A CHEVALLEY FORMULA FOR SEMI-INFINITE FLAG MANIFOLDS AND QUANTUM K-THEORY

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Abstract. We give a combinatorial Chevalley formula for an arbitrary weight, in the torus-equivariant K-theory of semi-infinite flag manifolds, which is expressed in terms of the quantum alcove model. As an application, we prove the Chevalley formula for anti-dominant fundamental weights in the (small) torus-equivariant quantum K-theory of the flag manifold G/B; this has been a longstanding conjecture about the multiplicative structure of QK(G/B). Moreover, in type An−1, we prove that the so-called quantum Grothendieck polynomials indeed represent Schubert classes in the (non-equivariant) quantum K-theory QK(SLn/B).

1. Introduction

This paper is concerned with a geometric application of the combinatorial model known as the quantum alcove model, introduced in [13]. Its precursor, the alcove model of the first author and Postnikov, was used to uniformly describe the highest weight Kashiwara crystals of symmetrizable Kac-Moody algebras [20], as well as the Chevalley formula in the equivariant K-theory of flag manifolds G/B [19]. More generally, the quantum alcove model was used to uniformly describe certain crystals of affine Lie algebras (single-column Kirillov-Reshetikhin crystals) and Macdonald polynomials specialized at \( t = 0 \) [17, 18]. The objects of the quantum alcove model (indexing the crystal vertices and the terms of Macdonald polynomials) are paths in the quantum Bruhat graph on the Weyl group [2]. In this paper we complete the above picture, by extending to the quantum alcove model the geometric application of the alcove model, namely the K-theory Chevalley formula.

To achieve our goal, we need to consider the so-called semi-infinite flag manifold \( QG \). We give a Chevalley formula for an arbitrary weight in the T-equivariant K-group \( K_T(QG) \) of \( QG \), which is described in terms of the quantum alcove model. In [10] and [23], the Chevalley formulas for \( K_{T\times\mathbb{C}^*}(QG) \) were originally given in terms of the quantum LS path model in the case of a dominant and an anti-dominant weight, respectively. For a general (not dominant nor anti-dominant) weight, there is no quantum LS path model, but there is a quantum alcove model. Hence, in order to obtain a Chevalley formula for an arbitrary weight, we first need to translate the formulas above to the quantum alcove model by using the weight-preserving bijection between the two models given by Propositions 25 and 28. Starting from these translated formulas (Theorems 26 and 29), we prove a Chevalley formula in \( K_T(QG) \) (Theorem 32, which is the case \( q = 1 \) of Conjecture 30) for an arbitrary weight, based on the combinatorics of the quantum alcove model. We are currently working on the proof of Conjecture 30 in full generality.

The study of the equivariant K-group of semi-infinite flag manifolds was started in [10]. A breakthrough in this study is [8] (see also [9]), in which Kato established a \( \mathbb{C}[P] \)-module isomorphism from the (small) T-equivariant quantum K-theory \( QK_T(G/B) \) of the finite-dimensional flag manifold \( G/B \) onto (a version of) the T-equivariant K-group \( K'_T(QG) \) of \( QG \); here \( P \) is the weight lattice generated by the fundamental weights \( \varpi_k, k \in I \). This isomorphism sends each (opposite) Schubert class in \( QK_T(G/B) \) to the corresponding semi-infinite Schubert class in \( K'_T(QG) \); moreover, it respects the quantum multiplication in \( QK_T(G/B) \) and the tensor product in \( K'_T(QG) \). Based on this result, a longstanding conjecture on the multiplicative structure of \( QK_T(G/B) \), i.e., the Chevalley
formula (Theorem 30) for anti-dominant fundamental weights $-\varpi_k$, $k \in I$, for $QK_T(G/B)$ is proved by our anti-dominant Chevalley formula for $K_T \times C^*(Q_G)$ under the specialization at $q = 1$.

As another application of our Chevalley formula, we can prove an important conjecture for the non-equivariant quantum $K$-theory $QK(SL_n/B)$ of the flag manifold of type $A_{n-1}$ (Theorem 11): the quantum Grothendieck polynomials, introduced in [15], indeed represent Schubert classes in $QK(SL_n/B)$. In this way, we generalize the results of [8], where the quantum Schubert polynomials are constructed as representatives for Schubert classes in the quantum cohomology of $SL_n/B$. Therefore, we can use quantum Grothendieck polynomials to compute any structure constant in the quantum Grothendieck polynomials (Theorem 41): for anti-dominant fundamental weights $\theta$, we have that $QK$-theory is notoriously difficult. In addition, still for $QK(SL_n/B)$, we obtain very explicit information about the coefficients in the respective Chevalley formula (Theorem 17, Proposition 18 and Corollary 50).

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2. Background on the Quantum Bruhat Graph and Its Parabolic Version

2.1. Root Systems. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$ with Cartan subalgebra $\mathfrak{h}$. Denote by $\{\alpha_i\}_{i \in I}$ and $\{\alpha_i^\vee\}_{i \in I}$ the set of simple coroots and simple roots of $\mathfrak{g}$, respectively, and set $Q^\vee := \bigoplus_{i \in I} \mathbb{Z}\alpha_i^\vee$ and $Q^{\vee,+} := \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i^\vee$. Let $\Phi, \Phi^+, \Phi^-$ be the set of roots, positive roots, and negative roots of $\mathfrak{g}$, respectively, with $\theta \in \Phi^+$ the highest root of $\mathfrak{g}$; we set $\rho := (1/2) \sum_{\alpha \in \Phi^+} \alpha$. For $\alpha \in \Phi$, we set

$$\text{sgn}(\alpha) := \begin{cases} 1 & \text{if } \alpha \in \Phi^+, \\ -1 & \text{if } \alpha \in \Phi^- \end{cases}$$

and denote by $\alpha^\vee$ the coroot of $\alpha$. Also, let $\varpi_i$, $i \in I$, denote the fundamental weights for $\mathfrak{g}$, and set $P := \bigoplus_{i \in I} \mathbb{Z}\varpi_i$ and $P^+ := \sum_{i \in I} \mathbb{Z}_{\geq 0}\varpi_i$. Let $W := \langle s_i \mid i \in I \rangle$ be the (finite) Weyl group of $\mathfrak{g}$, where $s_i$ is the simple reflection with respect to $\alpha_i$ for $i \in I$. We denote by $\ell : W \to \mathbb{Z}_{\geq 0}$ the length function on $W$, by $e \in W$ the identity element, and by $w_0 \in W$ the longest element. For $\alpha \in \Phi$, denote by $s_\alpha \in W$ the reflection with respect to $\alpha$; note that $s_{-\alpha} = s_\alpha$.

Let $J$ be a subset of $I$. We set $Q_J := \bigoplus_{i \in J} \mathbb{Z}\alpha_i$, $\Phi_J := \Phi \cap Q_J$, $\Phi_J^\pm := \Phi^\pm \cap Q_J$, $\rho_J := (1/2) \sum_{\alpha \in \Phi_J^+} \alpha$. We denote by $W_J := \langle s_i \mid i \in J \rangle$ the parabolic subgroup of $W$ corresponding to $J$, and we identify $W/W_J$ with the corresponding set of minimal coset representatives, denoted by $W^J$; note that if $J = \emptyset$, then $W^J = W^\emptyset$ is identical to $W$. For $w \in W$, we denote by $[w] = [w]^J \in W^J$ the minimal coset representative for the coset $wW_J$ in $W/W_J$.

Let $W_{af} := \langle s_i \mid i \in I_{af} \rangle$, with $I_{af} := I \sqcup \{0\}$, be the (affine) Weyl group of the untwisted affine Lie algebra $\mathfrak{g}_{af}$ associated to $\mathfrak{g}$. For each $\xi \in Q^\vee$, let $t_\xi \in W_{af}$ denote the translation by $\xi$ (see [7 Sect. 6.5]). Then, $\{t_\xi \mid \xi \in Q^\vee\}$ forms an abelian normal subgroup of $W_{af}$, in which $t_\xi t_\zeta = t_{\xi + \zeta}$ holds for $\xi, \zeta \in Q^\vee$. Moreover, we know from [7 Proposition 6.5] that

$$W_{af} \cong W \ltimes \{t_\xi \mid \xi \in Q^\vee\} \cong W \ltimes Q^\vee;$$

note that $s_0 = s_{gt_{-\theta^\vee}}$. We set $W_{af}^{\geq 0} := W \ltimes Q^\vee$.+
2.2. The quantum Bruhat graph. We take and fix a subset $J$ of $I$.

**Definition 1.** The (parabolic) quantum Bruhat graph $QB(W^J)$ is the $(Φ^+ \setminus Φ_J^+)$-labeled directed graph whose vertices are the elements of $W^J$, and whose directed edges are of the form: $w \xrightarrow{β} v$ for $w, v \in W^J$ and $β \in Φ^+ \setminus Φ_J^+$ such that $v = [ws_β]$, and such that either of the following holds: (i) $ℓ(v) = ℓ(w) + 1$; (ii) $ℓ(v) = ℓ(w) + 1 - 2(ρ - ρ_J, β^\vee)$. An edge satisfying (i) (resp., (ii)) is called a Bruhat (resp., quantum) edge.

Remarks 2 (see [16, Remark 6.13]). For each $v, w \in W^J$, there exists a directed path in $QB(W^J)$ from $v$ to $w$.

For a directed path $p : v = v_0 \xrightarrow{β_1} v_1 \xrightarrow{β_2} \cdots \xrightarrow{β_l} v_l = w$ in $QB(W^J)$, we define the weight $wt^J(p)$ of $p$ by

$$wt^J(p) := \sum_{1 \leq k \leq l; \atop v_{k-1} \xrightarrow{β_k} v_k \text{ is a quantum edge}} β_k^\vee \in Q^\vee_{J^+}.$$ 

when $J = \emptyset$, we write $wt(p)$ for $wt^0(p)$. We know the following from [16 Proposition 8.1].

**Proposition 3.** Let $v, w \in W^J$. If $p$ and $q$ are shortest directed paths in $QB(W^J)$ from $v$ to $w$, then $wt^J(p) \equiv wt^J(q) \mod Q^\vee_J$. In particular, if $J = \emptyset$, then $wt(p) = wt(q)$.

For $v, w \in W^J$, we denote by $ℓ^J(v \Rightarrow w)$ the length of a shortest directed path from $w$ to $v$. When $J = \emptyset$, we write $ℓ(v \Rightarrow w)$ for $ℓ^0(v \Rightarrow w)$.

Assume that $J = \emptyset$. In this case, we denote by $wt(w \Rightarrow v)$ the weight $wt(p)$ of a shortest directed path in $QB(W)$ from $w$ to $v$, which is independent of the choice of a shortest directed path by Proposition 3. Also, we will use the shellability of the quantum Bruhat graph $QB(W)$ with respect to a reflection ordering on the positive roots [5], which we now recall.

