Quillen superconnections and connections on supermanifolds

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Abstract

Given a supervector bundle $E = E_0 \oplus E_1 \to M$, we exhibit a parametrization of Quillen superconnections on $E$ by graded connections on the Cartan-Koszul supermanifold $(M, \Omega(M))$. The relation between the curvatures of both kind of connections, and their associated Chern classes, is discussed in detail. In particular, we find that Chern classes for graded vector bundles on split supermanifolds can be computed through the associated Quillen superconnections.

1 Introduction

Quillen superconnections are ordinary connections (thought as differential operators on vector-valued forms), but defined on a vector bundle which carries a $\mathbb{Z}_2$–grading, $E = E_0 \oplus E_1$, and having odd total degree. They have been widely used both in Physics and Mathematics, ever since their appearance in [17] to construct a representative for the Chern character of the index of a family of elliptic operators.

Ne’eman and Sternberg [14 [15] gave a formulation of a Yang-Mills theory based on a superconnection (rather than conventional connections), including the scalar Higgs field as one of its components. That idea was very interesting, as one of the main problems of Yang-Mills theories is precisely the inclusion of the Higgs as one of the basic bosonic fields. However, the model obtained this way does not include a quadratic term to provide spontaneous symmetry breaking, so it contains unwanted massless fields. Further steps to remedy this state of affairs
have been given in [12, 18] (for an alternative treatment within non-commutative
geometry, see [13]).

A second source of interest in superconnections is string theory. $D$–branes,
joint with fundamental string states and ordinary field theoretical solitons, form
duality multiplets in the context of duality conjectures that explain how the
different existing superstring theories are perturbation expansions of a single
theory, called $M$–theory. These $D$–branes can be understood as topological
defects in the worldvolumes of higher dimensional unstable systems of branes,
such as brane-antibrane pairs. This unstability shows itself in the presence of
tachyons, which can not removed by the usual Gliozzi-Scherk-Olive (or GSO)
projection. Indeed, it has been suggested that these tachyon fields could be in-
terpreted as Higgs fields, and that they could be incorporated, again, within the
components of certain superconnections, constructed from the supervector bun-
dles determined by the Chan-Paton gauge bundles of the brane-antibrane system.
The Chern character of these bundles, can be used to relate their cohomology to
the $K$–theory group, allowing a topological classification of $D$–brane charges,
see [26]. We strongly recommend [16, 21] for a detailed account of further devel-
opments based on these ideas.

The $D$–branes mentioned above are associated to $(p + 1)$–submanifolds of a
10–dimensional spacetime with $0 \leq p \leq 9$. The question arises of how to trans-
late these ideas to the general case of superbranes moving on a supermanifold,
as Quillen superconnections are defined over ordinary (commutative) manifolds.
Rather than trying to extend the notion of Quillen superconnection to this set-
ing, we show here that they can be put on correspondence with Koszul connec-
tions on a certain class of supermanifolds. This class can be thought as those
split supermanifolds endowed with an odd-degree differential (a homological vec-
tor field), so they can be seen also as Gerstenhaber algebras associated to Lie
algebroids (see [24]). But, for the sake of simplicity, in this paper we will restrict
ourselves to the case of the well-known Cartan-Koszul supermanifold $(M, \Omega(M))$,
whose sheaf of superfunctions is simply the sheaf of (ordinary) differential forms
on a manifold, so the homological vector field is simply the exterior differential
(notice that these split supermanifolds are examples of differential graded mani-
folds [25, 20], and they carry an additional $\mathbb{Z}$–grading, known in Physics as the
ghost number grading). We hope that, in this way, the geometric meaning of the
constructions presented here will be clearer.

The contents of the paper are as follows. Section 2 serves the dual purpose
of setting up the main notations and making the paper self-contained (perhaps
at the risk of irritating the expert reader). Sections 3, 4, 5 review the main
properties of Quillen superconnections and Koszul graded connections on su-
permanifolds, paying particular attention to the construction of graded vector
bundles (over a supermanifold) and their relation to supervector bundles (over
the base of the supermanifold—an ordinary manifold). The remaining sections develop a parametrization of Quillen superconnections by means of graded connections on \((M, \Omega(M))\), and to study the relationship between their associated Chern classes.

2 Differential operators on graded modules

Throughout this paper, \(\mathbb{K}\) will denote either \(\mathbb{R}\) or \(\mathbb{C}\). Let \(G\) be a commutative group endowed with a morphism \(\epsilon : G \to \mathbb{Z}/2\mathbb{Z}\), and let \(A = \oplus_{g \in G} A^g\) be a \(G\)-graded commutative \(\mathbb{K}\)-algebra with identity element \(1_A\). Elements \(a \in A^g\) will be called homogeneous of degree \(|a| = g\). The commutativity in \(A\) means \(ab = (-1)^{\epsilon(|a|)\epsilon(|b|)}ba\), for all homogeneous \(a, b \in A\). But in this paper \(G\) will be either \(\{0\}\), \(\mathbb{Z}\) or \(\mathbb{Z}_2\), so we will simplify this notation by writing \(ab = (-1)^{ab}ba\). Note that any \(\mathbb{Z}\)-grading induces a corresponding \(\mathbb{Z}_2\)-grading by collecting even and odd elements. In this case, \(A = A^0 \oplus A^1\) is called a superalgebra (and the \(\mathbb{Z}\)-modules over it are called supermodules).

Suppose \(A\) is simply a \(G\)-graded \(\mathbb{K}\)-algebra (not necessarily commutative). It can be viewed as a Lie superalgebra when endowed with the supercommutator:

\[ [a, b] = ab - (-1)^{ab}ba, \]  

for all homogeneous \(a, b \in A\), extended to all of \(A\) by \(\mathbb{K}\)-bilinearity. Notice that a \(G\)-graded algebra \(A\) is commutative precisely when the supercommutator vanishes identically. The properties of the supercommutator \([\ ]\) defining the Lie superalgebra structure are, besides its \(\mathbb{K}\)-bilinearity (for homogeneous \(a, b, c \in A\)):

(a) Graded skew-symmetry: \([a, b] = -(-1)^{ab}[b, a]\),

(b) Graded Jacobi identity: \([a, [b, c]] = [[a, b], c] + (-1)^{ab}[b, [a, c]]\).

If \(N, P\) are finitely-generated \(A\)-modules, the set \(\text{Hom}_\mathbb{K}(N, P)\) naturally inherits a \(G\)-graded \(A\)-module structure. Thus, given a \(\phi \in \text{Hom}_\mathbb{K}(N, P)\) and an element \(v \in N\), we will have \(|\phi(v)| = |\phi| + |v|\). In particular, for each \(k \in \mathbb{N}\) we have \(G\)-graded \(A\)-modules \(\Omega^k(N) \subset \text{Hom}_\mathbb{K}(N \times k), \times N, A\), and \(\Omega^k(N; P) \subset \text{Hom}_\mathbb{K}(N \times k), \times N, P\), whose elements are the alternate ones. Defining \(\Omega^0(N) := A\), \(\Omega^0(N; P) := P\), and \(\Omega^{-k}(N) := \{0\} =: \Omega^{-k}(N; P)\) for \(k \in \mathbb{N}\) or \(-k > \text{rank} N\), we will write

\[ \Omega(N) = \bigoplus_{i \in \mathbb{Z}} \Omega^i(N) \text{ and } \Omega(N; P) = \bigoplus_{j \in \mathbb{Z}} \Omega^j(N; P). \]

Thus, \(\Omega(N; P)\) not only has a \(G\)-grading, but also a \(\mathbb{Z}\)-grading. The total degree of an element \(\phi \in \Omega^k(N; P)\) is the pair \((k, |\phi|) \in \mathbb{Z} \times G\). When necessary, we will indicate this total degree by the notation \(\phi \in \Omega^{(k, |\phi|)}(N; P)\).
Example 1. Let $\pi : E \to M$ be a smooth real vector bundle over the smooth manifold $M$, whose dual bundle will be denoted $E^*$. The algebra of smooth functions $C^\infty(M)$ is commutative in the usual sense ($G = \{0\}$), the space of smooth sections $\Gamma E$ is a $C^\infty(M)$–module, and the exterior bundle $\Gamma(\Lambda E)$ is both a $C^\infty(M)$–module and a $\mathbb{Z}$–graded commutative algebra, $\Gamma(\Lambda E) = \oplus_{i \in \mathbb{Z}} \Gamma(\Lambda^i E)$ (with $\Gamma(\Lambda^0 E) = C^\infty(M)$ and $\Gamma(\Lambda^{-i} E) = \{0\}$ if $i \in \mathbb{N}$).

Taking as the algebra $A = C^\infty(M)$, we can consider the $C^\infty(M)$–module $\mathcal{X}(M) = \text{Der}(C^\infty(M))$ of vector fields on $M$. In this context, the notation $\Omega^k(M; F)$ means that $F$ is another smooth vector bundle over $M$, so its elements (the $F$–valued differential forms on the manifold $M$) are of the form $\phi : \mathcal{X}(M) \times \cdots \times \mathcal{X}(M) \to F$, $C^\infty(M)$–linear and alternate.

