EXTENDABILITY OF PARALLEL SECTIONS IN VECTOR BUNDLES

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ABSTRACT. We address the following question: Given a differentiable manifold $M$ what are the open subsets $U$ of $M$ such that, for all vector bundles $E$ over $M$ and all linear connections $\nabla$ on $E$, any $\nabla$-parallel section in $E$ defined on $U$ extends to a $\nabla$-parallel section in $E$ defined on $M$?

For simply connected manifolds $M$ (among others) we describe the entirety of all such sets $U$ which are, in addition, the complement of a $C^1$ submanifold (boundary allowed) of $M$; this delivers a partial positive answer to a problem posed by Antonio J. Di Scala and Gianni Manno. Furthermore, in case $M$ is an open submanifold of $\mathbb{R}^n$, $2 \leq n$, we prove that the complement of $U$ in $M$, not required to be a submanifold now, can have arbitrarily large $n$-dimensional Lebesgue measure.

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1. Introduction

In their very recent preprint [1] Antonio J. Di Scala and Gianni Manno address a question that, in a generalized form, reads as follows: given a vector bundle $E$ over a manifold $M$, a connection $\nabla$ on $E$, and a $\nabla$-parallel section $\sigma$ in $E$ defined on an open subset $U \subset M$, does there exist a $\nabla$-parallel section $\tilde{\sigma}$ defined on $M$ such that $\tilde{\sigma}|_U = \sigma$?\footnote{To be precise, Di Scala-Manno ask this for $M$ simply connected and $U$ dense and connected in $M$, in which case a comprehensive positive answer seems reasonable.}

Surely the answer to the latter question depends on each and every one of the various data points involved $(M, E, \nabla, U, \sigma)$. We concretize the question by posing the following problem: for a given (simply connected or not) manifold $M$ describe/characterize the set of all open subsets $U \subset M$ such that, for all vector bundles $E$ over $M$, all connections $\nabla$ on $E$, and all $\nabla$-parallel sections $\sigma$ in $E$ defined on $U$, there exists a $\nabla$-parallel extension $\tilde{\sigma}$ as above. As a matter of fact, we will rather try and characterize the universe of closed subsets $F \subset M$ whose complement $U = M \setminus F$ has the aforementioned property, but that appears to be

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just a question of taste. These closed subsets $F \subset M$ will be called *negligible in $M$* (see definition 2.3).

Our results come in two groups: First, in section 2, we derive *necessary conditions* for a set $F$ to be negligible in $M$. Specifically, we prove that when $F$ is negligible in $M$ (and $M$ is connected), then the complement $M \setminus F$ is necessarily connected (see proposition 2.10); when $M$ is of dimension 2 or higher, then, moreover, $F$ needs to be nowhere dense in $M$ (see corollary 2.13). Observe that these conditions already appear in [1]—their necessity however has remained unproven.

Second, in section 3, we derive *sufficient conditions* for a set $F$ to be negligible in $M$. This is probably the more interesting part (as compared to section 2) since here we prove that parallel extensions of parallel sections do in fact exist. The most striking result of section 3 is corollary 3.13 which asserts in particular that when $M$ is a simply connected (second-countable, Hausdorff) manifold and $F \subset M$ is a closed $C^1$ submanifold with boundary such that $M \setminus F$ is dense and connected in $M$, then $F$ is negligible in $M$ (cf. remark 3.14). Hence corollary 3.13 yields a partial (positive) answer to [1, Problem 1].

As a sideline in section 3, we will show that the Lebesgue measure of a set $F$ is quite unrelated to the negligibility of $F$. Indeed, we prove the existence of negligible subsets of $\mathbb{R}^n$ of arbitrarily large (even infinite) measure (see corollary 3.6). On the opposite side we show that the fact that $M \setminus F$ is dense and connected in $M = \mathbb{R}^n$, $2 \leq n$, does not imply that $F$ is negligible in $M$ for all connections of class $C^0$ on smooth vector bundles over $M$ (see corollary 3.4); we cannot however produce an example of a closed set $F \subset M = \mathbb{R}^n$ with $M \setminus F$ dense and connected such that $F$ is not negligible for some $C^\infty$ connection on a vector bundle. Nonetheless, the latter observation is somewhat negative in view of [1, Problem 1].

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## 2. Basic theory of negligible sets

By a *manifold* we mean a finite-dimensional real differentiable manifold of class $C^k$, $1 \leq k \leq \infty$, without boundary; we make no topological assumptions whatsoever. The sheaf of real-valued functions of class $C^m$ on $M$, $0 \leq m \leq k$, will be denoted $\mathcal{C}^m_M$, $\mathcal{C}^m$ when $M$ is clear from the context.

A *vector bundle* over a manifold is a finite-rank real vector bundle of class $C^l$, $0 \leq l \leq k$. When $E$ is a vector bundle over $M$ and $0 \leq m \leq l$, we write $\mathcal{C}^m(E)$ for the sheaf of sections of class $C^m$ in $E$; we write $\mathcal{C}^m(U,E)$ as a synonym for $\mathcal{C}^m(E)(U)$ when $U$ is open in $M$.

A *connection* on a vector bundle $E$ over $M$, $E$ of class $C^l$, $1 \leq l$, is a morphism of abelian sheaves on $M$

$$\nabla : \mathcal{C}^1(E) \to \mathcal{C}^0(T^\vee(M) \otimes E)$$
satisfying
\[ \nabla_U(f s) = df \otimes s + f \nabla_U(s) \]
for all open subsets \( U \) of \( M \), all \( f \in C^1(U) \), and all \( s \in C^1(U, E) \). We denote the kernel of \( \nabla \) by \( \Gamma^\nabla(E) \) and write \( \Gamma^\nabla(E, U) \) as a synonym for \( \Gamma^\nabla(E)(U) \). We say that \( \nabla \) is of class \( C^m \), \( 0 \leq m \leq \infty \), when \( m + 1 \leq l \) and \( \nabla \) maps \( C^l(E) \) into \( C^m(T^\nabla(M) \otimes E) \). Here, \( \infty + 1 := \infty \). Note that a connection is the same thing as a connection of class \( C^0 \).

A vector bundle with connection over \( M \) is a pair \((E, \nabla)\) such that \( E \) is a vector bundle over \( M \) and \( \nabla \) is a connection on \( E \).

**Remark 2.1** (Connections of class \( C^\infty \)). Typically connections are only dealt with when \( E \) and (consequently) \( M \) are of class \( C^\infty \). A connection is then a morphism of abelian sheaves
\[ \nabla: C^\infty(E) \to C^\infty(T^\nabla(M) \otimes E) \]
satisfying Leibniz’s rule eq. (2.0.1) for all \( f \in C^\infty(U) \) and all \( s \in C^\infty(U, E) \). We briefly reconcile this view with ours.

Evidently, when \( \nabla \) is a connection on \( E \) over \( M \) in our sense, \( \nabla \) induces, by restriction, a morphism of abelian sheaves
\[ \nabla': C^l(E) \to C^0(T^\nabla(M) \otimes E) \]
satisfying eq. (2.0.1) for all \( f \in C^l(U), s \in C^l(U, E) \), \( U \subset M \) open.

Conversely, given such a morphism \( \nabla' \), there exists a unique connection \( \nabla \) on \( E \) such that \( \nabla \) restricted to \( C^l(E) \subset C^1(E) \) equals \( \nabla' \). As a matter of fact, on an open subset \( U \) of \( M \) over which \( E \) is trivial (in the \( C^l \) sense) the values of \( \nabla_U \) are determined by \( \nabla'_U \) and the Leibniz rule eq. (2.0.1) for \( f \in C^l(U) \) and \( s \in C^l(U, E) \).

Moreover, under this correspondence, \( \nabla \) is of class \( C^m \), \( 0 \leq m \leq \infty \), if and only if \( \nabla' \) has image lying in the subsheaf \( C^m(T^\nabla(M) \otimes E) \) of \( C^0(T^\nabla(M) \otimes E) \). In particular, for \( E \) and \( M \) of class \( C^\infty \), the usual \( C^\infty \) connections on \( E \) correspond naturally to our connections on \( E \) which are of class \( C^\infty \).

**Remark 2.2** (The global section component). Let \( 0 \leq m \leq \infty \), \( m + 1 \leq l \), and
\[ \nabla_0: C^l(M, E) \to C^m(M, T^\nabla(M) \otimes E) \]
an additive map satisfying Leibniz’s rule eq. (2.0.1) \((\nabla_0 \text{ in place of } \nabla_U)\) for all \( f \in C^l(M) \), \( s \in C^l(M, E) \). Then, in case \( M \) is Hausdorff, there exists a unique morphism of abelian sheaves
\[ \nabla: C^l(E) \to C^m(T^\nabla(M) \otimes E) \]
such that \( \nabla_M = \nabla_0 \) and eq. (2.0.1) is satisfied for all \( U \subset M \) open, all \( f \in C^l(U) \), and all \( s \in C^l(U, E) \).

The main reason is the following. When \( M \) is Hausdorff, then for all \( U \subset M \) open and all \( p \in U \) there exists a function \( f \in C^k(M) \) such that \( a) \), \( f = 1 \) on an open neighborhood \( V \) of \( p \) in \( U \) and \( b) \), the support of \( f \) is a (compact) subset of \( U \). This fact implies that, for all \( s \in C^l(U, E) \), there exists \( \tilde{s} \in C^l(M, E) \) such that \( \tilde{s} = s \) on \( V \). Thus the calculation
\[ \nabla_U(s)|_V = \nabla_V(s)|_V = \nabla_V(\tilde{s})|_V = \nabla_M(\tilde{s})|_V = \nabla_0(\tilde{s})|_V \]
shows the uniqueness of \( \nabla \). The existence of \( \nabla \) follows noting that when \( \tilde{s}, \tilde{s}_1 \in C^l(M, E) \) agree on an open subset \( V \cap V_1 \) of \( M \), then \( \nabla_0(\tilde{s}) \) and \( \nabla_0(\tilde{s}_1) \) agree on \( V \cap V_1 \), too. In fact, for \( p \in V \cap V_1 \), there exists a function \( g \in C^k(M) \) such that \( a) \),
\[ g = 1 \text{ on a neighborhood } W \text{ of } p \text{ and } b, \] 
In particular, \( gt = 0 \) on \( M \), where \( t = \tilde{s}_1 - \tilde{s} \), and thus, on \( W \),

\[
0 = \nabla_0(gt) = dg \otimes t + g\nabla_0(t) = 0 \otimes t + 1\nabla_0(t) = \nabla_0(\tilde{s}_1) - \nabla_0(\tilde{s})
\]
due to the additivity of and eq. (2.0.1) for \( \nabla_0 \).

**Definition 2.3.** Let \( M \) be a manifold, \( F \) a closed subset of \( M \).

1. Let \( (E, \nabla) \) be a vector bundle with connection over \( M \). Then \( F \) is called

\[ \text{negligible in } M \text{ for } (E, \nabla) \]

when the restriction map

\[ \Gamma(E)(M) \to \Gamma(E)(M \setminus F) \]

of the sheaf \( \Gamma(E) \) is surjective.

2. Let \( 0 \leq m \leq \infty \). Then \( F \) is called \textit{negligible in } M \textit{ for all connections of class } \( C^m \)

when, for all vector bundles \( E \) over \( M \) and all connections \( \nabla \) of class \( C^m \) on \( E \), the set \( F \) is negligible in \( M \) for \( (E, \nabla) \).

Note that whether a closed set \( F \subset M \) is negligible for \( (E, \nabla) \) depends exclusively on the (pre)sheaf \( \Gamma(E) \). Following that philosophy, given an arbitrary topological space \( M \) and a presheaf (of sets) \( \mathcal{P} \) on \( M \), one might call a closed subset \( F \) of \( M \) \textit{negligible in } \( M \) \textit{ for } \( \mathcal{P} \)

when the restriction map \( \mathcal{P}(M) \to \mathcal{P}(M \setminus F) \) is surjective.

