THE MATCHING RAMSEY NUMBER OF HYPERGRAPHS, REVISITED

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ABSTRACT. For positive integers \( r \) and \( t \), the hypergraph matching Ramsey number \( R_r(s_1, \ldots, s_t) \) is the least integer \( n \) such that for any \( t \)-hyperedge coloring of complete \( r \)-uniform hypergraph \( K_n^r \), there exists at least one \( j \in \{1, 2, \ldots, t\} \) such that the subhypergraph of all hyperedges receiving color \( j \) contains at least one matching of size \( s_j \). This parameter was completely determined by Alon, Frankl, and Lovász in 1986.

In 2015, the first present author and Hossein Hajibabalahassan found a sharp lower bound for the chromatic number of general Kneser hypergraphs by means of the notion of alternation number of hypergraphs. In this paper, as our first result, we establish a theory to give a development of this result. Doing so, as our second result, we resolve the problem of characterizing \( R_r(s_1,s_2,\ldots, s_t) \). Our final result is introducing a generalization of \( \mathbb{Z}_p \)-Tucker Lemma together with an application of this generalized lemma in determining the matching Ramsey number \( R_r(s_1, s_2, \ldots, s_t) \).

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1. Introduction

A hypergraph \( \mathcal{H} \) is an ordered pair \( \mathcal{H} = (V, E) \), where \( V \) is a set (called the vertex set of \( \mathcal{H} \)) and \( E \) is a set of some nonempty subsets of \( V \) (called the hyperedge set of \( \mathcal{H} \)). By a proper vertex coloring of \( \mathcal{H} \), we mean a function \( c : V(\mathcal{H}) \rightarrow C \) such that for each \( S \in E(\mathcal{H}) \) with \( |S| \geq 2 \) we have \( |\{c(v) : v \in S\}| \geq 2 \). Such a set \( C \) is called the set of colors. By a singleton hyperedge of \( \mathcal{H} \), we mean a hyperedge \( e \) with \( |e| = 1 \). Whenever \( \mathcal{H} \) has no singleton hyperedges, we define the chromatic number of \( \mathcal{H} \), denoted by \( \chi(\mathcal{H}) \), as the minimum cardinality of a set \( C \) such that a proper vertex coloring \( f : V(\mathcal{H}) \rightarrow C \) exists. If \( \mathcal{H} \) has some singleton hyperedges, then we define \( \chi(\mathcal{H}) = +\infty \).

For any nonnegative integer \( n \), let the symbols \( [n] \) denote the set \( \{1, \ldots, n\} \). Also, for a positive integer \( r \geq 2 \), let \( \mathbb{Z}_r = \{\omega, \omega^2, \ldots, \omega^r\} \) be a cyclic group of order \( r \) with generator \( \omega \). For a vector \( X = (x_1, x_2, \ldots, x_n) \in (\mathbb{Z}_r \cup \{0\})^n \setminus \{0\} \), a subsequence of nonzero terms of \( X \) is called alternating if each two consecutive terms of this subsequence are different. The length of the longest alternating subsequence of \( X \) is denoted by \( \text{alt}(X) \). For example, if we set \( r = 3, n = 6 \), and \( X = (\omega^2, 0, \omega^2, \omega^3, 0, \omega^3) \), then \( \text{alt}(X) = 3 \). Also, we define \( \text{alt}(0, \ldots, 0) \) to be zero. For an \( X = (x_1, x_2, \ldots, x_n) \in (\mathbb{Z}_r \cup \{0\})^n \) and \( \epsilon \in \mathbb{Z}_r \), we define \( X^\epsilon \subset [n] \) to be the set of all indices \( j \) such that \( x_j = \epsilon \), i.e., \( X^\epsilon = \{j : x_j = \epsilon\} \). By abuse of language, we can write \( X = (X^\epsilon)_{\epsilon \in \mathbb{Z}_r} \).

