Filtering and identification of Markov jump discrete time systems using filter with unknown input

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Abstract. An algorithm for identifying the state of a Markov chain included in the description of a discrete stochastic system is considered. To construct the estimates, indirect observations and the Kalman filtering algorithm with an unknown input are used. An example is provided to illustrate the proposed approach.

1. Introduction

Systems with Markovian jump parameters are a special class of switching systems, and they are modeled by a set of systems with the transitions between the models determined by a Markov chain taking values in a finite set. There are many real applications of these systems, for example, economic systems [1], power systems [2], flight systems [3], communication systems and physical processes [4]. Such algorithms occupy a special place in the problem of the fault detection [5–8].

In this paper, we consider the problem of identifying the state of the Markov chain, which is included into the description of a linear stochastic system, and also considers the problem of estimating the state vector of this system. The solution was obtained using the separation principle, Kalman filtering and estimates vector of unknown input [9–12]. A filter transfer matrix is proposed to choose based on minimizing the sum of quadratic forms of estimation errors. A numerical example of solving the filtering and identification problem for a linear system of Markov jumps with three modes is given.

2. Problem formulation

Consider the following linear discrete-time stochastic system, having a jump parameter, of the form

\begin{equation}
    x(k+1) = A_{\gamma(k)} x(k) + B_{\gamma(k)} u(k) + q_{\gamma(k)}(k), \quad x(0) = x_0,
\end{equation}

where \( x(k) \in \mathbb{R}^n \) denote the state of the system; \( x_0 \) is random vector with known math expected value and covariance \( N_{0i} = \mathbb{E}\{ (x_0 - \bar{x}_0)(x_0 - \bar{x}_0)^T \} / \gamma = \gamma_i \}, \quad i = \{1, r\}; \quad A_{\gamma(k)}, \quad B_{\gamma(k)} \) are real-valued matrices; \( u(k) \in \mathbb{R}^m \) denote known input; \( \gamma = \gamma(k) \) is Markov chain with \( r \) states (\( \gamma_1, \gamma_2, \ldots, \gamma_r \)); \( q_{\gamma(k)}(k) \) is random perturbations with characteristics: \( \mathbb{E}\{ q_{\gamma(k)}(k) \} = 0 \), \( \mathbb{E}\{ q_{\gamma(k)}(k) q_{\gamma' (k)}^T (j) / (\gamma(\xi) = \gamma(k), k \leq \xi \leq j) \} = Q_{\gamma(k)} \delta_{\gamma_j} \) where \( \delta_\gamma \) denotes Kronecker delta.

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The probability of states of the jump process \( p_j(k) = P(\gamma(k) = j), \quad j = 1, r \) satisfies the equation
\[
p_j(k+1) = \sum_{i=1}^{r} p_i(k)p_{i,j}, \quad p_j(0) = p_{j,0}, \quad j = 1, r.
\] (2)

Here \( p_{i,j} \) is the probability of transition from state \( i \) to state \( j \) for one step, \( p_{j,0} \) is the initial probability of the \( j \)-th state.

It is required to find the estimate of a parameters \( \gamma(k) \) (identification problem) and find the estimate of the state vector \( x(k) \) based on the Kalman filtering algorithm according to indirect observations: An observation vector is:
\[
\gamma(k) = S_{\gamma(k)}x(k) + v_{\gamma(k)}(k),
\] (3)

where \( v_{\gamma(k)}(k) \) is the Gaussian random sequence independent of \( q_{\gamma(k)}(k) \) with characteristics: 
\( E[v_{\gamma(k)}(k)] = 0, E[v_{\gamma(k)}(k)v_{\gamma(k)}^T(j)/\gamma(\xi) = \gamma(k), k \leq \xi \leq j] = V_{\gamma(k)} \delta_{ij}. \) The pair of matrices \( A_i, S_{\gamma_i} \) \((i = 1, r)\) is detectable.

