ON FUJITA’S FREENESS CONJECTURE
FOR 3-FOLDS AND 4-FOLDS

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Abstract. We shall prove a conjecture of T. Fujita on the freeness of the adjoint linear systems in some cases: Let $X$ be a smooth projective variety of dimension $n$ and $H$ an ample divisor. Assume that $n = 3$ or $4$. Then $|K_X + mH|$ is free if $m \geq n + 1$. Moreover, we obtain more precise result in the case $n = 3$.

Introduction

T. Fujita raised the following:

Conjecture 1. Let $X$ be a smooth projective variety of dimension $n$ and $H$ an ample divisor. Then $|K_X + mH|$ is free if $m \geq n + 1$. Moreover, if $(H^n) \geq 2$, then $|K_X + nH|$ is also free. □

In the case $n = 3$, Ein and Lazarsfeld [EL1] gave an affirmative answer to the first part of Conjecture 1 and Fujita [F] the second part. A stronger version of Fujita’s freeness conjecture is the following:

Conjecture 2. Let $X$ be a normal projective variety of dimension $n$, $x_0 \in X$ a smooth point, and $L$ an ample Cartier divisor. Assume that there exist positive numbers $\sigma_p$ for $p = 1, 2, \ldots, n$ which satisfy the following conditions:

1. $\sqrt{(L^p \cdot W)} \geq \sigma_p$ for any subvariety $W$ of dimension $p$ which contains $x_0$,
2. $\sigma_p \geq n$ for all $p$ and $\sigma_n > n$.

Then $|K_X + L|$ is free at $x_0$. □

In the case $n = 3$, [F] proved that, if $\sigma_1 \geq 3, \sigma_2 \geq \sqrt{7}$ and $\sigma_3 \geq \sqrt{51}$, then $|K_X + L|$ is free at $x_0$. In an arbitrary dimension, Angehrn and Siu [AS] proved a weaker result that, if $\sigma_p > \frac{1}{2}n(n + 1)$ for any $p$, then $|K_X + L|$ is free at $x_0$. In particular, $|K_X + mH|$ is free if $m \geq \frac{1}{2}n(n + 1) + 1$ in the situation of Conjecture 1. Parallel arguments are possible in differential geometry and algebraic geometry; this proof is translated to algebraic geometry by Kollár. There is also a paper by Tsuji [T]. A paper of Smith [Sm] suggests that, even if $X$ is singular, we should have the spannedness of the reflexive sheaf $\mathcal{O}_X(K_X + L)$ under certain conditions.

We shall prove the following results in this paper:

1. (Theorem 3.1): In the case $n = 3$, Conjecture 2 is true.
2. (Theorem 4.1): In the case $n = 4$, if $\sigma_p \geq 5$ for all $p$, then $|K_X + L|$ is free at $x_0$. In particular, the first part of Conjecture 1 is true.

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In §§1 and 2, we shall explain the general strategy toward the freeness results. This is an application of the vanishing theorem of [K1] and [V], and has the origin in the proof of the base point free theorem ([K2], [Sh1]). The adjoint linear system appears naturally in the course of the proof. We note that there exists no universal bound of $m_0$ as a function of $n$ such that $|mH|$ is free if $m \geq m_0$. This justifies that we ask the effective freeness for the adjoint linear system $|K_X + mH|$. §§3 and 4 are devoted to the proof of the main results.

The author would like to thank Professors L. Ein and R. Lazarsfeld for showing the manuscript [EL2] to the author at AMS Summer Institute at Santa Cruz. In the first version of this paper, Theorem 2.2 was weaker so that we had to assume $\sigma_3 > \sqrt{3(\frac{2}{3})^3 + 24} = \sqrt{30.0183\cdots} = 3.107865\cdots$ in Theorem 3.1. After that, the author received two letters, one from S. Helmke and the other from L. Ein and R. Lazarsfeld, and both contained the optimal result stated as in Theorem 3.1 by using the idea of T. Fujita [F]. This paper follows the argument of the former. The author would like to express his gratitude to S. Helmke for allowing the author to reproduce his result.

1. Minimal center of log canonical singularities

We recall the standard notation (cf. [KMM]). Let $X$ be a normal variety of dimension $n$. A $\mathbb{Q}$-divisor is an element of $Z_{n-1}(X) \otimes \mathbb{Q}$, i.e., a finite formal sum $D = \sum_{j=1}^n d_j D_j$ of prime divisors $D_j$ with coefficients $d_j \in \mathbb{Q}$. We usually require implicitly that the $D_j$ are distinct. $D$ is said to be effective if $d_j \geq 0$ for all $j$. The round up of $D$ is defined by $\lceil D \rceil = \sum_j \lceil d_j \rceil D_j$. Two $\mathbb{Q}$-divisors $D_1, D_2$ are said to be $\mathbb{Q}$-linearly equivalent, and we write $D_1 \sim_\mathbb{Q} D_2$, if there exists a positive integer $m$ and a non-zero rational function $h$ such that $m(D_1 - D_2) = \text{div}(h)$. By abuse of notation, we sometimes write $D_1 = D_2$ instead of $D_1 \sim_\mathbb{Q} D_2$ when the canonical divisors are involved (e.g., the first paragraph of Definition 1.2).

$D$ is called a $\mathbb{Q}$-Cartier divisor if it is in the image of the natural injective homomorphism $\text{Div}(X) \otimes \mathbb{Q} \rightarrow Z_{n-1}(X) \otimes \mathbb{Q}$. If $D$ is an effective $\mathbb{Q}$-Cartier divisor and $x_0 \in X$ is a point, then the order $\text{ord}_{x_0} D \in \mathbb{Q}$ is defined by linearity. If $X$ is complete and $D$ is $\mathbb{Q}$-Cartier, we can define the intersection number $(D^s \cdot S) \in \mathbb{Q}$ for any subvariety $S$ of dimension $s$ on $X$. $D$ is said to be nef if $(D \cdot C) \geq 0$ for any curve $C$. In this case, $D$ is called big if $(D^n) > 0$.

Let $\mu : Y \rightarrow X$ be a birational morphism of normal varieties. The exceptional locus $\text{Exc}(\mu)$ of $\mu$ is the smallest closed subset of $Y$ such that $\mu|_{Y \setminus \text{Exc}(\mu)}$ is an isomorphism. If $D$ is a $\mathbb{Q}$-divisor (resp. a $\mathbb{Q}$-Cartier divisor) on $X$, we can define the strict transform (resp. total transform) $\mu^{-1}_* D$ (resp. $\mu^* D$). For a pair $(X, D)$ of a variety and a $\mathbb{Q}$-divisor, an embedded resolution or a log resolution is a proper birational morphism $\mu : Y \rightarrow X$ from a smooth variety $Y$ such that the union of the support of $\mu^{-1}_* D$ and $\text{Exc}(\mu)$ is a normal crossing divisor.