Let \( \mathcal{H} \) be a hypergraph with \( n \) vertices and \( \sigma : [n] \rightarrow V(\mathcal{H}) \) be an injective mapping. We define \( \text{alt}_r(\mathcal{H}, \sigma) \) to be the maximum possible value of \( \text{alt}(X) \) for \( X = (X^\epsilon)_{\epsilon \in \mathbb{Z}_r} \in (\mathbb{Z}_r \cup \{0\})^n \) such that none of \( \sigma(X^\epsilon) \) contains any hyperedge of \( \mathcal{H} \). In other words,

\[
\text{alt}_r(\mathcal{H}, \sigma) = \max \{\text{alt}(X) : X \in (\mathbb{Z}_r \cup \{0\})^n \text{ and for each } \epsilon \in \mathbb{Z}_r, \text{ we have } E(\mathcal{H}[\sigma(X^\epsilon)]) = \emptyset\}.
\]

The alternation number of \( \mathcal{H} \), \( \text{alt}_r(\mathcal{H}) \), is the minimum possible value for \( \text{alt}_r(\mathcal{H}, \sigma) \) where the minimum is taken over all injective mappings \( \sigma : [n] \rightarrow V(\mathcal{H}) \).

The general Kneser hypergraph \( KG_r(\mathcal{H}) \) is an \( r \)-uniform hypergraph which has \( E(\mathcal{H}) \) as vertex set and whose hyperedges are formed by \( r \) pairwise disjoint hyperedges of \( \mathcal{H} \), i.e.,

\[
E(KG_r(\mathcal{H})) = \{\{e_1, \ldots, e_r\} : e_i \cap e_j = \emptyset \text{ for all } i \neq j\}.
\]
Alishahi and Hajiabolhassan [3] presented a lower bound for the chromatic number of general Kneser hypergraphs $KG^r(\mathcal{H})$ in terms of $n$ and $\text{alt}_r(\mathcal{H})$.

**Theorem 1.** ([3]) For an integer $r \geq 2$ and a hypergraph $\mathcal{H}$, we have 
\[ \chi (KG^r(\mathcal{H})) \geq \left\lceil \frac{|V(\mathcal{H})| - \text{alt}_r(\mathcal{H})}{r-1} \right\rceil. \]

Theorem 1 was applied on various families of graphs to compute their chromatic numbers or investigating their coloring properties; see [1, 2, 4, 5, 6, 10, 14].

For two positive integers $n$ and $r$, the symbol $[n]^r$ denotes the set of all $r$-subsets of $[n]$. The **complete $r$-uniform hypergraph** $K_n^r$ is a hypergraph with vertex set $[n]$ and the hyperedge set $[n]^r$. A set consisting of $m$ pairwise disjoint hyperedges of a hypergraph is called an $m$-*matching*. For positive integers $r$ and $t$, the $(r,t)$-hypergraph matching Ramsey number, $R_r(s_1, s_2, \ldots, s_t)$, is the least integer $n$ such that for any $t$-hyperedge coloring of $K_n^r$, say $c : [n]^r \to [t]$, there exists at least one $j \in [t]$ and at least one matching of size $s_j$ such that each hyperedge of this matching receives $j$ as its color.

The hypergraph matching Ramsey number was studied in different languages in the literature, see [7, 8, 11, 12], and it was completely characterized by Alon, Frankl, and Lovász [7], as follows.

**Theorem 2.** ([7]) Let $s_1, s_2, \ldots, s_t$, and $r$ be positive integers with $s_1 \leq s_2 \leq \cdots \leq s_t$. Then 
\[ R_r(s_1, s_2, \ldots, s_t) = 1 + \sum_{i=1}^{t} s_i + s_t(r-1) - t. \]

**Outline of The Paper.** In the rest, first we establish a development of Theorem 1 in Section 2. Then, based on the results of Section 2, we resolve Theorem 2 in Section 3. Finally, in Section 4, we give a generalization of $\mathbb{Z}_p$-Tucker Lemma together with an application of this generalized lemma in determining the matching Ramsey number $R_r(s_1, s_2, \ldots, s_t)$.

## 2. Matching Coloring of Hypergraphs

Let $\mathcal{H}$ be a hypergraph with $n$ vertices and $r$ be a positive integer. A mapping $\tau : \mathbb{N} \to \{0, 1, \ldots, r-1\}$ is called a color-frequency mapping. For a subset $A$ of $\mathbb{N}$, an $(A, \tau)$-matching coloring of $\mathcal{H}$ is a mapping $c : E(\mathcal{H}) \to A$ such that for each $a \in A$, the hypergraph induced by the hyperedges receiving $a$ as their color has no matching of size $\tau(a) + 1$. In other words, there is no $a$-monochromatic $(\tau(a) + 1)$-matching. The **$\tau$-matching chromatic number** of $\mathcal{H}$, denoted by $\chi_M(\tau, \mathcal{H})$, is the least possible cardinality of a finite subset $A$ of $\mathbb{N}$ such that $\mathcal{H}$ admits an $(A, \tau)$-matching coloring. If there exists no such finite subset $A$, then we define $\chi_M(\tau, \mathcal{H})$ to be infinite.

