SINGULAR RIEMANNIAN FOLIATIONS AND THE PRESCRIBING SCALAR CURVATURE PROBLEM

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Abstract. An orbit-like foliation is a singular foliation on a complete Riemannian manifold $M$ whose leaves are locally equidistant (i.e., a singular Riemannian foliation) and (transversely) infinitesimally homogenous. This class of singular foliation contains not only the class of partition of the space into orbits of isometric actions, but also infinite many non homogenous examples and in particular the partition of $M$ into orbits of a proper groupoid.

In this paper we prove a version of Kondrakov Embedding Theorem and an analogous Principle of Symmetric Criticality of Palais for basic functions of orbit-like foliations. As proof of concepts, we study not only the corresponding Yamabe problem in the setting, but also to the case of fiber bundles with homogeneous fibers, seeking for the existence of metrics with constant scalar curvature that respect the respective Riemannian Foliation decomposition. An application to the existence of positive constant scalar curvature on exotic spheres is presented. In an upcoming version we shall extend the results to the corresponding Kazdan–Warner problem.

1. Introduction

It is classical nowadays (see for instance [Heb90, HV93, Heb00]) that geometric analytic problems modeled on manifolds equipped with symmetries coming from group actions are easier to deal given both, the existence of better compactness embeddings of Sobolev spaces in Lebesgue spaces [HV97], and the classical Principle of Symmetric Criticality due to Palais [Pal79].

A recent proof of concept of the aforementioned discussion is presented in [CaMdOS21], where the authors pose and solve the analogous Kazdan–Warner problem ([KW75a, KW75b, KW75c]) in the setting of Riemannian manifolds with isometric group actions: which invariant functions are the scalar curvature of Riemannian metrics?

On the other hand, a partition of the Riemannian manifolds into orbits of isometric actions are particular examples of singular Riemannian foliations (SRF for short), i.e., singular foliations locally equidistant (see Definition 2.1). This more general class of singular foliations, appear naturally in different context. In particular, the partition of orbits of a proper groupoid (recall Definition 2.9 and discussion in Section 2.1.3) are examples of the so called orbit-like foliation, i.e., SRF whose restriction to each slice is
homogenous, i.e., coming from an action of a compact group (see Definition 2.7). As far as we know, even the most classical examples of non homogenous orbit-like foliations of codimension one in spheres (see [Kar81]) and the more recent examples of orbit-like foliations of codimension greater than one in spheres (see [Rad14]) may not come from proper (global) groupoids. Thus, at least until to this date, orbit-like foliations are a broader class of singular foliations than those whose leaves are orbits of proper groupoids.

Pursuing to establish an analytic framework to consider both physical and geometric problems modeled on Riemannian manifolds with singular Riemannian foliations, and in particular orbit-like foliations, in this paper we synthesize and disseminate the concept of the Sobolev space of basic distributions, which consists in the Banach space of distributions that are constant along the leaves of a Singular Riemannian Foliation. In particular we prove a Kondrakov type result.

**Theorem 1.1** (Kondrakov-type theorem). Suppose that $M$ is a connected compact Riemannian manifold and $\mathcal{F}$ is a SRF on $M$ whose leaves are closed. Then there exists $p_0 > p^* := np/(n-p)$ such that given $1 < q < p_0$, the canonical embedding $W^{1, p}(M)^{\mathcal{F}} \hookrightarrow L^q(M)$ is compact.

The well definition of Sobolev space of basic distribution is possible given the existence of a natural basic projection operator, which plays the analogous role of the classical average operator in Riemannian manifolds with group actions (see [Bre72]). This operator also allows us to prove principle of Basic criticality for orbit-like foliations, see Lemma 2.19.

Since both Yamabe ([Tru68, Aub76, SY79a, SY79b, SY81]) and Kazdan–Warner problems have re-gaining huge interest one naturally uses the developed machinery to approach these in our scenario. Namely, we prove:

**Theorem 1.2.** Let $M^n$, $n \geq 3$, be a closed Riemannian manifold endowed with an orbit-like foliation $\mathcal{F}$ with closed leaves. Then $M$ has a Riemannian metric of constant scalar curvature for which $\mathcal{F}$ is an orbit-like foliation.

Once Riemannian submersions are a particular kind of manifold with Riemannian foliations, and metrics with curvature properties gain an additional constraint in this scenario given the isometry condition between horizontal space and the base Riemannian manifold, to stretch the range of the developed machinery, we also present a result concerning the existence of Riemannian submersion metrics of constant positive scalar curvature on fiber bundles with homogeneous fibers:

**Theorem 1.3.** Let $M^n$, $n \geq 3$, be a closed Riemannian manifold endowed with a Foliation $\mathcal{F}$ induced by a fiber bundle such that:

(i) The structure group $G$ is compact and has non-abelian Lie algebra;
(ii) The fiber $L$ is an homogeneous space.

Then $M$ has a Riemannian metric of positive constant scalar curvature for which $\mathcal{F}$ is Riemannian.
Then a simple combination of the classical Eells–Kuiper invariant ([ES64]), which determine the number of diffeomorphism classes of exotic spheres that can be realized as the total sphere bundles; with Theorem 1.3, allow us to obtain the following:

**Corollary 1.4.** 16 (resp. 4.096) from the 28 (resp. 16.256) diffeomorphisms classes of the 7-dimensional (resp .15)-exotic spheres admit metrics of positive constant scalar curvature. Moreover, these can be taken as Riemannian submersion metrics when such spaces are considered as the total space of sphere bundles.

This paper is organized as follows. In Section 2 we review the linear Lie groupoid associated to the semi-local description of an orbit-like foliation $\mathcal{F}$ near a closed leave $B$ of $\mathcal{F}$. We try to present this in a self contained presentation, hopping to make it accessible also to readers without previous training in groupoid or singular Riemannian foliations. Then a type of principle of symmetric criticality of Palais is proved in a neighborhood of a leaf $B \in \mathcal{F}$; see Lemma 2.19. In Section 3 we check that the operator $J$ associated to the Yamabe problem satisfies the basic criticality principle, once one considers a special basic metric. Then in Section 4 Sobolev spaces of basic functions of SRF $\mathcal{F}$ and the $\mathcal{F}$-average operator on $M$ are presented and Theorem 1.1 is proved. Finally Theorems 1.2 and 1.3 as well Corollay as 1.4 are proved in Section 5.

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2. **Linear Lie groupoid and criticality**

2.1. **A few facts about singular Riemannian foliations, orbit-like foliations and the Linear groupoid.** In this section we review serveral facts on singular Riemannian foliations $\mathcal{F}$, most of them can be found in [AR17] [ ARTMS21]. We also stress that, along this paper, the leaves of $\mathcal{F}$ are closed on a compact manifold $M$.

2.1.1. **Singular Riemannian foliations.**

**Definition 2.1** (SRF). A *singular Riemannian foliation* on a complete Riemannian manifold $M$ is a partition $\mathcal{F} = \{L\}$ of $M$ into immersed submanifolds without self-intersections (the *leaves*) that satisfies the following properties:
(a) $F$ is a singular foliation, i.e., for each $v_p$ tangent to $L_p$ (i.e., the leaf through $p \in M$) then there exists a local vector field $\vec{X}$ so that $\vec{X}(p) = v_p$ and $\vec{X}$ is tangent to the leaves;

(b) $F$ is Riemannian, i.e, each geodesic $\gamma$ that starts orthogonal to a leaf $L_{\gamma(0)}$ remains orthogonal to all leaves that it meets.

**Remark 2.2.** Item (a) is equivalent to saying that given a point $q \in M$, there exists a neighborhood $U$ of $q$ in $M$, a simple foliation $P = \{P\}$ on $U$ (i.e, given by fibers of a submersion on $U$) so that the leaf $P_q$ (the plaque through $q$) is a relative compact open set of the leaf $L_q$ and $P$ is a subfoliation of $F|U$, i.e, for each $x \in U$ we have $P_x \subset L_x$. From now on $P \subset F|U$ denotes to be a subfoliation. In particular item (a) implies that $F \cap S_q$ is a singular foliation for each transverse submanifold $S_q$, i.e., $T_qM = T_qS_q \oplus T_qL_q$. Roughly speaking item (b) says that the leaves are locally equidistant. In other words, item (b) is equivalent to saying that there exists $\epsilon > 0$ so that if $x \in \partial \text{tub}_\epsilon(P_q)$ (the cilinder of radius $\epsilon$ of the plaque $P_q$) then the connected component of $L_x \cap U$ containing $x$ is contained in $\partial \text{tub}_\epsilon(P_q)$.

Typical examples of singular Riemannian foliations (SRF for short) are, among others, the partition of $M$ into orbits of isometric actions; infinite many examples of nonhomogenous SRF on Euclidean spheres constructed by Radeschi using Clifford system [Rad14]; the holonomy foliation in a Euclidean fiber bundle with a connection compatible with the metric of the fibers (see Example 2.11).

Several properties of SRF are natural generalizations of classical properties of the partition of $M$ into orbits of isometric actions, see [AB15]. Let us review a few of them.

The first one is the generalization of the so called slice representation. Let $\pi : U \to L_q$ be the metric projection, and $S_q = \pi^{-1}(q)$ be the slice i.e., $S_q := \exp_q(\nu_qL \cap B_\epsilon(0))$ where $\nu_q(L_q)$ is the normal space. Then the infinitesimal foliation $F_q = \exp_q^{-1}(S_q \cap F)$ turns to be a SRF on the open set of the Euclidean space $(\nu_qL_q, g_q)$. The infinitesimal foliation $F_q$ on a neighborhood of $\nu_q(L_q)$ can be extended via the homothetic transformation $h^0_\lambda(v) = \lambda v$ to a SRF on $(\nu_q(L_q), g_q)$. The foliation $F_q$ plays a role in the theory of SRF similar to the role played by the slice representation in the theory of isometric actions.

Another general property of SRF that is analogous to the theory of isometric action, is that the partition of $M$ into the leaves of $F$ with the same dimension is a stratification. Recall that a stratification of $M$ is a partition of $M$ into embedded submanifolds $\{M_i\}_{i \in I}$ (called strata) such that:

(i) the partition is locally finite, i.e., each compact subset of $M$ only intersects a finite number of strata;
(ii) for each $i \in I$, there exists a subset $I_i \subset I/\{i\}$ such that the closure
of $M_i$ is $M_i = M_i \cup \bigcup_{j \in I_i} M_j$;
(iii) dim $M_j < \dim M_i$ for all $j \in I_i$

The stratum with the leaves of greatest dimension (the regular leaves) is a
open dense set, and its space of leaves is connected.

