Analysis of an exactly mass conserving space-time hybridized discontinuous
Galerkin method for the time-dependent Navier–Stokes equations

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Abstract

We introduce and analyze a space-time hybridized discontinuous Galerkin method for the evolutionary Navier–Stokes equations. Key features of the numerical scheme include point-wise mass conservation, energy stability, and pressure robustness. We prove that there exists a solution to the resulting nonlinear algebraic system in two and three spatial dimensions, and that this solution is unique in two spatial dimensions under a small data assumption. A priori error estimates are derived for the velocity in a mesh-dependent energy norm.

Keywords: Navier–Stokes, space-time, hybridized, discontinuous Galerkin, finite element method

1. Introduction

In this article, we are concerned with the numerical solution of the transient Navier–Stokes system posed on a convex polygonal ($d = 2$) or polyhedral ($d = 2$) domain $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$: given a suitably chosen body force $f$, kinematic viscosity $0 < \nu \leq 1$, and initial data $u_0$, find $(u, p)$ such that

\begin{align}
\partial_t u - \nu \Delta u + \nabla \cdot (u \otimes u) + \nabla p &= f, & \text{in } \Omega \times (0, T], \quad (1a) \\
\nabla \cdot u &= 0, & \text{in } \Omega \times (0, T], \quad (1b) \\
u \cdot u &= 0, & \text{on } \partial \Omega \times (0, T], \quad (1c) \\
u(x, 0) &= u_0(x), & \text{in } \Omega. \quad (1d)
\end{align}

We analyze a space-time hybridizable discontinuous Galerkin (HDG) scheme for the evolutionary Navier–Stokes system eq. (1) based on the exactly mass conserving, pointwise solenoidal discretization of Rhebergen and Wells [35]. Our analysis shows that the resulting nonlinear algebraic system of equations is solvable for $d \in \{2, 3\}$, uniquely solvable if $d = 2$ under mild conditions on the problem data, and that the numerical method converges optimally in a mesh dependent norm. Moreover, the velocity error estimates are independent of the pressure and inverse powers of $\nu$. 

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The article is organized as follows: in the remainder of Section 1, we give an overview of relevant literature, set notation, introduce the space-time HDG discretization, and introduce our main results. In Section 2, we study the conservation properties of the numerical scheme and introduce analysis tools that we will require. We consider the well-posedness of the nonlinear algebraic system arising from the numerical scheme in Section 3. Section 4 is dedicated to the error analysis for the velocity. We present a numerical test case with a manufactured solution in Section 5 to verify the theory, and finally we draw conclusions in Section 6.

1.1. Related results

Pressure-robust discretizations for the solution of the Stokes and Navier–Stokes equations have garnered much recent interest. These methods mimic at the discrete level the fundamental invariance property of incompressible flows that perturbing the external forcing term by a gradient field affects only the pressure, and not the velocity [26]. A consequence of this invariance property is that the velocity error estimates are independent of the best approximation error of the pressure scaled by the inverse of the viscosity, which is in contrast to non-pressure-robust finite element methods for incompressible flows. Classic examples of non-pressure-robust finite element methods include the Taylor–Hood finite element [21], Crouzeix–Raviart [13], MINI elements [3], and discontinuous Galerkin methods [12, 11, 14].

The use of the discontinuous Galerkin method as a higher-order time stepping scheme has had much success in application to both parabolic and hyperbolic equations [7, 8, 9, 18, 19, 15, 31, 36, 38, 40]. We refer to the book by Thomée [38] for an introduction to the subject. Moreover, when combined with the discontinuous Galerkin method in space, the resulting space-time discretization is fully conservative, allows for hp-additivity in space-time, and can achieve higher-order accuracy in both space and time [18, 19, 40]. Furthermore, the unified treatment of the spatial and temporal discretizations makes it an excellent candidate for the solution of partial differential equations on time-dependent domains; see e.g. [22, 23, 27, 29, 31, 39] for further developments along this direction.

Despite its advantages, use of the discontinuous Galerkin method in both the spatial and temporal discretizations introduces a significant computational burden compared to traditional time stepping methods. In particular, the number of globally coupled degrees of freedom is $O(p^{d+1})$, where $p$ denotes the polynomial order and $d$ is the spatial dimension of the problem under consideration. However, the number of globally coupled degrees of freedom may be reduced through hybridization [10]. Approximate traces of the solution on element facets are introduced as new unknowns in the problem. The resulting linear system may then be reduced through static condensation to a global system of size $O(p^d)$ for only these approximate traces. This strategy was employed by Rhebergen and Cockburn [29, 30] who introduced the space-time hybridizable DG (HDG) method to alleviate the computational burden of space-time DG.

For DG time stepping in the context of incompressible flow problems, we refer to [1, 9, 39, 29, 31, 22, 23]. The references [1, 9] introduce DG time stepping schemes combined with inf-sup stable conforming finite element methods for incompressible flow problems. The references [39, 31] introduce space-time DG schemes for incompressible flows on time-dependent domains, while [29, 22, 23] introduce space-time HDG schemes for incompressible flows on time-dependent domains. Of particular importance is the work of Chrysaftinos and Walkington [9], wherein fine properties of polynomials are used to develop tools essential to our analysis.
1.2. Notation

We use standard notation for Lebesgue and Sobolev spaces: given a bounded measurable set $D$, we denote by $L^p(D)$ the space of $p$-integrable functions. When $p = 2$, we denote the $L^2(D)$ inner product by $(\cdot, \cdot)_D$. We denote by $W^{k,p}(D)$ the Sobolev space of $p$-integrable functions whose distributional derivatives up to order $k$ are $p$-integrable. When $p = 2$, we write $W^{k,p}(D) = H^k(D)$. We denote by $\gamma : H^{s+1/2}(D) \to H^s(\partial D)$ the trace operator. We define $H^1_0(D)$ to be the subspace of $H^1(D)$ of functions with vanishing trace on the boundary of $D$. We denote the space of polynomials of degree $k \geq 0$ on $D$ by $P_k(D)$. We use standard notation for spaces of vector valued functions with $d$ components, e.g. $L^2(D)^d$, $H^k(D)^d$, $P^k(D)^d$, etc. At times we drop the superscript for convenience, e.g. we denote by $\|\cdot\|_{L^2(\Omega)}$ the norm on both $L^2(\Omega)$ and $L^2(\Omega)^d$.

Next, let $U$ be a Banach space, $I = [a,b]$ an interval in $\mathbb{R}$, and $1 \leq p < \infty$. We denote by $L^p(I;U)$ the Bochner space of $p$-integrable functions defined on $I$ taking values in $U$. When $p = \infty$, we denote by $L^\infty(I;U)$ the Bochner space of essentially bounded functions taking values in $U$ and by $C(I;U)$ the space of (time) continuous functions taking values in $U$. By $H^k(I;U)$, we denote the Bochner-Sobolev space for $k \geq 1$:

$$H^k(I;U) := \left\{ u \in L^2(I;U) \mid \frac{D^j u}{Dt^j} \in L^2(I;U), \ j = 1, \ldots, k \right\},$$

equipped with its usual norm. Finally, we let $P_k(I;U)$ denote the space of polynomials of degree $k \geq 0$ in time taking values in $U$.

1.3. The continuous problem

We will begin our discussion of the Navier–Stokes system with the theory of weak solutions [37]. We define two function spaces, $H$ and $V$, as follows:

$$H = \left\{ u \in L^2(\Omega)^d \mid \nabla \cdot u = 0 \text{ and } u \cdot n|_{\partial \Omega} = 0 \right\},$$

$$V = \left\{ u \in H^1_0(\Omega)^d \mid \nabla \cdot u = 0 \right\}.$$  \hfill (2a)

$$V = \left\{ u \in H^1_0(\Omega)^d \mid \nabla \cdot u = 0 \right\}.$$  \hfill (2b)

Note that $H \subset H(\text{div};\Omega) := \left\{ v \in L^2(\Omega)^d \mid \nabla \cdot u \in L^2(\Omega) \right\}$. We equip $H$ and $V$ with the standard norms on $L^2(\Omega)^d$ and $H^1_0(\Omega)^d$, respectively. By testing eq. (1a) with $v \in V$ and integrating by parts, we arrive at the following abstract ODE: find $u \in L^2(0,T;V) \cap L^\infty(0,T;H)$ such that for a.e. $t \in (0,T]$,

$$\begin{align*}
\frac{du}{dt} + \nu \Delta u + \nu \nabla u \cdot \nabla v + (u \cdot \nabla)u, v &= (f, v)_{V^* \times V}, \quad \forall v \in V, \quad (3a) \\
u(0) &= u_0. \quad (3b)
\end{align*}$$

Remark 1.1 (On the regularity of weak solutions and consistency). We require at least $(u,p) \in H^1(0,T;L^2(\Omega)^d) \cap L^2(0,T;H^1(\Omega)^d) \times L^2(0,T;H^1(\Omega)) \cap L^2(0,T;L^2_0(\Omega))$ for the consistency of the space-time HDG method. Thus, we restrict our attention to strong solutions. We assume that (at least) $f \in L^2(0,T;H)$ and $u_0 \in V$. Provided the problem data is sufficiently small, the following result on strong solutions taken from [9, Theorem 5.4] holds:

**Theorem 1.1.** Let $\Omega \subset \mathbb{R}^3$ be a convex polyhedral domain. There exists a $C > 0$, dependent on the final time $T$, such that for $u_0 \in V$ and $f \in L^2(0,T;H)$ satisfying

$$\|u_0\|^2_V + \frac{1}{\nu} \|f\|^2_{L^2(0,T;L^2(\Omega))} \leq C \nu^2,$$

$$\|u_0\|^2_V + \frac{1}{\nu} \|f\|^2_{L^2(0,T;L^2(\Omega))} \leq C \nu^2,$$

$$3$$
there exists a unique strong solution of eq. (3) with \((u, p) \in L^\infty(0, T; V) \cap L^2(0, T; H^2(\Omega)^d) \cap L^2(0, T; H^0(\Omega)^d)\) and \(\partial_t u \in L^2(0, T; H^1(\Omega))\) such that

\[
\|u\|^2_{L^\infty(0, T; V)} + \nu \|u\|^2_{L^2(0, T; H^2(\Omega))} \lesssim \nu^2, \quad \|\partial_t u\|^2_{L^2(0, T; L^2(\Omega))} \lesssim \nu^3.
\] (5)

The assumption on the problem data eq. (4) can be interpreted as small initial data and body force, or large viscosity and arbitrary data. In light of Theorem 1.1, we will make the following assumption on the problem data:

**Assumption 1.** We assume that eq. (4) holds. Note that, since \(u_0 \in V \subset H^1_0(\Omega)^d\), eq. (4) implies the existence of a constant \(C > 0\) such that

\[
\|u_0\|^2_{L^2(\Omega)} + \frac{1}{\nu} \|f\|^2_{L^2(0, T; L^2(\Omega))} \leq C\nu^2.
\] (6)

**Remark 1.2.** If \(\Omega \subset \mathbb{R}^2\) is a convex polygon, the existence of a global unique strong solution \((u, p)\) can be shown without any restriction on the problem data (see e.g. [37]). However, we will later require a similar restriction on the data to prove the uniqueness of the discrete solution in two dimensions. We therefore assume eq. (4) **even in the two dimensional case.**

Therefore, given \(f \in L^2(0, T; H)\) and \(u_0 \in V\) satisfying the small data assumption eq. (6), we consider the following space-time formulation for the strong solution to the Navier–Stokes system: for all \((v, q) \in L^2(0, T; H^1_0(\Omega)^d) \cap H^1(0, T; L^2(\Omega)^d) \times L^2(0, T; L^2_0(\Omega) \cap H^1(\Omega))\), find \((u, p) \in L^\infty(0, T; V) \cap L^2(0, T; H^2(\Omega)^d) \cap V) \cap H^1(0, T; H) \times L^2(0, T; L^2_0(\Omega) \cap H^1(\Omega))\) satisfying

\[
-\int_0^T (u, \partial_t v) \, dt + \int_0^T ((u \cdot \nabla)u, v) \, dt + \nu \int_0^T (\nabla u, \nabla v) \, dt + \int_0^T (\nabla p, v) \, dt \\
+ (u(T), v(T)) - \int_0^T (q, \nabla \cdot u) \, dt = (u_0, v(0)) + \int_0^T (f, v) \, dt.
\] (7)

### 1.4. The numerical method

To obtain a triangulation of the space-time domain \(\Omega \times (0, T)\), we first tessellate the spatial domain \(\Omega \subset \mathbb{R}^d\), \(d = \{2, 3\}\) with simplicial elements (if \(d = 2\)), or tetrahedral elements (if \(d = 3\)). We denote the resulting tessellation by \(\mathcal{T}_h = \{K\}\), and we assume that it is conforming and quasi-uniform. Furthermore, we let \(\mathcal{F}_h\) and \(\partial \mathcal{T}_h\) denote, respectively, the set and union of all edges of \(\mathcal{T}_h\). By \(h_K\), we denote the diameter of the element \(K \in \mathcal{T}_h\), and we let \(h = \max_{K \in \mathcal{T}_h} h_K\).

Next, we partition the time interval \([0, T]\) into a series of \(N + 1\) time-levels \(0 = t_0 < t_1 < \cdots < t_N = T\) of length \(\Delta t_n = t_{n+1} - t_n\). For simplicity of presentation, we assume a uniform time step size \(\Delta t_n = \Delta t\) for \(0 \leq n \leq N\). We remark, however, that a variable time step size poses no additional difficulty in the application nor the analysis of the method. A space-time slab is then defined as \(\mathcal{E}^n = \Omega \times I_n\), with \(I_n = (t_n, t_{n+1})\). We then tessellate the space-time slab \(\mathcal{E}^n\) with space-time prisms \(K \times I_n\), i.e. \(\mathcal{E}^n = \bigcup_{K \in \mathcal{T}_h} K \times I_n\). We denote this tessellation by \(\mathcal{T}^n_h\).

Combining each space-time slab \(n = 0, \ldots, N - 1\), we obtain a tessellation of the space-time domain \(\mathcal{T}_h = \bigcup_{n=0}^{N-1} \mathcal{T}^n_h\).

