New traveling wave rational form exact solutions for strain wave equation in micro structured solids

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Abstract
Strain wave equation is a fourth order non-linear partial differential equation that arises in the study of non-dissipative strain wave propagation in micro structured solids. This equation also represents the dynamics of several physical phenomena. This equation can also be consider as a generalization of Boussinesq equation with dual dispersion. In this paper, a general strain wave equation is considered and obtained several new exact solutions. A variant of F-expansion method is applied to obtain the required solutions. The available traveling wave exact solutions are primarily obtained by integrating the resulting fourth order ordinary differential equation twice. But, in this paper, we show that there exist several traveling wave solutions to strain wave equation which cannot be derived using the existing methods. Several families of new exact solutions in rational function form are derived using this novel method, without performing the initial integration.

1. Introduction

Evolution equations appearing in several applied fields such as fluid mechanics, plasma physics and quantum field theory are inherently nonlinear. Exact solutions to these equations are inevitable in understanding the physics behind the processes. Even though there are several methods for solving nonlinear partial differential equations, such as, Backlund transformation, Darboux transformation, inverse scattering method, the homogeneous balance method and Lie group method, the available exact solutions are very less in the case of several equations [1–3, 4–8]. Traveling wave solutions of nonlinear partial differential equations play a major role in analyzing the corresponding physical problems. The different methods to obtain such solutions are tanh-method, sech- method, Exp-function method [9–14].

The strain wave equation, which is a nonlinear partial differential equation that governs the micro strain waves in micro structured solids [15–17], is given by

\[ v_{tt} - v_{xx} = \epsilon \gamma_1 (v^2)_{xx} - \mu \gamma_2 v_{xxx} + \mu \gamma_3 v_{xxxx} - (\mu \gamma_4 - \nu \gamma_5) v_{xx} + \mu \mu (\gamma_6 v_{xxxx} + \gamma_7 v_{xxxxx}) = 0, \]  

(1)

where \( \epsilon \) is the coefficient corresponding to elastic strains, \( \mu \) gives the ratio between micro structure size and the wave length, \( \nu \) is the coefficient of dissipation and \( \gamma_i \)'s are nonzero real numbers. When \( \nu = 0 \), this equation describes the non-dissipative case of strain wave equation. This equation also generalizes the Boussinesq equation, which is a fourth order nonlinear equation with applications in several fields [15, 18, 16, 19, 20, 21–24, 10, 25, 26, 17, 27, 28–32, 33]. It represents the dynamics of physical phenomena such as shallow water waves in ocean regions, non-linear lattice waves and vibration of nonlinear strings. There are several variants for strain wave equation in the literature. One such equation is given by [34]
\( v_{tt} - v_{xx} - 3(v^2)_{xx} - v_{xxxx} = 0, \) 

(2)

where \( v \) is a function of the variables \( x \) and \( t \). The so called good Boussinesq equation and bad Boussinesq equation are also particular cases of strain wave equation [33], given by

\( v_{tt} - v_{xx} - (v^2)_{xx} \pm v_{xxxx} = 0, \)

(3)

Another particular case of strain wave equation with dual dispersion is given by [16, 35]

\( v_{tt} - v_{xx} - (v^2)_{xx} - v_{xxxx} + \alpha v_{xxtt} = 0, \)

(4)

where \( \alpha \) is any positive constant.

In this paper, we consider the general form of non-dissipative strain wave propagation in micro structure solids. This equation in dimensionless form is given by [15, 16, 35]

\[ v_{tt} + c_1 v_{xx} + c_2 (v^2)_{xx} + c_3 v_{xxxx} + c_4 v_{xxtt} = 0, \]

(5)

where \( c_1, c_2, c_3 \) and \( c_4 \) are real constants.

Certain exact solutions for the strain wave equation and some of its variants have been derived by several authors [15, 18, 16, 36, 35, 22–24, 10, 17, 27, 28–31, 33, 37]. The most common approach to solve strain wave equation is to assume the solution in traveling wave form and then convert the equation in to a fourth order ordinary differential equation. This is then integrated to obtain a second order equation and any of the several ansatz methods are applied to derive exact solutions. Even then, the available exact solutions for this equation are very less in the literature and there is a need to find out new exact solutions due to the wide range of applications of this equation in different physical models. Our aim is to derive several families of new exact solutions to the strain wave equation, which are not available in the literature, using a novel method explained here.

