BOUNDEDNESS, COMPACTNESS AND SCHATTEN-CLASS MEMBERSHIP OF WEIGHTED COMPOSITION OPERATORS

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Abstract. The boundedness and compactness of weighted composition operators on the Hardy space $H^2$ of the unit disc is analysed. Particular reference is made to the case when the self-map of the disc is an inner function. Schatten-class membership is also considered; as a result, stronger forms of the two main results of a recent paper of Gunatillake are derived. Finally, weighted composition operators on weighted Bergman spaces $A^2_\alpha(D)$ are considered, and the results of Harper and Smith, linking their properties to those of Carleson embeddings, are extended to this situation.

1. Introduction

Let $D$ denote the open unit disk of the complex plane. For $1 \leq p < \infty$, recall that the classical Hardy space $H^p$ consists of holomorphic functions $f$ on $D$ for which the norm

$$\|f\|_p = \left( \sup_{0 \leq r < 1} \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p}$$

is finite. If $p = \infty$, $H^\infty$ is the space of holomorphic functions $f$ on $D$ such that

$$\|f\|_\infty = \sup_D |f(z)| < \infty.$$

Fatou’s Theorem asserts that any Hardy function $f$ has radial limit at $e^{i\theta} \in \partial D$ except on a set Lebesgue measure zero (see [7], for instance). Throughout this work, $f(e^{i\theta})$ will denote the radial limit of $f$ at $e^{i\theta}$, i. e., $f(e^{i\theta}) = \lim_{r \to 1^-} f(re^{i\theta})$.

If $\varphi$ is an analytic function on $D$ which takes $D$ into itself, the Littlewood Subordination Principle [17] ensures that the composition operator induced by $\varphi$,

$$C_\varphi f = f \circ \varphi, \quad (f \in H^p),$$

is bounded on $H^p$, $1 \leq p \leq \infty$. 

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On the other hand, given \( h \in \mathcal{H}^p \) it is possible to define for those functions \( f \in \mathcal{H}^p \) for which it makes sense the weighted composition operator \( W_{h,\varphi} \):

\[
W_{h,\varphi}f(z) = h(z)f(\varphi(z)).
\]

Note that if \( f \) belongs to the domain of the operator, which will denoted by \( D(W_{h,\varphi}) \), the weighted composition operator may be expressed by

\[
W_{h,\varphi} = T_h C_\varphi,
\]

where \( T_h \) denotes the Toeplitz operator with symbol \( h \). Observe that the condition \( h \in \mathcal{H}^\infty \) is always a sufficient condition for boundedness of \( W_{h,\varphi} \). By considering the image of the constant functions, it is clear that \( h \in \mathcal{H}^p \) is a necessary condition. Nevertheless, the most interesting feature in this sense is that there exists symbols \( h \) (unbounded) and \( \varphi \in \mathcal{H}^\infty \) with \( \varphi(\mathbb{D}) \subseteq \mathbb{D} \) such that \( W_{h,\varphi} \) is a bounded operator in \( \mathcal{H}^p \).

Similar considerations, with the obvious modifications, apply to other spaces of analytic functions such as the Bergman space \( \mathcal{A}^2(\mathbb{D}) \). For some general properties of these operators on Hardy and Bergman spaces we refer to the survey [16], where it is remarked that characterising the boundedness of weighted composition operators on spaces of functions on the disc enables one to characterise the boundedness of such operators on Hardy and Bergman spaces of other simply-connected domains, such as the half-plane.

Though weighted composition operators have attracted recently the attention of operator theorists (see for instance the recent papers [3], [4], [11], [12]), we would like to point out that its study traces back to the sixties. Indeed, de Leeuw showed that the isometries in the Hardy space \( \mathcal{H}^1 \) are weighted composition operators and Forelli obtained the same result for the Hardy spaces \( \mathcal{H}^p \) when \( 1 < p < \infty, \ p \neq 2 \) (see [14] and [8]).

The aim of this work is taking further the study of weighted composition on spaces of analytic functions of \( \mathbb{D} \). In particular, we deal with the boundedness of weighted composition operators on Hardy and Bergman spaces of the unit disc. In this sense, we are primarily interested in the behaviour of the weights \( h \). Let us point out in this direction that an applicable characterization of those weights \( h \) so that \( W_{h,\varphi} \) is bounded on the Bergman space would solve the famous Brenner conjecture (see Shimorin’s work [22] and the related work by Smith [23]).

