Bounds on Codes Correcting Tandem and Palindromic Duplications

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Abstract. In this work, we derive upper bounds on the cardinality of tandem and palindromic duplication correcting codes by deriving the generalized sphere packing bound for these error types. We first prove that an upper bound for tandem or palindromic deletions is also an upper bound for inserting the respective type of duplications. Therefore, we derive the bounds based on these special deletions as this results in tighter bounds. We determine the spheres for tandem and palindromic duplications/deletions and the number of words with a specific sphere size. Our upper bounds on the cardinality directly imply lower bounds on the redundancy which we compare with the redundancy of the best known construction correcting arbitrary burst errors. Our results indicate that the correction of palindromic duplications requires more redundancy than the correction of tandem duplications. Further, there is a significant gap between the minimum redundancy of duplication correcting codes and burst insertion correcting codes.

1 Introduction

The increasing demand for high density and long-term data storage and the recent advance in biotechnological methodology has motivated the storage of digital data in DNA. One interesting application in this area involves the storage of data in the DNA of living organisms. Tagging genetically modified organisms, infectious bacteria, conducting biogenetical studies or storing data are only a few in a list of modern applications. However, the data is corrupted by errors during the replication of DNA and therefore an adequate error protection mechanism has to be found. Typical errors include point insertions, deletions, substitutions and tandem or palindromic duplications. While the correction of

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substitutions, insertions and deletions is well studied, knowledge about correcting tandem and palindromic duplication errors is relatively limited. In the former case, a subsequence of the original word is duplicated and inserted directly after the original subsequence. An example for a tandem duplication of length 3 in a DNA sequence \textit{GAT CAT G} is \textit{GAT ATCG}, where the underlined part highlights the duplication. Similarly, a palindromic duplication in the same word is \textit{GATCATAATG}. The focus of this paper is to determine the minimum redundancy needed to correct tandem and palindromic duplications.

The idea of upper bounding the sizes of insertion/deletion correcting codes by the fractional transversal number of the associated hypergraph has been introduced in [1]. In [2], this procedure has been analyzed and generalized to other error models, such as the Z-channel, grain-error channel, and projective spaces. Further, it has been shown that the average sphere packing value provides a valid upper bound on code sizes, if the associated hypergraph is regular and symmetric. Repetition errors form a related error model to tandem and palindromic duplications.

1.1 Preliminaries

We denote \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{F}_q^n \) to be a vector of \( n \) symbols, \( x_i \in \mathbb{F}_q \forall i \). The length of a word \( x \) is denoted by \( |x| \). A tandem duplication of length \( \ell \) at position \( i \) with \( 0 \leq i \leq n - \ell \) in a word \( x = (uvw) \), with \( |u| = i, |v| = \ell, |w| = n - \ell - i \) is defined by \( \tau_{i,\ell}(x) = (uvw) \in \mathbb{F}_q^{n+\ell} \) and a palindromic duplication is defined by \( \pi_{i,\ell}(x) = (uvv^Rw) \), where \( v^R = (v_1v_{\ell-1} \ldots v_1) \) is the reversal of \( v \). The inverse operation, a tandem deletion of length \( \ell \) at position \( 0 \leq i \leq n - 2\ell \) in a word \( x = (uvw) \) with \( |u| = i, |v| = \ell, |w| = n - 2\ell - i \) is denoted by \( \tau_{i,\ell}^\delta(x) = (uvw) \in \mathbb{F}_q^{n-\ell} \). Finally, we write a palindromic deletion of length \( \ell \) at position \( 0 \leq i \leq n - 2\ell \) in a word \( x = (uvw^Rw) \) with \( |u| = i, |v| = \ell, |w| = n - 2\ell - i \) as \( \pi_{i,\ell}^\delta(x) = (uvw) \in \mathbb{F}_q^{n-\ell} \). Note that the deletion operations are only defined at positions \( i \), where the word \( x \) is of the form \( (uvw) \), respectively \( (uvw^Rw) \) with \( |u| = i \). With these definitions, the sphere of a word \( x \) is the set of all vectors that are reached by a maximum of \( t \) tandem or palindromic duplications, respectively deletions, i.e.,

\[
S_{i,\ell,\theta,t}(x) = \{y/y_i = \epsilon_{i_1,\ell}(\epsilon_{i_2,\ell}(\epsilon_{i_3,\ell}(\cdots(\epsilon_{i_\theta,\ell}(x))\cdots), \theta \leq t)\},
\]