#### 1.4.1. The space-time hybridized DG method

We discretize the Navier–Stokes problem eq. (1) using the exactly mass conserving hybridized discontinuous Galerkin method developed in [35] combined with a high-order discontinuous Galerkin
time stepping scheme. We first introduce the following discontinuous finite element spaces on $\mathcal{T}_h$:

\[
\begin{align*}
V_h & := \{ v_h \in L^2(\Omega)^d \mid v_h \in P_{k_s}(K)^d \forall K \in \mathcal{T}_h \} , \\
Q_h & := \{ q_h \in L^2_0(\Omega) \mid q_h \in P_{k_s-1}(K) \forall K \in \mathcal{T}_h \} .
\end{align*}
\]

On $\partial \mathcal{T}_h$, we introduce the following facet finite element spaces:

\[
\begin{align*}
\bar{V}_h & := \{ \bar{v}_h \in L^2(\partial \mathcal{T}_h) \mid \bar{v}_h \in P_{k_s}(F)^d \forall F \in \mathcal{F}_h, \bar{v}_h|_{\partial \Omega} = 0 \} , \\
\bar{Q}_h & := \{ \bar{q}_h \in L^2(\partial \mathcal{T}_h) \mid \bar{q}_h \in P_{k_s}(F) \forall F \in \mathcal{F}_h \} .
\end{align*}
\]

From these spaces, we construct the following space-time finite element spaces in which we will seek our approximation on each space-time slab $\mathcal{E}^n_h$:

\[
\begin{align*}
V_h & := \{ v_h \in L^2(0,T;L^2(\Omega)^d) \mid v_h|_{(t_n,t_{n+1})} \in P_{k_s}(t_n,t_{n+1};V_h) \} , \\
Q_h & := \{ q_h \in L^2(0,T;L^2(\Omega)) \mid q_h|_{(t_n,t_{n+1})} \in P_{k_s}(t_n,t_{n+1};Q_h) \} , \\
V_h & := \{ \bar{v}_h \in L^2(0,T;L^2(\partial \mathcal{T}_h)^d) \mid \bar{v}_h|_{(t_n,t_{n+1})} \in P_{k_s}(t_n,t_{n+1};\bar{V}_h) \} , \\
\bar{Q}_h & := \{ \bar{q}_h \in L^2(0,T;L^2(\partial \mathcal{T}_h)) \mid \bar{q}_h|_{(t_n,t_{n+1})} \in P_{k_s}(t_n,t_{n+1};\bar{Q}_h) \} .
\end{align*}
\]

We note that, in general, the polynomial degree in space $k_s$ can be chosen independently of the polynomial degree in time $k_t$, but for ease of presentation we choose $k_t = k_s = k$. This choice forces us to consider $k_t \geq 1$, but the analysis herein is valid also for the case $k_t = 0$ (corresponding to a modified backward Euler scheme). We adopt the following notation for various product spaces of interest in this work:

\[
V_h = V_h \times \bar{V}_h, \quad Q_h = Q_h \times \bar{Q}_h, \quad \bar{V}_h = V_h \times \bar{V}_h, \quad \bar{Q}_h = Q_h \times \bar{Q}_h.
\]

Pairs in these product spaces will be denoted using boldface; for example, $\mathbf{v}_h := (v_h, \bar{v}_h) \in V_h$. On each space-time slab $\mathcal{E}^n$, the space-time HDG method for the Navier–Stokes problem reads: find $(\mathbf{u}_h, \mathbf{p}_h) \in V_h \times Q_h$ satisfying for all $(\mathbf{v}_h, \mathbf{q}_h) \in V_h \times Q_h$,

\[
\begin{align}
- \int_{I_n} (u_h, \partial_t v_h)_{\mathcal{T}_h} \, dt & + \int_{I_n} (\nu a_h(u_h, v_h) + o_h(u_h; u_h, v_h)) \, dt \\
& + (u_{n+1}, v_{n+1})_{\mathcal{T}_h} + \int_{I_n} b_h(p_h, v_h) \, dt = (u_n^-, v_n^-)_{\mathcal{T}_h} + \int_{I_n} (f, v_h)_{\mathcal{T}_h} \, dt,
\end{align}
\]

where $(u, v)_{\mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} \int_K uv \, dx$. We initialize the numerical scheme by choosing $u_0^- = P_h u_0$ on the first space-time slab $\mathcal{E}_0^0$, where $P_h : L^2(\Omega) \to V_h^{\text{div}}$ is the $L^2$-projection onto $V_h^{\text{div}} := \{ u_h \in V_h : b_h(q_h, u_h) = 0, \forall q_h \in Q_h \}$, the discretely divergence free subspace of $V_h$. Here, we denote by $u_{n\pm}$ the traces at time level $t_n$ from above and below, i.e. $u_{n\pm} = \lim_{\epsilon \to 0} u_h(t^n \pm \epsilon)$. We define the time jump operator at time $t_n$ by $[u_h]_{t_n} = u_{n+} - u_{n-}$.

The discrete multilinear forms $a_h(\cdot, \cdot) : V_h \times V_h \to \mathbb{R}, b_h(\cdot, \cdot) : Q_h \times V_h \to \mathbb{R}$, and $o_h(\cdot, \cdot, \cdot) : V_h \times V_h \times V_h \to \mathbb{R}$ appearing in eq. (8) serve as approximations to the viscous, pressure-velocity
coupling, and convection terms appearing in eq. (7), and are defined as:

\[
a_h(u, v) := \sum_{K \in T_h} \int_K \nabla u : \nabla v \, dx + \sum_{K \in T_h} \int_{\partial K} \frac{\alpha}{h_K} (u - \bar{u}) \cdot (v - \bar{v}) \, ds, \quad (9a)
\]

\[
- \sum_{K \in T_h} \int_{\partial K} [(u - \bar{u}) \cdot \partial_n v_h + \partial_n u \cdot (v - \bar{v})] \, ds,
\]

\[
o_h(w; u, v) := - \sum_{K \in T_h} \int_K u \otimes w : \nabla v \, dx + \sum_{K \in T_h} \int_{\partial K} \frac{1}{2} w \cdot n(u + \bar{u}) \cdot (v - \bar{v}) \, ds \quad (9b)
\]

\[
+ \sum_{K \in T_h} \int_{\partial K} \frac{1}{2} |w \cdot n| (u - \bar{u}) \cdot (v - \bar{v}) \, ds,
\]

\[
b_h(p, v) := - \sum_{K \in T_h} \int_K p \nabla \cdot v \, dx + \sum_{K \in T_h} \int_{\partial K} v \cdot n \bar{p} \, ds. \quad (9c)
\]

Here, we slightly abuse notation by using \( n \) to denote the outward unit normal \( n_K \) to the element \( K \) for brevity. To ensure stability of the numerical scheme, \( \alpha > 0 \) must be chosen sufficiently large [34].

1.4.2. Preliminaries

In this section, we present some preliminaries and rapidly recall the main properties of the multilinear forms eq. (9) that have appeared previously in the literature. Throughout this section and the rest of the article, we denote by \( C > 0 \) a generic constant independent of the mesh parameters \( h \) and \( \Delta t \) and the viscosity \( \nu \), but possibly dependent on the domain \( \Omega \), the polynomial degree \( k \), and the spatial dimension \( d \). At times we also use the notation \( a \lesssim b \) to denote \( a \leq Cb \).

To set notation, let

\[
V(h) := V_h + V \cap H^2(\Omega)^d, \quad \bar{V}(h) := \bar{V}_h + H^{3/2}(\partial T_h)^d
\]

and define the product space \( \mathbf{V}(h) := V(h) \times \bar{V}(h) \). We introduce the following mesh-dependent inner-products and norms:

\[
(u, v)_{0,h} := (u, v)_{T_h} + \sum_{K \in T_h} h_K (u - \bar{u}, v - \bar{v})_{\partial K}, \quad \forall u, v \in V(h),
\]

\[
\|v\|_{1,p,h}^1 := \sum_{K \in T_h} \|\nabla v\|_{L^p(K)}^p + \sum_{F \in F_h} \frac{1}{h_F} \|\nabla v\|_{L^p(F)}^p, \quad \forall v \in V(h),
\]

\[
\|v\|_{2,2}^2 := \sum_{K \in T_h} \|\nabla v\|_{L^2(K)}^2 + \sum_{K \in T_h} h_K^{-1} \|\bar{v} - v\|_{L^2(\partial K)}^2, \quad \forall v \in V(h),
\]

\[
\|v\|_{v,2}^2 := \|v\|_{v,2}^2 + \sum_{K \in T_h} h_K \|\nabla v\|_{L^2(\partial K)}^2, \quad \forall v \in V(h),
\]

\[
\|q\|_{p,2}^2 := \|q_h\|_{L^2(\Omega)}^2 + \sum_{K \in T_h} h_K \|q_h\|_{L^2(\partial K)}^2, \quad \forall q_h \in Q_h,
\]

where we note that the equivalence constants of \( \|\cdot\|_v \) and \( \|\cdot\|_{v'} \) on the finite-dimensional space \( V_h \) are independent of the mesh size; see [34]. The bilinear form \( a_h(\cdot, \cdot) \) is continuous and for
sufficiently large $\alpha$ enjoys discrete coercivity [34, Lemmas 4.2 and 4.3], i.e. for all $v_h \in \mathbf{V}_h$ and $u, v \in \mathbf{V}(h)$

$$a_h(v_h, v_h) \geq C \|v_h\|_v^2$$

and

$$|a_h(u, v)| \leq C \|u\|_{v'} \|v\|_{v'}.$$  \hfill (12)

The trilinear form $o_h(\cdot; \cdot; \cdot)$ satisfies [6, Proposition 3.6]

$$o_h(w_h; v_h, v_h) = \frac{1}{2} \sum_{K \in T} \int_{\partial K} |w_h \cdot n| |w_h - \tilde{v}_h|^2 \, ds \geq 0 \quad w_h \in \mathbf{V}_h^\text{div}, \forall \mathbf{v}_h \in \mathbf{V}_h.$$  \hfill (13)

Further, the trilinear form $o_h(\cdot; \cdot; \cdot)$ is Lipschitz continuous in its first argument [6, Proposition 3.4]: for all $w_1, w_2 \in \mathbf{V}(h)$, $u \in \mathbf{V}(h)$ and $v \in \mathbf{V}(h)$ it holds that

$$|o_h(w_1; u, v) - o_h(w_2; u, v)| \leq C \|w_1 - w_2\|_{1,h} \|u\|_v \|v\|_v.$$  \hfill (14)

1.5. Well-posedness and stability

To the best of the authors’ knowledge, a rigorous study of well-posedness for higher-order space-time Galerkin schemes applied to the Navier–Stokes equations has yet to appear in the literature. We remark that for the lowest order scheme ($k = 1$), uniqueness of the discrete solution is a consequence of the following energy estimate which we will derive in Section 3:

**Lemma 1.1.** Let $d = 2$ or $3$, $k \geq 1$, and suppose that $u_h \in \mathbf{V}_h$ is an approximate velocity solution of the Navier–Stokes equations computed using the space-time HDG scheme eq. (8) for $n = 0, \ldots, N - 1$. There exists a $C > 0$ such that

$$\|u_N^{-}\|_{L^2(\Omega)}^2 + \sum_{n=0}^{N-1} \|u_h\|_{n}^2 + \nu \int_0^T \|u_h\|_v^2 \, dt \leq C \left( \frac{1}{\nu} \int_0^T \|f\|_{L^2(\Omega)}^2 \, dt + \|u_0\|_{L^2(\Omega)}^2 \right).$$

For higher order schemes in time ($k \geq 2$), this energy bound is insufficient to prove the uniqueness of the discrete solution as $\text{ess sup}_{0 < t < T} u_h(t)$ need not be attained at the partition points of the time-interval. Consequently, Lemma 1.1 offers no uniform control over the discrete velocity solution $u_h$ in $L^\infty(0, T; L^2(\Omega)^d)$. We will show that such a bound is possible if $d = 2$ using the tools introduced by Chrysalinos and Walkington [7, 8, 9].

We begin by introducing the exponential interpolant from [9]. Let $(\mathbf{V}, (\cdot, \cdot)_V)$ be an inner-product space and let $\lambda > 0$ be given. The exponential interpolant $\tilde{v}$ of $v \in P_h(I_n; V)$ is defined by

$$\int_{I_n} (\tilde{v}, q)_V \, dt = \int_{I_n} (v, q)_V e^{-\lambda(t-t_n)} \, dt, \quad \forall q \in P_{k-1}(I_n, V).$$  \hfill (15)

such that $\tilde{v}(t_n^+) = v(t_n^+)$. By [9, Lemma 3.4 and Lemma 3.6] there exists a constant $C > 0$ such that for all $v \in P_h(I_n; V)$ and $1 \leq p \leq \infty$,

$$\|\tilde{v}\|_{L^p(I_n; V)} \leq C \|v\|_{L^p(I_n; V)}.$$  \hfill (16)

Next, the discrete characteristic function of $v \in P_h(I_n; V)$ for fixed $s \in (t_n, t_{n+1})$ is defined as the function $v_{\chi} \in P_h(I_n; V)$ satisfying $v_{\chi}(t_n^+) = v(t_n^+)$ and

$$\int_{t_n}^{t_{n+1}} (v_{\chi}, w)_V \, dt = \int_{t_n}^{s} (v, w)_V \, dt, \quad \forall w \in P_{k-1}(I_n; V),$$  \hfill (17)
By [9, Lemmas 3.1 and 3.2] the discrete characteristic function satisfies
\[ \|v\chi\|_{L^2(I_n, V)} \leq C\|v\|_{L^2(I_n, V)}. \] (18)

Moreover, if \( v(t) = z \) is constant in time, its discrete characteristic function can be characterized by \( v\chi = p(t)z \) for \( p \in P_k(I_n) \) satisfying \( p(t_n^+ - t_n^-) = 1 \) and
\[ \int_{t_n}^{t_n^+} pq \, dt = \int_{t_n}^{t_n^+} q \, dt, \quad \forall q \in P_{k-1}(I_n), \] (19a)
\[ \|p\|_{L^\infty(I_n)} \leq C. \] (19b)

With the help of these tools, it is possible to bound the discrete solution \( u_h \) in \( L^\infty(0, T; L^2(\Omega)^d) \) in two spatial dimensions:

**Lemma 1.2.** Let \( d = 2, k \geq 1, \) and suppose \( u_h \in \mathcal{V}_h \) is an approximate velocity solution of the Navier–Stokes equations computed using the space-time HDG scheme eq. (8) for \( n = 0, \ldots, N - 1. \) There exists a constant \( C > 0 \) such that
\[ \|u_h\|_{L^\infty(0, T; L^2(\Omega)^d)}^2 \leq C \left( \frac{1}{\nu} \int_0^T \|f\|_{L^2(\Omega)}^2 \, dt + \|u_0\|_{L^2(\Omega)}^2 \right) + C \left( \frac{1}{\nu^2} \int_0^T \|f\|_{L^2(\Omega)}^2 \, dt + \|u_0\|_{L^2(\Omega)}^2 \right)^2. \]

With this bound in hand, we can prove the following uniqueness result in two dimensions for the solution of the nonlinear system of algebraic equations arising from the discrete scheme eq. (8):

**Theorem 1.2** (Uniqueness in two dimensions). Let \( u_h \in \mathcal{V}_h \) be an approximate velocity solution of the Navier–Stokes equations computed using the space-time HDG scheme eq. (8) for \( n = 0, \ldots, N - 1. \) In two dimensions, if the problem data satisfies eq. (6) then \( u_h \) is the unique velocity solution to eq. (8).

We defer the proofs of Lemma 1.2 and Theorem 1.2 to Section 3. In addition to the bound on \( u_h \) in \( L^\infty(0, T; L^2(\Omega)^d) \), the other key ingredient for proving Theorem 1.2 is a novel discrete version of the classic Ladyzhenskaya inequalities (see e.g. [20, Section II.3]) valid for broken polynomial spaces. We will discuss this further in Section 2. Note that, similar to the continuous theory, the scaling of the exponents in the discrete Ladyzhenskaya inequality with respect to the spatial dimension prevents us from extending the proof of uniqueness to \( d = 3. \)

### 1.6. Error analysis

Our main result is a pressure-robust error estimate for the approximate velocity arising from the numerical scheme eq. (8) under the assumption that the problem data satisfies eq. (6):

**Theorem 1.3** (Velocity error). Let \( u \) be the strong velocity solution to the Navier–Stokes system eq. (1) guaranteed by Theorem 1.1 and assume it further satisfies
\[ u \in H^{k+1}(0, T; V \cap H^2(\Omega)^d) \cap H^1(0, T; H^{k+1}(\Omega)^d), \]
with initial data \( u_0 \in H^{k+1}(\Omega)^d \). Let \((u_h, \bar{u}_h) \in V_h \) be an approximate velocity solution to the Navier–Stokes system computed using the space-time HDG scheme eq. (8) for \( n = 0, \ldots, N - 1 \). Then, there exists a constant \( C > 0 \) such that the error \( e_h = (u - u_h, \gamma(u) - \bar{u}_h) \) satisfies

\[
\|e_h\|_{L^2(\Omega)}^2 + \sum_{n=0}^{N-1} \|\gamma(u)\|_{L^2(\Omega)}^2 + \nu \int_0^T \|e_h\|_{W^1_v}^2 \, dt \leq \exp(CT) \left( h^{2k} + \Delta t^{2k+2} \right) C(u),
\]

provided the time step satisfies \( \Delta t \lesssim \nu \). Here, \( C(u) \) depends on Sobolev–Bochner norms of the velocity \( u \), but is independent of the pressure \( p \).