If we consider $A = \Gamma(\Lambda E)$, an important $\mathbb{Z}$–graded $\Gamma(\Lambda E)$–module is the one of graded endomorphisms, $\text{End}_\mathbb{R}(\Gamma(\Lambda E)) = \oplus_{i \in \mathbb{Z}} \text{End}^i_{\mathbb{R}}(\Gamma(\Lambda E))$, as well as the $\Gamma(\Lambda E)$–modules that can be formed by taking tensor products with other vector bundles over $M$.

Remark 1. Note that any $a \in A^g$ can be viewed as an element $a \in \text{End}^g_{\mathbb{R}}(P)$ for any $G$–graded $A$–module $P$, acting as $a(v) = a \cdot v$, $v \in P$.

To define differential operators, we will need some facts about the algebraic structure of $\text{Hom}_\mathbb{K}(N, P)$, where $N, P$ are $G$–graded $A$–modules, with $A = \oplus_{g \in G} A^g$. First, note that if $D \in \text{Hom}_\mathbb{K}(N, P)$, and $a \in A$, then we can define $[D, a] \in \text{Hom}_\mathbb{K}(N, P)$ by putting, for all $v \in N$:

$$[D, a](v) := D(av) - (-1)^{aD}aD(v) \in P.$$  \hfill (2)

It is immediate to check that this bracket satisfies:

$$[[D, D'], a] = [D, [D', a]] - (-1)^{D'D'}[D', [D, a]],$$  \hfill (3)

for any pair $D, D' \in \text{Hom}_\mathbb{K}(N, P)$, and also:

$$[D, ab] = [D, a]b + (-1)^{aD}a[D, b],$$  \hfill (4)

for any $a, b \in A$.

Definition 1. Let $A$ be a $G$–graded commutative $\mathbb{K}$–algebra, and let $N, P$ be $G$–graded $A$–modules. A morphism $\Delta \in \text{Hom}_\mathbb{K}(N, P)$ is called a $(G$–graded) differential operator of order $\leq k$ (with $k \in \mathbb{N} \cup \{0\}$) if, for all $a_0, a_1, \ldots, a_k \in A$, the following holds:

$$[[\cdots [[\Delta, a_0], a_1], \cdots], a_k] = 0.$$  \hfill (5)

The set of all such operators will be denoted $\mathcal{D}_k(N; P)$ (or $\mathcal{D}_k(N)$ in the case $P = N$). It is clear that $\mathcal{D}_k(N; P)$ is an $A$–module, as for any $a, b \in A$, $\Delta \in \mathcal{D}_k(N; P)$:

$$[a\Delta, b] = a\Delta(b) - (-1)^{(a+\Delta)b}ba\Delta = a\Delta(b) - (-1)^{(a+\Delta)b+a}ab\Delta = a[\Delta, b].$$
Thus, we have a $\mathbb{Z}$–graded $A$–module $\mathcal{D}(N; P) := \bigoplus_{k \in \mathbb{Z}} \mathcal{D}_k(N; P)$, with the usual conventions for $k < 0$. To account for the $G$–grading, let us define

$$\mathcal{D}_k^p(N; P) := \mathcal{D}_k(N; P) \bigcap \text{Hom}_R^p(N; P).$$

**Example 2.** Let $E$ be a real vector bundle as in Example 1. A linear (Koszul) connection $\nabla$ on $E$ is an $\mathbb{R}$–bilinear mapping $\nabla: \mathfrak{X}(M) \times \Gamma E \to \Gamma E$ (with $\nabla(X, \sigma)$ denoted $\nabla_X \sigma$) such that $\nabla_X(f\sigma) = df(X)\sigma + f\nabla_X\sigma$, for all $f \in C^\infty(M)$. This can be written as $[\nabla, f](X)(\sigma) = df(X) \cdot \sigma$, or $[\nabla, f] = df$ as an $E$–valued 1–form. The mapping $\nabla$ can be extended to an operator $d\nabla : \Omega^p(M; E) \to \Omega^{p+1}(M; E)$ by defining (for $\phi \in \Omega^p(M; E)$, $X_0, \ldots, X_p \in \mathfrak{X}(M)$):

$$(d\nabla \phi)(X_0, \ldots, X_p) = \sum_{i=0}^p s_i(\nabla_{X_i} \phi)(X_0, \ldots, \hat{X}_i, \ldots, X_p) + \sum_{i<j} s_{i+j} \phi([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, X_p),$$

where, as usual, a hat over an index denotes that the corresponding term is omitted, and we have used the shorthand $s_j = (-1)^j$. Now, from $[\nabla, f] = df$ and the property (1) of the commutator (2), we get that $d\nabla$ must be a first-order differential operator on the $(M; \Omega)$–module $\Omega(M; E)$: $[d\nabla, \alpha, \beta] = 0$, for all $\alpha, \beta \in \Omega(M)$ (just write $\alpha = f_I dg_I$ for some multi-index $I = (i_1, \ldots, i_{|I|})$). Thus, we can characterize linear connections $\nabla$ on $E$ as the restrictions to $\Omega^1(M; E)$ of first-order differential operators on $\Omega(M; E)$ of degree 1, $D \in \mathcal{D}_1^1(\Omega(M; E))$, such that $[D, f] = df$, for all $f \in C^\infty(M)$. So, $D = d\nabla$.

A case of particular interest is obtained by taking $N = A$ in Definition 1

**Definition 2.** Let $A$ be a $G$–graded commutative $\mathbb{K}$–algebra, $P$ a $G$–graded $A$–module. A morphism $D \in \text{Hom}_g(A; P)$ is called a $(G$–graded $)P$–derivation on $A$, of $G$–degree $k$, if for all $a, b \in A$:

$$D(ab) = D(a)b + (-1)^{ak} aD(b) \in P.$$ 

If, moreover, $D$ satisfies $[D, a] = 0$, for all $a \in A$, we say that $D$ is an algebraic derivation.

The set of all $P$–derivations on $A$ is denoted $\text{Der}(A; P) = \bigoplus_{k \in G} \text{Der}^k(A; P)$. If $\Delta \in \mathcal{D}_1^1(A; P)$, then $\Delta - \Delta(1_A) \in \text{Der}(A; P)$, as it is readily checked. Thus, we have the decomposition of the space of first-order differential operators in this case: $\mathcal{D}_1^1(A; P) = \text{Der}(A; P) \oplus P$. When $P = A$, we simply write $\text{Der}A$, and refer to any $\delta \in \text{Der}A$ as a $(G$–graded $)$derivation of $A$. In particular, $\mathcal{D}_1^1(A) = \text{Der}A \oplus A$.

**Remark 2.** A case of particular interest appears when $Q$ is any $\mathbb{K}$–module and we form the left $A$–module $A \otimes Q$. If $\delta \in \text{Der}(A \otimes Q)$, and $a \in A$, in general we
have $[\delta, a] \in \text{End}(A \otimes Q)$. But if it happens that, for each $a \in A$, this morphism is of the form ‘multiplication by an element of $A$’, and we denote this element by $\overline{\delta}a$ (as it depends on both, $\delta$ and $a$), then $\overline{\delta} : A \to A$ is a derivation, as a consequence of (1):

$$\overline{\delta}(ab) = [\delta, ab] = [\delta, a]b + (-1)^{\alpha \delta} a[\delta, b] = \overline{\delta}(a)b + (-1)^{\alpha \delta} a\overline{\delta}(b).$$

In this case, we say that $\overline{\delta}$ is the derivation in $A$ induced by $\delta$.

**Example 3.** Let $E$ be a smooth real vector bundle over the manifold $M$.

(a) Given a linear connection $\nabla$ on $E$, its extension to a differential operator on $\Omega(M; E)$ (as in Example 2) is done in such a way that it acts as a derivation of degree 1.

(b) Given a $P \in \Omega^k(M; \text{End}_K(E))$, we can define a derivation of degree $k$ (which will be denoted also as $P$) by:

$$(P \sigma)(X_1, \ldots, X_{p+k}) = \frac{1}{p!k!} \sum_{\pi \in S_{p+k}} \text{sg}(\pi) P(X_{\pi(1)}, \ldots, X_{\pi(k)})(\phi(X_{\pi(k+1)}, \ldots, X_{\pi(p+k)})),$$

for any vector fields $X_1, \ldots, X_{p+k}$, and $\sigma \in \Omega^p(M; E)$ (here $\text{sg}(\pi)$ denotes the signature of the permutation $\pi$). In particular, if $P \in \Omega^1(M; E)$, then $P\sigma(X) = P(X)(\sigma) \in \Gamma(E)$. These are examples of algebraic derivations.

(c) Again, let $\nabla$ be a linear connection on $E$. Given any $a \otimes X \in \Gamma(\Lambda^a E \otimes TM)$, we have an endomorphism $\nabla_{a \otimes X} \in \text{End}^0(\Gamma(\Lambda E))$ defined by $\nabla_{a \otimes X}(b) := a \wedge \nabla_X(b)$. If $K \in \Gamma(\Lambda^a E \otimes TM)$ is an arbitrary element, we define $\nabla_K$ by its linear extension. This is a derivation on $\Gamma(\Lambda E)$ of $Z$–degree $a$.