with \( \epsilon \in \{\tau, \pi, \tau^\delta, \pi^\delta\} \). Here, \( \tau \) and \( \tau^\delta \) denote tandem duplications, respectively deletions and \( \pi, \pi^\delta \) denote palindromic duplications and deletions. By this definition \( x \in S_{i,\ell,\theta,t}(x) \) and the size of these sets depends on \( x \), which is the...
key complication when computing upper bounds on the code cardinality. For a word $x$, let $r(x)$ be the number of runs, respectively $r_i(x)$ the number of runs of length $i$ and $r_{\geq i}(x)$ be the number of runs of length at least $i$ in $x$. For example, the sequence $x = (ATTTAAC)$ has 4 runs, where 2 are of length 1, one is of length 2 and one of length 3. A codebook $C \subseteq F_q^n$ is called $t$-tandem duplication (palindromic duplication, tandem deletion, palindromic deletion) correcting, if $S_{c,t,t}(c_1) \cap S_{c,t,t}(c_2) \neq \emptyset$ implies $c_1 = c_2$ for all $c_1, c_2 \in C$. In the following we will omit the index $t$ for the case $t = 1$ for the above definitions and use the term single-error correcting.

2 Relationship of Duplication and Deletion Codes

2.1 Equivalence of Tandem Duplication and Deletion Codes

For insertion and deletion correcting codes, it has been shown that a code $C$ is $t$-insertion correcting if and only if it is $t$-deletion correcting [5]. A similar behavior can be shown for tandem duplications. To do so, we start with some terminology for tandem duplications, that has been introduced in [4].

**Definition 1 (ℓ-step derivative).** From $x \in F_q^n$ we define the $\ell$-step derivative $\phi_\ell(x) = (u, v)$ with $u = (x_1, x_2, \ldots, x_\ell)$ and $v = (x_{\ell+1}, x_{\ell+2}, \ldots, x_n) - (x_1, x_2, \ldots, x_{n-\ell})$.

It has been shown in [4] that a tandem duplication of length $\ell$ in $x$ corresponds to an insertion of $\ell$ consecutive zeros in $v$. This motivates the introduction of the trunk and zero-signature representation for $v$.

**Definition 2 (Trunk and ℓ-zero signature).** Let $0^m$ denote the $m$-fold repetition of 0 and let $v = (0^{m_0}, w_1, 0^{m_1}, w_2, \ldots, w_p, 0^{m_p})$ with $w_i \in F_q \setminus \{0\}$ and $p = wt_1(v)$ be the Hamming weight of $v$. We then define the trunk of $v$ to be $\mu_\ell(v) = (0^{m_0 \mod \ell}, w_1, 0^{m_1 \mod \ell}, w_2, \ldots, w_p, 0^{m_p \mod \ell})$ as the word that is obtained by shortening every zeros run of length $m$ to be of length $m \mod \ell$. Further, the $\ell$-zero signature of $v$ is defined as $\sigma_\ell(v) = (\lfloor \frac{m_0}{\ell} \rfloor, \lfloor \frac{m_1}{\ell} \rfloor, \ldots, \lfloor \frac{m_p}{\ell} \rfloor)$.

Note that $v$ is uniquely determined by its trunk $\mu_\ell(v)$ and zero-signature $\sigma_\ell(v)$. Further, let $\rho_\ell(x)$ be the root of $x$, which is defined as the tuple $\rho_\ell(x) = (u, \mu_\ell(v))$, where $(u, v)$ is the $\ell$-step derivative of $x$. Note that for notational reasons this definition of the root is slightly different from that in [4]. It is easy to see that a tandem duplication in $x$ corresponds to increasing an entry of $\sigma_\ell(v)$ by 1 (a tandem deletion corresponds to decreasing the entry by 1), but leaves the root $\rho_\ell(x)$ unchanged. With these definitions it is possible to introduce the following metric, that is closely related to tandem duplications.

**Definition 3 (Tandem duplication distance).** For two words $x_1, x_2 \in F_q^n$ with $\ell$-step derivatives $\phi_\ell(x_1) = (u_1, v_1)$ and $\phi_\ell(x_2) = (u_2, v_2)$ we define the tandem duplication distance to be

$$d_{\tau,\ell}(x_1, x_2) = \begin{cases} \infty, & \text{if } \rho_\ell(x_1) \neq \rho_\ell(x_2) \\ |\sigma_\ell(v_1) - \sigma_\ell(v_2)|_1, & \text{if } \rho_\ell(x_1) = \rho_\ell(x_2) \end{cases}$$

(2)
Proof. We denote \( C \subset \mathbb{F}_q^n \) a code. If \( d_{\pi_2}(\mathbf{e}_1, \mathbf{e}_2) \geq 2t + 1 \) for \( \mathbf{e}_1, \mathbf{e}_2 \in C \) then \( \mathbf{e}_1 \neq \mathbf{e}_2 \).