Most of the results of this paper are the applications of the following vanishing theorem:

**Theorem 1.1.** Let $X$ be a smooth projective variety and $D$ a $\mathbb{Q}$-divisor. Assume that $D$ is nef and big, and that the support of the difference $\lceil D \rceil - D$ is a normal crossing divisor. Then $H^p(X, K_X + \lceil D \rceil) = 0$ for $p > 0$. \hfill \Box

**Definition 1.2.** Let $X$ be a normal variety and $D = \sum_j d_j D_j$ an effective $\mathbb{Q}$-divisor such that $K_X + D$ is $\mathbb{Q}$-Cartier. If $\mu : Y \rightarrow X$ is an embedded resolution of the pair $(X, D)$, then by abuse of notation, we sometimes write $\mu_* (\lceil D \rceil) = \mu_* (D)$ instead of $\mu_* (\lceil D \rceil)$ and $\mu_* (D)$ respectively.
pair \((X, D)\), then we can write
\[
K_Y + \mu_*^{-1}D = \mu^*(K_X + D) + F
\]
with \(F = \sum_j e_j E_j\) for the exceptional divisors \(E_j\). We call \(F\) the discrepancy and \(e_i \in \mathbb{Q}\) the discrepancy coefficient for \(E_j\). We regard \(-d_i\) as the discrepancy coefficient for \(D_i\).

The pair \((X, D)\) is said to have only log canonical singularities (LC) (resp. kawamata log terminal singularities (KLT)) if \(d_i \leq 1\) (resp. \(< 1\)) for all \(i\) and \(e_j \geq -1\) (resp. \(> -1\)) for all \(j\) for an embedded resolution \(\mu : Y \to X\). One can also say that \((X, D)\) is LC (resp. KLT), or \(K_X + D\) is LC (resp. KLT), when \((X, D)\) has only LC (resp. KLT). If \(D = 0\), then \(X\) is called LC (resp. KLT) if so is \((X, D)\). The pair \((X, D)\) is said to be LC (resp. KLT) at a point \(x_0 \in X\) if \((U, D|_U)\) is LC (resp. KLT) for some neighborhood \(U\) of \(x_0\).

**Definition 1.3.** A subvariety \(W\) of \(X\) is said to be a center of log canonical singularities for the pair \((X, D)\), if there is a birational morphism from a normal variety \(\mu : Y \to X\) and a prime divisor \(E\) on \(Y\) with the discrepancy coefficient \(e \leq -1\) such that \(\mu(E) = W\). For example, if \(E = \mu_*^{-1}D_i\) for some \(i\), then we have \(e = -d_i\), so \(D_i\) is a center of log canonical singularities if and only if \(d_i \geq 1\). For another such \(\mu' : Y' \to X\), if the strict transform \(E'\) of \(E\) exists on \(Y'\), then we have the same discrepancy coefficient for \(E'\). The divisor \(E'\) is considered to be equivalent to \(E\), and the equivalence class of these prime divisors is called a place of log canonical singularities for \((X, D)\).

The set of all the centers (resp. places) of log canonical singularities is denoted by \(\text{CLC}(X, D)\) (resp. \(\text{PLC}(X, D)\)). Thus there is a natural surjective map \(\text{PLC}(X, D) \to \text{CLC}(X, D)\), which is not necessarily injective. If \((X, D)\) is LC, then \(\text{CLC}(X, D)\) is a finite set. The union of all the subvarieties in \(\text{CLC}(X, D)\) is denoted by \(\text{LLC}(X, D)\) and called the locus of log canonical singularities for \((X, D)\). \(\text{LLC}(X, D)\) is a closed subset of \(X\), and is empty if and only if \((X, D)\) is KLT. For a point \(x_0 \in X\), we define \(\text{CLC}(X, x_0, D) = \{W \in \text{CLC}(X, D); x_0 \in W\}\).

**Theorem 1.4.** (Connectedness Lemma, [Sh2], [Ko]). Let \(f : X \to Z\) be a proper surjective morphism of normal varieties with connected fibers, and \(D = \sum_i d_i D_i\) a \(\mathbb{Q}\)-divisor on \(X\) such that \(K_X + D\) is \(\mathbb{Q}\)-Cartier. Assume the following conditions:

1. if \(d_i < 0\), then \(\text{codim}(f(D_i)) \geq 2\);
2. \(- (K_X + D)\) is \(f\)-nef and \(f\)-big.

Then \(\text{LLC}(X, D) \cap f^{-1}(z)\) is connected for any point \(z \in Z\).

We include the proof for the convenience of the reader.

**Proof.** Let \(\mu : Y \to X\) be an embedded resolution of the pair \((X, D)\), and \(K_Y + D_Y = \mu^*(K_X + D)\). By definition, we have \(\text{LLC}(X, D) = \mu(\text{LLC}(Y, D_Y))\). So we may assume that \(X\) is smooth and \(\text{Supp}(D)\) is normal crossing. If we write \(D = S + D'\) with \(S = \sum_{d_i \geq 1} d_i D_i\) and \(D' = \sum_{d_i < 1} d_i D_i\), then \(\text{LLC}(X, D) = \text{Supp}(S)\).

By the condition (2), we have \(R^1 f_* \mathcal{O}_X(\lceil -D' \rceil) = 0\). From an exact sequence
\[
0 \to \mathcal{O}_X(\lceil -D' \rceil) \to \mathcal{O}_X(\lceil -D'' \rceil) \to \mathcal{O}_{\text{Supp}(S)}(\lceil -D' \rceil) \to 0
\]
we deduce that the natural homomorphism \(f_* \mathcal{O}_X(\lceil -D' \rceil) \to f_* \mathcal{O}_{\text{Supp}(S)}(\lceil -D' \rceil)\) is surjective. Since \(\lceil -D'' \rceil \geq 0\), there is a natural homomorphism \(\mathcal{O}_Z \to f_* \mathcal{O}_X(\lceil -D'' \rceil)\), which is an isomorphism by the condition (1). Therefore, we obtain our assertion.
\qed
Proposition 1.5. Let $X$ be a normal variety and $D$ an effective $\mathbb{Q}$-Cartier divisor such that $K_X + D$ is $\mathbb{Q}$-Cartier. Assume that $X$ is KLT and $(X, D)$ is LC. If $W_1, W_2 \in \text{CLC}(X, D)$ and $W$ an irreducible component of $W_1 \cap W_2$, then $W \in \text{CLC}(X, D)$. In particular, if $(X, D)$ is not KLT at a point $x_0 \in X$, then there exists the unique minimal element of $\text{CLC}(X, x_0, D)$.

