Terwilliger Algebras of Wreath Powers
of One-Class Association Schemes

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Abstract

In this paper, we study the subconstituent algebras, also called as Terwilliger algebras, of association schemes that are obtained as the wreath product of one-class association schemes $K_n = H(1, n)$ for $n \geq 2$. We find that the $d$-class association scheme $K_{n_1} \wr K_{n_2} \cdot \cdot \cdot \wr K_{n_d}$ formed by taking the wreath product of $K_{n_i}$ has the triple-regularity property. We determine the dimension of the Terwilliger algebra for the association scheme $K_{n_1} \wr K_{n_2} \cdot \cdot \cdot \wr K_{n_d}$. We give a description of the structure of the Terwilliger algebra for the wreath power $(K_n)^d$ for $n \geq 2$ by studying its irreducible modules. In particular, we show that the Terwilliger algebra of $(K_n)^d$ is isomorphic to $M_{d+1}(\mathbb{C}) \oplus M_1(\mathbb{C}) \oplus \frac{1}{2}d(d+1)$ for $n \geq 3$, and $M_{d+1}(\mathbb{C}) \oplus M_1(\mathbb{C}) \oplus \frac{1}{2}d(d-1)$ for $n = 2$.

1 Introduction

The subconstituent algebra, which is also known as the Terwilliger algebra of an association scheme was introduced by Terwilliger in 1992 as a new algebraic tool for the study of association schemes [19]. The Terwilliger algebra of a commutative association scheme is a finite dimensional, semi-simple $\mathbb{C}$-algebra, and is noncommutative in general. This algebra helps understanding the structure of the association schemes. It has been studied extensively for many classes of association schemes. For example, the Terwilliger algebra for $P$- and $Q$-polynomial association schemes has been studied in [20, 22, 21, 7]. The structure of Terwilliger algebra of group association schemes has been studied in [1] and [2]. In [15] the structure of the Terwilliger algebra of a Hamming scheme $H(d, n)$ is given as symmetric $d$-tensors of the Terwilliger algebra of $H(1, n)$ which are all isomorphic for $n > 2$. It is also shown that the Terwilliger algebra of $H(d, n)$ is decomposed as a direct sum of Terwilliger algebra of hypercubes $H(d, 2)$ in [15]. They deduce the decomposition into simple bilateral ideals using the representation of classical groups. There is a detailed study of the irreducible modules of the algebra for $H(d, 2)$ in [11], and for Doob schemes (the schemes coming as direct products of copies of $H(2, 4)$ and/or Shrikhande graphs,) in [18]. Both of these studies used elementary linear algebra and module theory.

In this paper, we study the Terwilliger algebras of association schemes which are obtained as wreath products of $H(1, n)$, also denoted $K_n$, for $n \geq 2$. We find that the $d$-class association scheme $K_{n_1} \wr K_{n_2} \cdot \cdot \cdot \wr K_{n_d}$ formed by taking the wreath product of one-class association schemes $K_{n_i}$ has the triple-regularity property in the sense of [16] and [14]. Based on this fact, we
determine the dimension of the Terwilliger algebra for the association scheme $K_{n_1} \wr K_{n_2} \cdots \wr K_{n_d}$. We then find that the wreath power $(K_n)^d = K_n \wr K_n \wr \cdots \wr K_n$, $d$ copies of $K_n$, is formally self-dual and the Terwilliger algebra is isomorphic to $M_{d+1}(\mathbb{C}) \oplus M_1(\mathbb{C})^{\oplus \frac{1}{2}d(d+1)}$ for $n \geq 3$, while $M_{d+1}(\mathbb{C}) \oplus M_1(\mathbb{C})^{\oplus \frac{1}{2}d(d-1)}$ for $n = 2$ in the notion of Wedderburn-Artin’s decomposition theorem of semisimple algebra. The case $(K_2)^d$ behaves a little differently from the general case $(K_n)^d$ for $n \geq 3$. We also study the corresponding irreducible modules for $n = 2$ in the course. We give the decomposition by following the elementary approach employed in [22, 18, 11].

The remainder of the paper is organized as follows. In Section 2, we provide the notation and terminology as well as a few basic facts on the Terwilliger algebra and wreath product of association schemes that will be used throughout. In Section 3, we discuss the structure of the wreath product of one-class association schemes and compute the dimension of its Terwilliger algebra. In Section 4, we study Terwilliger algebras of wreath powers of one-class association schemes and their irreducible modules. In Section 5, we make a concluding remark and mention a few related open problems.

2 Preliminaries

In this section, we first briefly recall the notation and some basic facts about association schemes and the Terwilliger algebra of a scheme that are needed to deduce our results. Then we recall the definition of wreath product of association schemes. For more information on the topics covered in this section, we refer the reader to [3, 5, 19, 20, 17].

2.1 Association schemes and their Terwilliger algebras

Let $X$ denote an $n$-element set, and let $M_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra of matrices whose rows and columns are indexed by $X$. Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ be a $d$-class commutative association scheme of order $n$. The $d + 1$ relations $R_i \subseteq X \times X := \{(x, y) : x, y \in X\}$ are conveniently described by their $\{0, 1\}$-adjacency matrices $A_0, A_1, \ldots, A_d$ defined by $(A_i)_{xy} = 1$ if $(x, y) \in R_i$; 0 otherwise. The intersection numbers $p_{i,j}^h$ are defined in terms of the relations for the scheme by

$$p_{i,j}^h = |\{z \in X : (x, z) \in R_i, (z, y) \in R_j\}|$$

where $(x, y)$ is a member of the relation $R_h$. The definition of an association scheme is equivalent to the following four axioms:

1. $A_0 = I$,
2. $A_0 + A_1 + \cdots + A_d = J$,
3. $A_i^t = A_{i'}$ for some $i' \in \{0, 1, \ldots, d\}$,
4. $A_i A_j = \sum_{h=0}^{d} p_{i,j}^h A_h$, for all $i, j \in \{0, 1, \ldots, d\}$

where $I = I_n$ and $J = J_n$ are the $n \times n$ identity matrix and all-ones matrix, respectively, and $A^t$ denotes the transpose of the matrix $A$. The scheme is symmetric if $A_i = A_{i'}$ for all $i$, and is commutative if $p_{i,j}^h = p_{j,i}^h$ for all $h, i, j$; and thus $A_i A_j = A_j A_i$. 

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If the scheme is commutative, the adjacency matrices generate a \((d + 1)\)-dimensional commutative subalgebra \(\mathcal{M} = \langle A_0, A_1, \ldots, A_d \rangle\) of the full matrix algebra \(M_X(\mathbb{C})\) over the field of complex numbers \(\mathbb{C}\). The algebra \(\mathcal{M}\) is known as the Bose-Mesner algebra of the scheme. The Bose-Mesner algebra for a commutative association scheme, being semi-simple, admits a second basis \(E_0, E_1, \ldots, E_d\) of primitive idempotents. Note that \(\mathcal{M}\) is also closed under the Hadamard (entrywise) multiplication “\(\circ\)” of matrices. So there are nonnegative real numbers \(q_{ij}^h\) called the Krein parameters, such that

\[
(4) \quad E_i \circ E_j = |X|^{-1} \sum_{h=0}^{d} q_{ij}^h E_h.
\]

There exist two sets of the \((d + 1)^2\) complex numbers \(p_j(i)\) and \(q_j(i)\) according to the \(d + 1\) expressions

\[
A_j = \sum_{i=0}^{d} p_j(i)E_i, \quad \text{and} \quad E_j = |X|^{-1} \sum_{i=0}^{d} q_j(i)A_i.
\]

The number \(p_j(i)\) is characterized by the relation \(A_j E_i = p_j(i)E_i\). That is \(p_j(i)\) is the eigenvalue of \(A_j\), associated with the eigenspace spanned by the columns of \(E_i\), occurring with the multiplicity \(m_i = \text{rank}(E_i)\). We define \(P\) to be the \((d + 1) \times (d + 1)\) matrix whose \((i, j)\)-entry is \(p_j(i)\). This \(P\) is referred to as the character table or first eigenmatrix of the scheme \(\mathcal{X}\).

Given an \(n\)-element set \(X\), and the \(\mathbb{C}\)-algebra \(M_X(\mathbb{C})\) (or \(M_n(\mathbb{C})\)), by the standard module of \(X\), we mean the \(n\)-dimensional vector space \(V = \mathbb{C}^X = \bigoplus_{x \in X} \mathbb{C} \hat{x}\) of column vectors whose coordinates are indexed by \(X\). Here for each \(x \in X\), we denote by \(\hat{x}\) the column vector with 1 in the \(x\)th position, and 0 elsewhere. Observe that \(M_X(\mathbb{C})\) acts on \(V\) by left multiplication. We endow \(V\) with the Hermitian inner product defined by \(\langle u, v \rangle = u^\top \overline{v} (u, v \in V)\). For a given association scheme \(\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})\), \(V\) can be written as the direct sum of \(V_i = E_i V\) where \(V_i\) are the maximal common eigenspaces of \(A_0, A_1, \ldots, A_d\). Given an element \(x \in X\), let \(R_i(x) = \{y \in X : (x, y) \in R_i\}\). The valencies \(k_0, k_1, \ldots, k_d\) of \(\mathcal{X}\) are denoted by \(k_i = |R_i(x)| = p_{ii}^0\). Let \(V^*_i = V^*_i(x) = \bigoplus_{y \in R_i(x)} \mathbb{C} y\). Both \(R_i(x)\) and \(V^*_i\) are referred to as the \(i\)th subconstituent of \(\mathcal{X}\) with respect to \(x\). Let \(E^*_i = E^*_i(x)\) be the orthogonal projection map from \(V = \bigoplus_{i=0}^{d} V^*_i\) to the \(i\)th subconstituent \(V^*_i\). So, \(E^*_i\) can be represented by the diagonal matrix given by

\[
(E^*_i)_{yy} = \begin{cases} 1 & \text{if } (x, y) \in R_i \\ 0 & \text{if } (x, y) \notin R_i \end{cases}.
\]

The matrices \(E^*_0, E^*_1, \ldots, E^*_d\) are linearly independent, and they form a basis for a subalgebra \(\mathcal{M}^* = \mathcal{M}^*(x) = \langle E^*_0, E^*_1, \ldots, E^*_d \rangle\) of \(M_X(\mathbb{C})\).

Let \(A^*_i = A^*_i(x)\) be the diagonal matrix in \(M_X(\mathbb{C})\) with \(yy\) entry \((A^*_i)_{yy} = n \cdot (E^*_i)_{yy}\). Then we have

\begin{enumerate}
\item \(A^*_0 = I\),
\item \(A^*_0 + A^*_1 + \cdots + A^*_d = nE^*_0\),
\item \(A^*_i A^*_j = \sum_{h=0}^{d} q_{ij}^h A^*_h = A_j A_i\) for all \(i, j\).
\end{enumerate}
Furthermore, we also have
\[ A_j^* = \sum_{i=0}^{d} q_j(i) E_i^* \quad \text{and} \quad E_j^* = \frac{1}{n} \sum_{i=0}^{d} p_j(i) A_i^* . \]
Thus, \( A_0^*, A_1^*, \ldots, A_d^* \) form a second basis for \( M^* \). The algebra \( M^* \) is shown to be commutative, semi-simple subalgebra of \( M_X(\mathbb{C}) \). This algebra is called the dual Bose-Mesner algebra of \( X \) with respect to \( x \).

Let \( X = (X, \{ R_i \}_{0 \leq i \leq d}) \) denote a scheme and fix a vertex \( x \in X \), and let \( T = T(x) \) denote the subalgebra of \( M_X(\mathbb{C}) \) generated by the Bose-Mesner algebra \( M \) and the dual Bose-Mesner algebra \( M^* \). We call \( T \) the Terwilliger algebra of \( X \) with respect to \( x \). The facts discussed in the rest of the section will be useful when we describe the irreducible \( T \)-modules for our schemes in the subsequent sections.

Lemma 2.1 \([19]\) Let \( X = (X, \{ R_i \}_{0 \leq i \leq d}) \) denote a \( d \)-class association scheme. For an arbitrary fixed vertex \( x \in X \), let \( T = T(x) \). There exists a set \( \Phi = \Phi(x) \) and a basis \( \{ e_\lambda : \lambda \in \Phi \} \) of the center of \( T \) such that
\[
\begin{align*}
(i) & \quad I = \sum_{\lambda \in \Phi} e_\lambda, \\
(ii) & \quad e_\lambda e_\mu = \delta_{\lambda \mu} e_\lambda \quad \text{(for all} \, \lambda, \mu \in \Phi). 
\end{align*}
\]

We refer to \( e_\lambda \) as the central primitive idempotents of \( T \). Let \( V = \mathbb{C}^X \) denote the standard module.