In Section 2 we use recent results of Harper and Smith [11, 12] (of which the main tool is the reproducing kernel of the Hilbert space \( \mathcal{H}^2 \)) to derive new results on the boundedness, compactness and Schatten-class membership of \( W_{h,\varphi} \). A case of particular interest is when \( \varphi \) is inner, and in this case the properties of the angular derivative are seen to play an important role. In particular, we give extensions of
recent results due to Jury [15] and Gunatillake [10]. Most of our results apply primarily to operators on the Hilbert space $H^2$, but some results and proofs make sense also for all $L^p$ with $1 \leq p < \infty$.

In Section 3 we extend some results concerning the link between weighted composition operators and Carleson embeddings, due to Harper and Smith [11] [12], to the case of weighted Bergman spaces.

2. BOUNDEDNESS AND COMPACTNESS ON HARDY SPACES

2.1. Conditions for boundedness. Contreras and Hernández-Díaz [3] gave a necessary and sufficient condition for boundedness of $W_{h,\varphi}$, namely that $\mu_{h,\varphi}$ is a Carleson measure on $D$, where

$$
\mu_{h,\varphi}(E) = \int_{\varphi^{-1}(E) \cap T} |h|^2 dm,
$$

for measurable subsets $E \subseteq D$.

Zen Harper [11, Thm. 3.3] exploited the theory of admissibility for semigroups to show (as part of a family of more general results) that a more explicit necessary and sufficient condition for boundedness of $W_{h,\varphi}$ is the following:

$$
\sup_{|w|<1} \left\| \frac{(1-|w|^2)^{1/2}h}{1-\overline{w}\varphi} \right\|_2 < \infty.
$$

Moreover, $W_{h,\varphi}$ is compact if and only if

$$
\left\| \frac{(1-|w|^2)^{1/2}h}{1-\overline{w}\varphi} \right\|_2 \to 0 \quad \text{as} \quad |w| \to 1.
$$

Some independent generalizations to mappings between $H^p$ and $H^q$ spaces were given by Cučković and Zhao [6].

In fact, (2) follows directly from (1) on using the standard reproducing kernel test for Carleson embeddings (see [9, p. 231] or [21, p. 105]), namely that an embedding $J : H^2 \to L^2(\mu)$ is bounded (i.e., $\mu$ is a Carleson measure on $D$) if and only if $J$ is uniformly bounded on the set of normalized reproducing kernels $\{\tilde{k}_w : w \in D\}$ on $H^2$, where

$$
\tilde{k}_w(z) = \frac{(1-|w|^2)^{1/2}}{1-\overline{w}z}, \quad \text{for} \quad z \in D.
$$

For

$$
\int_D |\tilde{k}_w|^2 d\mu_{h,\varphi} = \int_T |h|^2 |\tilde{k}_w \circ \varphi|^2 dm,
$$

from which (2) follows easily. Similar arguments involving vanishing Carleson measures establish (3).

On the other hand, Jury [15] has recently obtained the estimate $\|W_{h,\varphi}\| \leq \|h\|_{H(\varphi)}$, but only for $h \in H(\varphi)$, where $H(\varphi)$ is the de Branges–Rovnyak space, that is, the reproducing kernel Hilbert space on $D$ with kernel

$$
k^\varphi_w(z) = \frac{1-\varphi(w)\varphi(z)}{1-\overline{w}z}.
$$
Proof. If \( \varphi \) is inner, then \( H(\varphi) = K_{\varphi} := \mathcal{H}^2 \ominus \varphi \mathcal{H}^2 \), and \( P_{K_{\varphi}}h = \varphi P_{\mathcal{P}_h}h \), where \( P_{\mathcal{P}_h} \) is the orthogonal projection onto \( L^2 \ominus \mathcal{H}^2 \). In this case, it is easy to check that, if \( h \in K_{\varphi} \),

\[
\left\| \sum_{n=0}^{\infty} a_n \varphi^n h \right\|_2^2 = \|h\|_2^2 \sum_{n=0}^{\infty} |a_n|^2,
\]

since

\[
\langle \varphi^n h, \varphi^m h \rangle = \begin{cases} 
0 & \text{for } n \neq m, \\
\|h\|_2^2 & \text{for } n = m,
\end{cases}
\]

and so \( \|W_{h,\varphi}f\|_2 = \|h\|_2 \|f\|_2 \). Thus the condition is correct for \( h \in K_{\varphi} \), although not in general. Indeed, if \( \varphi(z) = z \), then \( \|W_{h,\varphi}\| = \|T_h\| = \|h\|_\infty \).

