Representation of Berry phase for spin system by the trajectories of Majorana stars

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The Majorana’s stellar representation, which represents a quantum state with the Majorana stars on the Bloch sphere, provides an intuitive way to study physical system with high dimensional projective Hilbert space. For a cyclic evolution of state, each star traces out an independent loop on the sphere. In this letter, we study the Berry phase of a spin-\(J\) state by these stars and loops. It is shown that the Berry phase not only contains the solid angles subtended by every Majorana star’s respective loop but also is related to the pair correlations between the stars, which collects the integrals of the pair solid angles weighted by the correlation factors of the star pairs. This pair solid angle contains all the motions of the star pair: the solid angles subtended by the loops of the two star themselves and the loops of the relative motions between the two stars. To demonstrate our theory, we study a two mode interacting boson system. Finally, the relation between stars’ correlations and quantum entanglement is discussed.

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Introduction. Berry phase, which reveals the gauge structure associated with cyclic evolution in Hilbert space\cite{1}, has become a central unifying concept of quantum theory\cite{2,3}. After introduced into quantum mechanics by Berry\cite{4}, such phase is founded to play an important role in the study of many important physics phenomena, such as quantum Hall effect\cite{5,6}, polarization of crystal insulators\cite{7} and topological phase transition\cite{8}. This phase, also known as geometric phase, reveals the fact that, a quantum eigenstate \(|\Psi\rangle\) will acquire an additional geometric phase factor \(\oint \! - \! \text{Im} \langle \Psi |dR|\Psi\rangle\) in cyclic adiabatic processes, where the integral is only depends on the geometric path of \(R\) in the parameter space.

For the simplest case of an arbitrary two-level state, the geometric path can be perfectly represented by the close trajectory of a point on the Bloch sphere, and the Berry phase is proportional to the solid angle subtended by it. This geometric interpret seems hard to be used for large spin system because it is difficult to imagine the trace of a state in the higher dimensional space. However, Majorana's stellar representation (MSR) give us a bridge between the high dimensional projective Hilbert space and the two-dimensional Bloch sphere\cite{9}. In the Majorana's stellar representation, one can describe a spin-\(J\) state (or equivalently a \(n\) body two-mode boson state with \(n = 2J+1\)) intuitively by 2\(J\) points on the two-dimensional Bloch sphere rather than one point on a high dimensional geometric structure, and these 2\(J\) points are called Majorana stars (MSs) of the system. This naturally provides an intuitive way to study the Berry phase for high spin system\cite{11}.

The reason for Majorana’s stellar representation drawing much more attention recently is the studying of spin-orbit coupling in cold atom physics\cite{12,13}. In cold atom physics, the large-spin atoms, such as lanthanide atoms, are introduced as candidates in the process of inducing synthetic gauge field by spin-orbit coupling, since their narrow linewidth transitions will suppress the additional heating\cite{14,15}. For high-spin condensates, spin-orbit coupling drive the Majorana stars moving periodically on the Bloch sphere, i.e., forming the so-called “Majorana spin helix”. Hence, one will naturally ask, can we have explicit relation between Berry phase and the Majorana stars’ helixes or loops? Recently, Bruno established a novel representation of the Berry phase of large-spin systems\cite{10} by introducing coherent state representation (CSR) into MSR, and the geometric phase has been viewed as the Aharonov-Bohm phase acquired by the Majorana stars as they move through the gas of Dirac strings. However, in the Bruno’s excellent work, the connection between Berry phase and geometric trajectories of the MSs on the Bloch sphere is still not clear or intuitive.

In this Letter, we present a novel formula for Berry phase of spin system which gives the intuitive relation between Berry phase and MSs’ trajectories on the Bloch sphere. We find the Berry phase can be decomposed into two contributions: one is from the sum of the solid angles subtended by every Majorana star’s close trajectory; the other is from pair correlations between the stars, which collects the solid angles by the relative motions of each pair stars. By using a two-mode boson system, which can be realized in cold atom physics\cite{12,13}, we calculate the Berry phase numerically to verify our results. We also find the pair correlations between stars are naturally related to the quantum entanglement of the particles. In this respect, it provides an intuitive way to study mea-
measurement and classification of multiparticle entanglement of n particles.

**Berry phase in MSR.** As we know, a spin-1/2 state can be described by a point on the Bloch sphere. For a spin-J system, its angular momentum operators can be described by the creation and annihilation operators of two mode bosons with Schwinger boson representation. Under Schwinger boson representation, the basis of the spin-J system $|jm\rangle$ is equivalent to a two mode boson state $|J+m, J-m\rangle$. Therefore a spin-J state $\sum_{m} C_{m}|jm\rangle$ equals to a generic state of a n-dimensional two mode boson system $|\Psi\rangle_{n} = \sum_{m} C_{m}\sqrt{\frac{1}{(J+m)!}}\sqrt{(J-m)!}|\Omega\rangle$ with $n = 2J$, which can be factorized as

$$|\Psi\rangle_{n} = \frac{1}{N_{n}(U)} \prod_{k=1}^{n} \hat{a}_{u_{k}}^{\dagger}|\Omega\rangle = \frac{1}{\sqrt{n!N_{n}(U)}} \sum_{P} |u_{P(1)}\rangle|u_{P(2)}\rangle \cdots |u_{P(n)}\rangle$$

