Design and Analysis of Transmit Beamforming for Millimetre Wave Base Station Discovery

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Abstract

In this paper, we develop an analytical framework for the initial access (a.k.a. base station (BS) discovery) in a millimeter wave (mm-wave) communication system and propose an effective strategy for transmitting the reference signals (RSs) used for BS discovery. Specifically, by formulating the problem of BS discovery at user equipments as hypothesis tests, we derive a detector based on the Generalised Likelihood Ratio Test (GLRT) and characterise the statistical behaviour of the detector. Theoretical results obtained allow analysis of the impact of key system parameters on the performance of BS discovery, and can provide guidance in choosing suitable system parameters for initial access. Based on the theoretical results, we identify the desirable beam patterns for RS transmission. Using the method of large deviations, we show that the desirable beam patterns minimise the average miss-discovery probability of cell-edge UEs with line-of-sight path to the BS. We then propose a strategy for RS transmission that uses a pre-designed beamforming codebook to approximate the desirable patterns. The proposed strategy is designed to provide flexible choices of different codebook sizes and the associated beam widths to better approximate the optimal patterns when the available hardware at BS is limited. Numerical results demonstrate the effectiveness of the proposed method.

I. INTRODUCTION

Millimetre-Wave (mm-wave) communication is an important ingredient in fifth generation cellular networks (5G) [1]–[3] due to the large contiguous available bandwidth in the 30-300 GHz spectrum, and has already attracted much research attention [4]–[7]. Due to the high free-space pathloss at mm-wave frequencies, it is well understood that beamforming facilitated by a large number of antennas at Base Stations (BSs) and/or User Equipments (UEs) is essential.

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to achieve reasonable coverage range and high spectrum efficiency \cite{8}. The reliance on high-gain beamforming at the data transmission phase, however, has posed challenges in the design of initial access for mm-wave systems, as conventional designs for systems operating at lower frequencies (e.g., below 6 GHz) may not work for mm-wave systems \cite{9}.

In the initial access of existing cellular systems such as the Long Term Evolution (LTE), BSs periodically broadcast reference signals (RSs) in synchronisation channels and UEs detect the presence of the RS and determine the correct frame timer of the BS. Due to the favourable propagation characteristics in the sub-6 GHz bands, the RS in LTE is transmitted omnidirectionally from the BSs \cite{10} without sophisticated beamforming. Such a transmission strategy, if adopted in mm-wave systems, will lead to a much smaller discoverable range than the intended coverage range. In other words, a UE that can achieve a reasonably high data-rate when high-gain beamforming is employed may not be able to discover and synchronise to the BS \cite{9}. In this case, communication link establishment fails at the first step and initial access will then be the bottleneck of a mm-wave cellular system. This paper tackles the problem of mm-wave BS discovery and proposes an effective strategy for transmitting the RSs in the initial access.

Preliminary studies on the mm-wave initial access problem can be found in \cite{11}, \cite{12}. Desai. et al. in \cite{11} considered a signal to noise ratio (SNR)-based criteria for mm-wave BS discovery, where a successful detection of a BS is declared by a UE when the received SNR is higher than a threshold. Based on this assumption, they developed a hierarchical search algorithm using the beamforming scheme from \cite{5} to improve the SNR of the received RSs. Abu-Surra. et al. in \cite{12} numerically showed that a certain received SNR of the RSs is required to guarantee a targeted probability of BS discovery. They then developed a beam-broadening technique to accommodate appropriate beamforming transmissions to attain the required SNR. While the SNR criteria to some extent can indicate the performance of BS discovery, it does not capture the fact that the performance of mm-wave BS discovery depends also on the length of RS sequence and the time a UE uses for BS searching.

Another approach to mm-wave BS discovery and synchronisation is to formulate the problem as a Generalised Likelihood Ratio Test (GLRT) \cite{13}. In \cite{13}, they derived a detector by assuming a single-path channel. While interesting observations were made that showed the superiority of an omnidirectional transmission (without beamforming) of the RSs over directional transmission employing random beamformers, they did not consider more sophisticated beamforming
strategies such as optimised beamformers for the transmission of RSs.

In this paper, we develop an analytical framework for mm-wave BS discovery and synchronisation. This framework provides a systematic approach to analysing the impact of system parameters on the performance of mm-wave BS discovery. Specifically, in contrast to the single-path channel model in [13], we derive a GLRT detector based on a general multipath channel model. We explicitly characterise the statistical properties of the GLRT detector, from which we establish the relationship between the performance of mm-wave BS discovery and key system design parameters, including the RS sequence length and the time used for BS discovery at the UEs.

Based on the analytical results we derive, we identify desirable beam patterns for transmitting the RSs in initial access. Using the method of large deviations [14], we show that the patterns identified minimise the average miss-discovery probability of cell-edge UEs who have line-of-sight path to the BS. We then propose an effective method for constructing the beamforming codebooks for RS transmission to approximate the desirable patterns.

The proposed method allows flexible choice of different codebook sizes and the corresponding beam widths, which is beneficial when hardware resources at BSs are limited. The effectiveness of the proposed method is validated through numerical examples that take into account practical hardware constraints at BSs and realistic mm-wave multipath channel models. For a practical choice of UE searching time, numerical results show that our proposed beamforming strategy provides orders of magnitude improvement compared to random beamformers.

**Notations**: boldface uppercase letters and boldface lowercase letters are used to denote matrices and vectors, respectively, e.g., $\mathbf{A}$ is a matrix and $\mathbf{a}$ is a vector. Notations $(\cdot)^T$ and $(\cdot)^\dagger$ denote transpose and conjugate transpose, respectively. $\mathbf{I}_M$ is an $M \times M$ identity matrix. Notation $\|\mathbf{a}\|_2$ stands for the $l_2$ norm of vector $\mathbf{a}$, $\|\mathbf{A}\|_F$ stands for the Frobenius norm of matrix $\mathbf{A}$, and $\text{Tr}\{\mathbf{A}\}$ stands for the trace of matrix $\mathbf{A}$. Finally, we use $\mathbb{E}\{\cdot\}$ to denote the expectation operation.

**II. System Model and Problem Description**

Consider a mm-wave communication system, in which each BS broadcasts its RS (synchronisation sequence) periodically to allow UEs in its coverage range to detect the presence of the RS and to synchronise their frame timing. Fig. 1(a) depicts the frame structure of the transmit signal, where $T_{\text{slot}}$ is the duration of a slot and $T_{\text{rs}} < T_{\text{slot}}$ is the duration of each RS.
We also consider that directional transmissions of the RS are possible and the transmission pattern is repeated for every $J$ slots. More specifically, as illustrated in Fig. 1 (a), a set of $J$ beamformers $w_j \in \mathbb{C}^{N_T \times 1}$, $j = 1, \ldots, J$, are sequentially employed for transmitting the RS sequence during $J$ consecutive slots. Here $N_T$ is the number of transmit antennas.

Denote the intended coverage angular space of a mm-wave BS as $\Omega$ and suppose that the beamformers come from a pre-designed codebook. To provide universal coverage of $\Omega$, a beamforming codebook consisting of multiple beamformers can be adopted, with narrow beams pointing in different directions, as illustrated in Fig. 1 (b). Alternatively, a codebook with a single beamformer of relatively wide beam may also be adopted, as illustrated in Fig. 1 (c). An omnidirectional transmission is thus a special case in such a setup and can be realised with $w_j = [1, 0, 0, \ldots, 0]^T$, $\forall j$.

