Navier–Stokes, Gross–Pitaevskii and generalized diffusion equations using the stochastic variational method

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Abstract

The stochastic variational method is applied to particle systems and continuum media. As a brief review of this method, we first discuss the application to particle Lagrangians and derive a diffusion-type equation and the Schrödinger equation with the minimum gauge coupling. We further extend the application of the stochastic variational method to Lagrangians of continuum media and show that the Navier–Stokes, Gross–Pitaevskii and generalized diffusion equations are derived. The correction term for the Navier–Stokes equation is also obtained in this method. We discuss the meaning of this correction by comparing it with the diffusion equation.

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1. Introduction

The variational principle plays a fundamental role in many branches in physics, such as analytic mechanics, field theory, statistical mechanics, quantum many-body problems, scattering theory, fluid dynamics and so on. Due to its elegant mathematical nature, the approach serves also as important guidance to formulate models, extracting the essential physical structure of the system in question. In particular, the method is indispensable when we deal with symmetries and the associated conservation laws of a system [1]. All the consequences of symmetries, at least in the classical level, are simply condensed in the fact that the Lagrangian (density) is invariant under the transformations of the corresponding symmetry.

On the other hand, once some dissipative processes are involved, the usual variational approach loses its elegance. To include dissipative effects, one has to introduce some extra function, such as the so-called Rayleigh dissipation function or a time-dependent external factor for a Lagrangian [2]. The choice of the functions is usually ambiguous. This modification is attributed to the fact that dissipation is essentially related to the energy conversion from
macroscopic degrees of freedom to internal microscopic motions, accompanying with the entropy production [1]. In this sense, dissipation is beyond the scope of the classical variational method.

Around three decades ago, a new variational method was proposed [5–7], where dynamical variables are extended to stochastic quantities. In this stochastic variational method (SVM), the effects of the microscopic degrees of freedom to the macroscopic dynamical variables are represented in terms of noises, leading naturally to dissipation of the dynamical variables. The advantage of this method is that we do not need to modify Lagrangians by introducing extra functions, and, once the noise is specified for a given Lagrangian, the dissipative dynamics is derived in a straightforward way.

Since then, the SVM has been studied for several cases, exclusively for particle Lagrangians. It was shown that diffusion-type equations [8–10] and the Schrödinger equation [11–14] can be derived in the framework of the SVM. In particular, in the latter case, the SVM approach has been extensively discussed in relation to Schrödinger and Nelson’s stochastic formulation of quantum mechanics [15, 16].

On the other hand, it seems that, in spite of the interesting aspects of the method, the potential possibility of the SVM has not been explored sufficiently. Recently, we have shown that the method can be used to derive the Navier–Stokes equation (NS) for compressible fluids [17]. However, the principal aim of that work was to show the essence of the derivation of the NS and the detailed discussion on the formulation was not presented. In this work, we follow up the previous report, exhibiting the correspondence between the classical and SVM. Furthermore, we show how the SVM can be applied to several examples. In this manner, we can grasp the current status of the SVM applications in a unified way, since the literatures on the known applications of the SVM are rather scattered. As new applications, we derive the Gross–Pitaevskii and generalized diffusion equations in the framework of the SVM. We show also that the present SVM approach naturally induces a correction term of the NS, and the physical meaning of the correction is discussed by comparing with the usual diffusion equation.

This paper is organized as follows. In section 2, we summarize the classical variational method to be contrasted to the SVM. In section 3, the application of the SVM to a particle system is discussed for four different cases. One of such cases leads naturally to the Schrödinger equation. This method is applicable even when there exists an external gauge field as is discussed in section 4. In section 5, the SVM is applied to continuum media, and the Navier–Stokes, Gross–Pitaevskii and generalized diffusion equations are derived. The possible generalization of the NS is further discussed by comparing with the diffusion equation. Section 6 is devoted to concluding remarks.

2. Classical variational method

To clarify the difference between the classical variational method and the SVM, let us restate the classical variational method in the following form. Suppose that there exists a vector field \( \mathbf{f} = \mathbf{f}(\mathbf{r}, t) \), where the velocity of a particle at the position \( \mathbf{x}(t) \) is given by

\[
\frac{d}{dt} \mathbf{x}(t) = \mathbf{f}(\mathbf{r}, t)|_{\mathbf{r}=\mathbf{x}(t)}.
\]

As is shown, this quantity coincides with the particle velocity at the position of the particle \( \mathbf{r} = \mathbf{x}(t) \), but is defined for all \( \mathbf{r} \) at a given \( t \). This is essential when we introduce the stochastic nature for the particle trajectory. Our strategy is to determine this field \( \mathbf{f} \) by the variational principle.

3 See, for example [3] and for another interesting approach see [4].

2
Let us consider the usual action for a particle, which is given as the difference of the kinetic and potential terms,

$$ I = \int_{t_a}^{t_b} \left[ \frac{m}{2} \left( \frac{dx(t)}{dt} \right)^2 - V(x(t)) \right] dt, $$

(2)

where $m$ is the mass of a particle and $V$ is the potential energy.

We further consider the variation of the position, $\delta x \rightarrow x + \delta x$, with the conditions $\delta x(t_a) = \delta x(t_b) = 0$ as usual. Here $t_a$ and $t_b$ represent the initial and final times of the particle trajectory, respectively. The variation of the action is then given by

$$ \delta I = \int_{t_a}^{t_b} \left[ m \frac{dx(t)}{dt} \delta \left( \frac{dx(t)}{dt} \right) - \delta V(x(t)) \right] dt = \int_{t_a}^{t_b} \left[ -m \frac{d}{dt} \left( \delta \frac{dx(t)}{dt} - \nabla V(x(t)) \right) \right] \cdot \delta x, $$

(3)

and the stationarity condition of the action leads to

$$ m(\partial_t + f(x(t), t) \cdot \nabla) f(x(t), t) + \nabla V(x(t)) = 0. $$

(4)

Of course, if we introduce the following variable,

$$ v(t) = f(x(t), t), $$

(5)

then equation (4) can be cast into the usual form of Newton’s equation of motion,

$$ \frac{dv(t)}{dt} = v(t), $$

$$ m \frac{dv(t)}{dt} + \nabla V(x(t)) = 0, $$

(6a)

(6b)

where we used the definition of the material derivative,

$$ \frac{dv(t)}{dt} = \frac{df(x(t), t)}{dt} = \left( \partial_t + f(x(t), t) \cdot \nabla \right) f(x(t), t). $$

(7)

On the other hand, we can regard equation (5) as a special value of the solution of a partial differential equation for the field $f(r, t)$,

$$ m(\partial_t + f(r, t) \cdot \nabla) f(r, t) + \nabla V(r) = 0, $$

(8)

with some appropriate initial boundary conditions. In the case of the classical variational principle, we do not need to calculate $f(r, t)$ for all $r$ at a given $t$ to determine the classical particle trajectory. However, when we extend $x(t)$ to a stochastic quantity, as we will see soon later, we need to know the values of $f(r, t)$ for general values of $r$.

