Supplementary Information:

A single ‘weight-lifting’ game covers all kinds of games

Short: A single game covers all games

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Keywords: game theory, prisoner’s dilemma game, chicken game, stag hunt game
Proof for the equivalence between the traditional pairwise game with cooperation and defection and the weight-lifting game.

In this proof, we first show that (1) all the pairwise games are covered by the weight-lifting game, and then we prove the opposite that (2) the full range of the weight-lifting game is covered by the pairwise games. In this proof, we denote the payoff matrix of pairwise games with two strategies: cooperation (C) and defection (D). Player rewards are determined by the payoff matrix and the strategies that they choose (i.e., (C) or (D)) as follows:

\[
A = \begin{pmatrix}
C & D \\
D & T \\
\end{pmatrix}
\]

(1)

where we consider an unlimited well-mixed population.

(1) Proof (all pairwise games → a weight-lifting game)

(i) If \( T \neq P \), any pairwise game can be transformed into the following equivalent game:

\[
A = \begin{pmatrix}
C & D \\
D & \left( \begin{array}{cc} r & S \\ 1 & 0 \end{array} \right) \\
\end{pmatrix}
\]

(1)

where \( r \geq s \) and \( 1 \geq s \).

The payoff matrix of the weight-lifting game is given by:

\[
A = \begin{pmatrix}
C & D \\
D & \left( \begin{array}{cc} bp_2 - c & bp_1 - c \\ bp_1 & bp_0 \end{array} \right) \\
\end{pmatrix}
\]

(2)

This is equivalent to the game with

\[
A = \begin{pmatrix}
C & D \\
D & \left( \begin{array}{cc} (b(p_2 - p_0) - c) / b / (p_1 - p_0) & (b(p_1 - p_0) - c) / b / (p_1 - p_0) \\ (bp_0) & 0 \end{array} \right) \\
\end{pmatrix}
\]

(3)

if \( p_1 \neq p_0 \). For \( r = (b(p_2 - p_0) - c) / b / (p_1 - p_0) = \frac{p_2}{p_1 - p_0} - p_0 / (p_1 - p_0) - c / b / (p_1 - p_0) \) and \( s = (b(p_1 - p_0) - c) / b / (p_1 - p_0) = 1 - c / b / (p_1 - p_0), \) we obtain \( c / b = (1 - s) / (p_1 - p_0) \geq 0 \) and \( r - s = (p_2 - p_0) / (p_1 - p_0) - 1 \geq 0 \). Therefore, for instance, \( p_0 = 0, p_2 = 1, \) \( p_1 = 1 / (r - s + 1) \) and \( c / b = (1 - s) / (r - s + 1) > 0 \). In this case, \( 0 < p_1 < 1, \) i.e., \( p_1 \neq p_0 \).

(ii) If \( T = P \), the pairwise game can be transformed into the following equivalent game:

\[
A = \begin{pmatrix}
C & D \\
D & \left( \begin{array}{cc} r & S \\ 0 & 0 \end{array} \right) \\
\end{pmatrix}
\]

(4)

where \( r \geq s \) and \( 0 \geq s \). The weight-lifting game is equivalent to

\[
A = \begin{pmatrix}
C & D \\
D & \left( \begin{array}{cc} b(p_2 - p_0) - c & -c \\ 0 & 0 \end{array} \right) \\
\end{pmatrix}
\]

(5)

when \( p_0 = p_1 \). For \( r = b(p_2 - p_0) - c \) and \( s = -c \), we obtain \( s = -c \leq 0 \) and \( r - s = b(p_2 - p_0) \geq 0 \). For instance, \( c / b = -s / (r - s) \geq 0 \) for \( p_0 = 0, p_1 = 0 \) and \( p_2 = 1 \).

Therefore, all pairwise games are covered as a special case of the weight-lifting game.
(2) Proof (all weight-lifting games → a pairwise game)

We employ the proof by contradiction. We assume that there exists a weight-lifting game that does not satisfy the conditions \( R \geq S, \ T \geq P, \) and \( T \geq S. \) The case \( b = r + f = 0 \) is omitted because this is a trivial pairwise game of

\[
A = \begin{pmatrix}
C & D \\
-c & -c \\
D & 0 & 0
\end{pmatrix}
\]

(i) Case \( R < S \)

If the condition \( R \geq S \) is not satisfied,

\[
R < S \iff (r - c)p_2 - (f + c)(1 - p_2) < (r - c)p_1 - (f + c)(1 - p_1)
\]

\[
\iff rp_2 - cp_2 + fp_2 - f - c + cp_2 < rp_1 - cp_1 + fp_1 - f - c + cp_1
\]

\[
\iff rp_2 + fp_2 < rp_1 + fp_1
\]

\[
\iff (r + f)p_2 < (r + f)p_1
\]

Owing to \( r + f > 0, \) we obtain

\[
p_2 < p_1
\]

This contradicts \( p_1 \leq p_2. \) Thus, the weight-lifting game satisfies the condition \( R \geq S. \)

(ii) Case \( T < P. \)

If the condition the condition \( T \geq P \) is not satisfied,

\[
T < P \iff rp_1 - f (1 - p_1) < rp_0 - f (1 - p_0)
\]

\[
\iff rp_1 - f + fp_1 < rp_0 - f + fp_0
\]

\[
\iff rp_1 + fp_1 < rp_0 + fp_0
\]

\[
\iff (r + f)p_1 < (r + f)p_0
\]

Owing to \( r + f > 0, \) we obtain

\[
p_1 < p_0
\]

This contradicts \( p_0 \leq p_1. \) Thus, the condition \( T \geq P \) is always satisfied.

(iii) Case \( T < S. \)

If the condition \( T \geq S \) is not satisfied,

\[
T < S \iff rp_1 - f (1 - p_1) < (r - c)p_1 - (f + c)(1 - p_1)
\]

\[
\iff rp_1 - f + fp_1 < rp_1 - cp_1 - f + fp_1 - c + cp_1
\]

\[
\iff c < 0
\]

This contradicts \( 0 \leq c. \) Thus, the condition \( T \geq S \) is always satisfied.

Therefore, the weight-lifting game is a pairwise game satisfying \( R \geq S, \ T \geq P, \) and \( T \geq S. \)

From (1) and (2), the weight-lifting game is equivalent to the pairwise game. Q.E.D.