The waveform of the RS in its complex baseband is denoted as $s(t)$, where $s(t) \neq 0$ iff $t \in [0, T_{rs}]$. The transmitted signal waveform $x(t)$ is then represented as:

$$x(t) = \sum_{l=-\infty}^{+\infty} w_{[l]J} s(t - (l - 1)T_{slot}),$$

where $[l]J$ is a modified modulo operation with $[l]J = j$, if $l = kJ + j$, and $1 \leq j \leq J$, $k \in \mathbb{Z}$. To facilitate the analysis, we assume that the channel used to transmit the RS is frequency flat and does not vary within each slot. Denoting the channel between the BS and a UE at slot $l$ as
\( \mathbf{H}_l \in \mathbb{C}^{N_R \times N_T} \) with \( N_R \) being the number of receive antennas, the received signal waveform at UE can be represented as:

\[
y(t) = \sum_{l=-\infty}^{+\infty} \mathbf{H}_l \mathbf{w}_{[l]} \mathbf{s}(t - (l - 1)T_{\text{slot}} - \tau_0) + \mathbf{z}(t),
\]

\[= \sum_{l=-\infty}^{+\infty} \mathbf{h}_l \mathbf{s}(t - (l - 1)T_{\text{slot}} - \tau_0) + \mathbf{z}(t), \tag{2}\]

where \( \mathbf{h}_l \triangleq \mathbf{H}_l \mathbf{w}_{[l]} \in \mathbb{C}^{N_R \times 1} \) is the effective channel (after transmit beamforming) of slot \( l \) and \( \tau_0 \) is a BS time offset and is unknown to UE initially. The noise \( \mathbf{z}(t) \) is assumed to be spatially independent complex Gaussian with zero-mean and an unknown variance \( \sigma^2 \). In the following, we consider a fully digital receiver at UEs and note that power-efficient digital mm-wave receivers are possible by adopting low-resolution analog-to-digital convertors at UE, with a minimal loss of the received SNR \([13]\). We also note that receiver combining using either analog combiner or hybrid combiner \([15]\) can be incorporated by (2) and thus our subsequent analysis applies. For instance, with a hybrid combiner, \( \mathbf{h}_l \in \mathbb{C}^{N_{RF} \times 1} \) is the effective channel after transmit beamforming and receive combining, where \( N_{RF} \) is the number of RF chains at the UE; in the case of adopting an analog combiner, the effective channel is a scalar (\( N_{RF} = 1 \)).

In the initial access, a UE attempts to detect the presence of \( \mathbf{s}(t) \) and find the correct synchronisation timer, i.e., \( \tau_0 \). We assume that each UE performs detection based on signals collected from \( L \geq J \) consecutive slots. Since the RS is broadcast in every slot, it is sufficient to assume that \( \tau_0 \) lies in the interval \([0, T_{\text{slot}}]\). The UEs therefore only need to examine signals collected at time lags \( \tau \in [0, T_{\text{slot}}] \) to determine if the RS is detected.

Without loss of generality, we consider that the signals from the observation interval \([\tau, \tau + LT_{\text{slot}}]\) are used for detection at time lag \( \tau \). The UE samples the signals received in the observation period at rate \( 1/T_s \), with \( N_{\text{slot}} \) samples per slot. We assume that \( \tau_0 \) is discrete and takes one of the \( N_{\text{slot}} \) possible values.

Since the RS has a duration \( T_{rs} < T_{\text{slot}} \), only the signals from the \( L \) subintervals, \([lT_{\text{slot}} + \tau, (l - 1)T_{\text{slot}} + T_{rs} + \tau], l = 1, \ldots, L\), are used to test the hypothesis that there is a reference signal and that the offset is \( \tau \). We denote

\[
y_{l,\tau}(t') \triangleq y(t + \tau) = \mathbf{h}_l \mathbf{s}(t - (l - 1)T_{\text{slot}} + \tau - \tau_0) + \mathbf{z}(t), \tag{3}\]
as the signals in the \( l \)th subinterval, where \( t' = t - ((l - 1)T_{\text{slot}} + \tau) \in [0, T_{\text{rs}}] \). The corresponding discrete-time samples are stored in matrix \( Y_\tau = [Y_{1,\tau}^T, \ldots, Y_{L,\tau}^T] \in \mathbb{C}^{L N_R \times N_s} \), where \( Y_{l,\tau} = [y_{l,\tau}(T_s), y_{l,\tau}(2T_s), \ldots, y_{l,\tau}(N_s T_s)] \in \mathbb{C}^{N_R \times N_s} \), and \( N_s \leq N_{\text{slot}} \) is the number of samples per slot for the RS, with \( T_{\text{rs}} = N_s T_s \). The sampled RS is \( s = [s(T_s), s(2T_s), \ldots, s(N_s T_s)]^T \in \mathbb{C}^{N_s \times 1} \). Similar to [16], assuming the channel follows:

\[
    h_l(\tau) = \begin{cases} 
        h_l, & \tau = \tau_0 \\
        0, & \tau \neq \tau_0 
    \end{cases}
\]

(4)

the problem of mm-wave BS detection and synchronisation is formulated as the following binary test:

\[
    \begin{align*}
    \mathcal{H}_1 : & \quad Y_\tau = h s^T + Z_\tau, \\
    \mathcal{H}_0 : & \quad Y_\tau = Z_\tau.
    \end{align*}
\]

(5)

where \( Z_\tau \in \mathbb{C}^{LN_R \times N_s} \) follows a similar construction to \( Y_\tau \) and \( h = [h_1^T, \ldots, h_L^T]^T \in \mathbb{C}^{LN_R \times 1} \).

The hypothesis test is repeated for all \( N_{\text{slot}} \) discrete values of \( \tau \).

### III. Generalised Likelihood Ratio Test for BS Discovery

In the initial access phase, it is reasonable to assume that UEs have no prior knowledge of any parameters in the test (5), including the channel \( h \) and the noise variance \( \sigma^2 \). For this reason, we employ the Generalised Likelihood Ratio Test (GLRT) method to perform the hypothesis testing.

The GLRT for this problem is represented as the following:

\[
    L_G'(\tau) = \frac{\max_{h, \sigma^2_1} p(Y_\tau|\mathcal{H}_1; h, \sigma^2_1)}{\max_{\sigma^2_0} p(Y_\tau|\mathcal{H}_0; \sigma^2_0)} \geq \gamma',
\]

(6)

where \( L_G'(\tau) \) is the test statistic, \( \gamma' \) is a threshold, \( p(Y_\tau|\mathcal{H}_1; h, \sigma^2_1) \) is the conditional probability density function (PDF) of \( Y_\tau \) under \( \mathcal{H}_1 \) with given channel \( h \) and noise variance \( \sigma^2_1 \), and \( p(Y_\tau|\mathcal{H}_0; \sigma^2_0) \) is the conditional PDF of \( Y_\tau \) under \( \mathcal{H}_0 \) for a given noise variance \( \sigma^2_0 \).

To derive the GLRT test statistic \( L_G'(\tau) \), we first need to obtain the maximum likelihood (ML) estimate of \( h \) and \( \sigma^2 \) under \( \mathcal{H}_1 \) and the ML estimate of \( \sigma^2_0 \) under \( \mathcal{H}_0 \). In the following, for notational simplicity, we drop the dependency of all metrics on \( \tau \) when no ambiguity is caused.

Since all entries of \( Z \) are i.i.d. complex Gaussian with zero mean and variance \( \sigma^2 \), the PDF
of $Z$ can be represented as:

$$p(Z) = \frac{1}{(\pi \sigma^2)^N} \exp \left\{ -\frac{\|Z\|_F^2}{\sigma^2} \right\},$$  \hspace{1cm} (7)

where $N = N_R N_S L$.

Therefore, it can be obtained that:

$$p(Y|\mathcal{H}_0; \sigma_0^2) = \frac{1}{(\pi \sigma_0^2)^N} \exp \left\{ -\frac{\|Y\|_F^2}{\sigma_0^2} \right\}$$  \hspace{1cm} (8)

and

$$p(Y|\mathcal{H}_1; h, \sigma_1^2) = \frac{1}{(\pi \sigma_1^2)^N} \exp \left\{ -\frac{\|Y - hs^T\|_F^2}{\sigma_1^2} \right\}.$$  \hspace{1cm} (9)

It is straightforward to show that under $\mathcal{H}_0$, the ML estimate of $\sigma_0^2$ is

$$\hat{\sigma}_0^2 = \frac{\|Y\|_F^2}{N} = \frac{1}{N} \sum_{l=1}^L \|Y_l\|_F^2,$$  \hspace{1cm} (10)

and under $\mathcal{H}_1$, the ML estimate of $h$ and $\sigma_1^2$ are:

$$\hat{h} = \frac{1}{\|s\|_2^2} Ys^*,$$  \hspace{1cm} (11)

and

$$\hat{\sigma}_1^2 = \frac{\|Y - \hat{h}s^T\|_F^2}{N},$$  \hspace{1cm} (12)

respectively, where $s^* = (s^\dagger)^T$.