Proof. Since the assertion is local, we may assume that $X$ is affine. Let $D_i$ ($i = 1, 2$) be a general member among effective Cartier divisors which contain $W_i$. Let $\mu : Y \to X$ be an embedded resolution of the pair $(X, D + D_1 + D_2)$. We choose $\mu$ so that there are divisors $E_i$ above the $W_i$ with the discrepancy coefficients $-1$. Let $e_i$ (resp. $e'_i$) be the coefficients of $E_i$ in $\mu^* D$ (resp. $\mu^* D_i$), and $a_i$ the positive numbers such that $e_i = a_i e'_i$. Then we have $\text{LLC}(X, (1 - \epsilon) D + a_1 \epsilon D_1 + a_2 \epsilon D_2) = W_1 \cup W_2$ for $0 < \epsilon \ll 1$. By the connectedness lemma, there exist divisors $F_i(\epsilon)$ on $Y$ for $i = 1, 2$ such that $F_i(\epsilon) \in \text{PLC}(X, (1 - \epsilon) D + a_1 \epsilon D_1 + a_2 \epsilon D_2)$, $\mu(F_i(\epsilon)) \subset W_i$, and $F_1(\epsilon) \cap F_2(\epsilon) \neq \emptyset$. Since there are only a finite number of the exceptional divisors for $\mu$, we have the common divisors $F_i = F_i(\epsilon)$ for a convergent sequence $\epsilon \to 0$. Then $\mu(F_1 \cap F_2) \in \text{CLC}(X, D)$. \hfill \qed

We shall control the singularities of the minimal center of log canonical singularities in the following theorem which are also obtained independently by Ein and Lazarsfeld ([EL2]). This is a variant of the connectedness lemma.

Theorem 1.6. Let $X$ be a normal variety, $x_0 \in X$ a KLT point, $D$ an effective $\mathbb{Q}$-Cartier divisor such that $(X, D)$ is LC at $x_0$, $W$ a minimal element of $\text{CLC}(X, x_0, D)$, and $E$ a place of log canonical singularities for $(X, D)$ on a blow-up $Y$ of $X$ which lies above $W$. Then $W$ is normal at $x_0$, and the projection $E \to W$ has connected fibers in a neighborhood of $x_0$.

Proof. Since the assertions are local, we may assume that $X$ is affine. We may assume that $\text{LLC}(X, D) = W$ as in the proof of Proposition 1.5. We may also assume that $\text{PLC}(X, D)$ consists of one element. Let $\mu : Y \to X$ be an embedded resolution of the pair $(X, D)$. We write

$$K_Y + E + F = \mu^* (K_X + D)$$

where $E$ is a prime divisor such that $\mu(E) = W$, and $F$ is a divisor whose coefficients are smaller than 1. By Theorem 1.1

$$H^1(Y, -E + F - F^\gamma) = 0$$

and we obtain a surjection

$$H^0(Y, F - F^\gamma) \to H^0(E, F^\gamma|_E)$$

Since $F - F^\gamma$ is effective and exceptional, we have

$$H^0(X, O_X) \xrightarrow{\sim} H^0(Y, F - F^\gamma)$$

Therefore, the natural injective homomorphism

$$H^0(W, O_W) \to H^0(E, F - F^\gamma)$$

is surjective. Thus $W$ is normal, and $\mu : E \to W$ has connected fibers. \hfill \qed
Proposition 1.7. Let $x_0 \in X$, $D$ and $W$ be as in Theorem 1.6. Assume that $W$ is a prime divisor. Then there exists an effective $\mathbb{Q}$-divisor $D_W$ on $W$ such that $K_X + D|_W = K_W + D_W$ and the pair $(W, D_W)$ is KLT at $x_0$.

Proof. By using the residue map $\omega_X(W) \to \omega_W$, we can define naturally the effective $\mathbb{Q}$-divisor $D_W$ on $W$ (cf. [KMM]). We have

$$
\mu^*_E(K_W + D_W) = (K_Y + E + F)|_E = K_E + F|_E
$$

where $\mu_E = \mu|_E : E \to W$, and the pair $(W, D_W)$ is KLT, since $\mu_E$ is birational. □

Question 1.8. Let $x_0 \in X$, $D$ and $W$ be as in Theorem 1.6. Does there exist an effective $\mathbb{Q}$-divisor $D_W$ on $W$ such that $K_X + D|_W = K_W + D_W$ and the pair $(W, D_W)$ is KLT at $x_0$?

We shall give an affirmative answer to Question 1.8 in the case $\operatorname{codim} X W = 2$ in [K3]. We also have a positive evidence in the case $\dim W = 2$:

Theorem 1.9. Let $x_0 \in X$, $D$ and $W$ be as in Theorem 1.6. Assume that $\dim W = 2$. Then $W$ has at most a rational singularity at $x_0$. Moreover, if $W$ is singular at $x_0$, and if $D'$ is an effective $\mathbb{Q}$-Cartier divisor on $X$ such that $\operatorname{ord}_{x_0} D'|_W \geq 1$, then $\{x_0\} \in \operatorname{CLC}(X, x_0, D + D')$.

Proof. We use the notation of Theorem 1.6. As in the proof of Proposition 1.5, we may assume that none of the coefficients of $F$ are integers and the projection $\mu : E \to W$ is factorized as $E \overset{\alpha}{\to} \tilde{W} \overset{\beta}{\to} W$ with $\tilde{W}$ smooth and $\beta$ birational. By the vanishing theorem

$$
H^p(Y, -E + \gamma - F|_E) = H^p(Y, \gamma - F|_E) = 0 \text{ for } p > 0
$$

where the latter is obtained by replacing $D$ by $(1 - \epsilon)D$ for a small $\epsilon > 0$. Then

$$
H^p(E, \gamma - F|_E) = 0 \text{ for } p > 0
$$

Similarly, we have $R^p\alpha_* \mathcal{O}_E(\gamma - F|_E) = 0$ for $p > 0$. Therefore, if we set $F = \alpha_* \mathcal{O}_E(\gamma - F|_E)$, then $R^p\beta_* F = 0$ for $p > 0$. We know that $\gamma - F|_E$ is effective and $\mu_* \mathcal{O}_E(\gamma - F|_E) = \mathcal{O}_W$. Therefore, we have an injective homomorphism $\mathcal{O}_{\tilde{W}} \to F$, and the support of its cokernel is contained in $\operatorname{Exc}(\beta) = \bigcup_i G_i$.

