Factorization of linear partial differential operators and Darboux integrability of nonlinear PDEs

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Abstract

Using a new definition of generalized divisors we prove that the lattice of such divisors for a given linear partial differential operator is modular and obtain analogues of the well-known theorems of the Loewy-Ore theory of factorization of linear ordinary differential operators. Possible applications to factorized Gröbner bases computations in the commutative and non-commutative cases are discussed, an application to finding criterions of Darboux integrability of nonlinear PDEs is given.

1 Introduction

Factorization is often used for simplification of solution procedures for polynomials (factorized Gröbner bases computations) and linear ordinary differential operators (LODO). It is well-known that every (multivariate) polynomial factors into product of irreducible polynomials (in the given coefficient field) in a unique way; for LODO an analogous result had been proved by E.Landau and in a more precise form by A.Loewy [16, 17] and in a more generalization of the notion of a factor (divisor) for LPDO.

For the non-commutative ring of LODO (with coefficients in some differential field, for simplicity we will suppose that coefficients belong to \( \mathbb{Q}(x) \) i.e. they are rational functions with arbitrary algebraic number coefficients) one can use the Euclid division algorithm to prove that any left or right ideal in this ring is principal ideal, we obtain an extension of the notion of a divisor and this suffices (cf. for example [11, Ch.12]) to obtain uniqueness of decomposition of any algebraic integer (i.e. of the principal ideal it generates) into product of prime “ideal” divisors.

For the non-commutative ring of LODO (with coefficients in some differential field, for simplicity we will suppose that coefficients belong to \( \mathbb{Q}(x) \) i.e. they are rational functions with arbitrary algebraic number coefficients) one can use the Euclid division algorithm to prove that any left or right ideal in this ring is principal ideal; there are no nontrivial two-sided ideals. So there is no possibility (and necessity) of “ideal” generalization of the notion of divisors, the only implication of non-commutativity results in “similarity” of factors in different factorizations of a given LODO.

Another well-known “unique factorization” theorem is the classical Jordan-Hölder theorem in the theory of finite groups (or finitely generated modules).

In the first half of the XX century a common approach to these (and many more) cases was proposed. Let us introduce the obvious partial order in the set of (left) ideals: \( I_1 \sqsubseteq I_2 \) if \( I_1 \supseteq I_2 \); we call \( I_1 \) a divisor of \( I_2 \) in such a case. Then instead of factorizations

\[
L = L_1 \circ \ldots \circ L_k
\]

and references therein), applications to the differential Galois group computation were given in [13].

Unfortunately very little is known about factorization properties of linear partial differential operators (LPDO). The following interesting example was given by E.Landau (see [1]):

\[
P = D_x + xD_y, \quad Q = D_x + 1, \quad R = D_y^2 + xD_xD_y + D_x + (2 + x)D_y,
\]

then \( L = Q \circ Q \circ P = R \circ Q \). On the other hand the operator \( R \) is absolutely irreducible, i.e. one can not factor it into product of first-order operators with coefficients in any extension of \( \mathbb{Q}(x,y) \).

This example shows that in order to develop a “good” theory of factorization of LPDO one shall try to use some generalization of the notion of a factor (divisor) for LPDO. Such tricks are very common in the commutative case, as the first example we may cite Kummer-Dedekind theory of divisors for algebraic number rings. As proposed by Dedekind we may use ideals of the ring of algebraic integers of a given (finite) extension of \( \mathbb{Q} \). Since not all ideals in this ring are principal ideals (i.e. they are not generated as multiples of a single element) we obtain an extension of the notion of a divisor and this suffices (cf. for example [11, Ch.12]) to obtain uniqueness of decomposition of any algebraic integer (i.e. of the principal ideal it generates) into product of prime “ideal” divisors.

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of an element $L$ of the ring we will consider chains $|L| > |L_2 \circ \cdots \circ L_k| > |L_3 \circ \cdots \circ L_k| > \cdots > |L_k| > 0 = |1|$ of corresponding (left) principal ideals. Irreducibility of factors corresponds to maximality of this chain, i.e. impossibility to insert intermediate ideals between any two elements of the chain. The partially ordered set $M$ (also called poset) of ideals in the cited above “good” cases has the following two fundamental properties:

a) for any two elements $A, B \in M$ one can find a unique $C = \text{sup}(A, B)$, i.e. such $C$ that $C \geq A, C \geq B$, and $\forall X \in M, (X \geq A, X \geq B) \Rightarrow X \geq C$. Analogously there exist a unique $D = \text{inf}(A, B)$. $D \leq A, D \leq B, \forall X \in M, (X \leq A, X \leq B) \Rightarrow X \leq D$. Such posets are called lattices, $\text{sup}(A, B)$ and $\text{inf}(A, B)$ correspond to the least common multiple and the greatest common divisor for the cases of number rings and LODO. In our cases a lattice will always have zero i.e. an element $0 \in M$ such that $\forall X \in M, X \geq 0$. For simplicity (and following the established tradition) $\text{sup}(A, B)$ will be hereafter denoted as $A + B$ and $\text{inf}(A, B)$— as $A \cdot B$;

b) For any three $A, B, C \in M$ the following modular identity holds:

$$(A \cdot C + B) \cdot C = A \cdot C + B \cdot C$$

(3)

This weaker form of distributivity was discovered by Dedekind. The theory of modular lattices (i.e. posets with the above two properties, such posets are also called “Dedekind structures”) has beautiful (for our purpose :) results. Namely these two simple properties are sufficient to prove the following four elegant theorems (cf. [1, 11, 12]):

**Theorem 1** (Jordan-Hölder-Dedekind chain condition) Any two finite maximal chains

$$L > L_1 > \cdots > L_k > 0 \quad (4)$$

for a given $L \in M$ have equal length: $k = r$.

