MIRROR SYMMETRY FOR QUASI-SMOOTH CALABI-YAU
HYPERSURFACES IN WEIGHTED PROJECTIVE SPACES

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Abstract. We consider a $d$-dimensional well-formed weighted projective space $\mathbb{P}(\overline{w})$ as a toric variety associated with a fan $\Sigma(\overline{w})$ in $\mathbb{N}^w \otimes \mathbb{R}$ whose 1-dimensional cones are spanned by primitive vectors $v_0, v_1, \ldots, v_d \in \mathbb{N}^w$ generating a lattice $\mathbb{N}^w$ and satisfying the linear relation $\sum_{i=0}^d w_i v_i = 0$. For any fixed dimension $d$, there exist only finitely many weight vectors $\overline{w} = (w_0, \ldots, w_d)$ such that $\mathbb{P}(\overline{w})$ contains a quasi-smooth Calabi-Yau hypersurface $X_w$ defined by a transverse weighted homogeneous polynomial $W$ of degree $w = \sum_{i=0}^d w_i$. Using a formula of Vafa for the orbifold Euler number $\chi_{\text{orb}}(X_w)$, we show that for any quasi-smooth Calabi-Yau hypersurface $X_w$ the number $(-1)^{d-1} \chi_{\text{orb}}(X_w)$ equals the stringy Euler number $\chi_{\text{str}}(X^*_w)$ of Calabi-Yau compactifications $X^*_w$ of affine toric hypersurfaces $Z_w$ defined by non-degenerate Laurent polynomials $f_w \in \mathbb{C}[\mathbb{N}^w]$ with Newton polytope $\text{conv}(\{v_0, \ldots, v_d\})$. In the moduli space of Laurent polynomials $f_w$ there always exists a special point $f^*_w$ defining a mirror $X^*_w$ with a $\mathbb{Z}/w\mathbb{Z}$-symmetry group such that $X^*_w$ is birational to a quotient of a Fermat hypersurface via a Shioda map.

1. Introduction

Many topologically different examples of smooth Calabi-Yau threefolds and many evidences for mirror symmetry were obtained from quasi-smooth Calabi-Yau hypersurfaces $X_w$ in 4-dimensional weighted projective spaces $\mathbb{P}(w_0, w_1, w_2, w_3, w_4)$ defined by weighted homogeneous polynomials $W \in \mathbb{C}[z_0, \ldots, z_4]$ of degree $w = \sum_{i=0}^4 w_i$ such that the differential $dW$ vanishes exactly in the origin $0 \in \mathbb{C}^5$ [CLS90, GP90, CdlOGP91, CdlOK95]. The Hodge numbers of two $d$-dimensional smooth Calabi-Yau varieties $V$ and $V^*$ that are mirror symmetric to each other must satisfy the equalities

$$h^{p,q}(V) = h^{d-p,q}(V^*)$$

for all $0 \leq p, q \leq d$ [Wit92]. In particular, the Euler number $\chi = \sum_{p,q} (-1)^{p+q} h^{p,q}$ must satisfy the equality

$$\chi(V) = (-1)^d \chi(V^*).$$

Unfortunately, quasi-smooth Calabi-Yau hypersurfaces $X_w \subset \mathbb{P}(w_0, w_1, w_2, w_3, w_4)$ defined by weighted homogeneous polynomials $W$ are usually singular. The first mathematical verifications of the above mentioned equalities for Hodge and Euler numbers were based on the existence of crepant desingularizations $\rho : Y \to X_w$ that allow to replace $X_w$ by smooth Calabi-Yau threefolds $Y$ [Roa90]. Note that crepant desingularizations $\rho : Y \to X_w$ of quasi-smooth Calabi-Yau hypersurfaces $X_w \subset \mathbb{P}(w_0, w_1, \ldots, w_d)$ do not exist in general if $d \geq 5$. 

Let us recall some definitions and facts on \(d\)-dimensional weighted projective spaces \(\mathbb{P}(\overline{w})\) and quasi-smooth Calabi-Yau hypersurfaces in \(\mathbb{P}(\overline{w})\), where \(\overline{w} := (w_0, w_1, \ldots, w_d)\).

**Definition 1.1.** A weighted projective space \(\mathbb{P}(\overline{w})\) is a quotient of \(\mathbb{C}^{d+1} \setminus \{0\}\) by the \(\mathbb{C}^*\)-action \(\lambda \cdot (a_0, a_1, \ldots, a_d) = (\lambda^{w_0} a_0, \lambda^{w_1} a_1, \ldots, \lambda^{w_d} a_d)\) defined by the weight vector \(\overline{w} = (w_0, w_1, \ldots, w_d) \in \mathbb{Z}_{>0}^{d+1}\), whose coordinates \(w_i\) are called weights. A weighted projective space \(\mathbb{P}(\overline{w})\) is called *well-formed* if

\[
gcd(w_0, \ldots, w_{i-1}, w_{i+1}, \ldots, w_d) = 1 \quad \forall i \in \{0, 1, \ldots, d\}.
\]

In this paper, we consider only well-formed weighted projective spaces \(\mathbb{P}(\overline{w})\). A weighted homogeneous polynomial \(W \in \mathbb{C}[z_0, z_1, \ldots, z_d]\) of degree \(k\) is characterized by the condition

\[
W(\lambda^{w_0} z_0, \lambda^{w_1} z_1, \ldots, \lambda^{w_d} z_d) = \lambda^k \cdot W(z_0, z_1, \ldots, z_d).
\]

Moreover, a weighted homogeneous polynomial \(W\) of degree \(k\) and the hypersurface \(X_k := \{W(z) = 0\} \subset \mathbb{P}(\overline{w})\) of degree \(k\) are called transverse if the common zeros of all partial derivatives

\[
\partial W/\partial z_i = 0 \quad (0 \leq i \leq d)
\]

is the point \(0 \in \mathbb{C}^{d+1}\). A weight vector \(\overline{w} = (w_0, w_1, \ldots, w_d) \in \mathbb{Z}_{>0}^{d+1}\) is called transverse if there exists at least one transverse Calabi-Yau hypersurface \(X_w \subset \mathbb{P}(\overline{w})\) of degree \(w = \sum_i w_i\). Another name, more often used by mathematicians, for transverse Calabi-Yau hypersurfaces is quasi-smooth Calabi-Yau hypersurfaces. The quasi-smoothness (or transversality) condition ensures that the hypersurface \(X_w\) has no singularities in addition to those coming from the singularities of the ambient space [KS98b], where the only singularities of weighted projective spaces are cyclic quotient singularities [CLS11, Definition 11.4.5].

**Definition 1.2.** A weight vector \(\overline{w} \in \mathbb{Z}_{>0}^{d+1}\) has the *IP-property* if

\[
\Delta(W) := \text{conv}(\{(u_0, u_1, \ldots, u_d) \in \mathbb{Z}_{>0}^{d+1} \mid \sum_{i=0}^{d} w_i u_i = w\}) \subseteq \mathbb{R}^{d+1}
\]

is a \(d\)-dimensional lattice polytope containing the lattice point \((1, 1, \ldots, 1) \in \mathbb{Z}^{d+1}\) in its interior.

By [Ska96, Lemma 2], any transverse weight vector \(\overline{w} \in \mathbb{Z}_{>0}^{d+1}\) has the IP-property. For any fixed dimension \(d\), there exist only finitely many transverse weight vectors and only finitely many weight vectors \(\overline{w} \in \mathbb{Z}_{>0}^{d+1}\) with IP-property. The numbers \(T(d)\) and \(\text{IP}(d)\) are known for \(d \leq 5\):

- \(T(2) = \text{IP}(2) = 3\): \(\{\overline{w}\} = \{(1, 1, 1), (1, 1, 2), (1, 2, 3)\}\);
- \(T(3) = \text{IP}(3) = 95\) [Rei80];
- \(T(4) = 7,555\) [KS92, KS94], \(\text{IP}(4) = 184,026\) [Ska96];
- \(T(5) = 1,100,055\) [LSk99, BK16], \(\text{IP}(5) = 322,383,760,930\) [SchS19].

The following statement is a general result of Artebani, Comparin, and Guibiot [ACG16, Theorem 1] in the particular case of weighted projective spaces:
Proposition 1.3. Let \( \overline{w} = (w_0, w_1, \ldots, w_d) \) be a weight vector with IP-property. Then a general hypersurface \( X_w \subset \mathbb{P}(\overline{w}) \) of degree \( w = \sum_i w_i \) is a Calabi-Yau variety.

