Infinite Hopf Families of Algebras and Yang-Baxter Relations

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Abstract

A Yang-Baxter relation-based formalism for generalized quantum affine algebras with the structure of an infinite Hopf family of (super-) algebras is proposed. The structure of the infinite Hopf family is given explicitly on the level of $L$ matrices. The relation with the Drinfeld current realization is established in the case of $4 \times 4$ $R$-matrices by studying the analogue of the Ding-Frenkel theorem. By use of the concept of algebra “comorphisms” (which generalize the notion of algebra comodules for standard Hopf algebras), a possible way of constructing infinitely many commuting operators out of the generalized $RLL$ algebras is given. Finally some examples of the generalized $RLL$ algebras are briefly discussed.

1 Introduction

Yang-Baxter relations ($RTT$ relations) have proved to be a central ingredient in 2D integrable models and quantum algebras over the last 20 years. In quantum algebras, the Yang-Baxter relations have been used in several different contexts, including

- standard finite quantum groups (FRT formalism \([8]\), i.e. the $RTT = TTR$ relation

$$R_{12}T_1T_2 = T_2T_1R_{12}$$

in which $R$ solves the spectral parameter-less Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

and $T$ also does not depend on any spectral parameters;

- quantum affine algebras (RS formalism \([24]\), i.e. the $RLL$ relations (here and below $u_\pm \equiv u \pm \frac{bc}{\hbar}$ unless otherwise specified)

$$R_{12}(u - v)L_1^+(u)L_2^+(v) = L_2^+(v)L_1^+(u)R_{12}(u - v),$$

$$R_{12}(u_+ - v_+)L_1^+(u)L_2^+(v) = L_2^+(v)L_1^+(u)R_{12}(u_+ - v_+),$$

in which the first equation is defined for any spectral parameter $u$ and $v$, while the second equation is defined only for $|u| < |v|$, and $R$ solves the spectral Yang-Baxter equation

$$R_{12}(u - v)R_{13}(v - w)R_{23}(u - w) = R_{23}(u - w)R_{13}(v - w)R_{12}(u - v);$$

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• the $RLL = LLR^*$ relations \cite{11, 12}

\[
R_{12}(u - v) L_1^+(u) L_2^+(v) = L_2^+(v) L_1^+(u) R_{12}(u - v),
\]
\[
R_{12}(u - v_+) L_1^+(u) L_2^-(v) = L_2^-(v) L_1^+(u) R_{12}^*(u_+ - v_-),
\]

where the first equation is defined for any spectral parameter $u$ and $v$, while the second equation is defined only for $|u| < |v|$, and $R$ and $R^*$ both solve the spectral Yang-Baxter equation

\[
R_{12}(u - v) R_{13}(v - w) R_{23}(u - w) = R_{23}(u - w) R_{13}(v - w) R_{12}(u - v),
\]

and are related by some analytical relations (such as modular transformations when the entries of $R$ and $R^*$ are elliptic functions);

• dynamical $RLL$ relations \cite{7, 8, 10}

\[
R_{12}(u - v, \lambda) L_1^+(u, \lambda) L_2^+(v, \lambda) = L_2^+(v, \lambda) L_1^+(u, \lambda) R_{12}(u - v, \lambda),
\]
\[
R_{12}(u - v_+, \lambda) L_1^+(u, \lambda) L_2^-(v, \lambda) = L_2^-(v, \lambda) L_1^+(u, \lambda) R_{12}^*(u_+ - v_-, \lambda),
\]

in which the first equation is defined for any spectral parameter $u$ and $v$, while the second equation is defined only for $|u| < |v|$, and $R$ solves the dynamical spectral Yang-Baxter equation

\[
R_{12}(u - v, \lambda - h(3)) R_{13}(v - w, \lambda) R_{23}(u - w, \lambda - h(1)) = R_{23}(u - w, \lambda) R_{13}(v - w, \lambda - h(2)) R_{12}(u - v, \lambda);
\]

• the combination of $RLL = LLR^*$ and the dynamical $RLL$ relations \cite{14, 17},

\[
R_{12}(u - v, \lambda) L_1^+(u, \lambda) L_2^+(v, \lambda) = L_2^+(v, \lambda) L_1^+(u, \lambda) R_{12}(u - v, \lambda),
\]
\[
R_{12}(u - v_+, \lambda) L_1^+(u, \lambda) L_2^-(v, \lambda) = L_2^-(v, \lambda) L_1^+(u, \lambda) R_{12}^*(u_+ - v_-, \lambda),
\]

where again the third relation is defined only for $|u| < |v|$. Of course, this list does not exhaust all of the uses of Yang-Baxter relations in defining quantum affine algebras — first of all there are versions of the above formalisms corresponding to deformations of Lie super-algebras, in which all $L$-matrices appearing on the left-hand side have to be multiplied by a numerical signature matrix from the right, while all $L$-matrices appearing on the right-hand side have to be multiplied by the same numerical signature matrix from the left \cite{13} — and there are some other rare cases in which some other strange deformed versions of $RLL$ relations are used.

Despite the numerous different forms of the Yang-Baxter relations in different quantum algebraic contexts, there are essential factors that are common to most of these: there are alternative, essentially Yang-Baxter relation-free definitions (or realizations) of the same quantum algebras (if the $R$-matrices are of the form $R_{ij} = a_{ij} \delta^k_i \delta_j^l + b_{ij} \delta^k_i \delta_j^k$, anyway), and every known such algebra (i.e. one which admits any one of the above formalisms) falls into the class of quasi-triangular quasi-Hopf algebras — that is, it is a certain twist of a standard triangular quasi-Hopf algebra \cite{3, 17}.

In practice, there are natural reasons to use many different realizations for the same quantum algebra. For instance, to apply these algebras in integrable models in 2D physics, the best formalism to use is often the $RLL$-like form; to study infinite dimensional representations, the best formalism is the so-called Drinfeld current realization; and to study structure and classification problems the best formalism is the Chevalley generating relations. For these reasons the issue of the relations between different formalisms is an important subject, and would be by no means an easy task were it not for the famous Ding-Frenkel theorem which establishes an explicit relationship between the Yang-Baxter realization and the Drinfeld current realization.

Now our problem arises: on the one hand, in the study of two-parameter deformations of affine Lie (super) algebras, we have accumulated a number of different two-parameter quantum affine (super-)algebras which are
members of the so-called infinite Hopf family of (super-)algebras [13, 16, 21, 27, 28] — a co-structure which generalizes the standard Drinfeld coproduct — for standard quantum affine algebras — and we know nothing about the corresponding RLL formalism; on the other hand, many authors have studied two-parameter quantum affine algebras [1, 2, 3, 4, 5, 6, 13, 15, 20], but from the point of view of Yang-Baxter realizations only, and the co-structures studied are standard Drinfeld twists of Hopf algebras. Can these results be related by use of an analogue of Ding-Frenkel theorem? That is, does there exist an RLL formalism for two-parameter quantum affine algebras with the structure of an infinite Hopf family of (super-)algebras? How far can we go in generalizing both the RLL formalism and the Ding-Frenkel theorem? In this paper, we will partly address these problems by introducing a generalized RLL relation which contains all the above listed formalisms as special cases. We shall consider some of the necessary conditions for such a generalized formalism to be consistent, which result in severe restrictions on the possible choices of the $R$-matrices. Then, we shall establish the infinite Hopf family structure over the generalized RLL formalism and study the analogue of the Ding-Frenkel theorem in the special case of $4 \times 4$ $R$-matrices. We also outline a way to construct infinitely many commuting operators out of our generalized RLL algebras and finally discuss briefly some possible examples of our construction.