**Theorem 2** (Kurosh & Ore) If $L = L_1 + L_2 + \cdots + L_p = L_{1_1} + L_{1_2} + \cdots + L_{1_r} = \text{sup}(L_1, \text{sup}(L_2, \cdots, \text{sup}(L_r, L_r)))$ and each $L_i$ can not be sup-represented: $L_i \neq A_1 + B_1$ for $A_1 \neq L_1, B_1 \neq L_1$, we call a sup-representation (1) a direct sum if the set $\{L_i\}$ is independent that is $\forall i, (L_1 + \cdots + L_{i-1} + L_{i+1} + \cdots + L_k) \neq L_i$ and each $L_i$ can not be represented as a sum of two independent elements. Direct sums are denoted $L = L_1 \oplus L_2 \oplus \cdots \oplus L_k$. An element $A \in M$ is called indecomposable if $A \neq B \cdot C, B \neq 0, C \neq 0$.

**Theorem 3** (Ore) Let an element $L$ of a modular lattice have finite maximal chains (1) and $L = L_1 \oplus \cdots \oplus L_k = M_1 \oplus \cdots \oplus M_j$, with indecomposable $L_1, M_j$. Then $r = r$ and $\forall L_0$ one can find $M_j$ such that $L = L_1 \oplus \cdots \oplus L_{i-1} \oplus M_1 \oplus \cdots \oplus L_{i-1} \oplus L_{i+1} \oplus \cdots \oplus L_p$.

We let call $l(L) := k + 1$ the length of $L \in M$ if $L$ has a finite maximal chain (1) (of length $k + 1$). We set $l(0) = 0$. The length of a LODO is equal to the number of irreducible factors in decomposition (2).

**Theorem 4** If all elements of a modular $M$ have finite length then $l(A + B) + l(A \cdot B) = l(A) + l(B)$.

These theorems give a unified approach to many well known facts in the theory of groups (and group representations), commutative and non-commutative rings; in particular they encompass many results of the Loevey-Ore theory of factorization of LODO.

Let us prove here for completeness that the poset of (left) ideals of a (non-commutative) ring is a modular lattice. Firstly we notice that $\text{sup}(A, B) = A + B$ corresponds to the intersection of ideals $A, B$; $\text{inf}(A, B) = A \cdot B$ corresponds to the ideal composed of the sums $a + b, a \in A, b \in B$. Then if $x \in A \cdot C + B \cdot C$ (the r.h.s of (1)) then $x \in C, x = a + b, a \in A, c \in C, b \in B$. Obviously $a + b \in C$, $a + b \in A \cdot C + B$, so $A \cdot C + B \cdot C \subset (A \cdot C + B) \cdot C$. Vice versa if $x \in (A \cdot C + B) \cdot C$ then $x \in C$, $x = a + b$, $a \in A, a \in C, b \in B \Rightarrow b = x - a \in C$ so $b \in B \cdot C$ and $x \in A \cdot C + B \cdot C$ which proves $(A \cdot C + B) \cdot C \subset C \cdot A \cdot C + B \cdot C$.

The basic notion of similarity also exists for modular lattices:

**Definition 1** Two elements $A, B$ of a modular lattice $M$ are called similar if one can find $C \in M$ such that $A \cdot C = B \cdot C = 0$ and $A + C = B + C$ (i.e. $A \oplus C = B \oplus C$).

We will need also the notion of similarity of intervals or quotients $[B/A] := \{X \in M|A \leq X \leq B\}$ for pairs $A \leq B$.

**Proposition 1** If $A, B$ are elements of a modular lattice $M$ then the intervals $I_1 = [A/(A \cdot B)]$ and $I_2 = [A \cdot B/B]$ are projective, $I_1 \sim I_2$, i.e. isomorphic with specific poset isomorphisms $\psi : I_1 \rightarrow I_2, \psi(X) = X + B, \psi = \phi^{-1}$.

**Definition 2** Two intervals $[B_1/A_1], [B_2/A_2]$ are called similar if there exists a finite sequence of projective intervals $[B_1/A_1] \sim I_1 \sim I_2 \sim \cdots \sim I_k = [B_2/A_2]$.

One can prove that in Theorems [1, 12] the corresponding factors (intervals) are similar (in some transposed order). Similarity of intervals in Theorem [1] gives similarity of the respective irreducible factors in [1] or isomorphism of the factor groups (form modules) for the modular lattice of normal subgroups of a given (finite) group (resp. submodules). The case of the ring of LDPO is more complicated. It has no two-sided ideals and left (right) ideals are no longer principal ideals in the general case. Certainly the poset of all left (right) ideals is a modular lattice. But unfortunately we can not use the above results: for a LDPO $L$ we get finite chains (1) of left ideals (the ring of LDPO is Noetherian, see [1]) but the intervals in any chain are not (as a rule) “irreducible” i.e. one can always insert intermediate ideals between some of them so the length of chains (1) for a given $L$ is not bounded. For example for arbitrary LODO $L \in Q(x)D_x \subset Q(x, y)D_y$ we can take $|L| > |L| + |D_x| > |L| + |D_y| > |L| + |D_y| > 0 = Q(x,y)D_x \cup D_y$. Even the simplest $D_x$ becomes “reducible”! Similar infinite examples exist for decompositions into (direct) sup-sums. So Theorems [1, 12] are useless.

