Moser-Trudinger inequality on conformal discs

G. Mancini* and K. Sandeep†

Abstract
We prove that a sharp Moser-Trudinger inequality holds true on a conformal disc if and only if the metric is bounded from above by the Poincaré metric. We also derive necessary and sufficient conditions for the validity of a sharp Moser Trudinger inequality on a simply connected domain in $\mathbb{R}^2$.

1 Introduction
In 1971 Moser, sharpening an inequality due to Trudinger, proved that
\[
\sup_{u \in H^1_0(\Omega)} \int_{\Omega} \left( e^{4\pi u^2} - 1 \right) dx < +\infty
\] (1.1)
for every bounded open domain $\Omega \subset \mathbb{R}^2$ (in [18]). This inequality is sharp, in the sense that the 'critical' constant $4\pi$ cannot be improved. Referred as 'Moser-Trudinger inequality', (1.1) also implies the estimate
\[
\ln \frac{1}{|\Omega|} \int_{\Omega} e^u \leq C + \frac{1}{16\pi} \int_{\Omega} |\nabla u|^2 \quad \forall u \in H^1_0(\Omega)
\] (1.2)

* Dipartimento di Matematica, Università degli Studi "Roma Tre", Largo S. Leonardo Murialdo, 1 - 00146 Roma, Italy. E-mail mancini@mat.uniroma3.it.
† TIFR Centre for Applicable Mathematics, Sharadanagar, Chikkabommasandra, Bangalore 560 065. E-mail sandeep@math.tifrbng.res.in
for some universal constant $C > 0$ and any bounded domain $\Omega$ and, again, the constant $\frac{1}{16\pi}$ is sharp (see [16]).

Inequality (1.1) has been extended to any 2-d compact Riemannian manifold with or without boundary (see [5], [12]) or even to some subriemannian manifolds (see [7] and references therein). However, little is known in case $\Omega$ is a non compact 2-d Riemannian manifold, even in the simplest cases $\Omega \subset \mathbb{R}^2$ with $|\Omega| = \infty$ (see [1], [21]) or $\Omega = \mathcal{H}^2$, the 2-d hyperbolic space.

We address here the case of conformal discs, i.e. $\Omega = D$, the unit open disc in $\mathbb{R}^2$, endowed with a conformal metric $g = \rho g_e$, where $g_e$ denotes the euclidean metric and $\rho \in C^2(D)$, $\rho > 0$. Denoted by $dV_g = \rho dx$ the volume form, by conformal invariance of the Dirichlet integral (1.1) takes the form

$$\sup_{u \in C^\infty_0(D), \int_D |\nabla u|^2 \leq 1} \int_D \left( e^{4\pi u^2} - 1 \right) dV_g < \infty$$

A relevant case is the hyperbolic metric $g_h := \left( \frac{2}{1 - |x|^2} \right)^2 g_e$. We will show that (1.3) holds true in this case. Actually, we have the following

**Theorem 1.1.** Given a conformal metric $g$ on the disc, (1.3) holds true if and only if $g \leq cg_h$ for some positive constant $c$.

After a personal communication, in [3] the inequality (1.3) with $g = g_h$ found an application in the study of blow up analysis and eventually a different proof of (1.3) when $g = g_h$.

As for (1.1) in case $|\Omega| = +\infty$, the supremum therein will be in general infinite. To have it finite, an obvious necessary condition is that

$$\lambda_1(\Omega) := \inf \left\{ \int_\Omega |\nabla u|^2 dx : u \in C^\infty_0(\Omega), \int_\Omega |u|^2 = 1 \right\} > 0$$

As a partial converse, it was shown by D.M. Cao [9] that $\lambda_1(\Omega) > 0$ implies subcritical exponential integrability, i.e. for every $\alpha < 4\pi$ it results

$$\sup_{u \in C^\infty_0(\Omega), \int_\Omega |u|^2 \leq 1} \int_\Omega \left( e^{\alpha u^2} - 1 \right) dx < \infty$$

(1.4)
(see also [19] and [1], for a scale invariant version of Trudinger inequality which implies (1.4) ). However, no information is provided for the critical case $\alpha = 4\pi$. We will show that $\lambda(\Omega) > 0$ is, on simply connected domains, also sufficient for (1.1) to hold true. To state our result, let

$$\omega(\Omega) := \sup\{r > 0 : \exists D_r(x) \subset \Omega\}$$

Theorem 1.2. Let $\Omega$ be a simply connected domain in $\mathbb{R}^2$. Then

$$(1.1) \text{ holds true } \iff \lambda(\Omega) > 0 \iff \omega(\Omega) < +\infty$$

Remark 1. The topological assumption on $\Omega$ cannot be dropped: in Appendix we exhibit domains $\Omega$ with $\omega(\Omega) < +\infty$ and $\lambda(\Omega) = 0$, for which, henceforth, (1.1) fails. However, we suspect that $\lambda(\Omega) > 0$ is sufficient to insure (1.1).

2 Proof of the main results and asymptotics for $L^p$ Sobolev inequalities

Proof of Theorem 1.1. Let us write $g := \rho g_e = \zeta g_h$ with $\zeta := \rho \left(1 - \frac{|x|^2}{4}\right)^2$. We first prove that if there is $x_n \in D$ such that $\zeta(x_n) \to_n +\infty$, then there are $u_n \in H^1_0(D)$ with $\int |\nabla u_n|^2 \leq 1$ such that $\int_D \left(e^{4\pi u_n^2} - 1\right) \zeta dV_h \to_n +\infty$.

