Comparison of relativity theories with observer-independent scales of both velocity and length/mass

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ABSTRACT

We consider the two most studied proposals of relativity theories with observer-independent scales of both velocity and length/mass: the one discussed by Amelino-Camelia as illustrative example for the original proposal (gr-qc/0012051) of theories with two relativistic invariants, and an alternative more recently proposed by Magueijo and Smolin (hep-th/0112090). We show that these two relativistic theories are much more closely connected than it would appear on the basis of a naive analysis of their original formulations. In particular, in spite of adopting a rather different formal description of the deformed boost generators, they end up assigning the same dependence of momentum on rapidity, which can be described as the core feature of these relativistic theories. We show that this observation can be used to clarify the concepts of particle mass, particle velocity, and energy-momentum-conservation rules in these theories with two relativistic invariants.
1 Introduction

One of us put forward in Refs. [1, 2] the proposal of special-relativistic theories (theories of the transformation rules that connect the observations of different inertial observers) with two observer-independent scales. Galilei's relativity principle peacefully coexists with the absence of observer-independent scales, as shown by the structure of the Galilei-Newton transformation rules. Einstein's Special Relativity relies on the ordinary Lorentz transformations, which host one observer-independent scale, the velocity scale $c$. In Refs. [1, 2] it was argued that there should also be some examples of special-relativistic theories with two (or more) observer-independent scales, which could be called [1] “Doubly Special Relativity” or “DSR”, and a first example, which we shall call DSR1 in the following, was analyzed in detail, including a careful formulation of the postulates, an explicit derivation of the \textit{finite} deformed Lorentz transformations and a study of the kinematical conditions for particle production in collision processes. Most of the quantitative results reported in Refs. [1, 2] were obtained in leading order (all orders in $c \sim 3 \cdot 10^8 \text{m/s}$ but only leading order in the second observer-independent scale, tentatively identified with the Planck length $L_p \sim 1.6 \cdot 10^{-35} \text{m}$ or the corresponding Planck mass $E_p \equiv 1/L_p$) and were later generalized to all orders in Refs. [3, 4, 5, 6]. While progress was coming quickly in the analysis of the first DSR example, for some time no other DSR example was identified in the literature. This changed when Magueijo and Smolin proposed another DSR theory in Ref. [7], which we shall call DSR2 in the following. More recently, other possible realization of the DSR idea have been considered (see, e.g., Refs. [8, 9]), but DSR1 and DSR2 remain the focus of the majority of what is becoming a rather large DSR literature (see, e.g., Refs. [10, 11, 12, 13, 14, 15])

DSR1 and DSR2 were originally formulated using somewhat different conventions and notation, and are being considered in the literature as significantly different [10] realizations of the DSR idea; most studies focusing one or the other [11, 12, 13, 14, 15]. But in the literature one can still not find a robust comparative study of these two relativistic theories. Our analysis will here focus on establishing the common features and the main differences between these two DSR proposals. We will argue that a DSR theory is primarily characterized by the dependence of (space) momentum on rapidity. The only other necessary ingredient is the $E(p)$ (energy/momentum) dispersion relation (once the dependence of $p_i$ on rapidity is fixed, the dispersion relation also fixes the dependence of energy on rapidity). And we find that remarkably, in spite of adopting a rather different formal description of the deformed boost generators, DSR1 and DSR2 predict the same dependence of momentum on rapidity. We therefore expose the fact that the difference between these two relativistic theories amounts to a simple redefinition of energy, a single function of a single variable, whereas it was so far assumed that some nontrivial nonlinear transformation $P^\text{DSR2}_\mu (P^\text{DSR1}_\mu)$ (four functions, each depending on four variables) should be involved in the connection between DSR1 and DSR2.

In light of the simplicity of the relation between DSR1 and DSR2, sharing the same core relativistic features, we find appropriate to reanalize some of the perspectives on these two theories which have been presented in the literature. In particular, some studies considering DSR2 have adopted a description of particle mass and of particle velocity which is different from the one adopted in most studies of DSR1. Since we are finding that DSR1 and DSR2 are connected in an elementary way, it
seems illogical that such differences in the conceptual analysis would be necessary. Indeed we will find that these claimed differences are due to misinterpretation of some differences which are purely in the realm of the choice of notation, and therefore cannot have physical consequences.

We also provide a careful analysis of kinematical conditions for particle production in collision processes, which are a key component of DSR theories. These kinematical conditions were already carefully analyzed within the DSR1 proposal in Refs. [1, 2, 4, 5, 6], whereas in the paper [7] announcing the DSR2 proposal there was no discussion of these kinematical conditions.

Importantly we find that, although the difference between DSR1 and DSR2 is of marginal conceptual significance, there are some contexts in which from a quantitative perspective the two theories lead to significantly different predictions. Research conducted over the last few years [17, 18, 19, 20, 21, 22] has led to the conclusion that for theories, like DSR1 and DSR2, predicting a Planck-scale deformation of the dispersion relation $E(p)$, the only two contexts in which the new effects could be observably large are: (i) astrophysical studies of signal dispersion, which are sensitive to the relativistic relation between velocity and momentum [17, 18, 19, 21], and (ii) analyses of cosmic-ray data, which are sensitive to the structure of the relativistic laws of energy-momentum conservation [20, 21, 22]. From the results here reported we conclude that DSR2 (just as previously established for DSR1 [1, 2]) only has negligibly small implications for cosmic-ray physics. We also find that the DSR2 relation between velocity and momentum of a particle does not lead to observably large new effects, while the corresponding DSR1 prediction is testable and will be tested [19] exploiting the remarkable sensitivity [18] of the GLAST space telescope.

Since the comparative study we are reporting is articulated over several points, for clarity we reserve the next Section to a description of the key characteristics of DSR1, while Section 3 discusses the corresponding characteristics of DSR2 and compares the two relativistic theories. Section 4 is an aside on the nonlinearities in the DSR framework. The closing Section 5 summarizes our key results and emphasizes some key open issues.

2 Main features of DSR1

(DSR1.a): general structure. As mentioned, work on DSR1 started already in the original papers [1, 2] that proposed the idea of theories with both a velocity scale and a length/momentum scale as relativistic invariants. The analysis preliminary analysis of DSR1 presented in Refs. [1, 2] intended to show that Galilei’s Relativity Principle can coexist with postulates introducing two observer-independent scales.

The guiding intuition came in part from research in quantum gravity where it is often assumed that the Planck length $L_p$ has a fundamental role in the short-distance structure of space-time. Such a structural role for the Planck length can easily come into conflict with one of the cornerstones of Einstein’s Special Relativity: FitzGerald-Lorentz length contraction. According to FitzGerald-Lorentz length contraction, different inertial observers would attribute different values to the same physical length. If the Planck length only has the role we presently attribute to it, which is basically the role
of a coupling constant (an appropriately rescaled version of the gravitational coupling), no problem arises for FitzGerald-Lorentz contraction, but if we try to promote $L_p$ to the status of an intrinsic characteristic of space-time structure it is natural to find conflicts with FitzGerald-Lorentz contraction. For example, it is very hard (perhaps even impossible) to construct discretized versions or non-commutative versions of Minkowski space-time which enjoy ordinary Lorentz symmetry. Therefore, unless the Relativity postulates are modified, it appears impossible to attribute to the Planck length a truly fundamental (observer-independent) intrinsic role in the microscopic structure of space-time.