Corollary 1. A code \( C \subset \mathbb{F}_q^n \) is \( t \)-tandem duplication correcting (length \( t \)) if and only if \( d_{\tau_2}(\mathbf{e}_1, \mathbf{e}_2) \geq 2t + 1 \) \( \forall \mathbf{e}_1, \mathbf{e}_2 \in C, \mathbf{e}_1 \neq \mathbf{e}_2 \).

Corollary 2. A code \( C \subset \mathbb{F}_q^n \) is \( t \)-tandem deletion correcting (length \( t \)) if and only if \( d_{\rho_2}(\mathbf{e}_1, \mathbf{e}_2) \geq 2t + 1 \) \( \forall \mathbf{e}_1, \mathbf{e}_2 \in C, \mathbf{e}_1 \neq \mathbf{e}_2 \).

Note that Corollary 1 and 2 imply that if for a codeword \( \mathbf{e}_1 \in C \) of a \( t \)-tandem duplication correcting code \( C \) there exists another codeword \( \mathbf{e}_2 \in C \) with \( \rho_2(\mathbf{e}_1) = \rho_2(\mathbf{e}_2) \), we have \( |\sigma_{\pi_2}(\mathbf{e}_1)|_1 = |\sigma_{\pi_2}(\mathbf{e}_2)|_1 \geq t \). Let us formulate the central theorem of the section which follows from Corollaries 1 and 2.

Theorem 1. A code \( C \subset \mathbb{F}_q^n \) is \( t \)-tandem duplication correcting if and only if it is \( t \)-tandem deletion correcting.

### 2.2 Relationship of Palindromic Duplication and Deletion codes

For palindromic duplication errors, an equivalence similar to Theorem 1 does not hold. A counter example for \( \ell = 2, t = 1 \) is presented here.

Example 1. Let \( C = \{\mathbf{e}_1, \mathbf{e}_2\} \) with \( \mathbf{e}_1 = (\text{AGAGAG}) \) and \( \mathbf{e}_2 = (\text{AGAAGG}) \). \( C \) is palindromic deletion correcting, since \( S_{\pi_2, 2}(\mathbf{e}_1) = \{\mathbf{e}_1\} \) and \( S_{\pi_2, 2}(\mathbf{e}_2) = \{\mathbf{e}_2, (\text{AGAG})\} \) and thus \( S_{\pi_2, 2}(\mathbf{e}_1) \cap S_{\pi_2, 2}(\mathbf{e}_2) = \emptyset \). On the other hand, \( C \) is not palindromic duplication correcting since \( S_{\pi_2}(\mathbf{e}_1) \cap S_{\pi_2}(\mathbf{e}_2) = \{\text{AGAAGGAG}\} \).

However, the following lemma can be shown (for its proof see Appendix B).

Lemma 1. \( \pi_{i,\ell}(\mathbf{x}_1) \neq \pi_{j,\ell}(\mathbf{x}_2) \Rightarrow \pi_{i,\ell}^{t}(\mathbf{x}_1) \neq \pi_{j,\ell}^{t}(\mathbf{x}_2) \forall \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{F}_q^n \).

In other words, if a palindromic duplication in \( \mathbf{x}_1 \) at position \( i \) and a palindromic duplication in \( \mathbf{x}_2 \) at position \( j \) do not result in the same descendant, neither do palindromic deletions at the same positions \( i \) and \( j \). This allows to formulate the following theorem.

Theorem 2. Every single palindromic duplication correcting code \( C \subset \mathbb{F}_q^n \) is a single palindromic deletion correcting code.

