On uniform asymptotic upper density in locally compact abelian groups

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Abstract

Starting out from results known for the most classical cases of \( \mathbb{N}, \mathbb{Z}^d, \mathbb{R}^d \) or for \( \sigma \)-finite abelian groups, here we define the notion of asymptotic uniform upper density in general locally compact abelian groups. Even if a bit surprising, the new notion proves to be the right extension of the classical cases of \( \mathbb{Z}^d, \mathbb{R}^d \). The new notion is used to extend some analogous results previously obtained only for classical cases or \( \sigma \)-finite abelian groups. In particular, we show the following extension of a well-known result for \( \mathbb{Z} \) of Fürstenberg: if in a general locally compact Abelian group \( G \) a set \( S \subset G \) has positive uniform asymptotic upper density, then \( S - S \) is syndetic.

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1 Measuring large, but not necessarily dense infinite sequences and sets in groups

Our aim here is to extend the notion of uniform asymptotic upper density, used in case of \( \mathbb{R} \) already by Beurling and Pólya in the analysis of entire functions. The same notion is frequently called by others as Banach density, c.f. e.g. \cite{3} p. 72.

The notion of uniform asymptotic upper density – u.a.u.d. – is a way to grab the idea of a set being relatively considerable, even if not necessarily dense or large in some other more easily accessible sense. In many theorems, in particular in Fourier analysis and in additive problems where difference sets or sumsets are considered, the u.a.u.d. is the right notion to express that a set becomes relevant in the question considered. However, to date the notions was only extended to sequences and subsets of the real line, and some immediate relatives like \( \mathbb{Z}^d, \mathbb{R}^d \), as well as to finite, or at least finitely constructed (e.g. \( \sigma \)-finite) cases.

A framework where the notion might be needed is the generality of LCA groups. In recent decades it is more and more realized that many questions e.g. in additive number theory can be investigated, even sometimes structurally better understood/described, if we leave e.g. \( \mathbb{Z} \), and consider the analogous questions in Abelian groups. In fact, when some analysis, i.e. topology also has a role – like in questions of Fourier analysis e.g. – then the setting of LCA groups seems to be the natural framework. And indeed several notions and questions, where in classical results u.a.u.d. played a role, have already been defined, even in some extent discussed in LCA groups. Nevertheless, it seems that no attempt has been made to extend the very notion of u.a.u.d. to this setup.

One of the more explicit attempts to really "measure sets in infinite groups" is perhaps the work of Borovik at al. \cite{2}, \cite{1}. Other papers, where some ideas close to ours can be seen, are \cite{5} – considering measures, not sets, although the investigation there is focused on local structure at small neighborhoods of points – and in \cite{6}, where at least the setup of LCA groups is apparent (although the interest is quite different).

For cases of \( \sigma \)-finite groups \( G \) it is easy to design the u.a.u.d., compare \cite{4}. In the more general framework of discrete groups, I.Z. Ruzsa \cite{9} had two constructions to define u.a.u.d.

However, neither of these constructions were the same as ours. Below we will explain, how one may construct notions of u.a.u.d., which finely extend the classical notion.

2 Some additive number theory flavored results for difference sets

Let us denote the upper density of \( A \subseteq \mathbb{N} \) as \( \overline{d}(A) := \limsup_{n \to \infty} A(n)/n > 0 \) with \( A(n) := \#(A \cap [1, n]) \). Erdős and Sárközy (seemingly unpublished, but quoted in \cite{4} and in \cite{8}) observed the following.

Proposition 1 (Erdős-Sárközy). If the upper density \( \overline{d}(A) \) of a sequence \( A \subseteq \mathbb{N} \) is positive, then writing the positive elements of the sequence \( D(A) := D_1(A) := A - A \) as \( D(A) \cap \mathbb{N} = \{(0 <)d_1 < d_2 < \ldots \} \) we have \( d_{n+1} - d_n = O(1) \).

This is analogous, but not contained in the following result of Hegyvári, obtained for \( \sigma \)-finite groups. An abelian group is called \( \sigma \)-finite (with respect to \( H_n \)), if there exists an
increasing sequence of finite subgroups \( H_n \) so that \( G = \bigcup_{n=1}^{\infty} H_n \). For such a group Hegyvári defines asymptotic upper density (with respect to \( H_n \)) of a subset \( A \subset G \) as

\[
\overline{d}_{H_n}(A) := \limsup_{n \to \infty} \frac{|A \cap H_n|}{#H_n}.
\]  

(1)

Note that for finite groups this is just \( \frac{|A \cap G|}{#G} \). Hegyvári proves the following [4, Proposition 1].

**Proposition 2** (Hegyvári). Let \( G \) be a \( \sigma \)-finite abelian group with respect to the increasing, exhausting sequence \( H_n \) of finite subgroups and let \( A \subset G \) have positive upper density with respect to \( H_n \). Then there exists a finite subset \( B \subset G \) so that \( A - A + B = G \). Moreover, we have \( #B \leq 1/\overline{d}_{H_n}(A) \).

Fürstenberg calls a subset \( S \subset G \) in a topological Abelian (semi)group a syndetic set, if there exists a compact set \( K \subset G \) such that for each element \( g \in G \) there exists a \( k \in K \) with \( gk \in S \); in other words, in topological groups \( \bigcup_{k \in K} Sk^{-1} = G \). Then he presents as Proposition 3.19 (a) of [3] the following.

