CNM Models, Holomorphic Functions and Projective Superspace $C$-Maps

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ABSTRACT

Continuing the investigation of CNM (chiral-nonminimal) hypermultiplet nonlinear $\sigma$-models, we propose extensions of the concept of the $c$-map which relate holomorphic functions to hyper-Kähler geometries. In particular, we show that a whole series of hyper-Kähler potentials can be derived by replacing the role of the 4D, $N=1$ tensor multiplet in the original $c$-map by 4D, $N=1$ non-minimal multiplets and auxiliary superfields. The resulting $N=2$ models appear to have interesting connections to Calabi-Yau manifolds and algebraic varieties. These models also emphasize the fact that special hyper-Kähler manifolds (the analogs of special Kähler manifolds) without isometries exist.

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1 Introduction

The study of supersymmetrical nonlinear \( \sigma \)-models has proven repeatedly to provide a way in which to derive new results in complex manifold theories with potentials. One such example of this is provided by the “c-map,” known for approximately a decade due to a work of Cecotti, Ferrara and Girardello \cite{1}. According to the Glossary in \cite{2} the c-map is “a method for constructing the hypermultiplet moduli space of a type-II string theory compactified on a Calabi-Yau three-fold from the vector multiplet moduli space of the other type-II theory on the same three-fold”. In \textit{rigid supersymmetry}, with which we only concern ourselves in this paper, the c-map sends a special Kähler manifold of the rigid type \( \mathcal{M}_d \) to a hyper-Kähler manifold \( \mathcal{H}_{2d} \) (with both subscripts denoting the complex dimensions) \cite{1}. One feature that arose due to the method of construction utilized is that the resultant hyper-Kähler metrics always possess a set of isometries. So clearly this construction leads to a restricted class of hyper-Kähler geometries. This naturally raises the question of whether there exist generalizations of the c-map that lead to a wider class of hyper-Kähler geometries.

Recently, we began an effort to investigate the structure of 4D, \( N = 2 \) nonlinear \( \sigma \)-models which have the property that the \( N = 2 \) supersymmetry is manifest \cite{3} by use of projective superspace. Projective superspace techniques made their first appearance in the literature via the work in \cite{4} but received their most complete development in the later works of \cite{5, 6}. One implication of our completed investigation is that essentially all 4D, \( N = 1 \) supersymmetric nonlinear \( \sigma \)-models possess 4D, \( N = 2 \) supersymmetric extensions. It is thus natural that we turn our attention to the question of whether the projective approach can be used to gain more insight into the issue of generalizations to the c-map. In this work we will present results from this effort. We begin by looking at the classical c-map \cite{1} and presenting a streamlined proof as compared to the original one. Here, using projective superspace allows the discovery of a simple reason why the original c-map necessarily relates a holomorphic function to a hyper-Kähler metric. We next observe that the original c-map is related to a particular projective multiplet representation, “the projective O(2)-multiplet.” This is followed with the observation that the role of the projective O(2)-multiplet is not unique and with a simple generalization, projective O(2n)-multiplets and finally the projective polar multiplet may be used instead.

The structure of our proposed generalizations to the c-map suggest interesting con-

\footnote{There is a mild restriction that in order to possess an \( N = 2 \) extension, the Kähler potential defining the \( N = 1 \) nonlinear \( \sigma \)-model must be an analytic function.}
nections to issues in algebraic geometry. In particular the models which emerge from this approach have the structure of being theories that are defined on hypersurfaces in Kähler spaces. These surfaces are defined by holomorphic constraints. An interesting case for the polar multiplets occurs when these holomorphic constraints are polynomial in nature.

2 The Aboriginal C-Map

The work of [1] started with a holomorphic prepotential $F(\Phi)$ defining the Kähler potential $K(\Phi, \bar{\Phi})$ for a special Kähler geometry with metric $g_{I \bar{J}}(\Phi, \bar{\Phi})$

$$K(\Phi^I, \bar{\Phi}^J) = \Phi^I F_I(\Phi) + \bar{\Phi}^I \bar{F}_I(\bar{\Phi}) \, , \, g_{I \bar{J}}(\Phi, \bar{\Phi}) = F_{I J}(\Phi) + \bar{F}_{I J}(\bar{\Phi}) \, . \quad (2.1)$$

Next, the following extension of the Kähler potential was introduced

$$H(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) = K(\Phi, \bar{\Phi}) + \frac{1}{2} g^{I \bar{J}}(\Phi, \bar{\Phi})(\Psi_I + \bar{\Psi}_I)(\Psi_J + \bar{\Psi}_J) \, . \quad (2.2)$$

In (2.1) and (2.2) the quantities $\Phi^I$, as usual, are chiral superfields that have the geometrical interpretation of being the coordinates of the complex manifold whose metric in given by $g_{I \bar{J}}$. In (2.2) the quantities $\Psi_I$ are also chiral superfields. However, their geometrical interpretation is very different. Since these appear in the expression multiplying the inverse Kähler metric, it follows that $\Psi_I$ must be co-vectors associated with the Kähler manifold. After considerable effort it was shown that $H(\Phi, \bar{\Phi}, \Psi, \bar{\Psi})$ defines a hyper-Kähler geometry. At this stage, we see that the $c$-map also has the interpretation of providing a mechanism for defining hyper-Kähler geometries in terms of a purely holomorphic function $F(\Phi)$. According to a well-known theorem regarding supersymmetric non-linear $\sigma$-models, all such 4D, $N=2$ (or 2D, $N=4$) supersymmetric models describe hyper-Kähler geometries. An apparent feature of (2.2) is that it is necessarily invariant under the change of variables $\Psi_I \rightarrow \Psi_I + i a_I$, if $a_I$ are a set of real quantities. This is equivalent to the appearance of a set of isometries of the hyper-Kähler metric. So it is an intrinsic feature of the $c$-map (as described above) to entail a procedure in which some hyper-Kähler manifolds that necessarily contain isometries are constructed from a single holomorphic function.