Suppose that \( \varphi \) is inner, and let \( \alpha \) be an automorphism of the disc such that \( \alpha(\varphi(0)) = 0 \). Then \( \alpha \circ \varphi \) is also inner, and \( W_{h,\varphi}C_\alpha = W_{h,\alpha \circ \varphi} \). For questions of boundedness and compactness, we may therefore assume without loss of generality that \( \varphi(0) = 0 \), and hence \( \{1, \varphi, \varphi^2, \ldots\} \) is an orthonormal set and \( C_\varphi \) is an isometry.

A further observation is that \( W_{h,\varphi} \) is bounded as soon as \( \varphi \) is inner (w.l.o.g. \( \varphi(0) = 0 \) again) and \( h \in \mathcal{H}^2 \) satisfies \( \langle h, h \varphi^n \rangle = 0 \) for all \( n \geq 1 \) (which is a weaker condition than \( h \in K_{\varphi} \)); for

\[
\left\| \frac{h}{1 - \overline{\varphi} w} \right\|_2^2 = \left\| \sum_{k=0}^{\infty} h \overline{\varphi^k} w^k \right\|_2^2 = \|h\|_2^2 \sum_{k=0}^{\infty} |w|^{2k},
\]

since \( \langle h \varphi^m, h \varphi^n \rangle = 0 \) for \( m \neq n \). Then by (2) it follows that \( W_{h,\varphi} \) is bounded.

In order to make a more systematic analysis of the weights that induce a bounded weighted composition operator, we make the following definition.

**Definition 2.1.** For \( \varphi : \mathbb{D} \to \mathbb{D} \) analytic, the multiplier space of \( \varphi \) is defined by

\[ \mathcal{M}_{\varphi} = \{ h \in \mathcal{H}^2 : W_{h,\varphi} := T_h C_\varphi \text{ is bounded} \}. \]

Evidently, \( \mathcal{H}^\infty \subseteq \mathcal{M}_{\varphi} \subseteq \mathcal{H}^2 \) for all analytic self-maps \( \varphi \) of the unit disc. It is easily verified that \( \mathcal{M}_{\varphi} \) is a Banach space with the norm \( \|h\|_{\mathcal{M}_{\varphi}} = \|W_{h,\varphi}\| \).

Contreras and Hernández-Díaz [4] and Matache [19] showed that \( \mathcal{M}_{\varphi} = \mathcal{H}^\infty \) if and only if \( \varphi \) is a finite Blaschke product. In Corollary [24] we shall present a stronger result.

The other extreme situation is the following.

**Theorem 2.2.** \( \mathcal{M}_{\varphi} = \mathcal{H}^2 \) if and only if \( \|\varphi\|_\infty < 1 \).

**Proof.** If \( \|\varphi\|_\infty < 1 \), and \( \|f\|_2 = 1 \), then

\[
\|C_\varphi f\|_\infty = \left\| \sum_{n=0}^{\infty} \hat{f}(n) \varphi^n \right\|_\infty \leq \left( \sum_{n=0}^{\infty} |\hat{f}(n)|^2 \right)^{1/2} \left( \sum_{n=0}^{\infty} \|\varphi\|_{\infty}^{2n} \right)^{1/2},
\]

where \( \hat{f}(n) \) is the Fourier coefficient of \( f \).
and we conclude easily that $C\phi$ maps $H^2$ boundedly into $H^\infty$, whence $M\phi = H^2$.

Conversely, if $M\phi = H^2$, then clearly $C\phi$ maps $H^2$ into $H^\infty$, and the mapping is bounded by the closed graph theorem. In particular the functions

$$\frac{(1 - |a|^2)^{1/2}}{1 - \overline{a}\phi(z)}, \quad (z \in \mathbb{D}),$$

are uniformly bounded for $a \in \mathbb{D}$. However, if $|\phi(z_0)| = 1 - \delta$ for some $z_0 \in \mathbb{D}$ and $0 < \delta < 1$, then for a suitable choice of $a$ of modulus $1 - \delta$ we obtain

$$\left\| \frac{(1 - |a|^2)^{1/2}}{1 - \overline{a}\phi} \right\|_\infty \geq \frac{(1 - (1 - \delta)^2)^{1/2}}{1 - (1 - \delta)^2} = (2\delta - \delta^2)^{-1/2},$$

which tends to $\infty$ as $\delta \to 0$. Hence if $M\phi = H^2$ we must have $\|\phi\|_\infty < 1$. □

There is a close relation between the angular derivative of $\phi$ and the boundedness of $W_{h,\phi}$, as we explain in the next section.

