ESTIMATES OF POSITIVE SUB AND SUPER SOLUTIONS OF SCHRÖDINGER EQUATIONS WITH SINGULAR POTENTIALS.

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Abstract. Consider operators \( L_V := \Delta + V \) in a bounded smooth domain \( \Omega \subset \mathbb{R}^N \). Assume that \( V \in C^1(\Omega) \) satisfies \( V(x) \leq \bar{a} \delta(x)^{-2} \) in \( \Omega \) and that \( L_V \) has a ground state \( \Phi_V \) in \( \Omega \). Assuming an additional condition on the behavior of \( \Phi_V \) (see Section 3) we derive sharp, two-sided estimates of weighted integrals of positive \( L_V \) harmonic functions and \( L_V \) potentials. These lead to a-priori estimates of positive \( L_V \) supersolutions and subsolutions assuming (in the latter case) existence of \( L_V \) boundary trace.

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1. Introduction

Let \( \Omega \) be a bounded Lip domain in \( \mathbb{R}^N, N \geq 3 \). We study the operator

\[ L_V := \Delta + V \]

where \( V \in C^1(\Omega) \). We assume that the potential \( V \) satisfies the conditions:

(A1) \[ \exists \bar{a} > 0 : \quad |V(x)| \leq \bar{a} \delta(x)^{-2} \quad \forall x \in \Omega \]

\[ \delta(x) = \delta_{\partial \Omega}(x) := \text{dist} (x, \partial \Omega). \]

and,

(A2) \[ \gamma_- < 1 < \gamma_+ . \]

Here, \( \gamma_+ = \sup A \) and \( \gamma_- = \inf A \) where

\[ A := \{ \gamma : \int_{\Omega} |\nabla \phi|^2 dx \geq \gamma \int_{\Omega} \phi^2 V dx \quad \forall \phi \in H_0^1(\Omega) \}. \]

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Condition (A1) and Hardy’s inequality imply that $\gamma_+ > 0$ and $\gamma_- < 0$. If $V$ is positive then $\gamma_- = -\infty$ and $\gamma_+$ is the Hardy constant relative to $V$ in $\Omega$, denoted by $c_H(V)$. If $V$ is negative, obviously $\gamma_+ = \infty$.

For every $\gamma \in (\gamma_-, \gamma_+)$ there exists a Green function for $L_{\gamma V}$ in $\Omega$. The Green function of $L_V$ in $\Omega$ is denoted by $G^\Omega_V$.

Conditions (A1) and (A2) imply:

(i) $L_V$ has a ground state in the sense of Agmon [1]. The ground state $\Phi_V$ is normalized by the condition $\Phi_V(x_0) = 1$ where $x_0$ is a fixed reference point in $\Omega$.

(ii) For every $y_0 \in \Omega$ and $\epsilon > 0$ there exists a constant $C > 0$ such that

\[
C^{-1}G_V(x, y_0) \leq \Phi_V(x) \leq CG_V(x, y_0) \quad \forall x \in \Omega : |x - y_0| \leq \epsilon.
\]

For a proof based on [20], see [13, Lemma 1.2].

(iii) A positive $L_V$ superharmonic $w$ is an $L_V$ potential (i.e., it does not dominate any positive $L_V$ harmonic function) if and only if $w = G_V[\tau]$ for some $\tau \in \mathcal{M}_+(\Omega; \Phi_V)$ (see [2]). Here $\mathcal{M}(\Omega; \Phi_V)$ denotes the space of Borel measures $\tau$ such that $\int_\Omega \Phi_V d|\tau| < \infty$.

$L_V$ is weakly coercive in the sense of Ancona. (A proof, due to [19], is provided in [13, Lemma 1.1].) Therefore, by Ancona [3]:

- $L_V$ possesses a Martin kernel $K_V$ such that, for every $y \in \partial \Omega$, $x \mapsto K_V(x, y)$ is positive $L_V$ harmonic in $\Omega$ and vanishes on $\partial \Omega \setminus \{y\}$ and the Representation Theorem holds:

  If $u$ is a positive $L_V$-harmonic function then there exists $\nu \in \mathcal{M}_+(\partial \Omega)$ (= the space of positive, bounded Borel measures) such that

  \[
  u(x) = \int_{\partial \Omega} K_V(x, y) d\nu(y) =: \mathbb{K}_V[\nu] \quad x \in \Omega.
  \]

- The **Boundary Harnack Principle** (briefly BHP) holds. (See also Ancona [4].)

The present paper is devoted to the derivation of weighted integral estimates of positive $L_V$ superharmonic and $L_V$ subharmonic functions. The weight $W$ is given by,

\[
W := \frac{\Phi_V}{\Phi_0}.
\]

The estimates are sharp and two sided (see Theorem 7.5 below). In this sense the weight $W$ is optimal. The derivation is based on assumptions (A1), (A2) and an additional condition on the behavior of the ground state, (see (C1) in section 3).
Such estimates have been derived in \cite{15} and \cite{16} when $V$ is the Hardy potential or $V$ is of the form

$$V = \gamma V_F, \quad V_F = \frac{1}{\delta_F^2}, \quad \delta_F(x) = \text{dist}(x, F)$$

where $F \subset \partial \Omega$ is a smooth k-dimensional manifold without boundary. In these cases a sharp two sided estimate of $\Phi_V$ is available. This fact was crucial for ) The estimates have been applied in the study of positive solutions of a family of semilinear boundary value problems involving $L_V$ and a nonlinear term.

Linear and nonlinear boundary value problems for operators $L_V$, with $V$ as in (1.4), have been investigated by many authors. The cases $F = \partial \Omega$ or $F$ a singleton have been most frequently investigated. Following is a list of some recent works in the area:

Bandle, Moroz and Reichel \cite{5}, \cite{6}, Marcus and P.T. Nguyen \cite{15}, \cite{16}, Gkikas and Veron \cite{10}, P.T. Nguyen \cite{18}, Y. Du and L. Wei \cite{7}, \cite{21}, Marcus and Moroz \cite{14}, Chen and Veron \cite{12}, Gkikas and Nguyen \cite{8}, \cite{9}.

The main tools used in the paper are potential theoretic results (mentioned above) and estimates of the Green and Martin kernels \cite{13}.

The plan of the paper: Section 2 is devoted to notations and statement of some results from the literature. The main results - two sided estimates of $K_V[\nu], \nu \in M_+(\partial \Omega)$ and $G_V[\tau], \tau \in M_+(\Omega; \Phi_V)$ - are stated in Section 3 and proved in Sections 4 and 5. In order to simplify the presentation we assume that $\Omega$ is a smooth domain. In section 6 we describe a family of potentials $V$ that satisfy conditions (A1), (A2), (C1) and include the potentials studied in \cite{15} and \cite{16}. Finally in Section 7 we discuss the notion of $L_V$ boundary trace and apply the previous estimates to derive estimates of positive $L_V$ superharmonic and $L_V$ subharmonic functions. In the first case we use the Riesz representation formula. In the second case we show that, assuming existence of the $L_V$ trace, a similar representation formula holds.

