Logical Characterizations of Fuzzy Bisimulations in Fuzzy Modal Logics over Residuated Lattices

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Abstract

There are two kinds of bisimulation, namely crisp and fuzzy, between fuzzy structures such as fuzzy automata, fuzzy labeled transition systems, fuzzy Kripke models and fuzzy interpretations in description logics. Fuzzy bisimulations between fuzzy automata over a complete residuated lattice have been introduced by Ćirić et al. in 2012. Logical characterizations of fuzzy bisimulations between fuzzy Kripke models (respectively, fuzzy interpretations in description logics) over the residuated lattice $[0,1]$ with the Gödel t-norm have been provided by Fan in 2015 (respectively, Nguyen et al. in 2020). There was the lack of logical characterizations of fuzzy bisimulations between fuzzy graph-based structures over a general residuated lattice, as well as over the residuated lattice $[0,1]$ with the Łukasiewicz or product t-norm. In this article, we provide and prove logical characterizations of fuzzy bisimulations in fuzzy modal logics over residuated lattices. The considered logics are the fuzzy propositional dynamic logic and its fragments. Our logical characterizations concern invariance of formulas under fuzzy bisimulations and the Hennessy-Milner property of fuzzy bisimulations. They can be reformulated for other fuzzy structures such as fuzzy labeled transition systems and fuzzy interpretations in description logics.

Keywords: bisimulation, fuzzy bisimulation, fuzzy modal logic, residuated lattice

1. Introduction

Bisimulation is a useful notion for characterizing equivalence of states in transition systems [27, 17]. It has been extensively studied in modal logic for characterizing logical indiscernibility of states and separating the expressive power of modal logics (see, e.g., [1, 2, 14]). Bisimulation can be used for minimizing structures. It can also be exploited for studying indiscernibility of individuals and concept learning in description logics [23].

To deal with vagueness, fuzzy structures are used instead of crisp ones. There are two kinds of bisimulation, namely crisp and fuzzy, between fuzzy structures such as fuzzy automata, fuzzy transition systems, fuzzy Kripke models and fuzzy interpretations in description logics. Researchers have studied crisp bisimulations for fuzzy transition systems [1, 5, 31, 28, 29], weighted automata [8], Heyting-valued modal logics [10], Gödel modal logics [12] and fuzzy description logics [21]. They
have also studied fuzzy bisimulations for fuzzy automata \cite{6, 7}, weighted/fuzzy social networks \cite{11, 18}, Gödel modal logics \cite{12} and fuzzy description logics \cite{21, 22, 24}.

This article is devoted to study logical characterizations of fuzzy bisimulations. In \cite{12} Fan introduced fuzzy bisimulations between fuzzy Kripke models over the lattice $[0, 1]$ using the Gödel semantics (i.e., the Gödel t-norm and its residuum). She provided logical characterizations of such bisimulations in the basic fuzzy monomodal logic and its extension with converse. The results concern invariance of modal formulas under fuzzy bisimulations and the Hennessy-Milner property of fuzzy bisimulations. In \cite{11} Fan and Liau studied fuzzy bisimulations under the name “regular equivalence relations” for weighted social networks. They provided logical characterizations for such bisimulations under the Gödel semantics, including invariance results and the Hennessy-Milner property. In \cite{21} Nguyen et al. defined and studied fuzzy bisimulations for a large class of fuzzy description logics under the Gödel semantics. The work \cite{21} contains results on invariance of concepts under such bisimulations and the Hennessy-Milner property of such bisimulations. In \cite{10} Eleftheriou et al. studied bisimulations for Heyting-valued modal logics. Bisimulations defined in \cite{10} are crisp and cut-based (i.e., using fuzzy values as thresholds). As discussed by Fan \cite{12}, such bisimulations give another representation of fuzzy bisimulations. The work \cite{10} contains results on logical characterizations of the studied bisimulations, including the Hennessy-Milner property.

Note that the results on fuzzy bisimulations of all the works \cite{11, 12, 21} are formulated and proved only for fuzzy structures over the lattice $[0, 1]$ using the Gödel semantics. The results of \cite{10} concern only modal logics over Heyting algebras of truth values. Such algebras are residuated lattices that use $\otimes = \wedge$, and therefore, are closely related to the Gödel semantics. There was the lack of logical characterizations of fuzzy bisimulations between fuzzy graph-based structures over a general residuated lattice, as well as over the residuated lattice $[0, 1]$ with the Lukasiewicz or product t-norm.

In this article, we provide and prove logical characterizations of fuzzy bisimulations in fuzzy modal logics over general residuated lattices. The considered logics are the fuzzy propositional dynamic logic and its fragments. Our logical characterizations concern invariance of formulas under fuzzy bisimulations and the Hennessy-Milner property of fuzzy bisimulations. Our results are significant from the theoretical point of view, as they solve the problem stated in the last sentence of the above paragraph. They would also have an impact on practical applications, e.g. for studying logical similarity of individuals and concept learning in fuzzy description logics, as they can be reformulated for other fuzzy structures such as fuzzy interpretations in description logics and fuzzy labeled transition systems, and moreover, residuated lattices cover the lattice $[0, 1]$ with any t-norm, including the product and Lukasiewicz t-norms.

The rest of this article is structured as follows. Section \ref{preliminaries} contains preliminaries for this work. In Section \ref{fuzzy bisimulations}, we define fuzzy bisimulations between fuzzy Kripke models and provide some of their basic properties. Sections \ref{invariance} and \ref{H-M property} contain our results on invariance of formulas under fuzzy bisimulations and the Hennessy-Milner property of fuzzy bisimulations, respectively. Section \ref{related work} contains a discussion on related work. Conclusions are given in Section \ref{conclusion}. The work also contains two appendices: the first one is the proof of a lemma, whereas the second one is a discussion on the relationship with fuzzy bisimulations between fuzzy automata \cite{6}. 

2
2. Preliminaries

In this section, we recall definitions and properties of residuated lattices and fuzzy sets, then present the syntax and semantics of the fuzzy modal logics considered in this article, together with some related notions.

2.1. Residuated Lattices and Fuzzy Sets

A residuated lattice \([15, 3]\) is an algebra \(\mathcal{L} = (L, \leq, \otimes, \Rightarrow, 0, 1)\) such that

- \((L, \leq, 0, 1)\) is a bounded lattice with the least element 0 and the greatest element 1,
- \((L, \otimes, 1)\) is a commutative monoid,
- \(\otimes\) and \(\Rightarrow\) form an adjoint pair, which means that, for every \(x, y, z \in L\),

\[
x \otimes y \leq z \iff x \leq (y \Rightarrow z).
\]  

The expression \(y \Rightarrow z\) is called the residual of \(z\) by \(y\). Given a residuated lattice \(\mathcal{L} = (L, \leq, \otimes, \Rightarrow, 0, 1)\), let \(\land\) and \(\lor\) denote the join and meet operators associated with the lattice. By \(x \equiv y\) we denote \((x \Rightarrow y) \land (y \Rightarrow x)\). We use the convention that \(\otimes\) and \(\land\) bind stronger than \(\lor\), which in turn binds stronger than \(\Rightarrow\) and \(\equiv\).

A residuated lattice \(\mathcal{L} = (L, \leq, \otimes, \Rightarrow, 0, 1)\) is complete (resp. linear) if the bounded lattice \((L, \leq, 0, 1)\) is complete (resp. linear). It is a Heyting algebra if \(\otimes\) is the same as \(\land\).

We will need the following lemma. Although most assertions of this lemma are well-known \([3, 15]\) and the proof of this lemma is straightforward, we present the proof in Appendix A to make this article self-contained.

**Lemma 2.1 (cf. \([3, 15]\)).** Let \(\mathcal{L} = (L, \leq, \otimes, \Rightarrow, 0, 1)\) be a residuated lattice. The following properties hold for all \(x, x', y, y', z \in L\):

\[
\begin{align*}
x & \leq x' \text{ and } y \leq y' \text{ implies } x \otimes y \leq x' \otimes y' \quad (2) \\
x' \leq x \text{ and } y \leq y' \text{ implies } (x \Rightarrow y) \leq (x' \Rightarrow y') \quad (3) \\
x & \leq y \iff (x \Rightarrow y) = 1 \quad (4) \\
x \otimes 0 & = 0 \quad (5) \\
x \otimes (y \lor z) & = x \otimes y \lor x \otimes z \quad (6) \\
x \otimes (x \Rightarrow y) & \leq y \quad (7) \\
x \otimes (y \Rightarrow z) & \leq (x \Rightarrow y) \Rightarrow z \quad (8) \\
x \otimes (y \Leftarrow z) & \leq (x \Rightarrow y) \Rightarrow z \quad (9) \\
x \otimes (y \Leftarrow z) & \leq y \Leftarrow x \otimes z \quad (10) \\
x \Rightarrow (y \Rightarrow z) & = y \Rightarrow (x \Rightarrow z) \quad (11) \\
x \Rightarrow (y \Rightarrow z) & \leq x \otimes y \Rightarrow z \quad (12) \\
x \Rightarrow (y \Leftarrow z) & \leq x \otimes y \Rightarrow z \quad (13) \\
(x \Rightarrow y) \otimes (y \Rightarrow z) & \leq x \Rightarrow z \quad (14) \\
(x \Leftarrow y) \otimes (y \Leftarrow z) & \leq x \Leftarrow z \quad (15) \\
(x \Leftarrow x') \land (y \Leftarrow y') & \leq x \land y \Leftarrow x' \land y' \quad (16)
\end{align*}
\]
\[(x \Leftrightarrow x') \land (y \Leftrightarrow y') \leq x \lor y \Leftrightarrow x' \lor y' \]  
\[(x \Leftrightarrow y) \leq (z \Rightarrow x) \Leftrightarrow (z \Rightarrow y) \]  
\[(x \Leftrightarrow y) \leq (x \Rightarrow z) \Leftrightarrow (y \Rightarrow z). \]

In addition, if \( \mathcal{L} \) is a Heyting algebra, then the following properties hold for all \( x, x', y, y', z \in \mathcal{L} \):
\[(x \Leftrightarrow x') \land (y \Leftrightarrow y') \leq (x \Rightarrow y) \Leftrightarrow (x' \Rightarrow y') \]  
\[x \leq (y \Leftrightarrow z) \implies x \otimes y = x \otimes z. \]

**Example 2.2.** Consider the case when \( \mathcal{L} \) is the unit interval \([0, 1]\). The most well-known operators \( \otimes \) are the Gödel, Lukasiewicz and product t-norms. They are specified below together with their corresponding residua \((\Rightarrow)\).

|         | Gödel          | Lukasiewicz        | Product        |
|---------|----------------|--------------------|----------------|
| \( x \otimes y \) | \( \min\{x, y\} \) | \( \max\{0, x + y - 1\} \) | \( x \cdot y \) |
| \( x \Rightarrow y \) | \( \begin{cases} 1 & \text{if } x \leq y \\ y/x & \text{otherwise} \end{cases} \) | \( \min\{1, 1 - x + y\} \) | \( \begin{cases} 1 & \text{if } x \leq y \\ y/x & \text{otherwise} \end{cases} \) |

Note that, for all of these cases of \( \otimes \), the considered residuated lattice is linear and complete, and the operator \( \otimes \) is continuous.

From now on, let \( \mathcal{L} = (\mathcal{L}, \leq, \otimes, \Rightarrow, 0, 1) \) be an arbitrary residuated lattice.

Given a set \( X \), a function \( f : X \to \mathcal{L} \) is called a fuzzy set, as well as a fuzzy subset of \( X \). If \( f \) is a fuzzy subset of \( X \) and \( x \in X \), then \( f(x) \) means the fuzzy degree in which \( x \) belongs to the subset. For \( \{x_1, \ldots, x_n\} \subseteq X \) and \( \{a_1, \ldots, a_n\} \subseteq \mathcal{L} \), we write \( \{x_1 : a_1, \ldots, x_n : a_n\} \) to denote the fuzzy subset \( f \) of \( X \) such that \( f(x_i) = a_i \) for \( 1 \leq i \leq n \) and \( f(x) = 0 \) for \( x \notin X \setminus \{x_1, \ldots, x_n\} \).

Given fuzzy subsets \( f \) and \( g \) of \( X \), we write \( f \leq g \) to denote that \( f(x) \leq g(x) \) for all \( x \in X \). If \( f \leq g \), then we say that \( g \) is greater than or equal to \( f \).

A fuzzy subset of \( X \times Y \) is called a fuzzy relation between \( X \) and \( Y \). A fuzzy relation between \( X \) and itself is called a fuzzy relation on \( X \).

Given \( Z : X \times Y \to \mathcal{L} \), the converse \( Z^- : Y \times X \to \mathcal{L} \) of \( Z \) is defined by \( Z^-(y, x) = Z(x, y) \).