The proof of Theorem 1.3 is deferred to Section 4. We remark that the time step restriction \( \Delta t \lesssim \nu \) in Theorem 1.3 is necessary in the proof of this theorem to use a discrete Grönwall inequality; it is not necessary for the stability of the space-time HDG method eq. (8), but rather to quantify the asymptotic rates of convergence.

2. Preliminary results

2.1. Properties of the numerical scheme

Let \( V^\operatorname{div} \) denote the subspace of \( V_h \) of discrete divergence free velocity fields:

\[
V^\operatorname{div}_h = \left\{ u_h \in V_h : \int_{I_n} b_h(q_h, u_h) \, dt = 0, \quad \forall q_h \in Q_h \right\}.
\]

The following result motivates the use of equal order polynomial degrees in time for both the velocity and pressure approximation spaces:

**Lemma 2.1.** \( V^\operatorname{div}_h = \left\{ v_h \in V_h \mid v_h \mid_{\mathcal{E}^n} \in P_k(I_n; V^\operatorname{div}_h) \right\} \).

**Proof.** The proof is very similar to that of [9, Lemma 2.3] with minor modifications and is therefore omitted. \( \square \)

An immediate consequence of Lemma 2.1 is that \( u_h(t) \in H \) a.e. \( t \in (0, T) \) where \( H \) is defined in eq. (2a). To see this we first expand \( u_h \) in terms of an orthonormal basis \( \{ \phi_i \}_{i=0}^k \) of \( P_k(I_n) \) with respect to the \( L^2(I_n) \) inner-product:

\[
u_h = \sum_{i=0}^k \phi_i(t) u_i(x), \quad u_i \in V_h. \tag{20}
\]

By Lemma 2.1, \( u_h \in P_k(I_n; V^\operatorname{div}_h) \), so \( u_i \in V^\operatorname{div}_h \) for each \( i = 0, \ldots, k \). Thus,

\[
0 = b_h(q_h, u_i) = - \sum_{K \in T_h} \int_K q_h \nabla \cdot u_i \, dx + \sum_{K \in T_h} \int_{\partial K} u_i \cdot n \tilde{q}_h \, ds, \quad \forall q_h \in Q_h.
\]

Following the same arguments as [35, Proposition 1] it follows that \( \nabla \cdot u_i = 0 \) for all \( x \in K \), \( [u_i \cdot n] = 0 \) on all \( F \in \mathcal{F}^\text{in}_h \), and \( u_i \cdot n = 0 \) on \( \partial \Omega \) for \( i = 0, \ldots, k \). By eq. (20) and since \( H \) is a linear space the result follows.
Lemma 2.2 (Consistency). Let \((u,p) = (u,\gamma(u))\) and \(p = (p,\gamma(p))\). Then, it holds that
\[
-\int_{I_n} (u, \partial_t v_h)_{\mathcal{T}_h} \, dt + (u(t_{n+1}), v_n^-)_{\mathcal{T}_h} + \int_{I_n} (\nu a_h(u, v_h) + a_h(u; u, v_h) + b_h(p, v_h)) \, dt \\
- \int_{I_n} b_h(q_h, u) \, dt = (u(t_n), v_n^+)_{\mathcal{T}_h} + \int_{I_n} (f, v_h)_{\mathcal{T}_h} \, dt, \quad \forall (v_h, q_h) \in \mathbf{V}_h \times \mathbf{Q}_h.
\]

2.2. Scalings and embeddings

We begin by recalling a number of results for piece-wise polynomials. First, for polynomials in time, let \((V, (\cdot,\cdot)_V)\) be an inner product space. Then, there exists \(C > 0\) such that for all \(v \in P_k(I_n, V)\) (see e.g. [9, Lemma 3.5]):
\[
\|v\|_{L^p(I_n,V)} \leq C \Delta t^{1/p-1/2} \|v\|_{L^2(I_n,V)}, \quad 1 \leq p \leq \infty, \tag{21a}
\]
\[
\|\partial_t v\|_{L^2(I_n,V)} \leq C \Delta t^{-1} \|v\|_{L^2(I_n,V)} \tag{21b}
\]
Next, we recall the following discrete version of the Sobolev embedding theorem valid for broken polynomial spaces \(P_r(T_h) = \{ f \in L^2(\Omega) \mid f|_K \in P_r(K), \forall K \in T_h \}\) where \(r \geq 0\). Let \(1 \leq p < \infty\), then for all \(q\) satisfying \(1 \leq q \leq pd/(d-p)\) if \(1 \leq p < d\), or \(1 \leq q < \infty\) if \(d \leq p < \infty\), there exists a constant \(C > 0\) such that [14, Theorem 5.3]:
\[
\|v_h\|_{L^q(\Omega)} \leq C \|v_h\|_{1,p,h}, \quad \forall v_h \in P_r(T_h). \tag{22}
\]
In the case \(p = 2\), we write \(\|\cdot\|_{1,2,h} = \|\cdot\|_{1,h}\). Note that choosing \(p = q = 2\) in eq. (22) yields the discrete Poincaré inequality: \(\|v_h\|_{L^2(\Omega)} \leq C_P \|v_h\|_{1,h} \) for all \(v_h \in \mathbf{V}_h\). By the triangle inequality, \(\|v_h\|_{1,h} \leq \|v_h\|_v\), so that
\[
\|v_h\|_{L^2(\Omega)} \leq C_P \|v_h\|_v, \quad \forall v_h \in \mathbf{V}_h. \tag{23}
\]
We now prove a discrete version of the Ladyzhenskaya inequalities valid for broken polynomial spaces. While the analogue of these inequalities are well known in the context of \(H^1\)-conforming finite element methods [17], to our knowledge they have yet to be extended to non-conforming finite element spaces.

Lemma 2.3 (Ladyzhenskaya inequalities for broken polynomial spaces). There exists a constant \(C > 0\) such that for \(d \in \{2, 3\}\):
\[
\|v_h\|_{L^4(\Omega)} \leq C \|v_h\|_{L^2(\Omega)}^{2(2-d)} \|v_h\|_{1,h}^{2(5-2d)}, \quad \forall v_h \in \mathbf{V}_h. \tag{24}
\]
Proof. It suffices to consider the scalar case. We focus first on the case \(d = 2\). Inserting \(v_h^2\) into eq. (22) with \(r = 2k, p = 1,\) and \(q = 2\) yields \(\|v_h\|_{L^4(\Omega)} \leq C \|v_h\|_{1,h}\). The result follows after noting that the right hand side can be bounded by applying the Cauchy–Schwarz inequality and a local discrete trace inequality \(\|v_h\|_{L^2(F)} \leq C h_K^{-1/2} \|v_h\|_{L^2(K)}\) [14, Lemma 1.46]:
\[
\frac{1}{2} \|v_h^2\|_{1,1,h} = \sum_{K \in T_h} \int_K (\nabla v_h v_h) \, dx + \sum_{F \in F_h} \int_F \|v_h\| \cdot \|v_h\| \, ds \leq C \|v_h\|_{L^2(\Omega)} \|v_h\|_{1,h}.
\]
For the case \(d = 3\), the result follows from the Cauchy–Schwarz inequality and eq. (22) with \(q = 6\) and \(p = 2\).

For \(d = 3\), interpolating between \(L^2(\Omega)^d\) and \(L^4(\Omega)^d\) and using eq. (24) yields:
\[
\|v_h\|_{L^3(\Omega)} \leq C \|v_h\|_{L^2(\Omega)} \|v_h\|_{1,h}^{1/2}, \quad \forall v_h \in \mathbf{V}_h. \tag{25}
\]
3. Well-posedness of the discrete problem

3.1. Existence of a discrete solution

We will begin by showing the existence of a solution to the nonlinear system of algebraic equations arising from eq. (8) by making use of the following topological degree result taken from [14, Lemma 6.42]:

**Lemma 3.1.** Let \((X, \|\cdot\|_X)\) be a finite-dimensional normed space. Let \(M > 0\) and let \(\Psi : X \times [0, 1] \to X\) satisfy

1. \(\Psi\) is continuous.
2. \(\Psi(\cdot, 0)\) is an affine function and the equation \(\Psi(v, 0) = 0\) has a solution \(v \in X\) such that \(\|v\|_X < M\).
3. For any \((v, \rho) \in X \times [0, 1]\), \(\Psi(v, \rho) = 0\) implies \(\|v\|_X < M\).

Then, there exists \(v \in X\) such that \(\Psi(v, 1) = 0\) and \(\|v\|_X < M\).

To apply Lemma 3.1 in section 3.1.3 we first require the proof of Lemma 1.1 and well-posedness of the space-time HDG discretization of the linear time-dependent Stokes problem as discussed next.

3.1.1. Proof of Lemma 1.1.

**Proof.** Testing eq. (8) with \((v_h, \bar{v}_h, q_h, \bar{q}_h) = (u_h, \bar{u}_h, p_h, \bar{p}_h)\), using the coercivity of \(a_h(\cdot, \cdot)\) and the fact that \(a_h(\cdot; u_h, u_h) \geq 0\), and integrating by parts in time, we find that there exists a constant \(C_1 > 0\) such that

\[
\frac{1}{2}\|u_{n+1}^-\|^2_{L^2(\Omega)} + \frac{1}{2}\|u_h\|^2_{L^2(\Omega)} - \frac{1}{2}\|u_n^-\|^2_{L^2(\Omega)} + C_1\nu \int_{I_n}\|u_h\|^2_v dt \leq \int_{I_n}(f, u_h)\tau_n dt.
\]

To bound the right hand side, we apply the Cauchy–Schwarz inequality, the discrete Poincaré inequality eq. (23), and Young’s inequality with \(\epsilon = C_1/(C_P\nu) > 0\). Rearranging, we see there is a constant \(C_2 > 0\) such that

\[
\|u_{n+1}^-\|^2_{L^2(\Omega)} + \|u_h\|^2_{L^2(\Omega)} - \|u_n^-\|^2_{L^2(\Omega)} + \nu C_1 \int_{I_n}\|u_h\|^2_v dt \leq \frac{C_2}{\nu} \int_{I_n}\|f\|^2_{L^2(\Omega)} dt.
\]

The result follows after summing over all space-time slabs.

3.1.2. A linearized problem

Before we can apply the topological degree argument, we will need to study the space-time HDG solution of the linear time-dependent Stokes problem:

\[
\begin{align*}
\partial_t u - \nu \Delta u + \nabla p &= f, &\text{in } \Omega \times (0, T], \\
\nabla \cdot u &= 0, &\text{in } \Omega \times (0, T], \\
u a_h(u_h, \bar{v}_h) = (u_h^+, \bar{v}^-)\tau_n + \int_{I_n} (f, v_h)\tau_n dt. &\text{in } \partial\Omega \times (0, T],
\end{align*}
\]

\[
\begin{align*}
u a_h(u_h, \bar{v}_h) = (u_h^+, \bar{v}^-)\tau_n + \int_{I_n} (f, v_h)\tau_n dt. &\text{in } \Omega.
\end{align*}
\]

**Lemma 3.2.** There exists a unique pair \(u_h \in V_h^{\text{div}} \times \bar{V}_h\) such that for all \(v_h \in V_h^{\text{div}} \times \bar{V}_h\):

\[
- \int_{I_n} (u_h, \partial_t v_h)\tau_n dt + (u_n^-, \bar{v}_n^-)\tau_n + \int_{I_n} \nu a_h(u_h, v_h) dt = (u_n^-, \bar{v}_n^+)\tau_n + \int_{I_n} (f, v_h)\tau_n dt.
\]

Note that this is simply the space-time HDG scheme applied to the time-dependent Stokes problem eq. (26).

**Proof.** The result follows from the Lax–Milgram theorem.
3.1.3. The topological degree argument

**Theorem 3.1.** Let $d \in \{2,3\}$ and $k \geq 1$. There exists at least one discrete velocity solution $u_h \in \mathcal{V}_h^{\text{div}} \times \tilde{V}_h$ to eq. (8) for $n = 0, \ldots, N - 1$ satisfying the energy estimate Lemma 1.1.

**Proof.** We set $X = \mathcal{V}_h^{\text{div}} \times \tilde{V}_h$ and equip it with the norm

$$\|u_h\|_X^2 := \|u_h^\nu\|_{L^2(\Omega)}^2 + \sum_{n=0}^{N-1} \|u_n^\nu\|_{L^2(\Omega)}^2 + \nu \int_0^T \|u_h\|^2 dt.$$

Define the continuous mapping $\Psi : X \times [0,1] \to X$ for $n = 0, \ldots, N - 1$, by

$$\int_{I_n} (\Psi(u_h, \rho), v_h)_{0,h} dt = -\int_{I_n} (u_h, \partial_t v_h)_{\tau_h} dt + (u_{n+1}^\nu, v_{n+1}^\nu)_{\tau_h} + \int_{I_n} \nu a_h(u_h, v_h) dt$$

$$+ \int_{I_n} \rho o_h(u_h; u_h, v_h) dt - (u_n^\nu, v_n^\nu)_{\tau_h} - \int_{I_n} (f, v_h)_{\tau_h} dt.$$

$\Psi$ is well-defined by the Riesz representation theorem, verifying item (1) in Lemma 3.1. Next, we choose $u_h \in X$ such that $\Psi(u_h, \rho) = 0$ for some $\rho \in [0,1]$. Since $o_h(u_h; u_h, u_h) \geq 0$, we can repeat the proof of Lemma 1.1 to bound $u_h$ uniformly with respect to $\rho$:

$$\|u_h^\nu\|_{L^2(\Omega)}^2 + \sum_{n=0}^{N-1} \|u_n^\nu\|_{L^2(\Omega)}^2 + \nu \int_0^T \|u_h\|^2 dt \leq C \left( \frac{1}{\nu} \int_0^T \|f\|_{L^2(\Omega)}^2 dt + \|u_0\|_{L^2(\Omega)}^2 \right),$$

which verifies item (3) in Lemma 3.1 with

$$M^2 = C \left( \frac{1}{\nu} \int_0^T \|f\|_{L^2(\Omega)}^2 dt + \|u_0\|_{L^2(\Omega)}^2 \right) + \epsilon,$$

for any $\epsilon > 0$. Finally, note that $\Psi(\cdot, 0) : X \to X$ is an affine function since the nonlinear convection term disappears for $\rho = 0$. By Lemma 3.2, there exists a solution to $\Psi(u_h, 0) = 0$, verifying item (2) in Lemma 3.1. Therefore, there exists a solution $u_h$ to $\Psi(u_h, 1) = 0$ satisfying $\|u_h\|_X < M$. Equivalently, $u_h \in \mathcal{V}_h^{\text{div}} \times \tilde{V}_h$ solves eq. (8) for all $v_h \in \mathcal{V}_h^{\text{div}} \times \tilde{V}_h$ and satisfies the energy bound in Lemma 1.1. \hfill \Box

3.2. Uniqueness of the discrete velocity solution in two dimensions

3.2.1. Bounds on the trilinear convection term

In the analysis that follows, we will require tighter bounds on the trilinear convection form than is provided by eq. (14). For this, we will make extensive use of the results of Section 2.2. We remark that, although we focus on $d = 2$ for the proof of uniqueness, the bound eq. (29) will be essential for the error analysis in both two and three spatial dimensions in Section 4.

