Three-generation Models from $E_8$
Magnetized Extra Dimensional Theory

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Abstract

We study 10D super Yang-Mills $E_8$ theory on the 6D torus compactification with magnetic fluxes. We study systematically the possibilities for realizing 4D supersymmetric standard models with three generations of quarks and leptons. We also study quark mass matrices.
1 Introduction

Extra dimensional field theories, in particular string-derived extra dimensional field theories, play an important role in particle phenomenology as well as cosmology. Realization of a 4D chiral theory is one of important issues when we start with higher dimensional theories. Non-trivial gauge and geometrical backgrounds would lead to various 4D chiral theories.

Introducing magnetic fluxes is one of interesting ways to realize a 4D chiral theory. Indeed, several studies on models with magnetic fluxes have been carried out in field theories and superstring theories [1–9]. Furthermore, magnetized D-brane models are T-duals of intersecting D-brane models and within the latter framework several interesting models have been constructed [4–6, 10–13].

In extra dimensional models with magnetic fluxes, the number of zero-modes is determined by the size of magnetic fluxes. Their wavefunction profiles are quasi-localized in extra dimensions. We can compute Yukawa couplings and higher order couplings in 4D effective theories by overlap integrals of zero-mode wavefunctions [7, 15–17]. When zero-modes are quasi-localized far away each other in extra dimensions, their couplings would be suppressed. On the other hand, when their localizing points are close to each other, their couplings would be large. Thus, extra dimensional models with magnetic fluxes would be quite interesting from the phenomenological viewpoint. In addition to the torus compactification, the orbifold compactification with magnetic fluxes also leads to various interesting models [18, 19].

In most of model building with magnetic fluxes, one has often started with $U(N)$ gauge groups. That is a reasonable starting point from the viewpoint of magnetized D-brane models. Then, several interesting models have been constructed as said above. On the other hand, gauge theories with the gauge groups, $E_6$, $E_7$ and $E_8$ are also interesting from the bottom-up phenomenological viewpoints. That is, those gauge theories are interesting as grand unified theories in particle physics and quarks and leptons are involved in adjoint representations of those gauge groups in group-theoretical sense. Thus, it would be interesting to study extra dimensional models with magnetic fluxes and these exceptional groups. Indeed, such a study has been carried out in [26], showing 4D interesting effective theories with semi-realistic massless spectra. In particular, the $E_8$ models have the most variety because the gauge group is largest. In this paper, we study 10D $E_8$ super Yang-Mills theory on the 6D torus compactification with magnetic fluxes. We systematically classify 4D effective theories with semi-realistic massless spectra.

This paper is organized as follows. In section 2, we review briefly zero-modes and their wavefunctions and Yukawa couplings. In section 3, we study systematically three-generation models. We also study quark mass matrices in our models. Section 4 is devoted to conclusion and discussion.

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1 See for a review [14] and references therein.

2 These exceptional gauge groups can be derived in heterotic string theory, type IIB string theory with non-perturbative effects and F-theory. For example, $E_8 \times E_8$ heterotic orbifold models lead to realistic models [20–24]. See also for another heterotic models e.g. Ref. [25].
2 Magnetized extra dimensions

Here we give a brief review on the torus models with magnetic fluxes [7]. We start with 10D super Yang-Mills theory, which has the gauge group $G$. We denote the vector fields and gaugino fields by $A_M$ ($M = 0, \cdots, 9$) and $\lambda$, respectively. Its Lagrangian is written as

$$
\mathcal{L} = -\frac{1}{4g^2} \text{Tr} (F^{MN} F_{MN}) + \frac{i}{2g^2} \text{Tr} (\bar{\lambda} \Gamma^M D_M \lambda),
$$

(1)

where $\Gamma^M$ is the gamma matrix for ten-dimensions and the covariant derivative $D_M$ is given as

$$
D_M \lambda = \partial_M \lambda - i[A_M, \lambda],
$$

(2)

where $A_M$ is the vector field. Furthermore, the field strength $F_{MN}$ is given by

$$
F_{MN} = \partial_M A_N - \partial_N A_M - i[A_M, A_N].
$$

(3)

2.1 Zero-modes on magnetized torus

We consider the background $R^{3,1} \times (T^2)^3$, whose coordinates are denoted by $x_{\mu}$ ($\mu = 0, \cdots, 3$) for the uncompact space $R^{3,1}$ and $y_m$ ($m = 4, \cdots, 9$) for the compact space $(T^2)^3$. We often use complex coordinations $z_d$ ($d = 1, 2, 3$) for the $d$-th torus $T^2_d$, e.g. $z_1 = y_4 + \tau_1 y_5$. Here, $\tau_d$ denote complex structure moduli of the $d$-th $T^2_d$, while the area of $T^2_d$ is denoted by $A_d$. The periodicity on $T^2_d$ is written as $z_d \sim z_d + 1_d$ and $z_d \sim z_d + \tau_d$.

The gaugino fields $\lambda$ and the vector fields $A_\mu$ and $A_m$ are decomposed as

$$
\lambda(x, z) = \sum_n \chi_n(x) \otimes \psi_n(z),
$$

$$
A_\mu(x, z) = \sum_n A_{n, \mu}(x) \otimes \phi_{n, \mu}(z),
$$

$$
A_m(x, z) = \sum_n \phi_{n, m}(x) \otimes \phi_{n, m}(z).
$$

(4)

Hereafter, we concentrate on zero-modes, $\psi_0(z)$, and we denote them as $\psi(z)$ by omitting the subscript “0”. Furthermore, the internal part $\psi(z)$ is decomposed as a product of the $T^2_d$ parts, i.e. $\psi_{(d)}(z_d)$. Each of $\psi_{(d)}(z_d)$ is two-component spinor,

$$
\psi_{(d)} = \begin{pmatrix} \psi_{+(d)} \\ \psi_{-(d)} \end{pmatrix},
$$

(5)

and their chirality for the $d$-th part is denoted by $s_d$. We use the following gamma matrix for $T^2_d$

$$
\tilde{\Gamma}^1_{(d)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tilde{\Gamma}^2_{(d)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.
$$

(6)
We introduce the magnetic flux along the $U(1)_a$ (Cartan) direction of $G$ on $T^2_d$,

$$F = \frac{\pi i}{\text{Im} \tau_d} m^a_{(d)} (dz_d \wedge d\bar{z}_d),$$

(7)

where $m^a_{(d)}$ is an integer [27]. Here, we normalize $U(1)_a$ charges $q^a$ such that all $U(1)_a$ charges are integers and the minimum satisfies $|q^a| = 1$. The above magnetic flux can be obtained from the vector potential,

$$A(z_d) = \frac{\pi m^a_{(d)}}{\text{Im} \tau_d} \text{Im}(\bar{z}_d \; dz_d).$$

(8)

This form of the vector potential satisfies the following relations,

$$A(z_d + 1) = A(z_d) + \frac{\pi m^a_{(d)}}{\text{Im} \tau_d} \text{Im}(dz_d),$$

$$A(z_d + \tau_d) = A(z_d) + \frac{\pi m^a_{(d)}}{\text{Im} \tau_d} \text{Im}(\bar{z}_d \; dz_d).$$

(9)

These relations can be represented as the following gauge transformations,

$$A(z_d + 1) = A(z_d) + d\chi_1^{(d)}, \quad A(z_d + \tau_d) = A(z_d) + d\chi_2^{(d)},$$

(10)

where

$$\chi_1^{(d)} = \frac{\pi m^a_{(d)}}{\text{Im} \tau_d} \text{Im}(z_d), \quad \chi_2^{(d)} = \frac{\pi m^a_{(d)}}{\text{Im} \tau_d} \text{Im}(\bar{z}_d \; z_d).$$

(11)

Then, the fermion field $\psi_{(d)}(z_d)$ with the $U(1)_a$ charge $q^a$ must satisfy

$$\psi_{(d)}(z_d + 1) = e^{i q^a \chi_1^{(d)}(z_d)} \psi_{(d)}(z_d), \quad \psi_{(d)}(z_d + \tau_d) = e^{i q^a \chi_2^{(d)}(z_d)} \psi_{(d)}(z_d).$$

(12)

By the magnetic flux [27] along the $U(1)_a$ direction, all of 4D gauge vector fields $A_\mu$ with non-vanishing $U(1)_a$ charges, become massive, that is, the gauge group is broken from $G$ to $G' \times U(1)_a$ without reducing its rank [3], where 4D gauge fields $A_\mu$ in $G' \times U(1)_a$ have vanishing $U(1)_a$ charges and their zero-modes $\phi_\mu(z)$ have a flat profile. Since the magnetic flux has no effect on the unbroken gauge sector, 4D N=4 supersymmetry remains in the $G' \times U(1)_a$ sector, that is, there are massless four adjoint gaugino fields and six adjoint scalar fields [4].

In addition, matter fields appear from gaugino fields corresponding to the broken gauge part, that is, they have non-trivial representations under $G'$ and non-vanishing $U(1)_a$ charges $q^a$. The Dirac equations for their zero-modes become

$$(\partial_{z_d} + \frac{\pi q^a m^a_{(d)}}{2 \text{Im}(\tau_d)} \bar{z}_d) \psi_{+(d)}(z_d, \bar{z}_d) = 0,$$

(13)

$$\left( \partial_{z_d} - \frac{\pi q^a m^a_{(d)}}{2 \text{Im}(\tau_d)} \bar{z}_d \right) \psi_{-(d)}(z_d, \bar{z}_d) = 0,$$

(14)

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3For example, when $G = SU(N)$, $G'$ would correspond to $SU(N - 1)$.

4In string terminology, these adjoint scalar fields correspond to open string moduli, that is, D-brane position moduli. How to stabilize these moduli is one of important issues.
for $T_d^2$. When $q^a m_{(d)}^0 > 0$, the component $\psi_{+(d)}$ has $M = q^a m_{(d)}^0$ independent zero-modes and their wavefunctions are written as \cite{11}

$$\Theta^{j,M}(z) = N_M e^{i\pi M z / \text{Im}(\tau)} \theta \left[ \begin{array}{c} j/M \\ 0 \end{array} \right] (M z, M \tau),$$ \hfill (15)

where $j$ denotes the flavor index, i.e. $j = 1, \cdots, M$ and

$$\theta \left[ \begin{array}{c} a \\ b \end{array} \right] (\nu, \mu) = \sum_n \exp \left[ \pi i (n+a) \nu + 2\pi i (n+a)(\nu+b) \right],$$

that is, the Jacobi theta-function. Here, the normalization factor $N_M$ is obtained as

$$N_M^{(d)} = \left( \frac{2\text{Im}\tau_d |M(d)|}{A_d^2} \right)^{1/4}.$$ \hfill (16)

Note that $\Theta^{0,M}(z) = \Theta^{M,M}(z)$. Furthermore, for $q^a m_{(d)}^0 > 0$, the other component $\psi_{-(d)}$ has no zero-modes. On the other hand, when $q^a m_{(d)}^0 < 0$, the component $\psi_{-(d)}$ has $|q^a m_{(d)}^0|$ independent zero-modes, but the other component $\psi_{+(d)}$ has no zero-modes.

As a result, we can realize a chiral spectrum when we introduce magnetic fluxes on all of three $T_d^2$. That is, since the ten-dimensional chirality of gaugino fields is fixed, zero-modes for either $q^a > 0$ and $q^a < 0$ appear with a fixed four-dimensional chirality.