If you do not like to work with closed sets, call an object \( U \) of a (small) category \( \mathcal{C} \) with terminal object \( T \) \textit{full in } \( \mathcal{C} \)

for \( \mathcal{P} \) (feel free to substitute this expression by one that is more to your taste) when the restriction map \( \mathcal{P}(T) \to \mathcal{P}(U) \) is surjective.

**Example 2.4 (Empty set).** Quite trivially, for all manifolds \( M \), the empty set is negligible in \( M \) for all connections of class \( C^0 \).

**Example 2.5 (Whole space).** For all manifolds \( M \), the set \( M \) itself is negligible in \( M \) for all connections of class \( C^0 \). Indeed, let \( (E, \nabla) \) be a vector bundle with connection over \( M \). Then, on the one hand,

\[ \Gamma(E)(M \setminus M, E) = \Gamma(E)(\emptyset, E) = \{\emptyset\} \]

On the other hand, \( \Gamma(E)(M, E) \) contains the zero section \( z \) and \( z|_0 = 0 \).

**Lemma 2.6.** Let \( M \) be a manifold, \( (E, \nabla) \) a vector bundle with connection over \( M, U \) an open, connected subset of \( M \), \( s, t \in \Gamma(E)(U, E), p \in U \) such that \( s(p) = t(p) \). Then \( s = t \).

**Proof sketch.** Let \( q \in U \). Then, due to the connectedness of \( U \), there exists a (piece-wise) \( C^1 \) path \( \gamma \colon [0, 1] \to M \) with \( \gamma(0) = p, \gamma(1) = q \), and \( \gamma([0, 1]) \subset U \). Trivializing \( E \) locally along \( \gamma \), one sees that \( s \circ \gamma \) and \( t \circ \gamma \) correspond to solutions of an \( r \)-dimensional system of linear ordinary differential equations defined on \([0, 1]\),

where \( r \in \mathbb{N} \) is the local rank of \( E \) along \( \gamma \) (cf. remark 3.7). By the uniqueness in the Picard-Lindelöf theorem we deduce the equality \( s \circ \gamma = t \circ \gamma \) from

\[ (s \circ \gamma)(0) = s(p) = t(p) = (t \circ \gamma)(0). \]

Specifically, \( s(q) = t(q) \). As \( q \in U \) was arbitrary, we are done.

Let \( M \) be a manifold, \( E \) a vector bundle over \( M \), \( U \) an open subset on \( M \). Then we have a natural notion of restriction of \( E \) to \( U \), denoted \( E|_U \), which is a vector bundle of class \( C^l \) over the manifold \( M|_U \) (most of the time \( M|_U \) is sloppily denoted \( U \) of class \( C^k \).
Assume $1 \leq l$, and let $\nabla$ be a connection on $E$. Then we may sheaf-theoretically restrict $\nabla$ to $U$ to obtain a morphism of abelian sheaves

$$\nabla|_U: C^1(E)|_U \to C^0(T^\vee(M) \otimes E)|_U$$

on $U$ (or better, on $M|_U$). However, since

$$C^1(E)|_U = C^1(E|_U),$$

$$C^0(T^\vee(M) \otimes E)|_U = C^0((T^\vee(M) \otimes E)|_U) = C^0(T^\vee(M)|_U \otimes E|_U)$$

$$\cong C^0(T^\vee(M|_U) \otimes E|_U),$$

and the Leibniz rule transfers from $\nabla$ down to $\nabla|_U$, we see that $\nabla|_U$ becomes a connection on $E|_U$ (modulo the identification of $T^\vee(M)|_U$ and $T^\vee(M|_U)$ which, in turn, amounts to identifying $T_p(M|_U)$ and $T_p(M)$ for $p \in U$). We write $(E, \nabla)|_U$ for the pair $(E|_U, \nabla|_U)$.

**Proposition 2.7.** Let $M$ be a manifold, $F$ a closed subset of $M$, $(E, \nabla)$ a vector bundle with connection over $M$, $\mathfrak{U}$ an open cover of $M$ such that

1. for all $U \in \mathfrak{U}$, the set $F \cap U$ is negligible in $U$ for $(E, \nabla)|_U$, and
2. for all connected components $C$ of $U \cap V$, where $U, V \in \mathfrak{U}$, $U \neq V$, there exists an element $p \in C \setminus F$.

Then $F$ is negligible in $M$ for $(E, \nabla)$.

**Proof.** Let $s \in \Gamma^\vee(M \setminus F, E)$. Then, for all $U \in \mathfrak{U}$, due to item 1, there exists an element $\bar{s}_U \in \Gamma^\vee(U, E)$ such that $\bar{s}_U|_{U \cap F} = s|_{U \cap F}$. Therefore there exists a family $(\bar{s}_U)_{U \in \mathfrak{U}}$ such that, for all $U \in \mathfrak{U}$, $\bar{s}_U$ has the aforementioned property. Note that the existence of $(\bar{s}_U)$ follows at once invoking the axiom of choice. However, the axiom of choice can be circumvented here. Indeed, for all $U \in \mathfrak{U}$, there exists a unique extension $\bar{s}_U \in \Gamma^\vee(U, E)$ of $s|_{U \cap F}$ such that $\bar{s}_U = 0$ on all connected components of $U$ that are contained in $F$ (use lemma 2.6 for those components of $U$ that are not contained in $F$).

Let $U, V \in \mathfrak{U}$, $U \neq V$. Let $C$ be a connected component of $U \cap V$. Then by item 2 there exists an element $p \in C \setminus F$. Since both $\bar{s}_U$ and $\bar{s}_V$ agree with $s$ on $(U \cap V) \setminus F$, we have $\bar{s}_U(p) = \bar{s}_V(p)$. So $\bar{s}_U$ and $\bar{s}_V$ agree on $C$ by lemma 2.6 as $C$ is connected and open in $M$. As $C$ was arbitrary, we see that $\bar{s}_U$ and $\bar{s}_V$ agree on $U \cap V$.

Thus as $\Gamma^\vee(E)$ is a sheaf, there exists one, and only one, $\bar{s} \in \Gamma^\vee(E)(M)$ such that $\bar{s}|_U = \bar{s}_U$ for all $U \in \mathfrak{U}$. In consequence, for all $U \in \mathfrak{U}$,

$$(\bar{s}|_{M \setminus F})|_{U \setminus F} = (\bar{s}|_U)|_{U \setminus F} = \bar{s}_U|_{U \setminus F} = s|_{U \setminus F}$$

implying that $\bar{s}|_{M \setminus F} = s$ as the $U \setminus F$, $U \in \mathfrak{U}$, furnish an open cover of $M \setminus F$. \hfill $\Box$

**Proposition 2.8.** Let $M$ be a manifold, $F$ a closed subset of $M$, $(E, \nabla)$ a vector bundle with connection over $M$. Then $F$ is negligible in $M$ for $(E, \nabla)$ if and only if, for all connected components $U$ of $M$, the set $F \cap U$ is negligible in $U$ for $(E, \nabla)|_U$.

**Proof.** The “if” part follows from proposition 2.7 taking $\mathfrak{U}$ to be the set of connected components of $M$. Note that item 2 of proposition 2.7 holds since any two connected components $U \neq V$ of $M$ have empty intersection, whence $U \cap V$ has itself no connected component at all.

The “only if” part is obtained as follows. Assume that $F$ is negligible in $M$ for $(E, \nabla)$. Let $U$ be a connected component of $M$, $s \in \Gamma^\vee(U \setminus F, E)$. Define $t: M \setminus F \to E$ to be $s$ on $U \setminus F$ and the zero section outside of $U$. Then $t \in \Gamma^\vee(M \setminus F, E)$,
specifically as $U$ as well as $M \setminus U$ are open in $M$. Thus there exists $\tilde{t} \in \Gamma^\nabla(M,E)$ such that $\tilde{t}|_{M\setminus U} = t$. In consequence, we have $\tilde{s} := \tilde{t}|_{U} \in \Gamma^\nabla(U,E)$ and

$$\tilde{s}|_{U\setminus F} = \tilde{t}|_{U\setminus F} = t|_{U\setminus F} = s,$$

which was to be demonstrated. \hfill \square

Remark 2.9 (Connection forms). Let $M$ be a manifold, $E$ a vector bundle over $M$, $r \in \mathbb{N}$, and $(e_1, \ldots, e_r)$ a frame (i.e., a global frame of class $C^1$, $1 \leq i \leq l$) for $E$. Then, for all $r \times r$ matrices $(\omega^\beta_\alpha)_{\alpha,\beta}$ with values in $C^0(M,T^\nabla(M))$, there exists one, and only one, connection $\nabla$ on $E$ such that

$$\nabla_M(e_\beta) = \omega^\beta_\alpha \otimes e_\alpha, \quad \forall \beta.$$

For the uniqueness of $\nabla$ let $U$ be an open subset of $M$, $\sigma \in C^1(U,E)$ and observe that there exists an $r$-tuple $(\sigma^\beta)$ of elements of $C^1(U)$ such that $\sigma = \sigma^\beta(e_\beta|_U)$. Therefore,

$$\nabla_U(\sigma) = \nabla_U(\sigma^\beta(e_\beta|_U)) = d\sigma^\beta \otimes (e_\beta|_U) + \sigma^\beta \nabla_U(e_\beta|_U) = d\sigma^\beta \otimes (e_\beta|_U) + \sigma^\beta \nabla_M(e_\beta)|_U
= d\sigma^\beta \otimes (e_\beta|_U) + \sigma^\beta(\omega^\beta_\alpha \otimes e_\alpha)|_U.$$

For the existence of $\nabla$ simply define $\nabla_U$, for $U \subset M$ open, by the latter identity—note that the tuple $(\sigma^\beta)$ is unique. Then the family $\nabla = (\nabla_U)$ is a morphism of sheaves on $M$

$$\nabla: C^1(E) \to C^0(T^\nabla(M) \otimes E)$$

mainly since, for $V \subset U$ open, $\sigma|_V = \sigma^\beta|_V(e_\beta|_V)$ and $(d\sigma^\beta)|_V = d(\sigma^\beta|_V)$. Furthermore, the morphism of sheaves $\nabla$ is additive; it satisfies Leibniz’s rule as, for $f \in C^1(U)$, we have

$$d(f \sigma^\beta) = \sigma^\beta df + fd\sigma^\beta, \quad \forall \beta.$$

We omit the details.

When $M$ is a manifold, $r \in \mathbb{N}$, the projection $M \times \mathbb{R}^r \to M$ becomes a vector bundle $E$ of class $C^k$ over $M$ the obvious way. $E$ is called the trivial bundle of rank $r$ over $M$. Let $(e_1, \ldots, e_r)$ be the standard frame for $E$, that is,

$$e_\alpha: M \to E, \quad e_\alpha(x) = (x, (0,\ldots,0,1,0,\ldots,0)),$$

where the “1” is placed in the $\alpha$’s component. Then, for all subsets $U$ of $M$, any (set-theoretic) section $s$ in $E$ defined on $U$ can be expressed uniquely in the form $s = s^\beta e_\beta$, where the $s^\beta$, $\beta = 1,\ldots,r$, are functions $s^\beta: U \to \mathbb{R}$; for $0 \leq m \leq k$, when $U \subset M$ is open, the section $s$ is of class $C^m$ if and only if $s^\beta$ is of class $C^m$ for all $\beta$.

The standard connection on $E$ is defined by the formula

$$\nabla_U(s) = ds^\beta \otimes (e_\beta|_U),$$

where $U \subset M$ is open and $s \in C^1(U,E)$. Observe that $\nabla = (\nabla_U)$ is indeed a connection on $E$—as a matter of fact, you obtain $\nabla$ taking $\omega^\beta_\alpha = 0$ for all $\alpha, \beta$ in remark 2.9. Moreover, $\Gamma^\nabla(E)$ is precisely the sheaf of locally constant sections in $E$, a section $s$ in $E$ being called locally constant if its composition with the projection $E \to \mathbb{R}^r$ is locally constant.
Proposition 2.10. Let \( M \) be a connected manifold, \( F \) a closed subset of \( M \), \( 1 \leq r \in \mathbb{N} \), such that \( F \) is negligible in \( M \) for \( (E, \nabla) \), where \( \nabla \) is the standard connection on the trivial bundle \( E \) of rank \( r \) over \( M \). Then \( M \setminus F \) is connected.