**Theorem 4 ([2]).** Fix a reflection ordering on $Φ^+$.

1. For any pair of elements $v, w \in W$, there is a unique directed path from $v$ to $w$ in the quantum Bruhat graph $QB(W)$ such that its sequence of edge labels is strictly increasing (resp., decreasing) with respect to the reflection ordering.
2. The path in (1) has the smallest possible length $ℓ(v \Rightarrow w)$.

2.3. Additional results. In this subsection, we fix a dominant weight $λ \in P^+$, and set $J = J_λ := \{i \in I \mid \langle λ, α_i^\vee \rangle = 0\} \subset I$. Let $v, w \in W^J$, and let $p$ be a shortest directed path in $QB(W^J)$ from $v$ to $w$. Then we deduce by Proposition 3 that $⟨λ, wt^J(p)⟩$ does not depend on the choice of a shortest directed path $p$. We write $⟨λ, wt^J(v \Rightarrow w)⟩$ for $⟨λ, wt^J(p)⟩$.

**Lemma 5 ([17 Lemma 7.2]).** Keep the notation and setting above. Let $σ, τ \in W^J$. Then, $⟨λ, wt^J(σ \Rightarrow τ)⟩ = ⟨λ, wt(v \Rightarrow w)⟩$ for all $v \in σW_J, w \in τW_J$.

**Definition 6.** For a rational number $b \in \mathbb{Q}$, we define $QB_{βλ}(W^J)$ (resp., $QB_{βλ}(W)$) to be the subgraph of $QB(W^J)$ (resp., $QB(W)$) with the same vertex set but having only those directed edges of the form $w \xrightarrow{β} v$ for which $b(λ, β^\vee) \in \mathbb{Z}$ holds.

**Lemma 7 ([17 Lemma 6.2]).** Keep the notation and setting above. Let $w \xrightarrow{γ} wγ$ be an edge in $QB_{βλ}(W)$ for some rational number $b$. Then there exists a directed path from $[w]$ to $[wγ]$ in $QB_{βλ}(W^J)$ (possibly of length 0).
Lemma 8 ([17 Lemma 6.7]). Consider two directed paths in QB(W) between some w and v. Assume that the first one is a shortest path, while the second one is in QB_{b\lambda}(W), for some rational number b. Then the first path is in QB_{b\lambda}(W) as well.

We now recall [16 Proposition 7.2], which constructs the analogue of (one version of) the so-called Deodhar lifts [3] for the quantum Bruhat graph; we will call them quantum right Deodhar lifts.

Proposition 9 ([16]). Given v, w ∈ W, there exists a unique element x ∈ vW_J such that ℓ(x ⇒ v) attains its minimum value as a function of x ∈ vW_J.

We refer also to [16 Theorem 7.1], stating that the mentioned minimum is, in fact, attained by the minimum of the coset vW_J with respect to the w-tilted Bruhat order ≤_w on W [2]. Therefore, it makes sense to denote it by min(vW_J, ≤_w), although we will not use this stronger result.

The quantum Bruhat graph analogue of the second version of the Deodhar lifts was given in [23 Proposition 2.25]; we will call these quantum left Deodhar lifts. The mentioned result is stated based on the so-called dual v-tilted Bruhat order ≤^*_v on W, introduced in [23 Definition 2.24]. It is proved by reduction to [16 Theorem 7.1].

Proposition 10 ([23]). Given v, w ∈ W, the coset uW_J has a unique maximal element with respect to ≤^*_v, which is denoted by max(uW_J, ≤^*_v).

For our purposes, the weaker version of this result, which is stated below, suffices; this is the analogue of Proposition 9.

Proposition 11. Given v, w ∈ W, there exists a unique element x ∈ wW_J such that ℓ(x ⇒ v) attains its minimum value as a function of x ∈ wW_J.

The mentioned element is max(uW_J, ≤^*_v). In [16] we gave a proof of Proposition 9, i.e., [16 Proposition 7.2], which is independent of [16 Theorem 7.1], mentioned above; this proof was based on [16 Lemmas 7.4, 7.5]. Likewise, Proposition 11 can be proved independently of Proposition 10 as an immediate consequence of the analogues of the mentioned lemmas. These analogues are stated as Lemmas 20 and 21 in Section 3.3 and are also needed in the proof of Lemma 22 in that section.

3. Background on the combinatorial models

Throughout this section, λ is a dominant weight whose stabilizer is the parabolic subgroup W_J of W for a subset J ⊂ I.

3.1. Quantum LS paths.

Definition 12 ([17]). A quantum LS path η ∈ QLS(λ) is given by two sequences

(1) \(0 = b_1 < b_2 < b_3 < \ldots < b_t < b_{t+1} = 1\); \((\phi(\eta) = \sigma_1, \sigma_2, \ldots, \sigma_t = \iota(\eta))\),

where \(b_k \in \mathbb{Q}\), \(\sigma_k \in W^J\), and there is a directed path in QB_{b_{k}\lambda}(W^J) from \(\sigma_{k-1}\) to \(\sigma_k\), for each \(k = 2, \ldots, t\). The elements \(\sigma_k\) are called the directions of \(\eta\), while \(\iota(\eta)\) and \(\phi(\eta)\) are the initial and final directions, respectively.

This data encodes the sequence of vectors

(2) \(v_t := (b_{t+1} - b_t)\sigma_t \lambda, \ldots, v_2 := (b_3 - b_2)\sigma_2 \lambda, \quad v_1 := (b_2 - b_1)\sigma_1 \lambda\).

We can view the quantum LS path \(\eta\) as a piecewise-linear path given by the sequence of points

0, \(v_t, v_{t-1} + v_t, \ldots, v_1 + \ldots + v_t\).
There is also a standard way to express $\eta$ as a map $\eta : [0, 1] \to \mathfrak{h}_R^*$ with $\eta(0) = 0$ (where $\mathfrak{h}_R^* = \mathbb{R} \otimes \mathbb{Z} \times$ is the real part of the dual Cartan subalgebra), but we do not need this here. The endpoint of the path, also called its weight, is $\text{wt}(\eta) := \eta(1) = v_1 + \ldots + v_t$.

We define the (tail) degree function (cf. [17, Corollary 4.8]) by

$$\text{deg}(\eta) := -\sum_{k=2}^{t} (1 - b_k) \langle \lambda, \text{wt}(\sigma_{k-1} \Rightarrow \sigma_k) \rangle.$$  

Given $w \in W$, we define $\iota(\eta, w) \in W$, called the initial direction of $\eta$ with respect to $w$, by the following recursive formula:

$$\begin{cases} 
  w_0 := w, \\
  w_k := \min(\sigma_k W, \prec w_{k-1}) & \text{for } k = 1, \ldots, t, \\
  \iota(\eta, w) := w_t.
\end{cases}$$

Also, we set

$$\xi(\eta, w) := \sum_{k=1}^{t} \text{wt}(w_{k-1} \Rightarrow w_k),$$

and

$$\text{deg}_{\iota}(\eta) := -\sum_{k=1}^{t} (1 - b_k) \langle \lambda, \text{wt}(w_{k-1} \Rightarrow w_k) \rangle.$$  

Given $v \in W$, we define $\kappa(\eta, v) \in W$, called the final direction of $\eta$ with respect to $v$, by the following recursive formula:

$$\begin{cases} 
  w_{t+1} := v, \\
  w_k := \max(\sigma_k W, \succ v_{k+1}) & \text{for } k = 1, \ldots, t, \\
  \kappa(\eta, v) := w_1.
\end{cases}$$

Also, we set

$$\zeta(\eta, v) := \sum_{k=1}^{t} \text{wt}(w_k \Rightarrow w_{k+1}).$$

### 3.2. The quantum alcove model.

We say that two alcoves are adjacent if they are distinct and have a common wall. Given a pair of adjacent alcoves $A$ and $B$, we write $A \overset{\beta}{\to} B$ for $\beta \in \Phi$ if the common wall is orthogonal to $\beta$ and $\beta$ points in the direction from $A$ to $B$. Recall that alcoves are separated by hyperplanes of the form

$$H_{\beta, l} = \{ \mu \in \mathfrak{h}_R^* | \langle \mu, \beta' \rangle = l \},$$

where $\mathfrak{h}_R^* = \mathbb{R} \otimes \mathbb{Z} \times$ is the real part of the dual Cartan subalgebra. We denote by $s_{\beta, l}$ the affine reflection in this hyperplane.

The fundamental alcove is defined as

$$A_0 = \{ \mu \in \mathfrak{h}_R^* | 0 < \langle \mu, \alpha' \rangle < 1 \text{ for all } \alpha \in \Phi^+ \}.$$  

**Definition 13** ([19]). An alcove path is a sequence of alcoves $(A_0, A_1, \ldots, A_m)$ such that $A_{j-1}$ and $A_j$ are adjacent, for $j = 1, \ldots, m$. We say that $(A_0, A_1, \ldots, A_m)$ is reduced if it has minimal length among all alcove paths from $A_0$ to $A_m$.  

Let $\lambda$ be any weight, although we will later focus on $\lambda$ being dominant or anti-dominant. Let $A_\lambda = A_0 + \lambda$ be the translation of the fundamental alcove $A_0$ by the weight $\lambda$.

**Definition 14** ([19]). The sequence of roots $\Gamma(\lambda) = (\beta_1, \beta_2, \ldots, \beta_m)$ is called a $\lambda$-chain, respectively reduced $\lambda$-chain, if

$$A_0 = A_0 \xrightarrow{-\beta_1} A_1 \xrightarrow{-\beta_2} \cdots \xrightarrow{-\beta_m} A_m = A_{-\lambda}$$

is an alcove path, respectively reduced alcove path.

A reduced alcove path $(A_0 = A_0, A_1, \ldots, A_m = A_{-\lambda})$ can be identified with the corresponding total order on the hyperplanes, to be called $\lambda$-hyperplanes, which separate $A_0$ from $A_{-\lambda}$. This total order is given by the sequence $H_{\beta_i,-l_i}$ for $i = 1, \ldots, m$, where $H_{\beta_i,-l_i}$ contains the common wall of $A_{i-1}$ and $A_i$. Note that $\langle \lambda, \beta_i' \rangle \geq 0$, and that the integers $l_i$, called heights, have the following ranges:

$$0 \leq l_i \leq \langle \lambda, \beta_i' \rangle - 1 \quad \text{if} \quad \beta_i \in \Phi^+, \quad \text{and} \quad 1 \leq l_i \leq \langle \lambda, \beta_i' \rangle \quad \text{if} \quad \beta_i \in \Phi^-.$$ 

Note also that a $\lambda$-chain $(\beta_1, \ldots, \beta_m)$ determines the corresponding reduced alcove path, so we can identify them as well.

