Lie symmetry analysis and invariant solutions of (3 + 1)-dimensional Calogero–Bogoyavlenskii–Schiff equation

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Abstract Lie group analysis is applied to carry out the similarity reductions of the (3 + 1)-dimensional Calogero–Bogoyavlenskii–Schiff (CBS) equation. We obtain generators of infinitesimal transformations of the CBS equation and each of these generators depend on various parameters which give us a set of Lie algebras. For each of these Lie algebras, Lie symmetry method reduces the (3 + 1)-dimensional CBS equation into a new (2 + 1)-dimensional partial differential equation and to an ordinary differential equation. In addition, we obtain commutator table of Lie brackets and symmetry groups for the CBS equation. Finally, we obtain closed-form solutions of the CBS equation by using the invariance property of Lie group transformations.

Keywords (3 + 1)-Dimensional Calogero–Bogoyavlenskii–Schiff equation · Lie symmetries · Similarity transformations method · Generators of infinitesimal transformations · Similarity solutions

1 Introduction

The nonlinear partial differential equations exhibit a rich variety of nonlinear phenomena and they arise in many physical fields like the stratified shear flow in ocean and atmosphere, condensed matter physics, fluid mechanics. The nonlinear characteristic abruptly changes due to small variation in given parameters including time. Therefore, closed-form solutions of nonlinear PDEs play an important role to look into the internal mechanism of convoluted physical phenomena. Many significant methods have been proposed for obtaining solutions of nonlinear PDEs (for details, see [1–5]). Lie group method of infinitesimal transformations [6,7] is an important tool to find analytical solutions of nonlinear PDEs and it is used by many authors [8–11].

The study of multidimensional integrable systems is one of the main themes in integrable systems. Several integrable models have been recently developed in the context of (3 + 1)-dimensional equations. In this paper, we study the generalized (3 + 1)-dimensional Calogero–Bogoyavlenskii–Schiff (CBS) equation given by

\[ v_t + \Phi(v)v_y + \Phi_1(v)v_z = 0, \]

where

\[ \Phi(v) = \partial_x^2 + av + bv_x \partial_x^{-1}, \]
\[ \Phi_1(v) = \partial_x^2 + cv + dv_x \partial_x^{-1}. \]
or equivalently
\[ v_t + avv_y + cvv_z + bu_x \partial_x^{-1} v_y + du_x \partial_x^{-1} v_z + v_y x y + v_z x z = 0, \]
where \( a, b, c \) and \( d \) are parameters. Using a dimensional reduction \( \partial_z = \partial y = \partial x \), Eq. (1) is reduced to the standard Korteweg–de Vries (KdV) equation. The \((3+1)\)-dimensional CBS equation (1) can be written in the potential form
\[ \Delta := u_{xt} + au_x u_{xy} + bu_x u_{xx} + cu_x u_{xz} + du_x u_{xx} + u_x x y + u_x x z = 0, \]
using the potential \( v = u_x \). The CBS equation was first introduced by Bogoyavlenskii and Schiff [12–14] in different ways. Bogoyavlenskii used modified Lax formalism, whereas Schiff derived the same equation by reducing self-dual Yang–Mills equation. In [12], Bogoyavlenskii proved that every equation equivalent to the CBS equation has an overturning soliton. In [13], several periodic and breaking solutions were constructed for the \((2+1)\)-dimensional CBS equation as well as for a modified equation related to the CBS equation via the Miura transformation \( v^2 u_x = u_x \). It was shown that the CBS equation can be transformed into a trilinear form in [14]. Toda and Yu [15] constructed some new integrable models using the Calogero method. In this method, the \((2+1)\)-dimensional equations are derived considering the Lax pair \( L \) and \( T \) and modifying the \( T \) operator to include another spatial dimension \( z \). They also thus derived the \((2+1)\)-dimensional CBS equation from the Korteweg-de Vries equation. Kobayashi [16] has applied Painlevé test to derive a modified version with variable coefficients of the \((2+1)\)-dimensional Korteweg-de Vries or CBS equation. They have also shown that a transformation which links the form with variable coefficients to the canonical one and its Lax pair with a nonisospectral condition in \((2+1)\)-dimensions. Bruzon et al. [17] applied the classical Lie group method of infinitesimal transformations to the CBS equation, as well as related potential equation. Li et al. [18] have investigated singular solutions and their limit forms for the generalized CBS equation by using bifurcation method of dynamical systems. Wang [19] implemented Hirota bilinear method to construct quasi-periodic wave solutions in terms of theta functions for Hirota equation and this results to the \((2+1)\)-dimensional generalized CBS equation to get quasi-periodic wave solutions under the Bäcklund transformation. Wazwaz [20] applied Hirota’s bilinear method to obtain multiple front-wave solutions of \((2+1)\)-dimensional CBS equation and two other nonlinear models. In [21,22], the authors derived exact solutions of \((2+1)\)-dimensional CBS equation with the help of group symmetry method.

In the present article, we apply Lie group method of infinitesimal transformations to the \((3+1)\)-dimensional CBS equation (2). We point out Lie symmetries and similarity reductions by constructing invariant solutions from the set of Lie algebras. We also obtain some new reduced PDEs and ODEs. By using these reduced ODEs, we obtain closed-form solutions for Eq. (2).

### 2 Method of symmetries

In this section, we recall the general procedure for determining symmetries for any system of PDEs. Let us consider the general case of a system of PDEs of order \( p \) with \( m \) dependent and \( n \) independent variables given as
\[ \Delta_s(x, u^p) = 0, \quad s = 1, \ldots, l, \]
where \( x = (x_1, x_2 \ldots x_n) \), \( u = (u_1, u_2 \ldots u_m) \) and the derivatives of \( u \) with respect to \( x \) up to order \( p \) is denoted by \( u^p \). We consider the one-parameter Lie group of infinitesimal transformations acting on the dependent variable \( u \) and the independent variable \( x \) of the system (3):
\[ \begin{align*}
\dot{x}^i &= x^i + \omega \xi^i(x, u) + O(\omega^2), & i = 1, 2, \ldots, n, \\
\dot{u}^j &= u^j + \omega \eta^j(x, u) + O(\omega^2), & j = 1, 2, \ldots, m,
\end{align*} \]
where \( \xi^i \) and \( \eta^j \) are generators of infinitesimal transformations for the independent and the dependent variables, respectively, and \( \omega \) is the group parameter which is admitted by the system (3).

