Massive-Conformal Dictionary

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The finite-volume spectrum of an integrable massive perturbation of a rational conformal field theory interpolates between massive multi-particle states in infinite volume (IR limit) and conformal states, which are approached at zero volume (UV limit). Each state is labeled in the IR by a set of ‘Bethe Ansatz quantum numbers’, while in the UV limit it is characterized primarily by the conformal dimensions of the conformal field creating it. We present explicit conjectures for the UV conformal dimensions corresponding to any IR state in the $\phi_{1,3}$-perturbed minimal models $\mathcal{M}(2, 5)$ and $\mathcal{M}(3, 5)$. The conjectures, which are based on a combinatorial interpretation of the Rogers-Ramanujan-Schur identities, are consistent with numerical results obtained previously for low-lying energy levels.
1. Introduction

Important properties of a quantum field theory can be learned from its spectrum in finite volume. The volume dependence of energy levels contains information about the particle content of the theory as well as its $S$-matrix \[1\]. Due to scaling, where the mass scale $M$ times the length $L$ of the system serves as a dimensionless scaling parameter, one can probe the UV (massless) limit of the theory by considering the small-volume regime. In this paper we consider only integrable theories in 1+1 dimensions, which have been investigated extensively in recent years. A variety of techniques – perturbative and non-perturbative, analytical and numerical – are employed in the study of their finite-volume spectrum \[1\] (It should be noted, though, that some of the methods used are applicable to theories in higher dimensions.)

Many interesting integrable theories can be formulated \[3\] as perturbations by a certain relevant spinless operator of rational conformal field theories (CFTs). The UV limit in this case is the CFT itself, which is fairly well understood; in particular, its finite-volume partition function is given (see e.g. \[4\]) as a bilinear combination of characters of irreducible highest-weight representations of the chiral algebra of the CFT. The spectrum of the perturbed theory can be studied perturbatively by the so-called conformal perturbation theory. The corresponding small-volume expansion is known \[5\] to have a finite nonzero radius of convergence (with a finite number, possibly zero, of terms which require short-distance regularization). However, due to the technical difficulties in computing high-order terms, it is useful only for very small volume, compared to the inverse mass scale. A numerical non-perturbative method, known as the truncated conformal space approach \[6\], is useful for getting pretty accurate estimates up to moderately large volume, but only for low-lying energy gaps.

On the other hand, if the perturbed CFT is purely massive the system in infinite volume is described by a relativistic scattering theory of massive particles \[3\] and the spectrum is built of multi-particle states. In finite volume the spectrum gets quantized, and at least at large volume it is still meaningful to talk about multi-particle states of definite particle content – they are stationary scattering states of these particles. If the theory is integrable, and so its $S$-matrix factorizable \[7\], then it is known \[2\] \[6\] \[8\] how to obtain the energies of these states from the $S$-matrix, up to off-shell corrections which decay exponentially with the volume. Unfortunately, information about these exponential

\[1\] Cf. sect. 1 of \[2\] for an overview and references.
corrections is very limited, and so the applicability of this approximation of the spectrum is restricted to large volume. Therefore, it seems that the interesting problem of explicitly describing the interpolation between the small and large volume regimes, in other words between the UV and the IR, remains intractable.

Here we make an attempt to partially solve this problem, namely to find the map between conformal states and the multi-particle states in the massive theory to which they evolve under an integrable perturbation. Specifically, we will present what we regard as very plausible conjectures for such massive-conformal dictionary in two simple – from the $S$-matrix point of view – nontrivial theories, namely the $\phi_{1,3}$-perturbed nonunitary minimal models $\mathcal{M}(2,5)$ (the perturbed Yang-Lee CFT) \cite{9} and $\mathcal{M}(3,5)$ \cite{10}.

Our attempt to tackle this problem, which was first addressed in \cite{6}, is motivated by recent developments in understanding the CFT spectrum from the point of view of the underlying lattice systems. The diagonalization of the hamiltonian of certain gapless spin chains using Bethe equations has been shown \cite{11} to lead in the appropriate scaling limit to a description of the CFT spectrum in terms of massless fermionic quasiparticles. This description is encapsulated in fermionic sum representations \cite{11}-\cite{13}, generalizing the sum side of the Rogers-Ramanujan-Schur (R-R-S) identities, for the CFT characters. (In the case of $\mathcal{M}(2,5)$ the characters are the two $q$-series of the R-R-S identities themselves \cite{14}-\cite{13}.) Our strategy is then to construct a map between the massive multi-particle states and the conformal multi-quasiparticle states; as we will see, the crucial mathematics involved is closely related to Schur’s combinatorial interpretation of the R-R-S identities.

The outline of the paper is as follows. In sect. 2 we review some general features of the finite-volume spectrum. Sect. 3 discusses in detail the perturbed $\mathcal{M}(2,5)$ model, while sect. 4 is devoted mainly to the perturbed $\mathcal{M}(3,5)$ model in the phase of spontaneously broken $\mathbb{Z}_2$ symmetry. Both theories involve a single type of massive particle, hence their simplicity. Conclusions and an outlook to generalizations are included in sect. 5.