The proof given in [1] that the potential (2.2) describes a hyper-Kähler geometry is quite involved. Below we will give an alternate and much more transparent proof of this, taking advantage of the fact that any nonlinear 4D $\sigma$-model that realizes $N=2$

\footnote{Here we use this term to refer to its more recent definition, not its original one.}
supersymmetry necessarily describes a hyper-Kähler geometry. A set of $d\ N = 2$ tensor multiplets is described in $N = 1$ superspace by $d$ chiral superfields $\Phi^I$ and $d$ real linear superfields $G^I$.

A way to make manifest $N = 2$ supersymmetry is to use the projective superspace techniques [5] which, in turn, naturally emerge from the fundamental concept of harmonic superspace [8]. In this approach to $N = 2$ supersymmetric theories, a whole sequence of scalar multiplets, known as $O(2n)$ projective multiplets [5, 6] has been found. The case of the $O(2)$ projective multiplet is directly relevant to the $N = 2$ tensor multiplet. The quantity defined by

$$\Xi^I(w) = \Phi^I + wG^I - w^2\Phi^I, \quad I = 1, \ldots, d, \quad (2.3)$$

is an $O(2)$ projective multiplet and is allowed to enter a projective $N = 2$ supersymmetric action of the form

$$S = \frac{1}{2\pi i} \int_C \frac{dw}{w} \int d^8z \, \mathcal{L}(\Xi^I(w), w), \quad (2.4)$$

where the Lagrangian may be specified to the form

$$S = -\frac{1}{2\pi i} \int_C \frac{dw}{w} \int d^8z \, \frac{F(\Xi^I(w))}{w^2} + \text{h.c.}, \quad (2.5)$$

in terms of a single holomorphic function $F$. A simple calculation gives

$$F(\Xi^I(w)) = F\left(\Phi^I + wG^I - w^2\Phi^I\right) = F(\Phi) + wF_I(\Phi)G^I - w^2\left(F_I(\Phi)\Phi^I - \frac{1}{2}F_{IJ}(\Phi) G^IG^J\right) + O(w^3).$$

Therefore, the action is equivalent to

$$S[\Phi, \Phi, G] = \int d^8z \left\{ K(\Phi, \Phi) - \frac{1}{2}g_{IJ}(\Phi, \Phi) G^IG^J \right\}. \quad (2.6)$$

The real linear superfields $G^I$ can be dualized (using an $N = 1$ superfield duality transformation) into pairs of (anti)chiral ones, $\Psi_I$ and $\bar{\Psi}_I$. As a result, the action turns into

$$S[\Phi, \Phi, \Psi, \bar{\Psi}] = \int d^8z \, H(\Phi, \Phi, \Psi, \bar{\Psi}), \quad (2.7)$$

with $H(\Phi, \Phi, \Psi, \bar{\Psi})$ given precisely as in (2.2). A special feature of the potential (2.2) is that the (anti) chiral superfields $\Psi$ and $\bar{\Psi}$ cannot be converted into complex (anti)linear ones $\Gamma$ and $\bar{\Gamma}$!

We thus see in Eqs. (2.3–2.7) a simple and straightforward proof of the existence of the $c$-map with the consequence that the potential (2.2) describes a hyper-Kähler geometry.


3 Construction of New $O(2n)$ $C$-Maps

As noted below Eq. (2.7), the hyper-Kähler geometry there described cannot be obtained from our previous discussion of CNM-hypermultiplet models [3]. The technical reason for this is because of the impossibility to perform an $N=1$ superfield duality transformation from chiral to complex linear superfields whenever the chiral superfields appear in an action solely via the linear combination $\Psi_I + \bar{\Psi}_I$ and CNM models necessarily involve complex linear multiplets. This leaves two options:

- It is impossible to eliminate all the auxiliary superfields in the CNM-hypermultiplet models appropriate for the case at hand;
- If it is possible to eliminate all the auxiliary superfields in the CNM-hypermultiplet models for Kähler potential of the form (2.1), we apparently arrive at a different hyper-Kähler manifold as compared to that defined by (2.2). This implies the $c$-map is not unique, at least in the case of rigid SUSY.

Let us repeat the above considerations for the case of $O(4)$ projective superspace multiplet (instead of the tensor multiplet). The basic superfields read

$$\Sigma^I(w) = \Phi^I + w\Gamma^I + w^2 V^I - w^3 \tilde{\Gamma}^I + w^4 \tilde{\Phi}^I , \quad \tilde{V}^I = V^I ,$$

(3.1)

where $\Phi^I$ are chiral, $\Gamma^I$ complex linear, $V^I$ real unconstrained superfields. Instead of the action (2.3), we now have

$$S_{O(4)} = \int \frac{d\tau}{2\pi} \int C \frac{d^8z}{w^4} F(\Sigma^I(w)) + \text{h.c.} ,$$

(3.2)

or, after performing the contour integral,

$$S[\Phi, \bar{\Phi}, \Gamma, \bar{\Gamma}, V] = \int d^8z \left\{ K(\Phi, \bar{\Phi}) - g_{IJK}(\Phi, \bar{\Phi}) \Gamma^I \Gamma^J + \frac{1}{2} g_{IJK}(\Phi, \bar{\Phi}) V^I V^J 
+ \frac{1}{2} \left[ F_{IJK}(\Phi) \Gamma^J \Gamma^K + \bar{F}_{IJK}(\bar{\Phi}) \bar{\Gamma}^J \bar{\Gamma}^K \right] V^I 
+ \frac{1}{4!} F_{IJKL} \Gamma^I \Gamma^J \Gamma^K \Gamma^L 
+ \frac{1}{4!} \bar{F}_{IJKL} \bar{\Gamma}^I \bar{\Gamma}^J \bar{\Gamma}^K \bar{\Gamma}^L \right\} .$$

(3.3)

The auxiliary superfields $V^I$ can be easily eliminated via

$$V^L = -\frac{1}{2} g^{IL}(\Phi, \bar{\Phi}) \left[ F_{IJK}(\Phi) \Gamma^J \Gamma^K + \bar{F}_{IJK}(\bar{\Phi}) \bar{\Gamma}^J \bar{\Gamma}^K \right] ,$$