The generator \( \mathbf{v} \) associated with the above group of transformations can be written as
\[ \mathbf{v} = \sum_{i=1}^{n} \xi^i(x, u) \partial_x^i + \sum_{a=1}^{m} \eta^a(x, u) \partial_{u^a}. \]
Lie group of transformations are such that if \( u \) is a solution of the system (3), then \( \dot{u} \) is also a solution.
The method for finding group symmetry [7] is by finding corresponding generators of Lie group of infinitesimal transformations. This leads to an overdetermined linear system of equations for generators $ξ^j(x, u)$ and $η^j(x, u)$. The invariance of the system (3) under the group transformations leads to the invariance conditions

$$P_{r^{(p)}} u = 0, \quad s = 1, \ldots, l$$

whenever $Δs(x, u^p) = 0$, (6)

where $P_{r^{(p)}}$ is called the $p$th-order extension of the generator of infinitesimal transformations $u$ given by

$$P_{r^{(p)}} u = ν + \sum_{α=1}^{n} \sum_{i=1}^{j} η^j(x, u^p) η_u^j,$$ (7)

where $j = (j_1, \ldots, j_k), 1 ≤ j_k ≤ n, 1 ≤ k ≤ p$, and sum is over all the orders of $j$’s, $0 < j_p ≤ p$. If $j = k$, then coefficients $η_u$ of $η_u^j$ depend only on derivatives of $u$ up to order $k$ and

$$η_u^j(x, u^p) = D_j \left( η_u - \sum_{i=1}^{n} η^i u_i^p \right) + \sum_{i=1}^{n} η^i u_{j, i}^p,$$ (8)

where $u_i^p = \frac{∂u_i^p}{∂x^i}$ and $u_{j, i}^p = \frac{∂u_{j, i}^p}{∂x^i}$. The set of all infinitesimal symmetries of this system have an important property that it forms a Lie algebra under the usual Lie bracket.

3 Lie symmetry analysis of (3 + 1)-dimensional CBS equation

First, we perform Lie similarity reductions of CBS equation using Lie symmetry analysis. We consider the one-parameter Lie group of infinitesimal transformations on $(x_1 = x, x_2 = y, x_3 = z, x_4 = t, u_1 = u)$,

$$\dot{x} = x + ωξ^1(x, y, z, t, u) + O(ω^2),$$

$$\dot{y} = y + ωξ^2(x, y, z, t, u) + O(ω^2),$$

$$\dot{z} = z + ωξ^3(x, y, z, t, u) + O(ω^2),$$

$$\dot{t} = t + ωτ(x, y, z, t, u) + O(ω^2),$$

$$\dot{u} = u + ωη(x, y, z, t, u) + O(ω^2),$$

where $ω$ is the continuous group parameter. The associated vector field has the form

$$\mathbf{v} = \xi^1(x, y, z, t, u) \frac{∂}{∂x} + \xi^2(x, y, z, t, u) \frac{∂}{∂y} + \xi^3(x, y, z, t, u) \frac{∂}{∂z} + \tau(x, y, z, t, u) \frac{∂}{∂t} + η(x, y, z, t, u) \frac{∂}{∂u}.$$
by taking linear combinations of generators $v_i$ number of subalgebras for this Lie algebra obtained linearly independent. In general, there is an infinite

\[ v_i \begin{array}{cccccc}
0 & -2v_2 & 2v_3 & 0 & 0 & 0 \\
2v_2 & 0 & \xi v_6 + v_5 & 0 & 0 & 0 \\
-2v_3 & -\xi v_6 - v_5 & 0 & 2v_3 & 0 & 0 \\
0 & 0 & -2v_3 & 0 & -2v_5 & -2v_6 \\
0 & 0 & 0 & 2v_5 & 0 & 0 \\
0 & 0 & 0 & 0 & 2v_6 & 0 \\
\end{array} \]


\[ v_2 = \lambda(t) \frac{\partial}{\partial x} + \frac{\partial}{\partial t} + \left( \frac{\lambda'(t)y}{b} + \gamma \left( \frac{bz - dy}{b} , t \right) \right) \frac{\partial}{\partial u}, \]

\[ v_3 = \lambda(t) \frac{\partial}{\partial x} + \frac{at}{c} \frac{\partial}{\partial y} + t \frac{\partial}{\partial z} + \left( \frac{x}{c} + \frac{\lambda'(t)y}{b} + \gamma \left( \frac{bz - dy}{b} , t \right) \right) \frac{\partial}{\partial u}, \]

\[ v_4 = (-x + \lambda(t)) \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y} + 2z \frac{\partial}{\partial z} + \left( u + \frac{\lambda'(t)y}{b} + \gamma \left( \frac{bz - dy}{b} , t \right) \right) \frac{\partial}{\partial u}, \]

\[ v_5 = \lambda(t) \frac{\partial}{\partial x} + \frac{\partial}{\partial z} + \left( \frac{\lambda'(t)y}{b} + \gamma \left( \frac{bz - dy}{b} , t \right) \right) \frac{\partial}{\partial u}, \]

\[ v_6 = \lambda(t) \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \left( \frac{\lambda'(t)y}{b} + \gamma \left( \frac{bz - dy}{b} , t \right) \right) \frac{\partial}{\partial u}. \]

(13)