**Remark 15.** An alcove path corresponds to the choice of a word for the affine Weyl group element sending $A_0$ to $A_{-\lambda}$ ([19] Lemma 5.3). For $\lambda$ dominant, another equivalent definition of a reduced alcove path/$\lambda$-chain, based on a root interlacing condition which generalizes a similar condition characterizing reflection orderings, can be found in ([20] Definition 4.1, Proposition 10.2).

When $\lambda$ is dominant, we have a special choice of a reduced $\lambda$-chain in ([20] Section 4), which we now recall.

**Proposition 16** ([20]). Given a total order $I = \{1 < 2 < \cdots < r\}$ on the set of Dynkin nodes, one may express a coroot $\beta_\vee = \sum_{i=1}^r c_i \alpha_i^\vee$ in the $\mathbb{Z}$-basis of simple coroots. Consider the total order on the set of $\lambda$-hyperplanes defined by the lexicographic order on their images in $\mathbb{Q}^{r+1}$ under the map

$$H_{\beta,-l} \mapsto \frac{1}{\langle \lambda, \beta_\vee \rangle}(l, c_1, \ldots, c_r).$$

This map is injective, thereby endowing the set of $\lambda$-hyperplanes with a total order, which is a reduced $\lambda$-chain. We call it the lexicographic (lex) $\lambda$-chain, and denote it by $\Gamma_{\text{lex}}(\lambda)$.

The rational number $l/\langle \lambda, \beta_\vee \rangle$ is called the relative height of the $\lambda$-hyperplane $H_{\beta,-l}$. By definition, the sequence of relative heights in the lex $\lambda$-chain is weakly increasing.

The objects of the quantum alcove model are defined next. This model was introduced in [13] and then used in [17] [18] in connection with Kirillov-Reshetikhin crystals and Macdonald polynomials specialized at $t = 0$. Here we consider a generalization of it, by letting $\lambda$ be any weight, as opposed to only a dominant weight, as originally considered; another aspect of the generalization is making the model depend on a fixed element $w \in W$, such that the initial model corresponds to $w$ being the identity. In addition to $w$, we fix an arbitrary reduced $\lambda$-chain $\Gamma(\lambda) = (\beta_1, \ldots, \beta_m)$, and let $r_i := s_{\beta_i}, t_i := s_{\beta_i,-l_i}$.

**Definition 17** ([13]). A subset $A = \{j_1 < j_2 < \cdots < j_s\}$ of $[m] := \{1, \ldots, m\}$ (possibly empty) is a $w$-admissible subset if we have the following directed path in the quantum Bruhat graph $QB(W)$:

$$\Pi(w, A) : \quad w \xrightarrow{\l_1} wr_{j_1} \xrightarrow{\l_2} wr_{j_1}r_{j_2} \xrightarrow{\l_3} \cdots \xrightarrow{\l_s} wr_{j_1}r_{j_2} \cdots r_{j_s} =: \text{end}(w, A).$$

We let $\mathcal{A}(w, \Gamma(\lambda))$ be the collection of all $w$-admissible subsets of $[m]$. 
We now associate several parameters with the pair \((w, A)\). The weight of \((w, A)\) is defined by
\[
\text{wt}(w, A) := -w\hat{r}_{j_1} \cdots \hat{r}_{j_s}(-\lambda).
\]

Given the height sequence \((l_1, \ldots, l_m)\) mentioned above, we define the complementary height sequence \((\tilde{l}_1, \ldots, \tilde{l}_m)\) by \(\tilde{l}_i := (\lambda, \tilde{\beta}_i) - l_i\). Given \(A = \{j_1 < \ldots < j_s\} \subseteq A(w, \Gamma(\lambda))\), we let
\[
A^- := \{j_i \in A \mid wr_{j_1} \cdots r_{j_i-1} > wr_{j_i} \cdots r_{j_i-r_{j_i}}\};
\]
in other words, we record the quantum steps in the path \(\Pi(w, A)\) defined in (11). We also define
\[
\text{down}(w, A) := \sum_{j \in A^-} |\beta_j|^\vee \in Q^{\vee, +}, \quad \text{height}(w, A) := \sum_{j \in A^-} \text{sgn}(\beta_j)\tilde{l}_j.
\]

For examples, we refer to [12] [17].

3.3. Additional shellability results. In [22] Section 4.3, we constructed a reflection ordering \(\prec_\lambda\) on \(\Phi^+\) which depends on \(\lambda\). The bottom of the order \(\prec_\lambda\) consists of the roots in \(\Phi^+ \setminus \Phi^+_j\). For two such roots \(\alpha\) and \(\beta\), define \(\alpha \prec \beta\) whenever the hyperplane \(H_{(\alpha, \theta)}\) precedes \(H_{(\beta, \theta)}\) in the lex \(\lambda\)-chain (see Proposition 10). This forms an initial section \([5]\) of \(\prec_\lambda\). The top of the order \(\prec_\lambda\) consists of the positive roots in \(\Phi^+_j\), and we fix any reflection ordering for them. We refer to the reflection ordering \(\prec_\lambda\) throughout.

Remark 18. It is not hard to see that, in the lex \(\lambda\)-chain, the order on the \(\lambda\)-hyperplanes \(H_{\beta, -s}\) with the same relative height (not necessarily equal to 0) is given by the order \(\prec_\lambda\) on the corresponding roots \(\beta\). We will use this fact implicitly below.

We recall [17] Lemma 6.6, which characterizes the quantum right Deodhar lifts in shellability terms.

**Lemma 19** ([17]). Consider \(\sigma, \tau \in W^J\) and \(w_j \in W_J\). Write \(\min(\tau W_J, \preceq_{\sigma W_j}) \in \tau W_J\) as \(\tau w'_j\), with \(w'_j \in W_J\).

1. There is a unique directed path in \(QB(W)\) from \(\sigma w_J\) to some \(x \in \tau W_J\) whose edge labels are increasing and lie in \(\Phi^+ \setminus \Phi^+_j\). This path ends at \(\tau w'_j\).

2. Assume that there is a directed path from \(\sigma\) to \(\tau\) in \(QB_{b, \lambda}(W^J)\) for some \(b \in \mathbb{Q}\). Then the path in (1) from \(\sigma w_J\) to \(\tau w'_J\) is in \(QB_{b, \lambda}(W)\).

In order to state the analogue of Lemma 19 for the left Deodhar lifts, namely Lemma 22, we need the reverse of the reflection order \(\prec_\lambda\), which is denoted \(\prec_\lambda^*\) (this has all the roots in \(\Phi^+_j\) at the beginning). It is well-known that \(\prec_\lambda^*\) is a reflection order as well. We also need the following two lemmas, which are proved in the same way as their counterparts in [16], namely Lemmas 7.4 and 7.5 in this paper.

**Lemma 20.** Assume that \(\ell(x \Rightarrow v)\), as a function of \(x \in wW_J\), has a minimum at \(x = x_0\). Then the path from \(x_0\) to \(v\) with increasing edge labels with respect to \(\prec_\lambda^*\) (cf. Theorem 4 (1)) has all its labels in \(\Phi^+ \setminus \Phi^+_j\).

**Lemma 21.** Assume that the paths with increasing edge labels from two elements \(x_0, x_1\) in \(wW_J\) to \(v\) (cf. Theorem 4 (1)) have all labels in \(\Phi^+ \setminus \Phi^+_j\). Then \(x_0 = x_1\).

**Lemma 22.** Consider \(\sigma, \tau \in W^J\) and \(w_j \in W_J\). Write \(\max(\sigma W_J, \preceq_{\tau w_j}) \in \sigma W_J\) as \(\sigma w'_J\), with \(w'_J \in W_J\).

1. There is a unique directed path in \(QB(W)\) from some \(x \in \sigma W_J\) to \(\tau w_J\) whose edge labels are increasing with respect to \(\prec_\lambda^*\) and lie in \(\Phi^+ \setminus \Phi^+_j\). This path starts at \(\sigma w'_J\).
(2) Assume that there is a directed path from $\sigma$ to $\tau$ in $QB(b\lambda(W^J))$ for some $b \in \mathbb{Q}$. Then the path in (1) from $\sigma w_J^i$ to $\tau w_J$ is in $QB(b\lambda(W^J))$.

**Proof.** The proof is completely similar to that of Lemma [19] i.e., [17] Lemma 6.6, based on Lemmas [20] [21] [22] and Theorem [4] (2). $\square$

4. **Chevalley formulas for semi-infinite flag manifolds**

Consider a simply-connected simple algebraic group $G$ over $\mathbb{C}$, with Borel subgroup $B = TN$, maximal torus $T$, and unipotent radical $N$. The full semi-infinite flag manifold $QB_G^{rat}$ is the reduced (ind-)scheme associated to $G(\mathbb{C}((z))) / (T(\mathbb{C}) \cdot N(\mathbb{C}((z))))$; in this paper, we concentrate on its semi-infinite Schubert subvariety $QB_G := QB_G(e) \subset QB_G^{rat}$ corresponding to the identity element $e \in W_{af}$, which we also call the semi-infinite flag manifold. The $T \times \mathbb{C}^*$-equivariant $K$-group $K_{T \times \mathbb{C}^*}(QB_G)$ of $QB_G$ has a (topological) $\mathbb{C}[q,q^{-1}][P]$-basis of semi-infinite Schubert classes, and its multiplicative structure is determined by a *Chevalley formula*, which expresses the tensor product of a Schubert class with the class of a line bundle. In [10] and [24], the Chevalley formulas were given in the case of a dominant and an anti-dominant weight $\lambda$, respectively. These formulas were expressed in terms of the quantum LS path model. We will express them in terms of the quantum alcove model based on the lexicographic $\lambda$-chain. The goal is to generalize these formulas for an arbitrary weight $\lambda$, and we will also see that an arbitrary $\lambda$-chain can be used. Throughout this section, $W_J$ is the stabilizer of $\lambda$, and we use freely the notation in Section 2.