(3.4)
which can be used to re-write the action totally in terms of the chiral and complex linear superfields. Then, the action will include terms of the second- and fourth-order in \( \Gamma \) and \( \bar{\Gamma} \). We explicitly find that the removal of the auxiliary superfields yields,

\[
S[\Phi, \bar{\Phi}, \Gamma, \bar{\Gamma}] = \int d^8z \left\{ K(\Phi, \bar{\Phi}) - g_{IJ}(\Phi, \bar{\Phi}) \Gamma^I \Gamma^J \right. \\
- \frac{1}{4} F_{MJK}(\Phi) g^{MN} \tilde{F}_{NRS}(\bar{\Phi}) \Gamma^J \Gamma^K \Gamma^R \Gamma^S \\
+ \frac{1}{4} \mathcal{F}_{IJKL} \Gamma^I \Gamma^J \Gamma^K \Gamma^L + \frac{1}{4} \tilde{\mathcal{F}}_{IJKL} \Gamma^I \Gamma^J \Gamma^K \Gamma^L \left\} ,
\]

\[ \mathcal{F}_{IJKL} \equiv F_{IJKL} - 3 F_{IJM} g^{MN} F_{KLN} \]  

We emphasize that this action despite its relative simplicity and written in terms of \( N=1 \) superfields actually possesses \( N=2 \) supersymmetry. This is directly analogous to the original \( N=1 \) superfield introduction of the \( N=2 \) supersymmetric Kählerian Vector Multiplet (KVM) model [3].

To obtain the hyper-Kähler potential, we have to dualize the (anti)linear scalars \( \Gamma, \bar{\Gamma} \) into (anti)chiral ones \( \Psi, \bar{\Psi} \). This amounts to implementing an \( N=1 \) superfield Legendre transform of \( \Gamma \) and \( \bar{\Gamma} \) in the action (3.5) by writing the master action

\[
S^*[\Phi, \bar{\Phi}, W, \bar{W}, \Psi, \bar{\Psi}] = S[\Phi, \bar{\Phi}, W, \bar{W}] + \int d^8z \left[ W^I \Psi_I + \bar{W}^I \bar{\Psi}_I \right] ,
\]

and then to determine the unconstrained complex superfields \( W \) and \( \bar{W} \) in terms of \( \Psi \) and \( \bar{\Psi} \) by solving their equations of motion

\[
\frac{\delta S^*}{\delta W^I} = 0 \rightarrow \Psi_I + \frac{\partial}{\partial W^I} \mathcal{L}(\Phi, \bar{\Phi}, W, \bar{W}) = 0 ,
\]

where \( \mathcal{L}(\Phi, \bar{\Phi}, \Gamma, \bar{\Gamma}) \) is the Lagrangian in (3.3). Since the Kähler metric \( g_{IJ}(\Phi, \bar{\Phi}) \) is non-singular and taking into account the explicit structure of \( \mathcal{L}(\Phi, \bar{\Phi}, \Gamma, \bar{\Gamma}) \), the above equations (3.7) are solved uniquely in a small neighborhood of the origin in \( W \)-space. Globally, the equations (3.7) are solved exactly although not uniquely, see below. The hyper-Kähler structure thus derived does not possess abelian isometries, in contrast to (2.2), obtained via the original \( c \)-map.

One of the interesting features of the O\((2n)\) (for \( n \geq 2 \)) projective multiplets, in contrast to the O\((2)\) projective multiplet, is that they can be coupled to Yang-Mills multiplets. The exception in the case of the O\((2)\) projective multiplet is due to the fact that only the O\((2)\) multiplet contains a component level 2-form gauge field. However, the O\((2n)\) (for \( n \geq 2 \)) projective multiplets can only provide representations of matter that are real representations of the Yang-Mills gauge group. This condition arises due
to the reality of the superfield that occurs as the coefficient of the $w^n$-term in the $O(2n)$ projective multiplet. Since the action \([3.3]\) depends on the holomorphic function $F$, it is interesting to conjecture that the $O(4)$ projective $\sigma$-model can be combined with the superfield Kählerian Vector Multiplet model \([3]\) by identifying the two a priori independent holomorphic functions as one and the same. This seems logical as a step toward the construction of hyper-Kählerian Vector Multiplet (HKVM) models. In the limit where $F = \frac{1}{2} \Phi^2$ and with a modification of the linearity condition, the combined KVM action and projective $O(4)$ $\sigma$-model \((3.3)\) describes the $N = 4$ Yang-Mills model. It is a topic for additional study to see if this $N = 4$ supersymmetry continues to exist with more general choices of $F$ such as the Seiberg-Witten prepotential \([10]\).

There is an obvious generalization to the results in \((3.1)\) and \((3.5)\). Consider the general $O(2n)$ multiplet of the form

$$\Sigma^I(w) = \Phi^I + w \Gamma^I + \sum_{\ell=2}^{n-1} w^\ell U_{\ell-1}^I + w^n V^I$$

$$+ (-1)^n \left\{ \sum_{\ell=2}^{n-1} (-1)^\ell w^{2n-\ell} \bar{U}_{\ell-1}^I - w^{2n-1} \bar{\Gamma}^I + w^{2n} \bar{\Phi}^I \right\}, \quad (3.8)$$

where $\Phi$ is a chiral superfield, $\Gamma$ is a complex linear superfield, $V^I$ is an arbitrary real general superfield and the remaining 4D, $N = 1$ superfields $U_{\ell}$ in \((3.8)\) are complex general ones. We consider the action of the form

$$S_{O(2n)} = (-1)^n \frac{1}{2\pi i} \oint_C \frac{dw}{w} \int d^8z \frac{F(\Sigma^I(w))}{w^{2n}} + \text{h.c.}. \quad (3.9)$$

