The twistor equation in Lorentzian spin geometry

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Abstract

In this paper we discuss the twistor equation in Lorentzian spin geometry. In particular, we explain the local conformal structure of Lorentzian manifolds, which admit twistor spinors inducing lightlike Dirac currents. Furthermore, we derive all local geometries with singularity free twistor spinors that occur up to dimension 7.

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1 Introduction

A classical object in differential geometry are conformal Killing fields. These are by definition infinitesimal conformal symmetries, i.e. the flow of such vector fields preserves the conformal class of the metric. The number of linearly independent conformal Killing fields measures the degree of conformal symmetry on the manifold. This number is bounded by \( \frac{1}{2}(n+1)(n+2) \), where \( n \) is the dimension of the manifold. If it is the maximal one the manifold is conformally flat. S. Tachibana and T. Kashiwada (cf. [TK69], [Kas68]) introduced a generalization of conformal Killing fields, the conformal Killing forms (or twistor forms). Conformal Killing forms are solutions of a conformally invariant twistor type equation on differential forms. They were studied in General Relativity mainly from the local viewpoint in order to integrate the equation of motion (e.g. [PW70]), furthermore they were used to obtain symmetries of field equations ([BC97], [BCK97]). Recently, U. Semmelmann ([Sem01]) started to discuss global properties of conformal Killing forms in Riemannian geometry. Another generalization of conformal Killing vectors is that of conformal Killing spinors (or twistor spinors), which are solutions of the conformally invariant twistor equation on spinors introduced by R. Penrose in General Relativity (cf. [PR86]). Whereas conformal Killing fields are classical symmetries, conformal Killing spinors define infinitesimal symmetries on supermanifolds (cf. [ACDS98]). Special kinds of such spinors, parallel and special Killing spinors, occur in supergravity and string theories. In 1989 A. Lichnerowicz and Th. Friedrich started a systematic study of twistor spinors in conformal Riemannian geometry. Whereas the global structure of Riemannian manifolds admitting twistor spinors is quite well understood (cf. e.g. [Lic85], [Lic88a], [Lic89], [Fri89], [Lic90], [BFGK91], [Hab90], [Hab93], [Hab94], [Hab96], [KR94], [KR96], [KR97], [KR98]), the state of art in its origin, Lorentzian geometry, is far from being satisfactory. We are mainly interested in the following problems:
1. Which Lorentzian geometries admit twistor spinors?

2. How are the properties of twistor spinors related to the geometric structures where they can occur?

J. Lewandowski ([Lew91]) described the local normal forms of 4-dimensional spacetimes with zero free twistor spinors. His results indicated that there are interesting relations between twistor spinors, different global contact structures and Lorentzian geometry that should be discovered. H. Baum ([Bau99], [Bau00]) described twistor spinors on Fefferman spaces and on Lorentzian symmetric spaces. Ch. Bohle ([Boh03]) and F. Leitner ([Le03]) studied Lorentzian geometries with Killing spinors, which are a special class of twistor spinors. In the present paper we describe the geometric structures, which appear up to dimension 7. We consider only the case of zero free twistor spinors. Results for twistor spinors with zeros can be found in [Le01].

After recalling the definition of twistor spinors we discuss in section 3 Brinkmann spaces, Lorentzian Einstein–Sasaki structures and Fefferman spaces and their relation to the problem in question. In chapter 4 we study the local conformal structure of Lorentzian manifolds that admit twistor spinors inducing lightlike Dirac currents (cf. Proposition 4.4). In chapter 5 we derive all local conformal structures of Lorentzian manifolds admitting singularity free solutions of the twistor equation in low dimensions $n \leq 7$ (cf. Theorem 5.1).

2 The twistor equation on spinors

In this section we recall the definition of twistor spinors and fix some notations. For more details we refer to [PRS6] or [BFGK91].

Let $(M^n, g)$ be a semi-Riemannian spin manifold of dimension $n \geq 3$. We denote by $S$ the spinor bundle and by $\mu : T^*M \otimes S \to S$ the Clifford multiplication. The 1-forms with values in the spinor bundle decompose into two subbundles

$$T^*M \otimes S = V \oplus Tw,$$

where $V$, being the orthogonal complement to the ‘twistor bundle’ $Tw := Ker \mu$, is isomorphic to $S$. Usually, we identify $TM$ and $T^*M$ using the metric $g$.

We obtain two differential operators of first order by composing the spinor derivative $\nabla^S$ with the orthogonal projections onto each of these subbundles, the Dirac operator $D$

$$D : \Gamma(S) \xrightarrow{\nabla^S} \Gamma(T^*M \otimes S) = \Gamma(S \oplus Tw) \xrightarrow{pr_S} \Gamma(S),$$

and the twistor operator $P$

$$P : \Gamma(S) \xrightarrow{\nabla^S} \Gamma(T^*M \otimes S) = \Gamma(S \oplus Tw) \xrightarrow{pr_Tw} \Gamma(Tw).$$

Locally, these operators are given by the following formulas

$$D \varphi = \sum_{i=1}^n \sigma^i \cdot \nabla^S_{s_i} \varphi$$

$$P \varphi = \sum_{i=1}^n \sigma^i \cdot \nabla^S_{s_i} \varphi$$
\[ P\varphi = \sum_{i=1}^{n} \sigma^i \otimes (\nabla_{s_i}^{S} \varphi + \frac{1}{n} s_i \cdot D\varphi), \]

where \((s_1, \ldots, s_n)\) is a local orthonormal basis, \((\sigma^1, \ldots, \sigma^n)\) its dual and \(\cdot\) denotes the Clifford multiplication. Both operators are conformally covariant. More exactly, if \(\tilde{g} = e^{2\sigma} g\) is a conformal change of the metric, the Dirac and the twistor operator satisfy

\[
D_{\tilde{g}} = e^{-\frac{n+1}{2}\sigma} D_g e^{\frac{n-1}{2}\sigma} \quad \text{and} \quad P_{\tilde{g}} = e^{-\frac{2}{\sigma}} P_g e^{-\frac{2}{\sigma}}.
\]

A spinor field is called \textit{twistor spinor} or \textit{conformal Killing spinor} if it lies in the kernel of the twistor operator \(P\). Using the local formula for the twistor operator one obtains the following characterization of a twistor spinor: A spinor field \(\varphi \in \Gamma(S)\) is a twistor spinor if and only if

\[
\nabla_{X}^{S} \varphi + \frac{1}{n} X \cdot D\varphi = 0 \quad \text{for all vector fields } X.
\]

Obviously, each parallel spinor \((\nabla^{S} \varphi = 0)\) is a twistor spinor. An other special class of twistor spinors are the Killing spinors \(\varphi\), which satisfy \(\nabla_{X}^{S} \varphi = \lambda X \cdot \varphi\) for some \(\lambda \in \mathbb{C} \setminus \{0\}\). It is a well-known fact, that – as in the case of conformal vector fields – the dimension of the space of twistor spinors is bounded and the maximal possible dimension is attained only for conformally flat manifolds. More exactly, it holds

\textbf{Proposition 2.1} \textit{(cf. [BFGK91])}

1. The dimension of the space of twistor spinors is a conformal invariant and bounded by

\[ \dim \ker P \leq 2 \text{rank} S = 2\left[\frac{n}{2}\right]+1 =: d_n. \]

2. If \(\dim \ker P = d_n\) then \((M^n, g)\) is conformally flat.

3. If \((M^n, g)\) is simply connected and conformally flat then \(\dim \ker P = d_n\).

Hence, for example, all simply connected space forms \(\mathbb{R}^n_k, \mathbb{H}^n_k, \tilde{S}^n_k\) admit the maximal number of linearly independent twistor spinors.