By successively blowing up along singular stratum we have a
desingularization of SRF. More more precisely:

**Theorem 2.3** ([Ale10]). Let $F$ be a singular Riemannian foliations on a
Riemannian compact manifold $(M, g)$ with compact leaves. For each small
$\epsilon > 0$ there exists a singular Riemannian foliation $F_\epsilon = \{(L_\epsilon)_x\} x \in M$, on a
Riemannian compact manifold $(M_\epsilon, g_\epsilon)$ and map $\pi_\epsilon : M_\epsilon \to M$ so that:

(a) $\pi_\epsilon$ projects leaves of $F_\epsilon$ to leaves of $F$;
(b) let $\Sigma$ be the singular stratum and $\Sigma_\epsilon = \pi_\epsilon^{-1}(\Sigma)$ then $\pi_\epsilon : M_\epsilon - \Sigma_\epsilon \to M - \Sigma$ is a foliated diffeomorphism;
(c) $d(L_{\pi_\epsilon(x)}, L_{\pi_\epsilon(y)})$ $-$ $d((L_\epsilon)_x, (L_\epsilon)_y) \leq \epsilon.

In particular the metric space $M/F$ is a Gromov-Hausdorff limit of a
sequence of Riemannian orbifolds $\{M_n/F_n\}$.

In the next section, we will consider a particular type of SRF (the so
called orbit-like foliation) that is fundamental to understand the semi-local
model of SRF; see Theorem 2.12.

2.1.2. Linearized foliations and orbit-like foliations. Given a closed leaf $B = L_q$ we can always find a $F$-saturated tubular neighborhood $U = \text{tub}_r(L_q)$ of
$L_q$. The foliation restricted to $U$, i.e., $F|_U$ (and in particular the partition
by plaques) are invariant by the homothetic transformation $h_\lambda : U \to U$
defined as $h_\lambda(\exp(v)) = \exp(\lambda v)$ for each $v \in \nu^\epsilon(B)$ where $\lambda \in (0, 1]$.

For each smooth vector field $\bar{X}$ in $U$ tangent to $F$, we associate a smooth
vector field $\bar{X}^\ell$, called the linearization of $\bar{X}$ with respect to $B$ as:

$$\bar{X}^\ell(q) = \lim_{\lambda \to 0} (h_\lambda^{-1})_*(\bar{X}) \circ h_\lambda(q)$$

Since the restricted foliation $F|_U$ is homothetic invariant, $\bar{X}^\ell$ is still tangent
to $F$. It is possible to prove that

**Lemma 2.4.** The flows of these vector fields, once identified with the normal
exponential map, induce isometries on the fibers of the normal $\delta$-fiber bundle
$E^\delta = \nu^\delta(B) = \{\xi \in \nu(B), \|\xi\| < \delta\}.$

**Example 2.5.** Given a SRF $F$ with compacts leaves on $\mathbb{R}^m$, and $B = 0$
we have for $\bar{X}$ tangent to the leaves of $F$ that $\bar{X}^\ell(v) = \lim_{\lambda \to 0} \frac{1}{\lambda} \bar{X}(\lambda v) = (\nabla_v \bar{X})_0$ i.e, $\bar{X}^\ell$ is in fact a linear vector field. In addition, one can check that
it is also a Killing vector field. This can be proved using the fact that the
leaves are tangent to the spheres and hence $0 = (\bar{X}^\ell(v), v) = (\nabla_v \bar{X})_0, v)$.

Note that the Killing vector fields $\bar{X}^\ell$ induce a Lie algebra of a connected
Lie subgroup $K^0 \subset O(n)$. Since by hypothesis $F$ is compact, it is possible
to check that $K^0$ is also compact. We have then in this example an homogenous subfoliation $\mathcal{F}^\ell = \{K^0(v)\}_{v \in \mathbb{R}^m} \subset \mathcal{F}$. This turns to be the maximal homogenous subfoliation of $\mathcal{F}$.

The above example illustrates a more general phenomenon.

**Definition 2.6.** Given a SRF $\mathcal{F}|_U$ with compact leaves, the composition of linearized flows tangent to $\mathcal{F}|_U$ induces a singular subfoliation $\mathcal{F}^\ell \subset \mathcal{F}|_U$ on $U$, the so called linearized foliation $\mathcal{F}^\ell$. $\mathcal{F}^\ell$ can also been seen as the maximal infinitesimal homogenous subfoliation of $\mathcal{F}|_U$. In other words, Let $\pi: U \to L_q$ be the metric projection, and $S_q = \pi^{-1}(q)$ be a slice, i.e., $S_q := \exp_q(L_q \cap B_\epsilon(0))$. Define $\mathcal{F}_q^\ell$ as the extension of $\exp_q^{-1}(S_q \cap \mathcal{F}^\ell)$ via the homothetic transformation $h_\lambda^q(v) = \lambda v$.

The foliation $\mathcal{F}_q^\ell$ is the maximal homogenous subfoliation of the infinitesimal foliation $\mathcal{F}_q$.

**Definition 2.7 (Orbit-like foliation).** A SRF $(M, \mathcal{F})$ with compact leaves is called orbit-like if for each leaf $B = L_q$ we have $\mathcal{F}^\ell = \mathcal{F}|_U$. In other words if for each $q \in M$ the infinitesimal foliation $\mathcal{F}_q$ is homogenous, and the leaves are orbits of a compact (isometric) group $K_0^q$.

**Remark 2.8.** To be orbit-like could be consider a topological property in the following sence: Let $(M_1, \mathcal{F}_1)$ be two SRF and $\psi: (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)$ be a foliated diffeomorphism. Then $(M_2, \mathcal{F}_2)$ is orbit-like if and only if $(M_1, \mathcal{F}_1)$ is orbit-like, see [AR17].

2.1.3. *Sasaki metrics, SRF and holonomy groupoid.* Let $E = \nu(B)$ be the normal bundle of a compact leaf $B = L$ of $\mathcal{F}$. Consider the Euclidean vector bundle $\mathbb{R}^k \to E \to B$ where the metric on each fiber $E_p$ is defined as the metric $g_p$ for $p \in B$.

By pulling back via the normal exponential map, we can identify the foliation $\mathcal{F}^\ell$ and $\mathcal{F}|_U$ to singular foliations on open set $(\exp^\nu)^{-1}(U)$ of $E$, and from now on we use the same notation for the foliations on $U$ or on $(\exp^\nu)^{-1}(U)$. By homothetic transformation we can extend $\mathcal{F}^\ell$ and $\mathcal{F}$ to $E$.

We recall that there exists a Sasaki metric $g^0$ on $E$ so that $\mathcal{F}^\ell$ and $\mathcal{F}$ are singular Riemannians foliations. In fact, we can find a distribution $\mathcal{T}$ homothetic invariant that is tangent to $\mathcal{F}^\ell$ and $\mathcal{F}$. This distribution can be constructed by finding a distribution $\hat{\mathcal{T}}$ tangent to $\mathcal{F}$ and then linearizing the vector fields tangent to $\hat{\mathcal{T}}$. In particular there may exist different ways to construct $\mathcal{T}$. Let $g^0$ be the associated Sasaki metric, i.e., the metric so that:

- $\mathcal{T}$ is orthogonal to the fibers $E$,
- the foot point projection, $\pi: (E, g^0) \to (B, g)$ is a Riemannian submersion
- and the fibers $E_p$ have the flat metric $g_p$. 

Let us denote $\nabla^r$ the connection associated to the distribution $\mathcal{T}$. It is possible to check that $\nabla^r$ is compatible with the Euclidean metric on the fibers of $E$.

As we are going to recall in Theorem 2.12, we need to recall the definition of Lie groupoid. We can use the connection $\nabla^r$ to describe $\mathcal{F}$ and its linearization $\mathcal{F}^r$. In order to better understand Theorem 2.12 we need to recall the definition of Lie groupoid.

**Definition 2.9.** A Lie groupoid $\mathcal{G} = \mathcal{G}_1 \rightrightarrows \mathcal{G}_0$ consist of:

1. a manifold $\mathcal{G}_0$ called the set of objects;
2. a (possible non Hausdorff) manifold $\mathcal{G}_1$ called the set of arrows (between objects);
3. submersions $s, t : \mathcal{G}_1 \to \mathcal{G}_0$ which associate to an arrow $g \in \mathcal{G}_1$ its source (i.e. $s(g)$) and its target (i.e. $t(g)$) respectively;
4. a multiplication map $m : \mathcal{G}_2 \to \mathcal{G}_1$, $m(g,h) = gh$ where $\mathcal{G}_2 = \{(g,h) \in \mathcal{G}_1 \times \mathcal{G}_1 | s(g) = t(h)\}$, that satisfies $s(gh) = s(h)$ and $t(gh) = t(g)$;
5. a global section $1 : \mathcal{G}_0 \to \mathcal{G}_1$ called unit that satisfies $t(1(x)) = x = s(1(x))$, $1_x h = h$ and $g1_x = g$ for all $h \in t^{-1}(x)$ and $g \in s^{-1}(x)$;
6. a diffeomorphism $\iota : \mathcal{G}_1 \to \mathcal{G}_1$, $\iota(g) = g^{-1}$ called inverse map that satisfies $s(g^{-1}) = t(g)$, $t(g^{-1}) = s(g)$, $gg^{-1} = 1_{t(g)}$, $g^{-1}g = 1_{s(g)}$.

A Lie groupoid $\mathcal{G} = \mathcal{G}_1 \rightrightarrows \mathcal{G}_0$ induces a singular foliation on $\mathcal{G}_0$ whose leaves are the connected components of the orbits of $\mathcal{G}$, i.e., $\mathcal{G}(x) = \{t(s^{-1}(x))\}$. A Lie groupoid $\mathcal{G} = \mathcal{G}_1 \rightrightarrows \mathcal{G}_0$ is called proper Lie groupoid if the map $\psi : \mathcal{G}_1 \times \mathcal{G}_1 \to \mathcal{G}_0 \times \mathcal{G}_0$ defined as $\psi(g) = (s(g), t(g))$ is proper.

