SUBADDITIVITY AND ADDITIVITY OF THE YANG-MILLS ACTION
FUNCTIONAL IN NONCOMMUTATIVE GEOMETRY

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Abstract. We formulate notions of subadditivity and additivity of the Yang-Mills action functional in noncommutative geometry. We identify a suitable hypothesis on spectral triples which proves that the Yang-Mills functional is always subadditive, as per expectation. The additivity property is much stronger in the sense that it implies the subadditivity property. Under this hypothesis we obtain a necessary and sufficient condition for the additivity of the Yang-Mills functional. An instance of additivity is shown for the case of noncommutative $n$-tori. We also investigate the behaviour of critical points of the Yang-Mills functional under additivity. At the end we discuss few examples involving compact spin manifolds, matrix algebras, noncommutative $n$-torus and the quantum Heisenberg manifolds which validate our hypothesis.

1. Introduction

Given a vector bundle $\mathbb{E}$ on a Riemannian manifold $(M, g)$ and a connection $\nabla$ on $\mathbb{E}$, the Yang-Mills functional is given by

$$\mathcal{YM}(\nabla) = \int_M ||\Theta_{\nabla}||^2 dV_g,$$

where $\Theta_{\nabla}$ denotes the curvature of $\nabla$. Atiyah-Bott ([1]) initiated the study of corresponding gradient flow to the Yang-Mills energy on a closed Riemann surface and proposed studying it as a means of understanding the topology of the space of connections using infinite-dimensional Morse theory. Immediately, one remarkable application of the flow appeared in Donaldson’s characterization of the correspondence between the algebraic and differential geometry on Kähler manifolds ([15]). He demonstrated that the stability of a bundle is equivalent to it admitting irreducible Hermitian-Einstein connection with respect to the Kähler metric. Around same time noncommutative differential geometry was invented by A. Connes ([7],[8]) for the purpose of extending differential geometry and topology beyond their classical framework in order to deal with ‘spaces’, such as leaf spaces of foliations and orbit spaces of discrete or Lie group actions on manifolds, which elude analysis by classical methods. The generalization of the Yang-Mills functional to the noncommutative context first appeared in ([13]) by the work of Connes-Rieffel for the case of $C^*$-dynamical systems. Latter Connes formulated this notion
more formally in the language of $\mathcal{K}$-cycles or spectral triples in ([9]), and investigated the case of noncommutative two-torus in great detail which suggests extensions of Yang-Mills theoretic techniques in the study of noncommutative differential (and possibly holomorphic) geometry of 'vector bundles' on $C^*$-algebras. It turns out that these two notions of Yang-Mills in noncommutative geometry, the older one for the $C^*$-dynamical systems (due to Connes-Rieffel in [13]) and the more formal one in the context of spectral triples (due to Connes in [9]), are equivalent for the case of noncommutative $n$-tori ([3]) and the quantum Heisenberg manifolds ([4]). However, the general case remains unanswered. But certainly, the formulation of Yang-Mills in the spectral triple setting is the adequate generalization of the classical Yang-Mills to the noncommutative framework.

In the Noncommutative Geometry programme of Connes, by a noncommutative topological space we mean an involutive subalgebra of a (unital) $C^*$-algebra. It is now widely accepted that geometry over a noncommutative space $A$ is governed by a triple $(A, H, D)$, called spectral triple. Here, $A$ is a unital associative $\star$-subalgebra of a $C^*$-algebra $\mathcal{A}$ faithfully represented on the separable Hilbert space $H$, and $D$ is an unbounded self-adjoint operator with compact resolvent acting on $H$ such that $[D, a]$ extends to a bounded operator on $H$ for all $a \in A$. If there exists a $\mathbb{Z}_2$-grading operator $\gamma \in \mathcal{B}(H)$ which commutes with $A$ and anticommutes with $D$, then the quadruple $(A, H, D, \gamma)$ is called an even spectral triple. Spectral triple generalizes classical spin manifolds to the noncommutative framework. Here, finitely generated projective modules equipped with Hermitian structure serve the role of complex vector bundles and the $L^2$-norm is specified by the Dixmier trace on spectral triples. The Yang-Mills action functional ([10]) on a finitely generated projective (f.g.p) module $E$ over $A$, equipped with a Hermitian structure, is a certain map $YM : C(E) \rightarrow \mathbb{R}_{\geq 0}$ generalizing ([11]), where $C(E)$ denotes the affine space of compatible connections on $E$. Here, compatibility is described with respect to the Hermitian structure on f.g.p module $E$. A crucial application in physics is observed in ([11]). Note that Yang-Mills represents energy functional and hence, critical points of it is of particular interest in mathematics as well as in physics literature. These have been investigated by Rieffel ([24]) on the noncommutative two-torus and Kang ([18]) on the quantum Heisenberg manifolds. It would not be an exaggeration to say that Yang-Mills is an important and active area of research in noncommutative geometry, and over the years it has been studied by various authors (e.g. [13],[24],[25],[18],[21],[3],[4]).

Since the domain of the Yang-Mills functional (henceforth briefly abbreviated as Y-M) is an affine space, the usual notions of subadditivity and additivity of a function do not make sense. We systematically formulate these notions and prove that under a suitable hypothesis on spectral triples Y-M is always subadditive. This is expected since Y-M represents energy functional. The notion of additivity turns out to be stronger than subadditivity in the sense that additivity implies subadditivity. Let us briefly describe our setting. Like in the classical case where forming the product between two geometric spaces is a basic operation in geometry, considering tensor product of noncommutative spaces is also of much relevant importance not only for construction of a would-be tensor category, but also bears interest for some applications.
in theoretical physics ([12]). For example, the almost commutative spectral triple corresponding to the standard model of particle physics ([2]) is a tensor product of a canonical commutative spectral triple with a finite-dimensional noncommutative one. Given two even spectral triples $(A_j, H_j, D_j, \gamma_j), j = 1, 2$, their product is defined by the following rule, due to Connes ([9]),

$$(A_1, H_1, D_1, \gamma_1) \otimes (A_2, H_2, D_2, \gamma_2) := (A_1 \otimes A_2, H_1 \otimes H_2, D = D_1 \otimes 1 + \gamma_1 \otimes D_2, \gamma_1 \otimes \gamma_2).$$

It is enough if one of the spectral triples, instead of both, is even. However, in our context, w.l.o.g. we always consider even spectral triples. This is explained at the beginning of Section [3]. Now, if $E_1$ and $E_2$ are two Hermitian f.g.p modules over $A_1$ and $A_2$ respectively, and $\nabla_j \in C(E_j)$ for $j = 1, 2$, then $\nabla := \nabla_1 \otimes 1 + 1 \otimes \nabla_2$ is a connection on $E = E_1 \otimes E_2$. Important observation is that it is a compatible connection, i.e. $\nabla \in C(E)$, with respect to a natural Hermitian structure on $E$. We use the structure theorem of Hermitian f.g.p modules obtained in ([3]) to prove this. A natural question is whether there is any relation between $YM(\nabla)$ and $YM(\nabla_j)$ for $j = 1, 2$. We define the notions of subadditivity and additivity of Y-M in this context. Under the following hypothesis on spectral triples $(A_j, H_j, D_j, \gamma_j), j = 1, 2$,

**Hypothesis:**

$$\frac{\pi_1(\partial^2(A_1)) \otimes A_2 + A_1 \otimes \pi_2(\partial^2(A_2))}{\pi_1(\partial_1^2(A_1)) \otimes A_2 + A_1 \otimes \pi_2(\partial_2^2(A_2))} \cong \Omega^2_{D_1}(A_1) \otimes A_2 \oplus A_1 \otimes \Omega^2_{D_2}(A_2)$$

as $A_1 \otimes A_2$-bimodules.

we prove that Y-M is always subadditive, as per expectation. To validate this hypothesis we discuss few examples involving compact spin manifolds, matrix algebras, noncommutative $n$-tori and the quantum Heisenberg manifolds. These are the cases for which the respective Dirac dga is known in the literature. Under the above hypothesis we also obtain a necessary and sufficient condition for additivity of Y-M. An instance of additivity of Y-M is shown for the case of noncommutative $n$-tori. It is also natural to ask if Y-M becomes additive then how its critical points behave. For this we obtain a useful necessary and sufficient condition which determines when the critical points of Y-M for two spectral triples give rise to a critical point of Y-M for the product spectral triple.

Organization of the paper is as follows. Section [2] is mainly preliminaries. Sections [3] is the main content where we define the notions of subadditivity and additivity of Y-M and prove the above discussed results. Section [4] contains an instance of additivity of Y-M. Section [5], [6], [7] discuss examples where our hypothesis is validated. These include the case of compact spin manifolds, matrix algebras, noncommutative $n$-torus and the quantum Heisenberg manifolds.

2. Spectral triples and the Yang-Mills functional

All algebras considered in this article will be assumed unital.

**Definition 2.1.** A spectral triple $(A, H, D)$ over a unital, associative $\ast$-algebra $A$ consists of the following:

1. a $\ast$-representation $\pi$ of $A$ on the separable Hilbert space $H$,
2. an unbounded self-adjoint operator $D$ acting on $H$ such that
(i) \([D, \pi(a)], \) initially defined on \(\text{Dom}(D)\), extends to a bounded operator on \(\mathcal{H}\),
(ii) \(D\) has compact resolvent.

It is said to be an even spectral triple if there exists a \(\mathbb{Z}_2\)-grading \(\gamma \in \mathcal{B}(\mathcal{H})\) (i.e. \(\gamma = \gamma^*\) and \(\gamma^2 = \text{id}\)) such that \(\gamma\) commutes with each element of \(\mathcal{A}\) and anticommutes with \(D\). If no such \(\gamma\) is present then the spectral triple \((\mathcal{A}, \mathcal{H}, D)\) is called odd. We will always assume that \(\pi\) is a unital faithful representation.

**Definition 2.2.** If \(|D|^{-d}\), for positive number \(d\), lies in the Dixmier ideal \(\mathcal{L}^{(1, \infty)} \subseteq \mathcal{B}(\mathcal{H})\), then the spectral triple \((\mathcal{A}, \mathcal{H}, D)\) is called \(d\)-summable spectral triple (this is sometimes referred as \((d, \infty)\) or \(d^+\)-summable in the literature).

Associated to every spectral triple \((\mathcal{A}, \mathcal{H}, D)\) there is a differential graded algebra (dga) \((\Omega^2_D(\mathcal{A}), d)\) defined by Connes, which we will call the Connes’ calculus or the Dirac DGA. Recall its definition from (Ch. [6] in [9]). However, for our purpose, the following is enough

\[
\Omega^1_D(\mathcal{A}) = \{ \sum a_j[D, b_j] : a_j, b_j \in \mathcal{A} \} \subseteq \mathcal{B}(\mathcal{H})
\]
\[
\pi(\Omega^2(\mathcal{A})) = \{ \sum a_j[D, b_j][D, c_j] : a_j, b_j, c_j \in \mathcal{A} \} \subseteq \mathcal{B}(\mathcal{H})
\]
\[
\pi(dJ^0(\mathcal{A})) = \{ \sum [D, b_j][D, c_j] : \sum b_j[D, c_j] = 0, b_j, c_j \in \mathcal{A} \} \subseteq \mathcal{B}(\mathcal{H})
\]
\[
\Omega^2_D(\mathcal{A}) = \pi(\Omega^2(\mathcal{A})) / \pi(dJ^0(\mathcal{A})).
\]

All of these are bimodules over \(\mathcal{A}\). We have the Dirac dga differentials \(d : \mathcal{A} \longrightarrow \Omega^1_D(\mathcal{A})\) given by \(a \mapsto [D, a]\), and \(d : \Omega^1_D(\mathcal{A}) \longrightarrow \Omega^2_D(\mathcal{A})\) given by \(a[D, b] \mapsto [[D, a][D, b]]\). Note that \((da)^* = -d(a^*)\) by convention. For the classical case of compact spin manifolds, where \(D\) is the classical Dirac operator, \(\Omega^2_D\) gives back the space of de-Rham forms (Page 551 in [9]). So, Dirac dga can be thought of as noncommutative space of forms. However, this dga is very hard to compute and not much of computation is known in the literature except ([6], [3], [4], [5]).

Using this Dirac dga, Connes extended the classical notion of Yang-Mills action functional to the noncommutative geometry framework in ([9]). Let us recall it now.

Let \(\mathcal{E}\) be a finitely generated projective right module over \(\mathcal{A}\). We will write \(\text{f.g.p}\) to mean finitely generated projective throughout the article. Unless explicitly mentioned, we will only consider right modules in this article. Let \(\mathcal{E}^* := \mathcal{H}\text{om}_{\mathcal{A}}(\mathcal{E}, \mathcal{A})\). Clearly, \(\mathcal{E}^*\) is also a right \(\mathcal{A}\)-module by the rule \((\phi, a)(\eta) := a^* \phi(\eta), \forall \eta \in \mathcal{E}, a \in \mathcal{A}\).

**Definition 2.3.** ([9]) A Hermitian structure on \(\mathcal{E}\) is an \(\mathcal{A}\)-valued positive-definite sesquilinear map \((\cdot, \cdot)_A\) satisfying the following :

(a) \((\xi, \xi')^*_A = (\xi', \xi)_A, \forall \xi, \xi' \in \mathcal{E}\).
(b) \((\xi, \xi' . a)_A = (\xi, \xi'\cdot a)_A, \forall \xi, \xi' \in \mathcal{E}, \forall a \in \mathcal{A}\).
(c) The map \(\xi \mapsto \Phi_\xi\) from \(\mathcal{E}\) to \(\mathcal{E}^*\), given by \(\Phi_\xi(\eta) := (\xi, \eta)_A\) for all \(\eta \in \mathcal{E}\), gives conjugate linear \(\mathcal{A}\)-module isomorphism between \(\mathcal{E}\) and \(\mathcal{E}^*\). This property will be referred as the self-duality of \(\mathcal{E}\).
Any free $\mathcal{A}$-module $\mathcal{E}_0 = \mathcal{A}^n$ has a Hermitian structure on it, given by $\langle \xi, \eta \rangle_\mathcal{A} = \sum_{j=1}^{n} \xi^*_j \eta_j$ for all $\xi = (\xi_1, \ldots, \xi_n), \eta = (\eta_1, \ldots, \eta_n) \in \mathcal{E}_0$. We refer this as the canonical Hermitian structure on $\mathcal{E}_0$. By definition, any f.g.p module $\mathcal{E}$ can be written as $\mathcal{E} = p\mathcal{A}^n$ for some idempotent $p \in M_n(\mathcal{A})$. If this idempotent $p$ is a projection, i.e., $p = p^2 = p^*$, one can restrict the canonical Hermitian structure on $\mathcal{A}^n$ to $\mathcal{E}$ and then $\mathcal{E}$ becomes a Hermitian f.g.p module. Moreover, it is proved in Lemma 2.2(b) in [16] that if $\mathcal{A}$ is stable under the holomorphic functional calculus in a unital $C^*$-algebra $\mathcal{A}$ (in which case the unit will belong to $\mathcal{A}$), then we have the following existence lemma of Hermitian structure.

**Lemma 2.4** ([3]). Every f.g.p module $\mathcal{E}$ over $\mathcal{A}$ is isomorphic as a f.g.p module with $p\mathcal{A}^n$ where $p \in M_n(\mathcal{A})$ is a self-adjoint idempotent, that is a projection. Hence, $\mathcal{E}$ has a Hermitian structure on it.

**Remark 2.5.** The above lemma proves the existence of Hermitian structure with the assumption of closure under holomorphic functional calculus. Without the assumption, the existence of Hermitian structure on arbitrary f.g.p module over $\mathcal{A}$ is not known.

With the assumption of closure under the holomorphic functional calculus, one has the following structure theorem of Hermitian f.g.p module. (Th. 3.3 in [3]).

**Theorem 2.6** ([3]). Let $\mathcal{E}$ be a f.g.p $\mathcal{A}$-module with a Hermitian structure on it. Suppose $\mathcal{A}$ is stable under the holomorphic functional calculus in a $C^*$-algebra $\mathcal{A}$. Then we have a self-adjoint idempotent $p \in M_n(\mathcal{A})$ such that $\mathcal{E} \cong p\mathcal{A}^n$ as f.g.p module, and $\mathcal{E}$ has the induced canonical Hermitian structure.

