THE ALEXANDER POLYNOMIAL OF (1, 1)-KNOTS

A. CATTABRIGA

Mathematics Department, University of Bologna,
P. zza di Porta S. Donato, 5, 40126 Bologna, Italy
cattabri@dm.unibo.it

ABSTRACT

In this paper we investigate the Alexander polynomial of (1, 1)-knots, which are knots lying in a 3-manifold with genus one at most, admitting a particular decomposition. More precisely, we study the connections between the Alexander polynomial and a polynomial associated to a cyclic presentation of the fundamental group of an n-fold strongly-cyclic covering branched over the knot \( K \), which we call the \( n \)-cyclic polynomial of \( K \). In this way, we generalize to all (1,1)-knots, with the only exception of those lying in \( S^2 \times S^1 \), a result obtained by J. Minkus for 2-bridge knots and extended by the author and M. Mulazzani to the case of (1,1)-knots in \( S^3 \). As corollaries some properties of the Alexander polynomial of knots in \( S^3 \) are extended to the case of (1,1)-knots in lens spaces.

Keywords: (1,1)-knots, cyclic branched coverings, Alexander polynomial, cyclically presented groups.

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1. Introduction

A knot \( K \) in a 3-manifold \( M \) is called a (1, 1)-knot, or a 1-bridge torus knot, if it can be embedded into a Heegaard torus of \( M \) except at one over (or under) bridge. This means that there exists a Heegaard splitting of genus one

\[
(M, K) = (H, A) \cup_\varphi (H', A'),
\]

where \( H \) and \( H' \) are solid tori, \( A \subset H \) and \( A' \subset H' \) are properly embedded trivial arcs, and \( \varphi : (\partial H', \partial A') \to (\partial H, \partial A) \) is an attaching homeomorphism (see Fig. 1.1).

Such a decomposition is called (1,1)-decomposition of the knot. By definition, the ambient manifold has genus less or equal than one, so \( M \), up to homeomorphism,

\footnote{This means that there exists a disk \( D \subset H \) (resp. \( D' \subset H' \)) with \( A \cap D = A \cap \partial D = A \) and \( \partial D - A \subset \partial H \) (resp. \( A' \cap D' = A' \cap \partial D' = A' \) and \( \partial D' - A' \subset \partial H' \)).}
is $S^2 \times S^1$ or a lens space $L(p, q)$, including the case $L(1, 0) = S^3$. Important examples of $(1, 1)$-knots in $S^3$ are torus knots (by definition) and 2-bridge knots (see [13]). This family of knots has interesting features and has recently been studied by many authors from different points of view. For instance, in [5, 7, 9], different representations of $(1, 1)$-knots are given, while in [1, 4, 6, 13, 17] their connections with manifolds with cyclically presented fundamental groups are studied.

![Fig. 1.1. A (1, 1)-decomposition.](image)

In this paper we investigate the Alexander polynomial of $(1, 1)$-knots. This class of knots seems to have very interesting features with regards to the Alexander polynomial. For example, in [12], the author shows that it is possible to realize each Alexander polynomial in $S^3$ using $(1, 1)$-knots. Moreover, he exhibits an infinite family of $(1, 1)$-knots in $S^3$ with a trivial Alexander polynomial. In this paper we focus especially on $(1, 1)$-knots in lens spaces. From the results of [17], each $n$-fold strongly-cyclic branched covering of a $(1, 1)$-knot admits a cyclic presentation for the fundamental group and a polynomial associated to this presentation. Moreover, if $K \subset L(p, q)$ and $n$ is coprime with $p$, the covering is unique, and so there is a natural way to associate to $K$ a polynomial that will be called the $n$-cyclic polynomial of $K$. Using the reduced Reidemeister torsion of the knot complement, we find the relation between this polynomial and the Alexander polynomial. In this way, we generalize to all $(1, 1)$-knots, with the only exception of those lying in $S^2 \times S^1$, a result of [16] for 2-bridge knots and of [11] for $(1, 1)$-knots in $S^3$. As corollaries, we extend to all $(1, 1)$-knots in lens spaces some properties of the Alexander polynomial of knots in $S^3$.