We conclude that the poset of all (left) ideals of LDPO is too “large”. For the commutative case of multivariate polynomials one can limit oneself to principal ideals and get the desired modular lattice with finite chains. Again for LDPO the poset of left (principal) ideals is too “small”: it does not form even a lattice. For example for the two operators $P, Q$ in (1) the intersection of the left principal ideals $\{P\} \cap \{Q\}$
their “LCM”) is no longer principal: one can easily check directly that there are no second-order common left multiples of both $P, Q$ but we have two linearly independent third-order operators divisible by $P, Q$:

$$L_{31} = (xD_x D_y + (x^2 - 1)D_y - D_x - 1) \ast P =$$

$$\left( x^2 D_y^2 + xD_y D_x - (x + 1)D_y - D_x \right) \ast Q,$$

$$L_{32} = (D_y^2 + 2D_x + 1) \ast P = Q \ast Q \ast P =$$

$$\left( D_y^2 + xD_x D_y + (x + 2)D_y + D_x \right) \ast Q = R \ast Q,$$

so there is no “least” common left multiple. Analogously we can directly check that these $L_{31}, L_{32}$ have only $Q, P$ as their common right divisors so $L_{31}, L_{32}$ have no “greatest” right common divisor. Also as the E. Landau’s example shows the Jordan-Hölder-Dedekind chain condition fails for principal ideals.

Below (section 3) we define an “intermediate” poset of “codimension 1” left ideals which is larger than the poset of principal left ideals but smaller than the lattice of all left ideals. This new poset of “generalized divisors” provides all the necessary properties: it is a modular lattice with finite maximal chains (for every element and finite decompositions into (direct) sup-sums so the basic Theorems 1–4 are applicable; any first-order LPDO is irreducible and any LODO $L$ irreducible in $Q[x][D_x]$ remains irreducible (as LPDO) in our poset. This is our main result.

For applications the most important property of our modular lattice of generalized divisors would be certainly the possibility to decompose operators into sup-sums in an overdetermined system of LPDO

$$\begin{align*}
L_1 f &= 0, \\
L_2 f &= 0, \\
\vdots \\
L_k f &= 0.
\end{align*}$$

(8)

Suppose that $L_1 = A_1, \ldots A_n$ for some left ideal divisors then each $A_i$ is finitely generated (see 3): $A_i = \langle L_1, \ldots, L_{i-1} \rangle$, we can decompose $\langle \rangle$ into union of systems

$$\begin{align*}
L_1 f &= 0, \\
L_{1,1} f &= 0, \\
L_2 f &= 0, \\
\vdots \\
L_k f &= 0.
\end{align*}$$

Sums of solutions of (9) are obviously solutions of (8) and we conjecture that they span the whole space of solutions of (8). Also we need an algorithm for such sup-decompositions of LPDO (see section 4 for the discussion).

Substitution of (8) with (9) is an analogue of the well-known factorization technique for commutative Gröbner bases computations. This technique considerably reduces the complexity of computations in many practical cases. The overdetermined systems of type (8) with one or many unknown functions are typical in many applications (13, 23): computation of conservation laws, symmetries and invariant solutions of systems and single nonlinear ODEs and PDEs. For any system (8) one may use the standard Janet-Riquier technique (13, 22, 22) of reduction of (8) to the so called passive (standard, normal) form. In the case of constant coefficient systems (8) (in the commutative case) this algorithm practically coincides with the Gröbner algorithm (for total degree+weight ordering). Unfortunately the complexity of Janet-Riquier algorithm is very high even for modest LPDO systems. Recently one interesting contribution to reduction of the complexity for computation of the genus (roughly speaking this is the “dimension” of the solution space) of (8) was given in (20). Our approach may help in decomposition of the solution space of (8) into “irreducible submanifolds”. Further possible generalizations and applications to the commutative case are discussed in section 5.

Another connection of our definition of factorization of LPDO and integrability properties of nonlinear PDEs is discussed in section 6; the established in (1, 14, 26) criterion of Darboux integrability (9) (“explicit” integrability) of such nonlinear PDEs is equivalent to generalized factorization of the corresponding linearized equation. This gives a new insight into possible generalizations of the notion of Darboux integrability of higher-order nonlinear PDEs which is now under investigation.

2 Divisor ideals of LPDO

We study general LPDO

$$L = \sum_{i=1}^{n} a_{i_1 \ldots i_n} (\vec{x}) D_{i_1}^{a_{i_1}} \ldots D_{i_n}^{a_{i_n}},$$

\[ \vec{a} = a_1 + \ldots + a_n, \quad \vec{x} = (x_1, \ldots, x_n), \quad D_{i_1} = \partial / \partial x_{i_1}, \quad \text{ord}(L) := m. \]

For simplicity and without loss of generality we will suppose that the number of the independent variables is $n = 2, x := x_1, y := x_2$, and the coefficients $a_{ij} (x, y)$ in (9) are rational functions with rational coefficients, $a_{ij} (x, y) \in Q(x, y)$, so $L \in Q(x, y)[D_y, D_x]$.

It is straightforward to check that for every finite set of LPDO $L_1, \ldots, L_k$ one may algorithmically find all their common left multiples (left c.m.) up to fixed order $N$: take $M_1 \circ L_1 = \ldots = M_k \circ L_k$ with $\text{ord}(M_i) = N - \text{ord}(L_i)$ and indefinite coefficients, then we get a linear algebraic (not differential!) system for the coefficients of $M_i$; the number of equations in this linear system will be less than the number of the unknown coefficients for sufficiently large $N$, so the set of left (right) c.m. is always nonempty.