Lemma 2.2 \([19]\) Let \( X = (X, \{ R_i \}_{0 \leq i \leq d}) \) denote a scheme and fix a vertex \( x \in X \), and let \( T = T(x) \). Let \( \{ e_\lambda : \lambda \in \Phi \} \) be the central primitive idempotents of \( T \).
\[
\begin{align*}
(i) & \quad V = \sum_{\lambda \in \Phi} e_\lambda V \quad \text{(Orthogonal direct sum). Moreover,} \; e_\lambda : V \to e_\lambda V \text{ is an orthogonal projection} \quad \text{for all} \, \lambda \in \Phi. \\
(ii) & \quad \text{For each irreducible} \; T \text{-module} \; W, \; \text{there is a unique} \; \lambda \in \Phi \; \text{such that} \; W \subseteq e_\lambda V. \; \text{We refer} \quad \text{to} \; \lambda \; \text{as the type of} \; W. \\
(iii) & \quad \text{Let} \; W \; \text{and} \; W' \; \text{denote irreducible} \; T \text{-modules.} \; \text{Then} \; W \; \text{and} \; W' \; \text{are} \; T \text{-isomorphic if and only if} \; W \; \text{and} \; W' \; \text{have the same type.} \\
(iv) & \quad \text{For all} \; \lambda \in \Phi, \; e_\lambda V \; \text{can be decomposed as an orthogonal direct sum of irreducible} \; T \text{-modules of type} \; \lambda. \\
(v) & \quad \text{Referring to (iv), the number of irreducible} \; T \text{-modules in the decomposition is independent of the decomposition.} \; \text{We shall denote this number by} \; \text{mult}(e_\lambda) \quad \text{(or simply} \; \text{mult}(\lambda)) \quad \text{and} \quad \text{refer to it as the multiplicity (in} \; V \text{) of the irreducible} \; T \text{-module of type} \; \lambda.
\end{align*}
\]

The set of triple products \( E_i^* A_j E_h^* \) in \( T(x) \) plays a special role in our study, so we look at them little closely. We can view \( E_i^* A_j E_h^* \) as a linear map from \( V_h^* \to V_i^* \) such that
\[ E_i^* A_j E_h^* \hat{y} = \sum_{z \in R_i(x) \cap R_j(y)} \hat{z} \]
for each \( \hat{y} \in V_h^* \). Terwilliger proved the following key fact in \([19]\) Lemma 3.2].
Proposition 2.3 For $0 \leq h, i, j \leq d$, $E^*_i A_j E^*_h = 0$ if and only if $p^h_{ij} = 0$.

Note that for every $i \in \{0, 1, \ldots, d\}$, $A_i$ and $E^*_i$ can be written in terms of the triple products $E^*_i A_j E^*_h$. Thus, the triple products $E^*_i A_j E^*_h$ generate the Terwilliger algebra. It is often easier to find the irreducible modules if we work with the triple products $E^*_i A_j E^*_h$ instead of $A_i$ and $E^*_i$.

2.2 The Wreath product of association schemes

We briefly recall the notion of the wreath product. Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ and $\mathcal{Y} = (Y, \{S_j\}_{0 \leq j \leq e})$ be association schemes of order $|X| = m$ and $|Y| = n$. The wreath product $\mathcal{X} \wr \mathcal{Y}$ of $\mathcal{X}$ and $\mathcal{Y}$ is defined on the set $X \times Y$; but we take $Y = \{y_1, y_2, \ldots, y_n\}$, and regard $X \times Y$ as the disjoint union of $n$ copies $X_1 \times \{y_1\}, X_2 \times \{y_2\}, \ldots, X_n \times \{y_n\}$. The relations on $X_1 \cup X_2 \cup \cdots \cup X_n$ is defined by the following rule:

For any $j$, the relations between the elements of $X_j$ are determined by the association relations between the first coordinates in $\mathcal{X}$. For any $i$ and $j$, the relations between $X_i$ and $X_j$ are determined by the association relation of the second coordinates $y_i$ and $y_j$ in $\mathcal{Y}$ and the relation is independent from the first coordinates.

We may arrange the relations $W_0, W_1, \ldots, W_{d+e}$ of $\mathcal{X} \wr \mathcal{Y}$ as follows:

- $W_0 = \{((x, y), (x, y)) : (x, y) \in X \times Y\}$
- $W_k = \{((x_1, y), (x_2, y)) : (x_1, x_2) \in R_k, y \in Y\}$, for $1 \leq k \leq d$; and
- $W_k = \{((x_1, y_1), (x_2, y_2)) : x_1, x_2 \in X, (y_1, y_2) \in S_{k-d}\}$ for $d+1 \leq k \leq d+e$.

Then the wreath product $\mathcal{X} \wr \mathcal{Y} = (X \times Y, \{W_k\}_{0 \leq k \leq d+e})$ is a $(d+e)$-class association scheme. It is clear that $\mathcal{X} \wr \mathcal{Y}$ is commutative (resp. symmetric) if and only if $\mathcal{X}$ and $\mathcal{Y}$ are. Let $A_0, A_1, \ldots, A_d$ and $C_0, C_1, \ldots, C_e$ be the adjacency matrices of $\mathcal{X}$ and those of $\mathcal{Y}$, respectively. Then the adjacency matrices $W_k$ of $\mathcal{X} \wr \mathcal{Y}$ are given by

$$W_0 = C_0 \otimes A_0, W_1 = C_0 \otimes A_1, \ldots, W_d = C_0 \otimes A_d, W_{d+1} = C_1 \otimes J_m, \ldots, W_{d+e} = C_e \otimes J_m,$$

where “$\otimes$” denotes the Kronecker product: $A \otimes B = (a_{ij}B)$ of two matrices $A = (a_{ij})$ and $B$.

With the above ordering of the association relations of $\mathcal{X} \wr \mathcal{Y}$, the relation table of the wreath product is described by

$$R(\mathcal{X} \wr \mathcal{Y}) = \sum_{k=0}^{d+e} k \cdot A_k = I_n \otimes R(\mathcal{X}) + \{R(\mathcal{Y}) + d(J_n - I_n)\} \otimes J_m.$$

3 The Dimension of the Terwilliger Algebra of $K_{n_1} \wr K_{n_2} \wr \cdots \wr K_{n_d}$

In this section, we first calculate the dimension of the Terwilliger algebra of the wreath product of one-class association schemes. We also show that these wreath product schemes satisfy the triple regularity property.
Throughout the paper, for the notational simplicity, by $[n]$ we denote the set of integers $\{1, 2, \ldots, n\}$, and by $K_n = H(1, n)$ we denote both the one-class association scheme $([n], \{R_0, R_1\})$ with $A_1 = J - I$, and the complete graph on $n$ vertices.

Let $X = (X, \{R_i\}_{0 \leq i \leq d})$ denote the $d$-class association scheme $K_{n_1} \bowtie K_{n_2} \bowtie \cdots \bowtie K_{n_d}$ with

$$X = [n_1] \times [n_2] \times \cdots \times [n_d] = \{(a_1, a_2, \ldots, a_d) : a_i \in [n_i], \text{ for } i = 1, 2, \ldots, d\}.$$  

Let $(1, 1, \ldots, 1) \in X$ be a fixed base vertex $x$ of $X$. Without loss of generality, we can arrange the association relations such that

- $R_1(x) = \{(a, 1, 1, \ldots, 1) : a \in \{2, 3, \ldots, n_1\}\}$,
- for $i = 2, 3, \ldots, d$,

$$R_i(x) = \{(a_1, a_2, \ldots, a_{i-1}, b, 1, 1, \ldots, 1) : a_k \in [n_k] \text{ for } k = 1, 2, \ldots, i-1, b \in [n_i] - \{1\}\}.$$  

We observe that $k_i = |R_i(x)| = (n_i - 1) \prod_{k=1}^{i-1} n_k$. We can arrange rows and columns of the relation table of $X$ by the order of parts in the partition $X = R_0(x) \cup R_1(x) \cup \cdots \cup R_d(x)$.

**Example 3.1** The following is the relation table for the wreath product of three association schemes $K_2$, $K_2$ and $K_3$ arranging the elements in the order described above.

$$R(K_2 \bowtie K_2 \bowtie K_3) = \begin{pmatrix}
0 & 1 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 \\
1 & 0 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 \\
2 & 2 & 0 & 1 & 3 & 3 & 3 & 3 & 3 & 3 \\
2 & 2 & 1 & 0 & 3 & 3 & 3 & 3 & 3 & 3 \\
3 & 3 & 3 & 3 & 0 & 1 & 2 & 2 & 3 & 3 \\
3 & 3 & 3 & 3 & 1 & 0 & 2 & 2 & 3 & 3 \\
3 & 3 & 3 & 3 & 2 & 2 & 0 & 1 & 3 & 3 \\
3 & 3 & 3 & 3 & 2 & 2 & 1 & 0 & 3 & 3 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 0 & 1 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 1 & 0 & 2 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 2 & 2 & 0 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 2 & 2 & 1 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 2 & 2 & 1
\end{pmatrix}$$

We note that any table obtained from this table by permuting the order of rows and corresponding columns simultaneously represents the same association scheme.

**Lemma 3.2** Let $X = (X, \{R_i\}_{0 \leq i \leq d}) = K_{n_1} \bowtie K_{n_2} \bowtie \cdots \bowtie K_{n_d}$. Then the complete list of nonzero $p_{ij}^h$ where $h, i, j \in \{0, 1, 2, \ldots, d\}$ is as follows.

1. For $h = 0$, $k_0 = p_{00}^0 = 1$, $k_1 = p_{11}^0 = n_1 - 1$, and

$$k_j = p_{jj}^0 = (n_j - 1) \prod_{l=1}^{j-1} n_l \quad \text{for } j = 2, 3, \ldots, d.$$  

2. For $h = 1, 2, \ldots, d$, 

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(a) \( p_{hh}^h = (n_h - 2) \prod_{l=1}^{h-1} n_l \).

(b) \( p_{jj}^h = (n_j - 1) \prod_{l=1}^{j-1} n_l \) for \( h + 1 \leq j \leq d \).

(c) \( p_{jh}^h = p_{hj}^h = (n_j - 1) \prod_{l=1}^{j-1} n_l \) for \( 1 \leq j \leq h - 1 \).

(d) \( p_{00}^h = p_{0h}^h = 1 \).

Proof: It is straightforward to calculate the intersection numbers.

Due to this lemma, we have the following list of non-zero triple products in \( T \).

**Theorem 3.3** The complete list of nonzero triple products \( E_i^* A_j E_h^* \) among all \( h, i, j \in \{0, 1, 2, \ldots, d\} \) in \( X = K_n_1 \wr K_n_2 \wr \cdots \wr K_n_d \) is given as follows.

1. \( E_i^* A_i E_0^* \) for \( 0 \leq i \leq d \)
2. \( E_h^* A_h E_h^* \) if and only if \( n_h \geq 3 \) for \( 1 \leq h \leq d \),
3. \( E_j^* A_j E_h^* \) for \( 2 \leq h + 1 \leq j \leq d \)
4. \( E_j^* A_h E_h^* \) for \( 1 \leq j + 1 \leq h \leq d \)
5. \( E_j^* A_j E_h^* \) for \( 1 \leq j + 1 \leq h \leq d \)

Proof: Immediate from the above lemma by Proposition 2.3.

In order to calculate the dimension of the Terwilliger algebra for the \( d \)-class association scheme \( X = (X, \{R_i\}_{0 \leq i \leq d}) = K_n_1 \wr K_n_2 \wr \cdots \wr K_n_d \). Let \( x \in X \) be a fixed vertex, and consider the subspace \( T_0 = T_0(x) \) of \( T = T(x) \) spanned by \( \{E_i^* A_j E_h^* : 0 \leq i, j, h \leq d\} \). It is easy to see that \( T \) is generated by \( T_0 \) as an algebra since \( T_0 \) contains \( A_i \) and \( E_i^* \) for all \( i \), but in general, \( T_0 \) may be a proper linear subspace of \( T \). However, we will see that \( T_0 = T \) for \( K_n_1 \wr K_n_2 \wr \cdots \wr K_n_d \) shortly. First we have the following dimension formula for \( T_0 \).