We show that certain exact solutions for strain wave equation in rational form are not able to be derived from the second order ordinary differential equation obtained by integrating the converted fourth order ordinary differential equation twice. We use the original fourth order ordinary differential equation, without performing the initial integration twice, to obtain several new exact solutions using a rational \( F \)-expansion method for the first time. First of all we derive eight families of generic solutions to this equation and then derive several doubly periodic particular solutions in terms of Jacobi elliptic functions.

The method employed in this paper is explained in the next section. Applying this method, several general forms for the new exact solutions are derived in the third section. In the fourth section, families of explicit exact solutions are derived from these general solutions using Jacobi elliptic functions. The paper is concluded with a discussion on the new exact solutions derived.

2. The method

The traveling wave ansatz method is applied to convert the given nonlinear partial differential equation (5) in to a nonlinear ordinary differential equation. Let

\[ v(t, x) = g(at + bx), \]

(6)

where \( a \) and \( b \) are real parameters. Substituting this traveling wave form in the equation (5), it becomes an ordinary differential equation

\[ b^2 g^{(4)}(u)(a^2 c_4 + b^2 c_3) + (a^2 + b^2 c_1)g''(u) + b^2 c_2 (g(u)^2)'' = 0, \]

(7)

where \( u = at + bx \). Now, on integration two times and assuming zero values to the constants of integration, we get the second order nonlinear ordinary differential equation

\[ b^2 (a^2 c_4 + b^2 c_3) g''(u) + g(u)(a^2 + b^2 c_1) + b^2 c_2 g(u)^2 = 0. \]

(8)

Almost all the traveling wave exact solutions available in the literature for the equation (5) or its variants, have been derived by solving this second order equation [15, 16, 35, 10, 17, 28, 30, 31, 33]. Since the order of the equation is reduced by two, different ansatz methods can be easily applied to this equation to derive exact solutions. If we apply the same methods to the original fourth order equation, the computations become more difficult and sometimes infeasible. Our main objective in this paper is to show that there are several nontrivial exact traveling wave solutions to strain wave equation which can be derived directly from the fourth order ordinary differential equation (7), which are not solutions to the second order equation (8). So, in this paper, we directly solve the equation (7) to obtain new traveling wave exact solutions in rational form to the strain wave equation (5), for the first time.

A new variant of \( F \)-expansion method is applied to derive the desired exact solutions to the strain wave equation. We seek solutions of the form
where \(A_1, B_1\) and \(B_2\) are parameters to be determined later. Here, \(F(u)\) satisfies the first order nonlinear differential equation

\[
g(u) = \frac{A_1}{B_1 + B_2 F(u)},
\]

(9)

where \(P, Q\) and \(R\) are constants. These constants can be selected in such a way that the above equation is solvable in terms of functions such as Jacobi elliptic functions. From equation (10), it follows that the function \(F(u)\) satisfies the following equations.

\[
F' = P F^3 + Q F, \quad F'' = 6 P F^2 F' + Q F', \quad F''' = 6 P (2 F'^2 + F^2 F'') + Q F''.
\]

(11)

These equations are used to convert the fourth order nonlinear ordinary differential equation (7) in to an algebraic equation in terms of \(F\) and this equation is then solved to extract the required exact solutions for strain wave equation.

3. General form of exact solutions

Exact traveling wave solutions for the non-dissipative strain wave equation (3) are achieved in this section. This is achieved by solving the fourth order nonlinear ordinary differential equation (7) in the form (9), using the equations (10) and (11).