Let $F^{**} = \mathcal{O}_{\tilde{W}}(G)$ be the double dual of $F$, where $G = \sum_i g_i G_i$ for some nonnegative integers $g_i$. Let $g = \sum_i g_i$. We define a sequence of effective divisors $G^j = \sum_i g_i^j G_i$ for $j = 0, 1, \ldots, g$ such that $\sum_i g_i^j = g - j$ inductively as follows. First, set $G^0 = G$. Assuming that $G^{j_0}$ is defined for a $j_0 < g$, pick an $i_{j_0}$ such that $(G^{j_0} \cdot G_{i_{j_0}}) < 0$, and set $G^{j_0+1} = G^{j_0} - G_{i_{j_0}}$. Let $F^j = F \cap \mathcal{O}_{\tilde{W}}(G^j)$. Then we have injective homomorphisms $F^j/F^{j+1} \to \mathcal{O}_{\tilde{W}}(G^j)/\mathcal{O}_{\tilde{W}}(G^{j+1}) \cong \mathcal{O}_{G_{i_j}}(G^j)$. Since $H^0(G_{i_j}, G^j) = 0$, we have $H^1(F^{j+1}) = 0$ if $H^1(F^j) = 0$. Therefore, we have $H^1(\tilde{W}, \mathcal{O}_{\tilde{W}}) = 0$.

In order to prove the second part, we shall prove that $g_{i_0} = 0$ for some $i_0$. Assume the contrary that $g_i > 0$ for all $i$. By the above argument, we have $H^1(F^j/F^{j+1}) = 0$, hence $H^1(\mathcal{O}_{\tilde{W}}(G^j)/\mathcal{O}_{\tilde{W}}(G^{j+1})) = 0$ for all $j$. Then $H^1(G, G|_G) = 0$. By the Serre duality, we have $H^0(G, K_W|_G) = 0$. But since $W$ has a rational singularity at $x_0$, it is a contradiction, hence $g_{i_0} = 0$ for some $i_0$. 
Let \( F|_E = \sum f_\ell F_\ell \) and \( \alpha^* G_{i_0} = \sum k_\ell F_\ell \). Since \( g_{i_0} = 0 \), there exists an \( \ell_0 \) such that \( \alpha(F_{i_0}) = G_{i_0} \) and \(-f_{\ell_0} < k_{\ell_0} \), hence \(-f_{\ell_0} \leq k_{\ell_0} - 1 \). We have \( K_Y + E + F + \mu^* D' = \mu^* (K_X + D + D') \). Since \( \text{ord}_{x_0} D|_W \geq 1 \), the coefficient of \( F + \mu^* D' \) on \( F_{i_0} \) is at least \( f_{\ell_0} + k_{\ell_0} \geq 1 \), and \( \{x_0\} \in \text{CLC}(X, x_0, D + D') \).

We shall replace the minimal center of log canonical singularities by a smaller subvariety by using the following theorem due to [EL2]. This is another evidence of Question 1.8 on the adjunction and the inverse adjunction.

**Theorem 1.10.** Let \( x_0 \in X, D \) and \( W \) be as in Theorem 1.6. Let \( D_1 \) and \( D_2 \) be effective \( \mathbb{Q} \)-Cartier divisors on \( X \) whose supports do not contain \( W \) and which induce the same \( \mathbb{Q} \)-Cartier divisor on \( W \). Assume that \( (X, D + D_1) \) is LC at \( x_0 \) and there exists an element of \( \text{CLC}(X, x_0, D + D_1) \) which is properly contained in \( W \). Then the similar statement holds for the pair \( (X, D + D_2) \).

**Proof.** Since the assertion is local, we may assume that \( X \) is affine. First, we assume that \( W \) is the only element of \( \text{CLC}(X, D) \) and there is only one place of log canonical singularities above it. Let \( \mu : Y \to X \) be an embedded resolution of the pair \( (X, D + D_1 + D_2) \), and \( E \) the divisor which represents the place of log canonical singularities for \( (X, D) \). We write \( K_Y + E + \sum f_i F_i = \mu^* (K_X + D) \) and \( K_Y + E + \sum f_i \alpha F_i = \mu^* (K_X + D + D_\alpha) \) for \( \alpha = 1, 2 \). Let \( D_W \) be the \( \mathbb{Q} \)-Cartier divisors on \( W \) which is induced by the \( D_\alpha \), and \( \mu^* D_W = \sum g_i G_i \) for \( G_i = F_i \cap E \). Then we have \( f_{i,1} = f_{i,2} = f_i + g_i \) if \( G_i \neq \emptyset \). By the connectedness lemma, there exists \( i_0 \) such that \( G_{i_0} \neq \emptyset \) and \( f_{i_0,1} \geq 1 \). Since the support of \( D_1 \) does not contain \( W \), we have \( \mu(G_{i_0}) \subseteq W \). Hence there exists an element of \( \text{CLC}(X, x_0, D + D_2) \) which is properly contained in \( W \).

Let \( c = \sup \{ t \in \mathbb{Q}; K_X + D + tD_2 \text{ is LC at } x_0 \} \)

Then the assumption of the theorem is satisfied by the pair \( (X, D + cD_2) \). By the preceding argument, there exists an element of \( \text{CLC}(X, x_0, D + cD_1) \) which is properly contained in \( W \). Since \( (X, D + D_1) \) is LC, we have \( c \geq 1 \), and \( (X, D + D_2) \) is LC at \( x_0 \).

Now we consider the general case. Let us take a general Cartier divisor \( D' \) which contains \( W \), and a positive number \( a \) such that \( K_X + (1 - \epsilon)D + aD' = K_X + D + \epsilon(-D + aD') \) is LC with the only one place of canonical singularities for any \( 0 < \epsilon \ll 1 \). There exists a function \( b(\epsilon) \) such that the assumption of the theorem holds for the pair \( (X, (1 - \epsilon)D + aD' + (1 + b(\epsilon))D_1) \). Being LC is a closed condition for the coefficients of divisors, so \( b(\epsilon) \) is a well defined continuous convex function such that \( \lim_{\epsilon \to 0} b(\epsilon) = 0 \). By the first part of the proof, the conclusion of the theorem holds for \( (X, (1 - \epsilon)D + aD' + (1 + b(\epsilon))D_2) \). Looking at the discrepancies, we conclude that \( c = 1 \). Then the minimal element of \( \text{CLC}(X, x_0, D + D_2) \) should be smaller than \( W \), since the support of \( D_2 \) does not contain \( W \). \( \square \)

## 2. General method

We shall consider the conditions for the existence of a member of the given linear system which has prescribed order at a given point.

**Proposition 2.1.** Let \( X \) be a normal and complete variety of dimension \( n \), \( L \) a nef and big Cartier divisor, \( x_0 \in X \) a point, and \( t \) a rational number such that \( t > 0 \) and \( t < 1 \). Then there exists an effective \( \mathbb{Q} \)-Cartier divisor \( D \) of dimension \( n - 1 \) and a rational number \( \epsilon \) such that

\[ K_X + D + tD \text{ is LC at } x_0 \]

and

\[ \text{ord}_{x_0} D|_W \geq \epsilon \]

where \( \epsilon \) is a positive rational number.