**Theorem 3.4** Let \( X = K_n_1 \wr K_n_2 \wr \cdots \wr K_n_d \). Then the dimension of \( T_0 \) is given by

\[
\dim(T_0) = (d + 1)^2 + \frac{1}{2}d(d + 1) - b
\]

where \( b \) is the number of \( K_2 \) factors in the wreath product. In particular,

\[
(d + 1)^2 + \binom{d}{2} \leq \dim(T_0) \leq (d + 1)^2 + \binom{d + 1}{2}
\]

Proof: In the Theorem 3.3, the number of non-zero triple products can be counted as \( d + 1 \) from (1), \( d - b \) from (2), \( \frac{1}{2}d(d - 1) \) from (3), and \( d(d + 1) \) from (4) and (5). As they are independent of each other we have the \( \dim(T_0(x)) = d + 1 + d - b + \frac{1}{2}d(d - 1) + d(d + 1) = (d + 1)^2 + \frac{1}{2}d(d + 1) - b \) as
desired. The case when all \( n_i = 2 \) gives the lower bound as in this case \( b = d \). The upper bound is given by the case where \( n_i \geq 3 \) for all \( i \). In such a situation \( b = 0 \). This completes the proof. 

We now show that \( T = T_0 \) for the wreath product scheme \( K_{n_1} \wr K_{n_2} \cdots \wr K_{n_d} \); so the scheme has the triple-regularity property. The concept of triple-regularity was first studied by Terwilliger. For more information on it, we refer to [14, p.120]). We use the following equivalent properties of triple-regularity observed by Munemasa.

**Proposition 3.5** ([16]) Let \( \mathcal{X} \) be a commutative association scheme. Then the following are equivalent.

1. \( \mathcal{X} \) is triply regular; i.e. \( \mathcal{X} \) has the property that the size of the set \( R_i(x) \cap R_j(y) \cap R_k(z) \) depends only on the set \( \{i, j, h, l, m, n\} \) where \( (x, y) \in R_i \), \( (x, z) \in R_m \) and \( (y, z) \in R_n \).
2. \( A_i E_h^* A_j \in T_0 \) for any \( h, i, j \).
3. \( T(x) = T_0(x) \) for \( x \in X \).

According to this proposition, it suffices to verify that all triple products \( A_i E_h^* A_j \) belong to \( T_0 \) in order to show that \( T = T_0 \) for \( K_{n_1} \wr K_{n_2} \cdots \wr K_{n_d} \).

**Lemma 3.6** For the \( d \)-class scheme \( K_{n_1} \wr K_{n_2} \cdots \wr K_{n_d} \), we have the following.

1. \( A_i E_h^* A_j = (A_i E_h^*) (E_h^* A_j) \) for all \( h, i, j \in \{0, 1, \ldots, d\} \).
2. \( A_h E_h^* = \sum_{j=0}^{h} E_j^* A_h E_h^* \) for all \( h \in \{0, 1, \ldots, d\} \).
3. For \( 0 \leq i < h \leq d \), \( A_i E_h^* = E_h^* A_i E_h^* \).
4. For \( 0 \leq h < i \leq d \), \( A_i E_h^* = E_i^* A_i E_h^* \).

Proof: (1) It is trivially true as \( E_h^* \) are idempotents.

(2) The nonzero entries of \( A_h E_h^* \) are the nonzero entries of the columns of \( A_h \) indexed by vertices in \( R_h(x) \). The rest of the entries are zero. The columns of \( A_h \) indexed by vertices in \( R_h(x) \) have 1 in rows indexed by the vertices in \( R_0(x) \cup R_1(x) \cup \cdots \cup R_{h-1}(x) \). In addition the rows indexed by the vertices \( R_h(x) \) have zero in the diagonal blocks of size \( (\prod_{i=1}^{h-1} n_i) \times (\prod_{i=1}^{h-1} n_i) \), and 1 elsewhere. These are essentially \( \sum_{j=0}^{h} E_j^* A_h E_h^* \).

(3) If \( i < h \), the nonzero entries of \( A_i E_h^* \) are the nonzero entries of the columns of \( A_i \) indexed by vertices in \( R_h(x) \). The rest of the entries are zero. The columns of \( A_i \) indexed by vertices in \( R_h(x) \) have 1 in rows indexed by the vertices in \( R_h(x) \). The rest of the entries are zero.

(4) If \( i > h \), the nonzero entries of \( A_i E_h^* \) are the nonzero entries of the columns of \( A_i \) indexed by vertices in \( R_h(x) \). The rest of the entries are zero. The columns of \( A_i \) indexed by vertices in \( R_h(x) \) have 1 in rows indexed by the vertices in \( R_i(x) \). The rest of the entries are zero. This completes the proof. 

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Lemma 3.7. In $K_{n_1} \ast K_{n_2} \ast \cdots \ast K_{n_d}$,

1. $E_h^* A_h = \sum_{j=0}^{h} E_h^* A_h E_j^*$ for $h \in \{0, 1, \ldots, d\}$;
2. For $0 \leq i < h \leq d$, $E_h^* A_i = E_h^* A_i E_h^*$;
3. For $0 \leq h < i \leq d$, $E_h^* A_i = E_h^* A_i E_i^*$.

Proof: It follows from the previous lemma and the fact that the transpose of $A_i E_h^*$ is $E_h^* A_i$. \hfill \blacksquare

Lemma 3.8. For $K_{n_1} \ast K_{n_2} \ast \cdots \ast K_{n_d}$, we have the following linear combinations for $A_i E_h^* A_j$.

1. For $h \in \{2, 3, \ldots, d\}$,
   \[
   A_h E_h^* A_h = (n_h - 1) \left( \prod_{k=1}^{h-1} \frac{1}{n_k} \right) \sum_{m=0}^{d} \sum_{n=0}^{h-1} E_i^* A_m E_n^* \\
   + (n_h - 2) \left( \prod_{k=1}^{h-1} \frac{1}{n_k} \right) \left\{ \sum_{m=0}^{d} \sum_{n=0}^{h-1} E_i^* A_m E_j^* + \sum_{m=0}^{d} \sum_{l=0}^{h} E_i^* A_m E_n^* \right\}.
   \]

2. (a) For $d \geq j > h \geq 2$,
   \[
   A_h E_h^* A_j = (n_h - 1) \left( \prod_{k=1}^{h-1} \frac{1}{n_k} \right) \sum_{m=0}^{d} \sum_{l=0}^{h-1} E_i^* A_{m+l} E_j^* + (n_h - 2) \left( \prod_{k=1}^{h-1} \frac{1}{n_k} \right) \sum_{m=0}^{d} E_i^* A_{m+l} E_j^*;
   \]
   (b) for $2 \leq j < h \leq d$,
   \[
   A_h E_h^* A_j = (n_j - 1) \left( \prod_{k=1}^{j-1} \frac{1}{n_k} \right) \sum_{m=0}^{d} \sum_{l=0}^{j-1} E_i^* A_{m+l} E_j^*.
   \]

3. (a) For $d \geq i > h \geq 2$,
   \[
   A_i E_h^* A_h = (n_h - 1) \left( \prod_{k=1}^{h-1} \frac{1}{n_k} \right) \sum_{m=0}^{d} \sum_{n=0}^{h-1} E_i^* A_m E_n^* \\
   + (n_h - 2) \left( \prod_{k=1}^{h-1} \frac{1}{n_k} \right) \sum_{m=0}^{d} E_i^* A_m E_h^*;
   \]
   (b) for $i < h$,
   \[
   A_i E_h^* A_h = n_i \cdots n_{i-1} (n_i - 1) \sum_{m=0}^{d} \sum_{n=0}^{h-1} E_i^* A_m E_n^*.
   \]

4. (a) For $d \geq i > h \geq 2$,
   \[
   A_i E_h^* A_i = n_i \cdots n_{i-1} (n_i - 1) \sum_{m=0}^{d} E_i^* A_m E_i^*;
   \]

\hfill 9
(b) for $2 \leq i < h \leq d$,

$$A_iE_h^*A_i = n_1 \cdots n_{i-1}(n_i - 1) \sum_{m=0}^{d} E_m^*A_mE_n^*.$$

Proof: (1) Applying the identities in Lemmas 3.6 and 3.7 to $A_iE_h^*A_j = (A_iE_h^*)(E_h^*A_j)$ we have

$$A_iE_h^*A_j = k_h \sum_{m=0}^{d} \sum_{l=0}^{h-1} \sum_{n=0}^{h-1} E_m^*A_mE_n^*$$

$$+(k_h - (k_0 + \cdots + k_{h-1})) \left\{ \sum_{m=0}^{d} \sum_{n=0}^{h-1} E_m^*A_mE_n^* + \sum_{m=0}^{D} \sum_{l=0}^{h} E_l^*A_mE_n^* \right\}$$

with $k_h = n_1n_2 \cdots n_h - n_1n_2 \cdots n_{h-1}$ as desired.

(2) The proof of part (a) follows from Lemma 3.6(1) and 3.7(3), while (b) follows from Lemma 3.6(1) and 3.7(2).

(3) The proof is a similar to part (2).

(4) The proof of (a) follows from Lemma 3.6(3) and 3.7(3), and (b) follows from Lemma 3.6(2) and 3.7(2).

Lemma 3.9

In $K_{n_1} \wr K_{n_2} \wr \cdots \wr K_{n_d}$, for $i, j, h \in \{0, 1, \ldots, d\}$, suppose that no two of $i, j, h$ are equal.

(i) If $i > h$ and $j > h$, then

$$A_iE_h^*A_j = (n_h - 1) \left( \prod_{k=1}^{h-1} n_k \right) \sum_{m=0}^{d} E_m^*A_mE_n^*.$$

(ii) If $i < h < j$, then

$$A_iE_h^*A_j = (n_i - 1) \left( \prod_{k=1}^{i-1} n_k \right) \sum_{m=0}^{d} E_m^*A_mE_n^*.$$

(iii) If $i > h > j$, then

$$A_iE_h^*A_j = (n_j - 1) \left( \prod_{k=1}^{j-1} n_k \right) \sum_{m=0}^{d} E_m^*A_mE_n^*.$$

(iv) If $i < h$ and $j < h$, then

(a) for $i < j$,

$$A_iE_h^*A_j = (n_i - 1) \left( \prod_{k=1}^{i-1} n_k \right) E_h^*A_jE_n^*;$$
(b) for $i > j$,\[ A_i E^*_h A_j = (n_j - 1) \left( \prod_{k=1}^{i-1} n_k \right) E^*_h A_i E^*_h.\]

Proof: (i), (ii) and (iii) are similar to Lemma 3.2.8

(iv)(a) Lemmas 3.6(ii) and 3.7(ii) gives us nonzero entries of $A_i E^*_h A_j$ occur in the rows and columns indexed by the vertices $R_h(x)$. Consider the diagonal blocks of size $k_j \times k_j$ inside $A_i E^*_h A_j$ indexed by the rows and columns of vertices in $R_h(x)$. These diagonal blocks have $k_i$ 1’s in each row and column. The off diagonal entries are all zero. In a similar manner consider diagonal blocks of size $k_j \times k_j$ inside $A_i E^*_h A_j$ indexed by the rows and columns of vertices in $R_h(x)$. These diagonal blocks are all zero and the off diagonal entries are 1. This observation gives us\[ A_i E^*_h A_j = n_1 \cdots n_{i-1} (n_i - 1) E^*_h A_j E^*_h.\]

(b) Similar to part (a). □

**Theorem 3.10** The $d$-class association scheme $K_{n_1} \wr K_{n_2} \wr \cdots \wr K_{n_d}$ is triply regular and $T = T_0$.

Proof: It is straightforward to check that each of the rest of $A_i E^*_h A_j$ that are not covered in Lemma 3.8 and Lemma 3.9 also can be expressed as linear combination of the generators of $T_0$. Thus the conclusion follows from Proposition 3.5. □

In summary, we have the following.

**Theorem 3.11** The dimension of the Terwilliger algebra $T$ of $K_{n_1} \wr K_{n_2} \wr \cdots \wr K_{n_d}$ is\[ \dim(T) = (d + 1)^2 + \frac{1}{2} d(d + 1) - b\]

where $b$ denotes the number of factors $K_{n_i}$ with $n_i = 2$.