(d) Given any $L \in \Gamma(\Lambda E \otimes E^*)$, it can be expressed as a sum of decomposable elements, each of them of the form $b \otimes \beta$, for some homogeneous section $b \in \Gamma(\Lambda^b E)$, and $\beta \in \Gamma(\Lambda^* E)$. For these, we define $i_{b \otimes \beta} : \Gamma(\Lambda E) \to \Gamma(\Lambda E)$ by putting $i_{b \otimes \beta}(a) = b \wedge i(\beta)(a)$, and then extend by linearity. In this way, we get an element of $\text{Der}^{b-1}\Gamma(\Lambda E)$.

**Remark 3.** An important consequence can be drawn from the preceding examples. Notice that once a linear connection $\nabla_0$ on $E$ has been chosen, any derivation $D \in \text{Der}^1\Omega(M; E)$ such that its induced derivation on $\Omega(M)$ is $\overline{D} = d$, decomposes as:

$$D = d\overline{\nabla}_0 + P,$$

for some $P \in \Omega^1(M; \text{End}_K(E))$ (see 3.8 in [9]). In particular, $d\nabla = d\overline{\nabla}_0 + P$. 


3 Quillen superconnections

In the examples of the preceding section, which were intended to illustrate the classical notions of differential geometry from the point of view of differential operators on modules, we chose a smooth vector bundle $E \to M$, whose sections are a module over the commutative algebra $C^\infty(M)$. In this way, we obtained a $\mathbb{Z}$–graded algebra $\Gamma(\Lambda E)$, but forgot the $G$–grading (which appears in Definitions 1 and 2) or, more precisely, took it to be trivial. Quillen’s definition of superconnection, takes full advantage of the $G \times \mathbb{Z}$–bigrading by considering a $\mathbb{K}$–supervector bundle $E = E_0 \oplus E_1$ over $M$. But, aside from this fact, the definition is formally the same as the classical one described in Example 2. In this section we give only the basic facts about Quillen superconnections that will be required to prove our main result. A detailed study of this notion can be found in the original paper [17] and in [3, 4].

Definition 3. A Quillen superconnection (or simply superconnection) on a $\mathbb{K}$–supervector bundle over $M$, $E = E_0 \oplus E_1$ is a derivation on $\Omega(M; E)$ (thus, a first-order differential operator), of odd total degree, which induces the exterior differential $d$ on $\Omega(M)$, that is:

$$D(\alpha \wedge \sigma) = d\alpha \wedge \sigma + (-1)^{\text{deg} \alpha} \alpha \wedge D\sigma,$$

(5)

for $\alpha \in \Omega(M)$ and $\sigma \in \Omega(M; E)$.

Remark 4. This definition is the one given in [17]. But note that condition (5) is equivalent to:

$$[D, \alpha](\phi) = d\alpha \wedge \phi,$$

in terms of the graded commutator (2).

As in the classical case, the difference between two superconnections is an $\text{End}(E)$–valued form on $M$. Indeed, it is an algebraic operator in the sense of Definition 2 as, if $D$ and $D'$ are superconnections, and $\alpha \in \Omega(M)$, then $[D - D', \alpha] = d\alpha - d\alpha = 0$. The following result, proved in [3], uses the identification mentioned in (b) of Example 3.

Proposition 3.1. The space of superconnections on $E$ is an affine space modeled on the vector space:

$$\Omega^-(M; \text{End}_\mathbb{K}(E)) := \bigoplus_{i+k, \text{odd}} \Omega^{(i,k)}(M; \text{End}_\mathbb{K}(E)).$$

Noticing that Definition 3 involves the total degree, we see that the space modeling Quillen superconnections is built from sections of the following vector
bundles:
\[ \bigoplus_{k \geq 0} \Gamma(\Lambda^{2k+1}T^*M \otimes \text{End}^0_{\mathbb{K}}(E)), \]
\[ \bigoplus_{k \geq 0} \Gamma(\Lambda^{2k}T^*M \otimes \text{End}^1_{\mathbb{K}}(E)), \]

where \( \text{End}^0_{\mathbb{K}}(E) = (E^*_0 \otimes E_0) \oplus (E^*_1 \otimes E_1) \) and \( \text{End}^1_{\mathbb{K}}(E) = (E^*_0 \otimes E_1) \oplus (E^*_1 \otimes E_0) \).

This fact, combined with Remark 3, shows that when the vector bundle \( E = M \times V \) is trivial, any superconnection can be written as \( D = d + P \), with \( P \in \Omega^-(M; \text{End}_{\mathbb{K}}(V)) \). This is the case for most applications in Physics, see \[1, 8, 14, 15, 22, 27\].

**Example 4.** Consider any Koszul connection \( \nabla \) on the vector bundle \( E \to M \), such that it preserves the \( \mathbb{Z}_2 \)-grading of \( E = E_0 \oplus E_1 \). Any superconnection \( D \in \text{Der}\Omega(M; E) \) of total degree 1, can be written, following Remark 3 as \( D = d^0 + P \), where \( P = A + L \in \Omega^0(M; \text{End}^1(E)) \oplus \Omega^1(M; \text{End}^0(E)) \). That means that \( A \) is just an element of \( \Gamma(\text{End}^1(E)) \), and, for any vector field \( X \in \mathcal{X}(M) \), we have \( L(X) \in \Gamma(\text{End}^0(E)) \). Thus, Quillen superconnections of total degree 1, can be parameterized by four tensor fields: \( T_1 \in \Gamma(T^*M \otimes E^*_0 \otimes E_0) \) and \( T_2 \in \Gamma(T^*M \otimes E^*_1 \otimes E_1) \) corresponding to \( L \), and \( T_3 \in \Gamma(E^*_0 \otimes E_0) \), \( T_4 \in \Gamma(E^*_1 \otimes E_1) \) corresponding to \( A \). This is the setting presented in \[17\].

**Definition 4.** The curvature of a superconnection \( D \) is the element \( R^D \in \text{End}_{\mathbb{K}}\Omega(M; E) \) given by:
\[ R^D = D^2 = \frac{1}{2}[D, D]. \]

The curvature \( R^D \) is thus a second-order differential operator on \( \Omega(M; E) \), of even total degree. Indeed, as for any \( \alpha \in \Omega(M) \) (applying \[4\]):
\[ \frac{1}{2}[[D, D], \alpha] = [D, [D, \alpha]] = [D, d\alpha] = d^2\alpha = 0, \]
we have that \( R^D \in \Omega^+(M; E) \) is an algebraic derivation in the sense of Definition 2.

**4 Connections on supermanifolds**

The basic idea underlying the definition of a supermanifold is the replacement of the commutative sheaf of differentiable functions \( C^\infty(M) \) of a manifold \( M \), by another one in which we can accommodate objects with a \( \mathbb{Z}_2 \)-grading. The axiomatic for such spaces was given by M. Rothstein in \[19\], here we follow the slightly modified version of \[2\].
**Definition 5.** A supermanifold of dimension $(m|n)$ and basis $(M, C^\infty(M))$ is given by a usual differential manifold $M$, with dimension $m$, and a sheaf $A$ of $\mathbb{Z}_2$-graded commutative $K$-algebras (called the structural sheaf) such that:

1. There is an exact sequence of sheaves
   
   $0 \to N \to A \xrightarrow{\sim} C^\infty(M) \to 0,$

   where $N$ is the sheaf of nilpotents of $A$ and $\sim$ is a surjective morphism of sheaves of graded commutative $K$-algebras (called the structural morphism).

2. $N/N^2$ is a locally free module with rank $n$ over $C^\infty(M) = A/N$, and $A$ is locally isomorphic, as a sheaf of $\mathbb{Z}_2$-graded $K$-commutative algebras, to the exterior bundle $\Lambda C^\infty(M)(N/N^2)$.

By definition, $A$ is locally isomorphic to the sheaf of sections of the exterior algebra bundle $\Lambda F \to M$, for some vector bundle $F \to M$ (canonically attached to $A$). When $A$ is globally isomorphic to $\Gamma(\Lambda F)$, the supermanifold is said to be split. In particular, any smooth real supermanifold is split, but this is not true for arbitrary holomorphic supermanifolds. Indeed, Koszul (see [7]) has proven that the existence of splittings of a given supermanifold is related to the existence of certain graded linear connections on it (see Definition 7 below). We will assume for the remainder of this paper that supermanifolds $(M, A)$ are split. Indeed, the complex structure will play no role from now on, so we will assume $K = \mathbb{R}$ for simplicity (and drop the corresponding subindexes), unless otherwise explicitly stated.

Supervector fields are defined as (graded) derivations of the structural sheaf $A = \Gamma(\Lambda E)$. The following result, involving the derivations (c) and (d) of Example 3, is obtained by adapting the classical proof of the decomposition of a derivation on $\Omega(M)$ (see [6, 10]), and gives us the structure of supervector fields.