It is clearly not necessary to introduce such a modification of the Relativity postulates, since we do not (yet?) have any conclusive experimental evidence that require us to attribute to $L_p$ an observer-independent role in the microscopic structure of space-time, but it is of course legitimate [1, 2] to explore this possibility.

Actually, there is some tentative encouragement from experiments for the idea that $L_p$ is also a feature of kinematics, rather than simply a coupling constant. In fact, over the last few years there have been attempts to interpret puzzling observations [16] of ultra-high-energy cosmic rays as a manifestation of new rules of kinematics for particle production in collision processes, and that these new rules might involve a new kinematical length scale. The relevant process is photopion production by high-energy protons colliding with soft photons. These processes are important because at sufficiently high energies, above a threshold energy $E_{th}$, pion-producing interactions between the high-energy cosmic-ray proton and one of the soft photons in the CMBR environment become kinematically allowed. It was shown (see, e.g., Refs. [20, 28, 22, 21]) that certain puzzling aspects of cosmic-ray observations could be explained if the conventional special-relativistic estimate of the photopion production threshold energy was modified at order $L_p E_{th}^{1+n}$, with $n \leq 1$.

Another class of observations relevant for Lorentz invariance which is improving very rapidly are the ones pertaining to a possible wavelength/energy dependence of the speed of photons [18, 29]. With experiments such as AMS [30] and GLAST [19] the expected sensitivity should allow to investigate the possibility of corrections of order $L_p E$ to the speed-of-light law, $v_\gamma \simeq c(1 \pm L_p E)$.

Since the objective of the present study is the one of comparing the two most studied DSR proposals, rather than providing any encouragement (or discouragement) for the DSR idea, we refer the interested reader to Refs. [1, 2] for a more detailed description of the motivation and the logical structure of DSR theories. It will become evident in the following that DSR1 proved to be a good choice as example of relativistic theory with two invariants to be used in illustrating the proposal put forward in Refs. [1, 2]. In fact, DSR1 attributes to the Planck length the role of inverse of the maximum value of momentum that can be held by fundamental particles, and in this sense may be an appealing possibility for work in quantum gravity in which some sort of quantization of spacetime at the planck scale is introduced. Moreover, DSR1 predicts corrections of order $L_p E$ to the speed-of-light law and is therefore appealing from a phenomenological perspective, since, as just mentioned, this effect can be tested in the not-so-distant future.

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1Pedagogical illustrative examples of this observation have been discussed, e.g., in Ref. [23] for the case of discretization and in Refs. [24, 25, 26, 27] for the case of non-commutativity.
(DSR1.b): generators of deformed Lorentz transformations. As argued in Refs. [1, 2] the introduction of the second (length/momentum) relativistic invariant does not naturally invite us to revise space rotations, but it appears inevitable that such relativistic theories should involve deformed boost generators. The DSR1 relativistic theory involves the following differential representation of the boost generators (without loss of generality we choose to focus on the boost that acts along the $z$ axis):

$$N_z = p_z \frac{\partial}{\partial E} + \left( \frac{\tilde{L}_p \hat{p}^2}{2} + 1 - e^{-2\tilde{L}_p E} \right) \frac{\partial}{\partial p_z} - \tilde{L}_p p_z \left( \frac{p_j}{\partial p_j} \right).$$

(1)

$\tilde{L}_p$ is here assumed to be of the order of the Planck length (but not necessarily given exactly by the Planck length).

We also note here, because of its usefulness for the study of possible experimental tests of DSR1 (which would of course be only sensitive to leading-order effects), the formula for boost generators approximated in leading order in the deformation scale

$$N_z \simeq p_z \frac{\partial}{\partial E} + \left( E + \frac{\tilde{L}_p \hat{p}^2}{2} - \tilde{L}_p E^2 \right) \frac{\partial}{\partial p_z} - \tilde{L}_p p_z \left( \frac{p_j}{\partial p_j} \right).$$

(2)

(DSR1.c): deformed Lorentz transformation rules. The DSR1 Lorentz generators of course give a direct description of infinitesimal Lorentz transformations. In order to obtain finite Lorentz transformations (the ones that truly describe Lorentz symmetry in physics) one needs to exponentiate these generators, thereby constructing candidate elements of a group of (deformed) Lorentz transformations. The generators (1) had already surfaced in the context of math-oriented studies of $\kappa$-Poincarè Hopf algebras. It was well known that for Lorentz-sector generators of generic $\kappa$-Poincarè Hopf algebras the exponentiation procedure does not actually lead to a group (one only obtains a quasigroup in the sense of Batalin [25, 31]). For the construction of DSR1 it was therefore a key technical point, from the math perspective, the observation [1] that the generators of the specific Lorentz sector of $\kappa$-Poincarè Hopf algebra described in (1) do lead to group structure upon exponentiation. The DSR1 generators are therefore consistent with a genuine group of deformed Lorentz symmetries, whereas generic boost generators within $\kappa$-Poincarè Hopf algebras are not.

For the purposes of the later comparison between DSR1 and DSR2 it is convenient to revisit here briefly how from the DSR1 generators one obtains the DSR1 transformation laws between inertial observers. This was discussed in detail in Refs. [1, 2, 5]. The mentioned exponentiation of generators actually means that energy and momentum satisfy some differential equations as function of rapidity. From (1) one straightforwardly obtains (for simplicity, in the case in which the components of momentum orthogonal to the boost vanish)

$$\begin{align*}
\frac{\partial p(\xi)}{\partial \xi} &= p'(\xi) = -\frac{\tilde{L}_p}{2} p^2(\xi) + \frac{1-e^{-2\tilde{L}_p E(\xi)}}{2\tilde{L}_p} \\
\frac{\partial E(\xi)}{\partial \xi} &= E'(\xi) = p(\xi)
\end{align*}$$

(3)

The emerging group is just the ordinary Lorentz group (same composition law) but nonlinearly realized. This is easily verified [1] using the Baker-Campbell-Hausdorff formula (applied to group elements constructed as exponentials of the generators) and exploiting the fact that the deformed Lorentz generators still satisfy the ordinary Lorentz algebra.
Here the rapidity $\xi$ is as usual intended as the coefficient of the boost generator in the exponential that describes a finite group transformation.

Eq. (3) is a system of two first-order differential equations in $p(\xi)$ and $E(\xi)$. They can be combined to obtain a single (second-order) differential equation for $p(\xi)$

$$p'' + \tilde{L}_p^2 p^3 + 3\tilde{L}_p p' - p = 0 .$$

This equation can be straightforwardly integrated [5]. For our purposes it is sufficient to note here the result that describes these finite transformations in a simple illustrative case\(^{3}\): we consider a finite Lorentz transformation which is purely a boost in the $z$ direction and acts on a four-momentum with components $(0, 0, p_z, E_0)$. One finds the solution [5]

$$p_z(\xi) = p_{z,0} \frac{\cosh(\xi) + A_0 \sinh(\xi)}{1 - L_p p_{z,0} (A_0 - A_0 \cosh(\xi) - \sinh(\xi))} ,$$

where $A_0$ codifies the information about the initial conditions ($A_0$ is \(\xi\)-independent), as described in Ref. [5].

