Extension of 2-forms and symplectic varieties

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Introduction

In this paper we shall prove two theorems (Stability Theorem, Local Torelli Theorem) for symplectic varieties.

Let us recall the notion of a symplectic singularity. Let $X$ be a good representative of a normal singularity. Then the singularity is symplectic if the regular locus $U$ of $X$ admits an everywhere non-degenerate holomorphic closed 2-form $\omega$ where $\omega$ extends to a regular form on $Y$ for a resolution of singularities $Y \to X$. Similarly we say that a normal compact Kaehler space $Z$ is a symplectic variety if the regular locus $V$ of $Z$ admits a non-degenerate holomorphic closed 2-form $\omega$ where $\omega$ extends to a regular form on $\tilde{Z}$, where $\tilde{Z} \to Z$ is a resolution of singularities of $Z$. When $Z$ has a resolution $\pi : \tilde{Z} \to Z$ such that $(\tilde{Z}, \pi^*\omega)$ is a symplectic manifold, we call $Z$ has a symplectic resolution.

Examples

(i) This is one of examples of symplectic singularities studied in [Be 1]. For details see [Be 1] and the references there. Let $Q \subset \mathbb{P}^{n-1}$ be a general quadratic hypersurface. Identify a point of the Grassmannian $Gr(2, n)$ with a line in $\mathbb{P}^{n-1}$. Let $Gr_{iso}(2, n)$ be the subvariety of $Gr(2, n)$ corresponding to the lines of $\mathbb{P}^{n-1}$ contained in $Q$. It is checked that $\dim Gr_{iso}(2, n) = \dim Gr(2, n) - 3 = 2n - 7$. Embed $Gr_{iso}(2, n)$ into $\mathbb{P}^{1/2n(n-1)-1}$ by the Plücker embedding $Gr(2, n) \to \mathbb{P}^{1/2n(n-1)-1}$. Now consider the cone $X$ over $Gr_{iso}(2, n)$. Then the germ $(X, 0)$ at the vertex is a symplectic singularity of dimension $2n - 6$. The $X$ is actually obtained as the closure $\overline{O_{min}}$ of the minimal nilpotent orbit $O_{min}$ of the Lie algebra $Lie(SO(n))$, and $\overline{O_{min}}$ has the Kostant-Kirillov symplectic 2-form.

(ii) Let $A := \mathbb{C}^{2l}/\Gamma$ be an Abelian variety of dimension $2l$. Let $(z_1, z_2, ..., z_{2l-1}, z_{2l})$ be the standard coordinates of $\mathbb{C}^{2l}$. Then $\mathbb{Z}/2\mathbb{Z}$ acts on $A$ by $z_i \to -z_i$ ($i = 1, ..., 2l$). The quotient $Z$ of $A$ by the action becomes a symplectic variety of dimension $2l$. A symplectic 2-form is, for example, given by $\Sigma_{1 \leq i < j \leq 2l} dz_i \wedge dz_{i+j}$. The $Z$ has singularities, and $Z$ has no symplectic resolution when $l > 1$.

(iii) These are symplectic varieties studied by O’Grady [O]. Let $S$ be a polarized K3 surface. Let $c$ be an even number with $c \geq 4$. Denote by $\overline{M}_{0,c}$ the moduli space of rank 2 semi-stable torsion free sheaves with $c_1 = 0$ and $c_2 = c$. $\overline{M}_{0,c}$ becomes a projective symplectic variety of dim = $4c - 6$. The singular locus $\Sigma$ has dimension $2c$. Moreover, O’Grady showed that $\overline{M}_{0,4}$ has a
symplectic resolution, however $\overline{M}_{0,c}$ has $\mathbb{Q}$-factorial terminal singularities when $c \geq 6$ (cf. section 3 of the e-print version of [O]: [alg-geom/9708009]). Therefore $\overline{M}_{0,c}$ have no symplectic resolution when $c \geq 6$.

A symplectic singularity / variety will play an important role in the generalized Bogomolov decomposition conjecture (cf. [Kata], [Mo]):

**Conjecture:** Let $Y$ be a smooth projective variety over $\mathbb{C}$ with Kodaira dimension 0. Then there is a finite etale cover $Y' \to Y$ such that $Y'$ is birationally equivalent to $Y_1 \times Y_2 \times Y_3$, where $Y_1$ is an Abelian variety, $Y_2$ is a symplectic variety, and $Y_3$ is a Calabi-Yau variety.

In this conjecture we hope that it is possible to replace $Y_2$ and $Y_3$ by their birational models with only $\mathbb{Q}$-factorial terminal singularities respectively. Main results are these.

**Theorem 7 (Stability Theorem):** Let $(Z, \omega)$ be a projective symplectic variety. Let $g : Z \to \Delta$ be a projective flat morphism from $Z$ to a 1-dimensional unit disc $\Delta$ with $g^{-1}(0) = Z$. Then $\omega$ extends sideways in the flat family so that it gives a symplectic 2-form $\omega_t$ on each fiber $Z_t$ for $t \in \Delta$, with a sufficiently small $\varepsilon$.

In the above, the result should also hold for a (non-projective) symplectic variety $(Z, \omega)$ and for a proper flat morphism $g$. But two ingredients remained unproved in the general case (cf. Remark below Theorem 7).

Let $Z$ be a symplectic variety. Put $\Sigma := \text{Sing}(Z)$ and $U := Z \setminus \Sigma$. Let $\pi : Z \to S$ be the Kuranishi family of $Z$, which is, by definition, a semi-universal flat deformation of $Z$ with $\pi^{-1}(0) = Z$ for the reference point $0 \in S$. When $\text{codim}(\Sigma \subset Z) \geq 4$, $S$ is smooth by [Na 1, Theorem 2.4]. $Z$ is not projective over $S$. But we can show that every member of the Kuranishi family is a symplectic variety (cf. Theorem 7'). Define $\mathcal{U}$ to be the locus in $Z$ where $\pi$ is a smooth map and let $\pi : \mathcal{U} \to S$ be the restriction of $\pi$ to $\mathcal{U}$. Then we have

**Theorem 8 (Local Torelli Theorem):** Assume that $Z$ is a $\mathbb{Q}$-factorial projective symplectic variety. Assume $h^1(Z, \mathcal{O}_Z) = 0$, $h^0(U, \Omega^2_Z) = 1$, $\dim Z = 2l \geq 4$ and $\text{Codim}(\Sigma \subset Z) \geq 4$. Then the following hold.

1. $R^2\pi_*\big(\pi^{-1}\mathcal{O}_S\big)$ is a free $\mathcal{O}_S$ module of finite rank. Let $\mathcal{H}$ be the image of the composite $R^2\pi_*\mathcal{O}_Z \to R^2\pi_*\mathcal{O}_Z' \to R^2\pi_*\big(\pi^{-1}\mathcal{O}_S\big)$. Then $\mathcal{H}$ is a local system on $S$ with $\mathcal{H}_s = H^2(U_s, \mathcal{C})$ for $s \in S$.

2. The restriction map $H^2(Z, \mathcal{C}) \to H^2(U, \mathcal{C})$ is an isomorphism. Take a resolution $\nu : \tilde{Z} \to Z$ in such a way that $\nu^{-1}(U) \cong U$. For $\alpha \in H^2(U, \mathcal{C})$ we take a lift $\tilde{\alpha} \in H^2(\tilde{Z}, \mathcal{C})$ by the composite $H^2(U, \mathcal{C}) \cong H^2(Z, \mathcal{C}) \to H^2(\tilde{Z}, \mathcal{C})$. Choose $\omega \in H^0(U, \Omega^2_U) = \mathcal{C}$. This $\omega$ extends to a holomorphic 2-form on $\tilde{Z}$. Normalize $\omega$ in such a way that $\int_{\tilde{Z}}(\omega\omega) = 1$. Then one can define a quadratic form $q : H^2(U, \mathcal{C}) \to \mathcal{C}$ as
\[ q(\alpha) := \frac{1}{2} \int_{\tilde{Z}} (\omega^{n-1})^2 + (1 - l)(\int_{\tilde{Z}} \omega^{n-1}) (\int_{\tilde{Z}} \omega^{l-1} \tilde{\alpha}) . \]

This form is independent of the choice of \( \nu : \tilde{Z} \to Z \).

(3) Put \( H := H^2(U, C) \). Then there exists a trivialization of the local system \( \mathcal{H} : \mathcal{H} \cong H \times S \). Let \( D := \{ x \in P(H); q(x) = 0, q(x + \pi) > 0 \} \). Then one has a period map \( p : S \to D \) and \( p \) is a local isomorphism.

Note that when \( Z \) is a symplectic variety with terminal singularities, the condition \( \text{codim}(\Sigma \subset Z) \geq 4 \) is always satisfied \((\text{Na 2})\).

The stability theorem will be proved by using the following theorem and the fact that a projective variety with rational singularities has Du Bois singularities \((\text{cf. [Ko]})\):

**Theorem 4.** Let \( X \) be a Stein open subset of a complex algebraic variety. Assume that \( X \) has only rational Gorenstein singularities. Let \( \Sigma \) be the singular locus of \( X \) and let \( f : Y \to X \) be a resolution of singularities such that \( f|_{Y \setminus f^{-1}(\Sigma)} : Y \setminus f^{-1}(\Sigma) \cong X \setminus \Sigma \). Then \( f_\ast \Omega^2_Y \cong i_\ast \Omega^2_U \) where \( U := X \setminus \Sigma \) and \( i : U \to X \) is a natural injection.

The same result was obtained by D. van Straten and Steenbrink [S-S] for an arbitrary isolated normal singularity with \( \text{dim} \geq 4 \), and later Flenner [Fl] proved it for arbitrary normal singularity with \( \text{Codim}(\Sigma \subset X) \geq 4 \). By Theorem 4 \( X \) is a symplectic singularity if and only if \( X \) is rational Gorenstein and the regular part of \( X \) admits an everywhere non-degenerate 2-form \((\text{cf. Theorem 6})\). This fact is often useful to determine that certain kinds of singularities given by G.I.T. quotient are symplectic \((\text{Example 6'})\).

The local Torelli theorem has been proved for non-singular symplectic varieties by Beauville \([\text{Be 2, Theoreme 5}]\). In our singular case, it is based on the Hodge decomposition \( H^2(U, C) = H^0(U, \Omega^0_U) \oplus H^1(U, \Omega^1_U) \oplus H^2(U, \Omega_U) \). We need the condition that \( \text{codim}(\Sigma \subset Z) \geq 4 \) to have this decomposition \((\text{cf. [Oh, Na 1, Lemma 2.5]})\). We shall prove that \( R^2\mathcal{F}, C \) is a constant sheaf around \( 0 \in S \) by using the \( Q \)-factoriality of \( Z \). When \( Z \) is not \( Q \)-factorial, the statement for \( \mathcal{H} \) in (1) does not hold as it stands because \( H^2(Z, C) \to H^2(U, C) \) is not surjective.

We have formulated the local Torelli theorem for a projective symplectic variety, but it is possible to get similar statements for a general \((\text{non-projective})\) symplectic variety. For a non-projective symplectic variety, \( Q \)-factoriality should be replaced by a certain condition which is equivalent to the \( Q \)-factoriality in the projective case \((\text{cf. Remark (2)} \text{ on the final page})\).

The following problem would be of interest in view of Global Torelli Problem.

**Problem.** Let \( Z \) be a \( Q \)-factorial projective symplectic variety with terminal singularities. Assume that \( Z \) is smoothable by a suitable flat deformation. Is \( Z \) then non-singular?
In the first section we shall prove Theorem 4. Other two theorems are proved in the second section.

**Notation.** Let $\mathcal{F}$ be a coherent sheaf on a normal crossing variety $D$. Assume that $\mathcal{F}$ is a locally free sheaf on the regular locus of $D$. Then $\mathcal{F} = \mathcal{F}/(\text{torsion})$ by definition. Here (torsion) means the subsheaf of the sections whose support are contained in the singular locus of $D$.

§1. Extension properties for Rational Gorenstein Singularities

**Proposition 1.** Let $X$ be a Stein open subset of a complex algebraic variety. Assume that $X$ has only rational Gorenstein singularities. Let $\Sigma$ be the singular locus of $X$ and let $f : Y \to X$ be a resolution of singularities such that $f|_{Y \setminus f^{-1}(\Sigma)} : Y \setminus f^{-1}(\Sigma) \cong X \setminus \Sigma$ and $D := f^{-1}(\Sigma)$ is a simple normal crossing divisor. Then $f_*\Omega^2_Y (\log D) \cong i_*\Omega^2_U$ where $U := X \setminus \Sigma$ and $i : U \to X$ is a natural injection.

**Proof.** Let $\omega \in H^0(U, \Omega^2_U)$. $\omega$ is a meromorphic 2-form on $Y$. In fact, $\text{Coker}[f_*\Omega^2 \to i_*\Omega^2_U]$ is a torsion sheaf whose support is contained in $\Sigma$. Hence $\phi \omega$ is an element of $\Gamma(X, f_*\Omega^2_Y )$ for a suitable holomorphic function $\phi$ on $X$.

