Weak $K \to \pi$ generalized form factors and transverse transition quark-spin density from the instanton vacuum

Hyeon-Dong Son,1, Seung-il Nam,2, and Hyun-Chul Kim1,3,4

1Department of Physics, Inha University, Incheon 402-751, Republic of Korea
2Department of Physics, Pukyong National University, Busan 608-737, Republic of Korea
3School of Physics, Korea Institute for Advanced Study (KIAS), Seoul 130-722, Republic of Korea
4Research Center for Nuclear Physics (RCNP), Osaka University, Ibaraki, Osaka, 567-0047, Japan

(Dated: June 10, 2015)

We investigate the generalized $K \to \pi$ transition vector and tensor form factors, from which we derive the transverse quark spin density in the course of the $K \to \pi$ transition, based on the nonlocal chiral quark model from the instanton vacuum. The results of the transition tensor form factor are in good agreement with recent data of lattice QCD. The behavior of the transverse quark spin density of the $K \to \pi$ transition turn out to be very similar to those of the pion and the kaon.

PACS numbers: 13.20.Eb, 14.40.Df, 12.39.Ki

Keywords: Semileptonic decay of the $K$ meson, transition tensor form factor, transverse spin density, nonlocal chiral quark model from the instanton vacuum

I. INTRODUCTION

Semileptonic decay of the $K$ mesons ($K_{l3}$ decay) provides a solid basis for testing various features of the Standard Model (SM). In particular, the $K_{l3}$ decay can be used for determining the Cabibbo-Kobayashi-Maskawa (CKM) matrix [1, 2] precisely within the Standard Model (SM) (see for example a recent analysis [3] and references therein). Since the $W$-boson exchange in the SM governs the physics of the $K_{l3}$ decay, the $K \to \pi$ vector transition elements have been mainly considered to describe the $K_{l3}$ decay, while other terms such as the tensor component were set aside. Several experimental collaborations have searched for possible nonzero values of the $K \to \pi$ tensor form factors but found that the results turned out to be more or less consistent with the SM prediction of the null value of the tensor form factors [4–8]. On the other hand, extensions beyond the SM (BSM) with supersymmetry shed new light on the role of the tensor operator in describing various weak decay processes of the kaon [9–13] (see also recent reviews [14, 15] and references therein). These tensor operators arising from the BSM extensions reveal new physics originating at the TeV scale, which may be checked due to recent experimental progress in the near future. In the meanwhile, lattice quantum chromodynamics (LQCD) can also test the reliability of these operators. Very recently, Baum et al. [16] computed the matrix elements of the electromagnetic operator $\bar{\psi} \sigma_{\mu\nu} \psi F^{\mu\nu}$ [9] between the pion and the kaon within LQCD, which may be related to the CP-violating part of the $K \to \pi^{+}\pi^{-}$ semileptonic decays.

The tensor operator has another prominent place on the transversity of hadrons [17–21]. While the transversity of hadrons provides us with essential information on the quark spin structure of hadrons, it is very difficult to be measured experimentally owing to its chiral-odd nature and the absence of its direct probe. However, using semi-inclusive deep inelastic scattering processes, Anselmino et al. were able to extract the transverse parton distribution functions of the nucleon and the corresponding tensor charges [22–25]. While the transversity of the nucleon was extensively studied, the transversities of the $\pi$ and $K$ mesons received little attention again because of experimental difficulties to measure them. In the meanwhile, it was found that the tensor form factors of hadrons can be understood as generalized form factors that are defined as the Mellin moments of generalized parton distributions (GPDs) (see reviews [20, 28] for details). Moreover, the tensor form factors unveil the transverse quark spin structure inside a hadron in the transverse plane [29] with the proper probabilistic interpretation of the transverse quark densities [30, 31]. Recently, QCDSF/UKQCD Collaborations announced the first results for the pion transversity on lattice [32]. They also presented the probability density of the polarized quarks inside the pion, combining the electromagnetic form factor of the pion [33] with its tensor form factor. It was demonstrated in Ref. [32] that when the quarks are transversely polarized, their spatial distribution in the transverse plane is strongly distorted. In addition, the $K \to \pi$ transitions

*E-mail: hdson@inha.edu
†E-mail: sinam@pknu.ac.kr
‡E-mail: hchkim@inha.ac.kr
can be also investigated from a different point of view. Exclusive or semi-exclusive weak processes may provide information on the $K \rightarrow \pi$ transitions via the weak GPDs. In fact, the weak GPDs of baryons have been already examined in Refs. [34–36]. It is thus worthwhile to study the GPDs in view of the $K \rightarrow \pi$ transitions. In Ref. [37], two of the authors have investigated the transition vector form factors of the $K_{13}$ decay, based on the low-energy effective chiral action from the instanton vacuum. However, Ref. [37] concentrated mainly on the $K_{13}$ decay. In this work, we extend the previous investigation by computing the $K \rightarrow \pi$ transition vector and tensor form factors also in the space-like region. Once we have these form factors, we can immediately study the transverse quark spin densities of the $K \rightarrow \pi$ transition.

In the present work, we want to utilize the nonlocal chiral quark model (NLχQM) from the instanton vacuum to compute the $K \rightarrow \pi$ transition vector and tensor form factors, aiming at examining the transverse quark spin densities in the course of the $K \rightarrow \pi$ transition. The NLχQM from the instanton was first derived by Diakonov and Petrov [38, 39] in the chiral limit and was extended beyond the chiral limit [40, 42]. Since the instanton vacuum realizes the spontaneous chiral symmetry breaking (SχSB) naturally via quark zero modes, the NLχQM from the instanton vacuum provides a good framework to study the vector and tensor form factors of the $K \rightarrow \pi$ transition. In fact, the model has been proven to be successful in reproducing experimental data or in comparison with the results of LQCD for the $K\pi$ transition. The NLχQM is characterized by the two phenomenological parameters, i.e., the average instanton size $(\bar{\rho})$ and the average inter-instanton distance $(\bar{R})$. An essential advantage of this approach lies in the fact that the normalization point is naturally given by the average size of instantons and is approximately equal to $\rho^{-1} \approx 0.6 \text{GeV}$. This fact is essential, in particular, when one calculates the matrix elements of the tensor current, since they are scale-dependent. To compare the results of the tensor form factors from any model, the normalization scale should be well defined such that the results can be compared to those from other models or from LQCD. The values of the $\bar{\rho}$ and $\bar{R}$ were estimated many years ago phenomenologically [55] as well as theoretically [58, 59]. Once the above-mentioned two parameters $\bar{\rho}$ and $\bar{R}$ are determined, the NLχQM from the instanton vacuum does not have any adjustable parameter. Furthermore, this approach was supported by several LQCD studies of the QCD vacuum [57, 59] and the momentum dependence of the dynamical quark mass from the instanton vacuum [59] is in a remarkable agreement with those from LQCD [60, 61].