(DSR1.d): dispersion relation. A key characteristic of DSR1, both conceptually and phenomenologically, is the deformed dispersion relation. Combining the deformed boosts of (1) and undeformed space-rotation generators one finds that the dispersion relation (the relation between energy and momentum which is kept invariant under the rotation/boost transformations) is given by

$$m^2 = \tilde{L}_p^{-2} \left( e^{\tilde{L}_p E} + e^{-\tilde{L}_p E} - 2 \right) - p^2 e^{\tilde{L}_p E} .$$

Especially for phenomenological applications (experimental searches) the key point is the leading correction to the ordinary special-relativistic dispersion relation that follows from (6)

$$m^2 \simeq E^2 - p^2 - \tilde{L}_p p^2 E .$$

Of course, using the dispersion relation, (6), one can obtain the formula that gives the dependence of energy on rapidity from the formula, (5), that gives the dependence of (space) momentum on rapidity. Therefore a compact way to codify the content of a DSR theory is provided by a pair of equations: a formula, of type (5), for the dependence of momentum on rapidity and a formula, of type (6), that expresses energy as a function of momentum.

(DSR1.e): mass. In (5) we have adopted the notation "$m^2$" in the dispersion relation by simple analogy with the corresponding special-relativistic equation. In Einstein’s special relativity the quadratic mass casimir $m^2$ coincides with the square of the rest energy $M$ and we commonly refer to both concepts as if they were equivalent. In the new type of relativistic theories here of interest this correspondence is lost. In particular, in DSR1, the rest energy, $M$ (which we shall also sometimes denote by $E_{\text{rest}}$), is related to the $m$ of (6) by the relation

$$m = \tilde{L}_p^{-1} \left( e^{\tilde{L}_p M/2} - e^{-\tilde{L}_p M/2} \right) .$$

\(^3\)More general discussions of the transformation rules can be found in Refs. [1, 2, 5].
Note that $M$ differs from $m$ only at order $\tilde{L}_p^2$.

(DSR1.f): maximum momentum. It is easy to verify from the formulas provided above that for positive $\tilde{L}_p$ the DSR1 transformation rules predict [1, 2, 5, 6] that in the infinite-rapidity limit the momentum of an on-shell particle saturates to a maximum value $p_{\text{max}} = \tilde{L}_p^{-1}$, while energy diverges (instead both diverge in the infinite-rapidity limit in special relativity).

The existence of a maximum momentum in DSR1 may suggest [1, 2, 6] the interpretation of $\tilde{L}_p$ as the observer-independent minimum value of wavelength.

(DSR1.g): velocity. The law that relates the velocity of particles to their (mass and) energy is a key aspect of a relativistic theory. Velocity is naturally introduced in the spacetime sector, while the DSR1 postulates (just like the DSR2 postulates described later) were formulated in energy-momentum space. In Refs. [1, 2] it was observed that the relation $v_{\text{particle}} = dE/dp$ holds both in Galilei relativity and in Einstein’s special relativity, and it was then argued that it is natural to assume the validity of this relation also in DSR theories. This basically amounts to the expectation that also DSR theories should allow an Hamiltonian formulation ($\dot{x} = dH/dp$). It also represents a plan for the construction of the spacetime sector of a DSR theory, a spacetime in which $v_{\text{particle}} = dE/dp$ would hold.

Particularly noteworthy is the DSR1 deformed speed-of-light law:

$$v_{\gamma} = \frac{dE}{dp} = e^{\tilde{L}_p E} \approx c \left( 1 + \tilde{L}_p c^{-1} E \right). \quad (9)$$

This law is fully consistent with the wavelength independence of the speed of light found in our low-energy observations (observations at energy scales that are much smaller than $1/\tilde{L}_p$) but would lead (see later) to observably large effects in forthcoming experiments.

(DSR1.h): magnitude of the deformation. A key aspect of a DSR theory is the magnitude of the new effects it predicts at high energies (in the low-energy regime all realistic DSR theories must reproduce ordinary special relativity, since Einstein’s theory is verified to very good accuracy in low-energy experiments). Of course, for applications in phenomenology (experimental searches) it is sufficient to establish the magnitude of the leading-order corrections that the DSR theory predicts with respect to ordinary special relativity. DSR1 is based on a deformed dispersion relation which involves a correction of order $\tilde{L}_p E^3$ to the ordinary $E^2 = p^2 + m^2$ dispersion relation. Therefore the dimensionless quantity that characterizes the strength of the deformation of the dispersion relation is $\tilde{L}_p E$. It is easy to verify that the same statement applies to all aspects of DSR1: all the modifications of ordinary special-relativistic results that are predicted by DSR1 have characteristic strength $\tilde{L}_p E$.

(DSR1.i): two-particle sector and conservation laws. As emphasized in Refs. [1, 2, 6], in DSR theories the step from the one-particle sector to multiparticle sectors is usually rather nontrivial. Ordinary special relativity is a linear theory and therefore no such complications are encountered; for example, one can meaningfully attribute a total momentum $\vec{p}_a + \vec{p}_b$ to a system composed of two particles, one with momentum $\vec{p}_a$ and the other with momentum $\vec{p}_b$. The concept of total momentum

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4As discussed later in the paper, it appears that the $\kappa$-Minkowski noncommutative spacetime would provide a spacetime realization of DSR1, and the structure of that spacetime is indeed consistent [32] with $v_{\text{particle}} = dE/dp$.

5The observation that photons of infinite energy would have infinite velocity in DSR1 was first reported in Ref. [3].
is instead highly nontrivial in DSR theories, as illustrated by the considerations on total momentum in DSR1 reported in Ref. [6]. For phenomenological applications a key aspect of multiparticle systems in DSR theories are the kinematical conditions (“energy-momentum conservation”) for particle production in collision processes. As mentioned, certain cosmic-ray observations might invite us to consider deformed conservation laws. As observed in Refs. [1, 2, 6], DSR theories will necessarily involve deformed rules for energy-momentum conservation; for example, in a process $a + b \rightarrow c + d$ it would be inconsistent to enforce the conditions $E_a + E_b = E_c + E_d$, $\vec{p}_a + \vec{p}_b = \vec{p}_c + \vec{p}_d$ since these conditions would not provide an observer-independent law. In ordinary special relativity the conditions $E_a + E_b = E_c + E_d$, $\vec{p}_a + \vec{p}_b = \vec{p}_c + \vec{p}_d$ can be derived from the (linear) structure of Lorentz transformations, and they do provide an observer-independent kinematical law for collision processes (they are either satisfied for all inertial observers or not satisfied for all inertial observers). The deformed (nonlinear) transformation laws of DSR theories would clearly not be consistent with the conditions $E_a + E_b = E_c + E_d$, $\vec{p}_a + \vec{p}_b = \vec{p}_c + \vec{p}_d$.