Since $(X,p) \cong (R.D.P) \times (\mathbb{C}^{n-2}, 0)$ for all $p \in X$ outside certain codimension 3 (in $X$) locus $\Sigma_0 \subset \Sigma ([\text{Re}])$, it is clear that $f_*\Omega^2_Y (\log D) \cong i_*\Omega^2_U$ at such points $p$. Let $F$ be an irreducible component of $D$ with $f(F) \subset \Sigma_0$. Put $k := \dim \Sigma_0 - \dim f(F)$. We shall prove that $\omega$ has at worst log pole along $F$ by the induction on $k$.

(a) $k = 0$:

(a-1): Put $l := \text{codim}(\Sigma_0 \subset X)$. Note that $l \geq 3$. Take a general $l$ dimensional complete intersection $H := H_1 \cap \ldots \cap H_{n-l}$. Let $p \in H \cap f(F)$. Since $H$ is general, $p \in f(F)$ is a smooth point. Replace $X$ by a suitable small open neighborhood of $p$. Then $H \cap f(F) = \{p\}$. $H$ has a unique dissident point $p$ and other singularities are locally isomorphic to $(R.D.P) \times (\mathbb{C}^{l-2}, 0)$. By perturbing $H$ we can define a flat holomorphic map $g : X \to \Delta^{n-l}$ such that the fiber $X_0$ over $0 \in \Delta^{n-l}$ coincides with $H$. We may assume that $g$ has a section passing through $p$ and each fiber $g^{-1}(t)$ intersects $f(F)$ only in this section. The map $f : Y \to X$ gives a simultaneous resolution of $X_t (t \in \Delta^{n-l})$. Since $H$ is general and $X$ is sufficiently small, $D_t := D \cap Y_t$ are normal crossing divisors of $Y_t$ for all $t \in \Delta^{n-l}$. Let $D'$ be the union of irreducible components of $D$ which are mapped in this section. $D' \to \Delta^{n-l}$ is a proper map. We put $\pi = g \circ f$. We often write $\Delta$ for $\Delta^{n-l}$.

(a-2): We shall prove the following.

**Claim.** By replacing $\Delta^{n-l}$ by a smaller disc and by restricting everything (e.g. $X$, $Y$, $D$, $D'$ ...) over the new disc, we have a subset $K \subset Y$ which contains $D'$ and which is proper over $\Delta^{n-l}$ with the following property:

The $\omega$ is mapped to zero by the comoposition of the maps.
\[ H^0(U, \Omega^2_U) \cong H^0(Y \setminus D', \Omega^2_{Y \setminus D'}(\log D)) \to H^0(Y \setminus K, \Omega^2_{Y \setminus K}(\log D)) \to H^1_k(Y, \Omega^2_Y(\log D)). \]

If the claim is verified, then \( \omega|_{Y \setminus K} \) extends to a logarithmic 2-form on \( Y \). It is clear that its restriction to \( U \) is \( \omega \).

**Proof of Claim.** We shall first prove that \( R^1 \pi_* \Omega^2_Y(\log D) = 0 \). There is a filtration \( \pi^* \Omega^2_\Delta \subset \mathcal{G} \subset \Omega^2_Y(\log D) \) which yields exact sequences

\[ 0 \to \mathcal{G} \to \Omega^2_Y(\log D) \to \Omega^2_{Y/\Delta}(\log D) \to 0. \]

By these sequences it suffices to prove that \( R^1 \pi_* \mathcal{G} = R^1 \pi_* \Omega^1_{Y/\Delta}(\log D) = R^1 \pi_* \Omega^2_{Y/\Delta}(\log D) = 0. \)

We shall use the relative duality theorem due to Ramis and Ruget [R-R] to prove these facts. Before applying the relative duality we note that \( R^i \pi_* \Omega^2_{Y/\Delta}(\log D)(-D) = R^i \pi_* \Omega^1_{Y/\Delta}(\log D)(-D) = 0 \) for \( i > l-1 \). To prove these, we only have to check that \( R^i \pi_* \Omega^2_{Y/\Delta}(\log D)(-D_i) = R^i \pi_* \Omega^1_{Y/\Delta}(\log D_i)(-D_i) = 0 \) for \( i \geq l - 1 \) and for \( t \in \Delta \), for example, by using [B-S, VI, Cor. 4.5, (i)]. Since \( l \geq 3 \), these follow from a vanishing theorem in [St] except for the vanishing of \( R^2 \pi_* \Omega^1_Y(\log D_l)(-D_l) \). But, by the same argument as the proof of [Na-St, Theorem (1.1)] we see that \( R^2 \pi_* \Omega^1_Y(\log D_l)(-D_l) = 0. \)

Now the relative duality says that

\[ R \pi_! R \text{Hom}(Y; \Omega^l_{Y/\Delta}(\log D)(-D) \otimes \pi^* \Omega^n_{\Delta}, \omega_Y) \]

\[ \cong R \text{Homtop}(\Delta, R \pi_*(\Omega^l_{Y/\Delta}(\log D)(-D) \otimes \pi^* \Omega^{n-l}_{\Delta}, \omega_\Delta) \]

for \( j = 0, 1, 2 \).

We have \( R^1 \pi_! R \text{Hom}(Y; \Omega^l_{Y/\Delta}(\log D)(-D) \otimes \pi^* \Omega^n_{\Delta}, \omega_Y) \cong R^1 \pi_! \Omega^l_{Y/\Delta}(\log D) \).

Therefore we have to prove that \( \mathcal{E} \text{xtop}^{1-n}(\Delta, R \pi_*(\Omega^l_{Y/\Delta}(\log D)(-D) \otimes \pi^* \Omega^{n-l}_{\Delta}, \omega_\Delta) = 0. \)

Choose a bounded complex of **FN** free \( \mathcal{O}_\Delta \) modules \( L \) representing \( R \pi_! \Omega^n_{Y/\Delta}(\log D)(-D) \) (cf. [R-R]). Since \( R^i \pi_* \Omega^n_{Y/\Delta}(\log D)(-D) = 0 \) for \( i > l - 1 \), we have \( \mathcal{H}^i(L) = 0 \) for \( i > l - 1 \). Let \( Q := \text{Ker}[L^{l-2} \to L^{l-1}] \). Then \( R \text{Homtop}(\Delta, R \pi_*(\Omega^l_{Y/\Delta}(\log D)(-D) \otimes \pi^* \Omega^{n-l}_{\Delta}, \omega_\Delta) \) is represented by the complex \( \text{Homtop}(... \to L^{-3} \to Q \to 0 ... \Omega^{n-l}[n-l]) \). Hence we know that \( \mathcal{E} \text{xtop}^{1-n}(\Delta, R \pi_*(\Omega^l_{Y/\Delta}(\log D)(-D) \otimes \pi^* \Omega^{n-l}_{\Delta}, \omega_\Delta) = 0. \)
We are now in a position to justify the claim. Let \( j : Y \setminus D' \to Y \). The \( \omega \) defines an element of \((\pi_{*}j_{*}\Omega^{2}_{Y}(\log D))_{0}\). By a coboundary map \((\pi_{*}j_{*}\Omega^{2}_{Y}(\log D))_{0} \to (R^{1}\Gamma_{\pi_{*}D'\Omega^{2}_{Y}(\log D)})_{0}\), it defines the obstruction class \( \text{ob}(\omega) \in (R^{1}\Gamma_{\pi_{*}D'\Omega^{2}_{Y}(\log D)})_{0} \).

By the observation above, \( \text{ob}(\omega) \) is sent to zero by the natural map \((R^{1}\Gamma_{\pi_{*}D'\Omega^{2}_{Y}(\log D)})_{0} \to (R^{1}\pi_{*}\Omega^{2}_{D}(\log D))_{0}\). Therefore there is a small disc \( \Delta \subset \Delta \) and a subset \( K \subset \pi^{-1}(\Delta)(= Y_{t}) \) which contains \((\pi|_{D'})^{-1}(\Delta)(= D'_{t})\) and which is proper over \( \Delta \), such that \( \text{ob}(\omega) \) is already zero in \( H^{1}_{K}(Y_{t}, \Omega^{2}_{Y}(\log D'_{t})) \). This is nothing but our claim.

(b) \( k \): general

(b-1): Take a general \( l + k \) dimensional complete intersection \( H := H_{1} \cap \ldots \cap H_{n-1} \). Let \( p \in H \cap f(F) \). \( p \in f(F) \) is a smooth point. Replace \( X \) by a small open neighborhood of \( p \). Then \( H \cap f(F) = \{p\} \). By perturbing \( H \), we can define a flat holomorphic map \( g : X \to \Delta^{n-k-1} \) with \( g^{-1}(0) = H \). We may assume that \( g \) has a section passing through \( p \) and each fiber \( g^{-1}(t) \) intersects \( f(F) \) only in this section. The map \( f : Y \to X \) gives a simultaneous resolution of \( X_{t} (t \in \Delta^{n-l-k}) \). Since \( H \) is general and \( X \) is sufficiently small, \( D_{i} := D_{i}Y_{t} \) are normal crossing divisors of \( Y_{t} \) for all \( t \in \Delta^{n-l-k} \). Let \( D' \) be the union of irreducible components of \( D \) which are mapped in the section. Then \( D' \to \Delta^{n-k-1} \) is a proper map. We put \( \pi = g \circ f \). We often write \( \Delta \) for \( \Delta^{n-k-1} \). By an induction hypothesis we have an isomorphism \( H^{0}(Y \setminus D', \Omega^{2}_{Y\setminus D'}(\log D)) \cong H^{0}(U, \Omega^{2}_{U}) \).

(b-2): We shall prove the following

Claim. By replacing \( \Delta^{n-l} \) by a smaller disc and by restricting everything (e.g. \( X, Y, D, D' \ldots \)) over the new disc, we have a subset \( K \subset Y \) which contains \( D' \) and which is proper over \( \Delta^{n-l} \) with the following property:

The \( \omega \) is mapped to zero by the comoposition of the maps

\[
H^{0}(U, \Omega^{2}_{U}) \cong H^{0}(Y \setminus D', \Omega^{2}_{Y\setminus D'}(\log D)) \to H^{0}(Y \setminus K, \Omega^{2}_{Y\setminus K}(\log D)) \to H^{1}_{K}(Y, \Omega^{2}_{Y}(\log D)).
\]

If the claim is verified, then \( \omega|_{Y\setminus K} \) extends to a logarithmic 2-form on \( Y \). It is clear that its restriction to \( U \) is \( \omega \).

The proof of the claim is similar to the claim in (a-2). When we apply the relative duality we need the vanishing: \( R^{i}\pi_{*}\Omega^{l+k-2}_{Y}(\log D_{t})(-D_{i}) = R^{i}\pi_{*}\Omega^{l+k-1}_{Y}(\log D_{t})(-D_{i}) = R^{i}\pi_{*}\omega Y_{i} = 0 \) for \( i \geq l + k - 1 \) and for \( t \in \Delta \).

Q.E.D.

Lemma 2. Let \( p \in X \) be a Stein open neighborhood of a point \( p \) of a complex algebraic variety. Assume that \( X \) is a rational singularity of \( \dim X \geq 3 \). Let \( f : Y \to X \) be a resolution of singularities of \( X \) such that \( E := f^{-1}(p) \) is a simple normal crossing divisor. Then \( H^{0}(Y, \Omega^{1}_{Y}) \to H^{0}(Y, \Omega^{1}_{Y}(\log E)) \) are isomorphisms for \( i = 1, 2 \).

Proof. By the assumption we can take a complete algebraic variety \( Z \) which
contains $X$ as an open set. We may assume that $f$ is obtained from a resolution $\hat{Z} \to Z$. Set $V := Z \setminus E$. Recall that the natural exact sequence

$$\to H^j(\hat{Z}, C) \to H^j(V, C) \to H^{j+1}_E(\hat{Z}, C) \to$$

is obtained from the following exact sequence of the complexes by taking hypercohomology

$$\to H^{j-i}(\Omega^i_Z) \to H^{j-i}(\Omega^i_Z(\log E)) \to H^{j-i}(\Omega^i_Z(\log E)/\Omega^i_Z) \to .$$

We know that this exact sequence coincides with the exact sequence

$$\to Gr^i_F(H^j(\hat{Z}, C)) \to Gr^i_F(H^j(V, C)) \to Gr^i_F(H^{j+1}_E(\hat{Z}, C))$$

which comes from the mixed Hodge structures. In particular, the map $H^{j-i}(\Omega^i_Z(\log E)/\Omega^i_Z) \to H^{j-i+1}(\Omega^i_Z)$ is interpreted as the map $Gr^i_F(H^{j+1}_E(\hat{Z}, C)) \to Gr^i_F(H^{j+1}(\hat{Z}, C))$.

We next consider the natural map of mixed Hodge structures: $H^{j+1}(\hat{Z}, C) \to H^{j+1}(E, C)$. We have $Gr^i_F(H^{j+1}(E, C)) \cong H^{j-i+1}(\hat{V}^i_E)$ (cf. [Fr]), (see Introduction, for the notation $\hat{V}^i_E$). Note that $\hat{V}^i_E$ is isomorphic to the cokernel of the injection $\Omega^i_Y(\log E)(-E) \to \Omega^i_Y$. Therefore the composed map

$$H^{j-i}(\Omega^i_Z(\log E)/\Omega^i_Z) \to H^{j-i+1}(\Omega^i_Z) \to H^{j-i+1}(\Omega^i_Y/\Omega^i_Y(\log E)(-E))$$

is interpreted as the map

$$Gr^i_F(H^{j+1}_E(\hat{Z}, C)) \to Gr^i_F(H^{j+1}(E, C)).$$

By the isomorphisms $H(Y, C) \cong H(E, C)$, $H(Y)$ have natural mixed Hodge structures. We put $U^r := Y \setminus E$.

