RUSCHEWEYH'S UNIVALENCE CRITERION AND QUASICONFORMAL EXTENSIONS

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Abstract

Ruscheweyh extended the work of Becker and Ahlfors on sufficient conditions for a normalized analytic function on the unit disk to be univalent there. In this paper we refine the result to a quasiconformal extension criterion with the help of Becker's method. As an application, a positive answer is given to an open problem proposed by Ruscheweyh.

1. Introduction

Throughout the paper, $D$ denotes the unit disk $\{ |z| < 1 \}$ in the complex plane $\mathbb{C}$ and $D^*$ the exterior domain of $D$ in the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{ \infty \}$.

Let $\mathcal{A}$ be a family of normalized analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ on $D$. We say that a sense-preserving homeomorphism $f$ of a plane domain $G \subset \mathbb{C}$ is $k$-quasiconformal if $f$ is absolutely continuous on almost all lines parallel to the coordinate axes and $|f_z| \leq k |f_z|$, almost everywhere $G$, where $f_z = \partial f / \partial z$ and $k$ is a constant with $0 < k < 1$.

Ahlfors [1] has shown that the following condition is sufficient for quasiconformal extensibility of univalent functions as an extension of Becker's univalence condition [2] (see also [7], p. 175);

**Theorem A** ([1], [3]). Let $f \in \mathcal{A}$. If there exists a $k$, $0 \leq k < 1$, such that for a constant $c \in \mathbb{C}$ satisfying $|c| \leq k$ and all $z \in D$

\[
|cz|^2 + (1 - |z|^2) \left| \frac{zf''(z)}{f'(z)} \right| \leq k
\]

then $f$ has a $k$-quasiconformal extension to $\mathbb{C}$.

The limiting case $k \to 1$ in the above theorem ensures univalence of $f$ in $D$. Ruscheweyh [8] extended this univalence condition in the following way;

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Theorem B ([8]). Let \( s = a + ib, \ a > 0, \ b \in \mathbb{R} \) and \( f \in \mathcal{A} \). Assume that for a constant \( c \in \mathbb{C} \) and all \( z \in \mathbb{D} \)

\[
|cz|^2 + s - a(1 - |z|^2)\left\{ s\left( 1 + \frac{zf''(z)}{f'(z)} \right) + (1 - s)\frac{zf'(z)}{f(z)} \right\} \leq M
\]

with

\[
M = \begin{cases} 
a|s| + (a - 1)|s + c|, & \text{if } 0 < a \leq 1, \\
|s|, & \text{if } 1 < a,
\end{cases}
\]

then \( f \) is univalent in \( \mathbb{D} \).

The case \( s = 1 \) with \( c \) replaced by \(-1 - c\) is the special case of Theorem A.

The purpose of this paper is to refine Ruscheweyh’s univalence condition to a quasiconformal extension criterion which includes Theorem A;

Theorem 1. Let \( s = a + ib, \ a > 0, \ b \in \mathbb{R} \), \( k \in [0, 1) \) and \( f \in \mathcal{A} \). Assume that

\[
|cz|^2 + s - a(1 - |z|^2)\left\{ s\left( 1 + \frac{zf''(z)}{f'(z)} \right) + (1 - s)\frac{zf'(z)}{f(z)} \right\} \leq M
\]

with

\[
M = \begin{cases} 
ak|s| + (a - 1)|s + c|, & \text{if } 0 < a \leq 1, \\
k|s|, & \text{if } 1 < a,
\end{cases}
\]

then \( f \) has an \( l \)-quasiconformal extension to \( \mathbb{C} \), where

\[
l = \frac{2ka + (1 - k^2)|b|}{(1 + k^2)a + (1 - k^2)|s|} < 1.
\]

Remark 1.1. If \( f \in \mathcal{A} \), then it is easy to verify that there exists a sequence \( \{z_n\} \subset \mathbb{D} \) with \( |z_n| \to 1 \) such that for each \( s \in \{z \in \mathbb{C} : \text{Re } z > 0\} \)

\[
\sup_n \left| s\left( 1 + \frac{z_n f''(z_n)}{f'(z_n)} \right) + (1 - s)\frac{z_n f'(z_n)}{f(z_n)} \right| < \infty
\]

which shows that (3) implies the inequality

\[
|c + s| \leq M.
\]

This inequality is needed for proving that \( f(z) \) has no zeros in \( 0 < |z| < 1 \) (see Lemma 7). In [8], it is mentioned that (3) implies \( f(z) \neq 0, \ 0 < |z| < 1 \), without proof. The part of (5) can be found in [8].