The present work is organized as follows: In Section II, we introduce the $K \rightarrow \pi$ transition GPDs based on which the generalized form factors are defined. We also present the definition of the transverse quark spin densities of the $K \rightarrow \pi$ transition. In Section III, we show how to compute the transition vector and tensor form factors within the framework of the NLχQM. In Section IV, we present the results and discuss them. The final section is devoted to the summary of the present work and discuss future perspectives related to the transition GPDs and generalized form factors.

II. GENERALIZED FORM FACTORS AND QUARK SPIN DENSITY OF THE $K \rightarrow \pi$ TRANSITION

The transition vector (tensor) GPDs $H_K^{\pi}(x, \xi, t) (E_T^{K\pi}(x, \xi, t))$ for the $K \rightarrow \pi$ transition are defined respectively in terms of the matrix element of the vector (tensor) nonlocal operators between the $K^0$ and the $\pi^+$ states:

$$2P^+ H_K^{\pi}(x, \xi, t) = \int \frac{d\lambda}{2\pi} e^{i\lambda(P \cdot n)} \langle \pi^- (p') | \bar{s}(-\lambda n/2) \gamma^+ \gamma^3 [-\lambda n/2, \lambda n/2] u(\lambda n/2) | K^0(p) \rangle,$$

$$\frac{P^+ \Delta j - \Delta j P^+}{m_K} E_T^{K\pi}(x, \xi, t) = \int \frac{d\lambda}{2\pi} e^{i\lambda(P \cdot n)} \langle \pi^- (p') | \bar{s}(-\lambda n/2) i\sigma^j \gamma^3 [-\lambda n/2, \lambda n/2] u(\lambda n/2) | K^0(p) \rangle,$$

where $n$ denotes the light-like auxiliary vector. The momenta $p$ and $p'$ correspond to those of the kaon and the pion, respectively. The $P$ represents the average momentum of the kaon and pion momenta $P^\mu = (p^\mu + p'^\mu)/2$, whereas $\Delta$ corresponds to the momentum transfer $\Delta^\mu = p' - p$, the square of which is expressed as $t = \Delta^2$. $P^+ = (P^0 + P^3)/\sqrt{2}$ and $\Delta^j$ are expressed in the light-cone basis. The index $j$ labels the transverse component, i.e. $j = 1$ or $j = 2$. The kaon mass in the denominator is introduced to define the tensor transition GPD $E_T^{K\pi}(x, \xi, t)$ to be dimensionless. The gauge connection $[-\lambda n/2, \lambda n/2] = P \exp[i \int_{-\lambda n/2}^{\lambda n/2} dx A^+(x^- n^-)]$ can be suppressed in the light cone gauge. The
generalized transition vector form factors $A_{n+1,i+1}^{K\pi}$ and $C_{n+1,i+1}^{K\pi}$ are defined by the following matrix elements:

$$\langle \pi^- | O^{\mu_1\cdots\mu_n}_V | K^0(p) \rangle = S \left[ 2P^\mu P^{\mu_1} \cdots P^{\mu_n} A_{n+1,0}^{K\pi}(t) + 2 \sum_{i=1, \text{odd}}^{n} \Delta^\mu \Delta^\mu_1 \cdots \Delta^\mu_i P^\mu_{i+1} \cdots P^n A_{n+1,i+1}^{K\pi}(t) \right. $$

$$+ 2 \sum_{i=0, \text{even}}^{n} \Delta^\mu \Delta^\mu_1 \cdots \Delta^\mu_i P^\mu_{i+1} \cdots P^n C_{n+1,i+1}^{K\pi}(t) \bigg] ,$$

(2)

where the generalized vector transition operator is expressed as

$$O_V^{\mu_1\cdots\mu_n} = S \left[ \hat{s}(\gamma^\mu \hat{K}_1 \tau) \cdots (\hat{A} \hat{D} \tau_{n\mu}) \right] u .$$

(3)

The operation $S$ means the symmetrization in $(\mu, \cdots, \mu_n)$ with the trace terms subtracted in all indices. $D_\mu$ indicates the hermitized covariant derivative $i \hat{D}_\mu \equiv (\hat{D}_\mu - i \hat{D} \mu) / 2$ in QCD. Note that the $A_{1,0}$ and $C_{1,1}$ are related to the form factors $f_{1+} = A_{1,0}^{K\pi}$ and $f_{1-} = 2C_{1,1}^{K\pi}$ of the $K_{3\ell}$ decay, which are defined as

$$\langle \pi^- | s\gamma^\mu u | K^0(p) \rangle = 2P^\mu f_{1+}(t) + \Delta^\mu f_{1-}(t),$$

(4)

where $s$ and $u$ denote the strange and up quark fields. The generalized transition tensor form factors $B_{T,n,i}^{K\pi}$ can be also defined by the following matrix element

$$\langle \pi^- | O_T^{\mu_1\cdots\mu_{n-1}} | K^0(p) \rangle = AS \left[ \frac{P^\mu \Delta^\mu - \Delta^\mu P^\mu}{m_K} \sum_{i=0}^{n-1} \Delta^\mu_1 \cdots \Delta^\mu_i P^\mu_{i+1} \cdots P^n B_{T,n,i}^{K\pi}(t) \right] ,$$