In his book ([9]), Connes has suggested that in the context of Hermitian structure and the Yang-Mills functional one should always work with spectrally invariant algebras, that is subalgebras of $C^*$-algebras stable under the holomorphic functional calculus. The reason is that all possible notions of positivity will coincide in that case. Moreover, we will also have Th. (2.6) which makes computation with the Hermitian structure much easier. Hence, incorporating Connes’ suggestion, throughout the article we will always work with spectrally invariant algebras. Note that in the classical situation of manifolds, $C^\infty(\mathbb{M})$ is indeed spectrally invariant subalgebra of $C(\mathbb{M})$ ([16]).

**Definition 2.7.** Let $\mathcal{E}$ be a f.g.p module over $\mathcal{A}$ equipped with a Hermitian structure $\langle \ldots \rangle_\mathcal{A}$. A compatible connection on $\mathcal{E}$ is a $\mathbb{C}$-linear map $\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes_\mathcal{A} \Omega^1_D(\mathcal{A})$ satisfying

(a) $\nabla(\xi a) = (\nabla \xi) a + \xi \otimes da, \quad \forall \xi \in \mathcal{E}, a \in \mathcal{A}$;

(b) $\langle \xi, \nabla \eta \rangle - \langle \nabla \xi, \eta \rangle = d(\langle \xi, \eta \rangle_\mathcal{A} \quad \forall \xi, \eta \in \mathcal{E}$ (Compatibility).

The meaning of the last equality in $\Omega^1_D(\mathcal{A})$ is, if $\nabla(\xi) = \sum \xi_j \otimes \omega_j \in \mathcal{E} \otimes \Omega^1_D(\mathcal{A})$, then $\langle \nabla \xi, \eta \rangle = \sum \omega^*_j \langle \xi_j, \eta \rangle_\mathcal{A}$. Any f.g.p right module has a connection. An example of a compatible connection is the Grassmannian connection $\nabla_0$ on $\mathcal{E} = p\mathcal{A}^n$, given by $\nabla_0(\xi) = pd\xi$, where $d\xi = (d\xi_1, \ldots, d\xi_n)$ and $p \in M_n(\mathcal{A})$ is a projection. This connection is compatible with the
The definition of $E$ describe the commutator of the subspace $\pi$ for all $[\nabla, \pi] \subseteq \pi(\Omega^2)$. This gives a well defined inner-product on $\Omega^2_{\pi}$. Now, for $\phi, \psi \in \text{Hom}_A(E, E \otimes_A \Omega^2_{\pi})$, define the inner-product as

\[ \langle \langle \phi, \psi \rangle \rangle := \sum_{k=1}^{n} \sum_{i=1}^{n} \langle \langle \phi(e_k)i \rangle, \psi(e_k)i \rangle_{\Omega^2_{\pi}} \]

where $\{e_1, \ldots, e_n\}$ is the standard canonical basis of the free module $A^n$ over $A$.

**Definition 2.9.** The Yang-Mills action functional is a map $\mathcal{YM} : C(E) \rightarrow \mathbb{R}_{\geq 0}$ given by

\[ \mathcal{YM}(\nabla) = \langle \langle \Theta, \Theta \rangle \rangle. \]

**Remark 2.10.** The definition of $\mathcal{YM}$ does not depend on the choice of the projection used to describe $E$. This is discussed in Remark [4.3] in [3].

Let $\nabla_t = \nabla + t\mu$ be a linear perturbation of a connection $\nabla$ on $E$ by an element $\mu \in \text{Hom}_A(E, E \otimes_A \Omega^2_{\pi}(A))$. One can check that the curvature $\Theta_t$ of the connection $\nabla_t$ becomes $\Theta + t[\nabla, \mu] + O(t^2)$. If we suppose that $\nabla$ is an extremum of the YangMills action functional, this linear perturbation should not affect the action. In other words, we should have

\[ \frac{d}{dt}|_{t=0} \mathcal{YM}(\nabla + t\mu) = 0. \]
From here it follows that \(\langle\langle [\nabla, \mu], \Theta \rangle\rangle = 0\), where \(\langle\langle \ , \ \rangle\rangle\) is the inner-product on \(\text{Hom}_A(E, E \otimes_A \Omega^2_D)\) described in (2.3). Here, \([\nabla, \mu] = \tilde{\nabla} \circ \mu + (1 \otimes \Pi) \circ (\mu \otimes 1) \circ \nabla\), with \(\Pi : \Omega^1_D \times \Omega^1_D \to \Omega^2_D\) being the product map of the Dirac dga. One can check that \([\nabla, \mu]\) is indeed \(A\)-linear. Since \(\mu\) is arbitrary, we derive the equation of motion \([\nabla^*, \Theta] = 0\), where the adjoint of \([\nabla, \cdot]\) is defined by
\[
\langle\langle [\nabla^*, \alpha], \beta \rangle\rangle = \langle\langle \alpha, [\nabla, \beta] \rangle\rangle.
\]
For detail on these we refer ([20]).

**Definition 2.11.** A compatible connection \(\nabla \in C(E)\) is called a critical point of Yang-Mills action functional if
\[
\left.\frac{d}{dt}\right|_{t=0} \mathcal{YM}(\nabla + t\mu) = 0
\]
for all \(\mu \in \text{Hom}_A(E, E \otimes_A \Omega^1_D(A))\).

### 3. Subadditivity and additivity of the Yang-Mills functional

Let \((A_1, \mathcal{H}_1, D_1, \gamma_1)\) and \((A_2, \mathcal{H}_2, D_2, \gamma_2)\) be two even spectral triples. The product of these, due to Connes ([9]), is given by the following even spectral triple
\[
(A_1, \mathcal{H}_1, D_1, \gamma_1) \otimes (A_2, \mathcal{H}_2, D_2, \gamma_2) := (A := A_1 \otimes A_2, \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2, D = D_1 \otimes 1 + \gamma_1 \otimes D_2, \gamma = \gamma_1 \otimes \gamma_2).
\]
At this point one should note the following.

1. One can also consider the following multiplication formula for even spectral triples
\[
(A_1, \mathcal{H}_1, D_1, \gamma_1) \otimes (A_2, \mathcal{H}_2, D_2, \gamma_2)
\]
\[
= (A_1 \otimes A_2, \mathcal{H}_1 \otimes \mathcal{H}_2, D' = D_1 \otimes \gamma_2 + 1 \otimes D_2, \gamma_1 \otimes \gamma_2).
\]
In this case there exists a unitary \(U \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)\) given by
\[
U := \frac{1}{2}(1 \otimes 1 + \gamma_1 \otimes 1 + 1 \otimes \gamma_2 - \gamma_1 \otimes \gamma_2),
\]
such that the spectral triples \((A_1 \otimes A_2, \mathcal{H}_1 \otimes \mathcal{H}_2, D)\) and \((A_1 \otimes A_2, \mathcal{H}_1 \otimes \mathcal{H}_2, D')\) become unitary equivalent, i.e. \(UDU^* = D'\) (26).

2. If one starts with an odd spectral triple \((A, \mathcal{H}, D)\), i.e. without the grading operator, then one can construct an even spectral triple \((A, \mathcal{H} = \mathcal{H} \otimes \mathbb{C}^2, D, \gamma)\) using any two \(2 \times 2\) Pauli spin matrices such that \(\Omega^*_D(A) \cong \Omega^*_D(A)\) as dgas (Lemma 2.7 in [5]). Therefore, when working with Dirac dga, one can w.l.o.g. assume that the spectral triple is always even.

3. If one of the spectral triple is even and the other is odd then the multiplication rule (3.4) is still well-defined. Only difference is that the resulting product spectral triple is now odd.

4. Let \(\sigma_1, \sigma_2, \sigma_3\) be the \(2 \times 2\) Pauli spin matrices. For two odd spectral triples \((A_j, \mathcal{H}_j, D_j), j = 1, 2\), one can also consider the following spectral triple
(\mathcal{A}_1 \otimes \mathcal{A}_2, \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathbb{C}^2, D = D_1 \otimes 1 \otimes \sigma_1 + 1 \otimes D_2 \otimes \sigma_2)

as their multiplication. It is an even spectral triple with the grading operator $1 \otimes 1 \otimes \sigma_3$.

However, observe that this is nothing but first making $(\mathcal{A}_1, \mathcal{H}_1, D_1)$ even as described above in point (2), and then following the multiplication rule given by $[3.4]$.

We fix the following notations throughout the article.

**Notation**:
(a) $A = \mathcal{A}_1 \otimes \mathcal{A}_2$, (b) $H = \mathcal{H}_1 \otimes \mathcal{H}_2$, (c) $D = D_1 \otimes 1 + \gamma_1 \otimes D_2$, (d) $E = \mathcal{E}_1 \otimes \mathcal{E}_2$, (e) $\pi = \pi_1 \otimes \pi_2$.

**Lemma 3.1**. $\Omega^1_D(A) = \Omega^1_{D_1}(A_1) \otimes \mathcal{A}_2 \oplus A_1 \otimes \Omega^1_{D_2}(A_2)$ as $A$-bimodules.

**Proof**. Note that $[D, \sum a_1 \otimes a_2] = \sum [D_1, a_1] \otimes a_2 + \gamma_1 a_1 \otimes [D_2, a_2]$. Since, $\gamma_1[D_1, a_1] = -[D_1, a_1] \gamma_1$ and $\gamma_1^2 = 1$, we have the following $A$-bimodule inclusion,

$$\Omega^1_D(A) \subseteq \Omega^1_{D_1}(A_1) \otimes \mathcal{A}_2 \oplus A_1 \otimes \Omega^1_{D_2}(A_2).$$

In order to show the equality, observe that any element $\sum a_0[D_1, a_1] \otimes a_2$ of $\Omega^1_{D_1}(A_1) \otimes \mathcal{A}_2$ can be written as $\sum (a_0 \otimes a_2)[D, a_1 \otimes 1]$. Similarly, $\sum a_0 \otimes a_1 D_2, a_2 \in A_1 \otimes \Omega^1_{D_2}(A_2)$ can be written as $\sum (a_0 \otimes a_1)[D, 1 \otimes a_2]$. \hfill $\square$

**Lemma 3.2**. The Dirac dga differential $d : A \rightarrow \Omega^1_D(A)$ is given by

$$d(a_1 \otimes a_2) = (d_1(a_1) \otimes a_2, a_1 \otimes d_2(a_2))$$

where $d_j : A_j \rightarrow \Omega^1_{D_j}(A_j)$, for $j = 1, 2$, are the Dirac dga differentials associated with $A_j$.

**Proof**. Follows from the previous Lemma (3.1). \hfill $\square$

**Lemma 3.3**. $\pi(\Omega^2(A)) = (\pi_1(\Omega^2(A_1)) \otimes \mathcal{A}_2 + A_1 \otimes \pi_2(\Omega^2(A_2))) \oplus \Omega^1_{D_1}(A_1) \otimes \Omega^1_{D_2}(A_2)$ as $A$-bimodules.

**Proof**. Arbitrary element of $\pi(\Omega^2(A))$ looks like

$$\sum_{i,j,k} (a_{0i} \otimes a_{1i})[D, b_{0j} \otimes b_{1j}][D, c_{0k} \otimes c_{1k}]$$

$$= \sum_{i,j,k} a_{0i}[D_1, b_{0j}][D_1, c_{0k}] \otimes a_{1i}b_{1j}c_{1k} + a_{0i}b_{0j}c_{0k} \otimes a_{1i}a_{1j}b_{1j}c_{1k}$$

$$+ \gamma_1 (a_{0i}b_{0j} b_{1j}[D_2, b_{1j}]c_{1k} - a_{0i}b_{0j}c_{0k} \otimes a_{1i}b_{1j} b_{2j} c_{1k} \otimes a_{1i}b_{1j} c_{1k}).$$

Since, $\gamma_1$ anticommutes with $[D_1, a]$ but commutes with $A_1$ and $[D_1, a][D_1, b]$ we have the inclusion ‘\subseteq’. Now,

(i) $\Omega^1_{D_1}(A_1) \otimes \Omega^1_{D_2}(A_1) \subseteq \pi(\Omega^2(A)) :$ Consider $\sum_{i,j} a_{0i}[D_1, b_{0j}] \otimes a_{1i}b_{1j} \in \Omega^1_{D_1}(A_1) \otimes \Omega^1_{D_2}(A_1)$. Observe that

$$\sum_{i,j} (a_{0i} \otimes a_{1i})[D, 1 \otimes b_{1j}][D, b_{0j} \otimes 1] = \sum_{i,j} \gamma_1 a_{0i}[D_1, b_{0j}] \otimes a_{1i}b_{1j}.$$
\( \pi_1(\Omega^2(A_1)) \otimes A_2 \subseteq \pi(\Omega^2(A)) \): Consider \( \sum_{i,j,k} a_{0i}[D_1, b_{0j}] [D_1, c_{0k}] \otimes a_{1i} \in \pi(\Omega^2(A_1)) \otimes A_2 \). Observe that
\[
\sum_{i,j,k} (a_{0i} \otimes a_{1i}) [D_1, b_{0j} \otimes 1] [D_1, c_{0k} \otimes 1] = \sum_{i,j,k} a_{0i}[D_1, b_{0j}] [D_1, c_{0k}] \otimes a_{1i}.
\]

(iii) \( A_1 \otimes \pi_2(\Omega^2(A_2)) \subseteq \pi(\Omega^2(A)) \): Take \( \sum_{i,j,k} a_{0i} \otimes a_{1i} [D_2, b_{1j}] [D_2, c_{1k}] \in A_1 \otimes \pi(\Omega^2(A_2)) \). Observe that
\[
\sum_{i,j,k} (a_{0i} \otimes a_{1i}) [D_1, b_{1j}] [D_1, c_{1k}] = \sum_{i,j,k} a_{0i} \otimes a_{1i} [D_2, b_{1j}] [D_2, c_{1k}].
\]

This gives the reverse inclusion `\( \supseteq \)` and completes the proof. \( \square \)

**Lemma 3.4.** As an \( \mathcal{A} \)-bimodule, we have
\[ \pi(dJ^1_0(A)) = \pi_1(d_1J^1_0(A_1)) \otimes A_2 + A_1 \otimes \pi_2(d_2J^1_0(A_2)) , \]
i.e. \( \pi(dJ^1_0(A)) \cap (\Omega^1_{D_1}(A_1) \otimes \Omega^1_{D_2}(A_2)) = \{0\} \) in \( \pi(\Omega^2(A)) \).

