On a multiplicative multivariate gamma distribution with applications in insurance

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Abstract

In a recent paper [Albrecher, Constantinescu and Loisel (2011). Explicit ruin formulas for models with dependence among risks. *Insurance: Mathematics and Economics* 48(2), 265 – 270] Professors Hansjörg Albrecher, Corina Constantinescu and Stephane Loisel noted - in passing - a way to employ exponential mixtures for formulating multivariate probability distributions with dependent univariate margins distributed gamma. The main message of our report is that the probabilistic construction in ibid., which has been arguably overlooked by the actuarial community, should be given very serious considerations when modelling dependent risks. In order to convey this message, we conduct a systematic study of the aforementioned construction. Specifically, we show, among other findings, that the model in Albrecher et al. (2011): (1) admits the interpretation of the Multiplicative Background Risk Model with risk components distributed gamma, and as such is easy to communicate to upper management; (2) is remarkably tractable, e.g., the risks aggregation within it is significantly simpler than in the case when the risk components are independent and distributed gamma; and (3) enjoys rich (tail) dependence characteristics.

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1 Introduction

Let $\mathcal{X}$ be a collection of actuarial risks, that is let it contain random variables (r.v.’s) $X : \Omega \rightarrow \mathbb{R}$ defined on the probability space $(\Omega, \mathcal{F}, P)$ and interpreted as the financial risks an insurer is exposed to. Often, for applications in insurance, actuaries would consider those $X \in \mathcal{X}$, whose distributions are supported on the non-negative real half-line, have positive skewness, and allow for a certain degree of heavy-tailness. One such distribution, which has been of prominent importance in insurance applications, is gamma (e.g., Lin, 2006; Willmot and Lin, 2001; Wüthrich and Merz, 2008, and references therein). We also refer to Hürlimann (2001), Dornheim and Brazauskas (2007), and Furman et al. (2018) for applications in solvency assessment, loss reserving, and aggregate risk approximations, respectively.

Further, let $\gamma \in \mathbb{R}_+$ and $\sigma \in \mathbb{R}_+$ denote, correspondingly, the shape and scale parameters, then the r.v. $X$ is said to be distributed gamma, succinctly $X \sim Ga(\gamma, \sigma)$, if it has the probability density function (p.d.f.)

$$f(x) = \frac{1}{\Gamma(\gamma)} e^{-x/\sigma} x^{\gamma-1} \sigma^{-\gamma} \text{ for all } x \in \mathbb{R}_+, \quad (1)$$

where $\Gamma(\cdot)$ stands for the complete gamma function. The popularity of the r.v.’s distributed gamma in insurance applications is not surprising: the p.d.f.’s of the (aggregate) insurance losses have as a rule the same shape as (1), i.e., they are positively skewed, unimodal and have positive supports; p.d.f. (1) is log-convex for $\gamma \in (0, 1)$ and so has decreasing failure rate, thus allowing for moderate heavy-tailness (Klugman et al., 2012); p.d.f. (1) has been very well studied and has turned out remarkably tractable.

When it comes to multivariate extensions of (1), there are ample of dependence structures with univariate margins distributed gamma to consider (e.g., Kotz et al., 2000, for a comprehensive reference). However, irrespective of whether the two-steps copula approach or the more ‘natural’ stochastic representation approach to formulate the desired multivariate gamma distribution is pursued, the tractability of the result is often an issue. For the former approach, the cumulative distribution function (c.d.f.) of (1) cannot be written in a closed form, and as a result intensive numerical algorithms are often needed to implement copula-based multivariate gamma models (e.g., Bargés et al., 2009; Cossette et al., 2013). For the latter approach, consider the following example. Let $Y_j \sim Ga(\gamma_j (\in \mathbb{R}_+), \sigma)$ for $j = 1, \ldots, n + 1$ be mutually independent r.v.’s, and set $X = (X_1, \ldots, X_n)' = (Y_1 + Y_{n+1}, \ldots, Y_n + Y_{n+1})'$. Then the distribution of the r.v. $X$ is the multivariate gamma of Mathai and Moschopoulos (1991) (also, e.g., Avanzi et al., 2016; Furman and Landsman, 2005, for recent applications in insurance). Consequently, for the p.d.f. of the r.v. $X$, we have

$$f(x_1, \ldots, x_n) \propto \int_0^{\Lambda_1 x_1} \cdots \int_0^{\Lambda_n x_n} e^{-x/\sigma} \prod_{i=1}^n (x_i - x)^{\gamma_i - 1} e^{-(x_i - x)/\sigma} dx \quad (2)$$
for all \((x_1, \ldots, x_n)' \in \mathbb{R}_+^n\), which inconveniently takes distinct forms for each of the \(n!\) orderings of \(x_1, \ldots, x_n\).

**Note 1.** The r.v.’s \(Y_1, \ldots, Y_n\) and \(Y_{n+1}\) are often interpreted as, respectively, the specific and systematic risk factors (r.f.’s). The systematic r.f., \(Y_{n+1}\), has also been referred to as the background risk (Gollier and Pratt, 1996), and so the distribution of the r.v. \(X = (X_1, \ldots, X_n)'\) can be associated with an Additive Background Risk Model with risk components (r.c.’s) distributed gamma (G-ABRM). Succinctly, for \(\gamma = (\gamma_1, \ldots, \gamma_n)'\), we write \(X \sim Ga_n^+(\gamma, \gamma_{n+1}, \sigma)\), where \(\gamma_{n+1}\) serves as the dependence parameter.

An alternative way to link the specific r.f.’s and the systematic (or background) r.f. is with the help of multiplication. Namely, in order to formulate a Multiplicative Background Risk Model with the r.c.’s distributed gamma (G-MBRM), we must find a sequence of \((n + 1)\) independent r.v.’s \(Z_1, \ldots, Z_n, Z_{n+1}\), say, such that \(X = (X_1, \ldots, X_n)' = (Z_1Z_{n+1}, \ldots, Z_nZ_{n+1})'\) results in that the coordinates of the r.v. \(X\) are distributed gamma. One solution of this exercise, which is of pivotal importance for this paper, can be found in Feller (1968) (also, Albrecher et al., 2011; Sarabia et al., 2017). We organize the rest of the paper as following: In Section 2 we explore the basic distributional properties of - what we call - the Multiplicative Multivariate Gamma (MMG) distribution. Then, in Sections 3 and 4, respectively, we discuss in detail and elucidate with examples of actuarial interest the aggregation and (tail) dependence properties of the MMG distribution. Section 5 concludes the paper.

### 2 Definition and basic properties

Multivariate distributions lay the very foundation of the successful (insurance) risk measurement - and thus of the consequent risk management - processes. However, the toolbox of the available stochastic dependencies that can be used to link stand-alone r.c.’s into risk portfolios (r.p.’s) is somewhat overwhelming. Indeed, there are infinitely many ways to formulate the joint distribution of two dependent risk r.v.’s, whereas there is a single way only to write this distribution under the assumption of independence. The case of the multivariate distributions with the margins distributed gamma is of course not an exception (e.g., Kotz et al., 2000).

Nevertheless, real applications impose significant constrains on the model choice. Namely, practitioners often opt for those multivariate distributions that: (1) admit meaningful and relevant interpretations; (2) allow for an adequate fit to the modelled data, be it in the ‘tail’, in the ‘body’, and/or in the dependence; and (3) can be readily implemented. We feel that the multivariate distribution
with the univariate margins distributed gamma that we put forward next (also, Albrecher et al., 2011; Sarabia et al., 2017) is exactly such.