(5)

where the generalized tensor transition operator is expressed as

$$O_T^{\mu_1\cdots\mu_{n-1}} = AS \left[ \hat{s}\sigma^{\mu\nu} \hat{D}_\mu \cdots (\hat{A} \hat{D} \tau_{n-1\mu}) \right] u .$$

(6)

The operations $A$ and $S$ mean the anti-symmetrization in $(\mu, \nu)$ and symmetrization in $(\nu, \cdots, \mu_{n-1})$ with the trace terms subtracted in all the indices. The antisymmetric tensor is defined as $\sigma_{\mu\nu} = i (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu) / 2$. Note that there are odd and even terms of $\xi$ in Eqs. (3) and (6), respectively, due to the fact that the matrix elements for the $K \to \pi$ transitions do not vanish under the time-reversal transformation. The leading-order transition vector and tensor form factors are then expressed as

$$\langle \pi^- | s\gamma^\mu u | K^0(p) \rangle = 2P^\mu A_{1,0}^{K\pi}(t) + 2\Delta^\mu C_{1,1}^{K\pi}(t),$$

(7)

$$\langle \pi^- | s\sigma^{\mu\nu} u | K^0(p) \rangle = \left( P^\mu \Delta^\mu_P - \Delta^\mu_P P^\mu \right) B_{T,1,0}^{K\pi}(t),$$

(8)

in which we are mainly interested. Combining Eqs. (1, 2) with Eq. (5), we find the formula for $n$th order Mellin moments of the vector and tensor transition GPDs:

$$\int dx x^n H^{K\pi}(x, \xi, t) = A_{n+1,0}^{K\pi}(t) + \sum_{i=1, \text{odd}}^{n} (-2\xi)^{i+1} A_{n+1,i+1}^{K\pi}(t) + \sum_{i=1, \text{odd}}^{n+1} (-2\xi)^i C_{n+1,i}^{K\pi}(t),$$

$$\int dx x^n E_T^{K\pi}(x, \xi, t) = \sum_{i=0}^{n} (-2\xi)^i B_{T,n+1,i}^{K\pi}(t).$$

(9)

so that the transition vector and tensor form factors can be identified respectively as the first moments of the vector and tensor transition GPDs

$$\int dx H^{K\pi}(x, \xi, t) = A_{1,0}^{K\pi} - 2\xi C_{1,1}^{K\pi}, \quad \int dx E_T^{K\pi}(x, \xi, t) = B_{T,1,0}^{K\pi}(t),$$

(10)

where the skewness parameter is defined as $\xi = -\Delta^+/(2P^+)$. Finally the spin distribution of the transversely polarized quark in the course of the $K \to \pi$ transition is written as follows

$$\rho_1^{K\pi}(b, s_\perp) = \frac{1}{2} \left[ A_{1,0}^{K\pi}(b^2) - \frac{s_\perp^2 e^{ij} b^i b^j}{m_K} \frac{\partial B_{T,1,0}^{K\pi}(b^2)}{\partial b^2} \right] ,$$

(11)
with the Fourier transformations of the transition vector and tensor form factors
\[ F_{1,0}^{K \pi}(b_\perp) = \frac{1}{\sqrt{2\pi}} \int d^2\Delta^{-ib_\perp} \Delta F_{1,0}^{K \pi}(t) = \frac{1}{\sqrt{2\pi}} \int_0^\infty Q dQ J_0(bQ) F_{1,0}^{K \pi}(Q^2). \tag{12} \]

The form factors \( F_{1,0}^{K \pi}(t) \) and densities \( F_{1,0}^{K \pi}(b_\perp) \) stand generically either for the transition vector ones or the tensor ones. The \( s_\perp = (s_x, s_y) \) stands for the fixed transverse spin of the quark. We choose the \( z \) direction for the quark longitudinal momentum for simplicity and select the \( x \) axis in the transverse plane for the quantization of the spin of the quark in the course of the \( K \to \pi \) transition in the transverse plane, that is, \( s_\perp = (\pm 1, 0) \).

### III. NONLOCAL CHIRAL QUARK MODEL FROM THE INSTANTON VACUUM

The NL\( \chi \)QM can be derived from the instanton liquid model for the QCD vacuum. Starting from the QCD partition function in the one-loop approximation, in which the classical background field (instantons) and one can express the following partition function \[ 33, 50, 61 \]
\[ Z_{\text{reg}}^{1 \text{-loop}} = \frac{1}{N_+!N_-!} \int d\xi I d_0(\rho_I) \exp(-U_{\text{int}}) \text{Det}[m, M_{\text{cut}}], \tag{13} \]
where \( N_+ \) and \( N_- \) denote the number of instantons and anti-instantons, respectively. \( \xi_I \) designates generically the collective coordinates including the center positions of the instantons \( z_I \), their sizes \( \rho_I \), and orientations of the instantons expressed in terms of SU\( (N_c) \) matrices in the adjoint representation. \( d_0(\rho_I) \) represents the one-instanton weight, which was originally derived by ’t Hooft in SU(2) \[ 62 \] and by Bernard in SU(3) and SU\( (N_c) \) \[ 63 \] in the one-loop approximation:
\[ d_0(\rho_I) = \frac{C_{N_c}}{\rho_I^2} \beta(M_{\text{cut}})^{2N_c} \exp[-\beta(\rho_I)], \tag{14} \]
where \( \beta(\rho_I) \) is the inverse of the strong coupling constant in the one-loop approximation
\[ \beta(\rho_I) = \frac{8\pi^2}{g^2(\rho_I)} = b \log \left( \frac{1}{\Lambda_{\text{PV}}(\rho_I)} \right) \tag{15} \]
with \( b = 11N_c/3 - 2N_f/3 \). The QCD scale parameter \( \Lambda_{\text{PV}} \) here is given in the Pauli-Villars regularization and is related to that in the \( \overline{\text{MS}} \) scheme \( \Lambda_{\text{PV}} = 1.09\Lambda_{\overline{\text{MS}}} \). The coefficient \( C_{N_c} \) depends on renormalization schemes and is given in the Pauli-Villars scheme as
\[ C_{N_c} = \frac{4.66\exp(-1.68N_c)}{\pi^2(N_c - 1)!/(N_c - 2)!}. \tag{16} \]

The effective instanton size distribution which is related to \( d_0(\rho_I) \) is reduced to a \( \delta \)-function in the large \( N_c \) limit because of the presence of \( b \) in Eq.\((15)\), which picks up the average size of the instanton \( \bar{\rho} \). The instanton interaction potential \( U_{\text{int}} \) was derived and studied in Ref. \[ 50 \]. The regularized and normalized fermionic determinant \( \text{Det} \) depends on the Pauli-Villars cut-off mass \( M_{\text{cut}} \).