If the underlying residuated lattice is complete, then the composition of fuzzy relations \( Z_1 : X \times Y \to \mathcal{L} \) and \( Z_2 : Y \times Z \to \mathcal{L} \), denoted by \( Z_1 \circ Z_2 \), is defined to be the fuzzy relation between \( X \) and \( Z \) such that \( (Z_1 \circ Z_2)(x, z) = \sup\{Z_1(x, y) \otimes Z_2(y, z) \mid y \in Y\} \) for all \( (x, z) \in X \times Z \).

Let \( Z \) be a set of fuzzy relations between \( X \) and \( Y \). If \( Z \) is finite or the underlying residuated lattice is complete, then by \( \sup Z \) we denote the fuzzy relation between \( X \) and \( Y \) specified by: \( (\sup Z)(x, y) = \sup\{Z(x, y) \mid Z \in Z\} \) for \( (x, y) \in X \times Y \). We write \( Z_1 \cup Z_2 \) to denote \( \sup\{Z_1, Z_2\} \).

A fuzzy relation \( Z : X \times X \to \mathcal{L} \) is

- reflexive if \( Z(x, x) = 1 \) for all \( x \in X \),
- symmetric if \( Z(x, y) = Z(y, x) \) for all \( x, y \in X \),
- transitive if \( Z(x, y) \otimes Z(y, z) \leq Z(x, z) \) for all \( x, y, z \in X \).

It is a fuzzy equivalence relation if it is reflexive, symmetric and transitive.
2.2. Fuzzy Modal Logics

Let $\Sigma_A$ denote a non-empty set of actions, which are also called atomic programs, and let $\Sigma_P$ denote a non-empty set of propositions, which are also called atomic formulas. The pair $\langle \Sigma_A, \Sigma_P \rangle$ forms the signature for the fuzzy modal logics considered in this article.

Let $\Phi \subseteq \{\cup, \rightarrow, ?\}$. By $fPDL^\Phi$ we denote the fuzzy propositional dynamic logic without the union program constructor if $\cup$ belongs to $\Phi$, without the test operator if $?$ belongs to $\Phi$, and without the full version of implication if $\rightarrow$ belongs to $\Phi$.

In the following, an expression like $\cup /\notin \Phi$ can be read as "$\cup$ is not excluded".

Programs and formulas of $fPDL^\Phi$ over a residuated lattice $L = \langle L, \leq, \otimes, \Rightarrow, 0, 1 \rangle$ are defined as follows:

- if $\rho \in \Sigma_A$, then $\rho$ is a program of $fPDL^\Phi$,
- if $\alpha$ and $\beta$ are programs of $fPDL^\Phi$, then
  - $\alpha \circ \beta$ and $\alpha^*$ are programs of $fPDL^\Phi$,
  - if $\cup /\notin \Phi$, then $\alpha \cup \beta$ is a program of $fPDL^\Phi$,
  - if $? /\notin \Phi$ and $\varphi$ is a formula of $fPDL^\Phi$, then $\varphi ?$ is a program of $fPDL^\Phi$,
- if $a \in L$, then $a$ is a formula of $fPDL^\Phi$,
- if $p \in \Sigma_P$, then $p$ is a formula of $fPDL^\Phi$,
- if $\varphi$ and $\psi$ are formulas of $fPDL^\Phi$, $\alpha$ is a program of $fPDL^\Phi$ and $a \in L$, then $\varphi \land \psi$, $\varphi \lor \psi$, $a \rightarrow \varphi$, $\varphi \rightarrow a$, $\neg \varphi$, $[\alpha] \varphi$ and $\langle \alpha \rangle \varphi$ are formulas of $fPDL^\Phi$,
- if $\varphi$ and $\psi$ are formulas of $fPDL^\Phi$ and $\rightarrow /\notin \Phi$, then $\varphi \rightarrow \psi$ is a formula of $fPDL^\Phi$.

Note that, even when $\rightarrow /\in \Phi$, $fPDL^\Phi$ allows implications of the form $a \rightarrow \varphi$ or $\varphi \rightarrow a$ with $a \in L$. By $fPDL$ we denote $fPDL^\Phi$ with $\Phi = \emptyset$. By $fK^0$ we denote the largest sublanguage of $fPDL^\Phi$ with $\Phi = \{\cup, \rightarrow, ?\}$ that disallows the remaining program constructors ($\alpha \circ \beta$ and $\alpha^*$) and the formula constructors $\neg \varphi$, $\varphi \lor \psi$ and $[\alpha] \varphi$. That is, formulas of $fK^0$ are of the form $a, p, \varphi \land \psi$, $a \rightarrow \varphi$, $\varphi \rightarrow a$ or $\langle \rho \rangle \varphi$, where $a \in L$, $p \in \Sigma_P$, $\rho \in \Sigma_A$, and $\varphi$ and $\psi$ are formulas of $fK^0$.

We use letters like

- $\rho$ to denote actions from $\Sigma_A$,
- $p$ and $q$ to denote propositions from $\Sigma_P$,
- $a$ and $b$ to denote values from $L$,
- $\varphi$ and $\psi$ to denote formulas,
- $\alpha$ and $\beta$ to denote programs.

Given a finite set $\Gamma = \{\varphi_1, \ldots, \varphi_n\}$ with $n \geq 0$, we denote

$\bigwedge \Gamma = \varphi_1 \land \ldots \land \varphi_n \land 1$,

$\bigotimes \Gamma = \varphi_1 \otimes \ldots \otimes \varphi_n \otimes 1$. 
Definition 2.3. A fuzzy Kripke model over a signature \( \langle \Sigma_A, \Sigma_P \rangle \) and a residuated lattice \( \mathcal{L} = \langle L, \leq, \otimes, \Rightarrow, 0, 1 \rangle \) is a pair \( \mathcal{M} = \langle \Delta^\mathcal{M}, \cdot^\mathcal{M} \rangle \), where \( \Delta^\mathcal{M} \) is a non-empty set, called the domain, and \( \cdot^\mathcal{M} \) is the interpretation function that maps each \( p \in \Sigma_P \) to a fuzzy set \( p^\mathcal{M} : \Delta^\mathcal{M} \to L \) and maps each \( \varrho \in \Sigma_A \) to a fuzzy relation \( \varrho^\mathcal{M} : \Delta^\mathcal{M} \times \Delta^\mathcal{M} \to L \). The interpretation function is extended to complex programs and formulas as follows, under the condition that the used suprema and infima each \( \cdot \) is witnessed as defined shortly.

\[
(\varphi \land \psi)^\mathcal{M}(x,y) = (\text{if } x = y \text{ then } \varphi^\mathcal{M}(x) \text{ else } 0)
\]

\[
(\alpha \lor \beta)^\mathcal{M}(x,y) = \alpha^\mathcal{M}(x,y) \lor \beta^\mathcal{M}(x,y)
\]

\[
(\alpha \Rightarrow \beta)^\mathcal{M}(x,y) = \sup\{\alpha^\mathcal{M}(x,z) \otimes \beta^\mathcal{M}(z,y) \mid z \in \Delta^\mathcal{M}\}
\]

\[
(\alpha^*\mathcal{M}(x,y) = \sup\{\otimes\{\alpha^\mathcal{M}(x_i, x_{i+1}) \mid 0 \leq i < n\} \mid n \geq 0, x_0, \ldots, x_n \in \Delta^\mathcal{M}, x_0 = x, x_n = y\}
\]

\[
\alpha^\mathcal{M}(x) = a
\]

\[
(\varphi \land \psi)^\mathcal{M}(x) = \varphi^\mathcal{M}(x) \land \psi^\mathcal{M}(x)
\]

\[
(\varphi \lor \psi)^\mathcal{M}(x) = \varphi^\mathcal{M}(x) \lor \psi^\mathcal{M}(x)
\]

\[
(\varphi \Rightarrow \psi)^\mathcal{M}(x) = (\varphi^\mathcal{M}(x) \Rightarrow \psi^\mathcal{M}(x))
\]

\[
(\neg \varphi)^\mathcal{M}(x) = (\varphi \Rightarrow 0)^\mathcal{M}(x)
\]

\[
([\alpha] \varphi)^\mathcal{M}(x) = \inf\{\alpha^\mathcal{M}(x,y) \Rightarrow \varphi^\mathcal{M}(y) \mid y \in \Delta^\mathcal{M}\}
\]

\[
(([\alpha] \varphi)^\mathcal{M}(x) = \sup\{\alpha^\mathcal{M}(x,y) \otimes \varphi^\mathcal{M}(y) \mid y \in \Delta^\mathcal{M}\}.
\]

Example 2.4. Let \( \Sigma_A = \{ \varrho \} \), \( \Sigma_P = \{ p \} \) and let \( L \) be the unit interval \([0,1]\). Consider the fuzzy Kripke model \( \mathcal{M} \) specified by \( \Delta^\mathcal{M} = \{ u, v, w \} \), \( p^\mathcal{M} = \{ u : 0.9, v : 0.5, w : 0.8 \} \), \( \varrho^\mathcal{M} = \{ \langle u, v \rangle : 0.6, \langle u, w \rangle : 0.7 \} \) and depicted below:

\[
\begin{array}{c}
0.6 \quad 0.7 \\
\hline
u : 0.9 \\
v : 0.5 \\
w : 0.8
\end{array}
\]

The values \( \varphi^\mathcal{M}(u) \) for some example formulas \( \varphi \) using the Gődel, Lukasiewicz or product t-norm \( \otimes \) are given below:

| \( \langle \varrho \rangle p \) \( \cdot^\mathcal{M}(u) \) | Gödel | Lukasiewicz | Product |
|---|---|---|---|
| \( \langle \varrho \rangle p \) \( \cdot^\mathcal{M}(u) \) | 0.7 | 0.5 | 0.56 |
| \( \langle \varrho^* \rangle p \) \( \cdot^\mathcal{M}(u) \) | 0.5 | 0.9 | 5/6 |
| \( \langle \varrho^* \rangle p \) \( \cdot^\mathcal{M}(u) \) | 0.9 | 0.9 | 0.9 |
| \( \langle \varrho^* \rangle p \) \( \cdot^\mathcal{M}(u) \) | 0.5 | 0.9 | 5/6 |

A fuzzy Kripke model \( \mathcal{M} \) is witnessed w.r.t. \( fPDL^{-\Phi} \) if every infinite set under the infimum (resp. supremum) operator in Definition 2.3 has a smallest (resp. biggest) element when considering only formulas and programs of \( fPDL^{-\Phi} \) (cf. [10]). The notion of whether a fuzzy Kripke model \( \mathcal{M} \) is witnessed w.r.t. \( fK^0 \) is defined analogously by restricting to formulas and programs of \( fK^0 \).
A fuzzy Kripke model $M$ is image-finite if, for every $x \in \Delta^M$ and every $\varrho \in \Sigma_A$, the set $\{y \in \Delta^M | \varrho^M(x, y) > 0\}$ is finite. It is finite if $\Delta^M$ is finite.

Observe that every finite fuzzy Kripke model is witnessed w.r.t. $fPDL^\Phi$ (and hence also w.r.t. $fK^0$) and every image-finite fuzzy Kripke model is witnessed w.r.t. $fK^0$. If the underlying residuated lattice is finite, then all fuzzy Kripke models are witnessed w.r.t. $fPDL^\Phi$.

3. Fuzzy Bisimulations between Kripke Models

In this section, we define fuzzy bisimulations between fuzzy Kripke models, then state and prove some of their basic properties. The relationship with fuzzy bisimulations between fuzzy automata is presented in Appendix B.

**Definition 3.1.** Given fuzzy Kripke models $M$ and $M'$, a fuzzy relation $Z : \Delta^M \times \Delta^{M'} \to L$ is called a fuzzy bisimulation between $M$ and $M'$ if the following conditions hold for all $p \in \Sigma_P$, $\varrho \in \Sigma_A$ and all possible values for the free variables:

$$Z(x, x') \leq (p^M(x) \Leftrightarrow p^{M'}(x'))$$

$$\exists y' \in \Delta^{M'} (Z(x, x') \otimes \varrho^M(x, y) \leq \varrho^{M'}(x', y') \otimes Z(y, y'))$$

In this case, $Z$ is a fuzzy bisimulation if and only if:

$$\exists y \in \Delta^M (Z(x, x') \otimes p^M(x) \leq p^{M'}(x', y) \otimes Z(y, y'))$$

**Example 3.2.** Let $\Sigma_A = \{\varrho\}$, $\Sigma_P = \{p\}$ and $L = [0, 1]$. Consider the fuzzy Kripke models $M$ and $M'$ depicted and specified below.