Proof: Immediate consequence of Lemma 3.4, Proposition 3.5 and Theorem 3.10. □

**4 Terwilliger Algebras of Wreath Powers of $K_m$**

In this section we describe the structure of the Terwilliger algebra of the wreath power $(K_m)^d = K_m \wr K_m \wr \cdots \wr K_m$ of $d$ copies of the one-class association scheme $K_m$. We begin the section with the description of the Terwilliger algebra of the one-class association scheme $K_m$ of order $m$. The nontrivial relation graph of $K_m$ is the adjacency matrix of the complete graph $K_m$ which is also viewed as Hamming graph $H(1, m)$. We then describe the Terwilliger algebra of $(K_m)^d$ whose first relation graph is the complete $m$-partite strongly regular graph with parameters $(v, k, \lambda, \mu) = (m^2, m(m-1), m(m-2), m(m-1))$. We then compare the combinatorial structures
of wreath square \((K_m)^2\) and cube \((K_m)^3\) of \(K_m\) to describe irreducible \(T\)-modules of \((K_m)^3\) by extending those of \((K_m)^2\). Similarly, the structure of the irreducible \(T\)-modules of \((K_m)^d\) for any higher \(d\) will be described from that of \((K_m)^{(d-1)}\). It is shown that all non-primary irreducible \(T\)-modules of wreath powers of \(K_m\) are of dimension 1. We conclude the section by describing the Terwilliger algebra of the \(d\)-power \((K_m)^d\) for an arbitrary \(d \geq 2\).

### 4.1 The Terwilliger algebra of \(K_m\)

Let \(K_m = ([m], \{R_0, R_1\})\), and let \(x = 1\). Then

\[ R_1(x) = \{2, 3, \ldots, m\} \]

and, we may denote \(A_1\) by

\[ A_1 = J - I = \begin{bmatrix} 0 & 1^t \\ 1 & J_{m-1} - I_{m-1} \end{bmatrix} \]

where \(1\) is the \((m - 1)\)-dimensional column vector all of whose entries are 1.

**Remark 4.1** By Theorem 3.11, we know that the dimension of the Terwilliger algebra of \(K_m\) is 5 if \(m > 2\) and 4 if \(m = 2\). Also by Theorem 3.3, all the matrices in the Terwilliger algebra of \(K_m\) is a linear combination of the matrices \(E_0^* A_0 E_0^*\), \(E_0^* A_1 E_1^*\), \(E_1^* A_1 E_0^*\), \(E_1^* A_0 E_1^*\), and \(E_1^* A_1 E_1^*\). (If \(m = 2\), then \(E_1^* A_1 E_1^* = 0\).)

If we set

\[ E_{11} = E_0^* A_0 E_0^*, \ E_{12} = E_0^* A_1 E_1^*, \ E_{21} = \frac{1}{m - 1} E_1^* A_1 E_0^*, \ E_{22} = \frac{1}{m - 1} (E_1^* A_0 E_1^* + E_1^* A_1 E_1^*), \]

then these matrices form a subalgebra \(U\) of \(T(x)\) as its multiplication table is given by

\[
\begin{array}{cccc}
E_{11} & E_{11} & E_{12} & E_{21} \\
E_{12} & 0 & 0 & E_{11} \\
E_{21} & E_{21} & E_{22} & 0 \\
E_{22} & 0 & E_{21} & E_{22} \\
\end{array}
\]

Considering the isomorphism between \(U\) and \(M_2(\mathbb{C})\) that takes

\[ E_{11} \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \ E_{12} \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \ E_{21} \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}; \ E_{22} \mapsto \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \]

we see that \(T(x) = M_2(\mathbb{C}) \oplus M_1(\mathbb{C})\) by Wedderburn-Artin’s Theorem (cf. [9, Sec. 2.4]). Specifically, if we set \(F = E_1^* A_0 E_1^* - E_{22}\), then it turns out that \(FX = 0\) for all \(X \in U\). This gives us \(T(x) = \mathbb{C}F \oplus U\). While \(F = 0\) for \(m = 2\), \(F \neq 0\) for all \(m > 2\). Therefore, we reasserted the following.

**Theorem 4.1** [13] The Terwilliger algebra of \(K_m\) can be described as follows:

\[
T(x) \cong \begin{cases} 
M_2(\mathbb{C}) & \text{if } m = 2 \\
M_2(\mathbb{C}) \oplus M_1(\mathbb{C}) & \text{if } m > 2 
\end{cases}
\]
4.2 The Terwilliger algebra of \((K_m)^2\) and its irreducible modules

Let \(\mathcal{X} = (K_m)^2 = (X, \{R_0, R_1, R_2\})\) be the wreath square of \(K_m\). Without loss of generality, we arrange the relations so that the first relation graph \((X, R_1)\) is to be the complete \(m\)-partite strongly regular graph with parameters \((v, k, \lambda, \mu) = (m^2, m(m-1), m(m-2), m(m-1))\), which is also the wreath square of the complete graph \(K_m\).

Let \(X = [m] \times [m] = \{(i, j) : i, j \in [m]\}\), and let \(x = (1, 1)\). We will refer to \((1, 1)\) as the base vertex \(x\). Then
\[
R_1(x) = \{(i, j) : i \in [m], j \in \{2, 3, \ldots, m\}\},
\]
\[
R_2(x) = \{(i, 1) : i \in \{2, 3, \ldots, m\}\}.
\]

Let the adjacency matrices \(A_i\) and the relation table \(R\) of \(\mathcal{X}\) be decomposed according to the partition \(X = R_0(x) \cup R_1(x) \cup R_2(x)\). Then,
\[
A_1 = \begin{bmatrix}
0 & 1_2 & 0_1 \\
1_2 & B_2 & L \\
0_1 & L^t & B_1
\end{bmatrix}, \quad A_2 = \begin{bmatrix}
0 & 0_2 & 1_1^t \\
0_2 & C_2 & N \\
1_1 & N^t & C_1
\end{bmatrix}, \quad R = \begin{bmatrix}
0 & 1_2^t & 21_1^t \\
1_2 & B_2 + 2C_2 & L \\
21_1 & L^t & 2C_1
\end{bmatrix},
\]
where \(1_2\) and \(0_1\) are all-ones column vectors of size \(m(m-1)\) and \((m-1)\), respectively; \(0_2\) and \(0_1\) are all-zeros column vectors of size \(m(m-1)\) and \(m-1\), respectively; \(L\) and \(N\) are \(m(m-1) \times (m-1)\) all-ones and all-zeros matrices, respectively; \(B_2 = J_{m(m-1)} - (I_{m-1} \otimes J_m)\), and \(B_1\) is a \((m-1) \times (m-1)\) zero matrix, while \(C_2 = I_{m-1} \otimes (J_m - I_m)\) and \(C_1 = J_{m-1} - I_{m-1}\).

We note that \(A_2 = \frac{1}{m(m-1)}A_1^2 - \frac{m-2}{m-1}A_1 - I\). We also note that \((K_m)^2\) is formally self-dual \(P\)- and \(Q\)-polynomial association scheme with its first and second eigenmatrices
\[
P = Q = \begin{bmatrix}
1 & m(m-1) & m-1 \\
1 & 0 & -1 \\
1 & -m & m-1
\end{bmatrix}.
\]

The characteristic polynomial of \(A_1\) is \(\theta^2 + (\mu - \lambda) \theta + (\mu - k) = 0\), and the eigenvalues of \(A_1\) are given by \(k = m(m-1)\), \(r = 0\), and \(s = -m\) with multiplicities, \(1\), \(m(m-1)\), and \(m-1\), respectively. The induced subgraphs of the graph \((K_m)^2\) on the vertex sets \(R_1(x)\) and \(R_2(x)\) with adjacency matrices \(B_2\) and \(B_1\) respectively, are called the subconstituents of the graph with respect to \(x\).

Set \(\lambda = p_{11}^1 = m(m-2), \mu = p_{11}^2 = m(m-1), k_1 = m_1 = p_{11}^0 = m(m-1),\) and \(k_2 = m_2 = m-1\). Furthermore, \(B_21_2 = m(m-1)1_2\) and \(C_11_1 = (m-1)1_1\). The eigenvalues of \(B_2\) are \(m(m-2), -m\) and 0 with multiplicities \(1, m-2\) and \(m^2 - 2m + 1\) respectively. For \(B_1\), 0 is the only eigenvalue with multiplicity \(m-1\).

Cameron, Goethals and Seidel introduced the concept of restricted eigenvalues and eigenvectors [6]. An eigenvalue of \(B_2\) (resp. \(B_1\)) is called restricted if it has an eigenvector orthogonal to the all-ones vector of size \(k_1\) (resp. \(k_2\)). Tomiyama and Yamazaki used restricted eigenvectors, the eigenvectors associated with the restricted eigenvalue of \(B_i\) that are orthogonal to \(1_1\), to describe the subconstituent algebra of a 2-class association scheme constructed from a strongly regular graph. In order to describe the irreducible \(T\)-modules of our scheme \((K_m)^2\), we will need the result of Theorem 5.1 in [6] for our scheme.

**Lemma 4.2** With the above notations for \((K_m)^2\) and its eigenvalues \(m(m-1), 0\) and \(-m\), we have the following.
1. Suppose \( y \) is a restricted eigenvector of \( B_2 \) with an eigenvalue \( \theta \). Then \( y \) is the eigenvector of \( LL^t \) with the eigenvalue \(-\theta(\theta+m)\), and \( L^t y \) is the zero vector or the restricted eigenvector of \( B_1 \) with the eigenvalue \(-m - \theta\). In particular, \( L^t y \) is zero if and only if \( \theta \in \{0, -m\} \).

2. Suppose \( z \) is a restricted eigenvector of \( B_1 \) with an eigenvalue \( \theta' \). Then \( z \) is the eigenvector of \( L^t L \) with the eigenvalue \(-\theta'(\theta'+m)\), and \( Lz \) is the zero vector or the restricted eigenvector of \( B_2 \) with the eigenvalue \(-m - \theta'\). In particular, \( Lz \) is zero if and only if \( \theta' \in \{0, -m\} \).

Proof: It is immediate from the fact that \( B_2 \) is the adjacency matrix of the strongly regular graph with parameters \((m(m-1), m(m-2), m(m-3), m(m-2))\).

The next three lemmas are results of Tomiyama and Yamazaki (reported in [22]) suited to our association scheme \((K_m)^2\).

**Lemma 4.3** Let \( T \) denote the Terwilliger algebra of \((K_m)^2\). Then with the above notations in this subsection, we have the following.

1. Suppose \( y \) is a restricted eigenvector of \( B_2 \) with an eigenvalue \( \theta \). Then the vector space \( W \) over \( \mathbb{C} \) which is spanned by \((0, y^t, 0^t_1)^t\) is a thin irreducible \( T \)-module over \( \mathbb{C} \) and \( \dim W = 1 \) if \( \theta \in \{0, -m\} \).

2. Suppose \( z \) is a restricted eigenvector of \( B_1 \) with an eigenvalue \( \theta' \). Then the vector space \( W' \) over \( \mathbb{C} \) which is spanned by \((0, 0^t_2, z^t)^t\) is a thin irreducible \( T \)-module over \( \mathbb{C} \) and \( \dim W' = 1 \) if \( \theta' \in \{0, -m\} \).

Proof: (1) Since \( y \) is the restricted eigenvector of \( B_2 \) associated with eigenvalue \( \theta \), so \((B_2 y)^t = (\theta y)^t\). Hence, \((0, (B_2 y)^t, 0^t_1) \in \text{span}\{(0, y^t, 0^t_1)\}\), and \( W \) is spanned by \((0, y^t, 0^t_1)^t\). Observe that \( A_1(0, y^t, 0^t_1)^t = (0, (B_2 y)^t, 0^t_1)^t \), and also \( y \) is orthogonal to \( 1_2 \) with the associated eigenvalue 0 or \(-m\). By Lemma 4.2 \( L^t y = 0_1 \). Therefore, \( W \) is \( A_1 \)-invariant and thus \( M \)-invariant. \( W \) is also \( M^* \)-invariant.

(2) The proof is similar to that of (1).

**Lemma 4.4** For the association scheme \((K_m)^2\), let \( V \) denote the standard \( T \)-module. There exist irreducible \( T \)-modules \( \{W_i\}_{1 \leq i \leq m(m-1)} \) and \( \{W'_j\}_{2 \leq j \leq m-2} \) such that \( V = (\oplus W_i) \oplus (\oplus W'_j) \).

