Error propagation in polarimetric demodulation

A. Asensio Ramos\textsuperscript{1,*} and M. Collados\textsuperscript{1}

\textsuperscript{1}Instituto de Astrofísica de Canarias, 38200, La Laguna, Spain

\textsuperscript{*}Corresponding author: aasensio@iac.es

The polarization analysis of the light is typically carried out using modulation schemes. The light of unknown polarization state is passed through a set of known modulation optics and a detector is used to measure the total intensity passing the system. The modulation optics is modified several times and, with the aid of such several measurements, the unknown polarization state of the light can be inferred. How to find the optimal demodulation process has been investigated in the past. However, since the modulation matrix has to be measured for a given instrument and the optical elements can present problems of repeatability, some uncertainty is present in the elements of the modulation matrix and/or covariances between these elements. We analyze in detail this issue, presenting analytical formulae for calculating the covariance matrix produced by the propagation of such uncertainties on the demodulation matrix, on the inferred Stokes parameters and on the efficiency of the modulation process. We demonstrate that, even if the covariance matrix of the modulation matrix is diagonal, the covariance matrix of the demodulation matrix is, in general, non-diagonal because matrix inversion is a nonlinear operation. This propagates through the demodulation process and induces correlations on the inferred Stokes parameters. © 2008 Optical Society of America

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1. Introduction

The majority of the information that we have obtained during the last years about the magnetism of the Sun and other astrophysical objects is based on the analysis of the polarization of the light. The polarization state of a light beam is typically described, using the Stokes formalism, by the so-called Stokes vector:

\vspace{0.5cm}

\[ \mathbf{S} = (I, Q, U, V)^T, \]

\vspace{0.5cm}

where $I$ refers to the total intensity of the beam, $V$ describes its circular polarization properties while $Q$ and $U$ are used to described linear polarization. When the light beam passes through optical elements or, in general, any medium that modifies its polarization properties, the emergent Stokes vector can be related to the input one by the following linear relation:

$$S^{\text{out}} = MS^{\text{in}}.$$  (2)

The $4 \times 4$ matrix, $M$, is the so-called Mueller matrix and it can be used to unambiguously represent any optically passive medium. It is important to note that these matrices have to obey certain properties so that they have physical meaning [1, 2].

The analysis of polarization of a given beam is usually carried out with the aid of modulation schemes. This is a requisite in the short-wavelength (optical) domain because no detectors that present the required polarization sensitivity are still available. Furthermore, the majority of the detectors that are used in the optical are only sensitive to the total amount of light or, in other words, to the total intensity given by Stokes $I$. This is not the case in the long-wavelength (microwave) domain, where it is relatively easy to build detectors that analyze directly the polarization properties of the light beam. Modulation schemes have been built to overcome the difficulties when measuring the polarization state of light beams in the short-wavelength domain. Any modulation scheme consists of a train of optical devices that produce a known modification of the input beam so that the observed intensity in the detector is a linear combination of the elements of the input Stokes vector. Carrying out several of these measurements, each one with a different combination of optical devices, it is possible to infer the input Stokes vector using a demodulation procedure. Assuming that the Mueller matrix of the $j$-th combination of optical devices is given by $M_j$, the detected intensity in each case is given by:

$$I_j^{\text{out}} = [M_j]_{00}S_0^{\text{in}} + [M_j]_{01}S_1^{\text{in}} + [M_j]_{02}S_2^{\text{in}} + [M_j]_{03}S_3^{\text{in}},$$  (3)

where we have used the standard notation $S = (S_0, S_1, S_2, S_3)^T$ for the Stokes vector, which will be used extensively in the rest of the paper. By putting several measurements together, the modulation scheme can be expressed as the following linear system:

$$I^{\text{out}} = OS^{\text{in}},$$  (4)

where each row of the matrix $O$ is built with the first rows of the different Mueller matrices of the different combinations of optical elements used in the modulation scheme. Therefore, the matrix $O$ has dimensions $N \times 4$, with $N$ the number of measurements. Note that, neither the vector $I^{\text{out}}$ is a Stokes vector nor the matrix $O$ is a Mueller matrix. The vector $I^{\text{out}}$ consists of different intensity measurements, while the matrix $O$ does not fulfill, in general, the conditions described by [1] and [2] to be considered a Mueller matrix.
Several different modulation techniques have been developed. One of the most widespread methods is to use a temporal modulation with a quarter-wave plate and a linear polarizer, with the fast axis of the quarter-wave plate being set to preselected angles [3,4]. This approach has the advantage that the different polarizations states are observed on the same pixel. However, modulation has to be carried out very fast in order not to be limited by atmospheric fluctuations. Another possibility is to use spatial modulation in which the measurements are carried out simultaneously by using a beamsplitter instead of a linear polarizer [5,6]. The advantage is that atmospheric fluctuations are avoided but potential problems appear because the optical path and the detector response are not the same for all the polarization states. The most advanced polarimeters now use a combination of both approaches in the so-called spatio-temporal modulation [7,8].