We can also introduce Wilson lines along the $U(1)_b$ direction of $G'$. That breaks further the gauge group $G'$ to $G'' \times U(1)_b$ without reducing its rank\footnote{For example, when $G' = SU(N - 1)$, the Wilson line breaks it to $SU(N - 2) \times U(1)_b$.}. All of the $U(1)_b$-charged fields including 4D vector, spinor and scalar fields become massive because of the Wilson line, when they are not charged under $U(1)_a$ and their zero-mode profiles are flat. On the other hand, the matter fields with non-trivial profiles due to magnetic flux have different behavior. For matter fields with $U(1)_a$ charge $q^a$ and $U(1)_b$ charge $q^b$, the Dirac equations of the zero-modes are modified by the Wilson line background, $C_d^b = C_{d,1}^b + \tau_d C_{d,2}^b$ as

$$\left( \partial_{z_d} + \frac{\pi}{2\text{Im} \tau_d} (q^a m_{(d)}^0 z_d + q^b C_{d}^b) \right) \psi_{+(d)}(z_d, \bar{z}_d) = 0,$$ \hfill (17)

$$\left( \partial_{z_d} - \frac{\pi}{2\text{Im} \tau_d} (q^a m_{(d)}^0 \bar{z}_d + q^b C_{d}^b) \right) \psi_{-(d)}(z_d, \bar{z}_d) = 0,$$ \hfill (18)

where $C_{d,1}^b$ and $C_{d,2}^b$ are real parameters. That is, we can introduce Wilson lines along the $U(1)_b$ direction by replacing $\chi_{(d)}^{(i)}$ in (11) as \cite{11}

$$\chi_{1}^{(d)} = \frac{\pi}{\text{Im} \tau_d} \text{Im}(m_{(d)}^0 z_d + q^b C_{d}^b / q^a), \quad \chi_{2}^{(d)} = \frac{\pi}{\text{Im} \tau_d} \text{Im}(\tau_d (m_{(d)}^0 z_d + q^b C_{d}^b / q^a)).$$ \hfill (19)

Because of this Wilson line, the number of zero-modes does not change, but their wave functions are shifted as

$$\Theta^{j,M}(z_d) \rightarrow \Theta^{j,M}(z_d + q^b C_{d}^b / (q^a m_{(d)}^0)).$$ \hfill (20)
Note that the shift of zero-mode profiles depend on \( U(1)_b \) charges of matter fields. We often denote the degree of Wilson lines as \( \zeta^{(d)} = q^b \xi^b_{d}/(q^a m^a_{(d)}) \). Also, we can introduce the Wilson line \( \xi^d_{a} \) along the \( U(1)_a \) direction.

Similarly we can analyze 4D massless scalar modes \([7]\). When the above magnetic fluxes satisfies the following relation,

\[
\sum_{d=1}^{3} \pm \frac{m^a_{(d)}}{A^a_d} = 0, \tag{21}
\]

for one combinations of signs, the 4D \( N=1 \) supersymmetry is preserved \([7,8,28]\) for the above fermion fields with the \( U(1) \) charge \( q^a \). That is, there is the same number of 4D scalar zero-modes and their wave function profiles are the same as the above fermion fields. For example, for Higgs fields, we study zero-modes and their profiles of Higgsino fields.

### 2.2 Yukawa couplings

The Yukawa couplings of zero-modes in 4D effective theory can be computed by overlap integral of their wavefunctions. For example, we consider the coupling among three fields, whose wavefunctions are written by \( \psi^i(z) \), \( \psi^j(z) \) and \( \psi^k(z)^* \). Their 4D Yukawa couplings are obtained by the overlap integral of wavefunctions

\[
Y_{ijk} = g \int d^6 z \; \psi^i(z) \psi^j(z) \left( \psi^k(z) \right)^*, \tag{22}
\]

on the 6D extra dimensions.

Suppose that the matter fields \( \psi^i(z) \), \( \psi^j(z) \) and \( \psi^k(z)^* \) have \( M_{1}^{(d)} \), \( M_{2}^{(d)} \) and \( M_{3}^{(d)} \) zero-modes on the \( d \)-th torus \( T_{(d)}^2 \) by certain magnetic fluxes and their wavefunctions are affected by Wilson lines, \( \zeta_1^{(d)}, \zeta_2^{(d)} \) and \( \zeta_3^{(d)} \) on the \( d \)-th torus \( T_{(d)}^2 \). Then, their wavefunctions on \( T_{(d)}^2 \) are written by \( \Theta^{i,M_{1}^{(d)}}(z_d + \zeta_1^{(d)}) \), \( \Theta^{j,M_{2}^{(d)}}(z_d + \zeta_2^{(d)}) \) and \( \Theta^{k,M_{3}^{(d)}}(z_d + \zeta_3^{(d)})^* \) as Eq. (15). In this case, the corresponding Yukawa coupling can be written as \([7]\)

\[
Y_{ijk} = g \prod_{d=1}^{3} \left( \frac{2 \text{Im} \tau_d M_{1}^{(d)} M_{2}^{(d)}}{M_{3}^{(d)}} \right)^{1/4} \times \exp \left( \frac{i \pi}{M_{1}^{(d)}} \text{Im} \zeta_1^{(d)} + M_{2}^{(d)} \text{Im} \zeta_2^{(d)} + M_{3}^{(d)} \text{Im} \zeta_3^{(d)}/\text{Im} \tau_d \right) \times \| \begin{pmatrix} \frac{1}{M_{1}^{(d)}} & \frac{j}{M_{2}^{(d)}} & \frac{k}{M_{3}^{(d)}} \\ \zeta_1^{(d)} & \tau_d M_{1}^{(d)} M_{2}^{(d)} & M_{3}^{(d)} \end{pmatrix} \right| (\tilde{\zeta}^{(d)}, \tau_d M_{1}^{(d)} M_{2}^{(d)} M_{3}^{(d)}), \tag{23}
\]

where \( \tilde{\zeta}^{(d)} = M_{2}^{(d)} M_{3}^{(d)} (\zeta_2^{(d)} - \zeta_3^{(d)}) \).

### 3 \( E_8 \) theory

Here, we study \( 10 \)D \( N=1 \) super Yang-Mills theory with the gauge group \( E_8 \).
3.1 Magnetic fluxes

When we decompose $E_8$ to $SU(3) \times SU(2) \times U(1)_Y \times U(1)^4$, $U(1)^5$ including $U(1)_Y$ appear. We can introduce magnetic fluxes along these $U(1)^5$ directions. The 248 adjoint representation of $E_8$ is decomposed to several representations under $SU(3) \times SU(2) \times U(1)_Y \times U(1)^4$. Certain representations are shown in Table 1 where we follow the notation in Ref. [29]. The total 248 representation consists of the representations in Table 1 their conjugate representations and the adjoint representations of $SU(3) \times SU(2) \times U(1)_Y \times U(1)^4$.

Now, we introduce magnetic fluxes $m_{i(d)}^I$ along five $U(1)_I$ ($I = a, b, c, d, Y$) directions on the $d$-torus. Then, the sum of magnetic fluxes $\sum_I q^I m_{i(d)}^I$ appears in the zero-mode Dirac equation for the matter fields with the $U(1)_I$ charges $q^I$. We require $\sum_I q^I m_{i(d)}^I$ to be integers for all of matter fields, that is, the quantization conditions of magnetic fluxes. For example, five $(3, 2)_1$ representations under $SU(3) \times SU(2) \times U(1)_Y$ as well as their conjugates appear from the 248 adjoint representation. In the zero-mode equations of these five $(3, 2)_1$ matter fields, $Q_i$ ($i = 1, \cdots, 5$), the following sum of magnetic fluxes $\sum_I q^I m_{i(d)}^I$ appear

\[
\begin{align*}
  m_{i(d)}^{Q_1} &= m_{i(d)}^a + m_{i(d)}^c - m_{i(d)}^d + m_{i(d)}^Y, \\
  m_{i(d)}^{Q_2} &= m_{i(d)}^b + m_{i(d)}^c - m_{i(d)}^d + m_{i(d)}^Y, \\
  m_{i(d)}^{Q_3} &= -m_{i(d)}^a - m_{i(d)}^b + m_{i(d)}^c - m_{i(d)}^d + m_{i(d)}^Y, \\
  m_{i(d)}^{Q_4} &= -3m_{i(d)}^c - m_{i(d)}^d + m_{i(d)}^Y, \\
  m_{i(d)}^{Q_5} &= 4m_{i(d)}^d + m_{i(d)}^Y.
\end{align*}
\]

Thus, we have to take the magnetic fluxes such that all of $m_{i(d)}^{Q_i}$ for $i = 1, \cdots, 5$ are integers.

We can write the sum of magnetic fluxes $\sum_I q^I m_{i(d)}^I$ for the other matter fields in terms of $m_{i(d)}^{Q_i}$. For example, the sum of magnetic fluxes $\sum_I q^I m_{i(d)}^I$ for the matter field $Q_Y$ can be written as

\[
m_{(d)}^{Q_Y} = m_{(d)}^{Q_1} + m_{(d)}^{Q_2} + m_{(d)}^{Q_3} + m_{(d)}^{Q_4} + m_{(d)}^{Q_5}.
\]

Thus, if all of $m_{i(d)}^{Q_i}$ for $i = 1, \cdots, 5$ are integers, $m_{(d)}^{Q_Y}$ are also integers. Similarly, the
splits of magnetic fluxes $\sum_I q^I m^I_{(d)}$ are written as
\[
\begin{align*}
m_{ui}^{(d)} &= m_{Q_i}^{(d)} - m_{Q_Y}^{(d)}, & \text{for } u_i (i = 1, \cdots, 5), \\
m_{ei}^{(d)} &= m_{Q_i}^{(d)} + m_{Q_Y}^{(d)}, & \text{for } e_i^c (i = 1, \cdots, 5), \\
m_{di}^{(d)} &= m_{Q_i}^{(d)} + m_{Q_5}^{(d)}, & \text{for } d_i^c (i = 1, \cdots, 4), \\
m_{Li}^{(d)} &= m_{Q_i}^{(d)} + m_{Q_5}^{(d)} - m_{Q_Y}^{(d)}, & \text{for } L_i (i = 1, \cdots, 4), \\
m_{vi}^{(d)} &= m_{Q_i}^{(d)} - m_{Q_5}^{(d)}, & \text{for } \nu_i^c (i = 1, \cdots, 4), \\
m_{Si}^{(d)} &= m_{Q_i}^{(d)} - m_{Q_4}^{(d)}, & \text{for } S_i (i = 1, 2, 3), \\
m_{Di}^{(d)} &= m_{Q_i}^{(d)} + m_{Q_4}^{(d)}, & \text{for } D_i^c (i = 1, 2, 3).
\end{align*}
\]

In addition, the sums $\sum_I q^I m^I_{(d)}$ are written as
\[
\begin{align*}
m_{D1}^{(d)} &= -m_{Q_2}^{(d)} - m_{Q_3}^{(d)}, & m_{D2}^{(d)} &= -m_{Q_3}^{(d)} - m_{Q_1}^{(d)}, & m_{D3}^{(d)} &= -m_{Q_1}^{(d)} - m_{Q_2}^{(d)}, \\
\text{for the matter fields } D_1, D_2 \text{ and } D_3,
\end{align*}
\]
\[
\begin{align*}
m_{N1}^{(d)} &= -m_{Q_1}^{(d)} - m_{Q_3}^{(d)}, & m_{N2}^{(d)} &= -m_{Q_2}^{(d)} - m_{Q_3}^{(d)}, & m_{N3}^{(d)} &= -m_{Q_1}^{(d)} - m_{Q_2}^{(d)}, \\
\text{for the matter fields } N_1, N_2 \text{ and } N_3.
\end{align*}
\]

