GENERALIZED DUALITY FOR $k$-FORMS

FRANK KLINGER

Abstract. We give the definition of a duality that is applicable to arbitrary $k$-forms. The operator that defines the duality depends on a fixed form $\Omega$. Our definition extends in a very natural way the Hodge duality of $n$-forms in $2n$ dimensional spaces and the generalized duality of two-forms. We discuss the properties of the duality in the case where $\Omega$ is invariant with respect to a subalgebra of $\mathfrak{so}(V)$. Furthermore, we give examples for the invariant case as well as for the case of discrete symmetry.

1. Introduction: Self duality and $\Omega$-duality

Given a Riemannian or semi-Riemannian space $(V, g)$ of dimension $D$, the metric $g$ induces isomorphisms $\ast : \Lambda^k V \rightarrow \Lambda^{D-k} V$. This so called Hodge operator has the property $\ast^2 = \varepsilon \mathbb{1}$ where the sign $\varepsilon$ depends on the dimension and the signature of the metric by $\ast^2 |_{\Lambda^k V} = (-)^{t+k(D-k)} \mathbb{1}$. If the dimension of $V$ is even, $D = 2n$, we have a particular automorphism $\ast : \Lambda^n V \rightarrow \Lambda^n V$. For $\varepsilon = 1$, i.e. $D \equiv 0 \mod 4$, we call the $n$-form $F$ self dual and anti-self dual if it is an eigenform of $\ast$ to the eigenvalue 1 and $-1$, respectively, i.e.

$$\ast F = \pm F. \quad (1)$$

Duality relations are in particular interesting for two-forms. Consider a vector bundle $E$ over the Riemannian base $(M, g)$. The curvature tensor of a connection on $E$ is a two-form on $M$ with values in the endomorphism bundle of $E$. So for $\dim M = 4$ we may consider connections with (anti-)self dual curvature tensor. In the case of $E = TM$ (anti-)self duality is connected to complex structures on $M$; see [3].

In dimension four we may use the volume form $\text{vol} = *1$ to rewrite (1) as

$$\ast (\ast \text{vol} \wedge F) = \pm F. \quad (2)$$

This motivates the introduction of $\Omega$-duality of two-forms in arbitrary dimension; see for example [1, 2, 4, 7, 9] and [19]. It is defined as follows. Let $\Omega$ be a four-form on $V$ and consider the symmetric operator $\ast \Omega : \Lambda^2 V \rightarrow \Lambda^2 V$ with

$$\ast \Omega F := \ast (\ast \Omega \wedge F). \quad (3)$$

Let us suppose that $\ast \Omega$ admits real eigenvalues, then a two-form $F$ is called $(\Omega, \beta)$-dual if it obeys

$$\ast (\ast \Omega \wedge F) = \beta F. \quad (4)$$
(see [1]). In local coordinates with $\Omega = \Omega_{ijkl}$ and $F = F_{ij}$ the left hand side of (4) is given by $*(\Omega \wedge F)_{ij} = \frac{1}{2} \Omega_{ijkl} F^{kl}$.

**Example 1.1.** Consider the three-form $\theta$ in seven dimensions that is given by $\theta_{ijkl} = 1$ for $(ijkl) = (123), (435), (471), (516), (572), (624), (673)$. We associate to this the four-form $\theta := *\theta$ in seven and the four-form $\Theta := \theta + \Theta_{ijkl} e_8$ in eight dimensions. The latter is self-dual, i.e. $\Theta = *\Theta$. The forms above are strongly related to the discussion of $\mathfrak{g}_2$ and $\mathfrak{spin}(7)$. In particular, the duality relations yield the decompositions of the adjoint representations of $\mathfrak{so}(7)$ and $\mathfrak{so}(8)$ into irreducible representations of $\mathfrak{g}_2$ and $\mathfrak{spin}(7)$, respectively:

- $*g : \Lambda^2 \mathbb{R}^7 \rightarrow \Lambda^2 \mathbb{R}^7$ has eigenvalues 1 and $-2$ and the eigenspace decomposition corresponds to the decomposition of $\Lambda^2 \mathbb{R}^7$ with respect to $\mathfrak{g}_2$. In particular $E(1, *g) = 14$ is the adjoint representation and $E(-2, *g) = 7$ is the vector representation of $\mathfrak{g}_2$.
- $*\Theta : \Lambda^2 \mathbb{R}^8 \rightarrow \Lambda^2 \mathbb{R}^8$ has eigenvalues 1 and $-3$. The eigenspace decomposition corresponds to the decomposition of $\Lambda^2 \mathbb{R}^8$ with respect to $\mathfrak{spin}(7)$. In particular $E(1, *\Theta) = 21$ is the adjoint representation and $E(-3, *\Theta) = 7$ is the vector representation of $\mathfrak{spin}(7)$.

**Example 1.2.** Consider the globally defined parallel four-form $\Omega := \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3$ on the quaternionic-Kähler manifold $(M, \omega_1, \omega_2, \omega_3)$.

Then the operator $*\Omega$ on $\Lambda^2 TM \otimes \mathbb{C}$ has eigenvalues $1$, $\frac{1}{3}$ and $-\frac{2m+1}{3}$ corresponding to the eigenspaces $E(1, *\Omega) = S^2 Y \otimes \sigma_Z$, $E(-\frac{1}{3}, *\Omega) = \Lambda_0^2 Y \otimes S^2 Z$ and $E(-\frac{2m+1}{3}, *\Omega) = \sigma_Y \otimes S^2 Z$. Here $TM \otimes \mathbb{C} = Y \otimes Z$ is the (local) decomposition of the complexified tangent bundle into a rank-2$m$ and a rank-2 bundle with respect to the $Sp(m)Sp(1)$-structure, and $g = \sigma_Y \otimes \sigma_Z$ the corresponding decomposition of the complexified Riemannian metric on $M$. The 1-eigenspace is connected to half-flatness introduced by the authors in [1, 2].

**Remark 1.3.** Equation (4) can be generalized further in a straight forward way. Let $\Omega$ be a 2$k$-Form and $F$ be a $k$-Form. Then $*\Omega(F) := *(*\Omega \wedge F)$ is also a $k$ form and the question whether or not $*\Omega$ has real eigenvalues is reasonable. Such operators are discussed in [5] and examples are given in [10] for $k = 4, 6$ in dimension ten.

### 2. Duality of $k$-forms

All examples in the previous section have in common that the $\ell$-form $\Omega$ yields a duality relation on the space $\Lambda^2 V$ only. It would be preferable to give for one fixed $\Omega$ a duality relation on each $\Lambda^k V$. As we will see in Lemma 2.5 this is possible at least up to some mild restrictions.

**Definition 2.1.** Let $\Omega \in \Lambda^4 V$ be an $\ell$-form on $V = \mathbb{R}^D$. The duality operator $b_\Omega$ is defined by

\[
(5) \quad b_\Omega : \Lambda^k V \rightarrow \Lambda^k V, \quad F \mapsto \pi_k(\Omega \otimes F).
\]

---

1The invariant four-forms are explicitly given by $\theta_{ijkl} = 1$ for $(ijkl) = (1245), (1276), (1346), (1357), (2566), (2437), (4567)$ and $\Omega_{ijkl} = 1$ for $(ijkl) = (1245), (1276), (1346), (1357), (2566), (2437), (4567), (1238), (4358), (4718), (5168), (5728), (6248), (6738).
Here \( \pi_k : \Lambda^i V \otimes \Lambda^k V \to \Lambda^k V \) denotes the projection in the decomposition of \( \Lambda^i V \otimes \Lambda^k V \) with respect to \( \mathfrak{so}(V) \); see (49). We call the duality operator \( b_\Omega \) of order \( N \) if it admits \( N \) distinct eigenvalues.

**Example 2.2.** In [10] the authors discuss two operators on three- and two-forms in dimension \( D = 10 \). These two are covered by the special case \( \ell = 2k \) in Definition 2.1.

**Example 2.3.** A very basic example is the following. Let \( \Omega \) be a complex structure on \( V = \mathbb{R}^{2n} \) interpreted as two-form, i.e. \( \Omega_{ij} \Omega_{k} = -\delta_{ik} \). Then \( b_\Omega \) on \( \Lambda^1 V \) has eigenvalues \( \pm i \) with eigenspaces \( \Lambda^1_{(i)} V = \Lambda^{1,0} V \) and \( \Lambda^1_{(-i)} V = \Lambda^{0,1} V \). The eigenvalues of \( b_\Omega \) on \( \Lambda^k V \) for \( k \leq n \) are then given by \( k = 2q \) for \( q = 0, \ldots, k \) with eigenspaces \( \Lambda^k_{(k-2q)} V = \Lambda^k_{-q(\Lambda^{1,0} V) \otimes \Lambda^q(\Lambda^{0,1} V)} = \Lambda^{k-\ell,q} V \). For \( k > n \) we refer the reader to Remark 2.12. For instance, the Hodge dual to \( \Lambda^{n-k}_{(k-2q)} V = \Lambda^{n-k,q} V \) is \( \Lambda^{n+k}_{(k-2q)} V = \Lambda^{n,q,k} V \).

**Remark 2.4.** The preceding example can be generalized. For \( \Omega \in \Lambda^2 V \) the action of \( b_\Omega \) on \( \Lambda^i V \) is just the action of \( \frac{1}{k} \Omega \in \mathfrak{so}(V) \) on \( \Lambda^k V \).

**Lemma 2.5.** Let \( \Omega \in \Lambda^i V \). Then \( b_\Omega \neq 0 \) only if \( \ell \) is even and \( \ell \leq 2k \).

**Proof.** Consider the \( \mathfrak{so}(V) \)-decomposition as given in (49). Then we have
\[
\Lambda^k V \subset \Lambda^i V \otimes \Lambda^k V = \bigoplus_{i=0}^{\min\{k,\ell\}} \bigoplus_{j=0}^{\min\{k,\ell\}} \Lambda^{k+i-j-j} \oplus \bigoplus_{i=0}^{\min\{k,\ell\}} \Lambda^{k+i-2i} V
\]
if and only if \( 2i = \ell \) for some \( i \in \{0, \ldots, \min\{k,\ell\} \} \). This is \( \ell \) even and \( \frac{\ell}{2} \leq \min\{k,\ell\} \) or \( \ell \leq 2k \).

**Remark 2.6.** Because of the restriction given in Lemma 2.5 it would be preferable, that \( \ell \) is not too big. Therefore, the case \( \ell = 4 \) is of particular interest. The main examples which have been cited so far are connected to this value.

**Example 2.7.** The examples from section 1, the Hodge duality and the \( \Omega \)-dualities, are operators of the form \( b_\Omega \). They are of order two (Hodge-duality and Example 1.1) or order three (Example 1.2) with \( \ell = 2k \).

**Lemma 2.8.** In local coordinates be may write \( \Omega = (\Omega_{i_1 \ldots i_2 m}) \). Then \( b_\Omega \) and \( b^2_\Omega \) are given by
\[
(b_\Omega)^{d_1 \ldots d_k}_{i_1 \ldots i_k} = (\Omega^{d_1 \ldots d_m}_{i_1 \ldots i_m} \delta_{i_{m+1} \ldots i_k}^{d_{m+1} \ldots d_k})
\]
and
\[
(b^2_\Omega)^{d_1 \ldots d_k}_{i_1 \ldots i_k} = \delta_{j_1 \ldots j_m}^{d_1 \ldots d_m} (\Omega^{j_1 \ldots j_m}_{i_1 \ldots i_m} \Omega^{d_1 \ldots d_m}_{j_1 \ldots j_m})
\]
respectively. In particular \( b_\Omega \) is trace free.