The form of the new energy-momentum-conservation-like laws will of course depend on the structure of the specific DSR theory (they must reflect the structure of the transformation laws). Much work has been done [1, 2, 6] on these laws. For the purpose of the present paper it is sufficient to note here one example of these conservation laws which is consistent [1] in leading order in $\tilde{L}_p$:

$$E_a + E_b - \tilde{L}_p c \vec{p}_a \cdot \vec{p}_b - E_c - E_d + \tilde{L}_p c \vec{p}_c \cdot \vec{p}_d \simeq 0 \ ,$$

$$\vec{p}_a + \vec{p}_b - \tilde{L}_p (E_a \vec{p}_b + E_b \vec{p}_a)/c - \vec{p}_c - \vec{p}_d + \tilde{L}_p (E_c \vec{p}_d + E_d \vec{p}_c)/c \simeq 0 \ .$$

It is easy to verify using the DSR1 boost generators that when satisfied in one inertial frame these conservation rules are also satisfied in all other inertial frames. It is noteworthy that these conservation laws “mix” the particles (the nonlinear correction terms involve properties of pairs of particles). Other consistent candidates mixing the particles and some consistent examples of non-mixing laws are discussed in Refs. [1, 6].

**DSR1.j**: **space-time sector.** The DSR1 postulates concern energy-momentum space, but of course one would like to identify a consistent space-time counterpart. This identification process must still be considered as “in progress”; however, various arguments [1, 6], and particularly the mentioned one-particle-sector energy-momentum-space connection with $\kappa$-Poincaré Hopf algebras, suggest that the $\kappa$-Minkowski noncommutative spacetime, $[x_j, x_k] = 0$, $[x_j, t] = i\tilde{L}_p x_j$, would provide a spacetime realization of DSR1. To provide motivation for this identification we just mention here that it is known [32] that propagation in $\kappa$-Minkowski satisfies the DSR1 dispersion relation and that products of “time-to-the-right-ordered wave exponentials” in $\kappa$-Minkowski satisfy the property

$$(e^{i\vec{p} \cdot \vec{x}} e^{-iEt})(e^{i\vec{k} \cdot \vec{x}} e^{-i\omega t}) = (e^{i\vec{q} \cdot \vec{x}} e^{-i(E + \omega)t})$$

with $\vec{q} = \vec{p} + e^{\tilde{L}_p E} \vec{k} \simeq \vec{p} + \vec{k} + \tilde{L}_p E \vec{k}$. This shows that the way to compose $\vec{p}$ and $\vec{k}$ into $\vec{q}$ resembles the structure of Eq. (11) (but the correspondence is not exact and this requires further study [6]).
(DSR1.k): testability. DSR1 is definitely testable with forthcoming experiments such as AMS [30] and GLAST [19], which, as mentioned in Section 2, will test very accurately the possibility of a wavelength/energy dependence of speed of photons. In fact, as discussed in point (DSR1.g), DSR1 predicts the type of $L_pE$ correction to the speed-of-light law that these experiments can verify.

For the other class of sensitive Lorentz-invariance tests mentioned in Section 2, studies of particle-production thresholds through observations of cosmic rays, it appears [1, 2, 6] instead that the effects predicted by DSR1 are not large enough for testing with planned experiments. This conclusion appears to be inevitable [1, 2, 6] if energy-momentum deformed-conservation laws, e.g. Eqs. (10),(11), are applied to the DSR1 description of photon-pion production (relevant for observations of cosmic rays).

3 Main features of DSR2

(DSR2.a): general structure. As mentioned, a second example of DSR theory, which we here call DSR2, was more recently proposed in Ref. [7] by Magueijo and Smolin. It appears that a key guiding intuition for this study was the one that a better example of DSR theory could be obtained by describing the deformed boost generators as a combination of the conventional special-relativistic boost generators and of the generator of dilatations.

(DSR2.b): generators of deformed Lorentz transformations. In DSR2 (just like in DSR1) boost generators are deformed but there is no deformation of the rotation generators. The differential representation of the DSR2 boost generators is [7] (again, without loss of generality, we choose to focus on the boost that acts along the $z$ axis):

$$N_z \approx p_z \frac{\partial}{\partial E} + E \frac{\partial}{\partial p_z} - \lambda p_z \left( E \frac{\partial}{\partial E} + p_j \frac{\partial}{\partial p_j} \right). \tag{12}$$

For conceptual clarity we are using a different notation, $\lambda$, for the DSR2 deformation scale, while for the analogous DSR1 scale we used $\tilde{L}_p$ (also notice that our $\lambda$ is the scale denoted by $l_0$ in Ref. [7]).

(DSR2.b): deformed Lorentz transformation rules. The DSR2 transformation rules were obtained explicitly in Ref. [7], using an elegant argument which allowed to avoid explicit integration of differential equations. In particular, it was shown that the DSR2 boost transformation rules for the simple illustrative case of a transformation which is purely a boost in the $z$ direction and acts on a four-momentum with components $(0, 0, p_{z,0}, E_0)$ should take the form:

$$p_z(v) = \frac{\gamma(v)[p_{z,0} - vE]}{1 + \frac{v^2}{\gamma(v) - 1}E_0 - \lambda \gamma(v)(v)p_{z,0}}, \tag{13}$$

where $\gamma(v) \equiv 1/\sqrt{1 - v^2}$.

The parameter $v$ was interpreted in Ref. [7] as the relative velocity between the observers connected by the boost (but this appears, as discussed below, to lead to a questionable result concerning the relation between velocity and energy of a particle for a given observer).
The DSR2 formula (13) appears to be very different from the corresponding DSR1 formula (5). Moreover, the DSR2 deformation, codified in (12), appears to be very different from the DSR1 deformation, codified in (1). As mentioned this has led to various speculations [10, 11, 12, 13, 14, 15] on the possible differences there might be present in the conceptual structure of the two relativistic theories. However, as we were approaching the study here reported it became soon obvious that a genuine comparison between DSR1 and DSR2 could not be based on naive inspection of Eqs.(13) and (5), which assume a different parametrization, and could not even be based on naive inspection of Eqs. (12) and (1), since the form of the generators of boosts is affected by various aspects of a DSR theory, some elementary aspects and some more delicate ones. [The generators of boosts encode (in an appropriate sense) at least two aspects of the DSR theory: the aspect here of interest (the dependence of momentum on rapidity) and the dispersion relation, which deserves being considered separately.]

In order to have more robust ground for the comparison, it appeared necessary to analyze the transformation laws predicted by the two theories from the same perspective. We then observed that the DSR2 boost generators (12) lead to the first-order differential equations (focusing again, for simplicity, on the case in which the components of momentum orthogonal to the boost vanish)

\[
\frac{\partial p(\xi)}{\partial \xi} = p'(\xi) = E(\xi) - \lambda p^2(\xi)
\]

\[
\frac{\partial E(\xi)}{\partial \xi} = E'(\xi) = p(\xi) - \lambda E(\xi)p(\xi)
\]

which specify the relation between energy and momentum and rapidity, i.e. they are the analog of the DSR1 differential equations (3).