It is convenient to display the commutators of Lie algebra by making arbitrary functions as constant (or zero), through its commutator table whose $(i, j)$th entry is $[v_i, v_j] = v_i \ast v_j = v_i \cdot v_j - v_j \cdot v_i$. The commutator table is antisymmetric with its diagonal elements all being zero as we have $[v_a, v_b] = -[v_b, v_a]$ (for more details, see [5,6]). The structure constants are easily read off from the commutator table. For (13), we have the following table of Lie brackets (Table 1):

| *  | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ |
|----|------|------|------|------|------|------|
| $v_1$ | 0    | $-2v_2$ | $2v_3$ | 0    | 0    | 0    |
| $v_2$ | $2v_2$ | 0    | $\xi v_6 + v_5$ | 0    | 0    | 0    |
| $v_3$ | $-2v_3$ | $-\xi v_6 - v_5$ | 0    | $2v_3$ | 0    | 0    |
| $v_4$ | 0    | 0    | $-2v_3$ | 0    | $-2v_5$ | $-2v_6$ |
| $v_5$ | 0    | 0    | 0    | $2v_5$ | 0    | 0    |
| $v_6$ | 0    | 0    | 0    | 0    | $2v_6$ | 0    |

The set of all these representatives is called an optimal system (for details, see [6,7]).

4 Symmetry group of $(3 + 1)$-dimensional
Calogero–Bogoyavlenskii–Schiff equation

In this section, to obtain the group transformations $X_i : (x, y, z, t, u) \rightarrow (\hat{x}, \hat{y}, \hat{z}, \hat{t}, \hat{u})$ which is generated by the generators of infinitesimal transformations $v_i$ for $i = 1, 2, \ldots, 6$, we need to solve following system of ODEs

\[ \frac{d(\hat{x}, \hat{y}, \hat{z}, \hat{t}, \hat{u})}{d\omega} = (\xi^1, \xi^2, \xi^3, \tau, \eta), \]

\[ (\hat{x}, \hat{y}, \hat{z}, \hat{t}, \hat{u})|_{\omega=0} = (\xi^1, \xi^2, \xi^3, \tau, \eta). \]

The one-parameter groups $X_i$ spanned by $v_i$ are given as follows

$X_1 : (x, y, z, t, u) \rightarrow (x + \omega(x + \lambda(t)), y, z, t + 2\omega t, u + \omega \left( -u + \frac{\lambda'(t)y}{b} + \gamma \left( \frac{bz - dy}{b} , t \right) \right)),$

$X_2 : (x, y, z, t, u) \rightarrow (x + \omega \lambda(t), y, z, t + \omega, u + \omega \left( \frac{\lambda'(t)y}{b} + \gamma \left( \frac{bz - dy}{b} , t \right) \right)),$

$X_3 : (x, y, z, t, u) \rightarrow (x + \omega \lambda(t), y + \omega \frac{\omega t}{c} + z + \omega t, u + \omega \left( \frac{x}{c} + \frac{\lambda'(t)y}{b} + \gamma \left( \frac{bz - dy}{b} , t \right) \right)),$

$X_4 : (x, y, z, t, u) \rightarrow (x + \omega(-x + \lambda(t)), y + 2\omega y, z + 2\omega z, t, u + \omega \left( -u + \frac{\lambda'(t)y}{b} + \gamma \left( \frac{bz - dy}{b} , t \right) \right)),$

$X_5 : (x, y, z, t, u) \rightarrow (x + \omega \lambda(t), y, z + \omega, t, u + \omega \left( \frac{\lambda'(t)y}{b} + \gamma \left( \frac{bz - dy}{b} , t \right) \right)).$
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$X_6 : (x, y, z, t, u) \rightarrow (x + \omega \lambda(t), y
+ \omega, z, t, u + \omega \left(\frac{\lambda'(t)y}{b} + \gamma \left(\frac{bz - dy}{b}, t\right)\right))$. (14)

The entries on the right side give the transformed point
exp$(x, y, z, t, u) = (\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}, \tilde{u})$. If $u = f(x, y, z, t)$
is a known solution of Eq. (2), then by using above

groups $X_i$ $(i = 1, 2, \ldots, 6)$ corresponding new solutions $u_i$ $(i = 1, 2, \ldots, 6)$ can be obtained as follows

$u_1 = f(x + \omega(x + \lambda(t)), y, z, t + 2\omega t, u
+ \omega \left(-u + \frac{\lambda'(t)y}{b} + \gamma \left(\frac{bz - dy}{b}, t\right)\right)),$

$u_2 = f(x + \omega\lambda(t), y, z, t + \omega, u
+ \omega \left(\frac{\lambda'(t)y}{b} + \gamma \left(\frac{bz - dy}{b}, t\right)\right)),$

$u_3 = f(x + \omega\lambda(t), y, \omega, t, u
+ \omega \left(\frac{\lambda'(t)y}{c} + \gamma \left(\frac{bz - dy}{b}, t\right)\right)),$

$u_4 = f(x + \omega(-x + \lambda(t)), y + 2\omega y, z + 2\omega z, t, u
+ \omega \left(u + \frac{\lambda'(t)y}{b} + \gamma \left(\frac{bz - dy}{b}, t\right)\right)),$

$u_5 = f(x + \omega\lambda(t), y, z + \alpha, t, u
+ \omega \left(\frac{\lambda'(t)y}{b} + \gamma \left(\frac{bz - dy}{b}, t\right)\right)),$

$u_6 = f(x + \omega\lambda(t), y + \omega, z, t, u
+ \omega \left(\frac{\lambda'(t)y}{b} + \gamma \left(\frac{bz - dy}{b}, t\right)\right)).$ (15)

5 Symmetry reduction and closed-form solutions

In this section, we assume different functions of $\lambda$ and $\gamma$ to get similarity variables for different vector fields. We obtain similarity solutions for Eq. (2) by solving reduction equations which can be found with the help of similarity variables. To find similarity variables, we first solve associated Lagrange’s system

$$
\frac{dx}{\xi^1(x, y, z, t, u)} = \frac{dy}{\xi^2(x, y, z, t, u)} = \frac{dz}{\xi^3(x, y, z, t, u)} = \frac{dt}{\tau(x, y, z, t, u)} = \frac{dv}{\eta(x, y, z, t, u)}.
$$