To this extent, let $\varphi_n$ be a conformal diffeomorphism of the disc such that $\varphi_n(0) = x_n$. Then

$$\exists \epsilon_n > 0 \text{ such that } |x| \leq \epsilon_n \Rightarrow \zeta(\varphi_n(x)) \geq \frac{1}{2} \zeta(\varphi_n(0)) \to_n +\infty$$

Let now $v_n(x) = v_n(|x|)$ be the Moser function defined as

$$v_n(r) = \sqrt{\frac{1}{2\pi}} \left[ \left(\log \frac{1}{\epsilon_n}\right)^{\frac{1}{2}} \chi_{[0,\epsilon_n]} + \left(\log \frac{1}{\epsilon_n}\right)^{-\frac{1}{2}} \log \frac{1}{r} \chi_{[\epsilon_n,1]} \right]$$

Notice that $\int_D |\nabla v_n|^2 = 1$. Let $u_n := v_n \circ \varphi_n$. Then, by conformal invariance and because $\varphi_n$ are hyperbolic isometries, $\int_D |\nabla u_n|^2 = 1$ and

$$\int_D \left(e^{4\pi u_n^2} - 1\right) \zeta dV_h = \int_D \left(e^{4\pi v_n^2} - 1\right) \zeta \circ \varphi_n dV_h$$
Hence a bound for \( g \) in terms of \( g_h \) is necessary for (1.1) to hold true.

We now prove that boundedness of \( \zeta \) is also sufficient for (1.1) to hold true.

Under this assumption, (1.1) reduces to

\[
\sup_{u \in C_C^\infty(D)} \int_D |\nabla u|^2 \leq 1 - e^{-2\pi^2 n} \int_D (e^{4\pi u^2} - 1) dV_h < \infty \quad (2.1)
\]

Let \( u^* \) be the symmetric decreasing hyperbolic rearrangement of \( u \), i.e.

\[
\mu_h(\{u^* > t\}) = \mu_h(\{u > t\})
\]

By the properties of the rearrangement (see [6]), it is enough to prove (2.1) for \( u \) radially symmetric. For \( u \) radial, inequality (2.1) rewrites, in hyperbolic polar coordinates \(|x| = \tanh \frac{t}{2}\), as

\[
\sup_{2\pi \int_0^\infty |u'|^2 \sinh t dt \leq 1} \int_0^\infty (e^{4\pi u^2} - 1) \sinh t dt < \infty \quad (2.2)
\]

To prove (2.2), observe first that from \( \int_D |\nabla u|^2 \leq 1 \) it follows, for \( t < \tau \),

\[
|u(\tau) - u(t)| = \left| \int_t^\tau u'(s) ds \right| \leq \left( \int_t^\tau |u'|^2 \sinh s ds \right)^{\frac{1}{2}} \times \left( \int_t^\infty \frac{ds}{\sinh s} \right)^{\frac{1}{2}} \leq \frac{1}{2\pi \sinh t} \]

and since \( \int_D u^2 dV_h = 2\pi \int_0^\infty u^2 \sinh t dt < \infty \) implies \( \lim_{T \to +\infty} u(\tau) = 0 \), we get

\[
\int_D |\nabla u|^2 \leq 1 \quad \Rightarrow \quad |u(t)| \leq \left( \frac{1}{2\pi \sinh t} \right)^{\frac{1}{2}} \quad \forall t \quad (2.3)
\]

Now, let \( 2\pi \sinh T > 1 \) so that \( \int_D |\nabla u|^2 \leq 1 \) implies \( u(T) < 1 \), and set \( v := u - u(T) \), \( w := \sqrt{1 + u^2(T)} v \) so that \( w(T) = 0 \) and \( 2\pi \int_0^T |w'|^2 t dt \)

\[
\leq [1 + u^2(T)] 2\pi \int_0^T |w'|^2 \sinh t dt \leq \left[ 1 + \left( \int_0^T |w'|^2 \sinh t dt \right) \left( \int_T^\infty \frac{dt}{\sinh t} \right) \right] \times \left[ 1 - \int_T^\infty \frac{dt}{\sinh t} \right] \leq \left[ 1 + \left( \int_T^\infty \frac{|w'|^2 \sinh t dt}{\sinh T} \right) \right] \times \left[ 1 - \int_T^\infty \frac{2\pi |w'|^2 \sinh t dt}{\sinh T} \right]
\]
\[ \leq \left[ 1 + 2\pi \int_T^\infty |u'|^2 \sinh td t \right] \times \left[ 1 - 2\pi \int_T^\infty |u'|^2 \sinh td t \right] \leq 1. \]

Now, an application of (1.1) gives
\[ 2\pi \int_0^T e^{4\pi u^2} t dt \leq cT^2, \]
and since
\[ u^2 = v^2 + 2\nu u(T) + u^2(T) \leq v^2 + v^2 u(T)^2 + 1 + u^2(T) \leq w^2 + 2 \]
implies
\[ 2\pi \int_0^T e^{4\pi u^2} \sinh td t \leq 2e^{8\pi} \int_0^T \frac{\sinh T}{T} e^{4\pi u^2} t dt \leq c(T)T^2, \]
we get
\[ 2\pi \int_0^T [e^{4\pi u^2} - 1] \sinh td t \leq c(T) \quad (2.4) \]
for some constant \( c(T) \) which does not depend on \( u \).