![Diagram](image)

- $\Delta^M = \{u, v, w\}$, $\Delta^{M'} = \{u', v', w'\}$,
- $p^M = \{u:0, v:0.5, w:0.8\}$, $p^{M'} = \{u':0, v':0.5, w':0.8\}$,
- $\varrho^M = \{(u, v):0.6, (u, w):1\}$, $\varrho^{M'} = \{(u', v'):1, (u', w'):0.8\}$.

In the case when $\otimes$ is the Gödel, Lukasiewicz or product t-norm, the greatest fuzzy bisimulation $Z$ between $M$ and $M'$ can be computed as follows:

- $Z(v, v') = (0.5 \Leftrightarrow 0.5) = 1$, $Z(w, w') = (0.8 \Leftrightarrow 0.8) = 1$;
- $Z(v, w') = (0.5 \Leftrightarrow 0.8)$, $Z(w, v') = (0.8 \Leftrightarrow 0.5)$;
- $Z(v, u') \leq (0.5 \Leftrightarrow 0) = 0$, $Z(w, u') \leq (0.8 \Leftrightarrow 0) = 0$;
- $Z(u, v') \leq (0 \Leftrightarrow 0.5) = 0$, $Z(u, w') \leq (0 \Leftrightarrow 0.8) = 0$;
- $(\varrho^{M'} \circ Z)(u', v) = 1$, $(\varrho^{M'} \circ Z)(u', w) = \max\{0.8, (0.8 \Leftrightarrow 0.5)\} = 0.8$.

Thus, the condition only requires $Z(u, u') \leq 0.8$.\]
• \((g^M \circ Z)(u, w') = 1\), \((g^M \circ Z)(u, v') = \max\{0.6, (0.8 \Leftrightarrow 0.5)\}\)

thus, the condition \((27)\) only requires \(Z(u, u') \leq \max\{0.6, (0.8 \Leftrightarrow 0.5)\}\);

• therefore, \(Z(u, u') = \max\{0.6, (0.8 \Leftrightarrow 0.5)\}\).

That is,

• if \(\otimes\) is the Gödel t-norm, then

\[
Z = \{(u, u') : 0.6, \langle v, v' \rangle : 1, \langle w, w' \rangle : 1, \langle v, w' \rangle : 0.5, \langle w, v' \rangle : 0.5\};
\]

• if \(\otimes\) is the Lukasiewicz t-norm, then

\[
Z = \{(u, u') : 0.7, \langle v, v' \rangle : 1, \langle w, w' \rangle : 1, \langle v, w' \rangle : 0.7, \langle w, v' \rangle : 0.7\};
\]

• if \(\otimes\) is the product t-norm, then

\[
Z = \{(u, u') : 0.625, \langle v, v' \rangle : 1, \langle w, w' \rangle : 1, \langle v, w' \rangle : 0.625, \langle w, v' \rangle : 0.625\}.\]

\begin{proposition}
Suppose that the underlying residuated lattice is complete. If \(Z\) is a fuzzy bisimulation between fuzzy Kripke models \(M\) and \(M'\), then it satisfies the following conditions for all \(x \in \Delta^M\), \(x' \in \Delta^M'\) and \(\varrho \in \Sigma_A:\)

\[
Z(x, x') \leq \inf\{p^M(x) \Leftrightarrow p^M'(x') \mid p \in \Sigma_P\} \quad (25)
\]

\[
Z^{-} \circ g^M \leq g^M' \circ Z^{-} \quad (26)
\]

\[
Z \circ g^M' \leq g^M \circ Z. \quad (27)
\]

Conversely, if the underlying residuated lattice is also linear, fuzzy Kripke models \(M\) and \(M'\) are image-finite and \(Z : \Delta^M \times \Delta^M' \rightarrow L\) is a fuzzy relation satisfying the conditions \((24)\)–\((27)\), then \(Z\) is a fuzzy bisimulation between \(M\) and \(M'\).

\begin{proof}
Suppose that \(Z\) is a fuzzy bisimulation between fuzzy Kripke models \(M\) and \(M'\). Let \(x \in \Delta^M, x' \in \Delta^M'\) and \(\varrho \in \Sigma_A\). We show that \(Z\) satisfies the assertions \((25)\)–\((27)\).

The assertion \((25)\) holds because \((22)\) holds for all \(p \in \Sigma_P\).

Consider the assertion \((26)\). We need to prove that, for every \(x' \in \Delta^M'\) and \(y \in \Delta^M\),

\[
(Z^{-} \circ g^M)(x', y) \leq (g^M' \circ Z^{-})(x', y).
\]

It is sufficient to show that, for every \(x \in \Delta^M\),

\[
Z(x, x') \otimes g^M(x, y) \leq (g^M' \circ Z^{-})(x', y).
\]

This inequality follows from the condition \((23)\).

Consider the assertion \((27)\). We need to prove that, for every \(x \in \Delta^M\) and \(y' \in \Delta^M'\),

\[
(Z \circ g^M')(x, y') \leq (g^M \circ Z)(x, y').
\]

It is sufficient to show that, for every \(x' \in \Delta^M'\),

\[
Z(x, x') \otimes g^M'(x', y') \leq (g^M \circ Z)(x, y').
\]

\end{proof}
This inequality follows from the condition (21).

For the converse, suppose that the underlying residuated lattice is linear, \( M \) and \( M' \) are image-finite fuzzy Kripke models and \( Z: \Delta^M \times \Delta^{M'} \rightarrow L \) is a fuzzy relation satisfying the conditions (23)–(27). We prove that \( Z \) is a fuzzy bisimulation between \( M \) and \( M' \). Let \( p \in \Sigma_P, q \in \Sigma_A \), \( x \in \Delta^M \) and \( x' \in \Delta^{M'} \). We need to show that \( Z \) satisfies the assertions (22)–(24), for any \( y \in \Delta^M \) or \( y' \in \Delta^{M'} \) if it is a free variable.

The assertion (22) follows from (25).

Consider the assertion (23) for any \( y \in \Delta^M \). By definition,
\[
Z(x, x') \otimes g^M(x, y) \leq (Z^- \circ g^M)(x', y).
\]
By (26), it follows that
\[
Z(x, x') \otimes g^M(x, y) \leq (g^{M'} \circ Z^-)(x', y).
\]
Since the underlying residuated lattice is linear and \( M' \) is image-finite, there exists \( y' \in \Delta^{M'} \) such that
\[
(g^{M'} \circ Z^-)(x', y) = g^{M'}(x', y') \otimes Z(y, y').
\]
Therefore, the following inequality holds, which implies (23):
\[
Z(x, x') \otimes g^M(x, y) \leq g^{M'}(x', y') \otimes Z(y, y').
\]

Consider the assertion (24) for any \( y' \in \Delta^{M'} \). By definition,
\[
Z(x, x') \otimes g^{M'}(x', y') \leq (Z \circ g^{M'})(x, y').
\]
By (27), it follows that
\[
Z(x, x') \otimes g^{M'}(x', y') \leq (g^M \circ Z)(x, y').
\]
Since the underlying residuated lattice is linear and \( M \) is image-finite, there exists \( y \in \Delta^M \) such that
\[
(g^M \circ Z)(x, y') = g^M(x, y) \otimes Z(y, y').
\]
Therefore, the following inequality holds, which implies (24):
\[
Z(x, x') \otimes g^{M'}(x', y') \leq g^M(x, y) \otimes Z(y, y').
\]

Example 3.4. Let the underlying residuated lattice use \( L = \{0, a, b, 1\} \) (with four pairwise distinct elements) where \( a \) and \( b \) are not comparable. Let \( \Sigma_A = \{q\} \) and \( \Sigma_P = \emptyset \). Consider the Kripke models \( M \) and \( M' \) specified and depicted below:

- \( \Delta^M = \{u, v\}, \ g^M = \{(u, v) : 1\}, \)
- \( \Delta^{M'} = \{u', v_1', v_2'\}, \ g^{M'} = \{(u', v_1') : a, (u', v_2') : b\} \).

\[
\begin{array}{c|c}
  \mathcal{M} & \mathcal{M}' \\
  \hline
  u & u' \\
  v & a \\
  1 & b \\
\end{array}
\]

\[
\begin{array}{c|c}
  \mathcal{M} & \mathcal{M}' \\
  \hline
  u & u' \\
  v & a \\
  1 & b \\
\end{array}
\]
Let \( Z : \Delta^M \times \Delta^{M'} \to L \) be the fuzzy relation specified by 
\[
Z = \{ \langle u, u' \rangle : 1, \langle v, v'_1 \rangle : 1, \langle v, v'_2 \rangle : 1 \}.
\]
Observe that \( Z \) satisfies the conditions (25)–(27), but it is not a fuzzy bisimulation between \( M \) and \( M' \), as it does not satisfy the condition (23). The reason is that the underlying residuated lattice is not linear.

Example 3.5. Let \( L \) be the unit interval \([0, 1]\) and \( \otimes \) the Gödel t-norm. Let \( \Sigma_A = \{ \varrho \} \) and \( \Sigma_P = \emptyset \). Consider the Kripke models \( M \) and \( M' \) specified and depicted below (cf. [4, 21]):

- \( \Delta^M = \{ u, v \}, \quad \rho^M = \{ \langle u, v \rangle : 1 \} \)
- \( \Delta^{M'} = \{ u', v'_i \mid i \in \mathbb{N} \setminus \{0\} \}, \quad \rho^{M'} = \{ \langle u', v'_i \rangle : \frac{i}{n+1} \mid i \in \mathbb{N} \setminus \{0\} \} \)

Let \( Z : \Delta^M \times \Delta^{M'} \to L \) be the fuzzy relation specified by 
\[
Z = \{ \langle u, u' \rangle : 1, \langle v, v'_i \rangle : 1 \mid i \in \mathbb{N} \setminus \{0\} \}.
\]
Observe that \( Z \) satisfies the conditions (25)–(27), but it is not a fuzzy bisimulation between \( M \) and \( M' \), as it does not satisfy the condition (23). The reason is that \( M' \) is not image-finite.

The above proposition is related to Remark 3.4 of [21]. In [12], Fan studied fuzzy bisimulations that are defined for fuzzy Kripke models (over a signature with \( |\Sigma_A| = 1 \)) using conditions like (25)–(27) and the Gödel semantics over the unit interval \([0, 1]\). In [21], Nguyen et al. studied fuzzy bisimulations that are defined for interpretations in description logics using conditions like (22)–(24) and the Gödel semantics over the unit interval \([0, 1]\). Note that the residuated lattices used in both the works [12, 21] are linear, complete and use a fixed operator \( \otimes \), which is the Gödel t-norm. The relationship between (22)–(24) and (25)–(27) is characterized by the above proposition. On one hand, the conditions (22)–(24) do not require the underlying residuated lattice to be complete and, as discussed in [21], the style is appropriate for the extension that deals with number restrictions (in description logics) and graded modalities. On the other hand, when restricting to complete residuated lattices and the case without graded modalities, the conditions (25)–(27) are weaker\(^1\) and, when used instead of (22)–(24), make the notion of fuzzy bisimulation stronger\(^2\) for non-image-finite fuzzy Kripke models.

A fuzzy bisimulation between \( M \) and itself is called a fuzzy auto-bisimulation of \( M \).

Proposition 3.6. Let \( M, M' \) and \( M'' \) be image-finite fuzzy Kripke models.

1. The fuzzy relation \( Z : \Delta^M \times \Delta^M \to L \) specified by \( Z(x, x') = (\text{if } x = x' \text{ then } 1 \text{ else } 0) \) is a fuzzy auto-bisimulation of \( M \).
2. If \( Z \) is a fuzzy bisimulation between \( M \) and \( M' \), then \( Z^- \) is a fuzzy bisimulation between \( M' \) and \( M \).

\(^1\) (23) implies (26), but not vice versa; similarly, (24) implies (27), but not vice versa.
\(^2\) in the sense that more fuzzy relations can be fuzzy bisimulations (see, e.g., Example 3.5).
3. If the underlying residuated lattice is linear and complete, $Z_1$ is a fuzzy bisimulation between $\mathcal{M}$ and $\mathcal{M}'$, and $Z_2$ is a fuzzy bisimulation between $\mathcal{M}'$ and $\mathcal{M}''$, then $Z_1 \circ Z_2$ is a fuzzy bisimulation between $\mathcal{M}$ and $\mathcal{M}''$.

4. If the underlying residuated lattice is linear and $Z$ is a finite set of fuzzy bisimulations between $\mathcal{M}$ and $\mathcal{M}'$, then $\sup Z$ is also a fuzzy bisimulation between $\mathcal{M}$ and $\mathcal{M}'$.

**Proof.** The proofs of the first two assertions are straightforward.

Consider the third assertion and assume that the premises hold. We have to show that $Z_1 \circ Z_2$ satisfies the conditions (22)–(24).