**Example 2.10 (Holonomy groupoid).** Consider an Euclidean bundle $\mathbb{R}^n \to E \to B$ with a compatible connection $\nabla^r$. Given a piece-wise smooth curve $\alpha : [0,1] \to B$, let $\|\alpha$ be the $\nabla^r$-parallel transport along $\alpha$. In this case:

(1) $\mathcal{G}_0 = B$;
(2) $\mathcal{G}_1 = \{\|\alpha, \forall \alpha : [0,1] \to B$ piecewise smooth$\};$
(3) the source is $s(\|\alpha) = \alpha(0)$, the target is $t(\|\alpha) = \alpha(1)$;
(4) the multiplication is produced with concatenation, i.e., $m(\varphi_\beta, \varphi_\alpha) = \varphi_{\beta \circ \alpha}$;
(5) the unit $1_{\alpha(0)} = \text{Id};$
(6) the inverse: $(\|\alpha)^{-1} = \|\alpha^{-1}$.

The above Lie groupoid is not appropriate to describe the parallel transport of vectors of $E$, since $\mathcal{G}_0 = B$. In order to correct this problem, we need to construct a new Lie groupoid so that the set of objects is $E$. At the same time, we briefly review (in a concrete example) how to construct a transformation Lie groupoid of a representation $\mu : \mathcal{G}_1 \times_\mathcal{G}_0 E \to E$ of a Lie groupoid $\mathcal{G}_1 \rightrightarrows \mathcal{G}_0$.

**Example 2.11 (Transformation holonomy groupoid).** Consider the notation of Example 2.10. Set $\mathcal{G}_1 \times_B E = \{(g,v_x) \in \mathcal{G}_1 \times E | s(g) = x = \pi(v_x)\}$. We define the representation $\mu : \mathcal{G}_1 \times_B E \to E$ as $\mu(\|\alpha, v_{\alpha(0)})) = \|\alpha v_{\alpha(0)}$ and the transformation holonomy groupoid $\text{Hol}^r = \mathcal{G} \ltimes E$ as:
We recall that the leaves of the foliation $F$ to the Sasaki metric. The orbits of $\text{Hol}_r$ is a singular Riemannian foliation $F^r$ on $E$ with respect to the Sasaki metric.

**Theorem 2.12** ([AldMS21]). Consider the foliations $F$ and $F^t$ on the normal bundle $E = \nu(B)$ of a closed leaf $B$. Let $\nabla^r$ be a Sasaki connection compatible with these foliations (i.e., so that the induced distribution $T$ is tangent to them). Let $K^0_p$ be the connected Lie group whose Lie algebra is associated to the Killing vector fields of the linearization of infinitesimal foliation $F_p$ on $E_p$. Then:

(a) $F = \text{Hol}^t(F_p)$

(b) $F^t = \text{Hol}^r\{K^0_p(v)\}_{v \in E_p}$

The above result already suggest that the leaves of $F^t$ may coincide with the orbits of a (Lie) groupoid $G^t$.

### 2.1.4. Linear Lie groupoid.

Let us recall how to construct the linear groupoid $G^t \rightrightarrows E = \nu(B)$ whose orbits are leaves of the orbit-like foliation $\widetilde{F}$.

We start by considering $O(E)$ the orthogonal frame bundle of $E = \nu(B)$. We recall that the leaves of the foliation $F^t$ are orbits of flows of linearized vector fields, that induce isometries between the fibers on $E$ (see Lemma 2.4), and hence can be lifted to flows on $O(E)$. The orbits of these flows on the frame bundle are leaves of a regular foliation $\widetilde{F}$ on $O(E)$.

Note that the action of each isotropic group $K^0_p$ induces a free action on $O(E_p)$ and the orbits of this action are tangent to the leaves of $\widetilde{F}$. Also the connection $\nabla^r$ induces a linear distribution $\widetilde{T}$ tangent to the leaf of $\widetilde{F}$. The tangent space of $\widetilde{F}$ through $\xi \in O(E)$ can be described as $T_{\tilde{\xi}}L_{\tilde{\xi}} = TK^0_p(\xi) \oplus \tilde{T}_{\xi}$. We can induces a Riemannian metric $\tilde{g}$ on $T_{\tilde{\xi}}\tilde{L}$ as follows: first define the metric on $\tilde{T}$ so that $d\pi : \tilde{T}_{\xi} \rightarrow (TB, g)$ turns to be an isometry, then we induces the metric on the orbits of $K^0_p(\xi)$ using a bi-invariant metric. It follows direct from this definition that the leaves of $\widetilde{F}$ are locally isometric. By construction (and using the fact that $B$ is a leaf) one can check that the leaves of $\widetilde{F}$ have trivial holonomy and are in fact diffeomorphic (and hence isometric) to each other.

Set $\mathcal{G} := \text{Hol}(\widetilde{F})/O(n) \rightrightarrows O(E)/O(n) = B$. Here $\text{Hol}(\widetilde{F})$ denotes the holonomy groupoid of the foliation $\widetilde{F}$, that is defined as follows: the set of objects is the ambient space of the foliation $\widetilde{F}$, an arrow $\tilde{g} \in (\text{Hol}(\widetilde{F}))_1$...
is a class of a path in a leaf of $\tilde{L} \in \tilde{\mathcal{F}}$ joining $\tilde{s}(\tilde{g}) \in \tilde{L}$ with $\tilde{t}(\tilde{g}) \in \tilde{L}$, where the equivalence relation identifies paths inducing the same germ of diffeomorphisms sliding transversals along the paths; the multiplication, unit and inverse maps are defined by concatenations, homotopy of a constant paths and the inverse paths.

The Lie algebroid of $G$ turns to be $\mathcal{A} = T\mathcal{F}/O(n) \rightarrow B$ and the metric $\tilde{g}$ induces a Riemannian metric on $\mathcal{A}$. Let us denote $\tilde{\nu}$ the density associated to the Riemannian metric.

Since $E = O(E) \times_B \mathbb{R}^n$, we can define the representation $\mu : \mathcal{G}_1 \times \mathcal{G}_0 \xrightarrow{} E$ as $\mu(g, [\xi, e]) = [\tilde{t}(\tilde{g}), e]$, where $\tilde{g} \in \text{Hol}(\tilde{\mathcal{F}})$ is the unique representative of $g$ so that $\tilde{s}(\tilde{g}) = \xi$. The linear Lie groupoid $G^t$ is defined as the transformation Lie groupoid of the representation $\mu$, i.e., $G^t = \mathcal{G} \times \mathcal{E}$. Recall that the target and source maps are $s^t((g, v)) = v$ and $t^t((g, v)) = \mu(g, v)$. Note that $(s^t)^{-1}(v_x) = (s^{-1}(x), v_x)$ where $s$ is the source map of $\mathcal{G}_1 \xrightarrow{} B$. Let us sum up the relation between the source fibers and leaves of $\tilde{\mathcal{F}}$.

**Lemma 2.13.**

1. The leaves of $\tilde{\mathcal{F}}$ have trivial holonomy and are isometric to each other.
2. The fiber of source map $s^t$ are isometric to (each) leaf of $\tilde{\mathcal{F}}$.

**2.2. Linearized vector fields and volume.**

**Lemma 2.14.** Given a Sasaki metric on $\mathbb{R}^k \rightarrow E \rightarrow B$ (possibly changing the metric on $B$ when $\dim B = 1$), there exists a module of linearized vector fields that preserve the volume $\nu$ (induced by the Sasaki metric). The composition of their flows is transitive on the leaves of $\mathcal{F}^t$.

**Proof.** It suffices to construct two types of linearized vector fields that preserve the volume $\nu$:

**Type 1:** $\tilde{F} \in \mathfrak{X}(E)$ whose orbits restricts to $B$ are transitive on $B$,

**Type 2:** $\tilde{G} \in \mathfrak{X}(E)$ whose orbits fix the fibers of $E$ and are transitive on the infinitesimal foliation on the fibers.

**Constructing type 1 vector fields**

First we consider a vector field $\tilde{F} \in \mathfrak{X}(B)$ that preserves the volume $\nu_B$ of $B$. If $\dim B = 0$ there is nothing to do. If $\dim B = 1$, i.e., if $B$ is the circle $S^1$, it is easy to see the existence of a global vector field that preserves the volume (changing the metric of $B$ if necessarily). So let us assume that $\dim B \geq 2$ and let us review the construction that given a point $p \in B$ and small neighborhood $W \subset B$ of $p$ there exists a vector field $\tilde{F} \in \mathfrak{X}(B)$ with a support on neighborhood $W$ that preserves the volume. Consider a coordinate system so that $p$ is identified with 0 and $\nu_B$ with $dx_1 \wedge \cdots \wedge dx_n$. Let $\tilde{F}$ be an (Euclidean) Killing vector field that fixes 0 and a smooth non negative function $\rho : \mathbb{R} \rightarrow \mathbb{R}$, with small compact support in $(-\epsilon, \epsilon)$ and $\rho(0) = 1$. Since the flow of $\tilde{F}$ preserves $\nu_B$, then the flow of $\tilde{F}(x) = \rho(||x||)\tilde{F}(x)$ also preserves $\nu_B$ and has compact support. By pulling
back $\bar{F}$ via the coordinate system, we identify the vector field $\bar{F}$ with a vector field on $B$, that we are also denoting as $\bar{F}$.

Now we can extend the vector field $\bar{F}$ on $B$ to a $\pi$-basic vector field on $E$ so that $\bar{F} \in \mathcal{X}(T)$. Since $T \subset T\mathcal{F}^\ell$ is homothetic, the vector field $\bar{F}$ is homothetic $\mathcal{F}^\ell$-vector field. Therefore $\bar{F}$ is linearized vector fields and hence it flows induces isometries between the fibers.