Formally, let \( E_\lambda \) and \( \Lambda \) denote, respectively, an exponentially distributed r.v. with the rate parameter \( \lambda \in \mathbb{R}_+ \) and an arbitrarily distributed r.v. with the range \( \Lambda \subseteq \mathbb{R}_+ \). Also, let ‘∗’ represent the mixture operator (e.g., Feller, 1968; Su and Furman, 2017a), such that, for ‘\( d \)’ denoting equality in distribution, it holds that \( E_\lambda \ast \Lambda \overset{d}{=} E_\Lambda \). We note in passing that the just-mentioned mixture operator is referred to as ‘randomization’ in Feller (1968), and is closely related - via the Bernstein-Widder theorem - to the notion of the Laplace transform of the p.d.f. of \( \Lambda \). More specifically, if \( f_\Lambda \) and \( L\{f_\Lambda\} \) denote, correspondingly, the p.d.f. of \( \Lambda \) and its Laplace transform, that is

\[
L\{f_\Lambda\}(x) = \int_\Lambda e^{-x\lambda}f_\Lambda(\lambda)d\lambda,
\]

then (3) establishes the decumulative distribution function (d.d.f.) of the r.v. \( E_\Lambda \).

Recall that in this paper we are interested in formulating a multivariate distribution with the univariate margins distributed gamma and a dependence. To this end, we assume that the r.v. \( \Lambda \) is distributed a special shifted inverse Beta, succinctly \( \Lambda \sim IB(\gamma) \) with p.d.f.

\[
f_\Lambda(\lambda) = \frac{\lambda^{-1}(\lambda - 1)^{-\gamma}}{\Gamma(1-\gamma)\Gamma(\gamma)} \quad \text{for all } \lambda > 1,
\]

where \( \gamma \in (0, 1) \) is the shape parameter. In our context the choice of (4) is unique that readily follows from the Bernstein-Widder theorem. The next simple facts are used frequently latter on in the paper, and are hence formulated as a lemma. In the following, the \( k \)-th order derivative of the Laplace transform is denoted by \( \psi^{(k)} \), \( k \in \mathbb{N} := \{1, 2, \ldots\} \), also \( \mathbb{R}_{0,+} := [0, \infty) \).

**Lemma 1.** Let \( \Lambda \sim IB(\gamma) \), \( \gamma \in (0, 1) \) with p.d.f. (4), then:

(i) The Laplace transform of (4) is

\[
L\{f_\Lambda\}(x) = \frac{\Gamma(\gamma)}{\Gamma(\gamma)} \quad \text{for all } x \in \mathbb{R}_{0,+},
\]

where \( \Gamma(\cdot, \cdot) \) denotes the upper incomplete gamma function.

(ii) The negative \( k \)-th order moment of the r.v. \( \Lambda \) is

\[
E[\Lambda^{-k}] = \frac{\Gamma(\gamma + k)}{\Gamma(\gamma)\Gamma(k + 1)} \quad \text{for all } k \in \mathbb{N}.
\]

(iii) The alternating sign \( k \)-th order derivative of \( L\{f_\Lambda\} \) is

\[
(-1)^k \psi_A^{(k)}(x) = \sum_{i=0}^{k-1} \binom{k-1}{i} \frac{\Gamma(i - \gamma + 1)}{\Gamma(1-\gamma)\Gamma(\gamma)} e^{-x} \left( x^i \right) \quad \text{for all } x \in \mathbb{R},
\]

where \( \psi_A^{(k)} \) is the alternating sign \( k \)-th order derivative of \( L\{f_\Lambda\} \).
Proof. The proof of (i) is due to Equation 3.383(9) in Gradshteyn and Ryzhik (2014); (ii) follows readily via the integral representation of the Beta function. In order to check (iii), we have
\[
(-1)^k \frac{d^k}{dx^k} \mathcal{L}\{f_\Lambda\}(x) = \frac{1}{\Gamma(1-\gamma)\Gamma(\gamma)} \int_1^\infty \lambda^{k-1} e^{-\lambda x} (\lambda - 1)^{-\gamma} d\lambda
\]
\[
= \frac{1}{\Gamma(1-\gamma)\Gamma(\gamma)} \int_0^\infty (1+\lambda)^{k-1} e^{-(1+\lambda)x} \lambda^{-\gamma} d\lambda
\]
\[
= \frac{1}{\Gamma(1-\gamma)\Gamma(\gamma)} \sum_{i=0}^{k-1} \binom{k-1}{i} e^{-x} \int_0^\infty e^{-\lambda x} \lambda^i d\lambda.
\]
This completes the proof of the lemma.

Let \(E_{\lambda,1}, \ldots, E_{\lambda,n}\) denote independent copies of \(E_\lambda\), and let \(\Lambda \sim IB(\gamma)\), \(\gamma \in (0,1)\). Also, let \(\sigma = (\sigma_1, \ldots, \sigma_n)'\) denote a positive parameters vector.

Definition 1. Set \(X_j = \sigma_j E_{\lambda,j} \star \Lambda\), \(j = 1, \ldots, n\), then the r.v. \(X = (X_1, \ldots, X_n)'\) has a multiplicative multivariate distribution with univariate margins distributed gamma, and we succinctly write \(X \sim Ga_n^\gamma(\gamma, \sigma)\), where \(\gamma \in (0,1)\) and \(\sigma \in \mathbb{R}_n^+\) are parameters.

Note 2. Let \(E_j := E_{1,j}\), \(j = 1, \ldots, n\) denote independent copies of a r.v. distributed exponentially with unit scale, then the joint distribution of the r.v. \(X = (X_1, \ldots, X_n)'\) in Definition 1 admits the following Multiplicative Background Risk Model representation
\[
X = (X_1, \ldots, X_n)' \stackrel{d}{=} (\sigma_1 E_1/\Lambda, \ldots, \sigma_n E_n/\Lambda)'
\]
(6)
(also, e.g., Asimit et al., 2016; Marshall and Olkin, 2007; Merz and Wüthrich, 2014; Meucci, 2005, for applications in portfolio construction, survival analysis, optimal insurance design, and risk allocation, respectively). Above, the r.v. \(\Lambda\) can be interpreted as the systematic r.f. that endangers every r.c. of the r.p. \(X = (X_1, \ldots, X_n)'\) in Equation (6). The Monte Carlo simulation of (6) is immediate.

Theorem 1. Let \(\Lambda \sim IB(\gamma)\), \(\gamma \in (0,1)\), and let \(\sigma_1, \ldots, \sigma_n\) be positive scale parameters, then the following assertions hold:

(i) The r.v. \(X = \sigma E_\lambda \star \Lambda\) has the d.d.f.
\[
\overline{F}(x) = \Gamma(\gamma, x/\sigma)/\Gamma(\gamma)\text{ for all }x \in \mathbb{R}_{0,+},
\]
that is \(X\) is distributed gamma with the shape and scale parameters equal to \(\gamma \in (0,1)\) and \(\sigma \in \mathbb{R}_+\), respectively.
(ii) The r.v. $X = (X_1, \ldots, X_n)'$ with the $j$-th coordinate $X_j = \sigma_j E_{\lambda_j} \ast \Lambda$, has the joint d.d.f.

$$
F(x_1, \ldots, x_n) = \frac{\Gamma(\gamma, x_1/\sigma_1 + \cdots + x_n/\sigma_n)}{\Gamma(\gamma)},
$$

for all $(x_1, \ldots, x_n)' \in \mathbb{R}_{d,+}^n$.

