The multi-physics metawedge: graded arrays on fluid-loaded elastic plates and the mechanical analogues of rainbow trapping and mode conversion

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Abstract

We consider the propagation and mode conversion of flexural-acoustic waves along a fluid-loaded graded array of elastic resonators, forming a metasurface. The multi-physics nature of the problem, coupling two disparate physical systems, brings both challenges and novel features not previously seen in so-called bifunctional metamaterials. In particular, by using an appropriately designed graded array of resonators, we show that it is possible to employ our metasurface to mode-convert sub-sonic surface flexural waves into bulk acoustic waves and vice-versa; transferring energy between two very different physical systems. Whilst the sub-sonic mechanical surface wave is dispersive, the bulk acoustic wave is dispersionless and radiates energy at infinity. We also show that this bifunctional metasurface is capable of exhibiting the classical effect of rainbow trapping for sub-sonic surface waves.

1. Introduction

The interaction of sound with compliant structures is a rich area of research that has generated significant interest, academic and otherwise, over the years with, at least, five monographs [1–5] on the subject. The interaction between acoustical and structural waves covers a very broad range of topics, with an equally broad range of techniques; from formal mathematical analyses (see, for example, [5]) to more applied research concerned with submarine acoustics [6]. In terms of physical problems, there are obvious applications, such as the generation of noise by flows past elastic structures (see, for example, [7]), flow in pipes [8], scattering of water waves by flexible ice sheets [9, 10], and the insonification of underwater elastic shells [11] amongst many others. The field itself has an illustrious history: the overview of the influence of fluid-loading on vibrating structures provided by Crighton’s 1998 Rayleigh Medal lecture [12] notes that it is almost impossible to find an area of research in wave motion where Lord Rayleigh has not worked and that fluid-loaded structures are no exception. In particular, Lord Rayleigh analysed the behaviour of air when excited by a vibrating circular plate [13] and his contemporaries, such as Lamb, also investigated similar problems [14].

Our interest here stems from recent advances in an apparently disconnected area, that of so-called metamaterials [15, 16] that are man-made composite structures that owe their properties to sub-wavelength structuration. These metamaterials can be designed to have effective properties not found in nature such as negative refractive index [17, 18] and these ideas have proved highly effective in other areas of wave physics, primarily in electromagnetism [19], in terms of controlling wave motion and are now influencing emergent areas such as acoustic [20, 21] and seismic [22–24] metamaterials. It is natural therefore to turn to fluid-loaded structures and evaluate whether metamaterials can be employed to any effect in this area. Since the structural acoustics of fluid-loading is inherently connected with a compliant surface such as an elastic plate or shell it is natural to consider metasurfaces rather than bulk metamaterials in this context.
An active area of research in metasurfaces [25] focuses upon graded resonator metasurfaces, where the general concept is that for a graded surface, or waveguide, different wavelengths are trapped at different spatial positions. Sub-wavelength microstructures are commonly employed in two ways: the first approach is to create effective macroscopic wavespeeds that vary spatially, thus achieving the required control of wave propagation. The second approach involves using deep sub-wavelength resonances to obtain the desired effects. Whilst the latter approach is significantly more difficult to create, it is far more powerful; hence our aim in the present paper is to extend this latter concept to fluid-loaded compliant structures. These ideas are being widely adopted in photonics and phononics due to their excellent abilities to control, manipulate and filter waves in compact devices. Graded and chirped designs include: trapping in rainbow devices [26–29], flat focussing mirrors and lenses in optics, plasmonics and acoustics [30–35], gradient index lens for acoustic and flexural waves focussing [36, 37], acoustic absorbers [38–40] and sound enhancement [41, 42].

In the absence of the fluid-loading, it is only very recently that graded sub-wavelength structures for thin elastic plates has been considered as a chirped graded array [43], thin beams [44] or in the context of gradient index (GRIN) lenses created by graded structuration to control elastic symmetric ($S_0$) and antisymmetric ($A_0$) waves [45] and to obtain deeply sub-wavelength focussing [46], cloaking [47] and GRIN lenses (e.g. Luneburg, Maxwell-fisheye and Eaton lenses [44, 48]) using resonator arrays [37].

