Yang-Mills Fields on the 4-dimensional torus.
Part I: Classical Theory

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Abstract We review some of the most important results obtained over the years on the study of Yang-Mills fields on the four dimensional torus at the classical level.

1This contains the first part of the lectures given by the author at the 1997 Advanced School on Non-perturbative Quantum Physics (Peñiscola)
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1 Introduction

Yang-Mills fields play a crucial role in our understanding of Particle Physics. Under a relatively simple and elegant formulation, a rich behaviour hides. Phenomena which are of great importance in the real world, such as Confinement and chiral symmetry breaking, are supposed to follow from the dynamics of these fields. However, these phenomena are highly non-perturbative and still defy ourselves to try to derive them properly from first principles. In trying to address such complicated issues a good deal of new ideas and strategies have been put forward by different authors over the years. Some of these techniques constitute a major breakthrough in our capacity to derive quantitative and qualitative approaches to Quantum Field Theory (QFT). What in our opinion is the most important new method which arose in this context was the formulation of Yang-Mills theories and other QFTs on a space-time lattice [1]. This method, not only gave rise to new non-perturbative methods of computation, but also provided a new illuminating way of understanding the phenomena taking place in Quantum Field Theory. Other major contribution followed the work of Polyakov, who argued about the relevance of classical configurations in computing and explaining some effects in QFT [2, 3]. Most notably the so-called pseudo-particles, of which the best known example is the instanton [4], are known to play an important role in some statistical mechanical systems and are expected to do the appropriate in QFT.

In this review we will study a different strategy put forward by ‘t Hooft [5, 6, 7]. He considered Yang-Mills fields living in an Euclidean space-time torus $T_4$. In ’t Hooft’s original presentation, the interest of this approach lies in the characterization of Confinement that it provides, and the information it supplies on the possible phases of Yang-Mills theory. In very brief terms, we can say that the topology of the torus gives rise to a non-trivial topology in the space of Yang-Mills fields which has an appealing physical interpretation. The way we see it nowadays, this is just one aspect of the interest that a formulation of Yang-Mills fields on the torus can have. First of all, the torus provides a gauge invariant infrared cut-off of the theory. For that purpose, we could have taken any other Riemannian 4-dimensional compact manifold, but the torus has several advantages which we will now list:

- The group of traslations is abelian, and hence one might still use Fourier decompositions. Furthermore, one can use a flat metric in all points of
space-time.

- One can regard all configurations on the torus as configurations in $\mathbb{R}^4$ which are periodic. In other words, one can glue up configurations defined on the torus, to make up configurations on $\mathbb{R}^4$.

- In the lattice formulation of gauge theories, numerical methods demand a finite number of degrees of freedom. Henceforth, one is forced to put the system in a finite volume. For reasons similar to the ones mentioned before, one usually imposes periodic boundary conditions, which is equivalent to having the system living on the torus. The study of Yang-Mills on the torus allows us to understand some of the observed finite size effects. One can take the continuum limit, in a way such that the size of the torus stays finite in physical units.

- Topology of the torus is non-trivial (both $\pi_1(T_4)$ and $H_i(T_4)$ are non-zero). This allows the topological characterization of some properties of the theory, including Confinement. This is basically ’t Hooft’s original motivation. In addition, as is the case in other theories, boundary conditions serve to stabilize several classical configurations.

- A curious phenomenon takes place in the large N limit. Eguchi and Kawai [8] argued, that in that limit, some quantities might have no finite-size effects. Hence, the torus would give results which are identical to those of $\mathbb{R}^4$.

- The size of the torus provides a parameter, which can be used to interpolate between the perturbative (small size) region and the non-perturbative (big size) one. This can provide a method for approximately calculating some non-perturbative quantities, as argued by Lüscher [4]. In addition, if the ideas of the author are correct, this can provide an insight into the structure of the Yang-Mills vacuum and the origin of Confinement [10].

To conclude, we can say that the study of Yang-Mills on the 4-D torus serves as an example of the behaviour of the fields on other non-simply connected manifolds. The latter can be relevant in other contexts.

In these lectures, we will review the most important results obtained by different authors on this topic. They can be split into two parts: classical and
quantum theory. In this paper we will present the first part. The quantum part, which will follow, will make use of the results presented here.

Our aim is to make a pedagogical and self-contained introduction to the subject. Accordingly, we will present all the ideas and results in the most simple language. We will nevertheless frequently comment upon the more precise mathematical statement of those results. In other cases, we will refer the readers to the relevant publications.

The layout of the paper is shown in the Contents.

2 Notation and general formulas

Let us consider a 4-dimensional torus $T_4$ of size $l_0 \times l_1 \times l_2 \times l_3$. The metric is Euclidean and we take coordinates $x (x_\mu$ for $\mu = 0 \ldots 3$) on the torus as follows:

$$0 \leq x_\mu < l_\mu .$$

Let us introduce the symbol $\hat{\mu} \equiv (0, \ldots, l_\mu, 0, \ldots)$ which has all but the $\mu$th component equal to zero.