**Proof.** Arbitrary element of \( \pi(dJ^1_0(A)) \) looks like
\[
\sum_{j,k} [D_1, b_{0j} \otimes b_{1j}] [D_1, c_{0k} \otimes c_{1k}]
\]
such that
\[(3.5) \quad \sum_{j,k} (b_{0j} \otimes b_{1j}) [D_1, c_{0k} \otimes c_{1k}] = 0. \]

This equation (3.5) gives us the following two equations (by Lemma 3.1)
\[
(3.6) \quad \sum_{j,k} b_{0j} [D_1, c_{0k}] \otimes b_{1j} c_{1k} = 0,
\]
\[
(3.7) \quad \sum_{j,k} b_{0j} c_{0k} \otimes b_{1j} [D_2, c_{1k}] = 0.
\]

For arbitrary \( \sum [D_1, b_{0j}] [D_1, c_{0k}] \otimes a \in \pi_1(d_1J^1_0(A_1)) \otimes A_2 \) and \( \sum b \otimes [D_2, b_{1j}] [D_2, c_{1k}] \in A_1 \otimes \pi_2(d_2J^1_0(A_2)) \) if we denote \( \xi \) to be their summation, then we see that \( \xi = (1 \otimes a) \xi_1 + (b \otimes 1) \xi_2 \) where,
\[
\xi_1 = \sum [D_1, b_{0j} \otimes 1] [D_1, c_{0k} \otimes 1] \quad \text{and} \quad \xi_2 = [D_1, 1 \otimes b_{1j}] [D_1, 1 \otimes c_{1k}] .
\]
Both \( \xi_1 \) and \( \xi_2 \) are in \( \pi(dJ^1_0(A)) \) by equations (3.6) and (3.7). Being bimodule over \( \mathcal{A} = A_1 \otimes A_2 \) we conclude that \( \xi \in \pi(dJ^1_0(A)) \). This proves the following,
\[
\pi_1(d_1J^1_0(A_1)) \otimes A_2 + A_1 \otimes \pi_2(d_2J^1_0(A_2)) \subseteq \pi(dJ^1_0(A)).
\]
To prove the reverse inclusion, first recall from Lemma 3.3 that \( J^1_0(\mathcal{A}) = J^1_0(\mathcal{A}_1) \otimes A_2 \oplus A_1 \otimes J^1_0(\mathcal{A}_2) \). Consider the element \( \omega = \sum_{j,k} b_{0j} d_1(c_{0k}) \otimes b_{1j} c_{1k} \) in \( J^1_0(\mathcal{A}_1) \otimes A_2 \subseteq J^1_0(\mathcal{A}) \). So, \((1 \otimes d_2)\omega \in J^1_0(\mathcal{A}_1) \otimes \Omega^1(\mathcal{A}_2)\). Thus,

\[
(\pi_1 \otimes \pi_2) \circ (1 \otimes d_2) \left( \sum_{j,k} b_{0j} d_1(c_{0k}) \otimes b_{1j} c_{1k} \right) = 0
\]

(3.8)

\[
\Rightarrow \sum_{j,k} b_{0j} [D_1, c_{0k}] \otimes ([D_2, b_{1j}] c_{1k} + b_{1j} [D_2, c_{1k}]) = 0.
\]

Similarly, for \( \omega = \sum_{j,k} b_{0j} c_{0k} \otimes b_{1j} d_2(c_{1k}) \) in \( A_1 \otimes J^1_0(\mathcal{A}_2) \subseteq J^1_0(\mathcal{A}) \) we get

\[
(\pi_1 \otimes \pi_2) \circ (d_1 \otimes 1) \left( \sum_{j,k} b_{0j} c_{0k} \otimes b_{1j} d_2(c_{1k}) \right) = 0
\]

(3.9)

\[
\Rightarrow \sum_{j,k} ([D_1, b_{0j}] c_{0k} + b_{0j} [D_1, c_{0k}]) \otimes b_{1j} [D_2, c_{1k}] = 0.
\]

Finally, equation 3.8–3.9 gives us

\[
\sum_{j,k} b_{0j} [D_1, c_{0k}] \otimes [D_2, b_{1j}] c_{1k} - [D_1, b_{0j}] c_{0k} \otimes b_{1j} [D_2, c_{1k}] = 0.
\]

This implies that any arbitrary element \( \xi = \sum_{j,k} [D, b_{0j} \otimes b_{1j}] [D, c_{0k} \otimes c_{1k}] \) of \( \pi(dJ^1_0(\mathcal{A})) \) is actually of the form

\[
\xi = \sum_{j,k} [D_1, b_{0j}] [D_1, c_{0k}] \otimes b_{1j} c_{1k} + b_{0j} c_{0k} \otimes [D_2, b_{1j}] [D_2, c_{1k}].
\]

That is,

\[
\pi(dJ^1_0(\mathcal{A})) \subseteq \pi_1(\Omega^2(\mathcal{A}_1)) \otimes A_2 + A_1 \otimes \pi_2(\Omega^2(\mathcal{A}_2)),
\]

in view of Lemma 3.3, i.e. \( \pi(dJ^1_0(\mathcal{A})) \cap (\Omega^1_d(\mathcal{A}_1) \otimes \Omega^1_d(\mathcal{A}_2)) = \{0\} \) in \( \pi(\Omega^2(\mathcal{A})) \). Now, recall a general result (Exercise 6, Part I, Chapter 2, Page 69 in [14]) that given two Hilbert spaces \( \mathcal{H}_1, \mathcal{H}_2 \), and operators \( T_i \in B(\mathcal{H}_1), T'_i \in B(\mathcal{H}_2), i = 1, \ldots, k \), such that the \( T_i \) are linearly independent and such that \( \sum_{i=1}^k T_i \otimes T'_i = 0 \), it follows that \( T'_i = 0 \) for all \( i = 1, \ldots, k \). Using the faithfulness of \( \pi_1, \pi_2 \) in our case, we see that the equation 3.9 implies \( \sum_{j,k} b_{0j} [D_1, c_{0k}] = 0 \) and equation 3.7 implies \( \sum_{j,k} b_{1j} [D_2, c_{1k}] = 0 \). Hence,

\[
\pi(dJ^1_0(\mathcal{A})) \subseteq \pi_1(d_1 J^1_0(\mathcal{A}_1)) \otimes A_2 + A_1 \otimes \pi_2(d_2 J^1_0(\mathcal{A}_2))
\]

which concludes the proof.

\[\square\]

**Remark 3.5.** In general, it is may not be true that

\[
\pi_1(\Omega^2(\mathcal{A}_1)) \otimes A_2 \cap A_1 \otimes \pi_2(\Omega^2(\mathcal{A}_2)) = \{0\},
\]

\[
\pi_1(d_1 J^1_0(\mathcal{A}_1)) \otimes A_2 \cap A_1 \otimes \pi_2(d_2 J^1_0(\mathcal{A}_2)) = \{0\}.
\]
Proposition 3.8. \(\nabla\in\mathcal{C}(\mathcal{E}_1)\) and \(\nabla_2\in\mathcal{C}(\mathcal{E}_2)\) define
\[
\nabla : \mathcal{E}_1 \otimes \mathcal{E}_2 \longrightarrow (\mathcal{E}_1 \otimes \mathcal{E}_2) \otimes \mathcal{A} \Omega_D^1(\mathcal{A})
\]
\[e_1 \otimes e_2 \longmapsto \nabla_1(e_1) \otimes e_2 + e_1 \otimes \nabla_2(e_2)\]

Proposition 3.8. \(\nabla\in\mathcal{C}(\mathcal{E})\), i.e. if \(\nabla_1\) and \(\nabla_2\) are compatible connections on \(\mathcal{E}_1\) and \(\mathcal{E}_2\) respectively, then so is \(\nabla\) on \(\mathcal{E}\).
Proof. Clearly, $\nabla$ is a $\mathbb{C}$-linear map. Now, for $e_1 \otimes e_2 \in \mathcal{E}$ and $x \otimes y \in \mathcal{A}$,

$$\nabla((e_1 \otimes e_2)(x \otimes y)) = \nabla(e_1 x \otimes e_2 y)$$

$$= \nabla_1(e_1 x) \otimes e_2 y + e_1 x \otimes \nabla_2(e_2 y)$$

$$= \nabla_1(e_1 x) \otimes e_2 y + (e_1 \otimes d_1 x) \otimes e_2 y + e_1 x \otimes \nabla_2(e_2)y + e_1 x \otimes (e_2 \otimes d_2 y)$$

$$= (\nabla_1(e_1)x \otimes e_2 y + e_1 x \otimes \nabla_2(e_2)y) + (e_1 \otimes e_2 y \otimes d_1 x \otimes 1 + e_1 x \otimes e_2 \otimes d_2 y)$$

$$= (\nabla_1(e_1) \otimes e_2 + e_1 \otimes \nabla_2(e_2))(x \otimes y) + (e_1 \otimes e_2) \otimes (d_1 x \otimes y + x \otimes d_2 y)$$

$$= \nabla(e_1 \otimes e_2)(x \otimes y) + (e_1 \otimes e_2) \otimes d(x \otimes y)$$

by Lemma (3.2). Hence, $\nabla$ is a connection on $\mathcal{E}$. Now, we show that $\nabla$ is compatible with respect to the Hermitian structure on $\mathcal{E}$. Let

$$\nabla_1(e_1) = \sum_i e_{1i} \otimes \omega_{1i} \in \mathcal{E}_1 \otimes \Omega^1_{D_1}(\mathcal{A}_1),$$

$$\nabla_2(e_2) = \sum_i e_{2i} \otimes \omega_{2i} \in \mathcal{E}_2 \otimes \Omega^1_{D_2}(\mathcal{A}_2),$$

$$\nabla_1(e_1') = \sum_i e_{1i}' \otimes \omega_{1i}' \in \mathcal{E}_1 \otimes \Omega^1_{D_1}(\mathcal{A}_1),$$

$$\nabla_2(e_2') = \sum_i e_{2i}' \otimes \omega_{2i}' \in \mathcal{E}_2 \otimes \Omega^1_{D_2}(\mathcal{A}_2).$$

Then,

$$\nabla_1(e_1) \otimes e_2 + e_1 \otimes \nabla_2(e_2) = \sum_i (e_{1i} \otimes e_2 \otimes \omega_{1i}, e_1 \otimes e_{2i} \otimes \omega_{2i}),$$

$$\nabla_1(e_1') \otimes e_2' + e_1' \otimes \nabla_2(e_2') = \sum_i (e_{1i}' \otimes e_2' \otimes \omega_{1i}', e_1' \otimes e_{2i}' \otimes \omega_{2i}').$$

and

$$d\langle e_1 \otimes e_2, e_1' \otimes e_2' \rangle = d(\langle e_1, e_1' \rangle \otimes \langle e_2, e_2' \rangle)$$

$$= d_1(\langle e_1, e_1' \rangle) \otimes (e_2, e_2') + (e_1, e_1') \otimes d_2(\langle e_2, e_2' \rangle).$$

Since, $\nabla_1 \in C(\mathcal{E}_1)$ and $\nabla_2 \in C(\mathcal{E}_2)$ we have

$$\langle e_1, \nabla_1 e_1' \rangle - \langle \nabla_1 e_1, e_1' \rangle = d_1(\langle e_1, e_1' \rangle)$$

$$\langle e_2, \nabla_2 e_2' \rangle - \langle \nabla_2 e_2, e_2' \rangle = d_2(\langle e_2, e_2' \rangle)$$

which further implies the following equations

$$\sum_i \langle e_1, e_1' \rangle \omega_{1i}' \otimes \omega_{1i}^* \langle e_1, e_1' \rangle = d_1(\langle e_1, e_1' \rangle),$$

$$\sum_i \langle e_2, e_2' \rangle \omega_{2i}' \otimes \omega_{2i}^* \langle e_2, e_2' \rangle = d_2(\langle e_2, e_2' \rangle).$$
Now,
\[
\langle e_1 \otimes e_2, \nabla (e_1' \otimes e_2') \rangle = \langle e_1 \otimes e_2, \sum_i e_{1i}' \otimes e_{2i}' \otimes \omega_{1i}' + e_{1}' \otimes e_{2i}' \otimes \omega_{2i}' \rangle
\]
\[
= \sum_{i,j} \langle e_1 \otimes e_2, e_{1i}' \otimes e_{2i}' \otimes a_{01ij}[D_1, a_{11ij}] + e_{1}' \otimes e_{2i}' \otimes a_{02ij}[D_2, a_{12ij}] \rangle
\]
\[
= \sum_{i,j} \langle e_1 \otimes e_2, e_{1i}' \otimes e_{2i}' \otimes ((a_{01ij} \otimes 1)[D, a_{11ij} \otimes 1]) + e_{1}' \otimes e_{2i}' \otimes ((1 \otimes a_{02ij})[D, 1 \otimes a_{12ij}]) \rangle
\]
\[
= \sum_{i,j} \langle e_1, e_{1i}' \rangle \otimes \langle e_2, e_{2i}' \rangle ((a_{01ij} \otimes 1)[D, a_{11ij} \otimes 1])
\]
\[
+ \sum_{i,j} \langle e_1, e_{1i}' \rangle \otimes \langle e_2, e_{2i}' \rangle ((1 \otimes a_{02ij})[D, 1 \otimes a_{12ij}])
\]
\[
= \sum_{i,j} \langle e_1, e_{1i}' \rangle a_{01ij}[D, a_{11ij} \otimes 1] + \langle e_1, e_{1i}' \rangle [D, 1 \otimes a_{12ij}]
\]
\[
= \sum_i \langle e_1, e_{1i}' \rangle \omega_{1i}' \otimes \langle e_2, e_{2i}' \rangle + \langle e_1, e_{1i}' \rangle \otimes \langle e_2, e_{2i}' \rangle \omega_{2i}'.
\]

Similarly,
\[
\langle \nabla (e_1 \otimes e_2), e_1' \otimes e_2' \rangle = \sum_i \omega_{i1}' \langle e_{1i}, e_1' \rangle \otimes \langle e_2, e_2' \rangle + \langle e_1, e_1' \rangle \otimes \omega_{2i}' \langle e_{2i}, e_2' \rangle.
\]

Subtracting, we get from equations \((3.11), (3.12)\) and \((3.10)\) that
\[
\langle e_1 \otimes e_2, \nabla (e_1' \otimes e_2') \rangle - \langle \nabla (e_1 \otimes e_2), e_1' \otimes e_2' \rangle = d(\langle e_1 \otimes e_2, e_1' \otimes e_2' \rangle)
\]

This proves that \(\nabla\) is a compatible connection i.e. \(\nabla \in C(\mathcal{E})\). \(\Box\)

**Remark 3.9.** Individually, \(\nabla_1 \otimes 1\) and \(1 \otimes \nabla_2\) are not connections on \(\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2\).

Now, we are in a position to define subadditivity and additivity of the Yang-Mills functional.

**Definition 3.10.** For any two even spectral triples \((\mathcal{A}_j, \mathcal{H}_j, D_j, \gamma_j)\) and Hermitian f.g.p modules \(\mathcal{E}_j\) over \(\mathcal{A}_j, j = 1, 2\), we say that the Yang-Mills action functional \(\mathcal{YM}\) is

1. **Subadditive** if \(\sqrt{\mathcal{YM}(\nabla_1)} \leq \sqrt{\alpha \mathcal{YM}(\nabla_1)} + \sqrt{\beta \mathcal{YM}(\nabla_2)}, \) for all \(\nabla_j \in C(\mathcal{E}_j)\),

2. **Additive** if \(\mathcal{YM}(\nabla) = \alpha \mathcal{YM}(\nabla_1) + \beta \mathcal{YM}(\nabla_2),\) for all \(\nabla_j \in C(\mathcal{E}_j);\)

for certain positive constants \(\alpha\) and \(\beta\), essentially determined by the summability of the individual spectral triples (These constants will be explicitly determined in Thm. \((3.19)\)).

**Remark 3.11.** Above definition \((3.10)\) is natural in the following sense. The Yang-Mills action functional is defined using certain inner-product (Def. \((2.9)\)). Hence, the square root is given by a certain norm. However, one should note that the domain of the Yang-Mills functional is an affine space instead of a vector space. Therefore, a suitable formulation of subadditivity and additivity was needed.
Now, we put an assumption on the individual spectral triples \((A_j, H_j, D_j), j = 1, 2\), to show that the Yang-Mills action functional is always subadditive.

**Assumption:** For \(A = A_1 \otimes A_2\),
\[
\frac{\pi_1(\Omega^2_1(A_1) \otimes A_2 + A_1 \otimes \pi_2(\Omega^2_1(A_2)))}{\pi_1(D_1 A_1) \otimes A_2 + A_1 \otimes \pi_2(D_2 A_2)} \cong \Omega^2_1(D_1 A_1) \otimes A_2 \oplus A_1 \otimes \Omega^2_1(D_2 A_2)
\]
as \(A\)-bimodules.

**Lemma 3.12.** The above assumption is equivalent to the fact that
\[
\Omega^2_D(A) \cong \Omega^2_D(A_1) \otimes A_2 \oplus A_1 \otimes \Omega^2_D(A_2) \oplus \Omega^1_D(A_1) \otimes \Omega^1_D(A_2)
\]
as \(A\)-bimodules.

**Proof.** Follows from Lemma \((3.3)\) and \((3.4)\). □

In general, it is not known whether Lemma \((3.12)\) is always true for any pair of spectral triples \((A_j, H_j, D_j), j = 1, 2\). One has to check this for each particular cases. After the end of this section we provide few examples to validate this assumption.

**Lemma 3.13.** The Dirac dga differential \(d : \Omega^1_D(A) \rightarrow \Omega^2_D(A)\) is given by the following
\[
d(\omega_1 \otimes a_2, a_1 \otimes \omega_2) = (d_1(\omega_1) \otimes a_2, a_1 \otimes d_2(\omega_2), \omega_1 \otimes d_2(a_2) - d_1(a_1) \otimes \omega_2)
\]
where, for \(j = 1, 2\), \(d_j : \Omega^1_D(A_j) \rightarrow \Omega^2_D(A_j)\) are the Dirac dga differentials associated with \(A_j\).