Let $\nu_E$ be the volume form on the fibers $E$, note that $\pi^*\nu_B \wedge \nu_E$ coincides with the volume $\nu$ of the Sasaki metric, because $\pi : E \to B$ is a Riemannian submersion. Since $\pi \circ \varphi^\ell_t = \varphi^\ell_t \circ \pi$ and $\varphi^\ell_t$ preserves $\nu_B$ we infer that

\[
(\varphi^\ell_t)^*(\pi^*\nu_B \wedge \nu_E) = \pi^*\nu_B \wedge \nu_E
\]

**Constructing type 2 vector fields:**

Consider the connected isotropy group $K_0^p$. By parallel transport with respect to $\nabla^\tau$ we can induce an action of $\mu : K_0^p \times \pi^{-1}(W) \to \pi^{-1}(W)$ on a neighborhood $W \subset B$ of $p$ that fixes the fibers, and act on each fiber isometrically. For $\xi$ in the Lie algebra of $K_0^p$, set $\bar{G}(x) = d\mu_x(f(\pi(x))\xi)$ for some smooth non negative function $f$ with compact support on $W$ and so that $f(p) = 1$. Since its flow acts isometrically on each fiber and its projection on $B$ is the identity we have that:

\[
(\varphi^G_t)^*(\pi^*\nu_B \wedge \nu_E) = \pi^*\nu_B \wedge \nu_E
\]

**Remark 2.15.** The construction of the Lie groupoid presented in Section 2.1.4 can be done using the flows of linearized vector fields that preserve the volume of the Sasaki metric.

### 2.3. Average operator of $\mathcal{F}^\ell$

We now define the average operator $Av : C^\infty_c(E^\delta) \to C^\infty_c(E^\delta)_b$ that projects (as we will see below) smooth functions with support on $E^\delta$ (identified via exponential map with $\text{Tub}_\delta(B)$) onto basic functions with support on $E^\delta$.

\[
Av(f)(v_x) = \frac{1}{\mathcal{V}(x)} \int_{(s^{-1})^{-1}(v_x)} f \circ t^\ell \bar{v} = \frac{1}{\mathcal{V}} \int_{(s^{-1})^{-1}(x)} f \circ \mu(g, v_x) \bar{v}_x
\]

where $\mathcal{V}(x) = \int_{(s^{-1})^{-1}(x)} \nu_z$. Here we used the fact that $(s^{-1})^{-1}(v_x) = (s^{-1}(x), v_x)$, where $s$ is the source map of $G_1 \Rightarrow B$.

**Lemma 2.16.** Consider $f \in C^\infty_c(E^\delta)$. Then $\mathcal{V}$ is constant and $Av(f)$ is a basic function.

**Proof.** The lemma follows from Lemma 2.13. In fact, Lemma 2.13 implies directly that $\mathcal{V}$ is constant. It also allows us to check that $Av(f)$ is a basic function, analogously to how this fact is demonstrated in the classic

\[1\text{This can be easily checked using an adapted orthonormal frame } \{\tau_i\} \text{ of } \tau \text{ and } \{e_\alpha\} \text{ tangent to the fibers of } E \text{ and see that the Sasaki volume and this volume form coincides.} \]
case where the groupoid comes from the action of a group on $M$. In other words, set $x = s(g_1)$ and $y = t(g_1)$ since the groupoid is transitive, i.e., $s^{-1}(x) = s^{-1}(y) = L_ξ,$

$$Av(f)(μ(g_1,v_x)) = \frac{1}{V} \int_{s^{-1}(y)} f(μ(g_2,μ(g_1,v_x)))\tilde{ν}$$

$$= \frac{1}{V} \int_{L_ξ} f(μ(g_2g_1,v_x))\tilde{ν}$$

$$= \frac{1}{V} \int_{s^{-1}(x)} f(μ(g_2g_1,v_x))\tilde{ν}$$

□

**Definition 2.17.** A linear functional $l : C^∞_c(E^δ) → \mathbb{R}$ is called symmetric with respect to the foliation $F^δ$ if it fulfills the following property: $l(f ∘ ϕ^F) = l(f)$ for each $f ∈ C^∞_c(E^δ)$ and for each $ϕ^F$ that is a composition of flows of linearized vector fields that preserve the Sasaki metric.

**Lemma 2.18.** Let $l$ be a linear functional on $C^∞_c(E^δ)$ symmetric with respect to the foliation $F^δ$. Then for each $f ∈ C^∞_c(E^δ)$

$$l(Av(f)) = l(f)$$

**Proof.** Consider a open cover $\{U_α\}$ of $L$ such that $U_α$ is diffeomorphic to a dim $L$-rectangle in Euclidean space, $\{ρ_α\}$ the partition of unit subordinate to $\{U_α\}$. Set $f_α(v_x) = f(v_x)ρ_α(x)$. In order to prove the lemma it suffices to prove

$$l\left(\int_{(s)^{-1}(x)} f_α ∘ μ(g,v_x)\tilde{ν}_x\right) = l\left(f_α\int_{(s)^{-1}(x)} \tilde{ν}_x\right)$$

Let $\{\tilde{U}_i\}$ be the connected component of $\{(t|_{(s)^{-1}(x)})^{-1}(U_α)\}$. Note that $\tilde{U}_i$ is a neighborhood of $s^{-1}(x)$ that can be isometric identified to a neighborhood of the leaf $L_ξ$ of the foliation $\tilde{F}$ (used in the construction of the groupoid $G_1$), i.e., $\tilde{U}_i$ does not depend on $x$ and hence, from now, the neighborhood $\tilde{U}_i$ and its isometric neighborhood on $L_ξ$ are going to be denoted by the same notation.

$$\int_{(s)^{-1}(x)} f_α ∘ μ(g,v_x)\tilde{ν}_x = \sum_i \int_{\tilde{U}_i} f_α ∘ μ(g,v_x)\tilde{ν}_i$$

where $\tilde{ν}_i = ρ_i ∘ t\tilde{ν}_x$. Also note that each neighborhood $\tilde{U}_i$ is diffeomorphic to $S × G_{x_i}$ where $G_{x_i}$ is the connected component of the isotropic group of a point $x_i ∈ U_i$ and $S$ is Euclidean rectangle on the Euclidian space with dimension of $L$. Since $G_{x_i}$ is diffeomorphic to a connected compact group $K$ we have that $\tilde{U}_i$ is diffeomorphic to $S × K$. Let $ψ_i : S × K → \tilde{U}_i$ be a
parametrization, \( f_\alpha(\cdot, \cdot) = f_\alpha \circ \mu(\psi(\cdot), \cdot) \) and \( \tilde{\nu}_i \omega_0 = \psi^*_i \tilde{\nu}_i \). Then writing in coordinates we have:

\[
(2.4) \quad \int_{\tilde{\mathcal{U}}_i} f_\alpha \circ \mu(g, v_x) \tilde{\nu}_i = \int_{S \times K} f_\alpha(s, k, x, v) \sigma_i(s, k) \tilde{\nu}_0
\]

Given an \( \epsilon \) we claim that there exists a partition \( \{P_{ij}\} \) of \( \tilde{\mathcal{U}}_i \) so that

\[
(2.5) \quad \int_{S \times K} f_\alpha(s, k, x, v) \sigma_i(s, k) \tilde{\nu}_0 = \sum_j f_\alpha(s_{ij}, k_{ij}, x, v) V_{ij} + R_i(x, v)
\]

where \( \|R_i\|_{W^{k,p}} < \epsilon \). Here \( V_{ij} = \sigma_i(s_{ij}, k_{ij}) \int_{\psi^{-1}(P_{ij})} \tilde{\nu}_0 \) and \( (s_{ij}, k_{ij}) \in \psi^{-1}(P_{ij}) \). In fact, define \( h : (S \times K) \times E_i \to \mathbb{R} \) as \( h((s, k), e) := f_\alpha((s, k), e) \cdot \sigma_i(s, k) \) where \( e = (x, v) \in E_i = \pi^{-1}(U_i) \cap E^{3/2} \). Since \( h \) is uniformly continuous on \( (S \times K) \times E_i \), we can check (by the definition of Riemann integral) that \( |R(x, v)| < \epsilon \) independent of \( e = (x, v) \in E_i \). Now if we set \( \hat{h} = \frac{\partial}{\partial \psi} f_\alpha(h(s, k), e) \cdot \sigma_i(s, k) \) and replace \( h \) in eq. (2.5) we infer, by similar argument that \( |\hat{R}(x, v)| < \epsilon_1 \) once we consider a refinement of \( \{P_{ij}\} \). This fact and the partial derivation \( \frac{\partial}{\partial \psi} \) of eq. (2.5) (using our original \( h \) ) imply that \( |\frac{\partial}{\partial \psi} R(x, v)| < \epsilon \). These facts allow us to conclude that \( \|R_i\|_{W^{k,p}} < \epsilon \) for a refinement of \( \{P_{ij}\} \).

Note that since the fibers of the source map are connected, each point \( g \in s^{-1}(x) \) is \( g = \varphi^\ell(i(x)) \) for some \( \varphi^\ell \), that is induced by composition of flows of linearized vector field \( \varphi^F \) that preserve volume of the Sasaki metric (see Remark 2.15). Hence we can rewrite \( f_\alpha \) intrinsically as follows:

\[
(2.6) \quad f_\alpha(s_{ij}, k_{ij}, x, v) = f_\alpha \circ \mu(\varphi^\ell_{ij}(x), v_x) = f_\alpha \circ t^\ell \circ \varphi^\ell_{ij} \circ i^\ell(v_x)
\]

Given \( \epsilon \), it follows from eq. (2.3), (2.4), (2.5), (2.6) that there is a partition \( \{P_{ij}\} \) of \( \tilde{\mathcal{U}}_i \) so that:

\[
(2.7) \quad \int_{s^{-1}(x)} f_\alpha \circ \mu(g, v_x) \tilde{\nu}_x = \sum_{ij} f_\alpha \circ t^\ell \circ \varphi^\ell_{ij} \circ i^\ell(v_x) V_{ij} + R(v_x)
\]

where \( \|R\|_{W^{k,p}} < \frac{\epsilon}{\pi} \). Note that \( l \) to be symmetric is equivalent to

\[
(2.8) \quad l(f_\alpha \circ t^\ell \circ \varphi^\ell \circ i) = l(f_\alpha \circ t^\ell \circ i^\ell)
\]

Applying \( l \) in the eq. (2.7), and using (2.8) we conclude:

\[
(2.9) \quad l \left( \int_{s^{-1}(x)} f_\alpha \circ \mu(g, v_x) \tilde{\nu}_x \right) = \sum_{ij} l(f_\alpha \circ t^\ell \circ i^\ell) V_{ij} + l(R)
\]

By the same argument as in eq. (2.5), and taking in consideration that \( f_\alpha(v_x) = f_\alpha \circ t^\ell \circ i^\ell(v_x) \) we have:

\[
(2.10) \quad f_\alpha(v_x) \int_{s^{-1}(x)} \tilde{\nu}_x = \sum_{ij} f_\alpha \circ t^\ell \circ i^\ell(v_x) V_{ij} + \tilde{R}(v_x)
\]
where \( \| \tilde{R} \|_{W^{k,p}} < \frac{\epsilon}{2} \). Applying \( l \) on the previous equation we have:

\[
(2.11) \quad l \left( f \int_{s^{-1}(x)} \tilde{\nu}_x \right) = \sum_{ij} l(f \circ t^\ell \circ i^\ell)V_{ij} + l(\tilde{R})
\]

Arbitrary of the choice of \( \epsilon \) and eq. (2.9) and (2.11) conclude the proof of eq. (2.2) and hence the proof of the lemma.

\[\square\]

### 2.4. Linear version of Principle of Symmetric Criticality of Palais.

**Lemma 2.19.** Let \( \mathcal{G}^\ell \Rightarrow E \) be the linear Lie groupoid whose orbits are the leaves of the orbit-like foliation \( \mathcal{F}^\ell \) on \( E \). Let \( l \) be a linear functional on \( C^\infty_c(E^\delta) \). Assume that

(a) \( l(b) = 0 \), for all \( \mathcal{F}^\ell \)-basic function \( b \in C^\infty_c(E^\delta) \)

(b) \( l \) is symmetric with respect to \( \mathcal{F}^\ell \) i.e., fulfills Definition 2.17.