We now consider Yang-Mills fields defined on $T_4$. For simplicity we will restrict ourselves to the $SU(N)$ gauge groups. We will express the vector potential fields in the usual matrix form:

$$A_\mu(x) = A_\mu^a(x) \lambda_a ,$$

where $\lambda_a$ are the generators of the $SU(N)$ Lie algebra in the fundamental representation, normalized by:

$$Tr(\lambda_a \lambda_b) = \frac{\delta_{ab}}{2}$$

$$[\lambda_a, \lambda_b] = i f_{abc} \lambda_c .$$

The expression for the field tensor $F_{\mu\nu}(x)$ in terms of the vector potential $A_\mu(x)$ is given by:

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - i [A_\mu(x), A_\nu(x)] .$$

Gauge transformations operate on the vector potential $A_\mu(x)$ as follows:

$$[\Omega(x)] A_\mu(x) = \Omega(x) A_\mu(x) \Omega^+(x) + i \Omega(x) \partial_\mu \Omega^+(x) .$$
In a more precise mathematical language, Yang-Mills fields on the torus are connections on a principal fiber bundle whose base space is $T_4$. A good introduction to this formulation aimed to an audience of physicists can be found in Ref. [11]. In order to formulate the theory, one has first to define an $SU(N)$ principal fiber bundle over $T_4$. This can be done by giving a covering of the torus by open contractible sets $U_\alpha$. These sets can be considered gauge-coordinate patches. In the overlapping regions ($x \in U_\alpha \cap U_\beta$), the consistency of the two descriptions implies that the two gauge-coordinates are related by an $SU(N)$ gauge transformation $\Omega_{\alpha\beta}(x)$, which is called a transition function. The open sets and the transition functions define the principal bundle.

Connections define a way to parallel transport from one point of space to another along some path. Within one patch $U_\alpha$ the parallel transport matrices are given by the well-known ordered exponential formulas in terms of the vector potentials $A^{(\alpha)}_\mu(x)$ for this patch:

$$U(\gamma : P \to Q) = T \exp \left( -i \int_\gamma A^{(\alpha)}_\mu(x) dx_\mu \right),$$

where ordering within the path goes from left to right. In the overlapping regions between two patches ($x \in U_\alpha \cap U_\beta$), one might use instead the vector potentials $A^{(\beta)}_\mu(x)$, related to the previous ones by a gauge transformation involving the transition matrices:

$$A^{(\beta)}_\mu(x) = [\Omega_{\beta\alpha}(x)] A^{(\alpha)}_\mu(x),$$

which can be considered a change of coordinates for the gauge fields. The link with our original presentation follows by identifying our vector potential $A_\mu(x)$ with that of a single patch which covers the whole torus.

### 3 Boundary Conditions and Twist

By continuity one can extend $A_\mu(x)$ and $F_{\mu\nu}(x)$ to the boundary of the hypercube $[0, l_0] \times [0, l_1] \times [0, l_2] \times [0, l_3]$. However, one must realize that for a point $x$ located on the face $x_\nu = 0$, there is a corresponding point $x + \hat{\nu}$ which labels the same point on the torus. Henceforth, the gauge fields $A_\mu(x)$ and $A_\mu(x + \hat{\nu})$ have to be physically equivalent. For gauge fields, equivalence
does not imply equal vector potentials. It is enough if they are equal modulo a gauge transformation:

\[ \mathbf{A}_\mu(x + \hat{\nu}) = [\Omega_\nu(x)]\mathbf{A}_\mu(x) \]  

where \( \Omega_\nu(x) \) are elements of the SU(\( N \)) group which depend only on the transverse coordinates \( x_\rho \) with \( \rho \neq \nu \). These matrices are known as twist matrices.

If we now consider a point \( x \) located at the two-dimensional surface \( x_\nu = x_\rho = 0 \), consistency of the previous boundary conditions demands:

\[ \mathbf{A}_\mu(\hat{\nu} + \hat{\rho}) = [\Omega_\nu(x + \hat{\rho})\Omega_\rho(x)]\mathbf{A}_\mu(0) = [\Omega_\rho(x + \hat{\nu})\Omega_\nu(x)]\mathbf{A}_\mu(0) . \]  