All these and subsequent results are certainly invariant w.r.t. substitution of left ideal with right ideals; application of the usual adjoint operation will suffice for this purpose. We will denote the left (right) principal ideal generated by LPDO $L$ with $\langle L \rangle$ (resp. $\langle L \rangle$).

Definition 3 The left LPDO ideal $\text{lLCM}(\langle L_1 \rangle, \ldots, \langle L_k \rangle) := \langle L_1 \rangle \cap \ldots \cap \langle L_k \rangle$ is called the left least common multiple of LPDO $L_i$.

This lLCM is always non-empty and (see Introduction) not principal in the general case.

Definition 4 The same ideal $\langle L_1 \rangle \cap \ldots \cap \langle L_k \rangle$ will be also called the left greatest common divisor of $L_i$.

Remark This is serious :-) Below the reader will see that this is the key to the whole trick.

Definition 5 We call two LPDO $L, R$ a (generalized) divisor operator couple for LPDO $M$ if there exist LPDO $X, Y, Q$ such that

$$X \circ M = Y \circ R,$$

$$X \circ L = Y \circ Q.$$
One may informally think that $M \cong L \circ R$ and if in fact $M$ factors into the product of $L$, $R$ then we may choose $X = Q = 1$, $Y = L$ in (1).

A divisor operator couple is called nontrivial if $\text{ord}(L) > 0$, $\text{ord}(R) > 0$ and $L$, $R$ are not divisible by $M$, i.e. $L \not\equiv M \circ P$, $R \not\equiv K \circ M$. In this case we will say that the operators $M$ and $R$ ($M$ and $L$) have nontrivial (generalized) right (resp. left) common divisor.

**Remark.** These definitions actually say that we can restore the (greatest) common divisor if we can find (least) common multiples; for the case of LODO if $Z = \text{ILCM}(M, R) = X \circ M = Y \circ R$ and $V = \text{rcmd}(X, Y) = X \circ L = Y \circ Q$ then $M = L \circ Q$, $R = Q \circ G$, $G = \text{rcmd}(M, R)$. This explains Definition 3. For integral domains (commutative rings with 1 and no zero divisors) if two elements have LCM (i.e. some common multiple such that this c.m. divides any other c.m.) then they automatically have GCD (i.e. some common divisor such that this divisor is divided by any other common divisor); the converse is not true in general; also $\text{LCM}(ac, bc) = c \cdot \text{LCM}(a, b)$ when one of them exist, this is again not true for GCD (the author thanks Dr. N.N. Osipov who communicated to him these facts for integral domains).

**Lemma 1** If (1) holds and for some $X_1$, $Y_1$ we have $X_1 \circ M = Y_1 \circ R$, then $X_1 \circ L = Y_1 \circ Q$.

**Proof.** Let us find some left c.m. of $X_1$, $X_1 = \tilde{X} \circ X_1$. Then $\tilde{X} \circ X_1 \circ M = \tilde{X} \circ Y_1 \circ R = \tilde{X} \circ X_1 \circ M = X_1 \circ Y \circ R$ so (since the ring of LPDO has no zero divisors) $\tilde{X} \circ X_1 \circ Y = \tilde{X} \circ Y_1 \circ Q$ and $X_1 \circ X \circ L = Y_1 \circ Y_1 \circ Q$ hence $X_1 \circ L = Y_1 \circ Q$. So (1) does not depend on the choice of the ILCM$(M, R) = X \circ M = Y \circ R$.

**Lemma 2** If (1) holds and some right c.m. of $M$, $L$ is chosen $M \circ \tilde{X} = L \circ \tilde{Y}$ then $M \circ \tilde{X} = L \circ \tilde{Y}$.

**Proof.** $Y \circ Q \circ \tilde{X} = X \circ L \circ \tilde{Y} = X \circ M \circ \tilde{X} = Y \circ R \circ \tilde{X} \Rightarrow Q = Y = \tilde{X}$.

From (1) and Lemma 1 we conclude that the set of operators $L$ forming a generalized divisor couple with fixed $R$, $M$ is a right ideal $\{L\}$; from (1) we see that for fixed $M$, $L$ operators $R$ form a left ideal $\{R\}$.

**Definition 6** Left ideal $\{R\}$ and right ideal $\{L\}$ form a (generalized) divisor ideal couple for an operator $M$ (we denote this fact as $\{M|L|R\}$) if:

a) any $R \in \{R\}$, $L \in \{L\}$ form a divisor operator couple for $M$ i.e. (1) holds;

b) if some LPDO $L$ forms divisor operator couples for $M$ with every $R \in \{R\}$ then $L \in \{L\}$;

c) if some LPDO $R$ forms divisor operator couples for $M$ with every $L \in \{L\}$ then $R \in \{R\}$.

**Lemma 3** Let $\{L|M|R\}$ be a divisor ideal couple for $M$. Then for every $M_1 \in \{R\}$ we can find a unique right ideal $\{Q_1\}$ such that $\{Q_1|M_1|R\}$.

**Proof.** Since $M_1 \in \{R\}$ then for every $L \in \{L\}$ we have the unique (Lemma 1) $Q_1$ such that $3X_1Y_1$, $X_1 \circ M = Y_1 \circ M_1$, $X_1 \circ L = Y_1 \circ Q_1$.