Remark 1.2. A similar argument to Remark 1.1 is also valid for Theorem A. It follows that the assumption \( |c| \leq k \) is embedded in the inequality (1).
The next application follows from Theorem 1. Let $a > 0$ and $b \in \mathbb{R}$. It follows from a result of Sheil-Small [9, Theorem 2] that

$$\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} + (a + i\beta - 1) \frac{zf'(z)}{f(z)} \right\} > 0 \quad (z \in \mathbb{D})$$

is sufficient for $f \in \mathcal{A}$ to be a Bazilevič function of type $(\alpha, \beta)$ (see also [5]). Here, a function $f \in \mathcal{A}$ is called Bazilevič of type $(\alpha, \beta)$ if

$$f(z) = \left[ (\alpha + i\beta) \int_0^z g(\zeta) h(\zeta) e^{i\beta - 1} d\zeta \right]^{1/(\alpha + i\beta)}$$

for a starlike univalent function $g \in \mathcal{A}$ and an analytic function $h$ with $h(0) = 1$ satisfying $\Re(e^{ih}) > 0$ in $\mathbb{D}$ for some $\lambda \in \mathbb{R}$. Together with this fact, the next theorem follows;

**Theorem 2.** Let $\alpha > 0$, $\beta \in \mathbb{R}$ and $k \in [0, 1)$. If $f \in \mathcal{A}$ satisfies

$$\left| 1 + \frac{zf''(z)}{f'(z)} + (a + i\beta - 1) \frac{zf'(z)}{f(z)} - \frac{\alpha^2 + \beta^2}{\alpha} \right| \leq M$$

for all $z \in \mathbb{D}$ with

$$M = \begin{cases} k & \text{if } \alpha < \alpha^2 + \beta^2, \\ k(\alpha^2 + \beta^2) \alpha & \text{if } \alpha^2 + \beta^2 \leq \alpha, \end{cases}$$

then $f$ is a Bazilevič function of type $(\alpha, \beta)$ and can be extended to a $\tilde{k}$-quasiconformal automorphism of $\mathbb{C}$, where

$$\tilde{k} = \frac{2k\alpha + (1 - k^2) |\beta|}{(1 + k^2) \alpha + (1 - k^2) \sqrt{\alpha^2 + \beta^2}}.$$

Next, we shall discuss quasiconformal extensibility of functions $g(z) = z + \frac{d}{z} + \cdots$ analytic in $\mathbb{D}^*$.

**Theorem 3.** Let $s = a + ib$, $a \geq 1$, $b \in \mathbb{R}$ and $k \in [0, 1)$ which satisfies $|b/s| \leq k$. Let $g(\zeta) = \zeta + \frac{d}{\zeta} + \cdots$ be analytic in $\mathbb{D}^*$ and fulfill

$$\left| ib + (1 - |\zeta|^2)a \left( 1 - \frac{\zeta g'(\zeta)}{g(\zeta)} \right) - s \frac{\zeta g''(\zeta)}{g'(\zeta)} \right| \leq ak|s| - |b|(a - 1)$$

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for all $\zeta \in D^*$. Then $g$ can be extended to an $l$-quasiconformal automorphism of $C$, where

$$l = \frac{2ka + (1 - k^2)|b|}{(1 + k^2)a + (1 - k^2)|s|}.$$ 

The case $k \to 1$ corresponds to a univalence criterion which is due to Ruscheweyh [8].

Theorem 3 yields the following corollary which gives a positive answer to an open problem proposed by Ruscheweyh [8], i.e., whether a function $g(\zeta) = \zeta + d/\zeta + \cdots$ with $(|\zeta|^2 - 1)[1 + (\zeta f''(\zeta)/f'(\zeta)) - (\zeta f''(\zeta)/f(\zeta))] \leq k$ for all $\zeta \in D^*$ admits a quasiconformal extension to $C$;

**Corollary 4.** Let $g(\zeta) = \zeta + d/\zeta + \cdots$ be analytic in $D^*$. If there exists $k \in [0, 1)$ such that

$$\left(\frac{|\zeta|^2}{|\zeta|^2 - 1}\right) \left|1 + \frac{\zeta g''(\zeta)}{g'(\zeta)} - \frac{\zeta g'(\zeta)}{g(\zeta)}\right| \leq k$$

for all $\zeta \in D^*$, then $g$ can be extended to a $k$-quasiconformal automorphism of $C - \{0\}$.