This implies

\[ \Omega_\rho(x + \hat{\nu})\Omega_\nu(x) = z_{\rho\nu} \Omega_\nu(x + \hat{\rho})\Omega_\rho(x) , \]  

where the constants \( z_{\rho\nu} \) are phases which can occur due to the quadratic way in which \( \Omega \) enters in the gauge transformation formula Eq. 5. Since \( \Omega_\rho(x + \hat{\nu})\Omega_\nu(x) \) and \( \Omega_\nu(x + \hat{\rho})\Omega_\rho(x) \) belong to SU(\( N \)), the phases \( z_{\rho\nu} \) are elements of the center of this group (\( \mathbb{Z}_N \)). Hence, we can express them as

\[ z_{\mu\nu} = \exp\{2\pi i \frac{n_{\mu\nu}}{N}\} , \]  

where \( n_{\mu\nu} \) is an antisymmetric tensor of integers defined modulo \( N \). This tensor has 6 independent coefficients, which can be expressed as usual in terms of two independent three vectors (\( \vec{k} \) and \( \vec{m} \)) as follows:

\[ n_{ij} = \epsilon_{ijk} m_k \]
\[ n_{0i} = k_i . \]  

We will refer to these boundary conditions as twisted boundary conditions, and to \( n_{\mu\nu} \) as the twist tensor. The elements of the tensor \( n_{\mu\nu} \), being integers, label topologically inequivalent sectors. The particular case \( n_{\mu\nu} = 0 \) is often referred as no-twist. In particular, choosing the twist matrices equal to the identity, corresponds to this case. The gauge fields are then said to be strictly periodic.

Under gauge transformations, the vector potentials transform as shown in Eq. 6, but the twist matrices also change as follows:

\[ \Omega_\mu(x) \rightarrow \Omega'_\mu(x) = \Omega(x + \hat{\nu})\Omega_\mu(x)\Omega^+(x) . \]  

7
However, notice that the change of twist matrices does not produce any change of the twist tensor $n_{\mu\nu}$.

In addition to gauge transformations, there is another symmetry group present. It is the group of transformations:

$$\Omega_\mu(x) \rightarrow \Omega'_\mu(x) = z_\mu \Omega_\mu(x),$$

where $z_\mu$ are elements of the center. This group is isomorphic to $\mathbb{Z}_N^4$ and plays an important role in what follows.

The previous introduction of twisted boundary conditions follows very closely the one made by 't Hooft in his first papers on the subject [5, 6]. Using the language of connections on fiber bundles, the twist matrices are simply compositions of different transition functions in the limit when one of the patches covers the whole torus [12, 13]. In this respect, the consistency condition Eq. 9 is related to the so-called co-cycle condition, that transition matrices have to satisfy:

$$\Omega_\alpha\gamma = \Omega_\alpha\beta \Omega_{\beta\gamma}.$$

The occurrence of the phases $z_{\mu\nu}$ and of the twist tensor, has to do with the fact that gauge transformations of the vector potentials are insensitive to the center of the group $\mathbb{Z}_N$. Henceforth, one could consider the gauge group (structure group) as being $SU(N)/\mathbb{Z}_N$. A non-zero twist tensor can be seen as an obstruction to go from an $SU(N)/\mathbb{Z}_N$ principal fiber bundle to an $SU(N)$ bundle. This obstruction can be related to the second homology class of the base manifold with coefficients in the center of the group $H_2(T_4, \mathbb{Z}_N)$. For a proof of these facts see for example Ref. [14].

4 Twist Matrices

The first problem which we will address is the form and existence of solutions to the twist matrix equations Eq. 10. We will consider two classes of solutions which will be relevant in what follows. The first class is given by abelian space-dependent matrices. The second class is given by constant non-abelian matrices: twist eaters.

4.1 Abelian twist matrices

Here we will show that there is always a solution to the problem of finding twist matrices satisfying Eq. 10. This solution is given in terms of commuting
matrices, and is hence referred as abelian. Let us set:

\[ \Omega_\mu(x) = \exp \{ i \omega_\mu(x) \}, \]  

(16)

with \([\omega_\mu(x), \omega_\nu(x)] = 0\). If we diagonalize the matrices \(\omega_\mu\), the twist condition Eq. [10] amounts to the following condition on its eigenvalues \(\omega_a^\mu\):

\[ \Delta_\nu \omega_a^\mu(x) - \Delta_\mu \omega_a^\nu(x) = \frac{2\pi n_{\mu\nu}}{N} + 2\pi q_{a\mu\nu}^\nu, \]  

(17)

where \(Q_{\mu\nu} = \text{diag}(q_{a\mu\nu})\) is an integer diagonal matrix satisfying \(\text{Tr}(Q_{\mu\nu}) = -n_{\mu\nu}\), and the symbol \(\Delta_\mu\) is defined as:

\[ \Delta_\mu \phi(x) = \phi(x + \hat{\mu}) - \phi(x). \]  

(18)

A particular solution of the previous equation is given by:

\[ \omega_\mu(x) = \frac{\pi}{N} n_{\mu\nu} \frac{x_\nu}{l_\nu} T, \]  