**Proof.** By the connectedness lemma, there exists an effective \( \mathbb{Q} \)-Cartier divisor \( E \) such that \( \text{ord}_{x_0} E|_W \geq \epsilon \). Let \( F \) be an effective \( \mathbb{Q} \)-Cartier divisor on \( W \) such that \( F \cdot E > 0 \). By the linear equivalence, there exists an effective \( \mathbb{Q} \)-Cartier divisor \( D \) of dimension \( n - 1 \) and a rational number \( \epsilon \) such that

\[ K_X + D + tD \text{ is LC at } x_0 \]

and

\[ \text{ord}_{x_0} D|_W \geq \epsilon \]

where \( \epsilon \) is a positive rational number. \( \square \)

**Theorem 2.2.** Let \( X \) be a normal and complete variety of dimension \( n \), \( L \) a nef and big Cartier divisor, \( x_0 \in X \) a point, and \( t \) a rational number such that \( t > 0 \) and \( t < 1 \). Then there exists an effective \( \mathbb{Q} \)-Cartier divisor \( D \) of dimension \( n - 1 \) and a rational number \( \epsilon \) such that

\[ K_X + D + tD \text{ is LC at } x_0 \]

and

\[ \text{ord}_{x_0} D|_W \geq \epsilon \]

where \( \epsilon \) is a positive rational number.

**Proof.** By the connectedness lemma, there exists an effective \( \mathbb{Q} \)-Cartier divisor \( E \) such that \( \text{ord}_{x_0} E|_W \geq \epsilon \). Let \( F \) be an effective \( \mathbb{Q} \)-Cartier divisor on \( W \) such that \( F \cdot E > 0 \). By the linear equivalence, there exists an effective \( \mathbb{Q} \)-Cartier divisor \( D \) of dimension \( n - 1 \) and a rational number \( \epsilon \) such that

\[ K_X + D + tD \text{ is LC at } x_0 \]

and

\[ \text{ord}_{x_0} D|_W \geq \epsilon \]

where \( \epsilon \) is a positive rational number. \( \square \)
t > 1. Then there exists an effective $\mathbb{Q}$-Cartier divisor $D$ such that $D \sim_{\mathbb{Q}} tL$ and

$$\text{ord}_{x_0} D \geq \sqrt[n]{\frac{(L^n)}{\text{mult}_{x_0} X}}.$$  

**Proof.** We shall prove that there exists a positive integer $m$ with $mt \in \mathbb{N}$ and a member $D_m \in |mtL|$ such that

$$\text{ord}_{x_0} D_m \geq m \sqrt[n]{\frac{(L^n)}{\text{mult}_{x_0} X}}.$$  

Since

$$\text{length } \mathcal{O}_{X,x_0}/m^d_{x_0} = \frac{d^n}{n!} \text{mult}_{x_0} X + \text{lower terms in } d$$

it is enough to prove that

$$h^0(X, mL) = \frac{m^n(L^n)}{n!} + \text{lower terms in } m$$

If we replace $X$ by its desingularization, we may assume that $X$ is smooth and projective. Let $H$ be a very ample Cartier divisor such that $H - K_X$ is ample. Then we have $H^p(X, mL + H) = 0$ for any $p > 0$ and $m > 0$. Hence

$$h^0(X, mL + H) = \chi(X, mL + H) = \frac{m^n(L^n)}{n!} + \text{lower terms in } m$$

If $Y$ is a general member of the linear system $|H|$, then

$$0 \to \mathcal{O}_X(mL) \to \mathcal{O}_X(mL + H) \to \mathcal{O}_Y(mL + H) \to 0$$

Since dim $Y = n - 1$, we obtain the result. \hfill $\square$

The following theorem is due to S. Helmke after the idea of Fujita [F].

**Theorem 2.2.** Let $X$ be a normal projective variety of dimension $n$, $x_0 \in X$ a smooth point, $L$ an ample Cartier divisor, $W$ a prime divisor with $\text{ord}_{x_0} W = d \geq 1$, and $e, k$ positive rational numbers such that $de \leq 1$ and $k^n < (L^n)$. Assume that for any effective $\mathbb{Q}$-divisor $D$, if $D \sim_{\mathbb{Q}} L$ and $\text{ord}_{x_0} D \geq k$, then it follows that $D \geq kW$. Then there exists a real number $\lambda$ with $0 \leq \lambda \leq 1 - de$ which satisfies the following condition: if $k'$ is a positive rational number such that $k' > k$ and

$$(\lambda k)^n + \left( \frac{1 - de - \lambda}{1 - \lambda} \right)^n - \left( \frac{\lambda de}{1 - de - \lambda} \right)^n < (L^n)$$

then there exist an effective $\mathbb{Q}$-divisor $D$ such that $D \sim_{\mathbb{Q}} L$ and $\text{ord}_{x_0} D \geq k'$. (If $\lambda = 1 - de$, then the left hand side of the above inequality should be taken as a limit.)

**Proof.** Let us define a function $\phi(q)$ for $q \in \mathbb{Q}_{\geq 0}$ to be the largest real number such that $D \geq \phi(q)W$ whenever $D \geq 0, D \sim_{\mathbb{Q}} L$ and $\text{ord}_{x_0} D = q$. Then $\phi$ is a convex function. In fact, if $\text{ord}_{x_0} D = q$, and $D = (\phi(q) + \epsilon)W$, then

$$\text{ord}_{x_0} (\phi(q) + \epsilon)W = q$$
for \( 0 < \epsilon_i \ll 1 \) and \( i = 1, 2 \), then \( \text{ord}_{x_0}(tD_1 + (1 - t)D_2) = tq_1 + (1 - t)q_2 \) and \( tD_1 + (1 - t)D_2 = (t(\phi(q_1) + \epsilon_1) + (1 - t)(\phi(q_2) + \epsilon_2))W + \) other components, hence \( \phi(tq_1 + (1 - t)q_2) \leq t\phi(q_1) + (1 - t)\phi(q_2) \). Since \( \phi(k) \geq ek \), there exists a real number \( \lambda \) such that \( 0 \leq \lambda < 1 \) and \( \phi(q) \geq \frac{\epsilon(q - \lambda k)}{1 - \lambda} \) for any \( q \). Since \( q \geq \phi(d) \), we have \( \lambda \leq 1 - de \).