The following theorem provides a sharp lower bound for the $\tau$-matching chromatic number of hypergraphs.

**Theorem 3.** Let $r \geq 2$ be an integer and $\tau : \mathbb{N} \to \{0, 1, \ldots, r-1\}$ be a color-frequency mapping. Then, for any hypergraph $\mathcal{H}$, we have 
\[ \chi_M(\tau, \mathcal{H}) \geq \min \left( \left\{ |A| : A \subset \mathbb{N}, \ A \text{ is finite, and } \sum_{a \in A} \tau(a) \geq |V(\mathcal{H})| - \text{alt}_r(\mathcal{H}) \right\} \cup \{+\infty\} \right). \]

Note that Theorem 3 is a generalization of Theorem 1. To see this, consider the color-frequency mapping $\tau : \mathbb{N} \to \{0, 1, \ldots, r-1\}$ such that $\tau(a) = r-1$ for each $a \in \mathbb{N}$. Now, it is clear that
\[ \chi_M(\tau, \mathcal{H}) = \chi(KG^r(\mathcal{H})) \] and
\[ \min \left\{ |A| : A \subseteq \mathbb{N} \text{ and } \sum_{a \in A} \tau(a) \geq |V(\mathcal{H})| - \text{alt}_r(\mathcal{H}) \right\} = \left\lceil \frac{|V(\mathcal{H})| - \text{alt}_r(\mathcal{H})}{r-1} \right\rceil. \]

**Proof of Theorem 3.** Let \( \mathcal{H} \) be a hypergraph. If there is no finite subset \( A \subseteq \mathbb{N} \) such that \( \mathcal{H} \) admits an \((A, \tau)\)-matching coloring, then \( \chi_M(\tau, \mathcal{H}) = +\infty \) and there is nothing to prove.

Let \( A \) be a finite subset of \( \mathbb{N} \). Our procedure is to show that if \( \mathcal{H} \) admits an \((A, \tau)\)-matching coloring \( c : E(\mathcal{H}) \to A \), then we have \( \sum_{a \in A} \tau(a) \geq |V(\mathcal{H})| - \text{alt}_r(\mathcal{H}) \). In this regard, for each \( a \) in \( \mathcal{H} \) with \( \tau(a) < r-1 \), we add \( r - 1 - \tau(a) \) additional vertices \( x_1^{(a)}, x_2^{(a)}, \ldots, x_{r-1-\tau(a)}^{(a)} \) to \( \mathcal{H} \) together with adding all additional singleton hyperedges \( \{x_1^{(a)}\}, \{x_2^{(a)}\}, \ldots, \{x_{r-1-\tau(a)}^{(a)}\} \) to \( E(\mathcal{H}) \) in order to obtain a new hypergraph \( \mathcal{H} \). Also, we extend the hyperedge coloring \( c : E(\mathcal{H}) \to A \) to a hyperedge coloring of \( \mathcal{H} \), say \( \tilde{c} : E(\mathcal{H}) \to A \), in such a way that for each \( a \) in \( \mathcal{H} \) with \( \tau(a) < r-1 \), all additional singleton hyperedges \( \{x_1^{(a)}\}, \{x_2^{(a)}\}, \ldots, \{x_{r-1-\tau(a)}^{(a)}\} \) are colored by \( a \). On one hand, since \( c : E(\mathcal{H}) \to A \) is an \((A, \tau)\)-matching coloring of \( \mathcal{H} \), for each \( a \) in \( A \), the size of each matching in \( E(\mathcal{H}) \) whose hyperedges are colored by \( a \) is less than or equal to \( \tau(a) \). On the other hand, there are exactly \( r - 1 - \tau(a) \) hyperedges in \( E(\mathcal{H}) - E(\mathcal{H}) \) that are colored by \( a \) under \( \tilde{c} : E(\mathcal{H}) \to A \). We conclude that for each \( a \) in \( A \), the size of any matching in \( E(\mathcal{H}) \) whose hyperedges are colored by \( a \) is at most \( r - 1 \). Therefore, \( \tilde{c} : E(\mathcal{H}) \to A \) is a proper vertex coloring of \( KG^r(\mathcal{H}) \). So, we have
\[ |A| \geq \chi(KG^r(\mathcal{H})) \geq \frac{|V(\mathcal{H})| - \text{alt}_r(\mathcal{H})}{r-1}; \]
and therefore,
\[ (r - 1)|A| \geq |V(\mathcal{H})| - \text{alt}_r(\mathcal{H}). \]