2 The generalized formalism, associativity and co-structure

2.1 The generalized RLL relation

The generalized RLL formalism we shall study is

$$R_{12}^{(i)(j)}(u-v,\lambda)L_1^{\pm(i,j)}(u,\lambda)L_2^{\pm(i,j)}(v,\lambda) = L_2^{\pm(i,j)}(v,\lambda)L_1^{\pm(i,j)}(u,\lambda)R_{12}^{(j)}(u-v,\lambda), \quad (1)$$

$$R_{12}^{(i)(j)}(u-v_-,\lambda)L_1^{+(i,j)}(u,\lambda)L_2^{-(-i,j)}(v,\lambda) = L_2^{-(-i,j)}(v,\lambda)L_1^{+(i,j)}(u,\lambda)R_{12}^{(j)}(u_+-v_-,\lambda), \quad |u| < |v|, \quad (2)$$

$$u_\pm = u \pm \frac{\hbar c^{(i,j)}}{4}, \quad (c^{(i,j)} \text{ is central})$$

Unlike the usual RLL formalisms listed in the introduction, the $R$-matrices are now labeled by an upper index $(i)$ and the $L$-matrices by two upper indices $(i,j)$ where $i,j \in \mathbb{Z}$. The parameters $u, v$ are the usual spectral parameters, and $\lambda$ is a dynamical variable (which may or may not appear at all).

At this moment we assume no restrictions on the possible form of the $R$-matrices, and in particular we have assigned no meaning to the extra upper indices of $R$ and $L$ and no relationship between the two $R$-matrices carrying different upper indices at all. Such restrictions will emerge as we consider the associativity and co-structure of the algebras defined by the above relation.

2.2 Associativity

Any work on quantum algebras should ensure that the algebra being studied is associative, and, in principle, could be equipped with a co-structure, so that tensor products of representations may be defined. So we shall first examine our generalized RLL relations to ensure that they really define associative algebras. To this end, let us consider the case of product of $3$ $L$-matrices $L_1^{(i,j)}(u,\lambda)L_2^{(i,j)}(v,\lambda)L_3^{(i,j)}(w,\lambda)$. If the algebra is associative, we should be able to apply this product in two different ways, i.e.

$$L_1^{(i,j)}(u,\lambda)L_2^{(i,j)}(v,\lambda)L_3^{(i,j)}(w,\lambda) = [L_1^{(i,j)}(u,\lambda)L_2^{(i,j)}(v,\lambda)]L_3^{(i,j)}(w,\lambda) = L_1^{(i,j)}(u,\lambda)[L_2^{(i,j)}(v,\lambda)L_3^{(i,j)}(w,\lambda)].$$

Now temporarily we omit the upper indices $\pm$ of $L^{(i,j)}$ and arguments of $R^{(i)}$ and $L^{(i,j)}$ (though the effects of these arguments are not neglected). We now apply the generalized RLL relations first to reverse the order
of $L_1^{(i,j)} L_2^{(i,j)} L_3^{(i,j)}$ and then to return to the original order. We have
\[ R_{32}^{(i)} R_{31}^{(i)} R_{23}^{(i)} R_{12}^{(i)} L_1^{(i,j)} L_2^{(i,j)} L_3^{(i,j)} = L_1^{(i,j)} L_2^{(i,j)} L_3^{(i,j)} R_{32}^{(j)} R_{31}^{(j)} R_{23}^{(j)} R_{12}^{(j)}, \tag{3} \]
where the order of lower indices of the $L$'s is changed in the following mane:
\[
(123) \rightarrow (213) \rightarrow (231) \rightarrow (321) \\
\rightarrow (312) \rightarrow (132) \rightarrow (123).
\]
On the first line we changed $L_1^{(i,j)} L_2^{(i,j)} L_3^{(i,j)}$ into $L_3^{(i,j)} L_2^{(i,j)} L_1^{(i,j)}$ using left-grouping of the successive products, while on the second line we changed $L_3^{(i,j)} L_2^{(i,j)} L_1^{(i,j)}$ back into $L_1^{(i,j)} L_2^{(i,j)} L_3^{(i,j)}$ using right-grouping, so associativity has been implied in the above operation. A sufficient but probably not too restrictive condition to ensure the correctness of the equation (3) is that
\[ R_{32}^{(i)} R_{31}^{(i)} R_{23}^{(i)} R_{12}^{(i)} = R_{32}^{(j)} R_{31}^{(j)} R_{23}^{(j)} R_{12}^{(j)} = 1. \]
Therefore, assuming unitarity of $R$-matrices, i.e.
\[ R_{12}^{(i)}(u, \lambda) R_{21}^{(i)}(-u, \lambda) = 1, \tag{4} \]
\[ R_{12}^{(j)}(u, \lambda) R_{21}^{(j)}(-u, \lambda) = 1, \tag{5} \]
we conclude that $R^{(i)}$ and $R^{(j)}$ must separately satisfy the (probably generalized) dynamical Yang-Baxter equation
\[ R_{12}^{(k)}(u - v, \lambda - h^{(i)}) R_{13}^{(k)}(v - w, \lambda) R_{23}^{(k)}(u - w, \lambda - h^{(i)}) = R_{23}^{(k)}(u - w, \lambda) R_{13}^{(k)}(v - w, \lambda - h^{(i)}) R_{12}^{(k)}(u - v, \lambda), \tag{6} \]
Later we shall see that unitarity of the $R$-matrices is also required if the Drinfeld current realization is considered, and will introduce a coproduct which is compatible with the infinite Hopf family in that realization.

Let us stress that so far we have not assumed any relationship between $R^{(i)}$ and $R^{(j)}$. The different upper indices only indicate that the two $R$-matrices may be different. Thus associativity has not resulted in any restrictions on the relation between $R$-matrices.

### 2.3 Co-structure

In contrast, however, the definition of a co-structure does require some relationship between $R^{(i)}$ and $R^{(j)}$, as we discuss below.

First we recall what we mean by a co-structure. Algebraically a co-structure is a property of algebras which allows the definition of a tensor product between two algebras (usually two copies of the same algebra — in that case the algebra is called co-closed). For instance, for algebras defined by the non-dynamical $RLL = LLR$ relations, the co-structure is just the standard Hopf algebra structure; in particular, the coproduct is simply an operation which creates a generalized $L$ matrix $\mathcal{L}^{(n)}$ which obey the $R\mathcal{L}^{(n)} \mathcal{L}^{(n)} = \mathcal{L}^{(n)} R \mathcal{L}^{(n)}$ relation with the same $R$, where
\[
\mathcal{L}^{(n)} = \Delta^{(n)} L = (id \otimes \ldots \otimes id \otimes \Delta) \circ (id \otimes \ldots \otimes \Delta) \circ \ldots \circ (id \otimes \Delta) \circ \Delta L = L \otimes L \otimes \ldots \otimes L
\]
is the $n$-th coproduct of $L$, in which $L$ can be either $L^+$ or $L^-$, and $L \otimes L$ is defined via $(L \otimes L)^{\alpha} = \sum_c L_c^\alpha \otimes L_c^\alpha$. The point is that applying $R_{12}$ to the left hand side of $L_1 \mathcal{L}_2$ would result in
\[ R_{12} \mathcal{L}_1^{(n)} \mathcal{L}_2^{(n)} = L_2 L_1 R_{12} \mathcal{L}_1^{(n-1)} \mathcal{L}_2^{(n-1)} = \ldots = \mathcal{L}_2^{(n)} \mathcal{L}_1^{(n)} R_{12}, \tag{7} \]
so that the $n$-th coproduct $\Delta^{(n)}$ is an algebra homomorphism.

For algebras defined by relations like $RLL = LLR^*$, this is impossible because after the first use of the $RLL$ relation, a different $R$-matrix $R^*$ appears and the iteration stops. One possible way round is to introduce a twist operation such that the $R^*$ which appears after each use of the $RLL$ relation is twisted back to $R$ and the iteration can be continued. This amounts to the notion of the Drinfeld twist in quasi-Hopf algebras, and the possibility of defining the twist operation imposes severe restrictions on the relations between $R$ and $R^*$.