Since we aim at deriving the \( K \to \pi \) transition form factors in the present work, we need to include the external sources for the vector and tensor fields in the fermionic determinant \( \text{Det} \), which is given as a functional of \( V_\mu \) and \( T_{\mu\nu} \) \[ 64 \]:
\[ \overline{\text{Det}} := \text{Det}(i\bar{\psi} + gA + \gamma^\nu \sigma_{\mu\nu}T_{\mu\nu} + i\hat{m}), \tag{17} \]
where \( A_\mu \) is the gluon field with the gauge coupling constant \( g \) and \( \hat{m} = \text{diag}(m_u, m_d, m_s) \) denotes the current quark mass matrix that shows explicit chiral and flavor SU(3) symmetry breaking, of which their numerical values are given as \( m_u = m_d = 5 \text{MeV} \) and \( m_s = 150 \text{MeV} \). The fermionic determinant \( \text{Det} \) can be divided into two parts corresponding to the low and high Dirac eigen-frequencies with respect to an arbitrary splitting parameter \( M_1 \):
\[ \text{Det}(m, M_{\text{cut}}) := \text{Det}_{\text{low}}(m, M_1) \text{Det}_{\text{high}}(M_1, M_{\text{cut}}). \]

The high-frequency part \( \text{Det}_{\text{high}} \) was shown to contribute to the statistical weights of individual instantons. That is, it influences mainly the renormalization of the coupling constant in a sense of the renormalization group equation. On the other hand, The low-frequency part \( \text{Det}_{\text{low}} \) can only be
treated approximately, the would-be zero modes being only taken into account. It was proven that the \(\tilde{\text{Det}}_{\text{low}}\) depends weakly on the scale \(M_1\) in a broad range of \(M_1\), so that the the matching between \(\text{Det}_{\text{high}}\) and \(\text{Det}_{\text{low}}\) turns out to be smooth \cite{38}. The natural choice of the parameter \(M_1\) can be taken to be roughly \(M_1 \sim 1/\bar{\rho}\), where \(\bar{\rho}\) is the average size of instantons \(1/\bar{\rho} \simeq 600\,\text{MeV}\). Thus, as mentioned already, \(1/\bar{\rho}\) can be considered as the natural scale of the present model. Of course the choice of \(\bar{\rho} \simeq 600\,\text{MeV}\) is not strict but has some ambiguity. We will discuss this ambiguity in the context of the tensor form factor later.

The low-frequency part \(\text{Det}_{\text{low}}\) was derived in Refs. \cite{38,64,65} and its explicit form is written as

\[
\text{Det}_{\text{low}} = (\det(i\partial + \mathcal{Y} + \sigma \cdot T + i\dot{\mathcal{m}}))^{-1} \int \prod_f D\psi_f D\bar{\psi}_f \times \exp \left( \int d^4x \psi_f^\dagger (i\partial + \mathcal{Y} + \sigma \cdot T + i m_f) \psi_f \right) \prod_f \left\{ \left( \prod_{N_+} V_{+,f}[\psi_f^\dagger, \psi_f] + N_- \prod_{V_{-,f}[\psi_f^\dagger, \psi_f] \right\}
\]

where

\[
\tilde{V}_{\pm,f}[\psi_f^\dagger, \psi_f] = \int d^4x \left( \psi_f^\dagger(x) L_f(x, z) i\partial \Phi_{\mp,0}(x; \xi_{\pm}) \right) \int d^4y \left( \Phi_{\mp,0}^\dagger(y; \xi_{\pm})(i\partial L_f^\dagger(y, z) \psi_f(y) \right).
\]

The \(\psi_f\) denotes the quark field, given flavor \(f\). The \(m_f\) is the current quark mass corresponding to \(\psi_f\). The \(N_+\) and \(N_-\) stand for the number of instantons and anti-instantons. The gauge connection \(L_f\) is defined as

\[
L_f(x, z) := \text{P} \exp \left( \int_z^y \text{d}z_\mu V_\mu(\zeta) \right),
\]

which is essential to make the nonlocal effective action gauge-invariant and should be attached to each fermionic line. The \(\Phi_{\mp,0}(x; \xi_{\pm})\) represents the zero-mode solution of the Dirac equation in the instanton \((A_{\mu,\pm})\) and anti-instanton \((A_{\mu,-})\) fields \((i\partial + A_{\pm})\Phi_{\mp,0}(x; \xi_{\pm}) = \lambda_n \Phi_{\mp,0}(x; \xi_{\pm})\). Having exponentiated and bosonized the fermionic interactions \(V_{\pm,f}\), and having averaged the low-frequency part of the fermionic determinant \(\text{Det}_{\text{low}}\) over collective coordinates \(\xi_{\pm}\), we arrive at the effective chiral partition function of which the detailed derivation can be found in Refs. \cite{38,64,65,70}.