2. Notation and preliminaries

Denote,

$$T(r, \rho) = \{\xi = (\xi_1, \xi') \in \mathbb{R} \times \mathbb{R}^{N-1} : |\xi_1| < \rho, \ |\xi'| < r\}.$$
Assuming that $\Omega$ is a bounded Lipschitz domain, there exist positive numbers $r_0$, $\kappa$ such that, for every $y \in \partial \Omega$, there exist: (i) a set of Euclidean coordinates $\xi = \xi_y$ centered at $y$ with the positive $\xi_1$ axis pointing in the direction of $\mathbf{n}_y$ and (ii) a function $F_y$ uniformly Lipschitz in $\mathbb{R}^{N-1}$ with Lipschitz constant $\leq \kappa$ such that

$$Q_y(r_0, \rho_0) := \Omega \cap T_y(r_0, \rho_0)$$

$$= \{ \xi = (\xi_1, \xi') : F_y(\xi') < \xi_1 < \rho_0, \ |\xi'| < r_0 \},$$

where $T_y(r_0, \rho_0) = y + T(r_0, \rho_0)$ in coordinates $\xi = \xi_y$ and $\rho_0 = 10\kappa r_0$. Without loss of generality, we assume that $\kappa > 1$. The set of coordinates $\xi_y$ is called a standard set of coordinates at $y$ and $T_y(r, \rho)$ with $0 < r \leq r_0$ and $\kappa r < \rho \leq 10\kappa r$ is called a standard cylinder at $y$.

If $\Omega$ is a bounded $C^2$ domain there exists $\bar{\beta} > 0$ such that for every $x \in \Omega$ there is a unique point $\sigma(x) \in \partial \Omega$ such that $|x - \sigma(x)| = \delta(x)$ and $x \mapsto \delta(x)$ is in $C^2(\Omega_{\bar{\beta}})$ while $x \mapsto \sigma(x)$ is in $C^1(\Omega_{\bar{\beta}})$. The set of coordinates $(\delta, \sigma)$ defined in this way in $\Omega_{\bar{\beta}}$ is called the flow coordinates set. We denote

$$D_{\beta} = \{ x \in \Omega : \delta(x) > \beta \}, \quad \Omega_{\beta} = \{ x \in \Omega : \delta(x) < \beta \},$$

$$\Sigma_{\beta} = \{ x \in \Omega : \delta(x) = \beta \}.$$

In the sequel we denote

$$\beta_0 := \min(r_0, \bar{\beta}).$$

Notation. Let $f_i$, $i = 1, 2$, be positive functions on some domain $X$. Then the notation $f_1 \sim f_2$ in $X$ means: there exists $C > 0$ such that

$$\frac{1}{C} f_1 \leq f_2 \leq C f_1 \quad \text{in} \ X.$$

The notation $f_1 \precsim f_2$ means: there exists $C > 0$ such that $f_1 \leq Cf_2$ in $X$. The constant $C$ will be called a similarity constant.

We state below the Boundary Harnack Principle in Lipschitz domains, due to Ancona [An87]:

**Theorem 2.1.** Let $y \in \partial \Omega$ and let $\xi_y$ and $T_y(r, \rho)$ be a standard set of coordinates and a standard cylinder at $y$. Let $\omega := \Omega \cap T_y(r, \rho)$ and $A = (0, ..., 0, \rho/2)$, $A' = (0, ..., 0, 3\rho/4)$ in the coordinates $\xi = \xi_y$.

\footnote{If $\Omega$ is smooth, $\mathbf{n}_y$ denotes the inward normal at $y$. If $\Omega$ is Lipschitz, $\mathbf{n}_y$ denotes an approximate normal.}
There exists a constant $c$ depending only on $N, \bar{a}$ and $\frac{F}{r}$ such that whenever $u$ is a positive $L_V$ harmonic function in $\omega$ that vanishes continuously in $\partial \omega \cap T_y(r, \rho)$ the following inequality holds:

\begin{equation}
(2.3) \quad c^{-1}r^{N-2} G_V(x, A') \leq \frac{u(x)}{u(A)} \leq c r^{N-2} G_V(x, A'),
\end{equation}

for every $x \in \omega \cap T_y(\frac{r}{2}, \frac{\rho}{2})$. In particular, for any pair $u, v$ of positive $L_V$ harmonic functions in $\omega$ that vanish on $\partial \omega \cap T_y(r, \rho)$:

\begin{equation}
(2.4) \quad u(x)/v(x) \leq Cu(A)/v(A), \quad \forall \ x \in \omega \cap T_y(r/2, \rho/2))
\end{equation}

where $C = c^2$.

The following is a well-known consequence of BHP (see e.g. [4, Lemma 3.5]). Here $x_0$ is a reference point in $\Omega \setminus T(r, \rho)$ and $n_y$ is the unit vector in the direction of the $(\xi_y)_1$ axis.

**Proposition 2.2.** There exists a constant $C$ such that for all $x = y + tn_y$, $|t| \leq \frac{3}{4} \rho$, \n
\begin{equation}
(2.5) \quad C^{-1}t^{2-N} \leq K_V(x, y)G_V(x, x_0) \leq C t^{2-N}.
\end{equation}

and $C$ can be chosen depending only on $\bar{a}, \frac{F}{r}$ and $N$.

Estimates of the Green and Martin kernels [13] will be frequently used in the sequel. These are valid in bounded Lipschitz domains.

**Theorem 2.3.** Assume $(A1), (A2)$ and $N \geq 3$.

Then, for every $b > 0$ there exists a constant $C(b)$, depending also on $N, r_0, \kappa, \bar{a}$, such that: if $x, y \in \Omega$ and

\begin{equation}
(2.6) \quad |x - y| \leq \frac{1}{b} \min(\delta(x), \delta(y))
\end{equation}

then

\begin{equation}
(2.7) \quad \frac{1}{C(b)}|x - y|^{2-N} \leq G_V(x, y) \leq C(b)|x - y|^{2-N}.
\end{equation}

In the next theorems, $C$ stands for a constant depending only on $r_0, \kappa, \bar{a}$ and $N$.

**Theorem 2.4.** Assume $(A1), (A2)$ and $N \geq 3$.

If $x, y \in \Omega$ and

\begin{equation}
(2.8) \quad \max(\delta(x), \delta(y)) \leq r_0/10\kappa
\end{equation}

\begin{equation}
(2.9) \quad \min(\delta(x), \delta(y)) \leq \frac{|x - y|}{16(1 + \kappa)^2}
\end{equation}
then
\[
\frac{1}{C}|x-y|^{2-N} \frac{\Phi_V(x)\Phi_V(y)}{\Phi_V(x_y)^2} \leq G_V(x,y)
\]
(2.10)
\[
\leq C|x-y|^{2-N} \frac{\Phi_V(x)\Phi_V(y)}{\Phi_V(x_y)^2}.
\]
The point \(x_y\) depends on the pair \((x,y)\). If
\[
\hat{r}(x,y) := |x-y| \vee \delta(x) \vee \delta(y) \leq r_0/10\kappa
\]
x\(y\) can be chosen arbitrarily in the set
(2.11)
\[
A(x,y) := \{z \in \Omega : \frac{1}{2}\hat{r}(x,y) \leq \delta(z) \leq 2\hat{r}(x,y)\} \cap B_{4\hat{r}(x,y)}(\frac{x+y}{2})\}
\]
Otherwise set \(x_y = x_0\) where \(x_0\) is a fixed reference point.