The $T \times \mathbb{C}^*$-equivariant $K$-group $K_{T \times \mathbb{C}^*}(QB_G)$ is the $\mathbb{C}[q,q^{-1}][P]$-submodule of the (Iwahori-) equivariant $K$-group $K_{1 \times \mathbb{C}^*}(QB_G^{rat})$ of $QB_G^{rat}$, introduced in [10], consisting of all (possibly infinite) linear combinations of the classes $[QB_G(x)]$, $x \in W_{af}^\geq = W \times Q_{\lambda^+}$, of the structure sheaf of the semi-infinite Schubert variety $QB_G(x) \subset QB_G$ with coefficient $a_x \in \mathbb{C}[q,q^{-1}][P]$ such that the sum $\sum_{x \in W_{af}^\geq} |a_x|$ of the absolute values $|a_x|$ lies in $\mathbb{C}(q^{-1})[P]$. Here $\mathbb{C}^*$ acts on $QB_G$ by loop rotation, and $\mathbb{C}[P] := \mathbb{C} \otimes_{\mathbb{Z}} \mathbb{Z}[P]$ is the group algebra of $P$, spanned by formal exponentials $e^\mu$, for $\mu \in P$, with $e^\mu e^\nu = e^{\mu + \nu}$; note that $\mathbb{Z}[P]$ is identified with the representation ring of $T$. We also consider the $\mathbb{C}[q,q^{-1}][P]$-submodule $K_{T \times \mathbb{C}^*}(QB_G)$ and $K_{T}^I(QB_G)$ consisting of all finite linear combinations of the classes $[QB_G(x)]$, $x \in W_{af}^\geq$, with coefficients in $\mathbb{C}[q,q^{-1}][P]$. The $T$-equivariant $K$-groups of $QB_G$, denoted by $K_T(QB_G)$ and $K_T^I(QB_G)$, are obtained from the $K_{T \times \mathbb{C}^*}(QB_G)$ and $K_{T}^I(QB_G)$ above, respectively, by the specialization $q = 1$. Hence the Chevalley formulas in $K_T(QB_G)$ (for arbitrary weights) and $K_T^I(QB_G)$ (for anti-dominant weights) are obtained from the corresponding one in $K_{T \times \mathbb{C}^*}(QB_G)$ by setting $q = 1$. Note that the $T$-equivariant $K$-group $K_T(QB_G)$ turns out to be the $\mathbb{C}[P]$-module consisting of all (possibly infinite) linear combinations of the classes $[QB_G(x)]$, $x \in W_{af}^\geq$, with coefficients in $\mathbb{C}[P]$. Also, $K_T^I(QB_G)$ is the $\mathbb{C}[P]$-submodule of $K_T(QB_G)$ consisting of all finite linear combinations of the classes $[QB_G(x)]$, $x \in W_{af}^\geq$, with coefficients in $\mathbb{C}[P]$.

4.1. **Chevalley formula for dominant weights.** We start with the Chevalley formula for dominant weights, which was derived in terms of semi-infinite LS paths in [10], and then restated in [23] Corollary C.3 in terms of quantum LS paths.

Let $\lambda = \sum_{i \in I} \lambda_i w_i$ be a dominant weight. We denote by $\overline{\text{Par}}(\lambda)$ the set of $I$-tuples of partitions $\chi = (\chi^{(i)}_{\nu} \mid i \in I)$ such that $\chi^{(i)}_{\nu}$ is a partition of length at most $\lambda_i$ for all $i \in I$. For $\chi = (\chi^{(i)}_{\nu} \mid i \in I) \in \overline{\text{Par}}(\lambda)$, we set $|\chi| := \sum_{i \in I} |\chi^{(i)}_{\nu}|$, with $|\chi^{(i)}_{\nu}|$ the size of the partition $\chi^{(i)}_{\nu}$. Also set $\nu(\chi) := \sum_{i \in I} \chi^{(i)}_{\nu} \in Q_{\lambda^+}^*$, with $\chi^{(i)}_{\nu}$ the first part of the partition $\chi^{(i)}_{\nu}$.
Theorem 23 \([10, 23]\). Let \(x = \text{wt}_\xi \in W_{af}^{\geq 0} = W \times Q^{1+} \). Then, in \(K_{T \times C^*}(Q_G)\), we have
\[
\mathcal{O}_{Q_G}(-w_\lambda) \cdot \mathcal{O}_{Q_G(w)} = \sum_{\eta \in QLS(\lambda)} \sum_{\chi \in \text{Par}(\lambda)} q^{\deg_w(\eta) - (\lambda, \xi) - |x|} e^{\text{wt}(\eta)} \mathcal{O}_{Q_G((\eta \circ \xi) t_\xi + \xi, \eta, w) + \chi(\xi)}.
\]

Remark 24. The original Chevalley formula for a dominant weight, as stated in \([23, \text{Corollary C.3}]\), is in terms of a slightly different version of quantum LS paths. They can be recovered from those in Definition 12 simply by replacing the numbers \(b_i\) with \(1 - b_i\) (arranged increasingly) and by reversing the second sequence in \((1)\). Indeed \(QB_{b\lambda}(W)\) is identical to \(QB_{(1-b)\lambda}(W)\). The same observation applies to the original Chevalley formula for an anti-dominant weight, as stated in \([23, \text{Theorem 1}]\), see Theorem 27 below.

We now translate this formula in terms of the quantum alcove model for the lex \(\lambda\)-chain \(\Gamma_{\text{lex}}(\lambda)\).

To this end, given \(w \in W\), we construct a bijection \(A \mapsto \eta\) between \(A(w, \Gamma_{\text{lex}}(\lambda))\) and \(QLS(\lambda)\), for which several properties are then proved.

In order to construct the forward map, let \(A = \{j_1 < \ldots < j_s\} \in A(w, \Gamma_{\text{lex}}(\lambda))\). The corresponding heights are within the first range in \((9)\). Consider the weakly increasing sequence of relative heights
\[(14) \quad h_i := \frac{l_{j_i}}{\langle \lambda, \beta_{j_i}^\vee \rangle} \in [0, 1) \cap \mathbb{Q}, \quad i = 1, \ldots, s.\]

Let \(0 < b_2 < \ldots < b_t < 1\) be the distinct nonzero values in the set \(\{h_1, \ldots, h_s\}\), and let \(b_1 := 0, b_{t+1} := 1\). For \(k = 1, \ldots, t\), let \(I_k := \{1 \leq i \leq s \mid h_i = b_k\}\); these sets are all nonempty, except perhaps \(I_1\).

Recall the path \(\Pi(w, A)\) in \(QB(W)\) defined in \((11)\). We divide this path into subpaths corresponding to the sets \(I_k\), and record the last element in each subpath; more precisely, for \(k = 0, \ldots, t\), we define the sequence of Weyl group elements
\[w_k := w \prod_{i \in I_1 \cup \ldots \cup I_k} r_{j_i},\]
where the non-commutative product is taken in the increasing order of the indices \(i\); in particular, \(w_0 := w\). For \(k = 1, \ldots, t\), let \(\sigma_k := [w_k] \in W^J\). We can now define the forward map as
\[(w, A) \mapsto \eta := ((b_1, b_2, \ldots, b_t, b_{t+1}); (\sigma_1, \ldots, \sigma_t)).\]

We will verify below that the image is in \(QLS(\lambda)\).

The inverse map is constructed using the quantum right Deodhar lift and the related shellability property of the quantum Bruhat graph. We begin with a quantum LS path \(\eta \in QLS(\lambda)\) in the form \((1)\). Letting \(w_0 = w\), define the lifts
\[(15) \quad w_k = \min(\sigma_k W_J, <_{w_{k-1}}) \quad \text{for} \ k = 1, \ldots, t.\]

By Lemma 19, for each \(k = 1, \ldots, t\) there is a unique directed path from \(w_{k-1}\) to \(w_k\) in \(QB_{b\lambda}(W)\) with labels in \(\Phi^+ \setminus \Phi_J^+\), which are increasing with respect to the reflection order \(<_\lambda\). Let us replace each label \(\beta\) in this path with the pair \((\beta, b_k(\lambda, \beta^\vee))\), where the second component is in \(\{0, 1, \ldots, \langle \lambda, \beta^\vee \rangle - 1\}\), by the definition of \(QB_{b\lambda}(W)\). Thus, each such pair defines a \(\lambda\)-hyperplane. By concatenating these paths we obtain a directed path in \(QB(W)\) starting at \(w\), together with a sequence of \(\lambda\)-hyperplanes. We will show that this sequence is lex-increasing, and thus it defines a \(w\)-admissible subset.
Theorem 27. Note that, if the relative height of the \( \lambda \) which is defined just as the reverse of the lex \((\lambda, \beta_j)\) is completely similar to that of the map in the mentioned -chain described in Proposition 16; note that the \( \lambda \)-hyperplanes increase because of the compatibility of the reflection order \(<_\lambda\) with the lex \( \lambda \)-chain (see Remark 18).

To show that the two maps are mutually inverse, the crucial fact to check is that the forward map is well-defined. By the definition of relative height, the subpath of \( \Pi(w, A) \) from \( w_{k-1} \) to \( w_k \) is in \( \text{QB}_{b_k \lambda}(\tilde{W}) \). Thus, Lemma 7 implies that \( \eta \in \text{QLS}(\lambda) \). For the well-definedness of the inverse map, it suffices to prove that the constructed sequence of \( \lambda \)-hyperplanes is lex-increasing. Indeed, the relative heights of the \( \lambda \)-hyperplanes are the numbers \( b_k \), so they weakly increase by construction; on the other hand, within the same relative height, the \( \lambda \)-hyperplanes increase because of the compatibility of the reflection order \(<_\lambda\) with the lex \( \lambda \)-chain (see Remark 18).

We now translate this formula in terms of the quantum alcove model via Proposition 25.

Theorem 26. Let \( \lambda \) be a dominant weight, \( \Gamma_{\text{lex}}(\lambda) \) the lex \( \lambda \)-chain, and let \( x = \text{wt}_\xi \in W_{af}^{\geq 0} \). Then, in \( K_{T \times C^*}(\mathbb{Q}_G) \), we have

\[
[\mathcal{O}_{\mathcal{Q}_G}(-w_\xi \lambda)] : [\mathcal{O}_{\mathcal{Q}_G}(x)] = \sum_{A \in \mathcal{A}(w, \Gamma_{\text{lex}}(\lambda))} \sum_{\chi \in \text{Far}(\lambda)} q^{-\text{height}(w, A) - (\lambda, \xi) - |\chi|} e^{\text{wt}(w, A)} \left[ \mathcal{O}_{\mathcal{Q}_G}(\text{end}(w, A)_{\xi + \text{down}(w, A) + |\chi|}) \right].
\]

4.2. Chevalley formula for anti-dominant weights. We continue with the Chevalley formula for an anti-dominant weight \( \lambda \), which was derived in terms of quantum LS paths in \cite[Theorem 1]{23}.

