Fractional Integration of the Product of Two Multivariable Gimel-Functions and A General Class of Polynomials

By Frederic Ayant

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I. Introduction and Preliminaries

The fractional integral operator involving various special functions has found significant importance and applications in various subfields of applicable mathematical analysis. Since the last four decades, some workers like Love [17], McBride [20], Kalla [8,9], Kalla and Saxena [10,11], Saxena et al. [29], Saigo [24,25], Kilbas [12], Kilbas and Sebastian [14] and Kiryakova [16,17] have studied in depth the properties, applications and different extensions of various hypergeometric operators of fractional integration. A detailed account of such operators along with their properties and applications can be found in the research monographs by Samko, Kilbas, and Marichev [26], Miller and Ross [22], Kilbas, Srivastava, and Trujillo [15] and Debnath and Bhatta [6]. A useful generalization of the hypergeometric fractional integrals, including the Saigo operators [23,24], has been introduced by Marichev [18], see Samko et al. [28] and also see Kilbas and Saigo [13] for more details. The generalized fractional integral operator of arbitrary order, involving Appell function $F_3$ in the kernel defined and studied by Saigo and Maeda [27, p. 393, Eq. (4.12)] and (4.13) in the following manner:

Let $\alpha, \alpha', \beta, \beta', \eta$ be complex numbers and, $x, Re(\eta) > 0$, we have, see Saigo and Maeda [28, p. 393, Eq (4.12)]

\[ f_{0,x}^{\alpha,\alpha',\beta,\beta',\eta} f(x) = \frac{z^{-\alpha}}{\Gamma(\eta)} \int_0^x (x-t)^{\eta-1} t^{-\alpha'} F_3 \left[ \begin{array}{c} \alpha, \alpha', \beta, \beta', \eta, 1 - \frac{t}{x}, 1 - \frac{x}{t} \end{array} \right] f(t) dt \]  \hspace{1cm} (1.1)

and

\[ f_{x,\infty}^{\alpha,\alpha',\beta,\beta',\eta} f(x) = \frac{x^{-\alpha}}{\Gamma(\eta)} \int_0^x (t-x)^{\eta-1} t^{-\alpha'} F_3 \left[ \begin{array}{c} \alpha, \alpha', \beta, \beta', \eta, 1 - \frac{t}{x}, 1 - \frac{x}{t} \end{array} \right] f(t) dt \]  \hspace{1cm} (1.2)

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We have the following two results due to Saigo [25] where $\Re(\eta) > 0$

**Definition 3**

\[
I_{0+}^{\alpha,\beta,\eta} f(z) = \frac{x^{-\alpha-\beta}}{\Gamma(\eta)} \int_0^x (x-t)^{\alpha-1} F \left[ \alpha + \beta, -\eta; \alpha; 1 - \frac{t}{x} \right] f(t) dt
\]

(1.3)

**Definition 4**

\[
I_{0-}^{\alpha,\beta,\eta} f(z) = \frac{1}{\Gamma(\eta)} \int_x^\infty t^{-\alpha-\beta} (x-t)^{\alpha-1} F \left[ \alpha + \beta, -\eta; \alpha; 1 - \frac{x}{t} \right] f(t) dt
\]

(1.4)

F is the Gaussian hypergeometric function. We obtain the following lemmas.

**Lemma 1.**

\[
\left( I_{0,x}^{\alpha,\alpha',\beta,\beta',\eta} \right) = \frac{\Gamma(u)\Gamma(\mu + \eta - \alpha - \alpha' - \beta)(\mu + \beta' - \alpha')}{\Gamma(\mu + \eta - \alpha - \alpha')\Gamma(\mu + \eta - \alpha - \beta)\Gamma(\mu + \beta')} x^{\mu-\alpha-\alpha'+\eta-1}
\]

(1.5)

where $\alpha, \alpha', \beta, \beta', \eta \in \mathbb{C}, \Re(\mu) > \max\{0, \Re(\alpha + \alpha' + \beta - \eta), \Re(\alpha' - \beta')\}$

**Lemma 2.**

\[
\left( I_{0,\infty}^{\alpha,\alpha',\beta,\beta',\eta} \right) = \frac{\Gamma(1 + \alpha + \alpha' - \eta - \mu)\Gamma(1 + \alpha + \beta' - \eta - \mu)\Gamma(1 - \beta - \mu)}{\Gamma(1 - \mu)\Gamma(1 - \mu - \eta + \alpha + \alpha' + \beta')\Gamma(1 + \alpha - \beta - \mu)} x^{\mu-\alpha-\alpha'+\eta-1}
\]

(1.6)

where $\alpha, \alpha', \beta, \beta', \eta \in \mathbb{C}, \Re(\eta) > 0, \Re(\mu) < \min\{\Re(-\beta), \Re(\alpha + \alpha' - \eta), \Re(\alpha' + \beta' - \eta)\}$

**Lemma 3.**

\[
\left( I_{0,x}^{\alpha,\beta,\eta} \right) = \frac{\Gamma(u)\Gamma(\mu - \beta)}{\Gamma(\mu + \eta + \alpha + \eta)\Gamma(\mu - \beta)} x^{\mu-\beta-1}
\]

(1.7)

where $\alpha, \beta, \eta \in \mathbb{C}, \Re(\alpha) > 0, \Re(\mu) > \max\{0, \Re(\beta - \eta), \Re(\alpha' - \beta')\}$

**Lemma 4.**

\[
\left( I_{0,\infty}^{\alpha,\beta,\eta} \right) = \frac{\Gamma(\beta - \mu + 1)\Gamma(\eta - \mu + 1)}{\Gamma(1 - \mu)\Gamma(\alpha + \beta + \eta - \mu + 1)} x^{\mu-\beta-1}
\]

(1.8)

where $\alpha, \beta, \eta \in \mathbb{C}, \Re(\alpha) > 0, \Re(\mu) < 1 + \min\{\Re(\beta), \Re(\eta)\}$

Recently, Gupta et al. [7] have obtained the images of the product of two H-functions in Saigo operator given by (1.3) and (1.4) and thereby generalized several results obtained earlier by Kilbas, Kilbas and Sebastian [14] and Saxena et al. [29] as mentioned in this paper cited above. It has recently become a subject of interest for many researchers in the field of fractional calculus and its applications. Motivated by these avenues of applications, a number of workers have made use of the fractional calculus operators to obtain the image formulas. The aim of this paper is to obtain four results that give the theorems of the product of two multivariable Gimel functions and a general class of multivariable polynomials [30] in Saigo-Maeda operators and Saigo operators.

