FAMILIES OF NON-TILING DOMAINS SATISFYING PÓLYA’S CONJECTURE

PEDRO FREITAS AND ISABEL SALAVESSA

Abstract. We show the existence of classes of non-tiling domains satisfying Pólya’s conjecture in any dimension, in both the Euclidean and non-Euclidean cases. This is a consequence of a more general observation asserting that if a domain satisfies Pólya’s conjecture eventually, that is, for a sufficiently large order of the eigenvalues, and may be partitioned into $p$ non-overlapping isometric sub-domains, with $p$ arbitrarily large, then there exists an order $p_0$ such that for $p$ larger than $p_0$ all such sub-domains satisfy Pólya’s conjecture. In particular, this allows us to show that families of sectors of domains of revolution with analytic boundary, and thin cylinders satisfy Pólya’s conjecture, for instance. We also improve upon the Li-Yau constant for general cylinders in the Dirichlet case.

1. Introduction

It is now more than sixty years since Pólya conjectured that Dirichlet and Neumann Laplace eigenvalues of Euclidean domains are bounded from below and above, respectively, by the first term in the corresponding Weyl asymptotics [P1]. More precisely, given a bounded domain $\Omega \subset \mathbb{R}^n$ and denoting these eigenvalues by

$$0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \cdots \to \infty, \text{ as } k \to \infty$$

[Dirichlet] and

$$0 = \mu_1(\Omega) \leq \mu_2(\Omega) \leq \cdots \to \infty, \text{ as } k \to \infty,$$

[Neumann]
Pólya’s conjecture states that

$$\mu_{k+1}(\Omega) \leq \frac{4\pi^2}{\omega_n^{2/n}} \left( \frac{k}{|\Omega|} \right)^{2/n} \leq \lambda_k(\Omega),$$

(1.1)

where $|\Omega|$ and $\omega_n$ denote, respectively, the $n$–measure of $\Omega$ and of the $n$–ball of unit radius. A few years later Pólya went on to prove that indeed this is the case for the Dirichlet problem for domains which tile the plane [P2] – see [1], for the extension of Pólya’s result to $\mathbb{R}^n$. Pólya also proved in [P2] that the result holds in the Neumann case, but only for the more restricted class of regularly plane covering domains, with the general case having been proved later by Kellner [K2].

Although the conjecture remains open to this day in both cases, it is at the origin of a number of results in the literature along several different directions. One of the first was what
may be called a proof-of-concept result by Lieb in 1980 [L], who showed that for general bounded domains there exists an absolute constant $C = C_n$, depending only on the dimension, such that

$$\lambda_k(\Omega) \geq C_n \times \frac{4\pi^2}{\omega_n^{2/n}} \left(\frac{k}{|\Omega|}\right)^{2/n}.$$  

(1.2)

Three years later, Li and Yau [LY] improved this result by showing that the constant $C_n$ above could be taken to equal $\frac{n}{n+2}$. A Neumann counterpart to this result was obtained by Kröger in 1992 [Kr], who showed that

$$\mu_k \leq \left(\frac{n+2}{2}\right)^{2/n} \frac{4\pi^2}{(\omega_n)^{2/n}} \left(\frac{k-1}{|\Omega|}\right)^{2/n}.$$  

(1.3)

Other results with different constants may be found, for instance, in Urakawa’s work [U], where the lattice packing density of a domain is used to measure how close to one the factor multiplying the constant in the Weyl asymptotics can be.

Another approach consists in identifying specific constructions that produce new domains satisfying (1.1). This was done by Laptev in [L], who obtained the first examples of non-tiling domains in dimensions four or higher. More generally, he considered $\Omega$ to be the Cartesian product between two domains, say $\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{R}^p \times \mathbb{R}^q$, such that $p$ is larger than or equal to two and satisfies Pólya’s conjecture in the Dirichlet case. Then the same holds for the Cartesian product $\Omega$ – see also [HS] for further results for Neumann boundary conditions. We note that the case of cylinders given by the Cartesian product of a general domain and an interval is still open.