**Theorem 4.1.** Let $\pi : F \to M$ be a smooth vector bundle over the differential manifold $M$. Let $\delta \in \text{Der} \Gamma(\Lambda F)$, and let $\nabla$ be a linear connection on $F^*$. Then, there exist unique tensor fields $K \in \Gamma(\Lambda F \otimes TM)$ and $L \in \Gamma(\Lambda F \otimes F^*)$, such that

$\delta = \nabla_K + i_L.$

**Remark 5.** By combining this theorem with Remark 2 we get a corresponding decomposition of the space of first-order superdifferential operators $\mathcal{D}_1(\Gamma(\Lambda F))$.

**Example 5.** If we consider the cotangent bundle of a manifold, $T^*M \to M$, and a linear connection on $TM, \nabla$, it is immediate to check that the Lie derivative with respect to a vector field $X \in \mathcal{X}(M)$, $\mathcal{L}_X \in \text{Der} \Gamma(\Lambda T^*M)$, admits the decomposition:

$\mathcal{L}_X = \nabla_X + i\nabla_{X+T(X,\cdot)}$,

where $T$ is the torsion of $\nabla$. 

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Differential superforms can be now defined on a supermanifold \((M, A)\) by (graded) duality; note that they have a natural structure of sheaves of right \(A\)–modules, denoted by \(\Omega^k(M, A)\), \(k \in \mathbb{Z}\) (with \(\Omega^{-p}(M, A) = \{0\}\) if \(p \geq 0\)). Thus, for instance, the elements of \(\Omega^1(M, A)\) are morphisms of sheaves of \(A\)–modules \(\text{Der}(A) \to A\). Other notions from the usual calculus on manifolds admit straightforward generalizations (see [6] for details).

**Example 6.** In particular, we have the graded differential, which we will denote \(d\), with bidegree \((1, 0)\). If \((\cdot, \cdot)\) denotes the pairing between supervector fields and superforms, it satisfies:

\[
\langle \delta; d\alpha \rangle = \delta(\alpha),
\]

for any \(0\)–superform \(\alpha \in A\), and \(\delta \in \text{Der}(A)\).

**Definition 6.** A graded vector bundle over \((M, A)\), is a locally free sheaf \(L\) of \(\mathbb{Z}_2\)-graded \(A\)-modules over \(M\), \(L = L_0 \oplus L_1\).

**Example 7.** Given a supermanifold \((M, A)\), the sheaf of derivations \(\text{Der}(A)\) is locally free and finitely generated (because of condition (2) in Definition [5]), and gives rise to a graded vector bundle over \((M, A)\), which is called the supertangent bundle.

The following result, which will be essential in what follows, explains the structure of such bundles. For its proof, see [11].

**Theorem 4.2.** Let \(L\) be a graded vector bundle over \((M, A)\), then there is a supervector bundle over \(M\), \(E = E_0 \oplus E_1\), such that \(L\) is isomorphic (as \(A\)-supermodules) to \(A \otimes \Gamma(E)\).

**Example 8.** Given any split supermanifold \((M, \Gamma(\Lambda F))\), we have the isomorphism \(\text{Der}\Gamma(\Lambda F) \simeq \Gamma(\Lambda F) \otimes \Gamma(TM \oplus F^*)\) (simply choose any connection \(\nabla\) on the vector bundle \(F \to M\), and make use of Theorem [4.1]).

**Example 9.** We will need also the notion of vector-valued \(k\)–superforms. These are morphisms of sheaves of \(A\)–supermodules \(S : \text{Der}(A) \times \cdots \times \text{Der}(A) \to L\), with \(L\) a graded vector bundle over \((M, A)\), and form the sheaf of \(A\)–modules \(\Omega^k((M, A); L)\). In particular, \(L\)–valued \(0\)–superforms are just elements of the sheaf of \(A\)–modules \(L\), and \(L\)–valued \(1\)–superforms can be identified with \(\Omega^1(M, A) \otimes L\).

Next, we define graded connections, mimicking the notion of classical Koszul connection (see [5], [7]).

**Definition 7.** Let \(L\) be a graded vector bundle over the supermanifold \((M, A)\). A graded connection on \(L\) is an even morphism of left \(A\)–modules from \(\mathbb{W} : \)
Der(\mathcal{A}) \rightarrow \text{End}(\mathcal{L})$, whose action is denoted \( \delta \mapsto \nabla \), satisfying the Leibniz property:

\[
\nabla_\delta (\alpha \phi) = \delta(\alpha) \phi + (-1)^{[\alpha][\delta]} \alpha \nabla_\delta \phi,
\]

where \( \delta \in \text{Der}(\mathcal{A}), \phi \in \mathcal{L} \) and \( \alpha \in \mathcal{A} \). In particular, a graded connection on the \( \mathcal{A} \)-module \( \mathcal{L} = \text{Der}(\mathcal{A}) \) is called a graded linear connection.

The graded curvature of \( \nabla \), \( R^\nabla \), is given by:

\[
R^\nabla (\delta, \delta') \phi = \nabla_\delta \nabla_{\delta'} \phi - (-1)^{[\delta][\delta']} \nabla_{\delta'} \nabla_\delta \phi - \nabla_{[\delta, \delta']} \phi,
\]

where \( \delta, \delta' \in \text{Der}(\mathcal{A}) \) and \( \phi \in \mathcal{L} \).

**Remark 6.** The Leibniz property above, can be rewritten as

\[
[\nabla, \alpha] = d\alpha,
\]

for any 0—superform \( \alpha \in \mathcal{A} \), in terms of the graded differential (see Example 6) and the graded commutator (2). Also, the graded curvature can be defined as

\[
R^\nabla (\delta, \delta') = [\nabla_\delta, \nabla_{\delta'}] - \nabla_{[\delta, \delta']},
\]

where now the graded commutators are those of \( \text{End}(\mathcal{L}) \) and \( \text{End}(\mathcal{A}) \), respectively, in the first and second terms of the right hand side (notice that the last bracket restricts to another one on \( \text{Der}(\mathcal{A}) \), that is: the bracket of two derivations is again a derivation, as can be readily checked).

## 5 The Cartan-Koszul supermanifold

From Remarks 4 and 6 it is clear that, although their definitions are formally similar, Quillen superconnections and graded connections are quite different objects: the module considered in the first case is \( \Omega(M; E) \), endowed with the total grading, while in the second, it is \( \Omega((M, \mathcal{A}); \mathcal{L}) \) with its natural \( \mathbb{Z} \times \mathbb{Z}_2 \)—bigrading. Of course, a reason for that difference is that (despite its name), Quillen superconnections are defined without reference to any supermanifold. However, the two notions can be brought closer if we restrict our attention to graded connections on a particular supermanifold, as we will now see.

Suppose we are given a supervector bundle over the ordinary manifold \( M \), \( E = E_0 \oplus E_1 \). Consider the split supermanifold whose sheaf of superalgebras is the module of differential forms on the base manifold, \( (M, \mathcal{A}) = (M, \Omega(M)) = (M, \Gamma(\Lambda T^* M)) \). This is called the Cartan-Koszul (or, simply, Cartan) supermanifold. Also, consider the graded vector bundle on \( (M, \Omega(M)) \) defined by \( E = E_0 \oplus E_1 \), that is, \( \mathcal{L} = \Omega(M) \otimes \Gamma(E) \) (recall Theorem 4.2). In this case, a graded connection \( \nabla \) on \( (M, \Omega(M)) \) gives, for each \( \delta \in \text{Der}\Omega(M) \), an even
morphism of $\Omega(M)$–modules $\nabla_\delta : \Omega(M) \otimes \Gamma(E) \to \Omega(M) \otimes \Gamma(E)$. We put on $\Gamma(E)$ a structure of $\Omega(M)$–module through the injection of $C^\infty(M)$–modules $\Gamma(E) \to 1 \otimes \Gamma(E) \subset \Omega(M) \otimes \Gamma(E)$, where 1 is the constant function on $M$, $1(x) = 1 \in \mathbb{R}$, for all $x \in M$. Then, if $\alpha \in \Omega(M)$, $\phi \in \Gamma(E)$, we can define $\alpha \phi = 1 \otimes \alpha \phi = \alpha \otimes \phi$, and:

$$\nabla_\delta(\alpha \phi) = \nabla_\delta(\alpha \otimes \phi)$$

$$= \delta(\alpha) \otimes \phi + (-1)^{|\delta||\alpha|} \alpha \otimes \nabla_\delta \phi$$

$$= \delta(\alpha) \phi + (-1)^{|\delta||\alpha|} \alpha \nabla_\delta \phi.$$  \hspace{1cm} (8)

Thus, we can think of a graded connection on $L = \alpha \phi$ as a map $\nabla \nabla \delta \in$ Leibniz rule: $\alpha \wedge \nabla \delta$ and to give the action of a graded connection generated sheaf of $\Omega(M)$ $\delta \in \Gamma(E)$, such that for each $\delta \in \text{Der}\Omega(M)$, it gives an even morphism of $\Omega(M)$–modules, $\nabla_\delta : \Gamma(E) \to \Gamma(E)$, satisfying the Leibniz rule: $\nabla_\delta(\alpha \phi) = \delta(\alpha) \phi + (-1)^{|\delta||\alpha|} \alpha \nabla_\delta \phi$.