Since we assumed no relationship between $R^{(i)}$ and $R^{(j)}$ for $i \neq j$ so far, our situation is very much like the case of $RLL = LLR^*$ relations. However, our aim here is not to resort to a Drinfeld twist but rather to incorporate the structure of the infinite Hopf family in the generalized $RLL$ formalism. Therefore, instead of retaining the tensor product of many copies of the same algebra, we can think about making tensor products between different algebras defined by our generalized $RLL$ formalism — this is exactly what we did in the Drinfeld current realization of infinite Hopf family of algebras.

In order to incorporate the infinite Hopf family structure directly at the level of $RLL$ formalism, however, we have to introduce a relationship between the two $R$-matrices $R^{(i)}$ and $R^{(j)}$ in our generalized $RLL$ formalism.

First we choose some unitary solution of the (dynamical) Yang-Baxter equation (8), denoted $R^{(0)}(u, \lambda)$. We then introduce invertible operators $\rho^\pm_0$ which act on this $R$-matrix to give $R^{(\pm 1)}(u, \lambda) = \rho^\pm_0 R^{(0)}(u, \lambda)$. The permitted operators $\rho^\pm_0$ are such that they preserve unitarity as well as the Yang-Baxter equation. On $R^{(\pm 1)}(u, \lambda)$ we may again act with $\rho^\pm_1$ (with the same property as $\rho^\pm_0$) to get $R^{(\pm 2)}(u, \lambda)$ etc. Assume that we have a large (possibly infinite) set $\Upsilon$ of $R$-matrices whose elements are obtained by recursively applying the relation $R^{(\pm (n+1))}(u, \lambda) \equiv (\rho^\pm_n) R^{(\pm n)}(u, \lambda)$. An additional requirement is that $\rho^\pm_{n+1} \circ \rho^\mp_n = \rho^\pm_{n-1} \circ \rho^\mp_n = id_n$, so that $\Upsilon$ forms a single $\rho^\pm_n$ orbit. We may then put these $R$-matrices into our generalized $RLL$ relations.

These relations define a huge family of associative algebras $\{A_{i,j} \equiv A(R^{(i)}, R^{(j)}), i, j \in \mathbb{Z} \}$, each labeled by two ordered integers corresponding to the upper indices of the two $R$-matrices in the above relations. Among these algebras we are particularly interested in the ones labeled by two successive integers, i.e. $A_{n,n+1} \equiv A_n$. For these algebras we simplify the notation as follows: $L^{\pm(n,n+1)} \rightarrow L^{\pm(n)}$, $c^{(n,n+1)} \rightarrow c_n$. It is remarkable that for each pair of algebras $A_n$ and $A_{n \pm 1}$ there exists an algebra morphism

$$\tau^\pm_n : A_n \rightarrow A_{n \pm 1}$$

defined via

$$\tau^+_n L^{\pm(n)} = L^{\pm(n+1)} , \quad \tau^-_n L^{\pm(n)} = L^{\pm(n-1)} ,$$

$$\tau^+_n R^{(n)}(u, \lambda) = \rho^+_n R^{(n)}(u, \lambda) = R^{(n+1)}(u, \lambda) ,$$

$$\tau^-_n R^{(n)}(u, \lambda) = \rho^-_{n+1} R^{(n+1)}(u, \lambda) = R^{(n+2)}(u, \lambda) .$$

The algebra morphisms $\tau^\pm_n$ also obey the condition

$$\tau^-_{n+1} \circ \tau^+_n = \tau^+_n \circ \tau^-_n = id_n ,$$

because they are essentially the lift of the action of $\rho^\pm_n$ onto the algebras $A_n$. For any pair of integers $n < m$, the composition of $\tau^\pm_n$

$$Mor(A_m, A_n) \ni \tau^{(m,n)} \equiv \tau^{(m,n)} \equiv \tau^{(m-1)} \circ \tau^{(m-1)} \circ \tau^+_n \circ \tau^-_{n+1} : A_m \rightarrow A_n ,$$

$$Mor(A_n, A_m) \ni \tau^{(m,n)} \equiv \tau^{(m,n)} \equiv \tau^{(n,m)} \tau^{(m,n)} = \tau^{(m,n)} = \tau^{(n,m)} \tau^{(m,n)} = \tau^{(m,n)}$$

are algebra morphisms between $A_n$ and $A_m$, and since $\tau^{(m,n)} \tau^{(n,m)} = id_m$, $\tau^{(n,m)} \tau^{(m,n)} = id_n$ and $\tau^{(m,p)} \tau^{(p,n)} = \tau^{(m,n)}$, they make the family of algebras $\{A_n, n \in \mathbb{Z} \}$ into a category.

Let us now define the following co-structure:
one can check that the following relations hold:

\[
\Delta^+_n L^{\pm(n)}(u, \lambda) = L^{\pm(n)} \left( u \pm \frac{\hbar c_n + 1}{4} \right) \otimes L^{\pm(n+1)} \left( u \pm \frac{\hbar c_n}{4} \right),
\]

\[
\Delta^-_n L^{\pm(n)}(u, \lambda) = L^{\pm(n-1)} \left( u \pm \frac{\hbar c_n - 1}{4} \right) \otimes L^{\pm(n)} \left( u \pm \frac{\hbar c_n}{4} \right);
\]

- **Antipode**

\[
S^+_n L^{\pm(n)}(u, \lambda) = L^{\pm(n+1)}(u, \lambda)^{-1},
\]

\[
S^-_n L^{\pm(n)}(u, \lambda) = L^{\pm(n-1)}(u, \lambda)^{-1} ;
\]

- **Counit**

\[
\epsilon_n L^{\pm(n)}(u, \lambda) = id_n.
\]

By straightforward verification, we find that the above co-structure obeys the following axioms for infinite Hopf family of algebras \[13, 22, 23, 25\]:

- \((\epsilon_n \otimes id_{n+1}) \circ \Delta^+_n = \tau^+_n\), \((id_{n-1} \otimes \epsilon_n) \circ \Delta^-_n = \tau^-_n\)

- \(m_{n+1} \circ (S^+_n \otimes id_{n+1}) \circ \Delta^+_n = \epsilon_{n+1} \circ \tau^+_n, m_{n-1} \circ (id_{n-1} \otimes S^-_n) \circ \Delta^-_n = \epsilon_{n-1} \circ \tau^-_n\)

- \((\Delta^-_n \otimes id_{n+1}) \circ \Delta^+_n = (id_{n-1} \otimes \Delta^+_n) \circ \Delta^-_n\)

in which \(m_n\) is the algebra multiplication for \(A_n\). Moreover, it is easy to check that the images of \(\Delta^\pm_n\) satisfy the defining relations of \(A_{n+2}\) with the center \(c_n + c_{n+1}\), and hence \(\Delta^\pm_n\) is an algebra homomorphism. By using compositions of the coproduct, \(\Delta^{(m)+}_n \equiv (id_n \otimes id_{n+1} \otimes \ldots \otimes id_{n+m-2} \otimes \Delta^+_n \otimes \Delta^+_n) \otimes (id_n \otimes id_{n+1} \otimes \ldots \otimes id_{n+m-3} \otimes \Delta^+_n \otimes \Delta^+_n) \cdots \otimes (id_n \otimes \Delta^+_n) \circ \Delta^+_n\), we can show that all the algebras \(A_{n,m}\) are inter-related by the co-structure. Indeed, denoting

\[
\mathcal{L}^{\pm(n,m)}(u, \lambda) \equiv \Delta^{(m)+}_n \otimes L^{\pm(n)}(u, \lambda),
\]

one can check that the following relations hold:

\[
R^{(n)}_{12}(u - v, \lambda) \mathcal{L}^{\pm(n,m)}_1(\lambda, v) \mathcal{L}^{\pm(n,m)}_2(\lambda, v, \lambda) = \mathcal{L}^{\pm(n,m)}_2(\lambda, v) \mathcal{L}^{\pm(n,m)}_1(\lambda, v, \lambda) R^{(m)}_{12}(u - v, \lambda), \quad |u| < \frac{1}{4}, \quad (9)
\]

\[
R^{(n)}_{12}(u - v_+, \lambda) \mathcal{L}^{+}(n,m)_{12}(\lambda, v) \mathcal{L}^{+}(n,m)_{12}(\lambda, v, \lambda) = \mathcal{L}^{+}(n,m)_{12}(\lambda, v) \mathcal{L}^{+}(n,m)_{12}(\lambda, v, \lambda) R^{(m)}_{12}(u - v_+, \lambda), \quad |u| < \frac{1}{4}, \quad (10)
\]

These relations are exactly the generating relations of \(A_{n,m}\) with the center \(c^{(n,m)}\) given by \(c^{(n,m)} = c_n + c_{n+1} + \ldots + c_{m-1}\). Equations (9) and (10) are the analogues of (8) in the case of infinite Hopf family of algebras.

