Regularity Results on the Flow Maps of Periodic Dispersive Burgers Type Equations and the Gravity–Capillary Equations

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Abstract
In the first part of this paper we prove that the flow associated to a dispersive Burgers equation with a non local term of the form $|D|^{\alpha-1} \partial_x u$, $\alpha \in [1, +\infty[$ is Lipschitz from bounded sets of $H^s_0(\mathbb{T}; \mathbb{R})$ to $C^0([0, T], H^{s-(2-\alpha)^+}_0(\mathbb{T}; \mathbb{R}))$ for $T > 0$ and $s > \left\lceil\frac{\alpha}{\alpha-1}\right\rceil - \frac{1}{2}$, where $H^s_0$ are the Sobolev spaces of functions with 0 mean value, proving that the result obtained in Said (A geometric proof of the quasi-linearity of the water-waves system and the incompressible Euler equations) is optimal on the torus. The proof relies on a paradifferential generalization of a complex Cole–Hopf gauge transformation introduced by Tao (J Hyperbol Differ Equ 1:27–49, 2004) for the Benjamin–Ono equation. For this we prove a generalization of the Baker–Campbell–Hausdorff formula for flows of hyperbolic paradifferential equations and prove the stability of the class of paradifferential operators modulo more regular remainders, under conjugation by such flows. For this we prove a new characterization of paradifferential operators in the spirit of Beals (Duke Math J 44:45–57, 1977). In the second part of this paper we use a paradifferential version of the previous method to prove that a re-normalization of the flow of the one dimensional periodic gravity–capillary equation is Lipschitz from bounded sets of $H^s$ to $C^0([0, T], H^{s-\frac{1}{2}})$ for $T > 0$ and $s > 3 + \frac{1}{2}$. This proves that the result obtained in Said (A geometric proof of the quasi-linearity of the water-waves system and the incompressible Euler equations) is optimal for the water waves system.

Keyword Flow map · Regularity · Quasi-linear · Nonlinear Burgers type dispersive equations · Water Waves system · Gravity–capillary equations · Cole–Hopf gauge transform
1 Introduction

In our study of the quasi-linearity of the water waves system in [35] we studied the flow map regularity for some model nonlinear dispersive equations of the form:

$$\partial_t u + u \partial_x u + |D|^{\alpha-1} \partial_x u = 0 \quad \text{on } \mathbb{D},$$  \hspace{1cm} (1.1)

where $\mathbb{D} = \mathbb{R}$ or $\mathbb{T}$, and $\mathbb{T}$ is taken of length $2\pi$, $\alpha \in [0, 2]$ and $|D|$ is the Fourier multiplier with symbol $|\xi|$. We proved that they are quasi-linear. We based our work on the following distinction between semi-linearity and quasi-linearity given in [29]:

- A partial differential equation is said to be semi-linear if its flow map is regular (at least $C^1$).
- A partial differential equation is said to be quasi-linear if its flow map is not Lipschitz.

More precisely we proved that:

- the flow map associated to (1.1) fails to be uniformly continuous from bounded sets of $H^s(\mathbb{D})$ to $C^0([0, T], H^s(\mathbb{D}))$ for $T > 0$ and $s > 2 + \frac{1}{2}$.

The drawback of this test of quasi-linearity, where we only analyse the uniform continuity of the flow map, is that it does not show the effect of the dispersive term. The natural question is then to ask if one can know $\alpha$ exactly by having a more refined analysis of the regularity of the flow map.
For this we can start by noticing that independently of $\alpha$ the flow map is Lipschitz from bounded sets of $H^s(D)$ to $C^0([0, T], H^{s-1}(D))$ and ask: can the space $H^{s-\mu}(D)$ with $\mu < 1$ depending on $\alpha$? Again in [35] we proved that the best $\mu$ one can hope for is $\mu = 1 - (\alpha - 1)^+$, where $a^+ := \max(a, 0)$, more precisely we showed that:

- the flow map cannot be Lipschitz from bounded sets of $H^s(D)$ to $C^0([0, T], H^{s-1+(\alpha-1)^+}(D))$ for $\epsilon > 0$.

We now turn to the literature to assess the optimality of the result. First in [37] when $D = \mathbb{R}$ the Eq. (1.1) is actually shown to be quasi-linear for $\alpha \in [0, 3]$ and becomes semi-linear for $\alpha = 3$. For $\alpha = 3$ this is the Korteweg-de Vries equation, suggesting that our results are sub-optimal. When $D = \mathbb{T}$, in [30], for the case $\alpha = 2$ that is the Benjamin–Ono equation, the flow map is shown to be Lipschitz (and even has analytic regularity) on bounded subsets of $H^0$ the (Sobolev spaces of functions with 0 mean value). This result implies that our results could be optimal, but there might be a subtlety regarding the behavior of low frequencies.

The aim of the current paper is to prove that the results obtained in [35] are optimal on the torus and for the full periodic water waves system with surface tension, that is the gravity–capillary equation.

Before we give the main results of this paper, it is important to place the question of the flow map regularity in the vast and rich literature that studies equations of the form (1.1), for a comprehensive and complete overview of those equations and their link to other problems coming from mechanical fluids and dispersive non linear equations in physics we refer to Saut’s [37, 38]. Beyond the starting point of hyperbolic local well-posedness of (1.1) in $H^s$, $s > 1 + \frac{1}{2}$, three natural questions are: (1) is the problem globally well-posed or is there blow up within finite time? (2) What is the smallest $s$ for which we still have local well-posedness? (3) Is the continuous dependence on the initial datum optimal, that is, is the equation semi/quasi-linear? The first two questions are usually closely connected due to the existence of conservation laws and form some of the most important problems in PDEs today. Understanding them demands a better understanding of the equation beyond the basic hyperbolic structure.

For the equations analysed in this paper, this means that a refined understanding of the interaction between the dispersive term and nonlinear transport term is needed to answer those questions. For $\alpha \leq 1$ and $\alpha \geq 2$ the problem for Eq. (1.1) is now very well understood. Indeed for $\alpha \leq 1$ the equation is known to exhibit finite time blow-up and does so through a wave breaking scenario [10, 17–19, 33, 39]. For $\alpha \geq 2$ the equation is globally well-posed and the optimal threshold $s_c$ is known [20, 21, 25, 27, 31, 41], for the special cases where (1.1) is integrable that is the Benjamin–Ono equation ($\alpha = 2$) and the KdV equation ($\alpha = 3$) the equation is even better understood due to the remarkable construction of Birkhoff coordinates [13, 22, 23]. For $1 < \alpha < 2$, the global Cauchy problem is not as well understood. A numerical study was carried out by Klein and Saut in [24] and they conjectured that there is blow-up for $1 < \alpha \leq \frac{3}{2}$ and that the equations are globally well-posed for $\alpha > \frac{3}{2}$. To the best of the author’s knowledge the only progress towards answering this conjecture was given in [32] where the authors proved global well posedness for $\alpha > 1 + \frac{6}{7}$. One of the main goals of the current work is to show that a refined understanding of
the third question sheds some light on some interactions between the dispersive and nonlinear terms. This understanding could later be used to give answers to the first two questions. Indeed determining $\alpha$ here through the understanding of the exact regularity of the flow map required us to construct a generalized Baker–Campbell–Hausdorff formula for hyperbolic paradifferential equations. We use those tools developed here in a subsequent work [36] to answer global Cauchy type questions.

1.1 On the Torus

We show that the flow map associated to (1.1) is Lipschitz from bounded sets of $H^s_0(\mathbb{T})$ to $C^0([0, T], H^{s-1+ (\alpha-1^+)}_0(\mathbb{T}))$. We begin by recalling a classical result in the literature [28, 37].

**Theorem 1.1** Consider three real numbers $\alpha \in [0, 2[$, $s \in ]2 + \frac{1}{2}, +\infty[$, $r > 0$ and $u_0 \in H^s(\mathbb{T})$. Then there exists $C_s > 0$ such that for $0 < T < \frac{C_s}{r + \|\partial_x u_0\|_{L^\infty(\mathbb{T})}}$ and all $v_0$ in the ball $B(u_0, r) \subset H^s(\mathbb{T})$ there exists a unique $v \in C^0([0, T], H^s(\mathbb{T}))$ solving the Cauchy problem:

$$\begin{align*}
\partial_t v + v \partial_x v + |D|^{\alpha-1} \partial_x v &= 0 \\
v(0, \cdot) &= v_0(\cdot).
\end{align*}$$

Moreover, for all $\mu \in [0, s]$, $\exists C_\mu \in \mathbb{R}_+$ such that:

$$\forall t \in [0, T], \quad \|v(t)\|_{H^\mu(\mathbb{T})} \leq e^{C_\mu \|\partial_x v\|_{L^1([0, T], L^\infty(\mathbb{T}))}} \|v_0\|_{H^\mu(\mathbb{T})}.$$  

(1.3)

Taking $v_0 \in B(u_0, r)$, and assuming moreover that $u_0 \in H^{s+1}(\mathbb{T})$ then:

$$\forall t \in [0, T], \quad \|(u - v)(t)\|_{H^s(\mathbb{T})} \leq C(t, u, v) \|u_0 - v_0\|_{H^s(\mathbb{T})},$$  

(1.4)

where $u$ is the solution emanating from $u_0$ and

$$C(t, u, v) = e^{C_s(\|\partial_x (u,v)\|_{L^1([0, T], L^\infty(\mathbb{T}))} + C_s\|u\|_{L^1([0, T], H^{s+1}(\mathbb{T}))})}. $$

In [35] we studied the regularity of the flow map and proved the following.

**Theorem 1.2** [35] Consider three real numbers $\alpha \in [0, 2[$, $s \in ]2 + \frac{1}{2}, +\infty[$, $r > 0$ and $u_0 \in H^s(\mathbb{T}; \mathbb{R})$. Let $C_s$ be given by Theorem 1.1 and $0 < T < \frac{C_s}{r + \|\partial_x u_0\|_{L^\infty(\mathbb{T})}}$.

- Then the flow map associated to the Cauchy problem (1.2):

$$B(u_0, r) \to C^0([0, T], H^s(\mathbb{T}; \mathbb{R}))$$

$$v_0 \mapsto v$$

is continuous but not uniformly continuous.
• Moreover for all $\epsilon > 0$ the flow map:

$$B(u_0, r) \rightarrow C^0([0, T], H^{s-1+(\alpha-1)+\epsilon}(\mathbb{T}; \mathbb{R}))$$

$$v_0 \mapsto v$$

is not Lipschitz.

Here we prove that these results are essentially optimal on the torus, more precisely we prove the following theorem.

**Theorem 1.3** Consider three real numbers $\alpha \in ]1, 2[, s \in ]\frac{\alpha}{\alpha-1} - \frac{1}{2}, +\infty[,$ $r > 0$ and $u_0 \in H^s_0(\mathbb{T}; \mathbb{R}).$ Let $C_s$ be given by Theorem 1.1 and $0 < T < \frac{C_s}{r + \|\partial_x u_0\|_{L^\infty(\mathbb{D})}}.$ Then the flow map associated to the Cauchy problem (1.2)

$$B(u_0, r) \cap H^s_0(\mathbb{T}; \mathbb{R}) \rightarrow C^0([0, T], H^{s-(2-\alpha)+}(\mathbb{T}; \mathbb{R}))$$

$$v_0 \mapsto v$$

is Lipschitz, more precisely we have the a priori estimate for $t \in [0, T]$

$$\| (u - v)(t) \|_{H^{s-(2-\alpha)+}(\mathbb{D})} \leq C(t, u, v) \| u_0 - v_0 \|_{H^{s-(2-\alpha)+}(\mathbb{D})},$$

with

$$C(t, u, v) = \exp \left( Ct \| u \|_{L^\infty_t H^s} \exp \left( C \| (u, v) \|_{L^\infty_t W^s_{\alpha-1, \infty}} \right) \right).$$

Several remarks are in order.

(1) As a corollary of Theorem 1.3 we prove in Sect. 2.2 the following.

**Corollary 1.1** Consider three real numbers $\alpha \in ]1, 2[, s > \left[ \frac{\alpha}{\alpha-1} - \frac{1}{2}, \ r > 0,$ and $u_0 \in H^s(\mathbb{T}; \mathbb{R}).$ Let $C_s$ be given by Theorem 1.1 and $0 < T < \frac{C_s}{r + \|\partial_x u_0\|_{L^\infty(\mathbb{D})}}.$

- Then the flow map associated to the Cauchy problem (1.2):

$$B(u_0, r) \rightarrow C^0([0, T], H^{s}(\mathbb{T}; \mathbb{R}))$$

$$v_0 \mapsto v$$

is continuous but not uniformly continuous.

- For all $\epsilon > 0$ the flow map:

$$B(u_0, r) \rightarrow C^0([0, T], H^{s-1+\epsilon}(\mathbb{T}; \mathbb{R}))$$

$$v_0 \mapsto v$$

is not $C^1.$
(2) The case \( \alpha = \frac{3}{2} \) is closely related to the system obtained after reduction and para-linearization of the periodic water waves system in dimension 1 obtained in [2, Proposition 3.3] by Alazard et al. which we will treat in the second part of this paper.

(3) The case \( \alpha = 2 \) and the Benjamin-Ono equation on the circle was obtained by Molinet in [30]. Though Molinet’s result extends to the Cauchy problem on \( L^2(\mathbb{T}) \) and only studied the flow map regularity for data with 0 mean value.

The main language and techniques used in this article is that of paraproducts, paracomposition paradifferential operators and paradifferential calculus for which a rigorous review is given in Appendix A.2. We give an intuitive interpretation of those concepts in the following paragraph so that the reader unfamiliar with this language can get a good grasp of the statements without having to go through Appendix A.2 first.

- **Paraproducts, paracomposition and paradifferential operators**

For the sake of this discussion let us pretend that \( \partial_x \) is left-invertible with a choice of \( \partial^{-1}_x \) that acts continuously from \( H^s \) to \( H^{s+1} \). We follow here analogous ideas to the ones presented by Shnirelman in [40]. One way to define the paraproduct of two functions \( f, g \in H^s \) with \( s \) sufficiently large is: we differentiate \( fg \) \( k \) times, using the Leibniz formula, and then restore the function \( fg \) by the \( k \)-th power of \( \partial^{-1}_x \):

\[
fg = \partial^{-k}_x \partial^k_x (fg) \\
= \partial^{-k}_x (g \partial^k_x f + k \partial_x g \partial^{k-1}_x f + \cdots + k \partial_x f \partial^k_x g + g \partial^k_x f) \\
= T_g f + T_f g + R,
\]

where,

\[
T_g f = \partial^{-k}_x (g \partial^k_x f), \quad T_f g = \partial^{-k}_x (f \partial^k_x g),
\]

and \( R \) is the sum of all remaining terms. The key observation is that if \( s > \frac{1}{2} + k \), then \( g \mapsto T_f g \) is a continuous operator in \( H^s \) for \( f \in H^{s-k} \). The remainder \( R \) is a continuous bilinear operator from \( H^s \) to \( H^{s+1} \). The operator \( T_f g \) is called the paraproduct of \( g \) and \( f \) and can be interpreted as follows. The term \( T_f g \) takes into play high frequencies of \( g \) compared to those of \( f \) and demands more regularity in \( g \in H^s \) than \( f \in H^{s-k} \) thus the term \( T_f g \) bears the "singularities" brought on by \( g \) in the product \( fg \). Symmetrically \( T_g f \) bears the "singularities" brought on by \( f \) in the product \( fg \) and the remainder \( R \) is a smoother function \( H^{s+1} \) and does not contribute to the main singularities of the product. Notice that this definition uses "general" heuristic from PDEs that is the worst terms are the highest order terms (ones involving the highest order of differentiation). Now to make such a definition rigorous, we quantify this frequency comparison. The starting point is the product formula:

\[
fg(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i x \cdot (\xi_1 + \xi_2)} \mathcal{F}(f)(\xi_1) \mathcal{F}(g)(\xi_2) d\xi_1 d\xi_2.
\]
Now if for some parameters $B > 1$, $b > 0$ one defines a cut-off function:

$$\psi^{B,b}(\eta, \xi) = 0 \quad \text{when } |\xi| < B|\eta| + b,$$
$$\psi^{B,b}(\eta, \xi) = 1 \quad \text{when } |\xi| > B|\eta| + b + 1,$$

then one can rigorously define the paraproduct as

$$T_{g}^{B,b} f(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \psi^{B,b}(\xi_2, \xi_1) e^{i x \cdot (\xi_1 + \xi_2)} \mathcal{F}(f)(\xi_1) \mathcal{F}(g)(\xi_2) d\xi_1 d\xi_2.$$

To get a good intuition of a paradifferential operator $T_p$ with symbol $p \in \Gamma_{\rho}^{\beta}$, as a first gross approximation, one can think of $T_p$ as the composition of a paraproduct $T_f$ with Fourier multiplier $m(D)$, that is:

$$T_p \approx T_f m(D), \quad \text{with } f \in W^{\rho, \infty} \text{ and } m \text{ is of order } \beta.$$ 

Indeed following Coifman and Meyer’s symbol reduction Proposition 5 of [11], one can show that linear combinations of composition of a paraproduct with a Fourier multiplier are dense in the space of paradifferential operators.

Finally for the paracomposition operation we again work with $f \in H^s$ and $g \in C^s$ with $s$ large and consider the composition of two functions $f \circ g$ which bears the singularities of both $f$ and $g$, and our goal is to separate them. We proceed as before by differentiating $f \circ g$ $k$ times, using the Faà di Bruno’s formula, and then restore the function $fg$ by the $k$-th power of $\partial^{-1}_x$:

$$f \circ g = \partial^{-k}_x \partial^k_x (f \circ g)$$
$$= \partial^{-k}_x ((\partial^k_x f \circ g) \cdot (\partial_x g)^k + \cdots + (\partial_x f \circ g) \cdot \partial^k_x g)$$
$$= g^* f + T_{\partial_x f \circ g} g + R,$$

where,

$$g^* f = \partial^{-k}_x ((\partial^k_x f \circ g) \cdot (\partial_x g)^k)$$

is the paracomposition of $f$ by $g$

and $R$ is the sum of all remaining terms. Again the key observation is that if $s > \frac{1}{2} + k$, then $f \mapsto g^* f$ is a continuous operator in $H^{s}$ for $g \in C^{s-k}$. Thus this term bears essentially the singularities of $f$ in $f \circ g$. As before $T_{\partial_x f \circ g} g$ bears essentially the singularities of $g$ in $f \circ g$. The remainder $R$ is a continuous bilinear operator from $H^s$ to $H^{s+1}$. Thus we have separated the singularities of the composition $f \circ g$.

### 1.2 The Periodic Gravity Capillary Equation

We follow here the presentation in [2, 4, 5].
1.2.1 Assumptions on the Domain

We consider a domain with free boundary, of the form:

$$\{(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R} : (x, y) \in \Omega_t\},$$

where $\Omega_t$ is the domain located between a free surface:

$$\Sigma_t = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = \eta(t, x)\}$$

and a given (general) bottom denoted by $\Gamma = \partial \Omega_t \setminus \Sigma_t$. More precisely we assume that initially ($t = 0$) we have the hypothesis $(H_t)$ given by:

• The domain $\Omega_t$ is the intersection of the half space, denoted by $\Omega_{1,t}$, located below the free surface $\Sigma_t$,

$$\Omega_{1,t} = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y < \eta(t, x)\} \quad (H_t)$$

and an open set $\Omega_2 \subset \mathbb{R}^{1+1}$ such that $\Omega_2$ contains a fixed strip around $\Sigma_t$, which means that there exists $h > 0$ such that,

$$\{(x, y) \in \mathbb{R} \times \mathbb{R} : \eta(t, x) - h \leq y \leq \eta(t, x)\} \subset \Omega_2. \quad (H_t)$$

We shall assume that the domain $\Omega_2$ (and hence the domain $\Omega_t = \Omega_{1,t} \cap \Omega_2$) is connected.

1.2.2 The Equations

We consider an incompressible inviscid liquid, having unit density. The equations of motion are given by the Euler system of the velocity field $v$:

$$\begin{cases}
\partial_t v + v \cdot \nabla v + \nabla P = -ge_y \\
\text{div } v = 0
\end{cases} \quad \text{in } \Omega_t, \quad (1.5)$$

where $-ge_y$ is the acceleration of gravity ($g > 0$) and where the pressure term $P$ can be recovered from the velocity by solving an elliptic equation. The problem is then coupled with the boundary conditions:

$$\begin{cases}
v \cdot n = 0 & \text{on } \Gamma, \\
\partial_t \eta = \sqrt{1 + (\partial_x \eta)^2} v \cdot v & \text{on } \Sigma_t, \\
P = -\kappa H(\eta) & \text{on } \Sigma_t,
\end{cases} \quad (1.6)$$

where $n$ and $v$ are the exterior normals to the bottom $\Gamma$ and the free surface $\Sigma_t$, $\kappa$ is the surface tension and $H(\eta)$ is the mean curvature of the free surface:

$$H(\eta) = \partial_x \left( \frac{\partial_x \eta}{\sqrt{1 + (\partial_x \eta)^2}} \right).$$
We are interested in the case with surface tension and take $\kappa = 1$. The first condition in (1.6) expresses the fact that the particles in contact with the rigid bottom remain in contact with it. As no hypothesis is made on the regularity of $\Gamma$, this condition makes sense in a weak variational meaning due to the hypothesis ($H_t$), for more details on this we refer to Sect. 2 in [2].