Take some left c.m. of $M_1$, $R$ for $R \in \{R\}$: $Z_1 = X_1 \circ M = Y_1 \circ M_1$, and some left c.m. of $Y_1$, $M_1$: $Z_2 = Y_1 \circ M_1 = X_1 \circ Y_1 \circ M_1$. Then $\tilde{X}_1 \circ Y_1 \circ M = \tilde{X}_1 \circ Y_1 \circ M_1 = \tilde{Y}_1 \circ M_1 \circ R$, so we get $\tilde{X}_1 \circ Y_1 \circ M = \tilde{Y}_1 \circ Y_1 \circ M_1$. Using Lemma 1 we conclude $\tilde{X}_1 \circ Y_1 \circ L = \tilde{Y}_1 \circ Y_1 \circ Q_1 = \tilde{Y}_1 \circ X_1 \circ Q_1$. Consequently $\tilde{Y}_1 \circ Y_1 \circ Q = \tilde{X}_1 \circ Y_1 \circ L = \tilde{X}_1 \circ Y_1 \circ Q_1 = \tilde{Y}_1 \circ X_1 \circ Q_1$. Canceling $\tilde{Y}_1$ we get $Y_1 \circ Q = X_1 \circ Q_1$ and finally

$$X_1 \circ Q_1 = Y_1 \circ Q_1 \circ R, \quad X_1 \circ Q_1 = Y_1 \circ Q_1 \circ Q,$$

(12)

which shows that any $Q_1$ in (1) (it depends on $L \in \{L\}$) form a divisor operator couple for $M_1$ with any $R \in \{R\}$ so the condition a) of Definition 4 holds. In order to prove the condition c) it of $\{M_1|L|R\}$ we suppose (12) to be true for some LPDO $Q_1$ and all $Q_1$ obtained from $\text{ILCM}(M_1, R)$ with $L \in \{L\}$. Fixing $X_1$, $Y_1$, $L$, $Q$ and using the same definition of $\tilde{X}_1$, $\tilde{Y}_1$, $\tilde{Q}$, $\tilde{X}_1 \circ Y_1 \circ M = \tilde{X}_1 \circ Y_1 \circ Q = \tilde{X}_1 \circ X_1 \circ L = \tilde{X}_1 \circ Y_1 \circ Q$ hence $\tilde{X}_1 \circ L = Y_1 \circ Q$.

So (1) does not depend on the choice of the ILCM$(M, R) = X \circ M = Y \circ R$.

In this case we see from (11) that actually we may set $\tilde{X}_1 = Y_1$, $\tilde{Y}_1 = X_1$, $L := \tilde{Q}$. So we have (11), (12) with $R \in \{R\}$ and we shall prove (12) for the constructed $L$, $Q$, $M$, $R$. Again $Z_2 = Y_1 \circ M_1 = \tilde{X}_1 \circ Y_1 \circ M_1$, $X_1 \circ Y_1 \circ M_1 = \tilde{X}_1 \circ Y_1 \circ Q_1 = \tilde{X}_1 \circ Y_1 \circ Q_1 = \tilde{X}_1 \circ X_1 \circ Q_1$.

Lemma 2 essentially says that the right parts of $\{L|M|R\}$ are internally characterizable as some special left ideals. The same is true for $\{L\}$. We will call such $\{R\}$ right divisor ideals or r.d.i. (they are left ideals of the ring of LPDO $-$ and $\{L\}$ — left divisor ideals or l.d.i. (they are right ideals). Any principal left ideal is a r.d.i.; $\{R_0\}$ with $R_0$ is obviously we have $\{R_0\}$ with $R = P = Q = \circ L$, $L$ being arbitrary LPDO.

A divisor ideal couple $\{L|M|R\}$ is called trivial if either $\{L = \{M (\text{then } R) = \circ |) \}$ or $|) = \{M\}$, $\{L\} = \circ$.

On the other hand as we will see in the next section there are divisor ideals which are not principal and not every (left) ideal is r.d.i.

Let us now prove that the set of r.d.i. with the natural ordering $\{R_1\} \geq \{R_2\}$ iff $\{R_1\} \subset \{R_2\}$ forms a lattice. Namely for two r.d.i. $\{R_1\}$, $\{R_2\}$ we take their intersection in the ILCM $(ILCM(R_1), \{R_2\}) = \sup(\{R_1\}, \{R_2\}) := (R_1 \cap \{R_2\})$. Then for $M \in \{R_1\} \cap \{R_2\}$ we find two corresponding l.d.i. (Lemma 3) $\{L_1|\{M\}|R_1\}$, $\{L_2|\{M\}|R_2\}$. Now let us take all LPDO $L$ such that (13) holds for every $R \in \{R_1\} \cap \{R_2\}$.

The right ideal $\{L\}$ of such operators forms the divisor ideal couple with $\{R_1\} = \{R_2\} \cap \{R_1\}$ for $M$ since a) and b) in Definition 4 hold automatically and $\{L\} \supset \{L_1\} \cup \{L_2\}$ so every operator $R$ such that (13) holds for all $L \in \{L\}$ forms a divisor operator couple with
all \( L \in \{ L_1 \} \) and \( L \in \{ L_2 \} \) so \( R \in [R_1] \cap [R_2] \) and the condition (c) holds. As we will see in the next section the constructed \( \{ L \} \) is in general greater than the set of all sums of elements of \( \{ L_1 \} \) and \( \{ L_2 \} \). Obviously this \( \{ L \} \) plays the role of IGCD(\( \{ L_1 \}, \{ L_2 \} \) = \( \{ L_1 \}, \{ L_2 \} \) in the poset of all \( d.i. \). The determination of \( \{ R \} = \text{rIGCD}(\{ R_1 \}, \{ R_2 \}) \) := inf(\( \{ R_1 \}, \{ R_2 \} \)) := \( \{ R_1 \}, \{ R_2 \} \) is obtained in the same way: we now take \( \{ L \} = \text{rLCM}(\{ L_1 \}, \{ L_2 \}) := \{ L_1 \}, \{ L_2 \} \) and the corresponding \( \{ R \} \supset \{ R_1 \} \cup \{ R_2 \} \) is defined using (4). So our lattice of \( d.i. \) does not form a sublattice of the lattice of all right ideals of \( LPDO \), it changes inf; such subsets are called “meet-sublattices”.