The present paper is inspired by mirror symmetry and the following formula of Vafa for the orbifold Euler number \( \chi_{\text{orb}}(X_w) \) of quasi-smooth Calabi-Yau hypersurfaces \( X_w \subset P(\overline{w}) = \mathbb{P}(w_0, w_1, \ldots, w_d) \) of degree \( w = \sum_{i=0}^{d} w_i \) for arbitrary \( d \geq 2 \):

\[
\chi_{\text{orb}}(X_w) = \frac{1}{w} \sum_{l,r=0}^{w-1} \prod_{0 \leq i \leq d \quad l_i, r_i \in \mathbb{Z}} \left( 1 - \frac{1}{q_i} \right),
\]

where \( q_i := \frac{w_i}{w} \) for all \( i \in I = \{0, 1, \ldots, d\} \) and one assumes \( \prod_{0 \leq i \leq d \quad l_i, r_i \in \mathbb{Z}} \left( 1 - \frac{1}{q_i} \right) = 1 \) if \( l_i, r_i \notin \mathbb{Z} \) for all \( i \in I \). It is remarkable that Formula (1) appeared without using algebraic geometry as the Witten index \( \text{Tr}(-1)^{F} \) of the \( N = 2 \) superconformal Landau-Ginzburg field theory defined by a weighted homogeneous superpotential \( W \) of degree \( w \) [Vaf89, Formula (28)].

A first general mathematical (\( K \)-theoretic) interpretation of Vafa’s Formula (1) was obtained by Ono and Roan:

Theorem 1.4 [OR93, Theorem 1.1]. Let \( S^{2d+1} \subset \mathbb{C}^{d+1} \setminus \{0\} \) be the unit sphere. Consider the compact smooth \((2d-1)\)-dimensional real manifold \( S_w := S^{2d+1} \cap \{W = 0\} \) together with the \( S^1 \)-fibration \( S_w \to X_w \) which is the restriction of the Seifert \( S^1 \)-fibration \( S^{2d+1} \to \mathbb{P}(w_0, w_1, \ldots, w_d) \) to a quasi-smooth Calabi-Yau hypersurface \( X_w \) of degree \( w = \sum_{i=0}^{d} w_i \). Then the \( S^1 \)-equivariant \( K \)-groups \( K^i_{S^1}(S_w) \) \((i = 0, 1)\) have finite rank and

\[
\text{rank } K^0_{S^1}(S_w) - \text{rank } K^1_{S^1}(S_w) = \frac{1}{w} \sum_{l,r=0}^{w-1} \prod_{0 \leq i \leq d \quad l_i, r_i \in \mathbb{Z}} \left( 1 - \frac{1}{q_i} \right).
\]

In particular, the right side of this equation is an integer.

Quasi-smooth Calabi-Yau hypersurfaces \( X_w \subset \mathbb{P}(\overline{w}) = \mathbb{P}(w_0, w_1, \ldots, w_d) \) can be considered as orbifolds, i.e., as geometric objects that are locally quotients of smooth manifolds by finite group actions. The orbifold cohomology theory developed by Chen and Ruan [CR04, ALR07] allows to define for any projective orbifold \( V \) certain vector spaces \( H^{p,q}_{CR}(V) \), whose dimensions \( h^{p,q}_{\text{orb}}(V) \) are called orbifold Hodge numbers. In particular, one obtains

\[
\chi_{\text{orb}}(V) := \sum_{p,q} (-1)^{p+q} h^{p,q}_{\text{orb}}(V).
\]

Another mathematical approach to orbifold Hodge and orbifold Euler numbers was suggested in [BD96, Bat98] via so-called stringy Hodge numbers \( h^{p,q}_{\text{str}}(X) \) that were defined for arbitrary projective algebraic varieties \( X \) with at worst Gorenstein canonical singularities. We note that such singularities do not necessarily admit a local orbifold structure. The mathematical definition of stringy Hodge numbers \( h^{p,q}_{\text{str}}(X) \)
uses an arbitrary desingularization $\rho : Y \to X$. Their definition immediately implies the equalities $h^{p,q}_{\text{str}}(X) = h^{p,q}(Y)$ if the desingularization $\rho$ is crepant [Bat98].

Our consideration of the orbifold Euler number is motivated by the combinatorial mirror duality for Calabi-Yau hypersurfaces $X$ in $d$-dimensional Gorenstein toric Fano varieties [Bat94]. This mirror duality is based on the polar duality $\Delta \leftrightarrow \Delta^*$ between $d$-dimensional reflexive lattice polytopes $\Delta$ and $\Delta^*$. Recall that a $d$-dimensional lattice polytope $\Delta \subseteq M_\mathbb{R} \cong \mathbb{R}^d$ is called reflexive if it contains the lattice point $0 \in M$ in its interior and the dual polytope $\Delta^* := \{ y \in N_\mathbb{R} | \langle x, y \rangle \geq -1 \ \forall \ x \in \Delta \} \subseteq N_\mathbb{R}$ is also a lattice polytope. If $\Delta$ is reflexive, then $\Delta^*$ is also reflexive and $(\Delta^*)^* = \Delta$.

The polar duality induces a natural one-to-one correspondence $\theta \leftrightarrow \theta^*$ between $k$-dimensional faces $\theta \preceq \Delta$ of $\Delta$ and $(d - k - 1)$-dimensional faces $\theta^* \preceq \Delta^*$.

If $X$ is a general Calabi-Yau hypersurface in the Gorenstein toric Fano variety $\mathbb{P}_\Delta$ associated with the normal fan $\Sigma_\Delta$ of $\Delta$, then the stringy Euler number $\chi_{\text{str}}(X)$ can be computed by the following combinatorial formula [BD96, Cor. 7.10]:

$$
\chi_{\text{str}}(X) = \sum_{k=1}^{d-2} (-1)^{k-1} \sum_{\theta \in \Delta, \dim(\theta) = k} \text{Vol}_k(\theta) \cdot \text{Vol}_{d-k-1}(\theta^*),
$$

where $\text{Vol}_{\dim(\cdot)}(\cdot) \in \mathbb{N}$ denotes the normalized volume (cf. Definition 2.9). Using Formula (2) and the bijection $\theta \leftrightarrow \theta^*$, one immediately obtains the equality

$$
\chi_{\text{str}}(X) = (-1)^{d-1} \chi_{\text{str}}(X^*),
$$

where $X^*$ denotes the mirror of $X$ obtained as a Calabi-Yau hypersurface in the Gorenstein toric Fano variety $\mathbb{P}_{\Delta^*}$. Moreover, by combinatorial methods one can show the following equalities of stringy Hodge numbers

$$
h^{p,q}_{\text{str}}(X) = h^{d-1-p,q}_{\text{str}}(X^*) \ (0 \leq p, q \leq d - 1)
$$

predicted by mirror duality [BB96].

Remark 1.5. The following combinatorial property of the polar duality for reflexive polytopes is crucial for the combinatorial Mirror Construction 1.7:

The set of generators of 1-dimensional cones $\sigma$ in the normal fan $\Sigma_{\Delta} \subseteq N_\mathbb{R}$ of a reflexive polytope $\Delta \subseteq M_\mathbb{R}$ is the set of vertices of the dual reflexive polytope $\Delta^* \subseteq N_\mathbb{R}$.

We consider a well-formed weighted projective space $\mathbb{P}(\vec{w})$ as a projective toric variety corresponding to a $d$-dimensional simplicial fan $\Sigma(\vec{w})$ in $N_{\mathbb{P}} \otimes \mathbb{R}$ whose 1-dimensional cones are spanned by primitive lattice vectors $v_0, v_1, \ldots, v_d \in N_{\mathbb{P}}$ generating the lattice $N_{\mathbb{P}}$ and satisfying the linear relation $\sum_{i=0}^{d} w_i v_i = 0$. In this way, the lattice $N_{\mathbb{P}}$ can be identified with the quotient lattice $\mathbb{Z}^{d+1}/\mathbb{Z} \vec{w}$.