Let \( m \) be a large and sufficiently divisible integer and \( v : H^0(X, mL) \to O_{X,x_0}(mL) \cong \mathcal{O}_{X,x_0} \) the evaluation homomorphism. We consider subspaces \( V_j = v^{-1}(m_{2n}^j) \) of \( H^0(X, mL) \) for integers \( j \) such that \( \lambda km \leq j \leq k'm \). First, we have

\[
\dim V_{\lambda km} \geq \dim H^0(X, mL) - \left(\frac{\lambda km}{n!}\right) + \text{lower terms in } m
\]

Let \( D \in mL \) be a member corresponding to \( h \in V_j \) for some \( j \). Since we have \( D \geq \phi(j/m)mW \), the number of conditions in order for \( h \in V_{j+1} \) is at most the number of homogeneous polynomials of order \( j - \phi(j/m)dm \) in \( n \) variables, i.e.,

\[
\frac{(j - \phi(j/m)dm)^{n-1}}{(n-1)!} + \text{lower terms in } m
\]

Therefore, our assertion follows from

\[
\frac{(\lambda km)^n}{n!} + \sum_{j=\lambda km}^{k'm-1} \left(\frac{j - \frac{de(j - \lambda km)}{1 - \lambda}}{(n-1)!}\right)^{n-1} + \text{lower terms in } m
\]

\[
= \frac{(\lambda km)^n}{n!} + \left(\frac{1 - \frac{de}{1 - \lambda}}{1 - \lambda}\right)^{n-1} \left\{ (k' + \frac{\lambda de}{1 - \lambda})^n - (\lambda k + \frac{\lambda de}{1 - \lambda})^n \right\} \frac{m^n}{n!} + \text{lower terms in } m
\]

\[
< \frac{m^n(L^n)}{n!} + \text{lower terms in } m
\]

\[ \square \]

We begin to state our freeness result in the ideal case in which the minimal center of canonical singularities is an isolated point.

**Proposition 2.3.** Let \( X \) be a normal projective variety of dimension \( n \), \( x_0 \in X \) a Gorenstein KLT point, and \( L \) an ample Cartier divisor. Assume that there exists an effective \( \mathbb{Q} \)-Cartier divisor \( D \) which satisfies the following conditions:

1. \( D \sim_{\mathbb{Q}} tL \) for a rational number \( t < 1 \),
2. \( (X, D) \) is LC at \( x_0 \),
3. \( \{x_0\} \in \text{CLC}(X, D) \).

Then \( |K_X + L| \) is free at \( x_0 \).

**Proof.** Let \( D' \) be a general member of \( mL \) for \( m \gg 0 \) which passes through \( x_0 \). If we replace \( D \) by \( (1 - \epsilon_1)(D + \epsilon_2D') \) for some \( 0 < \epsilon_i \ll \frac{1}{m} \), then we may assume that \( x_0 \) is an isolated point of \( \text{LLC}(X, D) \). Let \( \mu : Y \to X \) be an embedded resolution of the pair \( (X, D) \). Then

\[
K_Y + E + E = \mu^*(K_X + D)
\]
where $E$ is a reduced divisor such that $\mu(E) = \{x_0\}$, and $F$ is a divisor of the form $\sum_j f_j F_j$ with $f_j < 1$ if $x_0 \in \mu(F_j)$. Then

$$K_Y + (1 - t)\mu^* L \sim_\mathbb{Q} \mu^*(K_X + L) - E - F.$$ 

Thus

$$H^1(Y, \mu^*(K_X + L) - E + \gamma - F^\gamma) = 0$$

and we obtain a surjection

$$H^0(Y, \mu^*(K_X + L) + \gamma - F^\gamma) \to H^0(E, \mu^*(K_X + L)) \cong \mathbb{C}.$$ 

Since $\gamma - F^\gamma$ is effective and exceptional over a neighborhood of $x_0$,

$$H^0(X, K_X + L) \to H^0(E, \mu^*(K_X + L))$$

is also surjective.

By combining Propositions 2.1 and 2.3, we would obtain Fujita’s freeness conjecture if we would have only the ideal case.

3. Smooth 3-fold

**Theorem 3.1.** Let $X$ be a normal projective variety of dimension 3, $L$ an ample Cartier divisor, and $x_0 \in X$ a smooth point. Assume that there are positive numbers $\sigma_p$ for $p = 1, 2, 3$ which satisfy the following conditions:

1. $\sqrt[p]{(L^p \cdot W)} \geq \sigma_p$ for any subvariety $W$ of dimension $p$ which contains $x_0$,
2. $\sigma_1 \geq 3$, $\sigma_2 \geq 3$, and $\sigma_3 > 3$.

Then $|K_X + L|$ is free at $x_0$.

**Proof.** Step 0. Let $t$ be a rational number such that $t > \frac{3}{\sqrt[3]{(L^3)}}$. Since $\sigma_3 > 3$, we can take $t < 1$. By Proposition 2.1, there exists an effective $\mathbb{Q}$-Cartier divisor $D$ such that $D \sim_\mathbb{Q} tL$ and $\text{ord}_{x_0} D = 3$. Let $c \leq 1$ be the log canonical threshold of $(X, D)$ at $x_0$:

$$c = \sup \{t \in \mathbb{Q}; K_X + tD \text{ is LC at } x_0\}$$

and let $W$ be the minimal element of $\text{CLC}(X, x_0, cD)$. If $W = \{x_0\}$, then $|K_X + L|$ is free at $x_0$ by Propositions 2.3.

Step 1. We consider the case in which $W = C$ is a curve. Since $t < 1$, we have $ct + (1 - c) < 1$. Since $\sigma_1 \geq 3$, there exists a rational number $t'$ with $ct + (1 - c) < t' < 1$ and an effective $\mathbb{Q}$-Cartier divisor $D'_C$ on $C$ such that $D'_C \sim_\mathbb{Q} (t' - ct)L |_C$ and $\text{ord}_{x_0} D'_C = 3(1 - c)$. By the Serre vanishing theorem, there exists an effective $\mathbb{Q}$-Cartier divisor $D'$ on $X$ such that $D' \sim_\mathbb{Q} (t' - ct)L$ and $D'|_C = D'_C$. In fact, if we take a sufficiently large and divisible integer $m$ such that $mD'_C$ is a Cartier divisor in $|m(t' - ct)L|_C|$ and $H^1(X, \mathcal{I}_C(m(t' - ct)L)) = 0$, then there exists an extension $D'_m \in |m(t' - ct)L|_C|$ of $mD'_C$, so we set $D' = \frac{1}{m}D'_m$.

Let $D'_1$ be a general effective $\mathbb{Q}$-Cartier divisor on an affine neighborhood $U$ of $x_0$ in $X$ such that $D'_1|_{C \cap U} = D'_C|_{C \cap U}$ and $\text{ord}_{x_0} D'_1 = 3(1 - c)$. Then we have $\text{ord}_{x_0}(cD + D'_1) = 3$, hence $\{x_0\} \in \text{CLC}(U, cD + D'_1)$. Let

$$c' = \sup \{t \in \mathbb{Q}; K_X + (cD + tD'_1) \text{ is LC at } x_0\}.$$
By Theorem 1.10, we conclude that \((X, cD + c'D')\) is LC at \(x_0\), and \(CLC(X, x_0, cD + c'D')\) has an element which is properly contained in \(C\), i.e., \(\{x_0\}\).

