\alpha)$ depends on the choice of a root $z$, with $\text{Im}(S_{m-1}(z)\overline{S_m(z)}) \geq 0$, of $R_{\mathcal{L}_m}(e^{i\omega/2}, z) = 0$. In numerical approximations, we choose the root $z$ which gives the maximal volume.

It is known that the volume of the $k$-fold cyclic covering over a hyperbolic link $\mathcal{L}$ is $k$ times the volume of the hyperbolic cone-manifold of $\mathcal{L}$ with cone-angle $2\pi/k$. As a direct consequence of Theorem 1.1, we obtain the following.

**Corollary 1.2.** The hyperbolic volume of the $k$-fold cyclic covering over the two-bridge link $\mathcal{L}_m$, with $k \geq 3$, is given by the following formula

$$k \text{Vol} E_{\mathcal{L}_m}(\frac{2\pi}{k}) = k \int_{\frac{2\pi}{k}}^{\pi} \log \left| \frac{S_m(z) - e^{-i\omega}S_{m-1}(z)}{S_m(z) - e^{i\omega}S_{m-1}(z)} \right| d\omega$$

where $z$, with $\text{Im}(S_{m-1}(z)\overline{S_m(z)}) \geq 0$, is a certain root of $R_{\mathcal{L}_m}(e^{i\omega/2}, z) = 0$.

The A-polynomial of a knot in $S^3$ was introduced by Cooper, Culler, Gillet, Long and Shalen [CCGLS] in the 90’s. It describes the $SL_2(\mathbb{C})$-character variety of the knot complement as viewed from the boundary torus. The A-polynomial carries a lot of information about the topology of the knot. For example, the sides of the Newton polygon of the A-polynomial of a knot in $S^3$ give rise to incompressible surfaces in the knot complement [CCGLS]. A generalization of the A-polynomial to links in $S^3$ was proposed by Zhang [Zh]. For an $\ell$-component link in $S^3$, Zhang defined a polynomial $\ell$-tuple link invariant called the A-polynomial $\ell$-tuple. The A-polynomial 1-tuple of a knot is just its A-polynomial. The A-polynomial $\ell$-tuple also carries important information about the topology of the link. For example, it can be used to construct concrete examples of hyperbolic link manifolds with non-integral traces [Zh].

The A-polynomial 2-tuple has been computed for a family of two bridge links called twisted Whitehead links [Tr]. In this paper we compute the A-polynomial 2-tuple for the canonical component of the character variety of $\mathcal{L}_m = J(2m + 1, 2m + 1)$. 
Theorem 1.3. Let \( \{Q_j(s,w)\}_{j \in \mathbb{Z}} \) be the sequence of polynomials in two variables \( s, w \) defined by \( Q_{-1} = Q_0 = 2 \) and
\[
Q_j = \alpha Q_{j-1} - Q_{j-2} + \beta
\]
where
\[
\alpha = (s^8 + s^4)w^4 + (-2s^8 + 6s^6 + 6s^4 - 2s^2)w^3 + (s^8 - 12s^6 + 34s^4 - 12s^2 + 1)w^2 \\
+ (-2s^6 + 6s^4 + 6s^2 - 2)w + s^4 + 1,
\]
\[
\beta = -2(s^2 - 1)^2 (s^4 - s^2)w^3 - 6s^2w^2 - (s^2 + 1)w + 1.
\]
Then the A-polynomial 2-tuple corresponding to the canonical component of the character variety of \( L_m \) is \([A(M, L), A(M, L)]\) where \( A(M, L) = (L - 1)Q_m(M, LM^{2m})\).

The paper is organized as follows. In Section 2 we review the definition of the A-polynomial \( \ell \)-tuple of an \( \ell \)-component link in \( S^3 \). In Section 3 we compute the nonabelian \( SL_2(\mathbb{C}) \)-representations of the double twist link \( J(2m+1, 2n+1) \). In Section 4 we compute the volume of hyperbolic cone-manifolds of \( L_m = J(2m+1, 2m+1) \) and give a proof of Theorem 1.1. The last section is devoted to the computation of the A-polynomial 2-tuple for the canonical component of the character variety of \( L_m \) and a proof of Theorem 1.3.