An error identification \( \gamma(k) \) parameter values, when the system (1) is in the \( j \)-th state \((\gamma = \gamma_j)\), but this state is erroneously identified as the \( i \)-th \((j \neq i)\), equation (1) can be represented as a model with an unknown input:
\[
x(k+1) = A_i x(k) + B_i u(k) + f^{(i,j)}(k) + q_j(k), \quad x(0) = x_0,
\] (4)

where the unknown input vector is determined by the formula:
\[
f^{(i,j)}(k) = (A_j - A_i)x(k) + (B_j - B_i)u(k) + q_j(k) - q_i(k).
\] (5)

Here we introduce notations for the matrices \( A_{\gamma(k)}, B_{\gamma(k)}, S_{\gamma(k)}, Q_{\gamma(k)}, V_{\gamma(k)} \) for \( \gamma(k) = \gamma_i \): \( A_i, B_i, S_i, Q_i, V_i \) respectively \((i = 1, r)\).

It is required to find an estimate of a parameters \( \gamma(k) \) (identification problem) and find an estimate of state vector \( x(k) \) based on the Kalman filtering algorithm with unknown input for model (4) according to observations (3).

3. Synthesis of the filter

To solve the problem of estimating the state vector and the unknown input, we use the model representation in the form (4) and the information from the observation (3), also we use the separation principle. This means that we first construct the estimate of the vector \( \hat{x}(k) \) on the assumption that the vector \( f^{(i,j)}(k) \) and the value of the jump parameter \( \gamma(k) \) are known. Then we constructed vectors estimations \( \hat{f}_i(k) = \hat{f}^{(i,j)}(k) \) and \( \hat{\gamma}(k) \) on the assumption that the state vector estimation \( \hat{x}(k) \) is known.

We define the estimate \( \hat{x}(k) \) using the Kalman filter:
\[
\hat{x}(k+1) = A_i \hat{x}(k) + \hat{f}_i(k) + K_i(k)(y(k+1) - S_i(A_i \hat{x}(k) + \hat{f}_i(k))), \quad \hat{x}(0) = \bar{x}_0,
\] (6)

where \( K_i(k) \) \((i = 1, r)\) are transfer matrices of the filter, which we define from the minimum of a following criterion for \( k \in [0, T] \):
\[
J(0, T, \bar{\gamma}) = E\left\{ \sum_{k=0}^{T} e^T(k)H e(k) / \gamma(0) = \gamma_{\bar{\gamma}} \right\},
\] (7)

where \( H > 0 \) is the weighting matrix, \( e(k) = x(k) - \hat{x}(k) \) is the error vector.
The matrices $K_i(k)$ are determined based on minimization of a criterion (7). We write down criterion (7) as a sum

$$J(0, T, i) = \sum_{k=0}^{T-1} \text{tr} N_i(k)H,$$  

(8)

where tr is trace of a quadratic matrix, the matrix $N_i(k) = E[e(k)e(k)\mathbf{T}/\gamma = \gamma_i]$ is determined from the equation:

$$N_i(k+1) = (A_i - K_i(k)S_i A_i)(\sum_{j=1}^{r} p_{i,j} N_j(k))(A_i - K_i(k)S_i A_i)^T +$$

$$+ (I - K_i(k)S_i)Q_i (I - K_i(k)S_i)^T + K_i (k)V_i K_i(k)^T, N_i(0) = N_0,$$

(9)

where $I$ is identity matrix with according dimension.

We introduce the Lyapunov function of the following form:

$$W(k, N_i(k)) = \text{tr} N_i(k)H + \text{tr} \sum_{r_{j,k}} \Psi_i(t)L_r(t),$$

(10)

where $\Psi_i(t) = (I - K_i(t)S_i)Q_i (I - K_i(t)S_i)^T + K_i(t)V_i K_i(t)^T + \Psi_i(t)$ ($\Psi_i(t)$ are some positive definite matrices). It is assumed here that in (10) the $L_r(t)$ matrices positive definite and satisfies the equation:

$$L_r(t) = (A_i - K_i(t)S_i A_i)^T(\sum_{j=1}^{r} p_{i,j} L_j(t + 1))(A_i - K_i(t)S_i A_i) + H, L_r(T) = L_{2r},$$

(11)

where $L_r$ is some positive definite matrix.