There is also an active community in so-called platonics [49] studying the generalisations of phononic crystals to elastic plates in-vacuo with considerable progress in pulling out features associated with Dirac cones [50], dynamic anisotropy and lensing /shielding effects [51]. Much of this is concerned with mass-loaded or constrained plates, but recent work on resonator arrays on plates [52] has moved this into contemporary areas of physics where recent articles such as [53] have shown the existence topologically protected edge states for finite arrays.

This is all primarily for effectivly scalar wave systems and thus does not have the degrees of freedom necessary to demonstrate mode conversion from surface to bulk waves and so can only demonstrate trapped rainbow phenomena. More recently arrays of resonators atop elastic half-spaces have demonstrated, theoretically, numerically [23, 24] and experimentally in ultrasonics [54], that one can generate rainbow trapping phenomena and additionally manipulate surface Rayleigh waves and mode convert between Rayleigh and bulk shear waves. This mode conversion creates a surface device that is simultaneously reflectionless, yet has zero transmission into surface waves, and moreover that this occurs over an ultra-broadband range of frequencies. Rainbow trapping also has important implications as one can strongly enhance the surface wave amplitude at particular points along the interface and this is selectively achieved by altering the frequency.

The ability in elasticity to selectively use sub-wavelength graded resonator systems to influence both surface and bulk waves, as well as the coupling between them, motivates us to consider fluid-loaded structures.

Scattering and radiation by periodic systems in the fluid-loaded literature are natural models of ribbed and reinforced structures such as a ship’s hull and have been much studied [55–58]. In particular, from the mid 80’s till the early 2000s, the so-called ‘fuzzy structure model’ [59–61] has provided accurate estimates of the effective frequency dependent attenuation created by a resonant array of masses attached to plates and shells. Here, by using the recently developed theoretical and numerical models for locally resonant metamaterials [24, 54], we further explore the physics underpinning the propagation in such resonant structures and unveil new phenomena arising from the interaction between acoustic and elastic modes. Although the governing equations are scalar, there is a coupling between the elastic waves in the plate and the acoustic pressure field in the fluid; it is this coupling that allows the structure to support a sub-sonic surface wave in addition to the bulk acoustic wave.

Experiments, and theory, have been developed for acoustic, electromagnetic, and even elastic, rainbow devices but this has not been approached for fluid-loaded structures that support surface flexural waves coupled with the acoustic field in a bulk fluid; these form an exemplar of a multi-physics application, in this case coupling elastic plate waves on a surface, that exhibit dispersion, into an acoustic wave, that is dispersionless, in the bulk. The coupling between these two different wave systems can be used to create an additional effect: mode conversion between the two physical systems. By appropriately designing a graded metasurface, it is possible to mode convert mechanical surface waves to bulk acoustic waves. The multi-physical nature of this problem, coupling two disparate physical systems, the fluid with the structure, in conjunction with a metasurface brings both new challenges and novel exciting features.

We begin, in section 2, by assessing the influence of attaching periodically arranged identical resonators, modelled by the simplest system, that of a mass-spring, to the fluid-loaded structure (as shown in figure 1). We deduce the dispersion relations that characterise the behaviour of waves within this periodic fluid-loaded plate- resonator system. These are utilised in section 3 to interpret the influence of grading the resonators by, for instance, increasing or decreasing the masses spatially along the array (as shown in figure 2) and this interpretation is complemented by careful numerical simulations for an incoming sub-sonic surface plate wave incident upon graded arrays. Given the interest in underwater acoustics and maritime structures we choose our numerical examples to have aluminium as the plate (thickness $h = 0.01$ m, Young’s modulus $E_p = 69 \times 10^9$ N m$^{-2}$, Poisson ratio $\nu_p = 0.334$ and density
$\rho_p = 2700 \text{ kg m}^{-3}$ and water (density $\rho_f = 1000 \text{ kg m}^{-3}$, and sound speed $c_f = 1500 \text{ m s}^{-1}$) as the fluid. Finally, we draw together some comments and future perspectives on the use of metamaterial ideas for fluid-loaded structures.

