I. INTRODUCTION AND STATEMENT OF THE RESULTS

Let $A$ be a subset of an abelian group $G$, $\mu$ be a measure on $G$ and $C > 0$ be a real number. Inverse results in additive number theory refer to those in which starting from the small doubling condition $\mu(A + A) < C\mu(A)$, it is possible to deduce structural information on $A$ and its sumset $A + A$. For finite sets $A$, the measure of $A$ can be its cardinality. More generally for infinite sets in a locally compact group, $\mu(A)$ can be chosen as the Haar measure of $A$. For infinite sets in a discrete semigroup, we may use various notions of density instead of measure.

One of the most popular inverse result is Kneser’s theorem. In an abelian group with $\mu(\cdot) = |\cdot|$, the cardinality measure, and $C \leq 2$ it provides mainly a periodical structure for sumsets $A + B$ such that $|A + B| < |A| + |B| - 1$, yielding also a structure for $A, B$ themselves.

In the particular semigroup $\mathbb{N}$ of positive integers with $\mu = \mathbb{d}$ being the lower asymptotic density, there exists such result when $\mathbb{d}(A + B) < \mathbb{d}(A) + \mathbb{d}(B)$ (see [1] or [2]); basically, $A + B$ is then a union of residue classes modulo some integer $q$. When $\mu$ is the upper asymptotic density or the upper Banach density, some structural information is still available (see [3] and [2] respectively).

Our goal is to investigate the validity of similar results in abelian $\sigma$-finite groups. A group $G$ is said to be $\sigma$-finite if

$$G = \bigcup_{n \geq 1} G_n$$

where $(G_n)_{n \geq 1}$ is a non decreasing sequence of finite groups. As examples we have the polynomial ring $\mathbb{F}_p[x]$ for any prime $p$ and integer $r \geq 1$. More generally, let $(C_n)_{n \geq 1}$ be a sequence of finite groups, $G_N = \prod_{n \in \mathbb{N}} C_n$ and $G_n = \prod_{i \leq n} C_i \leq G_N$. Then $G = \bigcup_{n \geq 1} G_n$ is $\sigma$-finite. Another class of $\sigma$-finite groups is given by the $p$-Prüfer groups $\mathbb{Z}(p^\infty) = \bigcup_{r \geq 1} \mathbb{Z}_p$ where $p$ is a prime number and $\mathbb{Z}_p$ denotes the group of complex $p$-th roots of unity. Further, if $(d_n)$ is a sequence of integers satisfying $d_n | d_{n+1}$ and $G_n = \mathbb{U}_{d_n}$, then $G = \bigcup_{n \geq 1} G_n$ is $\sigma$-finite.

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For any \( A \subset G \), we define its lower and upper asymptotic densities as
\[
\underline{d}(A) := \liminf_{n \to \infty} \frac{|A \cap G_n|}{|G_n|} \quad \text{and} \quad \overline{d}(A) := \limsup_{n \to \infty} \frac{|A \cap G_n|}{|G_n|},
\]
respectively. If both limits coincide, we denote by \( d(A) \) their common value. This type of groups and densities were already studied in the additive combinatorics literature; Hamidoune and R"odseth [8] proved that if \( \langle A \rangle = G \) and \( \alpha = \overline{d}(A) > 0 \), then \( hA = G \) for some \( h = O(\alpha^{-1}) \). Hegyvari [4] showed that then \( h(A - A) = \langle A - A \rangle \) for some \( h = O(\log^{-1} \alpha) \), where again \( \alpha = \overline{d}(A) > 0 \); this was improved by Hegyvari and the second author [5].

A Følner sequence for an additive group \( G \) is a sequence \( \Phi = (F_n)_{n \geq 1} \) of finite subsets in \( G \) such that
\[
\lim_{n \to \infty} \frac{|(g + F_n) \Delta F_n|}{|F_n|} = 0, \quad \text{for any } g \in G,
\]
where \( \Delta \) is the symmetric difference operator. We let
\[
d^*(A) := \sup_{\Phi} \limsup_{n \to \infty} \frac{|A \cap F_n|}{|F_n|}
\]
be the upper Banach density of \( A \).