**II. Multivariable Gimel-Function**

We throughout this paper, let $\mathbb{C}, \mathbb{R}$, and $\mathbb{N}$ be set of complex numbers, real numbers and positive integers respectively. Also, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We define a generalized transcendental function of several complex variables.
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(2.1)

\[ \mathcal{I}(z_1, \ldots, z_r) = \sum_{j=1}^{n_1} \mathcal{I}(z_1, \ldots, z_r) \]

with

(2.2)

(2.3)

\[ \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \psi(s_1, \ldots, s_r) \prod_{k=1}^{r} \theta_k(s_k) z_k^{s_k} \, ds_1 \cdots ds_r \]  

with \( \omega = \sqrt{-1} \)

\[ \psi(s_1, \ldots, s_r) = \frac{\prod_{j=1}^{n_1} \Gamma_{A_{j1}}(1 - a_{2j} + \sum_{k=1}^{2} \alpha_{j2}^{(k)} s_k)}{\sum_{i=1}^{R_i} [\prod_{j=n_1+1}^{n_1+2} \Gamma_{A_{j2}}(2 - a_{2j} - \sum_{k=1}^{2} \alpha_{j2}^{(k)} s_k)]} \frac{\prod_{j=1}^{q_{j2}} \Gamma_{B_{j2}}(2 - b_{2j2} + \sum_{k=1}^{2} \beta_{j2}^{(k)} s_k)}{\prod_{j=n_1+1}^{n_1+3} \Gamma_{A_{j3}}(1 - a_{3j} + \sum_{k=1}^{3} \alpha_{j3}^{(k)} s_k)} \]  

\[ \frac{\prod_{j=1}^{r_{j4}} \Gamma_{A_{jr}}(1 - a_{rj} - \sum_{k=1}^{r} \alpha_{rj}^{(k)} s_k)}{\sum_{i=1}^{R_i} [\prod_{j=n_1+1}^{r_{j4}+1} \Gamma_{A_{jr}}(2 - a_{rj} - \sum_{k=1}^{r} \alpha_{rj}^{(k)} s_k)]} \frac{\prod_{j=1}^{q_{j4}} \Gamma_{B_{jr}}(2 - b_{jr} + \sum_{k=1}^{r} \beta_{jr}^{(k)} s_k)}{\prod_{j=n_1+1}^{r_{j4}+3} \Gamma_{A_{jr}}(1 - a_{rj} + \sum_{k=1}^{r} \alpha_{rj}^{(k)} s_k)} \]  

and

\[ \theta_k(s_k) = \frac{\prod_{j=1}^{r_{j4}} \Gamma_{D_{jr}}(d_{jr} - c_{jr}^{(k)} s_k) \prod_{j=1}^{q_{j4}} \Gamma_{C_{jr}}(1 - c_{jr}^{(k)} + \gamma_{jr}^{(k)} s_k)}{\sum_{i=1}^{R_i} [\prod_{j=n_1+1}^{r_{j4}+1} \Gamma_{D_{jr}}(1 - d_{jr} + \sum_{k=1}^{r} \delta_{jr}^{(k)} s_k)]} \frac{\prod_{j=n_1+1}^{r_{j4}+1} \Gamma_{C_{jr}}(1 - d_{jr}^{(k)} + \gamma_{jr}^{(k)} s_k) \prod_{j=n_1+1}^{r_{j4}+1} \Gamma_{C_{jr}}(c_{jr}^{(k)} s_k)}{\prod_{j=n_1+1}^{r_{j4}+3} \Gamma_{A_{jr}}(1 - a_{rj} + \sum_{k=1}^{r} \alpha_{rj}^{(k)} s_k)} \]  

Notes
The contour $L_k$ is in the $s_k (k = 1, \ldots, r)$-plane and runs from $\sigma - i\infty$ to $\sigma + i\infty$ where $\sigma$ if is a real number with loop, if necessary to ensure that the poles of 
\[ \Gamma^{A_{kj}} \left( 1 - a_{j2} + \sum_{k=1}^{r} \alpha_{k,j}(s) s_k \right) (j = 1, \ldots, n_2), \Gamma^{A_{k}} \left( 1 - a_{j} + \sum_{k=1}^{r} \alpha_{k,j}(s) s_k \right) (j = 1, \ldots, n_1) \]
the right of the contour $L_k$ and the poles of $\Gamma^{B_{k}} (d_{k} - \delta_{k}) (s_k) (j = 1, \ldots, m(k)) (k = 1, \ldots, r)$ lie to the left of the contour $L_k$. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.1) can be obtained of the corresponding conditions for multivariable H-function given by as

\[ |\arg(z_k)| < \frac{1}{2} A_i^{(k)} \pi \]
where

\[ A_i^{(k)} = \sum_{j=1}^{m^{(k)}} D_j^{(k)} \delta_j^{(k)} + \sum_{j=1}^{n^{(k)}} C_j^{(k)} \tau_j^{(k)} - \tau_i^{(k)} \left( \sum_{j=\max(j,k)+1}^{m^{(k)}} D_j^{(k)} \delta_j^{(k)} + \sum_{j=\max(j,k)+1}^{n^{(k)}} C_j^{(k)} \tau_j^{(k)} \right) \]

Following the lines of Braaksma ([4] p. 278), we may establish the asymptotic expansion in the following convenient form

\[ N(z_1, \ldots, z_r) = 0( |z_1|^{\alpha_1}, \ldots, |z_r|^{\alpha_r}, \max(|z_1|, \ldots, |z_r|) \to 0 \]
\[ N(z_1, \ldots, z_r) = 0( |z_1|^{\beta_1}, \ldots, |z_r|^{\beta_r}, \min(|z_1|, \ldots, |z_r|) \to \infty \]
where $i = 1, \ldots, r$:

\[ \alpha_i = \min_{1 \leq j \leq m^{(i)}} \text{Re} \left[ D_j^{(i)} \left( \frac{\delta_j^{(i)}}{\gamma_j^{(i)}} \right) \right] \]
\[ \beta_i = \max_{1 \leq j \leq m^{(i)}} \text{Re} \left[ C_j^{(i)} \left( \frac{\gamma_j^{(i)}}{\delta_j^{(i)}} - 1 \right) \right] \]

Remark 1.
If $n_2 = \cdots = n_r-1 = p_2 = q_2 = \cdots = p_{r-1} = q_{r-1} = 0$. $A_{2j} = A_{2j+i} = B_{2j+i} = \cdots = A_{rj} = A_{rj+i} = B_{rj+i} = A_{r} = A_{r+i} = B_{r+i} = 1$, then the multivariable Gimel-function reduces in multivariable Aleph-function defined by Ayant [3].

