COMPLETE REDUCIBILITY OF INTEGRABLE MODULES FOR THE AFFINE LIE (SUPER) ALGEBRAS

S. Eswara Rao
School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road
Mumbai - 400 005 India

email: senapati@math.tifr.res.in

Abstract

We prove complete reducibility for an integrable module for an affine Lie algebra where the canonical central element acts non-trivially. We further prove that integrable modules does not exists for most of the super affine Lie algebras where the center acts non-trivially. ¹

¹2000 Mathematics subject classification primary 17B65, 17B68
Key words and phrases: Affine Lie algebras, Super affine Lie algebras and Integrable modules.
Introduction

Let $G$ be simple finite dimensional Lie algebra. Let $\hat{G}$ be the corresponding affine Lie algebra and let $K$ be the canonical central element. A module $V$ of $\hat{G}$ is called integrable if the Chevalley generators act locally nilpotently on $V$. In $[C]$ the irreducible integrable modules for $\hat{G}$ with finite dimensional weight spaces has been classified. In particular any irreducible integrable module with finite dimensional weight spaces where $K$ acts by positive integer is isomorphic to an highest weight module. In this work we prove that any integrable module with finite dimensional weight spaces where $K$ acts by non-zero scalars is completely reducible (Theorem (1.10)).

The integrable modules where $K$ acts trivially, need not be completely reducible. For example consider the $\hat{G}$ (without the derivation) module $G \otimes \mathbb{C}[t, t^{-1}]/(t - 1)^2$ where $K$ acts by zero which is not completely reducible. (See [E1] for the graded version).

In section 2 we consider affine Lie super algebras and prove that most often integrable modules with finite dimensional weight spaces do not exist. We use stronger definition of the integrability than that of [KW]. Let $G$ be simple finite dimensional Lie super algebra. Let $\hat{G}$ be the corresponding affine Lie super algebra. Assume that it has non-degenerate symmetric invariant bilinear form. Assume that the semisimple part of the even part of $G$ is at least two components. Then integrable modules for $\hat{G}$ with finite dimensional weight spaces where center acts by non zero scalar does not exist. (Theorem 2.6) Certainly integrable modules with $K$ acting zero exists. For example loop modules. Our techniques work only with the notion of stronger integrability. We do not know whether such a result hold with the weaker integrability of [KW].

In Theorem (2.9), we prove that an integrable irreducible module for $\hat{G}$ with finite dimensional weight spaces where center $K$ acts by positive integer is necessarily a highest weight module, assuming the semisimple part of the
finite even part is only one component. In this case we note that (Remark
(2.11)) the module is completely reducible for the even part. That class
includes the affine Lie super algebras associated with basic Lie super algebras
of types $A(0,n), B(0,n)$ and $C(n)$.

Section 1

(1.1) We will fix some notations. All our algebras are over complex num-
bbers $\mathbb{C}$. Let $\hat{\mathcal{G}}$ be simple finite dimensional Lie algebra. Let $\hat{\mathfrak{h}}$ be a Cartan subalgebra. Let $\hat{Q}$ and $\hat{\Lambda}$ be root and weight lattice of $\hat{\mathcal{G}}$. Let $\hat{\Lambda}^+$ be dominant integral weights of $\hat{\mathcal{G}}$. Let $\alpha_1, \cdots, \alpha_n$ be simple roots and let $\beta$ be highest root of $\hat{\mathcal{G}}$; $\alpha_1', \cdots, \alpha_n'$ be the corresponding simple roots. We choose non-degenerate bilinear form on $\hat{\mathfrak{h}}^*$ such that $(\beta, \beta) = 2$.