(iii) The p.d.f. that corresponds to d.d.f. (7) is, for all $(x_1, \ldots, x_n)' \in \mathbb{R}_{+}^n$,

$$
f(x_1, \ldots, x_n) = \frac{1}{\prod_{i=1}^n \sigma_i} \sum_{i=0}^{n-1} \binom{n-1}{i} \frac{\Gamma(i+1)}{\Gamma(1-\gamma)\Gamma(\gamma)} e^{-\sum_{j=1}^n x_j/\sigma_j} \left(\sum_{j=1}^n \frac{x_j}{\sigma_j}\right)^{-(i+1)}.
$$

Proof. The d.d.f.'s of the r.v.'s $X$ and $X$ follow immediately from Lemma 1, Statement (i) and Chapter 4 in Joe (1997). The joint p.d.f. follows from Lemma 1, Statement (iii) since

$$
f(x_1, \ldots, x_n) = \frac{(-1)^n}{\prod_{i=1}^n \sigma_i} \psi^{(n)}_\Lambda \left(\sum_{i=1}^n \frac{x_i}{\sigma_i}\right) \text{ for all } (x_1, \ldots, x_n) \in \mathbb{R}_{+}^n.
$$

This completes the proof of the theorem. $\square$

We note in passing that the MMG distribution put forward in Definition 1 coincides - up to a scale transformation - with the multivariate distributions having univariate margins distributed gamma that are discussed in Albrecher et al. (2011); Sarabia et al. (2017). The following facts are immediate from Theorem 1: (i) the distribution of $X \sim Ga_n^\times(\gamma, \sigma)$ is ‘marginally closed’, namely, $X_j \sim Ga(\gamma, \sigma_j)$, $j = 1, \ldots, n$; (ii) the mathematical expectation of the $j$-th coordinate is $E[X_j] = \gamma \sigma_j$; and (iii) the variance of the $j$-th coordinate is $\text{Var}[X_j] = \gamma \sigma_j^2$.

We further explore more properties of the MMG/G-MBRM and note with satisfaction that the r.p.’s with the joint distributions within this class are often more tractable than the r.p.’s having stochastically independent r.c.’s distributed gamma. We look into the minima and maxima r.v.’s first; both are of evident importance in insurance (e.g., Bowers et al., 1997). To this end, denote by $X_\wedge = \bigwedge_{i=1}^n X_i \sim F_\wedge$ and by $X_\vee = \bigvee_{i=1}^n X_i \sim F_\vee$ the minima and maxima r.v.’s. Then we have - unlike in the independent case - that the coordinates of the r.v. $X = (X_1, \ldots, X_n)'$ in Definition 1 are closed under minima.

**Theorem 2.** Let $X \sim Ga_n^\times(\gamma, \sigma)$, then $X_\wedge$ is distributed gamma. More specifically, we have $X_\wedge \sim Ga(\gamma, \sigma^*)$, where $\sigma^* = \left(\sum_{j=1}^n 1/\sigma_j\right)^{-1}$ is the positive scale parameter, and $\gamma \in (0, 1)$ is the shape parameter. Also the d.d.f. of $X_\vee$ is a linear combination of the d.d.f.’s of the univariate r.v.’s distributed gamma, such that

$$
F_\vee(x) = \sum_{\mathcal{S} \subseteq \{1, \ldots, n\}} (-1)^{|\mathcal{S}|} F_{X_{\mathcal{S}}}(x) \text{ for all } x \in \mathbb{R}_{0,+},
$$

where $X_{\mathcal{S}} = \bigwedge_{s \in \mathcal{S}} X_s$ and $X_{\mathcal{S}} \sim Ga(\gamma, \sigma_{\mathcal{S}}^*)$ with $\sigma_{\mathcal{S}}^* = \left(\sum_{j \in \mathcal{S}} 1/\sigma_j\right)^{-1}$.  

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Proof. The closure under the minima operation is trivial by evoking Theorem 1, Statement (ii). The distribution of the r.v. $X_\vee$ follows immediately (e.g., Corollary 2.2 in Su and Furman, 2017a, for a similar result in the context of a multivariate Pareto distribution). This completes the proof of the theorem. \qed

Another r.v. of pivotal interest in insurance is the aggregate risk r.v. denoted by $X_+ = X_1 + \cdots + X_n$; in addition, let $X_+ \sim F_+$. It is well-known that if $X_1, \ldots, X_n$ are mutually independent and distributed gamma with arbitrary parameters, then $F_+$ admits an infinite sum representation (Moschopoulos, 1985). We further show that for $X \sim G_n^\times(\gamma, \sigma)$ and when all the scale parameters are distinct, then $F_+$ is noticeably more elegant. The derivation of $F_+$ in the general case - i.e., for arbitrary (possibly equal) scale parameters - is more cumbersome and is presented in Section 3.

Let

$$w_i(\sigma) = \prod_{j=1, j\neq i}^n \frac{1}{1 - \sigma_j / \sigma_i} \text{ for } i = 1, \ldots, n.$$  \hspace{1cm} (8)

We often write $w_i$ omitting the vector of scale parameters $\sigma$ for the simplicity of notation.

**Proposition 1.** Let $X \sim Ga_n^\times(\gamma, \sigma)$ and assume that all the scale parameters are distinct, that is $\sigma_i \neq \sigma_j$ for $i \neq j \in \{1, \ldots, n\}$, then the d.d.f. of $X_+ = X_1 + \cdots + X_n$ is

$$F_+(x) = \sum_{i=1}^n w_i \Gamma(\gamma, x / \sigma_i) / \Gamma(\gamma) \text{ for all } x \in \mathbb{R}_{0,+}.$$  \hspace{1cm} (9)

Proof. Recall (e.g., Akkouchi, 2008) that for the convolution of $\sigma_1 E_{\lambda,1}, \ldots, \sigma_n E_{\lambda,n}$ with $\sigma_i \neq \sigma_j$, $i \neq j \in \{1, \ldots, n\}$, we have

$$F_+(x \mid \Lambda = \lambda) = \sum_{i=1}^n w_i \exp\{-x \lambda / \sigma_i\} \text{ for all } x \in \mathbb{R}_{0,+}.$$  

Therefore we also have

$$F_+(x) = \sum_{i=1}^n w_i E[\exp\{-x \Lambda / \sigma_i\}] \text{ for all } x \in \mathbb{R}_{0,+},$$

and the assertion of the proposition follows evoking Lemma 1, Statement (i). \qed

The last result in this section provides an expression for the higher-order (product) moments of the r.v. $X \sim G_n^\times(\gamma, \sigma)$. We employ a special form of this expression latter on in Section 4 to derive the formula for the Pearson index of linear correlation.
Theorem 3. Let $X \sim G_n^n(\gamma, \sigma)$, then, for $h_1, \ldots, h_n \in \mathbb{N}$, we have

$$E \left[ \prod_{i=1}^{n} X_i^{h_i} \right] = \frac{\Gamma(\gamma + \sum_{i=1}^{n} h_i)}{\Gamma(\gamma) \Gamma(\sum_{i=1}^{n} h_i + 1)} \prod_{i=1}^{n} \sigma_i^{h_i} \Gamma(h_i + 1). \quad (10)$$

Proof. We immediately have

$$E \left[ \prod_{i=1}^{n} X_i^{h_i} \right] = E \left[ E \left[ \prod_{i=1}^{n} X_i^{h_i} \mid \Lambda \right] \right] \tag{1}$$

$$\equiv E \left[ \prod_{i=1}^{n} \left( \frac{\sigma_i}{\Lambda} \right)^{h_i} \Gamma(h_i + 1) \right]$$

$$= E \left[ \Lambda^{-\sum_{i=1}^{n} h_i} \right] \prod_{i=1}^{n} \sigma_i^{h_i} \Gamma(h_i + 1),$$

where ‘(1)’ holds due to the moments formula in the case of the exponentially distributed r.v.’s (see, e.g., Klugman et al., 2012). The proof is then completed by evoking Lemma 1, Statement (ii). □