where $N_{n}(U) = (n!)^{D}P_{k}$ is the normalization coefficient with $U \equiv \{u_{1}, \ldots, u_{2J}\}$ (its expression is given in the supplement material). The sum $\sum_{P}$ being over all permutations $P$, taking 1, 2, ..., $n$ to $P(1), P(2), \ldots, P(n)$. The creation operators $\hat{a}_{u_{i}}^{\dagger} = (\cos \frac{D}{2} \hat{a}_{u_{i}}^{\dagger} + \sin \frac{D}{2} e^{i\phi_{i}} \hat{b}_{u_{i}}^{\dagger})$ and the annihilation operators $\hat{a}_{u_{i}}$ satisfy $[\hat{a}_{u_{i}}^{\dagger}, \hat{a}_{u_{j}}] = [\hat{a}_{u_{i}}, \hat{a}_{u_{j}}] = 0$ and $[\hat{a}_{u_{i}}^{\dagger}, \hat{a}_{u_{j}}] = \langle u_{i}, u_{j}\rangle$. And $|u_{i}\rangle = \cos \frac{D}{2} \hat{a}_{u_{i}}^{\dagger}|\Omega\rangle + \sin \frac{D}{2} e^{i\phi_{i}} \hat{b}_{u_{i}}^{\dagger}|\Omega\rangle$. If one denotes $\hat{a}_{u_{i}}^{\dagger}|\Omega\rangle = |\uparrow\rangle$ and $\hat{b}_{u_{i}}^{\dagger}|\Omega\rangle = |\downarrow\rangle$ as the orthogonal basis of a spin-1/2 state respectively, then (2) can be also understood as a full symmetrized state of n spin-1/2 particles. Consequently, the above factorization will give out n pairs of parameters $\theta_{k}, \phi_{k} (k = 1, \ldots, n)$ which correspond to n points $u_{k}(\theta_{k}, \phi_{k})$ on the Bloch sphere. Therefore, the quantum state in Eq. (2) and its evolution can be depicted by these points so called Majorana stars[15].

In particular, for a adiabatic cyclic evolution of state $|\Psi\rangle_{n}$, each star $u_{k}$ traces out an independent loop on the sphere. As we mentioned, this process will naturally accumulate a Berry phase for $|\Psi\rangle_{n}$. Hence, the interesting task in our scheme, then, is to calculate the Berry phase in terms of these parameterized loops. According to Berry’s definition, the Berry phase for $|\Psi\rangle_{n}$ reads: $\gamma_{n} = \frac{\pi}{2} \text{Im}^{n} \langle \Psi | \Omega_{u_{i}} | \Psi \rangle$ . After a long but straightforward calculation, the Berry phase becomes (see supplemental material for details of calculations)

$$\gamma_{n} = \gamma_{0} + \gamma_{C}$$

where $\gamma$ can be decomposed into two parts. One part $\gamma_{0} = \sum_{i=1}^{n} \Omega_{u_{i}} / 2$ is the sum of the solid angles $\Omega_{u_{i}} = \frac{1}{2}(1 - \cos \theta_{i}) d\theta_{i}$ subtended by the closed evolution paths of the MSs on the Bloch sphere (as Fig. 1(a) shows).

The another part of Berry phase is

$$\gamma_{C} = \frac{1}{2} \int \int \sum_{i=1}^{n} \beta_{ij} \Omega_{(u_{ij})},$$

which characterized by the correlations between the stars (hereafter we call it correlation phase). Here, $\Omega_{(u_{ij})}$ is $u_{i} \times u_{j} \cdot (u_{j} - u_{i})/d_{ij}$ is the sum of solid angles of the infinitely thin triangle $(u_{i}, u_{j}, u_{j} - u_{i})$ and $(u_{j}, u_{i} - u_{j}, u_{i} - u_{j})$, we denote it as pair solid angle. $\beta_{ij}$, the correlation factor is defined as

$$\beta_{ij}(D) \equiv -\frac{d_{ij}}{N_{n}^{2}(D)} \frac{\partial N_{n}(D)}{\partial d_{ij}}$$

with $D = \{d_{ij}\} (i < j)$, in which $d_{ij} \equiv u_{i} \cdot u_{j}$ as the “distance” between two stars $u_{i}(\theta_{i}, \phi_{i})$ and $u_{j}(\theta_{j}, \phi_{j})$. Note that, the normalization coefficient $N_{n}^{2}(U)$ only contains the products of the first degrees of $d_{ij}$ (see supplement material), and then can be written as $N_{n}^{2}(D) = -d_{ij} \frac{\partial N_{n}^{2}(U)}{\partial d_{ij}} + \text{terms without pair}(u_{i}, u_{j})$. Therefore, correlation factor $\beta_{ij}(D)$ is nothing but the weight of the $d_{ij}$ dependent terms to $N_{n}^{2}(D)$. Hence, the correlation phase can be described by the solid angles between each pair of stars weighted by the correlation factor $\beta_{ij}$.

Indeed, the pair solid angle $\Omega_{(u_{ij})}$ can be expressed by the relative evolution between $u_{i}$ and $u_{j}$, and the absolute evolutions of themselves. Consider the moving frame in which the star $u_{j}(\theta_{j}, \phi_{j})$ is fixed and located at z-axis $z(0, 0, 1)$, the spherical of the other star $u_{i}(\theta_{i}, \phi_{i})$ changes into $u_{j}(\theta_{i}'(\theta_{j}), \phi_{i}'(\phi_{j}))$ in this frame accordingly (as Fig. 1 shows). On the contrary, we can also