- Consider the condition (22). Let $x \in \Delta^\mathcal{M}$, $x' \in \Delta^\mathcal{M}'$, $x'' \in \Delta^\mathcal{M}''$ and $p \in \Sigma_P$. We have that

$$Z_1(x, x') \leq p^\mathcal{M}(x) \Leftrightarrow p^\mathcal{M}'(x')$$

$$Z_2(x', x'') \leq p^\mathcal{M}'(x') \Leftrightarrow p^\mathcal{M}''(x'').$$

Due to (15), $(p^\mathcal{M}(x) \Leftrightarrow p^\mathcal{M}'(x')) \otimes (p^\mathcal{M}'(x') \Leftrightarrow p^\mathcal{M}''(x'')) \leq (p^\mathcal{M}(x) \Leftrightarrow p^\mathcal{M}''(x''))$. By (2), it follows that

$$Z_1(x, x') \otimes Z_2(x', x'') \leq (p^\mathcal{M}(x) \Leftrightarrow p^\mathcal{M}''(x'')).$$

Therefore, $(Z_1 \circ Z_2)(x, x'') \leq (p^\mathcal{M}(x) \Leftrightarrow p^\mathcal{M}''(x''))$, which completes the proof of (22).

- Consider the condition (23). Let $x, y \in \Delta^\mathcal{M}$, $x'' \in \Delta^\mathcal{M}''$ and $q \in \Sigma_A$. We need to show that there exists $y'' \in \Delta^\mathcal{M}''$ such that

$$(Z_1 \circ Z_2)(x, x'') \otimes q^\mathcal{M}(x, y) \leq q^\mathcal{M}''(x'', y'') \otimes (Z_1 \circ Z_2)(y, y').$$

(28)

Let $x'$ be an arbitrary element of $\Delta^\mathcal{M}'$. Since $Z_1$ is a fuzzy bisimulation between $\mathcal{M}$ and $\mathcal{M}'$, there exists $y' \in \mathcal{M}'$ such that

$$Z_1(x, x') \otimes q^\mathcal{M}(x, y) \leq q^\mathcal{M}'(x', y') \otimes Z_1(y, y').$$

(29)

Since $Z_2$ is a fuzzy bisimulation between $\mathcal{M}'$ and $\mathcal{M}''$, there exists $y'' \in \mathcal{M}''$ such that

$$Z_2(x', x'') \otimes q^\mathcal{M}(x', y') \leq q^\mathcal{M}''(x'', y'') \otimes Z_2(y', y'').$$

(30)

By (29), (30) and (2), we have that

$$Z_2(x', x'') \otimes Z_1(x, x') \otimes q^\mathcal{M}(x, y) \leq Z_2(x', x'') \otimes q^\mathcal{M}'(x', y') \otimes Z_1(y, y')$$

$$\leq q^\mathcal{M}''(x'', y'') \otimes Z_2(y', y'') \otimes Z_1(y, y')$$

$$\leq q^\mathcal{M}''(x'', y'') \otimes (Z_1 \circ Z_2)(y, y'),$$

which implies (28) because $x'$ is an arbitrary element of $\Delta^\mathcal{M}'$, $\mathcal{M}''$ is image-finite and the underlying residuated lattice is linear.

- The condition (24) can be proved analogously.

Consider now the fourth assertion and assume that the premises hold. It is sufficient to consider the case when $Z = \{Z_1, Z_2\}$. We need to prove that $Z_1 \cup Z_2$ satisfies the conditions (22)–(24).
Consider the condition [22]. Let \( x \in \Delta^M, x' \in \Delta^{M'} \) and \( p \in \Sigma_P \). Since \( Z_1 \) and \( Z_2 \) are fuzzy bisimulations between \( M \) and \( M' \), we have that \( Z_1(x, x') \leq (p^M(x) \Leftrightarrow p^M(x')) \) and \( Z_2(x, x') \leq (p^M(x) \Leftrightarrow p^{M'}(x')) \). Hence, \((Z_1 \cup Z_2)(x, x') \leq (p^M(x) \Leftrightarrow p^{M'}(x')).\)

Consider the condition [23]. Let \( x, y \in \Delta^M, x' \in \Delta^{M'} \) and \( \varphi \in \Sigma_A \). We need to show that there exists \( y' \in \Delta^{M'} \) such that

\[
(Z_1 \cup Z_2)(x, x') \otimes g^M(x, y) \leq g^{M'}(x', y') \otimes (Z_1 \cup Z_2)(y, y').
\]  

(31)

Without loss of generality, assume that \( Z_1(x, x') \leq Z_2(x, x') \). Since \( Z_2 \) is a fuzzy bisimulation between \( M \) and \( M' \), there exists \( y' \in \Delta^{M'} \) such that

\[
Z_2(x, x') \otimes g^M(x, y) \leq g^{M'}(x', y') \otimes Z_2(y, y').
\]

Thus,

\[
(Z_1 \cup Z_2)(x, x') \otimes g^M(x, y) = Z_2(x, x') \otimes g^M(x, y)
\]

\[
\leq g^{M'}(x', y') \otimes Z_2(y, y')
\]

\[
\leq g^{M'}(x', y') \otimes (Z_1 \cup Z_2)(y, y').
\]

The condition [24] can be proved analogously.

Corollary 3.7. Let \( M \) be an image-finite fuzzy Kripke model. If \( Z \) is the greatest fuzzy auto-bisimulation of \( M \) and the underlying residuated lattice is linear and complete, then \( Z \) is a fuzzy equivalence relation.

Proof. Suppose that \( Z \) is the greatest fuzzy auto-bisimulation of \( M \) and the underlying residuated lattice is complete. By the assertion 1 of Proposition 3.6, \( Z \) is reflexive. By the assertion 2 of Proposition 3.6, \( Z^- \) is a fuzzy auto-bisimulation of \( M \). Hence, \( Z^- \leq Z \) and \( Z \) is symmetric. By the assertion 3 of Proposition 3.6, \( Z \circ Z \) is a fuzzy auto-bisimulation of \( M \). Hence, \( Z \circ Z \leq Z \) and \( Z \) is transitive. Therefore, \( Z \) is a fuzzy equivalence relation.

4. Invariance Results

We say that a formula \( \varphi \) is invariant under fuzzy bisimulations w.r.t. \( fPDL^{-\Phi} \) if, for every fuzzy Kripke models \( M \) and \( M' \) that are witnessed w.r.t. \( fPDL^{-\Phi} \) and for every fuzzy bisimulation \( Z \) between \( M \) and \( M' \), \( Z(x, x') \leq (\varphi^M(x) \Leftrightarrow \varphi^{M'}(x')) \) for all \( x \in \Delta^M \) and \( x' \in \Delta^{M'} \).

Theorem 4.1. All formulas of \( fPDL^{-\Phi} \) are invariant under fuzzy bisimulations w.r.t. \( fPDL^{-\Phi} \) if the underlying residuated lattice \( \mathcal{L} \) satisfies the following conditions:

\[
\text{if } \cup \notin \Phi, \text{ then } \mathcal{L} \text{ is linear};
\]

(32)

\[
\text{if } \rightarrow \notin \Phi \text{ or } ? \notin \Phi, \text{ then } \mathcal{L} \text{ is a Heyting algebra}.
\]

(33)

This theorem is an immediate consequence of the following lemma.
Lemma 4.2. Let $M$ and $M'$ be fuzzy Kripke models that are witnessed w.r.t. $fPDL^{-\Phi}$ and $Z$ a fuzzy bisimulation between $M$ and $M'$. Suppose that the underlying residuated lattice satisfies the conditions $(32)$ and $(33)$. Then, the following properties hold for every formula $\varphi$ of $fPDL^{-\Phi}$, every program $\alpha$ of $fPDL^{-\Phi}$ and every possible values of the free variables:

$$Z(x, x') \leq (\varphi^M(x) \Leftrightarrow \varphi^{M'}(x'))$$  \hspace{1cm} (34)

$$\exists y' \in \Delta^{M'}(Z(x, x') \otimes \alpha^M(x, y) \leq \alpha^M(x', y') \otimes Z(y, y'))$$  \hspace{1cm} (35)

$$\exists y \in \Delta^{M}(Z(x, x') \otimes \alpha^M(x', y') \leq \alpha^M(x, y) \otimes Z(y, y')).$$  \hspace{1cm} (36)

**Proof.** We prove this lemma by induction on the structures of $\varphi$ and $\alpha$. First, consider the assertion $(35)$. Let $x, y \in \Delta^M$ and $x' \in \Delta^{M'}$. It suffices to show that there exists $y' \in \Delta^{M'}$ such that

$$Z(x, x') \otimes \alpha^M(x, y) \leq \alpha^M(x', y') \otimes Z(y, y').$$  \hspace{1cm} (37)

The base case occurs when $\alpha$ is an atomic program and follows from $(33)$. The induction steps are given below.

- **Case $\alpha = (\psi?)$ (and $? \notin \Phi$):** If $x \neq y$, then $\alpha^M(x, y) = 0$ and, by $(5)$, the assertion $(37)$ clearly holds. Suppose $x = y$ and take $y' = x'$. By the induction assumption about $(34)$, $Z(x, x') \leq (\psi^M(x) \Leftrightarrow \psi^{M'}(x'))$. Hence, by $(21),$

$$Z(x, x') \otimes \psi^M(x) = Z(x, x') \otimes \psi^{M'}(x'),$$

which implies $(37)$.

- **Case $\alpha = \beta \cup \gamma$ (and $\cup \notin \Phi$):** Without loss of generality, suppose $\beta^M(x, y) \geq \gamma^M(x, y)$. Thus, $\alpha^M(x, y) = \beta^M(x, y)$. By the induction assumption of $(34)$, there exists $y' \in \Delta^{M'}$ such that

$$Z(x, x') \otimes \beta^M(x, y) \leq \beta^M(x', y') \otimes Z(y, y').$$

Thus,

$$Z(x, x') \otimes \alpha^M(x, y) = Z(x, x') \otimes \beta^M(x, y) \leq \beta^M(x', y') \otimes Z(y, y') \leq \alpha^M(x', y') \otimes Z(y, y').$$

- **Case $\alpha = \beta \circ \gamma$:** Since $M$ is witnessed w.r.t. $fPDL^{-\Phi}$, there exists $z \in \Delta^M$ such that $\alpha^M(x, y) = \beta^M(x, z) \otimes \gamma^M(z, y)$. By the induction assumption of $(35)$, there exist $z'$ and $y'$ such that:

$$Z(x, x') \otimes \beta^M(x, z) \leq \beta^M(x', z') \otimes Z(z, z')$$

$$Z(z, z') \otimes \gamma^M(z, y) \leq \gamma^M(z', y') \otimes Z(y, y').$$

Since $\otimes$ is associative and due to $(2)$, it follows that

$$Z(x, x') \otimes \alpha^M(x, y) = Z(x, x') \otimes \beta^M(x, z) \otimes \gamma^M(z, y) \leq \beta^M(x', z') \otimes Z(z, z') \otimes \gamma^M(z, y) \leq \beta^M(x', z') \otimes \gamma^M(z', y') \otimes Z(y, y') \leq \alpha^M(x', y') \otimes Z(y, y').$$
• Case $\alpha = \beta^*$: Since $\mathcal{M}$ is witnessed w.r.t. $fPDL^{-\Phi}$, there exist $x_0, \ldots, x_k \in \Delta^\mathcal{M}$ such that $x_0 = x, x_k = y$ and

$$\alpha^\mathcal{M}(x, y) = \beta^\mathcal{M}(x_0, x_1) \otimes \cdots \otimes \beta^\mathcal{M}(x_{k-1}, x_k).$$

Let $x'_0 = x'$. By the induction assumption of (23), there exist $x'_1, \ldots, x'_k \in \Delta^{\mathcal{M}'}$ such that

$$Z(x_i, x'_i) \otimes \beta^\mathcal{M}(x_i, x_{i+1}) \leq \beta^{\mathcal{M}'}(x'_i, x'_{i+1}) \otimes Z(x_{i+1}, x'_{i+1})$$

for all $0 \leq i < k$. Since $\otimes$ is associative and due to (2), it follows that

$$Z(x_0, x'_0) \otimes \alpha^{\mathcal{M}}(x_0, x_k) = Z(x_0, x'_0) \otimes \beta^{\mathcal{M}}(x_0, x_1) \otimes \cdots \otimes \beta^{\mathcal{M}}(x_{k-1}, x_k) \leq \beta^{\mathcal{M}'}(x'_0, x'_1) \otimes Z(x_1, x'_1) \otimes \beta^{\mathcal{M}}(x_1, x_2) \otimes \cdots \otimes \beta^{\mathcal{M}}(x_{k-1}, x_k) \leq \cdots \leq \beta^{\mathcal{M}'}(x'_0, x'_1) \otimes \cdots \otimes \beta^{\mathcal{M}'}(x'_{k-1}, x'_k) \otimes Z(x_k, x'_k) \leq \alpha^{\mathcal{M}'}(x'_0, x'_k) \otimes Z(x_k, x'_k).$$

Taking $y' = x'_k$, we obtain (37).