3. SVM for particle systems

In the classical variational method, the evolution of a particle is deterministic. In the SVM, we consider the case where the trajectory of a particle is stochastic [5]. This is because we would like to approximately take into account the effects of some microscopic degrees of freedom, which do not appear as the dynamical variables in a given action, through the effect of noise. Therefore, instead of equation (1), the time evolution of the particle is described by the following stochastic differential equation (SDE):

$$ dx(t) = f(x(t), t) dt + \sqrt{2\nu} \, dW_t (dt > 0), $$

(9)

where the last term represents the noise. Properties of the noise will depend on the detailed nature of the underlying microscopic dynamics. However, if there is a clear separation of
microscopic and macroscopic timescales, we can assume that microscopic information is approximately replaced by the Gaussian white noise as the most standard and simplest choice. Thus, $d\mathbf{W}_t$ is differential of a Wiener process, satisfying

$$E[d\mathbf{W}_t] = 0,$$  \hspace{1cm} (10a)

$$E[d\mathbf{W}_t^i d\mathbf{W}_t^j] = \delta^{ij} dt,$$ \hspace{1cm} (10b)

where $E[\cdot]$ denotes the expectation value for the stochastic processes. The magnitude of the noise is characterized by $\nu$ which is a constant. We call equation (9) the forward SDE, because it is defined only for $dt > 0$ (see the discussion below).

In stochastic systems, the particle distribution function is introduced as

$$\rho(\mathbf{r}, t) = E[\delta(\mathbf{r} - \mathbf{x}(t))],$$ \hspace{1cm} (11)

where the expectation value is taken over all the trajectories $\{\mathbf{x}(t)\}$. Therefore, $\rho(\mathbf{r}, t)$ contains the information of the time evolution for a given initial distribution $\rho(\mathbf{r}, t_0)$. Of course, the distribution of initial values $\{\mathbf{x}(t_0)\}$ for equation (9) should be given by $\rho(\mathbf{r}, t_0)$ so that equation (11) is valid at $t = t_0$. From this definition and using the properties of the noise, the evolution equation of $\rho$ is expressed in terms of the following Fokker–Planck equation [18]:

$$\partial_t \rho(\mathbf{r}, t) = \nabla \cdot (-\mathbf{f}(\mathbf{r}, t) + \nu \nabla) \rho(\mathbf{r}, t).$$ \hspace{1cm} (12)

As discussed in the previous section, the field $\mathbf{f}(\mathbf{r}, t)$ is yet unknown and is determined by employing the variational principle. However, different from the classical deterministic case, we now face a new problem: the trajectories of stochastic variables are not smooth and are non-differentiable. Thus, the time derivative is not uniquely defined. For example, we may consider the following two possible definitions for the velocity of the particle at $t$,

$$\mathbf{v}(t) = \lim_{dt \to 0^+} \frac{\mathbf{x}(t + dt) - \mathbf{x}(t)}{dt},$$ \hspace{1cm} (13a)

$$\tilde{\mathbf{v}}(t) = \lim_{dt \to 0^-} \frac{\mathbf{x}(t + dt) - \mathbf{x}(t)}{dt}.$$ \hspace{1cm} (13b)

Obviously, when the particle trajectory $\mathbf{x}(t)$ is continuous and smooth, the two definitions coincide, $\mathbf{v}(t) = \tilde{\mathbf{v}}(t)$, as is the case of the classical variational method. On the other hand, for the stochastic trajectory, $\mathbf{v}(t) \neq \tilde{\mathbf{v}}(t)$ in general. In spite of this, it was shown that the variational approach can be consistently formulated by completing the information of the backward SDE in addition to the forward SDE, as is shown in [5].

Let us introduce a SDE for the time-reversed evolution of $\mathbf{x}(t)$ as

$$d\mathbf{x}(t) = \tilde{\mathbf{f}}(\mathbf{x}(t), t) dt + \sqrt{2\nu} d\tilde{\mathbf{W}}_t \quad (dt < 0),$$ \hspace{1cm} (14)

where we assume the properties of new noise term $d\tilde{\mathbf{W}}_t$ stay the same as before,

$$E[d\tilde{\mathbf{W}}_t] = 0,$$ \hspace{1cm} (15a)

$$E[d\tilde{\mathbf{W}}_t^i d\tilde{\mathbf{W}}_t^j] = \delta^{ij}|dt|.$$ \hspace{1cm} (15b)

We call equation (14) the backward SDE. Note that the two noise terms $d\mathbf{W}_t$ and $d\tilde{\mathbf{W}}_t$ do not have correlations. We can obtain the Fokker–Planck equation from this backward SDE similar to the case of the forward SDE as

$$\partial_t \rho(\mathbf{r}, t) = \nabla \cdot (-\tilde{\mathbf{f}}(\mathbf{r}, t) - \nu \nabla) \rho(\mathbf{r}, t).$$ \hspace{1cm} (16)

4 The stochastic process of this kind is known as the Bernstein process in mathematics. The Bernstein process can be one of the Markov processes, see, for example, [27].
If the above backward SDE, as a set of stochastic processes, describes the correct representation of the time-reversed process of the forward SDE, the Fokker–Planck equation (16) should coincide with equation (12). From this condition, we obtain

$$\mathbf{f}(\mathbf{r}, t) = \tilde{\mathbf{f}}(\mathbf{r}, t) + 2v\nabla \ln \rho(\mathbf{r}, t) + \nabla \times \mathbf{A}_1(\mathbf{r}, t) + \mathbf{A}_2(t),$$

(17)

where $\mathbf{A}_1(\mathbf{r}, t)$ and $\mathbf{A}_2(t)$ are arbitrary functions. In the absence of external forces and non-singular boundary conditions, we can simply omit $\mathbf{A}_1(\mathbf{r}, t)$ and $\mathbf{A}_2(t)$, so that $\mathbf{f}(\mathbf{r}, t)$ and $\tilde{\mathbf{f}}(\mathbf{r}, t)$ are related as

$$\mathbf{f}(\mathbf{r}, t) = \tilde{\mathbf{f}}(\mathbf{r}, t) + 2v\nabla \ln \rho(\mathbf{r}, t).$$

(18)

In order to incorporate the forward and backward SDEs consistently in the variational scheme of stochastic variables [5, 16], we introduce the mean forward derivative,

$$D_{\mathbf{x}}(t) \equiv \lim_{h \to 0^+} E \left[ \frac{x(t + h) - x(t)}{h} \mid \mathcal{P}_t \right],$$

(19)

and the mean backward derivative,

$$\tilde{D}_{\mathbf{x}}(t) \equiv \lim_{h \to 0^+} E \left[ \frac{x(t) - x(t - h)}{h} \mid \mathcal{F}_t \right],$$

(20)

instead of the two velocities, equations (13a) and (13b). Here, $E[F(t') \mid \mathcal{P}_t]$ denotes the conditional average of a time sequence of stochastic variables $\{F(t'), t_a < t' < t_b\}$ only for $t' > t$, fixing the values of $F(t')$ for $t' \leq t$ [16]. For the $\sigma$-algebra of all measurable events of $\mathbf{X}$, $\mathcal{P}_t$ and $\mathcal{F}_t$ represent an increasing and a decreasing family of sub-$\sigma$-algebras, respectively. See, for example, section 1 of [7]. Thus, $DF(t)$ is still a stochastic quantity, defined by the forward SDE. In fact, this definition leads to

$$D_{\mathbf{x}}(t) = f(x(t), t).$$

(21)

Similarly, $E[F(t') \mid \mathcal{F}_t]$ is the conditional average for the past sequence, defined by the backward SDE, fixing the future values, so that

$$\tilde{D}_{\mathbf{x}}(t) = \tilde{f}(x(t), t).$$

(22)

These velocities are used to replace the classical velocity in the kinetic term of the action. However, this replacement is not unique and each case corresponds to different physical scenarios. In the following, we discuss four different cases.

