WEIGHTED AND MAXIMALLY HYPOELLIPTIC ESTIMATES FOR THE FOKKER-PLANCK OPERATOR WITH ELECTROMAGNETIC FIELDS

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Abstract. We consider a Fokker-Planck operator with electric potential and electromagnetic fields. We establish the sharp weighted and subelliptic estimates, involving the control of the derivatives of electric potential and electromagnetic fields. Our proof relies on a localization argument as well as a careful calculation on commutators.

1. Introduction and main results

There have been several works on the Fokker-Planck operator with electric potential \( V(x) \) which is

\[
K = y \cdot \partial_x - \partial_x V(x) \cdot \partial_y - \triangle_y + \frac{|y|^2}{4} - \frac{n}{2}, \quad (x, y) \in \mathbb{R}^{2n},
\]

where \( x \) denotes the space variable and \( y \) denotes the velocity variable, and \( V(x) \) is a potential defined in the whole space \( \mathbb{R}^n \). It is a degenerate operator with the absence of diffusion in \( x \) variable, and can be seen as a Kolmogorov-type operator. The classical hypoelliptic techniques and their global counterparts have been developed recently to establish global estimates and to investigate the short and long time behavior and the spectral properties for Fokker-Planck operator \( K \) in (1.1). We refer to Helffer-Nier’s notes [3] for the comprehensive argument on this topic, seeing also the earlier work [5] of Hérau-Nier. In the first author’s work [10, 11] we improved the previous result and gave a new criterion involving the microlocal property of potential \( V \). Here we also mention the very recent progress made by Ben Said-Nier-Viola [14] and Ben Said [1]. Finally as a result of the global estimates it enables to answer partially a conjecture stated by Helffer-Nier [3] which says Fokker-Planck operator \( K \) has a compact resolvent if and only if Witten Laplacian has a compact resolvent. The necessity part is well-known and the reverse implication still remains open with some partial answers; in fact various hypoelliptic techniques, such as Kohn’ method and nilpotent approach (e.g. [4, 8, 13]), were developed to establish the resolvent criteria for these two different type operators (see [3, 10, 12]).

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Inspired by the recent work of Helffer-Karaki [2], we consider here a more general Fokker-Planck operator with electromagnetic fields besides the electric potential, which reads

\[ P = y \cdot \partial_x - \partial_x V(x) \cdot \partial_y - H(x) \cdot (y \wedge \partial_y) - \Delta_y + \frac{|y|^2}{4} - \frac{n}{2}, \quad (x, y) \in \mathbb{R}^{2n}, \]

where \( n = 2 \) or \( 3 \) and \( H(x) \) is a scalar function of \( x \) for \( n = 2 \) and a vector field \((H_1(x), H_2(x), H_3(x))\) of only \( x \)-variable for \( n = 3 \), and \( y \wedge \partial_y \) is defined by

\[ y \wedge \partial_y = \begin{cases} y_1 \partial_{y_2} - y_2 \partial_{y_1}, & n = 2, \\ (y_2 \partial_{y_3} - y_3 \partial_{y_2}, y_3 \partial_{y_1} - y_1 \partial_{y_3}, y_1 \partial_{y_2} - y_2 \partial_{y_1}), & n = 3. \end{cases} \]

The operator is initiated by Helffer-Karaki [2], where they established the maximal estimate by virtue of nilpotent approach, giving a criteria for the compactness of the resolvent. Here we aim to give another proof, basing on a localization argument and a careful calculation on commutators. Note the operator \( P \) in (1.2) is reduced to the operator \( K \) given \((1.1)\) for \( H \equiv 0 \); meanwhile the maximal estimates for the Fokker-Planck operator with pure electromagnetic fields (i.e., \( V \equiv 0 \)) was investigated by Zeinab Karaki [7].

Before stating our main result we first introduce some notations used throughout the paper. We will use \( \| \cdot \|_{L^2} \) to denote the norm of the complex Hilbert space \( L^2(\mathbb{R}^{2n}) \), and denote by \( C_0^\infty(\mathbb{R}^{2n}) \) the set of smooth compactly supported functions. Denote by \( \mathcal{F}_x \) the (partial) Fourier transform with respect to \( x \) and by \( \xi \) the Fourier dual variable of \( x \). Throughout the paper we use the notation \( \langle \cdot \rangle = (1 + |\cdot|^2)^{1/2} \) and let \( \langle D_x \rangle^r = (1 - \Delta_x)^{r/2} \) be the Fourier multiplier with symbol \( \langle \xi \rangle^r \), that is,

\[ \forall \ u \in C_0^\infty(\mathbb{R}^{2n}), \quad \mathcal{F}_x(\langle D_x \rangle^r u)(\xi) = \langle \xi \rangle^r \mathcal{F}_x u(\xi). \]

Similarly we can define \( \langle D_y \rangle \).