In Section 2 we give the definition of $n$-cyclic polynomials. In Section 3 we recall the definition of the Alexander polynomial of a knot, and some results concerned with it. The main result of this paper is established in Section 4, where some corollaries and examples are also collected.
2. Strongly-cyclic coverings and cyclic presentations of groups

Let $F_n$ denote the free group of rank $n$. A group $G$ is called cyclically presented if there exists $n > 0$ and $w \in F_n$ such that

$$G = G_n(w) = \langle x_1, \ldots, x_n \mid w, \theta_n(w), \ldots, \theta_n^{n-1}(w) \rangle,$$

(2.1)

where $\theta_n : F_n \to F_n$ is the automorphism defined by $\theta_n(x_i) = x_{i+1}$ (subscripts mod $n$), for $i = 1, \ldots, n$. Such a presentation is called a cyclic presentation. Obviously $G_n(w)$ has an automorphism of order $n$ induced by $\theta_n$.

The polynomial associated to the cyclic presentation $G_n(w)$ is

$$p_{n,w}(t) = \sum_{i=1}^{n} a_i t^{i-1},$$

(2.2)

where $a_i$ is the total exponent sum of the letter $x_i$ in the word $w$, for $i = 1, \ldots, n$.

For each $s = 0, \ldots, (n-1)$, we have $G_n(w) = G_n(\theta_n^s(w))$, while the polynomials $p_{n,\theta_n^s(w)}(t)$ are different elements of $\mathbb{Z}[t]$. However, their image in the quotient ring $\mathbb{Z}[t]/(t^n - 1)$ is the same, up to units. So, from now on, we will consider the polynomial associated to the cyclic presentation $G_n(w)$ as an element of $\mathbb{Z}[t]/(t^n-1)$.

Note that the abelianization of $G_n(w)$ has a circulant presentation matrix (as a $\mathbb{Z}$-module), whose first row is given by the coefficients of $p_{n,w}$.

An $n$-fold cyclic covering $f : \tilde{M} \to M$, branched over a knot $K \subset M$, is called strongly-cyclic if the branched index of $K$ is $n$. In other words\(^2\), if $\omega_f : H_1(M - \mathcal{N}(K)) \to \mathbb{Z}_n$ denotes the monodromy of $f$ and $\mathcal{N}$ is the first homology class of a meridian loop of $K$, $\omega_f(\mathcal{N})$ generates $\mathbb{Z}_n$. Moreover, two $n$-fold strongly-cyclic coverings $f$ and $f'$, branched over the same knot $K \subset M$, are equivalent if there exists an invertible element $u \in \mathbb{Z}_n$ such that the following diagram commutes

\[
\begin{array}{ccc}
H_1(L(p, q) - \mathcal{N}(K)) & \xrightarrow{\omega_f} & \mathbb{Z}_n \\
\downarrow{\omega_f} & & \\
\mathbb{Z}_n & \xrightarrow{\mu_u} & \mathbb{Z}_n
\end{array}
\]

(2.3)

where $\mu_u$ denotes the multiplication by $u$. So, up to equivalence, a cyclic branched covering $f$ is strongly-cyclic if $\omega_f(\mathcal{N}) = 1$. From now on, $C_n(K)$ will denote an $n$-fold strongly-cyclic branched covering of $K$.

If we are dealing with knots in $S^3$, every $n$-fold cyclic branched covering of a knot is strongly-cyclic, so it exists and is unique, up to equivalence. This is no longer true for a $(1,1)$-knot $K$ in a lens space $L(p, q)$. However, from the existence

\(^2\)We denote with $\mathcal{N}(K) \subset M$ an open tubular neighborhood of $K$ in $M$.\n
and uniqueness conditions of [4], it follows that for each \( n \) coprime with \( p \) there is a unique \( n \)-fold strongly-cyclic branched covering of \( K \).

The connection between (1, 1)-knots and cyclically presented groups is given by the following theorem.

**Theorem 2.1.** [17] Every \( n \)-fold strongly-cyclic branched covering of a (1, 1)-knot admits a cyclic presentation for the fundamental group with \( n \) generators.

Moreover, this presentation is obtained by lifting a (1, 1)-presentation of the knot, as follows (for details see [4]).