**Theorem 2.5.** Assume (A1), (A2) and \(N \geq 3\).

If \(x \in \Omega, y \in \partial\Omega\) and \(|x-y| < \frac{r_0}{10\kappa}\) then
\[
\frac{1}{C} \frac{\Phi_V(x)}{\Phi_V(x_y)^2} |x-y|^{2-N} \leq K^\gamma_V(x,y) \leq C \frac{\Phi_V(x)}{\Phi_V(x_y)^2} |x-y|^{2-N},
\]
where \(x_y\) is an arbitrary point in \(A(x,y)\).

3. Main results

In the results stated below \(\Omega\) is a bounded \(C^2\) domain in \(\mathbb{R}^N\). The results will be extended, in a separate note, to the case of bounded Lipschitz domains.

The first result provides a sharp estimate of positive \(L_V\) harmonic functions.

**Theorem 3.1.** Assume conditions (A1), (A2). In addition assume that, for every \(a > a_0 >> 1\) and every pair \(x, z \in \Omega_{\delta_0}\) such that \(z\) lies on the normal to \(\partial\Omega\) at \(\sigma(x)\) (= nearest point to \(x\) on \(\partial\Omega\));

\[
(C1) \quad a\delta(x) \leq \delta(z) \implies \frac{\Psi_V(x)}{\Psi_V(z)} \leq c(a) \frac{\delta(x)^{a^*}}{\delta(z)^{a^*}},
\]
where
\[
0 \leq \alpha - \alpha^* < 1/2.
\]

Then
\[
(3.1) \quad \frac{1}{C} \|\nu\| \leq \int_{\Omega} \frac{\Phi_V}{\Phi_0} \mathbb{K}_V[\nu]dx \leq C\|\nu\| \quad \forall \nu \in \mathcal{M}_+(\partial\Omega),
\]
where \(C\) depends on \(\bar{a}, \Omega\) and the constants in \((C1)\).
Remark 3.2. (i) The lower estimate requires only conditions (A1) and (A2). Condition (C1) is used in the derivation of the upper estimate when \( a\delta(x) \leq |x - y| \).
(ii) If (3.1) holds for \( \alpha \) and \( \alpha^* \) then, it also holds with \( \alpha + \epsilon, \alpha^* \) with \( \epsilon > 0 \) such that \( \epsilon + \alpha - \alpha^* < 1/2 \). Therefore, without loss of generality, we may assume that \( \alpha \neq 1/2 \).

In the following two theorems we present estimates of \( L_V \) potentials. Recall that \( w \) is an \( L_V \) potential if \( w = G_V[\tau] \) for a positive measure \( \tau \in M(\Omega; \Phi_V) \).

**Theorem 3.3.** Assume (A1) and (A2). Then there exists a constant \( c \) depending on \( \bar{a}, r_0 \) and \( \kappa \) such that, for every \( \tau \in M(\Omega; \Phi_V) \),

\[
\frac{1}{c} \int_{\Omega_{\beta_0/4}} \Phi_V d\tau \leq \int_{\Omega_{\beta_1}} \frac{\Phi_V}{\Phi_0} G_V[\tau] dx
\]

where \( \beta_1 = \beta_0/48(2 + \kappa)^2 \) and

\[
\frac{1}{c} \int_{\Omega} \Phi_V d\tau \leq \int_{\Omega} \frac{\Phi_V}{\Phi_0} G_V[\tau] dx
\]

**Theorem 3.4.** Assume (A1), (A2) and (C1).

Then there exists \( c' > 0 \), depending on \( \bar{a}, \Omega \) and the constants in (C1) such that for every \( \tau \in M_+(\Omega; \Phi_V) \)

\[
\int_{\Omega} \frac{\Phi_V}{\Phi_0} G_V[\tau] dx \leq c' \int_{\Omega} \Phi_V d\tau.
\]

**Remark 3.5.** (i) As before, without loss of generality we may assume that in (C1), \( \alpha \neq 1/2 \) (see Remark 3.2).
(ii) See also Lemmas 5.1 and 5.2 below for estimates of surface integrals on \( \Sigma_\beta = \{ x \in \Omega : \delta(x) = \beta \} \), \( \beta < \beta_0 \).

Estimates as above and a version of Theorem 7.5 have been proved in [15] for \( V = \gamma/\delta^2 \) and in [16] for \( V = \gamma V_k \) where \( V_k = \delta F_k^{-2} \) and \( F_k \) is a smooth \( k \)-dimensional manifold without boundary. The estimates in [16] required \( \gamma < \min(c_H(V_k), \frac{1}{4}(2(N-k)-1)) \). The present estimates apply to a family of potentials which include those mentioned above (see Section 6) and require only \( \gamma < c_H(V_k) \).

In [15], [16] and [14] the estimates have been applied to a study of semilinear boundary value problem with absorption nonlinearity. (In those papers the definition of boundary trace was different from the \( L_V \) trace used in Section 7. However, the two definitions are equivalent in the specific cases studied there.)
4. Estimates of $L_V$ harmonic functions

**Proof of Theorem 3.1.** Let $y \in \partial \Omega$ and $b > 1$. Put

$$C_b(y) = \{ x \in \Omega : |x - y| \leq b\delta(x), \ \delta(x) < \epsilon_b \}$$

where $0 < \epsilon_b < \beta_0$ and $b$ are chosen so that

$$C_{2b}(y) \subset \Omega_{\beta_0} \cup \{y\} \ \forall y \in \partial \Omega.$$

By Proposition 2.2 there exists $t_0 \in (0, \beta_0)$ such that, for every $y \in \partial \Omega$,

$$K(x, y) G_V(x, y) \sim G_0(x, y) G_0(x, x)$$

with similarity constant dependent on $\bar{a}, \Omega$ but independent of $y$.

Let $y \in \partial \Omega$ and $b > 1$. Taking $t_0$ sufficiently small (depending on $b$)

$$C_b(y) := \{ \langle x - y, n_y \rangle > 0, \ |x - y| \leq b\delta(x), \ \delta(x) < t_0 \} \subset \Omega$$

for every $y \in \partial \Omega$.