2. The A-polynomial \( \ell \)-tuple of a link

2.0.1. Character varieties. The set of characters of representations of a finitely generated group \( G \) into \( SL_2(\mathbb{C}) \) is known to be a algebraic set over \( \mathbb{C} \) if \( G \) is a free abelian group of rank two. An orientation of \( K \) is a choice of an oriented meridian \( \mu \) and an oriented longitude \( \lambda \) such that the linking number between the longitude \( \lambda_j \) and the knot \( K_j \) is 0. The pair provides an identification of \( \chi(\pi_1(T_j)) \) and \( (\mathbb{C}^*)^2/\tau_j \), where \( (\mathbb{C}^*)^2 \) is the set of non-zero complex pairs \( (M, L) \) and \( \tau_j \) is the involution \( \tau(M_j, L_j) = (M_j^{-1}, L_j^{-1}) \), which actually does not depend on the orientation of \( K_j \).

The inclusion \( T_j \hookrightarrow E_L \) induces the restriction map
\[
\rho_j : \chi(\pi_1(E_L)) \rightarrow \chi(\pi_1(T_j)) \equiv (\mathbb{C}^*)^2/\tau_j.
\]
For each \( \gamma \in \pi_1(E_L) \) let \( f_\gamma \) be the regular function on \( \chi(\pi_1(E_L)) \) defined by
\[
f_\gamma(\chi_\rho) = (\chi_\rho(\gamma))^2 - 4 = (\text{tr} \rho(\gamma))^2 - 4,
\]
where \( \chi_\rho \) denotes the character of a representation \( \rho : \pi_1(E_L) \rightarrow SL_2(\mathbb{C}) \). Let \( \chi_j(\pi_1(E_L)) \) be the subvariety of \( \chi(\pi_1(E_L)) \) defined by \( f_{\rho_k} = 0 \), \( f_{\rho_j} = 0 \) for all \( k \neq j \). Let \( Z_j \) be the image of \( \chi_j(\pi_1(E_L)) \) under \( \rho_j \) and \( \check{Z}_j \subset (\mathbb{C}^*)^2 \) the lift of \( Z_j \) under the projection \( (\mathbb{C}^*)^2 \rightarrow (\mathbb{C}^*)^2/\tau_j \). It is known that the Zariski closure of \( \check{Z}_j \subset (\mathbb{C}^*)^2 \subset \mathbb{C}^2 \) in \( \mathbb{C}^2 \) is an
algebraic set consisting of components of dimension 0 or 1 [Zh]. The union of all the 1-dimension components is defined by a single polynomial $A_j \in \mathbb{Z}[M_j, L_j]$ whose coefficients are co-prime. Note that $A_j$ is defined up to $\pm 1$. We will call $[A_1(M_1, L_1), \ldots, A_t(M_t, L_t)]$ the A-polynomial $t$-tuple of $\mathcal{L}$. For brevity, we also write $A_j(M, L)$ for $A_j(M_j, L_j)$. We refer the reader to [Zh] for properties of the A-polynomial $t$-tuple.

3. Double twist links $J(2m + 1, 2n + 1)$

In this section we compute nonabelian $SL_2(\mathbb{C})$-representations of the double twist link $J(2m + 1, 2n + 1)$. They are described by the Chebyshev polynomials of the second kind, and so we first recall some properties of these polynomials.

3.1. Chebyshev polynomials. Recall that $\{S_j(v)\}_{j \in \mathbb{Z}}$ is the sequence of the Chebyshev polynomials of the second kind defined by $S_0(v) = 1$, $S_1(v) = v$ and $S_j(v) = vS_{j-1}(v) - S_{j-2}(v)$ for all integers $j$. The following two lemmas are elementary, see e.g. [Tr].

Lemma 3.1. For any integer $j$ we have
\[ S_j^2(v) + S_{j-1}^2(v) - vS_j(v)S_{j-1}(v) = 1. \]

Lemma 3.2. Suppose $V \in SL_2(\mathbb{C})$ and $v = tr V$. For any integer $j$ we have
\[ V^j = S_j(v)1 - S_{j-1}(v)V^{-1} \]
where 1 denotes the $2 \times 2$ identity matrix.

We will need the following lemma in the last section of the paper.