Also, the sums $\sum_I q^I m^I_{(d)}$ are written as
\[
\begin{align*}
m_{H^u}^{(d)} &= m_{Dj}^{(d)} + m_{Q_Y}^{(d)}, & \text{for } H^u_i (i = 1, 2, 3), \\
m_{H^d}^{(d)} &= m_{Dji}^{(d)} - m_{Q_Y}^{(d)}, & \text{for } H^d_i (i = 1, 2, 3).
\end{align*}
\]_Note that the sums $\sum_I q^I m^I_{(d)}$ for all of matter fields can be written in terms of $m_{Q_i}^{(d)} (i = 1, \cdots, 5)$ with integer coefficients. Hence, when all of $m_{Q_i}^{(d)}$ are integers, the sums $\sum_I q^I m^I_{(d)}$ for the other matter fields are always integers. Hereafter, we show the magnetic fluxes in terms of $m_{Q_i}^{(d)} (i = 1, \cdots, 5)$. That is, we classify models by studying systematically combinations of $m_{Q_i}^{(d)} (i = 1, \cdots, 5)$. We use the notation $m_\Phi = \prod_{d=1}^3 m_{(d)}^{(d)}$ for the matter field $\Phi$ and it denotes the total zero-mode numbers of $\Phi$. When $m_\Phi < 0$, the matter fields with conjugate representations appear, i.e. the anti-generations of $\Phi$.\]

### 3.2 Three-generation models
If the condition (21) is satisfied, 4D N=1 supersymmetry is preserved and tachyonic modes do not appear. Thus, first we concentrate on combinations of $m_{Q_i}^{(d)} (i = 1, \cdots, 5)$, which satisfy the condition (21). Here, we consider the area $\mathcal{A}_d$ as free parameters. Only their ratios, e.g. $\mathcal{A}_2/\mathcal{A}_1$ and $\mathcal{A}_3/\mathcal{A}_1$, are important to satisfy the condition (21). That is, there are two free parameters. Thus, all of five vectors $(m_{Q_1}^{(1)}, m_{Q_2}^{(1)}, m_{Q_3}^{(1)}) (i = 1, \cdots, 5)$, are not independent of each other to satisfy the condition (21). However, when five vectors $(m_{Q_{1(i)}}^{(1)}, m_{Q_{2(i)}}^{(1)}, m_{Q_{3(i)}}^{(1)}) (i = 1, 2, 3)$ are written in terms of two (independent) vectors, we can...
choose areas $A_d$ such that they satisfy the condition (21). Furthermore, a tachyonic mode would appear if one of $m^\Phi_{(i)}$ ($i = 1, 2, 2$) is not vanishing and the other two are vanishing, e.g. $(m^\Phi_{(1)}, m^\Phi_{(2)}, m^\Phi_{(3)}) = (m', 0, 0)$ with $m' \neq 0$. We rule out such a case.

In addition, we concentrate our systematic study on the following regions of $m^{Q_i}_{(d)}$ ($i = 1, \cdots, 5$). The matter fields $Q_i$ would correspond to left-handed quark doublets. Thus, we require that there are three generations, i.e.

$$\sum_{i=1}^{5} m^{Q_i} = 3. \quad (30)$$

On top of that, we concentrate on

$$m^{Q_i} \geq 0 \quad (31)$$

for each $Q_i$. That means that there is no anti-generations for quark doublets.

For the other matter fields, we allow anti-generations, but we require the total numbers of (chiral) generations to be equal to three,

$$\sum_{i=1}^{5} m^{ui} = 3, \quad \sum_{i=1}^{5} m^{ei} = 3, \quad \sum_{i=1}^{4} m^{di} = 3, \quad \sum_{i=1}^{4} m^{Li} + \sum_{i=1}^{3} m^{H^u_i} - \sum_{i=1}^{3} m^{H^d_i} = 3. \quad (32)$$

Only by the representations under $SU(3) \times SU(2) \times U(1)_Y$, one can not distinguish $L_i$, $H^d_i$ and conjugates of $H^u_i$. Thus, the last equation means that the number of the total chiral generation for the matter fields $Q_{(1, 2, 3)}$ under $SU(3) \times SU(2) \times U(1)_Y$ is equal to three and there are some vector-like generations with such a representation, which would correspond to pairs of Higgino fields. For simplicity, we concentrate $m^{Di} = m^{D^i} = 0$. The matter field $Q_Y$ has a representation similar to $Q_i$, but its $U(1)_Y$ charge is different. Thus, this field would correspond to an exotic matter field, and we require

$$m^{Q_Y} = 0. \quad (34)$$

We do not put any constraints on the $SU(3) \times SU(2) \times U(1)_Y$ singlets. We will classify the three-generation models with the above conditions in what follows.

All possible combinations are classified into the following seven types,

I : \quad $(m^{Q_1}, m^{Q_2}, m^{Q_3}, m^{Q_4}, m^{Q_5}) = (1, 1, 1, 0, 0),$

II : \quad $(m^{Q_1}, m^{Q_2}, m^{Q_3}, m^{Q_4}, m^{Q_5}) = (1, 1, 0, 0, 1),$

III : \quad $(m^{Q_1}, m^{Q_2}, m^{Q_3}, m^{Q_4}, m^{Q_5}) = (2, 0, 0, 0, 1),$

IV : \quad $(m^{Q_1}, m^{Q_2}, m^{Q_3}, m^{Q_4}, m^{Q_5}) = (1, 0, 0, 0, 2),$

V : \quad $(m^{Q_1}, m^{Q_2}, m^{Q_3}, m^{Q_4}, m^{Q_5}) = (2, 1, 0, 0, 0),$

VI : \quad $(m^{Q_1}, m^{Q_2}, m^{Q_3}, m^{Q_4}, m^{Q_5}) = (0, 0, 0, 0, 3),$

VII : \quad $(m^{Q_1}, m^{Q_2}, m^{Q_3}, m^{Q_4}, m^{Q_5}) = (3, 0, 0, 0, 0).$
Note that the matter representations in Table 1 have the permutation symmetries among $(Q_i, u^c_i, e^c_i, d^c_i, L_i)$ for $i = 1, 2, 3$. Furthermore, our conditions for the three-generation models are symmetric under the permutations among $m_{(d)}^{Q_i}$ for $i = 1, 2, 3, 4$. Up to such permutation symmetries, each of possible combinations is equivalent to one of the above types.

First, let us study the type I in (35). We choose
\[(m_{(1)}^{Q_1}, m_{(2)}^{Q_1}, m_{(3)}^{Q_1}) = (1, 1, 1).\] (36)
Then, we can not take $m_{(d)}^{Q_2} = m_{(d)}^{Q_1}$, because that leads to $m^{D_3} \neq 0$. Thus, the possible values of $m_{(d)}^{Q_2}$ are
\[(m_{(1)}^{Q_2}, m_{(2)}^{Q_2}, m_{(3)}^{Q_2}) = (1, -1, -1),\] (37)
and permutations of the entries. Similarly, for (36) and (37), the condition leads to
\[(m_{(1)}^{Q_3}, m_{(2)}^{Q_3}, m_{(3)}^{Q_3}) = (-1, -1, 1), \quad (-1, 1, -1), \quad (1, 1, 1).\] (38)
However, these vectors, $(m_{(1)}^{Q_i}, m_{(2)}^{Q_i}, m_{(3)}^{Q_i})$ ($i = 1, \cdots, 3$), are independent of each other. Then, we can not find $A_d$, which satisfy the SUSY condition. Thus, the type I is not interesting.

Similarly, we can study the type II in (35). We chose the same $m_{(d)}^{Q_1}$ and $m_{(d)}^{Q_2}$ as (36) and (37). Both $m_{(d)}^{Q_3}$ and $m_{(d)}^{Q_4}$ must be written by linear combinations of $m_{(d)}^{Q_1}$ and $m_{(d)}^{Q_2}$. Also the products, $\prod_d m_{(d)}^{Q_3}$ and $\prod_d m_{(d)}^{Q_4}$, must vanish. Then, possible combinations are obtained as
\[(m_{(1)}^{Q_3}, m_{(2)}^{Q_3}, m_{(3)}^{Q_3}) = (0, 2m, 2m), \quad (m_{(1)}^{Q_4}, m_{(2)}^{Q_4}, m_{(3)}^{Q_4}) = (0, 2n, 2n),\]
\[(m_{(1)}^{Q_3}, m_{(2)}^{Q_3}, m_{(3)}^{Q_3}) = (0, 2m, 2m), \quad (m_{(1)}^{Q_4}, m_{(2)}^{Q_4}, m_{(3)}^{Q_4}) = (2n, 0, 0),\]
\[(m_{(1)}^{Q_3}, m_{(2)}^{Q_3}, m_{(3)}^{Q_3}) = (2m, 0, 0), \quad (m_{(1)}^{Q_4}, m_{(2)}^{Q_4}, m_{(3)}^{Q_4}) = (0, 2n, 2n),\]
\[(m_{(1)}^{Q_3}, m_{(2)}^{Q_3}, m_{(3)}^{Q_3}) = (2m, 0, 0), \quad (m_{(1)}^{Q_4}, m_{(2)}^{Q_4}, m_{(3)}^{Q_4}) = (2n, 0, 0),\] (39)
where $m$ and $n$ are integers. The first three combinations do not satisfy $m^{D_1} = - \prod (m_{(d)}^{Q_2} + m_{(d)}^{Q_3}) = 0$ and $m^{D_2} = - \prod (m_{(d)}^{Q_1} + m_{(d)}^{Q_3}) = 0$. The last combination satisfies $m^{D_1} = m^{D_2} = 0$ when $m = n = -1$. However, such a case does not lead to $m^{Q_5} = \prod_d \left( \sum_{i=1}^5 m_{(d)}^{Q_i} \right) = 0$, because $\sum_{i=1}^4 m_{(d)}^{Q_i} = 0$ and $m_{(d)}^{Q_5} \neq 0$ for any $d$. In addition, the last three combinations lead to tachyonic modes. Thus, the type II does not lead to three-generation models.

We study the type V in (35). We choose
\[(m_{(1)}^{Q_1}, m_{(2)}^{Q_1}, m_{(3)}^{Q_1}) = (2, 1, 1).\] (40)
For $m_{(d)}^{Q_2}$, we have two possibilities,
\[(m_{(d)}^{Q_2}, m_{(d)}^{Q_2}, m_{(d)}^{Q_2}) = (1, -1, -1), \quad (-1, -1, 1).\] (41)
Then, we require other magnetic fluxes \((m_{Q_i}^{Q_i(1)}, m_{Q_i}^{Q_i(2)}, m_{Q_i}^{Q_i(3)})\) for \(i = 3, 4, 5\) can be written by linear combinations of \((m_{Q_1}^{Q_1(1)}, m_{Q_1}^{Q_1(2)}, m_{Q_1}^{Q_1(3)})\) and \((m_{Q_2}^{Q_2(1)}, m_{Q_2}^{Q_2(2)}, m_{Q_2}^{Q_2(3)})\) with integer coefficients. However, any combinations of this type can not lead to the three-generation models. For example, the conditions \(m_{Q_1}^{Q_1} = -\prod (m_{Q_2}^{Q_2} + m_{Q_3}^{Q_3}) = 0\) and \(m_{Q_2}^{Q_2} = -\prod (m_{Q_1}^{Q_1} + m_{Q_3}^{Q_3}) = 0\) are not satisfied.