Theorem 27. Let \( \lambda \) be an anti-dominant weight, and let \( x = \text{wt}_\xi \in W_{af}^{\geq 0} \). Then, in \( K'_{T \times C^*}(\mathbb{Q}_G) \subset K_{T \times C^*}(\mathbb{Q}_G) \), we have

\[
[\mathcal{O}_{\mathcal{Q}_G}(-w_\xi \lambda)] : [\mathcal{O}_{\mathcal{Q}_G}(x)] = \sum_{v \in \tilde{W}} \sum_{w \in \text{QLS}(\lambda)} (-1)^{\ell(v) - \ell(w)} q^{-\deg(\eta) - (\lambda, \xi)} e^{-\text{wt}(\eta)} \left[ \mathcal{O}_{\mathcal{Q}_G}(\text{wt}_\xi + \zeta(\eta, v)) \right].
\]

We now translate this formula in terms of the quantum alcove model for the lex \( \lambda \)-chain \( \Gamma_{\text{lex}}(\lambda) \), which is defined just as the reverse of the lex \((-\lambda)\)-chain described in Proposition 16. Note that the alcove path corresponding to the former (ending at \( A_\circ - \lambda \)) is just the translation by \(-\lambda\) of the
paths we obtain a directed path in $QB(A)$.

To show that the two maps are mutually inverse, we use the uniqueness part in Lemma 22 (1).

**Proof.** For the inverse map, we start with $(w, A)$. Another difference is concerned with proving $\text{height}(w, A) = \text{deg}(\eta)$. Note first that

$$\text{height}(w, A) = \text{height}(w, A),$$

where $\overline{A}$ is the subset of $A$ which corresponds to ignoring the $\lambda$-hyperplanes of relative height equal to 1; indeed, the contribution of each such hyperplane is 0, see (13). Thus, $\text{height}(w, A)$ is defined based on shortest directed paths in $Q\beta(W)$ from $w_{k-1}$ to $w_k$, for $k = 2, \ldots, t$. Comparing with the definition (3) of $\text{deg}(\eta)$, where $\sigma_k := \lfloor w_k \rfloor$, and using Lemma 5 as well as the analogue of (17), the desired equality is proved.

We translate the formula in Theorem 27 to the quantum alcove model via Proposition 28. We use the notation $|A|$ to indicate the cardinality of the set $A$.

**Theorem 29.** Let $\lambda$ be an anti-dominant weight, $\Gamma_{\text{lex}}(\lambda)$ the lex $\lambda$-chain, and let $x = \text{wt}_x \in W_{af}^\geq 0$. Then, in $K_{T \times C}(Q_G) \subset K_{T \times C}(Q_G)$, we have

$$[\mathcal{O}_{Q_G}(-w_\alpha \lambda)] \cdot [\mathcal{O}_{Q_G}(x)] = \sum_{A \in \mathcal{A}(w, \Gamma_{\text{lex}}(\lambda))} (-1)^{|A|} q^{-\text{height}(w, A) - \langle \lambda, x \rangle} e^{\text{wt}(w, A)} \mathcal{O}_{Q_G}(\text{end}(w, A) | t_{\xi+\text{down}(w, A)}).$$
4.3. Chevalley formula for arbitrary weights. We now conjecture the Chevalley formula for an arbitrary weight $\lambda = \sum_{i \in I} \lambda_i \omega_i$. This is the natural common generalization of Theorems 26 and 29. The case $q = 1$ of the conjecture, which is stated as Theorem 32, is proved in Section 5. In order to exhibit the general formula, let $\operatorname{Par}(\lambda)$ denote the set of $I$-tuples of partitions $\chi = (\chi^{(i)})_{i \in I}$ such that $\chi^{(i)}$ is a partition of length at most $\max(\lambda_i, 0)$.

**Conjecture 30.** Let $\lambda$ be an arbitrary weight, $\Gamma(\lambda)$ an arbitrary reduced $\lambda$-chain, and let $x = wt \xi \in W_{af}^\geq 0$. Then, in $K_{T \times G}(Q_G)$, we have

$$[O_{Q_G}(-w_0 \lambda)] \cdot [O_{Q_G}(x)] = \sum_{A \in A(w, \Gamma(\lambda))} \sum_{\chi \in \operatorname{Par}(\lambda)} (-1)^{n(A)} q^{-\text{height}(w, A) - \langle \lambda, \xi \rangle - \chi^{(i)}} e^{wt(w, A)} [O_{Q_G}(\text{end}(w, A)_{\chi + \text{down}(w, A) + (\chi)}],$$

where $n(A)$, for $A = \{j_1 < \cdots < j_s\}$, is the number of negative roots in $\{\beta_{j_1}, \ldots, \beta_{j_s}\}$.

**Example 31.** Assume that $g$ is of type $A_2$, and $\lambda = \omega_1 - \omega_2$. Then, $\Gamma(\lambda) := \{\alpha_1, -\alpha_2\}$ is a reduced $\lambda$-chain. Assume that $w = s_1 = s_{\alpha_1}$. In this case, we see that $A(s_1, \Gamma(\lambda)) = \emptyset, \{1\}, \{2\}, \{1, 2\}$, and we have the following table.

| $A$   | $n(A)$ | height($s_1, A$) | wt($s_1, A$) | end($s_1, A$) | down($s_1, A$) |
|-------|--------|-----------------|--------------|--------------|---------------|
| $\emptyset$ | 1 | 0 | $s_1 \lambda$ | $s_1$ | 0 |
| $\{1\}$ | 1 | 0 | $\lambda$ | $\emptyset$ | $\alpha_1^+$ |
| $\{2\}$ | 1 | 0 | $s_1 \lambda$ | $s_1 s_2$ | 0 |
| $\{1, 2\}$ | 1 | 0 | $\lambda$ | $s_2$ | $\alpha_1^+$ |

Also, we identify $\operatorname{Par}(\lambda)$ with $Z_{\geq 0}$. Therefore, we obtain

$$[O_{Q_G}(-w_0 \lambda)] \cdot [O_{Q_G}(s_1 \xi)] = \sum_{m \in Z_{\geq 0}} q^{-\langle \lambda, \xi \rangle - m} \left\{ \begin{array}{lcl} e^{s_1 \lambda} [O_{Q_G}(s_1 t_{\xi + m, \gamma})] & \text{if } A = \emptyset \\
\frac{q^{-1} e^{\lambda} [O_{Q_G}(t_{\xi + m, \gamma})]}{A = \{1\}} & \\
\frac{(-1) e^{s_1 \lambda} [O_{Q_G}(s_2 t_{\xi + m, \gamma})]}{A = \{2\}} & + \frac{(-1) q^{-1} e^{\lambda} [O_{Q_G}(s_2 t_{\xi + m, \gamma})]}{A = \{1, 2\}} \end{array} \right\}.$$

**Theorem 32.** Conjecture 30 holds for $q = 1$.

5. Proof of Theorem 32

For a positive root $\beta \in \Phi^+$, we define the quantum Bruhat operator $Q_\beta$ on the $T$-equivariant $K$-group $K_T(Q_G)$ as follows; recall that, by the definition, $K_T(Q_G)$ consists of all (possibly infinite) linear combinations of the classes $[O_{Q_G}(x)]$, $x \in W_{af}^\geq 0 = W \times Q_\gamma^+$, with coefficients in $C[P]$. We define

$$Q_\beta[O_{Q_G}(wt \xi)] := \begin{cases} [O_{Q_G}(ws_\beta t \xi)] & \text{if } w \xrightarrow{\beta} ws_\beta \text{ is a Bruhat edge in } QB(W), \\
O_{Q_G}(ws_\beta t \xi + m, \gamma) & \text{if } w \xrightarrow{\beta} ws_\beta \text{ is a quantum edge in } QB(W), \\
0 & \text{otherwise}, \end{cases}$$

where $w \in W$ and $\xi \in Q_\gamma = \sum_{i \in I} Z_0 \alpha_i^\gamma$. Also, we set $Q_{-\beta} := -Q_\beta$ for $\beta \in \Phi^+$. For a weight $\nu \in P$, we define

$$X^\nu[O_{Q_G}(wt \xi)] = e^{u_\nu/h} [O_{Q_G}(wt \xi)].$$
where \( w \in W, \xi \in Q^\vee, \) and \( h := \langle \rho, \theta^\vee \rangle + 1 \) is the Coxeter number. For \( i \in I, \) we define
\[
t_i[O_{Q_G}(x)] = [O_{Q_G(x \eta_i^\vee)}] \quad \text{for } x \in W_{af}.
\]
The following lemma is easily shown; cf. [19, Eqs.(10.3)–(10.5)].

**Lemma 33.**

1. We have \( Q^2_{\pm,\beta} = 0 \) for \( \beta \in \Phi^+ \setminus \Pi. \) For \( i \in I, \) we have \( Q_{\pm,\alpha_i} = t_i; \) \( Q_{\alpha_i}Q_{-\alpha_i} = Q_{-\alpha_i}Q_{\alpha_i} = -t_i, \) and
\[
(X^{\alpha_i} + Q_{\alpha_i})(X^{-\alpha_i} + Q_{-\alpha_i}) = (X^{-\alpha_i} + Q_{-\alpha_i})(X^{\alpha_i} + Q_{\alpha_i}) = 1 - t_i.
\]
2. We have \( X^\mu X^\nu = X^{\mu + \nu} \) for \( \mu, \nu \in P. \)
3. We have \( Q_\beta X^\nu = X^{s_\nu}Q_\beta \) for \( \nu \in P \) and \( \beta \in \Phi. \)

We set
\[
R_\beta := X^\rho(X^{\beta} + Q_\beta)X^{-\rho} \quad \text{for } \beta \in \Phi.
\]

**Proposition 34.** The family \( \{R_\beta \mid \beta \in \Phi\} \) satisfies the Yang-Baxter equation. Namely, if \( \alpha, \beta \in \Phi \) satisfy \( \langle \alpha, \beta^\vee \rangle \leq 0, \) or equivalently, \( \langle \beta, \alpha^\vee \rangle \leq 0, \) then
\[
R_\alpha R_{s_\alpha(\beta)}R_{s_\alpha s_\beta(\alpha)} \cdots R_{s_\beta(\alpha)}R_\beta = R_\beta R_{s_\beta(\alpha)} \cdots R_{s_\alpha s_\beta(\alpha)}R_{s_\alpha(\beta)}R_\alpha.
\]

*Proof.* We set \( \tilde{R}_\beta := 1 + Q_\beta \) for \( \beta \in \Phi. \) It follows from [21, Corollary 4.4] that the family \( \{\tilde{R}_\beta \mid \beta \in \Phi^+\} \) satisfies the Yang-Baxter equation; to apply this corollary, we take a field \( k \) containing the ring \( \mathbb{C}[Q_{\eta_i}^\vee]\) of formal power series in the variables \( Q_i = Q^{\alpha_i^\vee}, i \in I, \) and a \( k \)-valued multiplicative function \( E \) on \( \Phi^+ \) given by \( E(\alpha_i) := Q_i \) for each \( i \in I. \)

In order to prove that the family \( \{\tilde{R}_\beta \mid \beta \in \Phi\} \) also satisfies the Yang-Baxter equation, we make use of the following observation. Noting that the leftmost operator (say \( \tilde{R}_\alpha \)) on the left-hand side of the Yang-Baxter equation is identical to the rightmost operator on the right-hand side of the equation, we multiply both sides of the Yang-Baxter equation by the operator \( R_{-\alpha} \) on the left and on the right. If \( \alpha \) is not a simple root (resp., \( \alpha = \alpha_i \) for some \( i \in I \)), then the leftmost two operators \( \tilde{R}_{-\alpha} \tilde{R}_\alpha \) on the left-hand side and the rightmost two operators \( \tilde{R}_\alpha \tilde{R}_{-\alpha} \) on the right-hand side are both identical to \( 1 \) (resp., \( 1 - t_i \)) by Lemma 33 (1). Here we remark that the operator \( 1 - t_i \) on \( K_T(Q_G) \) is invertible, with its inverse \( (1 - t_i)^{-1} = 1 + t_i + t_i^2 + \cdots \), and commutes with \( \tilde{R}_\gamma \) for all \( \gamma \in \Phi. \) Hence, in the case that \( \alpha = \alpha_i \), we can remove the operator \( 1 - t_i \) from both sides of the equation by multiplying both sides by the inverse \( (1 - t_i)^{-1} \). With this observation, the same argument as for [19, Lemma 9.2] shows that the family \( \{\tilde{R}_\beta \mid \beta \in \Phi\} \) also satisfies the Yang-Baxter equation.