FIG. 1: (color online) A schematic illustration of (a) the solid angles subtended by the parallel transports of $u_{i}$ and $u_{j}$ on the Bloch sphere: $\Omega_{u_{i}}$ (areas subtended by the blue solid lines) and $\Omega_{u_{j}}$ (areas subtended by the red solid lines); (b) the solid angle in moving frame $(u_{1}(\frac{\pi}{2} + \theta_{1}, \phi_{1}), u_{1}(\frac{\pi}{2} - \phi_{1})$, $u_{1}(\theta_{1}, \phi_{1}))$ subtended by the parallel transports of relative evolution path between the two stars in (a): $\Omega_{u_{i}}'$ (areas subtended by the black solid lines)
obtain the relative vector $\mathbf{u}'_{ij} = (\theta'_{ij}, \phi'_{ij})$ of $\mathbf{u}_j$ in the moving frame with $\mathbf{u}_i(\theta_i, \phi_i)$ fixed at $z$-axis. The pair solid angle $\Omega(\mathbf{u}_{ij})$ vector becomes (see supplement material for details)
\begin{equation}
\Omega(\mathbf{u}_{ij}) = [d\phi'_{ij} + d\phi'_{ji}] + (\cos \theta_i d\phi_l + \cos \theta_j d\phi_j). \tag{6}
\end{equation}
where $\theta' = \theta'_{ij} = \theta'_{ji}$ is the angle between $u_i$ and $u_j$. Therefore the meaning of correlation phase $\gamma^{(n)}_{C}$ is quite clear: it is consist of the collection of the weighted relative evolutions between the stars
\begin{equation}
\gamma^{(n)}_{Rij} = \frac{1}{2} \int \beta_{ij}(\mathbf{D}) \Omega(\mathbf{u}'_{ij}) + \Omega(\mathbf{u}'_{ji}) \tag{7}
\end{equation}
with $\Omega(\mathbf{u}'_{ij}) = (1 - \cos \theta')d\phi'_{ij}$ and the collection of the weighted absolute evolutions of the pairs of stars
\begin{equation}
\gamma^{(n)}_{Aij} = \frac{1}{2} \int \beta_{ij}(\mathbf{D})[\cos \theta_i d\phi_i + \cos \theta_j d\phi_j]. \tag{8}
\end{equation}
Namely, $\gamma^{(n)}_C = \gamma^{(n)}_R + \gamma^{(n)}_A = \sum_{i=1}^{n} \sum_{j \neq i}^{n} \gamma^{(n)}_{Rij} + \gamma^{(n)}_{Aij}$.

So far, we know that the Berry phase in MSR is consist of not only the solid angles subtended by the paths of the stars but also their correlations. These results in Eq. 3 and 4 can be proved to be in accordance with the marvelous one in Ref. [16] which is derived by introducing the coherent state representation into MSR of the spin-$J$ system.

There follows are several notes for some specific cases.

(i) All the stars are overlap on one point. For this special case $\beta_{ij}$ and $\Omega(\mathbf{u}_{ij})$ have value zero, the Berry phase in Eq. 3 will reduce to the sum of solid angles of all stars. For example, this corresponds to the spin coheren state $|\Psi_{J}^{(2)}\rangle$. 

(ii) This case is just for a spin-$J$ in a uniform magnetic field $\mathbf{B} = B(\sin \theta \cos \varphi, \sin \theta \cos \varphi, \cos \theta)$. Its eigenstate $|\Psi_{m}^{(2)}\rangle = e^{i\varphi} \xi_{m}^{(2)}(\chi_{m}^{(2)})$ can be represented by $J + m$ coincide stars $\mathbf{u}(\theta, \varphi)$ and their $J - m$ coincide antipodal stars $\mathbf{u}'(\pi - \theta, \pi + \varphi)$. The Berry phase thus becomes $\gamma^{(2J)}_C = \gamma^{(2J)}_R = -\frac{1}{2}[J(m + m)\Omega_{m} - (J - m)\Omega_{-m}] = -m\Omega_{0}$, which is perfectly matches the result in Ref. [20].

(iii) All the stars rotate with same angular velocity as a rigid body. In this case, all the distances between star pairs $\mathbf{u}_i$ and $\mathbf{u}_j$ are invariant. At this point, $\beta_{ij}$ becomes constant, and $\gamma^{(n)}_{C}$ in Eq. 1 changes into a sum of solid angles as $\gamma^{(n)}_{C} = \frac{1}{2} \sum_{i=1}^{n} \sum_{j \neq i}^{n} \beta_{ij} \Omega(\mathbf{u}_{ij})$, where $\Omega(\mathbf{u}_{ij}) \equiv \Omega(\mathbf{u}_{ji})$ is the solid angles accumulated by the infinitely small solid angles $\Omega(\mathbf{u}_{ij})$. By integrating Eq. 3, $\Omega(\mathbf{u}_{ij})$ is turned out to be composed by the solid angles accumulated by the relative evolution between $\mathbf{u}_i$ and $\mathbf{u}_j$ (as Fig 1(b) shows), and the solid angle accumulated by the evolutions of $\mathbf{u}_i$ and $\mathbf{u}_j$ themselves (as Fig 1(a) shows), i.e.

\begin{equation}
\Omega(\mathbf{u}_{ij}) = \frac{\Omega_{u'_{ij}} + \Omega_{u'_{ji}}}{1 + \mathbf{u}_i \cdot \mathbf{u}_j} - ([\Omega_{u_i} + \Omega_{u_j}] \mod 2\pi)] \tag{9}
\end{equation}
where $\Omega_{u'_{ij}} (\Omega_{u'_{ji}})$ is the solid angles subtended by the closed evolution paths of $\mathbf{u}'_i (\mathbf{u}'_j)$ relative to $\mathbf{u}_j (\mathbf{u}_i)$.