**Proof.** Let \(\omega_1 = \sum a_{0i}[D_1, a_{1i}]\) and \(\omega_2 = \sum b_{0i}[D_2, b_{1i}]\). Then,
\[
(\omega_1 \otimes a_2, a_1 \otimes \omega_2) = \sum_i \left( (a_{0i} [D_1, a_{1i}] \otimes a_2, a_1 \otimes b_{0i} [D_2, b_{1i}] \right)
\]
\[
= \sum_i \left( (a_{0i} \otimes a_2)[D_1, a_{1i}] + 1) + (a_1 \otimes b_{0i})[D_1, 1 \otimes b_{1i}] \right)
\]
as an element of \(\Omega^1_D(A)\) (Lemma \((3.1)\)). Hence,
\[
d(\omega_1 \otimes a_2, a_1 \otimes \omega_2) = \sum_i \left[ [D_1, a_{0i} \otimes a_2][D_1, a_{1i}] + [D_1, a_1 \otimes b_{0i}][D_1, 1 \otimes b_{1i}] \right)
\]
\[
= \sum_i \left[ [D_1, a_{0i}][D_1, a_{1i}] \otimes a_2 + a_1 \otimes [D_2, b_{0i}][D_2, b_{1i}] \right)
\]
\[
+ \gamma_1 a_{0i}[D_1, a_{1i}] \otimes [D_2, a_2] - \gamma_1 [D_1, a_{1i}] \otimes b_{0i}[D_2, b_{1i}] \right)
\]
\[
= d_1(\omega_1) \otimes a_2 + a_1 \otimes d_2(\omega_2) + \gamma_1(\omega_1 \otimes d_2(a_2) - d_1(a_1) \otimes \omega_2).
\]

Our conclusion now follows from the previous Lemma \((3.12)\). □

**Lemma 3.14.** The product map \(\Pi : \Omega^1_D(A) \times \Omega^1_D(A) \rightarrow \Omega^2_D(A)\) is given by the following
\[
i (\omega_1 \otimes a_2, 0, (\omega_1' \otimes a_2', 0) = (\omega_1 \omega_1' \otimes a_2 a_2', 0, 0).
\]
\[
i (0, a_1 \otimes \omega_2), (0, a_1' \otimes \omega_2') = (0, a_1 a_1' \otimes \omega_2 \omega_2', 0).
\]
\[
i (0, a_1 \otimes \omega_2), (\omega_1' \otimes a_2', 0) = (0, 0, a_1 \omega_1' \otimes \omega_2 a_2').
\]
\[
i (\omega_1 \otimes a_2, 0, (0, a_1' \otimes \omega_2) = (0, 0, -\omega_1 a_1' \otimes a_2 \omega_2').
\]
Proof. Part (i), (ii), (iii) are straightforward verification using Lemma \((3.1, 3.3, 3.12)\). We only explain part (iv) to show why the minus sign appears. Let \( \omega_1 = \sum_i x_0i[D_1, x_{1i}] \) and \( \omega'_2 = \sum_i y_{0i}[D_2, y_{1i}] \). Then \( \omega_1 \otimes a_2 = \sum_i (x_{0i} \otimes a_2)[D, x_{1i} \otimes 1] \) and \( a'_1 \otimes \omega'_2 = \sum_i (a'_1 \otimes y_{0i})[D, 1 \otimes y_{1i}] \) as elements of \( \Omega^1_D(A) \). Hence,

\[
(\omega_1 \otimes a_2, 0)(0, a'_1 \otimes \omega'_2) = \sum_{i,j} (x_{0i} \otimes a_2)[D, x_{1i} \otimes 1](a'_1 \otimes y_{0j})[D, 1 \otimes y_{1j}]
\]

\[
= \sum_{i,j} (x_{0i}[D_1, x_{1i}] \otimes a_2)(\gamma_1 a'_1 \otimes y_{0j}[D_2, y_{1j}])
\]

\[
= \sum_{i,j} -\gamma_1 x_{0i}[D_1, x_{1i}]a'_1 \otimes a_2 y_{0j}[D_2, y_{1j}]
\]

\[
= -\gamma_1 \omega_1 a'_1 \otimes a_2 \omega'_2
\]

This element is identified with \((0, 0, -\omega_1 a'_1 \otimes a_2 \omega'_2) \in \Omega^2_D(A)\) by Lemma \((3.12)\). \(\square\)

**Proposition 3.15.** The curvature \(\Theta\) of the connection \(\nabla\) is given by

\[
\Theta(e_1 \otimes e_2) = \Theta_1(e_1) \otimes e_2 + e_1 \otimes \Theta_2(e_2)
\]

where, \(\Theta_1, \Theta_2\) are the curvatures associated to the connections \(\nabla_1, \nabla_2\) respectively.

Proof. For \(\nabla_1(e_1) = \sum_i x_i \otimes \omega_i\) and \(\nabla_2(e_2) = \sum_i y_i \otimes v_i\), we have

\[
\nabla(e_1 \otimes e_2) = \nabla_1(e_1) \otimes e_2 + e_1 \otimes \nabla_2(e_2)
\]

\[
= \sum_i x_i \otimes \omega_i \otimes e_2 + e_1 \otimes y_i \otimes v_i
\]

\[
= \sum_i (x_i \otimes e_2) \otimes \omega_i + (e_1 \otimes y_i) \otimes v_i
\]

using Lemma \((3.6)\). Since \(\Theta = \nabla \circ \nabla\), we get using Lemma \((3.13)\)

\[
\nabla \circ \nabla (e_1 \otimes e_2) = \nabla \circ \nabla_1(e_1) \otimes e_2 + e_1 \otimes \nabla \circ \nabla_2(e_2)
\]

\[
= \sum_i \nabla_1(x_i \otimes e_2) \omega_i + x_i \otimes e_2 \otimes d \omega_i + \nabla_2(e_1 \otimes y_i) v_i + e_1 \otimes y_i \otimes dv_i
\]

\[
= \sum_i (\nabla_1(x_i) \otimes e_2) \omega_i + (x_i \otimes \nabla_2(e_2)) \omega_i + x_i \otimes e_2 \otimes d \omega_i + e_1 \otimes y_i \otimes dv_i
\]

\[
= \sum_i (\nabla_1(x_i) \omega_i \otimes e_2 + (x_i \otimes d_1 \omega_i) \otimes e_2 + e_1 \otimes \nabla_2(y_i) v_i + e_1 \otimes y_i \otimes d_2 v_i)
\]

\[
= \sum_i \nabla_1(x_i \otimes \omega_i) \otimes e_2 + e_1 \otimes \nabla_2(y_i \otimes v_i)
\]

\[
= \nabla_1(\nabla_1(e_1)) \otimes e_2 + e_1 \otimes \nabla_1(\nabla_2(e_2))
\]

\[
= \Theta_1(e_1) \otimes e_2 + e_1 \otimes \Theta_2(e_2).
\]
The third equality from below comes from part (iii) and (iv) of Lemma (3.14).

Lemma 3.16. Both $\Theta_1 \otimes 1$ and $1 \otimes \Theta_2$ are in $\mathcal{H}om_A(\mathcal{E}, \mathcal{E} \otimes_A \Omega^2_{D_2}(A))$.

Proof. It is easy to verify that both $\Theta_1 \otimes 1$ and $1 \otimes \Theta_2$ are $A$-linear, because $\Theta_j$ are $A_j$-linear for $j = 1, 2$. Conclusion now follows from Lemma (3.12).

Recall the inner-product on $\pi(\Omega^2(A))$ from (2.2). By Lemma (3.3), we have the induced inner-product on the subspaces $\pi_1(\Omega^2(A_1)) \otimes A_2$, $A_1 \otimes \pi_2(\Omega^2(A_2))$ and $\Omega^1_{D_1}(A_1) \otimes \Omega^1_{D_2}(A_2)$ of $\pi(\Omega^2(A))$.

Lemma 3.17. The induced inner-product on the subspaces $\pi_1(\Omega^2(A_1)) \otimes A_2$, $A_1 \otimes \pi_2(\Omega^2(A_2))$ and $\Omega^1_{D_1}(A_1) \otimes \Omega^1_{D_2}(A_2)$ are given, up to multiplication by a positive constant, by the following

(i) $\langle T_1 \otimes a_2, T_1' \otimes a_2' \rangle = Tr_\omega(T_1^* T_1' |D_1|^{-k}) Tr_\omega(a_2^* a_2'|D_2|^{-l})$
(ii) $\langle a_1 \otimes T_2, a_1' \otimes T_2' \rangle = Tr_\omega(a_1^* a_1'|D_1|^{-k}) Tr_\omega(T_2^* T_2'|D_2|^{-l})$
(iii) $\langle T_1 \otimes S_1, T_1' \otimes S_1' \rangle = Tr_\omega(T_1^* T_1'|D_1|^{-k}) Tr_\omega(S_1^* S_1'|D_2|^{-l})$

respectively, where $Tr_\omega$ denotes the Dixmier trace.

Proof. Assume that $(A_1, \mathcal{H}_1, D_1)$ is a $k$-summable spectral triple and $(A_2, \mathcal{H}_2, D_2)$ is a $\ell$-summable spectral triple. Then, $(A, \mathcal{H}, D)$ is a $(k+\ell)$-summable spectral triple, and we have

$$\frac{\Gamma(\frac{k+\ell}{2} + 1)}{\Gamma(\frac{k}{2} + 1) \Gamma(\frac{\ell}{2} + 1)} Tr_\omega \left( (T_1 \otimes T_2)|D|^{-(k+\ell)} \right) = Tr_\omega \left( T_1|D_1|^{-k} \right) Tr_\omega \left( T_2|D_2|^{-\ell} \right)$$

for all $T_j \in \mathcal{B}(\mathcal{H}_j)$, $j = 1, 2$, (Page 576 in [9]) where $Tr_\omega$ denotes the Dixmier trace. The number $\frac{\Gamma(\frac{k+\ell}{2} + 1)}{\Gamma(\frac{k}{2} + 1) \Gamma(\frac{\ell}{2} + 1)}$ is a positive real constant, and this completes the proof.

Proposition 3.18. The subspaces $\pi_1(\Omega^2(A_1)) \otimes A_2 + A_1 \otimes \pi_2(\Omega^2(A_2))$ and $\Omega^1_{D_1}(A_1) \otimes \Omega^1_{D_2}(A_2)$ of $\pi(\Omega^2(A))$ are orthogonal to each other.

Proof. Recall from Lemma (3.3) that arbitrary element of $\Omega^1_{D_1}(A_1) \otimes \Omega^1_{D_2}(A_2)$ is of the form $\sum_{i,j}(a_{ij} \otimes a_{ij})|D_1| \otimes b_{ij}|D_2|$, and this element. For any element $\eta = \eta_1 \otimes \eta_2 \in \pi_1(\Omega^2(A_1)) \otimes A_2 + A_1 \otimes \pi_2(\Omega^2(A_2)) \subseteq \pi(\Omega^2(A))$ it follows, in view of the inner-product given in (2.2) and the fact

$$\frac{\Gamma(\frac{k+\ell}{2} + 1)}{\Gamma(\frac{k}{2} + 1) \Gamma(\frac{\ell}{2} + 1)} Tr_\omega \left( (T_1 \otimes T_2)|D|^{-(k+\ell)} \right) = Tr_\omega \left( T_1|D_1|^{-k} \right) Tr_\omega \left( T_2|D_2|^{-\ell} \right),$$
that

\[
\langle \xi, \eta \rangle
= \text{Tr}_\omega \left( \xi^* \eta |D|^{-(k+\ell)} \right)
= \sum_{i,j} \left( \frac{\Gamma \left( \frac{k+i}{2} + 1 \right)}{\Gamma \left( \frac{k}{2} + 1 \right) \Gamma \left( \frac{i}{2} + 1 \right)} \right) \text{Tr}_\omega \left( [D_1, b_{0j}]^* a_{\alpha_i} \gamma_1 \eta_1 |D_1|^{-k} \right) \text{Tr}_\omega \left( [D_2, b_{1j}]^* a_{\beta_i} \eta_2 |D_2|^{-\ell} \right)
= \sum_{i,j} \left( \frac{\Gamma \left( \frac{k+i}{2} + 1 \right)}{\Gamma \left( \frac{k}{2} + 1 \right) \Gamma \left( \frac{i}{2} + 1 \right)} \right) \text{Tr}_\omega \left( [D_1, b_{0j}]^* a_{\alpha_i} \gamma_1 \eta_1 |D_1|^{-k} \gamma_1 \right) \text{Tr}_\omega \left( [D_2, b_{1j}]^* a_{\beta_i} \eta_2 |D_2|^{-\ell} \right)
= \sum_{i,j} \left( \frac{\Gamma \left( \frac{k+i}{2} + 1 \right)}{\Gamma \left( \frac{k}{2} + 1 \right) \Gamma \left( \frac{i}{2} + 1 \right)} \right) \text{Tr}_\omega \left( \gamma_1 [D_1, b_{0j}]^* a_{\alpha_i} \gamma_1 \eta_1 |D_1|^{-k} \right) \text{Tr}_\omega \left( [D_2, b_{1j}]^* a_{\beta_i} \eta_2 |D_2|^{-\ell} \right)
= - \sum_{i,j} \left( \frac{\Gamma \left( \frac{k+i}{2} + 1 \right)}{\Gamma \left( \frac{k}{2} + 1 \right) \Gamma \left( \frac{i}{2} + 1 \right)} \right) \text{Tr}_\omega \left( [D_1, b_{0j}]^* a_{\alpha_i} \gamma_1 \eta_1 |D_1|^{-k} \right) \text{Tr}_\omega \left( [D_2, b_{1j}]^* a_{\beta_i} \eta_2 |D_2|^{-\ell} \right)
= - \langle \xi, \eta \rangle
\]

Here, the minus sign appears at the end because of the facts that \( \text{Tr}_\omega \) is a trace, \( \gamma_1 \) commutes with \( \eta_1 \) and \( a_{\alpha_i} \), anticommutes with \( D_1 \), hence commutes with \( |D_1| \) and \( |D_1|^{-k} \), anticommutes with \( [D_1, b_{0j}] \). Since, \( \xi \) and \( \eta \) are arbitrary, our claim follows. \( \square \)

Conclusion of the above Propn. (3.18) is that the algebraic direct sum in Lemma (3.3) is an orthogonal direct sum with the respect to the inner-product (2.2) on \( \pi(\mathcal{O}_2^2(\mathcal{A})) \). That is, the second algebraic direct sum in Lemma (3.12) is always an orthogonal direct sum. However, in general, the first algebraic direct sum in Lemma (3.12) fails to be an orthogonal direct sum.

**Theorem 3.19.** The Yang-Mills action functional is always subadditive.

**Proof.** Let \( \mathcal{E}_1 = p_1 \mathcal{A}_1^m \) and \( \mathcal{E}_2 = p_2 \mathcal{A}_2^m \) be Hermitian f.g.p modules over \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) respectively, where the Hermitian structures are the induced canonical structure. Then, \( \mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2 \) is a Hermitian f.g.p module over \( \mathcal{A} \) where, the Hermitian structure is the induced canonical structure. Moreover, \( \mathcal{E} \) is \( p_1 \mathcal{A}_1^m \otimes p_2 \mathcal{A}_2^m \cong (p_1 \otimes p_2) \mathcal{A}^m \). Let \( \{\sigma_1, \ldots, \sigma_m\} \) be the standard basis of \( \mathcal{A}_1^m \) as free module over \( \mathcal{A}_1 \) and \( \{\mu_1, \ldots, \mu_n\} \) be that of \( \mathcal{A}_2^m \). Then, \( \{\sigma_i \otimes \mu_j\} \) is the standard basis of \( \mathcal{A}^m \) as free module over \( \mathcal{A} \). We assume \( (\mathcal{A}_1, \mathcal{H}_1, D_1) \) is a \( k \)-summable spectral triple and \( (\mathcal{A}_2, \mathcal{H}_2, D_2) \) is a \( \ell \)-summable spectral triple. Then \( (\mathcal{A}, \mathcal{H}, D) \) is a \( (k+\ell) \)-summable spectral triple, and recall from Propn. (3.15) that \( \Theta = \Theta_1 \otimes 1 + 1 \otimes \Theta_2 \). Since, both \( \Theta_1 \otimes 1 \) and \( 1 \otimes \Theta_2 \) are \( \mathcal{A} \)-linear maps (Lemma (3.16)), i.e. they are in \( \text{Hom}_\mathcal{A}(\mathcal{E}, \mathcal{E} \otimes \mathcal{A} \Omega^2_D(\mathcal{A})) \), we get
the following (using Lemma 3.17),
\[
\sqrt{\mathcal{YM}(\nabla)} = \sqrt{\langle \Theta, \Theta \rangle} = ||\Theta_1 \otimes 1 + 1 \otimes \Theta_2|| \leq ||\Theta_1 \otimes 1|| + ||1 \otimes \Theta_2||
\]
\[
= \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} \langle \Theta_1(\sigma_i) \otimes \mu_j, \Theta_1(\sigma_i) \otimes \mu_j \rangle + \sum_{i=1}^{m} \sum_{j=1}^{n} \langle \sigma_i \otimes \Theta_2(\mu_j), \sigma_i \otimes \Theta_2(\mu_j) \rangle}
\]
\[
= \sqrt{c n \langle \Theta_1(\sigma_i), \Theta_1(\sigma_i) \rangle Tr_\omega(|D_1|^{-k}) + \sum_{j=1}^{n} cmTr_\omega(|D_2|^{-\ell}) \sqrt{\mathcal{YM}(\nabla_1)} + \sqrt{cmTr_\omega(|D_1|^{-k}) \sqrt{\mathcal{YM}(\nabla_2)}}}
\]
\[
= \sqrt{\alpha \sqrt{\mathcal{YM}(\nabla_1)} + \beta \sqrt{\mathcal{YM}(\nabla_2)}}
\]
where,
\[
\alpha = cnTr_\omega(|D_2|^{-\ell}) = \frac{n\Gamma\left(\frac{k}{2} + 1\right)\Gamma\left(\frac{\ell}{2} + 1\right)}{\Gamma\left(\frac{k + \ell}{2} + 1\right)} Tr_\omega(|D_2|^{-\ell})
\]
\[
\beta = cmTr_\omega(|D_1|^{-k}) = \frac{m\Gamma\left(\frac{k}{2} + 1\right)\Gamma\left(\frac{\ell}{2} + 1\right)}{\Gamma\left(\frac{k + \ell}{2} + 1\right)} Tr_\omega(|D_1|^{-k})
\]
are two positive real constants. This concludes the proof. \(\Box\)

**Remark 3.20.** We will reserve the notations for the constants \(\alpha\) and \(\beta\) throughout the rest of this article.

