Reduced Vectorial Ribaucour Transformation for the Darboux-Egoroff Equations

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\textbf{Abstract}

The vectorial fundamental transformation for the Darboux equations is reduced to the symmetric case. This is combined with the orthogonal reduction of Lamé type to obtain those vectorial Ribaucour transformations which preserve the Egoroff reduction. We also show that a permutability property holds for all these transformations. Finally, as an example, we apply these transformations to the Cartesian background.

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1 Introduction

At the turn of this century a number of results on differential geometry were already well established [3, 4, 13, 1]. We are thinking of conjugate nets described by the Darboux equations [3, 13] and related transformations [15] of Laplace, Lévy [21] and fundamental [18, 14] type; and the orthogonal nets, described by the Lamé equations [20, 4], and their Ribaucour transformations [27, 4]. The Lamé equations describe flat diagonal metrics, among which we find a distinguished class: those of Egoroff type. These particular classes of flat diagonal metrics are described by the Darboux-Egoroff equations [4], that were first proposed by Darboux [5] and studied further in [28, 26, 16] and finally, as it was recognized by Darboux [4], Egoroff gave an almost definitive treatment in [12].

The mentioned results are deeply connected with the modern theory of integrable systems. It is well known that integrable equations like Liouville or sine-Gordon were first considered in the context of differential geometry: minimal and pseudo-spherical surfaces. It has been recently discovered that the \( N \)-component Kadomtsev-Petviashvili (KP) hierarchy [6] describes the iso-conjugate transformations of the Darboux equations, and its vertex operators correspond precisely to Laplace, Lévy and fundamental transformations [10]. Moreover, the \( N \)-component BKP hierarchy [7] models the iso-orthogonal deformations of the Lamé equations, being its vertex operator the Ribaucour transformation [25]. Hence, one easily concludes the strong relation between Soliton Theory and Classical Differential Geometry. Let us mention that discrete integrable systems have the same connection with modern discrete geometry [9].

Recently in [11] it was shown that the Egoroff metrics play a fundamental role in the classification of two dimensional topological conformal field theories. The partition function of the deformed semisimple theories satisfies certain associativity or Witten-Dijkgraaf-Verlinde-Verlinde equations, which are connected with an Egoroff metric.

The integration of the Darboux equations was performed in [31] through the \( \hat{\partial} \)-method, while very recently the Lamé equations have been integrated within the inverse scattering technique in [30] and by the algebro-geometrical approach in [19].

In previous papers we have considered the iteration of the mentioned transformations within the modern soliton theory: in [23] we studied the Lévy transformation and in [22] the Ribaucour transformation, using in this
latter a vectorial approach. We want to reduce the vectorial fundamental transformation \([24, 9]\) to the symmetric case, and further the vectorial Ribaucour transformation to the Darboux-Egoroff equations.

The layout out of this paper is as follows: in §2 we consider the symmetric reduction of the Darboux equations obtaining the vectorial symmetric fundamental transformations for all the geometrical data; next, in §3 we combine these results with the ones of \([22]\) to get the reduction of the vectorial Ribaucour to the Darboux-Egoroff equations, giving the expressions of all the transformed relevant geometrical data. In both sections we consider the dressing of the Cartesian background and prove that the permutability property is preserved under all the reductions considered. We must recall the reader that the permutability property of these transformations is an important issue that for the fundamental transformation was considered in \([8, 9]\) and for the Ribaucour transformation in \([2, 1, 22]\).

\section{The vectorial fundamental transformation for the symmetric Darboux equations}

The Darboux equations

\[
\frac{\partial \beta_{ij}}{\partial u_k} - \beta_{ik} \beta_{kj} = 0, \quad i, j, k = 1, \ldots, N, \text{ with } i, j, k \text{ different},
\]  

for the \(N(N - 1)\) functions \(\{\beta_{ij}\}_{i,j=1,\ldots,N}\) of \(u := (u_1, \ldots, u_N)\), characterize \(N\)-dimensional submanifolds of \(\mathbb{R}^D, N \leq D\), parametrized by conjugate coordinate systems \([3, 13]\), and are the compatibility conditions of the following linear system

\[
\frac{\partial X_j}{\partial u_i} = \beta_{ji} X_i, \quad i, j = 1, \ldots, N, \quad i \neq j,
\]

involving suitable \(D\)-dimensional vectors \(X_i\), tangent to the coordinate lines.