Sum over $k = 0, T - 1$ the finite differences of the function $W(k, N_i(k))$, taking into account the formula (11), we obtain

$$\sum_{k=0}^{T-1} \Delta W(k, N_i(k)) = \sum_{k=0}^{T-1} [W(k + 1, N_i(k + 1)) - W(k, N_i(k))]=$$

$$= \sum_{k=0}^{T-1} \text{tr}[N_i(k+1)L_i(k + 1) - N_i(k)L_i(k) - \Psi_i(k)L_i(k)].$$

(12)

On the other hand, this expression can be represented as follows:

$$\sum_{k=0}^{T-1} \Delta W(k, N_i(k)) = W(t + 1, N_i(t + 1)) - W(t, N_i(t)) + ... + W(T, N_i(T)) -$$

$$- W(T - 1, N_i(T - 1)) = \text{tr} N_i(T)H - \text{tr} N_i(T)L_i(t) - \sum_{k=0}^{T-1} \Psi_i(k)L_i(k).$$

(13)

Add to the formula (8) the difference of the right-hand sides (12) and (13). As a result, criterion (8) will take the form:

$$J(0, T, i) = \sum_{k=0}^{T-1} \text{tr} N_i(k)H - \sum_{k=0}^{T-1} \text{tr} N_i(k)L_i(k) + \sum_{k=0}^{T-1} \text{tr}[A_i - K_i(k)S_i A_i](\sum_{j=1}^{r} p_{i,j} N_j(k))(A_i - K_i(k)S_i A_i)^T +$$

$$+ (I - K_i(k)S_i)Q_i (I - K_i(k)S_i)^T + K_i(k)V_i K_i(k)^T]L_i(k + 1).$$

(14)

Applying the rules of differentiating a function tr from a matrix [13], we calculate the derivatives

$$\frac{\partial J(0, T, i)}{\partial K_i(k)} = \frac{\partial}{\partial K_i(k)} \left( \sum_{k=0}^{T-1} \text{tr} N_i(k)H - \sum_{k=0}^{T-1} \text{tr} N_i(k)L_i(k) + \sum_{k=0}^{T-1} \text{tr}[A_i - K_i(k)S_i A_i](\sum_{j=1}^{r} p_{i,j} N_j(k))(A_i - K_i(k)S_i A_i)^T +
$$

$$+ (I - K_i(k)S_i)Q_i (I - K_i(k)S_i)^T + K_i(k)V_i K_i(k)^T]L_i(k + 1) \right) = \sum_{j=1}^{r} [L_i(k + 1)\sum_{k=0}^{T-1} A_i^T S_i^T + L_i(k + 1)K_i(k)S_i A_i \times$$

$$\times (\sum_{j=1}^{r} p_{i,j} N_j(k))A_i^T S_i^T - L_i(k + 1)S_i Q_i S_i^T + L_i(k + 1)K_i(k)S_i Q_i S_i^T + L_i(k + 1)K_i(k)V_i].$$

(15)
Equating this derivative to zero, we obtain the formula for determining the matrices $K_i(k)$ ($i = 1, r$):

$$K_i(k) = (A_i - K_iS_iA_i)(\sum_{j=1}^{r} p_{i,j}N_j(k))A_i^T + (I - K_iS_i)Q_i[I - K_i(S_i)]^T + KV_iK_i^T, \quad (18)$$

$$N_i = (A_i - K_iS_iA_i)(\sum_{j=1}^{r} p_{i,j}N_j(k))A_i^T + (I - K_iS_i)Q_i[I - K_i(S_i)]^T + KV_iK_i^T, \quad (19)$$

Note that if there are positive definite solutions $N_i$ ($i = 1, r$) of the matrices equation (18), then from the condition $Q_i + KV_iK_i^T > 0$ follows the validity of Theorem 1.6 [14], and this means the stability of the stationary filter:

$$\hat{x}(k+1) = A_i\hat{x}(k) + f_i(k) + K_i(\gamma(k+1) - S_i(A_i\hat{x}(k) + f_i(k))), \quad \hat{x}(0) = \bar{x}_0. \quad (20)$$

The use of filter (20) will consist in switching the matrix $K_i$ depending on the value of the jump parameter. Note that there is no such possibility of calculating steady-state matrices $K_i$ for filtering with the known $\gamma(k)$ with using Classic Kalman filter.

As an algorithm for estimating an unknown input, we will use LSM-estimates; in this case, an estimate can be constructed on the basis of minimizing the additional criterion [9] under the assumption that the value of the jump parameter is known (here it is assumed that $\gamma = \gamma_j$):

$$I(f_j(k)) = \sum_{i=1}^{r} \left[ \|y(t) - S_j(A_j\hat{x}(t-1) + B_ju(t-1) + f_j(t-1))\|^2_W + \|f_j(t-1)\|^2_W \right], \quad (21)$$

where $\bar{W}, \bar{W}$ are positive definite weight matrices.