Proof. Let \( C \) be a single palindromic duplication correcting code. For every two distinct codewords \( \mathbf{e}_1, \mathbf{e}_2 \in C, \mathbf{e}_1 \neq \mathbf{e}_2 \), we have \( S_{\pi_2}(\mathbf{e}_1) \cap S_{\pi_2}(\mathbf{e}_2) = \emptyset \). This implies that \( \pi_{i,\ell}(\mathbf{e}_1) \neq \pi_{j,\ell}(\mathbf{e}_2) \) for all \( i, j \in \{0, 1, \ldots, n - \ell\} \). With Lemma 1 it follows directly that \( \pi_{i,\ell}^{t}(\mathbf{e}_1) \neq \pi_{j,\ell}^{t}(\mathbf{e}_2) \) for all \( i, j \in \{0, 1, \ldots, n - 2\ell\} \) and thus \( S_{\pi_2,\ell}(\mathbf{e}_1) \cap S_{\pi_2,\ell}(\mathbf{e}_2) = \emptyset \). \( \Box \)

### 3 Sphere Sizes for Tandem and Palindromic Duplications and Deletions

In the following we derive the size of the spheres \( S_{\tau_2,\ell}(\mathbf{x}) \), see (1), for tandem and palindromic duplication and deletion errors. Note that by the definition of the error sphere \( S_{\tau_2,\ell}(\mathbf{x}) \), \( S_{\tau_2,\ell}(\mathbf{x}) \) contains \( \mathbf{x} \), which results in a sphere size that is equal to the number of descendants plus one. For the subsequent two lemmas we denote \( \phi_2(\mathbf{x}) = (\mathbf{u}, \mathbf{v}) \), according to the definition from Section 2.4.
3.1 Tandem Duplication Sphere

Lemma 2. \[|S_{\tau,\ell,t}(x)| = \sum_{j=0}^{t} \left( \text{wt}_H(v) + j \right) = \left( t + \text{wt}_H(v) + 1 \right).\]

Proof. Recall that a tandem duplication error corresponds to increasing one entry of the zero signature \(\sigma_\ell(v)\) by one. Then, the duplication sphere size equals the number of zero signatures \(\sigma \in \mathbb{N}^{\text{wt}_H(v)+1}\) with \(\sigma_i \geq \sigma_\ell(v)_i\) and \(|\sigma|_1 - |\sigma_\ell(v)|_1 \leq t\). \(\square\)

3.2 Tandem Deletion Sphere

Lemma 3. \[|S_{\tau,\ell,1}(x)| = \text{wt}_H(\sigma_\ell(v)) + 1.\]

Proof. It is only possible to delete a tandem duplication at positions, where \(\sigma_\ell(v)_j > 0\). Further, we add one as \(S_{\tau,\ell,1}(x)\) contains \(x\). \(\square\)

3.3 Palindromic Duplication Sphere

The size of the palindromic duplication sphere is not straightforward to derive due to the knotted nature of (11a–11c) and (12a–12c). We start with deriving the palindromic duplication sphere size for the cases \(\ell = 1\) and \(\ell = 2\). For \(\ell = 1\), a palindromic duplication is a single duplication. Therefore, the sphere size is \[|S_{\pi,1}(x)| = r(x) + 1,\] as duplications in the same run yield the same outcome.

Lemma 4. The size of the palindromic duplication sphere \(|S_{\pi,2}(x)|\) is

\[|S_{\pi,2}(x)| = n - \sum_{i=3}^{n} (i-2) r_i(x) = 2r(x) - r_1(x).\]

Proof. We start with the observation that there are \(n-1\) possible positions \(i \in \{0, 1, \ldots, n-2\}\) for palindromic duplications. Now, for \(\ell = 2\), the conditions \(\pi_i,\ell(x) = \pi_{i+j,\ell}(x)\) \(11a\) and \(12a\) become \(x_1 = x_2 = \cdots = x_{2+j} \forall j > 0\). We therefore deduce that two palindromic duplications in \(x\) of length 2 only result in the same vector \(y = \pi_i,\ell(x) = \pi_{i+j,\ell}(x)\) if they appear in the same run in \(x\). Further, two palindromic duplications at two different positions \(i\) and \(i+j, j > 0\) can only duplicate symbols from the same run, if this run has length at least 3. Thus, every additional symbol to runs of length at least 2 does not increase the duplication sphere size and has to be subtracted from the palindromic duplication sphere size. Using \(\sum_{i=1}^{n} ir_i(x) = n\) and \(\sum_{i=1}^{n} r_i(x) = r(x)\) yields the statement. \(\square\)
For $\ell \geq 3$ and $j \geq 2$, (11a–11c) and (12a–12c) do not imply $x_1 = x_2 = \cdots = x_{\ell+j}$.
For example, consider $\ell = 3$ and the word $x = (ACAACA)$. Then, $\pi_{0,3}(x) = \pi_{3,3}(x) = (ACAACAACA)$. However, it is possible to find an upper bound on the size of the palindromic duplication sphere. For $j = 1$, (11a–11c) become $x_1 = x_2 = \cdots = x_{\ell+1}$. Therefore two neighboring palindromic duplications can only result in the same word if they appear in one run.