**Proposition 3** (Fürstenberg). Let \( S \subset \mathbb{Z} \) with positive upper Banach density. Then \( S - S \) is a syndetic set.

In the following we extend the notion of uniform asymptotic upper density, (also called as Banach density) to arbitrary LCA groups, and present various generalized versions of the above results, which cover all of them.

In fact, our interest in the problem of the definition of u.a.u.d. in general LCA groups came from another problem, the so-called Turán extremal problem for positive definite functions. In that question some results, already known for classical situations like \( \mathbb{R}^d \), \( \mathbb{Z}^d \) or compact groups, can also be extended. We discuss these questions in [7].

## 3 Various forms of the asymptotic density

We start with the frequently used definition of asymptotic upper density in \( \mathbb{R}^d \). Let \( K \subset \mathbb{R}^d \) be a fat body, i.e. a set with \( 0 \in \text{int}K \), \( K = \text{int}K \) and \( K \) compact. Then asymptotic upper density with respect to \( K \) is defined as

\[
\overline{d}_K(A) := \limsup_{r \to \infty} \frac{|A \cap rK|}{|rK|}.
\]  

(2)

The definition (1) is clearly analogous to (2). As is easy to see, both (2) and (1) depends on the choice of the fundamental set \( K \) or sequence \( H_n \), even if positivity of (2) is invariant for a large class of underlying sets including all convex, but also many other bodies. The similar notion of density applies and has the same properties also for the discrete group \( \mathbb{Z}^d \).

On the other hand, for a given subset \( A \) in a \( \sigma \)-finite group \( G \), (1) can easily be zero for some fundamental sequence \( H_n \), while being maximal (i.e., 1) for some other choice \( H'_n \) of fundamental sequence.
Example 1. Let $G := \mathbb{Q}/\mathbb{Z}$, which is a $\sigma$-finite additive abelian group. Let $H_n := \{ r \in G : r = \frac{p}{q}, q \leq n \}$; then $H_n$ is an increasing and exhausting sequence of finite subgroups of $G$. Note that  
\[ \#H_n = \sum_{j \leq n} \varphi(j) \sim \frac{6}{\pi^2} n^2. \]
Let then $A_k := \{ r \in G : r = \frac{p}{q}, (p, q) = 1, (k^2 + k)! < q \leq (k+1)^2! \}$ and $A := \cup_{k=1}^\infty A_k$. Then it is not hard to prove that $\liminf_{n \to \infty} \frac{\#A \cap H_n}{\#H_n} = 0$ but $\limsup_{n \to \infty} \frac{\#A \cap H_n}{\#H_n} = 1$. Then it is clear that the value of the upper density can be either 0 or 1 depending on the choice of an appropriate subsequence of $H_n$ as fundamental sequence. With a little modification an example with arbitrary numbers as possible upper densities can be derived.

However, results corresponding to the above ones of Erdős, Sárközi and Hegyvári are easily sharpened by using only a weaker notion, that of asymptotic uniform upper density.

It could be defined as

\[ D_{H_n}(A) := \limsup_{n \to \infty} \sup_{x \in G} \frac{\#A \cap (H_n + x)}{\#H_n}. \]

for $\sigma$-finite abelian groups and is defined as

\[ D_K(A) := \limsup_{r \to \infty} \sup_{x \in \mathbb{R}^d} \frac{|A \cap (rK + x)|}{|rK|}. \]

in $\mathbb{R}^d$. It is obvious that these notions are translation invariant, and $D_{H_n}(A) \geq d_{H_n}(A)$, $D_K(A) \geq d_K(A)$. It is also well-known, that $D_K(A)$ gives the same value for all nice - e.g. for all convex - bodies $K \subset \mathbb{R}^d$, although this fact does not seem immediate from the formulation. Actually, we will obtain this as a side result, being an immediate corollary of Theorem 1, see Remark 1.

Similar definitions can be used for $\mathbb{Z}^d$. However, dependence on the fundamental sequence $H_n$ makes the $\sigma$-finite case less appealing, and we lack a successful notion for abelian groups in general. In particular, a natural requirement is to find a common generalization of asymptotic upper density, which works both for $\mathbb{R}^d$ and $\mathbb{Z}^d$, and also for a larger class of (say, abelian) groups, including, but not restricted to $\sigma$-finite ones.

Note also the following ambiguity in the use of densities in literature. Sometimes even in continuous groups a discrete set $\Lambda$ is considered in place of $A$, and then the definition of the asymptotic upper density is

\[ D_K^\#(A) := \limsup_{r \to \infty} \sup_{x \in \mathbb{R}^d} \frac{\#\Lambda \cap (rK + x)}{|rK|}. \]

That motivates our further extension: we are aiming at asymptotic uniform upper densities of measures, say measure $\nu$ with respect to measure $\mu$, (whether related by $\nu$ being the trace of $\mu$ on a set or not). E.g. in (5) $\nu := \#$ is the cardinality or counting measure of a set $\Lambda$, while $\mu := | \cdot |$ is just the volume. The general formulation in $\mathbb{R}^d$ is thus

\[ D_K(\nu) := \limsup_{r \to \infty} \sup_{x \in \mathbb{R}^d} \frac{\nu(rK + x)}{|rK|}. \]

Of course, to extend these notions some natural hypotheses should apply. We are considering abelian groups (although non-abelian groups come to mind naturally, here we do not
consider this extension), and in accordance to the group settings only densities with respect to translation-invariant measures $\mu$ are suitable. Otherwise we want $\nu$ to be a measure, possibly infinite, and $\mu$ be another, translation-invariant, nonnegative (outer) measure with strictly positive, but finite values when applied to sets considered.