\(^2\)To simplify the formulas we feel free to consider the projection up to a constant factor. This leads to the fact, that for example \( b_\Omega = 2\pi_\Omega \) for a four-form \( \Omega \) acting on \( \Lambda^3 V \), compare Example 1.1 and Lemma 2.8.
Proof. Consider \( \Omega = (\Omega_1, \ldots, \Omega_m) \) and \( F = (F_1, \ldots, F_k) \). Then the definition of \( b_\Omega(F) \) as projection on \( \Lambda^k V \) in (6) yields

\[
(b_\Omega(F))_{i_1 \ldots i_k} = \Omega_{j_1 \ldots j_m [i_1 \ldots i_m F^{j_1 \ldots j_m}_{i_{m+1} \ldots i_k}]}
\]

\[
= \delta^{a_1 \ldots a_k}_{i_1 \ldots i_k} \delta^{d_1 \ldots d_m}_{j_1 \ldots j_m} \delta_{a_1 \ldots a_m}^{i_1 \ldots i_m} \Omega^{j_1 \ldots j_m}_{a_1 \ldots a_m} F_{d_1 \ldots d_k}
\]

\[
= \delta^{[d_{m+1} \ldots d_k]}_{[i_{m+1} \ldots i_k]} \Omega^{d_1 \ldots d_m}_{i_1 \ldots i_m} F_{d_1 \ldots d_k}
\]

The square of \( b_\Omega \) obeys

\[
(b_\Omega^2(F))_{i_1 \ldots i_k} = \delta^{a_1 \ldots a_k}_{i_1 \ldots i_k} \Omega^{j_1 \ldots j_m}_{a_1 \ldots a_m} (b_\Omega(F))_{j_1 \ldots j_m a_{m+1} \ldots a_k}
\]

\[
= \delta^{a_1 \ldots a_k}_{i_1 \ldots i_k} \delta^{b_{m+1} \ldots b_k}_{b_{m+1} \ldots b_k} \delta^{d_1 \ldots d_k}_{i_1 \ldots i_m} \delta^{e_1 \ldots e_m}_{i_1 \ldots i_m} \delta_{b_{m+1} \ldots b_k} F_{c_1 \ldots c_m}
\]

\[
\Omega^{j_1 \ldots j_m}_{a_{m+1} \ldots a_k} b_{m+1} \ldots b_k F_{d_1 \ldots d_k}
\]

\[
= \delta^{b_{m+1} \ldots b_k}_{b_{m+1} \ldots b_k} \delta^{d_1 \ldots d_k}_{d_1 \ldots d_k} \delta^{e_1 \ldots e_m}_{e_1 \ldots e_m} \Omega^{j_1 \ldots j_m}_{i_1 \ldots i_m} F_{d_1 \ldots d_k}
\]

\[
\Omega^{j_1 \ldots j_m}_{i_1 \ldots i_m} F_{d_1 \ldots d_k}
\]

If we use (7), we see that the trace of \( b_\Omega \) is given by

\[
\text{tr}(b_\Omega) = \Omega^{i_1 \ldots i_m [i_{m+1} \ldots i_k]} \delta_{i_{m+1} \ldots i_k}^{i_1 \ldots i_m} = 0.
\]

\( \square \)

Remark 2.9. Let \( \Omega \) be an \( \ell \)-form with \( \ell = 2m \). From (7) we see that the linear operator \( b_\Omega \) is skew symmetric if \( m \) is odd and that it is symmetric if \( m \) is even. In particular, \( b_\Omega \) is diagonalizable with purely imaginary eigenvalues if \( m \) is odd and real eigenvalues if \( m \) is even.

If \( b_\Omega \) is of order \( N \) with different eigenvalues \( \beta_1, \ldots, \beta_N \), then \( b_\Omega \) solves its minimal polynomial \( \lambda^N - (\beta_1 + \cdots + \beta_N) \lambda^{N-1} + \cdots + (-)^N \beta_1 \cdots \beta_N = 0 \).

Because \( b_\Omega^2 \) is symmetric, it is contained in \( S^2(\Lambda^k V) \). So the right hand side of (8) is an element in \( \Lambda^V \otimes \Lambda^V \) that is embedded in \( S^2(\Lambda^k V) \) via some \( \delta \)-tensor. If \( b_\Omega \) is of order two with eigenvalues \( \beta_1 \neq -\beta_2 \) then \( b_\Omega \) has to be symmetric, too. This is enough to state the following result on duality operators of order two.

**Proposition 2.10.** Let \( \Omega \) be an \( \ell \)-form on \( V \). The operator \( b_\Omega \) is of order two with two eigenvalues \( \beta_1 \neq -\beta_2 \) only if \( \Lambda^\ell V \subset S^2(\Lambda^k V) \). In particular \( \ell \equiv 0 \mod 4 \).

Moreover, the projections on the two respective eigenspaces are given by

\[
\pi_{\beta_1} = \frac{\beta_2}{\beta_2 - \beta_1} \left( I - \frac{1}{\beta_2} b_\Omega \right)
\]

\[
\pi_{\beta_2} = \frac{\beta_1}{\beta_1 - \beta_2} \left( I - \frac{1}{\beta_1} b_\Omega \right)
\]

Remark 2.11. The restriction to \( \ell \) in Proposition 2.10 is a consequence of the symmetry of \( b_\Omega \) or, equivalently, of (50). This is not a contradiction to example 2.7 where the Hodge duality operator is of degree 2 but \( \ell = \dim V \) may be equal to 2 mod 4, because in this case we have \( \beta_1 = -\beta_2 = 1 \).

We emphasize on the following compatibility of the duality operator with the Hodge operator.
Remark 2.12. \quad \bullet \text{ Consider } V \text{ to be of dimension } D \text{ and let } \Omega \in \Lambda^{2m} V \text{ such that } b_\Omega \text{ is defined on } \Lambda^k V \text{ as well as on } \Lambda^{D-k} V \text{. Then we have }

\begin{equation}
(D-k) * b_\Omega * = (-1)^k (D-k) (k/m) b_\Omega
\end{equation}

where the action is on \( \Lambda^k V \).

In particular, if \( F \in \Lambda^k V \) is an eigenform of \( b_\Omega \) to the eigenvalue \( \beta \), then

\( *F \in \Lambda^{D-k} V \) is an eigenform of \( b_\Omega \) to the eigenvalue \( \beta' = \frac{\beta}{(m/k)} \), i.e.

\( \Lambda^{k}_{(\beta)} \cong \Lambda^{n-k}_{(\beta')} \) via *.

\( \bullet \) If we consider \( V \) of dimension \( 4m \) and \( \Omega \in \Lambda^{2m} V \) then for all \( F \in \Lambda^{2m} V \) we have

\begin{equation}
* b_\Omega(F) = b_\Omega(F) = b_\Omega(\ast F).
\end{equation}

In the case that \( \Omega \) is either self-dual or anti self-dual, i.e. \( \ast \Omega = \pm \Omega \) we have

\( \ast b_\Omega(F) = \pm b_\Omega(F) \). Therefore, for \( F \in \Lambda^{2m} \) we have \( \ast F = \pm F \) or \( \beta = 0 \), i.e. \( (\Lambda^{2m} V)^\ast \subset \Lambda^m \). We will recall this fact in Proposition 3.5.

3. Invariant duality operators

3.1. General properties of invariant duality operators. Let \( b_\Omega : \Lambda^k V \to \Lambda^k V \) be a duality operator of order \( N \) associated to \( \Omega \in \Lambda^k V \). Consider \( \Omega \) to be invariant with respect to a subalgebra \( \mathfrak{h} \subset \mathfrak{so}(V) \). Then \( b_\Omega \) is invariant under \( \mathfrak{h} \) as well. If \( \Lambda^k V = W_1 \oplus \ldots \oplus W_r \) is the decomposition into irreducible representation spaces with respect to \( \mathfrak{h} \), then

\[ b_\Omega|_{W_\alpha} = \beta_\alpha 1 \]

for some number \( \beta_\alpha \) due to Schur’s Lemma, i.e. \( W_\alpha \subset \Lambda^k_{(\beta_\alpha)} \). In particular, \( b_\Omega \) is diagonalizable with (not necessarily distinct) eigenvalues \( \beta_1, \ldots, \beta_r \). Because \( b_\Omega \) is trace free, we have in this special situation

\begin{equation}
\sum_{\alpha=1}^r \beta_\alpha \dim(W_\alpha) = 0.
\end{equation}

Definition 3.1. Let \( \Omega \in \Lambda^l V \) be invariant under a subalgebra \( \mathfrak{h} \subset \mathfrak{so}(V) \). Then \( b_\Omega : \Lambda^k V \to \Lambda^k V \) is called perfect if it is of order \( r \) where \( r \) is the number of irreducible submodules of \( \Lambda^k V \).

If \( \Omega \in \Lambda^l V \) is invariant with respect to a subalgebra \( \mathfrak{h} \subset \mathfrak{so}(V) \), then this is the same as to say that it spans a singlet within the decomposition of the \( \mathfrak{so}(V) \)-representation \( \Lambda^l V \) into irreducible \( \mathfrak{h} \)-representations.

As noticed before the case \( \ell = 4 \) is of particular interest. On the one hand due to the \( \Omega \)-duality of two-forms as in (4), on the other hand due to the restriction cf. Lemma 2.5. An \( \mathfrak{h} \)-invariant four-form may be constructed via an \( \mathfrak{h} \)-invariant metric as the \( \Lambda^4 V \)-part of \( S^2 \mathfrak{h} \subset S^2(\Lambda^2 V) \). This is in particular possible in the cases where \( \mathfrak{h} \) is a holonomy algebra; see [2]. The four-forms from the examples in section 1, that deal with \( \text{spin}(7) \), \( \mathfrak{g}_2 \), and \( \mathfrak{sp}(n) \oplus \mathfrak{sp}(1) \), are of this type. How they occur as a singlet and that they are unique up to a multiple can be seen as follows.

For instance, the four-form \( \theta \in \Lambda^4 \mathbb{R}^7 \), or it Hodge-dual \( \theta \in \Lambda^3 \mathbb{R}^7 \), considered in Example 1.1 is the singlet in the \( \mathfrak{g}_2 \)-decomposition \( \Lambda^4 \mathbb{R}^8 \cong \Lambda^3 \mathbb{R}^7 = 27 \oplus 7 \oplus 1 \). The
same is true for the four-form $\Theta \in \Lambda^4 \mathbb{R}^8$ also from Example 1.1. It represents the singlet in the $\text{spin}(7)$-decomposition $\Lambda^4 \mathbb{R}^8 = 35 \oplus 27 \oplus 7 \oplus 1$. Moreover the four-form from example 1.2 represents the singlet in the $\mathfrak{sp}(n) \oplus \mathfrak{sp}(1)$-decomposition of $\Lambda^4 V$ for $V = \mathbb{C}^{4n}$. Let us recall the splitting $V = E \otimes H$ with $H = \mathbb{C}^2$ and $H = \mathbb{C}^{2n}$, then it can be located in the following way. The splitting yields $\Lambda^3 V = \Lambda^3 (H \otimes E) = (1 \otimes S^2 E) \oplus (S^3 H \otimes \Lambda^2 V) \oplus (S^3 E \otimes 1)$. Then the singlet in $\Lambda^4 V$ coincides with singlet in the $\text{sp}(n) \oplus \text{sp}(1)$-decomposition of $\Lambda^4 V$ for $V = \mathbb{R}^{4n}$. Let us recall the splitting $V = E \otimes H$ with $H = \mathbb{C}^{2n} \otimes \mathbb{R}$. Then the singlet in $\Lambda^4 V = \Lambda^4 (H \otimes E) = (1 \otimes S^2 E) \oplus (S^3 H \otimes \Lambda^2 V) \oplus (S^3 E \otimes 1)$ coming from the trace in $S^2 E \otimes S^2 E$. In particular, these three examples yield perfect duality operators on the space of two forms.