The supervector fields on $(M, \Omega(M))$ are the derivations of $\Omega(M)$. Theorem \[1.1\] in this case states that, once a linear connection $\nabla$ on $TM$ is fixed, any $\delta \in \text{Der}\Omega(M)$ can be written as $\delta = \nabla_K + i_L$, for some vector-valued forms $K, L \in \Gamma(\Lambda^*TM \otimes TM)$. By the linearity properties $\nabla_{\alpha \otimes X} = \alpha \wedge \nabla_X$, $i_{\alpha \otimes X} = \alpha \wedge i_X$ (where $\alpha \in \Omega(M)$ and $X \in \mathfrak{X}(M)$), a basis for the locally-finitely generated sheaf of $\Omega(M)$–modules $\text{Der}\Omega(M)$, consists of derivations $\{\nabla_X, i_X\}$, and to give the action of a graded connection $\nabla$, it suffices to know $\nabla_{\nabla_X} \phi$, and $\nabla_{i_X} \phi$, for any $\phi \in \Gamma(E)$.

**Proposition 5.1.** Let $\nabla$ be a graded connection on the graded vector bundle $\mathcal{L} = \Omega(M) \otimes \Gamma(E)$ over the supermanifold $(M, \Omega(M))$, $\nabla$ be a connection on $TM$, and $\nabla^E$ be a compatible connection on $E$. Then, there is a tensor field $K = K_0 + K_1 \in \Gamma(\Lambda^*TM \otimes \Lambda^*TM \otimes \text{End}(E))$, with:

$$K_0 : TM \to \bigoplus_{i \geq 0} \Gamma(\Lambda^{2i}TM \otimes \text{End}^0(E)) \oplus \bigoplus_{i \geq 0} \Gamma(\Lambda^{2i+1}TM \otimes \text{End}^1(E)),$$

$$K_1 : TM \to \bigoplus_{i \geq 0} \Gamma(\Lambda^{2i+1}TM \otimes \text{End}^0(E)) \oplus \bigoplus_{i \geq 0} \Gamma(\Lambda^{2i}TM \otimes \text{End}^1(E)),$$

and such that:

$$\begin{align*}
\nabla_{\nabla_X} \phi &= \nabla_\delta_X \phi + K_0(X; \phi), \\
\nabla_{i_X} \phi &= K_1(X; \phi).
\end{align*}$$  \hspace{1cm} (9)

Moreover the connection is completely determined by these two tensor fields and the connections $\nabla$, $\nabla^E$.

**Proof.** It is clear that $K_0$ and $K_1$ are determined by (9); in turn, once $K_0$, $K_1$, and $\nabla$, $\nabla^E$ are given, $\nabla$ is uniquely characterized. Thus, we only need to prove
that $K_0, K_1$ are tensor fields. We will show this for $K_0$ (the computations in the case of $K_1$ are analogous). For any $f \in C^\infty(M)$, $X \in \mathcal{X}(M)$, and $\phi \in \Gamma(E)$:

$$K_0(fX; \phi) = \nabla_f \nabla X \phi - f \nabla^E X \phi$$

Also, applying (8):

$$K_0(X; f\phi) = \nabla X (f\phi) - f \nabla^E X (f\phi)$$

As an immediate consequence, we get the structure of the space of graded connections on the Cartan-Koszul supermanifold.

**Proposition 5.2.** The space of connections on the graded vector bundle $\mathcal{L} = \Omega(M) \otimes \Gamma(E)$, over the supermanifold $(M, \Omega(M))$, is an affine space modeled on:

$$\Gamma(T^*M \otimes \Lambda^2 T^*M \otimes \text{End}(E)).$$

The similarity between this space and the one modeling superconnections (see (6)), will allow us to associate a Quillen superconnection to any graded connection, in the next Section. For this, we will need some facts about the covariant graded differential associated to a graded connection.

Given the supermanifold $(M, \Omega(M))$, and a graded vector bundle $\mathcal{L} = \Omega(M) \otimes \Gamma(E)$ over it, we can construct the $\mathcal{L}$—valued superforms, $\Omega((M, \Omega(M)); \mathcal{L})$, and their derivations, as in Definition 2. Thus, if $\mathcal{D} \in \text{Der}\Omega((M, \Omega(M)); \mathcal{L})$, there exists a graded derivation, $\mathcal{D} : \Omega(M, \Omega(M)) \to \Omega(M, \Omega(M))$, such that:

$$[\mathcal{D}, \lambda] = \mathcal{D}(\lambda).$$

As $\Omega((M, \Omega(M)); \mathcal{L}) \simeq \Omega(M, \Omega(M)) \otimes \mathcal{L}$, any $\mathcal{D} \in \text{Der}\Omega((M, \Omega(M)); \mathcal{L})$ is characterized by its associated derivation, $\overline{\mathcal{D}}$, and its action $\mathcal{D}(\phi)$, for $\phi \in \Gamma(E)$. Then, it is extended to all of $\Omega((M, \Omega(M)); \mathcal{L})$ as a derivation of $\mathbb{Z} \times \mathbb{Z}_2$—bidegree $(1, 0)$.  

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**Definition 8.** Given a graded connection $\nabla$ on the graded vector bundle $\mathcal{L} = \Omega(M) \otimes \Gamma(E)$ over $(M, \Omega(M))$, its covariant graded differential is the graded derivation $d^\nabla \in \text{Der}\Omega((M, \Omega(M)); \mathcal{L})$ given by

$$d^\nabla = d$$

where $d$ is the graded differential.

As a consequence of Proposition 5.2, every covariant graded differential can be written locally as $d^\nabla = d + P$, where $P \in \Omega((M, \Omega(M)); \text{End}(\mathcal{L}))$ has $\mathbb{Z} \times \mathbb{Z}_2$–bidegree $(1, 0)$. Indeed, in view of the isomorphisms $\mathcal{L} \simeq \Omega(M) \otimes \Gamma(E)$, and $\Omega((M, \Omega(M)); \text{End}(\mathcal{L})) \simeq \Omega(M, \Omega(M)) \otimes \text{End}(\mathcal{L})$, we can consider $P \in \Omega((M, \Omega(M)); \text{End}(E))$. These computations can be readily generalized to any split supermanifold $(M, \Gamma(\Lambda F))$, where $F \to M$ is a vector bundle over $M$. Theorems 4.1 and 4.2 can be applied in this case, and the following results, whose proof is obvious, hold (we will apply them in Section 7 to the computation of the Chern classes of graded vector bundles over split supermanifolds).

**Proposition 5.3.** The space of connections on a graded vector bundle $\mathcal{L}$, over the supermanifold $(M, \Gamma(\Lambda F))$, is an affine superspace modeled on:

$$\Omega^{(1, +)}((M, \Gamma(\Lambda F)); \text{End}(E)) = \bigoplus_{k \in \mathbb{N} \cup \{0\}} \Omega^{(1, 2k)}((M, \Gamma(\Lambda F)); \text{End}(E)),$$

where $E = E_0 \oplus E_1$ is a supervector bundle over $M$ such that $\mathcal{L} \simeq \Gamma(\Lambda F) \otimes \Gamma(E)$.

In particular, any graded connection $\nabla$ on $\mathcal{L}$ over $(M, \Gamma(\Lambda F))$, induces a graded covariant differential $d^\nabla$ which is an operator on $\Omega((M, \Omega(M)); E)$ of $\mathbb{Z} \times \mathbb{Z}_2$–bidegree $(1, 0)$, and any such operator can be locally written as

$$d^\nabla = d + P,$$

for some $P \in \Omega^{(1, +)}((M, \Gamma(\Lambda F)); \text{End}(E))$.  

(10)

**Proposition 5.4.** The $\mathbb{Z} \times \mathbb{Z}_2$–bigraded algebra of $\text{End}(E)$–valued superforms, $\Omega((M, \Gamma(\Lambda F)); \text{End}(E))$, is isomorphic to the $\mathbb{Z} \times \mathbb{Z}_2$–bigraded algebra of operators $\mathcal{F} \in \text{End}\Omega((M, \Gamma(\Lambda F)); E)$ which are $\Omega(M, \Gamma(\Lambda F))$–linear, that is:

$$\mathcal{F}(\mu(\lambda \otimes \phi)) = (-1)^{[\lambda [\mu]} \mu \otimes \mathcal{F}(\lambda \otimes \phi),$$

for every $\lambda, \mu \in \Omega(M, \Gamma(\Lambda F))$, and $\phi \in \Gamma(E)$.