The so called Lamé coefficients satisfy

\[
\frac{\partial H_j}{\partial u_i} = \beta_{ij} H_i, \quad i, j = 1, \ldots, N, \quad i \neq j,
\]

and the points of the surface \(x = (x_1, \ldots, x_N)\) can be found by means of

\[
\frac{\partial x}{\partial u_i} = X_i H_i, \quad i = 1, \ldots, N,
\]
which is equivalent to the Laplace equation
\[
\frac{\partial^2 x}{\partial u_i \partial u_j} = \frac{\partial \ln H_i}{\partial u_j} \frac{\partial x}{\partial u_i} + \frac{\partial \ln H_j}{\partial u_i} \frac{\partial x}{\partial u_j}, \quad i, j = 1, \ldots, N, \quad i \neq j.
\]

The fundamental transformation for the Darboux system was introduced in [18, 14], and its vectorial extension was given in [24, 9]. It requires the introduction of a potential in the following manner: given vector solutions \( \xi_i \in V \) and \( \zeta^*_i \in W^* \) of (2) and (3), respectively, where \( V, W \) are linear spaces and \( W^* \) is the dual space of \( W \), one can define a potential matrix \( \Omega(\xi, \zeta^*) : W \rightarrow V \) through the equations
\[
\frac{\partial \Omega(\xi, \zeta^*)}{\partial u_i} = \xi_i \otimes \zeta^*_i.
\]

**Vectorial Fundamental Transformation.** Given solutions \( \xi_i \in V \) and \( \xi^*_i \in V^* \) of (2) and (3), \( i = 1, \ldots, N \), respectively, new rotation coefficients \( \hat{\beta}_{ij} \), tangent vectors \( \hat{X}_i \), Lamé coefficients \( \hat{H}_i \) and points of the surface \( \hat{x} \) are given by
\[
\begin{align*}
\hat{\beta}_{ij} &= \beta_{ij} - \langle \xi^*_j, \Omega(\xi, \xi^*)^{-1} \xi_i \rangle, \\
\hat{X}_i &= X_i - \Omega(X, \xi^*) \Omega(\xi, \xi^*)^{-1} \xi_i, \\
\hat{H}_i &= H_i - \xi^*_i \Omega(\xi, \xi^*)^{-1} \Omega(\xi, H), \\
\hat{x} &= x - \Omega(X, \xi^*) \Omega(\xi, \xi^*)^{-1} \Omega(\xi, H).
\end{align*}
\]

Here we are assuming that \( \Omega(\xi, \xi^*) \) is invertible. We shall refer to this transformation as vectorial fundamental transformation with transformation data \( (V, \xi_i, \xi^*_i) \).

The symmetric reduction requests the rotation coefficients to be symmetric; i. e.,
\[
\begin{align*}
\frac{\partial \beta_{ij}}{\partial u_k} - \beta_{ik} \beta_{kj} &= 0, \quad i, j, k = 1, \ldots, N, \text{ with } i, j, k \text{ different}, \\
\beta_{ij} - \beta_{ji} &= 0, \quad i, j = 1, \ldots, N, \quad i \neq j.
\end{align*}
\]

This implies local existence of potentials \( V \) and \( \Phi \) such that \( |X_i|^2 = \frac{\partial V}{\partial u_i} \) and
\[
H_i^2 = \frac{\partial \Phi}{\partial u_i}, \quad i = 1, \ldots, N.
\]
We now consider which transformation data \((V, \xi_i, \xi_i^*)\) gives a vectorial fundamental transformation that preserves the symmetric Darboux equations \((7)\).

We have the following observations:

**Lemma 1.**

1. Given a solution \(\xi_i \in V\) of \((2)\) then
   \[
   \xi_i^* := \xi_i^t L
   \]
   where \(^t\) means transpose and \(L \in L(V)\) is a linear operator on \(V\), is a \(V^*\)-valued solution of \((3)\) if and only if \((7)\) holds. We shall say that \(L\) is the associated linear operator.