$(i = 2)$: We shall first prove that the natural map $\beta : H^3(E, C) \to H^3(Y, C)$ is an injection.

By a local cohomology exact sequence it suffices to show that $\alpha : H^2(Y, C) \to H^2(U^r, C)$ is a surjection. Because $X$ has only rational singularity, $H^1(Y, C) \otimes C \cong H^2(Y, C)$.

On the other hand, one can prove that $H^2(U^r, C) \otimes C \to H^2(U', C)$ is a surjection; in fact, since $H(U', C) \cong H(Y, \mathcal{O}_Y) \cong H(Y, j_\ast \Omega_{U^r}) \cong H(Y, \mathcal{O}_Y(\log E))$, we have a commutative diagram of Hodge spectral sequences.
\[
\begin{array}{c}
\text{Let } F_i (\text{resp. } F_i') \text{ be the filtration of } H^{p+q}(U', \mathbb{C}) \text{ by the first (resp. second) spectral sequence. In particular, when } p + q = 2, \text{ we have a surjection } Gr^0 F_i H^2(U', \mathbb{C}) \to Gr^0 F_i H^2(U', \mathbb{C}). \text{ Because } X \text{ has only rational singularity, } H^2(Y, \mathcal{O}_Y) = 0, \text{ hence } Gr^0 F_i H^2(U', \mathbb{C}) = 0. \text{ Therefore } Gr^0 F_i H^2(U', \mathbb{C}) = 0, \text{ which implies that the natural map } H^2(U', \mathbb{C}) \to H^2(U', \mathcal{O}_{U'}) \text{ is the zero map. It is now clear that } H^2(U', \mathbb{Z}) \to H^2(U', \mathcal{O}_{U'}) \text{ is also the zero map, and hence } H^1(U', \mathcal{O}_{U'}) \to H^2(U', \mathbb{Z}) \text{ is a surjection. Therefore one has a surjection } H^1(U', \mathcal{O}_{U'}) \otimes \mathbb{C} \to H^2(U', \mathbb{C}). \text{ It is enough to prove that } H^1(Y, \mathcal{O}_Y^*) \to H^1(U', \mathcal{O}_{U'}) \text{ is surjective in order to prove that } \alpha \text{ is surjective. Put } f^0 := f|_{U'} : U' \to U, \text{ where } U := X \setminus \{p\}. \text{ Let } D_i \text{ be irreducible exceptional divisors of } f \text{ where } f(D_i) \neq \{p\}. \text{ Let } L \in \text{Pic}(U'). \text{ Since } \dim X \geq 3 \text{ and } U \subset X \text{ is 1-concave at } p, \text{ there exists a reflexive coherent sheaf } F \text{ on } X \text{ of rank 1 such that } F|_U \cong (f^0_* L)^* \text{ by [S, Theorem 5], where }^* \text{ means the double dual. By taking the double dual of both sides of the natural map } (f^0)^* f^0_* L \to L \text{ we get an injection } ((f^0)^* f^0_* L)^* \to L, \text{ hence } L \cong ((f^0)^* f^0_* L)^* \otimes \mathcal{O}_{U'}(\Sigma a_i D_i) \text{ with some } a_i \geq 0. \text{ On the other hand, we have an injection } ((f^0)^* f^0_* L)^* \to (f^* f_*)^* |_{U'} \text{ and } M := (f^* f_*)^* \in \text{Pic}(Y). \text{ We have } ((f^0)^* f^0_* L)^* \cong M|_{U'} \otimes \mathcal{O}_{U'}(\Sigma (-b_i) D_i) \text{ with some } b_i > 0. \text{ Therefore } \begin{array}{c}
0 \to \Omega^2_Y / \Omega^2_Y (\log E)(-E) \to \Omega^2_Y (\log E)/\Omega^2_Y (\log E)(-E) \to \Omega^2_Y (\log E)/\Omega^2_Y (E) \to 0.
\end{array}
\]
\[
0 \to \Omega^2_Y / \Omega^2_Y (\log E)(-E) \to \Omega^2_Y (\log E)/\Omega^2_Y (\log E)(-E) \to \Omega^2_Y (\log E)/\Omega^2_Y (E) \to 0.
\]
\[
\text{From this sequence we have a map } \delta : H^0(E, \Omega^2_Y (\log E)/\Omega^2_Y (E)) \to H^1(E, \Omega^2_Y /\Omega^2_Y (\log E)(-E)).
\]
\[
\text{The map } \beta \text{ is a morphism of mixed Hodge structures and } \delta \text{ can be interpreted as the map } Gr^2_F (H^2_Y (Y, \mathbb{C}) \to Gr^2_F (H^2(Y, \mathbb{C}). \text{ We already proved that } \beta \text{ is an injection. Hence } \delta \text{ is also an injection by the strict compatibility of the filtrations } F. \text{ Note that } \delta \text{ is factorized as } H^0(E, \Omega^2_Y (\log E)/\Omega^2_Y (E)) \stackrel{\tau}{\to} H^1(Y, \Omega^2_Y ) \to H^1(E, \Omega^2_Y /\Omega^2_Y (\log E)(-E)) \text{ where } \gamma \text{ is the last map in the following exact sequence}
\]
\[
\begin{array}{c}
\text{H}^0(Y, \Omega^2_Y ) \stackrel{\tau}{\to} \text{H}^0(Y, \Omega^2_Y (\log E)) \to \text{H}^0(E, \Omega^2_Y (\log E)/\Omega^2_Y (E)) \to \text{H}^1(Y, \Omega^2_Y ).
\end{array}
\]
\[
\text{Since } \delta \text{ is injective, } \gamma \text{ is also injective. Hence } \tau \text{ is surjective by the exact sequence.}
\]
\]

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(i = 1): We shall first prove that the natural map \( \beta : H^2_\mathbb{K}(Y, \mathbb{C}) \to H^2(Y, \mathbb{C}) \)

is an injection. By a local cohomology exact sequence it suffices to show that \( \alpha : H^1(Y, \mathbb{C}) \to H^1(U', \mathbb{C}) \) is a surjection. Since \( X \) has rational singularities, the sequence

\[
H^0(Y, \mathcal{O}_Y) \to H^0(Y, \mathcal{O}_Y^\vee) \to H^1(Y, \mathbb{Z}) \to 0
\]

is exact. By the same argument as \( (i = 2) \), \( H^1(U', \mathbb{Z}) \to H^1(U', \mathcal{O}_{U'}) \) is the zero map because \( X \) has rational singularities. Hence the sequence

\[
H^0(U', \mathcal{O}_{U'}) \to H^0(U', \mathcal{O}_{U'})^\vee \to H^1(U', \mathbb{Z}) \to 0
\]

is also exact. The restriction map \( H^0(Y, \mathcal{O}_Y^\vee) \to H^0(U', \mathcal{O}_{U'}^\vee) \) is an isomorphism because it is factorized as \( H^0(Y, \mathcal{O}_Y^\vee) \cong H^0(X, \mathcal{O}_X^\vee) \cong H^0(X \setminus \{p\}, \mathcal{O}_{X \setminus \{p\}}^\vee) \cong H^0(U', \mathcal{O}_{U'}^\vee) \). Similarly the restriction map \( H^0(Y, \mathcal{O}_Y) \to H^0(U', \mathcal{O}_{U'}) \)

is also an isomorphism. Hence the restriction \( H^1(Y, \mathbb{Z}) \to H^1(U', \mathbb{Z}) \) is an isomorphism by the exact sequences above, and \( \alpha \) is an isomorphism.

Let us consider the exact sequence

\[
0 \to \Omega^1_Y / \Omega^1_Y(\log E)(-E) \to \Omega^1_Y(\log E) / \Omega^1_Y(\log E)(-E) \to \Omega^1_Y(\log E) / \Omega^1_Y(\log E) \to 0.
\]

From this sequence we have a map \( \delta : H^0(E, \Omega^1_Y(\log E) / \Omega^1_Y(\log E)(-E)) \to H^1(E, \Omega^1_Y(\log E) / \Omega^1_Y(\log E)(-E)) \).

The map \( \beta \) is a morphism of mixed Hodge structures and \( \delta \) can be interpreted as the map \( Gr^1_F(H^2_\mathbb{K}(Y, \mathbb{C}) \to Gr^1_F(H^2(Y, \mathbb{C})) \). We already proved that \( \beta \) is an injection. Hence \( \delta \) is also an injection by the strict compatibility of the filtrations \( F \). Note that \( \delta \) is factorized as \( H^0(E, \Omega^1_Y(\log E) / \Omega^1_Y(\log E)(-E)) \xrightarrow{\gamma} H^1(Y, \Omega^1_Y) \to H^1(E, \Omega^1_Y(\log E)(-E)) \) where \( \gamma \) is the last map in the following exact sequence

\[
H^0(Y, \Omega^1_Y) \xrightarrow{\gamma} H^0(Y, \Omega^1_Y(\log E)) \to H^0(E, \Omega^1_Y(\log E) / \Omega^1_Y(\log E)) \to H^1(Y, \Omega^1_Y).
\]

Since \( \delta \) is injective, \( \gamma \) is also injective. Hence \( \tau \) is surjective by the exact sequence. Q.E.D.

**Remark.** In the proof of Lemma 2 the map \( H^0(E, \Omega^1_Y(\log E) / \Omega^1_Y(\log E)(-E)) \) is surjective for \( i = 1, 2 \) because we have proved that \( \delta \) is injective.

**Proposition 3.** Let \( X \) be a Stein open subset of a complex algebraic variety. Assume that \( X \) has only rational Gorenstein singularities. Let \( \Sigma \) be the singular locus of \( X \) and let \( f : Y \to X \) be a resolution of singularities such that \( D := f^{-1}(\Sigma) \) is a simple normal crossing divisor and such that \( f|_{Y \setminus D} \) is a simple normal crossing divisor such that \( f|_{Y \setminus D} : Y \setminus D \cong X \setminus \Sigma \).

Then \( f_* \Omega^1_Y \cong f_* \Omega^1_Y(\log D) \).
Proof. Since $(X,p) \cong (R.D.P.) \times (\mathbb{C}^{n-2},0)$ for all $p \in X$ outside certain codimension 3 (in $X$) locus $\Sigma_0 \subset \Sigma (|\text{Re}|)$, it is clear that $f_*\Omega_Y^2 \cong f_*\Omega_Y^2 (\log D)$ at such points $p$.

Let $\omega \in H^0(Y,\Omega_Y^2 (\log D))$. Let $F$ be an irreducible component of $D$ with $f(F) \subset \Sigma_0$. Put $k := \dim \Sigma_0 - \dim f(F)$. We shall prove that $\omega$ is regular along $F$ by the induction on $k$.

(a) $k = 0$:

(a-1): Put $l := \text{codim}(\Sigma_0 \subset X)$. Note that $l \geq 3$. Take a general $l$ dimensional complete intersection $H := H_1 \cap H_2 \cap \ldots \cap H_{n-l}$. Let $p \in H \cap f(F)$. $p \in f(F)$ is a smooth point. Replace $X$ by a small open neighborhood of $p$. Then $H \cap f(F) = \{p\}$. Moreover, $H := f^{-1}(H)$ is a resolution of singularities of $H$. Since $X$ has canonical singularities, $H$ has also canonical singularities. $H$ has a unique dissident point $p$ and other singular points are locally isomorphic to $(R.D.P.) \times (\mathbb{C}^{l-2},0)$. By perturbing $H$ we can define a flat holomorphic map $g : X \to \Delta^{n-l}$ with $g^{-1}(0) = H$. We may assume that $g$ has a section passing through $p$ and each fiber $X_t := g^{-1}(t)$ intersects $f(F)$ only in this section. Denote by $p_t \in X_t$ this intersection point. By definition $p_0 = p$. The map $f : Y \to X$ gives a simultaneous resolution of $X_t$ for $t \in \Delta^{n-l}$. Let $D'$ be the union of irreducible components of $D$ which are mapped in this section. Since $H$ is general and $X$ is sufficiently small, every irreducible component of $D'$ is mapped onto the section. $D' \to \Delta^{n-l}$ is a proper map and every fiber $D'_t$ is a normal crossing variety. Note that $f^{-1}_t(p_t) = D'_t$. We put $\pi = g \circ f$. We often write $\Delta$ for $\Delta^{n-l}$. There are filtrations $(\pi|_{D'})^*\Omega_Y^2 \subset \mathcal{F} \subset \hat{\Omega}_{D'}^2$ and $(\pi|_{\Delta})^*\Omega_Y^2 \subset \mathcal{G} \subset \Omega_{\Delta}^2 (\log D')$ which yield the following exact sequences

$$0 \to \mathcal{F} \to \hat{\Omega}_{D'}^2 \to \hat{\Omega}_{D'/\Delta}^2 \to 0,$$

$$0 \to (\pi|_{D'})^*\Omega_Y^2 \to \mathcal{F} \to (\pi|_{D'})^*\Omega_Y^2 \otimes \hat{\Omega}_{D'/\Delta}^2 \to 0,$$

$$0 \to \mathcal{G} \to \Omega_Y^2 (\log D') \to \Omega_{D'/\Delta}^2 (\log D') \to 0,$$

$$0 \to (\pi|_{\Delta})^*\Omega_Y^2 \to \mathcal{G} \to (\pi|_{\Delta})^*\Omega_{\Delta}^1 \otimes \Omega_{\Delta}^2 (\log D') \to 0.$$