Let $\mathcal{G} = \mathcal{G} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$ be the corresponding untwisted affine Lie algebra. Let $\mathfrak{h} = \hat{\mathfrak{h}} \oplus \mathbb{C}K \oplus \mathbb{C}d$ be the Cartan subalgebra of $\mathcal{G}$. Let $Q$ and $\Lambda$ be the root and weight lattice of $\mathcal{G}$. Let $\delta$ be the null root. Let $\Lambda_0$ be an element of $\hat{\mathfrak{h}}^*$ such that $\Lambda_0(\hat{\mathfrak{h}}) = 0, \Lambda_0(K) = 1$ and $\Lambda_0(d) = 0$. An element $\lambda$ in $\hat{\mathfrak{h}}^*$ can be treated as an element of $\hat{\mathfrak{h}}^*$ by extending as $\lambda(K) = 0$ and $\lambda(d) = 0$. Let $\lambda_0$ be the restriction to $\mathfrak{h}$. Given $\lambda \in \mathfrak{h}^*$, $\lambda$ can be uniquely written as

(1.2) $\lambda = \lambda_0 + \lambda(d)\delta + \lambda(K)\Lambda_0$.

(1.3) Definition An element $\lambda$ in $\Lambda^+$ is called minimal if for every $\mu \in \Lambda^+$ such that $\mu \nparallel \lambda$ implies $\mu = \lambda$. Here $\mu \nparallel \lambda$ means $\lambda - \mu = \sum_{i=1}^{n} n_i \alpha_i, n_i \in \mathbb{N}$.

(1.4) Lemma [H] Let $\lambda$ be minimal in $\Lambda^+$. Then $\lambda(\beta') = 0$ or 1.

Proof. See Exercise 13 of Chapter III of [H].
Let $\lambda \in \Lambda^+$ and let $V(\lambda)$ be the irreducible integrable highest weight module for $\hat{G}$. Let $P(\lambda)$ be the set of weights of $V(\lambda)$. Define $\mu \leq \lambda$ if $\lambda - \mu = \sum_{i=0}^{n} n_i \alpha_i, n_i \in \mathbb{N} (\alpha_0$ is the additional simple root of $\hat{G}$). Let $\overline{P}(\lambda) = \{ \pi \mid \mu \in P(\lambda) \}$. Clearly $\overline{P}(\lambda)$ determines a unique coset in $\hat{\Lambda} / \hat{Q}$. Let $\mu_0$ be the minimal element in $\hat{\Lambda}^+$ in the above coset. Let $s$ be a complex number such that $\lambda(d) - s$ is a non-negative integer.

(1.6) Lemma \ \ Let $\mu_0 = \overline{\mu}_0 + s\delta + \lambda(K)\Lambda_0$. Then $\mu_0 \in P(\lambda)$.

Proof \ \ First note that by minimality of $\overline{\mu}_0$ we have $\overline{\mu}_0 \in \hat{\Lambda}$. Then clearly $\mu_0 \leq \lambda$.

Claim \ \ $\mu_0 \in \Lambda^+$. Consider $\alpha_i^\vee$ for $1 \leq i \leq n$. Then $\mu_0(\alpha_i^\vee) = \overline{\mu}_0(\alpha_i) \in \mathbb{N}$ as $\overline{\mu}_0 \in \hat{\Lambda}^+$. Now $\alpha_0^\vee = K - \beta^\vee$ and

$$
\mu_0(\alpha_0^\vee) = \overline{\mu}_0(K - \beta^\vee) = \lambda(K) - \overline{\mu}_0(\beta^\vee)
$$

Now $\lambda(K)$ is positive integer and hence $\lambda(K) \geq 1$. We know from Lemma (1.4) that $\overline{\mu}_0(\beta^\vee) = 0$ or $1$. That means

$$
\mu_0(\alpha_i^\vee) \in \mathbb{N}.
$$

This prove the claim. Thus $\mu_0$ is dominant integral and $\leq \lambda$. By Proposition 12.5 (a) of [K1] it follows that $\mu_0 \in P(\lambda)$.

We need the following from [E2].

(1.7) Lemma \ \ (Lemma (2.6) of [E2]). Let $V$ be integrable module for $\hat{G}$ with finite dimensional weight spaces. Let $P(V)$ be the set of weights of $V$. Let $\lambda \in P(V)$. Then

(1) There exists $\eta_0 \in \hat{Q}$ such that $\lambda + \eta_0 + \eta \notin P(V)$ for all $0 \neq \eta \in \hat{\Lambda}$.
(2) There exists $\eta_0 \leq 0, \eta_1 \in \Lambda$ such that $\lambda + \eta_0 + \eta \notin P(V)$ for all $0 \neq \eta \geq 0, \eta \in \Lambda$.