(16)

5.1 Vector field $v_1$

$$
v_1 = (x + \lambda(t)) \frac{\partial}{\partial x} + 2t \frac{\partial}{\partial t}
+ \left(-u + \frac{\lambda'(t)y}{b} + \gamma \left(\frac{bz - dy}{b}, t\right)\right) \frac{\partial u}{\partial u}.
$$

(17)

Assume $\lambda(t) = \alpha, \gamma \left(\frac{bz - dy}{b}, t\right) = \beta$. Thus, we find the associated Lagrange system using Eqs. (16) and (17)

$$
\frac{dx}{x + \alpha} = \frac{dy}{0} = \frac{dz}{0} = \frac{dt}{2t} = \frac{du}{-u + \beta}.
$$

Similarity reduction of Eq. (2) is

$$
u(x, y, t) = \beta - \frac{f(X, Y, Z)}{\sqrt{t}},
$$

where

$$
X = \frac{x + \alpha}{\sqrt{t}}, \quad Y = y \quad \text{and} \quad Z = z,
$$

(18)

are the three invariants that we obtained. Substituting Eq. (18) into Eq. (2), we get the following PDE with three independent variables

$$
-f_x - \frac{X}{2} f_{xx} + \alpha f_x f_{xy} + b f_y f_{xx}
+ c f_x f_{xx} + d f_x f_{xx} + f_{xxx} + f_{xxx} = 0.
$$

(19)

The new set of generators of infinitesimal transformations for Eq. (19) by applying similarity transformations method (STM) is

$$
\xi_X = -a_1 X + 2a_4, \quad \xi_Y = 2a_1 Y + a_3, \quad \xi_Z = 2a_2 Z + a_2, \quad \eta_f = a_1 f + \frac{a_4 Y}{b} + \mu \left(\frac{bZ - dY}{b}\right).
$$

where $\xi_X, \xi_Y, \xi_Z$ and $\eta_f$ denote generators of infinitesimal transformations with respect to indicated variable and $a_1, a_2, a_3$ and $a_4$ are arbitrary constants. Assume $\mu (\frac{bZ - dY}{b}) = C$, then we get associated Lagrange’s system as follows

$$
\frac{dX}{-a_1 X + a_4} = \frac{dY}{2a_1 Y + a_3} = \frac{dZ}{2a_1 Z + a_2} = \frac{df}{a_1 f + \frac{a_4 Y}{b} Y + C}.
$$
Further, the function \( f \) can be written in new similarity form as

\[
f(X, Y, T) = H(r, s)\sqrt{2a_1Y + a_3} + \frac{a_4}{2a_1^2b}(2a_1Y + a_3) + M,
\]

where \( r = (a_1X - 2a_4)\sqrt{2a_1Y + a_3} \) and \( s = \frac{2a_1Y + a_3}{2a_1Z + a_2} \). \( (20) \)

This equation can be solved numerically.

5.2 Vector field \( v_2 \)

\[
v_2 = \lambda(t) \frac{\partial}{\partial x} + \frac{\partial}{\partial t} + \gamma \left( \frac{bZ - dy}{b} \right) \frac{\partial}{\partial u}.
\]

(22)

Assume \( \gamma \left( \frac{bZ - dy}{b} \right) = \frac{bZ - dz}{b} \). Then the associated Lagrange’s system is found using Eqs. (16) and (22)

\[
\frac{dx}{\lambda(t)} = \frac{dy}{0} = \frac{dz}{0} = \frac{dt}{1} = \frac{du}{\frac{\lambda(t)y}{b} + \frac{bZ - dy}{b}}.
\]

(23)

The similarity reduction of Eq. (2) in similarity form with the following similarity variables is

\[
u(x, y, t) = f(X, Y, Z) + \frac{\lambda(t)y}{b} + \frac{(bZ - dy)t}{b}.
\]

(24)

From Eqs. (2) and (24), we get a PDE with three independent variables

\[
afx f_{XY} + bfy f_{XX} + cfX f_{XZ} + d fZ f_{XX} + f_{XXX} + f_{XXZ} = 0.
\]

(25)

The new set of generators of infinitesimal transformations for Eq. (25) by applying STM is

\[
\xi_x = a_4X + a_5,
\]

\[
\xi_y = a_1Y + a_3,
\]

\[
\tau_T = a_1Z + a_2,
\]

\[
\eta_f = -a_4f + \psi \left( \frac{bZ - dY}{b} \right),
\]

where \( a_1, a_2, a_3, a_4 \) and \( a_5 \) are arbitrary constants and \( \psi \left( \frac{bZ - dY}{b} \right) \) is an arbitrary function.

Assume \( \psi \left( \frac{bZ - dY}{b} \right) = 0 \). Then the associated Lagrange’s system for Eq. (25) is given by

\[
\frac{dX}{a_4X + a_5} = \frac{dY}{a_1Y + a_3} = \frac{dT}{a_1Z + a_2} = \frac{df}{-a_4f}.
\]

(26)

by solving these equations, we can write the function \( f \) in the term of new similarity solution with \( r \) and \( s \) as new similarity variables

\[
f(X, Y, Z) = \frac{H(r, s)}{(a_1Y + a_3)x}, \quad r = \frac{a_4X + a_5}{(a_1Y + a_3)x}.
\]

(27)

Further, Eq. (25) can be reduced into following \((1 + 1)\)-dimensional PDE as follows

\[
-(a + b)q H_r H_{rr} + (c - as) \frac{a_1}{a_4} H_r H_{rs} - bH H_{rrr}
\]

\[
+ (d - bs) \frac{a_1}{a_4} H_s H_{rr} - 2a_4 H_r - 4a_4 H_{rrr} - a_4 r H_{rrrs} + a_1 (1 - s) H_{rrr} = 0.
\]

(28)

Lie group analysis method gives the following generators of infinitesimal transformations when it is applied on Eq. (27).

\[
\xi_r = b_1 r,
\]

\[
\xi_t = 0,
\]

\[
\eta_H = -b_1 H - b_2 (bs - d) \frac{a_4}{a_1},
\]

where \( b_1 \) and \( b_2 \) are arbitrary constants.