Lemma 3.3. For any integer $j$ we have
\[ S_j(z)S_{j-1}(z) = (z^2 - 2)S_{j-1}(z)S_{j-2}(z) - S_{j-2}(z)S_{j-3}(z) + z. \]

Proof. We have
\[
S_j(z)S_{j-1}(z) + S_{j-2}(z)S_{j-3}(z) \\
= (zS_{j-1}(z) - S_{j-2}(z))S_{j-1}(z) + S_{j-2}(z)(zS_{j-2}(z) - S_{j-1}(z)) \\
= z(S_{j-1}^2(z) + S_{j-2}^2(z)) - 2S_{j-1}(z)S_{j-2}(z).
\]
The lemma follows, since $S_{j-1}^2(z) + S_{j-2}^2(z) = 1 + zS_{j-1}(z)S_{j-2}(z)$ by Lemma 3.1. \qed

3.2. Nonabelian representations. In this subsection we study representations of link groups into $SL_2(\mathbb{C})$. A representation is called nonabelian if its image is a nonabelian subgroup of $SL_2(\mathbb{C})$. Let $\mathcal{L} = J(2m + 1, 2n + 1)$ and $\mathcal{L} = S^3 \setminus \mathcal{L}$ the link exterior. By [PT] (and [MPL]) also the link group of $\mathcal{L}$ has a two-generator presentation
\[ \pi_1(\mathcal{L}) = \langle a, b \mid aw = wa \rangle, \]
where $w = (b^{-1}a)^m[(b^{-1}a)^mba(b^{-1}a)^m]^{-n}$ and $a, b$ are meridians depicted in Figure 1.

Suppose $\rho : \pi_1(\mathcal{L}) \to SL_2(\mathbb{C})$ is a nonabelian representation. Up to conjugation, we may assume that
\[ \rho(a) = \begin{bmatrix} s_1 & 1 \\ 0 & s_1^{-1} \end{bmatrix} \quad \text{and} \quad \rho(b) = \begin{bmatrix} s_2 & 0 \\ u & s_2^{-1} \end{bmatrix} \]
where $(u, s_1, s_2) \in (\mathbb{C}^*)^3$ satisfies the matrix equation $\rho(aw) = \rho(wa)$. For any word $v$ in 2 letters $a$ and $b$, we write $\rho(v) = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$. Then, by Riley [Ri], $w_{12}$ can be written
as \( w_{21} = uw'_{21} \) for some \( w'_{12} \in \mathbb{C}[s_1^{\pm 1}, s_2^{\pm 1}, u] \) and the matrix equation \( \rho(aw) = \rho(wa) \) is equivalent to the single equation \( w'_{12} = 0 \). We call \( w'_{12} \) the Riley polynomial of \( \mathcal{L} \).

We now compute \( w'_{12} \) explicitly. Let \( x = \text{tr} \rho(a) = s_1 + s_1^{-1} \), \( y = \text{tr} \rho(b) = s_2 + s_2^{-1} \) and \( z = \text{tr} \rho(ab^{-1}) = s_1s_2^{-1} + s_1^{-1}s_2 - u \).

Let \( c = (b^{-1}a)^m \) and \( d = (ba^{-1}m)ba(b^{-1}a)^m = bc^{-1}ac \). Then \( w = cd^n \). Since

\[
\rho(b^{-1}a) = \begin{bmatrix} s_1s_2^{-1} & s_2^{-1} \\ -s_1u & s_1^{-1}s_2 - u \end{bmatrix},
\]

by Lemma 3.2 we have \( \rho(c) = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \) where

\[
\begin{align*}
c_{11} &= S_m(z) - (s_1^{-1}s_2 - u)S_{m-1}(z), \\
c_{12} &= s_2^{-1}S_{m-1}(z), \\
c_{21} &= -s_1us_{m-1}(z), \\
c_{22} &= S_m(z) - s_1s_2^{-1}S_{m-1}(z).
\end{align*}
\]

By a direct computation we then have \( \rho(d) = \rho(bc^{-1}ac) = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \) where

\[
\begin{align*}
d_{11} &= s_1s_2S_m^2(z) - (s_1^2 + s_2^2)S_m(z)S_{m-1}(z) + (s_1s_2 + u)S^2_{m-1}(z), \\
d_{12} &= s_2S_m^2(z) - (s_1 + s_1^{-1})S_m(z)S_{m-1}(z) + s_2^{-1}S^2_{m-1}(z), \\
d_{21} &= u(s_1S_m^2(z) - (s_2 + s_2^{-1})S_m(z)S_{m-1}(z) + s_1^{-1}S^2_{m-1}(z)), \\
d_{22} &= (s_1^{-1}s_2^{-1} + u)S_m^2(z) - (s_1^{-2} + s_2^{-2})S_m(z)S_{m-1}(z) + s_1^{-1}s_2^{-1}S^2_{m-1}(z).
\end{align*}
\]