**Proof** (1) follows from the proof of Lemma (2.6) of [E2]. The proof of (2) is similar.

**Proposition (1.8)** Let $V$ be integrable $\hat{G}$–module with finite dimensional weight spaces. Assume the canonical central element $K$ acts by positive integers. Let $\lambda \in P(V)$. Then there exists $\eta \geq 0$ such that $\lambda + \eta \in \Lambda^+$ and the irreducible integrable highest weight module $V(\lambda + \eta) \subseteq V$.

**Proof** By previous lemma there exists $\eta_0 \geq 0$ such that $\lambda + \eta_0 + \eta \notin P(V)$ for $0 \neq \eta \geq 0$. Now by arguments similar to the proof of Theorem 2.4 (i) of [C] will produce an highest weight module with highest weight $\lambda + \eta_0 + \eta_1$ for some $\eta_1 \geq 0$. Note that $(\lambda + \eta_0 + \eta_1)(d) \geq \lambda(d)$.

In the above proof we need our Lemma (1.7) as the proof of Lemma 2.6 (ii) of [C] is incomplete. We now recall the following variation of a standard result from [K1].

**Proposition (1.9)** Let $V$ be integrable module for $\hat{G}$ with finite dimensional weight spaces. Let $K$ act by positive integer. Suppose for every $v$ in $V$, there exist $N > 0$ such that $U(\hat{G})_{n+\delta v} = 0$ for all $n > N$ and for $\alpha \in \Delta U\{0\}$. Then $V$ is completely reducible.

We need to recall some standard notations from [K1] and prove two lemmas.

The Cartan subalgebra $h$ carries a non-degenerate bilinear form $(\cdot | \cdot)$. Let $\nu : h \to h^*$ be an isomorphism such that $\nu(h_1)(h_1) = (h | h_1)$. Let $< , >$ be the induced bilinear form on $h^*$. Recall the Casimir operator from Section (2.1), $\rho$ in $h^*$ from (2.5) from [K1]. Also recall the notion of primitive weights.
from (9.3) of [K1]. Note that in an integrable module the primitive weights are dominant integral.

Lemma A  Let $V$ be as above. Suppose $\lambda, \mu$ are primitive weights such that $\lambda - \mu = \beta \in Q^+ - \{0\}$. Then $2 < \lambda + \rho, \nu^{-1}(\beta) = (\beta, \beta)$.

Proof  Follows from the proof of theorem (10.7) of [K1]. See 10.7.3 and the next equation in [K1].

Lemma B  Let $V$ be as above. Let $v$ be a weight vector of weight $\lambda$ such that $(\Omega_0 - aI_V)^k v = 0$ for some $k \in \mathbb{Z}_+$ and $a \in \mathbb{C}$. Presumably $v^1 \in U_{-\beta}(\mathcal{G})v, \beta \in Q$. Then

$$(\Omega_0 - (a + 2 < \lambda + \rho, \nu^{-1}(\beta) > - (\beta, \beta))I_V)v^1 = 0.$$

Proof  Follows from (2.6.1) and (3.4.1) of [K1]. Also see (9.10.2) of [K1]. Note that $V$ is restricted in the sense of [K1].

Proof of the Proposition  Let $\mathcal{G} = n^- \oplus h \oplus n^+$ be the standard triangular decomposition. Let $V^0 = \{v \in V \mid n^+ v = 0\}$. Clearly $V^0$ is $h-$ invariant and hence decomposes under $h$. Let $V^1 = U(\mathcal{G})V^0$. It is standard fact that in an integrable module, each highest weight generate an irreducible integrable module. Thus $V^1$ is completely reducible. We will now prove that $V = V^1$.

Clearly the Casimir operator $\Omega_0$ acts on $V$ and leaves each finite dimensional weight space invariant. Thus $\Omega_0$ is locally finite on $V$. Suppose $V \neq V^1$. Then there exists $v$ in $V - V^1$ of weight $\lambda$ such that $n^+ v \subseteq V^1$ and $(\Omega_0 - aI_V)^k v = 0$ for some $k \in \mathbb{Z}_+$ and $a \in \mathbb{C}$. Since, clearly $\Omega_0 v \in V^1$, we have $a = 0$ and hence $\Omega_0^k v = 0$.