The similarity solution \( H(r, s) \) can be written in the following similarity form

\[
H(r, s) = \frac{G(\zeta)}{r} + \frac{b_2 (bs - d) \frac{a_4}{a_1}}{b_1},
\]

(29)

with similarity variable \( \zeta = s \). This transformation reduces Eq. (27) into following ODE
Lagrange’s system can be calculated as
\[ (c + d - (a + b)s)G(\zeta)G'(\zeta) + 6a_4(1 - s)G'(\zeta) = 0. \]  
(29)

Therefore, we obtain following solutions of Eq. (29)
\[ G(\zeta) = \frac{6a_4(1 - \zeta)}{(a + b)\zeta - (c + d)}, \]  
(30)
\[ G(\zeta) = \beta. \]  
(31)

Hence, we obtain the solutions of Eq. (2) which are given by
\[ u(x, y, z, t) = \frac{\lambda(t)y}{b} + \frac{(b-z\ UY 4)}{b} \]
\[ + b\left(\frac{a}{a_1 y + a_3}\right) \frac{a}{a_1 + a_4 y + a_5} \]
\[ \times \frac{a_1 a_2}{a_1 y + a_3} \]
\[ + [a_5 + a_4 x - a_4 \int \lambda(t)dt] [(a + b)\frac{a_1 z + a_3}{a_1 y + a_3} - (c + d)], \]  
(32)
\[ u(x, y, z, t) = \frac{\lambda(t)y}{b} + \frac{(b-y\ UY 4)}{b} \]
\[ + b\left(\frac{a}{a_1 y + a_3}\right) \frac{a}{a_1 + a_4 y + a_5} \]
\[ \times \frac{a_1 a_2}{a_1 y + a_3} \]
\[ + \frac{a_5 + a_4 x - a_4 \int \lambda(t)dt}{T}. \]  
(33)

5.3 Vector field \( \nu_3 \)
\[ \nu_3 = \lambda(t) \frac{\partial}{\partial x} + \frac{at}{c} \frac{\partial}{\partial y} + t \frac{\partial}{\partial z} \]
\[ + \left( \frac{x}{c} + \frac{\lambda(t)y}{b} + \gamma \left( \frac{b-z\ UY 4}{b}, t \right) \right) \frac{\partial}{\partial u}. \]  
(34)

Assume \( \lambda(t) = \gamma \left( \frac{b-z\ UY 4}{b}, t \right) = 0 \). The corresponding Lagrange’s system can be calculated as
\[ \frac{dx}{0} = \frac{dy}{at} = \frac{dz}{t} = \frac{dt}{0} = \frac{du}{x}. \]  
(35)

Similarity reduction of Eq. (2) in similarity form is
\[ u(x, y, z, t) = \frac{xy}{at} + f(X, Y, T), \quad \text{where} \quad \]
\[ X = x, \quad T = t \quad \text{and} \quad Y = y - \frac{az}{c}, \]  
(36)
are the three invariants that we obtained.

A new PDE with three independent variables is obtained with the help of Eqs. (2) and (35).
\[ Tf_{XY} + f_X + AXf_{XX} + BTf_Y f_{XX} + CTf_{XXX} = 0, \]  
(37)
where \( A = b/a, B = (bc - ad)/c \) and \( C = (c - a)/c \). The new set of generators of infinitesimal transformations for Eq. (36) by applying Lie group method is
\[ \eta_X = (a_1 + a_2 - a_3)X + a_2XlogT + \psi(T), \]
\[ \eta_Y = 2a_3Y + a_4, \]
\[ \tau_T = 2a_2TlogT + 2a_1T, \]
\[ \eta_f = -a_2logT f - (a_1 + a_2 - a_3) f \]
\[ \psi'(T)TY + A\psi(T)Y + (2A - 1)a_2 XY \]
\[ = BT \]
\[ + \frac{a_5 X}{T} + \delta(T), \]
where \( a_1, a_2, a_3, a_4 \) and \( a_5 \) are arbitrary constants and \( \psi(T), \delta(T) \) are arbitrary functions.

Case(1) \( a_1 \neq 0 \), other parameters and arbitrary functions are zero.

In this case, the Lagrange’s system for Eq. (36) is given by
\[ \frac{dX}{a_1 X} = \frac{dY}{0} = \frac{dT}{2a_1 T} = \frac{df}{-a_1 f}. \]

Further, the symmetry reduction of the function \( f \) with new similarity variable is
\[ f(X, Y, T) = \frac{H(r, s)}{\sqrt{T}}, \quad r = \frac{X}{\sqrt{T}} \quad \text{and} \quad s = Y. \]  
(38)

Further, Eq. (36) can be reduced into the following (1 + 1)-dimensional PDE
\[ \left( A - \frac{1}{2} \right) r H_{rr} + B H_s H_{rs} + C H_{rss} = 0. \]  
(39)

Lie group similarity analysis method gives the following generators of infinitesimal transformations when it is applied to Eq. (38).
\[ \xi_r = b_1 r + b_2, \]
\[ \xi_s = \rho(s), \]
\[ \eta_H = -b_1 H \frac{B}{B} + b_2(1 - 2A)s \]
\[ + b_3 r + b_4 + \frac{r(1 - 2A)\rho(s)}{2B}, \]
where \( b_1, b_2, b_3, b_4 \) are arbitrary constants and \( B = (bc - ad)/c \).
where \( b_1, b_2, b_3 \) and \( b_4 \) are arbitrary constants and \( \rho(s) \) is an arbitrary function of \( s \).