(a-2): Let us consider the exact sequence

$$0 \to \Omega_Y^2 / \Omega_Y^2 \to \Omega_Y^2 (\log D') (\log D') (\log D') (\log D') / \Omega_Y^2 (\log D') (\log D') (\log D') (\log D') (\log D') \to 0.$$
0 & 0 \\
\downarrow & \downarrow \\
H^0(D', F) & \longrightarrow & H^0(D', G \otimes O_{D'}) \\
\downarrow & \downarrow \\
H^0(D', \hat{\Omega}^{n-1}_{D'}) & \longrightarrow & H^0(D', \hat{\Omega}^{n-1}_{Y/\Delta}) \\
\downarrow & \downarrow \\
H^0(D', \hat{\Omega}^{n-1}_{D'/\Delta}) & \longrightarrow & H^0(D', \hat{\Omega}^{n-1}_{Y/\Delta}) \oplus O_{D'}) \\
\downarrow & \downarrow \\
H^0(D', \hat{\Omega}^{n-1}_{D'/\Delta}) \oplus \mu^{-i} & \longrightarrow & H^0(D', \hat{\Omega}^{n-1}_{Y/\Delta}) \oplus O_{D')} \oplus \mu^{-i}

We shall show that $H^0(D', \hat{\Omega}^{n-1}_{D'/\Delta}) = H^0(D', \hat{\Omega}^{n-1}_{Y/\Delta} \otimes O_{D'}) = 0$ for $i = 1, 2$. After these are proved our claim easily follows from the commutative diagrams above.

First note that $H^0(D', \hat{\Omega}^{n-1}_{D'}) = 0$ for $t \in \Delta$ and for $i = 1, 2$. In fact, if $h^0(D', \hat{\Omega}^{n-1}_{D'}) > 0$, then $h^0(D', \hat{\Omega}^{n-1}_{D'}) > 0$ by the mixed Hodge structures on $H^i(D', C)$. On the other hand, $X_t$ has only canonical singularities, hence has only rational singularities. Therefore $H^i(Y_t, O_{Y_t}) = 0$. Then one can check that $H^i(D', \hat{\Omega}^{n-1}_{D'}) = 0$. This is a contradiction. For details of this argument, see [Na 1, Claim 1, (i) in the proof of Prop. (1.1)]. As a consequence we know that $H^0(D', \hat{\Omega}^{n-1}_{D'/\Delta}) = 0$ for $i = 1, 2$.

We next note that $\mu_i(t) : H^0(D', \hat{\Omega}^{n-1}_{D'}) \rightarrow H^0(D', \hat{\Omega}^{n-1}_{Y_t/\Delta} \otimes O_{D'})$ are surjective for $i = 1, 2$ by Remark below Lemma 2. Since $H^0(D', \hat{\Omega}^{n-1}_{D'}) = 0$, this implies that $H^0(D', \hat{\Omega}^{n-1}_{Y_t/\Delta} \otimes O_{D'}) = 0$, hence $H^0(D', \hat{\Omega}^{n-1}_{Y/\Delta} \otimes O_{D'}) = 0$ for $i = 1, 2$. Q.E.D.

(a-3): We shall continue the proof in the case $k = 0$. By taking cohomology of the exact sequence
$0 \to \Omega^n_Y/\Omega^n_Y(\log D')(\log D') \to \Omega^n_Y(\log D')/\Omega^{n-1}(\log D')(\log D') \to \Omega^n_Y(\log D')/\Omega^{n-2}(\log D') \to 0.$

and by applying Claim in (a-2) we see that $\delta : H^0(D', \Omega^n_Y(\log D')/\Omega^n_Y) \to H^1(D', \Omega^n_Y(\log D')/\Omega^n_Y)$ is an injection. The map $\delta$ is factorized as $H^0(D', \Omega^n_Y(\log D')/\Omega^n_Y) \xrightarrow{\gamma} H^1(Y, \Omega^n_Y) \to H^1(D', \Omega^n_Y(\log D')/\Omega^n_Y) \to H^1(Y, \Omega^n_Y).$

Since $\delta$ is injective, $\gamma$ is also injective. Hence $\tau$ is surjective by the exact sequence. Q.E.D.

(b-1): Take a general $l+k$ dimensional complete intersection $H := H_1 \cap H_2 \cap \ldots \cap H_{n-l-k}$. Let $p \in H \cap f(F)$. $p \in f(F)$ is a smooth point. Replace $X$ by a small open neighborhood of $p$. Then $H \cap f(F) = \{p\}$. Moreover, $\tilde{H} := f^{-1}(H)$ is a resolution of singularities of $H$. Since $X$ has canonical singularities, $\tilde{H}$ has also canonical singularities. By perturbing $H$ we can define a flat holomorphic map $g : X \to \Delta^{n-l-k}$ with $g^{-1}(0) = H$. We may assume that $g$ has a section passing through $p$ and each fiber $X_t := g^{-1}(t)$ intersects $f(F)$ only in this section. Denote by $p_i \in X_t$ this intersection point. By definition $p_0 = p$. The map $f : Y \to X$ gives a simultaneous resolution of $X_t$ for $t \in \Delta^{n-l-k}$. Let $D'$ be the union of irreducible components of $D$ which are mapped in this section. Since $H$ is general and $X$ is sufficiently small, every irreducible component of $D'$ is mapped onto the section. $D' \to \Delta^{n-l-k}$ is a proper map and every fiber $D'_t$ is a normal crossing variety. Note that $f^{-1}_t(p_i) = D'_t$. We put $\pi = g \circ f$. We often write $\Delta$ for $\Delta^{n-l-k}$. There are filtrations $(\pi|_{D'})^*\Omega^n_X \subset \mathcal{F} \subset \Omega^n_{D'}$ and $\pi^*\Omega^n_{\Delta} \subset \mathcal{G} \subset \Omega^n_{\Delta}$. By an induction hypothesis we have an isomorphism $H^0(Y, \Omega^n_Y(\log D') \cong H^0(Y, \Omega^n_Y(\log D))$. Therefore we have to prove that $H^0(Y, \Omega^n_Y) \to H^0(Y, \Omega^n_Y(\log D'))$ is surjective (hence an isomorphism).

(b-2): Let us consider the exact sequence

$$0 \to \Omega^n_Y/\Omega^n_Y(\log D')(\log D') \to \Omega^n_Y(\log D')/\Omega^{n-1}(\log D')(\log D') \to \Omega^n_Y(\log D')/\Omega^{n-2}(\log D') \to 0.$$
By combining Propositions 1 and 3 we have the following.

**Theorem 4.** Let $X$ be a Stein open subset of a complex algebraic variety. Assume that $X$ has only rational Gorenstein singularities. Let $\Sigma$ be the singular locus of $X$ and let $f : Y \to X$ be a resolution of singularities such that $f|_{Y\setminus f^{-1}(\Sigma)} : Y \setminus f^{-1}(\Sigma) \cong X \setminus \Sigma$. Then $f_*\Omega^2_Y \cong i_*\Omega^2_X$ where $U := X \setminus \Sigma$ and $i : U \to X$ is a natural injection.

\section*{§2. Symplectic Varieties}

We shall begin this section with the stability of Kaehlerity under small deformation.

**Proposition 5.** Let $Z$ be a compact normal Kaehler space with rational singularities. Then any small (flat) deformation $Z_t$ of $Z$ is also a Kaehler space.

**Proof.** By a theorem of Bingener \[B\] we only have to prove that the map $H^2(Z, \mathbb{R}) \to H^2(Z, \mathcal{O}_Z)$ induced by a sheaf homomorphism $\mathbb{R}_Z \to \mathcal{O}_Z$ is surjective. Let $f : \tilde{Z} \to Z$ be a resolution of singularities. Since $Z$ has only rational singularities, we know that $R^1f_*\mathbb{R}\tilde{Z} = 0$. Consider the commutative diagram with exact rows

$$
\begin{array}{cccc}
0 & \longrightarrow & H^2(Z, \mathbb{R}) & \longrightarrow & H^2(\tilde{Z}, \mathbb{R}) & \longrightarrow & H^0(Z, R^2f_*\mathbb{R}) \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^2(Z, \mathcal{O}_Z) & \longrightarrow & H^2(\tilde{Z}, \mathcal{O}_Z) & \longrightarrow & 0
\end{array}
$$

(4)

By Hodge theory the middle vertical map is surjective. The surjectivity of the left vertical map follows from the following claim.

**Claim.** $\text{im}[H^2(\tilde{Z}, \mathbb{R}) \to H^0(Z, R^2f_*\mathbb{R})] = \text{im}[H^2(\tilde{Z}, \mathbb{R}) \cap H^{1,1} \to H^0(Z, R^2f_*\mathbb{R})]$.

**Proof.** It suffices to show that, if an element $\alpha \in H^2(\tilde{Z}, \mathbb{R})$ has the Hodge decomposition $\alpha = \alpha^{(2,0)} + \alpha^{(0,2)}$, then $\phi(\alpha) = 0$. Since $\tilde{\alpha}^{(2,0)} = \alpha^{(0,2)}$, we only have to prove that $\phi_C(\alpha^{(2,0)}) = 0$. We shall show that, for every point $z \in Z$, $\phi_C(\alpha^{(2,0)})_z = 0$ in $(R^2f_*\mathcal{C})_z$. Let $\nu : W \to \tilde{Z}$ be a projective bimeromorphic map such that $W$ is smooth and $D := (f \circ \nu)^{-1}(x)$ is a simple normal crossing divisor of $W$. Put $h := f \circ \nu$. Since $R^1\nu_*\mathcal{C} = 0$, $R^2f_*\mathcal{C}$ injects to $R^2h_*\mathcal{C}$. Therefore we have to check that $\alpha^{(2,0)}$ is sent to zero by the composition of the maps $H^2(\tilde{Z}, \mathcal{C}) \to H^2(W, \mathcal{C}) \to (R^2h_*\mathcal{C})_z = H^2(D, \mathcal{C})$. Since the map $H^2(\tilde{Z}, \mathcal{C}) \to H^2(D, \mathcal{C})$ is a morphism of mixed Hodge structures, it preserves the Hodge filtration $F$. In particular, it induces $Gr^F_2(H^2(\tilde{Z}, \mathcal{C})) \to Gr^F_2(H^2(D, \mathcal{C}))$. Since $\alpha^{(2,0)} \in Gr^F_2(H^2(\tilde{Z}, \mathcal{C}))$, $\phi_C(\alpha^{(2,0)}) \in Gr^F_2(H^2(D, \mathcal{C}))$. On the other hand, $Gr^F_2(H^2(D, \mathcal{C})) = 0$ because $Z$ has rational singularities; if
Gr^2_F(H^2(D, C)) \neq 0$, then $Gr^0_F(H^2(D, C)) = H^2(D, \mathcal{O}_D) \neq 0$ which contradicts the fact that $Z$ has rational singularities. For details of the argument, see [Na, Claim 1, (i) in the proof of Prop. (1.1)]. Q.E.D.

By Theorem 4 we have another definition of a symplectic singularity.

**Theorem 6.** Let $X$ be a Stein open subset of a complex algebraic variety. Then $X$ is a symplectic singularity (for the definition, see Introduction) if and only if $X$ has rational Gorenstein singularities and the regular locus $U$ of $X$ admits a everywhere non-degenerate holomorphic closed 2-form.

**Proof.** By [Be 1] if $X$ is symplectic, then $X$ has rational Gorenstein singularities. Hence, ”only if” part holds. On the other hand, by Theorem 4, ”if” part holds. Q.E.D.