The situation in the type III of (35) is similar. We take the same \((m_{Q_1}^{Q_1(1)}, m_{Q_1}^{Q_1(2)}, m_{Q_1}^{Q_1(3)})\) as Eq. (41). We have three possibilities for \(m_{Q_5}^{Q_5(1)}\) as

\[
(m_{Q_5}^{Q_5(1)}, m_{Q_5}^{Q_5(2)}, m_{Q_5}^{Q_5(3)}) = (1, 1, 1), \quad (1, -1, -1), \quad (-1, -1, 1).
\]

Then, we require other magnetic fluxes \((m_{Q_i}^{Q_i(1)}, m_{Q_i}^{Q_i(2)}, m_{Q_i}^{Q_i(3)})\) for \(i = 2, 3, 4\) can be written by linear combinations of \((m_{Q_1}^{Q_1(1)}, m_{Q_1}^{Q_1(2)}, m_{Q_1}^{Q_1(3)})\) and \((m_{Q_5}^{Q_5(1)}, m_{Q_5}^{Q_5(2)}, m_{Q_5}^{Q_5(3)})\) with integer coefficients. However, any combinations of this type can not lead to three-generation models.

The situation in the type IV of (35) is similar. We choose

\[
(m_{Q_1}^{Q_1(1)}, m_{Q_1}^{Q_1(2)}, m_{Q_1}^{Q_1(3)}) = (1, 1, 1),
\]

and we have three possibilities for \(m_{Q_5}^{Q_5(1)}\) as

\[
(m_{Q_5}^{Q_5(1)}, m_{Q_5}^{Q_5(2)}, m_{Q_5}^{Q_5(3)}) = (2, 1, 1), \quad (2, -1, -1), \quad (-2, -1, 1).
\]

Then, we require other magnetic fluxes \((m_{Q_i}^{Q_i(1)}, m_{Q_i}^{Q_i(2)}, m_{Q_i}^{Q_i(3)})\) for \(i = 2, 3, 4\) can be written by linear combinations of \((m_{Q_1}^{Q_1(1)}, m_{Q_1}^{Q_1(2)}, m_{Q_1}^{Q_1(3)})\) and \((m_{Q_5}^{Q_5(1)}, m_{Q_5}^{Q_5(2)}, m_{Q_5}^{Q_5(3)})\) with integer coefficients. However, any combinations of this type can not lead to three-generation models.

Now, let us study the type VI in (35). We choose

\[
(m_{Q_5}^{Q_5(1)}, m_{Q_5}^{Q_5(2)}, m_{Q_5}^{Q_5(3)}) = (3, 1, 1).
\]

Other magnetic fluxes \(m_{Q_i}^{Q_i(1)}\) for \(i = 1, 2, 3, 4\) must have vanishing elements for one of \(d = 1, 2, 3\). Various combinations are possible as

\[
\begin{align*}
(m_{Q_1}^{Q_1(1)}, m_{Q_2}^{Q_2(1)}, m_{Q_3}^{Q_3(1)}, m_{Q_4}^{Q_4(1)}) &= (0, 0, 0, 0), \\
(m_{Q_1}^{Q_1(1)}, m_{Q_2}^{Q_2(1)}, m_{Q_3}^{Q_3(1)}, m_{Q_4}^{Q_4(1)}) &= (0, 0, 0, 0), \\
(m_{Q_1}^{Q_1(1)}, m_{Q_2}^{Q_2(1)}, m_{Q_3}^{Q_3(1)}, m_{Q_4}^{Q_4(1)}) &= (0, 0, 0, 0), \\
(m_{Q_1}^{Q_1(1)}, m_{Q_2}^{Q_2(1)}, m_{Q_3}^{Q_3(1)}, m_{Q_4}^{Q_4(1)}) &= (0, 0, 0, 0), \\
(m_{Q_1}^{Q_1(1)}, m_{Q_2}^{Q_2(1)}, m_{Q_3}^{Q_3(1)}, m_{Q_4}^{Q_4(1)}) &= (0, 0, 0, 0), \\
(m_{Q_1}^{Q_1(1)}, m_{Q_2}^{Q_2(1)}, m_{Q_3}^{Q_3(1)}, m_{Q_4}^{Q_4(1)}) &= (0, 0, 0, 0), \\
(m_{Q_1}^{Q_1(1)}, m_{Q_2}^{Q_2(1)}, m_{Q_3}^{Q_3(1)}, m_{Q_4}^{Q_4(1)}) &= (0, 0, 0, 0), \\
(m_{Q_1}^{Q_1(1)}, m_{Q_2}^{Q_2(1)}, m_{Q_3}^{Q_3(1)}, m_{Q_4}^{Q_4(1)}) &= (0, 0, 0, 0), \\
(m_{Q_1}^{Q_1(1)}, m_{Q_2}^{Q_2(1)}, m_{Q_3}^{Q_3(1)}, m_{Q_4}^{Q_4(1)}) &= (0, 0, 0, 0), \\
(m_{Q_1}^{Q_1(1)}, m_{Q_2}^{Q_2(1)}, m_{Q_3}^{Q_3(1)}, m_{Q_4}^{Q_4(1)}) &= (0, 0, 0, 0).
\end{align*}
\]
Most of them do not lead to the three-generation models with the required conditions, but certain combinations of $m_{Qi}^{(d)}$ lead to three-generation models. Such combinations are shown in Tables 4, 6, 8, 10, 12, 14, 16, 18, 20 and 22 as the model VI-1, $\cdots$, 10. In those tables, the second, third and fourth rows show each of magnetic fluxes on $T_{1}^{2}$, $T_{2}^{2}$ and $T_{3}^{2}$, respectively. The corresponding massless spectra are shown in Tables 5, 7, 9, 11, 13, 15, 17, 19, 21 and 23. In those tables, the second column shows the zero-mode numbers of $Q_{1}$, $u_{1}$, $d_{1}$, etc. The other columns except the last column show the corresponding zero-mode numbers. The last column shows the total number of zero-modes in each row. In the tables, negative numbers mean matter fields with conjugate representations. Note that in our analysis we do not distinguish $L_{i}$, $H_{d_{i}}$ and conjugates of $H_{u_{i}}$.

We consider the type VII in eq.(35). We choose $(m_{Q_{1}(1)}^{(1)}, m_{Q_{1}(2)}^{(1)}, m_{Q_{1}(3)}^{(1)}) = (3, 1, 1).$ (47)

Other magnetic fluxes $m_{Qi}^{(d)}$ for $i = 2, 3, 4, 5$ must have vanishing elements for one of $d = 1, 2, 3$. Various combinations are possible

\[
\begin{align*}
(m_{Q_{1}}^{(1)}, m_{Q_{2}}^{(1)}, m_{Q_{3}}^{(1)}, m_{Q_{4}}^{(1)}) &= (0, 0, 0), \\
(m_{Q_{1}}^{(1)}, m_{Q_{2}}^{(1)}, m_{Q_{3}}^{(2)}, m_{Q_{4}}^{(2)}) &= (0, 0, 0), \\
(m_{Q_{1}}^{(1)}, m_{Q_{2}}^{(2)}, m_{Q_{3}}^{(2)}, m_{Q_{4}}^{(2)}) &= (0, 0, 0), \\
(m_{Q_{1}}^{(1)}, m_{Q_{2}}^{(1)}, m_{Q_{3}}^{(3)}, m_{Q_{4}}^{(3)}) &= (0, 0, 0), \\
(m_{Q_{1}}^{(2)}, m_{Q_{2}}^{(2)}, m_{Q_{3}}^{(2)}, m_{Q_{4}}^{(2)}) &= (0, 0, 0), \\
(m_{Q_{1}}^{(2)}, m_{Q_{2}}^{(3)}, m_{Q_{3}}^{(3)}, m_{Q_{4}}^{(3)}) &= (0, 0, 0).
\end{align*}
\]

(48)

Among them, all the combinations of magnetic fluxes leading to the three-generation models are shown in Tables 26, 28, 30, 32 34, 36 and 38 as the models VII-1, $\cdots$, 8, where $n$ denotes arbitrary integer. Thus, this type includes many semi-realistic models. The corresponding massless spectra are shown in Tables 27, 29, 31, 33, 35, 37 and 39.

We have classified the three-generation models with the required aspect. All of the models shown in Tables 2-39 have three chiral generations of quarks and leptons, several vector-like generations and many singlets, but matter fields with exotic representations such as $Q_Y$ do not appear.

### 3.3 Yukawa couplings

In section 3.2, we have obtained various semi-realistic models. Here, we study their Yukawa couplings. Our models have the gauge group $SU(3) \times SU(2) \times U(1)_Y \times U(1)_4$. The top Yukawa coupling must be allowed by $SU(3) \times SU(2) \times U(1)_Y \times U(1)_4$. Most of our models include several singlets with vanishing $U(1)_Y$ charge. Their vacuum expectation values (VEVs) would break extra $U(1)_4$ symmetries. Higher order couplings would lead to effective Yukawa couplings through such breaking and such effective Yukawa couplings
may be small. Thus, we require that the top Yukawa coupling must appear as a 3-point coupling allowed by the $SU(3) \times SU(2) \times U(1)_Y \times U(1)_Y^4$.

In section 3.2, the semi-realistic massless spectra are obtained from the type VI and type VII. However, any of the models in the type VI shown in Tables 2-25 do not allow the top Yukawa coupling. The top Yukawa coupling is allowed in only the models VII-3, 6 and 8 shown in Tables 29, 35 and 39. Hence, these models are more interesting than others.

All of the models VII-3, 6 and 8 have many vector-like generations and singlets with vanishing $U(1)_Y$, in addition to the three chiral generations. For example, the model VII-6 has ten and fifteen vector-like generations for $u$ and $d$, respectively, and other models have more vector-like generations. We expect that such vector-like generations would gain effective mass terms from higher order couplings including singlets after the symmetry breaking due to VEVs of singlets. For example, in the model VII-6 we can show that all of the vector-like generations have 3-point and 4-point couplings with singlets, which would become mass terms of vector-like generations after the symmetry breaking. Phenomenological aspects of our models, e.g. quark/lepton mass matrices, depend on patterns of many singlet VEVs, i.e. which linear combinations would remain as three chiral generations and which higher order couplings would become effective Yukawa couplings after the symmetry breaking.

Here, for illustration, let us study quark mass matrices with rather simple assumptions. First, let us consider the model VII-6. We assume that three chiral generations of $u$ and $d$ are originated from $u_2$ and $d_3$. The Higgs fields $H_u^3$ ($H_d^d$) have allowed Yukawa couplings with $Q_1$ and $u_2$ ($d_3$). The numbers of zero-modes for the $Q_1$ fields are equal to (3, 1, 1) on $T^2_1$, $T^2_2$ and $T^2_3$. Similarly, the numbers of zero-modes for $u_2$ fields equal to (1, 1, 3) on $T^2_1$, $T^2_2$ and $T^2_3$. In addition, the numbers of zero-modes for $H_u^3$ fields equal to (2, 2, 2) on $T^2_1$, $T^2_2$ and $T^2_3$. Note that the flavor structure of $Q_1$ is determined by the first $T^2_1$, while the flavor structure of $u_2$ is determined by the third $T^2_3$. Thus, for one of Higgs fields $H_u^3$, the up-sector Yukawa coupling matrix is always written as

$$Y_{ij}^u = a_i b_j. \quad (49)$$

That is a matrix with the rank one. Only the third generation can be massive, but the other two generations are massless for one of VEVs of eight $H_u^3$ fields. Suppose that all of eight $H_u^3$ fields develop their VEVs. Note that the two zero-modes of $H_u^3$ on the second $T^2_2$ do not lead to variety of the mass matrix, because both $Q_1$ and $U_2$ have single zero-modes on $T^2_2$. Then, the mass matrix induced from the 3-point couplings would be written by the following form,

$$m_{ij}^u = a_i^{(1)} b_j^{(1)} v^{(1,1)} + a_i^{(2)} b_j^{(2)} v^{(2,1)} + a_i^{(1)} b_j^{(2)} v^{(1,2)} + a_i^{(2)} b_j^{(2)} v^{(2,2)}, \quad (50)$$

where $v^{(k,\ell)}$ for $k, \ell = 1, 2$ denote the VEVs of $H_u^3$ fields and $k$ and $\ell$ correspond to the zero-mode indices for $T^2_1$ and $T^2_3$. Note that for each of $v^{(k,\ell)}$ the Yukawa matrix has
the same form as Eq. (49). The mass matrix $m_{ij}^u$ can be written as

$$m_{ij}^u = v^{(1,1)} (a_i^{(1)} + a_i^{(2)} v^{(2,1)} v^{(1,1)}) \left(b_j^{(1)} + b_j^{(2)} v^{(1,2)} / v^{(1,1)}\right) + a_i^{(2)} b_j^{(2)} \left( v^{(2,2)} - v^{(2,1)} v^{(1,2)} / v^{(1,1)} \right).$$

This mass matrix $m_{ij}^u$ corresponds to the rank-two. That is, the second and third generations are massive, but the first generation is massless. It is straightforward to derive the ratio between the charm and top quark masses, because there are several parameters such as VEVs $v^{(k,\ell)}$, the complex structure moduli and Wilson lines.