Starting from Eq. (4), the input Stokes vector can be obtained by solving the previous linear system of equations, so that:

\[ \mathbf{S}^{\text{in}} = \mathbf{DI}^{\text{out}}, \]

where the matrix \( \mathbf{D} \) is the inverse (or Moore-Penrose pseudoinverse in the general case that more than four measurements are carried out to infer the input Stokes vector) of the matrix \( \mathbf{O} \) and has dimensions \( 4 \times N \). In order to measure the four Stokes parameters, the minimum number of measurements is 4. In such a case, the matrix \( \mathbf{D} \) is unique and can be calculated as the standard inverse of \( \mathbf{O} \). However, it is also possible to carry out more measurements than Stokes parameters. In this case, the linear system of Eq. (4) is overdetermined and, in general, no solution that satisfies simultaneously all the equations exist. Recently, it has been shown that it is still possible to choose a solution if we seek for the one that maximizes the efficiency of the modulation-demodulation scheme [9]. Such efficiency for each Stokes parameter, represented with the vector \( \mathbf{\epsilon} \), is obtained from the demodulation matrix [9,10]:

\[ \epsilon_i = \left( n \sum_{j=1}^{n} D_{ij}^2 \right)^{-1/2}. \]

In this case, it can be shown that the optimal demodulation matrix is given by the Moore-Penrose pseudoinverse [11,12]:

\[ \mathbf{D} = \left( \mathbf{O}^T \mathbf{O} \right)^{-1} \mathbf{O}^T. \]

The pseudoinverse can also be found with the singular value decomposition of the modulation matrix. Note that, while \( \mathbf{DO} = \mathbf{I} \) in the general case, \( \mathbf{OD} = \mathbf{I} \) is only valid when \( N = 4 \) (\( \mathbf{I} \) is the identity matrix).

We present a theoretical calculation of how the error propagates in the demodulation process. This is an important issue that has been partially treated in the past, either considering how the uncertainty in the measurement of \( \mathbf{I}^{\text{out}} \) propagates to the demodulated
Stokes vector when the modulation matrix is perfectly known [13, 14] or taking into account uncertainties in the knowledge of the modulation matrix, though the full covariance matrix is not obtained [15, 16]. Many papers also deal with how error is propagated in non-ideal Mueller matrix polarimeters [17, 18] while others consider the optimization of such polarimeters [19–21].

2. Error propagation

In practical situations, the modulation matrix has to be measured once the optics and the modulation scheme have been defined [14, 16, 17, 20, 21]. As a consequence, one expects the elements of the modulation matrix to have some uncertainty produced by the measurement procedure. Additionally, uncertainties can also be found when the rotation of the fast axis of the retarders or of the linear polarizers has some repeatability problems [22]. In this case, although the average modulation matrix can be known with great precision, random deviations from the average modulation matrix can occur. These random deviations induce a modification on the modulation scheme that directly affects the inferred polarization properties of the input light beam. For these reasons, it is fundamental to characterize the error propagation through the demodulation process and how they affect the measured Stokes profiles. In general, since the matrix inversion (or pseudoinverse) is a nonlinear process, one should expect to find a full covariance matrix for the elements of the matrix \( D \), with non-zero non-diagonal elements. These non-diagonal elements account for the statistical correlations that appear between elements of the demodulation matrix after the inversion process. More importantly, a full covariance matrix has to be expected even in the case that the elements of the modulation matrix are not correlated (i.e., \( \text{cov}(O_{ij}, O_{kl}) \propto \delta_{ik}\delta_{jl} \), with \( \delta_{ik} \) the Kronecker delta). Note that a diagonal covariance matrix for the matrix \( O \) comes out when the measurements of the elements of the modulation matrix are statistically independent, something that can be assumed in some cases if the measurements of the elements of the matrix are carried out with proper calibration. However, this is an issue that has to be carefully analyzed for each case.