The fluid motion is supposed to be irrotational and $\Omega_t$ is supposed to be simply connected thus the velocity $v$ field derives from some potential $\phi$ that is $v = \nabla \phi$ and:

$$\begin{cases}
\Delta \phi = 0 & \text{in } \Omega, \\
\partial_n \phi = 0 & \text{on } \Gamma.
\end{cases}$$

The boundary condition on $\phi$ becomes:

$$\begin{cases}
\partial_n \phi = 0 & \text{on } \Gamma, \\
\partial_t \eta = \partial_y \phi - \partial_x \eta \partial_x \phi & \text{on } \Sigma_t, \\
\partial_t \phi = -g \eta + H(\eta) - \frac{1}{2} \left| \nabla_{x,y} \phi \right|^2 & \text{on } \Sigma_t.
\end{cases} \quad (1.7)$$

Following Zakharov [44] and Craig and Sulem [12] we reduce the analysis to a system on the free surface $\Sigma_t$. If $\psi$ is defined by

$$\psi(t, x) = \phi(t, x, \eta(t, x)),$$

then $\phi$ is the unique variational solution of

$$\Delta \phi = 0 \quad \text{in } \Omega_t, \quad \phi|_{y=\eta} = \psi, \quad \partial_n \phi = 0 \quad \text{on } \Gamma.$$

Define the Dirichlet–Neumann operator by

$$
(G(\eta)\psi)(t, x) = \sqrt{1 + (\partial_x \eta)^2} \partial_n \phi|_{y=\eta}(t, x)
= (\partial_y \phi)(t, x, \eta(t, x)) - \partial_x \eta(t, x) \partial_x \phi(t, x, \eta(t, x)).
$$

For the case with rough bottom we refer to [1, 2] and [5] for the well-posedness of the variational problem and the Dirichlet–Neumann operator. Now $(\eta, \psi)$ (see for example [12]) solves:

$$\begin{align*}
\partial_t \eta &= G(\eta) \psi, \\
\partial_t \psi &= -g \eta + H(\eta) + \frac{1}{2} (\partial_x \psi)^2 + \frac{1}{2} \frac{\partial_x \eta \partial_x \psi + G(\eta) \psi}{1 + (\partial_x \eta)^2}.
\end{align*} \quad (1.8)$$

The system is completed with initial data

$$\eta(0, \cdot) = \eta_{in}, \quad \psi(0, \cdot) = \psi_{in}.$$

We consider the case when $\eta, \psi$ are $2\pi$-periodic in the space variable $x$. 

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1.2.3 Flow Map Regularity

In [2] and [5], Alazard et al. perform a paralinearization and symmetrization of the water waves system that takes the form:

$$\partial_t u + TV \cdot \nabla u + iT\gamma u = f,$$

where $\gamma$ is an elliptic symbol of order $\frac{3}{2}$ which closely resembles the model problem we presented on $\mathbb{T}$ but with an extra non-linearity in $\gamma$. The paralinearization and symmetrization of the system was used to prove the well-posedness of the Cauchy problem in the optimal threshold $s > 2 + \frac{1}{2}$ in which the velocity field $v$ is Lipschitz.

We will complete this and our result in [35] by giving the precise regularity of the flow map. First we recall some previously known results on the Cauchy problem from [2, 5].

**Theorem 1.4** (From [2, 5]) Consider two real numbers $r > 0$, $s \in \mathbb{R}$ and $(\eta_0, \psi_0) \in H^{s+\frac{1}{2}}(\mathbb{T}) \times H^s(\mathbb{T})$ such that,

$$\forall (\eta'_0, \psi'_0) \in B((\eta_0, \psi_0), r) \subset H^{s+\frac{1}{2}}(\mathbb{T}) \times H^s(\mathbb{T})$$

the assumption $\left(H_t\right)_{t=0}$ is satisfied. Then there exists $T > 0$ such that the Cauchy problem (1.8) with initial data $(\eta'_0, \psi'_0) \in B((\eta_0, \psi_0), r)$ has a unique solution

$$(\eta', \psi') \in C^0([0, T]; H^{s+\frac{1}{2}}(\mathbb{T}) \times H^s(\mathbb{T}))$$

and such that the assumption $\left(H_t\right)$ is satisfied for $t \in [0, T]$. Moreover the flow map $(\eta'_0, \psi'_0) \mapsto (\eta', \psi')$ is continuous.

In [35] we completed this by the following.

**Theorem 1.5** Consider two real numbers $r > 0$, $s \in \mathbb{R}$ and $(\eta_0, \psi_0) \in H^{s+\frac{1}{2}}(\mathbb{T}) \times H^s(\mathbb{T})$ such that,

$$\forall (\eta'_0, \psi'_0) \in B((\eta_0, \psi_0), r) \subset H^{s+\frac{1}{2}}(\mathbb{T}) \times H^s(\mathbb{T})$$

the assumption $\left(H_t\right)_{t=0}$ is satisfied.

Then for all $R > 0$ the flow map associated to the Cauchy problem (1.8):

$$B((\eta_0, \psi_0), R) \to C^0([0, T], H^{s+\frac{1}{2}}(\mathbb{T}) \times H^s(\mathbb{T}))$$

$$(\eta'_0, \psi'_0) \mapsto (\eta', \psi')$$

is not uniformly continuous.

And at least a loss of $\frac{1}{2}$ derivative is necessary to have Lipschitz control over the flow map, that is for all $\epsilon > 0$ the flow map

$$B((\eta_0, \psi_0), R) \to C^0([0, T], H^{s+\epsilon}(\mathbb{T}) \times H^{s-\frac{1}{2}+\epsilon}(\mathbb{T}))$$
$(\eta'_0, \psi'_0) \mapsto (\eta', \psi')$

is not Lipschitz.

Here it is shown that those results are sufficient after suitable re-normalization of the flow map.

**Theorem 1.6** Consider two real numbers $r > 0$, $s \in \mathbb{R} + \frac{1}{2}, +\infty[$ and $(\eta_0, \psi_0) \in H^{s+\frac{1}{2}}(\mathbb{T}) \times H^s(\mathbb{T})$ such that,

\[ \forall (\eta'_0, \psi'_0) \in B((\eta_0, \psi_0), r) \subset H^{s+\frac{1}{2}}(\mathbb{T}) \times H^s(\mathbb{T}) \]

the assumption $(H_1)^{t=0}$ is satisfied. Define $(\eta, \psi)$ and $(\eta', \psi')$ as the solutions to the Cauchy problem (1.8) on $[0, T]$, $T > 0$. Define the following change of variables:

\[ \chi(t, x) = \int_0^x \frac{1}{\sqrt{1 + (\partial_x \eta(t, y))^2}} dy - \int_0^t \int_{\Sigma_s} \left[ \frac{1}{1 + (\partial_x \eta)^2} + \partial_x \phi \right] d\Sigma_s ds \]

\[ = \int_0^x \frac{1}{\sqrt{1 + (\partial_x \eta(s, y))^2}} dy - \int_0^t 2\pi \int_0^\pi \left[ \frac{1}{1 + (\partial_x \eta(s, y))^2} + \partial_x \phi \right] \sqrt{1 + (\partial_x \eta(s, y))^2} ds \]

\[ (1.9) \]

where $d\Sigma_t$ is the surface measure on $\Sigma_t$ and $\chi'$ is defined analogously from $(\eta', \psi')$. Then for $r$ sufficiently small and $t \in [0, T]$ we have:

\[ \| (\eta, \psi)^* - (\eta', \psi')^*(t, \cdot) \|_{H^s \times H^{s-\frac{1}{2}}} \leq C \left( \| (\eta_0, \psi_0, \eta'_0, \psi'_0) \|_{H^{s+\frac{1}{2}} \times H^s} \right) \| (\eta_0, \psi_0)^* - (\eta'_0, \psi'_0)^* \|_{H^s \times H^{s-\frac{1}{2}}} , (1.10) \]

where $*$ and $^*$ are the paracomposition by $\chi$ and $\chi'$, which we recall it’s definition in A.3.

The time integral in the re-normalization (1.9) is to ensure that the mean value of the transport term vanishes. This re-normalization is used here to compensate the non-linearity in the dispersive term $T_\gamma$ of order $\frac{3}{2}$.

**1.3 Strategy of the Proof**

For Theorem 1.3, we first work on $H^s_0$ and the main idea is to conjugate (1.1) to a semi-linear dispersive equation of the form:

\[ \partial_t w + |D|^{\alpha-1} \partial_x w = R_u, \]
where \( R \) is continuous from \( H^s \) to itself. For the viscous Burgers equation such a result is obtained by the Cole–Hopf transformation that reduces the problem to a one dimensional heat equation. In [41], Tao used a complex version of the Cole–Hopf transformation to reduce the problem on the Benjamin–Ono equation to a one dimensional Schrödinger type equation, this idea was extensively used to lower the regularity needed for the well-posedness of the Cauchy problem as in the previously mentionned paper [30]. A generalized pseudodifferential form of this transformation was used in [3] to reduce the one dimensional water waves system to a one dimensional semi-linear Schrödinger type system.

Formally if we follow the same lines of those previous papers, the transformation we will have to use is a pseudodifferential transformation of the form:

\[
\begin{align*}
    w &= \text{Op}(a)u, \\
    a &= e^{\frac{\sqrt{1-\alpha}}{4} |\xi|^{1-\alpha}} U,
\end{align*}
\]  

(1.11)

where \( U \) is a real valued periodic primitive of \( u \) that exists because \( u \) has mean value 0 and \( \text{Op}(a) \) is the pseudodifferential operator with symbol \( a \).

The main problem is that such an operator belongs to a Hörmander symbol class of the form \( S^{0}_{\alpha-1,2-\alpha} \), see Remark 4.3 for a formal definition, for \( \alpha = \frac{3}{2} \) this becomes \( S^{0}_{\frac{3}{2}, \frac{1}{2}} \), which is a “bad” symbol class with no general symbolic calculus rules. Thus we have to treat this transformation with care.

The idea here is inspired by the particular form of the formal computation, we express the desired operator as the time one of a flow map associated to a hyperbolic equation, that is \( a = e^{iT_p} \) where \( (e^{i\tau T_p})_{\tau \in \mathbb{R}} \) is defined as the group generated by the paradifferential operator \( iT_p \) where \( p \) is a real valued symbol of order smaller than 1.

This is inspired by previous results of Alazard et al. [6].

Take a different operator \( T_b \). The main new idea is to apply a Baker–Campbell–Hausdorff formula. Formally this allows one to express \( e^{i\tau T_p}T_b e^{-i\tau T_p} \) as a series of successive Lie derivatives \([iT_p, \ldots, iT_p, T_b] \). The same kind of computations work for \([e^{i\tau T_p}, T_b] \). The convergence of such a series is a non trivial problem, equivalent to solving a linear ODE in the Fréchet space of paradifferential symbol classes \( \Gamma^m_{\infty} \) defined in Appendix A.2. Such an ODE is generally not well posed and to solve such a problem one usually has to look at a Nash–Moser type scheme. Though in our case we have an explicit ODE that can be solved locally with loss of derivative. Inspired by Hörmander’s [16] and Beals in [8], we prove the existence of a symbol \( b^\tau_p \) such that \( e^{i\tau T_p}T_b e^{-i\tau T_p} = \text{Op}(b^\tau_p) \), moreover \( b^\tau_p \) is shown to have an asymptotic expansion given by the Baker–Campbell–Hausdorff formula. The use of paradifferential operators is the key here, as in Hörmander’s [16], because the continuity of paradifferential operators given by Theorem A.2 ensures that we do not need to control an infinite number of semi-norms as would have been the case for pseudodifferential operators.

Finally the adequate choice of \( T_p \) eliminates the transport term of order 1 and gives a term of order \( 2 - \alpha \) which is enough to get the desired estimate.
Passing from $H_0^s$ to $H^s$ we use the following gauge transform:

$$\tilde{u}(t, x) = u(t, x - t\int u_0) - \int u_0, \text{ where } \int u_0 = \frac{1}{|T|} \int_0^T u_0,$$

which we will use to prove that the solution map is continuous on $H^s$ but not uniformly continuous and is $C^1$ from $H^s$ to $H^{s-\mu}$ if and only if $\mu \geq 1$.

For the gravity–capillary equation the problem is more delicate. Indeed the model problems we study are for the paralinearized and symmetrized system, though the change of variable from the original system to the paralinearized and symmetrized one is known to be Lipschitz on $H^s$ for $s > 2 + \frac{1}{2}$ [2, 5]. Thus the problem is reduced to the study of the flow map regularity of an equation of the form

$$\partial_t u + TV \cdot \partial_x u + iT u = f.$$

In the same spirit as [3, 4] we perform a para-change of variable, that is we para-compose with $\chi$ defined by (1.9), to get:

$$\partial_t [\chi^* u] + Tw \cdot \partial_x (\chi^* u) + iT^{\frac{1}{2}} \chi^* u = f, \text{ with } W = \frac{V \circ \chi}{\partial_x \chi} \text{ and } \int W = 0.$$

We then proceed exactly as for Eq. (1.1) (with the 0 mean value hypothesis ensured by the choice of $\chi$).

- Transformation (1.11), in which we use a primitive of the solution is called a gauge transform in the literature.
- As for the Cole–Hopf transformation, this gauge transform (1.11) is essentially one dimensional.
- It is interesting to note that the gauge transformation can be iterated to eliminate the term of order $2 - \alpha$ and get at the step of order $k$ a remainder of order $k + 1 - k\alpha$ which is an improvement at each step as $\alpha > 1$. Choosing $k = \lceil \frac{1}{\alpha-1} \rceil$, we get a residual term that is bounded from $H^s$ to $H^s$ when one pays the “price” of working in high enough regularity that is $s > 1 + \frac{1}{\alpha-1}$. In [36] we use this iteration to prove that for $2 < \alpha < 3$, the paradifferential version of (1.1) can be transformed to a semi-linear equation with a regularizing remainder, that is:

$$\partial_t u + T_\alpha \partial_x u + \partial_x |D|^{\alpha-1} u = 0 \Rightarrow \partial_t Au + \partial_x |D|^{\alpha-1} Au = R(u),$$

where the operator norm of $R(u)$ is controlled by $\|u\|_{L^\infty([0, T], C^2_{2-\alpha})}$. 

Regularity Results on the Flow...
2 Study of the Model Problems

2.1 Proof of Theorem 1.3, the Estimates on $H^s_0$

We keep the notations of Theorem 1.3, fixing $u_0 \in H^s_0(\mathbb{T}; \mathbb{R})$ and $r > 0$ and taking:

\[ v_0, w_0 \in B(u_0, r) \subset H^s_0(\mathbb{T}; \mathbb{R}). \]

As the mean value is conserved by the flow of (1.2) we consider the solutions $u, v, w \in C^0([0, T]; H^s_0(\mathbb{T}; \mathbb{R}))$ to (1.2) with initial data $u_0, v_0, w_0$ and on a uniform small interval $[0, T]$. In (2.7) below we will need a smallness hypothesis on $u$ and $r > 0$. At the end of the proof we will give the argument to do away with this extra smallness hypothesis.

The main goal of the proof is to show the following estimate:

\[ \| v(t, \cdot) - w(t, \cdot) \|_{H^s-(2-\alpha)^+} \leq C(v, w, t) \| v_0 - w_0 \|_{H^s-(2-\alpha)^+}, \quad (2.1) \]

with

\[ C(v, w) = e^{Cte} \left[ \frac{\alpha}{\alpha+1} \right]^{-1} \| v_0 \|_{L^\infty_{t} H^s}. \]

The final simplification we make in this paragraph is that given the well-posedness of the Cauchy problem in $H^s$, and the density of $H^{+\infty}$ in $H^s$, it suffice to prove (2.1) for $v_0, w_0 \in H^{+\infty}$, which henceforth we will suppose.

We start by applying the paralinearization Theorem A.2 to the term $u \partial_x v$ to get:

\[
\begin{cases}
\partial_t v + T_{\xi|\xi|^{s-1}} v = R_0(v), \\
v(0, \cdot) = v_0(\cdot),
\end{cases}
\]

(2.2)

where

\[ R_0(v) = -T_{\partial_x v} \cdot R(v, \partial_x \cdot) + T_{\xi|\xi|^{s-1} - \partial_x |D|^{s-1}} \cdot , \]

with $R(v, \partial_x \cdot)(u) = R(v, \partial_x u)$ and $R$ is the residual term in the paraproduct decomposition (A.5). Now we reduce $H^{s-(2-\alpha)^+}$ estimates to $L^2$ ones by defining $f_1 = \langle D \rangle^{s-(2-\alpha)^+} v$. Commuting $\langle D \rangle^{s-(2-\alpha)^+}$ with (2.2), using the symbolic calculus rules of Theorem A.2, we get that:

\[
\begin{cases}
\partial_t f_1 + T_{\xi|\xi|^{s-1}} f_1 = R_1(v) f_1, \\
f_1(0, \cdot) = \langle D \rangle^{s-(2-\alpha)^+} v_0(\cdot),
\end{cases}
\]

(2.3)

where

\[ R_1(v) = \left[ \langle D \rangle^{s-(2-\alpha)^+}, T_{\xi|\xi|^{s-1}} \right] \langle D \rangle^{s-(2-\alpha)^+} f_1 + \langle D \rangle^{s-(2-\alpha)^+} R_0(v) \langle D \rangle^{s+(2-\alpha)^+} f_1 . \]
Using Theorem A.2 to treat the commutator in $R_1(v)$, Theorem A.1 and (A.5) to treat the paraproduct terms in $R_0(v)$ and finally that the difference between a Fourier multiplier and the paradifferential operator associated with its symbol is a smoothing operator, that is is of order $-k$ for all $k$ we get the estimates

$$\| R_1(v) \|_{L^2 \to L^2} \leq C \left( \| v \|_{W^{1,\infty}} + 1 \right), \quad (2.4)$$

and

$$\| (R_1(v) - R_1(w)) f_1 \|_{L^2} \leq C \| v - w \|_{W^{1,\infty}} \| f_1 \|_{L^2}. \quad (2.5)$$

We define analogously $g_1 = \langle D \rangle^{s-(2-\alpha)+} w$ and notice that by definition:

$$\| f_1 - g_1 \|_{L^2} = \| v - w \|_{H^{s-(2-\alpha)+}},$$

thus the problem is reduced to getting $L^2$ estimates on $f_1 - g_1$. Here we give the full proof using estimates that will be proved in Sect. 4.

### 2.1.1 Gauge Transform and Energy Estimate

The goal of this section is to find an operator $A$ such that

$$\partial_t [Af_1] + T_{i|\xi|\alpha-1|\xi} Af_1 + AT_{i|\xi|\alpha-1|\xi} f_1 + [A, T_{i|\xi|\alpha-1|\xi}] f_1 = (\partial_t A) f_1 + AR_1(f_1) f_1,$$

and $AT_{i|\xi|\alpha-1|\xi} + [A, T_{i|\xi|\alpha-1|\xi}]$ is a hyperbolic operator of order $(2 - \alpha)^+ < 1$.

If we define $V = \partial_x^{-1} v$ which is the periodic zero mean value primitive of $v$, then

$$\hat{V}(0) = 0 \quad \text{and} \quad \hat{V}(\xi) = \frac{\hat{v}(\xi)}{i \xi}, \quad \text{for } \xi \in \mathbb{Z}^*,$$

and we define analogously $W$ from $w$. Then a formal computation shows that one can choose $A = e^{it\xi|\xi|^{-\alpha} V} \in S^0_{\alpha-1,2-\alpha}(\mathbb{T} \times \mathbb{Z})$ which is a symbol class with no general symbolic calculus rules. Here we will define $A$ differently$^1$ $A = e^{itpv}$, that is, it is defined as the time one of the flow map generated by $T_{ipv}$, given by Proposition 4.1, with

$$p_v = -\frac{1}{\alpha} |\xi|^{1-\alpha} V \in \Gamma^{2-\alpha}_{\alpha-1} (\mathbb{T}),$$

where we used the estimate

$$M_r^{2-\alpha} (p_v) \leq C \| v \|_{W^{r-1,\infty}}, \quad 1 \leq r \leq \left\lceil \frac{\alpha}{\alpha - 1} \right\rceil.$$

$^1$ Similar ideas were used in Appendix C of [3] to get estimates on a change of variable operator which are still in the usual symbol classes $S^{m}_{1,0}$, the difficulty here being that we are no longer in those symbol classes.
We define analogously $e^{iT_pv}$ and $p_w$ from $w$. Now introduce:

$$f_2 = e^{iT_pv} f_1, \quad g_2 = e^{iT_pv} g_1.$$  \hspace{1cm} (2.6)

As $e^{-it_{pv}}$ and $e^{-iT_pv}$ are the time $-1$ generated by the flow map $p_v, p_w$ respectively we write:

$$\|f_1 - g_1\|_{L^2} = \left\|e^{-iT_pv} f_2 - e^{-iT_pv} g_2\right\|_{L^2} \leq \left\|e^{-iT_pv} [f_2 - g_2]\right\|_{L^2} + \left\|(e^{-iT_pv} - e^{-iT_pv}) g_2\right\|_{L^2}.$$

Applying estimate (1) of Proposition 4.1 and estimate (4.4):

$$\|f_1 - g_1\|_{L^2} \leq C e^{C\|v\|_{L^\infty}} \|f_2 - g_2\|_{L^2} + C e^{C\|(v, w)\|_{L^\infty}} \|V - W\|_{L^\infty} \|g_2\|_{H^{(2-\alpha)^+}},$$

where henceforth $C$ denotes a generic universal constant and we used that $\alpha > 1$. Now to absorb the term containing $\|V - W\|_{L^\infty}$ into the right hand side we work with $\|u_0\|_{H^s}$ and $r$ sufficiently small such that

$$C \|V - W\|_{L^\infty} e^{C\|v, w\|_{L^\infty}} \|g_2\|_{H^{(2-\alpha)^+}} \leq \frac{1}{2} \|f_1 - g_1\|_{L^2}$$

for $t \in [0, T]$ which gives

$$\|f_1 - g_1\|_{L^2} \leq C e^{C\|v\|_{L^\infty}} \|f_2 - g_2\|_{L^2}.$$  \hspace{1cm} (2.7)

The goal now is getting an $L^2$ estimates on $f_2 - g_2$. To get the equations on $f_2$ and $g_2$ we commute $e^{iT_pv}$ and $e^{iT_pw}$ with (2.3), we make the computations for $f_2$, those for $g_2$ are obtained by symmetry:

$$e^{iT_pv} \partial_t f_1 + e^{iT_pv} T_{v\xi} f_1 + e^{iT_pv} T_{v|\xi|^{a-1}\xi} f_1 = e^{iT_pv} R_1(v) f_1,$$

$$\partial_t (e^{iT_pv} f_1) + T_{v|\xi|^{a-1}\xi} e^{iT_pv} f_1 + (e^{iT_pv} T_{v\xi} - [T_{v|\xi|^{a-1}\xi}, e^{iT_pv}]) f_1 - [e^{iT_pv}, \partial_t] f_1 = e^{iT_pv} R_1(v) f_1.$$

Thus

$$\partial_t f_2 + T_{v|\xi|^{a-1}\xi} f_2 = e^{iT_pv} R_1(v) e^{-iT_pv} f_2 - R_2(v) f_2 - R_3(v) f_2,$$  \hspace{1cm} (2.8)

where

$$R_2(v) = (e^{iT_pv} T_{v\xi} + [T_{v|\xi|^{a-1}\xi}, e^{iT_pv}]) e^{-iT_pv} \quad \text{and} \quad R_3(v) = [e^{iT_pv}, \partial_t] e^{-iT_pv}.$$