This obviously leads to a generalization of \((3.2)\) and it includes a term of the form

$$(-1)^n \frac{1}{2} g_{IJ}(\Phi, \bar{\Phi}) V^I V^J \quad (3.10)$$

where the superfield $V^I$ is the real $N = 1$ coefficient superfield of the $w^n$ term in the $O(2n)$ multiplet $\Sigma^I(w)$. The only other manner in which this superfield appears in \((3.9)\) is via a term that is linear in $V^I$. That is why $V^I$ can always be explicitly and uniquely removed by its algebraic equation of motion. Clearly, there are lots of other terms involving the other complex superfields $U_{\ell}^I$ and their conjugates $\bar{U}_{\ell}^I$ for $\ell = 1, ..., n - 2$. For the full theory we have

$$S_{O(2n)}[\Phi, \Gamma, U, V] = \int d^8z \left[ K(\Phi, \bar{\Phi}) - g_{IJ}(\Phi, \bar{\Phi}) \Gamma^I \bar{\Gamma}^J + \mathcal{P}(U_{\ell}, \bar{U}_{\ell}, V ; \Phi, \bar{\Phi}, \Gamma, \bar{\Gamma}) \right.$$

$$+ (-1)^n \frac{1}{(2n)!} \left\{ F_{I_1 I_2} \ldots I_{2n}(\Phi) \Gamma^{I_1} \bar{\Gamma}^{I_2} \ldots \bar{\Gamma}^{I_{2n}} + \text{h.c.} \right\} \bigg], \quad (3.11)$$

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where the Kähler potential and metric are determined as in (2.1) in terms of the holomorphic prepotential in (3.9). The same holomorphic prepotential $F$ in (3.9) also determines the function $\mathcal{P}$ being a multinomial in $U_\ell$, $\bar{U}_\ell$ and $V$ with the coefficients depending polynomially on $\Gamma$, $\bar{\Gamma}$ and functionally on $\Phi$, $\bar{\Phi}$ such that

$$\mathcal{P}(U, \bar{U}, V ; \Phi, \bar{\Phi}, \Gamma, \bar{\Gamma}) \bigg|_{U=\bar{U}=V=0} = 0$$

$$(3.12)$$

In particular, for the O(6) multiplet model one obtains

$$\mathcal{P}_{O(6)} = g_{IJ}(\Phi, \bar{\Phi})U^I U^J - \frac{1}{2} g_{IJ}(\Phi, \bar{\Phi})V^I V^J$$

$$- V^I \left\{ F_{IJK}(\Phi, \bar{\Phi})\Gamma^J U^K + \frac{1}{6} F_{IJKL}(\Phi, \bar{\Phi})\Gamma^J \Gamma^K \Gamma^L + \text{h.c.} \right\}$$

$$- \frac{1}{6} \left\{ F_{IJK}(\Phi, \bar{\Phi})U^I U^J U^K + \text{h.c.} \right\}$$

$$+ \frac{1}{2} \left\{ F_{IJK}(\Phi, \bar{\Phi})\Gamma^I \Gamma^J \Gamma^K U^L - \frac{1}{2} F_{IJKL}(\Phi, \bar{\Phi})\Gamma^I \Gamma^J U^K U^L$$

$$- \frac{1}{12} F_{IJKLM}(\Phi, \bar{\Phi})\Gamma^I \Gamma^J \Gamma^K \Gamma^L U^M + \text{h.c.} \right\}.$$ 

$$(3.13)$$

As is already clear from (3.13), for $n > 2$ we face the problem of eliminating the auxiliary superfields. Of course, one can develop a perturbation theory to solve the equations of motion for $U_\ell$, $\bar{U}_\ell$ and $V$:

$$\frac{\partial}{\partial U_\ell^I} \mathcal{P} = \frac{\partial}{\partial U_\ell^I} \mathcal{P} = \frac{\partial}{\partial V^I} \mathcal{P} = 0$$

$$(3.14)$$

by representing $F(\Phi) = F_0 + \Delta F$, where $F_0 = \frac{1}{2} \Phi^2$ represents the leading contribution to $F(\Phi)$ and $\Delta F = \mathcal{O}(\Phi^3)$ corresponds to a small perturbation. However, explicit solutions to equations (3.14) can be found only for special choices of $F(\Phi)$. In general, Eqs. (3.14) present an algebro-geometric problem, and we return to this below.

Suffice it here to recall that the coordinate and the vector superfields $\Phi, \Gamma$ parametrize the total space of the tangent bundle, $T_M$, over the manifold of special Kähler geometry whose metric is determined by the holomorphic function $F$. The auxiliary superfields $U_\ell$, $V$, $\bar{U}_\ell$ enlarge this space considerably. The $(2n - 3)$ $d$ constraint equations (3.14), for $n \geq 2$, then locate $T_M$ as an algebraic variety within the enlarged field space.

For a general holomorphic prepotential $F(\Phi)$, the O(2n) multiplets themselves, $\Sigma^I$ (3.8), possess a $\mathbb{Z}_{2n}$ symmetry defined as follows:

$$\Sigma^I(w) \rightarrow \Sigma^I(e^{i\alpha} w) \quad , \quad (e^{i\alpha})^{2n} = 1.$$ 

$$(3.15)$$
Here the restriction $\exp(2ni\alpha) = 1$ follows from the twisted reality condition (3.2) to which the projective superfields $\Sigma^I(w)$ are subject. It is the requirement of $Z_{2n}$ symmetry which forbids us to add terms of the form

$$F(\Sigma^I(w))/w^k, \quad k \neq 0 \pmod{2n},$$

(3.16)

to the Lagrangian in (3.9).

It should be stressed that for the $O(2)$ model we have not only the discrete $Z_2$ symmetry, but also $d$ abelian noncompact isometries (Peccei-Quinn symmetry) in addition. The point is the $O(2)$ action in (2.6) is invariant under shifts

$$G^I(z) \rightarrow G^I(z) + a^I, \quad I = 1, \ldots, d,$$

(3.17)

with the $a$’s being arbitrary real constants. Such isometries are characteristic of the $O(2)$ model only and they cannot be present for the general $O(2n)$ projective models with $n > 1$.