The down-sector mass matrix is also the rank-two matrix, when we consider only the 3-point couplings with eight $H^d_2$ fields. Thus, the mass ratios $m_c/m_t$ and $m_s/m_b$ as well as the mixing angle $\theta_{cb}$ can be realized by choosing proper values of parameters, but the masses of the first generation, $m_u$ and $m_d$, and the mixing angles, $\theta_{us}$ and $\theta_{ub}$, are vanishing. They may be induced by effective Yukawa couplings, which are obtained from higher order couplings through the symmetry breaking.

As another illustrating example, let us study the quark mass matrices in the model VII-8 with $n = 1$. This model has three generations of $Q$ fields from $Q_1$, which have three-zero modes on the first $T^2_1$. In addition, this model has many vector-like generations of $u$ and $d$. There are allowed 3-points couplings including singlets, such that all of the vector-like generations of $u$ and $d$ gain masses after those singlets develop their VEVs. The low-energy phenomenology depends on mass terms of those vector-like generations. For illustration, we study the quark mass matrices with rather simple assumptions, again. For example, if all of three chiral light generations of $u$ are originated from $u_2$, we would have the same result as in the previous model, i.e. the rank-two mass matrices. To illustrate another possibility, here we consider the case that three light generations of $u$ ($d$) are originated from one of $u_2$ ($d_2$), one of $u_3$ ($d_3$) and one of $u_4$ ($d_4$). For example, their zero-modes correspond to the $j = 0$ mode on each of $T^2_2$. The $u_2$, $u_3$ and $u_4$ ($d_2$, $d_3$ and $d_4$) fields have different extra $U(1)^4$ charges. Thus, they are affected by different Wilson lines. This model also has several Higgs fields, which have the allowed 3-point couplings with these quarks. The Higgs fields, $H^u_3$, $H^d_2$ and the conjugates of $H^d_2$ are allowed to couple with $(Q,u_2)$, $(Q,u_3)$ and $(Q,u_4)$, respectively. Similarly, the Higgs fields, $H^d_3$, $H^d_2$ and the conjugates of $H^u_3$ are allowed to couple with $(Q,d_2)$, $(Q,d_3)$ and $(Q,d_4)$, respectively. Each of these Higgs fields has eight total zero-modes. To reduce the number of free parameters, we consider only one zero-mode for each of these Higgs fields, $H^u_3$, $H^d_2$, $H^d_3$, $H^u_2$ and the conjugates of $H^d_2$ and $H^u_3$, e.g. the zero-mode corresponding to $j = 0$ for each of $T^2_1$. Furthermore, for simplicity we choose $\tau_d = i$ and assume that all of Higgs VEVs are the same.

The Yukawa couplings are given by Eq. (23). Since three generations of $Q$ are originated from the first $T^2_1$, the flavor structure is determined almost by the first $T^2_1$, while the other tori contribute to the overall factors. That is, the mass ratios and mixing angles are determined by only the first torus $T^2_1$. In particular, the Wilson lines are important. Recall that $u_2$, $u_3$ and $u_4$ fields have different extra $U(1)^4$ charges. Thus, different Wilson lines $\tilde{\zeta}^{(d)}$ appear in the Yukawa couplings (23) corresponding to $(Q,u_2), (Q,u_3), (Q,u_4)$. 13
The $d_2$ field has the same extra $U(1)^4$ charges as $u_2$, but obviously has the $U(1)_Y$ charge different from $u_2$. Thus, different Wilson lines $\tilde{\zeta}^{(d)}$ can appear in the Yukawa couplings (23) corresponding to $(Q, u_2)$ and $(Q, d_2)$. Similarly, the $d_3$ ($d_4$) field have the same extra $U(1)^4$ charges as $u_3$ ($u_4$), but has the $U(1)_Y$ charge different from $u_3$ ($u_4$). Thus, the Wilson lines appearing in the Yukawa couplings for $(Q, d_3)$ and $(Q, d_4)$ are not independent of the other Wilson lines. Hence, there are four free parameters for Wilson lines on the first torus $T^2_{(1)}$. For example, we choose

$$\tilde{\zeta}^{(1)} = 0.071 \text{ for } (Q, u_2),$$
$$\tilde{\zeta}^{(1)} = 0.011 \text{ for } (Q, u_3),$$
$$\tilde{\zeta}^{(1)} = -0.021 \text{ for } (Q, u_4),$$
$$\tilde{\zeta}^{(1)} = -0.16 \text{ for } (Q, d_2).$$

(52)

Then, we can derive the following values

$$m_t/m_c = 73, \quad m_c/m_u = 41,$$
$$m_b/m_s = 69, \quad m_s/m_d = 46,$$
$$V_{cb} = 0.034, \quad V_{us} = 0.19, \quad V_{ub} = 0.003.$$

(53)

These mixing angles are realistic and mass ratios except $m_c/m_u$ are similar to experimental values up to a few factors. We have assumed all of Higgs VEVs are the same and taken $\tau_d = i$. By varying them, we would obtain more realistic values. Indeed, the number of free parameters is larger than the number of observables.

### 3.4 Another attempt for realistic models

In section 3.2, we have classified the three-generation models with the supersymmetric condition (21). If this condition is not satisfied, the Fayet-Illiopoulos D-terms, which depend on magnetic fluxes and the area $A_d$, appears along extra $U(1)^4$ directions in the terminology of 4D N=1 global supersymmetry. Most of models have many $SU(3) \times SU(2)$ singlets with vanishing $U(1)_Y$ charges. Such singlets may develop their VEVs such that they cancel the Fayet-Illiopoulos D-terms and a stable vacuum is realized. Thus, let us systematically search realistic models without imposing the condition (21).

Here, we concentrate the three-generation models without vector-like generations for $Q, u, e, d$. That correspond to the following conditions

$$m^{ui} \geq 0, \quad m^{di} \geq 0, \quad m^{ei} \geq 0,$$

(54)

in addition to the conditions (30), (31) and (32). For $L$, $H^u$ and $H^d$, we require the same condition as Eq. (33). Furthermore, we require the condition (34) and $m^{Di} = m^{D^c_i} = 0$.

Under the above conditions, we can find many models, which realize exactly the massless spectrum of the minimal supersymmetric standard model (MSSM), up to singlets.
For example, we choose the following magnetic fluxes,

\[
\begin{align*}
(m_{(1)}^{Q1}, m_{(2)}^{Q1}, m_{(3)}^{Q1}) &= (0, -1, -1), \\
(m_{(1)}^{Q2}, m_{(2)}^{Q2}, m_{(3)}^{Q2}) &= (0, -2, -1), \\
(m_{(1)}^{Q3}, m_{(2)}^{Q3}, m_{(3)}^{Q3}) &= (0, 1, 0), \\
(m_{(1)}^{Q4}, m_{(2)}^{Q4}, m_{(3)}^{Q4}) &= (-1, -1, 1), \\
(m_{(1)}^{Q5}, m_{(2)}^{Q5}, m_{(3)}^{Q5}) &= (1, 2, 1).
\end{align*}
\] (55)

This model has the three generations of quarks and leptons, and one pair of Higgs fields as well as many singlets. Thus, this model would be quite interesting from the viewpoint of the massless spectrum. However, the top Yukawa coupling is not allowed in this model. We can find many similar models, where the massless spectrum of the MSSM is realized, but unfortunately the top Yukawa coupling is not allowed.

Indeed, we can show that there is no model with the allowed top Yukawa coupling under the above condition. For example, let us consider the model, where the top Yukawa coupling appears from the coupling among the fields, \(Q_1\), \(u_2\) and \(H^u_3\). Since at least one zero-mode must appear from \(Q_1\), the possible magnetic fluxes are classified into the following three cases,

\[
(m_{(1)}^{Q1}, m_{(2)}^{Q1}, m_{(3)}^{Q1}) = (1, 1, 1), \quad (2, 1, 1), \quad (3, 1, 1).
\] (56)

The gauge invariance requires that \(m_{(d)}^{Q1} + m_{(d)}^{u2} + m_{(d)}^{H^u_3} = 0\). We require that both \(u_2\) and \(H^u_3\) have at least one zero-modes. Then, the possible combinations of magnetic fluxes are classified into the following four combinations,

\[
\begin{align*}
\begin{cases}
(m_{(1)}^{Q1}, m_{(2)}^{Q1}, m_{(3)}^{Q1}) = (3, 1, 1), \\
(m_{(1)}^{u2}, m_{(2)}^{u2}, m_{(3)}^{u2}) = (-1, -3, 1), \\
(m_{(1)}^{H^u_3}, m_{(2)}^{H^u_3}, m_{(3)}^{H^u_3}) = (-2, 2, -2),
\end{cases}
\end{align*}
\] (57)

\[
\begin{align*}
\begin{cases}
(m_{(1)}^{Q1}, m_{(2)}^{Q1}, m_{(3)}^{Q1}) = (3, 1, 1), \\
(m_{(1)}^{u2}, m_{(2)}^{u2}, m_{(3)}^{u2}) = (-1, -2, 1), \\
(m_{(1)}^{H^u_3}, m_{(2)}^{H^u_3}, m_{(3)}^{H^u_3}) = (-2, 1, -2),
\end{cases}
\end{align*}
\] (58)

\[
\begin{align*}
\begin{cases}
(m_{(1)}^{Q1}, m_{(2)}^{Q1}, m_{(3)}^{Q1}) = (2, 1, 1), \\
(m_{(1)}^{u2}, m_{(2)}^{u2}, m_{(3)}^{u2}) = (-1, -3, 1), \\
(m_{(1)}^{H^u_3}, m_{(2)}^{H^u_3}, m_{(3)}^{H^u_3}) = (-1, 2, -2),
\end{cases}
\end{align*}
\] (59)

\[
\begin{align*}
\begin{cases}
(m_{(1)}^{Q1}, m_{(2)}^{Q1}, m_{(3)}^{Q1}) = (2, 1, 1), \\
(m_{(1)}^{u2}, m_{(2)}^{u2}, m_{(3)}^{u2}) = (-1, -2, 1), \\
(m_{(1)}^{H^u_3}, m_{(2)}^{H^u_3}, m_{(3)}^{H^u_3}) = (-1, 1, -2),
\end{cases}
\end{align*}
\] (60)

\footnote{This model has four zero-modes for \(L\), and one of them can be considered as \(H^d\).}
The magnetic fluxes $m_{Qi}^{Q_i}$ are constrained such that they realize the above values of $m_{Q_i}^{u_2}$ and $m_{Q_i}^{H_u}$. Among such constrained combinations of $m_{Qi}^{Q_i}$, we can not find the above three-generation massless spectrum without vector-like generations.