**Remark.** The co-structure of an infinite Hopf family of algebras may look quite unnatural at first sight to readers who are familiar with standard Hopf algebra co-structures, because it makes use of tensor products between different algebras. However, the idea of making tensor product between different algebras is actually not so surprising — it has already been studied in several different contexts \[2, 22, 23, 25\] which are not related to our purpose.

### 3 Gauss Decomposition

In this section, we assume that the \(R\)-matrices have the form

\[
[R^{(n)}]_{ij}(u - v, \lambda) = a^{(n)}_{ij}(u - v, \lambda) \delta^i_j + b^{(n)}_{ij}(u + v, \lambda) \delta^i_j.
\]
This is because it is known that for $R$-matrices of more complex form, there is no analogue of the Ding-Frenkel theorem (such “bad” cases include the 8-vertex $R$-matrix [11, 12]).

The proof of the Ding-Frenkel theorem for $R$-matrices of arbitrary size is a tedious and complicated piece of work, and should in general be accomplished by induction over the size of $R$-matrices [9]. In this paper, however, we shall not make the induction over the size of $R$-matrices, but rather will illustrate our generalized RLL formalism only in the simplest case of $4 \times 4$ $R$-matrices.

Before we proceed, let us remark that the construction of the last section works fine if we understand all the tensor products as graded tensor products. In this way we would have the RLL formalism for an infinite Hopf family of super-algebras. In practice, however, it is usually better to change the graded matrix product into the usual matrix products. For members of the infinite Hopf family of super-algebras, this amounts to altering the RLL relations into the form

$$R_{12}^{(i)}(u - v, \lambda)L_1^{\pm(i,j)}(u, \lambda)\varpi L_2^{\pm(i,j)}(v, \lambda)\varpi \equiv \varpi L_2^{\pm(i,j)}(v, \lambda)\varpi L_1^{\pm(i,j)}(u, \lambda)R_{12}^{(j)}(u - v, \lambda),$$

$$R_{12}^{(i)}(u - v, \lambda)L_1^{\pm(i,j)}(u, \lambda)\varpi L_2^{\pm(i,j)}(v, \lambda)\varpi \equiv \varpi L_2^{\pm(i,j)}(v, \lambda)\varpi L_1^{\pm(i,j)}(u, \lambda)R_{12}^{(j)}(u - v - \lambda, \rho), |u| < |v|$$

where $\varpi$ is a diagonal numerical matrix given by $\varpi_{ij}^k = (-1)^{[i][j]}\delta_k^i\delta_j^k$ which reflects the grading of the $R$ and $L$-matrices. We may also write the usual (non-super) RLL relations in the above form, provided we understand $\varpi$ as the identity matrix. This section will be based on RLL relations of the above form, putting the deformations corresponding to the usual and super root systems on an equal footing.

The most general $4 \times 4$ $R$-matrix of the form [11] is written

$$R^{(i)}(u, \lambda) = \begin{pmatrix} a^{(i)}(u, \lambda) & 0 & 0 & 0 \\ 0 & b^{(i)}(u, \lambda) & t^{(i)}(u, \lambda) & 0 \\ 0 & s^{(i)}(u, \lambda) & e^{(i)}(u, \lambda) & 0 \\ 0 & 0 & 0 & d^{(i)}(u, \lambda) \end{pmatrix}$$

The corresponding numerical signature matrix $\varpi$ is given by

$$\varpi = \begin{pmatrix} 1 & 1 \\ 1 & \epsilon \end{pmatrix}, \quad \epsilon = \pm 1.$$
To get the desired analogue of the Ding-Frenkel theorem, it is convenient to rewrite the RLL relations in several equivalent ways. These include

\[
L^\pm(u, \lambda)^{-1}\varpi R^{(i)}_{12}(u-v, \lambda)L^\pm(u, \lambda)\varpi = \varpi L^\pm(u, \lambda)R^{(j)}_{12}(u-v, \lambda)\varpi L^\pm_2(v, \lambda)^{-1},
\]

\[
L^\pm_2(v, \lambda)^{-1}\varpi R^{(i)}_{12}(u-w, \lambda)L^\pm_2(v, \lambda)\varpi = \varpi L^\pm_2(v, \lambda)R^{(j)}_{12}(u-v, \lambda)\varpi L^\pm(v, \lambda)^{-1};
\]

\[
L^\pm_1(u, \lambda)^{-1}\varpi R^{(i)}_{12}(u-w, \lambda)L^\pm_1(v, \lambda)\varpi = \varpi L^\pm_1(u, \lambda)R^{(j)}_{12}(u-v, \lambda)\varpi L^\pm_1(v, \lambda)^{-1},
\]

where the unitarity of the R-matrices is implied by the equivalence of these different equations.

Expanding the above equations into the components, we get, after some algebra, the following relations:

1. Relations among \(k^\pm(u)\):

\[
a^{(i)}(u-v, \lambda)k^\pm_1(u)k^\pm_1(v) = k^\pm_1(v)k^\pm_1(u)a^{(j)}(u-v, \lambda),
\]

\[
a^{(i)}(u-v, \lambda)k^\pm_1(u)k^\pm_1(v) = k^\pm_1(v)k^\pm_1(u)a^{(j)}(u-v, \lambda),
\]

\[
k^\pm_1(v)^{-1}b^{(i)}(u-v, \lambda)k^\pm_1(u) = k^\pm_1(u)b^{(j)}(u-v, \lambda)k^\pm_1(v)^{-1},
\]

\[
k^\pm_2(v)^{-1}b^{(i)}(u-v, \lambda)k^\pm_1(u) = d^{(j)}(u-v, \lambda)k^\pm_2(v)^{-1}k^\pm_1(v)^{-1},
\]

\[
k^\pm_2(u)^{-1}k^\pm_2(v)^{-1}d^{(i)}(u-v, \lambda) = d^{(j)}(u-v, \lambda)k^\pm_2(v)^{-1}k^\pm_2(v)^{-1},
\]

\[
k^\pm_2(u)^{-1}b^{(i)}(v-u, \lambda)k^\pm_1(v) = k^\pm_2(v)^{-1}(v-u, \lambda)k^\pm_2(v)^{-1},
\]

\[
k^\pm_2(u)^{-1}b^{(i)}(v-u, \lambda)k^\pm_1(v) = k^\pm_2(v)^{-1}(v-u, \lambda)k^\pm_2(v)^{-1},
\]