2. Finite-volume spectrum (generalities)

In order to set up a massive-conformal dictionary we need to recall some general facts about the finite-volume spectrum, in particular the characterization of conformal

\footnote{2 This reference contains many plots which illustrate the problem most vividly, and the reader is strongly encouraged to look at it.}
states on the one hand and the massive multi-particle states on the other. We consider a perturbed CFT on a cylinder whose circumference $L$ is the "volume" of space, and impose periodic boundary conditions (on some bosonic order parameter) around the cylinder. The momentum $P(L; M)$ of any state is therefore some integral multiple $p$ of $2\pi/L$, and is independent of $M$ since the momentum is a good quantum number. For dimensional reasons the energy of an arbitrary state can be written as

$$E(L; M) = \frac{2\pi}{L} e(\rho) \quad (\rho = ML). \quad (2.1)$$

The dimensionless scaling functions $e(\rho)$ will be referred to as scaled energies. We will also introduce the (scaled) energy gaps with respect to the ground state (the latter is often exactly calculable from the $S$-matrix using the thermodynamic Bethe Anstaz $[16]-[18]$, and is renormalized such that $e_0(\infty) = 0$),

$$\hat{E}(L; M) = E(L; M) - E_0(L; M) = \frac{2\pi}{L} \hat{e}(\rho). \quad (2.2)$$

The partition function of the theory is defined as the generating function

$$Z(\rho) = \sum_{\text{states}} |q|^{e(\rho)} \left( q \overline{q} \right)^{p/2}, \quad (2.3)$$

where $\overline{q}$ is the complex conjugate of $q$.

[Note that the energies, and hence also the gaps, are smooth for positive volume (the scaled gaps $\hat{e}(\rho)$ are analytic in $\rho^{2-d\Phi}$ around zero, where $d\Phi < 2$ is the scaling dimension of the perturbing field, and their singularities occur away from the real axis). Therefore it is meaningful to talk about levels as associated with given functions $\hat{e}(\rho \geq 0)$ despite the fact that in integrable theories many level-crossings occur at positive $\rho$ even within sectors of same momentum (cf. discussion in [14]).]

Now at the conformal point $\rho = 0$, the scaled energy-momentum of a state which is created by a conformal field of right-left conformal dimensions $(\Delta, \bar{\Delta})$ is

$$(\hat{e}(0), p) = (d - d_{\text{min}}, s) = (\Delta + \bar{\Delta} - (\Delta + \bar{\Delta})_{\text{min}}, \Delta - \bar{\Delta}), \quad (2.4)$$

3 Henceforth, the scale $M$ is taken to be the mass of the lightest particle in the spectrum, i.e. its rest energy in infinite volume. Also, state labels, other than 0 which corresponds to the ground state, will be suppressed for the sake of notational transparency.
where the subscript ‘min’ refers to the field of minimum scaling dimension which creates the vacuum of the CFT (in a unitary CFT this is the identity field, \((\Delta, \bar{\Delta})_{\text{min}} = (0,0))

Recall \[19\] that for the ground state

\[ e_0(0) = -\tilde{c}/12 = -(c - 12d_{\text{min}})/12, \]

where \(c\) is the Virasoro central charge and \(\tilde{c}\) is called the effective central charge. The CFT partition function is the generating function of conformal dimension \(s\), namely

\[
Z_{\text{CFT}} = Z(\rho = 0) = \sum_{\text{conf. states}} q^{\Delta - c/12} \bar{q}^{\bar{\Delta} - \bar{c}/12} = |q|^{-\tilde{c}/12} \sum_{\text{conf. states}} \left|\frac{q}{\bar{q}}\right|^{|d - d_{\text{min}}|/12} \left(\frac{q}{\bar{q}}\right)^{s/2}.
\]

It can be expressed as

\[
Z_{\text{CFT}} = \sum_{i,\bar{i}} N_{i,\bar{i}} \chi_i(q) \chi_{\bar{i}}(\bar{q}) ,
\]

where the \(\chi_i(q)\) are characters of irreducible highest-weight representations of the chiral algebra of the CFT, and the \(N_{i,\bar{i}}\) are non-negative integers. Eq. \((2.6)\) manifestly shows the decoupling at \(\rho = 0\) of sectors of right- and left-movers, corresponding to the \(\chi_i(q)\) and \(\chi_{\bar{i}}(\bar{q})\) respectively.

We now turn to the large-volume regime, and for simplicity consider the case of a factorizable scattering theory with a single type of particle (mass \(M\)), whose vacuum is non-degenerate in infinite volume (the latter assumption excludes the possibility of kinks and thus simplifies the analysis, cf. \[2\]; in sect. 4 we will encounter a model with a doubly-degenerate vacuum). The energy gaps – as well as the energies themselves, up to the accuracy stated – and momenta of \(N\)-particle states are given \[6\], \[8\] by

\[
\hat{E}(L; M) = \sum_{k=1}^{N} M \cosh \theta_k + \mathcal{O}(e^{-\sigma ML}) ,
\]

\[
P(L; M) = \sum_{k=1}^{N} M \sinh \theta_k ,
\]

where \(\sigma > 0\) and the real rapidities \(\theta_k\) are quantized via the following equations of the Bethe Ansatz type:

\[
e^{iML \sinh \theta_k} \prod_{k' = 1 \atop k' \neq k}^{N} S(\theta_k - \theta_{k'}) = 1 \quad (k = 1, 2, \ldots, N),
\]

where \(S(\theta)\) is the two-particle scattering amplitude written as a function of rapidity, as customary \[7\]. We assume that \(S(0) = -1\), which appears to be universally true in theories where the particle is created by an interacting bosonic field.\[8\] Consequently \[17\], there is

\[\text{4 This field statistics, in turn, is dictated by the periodic boundary conditions we impose, which correspond to a modular invariant partition function.}\]
an exclusion rule in rapidity (momentum) space: any solution \( \{ \theta_k \}_{k=1}^N \) of (2.8), for any given \( \rho = ML > 0 \), consists of \( \theta_k \) which are all distinct. This allows us to order the \( \theta_k(\rho) \) in any solution such that \( \theta_1(\rho) < \theta_2(\rho) < \ldots < \theta_N(\rho) \).