If $\vec{w} \in \mathbb{Z}_{>0}^{d+1}$ is a weight vector having the IP-property, then the $d$-dimensional lattice polytope $\Delta'(W) := \Delta(W) - (1, \ldots, 1) \subseteq M_{\mathbb{P}} \otimes \mathbb{R}$ contains the origin $0 \in M_{\mathbb{P}}$. 


in its interior. We define the spanning fan $\Sigma'(w)$ of $\Delta'(W)$ as the fan in $M_\mathbb{R} \otimes \mathbb{R}$ consisting of all cones $\mathbb{R}_{\geq 0} \theta'$ over faces $\theta' \leq \Delta'(W)$ of the lattice polytope $\Delta'(W)$. By the theory of toric varieties [CL11], the fan $\Sigma'(w)$ defines a $d$-dimensional $\mathbb{Q}$-Gorenstein toric Fano variety $\mathbb{P}^d(w)$, which is a compactification of the algebraic torus $T_\mathbb{R}$ with the lattice of characters $N_\mathbb{R}$, i.e.,

$$T_\mathbb{R} := \{(x_0, x_1, \ldots, x_d) \in (\mathbb{C}^*)^{d+1} \mid \prod_{i=0}^{d} x_i^{w_i} = 1 \} \subset (\mathbb{C}^*)^{d+1}.$$ 

Using several results from [Bat17], we prove the following main statement of our paper:

**Theorem 1.6.** Let $w = (w_0, w_1, \ldots, w_d)$ be a weight vector with IP-property and $Z_\mathbb{R} \subset T_\mathbb{R}$ a non-degenerate affine hypersurface defined by a Laurent polynomial $f_\mathbb{R}$ with Newton polytope $\Delta_\mathbb{R} = \text{conv}(\{v_0, \ldots, v_d\}) \subset N_\mathbb{R} \otimes \mathbb{R}$. Then the Zariski closure of $Z_\mathbb{R}$ in the $\mathbb{Q}$-Gorenstein toric Fano variety $\mathbb{P}^d(w)$ is a Calabi-Yau hypersurface $X^*_w$ and

$$\chi_{\text{str}}(X^*_w) = (-1)^{d-1} \frac{1}{w} \sum_{l=0}^{w-1} \prod_{0 \leq i \leq d} \left(1 - \frac{1}{q_i}\right),$$

where $q_i = \frac{w_i}{w}$ ($i \in I$). In particular, if $w$ is a transverse weight vector, i.e., there exists a quasi-smooth Calabi-Yau hypersurface $X_w \subset \mathbb{P}(w)$ of degree $w = \sum_i w_i$, then

$$\chi_{\text{str}}(X^*_w) = (-1)^{d-1} \chi_{\text{orb}}(X_w).$$

The last statement in Theorem 1.6 supports the following mirror construction:

**Mirror Construction 1.7.** Let $w \in \mathbb{Z}_{\geq 0}^{d+1}$ be an arbitrary transverse weight vector. Then mirrors of quasi-smooth Calabi-Yau hypersurfaces $X_w \subset \mathbb{P}(w)$ can be obtained as Calabi-Yau compactifications $X^*_w$ of non-degenerate affine hypersurfaces $Z_w \subset T_\mathbb{R}$ with Newton polytope $\Delta^*_w = \text{conv}(\{v_0, v_1, \ldots, v_d\})$.

**Remark 1.8.** If the weighted projective space $\mathbb{P}(w) = \mathbb{P}(w_0, w_1, \ldots, w_d)$ is not Gorenstein, then the lattice simplex $\Delta^*_w$ is not reflexive and its dual simplex

$$\Delta_w = \{(u_0, u_1, \ldots, u_d) \in \mathbb{R}_{\geq 0}^{d+1} \mid \sum_{i=0}^{d} u_i w_i = w \} - (1, \ldots, 1)$$

is a rational simplex containing the lattice polytope $\Delta'(W) \subset M_\mathbb{R} \otimes \mathbb{R}$. It follows from [ACG16, Theorem 2] that if a $\mathbb{Q}$-Gorenstein weighted projective space $\mathbb{P}(w)$ contains a Calabi-Yau hypersurface $X_w$ defined by an invertible weighted homogeneous polynomial $W$, then the lattice simplex $\Delta^*_w$ is the Newton polytope of the Berglund-Hübsch-Krawitz mirror of $X_w \subset \mathbb{P}(w)$ [CR18, Section 5], [Kra10, CR10, Bor13].

**Remark 1.9.** Note that a weighted projective space $\mathbb{P}(w) = \mathbb{P}(w_0, w_1, \ldots, w_d)$ is a Gorenstein if and only if each weight $w_i$ ($0 \leq i \leq d$) divides the degree $w = \sum_{i=0}^{d} w_i$. 
In the latter case, one can choose a quasi-smooth Calabi-Yau hypersurface $X_w$ defined by the following weighted homogeneous polynomial of *Fermat-type*:

$$W = z_0^{w_0} + z_1^{w_1} + \cdots + z_d^{w_d} \in \mathbb{C}[z_0, z_1, \ldots, z_d].$$

In this case, the two simplices $\Delta^+_w$ and

$$\Delta_w = \Delta'(W) = \Delta(W) - (1, \ldots, 1)$$

are reflexive and Mirror Construction 1.7 coincides with the orbifoldizing mirror construction of Greene-Plesser [GP90].

**Remark 1.10.** The lattice simplex $\Delta^+_w = \text{conv}\{v_0, v_1, \ldots, v_d\}$ is the Newton polytope of the special non-degenerate Laurent polynomial $f^0_w(t) := \sum_{i=0}^d t^{v_i} \in \mathbb{C}[N_w]$ defining the affine hypersurface

$$Z^0_w = \mathbb{T}_w \cap \{(x_0, x_1, \ldots, x_d) \in (\mathbb{C}^*)^{d+1} \mid \sum_{i=0}^d x_i = 0\} \subset \mathbb{T}_w,$$

i.e., $f^0_w$ is the restriction of the linear function $x_0 + x_1 + \ldots + x_d : (\mathbb{C}^*)^{d+1} \to \mathbb{C}$ to the $d$-dimensional subtorus $\mathbb{T}_w \subset (\mathbb{C}^*)^{d+1}$. Note that the Laurent polynomial $f^0_w$ appears in the Givental-Hori-Vafa mirror construction for the weighted projective space $\mathbb{P}(\overline{w})$ [CR18, Section 4.4], [HV00]. The restriction of the projection

$$(\mathbb{C}^*)^{d+1} \to (\mathbb{C}^*)^d, \ (x_0, x_1, \ldots, x_d) \mapsto (x_1/x_0, \ldots, x_d/x_0)$$

to the affine hypersurface $Z^0_w \subset \mathbb{T}_w \subset (\mathbb{C}^*)^{d+1}$ defines an unramified cyclic Galois covering $\gamma_w : Z^0_w \to Z_0$ of order $w$, where

$$Z_0 := \{(y_1, \ldots, y_d) \in (\mathbb{C}^*)^d \mid \sum_{i=1}^d y_i = -1\} \subset (\mathbb{C}^*)^d.$$

Therefore, the affine hypersurface $Z^0_w$ can be obtained as a $(\mathbb{Z}/w\mathbb{Z})^{d-1}$-quotient of the affine Fermat hypersurface

$$F_w := \{(y_1, \ldots, y_d) \in (\mathbb{C}^*)^d \mid \sum_{i=1}^d y_i^w = -1\} \subset (\mathbb{C}^*)^d.$$

Thus, the Givental-Hori-Vafa polynomial $f^0_w$ defines a special ”*Fermat-type”* point in the moduli space of mirrors of quasi-smooth Calabi-Yau hypersurfaces $X_w \subset \mathbb{P}(\overline{w})$. The Givental-Hori-Vafa mirrors $Z^0_w$ were considered by Kelly [Kel13] via so-called *Shioda maps* in the Berglund-Hübsch-Krawitz mirror constructions for Calabi-Yau hypersurfaces $X_w \subset \mathbb{P}(\overline{w})$ defined by invertible polynomials $W$.

**Example 1.11 [CR18, Example 53].** Let $\overline{w} = (w_0, w_1, \ldots, w_d)$ be a weight vector with $w_0 = 1$. Then $v_0 = -\sum_{i=1}^d w_i$ and the lattice vectors $v_1, \ldots, v_d \in \mathbb{Z}^d$ form a $\mathbb{Z}$-basis of $N_{\overline{w}}$. Thus, the Givental-Hori-Vafa polynomial $f^0_w \in \mathbb{C}[N_{\overline{w}}]$ can be written in the form

$$f^0_w(t) = \sum_{i=0}^d t^{v_i} = \frac{1}{t_1^{w_1} \cdots t_d^{w_d}} + t_1 + \cdots + t_d.$$
We note that we can not expect Mirror Construction 1.7 working for Calabi-Yau hypersurfaces $X_w \subset \mathbb{P}(\mathfrak{w})$ that are not quasi-smooth. The following Examples 1.12 and 1.13 were proposed by Harald Skarke:

**Example 1.12 [KS92, Example 2].** Consider the IP-weight vector $\mathfrak{w} = (1, 1, 6, 14, 21)$. One can show that the weight vector $\mathfrak{w}$ is not transverse. The 4-dimensional lattice polytope $\Delta'(W) \subset \Delta_{\mathfrak{w}}$ is reflexive [Ska96, Lemma 1]. Therefore, general Calabi-Yau hypersurfaces $X_{43} \subset \mathbb{P}(\mathfrak{w})$ are birational to smooth Calabi-Yau threefolds $Y$ with Hodge numbers $h^{1,1}(Y) = 21, h^{2,1}(Y) = 273$, and Euler number $\chi(Y) = -504$ [Bat94].