Again, even at this stage of comparison, a naive inspection of (3) and (14) would appear to indicate that there can be very little in common between the DSR1 transformation laws and the DSR2 transformation laws. However, if now we combine the equations in (14) into a single (second-order) differential equation for \( p(\xi) \) we obtain

\[
p'' + \lambda^2 p^3 + 3\lambda pp' - p = 0
\]

which is identical (upon the, already implicit, identification of constants \( \lambda \equiv \bar{L}_p \)) to the Eq. (4) which we encountered at the corresponding point of the analysis of DSR1.

Clearly DSR2, through a different path, turned out to reproduce the same differential equation connecting momentum and rapidity that was already at the basis of DSR1. This striking fact should be further investigated. A key point will be to establish how central is the role of the laws governing the dependence of momentum on rapidity. For some applications it is possible that the original linear differential equations (separately of energy and momentum) contain the relevant information. But in some cases, like in the calculation of the dependence of momentum on rapidity, one does not miss any information by combining the two original linear differential equations and using the dispersion relation to obtain a single (second-order) differential equation.

**DSR2.d): dispersion relation.** The key point in which DSR1 and DSR2 differ is the dispersion relation. Using (12) and the unmodified space-rotation generators one easily verifies that in DSR2 the
deformed dispersion relation is codified in
\[ m^2 = \frac{E^2 - |\vec{p}|^2}{(1 - \lambda E)^2}, \tag{16} \]
which we also note here (for convenience in the later study of the phenomenological implications) approximated to leading order in \( \lambda \):
\[ m^2 = E^2 - \vec{p}^2 + 2\lambda E(E^2 - \vec{p}^2). \tag{17} \]

It is now clear that DSR2 assigns the same relativistic law, \( p(\xi) \), as DSR1, but adopts a simple redefinition of the energy \( E_{\text{DSR}2} = f(E_{\text{DSR}1}) \),
\[ E_{\text{DSR}2} = \frac{1 - e^{-2\lambda E_{\text{DSR}1}} + \lambda^2 \vec{p}^2}{2\lambda}, \tag{18} \]
through the analysis. The DSR2 momentum depends on rapidity just as in DSR1 but the redefinition of energy leads to a different dispersion relation and a different dependence of energy on rapidity\(^7\). It is easy to verify that one obtains a solution of the DSR2 differential equations (14) by simply assuming that momentum depends on rapidity, \( p(\xi) \), as in the DSR1 case, \textit{i.e.} according to (15), and assuming that the dependence of energy on rapidity is also specified by (15) through the DSR2 dispersion relation (16) (basically obtaining \( E(\xi) \) by composition of the function \( E(p) \), which is codified in the DSR2 dispersion relation, with the function \( p(\xi) \), which is codified in (15)).

\textbf{(DSR2.e): mass.} From the DSR2 dispersion relation it is easy to obtain the relation between the “casimir mass” \( m \) and the rest energy \( M \)
\[ m = \frac{M}{1 - \lambda M}. \tag{19} \]
Notice that DSR2 (unlike DSR1) predicts that \( M \) would differ from \( m \) already in leading order:
\[ m = M + \lambda M^2. \tag{20} \]

In Ref. [7] it was argued that the physical mass of particles should be identified in DSR theories with the casimir mass. This concept of the “physical mass” is troublesome. We feel, adopting the viewpoint of Refs. [1, 2], that one could here contemplate 3 distinct concepts of physical mass: the “rest-energy mass”, the inertial mass and the gravitational mass. For each of these 3 concepts the physics community has introduced a dedicated operative procedure giving physical meaning to the concept. In our present familiar theories these 3 logically-independent concepts turn out to be identified. This may or may not be true as we investigate new physical theories, but we feel that

\(^7\)But of course the functions \( E_{\text{DSR}1}(\xi) \) and \( E_{\text{DSR}2}(\xi) \) differ in a very elementary way. Since the energy is constrained to momentum by the dispersion relation, \( E_{\text{DSR}1}(p) \), \( E_{\text{DSR}2}(p) \), and since momentum depends on rapidity in the same way in DSR1 and DSR2, one basically finds \( E_{\text{DSR}1}(\xi) = E_{\text{DSR}1}(p(\xi)) \) and \( E_{\text{DSR}2}(\xi) = E_{\text{DSR}2}(E_{\text{DSR}1}(p(\xi))) \), with \( p(\xi) \) satisfying (4) in both cases.
at this point in the development of DSR theories there is no reason to expect any anomaly in this respect: one can straightforwardly identify the rest energy, and it is legitimate to expect that the rest energy will still coincide with the inertial mass and the gravitational mass. The concept of “casimir mass” is probably even troublesome at the level of analysis in which one attributes operative meaning to mathematical symbols: one would like to define the casimir mass roughly as the invariant mass fixed by the dispersion relation, but actually this does not single out the $m$ of Eq. (16). $m$ is certainly an invariant combination of energy and momentum, but also any function of $m, m' \equiv f(m)$, is an invariant function of energy and momentum. $M$, the rest energy, is a function of $m$ and is itself an invariant function of energy and momentum. It appears therefore legitimate to adopt $M$ also as the “casimir mass”. But the key issue is that probably we shouldn’t dwell on the casimir mass because it does not have any specific (operatively meaningful) significance. We should focus on rest energy, inertial mass and gravitational mass. At this point only the concept of rest energy is clearly identifiable in a DSR theory. And as long as we lack a full understanding of the concept of inertial mass in DSR, it would be meaningless to advocate deformations of the celebrated $E = mc^2$ Einstein relation, which is most fundamentally a relation between the rest energy and the inertial mass (which can be trivially extended, by boosting, to a relation between the energy away from the rest frame and the so-called “relativistic mass”).

**DSR2.f:** maximum momentum and energy. It is noteworthy that the structure of the DSR2 transformation rules is such that, for positive $\lambda$, boosts saturate to a maximum momentum $1/\lambda$ and correspondingly also energy saturates to $1/\lambda$. This is easily understood: momentum saturates just like in DSR1 because, as it was exposed in our previous calculations, momentum behaves in DSR2 just like in DSR1; in addition, in DSR2 also energy saturates because the simple map $E_{DSR1} \to E_{DSR2}$ that allows to obtain DSR2 form DSR1 is a map such that $E_{DSR1} = \infty$ (which corresponds to $p = 1/\lambda$) is mapped into $E_{DSR2} = 1/\lambda$. Since we now see that DSR1 and DSR2 share the same law $p(\xi)$ and it was known that both saturate momentum at the inverse of the absolute length scale, it is natural to wonder whether the combination of these facts would be inevitable: If a DSR theory saturates momentum at the value $1/\lambda$ is it necessary that the theory is also characterized by the specific law $p(\xi)$ found in DSR1 (and DSR2)? Our tentative answer is negative: by fixing the saturating behaviour of a nonlinear function one is not able to specify the function. However, it is possible that the logical consistency of a DSR framework introduces (in a way not yet uncovered in the literature) some additional constraints on the form of the relevant nonlinear functions, and perhaps these constraints fully specify the function $p(\xi)$ when a maximum value $p = 1/\lambda$ is imposed.

