Many-Body Theory of Synchronization by Long-Range Interactions

Nariya Uchida

1Department of Physics, Tohoku University, Sendai, 980-8578, Japan

(Dated: July 1, 2010)

Abstract. Collective oscillations of active interacting elements are observed in a variety of physical, chemical, and biological systems far from equilibrium. Numerous studies have been devoted to the mutual entrainment of oscillators that have different intrinsic frequencies. A class of models with global (or mean-field) coupling have enjoyed deep theoretical understanding, while the phase of the oscillators become coherent with the system dimension increasing. In contrast to the sharp transition for global coupling, the contrast to the sharp transition for global coupling. We interpret it as the result of quenched spatial heterogeneity. In contrast, for α ≤ d, the heterogeneity is averaged out and the transition is exactly described by the mean-field theory.

Model. In our model, oscillators indexed by i = 1, 2, . . . , N are arrayed on a d-dimensional regular lattice with the unit grid size. The phase ˙φi of the i-th oscillator located at r̂ i obeys the dynamic equation,

\[ \frac{d\dot{\phi}_i}{dt} = \omega_i - \sum_{j \neq i} G(r_i - r_j) \sin(\phi_i - \phi_j), \]

where \( \omega_i \) is the intrinsic frequency that has the Gaussian distribution with the standard deviation \( \omega_0 \),

\[ P(\omega_i) = \frac{1}{\sqrt{2\pi}\omega_0} \exp \left( -\frac{\omega_i^2}{2\omega_0^2} \right). \]

We require the coupling function \( G(r) \) to be positive, slowly decreasing function of \(|r|\), so that its moments

\[ \sigma_n = \sum_{j \neq i} G(r_i - r_j)^n \]

rapidly decays with n. To be specific, let us consider the power-law coupling \( G(r) = g_0/r^\alpha \) with the constants \( g_0 \) and \( \alpha \geq 0 \). We normalize the coupling by rescaling time so that \( \sigma_1 = 1 \) without losing generality. For the global coupling (\( \alpha = 0 \)), we have \( G(r) = g_0 = 1/N \), and the moment \( \sigma_n \sim 1/N^{n-1} \) for \( n \geq 2 \) vanish as \( N \to \infty \). In more general, for \( \alpha < d \), the integral \( \int dr/r^\alpha \) diverges with the system dimension \( r_{\text{max}} \sim N^{1/d} \), which means that \( g_0 \sim N^{\alpha/d-1} \) and \( \sigma_n (n \geq 2) \) vanish as \( N \to \infty \).

Motivated by the numerical results, this Letter theoretically addresses synchronization of oscillators with a general class of long-range coupling. We will develop a systematic perturbation expansion around the mean field, taking the moments \( \sigma_n \) of the interaction \( G(r) \) as the small parameters (which is analogous in spirit to the cluster expansion in the classical gas theory). For the power-law coupling \( G(r) = g_0/r^\alpha \), it is equivalent to a series expansion in \( \epsilon = \alpha/d - 1 \). We will solve for the order parameter profile and correlation functions up to \( O(\epsilon^2) \).

PACS numbers: 05.45.Xt, 05.40.-a
Order Parameter. In order to describe the collective behavior, we introduce the site-dependent complex order parameter $\psi_i$ with its amplitude $\rho_i$ and phase $\theta_i$, \[\psi_i = \rho_i e^{i\theta_i} = \sum_{j \neq i} G(r_i - r_j) e^{i\phi_j}, \quad (2)\]

with which we can rewrite (1) as

\[\frac{\partial \phi_i}{\partial t} = \omega_i - \rho_i \sin(\phi_i - \theta_i). \quad (3)\]