To analyze eqs. (2.7)-(2.8) it is convenient to write 
\[
S(\theta) = -e^{i \tilde{\delta}(\theta)},
\]
where unitarity 
\[
S(\theta)S(-\theta) = 1
\]
enables a branch choice in which the (shifted) phase shift \( \tilde{\delta}(\theta) \) is an odd function of \( \theta \) which is real-valued when \( \theta \) is real. Taking the logarithm of eqs. (2.8) one obtains
\[
\frac{\rho}{2\pi} \sinh \theta_k = n_k - \frac{1}{2\pi} \sum_{k' \neq k}^N \tilde{\delta}(\theta_k - \theta_{k'}),
\]
with the \( n_k \) distinct half-odd-integers (integers) when \( N \) is even (odd). Since \( \tilde{\delta}(0)=0 \) and \( \tilde{\delta}(\theta) \) is analytic around 0, it follows that the \( \theta_k \) are analytic in \( 1/\rho \) around 0, and can be expanded as \( \theta_k = 2\pi n_k/\rho + \mathcal{O}(1/\rho^2) \). Thus, in particular, the ordering of the \( \theta_k \) implies that the \( n_k \) are similarly ordered, \( n_1 < n_2 < \ldots < n_N \). This argument also suggests that given such a set of \( n_k \), there exists a unique (ordered) solution \( \{ \theta_k(\rho) \} \) to eqs. (2.9) for any given \( \rho > 0 \). Such an assumption is customary in all Bethe-Ansatz-type analyses (for certain \( \tilde{\delta}(\theta) \) it has been actually proven, see e.g. [16]), and we will adopt it here. To summarize, we assume that the \( N \)-particle states of the massive theory in finite volume are unambiguously specified by sets of ‘Bethe Ansatz quantum numbers’ \( \{ n_k \}_{k=1}^N \subset \mathbb{Z} + \frac{N+1}{2} \) with strictly increasing elements.

Before proceeding to specific models, let us work out a simple but rather formal exercise whose result will nevertheless be important later on. Consider the UV limit \( \rho \to 0 \) of the Bethe Ansatz system of eqs. (2.7), (2.9) with \( \tilde{\delta}(\theta) = \tilde{\delta} \cdot \text{sgn}\theta \), where \( \tilde{\delta} \) is a real constant. One can easily see that in the UV limit \( \theta_k \in \{0, \pm \infty\} \), and since the \( \theta_k \) are ordered such that \( \theta_k - \theta_{k'} > 0 \) for \( k > k' \), obtain
\[
\lim_{\rho \to 0} \frac{\rho}{2\pi} \sinh \theta_k = n_k + (N - 2k + 1)\frac{\tilde{\delta}}{2\pi} \quad (k = 1, 2, \ldots, N).
\]
The scaled energy-momentum of the corresponding state (ignoring any exponential corrections) is
\[
(\hat{\mathcal{E}}(\rho = 0), p) = \left( \sum_{k=1}^N |p_k|, \sum_{k=1}^N p_k \right),
\]
where \( p_k = n_k + (N - 2k + 1)\frac{\tilde{\delta}}{2\pi} \).
3. Perturbed $\mathcal{M}(2,5)$

We now specialize the discussion to the perturbed $\mathcal{M}(2,5)$ model, and first summarize some known results for the unperturbed CFT. The minimal model $\mathcal{M}(2,5)$ is of Virasoro central charge $c = -\frac{22}{5}$, and except for the identity field $1$ it contains only one primary field $\varphi$, whose conformal dimensions are $(-\frac{1}{5}, -\frac{1}{5})$ so that $\tilde{c} = c - 12d_\varphi = \frac{2}{5}$. The partition function is

$$Z_{\mathcal{M}(2,5)} = |\chi_0(q)|^2 + |\chi_1(q)|^2 = |q|^{-1/30}(|\hat{\chi}_0(q)|^2 + |q|^{2/5}|\hat{\chi}_1(q)|^2), \quad (3.1)$$

where the normalized characters $\hat{\chi}_a$, with $a=0 (1)$ corresponding to $\varphi (1)$, are given by $\mathcal{M}(2,5)$

$$\hat{\chi}_a(q) = \frac{1}{(q)\infty} \sum_{k\in\mathbb{Z}} (q^{k(10k+2a+1)} - q^{2k+1}(5k-a+2))$$

$$= \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-4+a})(1-q^{5n-1-a})} = \sum_{m=0}^{\infty} \frac{q^{m(m+a)}}{(q)_m}. \quad (3.2)$$

Here

$$(q)_0 = 1, \quad (q)_m = \prod_{j=1}^{m} (1-q^j) \quad \text{for} \ m = 1, 2, 3, \ldots. \quad (3.3)$$

The equality of the three different-looking $q$-series in (3.2) constitutes the two R-R-S identities $\mathcal{M}(2,5)$

For us the most physically revealing $q$-series representation for the two characters $\hat{\chi}_a$ is the last one listed on (3.2), which is referred to as of fermionic form. The reason is that it has a natural interpretation in terms of massless (right-moving, say) fermionic quasiparticles occupying certain restricted grids of momentum states, as exhibited by the following manipulations:

$$\sum_{m=0}^{\infty} \frac{q^{m(m+a)}}{(q)_m} = \sum_{m=0}^{\infty} \sum_{r=0}^{\infty} Q_m(r) \frac{q^{r+m(m+2a+1)/2}}{(q)_m}$$

$$= \sum_{m=0}^{\infty} \left( (I_j)_{j=1}^{m} \subset \mathbb{Z}_{\geq 0} + (m+2a+1)/2 \right) \sum_{I_{j+1} - I_j \geq 1} q^{\sum_{j=1}^{m} I_j}, \quad (3.4)$$

where $Q_m(r)$ is the number of additive partitions of the non-negative integer $r$ into $m$ distinct non-negative integers, which are denoted by $I_j - \frac{m+2a+1}{2}$ on the second line of (3.4) (see $\mathcal{M}(2,5)$ for the identity used in the first line). A change $I_j \mapsto I'_j = I_j - \frac{m+1}{2} + j$ in
the summation variables leads to a form which is more suitable for considerations of the massive perturbation, namely

\[ \hat{\chi}_a(q) = \sum_{m=0}^{\infty} \sum_{\{I'_j\}_{j=1}^m \subset \mathbb{Z}_{\geq a+1}} q^{\sum_{j=1}^m I'_j} \quad (a = 0, 1). \quad (3.5) \]