**Corollary 3.21.** The additivity is stronger condition than the subadditivity.

**Proof.** The expression for \(\Theta\) in Propn. 3.15 and additivity of the Yang-Mills implies that
\[
\mathcal{YM}(\nabla) = ||\Theta_1 \otimes 1 + 1 \otimes \Theta_2||^2 = \alpha ||\Theta_1||_{\mathcal{E}_1}^2 + \beta ||\Theta_2||_{\mathcal{E}_2}^2.
\]
Therefore,
\[
\sqrt{\mathcal{YM}(\nabla)} = \sqrt{\alpha ||\Theta_1||_{\mathcal{E}_1}^2 + \beta ||\Theta_2||_{\mathcal{E}_2}^2} \leq \sqrt{\alpha ||\Theta_1||_{\mathcal{E}_1} + \beta ||\Theta_2||_{\mathcal{E}_2}}
\]
\[
= \sqrt{\alpha \sqrt{\mathcal{YM}(\nabla_1)} + \beta \sqrt{\mathcal{YM}(\nabla_2)}}
\]
and hence, the additivity implies the subadditivity i.e. it is a stronger condition. \(\Box\)

**Proposition 3.22.** A necessary and sufficient condition for additivity of the Yang-Mills action functional is the following
\[
Re \left( \left( \sum_{i=1}^{m} Tr_\omega ((\Theta_1(\sigma_i))|D_1|^{-k}) \right) \left( \sum_{j=1}^{n} Tr_\omega ((\Theta_2(\mu_j))|D_2|^{-\ell}) \right) \right) = 0
\]
where, $Tr_\omega$ denotes the Dixmier trace. Here, $Tr_\omega\left((\Theta_1(\sigma_i))_i|D_1|-^k\right)$ means $Tr_\omega\left(P_1v_i|D_1|-^k\right)$, where $P_1$ is the orthogonal projection onto the orthogonal complement of $\pi_1(d_1J_0^1(A_1))$, and $(\Theta_1(\sigma_i))_i = [v_i] \in \Omega^2_{D_1}(A_1)$ for $v_i \in \pi_1(\Omega^2(A_1))$. Similar meaning for $Tr_\omega\left((\Theta_2(\mu_j))_j|D_2|^{-\ell}\right)$.

**Proof.** Let $E_1 = p_1A_1^n$ and $E_2 = p_2A_2^n$ be Hermitian f.g.p modules over $A_1$ and $A_2$ respectively, where the Hermitian structures are the induced canonical structure, and assume that $(A_1, H_1, D_1)$ is a $k$-summable spectral triple and $(A_2, H_2, D_2)$ is a $\ell$-summable spectral triple. From the proof of Thm. \((3.19)\), using Lemma \((3.17)\), we get

$$\mathcal{YM}(\nabla) = \alpha \mathcal{YM}(\nabla_1) + \beta \mathcal{YM}(\nabla_2)$$

$$+ \sum_{i=1}^{m} \sum_{j=1}^{n} \langle \Theta_1(\sigma_i) \otimes \mu_j , \sigma_i \otimes \Theta_2(\mu_j) \rangle + \langle \sigma_i \otimes \Theta_2(\mu_j) , \Theta_1(\sigma_i) \otimes \mu_j \rangle$$

$$= \alpha \mathcal{YM}(\nabla_1) + \beta \mathcal{YM}(\nabla_2)$$

$$+ \sum_{i=1}^{m} \sum_{j=1}^{n} 2Re \left( \langle \sigma_i, \Theta_1(\sigma_i) \rangle \langle \mu_j, \Theta_2(\mu_j) \rangle \right)$$

$$= \alpha \mathcal{YM}(\nabla_1) + \beta \mathcal{YM}(\nabla_2)$$

$$+ \sum_{i=1}^{m} \sum_{j=1}^{n} 2Re \left( Tr_\omega\left((\Theta_1(\sigma_i))_i|D_1|-^k\right) Tr_\omega\left((\Theta_2(\mu_j))_j|D_2|^{-\ell}\right) \right)$$

$$= \alpha \mathcal{YM}(\nabla_1) + \beta \mathcal{YM}(\nabla_2)$$

$$+ 2Re \left( \sum_{i=1}^{m} Tr_\omega\left((\Theta_1(\sigma_i))_i|D_1|-^k\right) \sum_{j=1}^{n} Tr_\omega\left((\Theta_2(\mu_j))_j|D_2|^{-\ell}\right) \right)$$

Here, $(\Theta_1(\sigma_i))_i$ is the $i$-th co-ordinate of $\Theta_1(\sigma_i) \in (\Omega^2_{D_1}(A_1))^m$ and $(\Theta_2(\mu_j))_j$ is the $j$-th co-ordinate of $\Theta_2(\mu_j) \in (\Omega^2_{D_2}(A_2))^n$. This is because for $j = 1, 2$, range of $\Theta_j$ lies in $p_jA_j^{k_j} \otimes A_j \Omega^2_{D_j}(A_j)$ which is contained in $(\Omega^2_{D_j}(A_j))^{k_j}$, with $k_j = m$ if $j = 1$ and $k_j = n$ if $j = 2$. The meaning of the complex number $Tr_\omega\left((\Theta_1(\sigma_i))_i|D_1|-^k\right)$ is then clear. Choose any representative $v_i \in \pi_1(\Omega^2(A_1))$ of $(\Theta_1(\sigma_i))_i \in \Omega^2_{D_1}(A_1)$. Then, $Tr_\omega\left((\Theta_1(\sigma_i))_i|D_1|-^k\right)$ means the complex number $Tr_\omega\left(P_1v_i|D_1|-^k\right)$, where $P_1$ is the orthogonal projection onto the orthogonal complement of $\pi_1(d_1J_0^1(A_1))$. Similar meaning for $Tr_\omega\left((\Theta_2(\mu_j))_j|D_2|^{-\ell}\right)$. Let $\xi, \eta$ denote the following complex numbers

$$\xi = \sum_{i=1}^{m} Tr_\omega\left((\Theta_1(\sigma_i))_i|D_1|-^k\right),$$

$$\eta = \sum_{j=1}^{n} Tr_\omega\left((\Theta_2(\mu_j))_j|D_2|^{-\ell}\right).$$
Hence, $\text{Re}(\xi\eta) = 0$ is a necessary and sufficient condition for additivity of the Yang-Mills functional, and this condition depends only on the individual spectral triples.

An instance of additivity of the Yang-Mills functional is shown in the next section. If the Yang-Mills functional becomes additive then it is natural to ask when critical points (Def. 2.11) on the individual spectral triples give rise to a critical point on the product spectral triple. We obtain a necessary and sufficient condition for this.

**Proposition 3.23.** A necessary condition for $\nabla$ to be a critical point for the Yang-Mills functional under additivity is that both $\nabla_1, \nabla_2$ must be critical points for the Yang-Mills functional on the individual spectral triple.

**Proof.** Choose $\mu_j \in \text{Hom}_{\mathcal{A}_j}(\mathcal{E}_j, \mathcal{E}_j \otimes \mathcal{A}_j \Omega_{\mathcal{D}_j}^1(\mathcal{A}_j))$ for $j = 1, 2$. Define $\mu = \mu_1 \otimes 1 + 1 \otimes \mu_2$. Then $\mu \in \text{Hom}_\mathcal{A}(\mathcal{E}, \mathcal{E} \otimes \mathcal{A} \Omega_{\mathcal{D}}^1(\mathcal{A}))$. If $\nabla$ is a critical point for the Yang-Mills functional then, we have

\[
0 = \frac{d}{dt} |_{t=0} \mathcal{Y}\mathcal{M}(\nabla + t\mu) = \frac{d}{dt} |_{t=0} \mathcal{Y}\mathcal{M}(\nabla_1 \otimes 1 + 1 \otimes \nabla_2 + t(\mu_1 \otimes 1 + 1 \otimes \mu_2)) = \frac{d}{dt} |_{t=0} \mathcal{Y}\mathcal{M}((\nabla_1 \otimes 1 + t\mu_1 \otimes 1) + (1 \otimes \nabla_2 + t \otimes \mu_2)) = \frac{d}{dt} |_{t=0} \mathcal{Y}\mathcal{M}((\nabla_1 + t\mu_1) \otimes 1 + 1 \otimes (\nabla_2 + t\mu_2)) = \alpha \frac{d}{dt} |_{t=0} \mathcal{Y}\mathcal{M}(\nabla_1 + t\mu_1) + \beta \frac{d}{dt} |_{t=0} \mathcal{Y}\mathcal{M}(\nabla_2 + t\mu_2)
\]

(by Th. 3.19). Since, the range of $\mathcal{Y}\mathcal{M}$ is $\mathbb{R}_{\geq 0}$ and $\alpha, \beta$ are positive real constants, we see that $\nabla_1, \nabla_2$ both are critical points for the Yang-Mills functional.

Recall from Def. 2.11 that a connection $\nabla$ on a Hermitian f.g.p module $\mathcal{E}$ is a critical point for the Yang-Mills functional if and only if $\langle (\nabla, \mu), \Theta \rangle = 0, \forall \mu \in \text{Hom}_\mathcal{A}(\mathcal{E}, \mathcal{E} \otimes \mathcal{A} \Omega_{\mathcal{D}}^1(\mathcal{A}))$. Here, $[\nabla, \mu] = \nabla \circ \mu + (1 \otimes \Pi) \circ (\mu \otimes 1) \circ \nabla$. In our situation, $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$ and $\mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2$. Using Lemma 3.11, any $\mu \in \text{Hom}_\mathcal{A}(\mathcal{E}, \mathcal{E} \otimes \mathcal{A} \Omega_{\mathcal{D}}^1(\mathcal{A}))$ can be written as $\mu = \mu_1 \otimes \mu_2$ where,

\[
\mu_1 = \text{Pr}_1 \circ \mu : \mathcal{E}_1 \otimes \mathcal{E}_2 \to (\mathcal{E}_1 \otimes \mathcal{A}_1 \Omega_{\mathcal{D}_1}^1(\mathcal{A}_1)) \otimes \mathcal{E}_2,
\]

\[
\mu_2 = \text{Pr}_2 \circ \mu : \mathcal{E}_1 \otimes \mathcal{E}_2 \to \mathcal{E}_1 \otimes (\mathcal{E}_2 \otimes \mathcal{A}_2 \Omega_{\mathcal{D}_2}^1(\mathcal{A}_2)).
\]

For $\nabla_j \in C(\mathcal{E}_j), j = 1, 2$, we define

\[
[\nabla_1 \otimes 1, \mu_1] := (\nabla_1 \otimes 1) \circ \mu_1 + (1 \otimes \Pi) \circ (\mu_1 \otimes 1) \circ (\nabla_1 \otimes 1),
\]

\[
[1 \otimes \nabla_2, \mu_2] := (1 \otimes \nabla_2) \circ \mu_2 + (1 \otimes \Pi) \circ (\mu_2 \otimes 1) \circ (1 \otimes \nabla_2).
\]

**Lemma 3.24.** We have

\[
[\nabla_1 \otimes 1, \mu_1] \in \text{Hom}_\mathcal{A}(\mathcal{E}_1 \otimes \mathcal{E}_2, (\mathcal{E}_1 \otimes \mathcal{A}_1 \Omega_{\mathcal{D}_1}^2(\mathcal{A}_1)) \otimes \mathcal{A}_2 \mathcal{E}_2),
\]

\[
[1 \otimes \nabla_2, \mu_2] \in \text{Hom}_\mathcal{A}(\mathcal{E}_1 \otimes \mathcal{E}_2, \mathcal{E}_1 \otimes \mathcal{A}_1 \mathcal{E}_2 \otimes \mathcal{A}_2 \Omega_{\mathcal{D}_2}^2(\mathcal{A}_2)).
\]
i.e. both are elements of $\mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^2_D(\mathcal{A}))$.

**Proof.** For $\xi_1 \in \mathcal{E}_1$ and $\xi_2 \in \mathcal{E}_2$, let

$$
\mu_1(\xi_1 \otimes \xi_2) = \sum_j \xi_{1j} \otimes \omega_{1j} \otimes \xi_{2j} \in (\mathcal{E}_1 \otimes_{\mathcal{A}} \Omega^1_{D_j}) \otimes \mathcal{E}_2
$$

$$
\nabla_1(\xi_1) = \sum_j \tilde{\xi}_{1j} \otimes \tilde{\omega}_{1j} \in \mathcal{E}_1 \otimes_{\mathcal{A}} \Omega^1_{D_j}.
$$

Then,

$$
(\tilde{\nabla}_1 \otimes 1) \circ \mu_1(\xi_1 \otimes \xi_2)
= \sum_j (\tilde{\nabla}_1 \otimes 1)((\xi_{1j} \otimes \omega_{1j} \otimes \xi_{2j})(a \otimes b))
= \sum_j (\tilde{\nabla}_1 \otimes 1)(\xi_{1j} \otimes \omega_{1j}a \otimes \xi_{2j}b)
= \sum_j (\nabla_1(\xi_{1j})\omega_{1j}a + \xi_{1j} \otimes d_1(\omega_{1j}a)) \otimes \xi_{2j}b
= \sum_j (\nabla_1(\xi_{1j})\omega_{1j} \otimes \xi_{2j})(a \otimes b) + ((\xi_{1j} \otimes \xi_{2j}) \otimes (d_1(\omega_{1j}) \otimes 1))(a \otimes b)
- ((\xi_{1j} \otimes \xi_{2j}) \otimes (\omega_{1j}d_1a \otimes 1))(1 \otimes b)
$$

and

$$
(1 \otimes \Pi) \circ (\mu_1 \otimes 1) \circ (\nabla_1 \otimes 1)(\xi_1 a \otimes \xi_2 b)
= (1 \otimes \Pi) \circ (\mu_1 \otimes 1)((\nabla_1(\xi_1)a + \xi_1 \otimes d_1a) \otimes \xi_2b)
= (1 \otimes \Pi) \circ (\mu_1 \otimes 1) \left( \left( \sum_j (\tilde{\xi}_{1j} \otimes \xi_2) \otimes (\tilde{\omega}_{1j} \otimes 1) \right)(a \otimes b) + ((\xi_1 \otimes \xi_2) \otimes (d_1a \otimes 1))(1 \otimes b) \right)
= \sum_j (1 \otimes \Pi) \circ (\mu_1 \otimes 1) \left( \left( \tilde{\xi}_{1j} \otimes \xi_2 \otimes (\tilde{\omega}_{1j} \otimes 1) \right)(a \otimes b) + (1 \otimes \Pi)((\xi_{1j} \otimes \omega_{1j} \otimes \xi_{2j}) \otimes d_1a)(1 \otimes b) \right)
= \sum_j (1 \otimes \Pi) \circ (\mu_1 \otimes 1) \left( \left( \tilde{\xi}_{1j} \otimes \xi_2 \otimes (\tilde{\omega}_{1j} \otimes 1) \right)(a \otimes b) + ((\xi_{1j} \otimes \xi_{2j}) \otimes (\omega_{1j}d_1a \otimes 1))(1 \otimes b) \right)
$$

Adding these two we see that $[\nabla_1 \otimes 1, \mu_1]$ is $\mathcal{A}$-linear. Similarly, one can show for $[1 \otimes \nabla_2, \mu_2]$. $\square$