**Lemma 3.3.** If $d = 2$, there exists a $C > 0$ such that for all $w_h, u_h, v_h \in \mathcal{V}_h$,

$$|o_h(w_h; u_h, v_h)| \leq C \|w_h\|_{L^2(\Omega)}^{1/2} \|u_h\|_v^{1/2} \|v_h\|_v \left( \|w_h\|_{L^2(\Omega)}^{1/2} \|u_h\|_v^{1/2} + \|u_h\|_{L^2(\Omega)}^{1/2} \|v_h\|_v^{1/2} \right).$$

(28)

Moreover, if $d \in \{2,3\}$, there exists a $C > 0$ such that for all $w_h, u_h, v_h \in \mathcal{V}_h$,

$$|o_h(w_h; u_h, v_h)| \leq C \|w_h\|_{L^2(\Omega)}^{1/2} \|u_h\|_v^{1/2} \|u_h\|_v \|v_h\|_v.$$

(29)
Proof. This proof relies on the following scaling identity for \( \mu \in P_k(\partial K) \) between \( L^p \) and \( L^2 \) norms on element boundaries which can be obtained using standard arguments:

\[
\| \mu \|_{L^p(\partial K)} \leq C h^{(d-1)(1/p-1/2)} \| \mu \|_{L^2(\partial K)}, \quad 2 \leq p < \infty. \tag{30}
\]

Now, split \( o_h(w_h, u_h, v_h) \) into three terms and bound each separately. Using that \( u_h + \bar{u}_h = 2u_h + (\bar{u}_h - u_h) \), we find:

\[
|o_h(w_h, u_h, v_h)| \leq \sum_{K \in T_h} \int_K |(u_h \otimes w_h) : \nabla v_h| \, dx + \sum_{K \in T_h} \int_{\partial K} |w_h \cdot n (u_h) \cdot (u_h - \bar{u}_h)| \, ds \tag{31}
\]

\[
+ \sum_{K \in T_h} \int_{\partial K} |w_h \cdot n (u_h - \bar{u}_h) \cdot (v_h - \bar{v}_h)| \, ds = T_1 + T_2 + T_3.
\]

To show eq. (28), we first apply the generalized Hölder inequality to \( T_1 \) with \( p = q = 4 \) and \( r = 2 \), the Cauchy–Schwarz inequality, and eq. (24) to find:

\[
|T_1| \leq C \| u_h \|_{L^2(\Omega)}^{1/2} \| u_h \|_{L^2(\Omega)}^{1/2} \| w_h \|_{L^2(\Omega)} \| w_h \|_{L^2(\Omega)} \| v_h \|_{L^2(\Omega)}.
\]

To bound \( T_2 \), we apply the generalized Hölder inequality with \( p = q = 4 \) and \( r = 2 \) and use the local discrete trace inequality \( \| v_h \|_{L^p(\partial K)} \leq C h_K^{-1/p} \| v_h \|_{L^p(\partial K)} \) (see e.g. [14, Lemma 1.52]) for \( p = 4 \), we have

\[
|T_2| \leq C \sum_{K \in T_h} \| u_h \|_{L^4(K)} \| u_h \|_{L^4(K)} \| w_h \|_{L^4(K)} \| w_h \|_{L^4(K)} \| v_h \|_{L^4(K)}.
\]

Applying the Cauchy–Schwarz inequality and eq. (24) we find

\[
|T_2| \leq C \| u_h \|_{L^2(\Omega)}^{1/2} \| u_h \|_{L^2(\Omega)}^{1/2} \| w_h \|_{L^2(\Omega)} \| w_h \|_{L^2(\Omega)} \| v_h \|_{L^2(\Omega)}.
\]

To bound \( T_3 \), we again apply the generalized Hölder inequality with \( p = q = 4 \) and \( r = 2 \), the local discrete trace inequality \( \| v_h \cdot n \|_{L^2(\partial K)} \leq C h_K^{-1/2} \| v_h \|_{L^2(K)} \), eq. (30) with \( d = 2 \) and \( p = 4 \), and the Cauchy–Schwarz inequality to find

\[
|T_3| \leq C \| w_h \|_{L^2(\Omega)} \| u_h \|_{L^2(\Omega)} \| v_h \|_{L^2(\Omega)}.
\]

Summing the bounds on \( T_1, T_2, \) and \( T_3 \) yields the result.

The proof of eq. (29) differs in the cases of \( d = 2 \) and \( d = 3 \). We begin with \( d = 3 \). To bound \( T_1 \), we first apply the generalized Hölder inequality with \( p = 3, q = 6 \) and \( r = 2 \) followed by the Cauchy–Schwarz inequality to find:

\[
|T_1| \leq \| w_h \|_{L^3(\Omega)} \| u_h \|_{L^6(\Omega)} \| v_h \|_{L^2(\Omega)}.
\]

Now applying eq. (25) and eq. (22) with \( q = 6 \), we have

\[
|T_1| \leq C \| w_h \|_{L^2(\Omega)}^{1/2} \| w_h \|_{L^2(\Omega)}^{1/2} \| u_h \|_{L^2(\Omega)} \| v_h \|_{L^2(\Omega)}.
\]
To bound $T_2$, we apply the generalized Hölder inequality with $p = 3$, $q = 6$ and $r = 2$ to find
\[ |T_2| \leq \sum_{K \in T_h} \| w_h \cdot n \|_{L^3(\partial K)} \| u_h \|_{L^6(\partial K)} \| v_h - \bar{v}_h \|_{L^2(\partial K)}. \]

Next, using the local discrete trace inequality $\| v_h \|_{L^p(\partial K)} \leq Ch_K^{-1/p} \| v_h \|_{L^p(K)}$ (see e.g. [14, Lemma 1.52]) for $p = 3$ and $p = 6$, the Cauchy–Schwarz inequality, eq. (25), and eq. (22) with $q = 6$, we have
\[ |T_2| \leq C \| w_h \|_{L^2(\Omega)}^{1/2} \| \mathbf{w}_h \|_{V}^{1/2} \| u_h \|_{V} \| v_h \|_{V}. \]

To bound $T_3$, we again apply the generalized Hölder inequality with $p = 3$, $q = 6$ and $r = 2$:
\[ |T_3| \leq \sum_{K \in T_h} \| w_h \cdot n \|_{L^3(\partial K)} \| u_h - \bar{u}_h \|_{L^6(\partial K)} \| v_h - \bar{v}_h \|_{L^2(\partial K)}. \]

Now, applying the local discrete trace inequality $\| v_h \cdot n \|_{L^p(\partial K)} \leq Ch_K^{-1/p} \| v_h \|_{L^p(K)}$ with $p = 3$, eq. (30) with $d = 3$ and $p = 6$, the discrete Cauchy–Schwarz inequality, and eq. (25), we have
\[ |T_3| \leq C \| w_h \|_{L^2(\Omega)}^{1/2} \| \mathbf{w}_h \|_{V}^{1/2} \| u_h \|_{V} \| v_h \|_{V}. \]

Summing the bounds on $T_1$, $T_2$, and $T_3$ yields the result. The case for $d = 2$ follows similarly, instead using $p = q = 4$ and $r = 2$ in the generalized Hölder inequality, eq. (22) with $q = 4$, and eq. (30) with $d = 2$ and $p = 4$.

3.2.2. Proof of Lemma 1.2

Proof. Let $\bar{u}_h$ be the exponential interpolant of $u_h$ as defined in eq. (15). Testing eq. (8) with $\bar{u}_h$, integrating by parts in time, and using the defining properties of the exponential interpolant, we have
\[
\frac{1}{2} \int_{I_n} \frac{d}{dt} \| u_h(t) \|_{L^2(\Omega)}^2 e^{-\lambda(t-t_n)} dt + \| u_n^+ \|_{L^2(\Omega)}^2 + \int_{I_n} (\nu a_h(\mathbf{u}_h, \bar{u}_h) + o_h(u_h; \mathbf{u}_h, \bar{u}_h)) dt \\
= (u_n^-, u_n^+)_{T_h} + \int_{I_n} (f, \bar{u}_h)_{T_h} dt.
\]

Integrating by parts again in time and applying the Cauchy–Schwarz inequality and Young’s inequality to the first term on the right hand side, we have
\[
\frac{\lambda}{2} \int_{I_n} \| u_h(t) \|_{L^2(\Omega)}^2 e^{-\lambda(t-t_n)} dt + \frac{1}{2} \| u_{n+1}^- \|_{L^2(\Omega)}^2 e^{-\lambda\Delta t} \\
\leq \frac{1}{2} \| u_n^- \|_{L^2(\Omega)}^2 + \int_{I_n} (f, \bar{u}_h)_{T_h} dt - \int_{I_n} (\nu a_h(\mathbf{u}_h, \bar{u}_h) + o_h(u_h; \mathbf{u}_h, \bar{u}_h)) dt.
\]

We now focus on bounding the right hand side. By the boundedness of $a_h(\cdot, \cdot)$ eq. (12) and eq. (16), there exists a constant $C_1 > 0$ such that
\[
\int_{I_n} |a_h(\mathbf{u}_h, \bar{u}_h)| dt \leq C_1 \int_{I_n} \| u_h \|_{V}^2 dt.
\]
In two spatial dimensions, we can use Lemma 3.3, eq. (16), and hence Young’s inequality with some $\epsilon_1 > 0$ to find there exists a constant $C_2 > 0$ such that

$$\int_{I_n} o_h(u_h; u_h, \tilde{u}_h) \, dt \leq \frac{\epsilon_1}{2} \| u_h \|^2_{L^\infty(I_n; L^2(\Omega))} + \frac{C_2}{2\epsilon_1} \left( \int_{I_n} \| u_h \|_v^2 \, dt \right)^2.$$

Next by the Cauchy–Schwarz inequality, Young’s inequality, the discrete Poincaré inequality, and eq. (16), there exists a constant $C_3 > 0$ such that for some $\epsilon_2 > 0$,

$$\int_{I_n} (f, \tilde{u}_h)_{\tau_h} \, dt \leq \frac{1}{2\epsilon_2} \int_{I_n} \| f \|_{L^2(\Omega)}^2 \, dt + \frac{C_3\epsilon_2}{2} \int_{I_n} \| u_h \|_v^2 \, dt.$$

Thus,

$$\frac{\lambda}{2} \int_{I_n} \| u_h \|^2_{L^2(\Omega)} e^{-\lambda(t-t_0)} \, dt + \frac{1}{2} \| u_{n+1} \|_{L^2(\Omega)} e^{-\lambda\Delta t} \| u_h \|^2_{L^\infty(I_n; L^2(\Omega))} \leq \frac{1}{2} \| u_n \|^2_{L^2(\Omega)} + \frac{1}{2\epsilon_2} \int_{I_n} \| f \|^2_{L^2(\Omega)} \, dt + \frac{C_3\epsilon_2}{2} \int_{I_n} \| u_h \|_v^2 \, dt + C_1 \nu \int_{I_n} \| u_h \|^2 \, dt + \frac{C_2}{2\epsilon_1} \left( \int_{I_n} \| u_h \|_v^2 \, dt \right)^2.$$

Choosing $\lambda = 1/\Delta t$ and applying the scaling identity in eq. (21a) with $p = \infty$, we find there exists a constant $C_4 > 0$ such that

$$\left( \frac{C_4^{-1}e^{-1}}{2} - \frac{\epsilon_1}{2} \right) \| u_h \|^2_{L^\infty(I_n; L^2(\Omega))} \leq \frac{1}{2} \| u_n \|^2_{L^2(\Omega)} + \frac{1}{2\epsilon_2} \int_{I_n} \| f \|^2_{L^2(\Omega)} \, dt + \frac{C_3\epsilon_2}{2} \int_{I_n} \| u_h \|_v^2 \, dt + C_1 \nu \int_{I_n} \| u_h \|^2 \, dt + \frac{C_2}{2\epsilon_1} \left( \int_{I_n} \| u_h \|_v^2 \, dt \right)^2.$$

Choosing $\epsilon_1 = C_4^{-1}e^{-1}/2, \epsilon_2 = 2\nu$, using the a priori estimates on $u_h$ in Lemma 1.1, and rearranging, we see there exists a $C_5 > 0$ such that

$$\| u_h \|^2_{L^\infty(I_n; L^2(\Omega))} \leq C_5 \left( \frac{1}{\nu} \int_0^T \| f \|^2_{L^2(\Omega)} \, dt + \| u_0 \|^2_{L^2(\Omega)} \right) + \frac{C_5}{\nu^2} \left( \frac{1}{\nu} \int_0^T \| f \|^2_{L^2(\Omega)} \, dt + \| u_0 \|^2_{L^2(\Omega)} \right)^2.$$

This bound holds uniformly for every space-time slab, so the result follows. \qed

### 3.2.3. Proof of Theorem 1.2.

**Proof.** Consider an arbitrary space-time slab $\mathcal{E}^m$. Suppose $(u_1, \bar{u}_1) \in \mathcal{V}_h^{\text{div}} \times \mathcal{V}_h$ and $(u_2, \bar{u}_2) \in \mathcal{V}_h^{\text{div}} \times \mathcal{V}_h$ are two solutions to eq. (8) corresponding to the same problem data $f$ and $u_0$, and set $w_h = u_1 - u_2$. Then, for all $w_h \in \mathcal{V}_h^{\text{div}} \times \mathcal{V}_h$, it holds that

$$- \int_{I_m} \left( w_h, \partial_t \bar{v}_h \right) \tau_h \, dt + \left( w_{m+1}^-, v_{m+1}^- \right) \tau_h + \int_{I_m} \nu a_h(w_h, v_h) \, dt + \int_{I_m} \left( o_h(u_1, u_1), v_h \right) \, dt = \left( w_{m}^-, v_{m}^+ \right) \tau_h. \quad (32)$$
\textbf{Step one:} Testing eq. (32) with \( v_h = w_h \), integrating by parts in time, using the coercivity of \( a_h(\cdot, \cdot) \), and noting that \( o_h(u_2, u_2, w_h) - o_h(u_1, u_1, w_h) \leq -o_h(w_h, u_2, w_h) \) by eq. (13), we find
\[
\frac{1}{2} \|w_{m+1}\|^2_{L^2(\Omega)} + \frac{1}{2} \|w_m\|^2_{L^2(\Omega)} - \frac{1}{2} \|w_0\|^2_{L^2(\Omega)} + C \nu \int_{I_m} \|w_h\|^2_v \, dt \leq \int_{I_m} |o_h(w_h, u_2, w_h)| \, dt.
\]