More recently, a connection between this problem and that of determining optimal spectral domains was found by Colbois and El Soufi, who proved an equivalence between Pólya’s conjecture and the convergence of the sequence $k^{-2/n}\lambda_k^*$, where $\lambda_k^*$ denotes the smallest possible value of $\lambda_k(\Omega)$ for domains $\Omega$ with given $|\Omega|$ [CS] – see also [FLP] for a similar result for domains which are unions of scaled copies of a single domain, and for references to other similar problems.

In this note we explore the second approach described above, providing a way of obtaining families of domains satisfying Pólya’s conjecture but which do not tile the space, and without the need to assume the existence of other such domains a priori. Another case where such a domain has been shown to exist is in the recent proof for the important (and iconic) case of Euclidean balls [FLPS].

The idea behind our approach is quite simple, and may be thought of as replicating Pólya’s argument to the case where instead of tiling the whole space we start from a domain that may be tiled by an arbitrarily large number of isometric copies of a subdomain. It is quite straightforward to see that, should the larger domain satisfy Pólya’s conjecture, then the same is true for the smaller domain [FMS]. However, and under certain conditions, we are able to show that it is sufficient for Pólya’s conjecture to hold for large enough eigenvalues of the original domain, for this to imply that there is an order of division of that domain into tiling subdomains, after which they must satisfy Pólya’s conjecture. Using this approach, we are able to show that, for instance, planar circular sectors with a sufficiently small angle opening and sufficiently thin cylinders satisfy inequality (1.1), as do certain sufficiently thin tiling subsets of solids of revolution in $\mathbb{R}^n$. A related idea had been used previously by Hersch [H] and also by the first author of the current paper [F], but only as a way of obtaining bounds for the first eigenvalue.
In the next section we formalise these ideas and establish the basic lemma which then allows us to provide several examples of non-tiling domains satisfying Pólya’s conjecture. We then consider the case of general cylinders in more detail, for which we substantially improve the constant $C_n = n/(n + 2)$ in (1.2).

2. The basic lemma

In order to proceed, it is convenient to have the following two definitions.

Given domains $\Omega, \Omega_0 \subset \mathbb{R}^n$ and an integer $p$, we say that $\Omega$ is $p$-tiled by $\Omega_0$ if $\Omega$ is the interior of the closure of the union of $p$ nonoverlapping isometric copies of $\Omega_0$.

We say that $\Omega$ satisfies Pólya’s conjecture eventually (for Dirichlet eigenvalues), if there is an order $k_0$ such that $\lambda_k(\Omega)$ satisfies the right-hand side inequality in (1.1) for all $k$ larger than or equal to $k_0$. A similar definition applies to the Neumann eigenvalues and the left-hand side inequality in (1.1). A class of domains which fall into this category are those having a second positive (resp. negative) term in the Weyl asymptotics, for the Dirichlet (resp. Neumann) eigenvalues respectively. More precisely, under certain geometric conditions in $\Omega$ we have the following two-term Weyl asymptotics [SV, Theorem 1.6.1 and Example 1.6.16]

$$
\lambda_k(\Omega) = \frac{4\pi^2 k^{2/n}}{(\omega_n|\Omega|)^{2/n}} + \frac{2\pi^2 \omega_n-1|\partial\Omega| k^{1/n}}{n (\omega_n|\Omega|)^{1+1/n}} + o(k^{1/n})
$$

and

$$
\mu_k(\Omega) = \frac{4\pi^2 k^{2/n}}{(\omega_n|\Omega|)^{2/n}} - \frac{2\pi^2 \omega_n-1|\partial\Omega| k^{1/n}}{n (\omega_n|\Omega|)^{1+1/n}} + o(k^{1/n}),
$$

where $|\partial\Omega|$ denotes the $(n-1)$-measure of the boundary of $\Omega$.