**Lemma 5.**

$$|S_{\pi,\ell}(x)| \leq n - \ell + 2 - \sum_{i=\ell+1}^{n} (i - \ell)r_i(x)$$

**Proof.** There are $n - \ell + 1$ possible positions for palindromic duplications of length $\ell$. Now, as seen before, duplications in the same run result in the same descendant. We therefore subtract the additional $i - \ell$ entries of runs with length at least $\ell + 1$ from the number of possible positions for duplications to obtain an upper bound on the duplication sphere. $\Box$

### 3.4 Palindromic Deletion Sphere

Similar to the previous section, we start with deriving the size of the palindromic deletions spheres for $\ell = 1$ and $\ell = 2$. For $\ell = 1$, a palindromic deletion is a de-duplication of one symbol. Therefore, the size of the error sphere becomes

$$|S_{\pi^+1}(x)| = r_{\geq 2}(x) + 1,$$

where $r_{\geq 2}(x)$ is the number of runs of length at least 2. Further, we derive the following lemma for binary words.

**Lemma 6.** The size of the palindromic deletion sphere $|S_{\pi^+,2}(x)|$ for $q = 2$ is

$$|S_{\pi^+,2}(x)| = r_{\geq 4}(x) + 1,$$

where $r_{\geq 4}(x)$ is the number of runs of length at least 4, that are located at the interior of $x$, i.e., between $x_2$ and $x_{n-1}$. Further, $r_{\geq 4}(x)$ denotes the number of runs of length at least 4 in $x$.

**Proof.** There are 4 possible patterns (0000), (1111), (0110), (1001), at which palindromic deletions of length 2 can occur. Recall that, as we have seen in the proof of Lemma 4, two palindromic deletions of length 2 at two distinct positions in a word $x$ can only results in the same outcome, if they appear in the same run. Every run of length at least 4 contains one of the patterns (0000), (1111) and therefore will contribute one element to the palindromic deletion sphere. The patterns (0110), (1001) contain a run of length exactly 2, that is located in the interior of $x$, such that there is at least one symbol to the left and right of the run. Thus, every run of length 2, that is located in the interior of $x$ also contributes one unique element in the palindromic deletion sphere. Therefore, counting also the element $x$, which is contained in $|S_{\pi^+,2}(x)|$, the total size of the deletion sphere is $r_{\leq 2}(x) + r_{\geq 4}(x) + 1$. $\Box$
Let us define the matrix $A^\pi_\ell(x) \in \mathbb{F}_q^{\ell \times n-2\ell+1}$ to be

$$A^\pi_\ell(x) = \begin{bmatrix} x_{2\ell} - x_1 & x_{2\ell+1} - x_2 & \cdots & x_n - x_{n-2\ell+1} \\ x_{2\ell-1} - x_2 & x_{2\ell} - x_3 & \cdots & x_{n-1} - x_{n-2\ell} \\ \vdots & \vdots & \ddots & \vdots \\ x_{\ell+1} - x_\ell & x_{\ell+2} - x_{\ell+1} & \cdots & x_{n-\ell+1} - x_{n-\ell} \end{bmatrix}. \quad (7)$$

With this definition it is directly possible to establish the following upper bound on the size of the palindromic deletion spheres for arbitrary deletion length $\ell$.

**Lemma 7.** The palindromic deletion sphere $|S_{x,\ell}(x)|$ is upper bounded by

$$|S_{x,\ell}(x)| \leq r^{(0)}(A^\pi_\ell(x)) + 1,$$

where $r^{(0)}(A^\pi_\ell(x))$ is the number of runs of all zero columns in $(A^\pi_\ell(x))$.

**Proof.** Clearly, a palindrome of length $\ell$ in the word $x$ corresponds to a zero column in the matrix $A^\pi_\ell(x)$. Therefore palindromic deletions are only possible at positions $i$, where $A^\pi_\ell(x)$ has a zero-column. Further, it can be shown that two neighboring zero columns are only possible if $x_{i+1} = x_{i+2} = \cdots = x_{i+2\ell+1}$, i.e. for a run of length $2\ell + 1$. However, two palindromic deletions inside the same run result in the same words. Therefore, every run of all zero columns in $(A^\pi_\ell(x))$ contributes one unique element to $S_{x,\ell}(x)$. \hfill \Box

## 4 Bounds on the Code Size

In this section, we derive bounds for tandem and palindromic deletion correcting codes, which also provide a bound for the duplication correcting codes, since every tandem/palindromic duplication correcting code is a deletion correcting code, as we have shown in Section 2. For single duplication correcting codes, we deduce the following from [2].