\section{Twistor spinors on Lorentzian spin manifolds - Examples}

Now, we restrict our attention to the case of Lorentzian signature \((- + \ldots +)\). In this section we will explain the special geometries that occur in the Theorem \textbf{2.1}. Let \((M^n, g)\) be an oriented and time-oriented Lorentzian spin manifold. On the spinor bundle \(S\) there exists an indefinite non-degenerate inner product \(\langle \cdot, \cdot \rangle\) such that

\[
\langle X \cdot \varphi, \psi \rangle = \langle \varphi, X \cdot \psi \rangle \quad \text{and} \quad X(\langle \varphi, \psi \rangle) = \langle \nabla^{S}_{X} \varphi, \psi \rangle + \langle \varphi, \nabla^{S}_{X} \psi \rangle,
\]

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for all vector fields \( X \) and all spinor fields \( \varphi, \psi \) (cf. [Bau81]). Each spinor field \( \varphi \in \Gamma(S) \) defines a vector field \( V_\varphi \) on \( M \), the so-called Dirac current, by

\[
g(V_\varphi, X) := -\langle X \cdot \varphi, \varphi \rangle.
\]

A direct calculation shows the following properties of the Dirac current

**Proposition 3.1** (cf. [Bau99]) Let \( \varphi \) be a spinor field on a Lorentzian spin manifold \((M^n, g)\) with Dirac current \( V_\varphi \). Then

1. \( V_\varphi \) is causal and future-directed.
2. The zero sets of \( \varphi \) and \( V_\varphi \) coincide.
3. If \( \varphi \) is a twistor spinor, \( V_\varphi \) is a conformal Killing field.

Now, let us discuss 3 types of special Lorentzian geometries that admit twistor spinors.

### 3.1 Brinkmann spaces with parallel spinors

A Lorentzian manifold is called Brinkmann space if it admits a non-trivial lightlike parallel vector field. Let us consider two examples of such spaces.

**Example 3.1 (pp-manifolds.)** A Brinkmann space is called pp-manifold if its Riemannian curvature tensor \( \mathcal{R} \) satisfies

\[
\text{Trace}_{(3,5),(4,6)} \mathcal{R} \otimes \mathcal{R} = 0.
\]

Equivalently, pp-manifolds can be characterized as those Lorentzian manifolds \((M^n, g)\), where the metric has the following local normal form depending only on one function \( f \) of \((n - 1)\) variables

\[
g = dt \, ds + f(s, x_1, \ldots, x_{n-2}) ds^2 + \sum_{i=1}^{n-2} dx_i^2.
\]

(cf. [Sch74]). In terms of holonomy, pp-manifolds can be characterized as those Lorentzian manifolds, for which the restricted holonomy group is contained in the Abelian normal subgroup \( 1 \ltimes \mathbb{R}^{n-2} \) of the parabolic subgroup \( (\mathbb{R} \times SO(n - 2)) \ltimes \mathbb{R}^{n-2} \) in \( SO(1, n - 1) \) (see [Lei01], [Lei02]). Using the latter fact one can easily prove that for each simply connected pp-manifold

\[
\dim \ker P \geq \frac{d_n}{4}.
\]

Furthermore, on generic pp-manifolds each twistor spinor is parallel. An important example of geodesically complete pp-manifolds are the Lorentzian symmetric spaces with solvable transvection group (Cahen-Wallach-spaces) (cf. [CW70], [Neu02]).
Example 3.2 (Brinkmann spaces with special Kähler flag) Let \((M^n, g)\) be a Brinkmann space with the lightlike parallel vector field \(V\). Then \(V\) defines a flag of subbundles \(RV \subset V^\perp \subset TM\) in \(TM\), where \(V^\perp = \{Y \in TM \mid g(V, Y) = 0\}\). We equip the bundle \(E := V^\perp / RV\) with the positive definite inner product \(\tilde{g}\) induced by \(g\) and with the metric connection \(\tilde{\nabla}\) induced by the Levi-Civita connection of \(g\). We call \(RV \subset V^\perp \subset TM\) a Kähler flag, if in case of even \(n\) there is a parallel orthogonal almost complex structure \(J : E \rightarrow E\) and if in case of odd \(n\) there exists a parallel subbundle \(H \subset E\) of codimension 1 equipped with a parallel orthogonal almost complex structure \(J : H \rightarrow H\).

The Kähler flag \(RV \subset V^\perp \subset TM\) is called special Kähler flag if in addition
\[
\text{Trace } (J \circ R^{\tilde{\nabla}}(X, Y)) = 0 \text{ for all } X, Y \in TM.
\]

A Brinkmann space has a special Kähler flag iff its reduced holonomy representation is contained in \(SU(2m - 2) \ltimes \mathbb{R}^{2m-2}\) (if \(n = 2m\)) resp. in \((SU(2m - 2) \ltimes 1) \ltimes \mathbb{R}^{2m-1}\) (if \(n = 2m + 1\)). It was proved by I.Kath in [Kat99] that a Brinkmann space \((M^n, g)\) has a special Kähler flag if and only if \((M^n, g)\) has pure parallel spinors.

3.2 Twistor spinors on Lorentzian Einstein–Sasaki manifolds

An odd-dimensional Lorentzian manifold \((M^{2m+1}, g; \xi)\) equipped with a vector field \(\xi\) is called Lorentzian Sasaki manifold if

1. \(\xi\) is a timelike Killing field with \(g(\xi, \xi) = -1\).
2. The map \(J := -\nabla \xi : TM \rightarrow TM\) satisfies
\[
J^2 X = -X - g(X, \xi)\xi \quad \text{and} \quad (\nabla X J)(Y) = -g(X, Y)\xi + g(Y, \xi)X.
\]

Let us consider the metric cone \(C_-(M) := (\mathbb{R}^+ \times M, -dt^2 + t^2 g)\) with timelike cone axis over \((M, g)\). The cone metric has signature \((2, 2m)\). Then the following relations between properties of \(M\) and those of its cone are easy to verify

\[
(M^{2m+1}, g; \xi) \quad \text{cone } C_-(M)
\]

\[
\begin{align*}
\text{Lorentzian Sasaki} & \iff \text{(pseudo)-Kähler} \\
\text{Lorentzian Einstein–Sasaki } (R < 0) & \iff \text{Ricci-flat and (pseudo)-Kähler} \\
\text{Lorentzian Einstein–Sasaki } (R < 0) & \iff \text{Hol}_0(C_-(M)) \subset SU(1, m)
\end{align*}
\]

The standard example for regular Lorentzian Einstein–Sasaki manifolds are \(S^1\)- bundles over Riemannian Kähler–Einstein spaces of negative scalar curvature: Let \((X^{2m}, h)\) be a Riemannian Kähler–Einstein spin manifold of scalar curvature \(R_X < 0\) and let \((M^{2m+1}, \pi, X; S^1)\) denote the \(S^1\)-principal bundle associated to the square root \(\sqrt{\Lambda^{m,0}X}\) of the canonical bundle of \(X\) given by the spin structure. Furthermore, let \(A\) be the connection on \(M\) induced by the Levi-Civita connection of \((X, h)\). Then
\[
g := \pi^* h - \frac{16m}{(m + 1)R_X} A \otimes A
\]
defines a Lorentzian Einstein–Sasaki metric on the spin manifold $M^{2m+1}$. Lorentzian Einstein–Sasaki manifolds admit a special kind of twistor spinors.

**Proposition 3.2** (cf. [Kat99], [Boh03]) Let $(M, g)$ be a simply connected Lorentzian Einstein–Sasaki manifold. Then $M$ is spin and admits a twistor spinor $\varphi$ on $M$ such that

1. $V_\varphi$ is a timelike Killing field with $g(V_\varphi, V_\varphi) = -1$
2. $V_\varphi \cdot \varphi = -\varphi$
3. $\nabla_{V_\varphi} \varphi = -\frac{1}{2}i\varphi$.

Conversely, if $(M, g)$ is a Lorentzian spin manifold with a twistor spinor satisfying the conditions a), b) and c). Then $\xi := V_\varphi$ is a Lorentzian Einstein–Sasaki structure on $(M, g)$. Each twistor spinor $\varphi$ on a Lorentzian Einstein-Sasaki manifold is the sum of two Killing spinors.

### 3.3 Twistor spinors on Fefferman spaces

Fefferman spaces are Lorentzian manifolds which appear in the frame work of CR geometry. Let us first explain the necessary notations from CR geometry. Let $N^{2m+1}$ be a smooth oriented manifold of odd dimension $2m+1$. A CR structure on $N$ is a pair $(H, J)$ where

1. $H \subset TM$ is a real $2m$-dimensional subbundle.
2. $J : H \to H$ is an almost complex structure on $H$, i.e. $J^2 = -Id$.
3. If $X, Y \in \Gamma(H)$, then $[JX, Y] + [X, JY] \in \Gamma(H)$ and $J([JX, Y] + [X, JY]) - [JX, JY] + [X, Y] \equiv 0$ (integrability condition).