Proof: Using Gram Schmidt process we can find eigenvectors \( v_1, v_2, \ldots, v_{m(m-1)} \) of \( B_2 \) such that \( \{v_i\}_{1 \leq i \leq m(m-1)} \) span \( E_1^t V \cong \mathbb{C}^{m(m-1)} \), \( \langle v_i, v_j \rangle = 0 \) for \( i \neq j \) with \( v_1 \) being the all-ones vector \( 1_2 \). Let \( \theta_i \) be the eigenvalue of \( B_2 \) with respect to the eigenvector \( v_i \), for \( 2 \leq i \leq m(m-1) \). Then \( \theta_i \in \{0, -m\} \) for \( 2 \leq i \leq m(m-1) \). Let \( W_i \) denote the linear span of \( \langle v_1, v_i \rangle, 0^t_2, 0^t_i \rangle, (0, v^t_i, 0^t_1)^t, \) and \( (0, 0^t_2, (L^t v_1)^t)^t \) over \( \mathbb{C} \). Then \( W_i \) is a thin irreducible \( T \)-module and \( W_i \cap W_j = \{0\} \) for \( i \neq j \).

Also,

\[
dim W_i = \begin{cases} 
3 & \text{if } i = 1 \\
2 & \text{if } 2 \leq i \leq m(m-1) 
\end{cases}
\]

Note that \( W_1 \) is the primary module generated by \( \langle (v_1, v_1), 0^t_2, 0^t_1 \rangle, (0, v^t_1, 0^t_1)^t, \) and \( (0, 0^t_2, (L^t v_1)^t)^t \) over \( \mathbb{C} \). For each \( i, 2 \leq i \leq m(m-1), \) \( W_i \) is generated by \( 0, v^t_i, 0^t_1 \). Note that \( w_1 = L^t v_1 \).
is an eigenvector of $B_1$. Let $w_2, \ldots, w_{m-1}$ be the eigenvectors of $B_1$ such that $w_1, \ldots, w_{m-1}$ span $E_2V \cong \mathbb{C}^{m-1}$ and $\langle w_i, w_j \rangle = 0$ for $i \neq j$. Let $W_i$ be the linear span of $(0, 0, w_i^t)^t$ over $\mathbb{C}$ for $2 \leq i \leq m - 1$. $W_i$ is a thin irreducible $T$-module of dimension 1. Thus, we have $V = (\oplus W_i) \oplus (\oplus W_j^t)$ as desired. 

**Lemma 4.5** For $(K_m)^2$, let $\{\theta_i\}_{1 \leq i \leq 2(m-1)}$, $\{\theta_i'\}_{2 \leq i \leq m-1}$, $\{W_i\}_{1 \leq i \leq 2(m-1)}$ and $\{W_i'\}_{2 \leq i \leq m-1}$. Then the following hold.

1. For all $i$ with $2 \leq i \leq m(m-1)$, $W_1$ and $W_i$ are not $T$-isomorphic.
2. For all $i$ and $j$ with $1 \leq i \leq m(m-1)$ and $2 \leq j \leq m-1$, $W_i$ and $W_j'$ are not $T$-isomorphic.
3. For $i$ and $j$ with $2 \leq i, j \leq m(m-1)$, $W_i$ and $W_j$ are $T$-isomorphic if and only if $\theta_i = \theta_j$.
4. For $i$ and $j$ with $2 \leq i, j \leq m-1$, $W_i'$ and $W_j'$ are $T$-isomorphic.

Proof: It is straightforward from the previous lemma according to Lemma 3.4 in [22].

**Remark 4.2** In order to describe the Terwilliger algebra of $(K_m)^2$, let $\Lambda$ denote the index set for the isomorphism classes of irreducible $T$-modules. Following the results of Lemmas 4.4 and 4.5, we can group the isomorphic irreducible modules together, say $V_\lambda$ for $\lambda \in \Lambda$, so that $V = \bigoplus_{\lambda \in \Lambda} V_\lambda$.

By Lemma 2.2, we know that for each subspace $V_\lambda$ there is a unique central idempotent $e_\lambda$ such that $V_\lambda = e_\lambda V$. Let $W$ be an irreducible $T$-module in the decomposition of $V$. Then the map taking $A \in e_\lambda T$ to the endomorphism $w \mapsto Aw$ where $w \in W$ is an isomorphism. Hence we have $e_\lambda V \cong \text{End}_\mathbb{C} W$. Thus $T = \bigoplus_{\lambda \in \Lambda} e_\lambda T$ is isomorphic to a direct sum of complex matrix algebra $M_k(\mathbb{C})$ where $k = \dim(W)$ as in the following. In what follows we use the notation $M_1(\mathbb{C})^\oplus l$ for the direct sum $M_1(\mathbb{C}) \oplus M_1(\mathbb{C}) \oplus \cdots \oplus M_1(\mathbb{C})$ of $l$ copies of $M_1(\mathbb{C})$.

**Theorem 4.6** Let $T$ be the Terwilliger algebra of $(K_m)^2$. Then $\dim T = 12$ and

$$T \cong M_3(\mathbb{C}) \oplus M_1(\mathbb{C})^\oplus 3.$$ 

Proof: By Lemma 4.5, the list of non-isomorphic irreducible $T$-modules of $(K_m)^2$ consists of (i) the primary module $W_1$ of dimension 3, (ii) two one-dimensional non-isomorphic irreducible modules from $W_i$, $2 \leq i \leq m(m-1)$, that represent two isomorphism classes corresponding to the eigenvalues 0 and $-m$, and (iii) one one-dimensional irreducible module which representing the class of $W_j'$ for all $2 \leq j \leq m-1$. 

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4.3 The Terwilliger algebra of $(K_m)^{3}$ and its irreducible modules

We now extend the irreducible modules of $(K_m)^{2}$ to find irreducible modules of $(K_m)^{3}$ in this section, and then generalize it to describe the Terwilliger algebra for $(K_m)^d$ for an arbitrary $d \geq 3$ in the next section. We note that the Terwilliger algebra of $(K_m)^{2}$ is generated by $A_1$, $E_0^2$ and $E_1^3$. As we move on to three or higher wreath powers of $K_m$, the scheme is no longer $P$-polynomial, so, we must consider many more, but within $2d + 1$ generators for a $d$-power case. We will see that a ‘concrete Terwilliger algebra’ for the wreath powers can be described as well. (Here the term ‘concrete’ is used in comparison with an ‘abstract’ Terwilliger algebra described in [10].) We describe the irreducible $T$-modules of $(K_m)^{d}$ by extending the irreducible $T$-modules of $(K_m)^{(d-1)}$ for $d \geq 3$. This can be done by investigating the structural relations between the wreath square and the wreath cube to begin the iterative process.

Let $\mathcal{X} = (K_m)^3 = (X, \{R_t\}_{0 \leq t \leq 3})$ be a 3-class association scheme of order $m^3$. Let $X = \{(a, b, c) : a, b, c \in [m]\}$. Choose $x_1 = (1, 1, 1)$ and fix it as the base vertex and we will refer to it as $x$ henceforth. Let us order the association relations such that $R_0(x) = \{(1, 1, 1)\}$, and

$$R_1(x) = \{(a, b, c) : a, b \in [m], \quad c \in \{2, 3, \ldots, m\}\},$$

$$R_2(x) = \{(a, b, 1) : a \in [m], \quad b \in \{2, 3, \ldots, m\}\},$$

$$R_3(x) = \{(a, 1, 1) : a \in \{2, 3, \ldots, m\}\}.$$

Let the relation table of $\mathcal{X}$ be decomposed into block matrix form according to the partition $X = R_0(x) \cup R_1(x) \cup R_2(x) \cup R_3(x)$. We can see that the relation matrix of $(K_m)^{2}$ is embedded into that of $(K_m)^{3}$ by relabeling the association relations. To see this, consider the relation matrix $R$ for $(K_m)^{2}$ given in the previous subsection. Denote the block labeled by $B_2 + 2C_2$ of $R$ by $D$. Then $D$ has entries 0, 1, or 2. We now form a new block from $D$ by replacing entries 2 with 3, 1 with 2 and 0 with 0, and call this $D_2$. It is easy to see that $D_2$ is the block of the relation table $(K_m)^{3}$ that is indexed by the vertices in $R_2(x)$. The relation table of $(K_m)^{3}$ is given by

$$\begin{bmatrix}
0 & 1_3^t & 21_2^t & 31_1^t \\
1_3 & D_3 & L_2 & L_1 \\
21_2 & L_2^t & D_2 & 2L \\
31_1 & L_1^t & 2L^t & 3C_1
\end{bmatrix}$$

where $1_3, 1_2, 1_1$ are all-ones column vectors of dimension $m^2(m - 1)$, $m(m - 1)$, $m - 1$, respectively; $L_2, L_1, L$ are all-ones matrices of size $m^2(m - 1) \times m(m - 1)$, $m^2(m - 1) \times (m - 1)$, $m(m - 1) \times (m - 1)$, respectively; $D_3 = I_m \otimes D_2 + (J_m - I_m) \otimes J_{m(m-1)}$ and $C_1 = J_{m-1} - I_{m-1}$. We are now ready to see the descriptions of the irreducible $T$-modules of the wreath cube.

**Theorem 4.7** The primary irreducible $T$-module of $\mathcal{X} = (K_m)^{3}$ is spanned by the four vectors

$$(1, 0_3^t, 0_2^t, 0_1^t)^t, \quad (0, 1_3^t, 0_2^t, 0_1^t)^t, \quad (0, 0_3^t, 1_2^t, 0_1^t)^t, \quad (0, 0_3^t, 0_2^t, 1_1^t)^t$$

where $0_3, 0_2, 0_1$ are all-zeros vectors of dimensions $m^2(m - 1)$, $m(m - 1)$, $(m - 1)$, respectively; and $1_3, 1_2, 1_1$ are all-ones vectors of dimensions $m^3(m - 1)$, $m(m - 1)$, $(m - 1)$, respectively.
Theorem 4.8 Let \( V \) be the standard module of \((K_m)^3\). There exist irreducible \( T \)-modules \( \{W_i^1\}_{2 \leq i \leq (m-1)} \) such that \( (0, 0^t, 0^t_2, 1^t_1) \) and \( \{W_i^1\}_{2 \leq i \leq (m-1)} \) together constitute \( E_3^1V \).

Proof: We have seen in the proof of Lemma 4.4 that there exist vectors \( w_1, \ldots, w_{m-1} \) which span \( \mathbb{C}^{m-1} \) and \( \langle w_i, w_j \rangle = 0 \) for \( 1 \leq i \neq j \leq m-1 \). Let \( W_i^1 \) be the linear span of \((0, 0^t_3, 0^t_2, w_i^t)\) over \( \mathbb{C} \). For \((K_m)^2\), the generators \( E^*_i A_h E^*_i \) act on the modules \( W_i^1 \) and the significant nonzero actions are \( E^*_i A_0 E^*_i \) and \( E^*_i A_2 E^*_i \). The embedded structure of \((K_m)^3\) in \((K_m)^3\) ensures that for \((K_m)^3\) the generators \( E^*_i A_h E^*_i \) act on the linear spaces \( W_i^1, 2 \leq i \leq (m-1) \) and the only nonzero actions are \( E^*_i A_0 E^*_i \) and \( E^*_i A_2 E^*_i \). It is clear that \((0, 0^t_3, 0^t_2, w_i^t)\) are \( E^*_0 A_3 E^*_i, E^*_i A_1 E^*_3, E^*_2 A_2 E^*_3 \)-invariant, and each \( W_i^1 \) is an irreducible \( T \)-module of dimension 1. Thus, the result follows.

Theorem 4.9 In the standard module \( V \) of \((K_m)^3\), there exist irreducible modules \( \{W_i^2\}_{1 \leq i \leq (m-1)} \) such that \( (0, 0^t_3, 1^t_2, 0^t_i) \) and \( \{W_i^2\}_{2 \leq i \leq (m-1)} \) constitute \( E_3^2V \).

Proof: We have seen in Lemma 4.4 that there exist vectors \( v_1, \ldots, v_{m-1} \) such that \( v_1, \ldots, v_{m-1} \) span \( \mathbb{C}^{m(m-1)} \) and \( \langle v_i, v_j \rangle = 0 \) for \( 1 \leq i \neq j \leq m-1 \). Let \( W_i^2 \) be the linear span of \((0, 0^t_3, v_i^t, 0^t_1)\) over \( \mathbb{C} \). For \((K_m)^2\), the generators \( E^*_i A_h E^*_i \) act on the modules \( W_i^1 \) and the significant nonzero actions are \( E^*_i A_0 E^*_i \), \( E^*_i A_1 E^*_3 \), and \( E^*_i A_2 E^*_i \). The embedded structure of \((K_m)^3\) in \((K_m)^3\) ensures that for \((K_m)^3\) the generators \( E^*_i A_h E^*_i \) act on the linear spaces \( W_i^2, 2 \leq i \leq (m-1) \) and the nonzero actions are \( E^*_0 A_3 E^*_i, E^*_i A_2 E^*_3 \), and \( E^*_3 A_2 E^*_2 \). Also, we see that \((0, 0^t_3, v_i^t, 0^t_1)\) is invariant under the action of \( E^*_0 A_2 E^*_2, E^*_i A_1 E^*_2, \) and \( E^*_3 A_2 E^*_2 \). Each of \( \{W_i^2\}_{2 \leq i \leq (m-1)} \) is an irreducible \( T \)-module of dimension 1, and the result follows.