The assertion (36) can be proved analogously as for (35). Consider the assertion (34). The case when $\varphi = a$ is trivial. The case when $\varphi = p$ follows from the condition (22). The case when $\varphi = \neg \psi$ is reduced to the case when $\varphi = (\psi \to 0)$.

• Case $\varphi = \psi \land \xi$: We have $\varphi^\mathcal{M}(x) = \psi^\mathcal{M}(x) \land \xi^\mathcal{M}(x)$ and $\varphi^{\mathcal{M}'}(x') = \psi^\mathcal{M}'(x') \land \xi^\mathcal{M}'(x')$. By the induction assumption of (33),

$$Z(x, x') \leq \psi^\mathcal{M}(x) \leftrightarrow \psi^\mathcal{M}'(x') \quad (38)$$
$$Z(x, x') \leq \xi^\mathcal{M}(x) \leftrightarrow \xi^\mathcal{M}'(x'). \quad (39)$$

By (16),

$$(\psi^\mathcal{M}(x) \leftrightarrow \psi^\mathcal{M}'(x')) \land (\xi^\mathcal{M}(x) \leftrightarrow \xi^\mathcal{M}'(x')) \leq (\varphi^\mathcal{M}(x) \leftrightarrow \varphi^\mathcal{M}'(x')). \quad (40)$$

The assertion (34) follows from (38), (39) and (40).

• The case $\varphi = (\psi \lor \xi)$ is similar to the previous case, using (17) instead of (16).

• Case $\varphi = (\psi \to \xi)$ (and $\rightarrow \notin \Phi$): The proof is similar to the proof of the case when $\varphi = \psi \land \xi$, using (20) instead of (16).

• Case $\varphi = (a \to \psi)$: We have $\varphi^\mathcal{M}(x) = (a \Rightarrow \psi^\mathcal{M}(x))$ and $\varphi^{\mathcal{M}'}(x') = (a \Rightarrow \psi^\mathcal{M}'(x'))$. By the induction assumption of (31), $Z(x, x') \leq (\psi^\mathcal{M}(x) \leftrightarrow \psi^\mathcal{M}'(x'))$. The assertion (34) follows from this and (18).

• The case $\varphi = (\psi \to a)$ is similar to the previous case, using (19) instead of (18).
• Case $\varphi = \langle \alpha \rangle \psi$: Since $\mathcal{M}$ is witnessed w.r.t. $fPDL^{-\Phi}$, there exists $y \in \Delta^\mathcal{M}$ such that

$$\varphi^\mathcal{M}(x) = \alpha^\mathcal{M}(x, y) \otimes \psi^\mathcal{M}(y).$$

(41)

By the induction assumption about (35), there exists $y' \in \Delta^{\mathcal{M}'}$ such that

$$Z(x, x') \otimes \alpha^\mathcal{M}(x, y) \leq \alpha^{\mathcal{M}'}(x', y') \otimes Z(y, y').$$

(42)

By definition,

$$\alpha^{\mathcal{M}'}(x', y') \otimes \psi^{\mathcal{M}'}(y') \leq \varphi^{\mathcal{M}'}(x').$$

(43)

By the induction assumption of (34),

$$Z(y, y') \leq \psi^\mathcal{M}(y) \iff \psi^{\mathcal{M}'}(y').$$

(44)

By (42) and (1),

$$Z(x, x') \leq \alpha^\mathcal{M}(x, y) \Rightarrow Z(y, y') \leq \alpha^{\mathcal{M}'}(x', y') \otimes \psi^{\mathcal{M}'}(y').$$

(45)

By (44), (2) and (3), it follows that

$$Z(x, x') \leq \alpha^\mathcal{M}(x, y) \Rightarrow \varphi^{\mathcal{M}'}(x').$$

(46)

Analogously, it can be shown that

$$Z(x, x') \leq \varphi^{\mathcal{M}'}(x').$$

(47)

Therefore,

$$Z(x, x') \leq \varphi^\mathcal{M}(x) \Rightarrow \varphi^{\mathcal{M}'}(x').$$

• Case $\varphi = [\alpha] \psi$: Since $\mathcal{M}'$ is witnessed w.r.t. $fPDL^{-\Phi}$, there exists $y' \in \Delta^{\mathcal{M}'}$ such that

$$\varphi^{\mathcal{M}'}(x') = (\alpha^{\mathcal{M}'}(x', y') \Rightarrow \psi^{\mathcal{M}'}(y')).$$

(45)

By the induction assumption about (36), there exists $y \in \Delta^\mathcal{M}$ such that

$$Z(x, x') \otimes \alpha^{\mathcal{M}'}(x', y') \leq \alpha^\mathcal{M}(x, y) \otimes Z(y, y').$$

(46)

By definition,

$$\varphi^\mathcal{M}(x) \leq \alpha^\mathcal{M}(x, y) \Rightarrow \psi^\mathcal{M}(y).$$

(47)
By the induction assumption of (34),
\[ Z(y, y') \leq \psi^M(y) \iff \psi^{M'}(y'). \tag{48} \]

By (46) and (1),
\[ Z(x, x') \leq \alpha^{M'}(x', y') \Rightarrow Z(y, y') \otimes \alpha^M(x, y). \]

By (48), (2) and (3), it follows that
\[ Z(x, x') \leq \alpha^{M'}(x', y') \Rightarrow Z(y, y') \otimes \alpha^M(x, y). \]

By (47), (45) and (3), it follows that
\[ Z(x, x') \leq \alpha^{M'}(x', y') \Rightarrow Z(y, y') \otimes \alpha^M(x, y). \]

Analogously, it can be shown that
\[ Z(x, x') \leq \alpha^M(x) \Rightarrow \phi^{M'}(x'). \]

Therefore,
\[ Z(x, x') \leq \phi^M(x) \iff \phi^{M'}(x'). \]

This completes the proof. \[\Box\]

The following lemma is a counterpart of Lemma 4.2 for $fK^0$.

**Lemma 4.3.** Let $M$ and $M'$ be fuzzy Kripke models that are witnessed w.r.t. $fK^0$ and $Z$ a fuzzy bisimulation between $M$ and $M'$. Then, the following property holds for every $x \in \Delta^M$, $x' \in \Delta^{M'}$ and every formula $\varphi$ of $fK^0$:
\[ Z(x, x') \leq \varphi^M(x) \iff \varphi^{M'}(x'). \]

This lemma can be proved analogously as done for the assertion (34) of Lemma 4.2 by using (23) and (24) instead of (35) and (36), respectively. Roughly speaking, the proof is a simplification of the proof of Lemma 4.2.

**Remark 4.4.** Analyzing the proof of Lemma 4.2, it can be seen that the condition (33) ($\mathcal{L}$ is a Heyting algebra if $\rightarrow \notin \Phi$ or $? \notin \Phi$) can be replaced by the conditions (20) and (21). This also applies to Theorem 4.1. \[\Box\]

**Remark 4.5.** To justify that a condition like (33) (or (20) and (21) together) is essential for Lemma 4.2 and Theorem 4.1 we show that, if $\rightarrow \notin \Phi$ or $? \notin \Phi$, $L = [0, 1]$ and $\otimes$ is the Lukasiewicz or product t-norm, then there exist finite fuzzy Kripke models $M$ and $M'$, a fuzzy bisimulation $Z$ between $M$ and $M'$, $x \in \Delta^M$, $x' \in \Delta^{M'}$ and a formula $\varphi$ of $fPDL^{\Phi}$ such that $Z(x, x') \not\leq (\varphi^M(x) \iff \varphi^{M'}(x'))$. Let
• $\Sigma_A = \emptyset$, $\Sigma_P = \{p, q\}$, $\varphi = (p \rightarrow q)$, $\psi = [p?]q$
• $\Delta^M = \{v\}$, $p^M = \{v : 0.2\}$, $q^M = \{v : 0.2\}$
• $\Delta^{M'} = \{v'\}$, $p^{M'} = \{v' : 0.3\}$, $q^{M'} = \{v' : 0.1\}$
• $\otimes$ be the Lukasiewicz or product t-norm,
• $Z$ be the greatest fuzzy bisimulation between $M$ and $M'$.

If $\otimes$ is the Lukasiewicz t-norm, then
• $Z(v, v') = \min\{ (p^M(v) \otimes p^{M'}(v')), (q^M(v) \otimes q^{M'}(v')) \} = 0.9$,
• $\varphi^M(v) = \psi^M(v) = 1$, $\varphi^{M'}(v') = 0.8$,
• $(\varphi^M(v) \otimes \varphi^{M'}(v')) = (\psi^M(v) \otimes \psi^{M'}(v')) = 0.8$.

If $\otimes$ is the product t-norm, then
• $Z(v, v') = \min\{ (p^M(v) \otimes p^{M'}(v')), (q^M(v) \otimes q^{M'}(v')) \} = 0.5$,
• $\varphi^M(v) = \psi^M(v) = 1$, $\varphi^{M'}(v') = 1/3$.
• $(\varphi^M(v) \otimes \varphi^{M'}(v')) = (\psi^M(v) \otimes \psi^{M'}(v')) = 1/3$.

Hence, $Z(v, v') \not\leq (\varphi^M(v) \otimes \varphi^{M'}(v'))$ and $Z(v, v') \not\leq (\psi^M(v) \otimes \psi^{M'}(v'))$.

Remark 4.6. Consider the formula constructor $\varphi \& \psi$ whose meaning in a Kripke model $M$ is specified by $(\varphi \& \psi)^M(x) = \varphi^M(x) \otimes \psi^M(x)$ for $x \in \Delta^M$. We show that formulas with this constructor may not be invariant under fuzzy bisimulations w.r.t. $fPDL^{-\Phi}$ for $\Phi = \{\cup, \rightarrow, \?\}$.

Let $L = [0, 1]$, $\Sigma_A = \emptyset$, $\Sigma_P = \{p\}$, $\varphi = p \& p$ and let $M$ and $M'$ be Kripke models such that $\Delta^M = \{v\}$, $p^M(v) = 0.5$, $\Delta^{M'} = \{v'\}$ and $p^{M'}(v') = 1$. Consider the case when $\otimes$ is the Lukasiewicz or product t-norm. Observe that $Z : \Delta^M \times \Delta^{M'} \to L$ with $Z(v, v') = 0.5 \Leftrightarrow 1 = 0.5$ is a fuzzy bisimulation between $M$ and $M'$. We have $\varphi^M(v') = 1$. If $\otimes$ is the Lukasiewicz t-norm, then $\varphi^M(v) = 0$ and $(\varphi^M(v) \otimes \varphi^{M'}(v')) = 0$. If $\otimes$ is the product t-norm, then $\varphi^M(v) = 0.25$ and $(\varphi^M(v) \otimes \varphi^{M'}(v')) = 0.25$. Thus, $Z(v, v') \not\leq (\varphi^M(v) \otimes \varphi^{M'}(v'))$.

5. The Hennessy-Milner Property

In this section, we present and prove the Hennessy-Milner property of fuzzy bisimulations. It is formulated for the class of modally saturated models, which is larger than the class of image-finite models. Our notion of modal saturatedness is a counterpart of the ones given in [13, 2, 21].

A fuzzy Kripke model $M$ is said to be modally saturated (w.r.t. $fK^0$) and the underlying residuated lattice $L$ if, for every $a \in \Delta \setminus \{0\}$, every $x \in \Delta^M$, every $\varphi \in \Sigma_A$ and every infinite set $\Gamma$ of formulas in $fK^0$, if for every finite subset $\Lambda$ of $\Gamma$ there exists $y \in \Delta^M$ such that $\varphi^M(y) \otimes \varphi^M(y) \geq a$ for all $\varphi \in \Lambda$, then there exists $y \in \Delta^M$ such that $\varphi^M(x, y) \otimes \varphi^M(y) \geq a$ for all $\varphi \in \Gamma$.