(iv) The pair of stars $\mathbf{u}_i(\theta_i, \phi_i)$ and $\mathbf{u}_j(\theta_j, \phi_j)$ are always on the same circle of longitude or latitude. The former refers to $\phi_i - \phi_j = 0, \pm \pi$, i.e. the two stars and the $z$ axis $(0, 0)$ will always in the same plane. It will accumulate no loop by the relative motions between the two stars. For the latter, we have $\theta_1 = \theta_2$ and the sum of relative motions between the two stars will also equal to zero owing to the symmetry. Thus, $\gamma^{(n)}_{Rij}$ will vanish in both of the two situations. Besides, for $\theta_1 = \theta_2$, if we have $\phi_i + \phi_j = \text{const}$, $\gamma^{(n)}_{Aij}$ will also vanish, and this star pair will give no contribution to the correlation phase.

Two mode interacting boson system. To illustrate the above theoretical results, we now consider an interacting boson system described by Hamiltonian $H = \frac{R \sin \theta}{4} (e^{i\varphi} \hat{a}^{\dagger} \hat{b} + e^{-i\varphi} \hat{b}^{\dagger} \hat{a}) + \frac{R \cos \theta}{2} (\hat{a}^{\dagger} \hat{a} - \hat{b}^{\dagger} \hat{b}) + \frac{\lambda}{2} (\hat{a}^{\dagger} \hat{a} - \hat{b}^{\dagger} \hat{b})^2$, where $R \cos \theta$ is the energy offset between the two modes. The parameter $R \sin \theta e^{i\varphi}$ measures the coupling between the two modes, and $\lambda = g/V$ with $g$ being the interaction strength between the two modes and $V$ being the volume of the system. This model can be derived from the bosonic-field Hamiltonian [22] and has received extraordinary attention in the literature on BECs [23]. We calculate the cyclic motions of MSs and their related Berry phase for the eigenstates of $H$ with different number of bosons by numerical simulations and comparing the results with the above theoretical formula.

As Fig. 2 shows, the Berry phases $\gamma$, calculated directly by its original definition, perfectly matches with $\gamma_0 + \gamma_C$ in our theory. Moreover, it is easy to find that, the phase factor $e^{i\varphi}$ can be removed by the transformation $\hat{b} \rightarrow \hat{b}' = be^{i\varphi}$. Therefore, the differences between the spherical coordinates $\phi_i$ for different stars $\mathbf{u}_i$ are only 0, or $\pm \pi$ like in specific case (iii), then we have $\gamma^{(n)}_R = 0$ and $\gamma^{(n)}_C = \gamma^{(n)}_A$ (see Fig. 2(b), 2(d)).

For $\lambda = 0$, the Hamiltonian $H$ reduces to the one of spin-$n/2$ in a magnetic field $\mathbf{B} = B(\sin \theta \cos \varphi, \sin \theta \cos \varphi, \cos \theta)$ in case (i) above. Therefore, the trajectories of all stars of the ground states are coincide with each other on the spheres in Fig. 2(b) and 2(c), and we have $\gamma = \gamma_0 = 2\Omega_{0}$, and $3\Omega_{0}$ for two and three bosons, respectively. On contrary, as shown in Fig. 2(d), the first excited state of $H$ with three bosons have one star in the positive axis direction and two coincide anti-stars in the negative axis direction.

As the interacting constant $\lambda$ increase, the interactions between the bosons break all of these coincidences in Fig. 2(b)-(d). Therefore, the loop of coincide stars becomes several different loops and bring out the correlation phases. We can use these changes of the symmetry of the states to clarify the type of state. For the states with two bosons, there have two different type of states: state with two coincide stars (see the first sphere in Fig.
Horne-Zeilinger (GHZ) type of states can be written in the form of concurrence like two qubits: \[ C = \frac{1}{2} \oint \Omega \left( d\mathbf{u}_{12} \right) \]. Therefore, the correlation phase of the state is directly related to its concurrence: \[ \gamma_C^{(2)} = \frac{1}{2} \oint \Omega \left( d\mathbf{u}_{12} \right) \]. For \( n = 3 \), there exist different measurements for three different entanglement classes [21, 24] of states: the 3-tangle for the Greenberg-Horne-Zeilinger (GHZ) [20] type of states can be written as \( \tau = \frac{1}{3} \beta_{12} \beta_{13} \beta_{23} N_3^2 \) with three unequal stars; the concurrence for the W type [27] of states becomes \( \mathcal{C}_{12} = \frac{2d_{12}}{3N_3^2} \), where two of the three stars coincide with each other, and the correlation phase of the W type state can be written in the form of concurrence like two qubits: \[ \gamma_C^{(3)} = \frac{1}{2} \oint \Omega \left( d\mathbf{u}_{12} \right) \mathcal{C}_{12} \]; the three same stars at one point brings no entanglement and thus no correlation phase for the separable states. This means that the types of entanglement can be distinguished by the number of unequal stars (or diversity degree of the state [24]), and measured by a normalized product of the distance between unequal stars. Since the classification of entanglement by the number of unequal stars also hold for \( n \) qubits [24], such as separable type (\( n_s = 1 \)), W type (\( n_s = 2 \)), and GHZ type (\( n_s = n \)), the normalized product of distances between unequal stars \( \left( \prod_{i<j} d_{ij} \right) / N_n^{2(n_s-1)} \) may be a valid measure of entanglement. This theory of entanglement in MSR will be studied in detail in a forthcoming paper.

**Concluding Remarks.** The Majorana’s stellar representation and recently relevant applications have indicated that the evolution of a high spin state can be displayed intuitively by loops of MSs on the Bloch sphere. Our study here is to show how can we “read out” the physical effects of the state such as Berry phase and entanglement from these stars and loops. The discussion shows that the Berry phase of a spin-\( J \) state not only determined by the solid angles subtended by every Majorana star’s evolution path but also associated with the correlation between the stars. A two mode interacting boson system is used to demonstrate these results. Similar with the Majorana star’s loop (or helix in Ref. [12]) representation for Berry phase, the entanglement for symmetric \( n \)-qubit state can also be expressed by the correlation between Majorana stars. The different type of entanglement can be clarified by the number of unequal stars. Furthermore, the product of normalized distances between unequal stars may be a valid measure of the entanglement of different type. All of these theory can be used to study any finite quantum systems with high spin states. Besides, there are other physical effects can be characterized by the MSR, we will develop our theory in future paper.