Now, using (2.3) and Hardy inequality
\[ \int_D |\nabla u|^2 \geq \frac{1}{4} \int_D |u|^2 dV_h, \]
we get
\[ \int_0^\infty [e^{4\pi u^2} - 1] \sinh td t \leq \left[ 2 \int_D |u|^2 dV_h + \sum_{p=2}^\infty \frac{8\pi^p}{p!} \int_0^\infty |u|^{2p} \sinh t dt \right] \leq \left[ 8 \int_D |\nabla u|^2 dV_h + \sum_{p=2}^\infty \frac{8\pi^p}{p!} \int_0^\infty |\frac{1}{2\pi \sinh T}|^p \sinh t dt \right] \left[ 8 + \sum_{p=2}^\infty \frac{2p}{p!} \int_0^\infty \frac{dt}{(\sinh t)^p} \right]. \]

From
\[ \int_T^\infty \frac{dt}{(\sinh t)^p} = \int_T^\infty \left[ \frac{2}{e^t - e^{-t}} \right]^{p-1} dt = \frac{2}{e^t} \int_T^\infty \left[ \frac{e^{-p+1}t}{e^{2p-2} - e^{2(p-1)t}} \right] dt \leq \frac{2}{e^{p-1} - 2^{1-p}e^{2(p-1)t}} \]
\[ = \frac{1}{p-1} \left[ \frac{1}{\sinh T} \right]^{p-1} \leq \left[ \frac{1}{\sinh T} \right]^{p-1} \quad \text{if } p \geq 2 \quad \text{and the above inequality we get} \]
\[ 2\pi \int_T^\infty [e^{4\pi u^2} - 1] \sinh td t \leq 2\pi \left[ 8 + \sinh t e^{\frac{4\pi}{\sinh T}} \right] = c(T) \quad (2.5) \]

Inequalities (2.4) and (2.5) give (2.2) and hence (1.1).

**Proof of Theorem 1.2** (1.1) implies
\[ 4\pi \int_\Omega \frac{u}{\|\nabla u\|}^2 dx \leq \int_\Omega \left( e^{4\pi (\frac{u}{\|\nabla u\|})^2} - 1 \right) dx \]
\[ \leq c(\Omega) \] and hence \( \lambda_1(\Omega) \geq \frac{4\pi}{c(\Omega)} \). In turn, this clearly implies \( \omega(\Omega) < +\infty \).

To complete the proof, it remains to show that if \( \Omega \) is simply connected then
\[ \omega(\Omega) < +\infty \] implies (1.1). Let \( \varphi : D \rightarrow \Omega \) be a conformal diffeomorphism, so that (1.1) rewrites as (1.3) where \( g := \varphi^* g_e = |\det J_\varphi| g_e \). Let us show that
\[ \omega(\Omega) < R \quad \Rightarrow \quad |\det J_\varphi(x)| \leq \frac{16R^2}{(1 - |x|^2)^2} \quad (2.6) \]
so that Theorem 1.1 applies to give the conclusion. Now, (2.6) follows from Koebe’s covering Theorem (see [15]): if \( \psi : D \rightarrow \Omega \) is a conformal diffeomorphism and \( z \not\in \psi(D) \) for some \( z \in D_r(\psi(0)) \), then \( |\psi'(0)| \leq 4r \). In
fact, given \( w \in D \), let \( \varphi_w(z) := \varphi(w + (1 - |w|)z), \ z \in D \). By assumption, \( \varphi_w(D) = \varphi(D_{1-|w|}(w)) \) cannot cover the disc \( D_R(\varphi(w)) \), and hence \( |\text{det} J_\varphi(w)|^{\frac{1}{2}} |(1 - |w|)| = |\varphi'(w)(1 - |w|)| = |\varphi_w'(0)| \leq 4R \).

We end this Section deriving, from Moser-Trudinger inequalities, an asymptotic formula for best constants in \( L^p \) Sobolev inequalities on 2-d Riemannian manifolds \((M, g)\) (see [20], [2], for smooth bounded domains in \( \mathbb{R}^2 \)). For notational convenience, we say that \((M, g)\) is an MT-manifold if

\[
\sup_{u \in C^\infty_0(D), \int_M |\nabla_g u|^2 dV_g \leq 1} \int_M (e^{4\pi u^2} - 1) dV_g < \infty
\]  

(2.7)

**Proposition 2.1.** Let \((M, g)\) be an MT-manifold. Then

\[
S_p = S_p(M, g) := \inf_{u \in C^\infty_0(M), u \neq 0} \frac{\int_M |\nabla_g u|^2 dV_g}{(\int_M |u|^p dV_g)^{\frac{2}{p}}} = \frac{8\pi e + o(1)}{p}
\]  

(2.8)

**Proof.** Let us prove first

\[
\liminf_p pS_p \geq 8\pi e
\]  

(2.9)

By assumption, there is \( C > 0 \) such that, for every \( p \in \mathbb{N} \), it results

\[
\int_M |\nabla_g u|^2 dV_g \leq 1 \implies C \geq \int_M (e^{4\pi u^2} - 1) dV_g \geq \frac{(4\pi)^p}{p!} \int_M |u|^{2p} dV_g
\]

and hence

\[
\left( \int_M |u|^{2p} dV_g \right)^{\frac{1}{2p}} \leq \frac{C^{\frac{1}{2p}} (p!)^{\frac{1}{2p}}}{\sqrt{4\pi}} \left( \int_M |\nabla_g u|^2 dV_g \right)^{\frac{1}{2}} \quad \forall u \in C^\infty_0(D)
\]