2. Relations between \(k^\pm(u)\) and \(e^\pm(v)\), \(f^\pm(v)\):

\[
k^\pm_1(u)a^{(j)}(u-v, \lambda)e^{\pm}(v) - e^{\pm}(v)k^\pm_1(u)b^{(j)}(u-v, \lambda) - k^\pm_1(u)e^{\pm}(u)s^{(j)}(u-v, \lambda) = 0,
\]

\[
b^{(j)}(u-v, \lambda)k^\pm_1(v)f^{\pm}(v) + t^{(i)}(u-v, \lambda)f^{\pm}(u)k^\pm_1(u) = f^{\pm}(u)k^\pm_1(u) = 0;
\]

\[
k^\pm_1(u)a^{(j)}(u-v, \lambda)e^{\pm}(v) - e^{\pm}(v)k^\pm_1(u)b^{(j)}(u-v, \lambda) = k^\pm_1(u)e^{\pm}(u)s^{(j)}(u-v, \lambda) = 0,
\]

\[
b^{(j)}(u-v, \lambda)k^\pm_1(v)f^{\pm}(v) + t^{(i)}(u-v, \lambda)f^{\pm}(u)k^\pm_1(u) = f^{\pm}(u)k^\pm_1(u) = 0;
\]

\[
k^\pm_2(v)^{-1}d^{(i)}(u-v, \lambda)f^{\pm}(u) - f^{\pm}(u)k^\pm_2(v)^{-1}b^{(j)}(u-v, \lambda) - e^{\pm}(v)k^\pm_2(v)^{-1}f^{\pm}(v)s^{(j)}(u-v, \lambda) = 0;
\]

\[
b^{(j)}(u-v, \lambda)k^\pm_2(v)^{-1}e^{\pm}(u) - e^{\pm}(u)d^{(j)}(u-v, \lambda)k^\pm_2(v)^{-1} + e^{\pm}(u)k^\pm_2(v)^{-1} = 0;
\]

\[
b^{(j)}(u-v, \lambda)k^\pm_2(v)^{-1}e^{\pm}(u) - e^{\pm}(u)d^{(j)}(u-v, \lambda)k^\pm_2(v)^{-1} + e^{\pm}(u)k^\pm_2(v)^{-1} = 0;
\]

\[
b^{(j)}(v-u, \lambda)k^\pm_1(v)f^{\pm}(u) - f^{\pm}(u)a^{(j)}(v-u, \lambda)k^\pm_1(v) + t^{(i)}(v-u, \lambda)f^{\pm}(u)k^\pm_1(v) = 0;
\]

\[
e^{\pm}(u)k^\pm_2(v)b^{(j)}(v-u, \lambda) - k^\pm_1(v)a^{(j)}(v-u, \lambda)e^{\pm}(u) + k^\pm_2(v)e^{\pm}(v)s^{(j)}(v-u, \lambda) = 0;
\]

\[
ee^{\pm}(v)k^\pm_2(u)b^{(j)}(v-u, \lambda) - k^\pm_1(v)a^{(j)}(v-u, \lambda)e^{\pm}(u) + k^\pm_2(v)e^{\pm}(v)s^{(j)}(v-u, \lambda) = 0;
\]

3. Relations containing two \(e^\pm\)'s or two \(f^\pm\)'s:

\[
a^{(i)}(u-v, \lambda)k^\pm_1(u)e^{\pm}(u)k^\pm_1(v)e^{\pm}(v) - e^{\pm}(v)k^\pm_1(u)e^{\pm}(v)k^\pm_1(u) = 0,
\]

\[
a^{(i)}(u-v, \lambda)k^\pm_1(u)e^{\pm}(u)k^\pm_1(v)e^{\pm}(v) - e^{\pm}(v)k^\pm_1(u)e^{\pm}(v)k^\pm_1(u) = 0,
\]

\[
d^{(j)}(u-v, \lambda)f^{\pm}(u)k^\pm_1(u)f^{\pm}(v)k^\pm_1(v) - e^{\pm}(v)k^\pm_1(u)f^{\pm}(u)k^\pm_1(u)a^{(j)}(u-v, \lambda) = 0;
\]

\[
d^{(j)}(u-v, \lambda)f^{\pm}(u)k^\pm_1(u)f^{\pm}(v)k^\pm_1(v) - e^{\pm}(v)k^\pm_1(u)f^{\pm}(u)k^\pm_1(u)a^{(j)}(u-v, \lambda) = 0;
\]

\[
d^{(j)}(u-v, \lambda)f^{\pm}(u)k^\pm_1(u)f^{\pm}(v)k^\pm_1(v) - e^{\pm}(v)k^\pm_1(u)f^{\pm}(u)k^\pm_1(u)a^{(j)}(u-v, \lambda) = 0;
\]

\[
d^{(j)}(u-v, \lambda)f^{\pm}(u)k^\pm_1(u)f^{\pm}(v)k^\pm_1(v) - e^{\pm}(v)k^\pm_1(u)f^{\pm}(u)k^\pm_1(u)a^{(j)}(u-v, \lambda) = 0;
\]
4. Mixed relations between $e^\pm$ and $f^\pm$'s:

$$-ee^+(u)f^+(v) + f^+(v)e^+(u) = \frac{-k_1^+(u)}{k_1^+(v)} e^{(j)}(u,v,\lambda) f^{(j)}(u,v,\lambda) k_1^+(u),$$

$$ -ee^+(u)f^-(v) + f^-(v)e^+(u) = \frac{-k_1^+(u)}{k_1^-(v)} e^{(j)}(u,v,\lambda) f^{(j)}(u,v,\lambda) k_1^+(u),$$

$$ -ee^-(u)f^-(v) + f^+(v)e^-(u) = \frac{-k_1^+(u)}{k_1^-(v)} e^{(j)}(u,v,\lambda) f^{(j)}(u,v,\lambda) k_1^-(v),$$

where the equations in (26) are defined for both $|u| > |v|$ and $|u| < |v|$, equation (27) is defined only for $|u| > |v|$ and equation (28) is defined only for $|u| < |v|$.

The appearance of these can be drastically simplified by defining

$$E(u) = e^+(u) - e^-(u),$$

$$F(u) = f^+(u) - f^-(u),$$

giving

- Relations between the $k_i^\pm$'s and the $E$ and $F$:

$$k_1^+(u) a^{(j)}(u,v,\lambda) E(v) = E(v) k_1^+(u) a^{(j)}(u,v,\lambda),$$

$$b^{(j)}(u,v,\lambda) F(v) = F(v) b^{(j)}(u,v,\lambda),$$

$$E(u) k_1^+(v) b^{(j)}(v,-u,\lambda) = k_1^-(v) a^{(j)}(v,-u,\lambda) E(u),$$

$$b^{(j)}(v,-u,\lambda) k_1^-(v) F(u) = F(u) b^{(j)}(v,-u,\lambda),$$

$$b^{(j)}(v,-u,\lambda) k_2^+(v) = E(v) d^{(j)}(v,-u,\lambda) k_1^+(v),$$

$$k_2^-(v) d^{(j)}(v,-u,\lambda) F(v) = F(v) k_2^+(v) b^{(j)}(v,-u,\lambda).$$

- Relations of the kind $EE$ and $FF$:

$$a^{(j)}(v,-u,\lambda) E(u) b^{(j)}(v,-u,\lambda) = 0,$$

$$d^{(j)}(u,v,\lambda) F(u) b^{(j)}(u,v,\lambda) = 0;$$

- Exchange relation between $E$ and $F$:

$$[E(u), F(v)]_e = k_2^-(v,-u,\lambda) [\Phi^{(j)}(u_+ - v,\lambda) - \Phi^{(j)}(u_- - v,\lambda)] k_2^-(v,-u,\lambda).$$

where $[E(u), F(v)]_e = E(u) F(v) - \epsilon F(v) E(u)$ and $\Phi^{(j)}(u)$ are defined via

$$\Phi^{(j)}(u,v,\lambda) = b^{(j)}(u,v,\lambda) - 1 f^{(j)}(u,v,\lambda)$$

for $|u| > |v|$ and $|u| < |v|$. 

