SVARC-MILNOR LEMMA: A PROOF BY DEFINITION

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Abstract. The famous Švarc-Milnor Lemma says that a group $G$ acting properly and cocompactly via isometries on a length space $X$ is finitely generated and induces a quasi-isometry equivalence $g \to g \cdot x_0$ for any $x_0 \in X$. We redefine the concept of coarseness so that the proof of the Lemma is automatic.

Geometric group theorists traditionally restrict their attention to finitely generated groups equipped with a word metric. A typical proof of Švarc-Milnor Lemma (see [5] or [1], p.140) involves such metrics. Recently, the study of large scale geometry of groups was expanded to all countable groups by usage of proper, left-invariant metrics: in [6] such metrics were constructed and it was shown that they all induce the same coarse structure on a group (see also [2]). The point of this note is that a proper action of a group $G$ on a space ought to be viewed as a geometric way of creating a coarse structure on $G$. That structure is not given by a proper metric but by something very similar; a pseudo-metric where only a finite set of points may be at mutual distance 0. From that point of view the proof of Švarc-Milnor Lemma is automatic and the Lemma can be summarized as follows. There are two ways of creating coarse structures on countable groups: algebraic (via word or proper metrics) and geometric (via group actions), and both ways are equivalent.

Definition 0.1. A pseudo-metric $d_X$ on a set $X$ is called a large-scale metric (or ls-metric) if for each $x \in X$ the set $\{y \in X \mid d_X(x, y) = 0\}$ is finite.

$(X, d_X)$ is called a large-scale metric space (or an ls-metric space) if $d_X$ is an ls-metric.

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Definition 0.2. An ls-metric $d_G$ on a group $G$ is proper and left-invariant if $d_G(g, h) = d_G(f \cdot g, f \cdot h)$ for all $f, g, h \in G$ and \{h $|$ $d_G(g, h) < r$\} is finite for all $r > 0$ and all $g \in G$.

Notice $G$ must be countable if it admits a proper ls-metric.

One aspect of Švarc-Milnor Lemma is $G$ being finitely generated. That corresponds to $(G, d_G)$ being metrically connected, i.e. there is $M > 0$ such that any two points in $G$ can be connected by a chain of points separated by at most $M$.

Lemma 0.3. Suppose $d_G$ is a proper and left-invariant ls-metric on $G$. $(G, d_G)$ is metrically connected if and only if $G$ is finitely generated.

Proof. If $G$ is generated by a finite set $F$, put $M = \max\{d_G(1_G, f) \mid f \in F\}$. If $(G, d_G)$ is $M$-connected, put $F = B(1_G, M + 1)$.

Definition 0.4. A function $f : (X, d_X) \to (Y, d_Y)$ of ls-metric spaces is called large-scale uniform (or ls-uniform) if for each $r > 0$ there is $s > 0$ such that $d_X(x, y) \leq r$ implies $d_Y(f(x), f(y)) \leq s$.

Lemma 0.5. Suppose $(G, d_G)$ and $(H, d_H)$ are two groups equipped with proper and left-invariant ls-metrics. A function $f : (G, d_G) \to (H, d_H)$ is ls-uniform if and only if for each finite subset $F$ of $G$ there is a finite subset $E$ of $H$ such that $x^{-1} \cdot y \in F$ implies $f(x)^{-1} \cdot f(y) \in E$ for all $x, y \in G$.

Proof. Suppose $f$ is ls-uniform and $F$ is a finite subset of $G$. Let $r$ be larger that all $d_X(1_G, g)$. $g \in F$. Pick $s > 0$ such that $d_G(g, h) < r$ implies $d_H(f(g), f(h)) < s$ and put $E = \{x \in H \mid d_H(1_H, x) < s\}$. If $x^{-1} \cdot y \in F$, then $d_G(x, y) < r$. Therefore $s > d_H(f(x), f(y)) = d_H(1_H, f(x)^{-1} \cdot f(y))$ and $f(x)^{-1} \cdot f(y) \in E$. Conversely, if $r > 0$ put $F = \{x \in G \mid d_G(1_G, x) < r\}$ and consider $E$ so that $x^{-1} \cdot y \in F$ implies $f(x)^{-1} \cdot f(y) \in E$. If $s$ is bigger that all $d_H(1_H, g), g \in E$, then $d_G(x, y) < r$ implies $f(x)^{-1} \cdot f(y) \in E$ and $d_H(f(x), f(y)) < s$.

Corollary 0.6. Given two proper and left-invariant ls-metrics $d_1$ and $d_2$ on the same group $G$, the identity $id_G : (G, d_1) \to (G, d_2)$ is a coarse equivalence.

Proof. The choice of $E = F$ always works for $id_G$.
Lemma 0.7. Suppose \( G \) acts via isometries on \( X \) and \( x_0 \in X \). If \( d_G \) is defined by \( d_G(g, h) = d_X(g \cdot x_0, h \cdot x_0) \), then \( d_G \) is a proper left-invariant ls-metric on \( G \) if and only if the following conditions are satisfied:

1. The stabilizer \( \{ g \in G \mid g \cdot x_0 = x_0 \} \) of \( x_0 \) is finite.
2. \( G \cdot x_0 \) is topologically discrete.
3. Every bounded subset of \( G \cdot x_0 \) that is metrically discrete is finite.