3.1. Case 1. Pure forward derivative

Let us consider the case where the velocity $d\mathbf{x}/dt$ in the classical action is replaced only by the mean forward derivative $D_{\mathbf{x}}(t)$. Then, the action is given by

$$I = \int_{t_a}^{t_b} dt \left[ \frac{m}{2} D_{\mathbf{x}}(t) \cdot D_{\mathbf{x}}(t) - V(x(t)) \right].$$

(23)

Note that the products of the mean forward (backward) derivatives are independent of the choice of the discretization scheme such as the Ito, Stratonovich–Fisk and Hänggi–Klimontovich schemes by the definition of the mean forward (or backward) derivative [18].

Now we introduce the variation of the position of the particle as

$$\mathbf{x}(t) \longrightarrow \mathbf{x}_\lambda(t) = \mathbf{x}(t) + \lambda \mathbf{g}(\mathbf{x}(t), t),$$

(24)

where $\lambda$ is an expansion parameter and $\mathbf{g}(\mathbf{r}, t)$ is an arbitrary smooth function which is differentiable and satisfies the boundary condition

$$\mathbf{g}(\mathbf{x}(t_a, b), t_a, b) = 0.$$ 

(25)
Substituting the above expression into action (23), we obtain

\[ I_h = I + \lambda \int_{t_0}^{t_1} dt E[\mathbf{mD}(\mathbf{x}(t)) \cdot g(\mathbf{x}(t), t) - \nabla V(\mathbf{r})(\mathbf{r} = \mathbf{x}(t)) \cdot \mathbf{g}(\mathbf{x}(t), t)] + O(\lambda^2) \]

\[ = I + \lambda \int_{t_0}^{t_1} dt E[-m \tilde{D}(\mathbf{x}(t)) \cdot \nabla V(\mathbf{r})(\mathbf{r} = \mathbf{x}(t)) \cdot \mathbf{g}(\mathbf{x}(t), t)] + O(\lambda^2) \]

\[ = I + \lambda \int_{t_0}^{t_1} dt E[-m \tilde{D}(\mathbf{x}(t)) \cdot \nabla V(\mathbf{r})(\mathbf{r} = \mathbf{x}(t)) \cdot \mathbf{g}(\mathbf{x}(t), t)] + O(\lambda^2). \tag{26} \]

Here, we have used the stochastic partial integral formula [17, 19] and equation (21). Using the Ito formula (Ito’s lemma) [18], the first term in the integral can be calculated as

\[ \tilde{D}(\mathbf{x}(t), t) = (\partial_t + \tilde{f}(\mathbf{r}, t) \cdot \nabla - \nu \nabla^2)\tilde{f}(\mathbf{r}, t) \bigg|_{\mathbf{r} = \mathbf{x}(t)}. \tag{27} \]

Therefore, the stationary condition of the action under the variations leads to

\[ m[\partial_t + \tilde{f}(\mathbf{r}, t) \cdot \nabla - 2\nu(\nabla \ln \rho(\mathbf{r}, t)) \cdot \nabla - \nu \nabla^2] \tilde{f}(\mathbf{r}, t) = -\nabla V(\mathbf{r}). \tag{28} \]

Here, \( \tilde{f}(\mathbf{r}, t) \) is replaced by \( \tilde{f}(\mathbf{r}, t) \) using the consistency condition (18). Equation (28) and the Fokker–Planck equation (12) constitute a system of partial differential equations for the unknown functions \( f(\mathbf{r}, t) \) and \( \rho(\mathbf{r}, t) \).

As stressed in the previous section, in the classical variational method, the final equation of motion requires only the velocity \( \mathbf{v}(t) = \mathbf{f}(\mathbf{x}(t), t) \) and the knowledge of the field \( \mathbf{f}(\mathbf{r}, t) \) at arbitrary position \( \mathbf{r} \) is not necessary. In the SVM, we, however, have to calculate the field \( \mathbf{f}(\mathbf{r}, t) \) itself. In fact, \( \mathbf{f}(\mathbf{x}(t), t) \) which appears in equation (28) cannot be completely replaced by \( \mathbf{v}(t) \) because of the second spatial derivative term, which did not appear in the classical variational method. That is, the fundamental variable is not \( \mathbf{v}(t) \) but \( \mathbf{f}(\mathbf{r}, t) \) in SVM. The velocity on the trajectory is known only after solving the forward SDE (9) with the solution of equation (28).

From equation (21), \( \mathbf{f}(\mathbf{r}, t) \) gives the mean forward velocity of the particle, obeying the forward SDE equation. Thus, one might expect that \( \mathbf{f}(\mathbf{r}, t) \) is the velocity field corresponding to the probability current. From the Fokker–Planck equation (12), we identify such a probability current as

\[ \mathbf{J}(\mathbf{r}, t) = (\mathbf{f}(\mathbf{r}, t) - \nu \nabla)\rho(\mathbf{r}, t). \tag{29} \]

One can see that \( \mathbf{f}(\mathbf{r}, t) \) is not the velocity field parallel to the probability current, and neither \( \tilde{f}(\mathbf{r}, t) \) is. However, the average of \( \mathbf{f}(\mathbf{r}, t) \) and \( \tilde{f}(\mathbf{r}, t) \),

\[ f_m(\mathbf{r}, t) = \frac{\mathbf{f}(\mathbf{r}, t) + \tilde{f}(\mathbf{r}, t)}{2} = \mathbf{f}(\mathbf{r}, t) - \nu \nabla \ln \rho(\mathbf{r}, t) = \tilde{f}(\mathbf{r}, t) + \nu \nabla \ln \rho(\mathbf{r}, t), \tag{30} \]

is related to the probability current as

\[ \mathbf{J}(\mathbf{r}, t) = \rho(\mathbf{r}, t)f_m(\mathbf{r}, t). \tag{31} \]

That is, the velocity parallel to \( \mathbf{J}(\mathbf{r}, t) \) is given by \( f_m(\mathbf{r}, t) \).

Equation (28) is then re-expressed as

\[ \rho(\partial_t + f_m \cdot \nabla) f_m^i - \nabla \cdot (v \rho \partial_t f_m) - \nabla \cdot [(v \rho \nabla \cdot f_m^i) - \nabla \cdot (v \rho \nabla (v \partial_t \ln \rho))] = -\frac{\rho}{m} \nabla \cdot \mathbf{V}, \tag{32} \]

where \( f_m^i \) is the \( i \)th component of \( f_m \). In terms of the mean velocity field, \( f_m \), the Fokker–Planck equation for \( \rho \), which is given by equation (12) (or equivalently by equation (16)), is reduced simply to the continuity equation,

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho f_m) = 0. \tag{33} \]

Equations (32) and (33) constitute a closed set of differential equations for \( f_m(\mathbf{r}, t) \) and \( \rho(\mathbf{r}, t) \). Note that equation (32) contains the term which violates the time-reversal symmetry and describes a dissipative process.
3.2. Case 2. Pure backward derivative

Another possibility is to use only the mean backward derivative to express the kinetic term

\[ I = \int_{t_a}^{t_b} dE \left[ \frac{m}{2} \left( \frac{m}{2} D_x(t) \cdot \tilde{D}_x(t) - V(x(t)) \right) \right]. \]  

(34)

Repeating the variational procedure analogous to case 1, we obtain

\[ I_b = I + \lambda \int_{t_a}^{t_b} dE \left[ -m \tilde{\nabla} \bar{f}(x(t), t) - \nabla V(r(t)) \right] \cdot \mathbf{g}(x(t), t) + O(\lambda^2). \]  

(35)

The stationarity condition of the action leads to, instead of equation (28),

\[ m [\partial_t + \tilde{f}(r, t) \cdot \nabla + 2v(\nabla \ln \rho(r, t)) \cdot \nabla + v \nabla^2 \tilde{f}(r, t) = -\nabla V(r). \]  

(36)

Equation for \( f_m \) becomes

\[ \rho (\partial_t + f_m \cdot \nabla)^2 f_m + \nabla \cdot (v \rho \partial_t f_m) + \nabla \cdot (v \rho \nabla f_m) - \nabla \cdot [(v \rho) \nabla (v \partial_t \ln \rho)] = -\frac{\rho}{m} \nabla^4 V. \]  

(37)

One can see that this equation describes the time-reversed process of equation (32). In fact, equation (37) coincides with equation (32) by exchanging the sign of \( v \).