2.2. Relation with the angular derivative. Once more, let $k_w$ be the reproducing kernel at $w$ in $H^2$, that is,

$$k_w(z) = \frac{1}{1 - \overline{w}z}, \quad (z \in \mathbb{D}).$$

If $W_{h,\phi}^*$ denotes the adjoint of $W_{h,\phi}$, it is not difficult to see that

$$W_{h,\phi}^* k_w = \overline{h(w)} k_{\overline{\phi}(w)}, \quad (w \in \mathbb{D}).$$

Hence, if $W_{h,\phi}$ is bounded, it follows that

\begin{equation}
|h(w)| \leq \|W_{h,\phi}\| \left( \frac{1 - |\phi(w)|^2}{1 - |w|^2} \right)^{1/2}
\end{equation}

for any $w \in \mathbb{D}$. Note that from equation (4) one gets another proof of the fact that if $\phi$ is a finite Blaschke product then $M\phi = H^\infty$; for then we have

$$\sup_{w \in \mathbb{D}} \frac{1 - |\phi(w)|^2}{1 - |w|^2} < \infty,$$

(as noted in [19]), and so $h \in H^\infty$. The next result generalizes this observation giving a necessary condition for boundedness and compactness of weighted composition operators.

**Proposition 2.3.** Let $\phi$ be an analytic self-map of $\mathbb{D}$. Let $E_\phi = \{ \zeta \in \mathbb{T} : \phi \text{ has finite angular derivative at } \zeta \}$. Then:

i) If $W_{h,\phi}$ is bounded on $H^p$ for some $1 \leq p < \infty$, then $h$ is pointwise bounded on every Stolz domain whose vertex is a point of $E_\phi$.

ii) If $W_{h,\phi}$ is compact on $H^p$ for some $1 \leq p < \infty$, then $h$ tends to zero as $|z| \to 1$ in any Stolz domain whose vertex is a point of $E_\phi$.
Proof. Note that if \( \varphi \) has a finite angular derivative at \( \zeta \in \mathbb{T} \), then by the Julia–Carathéodory Theorem, the nontangential limit satisfies
\[
\angle \lim_{w \to \zeta} \frac{1 - |\varphi(w)|}{1 - |w|} < \infty
\]
(see [3] pp. 51] or [20]). Indeed, it holds that
\[
|\varphi'(\zeta)| = \lim_{r \to 1} \frac{|\varphi'(r\zeta)|}{1 - |w|} = \angle \lim_{w \to \zeta} \frac{1 - |\varphi(w)|}{1 - |w|} = \liminf_{w \to \zeta} \frac{1 - |\varphi(w)|}{1 - |w|}.
\]
This along with (4) and a standard argument on reproducing kernels yield the statement of the proposition in \( \mathcal{H}^2 \). The result in \( \mathcal{H}^p \) is analogous.

At this point, one may ask whether Proposition 2.3 is also sufficient for boundedness of weighted composition operators. The next example shows that this is no longer true.

Example 2.4. Let \( \varphi \) be the map \( \varphi(z) = 1 - \sqrt{1 - z} \). Note that \( \varphi \) sends \( D \) onto a subdomain of \( D \) shaped roughly like a (two dimensional) ice cream cone with vertex at 1. It is easy to see that this map has finite angular derivative nowhere on the unit circle. So for any \( h \in \mathcal{H}^2 \) the boundedness requirement on \( h \) given by Proposition 2.3(i) holds trivially.

Let us consider the \( \mathcal{H}^2 \) function \( h(z) = (1 - z)^{-3/8} \) and the weighted composition operator operator \( W_{h,\varphi} \). Observe that if \( f(z) = (1 - z)^{-1/4} \), then
\[
(W_{h,\varphi}f)(z) = (1 - z)^{-1/2},
\]
so \( W_{h,\varphi} \) does not take \( \mathcal{H}^2 \) into itself. Thus \( W_{h,\varphi} \) satisfies the conclusion of Proposition 2.3(i), but not the hypothesis.