Since our main concern is to compute the \(K \rightarrow \pi\) tensor generalized form factors in the present work, we set the stage for them by using the relevant effective chiral action of the NL\(\chi\)QM with the external tensor source field \(T_{\mu\nu}\) derived from Eq. \(18\):

\[
\mathcal{S}_{\text{eff}}[T] = -\text{Sp} \left[ i\partial + i\dot{\mathcal{m}} + i\sqrt{\mathcal{M}} U^{\gamma_5} \sqrt{\mathcal{M}} + T_{\mu\nu} \sigma_{\mu\nu} \right].
\]

Here, the functional trace \(\text{Sp}\) runs over the space-time, color, flavor, and spin spaces. Note that isospin symmetry is assumed. The nonlinear pseudo-Nambu-Goldstone boson field is written as

\[
U^{\gamma_5} = \exp \left( \frac{i\gamma_5}{F_\psi} \lambda \cdot \phi \right), \quad \phi = (\pi, K, \eta),
\]

where the pion and kaon weak-decay constants are chosen to be \((F_\pi, F_K) = (93, 113)\,\text{MeV}\) empirically. The pseudoscalar meson fields are defined by

\[
\lambda \cdot \phi = \sqrt{2} \begin{pmatrix} \frac{1}{\sqrt{2}} \pi^0 + \frac{1}{\sqrt{6}} \eta & \frac{1}{\sqrt{2}} \pi^+ + \frac{1}{\sqrt{6}} \eta & K^+ \\ \frac{1}{\sqrt{2}} \pi^- - \frac{1}{\sqrt{6}} \eta & K^- & K^0 \\ \frac{1}{\sqrt{2}} \pi^0 - \frac{1}{\sqrt{6}} \eta & K^0 & -\frac{2}{\sqrt{6}} \eta \end{pmatrix}.
\]

For the numerical calculations, we use the mass values for the pion and kaon as \((m_\pi, m_K) = (140, 495)\,\text{MeV}\) throughout the present work, taking the flavor \(\text{SU}(3)\) symmetry breaking into account. The momentum-dependent dynamical quark mass, which is induced from the nontrivial quark-instanton interactions and indicates \(\text{SB}\,\chi\,\text{S}\), is given by

\[
M_f(k) = M_0 F^2(k) \left[ \sqrt{1 + \frac{m_f^2}{d^2}} - \frac{m_f}{d} \right],
\]

where \(M_0\) is the constituent quark mass at zero quark virtuality, and is determined by the saddle-point equation, resulting in about 350\,\text{MeV} \cite{38,39}. The form factor \(F(k)\) arises from the Fourier transform of the quark zero-mode solution for the Dirac equation with the instanton and has the following form:

\[
F(k) = 2\pi \left[ I_0(\tau) K_1(\tau) - I_1(\tau) K_0(\tau) - \frac{1}{\tau} I_1(\tau) K_1(\tau) \right],
\]

where \(I_n(\tau)\) is the modified Bessel function of the first and second kind, and \(K_n(\tau)\) is the modified Bessel function of the second kind.
where \( \tau \equiv |k_0| \). In this work, however, we use the following parametrization for numerical convenience:

\[
F(k) = \frac{2\mu^2}{2\mu^2 + k^2},
\]

where \( \mu = 1/\bar{\rho} = 600 \text{ MeV} \) can be regarded as the renormalization scale of the model. In order to take into account the explicit flavor SU(3) symmetry breaking effects properly, we modify the dynamical quark mass with the \( m_f \)-dependent term given in the bracket in the right-hand side of Eq. (24) [69, 70] in such a way that the instanton-number density \( N/V \) is independent of the current-quark mass, where \( N \) and \( V \) denote the number of instantons and the four-dimensional volume, respectively. Polyvititsa took into account the sum of all planar diagrams in expanding the quark propagator in the instanton background in the large \( N_c \) limit [69]. Taking the limit of \( N/(VN_c) \to 0 \) leads to the term in the bracket of Eq. (24). The parameter \( d \) is chosen to be 0.193 GeV. It is worth noting that this modification gives a correct hierarchy of the strengths for the chiral condensates: \( \langle \bar{u}u \rangle \approx \langle \bar{d}d \rangle > \langle \bar{s}s \rangle \) [71].

\[
\text{FIG. 1: Relevant Feynman diagrams for the } K \to \pi \text{ transition tensor form factor. The solid, dash, and wavy lines denote the quark, the pseudoscalar meson, and the tensor operator, respectively. The four momenta of the quarks are defined and explicitly given in Eq. (28).}
\]

\[
\langle \pi^{-}(p')|\bar{s}\sigma_{\mu
u}u|K^{0}(p)\rangle = -\frac{8N_{c}}{F_{\pi}F_{K}} \int \frac{d^{4}l}{(2\pi)^{4}} \left[ \frac{\sqrt{M_{u}^{2}M_{d}M_{s}}}{G_{u}G_{d}G_{s}} \epsilon^{ijk}k_{i\mu}k_{j\nu}M_{k\ell} - \frac{\sqrt{M_{u}^{2}M_{s}}}{2G_{u}G_{s}} (k_{u\mu}k_{d\nu} - k_{s\nu}k_{u\mu}) \right],
\]

where we introduced \( M_{f}(k^2_{f}) = m_{f} + M_{f}(k^2_{f}) \) and \( G_{f} = k^2_{f} + M_{f}^{2} \) with \( f = (u, d, s) \). The first and second terms inside the squared bracket in the right-hand side of Eq. (27) correspond to the diagrams (a) and (b) of Fig. 1 respectively. The four momenta of the kaon at rest and the pion are defined in the center-of-mass frame as

\[
k_{u} = l + \frac{p}{2} + \frac{\Delta}{2}, \quad k_{d} = l - \frac{p}{2} - \frac{\Delta}{2}, \quad k_{s} = l + \frac{p}{2} - \frac{\Delta}{2}.
\]

The four-momenta of the kaon at rest and the pion are defined in the center-of-mass frame as

\[
p = (0, 0, 0, iE_{K}), \quad p' = \left( -\sqrt{\left( \frac{t + m_{K}^{2} + m_{\pi}^{2}}{2m_{K}} \right) - m_{\pi}^{2}}, 0, 0, iE_{\pi} \right).
\]

In order to compare our numerical results of the transition tensor form factor with those of other works, it is crucial to know the renormalization scale, since the tensor current is not the conserved one. Results at two different scales are related by the following the next-to-leading (NLO) order evolution equation [21, 67, 68]:

\[
B^{K_{\pi}}_{1,0}(\mu^2) = \left( \frac{\alpha_{s}(\mu_{f}^{2})}{\alpha_{s}(\mu_{T}^{2})} \right)^{4/27} \left[ 1 - \frac{337}{486\pi} (\alpha_{s}(\mu_{f}^{2}) - \alpha_{s}(\mu_{T}^{2})) \right] B^{K_{\pi}}_{1,0}(\mu_{1})
\]

with the NLO strong coupling constant

\[
\alpha_{s}^{NLO}(\mu^2) = \frac{4\pi}{9\ln(\mu^2/A_{QCD}^{2})} \left[ 1 - \frac{64}{81} \ln(\mu^2/A_{QCD}^{2}) \right].
\]
where \( \mu_i \) denotes the initial renormalization scale, and we take \( N_f = 3 \) in the present work. Note that the scale dependence of the tensor form factor given in Eq. (30) is rather mild. As will be shown explicitly in the next Section, the tensor form factor is changed approximately by 10% when one scales down from \( \mu = 2 \text{ GeV} \) to \( \mu = 0.6 \text{ GeV} \). It indicates that even though we choose some higher or lower value of the scale of the model, the result is not much changed. Thus, the ambiguity in choosing the scale of the present model will have only a tiny effect on the result of the tensor form factor when one scales it to another normalization point.