Note that $\rho_i \leq 1$ due to the normalization of $\sigma_i$. When the coupling is long-ranged, $\psi_i$ involves infinitely many oscillators and is expected to change much slower than $\phi_i$. Therefore, we approximate $\psi_i$ to be constant in time. Then Eq. (2) is replaced by its temporal average,

\[\psi_i = \rho_i e^{i\theta_i} = \sum_j G_{ij} e^{i\phi_j} E(\rho_j, \omega_j), \quad (4)\]

where $G_{ij} = G(r_i - r_j)$ for $i \neq j$, and $G_{ij} = 0$ for $i = j$. The function $E(\rho_j, \omega_j)$ is the temporal average of $e^{i(\phi_j - \theta_j)}$, and is calculated following the original prescription by Kuramoto \[a, b, c, d = R, I\], and $i, j, k, \ell = 1, 2, \ldots, N$ are implied. We decompose the zeroth and first order coefficients into their averages $f_a = f_{a}(\psi) = \langle F_{ja}(\psi) \rangle$, $f_{ab} = \frac{\partial f_a}{\partial \psi_b}$, and the deviations $\delta F_{ja} = F_{ja} - f_a$, $\delta F_{ja,b} = f_{ja,b} - f_{a,b}$. Subtracting $\psi_a$ from (7) and then multiplying by the inverse of the $2N \times 2N$ matrix $M_{ij} = \delta_{ij} \delta_{ab} - G_{ij} f_{a,b}$, which is expanded as $[M^{-1}]_{ij} = \delta_{ij} \delta_{ab} + \sum f_{ja,b} + G_{ij} \delta F_{ja,b} + \cdots$ with $G_{ij} = G_{ik} G_{kj}$, $f_{ja,b} = f_{a} f_{c} f_{j} f_{b}$, etc., we obtain

\[\delta \psi_a = \Delta_a + \Gamma_{ab} U_{jb}, \quad (8)\]

\[U_{jb} = \delta F_{jb} + \delta F_{jbe} \psi_{jc} + \frac{1}{2} \delta F_{jbe,cd} \delta \psi_{jc} \delta \psi_{jd} + \cdots, \quad (9)\]

\[\Delta_a = \left(I_2 - \frac{\partial f}{\partial \psi}\right)^{-1} (f_a - \psi_a), \quad (10)\]

\[\Gamma_{ab} = \left[M^{-1}(G \otimes I_2)\right]_{ab} = G_{ij} \delta_{ab} + G_{ij} f_{a,b} + \cdots, \quad (11)\]

Equating the last 2 matrices $I_2 = \{\delta_{ab}\}$ and $\partial f/\partial \psi = \{f_{a,b}\}$, Eqs. (8)-(11) can be diagrammatized as shown in Fig. (1b), by combining the symbols defined in Fig. (1a). Recursively using (5) for the $\delta \psi$’s in (9), we get an expansion of $\delta \psi_a$ in terms of $\Delta_a$, $\Gamma_{ab}$, $\delta F_{ja}(\psi)$, $F_{ja,b}(\psi)$, and their derivatives; see Fig. (1c). The terminators $\Delta_a$ and $\delta F_{ja}$ are connected by the vertices $\delta F_{ja,b}, F_{ja,b}, \cdots$, and propagators $\Gamma_{ab}$ to the site $i$. For example, the graph framed by solid lines reads $\Gamma_{ab,ij} F_{ja,b} \delta F_{ke} \delta F_{kd}$, and the dotted-framed graph reads $\Gamma_{ab,ij} \frac{1}{2} F_{jbe,cd} \Gamma_{ab,ij} \delta F_{ke} \cdot \delta F_{df}$. Now we take the average of Eq. (12) over the distribution of $\omega_i$’s. The LHS vanishes by the definition of $\delta \psi$. On the RHS, $\delta F_{ja}$ and its derivative $\delta F_{jbe}$ are averaged out unless they are correlated with a partner at the same site. Graphically, it means that the legs of the graphs (with the black dots at their ends) have to be attached to each other or to vertices to produce correlation terms. For example, the dot-framed graph in Fig. (1c) yields the corresponding graph in Fig. (1d), which reads $\Gamma_{ab,ij} \frac{1}{2} (F_{jbe,cd})_j \cdot \Gamma_{ab,ij} \delta F_{ke} \delta F_{kd}$, where $\langle \cdots \rangle_j$ means the average over the distribution $P(\omega_j)$. Using the expansion (11) with the trace $G_{ij} = \sigma_n$, we obtain the $O(\epsilon^2)$ expression of this graph as

\[\frac{\sigma_n}{2} \sum_{f_{a,cd}} (f_{ad} - f_{cd} f_{a} f_{d}),\]

