On the investigation of some non-linear boundary value problems with parameters

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ON THE INVESTIGATION OF SOME NON-LINEAR BOUNDARY VALUE PROBLEMS WITH PARAMETERS

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Abstract. A scheme of the numerical-analytic method based upon successive approximations for the investigation of non-linear two-point boundary value problems containing parameters both in the differential equation and in the boundary condition is given.

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1. Introduction

The so-called numerical-analytic method (shortly NAM) based upon successive approximations was introduced by the first author jointly with Professor A. Samoilenko [1,2] for the purpose of studying the existence of solutions of non-linear boundary value problems (BVP) and finding approximations to them. For a survey of the further application and development of the NAM to various types of BVPs, including periodic, two-point, multipoint, impulsive, and parametrised ones, one can consult our series of papers in the Ukrainian Mathematical Journal joint with Samoilenko and Trofimchuk. The most recently published paper [3] from this series contains the seventh part of the survey. Extentions of NAM to some types of parametrised boundary value problems (PBVPS) can be found in [4,5].

2. Main results

We consider the following two-point non-linear boundary value problem containing
parameters both in the given differential equation and in the boundary condition:

\[ \frac{dx}{dt} = f(t, x, \lambda_1), \]
\[ Ax(0) + C(\lambda_1)x(\lambda_2) = d(\lambda_1, \lambda_2), \]
\[ x_1(0) = x_{10}, \quad x_2(0) = x_{20}. \]

Here, we suppose that \( x : [0, T] \rightarrow \mathbb{R}^n, \ T > 0 \) is fixed, the functions \( f : \Omega := [0, T] \times D \times [a_1, b_1] \rightarrow \mathbb{R}^n \) and \( d : I_1 \times I_2 \rightarrow \mathbb{R}^n \) are continuous in their domains of definition, \( D \subset \mathbb{R}^n(n \geq 3) \) is a closed, connected, and bounded domain, and \( \lambda_1 \in I_1 := [a_1, b_1], \lambda_2 \in I_2 := (0, T) \) are unknown scalar parameters. The \( n \times n \) matrices \( A \) and \( C(\lambda_1) \) are supposed to be such that \( \det h(\lambda_1) \neq 0 \) and \( \text{rank} \left[ r_{11}(\lambda_1), r_{12}(\lambda_1) \right] = 2 \) for some real \( k_1 \) and \( k_2 \) \( (k_1 \neq k_2) \) and all \( \lambda_1 \in I_1, \) where \( h(\lambda_1) := k_1A + k_2C(\lambda_1), \)

\[ H(\lambda_1) := h(\lambda_1)^{-1}. \]

(In the equality above, the matrices \( r_{11}(\lambda_1), r_{12}(\lambda_1), r_{21}(\lambda_1), \) and \( r_{22}(\lambda_1) \) have dimension \( 2 \times 2, \ 2 \times (n - 2), \ (n - 2) \times 2, \ (n - 2) \times (n - 2), \) respectively.\)

We aim at obtaining the values \( \lambda_1^{*} \in I_1 \) and \( \lambda_2^{*} \in I_2 \) for which the BVP (2.1), (2.2) has a solution \( x^{*} \) satisfying the additional condition (2.3) for its first and second components. By a solution of (2.1)–(2.3), we thus mean the pair \( (\lambda, x) \), where \( \lambda = (\lambda_1, \lambda_2). \)

It is obvious that the right-hand side boundary in BVP (2.1)–(2.3) should also be regarded as a parameter.

Let us denote by \(|f|\) the column \((|f_1|, |f_2|, \ldots, |f_n|)\). The inequalities between the vectors will be understood component-wise.