From the hypothesis it follows that $U(n^+) v$ is finite dimensional. So it contains vector $u_\beta v$ such that $u_\beta \in U(\mathcal{G})_{1, \beta}$ and $n^+ u_\beta v = 0, \beta \in Q^+ - \{0\}$.
Let $\mu = \lambda + \beta$ and note that $\lambda, \mu$ are primitive weights. Thus by Lemma A. 

\[(\lambda, \mu) = 2 < \mu + \rho, \nu^-(\beta) \neq (\beta, \beta).\]

Now by Lemma B it follows that $2 < \lambda + \rho, -\nu^-(\beta) = (\beta, \beta)$ as $\Omega_0(u_\beta v) = 0$. This is a contradiction to *. Thus $V = V^1$ and $V$ is completely reducible.

**Theorem (1.10)** Let $V$ be integrable module with finite dimensional weight spaces for $\hat{G}$. Suppose all eigenvalues of $K$ are non-zero. Then $V$ is completely reducible as $\hat{G}$-module.

**Proof** First decompose $V$ with $K$ action. As $K$ commutes with $\hat{G}$, each eigenspace is $\hat{G}$-module. Thus we can assume that $K$ acts by single scalar. It is well known that the central element $K$ acts by integer (see for example [E2]). Without loss of generality we can assume that $K$ acts by positive integer. We now decompose the module 

$$V = \bigoplus_{\lambda \in \Lambda/Q} W_\lambda$$

where $\mu_1, \mu_2$ weight occurs in $W_\lambda$ then $\mu_1 - \mu_2 \epsilon Q$. Clearly each $W_\lambda$ is a $\hat{G}$-module. Thus we can assume that the weights $P(V)$ of $V$ lie in single coset of $\Lambda$.

**Claim** Let $\lambda \epsilon P(V)$. Then there exists $\eta \geq 0$ such that $\lambda + \alpha \notin P(V)$ for all positive roots $\alpha$ such that $\alpha > \eta$.

**Proof of the claim** Suppose there exists infinitely many positive roots $\alpha$ such that $\lambda + \alpha \epsilon P(V)$. First by Proposition (1.8) there exists $\eta \geq 0$ such that $\lambda + \eta \epsilon \Lambda^+, \lambda + \eta \epsilon P(V), (\lambda + \eta)(d) \geq \lambda(d)$ and the irreducible integrable highest module $V(\lambda + \eta) \subseteq V$. 

7
Let $\overline{P}(V) = \{ \overline{\lambda} \mid \lambda \in P(V) \}$. Clearly $\overline{P}(V)$ defines a unique coset in $\Lambda$. Let $\overline{\mu_0}$ be the minimal weight for this coset. Let $\mu_0 = \overline{\mu_0} + \lambda(d)\delta + \lambda(K)w \leq \lambda + \eta$. By lemma (1.6) we have $\mu_0 \in P(V)$.

First note that the number of positive roots $\alpha_1$ such that $\alpha_1 \neq \eta$ is finite.

Now choose positive root $\alpha_1 > \eta$ such that $\lambda + \alpha_1 \in P(V)$. (This is due to our supposition). Now by above arguments there exists $\eta_1 \geq 0$ such that $\lambda + \alpha_1 + \eta_1 \in P(V), \lambda + \alpha_1 + \eta_1 \in \Lambda^+$ and $V(\lambda + \alpha_1 + \eta_1) \subseteq V$. Further $\mu_0 \leq \lambda + \alpha_1 + \eta_1$ and $\mu_0 \in P(\lambda + \alpha_1 + \eta_1) \subseteq P(V)$. Note that $\lambda + \alpha_1 + \eta_1 > \lambda + \eta$ (note the strict inequality). Thus $V(\lambda + \alpha_1 + \eta_1) \neq V(\lambda + \eta)$. Both modules have common weight $\mu_0$. Thus we have proved that $\dim V_{\mu_0} \geq 2$. By repeating $n$ times the above argument we get $\dim V_{\mu_0} \geq n$. But $\dim V_{\mu_0}$ is finite and thus this process has to stop. This proves our claim. It follows from the claim and that the module $V$ satisfies the conditions of Proposition 1.9 and hence it is completely reducible.