**Theorem 3.** The maximum cardinality $|C^\ast(n, \ell)|$ of tandem duplication (palindromic duplication) correcting codes of length $n$ satisfies [2]

$$|C^\ast(n, \ell)| \leq \sum_{x \in \mathbb{F}_q^n \cup \mathbb{F}_q^{-n}} \frac{1}{|S_{x,\ell}(x)|}, \quad (8)$$

where $\epsilon = r^\delta$ for tandem and $\epsilon = r^\delta$ for palindromic duplication correcting codes.

Note that it is necessary to formulate the fractional transversal sum over all words of length $n$ and $n - \ell$, since the spheres $S_{x,\ell}(x), x \in \mathbb{F}_q^n$ contain words both of length $n$ and $n - \ell$. The bound (8) can be rephrased to

$$|C^\ast(n, \ell)| \leq \sum_{i=0}^{i_{\text{max}}} N_e(n, \ell, i) + N_e(n - \ell, \ell, i) \cdot i + 1, \quad (9)$$

where $N_e(n, \ell, i) = |\{ x \in \mathbb{F}_q^n : |S_{x,\ell}(x)| = i + 1 \}|$ is the number of words of length $n$ with sphere size $i + 1$ and $i_{\text{max}}$ is the maximum sphere size. The next sections are directed towards finding $N_e(n, \ell, i)$ for our error models.
4.1 Bound for Tandem Deletions

Similar to the strategy in [2], we have to compute the number of words of length \( n \) with sphere size \( i + 1 \).

Lemma 8.

\[
N_{τ, i}(n, ℓ, i) = |\{ x ∈ F_2^n : |S_{τ, i}(x)| = i + 1 \}| =
\]

\[
= \sum_{ν=i}^{n-1} \sum_{ω=1}^{ℓ} q^ν A(n - (ν + 1)ℓ, ℓ - 1, ω) \binom{ω + 1}{i} \binom{ν - 1}{i - 1},
\]

where \( A(n', ℓ', ω) \) is the number of all words \( x ∈ F_q^n \) that have zero-runs of length at most \( ℓ' \) and Hamming weight \( ω \).

Proof. We consider the \( ℓ \)-step derivative \( φ_ℓ(x) = (u, v) \). According to Lemma 3, the size of the tandem deletion sphere is given by \( |S_{τ, i}(x)| = wt_H(σ_ℓ(v)) + 1 \) and we therefore want to find the number of words \( x ∈ F_q^n \) with \( wt_H(σ_ℓ(v)) = i \).

Let \( ν \) be the number of length \( ℓ \) tandem duplications in \( x \), i.e. \( |σ_ℓ(v)| = ν \). Further let \( J \) denote the support set of \( σ_ℓ(v) \), i.e. \( J = \{ m : σ(v)_m ≠ 0 \} \), with \( |J| = i \). The number of possibilities to distribute the duplications into \( σ_ℓ(v) \) for a given support \( J \) is equal to the number of solutions of

\[
\sum_{j=1}^{i} y_j = ν, \quad y_j ∈ N, \; ∀ 1 ≤ j ≤ i.
\]

(10)

This number is given by \( \binom{ν - 1}{i - 1} \) [Lemma 2.2]. Further, let \( ω \) be the Hamming weight of the trunk, i.e. \( wt_H(μ_ℓ(v)) = ω \) and thus \( |σ_ℓ(v)| = ω + 1 \), which corresponds to the number of unambiguous positions for tandem duplications of length \( ℓ \). The number of possible support sets \( J \) of \( σ_ℓ(v) \) with \( |J| = i \) then is \( \binom{ω + 1}{i} \). The vector \( μ_ℓ(v) \) can be chosen to be any \( q \)-ary vector of length \( n - (ν + 1)ℓ \) that has zero-runs of length at most \( ℓ - 1 \) and Hamming weight \( ω \). The number of such vectors is given by \( A(n - (ν + 1)ℓ, ℓ - 1, ω) \). At last, the first \( ℓ \) symbols \( u ∈ F_q^ℓ \) can be chosen arbitrarily and thus have \( q^ℓ \) possibilities. \( \Box \)