With this conventions adopted, we set

\[ D_{\beta} := \{ x \in \mathbb{R}^n : B(x, \beta(x)) \subset D \}, \]

where \( \beta : \mathbb{R}^n \rightarrow \mathbb{R}^n, \) and \( B(x, \beta(x)) \) is the \( \beta(x) \)-neighbourhood of an \( x \in \mathbb{R}^n. \)

We also assume that the following three conditions hold for the BVP (2.1)–(2.3):

(i) \( f \) is continuous on \( \Omega \) and bounded by some vector \( M \in \mathbb{R}^n_+ : \)

\[ |f(t, x, \lambda_1)| \leq M \quad \text{for all} \ (t, x, \lambda_1) \in \Omega, \]

and is Lipschitzian in the last two variables, i.e.,

\[ |f(t, x', \lambda_1') - f(t, x'', \lambda_1'')| \leq K|x' - x''| + |\lambda_1' - \lambda_1''|M_1, \]

where \( K \) and \( M_1 \) are non-negative matrices of dimension \( n \times n \) and \( n \times 1, \) respectively;

(ii) The set \( D_{\beta}, \)

\[ \beta(x, \lambda) := \frac{1}{2} TM' + \beta_1(x, \lambda), \]
d_1(x, \lambda) := d(\lambda) - [A + C(\lambda_1)]x,
\beta_1(x, \lambda) := |(k_1 - k_2)H(\lambda_1)[d(\lambda_1, \lambda_2) - (A + C(\lambda_1))x]| + |k_1H(\lambda_1)d_1(x, \lambda)|,
and
M' := \frac{1}{2} \left[ \max_{(t,x,\lambda_1) \in \Omega} f(t, x, \lambda_1) - \min_{(t,x,\lambda_1) \in \Omega} f(t, x, \lambda_1) \right],
is not empty:
D_\beta \neq \emptyset;

(iii) The greatest eigen-value \lambda_{\text{max}}(K) of the matrix K satisfies the inequality
\lambda_{\text{max}}(K) < \frac{q}{T},
where q = \frac{3}{10}.

Let us introduce the sequence of functions
\begin{align*}
x_{m+1}(t, y, \lambda) &= z(y) + k_1H(\lambda_1)d_1(z(y), \lambda) + \int_0^t f(s, x_m(s, y, \lambda), \lambda_1)ds \\
&\quad - \frac{t}{\lambda_2} \int_0^{\lambda_2} f(\tau, x_m(\tau, y, \lambda), \lambda_1)d\tau \\
&\quad + \frac{t}{\lambda_2}(k_2 - k_1)H(\lambda_1)d_1(z(y), \lambda),
\end{align*}
where
\begin{align*}
z &= \text{col}(z_1, z_2, \ldots, z_i, \ldots, z_j, \ldots, z_n) \\
&= \text{col}(y_1, y_2, \ldots, y_i(y), \ldots, y_j(y), \ldots, y_{n-2}) = z(y), \\
y &= \text{col}(y_1, y_2, \ldots, y_{n-2}), 	ext{ and } y_i(y), y_j(y) \text{ are solutions of the first two equations in the system}
x_{m+1}(0, y, \lambda) = \text{col}(x_{10}, x_{20}, x_3(0), \ldots, x_n(0)),
\end{align*}
i.e., the system
\begin{equation}
[E - k_1H(\lambda_1)\{A + C(\lambda_1)\}]z = \text{col}(x_{10}, x_{20}, x_3(0), \ldots, x_n(0)) - d(\lambda_1, \lambda_2).
\end{equation}
(Here and above, i and j denote the numbers of components of the vector z with respect to which system (2.5) is solvable.)

We set \( G = \{ y \in \mathbb{R}^{n-2} : z(y) \in D_\beta \} \). One can verify by direct computation that sequence (2.4) depending on the parameters \( \lambda_1, \lambda_2 \) and on the additional \((n - 2)\)-dimensional vector \( y \), satisfies the boundary conditions (2.2), (2.3) for arbitrary \( \lambda_1 \in I_1, \lambda_2 \in I_2 \), and \( y \in G \).