**Theorem 1.1.** Let \( G \) be a \( \sigma \)-finite abelian group. Let \( A \subset G \) and \( B \subset G \) satisfy \( \underline{d}(A + B) < \underline{d}(A) + \underline{d}(B) \). Then there exists a subgroup \( H \leq G \) with \( q := [G : H] < \infty \) and finite sets \( C, D \subset G \) such that \( A \subset C + H \) and \( B \subset D + H \). Moreover, \( A + B = C + D + H \) and \( |C + D| - 1 = |C| + |D| - 1 \).

As we will see in the proof of the theorem, a subgroup of finite index \( q \) has a density, namely \( 1/q \). Further, let \( 0 < \epsilon \leq 1 - \underline{d}(A + B) \). Then we have
\[
(1 - \epsilon) \frac{|C| + |D|}{q} \geq (1 - \epsilon)(\underline{d}(A) + \underline{d}(B)) \geq \underline{d}(A + B) = \frac{|C| + |D| - 1}{q}.
\]
Therefore \( |C| + |D| \leq \epsilon^{-1} \). We also get \( q \leq (\epsilon \alpha)^{-1} \).

**Remark.** We will also show that the exact same theorem holds true with either upper asymptotic or upper Banach densities in place of the lower asymptotic density.

This is in contrast with the integers setting, where Kneser’s theorem for the upper asymptotic density [6] is seriously different from the original one.

**Remark.** If \( G \) has no proper subgroup of finite index, which is the case of \( \mathbb{Z}(p^\infty) \) for instance, the conclusion implies that \( A + B = G \), so \( \underline{d}(A + B) < \underline{d}(A) + \underline{d}(B) \) is impossible except in the trivial case \( \underline{d}(A) + \underline{d}(B) > 1 \).

## 2. Proofs

We detail the proof of Theorem 1.1. The remark will be deduced straightforwardly by the same arguments.

**Proof of Theorem 1.1.** Let \( \alpha := \underline{d}(A) \) and \( \beta := \underline{d}(B) \). Let us assume that there exists \( \epsilon > 0 \) such that
\[
\underline{d}(A + B) < (1 - \epsilon)(\alpha + \beta).
\]
By definition of the lower limit, there exists an increasing sequence \( (n_i)_{i \geq 1} \) of integers such that
\[
|(A + B) \cap G_{n_i}| < (1 - \epsilon/2)|G_{n_i}|(\alpha + \beta).
\]
Furthermore, for any \( i \) large enough, we have
\[
|A_{n_i}| + |B_{n_i}| > (\alpha + \beta)|G_{n_i}|(1 - \epsilon/2)/(1 - \epsilon/3).
\]
Lemma 2.1. Let $H$ and invoke Corollary 2.2 and the pigeonhole principle. Thus

Let $\ell$ upon extracting again a suitable subsequence of $(n_i)$, so $G = \{g \in G : \forall x \in X, g + x \in X\}$. Let $H_i = \text{Stab}(A_{n_i} + B_{n_i})$. We obtain for each $i \geq 1$

$$A_{n_i} \subseteq C_i + H_i$$
$$B_{n_i} \subseteq D_i + H_i$$
$$A_{n_i} + B_{n_i} = E_i + H_i$$

where $E_i = C_i + D_i$ satisfies $|E_i| = |C_i| + |D_i| - 1$. Taking cardinalities yields

$$(1 - \epsilon/3)(|C_i| + |D_i|)|H_i| \geq (1 - \epsilon/3)(|A_{n_i}| + |B_{n_i}|)$$
$$|A_{n_i} + B_{n_i}| = (|C_i| + |D_i| - 1)|H_i|$$

from which we infer that $|C_i| + |D_i| < 3/\epsilon$ and $H_i \neq \{0\}$. Moreover,

$$(\alpha + \beta)|G_{n_i}|/2 < |A_{n_i}| + |B_{n_i}| \leq (|C_i| + |D_i|)|H_i|$$

so $[G_{n_i} : H_i] = [G_{n_i}] / |H_i| < \frac{|G_{n_i}|}{\alpha + \beta} < 6/((\alpha + \beta)\epsilon)$. By the pigeonhole principle, upon extracting again a suitable subsequence of $(n_i)$, one may assume that $[G_{n_i} : H_i] = k$ for any $i \geq 1$ and some fixed $k < 6/((\alpha + \beta)\epsilon)$.