We get analogously on $g_2$,

$$\partial_t g_2 + T_{v|\xi|^{a-1}\xi} g_2 = e^{iT_pw} R_1(w) e^{-iT_pw} g_2 - R_2(w) g_2 - R_3(w) g_2.$$  \hspace{1cm} (2.9)
First for $e^{i T_p v} R_1(v) e^{-i T_p v}$ we have from (2.4)–(2.5) combined with Proposition 4.1 we get

$$\left\| e^{i T_p v} R_1(v) e^{-i T_p v} \right\|_{L^2 \rightarrow L^2} \leq C e^C \| v \|_{L^\infty} \left( \| v \|_{W^{1,\infty}} + 1 \right),$$

and the difference estimates

$$\left\| [e^{i T_p v} (R_1(v) - R_1(w)) e^{-i T_p w}] g_2 \right\|_{L^2} \leq C e^C \| v - w \|_{W^{1,\infty}} \| g_2 \|_{L^2}.$$

By Proposition 4.4 we also get

$$\left\| [e^{i T_p v} R_1(w) e^{-i T_p w} - e^{i T_p w} R_1(w) e^{-i T_p w}] g_2 \right\|_{L^2} \leq C e^{C \| (v, w) \|_{L^\infty}} \| V - W \|_{L^\infty} \| g_2 \|_{H^{(2-\alpha)^+}}.$$

By definition of $p_v$ and Proposition 4.4 we have:

$$\partial_\xi (\xi | \xi |^{\alpha-1}) \partial_x p_v = v \xi \quad \text{and} \quad [e^{i T_p v}, \partial_t] = -e^{i T_p v} \int_0^1 e^{-i r T_p v} T_i \partial_v e^{i T_p v} dr.$$ 

In Corollary 4.2 we show that we have the estimates

$$\| \text{Re}(R_2(v)) \|_{L^2 \rightarrow H^{(2-\alpha)^+}} \leq C e^{\| v \|_{W^\left[ \frac{\alpha}{\alpha-1} \right]^{-1,\infty}} \| v \|_{W^{1,\infty}},$$

and the difference estimate

$$\| R_2(v) - R_2(w) \|_{L^2} \leq C e^{\| (v, w) \|_{W^\left[ \frac{\alpha}{\alpha-1} \right]^{-1,\infty}} \| v - w \|_{W^\left[ \frac{\alpha}{\alpha-1} \right]^{-1+(2-\alpha)^+,\infty}} \| g_2 \|_{H^{(2-\alpha)^+}}.$$

Now to treat $R_3(v)$ we note that the PDE verified by $V$ is

$$\partial_t V + \frac{v^2}{2} + |D|^{\alpha-1} v = 0$$

thus

$$R_3(v) = \frac{1}{\alpha} \int_0^1 e^{i r T_p v} T_i \xi |\xi|^{-\alpha} \left( \frac{v^2}{2} + |D|^{\alpha-1} v \right) e^{-i r T_p v} dr.$$ 

By Proposition 4.1 $R_3(v)$ is of order $(2 - \alpha)^+$ and applying Corollary 4.3 we get the estimates

$$\| \text{Re}(R_3(v)) \|_{L^2 \rightarrow H^{2-\alpha}} \leq C e^{C \| v \|_{L^\infty}} \| v \|_{W^{\alpha-1,\infty}}.$$
and we get the same difference estimate as $R_2$ that is
\[
\| [R_3(v) - R_3(w)] g_2 \|_{L^2} \leq C e^{\alpha \alpha - 1} \| v - w \|_{W^{\alpha \alpha - 1} - 1, \infty} \| g_2 \|_{H^{(2 - \alpha) +}}.
\]

Thus taking the difference between (2.8)–(2.9) and making an energy estimate we get
\[
\frac{d}{dt} \| (f_2 - g_2)(t, \cdot) \|_{L^2} \leq C \| (v, w) \|_{W^{\alpha \alpha - 1} - 1, \infty} \| V - W \|_{L^\infty} \| g_2 \|_{H^{(2 - \alpha) +}},
\]

Now by (2.7) and the Sobolev embedding for $s > \left[ \frac{\alpha}{\alpha - 1} \right] - \frac{1}{2} - (2 - \alpha)^+$,
\[
\| v - w \|_{W^{\alpha \alpha - 1} - 1, \infty} \leq C \| f_1 - g_1 \|_{L^2} \leq C e^{C \| v \|_{L^\infty}} \| f_2 - g_2 \|_{L^2},
\]

thus we get the differential inequality on $\| f_2 - g_2 \|_{L^2}$
\[
\frac{d}{dt} \| (f_2 - g_2)(t, \cdot) \|_{L^2} \leq C e^{C \| (v, w) \|_{L^\infty}} \| f_2 - g_2(t, \cdot) \|_{L^2} \| g_2 \|_{H^{(2 - \alpha) +}},
\]

which integrated gives the estimate
\[
\| (f_2 - g_2)(t, \cdot) \|_{L^2} \leq C(v_0, w_0, t) \| (f_2 - g_2)(0, \cdot) \|_{L^2},
\]

with
\[
C(v_0, w_0, t) = e^{C t e^{C \| (v, w) \|_{L^\infty}} W^{\alpha \alpha - 1} - 1, \infty} \| w \|_{L^\infty H^s}.
\]

Injected back in (2.7) concludes the proof.

2.1.2 Dropping the Smallness Hypothesis

We fix $u_0^\epsilon \in H^{+\infty}$ sufficiently close to $u_0$ and define $u^\epsilon$ as it is associated solution. We consider the new variables $\tilde{v} = v - u^\epsilon$ and $\tilde{w} = w - u^\epsilon$ then $\tilde{v}$ solves
\[
\partial_t \tilde{v} + \tilde{v} \partial_x \tilde{v} + \partial_x (u^\epsilon \tilde{v}) + \partial_x |D|^{\alpha - 1} \tilde{v} = 0,
\]

which is the same as Eq. (1.1) with an extra linear skew symmetric term in $H^s$ given by $\partial_x (u^\epsilon \tilde{v})$. It can be treated exactly in the same fashion as in the previous paragraph with
now the smallness hypothesis on \( u_0 \) dropped as \( \tilde{v} \) and \( \tilde{w} \) can be chosen sufficiently small by the choice of \( r \). Indeed after parilinearisation and commuting with \( \langle D \rangle^{s-(2-\alpha)^+} \) we get for \( \tilde{f}_1 = \langle D \rangle^{s-(2-\alpha)^+} \tilde{v} \)

\[
\partial_t \tilde{v} + T_{i \xi} (\tilde{v} + u^\epsilon) \tilde{v} + T_{i \xi} |\xi|^{\alpha-1} \tilde{v} = R_1(\tilde{v}) \tilde{f}_1 - T_{\partial_x u^\epsilon} \cdot - R(u^\epsilon, \partial_x \cdot).
\]

The proof then follows in verbatim as previously with the choice of

\[
p_\tilde{v} = -\frac{1}{\alpha} \xi |\xi|^{1-\alpha} (\tilde{V} + U^\epsilon) \in \Gamma_{\alpha}^{2-\alpha} (\mathbb{T}),
\]

where \( \tilde{V} + U^\epsilon \) is the zero average primitive of \( \tilde{v} + u^\epsilon \).

### 2.2 Proof of Corollary 1.1, the Estimates on \( H^s \)

The starting point is noticing that the mean value is preserved by (1.2) and by doing the change of unknowns:

\[
\begin{aligned}
\tilde{u}(t, x) &= u(t, x - t \int u_0) - \int u_0 , \\
\tilde{v}(t, x) &= v(t, x - t \int v_0) - \int v_0 ,
\end{aligned}
\]  

(2.10)

where \( \int u_0 = \frac{1}{2\pi} \int u_0 \) is the mean value. We can reduce the Cauchy problem for general data to ones with 0 mean value by verifying that \( \tilde{u}, \tilde{v} \in H^s_0 \) still solve (1.2). Thus the main goal is to prove that the change of variable (2.10) is not regular. More precisely we will show that there exists a positive constant \( C \) and two sequences \((u^\lambda_\epsilon)\) and \((v^\lambda_\epsilon)\) solutions of (1.2) in \( C^0([0, 1], H^s(\mathbb{T})) \) such that for every \( 0 \leq t \leq T \), where \( T \) is a uniform small time,

\[
\sup_{\lambda, \epsilon} \left\| u^\lambda_\epsilon(t, \cdot) \right\|_{H^s(\mathbb{T})} + \left\| v^\lambda_\epsilon(t, \cdot) \right\|_{H^s(\mathbb{T})} \leq C,
\]

\((u^\lambda_\epsilon, \tau)\) and \((v^\lambda_\epsilon, \tau)\) satisfy initially:

\[
\lim_{\lambda \to +\infty} \lim_{\epsilon \to 0} \left\| u^\lambda_\epsilon(0, \cdot) - v^\lambda_\epsilon(0, \cdot) \right\|_{H^s(\mathbb{T})} = 0,
\]

but for any fixed \( t > 0 \),

\[
\lim_{\lambda \to +\infty} \lim_{\epsilon \to 0} \left\| u^\lambda_\epsilon(t, \cdot) - v^\lambda_\epsilon(t, \cdot) \right\|_{H^s(\mathbb{T})} \geq c > 0.
\]
Which proves the non uniform continuity. Considering a weaker control norm we want to get, for all $\delta > 0$ and for $t > 0$:

$$\liminf_{\lambda \to +\infty} \liminf_{\epsilon \to 0} \| u_\epsilon^\lambda(t, \cdot) - v_\epsilon^\lambda(t, \cdot) \|_{H^{s-1+s(T)}} = +\infty.$$ 

### 2.2.1 Definition of the Ansatz

Take $\omega \in C^\infty(T)$ such that for $x \in [-\pi, \pi]$:

$$\omega(x) = 1 \text{ if } |x| \leq \frac{1}{2}, \quad \omega(x) = 0 \text{ if } |x| \geq 1.$$ 

Let $(\lambda, \epsilon)$ be two positive real sequences such that:

$$\lambda \to +\infty, \quad \epsilon \to 0, \quad \lambda \epsilon \to +\infty. \quad (2.11)$$

Put for $x \in [-\pi, \pi]$,

$$u_0(x) = \lambda^{\frac{1}{2} - s} \omega(\lambda x), \quad v_0(x) = u_0(x) + \epsilon \omega(x), \quad (2.12)$$

and extend $u_0^0$ and $v^0$ periodically. The main trick here will be to use the time reversibility of Eq. (1.2) by defining $\tilde{u}$, $\tilde{v}$ as the solution of (1.2) with data fixed at time $t > 0$ given by

$$\begin{cases} 
\tilde{u}(t, x) = u_0 - \int u_0 \\
\tilde{v}(t, x) = v_0 - \int v_0 
\end{cases}, \quad (2.13)$$

where $t \leq t_0$ is chosen small enough for the equations to be well-posed. Finally, define $u$ and $v$ by (2.10).

### 2.2.2 Main Estimates

First the estimates at time 0, for $\left\lceil \frac{s}{a-1} \right\rceil - \frac{3}{2} \leq s - 1 \leq v \leq s$:

$$\| u(0, x) - v(0, x) \|_{H^v} = \left\| \tilde{u}(0, x) - \tilde{v}(0, x) + \int u_0 - \int v_0 \right\|_{H^v}$$

By the estimate (2.1) and the Cauchy–Schwartz inequality,

$$\| u(0, x) - v(0, x) \|_{H^v} \leq C e^{\lambda^{1-a} \frac{v-s}{a} \epsilon}.$$ 

(2.14)
Now the estimates at a fixed time $t > 0$, by construction:
\[
\|u(t, x) - v(t, x)\|_{H^\nu} = \left\| u_0(x + t \int u_0) - v_0(x + t \int v_0) \right\|_{H^\nu} + O_{H^\nu}(\epsilon).
\]

Now by hypothesis $\lambda \epsilon \to +\infty$ and $t \int \omega > 0$, thus $u_0(\cdot + t \int u_0)$ and $u_0(\cdot + t \int v_0)$ have disjoint supports, thus
\[
\|u(t, x) - v(t, x)\|_{H^\nu} = \left\| u_0(x + t \int u_0) \right\|_{H^\nu} + \left\| u_0(x + t \int v_0) \right\|_{H^\nu} + O_{H^\nu}(\epsilon).
\]

Now to conclude the proof we differentiate the cases:

- in the case of non uniform continuity we take $\epsilon$ such that $\epsilon e^{C\lambda (2-\alpha)} \to 0$ and apply the previous estimates with $\nu = s$.
- In the case of non Lipschitz control we take $\epsilon$ such that $\lambda - 1 + \delta \epsilon \to +\infty$ and apply the previous estimates with $\nu = s - 1 + \delta$ to get
\[
\frac{\|u(t, x) - v(t, x)\|_{H^{s-1+\delta}}}{\|u(0, x) - v(0, x)\|_{H^{s-1+\delta}}} \geq \frac{C \lambda^{-1+\delta} + O(\epsilon)}{C e^{C \lambda^{-1+\delta}(2-\alpha)}} \geq C \lambda^{-1+\delta} \epsilon^{-1} + O(1),
\]
which gives the desired result.

### 3 Flow Map Regularity for the Periodic Gravity–Capillary Equation

#### 3.1 Prerequisites from the Cauchy Problem

We start by recalling the a priori estimates given by Proposition 5.2 of [2] combined with the results of [5]. We keep the notations of Theorem 1.6.

**Proposition 3.1** (From [2] and [5]) Consider a real number $s > 2 + \frac{1}{2}$. Then there exists a non decreasing function $C$ such that, for all $T \in [0, 1]$ and all solution $(\eta, \psi)$ of (1.8) such that:

\[
(\eta, \psi) \in C^0([0, T]; H^{s+\frac{1}{2}}(\mathbb{T}) \times H^s(\mathbb{T})) \quad \text{and} \quad (H_t) \text{ is verified for } t \in [0, T],
\]

we have:
\[
\| (\eta, \psi) \|_{L^\infty(0, T; H^{s+\frac{1}{2}} \times H^s)} \leq C \left( (\| \eta_0, \psi_0 \|_{H^{s+\frac{1}{2}} \times H^s}) + TC \left( (\| \eta, \psi \|_{L^\infty(0, T; H^{s+\frac{1}{2}} \times H^s)}) \right) \right).
\]
The proof will relies on the para-linearized and symmetrized version of (1.8) given by Proposition 4.8 and corollary 4.9 of [2] which are valid on $\mathbb{T}$ as shown in [5]. Before we recall this, for clarity as in [2], we introduce a special class of operators $\Sigma^m \subset \Gamma_0^m$ given by:

**Definition 3.1** *(From [2, §4])* Given $m \in \mathbb{R}$, $\Sigma^m$ denotes the class of symbols $a$ of the form

$$ a = a^{(m)} + a^{(m-1)}, $$

with,

$$ a^{(m)} = F(\partial x \eta(t, x), \xi), $$

$$ a^{(m-1)} = \sum_{|k|=2} G_\alpha(\partial x \eta(t, x), \xi) \partial_x^k \eta(t, x), $$

such that

1. $T_a$ maps real valued functions to real-valued functions;
2. $F$ is of class $C^\infty$ real valued function of $(\zeta, \xi) \in \mathbb{R} \times (\mathbb{Z} \setminus 0)$, homogeneous of order $m$ in $\xi$; and such that there exists a continuous function $K = K(\xi) > 0$ such that

$$ F(\zeta, \xi) \geq K(\xi) \sqrt{\xi}^m, $$

for all $(\zeta, \xi) \in \mathbb{R} \times (\mathbb{Z} \setminus 0)$;
3. $G_\alpha$ is a $C^\infty$ complex valued function of $(\zeta, \xi) \in \mathbb{R} \times (\mathbb{Z} \setminus 0)$, homogeneous of order $m - 1$ in $\xi$.

$\Sigma^m$ enjoys all the usual symbolic calculus properties in the sense of Proposition A.2 modulo acceptable remainders that we define by the following:

**Definition-Notation 3.1** *(From [2, Def 4.2])* Let $m \in \mathbb{R}$ and consider two families of operators of order $m,$

$$ \{ A(t) : t \in [0, T] \}, \quad \{ B(t) : t \in [0, T] \}. $$

We shall say that $A \sim_B B$ if $A - B$ is of order $m - \frac{3}{2}$ and satisfies the following estimate: for all $\mu \in \mathbb{R},$ there exists a continuous function $C$ such that for all $t \in [0, T],$

$$ \| A(t) - B(t) \|_{H^\mu \to H^{\mu - m + \frac{3}{2}}} \leq C \left( \| \eta(t) \|_{H^{\mu + \frac{1}{2}}} \right). $$

In the next proposition we recall the different symbols that appear in the para-linearization and symmetrization of the water waves equations.
Proposition 3.2 (From [2, §4.2]) We work under the hypothesis of Proposition 3.1. Put

\[ \lambda = \lambda^{(1)} + \lambda^{(0)}, \quad l = l^{(2)} + l^{(1)} \]

with,

\[
\begin{align*}
\lambda^{(1)} &= |\xi|, \\
\lambda^{(0)} &= \frac{1 + |\partial_x \eta|^2}{2|\xi|^2} \left\{ \partial_x \left( \alpha^{(1)} \right) \partial_x \eta + i \frac{\xi}{|\xi|} \partial_x \alpha^{(1)} \right\}, \\
\alpha^{(1)} &= \frac{1}{\sqrt{1 + |\partial_x \eta|^2}} \left( |\xi| + i \partial_x \eta \xi \right), \\
l^{(2)} &= (1 + |\partial_x \eta|^2)^{\frac{3}{2}} \xi^2, \\
l^{(1)} &= -\frac{i}{2} (\partial_x \cdot \partial_\xi) l^{(2)}.
\end{align*}
\]  

(3.1)

(3.2)

Now let \( q \in \Sigma^0, \ p \in \Sigma^\frac{1}{2}, \ \gamma \in \Sigma^\frac{3}{2} \) be defined by

\[
\begin{align*}
q &= (1 + |\partial_x \eta|^2)^{-\frac{1}{2}}, \\
p &= \left( 1 + |\partial_x \eta|^2 \right)^{-\frac{5}{4}} \left| \xi \right|^\frac{1}{2} + p(-\frac{1}{2}), \\
:= p'(-\frac{1}{2}) \\
\gamma &= \sqrt{l^{(2)} \lambda^{(1)}} + \sqrt{\frac{l^{(2)}}{\lambda^{(1)}}} \Re \lambda^{(0)} - \frac{i}{2} (\partial_\xi \cdot \partial_x) \sqrt{l^{(2)} \lambda^{(1)}}, \\
\gamma' &= \gamma^\prime(-\frac{1}{2}) \\
p'(-\frac{1}{2}) &= \frac{1}{\gamma'(-\frac{1}{2})} \left\{ q l^{(1)} - \gamma'(\frac{1}{2}) p'(\frac{1}{2}) + i \partial_\xi \gamma'(\frac{1}{2}) \cdot \partial_x p'(\frac{1}{2}) \right\}.
\end{align*}
\]

Then

\[ T_q T_\lambda \sim T_\gamma T_q, \quad T_q T_l \sim T_\gamma T_p, \quad T_\gamma \sim (T_\gamma)^*, \]

where \((T_\gamma)^*\) is the adjoint of \(T_\gamma\).

Now we can write the paralinearization and symmetrization of the Eq. (1.8) after a change of variable:

Corollary 3.1 (From [2, Corollary 4.9]) Under the hypothesis of Proposition 3.1, introduce the unknowns,\(^2\)

\[ U = \psi - T_B \eta, \quad \Phi_1 = T_p \eta \quad \text{and} \quad \Phi_2 = T_q U, \]

\(^2\) U is commonly called the "good" unknown of Alinhac. Introduced by Alazard-Metivier in [1] following earlier works by Lannes in [26].
where we recall,
\[
\begin{align*}
B &= (\partial_y \phi)\big|_{y=\eta} = \frac{\partial_x \eta \cdot \partial_x \psi + G(\eta)\psi}{1 + (\partial_x \eta)^2}, \\
V &= (\partial_x \phi)\big|_{y=\eta} = \partial_x \psi - B \partial_x \eta.
\end{align*}
\]

Then \( \Phi_1, \Phi_2 \in C^0([0, T]; H^s(\mathbb{T})) \) and
\[
\begin{align*}
\partial_t \Phi_1 + TV \times \partial_x \Phi_1 - T \gamma \Phi_2 &= f_1, \\
\partial_t \Phi_2 + TV \times \partial_x \Phi_2 + T \gamma \Phi_1 &= f_2,
\end{align*}
\]
with \( f_1, f_2 \in L^\infty(0, T; H^s(\mathbb{T})) \), and \( f_1, f_2 \) have dependence on \( (\Phi_1, \Phi_2) \) verifying:
\[
\| (f_1, f_2) \|_{L^\infty(0, T; H^s(\mathbb{T}))} \leq C \left( \| (\eta, \psi) \|_{L^\infty(0, T; H^{s+\frac{1}{2}} \times H^s(\mathbb{T}))} \right).
\]

3.2 Proof of Theorem 1.6

Corollary 3.1 shows that the paralinearization and symmetrization of the equations (1.8) are of the form of the equations treated in Theorem 1.3, so the proof will follow the same lines but with more care in treating the non-linearity in the dispersive term.

We keep the notations of Theorem 1.6, fixing \( (\eta_0, \psi_0) \in H^{s+\frac{1}{2}}(\mathbb{T}) \times H^s(\mathbb{T}) \) and \( r > 0 \). We begin by taking \( (\tilde{\eta}_0, \tilde{\psi}_0) \in \mathcal{B}((\eta_0, \psi_0), r) \subset H^{s+\frac{1}{2}}(\mathbb{T}) \times H^s(\mathbb{T}) \) and consider the solutions \( (\eta, \psi), (\tilde{\eta}, \tilde{\psi}) \in C_0([0, T]; H^{s+\frac{1}{2}}(\mathbb{T}) \times H^s(\mathbb{T})) \) to (1.2) with initial data \( (\eta_0, \psi_0), (\tilde{\eta}_0, \tilde{\psi}_0) \), on a uniform small interval \([0, T]\) where the hypothesis \( (H_t) \) is also supposed to be verified. Define the following change of variables:
\[
\chi(t, x) = \int_0^x \frac{1}{\sqrt{1 + (\partial_x \eta(t, y))^2}} \, dy - \int_0^t \int_{\Sigma_s} \left[ \frac{1}{1 + (\partial_x \eta(s, y))^2} + \partial_x \phi \right] \, d\Sigma_s \, ds
\]
\[
= \int_0^x \sqrt{1 + (\partial_x \eta(t, y))^2} \, dy - \int_0^t \int_0^{2\pi} \left[ \frac{1}{1 + (\partial_x \eta)^2} + \partial_x \phi \right] \sqrt{1 + (\partial_x \eta)^2} \, dy \, ds
\]
(3.4)

and \( \tilde{\chi} \) is defined analogously from \( (\tilde{\eta}, \tilde{\psi}) \).