Substituting (11) into (12), it can be obtained that

$$\hat{\sigma}_1^2 = \frac{1}{N} \left( \|Y - \frac{1}{\|s\|_2^2} Ys^* s^T\|_F^2 \right),$$

$$= \frac{1}{N} \text{Tr} \left\{ \left( Y - \frac{1}{\|s\|_2^2} Ys^* s^T \right) \left( Y - \frac{1}{\|s\|_2^2} Ys^* s^T \right)^\dagger \right\}$$

$$= \frac{1}{N} \left( \|Y\|_F^2 - \frac{1}{\|s\|_2^2} \|Ys^*\|_F^2 \right),$$

$$= \frac{1}{N} \sum_{l=1}^L \left( \|Y_l\|_F^2 - \frac{1}{\|s\|_2^2} \|Y_l s^*\|_2^2 \right).$$  \hspace{1cm} (13)

Substituting (10), (11) and (13) into (6), the test statistics $L_G'(\tau)$ can be represented as:

$$L_G'(\tau) = \left( \frac{\hat{\sigma}_0^2}{\hat{\sigma}_1^2} \right)^N \left( \frac{\sum_{l=1}^L \|Y_{l,\tau}\|_F^2}{\sum_{l=1}^L \left( \|Y_{l,\tau}\|_F^2 - \frac{1}{\|s\|_2^2} \|Y_{l,\tau} s^*\|_2^2 \right)} \right)^N.$$  \hspace{1cm} (14)
Equivalently, the test (6) can be re-written as

\[ L_G(\tau) = \frac{\sum_{l=1}^{L} \frac{1}{\|s\|^2_2} \|Y_{l,\tau}s^*\|^2_2}{\sum_{l=1}^{L} \left( \|Y_{l,\tau}\|^2_2 - \frac{1}{\|s\|^2_2} \|Y_{l,\tau}s^*\|^2_2 \right)} \overset{\mathcal{H}_1}{\gtrless} \frac{\gamma_1}{\mathcal{H}_0} \]

where \( \gamma = (\gamma')^{1/N} - 1 > 0 \) is the test threshold.

Based on the GLRT detector derived, a UE attempting to discover a BS needs to calculate the test statistic in (15) and compare it with the threshold \( \gamma \), which involves computing the correlation between a candidate RS sequence and the received signal samples and calculating the received signal energy.

The performance of BS discovery based on the derived GLRT detector depends on system parameters including the length of RS sequence \( N_s \) and the UE searching time \( L \), and on the choice of the threshold \( \gamma \). To analyse the impact of these parameters on the performance of mm-wave BS discovery, we present the following proposition.

**Proposition 1.** The GLRT test statistic \( L_G(\tau) \) has the following statistical properties:

\[
(N_s - 1)L_G(\tau) \sim
\begin{cases}
\mathcal{F}(2N_R L, 2N_R L(N_s - 1), 0) & \text{under } \mathcal{H}_0 \\
\mathcal{F}(2N_R L, 2N_R L(N_s - 1), \lambda) & \text{under } \mathcal{H}_1
\end{cases}
\]

where

\[ \lambda = \frac{2\|s\|^2_2}{\sigma^2} \sum_{l=1}^{L} \|h_l\|^2_2 = \frac{2P_T N_s}{\sigma^2} \sum_{l=1}^{L} \|h_l\|^2_2 \]

and \( P_T \) is the average transmit power. Here \( \mathcal{F}(n_1, n_2, \lambda) \) denotes non-central F-distribution with degrees of freedom (DoFs) \( n_1 \) and \( n_2 \) and noncentrality parameter \( \lambda \).

**Proof.** See Appendix A

Proposition 1 characterises the distribution of the test statistic \( L_G(\tau) \) using key system parameters such as the length of RS sequence \( N_s \), the cell search interval \( L \) and the transmission power. Since the PDF of the F-distribution is known, it is convenient to evaluate the impact of these parameters on the performance of BS discovery. The results obtained can be used in choosing suitable values of these parameters to provide satisfactory BS discovery performance.

In particular, as the probability of false alarm, i.e., the probability of declaring \( \mathcal{H}_1 \) under \( \mathcal{H}_0 \),
is independent of the effective channels, it is convenient to choose $\gamma$ such that the probability of false alarm is below some target $P_{FA}$. Recall that a UE needs to examine $N_{\text{slot}}$ time lags, corresponding to $N_{\text{slot}}$ tests. For the $i$th test with a hypothesised time lag $\tau_i$, the false alarm event is denoted as $FA_i = \{L_G(\tau_i) \geq \gamma \text{ under } H_0\}$. Then the probability of false alarm can be represented as $Pr\{\bigcup_{i=1}^{N_{\text{slot}}} FA_i\}$. Using the union bound, it can be shown that

$$Pr\left\{\bigcup_{i=1}^{N_{\text{slot}}} FA_i\right\} \leq \sum_{i=1}^{N_{\text{slot}}} Pr\{FA_i\} = N_{\text{slot}} Pr\{FA\}, \quad (17)$$

It then becomes clear that to meet a false alarm target $P_{FA}$, the test threshold $\gamma$ can be chosen such that the following equation is satisfied

$$1 - F(\gamma|2NR_L, 2NR_L(N_s - 1), 0) = \frac{P_{FA}}{N_{\text{slot}}} \quad (18)$$

where $Pr\{FA\} = 1 - F(\gamma|2NR_L, 2NR_L(N_s - 1), 0)$ and $F(x|n_1, n_2, \lambda)$ is the cumulative distribution function of the $F$-distribution $F(n_1, n_2, \lambda)$.

We now focus on the hypothesis $H_1$. Denote the probability of miss-detection (which is the probability of claiming $H_0$ under $H_1$) as $P_{\text{miss}} \triangleq Pr\{L_G(\tau) \leq \gamma | H_1\}$. We note that under $H_1$, both channel fluctuations and noise randomness can cause miss-detection, since the non-centrality parameter $\lambda$ depends on the effective channel gains. In the following, we denote $\bar{h}_L \triangleq \frac{1}{L} \sum_{l=1}^{L} \|h_l\|^2_2$ as the sample mean of the effective channel gain and assume that for any $\xi > 0$, there exists an $\underline{h}$ such that $Pr\{\bar{h}_L < \underline{h}\} = \xi$. Based on this assumption, we derive an upper bound of $P_{\text{miss}}$ in the following lemma, which captures both channel fluctuations and noise randomness.

**Lemma 1.** The probability of miss detection is upper bounded:

$$P_{\text{miss}} \leq \inf_{0<\xi<1} \left\{ \xi + (1 - \xi)F(\gamma|2NR_L, 2NR_L(N_s - 1), \bar{h}_L) \right\}, \quad (19)$$

where $\eta = \frac{2PrN_s}{\sigma^2} \bar{h}_L$.

**Proof.** See Appendix B.

Eq. (19) provides a convenient way to assess the probability of miss-detection under different system configurations and can thus provide guidance in choosing system parameters. Fig. 2
provides an illustrative example, where the dashed lines are obtained directly from simulations and the solid lines are obtained from simulations.

In producing Fig. 2, the RS sequence length is fixed at $N_s = 100$ and the number of samples in a slot is $N_{slot} = 5000$, corresponding to a 2% overhead due to the transmissions of RSs. The test threshold $\gamma$ is determined according to (18) with $P_{FA} = 0.001$. The effective channels $h_l$ are Rician fading with a $K$ factor equal to 10 dB. Various values of the average SNR (of the RS signal) at each receive antenna, i.e., $SNR_{rs} = \frac{P_T N_R \sigma^2}{E\{\bar{h}_L\}}$, are considered. In evaluating Eq. (19), $\xi$ is chosen to be $10^{-4}$ for $SNR_{rs} \geq -23$ dB and at $\xi = 10^{-5}$ for $SNR_{rs} = -21$ dB. The corresponding values of $\eta_L$ are obtained numerically.

It can be seen that the theoretical results obtained capture well the behaviour of $P_{miss}$ under different system configurations. In particular, it provides satisfactory accuracy in choosing suitable system parameters to guarantee a certain miss-detection rate. For instance, the theoretical result for $SNR_{rs} = -23$ dB suggests $L = 28$ for a target miss-detection rate of $10^{-3}$, which is very close to the value ($L = 24$) obtained from simulations.

**Remark 1.** Conditioned on a sequence of channel realisations $H_l$, $l = 1, \ldots, L$, the probability of miss-detection is $F(\gamma|2N_R L, 2N_R L(N_s - 1), L\eta_L)$, where $\eta_L = \frac{\lambda}{L} = \frac{2P_T N_s \bar{h}_L}{\sigma^2}$ is the normalised non-centrality parameter depending on the transmit beamformers. Since the CDF of the F-distribution, $F(n_1, n_2, \lambda)$ is monotonically decreasing with respect to $\lambda$ when the DoFs $(n_1, n_2)$
are fixed \[17\], \( F(\gamma|2N_RL, 2N_RL(N_s - 1), L\eta|_L) \) is monotonically decreasing with respect to the mean channel gain \( \bar{h}_L \) when \( N_s \) and \( L \) are fixed. To enhance the reliability of BS discovery for UEs at a certain direction, the RS transmit beamformers can be steered towards this direction to yield a larger \( \bar{h}_L \). However, this comes at the expense of sacrificing the beamforming gain at other directions, leading to poorer BS discovery performance at those directions. To provide a universal coverage for all directions, we propose to design the RS transmit beamformers such that \( \eta_L \) for UEs at different directions are equalised. In the next section, we elaborate on this design in more detail.