**Case(1a)** \( b_1 \neq 0 \), other parameters and arbitrary functions are zero.

In this case, we find a new similarity form of solution \( G(\zeta) \) and hence, \( H(r, s) \) can be rewritten as

\[
H(r, s) = \frac{G(\zeta)}{r} + \frac{(1 - 2A)rs}{2B},
\]

(39)

where \( \zeta = s \) is a similarity variable. Therefore, Eq. (39) reduces Eq. (38) into the following ODE

\[
2BG(\zeta)G'(\zeta) - 6CG'(\zeta) = 0,
\]

(40)

where \( B = (bc - ad)/c \) and \( C = (c - a)/c \). Hence, we get following solutions of Eq. (40)

\[
G(\zeta) = \frac{3(c - a)}{bc - ad},
\]

(41)

\[
G(\zeta) = -\alpha,
\]

(42)

where \( \alpha \) is a new constant. By back substitution, we obtained the following solutions of Eq. (2)

\[
u(x, y, z, t) = \frac{(c - 2d)}{2(bc - ad)} \frac{xy}{t} - \frac{(a - 2b)}{2(bc - ad)} \frac{xz}{t} + \frac{3(c - a)}{(bc - ad)x},
\]

(43)

\[
u(x, y, z, t) = \frac{(c - 2d)}{2(bc - ad)} \frac{xy}{t} - \frac{(a - 2b)}{2(bc - ad)} \frac{xz}{t} + \frac{\alpha}{x} + \frac{\beta_3}{\sqrt{t}} \int \frac{b_2^{1/2}Y}{Y} dY + \frac{\beta_4}{\sqrt{t}} \int \frac{b_2dY}{\rho(Y)}
\]

(44)

**Case(1b)** \( b_2 \neq 0 \), \( \rho(s) \neq 0 \), other parameters are zero.

In this case, associated Lagrange’s system is as follows

\[
dr = \frac{ds}{\rho(s)} = \frac{dH}{2B}.
\]

Hence, invariant solution \( H(r, s) \) can be represented into a new similarity form of solution \( G(\zeta) \) as

\[
H(r, s) = G(\zeta) + \frac{(1 - 2A)sr}{2B},
\]

(45)

where \( \zeta = r - b_2 \int \frac{ds}{\rho(s)} \) is a similarity variable. Therefore, Eq. (38) has been reduced into the following ODE

\[
G'''(\zeta) = 0.
\]

Hence, a solution of Eq. (45) is

\[
G(\zeta) = \beta_1 + \beta_2\zeta + \beta_3\zeta^2 + \beta_4\zeta^3.
\]

(46)

By back substitution, a solution of Eq. (2) is

\[
u(x, y, z, t) = \frac{(c - 2d)}{2(bc - ad)} \frac{xy}{t} - \frac{(a - 2b)}{2(bc - ad)} \frac{xz}{t} + \frac{\beta_3}{\sqrt{t}} \int \frac{b_2^{1/2}Y}{Y} dY + \frac{\beta_4}{\sqrt{t}} \int \frac{b_2dY}{\rho(Y)}
\]

(47)

where \( Y = y - (az/c) \) and \( \beta_1, \beta_2, \beta_3 \) and \( \beta_4 \) are arbitrary constants.

**Case(2)** \( a_2 \neq 0 \), other parameters and arbitrary functions are zero.

The Lagrange’s system for Eq. (36) is given by

\[
\frac{dX}{a_1(X + X \log T)} = \frac{dY}{0} = \frac{dT}{2a_1T \log T} = -a_1 \left( (1 + \log T)f + \frac{(2A - 1)XY}{BT} \right).
\]

(48)

Further, the symmetry reduction of the function \( f \) with new similarity variable is

\[
f(X, Y, T) = \frac{(2A - 1)XY}{2BT} + \frac{H(r, s)}{\sqrt{T \log T}},
\]

\[
r = X \sqrt{T \log T} \quad \text{and} \quad s = Y.
\]

(49)

Further, Eq. (36) can be reduced into the following (1+1)-dimensional PDE

\[
- H_r - \frac{r}{2} H_{rr} + BH_r H_{rr} + CH_{rr} = 0.
\]

(50)

Lie group similarity analysis method gives the following generators of infinitesimal transformations when it is applied to Eq. (49).

\[
\xi_r = -\frac{b_1}{2} r + b_3,
\]

\[
\xi_s = b_1 s + b_2,
\]

\[
\eta_H = \frac{b_1}{2} H + \frac{b_3}{2B} s + b_4.
\]

(51)

where \( b_1, b_2, b_3 \) and \( b_4 \) are arbitrary constants.
Case (2a) $b_1 \neq 0$, other parameters and arbitrary functions are zero.

In this case, we find a new similarity form of solution $G(\xi)$ and hence, $H(r, s)$ can be rewritten as

$$H(r, s) = \sqrt{s} G(\xi), \quad (50)$$

where $\xi = r \sqrt{s}$ is a similarity variable. Therefore, Eq. (50) reduces Eq. (49) into the following ODE

$$-G'(\xi) - \xi G''(\xi)$$

where (16)

and (16)

The invariance property of Lie group transformations gives a solution of Eq. (2)

$$u(x, y, z, t) = \frac{xy}{at} - \frac{c(a - 2b)}{a(bc - ad)} \frac{x(cy - az)}{t} + \beta_1 \frac{cy - az}{ct \log t} \quad (52)$$

Case (2b) $b_2 \neq 0$, other parameters are zero.