\textbf{Theorem 1} Assume that the conditions (i)-(iii) hold.
Then:
1. The sequence (2.4) converges to the function \( x^* = x^*(t, y, \lambda) \) as \( m \to \infty \) uniformly in \((t, y, \lambda) \in [0, T] \times G \times I_1 \times I_2\);

2. The limit function \( x^* \) is a solution of the "perturbed" BVP (2.6), (2.2), (2.3),

\[
\frac{dx}{dt} = f(t, x, \lambda_1) + \Delta(y, \lambda),
\]

with the initial value \( x^*(0, y, \lambda) = z(y) + k_1 H(\lambda_1) d_1(z(y), \lambda) \), where

\[
\Delta(y, \lambda) := \frac{1}{\lambda_2} (k_2 - k_1) H(\lambda_1) d_1(z(y), \lambda) - \frac{1}{\lambda_2} \int_0^{\lambda_2} f(t, x^*(t, y, \lambda), \lambda_1) dt;
\]

3. The following error estimation holds:

\[
|x_m(t, y, \lambda) - x^*(t, y, \lambda)| \leq \alpha_1(t, \lambda_2)|Q_m(\lambda_2)(E - Q(\lambda_2))^{-1} M' + KQ(\lambda_2)^{m-1}(E - Q(\lambda_2))^{-1} H_1(z(y), \lambda)|, \tag{2.7}
\]

where \( \alpha_1(t, \lambda_2) := \frac{10}{9} \alpha_1(t, \lambda_2) \leq \frac{5}{9} \lambda_2, \quad \alpha_1(t, \lambda_2) := 2t \left(1 - t\lambda_2^{-1}\right), \quad Q(\lambda_2) := \frac{3\lambda_2}{10} K \).

The proof of Theorem 1 can be carried out by using the techniques from [2] (Theorems 16.1, 18.1, and 20.1) and Theorem 1 of [4].

The following statement establishes the relation of the limit function \( x^* \) to the solution of the original BVP (2.1)–(2.3).

**Theorem 2** Under the conditions of Theorem 1, the pair \((x^*(\cdot, y^*, \lambda^*), \lambda^*)\) is a solution of the BVP (2.1)–(2.3) if, and only if \((y^*, \lambda^*)\) satisfies the determining equation

\[
\Delta(y, \lambda) = \frac{1}{\lambda_2} (k_2 - k_1) H(\lambda_1) d_1(z(y), \lambda) - \frac{1}{\lambda_2} \int_0^{\lambda_2} f(t, x^*(t, y, \lambda), \lambda_1) dt = 0. \tag{2.8}
\]

The proof of Theorem 2 is analogous to the corresponding statements from [2] (Theorems 16.3 and 18.3).

### 3. Sufficient existence conditions

In what follows, we need to consider the \( m \)th approximation to the determining equation (2.8):

\[
\Delta_m(y, \lambda) := \frac{1}{\lambda_2} (k_2 - k_1) H(\lambda_1) d_1(z(y), \lambda) - \frac{1}{\lambda_2} \int_0^{\lambda_2} f(t, x_m(t, y, \lambda), \lambda_1) dt = 0. \tag{3.1}
\]

**Theorem 3** Suppose that, for PBVP (2.1)–(2.3), conditions (i)–(iii) hold and, furthermore,
On some non-linear boundary value problems with parameters

(iv) There exists a closed, convex subset

\[ \Omega_1 = G_1 \times I_1' \times I_2' \subset G \times I_1 \times I_2, \]

where, for some \( m \geq 1 \) fixed, the approximate determining equation (3.1) has only one solution \((\bar{y}, \bar{\lambda})\), which has non-zero topological index;

(v) The inequality

\[ \inf_{(y, \lambda) \in \partial \Omega_1} |\Delta_m(y, \lambda)| > \frac{10}{27} \sup_{\lambda \in I_1' \times I_2'} \{\lambda_2KW(y, \lambda)\} \]  

(3.2)

is satisfied on the boundary \( \partial \Omega_1 \) of the subset \( \Omega_1 \), where

\[ W(x, y) := Q^m(\lambda_2)(E - Q(\lambda_2))^{-1}M' + KQ(\lambda_2)^{m-1}(E - Q(\lambda_2))^{-1}\beta_1(z(y), \lambda). \]

Then, there exists a solution \((x^*, \lambda^*)\) of PBVP (2.1)-(2.3), and the initial value \( x^*(0) \) of this solution at \( t = 0 \) is equal to

\[ z(y^*) + k_1Hd_1(z(y^*), \lambda^*), \]

where \( y^* \in G_1, \lambda^*_1 \in I_1', \) and \( \lambda^*_2 \in I_2' \).