3.3. Case 3. Mixed product of two derivatives

Hasegawa [8] studied the case where the kinetic term is written with both of \( Dx \) and \( \tilde{D}x \) as

\[ I = \int_{t_a}^{t_b} dE \left[ \frac{m}{2} D_x(t) \cdot \tilde{D}_x(t) - V(x(t)) \right]. \]  

(38)

Different from the previous two cases, this kinetic term is symmetric for the exchange between \( D \) and \( \tilde{D} \). The stochastic variation, then, leads to

\[ (\partial_t + f_m \cdot \nabla)^2 f_m + 2v^2 \nabla^4 \rho^{-1/2} \sqrt{\rho} = -\frac{1}{m} \nabla^4 V. \]  

(39)

Different from equations (32) and (37), one can see that this equation is symmetric for the time-reversed operation \( t \leftrightarrow -t \). This is the consequence of the symmetric replacement in the kinetic term.

One can confirm that equation (39) is completely equivalent to the result of [8], by noting the relation \( f_m(r, t) = 2v \nabla S(r, t) \) in his notation.

3.4. Case 4. Average of cases 1 and 2—Schrödinger equation

As the last example, we consider that the kinetic term of the action is given by the average of the mean forward and backward derivatives [11],

\[ I = \int_{t_a}^{t_b} dE \left[ \frac{m}{2} \left( D_x(t) \cdot D_x(t) + \tilde{D}_x(t) \cdot \tilde{D}_x(t) - V(x(t)) \right) \right]. \]  

(40)

In this case, the kinetic term is symmetric for the exchange of \( D \) and \( \tilde{D} \), and hence the derived equation by the variation is symmetric for the time-reversed operation. As a matter of fact, SVM leads to the following equation:

\[ \frac{m}{2} (\partial_t \tilde{f} + (\tilde{f} \cdot \nabla) \tilde{f} + \nabla^2 \tilde{f} + \partial_t f + (f \cdot \nabla) f - \nabla^2 f) = -\nabla V. \]  

(41)

This equation can be re-expressed as

\[ (\partial_t + f_m \cdot \nabla)^2 f_m + 2v^2 \nabla^4 \rho^{-1/2} \sqrt{\rho} = -\frac{1}{m} \nabla^4 V. \]  

(42)
It is known that equation (42), together with the continuity equation (33), is (almost) equivalent to the Schrödinger equation [11]. To show this, we introduce the following real function \( \theta \):

\[
\nabla \theta = \frac{1}{2 \nu} f_m.
\]

(43)

Obviously, here we assumed that \( f_m \) is irrotational, \( \nabla \cdot f_m = 0 \). Then, from equation (42), we obtain

\[
\nabla \left[ \partial_t \theta + \nu (\nabla \theta)^2 - \nu (\rho^{-1/2} \nabla^2 \sqrt{\rho}) + \frac{1}{2 \nu m} V \right] = 0.
\]

(44)

This leads to

\[
\partial_t \theta = -\nu (\nabla \theta)^2 + \nu (\rho^{-1/2} \nabla^2 \sqrt{\rho}) - \frac{1}{2 \nu m} V + c(t),
\]

(45)

where \( c(t) \) is an arbitrary function of \( t \). However, this is always absorbed into the definition of \( \theta \). Thus, without loss of generality, we can set \( c(t) = 0 \). On the other hand, the Fokker–Planck equation (12) is reduced to

\[
\partial_t \ln \sqrt{\rho} = -2 \nu (\nabla \ln \sqrt{\rho}) \cdot \nabla \theta - \nu \nabla^2 \theta.
\]

(46)

Now we introduce the wavefunction as

\[
\psi = \sqrt{\rho} e^{i \theta}.
\]

(47)

Then, from equations (45) and (46), the evolution equation of \( \psi \) is given by

\[
i \partial_t \psi = \left[ -\nu \nabla^2 + \frac{1}{2 \nu m} V \right] \psi.
\]

(48)

By choosing \( \nu = \hbar / (2m) \), we obtain the Schrödinger equation,

\[
i \hbar \partial_t \psi = \left[ \frac{1}{2m} \left( \frac{\hbar i \nabla}{\sqrt{\alpha}} \right)^2 + V \right] \psi.
\]

(49)

Note that, by the definition of \( \psi \) given by equation (47), \( |\psi|^2 \) gives naturally the probability density of the particle.

Of course, if the flow is rotational, we cannot introduce the phase of the wavefunction as equation (43). Thus, the equation obtained in the SVM is not completely equivalent to the Schrödinger equation.

The above derivation of the Schrödinger equation in terms of the SVM opens an interesting possibility for a different formulation of quantum mechanics, and extensive works have been done so far [10, 11], but in this paper, we do not pursue further in this direction. We only mention that there is an ambiguity for the choice of Lagrangian in the derivation of the Schrödinger equation. Let us consider the following action:

\[
I = \int_a^b dt E \left[ \frac{m}{2} \left( \frac{Dx(t) \cdot Dx(t) + \bar{D}x(t) \cdot \bar{D}x(t)}{2} - \frac{\alpha}{8} (Dx(t) - \bar{D}x(t)) \cdot (Dx(t) - \bar{D}x(t)) \right) - V(x(t)) \right],
\]

(50)

where \( \alpha \) is a constant. This action is reduced to the previous action (40) in the vanishing \( \alpha \) limit. As is shown in [14], the stochastic variation of this action again leads to the Schrödinger equation by choosing

\[
\nu = \frac{1}{\sqrt{1 + \alpha / 2}} \frac{\hbar}{2m}.
\]

(51)
Here, we discussed the wavefunction for the action given by equation (40). The wavefunction \( \psi \) defined by equation (47) can be introduced even in the other cases discussed before. However, only this fourth case leads to the known form of the Schrödinger equation.

For example, when we apply the same argument to equation (39), we obtain (again assuming that \( f_m \) is irrotational)

\[
\text{i} \hbar \partial_t \psi = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V \right] \psi + \frac{\hbar^2}{2m} \left[ \frac{1}{|\psi|^2} \nabla^2 |\psi|^2 - \frac{1}{2} \frac{1}{|\psi|^4} (\nabla |\psi|^2)^2 \right] \psi. \tag{52}
\]

Since this equation is not linear in \( \psi \), the superposition principle required in quantum mechanics is not satisfied.

4. Particle interacting with gauge fields

The gauge symmetry is an important principle to describe interactions among elementary particles. Thus, whether the SVM can be applied consistently keeping the gauge invariance is a crucial question. In the case of the Schrödinger equation, it is known that the SVM leads to the correct wave equation with the minimal coupling of a charged particle, starting from the corresponding classical action as sketched below \([11, 13]\).