2. Given symmetric \(\beta\)'s, \(\xi_i \in V\) and \(\zeta_i \in W\) solutions of \((2)\) and \(\xi_i^*\) and \(\zeta_i^*\) as prescribed in \((9)\), \(i = 1, \ldots, N\), with associated linear operators \(L\) and \(M\), respectively; then:
   \[
   \frac{\partial}{\partial u_i}(L^t \Omega(\xi, \zeta^*) - \Omega(\zeta^*, \xi^*)^t M) = 0, \quad i = 1, \ldots, N.
   \]

3. Suppose given a solution \(\beta_{ij}\) of the symmetric Darboux equations \((3)\), \(\xi_i \in V\) and \(\zeta_i \in W\) solving \((2)\) and \(\xi_i^*\) and \(\zeta_i^*\) as prescribed in \((9)\), with associated linear operators \(L\) and \(M\), respectively. Then, if
   \[
   L^t \Omega(\xi, \xi^*) - \Omega(\xi^*, \xi^*)^t M = 0, \quad L^t \Omega(\zeta, \zeta^*) - \Omega(\zeta^*, \zeta^*)^t L = 0,
   \]
   the vectorial fundamental transformation \((8)\):
   \[
   \begin{align*}
   \hat{\beta}_{ij} &= \beta_{ij} - \langle \xi_i^*, \Omega(\xi, \xi^*)^{-1} \xi_i \rangle, \\
   \hat{\zeta}_i &= \zeta_i - \Omega(\zeta, \xi^*) \Omega(\xi, \xi^*)^{-1} \xi_i, \\
   \hat{\zeta}_i^* &= \zeta_i^* - \xi_i^* \Omega(\xi, \xi^*)^{-1} \Omega(\xi, \zeta^*)
   \end{align*}
   \]
   is such that
   \[
   \hat{\zeta}_i^* := \hat{\zeta}_i^t M.
   \]

**Proof.** Point 1 is trivial to check; for 2, using \((3)\) and the definition \((9)\) we have
   \[
   \frac{\partial}{\partial u_i}(L^t \Omega(\xi, \zeta^*) - \Omega(\zeta, \xi^*)^t M) = L^t \xi_i \otimes \xi_i^t M - L^t \xi_i \otimes \zeta_i^t M = 0.
   \]
For 3, using (5), (6) and (9) we find that
\[
\hat{\zeta}_i^* = \zeta_i^t M - \xi_i^t \Omega(\xi, \xi^*)^{-1} \Omega(\xi, \zeta^*),
\]
and the constraints (10) gives
\[
\hat{\zeta}_i^* = \zeta_i^t M - \xi_i^t (\Omega(\xi, \xi^*)^t)^{-1} L^t \Omega(\xi, \zeta^*) = \left[ \zeta_i - \Omega(\xi, \zeta^*) \Omega(\xi, \zeta^*)^{-1} \xi_i \right]^t M,
\]
that recalling (3) implies the statement.

Therefore, as \(\Omega(\xi, \zeta^*)\) and \(\Omega(\xi, \xi^*)\) are defined by (5) up to additive constant matrices, we may take them such that
\[
L^t \Omega(\xi, \zeta^*) - \Omega(\zeta, \xi^*)^t M = 0.
\]

These lemmas imply:

**Vectorial Symmetric Fundamental Transformation.** The vectorial fundamental transformation (6) preserves the symmetric Darboux equations (7) whenever the transformation data \((V, \xi_i, \xi_i^*)\) satisfies
\[
\xi_i^* = \xi_i^t L,
L^t \Omega(\xi, \zeta^*) - \Omega(\zeta, \xi^*)^t L = 0.
\]

We say that \((V, \xi_i, L)\) is the transformation data for this particular vectorial fundamental transformation that we shall call vectorial symmetric fundamental transformation.

### 2.1 Permutability of vectorial symmetric fundamental transformations

The vectorial fundamental transformations permute among them (8):

**Permutability of Vectorial Fundamental Transformations.** The vectorial fundamental transformation with transformation data
\[
\left( V_1 \oplus V_2, \begin{pmatrix} \xi_i^{(1)} \\ \xi_i^{(2)} \end{pmatrix}, \begin{pmatrix} \xi_i^{* (1)} \\ \xi_i^{* (2)} \end{pmatrix} \right)
\]
coincides with the following composition of vectorial fundamental transformations:
1. **First transform with data**

\((V_2, \xi_{i,(2)}, \xi_{i,(2)})\),

and denote the transformation by ‘\(\prime\).

2. On the result of this transformation apply a second one with data

\((V_1, \xi'_{i,(1)}, \xi'_{i,(1)})\).

Therefore, the composition of two vectorial fundamental transformations yields a new vectorial fundamental transformation. When these two transformations are done in different order the resulting composed vectorial fundamental transformation is equivalent, through conjugation by a permutation matrix, to the first composed vectorial fundamental transformation, so that all the geometrical data are identical for both composed transformations; hence, the permutability character of these transformations. Moreover, it also follows that the vectorial fundamental transformation is just a superposition of a number of fundamental transformations.