By (4.1) and the strong Harnack inequality,

$$\frac{K_0(x, y)}{K_V(x, y)} \sim \frac{G_V(x, x_0)}{G_0(x, x_0)} \sim \frac{\Phi_V(x)}{\Phi_0(x)} \ \forall x \in C_b(y) \cap \Sigma_\beta, \ \forall \beta \in (0, t_0).$$

Hence,

$$\int_{C_b(y) \cap \Sigma_\beta} \frac{\Phi_V(x)}{\Phi_0(x)} K_V(x, y) dS_x \sim \frac{1}{c_1} \leq \int_{\Sigma_\beta} \frac{\Phi_V(x)}{\Phi_0(x)} K_V(x, y) dS_x$$

and

$$\int_{C_b(y) \cap \Sigma_\beta} \frac{\Phi_V(x)}{\Phi_0(x)} K_V(x, y) dS_x \leq c_2(b).$$

By Fubini’s theorem, the lower estimate in (3.1) is a consequence of (4.1).

Next we show that there exists $\lambda > -1$ such that

$$\int_{\Sigma_\beta \setminus C_b(y)} \frac{\Phi_V(x)}{\Phi_0(x)} K_V(x, y) dS_x \leq c_3(b) \beta^\lambda \ \forall \beta \in (0, t_0)$$

where $c_3(b) > 0$ is independent of $\beta$ and $y$. 
By Theorem 2.5,

\[ \frac{\Phi_V(x)}{\Phi_0(x)} K_V(x, y) \leq C \frac{\Phi_V^2(x)}{\Phi_V^2(x, y)} |x - y|^{2-N} \delta(x)^{-1} \]  

where, if $|x - y| \leq \beta_0$, we choose $x_y = (\delta(x) + |x - y|)n_{\sigma(x)}$. Thus $x$ and $x_y$ are on the normal to $\partial \Omega$ at $\sigma(x)$. (Recall that $\sigma(x)$ denotes the closest point to $x$ on $\partial \Omega$.) Consequently, by (4.6) and (C1) (with $z = x_y$) if

\[ x \in E_y(\beta, t_0) := \{ \delta(x) = \beta : b_\beta < |x - y| \leq t_0 \}, \]

then

\[ \frac{\Phi_V(x)}{\Phi_0(x)} K_V(x, y) \leq C \beta^{2\alpha^*-1} |x - y|^{2-N-2\alpha}. \]

Hence,

\[ \int_{E_y(\beta, t_0)} \frac{\Phi_V(x)}{\Phi_0(x)} K_V(x, y) dS_x \leq C \beta^{2\alpha^*-1} \int_{b_\beta}^{\beta_0} s^{-2\alpha} ds \leq C' \beta^{2(\alpha^* - \alpha)}. \]

Here we assumed that $\alpha \neq 1/2$. As mentioned in Remark 3.2, this does not involve a loss of generality in Theorem 3.1.

On the other hand, if $|x - y| \geq t_0$, (4.6) holds with $x_y = x_0$ so that $\Phi_V(x_y) = 1$. Therefore, by (4.6),

\[ \int_{\Sigma_\beta \cap |x - y| \geq t_0} \frac{\Phi_V(x)}{\Phi_0(x)} K_V(x, y) dS_x \leq C. \]

The last two inequalities imply (4.5) with $\lambda := 2(\alpha^* - \alpha)$. By (3.1), $-1 < \lambda \leq 0$.

By (4.4) and (4.5),

\[ \int_{\Sigma_\beta} \frac{\Phi_V(x)}{\Phi_0(x)} K_V(x, y) dS_x \leq C(b) \beta^\lambda. \]

Combining this inequality with (4.3) and integrating over $\beta \in (0, \beta_0)$ we obtain

\[ \frac{1}{C} \leq \int_{\Omega_{\beta_0}} \frac{\Phi_V(x)}{\Phi_0(x)} K_V(x, y) dS_x \leq C. \]

Assuming that $\delta(x_0) > \beta_0$, the integral over $\Omega \setminus \Omega_{\beta_0}$ is also bounded by a constant independent of $y$. (Recall that $K_V(x_0, y) = 1$ for every $y \in \partial \Omega$ and use Harnack’s inequality for positive $L_V$ harmonic functions.) Therefore this inequality implies (3.2) in the case $\nu = \delta_y$. By Fubini’s theorem, (3.2) follows in the general case. \qed
**Remark 4.1.** Inequality (4.7) is of interest in itself. It should be noted that this inequality is valid even if $\alpha = 1/2$ but in that case $\lambda = 2(\alpha^* - \alpha) - \epsilon$ where $\epsilon > 0$ satisfies $\alpha + \epsilon - \epsilon^* < 1/2$.

### 5. Estimates of $L_V$ potentials

In this section we prove Theorems 3.3 and 3.4. The proofs are based on two lemmas.

**Lemma 5.1.** Assume that (A1) and (A2) hold. Let $\tau \in \mathfrak{M}_+(\Omega; \Phi_V)$ and denote

$$I_1(\beta) := \frac{1}{\beta} \int_{\Sigma_\beta} \Phi_V(x) \int_{\Omega} G_V(x, y) \chi_{a\beta}(|x - y|) d\tau(y) dS_x,$$

where $\chi_s(t) = 1_{(0, s)}(t)$ and $a \geq 16(\kappa + 2)^2$. Then there exists a constant $c$ depending only on $a$, $\bar{a}$ and $\Omega$ such that,

$$\frac{1}{c} \int_{\Omega_{3a/2}} \Phi_V d\tau \leq I_1(\beta) \leq c \int_{\Omega} \Phi_V d\tau \quad \forall \beta \in (0, \beta_0/3a).$$

**Proof.** The domain of integration in $I_1(\beta)$ is $\{(x, y) \in \Sigma_\beta \times \Omega : |x - y| < a\beta\}$. We partition the domain of integration into three parts and estimate each of the resulting integrals separately. Accordingly we denote:

$$I_{1,1}(\beta) := \frac{1}{\beta} \int_{\Sigma_\beta} \Phi_V(x) \int_{\beta/a \leq \delta(y) \leq \beta} G_V(x, y) \chi_{a\beta}(|x - y|) d\tau(y) dS_x,$$

$$I_{1,2}(\beta) := \frac{1}{\beta} \int_{\Sigma_\beta} \Phi_V(x) \int_{\delta(y) \leq \beta/a} G_V(x, y) \chi_{a\beta}(|x - y|) d\tau(y) dS_x,$$

$$I_{1,3}(\beta) := \frac{1}{\beta} \int_{\Sigma_\beta} \Phi_V(x) \int_{\beta \leq \delta(y)} G_V(x, y) \chi_{a\beta}(|x - y|) d\tau(y) dS_x,$$

so that $I_1 = I_{1,1} + I_{1,2} + I_{1,3}$.