4 Construction of the Minimal Polar Multiplet

$C$-Map

In our previous work [3], we discussed\(^6\) the existence of 4D, $N = 2$ nonlinear $\sigma$-models based on the use of the polar representation of projective superspace

$$\Upsilon^I(w) = \sum_{n=0}^{\infty} \Upsilon_n^I(z)w^n = \Phi^I(z) + w\Gamma^I(z) + O(w^2),$$

$$\equiv [\Phi^I + \Gamma^I + \mathcal{A}^I(w)],$$

$$\check{\Upsilon}^I(w) = \sum_{n=0}^{\infty} \check{\Upsilon}_n^I(z)(-\frac{1}{w})^n = \check{\Phi}^I(z) - \frac{1}{w}\check{\Gamma}^I(z) + O((\frac{1}{w})^2),$$

$$\equiv [\check{\Phi}^I - \check{\Gamma}^I \frac{1}{w} + \check{\mathcal{A}}^I(w)].$$

(4.1)

The polar multiplet is in a sense the $n = \infty$ limit of the $O(2n)$ projective multiplets. It is distinguished from them because, as mentioned in [3], the polar multiplet realizes a certain $U(1)$ symmetry that cannot occur in the $O(2n)$ projective multiplets. We have

\(^6\)Our notational conventions in this section are those in [3].
proposed that a minimal $N = 2$ supersymmetric extension be of the form

$$S_\sigma[\Upsilon, \bar{\Upsilon}] = \int d^8z \left\{ \frac{1}{2\pi i} \oint \frac{dw}{w} K(\Upsilon(w), \bar{\Upsilon}(w)) \right\}$$

$$= \int d^8z \left\{ \frac{1}{2\pi i} \oint \frac{dw}{w} \exp[\left( A^I + w \Gamma^I \right) \partial_I + \left( \bar{A}^\bar{I} - \frac{1}{w} \bar{\Gamma}^\bar{I} \right) \bar{\partial}_{\bar{I}}] K(\Phi, \bar{\Phi}) \right\}$$

(4.2)

to describe a 4D, $N = 2$ nonlinear $\sigma$-model. There was nothing in our previous discussion that prevents the Kähler potential (4.2) from taking the form given in (2.1). Since the potential has a special form, it follows that

$$S_\sigma[\Upsilon, \bar{\Upsilon}] = \int d^8z \left\{ \frac{1}{2\pi i} \oint \frac{dw}{w} \left[ \Upsilon^I(w) \bar{F}_{\bar{I}}(\bar{\Upsilon}(w)) + \bar{\Upsilon}^\bar{I}(w) F_I(\Upsilon(w)) \right] \right\}$$

$$= \int d^8z \left\{ \frac{1}{2\pi i} \oint \frac{dw}{w} \left[ \Phi^I + \Gamma^I w + A^I \right] \exp\left[ \left( \bar{A}^\bar{K} - \frac{1}{w} \bar{\Gamma}^\bar{K} \right) \bar{\partial}_{\bar{K}} \right] \bar{F}_{\bar{I}}(\bar{\Phi}) \right.$$

$$+ \text{h.c.} \right\}$$

(4.3)

So at least formally, it is possible to utilize the polar multiplet to define a $c$-map that connects a solely holomorphic function $F$ to a 4D, $N = 2$ supersymmetric nonlinear $\sigma$-model and therefore a hyper-Kähler model. The action (4.3) can be shown to look similar to that in (3.11), with the differences that (i) the real auxiliary superfield $V$ is absent; (ii) the set of complex auxiliary superfields $U_\ell$ includes infinitely many representatives, $\ell = 1, 2, \ldots, \infty$; (iii) the function $P(\Gamma, \bar{\Gamma}, U_\ell, \bar{U}_\ell)$ becomes transcendental.

A specific feature of the model (1.2) and its special version (4.3) we note, is a rigid U(1) symmetry defined by

$$\Upsilon^I(w) \rightarrow \Upsilon^I(e^{i\alpha} w).$$

(4.4)

This symmetry can be treated as a formal limit, $U(1) = \lim_{n \to \infty} \mathbb{Z}_{2n}$, of the discrete symmetry (3.13) in the O(2n) multiplet model. Since the models (3.9) and (4.2) possess different symmetries, it is natural to expect that these models lead to different hyper-Kähler structures.

We would be remiss if we did not mention that this form of the polar multiplet, since it depends also only on the holomorphic function $F$, can be combined with the KVM action [9] by identifying the two holomorphic functions to offer a second possible starting point for the construction of a hyper-Kählerian Vector Multiplet (HKVM) action. It also worthwhile to note that such a HKVM model might play a role for the effective action of $N = 4$ supersymmetric Yang-Mills theory just as the KVM model plays the critical role of encoding the effective action of $N = 2$ supersymmetric Yang-Mills theory.
5 Discussion

The models discussed in this note appear to be avatars for a number of interesting phenomena, most of which follow from the nice structure of projective superspace \([5]\). We now discuss some of them in turn.

5.1 Some General Field-Space Issues

All the \(N = 2\) multiplets considered in this note take the form of an order-\(2n\) polynomial in the complex variable \(w\) with (effectively) \(N = 1\) superfield coefficients (suppressing the \(I\) superscripts):

\[
\Sigma(w) = \sum_{k=0}^{2n} w^k \Sigma_k.
\]

(5.1)

The terminal superfields in the series, \(\Sigma_0, \Sigma_{2n}\), are chiral and antichiral, respectively, and the adjacent ones, \(\Sigma_1, \Sigma_{2n-1}\) are complex (anti)linear superfields. In addition, all but the \(n \to \infty\) limit of (5.1) are required to satisfy the twisted reality condition

\[
\hat{\Sigma}(w) = \Sigma(w), \quad \hat{\Sigma}(w) \overset{\text{def}}{=} (-1)^n w^{2n} \sum_{k=0}^{2n} \left(\frac{-1}{w}\right)^k \Sigma_k.
\]

(5.2)

This forces the ‘middle’ superfield, \(\Sigma_n\) to be real, and so for \(n=1\), \(\Sigma_1\) is a real, not complex, linear superfield.

The first superfields \(\Sigma_0 = \Phi\), are identified as the ‘coordinate superfields’ since their lowest components map the 4D spacetime into the target manifold \(M_d\) and serve as local coordinates. The next superfields, \(\Sigma_1 = \Gamma\), may be identified with tangent vectors (much as the fermions within the coordinate chiral superfields \(\Phi\) are), so that the pair \(\Phi, \Gamma\) provides a local (super)coordinate chart for the total space of the holomorphic (real in the \(n = 1\) case) tangent space \(T_{M}\). The interpretation that complex linear superfields most naturally are associated with Kähler manifold tangent vectors was first given in \([12]\) and provided a solution to the puzzling fact that the component form of nonlinear \(\sigma\)-models defined in terms of \(\Gamma\) possess \([13]\) a very different form from those defined in terms of \(\Phi\).