**Proposition 3.25.** Given two spectral triples $(\mathcal{A}_j, \mathcal{H}_j, D_j, \gamma_j)$, $j = 1, 2$, and Hermitian f.g.p modules $\mathcal{E}_j$ over $\mathcal{A}_j$, if $\nabla_j \in C(\mathcal{E}_j)$ satisfy the following equation

$$
\langle [\nabla_1 \otimes 1, \mu_1], \Theta_1 \otimes 1 \rangle + \langle [1 \otimes \nabla_2, \mu_2], 1 \otimes \Theta_2 \rangle = 0
$$

for all $\mu \in \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^2_D(\mathcal{A}))$, then $\nabla$ is a critical point for the Yang-Mills functional on product spectral triple. Moreover, the converse is also true.
Proof. We have \( \nabla = \nabla_1 \otimes 1 + 1 \otimes \nabla_2 \) and \( \Theta = \Theta_1 \otimes 1 + 1 \otimes \Theta_2 \) by Propon. \((3.3, 3.15)\). Now, for the standard basis element \( \sigma_i \otimes \tau_j, i = 1, \ldots , m, j = 1, \ldots , n \), of the free \( \mathcal{A} \)-module \( (\mathcal{A}_1 \otimes \mathcal{A}_2)^{m \times n} = \mathcal{A}_1^m \otimes \mathcal{A}_2^n \), suppose that
\[
\mu(\sigma_i \otimes \tau_j) = \sum_k \xi_{ijk} \otimes \omega_{ijk} \in \mathcal{E} \otimes_{\mathcal{A}} \Omega_D^1(\mathcal{A}), \\
\nabla(\sigma_i \otimes \tau_j) = \sum_k \eta_{ijk} \otimes v_{ijk} \in \mathcal{E} \otimes_{\mathcal{A}} \Omega_D^1(\mathcal{A}).
\]
Then,
\[
[\nabla, \mu](\sigma_i \otimes \tau_j)
= \sum_k \nabla(\xi_{ijk})\omega_{ijk} + \xi_{ijk} \otimes d\omega_{ijk} + \mu(\eta_{ijk})v_{ijk}
= \sum_k (\nabla_1(\xi_{ijk1}) \otimes \xi_{ijk2} + \xi_{ijk1} \otimes \nabla_2(\xi_{ijk2}))\omega_{ijk} + \xi_{ijk} \otimes d\omega_{ijk} + \mu(\eta_{ijk})v_{ijk}
= \sum_k \nabla_1(\xi_{ijk1})\omega_{ijk1} \otimes \xi_{ijk2}a_2 - \nabla_1(\xi_{ijk1})(a_1 \otimes \omega_{ijk2}) \otimes \xi_{ijk2} + \xi_{ijk1} \otimes \nabla_2(\xi_{ijk2})(\omega_{ijk1} \otimes a_2)
+ \xi_{ijk1}a_1 \otimes \nabla_2(\xi_{ijk2})\omega_{ijk2} + \xi_{ijk1} \otimes \xi_{ijk2}a_2 \otimes d_1\omega_{ijk1} + \xi_{ijk1}a_1 \otimes \xi_{ijk2} \otimes d_2\omega_{ijk2}
+ \xi_{ijk1} \otimes \xi_{ijk2} \otimes (\omega_{ijk1} \otimes d_2a_2 - d_1a_1 \otimes \omega_{ijk2}) + \mu(\eta_{ijk})v_{ijk}
\]
as an element of \( \mathcal{E} \otimes_{\mathcal{A}} \Omega_D^2(\mathcal{A}) \), by using Lemma \((3.13, 3.14)\). So, for \( \Theta = \Theta_1 \otimes 1 + 1 \otimes \Theta_2 \), we have the following,
\[
\langle \langle [\nabla, \mu], \Theta \rangle \rangle
= \sum_{i=1}^m \sum_{j=1}^n \langle [\nabla, \mu](\sigma_i \otimes \tau_j), \Theta(\sigma_i \otimes \tau_j) \rangle
= \sum_{i=1}^m \sum_{j=1}^n \langle [\nabla, \mu](\sigma_i \otimes \tau_j), \Theta_1(\sigma_i \otimes \tau_j) \rangle + \langle [\nabla, \mu](\sigma_i \otimes \tau_j), \sigma_i \otimes \Theta_2(\tau_j) \rangle
= \sum_{i=1}^m \sum_{j=1}^n \sum_k (\nabla_1(\xi_{ijk1})\omega_{ijk1} + \xi_{ijk1} \otimes d_1\omega_{ijk1}) \otimes \xi_{ijk2}a_2, \Theta_1(\sigma_i \otimes \tau_j)
+ \langle \mu(\eta_{ijk})v_{ijk}, \Theta_1(\sigma_i \otimes \tau_j) \rangle + \langle \xi_{ijk1}a_1 \otimes (\nabla_2(\xi_{ijk2})\omega_{ijk2} + \xi_{ijk2} \otimes d_2\omega_{ijk2}), \sigma_i \otimes \Theta_2(\tau_j) \rangle
= \sum_{i=1}^m \sum_{j=1}^n \langle ((\widetilde{\nabla}_1 \otimes 1) \circ P r_1 \circ \mu)(\sigma_i \otimes \tau_j), \Theta_1(\sigma_i \otimes \tau_j) \rangle + \langle \mu(\eta_{ijk})v_{ijk}, \Theta_1(\sigma_i \otimes \tau_j) \rangle
\]
as an element of \( \mathcal{E} \otimes_{\mathcal{A}} \Omega_D^2(\mathcal{A}) \), by using Lemma \((3.13, 3.14)\).
where, \( \mu_1 = Pr_1 \circ \mu \) and \( \mu_2 = Pr_2 \circ \mu \). Now,
\[
\langle \langle (1 \otimes \Pi) \circ (\mu \otimes 1) \circ \nabla, \Theta \rangle \rangle
= \sum_{i,j} \langle \langle (1 \otimes \Pi) \circ (\mu \otimes 1) \circ (\nabla(\sigma_i \otimes \tau_j), \Theta(\sigma_i \otimes \tau_j)) \rangle \rangle
= \sum_{i,j} \langle \langle (1 \otimes \Pi) \circ (\mu \otimes 1) \circ (\nabla_1(\sigma_i) \otimes \tau_j + \sigma_i \otimes \nabla_2(\tau_j)), \Theta(\sigma_i) \otimes \tau_j + \sigma_i \otimes \Theta_2(\tau_j) \rangle \rangle
= \sum_{i,j} \langle \langle (1 \otimes \Pi) \circ \mu_1(\nabla_1(\sigma_i) \otimes \tau_j), \Theta_1(\sigma_i) \otimes \tau_j \rangle \rangle + \langle \langle (1 \otimes \Pi) \circ \mu_2(\sigma_i \otimes \nabla_2(\tau_j)), \sigma_i \otimes \Theta_2(\tau_j) \rangle \rangle
= \langle \langle (1 \otimes \Pi) \circ \mu_1 \circ (\nabla_1 \otimes 1), \Theta_1 \otimes 1 \rangle \rangle + \langle \langle (1 \otimes \Pi) \circ \mu_2 \circ (1 \otimes \nabla_2), 1 \otimes \Theta_2 \rangle \rangle
\]

Hence, we have
\[
\langle \langle \nabla, \mu \rangle \rangle = \langle \langle \nabla_1 \otimes 1, \mu_1 \rangle \rangle + \langle \langle 1 \otimes \nabla_2, \mu_2 \rangle \rangle + \langle \langle 1 \otimes \Theta_2 \rangle \rangle
\]
and this concludes the proof. \( \square \)

Combining Propn. (3.23) and (3.25) we conclude the following final theorem.

**Theorem 3.26.** If the Yang-Mills functional is additive then a necessary and sufficient condition for \( \nabla \) to be a critical point for the Yang-Mills functional on the product spectral triple is that both \( \nabla_1, \nabla_2 \) are critical points for the Yang-Mills functional on the individual spectral triple, and they satisfy the following equation
\[
\langle \langle \nabla_1 \otimes 1, \mu_1 \rangle \rangle + \langle \langle 1 \otimes \nabla_2, \mu_2 \rangle \rangle = 0
\]
for all \( \mu \in \text{Hom}_A(\mathcal{E}, \mathcal{E} \otimes \Omega^1_D(\mathcal{A})) \), where \( \mu_1 = Pr_1 \circ \mu \) and \( \mu_2 = Pr_2 \circ \mu \).

### 4. An instance of additivity: The case of noncommutative tori

In this section we provide an instance of additivity of the Yang-Mills functional for the case of noncommutative tori.

**Definition 4.1.** Let \( \Theta \in M_n(\mathbb{R}) \) be any \( n \times n \) real skew-symmetric matrix. Denote by \( A_\Theta \) the universal \( C^* \)-algebra generated by \( n \) unitaries \( U_1, \ldots, U_n \) satisfying
\[
U_k U_m = e^{2\pi i \Theta_{mk}} U_m U_k
\]
for \( k, m \in \{1, \ldots, n\} \).

**Action of the Lie group** \( T^n \): On \( A_\Theta \), the compact connected Lie group \( T^n \) acts as follows:
\[
\alpha_{(z_1, \ldots, z_n)}(U_k) = z_k U_k, \quad k = 1, \ldots, n.
\]

**The smooth subalgebra** \( A_\Theta^\infty \): The smooth subalgebra of \( A_\Theta \) under this action is given by
\[
A_\Theta^\infty := \left\{ \sum a_r U^r : \{a_r\} \in S(\mathbb{Z}^n), \quad r = (r_1, \ldots, r_n) \in \mathbb{Z}^n \right\},
\]
where, $\mathbb{S}(\mathbb{Z}^n)$ denotes vector space of multisequences $(a_r)$ that decay faster than inverse of any polynomial in $r = (r_1, \ldots, r_n)$. This is a unital subalgebra of $\mathcal{A}_\Theta$ stable under the holomorphic functional calculus (16), and called the noncommutative $n$-torus.

**The Trace:** The subalgebra $\mathcal{A}_\Theta^\infty$ is equipped with a unique $\mathbb{T}^n$-invariant tracial state given by $\tau(a) = a_0$, where $0 = (0, \ldots, 0) \in \mathbb{Z}^n$.

**The G.N.S. Hilbert space:** The Hilbert space $L^2(\mathcal{A}_\Theta^\infty, \tau)$ obtained by applying the G.N.S. construction to $\tau$ can be identified with $\ell^2(\mathbb{Z}^n)$ (22).

**The spectral triple:** Consider the irreducible representation of $\mathcal{C}(\mathcal{H}(n))$ on $\mathbb{C}^N$, where $N = 2^{[n/2]}$. Then, there are $n$ many Clifford gamma matrices $\gamma_1, \ldots, \gamma_n$ in $M_N(\mathbb{C})$ satisfying $\gamma_r \gamma_s + \gamma_s \gamma_r = 2\delta_{rs}$, $r, s \in \{1, \ldots, n\}$, where $\delta_{rs}$ denotes the Kronecker delta function. Consider the densely defined unbounded symmetric operator $D_\Theta := \sum_{j=1}^n \delta_j \otimes \gamma_j$ where,

$$
\delta_j (\sum_r a_r U^r) := \sum_r 2\pi i r_j a_r U^r.
$$

It is known that $D_\Theta$ is self-adjoint with compact resolvent, acting on $\mathcal{H}_\Theta = L^2(\mathcal{A}_\Theta^\infty, \tau) \otimes \mathbb{C}^N$, $N = 2^{[n/2]}$. Moreover, $|D_\Theta|^{-n}$ lies in the Dixmier ideal $\mathcal{L}^{(1, \infty)}$ with $Tr_\omega(|D_\Theta|^{-n}) = 2 N \pi^{n/2}/(n(2\pi)^n \Gamma(n/2))$ (see [16], Page 545). The tuple $(\mathcal{A}_\Theta^\infty, \mathcal{H}_\Theta, D_\Theta)$ gives us a $n$-summable spectral triple on $\mathcal{A}_\Theta^\infty$. If $n$ is even then this is an even spectral triple and the grading operator comes from the irreducible representation of $\mathcal{C}(\mathcal{H}(n))$ on $\mathbb{C}^N$.

We will be working with $\mathcal{A}_\Theta^\infty$ and denote it simply by $\mathcal{A}_\Theta$ for notational brevity. Consider the product $\mathcal{A}_\Theta \otimes \mathcal{A}_\Phi$, where $\mathcal{A}_\Theta$ is a noncommutative $n$-torus and $\mathcal{A}_\Phi$ is a noncommutative $m$-torus. It is known that (Proposition 5.1 and 5.3 in [3]),

$$
\Omega_{D_\Theta}^1(\mathcal{A}_\Theta) = A_0^\Theta , \quad \Omega_{D_\Phi}^1(\mathcal{A}_\Phi) = A_0^\Phi ,
$$

$$
\pi_1(\Omega^2(\mathcal{A}_\Theta)) = A_0^{1+n(n-1)/2} , \quad \pi_2(\Omega^2(\mathcal{A}_\Phi)) = A_0^{1+m(m-1)/2} ,
$$

$$
\pi_1(d_1 J_0^1(\mathcal{A}_\Theta)) = A_\Theta , \quad \pi_2(d_2 J_0^1(\mathcal{A}_\Phi)) = A_\Phi ,
$$

$$
\Omega_{D_\Theta}^2(\mathcal{A}_\Theta) = A_0^{n(n-1)/2} , \quad \Omega_{D_\Phi}^2(\mathcal{A}_\Phi) = A_0^{m(m-1)/2} .
$$

If $(\mathcal{A}_\Theta, \mathcal{H}_\Theta, D_\Theta)$ is not an even spectral triple (unless $n$ is even) we apply the process described in point [2] in Section (2) to make it even with grading operator $\gamma$. Let $D = D_\Theta \otimes 1 + \gamma \otimes D_\Phi$. Intuitively, one can guess that the Yang-Mills functional is going to be additive in this case. The reason is that $\mathcal{A}_\Theta \otimes \mathcal{A}_\Phi$ can be identified with a noncommutative $(n+m)$-torus $\mathcal{A}_\Psi$ for an obvious choice of $\Psi$, and $D$ becomes $D_\Psi$ acting on $\mathcal{H}_\Psi = L^2(\mathbb{Z}^{n+m}) \otimes \mathbb{C}_2^{(n+m)/2}$. Hence, both $\Omega_{D_\Theta}^1(\mathcal{A}_\Psi)$ and $\Omega_{D_\Phi}^2(\mathcal{A}_\Psi)$ are free modules of rank $(n+m)$ and $(n+m)(n+m-1)/2$ respectively. So, the Yang-Mills functional on a Hermitian f.g.p module $\mathcal{E} = p A_\Psi^q$, with $p \in M_q(\mathcal{A}_\Psi)$ a projection, is given by

$$
\mathcal{Y} \mathcal{M}(\nabla) = \sum_{1 \leq i < j \leq n+m} \tau_q([\nabla_i, \nabla_j]^* [\nabla_i, \nabla_j])
$$

where, $\tau_q$ denotes the extended trace $\tau \otimes Trace$ on $M_q(\mathcal{A}_\Psi)$ (see Proposition 5.12 in [3] for detail). This expression actually proves the additivity of the Yang-Mills functional but we go through little detail to see why our hypothesis in Section (3), or equivalently Lemma (3.12), is justified in this case.
Proposition 4.2. $\Omega^2_D(A_\Theta) \cong \Omega^2_{D_{D_\Theta}}(A_\Theta) \otimes A_\Phi \otimes A_\Theta \otimes \Omega^2_{D_{D_\Phi}}(A_\Phi) \oplus \Omega^1_{D_{D_\Theta}}(A_\Theta) \otimes \Omega^1_{D_{D_\Phi}}(A_\Phi)$ as $A_\Psi = A_\Theta \otimes A_\Phi$-bimodules.