We now describe the irreducible modules that span \( E_3^1V \). We know that \( E_3^1V \cong \mathbb{C}^{m^2(m-1)} \). We observe that any \( m^2(m-1) \) dimensional column vector can be partitioned into \( m \) equal parts each with \( m(m-1) \) components. Let each part be denoted by the index \( j \) where \( j \in \{1, 2, \ldots, m\} \) so that the \( m^2(m-1) \) dimensional column vector is of the form \((u^t, 0^t_2, \ldots, u^t_2)\) where each \( u^t \) is an \( m(m-1) \) dimensional column vector for each \( j \in \{1, 2, \ldots, m\} \). Let \( u_{i,j} = (u^t_1, u^t_2, \ldots, u^t_2, u^t_1, \ldots, u^t_m) \) denote the \( m^2(m-1) \) dimensional column vector such that \( u_i = \delta_{ij} v_i \) where \( v_i \) are the ones given in Lemma 4.4 and \( \delta_{ij} \) is the Kronecker delta; i.e., \( \delta_{ij} \) is 1 if and only if \( i = j \), and it is zero otherwise.

Lemma 4.10 In the standard module \( V \) of \((K_m)^3\), for each pair \( j, i, 1 \leq j \leq m, 2 \leq i \leq (m-1) \), the linear subspace \( W_{j,i} \) of spanned by \((0, u_{j,i}^t, 0^t_2, 0^t_1)\) is an irreducible \( T \)-module contained in \( E_3^1V \).

Proof: In the Terwilliger algebra of \((K_m)^2\), the generators \( E^*_i A_h E^*_i \) act on the modules \( W_i \) and the only nonzero actions are due to \( E^*_0 A_0 E^*_i, E^*_i A_1 E^*_3 \) and \( E^*_i A_2 E^*_i \). For \((K_m)^3\), the submatrix \( D_3 = I_m \otimes D_2 + (J_m - I_m) \otimes J_m \) indicates that in the Terwilliger algebra of \((K_m)^3\) the generators \( E^*_i A_h E^*_i \) act on the linear spaces in \( \{W_{j,i} : 1 \leq j \leq m, 2 \leq i \leq (m-1)\} \) where \( W_{j,i} \)
is the linear span of \((0, u^t_{j,i}, 0^t_{2,1})^t\) and the only nonzero actions are due to \(E^*_1 A_0 E^*_1\), \(E^*_1 A_2 E^*_1\), \(E^*_1 A_3 E^*_1\), \(E^*_0 A_1 E^*_1\), \(E^*_2 A_1 E^*_1\), and \(E^*_3 A_1 E^*_1\). It follows that \(W_{j,i}\) is an irreducible module of dimension 1 for each \(j, i\).

### Remark 4.3

We note that if \(W^0\) denotes the linear span of \((1, 0^t_{1,1}, 0^t_{1,2})^t\) over \(\mathbb{C}\), then \(W^0\) is an irreducible module that spans \(E^*_0 V\). We recall that, by Lemma 4.3, among the irreducible modules \(\{W_i\}_{2 \leq i \leq m(m-1)}\) of \((K_m)^{\circ 2}\) there are two non-isomorphic \(T\)-modules of dimension 1. In sum, so far, for \((K_m)^{\circ 3}\) we have the following non-isomorphic \(T\)-modules:

1. The primary module of dimension 4.
2. Two non-isomorphic \(T\)-modules of dimension 1 in \(E^*_1 V\).
3. Two non-isomorphic \(T\)-modules of dimension 1 in \(E^*_2 V\).
4. One non-isomorphic \(T\)-modules of dimension 1 in \(E^*_3 V\).

It leaves us with one non-isomorphic irreducible \(T\)-module of dimension 1 which is not accounted for as the total dimension of the Terwilliger algebra must be 22 by Theorem 3.11. It is in the subconstituent \(E^*_1 V\) as in the following.

### Lemma 4.11

Pick any nonzero vector \(u \in E^*_1 V\) which is orthogonal to all the irreducible modules \(\{W_{j,i}: 1 \leq j \leq m, 2 \leq i \leq m(m-1)\}\). The \(u\) spans a one-dimensional irreducible \(T\)-module. The actions of \(E^*_1 A_0 E^*_1\), \(E^*_1 A_1 E^*_1\), \(E^*_1 A_2 E^*_1\), \(E^*_1 A_3 E^*_1\), \(E^*_0 A_1 E^*_1\), \(E^*_2 A_1 E^*_1\), and \(E^*_3 A_1 E^*_1\) on this \(T\)-module are nonzero and rest are all zero.

Proof: It is straightforward.

### Theorem 4.12

For \((K_m)^{\circ 3}\), \(T \cong M_4(\mathbb{C}) \oplus M_1(\mathbb{C})^{\otimes 6}\).

Proof: It follows from the above remark and lemmas.

### 4.4 The Terwilliger algebra of \((K_m)^{\circ d}\) for \(m \geq 3\)

The description of the concrete Terwilliger algebra of a scheme involves describing the irreducible modules that constitute different subconstituents of the algebra. Earlier in this section, we have already studied all the subconstituents of the 2-class and 3-class wreath power association schemes. From the irreducible modules of \((K_m)^{\circ 3}\) we can describe all irreducible modules for \((K_m)^{\circ 4}\), from \((K_m)^{\circ 4}\) we can describe all irreducible modules for \((K_m)^{\circ 5}\), and so on. We now develop a recursive method to describe the Terwilliger algebra of a \(d\)-class association scheme \((K_m)^{\circ d}\) from a \((d-1)\)-class scheme \((K_m)^{(d-1)}\). Let \(X = (X, \{R_i\}_{0 \leq i \leq d})\) denote the \(d\)-class scheme \((K_m)^{\circ d}\) with

\[
X = [m] \times [m] \times \cdots \times [m] = \{(a_1, a_2, \ldots, a_d) : a_i \in [m], \text{ for } i = 1, 2, \ldots, d\}
\]
Let \((1,1,\ldots,1) \in X\) be a fixed base vertex \(x\) of \(X\). Without loss of generality, we can arrange the label of associate relations and the vertices so that for \(i = 1,2,\ldots,d-1,\)

\[
R_i(x) = \{(a_1,a_2,\ldots,a_{d-i-1},a,1,1,\ldots,1): a_k \in [m] \text{ for } 1 \leq k \leq d-i-1, \ a \in [m]-\{1\}\};
\]

\[
R_0(x) = \{x\};
\]

\[
R_d(x) = \{(a,1,1,\ldots,1): a \in [m]-\{1\}\}.
\]

In the same manner, we can get the relation table for \(d-1\) wreath power as well. First let us look at the relation table of \((K_m)^{(d-1)}\).

\[
\begin{bmatrix}
0 & 1^t_{d-1} & 21^t_{d-2} & \cdots & (d-2)1^t_2 & (d-1)1^t_1 \\
1^t_{d-1} & T_{d-1} & J_{d-1,d-2} & \cdots & J_{d-1,2} & J_{d-1,1} \\
21^t_{d-2} & J_{d-1,d-2}^t & T_{d-2} & \cdots & 2J_{d-2,2} & 2J_{d-2,1} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
(d-2)1^t_2 & J_{d-1,2}^t & 2J_{d-2,2}^t & \cdots & T_2 & (d-2)J_{1,1}^t \\
(d-1)1^t_1 & J_{d-1,1}^t & 2J_{d-2,1}^t & \cdots & (d-2)J_{1,1}^t & T_1
\end{bmatrix}
\]

where \(1_1\) are all-ones column vectors of size \(m^{i-1}(m-1)\), \(J_{j,k}\) are all-ones matrices of size \(m^{j-1}(m-1)\times m^{k-1}(m-1)\), \(T_1 = (d-1)(J_{m-1} - I_m)\) and \(T_i = I_m \otimes T_{i-1} + (d-i)(J_{m-I_m}) \otimes J_{m^2(m-1)}\) for \(i \in \{2,3,\ldots,d-1\}\).

Now using this table, we can describe the relation table for \((K_m)^{id}\) as follows. In the diagonal blocks \(T_i\) make the following changes. The entry 0 is kept same, and the entries \(i\) are replaced with \(i+1\), for all \(i = 1,2,\ldots,d-1\). Let us name the resulting new blocks \(U_i\) for \(i \in \{1,2,\ldots,d-1\}\). It is not hard to see that \(U_i\) are the diagonal blocks of the relation table of \((K_m)^{id}\). Let \(U_d = I_m \otimes U_{d-1} + (J_m - I_m) \otimes J_{m^{d-2}(m-1)}\), \(1^t_d\) is the all-ones column vector of size \(m^{d-1}(m-1)\). Abusing notation and denoting all the all-ones matrices in the relation table as \(J\) for all dimensions the relation table of the \(d\)-class association scheme is

\[
\begin{bmatrix}
0 & 1^t_d & 21^t_{d-1} & 31^t_{d-2} & \cdots & (d-1)1^t_2 & d1^t_1 \\
1^t_d & U_d & J_{d,d-1} & J_{d,d-2} & \cdots & J_{d,2} & J_{d,1} \\
21^t_{d-1} & J_{d,d-1}^t & U_{d-1} & 2J_{d-1,d-2} & \cdots & 2J_{d-1,2} & 2J_{d-1,1} \\
31^t_{d-2} & J_{d,d-2}^t & 2J_{d-2,d-1}^t & U_{d-2} & \cdots & 3J_{d-2,2} & 3J_{d-2,1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
(d-1)1^t_2 & J_{d,2}^t & 2J_{d-1,2}^t & 3J_{d-2,2}^t & \cdots & U_2 & (d-1)J_{1,1}^t \\
d1^t_1 & J_{d,1}^t & 2J_{d-1,1}^t & 3J_{d-2,1}^t & \cdots & (d-1)J_{1,1}^t & U_1
\end{bmatrix}
\]

In the next couple of paragraphs we will discuss the subconstituents of the \(d\)-class association scheme \((K_m)^{id}\). Let \(0_i\) denote zero column vectors of size \(m^{i-1}(m-1)\) for \(1 \leq i \leq d\). Any \(m^d\) dimensional column vectors can be divided into subparts \(1, m^{d-1}(m-1), m^{d-2}(m-1), \ldots, m(m-1)\) and \((m-1)\) respectively so that any vector looks like \((p,p_i^t,\ldots,p_i^t)^t\) where \(p_i\) is a \(m^{i-1}(m-1)\) dimensional column vector for \(1 \leq i \leq d\). With these notations, the primary module of the \(d\)-class association scheme \((K_m)^{id}\) may be described as follows.

**Theorem 4.13** Let \(V\) be the standard module of \((K_m)^{id}\). The vector \((1,0_i^t,\ldots,0_i^t)^t\) and vectors \(q_i = (0,p_i^t,\ldots,p_i^t)^t\) for \(1 \leq i \leq d\) such that

\[
p_j = \begin{cases} 1_i & \text{if } i = j \\ 0_i & \text{if } i \neq j \end{cases}
\]
generates the primary $T$-module.

Proof: Straightforward. ■

Let us consider the $d$-class association scheme $(K_m)^d$ of order $m^d$ and let $V$ denote its standard module. Finding the irreducible modules of the subconstituents $E_i^*V$ for $2 \leq i \leq d$ and $E_0^*V$ is more routine. $E_1^*V$ needs to be treated differently than the other and we will come to that as we go along.