If \( n \leq p \leq n + 1 \), let \( \alpha = \frac{n(n+1-p)}{p} \) and get, by interpolation,

\[
\|u\|_{2p} \leq \frac{1}{\sqrt{4\pi}} C^{\frac{\alpha}{2\pi}} (n!)^{\frac{1}{2\pi}} \times C^{\frac{1-\alpha}{2(n+1)}} (n + 1)!^{\frac{1-\alpha}{2(n+1)}} \|\nabla_g u\|^{\frac{1}{2p}}
\]

and hence

\[
S_{2p} \geq \frac{4\pi}{C\pi (n!)^{\frac{1}{2\pi}} (n+1)!^{\frac{1}{2(n+1)}}} \geq \frac{4\pi}{C\pi (n!)^{\frac{1}{2\pi}} (n+1)!^{\frac{1}{2(n+1)}}}.
\]

By Stirling’s formula we obtain
We have
\[ 2pS_{2p} \geq \frac{8\pi e}{C^{1+o(1)}(n+1)^{1+\frac{n}{p}}} \geq \frac{8\pi e}{1+o(1)} \quad \text{and hence (2.9)}. \]

To prove the reverse inequality, we use again the Moser function. For fixed $R > 0$ and $0 < l < R$, define $M_t(x) = M_t(||x||)$ on $\mathbb{R}^2$ as follows:

\[ M_t(r) = \sqrt{\log \left( \frac{R}{l} \right)} \left[ \chi_{[0,l]} + \frac{\log(R/l)}{\log(R/l)} \chi_{[l,R]} \right], \quad r \geq 0 \]

Let $q \in M$ and choose $R > 0$ strictly less than the injectivity radius of $M$ at $q$ and define $u_t(z) := M_t(Exp_q^{-1}(z))$ where $Exp_q$ is the exponential map at $q$. Note that $u_t$ is well defined and in $H^1(M)$. Now calculating in normal coordinates we get

\[ \int_M |\nabla g u_t|^2 \, dV_g = \int_{B(0,R)} g^{i,j}(x)(M_t)_x(M_t)_x \sqrt{g(x)} \, dx \]

Since the metric is smooth and $g_{i,j}(0) = \delta_{i,j}$ we get $g^{i,j} = \delta_{i,j} + O(|x|)$ and $\sqrt{g(x)} = 1 + O(|x|)$. Using this we get

\[ \int_M |\nabla g u_t|^2 \, dV_g = 2\pi + O(1)(\log \frac{R}{l})^{-1} \]

Similarly

\[ \int_M |u_t|^p \, dV_g \geq \int_{B(0,l)} |M_t(x)|^p \sqrt{g(x)} \, dx = C(\log \frac{R}{l})^p l^2 \]

for some $C > 0$. Taking $\log \frac{R}{l} = \frac{p}{4}$ and sending $p$ to infinity, we get

\[ \limsup_{p \to \infty} pS_p \leq \lim_{p \to \infty} \left( \frac{\int_M |\nabla g u_t|^2 \, dV_g}{\int_M |u_t|^p \, dV_g} \right)^{\frac{2}{p}} \leq 8\pi e \]

\[ \square \]

**Corollary 2.1.** If $g \leq cg_h$ then $S_p(D, g) = \frac{8\pi e + o(1)}{p}$

**Remark 2.** Let $p \in [1, 2)$ and $u_p = (1 - |x|^2)^{\frac{p}{2}}$. Then $u_p \in H^1_0(D)$ and

\[ \int_D |u_p|^p \, dv_h = \int_D |e^u - 1| \, dv_h = +\infty. \]

In particular, $S_p = 0$ for $p \in [1, 2)$. 

7
Let us now derive from Proposition 2.1 an inequality analogous to (1.2).

**Corollary 2.2.** Let \((M, g)\) be an MT-manifold. Then, chosen \(\delta \in (0, 1)\), there is a constant \(C(\delta) > 0\) such that, for every \(u \in H^1_0(M)\), it results

\[
\ln \int_M [e^u - 1]^2 dV_g \leq \ln \int_M [e^{2u} - 2u - 1] dV_g \leq C(\delta) + \frac{1}{4\delta\pi} \int_M |\nabla_g u|^2 dV_g \tag{2.10}
\]

**Proof.** After fixing \(\delta \in (0, 1)\), we get, by Taylor expansion

\[
\int_M [e^u - u - 1] dV_g = \sum_{p=2}^{\infty} \frac{1}{p!} \int_M u^p dV_g \leq \sum_{p=2}^{\infty} \frac{1}{\sqrt{p!}} \left[ \frac{\|\nabla_g u\|^2}{8\pi\delta} \right]^{\frac{p}{2}} \left( \frac{8\pi\delta}{S_p} \right)^{\frac{1}{2}}
\]

Since, by Stirling’s formula and (2.9)

\[
\limsup_p \frac{1}{(p!)^{\frac{1}{p}}} \frac{8\pi\delta}{S_p} \leq \delta < 1
\]

we conclude, also using the inequality \((e^t - 1)^2 \leq e^{2t} - 2t - 1, \ \forall t \in \mathbb{R}\), that

\[
\int_M [e^u - 1]^2 dV_g \leq \int_M [e^{2u} - 2u - 1] dV_g \leq C(\delta) \left( e^{\frac{\|\nabla_g u\|^2}{2\pi\delta}} - \frac{\|\nabla_g u\|^2}{2\pi\delta} - 1 \right)^{\frac{1}{2}} \tag{2.11}
\]

**Remark 3.** We believe that (2.10) holds with \(\delta = 1\) (and \(\frac{1}{4\pi}\) is optimal). Actually, as it is clear from the proof, subcritical exponential integrability (1.4) is enough to get (2.10). In particular, (2.10) holds true if \(M = \Omega\), a smooth open subset of \(\mathbb{R}^2\) with \(\lambda_1(\Omega) > 0\).