From the above corollary we have another extension criterion for analytic functions on $D$;

**Corollary 5.** Let $f \in \mathcal{A}$ with $f''(0) = 0$. If there exists $k \in [0, 1)$ such that

$$\left(1 - |z|^2\right) \left|1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right| \leq k$$

for all $z \in D$, then $f$ can be extended to a $k$-quasiconformal automorphism of $C$.

**2. Preliminaries**

Our investigations are based on the theory of Löwner chains. A function $f_t(z) = f(z, t) = a_1(t)z + \sum_{n=2}^{\infty} a_n(t)z^n$, $a_1(t) \neq 0$, defined on $D \times [0, \infty)$ is called a **Löwner chain** if $f_t(z)$ is holomorphic and univalent in $D$ for each $t \in [0, \infty)$ and satisfies $f_t(D) \subseteq f_s(D)$ and $f(0, s) = f(0, t)$ for $0 \leq s \leq t < \infty$, and if $a_1(t)$ is locally absolutely continuous in $t \in [0, \infty)$ with $\lim_{t \to \infty} |a_1(t)| = \infty$. Then $f(z, t)$ is absolutely continuous in $t \in [0, \infty)$ for each $z \in D$ and satisfies the **Löwner differential equation**

$$\frac{\partial f}{\partial t}(z, t) = h(z, t)zf'(z, t)$$

for $z \in D$ and almost every $t \in [0, \infty)$. Here, $\frac{\partial f}{\partial t}(z, t) = \partial f(z, t)/\partial t$, $f''(z, t) = \partial^2 f(z, t)/\partial z^2$ and $h(z, t)$ is a function measurable on $t \in [0, \infty)$, holomorphic in $|z| < 1$ and $\text{Re} h(z, t) > 0$ ([6]).
An interesting method connecting the theory of quasiconformal extensions with Löwner chains was obtained by Becker;

**Theorem C ([2], see also [4]).** Suppose that \( f(z, t) \) is a Löwner chain for which \( h(z, t) \) of (9) satisfies the condition
\[
\frac{|h(z, t) - 1|}{|h(z, t) + 1|} \leq k
\]
for all \( z \in \mathbb{D} \) and almost all \( t \in [0, \infty) \). Then \( f_t(z) \) admits a continuous extension to \( \bar{\mathbb{D}} \) for each \( t \geq 0 \) and the map defined by
\[
f(r e^{i\theta}) = \begin{cases} 
  f(re^{i\theta}), & \text{if } r < 1, \\
  f(e^{i\theta}, \log r), & \text{if } r \geq 1,
\end{cases}
\]
is a \( k \)-quasiconformal extension of \( f_0 \) to \( \mathbb{C} \).

### 3. Proof of Theorem 1

The proof is divided into two parts. The first part of the proof is based on [8].

(i) First we assume that \( f(z)/z \neq 0 \) for all \( z \in \mathbb{D} \). Then we can define
\[
f(z, t) = f(e^{-st}z) \left\{ 1 - a \frac{e^{2it}}{c} \frac{e^{-st}zf'(e^{-st}z)}{f(e^{-st}z)} \right\}^s
\]
and let
\[
F(z, t) = f(z, t/|z|).
\]
A straightforward calculation shows
\[
h(z, t) = \frac{F(z, t)}{ZF'(z, t)} = s \frac{1 + P(e^{-st/|z|}z, t/|z|)}{|z|} \frac{1 - P(e^{-st/|z|}z, t/|z|)}{1 - P(e^{-st/|z|}z, t/|z|)},
\]
where
\[
P(z, t) = \frac{c}{a} e^{-2it} + \frac{1 + (e^{-2it} - 1)H_s(z)}{1 + (e^{-2it} - 1)H_s(z)}
\]
and
\[
H_s(z) = s \left( 1 + \frac{zf''(z)}{f'(z)} \right) + (1 - s) \frac{zf'(z)}{f(z)}.
\]
Since \( h(z, t) \) is holomorphic in \( z \in \mathbb{D} \) and measurable on \( t \in [0, \infty) \), applying Theorem C to (11), we see that the condition
\[
\frac{|s(1 + P(e^{-st/|z|}z, t/|z|))|}{|s(1 - P(e^{-st/|z|}z, t/|z|))|} \leq l
\]
for all \( z \in \mathbb{D} \) and almost all \( t \in [0, \infty) \), applying Theorem C to (11), we see that the condition
\[
\frac{|s(1 + P(e^{-st/|z|}z, t/|z|))|}{|s(1 - P(e^{-st/|z|}z, t/|z|))|} \leq l
\]
implies \( l \)-quasiconformal extensibility of \( f(z) \). This is equivalent to

\[
(12) \quad \left| P + \frac{(1 + l^2)b}{(1 + l^2)a + (1 - l^2)|s|} \right| \leq \frac{2l|s|}{(1 + l^2)a + (1 - l^2)|s|}.
\]