Let us start with the action of a classical particle interacting with the external gauge fields:

\[
I = \int dt \left[ m \frac{\dot{v}^2}{2} + \frac{e}{c} v \cdot A - V - e \phi \right], \tag{53}
\]

where \( \phi \) and \( A \) are the scalar and vector potentials of the electromagnetic field and \( V \) is another potential energy, respectively. Here, we used the CGI-Gauss unit. One can easily check that this action is invariant for the gauge transform as

\[
A \rightarrow A' = A + \nabla \chi, \tag{54a}
\]

\[
\phi \rightarrow \phi' = \phi - \frac{1}{c} \frac{\partial}{\partial t} \chi, \tag{54b}
\]

where \( \chi \) is an arbitrary function.

For the stochastic variations to derive the quantum mechanical wave equation, we replace this classical action by

\[
I = \int dt E \left[ m \frac{\dot{D}x^2}{2} + \frac{e}{c} \dot{D}x \cdot D\dot{x} + e \frac{\dot{D}x + D\dot{x}}{2} \cdot A - V - e \phi \right]. \tag{55}
\]

The reason for this choice is that the action should be symmetric under the time reversal. Following the standard procedure, we find that the stochastic variation of the action becomes

\[
I_\lambda = I + \lambda \int dt E \left[ -m \frac{\dot{\bar{D}}f + D\bar{\bar{f}}}{2} - \frac{e}{2c} (D + \bar{\bar{D}})A + \frac{e}{c} \sum_i \frac{f^i + \bar{\bar{f}}^i}{2} \nabla A^i - \nabla V - e \nabla \phi \right]
\cdot g + O(\lambda^2) \tag{56},
\]

so that we obtain

\[
(\partial_t + f_m \cdot \nabla) f_m = -2v^2 \nabla (\rho^{-1/2} \nabla \sqrt{\rho}) + \frac{1}{m} \nabla (V + e \phi) = -\frac{e}{cm} \partial_t A + \frac{e}{cm} f_m \times (\nabla \times A). \tag{57}
\]

The second term on the right-hand side represents the Lorentz force.

\(^5\text{Here, invariance means that the modification of the action by the gauge transform is absorbed into the total time derivative term of the Lagrangian.}\)
In order to derive the corresponding Schrödinger equation, we introduce the phase $\theta$ by
\[
\nabla \theta = \frac{f_m + \frac{e}{cm} A}{2v},
\]
instead of equation (43). Then, the equation corresponding to equation (45) becomes
\[
\partial_t \theta = -v \left( \nabla \theta - \frac{e}{2cmv} A \right)^2 + v(\rho^{-1/2} \nabla^2 \sqrt{\rho}) - \frac{1}{2vm}(V + e\phi).
\]
On the other hand, the equation of $\ln \sqrt{\rho}$ is
\[
\partial_t \ln \sqrt{\rho} = -2v(\nabla \ln \sqrt{\rho}) \cdot (\nabla \theta) - v\nabla^2 \theta + \frac{e}{cm} (\nabla \ln \sqrt{\rho}) \cdot A + \frac{e}{2cm}(\nabla \cdot A).
\]
Finally, we find the equation for the wavefunction $\psi$ defined in equation (47) becomes
\[
i\hbar \partial_t \psi = \left[ \frac{1}{2m} \left( -i\nabla - \frac{e}{c} A \right)^2 + V + e\phi \right] \psi,
\]
where again $v = \hbar/(2m)$. This is exactly the Schrödinger equation of a charged particle, with the minimal coupling of the external gauge field.

Note that, in the derivation of the Schrödinger equation, we have assumed that the vector field $f_m$ is irrotational. On the other hand, in the derivation here, as is seen from the definition of the phase (58), $f_m$ can be rotational but $f_m + eA/(cm)$ should be irrotational. This fact shows that the well-known Aharonov–Bohm effect is closely related to the rotational nature of the flow of probability density.

5. Continuum medium

So far, we have discussed the application of the SVM to a particle Lagrangian. In this section, we will apply it to a system composed of a continuum medium [17].

Let us consider a continuum medium described by the mass density $\rho_m(r, t)$ and the collective flow velocity field $v(r, t)$. Then, the classical action of this system (in the Euler coordinate) is given by a functional of the two fields, $\rho_m$ and $v$ as
\[
I[\rho_m, v] = \int_{t_0}^{t_a} dt \left[ \frac{\rho_m(r, t) v^2(r, t)}{2} - \varepsilon(\rho_m(r, t)) \right],
\]
where $\varepsilon$ is the potential energy (internal energy) density of the medium. For the sake of simplicity, we assume that the potential energy is a function only of $\rho_m$. To perform the variational procedure directly from equation (62), we have to add the constraints for $\rho_m$ and $v$ as is done in [1, 3]. Due to this, in order to introduce the SVM, it is more convenient to use the Lagrangian coordinate system $\{R\}$, rather than the Euler coordinate system $\{r\}$ to specify the elements of the continuum medium. As is well known, the Lagrange coordinate system is attached to the medium in such a way that the coordinate $R$ represents the initial position of an element. In terms of this Lagrangian coordinate system, the above action is expressed as a functional of the trajectory of every element:
\[
I[x(R, t)] = \int_{b}^{a} dt \int d^3r \rho_0(R) \left[ \frac{1}{2} \left( \frac{dx(R, t)}{dt} \right)^2 - J\varepsilon(\rho_m(x(R, t))) \right],
\]
where $x(R, t)$ represents the trajectory of the element specified by $R$, $\rho_0(R)$ is the initial mass density and $J$ is the Jacobian of the transformation:
\[
R \rightarrow r = x(R, t),
\]
that is, $J = J(R, t) = \det|\partial r/\partial R|$. Assuming that there is no chaotic flows, the conservation of the mass of the system is expressed as
\[
d^3r \rho_m(r, t) = d^3R \rho_0(R),
\]
so that we have the following relation:
\[ \rho_m(r, t) = \frac{1}{J} \rho_0(R). \]  
(66)

It is easy to see that the classical variational principle for action (63) with respect to the trajectories \( x(R, t) \) leads to the Euler equation [17].

In terms of the Lagrangian coordinate, the implementation of SVM is straightforward. First, the trajectory of the Lagrangian coordinate is determined by the following forward and backward SDEs,
\[ \text{dx} = f(x, t) \, dt + \sqrt{2\tau} \, dW(t), \quad (dt > 0), \]  
(67a)
\[ \text{dx} = \tilde{f}(x, t) \, dt + \sqrt{2\tau} \, d\tilde{W}(t), \quad (dt < 0), \]  
(67b)
where \( dW(t) \) and \( d\tilde{W}(t) \) are noise terms whose correlation properties are same as equations (10) and (15). As before, we require that unknown fields \( f \) and \( \tilde{f} \) satisfy the consistency condition (18). Then, the continuity equation obtained from these Fokker–Planck equations is expressed with the mass density as
\[ \frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{f}_m) = 0. \]  
(68)

From now on, to simplify the notation, we will omit showing explicitly the \( (R, t) \) dependence of the trajectories, \( x = x(R, t) \), and the density, \( \rho_m = \rho_m(R, t) \).