(1.11) Remark Theorem (1.10) imply that an integrable module with finite dimensional weight spaces in which $K$ acts by positive integer belongs to the category $\mathcal{O}$.

Section 2

Let $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ be simple finite dimensional Lie super algebra $\mathcal{G}_0$ (respectively, $\mathcal{G}_1$) being its even (respectively, odd) part. We assume that $\mathcal{G}_0$ is reductive. We further assume that $\mathcal{G}$ carries a non-degenerate invariant "symmetric" bilinear form. Such Lie super algebras are called basic. We give the list of basis Lie super algebras from Proposition (1.1) of [K].
In this section we study the integrable representations of the untwisted affine Lie super algebras of basic Lie super algebras.

Let $G$ be a basic Lie-super algebra. Then the restriction to the even part need not be positive definite. In fact we choose the form in such a way that the restriction to the first component of the even part is positive definite and the restriction to the second component is negative definite. (see section 6 of [KW]). We normalize the form in such a way that $(\alpha, \alpha) = 2$ where $\alpha$ is the highest root of the first component of the even part of $G_0$ and $(\beta, \beta) = -2$ where $\beta$ is the highest root of the second component. Let $h$ be the Cartan subalgebra of $G$ which is contained in the even part.

(2.1) Define affine super algebra $\hat{G}$.

$$\hat{G} = G \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d.$$ 

The Lie bracket is given by the following. Write $x(n) = x \otimes t^n$.

$$[x(n), y(m)] = [x, y](m+n) + n(x, y)\delta_{m+n,0}K$$

$$[d, x(n)] = nx(n) \quad x, y \in G, m, n \in \mathbb{Z}, \quad K \text{ is central}$$

Let $\hat{h} = h \oplus \mathbb{C}K \oplus \mathbb{C}d$

| $G$ | $G_0$ |
|-----|------|
| $A(m, n)$ | $A_m + A_n + \mathbb{C}$, |
| $C(n)$ | $C_n + \mathbb{C}$ |
| $B(m, n)$ | $B_m + C_n$ |
| $D(m, n)$ | $D_m + C_n$, |
| $D(2, 1 : a)$ | $D_2 + A_1$ |
| $F(4)$ | $B_3 + A_1$ |
| $G(3)$ | $G_2 + A_1$ |
(2.2) **Definition** A module $V$ of $\hat{\mathcal{G}}$ is called integrable if

1. $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_{\lambda}$, $V_{\lambda} = \{ v \in V | hv = \lambda(h)v, \forall h \in \mathfrak{h} \}$

2. $V$ is integrable as a $\hat{\mathcal{G}}_0$ module.

3. For any $v \in V$, $U(\mathcal{G})v$ is finite dimensional.

Here $U(\mathcal{G})$ is the universal enveloping algebra of $\mathcal{G}$.

(2.3) **Remark** In [KW] integrable modules are studied with weaker conditions. In [KW] integrability means the module is integrable only with the affinization of one simple part of $\mathcal{G}_0$. Then they have classified irreducible highest weight module which are integrable in the above sense. See Theorem 6.1 & 6.2 of [KW].

The purpose of this section is to classify irreducible integrable modules for $\hat{\mathcal{G}}$ where center $K$ acts non trivially.

Let $\mathcal{G}_{01}$ and $\mathcal{G}_{02}$ be the first and second and the simple component of $\mathcal{G}_0$ as above. (In case of $D(2, n)$ and $D(2, 1; \alpha)$ the first component is not simple. Then we take one of the simple component). Let $h_1$ and $h_2$ be the respective Cartan sub algebras. Let $\mathfrak{a}_1$ and $\mathfrak{a}_2$ be the corresponding root system. The following is very standard. Does not matter whether the form is positive definite or negative definite.