The main goal of the proof is to show the following estimate:
\[
\|(\eta, \psi)^* (t, \cdot) - (\tilde{\eta}, \tilde{\psi})^* (t, \cdot)\|_{H^s \times H^{s+\frac{1}{2}}} 
\leq C \left( \| (\eta_0, \psi_0, \tilde{\eta}_0, \tilde{\psi}_0) \|_{H^{s+\frac{1}{2}}(\mathbb{T}) \times H^s(\mathbb{T})} \right) \| (\eta_0, \psi_0)^* - (\tilde{\eta}_0, \tilde{\psi}_0)^* \|_{H^s \times H^{s+\frac{1}{2}}},
\]
(3.5)
where $\ast$ and $\tilde{\ast}$ are the paracomposition operators defined by $\chi$ and $\tilde{\chi}$ respectively. We recall that the definition of the paracomposition operator is given in Sect. A.3.

Put $\Phi = (\Phi_1, \Phi_2)$ the unknowns obtained from $(\eta, \psi)$ after paralinearization and symmetrization of the equations as in Corollary 3.1. Define analogously $\tilde{\Phi} = (\tilde{\Phi}_1, \tilde{\Phi}_2)$ from $(\tilde{\eta}, \tilde{\psi})$. Let us notice that, in order to prove (3.5), it suffice to get estimates on $\Phi - \tilde{\Phi}$. Indeed we write

$$\begin{cases}
\Phi_1^* = T_p^* \eta^* \\
\Phi_2^* = T_q^* U^*
\end{cases}
\quad \text{and} \quad
\begin{cases}
\eta^* = T_{\frac{1}{p^*}} \Phi_1^* - (T_{\frac{1}{p^*}} T_p^* - Id) \eta^* \\
U^* = T_{\frac{1}{q^*}} \Phi_2^* - (T_{\frac{1}{q^*}} T_q^* - Id) U^*
\end{cases}$$

then by the ellipticity of the symbols $p$ and $q$ combined with the immediate $L^2$ estimates (as $s > 2 + \frac{1}{2}$) we have:

$$\| (\eta, \psi)^*(t, \cdot) - (\tilde{\eta}, \tilde{\psi})^*(t, \cdot) \|_{H^s \times H^{s-\frac{1}{2}}} \leq C \left( \left\| (\eta_0, \psi_0, \tilde{\eta}_0, \tilde{\psi}_0) \right\|_{H^{s+\frac{1}{2}} \times H^s} \right) \left\| \Phi^*(t, \cdot) - \tilde{\Phi}^*(t, \cdot) \right\|_{H^{s-\frac{1}{2}} \times H^{s-\frac{1}{2}}} \quad (3.6)$$

$$\| \Phi^*(t, \cdot) - \tilde{\Phi}^*(t, \cdot) \|_{H^{s-\frac{1}{2}} \times H^{s-\frac{1}{2}}} \leq C \left( \left\| (\eta_0, \psi_0, \tilde{\eta}_0, \tilde{\psi}_0) \right\|_{H^{s+\frac{1}{2}} \times H^s} \right) \left\| (\eta, \psi)^*(t, \cdot) - (\tilde{\eta}, \tilde{\psi})^*(t, \cdot) \right\|_{H^s \times H^{s-\frac{1}{2}}} \quad (3.7)$$

### 3.2.1 Gauge Transform

Again, as $s > 2 + \frac{1}{2}$ we have an immediate $L^2$ estimates on $\Phi - \tilde{\Phi}$, thus we only need to get $\dot{H}^{s-\frac{1}{2}} \times \dot{H}^{s-\frac{1}{2}}$ estimates. Let us start by writing $\Phi = \Phi_1 + i \Phi_2$ in Eq. (3.3):

$$\partial_t \Phi + T_V \cdot \partial_x \Phi + iT_\gamma \Phi = R_1(\Phi) \Phi, \quad (3.8)$$

Where $R_1$ verifies

$$\begin{cases}
\left\| R_1(\Phi) \right\|_{H^{s-\frac{1}{2}} \rightarrow H^{s-\frac{1}{2}}} \leq C \left( \left\| (\eta_0, \psi_0, \tilde{\eta}_0, \tilde{\psi}_0) \right\|_{H^{s+\frac{1}{2}} \times H^s} \right), \\
\left\| [R_1(\Phi) - R_1(\tilde{\Phi})] \tilde{\Phi} \right\|_{H^{s-\frac{1}{2}}} \leq C \left( \left\| (\eta_0, \psi_0, \tilde{\eta}_0, \tilde{\psi}_0) \right\|_{H^{s+\frac{1}{2}} \times H^s} \right) \left\| \Phi - \tilde{\Phi} \right\|_{H^{s-\frac{1}{2}}}
\end{cases}$$

Indeed the first estimate on $R_1$ is a rephrasing of Corollary 3.1. The difference estimates on $R_1$ follow from the fact that the change of variable to paralinearize and symmetrize the water waves system involves only taking paraproducts and Fourier multipliers which is a Lipschitz operation under sufficient regularity which is the case here given the hypothesis $s > 2 + \frac{1}{2}$. This is immediate for all of the terms in (1.8) except for the Dirichlet–Neumann operator for which this follows from Proposition 3.14 of [2].
The next step is to preform the change of variable by $\chi$, by Theorem A.5 we get:
\[
\partial_t \Phi^* + T_W \cdot \partial_X \Phi^* + i T_W \chi_T \dot{\Phi}^* = (R_1')^* \Phi^*, \quad \text{with} \quad W = \frac{V \circ \chi}{\partial_X \chi} \text{ and } \int T W = 0. \tag{3.9}
\]
Where $R_1' = (R_1)^* + T W \chi_T = (R_1)^* \Phi^*$ where $(R_1)^*$ and $(R_1)^*$ are the pull-back by $\chi$, Then $R_1'$ verifies
\[
\left\| (R_1^*)^* - (R_1^*) \right\|_{H^{s-\frac{1}{2}}} \leq C \left\| (\eta_0, \psi_0, \tilde{\eta}_0, \tilde{\psi}_0) \right\|_{H^{s+\frac{1}{2} + \frac{1}{2}}} \left\| \Phi^* - \Phi^* \right\|_{H^{s-\frac{1}{2}}}.\]

We get the same equation on $\tilde{\Phi}$ by symmetry. The difference estimates on $R_1'$ follows from the fact that $R_1'$ is the pullbacks of $R_1$ by the paracomposition of $\chi$ and the structure of $R_1$ noted above.

Introduce the following gauge transform $e^{iT_{p\phi}}$ as the time one of the flow map defined by Propositions 4.1 with

\[
p_{\phi} = \frac{2}{3} |\xi|^2 \partial_x^{-1} W \in \Gamma_2^{2-\alpha}(\mathbb{T}),
\]
and put
\[
\theta = e^{iT_{p\phi}} \Phi^*. \tag{3.10}
\]

We define analogously $e^{iT_{p\phi}}$ and $\tilde{\theta}$ from $\tilde{\Phi}^*$. From Proposition 4.1 the change of variable (3.10) is Lipschitz from $H^{s-\frac{1}{2}}$ to $H^{s-\frac{1}{2}}$ but under $H^s$ control on $(\Phi, \tilde{\Phi})$ which is equivalent by Theorem A.2 to a control on $\left\| (\eta_0, \psi_0, \tilde{\eta}_0, \tilde{\psi}_0) \right\|_{H^{s+\frac{1}{2} + \frac{1}{2}} \times H^s}$. We have:
\[
\left\| \Phi^*(t, \cdot) - \Phi^*(t, \cdot) \right\|_{H^{s-\frac{1}{2}}} \leq C \left\| (\eta_0, \psi_0, \tilde{\eta}_0, \tilde{\psi}_0) \right\|_{H^{s+\frac{1}{2} + \frac{1}{2}}} \left\| \theta(t, \cdot) - \tilde{\theta}(t, \cdot) \right\|_{H^{s-\frac{1}{2}}}, \tag{3.11}
\]

and for $\left\| (\eta_0, \psi_0, \tilde{\eta}_0, \tilde{\psi}_0) \right\|_{H^{s+\frac{1}{2} + \frac{1}{2}} \times H^s}$ sufficiently small
\[
\left\| \theta(t, \cdot) - \tilde{\theta}(t, \cdot) \right\|_{H^{s-\frac{1}{2}}} \leq C \left\| (\eta_0, \psi_0, \tilde{\eta}_0, \tilde{\psi}_0) \right\|_{H^{s+\frac{1}{2} + \frac{1}{2}}} \left\| \Phi^*(t, \cdot) - \tilde{\Phi}^*(t, \cdot) \right\|_{H^{s-\frac{1}{2}}}. \tag{3.12}
\]

To get the equations on $\theta$ and $\tilde{\theta}$ we commute $e^{iT_{p\phi}}$ and $e^{iT_{\tilde{p}\phi}}$ with (3.9), we make the computations for $\theta$, those for $\tilde{\theta}$ are obtained by symmetry:
\[
e^{iT_{p\phi}} \partial_t \Phi^* + e^{iT_{p\phi}} T_W \cdot \partial_X \Phi^* + i e^{iT_{p\phi}} T_W \cdot \partial_X \chi \Phi^* + i e^{iT_{p\phi}} T_W \cdot \partial_X \Phi^* + i e^{iT_{p\phi}} T_W \cdot \partial_X \Phi^* + \left( e^{iT_{p\phi}} T_W \partial_X - \left[ i T \right] \frac{1}{2} e^{iT_{p\phi}} \right) \Phi^* - (\partial_t e^{iT_{p\phi}}) \Phi^* = e^{iT_{p\phi}} R_1' \Phi^*.
\]
By the definition of $p_\Phi$ and Proposition 4.4 we have:

$$\partial_\xi (|\xi|^{\frac{3}{2}}) \partial_x p_\Phi = W_\xi \quad \text{and} \quad \partial_t e^{iTp_\Phi} = e^{iTp_\Phi} \int_0^1 e^{-irTp_\Phi} T_0 i\partial p_\Phi e^{irTp_\Phi} \, dr.$$ 

thus by Corollary 4.2 we get:

$$\partial_t \theta + iT \frac{\theta}{|\xi|^{\frac{3}{2}}} = R_2(\theta) \theta + e^{iTp_\Phi} R_1(\Phi^*) e^{-iTp_\Phi} \Phi^*, \quad \text{with} \quad R_2(\theta) = -(e^{iTp_\Phi} T_0 \partial_x - [iT \frac{3}{2}, e^{iTp_\Phi}]) e^{-iTp_\Phi} \frac{1}{2}\int_0^1 e^{-irTp_\Phi} T_0 i\partial p_\Phi e^{irTp_\Phi} \, dr.$$ 

Now as in the case of the model problem above by Corollary 4.2 and 4.3, $R_2$ and $e^{iTp_\Phi} R_1(\Phi) e^{-iTp_\Phi}$ and as $s > 3 + \frac{1}{2}$ they verify:

$$\|\text{Re}(R_2(\theta))\|_{H^{s-\frac{1}{2}}} \leq C \left( \left\| (\eta_0, \psi_0, \tilde{\eta}_0, \tilde{\psi}_0) \right\|_{H^{s+\frac{1}{2}} \times H^s} \right),$$

$$\| [R_2(\theta) - R_2(\tilde{\theta})] \tilde{\theta} \|_{H^{1-\frac{1}{2}}} \leq C \left( \left\| (\eta_0, \psi_0, \tilde{\eta}_0, \tilde{\psi}_0) \right\|_{H^{s+\frac{1}{2}} \times H^s} \right) \| \tilde{\theta}(t, \cdot) - \tilde{\theta}(t, \cdot) \|_{H^{1-\frac{1}{2}}},$$

and,

$$\| [e^{iTp_\Phi} R_1^*(\Phi^*) e^{-iTp_\Phi} - e^{iTp_\Phi} R_1^*(\tilde{\Phi}^*) e^{-iTp_\Phi} ] \tilde{\theta} \|_{H^{1-\frac{1}{2}}} \leq C \left( \left\| (\eta_0, \psi_0, \tilde{\eta}_0, \tilde{\psi}_0) \right\|_{H^{s+\frac{1}{2}} \times H^s} \right) \| \tilde{\theta}(t, \cdot) - \tilde{\theta}(t, \cdot) \|_{H^{1-\frac{1}{2}}}.$$ 

Thus we have succeeded to eliminate the term $T_V \cdot \partial_x$ of order 1 in (3.9) and got a term of order $\frac{1}{2}$. The result then follows by an energy estimate as in the model problem under the smallness hypothesis of $\left\| (\eta_0, \psi_0, \tilde{\eta}_0, \tilde{\psi}_0) \right\|_{H^{s+\frac{1}{2}} \times H^s}$. To get rid of the smallness hypothesis the proof follows as in the case of the model where we the the difference with a well understood background solution.

### 4 Baker–Campbell–Hausdorff Formula: Composition and Commutator Estimates

We will start by giving the propositions defining the operators used in the gauge transforms and the symbolic calculus associated to them. From those propositions we will deduce the direct estimates used in Sects. 2.1.1 and 3.2.1.

**Notation 4.1** To compute the conjugation and commutation of operators with a flow map, we introduce Lie derivatives, i.e. commutators. More precisely we introduce the following notations for commutation between operators:

$$\mathcal{L}_a^0 b = b, \quad \mathcal{L}_a b = [a, b] = a \circ b - b \circ a, \quad \mathcal{L}_a^2 b = [a, [a, b]],$$
In the following proposition the variable $t \in [0, T]$ is the generic time variable that appeared in the previous section and a new variable $\tau \in \mathbb{R}$ will be used and they should not be confused.

**Remark 4.1** For clarity in this section and the appendix we will present the symbolic calculus on $\mathbb{R}$. All the results stated here extend tautologically to the case of $\mathbb{T}$ by applying the following rules. An integral on $Z$, i.e $\int_Z$, should be understood as $\sum_Z$. A function $a$ is said to be in $C^\infty(\mathbb{T} \times \mathbb{Z})$ if for every $\xi \in \mathbb{Z}$, $a(\cdot, \xi) \in C^\infty(\mathbb{T})$. The partial derivative $\partial_\xi$ should be understood as the forward difference operator, i.e $\partial_\xi a(\xi) = a(\xi + 1) - a(\xi), \xi \in \mathbb{Z}$.

We recall the following simple identities for the Fourier transform on the Torus:

\[
\begin{align*}
\mathcal{F}_\mathbb{T}(\partial^\alpha_x f)(\xi) &= \xi^\alpha \mathcal{F}_\mathbb{T}(f)(\xi), \xi \in \mathbb{Z}, \\
\mathcal{F}_\mathbb{T}((e^{-2i\pi x} - 1)^\alpha f)(\xi) &= \xi^\alpha \mathcal{F}_\mathbb{T}(f)(\xi), \xi \in \mathbb{Z}.
\end{align*}
\]

We start with the proposition defining the flow map and its standard properties.

**Proposition 4.1** Consider two real numbers $\delta < 1$, $s \in \mathbb{R}$ and a real valued symbol $p \in \Gamma^\delta_1(\mathbb{R})$. The following linear hyperbolic equation is globally well-posed:

\[
\begin{align*}
\partial_\tau h - iT_p h &= 0, \\
h(0, \cdot) &= h_0(\cdot) \in H^s(\mathbb{R}).
\end{align*}
\]

For $\tau \in \mathbb{R}$, define $e^{i\tau T_p}$ as the flow map associated to 4.1 i.e:

\[
e^{i\tau T_p} : H^s(\mathbb{R}) \to H^s(\mathbb{R})
\]

\[
h_0 \mapsto h(\tau, \cdot).
\]

Then for $\tau \in \mathbb{R}$ we have,

1. $e^{i\tau T_p} \in \mathcal{L}(H^s(\mathbb{R}))$ and

\[
\left\| e^{i\tau T_p} \right\|_{H^s \to H^s} \leq e^{C|\tau| |M^\delta_1(p)|}.
\]

2. $i T_p \circ e^{i\tau T_p} = e^{i\tau T_p} \circ i T_p, e^{i(\tau + \tau') T_p} = e^{i\tau T_p} e^{i\tau' T_p}$.

3. $e^{i\tau T_p}$ is invertible and,

\[
(e^{i\tau T_p})^{-1} = e^{-i\tau T_p}.
\]
Moreover,

\[(e^{i \tau T_p})^* = e^{-i \tau (T_p)^*} = e^{-i \tau T_p} + R,\]

where \(R\) is a \(\delta - 1\) regularizing operator and \(e^{i \tau (T_p)^*}\) is the flow generated by the Cauchy problem:

\[
\begin{align*}
\partial_\tau h - i (T_p)^* h &= 0, \\
h(0, \cdot) &= h_0(\cdot) \in H^s(\mathbb{R}).
\end{align*}
\]  

(4.3)

(4) Taking a real valued symbol \(\tilde{p} \in \Gamma_1^\delta(\mathbb{R})\) we have:

\[
\| [e^{i \tau T_p} - e^{i \tau \tilde{T}_p}] h_0 \|_{H^s} \leq C |\tau| e^{|\tau| M_1^\delta(p, \tilde{p})} M_0^\delta(p - \tilde{p}) \| h_0 \|_{H^{s+\delta}}. 
\]  

(4.4)

**Proof** To prove point (1) we commute \(\langle D \rangle^s\) with the equation and make an energy estimate to get:

\[
\frac{d}{dt} \| h(t, \cdot) \|_{H^s}^2 \leq \left[ \| (\langle D \rangle^s, i T_p) \|_{H^s \rightarrow H^s} + \| T_p - (T_p)^* \|_{H^s \rightarrow H^s} \right] \| h(t, \cdot) \|_{H^s}^2,
\]

From Appendix A.2 we have as \(\delta \leq 1\) and \(p \in \Gamma_1^\delta(\mathbb{R})\)

\[
\| (\langle D \rangle^s, i T_p) \|_{H^s \rightarrow H^s} \leq K M_1^\delta(p),
\]

moreover \(p\) being real valued we have

\[
\| T_p - (T_p)^* \|_{H^s \rightarrow H^s} \leq K M_1^\delta(p).
\]

The desired estimate then follows from the Gronwall lemma. The identities in point (2) and the inverse and adjoint identities in point (3) are the standard algebraic identities for semi-groups. As for the residual term \(R\) in point (3) it is given explicitly by

\[
R = i \int_0^\tau e^{i (r - \tau) T_p} [T_p - (T_p)^*] e^{-i r (T_p)^*} dr.
\]

and \(H^s \rightarrow H^{s+\delta-1}\) estimate again follows from Appendix A.2. Point (4) comes by writing:

\[
\partial_\tau [e^{i \tau T_p} - e^{i \tau \tilde{T}_p}] h_0 - i T_p [e^{i \tau T_p} - e^{i \tau \tilde{T}_p}] h_0 = i T_p - \tilde{p} e^{i \tau \tilde{T}_p} h_0,
\]

and making the usual energy estimate. \(\square\)
The hypothesis \( p \in \Gamma^p \) can be relaxed to \( p \in \Gamma \), which is the minimal hypothesis needed to ensure well posedness in Sobolev spaces of (4.1).

At the moment the only bounds we obtained on \( e^{ixT_p} \) are the continuity bounds on Sobolev spaces, in order to study it’s symbol and the symbol of conjugated operators we need to transfer those continuity bounds to estimates on the symbol’s seminorms. This was first done by Beals in [8] for pseudodifferential operators in the class \( S_{\rho,p}^m \) with \( \rho < 1, m \in \mathbb{R} \). The following lemma gives explicitly the key estimate adapted from [8] given by (4.5), and we give one new estimate (4.6) that can then be directly applied to the paradifferential setting.

**Lemma 4.1** Consider an operator \( A \) continuous from \( \mathcal{S}(\mathbb{R}) \rightarrow \mathcal{S}'(\mathbb{R}) \) \( (C^\infty(\mathbb{T}) \rightarrow \mathcal{S}(\mathbb{T})) \) and let \( a \in \mathcal{S}'(\mathbb{R} \times \mathbb{R}) \) \( (\mathcal{S}(\mathbb{T} \times \mathbb{Z}) \rightarrow \mathcal{S}(\mathbb{T})) \) be the unique symbol associated to \( A \) (cf [9] for the uniqueness), that is if you let \( K \) be the kernel associated to \( A \) then for \( u, v \in \mathcal{S}(\mathbb{R}) \) \( (C^\infty(\mathbb{T}) \rightarrow \mathcal{S}(\mathbb{T})) \) in the case of \( \mathbb{T} \)

\[
(Au, v) = K(u \otimes v) \quad \text{and} \quad a(x, \xi) = \mathcal{F}_{y \rightarrow \xi} K(x, x - y).
\]

- If \( A \) is continuous from \( H^m \) to \( L^2 \), with \( m \in \mathbb{R} \), and \( \left[ \frac{1}{i} \frac{d}{dx} \right] A \) is continuous from \( H^{m+\delta} \) to \( L^2 \) with \( \delta \in (-\infty, 1) \), then \( 1 + |\xi| \right)^{-m} a(x, \xi) \in L^\infty_{x,\xi} (\mathbb{R}^2) \) \( (L^\infty_{x,\xi} (\mathbb{T} \times \mathbb{Z})) \) in the case of \( \mathbb{T} \) and we have the estimate:

\[
\left\| (1 + \xi)^{-m} a \right\|_{L^\infty_{x,\xi}} \leq C_m \left[ \left\| A \right\|_{H^m \rightarrow L^2} + \left\| \left[ \frac{1}{i} \frac{d}{dx} \right] A \right\|_{H^{m+\delta} \rightarrow L^2} \right].
\]  

(4.5)

- If \( A \) is continuous from \( H^m \) to \( L^2 \), with \( m \in \mathbb{R} \), and \( [ix, A] \) is continuous from \( H^{m-\rho} \) to \( L^2 \) with \( \rho \geq 0 \), then \( 1 + |\xi| \right)^{-m} a(x, \xi) \in L^\infty_{x,\xi} (\mathbb{R}^2) \) \( (L^\infty_{x,\xi} (\mathbb{T} \times \mathbb{Z})) \) in the case of \( \mathbb{T} \) and we have the estimate:

\[
\left\| (1 + \xi)^{-m} a \right\|_{L^\infty_{x,\xi}} \leq C_m \left[ \left\| A \right\|_{H^m \rightarrow L^2} + \left\| [ix, A] \right\|_{H^{m-\rho} \rightarrow L^2} \right].
\]  

(4.6)

**Proof** First without loss of generality through a standard mollification argument we work with \( a \in \mathcal{S}(\mathbb{R}^2) \) \( (C^\infty(\mathbb{T} \times \mathbb{Z})) \) in the case of \( \mathbb{T} \). We study the cases on \( \mathbb{T} \) and \( \mathbb{R} \) separately.