Proof. One can conclude this by comparing the free module (over $A_\Psi$) dimensions of both sides. Since, $A_\Psi$ is a noncommutative $(n + m)$-torus, $\Omega^2_D(A_\Psi)$ has dimension $(n + m)(n + m - 1)/2$ as free module over $A_\Psi$ (Proposition 5.3 in [3]). The dimension of $\Omega^2_{D_{D_\Theta}}(A_\Theta) \otimes \Omega^1_{D_{D_\Phi}}(A_\Phi)$ is $nm$ as free module over $A_\Psi$ (Proposition 5.1 in [3]). Therefore, by Lemma (5.3) we see that

$$\frac{\pi(\Omega^2(A_\Theta)) \otimes A_\Phi + A_\Theta \otimes \pi(\Omega^2(A_\Phi))}{\pi(dJ^1_1(A_\Psi))} \leq \frac{\pi(\Omega^2(A_\Psi))}{\pi(dJ^1_1(A_\Psi))} = \Omega^2_D(A_\Psi)$$

must be a free module with dimension $(n + m)(n + m - 1)/2 - nm = n(n - 1)/2 + m(m - 1)/2$. Since, $\Omega^2_{D_{D_\Theta}}(A_\Theta) \otimes A_\Phi \otimes A_\Theta \otimes \Omega^2_{D_{D_\Phi}}(A_\Phi)$ is also a free module of dimension $n(n - 1)/2 + m(m - 1)/2$, we have a canonical isomorphism

$$\frac{\pi(\Omega^2(A_\Theta)) \otimes A_\Phi + A_\Theta \otimes \pi(\Omega^2(A_\Phi))}{\pi(dJ^1_1(A_\Psi))} \cong \Omega^2_{D_{D_\Theta}}(A_\Theta) \otimes A_\Phi \otimes A_\Theta \otimes \Omega^2_{D_{D_\Phi}}(A_\Phi)$$

of $A_\Psi$-bimodules, and this concludes the proof. \qed

Remark 4.3. The proof of above Lemma explains Remark (4.5) in Section (3). We see that $\pi(dJ^1_1(A_\Psi))$ is a free module over $A_\Psi$ of rank 1, whereas $\pi_1(dJ^1_1(A_\Theta)) \otimes A_\Phi \oplus A_\Theta \otimes \pi_2(dJ^1_1(A_\Phi))$ is a free module over $A_\Psi$ of rank 2.

Proposition 4.4. The Yang-Mills functional is additive in this case.

Proof. Let $E_1 = p_1 A_{\Theta}^{q_n}$ and $E_2 = p_2 A_{\Phi}^{q_m}$ be two Hermitian f.g.p modules over $A_\Theta$ ($n$ torus) and $A_\Phi$ ($m$ torus) respectively. Let $\nabla_1 \in C(E_1)$ and $\nabla_2 \in C(E_2)$ be two compatible connections. Since, $\Omega^1_{D_{D_\Theta}}(A_\Theta)$ and $\Omega^1_{D_{D_\Phi}}(A_\Phi)$ both are free modules of rank $n$ and $m$ respectively, we have

$$\nabla_1(\xi) = \sum_{j=1}^{n} \nabla_{1j}(\xi) \otimes \sigma_j$$

$$\nabla_2(\eta) = \sum_{k=1}^{m} \nabla_{2k}(\eta) \otimes \mu_k$$

for $C$-linear maps $\nabla_{1j} : E_1 \rightarrow E_1$ and $\nabla_{2k} : E_2 \rightarrow E_2$. Here, $\{\sigma_1, \ldots, \sigma_n\}$ is the standard basis of the free module $\Omega^1_{D_{D_\Theta}}(A_\Theta)$ over $A_\Theta$ and $\{\mu_1, \ldots, \mu_m\}$ is that of $\Omega^1_{D_{D_\Phi}}(A_\Phi)$ over $A_\Phi$. It is known that the Yang-Mills functional on $E_1 = p_1 A_{\Theta}^{q_n}$ is given by

$$\mathcal{YM}(\nabla_1) = \sum_{1 \leq i < j \leq n} \tau_{q_n}(\nabla_{1i}, \nabla_{1j}^* \nabla_{1i}, \nabla_{1j})$$

(4.13)

where, $\tau_{q_n}$ denotes the trace $\tau_\Theta \otimes \text{Trace}$ on $M_{q_n}(A_\Theta)$ (see Proposition 5.12 in [3]), and the Yang-Mills functional on $E_2 = p_2 A_{\Phi}^{q_m}$ is given by

$$\mathcal{YM}(\nabla_2) = \sum_{1 \leq i < j \leq m} \tau_{q_m}(\nabla_{2i}, \nabla_{2j}^* \nabla_{2i}, \nabla_{2j})$$

(4.14)
where, \( \tau_{q,m} \) denotes the trace \( \tau_\Phi \otimes \text{Trace} \) on \( M_{q,m}(A_\Phi) \). Now, \( \mathcal{E} = \mathcal{E}_1 \otimes \mathcal{E}_2 = (p_1 \otimes p_2)A_\Phi^{q,m} \) and the Yang-Mills functional on \( \mathcal{E} \) is given by

\[
\mathcal{Y} \mathcal{M}(\nabla) = \sum_{1 \leq i < j \leq m+n} \tau([\nabla_i, \nabla_j]^* [\nabla_i, \nabla_j])
\]

where, \( \nabla_k : \mathcal{E} \rightarrow \mathcal{E} \) are \( \mathbb{C} \)-linear maps and \( \tau \) denotes the trace \( \tau_\Phi \otimes \text{Trace} \) on \( M_{q,m}(A_\Phi) \).

Since, \( \nabla = \nabla_1 \otimes 1 + 1 \otimes \nabla_2 \) we have

\[
\nabla_k(e_1 \otimes e_2) = \nabla_{1k}(e_1) \otimes e_2, \quad 1 \leq k \leq n \\
\nabla_k(e_1 \otimes e_2) = e_1 \otimes \nabla_{2,k-n}(e_2), \quad n + 1 \leq k \leq n + m
\]

Then,

\[
\sum_{1 \leq i < j \leq m+n} [\nabla_i, \nabla_j]^* [\nabla_i, \nabla_j] \\
= \sum_{1 \leq i < j \leq n} [\nabla_{1i}, \nabla_{1j}]^* [\nabla_{1i}, \nabla_{1j}] + \sum_{1 \leq i < j \leq m} 1 \otimes [\nabla_{2i}, \nabla_{2j}]^* [\nabla_{2i}, \nabla_{2j}].
\]

In view of equations (4.13), (4.14), (4.15), and because \( \tau = \tau_\Phi \otimes \text{Trace} \), we can now conclude that the Yang-Mills functional is additive in this case, i.e. the condition described in Propn. (3.22) is satisfied.

\[
\square
\]

5. THE CASE OF SPIN MANIFOLDS AND MATRIX ALGEBRAS

Let \( \mathcal{M} \) be an even dimensional closed Riemannian spin manifold and \( A_1 = C^\infty(\mathcal{M}) \) be the algebra of smooth functions. It is known that \( C^\infty(\mathcal{M}) \) is spectrally invariant in the unital \( C^* \)-algebra \( C(\mathcal{M}) \) (16). Let \( A_2 \) be a matrix algebra. Consider \( A = A_1 \otimes A_2 \). This algebra is a generalization of the product system “four dimensional manifold \( \times \) 2-point space” considered in (9). This is the algebra appearing in many examples in Physics. Let \( \pi_1 : C^\infty(\mathcal{M}) \rightarrow B(\mathcal{H}_1 = L^2(S)) \) be the representation of smooth functions on the square-integrable spinors, and \( \pi_2 : A_2 \rightarrow B(\mathbb{C}^n) \) be a faithful representation of the matrix algebra \( A_2 \) on \( \mathbb{C}^n \) for some suitable \( n \). Let \( D_1 = i\partial_\mu \gamma^\mu \) be the Dirac operator associated to the spin manifold \( \mathcal{M} \), and \( D_2 \) be a \( n \times n \) self-adjoint matrix. Let \( \gamma \) denotes the grading automorphism of the Clifford algebra associated to \( \mathcal{M} \) (\( \gamma := i^{d/2}\gamma_1 \ldots \gamma_d \)). We have two even spectral triples \( (C^\infty(\mathcal{M}), \mathcal{H}_1, D_1, \gamma) \) and \( (A_2, \mathbb{C}^n, D_2, \gamma) \). Consider \( D = D_1 \otimes 1 + \gamma \otimes D_2 \). It is known that for all \( k \geq 2 \),

\[
\pi_1(\Omega^k(A_1)) \supseteq \pi_1(J^k(A_1)) = \pi_1(\Omega^{k-2}(A_1))
\]

and for all \( k \geq 1, \Omega^k_{D_1} \cong \Gamma(\mathcal{M}, \wedge^k T^* \mathcal{M}) \) (9,19).

Now, for two even spectral triples \( (A_j, \mathcal{H}_j, D_j, \gamma_j), \) \( j = 1, 2 \), there is an isomorphism of dgas between the Dirac dga \( \Omega^*_{D}(A_1 \otimes A_2) \) and the skew dga \( \tilde{\Omega}^*_{D}(A_1, A_2) \) (see 17 for detail). That is,

\[
\Omega^*_{D}(A_1 \otimes A_2) \cong \tilde{\Omega}^*_{D}(A_1, A_2) \forall n \geq 0,
\]
where the definition of $\tilde{\Omega}^\bullet_D(A_1, A_2)$ is given below.

**Definition 5.1.** Consider the reduced universal dgas $(\Omega^\bullet(A_1), d_1)$ and $(\Omega^\bullet(A_2), d_2)$ associated with the spectral triples $(A_1, \mathcal{H}_1, D_1, \gamma_1)$ and $(A_2, \mathcal{H}_2, D_2, \gamma_2)$ respectively. Now, consider the product dga $\left(\Omega^\bullet(A_1) \otimes \Omega^\bullet(A_2), \tilde{d}\right)$ where,

$$(\omega_i \otimes u_j). (\omega_p \otimes u_q) := (-1)^{jp} \omega_i \omega_p \otimes u_j u_q$$

$${\tilde{d}}(\omega_i \otimes u_j) := d_1(\omega_i) \otimes u_j + (-1)^i \omega_i \otimes d_2(u_j)$$

for $\omega_i \in \Omega^\bullet(A_1)$ and $u_j \in \Omega^\bullet(A_2)$. One can represent $\Omega^\bullet(A_1) \otimes \Omega^\bullet(A_2)$ on $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ by the following map

$${\tilde{\pi}}(\omega_i \otimes u_j) := \pi_1(\omega_i) \gamma_2^j \otimes \pi_2(u_j) .$$

Let

$$\tilde{J}^k_0 := \text{Ker} \left\{ {\tilde{\pi}} : \bigoplus_{i+j=k} \Omega^i(A_1) \otimes \Omega^j(A_2) \longrightarrow \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) \right\} ,$$

and $\tilde{J}^n = \tilde{J}^0_n + \tilde{d} \tilde{J}^{n-1}$. Define $\tilde{\Omega}^\bullet_D(A_1, A_2) := \frac{\bigoplus_{i+j=n} \Omega^i(A_1) \otimes \Omega^j(A_2)}{\bigoplus_{i+j=n} \tilde{J}^i(A_1, A_2)}$, $\forall n \geq 0$. We call it the skew dga.

One has to compute $\tilde{\Omega}^\bullet_D(A_2)$ first. Recall from (§ 4) in [17] that there are three cases. Let $A_2$ be given as the direct sum of the algebras $A_{2,1} = M_{\mu}(\mathbb{C})$ and $A_{2,2} = M_{q}(\mathbb{C})$. The representation and the Dirac operator takes the form

$$\pi_2(A_2) = \begin{pmatrix} A_{2,1} & 0 \\ 0 & A_{2,2} \end{pmatrix} , \quad D_2 = \begin{pmatrix} \mu & 0 \\ 0 & \mu^* \end{pmatrix}$$

where $\mu$ denotes an arbitrary (non-zero) complex $p \times q$ matrix. Then, one has the following three cases.

**Case 1 :** $\mu^* \mu \sim 1_{q \times q}$ and $\mu \mu^* \sim 1_{p \times p}$, which is possible only for $p = q$. In this case $A_{2,1} = A_{2,2}$ and

$$\tilde{\Omega}^{2k}_D(A_2) = \begin{pmatrix} A_{2,1} & 0 \\ 0 & A_{2,1} \end{pmatrix} , \quad \tilde{\Omega}^{2k+1}_D(A_2) = \begin{pmatrix} 0 & A_{2,1} \\ A_{2,1} & 0 \end{pmatrix}$$

The multiplication rule is just the ordinary matrix multiplication of $2p \times 2p$ matrices.

**Case 2 :** $\mu^* \mu \sim 1_{q \times q}$ and $\mu \mu^* \sim 1_{p \times p}$. In this case

$$\tilde{\Omega}^1_D(A_2) = \{ X \in M_{q \times p}(\mathbb{C}) \} \bigoplus \{ Y \in M_{p \times q}(\mathbb{C}) \}$$

and there is no non-trivial multiplication of elements in $\tilde{\Omega}^1_D$.

**Case 3 :** $q \leq p$, $\mu^* \mu \sim 1_{q \times q}$ and $\mu \mu^* \sim 1_{p \times p}$. In this case

$$\tilde{\Omega}^1_D(A_2) = \{ X \in M_{q \times p}(\mathbb{C}) \} \bigoplus \{ Y \in M_{p \times q}(\mathbb{C}) \} , \quad \tilde{\Omega}^2_D(A_2) = \begin{pmatrix} A_{2,1} & 0 \\ 0 & 0 \end{pmatrix}$$

and all higher degrees of $\tilde{\Omega}^\bullet_D(A_2)$ are trivial. The multiplication rule is given by

$$\Pi : \tilde{\Omega}^1_D(A_2) \times \tilde{\Omega}^1_D(A_2) \longrightarrow \tilde{\Omega}^2_D(A_2)$$

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \ast \begin{pmatrix} X' & 0 \\ 0 & Y' \end{pmatrix} = \begin{pmatrix} X.Y' & 0 \\ 0 & 0 \end{pmatrix}$$
where, $X.Y'$ denotes the usual matrix multiplication.

Using these three cases and equation (5.16), it is shown in (§(7) in [17]) that the dga $\tilde{\Omega}_d^0(A_1, A_2)$ is the tensor product of the Dirac dga of $A_1$ and $A_2$. From the isomorphism in (5.17) it now follows that Lemma (3.12) holds in this case, i.e. our hypothesis in Section (3) is justified.

6. The case of quantum Heisenberg manifolds

Recall the definition of quantum Heisenberg manifolds from ([23]). For $x \in \mathbb{R}$, $e(x)$ stands for $e^{2\pi i x}$, where $i = \sqrt{-1}$.

**Definition 6.1.** For any positive integer $c$, let $S^c$ denote the space of smooth functions $\Phi : \mathbb{R} \times T \times \mathbb{Z} \to \mathbb{C}$ such that

1. $\Phi(x + k, y, p) = e(ckpy)\Phi(x, y, p)$ for all $k \in \mathbb{Z}$,
2. For every polynomial $P$ on $\mathbb{Z}$ and every partial differential operator $\tilde{X} = \frac{\partial^{m+n}}{\partial x^m \partial y^n}$ on $\mathbb{R} \times T$ the function $P(p)(\tilde{X}\Phi)(x, y, p)$ is bounded on $K \times \mathbb{Z}$ for any compact subset $K$ of $\mathbb{R} \times T$.

For each $h, \mu, \nu \in \mathbb{R}, \mu^2 + \nu^2 \neq 0$, let $A_{h}^\infty$ denote $S^c$ with product and involution defined by

$$ (\Phi \star \Psi)(x, y, p) := \sum_q \Phi(x - h(q - p)\mu, y - h(q - p)\nu, q)\Psi(x - h\mu, y - h\nu, p - q), $$

$$ \Phi^*(x, y, p) := \overline{\Phi(x, y, -p)}. $$

Then, $\pi : A_{h}^\infty \to \mathcal{B}(L^2(\mathbb{R} \times T \times \mathbb{Z}))$ given by

$$ (\pi(\Phi)\xi)(x, y, p) = \sum_q \Phi(x - h(q - 2p)\mu, y - h(q - 2p)\nu, q)\xi(x, y, p - q) $$

gives a faithful representation of the involutive algebra $A_{h}^\infty$. Now, $A_{\mu,\nu}^c$ is the norm closure of $\pi(A_{h}^\infty)$ is called the quantum Heisenberg manifold.

We will identify $A_{h}^\infty$ with $\pi(A_{h}^\infty)$ without any mention. Since, we are going to work with fixed parameters $c, \mu, \nu, h$ we will drop them altogether and denote $A_{\mu,\nu}^c$ simply by $A_h$. Here the subscript remains merely as a reminiscent of Heisenberg only to distinguish it from a general algebra. Moreover, $A_{h}^\infty$ is spectrally invariant subalgebra of $A_h$.