**Step 2-1.** We consider the case in which \(W = S\) is a surface. \(S\) is smooth or has a rational double point at \(x_0\). Let \(d := \text{mult}_{x_0} S = 1\) or 2. We assume first that \(d = 1\). As in Step 1, we take a rational number \(t'\), an effective \(\mathbb{Q}\)-Cartier divisor \(D'\) on \(X\) and a positive number \(c'\) such that \(ct + (1-c) < t' < 1\), \(D' \sim Q (t' - ct)L\), \(\text{ord}_{x_0} D'|S = 3(1-c)\), \((X, cD + c'D')\) is LC at \(x_0\), and that the minimal element \(W'\) of \(CLC(X, x_0, cD + c'D')\) is properly contained in \(S\). Thus we have the theorem when \(W' = \{x_0\}\).

We consider the case in which \(W' = C\) is a curve. Since \(t, t' < 1\), we have \(ct + c'(t' - ct) + (1-c)(1-c') < 1\). As in Step 1, there exists a rational number \(t''\) with \(ct + c'(t' - ct) + (1-c)(1-c') < t'' < 1\) and an effective \(\mathbb{Q}\)-Cartier divisor \(D''\) on \(C\) such that \(D''|_C \sim Q (t'' - ct - c'(t' - ct))L|_C\) and \(\text{ord}_{x_0} D''|_S = 3(1-c)(1-c')\). Let \(D'' \sim Q (t'' - ct - c'(t' - ct))L\) be its extension to \(X\) as before. Let us take a general effective \(\mathbb{Q}\)-Cartier divisor \(D''|_C\) on an affine neighborhood \(U\) of \(x_0\) in \(X\) such that \(D''|_{CU} = D''|_{CU}\) and \(\text{ord}_{x_0} D''|_S = 3(1-c)(1-c')\). By Theorem 1.10, there exists \(c''\) such that \((U, cD + c'D' + c''D'')\) is LC at \(x_0\) and there exists an element of \(CLC(U, x_0, cD + c'D' + c''D'')\) which is properly contained in \(S\). Moreover, since \(D''\) is chosen to be general, we have \(c'' > 0\) and the minimal element should be properly contained in \(C\). By Theorem 1.10 again, \((X, cD + c'D' + c''D'')\) is LC at \(x_0\) and there exists an element of \(CLC(X, x_0, cD + c'D' + c''D'')\) which is properly contained in \(C\).

**Step 2-2.** We assume that \(d = 2\). As in Step 2-1, we take a rational number \(t'\) with \(ct + \sqrt{2}(1-c) < t'\) and an effective \(\mathbb{Q}\)-Cartier divisor \(D'\) on \(X\) with \(D' \sim Q (t' - ct)L\) and \(\text{ord}_{x_0} D'|S = 3(1-c)\). Here we need the factor \(\sqrt{2}\) because \(S\) has multiplicity 2 at \(x_0\). We take \(0 < e' \leq 1\) such that \((X, cD + c'D')\) is LC and \(CLC(X, x_0, cD + c'D')\) has an element which is properly contained in \(S\).

We shall prove that we may assume \(ct + \sqrt{2}(1-c) < 1\). Then we can take \(t' < 1\) as in Step 2-1, and the rest of the proof is the same. For this purpose, we apply Theorem 2.2. In the argument of Steps 0 through 2-1, the number \(t\) was chosen under the only condition that \(t < 1\). So we can take \(t = 1 - \epsilon_1\), where the \(\epsilon_n\) for \(n = 1, 2, \ldots\) will stand for very small positive rational numbers. Then \(k = \frac{3}{1 - \epsilon_1} = 3 + \epsilon_2\) and \(e = \frac{1}{3\epsilon_2}\). This means the following: for any effective \(D \sim Q tL\), if \(\text{ord}_{x_0} D \geq 3\), then \(cD \geq S\). We look for \(k' = 6\) so that \(D \sim Q tL\) with \(t = \frac{1}{2}\) and \(\text{ord}_{x_0} D \geq 3\). The equation for \(k'\) becomes

\[
\lambda^3 + \left(\frac{1 - 2e - \lambda}{1 - \lambda}\right)^2 \left\{ (2 + \frac{2\lambda e}{1 - 2e - \lambda})^3 - (\lambda + \frac{2\lambda e}{1 - 2e - \lambda})^3\right\} < 1.
\]

We have \(\lambda + 2e \leq 1\), \(\frac{1}{3} \leq e\), and in particular, \(0 \leq \lambda \leq \frac{1}{3}\). If we put \(\frac{2e}{1 - \lambda} = \alpha\), then \(\frac{2}{3(1-\lambda)} \leq \alpha \leq 1\), and

\[
(-\lambda^3 + 6\lambda^2 - 12\lambda + 8)\alpha^2 + (-\lambda^3 + 12\lambda - 16)\alpha + 8 < 1.
\]

For a fixed \(\lambda\), since \(-\lambda^3 + 6\lambda^2 - 12\lambda + 8 > 0\), the left hand side attains the maximum at \(\alpha = \frac{2}{3(1-\lambda)}\) or \(1\), and the values are given by \(\frac{1}{(1-\lambda)^2}(\frac{2}{3}\lambda^4 - \frac{10}{9}\lambda^3 + \frac{8}{3}\lambda^2 - \frac{8}{3}\lambda + \frac{8}{3})\) or \(-2\lambda^3 + 6\lambda^2\), respectively. Since both numbers are smaller than 1, we obtain a member of \(|mL|\) with \(k' = 6\) for some \(m\). Then we choose a new \(t\) as \(t = \frac{1}{2}\). We have
Therefore, if\( \frac{2}{3} \leq c \leq 1 \) and \( ct + \sqrt{2}(1-c) < 1 \).

**Corollary 3.2.** ([EL1], [F]). Let \( X \) be a smooth projective variety of dimension 3, and \( H \) an ample divisor. Then \( |K_X + mH| \) is free if \( m \geq 4 \). Moreover, if \( (H^3) \geq 2 \), then \( |K_X + 3H| \) is also free. \( \square \)

4. Smooth 4-fold

**Theorem 4.1.** Let \( X \) be a normal projective variety of dimension 4, \( L \) an ample Cartier divisor, and \( x_0 \in X \) a smooth point. Assume that there are positive numbers \( \sigma_p \) for \( p = 1, 2, 3, 4 \) which satisfy the following conditions:

1. \( \frac{\sqrt{L \cdot W}}{\sigma_p} \geq 1 \) for any subvariety \( W \) of dimension \( p \) which contains \( x_0 \),
2. \( \sigma_p \geq 5 \) for all \( p \).

Then \( |K_X + L| \) is free at \( x_0 \).