A list and the explicit construction of invariant four-forms in dimension $D \leq 8$ for subgroups of $\mathfrak{so}(D)$ is given in [8]. In particular, the authors give a four-form depending on three real parameters, that yield the decomposition of $\Lambda^3 \mathbb{R}^8$ for $\mathfrak{h} = \mathfrak{u}(4) = \mathfrak{su}(4) \oplus \mathfrak{u}(1)$, as well as $\mathfrak{su}(4)$. The decomposition for $\mathfrak{u}(4)$ is not perfect, whereas the remaining two are.

The authors in [10] discuss the $\Omega$-duality in dimension $D = 10$ in the generalized sense cf. Remark 1.3. They construct a six-form and its associated Hodge-dual four-form invariant under $\mathfrak{su}(4) \oplus \mathfrak{u}(1) \subset \mathfrak{so}(8) \oplus \mathfrak{u}(1) \subset \mathfrak{so}(10)$. The corresponding eigenspace decompositions of $\Lambda^3 \mathbb{R}^{10}$ and $\Lambda^2 \mathbb{R}^{10}$ are not perfect in the sense of Definition 3.1.

3.2. The $\text{spin}(7)$- and $\mathfrak{g}_2$-duality. The first two examples of this section, i.e. Propositions 3.2 and 3.5 make use of the $\text{spin}(7)$-invariant four-form to give the eigenspace decomposition of $\Lambda^3 \mathbb{R}^8$ and $\Lambda^4 \mathbb{R}^8$. In particular, the duality-operator is perfect in both cases and therefore, the eigenspace decomposition coincides with the decomposition into irreducible representations. This extends the result from Example 1.1 to all forms on $\mathbb{R}^8$. Of course, these $\text{spin}(7)$-decompositions in terms of the invariant tensor $\Theta$ are not new, but very common in the literature, see e.g. [13, 18, 11] or [6]. Nevertheless, they yield nice examples how the known results fit in our duality framework.

**Proposition 3.2.** Consider the $\text{spin}(7)$-invariant four-form $\Theta$ on $V = \mathbb{R}^8$. Then $b_\Theta : \Lambda^2 V \rightarrow \Lambda^3 V$ with $(b_\Theta)_{ij}{}^{mn}{}_{l} = \Theta_{[lm}{}^{[ij}{}_{k]}$ is a perfect duality operator of order two that obeys

$$(b_\Theta)^2 = \frac{8}{3} \text{id} - \frac{10}{3} b_\Theta.$$

The eigenvalues are $-4$ and $\frac{2}{3}$ corresponding to the eight-dimensional and 48-dimensional $\text{spin}(7)$-invariant subspaces of $\Lambda^3 V$.\(^3\)

\(^3\)For his description the author in [11] uses the concept of vector cross products, of which a nice discussion and classification is given in [12].

\(^4\)8 is the spin representation and the 48 is the spin-$\frac{3}{2}$ representation of $\mathfrak{so}(7)$. The latter is given by vector-spinors which obey $\gamma^\mu \psi_\mu = 0$. 
Proof: The traces of the eight-tensor $\tilde{\Theta} = \Theta \otimes \Theta$ have components in the skew-symmetric parts of $S^2(\Lambda^4 \mathbb{R}^8)$ only. They are explicitly given by

\begin{align}
\Theta_{ijklm} \Theta^{klmn} &= 6 \delta_{ijkl}^m - 9 \Theta_{ij} [lm] \delta_{[i}^l \delta_{j]k}^m, \\
\Theta_{ijklm} \Theta^{klij} &= 12 \delta_{ijkl}^m - 4 \Theta_{ij}^{kl}, \\
\Theta_{ijklm} \Theta^{jklm} &= 42 \delta_{i}^l, \\
\Theta_{ijkl} \Theta^{ijkl} &= 336.
\end{align}

This gives

\begin{align}
b_{\Theta}^2(F)_{i_1 i_2 i_3} &= \delta_{j_1 j_2 i_1 i_2}^b \delta_{j_1 j_2 i_1 i_2} \Theta_{b^1 b^2 i_1 i_2} \Theta_{b^3 b^4} F_{d_1 d_2 d_3} \\
&= \frac{1}{4} \left( \delta_{b^1 b^2} \delta_{b^3} \delta_{b^4} \Theta_{j_1 j_2 i_1 i_2} \Theta_{b^1 b^2 i_1 i_2} \Theta_{b^3 b^4} F_{d_1 d_2 d_3} \\
&\quad + 2 \delta_{b^1 b^2} \delta_{b^3} \delta_{b^4} \Theta_{j_1 j_2 i_1 i_2} \Theta_{b^1 b^2 i_1 i_2} \Theta_{b^3 b^4} F_{d_1 d_2 d_3} \right) \\
&= \frac{1}{4} \left( \Theta_{d_1 d_2} \Theta_{j_1 j_2} \Theta_{j_1 j_2 i_1 i_2} F_{i_1 i_2} \theta_{d_1 d_2} - 2 \Theta_{j_2 d_1} \Theta_{j_2 i_1 i_2} \Theta_{d_1 d_2} \Theta_{j_2 j_1 i_1 i_2} F_{i_1 i_2} \theta_{d_1 d_2} \right) \\
&= \frac{1}{4} \left( 12 \delta_{d_1 d_2} \delta_{j_1 j_2} \Theta_{j_1 j_2} \Theta_{j_1 j_2 i_1 i_2} F_{i_1 i_2} \theta_{d_1 d_2} - 2 \theta_{d_1 d_2} \delta_{d_1 d_2} \delta_{j_1 j_2} \Theta_{j_1 j_2} \Theta_{j_1 j_2 i_1 i_2} F_{i_1 i_2} \theta_{d_1 d_2} \right) \\
&= \frac{4}{3} F_{i_1 i_2 i_3} - 4 \left( \Theta_{d_1 d_2} \Theta_{j_1 j_2} \Theta_{j_1 j_2 i_1 i_2} F_{i_1 i_2} \theta_{d_1 d_2} \right) \\
&\quad + 2 \left( \frac{1}{3} \delta_{d_1 d_2} \delta_{j_1 j_2} \Theta_{j_1 j_2} \Theta_{j_1 j_2 i_1 i_2} F_{i_1 i_2} \theta_{d_1 d_2} \right) \\
&\quad + 4 \left( \frac{1}{3} \delta_{d_1 d_2} \delta_{j_1 j_2} \Theta_{j_1 j_2} \Theta_{j_1 j_2 i_1 i_2} F_{i_1 i_2} \theta_{d_1 d_2} \right) \\
&= \frac{8}{3} F_{i_1 i_2 i_3} - \frac{4}{3} (\Theta_{d_1 d_2} \Theta_{j_1 j_2} \Theta_{j_1 j_2 i_1 i_2} F_{i_1 i_2} \theta_{d_1 d_2}) \\
&\quad + \frac{4}{3} \Theta_{d_1 d_2} \Theta_{j_1 j_2} \Theta_{j_1 j_2 i_1 i_2} F_{i_1 i_2} \theta_{d_1 d_2} \\
&= \frac{4}{3} F_{i_1 i_2 i_3}.
\end{align}

The eigenvalues of $b_{\Theta}$ are the zeros of $\beta^2 + \frac{10}{3} \beta - \frac{8}{3}$ that are $\frac{2}{3}$ and $-4$. The eigenspaces are given by $\Lambda^3_{\frac{2}{3}} V = 48$ and $\Lambda^3_{-4} V = 8$ due to $(-4) . 8 + \frac{2}{3} . 48 = 0$. \hfill $\square$

Lemma 3.3. Let $\Theta$ and $V$ as before and consider the duality map $b_{\Theta} : \Lambda^4 V \to \Lambda^4 V$ given by $b_{\Theta}(F)_{ijkl} = \Theta_{mn} F_{ijkl}$. This operator obeys\footnote{We postpone the calculations to Appendix B.}

\begin{align}
(14) &\ b_{\Theta}^2(F)_{ijkl} = \frac{5}{6} \Theta_{mn} F_{ijkl} F_{ijklm} + \frac{7}{6} F_{ijkl} \Theta_{mn} F_{ijkl} - \frac{5}{3} b_{\Theta}(F)_{ijkl}, \\
(15) &\ b_{\Theta}(F)_{ijkl} = \frac{1}{4} F_{ijkl} + \frac{3}{4} b_{\Theta}(F)_{ijkl} - \frac{10}{9} b_{\Theta}^2(F)_{ijkl} + \frac{2}{3} \Theta_{ijkl} \Theta_{mn} F_{ijklm}, \\
(16) &\ b_{\Theta}^2(F)_{ijkl} = 4 b_{\Theta}(F)_{ijkl} - \frac{5}{3} b_{\Theta}^2(F)_{ijkl} - \frac{14}{3} b_{\Theta}(F)_{ijkl} + \frac{4}{3} \Theta_{ijkl} \Theta_{mn} F_{ijklm}, \\
(17) &\ b_{\Theta}^2(F)_{ijkl} = - \frac{25}{3} b_{\Theta}^2(F)_{ijkl} - 20 b_{\Theta}(F)_{ijkl} - \frac{20}{9} b_{\Theta}^2(F)_{ijkl} + 16 b_{\Theta}(F)_{ijkl}.
\end{align}

Remark 3.4. Equation (17) yields, that $b_{\Theta}$ is a null of the polynomial

\begin{align}
\beta^5 + \frac{25}{3} \beta^4 + 20 \beta^3 + \frac{20}{9} \beta^2 - 16 \beta &= \beta(\beta + 4)(\beta + 3)(\beta + 2)(\beta - \frac{2}{3}) .
\end{align}

so that the possible eigenvalues are $\beta = 0, -2, -3, -4, \frac{2}{3}$. 

Proposition 3.5. Let $\Theta$ and $V$ as before. The duality operator $b_{\Theta} : \Lambda^4 V \to \Lambda^4 V$ with $(b_{\Theta})_{i_1i_2i_3i_4}^{j_1j_2j_3j_4} = \Theta_{i_1i_2}^{[j_1j_2} \delta_{i_3i_4]^{j_3j_4}}$ is a perfect duality operator of order four. The eigenspaces of $b_{\Theta}$ and the irreducible representations of $\Lambda^4 V$ with respect to $\text{spin}(7)$ correspond in the following way:

\begin{equation}
(19) \quad \Lambda^4_{(0)} V = 35, \quad \Lambda^4_{(-4)} V = 1, \quad \Lambda^4_{(-2)} V = 7, \quad \Lambda^4_{(4)} V = 27.
\end{equation}

The minimal polynomial is consequently given by

\begin{equation}
(20) \quad \beta(\beta + 2)(\beta + 4)(\beta - \frac{4}{3}) = \beta^4 + \frac{16}{3} \beta^3 + 4\beta^2 - \frac{16}{3} \beta.
\end{equation}

We know that $\Lambda^4 V$ decomposes into four irreducible representations of dimension $1, 7, 27$ and $35$ with respect to $\text{spin}(7)$. Therefore, one of the values from Remark 3.4 is not an eigenvalue. In principle, we do not need this information to sort one of the values out. Nevertheless, the following proof of Proposition 3.5 will implicitly make use of it.