Here, we shall prove the following Lemma;

**Lemma 6.** Under the assumption of Theorem 1, we have

\[
(13) \quad |aP(e^{-st/|s|}z, t/|s|) + ib| < k|s|
\]

for \( z \in D \) and \( t \in [0, \infty) \).

**Proof.** We have

\[
|aP + ib| \leq m_1 + m_2
\]

by triangle inequality, where

\[
m_1 = (1 - e^{-2t/|s|}) \left| \frac{ce^{-2at/|s|} + s}{1 - e^{-2at/|s|}} - aH_s(e^{-st/|s|}z) \right|
\]

and

\[
m_2 = \left| (ce^{-2at/|s|} + s) \frac{1 - e^{-2t/|s|}}{1 - e^{-2at/|s|}} - (ce^{-2t/|s|} + s) \right|.
\]

Then it is enough to show that \( m_1 + m_2 < k|s| \). (3) implies

\[
\left| \frac{c|e^{st/|s|}z|^2 + s}{1 - |e^{st/|s|}z|^2} - aH_s(e^{-st/|s|}z) \right| \leq \frac{M}{1 - |e^{st/|s|}z|^2} \leq \frac{M}{1 - e^{-2at/|s|}}
\]

for \( z \in D \). Let \( q(t) = (1 - e^{-2t/|s|})/(1 - e^{-2at/|s|}) \). Applying the maximum modulus principle to the function

\[
\frac{ce^{-2at/|s|} + s}{1 - e^{-2at/|s|}} - aH_s(e^{-st/|s|}z)
\]

we have

\[
m_1 \leq q(t)M.
\]

On the other hand

\[
m_2 \leq |c + s| |1 - q(t)|.
\]

Since \( 1 \leq q(t) < 1/a \) if \( 0 < a \leq 1 \) and \( 1/a < q(t) \leq 1 \) if \( 1 < a \) for all \( t \in [0, \infty) \), we conclude that \( m_1 + m_2 < k|s| \) which is our desired inequality.

We now let \( \Delta \) and \( \Delta' \) be disks which are defined by replacing \( P \) in (12) and (13) to a complex variable \( w \). It remains to find the smallest \( l \) so that \( \Delta' \) is
contained by $\Delta$. Note that if $k = l = 1$ then these two disks coincide. The following condition is necessary and sufficient for $\Delta' \subset \Delta$;

\[
(14) \quad \frac{(1 + l^2)b}{(1 + l^2)a + (1 - l^2)|s|} - \frac{b}{a} \leq \frac{2l|s|}{(1 + l^2)a + (1 - l^2)|s|} - \frac{k|s|}{a}.
\]

Then we conclude

\[
l \leq \frac{2ka + (1 - k^2)|b|}{(1 + k^2)a + (1 - k^2)\sqrt{a^2 + b^2}}.
\]

which is suitable for our purpose.

(ii) In order to eliminate the additional assumption that $f(z)/z \neq 0$ in $D$, we need a sort of stability of the condition (3);

**Lemma 7.** If $f \in A$ satisfies the assumption of Theorem 1, then so does $f_r(z) = \frac{1}{r} f(rz), \ r \in (0, 1)$.

**Proof.** It follows from the assumption that $aH_s(rz)$ is contained in the disk

\[
\Delta = \left\{ w \in \mathbb{C} : \left| w - \frac{cr^2|z|^2 + s}{1 - r^2|z|^2} \right| \leq \frac{M}{1 - r^2|z|^2} \right\}.
\]

We want to deduce that $aH_s(rz)$ lies in the disk

\[
\Delta' = \left\{ w \in \mathbb{C} : \left| w - \frac{c|z|^2 + s}{1 - |z|^2} \right| \leq \frac{M}{1 - |z|^2} \right\}.
\]

Therefore it is enough to see that $\Delta \subset \Delta'$, that is,

\[
(15) \quad \left| \frac{c|z|^2 + s}{1 - |z|^2} - \frac{cr^2|z|^2 + s}{1 - r^2|z|^2} \right| \leq \frac{M}{1 - |z|^2} - \frac{M}{1 - r^2|z|^2}.
\]

In view of the identity

\[
\frac{|z|^2}{1 - |z|^2} - \frac{r^2|z|^2}{1 - r^2|z|^2} = \frac{1}{1 - |z|^2} - \frac{1}{1 - r^2|z|^2},
\]

the inequality (15) is equivalent to (5).