**Action of the Heisenberg group:** Let $c$ be a positive integer. Let us consider the group structure on $G = \mathbb{R}^3 = \{(r, s, t) : r, s, t \in \mathbb{R}\}$ given by the multiplication

$$(r, s, t)(r', s', t') = (r + r', s + s', t + t' +csr').$$

There is an explicit isomorphism ([4]) between $G$ and $H_3$, the Heisenberg group of $3 \times 3$ upper triangular matrices with real entries and 1’s on the diagonal. For $\Phi \in S^c, (r, s, t) \in \mathbb{R}^3 \equiv G$,

$$ (L_{(r, s, t)}\phi)(x, y, p) = e(p(t + cs(x - r)))\phi(x - r, y - s, p) $$
extends to an ergodic action of the Heisenberg group on $A_h$. 
The Trace: The linear functional $\tau : A^\infty_h \to \mathbb{C}$, given by $\tau(\phi) = \int_0^1 \int_{\mathbb{T}} \phi(x,y,0) dx dy$ is invariant under the Heisenberg group action. So, the group action can be lifted to $L^2(A^\infty_h)$. The action at the Hilbert space level is denoted by the same symbol.

The G.N.S. Hilbert space: The Hilbert space $L^2(A^\infty_h, \tau)$ obtained by applying the G.N.S. construction to $\tau$ is isomorphic to $L^2(\mathbb{T} \times \mathbb{T} \times \mathbb{Z}) \cong L^2([0,1] \times [0,1] \times \mathbb{Z})$ \([5]\).

The spectral triple: One fixes an inner product on the Lie algebra of the Heisenberg Lie group by declaring the following basis,

$$X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & c & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X_3 = \begin{pmatrix} 0 & 0 & c\alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

as orthonormal. Here, $\alpha > 1$ is a real number. Then, $D_h = \sum_{j=1}^3 i\delta_j \otimes \sigma_j$ is an unbounded self-adjoint operator with compact resolvent acting on $\mathcal{H}_h := L^2(A^\infty_h, \tau) \otimes \mathbb{C}^2$, where

$$\delta_1(f) = -\frac{\partial f}{\partial x},$$

$$\delta_2(f) = 2\pi icpxf(x,y,p) - \frac{\partial f}{\partial y},$$

$$\delta_3(f) = 2\pi ipcxf(x,y,p).$$

and

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

are the $2 \times 2$ Pauli spin matrices. The tuple $(A^\infty_h, \mathcal{H}_h, D_h)$ is a 3-summable spectral triple on $A^\infty_h$ \([5, 4]\).

We will consider $\{1, h\mu, \hbar\nu\}$ to be linearly independent over $\mathbb{Q}$. In that case $A_h$, and hence $A^\infty_h$, becomes a simple algebra \([23, 6]\). We need this simpleness otherwise computation of the Dirac dga $\Omega^*_D$ done in \([5]\) fails. Let $\phi_{mn} \in S^\infty$ be the function $\phi_{m,n}(x,y,p) = e(mx + ny)\delta_{\mu0}$. These functions are eigenfunctions for $\delta_j$’s and they satisfy

$$\delta_1(\phi_{10}) = 2\pi \phi_{10}, \quad \delta_2(\phi_{10}) = 0, \quad \delta_3(\phi_{10}) = 0,$$

$$\delta_1(\phi_{01}) = 0, \quad \delta_2(\phi_{01}) = 2\pi \phi_{01}, \quad \delta_3(\phi_{01}) = 0.$$

Using these functions $\phi_{mn}$, and simpleness of $A^\infty_h$, it is shown in \([5]\) that

$$\Omega^1_{D_h}(A^\infty_h) = (A^\infty_h)^3, \quad \pi(\Omega^2(A^\infty_h)) \cong (A^\infty_h)^4,$$

$$\pi(d\Omega^1)(A^\infty_h) \cong A^\infty_h, \quad \Omega^1_{D_h}(A^\infty_h) \cong (A^\infty_h)^3.$$

In this section we consider $A = A^\infty_h \otimes A^\infty_h$. Since, $(A^\infty_h, \mathcal{H}_h, D_h)$ is an odd spectral triple we apply the process described in point (2) in Section (2) to make it even with grading operator $\gamma$. Let $D = D_h \otimes 1 + 1 \otimes D_h$. Unlike the last section, here we go in a straightforward way to verify that our hypothesis in Section (3), or equivalently Lemma (3.12), is justified in this case.
Proposition 6.2. $\Omega^2_{D_h}(A) \cong \Omega^2_{D_h}(A^\infty_h) \otimes A^\infty_h \oplus A^\infty_h \otimes \Omega^1_{D_h}(A^\infty_h) \oplus \Omega^1_{D_h}(A^\infty_h) \otimes \Omega^1_{D_h}(A^\infty_h)$ as $\mathcal{A} = A^\infty_h \otimes A^\infty_h$-bimodules.

Proof. We first claim that $\pi_1(\Omega^2(A^\infty_h)) \otimes \mathcal{A}^\infty_h + \mathcal{A}^\infty_h \otimes \pi_2(\Omega^2(A^\infty_h))$ is a free bimodule of rank 7 over $\mathcal{A}^\infty_h \otimes \mathcal{A}^\infty_h$. Note that $\Omega^1_{D_h}(A^\infty_h) \otimes \Omega^1_{D_h}(A^\infty_h)$ is free of rank 9 and hence, in view of Lemma (3), it is enough to show that $\pi(\Omega^2(A))$ is free with rank 16. Arbitrary element of $\pi(\Omega^2(A))$ looks like

$$\sum_{i,j,k}(a_{0i} \otimes a_{1j})[D, b_{0j} \otimes b_{1j}][D, c_{0k} \otimes c_{1k}]$$

$$= \sum_{i,j,k} a_{0i} \left( \sum_{n=1}^{3} \delta_n(b_{0j}) \otimes \sigma_n \right) \left( \sum_{n=1}^{3} \delta_n(c_{0k}) \otimes \sigma_n \right) \otimes a_{1j}b_{1j}c_{1k}$$

$$+ a_{0i}b_{0j}c_{0k} \otimes a_{1i} \left( \sum_{n=1}^{3} \delta_n(b_{1j}) \otimes \sigma_n \right) \left( \sum_{n=1}^{3} \delta_n(c_{1k}) \otimes \sigma_n \right)$$

$$+ \gamma \left( a_{0i}b_{0j} \left( \sum_{n=1}^{3} \delta_n(c_{0k}) \otimes \sigma_n \right) \otimes a_{1i} \left( \sum_{n=1}^{3} \delta_n(b_{1j}) \otimes \sigma_n \right) c_{1k} \right)$$

$$- \gamma \left( a_{0i} \left( \sum_{n=1}^{3} \delta_n(b_{0j}) \otimes \sigma_n \right) c_{0k} \otimes a_{1i}b_{1j} \left( \sum_{n=1}^{3} \delta_n(c_{1k}) \otimes \sigma_n \right) \right)$$

$$\cong \sum_{i,j,k} a_{0i} \left( \sum_{n=1}^{3} \delta_n(b_{0j}) \delta_n(c_{0k}) \right) \otimes a_{1i}b_{1j}c_{1k} + a_{0i}b_{0j}c_{0k} \otimes a_{1i} \left( \sum_{n=1}^{3} \delta_n(b_{1j}) \delta_n(c_{1k}) \right) \otimes I_4$$

$$+ \sum_{1 \leq m < n \leq 3} (a_{0i} \delta_m(b_{0j}) \delta_n(c_{0k}) - \delta_n(b_{0j}) \delta_m(c_{0k})) \otimes a_{1i}b_{1j}c_{1k} \otimes (\sigma_m \sigma_n \otimes I_2)$$

$$+ \sum_{1 \leq m < n \leq 3} (a_{0i}b_{0j}c_{0k} \otimes a_{1i} \delta_m(b_{1j}) \delta_n(c_{1k}) - \delta_n(b_{1j}) \delta_m(c_{1k})) \otimes (I_2 \otimes \sigma_m \sigma_n)$$

$$+ \gamma \left( a_{0i}b_{0j} \sigma_m(c_{0k}) \sigma_n(b_{1j}) \delta_n(c_{1k}) - a_{0i} \delta_m(b_{0j}) \sigma_n(c_{0k}) \otimes a_{1i}b_{1j} \delta_n(c_{1k}) \right) \otimes (\sigma_m \otimes \sigma_n)$$

Here, we are using the canonical isomorphism

$$(L^2(A^\infty_h, \tau) \otimes \mathbb{C}^2) \otimes (L^2(A^\infty_h, \tau) \otimes \mathbb{C}^2) \rightarrow (L^2(A^\infty_h, \tau) \otimes L^2(A^\infty_h, \tau) \otimes (\mathbb{C}^2 \otimes \mathbb{C}^2)$$

of Hilbert spaces to push all the matrices $\{I_4 = I_2 \otimes I_2, \sigma_m \sigma_n \otimes I_2, I_2 \otimes \sigma_m \sigma_n, \sigma_m \otimes \sigma_n\}$ to the extreme right. Observe that $\{I_4, \sigma_m \sigma_n \otimes I_2, I_2 \otimes \sigma_m \sigma_n, \sigma_j \otimes \sigma_\ell : 1 \leq m < n \leq 3, 1 \leq j, \ell \leq 3\}$ is a linear basis of $M_4(\mathbb{C}) = M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$. Thus, we get an obvious injective bimodule map $\pi(\Omega^2(A)) \rightarrow (A^\infty_h \otimes A^\infty_h)^{16}$. We claim that this map is onto. For that first consider the following three elements of $\pi(\Omega^2(A))$ given respectively by setting

$$a_{1i} = b_{1j} = c_{1k} = 1 ; \quad a_{0i} = b_{0j} = c_{0k} = 1 ; \quad b_{0j} = c_{1k} = 1$$

for all $j, k$. Now, use the simpleness of $A^\infty_h$ and follow the proof of Proposition [21] in (3) (the proof of the fact that $\pi_1(\Omega^1(A^\infty_h)) = (A^\infty_h)^3$ and $\pi(\Omega^2(A^\infty_h)) = (A^\infty_h)^4$). So, we conclude that $\pi_1(\Omega^2(A^\infty_h)) \otimes A^\infty_h + A^\infty_h \otimes \pi_2(\Omega^2(A^\infty_h))$ is a free bimodule of rank 7 over $A^\infty_h \otimes A^\infty_h$. 
Now, we show that $\pi_1(d_1J_h^0(A^\infty_h)) \otimes A^\infty_h \cap A^\infty_h \otimes \pi_2(d_2J_h^0(A^\infty_h))$ is a free $A^\infty_h \otimes A^\infty_h$-module of rank one. Since, $\pi_j(d_jJ_h^0(A^\infty_h)) \cong A^\infty_h$ for $j = 1, 2$, we only need to prove that $A^\infty_h \otimes A^\infty_h \subseteq \pi_1(d_1J_h^0(A^\infty_h)) \otimes A^\infty_h \cap A^\infty_h \otimes \pi_2(d_2J_h^0(A^\infty_h))$, the other inclusion being obvious. For any $\xi = \sum_j a_j \otimes b_j \in A^\infty_h \otimes A^\infty_h$, we can write $\xi = \sum_j \omega_j \otimes b_j = \sum_j a_j \otimes v_j$ with each $\omega_j \in \pi_1(d_1J_h^0(A^\infty_h))$ and $v_j \in \pi_2(d_2J_h^0(A^\infty_h))$. This is because $\pi_j(d_jJ_h^0(A^\infty_h)) \cong A^\infty_h$ for $j = 1, 2$. Hence, $\xi \in \pi_1(d_1J_h^0(A^\infty_h)) \otimes A^\infty_h \cap A^\infty_h \otimes \pi_2(d_2J_h^0(A^\infty_h))$ and this concludes the claim. Hence, we have the following canonical isomorphism

$$
\pi_1(d_1J_h^0(A^\infty_h)) \otimes A^\infty_h + A^\infty_h \otimes \pi_2(d_2J_h^0(A^\infty_h))
\cong \pi_1(d_1J_h^0(A^\infty_h)) \otimes A^\infty_h \oplus A^\infty_h \otimes \pi_2(d_2J_h^0(A^\infty_h))
\cong A^\infty_h \otimes A^\infty_h \oplus A^\infty_h \otimes A^\infty_h
\cong A^\infty_h \otimes A^\infty_h
$$

of $A^\infty_h \otimes A^\infty_h$-bimodules. By Lemma (3.4), $\pi(dJ_h^0(A))$ now becomes a free bimodule of rank 1 over $A^\infty_h \otimes A^\infty_h$. Since, $\Omega^2_{D_h}(A^\infty_h) \otimes A^\infty_h \oplus A^\infty_h \otimes \Omega^2_{D_h}(A^\infty_h)$ is also a free bimodule of rank 6 over $A^\infty_h \otimes A^\infty_h$, we have a canonical isomorphism

$$
\frac{\pi_1(\Omega^2(A^\infty_h) \otimes A^\infty_h + A^\infty_h \otimes \pi_2(\Omega^2(A^\infty_h)))}{\pi(dJ_h^0(A))} \cong \Omega^2_{D_h}(A^\infty_h) \otimes A^\infty_h \oplus A^\infty_h \otimes \Omega^2_{D_h}(A^\infty_h)
$$

of $A$-bimodules. Since, $\Omega^2_D(A) = \pi(\Omega^2(A)) / \pi(dJ_h^0(A))$, final conclusion follows from Lemma (3.3). \hfill \Box

7. THE CASE OF NONCOMMUTATIVE TORI AND QUANTUM HEISENBERG MANIFOLDS

In this section we consider $A_\Theta \otimes A^\infty_h$, where $A_\Theta$ is a noncommutative $n$-torus and $A^\infty_h$ is a quantum Heisenberg manifold. Recall from (3.1, 3.6),

$$
\Omega^1_{D_\Theta}(A_\Theta) = A^\infty_\Theta, \quad \Omega^2_{D_\Theta}(A_\Theta) = A^{(n-1)/2},
\Omega^1_{D_h}(A^\infty_h) = (A^\infty_h)^3, \quad \Omega^2_{D_h}(A^\infty_h) = (A^\infty_h)^3.
$$

If $(A_\Theta, H_\Theta, D_\Theta)$ is not an even spectral triple (unless $n$ is even) then we apply the process described in point (2) in Section (2) to make it even with grading operator $\gamma$. Let $D = D_\Theta \otimes 1 + \gamma \otimes D_h$ and $A = A_\Theta \otimes A^\infty_h$. In this section also we assume that $\{1, h\mu, h\nu\}$ is linearly independent over $\mathbb{Q}$ so that $A^\infty_h$ is a simple algebra. Next Proposition shows that our hypothesis in Section (3), or equivalently Lemma (3.12), holds in this case also.

**Proposition 7.1.** $\Omega^2_{D_h}(A) \cong \Omega^2_{D_\Theta}(A_\Theta) \otimes A^\infty_h \oplus A_\Theta \otimes \Omega^2_{D_h}(A^\infty_h) \oplus \Omega^1_{D_\Theta}(A_\Theta) \otimes \Omega^1_{D_h}(A^\infty_h)$ as $A = A_\Theta \otimes A^\infty_h$-bimodules.

**Proof.** We only sketch the proof as computations are similar to the previous section. First note that $\Omega^2_{D_\Theta}(A_\Theta) \otimes A^\infty_h \oplus A_\Theta \otimes \Omega^2_{D_h}(A^\infty_h)$ is a free $A$-bimodule of rank $n(n-1)/2 + 3$, and $\Omega^1_{D_\Theta}(A_\Theta) \otimes \Omega^1_{D_h}(A^\infty_h)$ is free of rank $3n$. It can be shown that
\[ \pi_1(d_1 J_0^1(A_\Theta)) \otimes A^\infty_h + A_\Theta \otimes \pi_2(d_2 J_0^1(A^\infty_h)) \cong A_\Theta \otimes A^\infty_h \]
as \mathcal{A}\text{-bimodule, and hence by Lemma (3.4), } \pi(dJ_1^0(A)) \text{ is a free bimodule of rank 1 over } \mathcal{A}. 

Hence, we need to show that \( \pi(\Omega^2(A)) \) is a free \( \mathcal{A}\)-bimodule of rank \( 3n + 4 + n(n - 1)/2 \). As done in the proof of Proposition (6.2), by writing down any arbitrary element of \( \pi(\Omega^2(A)) \) explicitly, one can observe that the claim follows similarly by using the simpleness of \( A^\infty_h \) and the proof of Proposition [5.3] in (3) (the proof of the fact that \( \Omega^2_{D_\Theta}(A_\Theta) = A_\Theta^{n^2(n - 1)/2} \) for any \( n\)-torus \( A_\Theta \)). \( \square \)

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