3 Aggregation properties of the multiplicative multivariate gamma distribution

One of the key paradigms in the modern Enterprise Risk Management requires that all risks are treated on a holistic basis (Sweeting, 2011). As a result, risk aggregation is of fundamental importance for the effective conglomerate-wide risk management, risk-sensitive supervision, and a great variety of other business decision making processes. We further show that, within the MMG/G-MBRM class, that is for $X \sim Ga_n^n(\gamma, \sigma)$ with arbitrary scale parameters and $\gamma \in (0, 1)$, the d.d.f. of the aggregate risk r.v. $X_+$ admits an amiable finite sum representation. To this end, we employ the well-studied machinery of divided differences (e.g., Milne-Thomson, 2000, for a comprehensive treatment). We note in passing that divided differences have been recently utilized in actuarial science (e.g., Hendriks and Landsman, 2017, among others). The rest of the section is divided into two: (1) theoretical considerations, and (2) applications.

3.1 Theoretical considerations

We remind at the outset that the divided differences, denoted by $\omega(y_1, \ldots, y_m)$, on a grid $\Delta = \{y_1, \ldots, y_m\}$ for a function $\omega : \mathbb{R} \rightarrow \mathbb{R}$ can be written as (e.g., Milne-Thomson, 2000)

$$\omega(y_1, \ldots, y_m) = \sum_{i=1}^{m} \prod_{1 \leq j \neq i \leq m} \frac{\omega(y_i)}{y_i - y_j}. \quad (11)$$
Denote
\[
g(y) = \frac{\Gamma(\gamma, y)}{y\Gamma(\gamma)} \quad \text{for all } y \in \mathbb{R}_{0,+},
\] (12)
then the following corollary is merely a rearrangement of (9).

**Corollary 1** (of Proposition 1). The d.d.f. of the r.v. \( X_+ \) can be formulated, for distinct \( \sigma_j \), \( j = 1, \ldots, n \), as
\[
F_+(x) = \frac{(-1)^{n-1}x^n}{\prod_{i=1}^{n} \sigma_i} g(x/\sigma_1, \ldots, x/\sigma_n) \quad \text{for all } x \in \mathbb{R}_{0,+},
\]
where \( g(x/\sigma_1, \ldots, x/\sigma_n) \) is the divided differences representation of \( g(\cdot) \) defined as per (12).

Obviously Equation (11) does not yield sensible results when some of the scale parameters of the r.v. \( X \sim \text{Ga}_n(\gamma, \sigma) \) are equal. To circumvent this inconvenience, we formulate and prove the following lemma.

**Lemma 2.** Consider \( \omega : \mathbb{R} \to \mathbb{R} \), and the grid \( \Delta = \{y_1, \ldots, y_m\} \) as before. For \( n_i \in \mathbb{N}, i = 1, \ldots, m \), assume \( \omega \) is at least \( k = \sqrt{\sum_{i=1}^{m} n_i - 1} \) times differentiable, then we have
\[
\omega(y_1, \ldots, y_m) = \frac{1}{\prod_{i=1}^{m} \Gamma(n_i)} \prod_{i=1}^{m} \frac{\partial^{n_i - 1}}{\partial y_i^{n_i - 1}} \omega(y_i) \prod_{1 \leq j \neq i \leq m} (y_i - y_j)^{-n_j - h_j}.
\]
where \((p)_n = p(p-1) \ldots (p-n+1)\) for \( n \in \mathbb{N} \) denotes the falling factorial, \((p)_0 := 1\).

**Proof.** We start with Equation (6) of Kunz (1956) and have
\[
\omega(y_1, \ldots, y_m) = \frac{1}{\prod_{i=1}^{m} \Gamma(n_i)} \prod_{i=1}^{m} \frac{\partial^{n_i - 1}}{\partial y_i^{n_i - 1}} \omega(y_i) \prod_{1 \leq j \neq i \leq m} (y_i - y_j)^{-n_j - h_j}.
\]
Then we differentiate term-by-term to obtain
\[
\omega(y_1, \ldots, y_m) = \prod_{i=1}^{m} \frac{\partial^{n_i - 1}}{\partial y_i^{n_i - 1}} \omega(y_i) \prod_{1 \leq j \neq i \leq m} (y_i - y_j)^{-n_j}.
\]
Finally we apply the Leibniz rule and readily have, for \( i = 1, \ldots, m \),

\[
\frac{\partial^{n_i-1}}{\partial y_i^{n_i-1}} \omega(y_i) \prod_{1 \leq j \neq i \leq m} (y_i - y_j)^{-n_j} = \sum_{h_1 + \cdots + h_m = n_i-1} \left( \frac{n_i - 1}{h_1, \ldots, h_m} \right) \frac{\partial^{h_i}}{\partial y_i^{h_i}} w(y_i) \prod_{1 \leq j \neq i \leq m} \frac{\partial^{h_j}}{\partial y_j^{h_j}} (y_i - y_j)^{-n_j}
\]

\[
= \sum_{h_1 + \cdots + h_m = n_i-1} \left( \frac{n_i - 1}{h_1, \ldots, h_m} \right) \frac{\partial^{h_i}}{\partial y_i^{h_i}} w(y_i) \prod_{1 \leq j \neq i \leq m} (-n_j) h_j (y_i - y_j)^{-n_j - h_j}.
\]

This concludes the proof of the lemma.

\[\square\]

**Theorem 4.** Consider \( X \sim G_{n\gamma}(\gamma, \alpha) \), where \( \gamma \in (0, 1) \) and \( \sigma = (\sigma_1, \ldots, \sigma_n)' \) with arbitrary coordinates in the latter vector of parameters. Let \( \sigma = (\sigma_1, \ldots, \sigma_1, \ldots, \sigma_m, \ldots, \sigma_m)' \) for \( m \in \mathbb{N} \) and \( n_1 + \cdots + n_m = n \), then for \( x \in \mathbb{R}_{0,+} \), the d.d.f. of \( X_+ \) admits the following finite sum form:

\[
F_+(x) = \frac{1}{\prod_{i=1}^{m} \sigma_i^{n_i}} \sum_{h_1 + \cdots + h_m = n_i-1} \sum_{i=1}^{m} g^* \left( \frac{x}{\sigma_i} \right) \sigma_i^{h_i+1} \prod_{1 \leq j \neq i \leq m} \frac{(-n_j) h_j}{(\Gamma(h_j + 1))} \left( \frac{1}{\sigma_j} - \frac{1}{\sigma_i} \right)^{-n_j - h_j},
\]

where

\[
g^*(y) = \sum_{k=0}^{h_i} \frac{1}{\Gamma(k + 1)} y^k (-1)^k \psi^{(k)}(y).
\]

**Proof.** The proof follows by rearranging d.d.f. (9) using the divided differences operator and consequently evoking Lemma 2.

\[\square\]