On the other hand, by Theorem 1.6, the Givental-Hori-Vafa hypersurface $Z_{\mathfrak{w}0} \subset \mathbb{C}^4$ defined by the equation

$$\frac{1}{t_1^6t_2^{14}t_3^{21}} + t_1 + t_2 + t_3 + t_4 = 0$$

is birational to a 3-dimensional Calabi-Yau variety $X^*_w$ with stringy Euler number

$$\chi_{\text{str}}(X^*_w) = -\frac{1}{43} \sum_{l,r=0}^{42} \prod_{0 \leq i \leq 4} (1 - \frac{1}{q_i}) = 506 \neq 504 = -\chi_{\text{str}}(X_{43}) = -\chi(Y).$$

Therefore, $X^*_w$ is not mirror of $X_{43}$.

**Example 1.13.** The weight vector $\mathfrak{w} = (1, 1, 2, 4, 5)$ has IP-property. Since the 4-dimensional lattice polytope $\Delta'(W) \subset \Delta_{\mathfrak{w}}$ is reflexive, a general Calabi-Yau hypersurface $X_{13} \subset \mathbb{P}(\mathfrak{w})$ is birational to a smooth Calabi-Yau threefold. On the other hand, by Theorem 1.6, the Givental-Hori-Vafa hypersurface $Z_{\mathfrak{w}0} \subset \mathbb{C}^4$ defined by the equation

$$\frac{1}{t_1^2t_2^4t_3^{12}} + t_1 + t_2 + t_3 + t_4 = 0$$

is birational to a 3-dimensional Calabi-Yau variety $X^*_w$ with stringy Euler number

$$\chi_{\text{str}}(X^*_w) = -\frac{1}{13} \sum_{l,r=0}^{12} \prod_{0 \leq i \leq 4} (1 - \frac{1}{q_i}) = \frac{1032}{5} \notin \mathbb{Z}.$$ 

Therefore, the Calabi-Yau variety $X^*_w$ cannot have a Landau-Ginzburg description and $X^*_w$ has no mirror at all.

2. **Stringy Euler numbers of toric Calabi-Yau hypersurfaces**

Let $M \cong \mathbb{Z}^d$ be a lattice of rank $d$, $N := \text{Hom}(M, \mathbb{Z})$ its dual lattice, and

$$\langle \cdot, \cdot \rangle : M \times N \to \mathbb{Z}$$

the natural pairing. Consider a $d$-dimensional lattice polytope $\Delta \subseteq M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^d$ and denote by $\Sigma_{\Delta}$ the normal fan of $\Delta$ in the dual real vector space $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$.
There exists a natural one-to-one correspondence \( \theta \leftrightarrow \sigma \) between \( k \)-dimensional faces \( \theta \preceq \Delta \) and \((d-k)\)-dimensional cones \( \sigma \in \Sigma_{\Delta} \), where
\[
\sigma = \{ y \in \mathbb{N}_{\mathbb{R}} \mid \min_{x \in \Delta} \langle x, y \rangle = \langle x', y \rangle \ \forall x' \in \theta \}.
\]

We are interested in \( d \)-dimensional Newton polytopes \( \Delta \subseteq M_{\mathbb{R}} \) of Laurent polynomials \( f_{\Delta} \in \mathbb{C}[M] \) defining non-degenerate affine hypersurfaces
\[
Z_{\Delta} \subseteq \mathbb{T} := \text{Hom}(M, \mathbb{C}^*) \cong (\mathbb{C}^*)^d
\]
that are birational to \emph{Calabi-Yau varieties} \( X \), i.e., normal projective algebraic varieties with at worst canonical singularities and trivial canonical class. By [Kho78, Theorem on page 41], such a Newton polytope \( \Delta \) must be a \emph{canonical Fano polytope}, i.e., a lattice polytope containing exactly one interior lattice point that we can assume to be \( 0 \in M \).

We review two results from [Bat17]. The first is a combinatorial characterization of \( d \)-dimensional canonical Fano polytopes \( \Delta \) such that \( Z_{\Delta} \) is birational to a Calabi-Yau hypersurface \( X \):

\begin{theorem}[Bat17, Theorem 2.23]
Let \( \Delta \subseteq M_{\mathbb{R}} \) be a \( d \)-dimensional canonical Fano polytope and \( \Delta^* = \{ y \in \mathbb{N}_{\mathbb{R}} \mid \langle x, y \rangle \geq -1 \ \forall x \in \Delta \} \) its rational dual polytope. We set
\[
[\Delta^*] := \text{conv}(\Delta^* \cap \mathbb{N}) \subseteq \mathbb{N}_{\mathbb{R}}
\]
to be the convex hull of all lattice points in \( \Delta^* \). Then a non-degenerate affine hypersurface \( Z_{\Delta} \subseteq \mathbb{T} \) is birational to a Calabi-Yau variety \( X \) if and only if \( [\Delta^*] \) is also a \( d \)-dimensional canonical Fano polytope.
\end{theorem}

\begin{remark}
The "if"-part in Theorem 2.1 was independently proven by Artebani, Comparin, and Guilbot [ACG16, Theorem 1]. Moreover, they proved that a canonical Calabi-Yau model \( X \) of the hypersurface \( Z_{\Delta} \subseteq \mathbb{T} \) can be explicitly constructed, e.g., as the Zariski closure of \( Z_{\Delta} \) in the toric \( \mathbb{Q} \)-Fano variety associated with the spanning fan of the canonical Fano polytope \( [\Delta^*] \subseteq \mathbb{N}_{\mathbb{R}} \).
\end{remark}

\begin{remark}
Canonical Fano polytopes \( \Delta \subseteq M_{\mathbb{R}} \) satisfying the criterion in Theorem 2.1 can be equivalently characterized by the condition that the Fine interior of \( \Delta \) is exactly its interior lattice point \( \{0\} \) [Bat17, Section 2], [BKS19].
\end{remark}

\begin{remark}
A \( d \)-dimensional canonical Fano polytope \( \Delta \) is called \emph{pseudoreflexive} if it satisfies the condition
\[
[[\Delta^*]] = \Delta.
\]
It is easy to see that if \( \Delta \) is pseudoreflexive, then \( [\Delta^*] \) is also pseudoreflexive. Thus, pseudoreflexive polytopes satisfy the combinatorial duality \( \Delta \leftrightarrow [\Delta^*] \), which was proposed by Mavlytov as a generalization of the polar duality for reflexive polytopes [Mav11]. We note that any pseudoreflexive polytope \( \Delta \) of dimension \( d \leq 4 \) is reflexive. A simple example of a 5-dimensional pseudoreflexive polytope \( \Delta \) that is not reflexive can be obtained as the Newton polytope of a general 4-dimensional
Calabi-Yau hypersurface of degree 7 in the weighted projective space $\mathbb{P}(1, 1, 1, 1, 1, 2)$ [KS98a]. It was expected by Mavlyutov that if $(\Delta, [\Delta^*])$ is a pair of pseudoreflexive polytopes, then Calabi-Yau hypersurfaces in toric $\mathbb{Q}$-Fano varieties corresponding to rational polytopes $\Delta^*$ and $[[\Delta^*]^*]$ are mirror symmetric to each other. For Calabi-Yau hypersurfaces of degree 7 in $\mathbb{P}(1, 1, 1, 1, 1, 2)$ this is true, but in general this is false [Bat17, Section 5].

**Remark 2.5.** Let $\Delta$ be an arbitrary canonical Fano polytope satisfying the condition in Theorem 2.1, i.e., $[\Delta^*]$ is canonical. In general, $\Delta$ need not to be pseudoreflexive. However, the canonical polytopes $[\Delta^*]$ and $[[\Delta^*]^*]$ are always pseudoreflexive. For this reason, we call a canonical Fano polytope $\Delta$ *almost pseudoreflexive* if $[\Delta^*]$ is also a canonical Fano polytope. The name is motivated by the fact that in this case $[[\Delta^*]^*]$ is the smallest pseudoreflexive polytope containing $\Delta$. So it is natural to call $[[\Delta^*]^*]$ the *pseudoreflexive hull* of $\Delta$. Since pseudoreflexive polytopes $\Delta$ of dimension 3 and 4 are reflexive, we will use the expressions *almost reflexive* and *reflexive hull* instead of *almost pseudoreflexive* and *pseudoreflexive hull*, respectively.