**Proof.** Steps 0, 1 and 2-1. Let \( t \) be a rational number such that \( t > \frac{4}{\sqrt{L^4}} \). Since \( \sigma_4 > 4 \), we can take \( t < 1 \). By Proposition 2.1, there exists an effective \( \mathbb{Q} \)-Cartier divisor \( D \) such that \( D \sim_{\mathbb{Q}} tL \) and \( \text{ord}_{x_0} D = 4 \). Let \( c \leq 1 \) be the canonical threshold of \((X, D)\) at \( x_0 \), and \( W \) the minimal element of \( \text{CLC}(X, x_0, cD) \). If \( W \) is a point, then \( |K_X + L| \) is free at \( x_0 \) by Propositions 2.3. Since \( \sigma_p \geq 4 \) for \( p = 1, 2 \), the cases in which \( W \) is a curve or a smooth surface can be treated similarly as in Steps 1 and 2-1 of the proof of Theorem 3.1.

Step 2-2. We consider the case in which \( W = S \) is a surface. Since \( S \) has a rational singularity at \( x_0 \) and its embedding dimension is 3 or 4, we have \( d := \text{mult}_{x_0} S = 2 \) or 3 ([A]). We can take \( t < \frac{4}{5} + \epsilon \) for \( 0 < \epsilon \ll 1 \), so \( \frac{3t}{4} + \frac{\sqrt{3}}{5} < 1 \). Therefore, if \( c \geq \frac{3}{4} \), then

\[
(a) \quad ct + \frac{4\sqrt{d(1-c)}}{\sigma_2} < 1
\]

In this case, we can take a rational number \( t' \) and an effective \( \mathbb{Q} \)-Cartier divisor \( D' \) on \( X \) such that \( ct + \frac{4\sqrt{d(1-c)}}{\sigma_2} < t' < 1 \), \( D' \sim_{\mathbb{Q}} (t' - ct)L \) and \( \text{ord}_{x_0} D'|S = 4(1-c) \), and proceed as in Step 2-1. On the other hand, if \( c \leq \frac{3}{4} \), then

\[
(b) \quad ct + \frac{\sqrt{d}}{\sigma_2} < 1
\]

We take \( t' \) and \( D' \) with \( D' \sim_{\mathbb{Q}} (t' - ct)L \) and \( \text{ord}_{x_0} D'|S = 1 \), and use Theorem 1.9 in order to obtain a smaller center of log canonical singularities.

Let \( c' \) and \( W' \) as before. We consider the case in which \( W' = C \) is a curve. We have

\[
\text{ord}_{x_0}(cD + c'D')|S \geq \begin{cases} 
4c + 4c'(1-c) = 4 - 4(1-c)(1-c') & \text{in the case (a)} \\
4c + c' = 4 - 4(1-c + c') & \text{in the case (b)}
\end{cases}
\]

In the case (a), since \( t, t' < 1 \), we have \( ct + c'(t' - ct) + (1-c)(1-c') < 1 \). In the case (b), we have

\[
ct + c'(t' - ct) + \frac{4}{3} \left(1 - c - \frac{c'}{4}\right) \leq (\frac{4}{3} + \epsilon)c + \frac{4}{3}(1-c) + c'\left(\frac{\sqrt{3}}{3} - \frac{1}{3}\right) \leq \frac{3 + \sqrt{3}}{3} + \epsilon < 1
\]
The rest is the same as before.

**Step 3.** We consider the case in which \( W = V \) is a 3-fold. Let \( d := \text{mult}_{x_0} V \). By Proposition 1.7, \( V \) has only a canonical singularity at \( x_0 \), and we have \( d = 1, 2 \) or 3. Since \( t < \frac{4}{3} + \epsilon \) and \( c \geq \frac{d}{4} \), we have

\[
ct + \frac{4\sqrt[3]{d}(1-c)}{\sigma_3} < \frac{d + (4 - d)\sqrt[3]{d}}{5} + \epsilon < 1
\]

Therefore, there exists \( t' < 1 \) and \( D' \) as before, and we obtain \( c' \) and \( W' \). If \( W' = C \) is a curve, then

\[
ct + \frac{4\sqrt{d}(1-c)c'}{\sigma_3} + \frac{4(1-c)(1-c')}{\sigma_1} < 1
\]

and we obtain our assertion as before.

We assume that \( W' = S \) is a surface. Let \( d' := \text{mult}_{x_0} S = 1, 2 \) or 3. We have to prove that one of the followings hold:

(a) \[
ct + \frac{4\sqrt{d}(1-c)c'}{\sigma_3} + \frac{4\sqrt{d'}(1-c)(1-c')}{\sigma_2} < 1
\]

(b) \[
ct + \frac{4\sqrt{d}(1-c)c'}{\sigma_3} + \frac{\sqrt{d'}}{\sigma_2} < 1
\]

If \( \sqrt{d} \geq \sqrt{d'} \), then (a) holds. Otherwise, we have the following cases:

1. \( d = 1, d' = 2 \) or 3,
2. \( d = 2, d' = 2 \) or 3,
3. \( d = 3, d' = 3 \).

In the case (1), since the embedding dimension of \( S \) at \( x_0 \) is 3, we have \( d' = 2 \). Then we have \( 4c \geq 1 \) and \( 4(1-c)c' \geq 2 \), hence

\[
\text{l.h.s. of (a)} < \frac{1 + 2 + \sqrt{2}}{5} + \epsilon < 1
\]

In the case (2), since \( 4c \geq 2 \), if \( 4(1-c)c' \geq 1 \), then

\[
\text{l.h.s. of (a)} < \frac{2 + \sqrt{2} + \sqrt{3}}{5} + \epsilon = \frac{4.9919 \cdots}{5} + \epsilon < 1
\]

Otherwise, (b) holds. Finally, in the case (3),

\[
\text{l.h.s. of (a)} < \frac{3 + \sqrt{3}c' + \sqrt{3}(1-c')}{5} + \epsilon < \frac{3 + \sqrt{3}}{5} + \epsilon < 1
\]

In any case, we obtain \( t'' < 1 \), \( D'', c'' \) and \( W'' \) as before.

When \( W'' \) is a curve, we have still \( t''' < 1 \). In fact, we have

\[
ct + \frac{4\sqrt[3]{d}(1-c)c'}{\sigma_3} + \frac{4\sqrt{d'}(1-c)(1-c')(1-c'')}{\sigma_2} + \frac{4(1-c)(1-c')(1-c'')(1-c''')}{\sigma_1} < 1
\]

in the case (a), and

\[
ct + \frac{4\sqrt[3]{d}(1-c)c'}{\sigma_3} + \frac{\sqrt{d'c''}}{\sigma_2} + \frac{4(1-c)(1-c')(1-c'')}{\sigma_1} < 1
\]

in the case (b). The rest of the proof is similar to the previous steps.

\[\Box\]

**Corollary 4.2.** Let \( X \) be a smooth projective variety of dimension 4, and \( H \) an ample divisor. Then \( |K_X + mH| \) is free if \( m \geq 5 \).  

\[\Box\]
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