One can easily conclude that this result can be extended to the vectorial symmetric fundamental transformation:

**Proposition.** The vectorial symmetric fundamental transformation with transformation data

\[
\left( V_1 \oplus V_2, \begin{pmatrix} \xi_{i,(1)} \\ \xi_{i,(2)} \end{pmatrix}, \begin{pmatrix} L_{(1)} & 0 \\ 0 & L_{(2)} \end{pmatrix} \right),
\]

coinsides with the following composition of vectorial symmetric fundamental transformations:

1. **First transform with data**

\((V_2, \xi_{i,(2)}, L_{(2)})\),

and denote the transformation by ‘\(\prime\).

2. On the result of this transformation apply a second one with data

\((V_1, \xi'_{i,(1)}, L_{(2)})\).
Proof. Because the transformation data follows the prescription of our Theorem they must satisfy

\[
(\xi_i^{\ast(s)})^t = L(s)\xi_i(s), \quad s = 1, 2 \\
L_i^t(s)\Omega(\xi_i(s), \xi_i^{\ast(s)}) - \Omega(\xi_i(s), \xi_i^{\ast(s)})^t L(s) = 0, \quad s = 1, 2, \\
L_i^t(1)\Omega(\xi_i(1), \xi_i^{\ast(2)}) - \Omega(\xi_i(2), \xi_i^{\ast(1)})^t L(2) = 0.
\]

The first vectorial fundamental transformation is a vectorial symmetric one with data \((V, \xi_i(2), L(2))\). Lemma 3 implies that the vectorial fundamental transformation of point 2 is also a vectorial symmetric fundamental transformation. \qed

Form these results we conclude that the composition of scalar symmetric fundamental transformations results in a vectorial symmetric fundamental transformation with associated matrix of diagonal type. In fact, when the associated matrix \(L\) is not diagonal the corresponding transformation can not be obtained by means of composition only, we need also a suitable coalescence of eigen-values.

2.2 Dressing the Cartesian background

The Cartesian net has \(X_i = e_i\), being \(\{e_i\}_{i=1,...,N}\) a linear independent set of vectors of \(\mathbb{R}^D\), \(H_i = 1\), the coordinates are \(x(u) = u\) and vanishing rotation coefficients \(\beta_{ij} = 0\). Hence,

\[
\xi_i = \xi_i(u_i) \in \mathbb{R}^M,
\]

and

\[
\Omega(\xi, \xi^\ast)(u) = \sum_{k=1,...,N} \Omega_k(u_i)
\]

with

\[
\Omega_i(u_i) = \int_{u_{i,0}}^{u_i} d u_i \xi_i \otimes \xi_i^t L + \Omega_{i,0}, \\
L^t \Omega_{i,0} - \Omega_{i,0}^t L = 0.
\]
Thus
\[
\Omega(X, \xi^*)(u) = A + \sum_{i=1, \ldots, N} e_i \otimes \int_{u_i,0}^{u_i} d u_i \xi_i^t(u_i)L,
\]
\[
\Omega(\xi, H)(u) = c + \sum_{i=1, \ldots, N} \int_{u_i,0}^{u_i} d u_i \xi_i(u_i),
\]
where \(A\) is a constant \(D \times M\) matrix and \(c \in \mathbb{R}^M\) is a constant vector, and the symmetric conjugate net is given by
\[
x(u) = u - \left[ A + \sum_{i=1, \ldots, N} e_i \otimes \int_{u_i,0}^{u_i} d u_i \xi_i^t(u_i) \right] \left[ \sum_{i=1, \ldots, N} \Omega_i(u_i) \right]^{-1} \times \left[ c + \sum_{i=1, \ldots, N} \int_{u_i,0}^{u_i} d u_i \xi_i(u_i) \right].
\]

3 The vectorial fundamental transformation for the Egoroff metrics

The Lamé equations describe \(N\)-dimensional conjugate orthogonal systems of coordinates [20, 4, 29]:
\[
\frac{\partial \beta_{ij}}{\partial u_k} - \beta_{ik} \beta_{kj} = 0, \quad i, j, k = 1, \ldots, N, \quad \text{with} \quad i, j, k \text{ different,} \quad (11)
\]
\[
\frac{\partial \beta_{ij}}{\partial u_i} + \sum_{k=1, \ldots, N}^{N} \beta_{ki} \beta_{kj} = 0, \quad i, j = 1, \ldots, N, \quad i \neq j. \quad (12)
\]