Proof. First we show that $-4, 0$ and $-2$ occur as eigenvalues and that the spaces of the right hand sides of (19) are subsets of the respective eigenspaces.

In particular, at least on part of the zero-eigenspace is given by $35 = (\Lambda^4 V)^{-} \subset \Lambda^4_{(0)} V$ due to the self-duality of $\Theta$ and Remark 2.12.

From (13) we immediately get $b_{\Theta}(\Theta) = -4 \Theta$, such that $1 = \mathbb{R} \Theta \subseteq \Lambda^4_{(-4)} V$.

The next element we insert into $b_{\Theta}$ is $F_{ijkl} = \alpha_{m[i} \Theta^m_{jkl]}$ for $\alpha \in \Lambda^2 V$:

\[
b_{\Theta}(F)_{ijkl} = \Theta^{mn}_{[ij} \varepsilon^{abcd}_{kl]mn} \alpha_{oa} \Theta^a_{bcd} \\
= \frac{1}{2} \Theta^{mn}_{[ij} \varepsilon^{abcd}_{kl]mn} \alpha_{oa} \Theta^a_{bcd} \\
= \frac{1}{2} \Theta^{mn}_{[ij} \varepsilon^{abcd}_{kl]mn} \alpha_{oa} \Theta^a_{bcd} + \frac{1}{4} \Theta^{mn}_{[ij} \varepsilon^{abcd}_{kl]mn} \alpha_{oa} \Theta^a_{bcd} \\
= \frac{1}{2} \varepsilon^{abcd}_{ijkl} g_{dd'} (12 \delta_{aa'} - 4 \Theta_{ab} \delta_{aa'}) \alpha_{co} + \frac{1}{2} \varepsilon^{abcd}_{ijkl} g_{dd'} \Theta_{oa} \alpha_{m} \\
- 9 \Theta_{ocd} \delta_{d'}^{a'} \delta_{d'}^{d'} \alpha_{m} \\
- 2 \alpha_{o[i} \Theta^{o}_{jkl]} - \frac{1}{2} \Theta^{abcd}_{ijkl} g_{aa'} g_{bb'} \alpha_{m} \left(4 \Theta_{ocd} \delta_{d'}^{a'} + 2 \Theta_{ocd} a'^{b'} \delta_{d'}^{m} \right) \\
+ 2 \Theta_{ocd} a'^{b'} \delta_{d'}^{a'} + \Theta_{ocd} a'^{b'} \delta_{d'}^{m} \\
- 2 \alpha_{o[i} \Theta^{o}_{jkl]} - \delta_{ijkl} (- \alpha_{o[i} \Theta^{o}_{jkl]} + \alpha_{m[i} \Theta^{m}_{jkl]}) \\
- \alpha_{o[i} \Theta^{o}_{jkl]}
\]

therefore $7 = \{ \alpha_{m[i} \Theta^{m}_{jkl]} : \alpha \in \Lambda^2 V \} \subset \Lambda^4_{(-2)} V$.

There is a space of dimension $27$ left, which cannot be decomposed further without getting more singlets in $\Lambda^4 V$. Therefore it is irreducible, and has to be a subspace of one of the eigenspaces. The trace formula $0 \cdot 35 + (-4) \cdot 1 + (-2) \cdot 7 + \beta \cdot 27 = 0$ is only solved by $\beta = \frac{2}{3}$. Such that equality in (19) follows.

---

\textsuperscript{6}We recall the decomposition of $\Lambda^2 V$ as given in Example 1.1 and that we have to double the eigenvalues given there, when we consider $b_{\Theta}$. In particular, $\alpha_{m[i} \Theta^{m}_{jkl]} = 0$ for $\alpha \in \Lambda^2_{(2)} V$.}
The above calculations and (14)-(16) yield the following decomposition of \( \hat{\Theta} = \Theta \otimes \Theta \)

\[
\Theta_{ijkl} \Theta^{mnop} = -42 \Theta^{[mn}[ij]^{op}]_{kl] + 2 \Theta_{[ijkl]}^{[m} \Theta^{nop]}_{l]} + 3 \Theta_{ij}^{[mn} \Theta_{kl]}^{op}.
\]

In contrast to its traces, the full eight-tensor \( \hat{\Theta} \) has contributions not only from the skew-symmetric parts \( \Lambda^8V, \Lambda^4V, \) and \( \Lambda^0V \) but also from \([6,2]_0\) and \([4,4]_0\). \(\square\)

Remark 3.6. We complete the discussion of the invariant \( \mathfrak{spin}(7) \)-four-form by adding the missing result for the closely related invariant \( \mathfrak{g}_2 \)-four-form, \( \hat{\Theta} \); see Example 1.1.

The minimal polynomial of \( b_{\hat{\Theta}} : \Lambda^3\mathbb{R}^7 \rightarrow \Lambda^3\mathbb{R}^7 \) is \( \beta^3 + \frac{16}{3} \beta^2 + 4 \beta - \frac{16}{3} \) and the eigenspaces are \( \Lambda_{(-4)}^3 \mathbb{R}^7 = 1, \Lambda_{(-2)}^3 \mathbb{R}^7 = 7, \) and \( \Lambda_{(4)}^3 \mathbb{R}^7 = 27. \)

3.3. **Lifting to higher dimensions.** There are two straightforward ways to lift an \( \ell \)-form \( \Omega \) on \( \mathbb{R}^n \) to \( \mathbb{R}^D \) for \( D > n \). First we consider the trivial lift given by the same Symbol \( \Omega \). Secondly, we consider the \( *_D \)-dual to this first lift, i.e. the \( (D-\ell) \)-form \( \hat{\Omega} = *_D \Omega \). If \( \Omega \) is \( \mathfrak{g} \)-invariant, these lifts are invariant with respect to \( \mathfrak{g} \oplus \mathfrak{so}(D-n) \). We will discuss these two constructions for the \( \mathfrak{spin}(7) \)-invariant four-form in dimension eight from the preceding section and its lifts to dimension ten. The maximal invariant subalgebra is \( \mathfrak{spin}(7) \oplus \mathfrak{so}(2) = \mathfrak{spin}(7) \oplus \mathfrak{u}(1). \)

We specify the \( e_9 \wedge e_{10} \)-plane and we consider \( \Theta \) to live on \( \text{span}\{e_i\}_{i \leq 8} = \mathbb{R}^8. \) With respect to the decomposition \( \mathbb{R}^{10} = \mathbb{R}^8 \oplus \mathbb{R}^2 \) the \( k \)-forms split as

\[
\Lambda^k \mathbb{R}^{10} = \Lambda^k \mathbb{R}^8 \oplus \Lambda^{k-1} \mathbb{R}^8 \otimes \mathbb{R}^2 \oplus \Lambda^{k-2} \mathbb{R}^8 \otimes \Lambda^2 \mathbb{R}^2.
\]

The trivial lift of \( \Theta \) now yields for \( k \geq 3 \) a duality operator which is given by \( b_{\Theta} = b_{\Theta} \otimes 1 \) on each summand. The eigenspace decomposition for \( k = 3, 4 \) can immediately be read from the preceding sections. Moreover, in the case \( k = 5 \) we can furthermore use the symmetry \( *_{10}(\Lambda^5 \mathbb{R}^8) = \Lambda^3 \mathbb{R}^8 \otimes \Lambda^2 \mathbb{R}^2 \) such that the missing decomposition follows from \( b_{\Theta} \) on \( \Lambda^3 \mathbb{R}^8 \) alone, and the eigenvalues and eigenspaces correspond as in (10) from Remark 2.12. In particular, the duality operator is not perfect in all cases, due to the doubling from the second summand in the right hand side of (22).

Secondly we consider the six-form \( *_{10} \Theta \). Because \( \Theta \) lives on \( \mathbb{R}^8 \subset \mathbb{R}^{10} \) we have \( *_{10} \Theta = *_{8} \Theta \wedge \epsilon = \Theta \wedge \epsilon \) which we will denote by \( \hat{\Theta} \). Here \( \epsilon \) denotes the volume-form on \( \mathbb{R}^2 \subset \mathbb{R}^{10} \). Although this six-form is directly connected to the one before, we get a different behavior of the eigenspaces. In fact, it turns out, that the restriction of \( b_{\Theta} \) to \( \Lambda^k \mathbb{R}^{10} / \ker(b_{\Theta}) \) is perfect for \( k = 3, 4 \). For \( k = 5 \) the operator is not perfect, but the two basic \( \mathfrak{spin}(7) \)-representations of dimension seven and eight correspond to the same non-vanishing eigenvalue.

We will state the results for \( k = 5, 4, 3 \) and again postpone the calculations for the case \( k = 5 \) to the appendix. That hopefully will convince the reader that the calculations for the remaining cases can be performed similarly.

For the case \( k = 5 \) we need the following lemma.

Lemma 3.7. We consider the maps
\[ d_\Theta : \Lambda^5 \mathbb{R}^8 \to \Lambda^3 \mathbb{R}^8, \quad d_\Theta(F)_{lmn} = \Theta_{ijkl} F_{ijkl} \]
\[ d_\Theta : \Lambda^3 \mathbb{R}^8 \to \Lambda^5 \mathbb{R}^8, \quad d_\Theta(F)_{ijklm} = \Theta_{ijkl} F_{ijklm} \]
These maps are isomorphisms and connected to \( b_\Theta \) and to the Hodge operator via
\[ d_\Theta \circ d_\Theta = \frac{6}{5} \text{id} + \frac{3}{2} b_\Theta(F), \quad \text{and} \quad \ast \, d_\Theta \ast = -20 \hat{d}_\Theta \].
A consequence of this is \( \hat{d}_\Theta \circ d_\Theta = -\ast \, d_\Theta \circ \hat{d}_\Theta \ast \).

Proof. The identities in (24) are verified in the appendix. Due to Schur’s Lemma \( d_\Theta \) and \( \hat{d}_\Theta \) are proportional to the identity when restricted to the eigenspaces of \( b_\Theta \) and moreover they are non-vanishing due to (24).

If we use lemma 3.7 and the calculations from the appendix we get the next result.

Proposition 3.8. Let \( \hat{\Theta} \) be the lift of \( \Theta \) to ten dimension given by \( \hat{\Theta} = \Theta \wedge \epsilon \). If we consider the decomposition of \( \Lambda^5 \mathbb{R}^{10} \) given by (22), then \( b_\hat{\Theta} : \Lambda^5 \mathbb{R}^{10} \to \Lambda^5 \mathbb{R}^{10} \) is given by
\[ b_\hat{\Theta} = \left( \begin{array}{c}-\frac{3}{10} d_\Theta \otimes \ast + \frac{6}{5} \hat{d}_\Theta \otimes \ast \end{array} \right) \].