Let $(N, H, J)$ be a CR manifold. In order to define Fefferman spaces we fix a contact form $\theta \in \Omega^1(N)$ on $N$ such that $\theta|_H = 0$. Let us denote by $T$ the Reeb vector field of $\theta$, which is defined by the conditions $\theta(T) = 1$ and $T \cdot d\theta = 0$. In the following we suppose that the Levi form $L_\theta : H \times H \to \mathbb{R}$

$$L_\theta(X, Y) := d\theta(X, JY)$$

is positive definite. Then $(N, H, J, \theta)$ is called a strictly pseudoconvex manifold. The tensor field $g_\theta := L_\theta + \theta \otimes \theta$ defines a Riemannian metric on $N$. There is a special metric covariant derivative on a strictly pseudoconvex manifold, the Tanaka-Webster connection $\nabla^W : \Gamma(TN) \to \Gamma(TN^* \otimes TN)$, uniquely defined by the conditions

$$\nabla^W g_\theta = 0$$
$$\text{Tor}^W(X, Y) = L_\theta(JX, Y) : T$$
$$\text{Tor}^W(T, X) = -\frac{1}{2}([T, X] + J[T, JX])$$

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for \(X, Y \in \Gamma(H)\). This connection satisfies \(\nabla^W J = 0\) and \(\nabla^W T = 0\) (cf. [Tan75], [Web78]). Let us denote by \(T_{10} \subset TN^C\) the eigenspace of the complex extension of \(J\) on \(H^C\) to the eigenvalue \(i\). Then \(L_\theta\) extends to a Hermitian form on \(T_{10}\) by

\[
L_\theta(U, V) := -i d\theta(U, \bar{V}), \quad U, V \in T_{10}.
\]

For a complex 2-form \(\omega \in \Lambda^2 N^C\) we denote by \(\text{trace}_\theta \omega\) the \(\theta\)-trace of \(\omega\):

\[
\text{trace}_\theta \omega := \sum_{\alpha=1}^{m} \omega(Z_\alpha, \bar{Z}_\alpha),
\]

where \((Z_1, \ldots, Z_m)\) is a unitary basis of \((T_{10}, L_\theta)\). Let \(\mathfrak{R}^W\) be the \((4,0)\)-curvature tensor of the Tanaka-Webster connection \(\nabla^W\) on the complexified tangent bundle of \(N\), and let us denote by

\[
\mathfrak{R}^W(X, Y, Z, V) := g_\theta(([\nabla^W_X, \nabla^W_Y] - \nabla^W_{[X,Y]})Z, \bar{V}).
\]

and let us denote by

\[
\text{Ric}^W := \text{trace}_{\theta}^{(3,4)} := \sum_{\alpha=1}^{m} \mathfrak{R}^W(\cdot, \cdot, Z_\alpha, \bar{Z}_\alpha)
\]

the Tanaka-Webster Ricci curvature and by \(R^W := \text{trace}_{\theta} \text{Ric}^W\) the Tanaka-Webster scalar curvature. The Ricci curvature \(\text{Ric}^W\) is a \((1,1)\)-form on \(N\) with \(\text{Ric}^W(X, Y) \in i\mathbb{R}\) for real vectors \(X, Y \in TN\). The scalar curvature \(R^W\) is a real function.

Now, let us suppose that \((N^{2m+1}, H, J, \theta)\) is a strictly pseudoconvex spin manifold. The spin structure of \((N, g_\theta)\) defines a square root \(\sqrt{\Lambda^{m+1,0}N}\) of the canonical line bundle

\[
\Lambda^{m+1,0}N := \{\omega \in \Lambda^{m+1} N^C \mid V \cdot \omega = 0 \quad \forall V \in \bar{T}_{10}\}.
\]

We denote by \((F, \pi, N)\) the \(S^1\)-principal bundle associated to \(\sqrt{\Lambda^{m+1,0}N}\). Let \(A^W\) denote the connection form on \(F\) defined by the Tanaka-Webster connection \(\nabla^W\). Then

\[
h_\theta := \pi^*L_\theta - i \frac{8}{m+2} \pi^*\theta \otimes (A^W - \frac{i}{4(m+1)} R^W \cdot \theta)
\]

is a Lorentzian metric such that the conformal class \([h_\theta]\) is an invariant of the CR structure \((N, H, J)\). The metric \(h_\theta\) is \(S^1\)-invariant, the fibres of the \(S^1\)-bundle are lightlike. We call \((F^{2m+2}, h_\theta)\) with its canonically induced spin structure Fefferman space of the strictly pseudoconvex spin manifold \((N, H, J, \theta)\).

**Proposition 3.3** ([Bau99]) Let \((N, H, J, \theta)\) be a strictly pseudoconvex spin manifold with the Fefferman space \((F, h_\theta)\). Then there exist two linearly independent twistor spinors \(\varphi\) on \((F, h_\theta)\) such that

a) \(V_\varphi\) is a regular lightlike Killing field

b) \(V_\varphi \cdot \varphi = 0\)

c) \(\nabla_{V_\varphi} \varphi = ic \varphi\), where \(c \in \mathbb{R}\setminus\{0\}\).

Conversely, if \((M, g)\) is an even dimensional Lorentzian spin manifold with a twistor spinor satisfying a), b) and c), then there exists a strictly pseudoconvex spin manifold \((N, H, J, \theta)\) such that its Fefferman space is locally isometric to \((M, g)\).
4  Twistor spinors inducing lightlike Dirac currents

As we noticed in Proposition 3.1, each twistor spinor $\varphi$ induces a causal conformal vector field $V_\varphi$. In this section we study the case that the Dirac current $V_\varphi$ is lightlike. Aiming at a local conformal classification we may assume in addition that $V_\varphi$ is Killing. Let us start with some notations. Let $W: \Lambda^2 M \to \Lambda^2 M$ denote the Weyl tensor of $(M^n, g)$ considered as selfadjoint map on the space of 2-forms and $\text{Ric}$ denotes the Ricci tensor of $(M^n, g)$ considered here as $(1,1)$-tensor or as $(2,0)$-tensor whatever is needed. In conformal geometry, there are two further curvature tensors that play an important role, the Rho tensor $K$

$$K(X) := \frac{1}{n - 2} \left( \frac{R}{2(n - 1)} X - \text{Ric}(X) \right), \quad X \in TM$$

and the Cotton-York tensor

$$C(X,Y) := (\nabla_X K)(Y) - (\nabla_Y K)(X), \quad X,Y \in TM.$$ 

For twistor spinors we have the following properties of the Cotton-York and the Weyl tensor

**Proposition 4.1** Let $\varphi$ be an arbitrary twistor spinor. Then the Dirac current $V_\varphi$ annihilates the Cotton-York and the Weyl tensor: $V_\varphi \cdot C = 0$, $V_\varphi \cdot W = 0$.

**Proof.** We use the following well-known integrability conditions for twistor spinors $\varphi$ (cf. [BFGK91])

$$W(\eta) \cdot \varphi = 0 \quad \text{for all 2-forms } \eta \quad (2)$$

$$W(X \wedge Y) \cdot D\varphi = n C(X,Y) \cdot \varphi. \quad (3)$$

We deduce from this

$$C(V_\varphi, X, Y) = g(V_\varphi, C(X, Y)) = -\langle C(X, Y) \cdot \varphi, \varphi \rangle$$

$$= -\frac{1}{n} \langle W(X \wedge Y) \cdot D\varphi, \varphi \rangle = \frac{1}{n} \langle D\varphi, W(X \wedge Y) \cdot \varphi \rangle = 0.$$

Moreover, with the relation $X \cdot \eta = -X \cdot \eta + X^2 \wedge \eta$ in the Clifford algebra, where $X$ denotes a vector and $\eta$ a 2-form, we have

$$W(V_\varphi, X, Y, Z) = -\langle \varphi, W(X,Y,Z) \cdot \varphi \rangle = -\langle \varphi, Z^3 \wedge W(X,Y) \cdot \varphi \rangle \in \mathbb{R}.$$

Since $\langle \varphi, \rho^3 \cdot \varphi \rangle \in i\mathbb{R}$ for all 3-forms $\rho^3$, it follows that $V_\varphi \cdot W = 0$. $\square$

Now, let us mention a special property of spinor fields with lightlike Dirac current:

**Proposition 4.2** (cf. [Le01]) Let $\varphi$ be a spinor field on a Lorentzian manifold with lightlike Dirac current $V_\varphi$. Then

1. $V_\varphi \cdot \varphi = 0$. 