Remark 2.
If $n_2 = \cdots = n_r = p_2 = q_2 = \cdots = p_r = q_r = 0$. $\tau_2 = \cdots = \tau_r = \tau_{(1)} = \cdots = \tau_{(r)} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the multivariable Gimel-function reduces in multivariable I-function defined by Prathima et al. [23].

Remark 3.
If $A_{2j} = A_{2j+i} = B_{2j+i} = \cdots = A_{rj} = A_{rj+i} = B_{rj+i} = 1$. $\tau_2 = \cdots = \tau_r = \tau_{(1)} = \cdots = \tau_{(r)} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the generalized multivariable Gimel-function reduces in multivariable I-function defined by Prasad [22].

Remark 4.
If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function reduces in the H-function of several defined by Srivastava and Panda [32,33]. About the simplified notations, see Ayant ([4], page 248-255)

Now, we define the second Gimel function of s variables, the parameters are identical to the Gimel function of r variables with the prim sign and the validities conditions are equivalent.

The generalized polynomials of multivariable defined by Srivastava [30], is given in the following manner:

\[ S_{N_{1}, \ldots, N_{s}, \ldots, N_{s}}^{m_{1}, \ldots, m_{r}, \ldots, m_{r}} [y_1, \ldots, y_s] = \sum_{K_1 = 0}^{[N_1/m_1]!} \cdots \sum_{K_s = 0}^{[N_s/m_s]!} \frac{(-N_1)^{m_1} K_1! \cdots (-N_s)^{m_s} K_s!}{K_1! \cdots K_s!} A[N_1, K_1; \cdots; N_s, K_s] y_1^{K_1} \cdots y_s^{K_s} \]
where $\mathcal{M}_1, \ldots, \mathcal{M}_b$ are arbitrary positive integers and the coefficients $A[N_1, K_1; \cdots; N_v, K_v]$ are arbitrary constants, real or complex.

We shall note $a_v = \frac{(-N_1)_{\mathcal{M}_1 K_1}}{K_1!} \cdots \frac{(-N_v)_{\mathcal{M}_v K_v}}{K_v!} A[N_1, K_1; \cdots; N_v, K_v]

### III. Main Results

We shall note

\begin{align}
U &= 0, n_2, 0, n_3; \cdots; 0, n_r-1, 0, n'_2, 0, n'_3; \cdots; 0, n'_{s-1} \\
V &= m^{(1)}, n^{(1)}; m^{(2)}, n^{(2)}; \cdots; m^{(r)}, n^{(r)}; m^{(t_1)}, n^{(t_1)}; m^{(t_2)}, n^{(t_2)}; \cdots; m^{(t_s)}, n^{(t_s)} \\
X &= \tau_{t_1}^{(1)}; R_{t_1}^{(1)}; \cdots; \tau_{t_r}^{(r)}; R_{t_r}^{(r)}; \cdots; \tau_{t_s}^{(s)}; R_{t_s}^{(s)} \\
Y &= \phi_{t_1}; \phi_{t_2}; \tau_{t_1}^{(1)}; R_{t_1}^{(1)}; \cdots; \phi_{t_r}; \phi_{t_r}^{(r)}; \tau_{t_r}^{(r)}; R_{t_r}^{(r)}; \cdots; \phi_{t_s}; \phi_{t_s}^{(s)}; \tau_{t_s}^{(s)}; R_{t_s}^{(s)}
\end{align}

**Theorem 1.**

\[
\left\{ \left( \int_{0}^{\alpha} x^{\alpha-\beta, \beta, \eta; \mu-1} (b-\alpha)^{-(\delta+1)} \sum_{K_1=1}^{K_1} \ldots \sum_{K_v=1}^{K_v} a_v c_{\mu} K_1 \ldots c_{v} K_v \right) \cdot \left( z_1 t^{\sigma_1} (b-\alpha)^{-\omega_1} \right) \cdot \ldots \cdot \left( z_t t^{\sigma_t} (b-\alpha)^{-\omega_t} \right) \right\} (x) = b^{-\mu} x^{-\alpha-\eta-1} \sum_{K_1=1}^{K_1} \ldots \sum_{K_v=1}^{K_v} a_v c_{\mu} K_1 \ldots c_{v} K_v
\]

where

\[
A_v = \left[ \begin{array}{c} a_{ij}^{(1)} A_{ij}^{(2)} A_{ij}^{(3)} \end{array} \right] \ldots \left[ A_{ij}^{(r-1)} A_{ij}^{(r-1)} A_{ij}^{(r-1)} \right] A_{ij}^{(r-1)}
\]

\[
B_v = \left[ \begin{array}{c} \alpha_{ij}^{(1)} \alpha_{ij}^{(2)} \alpha_{ij}^{(3)} \end{array} \right] \ldots \left[ \alpha_{ij}^{(r-1)} \alpha_{ij}^{(r-1)} \alpha_{ij}^{(r-1)} \right] A_{ij}^{(r-1)}
\]
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\[(a_{2j}; \alpha_{2j}, \alpha_{2j}^{(2)}, A_{2j}) \in \mathbb{N}_2, \{\tau_{r,_{j}}^{(r)} (a_{2j}; \alpha_{2j}^{(2)}); \alpha_{2j}^{(r)}; A_{2j}^{(s)}\}_{v=1, p_{r,_{j}}^{(r)}} \in \mathbb{N}_2, \{\alpha_{3j}; \alpha_{3j}^{(2)}, \alpha_{3j}^{(3)}; A_{3j}\}_{v=1, p_{r,_{j}}^{(2)}}\]

\[\tau_{r,_{j}}^{(r)} (a_{2j}; \alpha_{2j}^{(2)}; \alpha_{2j}^{(r)}; A_{2j}^{(s)}; \alpha_{3j}; \alpha_{3j}^{(2)}, \alpha_{3j}^{(3)}; A_{3j})_{v=1, p_{r,_{j}}^{(2)}}\]