Proposition 2.5. (See, for example, [2].) Let \( \varphi \) be an inner function and \( \zeta_0 \in \mathbb{T} \).

Then the following assertions are equivalent:
(i) \( \varphi \) has an angular derivative in the sense of Carathéodory at \( \zeta_0 \).
(ii) \( \zeta_0 \in E_\varphi \).

If \( \varphi \) is an inner function, there is a well known characterization of the set where \( \varphi \) has finite angular derivative, essentially due to Carathéodory. Let
\[
\varphi(z) = e^{i\alpha} z^N \prod_{n \geq 1} \frac{|a_n|}{1 - \sum_{n \geq 1} a_n z^n} \exp \left( - \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta) \right),
\]
where \( a \in \mathbb{R} \) and \( N \in \mathbb{N} \cup \{0\} \), be its canonical representation in terms of the zeroes \( \{a_n\}_n \) and a singular measure \( \mu \). We define the Ahern–Clark set \( E_\varphi \) [1] by:
\[
E_\varphi := \left\{ \zeta \in \mathbb{T} : \sum_{n \geq 1} \frac{|a_n|^2}{|\zeta - a_n|^2} + 2 \int_{\mathbb{T}} \frac{d\mu(t)}{|t - \zeta|^2} < +\infty \right\}.
\]

Proposition 2.5. (See, for example, [2].) Let \( \varphi \) be an inner function and \( \zeta_0 \in \mathbb{T} \).

Then the following assertions are equivalent:
(i) \( \varphi \) has an angular derivative in the sense of Carathéodory at \( \zeta_0 \).
(ii) \( \zeta_0 \in E_\varphi \).
It is known that \( E_\varphi \supseteq T \setminus \sigma(\varphi) \), where \( \sigma(\varphi) \) denotes the spectrum of \( \varphi \), namely, \( \sigma(\varphi) = \text{clos}\{a_n\}_n \cup \text{supp} \mu \). In general we do not have equality.

The following result may be seen as a generalization of the result that \( \mathcal{M}_\varphi = \mathcal{H}^\infty \) whenever \( \varphi \) is a finite Blaschke product (in which case \( E_\varphi = T \) and \( \sigma(\varphi) = \emptyset \)).

**Corollary 2.6.** Suppose that \( \varphi \) is an inner function whose zero set, if infinite, forms a finite union of subsequences tending nontangentially to points of \( T \), and whose singular part is given by a measure of finite support. Then any function \( h \in \mathcal{M}_\varphi \) is essentially bounded on all relatively compact subsets of \( E_\varphi = T \setminus \sigma(\varphi) \).

**Proof.** The fact that \( E_\varphi = T \setminus \sigma(\varphi) \) under the given hypotheses is shown in [2, Thm. 5.3]. On sufficiently small neighbourhoods of any relatively compact subset \( K \) of \( T \setminus \sigma(\varphi) \), one may extend \( \varphi \) analytically across \( T \); we have \( |\varphi(\zeta)| = 1 \) and \( |\varphi'(\zeta)| \) is uniformly bounded for \( \zeta \in K \). It is now easily verified that \( (1 - |\varphi(w)|^2)/(1 - |w|^2) \) is uniformly bounded as \( w \) approaches \( K \). The result follows from (4).

**Example 2.7.** The following example shows that the condition of Corollary 2.6 is not sufficient for membership of \( \mathcal{M}_\varphi \), even for \( h \in \mathcal{H}^2 \). Let

\[
\varphi(z) = \exp\left( -\frac{1+z}{1-z} \right), \quad (z \in \mathbb{D}),
\]

which is a singular inner function with \( \sigma(\varphi) = \{1\} \). For \( w = re^{i\alpha} \in \mathbb{D} \), it is straightforward to verify that

\[
\frac{1 - |\varphi(w)|^2}{1 - |w|^2} = \frac{1}{1 - r^2} \left( 1 - \exp\left( -2 \frac{1 - r^2}{|1 - re^{i\alpha}|^2} \right) \right),
\]

which, as \( r \to 1 \), tends to \( 2/|\alpha|^2 \). Hence by [4], for each \( h \in \mathcal{M}_\varphi \) we have that \( |h(e^{i\alpha})| = O(|\alpha|^{-1}) \) as \( \alpha \to 0 \). It is easy to construct examples of functions \( h \in \mathcal{H}^2 \), bounded on any closed subarc of \( T \) disjoint from \( \{1\} \), that nonetheless do not satisfy this condition, and hence induce unbounded operators \( W_{h,\varphi} \).