Summing over all space-time slabs \( n = 0, \ldots, N-1 \), rearranging, and noting that \( w_0^- = 0 \), we see that there exists a \( C_1 > 0 \) such that
\[
\|w_N^-\|^2_{L^2(\Omega)} + \sum_{n=0}^{N-1} \|w_h\|^2_{L^2(\Omega)} + \nu \int_0^T \|w_h\|^2_v \, dt \leq C_1 \int_0^T |o_h(w_h, u_2, w_h)| \, dt. \tag{33}
\]

\textbf{Step two:} Fix an integer \( m \) such that \( 0 \leq m \leq N - 1 \). Testing eq. (32) with the discrete characteristic function \( v_h = w_\chi \) where \( s = \arg \sup_{t \in I_m} \|u_h(t)\|_{L^2(\Omega)} \), integrating by parts in time, using Young's inequality, first with all space-time slabs, and thus we can replace the supremum over \( t \) in eq. (33) with the supremum over \( [0, T] \). Doing so, and adding \( 2 \nu \int_0^T \|w_h\|^2_v \, dt \) to both sides we see there exists a constant \( C_2 > 0 \) such that
\[
\frac{1}{2} \sup_{t \in I_m} \|w_h(t)\|^2_{L^2(\Omega)} \leq C_2 \nu \int_0^T \|w_h\|^2_v \, dt + \int_0^T \|o_h(u_2, u_2, w_\chi) - o_h(u_1, u_1, w_\chi)\| \, dt, \tag{34}
\]
where we have used that \( \sup_{t \in I_{m-1}} \|w(s)\|_{L^2(\Omega)} \geq \|w_m^-\|_{L^2(\Omega)} \). Setting \( I_{-1} = \{t_0\} = \{0\} \), we can sum over the space-time slabs \( n = 0, \ldots, m \) and use the boundedness of \( a_h(\cdot, \cdot) \) and the bound eq. (18) to find there exists a constant \( C_3 > 0 \) such that
\[
\frac{1}{2} \|w_h\|^2_{L^\infty(0,T;L^2(\Omega))} + 2 \nu \int_0^T \|w_h\|^2_v \, dt \leq C_3 \int_0^T \|o_h(w_h, u_2, w_h)\| \, dt + \int_0^T \|o_h(u_2, u_2, w_\chi)\| \, dt + \int_0^T \|o_h(u_1, w_h, w_\chi)\| \, dt.
\]
Here, we have used the bound eq. (33) from step one, that \( o_h(u_2, u_2, w_\chi) - o_h(u_1, u_1, w_\chi) = -o_h(w_h, u_2, w_\chi) - o_h(u_1, w_h, w_\chi) \), and the triangle inequality. From Lemma 3.3 and two applications of Young's inequality, first with \( p = q = 2 \) and second with \( p = 4, q = 4/3 \), we find
\[
|o_h(w_h, u_2, w_\chi)| \leq C_4 \left( \frac{1}{2 \epsilon} \|w_h\|^2_{L^2(\Omega)} \|u_2\|^2_v + \frac{1}{4 \epsilon^3} \|w_h\|^2_{L^2(\Omega)} \|u_2\|^2_{L^2(\Omega)} \|u_2\|^2_v + \frac{5 \epsilon}{4} \|w_h\|^2_v \right).
\]
Similarly, from Lemma 3.3 and eq. (18), we have
\[
|o_h(w_h, u_2, w_\chi)| \leq C_5 \left( \frac{1}{2 \epsilon} \|w_h\|^2_{L^2(\Omega)} \|u_2\|^2_v + \frac{1}{4 \epsilon^3} \|w_h\|^2_{L^2(\Omega)} \|u_2\|^2_{L^2(\Omega)} \|u_2\|^2_v + \frac{5 \epsilon}{4} \|w_h\|^2_v \right),
\]
and finally,
\[ |o_h(u_1, w_h, w_\chi)| \leq C_6 \left( \|u_1\|_{L^2(\Omega)} \|w_h\|_v^2 + \frac{1}{4\epsilon^2} \|u_1\|_{L^2(\Omega)}^2 \|w_h\|_{L^2(\Omega)}^2 \|u_1\|_v^2 + \frac{3\epsilon}{4} \|w_h\|_v^2 \right), \]

where \( \epsilon > 0 \). Collecting the above bounds, choosing \( \epsilon = O(\nu) \) sufficiently small and rearranging, we can find a \( C_7 > 0 \) such that
\[ \|w_h\|_{L^\infty(0,T;L^2(\Omega))}^2 + \nu \int_0^T \|w_h\|_v^2 \, dt \]
\[ \leq \frac{C_7}{\nu^4} \left( \nu^3 \|u_1\|_{L^\infty(0,T;L^2(\Omega))} + \nu \|u_1\|_{L^\infty(0,T;L^2(\Omega))}^2 \right) \left( \nu^3 + \nu \|u_2\|_{L^\infty(0,T;L^2(\Omega))}^2 \right) \]
\[ \times \left( \|w_h\|_{L^\infty(0,T;L^2(\Omega))}^2 + \nu \int_0^T \|w_h\|_v^2 \, dt \right). \]

(35)

**Step three:** For notational convenience, let \( \Xi = \nu^{-1} \int_0^T \|f\|_{L^2(\Omega)}^2 \, dt + \|w_0\|_{L^2(\Omega)}^2 \). Applying the bounds in Lemma 1.1 and Lemma 1.2 to eq. (35), we find there exists a \( C_8 > 0 \) such that
\[ \|w_h\|_{L^\infty(0,T;L^2(\Omega))}^2 + \nu \int_0^T \|w_h\|_v^2 \, dt \]
\[ \leq \frac{C_8}{\nu^4} \left( \nu^3 \Xi^{1/2} + \nu^2 \Xi + \Xi^2 + \nu^{-2} \Xi^2 \right) \left( \|w_h\|_{L^\infty(0,T;L^2(\Omega))}^2 + \nu \int_0^T \|w_h\|_v^2 \, dt \right). \]

The result follows if \( \nu < 1 \) and \( \Xi \leq \frac{1}{4} \min \{C_8^{-1/3}, C_8^{-1/2}, C_8^{-1}, C_8^{-2}\} \nu^2 \).

3.3. Recovering the pressure

Existence of the pressure pair \((p_h, \bar{p}_h) \in Q_h\) satisfying eq. (8) will require the following inf-sup condition:

**Theorem 3.2** (Inf-sup condition). Suppose that the spatial mesh \( T_h \) is conforming and quasi-uniform. There exists a constant \( \beta > 0 \), independent of \( h \) and \( \Delta t \), such that for all \( q_h \in Q_h \),
\[ \sup_{0 \neq v_h} \frac{\int_{I_n} b_h(q_h, v_h) \, dt}{\left( \int_{I_n} \|v_h\|_v^2 \, dt \right)^{1/2}} \geq \beta \left( \int_{I_n} \|q_h\|_v^2 \, dt \right)^{1/2}. \]

(36)

The proof, which exploits the tensor-product structure of the finite element spaces in an essential way, is an extension of the proof of [32, Lemma 1] to the space-time setting.

3.3.1. Proof of Theorem 3.2.

By [24, Theorem 3.1], the inf-sup condition eq. (36) is satisfied if we can decompose \( b_h(\cdot, \cdot) \) into \( b_1(\cdot, \cdot) : V_h \times Q_h \to \mathbb{R} \) and \( b_2(\cdot, \cdot) : V_h \times Q_h \to \mathbb{R} \) such that, for some constants \( \alpha_1, \alpha_2 > 0 \), it holds
that

\[
\sup_{(v_h, \tilde{v}_h) \in \mathcal{Z}_{b_2} \times \mathcal{V}_h} \frac{\int_{I_n} b_1(q_h, v_h) \, dt}{\left( \int_{I_n} \|v_h\|^2_v \, dt \right)^{1/2}} \geq \alpha_1 \left( \int_{I_n} \|q_h\|^2_{L^2(\Omega)} \, dt \right)^{1/2}, \tag{37a}
\]

and

\[
\sup_{v_h \in \mathcal{V}_h} \frac{\int_{I_n} b_2(\tilde{q}_h, v_h) \, dt}{\left( \int_{I_n} \|v_h\|^2_v \, dt \right)^{1/2}} \geq \alpha_2 \left( \sum_{K \in \mathcal{T}_h} \int_{I_n} h_K \|\bar{q}\|^2_{\partial K} \, dt \right)^{1/2}, \tag{37b}
\]

where

\[
\mathcal{Z}_{b_2} = \left\{ v_h \in \mathcal{V}_h : \int_{I_n} b_2(v_h, \tilde{q}_h) \, dt = 0, \ \forall \tilde{q}_h \in \mathcal{Q}_h \right\}.
\]

We thus define

\[
b_1(q_h, v_h) = -\sum_{K \in \mathcal{T}_h} \int_K q_h \nabla \cdot v_h \, dx \quad \text{and} \quad b_2(\tilde{q}_h, v_h) = \sum_{K \in \mathcal{T}_h} \int_{\partial K} v_h \cdot n \tilde{q}_h \, ds.
\]

We begin by proving eq. (37a). The tensor-product structure of the space \( \mathcal{Q}_h \) ensures that we can expand any \( q_h \in \mathcal{Q}_h \) in terms of an orthonormal basis of \( P_k(I_n) \) with respect to the \( L^2(I_n) \) inner product:

\[
q_h = \sum_{i=0}^k \phi_i(t) q_i(x), \quad \phi_i \in P_k(I_n), \ q_i \in \mathcal{Q}_h. \tag{38}
\]

Since each \( q_i \in L^2_0(\Omega) \), there exist \( z_i \in H^1_0(\Omega)^d \), \( 0 \leq i \leq k \) and constants \( \beta_i, \ 0 \leq i \leq k \) such that

\[
\nabla \cdot z_i = -q_i \quad \text{and} \quad \beta_i \|z_i\|_{H^1(\Omega)} \leq \|q_i\|_{L^2(\Omega)} \quad \text{(see e.g. [14, Theorem 6.5]).}
\]

We construct the desired \( \psi_h = (\psi_h, \tilde{\psi}_h) \in \mathcal{Z}_{b_2} \times \mathcal{V}_h \) by choosing

\[
\psi_h = \sum_{i=0}^k \phi_i(t) \Pi_{\text{BDM}} z_i, \quad \text{and} \quad \tilde{\psi}_h = \sum_{i=0}^k \phi_i(t) \Pi_V z_i,
\]

where \( \Pi_{\text{BDM}} : \left[ H^1(\Omega) \right]^d \rightarrow \mathcal{V}_h \) is the BDM projection [4] and \( \Pi_V \) is the \( L^2 \) projection onto the space \( \mathcal{V}_h \). By the orthonormality of the basis \( \{ \phi_i \}_{i=0}^k \), definition of the BDM projection [4], the single-valuedness of \( z_i \cdot n \) and \( \tilde{q}_i \) across element faces, and the fact that \( z_i \in H^1_0(\Omega) \), we have

\[
b_2(\tilde{q}_h, \psi_h) = \sum_{i=0}^k \sum_{K \in \mathcal{T}_h} \int_{\partial K} z_i \cdot n \tilde{q}_i \, ds = 0,
\]

and thus \( \psi_h \in \mathcal{Z}_{b_2} \). We now show that \( \psi_h \) satisfies the inequality in eq. (37a) with some \( \alpha_1 > 0 \) independent of the mesh parameters \( h \) and \( \Delta t \). Given \( q_h \in \mathcal{Q}_h \), we can use the expansion eq. (38), the definition of \( z_i, \ 0 \leq i \leq k \), and the commuting diagram property of the BDM projection [4] to find

\[
\int_{I_n} \|q_h\|^2_{L^2(\Omega)} \, dt = \int_{I_n} b_1(q_h, \psi_h) \, dt. \tag{39}
\]
Next, we need to show existence of a constant $\alpha_1 > 0$, independent of the mesh parameters $h$ and $\Delta t$, such that

$$\alpha_1^2 \int_{I_n} \|\psi_h\|_v^2 \, dt \leq \int_{I_n} \|q_h\|_{L^2(\Omega)}^2 \, dt. \quad (40)$$

But, this can easily be reduced to the proof of [34, Lemma 4.5] by expanding $\psi_h$ in terms of an orthonormal basis of $P_k(I_n)$ with respect to the $L^2(I_n)$ inner-product. Combining eq. (39) and eq. (40), we have

$$\frac{\int_{I_n} b_1(q_h, \psi_h) \, dt}{\left( \int_{I_n} \|\psi_h\|_v^2 \, dt \right)^{1/2}} = \frac{\int_{I_n} \|q_h\|_{L^2(\Omega)}^2 \, dt}{\left( \int_{I_n} \|\psi_h\|_v^2 \, dt \right)^{1/2}} \geq \alpha_1^2 \left( \int_{I_n} \|q_h\|_{L^2(\Omega)}^2 \, dt \right)^{1/2},$$

where $\alpha_1 > 0$ depends on the constants $\beta_i$, $0 \leq i \leq k$.

What we have left to show is eq. (37b). It suffices to construct an $\omega_h \in \mathcal{V}_h$ such that for some $\alpha_2 > 0$ it holds that

$$\frac{\int_{I_n} b_2(\bar{q}_h, \omega_h) \, dt}{\left( \int_{I_n} \|\omega_h\|_v^2 \, dt \right)^{1/2}} \geq \alpha_2 \left( \sum_{K \in \mathcal{T}_h} \int_{I_n} h_K \|\bar{q}_h\|_{\partial K}^2 \, dt \right)^{1/2}, \quad \forall \bar{q}_h \in \mathcal{Q}_h. \quad (41)$$

The tensor-product structure of $\mathcal{Q}_h$ ensures that we can expand any $\bar{q}_h \in \mathcal{Q}_h$ in terms of an orthonormal basis of $P_k(I_n)$ with respect to the $L^2(I_n)$ inner-product:

$$\bar{q}_h = \sum_{i=0}^k \phi_i(t) q_i(x), \quad \phi_i \in P_k(I_n), \quad q_i \in \mathcal{Q}_h. \quad (42)$$

Given $\bar{q}_h \in \mathcal{Q}_h$, we construct the required $\omega_h$ by choosing $\bar{\omega}_h = 0$, and defining $\omega_h \in \mathcal{V}_h$ element-wise by:

$$\omega_h|_{K \times I_n} = \sum_{i=0}^k \phi_i(t) L^{BDM}(\bar{q}_i|_{\partial K}),$$

with $\bar{q}_i \in \mathcal{Q}_h$ defined as in eq. (42). Here, $L^{BDM} : P_k(\partial K) \to (P_k(K))^d$ is the local BDM lifting satisfying for all $\bar{q}_h \in P_k(\partial K)$ (see e.g. [16, Proposition 2.10]):

$$(L^{BDM}\bar{q}_h) \cdot n = \bar{q}_h, \quad \text{and} \quad \|L^{BDM}\bar{q}_h\|_{L^2(K)} \leq C h_K^{1/2} \|\bar{q}_h\|_{L^2(\partial K)}, \quad \forall \bar{q}_h \in P_k(\partial K), \quad (43)$$

where $n$ is the unit outward normal to $\partial K$. Using the first property in eq. (43), it can be shown that

$$\int_{I_n} b_2(\bar{q}_h, \omega_h) \, dt = \sum_{K \in \mathcal{T}_h} \int_{I_n} \|\bar{q}_h\|_{L^2(\partial K)}^2 \, dt. \quad (44)$$