## 2. Fluid-loaded thin elastic plate–resonator arrays

### 2.1. Formulation

We consider a thin elastic plate governed by the Kirchhoff–Love plate equations, as the simplest model of an elastic plate that captures the relevant physics for underwater acoustics [4] whilst recognising that this model can be complexified to allow the plate to be either anisotropic or thicker such that, say, the Mindlin equations become relevant; these generalisations are readily incorporated. We also restrict our attention to two-dimensions for clarity and as it will allow us to illustrate the key concepts without being over-burdened by excessive algebra.

Here, a thin elastic plate of thickness $h$, density $\rho_p$, Young’s modulus $E_p$ and Poisson’s ratio $\nu_p$ is located at $z = 0$, $-\infty < x < \infty$. It is bounded on its upper surface by an infinite acoustic fluid ($-\infty < x < \infty$, $0 < z < \infty$) of density $\rho_f$ and sound speed $c_f$. An infinite periodic array of mass-spring systems are attached to the lower surface of the plate, exerting point reaction forces, as shown in figure 1.

The $(2D)$ equation of motion for a thin fluid-loaded plate is written as

$$D \frac{\partial^4}{\partial x^4} u(x, t) + M \frac{\partial^2}{\partial t^2} u(x, t) = \sum_{n=-\infty}^{\infty} E_n \delta(x - nd) - p(x, 0, t),$$  \hspace{1cm} (1)

in which $D = E_p h^3 / 12 (1 - \nu_p^2)$ is the plate bending stiffness, and $M = \rho_p h$ is the plate displacement, $p(x, 0, t)$ is the acoustic pressure in the fluid [4] and $F_n$ is the reaction force of the $n$th mass-spring system located at $x_n = nd$, where $d$ is the periodicity of the mass-spring array. The equation satisfied by the pressure in the acoustic fluid, with sound speed $c_f$, is the wave equation

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c_f^2} \frac{\partial^2}{\partial t^2} \right) p(x, z, t) = 0,$$  \hspace{1cm} (2)

with the continuity condition at the plate surface such that the normal velocity in the fluid there is equal to the normal velocity of the plate. The reaction forces $F_n$ are determined from the equations of motion of the mass-
spring system:

\[ F_n = -m \frac{d^2}{dt^2} u_{mn}(t) = \Lambda (u_{mn}(t) - u(nd, t)), \]

where \( m \) is the mass, \( \Lambda \) is the spring constant and \( u_{mn} \) is the displacement of the attached mass at position \( x_n = nd \). Losses can be inserted in the mass–spring equation of motion using, for instance, a velocity dependent damping. While this could lead to a more realistic physical model, for the sake of clarity, we prefer to deal with strictly real dispersion curves, and not complex as would happen by including damping.

### 2.2. Dispersion relation

It is convenient to first consider time-harmonic motion, in which all the dependent variables are proportional to \( \exp(-i\omega t) \) and henceforth this exponential factor is suppressed. Thus, equations (1)–(3) become

\[ D \frac{\partial^4}{\partial x^4} u(x) - M \omega^2 u(x) = \sum_{n=-\infty}^{\infty} F_n \delta(x - nd) - p(x, 0), \]

\[ 0 = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k^2 \right) p(x, z), \]

\[ F_n = m \omega^2 u_{mn} = \Lambda (u_{mn} - u(nd)), \]

with \( k = \omega / c_f \) being the fluid wavenumber. Eliminating the mass displacements from (6) gives

\[ F_n = -m \omega^2 \Lambda u(nd) / (m \omega^2 - \Lambda) = -L(\omega) u(nd). \]

For the mass-spring system here \( L(\omega) = m \omega^2 \Lambda / (m \omega^2 - \Lambda) \) and resonance occurs at \( \omega = \sqrt{\Lambda / m} \); more complicated arrangements of masses and springs, or rods, can be considered leading to more complex functions encapsulated within \( L(\omega) \). Fourier transform pairs for the plate displacement and the acoustic pressure are introduced, for example, for the plate displacement:

\[ u(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(\alpha) \exp(i\alpha x) \, d\alpha, \]

\[ \hat{u}(\alpha) = \int_{-\infty}^{\infty} u(x) \exp(-i\alpha x) \, dx. \]