We will consider two generalizations here. The first applies for the class of abelian groups $G$, equipped with a topological structure which makes $G$ a LCA (locally compact abelian) group. Considering such groups are natural for they have an essentially unique translation invariant Haar measure $\mu_G$ (see e.g. [?]), what we fix to be our $\mu$. By construction, $\mu$ is a Borel measure, and the sigma algebra of $\mu$-measurable sets is just the sigma algebra of Borel measurable sets, denoted by $B$ throughout. Furthermore, we will take $B_0$ to be the members of $B$ with compact closure: note that such Borel measurable sets necessarily have finite Haar measure. This will be important for not allowing a certain degeneration of the notion: e.g. if we consider $G = \mathbb{R}$, $\nu$ is the counting measure $\#$ and $A$ is some sequence $A = \{a_k : k \in \mathbb{N}\}$, say tending to infinity, then it is easy to define a (non-compact, but still measurable) union $V$ of decreasingly small neighborhoods of the points $a_k$ such that the Haar measure of $V$ does not exceed 1, but all of $A$ stays in $V$, hence the relative density of $A$ with respect to the counting measure, is infinite. (Another way to deal with this phenomenon would have been to fix that $\infty/\infty = 0$, but we prefer not to go into such questions.)

Note if we consider the discrete topological structure on any abelian group $G$, it makes $G$ a LCA group with Haar measure $\mu_G = \#$, the counting measure. Therefore, our notions below certainly cover all discrete groups. This is the natural structure for $\mathbb{Z}^d$, e.g. On the other hand all $\sigma$-finite groups admit the same structure as well, unifying considerations. (Note that $\mathbb{Z}^d$ is not a $\sigma$-finite group since it is torsion-free, i.e. has no finite subgroups.)

The other measure $\nu$ can be defined, e.g., as the trace of $\mu$ on the given set $A$, that is, $\nu(H) := \nu_A(H) := \mu_G(H \cap A)$, or can be taken as the counting measure of the points included in some set $\Lambda$ derived from the cardinality measure similarly: $\gamma(H) := \gamma_\Lambda(H) := \#(H \cap \Lambda)$.

**Definition 1.** Let $G$ be a LCA group and $\mu := \mu_G$ be its Haar measure. If $\nu$ is another measure on $G$ with the sigma algebra of measurable sets being $S$, then we define

$$
\overline{D}(\nu; \mu) := \inf_{C \subseteq G} \sup_{V \in S \cap B_0} \frac{\nu(V)}{\mu(C + V)} .
$$

In particular, if $A \subseteq G$ is Borel measurable and $\nu = \mu_A$ is the trace of the Haar measure on the set $A$, then we get

$$
\overline{D}(A) := \overline{D}(\nu_A; \mu) := \inf_{C \subseteq G} \sup_{V \in B_0} \frac{\mu(A \cap V)}{\mu(C + V)} .
$$

If $\Lambda \subseteq G$ is any (e.g. discrete) set and $\gamma := \gamma_\Lambda := \sum_{\lambda \in \Lambda} \delta_\lambda$ is the counting measure of $\Lambda$, then we get

$$
\overline{D}(\gamma; \mu) := \overline{D}(\nu_\Lambda; \mu) := \inf_{C \subseteq G} \sup_{V \in B_0} \frac{\#(\Lambda \cap V)}{\mu(C + V)} .
$$

**Theorem 1.** Let $K$ be any convex body in $\mathbb{R}^d$ and normalize the Haar measure of $\mathbb{R}^d$ to be equal to the volume $|\cdot|$. Let $\nu$ be any measure with sigma algebra of measurable sets $S$. Then we have

$$
\overline{D}(\nu; |\cdot|) = \overline{D}_K(\nu) .
$$
The same statement applies also to $\mathbb{Z}^d$.

Remark 1. In particular, we find that the asymptotic uniform upper density $\overline{D}_K(\nu)$ does not depend on the choice of $K$. For a direct proof of this one has to cover the boundary of a large homothetic copy of $K$ by standard (unit) cubes, say, and after a tedious $\varepsilon$-calculus a limiting process yields the result. However, Theorem 1 elegantly overcomes these technical difficulties.

Furthermore, we also introduce a second notion of density as follows.