Let us set

$$\mathcal{A}_i = \{K < G_{n_i} : [G_{n_i} : K] = k\}.$$ 

Thus $H_i \in \mathcal{A}_i$ for any $n \geq 1$.

**Lemma 2.1.** Let $i \geq 1$. If $L \in \mathcal{A}_{i+1}$, there exists $K \in \mathcal{A}_i$ such that $K \subseteq L$.

**Proof.** Let $L \in \mathcal{A}_{i+1}$. Let us set $K' = L \cap G_{n_i}$. Thus

$$\frac{G_{n_i}}{K'} = \frac{L + G_{n_i}}{L} \leq \frac{G_{n_i+1}}{L}$$

so $[G_{n_i} : K']$ divides $k$. Let us write

$$[G_{n_i}] = kg, \ [G_{n_i} : K'] = \frac{k}{h}, \ [K'] = gh$$

for some positive integers $g$ and $h$. Since $K'$ is abelian and $h$ divides $[K']$, there exists a subgroup $K$ of $K'$ of index $h$. It satisfies

$$[G_{n_i} : K] = [G_{n_i} : K'] \times [K' : K] = k. \quad \square$$

We draw from Lemma 2.1 the following corollary by an easy induction.

**Corollary 2.2.** For all integers $i \geq 1$, there exists a sequence of subgroups $K_{\ell} \in \mathcal{A}_{\ell}$ for $\ell \in [1, i]$ such that $K_{\ell} \leq K_{\ell+1}$ for all $\ell \in [i, j]$ and $K_j = H_j$.

Borrowing terminology from graph theory, for $K \in \mathcal{A}_i$ and $L \in \mathcal{A}_j$ where $i \leq j$, we call a path from $K$ to $L$ any non decreasing sequence of subgroups $K_i \in \mathcal{A}_i$ for which $K_i = K$ and $K_j = L$. With this terminology, the conclusion of Corollary 2.2 is that there exists a path from $K_i$ to $H_j$.

We shall construct inductively a non decreasing subsequence of subgroups $K_i \in \mathcal{A}_i$ for $i \geq 1$ such that for any $i$, the set of integers $j \geq i$ for which there exists a path from $K_i$ to $H_j$ is infinite.

To construct $K_i \in \mathcal{A}_i$ with the desired property, let us observe that $\mathcal{A}_i$ is finite and invoke Corollary 2.2 and the pigeonhole principle.
Suppose that $K_1, \ldots, K_i$ are already constructed for some $i \geq 1$; one constructs $K_{i+1}$ by observing again that $A_{i+1}$ is finite and applying the pigeonhole principle.

The sequence $(K_i)_{i \geq 1}$ being non decreasing, the union

$$K = \bigcup_{i \geq 1} K_i$$

is a subgroup of $G$. In fact, $K \leq \text{Stab}(A + B) = H$; indeed, let $g \in K$ and $x = a + b \in A + B$ where $(a, b) \in A \times B$. Let $i$ satisfy $g \in K_i$ and $x \in A_{ni} + B_{ni}$. Since $K_i$ is included in $H_j$ for infinitely many $j \geq 1$, there exists in particular some $j \geq i$ for which $K_i \subseteq H_j$. Thus $x + g \in A_{ni} + B_{nj}$ so $g \in H$.

Since $K_i \subseteq H \cap G_{ni}$, the indices $[G_{ni} : H \cap G_{ni}]$ divide $k = [G_{ni} : K_i]$. Now

$$G_{ni}/(H \cap G_{ni}) \simeq (G_{ni} + H)/H,$$

and $([G_{ni} + H]/H)_{i \in \mathbb{N}}$ is a non decreasing sequence of subgroups of bounded indices of $G/H$, whose union is $G/H$. It is therefore stationary, so $(G_{ni} + H)/H = G/H$ for any large enough $i$. It follows that $[G : H]$ divides $k$.

