Octonions and $M$-theory

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Abstract. We explain how structures related to octonions are ubiquitous in $M$-theory. All the exceptional Lie groups, and the projective Cayley line and plane appear in $M$-theory. Exceptional $G_2$-holonomy manifolds show up as compactifying spaces, and are related to the $M2$ Brane and 3-form. We review this evidence, which comes from the initial 11-dim structures. Relations between these objects are stressed, when extant and understood. We argue for the necessity of a better understanding of the role of the octonions themselves (in particular non-associativity) in $M$-theory.

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1. Introduction

If the current $M$-theory is a unique theory, one should expect it to make use of singular, non-generic mathematical structures. Now it is known that many of the special objects in mathematics are related to octonions [1], and therefore it is not surprising that this putative theory-of-everything should display geometric and algebraic structures derived from this unique non-associative division algebra.

Special algebraic objects related to octonions are the five exceptional simple Lie groups $G_2$, $F_4$, $E_6$, $E_7$, and $E_8$. Some sporadic (finite simple) groups seem also related to octonions (see e.g. Thomson’s $E_2(3)$ [2]). The use of them in future directions of the theory is not to be discarded. The Cayley plane and its predecessor the octonionic projective line on one hand, and the geometries associated to the Magic Square [3] on the other, stand as fundamental octonionic geometries.

Very recently, the newly discovered (1996) compact manifolds of $G_2$-holonomy (a case of Joyce manifolds [4]) might play a fundamental role also in $M$-theory.

Also, the four coincidences in the list of simple Lie groups

$$
\begin{align*}
A_1 &= B_1 = C_1 \\
B_2 &= C_2 \\
A_3 &= D_3 \\
D_2 &= A_1 \times A_1 \\
SU(2) &= Spin(3) = Sp(1) \\
Spin(5) &= Sp(2) \\
SU(4) &= Spin(6) \\
Spin(4) &= SU(2) \times SU(2)
\end{align*}
$$

are related to some irreps. of the exceptional groups (Adams [5]); they also appear in physics and in compactifications in $M$-theory disguised in various forms [6].

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What we lack at the moment is an understanding of the role of the octonions themselves in \( M \)-theory; for a previous discussion of the history of the role of octonions in physics see [7]

2. Projective lines and planes

The first apparition of octonions in \( M \)-theory is as the fourth ladder in the “Brane Scan” of Townsend (1987) [8]. The four lists of classical supersymmetric \( p \)-Branes including instantons, embedded in \( D \)-dimensional space(-time) correspond precisely to the four division algebras \( \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O} \).

Real ladder: From \( D = 1 \) \( p = -1 \) instanton (kink) to \( D = 4 \) \( p = 2 \) membrane
(domain wall; codimension one)
Complex ladder: From \( D = 2 \) \( p = -1 \) instanton (vortex) to \( D = 6 \) \( p = 3 \) “universe”
codimension two)
\( \mathbb{H} \): Quaternionic ladder: From \( D = 4 \) \( p = -1 \) instanton to \( D = 10 \) \( p = 5 \) brane
codimension four)
Octonionic ladder: From \( D = 8 \) \( p = -1 \), Fubini-Nicolai instanton [9] to \( D = 11 \)
\( p = 2 \) \( M2 \) membrane
codimension eight)

**Table I.** The Brane Scan (Townsend [8])

These four ladders can be thought of as “oxidation” of the corresponding instanton, in its turn associated to the fundamental line bundle for the projective spaces \( \mathbb{K} \mathbb{P}^1 \), \( \mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O} \) [10]. The four are supersymmetrizable and are linked as the four Hopf bundles

| Group | Susy |
|-------|------|
| \( O(1) \) | \( 1 + 1 \) |
| \( U(1) \) | \( 2 + 2 \) |
| \( Sp(1) \) | \( 4 + 4 \) |
| \( Spin(8) \) | \( 8 + 8 \) |

**Table II.** Elementary Solitons and Projective Lines [10]

The relation of the series with the numbers \( \mathbb{R}, \ldots, \mathbb{O} \) can be made more precise [11], although the octonion case is more obscure [12]. The elementary objects (e.g. strings
for $p = 1$ are “thin” (i.e., strictly $p$ space dimensions) but the soliton has a width; as emphasized by Townsend ([8], [13]) the elementary “thin” membrane has a continuous excitation spectrum, but the membrane really grows a “core” due to gravitation. Notice in Table I the $O$-series appears as a “second coming” of the Real ladder: both end up in a membrane. Quantization seems to select only the $O$ ladder.