**Lemma 2.1.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain satisfying Pólya’s conjecture eventually and assume that for each $p \in \mathbb{N}$ there exists a domain $\Omega_p$ such that $\Omega_p$ $p$-tiles $\Omega$. Then there exists $p_0$ such that $\Omega_p$ satisfies Pólya’s conjecture for the Dirichlet eigenvalues for all $p$ larger than $p_0$. A similar conclusion holds for the Neumann eigenvalues.

**Proof.** If $\Omega$ contains $p$ nonoverlapping copies of $\Omega_p$, it follows by Lemma 1 in [P2] that $\lambda_{kp}(\Omega) \leq \lambda_k(\Omega_p)$ for all positive integer $k$. Hence,

$$
\lambda_k(\Omega_p) \geq \frac{4\pi^2}{(\omega_n|\Omega|)^{2/n}} (kp)^{2/n}
$$

for $kp$ sufficiently large, say $kp \geq k_0$. By taking $p \geq k_0$, we obtain that the above inequality is satisfied for all positive integer $k$. On the other hand, $|\Omega| = p|\Omega_p|$, and so

$$
\lambda_k(\Omega_p) \geq \frac{4\pi^2}{(\omega_n|\Omega_p|)^{2/n}} k^{2/n}
$$

for all $p \geq k_0$ and $k \in \mathbb{N}$, proving the result in the Dirichlet case.

Similarly, by Lemma 2 in [P2] we have $\mu_{(k-1)p+1}(\Omega) \geq \mu_k(\Omega_p)$, for all $k \geq 1$, and assuming the Neumann eigenvalues of $\Omega$ satisfy Pólya’s inequality for $k \geq k_0'$, we obtain for $(k-1)p+1 \geq k_0'$,

$$
\mu_k(\Omega_p) \leq \frac{4\pi^2}{(\omega_n|\Omega_p|)^{2/n}} (k-1)^{2/n}.
$$
If \( k'_{0} = 1 \) we are done for any \( p \) and \( k \geq 1 \). If \( k'_{0} \geq 2 \) we take \( p \geq k'_{0} \) and we see that Pólya’s inequality holds on \( \Omega_{p} \) for any \( k \geq 2 \). It also holds for \( k = 1 \). \( \square \)

**Remark 2.1.** The same proof may be used to obtain a similar result if \( \Omega \) is a manifold with boundary – see the examples in Sections 3.3 and 3.4 below, which also shows that the condition on the existence of the order \( p_{0} \) cannot be dropped in general.

**Remark 2.2.** Clearly it is possible to state the above result for a specific value of \( p_{0} \) for which the conditions are satisfied. However, and without the assumption that the tiling takes place for all \( p \), it cannot be asserted that the conjecture is satisfied for all \( p \) larger than \( p_{0} \), as it may happen that the domain \( \Omega \) cannot be tiled for all such values of \( p \). To see this, it is enough to consider a regular \( n \)-polygon which is tiled by \( n \) equal triangles (which satisfy Pólya, as they are also tiling domains), but which, except for \( 4k^{*}n \) or \( 2 \times 4k^{*}n \), with \( k \in \mathbb{N}_{0} \), will not be tiled with similar triangles for other integers larger than \( n \).

### 3. Examples

We shall now use Lemma 2.1 to provide some examples of non-tiling domains satisfying Pólya’s conjecture.

#### 3.1. Sectors of domains of revolution.

Let \( \Omega \) be a convex domain of revolution around an axis in \( \mathbb{R}^{n} \) and with analytic boundary. Then it satisfies the nonperiodicity condition [SV, Lemma 1.3.19], and thus also Pólya’s conjecture eventually. On the other hand, \( \Omega \) is clearly tiled by \( p \) sectors of \( \Omega \) with an angle opening \( 2\pi/p \), and Lemma 2.1 may then be applied to obtain that, for all \( p \) sufficiently large, these sectors satisfy Pólya’s conjecture.