Proof. Recall that \( A \) is metrically discrete if there is \( s > 0 \) such that \( d_X(a, b) > s \) for all \( a, b \in A, a \neq b \). Clearly, if one of Conditions 1-3 is not valid, then there is \( r > 0 \) such that \( B(1_G, r) \) is infinite and \( d_G \) is not proper. Thus, assume 1-3 hold. Suppose \( B(1_G, r) \) is infinite for some \( r > 0 \) and pick \( g_i \) in that set. Suppose \( \{g_n\}_{n=1}^{k} \subseteq B(1_G, 2r) \) is constructed so that \( d_X(g_i \cdot x_0, x_0) < \frac{1}{i} \). Put \( A = B(1_G, r) \setminus \{g_n\}_{n=1}^{k} \) and notice \( A \cdot x_0 \) is infinite (otherwise the stabilizer of \( x_0 \) is infinite). Hence there are two different elements \( g, h \in A \) such that \( g \cdot x_0 \neq h \cdot x_0 \) and \( d_X(g \cdot x_0, h \cdot x_0) < \frac{1}{k+1} \). Put \( g_{k+1} = g^{-1} \cdot h \). However, \( g_n \cdot x_0 \to x_0 \), a contradiction.

It turns out, for nice spaces \( X \), \( d_G \) being a proper ls-metric is equivalent to the action being proper.

Corollary 0.8. Suppose \((X, d_X)\) is a metric space so that all infinite bounded subsets of \( X \) contain an infinite Cauchy sequence. If a group \( G \) acts via isometries on \( X \) and \( x_0 \in X \), then \( d_G(g, h) = d_X(g \cdot x_0, h \cdot x_0) \) defines a proper left-invariant ls-metric on \( G \) if and only if there is a neighborhood \( U \) of \( x_0 \) such that the set \( \{ g \in G \mid g \cdot U \cap U \neq \emptyset \} \) is finite.

Proof. Suppose there is a neighborhood \( U \) of \( x_0 \) such that the set \( \{ g \in G \mid g \cdot U \cap U \neq \emptyset \} \) is finite. Notice there is no converging sequence \( g_n \cdot x_0 \to x_0 \) with \( g_n \)’s being all different.

If \( d_G \) is proper, then choose any ball \( U = B(x_0, r) \) around \( x_0 \). Now, \( g \cdot U \cap U \neq \emptyset \) means there is \( x_g \in U \) so that \( d_X(g \cdot x_0, x_0) < r \). Therefore \( d_G(g, 1_G) = d_X(g \cdot x_0, x_0) \leq d_X(g \cdot x_0, g \cdot x_g) + d_X(g \cdot x_g, x_g) + d_X(x_g, x_0) \leq r + 2r + r = 4r \) and there are only finitely many such \( g \)’s.

Corollary 0.9. If a group \( G \) acts cocompactly and properly via isometries on a proper metric space \( X \), then \( g \to g \cdot x_0 \) induces a coarse equivalence between \( G \) and \( X \) for all \( x_0 \in X \).

Proof. Define \( d_G(g, h) = d_X(g \cdot x_0, h \cdot x_0) \) for all \( g, h \in G \). Clearly, \( d_G \) is left-invariant. Since action is proper, \( d_G \) is a proper ls-metric. Since action is cocompact, \( X \) is within bounded distance from \( G \cdot x_0 \).
Corollary 0.10 (Švarc-Milnor). A group $G$ acting properly and co-compactly via isometries on a length space $X$ is finitely generated and induces a quasi-isometry equivalence $g \to g \cdot x_0$ for any $x_0 \in X$.

Proof. Consider the proper left-invariant metric $d_G$ induced on $G$ by the action. The cocompactness of the action implies $G \cdot x_0$ is metrically connected. So is $(G, d_G)$ and $G$ must be finitely generated. Both $X$ and a Cayley graph of $G$ are proper geodesic spaces. Therefore any coarse equivalence between them is a quasi-isometric equivalence. ■

Final comments.

Let us point out that Švarc-Milnor Lemma 0.9 for non-finitely generated groups is useful when considering spaces of asymptotic dimension 0. A large scale analog $M^0$ of 0-dimensional Cantor set is introduced in [3]: it is the set of all positive integers with ternary expression containing 0’s and 2’s only (with the metric from $\mathbb{R}_+$):

$$M^0 = \left\{ \sum_{i=-\infty}^{\infty} a_i 3^i \mid a_i = 0, 2 \right\}.$$

Proposition 0.11. [3, Theorem 3.11] The space $M^0$ is universal for proper metric spaces of bounded geometry and of asymptotic dimension zero.

Proposition 0.12. The space $M^0$ is coarsely equivalent to $\bigoplus_{i=1}^{\infty} \mathbb{Z}_2$.

Proof. Consider the subset $A = \left\{ \sum_{i=0}^{\infty} a_i 3^i \mid a_i = 0, 2 \right\}$ of $M^0$. Notice $M^0$ is within bounded distance from $A$, so $A \to M^0$ is a coarse equivalence. Also, there is an obvious action of $\bigoplus_{i=1}^{\infty} \mathbb{Z}_2$ on $A$ (flipping $a_i = 0$ to 2 or $a_i = 2$ to 0 if the corresponding term in $\bigoplus_{i=1}^{\infty} \mathbb{Z}_2$ is not zero) that is proper and cocompact. ■

Notice any infinite countable group $G$ of asymptotic dimension 0 is locally finite (see [6]). Thus it can be expressed as the union of a strictly increasing sequence of its finite subgroups $G_1 \subset G_2 \subset \ldots$. Put $n_1 = |G_1|$, $n_i = |G_i/G_{i-1}|$ for $i > 1$, and observe (using 0.5) that $G$ is coarsely equivalent to $\bigoplus_{i=1}^{\infty} \mathbb{Z}_{n_i}$. We do not know if any two infinite countable groups of asymptotic dimension 0 are coarsely equivalent.

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