Minimizing (21) we obtain estimates of the unknown input:

$$\hat{f}_j(k) = (S_j^T \bar{W}S_j + \bar{W})^{-1} S_j^T \bar{W}[y(k+1) - S_j(A_j\hat{x}(k) + B_ju(k))]. \quad (22)$$

4. Jump parameter estimation

The identification algorithm for the parameter $\gamma(k)$ uses the estimate of the unknown input (22), and an estimate constructed based on the formula (5) using the estimates of the state vector $\hat{x}(k)$:

$$\hat{f}(\hat{x}(k), k) = (A_j - A_j\hat{x}(k) + (B_j - B_j)u(k), \quad (23)$$

which depends on the true value of the parameter $\gamma = \gamma_j$ ($j = 1, n$).

The value of the parameter $\gamma(k)$ is determined by the following algorithm:
1) the norm \( \psi(i, k) = \| \hat{f}(k) - \hat{\hat{f}}(\hat{x}(k), k) \| \) is calculated for all \( \gamma = \gamma_i \) (\( i = 1, r \));
2) exponential smoothing is performed for \( \psi(i, k) \)

\[
\psi(i, k+1) = \alpha \psi(i, k) + (1 - \alpha) \psi_s(i, k), \quad i = 1, r,
\]

where \( \alpha \) is smoothing factor;
3) the value of \( i \) is determined for which the smoothed value of the norm \( \psi_s(i, k) \) will be minimal.

The value of the number \( i \) will give an estimate of the jump parameter \( \hat{\gamma}_i(k) \).

5. Numerical simulation
Consider the problem of modeling a filter for discrete-time stochastic system with two-dimensional state vector, 3-mode Markovian jump parameter \( \gamma(k) \) (\( \gamma_1 = 1, \gamma_2 = 2, \gamma_3 = 3 \)) with transition probability matrix \( [p_{ij}] = \begin{pmatrix} 0.4 & 0.2 & 0.4 \\ 0.3 & 0.5 & 0.2 \\ 0.3 & 0.3 & 0.4 \end{pmatrix} \). The simulation was performed on a time interval \( k \in [0, 400] \).

Consider system (1) with the data:

\[
A_1 = \begin{pmatrix} 0.85 & 0.1 \\ -0.05 & 0.94 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0.60 & 0.05 \\ -0.02 & 0.45 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0.89 & 0.12 \\ -0.04 & 0.62 \end{pmatrix}, \quad B_1 = B_2 = B_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},
\]

\[
u(k) = \begin{pmatrix} 0.35 \\ 0.27 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 0.02 & 0 \\ 0 & 0.03 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0.03 & 0 \\ 0 & 0.02 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 0.02 & 0 \\ 0 & 0.04 \end{pmatrix}.
\]

The data describing observation vector (3) are as follows:

\[
S_1 = (1 \quad 0.9), \quad S_2 = (1 \quad 1), \quad S_3 = (1 \quad 0.9), \quad V_1 = 0.08, \quad V_2 = 1, \quad V_3 = 0.08.
\]

Weight matrices of criterion (7) and (21) are:

\[
H = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \overline{W} = 1, \quad \overline{W} = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}.
\]

The filters calculating the estimates are implemented according to equations (6), (9), (16), in which the estimate of the unknown input is calculated by the formula (22). The estimate of jumping parameter was performed according to the algorithm described in Section 4.

The simulation results are presented in Figure 1 and Figure 2. These results illustrate the quality of estimation.

![Figure 1. The jump parameter \( \gamma(k) \) and there estimate \( \hat{\gamma}(k) \).](image1)

![Figure 2. The components of state vector \( x_1(k) \), \( x_2(k) \) and its estimates \( \hat{x}_1(k) \), \( \hat{x}_2(k) \).](image2)
Using the simulation method, the fraction of erroneous estimates of the jump parameter was estimated, which is 2.23%.

6. Conclusion
The solution of the problem of synthesizing algorithm of filtering and identification for a linear discrete systems with random Markov jump parameter under incomplete information was obtained. The problem is solved by introducing an unknown input vector into the model of the system, which appears when jump identification fault. The simulation results confirmed the efficiency of the algorithm.

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