Similar to the previous section, we will replace the kinetic term with the forward and/or backward mean derivatives. For example, we choose
\[ I = \int_a^b dt \int d^3R \rho_0(R) E \left[ \frac{1}{2} D\mathbf{x} \cdot D\mathbf{x} - J(\rho_m) \right]. \]  
(69)

As the variation, we consider
\[ x(R, t) \rightarrow x_i(R, t) = x(R, t) + \lambda g(x(R, t), t), \]  
(70)
where \( \lambda \) is an expansion parameter and \( g(r, t) \) is an arbitrary function. Under this variation, the mass density \( \rho_m \) is changed as
\[ \delta_i \rho_m = \rho_m(r, t) - \rho_m(R, t) = -\lambda \frac{\partial \rho_m}{\partial J_i} \sum_{i,j} A_{ij} \frac{\partial g}{\partial r_j} + O(\lambda^2). \]  
(71)

Here, \( A_{ij} \) is the cofactor \((ij)\) of the Jacobian \( J \), which satisfies the following properties:
\[ A_{im} = \left. \frac{\partial J}{\partial (r_j/R_m)} \right| = (\tilde{\nabla}r_j \times \tilde{\nabla}r_k)_m, \]  
(72a)
\[ \sum_j \frac{\partial}{\partial R_j} A_{ij} = \tilde{\nabla} \cdot (\tilde{\nabla}r_j \times \tilde{\nabla}r_k) = 0, \]  
(72b)
\[ \sum_i \frac{\partial r_i}{\partial R_i} A_{ij} = \sum_i \frac{\partial r_i}{\partial R_i} A_{ji} = J \delta_{ij}, \]  
(72c)
\[ \sum_j A_{ij} \frac{\partial}{\partial R_j} = \sum_j A_{ij} \frac{\partial r_l}{\partial R_j} \frac{\partial}{\partial r_l} = \sum_j J \delta_{ij} \partial_{r_l} = J \partial_{r_l}, \]  
(72d)
where \( \tilde{\nabla} \) denotes the gradient with respect to \( R \) which should be distinguished from \( \nabla \). The indices \((i, j, k)\) in equations (72a) and (72b) should be taken as cyclic. By using these properties, the action under the variation is expressed as
\[ I_{\lambda} = I - \lambda \int_a^b dt \int d^3R \rho_0 E \left[ \lambda g \cdot \left( \tilde{D} \mathbf{x}(R, t) + \frac{1}{\rho_m} \nabla \left( \frac{d}{d(1/\rho_m)} \frac{\rho_m}{\rho_m} \right) \right) \right] + O(\lambda^2). \]  
(73)
Then, the stationary condition for the variations
\[ \tilde{D}D x(R, t) + \frac{1}{\rho_m} \nabla \left( -\frac{d}{d(1/\rho_m)} \frac{\varepsilon}{\rho_m} \right) = 0, \] (74)
leads to (conf. equation (28))
\[ \rho_m [\dot{\rho} + f(R, t) \cdot \nabla \varepsilon - 2 \nu \nabla \ln \rho_m(R, t)] \cdot \nabla - \nu \nabla^2 f(R, t) + \nabla \left( -\frac{d}{d(1/\rho_m)} \frac{\varepsilon}{\rho_m} \right) = 0. \] (75)
By using equation (30), this can be re-expressed as
\[ \rho_m (\dot{\rho} + f_m \cdot \nabla) f_{m i} - \sum_j \partial_j \left( \nu \rho_m \partial_i f_{m j} + \nu \rho_m \partial_j f_{m i} \right) - \sum_j \partial_j \left( \nu \rho_m \partial_j \ln \rho_m \right) = -\nabla^i \left( -\frac{d}{d(1/\rho_m)} \frac{\varepsilon}{\rho_m} \right), \] (76)
where \( \rho_m \) is the solution of the continuity equation (68), as before. Because of the asymmetric entry of \( D \) and \( \tilde{D} \) in the action, the derived equation violates the time-reversal symmetry.

5.1. Navier–Stokes equation
The above scheme is applicable to derive the NS. A derivation of the NS for incompressible fluids in the SVM has already been worked out in [20]. Recently, in [17], the present authors have derived the complete (compressible) NS starting from an action different from the previous works. Then, it is found that the difference between \( f \) and \( \tilde{f} \) plays a crucial role and the velocity field in NS corresponds to \( f_m \). Below, we reproduce this derivation within the framework developed here.

As is discussed in [17], we consider that the potential term \( \varepsilon \) (internal energy density) depends not only on \( \rho_m \) but also on the other parameter \( \hat{s} \) representing the internal thermodynamical state of the fluid element. In thermal equilibrium, \( \hat{s} \) is identified as the specific entropy. Now, we write the variation as
\[ I_\lambda = I - \lambda \int_a^b dt \int d^3 R \sum_i \rho_{m 0} E \left[ \lambda g \cdot \left( \tilde{D}D x(R, t) + \frac{1}{\rho_m} \nabla P \right) + \frac{T}{m} \delta \hat{s} \right] + O(\lambda^2), \] (77)
where \( T \) and \( P \) are defined in analogy with the usual thermodynamic relation as
\[ T = \left. \frac{\partial (m\varepsilon/\rho_m)}{\partial \hat{s}} \right|_{\rho_m}, \] (78)
\[ P = -\left. \frac{\partial}{\partial(1/\rho_m)} \left( \frac{\varepsilon}{\rho_m} \right) \right|_{\hat{s}}. \] (79)
In the thermodynamic equilibrium, these quantities coincide with the usual temperature and pressure.

Because we are considering dissipative processes, the entropy of a fluid element is not a conserved quantity. Then, to complete the variations, we need information about \( \delta \hat{s}, \delta \hat{s} \) for variation (70). In general, \( \delta \hat{s} \) is a scalar function of local quantities (does not depend explicitly on \( R \)) determined only by the properties of the fluid element. Therefore, the most general form of \( \delta \hat{s} \) depends only on the density \( \rho_m \) and its derivatives, \( \dot{\rho}_m \) and \( \nabla \rho_m \). Due to the vector nature of \( \nabla \rho_m \), this does contribute only to the higher order corrections of the NS. Thus, the most general form of \( \delta \hat{s} \) for our purpose is expressed as \( \delta \hat{s} = \delta (\rho_m, \dot{\rho}_m) \).

Suppose that the hydrodynamic timescale \( \tau_{hyd} \equiv \rho_m/\dot{\rho}_m \) is much larger than that of the microscopic degrees of freedom inside a fluid element, \( \tau_{mic} \). Note that the local thermal

\[ \text{See [21] for a similar but different approach.} \]
equilibrium should be perfectly achieved for the ideal fluid where \( \dot{s} \) is constant (\( \delta s \dot{s} = 0 \)). Therefore, we expect that the specific entropy is expressed in powers of \( \tau_{\text{mic}}/\tau_{\text{hyd}} \) in such a way that the ideal fluid case is recovered in the vanishing limit of \( \tau_{\text{mic}} \). We then write
\[
\dot{s} = \dot{s}_\text{eq} + a_1 \tau_{\text{mic}} \rho_m / \rho_m + a_2 (\tau_{\text{mic}} \rho_m / \rho_m)^2 + \cdots,
\]
where \( \dot{s}_\text{eq} \) is a constant for the variation and \( a_i \)s are expansion coefficients. Thus, the lowest order truncation gives
\[
\delta s \dot{s} = \delta s_\text{eq} (\rho_m / \rho_m),
\]
where \( g_{\text{mic}} (\rho_m) \) is a some function of \( \rho_m \) which characterizes the microscopic response to the variation of the entropy. For the stochastic variation, \( \rho_m \) is interpreted as \( (D + \tilde{D}) \rho_m / 2 \). Note that \( \delta s \dot{s} \) is the virtual change of \( s \) associated with the variations and does not necessarily satisfy the thermodynamic principles such as \( \delta s \dot{s} \geq 0 \).

