A PERTURBATION RESULT OF M-ACCRETIVE LINEAR OPERATORS IN HILBERT SPACES

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Abstract. A new sufficient condition is given for the sum of linear m-accretive operator and accretive operator one in a Hilbert space to be m-accretive. As an application, an extended result to the operator-norm error bound estimate for the exponential Trotter-Kato product formula is given.

1. Introduction

A linear operator $T$ with domain $\mathcal{D}(T)$ in a complex Hilbert space $\mathcal{H}$ is said to be accretive if

$$\text{Re} < Tx, x > \geq 0 \quad \text{for all } x \in \mathcal{D}(T)$$

or, equivalently if

$$\| (\lambda + T)x \| \geq \lambda \| x \| \quad \text{for all } x \in \mathcal{D}(T) \text{ and } \lambda > 0.$$ Further, if $\mathcal{R}(\lambda + T) = \mathcal{H}$ for some (and hence for every) $\lambda > 0$, we say that $T$ is m-accretive. In particular, every m-accretive operator is accretive and closed densely defined, its adjoint is also m-accretive (cf. [7], p. 279). Furthermore,

$$(\lambda + T)^{-1} \in \mathcal{B}(\mathcal{H}) \quad \text{and} \quad \| (\lambda + T)^{-1} \| \leq \frac{1}{\lambda} \quad \text{for } \lambda > 0,$$

where, $\mathcal{B}(\mathcal{H})$ denote the Banach space of all bounded linear operators on $\mathcal{H}$. In particular, a bounded accretive operator is m-accretive.

Consider two linear operators $T$ and $A$ in the Hilbert space $\mathcal{H}$, such that $\mathcal{D}(T) \subset \mathcal{D}(A)$. Assume furthermore that $T$ is m-accretive and $A$ is an accretive operator. Then the question is:

Under which conditions the sum $T + A$ is m-accretive?

Many papers have been devoted to this problem and most results treat pairs $T, A$ of relatively bounded or resolvent commuting operators. We refer the reader to [2, 3, 5, 6, 15, 17, 18, 20, 21, 22]. Since $T$ is closed it follows that there are two nonnegative constants $a, b$ such that

$$\| Ax \|^2 \leq a \| x \|^2 + b \| Tx \|^2, \quad \text{for all } x \in \mathcal{D}(T) \subset \mathcal{D}(A). \quad (1.1)$$

In this case, $A$ is called relatively bounded with respect to $T$ or simply $T$-bounded, and refer to $b$ as a relative bound. Gustafson [4], generalizing basic work of Rellich, Kato, and others (cf. [7]), showed that that $T + A$ is also m-accretive if $A$ is $T$-bounded, with...
b < 1 (see [4, Theorem 2.]). Okazawa showed in [14] that the closure of the sum \( T + A \) is m-accretive, if the bounded operator \( A(t+T)^{-1} \) on \( \mathcal{H} \) is a contraction for some \( t > 0 \), [14, Theorem 1]. In particular, he also showed that the validity of \( \text{(1.1)} \) with \( b = 1 \) implies that the closure of \( T + A \) is m-accretive, [14, Corollary 1]. Later, the same author in [13] gave a variant of perturbation by assumed the existence of two nonnegative constants \( a \) and \( \beta \leq 1 \) such that

\[
\text{Re} <Tx, Ax> + a \|x\|^2 + \beta \|Tx\|^2 \geq 0, \quad \text{for all } x \in \mathcal{D}(T).
\]

(1.2)

If \( \beta < 1 \), then \( T + A \) is m-accretive and also the closure of \( T + A \) is m-accretive for \( \beta = 1 \), [13, Theorem 4.1]. Note that this result cover the case of relatively bounded perturbation, see [13, Remark 4.4]. There are many papers on the question of such perturbation, see [15, 16, 17, 19, 21] for more results.

The aim of this paper is to establish a new perturbation results on the m-accretivity of the operator \( T + A \). This may be viewed as a slight improvement and generalization of the perturbation results, particularly, those of Okazawa, [15, 13]. The following lemma is our partial answer to the question above.