**Remark 2.6.** Recall some classification results concerning special classes of canonical Fano polytopes:

- Any 2-dimensional canonical Fano polytope is reflexive. There exist exactly 16 of them.
- All 3-dimensional canonical Fano polytopes are classified by Kasprzyk [Kas10]. There exist exactly 665,599 3-dimensional almost reflexive polytopes among all 674,688 3-dimensional canonical Fano polytopes. This list extends the classification of all 4,319 reflexive 3-dimensional polytopes obtained by Kreuzer and Skarke [KS98b].
- All 4-dimensional reflexive polytopes are classified by Kreuzer and Skarke [KS00]. This list consists of 473,800,776 reflexive polytopes, but the list of all 4-dimensional almost reflexive polytopes is still unknown.

The second result from [Bat17] is a combinatorial formula for the stringy Euler number $\chi_{\text{str}}(X)$ of a Calabi-Yau variety $X$ generalizing Formula (2) for reflexive polytopes. We recall a general definition for the stringy Euler number of a $d$-dimensional normal projective $\mathbb{Q}$-Gorenstein variety $X$ with at worst log-terminal singularities:

**Definition 2.7.** Let $\rho : Y \to X$ be a log-desingularization of $X$ whose exceptional locus is the union of smooth irreducible divisors $D_1, \ldots, D_s$ with only simple normal crossing and

$$K_Y = \rho^* K_X + \sum_{i=1}^{s} a_i D_i$$

for some rational numbers $a_i > -1$ ($1 \leq i \leq s$). We set $D_\emptyset := Y$ and

$$D_J := \bigcap_{j \in J} D_j, \quad \emptyset \neq J \subseteq I := \{1, \ldots, s\}.$$
Then the stringy Euler number $\chi_{\text{str}}(X)$ is defined to be

$$\chi_{\text{str}}(X) := \sum_{\emptyset \subseteq J \subseteq I} \chi(D_J) \prod_{j \in J} \left( \frac{-a_j}{a_j + 1} \right),$$

where $\chi(D_J)$ denotes the usual Euler number of the smooth projective variety $D_J$ [Bat98].

**Remark 2.8.** It is important to note that the stringy Euler number is independent on the log-desingularization $\rho : Y \to X$ and $\chi_{\text{str}}(X)$ equals the usual Euler number of $Y$ if $\rho : Y \to X$ is crepant. More generally, if two projective algebraic varieties $X$ and $X'$ are $K$-equivalent, then $\chi_{\text{str}}(X) = \chi_{\text{str}}(X')$. In particular, the stringy Euler number of a given birational class of algebraic varieties of non-negative Kodaira dimension is well-defined.

In addition, we will need the notion of the normalized volume of a rational polytope:

**Definition 2.9.** Let $\Delta \subseteq N_{\mathbb{R}}$ be a $d$-dimensional rational polytope, i.e., $\Delta$ has vertices in $N_{\mathbb{Q}} := N \otimes \mathbb{Q}$. Then the positive rational number

$$\text{Vol}_d(\Delta) := \frac{1}{l^d} \text{Vol}_d(l\Delta) = \frac{1}{l^d} \cdot d! \cdot \text{vol}_d(l\Delta)$$

is called the normalized volume of $\Delta$, where $l$ is a positive integer such that $l\Delta$ is a lattice polytope and $\text{vol}_d(\cdot)$ denotes the $d$-dimensional volume of $\Delta$ with respect to the lattice $N$. Similarly, one obtains the normalized volume $\text{Vol}_k(\theta) := \frac{1}{l^k} \text{Vol}_k(l\theta)$ with respect to the sublattice $\text{span}(\theta) \cap N$ for a $k$-dimensional rational face $\theta \leq \Delta$.

**Theorem 2.10** [Bat17, Theorem 4.11]. Assume that a $d$-dimensional canonical Fano polytope $\Delta \subseteq M_{\mathbb{R}}$ satisfies the condition in Theorem 2.1, i.e., a non-degenerate affine hypersurface $Z_\Delta \subset \mathbb{T}$ is birational to a Calabi-Yau variety $X$. Then the stringy Euler number of $X$ can be computed by the following combinatorial formula

$$\chi_{\text{str}}(X) = \sum_{k=1}^{d} (-1)^{k-1} \sum_{\theta \leq \Delta, \text{dim}(\theta) = k} \text{Vol}_k(\theta) \cdot \text{Vol}_{d-k}(\sigma_\theta \cap \Delta^*),$$

where $\sigma_\theta \cap \Delta^*$ is the $(d-k)$-dimensional pyramid with vertex $0 \in N$ over the $(d-k-1)$-dimensional dual face $\theta^* \leq \Delta^*$ of the dual rational polytope $\Delta^*$.

**Example 2.11.** Let $\Delta \subseteq M_{\mathbb{R}}$ be a 3-dimensional almost reflexive polytope. Then $X$ is a $K3$-surface with $\chi_{\text{str}}(X) = 24$. In this case, the combinatorial formula in Theorem 2.10 is equivalent to the identity

$$24 = \text{Vol}_3(\Delta) - \sum_{\theta \leq \Delta, \text{dim}(\theta) = 2} \frac{1}{n_\theta} \cdot \text{Vol}_2(\theta) + \sum_{\theta \leq \Delta, \text{dim}(\theta) = 1} \text{Vol}_1(\theta) \cdot \text{Vol}_1(\theta^*),$$

where $n_\theta$ denotes the integral distance between a facet $\theta \leq \Delta$ and $0 \in M$ [BS19, Theorem A].
Example 2.12. Let $\Delta \subseteq M_\mathbb{R}$ be a $d$-dimensional reflexive polytope. Then the dual polytope $\Delta^*$ is also a reflexive polytope. Therefore, we obtain

$$\text{Vol}_{d-k}(\sigma_{\theta} \cap \Delta^*) = \text{Vol}_{d-k-1}(\theta^*)$$

for any $k$-dimensional face $\theta \leq \Delta$ of $\Delta$ $(1 \leq k \leq d - 2)$. Together with the equality $\text{Vol}_d(\Delta) = \sum_{\theta \leq \Delta, \dim(\theta) = d-1} \text{Vol}_{d-1}(\theta)$, Theorem 2.10 implies the already discussed combinatorial Formula (2).

3. Calabi-Yau hypersurfaces in weighted projective spaces

Let $(w_0, w_1, \ldots, w_d) \in \mathbb{Z}_{d+1}^d > 0$ be a well-formed weight vector. In this section, we apply Theorems 2.1 and 2.10 to the special $d$-dimensional lattice simplex

$$\Delta := \Delta_\mathbb{R}^* = \text{conv} \{ v_0, v_1, \ldots, v_d \} \subseteq \mathbb{R}^d,$$

where $v_0, \ldots, v_d$ generate the lattice $M := \mathbb{Z}_{d+1}^d$ and satisfy the linear relation $\sum_{i=0}^d w_i v_i = 0$. Consider the lattice $\tilde{M} := M \oplus \mathbb{Z}$ together with $d + 1$ linearly independent lattice vectors $\tilde{v}_i := (v_i, 1) \in \tilde{M}$ $(0 \leq i \leq d)$. Set $M' := \langle \tilde{v}_i \rangle_{0 \leq i \leq d}$ and identify $M'$ with $\mathbb{Z}^{d+1}$ via the basis $\tilde{v}_i$ $(0 \leq i \leq d)$. Since $M$ is generated by $v_i$ $(0 \leq i \leq d)$, the lattice $\tilde{M}$ is generated by $\tilde{v}_i$ together with the lattice vector $\tilde{v} := (0, 1) \in M \oplus \mathbb{Z} = \tilde{M}$ and

$$\tilde{v} = \sum_{i=0}^d \frac{w_i}{w} \cdot \tilde{v}_i.$$

So the quotient $\tilde{M}/M'$ is a cyclic group of order $w$ generated by $\tilde{v} + M'$ and we can write

$$\tilde{M} = \mathbb{Z} \left( \frac{w_0}{w}, \frac{w_1}{w}, \ldots, \frac{w_d}{w} \right) + \mathbb{Z}^{d+1} = \mathbb{Z}(q_0, q_1, \ldots, q_d) + \mathbb{Z}^{d+1}.$$

In certain cases, it will be convenient to use the multiplicative description of $\tilde{M}/M'$ by the cyclic group $G := \langle g \rangle = \{ g^s \mid s \in \mathbb{Z}/w\mathbb{Z} \} \subseteq SL(d+1, \mathbb{C})$ generated by the diagonal matrix

$$g := \text{diag}(g_0, g_1, \ldots, g_d) := \text{diag}(e^{2\pi i q_0}, e^{2\pi i q_1}, \ldots, e^{2\pi i q_d}).$$

In the language of toric varieties, the $(d+1)$-dimensional simplicial cone $\mathbb{R}_{\geq 0}(1, \Delta) \subseteq \tilde{M}_\mathbb{R}$ describes the Gorenstein cyclic quotient singularity $\mathbb{C}^{d+1}/G$ considered by Corti and Golyshev [CG11].