IV. BEAMFORMING DESIGN FOR RS TRANSMISSION

In this section, we design the RS transmit beamformers with a consideration of the worst-case UEs, i.e., the edge-UEs who are located at the maximum supportable range (MSR) of the BS. For a direction (with respect to the BS) with nearby blockage, the MSR is the distance between the BS and the obstacle causing blockage. For a direction without blockage, the MSR can be determined as the distance between a UE and the BS at which a minimum data-rate can be achieved \[13\]. The pathloss of an edge-UE at direction \( \phi \) (with respect to the BS) is denoted as \( \alpha(\phi) \). The angularly dependent \( \alpha(\phi) \) captures the possibility that the MSR of a mm-wave BS is circularly asymmetric due to uneven blockage across different directions. To provide a universal discovery rate for edge-UEs across all directions, we propose to synthesise the RS transmit beamformers \( w_j, j = 1, \ldots, J \) such that the average beamforming gain matches as close as possible to a desirable pattern:

\[
\frac{1}{J} \sum_{j=1}^{J} G_j(\phi) = G^*(\phi) = \begin{cases} \frac{\kappa^*}{\bar{\alpha}} \alpha(\phi), & \forall \phi \in \Omega \\ 0, & \text{otherwise} \end{cases}
\]  

(20)

where \( G_j(\phi) \) is the transmit beamforming gain of \( w_j \) in direction \( \phi \) and \( G^*(\phi) \) is the desirable beam pattern. Here \( \bar{\alpha} \triangleq \frac{1}{|\Omega|} \int_{\Omega} \alpha(\phi)d\phi \) and \( \kappa^* = \frac{4\pi}{|\Omega|} \) is a gain factor achieved when beamformers \( w_j, j = 1, \ldots, J \) all have zero energy leakage outside \( \Omega \), i.e., \( \frac{1}{4\pi} \int_{\Omega} G^*(\phi)d\phi = 1 \). Notation \( |\Omega| \) denotes the solid angle spanned by angular interval \( \Omega \).

It can be seen that any sets of the beamformers with an average pattern \( G^*(\phi) \) equalise the amount of energy received by all edge-UEs across the angular interval \( \Omega \). In the following, by considering a simplified, yet important, LOS channel model for the edge-UEs, we carry out an
error exponent analysis on the probability of miss-detection, from which we show that any set of beamformers that do not satisfy (20) is suboptimal in the regime $L \to \infty$. We later present a systematic method to synthesise the beamformers to approach $G^*(\phi)$.

A. Error exponent analysis of $F(\gamma|2N_RL, 2N_RL(N_s - 1), L\eta)$ as $L \to \infty$

In this subsection, for tractability, we assume single-path channels that capture the dominant path for the edge-UEs and that the channels do not change during the searching interval. Channels of UEs in a pure LOS environment belong to this category. We next consider the number of slots used for BS discovery is $L = KJ$. Here $K \in \mathbb{N}$ is a scaling factor we used in the subsequent asymptotic analysis (e.g., when $K \to \infty$, it follows that $L \to \infty$). The normalised non-centrality parameter, $\eta(\phi)$, for an edge-UE at direction $\phi$ can be written as:

$$\eta(\phi) \triangleq \frac{\lambda(\phi)}{L} = \frac{1}{KJ} \frac{2P_T N_R N_s}{\alpha(\phi) \sigma^2} \sum_{k=1}^{K} \sum_{j=1}^{J} G_j(\phi) = \frac{2P_T N_R N_s}{\alpha(\phi) \sigma^2} G(\phi),$$

where $G(\phi) = \frac{1}{J} \sum_{j=1}^{J} G_j(\phi)$ is the average beamforming gain in direction $\phi$. The probability of miss-detection for edge-UEs at direction $\phi$ can be represented as

$$p_{\text{miss}}(\phi) = F(\gamma|2N_RL, 2N_RL(N_s - 1), L\eta(\phi)).$$

For assessing the overall performance of BS discovery, we consider the average miss-detection probability

$$\bar{p}_{\text{miss}} = \mathbb{E}_{\phi \in \Omega \{ p_{\text{miss}}(\phi) \}} = \int_{\Omega} p_{\text{miss}}(\phi) p(\phi) d\phi$$

as the main performance metric, where $p(\phi)$ is the PDF of the directions of the edge-UEs with $\int_{\Omega} p(\phi) d\phi = 1$.

In the following, by revisiting the Gartner-Ellis theorem \[14\], Chapter 2.3, pp. 43], we establish the large deviation principle of the test statistic $L_G$ when $\eta$ is fixed, and then characterise the asymptotic behaviour of $\bar{p}_{\text{miss}}$.

**Proposition 2.** The probability of miss-detection $p_{\text{miss}} = F(\gamma|2N_RL, 2N_RL(N_s - 1), L\eta)$ satisfies

$$\lim_{L \uparrow \infty} - \frac{1}{L} \log p_{\text{miss}} = \begin{cases} I^*(\eta, \gamma), & \gamma < \frac{2N_R + \eta}{2N_R(N_s - 1)} \\ 0, & \text{otherwise} \end{cases}$$

(24)
where \( I^*(\eta, \gamma) \) is the rate function given by:

\[
I^*(\eta, \gamma) = \frac{\eta}{2} \left( 1 - \frac{\gamma v^*}{N_R + \sqrt{N_R^2 + \gamma \eta v^*}} \right) + N_R(N_s - 1) \log \frac{2N_R(N_s - 1)}{v^*} - N_R \log \frac{\gamma v^*}{N_R + \sqrt{N_R^2 + \gamma \eta v^*}}.
\]  

(25)

Here \( v^* = \frac{x^2 - N_R^2}{\eta \gamma} \) and \( x^* > 0 \) is a solution to the following equation:

\[
\frac{\gamma + 1}{\eta \gamma} (x^2 - N_R^2) - x - N_R - 2N_R(N_s - 1) = 0.
\]

(26)

Moreover, when \( \gamma < \frac{2N_R + \eta}{2N_R(N_s - 1)} \), \( I^*(\eta, \gamma) \) is monotonically increasing in \( \eta \).

Proof. See Appendix C.

Proposition 3. Let \( G(\phi) \) be the average beamforming gain of a set of \( J \) beamformers used for RS transmission and \( \eta(\phi) \) be the normalised non-centrality parameter defined by (21). If \( \exists \Omega^- \subset \Omega \) and \( \eta^- \) such that \( \forall \phi \in \Omega^- \), \( \eta(\phi) \leq \eta^- \) and \( \int_{\Omega^-} p(\phi) d\phi > 0 \), the average miss-detection probability \( \bar{p}_{miss} \) satisfies:

\[
\lim_{L \to \infty} \frac{1}{L} \log \bar{p}_{miss} \leq I^*(\eta^-, \gamma), \text{ if } \gamma < \frac{2N_R + \eta^-}{2N_R(N_s - 1)}
\]

(27)

where \( I^*(\eta, \gamma) \) is the rate function given by (25).

Proof. See Appendix D.

Proposition 2 demonstrates that an approximated upper bound to the miss-detection probability can be obtained as follows:

\[
p_{miss} \approx \begin{cases} e^{-LI^*(\eta, \gamma)}, & \text{if } \gamma < \frac{2N_R + \eta}{2N_R(N_s - 1)} \\ 1, & \text{otherwise} \end{cases}
\]

(28)

Although the approximation provided in (28) ignores all sub-exponential terms, we have found that it provides satisfactory characterisation of the behaviour of \( p_{miss} \), under various values of \( \eta \), as shown in Fig. 3 (In producing Fig. 3 the value of \( \eta \) is varied by considering different values of \( SNR_{rs} \). Other system parameters are the same as those for Fig. 2). In particular, for the range of \( L \) for which \( p_{miss} \) is small (e.g., \( p_{miss} < 10^{-2} \)), it can be seen that (28) provides an accurate approximation to the slope of \( p_{miss} \) as \( L \) increases. Moreover, a larger \( \eta \) leads to a
steeper slope, which is consistent with the monotonicity of $I^*(\eta, \gamma)$ with respect to $\eta$.