$H(r, s) = G(\xi)$ represents a new similarity form of the solution with similarity variable $\xi = r$. Therefore, Eq. (49) is reduced into the following ODE

$$-G'(\xi) - \xi G''(\xi) = 0. \quad (53)$$

Hence, a solution of Eq. (2) is

$$u(x, y, z, t) = \frac{xy}{at} - \frac{c(a - 2b)}{a(bc - ad)} \frac{x(cy - az)}{t} + \beta_1 \frac{cy - az}{\sqrt{t \log t}} + \beta_2 \frac{x}{x}, \quad (54)$$

where $\beta_1$ and $\beta_2$ are arbitrary constant.

5.4 Vector field $v_4$

$$v_4 = (-x + \lambda(t)) \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y}$$
$$+ 2z \frac{\partial}{\partial z} \left( u + \frac{\lambda(t) y}{b} + \gamma \left( \frac{bz - dy}{b}, t \right) \right) \frac{\partial}{\partial u}. \quad (55)$$

Assume $\lambda(t) = t$, $\gamma \left( \frac{bz - dy}{b}, t \right) = \beta$. Then the associated Lagrange’s system can be found using Eqs. (55) and (16)

$$\frac{dx}{-x + t} = \frac{dy}{2y} = \frac{dz}{z} = \frac{dt}{0} = \frac{du}{u + \frac{b}{b} + \beta}.$$

We reduce Eq. (2) into following invariant form with new invariant variables

$$u(x, y, t) = \beta - \sqrt{y} f(X, Y, T) + \frac{y}{b} + \beta, \quad \text{where}$$

$$X = \sqrt{y}(t - x), \quad Y = \frac{z}{y} \quad \text{and} \quad T = t. \quad (56)$$

From Eqs. (56) and (2), we get the following PDE with three independent variables

$$-f_{XT} + af^2_x + \frac{b}{2}ff_{xx} + \frac{a + b}{2}Xfxf_{xx}$$
$$- (aY - c)fx_{xy}$$
$$+ (d - bY)fy_{xx} - 2fx_{xx}$$
$$- \frac{X}{2}f_{xxxx} - (1 - y)fx_{xxyy} = 0. \quad (57)$$

The new set of generators of infinitesimal transformations for Eq. (57) by applying STM is

$$\xi_x = a_1 X, \quad \xi_y = 0, \quad \xi = a_2 T + a_2, \quad \eta_f = -a_1 f + \mu(T) \sqrt{yY - d},$$

where $\xi_x, \xi_y, \xi_T$ and $\eta_f$ are generators of infinitesimal transformations with respect to the indicated variables. $a_1, a_2$ are arbitrary constants and $\mu(T)$ is an arbitrary function.

Further, the function $f$ can be written in a new similarity form as follows

$$f(X, Y, T) = \frac{H(r, s)}{\sqrt{2a_1 T + a_2}}$$
$$+ \sqrt{\frac{bY - d}{2a_1 T + a_2}} \int \frac{\mu(T)}{\sqrt{2a_1 T + a_2}} dt, \quad (58)$$

where $r = \frac{X}{\sqrt{2a_1 T + a_2}}$ and $s = Y$ are similarity variables. Equation (57) can be rewritten into the following $(1 + 1)$-dimensional PDE

$$-2a_1 H_r + a_1 H_{rr} + a H^2 + \frac{b}{2} H H_{rr}$$
$$+ \frac{a + b}{2} r H_r H_{rr} - (as - c) H_r H_{rr}$$
$$- (bs - d) H_r H_{rr} - 2 H_{rr} - \frac{r}{2} H_{rrrr}$$
$$- (1 - s) H_{rrrr} = 0. \quad (59)$$
If we apply Lie similarity analysis to Eq. (59), we get a new set of generators of infinitesimal transformations as \( \xi_r = 0, \xi_s = 0 \) and \( \eta_H = b_1 \sqrt{b} s - d \). Solution cannot be obtained in this case, though this equation can be solved numerically.

5.5 Vector field \( v_5 \)

\[
v_5 = \lambda(t) \frac{\partial}{\partial x} + \frac{\partial}{\partial z} + \left( \frac{\lambda'(t)}{b} + \gamma \left( \frac{bz - dy}{b}, t \right) \right) \frac{\partial}{\partial u}. \tag{60}\]

Assume \( \lambda(t) = \alpha, \gamma \left( \frac{bz - dy}{b}, t \right) = \frac{bz - dy}{b} \). Then the associated Lagrange’s system can be obtained by using Eqs. (16) and (60)

\[
\frac{dx}{\alpha} = \frac{dy}{0} = \frac{dz}{1} = \frac{dt}{0} = \frac{du}{\frac{bz - dy}{b}}.
\]

Eq. (2) can be reduced into the following similarity form with new similarity variables

\[
u(x, y, t) = f(X, Y, T) - \frac{d}{b} yz + \frac{z^2}{2}, \quad \text{where} \quad X = x - \alpha z, \quad Y = y \quad \text{and} \quad T = t. \tag{61}\]

From Eqs. (2) and (61), we get the following PDE with three independent variables

\[
\begin{align*}
fxr + afxfxy + bfyfxx - \alpha(c + d)fxfxx & = \frac{d}{b} f_{fxx} + f_{fxyy} - \alpha f_{fxxxx} = 0.
\end{align*} \tag{62}\]

The new set of generators of infinitesimal transformations for Eq. (62) by applying STM is

\[
\begin{align*}
\xi_x &= a_1 X + \theta(T), \\
\xi_y &= a_1 Y + a_3 T + a_4, \\
\tau_T &= 3a_1 T + a_2, \\
\eta_f &= \frac{\theta'(T)}{b} Y + \kappa(T) - a_1 f + a_3 \frac{d^2}{b^2} Y T + \frac{3a_1 d^2}{2b^2} Y^2 \\
&\quad + a_4 \frac{d^2}{b^2} Y^2 + a_3 \frac{X b + \alpha(c + d)Y}{ab},
\end{align*}
\]

where \( \xi_x, \xi_y, \tau_T \) and \( \eta_f \) denote generators of infinitesimal transformations with respect to the indicated variables and \( a_1 \) and \( a_2 \) are arbitrary constants and \( \theta(T) \) and \( \kappa(T) \) are arbitrary functions.

