We generalize a regularization method of Stumpf in the case of non-linear spinor field models to fourth order theories and to non-scalar interactions. The involved discrete symmetries can be connected with C, P, T transformations.

I. INTRODUCTION

Beside being obviously necessary for obtaining results, regularization has to be included into the definition of a theory. Different methods of regularization may lead to different outcomes. Renormalization is a special kind of regularization for singularities which occur only perturbatively. Since non-linear spinor field models are non-renormalizable, we are exclusively interested in a regularization process.

Using higher order derivatives, one can make a theory more regular or even fully regular. As can be seen e.g. by power counting arguments. However, the interpretation of the involved fields becomes a difficult task, as also canonical quantization is no longer obvious (spin-statistic theorem). We will stay with quantization of first order fields and calculate the commutators of the higher order fields afterwards.

We extend the canonically quantized two-field theory with scalar interaction of Stumpf to a fully regular four-field theory with non-trivial V-A interaction.
II. STUMPF 2-FIELD THEORY

We define a non-linear Heisenberg [3] or Nambu–Jona-Lasinio [4] like first order spinor field model by

\[(i\gamma^\mu \partial_\mu - m)_{IJ}\psi_J = g V_{IJKL} : \psi_J \psi_K \psi_L :\]

using a compactified index notation

\[
\psi_I = \psi_{i\lambda} := \begin{cases} 
\psi_i &= \psi((x, t)_{\alpha, \ldots, \beta} \text{ if } \lambda = 0 \\
\psi_i^\dagger &= \psi^\dagger((x, t)_{\alpha, \ldots, \beta} \text{ if } \lambda = 1 
\end{cases}
\]

where \(\alpha, \ldots, \beta\) are some algebraic degrees of freedom and \(V_{IJKL}\) is a constant vertex function antisymmetric in the last three indices \(V_{IJKL} = V_{[JKL]}\) as and \(g\) is the coupling constant.

The Stumpf model of regularization is restricted to the massless theory using scalar interaction. Parity \(P\) is defined as \((\vec{r}, t) \mapsto \rightarrow (-\vec{r}, t)\) and \(\psi \mapsto \rightarrow i\gamma_0 \psi\) yielding

\[
\psi_i'((\vec{r}', t) := P \psi_i((\vec{r}, t) = i\gamma_0 \psi_i((\vec{r}, t), \quad \vec{r}' = -\vec{r}.
\]

Lagrangian and Hamiltonian are given as usual. One has the common invariance \(L[\psi'] = L[\psi]\) and \(H[\psi'] = H[\psi]\).

Let us introduce an auxiliary field which is only the spin-parity transform, but not the space-parity transform as

\[
\xi_i := \psi_i'(\vec{r}, t) = \psi_i(-\vec{r}, t) = i\gamma_0 \psi_i(\vec{r}, t).
\]

This is sufficient to prove an equivalence theorem \(H[\Psi] \equiv H[\Psi, \xi]|_{\xi = i\gamma_0 \Psi}\), where the two-field Hamiltonian reads

\[
H[\Psi, \xi] := \frac{i}{4} \int [\bar{\Psi}(t, \vec{r}) \vec{\gamma} \cdot \vec{\nabla} \Psi(t, \vec{r}) - \vec{\nabla} \bar{\Psi}(t, \vec{r}) \cdot \vec{\gamma} \Psi(t, \vec{r})] dr^3
\]

\[
- \frac{i}{4} \int [\bar{\xi}(t, \vec{r}) \vec{\gamma} \cdot \vec{\nabla} \xi(t, \vec{r}) - \vec{\nabla} \bar{\xi}(t, \vec{r}) \cdot \vec{\gamma} \xi(t, \vec{r})] dr^3
\]

\[
+ \frac{1}{4} g \int \{|[\bar{\Psi}(t, \vec{r}) + \bar{\xi}(t, \vec{r})][\Psi(t, \vec{r}) + \xi(t, \vec{r})] \}^2 dr^3
\]
The proof needs scalar interaction \[^{5}\]. Transforming back to the Lagrangian picture results in two inequivalent Lagrangians

\[ H[\Psi] = H[\Psi, \xi]|_{\xi = -\gamma_0 \Psi} \]

\[ \Uparrow \quad \Uparrow \]

\[ L[\Psi] \neq L[\Psi, \xi]|_{\xi = -\gamma_0 \Psi} \]