**Note 3.** A close look at Theorem 4 reveals that the distribution of the r.v. \( X_+ \) can be considered a finite mixture of the r.v.’s distributed Erlang with stochastic scale parameters. To see this, first note that

\[
g^*(x/\sigma_i) = \sum_{k=0}^{h_i} \frac{1}{\Gamma(k + 1)} (x/\sigma_i)^k (-1)^k \psi^{(k)}(x/\sigma_i)
\]

\[
= \mathbb{E} \left[ \sum_{k=0}^{h_i} \frac{1}{\Gamma(k + 1)} (x/\sigma_i)^k e^{-\Lambda x/\sigma_i} \right]
\]

\[
= F_{e_i, h_i}(x) \text{ for all } x \in \mathbb{R}_{0,+},
\]



in which \( e_i, h_i \) denotes the r.v. distributed Erlang with the shape parameter \( h_i + 1 \) and the random
scale parameter $\sigma_i/\Lambda$. Then rewrite $\bar{F}_+$ as

$$
\bar{F}_+(x) = \frac{1}{\prod_{i=1}^{m} \sigma_i^m} \sum_{h_i=0}^{n_i-1} \sum_{h_1+\cdots+h_m=n_i-1} \prod_{1 \leq i \neq j \leq m} \frac{(-n_j)h_j}{\Gamma(h_j+1)} \left( \frac{1}{\sigma_j} - \frac{1}{\sigma_i} \right)^{-n_j-h_j} \right] \bar{F}_{e_i,h_i}(x),
$$

where

$$
p_{i,h_i} = \frac{\sigma_i^{h_i+1}}{\prod_{i=1}^{m} \sigma_i^{n_i}} \left[ \sum_{h_1+\cdots+h_m=n_i-1} \prod_{1 \leq i \neq j \leq m} \frac{(-n_j)h_j}{\Gamma(h_j+1)} \left( \frac{1}{\sigma_j} - \frac{1}{\sigma_i} \right)^{-n_j-h_j} \right].
$$

By setting $x = 0$, it is clear that $p_{i,h_i}$ are generalized weights in the sense that $\sum_{h_i=0}^{n_i-1} p_{i,h_i} = 1$. However, these weights are not necessarily positive. For an example, consider the bivariate case with $n = m = 2$ and $n_1 = n_2 = 1$. A simple calculation yields

$$
\bar{F}_+(x) = p_{1,0} \bar{F}_{e_{1,0}}(x) + p_{2,0} \bar{F}_{e_{2,0}}(x) \text{ for all } x \in \mathbb{R}_{0,+},
$$

where $e_{i,0} \sim Ga(1,\sigma_i/\Lambda)$, $i = 1,2$, $p_{1,0} = \sigma_2^{-1}(1/\sigma_2 - 1/\sigma_1)^{-1}$, and $p_{2,0} = \sigma_1^{-1}(1/\sigma_1 - 1/\sigma_2)^{-1}$. Therefore, depending on the values of $\sigma_1$ and $\sigma_2$, one of the weights must be negative.

### 3.2 Applications

Herein we confine the discussion to the individual and collective risk models (IRM, and CRM, respectively). In this respect, recall that we call the r.v. $S_n = X_1 + \cdots + X_n$, $n \in \mathbb{N}$ the IRM, where we let the severity r.v.’s $X_j$, $j = 1,\ldots,n$ be possibly non-homogeneous. In the CRM case, for $N \in \mathbb{Z}_{0,+} := \{0,1,2,\ldots\}$ denoting the frequency r.v., we are interested in exploring the distribution of the random sum $S_N = X_1 + \cdots + X_N$. In the context of the MMG/G-MBRM, we have

$$
S_n = \sigma_1 E_1 + \cdots + \sigma_n E_n/\Lambda, \quad (13)
$$

and

$$
S_N = \sigma E_1/\Lambda + \cdots + \sigma E_N/\Lambda. \quad (14)
$$

P.d.f.’s - rather than d.d.f.’s - often play an important role in the IRM/CRM contexts. Therefore the p.d.f.’s of $S_n$ and $S_N$ engage us in the rest of this section. We start with the p.d.f. of the former r.v. in the following proposition. Recall that $p_{i,h_i}$ are given in (13), and $\psi^{(k)}$ denotes the $k$-th order derivative of the Laplace transform.
Proposition 2. Let $X \sim Ga_n^\times(\gamma, \sigma)$ with $\sigma = (\sigma_1, \ldots, \sigma_1, \ldots, \sigma_m, \ldots, \sigma_m)'$ for $m \in \mathbb{N}$ and $n_1 + \cdots + n_m = n$, then, for $x \in \mathbb{R}_+$, the p.d.f. of the r.v. $S_n$ is given by

$$f_{S_n}(x) = \sum_{i=1}^{m} \sum_{h_i=0}^{n_i-1} p_{i,h_i} \frac{(-\sigma_i)^{-h_i+1}x^{hi}}{\Gamma(h_i+1)} \psi^{(h_i+1)}(x/\sigma_i).$$

Proof. The proof of the proposition follows from Note 3 that reports the mixture representation. Namely

$$f_{S_n}(x) = \sum_{i=1}^{m} \sum_{h_i=0}^{n_i-1} p_{i,h_i} f_{\theta_i,h_i}(x) = \sum_{i=1}^{m} \sum_{h_i=0}^{n_i-1} p_{i,h_i} \left( \frac{(1/\sigma_i)^{h_i+1}x^{hi}}{\Gamma(h_i+1)} \psi^{(h_i+1)}(x/\sigma_i) \right)$$

for all $x \in \mathbb{R}_+$. This completes the proof. \Box

The following corollary follows immediately (also, e.g., Sarabia et al., 2017).

Corollary 2. Let $X \sim Ga_n^\times(\gamma, \sigma)$, where $\gamma \in (0, 1)$ and $\sigma_1 = \cdots = \sigma_n \equiv \sigma \in \mathbb{R}_+$. Then, for $x \in \mathbb{R}_+$, the p.d.f. of the r.v. $S_n$ is given by

$$f_{S_n}(x) = \frac{1}{\Gamma(i+1)\Gamma(n-i)} \sigma^{-(n-i+\gamma-1)}e^{-x/\sigma}x^{n-i+\gamma-2}. \quad (15)$$

Note 4. It is not difficult to see that p.d.f. (15) admits the following finite mixture representation

$$f_{S_n}(x) = \sum_{i=0}^{n-1} p_i f_{\theta_i}(x),$$

where $\theta_i \sim Ga(n-i+\gamma-1, \sigma)$ and the weights are given by

$$p_i = \frac{\Gamma(n-i+\gamma-1) \Gamma(i+1) \Gamma(n-i) \Gamma(1-\gamma) \Gamma(\gamma)}{\Gamma(n-i) \Gamma(1-\gamma) \Gamma(\gamma)}$$

for $i = 0, \ldots, n-1$. Remarkably, in this special case, the weights $p_i$ are ‘proper’ in the sense that $p_i > 0$ and $\sum_{i=0}^{n-1} p_i = 1$. This observation complements Theorem 6 in Sarabia et al. (2017).

We further derive the p.d.f. of the r.v. $S_N$.

Proposition 3. Let $X \sim Ga_n^\times(\gamma, \sigma)$ with $\sigma_1 = \cdots = \sigma_n \equiv \sigma \in \mathbb{R}_+$, then, for $x \in \mathbb{R}_+$, the p.d.f.’s of the r.v. $S_N$ is

$$f_{S_N}(x) = \begin{cases} \frac{x^{\gamma-1}e^{-x/\sigma}}{\Gamma(\gamma)\sigma^{\gamma}} \sum_{i=0}^{\infty} \sum_{m=0}^{\infty} \frac{(1-\gamma)_i}{i!m!} \left( \frac{x}{\sigma} \right)^{m} P[N = m + i + 1], & \text{for } x > 0 \\ P[N = 0], & \text{for } x = 0 \end{cases}, \quad (16)$$

where $(p)_n = p(p+1)\ldots(p+n-1)$ for $n \in \mathbb{N}$ denotes the rising factorial, $(p)_0 := 1.
Proof. We have the following string of equations, for all $x \in \mathbb{R}_+$,

$$f_{S_N}(x) = \sum_{n=1}^{\infty} f_{S_n}(x) P[N = n]$$

$$= \frac{x^{\gamma-1}e^{-x/\sigma}}{\Gamma(\gamma)\sigma^\gamma} \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} \frac{(1-\gamma)_i}{i!(n-i-1)!} \left( \frac{x}{\sigma} \right)^{n-i-1} P[N = n]$$

$$= \frac{x^{\gamma-1}e^{-x/\sigma}}{\Gamma(\gamma)\sigma^\gamma} \sum_{n=0}^{\infty} \frac{(1-\gamma)_i}{i!} \sum_{m=n+1}^{\infty} \frac{(x/\sigma)^{m-i-1}}{(n-i-1)!} P[N = m + i + 1].$$

This completes the proof of the proposition.