3 Coordination of divisor ideals

Any (non-commutative) ring \( R \) satisfying the so-called Ore condition (absence of zero divisors and existence of at least one common multiple for every two non-zero elements) may be imbedded into a skew field (non-commutative ring with division) built with formal quotients \( \frac{L_{-1}}{M} = \frac{L}{M} \). Let \( N_{\infty} \leq L, M, N \in R \) (\( \mathbb{R} \)). Let us take \( R = Q(x, y)[D_x, D_y] \) and the corresponding skew field \( Q(x, y)[D_x, D_y] \). We can form a ring \( Q[x, y, D_x] \) of operators of the type \( L = axD_x^n + \ldots + \alpha D_x + M \in Q(x, y)[D_x] \) if we define the corresponding \( D_y \)-differentiation for the coefficients:

\[
\begin{align*}
\frac{d(L_1^{-1} \circ M_1)}{dy} &= (L_1^{-1} \circ \partial M_1)/\partial y - (L_1^{-1} \circ L_1')/\partial y + (\partial M_1)/\partial y \\
&= (\partial M_1(x, y))/\partial y - (\partial M_1(x, y))/\partial y + (\partial M_1(x, y))/\partial y.
\end{align*}
\]

Let \( M = Q(x, y)[D_x, D_y] \) and \( M = Q(x, y)[D_x, D_y] \). For any left ideal \( I \subset Q(x, y)[D_x, D_y] \), we obtain two principal ideals \( P_x(I) = I_x \). Their generators \( I_x = I_y \) are called coordinates of \( I \). In \( Q(x, y)[D_x, D_y] \), \( I_y \) is called a left ideal in \( Q(x, y)[D_x, D_y] \). In \( Q(x, y)[D_x, D_y] \), we also normalize them \( I_{D_x}(I_y) = 1 \). The following lemma plays the role of the main key in the subsequent proofs.

Lemma 4 If a r.d.i. \( [R] \subset Q(x, y)[D_x, D_y] \) contains two elements \( A \circ P, B \circ P \) such that \( A \in Q(x, y)[D_x] \), \( B \in Q(x, y)[D_y] \) then \( P \in [R] \).

Proof. Using (10) we obtain \( A \circ P \supset \overline{Q} = Q \circ \overline{Y} \). B \circ P \circ \overline{X} = S \circ \overline{Y} \) for some \( S, Q \). Then \( P \circ \overline{X} = A^{-1} \circ Q \circ \overline{Y} \in Q(x, y)[D_x, D_y] \), \( B \circ P \circ \overline{X} = B \circ A^{-1} \circ Q \circ \overline{Y} = S \circ \overline{Y} \Rightarrow \)

\[
B \circ A^{-1} \circ Q = S \in Q(x, y)[D_x, D_y],
\]

for \( B \in Q(x, y)[D_x, D_y] \). We will prove that in such circumstances \( Q \) is divisible by \( A : Q = A \circ Q, \overline{Q} \in Q(x, y)[D_x, D_y] \). Without loss of generality we may suppose \( \text{lof}(B) = 1 \) so (13) reads

\[
(D^n_0 + b_1(x, y)D^{n-1}_0 + \ldots + b_n(x, y)) \circ \overline{Q} = (C_1D^n_0 + C_2D^{n-1}_0 + \ldots + C_n) = S,
\]

\[
A^{-1} \circ Q
\]

with \( C \in Q(x, y, D_y) \). The leading coefficient of the l.h.s. \( (Q(x, y, D_y)[D_x, D_y] \) of (16) is \( C_0 \) so since \( S \in Q(x, y)[D_x, D_y] \), \( C_0 \in Q(x, y)[D_x, D_y] \). Then the coefficient of \( D^n_0 + n - 1 \) in (16) will be \( C_1 + \partial C_0/\partial y + b_1 C_0 \in Q(x, y)[D_x, D_y] \) so also \( C_1 \in Q(x, y)[D_x, D_y] \). Using induction we get \( C_1 \in Q(x, y)[D_x, D_y] \) so \( A^{-1} \circ Q = \overline{Q} \in Q(x, y)[D_x, D_y] \). This gives us the possibility to cancel \( A \circ \overline{X} = Q \circ \overline{Y} \). Then the operator \( P \in [R] \) since \( P \circ \overline{X} = (P \circ Q) \circ \overline{Y} \) in (10). \( \square \)

Remark. Actually we used only the fact \( \text{lof}(D_y)(B) = 1 \).

Corollary 1 If a r.d.i. \( [R] \) contains two elements \( A, B \), \( A \in Q(x, y)[D_x], B \in Q(x, y)[D_y] \) then \( [R] \) is trivial, \( [R] = 1 \) if \( Q \in Q(y)[D_x, D_y] \).