Definition 3.1. For any non-empty subset $J \subseteq I$, we define two $|J|$-dimensional sublattices

$$M'_J := \sum_{j \in J} \mathbb{Z} \tilde{v}_j \quad \text{and} \quad \tilde{M}_J := \tilde{M} \cap (M' \otimes \mathbb{Q}),$$

and the subgroup

$$G_J := \{ g^s \in G \mid g_j^s = 1 \ \forall j \in J \} \subseteq G.$$
Each $k$-dimensional face $\theta \preceq \Delta = \Delta_w^*$ is a $k$-dimensional simplex
$$\theta_J := \text{conv}(v_{i_0}, v_{i_1}, \ldots, v_{i_k})$$
determined by a $(k+1)$-element subset $J := \{i_0, i_1, \ldots, i_k\} \subseteq I = \{0, 1, \ldots, d\}$. We set $\tilde{\theta}_J$ to be the $(k+1)$-dimensional simplex
$$\tilde{\theta}_J := \text{conv}(0, \tilde{v}_{i_0}, \tilde{v}_{i_1}, \ldots, \tilde{v}_{i_k}).$$

**Proposition 3.2.** Let $\emptyset \neq J \subseteq I$ be a non-empty subset of $I$. Then

(i) $|\tilde{M}_J/M'_J| = \text{Vol}_{|J|}(\tilde{\theta}_J)$;

(ii) $|G_J| = n_J := \text{gcd}(w, w_j \mid j \in J)$;

(iii) $\text{Vol}_{|J|-1}(\tilde{\theta}_J) = n_\tilde{\theta} = |G_\tilde{\theta}|$, where $\tilde{\theta} := I \setminus J$.

Proof. (i) The normalized volume $\text{Vol}_{|J|}(\tilde{\theta}_J)$ of $\tilde{\theta}_J$ equals the index of the sublattice $M'_J$ generated by the lattice vectors $\tilde{v}_j$ ($j \in J$) in the $|J|$-dimensional lattice $\tilde{M}_J = \tilde{M} \cap (M'_J \otimes \mathbb{Q})$.

(ii) Assume without loss of generality $J = \{0, 1, \ldots, k\}$. We set $u := w/n_J \in \mathbb{N}$. Then one has $uq_j = w_j/n_j \in \mathbb{Z}$ for all $j \in J$, i.e., $g^u \in G_J$. This implies the inclusion $\langle g^u \rangle \subseteq G_J$ and the inequality $n_J = |\langle g^u \rangle| \leq |G_J|$. In order to prove the opposite inclusion $G_J \subseteq \langle g^u \rangle$, we set $u_j := w_j/n_j \in \mathbb{N}$ for $j \in J$ and assume that $g^s \in G_J$. Then $su_j/u = sq_j \in \mathbb{Z}$ for all $j \in J$, i.e., $u$ divides $su_j$ for all $j \in J$. Since $\text{gcd}(u, u_0, u_1, \ldots, u_k) = 1$, there exist integers $a$ and $a_0, \ldots, a_k$ such that
$$au + \sum_{j \in J} a_j u_j = 1.$$ Therefore, $u$ divides $s = asu + \sum_{j \in J} a_j su_j$, i.e., $g^s \in \langle g^u \rangle$.

(iii) Note that the quotient $\tilde{M}_J/M'_J$ can be considered as a subgroup of the cyclic group $\tilde{M}/M' \cong \mathbb{Z}/w\mathbb{Z}$. Indeed, if we take the homomorphism $\varphi : \tilde{M}_J \to \tilde{M}/M'$ obtained from the embedding of $\tilde{M}_J$ into $\tilde{M}$, then the kernel of $\varphi$ is $\tilde{M}_J \cap M' = \tilde{M} \cap (M'_J \otimes \mathbb{Q}) \cap M' = M'_J$ because $(M'_J \otimes \mathbb{Q}) \cap M' = M'_J$. Furthermore, an element $s(\tilde{v} + M') \in \tilde{M}/M'$ belongs to the subgroup $\tilde{M}_J/M'_J$ if and only if the coefficients $sq_i$ in the equation
$$s\tilde{v} = \sum_{i=0}^{d} sq_i \tilde{v}_i$$
are integers for all $i \notin J$. By Proposition 3.2 (ii), the latter happens if and only if $g^s \in G_\tilde{\theta}$. This shows that $\tilde{M}_J/M'_J \cong G_\tilde{\theta}$ and $\text{Vol}_k(\tilde{\theta}_J) = \text{Vol}_{k+1}(\tilde{\theta}_J) = n_\tilde{\theta}$. \qed

For the lattice simplex $\Delta = \Delta_w^* = \text{conv}(\{v_0, v_1, \ldots, v_d\})$ the dual polytope $\Delta^*$ is the $d$-dimensional rational simplex
$$\Delta^* = \Delta_w^* = \{(u_0, u_1, \ldots, u_d) \in \mathbb{R}_{\geq 0}^{d+1} \mid \sum_{i=0}^{d} w_i u_i = w\} - (1, 1, \ldots, 1).$$

Let $J \subseteq I = \{0, 1, \ldots, d\}$ be a $(k+1)$-element subset of $I$. Then we denote by $\sigma_J := \sigma_{\theta_J}$ the normal cone in the normal fan $\Sigma_\Delta = \{\sigma_J \mid \emptyset \subseteq J' \subseteq I\}$ corresponding
to the face $\theta_J \preceq \Delta$ of the lattice simplex $\Delta$ with $\dim(\sigma_J) = d - \dim(\theta_J) = d - k$, i.e., $\sigma_J$ is the cone generated by all inward-pointing facet normals of facets containing the face $\theta_J \preceq \Delta$.

**Proposition 3.3.** Let $J \subseteq I = \{0, 1, \ldots, d\}$ be a $(k + 1)$-element subset of $I$. Then the normalized volume $\Vol_{d-k}(\sigma_J \cap \Delta^*)$ of the rational polytope $\sigma_J \cap \Delta^*$ is given by

$$\Vol_{d-k}(\sigma_J \cap \Delta^*) = \frac{n_{\overline{J}}}{w} \prod_{i \in \overline{J}} \frac{1}{q_i},$$

where $\overline{J} = I \setminus J$, $q_i = \frac{w}{w_i}$ ($i \in I$), and $n_{\overline{J}} = \gcd(w, w_i | i \in \overline{J})$.

**Proof.** Consider the $d$-dimensional rational simplex

$$\Delta' := \{(u_0, u_1, \ldots, u_d) \in \mathbb{R}^{d+1} | u_i \geq 0 \ \forall i \in I, w_0 u_0 + w_1 u_1 + \ldots + w_d u_d = w\}.$$

Then the shifted simplex $\Delta' - (1, 1, \ldots, 1)$ is the dual simplex $\Delta^* \subseteq N_\mathbb{R}$ of $\Delta$ because

$$\Delta^* = \{(u_0, u_1, \ldots, u_d) \in \mathbb{R}^{d+1} | u_i \geq -1 \ \forall i \in I, w_0 u_0 + w_1 u_1 + \ldots + w_d u_d = 0\}$$

(Figure 1). For simplicity, we assume without loss of generality $J = \{0, 1, \ldots, k\}$, i.e., $\overline{J} = \{k + 1, \ldots, d\}$ and denote by $\{e_0, e_1, \ldots, e_d\}$ the standard basis of $\mathbb{R}^{d+1}$. The dual rational face $\theta_J^* \preceq \Delta^*$ as well as the shifted dual rational face $\theta_J^* := \theta_J^* + (1, 1, \ldots, 1) \preceq \Delta^* + (1, 1, \ldots, 1) = \Delta'$ of the shifted simplex $\Delta'$ have dimension $d - k - 1$. Moreover, $\Delta'$ has vertices $\frac{1}{q_i} e_i$ ($i \in I$) and $\theta_J^* + (1, 1, \ldots, 1) \preceq \Delta' \cap \{i \in \overline{J}\}$ (Figure 1).