**Theorem 1.1.** Let \( V(x) \in C^2(\mathbb{R}^n) \) with \( n = 2 \) or \( 3 \) be a real-valued function and let \( H(x) \) be a continuous real vector-valued function. Suppose there exists a constant \( C_0 \) such that for any \( x \in \mathbb{R}^n \) we have

\[ |H(x)| \leq C_0 \langle \partial_x V(x) \rangle^\delta \quad \text{with} \quad \delta < 2/3, \]

and

\[ \forall \ |\alpha| = 2, \quad |\partial_x^\alpha V(x)| \leq C_0 \langle \partial_x V(x) \rangle^s \quad \text{with} \quad s < \frac{4}{3}. \]

Then we can find a constant \( C \), depending on the above \( C_0 \) and \( s \), such that

\[ \forall \ u \in C_0^\infty(\mathbb{R}^{2n}), \quad \| \langle D_x \rangle^{2/3} u \|_{L^2} \leq C \left\{ \| Pu \|_{L^2} + \| u \|_{L^2} \right\}. \]

Moreover if \( H \) satisfies additionally that \( H \in C^1(\mathbb{R}^n) \) and

\[ |\partial_x H(x)| \leq C_0 \langle \partial_x V(x) \rangle^{s/2} \]

with \( s \) given in \( (1.4) \), then we have following subelliptic estimate

\[ \forall \ u \in C_0^\infty(\mathbb{R}^{2n}), \quad \| \langle D_x \rangle^r u \|_{L^2} \leq C \left\{ \| Pu \|_{L^2} + \| u \|_{L^2} \right\}, \]
where

\[
\tau = \begin{cases} 
\frac{2}{3}, & \text{if } s \leq \frac{2}{3}, \\
\frac{4 - 3s}{3}, & \text{if } \frac{2}{3} < s \leq \frac{10}{9}, \\
\frac{4 - 3s}{6}, & \text{if } \frac{10}{9} < s < \frac{4}{3}.
\end{cases}
\]

Note if the number \(s\) in (1.4) is less than or equal to \(\frac{2}{3}\) then we obtain the sharp subelliptic exponent \(\tau = \frac{2}{3}\). This enables to obtain the maximal estimate stated as below (see Section 4).

**Corollary 1.2.** If \(V\) satisfies (1.4) with \(s \leq \frac{2}{3}\), and \(H\) satisfies the conditions (1.3) and (1.6). Then we have the following maximal estimate, for any \(u \in C^\infty_0(\mathbb{R}^n)\),

\[
\| \langle \partial_x V \rangle^{2/3} u \|_{L^2} + \| \langle D_x \rangle^{2/3} u \|_{L^2} + \| (y \cdot \partial_x - \partial_x V(x) \cdot \partial_y - H(x) \cdot (y \wedge \partial_y)) u \|_{L^2} \\
+ \| \langle D_y \rangle^2 u \|_{L^2} + \| (y^2) u \|_{L^2} \leq C \left\{ \| Pu \|_{L^2} + \| u \|_{L^2} \right\}.
\]

**Remark 1.3.** The above maximal estimate was established by Helffer-Karaki [2] under similar conditions as that in (1.3), (1.4) and (1.6), but therein they require \(\delta = 0\) and \(s < \frac{2}{3}\). The result in Theorem 1.1 generalizes the one established by the first author [10], considered therein is a specific case of \(H \equiv 0\).

Another consequence of Theorem 1.1 is to analyze the compact criteria for resolvent of Fokker-Planck operator \(P\) in (1.2). Due to the weighted estimate (1.5) we see the Fokker-Planck operator \(P\) admits a compact resolve if \(|\partial_x V(x)| \rightarrow +\infty\) as \(|x| \rightarrow +\infty\). Moreover as in the purely electric case (i.e., \(H \equiv 0\)), \(P\) is closed linked with Witten Laplace operator \(\Delta_V^{(0)}\) defined by

\[
\Delta_V^{(0)} = -\Delta_x + \frac{1}{4} |\partial_x V(x)|^2 - \frac{1}{2} \Delta_x V(x).
\]

In fact we can repeat the argument for proving [10, Corollary 1.3] to conclude the following

**Corollary 1.4.** Let \(H(x)\) and \(V(x)\) satisfy the conditions (1.3), (1.4) and (1.6). Then the Fokker-Planck operator \(P\) in (1.2) has a compact resolvent if the Witten Laplacian \(\Delta_V^{(0)}\) has a compact resolvent.

The paper is organized as follow. In Sections 2 and 3 we prove, respectively, the weighted estimate and the subelliptic estimate in Theorem 1.1. The last section is devoted to proving Corollary 1.2 the maximal estimate.

## 2. Weighted estimate

In this part we prove the weighted estimate (1.5) in Theorem 1.1. From now on we use the notation \(Q = y \cdot \partial_x - \partial_x V(x) \cdot \partial_y - H(x) \cdot (y \wedge \partial_y)\) and \(L_j = \partial_{y_j} + \frac{y_j}{2}, j = 1, \ldots, n\).
Then we can rewrite the Fokker-Planck operator \( P \) in (1.2) as

\[
P = Q + \sum_{j=1}^{n} L_{j}^{*}L_{j}.
\]

By direct verification the following estimates hold for any \( u \in C_{0}^{\infty}(\mathbb{R}^{2n}) \). Here and below to simplify the notation we will use the capital letter \( C \) to denote different suitable constants.