Let \( t = \text{tr} \rho(d) \). From the above computations we have

\[
\begin{align*}
t &= (s_1s_2 + s_1^{-1}s_2^{-1} + u)(S_m^2(z) + S_{m-1}^2(z)) - (s_1^2 + s_1^{-2} + s_2^2 + s_2^{-2})S_m(z)S_{m-1}(z) \\
&= (xy - z)(S_m^2(z) + S_{m-1}^2(z)) - (x^2 + y^2 - 4)S_m(z)S_{m-1}(z).
\end{align*}
\]

Since \( w = cd^n \), by Lemma 3.2 we have

\[
\rho(w) = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} S_n(t) - d_{22}S_{n-1}(t) & d_{12}S_{n-1}(t) \\ d_{21}S_{n-1}(t) & S_n(t) - d_{11}S_{n-1}(t) \end{bmatrix}.
\]

With \( \rho(w) = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} \) we obtain

\[
\begin{align*}
w_{11} &= c_{11}(S_n(t) - d_{22}S_{n-1}(t)) + c_{12}d_{21}S_{n-1}(t), \\
w_{21} &= c_{21}(S_n(t) - d_{22}S_{n-1}(t)) + c_{22}d_{21}S_{n-1}(t).
\end{align*}
\]

By direct computations we have \( w_{21} = us_1(S_m(z)S_{n-1}(t) - S_{n-1}(z)S_n(t)) \) and

\[
\begin{align*}
w_{11} &= -S_{n-1}(t)\{(s_1s_2^{-1} + s_1^{-1}s_2 + s_1^{-1}s_2^{-1} - z)S_m(z) - s_1^{-2}S_{m-1}(z) \\
&+ S_n(t)(S_m(z) + (s_1s_2^{-1} - z)S_{n-1}(z)) \}.
\end{align*}
\]

Hence, the Riley polynomial of \( \mathcal{L} = J(2m + 1, 2n + 1) \) is

\[
w'_{21} = S_m(z)S_{n-1}(t) - S_{n-1}(z)S_n(t).
\]
It determines the nonabelian $SL_2(\mathbb{C})$-character variety of $\mathcal{L}$, which is essentially the set of all nonabelian representations $\rho: \pi_1(E_\mathcal{L}) \to SL_2(\mathbb{C})$ up to conjugation. Moreover, for any nonabelian representation $\rho$ of the form \((3.1)\) we have $\rho(w) = \begin{bmatrix} w_{11} & * \\ 0 & (w_{11})^{-1} \end{bmatrix}$ where

\[
(3.2) \quad w_{11} = -S_{n-1}(t) \{(s_1^{-1}s_2 + s_1^{-1}s_2^{-1})S_m(z) - s_1^{-2}S_{m-1}(z)\} + S_n(t)S_m(z).
\]

Let $\overline{w}$ is the word obtained from $w$ by exchanging $a$ and $b$, namely $\overline{w} = (a^{-1}b)^m [(ab^{-1})^m ab(a^{-1}b)^m]^n.$ It is easy to see that the equation $aw = wa$ is equivalent to $\overline{w}b = b\overline{w}$. Moreover, for any nonabelian representation $\rho$ of the form \((3.1)\) we have $\rho(\overline{w}) = \begin{bmatrix} \overline{w}_{11} & 0 \\ * & (\overline{w}_{11})^{-1} \end{bmatrix}$ where

\[
(3.3) \quad \overline{w}_{11} = -S_{n-1}(t) \{(s_1s_2^{-1} + s_1^{-1}s_2^{-1})S_m(z) - s_2^{-2}S_{m-1}(z)\} + S_n(t)S_m(z).
\]

**Remark 3.4.** The above formula for the nonabelian $SL_2(\mathbb{C})$-character variety of the double twist link $\mathcal{L} = J(2m + 1, 2n + 1)$ was already obtained in [PT] by a different method. Moreover, it was also shown in [PT] that the nonabelian character variety of $\mathcal{L}$ is reducible if and only if $m = n$. In which case, it has exactly 2 irreducible components and the canonical component is determined by the equation $t = z$.