Proposition 3 further demonstrates that the asymptotic behaviour of $\bar{p}_{\text{miss}}$ is dominated by those directions with the smallest normalised non-centrality parameter. RS transmit beamformers that maximise the minimum possible $\eta^-$ yield the steepest slope, and thus will outperform beamformers with a smaller $\eta^-$ when $L$ is sufficiently large. This insight motivates designing RS transmit beamformers by solving the following max-min optimisation problem:

$$\max_{G(\cdot)} \min_{\phi \in \Omega} \eta(\phi), \quad \text{s.t.} \quad \frac{1}{4\pi} \int_{\Omega} G(\phi) d\phi \leq 1,$$

where the constraint follows from the law of energy conservation. When $N_R$, $N_s$ and $P_T$ are fixed, (29) is equivalent to:

$$\max_{G(\cdot)} \min_{\phi \in \Omega} \frac{G(\phi)}{\alpha(\phi)}, \quad \text{s.t.} \quad \frac{1}{4\pi} \int_{\Omega} G(\phi) d\phi \leq 1$$

(30)

It is easy to check that $G^*(\phi)$ given by (20) is a solution to (30).

Remark 2. Due to the requirement of zero-leakage outside the angular interval $\Omega$, the desired pattern (20) may not be achieved exactly with a finite number of transmit antennas. Beamformers that approximate the pattern (20) are thus expected in practical implementations. Moreover, the requirement of matching the beamforming gain to the cell-edge pathloss poses an additional challenge in getting a good approximation to (20) when $\alpha(\phi)$ is uneven across $\Omega$ and/or the
hardware resources at the BS are limited. Driven by these observations, in the following, we propose a beamformer synthesis strategy that uses \( M \leq J \) beamformers to approximate (20) and present an optimisation-based method to synthesise the beamformers. The proposed method is to select between several candidate values of \( M \) the one that provides the best approximation to (20).

B. Beamforming strategy for RS transmission

We propose to approximate (20) based on a beamforming codebook of size \( M \leq J \). The \( M \) beamformers are used in sequence to transmit the RS during the \( J \) consecutive slots. Since \( M \leq J \), some of the \( M \) beamformers could be used in more than one slot. We use the following three-step procedure to construct the beamformers and perform slot allocation to the beamformers:

1) The whole intended angular space is partitioned into \( M \) non-overlapping subintervals:
   \[
   \Omega^{(m)}, \quad m = 1, \ldots, M, \quad \bigcup_{m=1}^{M} \Omega^{(m)} = \Omega \quad \text{and} \quad \Omega^{(m_1)} \cap \Omega^{(m_2)} = \emptyset, \quad \forall m_1 \neq m_2.
   \]

2) Beamformer \( w^{(m)}, \quad m = 1, \ldots, M \) is synthesised to approximate the following pattern:
   \[
   G^{(m)}(\phi) = \begin{cases} 
   \kappa^{(m)} \frac{\alpha(\phi)}{\alpha}, & \forall \phi \in \Omega^{(m)} \\
   0, & \text{otherwise}
   \end{cases}
   \]
   \( \kappa^{(m)} = \frac{4\pi}{|\Omega^{(m)}|} \).

3) Beamformers \( w^{(m)} \)s are used in sequence to transmit the RS, i.e., during the \( J \) consecutive slots, with \( J_m \) slots using beamformer \( w^{(m)} \), where
   \[
   J_m = \frac{J |\Omega^{(m)}|}{|\Omega|}.
   \]

As an example, let us consider the scenario presented in Fig. 1(b), where \( M = 4 \) sub-intervals are chosen, each covering a quarter of the 60° sector, i.e., \( |\Omega^{(m)}| = |\Omega|/4, \quad m = 1, \ldots, 4 \). Further, assuming \( \alpha(\phi) = \bar{\alpha}, \quad \forall \phi \in \Omega \), the desired beamformer pattern as defined in (31) is uniform within each sub-interval with zero-leakage outside the interval, as illustrated by the polar plot of Fig. 4. Using (32), it can be obtained that \( J_m = \frac{J}{4}, \quad m = 1, \ldots, 4 \). This implies that one can sequentially transmit the RS using the \( M = 4 \) beamformers for 4 consecutive slots, and repeat for \( J/4 \) times to meet (20), as illustrated in Fig. 4.

The proposed procedure offers flexibility in synthesising the beamformers to provide the best approximation to \( G^*(\phi) \), by choosing different angular partitionings (the \( \Omega_m \)s and the value
Fig. 4. An illustration of the three-step procedure for designing RS transmission strategy using beamformers: (a) Partitioning of sub-intervals with $M = 4$ and the desired patterns in (31); (b) Frame allocations using the $M = 4$ beamformers for RS transmission.

of $M$). Such a flexibility is beneficial when $\alpha(\phi)$ is not uniform in $\phi$, since by slicing the whole angular interval into $M > 1$ sub-intervals, a smaller variation of the pathloss $\alpha(\phi)$ can be achieved within each sub-interval and a better approximation to the desired pattern can be achieved using practical beam synthesis techniques.

Even if $\alpha(\phi)$ is uniform such that an omnidirectional transmission within $\Omega$ is desirable, the offered flexibility is beneficial. For instance, consider the two sets of beamformers illustrated in Fig. 1 (b) and (c), with $M = 4$ and $M = 1$, respectively. In principle, they can achieve comparable approximations to (20) and thus comparable BS discovery performance if appropriate hardware resources are available. However, with limited hardware resources and imperfect approximations to (20), their real performance can be significantly different. As we will show later in Section V, the proposed strategy allows us to design different codebooks to attain the best BS discovery performance under different hardware limits.

C. Beam pattern synthesis methods

Various techniques may be used to synthesise beamformers to approximate the ones defined in (31). In one category of such techniques, only the phase of the signal at each antenna port is controlled [6], [12]. The transmitted signals across all antennas thus have a constant modulus. The other category controls both the phase and magnitude of the signal emitted from each antenna [15], [18] and as such, beamformers constructed are expected to provide better
approximations. However, extra hardware cost is required to implement such beamformers. In this section, we present beamformer synthesis methods for both the categories, namely, Constant modulus (CM) and Variable modulus (VM).

To flexibly control the beamwidth, we adopt the optimisation-based approach in [12] to synthesise the desired beam patterns. Specifically, when optimising \( w^{(m)} \), the problem is formulated as minimising a weighted mismatch between the desired pattern defined in (31) and the synthesised pattern from \( w^{(m)} \). For the CM case, beamformer \( w^{(m)} \) is represented as
\[
\frac{1}{\sqrt{N_T}} [e^{i\varphi(1)}, \ldots, e^{i\varphi(N_T)}]^T,
\]
where \( \varphi(n) \in [-\pi, \pi] \), \( n = 1, \ldots, N_T \), are the phase variables to be optimised and \( i = \sqrt{-1} \). For the VM case, beamformer \( w^{(m)} \) takes the form of
\[
[w_m(1)e^{i\varphi(1)}, \ldots, w_m(N_T)e^{i\varphi(N_T)}]^T,
\]
where \( w_m(n) \) are real-valued variables such that \( ||w^{(m)}||^2 = 1 \). An extra constraint, i.e., \( w_m(n) = w_m(N_T - n - 1) \), is imposed such that the synthesised pattern is symmetric with respect to the centre of the beam and to reduce the dimension of the optimisation. A genetic algorithm is applied to solve the corresponding global optimisation problems [19].

Fig. 5 gives an example of the synthesised patterns using a uniformly linear array (ULA) with \( N_T = 32 \) omnidirectional antenna elements and half wavelength antenna spacing. The designed beams are used to approximate a uniform pattern within a \( 60^\circ \) sector, with \( \phi \in [-30^\circ, 30^\circ] \) denoting the angle with respect to the normal direction of the ULA. Various beam widths, and thus partitioning of \( \Omega \) are considered by choosing different values of \( M \) among \( M = \{1, 2, 4\} \).

It can be seen that the VM beamformers provide better approximations to the desired pattern.

Fig. 5. Beam patterns with \( M \in \{1, 2, 4\} \): \( N_T = 32 \). Solid lines: synthesised patterns; Dashed lines: desirable patterns.
(indicated by dash lines) than the CM beamformers, with smaller leakage outside the intended interval and smaller variations on the beamforming gain within the intended interval. Both the two methods are able to provide good approximations to the desired pattern.

V. NUMERICAL RESULTS

In this section, we demonstrate the performance of BS discovery under the beamforming strategy proposed in Section IV. Throughout this section, uniformly linear arrays (ULAs) with $N_T = 32$ and $N_R = 16$ omnidirectional antenna elements at the BS and at the UE are considered. The intended coverage angular space is a $60^\circ$ sector with $\phi \in [-30^\circ, 30^\circ]$. Antenna spacing at both the BS and UEs is fixed at half a wavelength of the carrier frequency.