**Estimate of $I_{1,1}(\beta)$.**

By the Hardy (chain) inequality (see e.g. [1, Lemma 3.2]), there exists $C(a) > 0$ such that, if

$$\beta/a \leq \delta(y) \leq \beta, \quad x \in \Sigma_\beta, \quad |x - y| \leq a\beta$$

then

$$\frac{1}{C(a)} \Phi_V(x) \leq \Phi_V(y) \leq C(a) \Phi_V(x).$$

(5.2)
By Theorem 2.3, if (*) holds then
\[ \frac{1}{c}|x-y|^{2-N} \leq G_V(x, y) \leq c|x-y|^{2-N} \]
for some constant \( c = c(a) \). Hence, (5.3)
\[ I_{1,1}(\beta) \sim \frac{1}{\beta} \int_{\Sigma_{\beta}} \int_{|x-y| \leq \beta/\delta(y)} |x-y|^{2-N} \chi_{a\beta}(|x-y|) \Phi_V(y) d\tau(y) dS_x \]
\[ = \frac{1}{\beta} \int_{|x-y| \leq \beta/\delta(y)} \int_{\Sigma_{\beta}} |x-y|^{2-N} \chi_{a\beta}(|x-y|) dS_x \Phi_V(y) d\tau(y) \]
For every \( y \) such that \( \beta/a \leq \delta(y) \leq \beta \), the domain of integration contains the set \( \{ x \in \Sigma_{\beta} : \beta - \beta/a \leq |x-y| \leq a\beta \} \). Therefore, for \( y \) as above,
\[ (a-1) \beta \leq \int_{\Sigma_{\beta}} |x-y|^{2-N} \chi_{a\beta}(|x-y|) dS_x \leq \int_0^{a\beta} dr = a\beta, \]
Hence by (5.3), (5.4)
\[ I_{1,1}(\beta) \sim \int_{|x-y| \leq \beta/\delta(y)} \int_{\Sigma_{\beta}} \Phi_V d\tau \]
with similarity constant depending on \( a \) and \( \partial \Omega \).

*Estimate of \( I_{1,2}(\beta) \).* Here we assume that \( \beta < \beta_0/3a \). Since \( \delta(y) < \beta/a \) it follows that in the domain of integration of \( I_{1,2} \),
\[ (a-1) \beta \leq \int_{\Sigma_{\beta}} |x-y|^{2-N} \chi_{a\beta}(|x-y|) dS_x \leq \int_0^{a\beta} dr = a\beta, \]
Thus the pair \((x, y)\) satisfies the conditions of Theorem 2.4 and consequently,
\[ \Phi_V(x)G_V(x, y) \sim \frac{\Phi_V(x)^2}{\Phi_V(x_y)^2} \Phi_V(y)|x-y|^{2-N}, \]
where \( x_y \) may be chosen as follows: \( x_y := \eta + |x-y|n_\eta \) with \( \eta \in \partial \Omega \) the closest point to \( x \).
Then \( \delta(x_y) = |x-y|\) and, by (5.3),
\[ \beta(1 - \frac{1}{a}) \leq \delta(x_y) \leq a\beta, \quad |x-x_y| \leq |x-y| \leq a\beta. \]
Hence, by the Hardy (chain) inequality (see [13, Lemma 3.2]), there exists \( c'(a) > 0 \) such that
\[ \frac{1}{c'} \Phi_V(x) \leq \Phi_V(x_y) \leq c' \Phi_V(x). \]
Therefore, by (5.7),
\[
\Phi_V(x)G_V(x, y) \sim |x - y|^{2-N}\Phi_V(y).
\]
with similarity constant depending on \(a\). Consequently (using Fubini’s theorem) we obtain
\[
I_{1,2}(\beta) \sim \frac{1}{\beta} \int_{\delta(y) \leq \beta/a} \int_{\Sigma_{\beta} \cap |x-y| \leq a\beta} |x - y|^{2-N} dS_x \Phi_V(y) d\tau(y).
\]
Hence,
\[
I_{1,2}(\beta) \sim \int_{\delta(y) \leq \beta/a} \Phi_V(y) d\tau(y).
\]

**Estimate of \(I_{1,3}(\beta)\).** In this case, as \(|x - y| < a\beta, \delta(x) = \beta\) and \(\beta < \delta(y) < (1+a)\beta\), inequality (5.2) holds. Moreover, by Theorem 2.3,
\[
\frac{1}{c} |x - y|^{2-N} \leq G_V(x, y) \leq c |x - y|^{2-N}.
\]
Therefore, as in (5.3), we obtain
\[
I_{1,3}(\beta) \sim \frac{1}{\beta} \int_{\Sigma_{\beta}} \int_{\beta \leq \delta(y)} |x - y|^{2-N} \chi_{a\beta}(|x - y|) \Phi_V(y) d\tau(y) dS_x
\]
\[
\sim \frac{1}{\beta} \int_{\beta \leq \delta(y)} \int_{\Sigma_{\beta}} |x - y|^{2-N} \chi_{a\beta}(|x - y|) dS_x \Phi_V(y) d\tau(y)
\]
\[
\leq \int_{\beta \leq \delta(y)} \Phi_V d\tau
\]
We also have a (partial) estimate from below.
If \(y\) is a point such that \(\beta \leq \delta(y) < \frac{3a}{2}\beta\) then \(B_{a\beta}(y) \cap \Sigma_{\beta}\) contains an \((N-1)\) dimensional ball of radius \(\beta/2\) and consequently there exists a constant \(c_3(a) > 0\) such that
\[
\int_{\Sigma_{\beta}} |x - y|^{2-N} \chi_{a\beta}(|x - y|) dS_x > c_3.
\]
Therefore
\[
I_{1,3}(\beta) \sim \frac{1}{\beta} \int_{\beta \leq \delta(y) < \frac{3a}{2}\beta} \Phi_V d\tau \leq I_{1,3}(\beta)
\]
Inequalities (5.12), (5.5) and (5.10), imply the lower estimate in (5.1). The upper estimate follows from (5.11), (5.5) and (5.10).
Lemma 5.2. Assume that conditions (A1), (A2) and (C1) hold.

Then there exists $C > 0$ such that for every $\tau \in \mathcal{M}(\Omega; \Phi_V)$ and $\beta \in (0, \beta_0)$,

$$I_{2,\lambda}(\beta) := \frac{1}{\beta^\lambda} \int_{\Sigma_\beta} \frac{\Phi_V(x)}{\beta} \int_{\Omega} G_V(x, y)(1 - \chi_{a\beta}(|x - y|))d\tau(y)dS_x$$

$$\leq C \int_{\Omega} \Phi_V d\tau,$$

where $\lambda := 2(\alpha^* - \alpha)$ when $\alpha \neq 1/2$. By (3.1) $-1 < \lambda \leq 0$. If $\alpha = 1/2$ then (5.13) holds with $\lambda = 2(\alpha^* - \alpha - \epsilon)$ where $\epsilon > 0$ is sufficiently small so that $-1 < \lambda$.

Proof. Since $\delta(x) = \beta$ and $|x - y| \geq a\beta$,

$$a \inf(\delta(x), \delta(y)) \leq |x - y|.$$

Therefore, by Theorem 2.4, (5.7) holds. As before we choose $x_y = \eta + |x - y|n_\eta$ where $\eta := \sigma(x)$ is the nearest point to $x$ on $\partial \Omega$. Thus $x$ and $x_y$ are on a normal to $\partial \Omega$ and $|x - y| = \delta(x_y) \geq a\delta(x)$.