The relation of the Cayley plane $\mathbb{OP}^2$ (R. Moufang, 1933) to the 11 $D$ Sugra “corner” of $M$-theory is more recent: it comes from a nice paper of Ramond [14]. The Cayley plane is the 16 $D$ rank one symmetric space (compact form)

$$F_4/B_4 : 1 \to \text{Spin}(9) \to F_4(-52) \to \mathbb{OP}^2 \to 1 \quad 52 = 36 + 16$$

Now the ratio $\rho$ of the orders of the Weyl’s groups is 3, as $\# \text{Weyl}(F_4)=1152$ and $\text{Weyl}(B_4)=\mathbb{Z}_2^4 \circ S_4$, of order 384. But when a pair $H, H \subset G$ of semisimple Lie groups have the same rank, an important construction of Konstant et al. [15] generates, for each irrep of $G$, $\rho$ irreps of $H$, where $\rho = [\text{Weyl}(G) : \text{Weyl}(H)]$ ($= 3$ in our case). In particular the Id(-entity) irrep of $F_4$ generates

$$\text{Id}(F_4) \to +44 \text{(graviton)} - 128 \text{(gravitino)} + 84 \text{ (3-form)} \text{ of Spin}(9)$$

i.e. precisely the Spin(9) content of 11 $D$ Sugra, where Spin(9) is the little group in the light cone!

3. The $E$-series

The evidence for the $E_1 - E_{10}$ series in the descent of Sugra 11 $D$ down to $D = 1$ is well-known, and first stated by Julia in 1982 [16]. Compactifying 11 $D$ Sugra from the original 11 $D$ to 3 $D$, all of the $E$ series of split forms appear successively; the moduli spaces of scalar fields are the homogeneous spaces. We recall only the non-compact/compact scalars:

| Dimension | Scalar Representation | $\text{Spin}(9)$ Content |
|-----------|-----------------------|--------------------------|
| 5 $D$     | $E(6, +6)/Sp(4)$      | 42                       |
| 4 $D$     | $E(7, +7)/SU(8)$      | 35+35                    |
| 3 $D$     | $E(8, +8)/SO(16)$     | 128                      |

On the other hand, the Heterotic Exceptional string, of course, makes an important and direct use of $E_8 \times E_8$, and its descent from 11 $D$ $M$-theory has been clarified in the fundamental work of Witten and Horawa [17], through compactification in a segment. Finally, we recall that $E_6$, which appears naturally in some H-E string compactifications, is a strong candidate for Grand Unified Theories.

4. Manifolds of $G_2$ holonomy

In the “old” superstring compactification 10 $D \to 4 D$, Ricci-flat Calabi-Yau 3-folds were the objects of course. In $M$-theory with 11 $D$, their place is taken by manifolds with
exceptional holonomy (of the Berger 1955 list [18]). In particular, $G_2$ compact holonomy manifolds are still Ricci flat and conserve $32/8=4$ supercharges, that is, $N = 1$ Susy in 4 $D$ as we want. These beasts are fairly new even for the mathematicians (Joyce manifolds, 1996 [4]). From the reduction of the tangent bundle of a compactifying space $K_7$, $M$-theory(11) $\rightarrow$ $K_7 + $ Minkowski

$$
\begin{align*}
G_2 & \rightarrow & SO(7) & \rightarrow & P & \rightarrow & K_7 \\
& & \downarrow & & \downarrow & & \\
& & \mathbb{RP}^7 & \rightarrow & E & \rightarrow & K_7
\end{align*}
$$

we see that $G_2$ holonomy involves some torsion properties in $K_7$, a hot topic today. The structure is given in two diagrams

$$
\begin{align*}
SU(3) & \rightarrow & SU(4) = Spin(6) & \rightarrow & S^7 \\
& & \downarrow & & \downarrow & & \\
& & G_2 & \rightarrow & Spin(7) & \rightarrow & S^7 \\
& & \downarrow & & \downarrow & & \\
& & S^6 & \rightarrow & S^6 & & \\
& & \downarrow & & \downarrow & & \\
& & \mathbb{Z}/2 & \rightarrow & \mathbb{Z}/2 & & \\
& & \downarrow & & \downarrow & & \\
& & G_2 & \rightarrow & Spin(7) & \rightarrow & S^7 \\
& & \downarrow & & \downarrow & & \\
& & G_2 & \rightarrow & SO(7) & \rightarrow & \mathbb{RP}^7
\end{align*}
$$

They come from the curious fact that $G_2$ is subgroup of $Spin(7)$ (Adams [5]) and of $SO(7)$ (from the 7 irrep). For a review of the involved physics see [19]. The reason for the descent $SO(7)$ to $G_2$ is because the later conserves a 3-form, related to the octonionic product [20], and very likely to the 3-form extant in $M$-theory in the 11 $D$ Sugra limit (see before). This is the only rationale, so far, for the presence of the octonions themselves in $M$-theory; we hope these matters will be clarified soon.

For alternative and/or complementary views of the items exposed here see [21], [22].

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