In the above, we have considered the case that the top Yukawa coupling is originated from the coupling among the fields, $Q_1$, $u_2$ and $H_u^3$. However, the situation is the same for the other allowed couplings. Hence, we can not obtain the three generation spectrum without vector-like generations in the models with the allowed top Yukawa coupling.

4 Conclusion

We have studied 10D $N = 1$ super Yang-Mills $E_8$ theory on the $(T^2)^3$ background with magnetic fluxes. We have classified the models with semi-realistic massless spectra, that is, three chiral generations and several vector-like generations. We have obtained various three-generation models, but the top Yukawa coupling is forbidden in many of them.

We have obtained various semi-realistic models. However, many vector-like generations are, in general, included in those models. Although we have concentrated to integer magnetic fluxes, it is interesting to extend our analysis to the torus compactification with fractional fluxes and non-Abelian Wilson lines [7, 9, 27, 30, 31], and the orbifold compactifications [18, 19]. Since such backgrounds could project out some zero-modes, they would lead to more interesting models.

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\begin{table}
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
 & $SU(3) \times SU(2)$ & $U(1)_a$ & $U(1)_b$ & $U(1)_c$ & $U(1)_d$ & $U(1)_Y$ \\
\hline
$Q_1$ & (3, 2) & 1 & 0 & 1 & -1 & 1 \\
$u_1^c$ & (3, 1) & 1 & 0 & 1 & -1 & -4 \\
$e_1^c$ & (1, 1) & 1 & 0 & 1 & -1 & 6 \\
$d_1^c$ & (3, 1) & 1 & 0 & 1 & 3 & 2 \\
$L_1$ & (1, 2) & 1 & 0 & 1 & 3 & -3 \\
$\nu_1^c$ & (1, 1) & 1 & 0 & 1 & -5 & 0 \\
$D_1$ & (3, 1) & 1 & 0 & -2 & 2 & -2 \\
$H^u_1$ & (1, 2) & 1 & 0 & -2 & 2 & 3 \\
$D^c_1$ & (3, 1) & 1 & 0 & -2 & -2 & 2 \\
$H^d_1$ & (1, 2) & 1 & 0 & -2 & -2 & -3 \\
$S_1$ & (1, 1) & 1 & 0 & 4 & 0 & 0 \\
$Q_2$ & (3, 2) & 0 & 1 & 1 & -1 & 1 \\
$u_2^c$ & (3, 1) & 0 & 1 & 1 & -1 & -4 \\
$e_2^c$ & (1, 1) & 0 & 1 & 1 & -1 & 6 \\
$d_2^c$ & (3, 1) & 0 & 1 & 1 & 3 & 2 \\
$L_2$ & (1, 2) & 0 & 1 & 1 & 3 & -3 \\
$\nu_2^c$ & (1, 1) & 0 & 1 & 1 & -5 & 0 \\
$D_2$ & (3, 1) & 0 & 1 & -2 & 2 & -2 \\
$H^u_2$ & (1, 2) & 0 & 1 & -2 & 2 & 3 \\
$D^c_2$ & (3, 1) & 0 & 1 & -2 & -2 & 2 \\
$H^d_2$ & (1, 2) & 0 & 1 & -2 & -2 & -3 \\
$S_2$ & (1, 1) & 0 & 1 & 4 & 0 & 0 \\
$Q_3$ & (3, 2) & -1 & -1 & 1 & -1 & 1 \\
$u_3^c$ & (3, 1) & -1 & -1 & 1 & -1 & -4 \\
$e_3^c$ & (1, 1) & -1 & -1 & 1 & -1 & 6 \\
$d_3^c$ & (3, 1) & -1 & -1 & 1 & 3 & 2 \\
$L_3$ & (1, 2) & -1 & -1 & 1 & 3 & -3 \\
$\nu_3^c$ & (1, 1) & -1 & -1 & 1 & -5 & 0 \\
$D_3$ & (3, 1) & -1 & -1 & -2 & 2 & -2 \\
$H^u_3$ & (1, 2) & -1 & -1 & -2 & 2 & 3 \\
$D^c_3$ & (3, 1) & -1 & -1 & -2 & -2 & 2 \\
$H^d_3$ & (1, 2) & -1 & -1 & -2 & -2 & -3 \\
$S_3$ & (1, 1) & -1 & -1 & 4 & 0 & 0 \\
$N_1$ & (1, 1) & 2 & 1 & 0 & 0 & 0 \\
$N_2$ & (1, 1) & 1 & 2 & 0 & 0 & 0 \\
$N_3^c$ & (1, 1) & 1 & -1 & 0 & 0 & 0 \\
$Q_4$ & (3, 2) & 0 & 0 & -3 & -1 & 1 \\
$u_4^c$ & (3, 1) & 0 & 0 & -3 & -1 & -4 \\
$e_4^c$ & (1, 1) & 0 & 0 & -3 & -1 & 6 \\
$d_4^c$ & (3, 1) & 0 & 0 & -3 & 3 & 2 \\
$L_4$ & (1, 2) & 0 & 0 & -3 & 3 & -3 \\
$\nu_4^c$ & (1, 1) & 0 & 0 & -3 & -5 & 0 \\
$Q_5$ & (3, 2) & 0 & 0 & 0 & 4 & 1 \\
$u_5^c$ & (3, 1) & 0 & 0 & 0 & 4 & -4 \\
$e_5^c$ & (1, 1) & 0 & 0 & 0 & 4 & 6 \\
$Q_Y$ & (3, 2) & 0 & 0 & 0 & 0 & 5 \\
\hline
\end{tabular}
\end{table}

Table 1: Decomposition of the $E_8$ 248 adjoint representation in $SU(3) \times SU(2) \times U(1)_Y \times U(1)^4$
| \(d\) | \(m_{Q_1}^{(d)}\) | \(m_{Q_2}^{(d)}\) | \(m_{Q_3}^{(d)}\) | \(m_{Q_4}^{(d)}\) | \(m_{Q_5}^{(d)}\) | \(m_{Q_Y}^{(d)}\) |
|---|---|---|---|---|---|---|
| 1  | 0  | 0  | 0  | 0  | 3  | 3 |
| 2  | 0  | 1  | -1 | -1 | 1  | 0 |
| 3  | 0  | -1 | 1  | 1  | 1  | 2 |

Table 2: Magnetic fluxes in the type VI (model VI-1)

| \(i\) | \(Q_i\) | \(Q_i\) | \(Q_i\) | \(Q_i\) | \(Q_i\) | \(Q_i\) | \(Q_i\) |
|---|---|---|---|---|---|---|---|
| 1  | 0  | 0  | 0  | 0  | 3  | 3 |
| 2  | 0  | 9  | -3 | -3 | 0  | 3 |
| 3  | 0  | 3  | -9 | -9 | 18 | 3 |

Table 3: Massless spectrum in the model VI-1

| \(d\) | \(m_{Q_1}^{(d)}\) | \(m_{Q_2}^{(d)}\) | \(m_{Q_3}^{(d)}\) | \(m_{Q_4}^{(d)}\) | \(m_{Q_5}^{(d)}\) | \(m_{Q_Y}^{(d)}\) |
|---|---|---|---|---|---|---|
| 1  | -1 | -1 | -1 | 0  | 3  | 0 |
| 2  | 0  | 0  | 0  | 0  | 1  | 1 |
| 3  | -1 | -1 | -1 | 0  | 1  | -2 |

Table 4: Magnetic fluxes in the type VI (model VI-2)

| \(i\) | \(Q_i\) | \(Q_i\) | \(Q_i\) | \(Q_i\) | \(Q_i\) | \(Q_i\) |
|---|---|---|---|---|---|---|
| 1  | 0  | 0  | 0  | 0  | 3  | 3 |
| 2  | 1  | 1  | 1  | 0  | 0  | 3 |
| 3  | 3  | 3  | 3  | 0  | -6 | 3 |

Table 5: Massless spectrum in the model VI-2
### Table 6: Magnetic fluxes in the type VI (model VI-3)

| d  | $m_{Q1}^{(d)}$ | $m_{Q2}^{(d)}$ | $m_{Q3}^{(d)}$ | $m_{Q4}^{(d)}$ | $m_{Q5}^{(d)}$ | $m_{QY}^{(d)}$ |
|----|----------------|----------------|----------------|----------------|----------------|----------------|
| 1  | -3             | 2              | -1             | -1             | 3              | 0              |
| 2  | 0              | 0              | 0              | 0              | 1              | 1              |
| 3  | 3              | -2             | 1              | 1              | 1              | 4              |

### Table 7: Massless spectrum in the model VI-3

| i  | 1 | 2 | 3 | 4 | 5 | sum |
|----|---|---|---|---|---|-----|
| $Q_i$ | 0 | 0 | 0 | 0 | 3 | 3   |
| $u_i$  | -3 | 12 | -3 | -3 | 0 | 3   |
| $e_i$  | -21 | 4 | -5 | -5 | 30 | 3   |
| $d_i$  | 0 | -5 | 4 | 4 | 3 |     |
| $L_i$  | 0 | 0 | 0 | 0 | 0 |     |
| $H_i^u$ | -5 | 0 | 3 |    | -2 |     |
| $H_i^d$ | 0 | 5 | -4 |    | 1 |     |
| $\nu_i^c$ | 12 | -3 | 0 | 0 | 9 |     |
| $S_i$  | 0 | 0 | 0 |    | 0 |     |
| $N_i$  | 0 | 0 | 0 |    | 0 |     |

### Table 8: Magnetic fluxes in the type VI (model VI-4)

| d  | $m_{Q1}^{(d)}$ | $m_{Q2}^{(d)}$ | $m_{Q3}^{(d)}$ | $m_{Q4}^{(d)}$ | $m_{Q5}^{(d)}$ | $m_{QY}^{(d)}$ |
|----|----------------|----------------|----------------|----------------|----------------|----------------|
| 1  | -3             | -2             | 1              | 1              | 3              | 0              |
| 2  | 0              | 0              | 0              | 0              | 1              | 1              |
| 3  | 3              | 2              | -1             | -1             | 1              | 4              |

### Table 9: Massless spectrum in the model VI-4

| i  | 1 | 2 | 3 | 4 | 5 | sum |
|----|---|---|---|---|---|-----|
| $Q_i$ | 0 | 0 | 0 | 0 | 3 | 3   |
| $u_i$  | -3 | -4 | 5 | 5 | 0 | 3   |
| $e_i$  | -21 | -12 | 3 | 3 | 30 | 3   |
| $d_i$  | 0 | 3 | 0 | 0 |    | 3   |
| $L_i$  | 0 | 0 | 0 | 0 | 0 |     |
| $H_i^u$ | 3 | 4 | -5 |    | 2 |     |
| $H_i^d$ | -4 | -3 | 12 |    | 5 |     |
| $\nu_i^c$ | 12 | 5 | -4 | -4 | 9 |     |
| $S_i$  | 0 | 0 | 0 |    | 0 |     |
| $N_i$  | 0 | 0 | 0 |    | 0 |     |

Table 6: Magnetic fluxes in the type VI (model VI-3)

Table 7: Massless spectrum in the model VI-3

Table 8: Magnetic fluxes in the type VI (model VI-4)

Table 9: Massless spectrum in the model VI-4
Table 10: Magnetic fluxes in the type VI (model VI-5)

| $d$ | $m_{Q_1}^{Q_1}$ | $m_{Q_2}^{Q_2}$ | $m_{Q_3}^{Q_3}$ | $m_{Q_4}^{Q_4}$ | $m_{Q_5}^{Q_5}$ | $m_{Q_Y}^{Q_Y}$ |
|-----|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 1   | -2              | -1              | -1              | 1               | 3               | 0               |
| 2   | 0               | 0               | 0               | 0               | 1               | 1               |
| 3   | -6              | -3              | -3              | 3               | 1               | -8              |