By assumption (C1),

$$\frac{\Phi_V(x)}{\Phi_V(x_y)} \leq c(a) \frac{\beta^{\alpha^*}}{|x - y|^{\alpha}}.$$

Therefore, by (5.7),

$$I_{2,0} = \frac{1}{\beta} \int_{\Sigma_\beta} \Phi_V(x) \int_{\Omega} G_V(x, y)(1 - \chi_{a\beta}(|x - y|))d\tau(y)dS_x$$

$$\leq \frac{1}{\beta} \int_{\Sigma_\beta \cap \{|x - y| > a\beta\}} \frac{\Phi_V(x)^2}{\Phi_V(x_y)^2} |x - y|^{2-N} \int_{\Omega} \Phi_V(y)d\tau(y) dS_x$$

$$\leq \beta^{2\alpha^*-1} \int_{\Sigma_\beta \cap \{|x - y| > a\beta\}} |x - y|^{2-N-2\alpha} \int_{\Omega} \Phi_V(y)d\tau(y) dS_x$$

$$\leq \beta^{2\alpha^*-1} \int_{a\beta}^{1} r^{-2\alpha} dr \int_{\Omega} \Phi_V(y)d\tau(y)$$

$$\leq \beta^{2\alpha^*-2\alpha} \int_{\Omega} \Phi_V(y)d\tau(y).$$

Here we assumed that $\alpha \neq 1/2$. In this case (5.13) holds with $\lambda = 2\alpha^* - 2\alpha$. If $\alpha = 1/2$ we replace it by $\alpha + \epsilon$ with $\epsilon > 0$ sufficiently small so that $\lambda = 2(\alpha^* - \alpha - \epsilon) > -1$. \qed
Proof of Theorem 3.3. By Lemma 5.1,
\[ \frac{1}{c} \int_{\Omega_{3a\beta/2}} \Phi V d\tau \leq I_1(\beta) \]
for every \( \beta < \beta_0/3a \). Therefore for every \( \beta_0/6a < \beta < \beta_0/3a \),
\[ \frac{1}{c} \int_{\Omega_{3a/4}} \Phi V d\tau \leq I_1(\beta). \]
More precisely, there exists a constant \( c^* \) depending only on \( a, \bar{a}, \beta_0, \kappa \) such that
\[ (5.15) \quad c^* \int_{\Omega_{3a/4}} \Phi V d\tau \leq \int_{[\bar{\beta}_0/8 < \delta < \beta_0/3a]} \frac{\Phi V}{\Phi_0} G_V[\tau] d\tau. \]
A suitable constant is given by \( c^* = (\inf_{\Omega_{\beta_0/4}} H)^{-1} \frac{\beta_0}{6ac} \), \( H \) is the Jacobian of the transformation from Euclidean coordinates to flow coordinates \((\delta, \sigma)\). It is known that \( H(x) \to 1 \) as \( \delta(x) \to 0 \).

For \( a = 16(2 + \kappa)^2 \), (5.15) implies (3.3).

Put \( \tau' = \tau 1_{[\bar{\beta}_0/8 < \beta < \beta_0/4]} \). Then, using Theorem 2.3 we obtain,
\[ (5.16) \quad c_1 \int_{[\bar{\beta}_0/8 < \beta < \beta_0/4]} \int_{|x-y| < \beta_0/8} \frac{\Phi V}{\Phi_0} G_V(x, y) dxd\tau' \geq \]
\[ c_2 \int_{[\bar{\beta}_0/8 < \beta < \beta_0/4]} \int_{|x-y| < \beta_0/8} |x-y|^{2-N} dxd\tau \geq \]
\[ c_3 \int_{[\bar{\beta}_0/8 < \beta < \beta_0/4]} \int_{[\bar{\beta}_0/8 < \beta < \beta_0/4]} \Phi V d\tau, \]
the constants depending only on \( a, \bar{a}, \beta_0, \kappa \). Combining (5.15) and (5.16) we obtain (3.4). \( \square \)

Proof of Theorem 3.4. By (5.1) and (5.13)
\[ I_1(\beta) \leq c_1 \int_{\Omega} \Phi V d\tau, \quad I_{2,\lambda}(\beta) \leq c_2 \int_{\Omega} \Phi V d\tau \]
for every \( \beta \in (0, r_1) \) where \( r_1 := \beta_0/48(2 + \kappa)^2 \). Therefore
\[ \int_{\Sigma_{\beta}} \frac{\Phi V}{\beta} G_V[\tau] d\tau \leq I_1(\beta) + \beta^\lambda I_{2,\lambda}(\beta) \leq c \max(1, \beta^\lambda) \int_{\Omega} \Phi V d\tau \]
with $\lambda$ as in (5.13). Consequently, integrating over $\beta$ in $(0, r_1)$,

\begin{equation}
\int_{\Omega} \frac{\Phi_V}{\Phi_0} G_V[\tau] \, dx \leq C_1 \int_{\Omega} \Phi_V \, d\tau
\end{equation}

where $C_1$ depends on $\bar{a}$, $\beta_0$, $\kappa$, $\alpha^*$, $\alpha$.

Therefore, to obtain (3.5), it remains to show that

\begin{equation}
\int_{D_{r_1}} \frac{\Phi_V}{\Phi_0} G_V[\tau] \, dx \leq C_2 \int_{\Omega} \Phi_V \, d\tau
\end{equation}

with $C_2$ depending on the parameters mentioned above.

Let $r_2 = r_1/2$ and write,

\[
\int_{D_{r_1}} \frac{\Phi_V}{\Phi_0} G_V[\tau] \, dx = \int_{D_{r_1}} \frac{\Phi_V}{\Phi_0} G_V(x, y) \, d\tau(y) \, dx + \int_{D_{r_1}} \frac{\Phi_V}{\Phi_0} G_V(x, y) \, d\tau(y) \, dx =: J_1 + J_2.
\]

In $J_2$, $x \in D_{r_1}$ and $y \in \Omega \setminus D_{r_2}$. Therefore $G_V(x, y) \sim \Phi_V(y)$. Consequently

\begin{equation}
J_2 \lesssim \int_{\Omega \setminus D_{r_2}} \Phi_V \, d\tau.
\end{equation}

In $J_1$, $x, y \in D_{r_2}$. Therefore, by (5.1), $G_V(x, y) \sim |x - y|^{2-N}$. Consequently

\begin{equation}
J_1 \lesssim \int_{D_{r_1}} \int_{D_{r_2}} |x - y|^{2-N} \, d\tau(y) \, dx \lesssim \tau(D_{r_2}) \lesssim \int_{D_{r_2}} \Phi_V \, d\tau.
\end{equation}

Combining these inequalities we obtain (5.18). \hfill \square

6. A FEW EXAMPLES

We discuss some families of potentials satisfying conditions (A1), (A2), (C1) required in theorems 3.1 – 3.3.

Denote by $V_F$ a potential of the form

\[ V_F = \frac{1}{\delta_F^p}, \quad F \subset \partial \Omega \text{ a compact set, } \delta_F(x) = \text{dist} \, (x, F). \]

I. Obviously a potential $\gamma V_F$ such that $\gamma < c_H(V_F)$ satisfies (A1) and (A2).

If $\gamma \in [0, 1/4)$ then $\gamma V_F$ satisfies condition (C1).