Since \( (r - 1)|A| = \left( \sum_{a \in A} \tau(a) \right) + |V(\mathcal{H})| - |V(\mathcal{H})| \) and \( \text{alt}_r(\mathcal{H}) = \text{alt}_r(\mathcal{H}) \), we have
\[ \left( \sum_{a \in A} \tau(a) \right) + |V(\mathcal{H})| - |V(\mathcal{H})| \geq |V(\mathcal{H})| - \text{alt}_r(\mathcal{H}); \]
and therefore, \( \sum_{a \in A} \tau(a) \geq |V(\mathcal{H})| - \text{alt}_r(\mathcal{H}) \); as desired.

In the proof of Theorem 3, the procedure is showing that for every finite subset \( A \) of \( \mathbb{N} \) that an \((A, \tau)\)-matching coloring exists, we have
\[ \sum_{a \in A} \tau(a) \geq |V(\mathcal{H})| - \text{alt}_r(\mathcal{H}). \]

If \( \mathcal{H} \) has at least one hyperedge, then every \((A, \tau)\)-matching coloring of \( \mathcal{H} \) satisfies \( A \neq \emptyset \). So, we can rewrite Theorem 3 as follows.
Theorem 4. Let \( r \geq 2 \) be an integer and \( \tau : \mathbb{N} \rightarrow \{0,1,\ldots,r-1\} \) be a color-frequency mapping. Then, for any hypergraph \( H \) with \( E(H) \neq \emptyset \), we have

\[
\chi_M(\tau,H) \geq \min \left( \left\{ |A| : \emptyset \neq A \subset \mathbb{N}, A \text{ is finite, and } \sum_{a \in A} \tau(a) \geq |V(H)| - \text{alt}_r(H) \right\} \cup \{+\infty\} \right).
\]

In the rest of this section, we are concerned with showing that the lower bounds in Theorems 3 and 4 are sharp.

Proposition 1. Let \( n, k, \) and \( r \) be positive integers with \( r \geq 2 \) and \( n \geq k \) and \( n - r(k-1) \geq 0 \). Also, let \( \tau : \mathbb{N} \rightarrow \{0,1,\ldots,r-1\} \) be a color-frequency mapping such that \( r - 1 \in \tau(\mathbb{N}) \). We have

\[
\chi_M(\tau,K^k_n) \leq \min \left( \left\{ |A| : \emptyset \neq A \subset \mathbb{N}, A \text{ is finite, and } \sum_{a \in A} \tau(a) \geq n - r(k-1) \right\} \cup \{+\infty\} \right).
\]

Proof. If there is no nonempty finite subset \( A \) of \( \mathbb{N} \) such that \( \sum_{a \in A} \tau(a) \geq n - r(k-1) \), then the right-hand side of the inequality is \(+\infty\) and the proof is completed. So, let us suppose that there exists a nonempty finite subset \( A := \{a_1,a_2,\ldots,a_t\} \) of \( \mathbb{N} \) with \( \tau(a_1) \leq \tau(a_2) \leq \cdots \leq \tau(a_t) \) and \( \sum_{a \in A} \tau(a) \geq n - r(k-1) \). Our aim is to show that for each such \( A \), there exists some subset \( B \) of \( \mathbb{N} \) such that \( |B| = |A| \) and \( r - 1 \in \tau(B) \) and \( \sum_{b \in B} \tau(b) \geq \sum_{a \in A} \tau(a) \geq n - r(k-1) \) for which \( K^k_n \) admits a \((B,\tau)\)-matching coloring.