2.A. Analytical approach

The full covariance matrix for the demodulation matrix can be obtained analytically following the standard error propagation formulae. The starting point is to consider that elements of the demodulation matrix are known nonlinear functions of the elements of the modulation matrix:

\[
D_{\alpha\beta} = D_{\alpha\beta}(O_{ij}). \tag{8}
\]
The most general error propagation formula for such operation reads [23]:

$$\text{cov}(D_{\alpha\beta}, D_{ab}) = \sum_{ijkl} \frac{\partial D_{\alpha\beta}}{\partial O_{ij}} \frac{\partial D_{ab}}{\partial O_{kl}} \text{cov}(O_{ij}, O_{kl}).$$  \hspace{1cm} (9)

This covariance matrix gives information about the variances of the elements of the demodulation matrix (diagonal elements), as well as covariances between those elements (non-diagonal elements). Note that, for a $4 \times N$ matrix, the covariance matrix is $4N \times 4N$. If the covariance matrix of the modulation matrix is diagonal (i.e., $\text{cov}(O_{ij}, O_{kl}) = \sigma(O)^2_{ij} \delta_{ik} \delta_{jl}$, with $\delta_{ik}$ the Kronecker delta), the previous equation simplifies to:

$$\text{cov}(D_{\alpha\beta}, D_{ab}) = \sum_{ij} \frac{\partial D_{\alpha\beta}}{\partial O_{ij}} \frac{\partial D_{ab}}{\partial O_{ij}} \sigma(O)^2_{ij}. \hspace{1cm} (10)$$

As already presented above, this turns out to be a good approximation if the elements of the modulation matrix are carefully measured. However, our approach is general and can cope with non-diagonal covariance matrices as will be shown below.

The derivatives that appear in Eqs. (9) and (10) can be calculated, in the most general case, starting from Eq. (7). However, we distinguish the cases of a square modulation matrix from the general case, because the analytical procedure can be largely simplified. In any case, we will show that the general expressions are equivalent to those of the particular case of a square matrix $O$.

2.A.1. Square modulation matrix

Although it is possible to calculate directly the derivatives from Eq. (7), it is more advantageous to use the fact that the matrix product between $O$ and $D$ commute, so that $DO = OD = 1$. In this case, it is easy to show that [24]:

$$\frac{\partial D_{\alpha\beta}}{\partial O_{ij}} = -D_{ai}D_{j\beta}. \hspace{1cm} (11)$$

Therefore, the full covariance matrix of the matrix elements of the demodulation matrix can be calculated by substituting Eq. (11) into Eq. (9):

$$\text{cov}(D_{\alpha\beta}, D_{ab}) = \sum_{ijkl} D_{ai}D_{j\beta}D_{ak}D_{lb} \text{cov}(O_{ij}, O_{kl}). \hspace{1cm} (12)$$

In the particular case of a diagonal covariance matrix for $O$, the previous equation simplifies to:

$$\text{cov}(D_{\alpha\beta}, D_{ab}) = \sum_{ij} D_{ai}D_{j\beta}D_{ai}D_{j\beta} \sigma(O)^2_{ij}. \hspace{1cm} (13)$$
2.A.2. Non-square modulation matrix

When \( N \) is larger than four, it is not possible to follow the previous approach because \( O \) and \( D \) do not commute. However, inserting the intermediate matrix \( A = O^T O \), using the fact that \( D = A^{-1} O^T \) and that the matrix \( A \) is always invertible [9], the derivative can be expressed, after some algebra, as:

\[
\frac{\partial D_{\alpha\beta}}{\partial O_{ij}} = A_{\alpha j}^{-1} \delta_{\beta i} + \sum_k O_{\beta k} \frac{\partial A_{\alpha k}^{-1}}{\partial O_{ij}}. \tag{14}
\]

The derivative of the elements of the \( A^{-1} \) matrix with respect to the elements of \( O \) can be calculated using the chain rule:

\[
\frac{\partial A_{\alpha k}^{-1}}{\partial O_{ij}} = \sum_{mn} \frac{\partial A_{\alpha k}^{-1}}{\partial A_{mn}} \frac{\partial A_{mn}}{\partial O_{ij}}. \tag{15}
\]

The definition of the matrix \( A \) allows us to obtain the following derivative easily:

\[
\frac{\partial A_{mn}}{\partial O_{ij}} = O_{in} \delta_{mj} + O_{jm} \delta_{ni}. \tag{16}
\]

The derivative of the elements of \( A^{-1} \) with respect to the elements of \( O \) can also be calculated easily from Eq. (11) because \( A \) is a square matrix, that commutes with its inverse. Therefore:

\[
\frac{\partial A_{\alpha\beta}^{-1}}{\partial O_{ij}} = -A_{\alpha i}^{-1} A_{j\beta}^{-1}. \tag{17}
\]