**Lemma 2.1.** Let \( H \) and \( V \) satisfy (1.3) and (1.4) respectively. Then for any \( \varepsilon > 0 \) we can find a constant \( C_{\varepsilon} \) such that the following estimate

\[
\sum_{j=1}^{n} \left( \| L_{j} \partial_{x}V \|^{1/3}u \|^{2}_{L^{2}} + \| L_{j} \langle y \rangle u \|^{2}_{L^{2}} \right) \leq \varepsilon \| \langle \partial_{x}V \rangle^{2/3} u \|^{2}_{L^{2}} + C_{\varepsilon} \left\{ \| Pu \|^{2}_{L^{2}} + \| u \|^{2}_{L^{2}} \right\}
\]

holds for all \( u \in C_{0}^{\infty}(\mathbb{R}^{2n}) \).

**Proof of this lemma.** We use (2.2) to get

\[
\sum_{1 \leq j \leq n} \| L_{j} \langle \partial_{x}V \rangle^{1/3} u \|^{2}_{L^{2}} \leq \text{Re} \left\langle P \langle \partial_{x}V(x) \rangle^{1/3} u, \langle \partial_{x}V(x) \rangle^{1/3} u \right\rangle_{L^{2}}
\]

\[
\leq \varepsilon \| \langle \partial_{x}V \rangle^{2/3} u \|^{2}_{L^{2}} + C_{\varepsilon} \| Pu \|^{2}_{L^{2}} + \text{Re} \left\langle [P, \langle \partial_{x}V(x) \rangle^{1/3}] u, \langle \partial_{x}V(x) \rangle^{1/3} u \right\rangle_{L^{2}}.
\]

Moreover using (1.4) yields, for any \( \varepsilon_{1}, \varepsilon > 0 \),

\[
\text{Re} \left\langle [P, \langle \partial_{x}V(x) \rangle^{1/3}] u, \langle \partial_{x}V(x) \rangle^{1/3} u \right\rangle_{L^{2}}
\]

\[
\leq \varepsilon_{1} \| \langle y \rangle \langle \partial_{x}V \rangle^{1/3} u \|^{2}_{L^{2}} + C_{\varepsilon_{1}} \| \langle \partial_{x}V \rangle^{s-2/3} u \|^{2}_{L^{2}}
\]

\[
\leq \frac{1}{2} \sum_{j=1}^{n} \| L_{j} \langle \partial_{x}V \rangle^{1/3} u \|^{2}_{L^{2}} + \varepsilon \| \langle \partial_{x}V \rangle^{2/3} u \|^{2}_{L^{2}} + C_{\varepsilon} \| u \|^{2}_{L^{2}},
\]

the last inequality using (2.3) and the fact that \( s < 4/3 \). Combining the above estimates we obtain

\[
\sum_{j=1}^{n} \| L_{j} \langle \partial_{x}V(x) \rangle^{1/3} u \|^{2}_{L^{2}} \leq \varepsilon \| \langle \partial_{x}V \rangle^{2/3} u \|^{2}_{L^{2}} + C_{\varepsilon} \left\{ \| Pu \|^{2}_{L^{2}} + \| u \|^{2}_{L^{2}} \right\}
\]

(2.4)

Similarly, using again (2.2) and (2.3),

\[
\sum_{1 \leq j \leq n} \| L_{j} \langle y \rangle u \|^{2}_{L^{2}} \leq \text{Re} \left\langle P \langle y \rangle u, \langle y \rangle u \right\rangle_{L^{2}} \leq \| \langle P, \langle y \rangle \rangle u, \langle y \rangle u \|_{L^{2}} + \| Pu \|_{L^{2}} \| \langle y \rangle^{2} u \|_{L^{2}}
\]

\[
\leq \varepsilon_{1} \sum_{1 \leq j \leq n} \| L_{j} \langle y \rangle u \|^{2}_{L^{2}} + C_{\varepsilon_{1}} \left\{ \| Pu \|^{2}_{L^{2}} + \| u \|^{2}_{L^{2}} \right\} + \| \langle P, \langle y \rangle \rangle u, \langle y \rangle u \|_{L^{2}},
\]

(2.5)
with \( \varepsilon_1 > 0 \) arbitrarily small. Moreover it follows from the assumption (1.3) that, for any \( \varepsilon, \varepsilon_1 > 0 \),

\[
\langle [P, \langle y \rangle u, \langle y \rangle u]_{L^2} \rangle \leq C \| \partial_x V \|^{2/3} \| u \|_{L^2} \| \langle y \rangle \partial_x V \|^{1/3} \| u \|_{L^2} \\
+ C \| \langle y \rangle \|_{L^2} \| \partial_x V \| \delta \| u \|_{L^2} + C \| \langle y \rangle u \|_{L^2} \| \langle D_y \rangle u \|_{L^2}
\]

\[
\leq \varepsilon \| \partial_x V \|^{2/3} \| u \|_{L^2}^2 + \varepsilon_1 \sum_{1 \leq j \leq n} \| L_j \langle y \rangle \|_{L^2}^2 + \| P u \|_{L^2}^2 + \| u \|_{L^2}^2
\]

\[
+ C_{\varepsilon, \varepsilon_1} \left\{ \sum_{1 \leq j \leq n} \| L_j \partial_x V \|^{1/3} \| u \|_{L^2}^2 + \| P u \|_{L^2}^2 + \| u \|_{L^2}^2 \right\},
\]