Substituting (9) in the equation (7), we get the following ordinary differential equation

\[
2 A_1 B_1 B_2^2 R (12 b^2 B_1^2 R (a^2 c_4 + b^2 c_3) + B_1^3 (4 a b^2 c_4 Q + a^2 + 4 b c_4 Q + b^2 c_3) + 3 A_1 b^2 B_1 c_2 = 0, \]

\[-2 A_1 B_2^2 P (12 b^2 B_1^2 P (a^2 c_4 + b^2 c_3) + B_2^2 B_1 (4 a b^2 c_4 Q + a^2 + 4 b c_4 Q + b^2 c_3) - A_1 b^2 B_2^2 c_2 = 0, \]

\[A_1 B_2 (-2 B_1 P (10 a b^2 c_4 Q + a^2 + 10 b^2 c_4 Q + b^2 c_3) + B_1 B_2^2 (a^2 (Q - b^2 c_4 (12 P + 11 Q^2)) + b^2 c_3 (-12 P + 11 Q^2)) + b^2 c_3 Q) + 4 A_1 b^2 c_2 (B_2^2 Q - B_1^2 P) = 0, \]

\[B_1^3 (-a^2 (b^2 c_3 (12 P + Q^2) + Q) + b^4 c_3 (12 P + Q^2) + b^2 c_3 Q) + 4 B_1 B_2^2 R (a^2 (1 - 5 b^2 c_4 Q) - 5 a c_3 Q + b^2 c_3) - 2 A_1 b^2 c_2 (B_2^2 Q - 3 B^2_1 R) = 0, \]

\[A_1 B_2^2 (4 b^2 P (a^2 (5 b^2 c_4 Q - 1) + 5 b^2 c_4 Q - b^2 c_3) - 2 A_1 b^2 B_1 c_2 P - B_2^2 (a^2 (b^2 c_4 (12 P + Q^2) + Q) + b^4 c_3 (12 P + Q^2) + b^2 c_3 Q) = 0, \]

\[2 B_2^2 R (10 a b^2 c_4 Q + a^2 + 10 b^2 c_4 Q + b^2 c_3) = 0. \]

(13)

Solving this system of equations simultaneously using MATHEMATICA, we get the following set of eight different solutions

\[
A_1 = -\frac{3 b^2 B_1 (c_3 - c_4) \Delta}{c_2 (\sqrt{2} (2 b^2 c_4 Q - 1) - 3 b^2 c_3 \Delta)}, \quad B_2 = \pm \frac{B_1 \Theta}{\sqrt{2}}, \quad a = \pm \frac{\Phi}{\Gamma},
\]

(14)

\[
A_1 = -\frac{3 b^2 B_1 (c_3 - c_4) \Delta}{c_2 (\sqrt{2} (2 b^2 c_4 Q - 1) - 3 b^2 c_3 \Delta)}, \quad B_2 = \pm \frac{B_1 \Theta}{\sqrt{2}}, \quad a = \pm \frac{\Phi}{\Gamma},
\]

(15)
\[ A_1 = \frac{3b^2B_1(c_3 - c_4)\Delta}{c_2(3b^2c_4\Delta + \sqrt{(2b^2c_4Q - 1)})}, \quad B_2 = \pm \frac{B_1}{\sqrt{2}} \hat{\Phi}, \quad a = \mp \hat{\Phi}, \]  
\[ A_1 = \frac{3b^2B_1(c_3 - c_4)\Delta}{c_2(3b^2c_4\Delta + \sqrt{(2b^2c_4Q - 1)})}, \quad B_2 = \pm \frac{B_1}{\sqrt{2}} \hat{\Phi}, \quad a = \mp \hat{\Phi}, \]  
where

\[ \Delta = Q^2 - 4PR, \]
\[ \Theta = \sqrt{\frac{\Delta + Q}{R}}, \]
\[ \hat{\Theta} = \sqrt{\frac{\Delta - Q}{R}}, \]
\[ \Gamma = \sqrt{\frac{(\Delta + Q)b^2c_2(36PR - 5Q^2) - 4b^2c_4Q + 1}{R}}, \]
\[ \hat{\Gamma} = \sqrt{\frac{(Q - \sqrt{\Delta})b^2c_2(36PR - 5Q^2) - 4b^2c_4Q + 1}{R}}, \]
\[ \Phi = \left( \frac{b^2}{R} \right)(-Q(36b^4c_5c_4PR + c_3) + 5b^4c_5c_4Q^3)
- \sqrt{\Delta}(b^4c_3(36PR - 5Q^2) + Q) + c_1(1 - 5b^2c_4Q))
+ 12b^2(c_3 - c_4c_4)PR - b^2(c_3 - 5c_4c_4)Q^2), \]
\[ \hat{\Phi} = \left( \frac{b^2}{R} \right)(-Q(36b^4c_5c_4PR + c_3) + 5b^4c_5c_4Q^3)
+ \sqrt{\Delta}(b^4c_3(36PR - 5Q^2) + Q) + c_1(1 - 5b^2c_4Q))
+ 12b^2(c_3 - c_4c_4)PR - b^2(c_3 - 5c_4c_4)Q^2). \]