Now we shall show that the condition $f(z)/z \neq 0$ in $D$ follows from the assumption of Theorem 1. Suppose, to the contrary, that $f(z_0) = 0$ for some $0 < |z_0| < 1$. We may assume that $f(z) \neq 0$ for $0 < |z| < |z_0|$. Then by Lemma 7 we can apply Theorem 1 to the function $f_{r_0}(z) = f(r_0z)/r_0, \ r_0 = |z_0|$ to conclude that $f_{r_0}$ has a quasiconformal extension to $C$. In particular, $f_{r_0}$ is injective on $D$. This, however, contradicts the relation $f_{r_0}(z_0/r_0) = f_{r_0}(0) = 0$. \qed
Remark 3.1. We can replace $|s|$ in (10) to any positive real value and continue our argument. However, it will be found that $|s|$ gives the smallest $l$ by calculations.

Remark 3.2. We have $l \geq k$, where $l = k$ if and only if $b = 0$. Indeed, let $l = l(k)$. Then we have $l'(k) > 0$ and $l''(k) \leq 0$ which imply $l \geq k$. If we suppose $l = k \neq 0$, then the right-hand side of (14) is greater than or equal to 0 only if $b = 0$. In the case $l = k = 0$ we also have $b = 0$ by (14). It easily follows from (4) that $l = k$ if $b = 0$.

4. Proof of Theorem 2

It is easy to see from (6) that $f$ is a Bazilevič function of type $(\alpha, \beta)$ under our assumption since $M$ is always less than or equal to $(\alpha^2 + \beta^2)/\alpha$.

Let us now prove quasiconformal extensibility of $f$. Setting $1/s = z + \frac{1}{a}$ which implies $a = \text{Re} \ s = \frac{z}{(\alpha^2 + \beta^2)}$ and $b = \text{Im} \ s = -\frac{\beta}{(\alpha^2 + \beta^2)}$, (7) turns to

$$\left| 1 + \frac{zf''(z)}{f'(z)} + \left( \frac{1}{s} - 1 \right) \frac{zf'(z)}{f(z)} - \frac{1}{a} \right| \leq \begin{cases} 0 < a < 1, \\ k/a, & 1 \leq a. \end{cases}$$

Therefore, Theorem 2 follows from Theorem 1 with $c = -s$. \qed

5. Proof of Theorem 3

First let $s \neq 1$. In that case we may assume $g(\zeta) \neq 0$ for all $\zeta \in \mathbb{D}^*$ because of a similar discussion of the proof of Theorem 1;

**Lemma 8.** Let $g(\zeta) = \zeta + \frac{d}{\zeta} + \cdots$ be analytic in $\mathbb{D}^*$. If $g$ satisfies the same assumption of Theorem 3, then so does $g_R(\zeta) = \frac{1}{R} f(R \zeta)$, $R > 1$.

**Proof.** We need to prove

$$\left| \frac{ib}{|\zeta|^2 - 1} - a G_s(R\zeta) \right| \leq \frac{ak|s| - |b|(a - 1)}{|\zeta|^2 - 1}$$

by using

$$\left| \frac{ib}{R^2|\zeta|^2 - 1} - a G_s(R\zeta) \right| \leq \frac{ak|s| - |b|(a - 1)}{R^2|\zeta|^2 - 1},$$

where

$$G_s(\zeta) = (1 - s) \left( \frac{\zeta g'(\zeta)}{g(\zeta)} - 1 \right) + s \frac{\zeta g''(\zeta)}{g'(\zeta)}.$$
In a similar way to the proof of Lemma 7, it suffices to see that
\[
\left| \frac{ib}{|\zeta|^2 - 1} - \frac{ib}{R^2|\zeta|^2 - 1} \right| \leq \frac{ak|s| - |b|(a - 1)}{|\zeta|^2 - 1} - \frac{ak|s| - |b|(a - 1)}{R^2|\zeta|^2 - 1}.
\]
This is equivalent to \(|b| \leq k|s|\). \(\square\)