where the last inequality holds because of (2.3) and the fact that \( \delta < 2/3 \). As a result combining the above estimates and choosing \( \varepsilon_1 \) small enough, we conclude

\[
\sum_{1 \leq j \leq n} \| L_j \langle y \rangle \|_{L^2}^2 \leq \varepsilon \| \partial_x V \|^{2/3} \| u \|_{L^2}^2 + C_{\varepsilon} \left\{ \sum_{1 \leq j \leq n} \| L_j \partial_x V \|^{1/3} \| u \|_{L^2}^2 + \| P u \|_{L^2}^2 + \| u \|_{L^2}^2 \right\}
\]

\[
\leq \varepsilon \| \partial_x V \|^{2/3} \| u \|_{L^2}^2 + C_{\varepsilon} \left\{ \| P u \|_{L^2}^2 + \| u \|_{L^2}^2 \right\},
\]

the last inequality using (2.4). This with (2.4) completes the proof of Lemma 2.1. \( \square \)

**Proof of Theorem 1.1: weighted estimate.** Here we will prove the weighted estimate (1.5) in Theorem 1.1. Let \( M \in C^1(\mathbb{R}^{2n}) \) be a real-valued function given by

\[
M = M(x, y) = 2 \langle \partial_x V(x) \rangle^{-2/3} \partial_x V(x) \cdot y.
\]

We use the fact that \( |M(x, y)| \leq C |y| \langle \partial_x V(x) \rangle^{1/3} \) and

\[
\text{Re} \langle Pu, Mu \rangle_{L^2} = \text{Re} \langle Qu, Mu \rangle_{L^2} + \text{Re} \sum_{j=1}^{n} \langle L_j^* L_j u, Mu \rangle_{L^2}
\]

due to (2.1), to conclude, by virtue of (2.3),

\[
\text{Re} \langle Qu, Mu \rangle_{L^2} \leq \| Pu \|_{L^2}^2 + C \| \langle y \rangle \partial_x V \|^{1/3} \| u \|_{L^2}^2 + \sum_{j=1}^{n} \| L_j^* L_j u, Mu \|_{L^2}^2
\]

\[
\leq \| Pu \|_{L^2}^2 + C \sum_{1 \leq j \leq n} \left( \| L_j \partial_x V \|^{1/3} \| u \|_{L^2} + \| L_j \langle y \rangle u \|_{L^2} + \| \langle y \rangle \|_{L^2}^2 \right)
\]

\[
\leq \varepsilon \| \partial_x V \|^{2/3} \| u \|_{L^2}^2 + C_{\varepsilon} \left( \| Pu \|_{L^2}^2 + \| u \|_{L^2}^2 \right),
\]

where we use Lemma 2.1 in the last line. As for the term on left side we use the fact that \( \text{Re} \langle Qu, Mu \rangle_{L^2} = \frac{1}{2} \langle [M, Q]u, u \rangle_{L^2} \) and

\[
\frac{1}{2} [M, Q] = -y \cdot \partial_x \left( \langle \partial_x V \rangle^{-2/3} \partial_x V \cdot y \right) + \langle \partial_x V \rangle^{-2/3} |\partial_x V|^2 + \langle \partial_x V \rangle^{-2/3} H \cdot (y \wedge \partial_x V)
\]

to compute, using (1.3) and (1.4) as well as Lemma 2.1,

\[
\| \langle \partial_x V \rangle^{2/3} u \|_{L^2}^2 \leq \text{Re} \langle Qu, Mu \rangle_{L^2} + \| u \|_{L^2}^2 + \varepsilon \| \langle \partial_x V \rangle^{2/3} u \|_{L^2}^2 + C_{\varepsilon} \| \langle y \rangle \partial_x V \|^{1/3} u \|_{L^2}^2
\]

\[
\leq \text{Re} \langle Qu, Mu \rangle_{L^2} + \varepsilon \| \langle \partial_x V(x) \rangle^{2/3} u \|_{L^2}^2 + C_{\varepsilon} \left( \| Pu \|_{L^2}^2 + \| u \|_{L^2}^2 \right),
\]
and thus, letting $\varepsilon = 1/2$ above,
\[
\| (\partial_x V)^{2/3} u \|_{L^2}^2 \leq 2 \text{Re} \langle Qu, Mu \rangle_{L^2} + C \{ \| Pu \|_{L^2}^2 + \| u \|_{L^2}^2 \}.
\]
As a result, we combine the above estimate with (2.5) to get
\[
\| (\partial_x V)^{2/3} u \|_{L^2}^2 \leq \varepsilon \| (\partial_x V)^{2/3} u \|_{L^2}^2 + C \varepsilon \{ \| Pu \|_{L^2}^2 + \| u \|_{L^2}^2 \},
\]
which gives the desired weighted estimate (1.5) if we let $\varepsilon$ be small enough. \hfill \Box

As an immediate consequence of Lemma 2.1 and the weighted estimate (1.5) we see the estimate
\[
\sum_{j=1}^n \left( \| L_j (\partial_x V)^{1/3} u \|_{L^2}^2 + \| L_j \langle y \rangle u \|_{L^2}^2 \right) \leq C \{ \| Pu \|_{L^2}^2 + \| u \|_{L^2}^2 \}
\]
holds for all $u \in C_0^\infty(\mathbb{R}^n)$, provided $H$ and $V$ satisfy (1.3) and (1.4) respectively.