Suppose that the $m^{d-1}$ dimensional column vectors $(0, h_j^t)^t$ for $1 \leq j \leq m^{d-i}(m - 1) - 1$ generate the one dimensional modules of the subconstituent $E_{i-1}^*V$ for the $(d-1)$-class association scheme $(K_m)^{(d-1)}$. If we add the $m^{d-1}(m - 1)$ dimensional column vector $0_d^t$ right after 0 in the above vectors we land up with $m^d$ dimensional vectors. For $1 \leq j \leq m^{d-i}(m - 1) - 1$ the vectors $(0, 0_d^t, h_j^t)$ and $(0, 0_d^t, \ldots, 1_{d-i-1}^t, \ldots, 0_l^t)^t$ span $\mathbb{C}^{m^{d-i}(m - 1)}$. Also, $\langle (0, h_j^t), (0, h_k^t) \rangle = 0$ for $j, k \in \{1, 2, \ldots, m^{d-i}(m - 1) - 1\}$. For the scheme $(K_m)^{(d-1)}$ since $(0, h_j^t)^t$ generates a one dimensional module it is $E_i^* A_j E_h^*$ invariant for all $i, j, h \in \{0, 1, \ldots, d - 1\}$. For $1 \leq j \leq m^{d-i}(m - 1) - 1$, let $W_j^{d_i}$ be the linear span of the vector $(0, 0_d^t, h_j^t)^t$. The embedded structure of the $(d-1)$-class association scheme $(K_m)^{(d-1)}$ in the $d$-class scheme $(K_m)^d$ ensures that for $1 \leq j \leq m^{d-i}(m - 1) - 1$, $(0, 0_d^t, h_j^t)^t$ are $E_i^* A_j E_h^*$ invariant for all $i, j, h \in \{0, 1, \ldots, d\}$. Note that now we are talking about the triple products of the $d$-class scheme $(K_m)^d$. The vectors $(0, 0_d^t, h_j^t)^t$ for $1 \leq j \leq m^{d-i}(m - 1) - 1$ generate all the one dimensional non-primary irreducible modules of the subconstituent $E_i^*V$ for the $D$-class association scheme.

We have so far described the subconstituents $E_0^*V$, $E_2^*V$, $\ldots$, $E_d^*V$ for the scheme $(K_m)^d$. Now, $E_i^*V \cong \mathbb{C}^{m^{d-1}(m - 1)}$. Observe that any $m^{d-1}(m - 1)$ dimensional column vector can be partitioned into $m$ equal parts each of dimension $m^{d-2}(m - 1)$. Let each part be denoted by the index $j$ where $j \in \{1, 2, \ldots, m\}$ so that the $m^{d-1}(m - 1)$ dimensional vector is of the form $(r_1^t, r_2^t, \ldots, r_m^t)^t$ where each $r_j$, $j \in \{1, 2, \ldots, m\}$ is a $m^{d-2}(m - 1)$ dimensional column vector. For each $i \in \{2, 3, \ldots, m^{d-2}(m - 1)\}$ and $j \in \{1, 2, \ldots, m\}$ let $r_{j,i} = (r_1^t, r_2^t, \ldots, r_i^t, \ldots, r_m^t)^t$ denote the $m^{d-1}(m - 1)$ dimensional column vector such that $r_j = \delta_{ij} h_j$ where $(0, h_j^t)$ generates the modules of $E_1^*V$ for the $(d-1)$-class scheme $(K_m)^{(d-1)}$. It is easy to see that each of the vectors $(0, r_{j,i}, 0_{d-i-1}^t, \ldots, 0_m^t)^t$ generates one dimensional irreducible module in the subconstituent $E_1^*V$ for the scheme $(K_m)^d$. These do not constitute all the irreducible modules in $E_1^*V$. So far we have accounted for the following non-isomorphic $T$-modules for $(K_m)^d$.

1. The primary module of dimension $d + 1$.
2. $d - 1$ non-isomorphic $T$-modules of dimension 1 in $E_1^*V$.
3. $d - 1$ non-isomorphic $T$-modules of dimension 1 in $E_2^*V$.
4. $d - 2$ non-isomorphic $T$-modules of dimension 1 in $E_3^*V$.
5. $2$ non-isomorphic $T$-modules of dimension 1 in $E_{d-1}^*V$.
6. $1$ non-isomorphic $T$-modules of dimension 1 in $E_d^*V$.  

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From Theorem \ref{thm:dimT} we have the dimension of $T((K_m)^d)$ is $(d+1)^2 + \frac{1}{2}d(d+1)$. From our above discussion we have $(d+1)^2 + (d-1) + (d-1) + (d-2) + \cdots + 2 + 1$ of the dimension. That leaves us with one non-isomorphic $T$ module of dimension 1 in the subconstituent $E^*_iV$ of multiplicity $m-1$. Pick any nonzero vector $r \in E^*_iV$ which is orthogonal to all the irreducible modules. Then $r$ spans a one-dimensional irreducible $T$-module.

In the discussion above we saw how we could build the irreducible modules of the scheme $(K_m)^d$. We will conclude the section by collecting all our non-isomorphic $T$-modules and describing the Terwilliger algebra of $(K_m)^d$. (Further detailed explanation of the content of this section can be found in \cite{4}.)

**Theorem 4.14** Let $\mathcal{X} = (K_m)^d$ be a $d$-class association scheme of order $m^d$, $m \geq 3$. Then the dimension of $T$ is $(d+1)^2 + \frac{1}{2}d(d+1)$ and $T \cong M_{d+1}(\mathbb{C}) \oplus M_1(\mathbb{C})^\oplus \frac{1}{2}d(d+1)$.

### 4.5 The Irreducible $T$-modules of $(K_2)^d$

We note that for the case $(K_2)^d$ the number of nonzero triple products $E^*_iA_jE^*_h$ are fewer in number than in the general case $(K_m)^d$ with $m \geq 3$. What is nice for $(K_2)^d$ is that instead of just knowing the existence of vectors that generate the irreducible modules, we are actually able to get specific vectors that generate the irreducible $T$-modules.

Let $\mathcal{X} = (K_2)^d$ be a $d$-class association scheme of order $2^d$. Without loss of generality we label the $2^d$ elements of $X$ by $x_1, x_2, \ldots, x_{2^d}$. We fix $x_1$ as the base vertex and we will refer to it as $x$ henceforth. Then, without loss of generality, we can arrange the elements such that $X$ is partitioned with $R_0(x) = \{x\}$, $R_1(x) = \{x_2, \ldots, x_{2^{d-1}+1}\}$, consisting of the $2^{d-1}$ elements, $R_2(x) = \{x_{2^{d-1}+2}, \ldots, x_{2^{d-1}+2^{d-2}+1}\}$ consisting of the next $2^{d-2}$ elements, and so on, with the last part $R_d(x) = \{x_{2^d}\}$.

**Remark 4.4** Let $1$ denote the all-ones vector in the standard module. Then the vector space over $\mathbb{C}$ spanned by $\{E^*_i1 : 0 \leq i \leq d\}$ is a thin irreducible $T$-module of dimension $d+1$ and is the primary module denoted as $\mathcal{P}$; so, by setting $d_i = E^*_i1$, the set $\{d_i : 0 \leq i \leq d\}$ generates $\mathcal{P}$.

In order to describe the irreducible $T$-modules, whose orthogonal direct sum forms the standard module $V$, we employ a particular set of vectors. Let $\hat{x}$ denote the column vector with 1 in the $x$-th position and 0 elsewhere.

**Lemma 4.15** For $l \in \{1, 2, \ldots, d-1\}$ define set of vectors $\{d^l_i\}_{1 \leq i \leq 2^{d-l}-1}$ by

$$d^l_i = \sum_{k=0}^{2^{l-1}-1} \hat{x}_{i+j+k} - \sum_{k=2^{l-1}}^{2^{l-1}-1} \hat{x}_{i+j+k}. $$

For each $i$, the corresponding values of $j$ are successively

$$j = 1, \ 1 + 2^l - 1, \ 1 + 2(2^l - 1), \ 1 + 3(2^l - 1), \ \ldots, \ 1 + (2^{d-l} - 2)(2^l - 1).$$

Then

$$\{d^l_i\}_{1 \leq i \leq 2^{d-l}-1}; \ d^l_i = \hat{x}_{2i} - \hat{x}_{2i+1}$$
\{d_i^2\}_{1 \leq i \leq 2d-2-1}: \quad d_i^2 = \hat{x}_{i+j} + \hat{x}_{i+j+1} - \hat{x}_{i+j+2} - \hat{x}_{i+j+3}; \quad j = 1, 4, 7, \ldots
\{d_i^3\}_{1 \leq i \leq 2d-3-1}: \quad d_i^3 = \sum_{k=0}^{2^3-1-1} \hat{x}_{i+j+k} - \sum_{k=2^3-1}^{2^3-1} \hat{x}_{i+j+k}; \quad j = 1, 8, 15, \ldots
\vdots
\quad d_i^{d-1} = \sum_{i=2}^{2d-2+1} \hat{x}_i - \sum_{i=2d-2-2+2}^{2d-2+1} \hat{x}_i.

In particular, \langle d_i^1, d_h^k \rangle = 0 unless \ h = i \ and \ k = l.

Proof: Proof follows from the construction of the vectors. \hfill \blacksquare

Lemma 4.16 Let \( W_{d_i} \) be the linear span of \( d_i^1 \) for all the vectors described in Lemma 4.15. Then \( W_{d_i} \) is an irreducible \( T \)-module of dimension 1 for \((K_2)^d\).

Proof: To prove that \( W_{d_i} \) is an irreducible \( T \)-module, we look at the action of the nonzero generators \( E_i^* A_j E_h^* \) of the Terwilliger algebra on the vector \( d_i^1 \). We will consider three cases.

1. The case when \( l = 1 \):
   (a) For \( \{d_i^1\}_{1 \leq i \leq 2d-2} \) by the construction of \( d_i^1 \), the action of the triple products \( E_i^* A_i E_h^* \) on \( d_i^1 \) can be nonzero only for \( E_1^* A_1 E_1^* \), \( E_1^* A_1 E_2^* \), \ldots, \( E_1^* A_d E_1^* \), and \( E_i^* A_1 E_i^* \) for \( 2 \leq i \leq d \). Among them \( E_1^* A_1 E_1^* = 0 \). The generators \( E_i^* A_j E_h^* \) of \( T \) that act on \( d_i^1 \) in a nonzero manner are \( E_1^* A_0 E_i^* \) and \( E_i^* A_d E_i^* \) with
   \[
   E_1^* A_0 E_i^* d_i^1 = d_i^1, \quad E_i^* A_d E_i^* d_i^1 = -d_i^1.
   \]
   (b) For \( \{d_i^1\}_{2d-2+1 \leq i \leq 2d-2+2d-3} \) by the construction of \( d_i^1 \), the action of the triple products \( E_i^* A_h E_h^* \) on \( d_i^1 \) are nonzero only for \( E_2^* A_0 E_2^* \) and \( E_2^* A_d E_2^* \), and act in the following manner:
   \[
   E_2^* A_0 E_2^* d_i^1 = d_i^1, \quad E_2^* A_d E_i^* d_i^1 = -d_i^1.
   \]
   (c) For every set of vectors corresponding to the different subconstituents will follow the same pattern. The last case is the following. For \( d_{d-1+1}^1 \) the generators \( E_i^* A_j E_h^* \) of \( T \) that act on \( d_{d-1+1}^1 \) in a nonzero manner for
   \[
   E_{d-1}^* A_0 E_{d-1}^* d_i^1 = d_i^1, \quad E_{d-1}^* A_d E_{d-1}^* d_i^1 = d_i^1.
   \]

2. The case when \( l \in \{2, 3, \ldots, d-2 \} \):
   (a) For \( \{d_i^1\}_{1 \leq i \leq 2d-(l+1)} \) by the construction of \( d_i^1 \), the action of the triple products \( E_i^* A_j E_h^* \) on \( d_i^1 \) can be nonzero only for \( E_1^* A_0 E_i^* \), \( E_1^* A_1 E_1^* \), \ldots, \( E_1^* A_d E_i^* \). Among them \( E_1^* A_1 E_1^* = 0 \). The generators \( E_i^* A_j E_h^* \) of \( T \) that act on \( d_i^1 \) in a nonzero manner are \( E_1^* A_0 E_i^* \), \( E_1^* A_d E_i^* \), \( E_1^* A_{d-1} E_1^* \), \ldots, \( E_1^* A_{d-(l-1)} E_1^* \) in the following manner:
   \[
   E_1^* A_0 E_1^* d_i^1 = d_i^1, \quad E_i^* A_d E_i^* d_i^1 = d_i^1, \quad E_{d-1}^* A_{d-j} E_i^* d_i^1 = 2^j d_i^1 \text{ where } 1 \leq j \leq l-1.
   \]
(b) For \( \{d^l_i\}_{2^d-(l+1)+1 \leq i \leq 2^d-(l+1)+2^d-(l+2)} \) by the construction of \( d^l_i \), the action of the triple products \( E^*_A A_j E^*_h \) on \( d^l_i \) can be nonzero only for \( E^*_A A_0 E^*_2, E^*_A A_1 E^*_2, \ldots, E^*_A A_d E^*_2 \). Among them \( E^*_A A_1 E^*_2 = 0 \). The generators \( E^*_A A_j E^*_h \) of \( T \) that act on \( d^l_i \) in a nonzero manner are \( E^*_A A_0 E^*_2, E^*_A A_d E^*_2, E^*_A A_{d-1} E^*_2, \ldots, E^*_A A_{d-(l-1)} E^*_2 \) in the following manner:

\[
E^*_A A_0 E^*_2 d^l_i = d^l_i, \quad E^*_A A_d E^*_2 d^l_i = d^l_i, \quad E^*_A A_{d-j} E^*_2 d^l_i = 2^j d^l_i \quad \text{for } 1 \leq j \leq l - 2,
\]

and

\[
E^*_A A_{d-(l-1)} E^*_2 d^l_i = -2^{l-1} d^l_i.
\]