**Proof.** Since \( \Gamma^\nabla(E) \) is the sheaf of locally constant sections in \( E \), we see that, for all \( U \subset M \) open, \( \Gamma^\nabla(E)(U) \) is isomorphic to \( (\mathbb{R}^r)^C \) as a real vector space, \( C \) denoting the set of connected components of \( U \). In particular, \( \Gamma^\nabla(E)(M) \) is isomorphic to \( (\mathbb{R}^r)^0 \) or \( \mathbb{R}^r \) (depending on whether or not \( M \) contains an element). Thus the surjectivity of the restriction map

\[
\Gamma^\nabla(E)(M) \to \Gamma^\nabla(E)(M \setminus F)
\]

implies \(|C| \leq 1\) for the set \( C \) of connected components of \( M \setminus F \). In turn, \( M \setminus F \) is connected. \( \square \)

**Example 2.11** (Intervals). Let \( M \subset \mathbb{R} \) be an open interval endowed with its canonical manifold structure (of class \( C^k \), \( 1 \leq k \leq \infty \)). Then for a closed subset \( F \) of \( M \) the following are equivalent:

1. \( F \) is negligible in \( M \) for all connections of class \( C^0 \).
2. There exists a number \( r \in \mathbb{N} \), \( r \geq 1 \), such that \( F \) is negligible in \( M \) for \( (E, \nabla) \), where \( \nabla \) is the standard connection on the trivial bundle \( E \) of rank \( r \) over \( M \).
3. \( M \setminus F \) is connected, that is, \( F = F \setminus (M \setminus F) \) is the complement in \( M \) of an open subinterval of \( M \).

Item 1 implies item 2 since you can take \( r = 1 \). Item 2 implies item 3 by means of proposition 2.10. Now, assume item 3. When \( F = M \), then \( F \) is negligible in \( M \) by example 2.5. So, suppose \( F \neq M \), so that there exists an element \( t_0 \in M \setminus F \). Let \( (E, \nabla) \) be a vector bundle with connection over \( M \) and \( s \in \Gamma^\nabla(M \setminus F, E) \). We know that the bundle \( E \) is trivial over \( M \) (see footnote 5). In particular, there exists \( r \in \mathbb{N} \) and a global frame \( e = (e_1, \ldots, e_r) \) for \( E \). Write \( s = s^\beta e_\beta \) with functions \( s^\beta : M \setminus F \to \mathbb{R} \). Then

\[
0 = \nabla_{M \setminus F}(s) = (ds^\alpha + s^\beta \omega^\alpha_\beta) \otimes e_\beta
\]

for some

\[
\omega^\alpha_\beta = A^\alpha_\beta dx^1 \in C^0(M, T^\nabla(M)),
\]

\( x^1 : M \to \mathbb{R} \) being the identity function on \( M \) here. This amounts to saying that the \( r \)-tuple of functions \( (s^\beta) \) solves the linear differential equation given by the \( r \times r \) matrix \(-A\), i.e., the equation \( y' = -Ay \), where \( A = (A^\alpha_\beta)_{\alpha, \beta} \).

By the (global) Picard-Lindelöf theorem the linear differential equation given by \(-A\) possesses a unique vector solution \( (\tilde{s}^\beta) \) on \( M \) such that \( \tilde{s}^\beta(t_0) = s^\beta(t_0) \) for all \( \beta \); moreover, \( \tilde{s}^\beta|_{M \setminus F} = s^\beta \) since \( M \setminus F \) is connected (more so, an interval). Therefore, putting \( \tilde{s} = \tilde{s}^\beta e_\beta \), we have \( \tilde{s} \in \Gamma^\nabla(E)(M) \) and \( \tilde{s}|_{M \setminus F} = s \). The restriction map in eq. \((2.3.1)\) is hence surjective implying that \( F \) is negligible in \( M \) for \( (E, \nabla) \). As \( (E, \nabla) \) was arbitrary, we have deduced item 1.

Proposition 2.12. For all natural numbers \( n \geq 2 \) there exists a connection \( \nabla \) of class \( C^\infty \) on the trivial bundle \( E \) of rank 1 over \( \mathbb{R}^n \) such that the following assertions hold:

1. \( \nabla \) is trivial on \( \mathbb{R}^n \setminus [-1, 1]^n \), that is, \( \nabla_{\mathbb{R}^n}(e) = 0 \) on \( \mathbb{R}^n \setminus [-1, 1]^n \), where \( e : \mathbb{R}^n \to E \) is given by \( e(x) = (x, 1) \).
(2) When \( Q = [-1,1]^{n-1} \times [-1,0] \), there exists an element \( t \in \Gamma^\nabla(E)(\mathbb{R}^n \setminus Q) \) not lying in the image of the restriction map
\[
\Gamma^\nabla(E)(\mathbb{R}^n) \to \Gamma^\nabla(E)(\mathbb{R}^n \setminus Q).
\]

**Proof.** Let \( n \in \mathbb{N}, n \geq 2 \). Define \( b: \mathbb{R} \to \mathbb{R} \) (a “bump”) by
\[
b(x) = \begin{cases} 
e^{-\frac{1}{1-x^2}} & \text{when } |x| < 1, \\ 0 & \text{when } |x| \geq 1. \end{cases}
\]

Then \( b \in C^\infty(\mathbb{R}) \), the support of \( b \) is contained in \([-1,1]\), we have \( b \geq 0 \) everywhere, yet \( b \) is not identically zero (since \( b(0) = e^{-1} > 0 \)). We know there exist \( g, h \in C^\infty(\mathbb{R}) \) such that, for all \( x \in \mathbb{R} \), we have \( g(x), h(x) \geq 0 \),
\[
g(x) = \begin{cases} 0 & \text{when } x < -1, \\ 1 & \text{when } 0 < x, \end{cases} \quad \text{and} \quad \h(x) = \begin{cases} 0 & \text{when } 1 < x, \\ 1 & \text{when } x < 0. \end{cases}
\]

As a matter of fact, \( g \) can be obtained by substituting \( 2x + 1 \) for \( x \) in
\[
\bar{g}(x) = \int_{-1}^{x} b(y) \, dy \int_{1}^{x} b(y) \, dy;
\]

\( h \) can then be obtained by letting \( h(x) = g(-x) \).

Define
\[
f: \mathbb{R}^n \to \mathbb{R}, \quad f(x_1, \ldots, x_n) = b(x_1) \cdots b(x_{n-1})h(x_n).
\]

Moreover, for \( i = 1, \ldots, n \), define \( \omega_i: \mathbb{R}^n \to \mathbb{R} \) by
\[
\omega_i(x) = -g(x_n) \frac{D_i f(x)}{1 + f(x)} \quad \text{for } i \leq n - 1, \quad \omega_n(x) = -\frac{D_n f(x)}{1 + f(x)}.
\]

Note that \( \omega_i \in C^\infty(\mathbb{R}^n) \) for all \( i \). We set
\[
\omega := \omega_1 dx^1 + \cdots + \omega_n dx^n \in C^\infty(\mathbb{R}^n, T^\nabla(\mathbb{R}^n)).
\]

By remark 2.9 there exists a unique connection \( \nabla \) on \( E \) such that
\[
\nabla_{e^n}(e) = \omega \otimes e,
\]
where \( e: \mathbb{R}^n \to E \) is given by \( e(x) = (x,1) \); concretely, we have
\[
\nabla_U(s) = (ds_2 + s_2(\omega|_U)) \otimes (e|_U)
\]
for all \( U \subset \mathbb{R}^n \) open and all \( s \in C^1(U, E) \), where \( s_2 \) denotes the second component of the function \( s: U \to E = \mathbb{R}^n \times \mathbb{R} \) (the tensor product of sections is taken pointwise).

We claim that, for all \( i = 1, \ldots, n \), the function \( \omega_i \) has support in \([-1,1]^n\). As a matter of fact, the support of \( f \), whence the support of \( D_i f \), already lies inside \([-1,1]^{n-1} \times (-\infty,1] \). Thus for \( i \leq n - 1 \) it suffices to note that we have \( g(x_n) = 0 \) for \( x_n \in (-\infty,-1) \). For \( i = n \) it suffices to note that the support of \( h' \) lies inside \([0,1]\) as \( h \) is constant on \( \mathbb{R} \setminus [0,1] \); thus the support of \( \omega_n \) is not only contained in \([-1,1]^n \), but contained in \([-1,1]^{n-1} \times [0,1] \). Anyways, the connection \( \nabla \) is trivial on \( \mathbb{R}^n \setminus [-1,1]^n \), i.e., we have item 1.

It remains to verify item 2. For that matter, define
\[
t_2: \mathbb{R}^n \setminus Q \to \mathbb{R}, \quad t_2(x) = 1 + \begin{cases} 0 & \text{when } x_n \leq 0, \\ f(x) & \text{when } 0 < x_n \end{cases}
\]
and
\[ t: \mathbb{R}^n \setminus Q \to E, \quad t(x) = (x, t_2(x)) \]

Let \( x \in \mathbb{R}^n \setminus [-1, 1]^n \). Then \( t(x) = e(x) \). When \( x_n \leq 0 \), this is evident. When \( 0 < x_n \), use the fact that \( f \) has support lying inside \([-1, 1]^{n-1} \times (-\infty, 1] \). In particular, we see that \( t \) is \( \nabla \)-parallel on \( \mathbb{R}^n \setminus [-1, 1]^n \).

On \( \mathbb{R}^{n-1} \times (0, \infty) \), we have
\[ dt_2 + t_2 \omega = \sum_{i=1}^n (D_i f + (1 + f)\omega_i)dx^i = 0. \]

In order to see that the summands \( i = 1, \ldots, n-1 \) vanish, note that for \( 0 < x_n \) one has \( g(x_n) = 1 \). Thus \( t \) is \( \nabla \)-parallel on \( \mathbb{R}^{n-1} \times (0, \infty) \), too, whence \( t \) is \( \nabla \)-parallel on \( \mathbb{R}^n \setminus Q \), in symbols: \( t \in \Gamma(\nabla(E)(\mathbb{R}^n \setminus Q)) \).

Assume that \( v \in \Gamma(\nabla(E)(\mathbb{R}^n)) \) with \( v|\mathbb{R}^n \setminus Q = t \). Let \( 0' = (0, \ldots, 0) \in \mathbb{R}^{n-1} \). As noted above, we know that \( \omega_n(x_n) \) vanishes for \( x_n < 0 \). Thus by the uniqueness in Picard-Lindelöf’s theorem we infer that \( v_2(0', x_n) = 1 \) for all \( x_n < 0 \) since \( v_2(0', -2) = t_2(0', -2) = 1 \). On the other hand, for all \( x_n > 0 \), we have
\[ v_2(0', x_n) = t_2(0', x_n) = 1 + b(0)^{n-1}h(x_n), \]
where the right-hand side tends, for \( x_n \to 0 \), to
\[ 1 + b(0)^{n-1}h(0) = 1 + b(0)^{n-1} > 1. \]

This contradicts the continuity of \( v \) (or, more precisely, the continuity of the function \( v_2(0', \_): \mathbb{R} \to \mathbb{R} \)). Therefore, there does not exist an element of \( \Gamma(\nabla(E)(\mathbb{R}^n)) \) which yields \( t \) when restricted to \( \mathbb{R}^n \setminus Q \).

**Corollary 2.13.** Let \( M \) be a connected, Hausdorff manifold of dimension \( \geq 2 \), \( F \neq M \) a closed subset of \( M \) such that, for all connections \( \nabla \) of class \( C^{k-1} \) on the trivial bundle \( E \) of rank 1 over \( M \), the set \( F \) is negligible in \( M \) for \( (E, \nabla) \). Then \( F \) is nowhere dense in \( M \).