### 3 Application to a geometric PDE

Here we apply Moser-Trudinger inequality to the following problem. Let \(\Omega\) be a smooth open set in \(\mathbb{R}^2\). Let \(K \in C^\infty(\Omega)\).

**Is it** \(K\) **the Gauss curvature of a conformal metric** \(g = \rho g_e\) **in** \(\Omega\) **?**

It is known that solving this problem amounts to solve the equation

\[
\Delta v + Ke^{2v} = 0 \quad \text{in} \quad \Omega \tag{3.1}
\]
In fact, if \( v \in C^2(\Omega) \) solves (3.1) then \( e^{2v}g_e \) is a conformal metric having \( K \) as Gauss curvature. Equation (3.1) is not solvable in general, e.g. if \( \Omega = \mathbb{R}^2 \), \( K \leq 0 \) and \( K(x) \leq -|x|^{-2} \) near \( \infty \) (a result due to Sattinger, see [10] or [13]). In [13] it is also noticed, as a Corollary of a general result, that if \( \Omega \) is bounded and \( K \in L^p(\Omega) \) for some \( p > 2 \), then (3.1) is solvable. We prove

**Theorem 3.1.** Let \((M, g)\) be an MT-manifold. Let \( K_i \in L^2(M) \). Then equation

\[
\Delta_g v + K_1 + K_2 e^{2v} = 0
\]

has a solution in \( H^1_0(M) + \mathbb{R} \)

**Remark 4.** In view of Remark 3, Theorem 3.1 applies to any smooth open set \( \Omega \subset \mathbb{R}^2 \) for which \( \lambda_1(\Omega) > 0 \).

When \( \Omega \) is the unite disc, sharp existence/nonexistence results for (3.1) have been obtained by Kalka and Yang [14] in the case of nonpositive \( K \). The following result is a restatement of Theorem 3.1 in [14]:

**Theorem 3.2.** (Kalka and Yang) Let \( K \in C(D) \), \( K < 0 \) in \( D \). Assume

\[
\exists \alpha > 1, C > 0 \quad \text{such that} \quad K \geq -\frac{C}{(1 - |x|^2)^2 \log(1 - |x|^2)|x|^{2\alpha}}
\]

Then equation (3.1) has a \( C^2 \) solution. If

\[
K \leq -\frac{C}{(1 - |x|^2)^2 \log(1 - |x|^2)|x|^{2\alpha}} \quad \text{for } |x| \text{ close to } 1
\]

then (3.1) has no \( C^2 \) solution in \( D \).

Existence is proved by monotone iteration techniques. We present here a variational existence result without sign assumptions on \( K \).

**Theorem 3.3.** Let \( \int_D K^2(1 - |x|^2)^2dx < +\infty \). Then equation (3.1) has a solution in \( H^1_0(D) + \mathbb{R} \).

**Remark 5.** This result is far from being sharp. For instance, if one takes \( K_\alpha = -\frac{\alpha}{2}(\frac{2}{1-|x|^2})^{2-\alpha}, \quad \alpha \in \mathbb{R}, \) (3.1) has the solution \( v_\alpha = \frac{\alpha}{2} \log \frac{2}{1-|x|^2} \), so that \( K_\alpha \) is the curvature of \( g_\alpha = (\frac{2}{1-|x|^2})^{\alpha}g_e \). So, negative \( \alpha \) give examples of positive curvatures \( K_\alpha \) with arbitrary blow up.
Proofs of Theorems 3.1 and 3.3 rely on inequality (2.11). We state below some consequences of (2.11) that we need.

**Lemma 3.1.** Let \((M, g)\) be an MT manifold. Let \(K \in L^2(\mu_g)\). Then \(I_K(v) := \int_M K[e^v - 1]dV_g\) is uniformly continuous on bounded sets of \(H^1_0(M)\). Furthermore, 
v_n \rightharpoonup v in \(H^1_0(M)\) implies \(I_K(v_n) \to I_K(v)\).

**Proof.** Let \(|\nabla_g u| + |\nabla_g v| \leq R\). Writing \(e^t - e^s = (e^{t-s} - 1)(e^s - 1) + (e^{t-s} - 1)\), we see, using the inequality \((e^t - 1)^2 \leq |e^{2t} - 1| \forall t\) and (2.11), that

\[ |I_K(u) - I_K(v)| \leq \left( \int_M K^2 dV_g \right)^{\frac{1}{2}} \times \left( \int_M |e^u - e^v|^2 dV_g \right)^{\frac{1}{2}} \]

\[ c(K) \left[ \left( \int_M |e^{2u} - 1|^2 dV_g \right)^{\frac{1}{4}} \times \left( \int_M |e^{2(u-v)} - 1|^2 dV_g \right)^{\frac{1}{4}} + \left( \int_M |e^{u-v} - 1|^2 dV_g \right)^{\frac{1}{2}} \right] \leq \]

\[ c(K, R, \delta) \left( e^{\frac{2\|\nabla_g(u-v)\|^2}{\pi \delta}} - \frac{2\|\nabla_g(u-v)\|^2}{\pi \delta} - 1 \right)^{\frac{1}{8}} \leq C(K, R, \delta) \|\nabla_g(u - v)\|^{\frac{1}{2}} \]

Next, assume \(v_n \rightharpoonup v\) in \(H^1_0(M)\) and a.e. From sup \(\int_M |\nabla_g v_n|^2 < \infty\) and Lemma 2.2 we get sup \(\int_M (e^{v_n} - 1)^2 dV_g < +\infty\) and hence Vitali’s convergence theorem applies to get \(\int_M K(e^{v_n} - 1)dV_g \to \int_M K(e^v - 1)dV_g\). □

We state without proof the following property

**Corollary 3.4.** Let \((M, g)\) be an MT manifold. Let \(I(v) := \int_M [(e^v - 1)^2 + v - 1]dV_g\), \(J(v) := \int_M [(e^v - v - 1)dV_g\). Then \(I, J \in Lip_{loc}(H^1_0(M))\).