This should be not a miracle, since we let open the constraint that the second field \(\xi\) is the parity transformed field \(\psi\). Moreover, quantization breaks the equivalence. Introduce \(\varphi_1 \equiv \psi, \varphi_2 \equiv \xi\) for convenience and the corresponding canonical momenta \(\Pi_i\). We contrast the one-field quantization with commutation relations given as

\[ \Pi(t, \vec{r}) := i\Psi^\dagger(t, \vec{r}) = \frac{\delta L[\Psi]}{\delta \partial_t \Psi}, \]

\[ \{\Pi(t, \vec{r}'), \Psi(t, \vec{r})\} = \delta(\vec{r}' - \vec{r}) \]

with the \textit{two-field case} where we obtain from the two field Lagrangian and canonical quantization

\[ \Xi_i(t, \vec{r}) := \frac{\delta L[\varphi_1, \varphi_2]}{\delta \partial_t \varphi_i} = \frac{i}{\lambda_i} \varphi_i^\dagger(t, \vec{r}), \]

\[ \{\varphi_i^\dagger(t, \vec{r}'), \varphi_i(t, \vec{r})\} = \lambda_j \delta_{ji} \delta(\vec{r}' - \vec{r}). \]

These two different quantization schemes break explicitly the equivalence of the theories. One observes that \(\varphi_1(t, \vec{r}) \neq i\gamma^0 \varphi_2(t, \vec{r})\) and the auxiliary field particle number operators \(N[\varphi_i]\) do not commute with the two-field Hamiltonian \(H[\varphi_1, \varphi_2]\). However, their sum does commute and one introduces the physical field \(\Psi = \sum \varphi_i\) while extending the theory as a great canonical ensemble

\[ K[\varphi_1, \varphi_2] := H[\varphi_1, \varphi_2] + \mu_1 N[\varphi_1] + \mu_2 N[\varphi_2]. \]

with \(\mu_i\) as chemical potentials. Finally one obtains the \textit{second order} field equations

\[ \Pi_{i=1}^2 (i\gamma^\mu \partial_\mu - \mu_i) \Psi(t, \vec{r}) = (i\gamma^\mu \partial_\mu - \mu_1)(i\gamma^\nu \partial_\nu - \mu_2) \Psi(t, \vec{r}) = \frac{1}{4} g \{\bar{\Psi}(t, \vec{r}) \Psi(t, \vec{r})\} \Psi(t, \vec{r}) \]

with the commutation relations and masses (chemical potentials)

\[ \{\bar{\Psi}(t, \vec{r}'), \Psi(t, \vec{r})\} = 0 \quad \lambda_i = \frac{1}{\mu_i - \mu_j}, \quad i \neq j, \quad \mu_i \neq \mu_j. \]

The expression for the \(\lambda_i\) as function of the \(\mu_i\) follows also in the reversed argumentation by fraction into parts the Fourier transformed equation \[^{6}\].
III. EXTENSION TO 4 AUXILIARY FIELDS

The results presented in this section have been obtained using CLIFORD a Maple V package for Clifford algebras developed by R. Ab\'lamowicz \cite{1}. Even using computer algebra, we have not been able to exhaust all possible interactions, but used a reduced picture, where we allowed scalar $S$, vectorial $V$, axial-vectorial $A$, and pseudo scalar $P$, interactions only.

To extend the theory \cite{2}, we need to examine the mechanism behind the proof that $H[\Psi] \equiv H[\varphi_1, \ldots, \varphi_n]$ with $n$-auxiliary fields. It can be shown that

$$\sum_{i,j} \varphi_i M_{ij} \varphi_j = 0$$

is sufficient, where $\varphi_i := T_i \psi$ and the $T_i$ form a discrete group with $T_i^2 = \pm 1$.