This Corollary explains why r.d.i. are “codimension 1” ideals: if we will take a “codimension 2” left ideal generated by \( D^n_0, D^{n-1}_0 \) (solutions of the corresponding system \( D^n_0 f = 0, D^{n-1}_0 f = 0 \) are functions of 2 variables parameterized by several constants and not by functions of 1 variable), it is contained only in the trivial r.d.i. \( [1] \). The set of divisor ideals is a bit larger than the set of principal ideals (which are obviously “codimension 1” ideals — solutions of the corresponding system of 1 equation are parameterized by functions of 1 variable).

Theorem 5 Two r.d.i. \( [R_1], [R_2] \), coincide if their coordinates coincide i.e. if \( P_x([R_1]) = P_x([R_2]), P_y([R_1]) = P_y([R_2]) \).

Proof. Let some \( R \in [R_1] \). Since \( P_x([R_1]) = P_x([R_2]) \) we may find \( \overline{R} \in [R_2] \), such that \( C \circ R = \overline{R}, C \in (C_1) \), \( C_2 \in Q(x, y, D_y) \). Multiplying with \( C_1 \) we get \( C_2 \circ R = C_1 \circ R = \overline{R} \in [R_2] \). Hence for some \( C_2 \in Q(x, y)[D_y] \) we have \( C_2 \circ R \in [R_2] \). Analogous considerations w.r.t. \( P_y \) give \( R \in [R_2] \) for \( C_2 \in Q(x, y)[D_y] \), thus \( R \in [R_2] \) (Lemma 3) hence \( R \in [R_2] \) (Lemma 3) hence \( R \in [R_1] \).

Remark. This is obviously not true for arbitrary ideals: the ideal generated by \( D_x, D_y \) have the same projections as the trivial [1].

Corollary 2 The lattice of r.d.i. (l.d.i.) is modular.

Proof. Since the modular identity (5) holds for projections due to modularity of the lattice of the (principal) left ideals of \( Q(x, y, D_y)[D_x] \), \( Q(x, y, D_y)[D_x] \), and GCD, LCM are preserved after projections, we conclude \( P_x(A \cdot C + B \cdot C) = P_x(A \cdot C + B \cdot C) = P_y((A \cdot C + B) \cdot C) = P_y((A \cdot C + B) \cdot C) \) (so \( (A \cdot C + B) \cdot C = A \cdot C + B \cdot C \)).
Corollary 3 For any r.d.i. \( |R| \) the length \( k + 1 \) of a chain of r.d.i. \( |R| > |R_1| > \ldots > |R_k| > 0 \) is limited: \( k + 1 \leq \text{ord}_{D_x}(|R_1|) + \text{ord}_{D_y}(|R_2|) \).

Proof. Since \( \mathcal{P}_x(|R_1|), \mathcal{P}_y(|R_1|) \) give chains of divisors of \( \mathcal{P}_x(R_1), \mathcal{P}_y(R_1) \) in \( Q(x,y,D_x)[D_y] \) (resp. \( Q(x,y,D_y)[D_x] \)), the adjacent elements of \( \mathcal{P}_x \)-projected chain may differ only in \( \text{ord}_{D_x}(|R_1|) \) places (resp. in \( \text{ord}_{D_y}(|R_2|) \) places).

Analogous result is true for (direct) sup-sums of r.d.i.

Thus Theorems 3 and 3 are applicable to the constructed lattice of r.d.i. (l.d.i.) of a given LPDO.

Proposition 2 Any first order LPDO \( R \) is reducible (i.e. it has no nontrivial divisor ideal couples).

Proof. Let \( R = r_1(x,y)D_x + r_2(x,y)D_y + r_3(y,x) \) but nevertheless we have some r.d.i. \( |R_1|, |R_2| > |R_3| > 0 \). Necessary condition that \( \text{ord}_{D_x}(|R_1|) = \text{ord}_{D_y}(|R_2|) = 0 \) (up to transposition \( x \leftrightarrow y \)), there exist \( R_{1,1} = R_{1,2} = R_{3,1} = K_1 \in |R_1|, K_1 \in Q(x,y)[D_y] \).

Since \( |R| \subset |R_1|, R_{1,3} = R_{1,1} - \frac{K_1}{|R_2|} = K_2 \in |R_2|, K_2 \in Q(x,y)[D_y] \). Hence \( |R_1| + |R_2| + |R_3| = 0 \Rightarrow Q(x,y)[D_y] \).

Corollary 3 Any LODO \( M \subset Q(x,y)[D_x] \) irreducible in this ring is irreducible as an element of \( Q(x,y)[D_x,D_y] \).

Proof. Suppose we have a nontrivial divisor ideal couple \( \{L,M,R\} \) so for some \( L \in R \) we have \( |L| > |R| \). But then \( \text{ord}_{D_x}(|L|) = \text{ord}_{D_y}(|R|) = 0 \).

\( \text{ord}_{D_x}(|L|) = \text{ord}_{D_y}(|R|) = 0 \) if we will find \( \text{ord}_{D_x}(|L|) = \text{ord}_{D_y}(|R|) = 0 \).

Thus Theorems 1–4 are applicable to the constructed order LPDO \( L \) factors in the “usual” sense and the solutions of (18) may be found via quadratures. If both \( H, K \) vanish, \( L \) is a LCM of two first-order LPDO. If both \( H, K \) do not vanish, one can apply the two Laplace transformations: \( L - L_1, L - L_{-1} \), using the substitutions

\[ v_1 = (D_y - A)v, \quad v_{-1} = (D_x - B)v \]

These (invertible) transformations give two new second order LPDO \( L_1, L_{-1} \) of the same form with different coefficients iff \( H \neq 0 \) (resp. \( K \neq 0 \)). In the generic case one obtains two infinite sequences

\[ L \rightarrow L_1 \rightarrow L_2 \rightarrow \cdots, \quad L - L_1 \rightarrow L_{-1} \rightarrow L_{-2} \rightarrow \cdots \]

If one of these sequences is finite (i.e. the corresponding Laplace invariant vanishes and the Laplace transform cannot be applied once more) then the final LPDO \( L_i \) is trivially factorable.