We conclude this section by specializing the p.d.f. of the r.v. $S_N$ reported in Proposition 3 for particular choices of the frequency r.v. In actuarial science some popular choices of the r.v. $N$ are, e.g., the Poisson, negative binomial, and logarithmic (e.g., Klugman et al., 2012). Below we first remind in passing the probability mass functions (p.m.f.’s) of the just-mentioned r.v.’s, and we then present the p.d.f.’s of the aggregate r.v.’s within the framework of the corresponding CRM’s.

- If $N \sim \text{Poisson}(\lambda)$ with $\lambda \in \mathbb{R}_+$, then the p.m.f. is given by
  $$P[N = n] = \frac{\lambda^n}{n!} e^{-\lambda}$$
  for all $n \in \mathbb{Z}_{0,+}$.

- If $N \sim \text{NB}(\beta, p)$, the negative binomial distribution with $\beta \in \mathbb{R}_+$ and $p \in (0, 1)$, then
  $$P[N = n] = \frac{\Gamma(n + \beta)}{\Gamma(n + 1)\Gamma(\beta)} p^\beta (1 - p)^n$$
  for all $n \in \mathbb{Z}_{0,+}$.

- If $N \sim \text{Logm}(\theta)$, the logarithmic distribution with $\theta \in (0, 1)$, then the p.m.f. is
  $$P[N = n] = \frac{-\theta^n}{n \log(1 - \theta)}$$
  for all $n \in \mathbb{Z}_+$.

Let $\Phi_1$ and $\Phi_3$, respectively, denote the two-variable confluent hypergeometric series of the first and third kind (see, e.g., Srivastava and Karlsson, 1985), that is with $a_1, a_2, a_3 \in \mathbb{R}_+$,

$$\Phi_1(a_1, a_2; a_3; x, y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(a_1)_{(i+j)}(a_2)_{(j)}}{(a_3)_{(i+j)}i!j!} x^i y^j,$$

for $x \in \mathbb{R}$, $|y| < 1$, and

$$\Phi_3(a_1; a_2; x, y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(a_1)_{(j)}}{(a_2)_{(i+j)}i!j!} x^i y^j,$$

for $x, y \in \mathbb{R}$. The following corollary follows readily.
Corollary 3 (of Proposition 3). In the context of the Collective Risk Model, we have, for all $x \in \mathbb{R}_+, \gamma \in (0, 1)$ and $\sigma \in \mathbb{R}_+$, that

- If $N \sim \text{Poisson}(\lambda)$, then
  $$f_{SN}(x) = \lambda x^{\gamma - 1} e^{-x/\sigma} \frac{1}{\Gamma(\gamma)\sigma^\gamma} \Phi_3(1 - \gamma, 2, x\lambda/\sigma, \lambda).$$

- If $N \sim \text{NB}(\beta, p)$, then
  $$f_{SN}(x) = \beta p^\beta (1 - p) x^{\gamma - 1} e^{-x/\sigma} \frac{1}{\Gamma(\gamma)\sigma^\gamma} \Phi_1(1 + \beta, 1 - \gamma, 2, x(1 - p)/\sigma, 1 - p).$$

- If $N \sim \text{Logm}(\theta)$, then
  $$f_{SN}(x) = -\frac{\theta}{\log(1 - \theta)} x^{\gamma - 1} e^{-x/\sigma} \frac{1}{\Gamma(\gamma)\sigma^\gamma} \Phi_1(1, 1 - \gamma, 2, x\theta/\sigma, \theta).$$

4 Dependence properties of the multiplicative multivariate gamma distribution

At the first sight, the dependence structure that underpins the MMG distribution - that is d.d.f. (7) - is not as versatile as the one behind the additive counterpart of Mathai and Moschopoulos (1991). This is because the Pearson correlation, $\rho$, for the former class of distributions does not attain every value in the interval $[0, 1]$, whereas it does so in the context of the latter class of distributions (e.g., Das et al., 2007; Su and Furman, 2017a,c, for a similar constrain in the context of default risk). More formally, we have the following proposition, the proof of which is a direct application of Theorem 3 and is thus omitted.

Proposition 4. Let $X \sim \text{Ga}_n^\times(\gamma, \sigma)$, then the Pearson correlation between any pair of $X_i$ and $X_j$, for $i \neq j \in \{1, \ldots, n\}$ is

$$\rho[X_i, X_j] = (1 - \gamma)/2,$$

where $\gamma \in (0, 1)$. In addition, we have $\rho[X_i, X_j] \in (0, 1/2)$ and it is a decreasing function of $\gamma \in (0, 1)$.

In the rest of this section we show that the just-mentioned seeming shortcoming should be in fact attributed to the Pearson index of correlation, $\rho$, itself, rather than to the dependence structure of the MMG distribution. As hitherto, we divide our observations herein into two subsections.
4.1 Theoretical considerations

At the outset we observe that the dependence structure that underlies the MMG/G-MBRM is not linear in the - common - background r.v. $\Lambda$. Therefore the machinery of copulas lands itself very naturally when exploring the relevant dependence properties. The next theorem states the copula function (e.g., Nelsen, 2006) of $X \sim Ga^\times_n(\gamma, \sigma)$.

**Theorem 5.** Assume that $X \sim Ga^\times_n(\gamma, \sigma)$, then the copula function underlying the d.d.f. of $X$ is given, for $\gamma \in (0, 1)$, by

$$C_\gamma(u_1, \ldots, u_n) = \frac{1}{\Gamma(\gamma)} \Gamma \left( \gamma, \sum_{i=1}^n \Gamma^{-1}(\gamma, u_i \Gamma(\gamma)) \right),$$

where $(u_1, \ldots, u_n)' \in [0, 1]^n$, and $\Gamma^{-1}(\cdot, s)$ denotes the inverse incomplete gamma function evaluated at $s \in \mathbb{R}_+$. This completes the proof of the theorem. □

**Note 5.** Copula function (18) is a member of the encompassing class of the Archimedean copulas. Specifically, set

$$\phi(s) = \frac{\Gamma(\gamma, s)}{\Gamma(\gamma)}$$

for all $s \in \mathbb{R}_{0,+}$, and observe that (18) admits the following form, for $(u_1, \ldots, u_n)' \in [0, 1]^n$,

$$C_\phi(u_1, \ldots, u_n) = \phi(\phi^{-1}(u_1) + \cdots + \phi^{-1}(u_n)),$$

where $\phi : [0, \infty) \rightarrow [0, 1]$ is a legitimate $n$-monotonic function - known as the Archimedean generator - and $\phi^{-1}$ is its inverse (e.g., McNeil and Nešlehová, 2009). The applications of the Archimedean class

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of copulas are multifaceted, e.g., Marshall and Olkin (2007) - life distributions, Denuit et al. (2005); Su and Furman (2017b) - insurance loss modelling, Cossette et al. (2018) - risk capital allocations, and Cherubini et al. (2013) - general finance.