Proposition 5.1. All image-finite fuzzy Kripke models are modally saturated.
Let $M$ be an image-finite fuzzy Kripke model, let $a \in L \setminus \{0\}$, $x, y \in \Delta^M$, $\varphi \in \Sigma_A$ and let $\Gamma$ be an infinite set of formulas in $fK^0$. Assume that, for every finite subset $\Lambda$ of $\Gamma$, there exists $y \in \Delta^M$ such that $g^M(x, y) \otimes \varphi^M(y) \geq a$ for all $\varphi \in \Delta$. For a contradiction, suppose that, for every $y \in \Delta^M$, there exists $\varphi_y \in \Gamma$ such that $g^M(x, y) \otimes \varphi^M_y(y) \geq a$. Let $\varphi_0$ be an arbitrary formula of $\Gamma$ and let $\Lambda = \{\varphi_y \mid g^M(x, y) > 0\} \cup \{\varphi_0\}$. Since $M$ is image-finite, $\Lambda$ is finite. For every $y \in \Delta^M$, if $g^M(x, y) = 0$, then by (49), $g^M(x, y) \otimes \varphi^M_0(y) = 0 \geq a$, else $\varphi_y \in \Lambda$ and $g^M(x, y) \otimes \varphi^M_y(y) \geq a$. Hence, for every $y \in \Delta^M$, there exists $\varphi \in \Delta$ such that $g^M(x, y) \otimes \varphi^M_y(y) \geq a$. This contradicts the assumption. \hfill $\blacksquare$

Let $L$ be a complete residuated lattice. We say that the operator $\otimes$ is continuous (w.r.t. infima) if, for every $x \in L$ and $Y \subseteq L$, $x \otimes \inf Y = \inf\{x \otimes y \mid y \in Y\}$. Clearly, all the Gödel, Lukasiewicz and product t-norms (in particular when $L$ is the unit interval $[0, 1]$) are continuous. In addition, if $L$ is a Heyting algebra, then $\otimes$ is continuous.

**Theorem 5.2.** Let $M$ and $M'$ be fuzzy Kripke models that are witnessed w.r.t. $fK^0$ and modally saturated.\(^3\) Suppose that the underlying residuated lattice $L = \langle L, \leq, \otimes, \Rightarrow, 0, 1 \rangle$ is complete and $\otimes$ is continuous. Then, the fuzzy relation $Z : \Delta^M \times \Delta^{M'} \rightarrow L$ specified by $Z(x, x') = \inf\{\varphi^M(x) \Leftrightarrow \varphi^{M'}(x') \mid \varphi \text{ is a formula of } fK^0\}$ is the greatest fuzzy bisimulation between $M$ and $M'$.

**Proof.** By Lemma 4.3, it is sufficient to prove that $Z$ is a fuzzy bisimulation between $M$ and $M'$.

By definition, $Z$ satisfies the condition (22).

We prove that $Z$ satisfies the condition (23). Let $\varphi \in \Sigma_A$, $x, y \in \Delta^M$ and $x' \in \Delta^{M'}$. Let $a = Z(x, x') \otimes g^M(x, y)$. For a contradiction, suppose that, for every $y' \in \Delta^{M'}$, $a \not\leq g^{M'}(x', y') \otimes Z(y, y')$. Since $\otimes$ is continuous, by the definition of $Z(y, y')$, it follows that, for every $y' \in \Delta^{M'}$, there exists a formula $\varphi_{y'}$ of $fK^0$ such that $a \not\leq g^{M'}(x', y') \otimes (\varphi_{y'}^M(y) \Leftrightarrow \varphi_{y'}^{M'}(y'))$.

For every $y' \in \Delta^{M'}$, let $\psi_{y'} = (\varphi_{y'} \rightarrow \varphi_{y'}^M(y)) \land (\varphi_{y'}^M(y) \rightarrow \varphi_{y'})$. Let $\Gamma = \{\psi_{y'} \mid y' \in \Delta^{M'}\}$. Observe that, for every $y' \in \Delta^{M'}$, $\psi_{y'}^M(y) = 1$ (by (41)) and $a \not\leq g^{M'}(x', y') \otimes \psi_{y'}^{M'}(y')$. Since $M'$ is modally saturated, it follows that there exists a finite subset $\Psi$ of $\Gamma$ such that, for every $y' \in \Delta^{M'}$, there exists $\psi \in \Psi$ such that $a \not\leq g^{M'}(x', y') \otimes \psi^{M'}(y')$. Since $M$ is image-finite, there exists $\varphi \in \Psi$ such that $a \not\leq g^{M}(x, y) \otimes \varphi^M(x')$. (49)

Let $\varphi = \langle \varphi \rangle \land \Psi$. It is a formula of $fK^0$. Thus, $\varphi^M(x) \geq g^M(x, y)$ since $(\land \Psi)^M(y) = 1$. Since $M'$ is witnessed w.r.t. $fK^0$, by (49) and (2), we have that $a \not\leq \varphi^{M'}(x')$, which means $Z(x, x') \otimes g^M(x, y) \not\leq \varphi^{M'}(x')$.\(^3\) These conditions are satisfied, for example, when $M$ and $M'$ are image-finite.

18
Since $\varphi^M(x) \geq \rho(x, y)$, by (2), it follows that

\[ Z(x, x') \otimes \varphi^M(x) \not\leq \varphi^M(x'). \]

By (1), this implies that

\[ Z(x, x') \not\leq (\varphi^M(x) \Rightarrow \varphi^M(x')), \]

which contradicts the definition of $Z(x, x')$.

Analogously, it can be proved that $Z$ satisfies the condition (24). This completes the proof. ■

**Corollary 5.3.** Let $M$ be an image-finite fuzzy Kripke model. Suppose that the underlying residuated lattice $L = \langle L, \leq, \otimes, \Rightarrow, 0, 1 \rangle$ is linear and complete and $\otimes$ is continuous. Then, the greatest fuzzy auto-bisimulation of $M$ exists and is a fuzzy equivalence relation.

**Proof.** Let $Z : \Delta^M \times \Delta^M \to L$ be specified by

\[ Z(x, x') = \inf \{ \varphi^M(x) \Leftrightarrow \varphi^M(x') \mid \varphi \text{ is a formula of } fK^0 \}. \]

Since $M$ is image-finite, it is witnessed w.r.t. $fK^0$ and modally saturated. By Theorem 5.2, $Z$ is the greatest fuzzy bisimulation between $M$ and $M'$. By Corollary 3.7, $Z$ is a fuzzy equivalence relation. ■

**Corollary 5.4.** Let $M$ and $M'$ be fuzzy Kripke models that are witnessed w.r.t. $fPDL^{-\Phi}$ and modally saturated. Suppose that the underlying residuated lattice $L = \langle L, \leq, \otimes, \Rightarrow, 0, 1 \rangle$ is complete, satisfies the conditions (32) and (33), and $\otimes$ is continuous. Then, for every $x \in \Delta^M$ and $x' \in \Delta^{M'}$,

\[ \inf \{ \varphi^M(x) \Leftrightarrow \varphi^M(x') \mid \varphi \text{ is a formula of } fK^0 \} = \inf \{ \varphi^M(x) \Leftrightarrow \varphi^M(x') \mid \varphi \text{ is a formula of } fPDL^{-\Phi} \}. \]

**Proof.** Since $fK^0$ is a sublanguage of $fPDL^{-\Phi}$, it is sufficient to show that

\[ \inf \{ \varphi^M(x) \Leftrightarrow \varphi^M(x') \mid \varphi \text{ is a formula of } fK^0 \} \leq \inf \{ \varphi^M(x) \Leftrightarrow \varphi^M(x') \mid \varphi \text{ is a formula of } fPDL^{-\Phi} \}. \]

Let $Z : \Delta^M \times \Delta^{M'} \to L$ be specified by

\[ Z(x, x') = \inf \{ \varphi^M(x) \Leftrightarrow \varphi^M(x') \mid \varphi \text{ is a formula of } fK^0 \}. \]

By Theorem 5.2, $Z$ is the greatest fuzzy bisimulation between $M$ and $M'$. Let $x \in \Delta^M$, $x' \in \Delta^{M'}$ and let $\varphi$ be an arbitrary formula of $fPDL^{-\Phi}$. By Lemma 4.2, $Z(x, x') \leq (\varphi^M(x) \Leftrightarrow \varphi^{M'}(x'))$.

Therefore,

\[ Z(x, x') \leq \inf \{ \varphi^M(x) \Leftrightarrow \varphi^{M'}(x') \mid \varphi \text{ is a formula of } fPDL^{-\Phi} \}. \]

This completes the proof. ■
6. Related Work and Discussion

The works [11, 12, 21] on fuzzy bisimulations have been briefly discussed in the introduction. We give below some additional remarks on these works before discussing other works related to logical characterizations of fuzzy/crisp bisimulations or simulations.

The results on fuzzy bisimulations of [11] are formulated only for finite social networks over the residuated lattice $[0, 1]$ using the Gödel t-norm. In that work, Fan and Liau did consider extending their results on fuzzy bisimulations to the settings with the Lukasiewicz and product t-norms. However, in [11, Example 2] they claimed that the extension does not work. The problem with that claim is that the authors used the logical language with the additional conjunction $\&$ which is interpreted as $\otimes$ (see Remark 4.6). In [11] Fan and Liau also studied crisp bisimulations under the name “generalized regular equivalence relations” for finite weighted social networks. They provided logical characterizations for crisp bisimulations under the Gödel, Lukasiewicz and product semantics. The characterizations are formulated w.r.t. fuzzy multimodal logics possibly with converse, which are extended with involutive negation and/or the Baaz projection operator. They concern invariance of modal formulas under crisp bisimulations and the Hennessy-Milner property of crisp bisimulations.

In [12] Fan also studied crisp bisimulations for fuzzy monomodal logics under the Gödel semantics. She provided logical characterizations of such bisimulations in the basic fuzzy monomodal logics possibly with converse, which are extended with involutive negation and/or the Baaz projection operator. The results of [12] on invariance of modal formulas and the Hennessy-Milner property for both crisp and fuzzy bisimulations are formulated for image-finite fuzzy Kripke models over a signature with only one accessibility relation.

In [21] Nguyen et al. also provided logical characterizations of crisp bisimulations for fuzzy description logics under the Gödel semantics. For the case with such bisimulations, the considered logics are extended with the Baaz projection operator or involutive negation. Apart from results on invariance of concepts and the Hennessy-Milner property of crisp/fuzzy bisimulations, the work [21] also gives results on conditional invariance of TBoxes and ABoxes under crisp/fuzzy bisimulations, separation of the expressive power of fuzzy description logics, and minimization of fuzzy interpretations by using crisp bisimulations.

When restricting to invariance results and the Hennessy-Milner property of fuzzy bisimulations in logics that are sublogics of $fPDL$, our results are more general than the results of [11, 12, 21]. For example, Theorem 5.2 together with Corollary 5.3 is strictly more general than Theorem 3 of [12]. One of the reasons is that Theorem 5.2 and Corollary 5.3 are formulated for complete residuated lattices, while Theorem 3 of [12] is formulated only for the lattice $[0, 1]$ using the Gödel t-norm. The other reasons are that: the logic considered in Theorem 3 of [12] is a proper sublogic of $fPDL$; Theorem 5.2 and Corollary 5.3 are formulated for witnessed and modally saturated models, while Theorem 3 of [12] is formulated only for image-finite models. On the other hand, all the papers [11, 12, 21] contain results formulated for modal/description logics with converse/inverse and/or the Baaz projection operator, which are not studied in the current work. The paper [21] also allows other constructors of description logics.

In [30, 28, 29], Wu et al. provided logical characterizations of crisp bisimulations/simulations for a few variants of fuzzy transition systems. The results are formulated w.r.t. crisp Hennessy-Milner logics, which use values from the unit interval $[0, 1]$ as thresholds for modal operators.

In [25] Pan et al. provided logical characterizations of fuzzy simulations for finite fuzzy labeled transition systems over finite residuated lattices. They are formulated w.r.t. an existential...
Hennessy-Milner logic. In [26] Pan et at. provided logical characterizations of simulations for finite quantitative transition systems over finite Heyting algebras. Quantitative transition systems are transition systems without labels for states but extended with a fuzzy equality relation between actions. Simulations studied in [26] are either fuzzy simulations or crisp simulations parameterized by a threshold used as a cut for the fuzzy equality relation between actions. The logical characterizations of simulations provided in [26] are formulated w.r.t. an existential cut-based crisp Hennessy-Milner logic for the case of crisp simulations, and w.r.t. an existential fuzzy Hennessy-Milner logic for the case of fuzzy simulations.

In [20] we provided logical characterizations of crisp cut-based simulations and bisimilarity for a large class of fuzzy description logics under the Zadeh semantics. The results concern preservation of information by such simulations, conditional invariance of ABoxes and TBoxes under bisimilarity between witnessed interpretations, as well as the Hennessy-Milner property for fuzzy description logics under the Zadeh semantics.

In [19] Marti and Metcalfe studied logical characterizations of crisp bisimulations in chain-based modal logics. The considered logics are monomodal logics whose formulas are interpreted in many-valued Kripke models over a chain-based algebra with a crisp frame. A chain-based algebra is a linear and complete bounded lattice. The main results of [19] concern characterizations of classes of Kripke models that have (resp. do not have) the Hennessy-Milner property.