**DSR2.g:** velocity. In Ref. [7] a DSR2 relation between the velocity of a particle and its energy-momentum was only discussed implicitly. As mentioned in point (DSR2.e), the boost parameter $v$ of the transformation law (13) was interpreted in Ref. [7] as the relative velocity between observers. Then, later in Ref. [7], this velocity $v$ enters the relation between the energy-momentum of a particle for a first observer that sees the particle at rest and the energy-momentum of that same particle for a second observer moving with velocity $v$ with respect to the first observer. From that relation it follows straightforwardly that Magueijo and Smolin are adopting the formula $v_{\text{particle}} = p/E$ to relate the velocity of a particle to its energy-momentum.
We suspect that this relation might turn out to be inconsistent with the full structure (still to be developed) of the DSR2 theory. In fact, it is easy to verify (using the DSR2 dispersion relation) that the velocity formula $v_{\text{particle}} = p/E$ is not consistent with the relation $v_{\text{particle}} = dE/dp$. As already emphasized in point (DSR1.g), we feel that at this stage of development of DSR theories there appears to be no reason to assume that in DSR theories the relation $v_{\text{particle}} = dE/dp$, which holds in Galilei relativity and in Einstein’s special relativity\(^8\) (and, as mentioned, is crucial for the Hamiltonian formulation for DSR theories), should lose validity.

If our concerns on this aspect of the analysis reported in Ref. [7] are justified it would follow that the boost parameter $v$ of the transformation law (13) should not be interpreted as the relative velocity between the observers connected by a DSR2 boost. Consequently also the velocity law $v_{\text{particle}} = p/E$ should be incorrect in DSR2. We maintain, as done in Refs. [1, 2], that the most reasonable assumption at this stage of development of DSR theories should be $v_{\text{particle}} = dE/dp$, and notice that, according to this viewpoint, the DSR2 velocity law is

$$v_{\text{particle}} = \frac{dE}{dp} = \frac{\sqrt{E^2 - (1 - \lambda E)^2 m^2}}{E - \lambda^2 m^2 E + \lambda m^2} = \frac{v}{1 + \lambda m \sqrt{1 - v^2}} = \frac{p - \lambda E p}{E - \lambda p^2}, \quad (21)$$

instead of $v_{\text{particle}} = p/E$.

For what concerns attempts to identify doable experimental tests of DSR2 predictions it is important to notice that in any case, both adopting $v_{\text{particle}} = dE/dp$ and adopting $v_{\text{particle}} = p/E$, in DSR2 one finds that:

- the speed of massless particles (e.g. photons) is still $c$, as in Einstein’s theory;
- the speed of massive particles is related to energy in DSR2 through a relation that differs from the corresponding relation of Einstein’s theory only through corrections of magnitude $L_p m^2/E$ (generically much smaller than the $L_p E$ velocity corrections of DSR1).

(DSR2.h): magnitude of the deformation. As shown above, in DSR2 $m^2 \simeq E^2 - \vec{p}^2 + 2 \lambda E (E^2 - \vec{p}^2) \simeq E^2 - \vec{p}^2 + 2 \lambda E m^2$ i.e. the dimensionless quantity characterizing the strength of the deformation of the dispersion relation is $\lambda m^2/E$. However, DSR2 does not appear to be describable as a theory which in all its aspects deforms ordinary special relativity at order $\lambda m^2/E$. In particular, the boost generators chosen in Ref. [7] deform the ordinary special-relativity boost generators at order $\lambda E$ (but their order-$\lambda E$ correction terms balance each other in such a way that the DSR2 deformed dispersion relation, which is only deformed at order $\lambda m^2/E$, is left invariant).

So, at least at the formal level DSR2 is, like DSR1, a modification of ordinary special relativity with leading-order corrections of characteristic strength $\lambda E$. However, as we show later in the paper, while

\(^8\)In Einstein’s special relativity (but not in Galilei relativity) it happens to be true that $dE/dp = p/E$. Somehow the interpretation of $v$ adopted in Ref. [7] preserves the (derived) relation $v_{\text{particle}} = p/E$, violating the (fundamental) relation $v_{\text{particle}} = dE/dp$, which is deeply connected with the concept of group velocity for waves and the possibility of a conventional Hamiltonian formulation of the theory. The formal structure of DSR2 allowed to introduce a boost parameter $v$ playing a role with some formal analogies with the role played by the relative velocity between observers in Einstein’s theory; however, the intuitive (but unjustified) assumption that the boost parameter $v$ of DSR2 should coincide with the relative velocity between the observers connected by the boost leads to the peculiar result $v_{\text{particle}} \neq dE/dp$. 

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the structure of DSR1 does lead to effects that are observably large for certain planned experiments, none of the predictions of DSR2 appears to be testable in the foreseeable future, and this is partly due to the fact that the DSR2 dispersion relation only has $\lambda m^2/E$ deformation terms.

(DSR2.i): two-particle sector and conservation laws. All results on Lorentz transformations and relativistic invariance reported in Ref. [7] on the DSR2 scheme concerned the one-particle sector. As emphasized in Refs. [1, 2, 6] (and here mentioned already in our point (DSR1.i)) the step from the one-particle sector to multi-particle sectors is highly non-trivial in DSR theories (unlike ordinary special relativity) because of the nonlinearities.

It is actually quite hard to give a full description even just of the two-particle sector; however, as shown in Refs. [1, 2, 6] there is one aspect of the two-particle (and in general multi-particle) sector which can be easily studied and leads to important insight concerning the possible experimental tests for DSR theories. One can in fact establish which are the types of energy-momentum conservation laws that can be applied to collision processes, particularly the ones involving particle production such as the process $\gamma + \gamma \to e^+ + e^-$. We consider here DSR2 conservation laws in leading order in the deformation scale $\lambda$.

We find that in DSR2 in leading order in $\lambda$ acceptable energy-momentum conservation rules for a generic $a + b \to c + d$ are

$$E_a + E_b + \lambda E_a^2 + \lambda E_b^2 - E_c - E_d - \lambda E_c^2 - \lambda E_d^2 \simeq 0 \ , \quad (22)$$

$$\vec{p}_a + \vec{p}_b + \lambda E_a \vec{p}_a + \lambda E_b \vec{p}_b - \vec{p}_c - \vec{p}_d - \lambda E_c \vec{p}_c - \lambda E_d \vec{p}_d \simeq 0 \ . \quad (23)$$

In fact, it is easy to verify, using the DSR2 boost generators, that when satisfied in one inertial frame these conservation rules are also satisfied in all other inertial frames.

Together with the “no-particle-mixing” conservation laws (22),(23), we also found$^{10}$ (again to order $\lambda$) the “particle-mixing” conservation laws

$$E_a + E_b - 2\lambda E_a E_b - E_c - E_d + 2\lambda E_c E_d \simeq 0 \ , \quad (24)$$

$$\vec{p}_a + \vec{p}_b - \lambda E_a \vec{p}_a - \lambda E_b \vec{p}_b - \vec{p}_c - \vec{p}_d + \lambda E_c \vec{p}_c + \lambda E_d \vec{p}_d \simeq 0 \ , \quad (25)$$

which also have the property that they are necessarily satisfied in all inertial frames if satisfied in a given inertial frames.