If we denote the \( \pm i \)-eigenspaces of \( \ast_2 \) on \( \mathbb{R}^2 \) by \( \mathbb{R}_\pm \), the eigenvalues and eigenspaces of \( b_\hat{\Theta} \) and their dimensions are given by

| \( \pm \frac{32}{5} i \) | \( \Lambda^4_{(0)} \mathbb{R}^8 \otimes \mathbb{R}_+ \otimes \Lambda^4_{(0)} \mathbb{R}^8 \otimes \mathbb{R}_- \) | 35 + 35 = 70 |
| \( \pm \frac{64}{5} i \) | \( \left\{ \pm \frac{3}{2} i d_\Theta(F), F \wedge \epsilon \right\} \mid F \in \Lambda^3_{(-4)} \mathbb{R}^8 \} \otimes \Lambda^4_{(-2)} \mathbb{R}^8 \otimes \mathbb{R}_\pm \) | 2 \times (8 + 7) |
| \( \pm \frac{8}{5} i \) | \( \Lambda^4_{\left(\frac{1}{2}\right)} \mathbb{R}^8 \otimes \mathbb{R}_\pm \) | 2 \times 27 |
| \( \pm \frac{3}{5} i \) | \( \left\{ \left(\mp 10 i \hat{d}_\Theta(F), F \wedge \epsilon \right) \mid F \in \Lambda^3_{\left(\frac{4}{2}\right)} \mathbb{R}^8 \right\} \) | 2 \times 48 |

The first summand in the third row and the space in the last row are subspaces of \( \Lambda^5_{\left(\frac{1}{2}\right)} \mathbb{R}^8 \otimes \Lambda^3_{\left(-4\right)} \mathbb{R}^8 \otimes \epsilon \) and \( \Lambda^5_{\left(\frac{1}{2}\right)} \mathbb{R}^8 \otimes \Lambda^3_{\left(\frac{4}{2}\right)} \mathbb{R}^8 \otimes \epsilon \), respectively.

Similar to Lemma 3.7 we get the following.

Lemma 3.9. Consider the maps
\[ c_\Theta : \Lambda^4 \mathbb{R}^8 \to \Lambda^2 \mathbb{R}^8, \quad c_\Theta(F)_{ij} = \Theta_{klm} F_{ijkl} \]
\[ \tilde{c}_\Theta : \Lambda^2 \mathbb{R}^8 \to \Lambda^4 \mathbb{R}^8, \quad \tilde{c}_\Theta(F)_{ijkl} = \Theta_{mijkl} F_{mijkl} \]
Their kernels are \( \ker(c_\Theta) = \Lambda^4_{(0)} \mathbb{R}^8 \otimes \Lambda^4_{(-4)} \mathbb{R}^8 \otimes \Lambda^4_{(\frac{4}{2})} \mathbb{R}^8 \) and \( \ker(\tilde{c}_\Theta) = \Lambda^2_{(2)} \mathbb{R}^8 \) and the restrictions to \( \Lambda^4_{(0)} \mathbb{R}^8 \) and \( \Lambda^2_{(2)} \mathbb{R}^8 \) obey
\[ c_\Theta \circ \tilde{c}_\Theta \mid_{\Lambda^4_{(0)} \mathbb{R}^8} = -24 id \quad \text{and} \quad \tilde{c}_\Theta \circ c_\Theta \mid_{\Lambda^2_{(2)} \mathbb{R}^8} = -24 id \].
Proof. The statements follow from calculations similar to those for the case \( k = 5 \) and from Schur’s Lemma together with the decompositions in Example 1.1 and Proposition 3.5.

From Lemma 3.9 we get a result similar to the previous Proposition.

Proposition 3.10. Let \( \hat{\Theta} \) be the lift of \( \Theta \) as before and consider the decomposition of \( \Lambda^4\mathbb{R}^{10} \) given by (22). Then \( b_{\hat{\Theta}} : \Lambda^4\mathbb{R}^{10} \to \Lambda^4\mathbb{R}^{10} \) is given by

\[
b_{\hat{\Theta}} = \left( \begin{array}{ccc}
\frac{1}{2}c_{\Theta} \otimes * & 0 & 6\hat{c}_{\Theta} \otimes * \\
0 & b_{\Theta} \otimes * & \end{array} \right).
\]

the eigenvalues and eigenspaces of \( b_{\hat{\Theta}} \) as well as their dimensions are given by

| \( i \) | \( \Lambda^4_{(0)}\mathbb{R}^8 \oplus \Lambda^4_{(-4)}\mathbb{R}^8 \oplus \Lambda^4_{(4)}\mathbb{R}^8 \oplus \Lambda^2_{(2)}\mathbb{R}^8 \otimes \epsilon \) | \( 35 + 1 + 27 + 21 = 84 \) |
|---|---|---|
| \( \pm 9i \) | \( \Lambda^3_{(-4)}\mathbb{R}^8 \otimes \mathbb{R}_\mp \) | \( 2 \times 8 \) |
| \( \pm \frac{3}{2}i \) | \( \Lambda^3_{(4)}\mathbb{R}^8 \otimes \mathbb{R}_\pm \) | \( 2 \times 48 \) |
| \( \pm 6\sqrt{2}i \) | \( \left\{ \left( \mp \frac{1}{\sqrt{2}}c_{\Theta}(F), F \wedge \epsilon \right) \mid F \in \Lambda^2_{(-6)}\mathbb{R}^8 \right\} \) | \( 2 \times 7 \) |

The space in the last row is a subspace of \( \Lambda^4_{(-2)}\mathbb{R}^8 \oplus \Lambda^2_{(-6)}\mathbb{R}^8 \otimes \epsilon \) to and can also be written as \( \left\{ \left( F, \mp \frac{1}{12\sqrt{2}}c_{\Theta}(F) \wedge \epsilon \right) \mid F \in \Lambda^4_{(-2)}\mathbb{R}^8 \right\} \) due to (27).

To complete the discussion we add the result for \( k = 3 \).

Proposition 3.11. With \( \hat{\Theta} \) as before and with (22) the operator \( b_{\hat{\Theta}} : \Lambda^3\mathbb{R}^{10} \to \Lambda^3\mathbb{R}^{10} \) is given by

\[
b_{\hat{\Theta}} = \left( \begin{array}{ccc}
-3c_{\Theta} \otimes * & 6\hat{c}_{\Theta} \otimes * \\
12\sqrt{2} & c_{\Theta} \otimes * & \end{array} \right).
\]

Its eigenvalues, eigenspaces and their dimensions are

| \( i \) | \( \Lambda^3_{(4)}\mathbb{R}^8 \) | \( 48 \) |
|---|---|---|
| \( \pm 18i \) | \( \Lambda^2_{(-6)}\mathbb{R}^8 \otimes \mathbb{R}_\mp \) | \( 2 \times 7 \) |
| \( \pm 6i \) | \( \Lambda^2_{(2)}\mathbb{R}^8 \otimes \mathbb{R}_\mp \) | \( 2 \times 21 \) |
| \( \pm 6\sqrt{7}i \) | \( \left\{ \left( \mp \frac{1}{\sqrt{7}}c_{\Theta}(F), F \wedge \epsilon \right) \mid F \in \mathbb{R}^8 \right\} \) | \( 2 \times 8 \) |

Here we used the following Lemma similar to Lemmas 3.7 and 3.9.

Lemma 3.12. The maps

\[
\begin{align*}
\varepsilon_\Theta &: \Lambda^3\mathbb{R}^8 \to \mathbb{R}^8, & \varepsilon_\Theta(F)_l &= \Theta_{ijkl}F^{ijk} \\
\tilde{\varepsilon}_\Theta &: \mathbb{R}^8 \to \Lambda^3\mathbb{R}^8, & \tilde{\varepsilon}_\Theta(F)_{jkl} &= \Theta_{ijkl}F^i
\end{align*}
\]
The eigenspaces \( V \) and the eigenvalues of \( \epsilon, \eta \). Moreover, for \( \sigma \in \mathbb{Z}_8 \) such that

\[
\sigma \in \{ \pm 1 \} \text{ and } \sigma^2 = 1 \text{ or } \sigma^2 = 7 \mod 8.
\]

The minimal polynomial of \( b_\Omega \) is given by

\[
\tilde{\Omega} = \Theta \delta, \quad e \in \mathbb{C}.
\]

The minimal polynomial of \( b_\Omega \) is given by

\[
p(t) = t(t^2 - 1)(t^2 - 4)(t^2 - 1 + \sqrt{2})(t^2 - 1 - \sqrt{2})
\]

and the eigenvalues of \( \sigma \) on \( \Lambda^2 V \) have multiplicities 3 for \( \pm 1 \) and \( \pm i \), and 4 for \( \pm \frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}} \).

The eigenspaces \( V_\beta \) for \( \beta = 0, \pm 1, \pm 2, \pm \sqrt{2}, \text{ and } \pm 1 \pm \sqrt{2} \) as well as their behavior under \( \sigma \in \mathbb{Z}_8 \) are explicitly given as follows.

\begin{align*}
V_{\pm 1} &= \text{span}\{ v_{\pm 1} = (e_{56} - e_{12}) \pm (e_{38} + e_{47}) \}, \quad v_{\pm 1}^2 = (e_{67} - e_{23}) \pm (e_{58} - e_{14}), \\
v_{\pm 1}^3 &= (e_{78} - e_{34}) \pm (e_{16} - e_{25}), \quad v_{\pm 1}^4 = -(e_{18} + e_{45}) \pm (e_{27} + e_{36})
\end{align*}

with \( v_{\pm 1} \xrightarrow{\sigma^2} v_{\pm 1} \xrightarrow{\sigma} v_{\pm 1} \xrightarrow{\sigma^2} -v_{\pm 1} \) such that \( \sigma^4 + 1 \) is the minimal equation on \( V_{\pm 1} \).

\begin{align*}
V_{\pm 2} &= \text{span}\{ v_{\pm 1} = e_{13} - e_{17} + e_{35} + e_{57} \pm (e_{24} - e_{28} + e_{46} + e_{68}) \}, \\
\sigma(v_{\pm 1}) &= v_{\pm 1} \text{ such that } \sigma \pm 1 = 0 \text{ on } V_{\pm 2}.
\end{align*}

\begin{align*}
V_{\pm \sqrt{2}} &= \text{span}\{ v_{\pm 1} = e_{13} + e_{17} + e_{35} + e_{57} \mp \sqrt{2}(e_{28} + e_{46}), \\
v_{\pm 1}^2 &= e_{24} - e_{28} - e_{46} + e_{68} \pm \sqrt{2}(e_{17} + e_{35})
\end{align*}

with \( \sigma(v_{\pm 1}^2) = \mp \sqrt{2}v_{\pm 1}^2 \) and \( \sigma(v_{\pm 1}^2) = -v_{\pm 1}^2 \), i.e. \( \sigma^2 \pm \sqrt{2} \sigma + 1 = 0 \) is the minimal equation on \( V_{\pm \sqrt{2}} \).

Moreover, for \( \epsilon, \eta \in \{ \pm 1 \} \) we have

\begin{align*}
V_{\epsilon + \eta \sqrt{2}} &= \text{span}\{ v_{\pm 1} = e_{14} - \epsilon e_{27} + \epsilon e_{36} + e_{58} + (\epsilon + \eta \sqrt{2})(e_{23} - \epsilon e_{18} + \epsilon e_{45} + e_{67}), \\
\epsilon v_{\pm 1}^2 &= e_{25} - \epsilon e_{16} - e_{38} + e_{47} + (\epsilon + \eta \sqrt{2})(e_{12} + \epsilon e_{34} + e_{56} + e_{78})
\end{align*}

with \( v_{\pm 1} \xrightarrow{\sigma^2} \epsilon v_{\pm 1}^2 \xrightarrow{\sigma} v_{\pm 1}^2 \) such that \( \sigma^2 - \epsilon 1 = 0 \) is the minimal equation on \( V_{\epsilon + \eta \sqrt{2}} \).