where the functions $f_{a,cd} = \frac{\partial^2 f_a}{\partial \psi_a \partial \psi_d}$ and $g_{ab}(\psi) = \langle F_{ka} F_{kb}\rangle_k$ are introduced. Another graph that gives an equivalent to the ensemble average over $\omega_i$’s. Expecting that spatial fluctuation of the order parameter is small for long-range interactions, we expand the RHS of (12) with respect to the deviation $\delta \psi_j = \psi_j - \overline{\psi}$, as

\[\overline{\psi}_a = \frac{1}{N} \sum_i \overline{\psi}_{ia}, \quad a = R, I,\]

which is equivalent to the ensemble average over $\omega_i$’s.
which simplify calculations, we choose the coordinate frame in
its recursive expansion. (d) The ensemble average of (c).

\[ \psi = \int_{-\infty}^{\infty} d\omega P(\omega)E_\omega(\overline{\psi}, \omega). \]  

The calculations of the derivatives \( f_{a,b}, f_{a,b,c} \) and \( h_{a,b,c} \) are also straightforward. The non-vanishing components are found to be \( f_{R,R} = \epsilon'_{R}, f_{I,,I} = \epsilon'_{I}, f_{R,R} = \epsilon''_{R}, f_{R,,I} = f_{I,,R} = \epsilon'_{R}, \) and \( h_{R,R} = \epsilon''_{R}, h_{I,,I} = \epsilon_{II}/2, \) where \( \epsilon' = d/d\rho \) and the abbreviations \( \epsilon_R = \epsilon_R/\overline{\psi}, \epsilon_{II} = \epsilon_{II}/\overline{\psi}, \) are used. Substituting these into Eqs. (12,13), we obtain

\[ \overline{\psi}_R = \epsilon_R + \sigma_2(1 - \epsilon'_R)V_R, \]

\[ V_R = \frac{1}{2} \left\{ \epsilon''(\epsilon''_{R} - 2\epsilon'_R\epsilon'_R) - \overline{\psi}_R \right\}. \]

with the functions \( \overline{\psi}_R = \rho \cos \theta \) on the LHS via the expansion

\[ \overline{\psi}_R = \left( \sqrt{\overline{\psi}^2 + \epsilon'_R} \right) = \overline{\psi}_R + \left\{ \delta \overline{\psi}^2 \right\}/\sqrt{\overline{\psi} + \epsilon'(\epsilon'_{R}/(\epsilon''_{R} - \epsilon'_R) + \epsilon''_{R} \epsilon''_{II}).} \]

Note that \( C_{ij}^{ab} = C_{ik}G_{kj} \) is a function of \( \epsilon_{ij} = \epsilon_i - \epsilon_j. \) At large distance, it decays as \( C_{ij}^{ab} \propto \epsilon_{ij}^{-d(1+2\epsilon)} \) for \( \epsilon > 0, \) as we can see from a simple dimensional analysis. (For \( d = 2, \) \( G_{ij}^{2} \) depends also on the direction of \( \epsilon_{ij}, \) reflecting the lattice anisotropy). On the other hand, setting \( i = j \) in (13), we obtain the variance of the order parameter,

\[ \langle \delta \psi_{ia} \delta \psi_{ib} \rangle = \sigma_2 \left( g_{ab} - f_a f_b \right). \]

The complete order parameter profile is obtained by numerical computation of the functions \( e_a(\rho) \) and \( e_b(\rho), \) and is shown in Fig. 2a. Note that \( \overline{\psi}_R \) in the current coordinate frame corresponds to \( |\psi| \) in the general frame. As we can see, the deviation from the mean-field profile is significant even for relatively small values of \( \sigma_2. \) (For comparison, \( \sigma_2 = 0.2 \) for \( (d, \alpha) = (1,2) \) and \( \sigma_2 \approx 0.057 \) for \( (d, \alpha) = (2,3) \) (square lattice).) The macroscopic order parameter is larger than the mean-field value for
$\omega_0 > \omega_1 \simeq 0.504$, and smaller for $\omega_0 < \omega_1$ for any non-zero value of $\sigma_2$. The enhancement of synchronization for large $\omega_0$ might look counter-intuitive, but it is a natural result of the spatial heterogeneity; there are regions that are more uniform than the others in terms of the intrinsic frequencies of the oscillators they contain. These regions can remain synchronized when the other regions are desynchronized, and contribute to the long tail of the order parameter profile. This effect of quenched heterogeneity is averaged out in the global-coupling case. Note also that we have rescaled the timescale so that $\sigma_1 = 1$. If $\sigma_1$ is not normalized, we must divide the intrinsic frequency and the order parameter by $\sigma_1$, which modifies Eq. (21) as $\overline{\psi} \approx \sigma_1 \sigma_2 / \omega_0^2 = C \sigma_2 / \omega_0^2$, where $C$ is a function of $\alpha$ and $d$.