Now our assertion can be proved in exactly the same as [19, Theorem 10.1]; use the commutation relations in Lemma 33 instead of [19, Eqs.(10.3)–(10.5)] in the proof of [19, Theorem 10.1].

**Remark 35.** The Yang-Baxter property, as stated in Proposition 33, is a weaker version of the similar property in [19, Definition 9.1]. Indeed, the additional requirement in the mentioned definition is that \( R_{-\alpha} = (R_\alpha)^{-1} \). By Lemma 33 this still holds in our case if \( \alpha \) is not a simple root, whereas \( R_{-\alpha} = (1 - t_i)(R_\alpha)^{-1} \) when \( \alpha = \alpha_i \) for some \( i \in I. \)

Let \( \lambda \in P \) be an arbitrary weight. Recall that a reduced \( \lambda \)-chain \( \Gamma = (\beta_1, \ldots, \beta_m) \) corresponds to the following reduced alcove path:
\[
A_0 = A_0 \overset{\beta_1}{\longrightarrow} A_1 \overset{\beta_2}{\longrightarrow} \cdots \overset{\beta_m}{\longrightarrow} A_m = A_{-\lambda} \ (= A_0 - \lambda).
\]
Remark 36. Let \( \Gamma \) be a reduced \( \lambda \)-chain, and let \( \Gamma' \) be an arbitrary \( \lambda \)-chain (possibly, not reduced). We deduce from the proof of \([19] \) Lemma 9.3 that \( \Gamma \) can be obtained from \( \Gamma' \) by a sequence of the following two procedures (YB) and (D):

(YB) for \( \alpha, \beta \in \Phi \) such that \( \langle \alpha, \beta^\vee \rangle \leq 0 \), or equivalently, \( \langle \beta, \alpha^\vee \rangle \leq 0 \), one replaces a segment of the form \( \alpha, s_\alpha(\beta), s_\alpha s_\beta(\alpha), \ldots, s_\beta(\alpha) \) by \( \beta, s_\beta(\alpha), \ldots, s_\alpha s_\beta(\alpha), s_\alpha(\beta), \alpha \); (D) one deletes a segment \( \beta, -\beta \) for \( \beta \in \Phi \).

We define the following operator (cf. \([19] \) Eq.(12.1)):

\[
R^\lambda := R_{\beta_m} R_{\beta_{m-1}} \cdots R_{\beta_2} R_{\beta_1} = X^\rho (X^{\beta_m} + Q_{\beta_m})(X^{\beta_{m-1}} + Q_{\beta_{m-1}}) \cdots (X^{\beta_2} + Q_{\beta_2}) (X^{\beta_1} + Q_{\beta_1}) X^{-\rho}.
\]

Remarks 37.

(1) We deduce from Proposition 34 and Remark 36 that the operator \( R^\lambda \) does not depend on the choice of a reduced \( \lambda \)-chain.

(2) Let \( \Gamma' : A_0 = B_0 \xrightarrow{-\gamma_1} B_1 \xrightarrow{-\gamma_2} \cdots \xrightarrow{-\gamma_k} B_k = A_{-\lambda} \) be a \( \lambda \)-chain (possibly, not reduced). We see from Proposition 34, Remark 36, and Lemma 33 (1) that

\[
R_{\gamma_k} R_{\gamma_{k-1}} \cdots R_{\gamma_2} R_{\gamma_1} = \prod_{i \in I} (1 - t_i)^{a_i} R^\lambda,
\]

where for each \( i \in I \), \( a_i \in \mathbb{Z} \) is the number of times Deletion (D) for \( \beta = \pm \alpha_i \) is performed when one obtains a reduced \( \lambda \)-chain from \( \Gamma' \).

By the same argument as for \([19] \) Proposition 14.5, we can prove Proposition 38. We use the facts about the central points of alcoves obtained in \([19] \) Lemmas 14.1 and 14.2, and the commutation relations in Lemma 33 instead of \([19] \) Eqs. (10.3)–(10.5)] in the proof of \([19] \) Proposition 14.5.

Proposition 38. Let \( \lambda \in P \) be an arbitrary weight, and let \( \Gamma \) be a reduced \( \lambda \)-chain of the form (24). For \( w \in W \) and \( \xi \in Q^\vee \), we have

\[
R^\lambda [O_{Q_G(w \xi)}] = \sum_{J = \{j_1, \ldots, j_s\}} e^{-w \tilde{r}_{j_1} \tilde{r}_{j_2} \cdots \tilde{r}_{j_s}(-\lambda)} Q_{\beta_{j_s}} \cdots Q_{\beta_{j_2}} Q_{\beta_{j_1}} [O_{Q_G(w \xi)}],
\]

where \( J = \{j_1, \ldots, j_s\} \) runs over all subsets of \( \{1, 2, \ldots, m\} \), and \( \tilde{r}_j \) denotes the affine reflection with respect to the common face of \( A_{j-1} \) and \( A_j \) for \( 1 \leq j \leq m \).

Proposition 39. Let \( \lambda \in P \) be an arbitrary weight. For \( x = w \xi \in W_{af}^\geq 0 \) with \( w \in W \) and \( \xi \in Q^\vee \), we have

\[
[O_{Q_G(-w_0 \lambda)}] \cdot [O_{Q_G(x)}] = \sum_{\chi \in \text{Par}(\lambda)} R^{[\lambda]} [O_{Q_G(w \xi_{\lambda+\delta}(\chi))}].
\]

Proof. If \( \lambda \) is a dominant (resp., anti-dominant) weight, then we deduce by Theorem 26 (resp., Theorem 29) that (27) holds for \( R^{[\lambda]} \) defined by the lex \( \lambda \)-chain \( \Gamma_{\text{lex}}(\lambda) \). Hence we see by Remark 37 that Proposition 39 holds if \( \lambda \) is dominant (resp., anti-dominant).

Now, let \( \lambda \in P \), and write it as \( \lambda = \sum_{i \in I} m_i \varpi_i \) with \( m_i \in \mathbb{Z} \). Then, \( \lambda = \lambda^+ + \lambda^- \), where

\[
\lambda^+ := \sum_{i \in I} \max(m_i, 0) \varpi_i, \quad \lambda^- := \sum_{i \in I} \min(m_i, 0) \varpi_i.
\]
note that $\lambda^+$ is dominant, and $\lambda^-$ is anti-dominant, and that $\overline{\text{Par}}(\lambda) = \overline{\text{Par}}(\lambda^+)$. We deduce that

$$[O_{G^C}(-w_0\lambda)] \cdot [O_{G^C}(x)] = [O_{G^C}(-w_0\lambda^\circ)] \cdot \{[O_{G^C}(-w_0\lambda^\circ)] \cdot [O_{G^C}(x)]]
$$

$$= [O_{G^C}(-w_0\lambda^\circ)] \sum_{\chi \in \overline{\text{Par}}(\lambda^\circ)} R^{[\lambda^\circ]}[O_{G^C}(wt_{\xi+i}(x))].$$

Let $\Gamma' = (\gamma_1, \gamma_2, \ldots, \gamma_k)$ be the $\lambda$-chain obtained by concatenating a reduced $\lambda^+$-chain $\Gamma^+$ with a reduced $\lambda^-$-chain $\Gamma^-$; note that $\Gamma'$ is not reduced in general. Then we have

$$R_{\gamma_k}R_{\gamma_{k-1}} \cdots R_{\gamma_2}R_{\gamma_1} = R^{[\lambda^\circ]}R^{[\lambda^+]}.$$ 

In order to prove that this operator is identical to $R^{[\lambda]}$, it suffices to show that $a_i = 0$ for all $i \in I$ in the notation of Remark 37(2). Let $\Gamma$ be a reduced $\lambda$-chain. Then there exists a sequence $\Gamma' = \Gamma_0, \Gamma_1, \ldots, \Gamma_s = \Gamma$ of $\lambda$-chains such that $\Gamma_t$ is obtained from $\Gamma_{t-1}$ by performing either (YB) or (D) for each $t = 1, 2, \ldots, s$ (see Remark 39). We show by induction on $0 \leq t \leq s$ that $\Gamma_t$ does not contain both of the roots $\alpha_i$ and $-\alpha_i$ for any $i \in I$. Assume that $t = 0$. Let $i \in I$, and assume that $\langle \lambda, \alpha_i^\vee \rangle > 0$; note that $\langle \lambda^+, \alpha_i^\vee \rangle > 0$ and $\langle \lambda^-, \alpha_i^\vee \rangle = 0$. We see from [11] (see also [19] Lemma 6.2) that $\Gamma^+$ contains $\alpha_i$, but does not contain $-\alpha_i$, and that $\Gamma^-$ contains neither $\alpha_i$ nor $-\alpha_i$. Hence $\Gamma'$ contains $\alpha_i$, but does not contain $-\alpha_i$. Similarly, if $\langle \lambda, \alpha_i^\vee \rangle < 0$ (resp., $= 0$), then $\Gamma'$ contains $-\alpha_i$, but does not contain $\alpha_i$ (resp., $\Gamma'$ contains neither $\alpha_i$ nor $-\alpha_i$). Assume that $t > 0$. If $\Gamma_t$ is obtained by applying (D) to $\Gamma_{t-1}$, then it is obvious by the induction hypothesis and the definition of (D) that $\Gamma_t$ holds for $\Gamma_t$. Assume that $\Gamma_t$ is obtained by applying (YB) to $\Gamma_{t-1}$. Then we deduce by the definition of (YB) that the roots appearing in $\Gamma_t$ are the same as those appearing in $\Gamma_{t-1}$. Hence our claim holds for $\Gamma_t$. Thus we have shown our claim, and hence that $a_i = 0$ for all $i \in I$. This completes the proof of Proposition 39.

\[\square\]

\textbf{Proof of Theorem 32.} The statement follows directly from Propositions 38 and 39.

\[\square\]

\section{The Quantum $K$-Theory of Flag Manifolds}

Y.-P. Lee defined the (small) quantum $K$-theory of a smooth projective variety $X$, denoted by $QK(X)$ [11]. This is a deformation of the ordinary $K$-ring of $X$, analogous to the relation between quantum cohomology and ordinary cohomology. The deformed product is defined in terms of certain generalizations of Gromov-Witten invariants (i.e., the structure constants in quantum cohomology), called quantum $K$-invariants of Gromov-Witten type.