(2.4) For any root $\alpha \in \hat{\Delta}_i$, let $\alpha^\vee$ be the co-root. Let $x_\alpha$ be the corresponding root vector. Choose $x_{-\alpha}$ in the negative root space such that $(x_\alpha, x_{-\alpha}) = \frac{2}{(\alpha, \alpha)}$. Then $x_\alpha, x_{-\alpha}, \alpha^\vee$ is an $s\ell_2$ triple. Let $\gamma = \alpha + n\delta$, $\alpha \in \hat{\Delta}_i$. Let $\gamma^\vee = \alpha^\vee + \frac{2n}{(\alpha, \alpha)} K$ be the co-root. Then it is easy to check that $x_\alpha(n), x_{-\alpha}(-n), \gamma^\vee$ is an $s\ell_2$ triple.

2.5 **Lemma** Let $V$ be an integrable $\hat{\mathcal{G}}$-module. Let $\lambda$ be a weight of $V$. Let $\gamma = \alpha + n\delta$, $\alpha \in \Delta_i$ such that $\lambda(\gamma^\vee) > 0$. Then $\lambda - \gamma$ is a weight of $V$. 

10
Proof Follows from standard \( s\ell_2 \) theory.

\( (2.6) \) Theorem  Notation as above. Assume the semi simple part of \( G_0 \) has at least two components. Let \( V \) be integrable module with finite dimensional weight spaces. Let the central element \( K \) act by non-zero scalar. Then \( V \) is necessarily trivial module.

Without loss of generality we can assume that \( K \) acts by positive integer. We can establish the following by the arguments similar to the proof of theorem (1.10).

\( (2.7) \) For any \( \lambda \in P(V) \) there exists \( N > 0 \) such that

\[
\lambda + \alpha + n\delta \notin P(V) \text{ for all } n \geq N \text{ and for all } \alpha \in \hat{\Delta}_1 \cup \{0\}.
\]

\( (2.8) \) There is one problem. The module \( V \) need not have finite dimensional weight spaces for \( \hat{G}_{01} \) as \( h_1 \oplus C K \oplus C d \) could be much smaller than the Cartan \( h = h_1 \oplus h_2 \oplus C K \oplus C d \). To overcome this problem, first observe that \( \hat{G}_{01} \) commutes with \( h_2 \). Now decompose the module \( V \) with respect to \( h_2 \) and \( h_2 \) weight space is a \( \hat{G}_{01} \)-module with finite dimensional weight spaces. Now apply arguments similar to the proof of (1.10) to conclude (2.7).

Claim  There exists a weight vector \( v \) of weight \( \lambda \) such that

\[
x_\alpha(n)v = 0 \text{ for } n < 0 \text{ and for all } \alpha \in \Delta_2 \cup \{0\}.
\]

First we complete the proof assuming the claim. From the claim we have \( h(n)v = 0 \) for \( n < 0 \) and \( h_2h_2 \). From the standard Hisenbrg highest weight module theory it follows that \( h(n)v \neq 0 \) for all \( n > 0 \) and for all \( h \) in \( h_2 \). Thus it follows that \( \lambda + m\delta \) is a weight for all \( m > 0 \) contradicting (2.7). Thus the module \( V \) has to be trivial.
Proof of the claim: From Lemma 1.7 (2) it follows that there exists \( \lambda \in P(V) \) such that \( \lambda - \alpha \notin P(V) \) for all \( \alpha \in \Delta_2^+ \). Let \( \Delta_2^{-ar} \) be the negative real roots of \( \hat{G}_{02} \). Define \( \Delta(\lambda) = \{ \gamma \in \Delta_2^{-ar} \mid \lambda(\gamma^\vee) \leq 0 \} \). Then \( \Delta(\lambda) \) is finite set. Indeed, let \( \gamma = \alpha - n\delta, \alpha \in \Delta_2, n > 0 \) be an element of \( \Delta_2^{-ar} \). Then \( \lambda(\gamma^\vee) = \lambda(\alpha^\vee) - n\lambda(K)_{(\alpha, \alpha)} > 0 \) for \( n \) sufficiently large (recall \( (\alpha, \alpha) < 0 \) for all \( \alpha \in \Delta_2 \)). Fix a positive integer \( r \) such that \( \alpha - s\delta \in \Delta_2^{-ar} - \Delta(\lambda) \) for \( s \geq r \).