Transforming (8) we see the pressure is

\[ \hat{p}(\alpha, z) = \hat{p}(\alpha, 0) e^{i\alpha z}, \]

where \( \gamma = \sqrt{k^2 - \alpha^2} \) and the continuity condition at the plate surface couples the pressure to the plate displacement and

\[ i\gamma \hat{p}(\alpha, 0) = \rho \omega^2 \hat{u}(\alpha). \]

Making use of the Poisson Summation Formula [62], the reaction force terms of equation (4) can be written in the form

\[ \sum_{n=-\infty}^{\infty} F_n \delta(x - nd) = -\frac{L(\omega)}{d} u(x) \sum_{n=-\infty}^{\infty} \exp(2\pi inx/d), \]

from which the transformed equation (4) becomes

\[ S(\alpha, \omega) \hat{u}(\alpha) + \frac{L(\omega)}{d} \sum_{n=-\infty}^{\infty} \hat{u}(\alpha - 2n\pi/d) = 0, \]

where \( S(\alpha, \omega) = D\alpha^4 - \omega^2 M - i\rho \omega^2 / \gamma. \)

In the absence of any forcing or resonators, that is if we consider a pristine elastic plate with fluid loading, the dispersion relation is given by \( S(\alpha, \omega) = 0 \). Going one level even simpler, that is to an elastic plate in vacuo \( D\alpha^4 - \omega^2 M = 0 \) and \( \alpha = (M \omega^2 / D)^{1/4} \) giving the flexural plate wavenumber–frequency relationship that is nonlinear and which indicates that these plate waves are dispersive. The addition of the fluid coupling leads to these flexural plate waves acquiring a complex component and they become leaky waves, and we shall not consider them further here. The fluid-loaded system also allows for a sub-sonic (relative to the overlying fluid) surface wave that exists due to the coupling between the fluid and plate and which exponentially decays into the fluid. This wave is effectively the primary excitation of the arrays we describe and our aim is to investigate how this surface wave interacts with the resonant array.
Returning to the array of resonators, equation (13) becomes

\[
\left[ 1 + \frac{L(\omega)}{d} \sum_{n=-\infty}^{\infty} \frac{1}{S(\alpha - 2n\pi/d, \omega)} \right] \sum_{n=-\infty}^{\infty} \tilde{u}(\alpha - 2n\pi/d) = 0
\]

and the dispersion relation for the periodic system is therefore

\[
1 + \frac{L(\omega)}{d} \sum_{n=-\infty}^{\infty} \frac{1}{S(\alpha - 2n\pi/d, \omega)} = 0.
\]

There are limits of (15) that are of interest and which can be approached analytically. For no fluid-loading the infinite sum of (15) can be evaluated exactly, as in for example [24], allowing a closed form expression for the dispersion relation. With fluid present in the upper half-space the situation is more complicated and a closed form expression appears unavailable. Leading order asymptotic approximations for sums of this form are available [58] in the heavy fluid-loading limit for which \(\tilde{\alpha}_1 \gg \kappa\), and \(\tilde{\alpha}_1 \gg \tilde{\alpha}_0\),

\[
\sum_{n=-\infty}^{\infty} \frac{1}{S(\alpha - 2n\pi/d, \omega)} \sim -\frac{d}{S\tilde{D}\tilde{\alpha}_1^2} \left\{ \cot \left( \frac{2\pi}{5} \right) + \frac{\sin(\tilde{\alpha}_1d)}{\cos(\alpha d) - \cos(\tilde{\alpha}_1d)} \right\},
\]

in which \(\tilde{\alpha}_1 = (\omega^2/D)^{1/3}\) is the wavenumber in the heavily fluid-loaded plate and \(\tilde{\alpha}_0 = (\omega^2M/D)^{1/3}\), the wavenumber of the \textit{in vacuo} plate. Hence, in this heavy fluid-loading approximation, we rewrite \(\alpha\) as a function of \(\omega\) as

\[
\cos(\alpha d) = \cos(\tilde{\alpha}_1d) + \frac{\sin(\tilde{\alpha}_1d)}{\frac{S\tilde{D}\tilde{\alpha}_1^2}{L(\omega)} - \cot \left( \frac{2\pi}{5} \right)}.
\]

From this expression, for a given value of \(\omega\), real values of \(\alpha\) are only obtained when the absolute value of the right hand side of (17) is less than or equal to 1. Numerical examination of this dispersion relation shows that in this low frequency, heavy fluid-loading, asymptotic regime the dispersion diagram does not exhibit the band gaps necessary to provide interesting metamaterial properties. Thus, we proceed to investigate the full dispersion relation, away from the aforementioned asymptotic limits, numerically.