**Example 6’** (cf. [O, 1.5]): Let $c$ be an even positive integer with $c \geq 4$ and let $E$ be a $\mathbb{C}$ vector space with $\dim E = c$, equipped with a non-degenerate (alternative) 2-form $\omega: E \times E \to \mathbb{C}$. Let $W$ be a 3-dimensional $\mathbb{C}$ vector space with a non-degenerate symmetric form $\kappa: W \times W \to \mathbb{C}$. Let $SO(W)$ be the special orthogonal subgroup of $GL(W)$ with respect to $\kappa$. Put

$$\operatorname{Hom}^\omega(W, E) := \{ \phi \in \operatorname{Hom}(W, E); \phi^* \omega = 0 \}.$$

Let $X := \operatorname{Hom}^\omega(W, E)//SO(W)$. Identifying $\operatorname{Hom}(W, E) = W^* \otimes E$ with $W \otimes E$ by $\kappa$, we can define a 2-form $\tilde{\omega}$ on $\operatorname{Hom}(W, E)$ by $\tilde{\omega}(\alpha \otimes \beta, \alpha' \otimes \beta') := \kappa(\alpha, \alpha') \omega(\beta, \beta')$ for $\alpha, \alpha' \in W$, $\beta, \beta' \in E$. Let $\operatorname{Hom}^\omega(W, E)^s$ be the open subset of $\operatorname{Hom}^\omega(W, E)$ which consists of points with trivial isotropy group and with closed orbits. Then $U := \operatorname{Hom}^\omega(W, E)^s//SO(W)$ becomes the regular part of $X$. Put $\Sigma := \operatorname{Sing}(X)$. By calculations $\dim X = 3c - 6$ and $\dim \Sigma = c$. The 2-form $\tilde{\omega}|_{\operatorname{Hom}(W, E)^s}$ descends to a symplectic 2-form $\omega_U$ on $U$. By a theorem of Bousot [Bo] we see that $X$ has rational singularities because $\operatorname{Hom}^\omega(W, E)$ has only rational singularities. $\wedge^{(3/2)c-3}\omega_U$ gives a trivialization of the dualizing sheaf of $X$; hence $X$ has rational Gorenstein singularities. By Theorem 6 $X$ has symplectic singularities.

**Theorem 7.** Let $(Z, \omega)$ be a projective symplectic variety. Let $g: Z \to \Delta$ be a projective flat morphism from $Z$ to a 1-dimensional unit disc $\Delta$ with $g^{-1}(0) = Z$. Then $\omega$ extends sideways in the flat family so that it gives a symplectic 2-form $\omega_t$ on each fiber $Z_t$ for $t \in \Delta$ with a sufficiently small $\epsilon$.

**Proof.** We shall shrink $\Delta$ suitably in each step of arguments, but use the same notation $\Delta$ after shrinking. Put $\dim Z = 2l$.

Let $\nu_0: \tilde{Z} \to Z$ be a resolution. Let $\omega \in H^0(Z, \nu_0^* \Omega^2_Z)$ be a symplectic 2-form. We take the conjugate $\bar{\omega} \in H^2(\tilde{Z}, \mathcal{O}_{\tilde{Z}})$ by the Hodge decomposition $H^2(\tilde{Z}, \mathcal{O}_{\tilde{Z}}) = H^0(\tilde{Z}, \Omega^2_{\tilde{Z}}) \oplus H^1(\tilde{Z}, \Omega^1_{\tilde{Z}}) \oplus H^2(\tilde{Z}, \mathcal{O}_{\tilde{Z}})$. Since $Z$ has only rational singularities, $H^2(\tilde{Z}, \mathcal{O}_{\tilde{Z}}) \cong H^2(Z, \mathcal{O}_Z)$; by this isomorphism we regard $\bar{\omega}$ as an element of $H^2(Z, \mathcal{O}_Z)$. Since $Z_t$ are projective varieties with rational singulari-
ties, the natural maps $H^i(Z_t, C) \to H^i(Z_t, O_{Z_t})$ are surjective for all $i$ by [Ko, Theorem 12.3]; hence by [D-J] $R^i \alpha_* O_Z$ are locally free sheaves which are compatible with base change. Therefore $\bar{\omega}$ extends sideways and defines non-zero $\bar{\omega}_t \in H^2(Z_t, O_{Z_t})$ for each $t$. Since $\wedge^2 \bar{\omega}_t \in H^2(Z, O_Z) = C$ is not zero, we also have $\wedge^2 \bar{\omega}_t \neq 0$ in $H^2(Z_t, O_{Z_t}) = C$. Take a resolution $\tilde{Z}_t \to Z_t$, and identify $H^2(\tilde{Z}_t, O_{\tilde{Z}_t})$ with $H^2(Z_t, O_{Z_t})$. By Hodge decomposition of $H^2(\tilde{Z}_t, C)$, we take the conjugate $\omega_t \in H^0(\tilde{Z}_t, \Omega^2_{\tilde{Z}_t})$ of $\bar{\omega}_t$. Note that $0 \neq \wedge^2 \omega_t \in H^0(\tilde{Z}_t, \omega_{\tilde{Z}_t}) = C$. This implies that $\omega_t$ is everywhere non-degenerate at regular locus of $Z_t$ because $\omega_{Z_t}$ is trivial, and we know that $Z_t$ is a symplectic variety.

However, by the fiberwise argument above, it is not clear whether $\omega$ holomorphically extends sideways. We shall prove that $\omega$ actually extends sideways by using Theorem 4. Let $U := \{ z \in Z; g \text{ is smooth at } z \}$. Denote by $i$ the natural inclusion of $U$ to $Z$. Put $F := \Omega^2_{U/\Delta}$, $F_0 := \Omega^2_U$ and $U := U \cap Z$. $i_* F$ is a coherent torsion free sheaf on $\tilde{Z}$, and hence is flat over $\Delta$. By the exact sequence

$$0 \to i_* F \to i_* F \to i_* F_0$$

we know that $i_* F \otimes O_Z \to i_* F_0$ is an injection, hence $h^0(i_* F \otimes O_Z) \leq h^0(i_* F_0)$.

On the other hand, for general $t$, $h^0(i_* F \otimes O_{Z_t}) = h^0(i_* F_t)$. This is proved in the following way. Since $t \in \Delta$ is general, we may assume that $g : Z \to \Delta$ has a simultaneous resolution $\alpha : \tilde{Z} \to Z$ if we replace $\Delta$ by a suitable open neighborhood of $t$. Put $f = g \circ \alpha$. We have a commutative diagram:

$$\alpha_* \Omega^2_{\tilde{Z}/\Delta} \otimes O_{Z_t} \longrightarrow i_* F \otimes O_{Z_t}$$

$$\downarrow \quad \quad \downarrow$$

$$\alpha_* \Omega^2_{\tilde{Z}_t} \longrightarrow i_* F_t$$

The horizontal map at the bottom is an isomorphism by Theorem 4. We shall prove that $H^0(\alpha_* \Omega^2_{\tilde{Z}/\Delta} \otimes O_{Z_t}) \to H^0(\alpha_* \Omega^2_{\tilde{Z}_t})$ is surjective; if so, then $H^0(i_* F \otimes O_{Z_t}) \to H^0(i_* F_t)$ is also a surjection by the diagram, and hence is an isomorphism. Now apply base change theorem to $(\alpha_* \Omega^2_{\tilde{Z}/\Delta}; g)$ and $(\Omega^2_{\tilde{Z}/\Delta}; f)$.

Then we have a commutative diagram

$$g_* (\alpha_* \Omega^2_{\tilde{Z}/\Delta}) \otimes k(t) \longrightarrow H^0(Z_t, \alpha_* \Omega^2_{\tilde{Z}/\Delta} \otimes O_{Z_t})$$

$$\downarrow \quad \quad \downarrow$$

$$f_* \Omega^2_{\tilde{Z}/\Delta} \otimes k(t) \longrightarrow H^0(Z_t, \alpha_* \Omega^2_{\tilde{Z}_t})$$

The vertical map on the left hand side is clearly an isomorphism. The horizontal maps are both isomorphisms if $t$ is general. Hence the vertical map
on the right hand side is also an isomorphism by the diagram. Therefore, for general $t$, $h^0(i_* F \otimes_{\mathcal{O}_Z} \mathcal{O}_{Z_t}) = h^0(i_* F_t)$.

Let $\nu_t : Z_t \to Z_t$ be a resolution of singularities. By Theorem 4 $i_* F_t \cong \nu_t \Omega^2_{Z_t}$. Since $h^0(i_* F_t) = h^0(Z_t, \Omega^2_{Z_t}) = h^2(Z_t, \mathcal{O}_{Z_t})$, $h^0(i_* F_t)$ is a constant function of $t$. $h^0(i_* F \otimes_{\mathcal{O}_Z} \mathcal{O}_{Z_t})$ is an upper semi-continuous function of $t$. For any $t$, $h^0(i_* F \otimes_{\mathcal{O}_Z} \mathcal{O}_{Z_t}) \leq h^0(i_* F_t)$ and the equality holds for general $t$. Since $h^0(i_* F_t)$ is constant, this implies that $h^0(i_* F \otimes_{\mathcal{O}_Z} \mathcal{O}_{Z_t})$ is constant and $h^0(i_* F \otimes_{\mathcal{O}_Z} \mathcal{O}_{Z_t}) = h^0(i_* F_t)$ for all $t$. By a theorem of Grauert (cf. [Ha, Corollary 12.9]), $g_*(i_* F) \otimes_{\mathcal{O}_Z} k(0) \to H^0(i_* F \otimes_{\mathcal{O}_Z} \mathcal{O}_Z) \cong H^0(i_* F_0)$ is an isomorphism. This implies that there is a lift $\tilde{\omega} \in \Gamma(g^{-1}(\Delta), \Omega^2_{\mathcal{U}/\Delta})$ of $\omega$. Let us consider $\wedge^{n/2} \tilde{\omega}$ as a section of the dualizing sheaf $\omega_Z$. Then $\wedge^{n/2} \tilde{\omega}$ can be regarded as a section of $\omega_{Z_t/\Delta_t}$. Since $\wedge^{n/2} \omega$ generates the line bundle $\omega_Z$, $\wedge^{n/2} \tilde{\omega}$ also generates $\omega_{Z_t/\Delta_t}$, if necessary, by taking $\epsilon$ smaller. Therefore $\omega_t := \tilde{\omega}|_{\mathcal{U}_t}$ is a non-degenerate 2-form for $t \in \Delta_t$. Since $Z_t$ has only rational Gorenstein singularities, $(Z_t, \omega_t)$ is a symplectic variety by Theorem 6.

**Remark.** By virtue of Proposition 5, Theorem 7 seems true even when $g$ is a proper flat morphism and $Z$ is a symplectic variety. The missing ingredients consist of two parts; (a) for a compact Kaehler space $Z$ with rational singularities, are the natural maps $H^i(Z, \mathbb{C}) \to H^i(Z, \mathcal{O}_Z)$ surjective for all $i$? (b) does Theorem 4 hold for an arbitrary rational Gorenstein singularity? (in the proof of Theorem 4 we used a vanishing theorem of [St] and this vanishing theorem is only known for the case $X$ is embedded as an open subset in a complex projective variety.)

If these two questions are affirmative, Theorem 7 holds in this full generality.

The following will be used later.

**Theorem 7'.** Let $(Z, \omega)$ be a symplectic variety with $\text{codim}(\Sigma \subset Z) \geq 4$. Let $g : \mathcal{Z} \to \Delta^n$ be a proper flat morphism from $\mathcal{Z}$ to a $n$-dimensional unit disc $\Delta^n$ with $g^{-1}(0) = Z$. Then $\omega$ extends sideways in the flat family so that it gives a symplectic 2-form $\omega_t$ on each fiber $Z_t$ for $t \in \Delta^n$ with a sufficiently small $\epsilon$.

**Proof.** As in the proof of Theorem 7, we put $\mathcal{U} := \{z \in \mathcal{Z}; \ g \text{ is smooth at } z \}$. Denote by $i$ the natural inclusion of $\mathcal{U}$ to $\mathcal{Z}$. Write $\pi$ for $g \circ i$. Put $F := \Omega^2_{\mathcal{U}/\Delta}, F_0 := \Omega^2_{\Delta}$, and $U := \mathcal{U} \cap \Delta$. When $n > 1$, $i_* F$ is not necessarily flat over $\Delta^n$. Instead of using base change theorem we shall apply the comparison theorem between formal and analytic higher direct images by Banica [B-S, VI Proposition 4.2].

Let $t_1, t_2, ..., t_n$ be coordinates of $\Delta^n$. We assume that $Z_p$ is Kaehler and $\text{codim}(\text{Sing}(Z_p) \subset Z_p) \geq 4$ for all point $p \in \Delta^n$.

(i): For $n$-tuple of positive integers $(i_1, i_2, ..., i_n)$, and for a point $p = (p_1, ..., p_n) \in \Delta^n$, we put $A_{i_1, ..., i_n, p} = C[t_1, ..., t_n]/((t_1-p_1)^{i_1}, (t_2-p_2)^{i_2}, ..., (t_n-p_n)^{i_n})$.
with values in \( \mathbb{m} \) with \( q \in \mathcal{O}_{\Delta(i_1, \ldots, i_n)} \) and \( U_{(i_1, \ldots, i_n)} := U \times_{\Delta^n} \Delta_{(i_1, \ldots, i_n)} \). When \( p = (0, 0, \ldots, 0) \), we write \( A_{(i_1, \ldots, i_n)} \) (resp. \( \Delta_{(i_1, \ldots, i_n)} \), \( U_{(i_1, \ldots, i_n)} \)) for \( A_{(i_1, \ldots, i_n)} \) (resp. \( \Delta_{(i_1, \ldots, i_n)} \), \( U_{(i_1, \ldots, i_n)} \)).