**Operators defined on \( \mathbb{T} \)** The first key observation is the following:

\[
(x, \xi) \in \mathbb{T} \times \mathbb{Z}, \quad e^{-ix\xi} A e^{ix\xi} = a(x, \xi),
\]  

(4.7)

which one can write as \( e^{ix\xi} \in L^2_x(\mathbb{T}) \). Thus taking \( L^2 \) norms in \( x \) we get:

\[
\| a(\cdot, \xi) \|_{L^2_x} \leq \| A \|_{H^m \rightarrow L^2} \| e^{ix\xi} \|_{H^m_x} \leq C_m \| A \|_{H^m \rightarrow L^2} (1 + |\xi|)^m.
\]  

(4.8)
Now to get the analogue of (4.8) but in the $\xi$ variable we observe that the continuity hypothesis reads for $(u, v) \in \mathcal{F}$:

$$\left| \int_{\mathbb{T} \times \mathbb{Z}} e^{ix \cdot \xi} a(x, \xi)(1 + |\xi|)^{-m} \mathcal{F}(v)(\xi)u(x)dx d\xi \right| \leq \| A \|_{H^m \rightarrow L^2} \| \mathcal{F}(v) \|_{L^2_{\xi}} \| u \|_{L^2_x}.$$  

Thus for all $u \in L^2(\mathbb{T})$

$$\left\| \int_{\mathbb{T}} e^{ix \cdot \xi} (1 + |\xi|)^{-m} a(x, \xi)u(x)dx \right\|_{L^2_{\xi}} \leq C_m \| A \|_{H^m \rightarrow L^2} \| u \|_{L^2_x}.$$  

Fixing $x_0 \in \mathbb{T}$ and choosing $u(x) = \mathbb{1}_{x_0 \pm \epsilon}$, for $\epsilon$ sufficiently small we get by continuity of $a$ the analogous estimate to (4.8):

$$\left\| (1 + |\xi|)^{-m} a(x, \xi) \right\|_{L^\infty_{x}L^2_{\xi}} \leq C_m \| A \|_{H^m \rightarrow L^2}.$$  

The second key observation is:

$$e^{-ix \cdot \xi} \left[ \frac{1}{i} \frac{d}{dx}, A \right] e^{ix \cdot \xi} = \partial_x a(x, \xi), \quad e^{-ix \cdot \xi} [ix, A] e^{ix \cdot \xi} = \partial_{\xi} a(x, \xi).$$  

Making the previous computations again with $\partial_x a$ and $\partial_{\xi} a$ instead of $a$ we get:

$$\left\| \partial_x a(\cdot, \xi) \right\|_{L^2_{x}} \leq C_m (1 + |\xi|)^{m+\delta} \left\| \left[ \frac{1}{i} \frac{d}{dx}, A \right] \right\|_{H^{m+\delta} \rightarrow L^2_x},$$  

and as $\rho \geq 0$:

$$\left\| \partial_{\xi} [(1 + |\xi|)^{-m} a(x, \xi)] \right\|_{L^\infty_{x}L^2_{\xi}} \leq C_m [\| A \|_{H^m \rightarrow L^2} + \| [ix, A] \|_{H^{m-\rho} \rightarrow L^2_x}].$$  

By the Sobolev embedding (4.9) and (4.12) give the desired result (4.6) on the torus. To get (4.5) we introduce as in [8]:

$$b(x, \xi, \xi_0) = a \left( \frac{x}{(1 + |\xi_0|)^{\delta}}, (1 + |\xi_0|)^{\delta} \xi \right).$$

Thus (4.8) becomes

$$\left\| b(\cdot, \xi) \right\|_{L^2_x} \leq C_m \| A \|_{H^m \rightarrow L^2} (1 + |\xi|)^m.$$  

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As \( \delta < 1 \) we have that \((1 + |\xi|)^{\delta} \sim (1 + |\xi_0|)^{\delta} \) for \(|\xi - \xi_0| \leq c(1 + |\xi_0|)^{\delta} \) for some fixed \( c > 0 \). Thus (4.11) becomes
\[
\| \partial_x b(\cdot, \xi) \|_{L^2_x} \leq C_m \left( \frac{1 + |\xi|}{(1 + |\xi_0|)^{\delta}} \right) \left\| \left[ \frac{1}{i} \frac{d}{dx}, A \right] \right\|_{H^{m+\delta} \to L^2_x}^{H^{m+\delta} \to L^2_x},
\]

Considering \( b \) as a function of \((x, \xi)\) on \( T \times B(\xi_0, c) \) by the Sobolev embedding we get:
\[
\| b(x, \xi) \|_{L^\infty_x} \leq C_m \left[ \| A \|_{H^m \to L^2} + \left\| \left[ \frac{1}{i} \frac{d}{dx}, A \right] \right\|_{H^{m+\delta} \to L^2} \right](1 + |\xi|)^m, \tag{4.13}
\]
which transferred back to \( a \) give the desired result (4.5) on the torus.

**Operators defined on** \( \mathbb{R} \) The main problem we face on \( \mathbb{R} \) when adapting the previous proof is we can no longer use \( e^{ix \cdot \xi} \) as a test function as it no longer belongs to \( L^2(\mathbb{R}) \). One way to get over this was given by Beals in [8], we choose \( g \) in \( \mathcal{S}(\mathbb{R}) \) such that \( g(0) = 1, \mathcal{F}(g) \) is supported in \( \{|\xi| \leq 1\} \) and \( g(x) = g(-x) \). Let \( g_x(y) = g(y - x) \) and compute for \( u \in \mathcal{S} \):
\[
u(x) = u(x)g_x(x) = \int_{\mathbb{R} \times \mathbb{R}} e^{i(x-y) \cdot \xi} \mathcal{F}_x(y)u(y)d\xi d\eta = \int_{\mathbb{R} \times \mathbb{R}} e^{-iy \cdot \xi} e^{ix \cdot \xi} g_y(x)u(y)d\xi d\eta.
\]

We now compute an analogue of (4.7):
\[
Au(x) = \int_{\mathbb{R} \times \mathbb{R}} e^{-iy \cdot \xi} A(e^{ix \cdot \xi} g_y)(x)u(y)d\xi d\eta
\]
\[
= \int_{\mathbb{R} \times \mathbb{R}} e^{i(x-y) \cdot \xi} a_0(x, y, \xi)u(y)d\xi d\eta
\]
\[
= \int_{\mathbb{R}} e^{ix \cdot \xi} a(x, \xi) \mathcal{F}(u)(\xi)d\xi,
\]
where,
\[
a_0(x, y, \xi) = e^{-ix \cdot \xi} A(e^{ix \cdot \xi} g_y)(x),
\]
and,
\[
a(x, \xi) = \int_{\mathbb{R} \times \mathbb{R}} e^{i(x-y) \cdot (\eta - \xi)} a_0(x, y, \eta)d\eta d\xi.
\]
Applying the same arguments as in the periodic case we get:

\[
\left\Vert (1 + |\xi|)^{-m} \partial_y^k a_0 \right\Vert_{L^\infty_{x,y,\xi}} \leq C_{m,k} \left[ \| A \|_{H^m \rightarrow L^2} + \left\| \left[ \frac{1}{i} \frac{d}{dx}, A \right] \right\|_{H^{m+\delta} \rightarrow L^2} \right] , \quad k \in \mathbb{N}
\]

and,

\[
\left\Vert (1 + |\xi|)^{-m} \partial_y^k a_0 \right\Vert_{L^\infty_{x,y,\xi}} \leq C_{m,k} \left[ \| A \|_{H^m \rightarrow L^2} + \left\| \left[ i x, A \right] \right\|_{H^{m-\rho} \rightarrow L^2} \right] , \quad k \in \mathbb{N}.
\]

Thus to conclude the proof we need to transfer the information on the amplitude \( a_0 \) to the symbol \( a \) which is a simple application of Oscillatory Integrals. Indeed it suffices to write:

\[
a(x, \xi) = \int_{\mathbb{R} \times \mathbb{R}} e^{i(x-y) \cdot (\eta - \xi)} a_0(x, y, \eta) d\eta dy
\]

\[
= \int_{\mathbb{R} \times \mathbb{R}} \frac{1}{|\xi - \eta|^2} (I - \Delta_y) e^{i(x-y) \cdot (\eta - \xi)} a_0(x, y, \eta) d\eta dy
\]

\[
= \int_{\mathbb{R} \times \mathbb{R}} \frac{1}{|\xi - \eta|^2} e^{i(x-y) \cdot (\eta - \xi)} (I - \Delta_y)^* a_0(x, y, \eta) d\eta dy,
\]

which gives the desired result as \( a_0(x, y, \eta) \) and all of it’s \( y \) derivatives are \( y \) integrable.

\[
\text{Remark 4.2} \quad \text{To get a good intuition behind Beals type estimates, it is worth noting that if instead of Sobolev continuity bounds we had Hölder continuity estimates on } A \text{ with } m \notin \mathbb{N}, \text{ then combined with the identity on the torus }
\]

\[ e^{-ix \cdot \xi} A e^{ix \cdot \xi} = a(x, \xi). \]

we get directly the analogue of estimates (4.5) and (4.6) that is

\[
\left\Vert (1 + |\xi|)^{-m} a \right\Vert_{L^\infty_{x,\xi}} \leq C_m \| A \|_{W^{m,\infty} \rightarrow L^\infty}.
\]

Analogously to Beals characterization of pseudodifferential operators through the continuity of the successive commutators \( \text{Op}(x), \frac{1}{i} \frac{d}{dx} \) with \( A \) on Sobolev spaces, we give the following characterization of Paradifferential operators through estimate (4.6).

\[
\text{Corollary 4.1} \quad \text{Consider two real numbers } m \text{ and } \rho \geq 0 \text{ and the spaces of paradifferential symbols } \Gamma^m_{\rho} (\mathbb{R}) \text{ equipped with the topology induced by the seminorms } M^m_{\rho}(\cdot; k) \text{ for } k \in \mathbb{N} \text{ defined in A.5 giving it a Fréchet space structure.}
\]
For $p \in \Gamma^m_\rho (\mathbb{R})$ we introduce the following family of seminorms:

$$H^m_0 (p; k) = \sum_{j=0}^{k} \left\| \mathcal{L}_i T_p \right\|_{H^m \to H^j},$$

$$H^m_n (p; k) = \sum_{l=0}^{n} \sum_{j=0}^{k} \left\| \mathcal{L}_i \mathcal{L}_{1,d} \mathcal{L}_i T_p \right\|_{H^m \to H^j}, \quad n \in \mathbb{N}, n \leq \rho,$$

and if $\rho \notin \mathbb{N}$:

$$H^m_\rho (p; k) = H^m_{\lfloor \rho \rfloor} (p; k) + \sup_{n \in \mathbb{N}} 2^n \left( \rho - \lfloor \rho \rfloor \right) \sum_{j=0}^{k} \left\| \mathcal{L}_i \mathcal{L}_{1,d} T_p \mathcal{P}_{\leq n} (D) \right\|_{H^m \to H^j}.$$

Then $H^m_\rho (p; k)_{k \in \mathbb{N}}$ induces an equivalent Fréchet topology to $M^m_\rho (\cdot; k)_{k \in \mathbb{N}}$ on:

$$\psi^{B,b} \left( \Gamma^m_\rho (\mathbb{R}) \right) = \left\{ \sigma^B_p, p \in \Gamma^m_\rho (\mathbb{R}) \right\}.$$

**Proof** Taking $m \in \mathbb{R}$, $\rho \geq 0$ and $p \in \Gamma^m_\rho (\mathbb{R})$ then Theorem A.1 gives for $\rho \in \mathbb{N}$

$$H^m_\rho (p; k) \leq K M^m_\rho (p; k + 1), \quad \text{for } k \in \mathbb{N},$$

and the case $\rho \geq 0$ and $\rho \in \mathbb{R} \setminus \mathbb{N}$ is obtained from the characterization of $H^\rho$ and $W^\rho, \infty$ spaces by a dyadic decomposition on balls in the frequency space given by Propositions A.3 and A.4 and the observation that there exists an $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$

$$T_p \mathcal{P}_{\leq n} (D) = T_p \mathcal{P}_{\leq n-N(D)} \mathcal{P}_{\leq n} (D)$$

where $\mathcal{P}_{\leq n-N(D)} (D)$ is applied to $p$ in the $x$ variable. Now by Lemma 4.1 for $\rho \in \mathbb{N}$

$$M^m_\rho (\sigma^B_p; k + 1) \leq H^m_\rho (p; k), \quad \text{for } k \in \mathbb{N},$$

and the case $\rho$ non integer is treated exactly as before which gives the equivalence of the desired topologies on the subspace

$$\psi^{B,b} \left( \Gamma^m_\rho (\mathbb{R}) \right) = \left\{ \sigma^B_p, p \in \Gamma^m_\rho (\mathbb{R}) \right\},$$

by noticing that $\sigma^{B,b}_p = \sigma^{B,b}_p + \sigma^{B+\epsilon,b}_{1-\psi^{B,b}_p} \psi^{B,b}_p$, thus estimates on $\sigma^{B,b}_p$ and $\sigma^{B,b}_p$ are equivalent. \qed
In order to give the key symbolic calculus results in Propositions 4.2 and 4.3 we need to introduce the paradifferential analogue of the Hörmander symbol class $S_{1-\delta,\delta}^0$. For this we follow §1.1 Chapter 1 of [43] and introduce the space of non regular symbols:

**Definition-Proposition 4.1** Consider $s \in \mathbb{R}_+$, for $0 \leq \delta, \rho < 1$, we say:

$$p \in W^{s,\infty}_{\rho,\delta}(\mathbb{R}) \iff \begin{cases} 
\| D_{\xi}^{\delta} p(\cdot, \xi) \|_{W^{s,\infty}} \leq C_{k,0} \langle \xi \rangle^{m-\rho k}, \\
\| D_{x}^{\lceil x \rceil+n} D_{\xi}^{\rho} p(x, \xi) \| \leq C_{k,n} \langle \xi \rangle^{m-\rho k+(n+\lceil x \rceil-s)\delta},
\end{cases} \quad (4.16)$$

for $k \geq 0$ and $n \geq 1$. The best constants $C_{k,n}$ in (4.16) define a family of seminorms denoted by $\rho,\delta M^m_{\rho,\delta}(\cdot; k)$, $(k, n) \in \mathbb{N}^2$ where $k$ is the number of derivatives we make on the frequency variable $\xi$ and $n$ is that in the $x$ variable. We also define the seminorm $\rho,\delta M^m_{\rho,\delta}(\cdot; k) = \rho,\delta M^m_{1,0}(\cdot; k)$ and define analogously $W^{s,\infty}_{\rho,\delta}(\mathbb{R}^* \times \mathbb{R})$ where $\mathbb{R}^* = \mathbb{R}\{0\}$.

Analogously to Corollary 4.1 we introduce the following family of seminorms:

$$\rho,\delta H^m_{0,s}(p; k) = \sum_{l=0}^{\lceil x \rceil} \sum_{j=0}^{k} \left\| \xi_{ix}^j \xi_{1/dx}^l \text{Op}(p) \right\|_{H^m \rightarrow H^j} + \sup_{n \in \mathbb{N}} \sum_{j=0}^{\lceil x \rceil} \left\| \xi_{ix}^j \xi_{1/dx}^{\lceil x \rceil} [P_n(D) \text{Op}(p)] \right\|_{H^m \rightarrow H^j},$$

where $P_n(D)$ is applied to $p$ in the $x$ variable. And for $n \geq 1$

$$\rho,\delta H^m_{n,s}(p; k) = \rho,\delta H^m_{0,s}(p; k) + \sum_{l=1}^{n} \sum_{j=0}^{k} \left\| \xi_{ix}^j \xi_{1/dx}^{\lceil x \rceil+l} \text{Op}(p) \right\|_{H^m \rightarrow H^{j-\lceil x \rceil-s)}.$$  

Then $\rho,\delta H^m_{n,s}(\cdot; k)_{(n,k) \in \mathbb{N}^2}$ induces an equivalent Fréchet topology to $\rho,\delta M^m_{n,s}(\cdot; k)_{(n,k) \in \mathbb{N}^2}$ on $W^{s,\infty}_{\rho,\delta}$.

**Proof** The $L^2$ continuity of such operators and control by their symbol seminorms is given in Appendix B of [36] and the reciprocal is given by Lemma 4.1.

**Remark 4.3** We note that for the standard Hörmander symbol classes and paradifferential symbols classes we have respectively

$$S^m_{\rho,\delta} = \cap_{s \geq 0} W^{s,\infty}_{\rho,\delta} \quad \text{and} \quad \Gamma^m_\rho = W^{\rho,\infty}_1.$$  

Now all of the ingredients are in place to give the key commutation and conjugation result.
Proposition 4.2 Consider three real numbers $\delta < 1$, $s \in \mathbb{R}$, $\rho \geq 1$, a real valued symbol $p \in \Gamma_{\rho}^{\delta}(\mathbb{R})$. Let $e^{i \tau T_{p}}$, $\tau \in \mathbb{R}$ be the flow map defined by Proposition 4.1 and take a symbol $b \in \Gamma_{\rho}^{\beta}(\mathbb{R})$ with $\beta \in \mathbb{R}$.

(5) There exists $b_{\tau}^{p} \in W^{p, \infty} S_{1-\delta, \delta}(\mathbb{R})$ such that:

$$e^{i \tau T_{p}} \circ T_{b} \circ e^{-i \tau T_{p}} = \text{Op}(b_{\tau}^{p}).$$

(4.17)

Moreover we have the estimates:

$$\left\| \text{Op}(b_{\tau}^{p}) - \sum_{k=0}^{[\rho-1]} \frac{\tau^{k}}{k!} \alpha_{i T_{p}} T_{b} \right\|_{H^{s} \rightarrow H^{s-\beta-|\rho|\delta+\rho}} \leq C_{\rho} e^{C|\tau| M_{1}^{\delta}(p) M_{\rho}^{\beta}(b) M_{\rho}^{\delta}(p)[\rho]},$$

(4.18)

$$1-\delta, \delta H_{\rho}^{\beta}(b_{\tau}^{p}; k) \leq C_{k}(M_{1}^{\delta}(p)) H_{\rho}^{\beta}(b; k) \left[ \sum_{i=0}^{k-1} H_{\rho}^{\delta}(p; k)^{k-i} \right], k \in \mathbb{N}.\quad (4.19)$$

(6) There exists $c b_{\tau}^{p} \in W^{p, 1-\infty} S_{1-\delta, \delta}(\mathbb{R})$ such that:

$$[e^{i \tau T_{p}}, T_{b}] = e^{i \tau T_{p}} \text{Op}(c b_{\tau}^{p}) \iff \text{Op}(c b_{\tau}^{p}) = T_{b} - \text{Op}(b_{-\tau}^{p}).$$

(4.20)

Moreover we have the estimates:

$$\left\| \text{Op}(c b_{\tau}^{p}) - \sum_{k=1}^{[\rho-1]} (-1)^{k-1} \frac{\tau^{k}}{k!} \alpha_{i T_{p}} T_{b} \right\|_{H^{s} \rightarrow H^{s-\beta-|\rho|\delta+\rho}} \leq C_{\rho} e^{C|\tau| M_{1}^{\delta}(p) M_{\rho}^{\beta}(b) M_{\rho}^{\delta}(p)[\rho]},$$

(4.21)

$$1-\delta, \delta H_{\rho-1}^{\beta+\delta-1}(c b_{\tau}^{p}; k) \leq C_{k}(M_{1}^{\delta}(p)) H_{\rho-1}^{\beta}(b; k) \left[ \sum_{i=0}^{k-1} H_{\rho}^{\delta}(p; k)^{k-i} \right], k \in \mathbb{N}.\quad (4.22)$$

Remark 4.4 It is important to notice that the main result of this proposition is the factorization of the $e^{i \tau T_{p}}$ terms in (4.17) and (4.20) where the right hand sides contain symbols in the usual classes modulo a more regular remainder. This was not a priori the case of the left hand sides containing $e^{i \tau T_{p}}$. In other words we prove the stability of $\Gamma_{\rho}^{m}$ under the conjugation by $e^{i \tau T_{p}}$ modulo more regular remainders.

This is crucial when studying the regularity of the flow map for:

$$s > \left\lfloor \frac{\alpha}{\alpha - 1} \right\rfloor - \frac{1}{2},$$
Indeed if \( p \) depends on a parameter \( \lambda \), \( D_\lambda e^{itT_p} \circ T_b \circ e^{-itT_p} \) is a priori an operator of order \( \beta + \delta \) by (4.4), but \( D_\lambda Op(b_p^\lambda) \) is shown in Proposition 4.5 to be an operator of order \( \beta \).

- In the language of pseudodifferential operators, \( Op(b_p^\alpha) \) is the asymptotic sum of the series \( \sum_{k=0}^\infty \frac{\tau^k}{k!} \Omega_{iT_p}^k T_b \) i.e the Baker–Campbell–Hausdorff formal series. Though \( Op(b_p^\lambda) \) is not necessarily equal to this sum, for this sum need not converge.

Proof

The structure of the proof is as follows:

(I) We will give a proof of the estimate (4.18) assuming \( b_p^\lambda \) exists.

(II) We will prove the existence of \( b_p^\lambda \in W^{\rho, \infty}_{1-\delta, \delta}(\mathbb{R}) \) which is the subtle part of the proof.

(III) Finally we will deduce point (6) from point (5).