Returning to (15), and writing it as

\[
\sum_{n=-\infty}^{\infty} \frac{1}{S(\alpha_n, \omega)} = -\frac{d}{L(\omega)}
\]

in which \(\alpha_n = \alpha - 2n\pi/d\), we note that the right hand side is always real, whereas the left hand side will have an imaginary part unless \(\gamma_n = \sqrt{k^2 - \alpha_n^2}\) is imaginary for each value of \(n\). Thus, for any given value of \(\omega\), this restricts the possible values of \(\alpha\) to \(\kappa < \alpha < 2\pi/d - \kappa\) when \(0 < \alpha < 2\pi/d\) and \(0 < \kappa < \pi/d\).

Due to the underlying periodicity, the dispersion diagram repeats in \(\alpha\) with period \(2\pi/d\). To find the roots we note that for any given value of \(\omega\), \(S(\alpha, \omega)\) has one real positive root and therefore, \(1/S(\alpha, \omega)\) passes through an infinity there and changes sign. Thus, the left hand side of (18), as a function of \(\alpha\), periodically varies from \(-\infty\) to \(\infty\) and must therefore equal the right hand side at least once in each such interval. For heavy fluid loading such solutions are close to the location of the infinities, but are found numerically by starting the searches very close to the location of the infinity (where the sign is known) and looking for a change of sign.

We choose typical parameters in figure 3 that shows the fluid-loaded plate–resonator curves together with the resonator frequency \((\omega_0)\), the acoustic sound line \(\omega = \alpha c_f\) (dashed line) and the sub-sonic coupled fluid-plate surface wave \(\omega = \alpha c_p\) (dotted line). The effect of the resonator is to punch through the surface wave mode to create a very narrow band gap, shown as the grey region in the inset to figure 3, and the hybridisation of the mode. The hybridisation shows a transition, as frequency decreases and one approaches the band gap from above, in behaviour from the coupled fluid-solid surface wave to the bulk acoustic mode. This behaviour, arising from the avoided crossing between the surface wave dispersion relation and the local resonance is the acoustic manifestation of a surface plasmon polariton [63] in condensed matter physics which is one of the main approaches in nanofabricated devices used to design and control the propagation of light. A key difference in this fluid-loaded analogy is that the mode below resonance emerges from a surface wave that is dispersive and so the analogy is not exact.

For typical underwater parameters the band-gap thickness remains narrow over a wide range of frequencies and resonator mass, see figure 4, and such a narrow frequency band suggests that the effect of the band gap will be only restricted to very limited frequencies and be of little use. This is relatively common in graded resonator systems, the resonating rods placed on top of an elastic surface interacting with surface Rayleigh waves [24] have similar behaviour, and to take advantage of the band gap the key addition is to make the resonant array spatially dependent in some way. The spatial dependence can come through a gradual change in the array spacing, a graduation of spring constant or mass in each resonator along the array. We choose to alter the masses in the
array by either linearly ramping them up, or down, with distance and illustrate how this influences an incident surface wave coming from a pristine region of the plate not containing resonators striking a semi-infinite array of resonators. The choice of mass change with position means that the horizontal axis is equivalently the \( x \)-position of the resonators and we note that for a mass of 1 kg the position is 3 m and the frequency is 250 Hz.
and these choices are used later. Figure 4 allows us to read off the position, $x = (\Lambda/\omega^2 - m(0))/(dm/dx)$, as a function of $m(x)$, at which waves are no longer supported by the array as a function of position and thus this curve gives the turning point and thus shows the spatial selection by frequency.