**Definition 2.** Let $G$ be a LCA group and $\mu := \mu_G$ be its Haar measure. If $\nu$ is another measure on $G$ with the sigma algebra of measurable sets being $\mathcal{S}$, then we define

$$\overline{\Delta}(\nu; \mu) := \inf_{F \subset G, \#F < \infty} \sup_{V \in \mathcal{S}} \frac{\nu(V)}{\mu(F + V)}.$$  \hfill (11)

In particular, if $A \subset G$ is Borel measurable and $\nu = \mu_A$ is the trace of the Haar measure on the set $A$, then we get

$$\overline{\Delta}(A) := \overline{\Delta}(\nu_A; \mu) := \inf_{F \subset G, \#F < \infty} \sup_{V \in \mathcal{S}} \frac{\mu(A \cap V)}{\mu(F + V)}.$$  \hfill (12)

If $\Lambda \subset G$ is any (e.g. discrete) set and $\gamma := \gamma_\Lambda := \sum_{\lambda \in \Lambda} \delta_\lambda$ is the counting measure of $\Lambda$, then we get

$$\overline{\Delta}(\Lambda) := \overline{\Delta}(\gamma_\Lambda; \mu) := \inf_{F \subset G, \#F < \infty} \sup_{V \in \mathcal{S}} \frac{\#(\Lambda \cap V)}{\mu(F + V)}.$$  \hfill (13)

The two definitions are rather similar, except that the requirements for $\overline{\Delta}$ refer to finite sets only. Because all finite sets are necessarily compact in an LCA group, (7) of Definition 1 extends the same infimum over a wider family of sets than (11) of Definition 2; therefore we get

**Proposition 4.** Let $G$ be any LCA group, with normalized Haar measure $\mu$. Let $\nu$ be any measure with sigma algebra of measurable sets $\mathcal{S}$. Then we have

$$\overline{\Delta}(\nu; \mu) \geq \overline{D}(\nu; \mu).$$  \hfill (14)

This specializes to $\mathbb{R}^d$ as follows.

**Proposition 5.** Let us normalize the Haar measure of $\mathbb{R}^d$ to be equal to the volume $|\cdot|$. Let $\nu$ be any measure with sigma algebra of measurable sets $\mathcal{S}$. Then we have

$$\overline{\Delta}(\nu; |\cdot|) \geq \overline{D}(\nu; |\cdot|).$$  \hfill (15)

Moreover, the following is obvious, since in discrete groups the Haar measure is the counting measure and the compact sets are exactly the finite sets.

**Proposition 6.** Let $\nu$ be any measure on the sigma algebra $\mathcal{S}$ of measurable sets in a discrete Abelian group $G$. Take $\mu := \#$ the counting measure, which is the normalized Haar measure of $G$ as a LCA group. Then

$$\overline{\Delta}(\nu; \#) = \overline{D}(\nu; \#).$$  \hfill (16)

So there is no difference for $\mathbb{Z}$, e.g. In general, however, the two densities, defined above, may well be different: in fact, we would bet for that, but we have no construction to show this.
4 Proof of Theorem 1

Since \( r \) and according to (18) we have
\[
|x| < \frac{1}{r K} \quad \text{and} \quad \nu(x) > \frac{\tau}{r K}.
\]
proof that \( \Delta K(\nu) \geq \tau \), hence the assertion.

Proof of \( \Delta K(\nu) \geq \tau \), put \( C := rK \) with some \( r > 0 \) given, and pick up a measurable
set \( V \) satisfying \( \nu(V) > \tau|V + C| \). We can then write
\[
\int \chi_V(t) \, d\nu(t) > \tau|V + C|.
\]

If \( t \in V \), \( u \in C(= rK) \), then \( t + u \in V + C \), hence \( \chi_{V+C}(t + u) = 1 \), and we get
\[
\chi_V(t) \leq \frac{1}{|C|} \int \chi_{V+C}(t + u) \, d\nu(u) \quad (\forall t \in V).
\]

If \( t \not\in V \), this is obvious, as the left hand side vanishes: hence (17) implies
\[
\tau|V + C| < \int \frac{1}{|C|} \int \chi_{V+C}(t + u) \, d\nu(u) \, d\tau = \int \chi_{V+C}(y) \, \frac{1}{|C|} \int \chi_{C}(y - t) \, d\tau \, dy.
\]

Since \( C = -C \), the inner function is
\[
f(y) := \frac{1}{|C|} \int \chi_{C}(y - t) \, d\tau = \frac{\nu(C + y)}{|C|},
\]
and according to (18) we have \( \tau|V + C| < \int \chi_{V+C}(y) \, f(y) \, dy = \int_{V+C} f \), hence for some
appropriate point \( z \in V + C \) we must have \( \tau < f(z) \). That is, \( \nu(C + z) > \tau|C| \), and we get
by \( C := rK \) the estimate
\[
\nu(rK + z) > \tau|rK|.
\]

Since \( r \) was arbitrary, it follows that \( \Delta K(\nu) \geq \tau \), and applying this to all \( \tau < \Delta (\nu; \cdot \cdot) \) the
statement follows.

5 Extension of the propositions of Erdős-Sárközy, of Hegyvári, and of Fürstenberg

Theorem 2. If \( G \) is a LCA group and \( A \subseteq G \) has \( \Delta(\nu) < 0 \), then there exists a finite subset
\( B \subseteq G \) so that \( A - A + B = G \). Moreover, we can find \( B \) with \( \#B \leq 1/\Delta(\nu) \).