Now we have orthogonal tangent directions, \(X_i \cdot X_j = \delta_{ij}\). In fact, if \(N = D\), the Lamé coefficients \(H_i\) allows us to construct a flat diagonal metric:
\[
d s^2 = \sum_{i=1}^{N} H_i(u)^2 d u_i \otimes d u_i, \quad (13)
\]
for which \(x_1, \ldots, x_N\) are flat coordinates. The reduction of the vectorial fundamental transformation to the orthogonal case; i. e., the vectorial Ribaucour transformation, was studied by us in [22]. The main results of it are
Lemma 2.  
1. Given a solution $\xi_i \in V$ of (2) then
\[
\xi_i^* := \left( \frac{\partial \xi_i}{\partial u_i} + \sum_{k=1,...,N, \ k \neq i} \xi_k \beta_{ki} \right)^t,
\]
(14)
is a $V^*$-valued solution of (3) if and only if (12) holds.

2. Given $\beta$'s solving the Lamé equations (11) and (12), $\xi_i \in V$ and $\zeta_i \in W$ solutions of (2) and $\xi_i^*$ and $\zeta_i^*$ as prescribed in (14), $i = 1, ..., N$, then:
\[
\frac{\partial}{\partial u_i} \left( \Omega(\xi, \zeta^*) + \Omega(\zeta, \xi^*)^t - \sum_{k=1,...,N} \xi_k \otimes \zeta_k^t \right) = 0, \quad i = 1, ..., N,
\]

3. Suppose given a solution $\beta_{ij}$ of the Lamé equations (11) and (12), $\xi_i \in V$ and $\zeta_i \in W$ solving (2) and $\xi_i^*$ and $\zeta_i^*$ as prescribed in (14). Then, if
\[
\Omega(\xi, \zeta^*) + \Omega(\zeta, \xi^*)^t = \sum_{k=1,...,N} \xi_k \otimes \zeta_k^t,
\]
\[
\Omega(\xi, \xi^*) + \Omega(\xi, \xi^*)^t = \sum_{k=1,...,N} \xi_k \otimes \xi_k^t
\]
(15)
the vectorial fundamental transformation (6):
\[
\hat{\beta}_{ij} = \beta_{ij} - (\xi_j^*, \Omega(\xi, \xi^*)^{-1} \xi_i), \\
\hat{\zeta}_i = \zeta_i - \Omega(\zeta, \xi^*)\Omega(\xi, \xi^*)^{-1} \xi_i, \\
\hat{\zeta}_i^* = \zeta_i^* - \xi_i^*\Omega(\xi, \xi^*)^{-1} \Omega(\xi, \zeta^*),
\]
is such that
\[
\hat{\zeta}_i^* := \left( \frac{\partial \hat{\zeta}_i}{\partial u_i} + \sum_{k=1,...,N, \ k \neq i} \hat{\zeta}_k \hat{\beta}_{ki} \right)^t.
\]

4. Moreover, the inversion of (14) is $\xi_i = \Omega(X, \xi^*)^t X_i$, so that
\[
\Omega(X, \xi^*) = \sum_{k=1}^N X_i \otimes \xi_i^t.
\]
The symmetric reduction of the Darboux equations can be combined with the Lamé equations to obtain the so-called equivalent system of Darboux-Egoroff equations
\[
\frac{\partial \beta_{ij}}{\partial u_k} - \beta_{ik} \beta_{kj} = 0, \quad i, j, k = 1, \ldots, N, \quad \text{with } i, j, k \text{ different},
\]
\[
\beta_{ij} - \beta_{ji} = 0, \quad i, j = 1, \ldots, N, \quad i \neq j,
\]
\[
\sum_{k=1}^{N} \frac{\partial \beta_{ij}}{\partial u_k} = 0, \quad i, j = 1, \ldots, N, \quad i \neq j.
\]
This gives rise to the Egoroff metrics; i.e., flat diagonal metrics (13) such that (8) holds.

The reduction of the vectorial fundamental transformation to the Darboux-Egoroff case can be thought as a superposition of two reductions, namely the symmetric together with the orthogonal reduction. Thus, we must request to the transforming data and potential the constraints for both reductions. This implies that:

**Vectorial Symmetric Ribaucour Transformation.** The vectorial fundamental transformation (6) preserves the Darboux-Egoroff equations (16) whenever the transformation data \((V, \xi_i, \xi_i^*)\) satisfy
\[
\xi_i^* = \xi_i^t L = \left( \frac{\partial \xi_i}{\partial u_i} + \sum_{k=1, \ldots, N, k \neq i} \xi_k \beta_{ki} \right)^t, \\
L^t \Omega(\xi, \xi^*) - \Omega(\xi, \xi^*)^t L = 0, \\
\Omega(\xi, \xi^*) + \Omega(\xi, \xi^*)^t = \sum_{k=1, \ldots, N} \xi_k \otimes \xi_k^t,
\]
for some linear operator \(L \in L(V)\).