Figure 1 depicts the simulated scatter plots of the copula function $C_\gamma$ for varying values of the $\gamma$ parameter. Interestingly, the MMG copula (Theorem 5) behaves as a reflection of the Clayton copula (Joe, 1997, also, e.g., Figure 1), and it can serve as a meaningful alternative in the situations when the Clayton copula is not applicable. The MMG copula therefore enriches the encompassing toolbox of the distinct Archimedean copulas available to researchers and practitioners.

Figure 1: Scatter plots of the MMG copula (4,000 simulation points) for varying value of the $\gamma$ parameter: $\gamma = 0.05$ (top left), $\gamma = 0.2$ (top right), $\gamma = 0.5$ (bottom left), $\gamma = 0.8$ (bottom right).

Generally, one of the useful contributions of copulas to the vast literature on multivariate modelling is that they have given rise to a number of indices of dependence that circumvent the known fallacies
of the Pearson $\rho$. Such indices of dependence are, e.g., the Kendall $\tau$ and Spearman $\rho_S$ measures of rank correlation, and we derive these two in the next subsection in the context of the MMG copula function $C_\gamma$.

In the rest of this subsection we build up the theoretical groundwork necessary for exploring the tail dependence of $C_\gamma$. As tail dependence represents the co-movement of extreme risks, it is of particular importance in the era following the financial crisis of 2007 – 2009. We note in passing that since the majority of the existing methods for quantifying tail dependence mainly aim at random pairs, we specialize the discussion in this part of the present report to the bivariate case only.

Let $\hat{C}$ denote the survival copula that corresponds to $C$, that is $\hat{C}(u_1, u_2) := u_1 + u_2 - 1 + C(1 - u_1, 1 - u_2)$, for $u_1, u_2 \in [0, 1]$. Then the first order lower and upper tail dependence parameters (e.g., Joe, 1997) are given by

$$\lambda_L := \lim_{u \downarrow 0} \frac{C(u, u)}{u} \quad \text{and} \quad \lambda_U := \lim_{u \downarrow 0} \frac{\hat{C}(u, u)}{u},$$

(19)

whereas the second order tail dependence parameters (Coles et al., 1999) are given by

$$\chi_L := \lim_{u \downarrow 0} \frac{2 \log u}{\log C(u, u)} - 1 \quad \text{and} \quad \chi_U := \lim_{u \downarrow 0} \frac{2 \log u}{\log C(u, u)} - 1.$$  

(20)

Recently an argument has been put forward that all of (19) and (20) may underestimate the extent of the tail dependence inherent in a copula. More specifically, Furman et al. (2015) claim and elucidate with numerous examples that as (19) and (20) are computed along the main diagonal $(u, u), \ u \in [0, 1]$, their values are not necessarily maximal when alternative paths in $[0, 1]^2$ are considered. This motivated the following definitions of the admissible paths and the paths of maximal dependence in ibid.

**Definition 2.** A function $\varphi : [0, 1] \to [0, 1]$ is called *admissible* if it satisfies the following conditions:

(C1) $\varphi(u) \in [u^2, 1]$ for every $u \in [0, 1]$; and

(C2) $\varphi(u)$ and $u^2/\varphi(u)$ converge to 0 when $u \downarrow 0$.

Then the path $(\varphi(u), u^2/\varphi(u))_{0 \leq u \leq 1}$ is admissible whenever the function $\varphi$ is admissible. Also, we denote by $\mathcal{A}$ the set of all admissible functions $\varphi$.

**Definition 3.** The path(s) $(\varphi(u), u^2/\varphi(u))_{0 \leq u \leq 1}$ in $\mathcal{A}$ are called paths of maximal dependence if they maximize the probability

$$\Pi_\varphi(u) = C(\varphi(u), u^2/\varphi(u))$$

or, equivalently, the distance function

$$d_\varphi \left( C, C^\perp \right)(u) = C(\varphi(u), u^2/\varphi(u)) - C^\perp(\varphi(u), u^2/\varphi(u)).$$
where $C^\perp$ is the independence copula, i.e., $C^\perp(u_1,u_2) = u_1u_2$ for all $0 \leq u_1, u_2 \leq 1$.

Obviously, the function $\varphi_0(u) = u$ is admissible and yields the representation of the diagonal path that serves as a building block for classical indices (19) and (20). For the Archimedean class of copulas, the following property of the maximal dependence path holds.

**Lemma 3** (Furman et al. 2015). For an Archimedean copula with generator $\phi$, if $x \frac{\partial}{\partial x} \phi^{-1}(x)$ is increasing on $x \in (0,1)$, then the path of maximal dependence coincides with the main diagonal.

The next lemma on a L’Hospital type rule for monotonicity, plays an importantly auxiliary role when deriving the maximal dependence path for $C_\gamma$.

**Lemma 4** (Pinelis, 2002). Let $-\infty \leq a < b \leq \infty$, also $g_1$ and $g_2$ be differentiable functions over the interval $(a,b)$. Assume that $g_2'(s) < 0$ for $s \in (a,b)$, and $\lim_{s \downarrow a} g_1(s) = 0$ and $\lim_{s \downarrow a} g_2(s) = 0$. Then $g_1/g_2$ is increasing on $(a,b)$ if $g_1'/g_2'$ is increasing.

Our last result in this subsection implies that measures of tail dependence (19) and (20) are in fact maximal in the context of the MMG copula $C_\gamma$.

**Theorem 6.** The maximal dependence path of the copula function $C_\gamma$ in (18) is diagonal.

**Proof.** Let $\phi^{-1}(x) = \Gamma^{-1}(\gamma,x\Gamma(\gamma))$, for all $x \in (0,1)$, so

$$x \frac{\partial}{\partial x} \phi^{-1}(x) = \frac{x}{-f(\Gamma^{-1}(\gamma,x\Gamma(\gamma)))},$$

where $f(\cdot)$ is the p.d.f. of $Ga(\gamma,1)$. Note that for $\gamma \in (0,1)$, which is exactly the case in the present report, $f(s)$ is decreasing for all $s \in \mathbb{R}_+$. Now, set $g_1(x) = x$ and $g_2(x) = -f(\Gamma^{-1}(\gamma,x\Gamma(\gamma)))$. Clearly, $\lim_{x \downarrow 0} g_1(x) = 0$, $\lim_{x \downarrow 0} g_2(x) = 0$, and $g_2(x)$ is decreasing on $x \in (0,1)$. Moreover,

$$\frac{g_1'(x)}{g_2'(x)} = \left. \frac{f(s)}{f'(s)} \right|_{s=\Gamma^{-1}(\gamma,x\Gamma(\gamma))} = \frac{-1}{1 + (1 - \gamma)(\Gamma^{-1}(\gamma,x\Gamma(\gamma)))^{-1}}$$

is increasing on $x \in [0,1]$. Evoking Lemma 4, we conclude that $x \frac{\partial}{\partial x} \phi^{-1}(x)$ is increasing for $x \in (0,1)$. Finally, based on Lemma 3, the path of maximal dependence for $C_\gamma$ is the diagonal, and the proof is completed. \qed

4.2 Applications

The next assertion reports the Kendall tau and Spearman rho rank correlations, implied by the MMG copula (18). The hypergeometric function plays a pivotal role in deriving the Spearman rho correlation.
in the following proposition, and it is given in Gradshteyn and Ryzhik (2014)

\[ q+1 F(q; a_1, \ldots, a_{q+1}; b_1, \ldots, b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_{q+1})_k z^k}{(b_1)_k \cdots (b_q)_k k!}. \] (21)

For \( a_1, \ldots, a_{q+1} \) all positive, and these are the cases of interest in the present report, the radius of convergence of the series is the open disk \(|z| < 1\). On the boundary \(|z| = 1\), the series converges absolutely if \( d = b_1 + \cdots + b_q - a_1 - \cdots - a_{q+1} > 0\), and it converges except at \( z = 1 \) if \( 0 \geq d > -1\).