2.3. Graded array

The conventional dispersion diagram of figure 3 for frequency versus wavenumber comes from equation (15), with $L(\omega)$ defined in (7) and all parameters (plate and fluid properties, mass–spring properties and the spacing) remaining constant. An alternative, less common, but instructive diagram shows how for fixed frequency the modes vary with a different parameter, such as those diagrams of figure 5. As frequency increases, from top to bottom, in figure 5, one sees the band gap move downwards as predicted by figure 4 and the vertical asymptote move to the right, that is, to shorter wavelengths. Figure 5 gives a valuable interpretation, for graded structures, as one can follow a branch from, say, high mass to low mass or vice-versa, and then immediately see how the wave system behaves. Before embarking upon this interpretation we extract the exact mass–frequency relationship.

The mass and spring constants only enter into (15) via the $L(\omega)$ term, and the dispersion relation can be re-written as

$$\frac{\omega^2 - \Lambda/m}{\omega^2 \Lambda} = -\frac{1}{d} \sum_{n=-\infty}^{\infty} \frac{1}{S(\alpha - 2n\pi/d, \omega)}.$$

Hence, if $\omega$ and $\Lambda$ are held constant, then the relation between resonator mass $m$ and wavenumber $\alpha$ is explicit:

$$m = \frac{\Lambda}{\omega^2 \left(1 + \frac{\Lambda}{d} \sum_{n=-\infty}^{\infty} \frac{1}{S(\alpha - 2n\pi/d, \omega)}\right)}.$$

Figure 5. For fixed frequencies (200, 250 and 300 Hz from (a) to (c)), spring constant and spacing dispersion plot showing mass (kg) as a function of horizontal wavenumber for 1 cm aluminium plate in water. Asymptotes shown: dashed line is $\alpha = \omega/c$ (sound line), solid line is obtained numerically, dotted line is the wavenumber of the unconstrained fluid-loaded plate. Inset shows zoom-in on very narrow band gap (shaded grey).
and similarly, holding $\omega$ and $m$ constant, a relation between resonator spring constant $\Lambda$ and $\alpha$ also emerges:

$$\Lambda = \frac{\omega^2}{\left(\frac{1}{m} + \frac{\omega^2}{d} \sum_{n=-\infty}^{\infty} \frac{1}{S(\alpha - 2n\pi/d, \omega)}\right)}.$$  \hfill (21)

Plotting the mass versus wavenumber, using (20), in figure 5 allows us to predict and interpret the behaviour of waves associated with a plate with an array of mass-spring resonators for which the spacing and spring constant are held constant, but the mass of the resonators varies (smoothly and monotonically) with position.

We first consider what happens to a wave as it moves through an array of gradually decreasing masses, i.e. ramping down the mass. From figure 5 for fixed frequency we start from the top of the graph and follow the dispersion curve downwards. Initially the surface wave propagates there with a horizontal wavenumber in the plate close to that of the surface wave in a pristine plate, and in the fluid with corresponding exponential decay in the direction perpendicular to the plate; this is effectively the dashed line of figure 3. As the wave progresses through the array, it encounters ever smaller values of the resonator mass, and the wavenumber adapts by also decreasing see figure 5 and the exponential rate of decay of the wave in the fluid perpendicular to the plate also decreases. The surface wave hybridises and moves towards the bulk acoustic wave branch until, for say 250 Hz at $m \sim 1$, the curve arrives at the band gap and meets the acoustic branch. At this point the system can no longer support a wave propagating in the direction parallel to the plate as the resonator mass decreases further. At this point, the system only supports radiative acoustic waves in the fluid and, therefore, the energy contained in the mechanical surface wave must be radiated out into the fluid, the point at which this occurs can be explicitly predicted from figure 4. Thus we have constructed a situation where mode conversion from a surface wave to a bulk wave occurs and where, as far as the surface displacement is concerned, there is both vanishing reflection and transmission. The cosine of the angle at which waves leave the surface is simply given by the ratio of sound speeds in the fluid and the fluid-loaded plate, i.e. $\cos \theta = (c_f/c_p)$, as shown in figure 6, and verified from the numerical results.