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2. \( \langle \varphi, \varphi \rangle = 0 \).

**Proof.** The claimed properties for spinors in the lemma are purely algebraic. Therefore, it is sufficient and appropriate to prove these properties on the level of the corresponding representations. For this we use the usual concrete realization of the representation of the Clifford algebra \( Cl_{1,n-1} \) and its complexification \( Cl^C_{1,n-1} \) on the spinor module \( \Delta_{1,n-1} \) in terms of Kronecker products of matrices \([BFGK91]\). Let us consider the complex \((2 \times 2)\)-matrices

\[
E := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad T := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad g_1 := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad g_2 := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}
\]

and let \( \tau(1) = i \) and \( \tau(2) = \tau(3) = \ldots = \tau(n) = 1 \). We denote by \((e_1, \ldots, e_n)\) an orthonormal basis of the Minkowski space \( \mathbb{R}^{1,n-1} \). If \( n = 2m \) then a Clifford representation on the spinor module \( \Delta_{1,n-1} \cong \mathbb{C}^{2m} \) is realized by the map \( \Phi_{2m} : Cl^C_{1,n-1} \to \mathbb{C}(2^m) \), which is generated by

\[
\Phi_{2m}(e_{2j-1}) = \tau(2j-1) \cdot E \otimes \cdots \otimes E \otimes g_1 \otimes T \otimes \cdots \otimes T \quad \text{(j-1)-times}
\]

\[
\Phi_{2m}(e_{2j}) = \tau(2j) \cdot E \otimes \cdots \otimes E \otimes g_2 \otimes T \otimes \cdots \otimes T \quad \text{(j-1)-times}
\]

where \( j = 1, \ldots, m \). If \( n = 2m+1 \) then a representation \( \Phi_{2m+1} : Cl^C_{1,n-1} \to \mathbb{C}(2^m) \) is generated by

\[
\Phi_{2m+1}(e_j) = \Phi_{2m}(e_j) \quad \text{for } j = 1, \ldots, 2m \quad \text{and}
\]

\[
\Phi_{2m+1}(e_n) = i \cdot T \otimes \cdots \otimes T.
\]

Furthermore, let us consider the vectors \( u(\nu) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ -i \nu \end{array} \right) \in \mathbb{C}^2, \nu = \pm 1 \), and the unitary basis \( \{ u(\nu_1, \ldots, \nu_m) := u(\nu_1) \otimes \cdots \otimes u(\nu_m) \mid \nu_i \in \{ \pm 1 \} \} \) of \( \Delta_{1,n-1} \cong \mathbb{C}^{2m} \) with respect to the standard scalar product \( \langle \cdot, \cdot \rangle_\Delta \) of \( \mathbb{C}^{2m} \). The indefinite scalar product \( \langle \cdot, \cdot \rangle \) on \( S \) is defined by the \( Spin_0(1,n-1) \)-invariant inner product \( \langle v, w \rangle_\Delta = (e_1 \cdot v, w)_\Delta \) on \( \Delta_{1,n-1} \). Let \( \ell : \Delta_{1,n-1} \to \mathbb{R}^{1,n-1} \) denote the map, which maps a spinor \( v \) to its Dirac current \( \ell(v) = (v, e_1 v)e_1 - \sum_{i \geq 2} (v, e_i v)e_i \). We calculate now the inverse image \( \ell^{-1}(\mathbb{R}(e_1 + e_2)) \) of the lightlike direction \( \mathbb{R}(e_1 + e_2) \). For this let

\[
v = \sum_{(\nu_1, \ldots, \nu_m) \in \{ \pm 1 \}^m} a_{\nu_1, \ldots, \nu_m} \cdot u(\nu_1, \ldots, \nu_m), \quad a_{\nu_1, \ldots, \nu_m} \in \mathbb{C},
\]

be an arbitrary spinor represented in the unitary basis of \( \Delta_{1,n-1} \). It holds

\[
e_1 \cdot v = - \sum a_{\nu_1, \ldots, \nu_m} \cdot u(\nu_1, \ldots, -\nu_m) \quad \text{and}
\]

\[
e_2 \cdot v = \sum \nu_m \cdot a_{\nu_1, \ldots, \nu_m} u(\nu_1, \ldots, -\nu_m).
\]

Then we obtain

\[
\langle v, e_1 v \rangle_\Delta = (e_1 \cdot v, e_1 \cdot v)_\Delta = \sum |a_{\nu_1, \ldots, \nu_m}|^2 \quad \text{and}
\]

\[
\langle v, e_2 v \rangle_\Delta = (e_1 \cdot v, e_2 \cdot v)_\Delta = \sum -\nu_m \cdot |a_{\nu_1, \ldots, \nu_m}|^2.
\]
It is \( \ell(v) \in \mathbb{R}(e_1 + e_2) \) if and only if \( \langle v, e_1 v \rangle = -\langle v, e_2 v \rangle \). The latter condition is equivalent to \( a_{\nu_1, \ldots, \nu_{m-1}} = 0 \) for all \( (\nu_1, \ldots, \nu_{m-1}) \in \{\pm 1\}^{m-1} \). Hence, a spinor \( v \) with \( \ell(v) \in \mathbb{R}(e_1 + e_2) \) has the form

\[
v = a \otimes u(1), \quad a \in \bigotimes_{m-1} \mathbb{C}^2.
\]

Then \( \langle v, v \rangle_\Delta = (e_1 \cdot v, v)_\Delta = -(a \otimes u(-1), a \otimes u(1)) = 0 \) and \( (e_1 + e_2) \cdot v = a \otimes (ig_1 + g_2)u(1) = 0 \). This proves the desired properties in case that \( \ell(v) \in \mathbb{R}(e_1 + e_2) \). Since the map \( \ell \) is equivariant under the action of the spin group \( \text{Spin}_\sigma(1, n-1) \) and the spin group acts transitive on the lightlike directions in the lightcone of the Minkowski space \( \mathbb{R}^{1,n-1} \), we can conclude that the claimed properties for spinors \( v \in \Delta_{1,n-1} \) with arbitrary lightlike Dirac current are true in general.

Let \( V \) be a vector field and let \( \theta \) denote the dual 1-form \( \theta(X) = g(X, V) \). Then \( V \) is called \textit{twisting} if the 3-form \( d\theta \wedge \theta \) nowhere vanishes and \textit{non-twisting} if \( d\theta \wedge \theta = 0 \). By the Frobenius Theorem the latter means that the distribution \( V^\perp \subset TM \) is integrable. For twistor spinors with non-twisting Dirac current we have the following result.

**Proposition 4.3 (cf. [Le01])** Let \( \varphi \) be a twistor spinor with lightlike, non-twisting Dirac current. Then \( (M, g) \) is locally conformally equivalent to a Brinkmann space with parallel spinor.

**Proof.** The condition that \( V_\varphi \) has no zero and no twist implies by the Frobenius Theorem that locally there are functions \( \sigma, f \) such that \( V_\varphi = e^{-2\sigma} \text{grad} f \). Then, locally we have \( V_\varphi = \text{grad} f \) with respect to the metric \( \tilde{g} = e^{2\sigma} g \). Hence, without loss of generality, we may assume that \( V := V_\varphi \) is a lightlike conformal gradient field. Let \( V = \text{grad} f \).