In the particular case of an \( n \)-dimensional ball \( B \), it can be \( p \)-tilied by circular sectors \( S_{p} \) with the same radius and angle opening \( 2\pi/p \), for any positive integer \( p \), as described in the previous example. The proof that \( D \) satisfies Pólya’s conjecture allowed the authors in [FLPS] to deduce that the same will happen for any sector of this form Although Lemma 2.1 only yields the existence of sufficiently thin sectors for which Pólya’s conjecture holds, say \( S_{p_{0}} \), it does not require the angle to be a rational multiple of \( \pi \) and may be extended to sectors of shells.

More precisely, given any angle \( \alpha \) in \( (0, \pi] \), we may conclude that there exists \( q = q_{\alpha} \in \mathbb{N} \) such that Pólya’s conjecture holds for sectors with opening angle \( \alpha/(jq_{\alpha}) \) for all \( j \in \mathbb{N} \). That Weyl’s two-term asymptotics \( (2.1) \) are satisfied in these cases, stems from the fact that such domains allow for separation of variables \( [N] \).

#### 3.2. Solid cylinders.

Let \( \Omega \subset \mathbb{R}^{n} \) be a domain satisfying the nonperiodicity condition (Definition 1.3.7 in [SV]), and consider the cylinder obtained by the cartesian product \( C_{\ell} = \Omega \times J \subset \mathbb{R}^{n+1} \), where \( J \) is the interval \([0, \ell]\). By considering the projection of the trajectories in \( C_{\ell} \) onto the basis \( \Omega \) of the cylinder, it is not difficult to see that \( C_{\ell} \) also satisfies the nonperiodicity condition – see also [M], where the two-term asymptotics was established directly in the case when \( n \) is two. By Theorem 1.6.1 in [SV] (see also the Remark 1.6.2 following the theorem and Example 1.6.16), the cylinder \( C_{\ell} \) satisfies the two-term Weyl asymptotics \( (2.1) \) and \( (2.2) \), and hence it satisfies Pólya’s conjecture eventually. We thus obtain that for each of the Dirichlet
and Neumann cases there exist values of \( \ell_p \) sufficiently small such that the cylinder \( C_{\ell_p} \) satisfies Pólya’s conjecture.

Lemma 2.1 may also be applied non-Euclidean examples, where the main point is again whether or not the nonperiodicity condition is satisfied. However, now there are known examples where it actually fails, such as hemispheres – see [SV, Example 1.3.16] and [FMS].

3.3. Sectors of geodesic disks. Provided we can ensure both the nonblocking and nonperiodicity conditions, a similar approach to that of example 3.1 above may be used for sectors of geodesic disks on manifolds. Concerning the former condition, this will hold for geodesic disks in both \( \mathbb{S}^n \) and \( \mathbb{H}^n \).

The nonperiodicity condition will hold for strongly convex spherical caps of the standard \( n \)-dimensional sphere \( \mathbb{S}^n \), that is, those that are strictly contained in a hemisphere [SV, Example 1.3.16]. We thus obtain that sufficiently thin sectors on \( \mathbb{S}^n \) will satisfy Pólya’s conjecture. In the hyperbolic case, it is possible to derive from [SV, Lemma 1.3.19] that the nonperiodicity condition holds for geodesic disks of any radius, and thus again the conjecture will be satisfied for sufficiently thin sectors.

3.4. Cylindrical surfaces. This example shows that, in general, the condition that \( p \) be large enough in Lemma 2.1 cannot be dropped. Let \( M \) be a closed \((n - 1)\)-manifold and assume the conditions in the lemma are satisfied for the domain \( S_\ell = M \times J \), where again \( J \) is the interval \([0, \ell]\) – recall that, as pointed out in Remark 2.1, Lemma 2.1 also holds in this case. This means that \( S_\ell \) may be sliced up into \( p \) domains in the way described in the lemma, each being a cylindrical surface itself, of the form \( S_{\ell/p} \). Then, for large enough \( p \), \( S_{\ell/p} \) will satisfy Pólya’s conjecture.