### 2.3. Schatten class operators

In this section we provide stronger versions of the two main theorems of [10]. To begin with, in [10 Thm. 2], it was shown that if \( \|\varphi\|_\infty < 1 \), then for every \( h \in \mathcal{H}^2 \) the weighted composition operator \( T = W_{h,\varphi} \) is compact. Indeed, much more is true, as the next result shows.

**Theorem 2.8.** Suppose that \( \|\varphi\|_\infty < 1 \). Then for every \( h \in \mathcal{H}^2 \) the weighted composition operator \( T = W_{h,\varphi} \) is trace-class (i.e., lies in \( S_1 \)).

**Proof.** Note that for \( w \in \mathbb{D} \)

\[
\|T_{k_w}h\|_2 \approx (1 - |w|^2)^{3/2}\left\|\frac{h\varphi}{(1 - \overline{w}\varphi)^2}\right\|_2.
\]
and since $|1 - w\varphi| \geq 1 - \|\varphi\|_{\infty}$, this is bounded by a constant multiple of $(1 - |w|^2)^{3/2}\|h\|_2$. Now a result of Harper and Smith [12, Thm. 3.1] asserts that for $1 \leq p < 2$, a sufficient condition for a weighted composition operator $T$ to lie in $S_p(\mathcal{H}^2)$ is

$$\int_{\mathbb{D}} \|Tk_w\|^p \frac{dA(w)}{(1 - |w|^2)^2} < \infty,$$

(6)

where $\tilde{k}_w = k_w/\|k_w\|$ and $\tilde{k}_w(z) = \frac{z}{(1 - \overline{w}z)^2}$, so that $\|\tilde{k}_w\| \approx (1 - |w|^2)^{-3/2}$. We see now that the integral in (6) converges, even for $p = 1$, and the result follows. $\blacksquare$

The other main result of [10], its Theorem 1, asserts that if $W_{h,\varphi}$ is bounded on $\mathcal{H}^2$, if $\varphi$ lies in the disc algebra, and if $h$ is continuous at all such points $\zeta$, then $W_{h,\varphi}$ is compact.

Here again, we may give a significantly stronger result.

**Theorem 2.9.** Suppose that $\varphi : \mathbb{D} \to \mathbb{D}$ is holomorphic and that for some $\delta > 0$ and $c_\delta > 0$ the function $h \in \mathcal{H}^2$ satisfies $|h(z)| \leq c_\delta$ a.e. on the set

$$A_\delta := \{z \in \mathbb{T} : |\varphi(z)| \geq 1 - \delta\}.$$

Then $W_{h,\varphi}$ is a bounded operator. If, in addition, $c_\delta \to 0$ as $\delta \to 0$, then $W_{h,\varphi}$ is a compact operator.

**Proof.** We estimate the $\mathcal{H}^2$ norm of $(1 - |w|^2)^{1/2}h/(1 - \overline{w}\varphi)$ for $w \in \mathbb{D}$. Now

$$\int_{A_\delta} \frac{(1 - |w|^2)^2|h|^2}{|1 - \overline{w}\varphi|^2} \leq c_\delta^2(1 - |w|^2)\|C_{\varphi}k_w\|^2 \leq c_\delta^2\|C_{\varphi}\|^2,$$

whereas

$$\int_{\mathbb{T} \setminus A_\delta} \frac{(1 - |w|^2)^2|h|^2}{|1 - \overline{w}\varphi|^2} \leq \frac{(1 - |w|^2)^2\|h\|_2^2}{1 - |w|(1 - \delta)^2} \leq \frac{\|h\|_2^2(1 - |w|^2)}{\delta^2}.$$

Thus

$$\frac{(1 - |w|^2)^{1/2}|h|}{1 - \overline{w}\varphi} \|2 \leq c_\delta^2\|C_{\varphi}\|^2 + \|h\|_2^2(1 - |w|^2)/\delta^2.$$