### IV. NUMERICAL RESULTS AND DISCUSSIONS

In this Section, we present the numerical results and discuss them. We start with the \( K \to \pi \) transition vector form factors. While the kinematically accessible region for the \( K \to \pi \) semileptonic form factors \( f_{l+} \) and \( f_{l-} \) is restricted to \( m_l^2 \leq t \leq (m_K - m_\pi)^2 \), where \( m_l \) is the lepton mass involved in the decay, the generalized transition vector form factors \( A^{K\pi}_{1,0} \) and \( C^{K\pi}_{1,1} \) related to the transition GPDs can be also defined in the spacelike region, since they can be extracted in principle from exclusive weak processes. In Fig. 2, we show the results of the \( K \to \pi \) transition vector form factors \( A^{K\pi}_{1,0} \) and \( C^{K\pi}_{1,1} \) in the space-like region. Both form factors fall off as \(|t|\) increases. The magnitude of \( A^{K\pi}_{1,0} \) turns out to be much larger than that of \( C^{K\pi}_{1,1} \). This can be understood from the results for the \( K \to \pi \) semileptonic decay \[37\] in which the magnitude of \( f_{l+}(m_l^2) \) is approximately eight times larger than that of \( f_{l-}(m_l^2) \). It is the general tendency also known from other approaches.

In the left panel of Fig. 3, we depict the transition tensor form factors \( B^{K\pi}_{1,0} \) as a function of \(-t\) at two different scales. Since it depends on the renormalization scale, we examine the scale dependence of the transition tensor form factor, based on Eq. (33). The solid curve draws the present result, which is given at the renormalization scale \( \mu = 0.6 \text{ GeV} \) of the NL\( \chi \)QM, whereas the dashed one represents the form factor at \( \mu = 2.0 \text{ GeV} \), which corresponds to the scale of LQCD \[19\]. We observe that the transition tensor form factor depends mildly on \( \mu \). The value of the form factor at \( t = 0 \) is given respectively as \( B^{K\pi}_{1,0}^{\mu=0.6 \text{ GeV}}(0) = 0.792 \) at \( \mu = 0.6 \text{ GeV} \) and \( B^{K\pi}_{1,0}^{\mu=2 \text{ GeV}}(0) = 0.709 \) at \( \mu = 2 \text{ GeV} \). That is, the magnitude of the form factor is approximately reduced by 10%, when \( \mu \) is scaled up to \( \mu = 2 \text{ GeV} \) from \( \mu = 0.6 \text{ GeV} \). Since the scale factor is an overall one, the \( t \)-dependence of the form factor is not affected by the scaling. The right panel of Fig. 3 draws the transition tensor form factor normalized by its value at \( t = 0 \) in comparison with that of LQCD \[19\] at \( \mu = 2 \text{ GeV} \). Note that Ref. \[19\] computed the transition tensor form factor \( f_T^{K\pi}(t) \) defined as

\[
\langle \pi^0(p')|\bar{s}d_{\mu\nu}d|K^0(p)\rangle = (p'_{\mu}p_{\nu} - p_{\nu}p'_{\mu}) \frac{\sqrt{2}f_T^{K\pi}(t)}{m_K + m_\pi},
\]

which can be written in terms of \( B^{K\pi}_{T,1,0} \):

\[
f_T^{K\pi}(t) = \frac{m_K + m_\pi}{2m_K} B^{K\pi}_{T,1,0}(t).
\]
At \( t = 0 \), the value of the form factor \( f_{T}^{K\pi}(0) \) is obtained to be \( f_{T}^{K\pi}(0) = 0.45 \) at \( \mu = 2 \) GeV, while the lattice result becomes \( f_{T}^{K\pi}(0) = 0.417 \pm 0.014(\text{stat}) \pm 0.05(\text{syst}) \) at the physical pion mass after the extrapolation from \( m_{\pi} = 270 \) MeV. Hence, the present result is in good agreement with the lattice one. The present result of the form factor falls becomes \( f \) with that from lattice QCD at \( \mu = 2 \) GeV.

Once we have derived the transition vector and tensor form factors, we can proceed to the calculation of the transverse quark spin density in the course of the \( K \to \pi \) transition, using Eq. (11). In doing so, it is more convenient to parameterize the form factors in the \( p \)-pole type, which is usually employed in the lattice calculation [32]:

\[
\mathcal{F}_{1,0}^{K\pi}(t) = \mathcal{F}_{1,0}^{K\pi}(0) \left(1 + \frac{t}{pM_{p}^{2}}\right)^{-p},
\]

so that the Fourier transform can be easily carried out. Having fitted the results of the form factors shown in Fig. 2 and the left panel of Fig. 3, we are able to determine the parameters as \( p = 0.850 \) and \( M_{p} = 1.312 \) GeV for \( A_{1,0}^{K\pi} \) and \( p = 2.172 \) and \( M_{p} = 0.776 \) GeV for \( B_{1,0}^{K\pi} \), respectively, at \( \mu = 0.6 \) GeV. Using these values, we can easily derive the quark spin transverse density in the course of the \( K \to \pi \) transition, which is defined in Eq. (11).