Table 11: Massless spectrum in the model VI-5

| $i$  | 1 | 2 | 3 | 4 | 5 | sum |
|------|---|---|---|---|---|-----|
| $Q_i$ | 0 | 0 | 0 | 0 | 3 | 3   |
| $u_i$ | 4 | 5 | 5 | -11 | 0 | 3 |
| $e_i$ | 28 | 11 | 11 | -5 | -42 | 3 |
| $d_i$ | -5 | -4 | -4 | 16 | 3   |
| $L_i$ | 0 | 0 | 0 | 0 | 0   |
| $H^u_i$ | -4 | 3 | 3 | 2   |
| $H^d_i$ | 5 | 0 | 0 | 5   |
| $\nu^e_i$ | -35 | -16 | -16 | 4 | -63 |
| $S_i$ | 0 | 0 | 0 | 0 | 0   |
| $N_i$ | 0 | 0 | 0 | 0 | 0   |

Table 12: Magnetic fluxes in the type VI (model VI-6)

| $d$ | $m_{Q_1}^{Q_1}$ | $m_{Q_2}^{Q_2}$ | $m_{Q_3}^{Q_3}$ | $m_{Q_4}^{Q_4}$ | $m_{Q_5}^{Q_5}$ | $m_{Q_Y}^{Q_Y}$ |
|-----|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 1   | -3              | -1              | 1               | 0               | 3               | 0               |
| 2   | 0               | 0               | 0               | 0               | 1               | 1               |
| 3   | 9               | 3               | -3              | 0               | 1               | 10              |

Table 13: Massless spectrum in the model VI-6

| $i$  | 1 | 2 | 3 | 4 | 5 | sum |
|------|---|---|---|---|---|-----|
| $Q_i$ | 0 | 0 | 0 | 0 | 3 | 3   |
| $u_i$ | -3 | -7 | 13 | 0 | 0 | 3   |
| $e_i$ | -57 | -13 | 7 | 0 | 66 | 3   |
| $d_i$ | 0 | 8 | -8 | 3 | 3   |
| $L_i$ | 0 | 0 | 0 | 0 | 0   |
| $H^u_i$ | 0 | 8 | -8 | 0 | 0   |
| $H^d_i$ | -3 | -7 | 13 | 3 | 3   |
| $\nu^e_i$ | 48 | 8 | -8 | -3 | 45 |
| $S_i$ | 0 | 0 | 0 | -3 | 0   |
| $N_i$ | 0 | 0 | 0 | 0 | 0   |

Table 13: Massless spectrum in the model VI-6
Table 14: Magnetic fluxes in the type VI (model VI-7)

| $d$ | $m_{Q1}^{(d)}$ | $m_{Q2}^{(d)}$ | $m_{Q3}^{(d)}$ | $m_{Q4}^{(d)}$ | $m_{Q5}^{(d)}$ | $m_{QY}^{(d)}$ |
|-----|----------------|----------------|----------------|----------------|----------------|----------------|
| 1   | 3              | -3             | -3             | 0              | 3              | 0              |
| 2   | 0              | 0              | 0              | 1              | 1              | 1              |
| 3   | -1             | 1              | 1              | 0              | 1              | 2              |

Table 15: Massless spectrum in the model VI-7

| $i$ | $Q_i$ | $u_i$ | $e_i$ | $d_i$ | $L_i$ | $H_i^u$ | $H_i^d$ | $\nu_i^e$ | $S_i$ | $N_i$ |
|-----|-------|-------|-------|-------|-------|---------|---------|-----------|-------|-------|
|     | 0     | 9     | 3     | 0     | 0     | 0       | 0       | 0         | 0     | 0     |

Table 16: Magnetic fluxes in the type VI (model VI-8)

| $d$ | $m_{Q1}^{(d)}$ | $m_{Q2}^{(d)}$ | $m_{Q3}^{(d)}$ | $m_{Q4}^{(d)}$ | $m_{Q5}^{(d)}$ | $m_{QY}^{(d)}$ |
|-----|----------------|----------------|----------------|----------------|----------------|----------------|
| 1   | -2             | 1              | 1              | 1              | 3              | 4              |
| 2   | 0              | 0              | 0              | 0              | 1              | 1              |
| 3   | 2              | -1             | -1             | -1             | 1              | 0              |

Table 17: Massless spectrum in the model VI-8

| $i$ | $Q_i$ | $u_i$ | $e_i$ | $d_i$ | $L_i$ | $H_i^u$ | $H_i^d$ | $\nu_i^e$ | $S_i$ | $N_i$ |
|-----|-------|-------|-------|-------|-------|---------|---------|-----------|-------|-------|
|     | 0     | 12    | 4     | 3     | 0     | 0       | 0       | 0         | 0     | 0     |

Table 17: Massless spectrum in the model VI-8

| $i$ | $Q_i$ | $u_i$ | $e_i$ | $d_i$ | $L_i$ | $H_i^u$ | $H_i^d$ | $\nu_i^e$ | $S_i$ | $N_i$ |
|-----|-------|-------|-------|-------|-------|---------|---------|-----------|-------|-------|
|     | 0     | 12    | 4     | 3     | 0     | 0       | 0       | 0         | 0     | 0     |
| $d$ | $m_{Q_1}^{(d)}$ | $m_{Q_2}^{(d)}$ | $m_{Q_3}^{(d)}$ | $m_{Q_4}^{(d)}$ | $m_{Q_5}^{(d)}$ | $m_{Q_Y}^{(d)}$ |
|-----|----------------|----------------|----------------|----------------|----------------|----------------|
| 1   | 3              | 3              | -3             | 0              | 3              | 6              |
| 2   | 0              | 0              | 0              | 1              | 1              |                |
| 3   | -1             | -1             | 1              | 0              | 1              | 0              |

Table 18: Magnetic fluxes in the type VI (model VI-9)

| $i$  | 1 | 2 | 3 | 4 | 5 | sum |
|------|---|---|---|---|---|-----|
| $Q_i$| 0 | 0 | 0 | 0 | 3 | 3   |
| $u_i$| -3| -3| 9 | 0 | 0 | 3   |
| $e_i$| -9| -9| 3 | 0 | 18| 3   |
| $d_i$| 0 | 0 | 0 | 3 | 3 |     |
| $L_i$| 0 | 0 | 0 | 0 | 0 |     |
| $H_{u_i}^d$| 0 | 0 | 0 | 0 | 0 |     |
| $H_{d_i}^d$| -3| -3| 9 | 3 |     |     |
| $\nu_i^c$| 0 | 0 | 0 | -3| -3|     |
| $S_i$| 0 | 0 | 0 | 0 | 0 |     |
| $N_i$| 0 | 0 | 0 | 0 | 0 |     |

Table 19: Massless spectrum in the model VI-9

| $d$ | $m_{Q_1}^{(d)}$ | $m_{Q_2}^{(d)}$ | $m_{Q_3}^{(d)}$ | $m_{Q_4}^{(d)}$ | $m_{Q_5}^{(d)}$ | $m_{Q_Y}^{(d)}$ |
|-----|----------------|----------------|----------------|----------------|----------------|----------------|
| 1   | 2              | 1              | -1             | -1             | 3              | 4              |
| 2   | 0              | 0              | 0              | 0              | 1              | 1              |
| 3   | -2             | -1             | 1              | 1              | 1              | 0              |

Table 20: Magnetic fluxes in the type VI (model VI-10)

| $i$ | 1 | 2 | 3 | 4 | 5 | sum |
|-----|---|---|---|---|---|-----|
| $Q_i$| 0 | 0 | 0 | 0 | 3 | 3   |
| $u_i$| -4| -3| 5 | 5 | 0 | 3   |
| $e_i$| -12| -5| 3 | 3 | 14 | 3   |
| $d_i$| -5| 0 | 4 | 4 | 3 |     |
| $L_i$| 0 | 0 | 0 | 0 | 0 |     |
| $H_{u_i}^d$| 0 | 3 | 3 | 6 |     |     |
| $H_{d_i}^d$| -3| 0 | 12| 9 |     |     |
| $\nu_i^c$| -3| -4| 0 | 0 | -7|     |
| $S_i$| 0 | 0 | 0 | 0 | 0 |     |
| $N_i$| 0 | 0 | 0 | 0 | 0 |     |

Table 21: Massless spectrum in the model VI-10
Table 22: Magnetic fluxes in the type VII (model VII-1)

| d | $m^{Q1}_{(d)}$ | $m^{Q2}_{(d)}$ | $m^{Q3}_{(d)}$ | $m^{Q4}_{(d)}$ | $m^{Q5}_{(d)}$ | $m^{QY}_{(d)}$ |
|---|---|---|---|---|---|---|
| 1 | 3 | 0 | 0 | 0 | 3n | 3+3n |
| 2 | 1 | -1 | -1 | 1 | 0 | 0 |
| 3 | 1 | 1 | 1 | -1 | 2n | 2+2n |

Table 23: Massless spectrum in the model VII-1

| $i$ | $Q_i$ | $u_i$ | $e_i$ | $d_i$ | $L_i$ | $H^u_i$ | $H^d_i$ | $\nu^c_i$ | $S_i$ | $N_i$ | sum |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 3 | $6n^2 + 3n$ | $6n^2 + 21n + 18$ | $6n^2 + 9n + 3$ | 0 | $12n^2 + 12n$ | $12n^2 + 12n$ | $6n^2 - 9n + 3$ | 0 | 0 | 3 |
| 2 | 0 | $-6n^2 - 9n - 3$ | $-6n^2 - 15n - 9$ | $-6n^2 - 3n$ | -3 | 0 | 0 | $-6n^2 + 3n$ | 0 | 0 | 3 |
| 3 | 0 | $-6n^2 - 9n - 3$ | $-6n^2 - 15n - 9$ | $-6n^2 - 3n$ | -3 | 0 | 0 | $-6n^2 + 3n$ | 0 | 0 | 3 |
| 4 | 0 | 0 | 0 | 0 | 0 | $6n^2 + 15n + 9$ | 0 | 0 | 0 | 0 | 3 |
| 5 | 0 | 0 | 0 | 0 | 0 | 0 | $6n^2 + 9n + 3$ | 0 | 0 | 0 | 3 |
| sum | 3 | 3 | 3 | 3 | 3 | $12n^2 + 12n$ | $12n^2 + 12n$ | 3 | 0 | 0 | 3 |
Table 24: Magnetic fluxes in the type VII (model VII-2)

| $i$ | $Q_i$ | $u_i$ | $e_i$ | $d_i$ | $L_i$ | $H_i^u$ | $H_i^d$ | $\nu_i^c$ | $S_i$ | $N_i$ | sum |
|-----|-------|-------|-------|-------|-------|---------|---------|----------|-------|-------|------|
| 1   | 3     | $6n^2 + 3n$ | $-6n^2 - 9n - 3$ | $-6n^2 - 9n - 3$ | 0 | $12n^2 + 12n$ | $12n^2 + 12n$ | $6n^2 - 3n$ | 0 | 0 | 3   |
| 2   | -3    | $-6n^2 - 9n - 3$ | $-6n^2 - 9n - 3$ | $-6n^2 - 15n - 9$ | $-3$ | 0 | 0 | $-6n^2 + 3n$ | 0 | 0 | 3   |
| 3   | -3    | $-6n^2 - 15n - 9$ | $-6n^2 - 15n - 9$ | $-6n^2 - 3n$ | 0 | 0 | 0 | 0 | -6 | -6 | -6  |