Then

\[
\text{Hess} f = \frac{1}{2} L_V g = \frac{\Delta f}{2n}g \quad \text{and} \quad \nabla_X V = \frac{\text{div} V}{2n} X \quad \text{for all} \ X.
\]

Since \( V \) is lightlike, we obtain \( 0 = X(g(V, V)) = 2g(\nabla_X V, V) = \frac{\text{div} V}{n} g(X, V) \) for all \( X \) and therefore \( \text{div} V = 0 \). This shows that the Dirac current \( V_\varphi \) is parallel. Now we show that the spinor \( \varphi \) is parallel, too. To this end, let \( (s_1, \ldots, s_n) \) be a local orthonormal frame with \( V_\varphi = s_1 + s_2 \). By Proposition 4.2 we know that \( V_\varphi \cdot \varphi = 0 \). Since \( V_\varphi \) is parallel, we obtain from \( 0 = \nabla_X (V_\varphi \cdot \varphi) = \nabla_X V_\varphi \cdot \varphi + V_\varphi \cdot \nabla_X \varphi \) that

\[
V_\varphi \cdot \nabla_X \varphi = 0 \quad \text{and} \quad s_1 \cdot s_2 \cdot \nabla_X \varphi = -\nabla_X \varphi \quad \text{for all} \ X.
\]

Let \( X \) be a vector field with \( g(X, X) = \pm 1 \) and let \( \psi := g(X, X) X \cdot \nabla_X \varphi \). Since \( \varphi \) is a twistor spinor, the spinor \( \psi \) does not depend on the choice of \( X \). Choose \( X \in V_\varphi \). Then \( V_\varphi \cdot \psi = -g(X, X) X \cdot V_\varphi \cdot \nabla_X \varphi = 0 \). On the other hand, for \( X = s_1 \) we obtain

\[
0 = V_\varphi \cdot \psi = -(s_1 + s_2) \cdot s_1 \cdot \nabla_{s_1} \varphi = (-1 + s_1 \cdot s_2) \cdot \nabla_{s_1} \varphi = -2 \nabla_{s_1} \varphi.
\]

This shows that \( \psi = 0 \), which implies \( \nabla \varphi = 0 \).

Using the conditions on the twist one is able to characterize – at least locally – all
geometries that admit a zero free twistor spinor with lightlike Dirac current. In particular, the following Proposition explains the role, Fefferman spaces and Brinkmann spaces are playing among all Lorentzian geometries that admit twistor spinors.

**Proposition 4.4** Let \((M, g)\) be a Lorentzian spin manifold admitting a twistor spinor \(\varphi\) such that \(V_\varphi\) is lightlike and Killing. Then the function \(\text{Ric}(V_\varphi, V_\varphi)\) is constant and non-negative on \(M\). Furthermore,

1. \(\text{Ric}(V_\varphi, V_\varphi) > 0\) if and only if \((M, g)\) is locally isometric to a Fefferman space. In this case the Dirac current is twisting and the dimension of \(M\) is even.

2. \(\text{Ric}(V_\varphi, V_\varphi) = 0\) if and only if \((M, g)\) is locally conformal equivalent to a Brinkmann space with parallel spinors. In this case the Dirac current is non-twisting.

**Proof.** At first, we observe the following general relations for a lightlike Killing field \(V\).

It is \(g(\nabla_X \nabla_Y V, Z) = \mathcal{R}(Y, Z, X, V)\) for all vector fields \(X, Y\) and \(Z\) in \(TM\), especially \(g(\nabla_X V, \nabla_Y V) = \mathcal{R}(V, X, Y, V)\). If in addition \(\nabla \bullet W = 0\) then

\[
g(\nabla_X \nabla_Y V, Z) = g \ast K(Y, Z, X, V) = -g(Y, V)K(Z, X) - g(Z, X)K(Y, V) + g(Y, X)K(Z, V) + g(Z, V)K(Y, X),
\]

where \(\ast\) denotes the Kulkarni-Nomizu product. Next we show that

\[
X(\text{Ric}(V, V)) = 4K(\nabla_X V, V) = 0 \quad \text{for all} \quad X \in TM.
\]

Fix \(x \in M\) and let \((e_1, \ldots, e_n)\) denote an orthonormal frame, arising by parallel displacement from \(x\). Then in the point \(x\) we have the following identities. Using the skew-symmetry of \(g(\cdot, \nabla V)\) and the symmetry of \(K\) we obtain

\[
\sum_k g(e_k, e_k) \cdot K(\nabla_{e_k} V, e_k) = 0.
\]  

(4)

The second Bianchi identity for the Riemannian curvature tensor and \(V(R) = 0\) yields

\[
\sum_k g(e_k, e_k) e_k (K(V, e_k)) = \sum_k g(e_k, e_k) \cdot (\nabla_{e_k} K)(V, e_k) = 0.
\]  

(5)

Furthermore, from \(\nabla \bullet C = 0\) follows

\[
V(K(e_i, V)) = (\nabla_V K)(e_i, V) = (\nabla_{e_i} K)(V, V).
\]  

(6)

It is

\[
\mathcal{R}(e_i, \nabla_{e_k} V, e_k, V) + \mathcal{R}(e_k, \nabla_{e_i} V, e_k, V) = g(\nabla_{e_k} \nabla_{e_i} V, \nabla_{e_i} V) + g(\nabla_{e_k} \nabla_{e_k} V, \nabla_{e_i} V) = e_k(\mathcal{R}(V, e_i, e_k, V)),
\]

\[
\mathcal{R}(e_i, \nabla_{e_k} V, e_k, V) + \mathcal{R}(e_k, \nabla_{e_i} V, e_k, V) = g(e_i, e_k)K(\nabla_{e_k} V, V) - g(e_i, V)K(\nabla_{e_k} V, e_k) + g(e_k, e_k)K(\nabla_{e_i} V, V) - g(e_k, V)K(\nabla_{e_i} V, e_k) - g(\nabla_{e_i} V, e_k)K(e_k, V)\quad \text{and}
\]
\[ e_k(\mathcal{R}(V, e_i, e_k, V)) = g(V, e_k) \cdot e_k(K(e_i, V)) + g(e_i, \nabla_{e_k} V) \cdot K(V, e_k) + g(e_i, V) \cdot e_k(K(V, e_k)) - g(e_i, e_k) \cdot e_k(K(V, V)) \,.
\]

Summing up the latter equations and using (4), (5) and (6) results in
\[
\sum_k g(e_k, e_k) (\mathcal{R}(e_i, \nabla_{e_k} V, e_k, V) + \mathcal{R}(e_k, \nabla_{e_k} V, e_k, V)) = (n - 1)K(\nabla_{e_i} V, V) \quad \text{and}
\]
\[
\sum_k g(e_k, e_k) \cdot e_k(\mathcal{R}(V, e_i, e_k, V)) = -3K(\nabla_{e_i} V, V) \,.
\]

Hence, \(K(\nabla_{e_i} V, V) = 0\). Eventually,
\[
e_i(\text{Ric}(V, V)) = 2\sum_k g(e_k, e_k) \cdot g(\nabla_{e_i} \nabla_{e_k} V, \nabla_{e_k} V)
= 2\sum_k g(e_k, e_k)g \ast K(e_k, \nabla_{e_k} V, e_i, V)
= 4K(\nabla_{e_i} V, V) \,.
\]

In particular, \(\text{Ric}(V, V)\) is constant on \(M\).

Now, let us consider \(V = V_\varphi\). Then the condition \(\text{Ric}(V, V) = \text{const.}\) follows from Proposition 4.1. We denote by \(\eta\) and \(\theta\) the 1-forms
\[
\eta(X) := K(V, X), \quad \theta(X) := g(V, X).
\]

Furthermore, let \(T\) be the vector field dual to \(\eta\). For the following, we normalize the spinor \(\varphi\) such that \(\eta(V) = -(n - 2)\text{Ric}(V, V) =: \varepsilon \in \{0, 1, -1\}\). Let us consider the endomorphism
\[
J : TM \rightarrow TM \quad J(X) := \nabla_X V.
\]

The map \(J\) is skew-adjoint and satisfies \(JV = JT = 0\), since \(V_\varphi\) is lightlike Killing and \(g(JT, X) = -K(V, \nabla_X V) = 0\) for all \(X\). Then we obtain for arbitrary vectors \(X\) and \(Y\)
\[
g(J^2(X), Y) = -g(J(X), J(Y)) = -\mathcal{R}(V, X, Y, V)
= -g(V, Y)K(X, V) - g(X, V)K(Y, V) + g(V, V)K(Y, X) + g(X, Y)K(V, V)
= g(X, y)\eta(V) - g(V, Y)\eta(X) - \theta(X)g(T, Y),
\]
which shows that
\[
J^2(X) = \varepsilon X - \theta(X)T - \eta(X)V \,.
\]

Moreover, it is
\[
g(T, V) = \varepsilon, \quad g(V, V) = g(T, T) = 0 \quad \text{(8)}
\]
\[
d\theta(X, Y) = 2g(JX, Y) \quad \text{(9)}
\]