Point (I) For point (5) we compute,

\[
\partial_\tau[e^{itT_p} \circ T_b \circ e^{-itT_p}] = i T_p \circ e^{itT_p} \circ T_b \circ e^{-itT_p} - e^{itT_p} \circ T_b \circ i T_p \circ e^{-itT_p}
\]

Using (2),

\[
\partial_\tau[e^{itT_p} \circ T_b \circ e^{-itT_p}] = e^{itT_p} \circ i T_p \circ T_b \circ e^{-itT_p} - e^{itT_p} \circ T_b \circ i T_p \circ e^{-itT_p}
\]

As \( e^{i0T_p} = Id \), integrating on \([0, \tau]\) we get:

\[
e^{itT_p} \circ T_b \circ e^{-itT_p} = T_b + \int_0^\tau e^{itT_p}[i T_p, T_b] e^{-itT_p} \, dr.
\]

Iterating the computation in \(*\) we get for \( n \in \mathbb{N}^* \),

\[
e^{itT_p} \circ T_b \circ e^{-itT_p} = \sum_{k=0}^n \frac{\tau^k}{k!} \Omega_{iT_p}^k T_b + \int_0^\tau \frac{(\tau - r)^n}{n!} e^{irT_p} \Omega_{iT_p}^{n+1} T_b \, e^{-irT_p} \, dr.
\]

Now the key point is the continuity of paradifferential operators given by Theorem A.1 and the symbolic calculus rules given by Theorem A.2. By Lemma 4.2:

\[
\| \Omega_{iT_p}^{[\rho]} T_b \|_{H^s \to H^{s-[\rho]\delta+\rho}} \leq C_\rho M^{\beta}_p(b) M^{\delta}_p(p)^{[\rho]}, \text{ for } \rho \in \mathbb{R}_+
\]

where \([\rho]\) is the upper integer part of \( \rho \).

Thus applying point (1) combined with (4.25) we get (4.18).

Point (II) The constant \( C_\rho \) in (4.25) is estimated “brutally” by Lemma 4.2: \( O(2^\rho \times [\rho]!) \), thus even though one has a \( \frac{1}{[\rho]} \) in (4.24) the convergence result is non trivial.
To get past this let us express explicitly the difficulty in the problem. Rearranging the terms in (4.23) we see that:
\[
\partial\tau \left[ e^{i\tau T_p} \circ T_b \circ e^{-i\tau T_p} \right] = e^{i\tau T_p} \left[ i T_p, T_b \right] e^{-i\tau T_p} = [i T_p, e^{i\tau T_p} T_b e^{-i\tau T_p}],
\]
(4.26)
thus we have to solve the following Cauchy problem in \( L(H^s(\mathbb{R}), H^{s-\beta}(\mathbb{R})) \):
\[
\begin{align*}
\partial\tau f(\tau) &= [iT_p, f(\tau)] \in \Gamma^{\beta}_{\rho-1}(\mathbb{R}), \\
f(0) &= T_b \in \Gamma^\beta_\rho(\mathbb{R}).
\end{align*}
\]
(4.27)
This amounts to the non trivial problem of solving a linear ODE in the Fréchet space \( \Gamma^{\beta}_{+\infty}(\mathbb{R}) \), indeed such a problem need not have a solution in general, and even if it does, it need not be unique. To solve such a problem one usually has to look at a Nash–Moser type scheme, though in our case we have an explicit ODE that can be solved with a series and a loss of derivative. Thus inspired by Hörmander’s [16], we remark another key estimate given by the continuity of paradifferential operators given by Theorem A.1 and the symbolic calculus in Theorem A.2 summarised in the following lemma.

**Lemma 4.2** Consider two real numbers \( \delta, \beta, \rho \geq 0 \), and two symbols \( p \in \Gamma_\rho^{\delta}(\mathbb{R}) \) and \( b \in \Gamma^{\beta}_\rho(\mathbb{R}) \) then there exists a constant \( C > 0 \) such that:
\[
\|\mathcal{L}_T \|_{H^s \to H^{s-\beta}} \leq C^{[\rho]+1} \left[ \frac{1}{2^{\rho}} \right] ! M^{\beta}_\rho(b) M^{\delta}_\rho(p) \left[ \frac{1}{\rho} \right], \quad \text{for } \rho \in \mathbb{R}_+,
\]
(4.28)
\[
\|\mathcal{L}_T \|_{H^s \to H^{s-\beta}} \leq C^{[\rho]} M^{\beta}_0(b) M^{\delta}_0(p) \left[ \frac{1}{\rho} \right], \quad \text{for } \rho \in \mathbb{R}_+.
\]
(4.29)

**Proof of Lemma 4.2** For (4.29), expanding \( \mathcal{L}_T \) as polynomial in \( T_p \) and \( T_b \) and counting with repetitions, we see that it contains at most \( 2^{[\rho]} \) terms of the form:
\[
\pm T_p \circ \cdots \circ \underbrace{T_b \circ \cdots \circ T_p}_{\text{position } i}, \quad i \in [0, [\rho]].
\]
Now by the continuity of paradifferential operators given in Theorem A.1 and A.2 we have:
\[
\|T_p \circ \cdots \circ T_b \circ \cdots \circ T_p\|_{H^s \to H^{s-\beta}} \leq K^{[\rho]} M^{\beta}_0(b) M^{\delta}_0(p) \left[ \frac{1}{\rho} \right],
\]
which gives (4.29).

For (4.28), we start by the case \( k \in \mathbb{N}^* \) is a an integer. We first notice that again by Theorem A.1 we have:
\[
\begin{align*}
\mathcal{L}^1_{T_p} T_b &\in \Gamma^{\beta+\delta-1}_{k-1}, \\
M^{\beta+\delta-1}_{k-1}(\mathcal{L}^1_{T_p} T_b) &\leq CM^{\beta}_k(b) M^{\delta}_k(p).
\end{align*}
\]
Thus iterating this formula we get:
\[
\begin{aligned}
\Omega_{T_p}^k T_b \in \Gamma_0^{\beta+k\delta-k}, \\
M_0^{\beta+\delta-k}(\Omega_{T_p}^k T_b) &\leq C_k M_k^\beta(b) \prod_{i \geq 1} M_i^\delta(p),
\end{aligned}
\]
and \(C_k\) verifies:
\[
C_k = 2kC_{k-1} \Rightarrow C_k = C2^k k!,
\]
Thus giving the result in the case \(k\) integer. For \(\rho \geq 0\), it suffice to see that for \(\rho \leq 1\) again by Theorem A.1 we have:
\[
\begin{aligned}
\Omega_{T_p}^1 T_b \in \Gamma_0^{\beta+\delta-\rho}, \\
M_0^{\beta+\delta-\rho}(\Omega_{T_p}^1 T_b) &\leq CM_\rho^\beta(b)M_\rho^\delta(p),
\end{aligned}
\]
which concludes the proof.

This means that if we can compensate the loss of \(\beta + \lceil \rho \rceil \delta\) derivatives with a cost negligible in comparison to \(\lceil \rho \rceil!\), we would have a convergent series in (4.24). A first approach would be to interpolate (4.25) and (4.29) which gives:
\[
\| \Omega_{T_p}^\lceil \rho \rceil T_b \|_{H^{\beta} \to H^{\rho-\beta}} \leq C \frac{\rho^{\lceil \rho \rceil \delta}}{\rho!} M_\rho^\beta(b) \frac{\rho^{\lceil \rho \rceil \delta}}{\rho!} M_\rho^\delta(p) \frac{\rho^{\lceil \rho \rceil \delta}}{\rho!} [\rho] \\
\times C((\rho+1)\lceil \rho \rceil\delta)^{\lceil \rho \rceil \delta} 2^{\lceil \rho \rceil \delta} [\rho]! \frac{\rho^{\lceil \rho \rceil \delta}}{\rho!} M_\rho^\beta(b) \frac{\rho^{\lceil \rho \rceil \delta}}{\rho!} M_\rho^\delta(p) [\rho] [\rho]! [\rho], \quad (4.30)
\]
This indeed solves the cost \(\lceil \rho \rceil!\) of (4.25) but depends on \(M_\rho\) norms of \(b\) and \(p\). An idea to control those norms in a cost negligible in comparison to \(\lceil \rho \rceil!\) would be to mollify \(p\) and \(b\) using an analytic mollifier, this might work but we found it better to mollify differently.

For this we introduce a mollification with the Gaussian function \(\phi_\epsilon(D)\) with symbol:
\[
\phi_\epsilon(\xi) = \frac{1}{\epsilon \sqrt{2\pi}} e^{-\frac{\xi^2}{2\epsilon^2}}, \epsilon > 0. \quad (4.31)
\]
Other than the standard properties of mollifiers, we have the following properties:

- For \(h \in H^s(\mathbb{R})\), \(\phi_\epsilon(D)h\) is real analytic.
- The moments of the Gaussian can be explicitly computed by, for \(k \in \mathbb{N}\):
\[
\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |\xi|^k e^{-\frac{\xi^2}{2\epsilon^2}} d\xi = \frac{2^{k+1} \Gamma(\frac{k+1}{2})}{\sqrt{\pi}} = (k-1)!! \begin{cases} 1 \frac{1}{\sqrt{2\pi}} & \text{if } k \text{ is even} \\ 1 & \text{if } k \text{ is odd} \end{cases}.
\]
From the moments of the Gaussian we deduce that, for $h \in H^s(\mathbb{R})$ and $k \in \mathbb{N}$:

$$\| \partial^k_x \phi_\epsilon(D)h \|_{H^s} \leq C_k \epsilon^{-k} \| h \|_{H^s},$$

and $C_k$ verifies for all $K > 0$, $K^k C_k = o(k!)$.

Now by the symbolic calculus rules given by Theorem A.2 and the fact that the class $\Psi^0_{1,1}(\mathbb{R}) = S^0_{1,1}(\mathbb{R}) \cap \left( S^0_{1,1}(\mathbb{R}) \right) ^*$ is closed under composition, for $\epsilon > 0$, there exists $\epsilon b^p_\tau \in \Psi^0_{1,1}(\mathbb{R})$ such that:

$$\sum_{k=0}^{+\infty} \frac{\tau^k}{k!} \Omega^k_{iT_p} T_b \phi_\epsilon(D) = \text{Op}(\epsilon b^p_\tau), \quad \text{with,}$$

$$M^0_p(\epsilon b^p_\tau) \leq \sum_{k=0}^{+\infty} \frac{\tau^k}{k!} \epsilon^{-k\delta - \beta} M^0_p(b) M^0_p(p)^k. \quad (4.32)$$

In order to pass to the limit in $\epsilon$ we will express $\epsilon b^p_\tau$ differently, for all $\epsilon > 0$,

$$\sum_{k=0}^{+\infty} \frac{\tau^k}{k!} \Omega^k_{iT_p} T_b \phi_\epsilon(D) \text{ converges in } \mathcal{L}(H^s(\mathbb{R})), \quad \text{thus by uniqueness of the limit:}$$

$$e^{ixT_p} \circ T_b \circ e^{-ixT_p} \phi_\epsilon(D) = \sum_{k=0}^{+\infty} \frac{\tau^k}{k!} \Omega^k_{iT_p} T_b \phi_\epsilon(D). \quad (4.33)$$

Thus,

$$e^{ixT_p} \circ T_b \circ e^{-ixT_p} \phi_\epsilon(D) = \text{Op}(\epsilon b^p_\tau). \quad (4.34)$$

Now we estimate the $\delta - 1$ $H^\beta_p(\cdot; k)_{k \in \mathbb{N}}$ norms of $\epsilon b^p_\tau$. To do so we need, in the word of Hörmander [15], a result which interpolates between information on the norm a of an operator and bounds for the derivatives of its symbol. This was exactly the goal of Lemma 4.1.

By commuting $\frac{1}{i} \frac{d}{dx}$ and $ix$ with (4.34) we get:

$$\left[ \frac{1}{i} \frac{d}{dx}, \text{Op}(\epsilon b^p_\tau) \right] = \left[ \frac{1}{i} \frac{d}{dx}, e^{ixT_p} \right] \circ T_b \circ e^{-ixT_p} \phi_\epsilon(D)$$

$$+ e^{ixT_p} \circ \left[ \frac{1}{i} \frac{d}{dx}, T_b \right] \circ e^{-ixT_p} \phi_\epsilon(D)$$

$$+ e^{ixT_p} \circ T_b \circ \left[ \frac{1}{i} \frac{d}{dx}, e^{-ixT_p} \right] \phi_\epsilon(D) + e^{ixT_p} \circ T_b \circ e^{-ixT_p} \left[ \frac{1}{i} \frac{d}{dx}, \phi_\epsilon(D) \right].$$

and,

$$[ix, \text{Op}(\epsilon b^p_\tau)] = [ix, e^{ixT_p}] \circ T_b \circ e^{-ixT_p} \phi_\epsilon(D) + e^{ixT_p} \circ [ix, T_b] \circ e^{-ixT_p} \phi_\epsilon(D)$$
+ e^{i \tau T_p} \circ T_b \circ [ix, e^{-i \tau T_p}] \phi_\epsilon (D) + e^{i \tau T_p} \circ T_b \circ e^{-i \tau T_p} [ix, \phi_\epsilon (D)].

To estimate $[\frac{1}{i} \frac{d}{dx}, e^{i \tau T_p}]$ and $[ix, e^{i \tau T_p}]$ we get back to (4.1) and see that:

$$
\begin{align*}
[\frac{1}{i} \frac{d}{dx}, e^{i \tau T_p}] &= \int_0^\tau e^{i(\tau-r)T_p}[\frac{1}{i} \frac{d}{dx}, T_{ip}] e^{ir T_p} dr, \\
[ix, e^{i \tau T_p}] &= \int_0^\tau e^{i(\tau-r)T_p}[ix, T_{ip}] e^{ir T_p} dr.
\end{align*}
(4.35)
$$

Thus by iteration, the continuity of $e^{i \tau T_p}$ and Lemma 4.1 we get:

$$
1^{-\delta, \delta} H_{m,n}^\beta (\epsilon b^p_\tau ; k) \leq C_{n,k} (M^\delta_1 (p)) H_0^\beta (b; k) \left[ \sum_{i=0}^{k-1} H_\rho^\delta (p; k)^{k-i} \right], \quad (k, n, m) \in \mathbb{N}^3.
$$

Thus we can pass to the limit in $\epsilon$ in (4.34), there exist $b^p_\tau \in W^{0, \infty} S_{1-\delta, \delta}^\beta$ such that:

$$
e^{i \tau T_p} \circ T_b \circ e^{-i \tau T_p} = \text{Op}(b^p_\tau).
(4.36)
$$

Moreover if $\rho \in \mathbb{N}$ we get $b^p_\tau \in W^{\rho, \infty} S_{1-\delta, \delta}^\beta$.

The $1^{-\delta, \delta} H_{m,\rho}^\beta (b^p_\tau ; k), \rho \notin \mathbb{N}$ estimates are obtained by interpolation. Indeed by Proposition 4.1 the sequence of seminorms

$$
\left(1^{-\delta, \delta} H_{m,n}^\beta (\cdot ; k) \right)_{(k,m,n) \in \mathbb{N}^3} \quad \text{and} \quad \left(1^{-\delta, \delta} M_{m,n}^\beta (\cdot ; k) \right)_{(k,m,n) \in \mathbb{N}^3}
$$

are equivalent. Thus we deduce $\left(1^{-\delta, \delta} M_{m,n}^\beta (\cdot ; k) \right)$ estimates. Now for $\rho \in \mathbb{R}_+$, $\left(1^{-\delta, \delta} M_{m,\rho}^\beta (\cdot ; k) \right)$ are interpolation norms which give $\left(1^{-\delta, \delta} M_{m,\rho}^\beta (\cdot ; k) \right)$ estimates and the existence of $b^p_\tau \in W^{\rho, \infty} S_{1-\delta, \delta}^\beta$ for $\rho \in \mathbb{N}$.

Point (III) For point (6) we compute:

$$
\partial_\tau [e^{i \tau T_p}, T_b] = [i T_p \circ e^{i \tau T_p}, T_b] = i T_p [e^{i \tau T_p} , T_b] + [i T_p, T_b] e^{i \tau T_p}.
$$

Thus by the definition of $e^{i \tau T_p}$ as the flow map we get the following Duhamel formula,

$$
[e^{i \tau T_p}, T_b] = \int_0^\tau e^{i(\tau-r)T_p} [i T_p, T_b] e^{ir T_p} dr,
$$

$$
e^{i \tau T_p} \int_0^\tau e^{-ir T_p} [i T_p, T_b] e^{ir T_p} dr.
$$
Applying point (5) to \( \star \) we get:

\[
[e^{i\tau T_p}, T_b] = e^{i\tau T_p} \sum_{k=1}^{n} (-1)^{k-1} \frac{\tau^k}{k!} \Omega^k_{i T_p} \ L_{i T_p}^k \ L_{i T_p} \ T_b e^{i\tau T_p} \ dr.
\]

Again applying point (1) combined with (4.25) we get (4.21).

To get (4.20) we inject (4.17) in \( \star \), which concludes the proof. \( \square \)

We give a result on the symbol of \( e^{i\tau T_p} \)

**Proposition 4.3** Consider two real numbers \( \delta < 1, \rho \geq 1 \) and a real valued symbol \( p \in \Gamma^\delta_{\rho}(\mathbb{R}) \).

Let \( e^{i\tau T_p}, \tau \in \mathbb{R} \) be the flow map defined by Proposition 4.1, then there exists a symbol \( e^{i\tau p} \in W^\rho,\infty S^\rho_{1-\delta,\delta}(\mathbb{R}^n \times \mathbb{R}) \) such that:

\[
e^{i\tau T_p} = \text{Op}(e^{i\tau p}). \tag{4.37}
\]

Moreover we have the identity:

\[
e^{i\tau T_p} = \text{Id} + T_{e^{i\tau p}} + \int_0^\tau e^{i(\tau-s)T_p}(T_{ip} - T_{ie^{ip}p}) \ ds. \tag{4.38}
\]

**Proof** The idea is that morally \( \text{Op}(e^{i\tau p}) \) should be defined by the asymptotic series:

\[
\text{Op}(e^{i\tau p}) \sim \sum \frac{i^k \tau^k}{k!} (T_p)^k = \sum \frac{i^k \tau^k}{k!} T_k \otimes_p,
\]

where \( \otimes_p \) is defined by Theorem A.2. To make this series converge we again introduce the Gaussian multiplier \( \phi_\epsilon (D) \) defined by (4.31), as we still have the factor \( \frac{1}{k!} \) as in Proposition 4.2.

As in the proof of Proposition 4.2 there exists a symbol \( \epsilon^p e^{i\tau p} \in \Psi^0_{1,1}(\mathbb{R}) \) such that:

\[
\sum_{k=0}^{+\infty} \frac{i^k \tau^k}{k!} (T_p)^k \phi_\epsilon (D) = \sum_{k=0}^{+\infty} \frac{i^k \tau^k}{k!} T_k \otimes_p \phi_\epsilon (D) = \text{Op}(\epsilon^p e^{i\tau p}), \tag{4.39}
\]

\[
1 - \delta,\delta M^0_p(e^{i\tau p}) \leq \sum_{k=0}^{+\infty} C^k C_k \frac{|\tau|^k}{k!} e^{-k\delta} M^\delta_p(p)^k, \tag{4.40}
\]

where \( C_k \) verifies for all \( K > 0, K^k C_k = o(k!) \).
Now in order to pass to the limit in $\epsilon$ we need to get uniform estimates on $1 - \delta, \delta H^0_{n,s}(\epsilon e^{i\tau p}; k)$ for $(k, n) \in \mathbb{N}$. To do so we see that:

$$\frac{\partial}{\partial \tau} \left[ \text{Op}(\epsilon e^{i\tau p}) h_0 \right] = iT_p \text{Op}(\epsilon e^{i\tau p}) h_0,$$

for $h_0 \in H^s(\mathbb{R}), s \in \mathbb{R}$. (4.41)

Thus a standard energy estimate combined with the commutation identities (4.35) and Lemma 4.1 we get:

$$1 - \delta, \delta H^0_{m,n}(\epsilon e^{i\tau p}; k)_{k \in \mathbb{N}} \leq C k, n_{(\epsilon e^{i\tau p}; k)}, (k, m, n) \in \mathbb{N}^3.$$  

Thus if of $\rho \in \mathbb{N}$ we can pass to the limit in $\epsilon$ and get $e^{i\tau p} \in \mathcal{W}_{1,\infty}(\rho, S^0_0(\mathbb{R}^* \times \mathbb{R})$).

For $\rho \in \mathbb{R}^+ \backslash \mathbb{N}$ follows by interpolation as in point (II) of the proof of Proposition 4.2.

Identity (4.38) comes from the following computation. Fix an $h_0 \in H^s, s \in \mathbb{R}$, then $[e^{i\tau T_p} - e^{i\tau T_{p'}}] h_0$ solves:

$$\begin{cases}
\frac{\partial}{\partial \tau} (e^{i\tau T_p} - e^{i\tau T_{p'}}) h_0 = i T_p (e^{i\tau T_p} - e^{i\tau T_{p'}}) h_0,

(e^{i\tau T_p} - e^{i\tau T_{p'}})(0, \cdot) = (I d - T_1) h_0(\cdot),
\end{cases}
$$

(4.42)

which by definition of $e^{i\tau T_p}$ gives (4.38).

We will now compute the different Gateaux derivatives of the operators defined above.