From a (1, 1)-decomposition \( (L(p, q), K) = (H, A) \cup \varphi(H', A') \) of \( K \), we get the following presentation of the knot group:

\[
\pi_1(L(p, q) - N(K), \ast) = \langle \alpha, \gamma \mid r(\alpha, \gamma) \rangle, \quad (2.4)
\]

where the generators \( \alpha \) and \( \gamma \) represent, respectively, a longitude of \( \partial H \) and a meridian loop of \( K \), and the relator \( r(\alpha, \gamma) \) corresponds to the loop \( \varphi(\partial D') \), where \( D' \) is a meridian disk of \( H' \) that does not intersect \( A' \) (see Fig. 1).

If necessary, by replacing the covering \( f : \tilde{M} \to L(p, q) \) with an equivalent one, we can suppose that \( \omega_f(\gamma) = 1 \). Let \( \tilde{r}(x, \gamma) = r(x^\varepsilon \gamma^\omega_f(\alpha), \gamma) = x^{i_1} \gamma^{\delta_1} \cdots x^{i_s} \gamma^{\delta_s} \) for some \( \varepsilon_1, \ldots, \varepsilon_s, \delta_1, \ldots, \delta_s \in \mathbb{Z} \). We have \( \pi_1(\tilde{M}, \tilde{\ast}) = G_n(w) \) where:

\[
w = x^{i_1}_{\varepsilon_1} \cdots x^{i_s}_{\varepsilon_s} \quad \text{(subscripts mod } n) \quad (2.5)
\]

with \( i_k \equiv 1 + \sum_{j=1}^{k-1} \delta_j \mod n \), for \( k = 1, \ldots, s \).

For each \( n \) coprime with \( p \), the polynomial associated to the cyclic presentation of the (unique) \( n \)-fold strongly-cyclic branched covering of \( K \) obtained as above (i.e. lifting a (1, 1)-decomposition of \( K \) and under the condition \( \omega_f(\gamma) = 1 \)) will be called the \( n \)-cyclic polynomial of \( K \), and it will be denoted with \( \Gamma_{K,n} \).

**Remark 2.1.** In the case of a (1, 1)-knot \( K \subset S^3 \times S^1 \), which corresponds to \( p = 0 \), the situation is rather different. We do not have the uniqueness of the \( n \)-fold strongly-cyclic branched covering of \( K \) for any value of \( n > 1 \), and so we do not have a natural way of defining the \( n \)-cyclic polynomial. Moreover, there exists at most a finite number of finite strongly-cyclic branched covering of \( K \), for almost all the \( K \).

Let \( \Delta_K \in \mathbb{Z}[t, t^{-1}] \) be the Alexander polynomial of a knot \( K \subset S^3 \) and denote with \( \Delta_{K,n} \) its projection on \( \mathbb{Z}[t]/(t^n - 1) \). With these notations the results of [4] and [16] can be restated as follows.

**Proposition 2.1.** Let \( K \subset S^3 \). The following holds:

(i) [16] if \( K \) is a 2-bridge knot, for each \( n > 1 \), we have \( \Gamma_{K,n}(t) = \Delta_{K,n}(t) \), up to units of \( \mathbb{Z}[t]/(t^n - 1) \).

(ii) [4] if \( K \) is a (1, 1)-knot, for each \( n > 1 \), we have \( \Gamma_{K,n}(t) = \Delta_{K,n}(t) \), up to units of \( \mathbb{Z}[t]/(t^n - 1) \).
The main result of this article is the generalization of this relation to all \((1, 1)\)-knots in lens spaces.

3. Alexander polynomial

In this section we recall the definition of the Alexander polynomial of a knot \(K\) in a compact, connected 3-manifold, and give some of its characteristics.

Let \(M\) be a compact connected manifold, \(* \in M\) be a fixed point and denote with \(\Phi : \pi_1(M, *) \to H_1(M)\) the Hurewitz homomorphism. Consider the projection \(j : H_1(M) \to H_1(M)/\text{Tors}(H_1(M))\) and the induced ring homomorphism\(^3\) \(\tilde{j} : \mathbb{Z}[H_1(M)] \to \mathbb{Z}[H_1(M)/\text{Tors}(H_1(M))]\). Note that, if \(r\) is the first Betti number of \(M\), we have \(\mathbb{Z}[H_1(M)/\text{Tors}(H_1(M))] \cong \mathbb{Z}[t_1, t_1^{-1}, \ldots, t_r, t_r^{-1}]\), where \(t_1, \ldots, t_r\) are generators of \(H_1(M)/\text{Tors}(H_1(M))\). Let \(E_1(M) \subset \mathbb{Z}[H_1(M)]\) be the first elementary ideal of \(\pi_1(M, *)\) (see [10]) and denote with \(E_1(M)\) the smallest principal ideal of \(\mathbb{Z}[H_1(M)/\text{Tors}(H_1(M))]\) containing \(j(E_1(M))\). The generator \(\Delta_M\) of \(E_1(M)\) is well-defined up to multiplication by units of \(\mathbb{Z}[H_1(M)/\text{Tors}(H_1(M))]\) and is called the Alexander polynomial of \(M\).