(c) Following a similar reasoning the last case will be as follows. For \( d^l_{2^d-l-1} \) the generators \( E^*_A A_j E^*_h \) of \( T \) that act on \( d^l_{2^d-l-1} \) in a nonzero manner are \( E^*_A A_0 E^*_2 \) and \( E^*_A A_d E^*_2 \) in the following manner:

\[
E^*_A A_0 E^*_2 d^l_{2^d-l-1} = d^l_{2^d-l-1}, \quad E^*_A A_d E^*_2 d^l_{2^d-l-1} = d^l_{2^d-l-1},
\]

\[
E^*_A A_{d-j} E^*_2 d^l_{2^d-l-1} = 2^j d^l_{2^d-l-1} \quad \text{for } 1 \leq j \leq l - 2,
\]

and

\[
E^*_A A_{d-(l-1)} E^*_2 d^l_{2^d-l-1} = -2^{l-1} d^l_{2^d-l-1}.
\]

3. The case when \( l = d - 1 \):

By the construction of \( d^{d-1}_1 \), the action of the triple products \( E^*_A A_j E^*_h \) on \( d^{d-1}_1 \) can be nonzero only for \( E^*_A A_0 E^*_1, E^*_A A_1 E^*_1, \ldots, E^*_A A_d E^*_1 \). Among them \( E^*_A A_1 E^*_1 = 0 \). The generators \( E^*_A A_j E^*_h \) of \( T \) that act on \( d^{d-1}_1 \) in a nonzero manner are \( E^*_A A_0 E^*_1, E^*_A A_{d-1} E^*_1, \ldots, E^*_A A_2 E^*_1 \) in the following manner:

\[
E^*_A A_0 E^*_1 d^{d-1}_1 = d^{d-1}_1, \quad E^*_A A_{d-j} E^*_1 d^{d-1}_1 = 2^j d^{d-1}_1 \quad \text{for } 1 \leq j \leq d - 3,
\]

and

\[
E^*_A A_2 E^*_1 d^{d-1}_1 = -2^{d-2} d^{d-1}_1.
\]

It is clear from the above cases that each of \( W_{d^l} \) is a \( T \)-module. They are irreducible since they are vector spaces generated by a single vector. This completes the proof.

As a consequence, we have the following decomposition of the standard module.

**Theorem 4.17** Let \( (K_2)^d \) be a \( d \)-class association scheme of order \( 2^d \). For \( l \in \{1, 2, \ldots, d - 1\} \) define set of vectors \( \{d^l_i\}_{1 \leq i \leq 2^d-l-1} \) by

\[
d^l_i = \sum_{k=0}^{2^d-l-1} x_{i+j+k} - \sum_{k=2^d-l-1}^{2^d-1} x_{i+j+k}.
\]

For each \( i \) the corresponding values of \( j \) are successively

\[
j = 1, \; 1 + 2^l - 1, \; 1 + 2(2^l - 1), \; 1 + 3(2^l - 1), \; \ldots, \; 1 + (2^{d-l} - 2)(2^l - 1).
\]

Let \( W_{d^l} \) be the linear span of \( d^l_i \).
(1) Let \( V \) be the standard module and \( \mathcal{P} \) be the primary module. Then
\[
V = \mathcal{P} \oplus \sum W_\nu
\]
where \( \nu \) over all \( d_i^j \) defined above.

(2) For \( l \in \{1, 2, \ldots, d-1\} \)

(a) \( W_\nu \) where \( \nu \in \{d_i^j : 1 \leq i \leq 2^{d-l+1}\} \) are \( T \)-isomorphic.

(b) \( W_\nu \) where \( \nu \in \{d_i^j : 2^{d-l+1}+1 \leq i \leq 2^{d-l+1}+2^{d-l+2}\} \) are \( T \)-isomorphic. Following above we finally have

(c) \( W_{d_i^{2d-l-3}} \) and \( W_{d_i^{2d-l-2}} \) are \( T \)-isomorphic.

(d) Rest of \( W_\nu \) are not \( T \)-isomorphic.

Proof: (1) is straightforward by the construction of the modules which are mutually orthogonal. (2) The proof is similar when we choose \( T \)-modules from same groups as described in cases (a)-(c). We shall show that \( W_{d_i^1} \) and \( W_{d_i^2} \) are \( T \)-isomorphic. Define an isomorphism
\[
\sigma : W_{d_i^1} \rightarrow W_{d_i^2} \quad \text{by} \quad \sigma(d_i^1) = d_i^2.
\]
We need to show that \( (\sigma B - B \sigma)W = 0 \) for all \( B \in T \). Let us consider the action of nonzero \( E_i^1 A_j E_k^1 \) on \( \sigma \); i.e., \( E_i^1 A_0 E_i^1, E_i^1 A_d E_i^1, E_i^1 A_d-1 E_i^1, \ldots, E_i^1 A_d-(l-1) E_i^1 \). Now
\[
(\sigma E_i^1 A_0 E_i^1 - E_i^1 A_0 E_i^1 \sigma)W_{d_i^1} = (\sigma - E_i^1 A_0 E_i^1 \sigma)W_{d_i^1} = d_i^2 - E_i^1 A_0 E_i^1 (W_{d_i^2}) = 0.
\]
Similar kind of reasoning shows that for all \( E_i^1 A_0 E_i^1, E_i^1 A_d E_i^1, E_i^1 A_d-1 E_i^1, \ldots, E_i^1 A_d-(l-1) E_i^1, \)
\[
(\sigma E_i^1 A_j E_k^1 - E_i^1 A_0 E_i^1 \sigma)W_{d_i^1} = 0.
\]
Thus, \( W_{d_i^1} \) and \( W_{d_i^2} \) are \( T \)-isomorphic.

Next we shall show that modules selected from different groups are not \( T \)-isomorphic. In particular, let us show that \( W_{d_i^1} \) and \( W_{d_i^{2d-l+1}} \) are not \( T \)-isomorphic. Suppose we assume that there exists an isomorphism
\[
\sigma : W_{d_i^1} \rightarrow W_{d_i^{2d-l+1}} \quad \text{such that} \quad (\sigma B - B \sigma)W_{d_i^1} = 0 \quad \text{for all} \quad B \in T.
\]
Then for \( E_i^1 A_0 E_i^1 \in T \),
\[
(E_i^1 A_0 E_i^1)W_{d_i^1} = W_{d_i^1} \quad \text{and} \quad (E_i^1 A_0 E_i^1)W_{d_i^{2d-l+1}} = 0.
\]
Now,
\[
[\sigma(E_i^1 A_0 E_i^1) - (E_i^1 A_0 E_i^1)\sigma]W_{d_i^1} = [\sigma - (E_i^1 A_0 E_i^1)\sigma]W_{d_i^1} = W_{d_i^{2d-l+1}} - 0 \neq 0
\]
which is a contradiction to our assumption. Thus, \( W_{d_i^1} \) and \( W_{d_i^{2d-l+1}} \) are not \( T \)-isomorphic. The other cases can be proved with a similar approach.

\[\text{Theorem 4.18} \quad \text{Let} \ \mathcal{X} = (K_2)^d \ \text{be a} \ d \text{-class association scheme of order} \ 2^d. \]
\[
\mathcal{T} \cong M_{d+1}(\mathbb{C}) \oplus M_1(\mathbb{C})^{\otimes \frac{1}{2}d(d-1)}.
\]

Proof: It is straightforward.

\[\blacksquare\]
5 Concluding Remarks

There is further work that is needed on the theme related to our work. Here we state a few problems that are of our interest.

1. In our attempt to describe the Terwilliger algebra of the $d$-class association scheme $(K_m)^d$ our base was the Terwilliger algebra described by Tomiyama and Yamazaki [22] for a 2-class association scheme constructed from a strongly regular graph. Although our 3-class association scheme was neither strongly regular nor a $P$-polynomial scheme, we were able to describe the Terwilliger algebra concrete manner largely because of the fact that $d$-class association scheme $(K_m)^d$ turned out to be triply regular, and the structure of the $(d-1)$-class was beautifully embedded in the $d$-class association scheme. We demonstrated further how we could extend the same method used to describe the Terwilliger algebra of the 3-class association scheme to the $d$-class association scheme $(K_m)^d$ for $d > 3$.

The general $d$-class wreath product scheme $K_{n_1} \wr K_{n_2} \wr \cdots \wr K_{n_d}$ is also triply regular, and its Terwilliger algebra has the same dimension as in the case of the wreath power $(K_m)^d$. However, it seems to be much more involved to describe the irreducible $T$-modules for the general wreath product scheme. The method used for the case $(K_m)^d$ does not work with different $n_i$’s. We do not know how to find vectors that generate the irreducible $T$-modules. However, Paul Terwilliger believes that all non-primary modules still have dimension 1. If that is the case, then the structure of the Terwilliger algebra of $K_{n_1} \wr K_{n_2} \wr \cdots \wr K_{n_d}$ is also the same as that of $(K_m)^d$.

2. In a slightly different direction, it will be interesting to look at some specific schemes obtained by taking the wreath power of two association schemes, such as the Hamming $H(2,q)$ instead of $H(1,q)$. We know that the Terwilliger algebra of a Hamming scheme can be described as symmetric $d$-tensors on the Terwilliger algebra of $H(1,q)$ [15], although in general $H(d,q)$ is not realized as a product of $H(1,q)$. It would be interesting to see how the Terwilliger algebra changes when we take the wreath power of a Hamming scheme $H(d,q)$ for an arbitrary $d > 1$.

3. There are also other products besides the wreath product. It is also an interesting problem to look at the direct power of $H(1,q)$. We study the direct power first because the direct product of two association schemes has a lot more classes than the wreath product. Namely, the direct product of a $d$-class association scheme and a $e$-class association scheme is of class $de + d + e$ while the wreath product becomes $(d + e)$-class association scheme. So a study of direct power requires a lot more work than that of wreath power. However, it may be worthy to look at it now as we know more about the schemes related to $H(1,q)$.

4. In [15], the irreducible $T$-modules and Terwilliger algebra has been investigated for the Doob schemes. The Doob schemes are the association schemes obtained by taking the direct product of copies of $H(2,4)$ and copies of schemes coming from the Shrikhande graph. In this case, the direct product of these schemes preserves many properties of the original factor schemes. One important property that is remained as the same is $P$-polynomial property. In terms of graphs, the Hamming $H(2,4)$ and the Shrikhande graphs are the only distance-regular graphs whose direct product is also distance-regular. This property no longer holds for the direct product of other Hamming Schemes. Nevertheless, the description of the
Terwilliger algebra of Doob schemes in terms of those of $H(2,4)$ and Shrikhande scheme may shed a light in understanding how the Terwilliger algebra of the product behaves when we study the Terwilliger algebra of the direct product of $H(d,q)$s with various $d$.

5. As explained by Eric Egge [10] and introduction of [19] it is possible to define an “abstract version” of the Terwilliger algebra using generators and relations. In all cases the concrete Terwilliger algebra is a homomorphic image of the abstract Terwilliger algebra, and in some cases they are isomorphic. In the case of $(K_m)^{id}$ the entire structure of the Terwilliger algebra is determined by the intersection numbers and Krein parameters, so it may be easy to see what is going on. Once all the vanishing intersection numbers and Krein parameters are worked out, we can obtain the defining relations for the algebra and we no longer need to consider the combinatorial structure further. Terwilliger believes that for the association schemes considered in the current paper, the abstract Terwilliger algebra and the concrete Terwilliger algebra are isomorphic. It is remained to study the Terwilliger algebras (basis, irreducible modules, dimension) from the generators/relations alone for the association schemes considered in this paper.

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