Now, let \(\varepsilon = 1\). We consider the \(J\)-invariant subbundle \(H = \text{span}\{V_\varphi, T\} \subset TM\). The spin structure of \((M, g)\) reduces to a spin structure of the hermitian bundle \((H, J, g)\).
The spin structure, $H$, $\theta$ and $J$ project down to the (locally defined) manifold $N$ resulting from $M$ by factoring out the integral curves of $V$. Then it can be proved as in \cite{Gra87} that $(N, H, J, \theta)$ is a strictly pseudoconvex manifold and that $(M, g)$ is locally isometric to the Fefferman space of $(N, H, J, \theta)$. The case $\varepsilon = -1$ can not occur, since by (7) and (8) $J$ would be a skew-adjoint involution on the positive definite subbundle $\text{span}(V,T)^\perp$ of $TM$. In case that $\varepsilon = 0$ the vector field $T$ is parallel to $V$. This implies that $\text{Im}(J^2) \subset \mathbb{R}V$. Since $\text{Im}J \subset V^\perp$ we obtain $J(V^\perp) \subset \mathbb{R}V$. Then \cite{LM89} shows that $d\theta$ vanishes on $V^\perp$, which means that $\theta$ is non-twisting. Hence, the second assertion of the Theorem follows from Proposition 4.3. \hfill $\Box$

5 Twistor spinors in dimension $n \leq 7$

In this section we discuss the twistor equation on Lorentzian spin manifolds in the dimensions $n = 3, 4, 5, 6$ and 7. For solutions of the twistor equation without singularities we derive a complete list of possible underlying local Lorentzian geometries in these low dimensions. For dimension $n = 4$ this was already proved by J. Lewandowski. To begin with, we state some special properties for spinors on low dimensional Lorentzian spin manifolds. These properties can be derived from the representation theory of the spinor modules in low dimensions (cf. \cite{LM89}) and the discussion of their orbit structure, which can be found e.g. in \cite{Br00}.

Lemma 5.1 (Properties of spinors on low dimensional Lorentzian manifolds)

1. Let $n = 3$. Then there exists a real structure $\tau : S \rightarrow S$ on the spinor bundle such that $\tau(X \cdot \varphi) = -X \cdot \tau(\varphi)$ and $\langle \tau\varphi, \tau\psi \rangle = -\langle \psi, \varphi \rangle$. Each spinor field $\varphi \in \Gamma(S)$ satisfies $V_{\varphi} \cdot \varphi = \langle \varphi, \varphi \rangle \varphi$ and $g(V_{\varphi}, V_{\varphi}) = -\langle \varphi, \varphi \rangle^2$. If $\varphi$ is a real spinor field, i.e. $\tau(\varphi) = \varphi$, then the Dirac current $V_{\varphi}$ is lightlike.

2. Let $n = 5$. Then there exists a quaternionic structure $J : S \rightarrow S$ such that $J(X \cdot \varphi) = X \cdot J(\varphi)$ and $\langle J\varphi, J\psi \rangle = -\langle \psi, \varphi \rangle$. Again each spinor field $\varphi \in \Gamma(S)$ satisfies $V_{\varphi} \cdot \varphi = \langle \varphi, \varphi \rangle \varphi$ and $g(V_{\varphi}, V_{\varphi}) = -\langle \varphi, \varphi \rangle^2$.

3. Let $n = 7$. Then there exists a quaternionic structure $J : S \rightarrow S$ such that $J(X \cdot \varphi) = -X \cdot J(\varphi)$ and $\langle J\varphi, J\psi \rangle = \langle \psi, \varphi \rangle$. Each spinor field $\varphi \in \Gamma(S)$ satisfies $V_{\varphi} \cdot \varphi = \langle \varphi, \varphi \rangle \varphi + \langle \varphi, J\varphi \rangle J\varphi$ and $|\langle \varphi, \varphi \rangle|^2 + |\langle \varphi, J\varphi \rangle|^2 = g(V_{\varphi}, V_{\varphi})$.

4. Let $n = 2, 4, 6$. Then each spinor field $\varphi \in \Gamma(S^\pm)$ satisfies $V_{\varphi} \cdot \varphi = 0$ and $g(V_{\varphi}, V_{\varphi}) = 0$.

Proof. The existence of the real or quaternionic structures is clear from the representation theory of the spinor modules (cf. \cite{LM89}). The main point here is to prove the formulas concerning $V_{\varphi} \cdot \varphi$ and $g(V_{\varphi}, V_{\varphi})$. Although these formulas are very natural, it seems that there is no natural proof for them. Therefore, we explain in the following the orbit structure of the spinor modules $\Delta_{1,n-1}$ for $n = 3, 4, 5, 6$ and 7 with respect to the action of the spin group and calculate explicitly with respect to convenient normal forms of representatives in the various orbits. Let $n = 3$. There is a real structure $\tau$ on $\Delta_{1,2} \cong \mathbb{C}^2$, which is invariant under the action.
of $\text{Spin}_0(1, 2)$ and $\text{Cl}_{2,1} \subset \text{Cl}_{1,2}^\mathbb{C}$. The real spinor representation $\Delta^\pm_{1,2}$ is isomorphic to the standard representation of $\text{SL}(2, \mathbb{R})$ on $\mathbb{R}^2$. Beside the zero orbit, there are the orbit types to the representatives

$$\sigma_1 = \begin{pmatrix} 1 \\ id \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 + ic \\ 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} i \\ 0 \end{pmatrix}$$

parametrized by $0 \neq c \in \mathbb{R}$ and $d \in \mathbb{R}$. We choose the Clifford representation generated by

$$\Phi(e_1) := \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \Phi(e_2) := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \Phi(e_3) := \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}.$$ 

Then it is

$$V_{\sigma_1} = \begin{pmatrix} 1 + d^2 \\ 0 \end{pmatrix}, \quad V_{\sigma_2} = \begin{pmatrix} -1 + d^2 \\ 0 \end{pmatrix} \quad \text{and} \quad V_{\sigma_3} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \in \mathbb{R}^{1,2}$$

and calculating the Clifford product gives $V_{\sigma_1} \cdot \sigma_i = \langle \sigma_i, \sigma_i \rangle_{\Delta} \cdot \sigma_i$ for $i = 1, 2, 3$. In particular, $g(V_{\sigma_1}, V_{\sigma_1}) = -|\sigma_i, \sigma_i|^2$ and every real spinor has lightlike Dirac current.

Let $n = 5$. There is a $\text{Cl}_{1,4}$-invariant quaternionic structure on the spinor module $\Delta_{1,4} \cong \mathbb{H}^2$. The orbit types are determined by the representatives

$$\sigma_1 = \begin{pmatrix} r \\ 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 \\ r \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \mathbb{H}^2,$$

where $r \in \mathbb{R}_+$. We generate the Clifford representation by

$$\Phi(e_1) := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Phi(e_2) := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$\Phi(e_3) := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \Phi(e_4) := \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix} \quad \text{and} \quad \Phi(e_5) := \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix},$$

where $i, j, k$ denote the imaginary units. Then $V_{\sigma_1} = V_{\sigma_2} = r^2 \cdot e_1$ and $V_{\sigma_3} = e_1 + e_2$.

Executing the Clifford multiplication results in $V_{\sigma_1} \cdot \sigma_i = \langle \sigma_i, \sigma_i \rangle_{\Delta} \cdot \sigma_i$, which also shows that $g(V_{\sigma_1}, V_{\sigma_1}) = -|\sigma_i, \sigma_i|^2$.

In general, the even-dimensional spinor representations $\Delta^\pm_{1,2m-1}$ split into the half spinor representations $\Delta^\pm_{1,2m-1}$ and the half spinor modules are spanned by

$$\Delta^\pm_{1,2m-1} = \{ u(\nu_1, \ldots, \nu_m) : \prod_{i=1}^m \nu_i = \pm 1 \}.$$ 

In particular, this shows that in all even dimensions there are positive and negative half spinors inducing lightlike Dirac currents (cf. Proof of Proposition 1.2).

For $n = 2$ the half spinors are represented by $\varphi = r \cdot u(1)$ and $r \cdot u(-1)$, where $r \in \mathbb{R}_+$. These spinors have lightlike Dirac current and Proposition 1.2 implies that $V_{\varphi} \cdot \varphi = 0$.

For $n = 4$ the real half spinor representations $\Delta^\pm_{1,3}$ are isomorphic to the canonical representation of $\text{SL}(2, \mathbb{C})$ on $\mathbb{C}^2 \cong \mathbb{R}^4$. Hence, there is exactly one non-trivial orbit in each half spinor module. These two orbits are represented by $u(1, 1)$ and $u(1, -1)$, which give rise to lightlike Dirac currents.