If \(K \subset X\) is a knot in a compact connected 3-manifold, the Alexander polynomial of \(K\) is the Alexander polynomial of \(M = X - N(K)\).

For a knot \(K \subset S^3\) the Alexander polynomial has the following characterization.

**Theorem 3.1.** [10] Let \(\Delta_K(t) \in \mathbb{Z}[t, t^{-1}]\) be the Alexander polynomial of a knot \(K \subset S^3\). We have:

(i) \(\Delta_K(t) = \Delta_K(t^{-1})\).

(ii) \(\Delta_K(1) = \pm 1\).

Moreover, each polynomial \(q(t) \in \mathbb{Z}[t, t^{-1}]\) satisfying conditions 1 and 2 is the Alexander polynomial of a knot in \(S^3\).

Actually, from [12], each Alexander polynomial of a knot in \(S^3\) can be realized by a \((1, 1)\)-knot in \(S^3\).

Moreover, the Alexander polynomial of a knot \(K \subset S^3\) determines the order of the first homology group of \(C_n(K)\), the \(n\)-fold (strongly-)cyclic branched covering of \(K\).

**Theorem 3.2.** [11] The abelian group \(H_1(C_n(K))\) is finite if and only if no root of the Alexander polynomial \(\Delta_K(t)\) of \(K\) is an \(n\)-root of unity. In this case:

\[
\sharp H_1(C_n(K)) = | \prod_{\zeta^n=1} \Delta_K(\zeta)|.
\]

\(^3\)For a group \(G\) we denote with \(\mathbb{Z}[G]\) its integral group ring.
Theorem 3.3. \[20\] Let $M$ be a compact, connected, orientable 3-manifold with $\chi(M) = 0$. If $\partial M \neq \emptyset$ and $b_1(M) = 1$, then $r_M(t) = \Delta_M(t)/(t-1)$, where $r_M(t) \in \mathbb{Q}(t)$ denotes the reduced Reidemeister torsion of $M$ and $t$ is a generator of $H_1(M)/\text{Tors}(H_1(M))$.

From this theorem it follows (see \[20\]) that for the Alexander polynomial of $(1, 1)$-knots in lens spaces it is still true that $\Delta_K(t) = \Delta_K(t^{-1})$. However, we will see that the property $\Delta_K(1) = \pm 1$ is no longer true.

For details on the reduced Reidemeister torsion, and on its calculation, we refer to \[20\].

Remark 3.1. In the case of a $(1, 1)$-knot $K \subset S^2 \times S^1$ it could happen that the first Betti number is two. That is the case, for example, of the trivial knot.

4. Main theorem

Theorem 4.1. Let $K \subset L(p, q)$ be a $(1, 1)$-knot and denote with $d$ the order of the torsion subgroup of $H_1(L(p, q) - N(K))$. Then, for each $n > 1$ such that $\gcd(p, n) = 1$, we have:

$$
\Gamma_{K,n}(t^{\frac{1}{n}}) = \Delta_{K,n}(t) \sum_{i=0}^{n-1} t^i,
$$

up to units of $\mathbb{Z}[t]/(t^n - 1)$, where $\Gamma_{K,n}$ is the $n$-cyclic polynomial associated to $K$ and $\Delta_{K,n}$ is the projection of the Alexander polynomial of $K$ on $\mathbb{Z}[t]/(t^n - 1)$.

Proof. To prove the statement we will use the reduced Reidemeister torsion of $M = L(p, q) - N(K)$. In the cellular presentation \[24\] of $\pi_1(M, \ast)$, the total exponent sum of $\alpha$ in $r(\alpha, \gamma)$ is $p$, since $\alpha$ is a generator of $\pi_1(L(p, q), \ast)$. So, by abelianization, we get $H_1(M) = \langle \alpha, \gamma \mid p\alpha + q'\gamma \rangle \cong \mathbb{Z} \oplus \mathbb{Z}_d$, where $d = \gcd(p, q')$.