If \( \tau(a_t) = r - 1 \), then we set \( B = A \). Otherwise, if \( \tau(a_t) \neq r - 1 \), then due to \( r - 1 \in \tau(\mathbb{N}) \), we consider a positive integer \( x \) with \( \tau(x) = r - 1 \) and put \( B := (A - \{a_t\}) \cup \{x\} \). So, in both cases, we have \(|B| = |A|\) and \( r - 1 \in \tau(B) \) and \( \sum_{b \in B} \tau(b) \geq \sum_{a \in A} \tau(a) \geq n - r(k-1) \). For the sake of simplicity of symbols, put \( B = \{b_1,b_2,\ldots,b_t\} \) with \( \tau(b_t) = r - 1 \).

We consider \( t \) pairwise disjoint sets \( S_1,S_2,\ldots,S_t \) such that the following three conditions are hold:

- \( \bigcup_{i=1}^{t} S_i = [n] \),
- For each \( i \) in \( \{1,2,\ldots,t-1\} \) we have \( |S_i| \leq \tau(b_i) \),
- \( |S_t| \leq r(k-1) + \tau(b_t) = r(k-1) + r - 1 = rk - 1 \).

Define \( c : E(K^k_n) \rightarrow B \) such that each \( e \in E(K^k_n) = \binom{[n]}{k} \) is mapped to \( c(e) := \min\{i : e \cap S_i \neq \emptyset\} \). The mapping \( c \) is a \((B,\tau)\)-matching coloring of \( \binom{[n]}{k} \); and therefore, the assertion follows. \( \square \)

In Proposition 1, if we assume \( n \geq rk \) instead of \( n \geq r(k-1) \), we obtain the following proposition.

Proposition 2. Let \( n, k, \) and \( r \) be positive integers with \( r \geq 2 \) and \( n \geq rk \). Also, let \( \tau : \mathbb{N} \rightarrow \{0,1,\ldots,r-1\} \) be a color-frequency mapping such that \( r - 1 \in \tau(\mathbb{N}) \). We have

\[
\chi_M(\tau,K^k_n) \leq \min \left( \left\{ |A| : A \subset \mathbb{N}, A \text{ is finite, and } \sum_{a \in A} \tau(a) \geq n - r(k-1) \right\} \cup \{+\infty\} \right).
\]

It is worth pointing out that if \( n \geq r(k-1) \), then \( \text{alt}_r(K^k_n) = r(k-1) \). Therefore, combining Theorem 3 and Proposition 2, leads to the following corollary to show that the lower bound in Theorem 3 is sharp.
Corollary 1. Let $n$, $k$, and $r$ be positive integers such that $r \geq 2$ and $n \geq rk$. For any color-
frequency mapping $\tau : \mathbb{N} \rightarrow \{0, 1, \ldots, r - 1\}$ with $r - 1 \in \tau(\mathbb{N})$, the $\tau$-matching chromatic number
of $K_n^k$, say $\chi_M(\tau, K_n^k)$, is equal to

$$\min \left( \left\{ |A| : A \subset \mathbb{N}, A \text{ is finite, and } \sum_{a \in A} \tau(a) \geq \left| V \left( K_n^k \right) \right| - alt_r \left( K_n^k \right) \right\} \bigcup \{+\infty\} \right).$$

Also, combining Theorem 4 and Proposition 1, leads to the following corollary to show that the lower bound in Theorem 4 is sharp.

Corollary 2. Let $n$, $k$, and $r$ be positive integers with $r \geq 2$ and $n \geq k$ and $n - r(k - 1) \geq 0$. Also, let $\tau : \mathbb{N} \rightarrow \{0, 1, \ldots, r - 1\}$ be a color-frequency mapping such that $r - 1 \in \tau(\mathbb{N})$. Then, $\chi_M(\tau, K_n^k)$ equals

$$\min \left( \left\{ |A| : \emptyset \neq A \subset \mathbb{N}, A \text{ is finite, and } \sum_{a \in A} \tau(a) \geq \left| V \left( K_n^k \right) \right| - alt_r \left( K_n^k \right) \right\} \bigcup \{+\infty\} \right).$$

3. Resolving Alon-Frankl-Lovász’s Theorem

The aim of this Section is presenting another proof of Theorem 2, as follows.

Another proof of Theorem 2. If $r = 1$ or $t = 1$ or $s_t = 1$, then the assertion follows. Therefore, we may assume that $r \geq 2$ and $t \geq 2$ and $s_t \geq 2$.