**Proposition 5.** For the copula \( C_\gamma \), the Kendall \( \tau \) rank correlation is given by

\[ \tau(C_\gamma) = 1 - \frac{2 \Gamma(\gamma + 1/2)}{\sqrt{\pi} \Gamma(\gamma)}, \]

and the Spearman \( \rho_S \) rank correlation is given by

\[ \rho_S(C_\gamma) = 6 \left( \frac{8^{-\gamma} \Gamma(3\gamma)}{\Gamma(\gamma + 1) \Gamma(2\gamma)} 2 F_1(1, 3\gamma; 2\gamma + 1/2) - 1/2 \right). \]

**Proof.** Recall that copula function (18) is a special member of the Archimedean class of copulas having generator

\[ \phi(s) = \frac{1}{\Gamma(\gamma)} \Gamma(\gamma, s) \text{ for all } s \in \mathbb{R}_{0,+}. \]

As a result, according to Theorem 4.3 in Joe (1997), we have

\[
\tau(C_\gamma) = 1 - 4 \int_0^\infty s \left[ \frac{\partial}{\partial s} \phi(s) \right]^2 ds
= 1 - \frac{4}{\Gamma(\gamma)^2} \int_0^\infty s^{2\gamma - 1} e^{-s} ds
= 1 - 4^{-\gamma} \frac{\Gamma(2\gamma)}{\Gamma(\gamma)^2}.
\] (22)

The expression for the Kendall \( \tau \) is obtained by simplifying (22).

We further proceed to the case of the Spearman \( \rho_S \). For \( i \neq j \in \{1, \ldots, n\} \), denote by \( f_i \) and \( f_j \) the marginal p.d.f.'s of the random pair \((X_i, X_j)' \subseteq X\), then by definition (see, Section 2.1.9 in Joe, 1997), we have

\[ \rho_S(C_\gamma) = 12 \int_0^\infty \int_0^\infty F(x_i, x_j) f_i(x_i) f_j(x_j) dx_i dx_j - 3, \]

where

\[
\int_0^\infty \int_0^\infty F(x_i, x_j) f_i(x_i) f_j(x_j) dx_i dx_j
= \frac{1}{\Gamma(\gamma)} \int_0^\infty \int_0^\infty \Gamma\left(\gamma, \frac{x_i}{\sigma_i} + \frac{x_j}{\sigma_j}\right) f_i(x_i) f_j(x_j) dx_i dx_j
= \frac{1}{\Gamma(\gamma) \Gamma(2\gamma)} \int_0^\infty \Gamma(\gamma, s) s^{2\gamma - 1} e^{-s} ds
\overset{(1)}{=} \frac{1}{\Gamma(\gamma) \Gamma(2\gamma)} \Gamma(3\gamma) 2^{3\gamma + 1} \gamma^2 F_1(1, 3\gamma; 2\gamma + 1/2). \]
Figure 2: The plot of the Pearson rho, Kendall tau, and Spearman rho measures of correlation for varying values of $\gamma \in (0, 1)$.

Here, the equality $\rho = 0$ holds because of (6.455(1)) in Gradshteyn and Ryzhik (2014). This completes the proof of the proposition. $\square$

Figure 2 depicts the values for the Pearson $\rho$, Kendall $\tau$ and Spearman $\rho_S$ indices of correlation with varying $\gamma \in (0, 1)$. The figure confirms that while the Pearson $\rho$ does not attain all values in $[0, 1]$ for the MMG/G-MBRM distribution, the other two indices are able to achieve this goal.

**Proposition 6.** Assume that $X \sim Ga_2^\gamma(\gamma, \sigma)$ has copula $C_\gamma$, the lower maximal tail dependence of $C_\gamma$ is

$$\lambda_L(C_\gamma) = \chi_L(C_\gamma) = 0.$$ 

The upper maximal tail dependence of $C_\gamma$ is

$$\lambda_U(C_\gamma) = 2 - 2^\gamma, \text{ and } \chi_U(C_\gamma) = 1.$$
Proof. Let us first study the lower tail dependence of \( C_\gamma \). The following string of equations hold:

\[
\chi_L = \lim_{u \searrow 0} \frac{2 \log \phi^{-1}(u)}{\log \phi(2\phi^{-1}(u))} - 1 = \lim_{t \to \infty} \frac{2 \log \phi(t)}{\log \phi(2t)} - 1 = \lim_{t \to \infty} 2 \frac{- \log \Gamma(\gamma) + \log \Gamma(\gamma; t)}{- \log \Gamma(\gamma) + \log \Gamma(\gamma; 2t)} - 1.
\]

We know that as \( t \to \infty \), the following asymptotic expansion holds (Temme, 1996):

\[
\Gamma(\gamma; t) = t^{\gamma-1} e^{-t} (1 + R(\gamma, t)),
\]

with \( R(\gamma, t) = \mathcal{O}(t^{-1}) \). Then, we have

\[
\lim_{t \to \infty} \frac{- \log \Gamma(\gamma) + \log \Gamma(\gamma; t)}{- \log \Gamma(\gamma) + \log \Gamma(\gamma; 2t)} = \lim_{t \to \infty} \frac{- \log \Gamma(\gamma) + (\gamma - 1) \log t - t + \log((1 + R(\gamma, t)))}{- \log \Gamma(\gamma) + (\gamma - 1) \log 2t - 2t + \log((1 + R(\gamma, 2t)))} = 1/2,
\]

and thus \( \chi_L = 0 \), which automatically implies \( \lambda_L = 0 \).

We now turn to study the upper tail dependence of \( C_\gamma \). Note that the mixture r.v. \( \Lambda \) has d.d.f. \( F_\Lambda \in \text{RV}_{-\gamma} \) that varies regularly at infinity with order \( -\gamma \) (Bingham et al., 1987). The expressions for \( \lambda_U \) and \( \chi_U \) are readily obtained by evoking Corollary 3.3 in Su and Hua (2017). This completes the proof for this proposition.

Proposition 6 readily implies - recall to this end that the copula \( C_\gamma \) is in fact a survival copula (by construction) - that the coordinates of \( X \sim \text{Ga}_{n}^\times(\gamma, \sigma) \) are asymptotically dependent in the lower tail, but independent in the upper tail. Speaking bluntly this means that \( X \) is more likely to take smaller values simultaneously, but less likely to form a cluster of large values. This conforms to the already made intuitive observation that the copula \( C_\gamma \) can serve as a reflected variant of the well-studied Clayton copula.

5 Conclusions

In the present report, we have systematically studied a class of multivariate multiplicative gamma distributions. We have demonstrated that the MMG distributions admit a very meaningful background risk model representation, where the interdependencies among risks are implied by a common systematic risk factor. Moreover, we have shown that the MMG distributions enjoy a remarkable level of analytical tractability, that is, the risk r.v.’s distributed MMG are straightforward to simulate, easy to aggregate, and have attractive (tail) dependence properties. The potential applications of the
MMG distributions in actuarial science are vast, and we hope to draw the attention of the community to this class of distributions, which has been arguably overlooked.

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