The remainder of the proof of eq. (37b) can easily be reduced to the proof of [32, Lemma 3] by expanding $\omega_h$ in terms of an orthonormal basis of $P_k(I_n)$ with respect to the $L^2(I_n)$ inner-product. In particular, it can be shown that $\alpha_2 = C h_{\min}/h_{\max}$, which remains uniformly bounded below provided we assume quasi-uniformity of $\mathcal{T}_h$. \hfill \(\square\)

The Ladyzhenskaya-Babuška-Brezzi theorem [4] yields the following corollary:

**Corollary 3.1.** To each discrete velocity solution pair $(u_h, \bar{u}_h) \in \mathcal{V}_h$ guaranteed by Theorem 3.1, there exists a unique discrete pressure pair $(p_h, \bar{p}_h) \in \mathcal{Q}_h$ satisfying eq. (8).
4. Error analysis for the velocity

4.1. Space-time projection operators

Let $P_h : L^2(\Omega)^d \rightarrow V_h^{\text{div}}$ and $\bar{P}_h : L^2(\Gamma)^d \rightarrow \bar{V}_h$ denote the orthogonal $L^2$-projections onto, respectively, the spaces $V_h^{\text{div}}$ and $\bar{V}_h$. The approximation properties of $\bar{P}_h$ are well-known while the approximation properties of $P_h$ rely critically on the fact that $V_h^{\text{div}} \subset H$. In particular, we can exploit the best approximation property of the orthogonal projection along with the approximation properties of the BDM projection to prove (see e.g. [33]):

**Lemma 4.1.** Let $k \geq 1$, $0 \leq m \leq 2$, and $u \in H^{k+1}(\Omega)^d$. If the spatial mesh $T_h$ is quasi-uniform and consists triangles in two dimensions or tetrahedras in three dimensions, then the following estimates hold:

\[
\sum_{K \in T_h} \| u - P_h u \|_{H^m(K)}^2 \lesssim h^{2(k-m+1)} \| u \|_{H^{k+1}(\Omega)}^2, \tag{45}
\]

\[
\sum_{K \in T_h} h^{-1} \| u - P_h u \|_{L^2(\partial K)}^2 \lesssim h^{2k} \| u \|_{H^{k+1}(\Omega)}^2. \tag{46}
\]

**Proof.** We begin by proving eq. (45). For $m = 0$, we have by the best approximation property of the orthogonal $L^2$-projection onto $V_h^{\text{div}}$:

\[
\| u - P_h u \|_{L^2(\Omega)} = \min_{v_h \in V_h^{\text{div}}} \| u - v_h \|_{L^2(\Omega)}.
\]

Since $\Pi_{\text{BDM}} u \in V_h^{\text{div}}$, eq. (45) follows from standard approximation properties of the BDM projection [4]. The proof for $m = 1$ follows by noting that, by triangle inequality,

\[
\| u - P_h u \|_{H^m(K)} \leq \| u - \Pi_V u \|_{H^m(K)} + \| \Pi_V u - P_h u \|_{H^m(K)},
\]

where $\Pi_V$ is the orthogonal $L^2$-projection onto $V_h$. Using the local inverse inequality $\| u_h \|_{H^1(K)} \leq Ch^{-1} \| u_h \|_{L^2(K)}$, the quasi-uniformity of the spatial mesh $T_h$, eq. (45), and the approximation properties of $\Pi_V$ (see e.g. [14, Lemma 1.58]), the result follows. The bound for $m = 2$ follows similarly. To prove eq. (46), we note that by the local trace inequality for functions in $H^1(K)$, we have

\[
\| u - P_h u \|_{L^2(\partial K)} \leq C \left( h^{-1/2} \| u - P_h u \|_{L^2(K)} + \| u - P_h u \|^{1/2}_{L^2(K)} \right).
\]

The result now follows from the quasi-uniformity of the mesh, the Cauchy–Schwarz inequality, and eq. (45).

Following [9, Definition 4.2], we introduce a space-time projection operator much in the same spirit as the temporal “DG-projection” defined in [38, Eq. (12.9)] or [15, Section 6.1.4], but appropriately modified for divergence free fields. Additionally, we will need an analogue of this temporal DG-projection onto the facet space $\bar{V}_h$:

**Definition 4.1.** 1. $\mathcal{P}_h : C(I_n; L^2(\Omega)) \rightarrow V_h$ satisfying $(P_h u)(t^{-}_{n+1}) = (P_h u)(t^{-}_n)$, with $(P_h u)(t^{-}_0) = P_h u(t_0)$, and

\[
\int_{I_n} (u - P_h u, v_h) dt = 0 \quad \forall v_h \in P_{k-1}(I_n, V_h^{\text{div}}). \tag{47}
\]
2. \( \bar{P}_h : C(I_n; L^2(\Gamma)) \rightarrow P_k(I_n; \bar{V}_h) \) satisfying \((\bar{P}_h u)(t_{n+1}^-) = (\bar{P}_h u)(t_{n+1}^-) \)

\[
\int_{I_n} (u - \bar{P}_h u, \bar{v}_h)_{\partial T_n} dt = 0, \quad \forall \bar{v}_h \in P_{k-1}(I_n; \bar{V}_h). \tag{48}
\]

We summarize the approximation properties of \( P_h \) and \( \bar{P}_h \) in Appendix A.

### 4.2. Parabolic Stokes projection

Motivated by [9, Definition 4.2], we introduce a parabolic Stokes projection which will be crucial to our error analysis in Section 4:

**Definition 4.2** (Parabolic Stokes projection). Let \( u \) be the strong velocity solution to the Navier–Stokes system eq. (1) guaranteed by Theorem 1.1. We define the parabolic Stokes projection \((\Pi_h u, \bar{\Pi}_h u, \Pi_h p, \bar{\Pi}_h p) \in \mathcal{V}^{\text{div}}_h \times \bar{V}_h \times \mathcal{Q}_h\) to be the solution to the following space-time HDG scheme:

\[
- \int_{I_n} (\Pi_h u, \partial_t v_h)_{\mathcal{T}_n} dt + ((\Pi_h u)_{n+1}^-, v_{n+1}^-)_{\mathcal{T}_n} + \int_{I_n} (\nu a_h(\Pi_h u, v_h) + b_h(\Pi_h p, v_h)) dt \\
= ((\Pi_h u)_n^-, v_n^+)_{\mathcal{T}_n} + \int_{I_n} (\partial_u v_h)_{\mathcal{T}_n} dt + \int_{I_n} \nu a_h(u, v_h) dt \quad \forall v_h \in \mathcal{V}_h, \tag{49}
\]

\[
\int_{I_n} b_h(q_h, \Pi_h u) dt = 0 \quad \forall q_h \in \mathcal{Q}_h,
\]

where \((\Pi_h u)_0 = P_h u(t_0)\) and \((\bar{\Pi}_h u)_0^-\) may be arbitrarily chosen. Here, we have denoted \(\Pi_h u = (\Pi_h u, \Pi_h u)\) and \(\Pi_h p = (\Pi_h p, \bar{\Pi}_h p)\).

**Remark 4.1.** We remark that eq. (49) is simply a space-time HDG scheme for the evolutionary Stokes problem eq. (26) with \( f = u_t - \nu \Delta u \) and \( u_0 = u(0) \). Consequently, \( \Pi_h u \in \mathcal{V}^{\text{div}}_h \) and thus \( \Pi_h u \in H\).

### 4.3. Uniform bounds on the parabolic Stokes projection

To perform our error analysis in Section 4, we will require a uniform bound on the Stokes projection:

\[
\text{ess sup}_{0 \leq t \leq T} \| \Pi_h u \|_\nu \leq C(u, u_0, \nu).
\]

Our plan is to follow the proof of [9, Theorem 4.10]. Therein, an essential ingredient is a discrete Stokes operator. Unfortunately, as \((\cdot, \cdot)_{\mathcal{T}_n}\) is *not* an inner-product on \(\mathcal{V}_h\), we cannot leverage the Riesz representation theorem to infer the existence of a discrete Stokes operator in the HDG setting. Instead, we introduce a novel discrete Stokes operator by mimicking the static condensation that occurs for the HDG method at the algebraic level following ideas from [5].

#### 4.3.1. Discrete Stokes operator

Consider the variational problem: find \( \phi_h \in \mathcal{V}^{\text{div}}_h \times \bar{V}_h \) such that

\[
a_h(\phi_h, w_h) = (u_h, w_h)_{\mathcal{T}_n}, \quad \forall w_h \in \mathcal{V}^{\text{div}}_h \times \bar{V}_h.
\]

This problem is well-posed by the Lax–Milgram theorem, implying the existence of a well-defined solution operator \( S_h : \bar{V}_h \rightarrow \mathcal{V}^{\text{div}}_h \times \bar{V}_h \) such that \( \phi_h = S_h(u_h) \). Note that \( S_h \) need not be surjective.
onto the product space $V_h^{\text{div}} \times \bar{V}_h$. However, as in [5], we can split the solution operator $S_h$ into “element” and “facet” solution operators $S_h^e$ and $S_h^f$. We will show that $S_h^e$ is invertible.

Define the facet solution operator $S_h^f : V_h^{\text{div}} \rightarrow \bar{V}_h$ as the unique solution of

$$a_h((v_h, S_h^f(v_h)), (0, \bar{w}_h)) = 0, \quad \forall \bar{w}_h \in \bar{V}_h. \tag{50}$$

Since $a_h(\cdot, \cdot)$ is symmetric, $S_h^f$ is self-adjoint. Next, we introduce a new bilinear form on $V_h^{\text{div}} \times V_h^{\text{div}}$:

$$a_h^*(v_h, w_h) = a_h((v_h, S_h^f(v_h)), (w_h, S_h^f(w_h))), \tag{51}$$

for which we introduce the element solution operator $S_h^e : V_h^{\text{div}} \rightarrow V_h^{\text{div}}$ satisfying

$$a_h^*(S_h^e(u_h), w_h) = (u_h, w_h)_{T_h}, \quad \forall w_h \in V_h^{\text{div}}. \tag{52}$$

**Lemma 4.2.** Fix a space-time slab $\mathcal{E} = \Pi_h T$. Let $(\Pi_h u, \bar{\Pi}_h u)$ be the velocity components of the parabolic Stokes projection solving eq. (49), and let $S_h^f : V_h^{\text{div}} \rightarrow \bar{V}_h$ be the facet solution operator introduced in eq. (50). Then, it holds that

$$\bar{\Pi}_h u = S_h^f(\Pi_h u). \tag{53}$$

**Proof.** Set $(v_h, \bar{v}_h, q_h, \bar{q}_h) = (0, \bar{v}_h, 0, 0)$ in eq. (49) and expand $\Pi_h u, \bar{\Pi}_h u,$ and $\bar{v}_h$ in terms of an orthonormal basis $\{\phi_i(t)\}_{i=0}^k$ of $P_k(I_n)$ with respect to the $L^2(I_n)$ inner-product to find

$$\sum_{i=0}^k \sum_{K \in \mathcal{T}_h} \int_{\partial K} \frac{\alpha}{h_k}(\Pi_h u)_i \cdot \bar{v}_i \, ds = \sum_{i=0}^k \sum_{K \in \mathcal{T}_h} \int_{\partial K} \left( \frac{\alpha}{h_k}(\Pi_h u)_i - \frac{\partial (\Pi_h u)_i}{\partial n} \right) \cdot \bar{v}_i \, ds. \tag{54}$$

By the definition of the operator $S_h^f$, we have for each $i = 0, \ldots, k$,

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} \left( \frac{\alpha}{h_k}(\Pi_h u)_i - \frac{\partial (\Pi_h u)_i}{\partial n} \right) \cdot \bar{v}_i \, ds = \sum_{K \in \mathcal{T}_h} \int_{\partial K} \frac{\alpha}{h_k} S_h^f((\Pi_h u)_i) \cdot \bar{v}_i \, ds, \tag{55}$$

and moreover, each $S_h^f((\Pi_h u)_i)$ is unique. Choosing $\bar{v}_i = (\Pi_h u)_i - S_h^f((\Pi_h u)_i) \in \bar{V}_h$ and rearranging allows us to conclude $(\Pi_h u)_i = S_h^f((\Pi_h u)_i)$ for each $i = 0, \ldots, k$. The result follows by uniqueness of the expansions of $u_h$ and $\bar{u}_h$ with respect to the chosen basis of $P_k(I_n)$ and the linearity of $S_h^f$. \hfill \square

**Lemma 4.3.** Fix a space-time slab $\mathcal{E} = \Pi_h T$. Let $(\Pi_h u, \bar{\Pi}_h u) \in V_h^{\text{div}} \times \bar{V}_h$ be the velocity components of the parabolic Stokes projection solving eq. (49) and let $A_h : V_h^{\text{div}} \rightarrow V_h^{\text{div}}$ be the discrete Stokes operator satisfying eq. (52). For notational convenience, we denote $A_h \Pi_h u = (A_h \Pi_h u, A_h \bar{\Pi}_h u)$. Then, for all $t \in I_n$, it holds that:

$$a_h(\Pi_h u, A_h \Pi_h u) = \|A_h \Pi_h u\|_{L^2(\Omega)}^2, \tag{54}$$

$$\left( \partial_t \Pi_h u, A_h \Pi_h u \right)_{T_h} = \frac{1}{2} \frac{d}{dt} a_h(\Pi_h u, \Pi_h u). \tag{55}$$
Proof. By Lemma 4.2 and the linearity of $A_h$ and $S_F$, we find

$$S_F (A_h \Pi_h u) = A_h (S_F \Pi_h u) = A_h \Pi_h u, \quad S_F (\partial_t \Pi_h u) = \partial_t (S_F \Pi_h u) = \partial_t \Pi_h u.$$  

The conclusion follows from eq. (52) after some basic calculations. □

4.3.2. Bounding the Stokes projection

**Lemma 4.4** (Uniform bound on the Stokes projection). Let $u$ be the strong velocity solution to the Navier–Stokes system eq. (1) guaranteed by Theorem 1.1 and let $(\Pi_h u, \Pi_h u, \Pi h p, \Pi h p) \in V_h^{\text{div}} \times \bar{V}_h \times Q_h$ be the solution to eq. (49), where we set $\bar{u}_0 = \bar{P}_h u_0$. Then, it holds that

$$\|\Pi_h u\|_{L^\infty(0,T;\mathcal{V}_h)} \leq C \left( \frac{1}{\nu} \int_0^T \|\partial_t u - \nu \Delta u\|_{L^2(\Omega)}^2 \,dt + \|u_0\|_{H^1(\Omega)}^2 \right).$$

Here, we define $\|\Pi_h u\|_{L^\infty(0,T;\mathcal{V}_h)} \equiv \text{ess sup}_{0 \leq t \leq T} \|\Pi_h u\|_v$.

**Proof.** The proof will proceed in two steps. In the first step, we bound the Stokes projection at the partition points of the time-intervals. In the second step, we use the exponential interpolant, combined with the results of the first step, to obtain a uniform bound on the Stokes projection over $(0,T)$.