The remaining \(2n-3\) superfields \(\Sigma_k, \ k = 2, \ldots, 2n-2\) are auxiliary, since their equations of motion (3.14) are purely algebraic. Generally (and for \(n > 1\)), their geometrical interpretation is as follows. The total space, \(Y\), coordinatized by \(\Sigma_0, \ldots, \Sigma_{2n}\) enlarges \(T_{M}\) considerably by the introduction of the \(2n-3\) auxiliary superfields; in fact,
\( \dim_{\mathbb{R}}(Y) = (2n + 1)d \). The \((2n-3)d\) algebraic equations (3.14) then bring us back to the complex \(2d\)-dimensional total space of \(T_M\), embedded now in the much bigger \(Y\) as an algebraic variety!

This is not dissimilar to the approach of [14, 15], where (super)string compactifications on Calabi-Yau manifolds [16, 17] were discussed with the aid of the non-linear \(\sigma\)-model

\[
S_{\text{CYC}} = S_{\text{kin.}} + S_{\text{con.}}
\]

\[
S_{\text{kin.}} = \sum_{r=1}^{m} \int d^8 z \ w^r K^{(r)}(\Phi, \bar{\Phi}) \ , \ K^{(r)}(\Phi, \bar{\Phi}) = \log \left( \sum_{\mu=0}^{n_r} |\Phi^{(r)}_{\mu}|^2 \right) ,
\]

\[
S_{\text{con.}} = \left[ -i \int d^6 z \ \sum_{a=1}^{K} \Lambda^a P_a(\Phi) + \text{h.c.} \right] ; \quad (5.3)
\]

for notation see [15]. Here too, one starts from a larger field space which is then dynamically constrained to the Calabi-Yau algebraic subvariety, defined as the the common zero-set of the simultaneous (complex) algebraic equations \(P_a(\Phi) = 0\).

However, in (5.3), the algebraic constraints on the \(\Phi\)-space are enforced through the introduction of the additional Lagrange multiplier superfields, \(\Lambda^a\). The models considered in this note are more frugal: the constraints (3.14) are enforced through varying the very same superfields which are being eliminated. In this respect, the present situation is somewhat more similar to the Landau-Ginzburg approach of Ref. [14], where the projectivity of the kinetic term turns some of the fields into effectively auxiliary Lagrange multipliers. Ultimately, the approaches of Refs. [14] and [15] become analytic continuations of one another, within the context of Witten’s gauged linear \(\sigma\)-model [18], or different gauge choices in the non-linear \(\sigma\)-model [19]. In addition, this unifying approach firmly establishes relations to toric geometry [20]. Here, however, it is not clear how such a relation could be established, since the multiplets (5.1) are all real (5.2), except when \(n \to \infty\), precluding the usual coupling to \(U(1)\) gauge fields—essential for relations to toric geometry à la the works in references [18, 20].

So, the algebraic system (3.14) seeming inherently real, it seems quite dissimilar to the system \(P_a(\Phi) = 0\), produced by varying the chiral superpotential in (3.3) by the chiral \(\Lambda^a\). Note, however, that the whole Lagrangian superdensity \(\mathcal{L}(U_\ell, \bar{U}_\ell, V; \Phi, \bar{\Phi}, \Gamma, \bar{\Gamma})\) within the square brackets in (3.11), and so also the real function \(P\) in (3.3), may be regarded as (twice) the real part of a holomorphic function. That is, \(\mathcal{L} = L + \text{h.c.}\), where \(L\) is a holomorphic function of the \(2n+1\) \(N=1\) superfields \(\Sigma_k\) before the reality condition (5.2) is imposed. From \(L(\Sigma_k)\), we obtain \(\mathcal{L}\) by imposing the reality condition (5.2) on the \(\Sigma_k\)’s and then adding the hermitian conjugate. Furthermore, the functions being constrained
to vanish in Eqs. (3.14) are real parts of holomorphic functions—the same ones as in Eqs. (3.14), but before the reality condition (5.2) enforced.

The situation is roughly as follows:

\[
(\Sigma_0, \Sigma_1, \ldots, \Sigma_{2n-1}, \Sigma_{2n}) = Y^c \xrightarrow{(3.14)^c} T_M^c 
\]

\[
(\Phi, \Gamma, \ldots, \bar{\Gamma}, \bar{\Phi}) = Y \xrightarrow{(3.14)} T_M
\]

In fact, we may assign the (formal) weights, \(\deg(\Sigma_k) = k\), so that \(Y^c\) is a complex weighted affine space. Formally, we may assign also \(\deg(w) = -1\), so that \(\deg(\Sigma(w)) = 0\), whereupon the degree of quasihomogeneity of the action (3.13), and so also of the holomorphic function \(L\) is \(2n\). Of course, the same is obtained upon using the above weight assignments after the contour integral is evaluated: in the action (3.11), with (3.13). Thus, the variable \(w\) effectively projectivizes the whole superfield system \(Y^c = (\Sigma_0, \ldots, \Sigma_{2n})\), and the constraints (3.14) describe the real part of a complete intersection of algebraic hypersurfaces (each defined by a quasihomogeneous holomorphic polynomial equation) in a weighted \textit{projective} space, \(P(w^{-1}, \Sigma_0, \ldots, \Sigma_{2n})\). The actual (still complex) field space, \(Y^c\) where \(w\) has been integrated out, may be understood as the \(w=1\) (affine) coordinate patch; the compactification of \(Y^c\) into \(P(w^{-1}, \Sigma_0, \ldots, \Sigma_{2n})\) is a fairly standard method in algebraic geometry.

For reasons that we hope will be clearer shortly, the passage from \(T_M \hookrightarrow Y\) to \(Y^c\) is reminiscent of the fact that the cycles used in describing the moduli space of complex structures on Calabi-Yau weighted complete intersections turn out themselves to be real parts of algebraic subspaces \([21]\).