Subclaim 1 \( \lambda - s\delta \notin P(V) \) for \( s \geq r \). Suppose \( \lambda - s\delta \in P(V) \) for some \( s \geq r \) we have \( \lambda((\alpha - s\delta)^\vee) > 0 \) then by Lemma (2.5), \( \lambda - s\delta - (\alpha - s\delta) = \lambda - \alpha \epsilon P(V) \) which is a contradiction to the choice of \( \lambda \).

Fix a positive integer \( p \) such that \( \lambda - s\delta \notin P(V) \) for \( s > p \) and \( \lambda - p\delta \epsilon P(V) \).

Subclaim 2 \( \lambda - \alpha - (m + p)\delta \notin P(V) \) for \( m > 0 \) and \( \alpha \in \Delta_2^+ \). Suppose the claim is false. Consider \( (\lambda - \alpha - (m + p)\delta)(\alpha^\vee) < 0 \) since \( \lambda(\alpha^\vee) < 0 \) and \( \alpha(\alpha^\vee) = 2 \). Then by Lemma (2.5) we have \( \lambda - \alpha - (m + p)\delta + \alpha = \lambda - (m + p)\delta \epsilon P(V) \) contradiction the choice of \( p \).

Subclaim 3 \( \lambda + \alpha - (m + p + 1)\delta \notin P(V) \) for \( m > r \) and \( \alpha \in \Delta_2^+ \). Suppose the claim is false. Consider

\[
(\lambda + \alpha - (m + p + 1)\delta)(\alpha - m\delta)^\vee > 0 \text{ as } \alpha - m\delta \notin \Delta(\lambda).
\]

Thus by Lemma 2.5 we have

\[
\lambda + \alpha - (m + 1 + p)\delta - \alpha + m\delta = \lambda - (1 + p)\delta \epsilon P(V)
\]

contradicts the choice of \( p \).

Thus we have proved

\[
\hat{G}_{02, -r\delta} V_{\lambda - p\delta} = 0, r > 0 \text{ and } \hat{G}_{02, \alpha - s\delta} V_{\lambda - p\delta} = 0 \text{ for all}
\]
but finitely many negative roots. Since \( V \) is integrable \( \overline{W} = U(\hat{G}_{02})V_{\lambda - p\delta} \) is finite dimensional. Let \( \mu \) be the lowest weight of \( \overline{W} \). This weight satisfies all the requirements of the claim.

(2.9) Theorem Let \( \hat{G} \) be the affine super algebra defined earlier. Assume that the semisimple part of the finite even part has only one component. Further assume that the non-degenerate form restricted to this simple Lie algebra \( G_0 \) is positive definite. Let \( V \) be irreducible integrable module with finite dimensional weight spaces. Assume the central element \( K \) acts as positive integer. Then \( V \) is an highest weight module.

Proof From the proof of Theorem (2.6) we have (2.7). Let \( \beta_1, \cdots, \beta_k \) be odd roots of \( G \). Let \( v \) be a wieght vector of \( V \) of weight \( \lambda \).

Claim The following vectors span is a finite dimensional space \( W \)

\[
\{ x_{\beta_1}(m_1) \cdots x_{\beta_k}(m_k)v, i_j \leq i_k, m \geq 0 \}
\]

where \( x_{\beta_i} \) is a root vector for the odd root space \( G_{\pm \beta_i} \). The affine roots that are occuring in the product are all distinct. In the above we take negative roots first and postive roots next. The indices are decreasing order. It is sufficient to prove that the vector space \( T \) spanned by the following vector is finite dimensional.

\[
\{ x_{\beta}(m_1) \cdots x_{\beta}(m_k)w, m_i \geq 0, m_i \neq m_j \}
\]

where \( w \) is any weight vector of \( V \). This is because there are only finitely many odd roots in \( G \).