When quarks involved in the \( K \to \pi \) transition are not polarized in the transverse plane, the transverse quark spin density is defined only in terms of \( A_{1,0}^{K\pi} \):

\[
\rho_{T}^{K\pi}(b) = A_{1,0}^{K\pi} \rho_{T}^{K\pi}(b^2)/2,
\]

which is just the same as the transverse charge density apart from the factor 1/2. The left panel of Fig. 3 draws this transverse spin density of the unpolarized quark in the \( K \to \pi \) transition. The result shows that the transverse spin of the quarks are uniformly distributed. Note that the density is singular at \( b = 0 \), which is very similar to the transverse quark spin densities of the pion and the kaon [52, 53]. On the other hand, if one of the quarks is polarized along the \( b_{z} \) direction, that is, \( s_{\perp} = (\pm 1, 0) \), then the transverse quark spin density in the \( K \to \pi \) transition gets shifted to the positive \( b_{y} \) direction, as shown in the right panel of Fig. 3. It is of great use to compute the average shift of the density so that we may see how much the transverse quark spin density is distorted by the quark polarization. One can define the average shift of the density to the \( b_{y} \) direction as follows:

\[
\langle b_{y} \rangle_{K\pi} = \frac{\int d^{2}b_{y} b_{y} \rho_{T}^{K\pi}(b, s_{\perp})}{\int d^{2}b \rho_{T}^{K\pi}(b, s_{\perp})} = \frac{1}{2m_{K}} \frac{B_{1,0}^{K\pi}(0)}{A_{1,0}^{K\pi}(0)}.
\]

We obtain the numerical value \( \langle b_{y} \rangle_{K\pi} = 0.169 \) fm, which can be compared with those of the pion and the kaon. The average shift of the transverse quark spin density in the pion was obtained to be \( \langle b_{y} \rangle_{\pi} = 0.152 \) fm that was almost the same as the lattice calculation \( \langle b_{y} \rangle_{\pi} = 0.151 \pm 0.024 \) fm [52], whereas those in the kaon turned out to be \( \langle b_{y} \rangle_{K,u} = 0.168 \) fm and \( \langle b_{y} \rangle_{K,s} = 0.166 \) fm for the up and down quark components in Model I in Ref. [53]. Thus, we
FIG. 4: (Color online) Unpolarized (left) and polarized (right) transverse quark-spin densities (TQSD) for $K^0 \to \pi^-$ in the transverse impact-parameter plane ($b_x-b_y$), being calculated at $\mu = 0.6$ GeV. We take the quark spin polarization as $s_x = +1$.

FIG. 5: (Color online) The profile of the polarized transverse quark spin densities at $\mu = 0.6$ GeV (solid curve) and $\mu = 2$ GeV (dashed curve), with $b_x = 0$ fixed.

find that the transverse quark spin density of the $K \to \pi$ transition shows the largest shift in comparison with those in the pion and the kaon.

Figure 5 illustrates the profile of the polarized transverse quark spin density of the $K \to \pi$ transition at two different scales. It shows clearly the distortion of the density in the positive $b_y$ direction. The scaling effect turns out to be negligible for the transverse quark spin densities.

V. SUMMARY AND CONCLUSION

In the present work, we have studied the transition vector and tensor form factors for the $K \to \pi$ transition within the framework of the nonlocal chiral-quark model from the instanton vacuum. We presented the numerical results for the form factors and compared in particular the tensor form factor with that of lattice QCD, considering the renormalization group evolution. We also presented the results for the transverse quark spin density in the course of the $K \to \pi$ transition without and with quark polarization in the transverse direction. We summarize below the important theoretical observations in the this work:
• The vector and tensor form factors smoothly decrease as \(-t\) increases. The value of the tensor form factor at \(t = 0\) becomes \(B^K_\pi(0) = 0.792\) at the renormalization scale \(\mu = 0.6\) GeV and \(B^K_\pi(0) = 0.709\) at \(\mu = 2\) GeV, while its overall \(t\) dependence does not change much.

• The transition vector and tensor form factors can be parameterized by a \(p\)-pole type one, which is a function of \(M\) and \(p\), resulting in \((p, M) \approx (2.172, 0.776\) GeV).

• The present theoretical result \(f^{K\pi}_{\text{NLQCD}}(0) = 0.45\) for the transition tensor form factor at \(t = 0\) is in good agreement with that from lattice QCD \(f^{K\pi}_{\text{LQCD}}(0) = 0.417 \pm 0.014(\text{stat}) \pm 0.05(\text{syst})\) at \(\mu = 2\) GeV.

• The transverse quark spin density of the \(K \to \pi\) transition was also computed as a function of the impact parameter \(b\). When a quark in the course of \(K \to \pi\) transition is polarized in the \(b_e\) direction, the density becomes shifted to the positive \(b_y\) direction. The average shift of the density \(\langle b_y \rangle^{K\pi} = 0.169\) fm at \(\mu = 0.6\) GeV is larger than those of the pion and the kaon.

In the present work, we wrote explicitly the expressions for the weak transition generalized parton distributions that include all information about the \(K \to \pi\) transition. These generalized parton distributions can be studied within the same theoretical framework. The corresponding investigation is under way.

**Acknowledgments**

H.Ch.K is grateful to Atsushi Hosaka for the discussions and his hospitality during his visit to Research Center for Nuclear Physics, Osaka University, where part of the present work was carried out. He also wants to express his gratitude to Emiko Hiyama at RIKEN for the valuable discussions. This work is supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Korean government (MEST) (No. 2013S1A2A2035612 (H.D.S and H.Ch.K.)), respectively. The work of S.i.N. is supported in part by the Korea Foundation for the Advancement of Science and Creativity (KOFAC) grant funded by the Korea government (MEST) (20142169990).