**Lemma 1.1.** Let \( T \) and \( A \) two operators such that \( \mathcal{D}(T) \subset \mathcal{D}(A) \). Assume that \( T \) is m-accretive, \( A \) is accretive and there exists \( c \geq 0 \), such that

\[
\text{Re} <Tx, Ax> \geq c \|Ax\|^2, \quad \text{for all } x \in \mathcal{D}(T).
\]

(1.3)

If we take \( b = \min \{c \geq 0 : \text{(1.3) holds}\} \), we have,

1. if \( 0 \leq b \leq 1 \), then \( T + A \) is also m-accretive,
2. if \( b > 1 \) then \( T + A \) is m-\( \omega \)-accretive, with \( \omega = \pi/2 - \arcsin(b^{-1}) \).

Here, \( T \) is m-\( \omega \)-accretive if \( e^{\pm i\theta}T \) is m-accretive for \( \theta = \frac{\pi}{2} - \omega, \ 0 < \omega \leq \pi/2 \). In this case, \( -T \) generates an holomorphic contraction semigroup on the sector \( |\arg(\lambda)| < \omega \). In this connection, we note that for any \( \varepsilon > 0 \)

\[
\| (\lambda + T)^{-1} \| \leq \frac{M_\varepsilon}{|\lambda|}, \quad \text{for } |\arg(\lambda)| \leq \frac{\pi}{2} + \omega - \varepsilon
\]

with \( M_\varepsilon \) is independent of \( \lambda \) (see [4, pp. 490]).

The novelty of the lemma is the optimality of \( b \) such that \( \text{(1.3) holds} \). Clearly, \( \text{(1.3)} \) implies \( \text{Re} <Tx, Ax> \geq 0 \) for all \( x \in \mathcal{D}(T) \), this exactly the assumption of [14, Theorem 2.]. Hence, we conclude that \( T + A \) is also m-accretive. Our result is a refinement of this result by given a more precise sector containing the numerical range in function of the constant \( b \). Also, from \( \text{(1.3)} \), we have for \( b > 0 \),

\[
\|Ax\| \leq \frac{1}{b} \|Tx\|, \quad \text{for all } x \in \mathcal{D}(T).
\]

(1.4)

Thus the assumption \( \text{(1.3)} \) is stronger than the relative boundedness with respect to \( T \). In particular, if \( b > 1 \) the lower bound \( \frac{1}{b} < 1 \), so according to [4, Theorem 2.], \( T + A \) is m-accretive. Here, we say more, \( T + A \) is m-\( \omega \)-accretive with \( \omega \) depends of the lower bound \( \frac{1}{b} < 1 \).
2. Proof of the Lemma

Proof of Lemma 1.1. Let \( b = \min \{ c \geq 0 : (1.3) \text{ holds} \} \). If \( b = 0 \), this exactly the Theorem 2. Assume that \( 0 \leq b \leq 1 \). We obtain from (1.3)

\[
0 \leq \Re \langle Tx, Ax \rangle - b \|Ax\|^2 \\
\leq \Re \langle Tx, Ax \rangle + (\alpha - b) \|Ax\|^2
\]

for some \( \alpha > 1 \). Using (1.2), we get

\[
0 \leq \Re \langle Tx, Ax \rangle + \frac{\alpha - b}{b^2} \|Tx\|^2.
\]

Choosing \( \alpha \) such that \( \beta = \frac{\alpha - b}{b^2} < 1 \), by (1.2) we conclude that \( T + A \) is m-accretive (cf. [13, Theorem 4.1]).

Now, suppose that \( b > 1 \). Let \( x \in D(T) \), then for every \( t > 0 \), we have

\[
\Re \langle tx + Tx, Ax \rangle = t \Re \langle x, Ax \rangle + \Re \langle Tx, Ax \rangle \\
\geq b \|Ax\|^2.
\]

Thus we have

\[
\|Ax\| \leq \frac{1}{b} \|tx + Tx\|. \tag{2.1}
\]

Since \( T \) is m-accretive, then

\[
\|A(t + T)^{-1}x\| \leq \frac{1}{b} \|x\|, \quad \text{for all } x \in \mathcal{H}.
\]

Hence it follows that

\[
\|A(t + T)^{-1}\| \leq \frac{1}{b} < 1. \tag{2.2}
\]

Then the operator \( I + A(t + T)^{-1} \) is invertible and

\[
\|(I + A(t + T)^{-1})^{-1}\| \leq \frac{b}{b - 1}.
\]