Then \( l = 0 \).

**Proof.** Consider \( f \in C^\infty_c(E^\delta) \).

\[
0 \overset{(*)}{=} l(Av(f)) \overset{(**)}{=} l(f)
\]

where (*) follows from item (a) and Lemma 2.16 and (**) follows from Lemma 2.18. The arbitrariness of choice of \( f \) implies that \( l = 0 \). \[\square\]

Let us illustrate the principle described above. Let \( J : C^\infty_c(E^\delta) \rightarrow \mathbb{R} \) be the functional:

\[
J(f) = c_1 \int_{E^\delta} \langle \nabla f, \nabla f \rangle \nu + \frac{1}{2} \int_{E^\delta} kf^2 \nu + c_2 \int_{E^\delta} f c_3 \nu
\]

where \( \langle \cdot, \cdot \rangle = g^0 \) is the Sasaki metric, \( \nu \) is the volume induced by the Sasaki metric, \( k : E^\delta \rightarrow \mathbb{R} \) is a \( \mathcal{F}^\ell \)-smooth function and \( c_i \) and \( c_3 > 0 \) are constant.

**Lemma 2.20.** Let \( f \in C^\infty_c(E^\delta) \) be a smooth function and \( b \in C^\infty_c(E) \) a smooth \( \mathcal{F}^\ell \) basic function. Set

\[
l(f) = dJ(b)(f) = Q(b, f) = 2c_1 \int_{E^\delta} \langle \nabla b, \nabla f \rangle \nu + \int_{E^\delta} kbf \nu + c_2 c_3 \int_{E^\delta} b c_3 - 1 f \nu,
\]

Then \( l : C^\infty_c(E^\delta) \rightarrow \mathbb{R} \) is symmetric with respect to volume preserves linearized vector fields, i.e., fulfills Definition 2.17.
Proof.

\[
Q(b \circ \varphi_t, f \circ \varphi_t) = 2c_1 \int_{E^x} \langle \nabla (b \circ \varphi_t), \nabla (f \circ \varphi_t) \rangle \nu + \int_{E^x} k(b \circ \varphi_t)(f \circ \varphi_t) \nu + c_2 c_3 \int_{E^x} (b \circ \varphi_t)c_3^{-1}(f \circ \varphi_t) \nu \tag{I}
\]

\[
= 2c_1 \int_{E^x} \langle \nabla (b \circ \varphi_t), \nabla (f \circ \varphi_t) \rangle \nu + \int_{E^x} kbf + c_2 c_3 b^c_3 - 1 f \nu \tag{II}
\]

\[
= 2c_1 \int_{E^x} \langle \nabla E (b \circ \varphi_t), \nabla (f \circ \varphi_t) \rangle \nu + \int_{E^x} kbf + c_2 c_3 b^c_3 - 1 f \nu \tag{III}
\]

\[
= 2c_1 \int_{E^x} \langle \nabla b, \nabla f \rangle \nu + \int_{E^x} kbf + c_2 c_3 b^c_3 - 1 f \nu \tag{IV}
\]

\[
= Q(b, f),
\]

where (I) follows from the fact that \( \varphi_t \) preserves \( \nu \), (II) from the fact that \( \nabla^E b = \nabla b \) (where \( \nabla^E \) is the induced Riemannian connection on the fibers) and (III) from the fact that \( \varphi_t \) induces isometries between the fibers of \( E \), see Lemma 2.4

2.5. A few words about the average of a SRF \( F \) on \( E \). We end this section briefly presenting an average operator \( \text{Av}_F \) for a general SRF \( F \) on \( (E, \langle \cdot, \cdot \rangle) \) where \( \langle \cdot, \cdot \rangle \) is a Sasaki metric. Although we can not (at least until now) use this operator to have the Palais principle, that will be a fundamental ingredient to construct the constant scalar curvature, the operator \( \text{Av}_F \) will be used to define \( F \)-Sobolev spaces.

Since in this section, we are dealing with the two foliations \( F^x \subset F \), we need two different notations for basic functions with respect to these foliations. Let \( C_c^\infty(E^x_b) \) denote (as usual) the space of basic functions with respect to \( F^x \) and \( C_c^\infty(E^x) \) denote basic functions with respect to \( F \). As we saw in previous sections \( Av : C_c^\infty(E^x) \rightarrow C_c^\infty(E^x_b) \). Note that for a fixed \( x_0 \in B \) the foliation \( F^x \) intersects the fiber \( E_{x_0} \). This allow us to project functions of \( C_c^\infty(E^x_b) \) into functions of \( C_c^\infty(E^x_{x_0}) \) i.e., into the space of basic functions of \( (F_{x_0}, E_{x_0}) \) with compact support on the fiber \( E^x_{x_0} \). Hence we can define the restriction operator \( R_{x_0}^x : C_c^\infty(E^x_b) \rightarrow C_c^\infty(E^x_{x_0}) \). Since \( E_{x_0} \) is a vector space, we have the Mendes-Radeschi average on this space [MRT18, Lemma 21], i.e, \( \text{Av}_{F_{x_0}} : C_c^\infty(E^x_{x_0}) \rightarrow C_c^\infty(E^x_{x_0}) \). Note that two different leaves of \( F_{x_0} \) (let us call them plaques) may belong to the same leaf \( L \in F \), but in this case they are isometric to each other, because the linearized holonomy sends one plaque to other isometrically. This observation allow us to extend \( F_{x_0} \)-basic function on \( E_{x_0} \) to \( F \)-basic function on \( E \), i.e., to define
the operator $\mathcal{E} : C^\infty_c(E_x^\delta)^\mathcal{F} \to C^\infty_c(E^\delta)^\mathcal{F}$. Finally we define the desired operator: $\text{Av}_F : C^\infty_c(E^\delta) \to C^\infty_c(E^\delta)^\mathcal{F}$ as $\text{Av}_F = \mathcal{E} \circ \text{Av}_{F_{x_0}} \circ \mathcal{R}_x^\ell \circ \text{Av}$.

\section{Principle of Symmetric Criticality of Palais on $M$}

We start by presenting a metric constructed with Sasaki metrics on tubular neighborhoods and partition of unity, that is analogous to the proof of \cite[Proposition 1.29]{MSNN01}.

\begin{lemma}
Consider an open cover of $\mathcal{F}$-tubular neighborhoods $\{\text{tub}_{r_a}(L_{q_a})\}$ of $M$. Then

(a) there exists a locally finite open cover of tubular neighborhoods $\text{tub}_{\delta_i}(L_{p_i})$ that is a refinement of $\{\text{tub}_{r_a}(L_{q_a})\}$. In addition $\text{tub}_{\delta_i/3}(L_{p_i})$ is still an open covering of $M$.

(b) There exists a $\mathcal{F}$-partition of unity $\{\rho_i\}$ with support on $\text{tub}_{\delta_i}(L_{p_i})$ so that $\rho_i = 1$ when restrict to $\text{tub}_{\delta_i/3}(L_{p_i})$. The partition of unity $\{\rho_i\}$ is said to be subordinate to $\{\text{tub}_{r_a}(L_{q_a})\}$.

Given an open cover of tubular neighborhoods $\{\text{tub}_{r_a}(L_{q_a})\}$ and a subordinate partition of unity $\{\rho_i\}$, we can define, via normal exponential map, a Sasaki metric $\langle \cdot, \cdot \rangle$ on $\text{tub}_{\delta_i}(L_{p_i})$ (see notation in Lemma 3.1) and a $\mathcal{F}$-basic metric on $M$ as:

\begin{equation}
\langle \cdot, \cdot \rangle = \sum_i \rho_i \langle \cdot, \cdot \rangle_i
\end{equation}

\begin{proposition}
Let $(M, \mathcal{F})$ be an orbit-like foliation and $k : M \to \mathbb{R}$ be a $\mathcal{F}$-smooth function. Let $\langle \cdot, \cdot \rangle$ be the metric construct above. Set

\[ l(f) = dJ_g(b)(f) = Q(b, f) = 2c_1 \int_M \langle \nabla b, \nabla f \rangle \nu + \int_M kb_f \nu + c_2 c_3 \int_M b^{c_3 - 1} f \nu, \]

Assume that for a given smooth $\mathcal{F}$-basic function $b \in C^\infty(M)$ we have that $l(\hat{b}) = dJ_g(b)(\hat{b}) = 0$ for all basic functions $\hat{b}$. Then $dJ(b) = 0$

\begin{proof}
Let $\{\hat{\rho}_j\}$ be a partition of unity subordinate to $\{\text{tub}_{\delta_i/3}(L_{p_i})\}$. Given an $f$, by setting $f_j = \hat{\rho}_j f$ we have that $f = \sum_j f_j$. In order to check that $l(f) = 0$ it suffices to check that $l(f_j) = 0$. For a fixed $j$ there exists $i$ so that $\text{supp}(f_j) \subset U_i := \text{tub}_{\delta_i/3}(L_{p_i})$. By construction $\langle \cdot, \cdot \rangle$ restrict to $\text{tub}_{\delta_i/3}(L_{p_i})$ coincides with the Sasaki metric $\langle \cdot, \cdot \rangle_i$. Therefore:

\[ l(f_j) = dJ_g(b)(f_j) = 2c_1 \int_M \langle \nabla b, \nabla f_j \rangle \nu + \int_M kb_f \nu + c_2 c_3 \int_M b^{c_3 - 1} f_j \nu = \int_{U_i} 2c_1 \langle \nabla b, \nabla f_j \rangle_i + kb_f + c_2 c_3 b^{c_3 - 1} f_j \nu \]