Moreover, if we set $\bar{p} = p/d$ and $\bar{q} = q'/d$ we have that $\bar{\eta} = \bar{p}\alpha + q\gamma$ and $\bar{\xi} = -s\alpha + r\gamma$ generate, respectively, the torsion part and the free part of $H_1(M)$, where $r$ and $s$ are integers such that $r\bar{\bar{p}} + s\bar{\bar{q}} = 1$. Now, consider the covering $F : \widetilde{M} \to M$ corresponding to the subgroup $\Phi^{-1}(\text{Tors}(H_1(M)))$ of $\pi_1(M, \ast)$, where $\Phi : \pi_1(M, \ast) \to H_1(M)$ denotes the Hurewitz homomorphism. The Betti number of $H_1(M)$ is one, so the covering $F$ is infinite cyclic with monodromy $\omega_F : H_1(M) \to \langle t \rangle \cong \mathbb{Z}$ defined by $\omega_F(\bar{\xi}) = t$ and $\omega_F(\bar{\eta}) = 1$.

Inverting the defining relations for $\bar{\xi}$ and $\bar{\eta}$ we have that $\omega_F(\gamma) = t\bar{\bar{p}}$ and $\omega_F(\alpha) = t\bar{\bar{q}}$. Up to contraction, which does not alter the torsion, the cell complex $C_\ast(M)$

$$
0 \to C_2 = \langle \varphi(D^+) \rangle \to C_1 = \langle \alpha, \gamma \rangle \to C_0 = \langle \ast \rangle \to 0,
$$

is a cellular decomposition for $M$. The reduced Reidemeister torsion of $M$ is, up to multiplication by $\pm t^h$, the torsion of the complex $\overline{C}_\ast(M) = \mathbb{Q}(t) \otimes_{\mathbb{Z}[t, t^{-1}]} C_\ast(\widetilde{M})$.
where $C_*(\overline{M})$ is the lifting of $C_*(M)$, with a fundamental family of cells of $\overline{M}$ as a base. Let $\delta_1 : \overline{C}_1 \to \overline{C}_0$ and $\delta_2 : \overline{C}_2 \to \overline{C}_1$ be the boundary operators. If $\ast$ is the fixed 0-cell over $\ast$, and $\tilde{\alpha}$ and $\tilde{\gamma}$ are the lifting of, respectively, $\alpha$ and $\gamma$ with starting point $\ast$ then, by the action of the monodromy map, we have $\delta_1(\tilde{\alpha}) = (t^{-q} - 1)\tilde{\ast}$ and $\delta_1(\tilde{\gamma}) = (t^p - 1)\tilde{\ast}$. Moreover it is well-known (see for example [2, § 9A-9B]) that, the matrix of $\delta_2$, is the Alexander matrix of the presentation $\Sigma$. So $\delta_1$ and $\delta_2$ are represented by the matrices $t(t^{-q} - 1 \ t^p - 1)$ and $(Q_\alpha(t) \ Q_\gamma(t))$, where $Q_\alpha(t)$ (resp. $Q_\gamma(t)$) is obtained from the Fox derivative $\partial r(\alpha, \gamma)/\partial \alpha$ (resp. $\partial r(\alpha, \gamma)/\partial \gamma$) by the substitutions $\alpha = t^{-q}$ and $\gamma = t^p$. The collection $\{ 1, 1, 0; t^p - 1, Q_\alpha(t) \}$ is a torsion chain for this complex (see [20, 21]), so $\delta_2$ is a torsion chain for this complex (see [20, 21]), so $r_M(t) = Q_\alpha(t)/(t^p - 1) \in \mathbb{Q}(t)$. By Theorem 3.1 $r_M(t) = \Delta_K(t)/(t - 1)$ and so $Q_\alpha(t) = \Delta_K(t)/(\sum_{i=0}^{p-1} t^i)$, where the equality holds up to units of $\mathbb{Z}[t, t^{-1}]$. To complete the proof we have to show that $Q_\alpha(t) = \Gamma_{K,n}(t^p)$ in $\mathbb{Z}[t]/(t^n - 1)$. Observe that, for each $n > 1$ coprime with $p$, the monodromy of the unique $n$-fold strongly-cyclic branched covering of $K$, $F_n$, is the composition of $\omega'$ with the epimorphism $\mathbb{Z} \to \mathbb{Z}_n$, given by $t \to 1$. Then we have $\omega_F(\alpha) = \tilde{-} \tilde{\omega}'$ and $\omega_F(\gamma) = \tilde{\gamma}$. If $\tilde{\gamma} \neq 1$, to calculate $\Gamma_{K,n}$ we have to replace $F_n$ with the equivalent covering with monodromy $\omega'$, such that $\omega'(\gamma) = 1$, and so $\omega'(\alpha) = \tilde{-} \tilde{\omega}_0^{-1}$, where $\tilde{\omega}_0^{-1}$ is the inverse of $\tilde{\omega}$ in $\mathbb{Z}_n$. Let $\tilde{r}(x, \gamma) = r(x \gamma, \omega'(\alpha), \gamma)$. It is easy to check, using formula (2.4), that $\Gamma_{K,n}(u) = \tilde{r}(x, \gamma) = \tilde{-} \tilde{\omega}_0^{-1} \tilde{-} \tilde{\omega}_0^{-1}$, and $\gamma = u$. So, setting $u = t^p$ in $\Gamma_{K,n}(u)$, we get $Q_\alpha(t)$, which ends the proof. □