Note the restriction \( I'_{j+1} - I'_j \geq 2 \) here, in contrast to the standard fermionic exclusion rule \( I_{j+1} - I_j \geq 1 \) in (3.4). This ‘difference 2 condition’ follows from the ‘generalized commutation relations’ of a so-called \( \mathbb{Z} \)-algebra in the Lie theoretic interpretation (and proof) of the R-R-S identities given in [24]. It also plays a prominent role in theorem 3.6 of [22], which states that the set

\[ \bigcup_{m=0}^{\infty} \{ L_{-I_m} \cdots L_{-I_1} v_a \mid \{I'_j\}_{j=1}^m \subset \mathbb{Z}_{\geq a+1}, \quad I'_{j+1} - I'_j \geq 2 \} \quad (3.6) \]

forms a basis for the irreducible Virasoro Verma modules of \( \mathcal{M}(2,5) \). (In (3.6) the \( L_m \) are the Virasoro generators and \( v_a, a = 0, 1, \) are the highest-weight states corresponding to the primary fields \( \varphi \) and \( 1 \), respectively.) As another side remark, note that a further change of variables in (3.5), to \( \{\sigma_\ell\}_{\ell=1}^\infty \) with \( \sigma_\ell = 1 \) if \( \ell \in \{I'_j\}_{j=1}^m \) and 0 otherwise, results in the one-dimensional configuration sum representation for the characters

\[ \hat{\chi}_a(q) = \sum_{\{\sigma_\ell\}_{\ell=1}^\infty} q^{\sum_{\ell=1}^\infty \ell \sigma_\ell}, \quad (3.7) \]

which appeared in [25]. To conclude this digression, let us mention that the equality of the rhs of (3.5) and the infinite product in (3.2) is equivalent to an interesting equality between different restricted partitions of integers [15].

The importance of eq. (3.5) for us is revealed when it is used in (3.4). Writing first

\[ \hat{\chi}_a(q) = \sum_{\bar{m}=0}^{\infty} \sum_{\{\bar{I}'_j\}_{j=1}^{\bar{m}} \subset \mathbb{Z}_{\leq -(a+1)}} \bar{q}^{-\sum_{j=1}^{\bar{m}} \bar{I}'_j}, \quad (3.8) \]

in order to accommodate the left-mover sector, consider the expression obtained for \(|\hat{\chi}_a(q)|^2\) by multiplying eqs. (3.5) and (3.8) together. It is a (restricted) sum over two sets \( \{\bar{I}'_j\} \) and
\{I_j^i\}, which we can combine into a sum over \{p_k\}, defined as \{I_j^i\} \cup \{I_j^i\} for \(a=0\) and \{I_j^i\} \cup \{0\} \cup \{I_j^i\} for \(a=1\). This leads to

\[
Z_{\mathcal{M}(2,5)} = |q|^{-1/30} \sum_{N=0}^{\infty} \sum_{\substack{(p_k)_{k=1}^N \in \mathbb{Z}+N+1/2 \in \mathbb{Z} \atop p_{k+1} - p_k \geq 2}} |q|^{\sum_{k=1}^{N}|p_k|+\frac{2}{5}N} \left(\frac{q}{q}\right)^{\sum_{k=1}^{N}p_k/2} . \tag{3.9}
\]

Eq. (3.9) suggests an interpretation of the full Hilbert space of the CFT \(\mathcal{M}(2,5)\) as built of multi-particle states characterized by sets of integer momenta \(p_k\) satisfying \(p_{k+1} - p_k \geq 2\), with a state belonging to the conformal family [1] or [\(\bar{\phi}\)] depending on whether or not, respectively, \(0 \in \{p_k\}\). The scaling dimension and spin of the state corresponding to \(\{p_k\}_{k=1}^N\) are given by

\[
(d(\{p_k\}), s(\{p_k\})) = \left(-\frac{2}{5} + \sum_{k=1}^{N} \epsilon(p_k), \sum_{k=1}^{N} p_k\right) , \tag{3.10}
\]

where the “dispersion relation of the conformal particles” reads

\[
\epsilon(p) = |p| + \frac{2}{5} \delta_{p,0} \quad (p \in \mathbb{Z}). \tag{3.11}
\]

We now observe that except for the zero-mode energy (the Kronecker-delta term in (3.11)), the above description of the spectrum of \(\mathcal{M}(2,5)\) is identical to the content of eqs. (2.10)-(2.11), provided we take \(\tilde{\delta} = -\pi\) there. To see this, note that \(\{n_k\}_{k=1}^N \in \mathbb{Z} + \frac{N+1}{2}\) with \(n_{k+1} - n_k \geq 1\) implies \(\{p_k\}_{k=1}^N \in \mathbb{Z}\) with \(p_{k+1} - p_k \geq 2\), if \(p_k = n_k - \frac{N+1}{2} + k\). Hence we can rewrite (3.9) as

\[
Z_{\mathcal{M}(2,5)} = |q|^{-1/30} \sum_{N=0}^{\infty} \sum_{\substack{(n_k)_{k=1}^N \in \mathbb{Z}+(N+1)/2 \in \mathbb{Z} \atop n_{k+1} - n_k \geq 1}} |q|^{\sum_{k=1}^{N}|n_k-\frac{N+1}{2}+k|+\frac{2}{5}N} \left(\frac{q}{q}\right)^{\sum_{k=1}^{N}n_k/2} . \tag{3.12}
\]