Then, the condition of \( \delta s I = 0 \) leads to
\[
\rho_m \left[ \frac{\partial}{\partial t} + \mathbf{f}_m \cdot \nabla \right] \mathbf{f} + \nabla (P - \mu \nabla \cdot \mathbf{f}_m) - (\nabla \cdot \eta \nabla) \mathbf{f} = 0,
\]
where \( \eta \) is the shear viscosity. Here, we used that \( \nu = \eta / \rho_m \) and \( \mu = -\rho_m g(\rho_m)/m(\partial T / \partial \rho_m) \).

Equation (81) contains the two different velocity fields, \( \mathbf{f}_m \) and \( \mathbf{f} \), but the latter can be eliminated by using equation (30). As a result, we obtain
\[
\rho \left[ \frac{\partial}{\partial t} + \mathbf{f}_m \cdot \nabla \right] \mathbf{f}_m + \nabla (P - \zeta \nabla \cdot \mathbf{f}_m) - \nabla \cdot (\eta \tilde{\nabla}^m) - (\nabla \cdot \eta \nabla) \left( \frac{\eta}{\rho_m} \nabla \ln \rho_m \right) = 0,
\]
where \( \tilde{\nabla}^m = \{e_{ij}^m\} \) is a (3 x 3) irreducible symmetric tensor, defined by
\[
e_{ij}^m = \partial_j f_i^m + \partial_i f_j^m - \frac{2}{3} (\nabla \cdot \mathbf{f}_m) \delta_{ij},
\]
and \( \zeta = \mu + 2\eta / 3 \). Identifying \( \zeta \) as the bulk viscosity, equation (82) is nothing but the NS, except for the last term on the left-hand side.

The last term is not only of the second order for the magnitude of fluctuations (noise) \( \nu = \eta / \rho_m \), but also of the third order for the spatial derivative \( \nabla \). The NS does not contain such a term since, by construction, only the first- and second-order spatial derivative terms are maintained. In the case of rarefied gases, this corresponds to the first-order truncation in the derivative (Chapman–Enskog) expansion of one-particle distribution functions.

On the other hand, in our case, the viscosity coefficients are directly related to the size of noise. When the effect of fluctuation is large, we cannot neglect the last term of equation (82). This term comes from the difference between \( \mathbf{f} \) and \( \mathbf{f}_m \). Therefore, in such a case, hydrodynamics is described by using the two velocities as equation (81). This feature resembles that of the generalized hydrodynamics proposed by Brenner [22, 23].

5.2. Gross–Pitaevskii equation

The Gross–Pitaevskii equation (GP) is a nonlinear Schrödinger equation and has been used to study some aspects of the Bose–Einstein condensates [24]. The derivation of the GP also has been studied from various points of view [25]. In particular, it is shown that GP can be derived in terms of the SVM, extending the method of section 3.4 to a system of a many-body Lagrangian [12]. In this subsection, we show that the formalism of the SVM for a continuum medium can readily be used to derive GP in a direct way.

We consider a system composed of identical bosons with mass \( m \), under the external trapping potential \( V(\mathbf{r}) \). Supposing the interaction among bosons is local, we write the potential energy density of the system as
\[
\varepsilon = V(\mathbf{r}) \rho_m / m + \frac{1}{2} U_0 \frac{\rho_m^2}{m^2},
\]
(84)
where \( \rho_m = \rho_m(\mathbf{r}, t) \) is the mass density of the boson and \( U_0 \) represents the strength of interaction among them. To obtain the equation for the wavefunction (order parameter), the kinetic term of the action should be replaced by the average of \( D\mathbf{x} \) and \( \bar{D}\mathbf{x} \) as was done in section 3.4. Then, in the Lagrangian coordinate system, we can define the action of the SVM as

\[
I = \int_b^0 dt \int d^3\mathbf{R} \rho_{m0}(\mathbf{R}) \times E \left[ \frac{1}{2} \frac{D\mathbf{x} \cdot D\mathbf{x} + \bar{D}\mathbf{x} \cdot \bar{D}\mathbf{x}}{m} - \frac{1}{m} V(\mathbf{x}) - \frac{1}{2m^2} U_0 \rho_m(\mathbf{x}, t) \right].
\]  

(85)

Following the similar procedure in the previous subsection, the SVM leads to

\[
(\partial_t + \mathbf{f}_m \cdot \nabla) f_m^0 = -\frac{1}{m} \nabla \cdot \mathbf{f}_m - \frac{U_0}{m^2} \nabla \rho_m + \frac{1}{m} \nabla \left[ V + U_0 |\psi|^2 \right],
\]

(86)

which should be solved together with the Fokker–Planck equation for \( \rho_m \), i.e. equation (68). This equation is the same as that discussed in [26] and known as the quantum Bernoulli equation.

To convert equation (86) into the corresponding Schrödinger-type equation, we follow the same procedure discussed in section 3.4, introducing the wavefunction (see equations (43) and (47)) with a modification of equation (47) by

\[
\psi = \sqrt{\frac{\rho_m}{m}} e^{i\theta}.
\]

(87)

because \( \rho_m \) is the mass density. We again assume that the flow is irrotational. The equation corresponding to equation (48) becomes

\[
i\hbar \partial_t \psi = \left[ -\nabla^2 + \frac{1}{2m^2} \left( V + U_0 |\psi|^2 \right) \right] \psi,
\]

(88)

and setting \( \nu = \hbar/(2m) \), we arrive at GP,

\[
i\hbar \partial_t \psi = \left[ -\frac{\hbar^2}{2m^2} \nabla^2 + V + U_0 |\psi|^2 \right] \psi.
\]

(89)

As before, from the definition of \( \psi \), \( |\psi|^2 = \rho_m/m \) gives the particle distribution function.

Note that the classical limit of GP is easily obtained by taking the vanishing limit of \( \hbar \) (or \( \nu \)) in equation (86):

\[
(\partial_t + \mathbf{f}_m \cdot \nabla) f_m^0 = -\frac{1}{m} \nabla \cdot \mathbf{f}_m - \frac{U_0}{m^2} \nabla \rho_m,
\]

(90)

together with equation (68). Of course, this equation is the same as those obtained from action (85) applying the classical variational method.

The above argument is easily extended to include the external gauge field via the minimum coupling as described in section 4. The required action in the Lagrangian coordinate system is

\[
I = \int_b^0 dt \int d^3\mathbf{R} \rho_0 \left[ \frac{1}{2} \mathbf{v}^2 + \frac{e}{cm} \mathbf{v} \cdot \mathbf{A} - \frac{1}{m} V(\mathbf{x}) - \frac{1}{2m^2} U_0 \rho_m(\mathbf{x}, t) - \frac{e}{m} \phi \right].
\]

(91)

and the equation corresponding to equation (57) becomes

\[
(\partial_t + \mathbf{f}_m \cdot \nabla) \mathbf{f}_m = -\frac{e}{cm} \partial_t \mathbf{A} + \frac{e}{cm} \mathbf{f}_m \times (\nabla \times \mathbf{A}).
\]

(92)

The corresponding equation for the wavefunction is given by

\[
i\hbar \partial_t \psi = \left[ \frac{1}{2m} \left( -i\hbar \nabla - \frac{e}{c} \mathbf{A} \right)^2 + (V + e\phi + U_0 |\psi|^2) \right] \psi.
\]

(93)

where \( \psi \) is given by equation (87) with the phase function given by equation (58) and the noise is set as \( \nu = \hbar/(2m) \).
5.3. Generalized diffusion equation and physical meaning of correction to NS

In section 5.1, we found that the NS derived using the SVM involves the higher order correction term, which was neglected for deriving the NS. In this section, we discuss the physical meaning of this correction term.