Observe that in the scalar case, \(M = 1\), the second equation above is trivial and the third one determines the potential completely. The associated transformation in this case can be found in [29].

**Permutability** The vectorial Ribaucour transformation was shown to have the permutability property in [22], moreover it was done as in our proof of the permutability for the symmetric case. This implies that the combination of both reductions should share the permutability character of the symmetric and orthogonal reduction.
Dressing the Cartesian background  Now we have:

$$\xi_i = \exp(L^i u_i) a_i, \quad a_i \in \mathbb{R}^M$$

constant vectors, and

$$\Omega(\xi, \xi')(u) = \sum_{i=1,\ldots,N} \Omega_i(u_i)$$

with

$$\Omega_i(u_i) = \int_{u_{i,0}}^{u_i} d u_i \exp(L^i u_i) a_i \otimes a_i^t \exp(L u_i) L + \Omega_{i,0},$$

$$L^t \Omega_{i,0} - \Omega_{i,0} L = 0,$$

$$\Omega_{i,0} = \exp(L^i u_{i,0}) a_i \otimes a_i^t \exp(L u_{i,0}) L.$$

Thus

$$\Omega(X, \xi')(u) = \sum_{i=1,\ldots,N} e_i \otimes a_i^t \exp(L u_i),$$

$$\Omega(\xi, H)(u) = c + \sum_{i=1,\ldots,N} (L^{-1})^t \exp(L^i u_i) a_i.$$

where $c \in \mathbb{R}^M$ is a constant vector and we assume that $L$ is invertible. The coefficients of the corresponding transformed flat line element and its flat coordinates are

$$H_i(u)^2 = \left(1 - a_i^t \exp(L^i u_i) L \left[ \sum_{k=1,\ldots,N} \Omega_k(u_k) \right]^{-1} \left[ c + \sum_{l=1,\ldots,N} (L^{-1})^t \exp(L^l u_l) a_l \right] \right)^2,$$

$$x_i(u) = u_i - a_i^t \exp(L u_i) \left[ \sum_{k=1,\ldots,N} \Omega_k(u_k) \right]^{-1} \left[ c + \sum_{l=1,\ldots,N} (L^{-1})^t \exp(L^l u_l) a_l \right].$$
In the diagonal case, $L = \text{diag}(\ell_1, \ldots, \ell_N)$ we find

$$\Omega(\xi, \xi^*) = (\Omega_{ij}), \quad \Omega_{ij} := \frac{\ell_j}{\ell_i + \ell_j} \sum_{k=1}^{N} \exp((\ell_i + \ell_j)u_k)a_{k,i}a_{k,j},$$

$$H_i(u)^2 = \left( 1 - \frac{1}{|\Omega(u)|} \sum_{k,l=1,\ldots,N} a_{i,k} \exp(\ell_ku_i) \ell_k \text{cofac}(\Omega(u))_{kl} \left( c_l + \frac{1}{\ell_l} \sum_{j=1,\ldots,N} \exp(\ell_ju_j)a_{j,l} \right) \right)^2,$$

$$x_i(u) = u_i - \frac{1}{|\Omega(u)|} \sum_{k,l=1,\ldots,N} a_{i,k} \exp(\ell_ku_i) \text{cofac}(\Omega(u))_{kl} \left( c_l + \frac{1}{\ell_l} \sum_{j=1,\ldots,N} \exp(\ell_ju_j)a_{j,l} \right),$$

$$\beta_{ij}(u) = -\frac{1}{|\Omega(u)|} \sum_{k,l=1,\ldots,N} \ell_k a_{j,k} \text{cofac}(\Omega(u))_{kl} a_{i,l} \exp(\ell_ku_j + \ell_lu_i).$$

Where $a_i^l := (a_{i,1}, \ldots, a_{i,N})$, and we are using the cofactor matrix $\text{cofac}(A)$; i.e., $A^{-1} = |A|^{-1} \text{cofac} A$. These type of solutions are the extension to multidimensions of the bright multi-soliton solutions of the attractive Nonlinear Schrödinger equation, which describes the propagation of optical pulses in nonlinear fibres.

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