By now it is only natural to state a

**Conjecture:** The \(N\)-particle state labeled by \(\{n_k\}_{k=1}^N \in \mathbb{Z} + \frac{N+1}{2}\) \((n_1 < n_2 < \ldots < n_N)\) in the massive perturbation of \(\mathcal{M}(2,5)\), goes over in the massless limit to a conformal state whose scaling dimension and spin are

\[
(d, s) = \left(-\frac{2}{5} + \sum_{k=1}^{N} \left(|n_k-\frac{N+1}{2}+k|+\frac{2}{5}N\right) + \sum_{k=1}^{N} n_k\right) . \tag{3.13}
\]
Following our analysis in reverse it is possible to further map the \( \{ n_k \} \) onto the massless quasiparticle labels \( \{ \tilde{I}_j \}_{j=1}^m \cup \{ I_j \}_{j=1}^n \), where the \( \tilde{I}_j \) and \( I_j \) correspond to the left- and right-moving quasiparticles, respectively, and are both restricted as in (3.4). In the \( [\varphi] \) sector, characterized by \( 0 \notin \{ p_k = n_k - \frac{N+1}{2} + k \} \), we have \( m = N - \tilde{m} = \# \{ p_k > 0 \} \) and \( \tilde{I}_j = -n_j + \frac{m}{2} \) for \( j = 1, \ldots, \tilde{m} \) while \( I_j = n_{\tilde{m}+j} + \frac{\tilde{m}}{2} \) for \( j = 1, \ldots, m \). In the sector \([I]\), on the other hand, where \( 0 \in \{ p_k = n_k - \frac{N+1}{2} + k \} \), we have \( m = N - \tilde{m} - 1 = \# \{ p_k > 0 \} \) and \( \tilde{I}_j = -n_j + \frac{m+1}{2} \) for \( j = 1, \ldots, \tilde{m} \) while \( I_j = n_{\tilde{m}+1+j} + \frac{\tilde{m}+1}{2} \) for \( j = 1, \ldots, m \).

Let us now mention the evidence we have in support of the conjecture stated above. Introducing the truncated conformal space approach, Yurov and Al. Zamolodchikov [6] studied numerically the finite-volume spectrum of the perturbed \( M(2,5) \) model in the sectors of total scaled momentum \( p=0,1,2, \) and compared a total number of 19 low-lying levels in these sectors with the Bethe Ansatz predictions (2.7)-(2.8). This allowed them to identify the massive labels \( \{ n_k \} \) corresponding to these levels, and our conjecture is consistent with their findings.

It is interesting to see the relation between the Bethe Ansatz equations (2.8), which provide a good approximation for the large-volume spectrum of the perturbed theory, and the UV (zero volume) quantization condition, eq. (2.10) with \( \tilde{\delta} = -\pi \), which was instrumental for obtaining our conjectured massive-conformal dictionary. The \( S \)-matrix of the perturbed \( M(2,5) \) model, which should be used in (2.8), is [4] \( S(\theta) = \frac{\sinh \theta + i\sqrt{3/2}}{\sinh \theta - i\sqrt{3/2}} \) and the shifted phase-shift appearing in (2.9) is therefore given by \( \tilde{\delta}(\theta) = -2\arctan\left( \frac{2}{\sqrt{3}} \sinh \theta \right) \). The effect of the formal exercise leading to (2.10) can be interpreted as replacing \( \tilde{\delta}(\theta) \) by \( \tilde{\delta}(\infty) \cdot \text{sgn} \theta = -\pi \cdot \text{sgn} \theta \). We stress that in general this replacement changes the result for the \( \rho \to 0 \) limit of the scaled gaps as given by (2.7)-(2.8) (not that \textit{a priori} there is any good reason to trust these equations at \( \rho = 0 \) !): even though all \( \theta_k \in \{ 0, \pm \infty \} \) in this limit, still differences between \( \theta_k \) of the same sign do not diverge, and so the \( \tilde{\delta}(\theta_k - \theta_{k'}) \) in (2.9) are not necessarily evaluated at \( \pm \infty \). Note, however, that for one-particle states, and two-particle states where \( \theta_1 \) and \( \theta_2 \) are not of the same sign in the UV limit, the above replacement \textit{is} harmless.\(^5\) For such states our conjecture implies that the scaled exponential corrections to the energy gaps (2.7) approach exactly 0 or \( -d_{\varphi} = \frac{2}{\pi} \) in the UV limit. (A similar observation concerning two-particle states in the zero-momentum sector of several perturbed rational CFTs was made in [26].)