For $|J| = |I| = d + 1$, i.e., $k = d$ and $\overline{J} = \emptyset$, we obtain $\theta_J = \Delta$, $\sigma_J = \{0\}$, and

$$\Vol_{d-k}(\sigma_J \cap \Delta^*) = \Vol_0(\{0\}) = 1 = \frac{w}{w} \prod_{i \in \emptyset} \frac{1}{q_i} = \frac{n_{\overline{J}}}{w} \prod_{i \in \overline{J}} \frac{1}{q_i}.$$ 

For $|J| = d$, i.e., $k = d - 1$ and $\overline{J} = \{w_d\}$, the associated dual rational face $\theta_J^* \preceq \Delta^*$ has dimension 0, i.e., $\theta_J^* := \{v\}$ is a (rational) vertex and $P_v := \text{conv}(\{(0, 0, \ldots, 0), v\})$ a 1-dimensional (rational) polytope. Therefore,

$$\Vol_{d-k}(\sigma_J \cap \Delta^*) = \Vol_1(P_v) = \frac{\gcd(w, w_d)}{w_d} = \frac{n_{\overline{J}}}{w} \prod_{i \in \overline{J}} \frac{1}{q_i}.$$ 

Let $k \leq d - 2$, i.e., $|J| \leq d - 1$. Moreover, let $\Pi_\overline{J}$ be the pyramid with vertex $0 \in \mathbb{R}^{d+1}$ over the shifted dual face $\theta_J^* \preceq \Delta'$, i.e., $\Pi_\overline{J} = \text{conv}(\{0, \frac{1}{q_i} e_i | i \in \overline{J}\})$ with $\dim(\Pi_\overline{J}) = d - k$ and

$$\Vol_{d-k}(\Pi_\overline{J}) = \prod_{i \in \overline{J}} \frac{1}{q_i}$$

(Figure 2). The pyramid $\Pi_\overline{J}$ is contained in the $(d - k)$-dimensional subspace generated by $e_{k+1}, \ldots, e_d$. The basis of the pyramid is the simplex $\theta_J^*$ that belongs to a hyperplane in the affine linear subspace defined by the equation

$$w_{k+1} u_{k+1} + \ldots + w_d u_d = w.$$
Figure 1. Illustration of Proposition 3.3. Grey coloured 2-dimensional rational simplex $\Delta'$ with unique interior lattice point $(1, 1, 1) \in \mathbb{Z}^3$ and vertices $\frac{1}{q_i} (i \in I = \{0, 1, 2\})$ together with a 1-dimensional dotted face $\theta'_J \preceq \Delta'$, i.e., $J = \{0\} \subseteq I$ and $d = 2$. Light grey coloured shifted dual simplex $\Delta^* = \Delta' - (1, 1, 1)$ with unique interior lattice point $(0, 0, 0) \in \mathbb{Z}^3$ together with a 1-dimensional dotted shifted face $\theta^*_J \preceq \Delta^*$. The grey shaded associated 2-dimensional normal cone $\sigma_J$ with two dotted rays corresponding to the (rational) vertices of $\theta^*_J$, where $\theta_J = \{v_0\} \subseteq \Delta$ is a vertex. Moreover, the area $\sigma_J \cap \Delta^*$ is crosshatched grey.

The integral distance between this hyperplane and the origin equals $w/n_J$, where $n_J = \gcd(w, w_{k+1}, \ldots, w_d)$. Therefore, we obtain

$$\text{Vol}_{d-k}(\sigma_J \cap \Delta^*) = \text{Vol}_{d-k-1}(\theta'_J) = \text{Vol}_{d-k-1}(\theta_J) = \text{Vol}_{d-k}(\Pi_J) \cdot \frac{n_J}{w} = \frac{n_J}{w} \prod_{i \in J} \frac{1}{q_i}. $$

\[ \square \]

**Theorem 3.4.** Let $\overline{w} = (w_0, w_1, \ldots, w_d)$ be a weight vector with IP-property and $Z_{\overline{w}} \subset \mathbb{T}_{\overline{w}}$ a non-degenerate affine hypersurface defined by a Laurent polynomial $f_{\overline{w}}$
Figure 2. Illustration for Proposition 3.3. Shaded faces are occluded. Grey coloured 2-dimensional rational simplex $\Delta'$ with unique interior lattice point $(1, 1, 1) \in \mathbb{Z}^3$ and vertices $\frac{1}{q_i} (i \in I = \{0, 1, 2\})$ together with a 1-dimensional dotted face $\theta'_J \preceq \Delta'$ with vertices $\frac{1}{q_i} (i \in J)$, i.e., $J = \{0\} \subseteq I$, $\overline{J} = \{1, 2\}$, and $d = 2$. The 2-dimensional grey crosshatched pyramid $\Pi_J$ with vertex $(0, 0, 0) \in \mathbb{Z}^3$ and basis $\theta'_J$. with Newton polytope $\Delta'_\overline{\pi}$. Then the Zariski closure of $Z_{\pi}$ in the $\mathbb{Q}$-Gorenstein toric Fano variety $\mathbb{P}^\vee(\overline{\pi})$ is a $(d - 1)$-dimensional Calabi-Yau variety $X^*_\overline{\pi}$ and

$$\chi_{str}(X^*_\overline{\pi}) = \frac{1}{w} \sum \frac{(-1)^{|J|} n^2}{q_i} \prod_{i \in J} \frac{1}{q_i},$$

where $\overline{J} = I \setminus J$ and $q_i = \frac{w_i}{w} (i \in I)$.

Proof. By assumption, $\Delta(W) \cap \mathbb{Z}^{d+1} = (1, 1, \ldots, 1)$, i.e., the lattice polytope $[\Delta^*] = [(\Delta'_\overline{\pi})^*] = \Delta(W) - (1, 1, \ldots, 1) = \Delta'(W')$ is a $d$-dimensional canonical Fano polytope. By Theorem 2.1 and Remark 2.2, the Zariski closure of the non-degenerate hypersurface $Z_{\pi} \subset \mathbb{T}_{\overline{\pi}}$ in the toric variety $\mathbb{P}^\vee(\overline{\pi})$ associated with the spanning fan $\Sigma^\vee(\overline{\pi})$ of the canonical Fano polytope $[\Delta^*]$ is a Calabi-Yau variety $X^*_\overline{\pi}$. Therefore, we are able to apply Theorem 2.10 to compute the stringy Euler number $\chi_{str}(X^*_\overline{\pi})$ of $X^*_\overline{\pi}$ via

$$\chi_{str}(X^*_\overline{\pi}) = \sum_{k=1}^d (-1)^{k-1} \sum_{\dim(\theta) = k} \text{Vol}_k(\theta) \cdot \text{Vol}_{d-k}(\sigma_\theta \cap \Delta^*).$$

Moreover, the $k$-dimensional faces of $\Delta$ are simplices $\theta_J = \text{conv}\{v_j \mid j \in J\}$ parametrized by subsets $J \subseteq I$ with $|J| = k + 1$, where $k \geq 1$ if and only if $|J| \geq 2$. 