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Supplement Material: Multiparticle correlation and Berry phase in Majorana’s stellar representation

DERIVATION OF THE SPHERICAL COORDINATES OF THE MAJORANA STARS

For the state \(|\Psi\rangle^n = \sum_{-n/2}^{n/2} C_m \hat{a}^{(\frac{m}{2}+m)} \hat{b}^{(\frac{m}{2}-m)} |\emptyset\rangle\), assuming \(x_1, x_2, \ldots, x_n\) are the roots of the equation

\[
\sum_{k=1}^{n} \frac{(-1)^k C_{n/2-k}}{\sqrt{(n-k)!k!}} x^{n-k} = 0.
\]

(1)

The spherical coordinates \(\theta_k\) and \(\phi_k\) of \(u_k\) can then be given by \(x_k = \tan \frac{\theta_k}{2} e^{i\phi_k} \). In this supplementary material, we give out the detail about the Derivation of Berry phase \(\gamma\) in Eq. (2) and (3) in the letter.

DERIVATION OF THE NORMALIZATION COEFFICIENT \(N_n^2\)

We notice that the states of MS \(|u_k\rangle\) have exchange symmetry. Therefore, its normalized coefficient can be calculated by picking any single star to interact with other stars

\[
N_n^2 = \langle \sum_{P} |u_{P(1)}\rangle |u_{P(2)}\rangle \cdots |u_{P(n)}\rangle
\]

\[
\cdot \langle \sum_{P} |u_{P(n-1)}\rangle |u_{P(n)}\rangle \rangle^2
\]

\[
\cdot \langle u_i | \sum_{P} |u_{P(1)}\rangle |u_{P(2)}\rangle \cdots |u_{P(n-2)}\rangle |u_{P(n-1)}\rangle \rangle^2
\]

\[
\cdot \cdots
\]

\[
\cdot \langle u_i | \sum_{P} |u_{P(1)}\rangle |u_{P(2)}\rangle \cdots |u_{P(n)}\rangle \rangle^2
\]

where The notation ‘\(\cdot \)’ indicates that removing star \(u_i\), (\(u_i\) and \(u_j\)) and (\(u_i, u_j\) and \(u_k\)) from the product in the first line of Eq. (2), respectively. Notice that, \(\langle u_j | u_i \rangle = (1 + \sigma \cdot u_i) / 2\) and the relation \(\langle \sigma \cdot u_j | \sigma \cdot u_i \rangle = u_i \cdot u_j + i \sigma \cdot (u_i \times u_j)\). For exchange symmetry, the item with cross product will vanish in the expression of normalization coefficient and leave the items with the product of \(u_i \cdot u_j\). Therefore, the normalization coefficient takes the form

\[
N_n(U) = (n!)^2 \prod_k |u_{P(k)}\rangle^2 = \frac{(n+1)!}{2^n} \sum_{k=0}^{n/2} \frac{n^2}{(2k+1)!!} \frac{D^n_k}{D^n_k}
\]

(3)

where \(D^n_k\) is a symmetric function\(^{[2]}\). its expression is

\[
D^n_k \equiv \sum_{i_1=1}^{n} \sum_{j_1>1}^{n} \cdots \sum_{i_k>\cdots>i_k}^{n} \sum_{j_k>\cdots>j_k}^{n} (u_{i_1} \cdot u_{j_1}) \cdots (u_{i_k} \cdot u_{j_k}),
\]

(4)

where the * indicates a restriction on the summations so that all non-repeated indices in each term take different values (e.g. \(D^2 = (u_1 \cdot u_2)(u_3 \cdot u_4) + (u_1 \cdot u_3)(u_2 \cdot u_4) + (u_1 \cdot u_4)(u_2 \cdot u_3)\)). For example, the normalization coefficient
After a straightforward calculation, the contribution of $\Psi$ to the state reduce to $\ket{\Psi} = \ket{\Psi_{(1)}} \cdots \ket{\Psi_{(n-2)}} \ket{\Psi_{(n-1)}}$.

Assuming Eq. (3) holds for natural number $n$. Then Eq. (2) becomes

\[
N_n = \frac{1}{2} \sum_{i,j} \frac{5 (u_1 \cdot u_2 + u_2 \cdot u_3 + u_1 \cdot u_3 + u_1 \cdot u_4 + u_2 \cdot u_4 + u_3 \cdot u_4)}{2} + \frac{(u_1 \cdot u_2) (u_3 \cdot u_4) + (u_1 \cdot u_3) (u_2 \cdot u_4) + (u_1 \cdot u_4) (u_2 \cdot u_3)}{2} \right)^{\frac{1}{2}},
\]

respectively. Next, we use mathematical induction to prove the expression of $N_n(U)$.