\(^5\) It is also harmless, at least formally, in the thermodynamic Bethe Ansatz computation of the UV effective central charge, whose value turns out to depend only on \( \tilde{\delta}(\infty) \); explicitly [17] [18], \( \tilde{c} = \frac{6}{\pi^2} \mathcal{L}(\frac{\pi}{1+x}) \) where \( \mathcal{L}(z) \) is the Rogers dilogarithm and \( x \geq 0 \) satisfies \( x = (1 + x)^{\tilde{\delta}(\infty)/\pi} \).
4. Other single-particle models

We know of only two other integrable perturbed rational CFTs with a single type of particle in the spectrum, namely the minimal models $\mathcal{M}(3,4)$ and $\mathcal{M}(3,5)$ perturbed by the $\phi_{1,3}$ operator. The former is the Ising field theory, describing the off-critical Ising model (at zero magnetic field) in the scaling limit [27]. This theory is rather trivial from the viewpoint of the spectrum, which can be constructed from that of a free Majorana fermion by performing the GSO projection [28]. The full finite-volume spectrum is known exactly. It has been discussed in detail in [2], where the massive-conformal dictionary in both the high- and low-temperature phases of the theory was also given. For comparison with eq. (3.12), which summarizes the dictionary in the case of perturbed $\mathcal{M}(2,5)$, let us just write down the analogous representation of the partition function of the Ising CFT (from which the dictionary in the high-temperature phase of the Ising field theory is easily read off):

$$Z_{\mathcal{M}(3,4)} = |q|^{-1/24} \sum_{N=0}^{\infty} \sum_{\{n_k\}_{k=1}^{N} \subset \mathbb{Z}^{+} \cap \mathbb{Z}^{+}/2} \sum_{n_{k+1} - n_k \geq 1} \sum_{n_k = 1}^{\infty} \frac{|q|^{(1-(-1)^N)/16+\sum_{k=1}^{N} |n_k|}}{q^{\sum_{k=1}^{N} n_k/2}}. \tag{4.1}$$

We will therefore concentrate on $\mathcal{M}(3,5)$ in the rest of this section.

The perturbed nonunitary model $\mathcal{M}(3,5)$ has a $\mathbb{Z}_2$ symmetry. Like the Ising field theory it has two phases (call them the ‘$\pm$’-phases), depending on the sign of the coupling to the perturbing field $\phi_{1,3}$, and the $\mathbb{Z}_2$ symmetry is spontaneously broken in one of them (the ‘$-$’-phase, say) [26]. Unlike the Ising field theory, however, the two phases are not related by some duality. Little is known about the ‘$+$’-phase and we will not discuss it here. The particle spectrum in the ‘$-$’-phase consists of a kink and an antikink, which interpolate between two degenerate vacua. The amplitude for scattering of a kink on an antikink (or vice versa) is given in [10] as $S(\theta) = -i \tanh(\frac{\theta}{2} - i\frac{\pi}{4})$, so that $\tilde{\delta}(\theta) = \arctan(\sinh \theta)$ with $\tilde{\delta}(\pm \infty) = \pm \frac{\pi}{2}$.

In [26] the finite-volume spectrum of the theory was studied using the truncated conformal space approach. Results for several low-lying levels in the zero-momentum sector were compared with predictions of the Bethe Ansatz equations (2.7)-(2.8) for two-kink states. With periodic boundary conditions, which we restrict attention to, there are only even-$N$-kink states ($\frac{N}{2}$ kinks and $\frac{N}{2}$ antikinks) in the spectrum, and eq. (2.3) should be modified (cf. [2]) to allow for all sets $\{n_k\}_{k=1}^{N} \subset \frac{1}{2} \mathbb{Z}$ with $N$ even and $n_{k+1} - n_k \in \mathbb{Z}_{\geq 1}$. The
results of \([20]\) enable identification of the UV scaling dimensions \(d = \frac{1}{3}, \frac{2}{3}, \frac{1}{2}, \frac{3}{2}, 2\frac{1}{2}, 2\frac{3}{4}\) as corresponding to the two-kink states \([-n, n]\) with \(n = \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3\), respectively, in our notation.

Now the central charge of \(\mathcal{M}(3, 5)\) is \(-\frac{3}{5}\), and there are four primary fields \(\phi_{1,r}\) whose left-right conformal dimensions and \(\mathbb{Z}_2\)-parities are \(\Delta^C = 0^+, -\frac{1}{20}, -\frac{1}{5}, \frac{1}{4}\) for \(r = 1, 2, 3, 4\), respectively. Hence the vacuum is created by the \(\mathbb{Z}_2\)-odd field \(\phi_{1,2}\), and the effective central charge is \(\frac{3}{5}\). The CFT partition function is

\[
Z_{\mathcal{M}(3,5)} = \sum_{r=1}^{4} |\chi_{1,r}(q)|^2 = |q|^{-1/20} \left( \sum_{\ell=0}^{1} |\hat{\chi}_0^{(\ell)}(q)|^2 + |q|^{1/10} \sum_{\ell=0}^{1} |\hat{\chi}_1^{(\ell)}(q)|^2 \right),
\]

where (see [13] and references therein)

\[
\hat{\chi}_a^{(\ell)}(q) = \sum_{m=0}^{\infty} \frac{q^{m(m+2a)/4}}{(q)_m} \quad (a, \ell = 0, 1).
\]

The labels \((a, \ell) = (0,0), (0,1), (1,0), (1,1)\) correspond to \(r = 2, 3, 1, 4\). As in the case of \(\mathcal{M}(2, 5)\), the fermionic quasiparticle representation (4.3) for the characters of \(\mathcal{M}(3, 5)\) is our starting point for obtaining the conjectured massive-conformal dictionary.