**Proof of Theorem 3.1** Let \(O := \{v \in H^1_0(M) : \int_M K_2(e^{2v} - 1)dV_g > 0\}\). By Lemma 3.1, \(O\) is open. Let

\[ E_K(v) = \int_M |\nabla_g v|^2 dV_g - 2 \int_M K_1 v dV_g - \log \int_M K_2(e^{2v} - 1)dV_g \quad v \in O \]
Since (2.7) implies \( \lambda_1(g) := S_2(M, g) > 0 \), we get from Corollary 2.2 and the assumption on \( K_i \),

\[
E_K(v) \geq \int_M |\nabla_g v|^2 dV_g -
\]

\[
- \left[ \frac{2}{\lambda_1(M)} \left( \int_M K_1^2 v^2 dV_g \right)^{\frac{1}{2}} \left( \int_M |\nabla_g v|^2 dV_g \right)^{\frac{1}{2}} + c(K_2) + \frac{1}{2} \log \int_M (e^{2v} - 1)^2 dV_g \right] \geq
\]

\[
\geq \left( \int_M |\nabla v|^2 dV_g \right)^{\frac{1}{2}} \left[ \left( 1 - \frac{1}{\pi \delta} \right) \left( \int_M |\nabla v|^2 dV_g \right)^{\frac{1}{2}} - c(K_1, M) \right] - c(K_2, \delta)
\]

for every \( v \in O \). Thus \( E_K \) is bounded below and coercive on \( O \). Hence, if \( v_n \in O \), \( E_K(v_n) \to \inf_{O} E_K \), we can assume \( v_n \) converges weakly to some \( v \). By Lemma 3.1 and boundedness of \( E_K(v_n) \) we infer that \( v \in O \) and \( E_K(v) = \inf_{O} E_K \). Since \( O \) is open, we see that

\[
\int_M [\nabla_g v \nabla_g \varphi - K_1 \varphi] dV_g - \frac{\int_M K_2 e^{2v} \varphi dV_g}{\int_M K_2 (e^{2v} - 1) dV_g} = 0 \quad \forall \varphi \in C_0^\infty(M)
\]

and hence \( v - \frac{1}{2} \log \int_M K_2 (e^{2v} - 1) dV_g \) solves (3.1).

**Proof of Theorem 3.3** It goes like above, with the obvious modification

\[
E_K(v) := \int_D |\nabla v|^2 dx - \log \int_D K(2v - 1) dx \geq
\]

\[
\int_D |\nabla v|^2 dx - \log(\int_D K^2 (1 - |x|^2)^2 dx)^{\frac{1}{2}} \left( \int_D \frac{(e^{2v} - 1)^2}{(1 - |x|^2)^2} dx \right)^{\frac{1}{2}} \geq
\]

\[
(1 - \frac{1}{2\pi \delta}) \int_D |\nabla v|^2 dx - c(K, \delta) \quad \forall v \in O
\]

**Remark 6.** In [22] a similar result is proven, but under the stronger assumption \( |K(x)| \leq \frac{C}{(1 - |x|)^\alpha} \) with \( \alpha \in (0, 1) \).

The result in Theorem 3.3, when applied to negative \( K \), is weaker than the one in Kalka-Yang. But, even more, the solutions we find don’t address the
main point in [14], which is to find complete metrics of prescribed (nonpositive) Gaussian curvature on noncompact Riemannian surfaces: a solutions of (3.1) has to blow to $+\infty$ along $\partial D$ to give rise to a complete metric, and this is not the case for the solutions obtained in Theorem 3.3. A first step in this direction is to build solutions of (3.1) with prescribed boundary values. Since without sign assumptions on $K$ one cannot expect $K$ to be the curvature of a complete metric (e.g., if $K \geq 0$ around $\partial D$, then $K$ cannot be the curvature of a complete conformal metric on the disc (see [14])) we restrict our attention to $K < 0$. Assuming again $\int_D K^2(1 - |x|^2)^2 dx < +\infty$, we see that the strictly convex functional

$$J_K(v) = \frac{1}{2} \int_D |\nabla v|^2 dx - \int_D K(e^{2v} - 1) dx \quad v \in H^1_0(D)$$

is well defined, uniformly continuous and weakly lower semicontinuous by Lemma 3.1. Furthermore, by Hardy’s inequality,

$$J_K(v) = \frac{1}{2} \int_D |\nabla v|^2 dx - \int_D Kvdx - \frac{1}{2} \int_D K(e^{2v} - 2v - 1) dx \geq$$

$$\frac{1}{2} \int_D |\nabla v|^2 dx - \frac{1}{2} \left( \int_D K^2(1 - |x|^2)^2 \right)^\frac{1}{2} \left( \int_D |\nabla v|^2 dx \right)^\frac{1}{2} \quad \forall v \in H^1_0(D)$$

Thus $J_K$ achieves its global minimum, which is the unique $H^1_0(D)$ solution of (3.1). The same arguments, applied to $K \Phi = Ke^{2\Phi}$, where $\Phi$ is the harmonic extension of some boundary data $\varphi$, lead to the following

**Theorem 3.5.** Let $K \leq 0$ and $\int_D K^2(1 - |x|^2)^2 dx < +\infty$.

Given a smooth boundary data $\varphi$, (3.1) has a unique solution which takes the boundary data $\varphi$ and which writes as $u = v + \Phi$, $v \in H^1_0(D)$.