(ii): For the later use we shall extend the notation above to the case where some indices \( i_t \) are infinity. For simplicity, we assume that \( i_1 = \infty, \ldots, i_{k-1} = \infty \) and \( i_k, \ldots, i_n \) are positive integers. The notation in the general case would be clear from the explanation below. Write \( \Delta^n = \Delta_{< k} \times \Delta_{\geq k} \), where \( \Delta_{< k} \) is the \( k-1 \)-dimensional polydisc with coordinates \( t_1, \ldots, t_{k-1} \) and \( \Delta_{\geq k} \) is the \( n-k+1 \)-dimensional polydisc with coordinates \( t_k, \ldots, t_n \). For \( n-k+1 \)-tuples of positive integers \( (i_k, \ldots, i_n) \), and for a point \( p = (p_{i_k}, \ldots, p_n) \in \Delta_{< k} \), we define \( \Delta_{(\infty, \ldots, i_k, \ldots, i_n)} \) as \( \Delta_{< k} \times \text{Spec} \mathbb{C}[t_k, \ldots, t_n]/((t_k - p_k)^{i_k}, (t_{k+1} - p_{k+1})^{i_{k+1}}, \ldots, (t_n - p_n)^{i_n}) \). Now \( F_{(\infty, \ldots, i_k, \ldots, i_n)} \) and \( U_{(\infty, \ldots, i_k, \ldots, i_n)} \) are defined in a similar way as the case where \( i_1, \ldots, i_n \) are all finite. We denote by \( \mathcal{O}_{(i_1, \ldots, i_n)} \) the structure sheaf of \( \Delta_{(i_1, \ldots, i_n)} \).

Claim. (1) \( R^j \pi_* F_{(i_1, \ldots, i_n)} \) are coherent for \( j = 0, 1 \) and for all \( (i_1, \ldots, i_n) \) with \( 0 < i_k \leq \infty \) (\( 1 \leq k \leq n \)).

(2) The natural maps \( \pi_* F_{(i_1+1, i_2, \ldots, i_n)} \rightarrow \pi_* F_{(i_1, i_2, \ldots, i_n)} \), \( \pi_* F_{(i_1, i_2+1, \ldots, i_n)} \rightarrow \pi_* F_{(i_1, i_2, \ldots, i_n)} \), \( \ldots, \pi_* F_{(i_1, \ldots, i_{k-1}+1, i_k, \ldots, i_n)} \rightarrow \pi_* F_{(i_1, \ldots, i_{k-1}, i_k, \ldots, i_n)} \) are surjective for all \( (i_1, \ldots, i_n) \) with \( 0 < i_k < \infty \) (\( 1 \leq k \leq n \)).

(3) \( \pi_* F \) is locally free and \( \pi_* F \otimes k(p) \cong H^0(U, F_p) \) for each \( p \in \Delta^n \).

Proof. We shall first prove (1) and (2). We finally conclude (3) by combining the comparison theorem [B-S, VI, Proposition 4.2] with (1) and (2).

(1): Since \( \text{codim}(\Sigma \subset Z) \geq 3 \) and \( \text{depth}(F_{(i_1, \ldots, i_n)}) = \dim U_{(i_1, \ldots, i_n)} \) for \( q \in U_{(i_1, \ldots, i_n)} \), \( i_* F_{(i_1, \ldots, i_n)} \) and \( R^1 i_* F_{(i_1, \ldots, i_n)} \) are both coherent. By the exact sequence

\[
0 \rightarrow R^1 g_*(i_* F_{(i_1, \ldots, i_n)}) \rightarrow R^1 \pi_* F_{(i_1, \ldots, i_n)} \rightarrow g_*(R^1 i_* F_{(i_1, \ldots, i_n)}) \rightarrow R^2 g_*(i_* F_{(i_1, \ldots, i_n)})
\]

we know that \( R^j \pi_* F_{(i_1, \ldots, i_n)} \) are coherent for \( j = 0, 1 \).

(2): We assume that \( p = (0, 0, \ldots, 0) \) because the proof are the same for all points \( p \). Since \( \text{codim}(\Sigma \subset Z) \geq 4 \), the spectral sequence

\[
E_1^{p,q} := H^q(U, \Omega^p_U) \Rightarrow H^{p+q}(U, \mathcal{C})
\]

degenerates at \( E_1 \) terms when \( p+q = 2 \) ([Oh, Na 1, Lemma 2.5]). Let \( U_m \rightarrow S_m \) be a flat deformation of \( U \) over \( S_m := \text{Spec} A_m \) where \( A_m := \mathbb{C}[t]/(t^{m+1}) \) with \( m \in \mathbb{Z}_{>0} \). Then, by [Na 1, Lemma 2.6], we see that the spectral sequence

\[
E_1^{p,q} := H^q(U, \Omega^p_{U_m/S_m}) \Rightarrow H^{p+q}(U, A_m)
\]

degenerates at \( E_1 \) terms with \( p+q = 2 \). Here \( A_m \) means the constant sheaf on \( U \) with values in \( A_m \). If we put \( U_{m-1} := U_m \times_{S_m} S_{m-1} \), then the restriction
map $H^q(U, \Omega^p_{U_m/S_m}) \to H^q(U, \Omega^p_{U_{m-1}/S_{m-1}})$ is surjective for $(p, q)$ with $p+q = 2$ ([Na 1, lemma 2.6]).

To prove (2) we only have to check that $\pi_* F_{(i_1+1, i_2, \ldots, i_n)} \to \pi_* F_{(i_1, i_2, \ldots, i_n)}$ is surjective by symmetry. One can split up the surjection $A_{(i_1+1, i_2, \ldots, i_n)} \to A_{(i_1, i_2, \ldots, i_n)}$ into a finite sequence of small extensions: $A_{(i_1+1, \ldots, i_n)} = A^{(N)} \to A^{(N-1)} \to \ldots \to A^{(1)} \to A_{(i_1, \ldots, i_n)}$ where $N = \dim_{\mathbb{C}} A_{(i_1, \ldots, i_n)}$. By definition, $K_j := \ker[A^{(j+1)} \to A^{(j)}]$ are one dimensional $\mathbb{C}$ vector spaces. For each $A^{(j+1)} \to A^{(j)}$ we can choose homomorphisms of local $\mathbb{C}$ algebras $\mathbb{C}[t]/(t^{m_j+1}) \to A^{(j+1)}$ and $\mathbb{C}[t]/(t^{m_j}) \to A^{(j)}$ in such a way that the diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & K_j & \longrightarrow & A^{(j+1)} & \longrightarrow & A^{(j)} & \longrightarrow & 0 \\
\phi_j & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathbb{C}[t]/(t^{m_j+1}) & \longrightarrow & \mathbb{C}[t]/(t^{m_j}) & \longrightarrow & 0
\end{array}
$$

(7)

commutes and $\phi_j$ is an isomorphism. We put $Z^{(j)} := Z \times_{\Delta^n} \text{Spec}\Lambda^{(j)}$, $F^{(j)} := F \otimes_{\mathcal{O}_{Z^{(j)}}} \mathcal{O}_{Z^{(j)}}$ and $F_{m_j} := F \otimes_{\mathcal{O}_{Z^{(j)}}} \mathcal{O}_{Z^{(j)}}$. By the previous observation, $\pi_* F_{m_j} \to \pi_* F_{m_{j-1}}$ is surjective. We see that $\pi_* F_{(j+1)} \to \pi_* F_{(j)}$ is surjective by the commutative diagram. Hence we know that $\pi_* F_{(i_1+1, i_2, \ldots, i_n)} \to \pi_* F_{(i_1, i_2, \ldots, i_n)}$ is surjective.

(3): We shall prove, by induction on $k$, that $\pi_* F_{(\infty, \ldots, \infty, i_k, \ldots, i_n; p)}$ are free $O_{(\infty, \ldots, \infty, i_k, \ldots, i_n; p)}$ modules and $\pi_* F_{(\infty, \ldots, \infty, i_k+1, \ldots, i_n; p)} \to \pi_* F_{(\infty, \ldots, \infty, i_k, \ldots, i_n; p)}$ are surjective for all $n-k+1$ tuple $(i_k, \ldots, i_n)$ without infinity. For $k = 1$, they are nothing but (2) of Claim. Let $\pi_* F_{(\infty, \ldots, \infty, i_k, \ldots, i_n; p)} \{p_{k-1} \}$ be the completion of $\pi_* F_{(\infty, \ldots, \infty, i_k, \ldots, i_n; p)}$ along the divisor $\{t_{k-1} = p_{k-1}\}$ of $\Delta^n$.

It suffices to prove that $\pi_* F_{(\infty, \ldots, \infty, i_k+1, \ldots, i_n; p)} \{p_{k-1} \} \to (\pi_* F_{(\infty, \ldots, \infty, i_k, \ldots, i_n; p)} \{p_{k-1} \})$ are surjective for all $p_{k-1}$ in order to prove that $\pi_* F_{(\infty, \ldots, \infty, i_k+1, \ldots, i_n; p)} \{p_{k-1} \}$ are surjective. By the comparison theorem [B-S, VI, Proposition 4.2] and by (1), we have

$$
(\pi_* F_{(\infty, \ldots, \infty, i_k, \ldots, i_n; p)} \{p_{k-1} \} \cong \lim_{\longrightarrow} \pi_* F_{(\infty, \ldots, \infty, m, i_k, \ldots, i_n; p_{k-1}, p)}.
$$

Note that $p = (p_k, \ldots, p_n) \in \Delta_{\geq k}$ and $p_{k-1} \in \Delta^1(t_{k-1})$, where $\Delta^1(t_{k-1})$ is the 1-dimensional disc with a coordinate $t_{k-1}$. By the induction hypothesis, the maps

$$
\pi_* F_{(\infty, \ldots, \infty, m, i_k+1, \ldots, i_n; p_{k-1}, p)} \to \pi_* F_{(\infty, \ldots, \infty, m, i_k, \ldots, i_n; p_{k-1}, p)}
$$

and

$$
\pi_* F_{(\infty, \ldots, \infty, m+1, i_k, \ldots, i_n; p_{k-1}, p)} \to \pi_* F_{(\infty, \ldots, \infty, m, i_k, \ldots, i_n; p_{k-1}, p)}
$$

are both surjective for all $p_{k-1}$. Therefore we conclude that

$$
(\pi_* F_{(\infty, \ldots, \infty, i_k+1, \ldots, i_n; p_{k-1} \}) \{p_{k-1} \} \to (\pi_* F_{(\infty, \ldots, \infty, i_k, \ldots, i_n; p_{k-1} \}) \{p_{k-1} \}
$$

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are surjective for all $p_{k-1}$.

We shall next prove that $\pi_* F_{(i_1, \ldots, i_n; p)}$ are free $O_{(i_1, \ldots, i_n; p)}$ modules.

We shall prove that $\pi_* F_{(\infty, \ldots, \infty; i_1, \ldots, i_n; p)}$ is a free $O_{(\infty, \ldots, \infty; i_1, \ldots, i_n; p)}$ module by assuming that $\pi_* F_{(\infty, \ldots, \infty; k-1, \ldots, i_n; p')}$ are free $O_{(\infty, \ldots, \infty; k-1, \ldots, i_n; p')}$ modules for all $* \in \mathbb{Z}_{\geq 0}$ and all $p' \in \Delta_{\geq 1}$.

We use the induction on the lexicographic order of $(i_k, \ldots, i_n)$. First $\pi_* F_{(\infty, \ldots, \infty, 1, \ldots, 1; p)}$ is a free $O_{(\infty, \ldots, \infty, 1, \ldots, 1; p)}$ module; in fact, let $(\pi_* F_{(\infty, \ldots, \infty, 1, \ldots, 1; p)})_{p_k-1}$ be the completion of $\pi_* F_{(\infty, \ldots, \infty, 1, \ldots, 1; p)}$ along the divisor $\{i_k = p_k\}$ of $\Delta^n$. Then

$$(\pi_* F_{(\infty, \ldots, \infty, 1, \ldots, 1; p)})_{p_k-1} \cong \lim_{\longrightarrow} \pi_* F_{(\infty, \ldots, \infty, m, 1, \ldots, 1; p_k-1; p)}.$$

Since $\pi_* F_{(\infty, \ldots, \infty, m, 1, \ldots, 1; p_k-1; p)}$ are free $O_{(\infty, \ldots, \infty, m, 1, \ldots, 1; p_k-1; p)}$ modules by assumption and $\pi_* F_{(\infty, \ldots, \infty, m, 1, \ldots, 1; p_k-1; p)} \rightarrow \pi_* F_{(\infty, \ldots, \infty, m-1, 1, \ldots, 1; p_k-1; p)}$ are surjective, the right hand side is a free $O_{(\infty, \ldots, \infty, 1, \ldots, 1; p)}$ module. Therefore, $\pi_* F_{(\infty, \ldots, \infty, 1, \ldots, 1; p)}$ is a free $O_{(\infty, \ldots, \infty, 1, \ldots, 1; p)}$ module.

Next consider $\pi_* F_{(\infty, \ldots, \infty, i_k, \ldots, i_n; p)}$. By induction $\pi_* F_{(\infty, \ldots, \infty, i_k, \ldots, i_n; p)} = \pi_* F_{(\infty, \ldots, \infty, i_k, \ldots, i_n; p)} \otimes O_{(\infty, \ldots, \infty, i_k, \ldots, i_n; p)}$ is isomorphic to $(O_{(\infty, \ldots, \infty, i_k, \ldots, i_n; p)})^r$.