Then we let
\[
f(1/\zeta, t) = \frac{1}{g(e^{st}\zeta)} \left\{ 1 - (1 - e^{-2t})e^{st}\zeta \frac{g'(e^{st}\zeta)}{g(e^{st}\zeta)} \right\}^{-\delta}
\]
and
\[
F(1/\zeta, t) = f(1/\zeta, t/|s|).
\]
Since
\[
h(1/\zeta, t) = \frac{\dot{F}(1/\zeta, t)}{(1/\zeta)F(1/\zeta, t)} = \frac{s}{|s|} \cdot \frac{1 + P(e^{st/|s|}\zeta, t/|s|)}{1 - P(e^{st/|s|}\zeta, t/|s|)}
\]
where
\[
P(\zeta, t) = (e^{2t/|s|} - 1)G_{s}(\zeta),
\]
it is sufficient to see that
\[
(16) \quad |aP(e^{st/|s|}\zeta, t/|s|) + ib| < k|s|
\]
for all \(\zeta \in D^*\) and \(t \in [0, \infty)\) under the assumption of the theorem. By triangle inequality we have
\[
|aP + ib| \leq \left| \frac{1 - e^{2t/|s|}}{1 - e^{2at/|s|}} (ib + (1 - e^{2at/|s|})aG_{s}(e^{st/|s|}\zeta)) \right| + \left| ib \left( \frac{1 - e^{2t/|s|}}{1 - e^{2at/|s|}} \right) \right|
\]
for \(\zeta \in D^*\) and \(t \in [0, \infty)\). Following the lines of the proof of Lemma 6, one can obtain that (8) implies (16). Therefore, a similar argument of the proof of Theorem 1 implies our assertion. The case \(s = 1\) follows from a theorem of Becker [2]. \(\square\)

6. Proof of Corollary 4 and 5

Proof of Corollary 4. Let \(R > 1\) be an arbitrary but fixed number. We would like to show that \(g_{R}(\zeta) = g(R\zeta)/R\) can be extended to a \(k\)-quasiconformal mapping of \(\hat{C} - \{0\}\). Since \(g(\zeta) \neq 0\) in \(\zeta \in D^*\) from the assumption, there exists a certain constant \(A\) such that
\[
\left| |\zeta|^2 - 1 \right| \left| 1 - \frac{\zeta g'_{R}(\zeta)}{g_{R}(\zeta)} \right| \leq A < \infty
\]
for all $\zeta \in \overline{D}$. We also have
\[
1 - \frac{\zeta g''_R(\zeta)}{g'_R(\zeta)} + \frac{\zeta g'_R(\zeta)}{g_R(\zeta)} \leq \frac{k}{|\zeta R|^2 - 1}
\]
for $\zeta \in D^*$. Thus we obtain with $s = R^2 A/k(R^2 - 1)$
\[
(|\zeta|^2 - 1) \left| \frac{1}{s} \left(1 - \frac{\zeta g_R(\zeta)}{g_R(\zeta)} \right) - 1 - \frac{\zeta g''_R(\zeta)}{g'_R(\zeta)} + \frac{\zeta g'_R(\zeta)}{g_R(\zeta)} \right| \leq \frac{A}{s} + k \frac{|\zeta|^2 - 1}{|\zeta R|^2 - 1} \leq k
\]
which implies quasiconformal extensibility of $g_R$ by Theorem 3. A limiting procedure proves Corollary 4.

Proof of Corollary 5. Note that the function $1 + (zf''(z)/f'(z)) - (zf'(z)/f(z))$ is analytic in $D$ and has a zero of order 2 at the origin by the condition $f''(0) = 0$. Thus, we obtain from the assumption that
\[
\frac{1}{|z|^2} \left(1 - |z|^2\right) \left| 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right| \leq k
\]
by the maximum modulus principle. Let $g(\zeta)$ be a function defined by
\[
g(\zeta) = \frac{1}{f(z)}
\]
where $\zeta = 1/z$. Then $g$ is analytic in $D^*$ and has the form $g(\zeta) = \zeta + d/\zeta + \cdots$. From the relations
\[
\frac{zf'(z)}{f(z)} = \frac{\zeta g'(\zeta)}{g(\zeta)}
\]
and
\[
1 + \frac{zf''(z)}{f'(z)} = -1 - \frac{\zeta g''(\zeta)}{g'(\zeta)} + 2 \frac{\zeta g'(\zeta)}{g(\zeta)}
\]
we can deduce our assertion by applying Corollary 4. 

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