Finally, as the equations (3.14) are used to eliminate the auxiliary superfields \(U_\ell, V, \bar{U}_\ell\), it is easy to determine their degrees. Translating from \(\deg(\Sigma_k)\), we have

\[
\deg(U_\ell) = \ell + 1 \quad , \quad \deg(V) = n \quad , \quad \deg(\bar{U}_\ell) = 2n - \ell - 1 ,
\]

and therefore with \(P = P + h.c.\),

\[
\deg_Z \left( \frac{\partial P}{\partial X} \right) = \left\lfloor \frac{2n - \deg(X)}{\deg(Z)} \right\rfloor ,
\]

where \(X, Z\) range over \(U_\ell, V, \bar{U}_\ell\), and “\(\lfloor \cdot \rfloor\)” indicates truncation to the integral part.

Given the above general observations, one may expect to obtain, after eliminating the auxiliary superfields \(U_\ell, V, \bar{U}_\ell\), the same description of \(T_M\), regardless of which of the \(O(2n)\) projective multiplets one used. This, however, is not true.
Consider, for example, the O(6) case. The three equations of the system (3.14) are of tri-degree (2,1,1), (1,1,2) and (1,1,1), respectively, with respect to the three auxiliary fields $U, V, \bar{U}$. Upon projectivization, these are simply three quasihomogeneous algebraic equations in three variables, where the coefficients depend polynomially and functionally on $\Phi, \bar{\Phi}$ and $\Gamma, \bar{\Gamma}$, respectively; the general methods discussed in Ref. [17] apply, although one must take into account that the coefficient functions are all (real parts of) derivatives of a single function. In particular, as a special case of Bézout’s theorem, the system (3.14) is expected to have a 4-fold (degenerate) solution: the $\frac{\partial P}{\partial V}$ equation is linear in $V$ and yields a unique solution for $V$ in terms of $U, \bar{U}$ (and of course, $\Phi, \bar{\Phi}, \Gamma, \bar{\Gamma}$). But then, the second equation is quadratic in $\bar{U}$ and linear in $U$, so that a substitution into the first one yields a quartic equation for $\bar{U}$, in terms of $\Phi, \bar{\Phi}, \Gamma, \bar{\Gamma}$. The four solutions are related by a $\Phi, \bar{\Phi}, \Gamma, \bar{\Gamma}$-dependent $Z_4$ action, $Z_4$. So, upon elimination of the auxiliary fields $U, V, \bar{U}$, the O(6) model describes four $Z_4$-related copies of $T_M$. Furthermore, as $Z_4$ is a $\Phi, \bar{\Phi}, \Gamma, \bar{\Gamma}$-dependent $Z_4$-action, the multiple copies may coincide at special subspace, $B \subset T_M$. That is, the target space of the O(6) model is $C^4_B(T_M)$, a 4-fold cover of $T_M$ branched over $B \subset T_M$! The case of no branching is included, in which case $B = \emptyset$. (For more information about branched coverings, see Ref. [17] and the references therein.)

By contrast, the O(4) model has no such degeneracy, since the only auxiliary superfield, $V$, appears linearly in its equation of motion, $\frac{\partial P}{\partial V} = 0$, and is eliminated uniquely. On the other hand, even without explicit calculations, it is clear that for $n > 3$ the target space of the O(2n) models will become an even higher (branched) cover of $T_M$.

Altogether another, but related, issue is that of dualizing the complex (real when $n = 1$) linear superfields $\Gamma$ into $\Psi$. The $\Psi$’s being dual to the $\Gamma$, the resulting model, in terms of $\Phi, \Psi$ (and their conjugates), has $T^*_M$ as the target space. In passing from the action (2.6) to (2.7), one solves Eq. (3.7) which for $n = 1$ is linear in $\Gamma$ (and obviously linear in $\Psi$). This ensures a 1–1 mapping $\Gamma \leftrightarrow \Psi$, and so also of the respective models’ target spaces $T_M \leftrightarrow T^*_M$.

Consider now what happens when $n > 1$. For $n = 2$, Eq. (3.7) is cubic in $\Gamma, \bar{\Gamma}$, and so assigns three values of $\Gamma$ for every value of $\Psi$, in a $\Phi, \bar{\Phi}$-dependent fashion. That is, in solving (3.7) for $\Gamma, \bar{\Gamma}$ in terms of $\Psi, \bar{\Psi}$, we obtain three Riemann sheets for each $\Psi$. They all coincide at least at $\Gamma, \bar{\Gamma} = 0 = \Psi, \bar{\Psi}$; other coincidences may appear depending on the Lagrangian $L(\Phi, \bar{\Phi}, \Gamma, \bar{\Gamma})$, and so ultimately depending on the choice of the function $F(\Sigma(w))$ in (3.2); the loci of such coincidences introduce branching. Since the $\Phi, \bar{\Phi}$-dependent discrete ‘jump’ from one to another Riemann sheet is not an isometry of the hyper-Kähler structure (and which we wish to preserve), in solving (3.7) we must not
identify them (akin to orbifolding) but must include them all. With the three \( \Phi, \bar{\Phi} \)-dependent \( \Psi \)-Riemann sheets, the \( \Phi, \bar{\Phi}, \Psi, \bar{\Psi} \)-model’s target space is then a (possibly branched) triple cover of the dual of the \( \Phi, \bar{\Phi}, \Gamma, \bar{\Gamma} \)-model’s target space. Now, as the \( \Psi \)- and the \( \Gamma \)-spaces are contractible (as (co)tangent spaces to \( \mathcal{M} \)), it may be possible to isolate any one of the \( \Psi \)-Riemann sheets\(^7\). If so, we would obtain 3 a priori distinct \( \Phi, \bar{\Phi}, \Psi, \bar{\Psi} \)-models, the target space of each being the dual of the \( \Phi, \bar{\Phi}, \Gamma, \bar{\Gamma} \)-model’s target space.