**Proposition 4.4** Consider two real numbers $\delta < 1, \rho \geq 1$, two real valued symbols $p, p' \in \Gamma^\delta_\rho(\mathbb{R})$. Let $e^{i\tau T_p}, e^{i\tau T_{p'}} , \tau \in \mathbb{R}$ be the flow maps defined by Proposition 4.1, then for $\tau \in \mathbb{R}$ we have:

$$e^{i\tau T_p} - e^{i\tau T_{p'}} = \int_0^\tau e^{i(\tau - r) T_p} iT_p e^{i\tau T_{p'}} dr.$$  

(4.43)

Another way to see this is with the Gateaux derivative of $p \mapsto e^{i\tau T_p}$ on the Fréchet space $\Gamma^\delta_\rho(\mathbb{R})$ is given by:

$$D_p e^{i\tau T_p} (h) = \int_0^\tau e^{i(\tau - r) T_p} T_1 h e^{i\tau T_{p'}} dr.$$  

(4.44)

Moreover consider an open interval $I \subset \mathbb{R}$, and a real valued symbols $p \in C^1(I, \Gamma^\delta_\rho(\mathbb{R}))$. Let $e^{i\tau T_p}, \tau \in \mathbb{R}$ be the flow map defined by Proposition 4.1 then for $\tau \in \mathbb{R}, z \in I$ we have:
\[ \partial_{\varepsilon} e^{i \tau T_{p}} = \int_{0}^{\tau} e^{i(\tau - r)T_{p}} T_{\partial_{\varepsilon} p} e^{i\tau T_{p}} dr. \] (4.45)

**Proof** Fix \( h_0 \in H^s, s \in \mathbb{R} \) then:
\[
\partial_{\tau} [e^{i \tau T_{p}} h_0] - i T_{p} [e^{i \tau T_{p}} h_0] = 0 \Rightarrow \partial_{\tau} [\partial_{\varepsilon} e^{i \tau T_{p}} h_0] - i T_{p} [\partial_{\varepsilon} e^{i \tau T_{p}} h_0] - T_{i \partial_{\varepsilon} p} [e^{i \tau T_{p}} h_0] = 0,
\]
which gives (4.45) by the definition of \( e^{i \tau T_{p}} \) and the Duhamel formula. The identities (4.43) and (4.44) are obtained in the same way. \( \square \)

**Proposition 4.5** Consider two real numbers \( \delta < 1, \rho \geq 1 \), two real valued symbols \( p, p' \in \Gamma_{\rho}^{\delta}(\mathbb{R}) \). Let \( e^{i \tau T_{p}}, e^{i \tau T_{p}'} \), \( \tau \in \mathbb{R} \) be the flow maps defined by Proposition 4.1 and take a symbol \( b \in \Gamma_{\rho}^{\beta}(\mathbb{R}) \) with \( \beta \in \mathbb{R} \) then for \( \tau \in \mathbb{R} \) we have:
\[
\text{Op}(b_{\tau}^{p}) - \text{Op}(b_{\tau}^{p'}) = \int_{0}^{\tau} e^{i(\tau - r)T_{p}} \mathcal{L}_{i T_{p} - p'} \text{Op}(b_{r}^{p'}) e^{i(\tau - r)T_{p}} dr
\]
\[= i \int_{0}^{\tau} \mathcal{L}_{T_{p} - \text{Op}(p'_{\tau - r})} \text{Op}((b_{r}^{p'})_{p}^{p}) dr.\] (4.46)

Another way to see this is with the Gateaux derivative of \( p \mapsto \text{Op}(b_{\tau}^{p}) \) on the Fréchet space \( \Gamma_{\rho}^{\delta}(\mathbb{R}) \) is given by:
\[
D_{p} \text{Op}(b_{\tau}^{p})(h) = \int_{0}^{\tau} \mathcal{L}_{i \text{Op}(h^{p}_{\tau - r})} \text{Op}(b_{r}^{p}) dr = \mathcal{L}_{i \int_{0}^{\tau} \text{Op}(h^{p}_{\tau - r}) dr} \text{Op}(b_{\tau}^{p}).\] (4.48)

Writing, \( \text{Op}(c b_{\tau}^{p}) = T_{b} - \text{Op}(b_{\tau}^{p}) \), and, \( \text{Op}(c b_{\tau}^{p'}) = T_{b} - \text{Op}(b_{\tau}^{p'}) \) we get:
\[
\text{Op}(c b_{\tau}^{p}) - \text{Op}(c b_{\tau}^{p'}) = - \int_{0}^{\tau} e^{-i(\tau + r)T_{p}} \mathcal{L}_{i T_{p} - p'} \text{Op}(b_{r}^{p'}) e^{i(\tau + r)T_{p}} dr
\]
\[-i \int_{0}^{\tau} \mathcal{L}_{T_{p} - \text{Op}(p'_{\tau - r})} \text{Op}((b_{r}^{p'})_{p}^{p}) dr.\] (4.49)
\[
D_{p} \text{Op}(c b_{\tau}^{p})(h) = - \int_{0}^{\tau} \mathcal{L}_{i \text{Op}(b^{p}_{\tau - r})} \text{Op}(b_{\tau}^{p}) dr = - \mathcal{L}_{i \int_{0}^{\tau} \text{Op}(b^{p}_{\tau - r}) dr} \text{Op}(b_{\tau}^{p}).\] (4.50)
\textbf{Proof} From (4.26) and (4.27) we have:

$$\begin{align*}
\partial_\tau [\text{Op}(b^p_\tau) - \text{Op}(b^{p'}_\tau)] &= \mathcal{L}_{iTp}(\text{Op}(b^p_\tau) - \text{Op}(b^{p'}_\tau)) + \mathcal{L}_{iT_{p-p'}} \text{Op}(b^{p'}_\tau), \\
\text{Op}(b^p_0) - \text{Op}(b^{p'}_0) &= 0.
\end{align*}$$

(4.52)

Thus the Duhamel formula gives (4.46) and (4.47). For the Gateaux derivative passing to the limit in (4.46) we have:

$$D_p \text{Op}(b^p_\tau)(h) = \int_0^\tau e^{i(\tau-r)T_{p}} \mathcal{L}_{iT_{r}} \text{Op}(b^p_r)e^{i(r-\tau)T_{p}} dr,$$

which gives (4.48).

\[\square\]

\textbf{Corollary 4.2} Consider two real numbers $1 < \alpha$, and $s \in \mathbb{R}$, two real valued symbols $a \in \Gamma^{\frac{\alpha}{\alpha-1}}(\mathbb{R})$ and $b \in \Gamma^{1}(\mathbb{R})$. Suppose that there exists a real valued symbol $p \in \Gamma^{2-\alpha}\left[\frac{\alpha}{\alpha-1}\right]^{-1}(\mathbb{R})$ such that:

$$b = -\partial_\xi p \partial_x a + \partial_x p \partial_\xi a.$$  

(4.53)

Define for $\tau \in \mathbb{R}$, $e^{i\tau T_{p}}$ as the flow map generated by $iT_{p}$ from Proposition 4.1,

$$R_\tau = \tau T_{ib} + \int_0^\tau e^{-isT_{p}}[T_{ip}, T_{ia}]e^{i\tau T_{p}} ds,$$

(4.54)

and,

$$\tilde{R}_\tau = e^{i\tau T_{p}} R_\tau e^{-i\tau T_{p}} = \tau e^{i\tau T_{p}} i T_{b} e^{-i\tau T_{p}} + [e^{i\tau T_{p}}, T_{ia}] e^{-i\tau T_{p}}.$$  

(4.55)

First $R_\tau \in \mathcal{L}(H^{s+(2-\alpha)^+}(\mathbb{R}), H^{s}(\mathbb{R}))$ and we have the estimate

$$\|R_\tau\|_{H^{s+(2-\alpha)^+} \rightarrow H^{s}} \leq C e^{CM^2_1(p)} M^\alpha_2(a) M^{2-\alpha}_2(p).$$

(4.56)

Second $R_\tau$ is skew symmetric and we have the estimate

$$\|\text{Re}(R_\tau)\|_{H^s \rightarrow H^s} \leq C e^{CM^2_1(p)} \left[\frac{\alpha}{\alpha-1}\right]^{(p)} M^2_2(p) M^{\alpha}_2(a).$$

(4.57)

Moreover taking two different symbols $b'$ and $p'$ and defining analogously $R'_\tau$, $\tilde{R}'_\tau$, we have for $h \in H^3$:

$$\|\left[R_\tau - R'_\tau\right]h\|_{H^s} \leq C M^\alpha_2 \left[\frac{\alpha}{\alpha-1}\right]^{(p,p')}.$$
Putting all of the \((2 - \alpha)\) derivatives on \(h\) gives the cruder estimate

\[
\| [R_\tau - R_\tau'] h \|_{H^s} \leq C M^{2-\alpha} \left( \frac{a}{a-1} \right) (p, p') M^{2-\alpha} \left( \frac{a}{a-1} \right) - (2-\alpha) (p - p') \| h \|_{H^{s+1(2-\alpha)}}.
\]

Analogous estimates hold for \(\tilde{R}_\tau\) and \(\tilde{R}_\tau - \tilde{R}_\tau'\).

**Proof** First we notice that by definition \(R_\tau, \tilde{R}_\tau\) are of order 1. Now to show that they are actually of order \(2 - \alpha\) we write by \((4.24)\)

\[
R_\tau = \tau \left( T_{ib} + [T_{ip}, T_{ia}] \right) - \int_0^\tau \frac{(\tau - r)^2}{(n + 1)!} e^{irT_p} \sum_{l=0}^{2} \sum_{i=0}^{2} T_{ia} e^{-irT_p} dr.
\]

By Theorem A.2, \(T_{ib}^0\) is of order 0 and \(T_{r_\tau^{2-\alpha}}\) is of order \(2 - \alpha\), thus we get estimate \((4.56)\).

To show skew symmetry and the difference estimate we need to expand further to the order \(n = \left\lceil \frac{a}{a-1} \right\rceil - 1 \geq 1\) by Proposition \((4.24)\):

\[
R_\tau = \tau \left( T_{ib} + [T_{ip}, T_{ia}] \right) + \sum_{k=1}^{n-1} \frac{(-1)^k \tau^{k+1}}{(k + 1)!} \sum_{l=0}^{2} \sum_{i=0}^{2} T_{ia} e^{-irT_p} dr.
\]

where again by Theorem A.2, \(T_{r_\tau^{2(1-\alpha)+1}}\) are of order \(k(1 - \alpha) + 1\) and by this choice of \(n, T_{r_\tau^{n(1-\alpha)+1}}\) is of order 0.

For skew symmetry we analyse each term separately, first as \(T_{r_\tau^0}\) and \(T_{r_\tau^{n(1-\alpha)+1}}\) are of order 0 we have

\[
\|\text{Re}(T_{r_\tau^0})\|_{L^2 \to L^2} \leq \| T_{r_\tau^0} \|_{L^2 \to L^2} \leq C M^{2-\alpha} \left( \frac{a}{a-1} \right) M^{2-\alpha} \left( \frac{a}{a-1} \right) (a),
\]

and

\[
\| e^{irT_p} \sum_{l=0}^{2} T_{ia} e^{-irT_p} \|_{L^2 \to L^2} \leq C e^{CM^{2-\alpha} \left( \frac{a}{a-1} \right) (p)} \left[ \frac{a}{a-1} \right] M^{2-\alpha} \left( \frac{a}{a-1} \right) (a).
\]
Now for $T_{r^k (1-\alpha)+1}$ we note that

$$
(T_{r^k (1-\alpha)+1})^* = (-1)^{k+1} \sum_{i(T_p)}^{n+1} i(T_a)^* = -\sum_{i(T_p)}^{n+1} i(T_a)^* = -T_{r^k (1-\alpha)+1} + T_{r^k 0}
$$

where $T_{r^k 0}$ is of order 0 and verifies the estimate

$$
\left\| T_{r^k 0} \right\|_{L^2 \rightarrow L^2} \leq CM^{2-\alpha} (p) M^{2-\alpha}_{k+1}(p) (p) M^{\alpha}_{k+1}(a) + C M^{2-\alpha} (p) M^{\alpha}_{k+1}(p) (p) M^{\alpha}_{k+1}(a)
$$

which gives the desired estimate (4.57) where estimated each term by it’s highest order counterpart.

For the difference estimate (4.58) we note that $T_{r^k (1-\alpha)+1}$ are operators in the usual paradifferential classes and thus their differential with respect to $p$ do not generate the undesired loss of $2 - \alpha$ derivatives. we take $n = \left\lfloor \frac{\alpha}{\alpha-1} \right\rfloor - 1 \geq 1.$

By this choice of $n$, $T_{r^k (1-\alpha)+1}$ is of order 0 and thus $D^i \tau T_p \sum_{i(T_p)}^{n+1} T_{i} e^{-irT_p}$ is of order $2 - \alpha$ by Proposition 4.5. Thus, estimating each term separately, we get the following difference estimates

$$
\left\| [T_{r^k 0} - T_{r^k}] h \right\|_{H^s} \leq C M^{2-\alpha} (p - p') M^{\alpha}_{2}(a) \| h \|_{H^s},
$$

and

$$
\left\| e^{irT_p} \sum_{i(T_p)}^{n+1} T_{i} e^{-irT_p} - e^{irT_p} \sum_{i(T_p)}^{n+1} T_{i} e^{-irT_p} \right\|_{H^s} \leq C e^{CM^{2-\alpha} (p, p')} M^{2-\alpha}_{0} (p - p') \left( M^{2-\alpha}_{\left\lfloor \frac{\alpha}{\alpha-1} \right\rfloor} (p) \left[ \alpha \right]_{\frac{\alpha}{\alpha-1}} + M^{2-\alpha}_{\left\lfloor \frac{\alpha}{\alpha-1} \right\rfloor} (p') \left[ \alpha \right]_{\frac{\alpha}{\alpha-1}} \right) M^{\alpha}_{\left\lfloor \frac{\alpha}{\alpha-1} \right\rfloor} (a) \| h \|_{H^s},
$$

Estimating factors by their highest order counterpart we get the desired estimates. We get the estimates on $R_{\tau}$ by writing analogously

$$
R_{\tau} = \tau \left( T_{ib} [T_{ip}, T_{ia}] \right) + \sum_{k=1}^{n-1} \frac{\tau^{k+1}}{k!} \sum_{i(T_p)}^{n+1} T_{ib} - \int_{0}^{\tau} \frac{(\tau - r)^n}{n!} e^{irT_p} \sum_{i(T_p)}^{n+1} T_{ib} e^{-irT_p} dr
$$

$$
+ \sum_{k=1}^{n-1} \frac{\tau^{k+1}}{(k+1)!} \sum_{i(T_p)}^{n+1} T_{ia} - \int_{0}^{\tau} \int_{0}^{s} \frac{(\tau - s - r)^n}{n!} e^{irT_p} \sum_{i(T_p)}^{n+1} T_{ia} e^{-irT_p} dr ds.
$$

$\square$
Corollary 4.3 Consider two real numbers $1 < \alpha$, and $s \in \mathbb{R}$, two real valued symbols $a \in \Gamma^{2-\alpha} \left( \left[ \frac{\alpha}{\alpha-1} \right] + 1 - \alpha \right)$ and $p \in \Gamma^{2-\alpha} \left( \left[ \frac{\alpha}{\alpha-1} \right] \right)$. Define for $\tau \in \mathbb{R}$, $e^{i\tau T_p}$ as the flow map generated by $iT_p$ from Proposition 4.1,

$$R_\tau = \int_0^\tau e^{isT_p} T_i a e^{-isT_p} ds,$$  

(4.59)

First $R_\tau \in \mathcal{L}(H^{s+2-\alpha})$ and we have the estimate

$$\| R_\tau \|_{H^{s+2-\alpha}} \leq C e^{CM_1^{2-\alpha}(p)M_1^{2-\alpha}(a)}.$$  

(4.60)

Second $R_\tau$ is skew symmetric and we have the estimate

$$\| \text{Re}(R_\tau) \|_{H^s} \leq C e^{CM_1^{2-\alpha}(p)M_1^{2-\alpha}(a)}.$$  

(4.61)

Finally taking two different symbols $a'$ and $p'$ and defining analogously $R'_\tau$, we have for $h \in H^s$:

$$\| [R_\tau - R'_\tau] h \|_{H^{s+2-\alpha}} \leq CM_1^{2-\alpha} \left( \left[ \frac{\alpha}{\alpha-1} \right] \right) e^{CM_1^{2-\alpha}(p,p')} \left( \left[ \frac{\alpha}{\alpha-1} \right] \right)^{-(2-\alpha)} (p - p') + CM_1^{2-\alpha} \left( \left[ \frac{\alpha}{\alpha-1} \right] - 1 \right) e^{CM_1^{2-\alpha}(p,p')} \left( \left[ \frac{\alpha}{\alpha-1} \right] \right)^{-(2-\alpha)} (p).$$

Proof We expand by Proposition (4.24) with $n = \left( \left[ \frac{\alpha}{\alpha-1} \right] \right) - 1$,

$$R_\tau = \sum_{k=0}^n \frac{1}{k!} \left[ \frac{\alpha}{\alpha-1} \right]^k T_{r_\tau}^k T_i a + \int_0^\tau \frac{(1-r)^n}{n!} e^{irT_p} T_{r_\tau}^{n+1} T_i a e^{-irT_p} dr,$$

Again by Theorem A.2, $T_{r_\tau}^{2-\alpha+k(1-\alpha)}$ are of order $2 - \alpha + k(1 - \alpha)$ for $k \leq \left( \left[ \frac{\alpha}{\alpha-1} \right] \right) - 1$. On the other hand, treating $\sigma_\alpha$ as a symbol in $\Gamma^{1} \left( \left[ \frac{\alpha}{\alpha-1} \right] \right)$, we get that $T_{r_\tau}^{n+1(1-\alpha)+1}$ is of order $(n + 1)(1 - \alpha) + 1 \leq 0$ by definition of $n$. Thus $D_p^{irT_p} T_{r_\tau}^{n+1} T_i a e^{-irT_p}$ is of order $2 - \alpha$ by Proposition 4.5. Thus, estimating each term separately, we get the following difference estimates, first for $k \leq n$

$$\| [T_{r_\tau}^{2-\alpha+k(1-\alpha)} - T_{r_\tau}^{2-\alpha+k(1-\alpha)'}] h \|_{H^s} \leq CM_1^{2-\alpha} (p - p') \left( M_1^{2-\alpha}(p)^{k-1} + M_1^{2-\alpha}(p')^{k-1} \right) M_1^{2-\alpha}(a) \| h \|_{H^{s+k(1-\alpha)+2-\alpha}}$$
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\[ +C \left( M_k^{2-\alpha} (p)^k + M_k^{2-\alpha} (p')^k \right) M_k^{2-\alpha} (a - a') \|h\|_{H^{r+k(1-\alpha)+2-\alpha}}, \]

\[
\left\| [T_{r+1}^{1+(n+1)(1-\alpha)} - T_{r+1}^{1+(n+1)(1-\alpha)'})] h \right\|_{H^s}
\leq CM_{n+1}^{2-\alpha} (p - p') \left( M_{n+1}^{2-\alpha} (p)^{k-1} + M_{n+1}^{2-\alpha} (p')^n \right) M_{n+1}^{2-\alpha} \left( \left\lfloor \frac{a}{\alpha-1} \right\rfloor +1-\alpha \right) (a) \|h\|_{H^s}
\]

\[ +C \left( M_{n+1}^{2-\alpha} (p)^n + M_{n+1}^{2-\alpha} (p')^{n+1} \right) M_{n+1}^{2-\alpha} \left( \left\lfloor \frac{a}{\alpha-1} \right\rfloor +1-\alpha \right) (a - a') \|h\|_{H^s}, \]

and

\[
\left\| e^{i\alpha T_p} \Sigma_{T_p}^{n+1} T_{ia} e^{-i\alpha T_p} - e^{i\alpha T_p} \Sigma_{T_p}^{n+1} T_{ia} e^{-i\alpha T_p'} \right\|_{H^s}
\leq C e^{CM_1^{2-\alpha} (p, p')} M_0^{2-\alpha} (p - p') \left( M_1^{2-\alpha} \left( \left\lfloor \frac{a}{\alpha-1} \right\rfloor +1-\alpha \right) \right) \times M_{2-\alpha} \left( \left\lfloor \frac{a}{\alpha-1} \right\rfloor +1-\alpha \right) (a) \|h\|_{H^{s+(2-\alpha)+}},
\]

which gives the desired estimate by estimating each factor by its highest order counterpart. To get the skew symmetry we note that for \( k \leq n \),

\[
\left( T_{r+1}^{2-\alpha+k(1-\alpha)} \right)^* = -\Sigma_{i(T_p)}^{n+1} i(T_a)^* = -T_{r+1}^{2-\alpha+k(1-\alpha)} + T_{r+1}^{k,0}
\]

here \( T_{r+1}^{k,0} \) is of order 0 and verifies the estimate

\[
\left\| T_{r+1}^{k,0} \right\|_{L^2 \to L^2} \leq CM_k^{2-\alpha} (p)^{k-1} M_{k+\max(k(1-\alpha)+2-\alpha,0)} (p) M_k^{2-\alpha} (a)
\]

\[ +C M_k^{2-\alpha} (p)^k M_{k+\max(k(1-\alpha)+2-\alpha,0)} (a), \]

and

\[
\left\| e^{i\alpha T_p} \Sigma_{T_p}^{n+1} T_{ia} e^{-i\alpha T_p} \right\|_{L^2 \to L^2} \leq C e^{CM_1^{2-\alpha} (p)} M_0^{2-\alpha} \left( \left\lfloor \frac{a}{\alpha-1} \right\rfloor \right) M_{2-\alpha} \left( \left\lfloor \frac{a}{\alpha-1} \right\rfloor +1-\alpha \right) (a).
\]

\[ \square \]

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Appendix A. Paradifferential Calculus

In this paragraph we review classic notations and results about paradifferential and pseudodifferential calculus that we need in this paper. We follow the presentations in [14, 15, 28, 42] which give an accessible and complete presentation.

Notation A.1 In the following presentation we will use the usual definitions and standard notations for regular functions $C^k$, $C^k_b$ for bounded ones and $C^k_0$ for those with compact support, the distribution space $\mathcal{D}'$, $\mathcal{E}'$ for those distribution with compact support, $\mathcal{D}^k$, $\mathcal{E}^k$ for distributions of order $k$, $L^p$ Lebesgue spaces, $H^s$ and $W^{p,q}$ Sobolev spaces and finally $\mathcal{S}$ for the Schwartz class and its dual $\mathcal{S}'$. All of those spaces are equipped with their standard topologies. We also use the Landau notation $O(\|X\|)$.

For the definition of the periodic symbol classes we will need the following definitions and notations.

Remark A.1 For clarity in this section and the appendix we again present the symbolic calculus on $\mathbb{R}$. All the results stated here extend tautologically to the case of $\mathbb{T}$ by applying the rules of Remark 4.1.

A.1. Littlewood–Paley Theory

Definition A.1 (Littlewood–Paley decomposition) Pick $P_0 \in C^\infty_0(\mathbb{R}^d)$ so that:

$$P_0(\xi) = 1 \quad \text{for } |\xi| < 1 \quad \text{and} \quad P_0(\xi) = 0 \quad \text{for } |\xi| > 2.$$ 

We define a dyadic decomposition of unity by:

$$P_{\leq k}(\xi) = P_0(2^{-k}\xi), \quad P_k(\xi) = P_{\leq k}(\xi) - P_{\leq k-1}(\xi).$$

Thus,

$$P_{\leq k}(\xi) = \sum_{0 \leq j \leq k} P_j(\xi) \quad \text{and} \quad 1 = \sum_{j=0}^{\infty} P_j(\xi).$$

Introduce the operator acting on $\mathcal{S}'(\mathbb{R}^d)$:

$$P_{\leq k}u = \mathcal{F}^{-1}(P_{\leq k}(\xi)u) \quad \text{and} \quad u_k = \mathcal{F}^{-1}(P_k(\xi)u).$$

Thus,

$$u = \sum_k u_k.$$ 

Finally put for $k \geq 1$, $C_k = \text{supp } P_k$ the set of rings associated to this decomposition.
An interesting property of the Littlewood–Paley decomposition is that even if the decomposed function is merely a distribution the terms of the decomposition are regular, indeed they all have compact spectrum and thus are entire functions. On classical functions spaces this regularization effect can be "measured" by the following inequalities due to Bernstein.