First note that if \( k\beta \) is a root for \( k > 0 \) then \( k = 1 \) or \( 2 \). Consider
\[
x_{\beta}(m)w = [h(m), x_{\beta}]w = h(m)x_{\beta}(w) \pm x_{\beta}h(m)w.
\]
By (2.7) both vectors are
zero for large \( m \). Let \( m_0 \) be such that \( x_{k\beta}(m)w = 0 \) for \( m > m_0 \) and \( k = 1, 2 \).
Then it is easy to see that \( T \) is spanned by

\[
\{ x_{\beta}(m_1) \cdots x_{\beta}(m_k)w; 0 \leq m_i \leq m_0, m_i \neq m_j, i \neq j \}
\]

which is clearly finite dimensional.

Let \( H \) be the center of the reductive Lie algebra \( \mathcal{G}_0 \). Consider

\[
(2.10) \quad S = U(\hat{\mathcal{G}}_0^-)U(h)U(\hat{\mathcal{G}}_0^+)U(\bigoplus_{n>0} H \otimes t^n)W.
\]

Then \( V = U(\bigoplus_{n<0} H \otimes t^n)U(\hat{\mathcal{G}}_-)S \)
by PBW basis theorem.

By (2.7) we conclude that \( U(\bigoplus_{n>0} H \otimes t^n)W = W_1 \) is finite dimensional. Clearly \( S \) is \( \hat{\mathcal{G}}_0 \)-module and by Theorem (1.10) \( S \) is completely reducible. In fact it is direct sum of highest weight modules. Since \( W_1 \) is finite dimensional it intersects only finitely many of them. Say \( V(\lambda_1) \cdots V(\lambda_k) \). Thus \( S = \bigoplus V(\lambda_i) \) a finite sum. Thus \( S \) has a maximal weight. (Here the ordering is the following \( \mu_1 \leq \mu_2 \) means \( \mu_2 - \mu_1 = \sum n_i \alpha_i, n_i \in \mathbb{N}, \alpha_i \)'s are small roots of \( \hat{\mathcal{G}} \). \( \hat{\mathcal{G}} \) is a generalized Kac-Moody Lie super algebra and it does admit simple roots. See [KW]). The maximal weight is in fact maximal for \( V \) as the rest of the space brings the weights down.

The maximal weight is in fact highest weight. As \( V \) is irreducible, it is irreducible highest weight module.

**(2.11) Remark** In the process we also established that an irreducible integrable highest weight module for \( \hat{\mathcal{G}} \) is completely reducible for \( \hat{\mathcal{G}}_0 \oplus \hat{H} \).

**Proof** Let \( V \) be irreducible highest weight module for \( \hat{\mathcal{G}} \). Let

\[
\Omega(V) = \{ v \in V \mid h(k)v = 0 \text{ for all } h \in H, k > 0 \}
\]
Let $M(k)$ be the irreducible highest weight module for $H \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$ where $K$ acts by $k$. Then by Theorem (1.7.3) of [FLM] we have $V = \Omega(V) \otimes M(k)$. Now $\Omega(V)$ is an integrable module and hence by Theorem (1.10) decomposes into irreducible modules for $\widehat{G_0}$. Thus the Remark follows.
REFERENCES

[C] Chari, V. Integrable representations of Affine Lie-Algebras. Invent Math. 85, 317-335 (1986).

[E1] Eswara Rao, S. Classification of Loop Modules with finite dimensional weight spaces. Math. Ann. 305, 651-663 (1996).

[E2] Eswara Rao, S. Classification of irreducible integrable modules for Toroidal Lie algebras with finite dimensional weight spaces preprint 2001.

[FLM] Frenkel, I., Lepowsky, J. and Mueurman, A. Vertex operator algebras and the Monster, Academic Press (1989).

[H] Humphreys, J.E., Introduction to Lie-algebras and representation theory, Springer Berlin, Hidelberg, New York (1972).

[K] Kac, V.G., Representations of classical Lie super algebras Lecture note in Math. 676 (1978), 597-626.

[K1] Kac, V. Infinite dimensional Lie-algebras, Cambridge University Press, 3rd edition (1990).

[KW] Kac, V.G. and Minoru Wakimoto Integrable highest weight modules over affine superalgebras and appell’s function, Communications of Mathematical Physics, 215, 631-682 (2001).