Remark 4.1. In the case of $S^2 \times S^1$, the previous theorem does not hold, since, as we have already observed, in this case $p = 0$, so for each $n > 1$, $\gcd(n, p) = n \neq 1$. In fact, if we look at the proof, it is easy to see that, since $\tilde{\gamma} = \tilde{\omega}' \tilde{\omega}_0^{-1} = 0$, the monodromy $\omega_F$ of the covering that leads to the calculation of the Alexander polynomial sends a meridian of the knot into $t^0$. So, the covering whose monodromy is the composition of $\omega_F$ with the projection $\mathbb{Z} \to \mathbb{Z}_n$, defined by $t \to 1$, is not strongly-cyclic for any value of $n > 1$.

Observe that, when $p = 1$, $K$ is a $(1, 1)$-knot in $S^3$, and we get the same statement as in Proposition 2.1. Moreover, if $\text{Tors}(H_1(L(p, q) - N(K))) \cong \mathbb{Z}_p$, then the projection of the Alexander polynomial on $\mathbb{Z}[t]/(t^n - 1)$ is equal to the $n$-cyclic polynomial, up to units.

As a corollary of this theorem, we get a generalization of Theorem 3.1.

Corollary 4.1. Let $K \subset L(p, q)$ be a $(1, 1)$-knot, and let $C_n(K)$ be the $n$-fold strongly-cyclic branched covering of $K$, with $n$ coprime with $p$. Denote with $d$ the order of the torsion subgroup of $H_1(L(p, q) - N(K))$. Then $H_1(C_n(K))$ is finite if and only if no root of $\Delta_{K,n}$ is an $n$-root of unity. Moreover if $d_n$ denote the order...
of the torsion subgroup of $H_1(C_n(K))$, we have:
\[
d_n = | \prod_{\zeta^n=1, \Delta_{K,n}(\zeta) \neq 0} \left( \frac{\Delta_{K,n}}{\Phi} \right) (\zeta) \sum_{j=0}^{\frac{n}{d}-1} \zeta^j|,
\]
(4.3)

where $\Phi$ is the product of the distinct cyclotomic polynomials $\Phi_s$ such that $s$ divides $n$ and $\Phi_s$ divides $\Delta_{K,n}$. In particular, if $H_1(C_n(K))$ is finite, we have:
\[
\sharp H_1(C_n(K)) = \prod_{\zeta^n=1} \Delta_{K,n}(\zeta) \sum_{j=0}^{\frac{n}{d}-1} \zeta^j.
\]
(4.4)