\[\{\alpha_{3j}; \alpha_{3j}^{(2)}, \alpha_{3j}^{(3)}; A_{3j}\}_{v=1, p_{r,_{j}}^{(2)}}\]

\[\{\alpha_{3j}; \alpha_{3j}^{(2)}, \alpha_{3j}^{(3)}; A_{3j}\}_{v=1, p_{r,_{j}}^{(2)}}\]

\[A_1 = (1 - \nu + \sum_{j=1}^{n} \lambda_j K_j; \sigma_1, \cdots, \sigma_r, \sigma_1', \cdots, \sigma_s', 1; 1), (1 + \alpha + \beta - \nu - \eta - \sum_{j=1}^{K} \delta_j K_j; \omega_1, \cdots, \omega_r, \omega_1', \cdots, \omega_s', 1; 1)\]

\[A = [(a_{rj}; \alpha_{rj}^{(1)}, \cdots, \alpha_{rj}^{(r)}; A_{rj})_{v=1, p_{r,_{j}}^{(1)}}; (a_{rj}; \alpha_{rj}^{(1)}, \cdots, \alpha_{rj}^{(r)}; A_{rj})_{v=1, p_{r,_{j}}^{(r)}}; \cdots; (a_{rj}; \alpha_{rj}^{(s)}, A_{rj}; A_{rj})_{v=1, p_{r,_{j}}^{(s)}}; \cdots; (a_{rj}; \alpha_{rj}^{(s)}; A_{rj})_{v=1, p_{r,_{j}}^{(s)}}]_{v=1, p_{r,_{j}}^{(s)}}\]

\[A = [(c_{j}^{(1)}, \gamma_{j}^{(1)}; C_{j}^{(1)}); (c_{j}^{(1)}, \gamma_{j}^{(1)}; C_{j}^{(1)}); \cdots; (c_{j}^{(s)}, \gamma_{j}^{(s)}; C_{j}^{(s)})]_{v=1, p_{r,_{j}}^{(s)}}\]

\[B = [(b_{2j}; \beta_{2j}^{(1)}, \beta_{2j}^{(2)}; B_{2j})_{v=1, p_{r,_{j}}^{(1)}}; (b_{2j}; \beta_{2j}^{(1)}, \beta_{2j}^{(2)}; B_{2j})_{v=1, p_{r,_{j}}^{(2)}}; \cdots; (b_{2j}; \beta_{2j}^{(s)}, B_{2j})_{v=1, p_{r,_{j}}^{(s)}}]_{v=1, p_{r,_{j}}^{(s)}}\]

\[B = [(b_{2j}; \beta_{2j}^{(1)}, \beta_{2j}^{(2)}; B_{2j})_{v=1, p_{r,_{j}}^{(1)}}; (b_{2j}; \beta_{2j}^{(1)}, \beta_{2j}^{(2)}; B_{2j})_{v=1, p_{r,_{j}}^{(2)}}; \cdots; (b_{2j}; \beta_{2j}^{(s)}, B_{2j})_{v=1, p_{r,_{j}}^{(s)}}]_{v=1, p_{r,_{j}}^{(s)}}\]

\[B_1 = (1 - \nu + \sum_{j=1}^{n} \lambda_j K_j; \sigma_1, \cdots, \sigma_r, \sigma_1', \cdots, \sigma_s', 1; 1), (1 + \alpha + \beta - \nu - \eta - \sum_{j=1}^{K} \delta_j K_j; \omega_1, \cdots, \omega_r, \omega_1', \cdots, \omega_s', 1; 1)\]

\[B_1 = (1 - \nu + \sum_{j=1}^{n} \lambda_j K_j; \sigma_1, \cdots, \sigma_r, \sigma_1', \cdots, \sigma_s', 1; 1), (1 + \alpha + \beta - \nu - \eta - \sum_{j=1}^{K} \delta_j K_j; \omega_1, \cdots, \omega_r, \omega_1', \cdots, \omega_s', 1; 1)\]
\[ B = [(d_{j1}^{(1)}, d_{j2}^{(1)}; D_{j1}^{(1)}), (\tau_{11}^{(1)}; d_{j1}^{(1)}; D_{j1}^{(1)}))]_{m^{(1)}} q_{m^{(1)}}^{(1)}; \cdots; \]

\[ [(d_{j1}^{(r)}, d_{j2}^{(r)}; D_{j1}^{(r)}))]_{1,m^{(r)}}, \tau_{j1}^{(r)}(d_{j1}^{(r)}; D_{j1}^{(r)}))_{m^{(r)}}, q_{m^{(r)}}^{(r)}; \]

\[ [(\delta_{j1}^{(1)}, \delta_{j2}^{(1)}; D_{j1}^{(1)}))]_{1,m^{(1)}}, \tau_{j1}^{(1)}(\delta_{j1}^{(1)}; D_{j1}^{(1)}))_{m^{(1)}}, q_{m^{(1)}}^{(1)}; \cdots; \]

\[ [(\delta_{j1}^{(s)}, \delta_{j2}^{(s)}; D_{j1}^{(s)}))_{1,m^{(s)}}, \tau_{j1}^{(s)}(\delta_{j1}^{(s)}; D_{j1}^{(s)}))_{m^{(s)}}, q_{m^{(s)}}^{(s)}; \cdots; \]

\[(3.12)\]

In our investigation, we will use these simplified notations cited above.

Proof

To prove (3.1), we first express the class of multivariable polynomials \( A_{i}^{(k)} \) in series with the help of (2.13), the multivariable Gimel-functions regarding Mellin-Barnes type integrals contour with the help of (2.1). Now interchange the order of summations and two multiple Mellin-Barnes integrals contour, respectively and taking the fractional integral operator inside (which is permissible under the stated conditions) and make simplifications. Next, we express the terms \( A_{i}^{(k)} \) in terms of Mellin-Barnes integrals contour (Srivastava et al. [31], page 18, (2.6.3) and after algebraic manipulations, we obtain

\[ \text{L.H.S} = \left( b - \sum_{k=1}^{r+s+1} \frac{1}{(2\pi\omega)} \int_{L_1} \cdots \int_{L_s} \psi(s_1, \cdots, s_r) \psi(t_1', \cdots, t_s') \prod_{k=1}^{r} \sum_{k=1}^{s} \theta_k(s_k) a_k^{s_k} \right) \]

Now using the lemma 1. Finally interpreting the resulting Mellin-Barnes integrals contour as a multivariable Gimel-function of \((r+s+1)\)-variables, we obtain the desired result (3.1).