**Example 10.** Given a graded connection $\nabla$ on $\mathcal{L} = \Gamma(\Lambda F) \otimes \Gamma(E)$ over $(M, \Gamma(\Lambda F))$, if $Q \in \Omega((M, \Gamma(\Lambda F)); \text{End}(E))$, then $[d^\nabla, Q]$ is $\Omega(M, \Gamma(\Lambda F))$–linear as a graded operator on $\Omega((M, \Gamma(\Lambda F)); E)$. Indeed, if we take any elements $\lambda \in \Omega(M, \Gamma(\Lambda F))$, $s \in \Omega((M, \Gamma(\Lambda F)); E)$, then it is clear that:

$$Q(\lambda s) = (-1)^{\lambda Q} \lambda Q(s).$$
Therefore (noticing that the $\mathbb{Z} \times \mathbb{Z}_2$-bidegree of $d^\nabla$ is $(1,0)$, the same as that of $d$):

\[
[d^\nabla, Q](\lambda s) = d^\nabla (Q(\lambda s)) - (-1)^{Qd}Q(d^\nabla (\lambda s)) \\
= (-1)^{\lambda Q}d^\nabla (\lambda Q(s)) - (-1)^{Qd}Q\left( (d\lambda)s + (-1)^{\lambda d}\lambda d^\nabla s \right) \\
= \lambda \left( (-1)^{\lambda(Q+\nabla)}d^\nabla (Q(s)) - (-1)^{Qd+\lambda(Q+\nabla)}Q(d^\nabla s) \right) \\
+ (-1)^{\lambda Q}d\lambda Q(s) - (-1)^{\lambda Q}d\lambda Q(s) \\
= (-1)^{\lambda(Q+\nabla)}\lambda [d^\nabla, Q](s).
\]

6 Quillen superconnections induced by graded connections

Let $E = E_0 \oplus E_1 \to M$ be a supervector bundle on $M$, and consider the graded vector bundle $L = \Omega(M) \otimes \Gamma(E)$ over the supermanifold $(M, \Omega(M))$. The main results of this paper, are based on the following remarks.

1. Differential forms on $M$ with values on $E$ can be identified with $L$-valued graded differential 0-forms on $(M, \Omega(M))$, $\Omega^0((M, \Omega(M)); L)$, through (see Example 9):

\[
\Omega(M; E) \simeq \Omega(M) \otimes \Gamma(E) = L \simeq \Omega^0((M, \Omega(M)); L).
\]

Explicitly, we will write $\alpha \otimes \phi \mapsto \alpha \phi$, where $\alpha \in \Omega(M)$ and $\phi \in \Gamma(E)$.

2. If $\nabla$ is a graded connection on the graded vector bundle $L = \Omega(M) \otimes \Gamma(E)$, over the supermanifold $(M, \Omega(M))$, the covariant graded differential associated to $\nabla$ acts on $E$-valued 0-superforms:

\[
d^\nabla : \Omega^0((M, \Omega(M)); E) \to \Omega^1((M, \Omega(M)); E)
\]

Explicitly, if $\alpha \phi \in \Omega^0((M, \Omega(M)); E)$, then $d^\nabla(\alpha \phi)$ is defined by

\[
\langle \delta; d^\nabla(\alpha \phi) \rangle = \nabla_\delta (\alpha \phi) = \delta(\alpha) \phi + (-1)^{|\delta||\alpha|} \alpha \nabla_\delta \phi \in \Omega^0((M, \Omega(M)); E),
\]

for any $\delta \in \text{Der}\Omega(M)$.

3. The usual exterior differential on $M$, $d$, is a derivation of $\Omega(M)$. Thus (as in Example 3), we can define the insertion (or evaluation) operator associated to it:

\[
\iota \alpha : \Omega^1((M, \Omega(M)); E) \to \Omega^0((M, \Omega(M)); E).
\]
**Definition 9.** The composition of the previous three maps define an operator
\[ D^\nabla : \Omega(M; E) \to \Omega(M; E), \]
given by:
\[ D^\nabla (\alpha \otimes \phi) = \nabla_d(\alpha\phi) \in \Omega^0((M, \Omega(M)); E) \simeq \Omega(M; E). \]

**Theorem 6.1.** *The operator so defined, \( D^\nabla \), is a Quillen superconnection.*

*Proof.* First notice that, since the graded connection is an even map, and the \( \mathbb{Z} \)-degree of the exterior derivative is 1, the \( \mathbb{Z}_2 \)-degree of \( \nabla_d(\alpha\phi) \) is
\[ |\nabla_d(\alpha\phi)| = 1 + |\alpha\phi|. \]
Therefore \( D^\nabla \) is of odd total degree. Moreover, \( D^\nabla \) is a Quillen superconnection, because
\[ D^\nabla (\alpha \wedge (\beta \otimes \phi)) = \nabla_d(\alpha \wedge (\beta \phi)) \]
\[ = d\alpha \wedge (\beta \phi) + (-1)^{|\alpha|} \alpha \wedge \nabla_d(\beta \phi) \]
\[ = d\alpha \wedge (\beta \phi) + (-1)^{|\alpha|} \alpha \wedge D^\nabla (\beta \otimes \phi), \]
and this is (5) in Definition 3. \( \square \)

Not all Quillen superconnection can be generated in this way, but we can prove the following result.

**Theorem 6.2.** *Any Quillen superconnection, \( D \), can be written as*
\[ D = D^\nabla + N, \]
*where*

1. \( \nabla \) is a graded connection on the graded vector bundle \( \mathcal{L} = \Omega(M) \otimes \Gamma(E) \) on the supermanifold \((M, \Omega(M))\),
2. \( N \in \text{End}^1(E) \).

Moreover, the decomposition is unique in the following sense: if \( \nabla' \) is another connection and \( N' \) another tensor such that:
\[ D = D^\nabla' + N', \]
then, \( D^\nabla = D^\nabla' \), and \( N = N' \).
Proof. Let us recall (see Example 5) that in a local coordinate system \( \{x^k\}_{1 \leq k \leq \dim M} \) on \( M \),

\[
d = \mathcal{L}_\text{id} = dx^k \land \mathcal{L}_{\partial_k}
\]

\[
= dx^k \land (\nabla_{\partial_k} + i\nabla_{\partial_k} + T(\partial_k, \cdot))
\]

Therefore, according to Proposition 5.1, for \( \phi \in \Gamma(E) \):

\[
\nabla \phi = dx^k \land \left( \nabla_{\partial_k} \phi + \nabla_{i\nabla_{\partial_k} + T(\partial_k, \cdot)} \phi \right)
\]

\[
= dx^k \land \left( \nabla^E_{\partial_k} \phi + K_0(\partial_k; \phi) + K_1(\nabla \partial_k + T(\partial_k, \cdot); \phi) \right)
\]

\[
= d\nabla \phi + K_0^a(\partial_k; \phi) + \widetilde{K}_1(\partial_k; \phi),
\]

where \( K_0^a \) denotes the antisymmetrization of \( K_0 \), and \( \widetilde{K}_1 \) is given by the expression \( \widetilde{K}_1(\partial_k; \phi) = dx^k \land K_1(\nabla \partial_k + T(\partial_k, \cdot); \phi) \). Let us check that they are tensor fields. If \( \{y^\ell\}_{1 \leq \ell \leq \dim M} \) is another set of coordinate functions, then, by the change of coordinates formulae

\[
\partial_k = \frac{\partial y^\ell}{\partial x^k} \partial_{y^\ell}, \quad dx^k = \frac{\partial x^k}{\partial y^\ell} dy^\ell,
\]

we get:

\[
dx^k \land K_0(\partial_k; \phi) = \frac{\partial x^k}{\partial y^\ell} dy^\ell \land \frac{\partial y^m}{\partial x^k} K_0(\partial_{y^m}; \phi)
\]

\[
= \frac{\partial x^k}{\partial y^\ell} \frac{\partial y^m}{\partial x^k} dy^\ell \land K_0(\partial_{y^m}; \phi)
\]

\[
= \delta^m_{\ell} dy^\ell \land K_0(\partial_{y^m}; \phi)
\]

\[
= dy^\ell \land K_0(\partial_{y^\ell}; \phi).
\]

The computation needed for proving that \( \widetilde{K}_1 \) is also a tensor field, is longer but equally straightforward.

Now, we can readily check that the operations \( K_0 \mapsto K_0^a \) and \( K_1 \mapsto \widetilde{K}_1 \) are surjective. For instance, let \( T \in \Gamma(\Lambda^{2k+3}T^*M) \) be given, and let us write it locally as:

\[
T = T_{i_1\cdots i_{2k+3}} dx^{i_1} \land \cdots \land dx^{i_{2k+3}}.
\]

To find a \( K \in \Gamma(T^*M \otimes \Lambda^{2k+1}T^*M) \) such that \( T = dx^p \land K(\nabla \partial_p + T(\partial_p, \cdot); \phi) \), we also express it locally, as \( K = K_{j_1\cdots j_{2k+1}} dx^{j_1} \land \cdots \land dx^{j_{2k+1}} \otimes a_k dx^k \), and write a system of equations for the unknowns \( K_{j_1\cdots j_{2k+1}}, a_k, \Gamma^r_{pq} \), where \( \Gamma^r_{pq} \) are the Christoffel symbols of the linear connection \( \nabla \) on \( TM \). To this end, note:

\[
\nabla \partial_p + T(\partial_p, \cdot) = \Gamma^r_{qp} \partial_r \otimes dx^q + 2(\Gamma^r_{pq} - \Gamma^r_{qp}) \partial_r \otimes dx^q = (2\Gamma^r_{pq} - \Gamma^r_{qp}) \partial_r \otimes dx^q.
\]
Call \( B_{pq}^k = 2\Gamma_{pq}^r - \Gamma_{qp}^r \). Then (denoting by \([ab]\) the antisymmetrization in the indexes \(a, b\)):

\[
dx^p \wedge K(\nabla_p + T(\partial_p, \cdot); \phi) = K_{j_1 \cdots j_{2k+1}} a_k B_{[pq]}^k dx^p \wedge \cdots \wedge dx^{j_{2k+1}} \wedge dx^q = K_{j_1 \cdots j_{2k+1}} a_k B_{[pq]}^k dx^{j_1} \wedge \cdots \wedge dx^{j_{2k+1}} \wedge dx^q \wedge dx^p.
\]

Thus, we are led to the systems of equations

\[
T_{i_1 \cdots i_{2k+3}} = K_{i_1 \cdots i_{2k+1}} a_k B_{[i_{2k+2}i_{2k+3}]}^k,
\]

which clearly admits a solution.