**Proof.** Assume that \( p \) is an interior point of \( F \) in \( M \). Then there exists a natural number \( n \) and a coordinate chart \( x: U \to \mathbb{R}^n \) on \( M \) at \( p \) such that \([-1, 1]^n \subset x(U)\) and \( K := x^{-1}([-1, 1]^n) \subset F \) (in fact, we can achieve \( x(U) = \mathbb{R}^n \) and \( U \subset F \)). As \( M \) is connected and \( \dim M \geq 2 \), we have \( n \geq 2 \). Since \( K \) is compact in \( M \) and \( M \) is Hausdorff, \( K \) is closed in \( M \). Thus \( \{ M \setminus K, U \} \) constitutes an open cover of \( M \).

In consequence, there exists a (unique) \( \tilde{\omega} \in C^{k-1}(M, T^\omega(M)) \) such that \( \tilde{\omega} \) is zero on \( M \setminus K \) and
\[ \tilde{\omega}|_U = x^\ast(\omega) = (\omega_1 \circ x)dx^1 + \cdots + (\omega_n \circ x)dx^n, \]
where \( \omega \) and the \( \omega_i, i = 1, \ldots, n \), stem from proposition 2.12—note that these are \( C^\infty \). By remark 2.9 there exists a unique connection \( \nabla \) on \( E \) such that
\[ \nabla_M(e) = \tilde{\omega} \otimes e, \]
the section \( e: M \to E \) being given by \( e(q) = (q, 1) \). Observe that \( \nabla \) is of class \( C^{k-1} \) as \( \tilde{\omega} \) is. Define a function \( t: M \setminus A \to E, A := x^{-1}([-1, 1]^{n-1} \times [-1, 0]) \), by
\[ \tilde{t}(q) = \begin{cases} e(q) & \text{when } q \in M \setminus K, \\ t(x(q)) & \text{when } q \in U \setminus A, \end{cases} \]
where \( t \) is as in item 2 of proposition 2.12. Then \( \tilde{t} \) is \( \nabla - \)parallel since it is so on \( M \setminus K \) and \( U \setminus A \). Therefore,
\[
\tilde{t}|_{M \setminus F} \in \Gamma \nabla (M \setminus F, E).
\]
So as \( F \) is negligible in \( M \) for \( (E, \nabla) \), there exists a \( \nabla - \)parallel section \( \tilde{v} \) on \( M \) such that \( \tilde{v} \) restricted to \( M \setminus F \) yields \( \tilde{t}|_{M \setminus F} \).

Now since \( F \neq M \), there exists an element \( p' \in M \setminus F \). Thus \( \tilde{t}(p') = \tilde{v}(p') \). As \( M \) is connected, the set \( M \setminus A \) is connected in \( M \).\(^2\) So by lemma 2.6, we see that
\[
\tilde{v}|_{M \setminus A} = \tilde{t}.
\]
Thus \( \tilde{v} \) furnishes a \( \nabla - \)parallel extension of \( \tilde{t} \) to all of \( M \). In turn, \( \tilde{v} \) furnishes a \( \nabla - \)parallel extension of \( t \) to all of \( \mathbb{R}^n \). This, however, contradicts proposition 2.12, item 2.\( \square \)

3. **Negligible Hyperspaces**

In what follows a **Banach space** is a real Banach space. Given a Banach space \( E \), we write \( \mathcal{L}(E) \) for the Banach space of continuous linear operators on \( E \).

**Theorem 3.1.** Let \( J \subset \mathbb{R} \) be an open interval, \( t_0 \in J \), \( E \) and \( F \) Banach spaces,\(^3\) \( V \subset F \) open,
\[
A: J \times V \to \mathcal{L}(E)
\]
a continuous map.

1. There exists a unique continuous map
\[
X: J \times V \to \mathcal{L}(E)
\]
such that
\[
D_1 X = AX \quad \text{and} \quad X(t_0, y) = \operatorname{id}_E, \quad \forall y \in V.
\]

2. When \( X \) is as in item 1 and \( A \) is of class \( C^1 \), then \( X \) is of class \( C^1 \).

**Proof.** When \( t_0 = 0 \), item 1 is precisely [4, IV, Proposition 1.9]. The general statement (arbitrary \( t_0 \)) follows considering the interval \( J - t_0 \) instead of \( J \) and translating \( A \) and \( X \) accordingly.

Now let \( X \) be as described in item 1. Set \( U := V \times \mathcal{L}(E) \) and define
\[
f: J \times U \to F \times \mathcal{L}(E), \quad f(t, (y, x)) = (0, A(t, y)x).
\]
Regard \( f \) as a time-dependent vector field on the open subset \( U \) of the Banach space \( F \times \mathcal{L}(E) \). Then, as one verifies easily, the map
\[
\alpha: J \times U \to U, \quad \alpha(t, (y, x)) = (y, X(t, y)x)
\]
constitutes a (global) flow for \( f \) with initial time \( t_0 \), that is, for all \( (y, x) \in U \) we have
\[
D_1 \alpha(t, (y, x)) = f(t, \alpha(t, (y, x))), \quad \forall t \in J, \quad \text{and} \quad \alpha(t_0, (y, x)) = (y, x).
\]

\( ^2 \)An elementary way to see this is the following: Take \( p_1, p_2 \in M \setminus A \). Since \( M \) is path-connected, there exists a path \( \gamma \) in \( M \) from \( p_1 \) to \( p_2 \). When \( \gamma \) does not meet \( A \), we are done. When \( \gamma \) meets \( A \), let \( t_1, t_2 \) be the infimum and supremum of \( \gamma^{-1}(A) \), respectively. Then modify \( \gamma \) from a little bit left of \( t_1 \) to a little bit right of \( t_2 \) continuously inside \( U \) so as to go around \( A \); use the chart \( x \) for that matter.

\( ^3 \)In our applications (lemma 3.3) \( E \) and \( F \) will both be equal to some \( \mathbb{R}^d, d \in \mathbb{N} \).
Lemma 3.2. Let \( I, J \subseteq \mathbb{R} \) be open intervals, \( E \) a Banach space, \( f \in C^0(I \times J, E) \).

1. For all \( x_0 \in I, \epsilon > 0, K \subset J \) compact, there exists a number \( \delta > 0 \) such that

\[
\| f(x, y) - f(x_0, y) \| < \epsilon 
\]

for all \( x \in I \) with \( |x - x_0| < \delta \) and all \( y \in K \).

2. When \( C \) is nowhere dense in \( I \), \( g \in C^0(I \times J, E) \) such that \( D_2 f = g \) on \((I \setminus C) \times J\) (in particular, \( f \) is assumed to be partially differentiable in its second variable on \((I \setminus C) \times J\)), then we have \( D_2 f = g \) everywhere on \( I \times J \).

Proof. Item 1 is fairly standard. The continuity of \( f \) in the points of \( \{ x_0 \} \times K \) entails that the set of all sets \( U = B_\delta(x_0) \times V \) with \( \delta > 0 \) and \( V \subseteq \mathbb{R} \) open such that eq. (3.2.1) holds for all \( (x, y) \in U \cap (I \times J) \) furnishes an open cover of \( \{ x_0 \} \times K \).

Since \( \{ x_0 \} \times K \) is compact in \( \mathbb{R}^2 \), there exists a finite subcover. The minimal \( \delta \) of the sets \( U \) in this subcover then possess the desired property.

Now, let \( C \) and \( g \) be as in item 2. Then we have

\[
f(x, y) - f(x, y_0) = \int_{y_0}^{y} g(x, \lambda) \, d\lambda
\]

for all \( x \in I \setminus C \) and all \( y_0, y \in J \). Let \( x_0 \in I, J' \subset J \) a compact subinterval, \( \epsilon > 0 \). Then by item 1 there exists a number \( \delta > 0 \) such that eq. (3.2.1) holds for all \( x \in I \) with \( |x - x_0| < \delta \) and all \( y \in J' \) as well as for \( f \) replaced by \( g \) (i.e., you apply item 1 twice, once for \( f \), once for \( g \) in place of \( f \), then you pass to the minimum of the two \( \delta \)'s). Since \( C \) is nowhere dense in \( I \), there exists \( x \in I \setminus C \) such that \( |x - x_0| < \delta \). Therefore, using eq. (3.2.2), we obtain

\[
\left\| f(x_0, y) - f(x_0, y_0) - \int_{y_0}^{y} g(x_0, \lambda) \, d\lambda \right\| < (2 + \lambda(J'))\epsilon
\]

for all \( y_0, y \in J' \). As \( \epsilon > 0 \) was arbitrary, we deduce

\[
f(x_0, y) - f(x_0, y_0) = \int_{y_0}^{y} g(x_0, \lambda) \, d\lambda
\]

for all \( y_0, y \in J' \). Indeed, the latter equality holds for all \( y_0, y \in J \) since we can pick \( J' = [y_0, y] \) or \( J' = [y, y_0] \) depending on whether \( y_0 \leq y \) or \( y < y_0 \) (note that these \( J' \) are subsets of \( J \) as \( J \) was assumed to be an interval). In turn, the function \( f(x_0, \_): J \rightarrow E \) is differentiable on \( J \), its derivative being equal to \( g(x_0, \_) \). As \( x_0 \in I \) was arbitrary, we infer that the function \( f \) is partially differentiable on \( I \times J \) with respect to the second variable; moreover, \( D_2 f = g \) holds on \( I \times J \), which was to be demonstrated. \( \square \)

\[\footnote{The decisive point here is to pass from the time-dependent vector field \( f \) to its associated time-independent vector field \( \tilde{f} \) as explained on [1, p. 71]. Then you use the existence of local flows of class \( C^1 \) for \( \tilde{f} \), i.e., [4, IV, Theorem 1.14].} \]
Lemma 3.3. Let \( n \in \mathbb{N}, 2 \leq n, M \) an open \( n \)-dimensional interval, i.e., \( M = I_1 \times \cdots \times I_n \) for open intervals \( I_i \subseteq \mathbb{R}, i = 1, \ldots, n \), \( F \) a closed subset of \( M \).

(1) When \( C_2 \subseteq I_2 \) is discrete, \( b_1 \in I_1 \), and

\[
(3.3.1) \quad F = \{ x \in M : b_1 \leq x_1, x_2 \in C_2 \},
\]

then \( F \) is negligible in \( M \) for all connections of class \( C^0 \).

(2) When \( C_2 \subseteq I_2 \) is nowhere dense, \( b_1 \in I_1 \), and we have eq. \((3.3.1)\), then \( F \) is negligible in \( M \) for all connections of class \( C^1 \).

(3) When \( C_1 \subseteq I_1, C_2 \subseteq I_2 \) are nowhere dense, \( b_1 \in I_1, b_2 \in I_2 \), and

\[
F = \{ x \in M : b_1 \leq x_1, x_2 \in C_2 \} \cap \{ x \in M : b_2 \leq x_2, x_1 \in C_1 \},
\]

then \( F \) is negligible in \( M \) for all connections of class \( C^0 \).

Proof. The arguments for the three parts of the lemma all start the same. Let \((E, \nabla)\) be a vector bundle with connection over \( M \). Since \( M \) is paracompact, Hausdorff, and \( C^k \)-contractible (that is, there exists a homotopy \( M \times [0,1] \rightarrow M \) of class \( C^k \), in the sense of manifolds with boundary, from the identity on \( M \) to a constant map \( M \rightarrow \{ c \}, c \in M \), the vector bundle \( E \) is trivial as a vector bundle of class \( C^l \).\(^5\)

In particular, there exists a global frame \( e = (e_1, \ldots, e_r) \) for \( E \). Denote

\[ \omega = \omega_1 dx^1 + \cdots + \omega_n dx^n : M \rightarrow \Gamma(M, T^\vee(M))^{r \times r} \]

the connection form of \( \nabla \) with respect to \( e \) (see remark 2.9); here the \( \omega_i \) are continuous functions on \( M \) with values in \( \mathbb{R}^{r \times r} \cong \mathcal{L}(\mathbb{R}^r) \).