Proof. Based on inequalities (2.7) and (3.2), similarly to Theorems 3.1 and 17.1 of [2], one can show that the vector fields \( \Delta(\cdot, \lambda) \) and \( \Delta_m(\cdot, \lambda) \) are homotopic for all \( \lambda \), which, by the well-known result of degree theory, immediately implies the assertion of Theorem 3. □

4. Necessary Existence Conditions

The following subsidiary statements will be used in the sequel.

Lemma 4 Under conditions (i)–(iii), for an arbitrary pair

\[ \{(z', \lambda'), (z'', \lambda'')\} \subset D_\beta \times I_1 \times I_2, \]  

(4.1)

the inequality

\[ |x^*(t, y', \lambda') - x^*(t, y'', \lambda'')| \leq [E + \bar{\sigma}_1(t, \gamma_2)K[E - Q(\gamma_2)]^{-1}] \left\{ |z(y') - z(y'')| + b_1(y', y'', \lambda', \lambda'') \right\} + \bar{\sigma}_1(t, \gamma_2)K[E - Q(\gamma_2)]^{-1}|\lambda'_1 - \lambda''_1|M_1 \]  

(4.2)

holds, where

\[ b_1(y', y'', \lambda', \lambda'') := |k_1[H(\lambda'_1)d_1(z(y'), \lambda') - H(\lambda''_1)d_1(z(y''), \lambda'')]| \]

\[ + T|k_2 - k_1| \left| \frac{1}{\lambda'_2}H(\lambda'_1)d_1(z(y'), \lambda') - \frac{1}{\lambda''_2}H(\lambda''_1)d_1(z(y''), \lambda'') \right| + 2TM, \]
\[z(y') = \text{col}(y_1', y_2', \ldots, y_{i-1}', y_i'(y'), \ldots, y_{n-2}'),\]

\[z(y'') = \text{col}(y_{1}'', y_2'', \ldots, y_{i-1}'', y_i''(y''), \ldots, y_{n-2}''),\]

and \(\gamma_2 = \max\{\lambda_2', \lambda_2''\} \).

**Proof.** By virtue of (2.4), we have

\[x_1(t, y', \lambda') - x_1(t, y'', \lambda'') = z(y') - z(y'')
+ k_1[\mathcal{H}(\lambda_1')d_1(z(y'), \lambda') - \mathcal{H}(\lambda_1'')d_1(z(y''), \lambda'')]
+ \int_0^t [f(s, z(y'), \lambda_1') - f(s, z(y''), \lambda_1'')] ds
- \frac{t}{\lambda_2'} \int_0^{\lambda_2'} f(\tau, z(y'), \lambda_1') d\tau + \int_0^{\lambda_2'} f(\tau, z(y''), \lambda_1'') d\tau
+ \frac{t}{\lambda_2''} (k_2 - k_1) \mathcal{H}(\lambda_1'')d_1(z(y'), \lambda')
- \frac{t}{\lambda_2''} (k_2 - k_1) \mathcal{H}(\lambda_1'')d_1(z(y''), \lambda'')
= z(y') - z(y'')
+ k_1[\mathcal{H}(\lambda_1')d_1(z(y'), \lambda') - \mathcal{H}(\lambda_1'')d_1(z(y''), \lambda'')]
+ \int_0^t \left\{ f(s, z(y'), \lambda_1') - f(s, z(y''), \lambda_1'') \right\} ds
- \frac{1}{\lambda_2'} \int_0^{\lambda_2'} [f(\tau, z(y'), \lambda_1') - f(\tau, z(y''), \lambda_1'')] d\tau ds
+ \frac{t}{\lambda_2'} \int_0^{\lambda_2'} f(\tau, z(y''), \lambda_1'') d\tau - \frac{t}{\lambda_2''} \int_0^{\lambda_2''} f(\tau, z(y''), \lambda_1'') d\tau
- t(k_2 - k_1) \left[ \frac{1}{\lambda_2'} \mathcal{H}(\lambda_1')d_1(z(y'), \lambda') - \frac{1}{\lambda_2''} \mathcal{H}(\lambda_1'')d_1(z(y''), \lambda'') \right].\]