Similarly to (3.4)-(3.5), we first write

\[
\frac{q^{m(m+2a)/4}}{(q)_m} = \sum_{\{I_j\} \in \mathbb{Z}_{\geq 0}^{m-(m-2a-2)/4}} q^{\sum_{j=1}^{m} I_j} \quad \sum_{I_{j+1} - I_j \geq 1} \quad \sum_{I_{j'} + 1 - I_{j'} \geq 1/2}
\]

where the change of variables \(I_j \mapsto I'_{j} = I_j + \frac{m+1}{4} - \frac{1}{2}\) has been performed. Putting together the sectors of left- and right-movers we arrive at

\[
Z_{\mathcal{M}(3,5)} = |q|^{-\frac{1}{20}} \sum_{N=0}^{\infty} \sum_{\{p_k\} \in \mathbb{Z}_{\geq 0}^{N} \cap \mathbb{Z}_{\geq 0}^{+}} |q|^{D(\{p_k\})} \left( \frac{q}{q} \right)^{\frac{1}{4}} \sum_{k=1}^{N} p_k \left( \frac{p_k}{q} \right)^{\frac{1}{2}},
\]

where

\[
D(\{p_k\}) = \begin{cases} 0 & \text{if } p_k \in \mathbb{Z} - \frac{k}{2} - \frac{1}{4} \quad \text{and} \quad \# \{p_k > 0\} \text{ is even} \leftrightarrow [\phi_{1,2}] \\ \frac{1}{10} & \text{if } p_k \in \mathbb{Z} - \frac{k}{2} - \frac{1}{4} \quad \text{and} \quad \# \{p_k > 0\} \text{ is odd} \leftrightarrow [\phi_{1,4}] \\ \frac{1}{10} & \text{if } p_k \in \mathbb{Z} - \frac{k}{2} + \frac{1}{4} \quad \text{and} \quad \# \{p_k > 0\} \text{ is even} \leftrightarrow [\phi_{1,1}] \\ 0 & \text{if } p_k \in \mathbb{Z} - \frac{k}{2} + \frac{1}{4} \quad \text{and} \quad \# \{p_k > 0\} \text{ is odd} \leftrightarrow [\phi_{1,3}]. \end{cases}
\]
Compare this result with
\[ Z_{\mathcal{M}(3,4)} = |q|^{-1/24} \sum_{N = 0}^{\infty} \sum_{\substack{\{n_k\}_{k=1}^N \subset \mathbb{Z}/2 \atop n_{k+1} - n_k \in \mathbb{Z}_{\geq 1}}} \sum_{k=1}^{N} n_k/2 \times |q|^D(\{n_k\}) \sum_{k=1}^{N} \frac{|n_k|}{q} \sum_{k=1}^{N} n_k/2 , \] (4.7)
where \( D(\{n_k\}) = \frac{1}{8} \) if \( n_k \in \mathbb{Z} \) and 0 otherwise, which gives the massive-conformal dictionary for the low-temperature phase of the Ising field theory.

Eq. (4.3) represents the partition function of the CFT as a sum over even-\( N \)-“particle” states. To complete the translation to the (UV limit of the) massive \( N \)-kink states we furthermore need a 1–1 map between the sets \( \{p_k\}_{k=1}^N \) in (4.5) and the sets \( \{n_k\}_{k=1}^N \subset \mathbb{Z} \) with \( n_{k+1} - n_k \in \mathbb{Z}_{\geq 1} \). The experience of sect. 3 suggests using (2.10) with \( \tilde{\delta} = \tilde{\delta}(\infty) \), which is equal to \( \pi/2 \) in our model. Indeed,
\[ p_k = n_k + \frac{N + 1}{4} - \frac{k}{2} \quad (k = 1, 2, \ldots, N \text{ even}) \] (4.8)
implements such a map of the allowed \( \{n_k\}_{k=1}^N \) onto \( \{p_k\}_{k=1}^N \) with \( p_k \in \frac{1}{2} \mathbb{Z} - \frac{k}{2} - \frac{1}{4} \) and \( p_{k+1} - p_k \in \mathbb{Z}_{\geq 0} + \frac{1}{2} \), which are the sets summed over in (4.5).

Hence we conjecture that the massive multi-kink state labeled by \( \{n_k\}_{k=1}^N \) in the ‘−’-phase of the \( \phi_{1,3} \)-perturbed \( \mathcal{M}(3,5) \) model comes from a conformal state of scaling dimension and spin
\[ (d, s) = \left( D(\{p_k\}) + \sum_{k=1}^{N} |p_k|, \sum_{k=1}^{N} n_k \right) , \] (4.9)
where the \( \{p_k\} \) and \( D(\{p_k\}) \) are given by eqs. (4.8) and (4.6). (The map between the \( \{n_k\}_{k=1}^N \) and the quasiparticle labels by \( \{I_j\}_{j=1}^\bar{m} \cup \{I_j\}_{j=1}^m \), where \( \bar{m} + m = N \), can be also obtained, as for the perturbed \( \mathcal{M}(2,5) \) model.) This correspondence is consistent with the results of [20] for the six lowest 2-kink states in the zero-momentum sector, as well as with the observation that the spinless conformal states of scaling dimension 1, 3/10, 9/10 and 4 evolve into 4-kink states, which can be deduced from the plots given there. (According to (4.6), the corresponding \( \{n_k\} \) are \( \{-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\} \), \( \{-\frac{5}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{5}{2}\} \), and \( \{-2, -1, 1, 2\} \).)

Before closing this section, let us make an amusing observation about two more integrable single-particle theories which however cannot be formulated as perturbations of
rational CFTs. These are the real-coupling affine Toda field theories based on the affine Lie algebras $A_1^{(1)}$ and $A_2^{(2)}$, whose UV limit is rather singular, being described by a free massless uncompactified scalar field [18]. The shifted phase-shifts, as determined from the factorizable $S$-matrices [29] of the two theories, satisfy $\tilde{\delta}(\infty) = \pi$ independently of the coupling. Using this as the value of $\tilde{\delta}$ in eq. (2.10), which has proven to be useful in the models studied earlier, we obtain $p_k = n_k + \frac{N+1}{2} - k$ for the “single-particle momenta” in the UV limit. This relation maps the allowed massive labels $\{n_k\}_{k=1}^N \in \mathbb{Z} + \frac{N+1}{2}$ with $n_{k+1} \geq n_k + 1$ onto $\{p_k\}_{k=1}^N \in \mathbb{Z}$ with $p_{k+1} \geq p_k$, which is the quantization condition appropriate for free bosons in a box with periodic boundary conditions.