[1] N. Cabibbo, Phys. Rev. Lett. 10 (1963) 531.
[2] M. Kobayashi and T. Maskawa, Prog. Theor. Phys. 49 (1973) 652.
[3] M. Antonelli et al. [Decays for the FlaviaNet Working Group on Kaon Collaboration], Eur. Phys. J. C 69 (2010) 399.
[4] S. A. Akimenko, V. I. Belousov, G. S. Bitsadze, A. M. Blick, Y. A. Budagov, I. E. Chirikov- Zorin, G. A. Chlachidze and Y. I. Davydov et al., Phys. Lett. B 259 (1991) 225.
[5] I. V. Ajinenko, S. A. Akimenko, G. I. Britvich, I. G. Britvich, K. V. Datsko, A. P. Filin, A. V. Inyakin and A. S. Konstantinov et al., Phys. Atom. Nucl. 66 (2003) 105 [Yad. Fiz. 66 (2003) 107].
[6] I. V. Ajinenko, S. A. Akimenko, G. A. Akopdzhian, K. S. Belous, I. G. Britvich, G. I. Britvich, A. P. Filin and V. N. Govorun et al., Phys. Atom. Nucl. 65 (2002) 2064 [Yad. Fiz. 65 (2002) 2125].
[7] I. V. Ajinenko, S. A. Akimenko, K. S. Belous, G. I. Britvich, I. G. Britvich, K. V. Datsko, A. P. Filin and A. V. Inyakin et al., Phys. Lett. B 574 (2003) 14.
[8] A. Lai et al. [NA48 Collaboration], Phys. Lett. B 604 (2004) 1.
[9] F. Gabbiani, E. Gabrielli, A. Masiero and L. Silvestrini, Nucl. Phys. B 477 (1996) 321.
[10] G. Martinelli, Nucl. Phys. Proc. Suppl. 73 (1999) 58.
[11] P. Herczeg, Prog. Part. Nucl. Phys. 46 (2001) 413.
[12] V. Cirigliano, J. Jenkins and D. E. Soper, Nucl. Phys. B 71 (1979) 109.
[13] J. P. Ralston and D. E. Soper, Phys. Rev. Lett. 95 (1992) 409.
[14] R. L. Jaffe and X. D. Ji, Phys. Rev. Lett. 67 (1991) 552.
[15] R. L. Jaffe and X. D. Ji, Nucl. Phys. B 375 (1992) 527.
[16] V. Barone, A. Drago and P. G. Ratcliffe, Phys. Rept. 359 (2002) 1.
[17] M. Anselmino, M. Boglione, U. D’Alesio, A. Kotzinian, F. Murgia, A. Prokudin and C. Turk, Phys. Rev. D 75 (2007) 054032.
[18] M. Anselmino, M. Boglione, U. D’Alesio, A. Kotzinian, F. Murgia, A. Prokudin and S. Melis, Nucl. Phys. Proc. Suppl. 191 (2009) 98.
[24] M. Anselmino, M. Boglione, U. D’Alesio, S. Melis, F. Murgia and A. Prokudin, Phys. Rev. D **87** (2013) 094019.
[25] A. Bacchetta, A. Courtoy and M. Radici, JHEP **1303** (2013) 119.
[26] K. Goeke, M. V. Polyakov and M. Vanderhaeghen, Prog. Part. Nucl. Phys. **47** (2001) 401.
[27] M. Diehl, Phys. Rept. **388** (2003) 41.
[28] A. V. Belitsky and A. V. Radyushkin, Phys. Rept. **418** (2005) 1.
[29] M. Diehl and P. Hagler, Eur. Phys. J. C **44** (2005) 87.
[30] M. Burkardt, Phys. Rev. **D62** (2000) 071503.
[31] M. Burkardt, Phys. Lett. **B272** (1986) 457.
[32] M. Diehl, Phys. Rept. **388** (2003) 173-208.
[33] M. Burkardt, Int. J. Mod. Phys. **A18** (2003) 173-208.
[34] D. Brommel et al. [QCDSF and UKQCD Collaborations], Phys. Rev. Lett. **101** (2008) 122001.
[35] D. Brommel et al. [QCDSF/UKQCD Collaboration], Eur. Phys. J. C **51** (2007) 335.
[36] A. Psaker, W. Melnitchouk and A. V. Radyushkin, Phys. Rev. D **75** (2007) 054001.
[37] B. Z. Kopeliovich, I. Schmidt and M. Siddikov, Phys. Rev. D **86** (2012) 113018.
[38] B. Z. Kopeliovich, I. Schmidt and M. Siddikov, Phys. Rev. D **89** (2014) 5, 053001.
[39] S. i. Nam and H. -Ch. Kim, Phys. Rev. D **75** (2007) 094011.
[40] D. Diakonov and V. Y. Petrov, Nucl. Phys. **B272** (1986) 457.
[41] D. Diakonov, Nucl. Phys. **B272** (1986) 457.
[42] M. Musakhanov, Phys. Rev. D **59** (1999) 114018.
[43] M. Musakhanov, Phys. Rev. D **59** (1999) 091502.
[44] M. Musakhanov, Phys. Rev. D **60** (1999) 114017.
[45] S. i. Nam and H.-Ch. Kim, Phys. Lett. B **700** (2011) 305.
[46] M. Musakhanov, Nucl. Phys. **A699** (2002) 340.
[47] M. Musakhanov, Nucl. Phys. **A699** (2002) 235.
[48] M. Musakhanov, Nucl. Phys. **A18** (2003) 173-208.
[49] M. Musakhanov, Nucl. Phys. **A707** (2012) 546.
[50] M. Musakhanov, Nucl. Phys. **A699** (2002) 541.
[51] M. Musakhanov, Nucl. Phys. **A700** (2011) 887.
[52] M. Musakhanov, Nucl. Phys. **B203** (1982) 93.
[53] M. Musakhanov, Nucl. Phys. **B245** (1984) 259.
[54] M. Musakhanov, Nucl. Phys. **B245** (1984) 259.
[55] M. Musakhanov, Nucl. Phys. **B245** (1984) 259.
[56] M. Musakhanov, Nucl. Phys. **B245** (1984) 259.
[57] M. Musakhanov, Nucl. Phys. **B245** (1984) 259.
[58] M. Musakhanov, Nucl. Phys. **B245** (1984) 259.
[59] M. Musakhanov, Nucl. Phys. **B245** (1984) 259.
[60] M. Musakhanov, Nucl. Phys. **B245** (1984) 259.
[61] M. Musakhanov, Nucl. Phys. **B245** (1984) 259.
[62] M. Musakhanov, Nucl. Phys. **B245** (1984) 259.