**Step one:** Integrating by parts in time in the term containing the temporal derivative in eq. (49), testing with $v_h = A_h \Pi_h u$ and using eq. (54) and eq. (55) in Lemma 4.3, we have

$$\frac{1}{2} \int_{I_n} \frac{d}{dt} a_h((\Pi_h u)_h, (\Pi_h u)_h) + \nu \int_{I_n} \|A_h \Pi_h u\|_{L^2(\Omega)}^2 dt$$

$$= \int_{I_n} (\partial_t u - \nu \Delta u, A \Pi_h u)_{\Omega_n} dt + a_h((\Pi_h u)_n, (\Pi_h u)_n^+) dt.$$

Using the coercivity of $a_h(\cdot,\cdot)$, the Cauchy–Schwarz inequality, Young’s inequality, and summing over all space-time slabs, we find

$$\left\| (\Pi_h u)^N\right\|_v^2 + \sum_{n=0}^{N-1} \left\| (\Pi_h u)_n\right\|_v^2 + \nu \int_0^T \|A_h \Pi_h u\|_{L^2(\Omega)}^2 dt$$

$$\leq C \left( \frac{1}{\nu} \int_0^T \|\partial_t u - \nu \Delta u\|_{L^2(\Omega)}^2 \,dt + \|\Pi_h u_0\|_v^2 \right).$$

As $(\Pi_h u_0)^- = P_h u_0$ and $(\Pi h u_0) = \bar{P}_hu_0$, we have from Lemma 4.1 and the approximation properties of $\bar{P}_h$ (see e.g. [33]) that

$$\left\| (\Pi_h u)^N\right\|_v^2 + \sum_{n=0}^{N-1} \left\| (\Pi_h u)_n\right\|_v^2 + \nu \int_0^T \|A \Pi_h u\|_{L^2(\Omega)}^2 dt \leq C \left( \frac{1}{\nu} \int_0^T \|\partial_t u - \nu \Delta u\|_{L^2(\Omega)}^2 \,dt + \|u_0\|_{H^1(\Omega)}^2 \right).$$

Note that for the lowest order scheme ($k = 1$), we can already infer the result.
Step two: It remains to obtain a bound for higher order polynomials in time. For this, we use the exponential interpolant of the pair \( A_h \Pi_h u = (A_h \Pi_h u, A_h \Pi_h u) \), which we denote by \( \tilde{A}_h \Pi_h u = (\tilde{A}_h \Pi_h u, \tilde{A}_h \Pi_h u) \). Integrating the first term on the right hand side of eq. (49) by parts in time, choosing \( v_h = A_h \Pi_h u \), and using eq. (15), eq. (55), and that \( \tilde{A}_h \Pi_h u \in \mathcal{V}_h^{\text{div}} \), we have

\[
\frac{1}{2} \int_{I_n} e^{-\lambda(t-t_n)} \frac{d}{dt} a_h(\Pi_h u, \Pi_h u) dt + ((\Pi_h u)_{n+1}^-, (A_h \Pi_h u)_{n}^+)_{T_n} + \nu \int_{I_n} a_h(\Pi_h u, \tilde{A}_h \Pi_h u) dt
= \int_{I_n} (\partial_t u - \nu \Delta u, \tilde{A}_h \Pi_h u)_{T_n} dt + ((\Pi_h u)_{n}^-, (A_h \Pi_h u)_{n}^+)_{T_n}.
\]

Proceeding in an identical fashion as in the proof of Lemma 1.2, and using eq. (16), we obtain

\[
\frac{e^{-C}}{2} \| \Pi_h u \|_{L^\infty(I_n; \mathcal{V}_h)}^2 + \frac{e^{-C}}{2} \| (\Pi_h u)_{n+1}^- \|_{V}^2
\leq C \left( \frac{1}{\nu} \int_{I_n} \| \partial_t u - \nu \Delta u \|_{L^2(\Omega)}^2 dt + \nu \int_{I_n} \| A \Pi_h u \|_{L^2(\Omega)}^2 dt + \| (\Pi_h u)_{n}^- \|_{V}^2 \right). \tag{57}
\]

Bounding the last two terms on the right hand side of eq. (57) using eq. (56) and omitting the second (positive) term on the left hand side, we see that there exists a constant \( C > 0 \) such that

\[
\| \Pi_h u \|_{L^\infty(I_n; \mathcal{V}_h)}^2 \leq C \left( \frac{1}{\nu} \int_0^T \| \partial_t u - \nu \Delta u \|_{L^2(\Omega)}^2 dt + \| u_0 \|_{H^1(\Omega)}^2 \right).
\]

This bound holds uniformly for every space-time slab, so the result follows.

4.3.3. Approximation properties of the parabolic Stokes projection

Lemma 4.5. Let \( u \) be the strong velocity solution to the Navier–Stokes system eq. (1) guaranteed by Theorem 1.1, let \((\Pi_h u, \Pi_h u) \in \mathcal{V}_h^{\text{div}} \times \mathcal{V}_h \) be the velocity pair of the Stokes projection eq. (49) for \( n = 0, \ldots, N-1 \), and let \( \mathcal{P}_h \) and \( \bar{\mathcal{P}}_h \) denote the projections introduced in Definition 4.1. Let \( \zeta_h = \bar{\mathcal{P}}_h u - \Pi_h u, \xi_h = u - \Pi_h u, \hat{\zeta}_h = \mathcal{P}_h u - \bar{\mathcal{P}}_h u \) and \( \hat{\xi}_h = u - \mathcal{P}_h u \). There is a constant \( C > 0 \) such that

\[
\| \zeta_h(t_{n+1}) \|_{L^2(\Omega)}^2 + \sum_{n=0}^{N-1} \| \zeta_h \|_{L^2(\Omega)}^2 + \nu \int_0^T \| \hat{\zeta}_h \|_{V}^2 dt \leq C \nu \int_0^T \| \zeta_h \|_{V}^2 dt.
\]

Proof. Our starting point will be the definition of the parabolic Stokes projection eq. (49). We will introduce the splitting \( u - \Pi_h u = \xi_h + \zeta_h \), where \( \xi_h = (\xi_h, \hat{\xi}_h) \) and \( \zeta_h = (\zeta_h, \hat{\zeta}_h) \). Testing eq. (49) with \( v_h = \zeta_h \in \mathcal{V}_h^{\text{div}} \times \mathcal{V}_h \), integrating by parts in time, using the defining properties of the projection \( \mathcal{P}_h \) Definition 4.1, the coercivity and boundedness of \( a_h(\cdot, \cdot) \) eq. (12), the Cauchy–Schwarz inequality and Young’s inequality with some sufficiently small \( \epsilon > 0 \), we have

\[
\| \zeta_h(t_{n+1}) \|_{L^2(\Omega)}^2 + \| \zeta_h(t_n) \|_{L^2(\Omega)}^2 - \| \zeta_h(t_n) \|_{L^2(\Omega)}^2 + C \nu \int_{I_n} \| \hat{\zeta}_h \|_{V}^2 dt \leq C \nu \int_{I_n} \| \hat{\zeta}_h \|_{V}^2 dt.
\]

We conclude by summing over all space-time slabs and noting that \( \zeta_h(t_{0}^+) = 0 \). □
4.4. Error analysis for the velocity

4.4.1. The error equation

We introduce the notation \( e_h = (e_h, \bar{e}_h) = (u - u_h, \gamma(u) - \bar{u}_h) \). From Lemma 2.2, we have the following Galerkin orthogonality result:

\[
- \int_{I_n} (e_h, \partial_t \psi_h) \tau_h \, dt + (\epsilon_{n+1}^-, \psi_{n+1}^-) \tau_h + \nu \int_{I_n} a_h(e_h, \psi_h) \, dt + \int_{I_n} b_h(p - p_h, \psi_h) \, dt \\
+ \int_{I_n} (o_h(u; u, \psi_h) - o_h(u_h; u_h, \psi_h)) \, dt - (e_n^+, v^+_h) \tau_h = 0, \quad \forall \psi_h \in \mathcal{V}_h.
\]

Introducing the splitting \( e_h = (u - \Pi_h u) + (\Pi_h u - u_h) = \eta_h + \theta_h \), integrating by parts in the first term on the left hand side, using the definition of the parabolic Stokes projection eq. (49), and choosing \( \psi_h = \theta_h \in \mathcal{V}_h^{\text{div}} \times \bar{\mathcal{V}}_h \), eq. (58) reduces to

\[
\int_{I_n} (\partial_t \theta_h, \theta_h) \tau_h \, dt + \nu \int_{I_n} a_h(\theta_h, \theta_h) \, dt + (\bar{\theta}_n^+ - \theta_n^-, \bar{\theta}_n^+) \tau_h = - \int_{I_n} (o_h(u; u, \theta_h) - o_h(u_h; u_h, \theta_h)) \, dt,
\]

where we have used that \( u_h, \Pi_h u \in P_k(I_n, H) \).

**Lemma 4.6.** Let \( (\Pi_h u, \bar{\Pi}_h u) \in \mathcal{V}_h^{\text{div}} \times \bar{\mathcal{V}}_h \) be the velocity pair of the Stokes projection eq. (49) and let \( (u_h, \bar{u}_h) \in \mathcal{V}_h \) be an approximate velocity solution to the Navier–Stokes system computed using the space-time HDG scheme eq. (8) for \( n = 0, \ldots, N - 1 \). Let \( \theta_h = \Pi_h u - u_h \), \( \eta_h = u - \Pi_h u \), \( \bar{\theta}_h = \bar{\Pi}_h u - u_h \) and \( \bar{\eta}_h = u - \bar{\Pi}_h u \). There exists a constant \( C > 0 \) such that

\[
\int_{I_n} \| \theta_h \|_{L^2(\Omega)}^2 \, dt \leq C \left( \nu^{1/2} \Delta t^{1/2} \int_{I_n} \| \theta_h \|_{V}^2 \, dt + \nu \Delta t \int_{I_n} \| \eta_h \|_{V}^2 \, dt \right) + \Delta t \| \bar{\theta}_n^- \|_{L^2(\Omega)}^2.
\]

**Proof.** We will proceed as in the proof of [9, Theorem 5.2]. Choose \( z_h \in \mathcal{V}_h^{\text{div}} \times \bar{\mathcal{V}}_h \) independent of time. We test eq. (58) with the discrete characteristic function \( z_h \in \mathcal{V}_h^{\text{div}} \times \bar{\mathcal{V}}_h \) of \( z_h \). Recall from Equation (18) that we can write \( z_h = \varphi(t) z_h \), with \( \varphi(t) \) satisfying \( \varphi(t_n^+) = 1 \) as well as eq. (19a) and eq. (19b). Then, we have

\[
(\theta_h(s), z_h)_{\tau_h} = - \int_{I_n} (o_h(u; u, z_h) - o_h(u_h; u_h, z_h) + \nu a_h(\theta_h, z_h)) \, dt + (\bar{\theta}_n^-, z_h)_{\tau_h}.
\]

By the boundedness of \( a_h(\cdot, \cdot) \) eq. (12), the bound on \( \varphi \) eq. (19b), and the Cauchy–Schwarz inequality,

\[
\int_{I_n} |a_h(\theta_h, z_h)| \, dt \leq C \Delta t^{1/2} \| z_h \|_V \left( \int_{I_n} \| \theta_h \|_{V}^2 \, dt \right)^{1/2}.
\]

After a few algebraic manipulations, we apply eq. (14), followed by eq. (19b), the energy estimate Lemma 1.1, the assumption eq. (6) on the problem data, and the Cauchy–Schwarz inequality, to
find
\[
\int_{I_n} \left| o_h(u; u, z_h) - o_h(u_h; u_h, z_h) \right| dt \\
= \int_{I_n} \left| o_h(u; \eta_h, z_h) + o_h(\eta_h; \Pi_h u, z_h) + o_h(u_h; \theta_h, z_h) - o_h(\theta_h; \Pi_h u, z_h) \right| dt
\]
\[
\leq C \| z_h \|_v \int_{I_n} \left( \| u \|_{H^1(\Omega)} \| \eta_h \|_v + \| \Pi_h u \|_v + \| u_h \|_v \| \theta_h \|_v + \| \Pi_h u \|_v \| \theta_h \|_v \right) dt
\]
\[
\leq C^\nu t^{1/2} \| z_h \|_v \left( \nu' t^{1/2} \left( \int_{I_n} \| \eta_h \|_v^2 dt \right)^{1/2} + (\nu' t^{1/2} + 1) \left( \int_{I_n} \| \theta_h \|_v^2 dt \right)^{1/2} \right)
\]
Combining eq. (60), eq. (61), and eq. (62),
\[
(\theta_h(s), z_h)_{T_h} \leq C \| z_h \|_v \left( \nu t^{1/2} + \nu' t^{1/2} \right) \left( \int_{I_n} \| \theta_h \|_v^2 dt \right)^{1/2}
\]
\[
+ C \| z_h \|_v \nu t^{1/2} \left( \int_{I_n} \| \eta_h \|_v^2 dt \right)^{1/2} + (\theta_0, z_h)_{T_h}
\]
This holds for any \( z_h \in V^\text{div}_h \times \tilde{V}_h \), so fix \( s \in I_n \) and select \( z_h = (\theta_h(s), \tilde{\theta}_h(s)) \in V_h \) to find
\[
\| \theta_h(s) \|_{L^2(\Omega)}^2 \leq C \left( \nu t^{1/2} + \nu' t^{1/2} \right) \left( \| \theta_h(s) \|_v \left( \int_{I_n} \| \theta_h \|_v^2 dt \right)^{1/2} +
\]
\[
C \nu t^{1/2} \| \theta_h(s) \|_v \left( \int_{I_n} \| \eta_h \|_v^2 dt \right)^{1/2} + (\theta_0, \theta_h(s))_{T_h}
\]
This holds for all \( s \in I_n \), so the result follows after integrating both sides over \( I_n \) and applying the Cauchy–Schwarz inequality and Young’s inequality. \( \square \)

**Lemma 4.7.** Let \( u \in L^\infty(0, T; V) \cap L^2(0, T; V \cap H^2(\Omega)^d) \cap H^1(0, T; H) \) be the strong solution to the continuous Navier–Stokes problem, let \( (\Pi_h u, \Pi_h u) \in V^\text{div}_h \times \tilde{V}_h \) be the velocity pair of the Stokes projection eq. (49), and let \( (u_h, \tilde{u}_h) \in V_h \) be an approximate velocity solution to the Navier–Stokes system computed using the space-time HDG scheme eq. (8) for \( n = 0, \ldots, N-1 \). Let \( \theta_h = \Pi_h u - u_h, \eta_h = u - \Pi_h u, \tilde{\theta}_h = \Pi_h u - \tilde{u}_h \) and \( \tilde{\eta}_h = u - \Pi_h u \). There exists a constant \( C > 0 \) such that
\[
\| \theta_N^+ \|_{L^2(\Omega)}^2 + \sum_{n=0}^{N-1} \| \theta_n \|_{L^2(\Omega)}^2 + \nu \int_0^T \| \theta_h \|_v^2 dt \leq C \exp(CT) \nu t \int_0^T \| \eta_h \|_v^2 dt,
\]
provided the time step satisfies \( \Delta t \leq C \nu \).