From now on we consider only the double twist link $\mathcal{L}_m = J(2m + 1, 2m + 1)$, where $m \neq -1, 0$. As mentioned above, the canonical component of the character variety of $\mathcal{L}_m$ is given by the equation $t = z$ where

\[
(3.4) \quad t = (xy - z)\Big((S_m^2(z) + S_{m-1}^2(z)) - (x^2 + y^2 - 4)S_m(z)S_{m-1}(z)\Big).
\]

4. **Volume of hyperbolic cone-manifolds of $\mathcal{L}_m$**

Recall that $E_{\mathcal{L}_m}(\alpha)$ is the cone-manifold of $\mathcal{L}_m$ with cone angles $\alpha_1 = \alpha_2 = \alpha$. There exists an angle $\alpha_{\mathcal{L}_m} \in [\frac{2\pi}{3}, \pi]$ such that $E_{\mathcal{L}_m}(\alpha)$ is hyperbolic for $\alpha \in (0, \alpha_{\mathcal{L}_m})$, Euclidean for $\alpha = \alpha_{\mathcal{L}_m}$, and spherical for $\alpha \in (\alpha_{\mathcal{L}_m}, \pi)$.

For $\alpha \in (0, \alpha_{\mathcal{L}_m})$, by the Schlafli formula we have

$$
\text{Vol } E_{\mathcal{L}_m}(\alpha) = \int_0^{\pi} 2\log |w_{11}| \, d\omega
$$

where $w_{11}$ is the $(1,1)$-entry of the matrix $\rho(w)$ and $\rho: \pi_1(\mathcal{L}_m) \to SL_2(\mathbb{C})$ is a representation of the form \((3.1)\) such that the following 3 conditions hold:

(i) $s_1 = s_2 = s = e^{i\omega/2}$,
(ii) the character $\chi_\rho$ of $\rho$ lies on the canonical component of the character variety of $\mathcal{L}_m$,
(iii) $|w_{11}| \geq 1$.

We refer the reader to [HLM], [HLMR] and references therein for the volume formula of hyperbolic cone-manifolds of links using the Schlafli formula.

We now simplify $w_{11}$ for representations $\rho$ of the form \((3.1)\) satisfying the conditions (i)–(iii). Consider the canonical component $t = z$ of the character variety of $\mathcal{L}_m$. With $s_1 = s_2 = s = e^{i\omega/2}$, equation \((3.2)\) implies that

$$
w_{11} = -S_{m-1}(z)\{(1 + s^{-2})S_m(z) - s^{-2}S_{m-1}(z)\} + S_m^2(z)
= (S_m(z) - S_{m-1}(z))(S_m(z) - s^{-2}S_{m-1}(z)).$$
Moreover, the equation \( t = z \) can be written as

\[
(s^2 + s^{-2} + 2 - z)(S_m^2(z) + S_{m-1}^2(z)) - 2(s^2 + s^{-2})S_m(z)S_{m-1}(z) = z.
\]

This, together with \( S_m^2(z) + S_{m-1}^2(z) = 1 + zS_m(z)S_{m-1}(z) \) (by Lemma 3.2), implies that

\[
S_m(z)S_{m-1}(z) = \frac{2z - (s^2 + s^{-2} + 2)}{(z - 2)(s^2 + s^{-2} - z)},
\]

\[
S_m^2(z) + S_{m-1}^2(z) = \frac{z^2 - 2(s^2 + s^{-2})}{(z - 2)(s^2 + s^{-2} - z)}.
\]

Then \((S_m(z) - S_{m-1}(z))^2 = S_m^2(z) + S_{m-1}^2(z) - 2S_m(z)S_{m-1}(z) = \frac{z^2 - 2}{s^2 + s^{-2} - z}\) and

\[
(S_m(z) - s^2S_{m-1}(z))(S_m(z) - s^{-2}S_{m-1}(z)) = S_m^2(z) + S_{m-1}^2(z) - (s^2 + s^{-2})S_m(z)S_{m-1}(z)
\]

\[
= \frac{s^2 + s^{-2} - z}{z - 2}.
\]

It follows that \((S_m(z) - S_{m-1}(z))^2 (S_m(z) - s^2S_{m-1}(z))(S_m(z) - s^{-2}S_{m-1}(z)) = 1\) and

\[
w_{11}^2 = (S_m(z) - S_{m-1}(z))^2 (S_m(z) - s^{-2}S_{m-1}(z))^2 = \frac{S_m(z) - s^{-2}S_{m-1}(z)}{S_m(z) - s^2S_{m-1}(z)}.
\]