For the sake of simplicity, we consider the case where there is no potential energy density in action (63). Physically, this situation will correspond to the free diffusion process. Then, from equation (76), together with the continuity equation, we have

\[ \partial_t \rho_m + \nabla \cdot (\rho_m f_m) = 0, \]  
\[ \rho_m \left( \partial_t f_m + \nabla f_m \cdot \nabla \right) f_m - \sum_j \partial_j \left\{ \nu \rho_m \partial_i f_m f_m + \nu \rho_m \partial_j \partial_i f_m \right\} - \partial_j \left( \nu \rho_m \partial_j (\nu \partial_i \ln \rho_m) \right) = 0. \]

The first equation is the continuity equation as before, and the second one is obtained by the SVM, that is, equation (76). In the discussion of section 5.1, the third term on the left-hand side of equation (94b) was neglected as the higher order term of the NS, but now we will keep it in this subsection.

This equation describes the generalized diffusion process. Suppose that there is a clear separation of timescales between \( \rho_m \) and \( f_m \) in the Lagrangian coordinate, and we observe the macroscopic dynamics of \( \rho_m \) where the material derivative of \( f_m \) is negligibly small. Then, from the second equation, we obtain

\[ f_m = -\frac{\nu}{2} \nabla \ln \rho_m. \]  

This corresponds to Fick’s law of the diffusion equation. In fact, by substituting it into the first equation, we obtain the usual diffusion equation (see also the appendix),

\[ \partial_t \rho_m = \frac{\nu}{2} \nabla^2 \rho_m. \]

Different from the diffusion equation, equation (94b) involves a memory effect through the time derivative of \( f_m \). In this form, we see that equation (94b) is interpreted as the generalized diffusion equation, and it reproduces the usual diffusion equation in the Markov limit where the material derivative of \( f_m \) is neglected.

In other words, to reproduce the diffusion equation, we cannot neglect the third term on the left-hand side of the second equation, which corresponds to the higher order correction term for the NS. In this sense, this higher order term may have a special meaning and be an important correction even in the NS.

Additionally, the last term of equation (82) is re-expressed as

\[ -\nu^2 \sum_j \partial_j \left( \rho_m \partial_j \partial_i \ln \rho_m \right) = -2\nu^2 \rho_m \partial_i \left( \rho_m^{-1/2} \nabla^2 \sqrt{\rho_m} \right) \rho_m \partial_i \frac{\delta F(\rho_m)}{\delta \rho_m}, \]  

where we introduced a kind of a free energy,

\[ F(\rho_m) = 2\nu^2 \int d^3x (\nabla \sqrt{\rho_m})^2. \]

Suppose that the potential term is only a function of \( \rho_m \). From equation (76), one can see that any contribution from the potential term is expressed by the form of \( \partial_j h(\rho_m) \) where \( h(\rho_m) \) is an arbitrary function. However, the higher order term discussed here cannot be cast into this from. Thus, this term is an inherent effect in the SVM.
6. Concluding remarks

In this paper, we discussed the application of the stochastic variational method (SVM) to particle systems and continuum media. The application to the particle systems has been studied from various points of view. First, we briefly summarized these results showing the derivation of the diffusion-type equation and the Schrödinger equation with and without the minimal gauge coupling.

We further generalized the idea of the SVM to the calculations of continuum media, and showed that the Navier–Stokes, Gross–Pitaevskii and generalized diffusion equations are derived.

As is known, the usual diffusion equation is inconsistent with underlying microscopic dynamics and should be modified [28, 29]. In fact, it is shown that the coarse-grained equation obtained from a microscopic Hamiltonian using the projection operator method is not the usual diffusion equation but the Maxwell–Cattaneo–Vernotte-type generalized diffusion equation. This equation has several advantages compared to the usual diffusion equation. For example, this generalized equation is consistent with an exact relation obtained from microscopic dynamics, while the diffusion equation is not [29]. Here, we proposed a new generalized diffusion equation in the framework of the SVM. It is interesting to ask the relation between this generalized equation from the SVM and the Maxwell–Cattaneo–Vernotte-type equation.

We found that the SVM predicts the higher order correction term to the NS. This term becomes important when there is strong inhomogeneity of the mass density. In other words, we cannot recognize this effect from the study of incompressible fluids. The importance of the correction term becomes clearer in the discussion of the generalized diffusion equation. In order to reproduce the usual diffusion equation in the Markov limit, the same higher order term in the generalized diffusion equation plays an important role. In this sense, this term may be an important correction even to NS.

There are several attempts to derive macroscopic equations for high-density matters by assuming a Langevin dynamics for underlying microscopic degrees of freedom [31]. There, the interaction of the Langevin dynamics is assumed in advance, while it is determined by applying the variational principle in our approach. It is thus interesting to discuss the meaning of the correction term which we found, comparing it with the behavior of such a dynamical density functional theory. More detailed study of this correction term is a future task.

The advantage of the SVM is that dissipative equations can be derived systematically once the action of the corresponding reversible dynamics is given. For example, in the physics of soft matter, dissipative equations are usually derived by applying Onsager’s variational principle [32, 33]. The SVM may serve another convenient approach to discuss the dynamics of such a complex fluid.

So far, we have assumed that the noise of the SVM is a Gaussian white noise. It might be possible to generalize this discussion to the case of multiplicative noises. For this work, we assume that the property of the noise is unchanged both for the forward and backward stochastic differential equations (SDEs). This seems to be a reasonable assumption because the properties of the noise are completely independent of the dynamics of macroscopic variables (Brownian particles) in this case of the additive noise. This is, however, not trivial for the multiplicative noise. If this assumption is still applicable, the extension of the present argument to the multiplicative noise will be straightforward.

As was mentioned in the introduction, the essence of the classical variational principle becomes more fundamental in the application of the approach to field theories and relativistic

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7 See also the short review paper [30].
systems. As far as we know, the SVM has not yet been applied to such cases, however see [34]. Studies of relativistic heavy-ion collisions revealed that it will be of the fundamental importance to discuss dissipative mechanisms in relativistic systems. Thus, an extension of the present approach to relativistic field theoretical systems is expected to furnish an important clue to formulate such processes from the fundamental concept of the coarse-graining procedure to arrive at the relativistic SDE. However, the Lorentz covariant SDE still remains an open question8.

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Appendix. Another limit to diffusion equation

In section 5.3, the generalized diffusion equation is derived and the equation is reduced to the usual diffusion equation in the Markov limit. This usual diffusion equation is still reproduced when a special initial condition is chosen.

Suppose that, as the initial condition of $f_m$, we have

$$v_m = -v \nabla \ln \rho.$$  \hspace{1cm} (A.1)

Then, we can easily check that both the first and second parts of equation (94) are reduced to the same equation:

$$\partial_t \rho_m = v \nabla^2 \rho_m.$$  \hspace{1cm} (A.2)

That is, the generalized diffusion equation is reduced to the usual diffusion equation even when we use the special initial condition. The difference from equation (96) is only the magnitude of the diffusion coefficient.

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