\footnote{We use the short notation \( e_{ijkl} = e_i \wedge e_j \wedge e_k \wedge e_l \).}
Last but not least,

\begin{equation}
V_0 = \text{span}\{w_1 = e_{24} + e_{28} - e_{46} + e_{68}, w_2 = e_{13} + e_{17} - e_{35} + e_{57}, \\
e_{15}, e_{26}, e_{37}, e_{48}\}
\end{equation}

with \(e_{15} \xrightarrow{\sigma} e_{26} \xrightarrow{\sigma} e_{37} \xrightarrow{\sigma} e_{48} \xrightarrow{\sigma} -e_{15}\) and \(w_1 \xrightarrow{\sigma} -w_2 \xrightarrow{\sigma} -w_1\). I.e. \(\sigma^4 + 1 = 0\) and \(\sigma^2 + 1 = 0\) are the minimal equations on \(E = \text{span}\{e_{15}, e_{26}, e_{37}, e_{48}\}\) and \(W = \text{span}\{w_1, w_2\}\), respectively.

\([k = 3]\). On \(A^3 V\) the duality operator \(3b_\Omega\) has minimal polynomial

\begin{equation}
p(t) = t(t^2 - 4)(t^2 - 2)(t^4 - 14t^2 + 16)
\end{equation}

such that the eigenvalues are given by \(0, \pm 2, \pm \sqrt{2}, \) and \(\pm \sqrt{7} \pm \sqrt{2}\). Moreover, the eigenvalues of \(\sigma\) have multiplicities 7 each.

The respective eigenspaces and the action of \(\sigma\) are given as follows.

\begin{equation}
V_{1,2} = \text{span}\{w_1^\pm = e_{237} - e_{125} - e_{156} + e_{367} \pm (e_{138} - e_{143} + e_{457} - e_{578}), \\
w_2^\pm = e_{348} - e_{236} - e_{267} + e_{478} \pm (e_{124} - e_{168} - e_{245} + e_{568}), \\
w_3^\pm = e_{145} + e_{158} - e_{347} - e_{378} \pm (e_{167} - e_{127} + e_{235} - e_{356}), \\
w_4^\pm = e_{126} - e_{148} - e_{256} - e_{458} \pm (e_{278} - e_{238} - e_{346} - e_{467}), \\
u_1^\pm = e_{257} - e_{213} - e_{136} + e_{657} \pm (e_{145} - e_{143} - e_{347} - e_{378}),  \\
u_2^\pm = e_{368} - e_{234} + e_{427} - e_{678} \pm (e_{126} - e_{148} - e_{256} + e_{458}), \\
u_3^\pm = e_{147} - e_{178} - e_{345} - e_{358} \pm (e_{156} - e_{125} + e_{237} - e_{367}), \\
u_4^\pm = e_{128} - e_{146} + e_{258} - e_{456} \pm (e_{267} - e_{236} + e_{348} - e_{478})\}
\end{equation}

This basis is well adapted in the way that \(w_1^\pm \xrightarrow{\sigma} w_2^\pm \xrightarrow{\sigma} w_3^\pm \xrightarrow{\sigma} w_4^\pm \xrightarrow{\sigma} -u_1^\pm,\) i.e. \(\sigma^4 - 1 = 0\) and \(\sigma^2 + 1 = 0\) are the respective minimal equations on \(W^\pm = \text{span}\{w_1^\pm\}\) and \(U^\pm = \text{span}\{u_1^\pm\}\).

For the zero eigenvalue we have

\begin{equation}
V_0 = \text{span}\{x_1 = e_{236} - e_{267} + e_{348} - e_{478}, x_2 = e_{145} - e_{158} + e_{347} - e_{478}, \\
x_3 = e_{256} - e_{126} - e_{148} + e_{458}, x_4 = e_{156} - e_{125} - e_{237} + e_{367}, \\
y_1 = e_{123} - e_{136} + e_{257} - e_{567}, y_2 = e_{234} - e_{247} + e_{368} - e_{678}, \\
y_3 = e_{147} - e_{178} + e_{345} - e_{358}, y_4 = e_{258} - e_{128} - e_{146} + e_{456}, \\
u_1 = e_{278} - e_{238} - e_{346} + e_{467}, u_2 = e_{138} - e_{134} - e_{457} + e_{578}, \\
u_3 = e_{124} + e_{168} - e_{245} - e_{568}, u_4 = e_{127} - e_{167} + e_{235} - e_{356}, \\
u_1 = e_{127} - e_{123} - e_{134} + e_{136} - e_{138} + e_{147} + e_{167} + e_{178}, \\
+ e_{235} - e_{257} + e_{345} + e_{356} + e_{358} - e_{457} - e_{567} - e_{678}, \\
u_2 = e_{128} - e_{124} + e_{146} - e_{168} - e_{234} + e_{238} - e_{245} - e_{247} + e_{258} + e_{346} - e_{368} + e_{456} + e_{467} - e_{568} - e_{678}, \\
u_1 = e_{127} - e_{123} + e_{134} - e_{136} - e_{138} - e_{147} + e_{167} - e_{178} + e_{235} - e_{257} - e_{345} + e_{356} - e_{358} + e_{457} - e_{567} + e_{578}, \\
u_2 = e_{124} - e_{128} - e_{146} + e_{168} - e_{234} + e_{238} + e_{245} - e_{247} - e_{258} + e_{346} - e_{368} + e_{456} + e_{467} + e_{568} - e_{678}\}
\end{equation}
Furthermore, the minimal equation is $\sigma^4 + 1 = 0$ on $X = \text{span}\{x_i\}$, $Y = \text{span}\{y_i\}$ and $U = \text{span}\{u_i\}$ as well as $\sigma^4 - 1 = 0$ on $V \oplus W$ for $V = \text{span}\{v_1, v_2\}$ and $W = \text{span}\{w_1, w_2\}$ - more precisely $\sigma^2 + 1 = 0$ on $V$ and $\sigma^2 - 1 = 0$ on $W^\pm = \text{span}\{w_1 \pm w_2\}$.

Furthermore,

$$V_{\pm \sqrt{2}} = \text{span}\{v_1^\pm = e_{168} - e_{124} - e_{245} + e_{568} \pm \sqrt{2}(e_{135} - e_{157}), \newline v_2^\pm = e_{127} + e_{167} - e_{235} - e_{356} \pm \sqrt{2}(e_{246} - e_{268}), \newline v_3^\pm = e_{238} + e_{278} - e_{346} - e_{467} \pm \sqrt{2}(e_{347} - e_{317}), \newline v_4^\pm = e_{134} + e_{138} - e_{457} - e_{578} \pm \sqrt{2}(e_{468} - e_{428})\}.$$  \hspace{1cm} (41)

This basis is chosen in such a way that $v_1^+ \xrightarrow{\sigma} v_2^+ \xrightarrow{\sigma} v_3^+ \xrightarrow{\sigma} v_4^+ \xrightarrow{\sigma} -v_1^+$.

Therefore, the minimal equation is $\sigma^4 + 1 = 0$ on both spaces.

Last but not least for $\beta \in \{ \pm \sqrt{2} \pm \sqrt{3} \}$ we have

$$V_\beta = \text{span}\{v_1^\beta = (e_{126} + e_{148} + e_{256} + e_{458}) + \frac{\beta}{4}(e_{238} + e_{278} + e_{346} + e_{467}) \newline + \frac{8 - \beta^2}{4}(e_{137} + e_{357}) + \frac{2}{\beta}(e_{243} + e_{678}) + \frac{\beta^2 - 4}{2\beta}(e_{247} + e_{368}), \newline v_2^\beta = (e_{125} + e_{156} + e_{237} + e_{367}) + \frac{\beta}{4}(e_{134} + e_{138} + e_{457} + e_{578}) \newline + \frac{8 - \beta^2}{4}(e_{248} + e_{468}) + \frac{2}{\beta}(e_{178} + e_{345}) + \frac{\beta^2 - 4}{2\beta}(e_{147} + e_{358}), \newline v_3^\beta = (e_{236} + e_{267} + e_{348} + e_{478}) + \frac{\beta}{4}(e_{124} + e_{168} + e_{245} + e_{568}) \newline + \frac{8 - \beta^2}{4}(e_{135} + e_{157}) + \frac{2}{\beta}(e_{128} + e_{456}) + \frac{\beta^2 - 4}{2\beta}(e_{146} + e_{258}), \newline v_4^\beta = (e_{145} + e_{158} + e_{347} + e_{378}) + \frac{\beta}{4}(e_{127} + e_{167} + e_{235} + e_{356}) \newline + \frac{8 - \beta^2}{4}(e_{246} + e_{268}) + \frac{2}{\beta}(e_{123} + e_{567}) + \frac{\beta^2 - 4}{2\beta}(e_{136} + e_{257})\}.$$  \hspace{1cm} (42)

This choice of basis obeys $v_1^\beta \xrightarrow{\sigma} v_2^\beta \xrightarrow{\sigma} v_3^\beta \xrightarrow{\sigma} v_4^\beta \xrightarrow{\sigma} -v_1^\beta$, such that $\sigma^4 - 1 = 0$ is the minimal equation for $\sigma$ on $V_\beta$.

$|k = 4|$. On $\Lambda^4 V$ the minimal polynomial of $6b_{13}$ is given by

$$p(t) = t(t^2 - 4)(t^2 - 16)(t^2 - 8)$$

and the eigenvalues $0$, $\pm 2$, $\pm 4$, and $\pm 2\sqrt{2}$ have multiplicities $26$, $16$, $4$ and $2$, respectively. Moreover, the multiplicities of the eigenvalues of $\sigma$ are $10$ for $\pm 1$, $9$ for $\pm i$, and $8$ for $\pm \frac{1}{\sqrt{2}} \pm \frac{i}{\sqrt{2}}$. We will list here the low dimensional eigenspaces and we will show, how $\Omega$ is related to the eigenvalues $\pm 2\sqrt{2}$.
The eigenspaces to the eigenvalues \( \pm 4 \) are given by
\begin{equation}
V_{\pm 4} = \text{span}\left\{ v_1^\pm = e_{1257} + e_{1356} + e_{2478} + e_{3468} \pm (e_{1347} - e_{1246} + e_{2568} - e_{3578}),
\quad v_2^\pm = e_{2368} + e_{2467} - e_{1358} - e_{1457} \mp (e_{1367} - e_{1468} + e_{2357} - e_{2458}),
\quad w_1^\pm = e_{1357} - e_{1458} - e_{2367} + e_{2468} \mp (e_{1368} + e_{1467} + e_{2358} + e_{2457}),
\quad w_2^\pm = e_{1256} - e_{1357} + e_{2468} - e_{3478} \pm (e_{1247} + e_{1346} - e_{2578} - e_{3568}) \right\}
\end{equation}
with \( v_1^\pm \to v_2^\pm \to \mp v_1^\pm \) and \( w_1^\pm \to -w_2^\pm \to -w_1^\pm \) such that the minimal equation of \( \sigma \) is \( \sigma^2 + 1 = 0 \) on \( V_4 \), and \( \sigma^4 - 1 = 0 \) on \( V_{-4} \).

The eigenspaces to the eigenvalues \( \pm 2\sqrt{2} \) are given by
\begin{equation}
V_{\pm 2\sqrt{2}} = \text{span}\left\{ u_1^\pm = e_{2345} - e_{1238} - e_{1678} + e_{4567} \pm \sqrt{2}(e_{2367} - e_{1458}),
\quad u_2^\pm = e_{1234} + e_{1278} + e_{3456} + e_{5678} \pm \sqrt{2}(e_{1256} - e_{3478}) \right\}
\end{equation}
with \( u_1^\pm \to u_2^\pm \to u_1^\pm \) such that \( \sigma \) has eigenvalues \( \pm 1 \) on \( V_{\pm 2\sqrt{2}} \).