**Proof.** Our starting point for deriving an error estimate for the velocity will be the error equation eq. (59). We begin by bounding the nonlinear convection terms. A few algebraic manipulations yield
\[- \int_{I_n} (o_h(u; u, \theta_h) - o_h(u_h; u_h, \theta_h)) dt \]
\[
\leq \int_{I_n} |o_h(u; \eta_h, \theta_h)| dt + \int_{I_n} |o_h(\eta_h; \Pi_h u, \theta_h)| dt + \int_{I_n} |o_h(\theta_h; \Pi_h u, \theta_h)| dt = T_1 + T_2 + T_3,
\]
where we have used that $o_h(u_h; \theta_h, \theta_h) \geq C \nu$. We now bound $T_1$ and $T_2$. By eq. (14), the assumption eq. (6) on the problem data, and Young’s inequality with some $\epsilon_1 > 0$, we find
\begin{equation}
\int_{I_n} |o_h(u; \eta_h, \theta_h)| \, dt \leq \frac{C \nu}{2 \epsilon_1} \int_{I_n} \| \eta_h \|_v^2 \, dt + \frac{\nu \epsilon_1}{2} \int_{I_n} \| \theta_h \|_v^2 \, dt,
\end{equation}
and similarly,
\begin{equation}
\int_{I_n} |o_h(\eta_h; \Pi_h u, \theta_h)| \, dt \leq \frac{C \nu}{2 \epsilon_1} \int_{I_n} \| \eta_h \|_v^2 \, dt + \frac{\nu \epsilon_1}{2} \int_{I_n} \| \theta_h \|_v^2 \, dt.
\end{equation}
The bound on $T_3$ is more complicated. To begin, we use Lemma 3.3 and Hölder’s inequality with $p = 4$ and $q = 4/3$ to find
\begin{equation}
\int_{I_n} |o_h(\theta_h; \Pi_h u, \theta_h)| \, dt \leq C \left( \int_{I_n} \| \theta_h \|_{L^2(\Omega)} \| \Pi_h u \|_v \, dt \right)^{1/4} \left( \int_{I_n} \| \theta_h \|_v^2 \, dt \right)^{3/4}.
\end{equation}
Recall Young’s inequality in the form $ab \leq \epsilon_2 a^p/p + \epsilon_2 (p+1) \nu b^q$ where $1/p + 1/q = 1$, $1 < q, p < \infty$, $a, b > 0$, and $\epsilon_2 > 0$ (see e.g. [25, Appendix A]). Choosing $p = 4$ and $q = 4/3$ we find
\begin{equation}
\int_{I_n} |o_h(\theta_h; \Pi_h u, \theta_h)| \, dt \leq C \left( \epsilon_2 \int_{I_n} \| \theta_h \|_{L^2(\Omega)}^4 \, dt \right)^{1/4} \left( \int_{I_n} \| \theta_h \|_v^2 \, dt \right)^{3/4}.
\end{equation}
Here, we have used the uniform bound on the Stokes projection in Lemma 4.4 and the assumption eq. (4) on the problem data. Next, we consider the error equation eq. (59). Integrating by parts in time on the left hand side of eq. (59), combining the result with eq. (65), eq. (66), eq. (67), and using the coercivity of $a_h(\cdot, \cdot)$ eq. (12), we have for some constants $C_1, C_2 > 0$:
\begin{equation}
\| \theta_{n+1} \|^2_{L^2(\Omega)} + \| \theta_n \|^2_{L^2(\Omega)} - \| \theta_n \|^2_{L^2(\Omega)} + C_1 \nu \int_{I_n} \| \theta_h \|_v^2 \, dt
\leq C_2 \left( \epsilon_1^{-1} \int_{I_n} \| \eta_h \|_v^2 \, dt + \epsilon_1 \nu \int_{I_n} \| \theta_h \|_v^2 \, dt + \epsilon_1 \int_{I_n} \| \theta_h \|_{L^2(\Omega)}^2 \, dt + \epsilon_1 \int_{I_n} \| \theta_h \|_v^2 \, dt \right).
\end{equation}
Choosing $\epsilon_1 = C_1/(2C_2)$ and $\epsilon_2 = C_3 \nu^{-1}$ where $C_3 > 2C_2/C_1$ in eq. (68), letting $C_4 = C_1/2 - C_2/C_3 > 0$, and using Lemma 4.6, we have upon rearranging that
\begin{align*}
\| \theta_{n+1} \|^2_{L^2(\Omega)} + \| \theta_n \|^2_{L^2(\Omega)} + (C_4 \nu - C_5 \nu^{1/2} \Delta t^{1/2}) \int_{I_n} \| \theta_h \|_v^2 \, dt
\leq (1 + C_5 \Delta t) \| \theta_n \|^2_{L^2(\Omega)} + C_5 \nu \Delta t \int_{I_n} \| \eta_h \|_v^2 \, dt.
\end{align*}
Summing over all space-time slabs and noting that $\theta_0 = 0$, we have
\begin{align*}
\| \theta_N \|^2_{L^2(\Omega)} + \sum_{n=0}^{N-1} \| \theta_n \|^2_{L^2(\Omega)} + (C_4 \nu - C_5 \nu^{1/2} \Delta t^{1/2}) \int_0^T \| \theta_h \|_v^2 \, dt
\leq C_5 \Delta t \left( \sum_{n=0}^{N-1} \| \theta_n \|^2_{L^2(\Omega)} + \nu \int_0^T \| \eta_h \|_v^2 \, dt \right),
\end{align*}
The result follows by a discrete Grönwall inequality [15, Lemma 1.11] for $\Delta t < C_4 \nu/(2C_5)$ and using that $\prod_{j=0}^{N-1} (1 + C \Delta t) \leq \exp \left( C \sum_{j=0}^{N-1} \Delta t \right) \leq \exp \left( C \Delta t \right)$.
4.5. Proof of Theorem 1.3

Proof. Let \( e_h = u - u_h \). We introduce the splitting \( e_h = \xi_h + \zeta_h + \theta_h \), where \( \theta_h = \Pi_h u - u_h \), \( \zeta_h = P_h u - \Pi_h u \), and \( \xi_h = u - P_h u \). Using the triangle inequality, Lemma 4.7, Lemma 4.5, and noting that \( [\xi_h]_n = 0 \) for \( n = 0, \ldots, N - 1 \), we find there exists a constant \( C > 0 \) such that

\[
\| e_h \|_{L^2(\Omega)}^2 + \sum_{n=0}^{N-1} \| [e_h]_n \|_{L^2(\Omega)}^2 + \sum_{n=0}^{N-1} \nu \int_0^T \| e_h \|_{v'}^2 \, dt \\
\leq \exp(CT) \left( \| \xi_h \|_{L^2(\Omega)}^2 + \nu \int_0^T \| \zeta_h \|_{v'}^2 \, dt \right).
\] (69)

To bound the last term on the right hand side of eq. (69), we employ Theorem A.2 to find

\[
\int_0^T \| \xi_h \|_{v'}^2 \, dt \lesssim h^{2k} \| u \|_{L^2(0,T;H^{k+1}(\Omega))}^2 + \Delta t^{2k+2} \| u \|_{H^{k+1}(0,T;H^2(\Omega))}^2.
\] (70)

The result will follow after bounding \( \| \xi_h \|_{L^2(\Omega)} \). By Lemma 4.1, there exists a constant \( C > 0 \) such that

\[
\| \xi_h \|_{L^2(\Omega)}^2 = \| u(T) - (P_h u)(T) \|_{L^2(\Omega)}^2 \lesssim h^{2k+2} \| u \|_{C(0,T;H^{k+1}(\Omega))}^2.
\]

5. Numerical results

In this section, we consider a simple test case with a manufactured solution to verify the theoretical results of the previous sections. We solve the Navier–Stokes equations on the space-time domain \( \Omega \times [0,T] = [0,1]^3 \). We impose Dirichlet boundary conditions along the boundaries \( x = 0, x = 1, y = 0 \), and Neumann boundary conditions along \( y = 1 \). We choose the problem data such that the exact solution is given by

\[
u = \sin(2\pi(x-t)) \cos(2\pi(y-t)), \quad p = \sin(2\pi(x-t)) \cos(2\pi(y-t)).
\]

This example was implemented using the Modular Finite Element Methods (MFEM) library \([2, 28]\) on prismatic space-time meshes.

We present the velocity and pressure errors, measured in the mesh-dependent \( \| \cdot \|_{v'} \)-norm and \( \| \cdot \|_{L^2(0,T;L^2(\Omega))} \)-norm, respectively, and rates of convergence for different levels of space-time refinement with polynomial degrees \( k = 2 \) and \( k = 3 \) in Table 1. We observe optimal rates of convergence for the velocity in the \( \| \cdot \|_{v'} \)-norm as expected from Theorem 1.3 as well as optimal rates of convergence for the pressure in the \( L^2(0,T;L^2(\Omega)) \)-norm.
| Cells per slab | Nr. of slabs | $\|u - u_h\|_{\nu'}$ | Rate | $\|p - p_h\|_{L^2(\Omega \times [0,T])}$ | Rate |
|---------------|--------------|-----------------|------|----------------------------------|------|
| 128           | 20           | 8.6e-01         | -    | 7.9e-03                          | -    |
| 512           | 40           | 2.1e-01         | 2.0  | 2.6e-03                          | 1.6  |
| 2048          | 80           | 5.2e-02         | 2.0  | 6.7e-04                          | 1.9  |
| 8192          | 160          | 1.3e-02         | 2.0  | 1.7e-04                          | 2.0  |
| 128           | 20           | 2.0e-01         | -    | 6.9e-04                          | -    |
| 512           | 40           | 2.7e-02         | 2.9  | 5.2e-05                          | 3.7  |
| 2048          | 80           | 3.5e-03         | 3.0  | 4.7e-06                          | 3.5  |
| 8192          | 160          | 4.3e-04         | 3.0  | 5.1e-07                          | 3.2  |

Table 1: Rates of convergence when solving eq. (1) with $\nu = 10^{-4}$. Note that $\Delta t = 1/(\text{Nr. of slabs})$. Top: using polynomials of degree $k = 2$, bottom: using polynomials of degree $k = 3$.

6. Conclusion

In this paper we have analyzed a space-time hybridized discontinuous Galerkin method for the evolutionary Navier–Stokes equations that is pointwise mass conserving, energy-stable, and pressure-robust. We have proven that there exists a solution to the resulting nonlinear algebraic system of equations in both two and three spatial dimensions, and under a small data hypothesis the solution is unique in two spatial dimensions. A priori error estimates in a mesh-dependent energy norm for the velocity which are anisotropic in space and time and independent of the pressure have been derived. Finally, we have verified our theoretical error bounds for the velocity with a numerical example.

A. Projection estimates

Here, we briefly outline the approximation properties of the projections $P_h$ and $\bar{P}_h$ introduced in Definition 4.1. We require some results from [15] adapted to the present setting:

**Theorem A.1.** [15, Theorem 6.9] The projections defined in Definition 4.1 exist and are unique. Furthermore, for all $n = 0, \ldots, N - 1$,

$$(P_h v)|_{I_n} = P_h(P_h v)|_{I_n} = P_h(P_h v)|_{I_n}, \quad (\bar{P}_h v)|_{I_n} = \bar{P}_h(\bar{P}_h v)|_{I_n} = \bar{P}_h(\bar{P}_h v)|_{I_n}.$$  

We first introduce a temporal “DG-projection” $P^t : C(I_n) \to P_k(I_n)$ satisfying $\int_{\Omega} (P^t w(t) - w(t)) v dt = 0$ for all $v \in P_{k-1}(I_n)$ and such that $P^t w(t_{n+1}) = w(t_{n+1})$ (see, e.g., [38, Chapter 12] or [15, Lemma 6.11]). For $u \in H^{k+1}(I_n)$ this projection satisfies

$$\|u - P^t u\|_{H^s(I_n)} \lesssim \Delta t^{r-s} |u|_{H^r(I_n)}, \quad 0 \leq s \leq 1 \leq r \leq k. \quad (A.1)$$

**Lemma A.1.** Let $\varphi \in C(I_n; V_{h,\text{div}})$ and $\tilde{\psi} \in C(I_n; \bar{V}_h)$. Then

$$P_h \varphi(x,t) = P^t \varphi(x,t) \quad \forall x \in \{K,F\}, \forall \{K \in \mathcal{T}_h, F \in \mathcal{F}_h\}, \quad (A.2a)$$

$$\bar{P}_h \tilde{\psi}(x,t) = P^t \tilde{\psi}(x,t) \quad \forall x \in F, \forall F \in \mathcal{F}_h. \quad (A.2b)$$

**Proof.** The proofs of eq. (A.2a) and eq. (A.2b) follow the proof of [15, Lemma 6.11] with minor modifications. \qed
With Lemma A.1 in hand, we can prove the following results:

**Theorem A.2.** Let \( k \geq 1, m \in \{0, 1\}, \) and \( u \in H^{k+1}(I_n; H_0^1(\Omega)^d \cap H^2(\Omega)^d) \cap C(I_n; H^{k+1}(\Omega)^d). \) Let \( P_h \) and \( \bar{P}_h \) be the projections defined in Definition 4.1. Let \( l = 0 \) if \( m \leq 1 \) and \( l = m \) if \( m = 2. \) Then, the following estimates hold:

\[
\sum_{K \in T_h} \int_{I_n} |u - P_h u|_{H^m(K)}^2 \, dt \leq h^{2(k-m+1)} \|u\|_{L^2(I_n; H^{k+1}(\Omega))}^2 + h^{-l} \Delta t^{2k+2} \|u\|_{H^{k+1}(I_n; H^m(\Omega))}^2, \tag{A.3a}
\]

\[
\sum_{K \in T_h} \frac{1}{h_K} \int_{I_n} \|P_h u - \bar{P}_h u\|_{L^2(\partial K)}^2 \, dt \leq \sum_{K \in T_h} \frac{1}{h_K} \int_{I_n} \|P_h u - \bar{P}_h u\|_{L^2(\partial K)}^2 \, dt \leq \sum_{K \in T_h} \frac{1}{h_K} \int_{I_n} \|P_h u - \bar{P}_h u\|_{L^2(\partial K)}^2 \, dt \leq C \sum_{K \in T_h} \frac{1}{h_K} \int_{I_n} \|P_h u - \bar{P}_h u\|_{L^2(\partial K)}^2 \, dt. \tag{A.3b}
\]

\[
\sum_{K \in T_h} \int_{I_n} \|\nabla(u - P_h u)n\|_{L^2(\partial K)}^2 \, dt \leq h^{2k} \|u\|_{L^2(I_n; H^{k+1}(\Omega))}^2 + \Delta t^{2k+2} \|u\|_{H^{k+1}(I_n; H^2(\Omega))}^2. \tag{A.3c}
\]

**Proof.** First, eq. (A.3a) can be shown in a similar fashion to [15, Lemmas 6.17 and 6.18] using the approximation properties of the spatial projection \( P_h \) given in Lemma 4.1 and the approximation properties of \( P^l \) given in eq. (A.1). For eq. (A.3b) we recall that \( (P_h v)|_{I_n} = P_h(P_h v)|_{I_n} \) and \( (P_h v)|_{I_n} = \bar{P}_h(P_h v)|_{I_n} \) by Theorem A.1, so by Lemma A.1, Fubini’s theorem, and the stability of \( P^l \) in the \( L^2(I_n) \) norm, we have

\[
\sum_{K \in T_h} \int_{I_n} \|P_h u - \bar{P}_h u\|_{L^2(\partial K)}^2 \, dt \leq C \sum_{K \in T_h} \frac{1}{h_K} \int_{I_n} \|P_h u - \bar{P}_h u\|_{L^2(\partial K)}^2 \, dt.
\]

We conclude using the triangle inequality and the approximation properties of the spatial projections \( P_h \) and \( \bar{P}_h \). Finally, eq. (A.3c) follows from eq. (A.3a) after noting that a local trace inequality yields

\[
h_K \|\nabla(u - P_h u)n\|_{L^2(\partial K)} \leq |u - P_h u|_{H^1(K)} + h_K^2 \|u - P_h u\|_{H^2(K)}.
\]

\( \square \)

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