Substituting Eq. (17) into Eq. (14) and after some algebra, we end up with the following expression for the derivative:

\[
\frac{\partial D_{\alpha\beta}}{\partial O_{ij}} = A_{\alpha j}^{-1} \delta_{\beta i} - A_{\alpha j}^{-1} \sum_n D_{n\beta} O_{in} - D_{\alpha i} D_{j\beta}. \tag{18}
\]

This expression reduces to Eq. (11) if \( O \) is square because the product of \( D \) and \( O \) commutes, so that:

\[
\sum_n D_{n\beta} O_{in} = \sum_n O_{in} D_{n\beta} = \delta_{i\beta}, \tag{19}
\]

After substitution in Eq. (18), we recover Eq. (11).

The final expression for the covariance of the demodulation matrix is obtained by plugging Eq. (18) into Eq. (9) [or Eq. (10)]:

\[
\text{cov}(D_{\alpha\beta}, D_{ab}) = \sum_{ijkl} \text{cov}(O_{ij}, O_{kl}) \times \left[ A_{\alpha j}^{-1} \delta_{\beta i} - A_{\alpha j}^{-1} \sum_n D_{n\beta} O_{in} - D_{\alpha i} D_{j\beta} \right] \times \left[ A_{ak}^{-1} \delta_{bk} - A_{ak}^{-1} \sum_m D_{mb} O_{km} - D_{ak} D_{lb} \right]. \tag{20}
\]
When both the modulation matrix $\mathbf{O}$ and the measured intensities $\mathbf{I}^{\text{out}}$ present uncertainties, the resulting uncertainty in the demodulated Stokes vector has contributions coming from both origins. Using standard error propagation formulae applied to Eq. (5), the covariance matrix can be written as:

$$
\text{cov}(S_i^{\text{in}}, S_j^{\text{in}}) = \sum_{\alpha\beta ab} \frac{\partial I_i^{\text{in}}}{\partial D_{\alpha\beta}} \frac{\partial I_j^{\text{in}}}{\partial D_{ab}} \text{cov}(D_{\alpha\beta}, D_{ab})
$$

$$
+ \sum_{kl} \frac{\partial S_i^{\text{in}}}{\partial I_k^{\text{out}}} \frac{\partial S_j^{\text{in}}}{\partial I_l^{\text{out}}} \text{cov}(I_k^{\text{out}}, I_l^{\text{out}}).
$$

The first contribution takes into account the uncertainty in the knowledge of the modulation matrix while the second contribution takes into account the uncertainty in the measurement of the intensities arriving to the detector [13, 25]. In the field of optimization of polarimeters, a suitable norm of this very last term (often neglecting the non-diagonal covariances in the vector $\mathbf{I}^{\text{out}}$ that we include here) is the chosen one to measure the efficiency of a polarimeter [13, 19, 25, 26]. Particularizing to our problem and calculating the partial derivatives, we end up with:

$$
\text{cov}(S_i^{\text{in}}, S_j^{\text{in}}) = \sum_{\alpha\beta} I_i^{\text{out}} I_j^{\text{out}} \text{cov}(D_{i\alpha}, D_{j\beta})
$$

$$
+ \sum_{kl} D_{ik} D_{jl} \text{cov}(I_k^{\text{out}}, I_l^{\text{out}}).
$$

The covariance matrix $\text{cov}(\mathbf{I}^{\text{out}})$ is typically assumed to be diagonal, since no statistical dependence is assumed between consecutive intensity measurements in the detector, so that $\text{cov}(I_k^{\text{out}}, I_l^{\text{out}}) \propto \delta_{kl}$. Since this is not generally the case for the quantities $\text{cov}(D_{i\alpha}, D_{j\beta})$, a non-zero (in general) covariance between different Stokes profiles appears as a consequence of some degree of correlation in the demodulation process.

In principle, it is possible to diagonalize the covariance matrix given by Eq. (22) to obtain its principal components. Such principal components define the directions along which the correlation between different Stokes parameters is minimized. Since the principal components define a new reference system, it is possible to rotate the original reference system in which the four-dimensional Stokes vector is defined. This rotation minimizes in a statistical sense the cross-talk. This could be of interest when synthetic calculations have to be compared with polarimetric observations. In this case, in order to minimize the effect of cross-talk, it could be advantageous to compare the observations with projections along the eigenvectors of the synthetic calculations.