If we now consider the contrasting case, that of ramping up the mass with a wave moving through an array of gradually increasing masses, that is we start from the lower branch of the curve, which cuts on at zero mass and $\alpha = \alpha_1$. As this wave progresses it encounters ever larger values of resonator mass and the wavenumber adapts and increases, correspondingly decreasing the wave speed, until when $\alpha = \pi/d$ the curve arrives at the lower edge of the band gap and the group velocity vanishes. This is the origin of so-called rainbow trapping as the position at which this reflection occurs is frequency dependent and so one can design selective spatial frequency separation of waves.
3. Numerical results

We now turn to numerical simulations to illustrate the mode conversion and rainbow trapping effects. The numerical simulations use the finite difference method together with perfectly matched layers (PML) [64, 65] which are used to mimic an infinite domain; this is done by surrounding a finite area of fluid, and the corresponding finite length of plate, with a finite thickness of PML. The forcing is taken to be at the origin and of form $\delta(x)\exp(-i\omega t)$. This is implemented in finite differences in the frequency domain with at least 10 grid points between each resonator to ensure sufficient resolution per wavelength.

We first consider the effect of ramping down the masses and the potential for mode conversion. We use 120 mass-spring resonators with spacing $d = 0.05$ m and graduate the masses (per unit length) by linearly ramping them down from $1.3$ to $0.75$ kg m$^{-1}$ in graduations of $0.005$ kg m$^{-1}$. The spring constant $\Lambda$ is fixed as $2.4674 \times 10^6$ kg m s$^{-2}$ which is chosen to give a resonant frequency of $250$ Hz for $1$ kg m$^{-1}$. The mode conversion is strikingly seen in figure 7, which shows the surface wave propagating along the array with a gradually increasing wavelength, see figure 8, until it reaches the critical point, here $x \approx 3$ m, where the surface wave mode has hybridised into a bulk wave and the energy radiates into the bulk as highly directional long waves; clearly the resonator spacing (of $0.05$ m) is sub-wavelength and for comparison the wavelength for the unconstrained fluid-loaded plate is $0.4508$ m and the wavelength in the fluid $6$ m. This striking mode conversion is accompanied by decreasing plate displacements as the wave mode converts. The resonator displacements increase initially, but close to the position of the resonant frequency mass, i.e. from $x = 2.5$ m onwards, there is a rapid decrease in amplitude.

The rainbow trapping phenomena is illustrated in figure 9, and the masses now increase in amplitude. This figure also shows the wavefield in the absence of resonators as a reference for comparison with figure 7, as well as the varying masses case of figure 9. We again use 120 mass–spring resonators with spacing $d = 0.05$ m and reverse the graduation with the masses linearly ramping up from $0.705$ to $1.3$ kg m$^{-1}$ in increments of $0.005$ kg m$^{-1}$ and with the spring constant $\Lambda$ is fixed as $2.4674 \times 10^6$ kg m$^{-1}$ s$^{-2}$. The surface wave has gradually decreasing amplitude as it passes through the array and gets cut-off at $x \approx 3$; for different frequencies

![Figure 7. Time-harmonic (250 Hz) snapshot of pressure in water above 1 cm aluminium plate with mass-spring resonators. Resonator mass decreases with x. Resonant frequency of mass-spring located at x = 3 m is 250 Hz; both x and z axes are measured in metres.](image.png)
this position varies and this position can be read off directly from figure 4. Here we witness almost total reflection, and notably the plate surface displacement tends to zero, but the amplitude of the mass resonators increases dramatically, figure 10, thus the motion is spatially localised on the resonators which is ideal for applications such as energy harvesting.

To summarise, mode conversion cannot occur for increasing masses, at least in the scenario we consider. The interpretation is most readily seen from figure 5 when decreasing the masses the curve intersects the top of the band gap at the sound line \( \alpha = \omega/c_f \) and stops; there is a narrow gap in this curve between \( \alpha = 0 \) and \( \alpha = \omega/c_f \). At this intersection point the group velocity is non-zero and the energy cannot continue into a coupled flexural plate wave so it mode converts into a fluid-only wave with \( \alpha < \omega/c_f \). By contrast, when the masses ramp up, we see from figure 5 that the curve reaches the bottom of the band gap at \( \alpha = \pi/d \gg \omega/c_f \), with zero group velocity. For \( \alpha > \pi/d \) the graph would continue as a reflection about \( \alpha = \pi/d \) so the curve is continuous there. Zero group velocity may be achieved by total reflection of the coupled flexural plate wave. Both because the curve is continuous and also since \( \alpha \gg \omega/c_f \) there is no possibility of mode conversion to a bulk fluid wave.