This is uniformly bounded, independently of $w$, and so $W_{h,\varphi}$ is a bounded operator; moreover, if $c_\delta \to 0$ as $\delta \to 0$, then given $\varepsilon > 0$, we may choose $\delta > 0$ sufficiently small such that $2c_\delta^2\|C_{\varphi}\|^2 < \varepsilon$, and then find $\eta > 0$ such that, if $|w| > 1 - \eta$, one has $\|h\|_2^2(1 - |w|^2)/\delta^2 < c_\delta^2\|C_{\varphi}\|^2$. Hence condition (3) is satisfied, and $W_{h,\varphi}$ is compact. $\blacksquare$
3. Weighted Bergman spaces

For $\alpha > -1$ the weighted Bergman space $\mathcal{A}_\alpha^2 = \mathcal{A}_\alpha^2(D)$ consists of all analytic functions in $D$ for which the norm, given by

$$\|f\|_{\mathcal{A}_\alpha^2} = \left(\frac{1}{\pi} \int_D |f(z)|^2 (1 - |z|^2)^\alpha \, dA(z)\right)^{1/2}$$

is finite. It is a reproducing kernel Hilbert space with kernel function given by

$$h_w(z) = \frac{\alpha + 1}{(1 - wz)^{\alpha+2}}$$

(cf. [5, p. 27] and [25, p. 135]). We shall also require the normalized functions $	ilde{h}_w = h_w/\|h_w\|_{\mathcal{A}_\alpha^2}$, given by

$$\tilde{h}_w(z) = \frac{(1 - |w|^2)^{(\alpha+2)/2}}{(1 - wz)^{\alpha+2}}.$$

A necessary and sufficient condition for boundedness of $W_{h,\varphi}$ on the Bergman space $\mathcal{A}^2(D) = \mathcal{A}_0^2(D)$ is given in [16, Thm. 3.1], namely that the measure $\nu_{h,\varphi}$ defined by

$$\nu_{h,\varphi}(E) = \int_{\varphi^{-1}(E)} |h(z)|^2 \, dA(z),$$

should be a Hastings–Carleson measure, i.e., the embedding $J : \mathcal{A}^2(D) \rightarrow L^2(D, d\nu_{h,\varphi})$ should be bounded. Such measures were characterised in [13, 18]. These can also be tested on normalized reproducing kernels $\tilde{h}_w$, for $w \in D$ as in [25, Thm. 7.5].

Zen Harper [11, Thm. 3.3] shows (as part of a more general theory, which we describe in Theorem below) that $W_{h,\varphi}$ is bounded on $\mathcal{A}^2(D)$ if and only if

$$\sup_{|w| < 1} \left\| \frac{(1 - |w|^2) h}{(1 - wz)^{\alpha+2}} \right\|_{\mathcal{A}_\alpha^2} < \infty. \tag{7}$$

This result generalizes to $\mathcal{A}_\alpha^2(D)$ as follows.

**Proposition 3.1.** Let $\varphi$ be a holomorphic self-map of the disc and $h \in \mathcal{A}_\alpha^2(D)$. Then the following conditions are equivalent:

(i) The weighted composition operator $W_{h,\varphi}$ is bounded on $\mathcal{A}_\alpha^2(D)$.

(ii) The measure $\nu_{h,\varphi}$ defined by

$$\nu_{h,\varphi}(E) = \int_{\varphi^{-1}(E)} |h(z)|^2 (1 - |z|^2)^\alpha \, dA(z)$$

is an $\mathcal{A}_\alpha^2$-Carleson measure, in the sense that the canonical embedding $J : \mathcal{A}_\alpha^2(D) \rightarrow L^2(D, d\nu_{h,\varphi})$ is bounded.

(iii) One has

$$\sup_{|w| < 1} \left\| \frac{(1 - |w|^2)^{1+\alpha/2} h}{(1 - wz)^{\alpha+2}} \right\|_{\mathcal{A}_\alpha^2} < \infty. \tag{8}$$
Moreover, $W_{h,\varphi}$ is compact if and only if $\nu_{h,\varphi}$ is a vanishing Carleson measure, or, equivalently, if
\begin{equation}
\left\| \frac{(1-|w|^2)^{1+\alpha/2}}{(1-w\varphi)^{\alpha+2}} \right\|_{A_2^\alpha} \to 0 \quad \text{as} \quad |w| \to 1.
\end{equation}