In order to describe the (small) $T$-equivariant quantum $K$-algebra $QK_T(G/B)$, for the finite-dimensional flag manifold $G/B$, we associate a variable $Q_k$ to each simple coroot $\alpha_k^\vee$, with $k \in I = \{1, \ldots, r\}$, and let $\mathbb{Z}[Q] := \mathbb{Z}[Q_1, \ldots, Q_r]$. Given $\xi = d_1\alpha_1^\vee + \cdots + d_r\alpha_r^\vee$ in $Q^{\vee,+}$, let $Q^\xi := Q_1^{d_1} \cdots Q_r^{d_r}$. Let $\mathbb{Z}[Q][P] := \mathbb{Z}[Q] \otimes_{\mathbb{Z}} \mathbb{Z}[P]$, where the group algebra $\mathbb{Z}[P]$ of $P$ was defined at the beginning of Section 1. As a $\mathbb{Z}[Q][P]$-module, $QK_T(G/B)$ is defined as $K_T(G/B) \otimes_{\mathbb{Z}[P]} Q[Q][P]$. The algebra $QK_T(G/B)$ has a $\mathbb{C}[Q][P]$-basis given by the classes $[O^w]$, $w \in W$, of the structure sheaf of the (opposite) Schubert variety $X^w \subset G/B$ of codimension $\ell(w)$.

\subsection{Main Results}

It is proved in [8] (see also [9]) that there exists a $\mathbb{C}[P]$-module isomorphism from $QK_T(G/B)$ onto $K_T(Q_G)$ that respects the quantum multiplication in $QK_T(G/B)$ and the tensor product in $K_T(Q_G)$; in particular, it respects the quantum multiplication with the line bundle $O_{G/B}(-w_k)$ and the tensor product with the line bundle $[O_{Q_G}(w_k w_k)]$, for $k \in I$. Note that this isomorphism sends each (opposite) Schubert class $[O^w]Q^\xi$ in $QK_T(G/B)$ to the corresponding
The above result and the formula in Theorem 29 imply an important conjecture in [19]: the Chevalley formula for \(QK\) and results in [12], we describe more explicitly the quantum \(K\)-theory \(QK\), the (non-equivariant) \(K\)-theory \(K\), and the index \(k\) is fixed in this section. We need more background and notation.

Theorem 40. Let \(k \in I\), and fix a reduced \((-\varpi_k)\)-chain \(\Gamma(-\varpi_k)\). Then, in \(QK_T(G/B)\), we have for \(w \in W\)

\[
[O^{s_k}] \cdot [O^w] = (1 - e^{w(\varpi_k) - \varpi_k})[O^w] + \sum_{A \in A(w, \Gamma(-\varpi_k)) \setminus \{\emptyset\}} (-1)^{\overline{A} - 1} Q^\down(w, A) e^{-\varpi_k - wt(w, A)}[O^{\text{end}(w, A)}].
\]

Let us now turn to the type \(A_{n-1}\) flag manifold \(Fl_n = SL_n/B\) and its (non-equivariant) quantum \(K\)-theory \(QK(\text{Fl}_n)\). In [15], the first author and Maeno defined the so-called quantum Grothendieck polynomials. According to [15], Theorem 6.4, whose proof is based on intricate combinatorics, these polynomials multiply precisely as stated by the quantum \(K\)-Chevalley formula (29); note that in the (non-equivariant) \(K\)-theory \(K(G/B)\), the (opposite) Schubert class \([O^w]\) is identical to the class of the structure sheaf of the Schubert variety \(X_{w, w} \subseteq G/B\) of codimension \(\ell(w)\) for each \(w \in W\).

It was proved in [3, Section 5.3] that the quantum \(K\)-theory ring of an arbitrary flag variety is uniquely determined by multiplication with the divisor classes. Based on the mentioned property of the quantum Grothendieck polynomials, we derive the following result, settling the main conjecture in [15].

Theorem 41. The quantum Grothendieck polynomials represent Schubert classes in \(QK(\text{Fl}_n)\).

Theorem 41 leads to an important application of quantum Grothendieck polynomials: computing the structure constants in \(QK(\text{Fl}_n)\) with respect to the Schubert basis. More precisely, the computation reduces to expanding the products of these polynomials in the basis they form. This is achieved by [15], Algorithm 3.28, which can be easily programmed; see also [15], Example 7.4. This application extends the similar one of Schubert polynomials, Grothendieck polynomials, and quantum Schubert polynomials, which was the main motivation for defining these polynomials.

6.2. The type \(A\) quantum \(K\)-Chevalley coefficients. This section refers entirely to type \(A_{n-1}\), more precisely to \(QK(\text{Fl}_n)\). Given a degree \(d = (d_1, \ldots, d_n-1)\), let \(N_{s_k, w}^{r, d}\) be the coefficient of \(Q_1^{d_1} \cdots Q_{n-1}^{d_{n-1}}[O^w]\) in the expansion of \([O^{s_k}] \cdot [O^w]\), for \(k \in I = \{1, \ldots, n-1\}\). Based on Theorem 40 and results in [12], we describe more explicitly the quantum \(K\)-Chevalley coefficients \(N_{s_k, w}^{r, d}\), where the index \(k\) is fixed in this section. We need more background and notation.

We start by recalling an explicit description of the edges of the quantum Bruhat graph on the Weyl group \(W\) of type \(A_{n-1}\), namely the symmetric group \(S_n\). The permutations \(w \in S_n\) are written in one-line notation \(w = w(1) \ldots w(n)\). For simplicity, we use the same notation \((i, j)\) with \(1 \leq i < j \leq n\) for the positive root \(\alpha_{ij} = \varepsilon_i - \varepsilon_j\) and the reflection \(s_{\alpha_{ij}}\), which is the transposition \(t_{ij}\) of \(i\) and \(j\). We need the circular order \(\prec_i\) on \([n] := \{1, \ldots, n\}\) starting at \(i\), namely \(i \prec_i i + 1 \prec_i \ldots \prec_i n \prec_i 1 \prec_i \ldots \prec_i i - 1\). It is convenient to think of this order in terms of the numbers \(1, \ldots, n\) arranged clockwise on a circle. We make the convention that, whenever we write \(a \prec b \prec c \prec \ldots\), we refer to the circular order \(\prec = \prec_a\). We also consider the reverse circular order \(\prec_i^r\) starting at \(i\), namely \(i \prec_i^r i - 1 \prec_i^r \ldots \prec_i^r 1 \prec_i^r n \prec_i^r \ldots \prec_i^r i + 1\), and use the same conventions.

Proposition 42. [12] For \(w \in S_n\) and \(1 \leq i < j \leq n\), we have an edge \(w \xrightarrow{(i,j)} w(i, j)\) if and only if there is no \(l\) such that \(i < l < j\) and \(w(i) \prec w(l) \prec w(j)\).
It is proved in [19] Corollary 15.4 that, given our fixed \( k \in I \), we have the following reduced \( \omega_k \)-chain \( \Gamma(\omega_k) \):

\[
( (k, k + 1), \ (k, k + 2), \ \ldots, \ (k, n) , \\
(k - 1, k + 1), \ (k - 1, k + 2), \ \ldots, \ (k - 1, n) , \\
(1, k + 1), \ (1, k + 2), \ \ldots, \ (1, n) ) .
\]

Alternatively, we can use the following reduced \( \omega_k \)-chain \( \Gamma'(\omega_k) \):

\[
( (k, k + 1), \ (k - 1, k + 1), \ \ldots, \ (1, k + 1) , \\
(k, k + 2), \ (k - 1, k + 2), \ \ldots, \ (1, k + 2) , \\
(k, n), \ (k - 1, n), \ \ldots, \ (1, n) ) .
\]

We will also need a reduced \(( - \omega_k )\)-chain \( \Gamma( - \omega_k ) \), and we choose it to be just the reverse of \( \Gamma(\omega_k) \).

Given \( v, w \in S_n \), we write \( v \rightarrow w \) whenever there is a path from \( v \) to \( w \) in \( QB(S_n) \) with edges labeled by a subsequence of \( \Gamma(\omega_k) \). We also write \( v \leftarrow w \) (or \( w \rightarrow v \)) whenever there is a path from \( w \) to \( v \) in \( QB(S_n) \) with edges labeled by a subsequence of \( \Gamma( - \omega_k) \).

We consider the following conditions on a pair \((v, w)\) of permutations in \( S_n \); the first two appeared in [12] Section 4.1.

**Condition A1.** For any pair of indices \( 1 \leq i < j \leq k \), both statements below are false:

\[
v(i) = w(j), \quad v(i) \prec w(j) \prec w(i).
\]

**Condition A2.** For every index \( 1 \leq i \leq k \), we have

\[
w(i) = \min \{ w(j) \mid i \leq j \leq k \},
\]

where the minimum is taken with respect to the circular order \( \prec_{v(i)} \) on \([n]\) starting at \( v(i) \).

**Condition A1’.** For any pair of indices \( n \geq i > j \geq k + 1 \), both statements below are false:

\[
v(i) = w(j), \quad v(i) \lessdot w(j) \lessdot w(i).
\]

**Condition A2’.** For every index \( n \geq i \geq k + 1 \), we have

\[
w(i) = \min \{ w(j) \mid i \geq j \geq k + 1 \},
\]

where the minimum is taken with respect to the circular order \( \lessdot_{v(i)} \) on \([n]\) starting at \( v(i) \).

It is clear that Conditions A1 and A2, respectively A1’ and A2’, are equivalent. We also consider similar conditions obtained by swapping the orders \( \prec \) and \( \lessdot \), which we label B1, B2, B1’, B2’, respectively.