Let $n = 6$. There is a $\text{Cl}_{5,1}$-invariant quaternionic structure on $\Delta_{1,5} \cong \mathbb{H}^4$. The occurring orbit types are

$$\sigma_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \sigma_4 = \begin{pmatrix} 1 \\ 0 \\ \lambda \\ 0 \end{pmatrix},$$

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where \( r \in \mathbb{R}_+ \) and \( \lambda \in \mathbb{H} \). Both half spinor modules admit exactly one non-trivial orbit and these are represented by \( \sigma_1 \) and \( \sigma_2 \), which then must have lightlike Dirac current. Finally, let \( n = 7 \). The module \( \Delta_{1,6} \) admits a \( Cl_{6,1} \)-invariant quaternionic structure \( J \) and the restriction of \( \Delta_{1,6} \) to the action of \( Spin(1,5) \) is isomorphic to the spinor representation \( \Delta_{1,5} \). From the orbit type classification of \( \Delta_{1,5} \) we derive the orbit types in \( \Delta_{1,6} \), which are then parametrized by

\[
\sigma_\lambda = \begin{pmatrix}
1 \\
0 \\
\lambda \\
0
\end{pmatrix}, \quad \lambda = \lambda_1 + \lambda_2 j \in Im\mathbb{H}.
\]

It is \( \langle \sigma_\lambda, \sigma_\lambda \rangle = -2i\lambda_1 \). Moreover, \( J\sigma_\lambda = \begin{pmatrix} j \\ -\lambda_2 + \lambda_1 j \\ 0 \\ 0 \end{pmatrix} \) and \( \langle \sigma_\lambda, J\sigma_\lambda \rangle = 2i\lambda_2 \). For a suitable realization of the Clifford representation we have \( V_{\sigma_\lambda} = (1+|\lambda|^2)e_1 + (1-|\lambda|^2)e_2 \), where \( e_1 \) is a timelike unit vector, and \( V_{\sigma_\lambda} \cdot \sigma_\lambda = \sigma_\lambda \cdot (-2i\lambda) \). This proves that \( V_{\sigma_\lambda} \cdot \sigma_\lambda = \sigma_\lambda \cdot \langle \sigma_\lambda, \sigma_\lambda \rangle + J\sigma_\lambda \cdot \langle \sigma_\lambda, J\sigma_\lambda \rangle \).

**Lemma 5.2** The dimension of the space of twistor spinors on a non-conformally flat Lorentzian manifold \((M^n, g)\) is bounded by

(1) \( n = 3 \): \( \dim_{\mathbb{C}} \Ker P \leq 1 \).
(2) \( n = 4 \): \( \dim_{\mathbb{C}} \Ker P \leq 2 \).
(3) \( n = 5 \): \( \dim_{\mathbb{C}} \Ker P \leq 2 \).

**Proof.** Using the twistor equation one shows that in case \( \dim_{\mathbb{C}} \Ker P > 1 \) there exists a dense set \( A \subset M \) such that \( \dim_{\mathbb{C}} \{ \varphi(x) \in S_x \mid \varphi \in \Ker P \} \geq 2 \) for all \( x \in A \).

In dimension 3, the Weyl tensor \( W \) vanishes. Hence, for each twistor spinor \( \varphi \) and all vector fields \( X, Y \) the condition \( C(X, Y) \cdot \varphi = 0 \) holds (cf. (3)). Let \( \dim_{\mathbb{C}} \Ker P > 1 \) and let \( x \in A \) be an arbitrary point. Since \( \dim_{\mathbb{C}} S_x = 2 \), the vectors \( C_x(U, W) \) annihilate \( S_x \) for all \( U, W \in T_xA \). This implies \( C = 0 \). A 4-dimensional Lorentzian manifold with a twistor spinor \( \varphi \) is of Petrov type N or 0 in each point \( x \in M \) where \( \varphi(x) \neq 0 \). Let \( \varphi^+ \) be the positive part of \( \varphi \) and suppose \( \varphi^+(x) \neq 0 \). Then \( W_x = 0 \) or \( L_x := \mathbb{R} \varphi^+(x) \) is the uniquely determined 4-fold principal null direction of \( W_x \) and \( L_x \cdot \varphi^+(x) = 0 \). Let \( \dim_{\mathbb{C}} \Ker P > 2 \). Then without loss of generality we may assume that \( \dim_{\mathbb{C}} (\Ker P^+ := \Ker P \cap \Gamma(S^+) > 1 \). Consider a dense set \( A \subset M \) with \( \dim_{\mathbb{C}} \{ \varphi^+(x) \in S_x^+ \mid \varphi^+ \in \Ker P^+ \} = \dim_{\mathbb{C}} S_x^+ = 2 \). Assume \( x \in A \) and \( W_x \neq 0 \). Then the 4-fold principle null direction \( L_x \) of \( W_x \) annihilates \( S_x^+ \). Hence \( W = 0 \). In dimension 5, \( \Ker P \) is a quaternionic space (Lemma 5.1). If \( \dim_{\mathbb{C}} \Ker P > 2 \), than \( \dim_{\mathbb{I}} \Ker P > 1 \) and there exists a dense set \( A \subset M \) such that \( \dim_{\mathbb{I}} \{ \varphi(x) \in S_x \mid \varphi \in \Ker P \} = \dim_{\mathbb{I}} S_x = 2 \). The integrability condition \( W(\eta) \cdot \varphi = 0 \) for all 2-forms \( \eta \) shows that the 2-forms \( W_x(\eta_x) \) annihilate \( S_x \) for all \( x \in A \). Hence, \( W = 0 \). \( \square \)

We call a twistor spinor \( \varphi \) *singularity free* if it has no zeros and the Dirac current \( V_\varphi \) does not changes the causal type. There are the following geometric structures of Lorentzian manifolds with singularity free twistor spinors in dimension \( n \leq 7 \).
Theorem 5.1 ([Lew91], [Le01]) Let $(M^n, g)$ be a Lorentzian manifold with a singularity free twistor spinor. Then $(M^n, g)$ is locally conformally equivalent to one of the following kinds of Lorentzian structures.

$n = 3$:
- $pp$-manifold

$n = 4$:
- $pp$-manifold
- Fefferman space

$n = 5$:
- $pp$-manifold
- Lorentzian Einstein–Sasaki manifold
- $\mathbb{R}^{1,0} \times (N^4, h)$, where $(N^4, h)$ is Riemannian Ricci-flat Kähler

$n = 6$:
- $pp$-manifold
- Fefferman space
- $\mathbb{R}^{1,1} \times (N^4, h)$, where $(N^4, h)$ is Riemannian Ricci-flat Kähler
- Brinkmann space with special Kähler flag

$n = 7$:
- $pp$-manifold
- Lorentzian Einstein-Sasaki manifold
- $\mathbb{R}^{1,2} \times (N^4, h)$, where $(N^4, h)$ is Riemannian Ricci-flat Kähler
- $\mathbb{R}^{1,0} \times (N^6, h)$, where $(N^6, h)$ is Riemannian Ricci-flat Kähler
- $\mathbb{R}^{1,0} \times (N^6, h)$, where $(N^6, h)$ is nearly Kähler, non-Kähler
- Brinkmann space with special Kähler flag

Proof. Let $n = 3$ and let denote by $\tau : S \to S$ the real structure on $S$ (Lemma 5.1). If $\varphi$ is a zero free twistor spinor then locally the real part $\text{Re} \varphi := \frac{1}{2}(\varphi + \tau(\varphi))$ or the imaginary part $\text{Im} \varphi := \frac{i}{2}(\tau(\varphi) - \varphi)$ is a zero free real spinor. Hence, according to Lemma 5.1, we have a zero free twistor spinor with lightlike Dirac current, which becomes a Killing vector field after a conformal change of the metric. Since the dimension is odd, $(M, g)$ is locally conformally equivalent to a Brinkmann space with parallel spinor (Proposition 4.4). The 3-dimensional Brinkmann spaces with parallel spinors have restricted holonomy group contained in $1 \ltimes \mathbb{R} \subset SO(1, 2)$, hence they are $pp$-manifolds (cf. Example 3.1).