Substituting these solutions in equation (9), we get, eight different solutions for the fourth order ordinary differential equation (7), using any function \( F \) satisfying the requirements given in equations (10) and (11). These general form of solutions for any such function \( F(at + bx) \) are given by

\[ v(t, x) = \frac{3b^2(c_3 - c_4)\Delta}{c_2(3b^2c_4Q - 1) - 3b^2c_4\Delta} \left( 1 \pm \frac{\theta}{\sqrt{2}} F \left( bx \pm \hat{\Phi} \frac{\theta}{\Gamma} t \right) \right), \]  
\[ v(t, x) = \frac{3b^2(c_3 - c_4)\Delta}{c_2(3b^2c_4Q - 1) - 3b^2c_4\Delta} \left( 1 \pm \frac{\theta}{\sqrt{2}} F \left( bx \mp \hat{\Phi} \frac{\theta}{\Gamma} t \right) \right), \]  
\[ v(t, x) = \frac{3b^2(c_3 - c_4)\Delta}{c_2(3b^2c_4Q + \sqrt{(2b^2c_4Q - 1)})} \left( 1 \pm \frac{\theta}{\sqrt{2}} F \left( bx \pm \hat{\Phi} \frac{\theta}{\Gamma} t \right) \right), \]
and

\[ v(t, x) = \frac{3b^2(c_3 - c_4)\Delta}{c_2(3b^2c_4Q + \sqrt{(2b^2c_4Q - 1)})} \left( 1 \pm \frac{\theta}{\sqrt{2}} F \left( bx \mp \hat{\Phi} \frac{\theta}{\Gamma} t \right) \right). \]

The above solutions are not able to derive from the second order ODE (8), since the corresponding resulting system of algebraic equations are having only trivial solutions. It can also be easily verified that the above solutions do not satisfy this second order ODE. So, all the above solutions can be derived only by using the fourth order ODE (7).

**4. Results and discussion**

The general form of solutions of strain wave equation (5) are given by equations (18)-(21). Depending on the values of different parameters appearing in the general form of solutions, we get real solutions or complex solutions. For getting real solutions a necessary condition is that \( \Delta \geq 0 \) and we consider only such cases. It is known that all the twelve Jacobi elliptic functions satisfy equation (10) and \( \Delta > 0 \). So, corresponding to every Jacobi elliptic function we can find exact solutions for the Boussinesq equation (5) in the general forms (18)-(21). A total of ninety six new traveling wave exact solutions, eight in terms of each Jacobi elliptic functions, can be derived from these general solutions. All these solutions will generate the periodic and hyperbolic solutions in the limiting case of the modulus of Jacobi elliptic functions as the modulus tends to zero or one respectively.
We illustrate explicitly the exact solutions for strain wave equation given by equation (18) in terms of Jacobi ellips
tic functions $u_{sn}$ and $cd(u)$, where $k$ is the modulus of the Jacobi elliptic functions with $0 \leq k^2 \leq 1$ [38] and the case of remaining ten Jacobi elliptic functions can be done in a similar manner. Here we take $e = k^2$ for our convenience, where $0 < e < 1$. Let $P = e, Q = -1 - e, R = 1$ in equation (10), then the function $F(u)$ becomes $F(u) = sn(u)$ or $F(u) = cd(u)$. Then, clearly $\Delta > 0$. Substituting these values in the equations (18), following exact solutions for strain wave equation are obtained.