The work [9] by Diaconescu concerns logical characterizations of crisp bisimulations in fuzzy modal logics over complete MTL-chains. A complete MTL-chain is a linear and complete residuated lattice. The considered logics are fuzzy monomodal logics that allow many-valued formulas and accessibility relations. The main result of [9] gives a necessary and sufficient algebraic condition for the class of image-finite Kripke models for such logics to admit the Hennessy-Milner property.

7. Conclusions

We have provided and proved logical characterizations of fuzzy bisimulations in the fuzzy propositional dynamic logic $fPDL$ and its sublogics over residuated lattices. The results concern invariance of formulas under fuzzy bisimulations and the Hennessy-Milner property of fuzzy bisimulations. The first theorem is formulated for fuzzy Kripke models that are witnessed, whereas the second theorem is formulated for fuzzy Kripke models that are witnessed and modally saturated.

Our results can be reformulated for other fuzzy structures such as fuzzy labeled transition systems and fuzzy interpretations in description logics. It is worth emphasizing that our results concern fuzzy bisimulations over general residuated lattices. They are interesting from the theoretical point of view, as the previous results on fuzzy bisimulations are formulated and proved only for the residuated lattice $[0, 1]$ using the Gödel t-norm or Heyting algebras.

In certain applications, the product t-norm is more suitable than the Gödel t-norm. For example, the closeness of a person to his/her great-grandmother can be assumed to be smaller than the closeness of that person to his/her mother. Furthermore, the product residuum is continuous w.r.t. both the arguments, whereas the Gödel residuum is not. This causes that the product residuum is more resistant to noise than the Gödel residuum. Our logical characterizations of fuzzy bisimulations open the way for studying logical similarity between individuals and concept learning in fuzzy description logics under the product semantics by applying fuzzy bisimulations.

On the technical matters, our results are formulated on a general level. Residuated lattices considered in this work may be infinite, whereas the work [27] considers only finite residuated lattices. The class of fuzzy Kripke models that are witnessed and modally saturated is larger than
the class of image-finite fuzzy Kripke models studied in \[10, 12\], the class of finite weighted social networks studied in \[11\] and the class of finite fuzzy labeled transition systems studied in \[25, 26\]. The considered fuzzy logic \textit{fPDL} contains the program constructors of propositional dynamic logic, which are absent in \[10, 11, 12, 25, 26\]. They correspond to role constructors in description logics.

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Appendix A. Proof of Lemma 2.1

Note that $\otimes$ is commutative and associative. Let $x, x', y, y', z \in L$.

- Since $\otimes$ is commutative, to prove (2), it is sufficient to show that, if $x \leq x'$, then $x \otimes y \leq x' \otimes y$.

  Assume that $x \leq x'$. By (1), $x' \leq (y \Rightarrow (x' \otimes y))$. Hence, $x \leq (y \Rightarrow (x' \otimes y))$. By (1), it follows that $x \otimes y \leq x' \otimes y$, which completes the proof of (2).

- Consider the assertion (3) and assume that $x' \leq x$ and $y \leq y'$. By (2), $(x \Rightarrow y) \otimes x' \leq (x \Rightarrow y) \otimes x \leq y$. Hence, $(x \Rightarrow y) \otimes x' \leq y$, which implies $(x \Rightarrow y) \leq (x' \otimes y)$ by using (1). By (1), $(x' \Rightarrow y) \otimes x' \leq y$, which implies $(x' \Rightarrow y) \leq (x' \otimes y)$. Hence, $(x' \Rightarrow y) \leq (x' \otimes y)$ by using (1). We have proved that $(x \Rightarrow y) \leq (x' \Rightarrow y)$ and $(x' \Rightarrow y) \leq (x' \otimes y)$, which together imply (3).

- Consider the assertion (4). We have that $x \leq y$ iff $1 \otimes x \leq y$ iff $1 \leq (x \Rightarrow y)$, by using (1).

The last inequality implies (4).

- Since $0 \leq (x \Rightarrow 0)$, by (1), $0 \otimes x \leq 0$. Hence, $x \otimes 0 = 0 \otimes x \leq 0$ and the assertion (5) holds.

- Consider the assertion (6). By (2), $x \otimes y \leq x \otimes (y \lor z)$ and $x \otimes z \leq x \otimes (y \lor z)$. Hence, $x \otimes y \lor x \otimes z \leq x \otimes (y \lor z)$. It remains to prove the converse. By (1), $y \leq (x \Rightarrow x \otimes z)$, By (2), it follows that $y \leq (x \Rightarrow x \otimes (y \lor z))$. Similarly, it can be shown that $z \leq (x \Rightarrow x \otimes (y \lor z))$. Hence, $y \lor z \leq (x \Rightarrow x \otimes (y \lor z))$. By (1), it follows that $x \otimes (y \lor z) \leq x \otimes y \lor x \otimes z$. This completes the proof of (6).

- The assertion (7) follows from (1) and the commutativity of $\otimes$.

- Consider the assertions (8) and (9). By (1) and (2), $x \otimes (x \Rightarrow y) \otimes (y \Rightarrow z) \leq y \otimes (y \Rightarrow z) \leq z$. Hence, $x \otimes (x \Rightarrow y) \otimes (y \Rightarrow z) \leq (y \Rightarrow (x \otimes z))$. This implies (8), by using (1). The assertion (9) follows from (8), by using (2).

- Consider the assertion (10). We need to prove that $x \otimes (y \leq z) \leq (y \Rightarrow (x \otimes z))$ (A.1) $x \otimes (y \leq z) \leq (x \otimes z \Rightarrow y)$ (A.2)
Consider the assertions (11)–(13). By (1), \((z \Rightarrow y) \otimes z \leq y\). By (2), it follows that \(x \otimes (y \Rightarrow z) \otimes y \leq x \otimes y\). By (1), it follows that \(x \otimes (y \Rightarrow z) \otimes z \leq y\). By (1), it follows that \(x \otimes (z \Rightarrow y) \otimes x \otimes z \leq y\). By (1), it follows that \(x \otimes (z \Rightarrow y) \otimes (x \otimes z) \leq y\). This implies (A.2), by using (2).

Consider the assertion (14). By (7) and (2), we have that \((x \Rightarrow y) \otimes x \leq (y \Rightarrow z) \otimes y \leq z\). By (2), it follows that (A.3) also implies (12), by using (1) and the commutativity of \(\otimes\). The assertion (13) follows from (12), by using (3).

Consider the assertion (14). By (7) and (2), we have that \(x \otimes (y \Rightarrow z) \otimes (x \otimes z) \leq y\). The assertion (14) follows from this, using (1) and the commutativity of \(\otimes\).

The assertion (15) follows from (14), using (2) and the commutativity of \(\otimes\) and \(\otimes\).

Due to the commutativity of \(\otimes\), to prove (16) it is sufficient to show that \((x \Leftrightarrow x') \land (y \Leftrightarrow y') \leq (x \land y \Rightarrow x' \land y')\).

By (1), this is equivalent to \(((x \Leftrightarrow x') \land (y \Leftrightarrow y')) \otimes (x \land y) \leq x' \land y'\). We need to prove that
\[
((x \Leftrightarrow x') \land (y \Leftrightarrow y')) \otimes (x \land y) \leq x'.
\]
By (1), this is equivalent to \(((x \Leftrightarrow x') \land (y \Leftrightarrow y')) \otimes (x \land y) \leq y'.

By (1), \((x \Rightarrow y) \otimes x \leq x'\). This implies (A.4), by using (2). Similarly, (A.5) also holds.

Due to the commutativity of \(\otimes\), to prove (17) it is sufficient to show that \((x \Leftrightarrow x') \land (y \Leftrightarrow y') \leq (x \lor y \Rightarrow x' \lor y')\).

By (1), this is equivalent to \(((x \Leftrightarrow x') \land (y \Leftrightarrow y')) \otimes (x \lor y) \leq x' \lor y'\). By (1), it is sufficient to prove that
\[
((x \Leftrightarrow x') \land (y \Leftrightarrow y')) \otimes x \leq x',
\]
\[
((x \Leftrightarrow x') \land (y \Leftrightarrow y')) \otimes y \leq y'.
\]
By (1), \((x \Rightarrow y) \otimes x \leq x'\). This implies (A.6), by using (2). Similarly, (A.7) also holds.

To prove (18), it is sufficient to show that \((x \Rightarrow y) \leq ((z \Rightarrow x) \Rightarrow (z \Rightarrow y))\). By (1), this is equivalent to \((x \Rightarrow y) \otimes (x \Rightarrow z) \otimes z \leq y\). This latter inequality holds because, by (7) and (2),
\[
z \otimes (z \Rightarrow x) \otimes (x \Rightarrow y) \leq x \otimes (x \Rightarrow y) \leq y.
\]

To prove (19), it is sufficient to show that \((y \Rightarrow x) \leq ((x \Rightarrow z) \Rightarrow (y \Rightarrow z))\). By (1), this is equivalent to \((x \Rightarrow y) \otimes (x \Rightarrow z) \otimes y \leq z\). This latter inequality holds because, by (7) and (2),
\[
y \otimes (y \Rightarrow x) \otimes (x \Rightarrow z) \leq x \otimes (x \Rightarrow z) \leq z.
\]
• Consider the assertion \((20)\) and suppose that \(L\) is a Heyting algebra. Due to the commutativity of \(\equiv\), to prove \((20)\) it is sufficient to prove that

\[(x \equiv x') \otimes (y \equiv y') \leq ((x \Rightarrow y) \Rightarrow (x' \Rightarrow y')).\]

By \((2)\) and \((11)\), it is sufficient to prove that \((x' \Rightarrow x) \otimes (y \Rightarrow y') \otimes (x \Rightarrow y) \otimes x' \leq y'.\) This holds because, by \((7)\) and \((2)\),

\[x' \otimes (x' \Rightarrow x) \otimes (y \Rightarrow y') \leq x \otimes (x \Rightarrow y) \otimes (y \Rightarrow y') \leq y \otimes (y \Rightarrow y') \leq y'.\]

• Consider the assertion \((21)\) and suppose that \(L\) is a Heyting algebra and \(x \leq (y \Leftrightarrow z)\). Since \(x \leq (y \Leftrightarrow z)\), we have that \(x \leq (y \Leftrightarrow z)\). By \((11)\), it follows that \(x \land y = x \otimes y \leq z\). Hence, \(x \land y \leq x \land z\) since \(\land\) is idempotent. Similarly, it can also be shown that \(x \land z \leq x \land y\).

Therefore, \(x \land y = x \land z\), which means \(x \otimes y = x \otimes z\).

Appendix B. The Relationship with Fuzzy Bisimulations between Fuzzy Automata

Clearly, a fuzzy Kripke model can be treated as a fuzzy labeled transition system (FLTS) and Definition 3.1 (which specifies fuzzy bisimulations) can be applied to FLTSs. In [1], Čirić et al. introduced a few kinds of fuzzy bisimulations (and simulations) for fuzzy automata over complete residuated lattices. Among them the one that researchers would have in mind as the default is called “forward bisimulation”. We recall it below and simply refer to it as fuzzy bisimulation between fuzzy automata. After that we relate it to the notion of fuzzy bisimulation between fuzzy Kripke models.

In this appendix, suppose that the underlying residuated lattice \(L\) is complete.

Given fuzzy sets \(R : X \to L\), \(S : Y \to L\) and \(Z : X \times Y \to L\), we define \((R \circ Z) : Y \to L\) and \((Z \circ S) : X \to L\) to be the fuzzy sets such that

\[(R \circ Z)(y) = \sup \{R(x) \otimes Z(x, y) \mid x \in X\} \quad \text{for } y \in Y;\]

\[(Z \circ S)(x) = \sup \{Z(x, y) \otimes S(y) \mid y \in Y\} \quad \text{for } x \in X.\]

A fuzzy automaton over an alphabet \(\Sigma\) (and \(L\)) is a tuple \(A = \langle A, \delta^A, \sigma^A, \tau^A \rangle\), where \(A\) is a non-empty set of states, \(\delta^A : A \times \Sigma \times A \to L\) is the fuzzy transition function, \(\sigma^A : A \to L\) is the fuzzy set of initial states, and \(\tau^A : A \to L\) is the fuzzy set of terminal states. For \(\varrho \in \Sigma\), by \(\delta^\varrho_A\) we denote the fuzzy relation on \(A\) such that \(\delta^\varrho_A(x, y) = \delta^A(x, \varrho, y)\) for \(x, y \in A\).

A fuzzy automaton \(A = \langle A, \delta^A, \sigma^A, \tau^A \rangle\) is image-finite if:

• the set \(\{x \in A \mid \sigma^A(x) > 0\}\) is finite, and

• for every \(\varrho \in \Sigma\) and every \(x \in A\), the set \(\{y \in A \mid \delta^\varrho_A(x, y) > 0\}\) is finite.