On the other hand, if:

\[
K_0 \in \Gamma(T^*M \otimes \Lambda^{2k}T^*M \otimes \text{End}^0(E)), \quad K_0^a \in \Gamma(\Lambda^{2k+1}T^*M \otimes \text{End}^0(E)),
\]

\[
K_0 \in \Gamma(T^*M \otimes \Lambda^{2k+1}T^*M \otimes \text{End}^1(E)), \quad K_0^a \in \Gamma(\Lambda^{2k+2}T^*M \otimes \text{End}^1(E)),
\]

\[
K_1 \in \Gamma(T^*M \otimes \Lambda^{2k}T^*M \otimes \text{End}^1(E)), \quad \tilde{K}_1 \in \Gamma(\Lambda^{2k+2}T^*M \otimes \text{End}^1(E)),
\]

\[
K_1 \in \Gamma(T^*M \otimes \Lambda^{2k+1}T^*M \otimes \text{End}^0(E)), \quad \tilde{K}_1 \in \Gamma(\Lambda^{2k+3}T^*M \otimes \text{End}^0(E)).
\]

So, by comparing these tensor fields with those that model Quillen superconnections (see (5)), we see that all of them can be obtained this way, except one. The term that never appears is \( \Gamma(\Lambda^0T^*M \otimes \text{End}^1(E)) = \Gamma(\text{End}^1(E)) \). This is why we need to add an \( N \in \text{End}^1(E) \) to get an arbitrary Quillen superconnection. \( \square \)

**Example 11.** Quillen’s paper [17] deals mainly with the particular case of superconnections \( D \) (on the superbundle \( E = E_0 \oplus E_1 \to M \)) of total degree not only odd, but exactly 1. As seen in Example 4, to characterize one of these superconnections we need a connection \( \nabla^E \) on the vector bundle \( E \to M \), and an odd vector-valued form \( P = A + L \in \Omega^0(M; \text{End}^1(E)) \oplus \Omega^1(M; \text{End}^0(E)) \). In view of the theorem above, all of these superconnections are induced by graded connections on the graded vector bundle \( \mathcal{L} = \Omega(M) \otimes \Gamma(E) \), over the supermanifold \( (M, \Omega(M)) \). It suffices to take \( N = A, K_0 = L, \) and \( K_1 = 0 \).

The next result, which is an immediate corollary of the preceding Theorem, tells us when two graded connections induce the same Quillen superconnections.

**Proposition 6.3.** \( D^\nabla = D^{\nabla'} \) if and only if

1. \( (K_0 - K_0')^a = 0 \) and,

2. \( \tilde{K}_1 - K_1' = 0 \).

There is a nice relation between the curvature of a graded connection and that of its associated Quillen superconnection, involving the exterior differential of \( M \), which is expressed in the following result.
Theorem 6.4. For any graded connection, $\nabla$:

$$R^{D\nabla} = \frac{1}{2} R^{\nabla}(d, d).$$

Proof. By definition, for any $\phi \in \Omega^0((M, \Omega(M)); E) \simeq \Gamma(E)$,

$$R^{\nabla}(d, d)\phi = \nabla_d \nabla_d \phi - (-1)^{|d||d|} \nabla_d \nabla_d \phi - \nabla_{[d,d]}\phi,$$

but $|d| = 1$ and $[d,d] = 2d^2 = 0$, so this gives:

$$R^{\nabla}(d, d)\phi = 2 \nabla^2_d \phi.$$

The extension to all of $\Omega^k((M, \Omega(M)); E)$ is then immediate. \qed

A straightforward consequence is the following.

Corollary 6.5. If $D = D\nabla + N$, then:

$$R^{D} = \frac{1}{2} R^{\nabla}(d, d) + \nabla_d N + N^2.$$

Example 12. Returning to the case presented in Example 11 for a complex superbundle $E = E_0 \oplus E_1 \to M$, if $D$ is a Quillen superconnection given by $D = d\nabla^E + A + L$, we have seen that it can be expressed as $D = D\nabla + A$, where $\nabla$ is defined by:

$$\nabla_X \phi = \nabla^E_X \phi + L(X; \phi)$$

$$\nabla_{1X} \phi = 0,$$

for any $\phi \in \Gamma(A) \simeq \Omega^0((M, \Omega(M)); E)$, with $L \in \Omega^1(M; \text{End}^0(E))$. By the preceding corollary, $R^{D} = \frac{1}{2} R^{\nabla}(d, d) + \nabla_d A + A^2$. Moreover, in the particular case of $A = 0$, and $L$ of the form:

$$L = i \begin{pmatrix} 0 & u^* \\ u & 0 \end{pmatrix},$$

with $u : E_0 \to E_1$, and $u^*$ its adjoint with respect to some metric, we have $D = D\nabla$, and so $R^{D} = \frac{1}{2} R^{\nabla}(d, d)$. The complex scalar field $u$ is identified, in Physics, with the tachyon field condensing between a brane-antibrane system.
7 Chern classes of graded vector bundles

Throughout this Section, \((M, \Gamma(\Lambda F))\) will be a split supermanifold, and \(\mathcal{L} = \Gamma(\Lambda F) \otimes \Gamma(E)\) a graded vector bundle over it, with \(E = E_0 \oplus E_1 \to M\) a supervector bundle over \(M\).

We want to associate to any graded connection \(\nabla\) on \(\mathcal{L}\), a set of closed superforms. As in the case of Quillen superconnections, this can be done with the aid of the supertrace: if \(A\) is any \(G\)-graded \(\mathbb{K}\)-algebra, a supertrace on \(A\) is any \(\mathbb{K}\)-linear form on \(A\) such that it vanishes on the supercommutator \([\nabla, \nabla]\).

The supertrace is not unique, in general, but for endomorphisms of a supervector bundle, \(A = \text{End}(E)\), a canonical supertrace \(\text{Str}\) can be defined as follows: any \(T \in \text{End}(E)\) decomposes as \(T = T_0 \oplus T_1 \in \text{End}^0(E) \oplus \text{End}^1(E)\), with a further decomposition \(T_0 = T_0^0 \oplus T_0^1 \in (E_0^0 \otimes E_0) \oplus (E_0^1 \otimes E_1)\). Then:

\[
\text{Str}(T) = \begin{cases} 
\text{Tr}(T_0^0) - \text{Tr}(T_0^1), & \text{if } T = T_0 \\
0, & \text{if } T = T_1
\end{cases}
\]

This linear form on the fibers of the supervector bundle \(\text{End}(E)\), can be seen as a \(\Omega^0(M, \Gamma(\Lambda F))\)-module morphism \(\Omega^0((M, \Gamma(\Lambda F)); \text{End}(E)) \to \Omega^0(M, \Gamma(\Lambda F))\), which can be extended to a \(\Omega(M, \Gamma(\Lambda F))\)-module morphism:

\[
\text{Str} : \Omega((M, \Gamma(\Lambda F)); \text{End}(E)) \to \Omega(M, \Gamma(\Lambda F)).
\]

That property of \(\Omega(M, \Gamma(\Lambda F))\)-linearity, guarantees that we can take the supertrace \([d\nabla, Q]\) (recall Example 10).