Figure 8. Decreasing resonator mass: time-harmonic (250 Hz) displacements of 1 cm aluminium plate with water-loading and mass-spring resonators with the resonant frequency of mass-spring located at \( x = 3 \) m at 250 Hz. The absolute values of the plate and mass displacements are shown in the left and right panels, respectively.

Figure 9. Time-harmonic (250 Hz) snapshot of pressure in water above 1 cm aluminium plate: top, a pristine plate without resonators and bottom, with mass-spring resonators. Resonator mass increases with \( x \). Resonant frequency of mass-spring located at \( x = 3 \) m is 250 Hz; both \( x \) and \( z \) axes are measured in metres.
4. Discussion

We have illustrated that metamaterial ideas, that is using arrays of local sub-wavelength structuration to influence global wave behaviour, are relevant to fluid-loaded structures. We have concentrated upon a feature, surface graded arrays of resonators, pertinent to energy harvesting, structural vibration damping, mode conversion and the control of waves on surfaces. We envisage, as in the theory of surface elastic wave mode conversion and rainbow trapping \cite{24} which prompted experimental verification \cite{54}, that the theory we present here will drive experimental work. In more general terms, it is clear that other features such as effective negative parameters, analogies of transformation optics and much more are transferable from metamaterials to the field of structural acoustics. It is also important to note that we have treated the simplest possible type of resonator, masses and springs, but the analysis easily generalises to more realistic rod-resonator arrays as utilised in, the non-fluid loaded plate, analyses of \cite{24,66}.

It is also notable that metamaterials can be used to convert acoustic waves into flexural waves. For example, if a highly focused acoustic beam is incident on a localised area of the plate, plots of the scattered field very similar to those of figures 7 and 9(b) may be generated, in which the incident acoustic energy is converted locally into flexural plate waves which are then either reflected or partially reflected and partially re-radiated at the resonator whose mass corresponds to edge of the band gap for the incident frequency. It is clear that such scattering is modified by the presence of the resonators, and their effects on the acoustic scattering are worthy of further investigation.

We also remark that, in recent years, there has been significant interest in so-called bifunctional metamaterials which aim to harness the power of two different physical phenomena to achieve metamaterial effects. Thus far, however, these approaches have focused on the control of fields that, fundamentally, are governed by the same physics and have the same underlying mathematical structure (see, for example, \cite{67–70}); thermal and electrical conduction are both governed by the Helmholtz operator, for example. In contrast, the present paper represents a step change in the study of bifunctional metamaterials. Here we demonstrate a truly bifunctional metamaterial, synthesising two fundamentally different physical systems, acoustic fluid (governed by the Helmholtz operator) and flexural plate waves (governed by the biharmonic operator), to create the striking mode conversion illustrated in section 2.3.

We emphasise that, whilst motivated by the seismic metawedge \cite{23, 24}, the present work goes far beyond the mode conversion demonstrated by the metawedge. The elastic metawedge examined in \cite{23, 24} is capable of converting elastic surface waves into elastic bulk waves, that is, conversion from one elastic mode to another. In contrast, the fluid-loaded metasurface is capable of converting flexural plate waves to acoustic waves, that is, converting mechanical waves into acoustic waves. In this sense, the fluid-loaded metasurface studied in the present paper transcends the boundaries between disparate physical phenomena creating a truly bifunctional metasurface.

Finally we believe that a thorough evaluation of the effect of losses in such a metasurface is another priority for future works. It is known for instance that in highly damped systems the flat branches of the dispersion curves tend to disappear (i.e. they are non-propagative solutions) \cite{71} while band gaps show a lower attenuation capacity. This could affect also the efficiency of the conversion presented in this article. However such an analysis would be better carried out using a time domain formulation (2D or 3D finite element for instance) rather than a frequency one. In this way the difficulties related to the interpretation of complex frequency and wavenumber solutions will be avoided.
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