From Lemma 2.20 $l_i : C^\infty_c(U_i) \to \mathbb{R}$ is symmetric. Since, by hypothesis $l(\hat{b}) = 0$ for all basic functions, it is also true that $l_i(\hat{b}) = 0$, $\forall \hat{b} \in C^\infty_c(U_i)$.
From Lemma 2.19 we infer that $l_i = 0$. Hence $l(f_j) = 0$ and this finishes the proof. □

4. Sobolev spaces of basic functions on SRF spaces

Let us start by extending the operator defined in Section 2.5 to the manifold $M$ using partition of unity. More precisely, given a covering of tubular neighborhoods $\{\text{tub}_{\alpha}(L_{\rho_i})\}$, we have a partition of unity $\{\rho_i\}$ subordinate to it; see Lemma 3.1 for notations. We define $\text{Av}_i : C^\infty(U_i) \rightarrow C^\infty(U_i)^F$ as the operator $\text{Av}_F$ defined on $E^\delta_i$ once we have identified, via normal exponential map, the tubular neighborhood $U_i = \text{tub}_{\alpha}(L_{\rho_i})$ with normal space $E^\delta_i = \nu^\delta_i(L_{\rho_i})$ of $L_{\rho_i}$. We finally define $\text{Av}_F : C^\infty(M) \rightarrow C^\infty(M)^F$ as:

$$\text{Av}_F(f) := \sum_i \text{Av}_i(\rho_i f).$$

Following [MR18, Lemma 21] it is possible to check that $\text{Av}_i : C^\infty(U_i) \rightarrow C^\infty(U_i)^F$ is a continuous operator with respect to the Sobolev metrics.

Once established the average operator we proceed by formalizing the construction of the Sobolev Space of basic distributions and prove some the corresponding Kondrakov Theorem 4.4 in the context.

Let $(M, g)$ be a Riemannian manifold, $F$ be a SRF on $(M, g)$ and denote by $\mu_g$ the Radon measure on $M$ induced by $g$. Fix a nonnegative integer $k$ and $1 \leq p < \infty$.

We define

$$C^{k,p}(M, g) = \left\{ u \in C^\infty(M) : \|u\|_{k,p,g} := \left[ \sum_{j=0}^{k} \int_M |\nabla^j u|^p \mu_g \right]^{1/p} < \infty \right\},$$

where $\| \cdot \|_{k,p,g} : C^{k,p}(M, g) \rightarrow [0, \infty[$ is called the $(k,p)$-Sobolev norm on $(M, g)$. In this context, we can define the Sobolev Space of $F$-basic distributions on $(M, g)$:

**Definition 4.1.** The Sobolev space $W^{k,p}(M, g)^F$ of $F$-basic distributions in $M$ is the Banach space of distributions in $W^{k,p}(M, g)$ that are constant along the leaves of the Singular Riemannian Foliation $F$.

**Remark 4.2.** For each $u \in W^{k,p}(M, g)^F$ we can find a sequence of smooth $F$ basic functions $\{u_n\}$ that converges (with respect to the Sobolev norm) to $u$. In fact consider a sequence of smooth functions $\{\tilde{u}_n\}$ that converges to $u$. Then $u_n = \text{Av}_F(\tilde{u}_n)$ converges to $u = \text{Av}_F(u)$.

This definition is compatible with the usual definition of $W^{k,p}_g(M)$ (see [Heb00, Def. 2.1, p.21]) in the sense that the arrows in the commutative
diagram below correspond to linear $∥ \cdot ∥_{k,p,g}$-continuous embeddings:

$W^{k,p}(M,g)^\mathcal{F} \hookrightarrow C^{k,p}(M,g)^\mathcal{F}$

$W^{k,p}(M,g) \hookrightarrow C^{k,p}(M,g)$

Remark 4.3. Suppose that $M$ is a compact manifold. Any two Riemannian metrics on $M$ yield equivalent $(k,p)$-Sobolev norms (see [Heb00, Prop. 2.2, p.22]). For such reason, we omit mention to Riemannian metrics since we only consider compact manifolds. Moreover, $W^{k,p}(M)$ can be equivalently defined as the completion of $C^\infty(M)$ with respect to any $(k,p)$-Sobolev norm.

It is classical nowadays that when considering manifolds with isometric actions, assuming that each orbit has infinity cardinality leads to better compact embeddings coming from Kondrakov’s Theorem (see [Heb00, Theorem 9.1, p.252]). More recently, [CaMdOS21] it was observed that when the group acts properly it suffices indeed that there exist at least on orbit of positive dimension to guarantee better compact embeddings, leading, for instance, to new proofs for the Yamabe problem in this scenario, but also to analogous results to Kazdan–Warner in the $G$-invariant setting.

Here we obtain a Kondrakov-type theorem for SRF with compact leaves on compact manifolds:

Theorem 4.4 (Kondrakov-type theorem). Suppose that $M$ is a connected compact Riemannian manifold and $\mathcal{F}$ is a SRF on $M$ whose leaves are closed. Then there exists $p_0 > p^* := np/(n - p)$ such that given $1 < q < p_0$, the canonical embedding $W^{1,p}(M)^\mathcal{F} \hookrightarrow L^q(M)$ is compact.

Before we proceed to the proof, recall that the regular stratum $M^{reg}$ of $\mathcal{F}$ on $M$ is open and dense in $M$, see Section 2.1.1. Let $k := \dim L_x$ for an $x \in M^{reg}$. Using Theorem 2.3, it is possible to check that there exist trivializing coordinate charts $\{ (\Omega, \varphi) \}$ on $M^{reg}$ with properties

(i) $\varphi(\Omega) = U \times V \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$;

(ii) $\forall y \in \Omega, U \times \text{pr}_2(\varphi(y)) \subset \varphi(Gy \cap \Omega)$, where $\text{pr}_2 : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ is the second projection.

Property (ii) implies that if $f : M \to \mathbb{R}$ is a $\mathcal{F}$-invariant function (hence constant along the leaves), then $f \circ \varphi^{-1}$ is constant on its first coordinate.

Proof of Theorem 4.4. Let $\mu$ be a Radon measure on $M$ induced by any Riemannian metric on $M$. Since $\mu(M \setminus M^{reg}) = 0$, then

$$\int_M |f|^q \, d\mu = \int_{M^{reg} \cup (M \setminus M^{reg})} |f|^q \, d\mu = \int_{M^{reg}} |f|^q \, d\mu$$

for any $1 \leq q < \infty$, $f \in L^q(M) = W^{0,q}(M)$. Since there is a leaf of dimension at least 1, if $d^*$ denotes the dimension of a principal orbit, then
d^* \geq 1. Cover M^{reg} with finitely many trivializing charts \{(\Omega, \varphi)\} such that 
\varphi(\Omega) = U \times V \subset \mathbb{R}^{d^*} \times \mathbb{R}^{n-d^*} and U \ni x \mapsto f \circ \varphi^{-1}(x, \cdot) is constant.

By the Fubini Theorem on U \times V and the Sobolev Embedding Theorem for open sets in \mathbb{R}^{n-d^*}, we conclude that there are constants C, K > 0 such that
\begin{equation}
\left( \int_{\Omega} |f|^q \, d\mu \right)^{1/q} = \left( \int_{U \times V} |f \circ \varphi^{-1}|^q (\varphi^{-1})^*(d\mu) \right)^{1/q},
\end{equation}
\begin{equation}
\left( \int_{U \times V} |f \circ \varphi^{-1}|^q (\varphi^{-1})^*(d\mu) \right)^{p/q} \leq C \int_V \left( |f \circ \varphi^{-1}|^p + |\nabla (f \circ \varphi^{-1})|^p \right) (\text{pr}_2 \circ \varphi^{-1})^*(d\mu) = K \int_{\Omega} (|f|^p + |\nabla f|^p) \, d\mu.
\end{equation}

Since there are finitely many open sets on the trivialization atlas, it follows that there exists C > 0 such that
\[ \|f\|_q = \left( \int_{M^{reg}} |f|^q \, d\mu \right)^{1/q} \leq C \|f\|_{1,p}. \]

Therefore, the canonical inclusion \( W^{1,p}(M)^F \hookrightarrow L(M) \) is continuous whenever \( 1 \leq q \leq p(n-d)/(n-d-p) \).

If \( \epsilon > 0 \) is sufficiently small, then the classic Kondrakov Theorem implies that this inclusion is compact whenever \( 1 \leq q \leq p(n-d+\epsilon)/(n-d+\epsilon-p) \).

The continuous function \( [1, \infty] \setminus \{p\} \ni t \mapsto pt/(t-p) \) is decreasing, so \( p(n-d+\epsilon)/(n-d+\epsilon-p) > p^* = np/(n-p) \).

\[ \square \]

5. The Yamabe problem on manifolds with orbit-like foliation and bundles

In this section we explore the developed machinery for, as proof of concept, study the Yamabe problem \[ \text{Yam60} \quad \text{Tru68} \quad \text{SY79a} \quad \text{SY79b} \quad \text{SY81} \] in the setting of both orbit-like foliation and fiber bundles, aiming prescribing constant scalar curvature metrics adapted to the induced foliation on both cases. In a near future we shall make it public the respective results on the Kazdan–Warner \[ \text{KW75a} \quad \text{KW75b} \quad \text{KW75c} \] in this scenario.