**Remark 4.1.** The two-dimensional \( +1 \)-eigenspace of \( \sigma \) within \( V_{2\sqrt{2}} \oplus V_{-2\sqrt{2}} \) is given by \( \text{span}\{ \Omega, \omega \} \) where
\begin{equation}
\begin{aligned}
\Omega &= \frac{1}{2}(u_1^+ + u_1^- + u_2^+ + u_2^-), \\
\omega &= \frac{1}{2}(u_1^+ - u_1^- + u_2^+ - u_2^-).
\end{aligned}
\end{equation}

These forms fulfill \( b_{\Omega}(\Omega) = \frac{\sqrt{2}}{3}\omega \) and \( b_{\Omega}(\omega) = \frac{\sqrt{2}}{3}\Omega \). In particular, \( \Omega \) itself is not an eigenform with respect to \( b_{\Omega} \), in contrast to the discussion following Definition 3.1.

We conclude this example by adding some comments on the eigenspaces of \( b_{\Omega} \) to the remaining eigenvalues \( 0 \) and \( \pm 2 \) which we as usual denote by \( V_0 \) and \( V_{\pm 2} \). This explains the so far unusual asymmetry in the behavior of \( \sigma \) on \( V_{\pm 4} \).

The map \( \sigma \) acting \( V_0 \) has eigenvalues \( \pm \frac{1}{\sqrt{5}} \pm \frac{1}{\sqrt{2}} \) with multiplicity 4, \( \pm i \) with multiplicity 3, as well as \( \pm 1 \) with multiplicity 2. Restricted to \( V_{\pm 2} \) the eight eigenvalues of \( \sigma \) come with multiplicity 2, each.

### 5. Outlook

The duality operator we defined here in flat space can be defined in the same way on a Riemannian or semi-Riemannian manifold. In particular, all that has been discussed for \( g \)-invariant duality operators can be transferred to manifolds with a \( g \)-structure. In this case the \( g \)-invariant differential form \( \Omega \in \Omega^1(M) \) is parallel with respect to a connection associated to the given \( g \)-structure.

One application of our duality relations may be the following. Let the manifold under consideration be spin, and take a connection on the spinor bundle \( S \) on \( M \). This connection and its curvature are locally described by elements in the exterior algebra of \( M \), the so called \( k \)-form potentials and fluxes; see for example [14, 15, 16]. The duality relation presented here may be a candidate to generalize the duality for metric connections on the base manifold \( M \).
APPENDIX A. USEFUL DECOMPOSITIONS

We are interested in the decomposition of certain tensor products of irreducible representations of \( \mathfrak{so}(n) \). We recall the decomposition of the tensor product of anti-symmetric powers of \( V = \mathbb{R}^n \) into irreducible \( \mathfrak{gl}(n) \)-modules. Let \( k, \ell \leq \frac{n}{2} \) then

\[
\Lambda^\ell V \otimes \Lambda^k V = \bigoplus_{i=0}^{\min\{k, \ell\}} [k + \ell - i, i].
\]

Here \([k + \ell - i, i]\) denotes the irreducible representation space of weight \( e_i + e_{k+\ell-i} \).

With respect to \( \mathfrak{so}(n) \) these spaces are reducible for \( i \neq 0 \). The irreducible components are obtained by contraction with the metric. If we denote the trace free parts by \([\cdot, \cdot]_0\) we get

\[
[k + \ell - i, i] = \bigoplus_{j=0}^{i} [k + \ell - i - j, i - j]_0
\]

which yields the final \( \mathfrak{so}(n) \)-decomposition

\[
\Lambda^\ell V \otimes \Lambda^k V = \bigoplus_{i=0}^{\min\{k, \ell\}} \bigoplus_{j=0}^{i} [k + \ell - i - j, i - j]_0.
\]

Due to Hodge duality the preceding formula can be used for, say, \( \ell > \frac{n}{2} \), too, we only have to insert \( \Lambda^{n-\ell} V \approx \Lambda^\ell \) instead. For a more systematic treatment of such decompositions we refer the reader to the nice article \([17]\).

By \( \pi_m \) we denote the projection \( \Lambda^k V \otimes \Lambda^\ell V \rightarrow \Lambda^m V = [m, 0] \).

We are in particular interested in the second symmetric power of \( \Lambda^k V \). With the above notation for \( k = \ell \) we have the following \( \mathfrak{so}(n) \)-decomposition

\[
S^2(\Lambda^k V) = \bigoplus_{j=0}^{\lfloor \frac{k}{2} \rfloor} \left( \bigoplus_{i=0}^{k-2j} [k + 2j - i, k - 2j - i]_0 \right)
\]

\[
= \bigoplus_{j=0}^{\lfloor \frac{k}{2} \rfloor} \left( \bigoplus_{i=0}^{k-2j-1} [k + 2j - i, k - 2j - i]_0 \right) \oplus \bigoplus_{j=0}^{\lfloor \frac{k}{2} \rfloor} \Lambda^{2j} V.
\]

In particular \( \Lambda^\ell V \subset S^2(\Lambda^k V) \) only if \( \ell \equiv 0 \mod 4 \).

APPENDIX B. SOME CALCULATIONS

In this appendix we add the calculations for equations (14) to (17) that we left out in Lemma 3.3 as well as the calculations for Lemma 3.7 and Proposition 3.8.

B.1. Calculations for Lemma 3.3. We recall the content of Lemma 3.3: The duality map \( b_\Theta^\ell(F)_{ijkl} = \Theta^\ell_{ij} \Theta^{\ell p q r s} F_{ijkl} \) given by \( b_\Theta(F)_{ijkl} = \Theta^\ell_{ij} \Theta^{\ell p q r s} F_{ijkl} \), obeys

\[
b_\Theta^2(F)_{ijkl} = \frac{1}{6} \Theta^m_{ij} \Theta^{mpq}_{kl} F_{mnop} + \frac{2}{3} \delta_{ijkl} - \frac{8}{3} b_\Theta(F)_{ijkl}
\]

\[
b_\Theta^3(F)_{ijkl} = \frac{1}{4} \delta_{ijkl} + \frac{3}{2} b_\Theta(F)_{ijkl} - \frac{12}{3} b_\Theta^2(F)_{ijkl} + \frac{4}{3} \Theta_{ijkl} \Theta^{prsn} F_{prsn}
\]

\[
b_\Theta^4(F)_{ijkl} = 4 b_\Theta(F)_{ijkl} - \frac{8}{3} b_\Theta^2(F)_{ijkl} - \frac{16}{3} b_\Theta^3(F)_{ijkl} + \frac{1}{3} \Theta_{ijkl} \Theta^{prsn} F_{prsn}
\]
(17) \[ b_\Theta^2(F)_{ijkl} = -\frac{2a}{3}b_\Theta^4(F)_{ijkl} - 20b_\Theta^6(F)_{ijkl} - \frac{2a}{3}b_\Theta^2(F)_{ijkl} + 16b_\Theta(F)_{ijkl} \]

To get (14) we calculate

\[ b_\Theta(F)_{ijkl} = \Theta^{mn}_{[ij}b_\Theta(F)_{kl]mn} \]

\[ = \Theta^{mn}_{[ij}c^{abcd}_{kl]mn} \Theta^{op}_{ab}F_{cdop} \]

\[ = \frac{1}{6}\Theta^{mn}_{[ij}c^{abcd}_{kl]mn} \Theta^{op}_{ab}F_{cdop} + \frac{1}{6}\Theta^{mn}_{[ij}c^{ad}_{kl]mn} \Theta^{op}_{ab}F_{cdop} - \frac{2}{3}\Theta^{mn}_{[ij}c^{bd}_{kl]mn} \Theta^{op}_{ab}F_{cdop} \]

\[ = \frac{1}{6}\Theta^{mn}_{[ij}c^{op}_{kl]F_{mnop} + \frac{1}{6}\left(12\delta^{op}_{[ij} - 4\Theta^{op}_{[ij}}F_{kl]op \right) \]

\[ + \frac{1}{3}f_{ijkl} \left(6\delta^{op}_{[ij} - 9\Theta^{[op}_{[ij} \Theta^{[op}_{]j'] F_{k'l} \right) \]

\[ + 4\delta^{ij}k'k'' \left(3\delta^{op}_{n'j'} + 2\delta^{op}_{n'j'} \right) \]

\[ - 6\delta^{ij}k'k'' \left(\frac{1}{2} \cdot \frac{1}{2} \Theta^{ij} \delta^{kn}_{n'} + \frac{1}{2} \cdot \frac{1}{2} \Theta_{ij} \delta^{kn}_{n'} + \frac{1}{2} \cdot \frac{1}{2} \Theta_{ij} \delta^{kn}_{n'} + \frac{1}{2} \cdot \frac{1}{2} \Theta_{ij} \delta^{kn}_{n'} \right) \]

\[ + \frac{2}{3} \cdot \frac{1}{3} \Theta_{ij} \delta^{kn}_{n'} \]
To evaluate \( b^3 \), we need the image of \( \Theta^{-1} F_{ij} \) under \( b_\Theta \), i.e. (16), we need the image of \( \Theta^{-1} F_{ij} \) under \( b_\Theta \),.

\[
\Theta_{ijkl}^{mn} \delta_{abcd}^{efgh} \Theta_{abc}^{opq} \Theta_{pqrs}^{efgh} F_{oprs} = \frac{1}{4} \Theta_{ijkl}^{mn} \delta_{c}^{efgh} \Theta_{abc}^{opq} \Theta_{pqrs}^{efgh} F_{oprs} - \frac{3}{4} \Theta_{ijkl}^{mn} \delta_{c}^{efgh} \Theta_{abc}^{opq} \Theta_{pqrs}^{efgh} F_{oprs}
\]

So we get for the third power of \( b_\Theta \)

\[
b^3_\Theta(F)_{ijkl} = b_\Theta(b^2_\Theta(F))_{ijkl} = \frac{4}{3} b_\Theta(F)_{ijkl} - \frac{8}{3} b_\Theta^2(F)_{ijkl} + \frac{8}{9} F_{ijkl} + \frac{16}{9} b_\Theta(F)_{ijkl} - \frac{2}{3} b_\Theta(F)_{ijkl} + \frac{8}{9} b_\Theta(F)_{ijkl} + \frac{4}{3} \Theta_{ijkl}^{mn} \Theta_{pqrs}^{efgh} F_{oprs}.
\]
This is
\[
\begin{align*}
b^4_\Theta(F)_{ijkl} &= b_\Theta(b^3_\Theta(F))_{ijkl} \\
&= \frac{1}{4} b_\Theta(F)_{ijkl} + \frac{2}{3} b^2_\Theta(F)_{ijkl} - \frac{10}{3} b^3_\Theta(F)_{ijkl} \\
&\quad + \frac{4}{3} \left( - \Theta_{ijkl} \Theta^{qrst} F_{oprs} + \frac{3}{2} \Theta_{ijkl} \Theta^{prsn} F_{prsn} - 6 b_\Theta(F)_{ijkl} \right) \\
&= - \frac{4}{3} F_{ijkl} + \frac{10}{3} b_\Theta(F)_{ijkl} + \frac{3}{2} b^2_\Theta(F)_{ijkl} - \frac{10}{3} b^3_\Theta(F)_{ijkl} \\
&\quad - \left( b^3_\Theta(F)_{ijkl} - \frac{4}{3} F_{ijkl} - \frac{2}{3} b_\Theta(F)_{ijkl} + \frac{10}{3} b^3_\Theta(F)_{ijkl} \right) \\
&\quad + \frac{4}{3} \Theta_{ijkl} \Theta^{prsn} F_{prsn} \\
&= 4 b_\Theta(F)_{ijkl} - \frac{8}{3} b^3_\Theta(F)_{ijkl} - \frac{12}{3} b^3_\Theta(F)_{ijkl} - \frac{1}{3} \Theta_{ijkl} \Theta^{prsn} F_{prsn} + \frac{4}{9} \Theta_{ijkl} \Theta^{prsn} F_{prsn}.
\end{align*}
\]