Define a color-frequency mapping $\tau : \mathbb{N} \rightarrow \{0, 1, \ldots, s_t - 1\}$ such that for each $i \in \mathbb{N}$,

$$\tau(i) = \begin{cases} 
    s_i - 1 & \text{if } i \in [t] \\
    0 & \text{otherwise}.
\end{cases}$$

In view of Theorem 3, if $\sum_{a \in \mathbb{N}} \tau(a) = \sum_{i=1}^{t} (s_i - 1) < n - alt_s(K_n^r)$, then there is no $(A, \tau)$-matching
coloring of $K_n^r$ for each finite set $A \subset \mathbb{N}$. This implies that since $alt_s(K_n^r) \leq s_t(r - 1)$, for each
$n \geq 1 + \sum_{i=1}^{t} (s_i - 1) + s_t(r - 1)$ and any $t$-hyperedge coloring $c : \binom{[n]}{r} \rightarrow [t]$, there are $j \in [t]
and s_j$ pairwise disjoint hyperedges $e_1, e_2, \ldots, e_{s_j} \in E(K_n^r)$ such that $c(e_1) = \cdots = c(e_{s_j}) = j$. In
other words, there is a matching $M$ of size $s_j$ whose hyperedges are colored with the same color $j$. Consequently, we have

$$R_r(s_1, s_2, \ldots, s_t) \leq 1 + \sum_{i=1}^{t} s_i + s_t(r - 1) - t.$$ 

So, it suffices to show that

$$R_r(s_1, s_2, \ldots, s_t) \geq 1 + \sum_{i=1}^{t} s_i + s_t(r - 1) - t.$$ 

In this regard, we consider $t$ pairwise disjoint sets $S_1, S_2, \ldots, S_t$ such that the following three conditions are hold :

- $\bigcup_{i=1}^{t} S_i = \left[ \sum_{i=1}^{t} s_i + s_t(r - 1) - t \right]$,
- For each $i$ in \{1, 2, \ldots, t - 1\} we have $|S_i| = s_i - 1$,
- $|S_t| = s_t(r - 1) + s_t - 1 = s_tr - 1$. 

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For the sake of simplicity, put \( \Lambda := \sum_{i=1}^{t} s_i + s_t(r - 1) - t \). Define a \textit{bad} \( t \)-hyperedge coloring \( C_{\text{bad}} : E(K_1^n) \to [t] \) such that each \( e \in E(K_1^n) = \left( \frac{[\Lambda]}{r} \right) \) is mapped to \( C_{\text{bad}}(e) := \min \{ i : e \cap S_i \neq \emptyset \} \). Since for each \( i \) in \([t]\) there are not any matchings of size \( s_t \) whose all hyperedges are colored by \( i \), we conclude that

\[
R_r(s_1, s_2, \ldots, s_t) \geq 1 + \sum_{i=1}^{t} s_i + s_t(r - 1) - t;
\]

which is desired. \( \Box \)

4. The \( Z_p \)-Tucker Lemma : A Generalization

This section is devoted to present a generalization of \( Z_p \)-Tucker Lemma.

The \( Z_p \)-Tucker Lemma is a development of the celebrated Tucker’s Lemma (which is equivalent to the famous Borsuk-Ulam Theorem). There are various important and surprising applications of Tucker’s Lemma. For a very detailed discussion in this matter, one can see the interesting reference [13].

First, we present the \( Z_p \)-Tucker Lemma. In this regard, we should note that for \( X_1 \) and \( X_2 \) in \((\mathbb{Z}_p \cup \{0\})^n\), we write \( X_1 \subset X_2 \) whenever for each \( \epsilon \in \mathbb{Z}_p \) we have \( X_1^\epsilon \subseteq X_2^\epsilon \).

**Lemma 1.** ([15, 16]) \( (\mathbb{Z}_p \)-Tucker Lemma) Let \( m \) and \( n \) be positive integers, \( p \) be a prime number, and \( \alpha \) be nonnegative integer with \( 0 \leq \alpha \leq m \). Also, let

\[
\lambda : (\mathbb{Z}_p \cup \{0\})^n \setminus \{0\} \to \mathbb{Z}_p \times [m]
\]

be a mapping that satisfies all of the following three properties simultaneously:

1. The mapping \( \lambda \) is a \( \mathbb{Z}_p \)-equivariant mapping, that is, for each \( \omega \in \mathbb{Z}_p \), we have
   \[
   \lambda_1(\omega^j X) = \omega^j \lambda_1(X) \text{ and } \lambda_2(\omega^j X) = \lambda_2(X).
   \]
2. For all \( X_1 \subset X_2 \in (\mathbb{Z}_p \cup \{0\})^n \setminus \{0\} \), the condition \( \lambda_2(X_1) = \lambda_2(X_2) \leq \alpha \) implies \( \lambda_1(X_1) = \lambda_1(X_2) \).
3. For all \( X_1 \subset X_2 \subset \cdots \subset X_p \in (\mathbb{Z}_p \cup \{0\})^n \setminus \{0\} \), if \( \lambda_2(X_1) = \lambda_2(X_2) = \cdots = \lambda_2(X_p) \geq \alpha + 1 \), then
   \[
   |\{\lambda_1(X_1), \lambda_1(X_2), \ldots, \lambda_1(X_p)\}| \leq p - 1.
   \]

Then, we have

\[
\alpha + (m - \alpha)(p - 1) \geq n.
\]

The next lemma is a generalization of \( \mathbb{Z}_p \)-Tucker Lemma. The proof is similar to the one by Meunier [15].

**Lemma 2.** Let \( p \) be a prime number and let \( \gamma_1, \gamma_2, \ldots, \gamma_m \in [p - 1] \). Also, let

\[
\lambda : (\mathbb{Z}_p \cup \{0\})^n \setminus \{0\} \to \mathbb{Z}_p \times [m]
\]

be a \( \mathbb{Z}_p \)-equivariant mapping such that for each \( i \in [m] \) and for each chain \( X_1 \subset X_2 \subset \cdots \subset X_i \in (\mathbb{Z}_p \cup \{0\})^n \setminus \{0\} \), if \( \lambda_2(X_1) = \lambda_2(X_2) = \cdots = \lambda_2(X_i) = i \), then

\[
|\{\lambda_1(X_1), \lambda_1(X_2), \ldots, \lambda_1(X_i)\}| \leq \gamma_i.
\]

Then, we have

\[
\sum_{i=1}^{m} \gamma_i \geq n.
\]
Proof. Clearly, the mapping $\lambda$ can be considered as a $\mathbb{Z}_p$-equivariant simplicial mapping from $\mathrm{sd} (\mathbb{Z}_p^n)$ to

$$C = \left( \sigma_{y_1-1}^{p-1} \right) \ast \cdots \ast \left( \sigma_{y_m-1}^{p-1} \right).$$

Therefore, by Dold’s Theorem [9, 13], the dimension of $C$ must be strictly larger than the connectivity of $\mathrm{sd} (\mathbb{Z}_p^n)$; which implies

$$\sum_{i=1}^{t} \gamma_i - 1 > n - 2;$$

as desired. \hfill $\square$

One can observe that Lemma 2 is a development of $\mathbb{Z}_p$-Tucker Lemma by regarding

$$\gamma_1 = \gamma_2 = \cdots = \gamma_\alpha = 1, \text{ and } \gamma_{\alpha+1} = \gamma_{\alpha+2} = \cdots = \gamma_m = p - 1.$$

The following proposition is a special case of Theorem 3; nevertheless, its proof shows an explicit application of Lemma 2.

Proposition 3. Let $p$ be a prime number and $\tau : \mathbb{N} \rightarrow \{0, 1, \ldots, p - 1\}$ be a color-frequency mapping. Then, for any hypergraph $\mathcal{H}$, we have

$$\chi_M (\tau, \mathcal{H}) \geq \min \left( \left\{ |A| : A \subset \mathbb{N}, \text{ A is finite, and } \sum_{a \in A} \tau(a) \geq |V(\mathcal{H})| - \mathrm{alt}_p(\mathcal{H}) \right\} \right).$$