Let
\[ A_2 = (1 - \nu - \sum_{j=1}^{v} \lambda_j K_j; \sigma_1, \ldots, \sigma_r, \sigma'_1, \ldots, \sigma'_s, 1; 1), (\eta + \mu - \alpha - \alpha' + \sum_{j=1}^{K} \delta_j K_j; \omega_1, \ldots, \omega_r, \omega'_1, \ldots, \omega'_s, 1; 1) \]

\[ (\beta + \mu + \sum_{j=1}^{v} \lambda_j K_j - \alpha - \alpha'; \omega_1, \ldots, \omega_r, \omega'_1, \ldots, \omega'_s, 1; 1), (\mu + \eta - \alpha - \beta' + \sum_{j=1}^{K} \delta_j K_j; \omega_1, \ldots, \omega_r, \omega'_1, \ldots, \omega'_s, 1; 1) \quad (3.13) \]

\[ B_2 = (\mu + \sum_{j=1}^{v} \lambda_j K_j; \sigma_1, \ldots, \sigma_r, \sigma'_1, \ldots, \sigma'_s, 1; 1), (\beta + \mu - \alpha + \sum_{j=1}^{K} \delta_j K_j; \omega_1, \ldots, \omega_r, \omega'_1, \ldots, \omega'_s, 1; 1) \]

\[ (1 - \mu - \beta' - \sum_{j=1}^{v} \lambda_j K_j; \sigma_1, \ldots, \sigma_r, \sigma'_1, \ldots, \sigma'_s, 1; 1), (1 + \alpha' + \beta - \mu - \eta - \sum_{j=1}^{K} \delta_j K_j; \omega_1, \ldots, \omega_r, \omega'_1, \ldots, \omega'_s, 1; 1) \quad (3.14) \]

We have the following resulting

Theorem 2.

\[
\begin{pmatrix}
\frac{c_1 t^{\lambda_1} (b - at)^{-\delta_1}}{z_1 t^{\sigma_1} (b - at)^{-\omega_1}} \\
\vdots \\
\frac{c_v t^{\lambda_v} (b - at)^{-\delta_v}}{z_v t^{\sigma_v} (b - at)^{-\omega_v}}
\end{pmatrix}
\]

\[
\left( b - \frac{a^2}{x^{\mu - \alpha - \alpha' + \eta - 1}} \sum_{K_1=1}^{[N_1/M_1]} \cdots \sum_{K_v=1}^{[N_v/M_v]} a_{K_1} \cdots a_{K_v} \right)
\]

\[
\begin{pmatrix}
\frac{z_1^{\sigma_1}}{b^{\frac{s_1}{b}}}
\\
\vdots
\\
\frac{z_v^{\sigma_v}}{b^{\frac{s_v}{b}}}
\end{pmatrix}
\]

Provided

\[
a, b, \alpha, \beta, \eta, \mu, v, \delta_k, \omega_i, \omega'_j \in \mathbb{C}, k = 1, \ldots, v; i = 1 \cdots, r; j = 1, \ldots, s
\]

\[
\lambda_k, \sigma_i, \sigma'_j > 0; \quad k = 1, \ldots, v; i = 1 \cdots, r; j = 1, \ldots, s
\]

\[
|\arg(z_i)| < \frac{1}{2} \pi A_i^{(k)} \quad \text{and} \quad A_i^{(k)} \text{is defined by (2.4)}, \quad |\arg(z'_j)| < \frac{1}{2} \pi A'_j^{(k)}; \quad \left| \frac{a}{b} \right| < 1
\]
To prove the equation (3.14), we use the similar method that formula (3.5) by using the lemma 2.

Let

\[ A_3 = (1 - \mu - \sum_{j=1}^{v} \lambda_j K_j; \sigma_1, \ldots, \sigma_r, \sigma'_1, \ldots, \sigma'_s, 1; 1) = (1 - v - \sum_{j=1}^{v} \delta_j K_j; \omega_1, \ldots, \omega_r, \omega'_1, \ldots, \omega'_s, 1; 1) \]

(3.15)

\[ B_3 = (1 - v - \sum_{j=1}^{v} \delta_j K_j; \omega_1, \ldots, \omega_r, \omega'_1, \ldots, \omega'_s, 0; 1) (1 + \beta - \mu - \sum_{j=1}^{K} \lambda_j K_j; \sigma_1, \ldots, \sigma_r, \sigma'_1, \ldots, \sigma'_s, 1; 1) \]

(3.16)

Theorem 3.

\[ \left\{ \left( \int_0^\infty \cdots \int_0^\infty \, d^{\sum_{i=1}^{\nu} N_i} \cdots \, d^{\sum_{i=1}^{\nu} N_r} \, d^{\sum_{i=1}^{\nu} N_s} \right) \left( \begin{array}{c} c_1 t^{\lambda_1 (b - at)}^{-\delta_1} \\ \vdots \\ c_{\nu} t^{\lambda_\nu (b - at)}^{-\delta_\nu} \end{array} \right) \right\} (x) \equiv \int_0^\infty \cdots \int_0^\infty \, d^{\sum_{i=1}^{\nu} N_i} \cdots \, d^{\sum_{i=1}^{\nu} N_r} \, d^{\sum_{i=1}^{\nu} N_s} \, \sum_{K_1=1}^{[N_1/M_1]} \cdots \sum_{K_\nu=1}^{[N_\nu/M_\nu]} a_{\nu} c_1^{K_1} \cdots c_\nu^{K_\nu} \left( \begin{array}{c} z_1 t^{\sigma_1 (b - at)}^{-\omega_1} \\ \vdots \\ z_\nu t^{\sigma_\nu (b - at)}^{-\omega_\nu} \end{array} \right) \]

(3.17)

Provided

\[ a, b, \alpha, \beta, \eta, \mu, v, \delta_k, \omega_i, \omega'_i \in \mathbb{C}, k = 1, \ldots, v; i = 1, \ldots, r; j = 1, \ldots, s \]

\[ \lambda_k, \sigma_i, \sigma'_j > 0; k = 1, \ldots, v; i = 1, \ldots, r; j = 1, \ldots, s \]
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\[ |\arg(z_i)| < \frac{1}{2} \pi A_i^{(k)} \] and \( A_i^{(k)} \) is defined by (2.4), \( |\arg(z'_i)| < \frac{1}{2} \pi A'_i^{(k)} \).