**Proof.** The equivalence of (i) and (ii) follows from the formula
\begin{equation}
\int_{\mathbb{D}} |h(z)|^2 |f(\varphi(z))|^2 (1-|z|^2)^\alpha \, dA(z) = \int_{\mathbb{D}} |f(z)|^2 \, d\nu_{h,\varphi}(z),
\end{equation}
adapted from the calculation for $A_2^\alpha(\mathbb{D})$, given in [16], without significant alteration. Next, observing that
\begin{equation}
\int_{\mathbb{D}} |\tilde{h}_w|^2 \, d\mu_{h,\varphi} = \int_{\mathbb{D}} |h(z)|^2 |\tilde{h}_w \circ \varphi(z)|^2 (1-|z|^2)^\alpha \, dA(z),
\end{equation}
and using the fact that $A_2^\alpha$-Carleson measures can be tested on normalized reproducing kernels (which may be found in [25, Sec. 7.2]), we see that (iii) implies (ii). Trivially, (i) implies (iii), this being the property that the weighted composition operator is uniformly bounded on the normalized kernels.

The proof of the characterization of compactness is similar, and is omitted. \qed

To conclude, we mention briefly how one may also derive results that provide characterizations of Schatten class weighted composition operators on $A_2^\alpha(\mathbb{D})$. The key is the following result of Harper [11, Thm. 3.1, Cor. 3.2].

**Theorem 3.2.** Let $T$ be a subnormal operator on a Hilbert space $\mathcal{H}$, with spectral radius $r(T) \leq 1$. Let $h \in \mathcal{H}$. Then there exists a finite positive Borel measure $\mu$ on $\mathbb{D}$ such that
\begin{equation}
\|f(T)h\|^2 = \int_{\mathbb{D}} |f(z)|^2 \, d\mu(z)
\end{equation}
for all functions $f$ analytic in a neighbourhood of $\mathbb{D}$.

The application to weighted composition operators is the following. Let $T = T_{\varphi}$ denote the operator of multiplication by $\varphi$ on $A_2^\alpha(\mathbb{D})$, which is subnormal, since $A_2^\alpha(\mathbb{D})$ is a closed subspace of $L^2(\mathbb{D}, (1-|z|^2)^\alpha \, dA(z))$. Then $f(T)h = h(f \circ \varphi) = W_{h,\varphi}f$. We thus have the following corollary.

**Corollary 3.3.** For $\varphi : \mathbb{D} \to \mathbb{D}$ holomorphic and $h \in A_2^\alpha(\mathbb{D})$, there exists a finite positive Borel measure $\mu$ on $\mathbb{D}$ such that
\begin{equation}
\|W_{h,\varphi}f\|_{A_2^\alpha}^2 = \int_{\mathbb{D}} |f(z)|^2 \, d\mu(z)
\end{equation}
for all functions $f$ analytic in a neighbourhood of $\mathbb{D}$. Thus $\|W_{h,\varphi}f\| = \|I_\mu f\|$, where $I_\mu : A_2^\alpha(\mathbb{D}) \to L^2(\mathbb{D}, \mu)$ is the canonical Carleson embedding. It follows that $W_{h,\varphi}$ is respectively bounded, compact or of Schatten class $S_p$ if and only if $I_\mu$ has the same property.
We shall omit the detailed calculations, which are entirely analogous to those of Prop. 2.4, but it may now be verified that the Schatten class membership of $I_\mu$, and hence $W_{h,\varphi}$, can be tested on reproducing kernels. Namely $I_\mu \in S_p(A^2_\alpha(D), L^2(\mu))$ for $1 < p < \infty$ if and only if $\int_D \|I_\mu \tilde{h}_w\|^p \frac{dA(w)}{(1-|w|^2)^2} < \infty,$ and $I_\mu \in S_1(A^2_\alpha(D), L^2(\mu))$ for $1 < p < \infty$ if and only if $\int_D \|I_\mu \tilde{h}_w\|^p \frac{dA(w)}{(1-|w|^2)^2} < \infty.$

where we write $\tilde{h}_w = \partial_w h_w$ and $\tilde{h}_w = \tilde{h}_w/\|\tilde{h}_w\|$. To within irrelevant constants, we have $\tilde{h}_w(z) \approx \frac{z}{(1-\overline{w}z)^{\alpha+3}}$ and $\tilde{h}_w(z) \approx \frac{(1-|w|^2)^{(\alpha+4)/2}z}{(1-\overline{w}z)^{\alpha+3}}$.

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