A quantity that is also affected by uncertainties in the knowledge of $\mathbf{O}$ is the efficiency defined in Eq. (6). Applying the error propagation formula, we get:

$$
\text{cov}(\epsilon_\alpha, \epsilon_\beta) = \sum_{ijkl} \frac{\partial \epsilon_\alpha}{\partial O_{ij}} \frac{\partial \epsilon_\beta}{\partial O_{kl}} \text{cov}(O_{ij}, O_{kl}).
$$
If we assume that the covariance matrix of the elements of the modulation matrix is diagonal, we get:

$$\text{cov}(\epsilon_\alpha, \epsilon_\beta) = \sum_{ij} \frac{\partial \epsilon_\alpha}{\partial O_{ij}} \frac{\partial \epsilon_\beta}{\partial O_{ij}} \sigma(O)_{ij}^2.$$  \hspace{1cm} (24)

The derivatives can be calculated using the definition of the efficiency and the chain rule:

$$\frac{\partial \epsilon_\alpha}{\partial O_{ij}} = \sum_{pq} \frac{\partial \epsilon_\alpha}{\partial D_{pq}} \frac{\partial D_{pq}}{\partial O_{ij}},$$ \hspace{1cm} (25)

where

$$\frac{\partial \epsilon_\alpha}{\partial D_{pq}} = -D_{aq} \delta_{pq} n^{-1/2} \left( \sum_l D_{al}^2 \right)^{-3/2}.$$ \hspace{1cm} (26)

Plugging this expression into Eq. (25), we can simplify it to read:

$$\frac{\partial \epsilon_\alpha}{\partial O_{ij}} = -n^{-1/2} \left( \sum_l D_{al}^2 \right)^{-3/2} \sum_q D_{aq} \frac{\partial D_{aq}}{\partial O_{ij}}.$$ \hspace{1cm} (27)

The derivative $\partial D_{aq}/\partial O_{ij}$ is obtained from Eq. (18) in the general case and from Eq. (11) in the case of a square modulation matrix.

According to the previous results, when the covariance matrix of the modulation matrix is diagonal and the variance is the same for all the elements, all the covariance matrices calculated in this section are proportional to the variance of the elements of $O$. As a consequence, reducing an order of magnitude the uncertainty in the elements of the modulation matrix reduces an order of magnitude the uncertainty in the demodulation matrix and in the inferred Stokes vector.

2.B. Monte Carlo approach

The previous analytical approach assumes that the error propagates normally. This is typically a good approximation when the modulation matrix is far from singular. In other words, the rows of the modulation matrix have to be as linearly independent as possible. However, since the inversion is a nonlinear process, it would be possible to obtain distributions in the demodulation matrix (and as a consequence, in the inferred Stokes parameters) that are far from Gaussians. In order to verify this issue and also to test that the previous analytical approach gives the correct answer, we have carried out a Monte Carlo experiment. For simplicity, we assume that the measurement uncertainty is the same for all the matrix elements of the modulation matrix and that it is characterized by the variance $\sigma^2$, so that the full variance matrix is given by:

$$\sigma(O)_{ij}^2 = \sigma^2.$$ \hspace{1cm} (28)

This selection makes the presentation of the results easier, but the previous approach is general and can cope with non-diagonal covariance matrices. The Monte Carlo experiment,
given an initial modulation matrix, generates $N$ instances of such a matrix with added uncertainties:

$$\hat{O}_i = O + \gamma_i \sigma(O),$$

(29)

where $\gamma_i$ is a normally distributed random constant with zero mean and unit variance. For each modulation matrix, we obtain the optimal demodulation matrix following Eq. (7). Finally, the statistical properties of the demodulation matrix are characterized by the full covariance matrix, given by:

$$\text{cov}(D_{ij}, D_{kl}) = E[D_{ij}D_{kl}] - E[D_{ij}]E[D_{kl}],$$

(30)

where $E[x]$ stands for the expected value of the $x$ random variable. In such a Monte Carlo approach, we estimate it using the sample mean, so that $E[x] = \langle x \rangle$.

3. **Illustrative examples**

The previous analytical approach is applied to several modulation matrices obtained from the literature and that represent different typical modulation schemes. We analyze the uncertainties in the demodulation matrix and in the inferred Stokes parameters using the error propagation formulation presented in the previous section. For illustration purposes, the results are also accompanied in some cases with plots obtained using the Monte Carlo approach. One of the reasons for that is to demonstrate whether the resulting distributions can be correctly assumed to be Gaussians. The Monte Carlo results have been obtained using $N = 60000$. We assume that the variance that characterizes the uncertainty in the measurement of the modulation matrix is $\sigma^2 = 6.25 \times 10^{-6}$ (the standard deviation is equal to $\sigma = 2.5 \times 10^{-3}$), which is a reasonable value for a standard calibration of the modulation.