Remark 2. We need a translation-invariant (Haar) measure, but not the topology or compactness.
Proof. Assume that $H \subset G$ satisfies $(A - A) \cap (H - H) = \{0\}$ and let $L = \{b_1, b_2, \ldots, b_k\}$ be any finite subset of $H$. By condition, we have $(A + b_i) \cap (A + b_j) = \emptyset$ for all $1 \leq i < j \leq k$. Take now $C := L$ in the definition of density (13) and take $0 < \tau < \rho := \Delta(A)$. By Definition 2 of the density $\overline{\Delta}(A)$, there are $x \in G$ and $V \subset G$ open with compact closure – or, a $V \in \mathcal{S}$ with $0 < |V| < \infty$ – satisfying

$$|A \cap (V + x)| > \tau|V + L|. \quad (20)$$

On the other hand

$$V + L = \bigcup_{j=1}^{k} (V + x + (b_j - x)) \supset \bigcup_{j=1}^{k} (((V + x) \cap A) + b_j) - x \quad (21)$$

and as $A + b_j$ (thus also $((V + x) \cap A) + b_j$) are disjoint, and the Haar measure is translation invariant, we are led to

$$|V + L| \geq k|(V + x) \cap A|. \quad (22)$$

Comparing (20) and (22) we obtain

$$|A \cap (V + x)| > \tau k|(V + x) \cap A| \quad \text{and also} \quad |V + L| > k\tau |V + L|, \quad (23)$$

hence after cancellation by $|V + L| > 0$ we get $k < 1/\tau$ and so in the limit $k \leq K := \lfloor 1/\rho \rfloor$. It follows that $H$ is necessarily finite and $\# H \leq K$.

So let now $B = \{b_1, b_2, \ldots, b_k\}$ be any set with the property $(A - A) \cap (B - B) = \{0\}$ (which implies $\# B \leq K$) and maximal in the sense that for no $b' \in G \setminus B$ can this property be kept for $B' := B \cup \{b'\}$. In other words, for any $b' \in G \setminus B$ it holds that $(A - A) \cap (B' - B') \neq \{0\}$.

Clearly, if $A - A = G$ then any one point set $B := \{b\}$ is such a maximal set, and if $A - A \neq G$, then a greedy algorithm leads to one in $\leq K$ steps.

Now we can prove $A - A + B = G$. Indeed, if there exists $y \in G \setminus (A - A + B)$, then $(y - b_j) \notin A - A$ for $j = 1, \ldots, k$, hence $B' := B \cup \{y\}$ would be a set satisfying $(B' - B') \cap (A - A) = \{0\}$, contradicting maximality of $B$.

Corollary 1. Let $A \subset \mathbb{R}^d$ be a (measurable) set with $\overline{\Delta}(A) > 0$. Then there exists $b_1, \ldots, b_k$ with $k \leq K := \lfloor 1/\overline{\Delta}(A) \rfloor$ so that $\bigcup_{j=1}^{k} (A - A + b_j) = \mathbb{R}^d$.

This is interesting as it shows that the difference set of a set of positive Banach density $\overline{\Delta}$ is necessarily rather large: just a few translated copies cover the whole space.

Observe that we have Proposition 3 as an immediate consequence, since $\mathbb{Z}$ is discrete, and thus the two notions $\overline{\Delta}$ and $\overline{D}$ of Banach densities coincide; moreover, the finite set $B := \{b_1, \ldots, b_k\}$ is a compact set in the discrete topology of $\mathbb{Z}$. But in fact we can as well formulate the following extension.

Corollary 2. Let $G$ be a LCA group and $S \subset G$ a set with positive upper Banach density, i.e. $\overline{D}(S) > 0$, where here $\overline{D}(S) = \overline{D}(|S; \mu|)$. Then the difference set $S - S$ is a syndetic set: moreover, the set of translations $K$, for which we have $G = KS$, can be chosen not only compact, but even to be a finite set with $\# K \leq \lfloor 1/\overline{D}(S) \rfloor$ elements.
This corollary is immediate, because \( \bar{\Delta}(S) \geq \bar{D}(S) \) according to Proposition 4.

This indeed generalizes the proposition of Fürstenberg. Also this result contains the result of Hegyvári: for on \( \sigma \)-finite groups the natural topology is the discrete topology, whence the natural Haar measure is the counting measure, and so on \( \sigma \)-finite groups Corollary 2 and Theorem 2 coincides. Finally, this also generalizes and sharpens the Proposition of Erdős and Sárközy. Indeed, on \( \mathbb{Z} \) or \( \mathbb{N} \) we naturally have \( \bar{\Delta}(A) = \bar{D}(A) \geq \overline{d}(A) \), so if the latter is positive, then so is \( \bar{D}(A) \); and then the difference set is syndetic, with finitely many translates belonging to a translation set \( K \), say, covering the whole \( \mathbb{Z} \). Hence \( d_{n+1} - d_n \) is necessarily smaller than the maximal element of the finite set \( K \) of translations.

\textbf{Theorem 3.} Let \( G \) be a LCA group and \( S \subset G \) a set with a positive, (but finite) uniform asymptotic upper density, regarding now the counting measure of elements of \( S \) in the definition of Banach density, i.e.\( \overline{D}(S) = \overline{D}(\#|S; \mu|) > 0 \). Then the difference set \( S - S \) is a syndetic set.