By using the Lipschitz condition on \(f\), similarly to Lemma 19.1 from [2, p. 154], we obtain

\[|x_1(t, y', \lambda') - x_1(t, y'', \lambda'')| \leq \left[ E + \alpha_1(t, \gamma_2)K |z(y') - z(y'')| + \frac{\alpha_1(t, \gamma_2)|\lambda_1' - \lambda_1''|M_1}{\lambda_2'} \right.
+ b_1(y', \lambda', \lambda''),\]
One can prove by induction that

\[ |x_m(t, y', \lambda') - x_m(t, y'', \lambda'')| \leq \sum_{i=0}^{m} \alpha_i(t, \gamma_2) K^i |z(y') - z(y'')| \]

\[ + \sum_{i=0}^{m} \alpha_i(t, \gamma_2) K^{i-1} |\lambda'_1 - \lambda''_1| M_1 \]

\[ + \sum_{i=0}^{m-1} \alpha_i(t, \gamma_2) K^i b_1(y', y'', \lambda', \lambda''), \] (4.3)

where (see, e.g., [2, p. 148] or [5])

\[ \alpha_{m+1}(t, \gamma) := \left(1 - \frac{t}{\gamma}\right) \int_0^t \alpha_m(s, \gamma) ds + \frac{t}{\gamma} \int_t^\gamma \alpha_m(s, \gamma) ds, \]

and \( \alpha_0(t, \gamma) \equiv 1. \)

Taking into account estimate (see Lemma 4 in [6])

\[ \alpha_{m+1}(t, \gamma) \leq \left(\frac{3}{10} \gamma\right)^m \alpha_1(t, \gamma), \]

\[ \alpha_1(t, \gamma) = \left(\frac{10}{9} \gamma\right)^m \alpha_1(t, \gamma) \leq \frac{5}{9} \gamma \]

and passing to the limit as \( m \to \infty \) in (4.3), we obtain the required inequality (4.2).

\[ \text{Lemma 5} \]

Let us suppose that BVP (2.1)–(2.3) satisfies conditions (i)–(iii).

Then the determining function \( \Delta \) is continuous in the domain \( G \times I_1 \times I_2 \) and, for arbitrary pairs (4.1), the following relation holds:

\[ |\Delta(y', \lambda') - \Delta(y'', \lambda'')| \leq b_2(y', y'', \lambda', \lambda'') + \frac{\gamma_2}{\gamma_1} |\lambda'_1 - \lambda''_1| M_1 \]

\[ + \frac{\gamma_2}{\gamma_1} K \left[ E + \frac{10}{27} \gamma_2 K (E - Q(\gamma_2))^{-1} \right] \left( |z(y') - z(y'')| \right. \]

\[ \left. + b_1(y', y'', \lambda', \lambda'') \right) =: \epsilon(\Delta(y', \lambda'), \Delta(y'', \lambda'')), \] (4.4)

where

\[ b_2(y', y'', \lambda', \lambda'') := |k_2 - k_1| \left| \frac{1}{\lambda'_{1}} H(\lambda'_{1}) d_1(z(y'), \lambda') - \frac{1}{\lambda'_{2}} H(\lambda'_{2}) d_1(z(y'), \lambda') \right| + 2M \]

and \( \gamma_1 := \min\{\lambda'_{2}, \lambda''_{2}\} \).