3. Subelliptic estimate

In this section we will prove the subelliptic estimate (1.7) in Theorem 1.1. The proof relies on a localization argument. Firstly we recall some standard results concerning the partition of unity. For more detail we refer to [6, 9] for instance. Let $g$ be a metric of the following form
\[
g_x = (\partial_x V(x))^s \, |dx|^2, \quad x \in \mathbb{R}^n,
\]
where $s$ is the real number given in (1.4).

**Lemma 3.1** (Lemma 4.2 in [10]). Suppose $V$ satisfies the condition (1.4). Then the metric $g$ defined by (3.1) is slowly varying, i.e., we can find two constants $C_*, r > 0$ such that if $g_x(x - \tilde{x}) \leq r^2$ then
\[
C_*^{-1} \leq \frac{g_x}{g_{\tilde{x}}} \leq C_*.
\]

**Lemma 3.2** ((Lemma 18.4.4. in [6])). Let $g$ be a slowly varying metric. We can find a constant $r_0 > 0$ and a sequence $x_\mu \in \mathbb{R}^n, \mu \geq 1$, such that the union of the balls
\[
\Omega_{\mu, r_0} = \{ x \in \mathbb{R}^n; \ g_x(x - x_\mu) < r_0^2 \}
\]
covers the whole space $\mathbb{R}^n$. Moreover there exists a positive integer $N$, depending only on $r_0$, such that the intersection of more than $N$ balls is always empty. One can choose a family of nonnegative functions $\{ \varphi_\mu \}_{\mu \geq 1}$ such that
\[
\text{supp } \varphi_\mu \subset \Omega_{\mu, r_0}, \quad \sum_{\mu \geq 1} \varphi_\mu^2 = 1 \quad \text{and} \quad \sup_{\mu \geq 1} |\partial_x \varphi_\mu(x)| \leq C (\partial_x V(x))^\frac{s}{2}.
\]

By Lemmas 3.1 and 3.2 we can find a constant $C$, such that for any $\mu \geq 1$ one has
\[
\forall \ x, \tilde{x} \in \text{supp } \varphi_\mu, \quad C^{-1} \langle \partial_x V(\tilde{x}) \rangle \leq \langle \partial_x V(x) \rangle \leq C \langle \partial_x V(\tilde{x}) \rangle.
\]
Lemma 3.3 (Lemma 4.6 in [10]). Let \( \{\varphi_\mu\}_{\mu \geq 1} \) be the partition given in Lemma 3.2, and let \( a \in [0, 1/2] \) be a real number. Then there exists a constant \( C \), depending on the integer \( N \) given in Lemma 3.3, such that for any \( u \in C_0^\infty (\mathbb{R}^{2n}) \) we have
\[
\| (1 - \Delta_x)^a u \|^2_{L^2} \leq C \sum_{\mu \geq 1} \| (1 - \Delta_x)^a \varphi_\mu u \|^2_{L^2} + C \| P u \|^2_{L^2} + C \| u \|^2_{L^2}.
\]

Let \( \{\varphi_\mu\}_{\mu \geq 1} \) be the partition of unity given in Lemma 3.2. For each \( x_\mu \in \mathbb{R}^n \) we define the operator
\[
P_{x_\mu} = y \cdot \partial_x - \partial_x V(x_\mu) \cdot \partial_y - H(x_\mu) \cdot (y \wedge \partial_y) - \Delta_y + \frac{|y|^2}{4} - \frac{n}{2}.
\]
Then
\[
\varphi_\mu P u = P_{x_\mu} \varphi_\mu u + R_\mu u
\] (3.4)
with
\[
R_\mu = -y \cdot \partial_x \varphi_\mu (x) - \varphi_\mu (\partial_x V(x) - \partial_x V(x_\mu)) \cdot \partial_y - \varphi_\mu (H(x) - H(x_\mu)) \cdot (y \wedge \partial_y).
\] (3.5)

Lemma 3.4. Suppose \( H(x) \) and \( V(x) \) satisfy the conditions (1.3)-(1.4) and (1.6). Let \( R_\mu \) be the operator given in (3.5). Then
\[
\forall u \in C_0^\infty (\mathbb{R}^{2n}), \quad \sum_{\mu \geq 1} \| R_\mu u \|^2_{L^2} \leq C \left\{ \| P (\partial_x V(x))^\ddot{s} u \|^2_{L^2} + \| P u \|^2_{L^2} + \| u \|^2_{L^2} \right\},
\] (3.6)
where \( \ddot{s} = \frac{2}{3} - \tau \) with \( \tau \) given in (1.7), i.e., \( \ddot{s} \) equals to 0 if \( s \leq 2/3 \), \( s - 2/3 \) if \( 2/3 < s \leq 10/9 \), and \( s/2 \) if \( 10/9 < s < 4/3 \).