In particular, $K$ is the curvature of the conformal metric $g = e^{2(v+\Phi)}g_e$.

To get a complete conformal metric with curvature $K$, one can build, following [17], a sequence $u_n$ of solutions of (3.1) taking $\varphi \equiv n$ and try to show that it converges to a solution $u$ of (3.1) such that $u(x) \to +\infty$ suitably fast as $|x| \to 1$. We don’t pursue the details.
A more natural approach to find a complete conformal metric with curvature $K$, is to look for a bounded $C^2$ solution of the equation

$$\Delta_H u + 1 + Ke^{2u} = 0 \quad (3.3)$$

where $\Delta_H$ denotes the hyperbolic laplacian (notice that solutions $u$ of (3.3) and $v$ of (3.1) are simply related: $v - u = \log \frac{2}{1 - |x|^2}$). We recall the following pioneering result ([4], see also [8])

**Theorem 3.6.** (Aviles-McOwen) Let $K \in C^\infty(D)$, $K \leq 0$ in $D$ and such that $-\frac{1}{c} \leq K \leq -c$ in $\{c \leq |x| < 1\}$ for some $c \in (0,1)$. Then there is a unique metric conformal and uniformly equivalent to the hyperbolic metric having $K$ as its Gaussian curvature.

We end this section with a result which might provide complete conformal metrics with prescribed nonpositive gaussian curvature. Given a conformal metric $g$ on the disc, let us denote by $K_g$ its curvature. Given $K$, $e^{2u}g$ is a conformal metric with curvature $K$ if $u \in C^2(D)$ satisfies the equation

$$\Delta_g u - K_g + Ke^{2u} = 0 \quad (3.4)$$

If, in addition, $u$ is bounded, then $e^{2u}g$ is quasi isometric to $g$. In this case, if $g$ is complete then $e^{2u}g$ is complete as well.

**Theorem 3.7.** Let $g \leq cg_h$ be a conformal metric. Let $K = K_g + H$ be nonpositive in $D$.
Assume $H \in L^2(D, \mu_g)$ . Then (3.4) has a solution in $H^1_0$.

**Proof.** Solutions for (3.4) can be obtained as critical points of the functional

$$J_K(v) = \frac{1}{2} \int_D |\nabla v|^2 dx - \int_D Hv dV_g - \frac{1}{2} \int_D K(e^{2v} - 2v - 1) dV_g \quad v \in H^1_0(D)$$

The assumption on $g$ implies $\lambda_1(g) := S_2(D, g) > 0$ and hence

$$J_K(v) \geq \frac{1}{2} \int_D |\nabla v|^2 dx - \frac{1}{\sqrt[4]{\lambda_1(g)}} \left( \int_D H^2 dV_g \right)^{\frac{1}{2}} \left( \int_D |\nabla v|^2 dx \right)^{\frac{1}{2}} \quad \forall v \in H^1_0(D)$$
Thus $J_K$ is a (possibly infinite somewhere) convex coercive functional in $H^1_0(D)$. By Fatou’s Lemma it is also weakly lower semicontinuous, and hence it achieves its infimum at some $v$.

Notice that $J_K(v + t\varphi) < +\infty$ for all $\varphi \in C_0^\infty(D)$ because

$$
\int_{\text{supp}(\varphi)} (-K)(e^{2(v+t\varphi)} - 2(v+t\varphi) - 1) dV_g \leq \sup_{\text{supp}(\varphi)} (-K) \int_D (e^{2v} - 2v - 1) dV_g < +\infty
$$

by Trudinger exponential integrability. Hence

$$
0 = \frac{d}{dt} J_K(v + t\varphi)_{t=0} = \int_D \nabla_g v \nabla_g \varphi - (K - K_g) \varphi - K(e^{2v} - \varphi) dV_g
$$

i.e. $v$ solves (3.4).

\[\square\]

**Remark 7.** In particular, following [4], one can take $K = f + H$, $f \in L^2(D, \mu_h)$ and $H \leq 0$ bounded and bounded away from zero around $\partial D$.

**Remark 8.** The above result slightly improves a result by D.M. Duc [11], where, in addition, conditions are given to insure the metric is complete.