Note that \(|S_m(z) - e^{-i\omega}S_{m-1}(z)| \geq |S_m(z) - e^{i\omega}S_{m-1}(z)|\) if and only if \(\text{Im}(S_{m-1}(z)S_m(z)) \geq 0\). Hence, for \(\alpha \in (0, \alpha_{L_m})\), by the Schlaefli formula we have

\[
\text{Vol } E_{L_m}(\alpha) = \int_{\alpha}^{\pi} 2 \log |w_{11}| \, d\omega = \int_{\alpha}^{\pi} \log \left| \frac{S_m(z) - s^{-2}S_{m-1}(z)}{S_m(z) - s^2S_{m-1}(z)} \right| \, d\omega
\]

where \(s = e^{i\omega/2}\) and \(z\), with \(\text{Im}(S_{m-1}(z)S_m(z)) \geq 0\), satisfy

\[
(s^2 + s^{-2} + 2 - z)(S_m^2(z) + S_{m-1}^2(z)) - 2(s^2 + s^{-2})S_m(z)S_{m-1}(z) - z = 0.
\]

This completes the proof of Theorem 1.1.

5. The A-polynomial 2-tuple of \(L_m\)

The canonical longitudes corresponding to the meridians \(a\) and \(b\) of \(J(2m + 1, 2n + 1)\) are respectively \(\lambda_a = wa^{-2n}\) and \(\lambda_b = wb^{-2n}\), where \(w = (a^{-1}b)^m[(ab^{-1})^m(ab(a^{-1}b)^m)^n]^{2n}\) is the word obtained from \(w\) by exchanging \(a\) and \(b\).

Consider the canonical component \(t = z\) of the character variety of \(L_m = J(2m + 1, 2m + 1)\). To compute the A-polynomial 2-tuple for this component, we first consider a representation \(\rho : \pi_1(L_m) \to SL_2(\mathbb{C})\) of the form (3.1) and find a polynomial relating \(s_1\) and \(w_{11}\) when both \(t = z\) and \(s_2 = (w_{11})^2 = 1\) occur. Recall from Subsection 3.2 that \(w_{11}\) and \(\bar{w}_{11}\) are upper left entries of \(\rho(w)\) and \(\rho(\bar{w})\) respectively.

With \(t = z\) and \(s_2 = 1\), by equations (3.2) and (3.3) we have

\[
w_{11} = -S_{m-1}(m)\{2s_1^{-1}S_m(z) - s_1^{-2}S_{m-1}(z)\} + S_m^2(z)
\]

\[
= (S_m(z) - s_1^{-1}S_{m-1}(z))^2
\]

and

\[
\bar{w}_{11} = -S_{m-1}(z)\{(s_1 + s_1^{-1})S_m(z) - S_{m-1}(z)\} + S_m^2(z).
\]

\[
= (S_m(z) - s_1S_{m-1}(z))(S_m(z) - s_1^{-1}S_{m-1}(z)).
\]
Moreover, since $S_m^2(z) + S_{m-1}^2(z) = 1 + zS_m(z)S_{m-1}(z)$, the equation $t = z$ becomes

$$0 = (2x - z)(S_m^2(z) + S_{m-1}^2(z)) - x^2S_m(z)S_{m-1}(z) - z$$

$$= (2x - z)(1 + zS_m(z)S_{m-1}(z)) - x^2S_m(z)S_{m-1}(z) - z$$

$$= (x - z)(2 + (z - x)S_m(z)S_{m-1}(z)).$$

Suppose $z - x = 0$. Then $w_{11} = -S_{m-1}(z)\{zS_m(z) - S_{m-1}(z)\} + S_m^2(z) = 1$ and

$$w_{11} = (S_m(x) - s^{-1}S_{m-1}(x))^2 = s^{2m}.$$ 

Here we use the fact that $S_j(s_1 + s_1^{-1}) = (s_1^{j+1} - s_1^{-j-1})/(s_1 - s_1^{-1})$ for all integers $j$.