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Notice that equation (39) can only be understood as an analytical continuation, because $\Phi^{\pm(i)}(u)$ are never simultaneously well-defined for the same $u$. In the sense of analytical continuations, $\Phi^{-i}(u)$ can be regarded as the same as $\Phi^{+i}(u)$ for almost any value of $u$ except $u = 0$, where there is a singularity. Therefore, it is convenient to express the difference between $\Phi^{+i}(u)$ and $\Phi^{-i}(u)$ in terms of Dirac delta function:

$$\Phi^{+i}(u) - \Phi^{-i}(u) = N^{(i)}(h, \lambda)\delta(u),$$

where $N^{(i)}(h, \lambda)$ is a normalization coefficient which may depend on the dynamical variable $\lambda$.

Using this notation, the exchange relation between $E(u)$ and $F(v)$ can be rewritten as

$$[E(u), F(v)]_\epsilon = \delta(u_+ - v_-)k_2^-(v_-)N^{(j)}(h, \lambda)k_1^-(v_-)^{-1} - \delta(u_- - v_+)k_2^+(v_+)N^{(j)}(h, \lambda)k_1^+(v_+)^{-1}.$$

(40)

Summarizing the above results, we conclude that the generating relations for the algebra defined by our generalized RLL relations [12,13] with the $4 \times 4$ $R$-matrices [14] can be written in terms of the Drinfeld currents $k_{1,2}^{\pm}(u)$ and $E(u), F(u)$ via equations [18,23], [26,36], [37,38] and (40), provided the $R$-matrices [14] satisfy the (dynamical) Yang-Baxter equation (3) and the unitarity conditions (4-5).

Let us stress that in the above calculations we made no use of explicit formulae for the $R$-matrices and the Drinfeld currents, so our result should hold for all unitary dynamical $R$-matrices. Such a result is very useful when we consider concrete examples of the generalized RLL algebras — to get the Drinfeld current realization, for example, the only thing we need to do is to substitute the concrete $R$-matrix entries into the equations (18-25), (29-36), (37-38) and (40).

Now let us make some more comments about the essential role of the unitarity conditions [12,13]. In section two, these conditions are assumed only because they are sufficient to make the generalized RLL relations associative. In this section, however, we see that to have the Drinfeld current realization, we need these conditions to hold. Actually there are other reasons to impose the unitarity conditions on the $R$-matrices. For instance, let us consider the cases when the $R$-matrices are non-dynamical and $\epsilon = +1$. In such cases, the equations [17] and (38) can be written as

$$E(u)E(v) = \frac{b^{(j)}(v - u)d^{(j)}(u - v)}{a^{(j)}(v - u)b^{(j)}(u - v)}E(v)E(u),$$

(41)

$$F(u)F(v) = \frac{a^{(i)}(v - u)b^{(i)}(u - v)}{b^{(i)}(v - u)d^{(i)}(u - v)}F(v)F(u).$$

(42)

It is proven in [27] that the co-structure of the Drinfeld current realization of the infinite Hopf family of algebras is characterized solely by the structure functions appearing in the commutation relations

$$E_i(u)E_j(v) = \Psi_{ij}(u - v|q)E_j(v)E_i(u),$$

$$F_i(u)F_j(v) = \Psi_{ij}(u - v|q)^{-1}F_j(v)F_i(u),$$

and the only condition that these structure functions have to obey is

$$\Psi_{ij}(u|q) = \Psi_{ji}(-u|q)^{-1}.$$

Now looking at the equations (41) and (42) we find that the condition imposed above on $\Psi_{ij}(u|q)$ is just

$$\frac{b^{(j)}(-u)d^{(j)}(u)}{a^{(j)}(-u)b^{(j)}(u)} = \frac{a^{(i)}(u)b^{(i)}(-u)}{b^{(i)}(u)d^{(i)}(-u)},$$

which holds trivially if the unitarity condition for $R^{(j)}$ is satisfied. Thus the unitarity condition is not merely among the sufficient conditions for the RLL relations to be associative, but also necessary for the Drinfeld current realization.
4 Comorphisms and infinitely many commuting operators

In applications of ordinary Hopf algebras, the image of the $n$-th coproduct $\Delta^{(n)}$ is the building block for constructing infinitely many commuting operators in integrable/solvable models. Since $R(u-v)\mathcal{L}^{(n)}(u)\mathcal{L}^{(n)}(v) = \mathcal{L}^{(n)}(v)\mathcal{L}^{(n)}(u)R(u-v)$, one can easily see that $[\text{tr}\mathcal{L}^{(n)}(u), \text{tr}\mathcal{L}^{(n)}(v)] = 0$, which can be subsequently expanded over the spectral parameter to yield infinitely many commuting operators.

For the generalized RLL algebras given by (12-13), however, we cannot apply the simple scenario above because the $R$ matrices on the two sides of eqs. (12-13) are different. In order to get infinitely many commuting operators associated with the generalized RLL algebras, we have to look at some new algebraic structures which we call comorphisms, the analogue of comodules $\mathcal{L}^{(n)}$ for standard Hopf and quasi-Hopf algebras.

Throughout this section, $\mathcal{L}^{(i,j)}$ can be either $\mathcal{L}^{+ij}$ or $\mathcal{L}^{-ij}$.

Now let $\{\mathcal{F}^{(n)}, n \in \mathbb{Z}\}$ be another family of algebras associated with the same $R$-matrices used in (12-13), whose member $\mathcal{F}^{(i)}$ is defined by the relation

$$R^{(i)}(u-v, \lambda)X^{(i)}_1(u, \lambda)X^{(i)}_2(v, \lambda) = X^{(i)}_2(v, \lambda)X^{(i)}_1(u, \lambda).$$

Unlike the case of $\mathcal{L}^{(i,j)}(u, \lambda) = (\mathcal{L}^{(i,j)}(u, \lambda)^a)^a$ which are matrices, now $X^{(i)}(u, \lambda)$ are only vectors with components labeled by one index, $X^{(i)}(u, \lambda) = (X^{(i)}(u, \lambda)^a)$. Clearly the action of the operators $\rho_n^{\pm}$ can also be lifted to the algebras $\mathcal{F}^{(n)}$ to give algebra morphisms. We denote the lifted action of $\rho_n^\pm$ on $\mathcal{F}^{(n)}$ by $\kappa_n^{\pm},$

$$\kappa_n^{\pm} : \mathcal{F}^{(n)} \rightarrow \mathcal{F}^{(n \pm 1)},$$

$$X^{(n)}(u, \lambda) \rightarrow X^{(n \mp 1)}(u, \lambda),$$

$$R^{(n)}(u-v, \lambda) \rightarrow \rho_n^\pm R^{(n)}(u-v, \lambda).$$

Let

$$\varphi^{(n)} : \mathcal{F}^{(n)} \rightarrow A_n \otimes \mathcal{F}^{(n+1)},$$

$$X^{(n)}(u, \lambda) \rightarrow L^{(n)}(u, \lambda) \otimes X^{(n+1)}(u, \lambda)$$

(where $L^{(n)}(u, \lambda) \otimes X^{(n+1)}(u, \lambda)^a \equiv \sum_b L^{(n)}(u, \lambda)^a_b \otimes X^{(n+1)}(u, \lambda)^b$) be algebra comorphisms in the sense that there is an algebra homomorphism $\phi^{(n, n+1)} : A_n \otimes \mathcal{F}^{(n+1)} \rightarrow \mathcal{F}^{(n+1)}$ induced by $\varphi^{(n)},$

$$\phi^{(n, n+1)} \circ \varphi^{(n)} = \kappa_n^+.\,$$

$\varphi^{(n)}$ also obeys

$$(\Delta_n^+ \otimes \kappa_{n+1}^-) \circ \varphi^{(n)} = (id_n \circ \varphi^{(n+1)}) \circ \varphi^{(n)}.$$