**Proposition A.1** (Bernstein’s inequalities) Suppose that \( a \in L^p(\mathbb{R}^d) \) has its spectrum contained in the ball \( \{ |\xi| \leq \lambda \} \).

Then \( a \in C^\infty \) and for all \( \alpha \in \mathbb{N}^d \) and \( 1 \leq p \leq q \leq +\infty \), there is \( C_{\alpha,p,q} \) (independent of \( \lambda \)) such that

\[
\| \partial_\alpha x a \|_{L^q} \leq C_{\alpha,p,q} \lambda^{\frac{|\alpha|}{p} + \frac{d}{q}} \| a \|_{L^p}.
\]

In particular,

\[
\| \partial_\alpha x a \|_{L^q} \leq C_{\alpha,p,q} \lambda^{\frac{|\alpha|}{p}} \| a \|_{L^p}, \text{ and for } p = 2, p = \infty
\]

\[
\| a \|_{L^\infty} \leq C \lambda^{\frac{d}{2}} \| a \|_{L^2}.
\]

If moreover \( a \) has its spectrum is included in \( \{ 0 < \mu \leq |\xi| \leq \lambda \} \) then:

\[
C_{\alpha,q}^{-1} \lambda^{\frac{|\alpha|}{p}} \| a \|_{L^q} \leq \| \partial_\alpha x a \|_{L^q} \leq C_{\alpha,q} \lambda^{\frac{|\alpha|}{p}} \| a \|_{L^q}.
\]

**Proposition A.2** For all \( \mu > 0 \), there is a constant \( C \) such that for all \( \lambda > 0 \) and for all \( \alpha \in W^{\mu,\infty} \) with spectrum contained in \( \{ |\xi| \geq \lambda \} \), one has the following estimate:

\[
\| a \|_{L^\infty} \leq C \lambda^{-\mu} \| a \|_{W^{\mu,\infty}}.
\]

**Definition A.2** (Zygmund spaces on \( \mathbb{R}^d \)) For \( r \in \mathbb{R} \) we define the space:

\[
C^r_s(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d), \quad C^r_s(\mathbb{R}^d) = \left\{ u \in \mathcal{S}'(\mathbb{R}^d), \| u \|_{C^r_s} = \sup_{q} 2^{qr} \| u_q \|_{L^\infty} < \infty \right\}
\]

equipped with its canonical topology giving it a Banach space structure. It’s a classical result that for \( r \notin \mathbb{N} \), \( C^r_s(\mathbb{R}^d) = W^{r,\infty}(\mathbb{R}^d) \) the classic Hölder spaces.

**Proposition A.3** Let \( B \) be a ball with center 0. There exists a constant \( C \) such that for all \( r > 0 \) and for all \( (u_q)_{q \in \mathbb{N}} \) in \( \mathcal{S}'(\mathbb{R}^d) \) verifying:

\[
\forall q, \ \text{supp} \ \hat{u}_q \subset 2^q B \quad \text{and} \quad (2^{qr} \| u_q \|_\infty)_{q \in \mathbb{N}} \text{ is bounded},
\]

then, \( u = \sum_q u_q \in C^r_s(\mathbb{R}^d) \) and \( \| u \|_{C^r_s} \leq \frac{C}{1 - 2^{-r}} \sup_{q \in \mathbb{N}} 2^{qr} \| u_q \|_{L^\infty} \).
Definition A.3 (Sobolev spaces on $\mathbb{R}^d$) It is also a classical result that for $s \in \mathbb{R}$:

$$H^s(\mathbb{R}^d) = \left\{ u \in \mathcal{S}'(\mathbb{R}^d), \|u\|_s = \left( \sum_q 2^{qs} \|u_q\|_{L^2}^2 \right)^{\frac{1}{2}} < \infty \right\}$$

with the right hand side equipped with its canonical topology giving it a Hilbert space structure and $\| \|$ is equivalent to the usual norm on $H^s$.

Proposition A.4 Let $B$ be a ball with center 0. There exists a constant $C$ such that for all $s > 0$ and for all $(u_q)_{q \in \mathbb{N}}$ in $\mathcal{S}'(\mathbb{R}^d)$ verifying:

$$\forall q, \ \text{supp} \hat{u}_q \subset 2^q B \quad \text{and} \quad (2^{qs} \|u_q\|_{L^2})_{q \in \mathbb{N}} \text{ is in } L^2(\mathbb{N}),$$

then, $u = \sum_q u_q \in H^s(\mathbb{R}^d)$ and $\|u\|_s \leq \frac{C}{1 - 2^{-s}} \left( \sum_q 2^{qs} \|u_q\|_{L^2}^2 \right)^{\frac{1}{2}}$.

We recall the usual nonlinear estimates in Sobolev spaces:

- If $u_j \in H^{s_j}(\mathbb{R}^d)$, $j = 1, 2$, and $s_1 + s_2 > 0$ then $u_1 u_2 \in H^{s_0}(\mathbb{R}^d)$ and if

$$s_0 \leq s_j, \ j = 1, 2 \quad \text{and} \quad s_0 \leq s_1 + s_2 - \frac{d}{2},$$

then $\exists K \in \mathbb{R}, \\|u_1 u_2\|_{H^{s_0}} \leq K \|u_1\|_{H^{s_1}} \|u_2\|_{H^{s_2}}$,

where the last inequality is strict if $s_1$ or $s_2$ or $-s_0$ is equal to $\frac{d}{2}$.

- For all $C^\infty$ function $F$ vanishing at the origin, if $u \in H^s(\mathbb{R}^d)$ with $s > \frac{d}{2}$, then

$$\|F(u)\|_{H^s} \leq C(\|u\|_{H^s}),$$

for some non decreasing function $C$ depending only on $F$.

A.2. Paradifferential Operators

We start by the definition of symbols with limited spatial regularity. Let $\mathcal{W} \subset \mathcal{S}'$ be a Banach space.

Definition A.4 Given $m \in \mathbb{R}$, $\Gamma^m_\mathcal{W}(\mathbb{R})$ denotes the space of locally bounded functions $a(x, \xi)$ on $\mathbb{R} \times (\mathbb{R} \setminus \{0\})$, which are $C^\infty$ with respect to $\xi$ for $\xi \neq 0$ and such that, for all $\alpha \in \mathbb{N}$ and for all $\xi \neq 0$, the function $x \mapsto \partial^\alpha a(x, \xi)$ belongs to $\mathcal{W}$ and there exists a constant $C_\alpha$ such that, for all $\epsilon > 0$:

$$\forall |\xi| > \epsilon, \ \left\| \partial^\alpha a(., \xi) \right\|_{\mathcal{W}} \leq C_\alpha,\epsilon (1 + |\xi|)^{m-|\alpha|}. \quad (A.1)$$

The spaces $\Gamma^m_\mathcal{W}(\mathbb{R})$ are equipped with their natural Fréchet topology induced by the semi-norms defined by the best constants in (A.1).
For quantitative estimates we introduce as in [28]:

**Definition A.5** For \( m \in \mathbb{R} \) and \( a \in \Gamma^m_{\mathcal{W}}(\mathbb{R}) \), we set

\[
M^m_{\mathcal{W}}(a; n) = \sup_{|\alpha| \leq n} \sup_{|\xi| \geq \frac{1}{2}} \left\| (1 + |\xi|)^{m - |\alpha|} \partial_\xi^\alpha a(., \xi) \right\|_{\mathcal{W}}, \text{ for } n \in \mathbb{N}.
\]

For \( \mathcal{W} = W^{\rho, \infty}, \rho \geq 0 \), we write:

\[
\Gamma^m_{W^{\rho, \infty}}(\mathbb{R}) = \Gamma^m_{\rho}(\mathbb{R}) \quad \text{and} \quad M^m_{\rho}(a) = M^m_{W^{\rho, \infty}}(a; 1).
\]

Moreover we introduce the following spaces equipped with their natural Fréchet space structure:

\[
C^\infty_B(\mathbb{R}) = \bigcap_{\rho \geq 0} W^\rho, \infty, \Gamma^m(\mathbb{R}) = \bigcap_{\rho \geq 0} \Gamma^m_{\rho}(\mathbb{R}), \quad \Gamma^{-\infty}(\mathbb{R}) = \bigcap_{\rho \geq 0} \bigcap_{m \in \mathbb{R}} \Gamma^m_{\rho}(\mathbb{R}),
\]

In higher dimensions the 1 in the definition of \( M^m_{\rho} \) should be replaced by \( 1 + \lfloor \frac{d}{2} \rfloor \).

**Definition A.6** Define an admissible cutoff function as a function \( \psi^{B, b} \in C^\infty(\hat{D}^2), \) \( B > 1, b > 0 \) that verifies:

\( 1 \) \( \psi^{B, b}(\xi, \eta) = 0 \) when \( |\xi| < B |\eta| + b \), and \( \psi^{B, b}(\xi, \eta) = 1 \) when \( |\xi| > B |\eta| + b + 1 \).

\( 2 \) for all \( (\alpha, \beta) \in \mathbb{N}^d \times \mathbb{N}^d \), there is \( C_{\alpha, \beta} \), with \( C_{0, 0} \leq 1 \), such that:

\[
\forall (\xi, \eta) : \left| \partial_\xi^\alpha \partial_\eta^\beta \psi^{B, b}(\xi, \eta) \right| \leq C_{\alpha, \beta} (1 + |\xi|)^{-|\alpha| - |\beta|}. \tag{A.2}
\]

**Definition A.7** Consider a real numbers \( m \in \mathbb{R} \), a symbol \( a \in \Gamma^m_{\mathcal{W}} \) and an admissible cutoff function \( \psi^{B, b} \) define the paradifferential operator \( T_a \) by:

\[
\widehat{T_a u}(\xi) = (2\pi)^{-1} \int_\mathbb{R} \psi^{B, b}(\xi - \eta, \eta) \hat{a}(\xi - \eta, \eta) d\eta,
\]

where \( \hat{a}(\eta, \xi) = \int e^{-ix \cdot \eta} a(x, \xi) dx \) is the Fourier transform of \( a \) with respect to the first variable. In the language of pseudodifferential operators:

\[
T_a u = \text{Op}(\sigma_a)u, \quad \text{where} \quad \mathcal{F}_x \sigma_a(\xi, \eta) = \psi^{B, b}(\xi, \eta) \mathcal{F}_x a(\xi, \eta).
\]

Let \( G_{\psi^{B, b}}(x, \eta) = \mathcal{F}_x^{-1} \psi^{B, b}(\cdot, \eta) \) then \( \sigma_a(\cdot, \eta) = G_{\psi^{B, b}}(\cdot, \eta) * a(\cdot, \eta) \) in particular for a Fourier multiplier \( m(D), T_{m(D)} \neq 0 \).

An important property of paradifferential operators is their action on functions with localized spectrum.
Lemma A.1 Consider two real numbers \( m \in \mathbb{R}, \rho \geq 0 \), a symbol \( a \in \Gamma^m_0(\mathbb{R}) \), an admissible cutoff function \( \psi^{B,b} \) and \( u \in \mathcal{S}(\mathbb{R}^d) \).

- For \( R >> b \), if \( \text{supp} \mathcal{F} u \subset \{|\xi| \leq R\} \), then:
  \[
  \text{supp} \mathcal{F} T_a u \subset \left\{ |\xi| \leq (1 + \frac{1}{B}) R - \frac{b}{B} \right\},
  \tag{A.3}
  \]

- For \( R >> b \), if \( \text{supp} \mathcal{F} u \subset \{|\xi| \geq R\} \), then:
  \[
  \text{supp} \mathcal{F} T_a u \subset \left\{ |\xi| \geq \left(1 - \frac{1}{B}\right) R + \frac{b}{B} \right\},
  \tag{A.4}
  \]

The main features of symbolic calculus for paradifferential operators are given by the following theorems taken from [28] and [34].

Theorem A.1 Let \( m \in \mathbb{R} \). if \( a \in \Gamma^m_0(\mathbb{R}) \), then \( T_a \) is of order \( m \). Moreover, for all \( \mu \in \mathbb{R} \) there exists a constant \( K \) such that:

\[
\|T_a\|_{H^\mu \rightarrow H^{\mu-m}} \leq KM^m_0(a) \quad \text{and,} \quad \|T_a\|_{W^\mu,\infty \rightarrow W^{\mu-m,\infty}} \leq KM^m_0(a), \mu \in \mathbb{N}.
\]

Theorem A.2 Let \( m, m' \in \mathbb{R} \), and \( \rho > 0 \), \( a \in \Gamma^m_\rho(\mathbb{R}) \) and \( b \in \Gamma^{m'}_\rho(\mathbb{R}) \).

- Composition: Then \( T_a T_b \) is a paradifferential operator with symbol:

\[
a \otimes b \in \Gamma^{m+m'}_\rho(\mathbb{R}), \quad \text{more precisely},
\]

\[
T^{\psi^{b,b}}_a T^{\psi^{b',b}}_b = T^{\psi^{b\otimes b,b}}_{a \otimes b}.
\]

Moreover \( T_a T_b - T_{a \# b} \) is of order \( m + m' - \rho \) where \( a \# b \) is defined by:

\[
a \# b = \sum_{|\alpha| < \rho} \frac{1}{i^{\alpha} \alpha!} \partial^\alpha_x a \partial^\alpha_x b,
\]

and there exists \( r \in \Gamma^{m+m'-\rho}_0(\mathbb{R}) \) such that:

\[
M^m_0+m'-\rho(r) \leq K (M^m_\rho(a)M^{m'}_0(b) + M^m_0(a)M^{m'}_\rho(b)),
\]

and we have

\[
T^{\psi^{b,b}}_a T^{\psi^{b',b}}_b - T^{\psi^{b\otimes b,b}}_{a \# b} = T^{\psi^{b+b',b}}_r.
\]
• Adjoint: The adjoint operator of $T_a$, $T_a^*$ is a paradifferential operator of order $m$ with symbol $a^*$ defined by:

$$a^* = \sum_{|\alpha| < \rho} \frac{1}{i|\alpha|!} \partial_x^\alpha \partial_x a.$$

Moreover, for all $\mu \in \mathbb{R}$ there exists a constant $K$ such that

$$\|T_a^* - T_a^*\|_{H^{\mu-m} \to H^{\mu-m+\rho}} \leq K M_{\rho}^m(a).$$

If $a = a(x)$ is a function of $x$ only then the paradifferential operator $T_a$ is called a paraproduct. It follows from Theorem A.2 and the Sobolev embedding that:

- If $a \in H^\alpha(\mathbb{R})$ and $b \in H^{\beta}(\mathbb{R})$ with $\alpha, \beta > \frac{d}{2}$, then

$$T_a T_b - T_{ab}$$

is of order $-\left(\min\{\alpha, \beta\} - \frac{d}{2}\right)$.

- If $a \in H^\alpha(\mathbb{R})$ with $\alpha > \frac{d}{2}$, then

$$T_a^* - T_a$$

is of order $-\left(\alpha - \frac{d}{2}\right)$.

- If $a \in W^{r,\infty}(\mathbb{R})$, $r \in \mathbb{N}$ then:

$$\|au - T_a u\|_{H^r} \leq C \|a\|_{W^{r,\infty}} \|u\|_{L^2}.$$  

An important feature of paraproducts is that they are well defined for function $a = a(x)$ which are not $L^\infty$ but merely in some Sobolev spaces $H^r$ with $r < \frac{d}{2}$.

**Proposition A.5** Let $m > 0$. If $a \in H^{\frac{d}{2}-m}(\mathbb{R})$ and $u \in H^\mu(\mathbb{R})$ then $T_a u \in H^{\mu-m}(\mathbb{R})$. Moreover,

$$\|T_a u\|_{H^{\mu-m}} \leq K \|a\|_{H^{\frac{d}{2}-m}} \|u\|_{H^\mu}.$$

A main feature of paraproducts is the existence of paralinearisation theorems which allow us to replace nonlinear expressions by paradifferential expressions, at the price of error terms which are smoother than the main terms.

**Theorem A.3** Let $\alpha, \beta, \kappa \in \mathbb{R}$ be such that $\alpha, \beta > \frac{d}{2}$ and $\kappa \geq 0$, then

- Bony’s Linearization Theorem: For all $C^\infty$ function $F$, if $a \in H^\alpha(\mathbb{R})$ then;

$$F(a) - F(0) - T_{F'(a)} a \in H^{2\alpha - \frac{d}{2}}(\mathbb{R}).$$
Theorem A.5

Let \( u \) be a \( W^{\alpha}(\mathbb{R}) \) and \( c \in W^{k,\infty} \), then \( R(a, b) = ab - T_a b - T_b a \in H^{\alpha + \beta - \frac{d}{2}}(\mathbb{R}) \) and \( R(a, c) = ac - T_a c - T_c a \in H^{\alpha + \kappa}(\mathbb{R}) \). Moreover there exists a positive constant \( K \) independent of \( a, b \) and \( c \) such that:

\[
\begin{align*}
\| R(a, b) \|_{H^{\alpha + \beta - \frac{d}{2}}} &= \| ab - T_a b - T_b a \|_{H^{\alpha + \beta - \frac{d}{2}}} \leq K \| a \|_{H^{\alpha}} \| b \|_{H^{\beta}}, \\
\| R(a, c) \|_{H^{\alpha + \kappa}} &\leq K \| a \|_{H^{\alpha}} \| c \|_{W^{k,\infty}}.
\end{align*}
\] (A.5)

A.3. Paracomposition

We recall the main properties of the paracomposition operator first introduced by Alinhac in [7] to treat low regularity change of variables. Here we present the results we reviewed and generalized in some cases in [34].

**Theorem A.4** Let \( \chi : \mathbb{R}^d \rightarrow \mathbb{R}^d \) be a \( C^{1+r} \) diffeomorphism with \( D\chi \in W^{r,\infty} \), \( r > 0, r \notin \mathbb{N} \) and take \( s \in \mathbb{R} \) then the following map is continuous:

\[
H^s(\mathbb{R}^d) \rightarrow H^s(\mathbb{R}^d)
\]

\[
u \mapsto \chi^* u = \sum_{k \geq 0} \sum_{l \geq 0} \rho_l(D) u_k \circ \chi,
\]

where \( N \in \mathbb{N} \) is such that \( 2^N > \sup_{k, \mathbb{R}^d} |\Phi_k D\chi|^{-1} \) and \( 2^N > \sup_{k, \mathbb{R}^d} |\Phi_k D\chi| \).

Taking \( \tilde{\chi} : \mathbb{R}^d \rightarrow \mathbb{R}^d \) a \( C^{1+\tilde{r}} \) diffeomorphism with \( D\tilde{\chi} \in W^{\tilde{r},\infty} \) map with \( \tilde{r} > 0 \), then the previous operation has the natural fonctorial property:

\[
\forall u \in H^s(\mathbb{R}^d), \chi^* \tilde{\chi}^* u = (\chi \circ \tilde{\chi})^* u + Ru,
\]

with, \( R : H^s(\mathbb{R}^d) \rightarrow H^{s + \min(r, \tilde{r})}(\mathbb{R}^d) \) continous.

We now give the key paralinearization theorem taking into account the paracomposition operator.

**Theorem A.5** Let \( u \) be a \( W^{1,\infty}(\mathbb{R}^d) \) map and \( \chi : \mathbb{R}^d \rightarrow \mathbb{R}^d \) be a \( C^{1+r} \) diffeomorphism with \( D\chi \in W^{r,\infty} \), \( r > 0, r \notin \mathbb{N} \). Then:

\[
u \mapsto \chi^* u(x) = \chi^* u(x) + T_{D\chi(x)} \chi(x) + R_0(x) + R_1(x) + R_2(x)
\]

where the paracomposition given in the previous theorem verifies the estimates:

\[
\forall s \in \mathbb{R}, \| \chi^* u(x) \|_{H^s} \leq C(\| D\chi \|_{L^{\infty}}) \| u(x) \|_{H^s},
\]

\[ u' \circ \chi \in \Gamma^0_{W^{0,\infty}(\mathbb{R}^d)}(\mathbb{R}^d) \text{ for } u \text{ Lipchitz}, \]

and the remainders verify the estimates:

\[
\| R_0 \|_{H^{1+r+\min(1+\rho,s-d/2)}} \leq C \| D\chi \|_{C^\rho_x} \| u \|_{H^{1+s}}
\]
\[
\| R_1 \|_{H^{1+r+s}} \leq C (\| D\chi \|_{L^\infty}) \| D\chi \|_{C^s} \| u \|_{H^{1+s}} . \\
\| R_2 \|_{H^{1+r+s}} \leq C \left( \| D\chi \|_{L^\infty}, \| D\chi^{-1} \|_{L^\infty} \right) \| D\chi \|_{C^s} \| u \|_{H^{1+s}} .
\]

Finally the commutation between a paradifferential operator \( a \in \Gamma^m_{\beta}(\mathbb{R}^d) \) and a paracomposition operator \( \chi^* \) is given by the following

\[
\chi^* T_a u = T_{a^*} \chi^* u + T_{q^*} \chi^* u \text{ with } q \in \Gamma^m_{0-\beta}(\mathbb{R}^d),
\]

where \( a^* \) has the local expansion:

\[
a^*(x, \xi) \sim \sum_{|\alpha| \leq [\min(r, \rho)]} \frac{1}{\alpha!} \partial^\alpha a(x, D\chi^{-1}(\chi(x)))^T \xi) Q_\alpha(x, \xi) \in \Gamma^m_{\min(r, \beta)}(\mathbb{R}^d),
\]

where,

\[
P_\alpha(x', \xi) = D^\alpha_{x'} (e^{i(\chi^{-1}(y')-\chi^{-1}(x')) - D\chi^{-1}(x')(y'-x')) \xi)|_{y'=x'}
\]

and \( Q_\alpha \) is polynomial in \( \xi \) of degree \( \frac{|\alpha|}{2} \), with \( Q_0 = 1 \), \( Q_1 = 0 \).

**Remark A.2** The simplest example for the paracomposition operator is when \( \chi(x) = Ax \) is a linear operator and in that case we see that if \( N \) is chosen sufficiently large in the definition of \( \chi^* \):

\[
u(Ax) = (Ax)^* u, \text{ and } T_{u'(Ax)} Ax = 0.
\]

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