(DSR2.j): space-time sector. All results on DSR2 Lorentz transformations and relativistic invariance reported in Ref. [7] concerned energy-momentum space. A key issue for future DSR2 studies will be the identification of a spacetime that is consistent with the structure of the DSR2 energy-momentum.

$^9$As emphasized already above, in relation with experimental tests of DSR theories it is always sufficient to analyze the theories in leading order in the second observer-independent scale; in fact, the new effects are extremely small and therefore we might only have a chance (if any) to see the leading-order corrections (absolutely impossible to gain experimental insight on the higher order corrections).

$^{10}$The reasons for the availability of alternative choices of conservation laws in DSR theories have been discussed in Refs. [1, 6].
space. We conjecture that this spacetime realization of DSR2 will require the introduction of a quantum spacetime supporting a nontrivial differential calculus. If this conjecture is correct it would affect severely some of the remarks on DSR2 field theories made in Ref. [7].

The fact that also the DSR2 one-particle-sector energy-momentum space can be cast [8] in a way that is consistent with the general structure of $\kappa$-Poincaré Hopf algebras could suggest that perhaps some formulation of $\kappa$-Minkowski noncommutative spacetime would also provide a candidate noncommutative spacetime for DSR2, in the same sense which we already discussed earlier for DSR1. This remains an open issue for future research. We only observe that, while for DSR1 there is the connection with “time-to-the-right-ordered wave exponentials” mentioned earlier, in the literature there has been so far no discussion of a $\kappa$-Minkowski ordering prescription which would lead to DSR2-type equations.

**DSR2.k): testability.** As mentioned above, the only two classes of observations with some chance of seeing DSR effects are: (a) velocity-law tests (most notably the ones based on observations of gamma-ray bursts [18]), and (b) threshold tests (in cosmic-ray physics).

Since DSR2 predicts no modification of the velocity law for photons, clearly the velocity-law tests planned by AMS and GLAST would not be sensitive to DSR2 effects. Even for massive particles the DSR2 velocity law is only different from the ordinary one at order $\lambda m^2/E$, well below the reach of any foreseeable related experiment (if indeed $\lambda$ is of the order of the Planck length).

Having here derived (in point (DSR2.h)) conservation laws for collision processes for the DSR2 scheme, we are now also in a position to study the predictions of DSR2 concerning particle-production thresholds and verify whether DSR2 gives rise to effects that would be observable in the physics of cosmic rays. This can be done in complete analogy with the discussion we gave above for the corresponding analysis with DSR1, and the reader can easily verify that the result is again negative: one finds a modification of the photopion-production threshold but the modification is not large enough to be noticeable in cosmic-ray observations.

We conclude that neither velocity-law tests nor threshold tests can reveal DSR2 effects (DSR2 effects are too small in those contexts). It appears that none of the predictions of DSR2 can be tested in the foreseeable future.

**4 Aside on nonlinearity in DSR**

11 In our analysis clearly a key role was played by the fact that both DSR1 and DSR2 are based, in the one-particle sector, on nonlinear realizations of the Lorentz group. The fact that the Lorentz group is still present, but is nonlinearly realized in some DSR proposals, has led some authors [33, 34] to argue that essentially the nonlinearity might be a formal artifact, that one should unravel the nonlinearity by performing some nonlinear redefinition of the energy-momentum variables, and that “true” translations would still have to satisfy linear commutation relations with boosts.

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11 This Section, which was not present in the first version of this manuscript (http://arXiv.org/abs/hep-th/0201245v1), has been added in order to address some of the issues raised in Refs. [33, 34].
While our focus has been on the DSR1/DSR2 comparison, rather than going back to considerations of the conceptual content of the DSR framework (which has been discussed in detail in many publications), we should stress here that some recent studies, perhaps most notably the ones reported in Refs. [35, 36, 37], have exposed the fact that the arguments advocated in Refs. [33, 34] clearly assuming that the underlying spacetime be classical (or at least commutative). The studies reported in Refs. [35, 36], elaborating on points already explored in the early references [1, 3, 5], show that in certain noncommutative spacetimes boosts must act nontrivially on the one-particle sector (which is faithfully described by the algebra relations) and on the multi-particle sector (which in these noncommutative spacetimes is described through the coalgebra sector of a Hopf algebra of symmetries of the type considered in Refs. [24, 26, 27]). The relativistic-invariant length scale $\lambda$ (or $\tilde{L}_p$) cannot be fully removed\(^{12}\), since this is the natural energy-momentum manifestation of the fact that the commutation relations among spacetime coordinates, characterized by the length scale $\lambda$, are being introduced as a fundamental feature of spacetime. For example in the $\kappa$-Minkowski noncommutative spacetime a proper description of translations necessarily requires some new structures, as most elementarily seen [35, 36] by looking at the product of plane waves in $\kappa$-Minkowski:

$$\left[e^{-i k_x x} e^{i Et}\right] \cdot \left[e^{-i q_x x} e^{i \omega t}\right] = \left[e^{-i(k+e^{\lambda E}q)x} e^{i(E+\omega)t}\right].$$

Clearly the spacetime noncommutativity is leading to a law of composition of energy momentum ($((k, E), (q, \omega)) \rightarrow ((k+e^{\lambda E}q), (E+\omega))$) which is not covariant under ordinary boosts and involves a clear nonlinearity.

And there starts to be growing evidence that noncommutativity is not the only realization of the idea of “spacetime quantization” that leads to DSR-type implementation of the Relativity Principle. In particular, in Ref. [37] it was recently shown that in certain quantum-gravity theories a quantum (Hopf-algebra), rather than classical, symmetry is realized when the cosmological constant is nonzero. And the symmetries are still quantum rather than classical in the limit of vanishing cosmological constant (the limit in which these quantum symmetries, of course, are Poincaré-like). Again the nonlinearities of the boost action turn out to be unavoidable: one appears to have some freedom to transfer some of the nonlinearity from the one-particle to the two-particle sector or viceversa, but at least some dependence on a new relativistic-invariant length (momentum) scale is unavoidable.

The DSR framework was introduced [1] as a contribution to quantum-gravity research and it implicitly assumes that spacetime has strikingly nonclassical features, such as noncommutativity. By focusing, as essentially done in Refs. [33, 34], on the inadequacy of the DSR framework for theories in a classical spacetime one is basically missing the key objective of the DSR proposal. DSR was never intended to apply in a classical spacetime. In an ordinary classical spacetime, in which a familiar concept of “translations” should be applicable, we cannot imagine a good reason for the proper concept of energy-momentum to require a nonlinear action of boosts on energy-momentum. One could formally introduce by hand a nonlinearity (just by mapping the standard energy-momentum into “new” energy-momentum), but this should not have any observable consequences since it is

\(^{12}\)At the cost of an awkward nonlinear redefinition of energy-momentum [10], one can remove $\lambda$ completely from the one-particle sector and could be misled into believing that the scale $\lambda$ would have been “redefined away”. But instead the same redefinition of energy-momentum that trivializes the one-particle sector requires that the action of boosts on two-particle states, described by the coalgebra sector [35, 36], involves nontrivially the scale $\lambda$. 