For $n = 1$, the state reduce to $\ket{u}$ and $N_n(U) = \langle u \mid u \rangle = 1$. The expression (3) holds. Assuming Eq. (3) holds for natural number $n$. For the situation of $n + 1$, we can add a new star $u_{n+1}$ to the state $\ket{\Psi}_{(n)}$. Then Eq. (2) becomes

\[
N_{n+1} = \left( \sum_{P} |u_{P(1)} \rangle \cdots |u_{P(n)} \rangle \right) = \frac{N_n}{2} + \sum_{i=1}^{n} \langle u_{i} \rangle \left( \sum_{P} |u_{P(1)} \rangle \cdots |u_{P(n-2)} \rangle \langle u_{P(n-1)} \rangle \right) \left( \frac{1 + \sigma \cdot u_{n+1}}{2} \right) |u_{i} \rangle
\]

After a straightforward calculation, the contribution of $u_{n+1}$ to the products of $l$ pair like $u_{i} \cdot u_{j}$ is

\[
\frac{(n - 2l + 1)!!}{(2l - 1)!!2^{n-l+1}} \sum_{m=2l-1}^{n} m!(n-m+1) = \frac{(n+2)!!}{2^{n-l+1}(2l+1)!} = \frac{(n+2)!}{2^{n+1}(2l+1)!!}.
\]
Substituting Eq. (2) and (7) into Eq. (6), we finally have

\[ N_{n+1} = \left(1 + \frac{n}{2}\right) N_n \]

\[ + \frac{(n + 2)!}{3 \times 2^{n+1}} \sum_i^n (u_{n+1} \cdot u_i) \]

\[ + \frac{(n + 2)!}{15 \times 2^{n+1}} \sum_{i,j,k} (u_{n+1} \cdot u_i)(u_j \cdot u_k)(N_n)^2_{ij} \]

\[ + \cdots \]

\[ = \frac{(n + 2)! \cdot (n+1)/2}{2^{n+1}} \sum_{k=0} D_k^{n+1} (2k + 1)!! \]

Thus, the expression hold for \( n + 1 \) and Eq. (2) is proved.

**DERIVATION OF THE BERRY PHASE \( \gamma^{(n)} \)**

In particular, for an adiabatic cyclic evolution of state \( |\Psi^{(n)}\rangle \), each star \( u_k \) traces out an independent loop on the sphere (see also note (iv) in Ref. [2]). As we mentioned, this process will naturally accumulate a Berry phase for \( |\Psi^{(n)}\rangle \). According to Berry's definition, the Berry phase for \( |\Psi^{(n)}\rangle \) reads: \( \gamma^{(n)} = \oint -\text{Im}\langle \Psi|dU|\Psi^{(n)}\rangle \), and \( A(U) \) takes the form

\[ n \langle \Psi|d\Psi \rangle^n = \sum_{i=1}^n \frac{(N^2_n)^2}{N_n^2} \frac{|u_i|d|u_i|}{N_n^2} \]

\[ + \sum_{i,j=1}^n \frac{\langle u_i|d|u_j\rangle \langle u_j|u_i\rangle (N^2_n)^2}{N_n^2} \]

\[ + \sum_{i,j=1}^n \sum_{k(\neq i,j)} \frac{\langle u_i|d|u_j\rangle \langle u_j|u_k\rangle \langle u_k|u_i\rangle (N^2_n)^2}{N_n^2} \]

\[ + \cdots \]

\[ + \sum_{i,j=1}^n \frac{\langle u_i|d|u_j\rangle \langle u_j|u_i\rangle \left( \sum_p \prod_{i=1}^n \langle u_p|u_i \rangle \right)}{N_n^2} \]

Compare with Eq. (2), the differential element \( \langle u_i|d|u_j\rangle \) are important for the calculation of Eq. (9). Consider an infinity small cyclic product of stars. By calculating at the leading order, we have

\[ \langle u_i|d|u_j\rangle \langle u_j|d|u_j\rangle = \langle u_i|d|u_j\rangle \langle u_j|u_i\rangle - \langle u_i|u_j\rangle \langle u_j|d|u_j\rangle + \langle u_i|u_j\rangle \langle u_j|u_i\rangle \]

\[ = \frac{2 + 2u_i \cdot u_j - u_i + u_j \cdot d|u_j| - iu_i \times u_j \cdot d|u_j|}{4}, \]

notice that \( |\langle u_i|u_j\rangle|^2 = \frac{1+u_i \cdot u_j}{2} \), then we obtain

\[ \langle u_i|d|u_j\rangle = \langle u_i|u_j \rangle \left[ \langle u_j|d|u_j \rangle + \frac{(u_i + u_j) \cdot d|u_j|}{2(1 + u_i \cdot u_j)} - \frac{iu_i \times u_j \cdot d|u_j|}{2(1 + u_i \cdot u_j)} \right]. \]

(11)
Substituting (11) into Eq. (9), and using the relations $(\sigma \cdot A)(\sigma \cdot B) = A \cdot B + i\sigma \cdot (A \times B)$ and $(A \times B) \cdot (C \times D) = (A \cdot C)(B \cdot D) - (A \cdot D)(B \cdot C)$. The imaginary part of Eq. (9) can be calculated directly as

$$\text{Im}^n(\Psi|d\Psi|\Psi) = \sum_{i=1}^{n} \left( \frac{N_n^2}{N_n^2} \text{Im}(u_i|d\psi_i) \right)$$

$$+ \text{Im} \sum_{i,j=1, i \neq j}^{n} \left[ \langle u_j|d\psi_j \rangle + \frac{(u_i + u_j) \cdot d\psi_j}{2(1 + u_i \cdot u_j)} - \frac{i u_i \times u_j \cdot d\psi_j}{2(1 + u_i \cdot u_j)} \right].$$

$$\left\langle u_i|d\psi_i \right\rangle \left[ \langle u_j|u_i \rangle \langle N_n^2 \rangle_{ij} + \sum_{k \neq i,j} \langle u_j|u_k \rangle \langle u_k|u_i \rangle \langle N_n^2 \rangle_{ijk} + \cdots + \langle u_j|u_i \rangle \left( \prod_{l=1}^{n} \langle u_{P(l)}|u_{P(l)} \rangle \right)_{ij} \right].$$