3.A. **Square modulation matrix**

The first example is representative of a square modulation matrix in which four measurements are carried out to infer the four Stokes parameters. The matrix is that used by the Tenerife Infrared Polarimeter (TIP; [27]) as presented in [9]. Although this matrix is not the experimental one for TIP, it presents the desired structure and can be used to gain some insight on the properties of the error propagation. The chosen modulation matrix is:

$$O_{\text{TIP}} = \begin{bmatrix}
1 & 0.47 & -0.68 & 0.48 \\
1 & -0.91 & -0.19 & -0.13 \\
1 & -0.11 & 0.57 & 0.72 \\
1 & 0.68 & 0.27 & -0.58
\end{bmatrix}$$

(31)
The optimal demodulation matrix, obtained from the application of Eq. (7), is just the inverse of \( O \):

\[
D_{\text{TIP}} = \begin{bmatrix}
0.19 & 0.31 & 0.18 & 0.31 \\
0.31 & -0.66 & -0.03 & 0.37 \\
-0.72 & -0.24 & 0.67 & 0.29 \\
0.36 & -0.34 & 0.60 & -0.61 \\
\end{bmatrix}
\]  
\quad (32)

As shown in Fig. 1, the distribution of the elements of the demodulation matrix are Gaussians whose standard deviation can be obtained from the diagonal elements of the matrix defined in Eq. (13):

\[
\sigma(D_{\text{TIP}}) = \begin{bmatrix}
1.14 & 1.08 & 1.19 & 1.07 \\
1.81 & 1.71 & 1.88 & 1.70 \\
2.34 & 2.22 & 2.43 & 2.20 \\
2.19 & 2.07 & 2.27 & 2.06 \\
\end{bmatrix} \times 10^{-3}.
\]  
\quad (33)

Although we only present the diagonal elements of the covariance matrix, note that the covariance matrix is a full matrix in which all the elements are different from zero.

As a direct consequence of the uncertainties in the modulation matrix, the efficiency of the proposed scheme presents uncertainties. They can be calculated using Eq. (24), so that the average value and their standard deviations are:

\[
\epsilon_{\text{TIP}} = (0.969, 0.612, 0.473, 0.506)
\]

\[
\sigma(\epsilon_{\text{TIP}}) = (1.38, 1.26, 1.25, 1.27) \times 10^{-3},
\]  
\quad (34)

where non-diagonal elements in \( \text{cov}(\epsilon_\alpha, \epsilon_\beta) \) are non-zero, but typically much smaller than the elements in the diagonal.

Now we investigate the propagation of uncertainties to the inferred Stokes parameters. To this end, we choose an initial light beam defined by \( S_{\text{in}} = (1, 10^{-3}, 10^{-3}, 10^{-3}) \). Such Stokes vector is representative of what one would observe in relatively low magnetic flux regions of the solar surface. The application of Eqs. (22) and (13) gives the following covariance matrix:

\[
\text{cov}(S_{\text{in}}) = \begin{bmatrix}
1.66 & -0.21 & 0.01 & -0.76 \\
-0.21 & 4.17 & 0.14 & 0.59 \\
0.01 & 0.14 & 6.99 & 0.32 \\
-0.76 & 0.59 & 0.32 & 6.09 \\
\end{bmatrix} \times 10^{-6}
\]  
\quad (35)

This result has been obtained assuming that there is no uncertainty in the measurement of the intensity arriving to the detector, so that \( \text{cov}(I_{\text{out}}^k, I_{\text{out}}^l) = 0 \). In so doing, we isolate the effect of the modulation on the inferred Stokes vectors. The results of the Monte Carlo simulation are shown in Fig. 2 demonstrating that the values are quasi-normally distributed around the original value with a dispersion that is given in each panel of the plot. Note that the
standard deviations are large, almost of the order of the standard deviation of the precision in the measurement of the modulation matrix. Note also that the largest value of \( \text{cov}(S_{i}^{\text{in}}, S_{j}^{\text{in}}) \) are on the diagonal of the matrix, but sizable non-diagonal elements also appear, indicating a certain degree of correlation between the inferred Stokes parameters. For instance, the value \( \text{cov}(S_{0}^{\text{in}}, S_{3}^{\text{in}}) = -0.76 \times 10^{-6} \) indicates that when the inferred Stokes \( I \) \( (S_{0}^{\text{in}}) \) increases, the inferred Stokes \( V \) \( (S_{3}^{\text{in}}) \) tends to systematically decrease. This can be understood as a cross-talk between all the Stokes parameters induced by the demodulation process due to the special structure of the modulation matrix. The two-dimensional distribution of the inferred Stokes \( I \) and \( V \) is shown in the left panel of Fig. 3. In specific cases, part of this cross-talk can be corrected for easily based on physical arguments. This happens, for instance, when observing the Stokes profiles induced by the Zeeman effect in magnetized regions of the solar surface with the aid of spectropolarimeters. In such a case, one can assume that Stokes \( Q \), \( U \) and \( V \) are zero away from the spectral line and carry out a correction of the cross-talk from Stokes \( I \) to Stokes \( Q \), \( U \) and \( V \).