5.1. The proof strategy and its self motivation. To proceed, fix a closed Riemannian manifold \((M, g)\) with an orbit-like foliation \(\mathcal{F}\) of closed leaves. Also assume that the scalar curvature of \(g\) is basic. We start making it clear that given some constant \(c\), on both Theorems \[ \text{1.2} \] and \[ \text{1.3} \] to search for a smooth basic and positive function \(u : M \to \mathbb{R}\) such that \(\text{scal}_{\ t/\ u^{n-2}} = c\) is equivalent to solve the following elliptic PDE:
\begin{equation}
4b_n \Delta_g u - \text{scal}_g u + cu^{7n} = 0, \tag{5.1}
\end{equation}
where \( c > 0 \) and \( \gamma_n := \frac{n+2}{n-2} \) and \( b_n := \frac{n-1}{n-2} \).

To begin with, it suffices to assume that \( \text{scal}_g \) is only continuous. This manner, we proceed by considering the following functional

\[
J(u) = 2b_n \int_M |\nabla u|^2_g + \frac{1}{2} \int_M \text{scal}_g u^2 - \frac{c}{2^*} \int_M u^{2^*}
\]
a priori defined in the Sobolev space \( W^{1,2}(M) \). To weakly solve the PDE (5.1) consists of finding a critical point \( u \) of \( J \), namely,

\[
dJ(u)(v) = 0 \quad \forall v \in W^{1,2}(M).
\]

For both the proofs of Theorems 1.2 and 1.3 these critical points correspond to local minima.

Note however that, since \( J \) is a basic functional in the sense of Definition 2.17, it is possible to restrict \( J \) to \( W^{1,2}(M)_F \). On the one hand, to find a critical point for \( J \) restricted to \( W^{1,2}(M)_F \) means to exist \( u \in W^{1,2}(M)_F \) such that \( dJ(u)(v) = 0 \forall v \in W^{1,2}(M)_F \). It is in this very point that a result such as the principle of symmetric criticality is needed. Observe that under this former hypothesis, Proposition 3.2 implies that \( dJ(u)(v) = 0 \forall v \in W^{1,2}(M)_F \), meaning that this basic critical point is a critical point indeed.

On the other hand, we justify our interests on finding a basic solution to ensure that the metric with constant scalar curvature is basic, hence preserving the foliation geometric structure. It turns out however that such a restriction plays a huge role in the argumentation of finding a minimum for \( J \) via variational methods due to the existence of better compactness results such as Theorem 4.4. Note for instance that \( \gamma_n \) is a critical exponent for the classical Kondrakov Theorem, i.e, it is such that it is not necessarily true to exist a compact embedding of \( W^{1,2}(M) \) in \( L^{\gamma_n+1}(M) \). So we naturally proceed to find a basic critical point.

To do so, we shall look for a local minima for \( J \) with specific constraints.

For both Theorems 1.2 and 1.3 we restrict the analysis to a codimension 1 submanifold \( M_F \) of \( W^{1,2}(M)_F \):

\[
M_F := \left\{ u \in W^{1,2}(M)_F : C \geq u \geq 0 \text{ a.e.} , c \frac{1}{2^*} \int_M u^{2^*} = \epsilon \right\}.
\]

A routine argument in variational methods, trivial in this scenario given the Kondrakov type result, then ensure the existence of such a minimum point \( u \) for both cases. Since the constraint is a submanifold, we observe that the Lagrange Multiplier equation obtained for this problem can be reduced to the original one, \( dJ(u)(v) = 0 \forall v \in W^{1,2}(M)_F \) after scaling the original metric, concluding that \( u \) is then a basic critical point.

Finally, due to the regularity theory of elliptic PDE’s one concludes that the solution \( u \) is smooth as long as \( \text{scal}_g \) and \( f \) are smooth. In fact, note that the non-linearity on the PDE (5.1) corresponds to the term \( u^{\gamma_n} \). Since the function \( F : x \rightarrow x^{\gamma_n} \) is of class \( C^1 \) and the solution \( u \) has finite essential supremum, then \( F(u) \in W^{1,2}(M) \). An iterative application of Theorem 3.58 in [Aub98, p. 87] implies the result. The maximum principle [Aub98, Proposition 3.75, p.98] implies that the obtained solution is positive.
5.2. The Yamabe problem on manifolds with orbit-like foliation. We now use the developed machinery to prove Theorem 1.2. We restate it here for convenience:

**Theorem 5.1.** Let $M^n, n \geq 3$, be a closed Riemannian manifold endowed with a orbit-like foliation $\mathcal{F}$ with closed leaves. Then $M$ has a Riemannian metric of constant scalar curvature for which $\mathcal{F}$ is a orbit-like foliation.

**Proof.** Equip $M$ with the basic metric $g = \langle \cdot, \cdot \rangle$ defined in Eq. (3.1). We claim that there is $c \geq 0$ such that for any $c' \geq c$ there exists a Riemannian metric $\tilde{g}$ with basic scalar curvature such that $\text{scal}_{\tilde{g}} = -c'$. As we already pointed out in Section 5.1, such a $\tilde{g}$ comes from a conformal change. Also taking in count this section, we proceed finding a basic critical point, what shall finishes the proof.

Take $c \geq 0$ such that
\[
\left(\frac{2^*}{2}\right)^{\frac{1}{2}} \min_M \text{scal}_g \text{vol}(M)^{1-2^*/2} + c \geq 0,
\]
and consider the functional
\[
J(u) = 2b_n \int_M |\nabla u|^2_g + \frac{1}{2} \int_M \text{scal}_g u^2 + \frac{c}{2^*} \int_M u^{2^*}
\]
defined in $W^{1,2}(M)^\mathcal{F}$. Let us show that $J$ is coercive.

To do so, note that the Hölder inequality implies that
\[
\left(\int_M u^2\right) \leq \text{vol}(M)^{1-2^*/2} \left(\int_M u^{2^*}\right)^{2/2^*}.
\]
(5.2)

We now consider two separate cases depending if $\min \text{scal}_g \leq 0$ or $\min \text{scal}_g > 0$. Respectively we have
\[
J(u) \geq 2b_n \int_M |\nabla u|^2_g + \frac{1}{2} \int_M \text{min scal}_g \text{vol}(M)^{1-2^*/2} \left(\int_M u^{2^*}\right)^{2/2^*} + \frac{c}{2^*} \int_M u^{2^*};
\]
\[
J(u) \geq 2b_n \int_M |\nabla u|^2_g + \frac{c}{2^*} \int_M u^{2^*}.
\]
We pass to the submanifold
\[
M_\mathcal{F} := \left\{ u \in W^{1,2}(M)^\mathcal{F} : C \geq u \geq 0 \text{ a.e.}, \frac{c}{2^*} \int_M u^{2^*} = \epsilon \right\},
\]
where we have liberty on the choice of $\epsilon$. We then check that $J|_{M_\mathcal{F}}$ is coercive and weakly lower semicontinuous. To do so, observe that considering the imposed restrictions one has
\[
J(u) \geq 2b_n \int_M |\nabla u|^2_g + \frac{1}{2} \int_M \min \text{scal}_g \text{vol}(M)^{1-2^*/2} \left(\frac{2^* \epsilon}{c}\right)^{2/2^*} + \epsilon; \quad (5.3)
\]
\[
J(u) \geq 2b_n \int_M |\nabla u|^2_g + \epsilon. \quad (5.4)
\]
from where it follows that $J\bigg|_{\mathcal{M}_F}$ is coercive.

It is also immediate to conclude that $\mathcal{M}_F$ is weakly closed. Indeed, take $\mathcal{M}_F \ni \{u_m\} \rightharpoonup u \in W^{1,2}(M)$. Once $\{u_m\} \subset W^{1,2}(M)^F$ and this is a Banach space one has that $u \in W^{1,2}(M)^F$. According to Theorem 4.4 one has the compact embedding of $W^{1,2}(M)^F$ in $L^{2^*}(M)$, from where it follows that $c\int_M u^{2^*} = \epsilon 2^*$. Moreover, the sequence $\{u_m\}$ has a pointwise convergent subsequence, so that $C \geq u \geq 0$ almost everywhere. Therefore, $u \in \mathcal{M}_F$ and $\mathcal{M}_F$ is weakly closed.

As a last step we observe that $J\bigg|_{\mathcal{M}_F}$ is weakly lower semicontinuous since: due to Theorem 4.4, any weakly converging sequence $\{u_m\} \subset W^{1,2}(M)^F$ strongly converges in $L^p(M) \cap \mathcal{M}_F$ for every $p \in [1, 2^*]$; moreover, once the weak convergence implies that $\liminf_{m \to \infty} \|u_m\|_{1,2} \geq \|u\|_{1,2}$, and since $\int_M u_m^2 \to \int_M u^2$ is a convergent sequence, it holds that

$$\liminf_{m \to \infty} \|u_m\|_{1,2}^2 = \int_M u^2 + \liminf_{m \to \infty} \int_M |\nabla u_m|^2,$$

and hence

$$\liminf_{m \to \infty} \int_M |\nabla u_m|^2 \geq \int_M |\nabla u|^2.$$  

Finally, once $\text{scal}_g$ is continuous and $u \in \mathcal{M}_F$ one concludes that

$$\liminf_{m \to \infty} J(u_m) \geq 2b_n \int_M |\nabla u|^2 + \frac{1}{2} \int_M \text{scal}_g u^2 - \epsilon = J(u).$$

It then follows that the restriction $J\bigg|_{\mathcal{M}_F}$ has a minimum $u \in \mathcal{M}_F$. Since we have obtained a critical pointed subjected to the artificial constraint $\mathcal{M}_F$, we must proceed by looking to the corresponding Lagrange Multiplier associated to this problem. To do so, note now that if $v \in W^{1,2}(M)^F$, the Lagrange Multiplier Theorem states that there is $\lambda \in \mathbb{R}$ such that $J'(u)(v) = 4b_n \int_M \langle \nabla u, \nabla v \rangle + \int_M \text{scal}_g u v - c \int_M u^m v = \lambda c \int_M u^m v = H'(u)(v),$

where $H^{-1}(0) = \mathcal{M}_F \setminus \partial \mathcal{M}_F$. We reinforce that $u$ does not lie in $\partial \mathcal{M}_F$ since both: we can assume that $u$ is not constant equal to $C$, otherwise it would imply that the original metric already has constant scalar curvature; moreover, the integral constraint avoid $u$ to be identically zero.