On the other hand, since \( M \) is closed, its first eigenvalue is zero and the first eigenvalue of \( S_h \) equals \( \pi^2/h^2 \). This eigenvalue will satisfy Pólya’s conjecture if and only if

\[
\frac{\pi^2}{h^2} \geq \frac{4\pi^2}{(\omega_n h |M|)^{2/n}} \Leftrightarrow h \leq \frac{1}{2^{n/(n-1)}} (\omega_n |M|)^{1/(n-1)}.
\]

We thus see that, given a value of \( \ell \), unless the height \( h = \ell/p \) is small enough, the first eigenvalue will not satisfy the above inequality.

A specific example may be obtained by considering the two-dimensional cylindrical surface \( S_\ell = S^1 \times [0, \ell] \), where \( S^1 \) denotes the unit circle. This satisfies the necessary conditions for the two-term Weyl asymptotics given by (2.1) to hold, and thus, given large enough \( \ell \), \( S_\ell \) does not satisfy Pólya’s conjecture, while \( S_{\ell/p} \) will do so, provided \( p \) is a sufficiently large integer.

4. Improved Constants for Cylinders

The results in [L] require the space dimension of the domain satisfying Pólya’s conjecture to be at least two, and thus do not apply to general cylinders of the form \( C_\ell = \Omega \times [0, \ell] \subset \mathbb{R}^{n+1} \), where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) and \( \ell \) a positive real number – note that the interval satisfies Pólya’s inequality which in this case becomes an identity for all eigenvalues. Thus, and to the best of our knowledge, the best result so far in this case is the Li and Yau bound \( (1.2) \) with constant given by \((n + 1)/(n + 3)\). This is a consequence of a result for the sum of the first \( k \)
eigenvalues, namely, that Dirichlet eigenvalues of an Euclidean domain $\Omega$ in $\mathbb{R}^n$ satisfy

\begin{equation}
\frac{1}{k} \left( \sum_{j=1}^{k} \lambda_j(\Omega) \right) \geq \beta_n \frac{4\pi^2 k^{2/n}}{(\omega_n|\Omega|)^{2/n}}, \quad \text{with} \quad \beta_n := \frac{n}{n+2}.
\end{equation}

which is asymptotically sharp as $k$ goes to infinity – as was mentioned in \cite{L}, this inequality is also related to a previous result by Berezin \cite{B}. In the next result we use (4.1) to improve the constant $C_n$ in (1.2) for general cylinders.

**Theorem 4.1.** Consider the cylinder $C_\ell = \Omega \times [0, \ell] \subset \mathbb{R}^{n+1}$, where $\Omega$ is a bounded domain in $\mathbb{R}^n$, and $\ell$ any positive real number. Then its Dirichlet eigenvalues satisfy

\begin{equation}
\lambda_k(C_\ell) \geq \alpha_{n+1} \left( \frac{4\pi^2}{\omega_{n+1}} \right)^{2/(n+1)} \left( \frac{k}{\ell|\Omega|} \right)^{2/(n+1)},
\end{equation}

where

\begin{equation}
\alpha_n := \frac{n}{(n+1)^{1-\frac{1}{n}}} \left( \frac{\pi}{4} \right)^{1/n} \left[ \frac{n+1}{\Gamma \left( \frac{n+1}{2} \right)} \right]^{2/n} \left[ \frac{n+2}{\Gamma \left( \frac{n+2}{2} \right)} \right]^{2/n}.
\end{equation}

For any $n \geq 2$, the coefficients $\alpha_n$ satisfy

\begin{equation}
\beta_{n+1} < \frac{n+1}{n+2} \left( \frac{\pi}{2} \times \frac{n+3}{n+2} \right)^{1/(n+1)} < \alpha_{n+1} < \frac{n+1}{n+2} \left( \frac{\pi}{2} \times \frac{n+2}{n+1} \right)^{1/(n+1)} < 1,
\end{equation}

where $\beta_n$ denotes the Berezin-Li-Yau coefficients defined in (4.1).