Let \( b_1 \in I_1 \) and \( C_2 \subseteq I_2 \). Since \( I_1 \) is open, there exists \( a_1 \in I_1 \) such that \( a_1 < b_1 \).

By theorem 3.1, item 1, we know that there exists a unique continuous function

\[ X : M = I_1 \times (I_2 \times \cdots \times I_n) \rightarrow \mathcal{L}(\mathbb{R}^r) \]

such that, for all \( x' = (x_2, \ldots, x_n) \in I_2 \times \cdots \times I_n \), we have

\[ X(a_1, x') = \text{id}_{\mathbb{R}^r} \]

and, for all \( x_1 \in I_1 \),

\[ D_1 X(x_1, x') + \omega_1(x_1, x') X(x_1, x') = 0. \]

Assume eq. \((3.3.1)\) and let \( s \in \Gamma^\nabla(M \setminus F, E) \). By abuse of notation, we write \( s \) also for the \( C^1 \) function \( M \setminus F \rightarrow \mathbb{R}^r \) that represents \( s \) with respect to \( e \). We define

\[ \tilde{s} : M \rightarrow \mathbb{R}^r, \quad \tilde{s}(x_1, x') = X(x_1, x') s(a_1, x'). \]

Obviously \( \tilde{s} : M \rightarrow \mathbb{R}^r \) is continuous and partially differentiable in the direction of \( x_1 \) so that

\[ D_1 \tilde{s} + \omega_1 \tilde{s} = 0 \]

on \( M \)—in particular, we see that \( D_1 \tilde{s} \) is continuous. Since \( \nabla_{M \setminus F}(s) = 0 \), we have

\[ D_1 s + \omega_1 s = 0 \]

on \( M \setminus F \). Since \( F \subseteq \{ x \in M : x_2 \in C_2 \} \) by eq. \((3.3.1)\), we have

\[ I_1 \times \{ x' \} \subseteq M \setminus F \]

\(^5\)Note that by \([3, \text{Theorem 1.6}]\) the vector bundle \( E \) is trivial in the sense of topological vector bundles. However, the pivotal \([3, \text{Proposition 1.7}]\) carries over nicely to the \( C^l \) manifold context; the critical point is to see that in “preliminary facts (1)” of the proof, the patching together of the two trivializations can be realized within class \( C^l \), at least for nice \( X \).
for all \( x' \in I_2 \times \cdots \times I_n \) with \( x_2 \notin C_2 \). Since \( \bar{s}(a_1, x') = s(a_1, x') \), the Picard-Lindelöf theorem implies that \( \bar{s} = s \) on \( I_1 \times \{x'\} \) for all such \( x' \). That is, \( \bar{s} \) and \( s \) agree on \( \{ x \in M : x_2 \notin C_2 \} \).

Suppose that \( C_2 \) is nowhere dense in \( I_2 \). Then \( I_1 \times C_2 \times I_3 \times \cdots \times I_n \) is nowhere dense in \( M \). Thus the set where \( \bar{s} \) and \( s \) agree is dense in \( M \) and, in turn, dense in \( M \setminus F \). Since \( \bar{s} \) and \( s \) are both continuous, this implies \( \bar{s}|_{M \setminus F} = s \). In other words, we have found a continuous extension of \( s \) to all of \( M \) which already satisfies \( \nabla = 0 \) in the direction of \( x_1 \).

Let \( i \in \mathbb{N}, \ 3 \leq i \leq n \). We claim that \( \bar{s} \) is partially differentiable in the direction of \( x_i \) such that

\[
D_i \bar{s} + \omega_i \bar{s} = 0
\]

holds on \( M \). For that matter, let \( x^0 = (x_1^0, \ldots, x_n^0) \in M \). Define

\[
j : I_2 \times I_i \to M, \quad j(x_2, x_i) = (x_1^0, x_2, x_3^0, \ldots, x_{i-1}^0, x_i, x_{i+1}^0, \ldots, x_n^0)
\]

and

\[
f, g : I_2 \times I_i \to \mathbb{R}^r, \quad f = \bar{s} \circ j, \quad g = -(\omega_i \bar{s} \circ j).
\]

Then \( f \) and \( g \) are both continuous. Moreover, \( f \) is partially differentiable with respect to its second variable on \( (I_2 \setminus C_2) \times I_i \) so that

\[
D_2 f = (D_i \bar{s}) \circ j = -(\omega_i \bar{s} \circ j) = g.
\]

Thus by means of lemma 3.2, item 2, we see that \( f \) is partially differentiable with respect to its second variable on \( I_2 \times I_i \) such that \( D_2 f = g \). Specifically, we have this identity in \( (x_2^0, x_i^0) \in I_2 \times I_i \). In consequence, \( \bar{s} \) is partially differentiable in the direction of \( x_i \) at \( x^0 \), and we have

\[
(D_i \bar{s})(x^0) = (D_2 f)(x_2^0, x_i^0) = g(x_2^0, x_i^0) = -(\omega_i \bar{s})(x^0).
\]

As \( x^0 \in M \) was arbitrary, our claim is proven.

Under the current assumptions we cannot conclude that \( \bar{s} \) is partially differentiable in the direction of \( x_2 \) in points of \( F \). We need further suppositions. So, let \( x^0 \in F \) and assume that \( x_2^0 \) is an isolated point of \( C_2 \). Then an easy application of the mean value theorem shows that \( \bar{s} \) is partially differentiable in the direction of \( x_2 \) at \( x^0 \) with

\[
(D_2 \bar{s})(x^0) = -(\omega_2 \bar{s})(x^0).
\]

This observation proves item 1 (all points of \( C_2 \) are isolated).

When the connection \( \nabla \) is of class \( C^1 \), then the \( \omega_1 : M \to \mathbb{R}^r \times r \cong \mathcal{L}(\mathbb{R}^r) \) are all of class \( C^1 \) \((i = 1, \ldots, n)\). The fact that \( \omega_1 \) is of class \( C^1 \) implies (by means of theorem 3.1, item 2) that \( X \) and, in turn, \( \bar{s} \) is of class \( C^1 \). Specifically, \( \bar{s} \) is partially differentiable in the direction of \( x_2 \) with continuous derivative \( D_2 \bar{s} \). Since \( D_2 \bar{s} + \omega_2 \bar{s} = 0 \) holds on \( M \setminus F \) and \( M \setminus F \) lies dense in \( M \), we infer that the latter equation holds on all of \( M \) by the continuity of its left-hand side. This observation proves item 2.

In order to prove item 3 you conduct the arguments that lead up to item 1 once again, only with indices 1 and 2 swapped. This procedure yields a second extension \( \bar{s}_2 \) of \( s \) to \( M \), of which we know a priori that it is partially differentiable in the direction of \( x_2 \). Since \( \bar{s}_2 \) is, just as \( \bar{s} \), continuous, we conclude that \( \bar{s}_2 = \bar{s} \) by means of the density of \( M \setminus F \) in \( M \). \( \square \)

**Corollary 3.4.** Let \( n \) and \( M \) be as in lemma 3.3. Then there exists \( F \) such that
(1) $F$ is negligible in $M$ for all connections of class $C^1$—in particular, $F$ is a closed nowhere dense subset of $M$ with $M \setminus F$ being connected—.

(2) there exists a connection $\nabla$ of class $C^0$ on the trivial bundle $E$ of rank 1 over $M$ such that $F$ is not negligible in $M$ for $(E, \nabla)$—in particular, $F$ is not negligible in $M$ for all connections of class $C^0$—.

(3) $\lambda^n(F) = 0$.

Proof. Purely for convenience (i.e., nicer formulas below) let us assume that $0 \in I_1$ and $[0, 1] \subset I_2$. Then we can take

$$F = \{x \in M : 0 \leq x_1, x_2 \in C\},$$

where $C$ denotes the habitual Cantor set. Then $F$ is obviously closed in $M$ and we have item 3 (since $\lambda^1(C) = 0$). Since $C$ is nowhere dense (and closed) in $[0, 1]$, it is so in $I_2$. Thus we infer item 1 from lemma 3.3, item 2.

In order to see item 2, define

$$f: I_1 \to \mathbb{R}, \quad f(x) = \begin{cases} 0 & \text{when } x < 0, \\ x^2 & \text{when } 0 \leq x, \end{cases}$$

and let $g: I_2 \to \mathbb{R}$ be the extension of the Cantor function such that $g(x) = 0$ for $x < 0$ and $g(x) = 1$ for $1 < x$. Moreover, let $\nabla$ (on $E$) be given by $\omega_1 dx^1$ with respect to the frame $e = e_1$, where $e(x) = (x, 1)$ and

$$\omega_1(x_1, \ldots, x_n) = \frac{f'(x_1)g(x_2)}{1 + f(x_1)g(x_2)}.$$

More explicitly,

$$\nabla_M (se) = (ds + s\omega_1 dx^1) \otimes e$$

for all $s \in C^1(M)$. Then $\nabla$ is a connection of class $C^0$ on $E$ (cf. remark 2.9). Define

$$s: M \to \mathbb{R}, \quad s(x_1, \ldots, x_n) = 1 + f(x_1)g(x_2).$$

Then $se|_{M \setminus F} \in C^1(M \setminus F, E)$ with $\nabla_{M \setminus F} (se|_{M \setminus F}) = 0$; note that $g$ is differentiable on $I_2 \setminus C$ with $g' = 0$. However, there exists no $\tilde{s} \in C^1(M)$ such that $\tilde{s}|_{M \setminus F} = s|_{M \setminus F}$ since such a $\tilde{s}$ would necessarily agree with $s$ (by continuity and the fact that $M \setminus F$ lies dense in $M$), yet the function $s$ is not partially differentiable with respect to its second variable in points $x \in M$ with $0 < x_1$ and $x_2 \in C$ (of which there exists at least one). 

\[
\Box
\]

Proposition 3.5. Let $n \in \mathbb{N},$ $2 \leq n,$ $M$ an open $n$-dimensional interval, and $\lambda_0 < \lambda^n(M) < \infty$. Then there exists $F$ such that $F$ is compact in $M$, $\lambda_0 < \lambda^n(F)$, and $F$ is negligible in $M$ for all connections of class $C^0$.

Proof. In case $\lambda^n(M) = 0$ take $F = \emptyset$. Assume $0 < \lambda^n(M) = \lambda(I_1) \cdot \ldots \cdot \lambda(I_n)$ now. Evidently, there exists a number $\delta > 0$ such that

$$\lambda_0 < (\lambda(I_1) - 2\delta) \cdot \ldots \cdot (\lambda(I_n) - 2\delta) \quad \text{and} \quad 0 < \lambda(I_i) - 2\delta, \quad \forall i.$$  

Moreover, for all $i \in \{1, \ldots, n\}$, there exists a compact subinterval $I'_i \subset I_i$ such that $\lambda(I_i) - \delta < \lambda(I'_i)$; also, there exists a closed, nowhere dense subset $C_i \subset I'_i$ such that $\lambda(I'_i) - \delta < \lambda(C_i)$ (take a “fat” Cantor set, for instance, translated and rescaled to fit inside $I'_i$). Put $F := C_1 \times \ldots \times C_n$. Then $F$ is obviously compact in $M$ with

$$(\lambda(I_1) - 2\delta) \cdot \ldots \cdot (\lambda(I_n) - 2\delta) < \lambda(C_1) \cdot \ldots \cdot \lambda(C_n) = \lambda^n(F).$$
For all \( i \), since \( I_i \) is open and \( I_i' \subset I_i \) compact, there exists \( b_i \in I_i \) such that \( I_i' \subset \{ x_i \in I_i : b_i \leq x_i \} \). Therefore, \( F \) is negligible in \( M \) for all connections of class \( C^0 \) by means of lemma 3.3, item 3. \( \square \)

**Corollary 3.6.** Let \( n \in \mathbb{N}, 2 \leq n, M \subset \mathbb{R}^n \) open (endowed with its canonical manifold structure of class \( C^k, 1 \leq k \leq \infty \)).