Given fuzzy automata \(A = \langle A, \delta^A, \sigma^A, \tau^A \rangle\) and \(A' = \langle A', \delta'^{A'}, \sigma'^{A'}, \tau'^{A'} \rangle\) over an alphabet \(\Sigma\), a fuzzy bisimulation (called “forward bisimulation” in \([3]\)) between \(A\) and \(A'\) is a fuzzy relation \(Z : A \times A' \to L\) satisfying the following conditions for all \(\varrho \in \Sigma\):

\[\sigma^A \leq \sigma'^{A'} \circ Z^- \quad \text{(B.1)}\]

\[Z^- \circ \delta^\varrho_A \leq \delta'^{\varrho'}_A \circ Z^- \quad \text{(B.2)}\]
Given a fuzzy automaton $A = (A, \delta^A, \sigma^A, \tau^A)$ over an alphabet $\Sigma$, we define the fuzzy Kripke model corresponding to $A$ to be the fuzzy Kripke model $M$ over the signature $\langle \Sigma_A, \Sigma_P \rangle$ with $\Sigma_A = \Sigma$ and $\Sigma_P = \{i, f\}$ such that:

- $\Delta^M = A \cup \{s_i, s_f\}$, where $s_i$ and $s_f$ are new states;
- $i^M = \{s_i : 1\}$ and $f^M = \{s_f : 1\}$;
- for every $\varrho \in \Sigma_A$, $x, y \in A$ and $z \in \Delta^M$:
  - $g^M(x, y) = \delta^A(x, \varrho, y)$,
  - $g^M(s_i, x) = \sigma^A(x)$ and $g^M(x, s_f) = \tau^A(x)$,
  - $g^M(z, s_i) = g^M(s_f, z) = g^M(s_i, s_f) = 0$.

Thus, $s_i$ (resp. $s_f$) stands for the new unique initial (resp. terminal) state; the propositions $i$ and $f$ are used to identify $s_i$ and $s_f$, respectively. The given definition is a counterpart of the definition of the fuzzy interpretation (in description logic) that corresponds to a fuzzy automaton [24].

Recall that, in this appendix, the underlying residuated lattice is assumed to be complete. The following proposition relates our notion of fuzzy bisimulation between fuzzy Kripke models to the notion of fuzzy bisimulation between fuzzy automata, which is defined and called “forward bisimulation” by Ćirić et al. [6].

**Proposition Appendix B.1.** Let $A = \langle A, \delta^A, \sigma^A, \tau^A \rangle$ and $A' = \langle A', \delta^{A'}, \sigma^{A'}, \tau^{A'} \rangle$ be fuzzy automata over the same alphabet, $M$ and $M'$ the fuzzy Kripke models corresponding to $A$ and $A'$, respectively. Let $s_i, s_f \in \Delta^M$ and $s'_i, s'_f \in \Delta^{M'}$ be the states such that $i^M(s_i) = f^M(s_f) = i^{M'}(s'_i) = f^{M'}(s'_f) = 1$. Let $Z$ be a fuzzy relation between $A$ and $A'$, $Z_2$ the fuzzy relation between $\Delta^M$ and $\Delta^{M'}$ such that $Z_2 = Z \cup \{(s_i, s'_i) : 1, (s_f, s'_f) : 1\}$.

- If $Z_2$ is a fuzzy bisimulation between $M$ and $M'$, then $Z$ is a fuzzy bisimulation between $A$ and $A'$.
- Conversely, if the underlying residuated lattice is also linear, $A$ and $A'$ are image-finite and $Z$ is a fuzzy bisimulation between $A$ and $A'$, then $Z_2$ is a fuzzy bisimulation between $M$ and $M'$.

This proposition uses conditions similar to the ones of Proposition 3.3. The reason is that the definition of fuzzy bisimulations between fuzzy automata uses conditions similar to (25)–(27).

**Proof.** Suppose that $Z_2$ is a fuzzy bisimulation between $M$ and $M'$. We show that $Z$ is a fuzzy bisimulation between $A$ and $A'$. Let $\varrho \in \Sigma$. We prove that $Z$ satisfies the conditions (B.1)–(B.3). The proof of that $Z$ satisfies the conditions (B.4)–(B.6) is similar and omitted.

Consider the condition (B.1) and let $y \in A$. We need to prove that

$$
\sigma^A(y) \leq (\sigma^{A'} \circ Z^{-})(y).
$$

(B.7)
Since $Z_2$ is a fuzzy bisimulation between $\mathcal{M}$ and $\mathcal{M}'$, by the condition (23) for $Z_2$,  

$$\exists y' \in \Delta^{\mathcal{M}'} \ (Z_2(s_i, s_i') \otimes g^\mathcal{M}(s_i, y) \leq g^\mathcal{M}'(s_i', y') \otimes Z_2(y, y')).$$

By the assumptions about $Z_2$, $\mathcal{M}$ and $\mathcal{M}'$, this means that there exists $y' \in \Delta^{\mathcal{M}'}$ such that  

$$\sigma^A(y) \leq g^\mathcal{M}'(s_i', y') \otimes Z_2(y, y').$$

(B.8)

Since $y \in A$, if $y' \notin A'$, then $Z_2(y, y') = 0$, and by (B.8), (B.7) implies (B.9). If $y' \in A'$, then $g^\mathcal{M}'(s_i', y') = \sigma^A(y')$, $Z_2(y, y') = Z(y, y')$ and (B.8) also implies (B.7).

Consider the condition (B.2). Let $x' \in A'$ and $y \in A$. We need to prove that  

$$(Z^{-} \circ \delta^A_e)(x', y) \leq (\delta^A_e \circ Z^{-})(x', y).$$

Let $x$ be an arbitrary element of $A$. It is sufficient to show that  

$$Z(x, x') \otimes \delta^A_e(x, y) \leq (\delta^A_e \circ Z^{-})(x', y).$$

(B.9)

Since $Z_2$ is a fuzzy bisimulation between $\mathcal{M}$ and $\mathcal{M}'$, by the condition (23) for $Z_2$,  

$$\exists y' \in \Delta^{\mathcal{M}'} \ (Z_2(x, x') \otimes g^\mathcal{M}(x, y) \leq g^\mathcal{M}'(x', y') \otimes Z_2(y, y')).$$

By the assumptions about $Z_2$, $\mathcal{M}$ and $\mathcal{M}'$, this means that there exists $y' \in \Delta^{\mathcal{M}'}$ such that  

$$Z(x, x') \otimes \delta^A_e(x, y) \leq g^\mathcal{M}'(x', y') \otimes Z_2(y, y').$$

(B.10)

Since $y \in A$, if $y' \notin A'$, then $Z_2(y, y') = 0$, and by (B.10), (B.11) implies (B.10). If $y' \in A'$, then $g^\mathcal{M}'(x', y') = \delta^A_e(x', y')$, $Z_2(y, y') = Z(y, y')$ and (B.10) also implies (B.11).

Consider the condition (B.3) and let $x' \in A'$. We need to prove that  

$$(Z^{-} \circ \tau^A_e)(x') \leq \tau^A_e(x').$$

Let $x$ be an arbitrary element of $A$. It is sufficient to show that  

$$Z(x, x') \otimes \tau^A_e(x) \leq \tau^A_e(x').$$

(B.11)

Since $Z_2$ is a fuzzy bisimulation between $\mathcal{M}$ and $\mathcal{M}'$, by the condition (23) for $Z_2$,  

$$\exists y' \in \Delta^{\mathcal{M}'} \ (Z_2(x, x') \otimes g^\mathcal{M}(x, s_f) \leq g^\mathcal{M}'(x', y') \otimes Z_2(s_f, y')).$$

By the assumptions about $Z_2$, $\mathcal{M}$ and $\mathcal{M}'$, this means that there exists $y' \in \Delta^{\mathcal{M}'}$ such that  

$$Z(x, x') \otimes \tau^A_e(x) \leq g^\mathcal{M}'(x', y') \otimes Z_2(s_f, y').$$

(B.12)

If $y' \neq s_f'$, then $Z_2(s_f, y') = 0$, and by (B.12), (B.11) implies (B.12). If $y' = s_f'$, then $g^\mathcal{M}'(x', y') = \tau^A_e(x')$, $Z_2(s_f, y') = 1$ and (B.12) also implies (B.11).

For the converse, suppose that the underlying residuated lattice is linear, $\mathcal{A}$ and $\mathcal{A}'$ are image-finite and $Z$ is a fuzzy bisimulation between $\mathcal{A}$ and $\mathcal{A}'$. We show that $Z_2$ is a fuzzy bisimulation.
between $\mathcal{M}$ and $\mathcal{M}'$. Recall that $\Sigma_P = \{i, f\}$. Let $p \in \Sigma_P$ and $\varrho \in \Sigma_A$. We need to prove that the following assertions hold for all possible values of the free variables:

\begin{align}
Z_2(x, x') &\leq (p^\mathcal{M}(x) \iff p^\mathcal{M}'(x')) \\
\exists y' \in \Delta^\mathcal{M}' (Z_2(x, x') \otimes \varrho^\mathcal{M}(x, y) \leq \varrho^\mathcal{M}'(x', y') \otimes Z_2(y, y')) \\
\exists y \in \Delta^\mathcal{M} (Z_2(x, x') \otimes \varrho^\mathcal{M}(x', y') \leq \varrho^\mathcal{M}(x, y) \otimes Z_2(y, y')).
\end{align}

Let $x \in \Delta^\mathcal{M}$, $x' \in \Delta^\mathcal{M}'$ and consider the assertion (B.13). If $\langle x, x' \rangle \in A \times A'$, then $p^\mathcal{M}(x) = p^\mathcal{M}'(x') = 0$ and (B.13) clearly holds. If $Z_2(x, x') = 0$, then (B.13) also holds. Suppose that $\langle x, x' \rangle \notin A \times A'$ and $Z_2(x, x') > 0$. Thus, $\langle x, x' \rangle = \langle s_i, s'_i \rangle$ or $\langle x, x' \rangle = \langle s_f, s'_f \rangle$. In both of these cases, for any $p \in \Sigma_P = \{i, f\}$, $p^\mathcal{M}(x) = p^\mathcal{M}'(x')$. Hence, (B.13) holds.

Let $x, y \in \Delta^\mathcal{M}$, $x' \in \Delta^\mathcal{M}'$ and consider the assertion (B.14). If $Z_2(x, x') \otimes \varrho^\mathcal{M}(x, y) = 0$, then (B.14) clearly holds. Suppose that $Z_2(x, x') \otimes \varrho^\mathcal{M}(x, y) > 0$. By (B.13), it follows that $Z_2(x, x') > 0$ and $\varrho^\mathcal{M}(x, y) > 0$. There are the following cases.

- **Case** $x, y \in A$ and $x' \in A'$: We have

\[ Z_2(x, x') \otimes \varrho^\mathcal{M}(x, y) = Z(x, x') \otimes \delta^A_\varrho(x, y). \]  

(B.16)

By (B.2),

\[ Z(x, x') \otimes \delta^A_\varrho(x, y) \leq (\delta^A_\varrho \circ Z^-)(x', y'). \]  

(B.17)

Since $\mathcal{A}'$ is image-finite and the underlying residuated lattice is linear and complete, there exists $y' \in A'$ such that

\[ (\delta^A_\varrho \circ Z^-)(x', y') = \delta^A(x', y') \otimes Z(y, y') = \varrho^\mathcal{M}'(x', y') \otimes Z_2(y, y'). \]  

(B.18)

The assertions (B.16)–(B.18) together imply (B.14).

- **Case** $x = s_i, y \in A$ and $x' = s'_i$: We have

\[ Z_2(x, x') \otimes \varrho^\mathcal{M}(x, y) = 1 \otimes \sigma^A(y) = \sigma^A(y). \]  

(B.19)

By (B.1),

\[ \sigma^A(y) \leq (\sigma^A \circ Z^-)(y). \]  

(B.20)

Since $\mathcal{A}'$ is image-finite and the underlying residuated lattice is linear and complete, there exists $y' \in A'$ such that

\[ (\sigma^A \circ Z^-)(y) = \sigma^A(y') \otimes Z(y, y') = \varrho^\mathcal{M}'(x', y') \otimes Z_2(y, y'). \]  

(B.21)

The assertions (B.19)–(B.21) together imply (B.14).

- **Case** $x \in A$, $y = s_f$ and $x' \in A'$: We have

\[ Z_2(x, x') \otimes \varrho^\mathcal{M}(x, y) = Z(x, x') \otimes \tau^A(x). \]  

(B.22)

By (B.3),

\[ Z(x, x') \otimes \tau^A(x) \leq \tau^A(x') = \varrho^\mathcal{M}'(x', s'_f) \otimes Z_2(y, s'_f). \]  

(B.23)

The assertions (B.22) and (B.23) together imply (B.14).

The assertion (B.13) can be proved analogously.