Suppose $2 + (z - x)S_m(z)S_{m-1}(z) = 0$. This is equivalent to

$$w_{11} = (S_m(z) - s_1^{-1}S_{m-1}(z)) = 1,$$

since $S_m^2(z) + S_{m-1}^2(z) = 1 + zS_m(z)S_{m-1}(z)$. It follows that $w_{11} = -1$ and

$$w_{11} = (S_m(z) - s_1^{-1}S_{m-1}(z))^2 = -S_m(z) - s_1^{-1}S_{m-1}(z).$$

Hence $S_m(z) = rS_{m-1}(z)$ where $r = \frac{s_1 w_{11} + s_1^{-1}}{w_{11} + 1}$. We have

$$1 = S_m^2(z) + S_{m-1}^2(z) - zS_m(z)S_{m-1}(z) = S_{m-1}^2(z)(1 - zr + r^2),$$

which implies that $S_{m-1}^2(z) = (1 - zr + r^2)^{-1}$. Equation (5.1) then becomes

$$-1 = S_{m-1}^2(z)(r - s_1)(r - s_1^{-1}) = (r - s_1)(r - s_1^{-1})/(1 - zr + r^2).$$

By solving for $z$ from the above equation, we obtain

$$z = 2\left(r + \frac{1}{r}\right) - (s_1 + s_1^{-1}) = 2\left(\frac{s_1 w_{11} + s_1^{-1}}{w_{11} + 1} + \frac{w_{11} + 1}{s_1 w_{11} + s_1^{-1}}\right) - (s_1 + s_1^{-1}).$$

Now, by plugging this expression of $z$ into the equation $2 + (z - x)S_m(z)S_{m-1}(z) = 0$ we obtain a polynomial (depending on $m$) relating $s_1$ and $w_{11}$. Moreover, we can find a recurrence relation between these polynomials as follows.

Let $P_m(x, z) = 2 + (z - x)S_m(z)S_{m-1}(z)$. By Lemma 3.3 we have $S_m(z)S_{m-1}(z) = (z^2 - 2)S_{m-1}(z)S_{m-2}(z) - S_{m-2}(z)S_{m-3}(z) + z$. This implies that

$$P_m = 2 + (z^2 - 2)(P_{m-1} - 2) - (P_{m-2} - 2) + z(z - x)$$

$$= (z^2 - 2)P_{m-1} - P_{m-2} + 8 - z(z + x).$$

Let $Q_m(s_1, w_{11}) = s_1^2(w_{11} + 1)^2(s_1^2 w_{11} + 1)^2P_m(x, z)$. By replacing

$$z = 2\left(\frac{s_1 w_{11} + s_1^{-1}}{w_{11} + 1} + \frac{w_{11} + 1}{s_1 w_{11} + s_1^{-1}}\right) - (s_1 + s_1^{-1})$$

into the above recurrence relation for $P_m$ we have

$$Q_m = \alpha Q_{m-1} - Q_{m-2} + \beta$$

where

$$\alpha = (s_1^8 + s_1^4)w_{11}^4 + (-2s_1^8 + 6s_1^6 + 6s_1^4 - 2s_1^2)w_{11}^3 + (s_1^8 - 12s_1^6 + 34s_1^4 - 12s_1^2 + 1)w_{11}^2$$

$$+ (-2s_1^6 + 6s_1^4 + 6s_1^2 - 2)w_{11} + s_1^4 + 1,$$

$$\beta = -2(s_1^4 - 1)^2(s_1^4 w_{11}^4 - (s_1^4 + s_1^2)w_{11}^3 - 6s_1^2 w_{11}^2 - (s_1^2 + 1)w_{11} + 1).$$
We have shown that \((\overline{w}_{11})^2 = 1\) and \((w_{11} - s_{12}^2m)Q(s_1, w_{11}) = 0\) when both \(t = z\) and \(s_2 = 1\) occur. The same holds true when both \(t = z\) and \(s_2 = -1\) occur. This implies that \((w_{11} - s_{12}^2m)Q(s_1, w_{11}) = 0\) when both \(t = z\) and \(s_2 = (\overline{w}_{11})^2 = 1\) occur.

Similarly, we have \((\overline{w}_{11} - s_{22}^2m)Q(s_2, \overline{w}_{11}) = 0\) when both \(t = z\) and \(s_1 = (\overline{w}_{11})^2 = 1\) occur. Since the canonical longitudes corresponding to the meridians \(a\) and \(b\) of \(L_m = J(2m + 1, 2m + 1)\) are respectively \(\lambda_a = w_0^a - 2m\) and \(\lambda_b = \overline{w}_b - 2m\), we conclude that the A-polynomial 2-tuple corresponding to the canonical component of the character variety of \(L_m\) is \([A(M, L), A(M, L)]\) where \(A(M, L) = (L - 1)Q_m(M, LM^{2m})\).

This completes the proof of Theorem 1.3.

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