We thus conclude that $u$ is a weak solution of (5.1) with $c$ replaced by $c' = (1 + \lambda)c$. On the other hand, by computing $J'(u)(u)$, we conclude that $1 + \lambda > 0$, thus $c' > 0$. We then proceed with a rescaling of the resulting metric to obtain the right constant scalar curvature metric. Note that such a scaling is possible given the liberty in the choice of $\epsilon$ in the definiton of $\mathcal{M}_F$. \qed
5.3. Prescribing Riemannian submersion metrics on fiber bundles.

In this section we discuss the Yamabe problem to the setting of fiber bundles whose fibers are homogeneous spaces. More precisely, we prove:

**Theorem 5.2.** Let $M^n$, $n \geq 3$, be a closed Riemannian manifold endowed with a Foliation $\mathcal{F}$ induced by a fiber bundle such that:

(i) The structure group $G$ is compact and has non-abelian Lie algebra;
(ii) The fiber $L$ is an homogeneous space.

Then $M$ has a Riemannian metric of positive constant scalar curvature for which $\mathcal{F}$ is Riemannian.

To unify the discussion in terms of Riemannian foliations, let $\pi : L \hookrightarrow (M, g) \rightarrow (B, h)$ be a Riemannian fiber bundle with compact structure group $G$ and total space $M$ closed. In this scenario, the decomposition of $M$ with respect to the Riemannian foliation induced by the fibers $\{L_x\} = \mathcal{F}$ is an example of SRF which leaves are diffeomorphic.

Note that $\pi$ can always be obtained as an associated bundle construction for some principal bundle $P$: there is a $G$-manifold $P$ for which the corresponding $G$-action is free and such that $M = P \times_G L$. We consider the particular case where the $G$-action on $L$ is transitive. This way, $L$ can be seen as the homogeneous space $G/G_l$, where $G_l$ is the isotropy subgroup at some $l \in L$. Hence, $L$ coincides with the orbit of $G$ through $l$.

**Proof of Theorem 5.2.** Given any $G$-invariant Riemannian metric $g_L$ in the fiber $L$, according to [CeSS18, Theorem F] it follows that $g_L$ develops positive scalar curvature after a finite Cheeger deformation (see [Zil] for more on such deformations). This manner, for simplicity we assume that $g_L$ itself has positive scalar curvature.

Now consider on $M$ the unique Riemannian submersion metric $g$ whose fibers are totally geodesic and isometric to $(L, g_L)$. Since we can shrink sufficiently the fibers by the means of a Canonical Variation we shall assume that $g$ has positive scalar curvature. Since $\text{scal}_{g_L}$ is $G$-invariant and $L$ is a homogeneous space it means that $\text{scal}_{g_L}$ is constant and so $g$ is a metric with basic scalar curvature.

Once more we rely in the discussion presented in section 5.1. More precisely, by the means of a conformal change we shall prescribe $c > 0$ as the scalar curvature of a metric $\bar{g} = u^{4/n-2}g$. Therefore, without further preliminaries, let $\epsilon > 0$ to be chosen conveniently. We search for a critical point of $J$ in

$$M_\mathcal{F} := \left\{ u \in W^{1,2}(M) : C \geq u \geq 0 \ a.e. : \frac{c}{2^*} \int_M u^{2^*} = \epsilon \right\},$$

where $C \geq \left( \frac{\epsilon 2^*}{c \text{vol}_{g}(M)} \right)^{\frac{1}{2^*}}$. 
As we have already seen in the proof of Theorem 5.1, the manifold $M_F$ is weakly closed and $J|_{M_F}$ is weakly lower semicontinuous. It only remains to prove that $J|_{M_F}$ is coercive:

Note that if $u \in M_F$ then

$$J(u) \geq 2b_n \int_M |\nabla u|^2_g + \frac{\min_M \text{scal}_g}{2} \int_M u^2 - \frac{c}{2^*} \int_M u^{2^*}$$

$$= 2b_n \int_M |\nabla u|^2_g + \frac{\min_M \text{scal}_g}{2} \int_M u^2 - \epsilon.$$  

Therefore, according to Poincaré inequality, since $\min_M \text{scal}_g \geq 0$, $J(u) \to \infty$ if $\|u\|_{W^{1,2}(M_F)} \to \infty$.

The remaining of the argument follows exactly to the ones in the proof of Theorem 5.1.

As a last step, we argue that we can indeed obtain a Riemannian submersion metric. To do so, first note that since $u$ is a basic function, then the obtained metric $u^{\frac{4}{n-2}} g$ makes the foliation $\mathcal{F}$ Riemannian. Moreover, since $g$ is a Riemannian submersion metric there is a Riemannian metric $\bar{g}$ in $M/\mathcal{F}$ for such that $\pi^* g = \bar{g}$. Hence, since $u$ is basic one concludes that $\pi^* (u^{\frac{4}{n-2}} g) = u^{\frac{4}{n-2}} \pi^* g = u^{\frac{4}{n-2}} \bar{g} = h$.

Finally, the fiber bundle $L \hookrightarrow (M, \bar{g}) \to (M/\mathcal{F}, h)$ with $\bar{g} = u^{\frac{4}{n-2}} g$ satisfies the thesis.

We finish this section with a simply application of our results to the prescription of constant scalar curvature metrics on exotic spheres.

5.3.1. Applications to bundles whose total space are exotic spheres. Eells and Kuiper in [EK62] computed the number of 7 (respectively 15)-exotic spheres that are realized as total spaces of sphere bundles. Therefore, by setting $G = O(n+1)$, $n = 7, 15, F = S^n$, considering this and the discussion in section 5.3, a simply application of Theorem 5.2 gives:

Theorem 5.3. 16 (resp. 4.096) from the 28 (resp. 16.256) diffeomorphisms classes of the 7-dimensional (resp. 15)-exotic spheres admit metrics of positive constant scalar curvature. Moreover, these can be taken as Riemannian submersion metrics when such spaces are considered as the total space of sphere bundles.

REFERENCES

[AB15] M.M. Alexandrino and R.G. Bettiol. Lie Groups and Geometric Aspects of Isometric Actions. Springer International Publishing, 2015.

[AldMS21] Marcos M. Alexandrino, Marcelo K. Inagaki, Mateus de Melo, and Ivan Struchiner. Lie groupoids and semi-local models of Singular Riemannian foliations, 2021.

[Ale10] Marcos M. Alexandrino. Desingularization of singular riemannian foliation. Geometriae Dedicata, 149(1):397–416, 2010.
Marcos M. Alexandrino and Marco Radeschi. Closure of singular foliations: the proof of Molinos conjecture. *Compositio Mathematica*, 153(12):2577–2590, 2017.

T. Aubin. Equations différentielles non linéaires et problème de yamabe concernant la courbure scalaire. *J. Math. Pures Appl.*, 55:269–296, 1976.

T. Aubin. *Some Nonlinear Problems in Riemannian Geometry*. Springer Monographs in Mathematics. Springer Berlin Heidelberg, 1998.

G. Bredon. *Introduction to Compact Transformation Groups*. Academic press, 1972.

Leonardo F. Cavenaghi, João Marcos do Ó, and Llohan D. Sperança. The symmetric Kazdan–Warner problem and applications, 2021.

Leonardo F. Cavenaghi, Renato J. M. e Silva, and Llohan D. Sperana. Positive ricci curvature through cheeger deformation, 2018.

J. Eells and N. Kuiper. An invariant of certain smooth manifolds. *Annali Mat. Pura e Appl.*, 60:413–443, 1962.

James Eells and J. H. Sampson. Harmonic mappings of riemannian manifolds. *American Journal of Mathematics*, 86(1):109–160, 1964.

E. Hebey. Changements de métriques conformes sur la sphère - Le problème de Nirenberg. *Bulletin des Sciences Mathématiques*, 114:215–242, 1990.

Emmanuel Hebey. *Nonlinear analysis on manifolds: Sobolev spaces and inequalities*. Courant lecture notes in mathematics. Courant Institute of Mathematical Sciences; American Mathematical Society, 2000.

E. Hebey and M. Vaugon. Le problème de Yamabe équivariant. *Bulletin des Sciences Mathématiques*, 117:241–286, 1993.

E. Hebey and M. Vaugon. Sobolev spaces in the presence of symmetries. *Journal de Mathématiques Pures et Appliquées*, 76:859–881, 1997.

Ferus D., Münzner H.-F., Karcher, H. Cliffordalgebren und neue isoparametrische hyperflächen. *Mathematische Zeitschrift*, 177:479–502, 1981.

Jerry L. Kazdan and F. W. Warner. A direct approach to the determination of gaussian and scalar curvature functions. *Inventiones mathematicae*, (28):227–230, 1975.

Jerry L. Kazdan and F. W. Warner. Existence and conformal deformation of metrics with prescribed gaussian and scalar curvatures. *Annals of Mathematics*, 101(2):317–331, 1975.

Jerry L. Kazdan and F. W. Warner. Scalar curvature and conformal deformation of riemannian structure. *J. Differential Geom.*, 10(1):113–134, 1975.

Ricardo Mendes and Marco Radeschi. A slice theorem for singular riemannian foliations, with applications, 2018.

S. Morita, American Mathematical Society, T. Nagase, and K. Nomizu. *Geometry of Differential Forms*. Iwanami series in modern mathematics. American Mathematical Society, 2001.

Richard S. Palais. The principle of symmetric criticality. *Comm. Math. Phys.*, 69(1):19–30, 1979.

M. Radeschi. Clifford algebras and new singular riemannian foliations in spheres. *Geometric and Functional Analysis*, 24(5):1660–1682, 2014.

R. Schoen and S-T. Yau. On the proof of the positive mass conjecture in general relativity. *Communications in Mathematical Physics*, 65(1):45 – 76, 1979.

R. M. Schoen and S-T. Yau. Proof of the Positive-Action Conjecture in Quantum Relativity. *Physical Review Letters*, 42:547–548, Feb 1979.

R. Schoen and S-T. Yau. Proof of the positive mass theorem. II. *Communications in Mathematical Physics*, 79(2):231 – 260, 1981.
[Tru68] N. S. Trudinger. Remarks concerning the conformal deformation of Riemannian structures on compact manifolds. *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze*, Ser. 3, 22(2):265–274, 1968.

[Yam60] H. Yamabe. On a deformation of Riemannian structures on compact manifolds. *Osaka Mathematical Journal*, 12(1):1–37, 1960.

[Zil] W. Ziller. On M. Müters PhD thesis. https://www.math.upenn.edu/~wziller/papers/SummaryMueter.pdf.

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