Proof. We write
\[
\sum_{\mu \geq 1} \| R_\mu u \|^2_{L^2} \leq I_1 + I_2 + I_3
\] (3.7)
with
\[
I_1 = 2 \sum_{\mu \geq 1} \| (y \cdot \partial_x \varphi_\mu) u \|^2_{L^2},
\]
\[
I_2 = 2 \sum_{\mu \geq 1} \| \varphi_\mu (\partial_x V(x) - \partial_x V(x_\mu)) \cdot \partial_y u \|^2_{L^2},
\]
\[
I_3 = 2 \sum_{\mu \geq 1} \| \varphi_\mu (H(x) - H(x_\mu)) \cdot (y \wedge \partial_y) u \|^2_{L^2}.
\]

Note it is just finite sum of at most \( N \) terms for each \( I_k, 1 \leq k \leq 3 \), recalling \( N \) is the integer given in Lemma 3.2. It follows from the last inequality in (3.2) that
\[
I_1 \leq C \| (y \cdot (\partial_x V(x)))^s/2 u \|^2_{L^2}.
\]
Similarly observing \( |x - x_\mu| \leq C \| (\partial_x V(x_\mu))^{-\ddot{s}/s} \) for any \( x \in \text{supp} \varphi_\mu \), we use the conditions (1.3) and (1.6) as well as (3.3) to compute
\[
I_2 \leq C \| (D_y \cdot (\partial_x V(x)))^{s/2} u \|^2_{L^2}.
\]
and
\[ I_3 \leq C \left\| \langle D_y \rangle \langle y \rangle u \right\|^2_{L^2} \leq C \sum_{1 \leq j \leq n} \left\| L_j \langle y \rangle u \right\|^2_{L^2} + C \left\| \langle y \rangle u \right\|^2_{L^2} \leq C \left\{ \left\| Pu \right\|^2_{L^2} + \left\| u \right\|^2_{L^2} \right\}, \]
the last inequality using the estimates (1.5) and (2.6) that were established in the previous section. As a result plugging the estimates on \( I_k \) into (3.7) yields
\[ \sum_{n \geq 1} \| R_n u \|^2_{L^2} \leq C \left( \left\| \langle y \rangle \langle \partial_x V \rangle \right\|^2_{L^2} + \left\| \langle D_y \rangle \langle \partial_x V \rangle \right\|^2_{L^2} \right) \]
the last inequality using again the weighted estimate (1.5) that were established in the previous section. This with (3.8) yields the validity of (3.6) for \( s \leq 2/3 \).

(a) The case of \( s \leq 2/3 \). In such a case we have
\[ \left\| \langle y \rangle \langle \partial_x V \rangle \right\|^2_{L^2} + \left\| \langle D_y \rangle \langle \partial_x V \rangle \right\|^2_{L^2} \leq C \sum_{1 \leq j \leq n} \| L_j \langle \partial_x V \rangle \|^2_{L^2} + C \| \langle \partial_x V \rangle \|^2_{L^2} \leq C \left\{ \left\| Pu \right\|^2_{L^2} + \left\| u \right\|^2_{L^2} \right\}, \]
the last inequality using the estimates (1.5) and (2.6) that were established in the previous section. This with (3.8) yields the validity of (3.6) for \( s \leq 2/3 \).

(b) The case of \( 10/9 < s < 4/3 \). We use (2.2) and (2.3) to conclude, for any \( u \in C^{\infty}_0 (\mathbb{R}^{2n}) \),
\[ \left\| \langle y \rangle \langle \partial_x V \rangle \right\|^2_{L^2} + \left\| \langle D_y \rangle \langle \partial_x V \rangle \right\|^2_{L^2} \leq C \left\| P \langle \partial_x V \rangle \right\|^2_{L^2} + C \| \langle \partial_x V \rangle \|^2_{L^2} \]
the last inequality using again the weighted estimate (1.5) since \( s < 4/3 \). This gives the validity of (3.6) for \( 10/9 < s < 4/3 \).

(c) The case of \( 2/3 < s \leq 10/9 \). In this case we use (2.2) and (2.3) to compute
\[ \left\| \langle y \rangle \langle \partial_x V \rangle \right\|^2_{L^2} + \left\| \langle D_y \rangle \langle \partial_x V \rangle \right\|^2_{L^2} \leq C \left\| P \langle \partial_x V \rangle \right\|^2_{L^2} + C \| \langle \partial_x V \rangle \|^2_{L^2} \]
Using the fact that \( 2s - \frac{5}{3} \leq \frac{s}{2} \) for \( s \leq 10/9 \), and letting \( \varepsilon \) above small enough, we get, in view of the weighted estimate (1.3),
\[ \left\| \langle y \rangle \langle \partial_x V \rangle \right\|^2_{L^2} + \left\| \langle D_y \rangle \langle \partial_x V \rangle \right\|^2_{L^2} \leq C \left\{ \left\| P \langle \partial_x V \rangle \right\|^2_{L^2} + \left\| Pu \right\|^2_{L^2} \right\}. \]
Inserting the above inequality into (3.8) we get the desired estimate (3.6) for \( 2/3 < s \leq 10/9 \). Thus the proof of Lemma 3.4 is completed. \( \square \)
Lemma 3.5. There is a constant $C$ independent of $x_\mu$, such that for any $u \in C_0^\infty(\mathbb{R}^{2n})$, one has
\[
\| \langle \partial_x V(x_\mu) \rangle^2 u \|_{L^2}^2 + \| \langle D_x \rangle^{2/3} u \|_{L^2}^2 \leq C \left\{ \| P_{x_\mu} u \|_{L^2}^2 + \| u \|^2_{L^2} \right\},
\]
or equivalently,
\[
\| \left( 1 + |\partial_x V(x_\mu)|^2 - \Delta_x \right)^{1/3} u \|_{L^2}^2 \leq C \left\{ \| P_{x_\mu} u \|_{L^2}^2 + \| u \|^2_{L^2} \right\},
\]
where the fractional Laplacian is defined by
\[
\mathcal{F}_x \left\{ \left( 1 + |\partial_x V(x_\mu)|^2 - \Delta_x \right)^{1/3} u \right\} = \left( 1 + |\partial_x V(x_\mu)|^2 + \xi^2 \right)^{1/3} \mathcal{F}_x u.
\]