The standard deviation $\text{std}(\psi) = \langle |\delta \psi|^2 \rangle^{1/2}$ is readily calculated from Eqs. (14,15,16,17), and is plotted in Fig. 2(b). For any non-zero value of $\sigma_2$, it exhibits a peak near $\omega = \omega_c$ and a long-tail for $\omega_0 \gg \omega_c$. The asymptotic behavior for large $\omega_0$ is obtained via the Taylor expansion of $e_R(\rho)$, $e_R(\rho)$ and $e_{II}(\rho)$, as $\text{std}(\psi) \approx \sigma_2 / 3 \sqrt{\omega_c \omega_0}$.

Summary. We have found that the mean-field picture of sharp synchronization transition is valid only for $\alpha \leq d$, and the transition is broadened for $\alpha > d$. It could be regarded as a novel example of smeared transition in random systems, which usually requires spatially correlated disorder [18]. The limitations of the perturbation theory for large $\alpha$ should be assessed by analysis of higher order corrections and comparison with numerical results, which are beyond the scope of the present paper and will be discussed elsewhere.

I wish to thank Ramin Golestanian for useful comments, discussions, and collaborated works that motivated the present study.

[1] Y. Kuramoto, Chemical Oscillations, Waves, and Turbulence, (Springer, New York, 1984).
[2] S. H. Strogatz, Physica D 143, 1 (2000).
[3] J. A. Acebrón et al., Rev. Mod. Phys. 77, 137 (2005).
[4] Y. Kuramoto, in Lecture Notes in Physics No. 30 (Springer, New York, 1975), p. 420.
[5] J. D. Crawford and K. T. R. Davies, Physica D 125, 1 (1999).
[6] H. Daido, Phys. Rev. Lett. 73, 760 (1994).
[7] J. D. Crawford, 1995, Phys. Rev. Lett. 74, 4341 (1995).
[8] H. Hong, H. Chaté, H. Park and L.-H. Tang, Phys. Rev. Lett. 99, 184101 (2007).
[9] H. Sakaguchi, S. Shimomoto and Y. Kuramoto, Prog. Theor. Phys. 77, 1005 (1987).
[10] S. H. Strogatz and R. E. Mirollo, J. Phys. A 21, L699 (1988); Physica D 31, 143 (1988).
[11] Y. Kuramoto, Prog. Theor. Phys. 94, 321 (1995); Y. Kuramoto and D. Battogtokh, Nonlin. Phenom. Complex Syst. 5, 380 (2002).
[12] D. M. Abrams and S. H. Strogatz, Phys. Rev. Lett. 93, 174102 (2004).
[13] J. L. Rogers and L. T. Wille, Phys. Rev. E 54, R2193 (1996).
[14] M. Maródi, F. d’Ovidio, and T. Vicsek, Phys. Rev. E 66, 011109 (2002).
[15] V. E. Tarasov and G. M. Zaslavsky, Chaos 16, 023110 (2006); N. Korabel, G. M. Zaslavsky and V. E. Tarasov, Commun. Nonlin. Sci. Numer. Simul. 12, 1405 (2007).
[16] N. Uchida and R. Golestanian, Phys. Rev. Lett. 104, 178103 (2010).
[17] N. Uchida and R. Golestanian, Europhys. Lett. 89, 50011 (2010).
[18] T. Vojta, J. Phys. A: Math. Gen. 36, 10921-10935 (2003); Phys. Rev. E 70, 026108 (2004).