Whenever the sector is open at direction $\phi$, the corresponding pathloss $\alpha(\phi)$ at the MSR is determined by making the discoverable area matched with the supportable area in the subsequent phase of data transmission as in [13]. Specifically, we use the notion of minimum achievable data-rate $R_{th}$ when a UE is served by a mm-wave BS and is allocated by a typical bandwidth $W$: $R_{th}(\phi) = \rho W \log_2 (1 + \text{SNR}_{th}(\phi))$, where $\rho$ is a parameter that reflects the overhead level of the system. By considering the case that $R_{th}(\phi)$ is only achievable when the maximum transmit and receive beamforming gains (denoted as $G_{T}^{\text{max}}$ and $G_{R}^{\text{max}}$) are attained, the pathloss $\alpha(\phi)$ can be estimated as $\alpha(\phi) = \frac{P_T \sigma^2 G_{T}^{\text{max}} G_{R}^{\text{max}}}{\text{SNR}_{th}(\phi) W_{rs}} W$, where $W_{rs}$ is the bandwidth used for RS transmission. The channel is considered as Rician channel, where the $K$ factor (the ratio of the energy of the dominant path to the sum of the energy of the non-dominant paths) is to 13.2 dB, according to the measurement results presented in [20]. We further assume a uniform distribution of the directions of the dominant path for the edge-UEs in the angular domain $\Omega$. Details of the relevant system parameters are provided in Table I.

A. BS discovery performance with blockage

As we mentioned earlier in Section IV, the coverage of a mm-wave BS could be asymmetric, e.g., some directions have smaller MSR distances than others. In this subsection, we consider such an example, where half of the sector, e.g., $\phi \in [-30^\circ, 0^\circ]$, is blocked, as illustrated in Fig. 6. The calculation of the edge pathloss for the open directions, i.e., $\phi \in [0^\circ, 30^\circ]$ has been explained in this section. As for the directions with blockage, for simplicity, the edge pathloss is assumed to be one half of that in the open directions.
TABLE I
SIMULATION PARAMETERS

| Parameter                              | Value       |
|----------------------------------------|-------------|
| Duration of RS:                         | $T_{rs} = 100\mu s$ |
| Duration of slot:                      | $T_{slot} = 0.5\, ms$ |
| RS Bandwidth:                          | $W_{rs} = 10\, MHz$ |
| Sampling interval:                     | $T_s = 1/W_{rs}$ |
| Data bandwidth:                        | $W = 1\, GHz$ |
| Minimum data-rate:                     | $R_{th} = 10\, Mbps$ |
| False alarm target:                   | $P_{FA} = 0.001$ |
| Overhead parameter:                   | $\rho = 0.4$ |
| Maximum beamforming Gain               | $G_{max}^T = N_T, G_{max}^R = N_R$ |

Fig. 6. A 60$^\circ$ sector: half of the angular coverage interval has blockage.

For this topology considered, as implied by (20), a higher average beamforming gain is required at the open directions ($[0^\circ, 30^\circ]$) compared to the directions with blockage ($[-30^\circ, 0^\circ]$). Using the three-step method and the beamformer synthesis techniques proposed in Section IV, we synthesised three sets of the beamformers with $M = 2, 3, 4$, each with an approximation to (20). It is noted that we do not have $M = 1$ for this example due to the difficulty of synthesising a single beamformer to match the desired non-uniform beam pattern. The synthesised beam patterns, along with optimised slot allocations, are illustrated in Fig. 7. We refer to these as the optimised schemes. (For brevity, we only present RS transmission designs using VM beamformers and their corresponding BS discovery performance.)

The performance of BS discovery from the optimised beamforming strategies are compared to a set of baseline strategies. For these baselines, the beamformers are obtained in the same way as those for the optimised cases. However, an equal slot allocation is used. It is easy to check that the resultant average beam pattern does not satisfy condition (20). In Fig. 7 these baselines
Variable modulus: $M=1$

Variable modulus: $M=2$

Variable modulus: $M=3$

Variable modulus: $M=4$

Fig. 7. Beam patterns and slot allocation for the synthesised beamformers using VM method. Optimised slot allocation is obtained by applying (32), while Non-optimised slot allocation takes an equal slot allocation to the $M$ beamformers. are also illustrated and we refer them as non-optimised schemes. RS transmission using random beamformers as considered in [13] is also included as a baseline.

Fig. 8 presents the corresponding BS discovery performance as a function of the number of slots used for BS discovery ($L$). The average miss-detection probability is obtained by averaging
all possible directions within the sector and over 500 channel realizations for each $L$. Clearly, the optimised RS transmissions using the beamformers constructed according to (20) significantly outperform all the baselines. For a reasonable UE searching time of 10 ms (e.g., $L = 10$), the optimised RS transmissions provide orders of magnitude improvement over the RS transmission with random beamforming. For a targeted average miss-detection rate, e.g., at $10^{-3}$, the optimised RS transmission with $M = 2$ requires a searching time $L = 12$, which is 20% faster than that required by the best non-optimised baseline ($M = 1$). These results clearly demonstrate the benefit of designing the RS transmissions that take into account asymmetric coverage of a mm-wave BS.

B. BS discovery performance without blockage

In this subsection, we consider an example in which there is no blockage at all in the $60^\circ$ sector to further demonstrate the effect of different choices of $M$ and beamformer synthesis techniques on the discovery performance when a limited number of transmit antennas are available at BS.

Fig. 9 shows the average miss-detection probability of edge-UEs versus the number of slots ($L$) that UEs use for BS discovery, for both VM and CM beamformers. As expected, RS transmission with the VM beamformers results in significantly superior BS discovery performance than that using the CM beamformers, since the VM beamformers are more capable of approximating the desired perfect patterns provided in (31).
In addition, the BS discovery performance exhibits slightly different behaviours for VM beamformers and CM beamformers when different codebook sizes are considered. For the VM method, as shown in Fig. 9, $M = 1$ achieves the best detection performance for all the values of $L$ examined. On the contrary, for the CM method, $M = 2$ provides the best performance when $L > 20$. This is mainly due to the imperfection of the synthesised beams. The result demonstrates that a different beam synthesis method may yield a different choice of $M$. Allowing a flexible choice of $M$ in the design, as we propose, can thus offer a better opportunity to approximate the desirable patterns and improve the performance of BS discovery, when there are limited hardware resources.

C. The impact of per-antenna power constraint

Up until now, we have assumed that the dynamic ranges of the power amplifiers (PAs) for the BS antennas are large enough to accommodate the VM beamformers. However, due to the poor power efficiency of the PAs in mm-wave bands, this may not be true since each PA may be designed to operate at full power in order to provide sufficient transmit power [21]. In this context, per-antenna power constraints are important, and may influence the implementation of the VM beamformers that require variable amplitudes at different antennas.

In the following, we demonstrate the impact of per-antenna power constraints on the construction of RS transmit beamformers and on the performance of BS discovery. In the simulation, we introduce a normalised per-antenna power constraint, denoted by $\beta$, as the ratio of per-antenna power limit to the maximum total transmit power $P_T$. We consider that $\beta$ is in the
range $\beta \in [1/N_T, 1]$. In this case, for the CM beamformers, the transmit power at each antenna is $P_T/N_T$. For the VM beamformers, the beamforming coefficients are scaled up/down by the same factor such that both the per-antenna power constraints and the total power constraint are satisfied. In particular, when $\beta = 1/N_T$, because of the uniform scaling, only the antenna with the largest beamforming weight transmits with power $P_T\beta = P_T/N_T$, all other antennas are transmitting with smaller powers. This means that the total transmit power is less than $P_T$, i.e., there is a loss in the total transmission power. As $\beta$ increases, the power loss becomes smaller and ultimately reduces to zero for sufficiently large $\beta$s.

Fig. 10 shows the average miss-detection probability versus per-antenna power constraint $\beta$. It can be seen that when $\beta$ is small (e.g., $\beta < 0.1$), the VM beamformers perform significantly worse than the CM beamformers due to the power loss. Increasing the dynamic range of the power amplifiers will favor the VM implementation. However, this comes at the price of increased cost and difficulty in implementing mm-wave transceivers. The trade-off between achievable performance and the hardware cost is clearly shown via these numerical results.

Moreover, the results also demonstrate the benefit of the proposed beamformer construction method due to a flexible choice of $M$. For instance, under a strict per-antenna power constraint, e.g., $\beta < 0.15$, the VM beamformer with $M = 2$ outperforms the alternatives with $M = 1$ and $M = 4$. In the absence of per-antenna power constraint, the VM beamformer with $M = 1$ performs the best. However, this benefit comes at the price of 5 dB dynamic range increase as compared with that required by $M = 2$. 

Fig. 10. Average miss-detection probability with different beamformers and per antenna power constraint: $L = 20$. 