By Nakayama’s lemma, we have a surjection from $(O_{(\infty, \ldots, \infty, i_k, \ldots, i_n; p)})^r$ to $\pi_* F_{(\infty, \ldots, \infty, i_k, \ldots, i_n; p)}$. We then have a commutative diagram with exact rows:

$$
\begin{array}{ccc}
(O_{(\infty, i_k+1, \ldots, i_n; p)})^r & \longrightarrow & (O_{(\infty, i_k, \ldots, i_n; p)})^r \\
\downarrow & & \downarrow \\
\pi_* F_{(\infty, i_k+1, \ldots, i_n; p)} & \longrightarrow & \pi_* F_{(\infty, i_k, \ldots, i_n; p)} \\
\end{array}
$$

On each row, the first map is injective and the second one is surjective. The right vertical map is an isomorphism. The middle vertical map is surjective; hence, by the Snake Lemma, the left vertical map is surjective. On the other hand, by the induction hypothesis, $\pi_* F_{(\infty, \ldots, \infty, i_k, \ldots, i_n; p)}$ is a free $O_{(\infty, \ldots, \infty, i_k, \ldots, i_n; p)}$ module of rank $r$. This implies that the left vertical map is an isomorphism. Again, by the Snake lemma, the middle vertical map is an injection, hence an isomorphism. Q.E.D.

Proof of Theorem 7’ continued. By Claim (3) the symplectic 2-form $\omega$ on $U$ extends sideways. Since Theorem 4 (hence Theorem 6) holds for a (non-algebraic) singularity with $\text{codim}(\Sigma \subset X) \geq 4$ by [Fl], the rest of the argument is the same as Theorem 7. Q.E.D.

We now consider the following situation: Let $Z$ be a symplectic variety. Put $\Sigma := \text{Sing}(Z)$ and $U := Z \setminus \Sigma$. Let $\pi : Z \rightarrow S$ be the Kuranishi family of $Z$, which is, by definition, a semi-universal flat deformation of $Z$ with $\pi^{-1}(0) = Z$ for the reference point $0 \in S$. When $\text{codim}(\Sigma \subset Z) \geq 4$, $S$ is smooth by [Na 1, Theorem 2.4]. Define $\mathcal{U}$ to be the locus in $Z$ where $\pi$ is a smooth map and let $\pi : \mathcal{U} \rightarrow S$ be the restriction of $\pi$ to $\mathcal{U}$. The following is a generalization of the Local Torelli Theorem [Be 2, Theorem 5] to singular symplectic varieties.
Theorem 8 Assume that \( Z \) is a \( \mathbb{Q} \)-factorial projective symplectic variety. Assume \( h^1(Z, \mathcal{O}_Z) = 0, h^0(U, \Omega^2_U) = 1 \), \( \dim Z = 2l \geq 4 \) and \( \text{Codim}(\Sigma \subset Z) \geq 4 \). Then the following hold.

1. \( R^2\pi_*(\pi^{-1}\mathcal{O}_S) \) is a free \( \mathcal{O}_S \) module of finite rank. Let \( \mathcal{H} \) be the image of the composite \( R^2\pi_*\mathcal{C} \to R^2\pi_*\mathcal{C} \to R^2\pi_*(\pi^{-1}\mathcal{O}_S) \). Then \( \mathcal{H} \) is a local system on \( S \) with \( \mathcal{H}_s = H^2(\mathcal{U}_s, \mathcal{C}) \) for \( s \in S \).

2. The restriction map \( H^2(Z, \mathcal{C}) \to H^2(U, \mathcal{C}) \) is an isomorphism. Take a resolution \( \nu: \tilde{Z} \to Z \) in such a way that \( \nu^{-1}(U) \cong U \). For \( \alpha \in H^2(U, \mathcal{C}) \) we take a lift \( \tilde{\alpha} \in H^2(\tilde{Z}, \mathcal{C}) \) by the composite \( H^2(U, \mathcal{C}) \cong H^2(Z, \mathcal{C}) \to H^2(\tilde{Z}, \mathcal{C}) \). Choose \( \omega \in H^0(U, \Omega^2_{\mathcal{U}_s}) = \mathcal{C} \). This \( \omega \) extends to a holomorphic 2-form on \( \tilde{Z} \). Normalize \( \omega \) in such a way that \( \int_{\tilde{Z}}(\omega^2)^l = 1 \). Then one can define a quadratic form \( q: H^2(U, \mathcal{C}) \to \mathcal{C} \) as

\[
q(\alpha) := 1/2 \int_{\tilde{Z}} (\omega^2)^l \tilde{\alpha}^2 + (1-l) \left( \int_{\tilde{Z}} \omega^{-1} \tilde{\alpha} \right) \left( \int_{\tilde{Z}} \omega^{-1} \tilde{\alpha} \right).
\]

This form is independent of the choice of \( \nu: \tilde{Z} \to Z \).

3. Put \( H := H^2(U, \mathcal{C}) \). Then there exists a trivialization of the local system \( \mathcal{H}: \mathcal{H} \cong H \times S \). Let \( \mathcal{D} := \{ x \in \mathbf{P}(H): q(x) = 0, q(x + \pi) > 0 \} \). Then one has a period map \( p: S \to \mathcal{D} \) and \( p \) is a local isomorphism.

Remark. (1) The assumption that \( \text{Codim}(\Sigma \subset Z) \geq 4 \) is always satisfied when \( Z \) has only terminal singularities [Na 2, Theorem].

(2) One can apply Theorem 8 for \( Z \) for irreducible symplectic \( V \) manifolds (cf. [Fu]); but the results seems rather trivial by Schlessinger’s rigidity theorem for quotient singularities.

Most interesting objects are \( \overline{M}_{0,c} \) in Example (iii) \( (c \geq 6) \) of Introduction. \( \overline{M}_{0,c} \) conjecturally satisfy the assumption of Theorem 8. Since they have more complicated singularities than quotient singularities, one expects that they give non-trivial examples for Theorem 8. Actually Theorem 8 is motivated by them.

Proof of (1): The first statement follows from the following result.

Proposition 9. The spectral sequence

\[
E_1^{p,q} = R^p\pi_*\Omega^q_{U/S} \Rightarrow R^{p+q}\pi_*(\pi^{-1}\mathcal{O}_S)
\]

degenerates at \( E_1 \) terms for \( p+q = 2 \). Moreover, \( E_1^{p,q} \) are locally free sheaves for \( p+q = 2 \).

Proof of Proposition 9. (a): \( E_1^{2,0} \) is locally free and compatible with base change by Theorem 7. Since \( Z \) is Gorenstein and \( \text{Codim}(\Sigma \subset Z) \geq 4 \), \( H^2(Z, \mathcal{O}_Z) \cong H^2(\mathcal{U}_s, \mathcal{O}_{\mathcal{U}_s}) \). Therefore \( E_1^{0,2} \) is also locally free and compatible with base change by the proof of Theorem 7. By [Na 1, Theorem 2.4] \( S \) is smooth. Since \( h^1(Z, \mathcal{O}_Z) = 0 \), we have \( h^0(Z, \Theta_Z) = 0 \), hence \( \pi: Z \to S \) is the universal
family. This implies that $T_{S,s} \cong H^1(U_s, \Theta_{U_s})$ for $s \in S$, where $T_{S,s}$ is the tangent space of $S$ at $s$. On the other hand, there are natural identifications $H^1(U_s, \Theta_{U_s}) \cong H^1(U_s, \Omega^1_{U_s})$ by a relative symplectic 2-form of $\pi$ (such a 2-form exists by Theorem 7'). Therefore, $E_{1,1}$ is locally free and compatible with base change.

(b): We shall prove that the composed map $H^2(Z, \mathbb{C}) \to H^2(U, \mathbb{C}) \to H^2(U, \mathbb{C})$ is surjective when we choose $S$ small enough. Since $H^2(Z, \mathbb{C}) \cong H^2(Z, \mathbb{C})$, it suffices to show that the restriction map $\alpha : H^2(Z, \mathbb{C}) \to H^2(U, \mathbb{C})$ is surjective. Let $\nu : Z \to Z$ be a resolution of singularities whose exceptional locus $E$ is a simple normal crossing divisor. Then $\alpha$ can be factorized as $H^2(Z, \mathbb{C}) \to H^2(Z, \mathbb{C}) \to H^2(U, \mathbb{C})$. There is an exact sequence of mixed Hodge structures

$$H^2_{\tilde{Z}}(\tilde{Z}, \mathbb{C}) \to H^2(\tilde{Z}, \mathbb{C}) \to H^2(U, \mathbb{C}) \to H^2_{\tilde{Z}}(\tilde{Z}, \mathbb{C}).$$

Note that $H^2_{\tilde{Z}}(\tilde{Z})$ has the mixed Hodge structure with weights $\geq 3$. On the other hand, $H^2(U)$ has the pure Hodge structure of weight 2 because $\text{Codim}(\Sigma \subset Z) \geq 4$. Therefore $H^2_{\tilde{Z}}(\tilde{Z}, \mathbb{C}) \to H^2(U, \mathbb{C})$ is a surjection.

We shall prove that, as a $\mathbb{C}$ vector space, $H^2_{\tilde{Z}}(\tilde{Z}, \mathbb{C})$ is generated by the image of $H^2(Z, \mathbb{C})$ and the image of $H^2_{\tilde{Z}}(\tilde{Z}, \mathbb{C})$: if this is proved, then $\alpha$ is surjective by the exact sequence. Let $E = \Sigma E_i$ be the irreducible decomposition and denote by $[E_i] \in H^2(Z, \mathbb{C})$ the cohomology class corresponding to the divisor $E_i$. Then we have $H^2_{\tilde{Z}}(\tilde{Z}, \mathbb{C}) \cong \oplus \mathbb{C}[E_i]$. Since $Z$ is $\mathbb{Q}$-factorial and $Z$ has only rational singularities, $\text{im}[H^2(\tilde{Z}, \mathbb{C}) \to H^0(Z, \mathbb{R}^2\nu_*\mathbb{C})] = \text{im}[\oplus \mathbb{C}[E_i] \to H^0(Z, \mathbb{R}^2\nu_*\mathbb{C})]$ by [Ko-Mo, (12.1.6)]. Note that $H^2(Z, \mathbb{C}) = \text{Ker}[H^2(\tilde{Z}, \mathbb{C}) \to H^0(Z, \mathbb{R}^2\nu_*\mathbb{C})]$ because $Z$ has only rational singularities. The argument here shows that the restriction map $H^2(Z, \mathbb{C}) \to H^2(U, \mathbb{C})$ is, in fact, an isomorphism because $H^2(Z, \mathbb{C}) \to H^2(\tilde{Z}, \mathbb{C})$ is an injection. (Note that $R^1\nu_*\mathbb{C} = 0$ because $Z$ has rational singularities.)

(c): Let $k(0)$ be the skyscraper sheaf supported at $0 \in S$ which is defined as the quotient $\mathcal{O}_S/m_0$, where $m_0$ is the ideal sheaf of $0 \in S$. Then the natural map $(R^2\pi_*(\pi^{-1}\mathcal{O}_S))_0 \to R^2\pi_*(\pi^{-1}k(0))$ is surjective. In fact, $(R^2\pi_*(\pi^{-1}\mathcal{O}_S))_0$ factors the map $(R^2\pi_*\mathcal{O}^{-1}_S)_0 \to R^2\pi_*(\pi^{-1}k(0)))$. This map is nothing but the map $H^2(Z, \mathbb{C}) \to H^2(U, \mathbb{C})$ which is a surjection by (b).

---

1Since $\text{Codim}(\Sigma \subset Z) \geq 4$, by [Oh, Na 1, Lemma 2.5] the Hodge spectral sequence $H^q(U, \Omega^p_U) = H^{p+q}(U, \mathbb{C})$ degenerates at $E_1$ terms when $p + q = 2$. Moreover, $h^2(U, \mathcal{O}_U) = h^0(U, \Omega^2_U)$. This filtration coincides with the Hodge filtration of the mixed Hodge structure on $H^2(U, \mathbb{C})$. In fact, there is a natural map $\phi_{p,q} : H^q(\tilde{Z}, \Omega^p_Z(\log E)) \to H^q(U, \Omega^p_U)$ for each $p$ and $q$. By [Fl] or Proposition 1, we have $\nu_*\Omega^2_Z(\log E) \cong i_*\Omega^2_U$, hence $\phi_{2,0}$ is an isomorphism. On the other hand, since $Z$ has only rational singularities and $\text{Codim}(\Sigma \subset Z) \geq 4$, $\phi_{0,2}$ is an isomorphism. Recall that the spectral sequence $H^q(Z, \Omega^p_Z(\log E)) = H^p(U, \Omega^q_U)$ degenerates at $E_1$ terms. Then we know that $\phi_{1,1}$ is also an isomorphism. Thus our filtration coincides with the Hodge filtration of the mixed Hodge structure of $H^2(U, \mathbb{C})$. Since $h^2(U, \mathcal{O}_U) = h^0(U, \Omega^2_U)$, the mixed Hodge structure is pure.
We are now in a position to prove the $E_1$ degeneracy of the spectral sequence. Let $0 \subset F^2 \subset F^1 \subset F^0 = R^2\pi_*(\pi^{-1}\mathcal{O}_S)$ be the decreasing filtration defined by the spectral sequence. By checking the coherence of each $E^{p,q}_k$ term (under the assumption $\text{codim}(\Sigma \subset Z) \geq 4$), we can see that $F^i$ are coherent. We shall prove that $(\text{Gr}^i\mathcal{F})_0 = (R^2\pi_*(\pi^{-1}\mathcal{O}_S))_0$ for $i = 0, 1, 2$.