Similarly we also introduce a family of algebras $\{\tilde{\mathcal{F}}^{(n)}, n \in \mathbb{Z}\}$ in which $\tilde{\mathcal{F}}^{(i)}$ is given by the relation

$$Y^{(i)}_2(u, \lambda)Y^{(i)}_1(v, \lambda)R^{(i+1)}(u-v, \lambda) = Y^{(i+1)}_2(u, \lambda)Y^{(i)}_1(v, \lambda),$$

where $Y^{(i)}(u, \lambda) = (Y^{(i)}(u, \lambda)^a).$ The analogue of $\varphi^{(n)}$ is now denoted $\tilde{\varphi}^{(n)},$

$$\tilde{\varphi}^{(n)} : \tilde{\mathcal{F}}^{(n)} \rightarrow \tilde{\mathcal{F}}^{(n-1)} \otimes A_n,$$

$$Y^{(n)}(u, \lambda) \rightarrow Y^{(n-1)}(u, \lambda) \otimes L^{(n)}(u, \lambda).$$

We require that $\tilde{\varphi}^{(n)}$ obey

$$\tilde{\varphi}^{(n-1, n)} \circ \tilde{\varphi}^{(n)} = \kappa_n^-,\,$$

$$(\kappa_{n-1}^- \otimes \Delta_n^-) \circ \tilde{\varphi}^{(n)} = (\tilde{\varphi}^{(n-1)} \circ id_n) \circ \tilde{\varphi}^{(n)}.$$
we can design in mimicking the structure of the comodule. For our infinite Hopf family of algebras, co-closure necessarily fails (otherwise we can act on $T_i$: i.e. commute among themselves, equations (44) and (43) with upper indices adapted), inserting the unitarity condition for the $\tilde{\kappa}_n^{\pm}$: $
abla$ is the lift of $\rho_{n+1}$ onto $\tilde{\mathcal{F}}(n)$.

There is a natural pairing $\langle , \rangle : \tilde{\mathcal{F}}(n) \otimes \mathcal{F}(n+1) \rightarrow \text{End}(\mathbb{C})$ between elements of the algebras $\tilde{\mathcal{F}}(n)$ and $\mathcal{F}(n+1)$ given by
\[
\langle Y(n)_i(u, \lambda), X^{(n+1)}(u, \lambda) \rangle = \sum_a Y_{a\dagger}(u, \lambda) X^{(n+1)a}(u, \lambda).
\]
A crucial observation is that the operators $T(n)(u, \lambda)$ commute among themselves, i.e.
\[
[T(n)(u, \lambda), T(n)(v, \lambda)] = 0. \tag{45}
\]
This equation can easily be obtained by multiplying together the defining relations for $\tilde{\mathcal{F}}(n)$ and $\mathcal{F}(n+1)$ (i.e. equations (14) and (15) with upper indices adapted), inserting the unitarity condition for the $R$-matrix, and taking the pairing $\langle , \rangle$ in both spaces labeled by suffices 1 and 2.

Note that the commutation relation (45) will not be spoiled by the action of $\varphi^{(n+1)}(\tilde{\varphi}(n))$, due to the existence of the algebra homomorphisms $\varphi^{(n,n+1)}$ and $\tilde{\varphi}(n-1,n)$ and the fact that $\kappa_n^{-} \circ \kappa_n^{+} = \kappa_{n-1}^{+} \circ \kappa^{-} = id_n$. Therefore, we can act on $T(n)(u, \lambda)$ successively with the operators $\varphi^{(n+1)}, \varphi^{(n+2)}, \ldots, \varphi^{(m-1)}$ as follows:
\[
T(n,m)(u, \lambda) = (id_{n+1} \circ \cdots \circ id_{m-2} \circ \varphi^{(m-1)} \circ (id_{n+1} \circ \cdots \circ id_{m-3} \circ \varphi^{(m-2)} \circ \cdots \circ (id_{n+1} \circ \varphi^{(n+2)}) \circ \varphi^{(n+1)}[T(n)(u, \lambda)]),
\]
which results in operators $T(n,m)(u, \lambda)$ of the form
\[
T(n,m)(u, \lambda) = \langle Y(n)(u, \lambda), L^{(n+1)}(u, \lambda) \hat{\otimes} \cdots \hat{\otimes} L^{(m-1)}(u, \lambda) \hat{\otimes} X^{(m)}(u, \lambda) \rangle, \tag{46}
\]
where $m > n \in \mathbb{Z}$. The conclusion is that the operators $T(n,m)(u, \lambda)$ still commute among themselves:
\[
[T(n,m)(u, \lambda), T(n,m)(v, \lambda)] = 0.
\]

Since these operators carry a spectral parameter dependence, we can expand them with respect to this parameter to get infinitely many commuting operators. The operators $T(n,m)(u, \lambda)$ may be viewed as generalizations of the transfer matrix appearing in the usual quantum inverse scattering method and thus are expected to yield novel integrable/solvable models when the $L$-matrices and the $Y(n)$ and $X^{(m)}$ are all specified in a specific representation.

Remark. In the case of standard (quasi-)Hopf algebras, the structure which is analogous to the result of the present section, known as the comodule, already exists [24], and played an important role in constructing integrable models with open boundaries. We emphasize that the notion of comodule is only meaningful for algebras which are co-closed. For our infinite Hopf family of algebras, co-closure necessarily fails (otherwise we are back to the conventional Hopf or quasi-Hopf algebra frameworks). So the comorphism is the best structure we can design in mimicking the structure of the comodule.
5 Examples

So far our study of the generalized RLL algebras has remained on the abstract level: we have not specified any concrete examples of the operators $\rho_n^\pm$ and hence the algebra morphisms $\tau^{(n,m)}$. In this section, we provide such examples, which may aid understanding of our earlier abstract constructions.

For any input $R$-matrix $R^{(0)}$ there is of course the trivial example given by $\rho_n^\pm = id$ for all $n$. Actually $\rho_n^\pm = id$ implies $\tau^{(n,m)} = id$ for all $n, m$ and hence the corresponding co-structure degenerates into the standard Hopf algebra structure. The following examples go beyond these trivial cases. In particular, we are interested in the cases which cannot be formulated as standard Hopf or quasi-Hopf algebras or for which the formulations as standard Hopf or quasi-Hopf algebras (if possible) are unknown.

First, let us discuss the possible actions of the operators $\rho_n^\pm$. Since these operators preserve unitarity and the Yang-Baxter equation, they cannot be chosen arbitrarily. However, since the $R$-matrices have several arguments, including the phase relative to the spectral parameter $u$, the deformation parameter $\hbar$ and/or $\eta$ (which, if present, plays the role of the trigonometric or elliptic period(s)) and the dynamical variable $\lambda$, and so on, the actions of $\rho_n^\pm$ may be chosen in such a way that they change these in a consistent way. For instance, the operators $\rho_n^\pm$ may

- replace the trigonometric or elliptic period(s) by some other values $\eta^{(n\pm 1)}$;
- shift the phase parameter by some amount $\pm \xi_n$;
- change the dynamical variable $\lambda$ to some other values $\lambda^{(n\pm 1)}$, etc.

In the following, we will not consider any dynamical $R$-matrices, and hence no examples of the last kind will occur. However, as we shall see, there are very rich choices of operators $\rho_n^\pm$ even for the first two kinds only.

5.1 in which $\rho_n^\pm$ act on the trigonometric or elliptic period(s)

In all the known cases of infinite Hopf families of algebras realized through Drinfeld currents, the structure functions are trigonometric or elliptic functions and the algebra morphisms $\tau^\pm$ act by changing the period (or one of the two periods) of the structure functions. From the point of view of the RLL realization, these cases correspond to operators $\rho_n^\pm$ which change the trigonometric or elliptic period(s) of the $R$-matrix entries. For instance, in the case of the trigonometric algebras $A_{h,\eta}(\hat{g})$, the operators $\rho_n^\pm$ act on $R^{(n)}(u)$ by changing the period $\eta^{(n)}$ into $\eta^{(n\pm 1)}$, where

$$\frac{1}{\eta^{(n+1)}} - \frac{1}{\eta^{(n)}} = h\epsilon_n.$$  \hfill (47)

In the case of $c_n = 0$ for all $n$, all the periods $\eta^{(n)}$ become identical, and the operators $\rho_n^\pm$ act as the identity. Correspondingly the structure of the infinite Hopf family degenerates into that of the standard Hopf algebra.