Case(1) \( a_1 \neq 0 \), other parameters and arbitrary functions are zero.

Now, with the help of Lie group analysis, the function \( f \) can be written into a new similarity form

\[
f(X, Y, T) = \frac{H(r, s)}{Y} + \frac{d^2}{2b^2} Y^2, \tag{63}\]

where \( r = X/Y \), and \( s = Y/T \frac{1}{4} \) are similarity variables.

Further, Eq. (62) can be reduced into the following \((1 + 1)\)-dimensional PDE

\[
\begin{align*}
s^4 H_{rst} + 2a H^2_r + [(a + b)r + \alpha(c + d)] H_r H_{rr} & = \alpha s H_r H_{rs} - bs H_s H_{tt} \\
& - H_r H_{rs} + 4 H_{rrr} + (\alpha + r) H_{rrrr} + s H_{rrrs} = 0.
\end{align*} \tag{64}\]

If we apply Lie similarity analysis to Eq. (64), we obtain a new set of generators of infinitesimal transformations such as \( \xi_r = 0, \xi_s = 0 \) and \( \eta_H = b_1 \). Solution cannot be found in this case, though this equation can be solved numerically.

5.6 Vector field \( v_6 \)

\[
v_6 = \lambda(t) \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \left( \frac{\lambda'(t)}{b} + \gamma \left( \frac{bz - dy}{b}, t \right) \right) \frac{\partial}{\partial u}. \tag{65}\]

Assume \( \lambda(t) = 1, \gamma \left( \frac{bz - dy}{b}, t \right) = 0 \). Then, the associated Lagrange’s system can be found by using Eqs. (16) and (65)

\[
\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{0} = \frac{dt}{0} = \frac{du}{0}.
\]

The invariant form of a solution of Eq. (2) with new symmetry variables is

\[
u(x, y, z, t) = f(X, Z, T), \quad \text{where} \quad X = x - y, \quad Z = z \quad \text{and} \quad T = t. \tag{66}\]

From Eqs. (2) and (66), we obtain a reduced PDE with three independent variables

\[
\begin{align*}
fxr - (a + b) fx fxx + cfxfxz + dfyfxx & = f_{fxxx} + f_{fxxx} = 0.
\end{align*} \tag{67}\]

By applying STM, the new set of generators of infinitesimal transformations for Eq. (67) are
\[ \xi_X = \frac{a_2}{3} X + \vartheta(T), \]
\[ \xi_Z = \frac{a_2}{3} Z + a_3 T + a_4, \]
\[ \tau_T = a_2 T + a_1, \]
\[ \eta_f = -\frac{a_2}{3} f + \frac{\vartheta'(T)}{d} Z + a_3 \frac{dX + (a + b) Z}{dc} + \nu(T), \]

where \( \xi_X, \xi_Z, \tau_T \) and \( \eta_f \) denote generators of infinitesimal transformations. \( a_1, a_2 \) and \( a_3 \) are arbitrary constants while \( \vartheta(T) \) and \( \nu(T) \) are arbitrary functions.

**Case(1)** \( a_1 \neq 0 \) and other parameters and arbitrary functions are zero.

Therefore, with the help of Lie group analysis, the function \( f \) can be written in a new similarity form of solution

\[ f(X, Y, T) = H(r, s), \quad \text{where} \quad r = X, \quad \text{and} \quad s = Z \quad \text{are similarity variables.} \quad (68) \]

Further, Eq. (67) can be reduced into the following \((1 + 1)\)-dimensional PDE

\[ -(a + b) H_r H_{rr} + c H_r H_{rs} + d H_s H_{rr} \]
\[ -H_{rrr} + H_{rrrs} = 0. \quad (69) \]

If we apply Lie similarity analysis to Eq. (69), we obtain a new set of generators of infinitesimal transformations

\[ \xi_r = b_1 r + b_2, \]
\[ \xi_s = b_1 s + b_3, \]
\[ \eta_H = -b_1 H + b_4. \]

Hence, we get the following invariant solution with similarity variable \( \xi \)

\[ H(r, s) = \frac{G(\xi)}{s + \frac{b_1}{b_0}}, \quad \xi = \frac{r + \frac{b_2}{b_1}}{s + \frac{b_3}{b_1}}. \]

Therefore, Lie similarity reduces Eq. (2) into fourth-order nonlinear ODE

\[ (a + b + (c + d) \xi) G'''' + 2c G'' + dGG'' + 4G'''' + (1 + \xi) G''''' = 0, \quad (70) \]

where \( G' \) denotes the differential of \( G \) with respect to \( \xi \). We get following two solutions of Eq. (2) using Eqs. (69) and (70), respectively,

\[ u(x, y, z, t) = \frac{A(x - y)}{a + b} - \frac{12}{(a + b)(x - y - 2C_0)} \]
\[ + Az + B, \quad (71) \]
\[ u(x, y, z, t) = \frac{b_4}{b_1} + \frac{C_0}{z + \frac{b_3}{b_1}}, \quad (72) \]

where \( C_0, A \) and \( B \) are arbitrary constants.

**6 Conclusion**

In this paper, we have shown that the CBS equation can be transformed by the Lie group of transformations method to new PDEs with less independent variables and again Lie group symmetry reduces these equations into ordinary differential equations. Generators of infinitesimal transformations and different vector fields for the CBS equation were obtained. Using the criterion of invariance for Eq. (2) under the extended generalized vector field, we found the Lie symmetry groups of \((3 + 1)\)-dimensional CBS equation and similarity variables which played an important role in the reduction of equation. This work is significant as the solutions obtained shall be helpful in other applied sciences such as field theory, fluid dynamics, nonlinear optics. The method which we have proposed in this article is also a standard, direct and computer literate method which allows us to avoid complicated and tedious algebraic calculation.

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