**Proof.** If $\omega$ denotes the monodromy of $C_n(K)$, up to equivalence, we can suppose that $\omega(\gamma) = 1$. Consider the equivalent $n$-fold strongly-cyclic branched covering $C'_n(K)$ with monodromy $\omega' = \mu_{p/d} \omega$, where $\mu_{p/d} : \mathbb{Z}_n \to \mathbb{Z}_n$ denotes the multiplication by $p/d$ which is invertible in $\mathbb{Z}_n$. Then the isomorphism $H_1(C_n(K)) \to H_1(C'_n(K))$ is given by $x_i \to x_j$ where $j \equiv i(p/d) \mod n$. As previously observed, the circulant matrix whose first row is given by the coefficients of $\Gamma_{K,n}(t)$ is a presentation matrix for $H_1(C_n(K))$ as a $\mathbb{Z}$-module, so, the circulant matrix $B$ whose first row is given by the coefficients of $G(t) = \Gamma_{K,n}(t^{p/d})$ is a presentation matrix for $H_1(C'_n(K))$ as a $\mathbb{Z}$-module. Obviously $H_1(C'_n(K))$ is finite if and only if $H_1(C_n(K))$ is finite and the order of the torsion subgroup of $H_1(C'_n(K))$ is $d_n$. By the theory of circulant matrices (see [2]), there exists a complex unitary matrix $F$, called Fourier matrix, such that $F B F^* = D = \text{Diag}(G(\zeta_1), G(\zeta_2), \ldots, G(\zeta_n))$, where $\zeta_1, \zeta_2, \ldots, \zeta_n$ are the $n$-roots of the unity. So $H_1(C'_n(K))$ is finite if and only if the rank of $B$ is $n$, and so if and only if $G(\zeta_i) = \Delta_{K,n}(\zeta_i) \sum_{j=0}^{\frac{n}{d}-1} \zeta^j \neq 0$ for each $i = 1, \ldots, n$. The first statement follows from the fact that, since $\gcd(p,n) = 1$, we have $\gcd(p/d, n) = 1$, and so $q_{p/d}(\zeta_i) \neq 0$, for $i = 1, \ldots, n$, where $q_{p/d}(t) = \sum_{j=0}^{\frac{n}{d}-1} t^j$. Moreover, by [19] Theorem 3.3, we have that $d_n = | \prod_{\zeta^n=1, G(\zeta) \neq 0} \left( \frac{G(\zeta)}{\Phi(\zeta)} \right) |$, where $\Phi$ is the product of the distinct cyclotomic polynomials $\Phi_s$ such that $s$ divides $n$ and $\Phi_s$ divides $G$. To end the proof, we have only to show that $\Phi_s$ is a divisor of $\Delta_{K,n} q_{p/d}$ if and only if $s$ is a divisor of $\Delta_{K,n}$. This follows from the fact that, if $s$ is a divisor of $n$, we have $\gcd(s, p/d) = 1$, so for each $\zeta$ primitive $s$-root of the unity, $q_{p/d}(\zeta) \neq 0$. □

Another straightforward corollary is the following.

**Corollary 4.2.** The Alexander polynomial of a $(1,1)$-knot $K \subset L(p,q)$ satisfies the relation $\Delta_K(1) = \pm d$, where $d$ is the order of the torsion subgroup of $H_1(L(p,q) - N(K))$.

**Proof.** It is enough to observe that $\Gamma_{K,n}(1) = \pm p$. □

An interesting question for future study could be whether any symmetric polynomial in $\mathbb{Z}[t, t^{-1}]$ can be realized as the Alexander polynomial of a $(1,1)$-knot in a lens space.
Example 4.1. Let $K_{p,q}$ be the trivial knot in $L(p,q)$. We have that $\pi_1(L(p,q) - N(K_{p,q}), \ast) = \langle \alpha, \gamma \mid \alpha^p \rangle$, (for reference see [5]). Then $p = d = \sharp \text{Tors}(H_1(L(p,q) - N(K_{p,q})))$. So, for each $n > p$ with $\gcd(n,p) = 1$, we have $\Gamma_{K_{p,q},n} = p = \Delta_{K_{p,q}}$.

We end this paper by observing that formula (2.5) can easily be implemented to find $\Gamma_{K,n}$, and so, by Theorem 4.1, to calculate $\Delta_K$, as shown in the following example.