1. For all numbers \( \lambda_0 < \lambda^n(M) \), there exists a compact subset \( F \) of \( M \) such that \( \lambda_0 < \lambda^n(F) \), yet \( F \) is negligible in \( M \) for all connections of class \( C^0 \).
2. When \( \lambda^n(M) = \infty \), there exists a closed subset \( F \) of \( M \) such that \( \lambda^n(F) = \infty \), yet \( F \) is negligible in \( M \) for all connections of class \( C^0 \).

**Proof.** We know there exists an at most countable set \( \Omega \) of compact \( n \)-dimensional cubes of strictly positive measure such that \( \bigcup \Omega = M \), the collection \( \Omega \) is locally finite in \( M \), and, for all \( Q_1, Q_2 \in \Omega, Q_1 \neq Q_2 \), the interiors of \( Q_1 \) and \( Q_2 \) are disjoint.\(^6\) In particular,

\[
\lambda^n(M) = \lambda^n(\bigcup \Omega) \leq \sum_{Q \in \Omega} \lambda^n(Q) = \sum_{Q \in \Omega} \lambda^n(Q^o) = \lambda^n(\bigcup_{Q \in \Omega} Q^o) \leq \lambda^n(M).
\]

Let \( \lambda_0 < \lambda^n(M) \) be given. Then there exists a finite subset \( \Omega' \subset \Omega \) such that

\[
\lambda_0 < \sum_{Q \in \Omega'} \lambda^n(Q^o) =: \lambda_1.
\]

Assume \( 0 \leq \lambda_0 \). Then \( 0 < \lambda_1 \) and, for all \( Q \in \Omega' \), we have

\[
\frac{\lambda_0}{\lambda_1} \lambda^n(Q^o) < \lambda^n(Q^o).
\]

Therefore by proposition 3.5, for all \( Q \in \Omega' \), there exists a compact subset \( F_Q \) of \( Q^o \) such that

\[
\frac{\lambda_0}{\lambda_1} \lambda^n(Q^o) < \lambda^n(F_Q)
\]

and \( F \) is negligible in \( Q^o \) for all connections of class \( C^0 \). In turn, there exists a corresponding tuple \( (F_Q)_{Q \in \Omega'} \). Put \( F := \bigcup_{Q \in \Omega'} F_Q \). Then \( F \) is compact in \( M \) as a finite union of compact subsets of \( M \). Moreover,

\[
\lambda_0 = \frac{\lambda_0}{\lambda_1} \sum_{Q \in \Omega'} \lambda^n(Q^o) < \sum_{Q \in \Omega'} \lambda^n(F_Q) = \lambda^n(\bigcup_{Q \in \Omega'} F_Q) = \lambda^n(F).
\]

Now let \( (E, \nabla) \) be a vector bundle with connection over \( M \). Set

\[
\mathcal{U} := \{ Q^o : Q \in \Omega' \} \cup \{ M \setminus F \}.
\]

Then \( \mathcal{U} \) is an open cover of \( M \), evidently. Furthermore, items 1 and 2 of proposition 2.7 hold (make distinctions as to whether \( U \), and possibly \( V \), are equal to some \( Q^o \) or \( M \setminus F \)). Hence \( F \) is negligible in \( M \) for \( (E, \nabla) \) by proposition 2.7. As \( (E, \nabla) \) was arbitrary, \( F \) is negligible in \( M \) for all connections of class \( C^0 \). This proves item 1 in case \( 0 \leq \lambda_0 \); in case \( \lambda_0 < 0 \) take \( F = \emptyset \) (see example 2.4).

Now assume \( \lambda^n(M) = \infty \). By proposition 3.5 there exists a family \( (F_Q)_{Q \in \Omega} \) of compact subsets \( F_Q \subset Q^o \) satisfying

\[
\frac{1}{2} \lambda^n(Q^o) < \lambda^n(F_Q)
\]

\(^6\)One can take \( \Omega \) to be the set of all \( Q \subset M \) such that, for some \( k \in \mathbb{N}, 2^k Q \) is a unit cube with integral vertices and \( Q \) is maximal with respect to set inclusion among all of those \( Q \).
such that $F_Q$ is negligible in $Q^e$ for all connections of class $C^0$.\footnote{Naively one would invoke the axiom of choice in order to conclude here. However, refining the statement of proposition 3.5, the axiom of choice can be bypassed. The main point is that in the proof of proposition 3.5 the choices of $4$, $P_i'$, and $C_i$, which lead to $F_i$, can be made explicit. For instance, $\delta$ may be chosen as the minimum of $\frac{1}{t} \lambda(I_i)$, $i = 1, \ldots, n$, and one third of the infimum of $\{ t \geq 0 : \lambda_0 = (\lambda(I_1) - t) \cdot \ldots \cdot (\lambda(I_{n}) - t) \}$ if such a minimum exists.} Define $F := \bigcup_{Q \in \Omega} F_Q$. Then as
\[
\sum_{Q \in \Omega} \lambda^n(Q^e) = \lambda^n(M) = \infty,
\]
we have
\[
\lambda^n(F) = \sum_{Q \in \Omega} \lambda^n(F_Q) = \infty.
\]
Since $\Omega$ is locally finite in $M$, the family $(F_Q)$ is locally finite in $M$, too. As the $F_Q$’s are compact whence closed in $M$, their union $F$ is closed in $M$. That $F$ is negligible in $M$ for all connections of class $C^0$ is inferred just like above employing proposition 2.7. Therefore we have item 2. \hfill \Box

**Remark 3.7 (Pullback connections).** Let $M$ and $M'$ be manifolds of classes $C^k$ and $C^{k'}$ respectively, $1 \leq k, k' \leq \infty$, $\phi : M' \to M$ a morphism of class $C^1$, $E$ a vector bundle of class $C^l$ over $M$, $1 \leq l \leq k$. Then by a pullback of $E$ by $\phi$ we mean a pullback of $E$ by $\phi$ in the sense of $C^1$ manifolds, that is, you first pass from $M$, $M'$, and $E$ to their corresponding manifolds of class $C^1$ (by possibly enlarging the atlases), then you speak of the pullback.

Let $(E', \phi')$ be a pullback of $E$ by $\phi$. Moreover, let $\nabla$ be a connection on $E$. We contend that there exists a unique connection $\nabla'$ on $E'$ such that the following holds: when $U$ is open in $M$, $r \in \mathbb{N}$, $(e_1, \ldots, e_r)$ is a frame for $E$ over $U$, and $(\omega^\beta)$ an $r \times r$ matrix with values in $C^0(U, T^\vee(M))$ such that
\[
\nabla_U(e_\beta) = \omega^\beta \otimes e_\alpha, \quad \forall \beta,
\]
then
\[
\nabla'_{U'}(\phi^*e_\beta) = \phi^*\omega^\beta \otimes \phi^*e_\alpha, \quad \forall \beta, \quad U' := \phi^{-1}(U).
\]
Here, $\phi^*$ has two meanings: for one, $\phi^*\omega^\beta$ denotes the pullback of $\omega^\beta$ in the sense of (degree 1) differential forms; for another, $\phi^*e_\beta$ denotes the pullback section of $e_\beta$ by $\phi$ with respect to the pullback bundle $(E', \phi')$ of $E$, that is, $\phi^*e_\beta$ is the unique section of class $C^1$ in $E'$ defined on $U'$ such that $\phi' \circ \phi^*e_\beta = e_\beta \circ \phi$.

As a matter of fact, for all $U$ and $e = (e_\beta)$ as above, the $r$-tuple $(\phi^*e_\beta)$ constitutes a frame for $E'$ over $U'$, i.e., a global frame for $E'|_{U'}$. Therefore, by remark 2.9, there exists a unique connection $\nabla'_{U,e}$ on $E'|_{U'}$ such that
\[
(\nabla'_{U,e})_{U'}(\phi^*e_\beta) = \phi^*\omega^\beta \otimes \phi^*e_\alpha, \quad \forall \beta.
\]
We may view $\nabla'_{U,e}$ as a morphism of abelian sheaves
\[
\nabla'_{U,e} : C^1(E')|_{U'} \to C^0(T^\vee(M') \otimes E')|_{U'}
\]
on $M'|_{U'}$—note that this is up to the isomorphism $T^\vee(M')|_{U'} \to T^\vee(M'|_{U'})$ which is induced by the inclusion map $U' \to M'$. For another pair $(V, f)$ consisting of an open set $V$ in $M$ and a local frame $f$ for $E$ over $V$ an easy calculation with transition functions shows that the morphisms of sheaves $\nabla'_{U,e}$ and $\nabla'_{V,f}$ agree on $U' \cap V'$, $V' = \phi^{-1}(V)$. Thus (since the internal hom of the abelian sheaves $C^1(E')$
and $C^0(T'(M') \otimes E')$ on $M'$ is a sheaf on $M'$ and since the $U'$ cover $M'$ there exists a unique morphism of abelian sheaves on $M'$

$$\nabla': C^1(E') \to C^0(T'(M') \otimes E')$$

such that $\nabla'|_{U'} = \nabla_{U', e}$ for all pairs $(U, e)$. The Leibniz rule readily extends from the individual $\nabla_{U', e}$ to $\nabla'$. This proves our claim.

$\nabla'$ is called the pullback connection associated to the given pullback diagram. The pullback connection $\nabla'$ has the following decisive property (slightly generalizing the property used to characterize $\nabla'$ above): for all open subsets $W$ of $M$ and all $\sigma \in C^1(E)(W)$ we have

$$\nabla_{\phi^{-1}(W)}(\phi^*\sigma) = \phi^*(\nabla_W(\sigma)),$$

where the $\phi^*$ on the right-hand side takes a section $\xi$ in $T'(M) \otimes E$ defined on $W$ to the unique section $\phi^*\xi$ in $T'(M') \otimes E'$ defined on $\phi^{-1}(W)$ such that, for all $p' \in \phi^{-1}(W)$, the value $(\phi^*\xi)(p')$ is the image of $\xi(\phi(p'))$ under the evident tensor product map

$$T'_\phi(p')(M) \otimes E_{\phi(p')} \to T'_{\phi^{-1}(W)}(M') \otimes E'_p;$$

writing $\xi = \xi^\alpha \otimes e_\alpha$ with respect to a local frame $e = (e_\alpha)$ for $E$, the $\xi^\alpha$ being local sections in $T'(M)$, we have $\phi^*\xi = \phi^*\xi^\alpha \otimes \phi* e_\alpha$. As a consequence, we see that the pullback of sections

$$\phi^*: C^1(E) \to \phi_*(C^1(E')),$$

viewed as a morphism of sheaves on $M$, maps sub(pre)sheaf $\Gamma^\nabla(E) \subset C^1(E)$ into the sub(pre)sheaf $\Gamma^\nabla'(E') \subset C^1(E')$.

**Proposition 3.8.** Let $M$ and $M'$ be manifolds of classes $C^k$ and $C^{k'}$ respectively, $1 \leq k, k' \leq \infty$, $\phi: M \to M'$ a $C^1$ diffeomorphism, $F \subset M$ such that $F' := \phi(F)$ is negligible in $M'$ for all connections of class $C^0$. Then $F$ is negligible in $M$ for all connections of class $C^0$.

**Proof.** Let $(E, \nabla)$ be a vector bundle with connection over $M$. As $\phi$ is a $C^1$ diffeomorphism, it has a $C^1$ inverse $\psi: M' \to M$. We know there exists a pullback bundle $(E', \nabla')$ of $E$ by $\psi.$\footnote{As a matter of fact, here, the general existence of pullbacks is not needed. When $\pi: E \to M$ is the projection of the vector bundle $E$, then take $E'$ to be given by $E$ (as the total space, with its induced $C^1$ structure), $\phi \circ \pi: E \to M'$ (as the projection), and the vector space structures that the fibers of $E$ already have; moreover, take $\phi' = \text{id}_E$.} Also, there exists a pullback connection $\nabla'$ of $\nabla$ (cf. remark 3.7). Let $\sigma$ be a $\nabla$-parallel section in $E$ defined on $M \setminus F$. Then there exists a pullback section $\sigma'$ of $\sigma$, which is a $\nabla'$-parallel section in $E'$ defined on $\psi^{-1}(M \setminus F) = M' \setminus F'$. As $F'$ is negligible in $M'$ for $(E', \nabla')$, there exists a $\nabla'$-parallel section $\tilde{\sigma}'$ in $E'$ defined on $M'$ such that $\tilde{\sigma}'|_{M' \setminus F'} = \sigma'$.