Proof. This follows from classical hypoelliptic technique, seeing for instance [3, Proposition 5.22]. We omit it here for brevity.

Completeness of the proof of Theorem 1.4: subelliptic estimate. In this part we will prove the subelliptic estimate (1.7). Let $\varphi_\mu, \mu \geq 1$, be the partition of unit given in Lemma 3.2 and let $\tau$ be given in (1.7). Then we use Lemma 3.5 to compute
\[
\| \langle D_x \rangle^{\tau} u \|_{L^2}^2 \leq C \sum_{\mu \geq 1} \| (1 - \Delta_x)^{\tau/2} \varphi_\mu u \|_{L^2}^2 + C \| P u \|_{L^2}^2 + C \| u \|_{L^2}^2,
\]
and moreover, observing $\tau \leq 2/3$ and using Fourier transform in $x$ if necessary,
\[
\| (1 - \Delta_x)^{\tau/2} \varphi_\mu u \|_{L^2}^2 \leq \| \left( 1 + |\partial_x V(x_\mu)|^2 - \Delta_x \right)^{\tau/2} \varphi_\mu u \|_{L^2}^2
\]
\[
= \| \left( 1 + |\partial_x V(x_\mu)|^2 - \Delta_x \right)^{1/3} \left( 1 + |\partial_x V(x_\mu)|^2 - \Delta_x \right)^{\tau/2} \varphi_\mu u \|_{L^2}^2
\]
\[
\leq \| \left( 1 + |\partial_x V(x_\mu)|^2 - \Delta_x \right)^{1/3} \langle \partial_x V(x_\mu) \rangle^{-\frac{\tau}{2}} \varphi_\mu u \|_{L^2}^2
\]
\[
\leq C \| P_{x_\mu} \langle \partial_x V(x_\mu) \rangle^{-\frac{s}{2}} \varphi_\mu u \|_{L^2}^2 + C \| \varphi_\mu u \|_{L^2}^2
\]
where $\hat{s} = \frac{3}{2} - \tau \geq 0$, and the last inequality follows from Lemma 3.5. As a result, combining the above estimates yields
\[
\| \langle D_x \rangle^{\tau} u \|_{L^2}^2 \leq C \sum_{\mu \geq 1} \| P_{x_\mu} \langle \partial_x V(x_\mu) \rangle^{-\hat{s}} \varphi_\mu u \|_{L^2}^2 + C \| P u \|_{L^2}^2 + C \| u \|_{L^2}^2.
\]

Thus the desired subelliptic estimate (1.7) will follow if the following
\[
\sum_{\mu \geq 1} \| P_{x_\mu} \langle \partial_x V(x_\mu) \rangle^{-\hat{s}} \varphi_\mu u \|_{L^2}^2 \leq C \left\{ \| P u \|_{L^2}^2 + \| u \|_{L^2}^2 \right\} \tag{3.9}
\]
holds for all $u \in C_0^\infty(\mathbb{R}^{2n})$, recalling $\hat{s} = \frac{3}{2} - \tau$ with $\tau$ given in (1.7). To prove (3.9) we write
\[
\langle \partial_x V(x_\mu) \rangle^{-\hat{s}} \varphi_\mu u = \left( \langle \partial_x V(x) \rangle^s \langle \partial_x V(x_\mu) \rangle^{-\hat{s}} \right) \varphi_\mu \langle \partial_x V(x) \rangle^{-\hat{s}} u.
\]
Then
\[
\sum_{\mu \geq 1} \| P_{x_\mu} \langle \partial_x V(x_\mu) \rangle^{-\hat{s}} \varphi_\mu u \|_{L^2}^2 \leq S_1 + S_2,
\]
with
\[ S_1 = 2 \sum_{\mu \geq 1} \| (\partial_x V(x)^{\frac{s}{2}} (\partial_x V(x)^{-\frac{s}{2}}) P_{x_{\mu}} \varphi_{\mu} (\partial_x V(x)^{-\frac{s}{2}}) u \|^2_{L^2} \]
\[ S_2 = 2 \sum_{\mu \geq 1} \| [P_{x_{\mu}}, (\partial_x V(x)^{\frac{s}{2}} (\partial_x V(x)^{-\frac{s}{2}})] \varphi_{\mu} (\partial_x V(x)^{-\frac{s}{2}}) u \|^2_{L^2}. \]