$(i = 0)$: Since $E^{0,2}_\infty \subset E^{0,2}_1$, it is enough to prove that the map $(R^2\pi_*(\pi^{-1}\mathcal{O}_S))_0 \rightarrow (R^2\pi_*\mathcal{O}_U)_0$ is surjective. We have a commutative diagram

\[
\begin{array}{ccc}
(R^2\pi_*(\pi^{-1}\mathcal{O}_S))_0 & \longrightarrow & (R^2\pi_*\mathcal{O}_U)_0 \\
\downarrow & & \downarrow \\
H^2(U, \mathcal{C}) & \longrightarrow & H^2(U, \mathcal{O}_U)
\end{array}
\] (9)

By (c) the vertical map on the left hand side is surjective. By (a), $(R^2\pi_*\mathcal{O}_U)_0 \otimes k(0) \cong H^2(U, \mathcal{O}_U)$. Since $\text{codim}(\Sigma \subset Z) \geq 4$, the spectral sequence

\[H^q(U, \Omega^p_U) = \Rightarrow H^{p+q}(U, \mathcal{C})\]

degenerates at $E_1$ terms when $p + q = 2$ ([Oh, Na 1, Lemma 2.5]). Hence the horizontal map at the bottom is surjective. By Nakayama’s Lemma we see that the horizontal map on the top is also surjective.

$(i = 1)$: By the assumption, $E^{0,1}_\infty = 0$, hence $E^{1,1}_\infty \subset E^{1,1}_1$. It is enough to prove that $(\mathcal{F})_0 \rightarrow (R^1\pi_*\Omega^1_U/S)_0$ is surjective. We have two commutative diagrams:

\[
\begin{array}{ccc}
(\mathcal{F})_0 \otimes k(0) & \longrightarrow & (R^2\pi_*(\pi^{-1}\mathcal{O}_S))_0 \otimes k(0) \\
\downarrow & & \downarrow \\
F^1 & \longrightarrow & H^2(U, \mathcal{C}) \\
\downarrow & & \downarrow \\
F^1 & \longrightarrow & H^2(U, \mathcal{O}_U) \\
(\mathcal{F})_0 & \longrightarrow & (R^1\pi_*\Omega^1_U/S)_0 \\
\downarrow & & \downarrow \\
F^1 & \longrightarrow & H^1(U, \Omega^1_U).
\end{array}
\] (10)

The $F^1$ in the first diagram is the filtration of $H^2(U, \mathcal{C})$ induced by the spectral sequence

\[H^q(U, \Omega^p_U) = \Rightarrow H^{p+q}(U, \mathcal{C}).\]

The rows in the first diagram are exact. Moreover, $F_1 \rightarrow H^1(U, \Omega^1_U)$ is injective by definition of $F_1$. Let us look at the first diagram. By (a) the
vertical map on the right hand side is an isomorphism. The middle vertical map is surjective by (c). Hence the vertical map on the left hand side is surjective.

We next observe the second diagram. The map $(F^1)_0 \to F^1$ is surjective because it is factorized as $(F^1)_0 \to (F^1) \otimes k(0) \to F^1$. Since $Gr^1_P = H^1(U, \Omega^1_U)$, the horizontal map at the bottom is surjective. Since $(R^1\pi_*\Omega^1_{U/S})_0 \otimes k(0) \cong H^1(U, \Omega^1_U)$, the map $(F^1)_0 \to (R^1\pi_*\Omega^1_{U/S})_0$ is surjective by Nakayama’s lemma.

$(i = 2)$: By the assumption $E^{1,0}_1 = 0$ and $E^{0,1}_1 = 0$; hence $E^{2,0}_1 \subset E^{2,0}_1$. We shall prove that $(F^2)_0 \to (\pi_*\Omega^2_{U/S})_0$ is surjective. We have two commutative diagrams:

$$
\begin{align*}
(F^2)_0 \otimes k(0) & \longrightarrow (F^1) \otimes k(0) \longrightarrow (R^1\pi_*\Omega^1_{U/S}) \otimes k(0) \longrightarrow 0 \\
F^2 & \longrightarrow F^1 \longrightarrow H^1(U, \Omega^1_U) \longrightarrow 0 \quad \text{(12)} \\
(F^2)_0 & \longrightarrow (\pi_*\Omega^2_{U/S})_0 \\
F^2 & \longrightarrow H^0(U, \Omega^2_U). \quad \text{(13)}
\end{align*}
$$

In the first diagram the vertical map on the right hand side is an isomorphism, and the middle vertical map is surjective by the argument of $(i = 1)$. Therefore $F^2 \otimes k(0) \to F^2$ is surjective; this implies that, in the second diagram, the vertical map on the left hand side is surjective. Look at the second diagram. Since the horizontal map at the bottom is an isomorphism and the vertical map on the right hand side is surjective by (a), we conclude that $(F^2)_0 \to (\pi_*\Omega^2_{U/S})_0$ is surjective by Nakayama’s lemma. Q.E.D.

Let us prove that $\mathcal{H}$ is a local system on $S$ with $\mathcal{H}_s = H^2(\mathcal{H}_s, \mathcal{C})$ for $s \in S$. First note that $H^2\pi_*(\pi^{-1}\mathcal{O}_S) \otimes k(s) \cong H^2(\mathcal{H}_s, \mathcal{C})$ by Lemma 9 because $\mathcal{O}_{U/S}, p + q = 2$ are compatible with base change, and the spectral sequence $H^q(\mathcal{H}_s, \mathcal{O}_{U/s}) \Rightarrow H^2(\mathcal{H}_s, \mathcal{C})$ degenerates at $E_1$ terms with $p + q = 2$. We take $S$ small enough so that $Z$ has strong deformation retract to $Z$. Choose $s \in S$. Then we have a diagram:

$$
H^2(U, C) \leftarrow H^2(Z, C) \to H^2(Z_s, C) \to H^2(U_s, C).
$$

The map $H^2(Z, C) \to H^2(U, C)$ is an isomorphism by (b); hence we have a map $\phi_s : H^2(U, C) \to H^2(U_s, C)$. Consider the map $\iota : \Gamma(S, R^2\pi_*\mathcal{C}) \to \Gamma(S, R^2\pi_*(\pi^{-1}\mathcal{O}_S))$ induced by the sheaf homomorphism $R^2\pi_*\mathcal{C} \to R^2\pi_*\mathcal{C} \to R^2\pi_*(\pi^{-1}\mathcal{O}_S)$. Denote by $\iota(s)$ the composite of $\iota$ with the evaluation map at $s$: $\Gamma(S, R^2\pi_*(\pi^{-1}\mathcal{O}_S)) \to H^2(U_s, C)$. For $s \in \Gamma(S, R^2\pi_*(\pi^{-1}\mathcal{O}_S))$ we see
that \((i(s))(\sigma) = \phi_s((i(0))(\sigma))\). If we choose \(\sigma_1, ..., \sigma_r \in \Gamma(S, R^2\pi_*C)\) in such a way that \((i(0))(\sigma_1), ..., (i(0))(\sigma_r)\) span \(H^2(U, C)\) as a \(C\) vector space, then \((i(s))(\sigma_1), ..., (i(s))(\sigma_r)\) also span \(H^2(U_s, C)\) for each \(s \in S\), if necessary, by replacing \(S\) by a smaller one. This implies that \(H^2(Z_s, C) \to H^2(U_s, C)\) is surjective in the diagram. Since \(H^2(Z_s, C) \to H^2(U_s, C)\) is injective (cf. the final statement of (b)), it is an isomorphism. Moreover, since \(\dim H^2(U_s, C)\) are constant, we know by the diagram that \(H^2(Z, C) \cong H^2(Z_s, C)\). This completes the proof of (1) of Theorem 8.

**Proof of (2) and (3):** The statement of (2) is now clear from the arguments in the proof of (1). In the proof of (1), we have constructed an isomorphism \(\phi_s : H^2(U, C) \to H^2(U_s, C)\) for each \(s \in S\). The trivialization \(H \times S \to \mathcal{H}\) is given by \((x, s) \to \phi_s(x) \in \mathcal{H}_s = H^2(U_s, C)\). By Theorem 7 and the assumption, \(\pi_*\Omega^2_{U/S}\) is a line bundle on \(S\) and is compatible with base change. Then we can choose \(\hat{\omega} \in \Gamma(S, \pi_*\Omega^2_{U/S})\) in such a way that \(\hat{\omega}_0 = \omega\) and \(\int_{\hat{\omega}} \omega_s^2 = 1\) for \(s \in S\). By Hodge decomposition \(H^2(U_s, C) = H^0(U_s, \Omega^2_{U_s/C}) \oplus H^1(U_s, \Omega^1_{U_s/C}) \oplus H^2(U_s, \Omega^0_{U_s/C})\), \(\hat{\omega}_s\) becomes an element of \(H^2(U_s, C)\). Now the period map \(p : S \to \mathbf{P}(H)\) is given by \(s \to \hat{x}_s^{\tau}(\hat{\omega}_s)\). The image of \(p\) is contained in \(D\), and \(D\) is a local isomorphism between \(S\) and \(D\). The proofs are similar to [Be 2, Theorem 5].

**Remark (1):** In (2) the quadratic form \(q(\alpha)\) can be defined by an arbitrary lift \(\hat{\alpha} \in H^2(\bar{Z}, C)\) of \(\alpha \in H^2(U, C)\). The proof is as follows. Note that (cf. (b) of the proof of Proposition 9)

\[
\ker[H^2(\bar{Z}, C) \to H^2(U, C)] = \im[\oplus_i C[E_i] \to H^2(\bar{Z}, C)].
\]

Since \(\int_{\bar{Z}} \omega^2 \beta^{-1} E_i = \int_{\bar{Z}} \omega^2 \beta^{-1} \in [E_i] = 0\), we only have to prove that \(\int_{\bar{Z}} \omega^2 \beta^{-1} E_i = 0\). Set \(S_i = \nu(E_i)\). We blow up \(\bar{Z}\) further and replace \(\nu\) by a new resolution for which the inverse image of \(S_i\) is a simple normal crossing divisor. Hereafter we call this new resolution \(\nu : \bar{Z} \to Z\) and put \(F := \nu^{-1}(S_i)\). By definition, \(F\) contains an irreducible component which is birational to the original \(E_i\). By abuse of notation we call this component \(E_i\). We only have to check that \(\omega^2 \beta^{-1} E_i = 0\) for the new \(E_i\). We shall derive a contradiction by assuming that \(\omega^2 \beta^{-1} E_i \neq 0\). Consider the map \(F \to S_i\) induced by \(\nu\). For \(p \in S_i\), denote by \(F_p\) the fiber over \(p\). For a general point \(p \in S_i\), \(F_p\) is a normal crossing variety. Since \(\text{Codim}(S_i \subset Z) \geq 3\) by assumption, we have no non-zero holomorphic \(2l - 2\) forms on \(S_i\). Therefore, if \(\omega^2 \beta^{-1} E_i \neq 0\), then, for a general point \(p \in S_i\), \(H^0(F_p, \Omega^0_{F_p/C}) \neq 0\) for some \(i > 0\). Put \(k = \dim S_i\) and take a general complete intersection of \(Z\) by \(k\) hyperplanes: \(H = H_1 \cap ... \cap H_k\). Put \(\tilde{H} := \nu^{-1}(H)\). \(H\) has canonical singularities, hence has rational singularities. Moreover, \(f : \tilde{H} \to H\) is a resolution of singularities. Choose a point \(p_i\) from \(H \cap S_i\). We may assume that this \(p_i\) is general in the above sense. Note that \(f^{-1}(p_i) = F_p\). Since \(R^j f_* \mathcal{O}_{F_p} = 0\) for \(j > 0\), we see that \(H^j(F_p, \Omega^j_{F_p/C}) = 0\) for \(j > 0\). By the mixed Hodge structure on \(H^j(F_p)\) we conclude that \(H^0(F_p, \Omega^0_{F_p/C}) = 0\) for all \(j > 0\). This is a contradiction.
Remark (2): In Theorem 8, if we replace the $\mathbb{Q}$-factoriality condition by the next condition (*), then it is also valid for a non-projective symplectic variety:

\[(*) \operatorname{im}[H^2(Z, \mathbb{Q}) \rightarrow H^0(Z, R^2\nu_! \mathbb{Q})] = \operatorname{im}[\oplus \mathbb{Q}[E_i] \rightarrow H^0(Z, R^2\nu_* \mathbb{Q})].\]

This condition is equivalent to the $\mathbb{Q}$-factoriality when $Z$ is projective [Ko-Mo, (12.1.6)]. But when $Z$ is non-projective, they do not seem equivalent; for example, when $Z$ has no Weil divisors, $\mathbb{Q}$-factoriality is meaningless. The condition (*) is an open condition for a family of symplectic varieties with terminal singularities.

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