It can be deduced from the results in [3] that for \( ω ≥ 2 \) the number of all \( q \)-ary vectors of length \( n' \), maximum zero-run length \( ℓ' \) and weight \( ω \) is given by

\[
A(n', ℓ', ω) = (q - 1)^ω \left\{ \begin{array}{c}
\sum_{p=0}^{ℓ'} \frac{ω - 1}{j} \binom{ω - 1}{j} \binom{n' - p - 1 - (j + 1)ℓ'}{ω - 1} - n' > ℓ' \\
\binom{n' - p - 1 - (j + 1)ℓ'}{ω - 1},
\end{array} \right.
\]

(11)

For \( ω = 0 \) and \( ω = 1 \), it is easy to verify that \( A(n', ℓ', 0) = 1 \) if \( n' ≤ ℓ' \), 0 otherwise, and \( A(n', ℓ', 1) = (q - 1) \max\{0, 2(ℓ' + 1) - n'\} \).
4.2 Bound for Palindromic Deletions of Length $\ell = 2$

Lemma 9.

$$N_{\pi}(n, 2, i) = \{|x \in \mathbb{F}_2^n : |S_{\pi_{1,2}}(x)| = i + 1\} = 2 \sum_{r_1=0}^{n} \sum_{r_2=0}^{\frac{n}{2}} \sum_{r_3=0}^{2} \sum_{r_4=0}^{2} \left( \frac{2}{r_{B,2}} \right) \left( \frac{2}{r_3} \right) \left( \frac{r_1 + r_3}{i - r_{I,2}} \right) \left( \frac{r_1 + r_3 + i - r_{I,2}}{r_{I,2}} \right) \left( \frac{r_1 + r_3 + i + r_{B,2} - 2}{i - r_{I,2}} \right) \left( \frac{n - r_1 + r_{I,2} - 3(r_3 + i) - 2r_{B,2} - 1}{i - r_{I,2}} \right).$$

Proof. By Lemma 6 we have to find the number of words $x \in \mathbb{F}_2^n$ with $r_{I,2}(x) + r_{\geq 4}(x) = i$. Let $r_1, r_2, r_3, r_4$ denote the number of runs of length 1, 2, 3, respectively length at least 4 in $x$. Further, let $r_{I,2}$ denote the number of runs of length 2 in the interior of $x$ and $r_{B,2} \in \{0, 1, 2\}$ the number of runs of length 2, that are located at the boundaries of $x$. Then, $r_2 = r_{I,2} + r_{B,2}$.

We start with counting the number of words with a given run-distribution $r_1, r_2, r_3$ and $r_{\geq 4}$. To begin with, we insert runs of length 3 between the runs of length 1. In total, there are $(\binom{\ell}{r_1 + r_3})$ possible such arrangements. We then insert $r_{\geq 4}$ runs of length at least 4 between these runs of length 1, respectively 3. There are $(\binom{r_1 + r_3 + r_{\geq 4}}{r_{\geq 4}})$ possibilities to do so. Next, we insert the $r_{I,2}$ runs of length 2 into a given constellation of runs of length 1, 3 and length at least 4. As those runs cannot be inserted at the beginning or ending of $x$, there are henceforth $(\binom{r_1 + r_2 + r_3 + r_{\geq 4} - 2}{r_{I,2}})$ possible combinations. As a final assembling step, we append the $r_{B,2}$ runs of length 2 to the left and right of $x$, where we have $(\binom{2}{r_{B,2}})$ possibilities of choosing positions for the runs at the boundaries. Finally, using [1] Lemma 2.2, there are

$$\left( \frac{n - r_1 - 2r_2 - 3r_3 - 3r_{\geq 4} - 1}{r_{\geq 4} - 1} \right)$$

possibilities to choose the lengths of the runs of length at least 4, since there are $n - r_1 - 2r_2 - 3r_3$ symbols that can be distributed onto these runs. Substituting $r_{\geq 4} = i - r_{I,2}$ and $r_2 = r_{I,2} + r_{B,2}$ and multiplying by 2, since the first run can either start with 0 or 1 yields the statement. \hfill \Box

4.3 Comparison with Burst Insertion Correcting Codes

Figure 1 shows the lower bounds (LB) on the redundancy for binary codes and different duplication lengths $\ell$. We compare our results with maximum redundancies of single burst insertion correcting codes from [7]. To the best of our knowledge, these constructions have the largest codebooks that can correct a single burst insertion. The figure also includes the redundancies from a single tandem duplication correcting construction with cardinality at least

$$|C_{VT}(n, \ell)| \geq 2^\ell \cdot \sum_{\omega=0}^{n-\ell} \left| \frac{n-\ell}{\omega + 2} \right|,$$
which can be shown to exist based on the principle from [4] using Varshamov-
Tenengolts codes [5]. Interestingly, there is a significant gap between the redun-
dancies of existing burst insertion constructions, which motivates a specialized
code construction that corrects tandem and palindromic duplication errors.