**Proposition 7.1.** For \(\nabla\) any graded connection on \(\mathcal{L} = \Gamma(\Lambda F) \otimes \Gamma(E)\), and \(Q \in \Omega^k((M, \Gamma(\Lambda F)); \text{End}(E))\), the following holds:

\[
\text{Str}[d\nabla, Q] = d\text{Str}(Q).
\]

**Proof.** Working locally, by the decomposition \([10]\), we can write \(d\nabla = d + P\). Put also \(Q = \alpha T\), where \(\alpha \in \Omega^k(M, \Gamma(\Lambda F))\) and \(T \in \Gamma(\text{End}E)\). Then, as \(\text{Str}\) vanishes on supercommutators:

\[
\text{Str}[d\nabla, Q] = \text{Str}[d + P, \alpha T] = \text{Str}[d, \alpha T] + \text{Str}[P, Q] = \text{Str}(d\alpha T).
\]

Finally, again because of the \(\Omega(M, \Gamma(\Lambda F))\)-linearity, \(d\) commutes with \(\text{Str}\), giving the statement. \(\square\)

**Definition 10.** Let \(\nabla\) be a graded connection on the graded vector bundle \(\mathcal{L} = \Gamma(\Lambda F) \otimes \Gamma(E)\) over the supermanifold \((M, \Gamma(\Lambda F))\). For any \(k \in \mathbb{N}\), the characteristic superform (or Chern superform) of \(\nabla\), of degree \(2k\), is the superform on \((M, \Gamma(\Lambda F))\) given by \(\text{Str}\left(d\nabla^{2k}\right)\). For the sake of simplicity, sometimes it will be denoted by \(\text{Str}(\nabla^{2k})\).
The next result is proved following the ideas presented in [17], but with some modifications to take into account that we are now dealing with vector-valued superforms (not just forms), and that the natural grading in this context is given by the \(\mathbb{Z} \times \mathbb{Z}_2\)-degree, not the total degree.

**Theorem 7.2.** Let \(\nabla\) be a graded connection on the graded vector bundle \(\mathcal{L} = \Gamma(\Lambda F) \otimes \Gamma(E)\) over \((M, \Gamma(\Lambda F))\), and let \(k \in \mathbb{N}\). Then:

1. The characteristic forms \(\text{Str}(\nabla^{2k})\) are closed even superforms.
2. If \(\nabla_0\) and \(\nabla_1\) are two graded connections on \(\mathcal{L}\), then the superforms \(\text{Str}(\nabla_0^{2k})\) and \(\text{Str}(\nabla_1^{2k})\) lie in the same graded De Rham cohomology class.

**Proof.**

(1) By applying proposition 7.1 we get:

\[
d\text{Str}(\nabla^{2k}) = \text{Str} \left[ d\nabla, d\nabla^{2k} \right] = \text{Str} \left[ d\nabla, \left( d\nabla^2 \right)^k \right] = 0,
\]

because \(d\nabla^{2k}\) has bidegree \((2k, 0)\), so

\[
\left[ d\nabla, d\nabla^{2k} \right] = (d\nabla)^{2k+1} - (d\nabla)^{2k+1} = 0 \quad (11)
\]

(this can be thought of as a form of Bianchi identity).

(2) Given \(\nabla_0\) and \(\nabla_1\), define for \(t \in [0, 1]\) the family of graded connections \(\nabla_t = (1 - t)\nabla_0 + t\nabla_1\). Then:

\[
d\nabla_t = (1 - t)d\nabla_0 + td\nabla_1 = d\nabla_0 + tP,
\]

with \(P \in \Omega^{(1, +)}((M, \Gamma(\Lambda F)); \text{End}(E))\). It is immediate that

\[
\frac{d}{dt} \left( d\nabla_t \right)^2 = \frac{d}{dt} \left( \frac{1}{2} \left[ d\nabla_t, d\nabla_t \right] \right)
= \frac{d}{dt} \left( \frac{1}{2} \left[ d\nabla_0 + tP, d\nabla_0 + tP \right] \right)
= \left[ d\nabla_0 + tP, P \right]
= \left[ d\nabla_t, \frac{d}{dt} d\nabla_t \right],
\]

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so, substituting this identity, using (11) for $d^{\nabla t}$, and Proposition 7.1:

$$\frac{d}{dt} \text{Str} \left( d^{\nabla_t} \right)^k = \text{Str} \left( k \left( d^{\nabla_t} \right)^{k-1} \frac{d}{dt} d^{\nabla_t} \right)$$

$$= \text{Str} \left( k \left[ d^{\nabla_t}, (d^{\nabla_t})^{k-1} \frac{d}{dt} d^{\nabla_t} \right] \right)$$

$$= d \text{Str} \left( k \left( d^{\nabla_t} \right)^{k-1} \right)$$

$$= d \text{Str} \left( k \left( d^{\nabla_t} \right)^{k-1} P \right).$$

Finally, integrating on the interval $[0, 1]$, we get:

$$\text{Str} \left( d^{\nabla_1} \right)^{2k} - \text{Str} \left( d^{\nabla_0} \right)^{2k} = d \int_0^1 \text{Str} \left( k \left( d^{\nabla_t} \right)^{k-1} P \right) \, dt.$$

Thus, $\text{Str}(\nabla^{2k}_0)$ and $\text{Str}(\nabla^{2k}_1)$ lie in the same graded De Rham cohomology class.

\[\square\]

**Definition 11.** For any $k \in \mathbb{N}$, the Chern class of degree $2k$ of the graded vector bundle $L$ over the supermanifold $(M, \Gamma(\Lambda F))$, denoted $c_h(L)$, is the cohomology class of any superform $\text{Str}(\nabla^{2k})$ (with $\nabla$ any graded connection on $L$).

On any supermanifold $(M, A)$, the structural morphism $A \simeq \mathcal{C}^\infty(M)$ gives rise to another morphism $\kappa : \Omega(M, A) \to \Omega(M)$, which, in turn, induces an isomorphism between the De Rham cohomologies of $(M, A)$ and $M$ (see [6], sec. 4.6). In the case of a split supermanifold $(M, \Gamma(\Lambda F))$, the structural morphism $\sim$ is just the projection onto the $\mathbb{Z}$–degree 0, so the computations can be easily done.

**Theorem 7.3.** The Chern classes of the supervector bundle $E = E_0 \oplus E_1 \to M$, over the manifold $M$, and those of the graded vector bundle $\mathcal{L} = \Gamma(\Lambda F) \otimes \Gamma(E)$, over the supermanifold $(M, \Gamma(\Lambda F))$, are the same.

**Proof.** Consider $\nabla^E$, a degree-preserving connection on $E$, and $\nabla$ a linear connection on $TM$. As the Chern classes are independent of the connection, we will choose the superconnection $D = d^{\nabla^E}$ to compute those of $E$, and the graded connection given by

\[
\begin{cases}
\nabla_{\nabla X} \phi = \nabla^E_{\nabla X} \phi \\
\nabla_{iX} \phi = 0
\end{cases}
\]
to compute those of $\mathcal{L}$. Then, it is clear that
\[ \kappa(d^\nabla \phi) = \nabla^E \phi = d^{\nabla E} \phi, \]
so the statement follows.

The Chern classes of some graded vector bundles of interest can be computed directly from this result.

**Corollary 7.4.** The Chern classes of the graded vector bundle $\mathcal{L} = \Gamma(\Lambda F) \otimes \Gamma(E)$ over $(M, \Gamma(\Lambda F))$, such that $E = E_0 \oplus E_1 \to M$ is a supervector bundle, are:
\[ \text{ch}_k(\mathcal{L}) = \text{ch}_k(E_0) - \text{ch}_k(E_1). \]

**Proof.** The statement is true for Quillen superconnections (see [17]). So it is true for graded vector bundles, by Theorem 7.3.

Recall that the supertangent bundle of a split supermanifold $(M, \Gamma(\Lambda F))$ is given by the sheaves of derivations $\text{Der}(\Gamma(\Lambda F))$, and that we have the isomorphism $\text{Der}(\Lambda F) \simeq \Gamma(\Lambda F) \otimes \Gamma(TM \otimes F^*)$ (see Examples 4 and 5). The supercotangent structural sheaf is then isomorphic to the dual sheaf $\text{Der}^*(\Lambda F) \simeq \Gamma(\Lambda F) \otimes \Gamma(T^*M \oplus F)$. This, together with the fact that usual Chern classes satisfy the duality property $\text{ch}_k(F^*) = (-1)^k \text{ch}_k(F)$ for any vector bundle $F$, imply an analogous result for the supertangent and supercotangent bundles.

**Corollary 7.5.** For any split supermanifold $(M, \Gamma(\Lambda F))$, the Chern classes of the supertangent bundle satisfy
\[ \text{ch}_k(\text{Der}(\Lambda F)) = \text{ch}_k(TM) - \text{ch}_k(F^*), \]
and those of the cotangent superbundle
\[ \text{ch}_k(\text{Der}^*(\Lambda F)) = \text{ch}_k(T^*M) - \text{ch}_k(F) = (-1)^k \text{ch}_k(\text{Der}(\Lambda F)). \]

**Example 13.** The Chern classes of both, the supertangent and the supercotangent bundle of the Cartan-Koszul supermanifold, all vanish.

In the non-graded case, Chern classes of degree greater than the dimension of the base manifold vanish, as a consequence of the existence of a top degree in the De Rham cohomology, given precisely by $\dim M$. For supermanifolds, it is a well-known fact that this top degree does not exist (see [6]) so, *a priori*, they do not necessarily vanish. However, it is a direct consequence of Theorem 7.3 that the classical result is still valid in the super context.

**Corollary 7.6.** The Chern classes of degree $2k$ of the graded vector bundle $\mathcal{L} = \Gamma(\Lambda F) \otimes \Gamma(E)$ over $(M, \Gamma(\Lambda F))$, such that $2k > \dim M$, are zero.

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