\textbf{Remark 3.} One would like to say that a density \( +\infty \) is ”even the better”. However, in non-discrete groups this is not the case: such a density can in fact be disastrous. Consider e.g. the set of points \( S := \{ 1/n : n \in \mathbb{N} \} \) as a subset of \( \mathbb{R} \). Clearly for any compact \( C \) of positive Haar /i.e. Lebesgue/ measure \( |C| > 0 \), and for any \( V \in \mathcal{B}_0 \) of finite measure and compact closure, \( |V + C| \) is positive but finite: whence whenever \( 0 \in \text{int} V \), we automatically have \( \#(S \cap V) = \infty \) and also \( \#(S \cap V)/|C + V| = \infty \), therefore \( \overline{D}(\#|S; \cdot|) = \infty \); but \( S - S \subset [-2, 2] \) and thus with a compact \( B \) it is not possible that \( B + S - S \) covers \( G = \mathbb{R} \), whence \( S - S \) is not syndetic.

\textbf{Problem 1.} The implicitly occurring set of translations \( K \), for which we have \( G = K + (S - S) \), is not controlled in size by the proof below. However, one would like to say that there must be some bound, hopefully even \( \mu(K) \leq [1/\overline{D}(S)] \), for an appropriately chosen compact set of translates \( K \). This we cannot prove yet.

\textbf{Proof.} We are not certain that our argument is the simplest possible: also, it does not give a good estimate for the measure of the required compact set exhibiting the syndetic property of \( S - S \). Nevertheless, we consider it worthwhile to present it in full detail, since the various steps, eventually leading to the result, seem to be rather general and useful auxiliary statements, having their own independent interest. Correspondingly, we break the argument in a series of lemmas.

\textbf{Lemma 1.} Let \( S \subset G \) and assume \( \overline{D}(\#|S; \mu|) = \rho \in (0, \infty) \). Consider any compact set \( H \subset G \) satisfying the ”packing type condition” \( H - H \cap S - S = \{0\} \) with \( S \). Then we necessarily have \( \mu(H) \leq 1/\overline{D}(S) \).

\textbf{Proof.} Let \( 0 < \tau < \rho \) be arbitrary. By definition of \( \overline{D}(S) \), (using \( H \) in place of \( C \)) there must exist a measurable set \( V \in S \cap \mathcal{B}_0 \), with compact closure so that \( \infty > \#(S \cap V) > \tau \mu(V + H) \), therefore also \( \#(S \cap V) > \mu((S \cap V) + H) \). However, for any two elements \( s \neq s' \in (S \cap V) \subset S \), \( (s + H) \cap (s' + H) = \emptyset \), since in case \( g \in (s + H) \cap (s' + H) \) we have \( g = s + h = s' + h' \), i.e. \( s - s' = h - h' \), which is impossible for \( s \neq s' \) and \( (H - H) \cap (S - S) = \{0\} \). Therefore for each \( s \in (S \cap V) \) there is a translate of \( H \), totally disjoint from all the others: i.e. the union \( (S \cap V) + H = \bigcup_{s \in (S \cap V)} (s + H) \) is a disjoint union. By the properties of the Haar measure, we thus have \( \mu((V \cap S) + H) = \sum_{s \in (S \cap V)} \mu(s + H) = \#((V \cap S)H)\mu(H) \).
Whence we find \( \#(S \cap V) \geq \tau \#(S \cap V)\mu(H) \), and, since \( \#(S \cap V) > \tau \mu(V + H) \) was positive, we can cancel with it and infer \( \mu(H) < 1/\tau \). This holding for all \( \tau < \rho = \overline{D}(S) \), we obtained that any compact set \( H \), satisfying the packing type condition with \( S \), is necessarily bounded in measure by \( 1/\overline{D}(S) \). \( \Box \)

**Lemma 2.** Suppose that \( S - S \cap H - H = \{0\} \) with \( \overline{D}(\#|S; \mu) = \rho \in (0, \infty) \) and \( H \subset G \) with \( 0 < \mu(H - H) \). Then the set \( A := S + (H - H) \) has the uniform asymptotic upper density \( \overline{D}(\mu|A; \mu) \), with respect to the Haar measure (restricted to \( A \), not less than \( \rho \cdot \mu(H - H) \).

**Proof.** Let \( C \subset G \) be arbitrary and denote \( Q := H - H \). We want to estimate from below the ratio \( \mu(A \cap V)/\mu(C + V) \) for an appropriately chosen \( V \in \mathcal{B}_0 \). Let us fix that we will take for \( V \) some set of the form \( U + Q \) with \( U \in \mathcal{B}_0 \). Clearly \( A \cap V = (S + Q) \cap (U + Q) \supset (S \cap U) + Q \). Now for any two elements \( s \neq t \in S \), thus even more for \( s, t \in (S \cap V) \), the sets \( s + Q \) and \( t + Q \) are disjoint, this being an easy consequence of the packing property because \( s + q = t + q' \Leftrightarrow s - t = q - r \), which is impossible for \( s - t \neq 0 \) by condition. Therefore by the properties of the Haar measure we get \( \mu((S \cap U) + Q) = \sum_{s \in (S \cap U) \mu(s + Q) = \#(S \cap U) \cdot \mu(Q) \). In all, we found \( \mu(A \cap V) \geq \#(S \cap U) \cdot \mu(Q) \).