$$= \sum_{i=1}^{n} \left[ \frac{N_n^2}{N_n^2} \text{Im}(u_i|d\psi_i) \right] + \text{Im} \sum_{i,j=1, i \neq j}^{n} \left[ \frac{(u_i + u_j) \cdot d\psi_j}{2(1 + u_i \cdot u_j)} - \frac{i u_i \times u_j \cdot d\psi_j}{2(1 + u_i \cdot u_j)} \right].$$

$$\cdot \text{Tr} \left\{ \frac{1 + \sigma \cdot u_i}{4N_n^2} \left[ \langle N_n^2 \rangle_{ij} + \sum_{k \neq i,j} \frac{(1 + \sigma \cdot u_k)(\langle N_n^2 \rangle_{ijk})}{2} + \cdots + \left( \prod_{l=1}^{n} \frac{1 + \sigma \cdot u_{P(l)}}{2} \right)_{ij} \right] \right\}$$

$$= \sum_{i=1}^{n} \text{Im}(u_i|d\psi_i).$$

$$+ \frac{1}{N_n^2} \sum_{i,j=1, i \neq j}^{n} \text{Tr} \left\{ \left[ \frac{\sigma \cdot (u_i \times u_j)(u_i + u_j) \cdot d\psi_j}{8(1 + u_i \cdot u_j)} - \frac{(u_i + u_j) \cdot u_i \times u_j \cdot d\psi_j}{8(1 + u_i \cdot u_j)} \right] \right\}$$

$$\cdot \left[ \langle N_n^2 \rangle_{ij} + \sum_{k \neq i,j} \frac{(1 + \sigma \cdot u_k)(\langle N_n^2 \rangle_{ijk})}{2} + \cdots + \left( \prod_{l=1}^{n} \frac{1 + \sigma \cdot u_{P(l)}}{2} \right)_{ij} \right]$$

$$= \sum_{i=1}^{n} \text{Im}(u_i|d\psi_i).$$

$$- \frac{1}{N_n^2} \sum_{i,j=1, i \neq j}^{n} \frac{u_i \times u_j \cdot d\psi_j}{8} \text{Tr} \left[ \left[ \langle N_n^2 \rangle_{ij} + \sum_{k \neq i,j} \frac{(1 + \sigma \cdot u_k)(\langle N_n^2 \rangle_{ijk})}{2} + \cdots + \left( \prod_{l=1}^{n} \frac{1 + \sigma \cdot u_{P(l)}}{2} \right)_{ij} \right] \right]$$

$$= \sum_{i=1}^{n} \text{Im}(u_i|d\psi_i) + \frac{1}{2} \sum_{i,j=1, i < j}^{n} \frac{u_i \times u_j \cdot d\psi_j \partial N_n^2}{N_n^2} \frac{\partial N_n^2}{\partial u_j}$$

$$= \sum_{i=1}^{n} \text{Im}(u_i|d\psi_i) + \frac{1}{2} \sum_{i,j=1, i < j}^{n} \frac{u_i \times u_j \cdot d(u_j - u_i) \partial N_n^2}{N_n^2} \frac{\partial N_n^2}{\partial u_j}$$

(12)
where
\[ (u_j|du_j) = i\left(\frac{1 - \cos \theta_j}{2}\right)d\phi_j, \]  
(13)

with spherical coordinates \(\theta_j\) and \(\phi_j\) of \(u_j\). And
\[
\frac{\partial N_n^2}{\partial d_{ij}} \equiv -\text{Tr}\left\{ \frac{1}{4} \left[ N_{ij}^2 + \sum_{k(\neq i,j)} \frac{(1 + \sigma \cdot u_k)N_{ij}^2}{2} + \cdots + \left( \sum_p \prod_{l=1}^p (1 + \sigma \cdot u_{P(l)}) \right)^n_{ij} \right]\right\}
\]
(14)

with the distance \(d_{ij} = 1 - u_1 \cdot u_2\). Here we notice that although Eq. (1) in the letter is a symmetric state where the MSs possess the exchange symmetry, and the spin-\(J\) system in MSR can be treated as a 2-\(J\)-boson system, it still differs from the regular identical boson system. It is because the states \(|u_i\rangle\) are nonorthogonal and thus \(|\Psi\rangle\) is unnormalized. This is the reason why we have the term \(\frac{\partial N_n^2}{\partial d_{ij}}\), which is not exist in the regular identical boson system. Compare Eq. (3) with Eq. (14), we find that
\[
N_n^2 = d_{ij} \frac{\partial N_n^2}{\partial d_{ij}} + \text{terms without pair}(u_i, u_j)
\]
(15)

By substituting Eq. (12) into the definition of \(A(u)\), the Berry phase becomes
\[
\gamma^{(n)} = \gamma_0^{(n)} + \gamma_C^{(n)},
\]
(16)

where
\[
\gamma_0^{(n)} = -\sum_{i=1}^{n} \Omega_{u_i}/2
\]
(17)

is the collection of the solid angles \(\Omega_{u_i} = \oint (1 - \cos \theta_i)d\phi_i\) of the closed evolution paths of the MSs on the Bloch sphere. And
\[
\gamma_C^{(n)} = \frac{1}{2} \oint \sum_{i=1}^{n} \sum_{j(\neq i)}^{n} \beta_{ij}(D)\Omega(du_{ij}),
\]
(18)

with the correlation factor
\[
\beta_{ij}(D) \equiv -\frac{d_{ij}}{N_n^2(D)} \frac{\partial N_n^2(D)}{\partial d_{ij}}.
\]
(19)