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simply a formal statement. Instead, as we just illustrated above in considering the product of plane waves in a noncommutative spacetimes, in certain quantum spacetimes the concept of translations may require profound revisions and the concept of action of boosts on energy-momentum must also be revised accordingly. The type of quantum spacetimes considered in Refs. [35, 36, 37] provide examples of natural application of the DSR framework.

5 Summary and open issues

DSR theories, theories of the relativistic transformation laws between inertial frames which are characterized by two observer-independent scales \[^1\], had been often discussed in terms of three features: the dependence of energy and momentum on rapidity, \(E(\xi)\) and \(p(\xi)\), and the dispersion relation \(E(p)\). This is clearly redundant since the functions \(E(p)\) and \(p(\xi)\) completely fix the function \(E(\xi)\). We have observed that the deformed boost generators of a DSR theory actually codify simultaneously the two functions \(E(p)\) and \(p(\xi)\). Two DSR theories can have very different form of the boost generators, but in some cases this formal difference might hide the fact that the two theories share the same dispersion relation or the same dependence of momentum on rapidity. We have shown that the relation between DSR1 and DSR2, the most studied examples of DSR theories, is actually of the special type in which the two theories describe the dependence momentum on rapidity through exactly the same differential equation. This came as a surprise because the DSR2 boost generators appear to be very different from the DSR1 boost generators. A posteriori, we understand the difference between DSR2 boosts and DSR1 boosts as due exclusively to a simple map between the concept of energy adopted in the two theories \(E_{\text{DSR2}} = f(E_{\text{DSR1}})\). All results previously obtained for the DSR1 theory also apply to DSR2, upon this simple redefinition of energy.

We now understand that the fact that momentum saturates to \(1/L_p\) in the infinite rapidity limit in both theories is not accidental: the differential equation that governs the dependence of momentum on rapidity is exactly the same in DSR1 and DSR2. The fact that in DSR1 energy diverges in the infinite-rapidity limit (just like in ordinary special relativity) while in DSR2 energy saturates to \(1/L_p\) in the infinite rapidity limit is also easily understood: the map \(E_{\text{DSR2}} = f(E_{\text{DSR1}})\) is such that an infinite value of \(E_{\text{DSR1}}\) corresponds to the value \(1/L_p\) for \(E_{\text{DSR2}}\). (For simplicity of discourse, we are here again implicitly assuming \(\tilde{L}_p = \lambda = L_p\).)

Similarly, acceptable laws of deformed energy-momentum conservation for DSR2 can be obtained, as here shown, by simply taking the corresponding DSR1 laws and performing the straightforward redefinition of energy.

The fact that DSR1 and DSR2 are connected by such a simple relation naturally suggests further scrutiny of the points made in previous studies in which a sharp difference, even of conceptual nature, was drawn between DSR1 and DSR2. These two DSR theories are very similar, perhaps too similar to be treated separately. We have here provided a unified language of notations and conventions which we hope to prove useful as these theories are investigated.

The fact that we uncovered the very simple relation between DSR1 and DSR2 also put us in the position to question the puzzling fact that some key concepts, such as particle mass and particle
velocity, were being investigated from very different perspective in some of the studies focusing on DSR1 and in some of the studies focusing on DSR2. We found that the differences originated in unjustified physical interpretation of the different choices of notations and conventions which had so far been used in studies of DSR1 and DSR2. In particular, we argued that also in DSR2, just like in DSR1, the only meaningful concept of physical mass that can be so far discussed is the one of rest-energy mass, and that in neither of the two theories there is at present any ground for speculating about modifications of the relation $E_{\text{rest}} = m I c^2$ between rest energy and inertial mass (or speculating that there should be differences between the rest-energy mass, the inertial mass and the gravitational mass). And we argued that in DSR2, just like in DSR1, there is so far no scientifically robust argument in support of the idea that the relation $v_{\text{particle}} = dE/dp$ (which has been so far always valid, both in Galilei relativity and in Einstein’s special relativity) should be abandoned. We have clarified that a contrary expectation which had been presented in the literature actually originated from the assumption that a DSR2-boost parameter, suggestively denoted by “$v$”, should coincide with the relative velocity of the two observers connected by the boost. The only support for this interpretation is a vague analogy with a corresponding parameter in the description of ordinary special-relativistic boosts, but we have shown that there is no robust argument that could support this identification in the context of the DSR2 theory.

Although the connection between DSR1 and DSR2 is of elementary conceptual nature, we have shown that the energy redefinition that specifies the connection can be quantitatively important in certain contexts, leading to significantly different predictions. Both DSR1 and DSR2, as so far constructed, are unable to introduce a significant modification of the photopion-production threshold relevant in the analysis of cosmic rays, but DSR1 does predict a modification of the speed-of-light law that could be observed in forthcoming experimental studies [18, 19, 21], whereas DSR2 does not predict a modification of the speed-of-light law. It appears that none of the characteristic features of DSR2 could be tested in the foreseeable future.

Among the key open issues for the research here of interest central importance should be attributed to the need of endowing these relativistic theories with a suitable spacetime sector. DSR theories of the DSR1/DSR2 type are introduced in the energy-momentum sector. This is of course acceptable, but eventually they must be extended to an associated spacetime sector (just like in textbook presentations of ordinary special relativity one introduces the concepts in the spacetime sector, but then derives an associated energy-momentum sector). We have stressed that these “DSR spacetimes” should be “quantum” in an appropriate sense. In fact, while the DSR idea of two relativistic-invariant scales could have other realizations [1], DSR1 and DSR2, like basically all DSR theories in the literature, are based on a nonlinear realization of the Lorentz group on energy-momentum space. We emphasized that a nonlinear realization of the Lorentz group on energy-momentum space should find support in the structure of the associated spacetime. It should be a manifestation of the fact that the ordinary concept of translations is inadequate. In an ordinary classical spacetime, in which a familiar concept of translations should be applicable, we cannot imagine a good reason for the proper concept of energy-momentum to require a nonlinear action of boosts on energy-momentum. It might be formally introduced, but only to eventually discover that it didn’t carry any physical significance. Instead,
as discussed in Section 4, in a genuinely “quantum” spacetime, for example in a noncommutative spacetime, the concept of translations can naturally require some revisions. In some noncommutative spacetimes the structure of plane waves and of the commutators of coordinates introduces an unavoidable nonlinearity in the law of composition of momenta, and such a nonlinear composition law cannot possibly be described covariantly in terms of a linear realization of the Lorentz group, while it may well be covariant in the DSR sense. For DSR1 the connection with “time-to-the-right-ordered wave exponentials” in $\kappa$-Minkowski noncommutative spacetimes (which was here mentioned in Section 2) should be further explored. For DSR1, as mentioned in Section 3, we are at an even more preliminary status in the search of a realization in a quantum spacetime.

Our results could also motivate further studies of the concept of maximum momentum in DSR theories, the aspect of DSR theories which is most appealing from a quantum-gravity perspective [1, 7]. Both in DSR1 and DSR2 momenta saturate to $1/L_p$. This is achieved through the same differential equation governing the dependence of momentum on rapidity. Are there other ways to achieve this saturation consistently with the logical structure of DSR theories? Assuming momenta do saturate to $1/L_p$ and that we are in the context of a DSR theory, is the differential equation (4) a robust general feature?

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