Table 25: Massless spectrum in the model VII-2

| $d$ | $m_{Q1}^{(d)}$ | $m_{Q2}^{(d)}$ | $m_{Q3}^{(d)}$ | $m_{Q4}^{(d)}$ | $m_{Q5}^{(d)}$ | $m_{QY}^{(d)}$ |
|-----|---------------|---------------|---------------|---------------|---------------|---------------|
| 1   | 3             | -3            | -3            | 3             | 0             | 0             |
| 2   | 1             | 0             | 0             | 0             | n             | 1+n           |
| 3   | 1             | 1             | 1             | -1            | 2n            | 2+2n          |

Table 26: Magnetic fluxes in the type VII (model VII-3)

| $i$ | $Q_i$ | $u_i$ | $e_i$ | $d_i$ | $L_i$ | $H_i^u$ | $H_i^d$ | $\nu_i^c$ | $S_i$ | $N_i$ | sum |
|-----|-------|-------|-------|-------|-------|---------|---------|----------|-------|-------|------|
| 1   | 3     | $-42$ | $27$  | $27$  | $15$  | $-24$   | 3       | -10      | 8     | 24    | 48   |
| 2   | -60   | 81    | 81    | 21    | $-120$| 3       | 3       | -30      | 8     | 24    | 24   |
| 3   | 21    | $-6$  | $-6$  | -6    | -6    | 3       | 3       | -3       | 8     | -93   | -93  |

Table 27: Massless spectrum in the model VII-3

| $d$ | $m_{Q1}^{(d)}$ | $m_{Q2}^{(d)}$ | $m_{Q3}^{(d)}$ | $m_{Q4}^{(d)}$ | $m_{Q5}^{(d)}$ | $m_{QY}^{(d)}$ |
|-----|---------------|---------------|---------------|---------------|---------------|---------------|
| 1   | 3             | -3            | -3            | 3             | 0             | 0             |
| 2   | 1             | 0             | 0             | 0             | n             | 1+n           |
| 3   | 1             | 1             | 1             | -1            | 2n            | 2+2n          |
| \(d\) | \(m^{Q_1}_{(d)}\) | \(m^{Q_2}_{(d)}\) | \(m^{Q_3}_{(d)}\) | \(m^{Q_4}_{(d)}\) | \(m^{Q_5}_{(d)}\) | \(m^{Q_Y}_{(d)}\) |
|------|----------------|----------------|----------------|----------------|----------------|----------------|
| 1    | 3              | -3             | 3              | 3              | 6n             | 6n+6           |
| 2    | 1              | 0              | 0              | 0              | n              | 1+n            |
| 3    | 1              | 1              | -1             | -1             | 0              | 0              |

Table 28: Magnetic fluxes in the type VII (model VII-4)

| \(i\) | 1 | 2 | 3 | 4 | 5 | sum |
|------|---|---|---|---|---|-----|
| \(Q_i\) | 3 | 0 | 0 | 0 | 0 | 3   |
| \(u_i\) | \(6n^2 + 3n\) | \(6n^2 + 15n + 9\) | \(-6n^2 - 9n - 3\) | \(-6n^2 - 9n - 3\) | 0 | 3   |
| \(e_i\) | \(6n^2 + 21n + 18\) | \(6n^2 + 9n + 3\) | \(-6n^2 - 15n - 9\) | \(-6n^2 - 15n - 9\) | 0 | 3   |
| \(d_i\) | \(6n^2 + 9n + 3\) | \(6n^2 - 3n\) | \(-6n^2 - 3n\) | \(-6n^2 - 3n\) | 3   |       |
| \(L_i\) | 0 | 9 | -3 | -3 | 3   |       |
| \(H^u_i\) | 0 | 0 | \(-12n^2 - 12n\) | \(-12n^2 - 12n\) | -12n^2 - 12n |       |
| \(H^d_i\) | 0 | 0 | \(-12n^2 - 12n\) | \(-12n^2 - 12n\) | -12n^2 - 12n |       |
| \(\nu^c_i\) | \(6n^2 - 9n + 3\) | \(6n^2 + 3n\) | \(6n^2 + 3n\) | \(-6n^2 + 3n\) | 3   |       |
| \(S_i\) | 0 | 0 | 0 | 0 | 0   |       |
| \(N_i\) | 0 | 0 | 0 | 0 | 0   |       |

Table 29: Massless spectrum in the model VII-4

| \(d\) | \(m^{Q_1}_{(d)}\) | \(m^{Q_2}_{(d)}\) | \(m^{Q_3}_{(d)}\) | \(m^{Q_4}_{(d)}\) | \(m^{Q_5}_{(d)}\) | \(m^{Q_Y}_{(d)}\) |
|------|----------------|----------------|----------------|----------------|----------------|----------------|
| 1    | 3              | -3             | 1              | 1              | -2             | 0              |
| 2    | 1              | 0              | 0              | 0              | 0              | 1              |
| 3    | 1              | 3              | -1             | -1             | 2              | 4              |

Table 30: Magnetic fluxes in the type VII (model VII-5)

| \(i\) | 1 | 2 | 3 | 4 | 5 | sum |
|------|---|---|---|---|---|-----|
| \(Q_i\) | 3 | 0 | 0 | 0 | 0 | 3   |
| \(u_i\) | 0 | -3 | 5 | 5 | -4 | 3   |
| \(e_i\) | 30 | -21 | 3 | 3 | -12 | 3   |
| \(d_i\) | 3 | 0 | 0 | 0 | 0 | 3   |
| \(L_i\) | 0 | 5 | -3 | -3 | -1 |     |
| \(H^u_i\) | 4 | 0 | 0 | 0 | 4 |     |
| \(H^d_i\) | 0 | -4 | 12 | 8 |     |     |
| \(\nu^c_i\) | -5 | 0 | 0 | 0 | -5 |     |
| \(S_i\) | 4 | 0 | 0 | 0 | 4 |     |
| \(N_i\) | 4 | 0 | -12 | -8 |     |     |

Table 31: Massless spectrum in the model VII-5

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| $d$ | $m_{Q1}^{(d)}$ | $m_{Q2}^{(d)}$ | $m_{Q3}^{(d)}$ | $m_{Q4}^{(d)}$ | $m_{Q5}^{(d)}$ | $m_{QY}^{(d)}$ |
|-----|----------------|----------------|----------------|----------------|----------------|----------------|
| 1   | 3              | -3             | -1             | -1             | 0              | -2             |
| 2   | 1              | 0              | 0              | 0              | -2             | -1             |
| 3   | 1              | -3             | -1             | -1             | 4              | 0              |

Table 32: Magnetic fluxes in the type VII (model VII-6)

| $i$ | 1 | 2 | 3 | 4 | 5 | sum  |
|-----|---|---|---|---|---|------|
| $Q_i$ | 3 | 0 | 0 | 0 | 0 | 3    |
| $u_i$  | 10 | 3 | -1 | -1 | -8 | 3    |
| $e_i$  | 0 | -15 | -3 | -3 | 24 | 3    |
| $d_i$  | -15 | 6 | 6 | 6 | 3 | 3    |
| $L_i$  | 0 | 1 | -3 | -3 | 8 | 8    |
| $H_i^u$ | -8 | 0 | 8 | 0 | 0 | 0    |
| $H_i^d$ | 0 | 8 | 0 | 0 | 8 | 8    |
| $\nu_i^c$ | -27 | 42 | 10 | 10 | 35 | 35  |
| $S_i$  | 8 | 0 | 0 | 0 | 0 | 0    |
| $N_i$  | 8 | 0 | 24 | 0 | 0 | 0    |

Table 33: Massless spectrum in the model VII-6

| $d$ | $m_{Q1}^{(d)}$ | $m_{Q2}^{(d)}$ | $m_{Q3}^{(d)}$ | $m_{Q4}^{(d)}$ | $m_{Q5}^{(d)}$ | $m_{QY}^{(d)}$ |
|-----|----------------|----------------|----------------|----------------|----------------|----------------|
| 1   | 3              | 1              | 1              | 1              | -2             | 4              |
| 2   | 1              | 0              | 0              | 0              | 0              | 1              |
| 3   | 1              | -1             | -1             | -1             | 2              | 0              |

Table 34: Magnetic fluxes in the type VII (model VII-7)

| $i$ | 1 | 2 | 3 | 4 | 5 | sum  |
|-----|---|---|---|---|---|------|
| $Q_i$ | 3 | 0 | 0 | 0 | 0 | 3    |
| $u_i$  | 0 | -3 | -3 | -3 | 12 | 3    |
| $e_i$  | 14 | -5 | -5 | -5 | 4 | 3    |
| $d_i$  | 3 | 0 | 0 | 0 | 0 | 3    |
| $L_i$  | 0 | 5 | 5 | 5 | 15 | 15   |
| $H_i^u$ | 4 | 0 | 0 | 0 | 0 | 4    |
| $H_i^d$ | 0 | -4 | -4 | -4 | -8 | -8   |
| $\nu_i^c$ | -5 | 0 | 0 | 0 | 0 | -5   |
| $S_i$  | 4 | 0 | 0 | 0 | 0 | 4    |
| $N_i$  | 4 | 0 | 4 | 0 | 0 | 8    |

Table 35: Massless spectrum in the model VII-7
Table 36: Magnetic fluxes in the type VII (model VII-8)

|   | \( m_{(d)}^{Q_1} \) | \( m_{(d)}^{Q_2} \) | \( m_{(d)}^{Q_3} \) | \( m_{(d)}^{Q_4} \) | \( m_{(d)}^{Q_5} \) | \( m_{(d)}^{Q_Y} \) |
|---|---------------|---------------|---------------|---------------|---------------|---------------|
| 1 | 3             | -1            | -1            | -1            | 0             | 0             |
| 2 | 1             | 0             | 0             | 0             | n             | 1+n           |
| 3 | 1             | -1            | -1            | -2n           | -2-2n         |

Table 37: Massless spectrum in the model VII-8

|   | 1 | 2 | 3 | 4 | 5 | sum |
|---|---|---|---|---|---|-----|
| \( Q_i \) | 3 | 0 | 0 | 0 | 0 | 3   |
| \( u_i \) | \(-6n^2 - 9n\) | \(2n^2 + 3n + 1\) | \(2n^2 + 3n + 1\) | \(2n^2 + 3n + 1\) | 0 | 3   |
| \( e_i \) | \(-6n^2 - 15n - 6\) | \(2n^2 + 5n + 3\) | \(2n^2 + 5n + 3\) | \(2n^2 + 5n + 3\) | 0 | 3   |
| \( d_i \) | \(-6n^2 - 3n + 3\) | \(2n^2 + n\) | \(2n^2 + n\) | \(2n^2 + n\) | 1 | 3   |
| \( L_i \) | 0 | 1 | 1 | 1 | 1 | 3   |
| \( H_{i}^{u} \) | \(-4n^2 - 4n\) | \(4n^2 + 4n\) | \(4n^2 + 4n\) | \(4n^2 + 4n\) | \(4n^2 + 4n\) | 4   |
| \( H_{i}^{d} \) | \(-4n^2 - 4n\) | \(4n^2 + 4n\) | \(4n^2 + 4n\) | \(4n^2 + 4n\) | \(4n^2 + 4n\) | 4   |
| \( \nu_{i}^{c} \) | \(-6n^2 + 3n + 3\) | \(2n^2 - n\) | \(2n^2 - n\) | \(2n^2 - n\) | \(2n^2 - n\) | 3   |
| \( S_{i} \) | 8 | 0 | 0 | 0 | 0 | 8   |
| \( N_{i} \) | 8 | 0 | 8 | 8 | 16 |