Indeed in this case,

\[ 0 \leq \gamma V_F \leq \gamma V_{\partial \Omega}. \]
Consequently, \[ G_0 \leq G_{\gamma F} \leq G_{\gamma \partial \Omega} \]
which implies
\[ c_1 \delta \leq \Phi_{\gamma F} \leq c_2 \delta^{\alpha} \]
for some \( \alpha \in (\frac{1}{2}, 1) \). Thus \( \gamma F \) satisfies condition (C1).

Similarly, let \( V = \gamma V_{\partial \Omega} \) where \( \gamma \) is a bounded measurable function in \( \Omega \) such that
\[ a_1 \leq \gamma \leq a_2 < 1/4. \]
Put \( \alpha_i := \frac{1}{2} + \sqrt{\frac{1}{4} - a_i}, i = 1, 2 \) and assume that
\[ 0 \leq \alpha_1 - \alpha_2 < 1/2. \]
Then \( 1/2 < \alpha_2 \leq \alpha_1 \), and
\[ \delta^{\alpha_1} \lesssim \Phi_V \lesssim \delta^{\alpha_2}. \]
Consequently \( V \) satisfies condition (C1).

II. Let \( V \) be a potential satisfying (A1), (A2) such that
\[ c_1 \delta^{\alpha_1} \delta_F^{\sigma} \leq \Phi_V \leq c_2 \delta^{\alpha_2} \delta_F^{\sigma}. \]
where \( \alpha_1, \alpha_2 \) satisfy (6.2).

Such an estimate holds, for instance, when \( V = \gamma V_F, F \subset \partial \Omega \) is a smooth \( k \)-dimensional manifold without boundary, \( 0 \leq k \leq N - 2 \) and \( \gamma < c_H(V) \). In this case, it is known that \( c_H(V) \leq \frac{1}{4} (N - k)^2 \) and, by [MT2],
\[ \Phi_V \sim \delta \delta_F^{\sigma}, \quad \sigma = \frac{1}{2} \left( k - N + \sqrt{(N - k)^2 - 4 \gamma} \right). \]
Note that \( \sigma > 0 \) when \( \gamma < 0 \) and \( \sigma < 0 \) when \( 0 < \gamma < c_H(V)^2 \).

Next we prove,

**Lemma 6.1.** Let \( V = \gamma V_F \) where \( F \subset \partial \Omega \) is a compact set and \( \gamma < c_H(V_F) \). Assume that (6.3) and (6.2) hold. Then \( V \) satisfies condition (C1).

**Remark.** Without loss of generality we may assume that \( \alpha_1 + \sigma \neq 1/2 \). Otherwise we replace \( \alpha_1 \) by \( \alpha_1 + \epsilon \) with \( \epsilon > 0 \) such that \( \alpha_1 + \epsilon - \alpha_1 < 1/2 \).

\footnote{Using this example it is possible to construct a family of potentials \( V \) for which \( \Phi_V \) satisfies (6.3) or related estimates.}
Proof. Let \( x, y \in \Omega_{\beta_0} \) and suppose that \( \beta_0 \geq |x - y| > a \delta(x) \) for some \( a \geq 1 + a_0 \) where \( a_0 := 16(2 + \kappa)^2 \). Obviously \( \delta(x) \leq \delta_F(x) \); we consider the following cases separately:

(i) \( a|x - y| \leq \delta_F(x) \), (ii) \( |x - y| \geq a \delta_F(x) \), (iii) \( |x - y| \sim \delta_F(x) \).

Let \( z := (\delta(x) + |x - y|)n_{\sigma(x)} \). Then \( \delta(z) \geq |x - y| > a \delta(x) \) and

\[
\begin{align*}
|\delta_F(x) - \delta_F(z)| & \leq |x - z| = |x - y|, \\
a \delta(x) & \leq |x - y| \leq \delta(z)
\end{align*}
\]  

(6.5)

(i) In this case (6.5) implies

\[
\delta_F(z) \leq |x - y| + \delta_F(x) \implies \delta_F(z) \lesssim \delta_F(x)
\]

and

\[
\delta_F(x) - |x - y| \leq \delta_F(z) \implies (1 - \frac{1}{a}) \delta_F(x) \leq \delta_F(z).
\]

Thus

\[
\delta_F(z) \sim \delta_F(x).
\]

Therefore, by (6.3),

(6.7)

\[
\Phi_V(x)/\Phi_V(z) \lesssim \delta(x)^{a_2}/\delta(z)^{a_1}.
\]

(ii) In this case (6.5) and the definition of \( z \) yield

\[
\begin{align*}
|x - y| & \leq \delta_F(z) \leq (1 + \frac{1}{a})|x - y|, \\
\delta_F(x) & \leq \delta_F(z) + |x - y| \leq 3|x - y|.
\end{align*}
\]

If \( \sigma < 0 \) then, by (6.5) and (6.8),

(6.9) \( \Phi_V(x)/\Phi_V(z) \lesssim \delta(x)^{a_2+\sigma}/(\delta(z)^{a_1}|x - y|^\sigma) \lesssim \delta(x)^{a_2+\sigma}/\delta(z)^{a_1+\sigma} \).

If \( \sigma > 0 \),

(6.10) \( \Phi_V(x)/\Phi_V(z) \lesssim \delta(x)^{a_2}|x - y|^\sigma/\delta(z)^{a_1}|x - y|^\sigma \).

Thus, for every \( \sigma \), (6.7) holds. (iii) In this case,

\[
|x - y| \leq \delta_F(z) \lesssim |x - y|.
\]

Consequently \( \delta_F(x) \sim \delta_F(z) \) and (6.1) follows. \( \square \)
7. THE $L_V$ BOUNDARY TRACE AND POSITIVE $L_V$ SUPERHARMONIC AND SUBHARMONIC FUNCTIONS

7.1. The boundary trace. Assume that $V$ satisfies conditions (A1), (A2).

Let $D \Subset \Omega$ be a Lipschitz domain, denote by $P_V^D$ the Poisson kernel of $L_V$ in $D$ and by $\omega_{x_0}^{x_0,D}$ the harmonic measure of $L_V$ in $D$ relative to a fixed reference point $x_0 \in D$. Then,

\[(7.1)\]
\[d\omega_{x_0}^{x_0,D} = P_V^D(x_0, \cdot) dS \text{ on } \partial D.\]

Let $\{D_n\}$ be a uniformly Lipschitz exhaustion of $\Omega$. It is well known that if $u$ is a positive $\Delta$-harmonic function then

\[(7.2)\]
\[u|_{\partial D_n} d\omega_{x_0}^{x_0,D_n} \rightarrow \nu \]
where $\nu \in M(\partial \Omega)$ is the boundary trace of $u$ and $\rightarrow$ indicates weak convergence in measure. In [17, Definition 3.6], (7.2) was used as a definition of boundary trace for solutions of certain semilinear equations with absorption. In this spirit we define,

**Definition 7.1.** A non-negative Borel function $u$ defined in $\Omega$ has an $L_V$ boundary trace $\nu \in M(\partial \Omega)$ if

\[(7.3)\]
\[\lim_{n \to \infty} \int_{\partial D_n} hud\omega_{x_0}^{x_0,D_n} = \int_{\partial \Omega} h d\nu \quad \forall h \in C(\overline{\Omega}),\]

for every uniformly Lipschitz exhaustion $\{D_n\}$ of $\Omega$. The $L_V$ trace will be denoted by $\text{tr}_V(u)$. Here we assume that $D_{\beta_0} \subset D_1$ and $x_0 \in D_{\beta_0}$.