**Theorem 43.** We have \( v \rightarrow w \) if and only if the pair \((v, w)\) satisfies Conditions A1 and A1’. Moreover, the corresponding path \( v = v_0, v_1, \ldots, v_m = w \) in \( QB(S_n) \) (with edges labeled by a subsequence of \( \Gamma(\omega_k) \)) is unique, and we have

\[
v_0(i) \leq v_1(i) \leq \ldots \leq v_m(i) \quad \text{for} \quad 1 \leq i \leq k ,
\]

\[
v_0(i) \lessdot v_1(i) \lessdot \ldots \lessdot v_m(i) \quad \text{for} \quad n \geq i \geq k + 1 .
\]

**Proof.** Given \( v \rightarrow w \), the pair \((v, w)\) satisfies Condition A1 by [12] Lemma 4.8 (1)]. On another hand, the given path from \( v \) to \( w \) in \( QB(S_n) \) with edges labeled by a subsequence of \( \Gamma(\omega_k) \) can be transformed into a similar path with edges labeled by a subsequence of \( \Gamma'(\omega_k) \). Indeed, by comparing the structures of \( \Gamma(\omega_k) \) and \( \Gamma'(\omega_k) \), we can see that the mentioned transformation only involves commutations of transpositions. We will prove Condition A1’ based on the new path. The first part is immediate. Now assume for contradiction that \( v(i) \lessdot w(j) \lessdot w(i) \) for
$n \geq i > j \geq k + 1$. Examining the sequence of transpositions that involve position $i$, we can see that one of them violates the criterion in Proposition 12.

Now assume that the pair $(v, w)$ satisfies Conditions A1 and A1'. By [12, Lemma 4.8 (2)], there exists a unique path $v = v_0, v_1, \ldots, v_m$ in $\text{QB}(S_n)$ with $v_m(i) = w(i)$ for $i = 1, \ldots, k$. Moreover, the stated property of the entries $v_j(i)$ for a fixed $i \in \{1, \ldots, k\}$ is part of the same lemma. Meanwhile, the case of $i \in \{k+1, \ldots, n\}$ is proved in a similar way, by using the path labeled by a subsequence of $\Gamma'(\varpi_k)$ which is obtained from the above one. Finally, based on the first part of this proof, the pair $(v, v_m)$ satisfies Condition A1', which further implies that $v_m = w$. \hfill $\square$

**Remarks 44.** (1) Fix $v \in W = S_n$ and a representative $\sigma$ of a parabolic coset modulo $W_{I\setminus\{k\}}$. It is easy to see that there is a unique permutation $w \in \sigma W_{I\setminus\{k\}}$ such that the pair $(v, w)$ satisfies Conditions A1 and A1'. Indeed, the equivalent Conditions A2 and A2' lead to an algorithm which suitably reorders the entries $\sigma(1), \ldots, \sigma(k)$ and $\sigma(k+1), \ldots, \sigma(n)$, respectively. More precisely, we iterate the construction of $w(i)$ given by Condition A2 for $i = 1, \ldots, k$, and the construction given by Condition A2' for $i = n, \ldots, k + 1$. This reordering algorithm is explained in more detail in [12], see Remark 4.5 and Example 4.6 there.

(2) Given a pair $(v, w)$ which satisfies Conditions A1 and A1', the construction of the unique path from $v$ to $w$ in Theorem 43 is given by [12, Algorithm 4.9]; this is a greedy type algorithm.

We have the following corollary of Theorem 43.

**Corollary 45.** We have $v \trianglerightequal w$ if and only if the pair $(v, w)$ satisfies Conditions B1 and B1'. Moreover, the corresponding path $w = v_m, v_{m-1}, \ldots, v_0 = v$ in $\text{QB}(S_n)$ (with edges labeled by a subsequence of $\Gamma(-\varpi_k)$) is unique, and we have

\[(30) \quad v_0(i) \trianglerightequal v_1(i) \trianglerightequal \ldots \trianglerightequal v_m(i) \quad \text{for } 1 \leq i \leq k,\]

\[(31) \quad v_0(i) \trianglerightequal v_1(i) \trianglerightequal \ldots \trianglerightequal v_m(i) \quad \text{for } n \geq i \geq k + 1.\]

**Proof.** We use the involution on $W$ given by $w \mapsto w^\circ := w_\circ w$, which maps (quantum) edges of $\text{QB}(W)$ to reverse (quantum) edges with the same labels, cf. [16, Proposition 4.4.1]. Therefore, we have $v \trianglerightequal w$ if and only if $v^\circ \trianglerightequal w^\circ$. Letting $i^\circ := n + 1 - i$, we have $w^\circ(i) = w(i)^\circ$, and we note that the involution $i \mapsto i^\circ$ on $[n]$ maps the order $<_i$ to $<_i^\circ$. Therefore, Conditions A1 and A1' correspond to Conditions B1 and B1' under this involution. We conclude that the statements of the corollary are translations of those in Theorem 43. \hfill $\square$

**Remark 46.** By analogy with Remark 14 (1), Conditions B2 and B2' lead to a corresponding reordering algorithm. Furthermore, there is an algorithm for constructing the unique path in $\text{QB}(S_n)$ in Corollary 45 which is completely similar to [12, Algorithm 4.9], cf. Remark 44 (2).

We are now ready to prove the main result of this section.

**Theorem 47.** For $\text{QK}(F_{I_n})$, we always have $N_{s_k, w}^{v, d} \in \{0, \pm 1\}$. Moreover, for every $v$ and parabolic coset $\sigma W_{I\setminus\{k\}}$, there are unique $d$ and $w \in \sigma W_{I\setminus\{i\}}$ (determined via the algorithms in Remark 46 and (29), cf. also Proposition 48) such that $N_{s_k, w}^{v, d} = \pm 1$ (the sign is as in (29)).

**Proof.** By (29) and Corollary 45, we need $w$ to satisfy Conditions B1 and B1'. The unique such $w$ in $\sigma W_{I\setminus\{i\}}$ is constructed via the reordering algorithm mentioned in Remark 46. Using Corollary 45 again, we know that there is a unique $w$-admissible subset $A \in \mathcal{A}(w, \Gamma(-\varpi_k))$ with $\text{end}(w, A) = v$; this can be constructed via the second algorithm mentioned in Remark 46. Then the corresponding degree $d = (d_1, \ldots, d_{n-1})$ and the sign of $N_{s_k, w}^{v, d}$ are calculated based on (29). \hfill $\square$

We will now show that the degree $d$ in Theorem 47 can be determined based on $v$ and $w$ only, that is, without constructing the respective path in $\text{QB}(S_n)$ from $w$ to $v$. As a corollary, we find
the maximum degree in the quantum $K$-Chevalley formula for a given $k \in I$. We use the notation $|\cdot|$ to indicate the cardinality of a set.

**Proposition 48.** Given a pair $(v, w)$ satisfying Conditions B1 and B1', the unique degree $d = (d_1, \ldots, d_{n-1})$ for which $N_{v,d}^{v,d} = \pm 1$ is expressed as follows:

$$d_i = \begin{cases} \{|j| \leq i, v(j) < w(j)\} & \text{if } i \in \{1, \ldots, k\} \\ \{|j| > i, v(j) > w(j)\} & \text{if } i \in \{k, \ldots, n-1\} \end{cases}.$$  

**Proof.** Consider $i, j \in \{1, \ldots, k\}$. It follows from Corollary 45 and particularly (30), that at most one of the roots $(j, l)$ in $\Gamma(-w_k)$ (for $l > k$) labels a quantum edge in the corresponding path from $w$ to $v$; moreover, this happens if and only if $v(j) < w(j)$. On the other hand, the simple root $\alpha_i = (i, i + 1)$ appears in the decomposition of $(j, l)$ if and only if $j \leq i$. This gives the formula for $d_i$ with $i \in \{1, \ldots, k\}$. The proof is completely similar for $i \in \{k, \ldots, n-1\}$, based on (31). □

**Example 49.** Consider $v = 12534$ in $S_5$, $k = 2$, and $\sigma = 34125$ in $W^I(2)$. The reordering algorithm in Remark 46 outputs $w = 43215 \in \sigma W^I(2)$. The second algorithm mentioned in Remark 46 determines the following path from $w$ to $v$ in $QB(S_5)$; its edges are labeled by a subsequence of $\Gamma(-w_2)$, which corresponds to an admissible subset $A$:

$$\begin{array}{cccccccc}
4 & 3 & < & 5 & 3 & > & 1 & 4 & < & 1 & 3 & > & 1 & 5 & > & 1 & 2 & = v.
\end{array}$$

Thus, we have $\downarrow(w, A) = (1, 4) + (2, 3)$, so $d = (1, 2, 1, 0)$; in fact, it is easier to determine $d$ based on Proposition 48. Finally, we have $N_{v,d}^{v,d} = -1$.

**Corollary 50.** Among all quantum $K$-Chevalley formulas for a fixed $k \in I$, there is a maximum degree (with respect to the componentwise order), namely

$$d_{\text{max}} = (1, 2, \ldots, k-1, \underbrace{k, \ldots, k}_{n+1-2k \text{ times}}, k-1, \ldots, 2, 1),$$

where $k := \min(k, n-k)$. The maximum is attained for $v = \text{id}$ and $w$ given by

$$w(i) = \begin{cases} n + 1 - i & \text{if } i \leq k \text{ or } i > n-k \\ i & \text{otherwise} \end{cases}.$$  

The maximum total degree is $k(n-k)$.

**Proof.** The upper bounds for $d_i$ with $i \in \{1, \ldots, k\} \cup \{n-k, \ldots, n-1\}$ are immediate based on Proposition 48. We will now check that $d_i \leq k$ for any $i$, which is equivalent to the fact that there are no more than $k$ quantum steps in the corresponding path from $w$ to $v$ in $QB(S_n)$. Like in the proof of Proposition 48 we can see that, whenever we have a root $(i, j)$ in $\Gamma(-w_k)$ (so $i \leq k$ and $j > k$) labeling such a quantum step, it follows that $v(i) < w(i)$ and $v(j) > w(j)$. Therefore, these increases for $i \leq k$, respectively decreases for $j > k$, come in pairs. It means that there are no more than $k$ such pairs, which verifies our claim. The remaining part of the corollary is straightforward to check. □

**Remarks 51.** (1) The maximum total degree in Corollary 50 is equal to the length of the maximum element in $W^I(\bar{k})$, that is, the largest Grassmannian permutation corresponding to the given $k$.

(2) Analogous results to Theorem 43 and the algorithms in Remark 44 were given for type $C$ in [12], and for types $B, D$ in [21]; they were used in connection with affine crystals and Macdonald
polynomials. In addition, they can be used to obtain more explicit information about the quantum $K$-Chevalley coefficients in the respective types, by analogy with the above approach in type $A$.

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