In dimension $n = 4$ and $n = 6$ we may assume that $\varphi$ is a zero free half-spinor in $S^\pm$, since we are only interested in local considerations. Since $V\varphi$ has no zero (cf. Proposition 3.1(2)) and is lightlike (cf. Lemma 5.1) there is a local conformal transformation of the metric such that $V\varphi$ is Killing and lightlike with respect to the conformally changed metric. Hence, according to Proposition 4.4, $(M, g)$ is locally conformal equivalent to a Fefferman space or to a Brinkmann space with parallel spinors.

The restricted holonomy group of a 4-dimensional Brinkmann space with parallel spinor is contained in $1 \ltimes \mathbb{R}^2 \subset SO(1, 3)$, hence it is a $pp$-manifold.

The restricted holonomy group of a 6-dimensional Brinkmann space with parallel spinor is contained in $1 \ltimes \mathbb{R}^4 \subset SO(1, 5)$, $SU(2) \ltimes \mathbb{R}^4 \subset SO(1, 5)$ or in $1 \ltimes SU(2) \subset SO(1, 1) \times SO(4) \subset SO(1, 5)$. Hence it is a $pp$-manifold, a Brinkmann space with special Kähler flag or the metric product of the hyperbolic plane $\mathbb{R}^{1,1}$ with a 4-dimensional Riemannian Ricci-flat Kähler manifold.

Now, let us consider $n = 5$. Let $\varphi$ be a zero free twistor spinor such that $V\varphi$ is a lightlike or a timelike vector field. In case that the Dirac current $V\varphi$ is lightlike we may assume...
as above that $V_\varphi$ is Killing as well. Then, since the dimension is odd, Proposition 4.4 shows that $(M,g)$ is locally conformal equivalent to a Brinkmann space with parallel spinors. The restricted holonomy group of a 5-dimensional Brinkmann spaces with parallel spinors is contained in $1 \times \mathbb{R}^3 \subset SO(1,4)$, hence it is a pp-manifold. Now, suppose that $V_\varphi$ is timelike. Then, by Lemma 5.1 the length function $\langle \varphi, \varphi \rangle$ of $\varphi$ has no zeros. In this case we can consider the conformally changed metric $\tilde{g} := \langle \varphi, \varphi \rangle^{-2}g$. Then the spinor field $\psi := |\langle \varphi, \varphi \rangle|^{-1/2}\varphi$ is a twistor spinor of constant length $\pm 1$ for $\tilde{g}$. Therefore, we may suppose that we have a twistor spinor $\varphi$ of constant length $\pm 1$ on $(M,g)$. In that case $\langle D \varphi, D \varphi \rangle$ is constant as well and $(M,g)$ is an Einstein space of scalar curvature

$$R = -\frac{4(n-1)}{n} \frac{\langle D \varphi, D \varphi \rangle}{\langle \varphi, \varphi \rangle}$$

(cf. [BFGK91]). If $R = 0$ then either $\varphi$ or $D \varphi$ is a non-vanishing parallel spinor with timelike Dirac current. Hence, $(M,g)$ is a product of the timelike line $\mathbb{R}^{1,0}$ and a Riemannian spin manifold with parallel spinor, which is then known to be Ricci-flat and Kähler. If $R$ is non-zero, either $\varphi$ is a Killing spinor or the spinors

$$\psi_{\pm} = \frac{1}{2} \varphi \pm \sqrt{\frac{n-1}{nR}} \cdot D \varphi$$

are Killing spinors to different Killing numbers. In the first case the Killing number of $\varphi$ has to be imaginary, since the length of the spinor $\varphi$ is constant. Using Lemma 5.1 one can show that (after normation and a change of the time-orientation if the length of the spinor is $+1$) the conditions of Proposition 3.2 are satisfied and therefore, $(M,g)$ has to be Einstein–Sasaki. In the second case when $\psi_{\pm}$ are imaginary Killing spinors we can proceed in the same way. If $\psi_{\pm}$ are real Killing spinors then $\psi_{\pm}, J \psi_{\pm}$ are four $\mathbb{C}$-linearly independent Killing spinors (here $J$ denotes the quaternionic structure on the spinor bundle) and, by Lemma 5.2, we can conclude that $(M,g)$ is conformally flat.

Let $n = 7$ and let $\varphi$ be a twistor spinor without singularity. We consider the quaternionic structure $J : S \rightarrow S$ on $S$ (Lemma 5.1). In case that $\langle \varphi, \varphi \rangle = 0$ and $\langle \varphi, J \varphi \rangle = 0$ the Dirac current $V_\varphi$ is lightlike and $(M,g)$ has to be conformally equivalent to a Brinkmann space with parallel spinor. The restricted holonomy group of a 7-dimensional Brinkmann space with parallel spinor is contained in $1 \times \mathbb{R}^5 \subset SO(1,6)$, $(SU(2) \times 1) \ltimes \mathbb{R}^5 \subset SO(1,6))$ or $1 \times SU(2) \subset SO(1,2) \times SO(4) \subset SO(1,6)$. Hence, it is a pp-manifold, a Brinkmann space with special Kähler flag or a metric product of a 3-dimensional Minkowski space with a 4-dimensional Riemannian Ricci-flat Kähler manifold.

In case that $\langle \varphi, \varphi \rangle \neq 0$ or $\langle \varphi, J \varphi \rangle \neq 0$ at least one of the following conformally changed metrics is Einstein: $\tilde{g} = \langle \varphi, \varphi \rangle^{-2}g$, $\tilde{g} = (Re \langle \varphi, J \varphi \rangle)^{-2} \tilde{g}$ or $\tilde{g} = (Im \langle \varphi, J \varphi \rangle)^{-2} \tilde{g}$. Therefore, we may assume that $(M,g)$ is an Einstein space if the Dirac current $V_\varphi$ is not lightlike. If $(M,g)$ is Ricci-flat, then $\varphi$ or $D \varphi$ are parallel with timelike Dirac current. Hence $(M,g)$ is the product of the timelike line $\mathbb{R}^{1,0}$ and a 6-dimensional Riemannian Ricci-flat Kähler manifold. If the scalar curvature is non-zero, then $(M,g)$ admits a Killing spinor $\varphi$. First, we consider the case that the Killing number $\lambda$ is imaginary. The spinor $\psi_{a,b} = a \varphi + b J \varphi$ is an imaginary Killing spinor with the same Killing number for all $a, b \in \mathbb{C}$. One easily checks that $V_{\psi_{a,b}} = (|a|^2 + |b|^2) \cdot V_\varphi$. Therefore, the Clifford multiplication with $V_\varphi$ acts as $\mathbb{C}$-linear isomorphism on $span_{\mathbb{C}} \{ \varphi, J \varphi \}$. This
implies that there are complex numbers \( \hat{a}, \hat{b} \) such that 
\[ V_{\bar{a}, \bar{b}} \cdot \psi_{\bar{a}, \bar{b}} = \pm \psi_{\bar{a}, \bar{b}}. \]
With Proposition 3.2 (after a possible change of time-orientation) we can conclude that \((M, g)\) is a Lorentzian Einstein-Sasaki space.

Now, let \( \varphi \) be a real Killing spinor on \((M, g)\) (i.e. the Killing number is real) with timelike Dirac current. Then the function \( Q_{\varphi} = \langle \varphi, \varphi \rangle^2 + g(V_{\varphi}, V_{\varphi}) \) is constant on \( M \). From Lemma 5.1 follows \( Q_{\varphi} = -|\langle \varphi, J\varphi \rangle|^2 \leq 0 \). In case \( Q_{\varphi} = 0 \), \((M, g)\) is a warped product \((\mathbb{R} \times F, -dt^2 + f(t)^2 h)\), where \((F, h)\) is a Riemannian manifold with parallel spinors and \( f(t) = c \exp(\pm t) \) (cf. \[ Boh03 \]). Hence, \((M, g)\) is locally conformal equivalent to a product of a timelike line with a 6-dimensional Riemannian Ricci-flat Kähler manifold. If \( Q_{\varphi} < 0 \), then \((M, g)\) is a warped product \((\mathbb{R} \times F, -dt^2 + f(t)^2 h)\), where \((F, h)\) is Riemannian manifold with real Killing spinors and \( f(t) = d \cosh(t + c) \). Hence, \((M, g)\) is locally conformal equivalent to a product of a timelike line and a 6-dimensional nearly Kähler, non-Kähler manifold (cf. \[ BFGK91 \]). \( \blacksquare \)

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