**Proof.** Denoting the eigenvalues of $\Omega$ in increasing order and repeated according to multiplicities by $\eta_j$, and those of the interval $J = [0, \ell]$ by $\rho_j = \pi^2 j^2 / \ell^2$, the eigenvalues of the Cartesian product $C_\ell = \Omega \times [0, \ell]$, are given by $\eta_j + \rho_l$. Given $\lambda > 0$, let $j_\lambda$ be the largest integer such that $\lambda - \eta_{j_\lambda} \geq 0$. Then we have

\begin{align}
N_{\Omega \times J}(\lambda) &= \# \{(j, l) : \eta_j + \rho_l \leq \lambda\} \\
&= \# \{(j, l) : \rho_l \leq \lambda - \eta_j\} \\
&= \sum_j \# \{l : \rho_l \leq (\lambda - \eta_j)_+\} \\
&= \sum_{j=1}^{j_\lambda} N_J((\lambda - \eta_j)) \\
&= \sum_{j=1}^{j_\lambda} \left[ \frac{\ell}{\pi} \sqrt{\lambda - \eta_j} \right] \\
&\leq \sum_{j=1}^{j_\lambda} \frac{\ell}{\pi} \sqrt{\lambda - \eta_j}.
\end{align}

(4.4)
Using the Cauchy-Schwarz inequality we obtain

\[ N_{\Omega \times J}(\lambda) \leq \frac{\ell}{\pi} \sqrt{j\lambda} \left( \sum_{j=1}^{j_\lambda} (\lambda - \eta_j) \right)^{1/2} \]

(4.5) \[ = \frac{\ell}{\pi} \sqrt{j\lambda} \left( \lambda j_\lambda - \sum_{j=1}^{j_\lambda} \eta_j \right)^{1/2} . \]

The Berezin-Li-Yau inequality (4.1) applied to the eigenvalues of \( \Omega \) reads

\[ \sum_{j=1}^{k} \eta_j \geq \beta_n \frac{4\pi^2 k^{2/n+1}}{(\omega_n |\Omega|)^{2/n}} , \]

and plugging this in (4.5) then yields

\[ k := N_{\Omega \times J}(\lambda) \leq \frac{\ell j_\lambda}{\pi} \left( \lambda - \beta_n \frac{4\pi^2 j_\lambda^{2/n}}{(\omega_n |\Omega|)^{2/n}} \right)^{1/2} . \]

Thus, for \( \lambda = \lambda_k \), we have

(4.6) \[ \lambda_k \geq \frac{\pi^2 k^2}{\ell^2 j_\lambda^2} + \beta_n \frac{4\pi^2 j_\lambda^{2/n}}{(\omega_n |\Omega|)^{2/n}} . \]

The right-hand side may now be viewed as a function on \( j_\lambda^2 \), having its minimum at

\[ j_\lambda^2 = \left( \frac{nk^2}{4\ell^2 \beta_n} \right)^{n/(n+1)} (\omega_n |\Omega|)^{2/(n+1)} . \]

Plugging this back in (4.6) we obtain the following lower bound for \( \lambda_k \)

\[ \lambda_k \geq \frac{\pi^2 k^2}{\ell^2} \left( \frac{4\beta_n \ell^2}{nk^2} \right)^{n/(n+1)} \frac{1}{(\omega_n |\Omega|)^{2/(n+1)}} + 4\pi^2 \beta_n \left( \frac{nk^2}{4\ell^2 \beta_n} \right)^{1/(n+1)} \frac{1}{(\omega_n |\Omega|)^{2/(n+1)}} \]

\[ = \left( \frac{4\beta_n}{\omega_n \ell |\Omega|} \right)^{n/(n+1)} \pi^2 \left( \frac{1}{n^{n/(n+1)}} + n^{1/(n+1)} \right) k^{2/(n+1)} . \]