The last step is easy. For \( b^5_\Theta \) we need the image of \( \Theta_{ijkl} \Theta^{prsn} F_{prsn} \). This is a multiple of \( \Theta_{ijkl} \) for which we have \( b_\Theta(\Theta)_{ijkl} = \Theta_{mn[ij} \Theta^{mnkl]} = -4 \Theta_{ijkl} \). This yields
\[
\begin{align*}
b^5_\Theta(F)_{ijkl} &= b_\Theta(b^4_\Theta(F))_{ijkl} \\
&= 4 b^4_\Theta(F)_{ijkl} - \frac{8}{3} b^3_\Theta(F)_{ijkl} - \frac{12}{3} b^3_\Theta(F)_{ijkl} - \frac{1}{3} \Theta_{ijkl} \Theta^{prsn} F_{prsn} - \frac{4}{9} \Theta_{ijkl} \Theta^{prsn} F_{prsn}.
\end{align*}
\]

B.2. Calculations for Lemma 3.7 and Proposition 3.8. The proof of Lemma 3.7 is a straightforward calculation. The maps
\[
\begin{align*}
d_\Theta &: \Lambda^5 \mathbb{R}^8 \to \Lambda^3 \mathbb{R}^8, \\
d_\Theta(F)_{lmn} &= \Theta_{ijkl} F^{ijk}_{mn}, \\
\check{d}_\Theta &: \Lambda^3 \mathbb{R}^8 \to \Lambda^5 \mathbb{R}^8, \\
\check{d}_\Theta(F)_{jklnmn} &= \Theta_{ijkl} F^{ijk}_{lnmn}.
\end{align*}
\]
are isomorphisms and connected to \( b_\Theta \) and to the Hodge operator via
\[
\begin{align*}
d_\Theta \circ \check{d}_\Theta &= -\frac{6}{5} id + \frac{3}{2} b_\Theta(F), \\
* d_\Theta * &= -20 \check{d}_\Theta.
\end{align*}
\]
A consequence of this is \( \check{d}_\Theta \circ d_\Theta = -* d_\Theta \circ \check{d}_\Theta, * \).

We make use of (13) and get
\[
\begin{align*}
d_\Theta \check{d}_\Theta(F)_{lmn} &= \Theta_{ijkl} \check{d}_\Theta(F)_{ijkl} \circ_{mn} \\
&= \delta_{lmn}^{abc} \Theta_{ijkl} \check{d}_\Theta(F)_{ijkl} \\
&= \delta_{lmn}^{abc} \left( \frac{1}{10} \Theta_{ijkl} F^{sbc} + \frac{3}{10} \Theta_{ijkl} \Theta_{sbcj} F^{s} \right) \\
&\quad - \frac{1}{10} \Theta_{ijkl} \Theta_{sbcj} F^{s} \\
&= - \frac{21}{5} F_{lmn} - \frac{3}{5} \delta_{lmn}^{abc} \left( - 6 \delta_{mnk} \delta_{s} - 4 \Theta_{sblk} \right) F^{sk}_{n} \\
&\quad - \frac{3}{5} \delta_{lmn}^{abc} \left( \delta_{bc} \delta_{sa} - 4 \Theta_{a} j_{s} \delta_{c} \delta_{sa} - \Theta_{sckj} \delta_{sa} \right) F_{sk}_{j} \\
&= - \frac{21}{5} F_{lmn} - \frac{3}{5} F_{lmn} - \frac{6}{5} \Theta_{j} k_{lm} F_{j} k_{n} + \frac{3}{10} \Theta_{j} k_{lm} F_{j} k_{n} \\
&\quad + \frac{18}{5} F_{lmn} + \frac{12}{5} \Theta_{j} k_{lm} F_{j} k_{n} \\
&= - \frac{4}{5} F_{lmn} + \frac{2}{5} b_\Theta(F)_{lmn}.
\end{align*}
\]
Furthermore we have

\[ s d\Theta(F)_{ijklm} = -\frac{1}{6} \epsilon_{ijklmnop} s \Theta(F)^{nop} = -\frac{1}{6} \epsilon_{ijklmnop} \Theta_{abc} \ast (sF)^{bcop} \]

\[ = \frac{1}{36} \epsilon_{ijklmnop} \epsilon_{abcopqrst} \Theta_{abc} \ast F_{rst} = 40 \delta_{abcopqrst} \Theta_{abc} \ast F_{rst} \]

\[ = 20 \delta_{ijklmnop} \epsilon_{n} \Theta_{abc} \ast F_{rst} = -20 \Theta_{n[ijk} F_{lm]n} \]

\[ = -20 d\Theta(F)_{ijklm} \]

The proof of Proposition 3.8 is divided into three cases \( b_{\hat{\Theta}}(F) \) where we consider \( \hat{F} = F^5, \hat{F} = F^3 \wedge V, \) and \( \hat{F} = F^3 \wedge \epsilon \) with \( F^k \in \Lambda^k \mathbb{R}^8 \) and \( V \in \mathbb{R}^2, \) separately.

First, we consider \( \hat{F} = F^5 \) and get

\[ b_{\hat{\Theta}}(F)_{lmnop} = \hat{\Theta}_{ijklmn} F^{ijkl} \]

\[ = 15 \Theta_{ijklmn} \epsilon_{ijklmnop} = 3 \Theta_{ijklmn} \epsilon_{ijklmnop} \]

\[ = 3b_{\Theta}(F)_{lmnop} = \frac{3}{10} (d\Theta(F) \wedge \epsilon)_{lmnop} \]

Second, we insert \( \hat{F} = F \wedge \epsilon \) which yields

\[ b_{\hat{\Theta}}(F \wedge \epsilon)_{lmnop} = \hat{\Theta}_{ijklmn} (F \wedge \epsilon)_{ijklmnop} \]

\[ = 15 \Theta_{ijklmn} \epsilon_{ijklmnop} \]

\[ = 30 (\Theta_{ijklmn} \epsilon_{ijklmnop} + 3 \Theta_{ijklmnop} \epsilon_{ijklmnop}) \]

\[ = 3 \Theta_{ijklmn} \epsilon_{ijklmnop} + 54 \Theta_{ijklmnop} \epsilon_{ijklmnop} \]

\[ = 3 \Theta_{ijklmnop} \epsilon_{ijklmnop} + 9 \Theta_{ijklmnop} \epsilon_{ijklmnop} \]

\[ = -3 \Theta_{ijklmnop} \epsilon_{ijklmnop} + 6 \hat{d}\Theta(F)_{lmnop} \]

\[ = 6 (d\Theta \otimes \ast)(F \wedge \epsilon)_{lmnop} \]

Last but not least, for \( \hat{F} = F \wedge V \) we get

\[ b_{\hat{\Theta}}(F \wedge V)_{lmnop} = \hat{\Theta}_{ijklmn} (F \wedge V)_{ijklmnop} \]

\[ = 15 \Theta_{ijklmn} \epsilon_{ijklmnop} \]

\[ = 15 (\Theta_{ijklmnop} \epsilon_{ijklmnop} + 3 \Theta_{ijklmnop} \epsilon_{ijklmnop}) \]

\[ = 54 \Theta_{ijklmnop} \epsilon_{ijklmnop} \]

\[ = 9 \Theta_{ijklmnop} \epsilon_{ijklmnop} + 9 \theta_{ijklmnop} \epsilon_{ijklmnop} \]

\[ = \frac{9}{2} (d\Theta \otimes \ast)(F \wedge V)_{lmnop} \]

The result on the eigenspaces and eigenvalues may be checked by applying \( b_{\hat{\Theta}} \) and using Lemma 24.

References

[1] D.V. Alekseevsky, V. Cortes, and C. Devchand. Partially-flat gauge fields on manifolds of dimension greater than four. hep-th/0205030, 2002.

[2] D.V. Alekseevsky, V. Cortes, and C. Devchand. Yang-Mills connections over manifolds with Grassmann structure. J. Math. Phys. 44 (2003), no. 12, 6047-6076.

[3] M. F. Atiyah, N. J. Hitchin, and I. M. Singer. Self-duality in four-dimensional Riemannian geometry. Proc. Roy. Soc. London Ser. A 362 (1978), no. 1711, 425-461.
[4] Laurent Baulieu, Hiroaki Kanno, and I. M. Singer. Special quantum field theories in eight and other dimensions. Commun. Math. Phys. 194 (1998), no. 1, 149-175.
[5] Laurent Baulieu and Céline Laroche. On generalized self-duality equations towards supersymmetric quantum field theories of forms. Modern Phys. Lett. A 13 (1998), no. 14, 1115-1132.
[6] Robert L. Bryant. Metrics with exceptional holonomy. Ann. Math. 126 (1987), no. 3, 525-576.
[7] E. Corrigan, P. Goddard, and A. Kent. Some comments on the ADHM construction in 4k dimensions. Commun. Math. Phys. 100 (1985), no. 1, 1-13.
[8] E. Corrigan, C. Devchand, D. B. Fairlie, and J. Nuys. First-order equations for gauge fields in spaces of dimension greater than four. Nucl. Phys. B 214 (1983), 452-464.
[9] Y. Brihaye, C. Devchand, and J. Nuys. Self-duality for eight-dimensional gauge theories. Phys. Rev. D (3) 32 (1985), no. 4, 990-994.
[10] C. Devchand, J. Nuys, and G. Weingart. Matryoshka of Special Democratic Forms. Comm. Math. Phys. 293 (2010), no. 2, 545-562.
[11] M. Fernandez. A classification of Riemannian manifolds with structure group Spin(7). Ann. Mat. Pura Appl. (4) 143 (1986), no. 1, 101-122.
[12] R. B. Brown and A. Gray. Vector cross products. Comment. Math. Helv. 42 (1967), 222-236.
[13] Jerome P. Gauntlett, Jan B. Gutowski, and Stathis Papadopoulos. The geometry of D = 11 null Killing spinors. J. High Energy Phys. 2003, no. 12, 049, 29 pp.
[14] U. Gran, J. Gutowski, G. Papadopoulos, and D. Roest. Aspects of spinorial geometry. Modern Phys. Lett. A 22 (2007), no. 1, 116.
[15] F. Klinker. The torsion of spinor connection and related structures. SIGMA Symmetry Integrability Geom. Methods Appl. 2 (2006), Paper 077, 28 pp.
[16] F. Klinker. SUSY structures on deformed supermanifolds. Differential Geom. Appl. 26 (2008), 566-582.
[17] Kazuhiko Koike and Itaru Terada. Young-Diagrammatic Methods for the Representation Theory of the Classical Groups of type $B_n$, $C_n$, $D_n$. J. Algebra 107 (1987), no. 2, 466-511.
[18] Dimitros Tsimpis. M-theory on eight-manifolds revisited: $N = 1$ supersymmetry and generalized Spin(7) structures. J. High Energy Phys. 2006, no. 4, 027, 26 pp.
[19] R. S. Ward. Completely solvable gauge-field equations in dimensions greater than four. Nucl. Phys. B236 (1984), 381-396.

Faculty of Mathematics, TU Dortmund University, 44221 Dortmund, Germany

E-mail address: frank.klinker@math.tu-dortmund.de