![Graph showing average miss-detection probability versus per-antenna power constraint $\beta$.]
VI. CONCLUSIONS

In this paper, we have developed an analytical framework for mm-wave BS discovery and proposed an effective beamforming strategy for transmitting the Reference Signals (RSs) for mm-wave BS discovery. We have established the relationship between the performance of mm-wave BS discovery and a set of key system parameters (including the beamformers used for RS transmission). The analytical results provide general guidance of choosing suitable system parameters to ensure satisfactory BS discovery performance in the emerging mm-wave cellular communications. Based on the results, we have also identified the desirable beam patterns for RS transmission, which takes into account asymmetric coverage. We have shown that the desirable beam patterns are asymptotically optimal and yield the minimum average miss-discovery probability for cell-edge UEs. To approximate the beam patterns, we have proposed a systematic approach to design the beamforming codebooks. Numerical results have demonstrated the effectiveness of the proposed method.

APPENDIX A

PROOF OF PROPOSITION [1]

We first show that random variables

\[
U_l \triangleq \frac{1}{\|s\|^2} \|Y_l s^*\|^2
\]

and

\[
V_l \triangleq \|Y_l\|^2_F - \frac{1}{\|s\|^2} \|Y_l s^*\|^2
\]

are statistically independent, where we recall that \( s^* = (s^T)^T \).

Under event \( H_1 \), it can obtained that

\[
Y_l = h_l s^T + Z_l.
\]

Denote \( P_T \) as the average transmit power with \( \|s\|^2 = N_s P_T \) and \( \bar{z}_{l,N_s} = \frac{1}{\sqrt{P_T N_s}} Z_l s^* \), it can be obtained that

\[
U_l = \| \sqrt{P_T N_s} h_l + \bar{z}_{l,N_s} \|^2
\]

\[
= P_T N_s \| h_l \|^2 + \| \bar{z}_{l,N_s} \|^2 + \sqrt{P_T N_s} h_l^\dagger \bar{z}_{l,N_s} + \sqrt{P_T N_s} \bar{z}_{l,N_s}^\dagger h_l,
\]
and

\[ V_l = P_T N_s |h_l|^2 + |Z_l|^2 + \sqrt{P_T N_s} h_l^T \bar{z}_{l,N_s} + \sqrt{P_T N_s} \bar{z}_{l,N_s}^T h_l - U_l \]

\[ = |Z_l|_F^2 - ||\bar{z}_{l,N_s}||_2^2. \]  

(38)

Construct a unitary matrix \( \tilde{S} \triangleq [\tilde{s}_1, \ldots, \tilde{s}_{N_s-1}, s^*/\sqrt{P_T N_s}] \in \mathbb{C}^{N_s \times N_s} \) with the last column being \( s^*/\sqrt{P_T N_s} \), then \( V_l \) can further be represented as:

\[ V_l = ||Z_l \tilde{S}||_F^2 - ||\bar{z}_{l,N_s}||_2^2 = \sum_{n=1}^{N_s-1} ||\bar{z}_{l,n}||_2^2, \]  

(39)

where \( \bar{z}_{l,n} = Z_l \tilde{s}_n, n = 1, \ldots, N_s - 1 \). It is easy to check that the vectors \( \bar{z}_{l,n}, n = 1, \ldots, N_s \), are jointly Gaussian. Since by construction, \( \bar{z}_{l,n}, n = 1, \ldots, N_s \), are mutually uncorrelated, they are statistically independent. As \( U_l \) is only a function of \( \bar{z}_{l,N_s} \) and \( V_l \) is a function of \( \bar{z}_{l,n}, n = 1, \ldots, N_s - 1 \), \( U_l \) and \( V_l \) are independent.

As the noises are i.i.d., \( U_l \)'s and \( V_l \)'s are statistically independent. It can therefore be concluded that \( U \triangleq \sum_{l=1}^{L} U_l \) and \( V \triangleq \sum_{l=1}^{L} V_l \) are independent.

From (37) and (39), it is clear that under \( H_1 \), \( 2U_l/\sigma^2 \) has a non-central chi-square distribution with \( 2N_R \) degrees of freedom (DoFs) and a non-centrality parameter \( 2N_s P_T ||h_l||_2^2/\sigma^2 \), and \( 2V_l/\sigma^2 \) admits a central chi-square distribution with \( 2N_R(N_s - 1) \) DoFs. Thus, \( 2U/\sigma^2 \) has a non-central chi-square distribution with \( 2LN_R \) DoFs and a non-centrality parameter

\[ \lambda = 2 P_T N_s/\sigma^2 \sum_{l=1}^{L} ||h_l||_2^2 \]

and \( 2V/\sigma^2 \) admits a central chi-square distribution with \( 2LN_R(N_s - 1) \) DoFs:

\[ \begin{align*}
\frac{2U}{\sigma^2} &\sim \chi^2_{2LN_R}(\lambda), \\
\frac{2V}{\sigma^2} &\sim \chi^2_{2LN_R(N_s - 1)}.
\end{align*} \]  

(40)

The proof of (16) under \( H_1 \) is immediate by considering that \( (N_s - 1)L_G(\tau) = \frac{2U/(2LN_R\sigma^2)}{2V/(2LN_R(N_s - 1)\sigma^2)}. \)

The proof of (16) under hypothesis \( H_0 \) is trivial by putting \( h = 0 \) in the \( H_1 \) case. This concludes the proof of Proposition 1.
The probability of miss-detection can be represented as \( P_{\text{miss}} = Pr\{L_G(t) \leq \gamma| \mathcal{H}_1, \bar{h}_L < \bar{h}\} \times \xi + Pr\{L_G(t) \leq \gamma| \mathcal{H}_1, \bar{h}_L \geq \bar{h}\} \times (1-\xi) \). Since \( F(x|n_1, n_2, \lambda) \) is monotonically decreasing with respect to \( \lambda \), provided that \( n_1 \) and \( n_2 \) are fixed \([17]\), it can be seen that \( Pr\{L_G(t) \leq \gamma| \mathcal{H}_1, \bar{h}_L \geq \bar{h}\} \leq Pr\{L_G(t) \leq \gamma| \mathcal{H}_1, \bar{h}_L = \bar{h}\} = F(\gamma|2N_RL, 2NR_L(N_s-1), L\eta) \). It can then be obtained that for an arbitrary \( \xi > 0 \) with \( Pr\{\bar{h}_L < \bar{h}\} = \xi \),

\[
P_{\text{miss}} \leq \xi + (1-\xi) \times \underset{\xi}{\text{Pr}}\{L_G(t) \leq \gamma| \mathcal{H}_1, \bar{h}_L \geq \bar{h}\}
\leq \xi + (1-\xi)F(\gamma|2N_RL, 2NR_L(N_s-1), L\eta).
\] (41)

### Appendix C

**Proof of Proposition**

Let \( U \) and \( V \) be the random variables defined in Appendix A and denote \( \bar{U} \triangleq \frac{2U}{\sigma^2}, \bar{V} \triangleq \frac{2V}{\sigma^2}, U_L \triangleq \bar{U}_L, V_L \triangleq \bar{V}_L \). Random variables \( U_L \) and \( V_L \) are independent as \( U \) and \( V \) are independent.

We first prove (24) by showing that \((U_L, V_L)\) satisfy the LDP, using the well known Gartner-Ellis Theorem. To demonstrate this, we need to show that the limiting logarithmic moment generation function (MGF)

\[
\Lambda(t) = \lim_{L \uparrow \infty} \frac{1}{L} \Lambda_L(Lt)
\] (42)

exists as an extended real number \([14\text{ Chapter 2.3, pp. 43}]\), where \( \Lambda_L(t) \triangleq \log M_{(U_L, V_L)}(t) \) is the logarithmic MGF of \((U_L, V_L)\) and \( t = [t_1, t_2] \).