One shall certainly take into consideration the original equation (17) performing all the computations of the Laplace invariants and Laplace transforms (which allows us to express all the mixed derivatives of \( u \) via \( x, y, u, \) and the non-mixed \( u_{x}, u_{y}, u_{x,y} \)).

Theorem 6 (1, 5, 15, 25) A second order, scalar, hyperbolic partial differential equation (14) is Darboux integrable if and only if both Laplace sequences are finite.

In [13, 14, 25] this method was also generalized for the case of a general second-order nonlinear PDE

\[ F(x,y,u,u_x,u_y,u_{xx},u_{xy},u_{yy}) = 0 \]

What can be said about factorizability of the operator \( L \) in our generalized sense?
Theorem 8 \[ L = \mathcal{D}_x \circ \mathcal{D}_y - a(x,y)\mathcal{D}_x - b(x,y)\mathcal{D}_a - c(x,y) \] is an LCM of two generalized right divisor ideals iff both Laplace sequences are finite.

The detailed proofs will be given elsewhere.

In fact these theorems demonstrate again that the "generalized" factorization introduced here enjoys the necessary natural properties: it is invariant w.r.t. the differential substitutions (\[\mathcal{D}_x\]) which destroy the "trivial" factorizations \[ L = (\mathcal{D}_x - B) \circ (\mathcal{D}_y - A) \text{ or } L = (\mathcal{D}_y - A) \circ (\mathcal{D}_x - B). \]

5 Conclusion

An obvious and important generalization of our definition of reducibility (or sup-decomposition) of single LPDO would be a proper definition of decomposition of systems of LPDO \[\mathcal{M}.\] Actually a formal generalization should be formulated inductively; for example if a system of 2 equations has indecomposable first equation \[L_{1} \mathcal{M} f = 0\] then we may try to find (generalized) divisor operator couples for the second LPDO \[L_{2}\] (i.e. the second equation \[L_{2} \mathcal{M} f = 0\]) forming for \[M = L_{2}\] equations (\[\mathcal{M}\]) modulo the left principal ideal generated by \[L_{1}.\]

The problem of zero-divisors (actually any LPDO is zero divisor now since it always has a multiple which belongs to the ideal \(L_{1} \mathcal{M}\)) is (apparently) solved using the fact that for non-decomposable \(L_{1}\) there are no "LCM-zero divisors", i.e. if \(M \notin L_{1} \mathcal{M}, R \notin L_{1} \mathcal{M}\) then some their right c.m. \(M \mathcal{M} \mathcal{D} X = R \mathcal{M} \mathcal{D} Y \notin L_{1} \mathcal{M}\). In fact our proofs rely only on absence of "LCM-zero divisors". Certainly, this approach deserves further thorough study in another publication. Especially interesting is the possibility to apply such a generalization to the commutative case (factorized Gröbner bases).

It would be interesting to compare our definitions of decomposition of ideals with the known results on decomposition of ideals in non-commutative rings with the Ore condition (existence of at least one common multiples for every two elements) \[\mathcal{M}.\]

A more challenging generalization is required for treatment of overdetermined linear partial differential systems with several unknown functions \(f_{k}\).

The algebraic nature of the set of operators \(\{Q\}\) in \[\mathcal{M}\] also is of interest: we can multiply \(Q\) on the left and on the right with arbitrary LPDO, but addition of different \(Q\) is doubly "stratified": only \(Q\) which belong to a fixed \(R\) or a fixed \(L\) may be added. As we have seen in the proof of Lemma \[\mathcal{M}\] each "stratum" of \(\{Q\}\) is actually some \(\mathcal{M} R\) (resp. \(\mathcal{M} L\)).

An algorithm of computation of divisor ideals for a given LPDO would be of big practical interest. As we have explained in Introduction an algorithm for sup-decompositions is much more important for applications (Theorems \[\mathcal{M}\] \[\mathcal{M}\]. One possible approach for algorithmization of sup-decompositions may mimic the methods of \[\mathcal{M}\]. For this purpose we have to generalize the eigering algorithm of \[\mathcal{M}\] to the case of skew differential fields of coefficients \(Q(x,y,\mathcal{D}_x)\) with greater (not algebraically closed) constant subfield \(Q(\mathcal{D}_x)\). Another approach to reducibility testing may rely on possible generalization of estimates of complexity of coefficients of factors given in \[\mathcal{M}\] for commutative coefficient fields to the case of the ring \(Q(x,y,\mathcal{D}_x)[\mathcal{D}_y]\). These difficult problems are far beyond the scope of this short communication. The theorems proved in Section \[\mathcal{M}\] may provide a basis for algorithmic checking of Darboux integrability of nonlinear PDEs (provided a suitable factorisation algorithm for corresponding linearized equations with coefficients depending on solutions of another PDEs will be found). Also we may conjecture that a generalization of the Darboux integrability method to PDEs of higher order with arbitrary number of independent variables may be given: such integrability should be related to representation of the corresponding linearized LPDO as an LCM of "first-order" generalized divisor ideals.

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