For the record, the dualizing map (3.7) for the \( \text{O}(2n) \) model is

\[
\{\Psi, \bar{\Psi}\} \xrightarrow{1-(2n-1)} \{\Gamma, \bar{\Gamma}\},
\]

implying that there are \( 2n-1 \) \( \Psi \)-Riemann sheets. This leads to a \( \Phi, \bar{\Phi}, \Psi, \bar{\Psi} \)-model the target space of which is a \( 2n-1 \)-fold (branched) cover of the dual of the \( \Phi, \bar{\Phi}, \Gamma, \bar{\Gamma} \)-model’s target space, or to \( 2n-1 \) a priori distinct dualizations of the \( \Phi, \bar{\Phi}, \Gamma, \bar{\Gamma} \)-model, using the isolated various \( \Psi \)-Riemann sheets. In the \( \text{O}(6) \) case, the target space of the latter we showed was \( \mathcal{C}_B^4(T_{\mathcal{M}}) \), a (possibly) branched 4-fold cover of \( T_{\mathcal{M}} \). Eq. (3.7) the produces either a \( \Phi, \bar{\Phi}, \Psi, \bar{\Psi} \)-model with target space \( \mathcal{C}_B^4(\mathcal{C}_B^4(T_{\mathcal{M}}^*) ) \), or 5 a priori distinctly dualized, \( n = 2 \) \( \Phi, \bar{\Phi}, \Psi, \bar{\Psi} \)-models with target space \( \mathcal{C}_B^4(T_{\mathcal{M}}^*) \), a single \( \Psi \)-Riemann sheets can be isolated.

Notice that both the system (3.14) and the dualizing map (3.7) are algebraic. This ensures that the above multiplicity counting is correct, except for possible degenerations and loss of some of the solutions owing to the real nature of the \( \frac{\partial P}{\partial V} = 0 \) equation, and the non-holomorphy of (3.7). These can be better controlled in the complex category, i.e., working in the upper row of the diagram (5.4). Of course, explicit solutions will be possible only for rather special choices of the function \( F(\Sigma(w)) \), as noted above.

Finally, much of the relations between (super)field theory and geometry stems from a cohomological interpretation of the massless sector of the Hilbert space. In the finite-n models considered here, we note that the discrete symmetry (3.15) produces (super)sectors in the Hilbert space and so induces (super)selection rules.

The \( N=1 \) ‘component’ superfields \( \Sigma_k \) have distinct transformations with respect to the \( \mathbb{Z}_{2n} \) specified in (3.15): \( \Phi, \bar{\Phi} \) are invariant, while \( \Gamma, \bar{\Gamma} \) transform with \( e^{\pm i\alpha} \). Correlation functions must have a total ‘\( \alpha \)-charge’ of 0 (mod \( 2n \)), which becomes strictly 0 in the \( n \to \infty \) limit. This symmetry may be further twisted by \( N=1 \) supersymmetric \( R \)-symmetries, complicating the selection rules accordingly.

\(^7\)Amusingly, this logical possibility arises because of the real projection (5.2).
5.2 Some Specific Field-Space Issues

While the above discussion, appropriately generalized, applies even to \( N = 2 \) models with more general Lagrangians, some of the features we wish to discuss depend exclusively on the special geometry determined by the single holomorphic function \( F(\Sigma(w)) \).

Above, we have drawn some analogies between the present models and the Calabi-Yau models (5.3) and their Landau-Ginzburg relatives [14]. Taking these seriously implies a novel concept for Calabi-Yau manifolds: that they admit a special geometry metric, perhaps even one where \( K^{(r)}(\Phi, \bar{\Phi}) \) and \( P_a(\Phi) \) in (5.3) are all determined by a single holomorphic function. Whether this is in any (useful) way related to the ‘standard’ Einstein-Kähler-Yau metric remains an open question, but seems well worth a detailed study, since there is only an infinite iteration procedure for constructing the ‘standard’ metric. Moreover, the physically relevant metric receives further world-sheet perturbation and instanton corrections. While it may seem nothing short of a miracle, it is logically possible that the special geometry metric, suggested by this \( N = 2 \) approach and with a suitable choice of \( F(\Sigma(w)) \) may shortcut this doubly infinite iterative procedure and produce a useful metric on Calabi-Yau manifolds. Furthermore, as we discussed, the dualizing map (3.7) is a multi-valued map \( T_M \leftrightarrow T_M^* \). Now, variations of the complex structure of a Calabi-Yau manifold are parametrized by \( T_M \)-valued 1-forms, while variations of the complexified Kähler class are parametrized by \( T_M^* \)-valued 1-forms. The dualizing map (3.7) then naturally maps between these two types of variations, and therefore also between the two corresponding moduli spaces. However, as we have shown above, this map is multi-valued, and this must be taken into account.

On the other hand, the special geometry is naturally given on the moduli space of Calabi-Yau manifolds [22, 23]. As discussed above, the target space of \( O(2n) \) models is closely related to \( T_M \), and we can now take \( \mathcal{M} \) to be the complex structure moduli space for a Calabi-Yau manifold rather than the manifold itself. As Ref. [23] shows, special geometry is very closely related to the so-called Hodge fibration over a complex moduli space \( \mathcal{M} \) of a larger space containing \( T_M \), the holomorphic tangent space to \( \mathcal{M} \)—indeed very similar to our situation as described above. The special geometry metric turns out to be remarkably well determined in terms of the periods of the holomorphic volume form [22, 23], for which a natural choice of cycles includes the real parts of some straightforward algebraic subspaces of the Calabi-Yau manifold [21]. As alluded to above, this is very reminiscent of the role of the reality ‘projection’ as shown in the diagram (5.4). How far these parallels can be exploited further remains another open question.
Finally, if the chiral superfields are valued in the adjoint representation of some compact Lie algebra which possesses a number of non-trivial Casimir invariants, then one obvious choice for the \(F\)-functions is given by a (non)linear combination of such invariants. Remarkably, this again seems to draw parallels with the description of Calabi-Yau complete intersection manifolds \[13\]. There, the sought after manifolds were embedded into products of complex projective spaces, which in turn may be represented as cosets \(\mathbb{C}P^n = SU(n+1)/U(n)\). Whether a suitable generalization of the standard coset construction, and furthermore subject to algebraic constraints to describe embedded complete intersections is possible with the \(N=2\) multiplets used here remains an open but intriguing question, ultimately leading to an \(N=2\) (and non-abelian) generalization of the linear \(\sigma\)-model à la Witten \[18\].

“Originality does not consist in saying what no one has ever said before, but in saying exactly what you think yourself.”

James Fitzjames Stephen

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