As $(\phi$ whence $)$ is a $C^1$ diffeomorphism, $\psi'$ is a $C^1$ diffeomorphism and, passing from the $C^1$ manifold structure of $E$ to its induced $C^1$ manifold structure, $(E, \psi'^{-1})$ is a pullback of $E'$ by $\psi^{-1} = \phi$. Hence there exists an associated pullback section $\tilde{\sigma}$ of $\tilde{\sigma}'$, which is $\nabla$-parallel since $\nabla$ is the pullback connection of $\nabla'$. Moreover, $\tilde{\sigma}|_{M \setminus F}$ is the pullback section of $\sigma$ with respect to the identity diagram. Thus $\tilde{\sigma}|_{M \setminus F} = \sigma$.

As $\sigma$ was arbitrary, this proves that $F$ is negligible in $M$ for $(E, \nabla)$. As $(E, \nabla)$ was arbitrary, this proves in turn that $F$ is negligible in $M$ for all connections of class $C^0$. \hfill \Box
We recall some terminology on (sub)manifolds with boundary. For that matter, let $M$ be a manifold, $F \subset M$, $0 \leq m \leq k$.

Let $p \in F$. Then $F$ is a $C^m$ submanifold with boundary of $M$ at $p$ when there exist $d, c \in \mathbb{N}$, an open neighborhood $U$ of $p$ in $M$, an open subset $V$ of $\mathbb{R}^d \times \mathbb{R}^c$, and an isomorphism $\phi: U \to V$ of class $C^m$ such that

$$\phi(F \cap U) = (H \times \{(0, \ldots, 0)\}) \cap V,$$

where

$$H = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : 0 \leq x_d\}.$$

In that case we write

$$\text{codim}_p(F, M) = c.$$

We say that $F$ is a $C^m$ submanifold with boundary of $M$ if, for all $p \in F$, $F$ is a $C^m$ submanifold with boundary of $M$ at $p$. In that case we set

$$\text{codim}(F, M) := \inf\{\text{codim}_p(F, M) : p \in F\},$$

where the infimum of the empty set is taken to be $\infty$.

**Remark 3.9 (Density and codimension).** Let $M$ be a manifold, $F$ a closed $C^0$ submanifold with boundary of $M$. Then the following are equivalent:

1. $F$ is nowhere dense in $M$.
2. $1 \leq \text{codim}(F, M)$.

Assume item 1. Let $p \in F$. Assume $\text{codim}_p(F, M) = 0$. Then there exists an open neighborhood $U$ of $p$ in $M$, a number $n \in \mathbb{N}$, and a homeomorphism $\phi: U \to \mathbb{R}^n$ such that $\phi(F \cap U) = \{x \in \mathbb{R}^n : 0 \leq x_n\}$. The preimage of $(0, \ldots, 0, 1)$ under $\phi$ would be an interior point of $F$ in $M$, which is impossible. So, $1 \leq \text{codim}_p(F, M)$.

As $p \in F$ was arbitrary, we have item 2.

Assume item 2, and let $p \in F$. Then there exists an open neighborhood $U$ of $p$ in $M$, numbers $d, c \in \mathbb{N}$, an open subset $V$ of $\mathbb{R}^d \times \mathbb{R}^c$, and a homeomorphism $\phi: U \to V$ such that $\phi(F \cap U) \subset (\mathbb{R}^d \times \{(0, \ldots, 0)\}) \cap V$. If $p$ were an interior point of $F$ in $M$, the point $\phi(p)$ would be an interior point of $\mathbb{R}^d \times \{(0, \ldots, 0)\}$ in $\mathbb{R}^d \times \mathbb{R}^c$. Yet as $1 \leq \text{codim}(F, M) \leq \text{codim}_p(F, M) = c$, this is absurd.

**Remark 3.10 (Maximal extensions).** Let $M$ be a manifold, $F$ a closed, nowhere dense subset of $M$, $(E, \nabla)$ a vector bundle with connection over $M$, $s \in \Gamma^\nabla(E)(M \setminus F)$. Let, for $i = 0, 1, U_i \subset M$ be open and $s_i \in \Gamma^\nabla(E)(U_i)$ such that $s_i|_{U_i \setminus F} = s|_{U_i \setminus F}$. Then $s_0$ and $s_1$ agree on $(U_0 \setminus F) \cap (U_1 \setminus F) = (U_0 \cap U_1) \setminus F$. Since $F$ nowhere dense in $M$, we know that $(U_0 \cap U_1) \cap F$ is nowhere dense in $U_0 \cap U_1$, whence $(U_0 \cap U_1) \setminus F$ is dense in $U_0 \cap U_1$. Thus we have $s_0 = s_1$ on all of $U_0 \cap U_1$ as $s_0$ and $s_1$ are continuous—note that this holds even though $E$ might be non-Hausdorff since on an open set over which $E$ is trivial, the $s_i$ correspond to continuous maps to the Hausdorff $\mathbb{R}^r$, $r \in \mathbb{N}$ being the local rank of $E$.

As $\Gamma^\nabla(E)$ is a sheaf, this argument shows that there exists one, and only one, $\tilde{s} \in \Gamma^\nabla(E)(\tilde{U})$ such that $\tilde{s}|_{U_i} = s_i$ holds for all $s_0$ as above. Note that $M \setminus F \subset \tilde{U}$ and $\tilde{s}|_{M \setminus F} = s$ since we can take $U_0 = M \setminus F$ and $s_0 = s$. We call $\tilde{s}$ the maximal $\nabla$-parallel extension of $s$.

$\tilde{s}$ has the property that when $p \in M \setminus \tilde{U}$, then there exists no $s_0 \in \Gamma^\nabla(E)(U_0)$, where $U_0 \subset M$ is open, $p \in U_0$, and $s_0 = s$ on $U_0 \setminus F$; otherwise we had $\tilde{s}|_{U_0} = s_0$ implying $U_0 \subset \tilde{U}$ and thus $p \in \tilde{U}$, in particular.
Let $F$ be an arbitrary topological space, $p \in F$. We say $p$ is an interior point (in the manifold sense) of $F$ when there exist $d \in \mathbb{N}$, an open neighborhood $U'$ of $p$ in $F$, and a homeomorphism $\phi' : U' \to \mathbb{R}^d$. We say $p$ is a boundary point (in the manifold sense) of $F$ when there exist $d \in \mathbb{N}$, an open neighborhood $U'$ of $p$ in $F$, and a homeomorphism $\phi' : U' \to H$, where

$$H = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : 0 \leq x_d\}$$

topologized by $\mathbb{R}^d$, such that $\phi'(p)$ lies in the topological boundary of $H$ in $\mathbb{R}^d$. The set of boundary points (in the manifold sense) of $F$ will be denoted $\partial F$.

**Theorem 3.11.** Let $M$ be a manifold, $F$ a closed $C^1$ submanifold with boundary of $M$ such that $1 \leq \text{codim}(F, M)$ and, for all connected components $F'$ of $F$ with $\text{codim}(F', M) = 1$, we have $F' \cap \partial F \neq \emptyset$. Then $F$ is negligible in $M$ for all connections of class $C^0$.

**Proof.** Let $(E, \nabla)$ be a vector bundle with connection over $M$, $s \in \Gamma^\nabla(E)(M \setminus F)$. Denote $s : \tilde{U} \to E$ the maximal $\nabla$-parallel extension of $s$ (see remark 3.10 for the definition). We want to show that $U = M$, or, more specifically, that $M \subset \tilde{U}$.

Assume that $p \in M$. When $p \notin F$, then $p \in \tilde{U}$ since $M \setminus F \subset \tilde{U}$. Suppose $p \in F$ from now on. As $F$ is a $C^1$ submanifold with boundary of $M$ at $p$, there exist $d, c \in \mathbb{N}$, an open neighborhood $U$ of $p$ in $M$, an open subset $V$ of $\mathbb{R}^d \times \mathbb{R}^c$, and a $C^1$ diffeomorphism $\phi : U \to V$, the sets $U$ and $V$ equipped with their induced manifold structures from $M$ and $\mathbb{R}^d \times \mathbb{R}^c$, respectively, such that

$$\phi(U \cap V) = (H \times \{(0, \ldots, 0)\}) \cap V.$$ 

Here,

$$H = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : 0 \leq x_d\};$$

when $d = 0$, that is, to mean $H = \mathbb{R}^0$. By further restricting $\phi$, we can achieve that $V$ is an open hyperrectangle in $\mathbb{R}^d \times \mathbb{R}^c \cong \mathbb{R}^{d+c}$, that is, $V = I_1 \times \cdots \times I_n$ with $n = d + c$ for open intervals $I_i \subset \mathbb{R}$, $i = 1, \ldots, n$, such that $V$ contains $\phi(p)$. Write $\phi(p) = (\phi_1(p), \ldots, \phi_n(p))$. Observe that

$$1 \leq \text{codim}(F, M) \leq \text{codim}_p(F, M) = c.$$

So we have $2 \leq c$ or $c = 1$.

Assume $2 \leq c$. Then $0 = \phi_{n-1}(p) \in I_{n-1}$, so that there exists $b_{n-1} \in I_{n-1}$ with $b_{n-1} < 0$. Put $C_n = \{0\} \setminus I_n$ (actually, $C_n = \{0\}$ since $0 = \phi_n(p) \in I_n$). Then $C_n$ is nowhere dense in $I_n$ and

$$\phi(F \cap U) \subset \{x \in V : x_{n-1} = x_n = 0\} \subset \{x \in V : b_{n-1} \leq x_{n-1}, x_n \in C_n\}.$$ 

Thus $\phi(F \cap U)$ is negligible in $V$ for all connections of class $C^0$ by lemma 3.3, item 1. Therefore by proposition 3.8, $F \cap U$ is negligible in $U$ for all connections of class $C^0$; in particular, $F \cap U$ is negligible in $U$ for $(E, \nabla)|_U$. Hence there exists an element $s_0 \in \Gamma^\nabla(E)(U)$ which agrees with $s$ on $U \setminus F$. So $U \subset \tilde{U}$ by the definition of $\tilde{s}$. As a result, $p \in \tilde{U}$.

Now, assume $c = 1$. If $d$ was equal to $0 \in \mathbb{N}$, the point $p$ would be an isolated point of $F$ and, in consequence, $F' = \{p\}$ would be a connected component of $F$ with $\partial F' = \emptyset$, contradicting our premises. Thus $d \neq 0$, that is, $1 \leq d$.