The fact that

\[
t + T + A = [I + A(t + T)^{-1}](t + T),
\]

it follows that \( -t \in \rho(T + A) \) and

\[
\|t(t + T + A)^{-1}\| \leq \frac{b}{b - 1} = M, \quad \text{for all } t > 0,
\]

with \( M > 1 \). Since \( T + A \) is accretive, \( \rho(T + A) \) contains also the half plane \( \{ z \in \mathbb{C} : Re(z) < 0 \} \). Put \( S = \{ z \in \mathbb{C} : z \neq 0; |\arg(z)| < \pi/2 - \arcsin(\frac{1}{M'}) = \theta \} \) and \( M' := 1/\sin(\pi/2 - \theta') \) with \( 0 < \theta < \theta' < \pi/2 \), clearly \( M' > M \). Let \( \mu \in \mathbb{C} \) such that \( |\arg(\mu)| \leq \theta' \) and fix \( \lambda \) with \( Re\lambda = -t < 0 \). Let \( |\mu - \lambda| \leq \frac{\lambda}{M'} \), we have that

\[
\|(\mu - \lambda)(t + T + A)^{-1}\| \leq \frac{M}{M'} < 1.
\]

Hence it follows that \( \mu \in \rho(T + A) \) and

\[
(\mu + T + A)^{-1} = (\lambda + T + A)^{-1}[I + (\mu - \lambda)(\lambda + T + A)^{-1}]^{-1}.
\]
Thus
\[
\|\mu(\mu + T + A)^{-1}\| \leq \frac{\|\mu\|}{|\lambda|} \frac{1}{1 - \frac{M}{M'}} M
\]
\[
\leq (1 + \frac{1}{M'}) \frac{1}{1 - \frac{M}{M'}} M.
\]

On the other hand,
\[
(1 + \frac{1}{M'}) \frac{1}{1 - \frac{M}{M'}} M = \frac{1 + \sin(\pi/2 - \theta')}{\sin(\pi/2 - \omega) - \sin(\pi/2 - \theta')}
\]
\[
\leq \frac{1}{\sin((\theta' - \theta)/2) \sin((\theta' + \theta)/2)}
\]
\[
\leq \frac{1}{\sin(\theta' - \theta) \sin(\theta)}
\]
\[
\leq \frac{1}{\sin(\theta' - \theta) \sin(\pi/2 - \arcsin(\frac{1}{M}))}
\]
\[
\leq \frac{1}{\sin(\theta' - \theta) \cos(\arcsin(\frac{1}{M}))}
\]
\[
\leq \frac{1}{\sin(\theta' - \theta) \sqrt{1 - \frac{1}{M^2}}}
\]
\[
\leq \frac{M}{\sin(\theta' - \theta) \sqrt{M^2 - 1}}.
\]

This implies that
\[
\|\mu + T + A\|^{-1} \leq \frac{M}{|\mu| \sin(\theta' - \theta) \sqrt{M^2 - 1}}.
\]

This shows that the sector \(S\) belongs to \(\rho(T + A)\) and for any \(\varepsilon > 0\),
\[
\|\mu + T + A\|^{-1} \leq \frac{M_{\varepsilon}}{|\mu|} \text{ for } |\arg(\mu)| \leq \pi/2 - \arcsin(\frac{1}{M}) + \varepsilon,
\]
with \(M_{\varepsilon} = \frac{M}{\sin(\varepsilon) \sqrt{M^2 - 1}}\) and \(\theta' - \theta = \varepsilon\). Clearly, \(M_{\varepsilon}\) is independent of \(\mu\). Hence, \(T + A\) is \(m\)-\(\omega\)-accretive, with \(\omega = \pi/2 - \arcsin(\frac{b - 1}{b})\). \(\square\)

Remark 2.1. (1) As seen in the last paragraph of the proof, the condition \(1.2\) implies \(1.3\) at least for \(0 \leq b \leq 1\). Thus \(1.3\) is covered by Lemma 1.1.
(2) If the assumptions of Lemma 1.1 are satisfied, we can see that \(Re < tx + Tx, Ax >\geq 0\) for all \(x \in D(T)\). Therefore \(A(t + T)^{-1}\) is bounded accretive operator for any \(t > 0\).
Corollary 2.2. Let $T$ and $A$ as in Lemma 1.1 obeying (1.3). Then
(1) $-(T + A)$ generates contractive one-parameter semigroup for $0 \leq b \leq 1$.
(2) $-(T + A)$ generates contractive holomorphic one-parameter semigroup with angle
$\omega = \arcsin(\frac{b-1}{b})$ for $b > 1$.