**Lemma 7.2.** If $\{D_n\}$ is a uniformly Lipschitz exhaustion of $\Omega$ then, for every positive $L_V$ harmonic function $u = \mathbb{K}_V[\nu],$

\[(7.4)\]
\[\lim_{n \to \infty} \int_{\partial D_n} hu d\omega_{x_0}^{x_0,D_n} = \int_{\partial \Omega} h d\nu \quad \forall h \in C(\overline{\Omega}).\]

**Remark.** The proof is similar to that of [17, Lemma 2.2]. We omit details.

**Lemma 7.3.** Assume (A1) and (A2). Then

\[(a)\]
\[\text{tr}_V(\mathbb{K}_V[\nu]) = \nu \quad \forall \nu \in M_+(\partial \Omega)\]

\[(b)\]
\[\text{tr}_V(\mathbb{G}_V[\tau]) = 0 \quad \forall \tau \in M_+(\Omega; \Phi_V).\]

**Proof.** (a) is a restatement of Lemma 7.2

(b) follows from the fact that $\mathbb{G}_V[\tau]$ is an $L_V$ potential, i.e., it does not dominate any positive $L_V$ harmonic function (see [2]).
Let \( \{D_n\} \) be a Lipschitz exhaustion of \( \Omega \) and let
\[
 u_n(x) = \int_{\partial D_n} G_V[\tau] P_{D_n}^V(x, \xi) dS_\xi \quad \forall x \in D_N.
\]

Then \( u_n \) is \( L_V \) harmonic in \( D_n \) and it is dominated by \( G_V[\tau] \) which is \( L_V \) superharmonic. Hence \( u = \lim u_n \) is an \( L_V \) harmonic function dominated by \( G_V[\tau] \). By the previous observation, \( u \equiv 0 \).

\( \square \)

### 7.2. Estimates of positive \( L_V \) superharmonic and \( L_V \) subharmonic functions.

**Proposition 7.4.** Assume (A1), (A2).

(i) If \( u \) is a positive \( L_V \) superharmonic function then
\[
 -L_V u = \tau \in M_+(\Omega; \Phi_V)
\]
and there exists a non-negative measure \( \nu \in M(\partial \Omega) \) such that
\[
 u = G_V[\tau] + K_V[\nu].
\]

(ii) Let \( u \) be a non-negative \( L_V \) subharmonic function and \( \tau := L_V u \). (Thus \( \tau \geq 0 \) is a Radon measure.) Then,
\[
 \tau \in M(\Omega; \Phi_V) \iff u \text{ has an } L_V \text{ boundary trace}
\]

If \( \nu := \text{tr}_V u \) exists then,
\[
 u + G_V[\tau] = K_V[\nu].
\]

**Proof.** (i) This statement is an immediate consequence of the Riesz decomposition lemma and the fact that \( u \) is an \( L_V \) potential if and only if \( u = G_V[\sigma] \) for some \( \sigma \in M_+(\Omega; \Phi_V) \).

(ii) If \( \tau \in M(\Omega; \Phi_V) \) then \( u + G_V[\tau] \) is positive \( L_V \) harmonic. By the Representation Theorem \( \exists \nu \in M(\partial \Omega) \) such that \( u + G_V[\tau] = K_V[\nu] \). By Lemma 7.3 \( \nu = \text{tr}_V u \).

Conversely suppose that \( \nu = \text{tr}_V u \) exists.

Let \( \{D_n\} \) be a Lipschitz exhaustion of \( \Omega \). The function \( u_n = u1_{D_n} \) satisfies,
\[
 u_n + G_V^{D_n}[\tau 1_{D_n}] = \int_{\partial D_n} uP_{D_n}^V(x, \xi) dS_\xi.
\]

Since \( \nu := \text{tr}_V u \) exists, the last term converges at the point \( x_0 \) to \( \nu \). Consequently \( \{G_V^{D_n}[\tau 1_{D_n}](x_0)\} \) is bounded and (by the monotone convergence theorem) converges to \( G_V[\tau](x_0) \). Therefore \( \tau \in M(\Omega; \Phi_V) \).

\( \square \)
Combining Proposition \[ \text{7.4} \] with Theorems \[ \text{3.1, 3.3 and 3.4} \] we obtain the following two sided estimates.

**Theorem 7.5.** Assume \((A1), (A2), (C1)\).

(i) Let \( u \) be a positive \( L_V \) superharmonic function and let \( \tau, \nu \) be as in Proposition \[ \text{7.4} \]. Then there exists a constant \( C \) depending only on \( \bar{a}, \alpha^*, \alpha, \beta_0, \Omega \) such that

\[
C^{-1} \left( \int_{\Omega} \Phi_V d\tau + \|\nu\| \right) \leq \int_{\Omega} \frac{\Phi_V}{\Phi_0} u dx \leq C \left( \int_{\Omega} \Phi_V d\tau + \|\nu\| \right).
\]

(ii) Let \( u \) be a positive \( L_V \) subharmonic function and assume that

\[
\tau := L_V u \in \mathcal{M}(\Omega; \Phi_V).
\]

Then \[ \text{7.9} \] holds and there exists a constant \( C \) depending only on \( \bar{a}, \alpha^*, \alpha, \beta_0, \Omega \) such that

\[
C^{-2} \|\nu\| \leq \int_{\Omega} \frac{\Phi_V}{\Phi_0} u dx + \int_{\Omega} \Phi_V d\tau \leq C^2 \|\nu\|.
\]

**Proof.** Inequality \[ \text{7.11} \] is an immediate consequence of \[ \text{7.7} \] and the estimates stated in Theorems \[ \text{3.1, 3.3 and 3.4} \].

For part (ii) we use \[ \text{7.9} \] and the above mentioned estimates to obtain:

\[
\frac{1}{C} \int_{\Omega} \Phi_V \Phi_0 u dx + \frac{1}{C} \int_{\Omega} \Phi_V d\tau \leq \int_{\Omega} \frac{\Phi_V}{\Phi_0} (u + G[\tau]) dx \leq C \|\nu\|,
\]

\[
\frac{1}{C} \|\nu\| \leq \int_{\Omega} \frac{\Phi_V}{\Phi_0} (u + G[\tau]) dx \leq \int_{\Omega} \frac{\Phi_V}{\Phi_0} u dx + C \int_{\Omega} \Phi_V d\tau.
\]

\[ \square \]

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