and the pair solid angle
\[
\Omega(du_{ij}) \equiv u_i \times u_j \cdot d(u_j - u_i)/d_{ij}
\]
(20)

which is the sum of solid angles of the infinitely thin triangle \((u_i, -u_j, -u_j - du_j)\) and \((u_j, -u_i, -u_i - du_i)\). For example, the correlation phase and correlation factor for the state of two, three and four stars
\[
\gamma_C^{(2)} = \frac{1}{2} \oint \frac{u_1 \times u_2 \cdot (du_2 - du_1)}{3 + u_1 \cdot u_2}, \quad \beta_{12} = \frac{d_{12}}{2N_2^2}
\]
\[
\gamma_C^{(3)} = \frac{1}{2} \oint \frac{u_1 \times u_2 \cdot (du_2 - du_1) + u_2 \times u_3 \cdot (du_3 - du_2) + u_1 \times u_3 \cdot (du_3 - du_2)}{3 + u_1 \cdot u_2 + u_1 \cdot u_3 + u_2 \cdot u_3}, \quad \beta_{ij} = \frac{d_{ij}}{N_3^2}
\]
\[
\gamma_C^{(4)} = \frac{1}{4} \oint \sum_{i,j=1}^{3} \frac{u_i \times u_j \cdot (du_j - du_i)(6 - d_{kl})}{N_4^2}, \quad \beta_{ij} = \frac{(6 - d_{kl})d_{ij}}{2N_4^2} \quad \text{(with the restrict of } i < j \text{ and } k, l \neq i, j) \]
(21)
If we rotate \( u_i(\theta_i, \phi_i) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \) to \( z(0,0) \) by

\[
T_i = \begin{pmatrix}
\cos \theta_i & 0 & -\sin \theta_i \\
0 & 1 & 0 \\
\sin \theta_i & 0 & \cos \theta_i
\end{pmatrix}
\begin{pmatrix}
\cos \phi_i & \sin \phi_i & 0 \\
-\sin \phi_i & \cos \phi_i & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

Equation (22)

\( u_j(\theta_j, \phi_j) \) will change into \( u'_j(\theta'_j, \phi'_j) \). The vector calculation in \( \Omega(du_{ij}) \) becomes

\[
u_i \times u_j \cdot du = \frac{[T_i^{-1}z] \times (T_i^{-1}u'_{ij}) \cdot d(T_i^{-1}u'_{ij})}{1 - u_i \cdot u_j} = T_i^{-1}(z \times u'_j) \cdot (T_i^{-1}u'_j + T_i^{-1}(z \times u'_j) \cdot (dT_i^{-1}u'_{ij}) = z \times u'_j + z \times u'_j \cdot (T_1dT^{-1}u'_{ij}).
\]

Similarly, the rotation

\[
T_j = \begin{pmatrix}
\cos \theta_j & 0 & -\sin \theta_j \\
0 & 1 & 0 \\
\sin \theta_j & 0 & \cos \theta_j
\end{pmatrix}
\begin{pmatrix}
\cos \phi_j & \sin \phi_j & 0 \\
-\sin \phi_j & \cos \phi_j & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

Equation (24)

will change \( u_j \) and \( u_i \) into \( z(0,0) \) and \( u'_j(u'_{ij}, \theta''_{ij}, \phi''_{ij}) \) respectively. We have

\[
u_j \times u_i \cdot du = z \times u'_{ij} \cdot du'_{ij} + z \times u'_{ij} \cdot (TjdT_{ij}^{-1}u'_{ij}).
\]

Substituting Eq. (23) and (24) in to the definition of \( \Omega(du_{ij}) \), and notice the distance \( d_{ij} = 1 - u_i \cdot u_j \) is invariant under the two rotations, we have

\[
\Omega(du_{ij}) = \frac{u_i \times u_j \cdot du + u_j \times u_i \cdot du}{1 - u_i \cdot u_j} = \frac{z \times u'_{ij} \cdot du' + z \times u'_{ij} \cdot (T_{ij}dT_{ij}^{-1}u'_{ij})}{1 - u_i \cdot u_j} = \frac{(1 - \cos \theta'_{ij})d\phi'_{ij} + (1 - \cos \theta'_{ij})d\phi'_{ij}}{1 - u_i \cdot u_j} + [\cos \theta_i d\phi_i + \cos \theta_j d\phi_j].
\]

Equation (26)

Note that, the form \( 1 - \cos \theta )d\phi \) is precisely the integration element for the solid angle \( \Omega_u \) subtend by the path of the star \( (\theta, \phi) \). If we integrate Eq. (26), the geometric meaning of \( \Omega(du_{ij}) \) emerges immediately. Therefore the meaning of correlation phase \( \gamma^{(n)}_{Cj} \) is quit clear: it is consist of the collection of the weighted relative evolutions between the stars

\[
\gamma^{(n)}_{Rij} = \frac{1}{2} \int \beta_{ij}(D) \frac{\Omega(du'_{ij}) + \Omega(du'_{ij})}{1 + u_i \cdot u_j}
\]

Equation (27)

with \( \Omega(du'_{ij}) = (1 - \cos \theta'_{ij})d\phi'_{ij} \) and the collection of the weighted absolute evolutions of the pairs of stars

\[
\gamma^{(n)}_{Aij} = \frac{1}{2} \int \beta_{ij}(D)[\cos \theta_i d\phi_i + \cos \theta_j d\phi_j].
\]

Equation (28)

Namely, \( \gamma^{(n)}_C = \sum_{i=1}^n \sum_{j \neq i}^n (\gamma^{(n)}_{Rij} + \gamma^{(n)}_{Aij}) \).