3. An application

One of interest is the operator-norm error bound estimate for the exponential Trotter-Kato product formula in the case of accretive perturbations, see [10, 11] and [12] for a short survey. Let $A$ be a semibounded from below densely defined self-adjoint operator and $B$ an $m$-accretive operator in a Hilbert space $\mathcal{H}$.

In [1, Theorem 3.4] it has been shown that if $B$ is $A$-bounded with lower bound $< 1$ and

$$\mathcal{D}((A + B)^\alpha) \subset \mathcal{D}(A^\alpha) \cap \mathcal{D}((B^*)^\alpha) \neq \{0\} \quad \text{for some } \alpha \in (0, 1],$$

then there is a constant $L_\alpha > 0$ such that the estimates

$$\left\| (e^{-tB/n}e^{-tA/n})^n - e^{-t(A+B)} \right\| \leq L_\alpha \frac{\ln n}{n^\alpha}$$

and

$$\left\| (e^{-tA^*/n}e^{-tB^*/n})^n - e^{-t(A+B)^*} \right\| \leq L_\alpha \frac{\ln n}{n^\alpha}$$

hold for some $\alpha \in (0, 1]$ and $n = 1, 2, \ldots$ uniformly in $t \geq 0$. Here $T^\alpha$ denotes the fractional powers of an $m$-accretive operator, see [8, 9].

The aim of the present result is to extend [1, Theorem 3.4]. This extension is accomplished by replacing the relative boundedness by the assumption (1.3). More precisely, we have

Theorem 3.1. Let $A$ be a semibounded from below densely defined self-adjoint operator and $B$ an $m$-accretive operator with (1.3) for some $b > 1$. Assume that (3.1) holds. Then there is a constant $L_\alpha > 0$ such that the estimates

$$\left\| (e^{-tB/n}e^{-tA/n})^n - e^{-t(A+B)} \right\| \leq L_\alpha \frac{\ln n}{n^\alpha}$$

and

$$\left\| (e^{-tA^*/n}e^{-tB^*/n})^n - e^{-t(A+B)^*} \right\| \leq L_\alpha \frac{\ln n}{n^\alpha}$$

hold for some $\alpha \in (0, 1]$ and $n = 1, 2, \ldots$ uniformly in $t \geq 0$.

Proof. From (1.3), we have for $b > 1$,

$$\|Bx\| \leq a \|Ax\|, \quad \text{for all } x \in \mathcal{D}(A),$$

with $a = \frac{1}{b} < 1$. Hence $B$ is $A$-bounded with lower bound $a < 1$. Also, by lemma 2.4, $A + B$ is $m$-\(\omega\)-accretive, with $\omega = \pi/2 - \arcsin(\frac{b-1}{b})$. Now, all assumptions of [1, Theorem 3.4] are fulfilled. Hence we obtain the desired result. □

Remark 3.2. It well known that, for an $m$-accretive operator $T$, the fractional powers $T^\alpha$ are $m-(\alpha \pi)/2$-accretive and, if $\alpha \in (0, 1/2)$, then $\mathcal{D}(T^\alpha) = \mathcal{D}(T^{\alpha*})$, see [8, Theorem 1.1]. Since $A$, $B$ and $A + B$ are $m$-accretive operators, we deduce that

$$\mathcal{D}((A + B)^\alpha) = \mathcal{D}((A + B)^\alpha) \subset \mathcal{D}(A^\alpha) \cap \mathcal{D}(B^\alpha) = \mathcal{D}(A^\alpha) \cap \mathcal{D}((B^*)^\alpha),$$

for some $\alpha \in (0, 1/2]$. Thus, the condition (3.1) may be omitted in Theorem 5.1 if we take $\alpha \in (0, 1/2]$ (cf. [1, Theorem 4.1]).
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