Example 4.2. By results of [7], each $(1,1)$-knot $K$ can be represented as $K(a,b,c,r)$, where $a, b, c, r$ are nonnegative integer parameters determining the Heegaard diagram of a $(1,1)$-decomposition of $K$. For each $m > 2$, let $K_m = K(1, m-2, 0, 1) \subset L(m-2, 1)$. This family of knots is very interesting since, as proved in [14], the $m$-fold strongly-cyclic branched covering of $K_m$, with monodromy that sends $\gamma$ into 1 and $\alpha$ into 0, is the Neuwirth manifold $N_m$ of type $m$. These manifolds were introduced by L. Neuwirth in [18], while in [8] it is proved that $N_m$ is a Seifert manifold of type $(0; -1; (2,1), \ldots, (2,1))$, with base $S^2$, Euler number $-1$ and $m$ exceptional fibers. From the parametric representation of $K_m$ (see [1] for details) we get

$$\pi_1(L(m-2,1) - N(K_m), \ast) = \langle \alpha, \gamma \mid (a \gamma)^{m-1} \alpha^{-1} \gamma \rangle,$$

and so

$$H_1(L(m-2,1) - N(K_m)) = \langle \alpha, \gamma \mid (m-2)\alpha + m \gamma \rangle \cong \begin{cases} \mathbb{Z} & \text{if } m \text{ is odd,} \\ \mathbb{Z} \oplus \mathbb{Z}_2 & \text{if } m \text{ is even.} \end{cases}$$

Notice that $K_3$ is the trefoil knot in $S^3$.

If we denote with $\omega$ the monodromy of the $n$-fold strongly-cyclic branched covering of $K_m$, with $\gcd(n,m-2) = 1$ and satisfying the condition $\omega(\gamma) = 1$, from the homology relation we get

$$c = \omega(\alpha) = \begin{cases} -m \cdot \frac{m-2}{2}, & \text{if } m \text{ is odd,} \\ -\frac{m}{2} \cdot \frac{(m-2)}{2}, & \text{if } m \text{ is even,} \end{cases}$$

where $x$ denote the inverse of $x$ in $\mathbb{Z}_n$. From the substitution $x = \alpha \gamma^{-c}$ in the presentation (1.5), we get the relator $\tilde{r}(x, \gamma) = (x \gamma^{c+1})^{m-1} \gamma^{-c} x^{-1} \gamma$. By formula (2.5), $\pi_1(C_n(K_m), \tilde{\ast}) = G_n(w)$, where

$$w = \left( \prod_{i=0}^{m-2} x_{1+i(1+c)} x_{2+i(m-2)(1+c)} \right)$$

(subscripts $\mod n$). So, we get:

$$\Gamma_{K_m,n}(t) = \left( \sum_{i=0}^{m-2} t^{i(1+c)} \right) - t^{1+(m-2)(1+c)}.$$
If $m$ is odd then $p/d = m - 2$ and we get

$$\Gamma_{K_m,n}(t^{m-2}) = (\sum_{i=0}^{m-2} t^{-2i}) - t^{-(m-2)} = (\sum_{i=0}^{m-2} t^{2i}) - t^{(m-2)} = \Delta_{K_m,n}(t) \sum_{j=0}^{m-3} t^j,$$

(where the second equality holds up to multiplication for $t^{2(m-2)}$). This means that for each $n > 2(m-2)$ coprime with $m-2$, the representative of $\Delta_{K_m}$ in $\mathbb{Z}[t]/(t^n - 1)$ does not depend on $n$. So, for each $m > 2$ odd

$$\Delta_{K_m}(t) = \frac{(\sum_{i=0}^{m-2} t^{2i}) - t^{(m-2)}}{\sum_{j=0}^{m-3} t^j} = \sum_{i=0}^{m-1} (-t)^i.$$

Analogously, if $m$ is even we have $p/d = \frac{m-2}{2}$ and therefore

$$\Gamma_{K_m,n}(t^{(m-2)/2}) = (\sum_{i=0}^{m-2} t^{-i}) - t^{-(m-2)/2} = (\sum_{i=0}^{m-2} t^i) - t^{(m-2)/2} = \Delta_{K_m,n}(t) \sum_{j=0}^{(m-4)/2} t^j.$$

By the same considerations, for each $m > 2$ even

$$\Delta_{K_m}(t) = \frac{(\sum_{i=0}^{m-2} t^i) - t^{(m-2)/2}}{\sum_{j=0}^{(m-4)/2} t^j} = t^{m/2} + 1.$$

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