One should notice, however, that the difference between $\eta^{(n+1)}$ and $\eta^{(n)}$ need not necessarily depend on $c_n$ in order to give rise to an infinite Hopf family of algebras. The operators $\rho_n^\pm$ could just replace $\eta^{(n)}$ by some arbitrarily chosen $\eta^{(n\pm 1)}$, which are completely independent of $\eta^{(n)}$. The infinite Hopf family of algebras thus given will not degenerate into the standard Hopf algebra for any value of $c_n$. The only known example of this kind that has been studied in the past is the second family of algebras studied in [2].

Notice also that the infinite Hopf family of algebras given by operators $\rho_n^\pm$ can only be defined for trigonometric or elliptic $R$-matrices.
5.2 in which $\rho_n^\pm$ act on the phase parameter

This is a novel class of examples which have not yet been studied in the literature. To give readers a flavor of what the corresponding algebras may look like, we give here an explicit example. Take the trigonometric $R$-matrix

\[
R^{(0)}(u, \hbar^{(0)}, \eta) = \begin{pmatrix}
1 & \sinh \pi\eta \sinh \pi\eta(u + \hbar^{(0)}) & \sinh \pi\eta \sinh \pi\eta(u + \hbar^{(0)}) & \sinh \pi\eta \sinh \pi\eta(u + \hbar^{(0)}) \\
\sinh \pi\eta \sinh \pi\eta(u + \hbar^{(0)}) & 1 & \sinh \pi\eta \sinh \pi\eta(u + \hbar^{(0)}) & \sinh \pi\eta \sinh \pi\eta(u + \hbar^{(0)}) \\
\sinh \pi\eta \sinh \pi\eta(u + \hbar^{(0)}) & \sinh \pi\eta \sinh \pi\eta(u + \hbar^{(0)}) & 1 & \sinh \pi\eta \sinh \pi\eta(u + \hbar^{(0)}) \\
\sinh \pi\eta \sinh \pi\eta(u + \hbar^{(0)}) & \sinh \pi\eta \sinh \pi\eta(u + \hbar^{(0)}) & \sinh \pi\eta \sinh \pi\eta(u + \hbar^{(0)}) & 1
\end{pmatrix}
\]

as input. Let the operators $\rho_n^\pm$ act by changing $\hbar^{(n)}$ into $\hbar^{(n)\pm 1}$, where $\hbar^{(n)\pm 1} \equiv \hbar^{(n)} \pm \xi_n$. (48)

Then, using the result of section three we can show that the commutation relations for the Drinfeld currents $E(u)$ and $F(u)$ of the algebra $\mathcal{A}(R^{(i)}, R^{(j)})$ take the form

\[
E(u)E(v) = \frac{\sinh \pi\eta(u - v + \hbar^{(j)})}{\sinh \pi\eta(u - v - \hbar^{(j)})} E(v)E(u),
\]
\[
F(u)F(v) = \frac{\sinh \pi\eta(u - v - \hbar^{(i)})}{\sinh \pi\eta(u - v + \hbar^{(i)})} F(v)F(u).
\]

Such algebras also do not degenerate into the standard Hopf algebra $c = 0$, provided $\xi_n$ are not proportional to $c_n$.

Notice that this kind of infinite Hopf family of algebras can be defined for any type of $R$-matrices, elliptic, trigonometric, or rational.

5.3 in which all but a few of $\rho_n^\pm$ act as the identity

The examples given in the last two subsections have the common property that all the operators $\rho_n^\pm$ act in the same way (though with different parameters for different $n$). One can also think of cases in which the actions of the operators $\rho_n^\pm$ vary drastically, so that one cannot describe the actions of these operators by simple recursion formulae like (47) or (48).

As a particular example of this kind, suppose that only a few of $\rho_n^\pm$ are different from identity. To be more concrete, let $\rho_i^+ (i = n, n + 1, \ldots, m - 1)$ and $\rho_j^- (j = n + 1, \ldots, m - 1, m)$ be given as in subsections 5.1 or 5.2, and the other $\rho_k^\pm$ be trivial, so that the co-structure of the infinite Hopf family differs from the standard Hopf algebra structure for only a few members of the family, i.e. $\mathcal{A}_n, \mathcal{A}_{n+1}, \ldots, \mathcal{A}_{m-1}$. As an extreme case, if $m = n + 1$ there is only one algebra in the family which has a non-trivial Hopf family structure; all the other algebras (which actually degenerate into two different algebras) are standard Hopf algebras.

*   *   *

The special cases described in the above three subsections are provided only as illustrative examples for the rich algebraic structures which can be incorporated into the framework of infinite Hopf families of algebras. We are far from being able to list the variations for the choices of $\rho_n^\pm$ exhaustively. In some sense, the word “infinite” might best be understood to refer to the infinite variation in choice of the $\rho_n^\pm$, rather than to the size of each algebra family (which may actually be finite in certain cases, as described in the examples in subsection 5.3).
From the point of view of physical applications, one might try to apply the examples given above in the framework of section 4 to get infinitely many commuting operators. As mentioned earlier, the operators $T^{(n,m)}(u, \lambda)$ can be regarded as analogues of the transfer matrices in integrable/solvable models. Their logarithmic derivatives in finite dimensional representations are thus expected to give rise to Hamiltonians of solvable spin chains. From the examples of subsections 5.1 and 5.2, we expect that certain solvable spin chains with site-dependent couplings would arise. In contrast, the examples of subsection 5.3 are expected to give rise to spin chains with local impurities or dislocations. Of course, these remarks make sense only if a finite dimensional representation for the matrices $T^{(n,m)}(u, \lambda)$ is available.

6 Concluding remarks

The structure and representation theory of infinite Hopf families of (super-)algebras have been studied in a number of papers over the last few years. Most of the known results are concerned with representations at $c = 1$ in the Drinfeld current realization. By use of the co-structure, it can be seen that representations of such algebras at any $c \in \mathbb{Z}_+$ should exist. However, the Yang-Baxter type realization for the infinite Hopf family of (super) algebras has remained unknown until now.

In this paper, we proposed a generalized RLL formalism for generalized quantum affine algebras which are members of an infinite Hopf family of (super-)algebras. The construction shows that there is a very rich structure hidden in the generalized formalism — most of the known quantum affine algebras can easily be seen to be special cases of the present formalism, and new examples of infinite Hopf family of algebras can also be obtained by specifying a particular set of operators $\rho^+_n$.

As we have stressed in the text, the co-structure of an infinite Hopf family provides the possibility of defining novel classes of commuting operators, which are essential ingredients in the theories of integrable fields and/or completely solvable lattice statistical models.

Despite the progress made in this paper, we would like to point out some important open problems. One problem is the connection with universal $R$-matrices. For standard Hopf algebras and quasi-Hopf algebras, the universal $R$ matrix, which lives in the tensor product space of the corresponding quantum algebra, provides the algebraic foundation of RLL formalism. The analogous construction in our case is still unknown. Another related open problem is the existence of representations at $c = 0$ (here we are concerned with the algebras which do not degenerate into standard Hopf algebras at $c = 0$). For standard quantum affine algebras, such (evaluation) representations play a very important role because, on the one hand, these representations are exactly where the universal $R$-matrix evaluates to give rise to the numerical $R$-matrices, and on the other hand the transfer matrices of solvable lattice statistical models also take values in such representations. With regard to physical applications we expect that the values of the operators $T^{(n,m)}(u, \lambda)$ studied in section 4 would indeed become the transfer matrices of certain lattice statistical models. If this is true, our generalized RLL algebras would be an ideal algebraic tool to study spin chains with site-dependent couplings and lattice statistical models with local impurities or dislocations. We leave the detailed study of this problem to future work.

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