## 4 Appendix

We present an example of a domain for which $\omega(\Omega) < +\infty$ and $\lambda_1(\Omega) = 0$. Let

$$
\Omega = \mathbb{R}^2 \setminus \bigcup_{n,m \in \mathbb{Z}} D_{n,m}, \quad D_{n,m} = D_{r_{n,m}}(n,m), \quad \log \frac{1}{r_{n,m}} = 2^{|n|+|m|}
$$

We are going to exhibit a sequence $u_k \in H^1_0(\Omega)$ such that

$$
\sup_k \int_\Omega |\nabla u_k|^2 < \infty \quad \int_\Omega u_k^2 \to_k +\infty
$$

Let $\psi_k \in C_0^\infty(D_{3k}, [0, 1])$, $\psi_k \equiv 1$ in $D_k$, be radial with $|\nabla \psi_k| \leq \frac{1}{k}$, so that

$$
\int_{\mathbb{R}^2} |\nabla \psi_k|^2 \leq 8\pi \quad \text{and} \quad \int_{\mathbb{R}^2} |\psi_k|^2 \geq \pi k^2
$$
Let $\varphi_{\epsilon}(x) = 2(1 - \frac{\log |x|}{\log \epsilon})$ in $A_{\epsilon} := \{ \epsilon \leq |x| \leq \sqrt{\epsilon} \}$ and $\varphi_{\epsilon} \equiv 0$ in $|x| \leq \epsilon$, so that
\[
\int_{|x| \leq \sqrt{\epsilon}} |\nabla \varphi_{\epsilon}|^2 \leq -\frac{4\pi}{\log \epsilon} \quad \text{and} \quad \int_{|x| \leq \sqrt{\epsilon}} |\varphi_{\epsilon}|^2 \leq \epsilon \pi
\]
Finally, let $\varphi = \varphi_{r_{n,m}}(x - (n,m))$ in $D_{\sqrt{r_{n,m}}}(n,m)$, $\varphi = 1$ elsewhere, and let
\[
\psi_k(x) = \min\{ \varphi(x), \psi_k(x) \}
\]
so that $u_k \in H^1_0(\Omega)$ and
\[
\int_{\Omega} |\nabla u_k|^2 \leq 8\pi + 4\pi \sum_{n,m} \frac{1}{2|n|+|m|} \leq 44\pi \quad \text{and} \quad \int_{\Omega} |u_k|^2 \geq \pi k^2 - \pi \sum_{n,m} r_{n,m}
\]

References

[1] S. Adachi and K. Tanaka, *Inequalities in $\mathbb{R}^N$ and their best exponents*, Proc. A.M.S., Vol. 128, n.o 7 (2000) 2051-2057

[2] Adimurthi and M.Grossi, *Asymptotic estimates for a two-dimensional problem with polynomial nonlinearity*, Proc. Amer. Math. Soc., 132, n. 4, 1013-1019

[3] Adimurthi and Tintarev, *On a version of Trudinger Moser Inequality with Mobius shift invariance*, Preprint,2009.

[4] P.Aviles, R. McOwen, *Conformal deformations of complete manifolds with negative curvature*, J. Differential Geom. 21 (1985), 269-281

[5] W. Beckner, *Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality*, Ann. of math. 138 (1993), 213-242

[6] Baernstein, Albert, II *A unified approach to symmetrization*, Partial differential equations of elliptic type (Cortona, 1992), 47-91, Sympos. Math., XXXV, Cambridge Univ. Press, Cambridge, 1994.

[7] T.P.Branson, L.Fontana and C.Morpurgo, *Moser-Trudinger and Beckner-Onofri’s inequalities on the CR sphere*, arXiv:0712.3905v2 [math.AP] 21 May 2008
[8] J. Bland-M. Kalka, *Complete metrics conformal to the hyperbolic disc* Proc. AMS, 97 (1986) 128-132.

[9] D.M. Cao, *Nontrivial solutions of semilinear elliptic equations with critical exponent in $\mathbb{R}^2$*, Comm. Partial Diff. Equations, 17 (1992) 407-435

[10] K.S. Cheng and C.S. Lin, *Conformal metrics with prescribed nonpositive Gaussian curvature on $\mathbb{R}^2$*, Cal. Var., II, 203-231 (2000)

[11] D.M. Duc *Complete metrics with nonpositive curvature on the disc*, Proc. AMS, 113, n. 1, 1991

[12] L. Fontana *Sharp borderline Sobolev inequalities on compact Riemannian manifolds*, Comment. Math. Helvetici 68 (1993) 415-454

[13] J. Kazhdan and Warner *Curvature functions for open 2-manifolds*, Ann. of Math., Vol. 99, n. 2 (1974) 203-219

[14] M. Kalka, D. Yang, *On conformal deformation of nonpositive curvature on noncompact surfaces*, Duke Math. J., 72, no. 2, 405-430 (1993)

[15] G.M. Goluzin, *Geometric Theory of functions of one complex variable* Translations of Math. Monographs, Vol. 26, AMS, 1969

[16] B. Kawohl, M. Lucia *Best constants in some exponential sobolev inequalities*, Indiana Univ. Math. Journ., vol 57, 4, (2008) 1907-1927.

[17] C. Loewner and L. Nirenberg, *Partial differential equations invariant under conformal or projective transformations*, in: L. Ahlfors, et al., (Eds.), Contributions to Analysis, Academic Press, New York, 1974, pp. 245-272.

[18] J. Moser, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. 20 (1970/71), 1077-1092

[19] T. Ogawa, *A proof of Trudinger’s inequality and its application to nonlinear Schrodinger equation*, Nonlinear Anal. 14 (1990), 765-769

[20] X. Ren, J. Wei *On a two dimensional elliptic problem with large exponent in nonlinearity*, Trans. Amer. Math. Soc. 343 (1994), 749-763

16
[21] B. Ruf, *A sharp Trudinger-Moser type inequality for unbounded domains in \( \mathbb{R}^2 \)*, J. Funct. Anal. 219 (2005), no. 2, 340–367.

[22] Sanxing Wu and Hongying Liu, On the elliptic equation \( \Delta u + K(x)e^{2u} = 0 \) on \( B^2 \), Proceedings of the AMS, 132, 10, pages 3083-3088, 2004.