To obtain 4 auxiliary fields we need the (pseudo) quaternion group $G$ of two generators $T_1, T_2$. $G$ is spanned by $G = < T_0 = 1d, T_1, T_2, T_3 = T_1 T_2 >$. Defining $\varphi_i = T_i \psi$ we have to investigate 5 different signatures which remain after utilization of permutation symmetry on the labels:

$$\{T_0^2, T_1^2, T_2^2, T_3^2\} \in (\pm 1, \pm 1, \pm 1, \pm 1); \quad \text{sig} \in (+4, +2, 0, -2, -4).$$

Examining the different signatures to obtain 4 eigenvectors, suitable normalized and fulfilling the desired equation to perform the equivalence proof $H[\Psi] \equiv H[\varphi_1, \ldots, \varphi_3]$ we find that only sig = 0 yields a reasonable result. The corresponding Hamiltonian is calculated to be

$$H[\psi_i] = \frac{i}{2} \sum_{j=1}^{4} \frac{1}{\lambda_j} \left[ \bar{\psi}_j \gamma^k \partial_k \psi_j - \partial_k \bar{\psi}_j \gamma^k \psi_j \right] dr^3$$

$$+ \frac{g}{16} \int \left[ \sum_{j,k,l,m=1}^{4} \bar{\psi}_j \gamma^k \bar{\psi}_k \gamma^l \psi_l \gamma^m \psi_m \right] dr^3$$

$$+ \sum_{j,k,l,m=1}^{4} \bar{\psi}_j \gamma^k \gamma^5 \psi_k \bar{\psi}_l \gamma^5 \psi_m \right] dr^3$$

This regular case has $\lambda_1 = \lambda_2 = -\lambda_3 = -\lambda_4 = 8$. The canonical commutation relation yields rewritten in $\Psi^\dagger, \Psi$ once more $\{\psi_i^\dagger, \psi_j\} = \lambda_i \delta_{ij}$ where two $\lambda_i$'s are negative. We define
the sum-field $\Psi = \sum_{i=1}^{4} \psi_i$ with $\lambda_i = \prod_{j, j \neq i} 1/(\mu_i - \mu_j)$. Calculating the corresponding field equation yields

$$\left[ \prod_{i=1}^{4} (i\gamma^\mu \partial_\mu - \mu_i) \right] \Psi = \frac{g}{16} \left[ \bar{\Psi} \gamma^\mu \gamma_\mu \Psi + \bar{\Psi} \gamma^\mu \gamma^5 \Psi \gamma_\mu \gamma^5 \Psi \right]$$

which is of fourth order and obeys the Heisenberg non-canonical quantization rule for the sum-field.

$$\{\Psi, \Psi\}_+ = 0.$$

Identifying the discrete symmetries with C, P, T, as already done in the two-field case with P, the 4 auxiliary subfields are given up to a phase by $\varphi_0 = P\psi$, $\varphi_1 = CP\psi$, $\varphi_2 = CPT\psi$, $\varphi_3 = TP\psi$. The theory with 4 auxiliary fields leads to a canonically quantized sub-field theory which is fully regular and of forth order in the physical sum-field $\Psi$.

**IV. DISCUSSION**

Our constructive model supports the approach given by Stumpf postulating higher order field equations with non-scalar vertex and factorizing them into canonically quantized non-particle auxiliary fields. The benefits of this method over the pragmatic introduction of auxiliary fields in [5, 6] are:

We get a dynamical regular theory by constructive methods. Many calculations have shown that no divergent integrals occur [6].

The ‘masses’ have to be addressed as chemical potentials, even if they occur in Dirac spinor theory at the place where mass-terms are convenient. This is important, since at least one of these chemical potentials has to be negative if a regularization shall take place. This is a reasonable situation only if the $\mu_i$ are chemical potentials not masses.

The auxiliary or sub-fermion fields possess not a proper particle interpretation. This is not an artifact, but a main ingredient in the Heisenberg-Stumpf theory of bound states which constitute the physical observable particles like quarks and leptons, but also bosons like gluons, the photon and electro weak W and Z bosons [3, 6].

Viewed in a more conventional setting [4], the additional parameters are exactly the counter-terms in a Pauli-Villars regularization [2, 3].
We derive from the canonical quantization of the sub-fields a non-canonical quantization of the physical sum-field which is of Heisenberg type. This usually ad hoc quantization rule is known to be of regular nature.

Most interesting is that the $\varphi_i$ fields are related by the discrete subgroup of the $\text{pin}(1, 3)$ group. The relation of this fact to the framework of Clifford analysis will be shown elsewhere.

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