![Fig. 1. Tandem/palindromic duplication bounds vs. burst insertion redundancies](image)

A Conditions for Equivalence of Palindromic Duplications in one Word

In this section we derive conditions that two palindromic duplications, respec-
tively deletions at two different positions $i$ and $i + j$ with $j > 0$ result in the
same word $\epsilon_{i,\ell}(x) = \epsilon_{i+j,\ell}(x)$ for $\epsilon \in \{\pi, \pi^3\}$. These conditions help to find the
sphere sizes $|S_{\epsilon,\ell,1}(x)|$, as it has been illustrated in Section 3.

A.1 Palindromic Duplications

For $j < \ell$ the condition $\pi_{i,\ell}(x) = \pi_{i+j,\ell}(x)$ can be expressed as (the left hand
side of the equations corresponds to $\pi_{i+j,\ell}(x)$ and the right hand side to $\pi_{i,\ell}(x)$)

\[ x_{i+\ell+1+m} = x_{i+\ell-m}, \quad m \in \{0, \ldots, j-1\}, \] (11a)
\[ x_{i+\ell+2j-m} = x_{i+\ell-m}, \quad m \in \{j, \ldots, \ell-1\}, \] (11b)
\[ x_{i+\ell+2j-m} = x_{i+1+m}, \quad m \in \{\ell, \ldots, \ell+j-1\}. \] (11c)

For $j \geq \ell$ these conditions are

\[ x_{i+\ell+1+m} = x_{i+\ell-m}, \quad m \in \{0, \ldots, \ell-1\}, \] (12a)
\[ x_{i+\ell+1+m} = x_{i+1+m}, \quad m \in \{\ell, \ldots, j-1\}, \] (12b)
\[ x_{i+\ell+2j-m} = x_{i+1+m}, \quad m \in \{j, \ldots, \ell+j-1\}. \] (12c)
A.2 Palindromic Deletions

The conditions $\pi^j_i(x) = \pi^j_{i+j}(x)$ for $j > 0$ are

\[
\begin{align*}
&x_{i+\ell+1+m} = x_{i+\ell-m}, \quad m \in \{0, \ldots, \ell - 1\}, \\
&x_{i+\ell+j+1+m} = x_{i+\ell+j-m}, \quad m \in \{0, \ldots, \ell - 1\}, \\
&x_{i+2\ell+1+m} = x_{i+\ell+1+m}, \quad m \in \{0, \ldots, j - 1\}.
\end{align*}
\]

(13a)

(13b)

(13c)

B Proof of Lemma 1

Consider the conditions for $x_1, x_2, \ldots, x_n$ that result from $\pi_{i,\ell}(x) = \pi_{i+j,\ell}(x)$ and $\pi^j_i(x) = \pi^j_{i+j}(x)$ in Appendix A. The lemma can then be shown by proving that (13a–13c) imply both (11a–11c) and (12a–12c). Note that for simplicity, we set $i = 0$ w.l.o.g. Let us start with the case $j < \ell$. (13a) directly implies (11a).

Rewriting (13b) yields $x_{\ell+1+m} = x_{\ell+2j-m}, \ m \in \{j, \ldots, \ell + j - 1\}$ and inserting it into (13a) results in $x_{\ell+2j-m} = x_{\ell-m}, \ m \in \{j, \ldots, \ell - 1\}$, which is (11b).

To obtain (11c), we use the modified version of (13b) from above and further observe that from (13c), we get $x_{\ell+1+m} = x_{1+m}, \ m \in \{\ell, \ldots, \ell + j - 1\}$.

The case $j \geq \ell$ can be shown in a similar way. (13a) directly implies (12a) and from (13c) follows (12b). Again, rewriting (13b) yields $x_{\ell+1+m} = x_{\ell+2j-m}, \ m \in \{j, \ldots, \ell + j - 1\}$. Writing additionally (13c) as $x_{\ell+1+m} = x_{1+m}, \ m \in \{\ell, \ldots, \ell + j - 1\}$, and noticing $j \geq \ell$, yields (12c).

\[\square\]

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