It remains to choose \( V \), that is, \( U \), appropriately. For the compact set \( C + Q \subset G \) and for any given small \( \varepsilon > 0 \), by definition of \( \overline{D}(\#|S; \mu) = \rho \) there exists some \( U \in \mathcal{B}_0 \) such that \( \#(S \cap U) > (\rho - \varepsilon)\mu((C + Q) + U) \). Choosing this particular \( U \) and combining the two inequalities we are led to \( \mu(A \cap V) \geq (\rho - \varepsilon)\mu(C + Q + U) \mu(Q) \), that is, for \( V := U + Q \) written in \( \mu(A \cap V)/\mu(C + V) \geq (\rho - \varepsilon)\mu(H - H) \).

As we find such a \( V \) for every positive \( \varepsilon \), the sup over \( V \in \mathcal{B}_0 \) is at least \( \rho \mu(H - H) \), and because \( C \subset G \) was arbitrary, we infer the assertion. \( \Box \)

**Lemma 3.** Suppose that \( S - S \cap H - H = \{0\} \) with \( \overline{D}(\#|S; \mu) = \rho \in (0, \infty) \) and \( H \subset G \) with \( 0 < \mu(H - H) \). Then there exists a finite set \( B = \{b_1, \ldots, b_k\} \subset G \) of at most \( k \leq [1/(\rho \mu(H - H))] \) elements so that \( B + (H - H) - (H - H) + (S - S) = G \). In particular, the set \( S - S \) is syndetic with the compact set of translates \( B + (H - H) + (H - H) \).

**Proof.** By the above Lemma 2 we have an estimate on the density of \( A := S + (H - H) \) with respect to Haar measure. But then we may apply Corollary 2 to see that the difference set \( S + (H - H) - (S + (H - H)) \) is a syndetic set with the set of translates \( B \) admitting \( \#B \leq [1/\overline{D}(\mu|A; \mu)] \leq [1/(\rho \mu(H - H))] \). Because also the set \( H \) is compact, this yields that \( S \) is syndetic as well, with set of translations being \( B + (H - H) + (H - H) \). \( \Box \)

One may think that it is not difficult, for a discrete set \( S \) of finite density with respect to counting measure, to find a compact neighborhood \( R \) of 0, so that \( R \cap (S - S) \) be almost empty with 0 being its only element. If so, then by continuity of subtraction, also for some compact neighborhood \( H \) of zero with \( (H - H) \subset R \) (and, being a neighborhood, with \( \mu(H) > 0 \), too) we would have \( (H - H) \cap (S - S) = \{0\} \), the packing type condition, whence concluding the proof of Theorem 3.

Unfortunately this idea turns to be naive. Consider the sequence \( S = \{n + 1/n : n \in \mathbb{N}\} \cup \mathbb{N} \) (in \( \mathbb{R} \)), which has uniform asymptotic upper density 2 with the cardinality measure, whilst \( S - S \) is accumulating at 0.

Nevertheless, this example is instructive. What we will find, is that sets of finite positive uniform asymptotic upper density cannot have a too dense difference set: it always splits into
a fixed, bounded number of disjoint subsets so that the difference set of each subset already leaves out a fixed compact neighborhood of 0. This will be the substitute for the above naive approach to finish our proof of Theorem 3 through proving also some kind of subadditivity of the uniform asymptotic upper density – another auxiliary statement interesting for its own right.

**Lemma 4.** Let $Q \subseteq G$ be any symmetric compact neighborhood of 0 and let $S$ have positive but finite uniform asymptotic upper density with respect to cardinality measure, i.e. $\overline{D}(\#S; \mu) = \rho \in (0, \infty)$. Then there exists a finite disjoint partition $S = \bigcup_{j=1}^{n} S_j$ of $S$ such that $(S_j - S_j) \cap Q = \{0\}$. Moreover, choosing an appropriate symmetric compact neighborhood $Q$ of 0, depending on $\varepsilon > 0$, we can even guarantee that the number of subsets in the partition is not more than $k \leq [(1 + \varepsilon)\mu(Q)]$.

**Proof.** Let $s \in S$ be arbitrary, consider $R := s + Q$, and let us try to estimate the number of other elements of $S$ falling in $R$. Clearly for any $C \subseteq G$ we have $\#(S \cap R)/\mu(C + R) \leq \sup_{V \in B_r} (\#(S \cap V)/\mu(C + V))$ so for any $\varepsilon > 0$ and with some appropriate $C \subseteq G$ this is bounded by $\rho + \varepsilon$ according to the density condition. Note that the choice of $C$ depends only on $\varepsilon$, but not on $R$. That is, we already have a bound $k := \#(S \cap R) \leq (\rho + \varepsilon)\mu(C + R)$ with the given $C = C(\varepsilon)$, independently of $R$, i.e. of $Q$.