\[
\text{Re}(\mu) + \sum_{i=1}^{r} \sigma_i \min_{1 \leq j \leq m_{(i)}} \text{Re} \left[ D_j \left( \frac{d_i^{(i)}}{d_j^{(i)}} \right) \right] + \sum_{i=1}^{s} \sigma'_i \min_{1 \leq j \leq m_{(i)}} \text{Re} \left[ D'_j \left( \frac{d_i^{(i)}}{d'_j^{(i)}} \right) \right] > \max\{0, \text{Re}(\beta - \eta)\}
\]

\[
\text{Re}(\nu) + \sum_{i=1}^{r} \omega_i \min_{1 \leq j \leq m_{(i)}} \text{Re} \left[ D_j \left( \frac{d_i^{(i)}}{d_j^{(i)}} \right) \right] + \sum_{i=1}^{s} \omega'_i \min_{1 \leq j \leq m_{(i)}} \text{Re} \left[ D'_j \left( \frac{d_i^{(i)}}{d'_j^{(i)}} \right) \right] > \max\{0, \text{Re}(\beta - \eta)\}
\]

To prove the formula (3.17), we use the similar method that the theorem 1 by using the lemma 3.

Let

\[
A_4 = (1 - \mu - \sum_{j=1}^{v} \delta_j K_j; \omega_1, \cdots, \omega_r, \omega'_1, \cdots, \omega'_s, 1; 1), \quad (-\eta + \mu + \sum_{j=1}^{v} \lambda_j K_j; \sigma_1, \cdots, \sigma_r, \sigma'_1, \cdots, \sigma'_s, 1; 1),
\]

\[
(1 - \mu - \eta + \beta - \sum_{j=1}^{v} \lambda_j K_j; \sigma_1, \cdots, \sigma_r, \sigma'_1, \cdots, \sigma'_s, 1; 1)
\]

\[ B_4 = (-\nu - \sum_{j=1}^{v} \delta_j K_j; \eta_1, \cdots, \eta_r, \eta'_1, \cdots, \eta'_s, 0; 1), \quad (\mu + \sum_{j=1}^{K} \lambda_j K_j; 1, \cdots, \sigma_r, \sigma'_1, \cdots, \sigma'_s, 1; 1)
\]

\[ (-\alpha - \beta - \eta + \mu + \sum_{j=1}^{K} \lambda_j K_j; \sigma_1, \cdots, \sigma_r, \sigma'_1, \cdots, \sigma'_s, 1; 1)
\]

We have the formula.

Theorem 4.

\[
\begin{pmatrix}
\ell_1 t^{\lambda_1} (b - at)^{-\delta_1} \\
\vdots \\
\ell_v t^{\lambda_v} (b - at)^{-\delta_v}
\end{pmatrix} = t^{\omega_1} t^{\sigma_1'} (b - at)^{-\omega_1'}
\]

\[
\begin{pmatrix}
\ell_1 t^{\lambda_1} (b - at)^{-\delta_1} \\
\vdots \\
\ell_v t^{\lambda_v} (b - at)^{-\delta_v}
\end{pmatrix} = b^{v-1} a_{\alpha}^{K_1} \cdots c_{\nu}^{K_v}
\]

\[
\begin{pmatrix}
\ell_1 t^{\lambda_1} (b - at)^{-\delta_1} \\
\vdots \\
\ell_v t^{\lambda_v} (b - at)^{-\delta_v}
\end{pmatrix} = \begin{pmatrix}
A_1, A_4, A : A \\
B, B : B, (0, 1; 1)
\end{pmatrix}
\]

Notes
Provided

$$a, b, \alpha, \beta, \eta, \mu, v, \delta_k, \omega_i, \omega'_j \in \mathbb{C}, k = 1, \ldots, r; i = 1, \ldots, s$$

$$\lambda_k, \sigma_i, \sigma'_j > 0; \ k = 1, \ldots, r; i = 1, \ldots, s$$

$$|\arg(z_i)| < \frac{1}{2} \pi A^{(k)}_i$$ and $$A^{(k)}_i$$ is defined by (2.4), $$|\arg(z'_i)| < \frac{1}{2} \pi A^{(k)}; \left|\frac{a}{b}x\right| < 1$$

$$\text{Re(}\mu\text{)} - \sum_{i=1}^{r} \sigma_i \min_{1 \leq j \leq m(i)} \text{Re} \left(D_{j} \left(\frac{d_i}{\delta_j}\right)^{(i)}\right) - \sum_{i=1}^{s} \sigma'_i \min_{1 \leq j \leq m'_(i)} \text{Re} \left(D'_j \left(\frac{d'_i}{\delta'_j}\right)^{(i)}\right) < 1 + \min[\text{Re}(\beta), \text{Re}(\alpha + \alpha' - \eta), \text{Re}(\alpha + \beta' - \eta)]$$

$$\text{Re(}\nu\text{)} - \sum_{i=1}^{r} \omega_i \min_{1 \leq j \leq m(i)} \text{Re} \left(D_{j} \left(\frac{d_i}{\delta_j}\right)^{(i)}\right) - \sum_{i=1}^{s} \omega'_i \min_{1 \leq j \leq m'_(i)} \text{Re} \left(D'_j \left(\frac{d'_i}{\delta'_j}\right)^{(i)}\right) < 1 + \max[\text{Re}(\beta), \text{Re}(\alpha + \alpha' - \eta), \text{Re}(\alpha + \beta' - \eta)]$$

Provided

$$a, b, \alpha, \beta, \eta, \mu, v, \delta_k, \omega_i, \omega'_j \in \mathbb{C}, k = 1, \ldots, r; i = 1, \ldots, s$$

$$|\arg(z_i)| < \frac{1}{2} \pi A^{(k)}_i$$ and $$A^{(k)}_i$$ is defined by (2.4), $$|\arg(z'_i)| < \frac{1}{2} \pi A; \left|\frac{a}{b}x\right| < 1$$

$$\text{Re(}\mu\text{)} - \sum_{i=1}^{r} \sigma_i \min_{1 \leq j \leq m(i)} \text{Re} \left(D_{j} \left(\frac{d_i}{\delta_j}\right)^{(i)}\right) - \sum_{i=1}^{s} \sigma'_i \min_{1 \leq j \leq m'_(i)} \text{Re} \left(D'_j \left(\frac{d'_i}{\delta'_j}\right)^{(i)}\right) < 1 + \min[\text{Re}(\mu), \text{Re}(\eta), \text{Re}(\beta)]$$

$$\text{Re(}\nu\text{)} - \sum_{i=1}^{r} \omega_i \min_{1 \leq j \leq m(i)} \text{Re} \left(D_{j} \left(\frac{d_i}{\delta_j}\right)^{(i)}\right) - \sum_{i=1}^{s} \omega'_i \min_{1 \leq j \leq m'_(i)} \text{Re} \left(D'_j \left(\frac{d'_i}{\delta'_j}\right)^{(i)}\right) < 1 + \max[\text{Re}(\beta), \text{Re}(\eta)]$$

To prove the theorem 4, we use the similar method that the equation (3.5) by using the lemma 4.