Finally, since the cylinder is now in \( \mathbb{R}^{n+1} \), and recalling that

\[ \omega_{n+1} = \omega_n \sqrt{\pi} \frac{\Gamma \left( \frac{n}{2} + 1 \right) \Gamma \left( \frac{n+1}{2} + 1 \right)}{\Gamma \left( \frac{n+1}{2} + 1 \right)} , \]
we obtain, by replacing for $\omega_n$ in the previous expression,

$$\lambda_k \geq \frac{(4\beta_n)^{n/(n+1)}}{(\omega_{n+1}\ell(\Omega))^{2/(n+1)}} \left[ \sqrt{\pi} \frac{\Gamma\left(\frac{n}{2} + 1\right)}{\Gamma\left(\frac{n+1}{2} + 1\right)} \left( \frac{1}{n^{n/(n+1)}} + n^{1/(n+1)} \right) k^{2/(n+1)} \right]^{2/(n+1)}$$

$$= \frac{n + 1}{(n + 2)^{1-1/(n+1)}} \left( \frac{\pi}{4} \right)^{1/(n+1)} \left[ \frac{\Gamma\left(\frac{n}{2} + 1\right)}{\Gamma\left(\frac{n+1}{2} + 1\right)} \right]^{2/(n+1)} \frac{4\pi^2}{(\omega_{n+1}\ell(\Omega))^{2/(n+1)}} k^{2/(n+1)}$$

$$= \alpha_{n+1} \frac{4\pi^2}{(\omega_{n+1}\ell(\Omega))^{2/(n+1)}} k^{2/(n+1)}$$
as desired.

It remains to prove the inequalities in the second part of the theorem. The left- and right-most inequalities follow in a straightforward way by algebraic manipulation. To prove the upper and lower bounds for the coefficients $\alpha_n$, we make use of Wendel’s inequalities for quotients of gamma functions [W], which in this case read as

$$\frac{1}{(n + 1)^{1/2}} \leq \frac{\Gamma\left(\frac{n}{2} + 1\right)}{\Gamma\left(\frac{n+1}{2} \right)} \leq \frac{n + 1/2}{n + 1/2}.$$

Replacing these in the expression for $\alpha_n$ yields the two inequalities.

**Remark 4.1.** The first two terms in the asymptotic behaviour of the coefficients $\alpha_n$ defined in Theorem 4.1 are given by $1 - \frac{1 - \log(\pi/2)}{n}$, which may still be seen to be smaller than $\alpha_{n+1}$, while remaining larger than $\beta_{n+1}$.

**Remark 4.2.** As we see from the last part of the result, our constant is larger than the corresponding Berezin-Li-Yau constant $\beta_n$. For instance, when $n = 2$ we obtain $\alpha_3 = (3/4)^{1/3} \approx 0.90856$ while the corresponding $\beta_3$ equals $3/5$.

**Remark 4.3.** The essential property of the domain $\Omega$ which is used in the proof of Theorem 4.1 is that it satisfies the Berezin-Li-Yau estimate (4.1). Provided this is the case, a similar proof may be used to obtain the same result when $\Omega$ is a domain (with boundary) on a manifold.

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Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais, 1049-001 Lisboa, Portugal & Grupo de Física Matemática, Faculdade de Ciências, Universidade de Lisboa, Campo Grande, Edifício C6, 1749-016 Lisboa, Portugal

Email address: pedrodefreitas@tecnico.ulisboa.pt

Grupo de Física Matemática, Faculdade de Ciências, Universidade de Lisboa, Campo Grande, Edifício C6, 1749-016 Lisboa, Portugal & Departamento de Física, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais, 1049-001 Lisboa, Portugal

Email address: isabel.salavessa@tecnico.ulisboa.pt