Next we show how to obtain the bound $k \leq \lfloor \mu(Q) \rfloor + 1$ for some appropriate choice of $Q$. This hinges upon a lemma of Rudin, stating that for any given compact set $C \subseteq G$ and $\varepsilon > 0$ there exists another Borel set $V$, also with compact closure, so that $\mu(C + V) < (1 + \varepsilon)\mu(V)$, c.f. Theorem 2.6.7 on page 52 of [?]; moreover, Rudin remarks that this can even be proved (actually, read out from the proof) with open sets $V$ having compact closure. It is a matter of invariance of Haar measure with respect to translations to ascertain that (some) of the interior points of $V$ be 0, so that $V$ is a neighborhood of 0: also, by regularity of the Borel measure, and by compactness of the closure, we can as well take $V$ to be its own closure. Furthermore, the same proof also shows that $V$ can even be taken symmetric. In all, for an appropriate choice of $V$ for $Q$, we even have $k := \#(S \cap R) \leq (\rho + \varepsilon)\mu(C + R) < (\rho + \varepsilon)(1 + \varepsilon)\mu(Q)$. Note that here the dependence on $C$ disappears from the end formula, but there is a dependence of $Q$ on $\varepsilon$. This is equivalent to the estimate in the Lemma.

It remains to construct the partition once we have a compact neighborhood $Q$ of 0 and a finite number $k \in \mathbb{N}$ such that $\#(S \cap (Q + s)) \leq k$ for all $s \in S$. this is standard argument. Consider a graph on the points of $S$ defined by connecting two points $s$ and $t$ exactly when $t \in s + Q$. Since $Q$ is symmetric, this is indeed a good definition for a graph (and not for a directed graph only).

In this graph by condition the degree of any point of $s \in S$ is at most $k - 1$: there are at most $k - 1$ further points of $S$ in $s + Q$. But it is well-known that such a graph can be partitioned into $k$ subgraphs with no edges within any of the induced subgraphs. That is, the set of points split into the disjoint union of some $S_j$ with no two points $s, t \in Q$ being in the relation $t \in s + Q$, defining an edge between them.

The proof of this is very easy for finite or countable graphs: just start to put the points, one by one, inductively into $k$ preassigned sets $S_j$ so that each point is put in a set where no neighbor of it stays; since each point has less than $k$ neighbors, this simple greedy algorithm can not be blocked and the points all find a place. Same for countable many points, while for larger cardinalities transfinite induction is needed to carry out the same reasoning.
It is easy to see that now we constructed the required partition: the $S_j$ are disjoint, and so are $(S_j - S_j)$ and $Q \setminus \{0\}$, for any $j = 1, \ldots, k$, too. This concludes the proof.

Lemma 5 (subadditivity). Let $\nu_0 = \sum_{j=1}^n \nu_j$ be a sum of measures, all on the common set algebra $S$ of measurable sets. Then we have $\mathcal{D}(\nu_0, \mu) \leq \sum_{j=1}^n \mathcal{D}(\nu_j, \mu)$. In particular, this holds for one given measure $\nu$ and a disjoint union of sets $A_0 = \bigcup_{j=1}^n A_j$, with $\nu_j := \nu|_{A_j}$, for $j = 0, 1, \ldots, k$.

Proof. Uniform asymptotic upper density is clearly monotone in the sets considered, therefore all $S_j$ have a density $0 \leq \rho_j \leq \rho < \infty$. Let $\varepsilon > 0$ be arbitrary, and take $C_j \subset G$ so that for all $V \in B_0$ in the definition of $\mathcal{D}(\nu_j|_{A_j}, \mu)$ we have $\nu_j(V) \leq (\rho_j + \varepsilon) \mu(C_j + V)$. Such $C_j$ exists in view of the infinum on $C \subset G$ in the definition of u.a.u.d.

Consider the (still) compact set $C := C_1 + \cdots + C_n$. By definition of u.a.u.d. there is $V \in B_0$ such that $\nu(V) \geq (\rho - \varepsilon) \mu(C + V)$. Obviously, $\mu(C_j + V) \leq \mu(C + V)$, so on combining these we obtain

$$\rho - \varepsilon \leq \nu(V) \leq \frac{\sum_{j=1}^n \nu_j(V)}{\mu(C + V)} \leq \sum_{j=1}^n \frac{\nu_j(V)}{\mu(C_j + V)} \leq \sum_{j=1}^n (\rho_j + \varepsilon),$$

that is, $\rho - \varepsilon \leq \sum_j (\rho_j + \varepsilon)$ holding for all $\varepsilon$, we find $\rho \leq \sum_j \rho_j$, as was to be proved.

Continuation of the proof of Theorem 3. We take now an arbitrary compact neighborhood $H \subset G$ of 0, with of course $\mu(H) > 0$, and also $Q := H - H$ again with $0 < \mu(Q) < \infty$ and $Q$ a symmetric neighborhood of 0. By Lemma 4 there exists a finite disjoint partition $S = \bigcup_{j=1}^n S_j$ with $(S_j - S_j) \cap (H - H) = \{0\}$. By subadditivity of u.a.u.d. (that is, Lemma 5 above), at least one of these $S_j$ must have positive u.a.u.d. $\rho_j$ (with respect to the counting measure), namely of density $0 < \rho/n \leq \rho_j \leq \rho < \infty$, with $\rho := \mathcal{D}(\#|_S, \mu)$.

Selecting such an $S_j$, we can apply Lemma 3 to infer that already $S_j$ – hence also $S \supset S_j$ – is syndetic.

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