The diagonalization of the \( \text{cov}(S^{\text{in}}) \) matrix gives the following eigenvector matrix ordered in row format:

\[
V = \begin{bmatrix}
0.986 & 0.042 & -0.012 & 0.160 \\
0.002 & 0.963 & -0.018 & -0.268 \\
-0.057 & 0.125 & 0.913 & 0.385 \\
-0.156 & 0.234 & -0.408 & 0.868
\end{bmatrix}
\]  

Each row represents the linear combination of Stokes parameters that one infers due to the induced cross-talk contamination. Note that the weight of one of the Stokes parameters is larger in each row, but the contamination from the other parameters is still large, a consequence of the chosen modulation matrix.

3.B. Non-square modulation matrix

We present here results for a typical non-square modulation matrix. We have chosen the one belonging to the Advanced Stokes Polarimeter (ASP; [28]) as presented in [9]. As it happens for the TIP matrix, this is probably not the exact matrix used in the polarimeter but is representative of what happens in a scheme in which more than four measurements are used.
to obtain the four Stokes parameters. The modulation matrix we choose is:

\[
O_{\text{ASP}} = \begin{bmatrix}
1 & 0.77 & 0.41 & -0.36 \\
1 & -0.06 & 0.41 & -0.86 \\
1 & -0.06 & -0.41 & -0.86 \\
1 & 0.77 & -0.41 & -0.36 \\
1 & 0.77 & 0.41 & 0.36 \\
1 & -0.06 & 0.41 & 0.86 \\
1 & -0.06 & -0.41 & 0.86 \\
1 & 0.77 & -0.41 & 0.36 \\
\end{bmatrix}.
\] (37)

After calculating the demodulation matrix and its covariance matrix, we can also calculate the diagonal of the covariance matrix for the efficiency which, transformed into standard deviations, give:

\[
\epsilon_{\text{ASP}} = (0.760, 0.415, 0.410, 0.659)
\]
\[
\sigma(\epsilon_{\text{ASP}}) = (1.37, 0.94, 0.88, 0.88) \times 10^{-3};
\] (38)

and the full covariance matrix for the inferred Stokes parameters using \( S^{\text{in}} = (1, 10^{-3}, 10^{-3}, 10^{-3}) \) as the input Stokes vector:

\[
\text{cov}(S^{\text{in}}) = \begin{bmatrix}
1.35 & -1.61 & 0 & 0 \\
-1.61 & 4.54 & 0 & 0 \\
0 & 0 & 4.65 & 0 \\
0 & 0 & 0 & 1.80 \\
\end{bmatrix} \times 10^{-6}.
\] (39)

Note that the elements of the diagonal of this covariance matrix are smaller than for the case of only four measurements (except for the case of \( \text{cov}(S^{\text{in}}_1, S^{\text{in}}_2) \)), probably induced by the larger number of measurements carried out. Furthermore, it is important to point out that this modulation scheme induces no correlations between \( S^{\text{in}}_2 \) (Stokes \( U \)) and \( S^{\text{in}}_3 \) (Stokes \( V \)) and any other Stokes parameter. On the contrary, there is a large correlation between Stokes \( I \) and Stokes \( Q \), something that can be clearly seen in the right panel of Fig. 3.