The normalized volumes Vol$_{\mid J \mid -1}(\theta_J)$ and Vol$_{d-\mid J \mid +1}(\sigma_J \cap \Delta^*)$ have been computed for every subset $J \subseteq I$ with $\mid J \mid \geq 2$ in Proposition 3.2 (iii) and Proposition 3.3, respectively. By substitution, we obtain

$$\chi_{\text{str}}(X^*_w) = \sum_{\mid J \mid \geq 2} (-1)^{\mid J \mid - 2} \cdot \text{Vol}_{\mid J \mid -1}(\theta_J) \cdot \text{Vol}_{d-\mid J \mid +1}(\sigma_J \cap \Delta^*)$$

$$= \sum_{0 \leq \mid J \mid \leq 2} (-1)^{\mid J \mid} \cdot \frac{n_J}{\prod_{l \in J} 1} = \frac{1}{\prod_{l \in J} 1} \sum_{0 \leq \mid J \mid \leq 2} (-1)^{\mid J \mid} \cdot \frac{n_J}{\prod_{l \in J} 1}.$$

\[\square\]

4. PROOF OF THEOREM 1.6

Let $(w_0, w_1, \ldots, w_d) \in \mathbb{Z}^{d+1}_{\geq 0}$ be a well-formed weight vector. We consider the cyclic group $G = \langle g \rangle = \{g^s \mid s \in \mathbb{Z}/w\mathbb{Z}\} \leq SL(d + 1, \mathbb{C})$, whose generator $g = \text{diag}(g_0, g_1, \ldots, g_d)$ acts linearly on $\mathbb{C}^{d+1}$ by the diagonal matrix

$$\text{diag}(e^{2\pi i q_0}, e^{2\pi i q_1}, \ldots, e^{2\pi i q_d}).$$

We aim at a combinatorial version of Vafa’s Formula (1). This version has been proven in [GRY91, Formula (4.1), page 255] for Calabi-Yau hypersurfaces in 4-dimensional weighted projective spaces:

**Theorem 4.1.** Let $(w_0, w_1, \ldots, w_d) \in \mathbb{Z}^{d+1}_{\geq 0}$ be as above. Then

$$\frac{1}{w} \sum_{l=0}^{w-1} \prod_{0 \leq i \leq d, l_qi \neq q_i \neq \in \mathbb{Z}} \left(1 - \frac{1}{q_i}\right) = \frac{1}{w} \sum_{\mid J \mid \leq d-1} (-1)^{\mid J \mid} \cdot \frac{n_J}{\prod_{l \in J} 1}.$$

**Proof.** Obviously, we have

$$\frac{1}{w} \sum_{l=0}^{w-1} \prod_{0 \leq i \leq d, l_qi \neq q_i \neq \in \mathbb{Z}} \left(1 - \frac{1}{q_i}\right) = \frac{1}{\mid G \mid} \sum_{g^s \in G^2} \prod_{0 \leq i \leq d, g_i^s = g_i} \left(1 - \frac{1}{q_i}\right).$$

First consider the trivial pair $(l, r) = (0, 0) \in (\mathbb{Z}/w\mathbb{Z})^2$. Then $g_i^s = g_i^r = 1$ for all $i \in I$ and we obtain the product

$$\prod_{i=0}^{d} \left(1 - \frac{1}{q_i}\right) = \sum_{0 \leq \mid J \mid \leq d} \prod_{j \in J} \left(1 - \frac{1}{q_j}\right) = \sum_{0 \leq \mid J \mid \leq d} (-1)^{\mid J \mid} \prod_{j \in J} \frac{1}{q_j}.$$

that appears as one summand in Vafa’s Formula (1). For $s \in \mathbb{Z}/w\mathbb{Z}$, we define the subset $J_s \subseteq I$:

$$J_s := \{i \in I \mid g_i^s = 1\}.$$

Then for any pair $(g^l, g^r) \in G^2$, we obtain

$$\prod_{0 \leq i \leq d, g_i^l = g_i^r} \left(1 - \frac{1}{q_i}\right) = \prod_{i \in J_r \cap J_s} \left(1 - \frac{1}{q_i}\right) = \sum_{0 \leq \mid J \mid \leq d} (-1)^{\mid J \mid} \prod_{j \in J} \frac{1}{q_j}.$$
and

$$\frac{1}{|G|} \sum_{(g',g') \in G^2} \prod_{0 \leq i \leq d} (1 - \frac{1}{q_i}) = \frac{1}{|G|} \sum_{(g',g') \in G^2} \left( \sum_{\emptyset \leq J \subseteq I} (-1)^{|J|} \prod_{j \in J} \frac{1}{q_j} \right)$$

$$= \frac{1}{|G|} \sum_{\emptyset \leq J \subseteq I} \left( \sum_{(g',g') \in G^2} (-1)^{|J|} \prod_{j \in J} \frac{1}{q_j} \right)$$

$$= \frac{1}{|G|} \sum_{\emptyset \leq J \subseteq I} (-1)^{|J|} n_J^2 \prod_{j \in J} \frac{1}{q_j},$$

where the last equality holds by Proposition 3.2 (ii). Moreover, we note that $n_J = 1$ if $|J| \in \{d, d+1\}$, since we assumed $\gcd(w_0, \ldots, w_{i-1}, w_{i+1}, \ldots, w_d) = 1 \ \forall i \in I$.

Using the equality $\sum_{i=0}^d q_i = 1$, we obtain

$$(-1)^{d+1} \prod_{i=0}^d \frac{1}{q_i} + (-1)^d \prod_{i=0}^d \frac{1}{q_j} = 0$$

and this implies

$$\frac{1}{w} \sum_{l,r=0}^{w-1} \prod_{0 \leq i \leq d} (1 - \frac{1}{q_i}) = \frac{1}{w} \sum_{\emptyset \leq J \subseteq I} (-1)^{|J|} n_J^2 \prod_{j \in J} \frac{1}{q_j}.$$

Example 4.2 [Vaf89, page 1182]. Let $X_{15} \subset \mathbb{P}(1,2,3,4,5)$ be the quasi-smooth hypersurface of degree 15 defined by the weighted homogeneous polynomial $W = z_1^3 + z_4z_5^2 + z_2^3 + z_2z_3^3 + z_0^{15}$. Then $q_0 = \frac{1}{15}$, $q_1 = \frac{2}{15}$, $q_2 = \frac{1}{3}$, $q_3 = \frac{4}{15}$, $q_4 = \frac{1}{3}$, and

$$\chi_{\text{orb}}(X_w) = \frac{1}{15} \sum_{l,r=0}^{14} \prod_{0 \leq i \leq 4} (1 - \frac{1}{q_i}) = \frac{1}{15} \sum_{\emptyset \leq J \subseteq I} (-1)^{|J|} n_J^2 \prod_{j \in J} \frac{1}{q_j}$$

$$= \frac{1}{15} \cdot \left( 225 - \frac{585}{4} + \frac{3375}{8} - \frac{19125}{8} \right) = -126.$$

Now we prove the main theorem of our paper:

Proof of Theorem 1.6. By assumption, $\mathbf{w} = (w_0, w_1, \ldots, w_d)$ is an arbitrary weight vector with IP-property and $Z_\mathbf{w} \subset \mathbb{T}_\mathbf{w}$ a non-degenerate affine hypersurface defined by a Laurent polynomial $f_\mathbf{w}$ with Newton polytope $\Delta^*_\mathbf{w}$. Applying Theorem 3.4, the Zariski closure of $Z_\mathbf{w}$ in the $\mathbb{Q}$-Gorenstein toric Fano variety $\mathbb{P}^\vee(\mathbf{w})$ is a Calabi-Yau
variety $X_w^*$ and

$$
\chi_{\text{str}}(X_w^*) = \frac{1}{w} \sum_{\emptyset \subseteq J \subseteq I, |J| \geq 2} (-1)^{|J|} n_J^2 \prod_{i \in J} \frac{1}{q_i}.
$$

Since $|J| + |\mathcal{J}| = |I| = d + 1$, we obtain

$$
\chi_{\text{str}}(X_w^*) = (-1)^{d-1} \frac{1}{w} \sum_{\emptyset \subseteq J \subseteq I, |J| \leq d-1} (-1)^{|J|} n_J^2 \prod_{i \in J} \frac{1}{q_i} = (-1)^{d-1} \frac{1}{w} \sum_{\emptyset \subseteq J \subseteq I, |J| \leq d-1} (-1)^{|J|} n_J^2 \prod_{j \in J} \frac{1}{q_j}
$$

and Theorem 4.1 implies

$$
\chi_{\text{str}}(X_w^*) = (-1)^{d-1} \frac{1}{w} \sum_{\emptyset \subseteq J \subseteq I, |J| \leq d-1} (-1)^{|J|} n_J^2 \prod_{j \in J} \frac{1}{q_j} = (-1)^{d-1} \frac{1}{w} \sum_{l,r=0}^{w-1} \prod_{0 \leq i \leq d} \left(1 - \frac{1}{q_i}\right).
$$

If $\overline{w} = (w_0, w_1, \ldots, w_d)$ is a transverse weight vector, Vafa’s Formula (1) yields

$$
\chi_{\text{orb}}(X_w) = \frac{1}{w} \sum_{l,r=0}^{w-1} \prod_{0 \leq i \leq d} \left(1 - \frac{1}{q_i}\right).
$$

Since $\overline{w}$ has IP-property [Ska96, Lemma 2], a combination of the last two equations implies

$$
\chi_{\text{str}}(X_w^*) = (-1)^{d-1} \chi_{\text{orb}}(X_w).
$$

\[\square\]

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