Assume that $p \in \partial F$. Then $0 = \phi_d(p) \in I_d$, whence $b_d < 0$ for some $b_d \in I_d$. Just like above we derive $p \in \tilde{U}$ invoking item 1 of lemma 3.3 and proposition 3.8.
Assume that \( p \) belongs to the interior of \( F \). We claim that \( F \cap U \subset M \setminus \tilde{U} \). For that matter, suppose \( q \in (F \cap U) \cap \tilde{U} \). Since there exists a \( C^1 \) (a \( C^\infty \), in fact) diffeomorphism \( V \to \mathbb{R}^n \) taking the set \( \{ x \in \mathbb{R}^n : 0 \leq x_\alpha, x_n = 0 \} \) to itself, we can assume that \( V = \mathbb{R}^n \). Now, since \( \phi(U \setminus \tilde{U}) \) is closed in \( V = \mathbb{R}^n \) and contains at least one element (namely \( \phi(p) \)), there exists an element \( x^0 \in \phi(U \setminus \tilde{U}) \) such that 
\[
\epsilon := |x^0 - \phi(q)| = \inf \{ |x - \phi(q)| : x \in \phi(U \setminus \tilde{U}) \}.
\]
In consequence, there exists an element \( p^0 \in U \setminus \tilde{U} \) such that \( \phi(p^0) = x^0 \). Observe that \( x^0 \neq \phi(q) \) as otherwise the injectivity of \( \phi \) would imply \( p^0 = q \) and thus \( p^0 \in \tilde{U} \). So, \( 0 < \epsilon \). Moreover, observe that \( x_n^0 = 0 \) since \( p^0 \in F \). Therefore, there exists an \( n \)-dimensional Euclidean move (a translation followed by an orthogonal transformation) \( \tau : \mathbb{R}^n \to \mathbb{R}^n \) taking \( \phi(q) \) to the origin, \( x^0 \) to \( (0, \ldots, 0, \epsilon, 0) \), and the set \( \{ x \in \mathbb{R}^n : x_n = 0 \} \) to itself. Define \( J_n := \mathbb{R}, J_{n-1} := (0, \infty) \), and, in case \( 2 < n \), \( J_i := (-\delta, \delta) \) with \( \delta = \sqrt{\frac{1}{2(n-2)} \epsilon} \) for \( i = 1, \ldots, n - 2 \). Moreover, let 
\[
V_0 := J_1 \times \cdots \times J_n, \quad U_0 := (\tau \circ \phi)^{-1}(V_0).
\]
Then 
\[
(\tau \circ \phi)(U_0 \setminus \tilde{U}) \subset \{ y \in V_0 : \sqrt{\frac{1}{2}} \leq y_n-1, y_n = 0 \}.
\]
As a matter of fact, when \( y \in V_0 \) such that \( y = \tau(x) \) for an \( x \in \phi(U \setminus \tilde{U}) \), then we have \( y_n = 0 \) (note that \( x_n = 0 \) as \( U \setminus \tilde{U} \subset F \)) and 
\[
eq |x - \phi(q)|^2 = |y|^2 = y_1^2 + \cdots + y_{n-1}^2 \leq (n-2)\delta^2 + y_{n-1}^2 = \frac{1}{2} \epsilon^2 + y_{n-1}^2.
\]
Accordingly, by item 1 of lemma 3.3, \( (\tau \circ \phi)(U_0 \setminus \tilde{U}) \) is negligible in \( V_0 \) for all connections of class \( C^0 \). In consequence, by proposition 3.8, \( U_0 \setminus \tilde{U} \) is negligible in \( U_0 \) for all connections of class \( C^0 \); note that \( (\tau \circ \phi)|_{U_0} : U_0 \to V_0 \) is a \( C^1 \) diffeomorphism. Specifically, \( U_0 \setminus \tilde{U} \) is negligible in \( U_0 \) for \( (E, \nabla)|_{U_0} \), whence there exists an element \( s_0 \in \Gamma^V(E)|_{U_0} \) such that \( s_0 = \bar{s} \) on \( U_0 \cap \tilde{U} \), i.e., \( s_0 = s \) on \( U_0 \setminus F \). The maximality of \( \bar{s} \) implies \( U_0 \subset \tilde{U} \). As \( p^0 \in U_0 \), we deduce \( p^0 \in \tilde{U} \)—a contradiction. Therefore, for all \( q \in F \setminus U \), we have \( q \notin \tilde{U} \).

Let \( Z \) be the connected component of the interior of \( F \) that contains \( p \). Then, on the one hand, the arguments of the preceding paragraph, applied to an arbitrary \( p' \in Z \) instead of \( p \), show that \( Z \setminus \tilde{U} \) is open in \( Z \). On the other hand, \( Z \setminus \tilde{U} \) is certainly closed in \( \tilde{U} \) since \( U \) is open in \( M \), thus \( Z \cap \tilde{U} \) open in \( Z \). As \( p \in Z \setminus \tilde{U} \), we infer \( Z = Z \setminus \tilde{U} \) from the connectedness of \( Z \). Let \( \overline{Z} \) be the closure of \( Z \) in \( F \). If \( Z = \overline{Z} \), then \( Z \) would be a connected component of \( F \) with \( \text{codim}(Z, M) = 1 \) and \( \partial Z = \emptyset \), which is impossible under our assumptions. Thus there exists an element \( p'' \in \overline{Z} \) such that \( p'' \notin Z \). Suppose that \( p'' \) is an interior point of \( F \). Then there exists a connected, open neighborhood \( U'' \) of \( p'' \) in \( F \) such that \( U'' \) contains only interior points of \( F \). As \( p'' \) lies in the closure of \( Z \), the intersection \( Z \cap U'' \) contains an element. In turn, \( Z \cap U'' \subset Z \) by the maximality of \( Z \). In particular, we conclude \( p'' \in Z \)—a contradiction. Therefore, \( p'' \) is not an interior point of \( F \), but a boundary point of \( F \). Moreover, \( \text{codim}_F(F, M) = 1 \) as the codimension of \( F \) in \( M \) is constant, i.e., constantly equal to 1, on \( Z \). Thus we find \( p'' \in \tilde{U} \) (just as we did for \( p \) in place of \( p'' \) above). However, we also have \( \overline{Z} \subset F \setminus \tilde{U} \) because \( F \setminus \tilde{U} \) is closed in \( F \) and a superset of \( Z \). This is a contradiction.
In conclusion, we see that \( p \in F \) or, more generally, \( p \in M \) implies \( p \in \widetilde{U} \). Thus \( M = \widetilde{U} \) and \( \tilde{s} \in \Gamma^\nabla(E)(M) \) so that \( \tilde{s}|_{M \setminus F} = s \). As \( s \) was arbitrary, this tells that \( F \) is negligible in \( M \) for \((E, \nabla)\). As \((E, \nabla)\) was arbitrary, we have deduced that \( F \) is negligible in \( M \) for all connections of class \( C^0 \), which was to be demonstrated. \( \square \)

**Definition 3.12.** We say that a connected manifold \( M \) is dissected by \( C^1 \) hypersurfaces when, for all closed, connected \( C^1 \) submanifolds \( F \) of \( M \) with \( \text{codim}(F, M) = 1 \), the space \( M \setminus F \) is disconnected (i.e., equal to the disjoint union of two nonempty, open subsets).

**Corollary 3.13.** Let \( M \) be a connected, Hausdorff manifold which is dissected by \( C^1 \) hypersurfaces, \( 2 \leq \dim M, F \neq M \) a closed \( C^1 \) submanifold with boundary of \( M \). Then the following are equivalent:

1. \( F \) is negligible in \( M \) for all connections of class \( C^0 \).
2. \( F \) is negligible in \( M \) for all connections of class \( C^{k-1} \).
3. \( F \) is negligible in \( M \) for all connections of class \( C^{k-1} \) on the trivial bundle of rank 1 over \( M \).
4. \( F \) is nowhere dense in \( M \) and \( M \setminus F \) is connected.
5. \( 1 \leq \text{codim}(F, M) \) and there exists no connected component \( Z \) of \( F \) such that \( \text{codim}(Z, M) = 1 \) and \( \partial Z = \emptyset \).

**Proof.** Clearly item 1 implies item 2, and item 2 implies item 3. Item 3 implies item 4 according to corollary 2.13 (here we use that \( M \) is Hausdorff, connected, and of dimension 2 or greater—observe that for \( M = \mathbb{R} \), equipped with its canonical \( C^k \) manifold structure, the conclusion fails as shown by example 2.11) and proposition 2.10 (observe that the standard connection on the trivial bundle of rank 1 over \( M \) is of class \( C^{k-1} \)).

Assume item 4. Then the codimension of \( F \) in \( M \) is \( \geq 1 \) by remark 3.9 (as a \( C^1 \) submanifold, \( F \) is also a \( C^0 \) submanifold). Let \( Z \) be a connected component of \( F \) such that \( \text{codim}(Z, M) = 1 \) and \( \partial Z = \emptyset \). Then, in particular, \( Z \) is a closed, connected \( C^1 \) submanifold of \( M \). As \( M \) is dissected by \( C^1 \) hypersurfaces, we infer, on the one hand, that \( M \setminus Z \) is disconnected. On the other hand, since \( F \) is nowhere dense in \( M \), we know that \( F \cap (M \setminus Z) = F \setminus Z \) is nowhere dense in \( M \setminus Z \). In turn, \( M \setminus F = (M \setminus Z) \setminus (F \setminus Z) \) is dense in \( M \setminus Z \), whence \( M \setminus Z \) is connected as the closure of a connected subspace—contradiction. Therefore, \( A \) as above cannot exist. Thus we have item 5.

Finally, from item 5 we obtain item 1 by means of theorem 3.11. \( \square \)

**Remark 3.14** (Manifolds dissected by hypersurfaces). Let \( M \) be a connected, second-countable, Hausdorff manifold with \( H_1(M; \mathbb{Z}/2\mathbb{Z}) \cong 0 \) (\( H \) denoting singular homology here). We contend that \( M \) is dissected by \( C^1 \) hypersurfaces. In particular, corollary 3.13 applies to all such \( M \) (assuming \( 2 \leq \dim M \) in addition, of course); the blatant examples are: \( M = \mathbb{R}^n \) or \( M = S^n \) for \( n \in \mathbb{N} \), \( n \geq 2 \). Note that the condition \( H_1(M; \mathbb{Z}/2\mathbb{Z}) \cong 0 \) can be strengthened to \( H_1(M; \mathbb{Z}) \cong 0 \). One might also require \( M \) to be simply connected.

The proof of our assertion consists in a twofold application of the following version of the Poincaré-Lefschetz duality theorem (see [2, VIII, 7.12]): When \( X \) is a second-countable\(^9\), Hausdorff topological \( m \)-manifold, \( m \in \mathbb{N} \), \( A \subset X \) a closed

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\(^9\)We include the hypothesis of second-countability mainly because the definition of Čech cohomology \( H(A, B) \) in [2, VIII, §6] requires \( A \) and \( B \) to be locally compact subspaces of some...
subspace, then there exists a sequence \((\gamma_i)_{i \in \mathbb{Z}}\) of isomorphisms (of abelian groups or else \(R\)-modules)
\[
\gamma_i : \hat{H}^i_c(A; R) \to H_{m-i}(X, X \setminus A; R),
\]
where \(R := \mathbb{Z}/2\mathbb{Z}\) (as a group or ring) and \(\hat{H}_c\) signifies Čech cohomology with compact supports.

As a matter of fact, take \(F\) to be a closed, connected \(C^1\) submanifold of \(M\) such that \(\text{codim}(F, M) = 1\). Set \(n := \dim M\). Then \(1 \leq n\) and \(F\) itself is a (second-countable, Hausdorff) topological \((n - 1)\)-manifold. Thus employing the duality theorem (for \(X = F, m = n - 1, A = F, i = n - 1\)—observe that this is plain Poincaré duality now—, we obtain
\[
\hat{H}^{n-1}_c(F) \cong H_0(F) \cong R = \mathbb{Z}/2\mathbb{Z},
\]
where we suppress the coefficient group (or coefficient ring) \(R\) in our notation of (co-)homology. The last isomorphism is due to the fact that \(F\) is nonempty and pathwise connected. Employing the duality theorem for \(X = M, m = n, A = F, i = n - 1\) yields
\[
\hat{H}^{n-1}_c(F) \cong H_1(M, M \setminus F).
\]
Yet as \(\tilde{H}_1(M) \cong H_1(M) \cong 0\) (by assumption) and \(\tilde{H}_0(M) \cong 0\) (since \(M\) is pathwise connected), the long exact sequence in reduced homology associated to the pair \((M, M \setminus F)\)—note that \(M \setminus F \neq \emptyset\) —implies
\[
H_1(M, M \setminus F) \cong \tilde{H}_0(M \setminus F).
\]
Therefore \(M \setminus F\) has precisely two (path-)connected components whence is disconnected, which was to be demonstrated.

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