Proof of Proposition 3 by means of Lemma 2. Let $|V(\mathcal{H})| = n$ and consider a bijection $\sigma : [n] \rightarrow V(\mathcal{H})$ such that $\mathrm{alt}_p (\mathcal{H}, \sigma) = \mathrm{alt}_p (\mathcal{H})$. First note that if there is no finite subset $A \subset \mathbb{N}$ such that $\mathcal{H}$ admits an $(A, \tau)$-matching coloring, then $\chi_M (\tau, \mathcal{H}) = +\infty$ and there is nothing to prove. Therefore, we can assume that $\chi_M (\tau, \mathcal{H}) = t$ is finite. If $t = 0$, then $\mathcal{H}$ has not any hyperedges; and therefore, $\mathrm{alt}_p (\mathcal{H}) = |V(\mathcal{H})|$. Hence, the empty set satisfies $\sum_{a \in \emptyset} \tau(a) = |V(\mathcal{H})| - \mathrm{alt}_p (\mathcal{H}) = 0$; and we are done. So, we may assume that $t$ is a positive integer. Let $c : E(\mathcal{H}) \rightarrow A$ be an $(A, \tau)$-matching coloring of $\mathcal{H}$ where $A = \{a_1, a_2, \ldots, a_t\}$ and $a_1 < a_2 < \cdots < a_t$. To prove the assertion, we need to show that $n \leq \mathrm{alt}_p (\mathcal{H}) + \sum_{a \in A} \tau(a)$.

Let $m = \mathrm{alt}_p (\mathcal{H}) + t$. For each $i \in [m]$, define

$$\gamma_i = \begin{cases} 1 & \text{if } i \leq \mathrm{alt}_p (\mathcal{H}) \\ \tau(a_{i-\mathrm{alt}_p (\mathcal{H})}) & \text{if } i \geq \mathrm{alt}_p (\mathcal{H}) + 1. \end{cases}$$

Now, we consider a mapping

$$\lambda : (\mathbb{Z}_p \cup \{0\})^n \setminus \{0\} \rightarrow \mathbb{Z}_p \times [m]$$

in order to apply Lemma 2. We endow $2^{[n]}$ with an arbitrary total ordering $\leq$.

- If $X \in (\mathbb{Z}_p \cup \{0\})^n \setminus \{0\}$ satisfies $\mathrm{alt}(X) \leq \mathrm{alt}_p (\mathcal{H}, \sigma)$, then set $\lambda_1 (X)$ to be the first nonzero component of $X$ and put $\lambda_2 (X) = \mathrm{alt}_p (X)$.
- If $X \in (\mathbb{Z}_p \cup \{0\})^n \setminus \{0\}$ satisfies $\mathrm{alt}(X) > \mathrm{alt}_p (\mathcal{H}, \sigma)$, then there exists some $\epsilon \in \mathbb{Z}_p$ such that $\sigma(X')$ is a superset of at least one hyperedge $e$ of $\mathcal{H}$, i.e., $e \subseteq \sigma(X')$. We define $\zeta(X)$ as the maximum positive integer $j$ for which there exists an $\epsilon \in \mathbb{Z}_p$ and some hyperedge $e \in E(\mathcal{H})$ such that $e \subseteq \sigma(X')$ and $c(e) = a_j$. Set $\lambda_2 (X) = \zeta(X) + \mathrm{alt}_p (\mathcal{H})$. For determining $\lambda_1 (X)$, put $\zeta(X) = j$ and choose $X^{\lambda_1 (X)}$ as the maximum $X'$ with respect to the ordering $\leq$ such that $\sigma(X')$ is a superset of some hyperedge $e$ with $c(e) = a_j$. 
Since the mapping $\lambda$ satisfies the condition of Lemma 2 with respect to prior presented $\gamma_i$‘s, we conclude that

$$n \leq \sum_{i=1}^{m} \gamma_i$$

$$= \text{alt}_p(H) + \sum_{a \in A} \tau(a);$$

as desired. \hfill \Box

Using the previous proposition, we can prove the following proposition, which shows an application of Lemma 2 in determining $R_r(s_1, s_2, \ldots, s_t)$ for some special cases.

**Proposition 4.** Let $s_1, s_2, \ldots, s_t, p$, and $r$ be positive integers, where $p$ is a prime number and $s_1 \leq s_2 \leq \cdots \leq s_t \leq p$. Then

$$1 + \sum_{i=1}^{t} s_i + s_t(r - 1) - t \leq R_r(s_1, s_2, \ldots, s_t) \leq 1 + \sum_{i=1}^{t} s_i + p(r - 1) - t.$$

In particular, we have

$$R_r(s_1, s_2, \ldots, s_t) = 1 + \sum_{i=1}^{t} s_i + s_t(r - 1) - t,$$

provided that $s_t$ is a prime number.

**Remark.** In a forthcoming paper, the second author extends the familiar results for all variations of colorability defect and equitable colorability defects of hypergraphs and multihypergraphs.

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