Using (3.3) gives
\[ S_1 \leq C \sum_{\mu \geq 1} \| P_{x_{\mu}} \varphi_{\mu} (\partial_x V(x)^{-\frac{s}{2}}) u \|^2_{L^2} \]
\[ \leq C \left\{ \| Pu \|^2_{L^2} + \| P (\partial_x V(x)^{-\frac{s}{2}}) u \|^2_{L^2} + \| (\partial_x V(x)^{-\frac{s}{2}}) u \|^2_{L^2} \right\} \]
\[ \leq C \| Pu \|^2_{L^2} + C \| u \|^2_{L^2}, \]
where in the first inequality we use (3.4) and Lemma 3.4, and the last inequality holds because
\[ \| P (\partial_x V(x)^{-\frac{s}{2}}) u \|^2_{L^2} \leq 2 \| Pu \|^2_{L^2} + 2 \| (\partial_x V(x)^{-\frac{s}{2}}) u \|^2_{L^2} \]
\[ \leq C \| Pu \|^2_{L^2} + C \| (\partial_x V(x)^{-\frac{s}{2}}) u \|^2_{L^2} \leq C \| Pu \|^2_{L^2} + C \| u \|^2_{L^2} \quad (3.10) \]
due to (2.2) and (2.3) as well as the fact that \(-s + s - 1 \leq 0\) and \(s \geq 0\). Similarly, following the argument in (3.10) we have
\[ S_2 \leq C \sum_{\mu \geq 1} \| \varphi_{\mu} (y) u \|^2_{L^2} \leq C \| (y) u \|^2_{L^2} \leq C \| Pu \|^2_{L^2} + C \| u \|^2_{L^2}. \]
This with the estimate on \( S_1 \) yields (3.9), and thus the subelliptic estimate (1.7) follows. The proof of Theorem 1.1 is completed. \( \square \)

4. Maximal estimate

In this part we investigate the maximal estimate, i.e., Corollary 1.2. First we list some commutation relations to be used below. Let \( Q \) and \( L_j, 1 \leq j \leq n, \) be given at the beginning of Section 2. By direct verification we have
\[ [L_j, L_k] = [L_j^*, L_k^*] = 0, \quad [L_j, L_k^*] = 1 \text{ if } j = k \text{ and } 0 \text{ otherwise}, \quad (4.1) \]
and moreover
\[ [Q, L_j^*] = \partial_{x_j} - \frac{1}{2} \partial_{x_j} V(x) + H(x) \cdot (e_j \wedge \partial_y) - \frac{1}{2} H(x) \cdot (y \wedge e_j), \quad (4.2) \]
where \( e_j = (0, \cdots, 1, \cdots, 0) \in \mathbb{R}^n \) with only \( j \)-th component equal to 1.

**Proof of Corollary 1.2** Using (2.2) gives
\[ \| L_j L_j^* u \|^2_{L^2} \leq \text{Re} \langle P L_j^* u, L_j^* u \rangle_{L^2} \]
\[ = \text{Re} \langle [P, L_j^*] u, L_j^* u \rangle_{L^2} + \text{Re} \langle Pu, L_j L_j^* u \rangle_{L^2} \]
\[ \leq \text{Re} \langle [P, L_j^*] u, L_j^* u \rangle_{L^2} + \frac{1}{2} \| L_j L_j^* u \|^2_{L^2} + 2 \| Pu \|^2_{L^2}. \]
Hence
\[ \| L_j L_j^* u \|^2_{L^2} \leq 2 \left( \| [P, L_j^*] u, L_j^* u \|_{L^2} \right) + 4\| Pu \|^2_{L^2}. \tag{4.3} \]

Moreover it follows from (1.4) and the estimates (1.5) and (1.7) with \( \tau = \frac{\varepsilon}{2} \), the last inequality following from Lemma 2.1 and (2.2). For the last term on the right side we use (1.3),
\[ \left| \left( [P, L_j^*] u, L_j^* u \right)_{L^2} \right| \leq C \left\{ \| (\partial_x V)^{2/3} u \|^2_{L^2} + \| (D_x)^{2/3} u \|^2_{L^2} \right\} + C \left\{ \| L_j (\partial_x V)^{1/3} u \|^2_{L^2} + \| L_j (D_x)^{1/3} u \|^2_{L^2} \right\} \]

As a result, combining the above estimates we obtain
\[ \left| \left( [P, L_j^*] u, L_j^* u \right)_{L^2} \right| \leq \varepsilon \left\{ \| L_j L_j^* u \|^2_{L^2} + \varepsilon \left( \| (D_x)^{\frac{2}{3}} u \|^2_{L^2} + \| Pu \|^2_{L^2} + \| u \|^2_{L^2} \right) \right\} \]

which, together with (1.3) and the estimates (1.5) and (1.7) with \( \tau = 2/3 \) therein, yields
\[ \sum_{1 \leq j \leq n} \| L_j L_j^* u \|^2_{L^2} \leq C \left\{ \| (\partial_x V)^{\frac{2}{3}} u \|^2_{L^2} + \| (D_x)^{\frac{2}{3}} u \|^2_{L^2} + \| Pu \|^2_{L^2} + \| u \|^2_{L^2} \right\} \]
\[ \leq C \left\{ \| Pu \|^2_{L^2} + \| u \|^2_{L^2} \right\} \]

The gives the assertion in Corollary 1.2, completing the proof. \( \Box \)

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