### IV. Particular Cases

In this section, we shall see four particular cases.

If we put $$\beta = -\alpha$$ in the theorem three, we get

**Corollary 1.**

$$\left\{ \begin{array}{c}
\left( \begin{array}{c}
\frac{c_1 t^{\lambda_1} (b - at)^{-\delta_1}}{\varepsilon_{N_1}} \\
\vdots \\
\frac{c_{N_1} t^{\lambda_{N_1}} (b - at)^{-\delta_{N_1}}}{\varepsilon_{N_1}}
\end{array} \right) \\
\left( \begin{array}{c}
\frac{z_1 t^{\sigma_1} (b - at)^{-\omega_1}}{\varepsilon_{M_1}} \\
\vdots \\
\frac{z_{N_1} t^{\sigma_{N_1}} (b - at)^{-\omega_{N_1}}}{\varepsilon_{M_1}}
\end{array} \right)
\end{array} \right\} \left( x \right) = b^{-\frac{v}{\alpha} + \frac{v}{\alpha'} - \beta - 1} \sum_{K_1=1}^{[N_1/M_1]} \ldots \sum_{K_{N_1}=1}^{[N_{N_1}/M_{N_1}]} a_\nu c_{K_1}^{N_{K_1}} \ldots c_{K_{N_1}}^{N_{K_{N_1}}}
$$

$$\left( \begin{array}{c}
\frac{z'_1 t^{\sigma'_1} (b - at)^{-\omega'_1}}{\varepsilon_{M_1}'} \\
\vdots \\
\frac{z'_{N_1} t^{\sigma'_{N_1}} (b - at)^{-\omega'_{N_1}}}{\varepsilon_{M_1}'}
\end{array} \right)
\right\}
where

\[ A_3 = (1 - v - \sum_{j=0}^{v} \delta_j K_j; \omega_1, \cdots, \omega_r, \omega'_1, \cdots, \omega'_s; 1, 1), (1 - \mu - \sum_{j=1}^{v} \lambda_j K_j; \sigma_1, \cdots, \sigma_r, \sigma'_1, \cdots, \sigma'_s; 1, 1) \]  \hspace{1cm} (4.2)

under the same existence conditions that formula (3.17) with \( \beta = -\alpha \).

If \( \beta = 0 \) in theorem three, we have

Corollary 2.

\[
\left( I_{q, \alpha}^{+} t^{\mu-1} (b - at)^{-\omega_1} \right) \begin{pmatrix}
    c_1 t^{\lambda_1} (b - at)^{-\delta_1} \\
    \vdots \\
    c_v t^{\lambda_v} (b - at)^{-\delta_v}
\end{pmatrix} \begin{pmatrix}
    z_1 t^{\sigma_1} (b - at)^{-\omega_1} \\
    \vdots \\
    z_v t^{\sigma_v} (b - at)^{-\omega_v}
\end{pmatrix}
\]

\[
\begin{pmatrix}
    z'_1 t^{\sigma'_1} (b - at)^{-\omega'_1} \\
    \vdots \\
    z' v t^{\sigma'_v} (b - at)^{-\omega'_v}
\end{pmatrix}
\]

\[b^{v-\mu-\beta-1} \sum_{K_1=1}^{[N_1/M_1]} \cdots \sum_{K_v=1}^{[N_v/M_v]} a_v c_1 K_1 \cdots c_v K_v
\]

where

\[ B_3 = (1 - \alpha - \eta - \sum_{j=1}^{v} \lambda_j K_j; \sigma_1, \cdots, \sigma_r, \sigma'_1, \cdots, \sigma'_s; 0, 1) \]  \hspace{1cm} (4.3)

(4.1)

\[
\begin{pmatrix}
    z_1 t^{\sigma_1} \\
    \vdots \\
    z_v t^{\sigma_v}
\end{pmatrix} \begin{pmatrix}
    A_1, A_5, A : A \\
    \vdots \\
    B, B_6 : B, (0, 1; 1)
\end{pmatrix}
\]

(4.4)
where
\[ A_0 = \left(1 - v - \sum_{j=0}^{v} \delta_j K_j; \omega_1, \cdots, \omega_r, \omega'_1, \cdots, \omega'_s, 0; 1\right), \left(1 - \mu - \eta - \sum_{j=1}^{u} \lambda_j K_j, \sigma_1, \cdots, \sigma_r, \sigma'_1, \cdots, \sigma'_s, 1; 1\right) \]  
(4.5)

\[ B_0 = \left(1 - v - \sum_{j=0}^{v} \delta_j K_j; \omega_1, \cdots, \omega_r, \omega'_1, \cdots, \omega'_s, 0; 1\right), \left(1 - \mu - \alpha - \eta - \sum_{j=1}^{u} \lambda_j K_j, \sigma_1, \cdots, \sigma_r, \sigma'_1, \cdots, \sigma'_s, 1; 1\right) \]  
(4.6)

provided that
\[ \lambda_k, \sigma_i, \sigma_j' > 0; \ k = 1, \cdots, v; i = 1 \cdots, r; j = 1, \cdots, s \]
\[ |arg(z_i)| < \frac{1}{2} \pi A_i^{(k)} \text{ and } A_i^{(k)} \text{ is defined by (2.4), } |arg(z_i')| < \frac{1}{2} \pi A_i^{(k)} \left|\frac{a}{b}\right| < 1 \]
\[ Re(\mu) + \sum_{i=1}^{r} \sigma_i \min_{1 \leq j \leq m^{(i)}} Re \left[D_j^{(i)} \left(\frac{\delta_j^{(i)}}{\delta_j^{(i)}}\right)\right] + \sum_{i=1}^{s} \sigma_i' \min_{1 \leq j \leq m^{(s)}} Re \left[D_j^{(s)} \left(\frac{\delta_j^{(s)}}{\delta_j^{(s)}}\right)\right] > \max[0, Re(\eta)] \]