$$v_{1,2}(t, x) = \frac{3b^2(c_3 - c_1)\Delta}{c_2(\sqrt{2b^2c_4(1 + e) + 1} + 3b^2c_4\Delta)\left(1 \pm \frac{e}{\sqrt{e}}\frac{\frac{dx}{\sqrt{e}}}{\frac{dt}{\sqrt{e}}}\right)}, \quad (22)$$

$$v_{3,4}(t, x) = \frac{3b^2(c_3 - c_1)\Delta}{c_2(\sqrt{2b^2c_4(1 + e) + 1} + 3b^2c_4\Delta)\left(1 \pm \frac{e}{\sqrt{e}}\frac{\frac{dx}{\sqrt{e}}}{\frac{dt}{\sqrt{e}}}\right)}, \quad (23)$$

where, now, the values of $\Delta, \Theta, \hat{\Theta}, \Gamma, \hat{\Gamma}, \Phi$ and $\hat{\Phi}$ are given below.

$$\Delta = (e - 1)^2, \quad \Theta = \sqrt{e - \sqrt{(e - 1)^2 + 1}}, \quad \hat{\Theta} = \sqrt{e + \sqrt{(e - 1)^2 + 1}},$$

$$\Gamma = \sqrt{(-e + \sqrt{(e - 1)^2 - 1})(b^4c_4^2(-e - 5)(5e - 1) + 4b^2c_4(e + 1) + 1)},$$

$$\hat{\Gamma} = \sqrt{e + \sqrt{(e - 1)^2 + 1})(b^4c_4^2(-e - 5)(5e - 1) - 4b^2c_4(e + 1) - 1)},$$

$$\Phi = ((b^2(5b^4c_4c_1(e + 1)^3 - (e + 1)(36b^4c_4c_1c_3 + c_1)$$

$$- 12b^2(c_3 - c_1c_3)e + b^2(c_3 - 5c_1c_4)(e + 1)^2 + \sqrt{(e - 1)^2(5b^2c_1c_4(e + 1)$$

$$+ b^2c_3(b^2c_4(-e - 5)(5e - 1) - e - 1) + c_1))))^\frac{1}{2},$$

$$\hat{\Phi} = ((b^2(5b^4c_4c_1(e + 1)^3 + (e + 1)(36b^4c_4c_1c_3 + c_1)$$

$$+ 12b^2(c_3 - c_1c_3)e - b^2(c_3 - 5c_1c_4)(e + 1)^2 + \sqrt{(e - 1)^2(5b^2c_1c_4(e + 1)$$

$$+ b^2c_3(b^2c_4(-e - 5)(5e - 1) - e - 1) + c_1))))^\frac{1}{2}.$$

A graphical representation of the above solutions for Boussinesq equation with dual dispersion is given in figure 1.

5. Conclusion

Several families of new generic exact solutions for general strain wave equation (5) in micro-structured solids are derived in this paper. From the derived solutions we can easily generate exact solutions for several variants of strain wave equation, such as equations (2), (3) and (4), by properly assigning values for the parameters. Since there appear several radicals in the solutions, we have to give the domains of existence of real solutions for all these important equations in terms of the modulus of the Jacobi elliptic functions and the coefficient $b$ of the variable $x$. Graphical representations of two sample solutions are given in figure 1, where first one represents the exact solution (22) in terms of Jacobi elliptic function $sn(u)$ and the second one represents the solution (23) in
terms of the Jacobi elliptic function \( cd(u) \). This shows that even though representation of the solutions are involved, the behavior of the solutions are nice.

We can obtain a total of ninety six such solutions in terms of the different Jacobi elliptic functions, all of which are new in the literature. It is possible to obtain trigonometric and hyperbolic solutions to the strain wave equation by simply letting the modulus of Jacobi elliptic functions tends to zero or one respectively. Some of the available rational exact trigonometric and hyperbolic solutions for such equations in [37] are limiting cases of the solutions obtained in this paper. The several families of exact solutions derived in this paper are useful in the study of micro structured solids, water waves in fluid dynamics and in analysing the phenomena in ocean engineering and other physical phenomena such as the dynamics of thin inviscid layers, non-linear lattice waves and vibration in nonlinear strings. They can be also utilized to check the accuracy of several approximate methods used to derive solutions for strain wave equation and its variants. The new rational \( F \)-expansion method applied in this paper can also be used to derive several exact solutions of nonlinear partial differential equations representing other physical phenomena.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Conflict of interest

The author declare that he has no conflict of interest.

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