\[ \text{Proof.} \] For every \( \{y', y''\} \subset G \) such that \( \{z(y'), z(y'')\} \subset D_\beta, \) there exists a continuous limit function of the uniformly convergent function sequence (2.4). The
Theorem 6 Assume that conditions (i)–(iii) hold. Then the subset
\[ \Omega_2 = G_2 \times I_1'' \times I_2'' \subset G \times I_1 \times I_2 \]
may contain a pair \((y^*, \lambda^*)\) generating a solution
\[ x^*(t, y^*, \lambda^*) = \lim_{m \to \infty} x_m(t, y^*, \lambda^*) \]
of PBVP (2.1)–(2.3) only if, for every \(m \geq 1\) and every pair \((\bar{y}, \bar{\lambda})\), the following
relation holds true:

\[
\Delta_m(\tilde{y}, \tilde{\lambda}) \leq \sup_{(y, \lambda) \in \Omega_2} \left\{ b_2(\tilde{y}, y, \tilde{\lambda}, \lambda) + \frac{\gamma_2}{\gamma_1} |\tilde{\lambda}_1 - \lambda_1| M_1 \\
+ \frac{\gamma_2}{\gamma_1} K \left[ E + \frac{10}{27} \gamma_2 K(E - Q(\gamma_2))^{-1} \right] \left( |z(\tilde{y}) - z(y)| \\
+ b_1(\tilde{y}, y, \tilde{\lambda}, \lambda) \right) + \epsilon(\Delta(\tilde{y}, \tilde{\lambda}), \Delta_m(\tilde{y}, \tilde{\lambda})) \right\}.
\] (4.5)

**Proof.** Let the determining function \( \Delta \) vanish at \( y = y^*, \lambda = \lambda^* \), i.e., that \( x^*(\cdot, y^*, \lambda^*) \) is a solution of the PBVP (2.1)–(2.3). Rewriting inequality (4.4) for the pairs \((y', \lambda') = (\tilde{y}, \tilde{\lambda})\) and \((y'', \lambda'') = (y^*, \lambda^*)\), we obtain

\[
|\Delta(\tilde{y}, \tilde{\lambda})| \leq b_2(\tilde{y}, y^*, \tilde{\lambda}, \lambda^*) + \frac{\gamma_2}{\gamma_1} |\tilde{\lambda}_1 - \lambda_1^*| M_1 \\
+ \frac{\gamma_2}{\gamma_1} K \left[ E + \frac{10}{27} \gamma_2 K(E - Q(\gamma_2))^{-1} \right] \left( |z(\tilde{y}) - z(y^*)| + b_1(\tilde{y}, y^*, \tilde{\lambda}, \lambda^*) \right).
\]

Relations (2.8) and (3.1) yield

\[
|\Delta(y, \lambda) - \Delta_m(y, \lambda)| = \left| \frac{1}{\lambda_2} \int_0^{\lambda_2} [f(t, x^*(t, y, \lambda), \lambda_1) - f(t, x_m(t, y, \lambda), \lambda_1)] dt \right|
\leq \frac{1}{\lambda_2} KW(y, \lambda) \int_0^{\lambda_2} \omega_1(t, \lambda_2) dt = \frac{10}{27} \lambda_2 KW(y, \lambda) = \epsilon(\Delta_m(y, \lambda), \Delta_m(y, \lambda)).
\] (4.6)

Relation (4.6) with \((y, \lambda) = (\tilde{y}, \tilde{\lambda})\) implies

\[
|\Delta_m(\tilde{y}, \tilde{\lambda})| \leq |\Delta(\tilde{y}, \tilde{\lambda})| + \epsilon(\Delta_m(\tilde{y}, \tilde{\lambda}), \Delta_m(\tilde{y}, \tilde{\lambda})).
\] (4.7)

Combining (4.6) and (4.7), we obtain the desired necessary condition (4.5). \( \blacksquare \)

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