If we put \( \beta = -\alpha \) in the equation (3.20), we get

Corollary 3.

\[ \begin{pmatrix} c_1 t^{\lambda_1} (b - at)^{-\delta_1} \\ \vdots \\ c_v t^{\lambda_v} (b - at)^{-\delta_v} \end{pmatrix} \begin{pmatrix} z_1 t^{\sigma_1} (b - at)^{-\omega_1} \\ \vdots \\ z_r t^{\sigma_r} (b - at)^{-\omega_r} \end{pmatrix} \]

\[ b^{-u - \mu + \alpha - 1} \sum_{K_1=1}^{[N_1/M_1]} \cdots \sum_{K_v=1}^{[N_v/M_v]} a_v c_1^{K_1} \cdots c_v^{K_v} \]

\[ \begin{pmatrix} z_1 t^{\sigma_1} (b - at)^{-\omega_1} \\ \vdots \\ z_r t^{\sigma_r} (b - at)^{-\omega_r} \end{pmatrix} \]

\[ \begin{pmatrix} A_3, A_7, A : A \\ \vdots \\ \vdots \\ \vdots \\ B, B_7 : B, (0, 1; 1) \end{pmatrix} \]  
(4.7)
where

\[ A_7 = \left(1 - \nu - \sum_{j=0}^{v} \delta_j K_j; \omega_1, \cdots, \omega_r, \omega_{r1}', \cdots, \omega_{r1}', 1; 1\right) \left(\alpha + \mu + \sum_{j=1}^{v} \lambda_j K_j; \sigma_1, \cdots, \sigma_r, \sigma_{r1}', \cdots, \sigma_{r1}', 1; 1\right) \] (4.8)

\[ A_7 = \left(1 - \nu - \sum_{j=0}^{v} \delta_j K_j; \omega_1, \cdots, \omega_r, \omega_{r1}', \cdots, \omega_{r1}', 0; 1\right) \left(\mu + \sum_{j=1}^{v} \lambda_j K_j; \sigma_1, \cdots, \sigma_r, \sigma_{r1}', \cdots, \sigma_{r1}', 1; 1\right) \] (4.9)

under the same existence conditions that formula (3.20) with \( \beta = -\alpha \).

If \( \beta = 0 \) in theorem four, we have

Corollary 4.

Let

\[ A_8 = \left(1 - \nu - \sum_{j=0}^{v} \delta_j K_j; \omega_1, \cdots, \omega_r, \omega_{r1}', \cdots, \omega_{r1}', 1; 1\right) \left(-\alpha - \eta + \mu + \sum_{j=1}^{v} \lambda_j K_j; \sigma_1, \cdots, \sigma_r, \sigma_{r1}', \cdots, \sigma_{r1}', 1; 1\right) \] (4.10)

\[ B_8 = \left(1 - \nu - \sum_{j=0}^{v} \delta_j K_j; \omega_1, \cdots, \omega_r, \omega_{r1}', \cdots, \omega_{r1}', 0; 1\right) \left(\mu - \alpha - \eta + \sum_{j=1}^{v} \lambda_j K_j; \sigma_1, \cdots, \sigma_r, \sigma_{r1}', \cdots, \sigma_{r1}', 1; 1\right) \] (4.11)

\[ \left\{ K_{\eta, \alpha}^{-\mu-1}(b - at)^{(-\delta_1)} \right\} \left( \begin{array}{c} c_1 t^{\lambda_1} (b - at)^{-\delta_1} \\ \vdots \\ c_v t^{\lambda_v} (b - at)^{-\delta_1} \end{array} \right) \left( \begin{array}{c} z_1 t^{\sigma_1} (b - at)^{-\omega_1} \\ \vdots \\ z_r t^{\sigma_r} (b - at)^{-\omega_r} \end{array} \right) \]

\[ \left( \begin{array}{c} z_1' \sigma_1 t^{\sigma_1} (b - at)^{-\omega_1'} \\ \vdots \\ z_r' \sigma_r t^{\sigma_r} (b - at)^{-\omega_r'} \end{array} \right) \]

\[ \left( x \right) = b^{-\nu} x^{\mu-1} \sum_{K_1=1}^{[N_1/M_1]} \cdots \sum_{K_v=1}^{[N_v/M_v]} a_v c_1 K_1 \cdots c_v K_v \]

\[ \left( \begin{array}{c} z_1 \frac{\pi_1}{\beta_1} \\ \vdots \\ z_r \frac{\pi_r}{\beta_r} \\ z_1' \frac{\pi_1'}{\beta_1'} \\ \vdots \\ z_r' \frac{\pi_r'}{\beta_r'} \end{array} \right) \left( \begin{array}{c} A, B, B_8 : A \\ 0, 1, 1 \\ 0, 1 \\ 0, 1 \\ 0, 1 \end{array} \right) \] (4.12)

Provided

\[ a, b, \alpha, \beta, \eta, \mu, \delta_k, \omega_i, \omega_j', \in \mathbb{C}, k = 1, \cdots, v; i = 1, \cdots, r; j = 1, \cdots, s \]

\[ \lambda_k, \sigma_i, \sigma_j' > 0; k = 1, \cdots, v; i = 1, \cdots, r; j = 1, \cdots, s \]
under the same existence conditions that equation (3.20) with $\beta = 0$.

Remark: By the similar procedure, the results of this document can be extended to the product of any finite number of multivariable Gimel-functions and a class of multivariable polynomials defined by Srivastava [30].

Agarwal [1,2] has studied the fractional integration about the multivariable H-function.

V. Conclusion

In this paper, we have obtained several theorems of the generalized fractional integral operators given by Saigo-Maeda and Saigo. The images have been developed regarding the product of the two multivariable Gimel-functions and a general class of multivariable polynomials in a compact and elegant form with the help of Saigo-Maeda and Saigo operators. Most of the results obtained in this paper are useful in deriving the composition formulae involving Riemann–Liouville, Erdelyi–Kober fractional calculus operators and multivariable Gimel functions. The findings of this paper provide an extension of the results given earlier by Kilbas [12], Kilbas and Saigo [13], Kilbas and Sebastian [14], Saxena et al.[29] and Gupta et al.[7] as mentioned earlier.

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