We infer that

$$A + B = F + H$$

where $F$ is a finite set and $[G : H] \leq k < 6/((\alpha + \beta)\epsilon)$.

Furthermore, if $i$ is large enough, we have

$$\frac{|H \cap G_{ni}|}{|G_{ni}|} = \frac{|G_{ni} + H|}{|H|},$$

whence we deduce that $d(H)$ exists and equals $1/q$. We conclude that $d(A + B)$ exists and equals $|F|/q \in \mathbb{Q} \cap (0, 1]$.

Let $C$ and $D$ be sets of representatives of $A$ and $B$, respectively, modulo $H$. Then $F = C + D$ and $A \subseteq C + H$ whereas $B \subseteq D + H$. In view of the previous paragraph and the inequality $d(A + B) < d(A) + d(B)$, we obtain that $|C + D| < |C| + |D|$. Now $|C + D| \geq |C| + |D| - 1$, since otherwise by Kneser’s theorem applied in the finite abelian group $G/H$, the set $F$ would admit a non trivial period and $\text{Stab}(A + B)$ would be strictly larger $H$. Therefore $|C + D| = |C| + |D| - 1$, and we conclude. \qed

We now prove the same result with the upper density. Let $\alpha := \overline{d}(A)$ and $\beta := \overline{d}(B)$ such that $\overline{d}(A + B) < (\alpha + \beta)(1 - \epsilon)$ for some $\epsilon > 0$. Then on the one hand we have for any large enough $n$

$$|A_{ni} + B_{ni}| \leq |(A + B) \cap G_{ni}| < (\alpha + \beta)(1 - \epsilon/2)|G_{ni}|.$$

On the other hand there exists infinitely many $n$ such that

$$|A_{ni}| + |B_{ni}| > \frac{1 - \epsilon/2}{1 - \epsilon/3}(\alpha + \beta)|G_{ni}|.$$

Hence for all those $n$

$$|A_{ni} + B_{ni}| < \overline{d}(A + B)(1 - \epsilon/2)|G_{ni}| < (1 - \epsilon/3)(|A_{ni}| + |B_{ni}|).$$

From there we are in position to conclude as in the proof of Theorem 1.1.

We now turn to the upper Banach density. By Remark 1.1 of [11] and the fact that $(G_{ni})_{n \in \mathbb{N}}$ is a Følner sequence we have

$$d^*(A) := \sup_{(x_n) \in G^n} \limsup_{n \to \infty} \frac{|A \cap (x_n + G_{ni})|}{|G_{ni}|},$$

where the supremum is in fact attained. Therefore let $x \in G^n$ and $y \in G^n$ satisfy $d^*(A) = \limsup_{n \to \infty} \frac{|G_{ni} \cap (x_n + A)|}{|G_{ni}|}$ and $d^*(B) = \limsup_{n \to \infty} \frac{|G_{ni} \cap (y_n + B)|}{|G_{ni}|}$, whereas

$$d^*(A + B) \geq \limsup_{n \to \infty} \frac{|(A + B) \cap (x_n + y_n) \cap G_{ni}|}{|G_{ni}|},$$

where

$$d^*(A + B) \geq \limsup_{n \to \infty} \frac{|(A + B) \cap (x_n + y_n) \cap G_{ni}|}{|G_{ni}|}.$$
If $d^*(A + B) < (1 - \epsilon)(d^*(A) + d^*(B))$, for infinitely many $n$ we have

$$|\langle A + x_n \rangle \cap G_n + \langle B + y_n \rangle \cap G_n| \leq |\langle A + B + x_n + y_n \rangle \cap G_n|$$

$$\leq \frac{1 - \epsilon/2}{1 - \epsilon} d^*(A + B)|G_n|$$

$$\leq (1 - \epsilon/2)(d^*(A) + d^*(B))|G_n|$$

$$< (1 - \epsilon/3)(|\langle A + x_n \rangle \cap G_n| + |\langle B + y_n \rangle \cap G_n|).$$

We then argue as in Theorem 1.1 with the sequences $(A'_{n})$ and $(B'_{n})$ where $A'_{n} = (A + x_n) \cap G_n$ and $B'_{n} = (B + y_n) \cap G_n$ to conclude.

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