5. Discussion

The eigenstates of the hamiltonian of an integrable, massive, perturbed rational CFT in finite volume can be characterized in two alternative ways. One is adequate for the large-volume (IR) regime, where states are labeled (schematically) by the Bethe Ansatz quantum numbers $\{n_k\}$. The other is tailored for the CFT (UV) limit, and employs the massless fermionic quasiparticle labels $(\phi | \{\bar{I}_j\} \cup \{I_j\})$, where $\phi$ specifies the conformal family, or a more algebraic description in terms of generators of the chiral algebra of the CFT acting on highest-weight (primary) states; either way, the conformal label directly gives the conformal dimensions (equivalently scaling dimension and spin) of the field creating the state in the unperturbed CFT.

Lacking the tools for computing the full exact spectrum at all volume, it seems more viable and still very interesting to find the correspondence between the two alternative characterizations. In this paper we conjectured the explicit dictionary between the IR and UV labels in two simple – yet nontrivial – theories. The simplicity of these two theories lies in the fact that there is a single (quasi)particle in their spectrum. Although generalizations of our work to models with more particles and bigger internal symmetry do not look quite straightforward, we still think that some useful and general insight has been gained.

The basic feature of the IR description of the spectrum is that the quantum numbers $n_k$ are those of (GSO-projected) free fermions. The interaction in this description shows itself in the $S$-matrix, which is not constant as a function of rapidity if the theory is nontrivial (from the point of view of the spectrum, at least). The single-particle momenta $p_k$ are then shifted at order $O(1/L^2)$ from their free-quantized values $2\pi n_k/L$, but the dispersion relation remains $E(p) = \sqrt{p^2 + m^2}$ as in a free theory. (It is important to
remember, however, that the total energy of a state is equal to the sum of the single-particle energies only up to off-shell exponential corrections.) This picture conforms with the usual (perturbative) quantum field theoretical viewpoint of trivial statistics and short-range interactions which are probed in scattering processes.

The CFT description, on the other hand, is manifestly non-perturbative. Nontriviality of a theory is indicated already by the presence of non-half-integral conformal dimensions. It is further reflected in the quasiparticle picture by nontrivial restrictions on the allowed quasiparticle labels \( I_j \); these restrictions are collective in nature, being dependent on the total number of quasiparticles in a state. This feature, combined with the fact that within each conformal family all levels are equally spaced and the total scaled energy is exactly the sum of the single-quasiparticle ones (which are linear in the \( I_j \)), suggests an interpretation of a nontrivial CFT as describing free massless “particles” obeying “generalized statistics”.

Hence a massive-conformal dictionary reconciles and provides a bridge between the two pictures. In particular, such a dictionary can be used to represent the CFT partition function as a sum over multi-particle states with the momenta of the “constituent particles” being quantized like ordinary free fermions, but with a nontrivial dispersion relation (cf. (3.13) and (4.9)). This observation may provide a clue for generalizations to other models. The next simplest class of models, after the ones considered here, consists of theories with diagonal \( S \)-matrix but more than one type of particle. The strategy we propose for obtaining a (conjecture for a) massive-conformal dictionary in such theories is summarized by the following vague prescription: find a “nice” representation of the partition function of the rational UV CFT as a sum over multi-particle states labeled by quantum numbers of several types of free fermions.

In the single-particle cases we studied, such “nice” representations – whose validity is independent of the question of the massive-conformal correspondence – were obtained in two steps. The first involved the recasting of the CFT partition function, as expressed via fermionic quasiparticle representations for the characters, in a form in which the separate sums over left- and right-movers are combined into a single sum. The states summed over in this single sum, which are still restricted by some “generalized fermionic statistics”, were then mapped in the second step onto the required ordinary fermionic states, using inspiration from the Bethe Ansatz description of the large-volume levels in the perturbed theory.

As an example of how the first step can be implemented consider the case of the minimal models \( \mathcal{M}(2, 2n+3) \). Using Gordon’s theorem \([30]\), whose analytical version \([31]\) is
encountered in the fermionic $n$-quasiparticle representations for the relevant Virasoro characters, the following generalization of eq. (3.9) is obtained:

$$Z_{\mathcal{M}(2,2n+3)} = |q|^{-\tilde{c}(2,2n+3)/12} \sum_{N=0}^{\infty} \sum_{\{p_k\}_{k=1}^{N} \subset \mathbb{Z}, \ p_{k+1} \geq p_k} \sum_{p_{k+n} \geq 1} 1 \sum_{p_k \geq 2} \sum_{\{p_k\}_{k=1}^{N}} \left( \frac{q}{\bar{q}} \right) \sum_{p_k} / 2,$$

(5.1)

where $\tilde{c}(2,2n+3) = \frac{2n}{2n+3}$ and $\tilde{d}_{1,r}^{(2,2n+3)} = \frac{n(n+1)-(r-1)(2n+2-r)}{2n+3} \quad (r = 1, 2, \ldots, n + 1)$. The restriction of having no more than $n$ "momenta" $p_k$ of the same value, imposed in (5.1), suggests that we are dealing with $n$ different types of particles, which is indeed the case in both the integrable $\phi_{1,3}$- and $\phi_{1,2}$-perturbations of $\mathcal{M}(2, 2n + 3)$ \textsuperscript{[32]} [33]. However, in (5.1) these $n$ types of particles appear "indistinguishable", which unfortunately prevents a straightforward use of this formula for implementing the second step mentioned above. Hence disentangling the massive-conformal dictionary in these and other theories remains an intriguing challenge.

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