The diagonalization of the previous matrix gives the following eigenvector matrix (in row order):

\[
V = \begin{bmatrix}
0.92 & 0.39 & 0 & 0 \\
-0.39 & 0.92 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}.
\] (40)

The demodulation process has induced a cross-talk between Stokes \( I \) and \( Q \) which can be represented by a rotation of \( \sim 23^\circ \).
3.C. Ideal modulation matrix

It is instructive to present results for the following ideal modulation matrix:

\[
O_{\text{IDEAL}} = \begin{bmatrix}
1 & 1 & 0 & 0 \\
1 & -1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
1 & 0 & -1 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & 0 & -1 \\
\end{bmatrix}.
\]  

(41)

The efficiency in this case is the maximum that can be reached, while the diagonal of the covariance matrix for the efficiency given as standard deviations is:

\[
\epsilon_{\text{IDEAL}} = (1, \sqrt{3}, \sqrt{3}, \sqrt{3})
\]

\[
\sigma(\epsilon_{\text{IDEAL}}) = (1.02, 1.02, 1.02, 1.02) \times 10^{-3},
\]  

(42)

and the full covariance matrix for the inferred Stokes parameters using \(S^{\text{in}} = (1, 10^{-3}, 10^{-3}, 10^{-3})\) as the input Stokes vector is:

\[
\text{cov}(S^{\text{in}}) = \begin{bmatrix}
1.04 & 0 & 0 & 0 \\
0 & 3.13 & 0 & 0 \\
0 & 0 & 3.13 & 0 \\
0 & 0 & 0 & 3.13 \\
\end{bmatrix} \times 10^{-6}.
\]  

(43)

The covariance matrix is diagonal with equal uncertainties in Stokes \(Q\), \(U\) and \(V\). These values are smaller than for the ASP example except for Stokes \(V\), where the uncertainty for the ideal modulation matrix is slightly larger. More important is the fact that, since the covariance matrix is diagonal, no correlation is found between the inferred Stokes parameters, so that no residual cross-talk induced by the modulation is induced in the demodulation.

4. Concluding remarks

We have presented analytical expressions for the calculation of the propagation of errors in the demodulation process when the modulation matrix is not known with infinite precision. This can happen when the modulation system has not been calibrated with enough precision or when the repeatability of the modulation system induces uncertainties in the modulation scheme. The formulae that we have presented allows polarimeter designers to calculate the errors in the demodulation matrix, the efficiency of the modulation scheme and the inferred Stokes parameters. They are simple to calculate and require only the knowledge of the modulation matrix, together with its covariance matrix. We have pointed out the fact that, in general, since matrix inversion (or Moore-Penrose pseudoinversion) is a nonlinear operation,
non-zero non-diagonal covariances have to be expected in the demodulation matrix even if such non-diagonal correlations are not present in the modulation matrix. This has the important consequence of generating spurious correlations (cross-talk) between the inferred Stokes parameters. We calculate the induced cross-talk by diagonalizing the covariance matrix. The matrix of eigenvectors represent the reference system in which the cross-talk is minimized. We have illustrated these points with three different modulation matrices representing three different ways of measuring the polarization state of light beams. Each method has its own advantages and disadvantages. It is up to the polarimeter designer to choose the modulation scheme depending on the desired precision. We hope that the formulae present in this paper are of interest for improving the quality of modulation based polarimeters. Routines in FORTRAN 90 and IDL\textsuperscript{1} for the calculation of the formulae presented in the paper can be obtained after an e-mail request to the authors of this paper.

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Fig. 1. Distribution of the $D_{00}$ matrix element of the TIP demodulation matrix when the modulation matrix elements are known with the uncertainty indicated in each panel. Note that the distribution is centered at the value obtained from applying Eq. (7) but a dispersion is present around this value.

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Fig. 2. Distribution of the inferred Stokes parameters when no uncertainty is assumed in the measurement of the $I_{\text{out}}$ vector but only in the measurement of the TIP modulation matrix, whose uncertainty is $\sigma = 2.5 \times 10^{-3}$. The input Stokes vector was $S_{\text{in}} = (1, 10^{-3}, 10^{-3}, 10^{-3})$. The distributions are close to Gaussians centered on the original values. The standard deviation of the inferred value is given in each panel. Note that the errors are roughly similar or slightly smaller than the uncertainty in the modulation matrix. Note also that correlations between the elements of the inferred vector are also expected and are produced by the non-diagonal covariance matrix of the demodulation matrix.
Fig. 3. Left panel: two-dimensional distribution obtained from the Monte Carlo simulation for $S_0^{\text{in}}$ and $S_3^{\text{in}}$ using TIP’s modulation scheme, i.e., the inferred Stokes $I$ and $V$, respectively. The plot shows the presence of a small correlation between the two results, a direct consequence of the appearance of correlations during the inversion process of the modulation matrix. Right panel: the same result but for the ASP case and for $S_0^{\text{in}}$ and $S_1^{\text{in}}$, i.e., the inferred Stokes $I$ and $Q$. Note that the correlation between both is even larger, as indicated by the respective covariance matrices given by Eqs. (35) and (39).