Due to the independence between \( U_L \) and \( V_L \), the MGF of \((U_L, V_L)\) is simply \( M_{(U_L, V_L)}(t) = M_{U_L}(t_1)M_{U_L}(t_2) \), where \( M_{U_L}(t) = \mathbb{E}_{U_L}\{e^{tU_L}\} \) and \( M_{V_L}(t) = \mathbb{E}_{V_L}\{e^{tV_L}\} \) are the MGFs for \( U_L \) and \( V_L \) respectively. We can therefore obtain that

\[
\Lambda_L(Lt) = \log M_{U_L}(Lt_1) + \log M_{V_L}(Lt_2)
= \log \mathbb{E}_{\bar{U}}\{e^{t_1\bar{U}}\} + \log \mathbb{E}_{\bar{V}}\{e^{t_2\bar{V}}\}.
\] (43)

Recall the facts shown in Appendix A that \( \bar{U} = \frac{2U}{\sigma^2} \sim \chi^2_{2LN_R}(\lambda) \) and \( \bar{V} = \frac{2V}{\sigma^2} \sim \chi^2_{2LN_R(N_s-1)} \),
we can obtain $M_{\tilde{U}}(t) = \mathbb{E}_{\tilde{U}}\{e^{t\tilde{U}}\}$ and $M_{\tilde{V}}(t) = \mathbb{E}_{\tilde{V}}\{e^{t\tilde{V}}\}$ as follows:

$$
\mathbb{E}_{\tilde{U}}\{e^{t_1\tilde{U}}\} = \begin{cases} 
\frac{e^{\frac{\lambda t_1}{1-2t_1}}}{(1-2t_1)^{LN_R}}, & t_1 < \frac{1}{2} \\
+\infty, & \text{otherwise}
\end{cases}
$$

(44)

$$
\mathbb{E}_{\tilde{V}}\{e^{t_2\tilde{V}}\} = \begin{cases} 
(1 - 2t_2)^{-LN_R(N_s-1)}, & t_2 < \frac{1}{2} \\
+\infty. & \text{otherwise}
\end{cases}
$$

(45)

Using (44), (45), (43) and (42), it can be shown that

$$
\Lambda(t) = \lim_{L \to \infty} \frac{1}{L} \Lambda_L(Lt)
$$

$$
= \begin{cases} 
\frac{\eta t}{1-2t} - N_R \log(1 - 2t_1) - N_R(N_s - 1) \log(1 - 2t_2), & t_1, t_2 < \frac{1}{2} \\
+\infty. & \text{otherwise}
\end{cases}
$$

(46)

Further, it can be shown that $\Lambda(t) = 0$ only when $t_1 < \frac{1}{2}$ and $t_2 < \frac{1}{2}$. This verifies that $(U_L, V_L)$ satisfy the Gartner-Ellis conditions [14, Assumption 2.3.2, pp. 43].

The rate function of $(U_L, V_L)$, i.e., $I_L(u, v)$ can then be obtained as

$$
I_L(u, v) \triangleq \sup_{t_1, t_2 \in \mathbb{R}} \{t_1 u + t_2 v - \Lambda(t)\}
$$

$$
= \sup_{t_1, t_2 < \frac{1}{2}} \{t_1 u + t_2 v - \frac{\eta t_1}{1-2t_1} + N_R \log(1 - 2t_1) + N_R(N_s - 1) \log(1 - 2t_2)\}
$$

(47)

$$
= t_1^* u + t_2^* v - \frac{\eta t_1^*}{1-2t_1^*} + N_R \log(1 - 2t_1^*) + N_R(N_s - 1) \log(1 - 2t_2^*),
$$

(48)

where $t_1^* = \frac{1}{2} - \frac{N_R + \sqrt{N_R^2 + \eta u}}{2u}$ and $t_2^* = \frac{1}{2} - \frac{N_R(N_s - 1)}{v}$.

With the rate function given in (48), the LDP tells us that

$$
\lim_{L \to \infty} \frac{1}{L} \log \Pr\{(U_L, V_L) \in A\} = - \inf_{(u, v) \in A} I_L(u, v),
$$

(49)

if the set $A \subseteq \mathbb{R}^2$ is continuous.

Consider $A = \{(u, v) | u \leq \gamma v\}$ as the collection of miss-detection events. This leads to the following rate function:

$$
I^*(\eta, \gamma) = \inf_{u/v \leq \gamma} I_L(u, v).
$$

(50)

Using the Karush-Kuhn-Tucker (KKT) conditions of (50), it can be shown that:
1) If \( u/v < \gamma \), then \( I^*(\eta, \gamma) = I_L(u^*, v^*) = 0 \), where \( u^* = 2N_R + \eta \), \( v^* = 2N_R(N_s - 1) \). In this case, \( \gamma > \frac{2N_R + \eta}{2N_R(N_s - 1)} \).

2) If \( u/v = \gamma \), then
\[
I^*(\eta, \gamma) = I_L(u^*, v^*) = I_L(\gamma v^*, v^*),
\]
(51)

where \( v^* \) is obtained by solving \( \frac{\partial I_L(\gamma v, v)}{\partial v} = 0 \), or equivalently by solving the following equation
\[
\frac{\gamma + 1}{2} - \frac{N_R + \sqrt{N_R^2 + \eta \gamma v}}{2v} - \frac{N_R(N_s - 1)}{v} = 0.
\]
(52)

Let \( x = \sqrt{N_R^2 + \eta \gamma v} \), (52) can be rewritten into:
\[
\frac{\gamma + 1}{\eta \gamma} (x^2 - N_R^2) - x - N_R - 2N_R(N_s - 1) = 0.
\]
(53)

Denote \( x^* > 0 \) is a solution of (53), then \( v^* \) can be obtained as follows:
\[
v^* = \frac{x^* v^2 - N_R^2}{\eta \gamma}.
\]
(54)

Substituting \( u^*, v^* \) into (48) yields (25).

This has concluded the proof of (24).

We now prove the monotonicity of \( I^*(\eta, \gamma) \) with respect to \( \eta \) when \( \gamma < \frac{2N_R + \eta}{2N_R(N_s - 1)} \). Towards this end, we note
\[
\frac{\partial I^*(\eta, \gamma)}{\partial \eta} = \frac{1}{2} \left( 1 - \frac{\gamma v^*}{N_R + \sqrt{N_R^2 + \gamma \eta v^*}} \right)
\]
(55)

\[
= \frac{1}{2} \frac{v^* - 2N_R(N_s - 1)}{2N_R + \sqrt{N_R^2 + \gamma \eta v^*}}.
\]
(56)

where (56) follows from (52). To show \( \frac{\partial I^*(\eta, \gamma)}{\partial \eta} > 0 \), it is sufficient to show
\[
v^* > 2N_R(N_s - 1).
\]
(57)

Considering (54) and noticing the fact that \( x^* > 0 \), showing (57) is equivalent to show
\[
x^* > \sqrt{N_R^2 + 2\gamma \eta N_R(N_s - 1)}.
\]
(58)

Denote \( f(x) = \frac{x + 1}{\eta}(x^2 - N_R^2) - x - N_R - 2\gamma \eta N_R(N_s - 1) \) as the left side of (53). Noticing that
$f(x^*) = 0$ and $x^* > 0$, it is sufficient to show $f(x = \sqrt{N^2_R + 2\gamma\eta N_R(N_s - 1)}) = 2\gamma N_R(N_s - 1) - N_R - \sqrt{N^2_R + 2\gamma\eta N_R(N_s - 1)} < 0$.

Since $\gamma < \frac{2N_R + \eta}{2N_R(N_s - 1)}$, it can be obtained that

$$\begin{align*}
(2\gamma N_R(N_s - 1) - N_R)^2 - \left(\sqrt{N^2_R + 2\gamma\eta N_R(N_s - 1)}\right)^2 \\
= 2\gamma N_R(N_s - 1) [2\gamma N_R(N_s - 1) - (2N_R + \eta)] < 0.
\end{align*}\quad(59)$$

It is therefore concluded that $\frac{\partial I^*(\eta, \gamma)}{\partial \eta} > 0$ when $\gamma < \frac{2N_R + \eta}{2N_R(N_s - 1)}$.

APPENDIX D

PROOF OF PROPOSITION 3

The average miss-detection probability $\bar{p}_{miss}$

$$\bar{p}_{miss} = \int_\Omega p_{miss}(\phi)p(\phi)d\phi$$

$$\geq \int_{\Omega^-} p_{miss}(\phi)p(\phi)d\phi$$

$$\geq \int_{\Omega^-} P_{miss}(\eta-, L, \gamma)p(\phi)d\phi$$

$$= P^- P_{miss}(\eta-, L, \gamma),$$

where $P^- = \int_{\Omega^-} p(\phi)d\phi > 0$ and (60) is due to the facts that $P_{miss}(\eta, L, \gamma)$ is monotonically decreasing with respect to $\eta$ when both $N_s$ and $L$ are fixed [17], and $\eta(\phi) \leq \eta^-, \forall \phi \in \Omega^-.$

According to (61) and applying Proposition 2, it can be obtained that:

$$\lim_{L \uparrow \infty} -\frac{1}{L} \log \bar{p}_{miss}$$

$$\leq \lim_{L \uparrow \infty} -\frac{1}{L} \left(\log P^- + \log P_{miss}(\eta-, L, \gamma)\right)$$

$$= I^*(\eta^-, \gamma),$$

where $I^*(\eta, \gamma)$ is the rate function given by (25). This concludes the proof.

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