(19)

with \(T = \text{diag}(1, \ldots , 1, 1 - N)\). This solution corresponds to the choice \(q_{a\mu\nu}^a = 0\) for \(a \neq N\) and \(q_{a\mu\nu}^N = -n_{\mu\nu}\). For arbitrary value of the integers \(q_{a\mu\nu}^a\) one gets:

\[ \omega_\mu(x) = \frac{\pi}{N} n_{\mu\nu} \frac{x_\nu}{l_\nu} T_{\mu\nu}, \]  

(20)

with

\[ T_{\mu\nu} = n_{\mu\nu} I + N Q_{\mu\nu}. \]  

(21)

A general solution can be obtained by adding to the previous solutions a term \(\Delta_\mu \phi(x)\), which is a general solution of the homogenous equation. This general solution can be easily seen to be a gauge transformation of the solution expressed in Eqs. 20-21.

4.2 Twist eaters

A class of solutions which plays an important role in the following are the so-called twist eaters or twist eating solutions. These are constant matrices \(\Omega_\mu(x) = \Gamma_\mu\) satisfying:

\[ \Gamma_\mu \Gamma_\nu = \exp \left\{ \frac{2\pi i n_{\mu\nu}}{N} \right\} \Gamma_\nu \Gamma_\mu. \]  

(22)
The previous equation defines a group, whose generators are $\Gamma_{\mu}$ and the phase $\exp\{2\pi i/N\}$, and satisfying the relations Eq. 22 involving $\varphi_{\mu\nu} = \frac{n_{\mu\nu}}{N}$. We will call this group the twist group $G$. Notice that the Clifford Algebra is a particular case of this class of groups. Our solutions can be looked at as unitary representations of this group. Hence, we can use the standard methods and ideas on representations. We will actually be more general that required and work in arbitrary space-time dimension $d$ ($\mu = 0 \ldots (d - 1)$).

Given one solution $\Gamma_{\mu}$ one can construct another one by means of:

$$\Gamma_{\mu} \rightarrow \Gamma'_{\mu} = \Omega \Gamma_{\mu} \Omega^+ \quad \text{(23)}$$

$$\Gamma_{\mu} \rightarrow \Gamma'_{\mu} = z_{\mu} \Gamma_{\mu} \quad \text{(24)}$$

with $\Omega \in SU(N)$ and $z_{\mu} \in \mathbb{Z}_N$. Furthermore, given two solutions with $\varphi_{\mu\nu} = \frac{n_{\mu\nu}^{(1)}}{N_1} = \frac{n_{\mu\nu}^{(2)}}{N_2}$, one can construct the direct sum, which is an $N_1 + N_2$ dimensional representation. The new representation is reducible. One needs only to find the irreducible representations and the rest can be formed by direct sum.

The choice of generators is not unique, and thus different sets of $\varphi_{\mu\nu}$ correspond to the same group. To investigate this point let us introduce the following notation for the product of the generators:

$$\Gamma(n) = \Gamma_0^n \Gamma_1^{n_1} \ldots \Gamma_{d-1}^{n_{d-1}} \quad \text{(25)}$$

where $n_{\mu}$ are integers. The matrices $\Gamma(n)$ satisfy:

$$\Gamma(n) \Gamma(m) = \Gamma(m + n) \exp\{\frac{2\pi i}{N} \sum_{\mu > \nu} n_{\mu} m_{\nu} n_{\mu\nu} \} \quad \text{(26)}$$

The matrices of the form $z \Gamma(n)$ form the group $G$. One might choose other generators $\Gamma'_{\mu} = \Gamma(s^{(\mu)})$ provided the matrix $s^{(\mu)}$ is invertible. Notice that if we use $\Gamma'_{\mu}$ instead of $\Gamma_{\mu}$, the twist tensor changes to:

$$n_{\mu\nu} \rightarrow n'_{\mu\nu} = \sum s_{\mu}^{(\rho)} s_{\nu}^{(\sigma)} n_{\rho\sigma} \quad \text{(27)}$$

For irreducible representations, by Schur lemma, all the matrices which commute with the generators (elements of the center) must be multiples of the identity. In particular, any matrix of the group taken to the $N$th power belongs to the center. This allows the numbers $s_{\mu}^{(\mu)}$ to be considered integers mod$N$. This enlarges the group of transformations of generators from
$SL(d, \mathbb{Z})$ to $GL(d, \mathbb{Z}_N)$. One can use these transformations to bring $\varphi_{\mu\nu}$ to a canonical form:

$$
\varphi_{\mu\nu}(\text{canonical}) = \begin{pmatrix}
0 & \Delta & 0 \\
-\Delta & 0 & 0 \\
0 & \ddots & 0 \\
0 & 0 & 0
\end{pmatrix},
$$

(28)

where $\Delta = \text{diag}(\frac{1}{p_1}, \ldots, \frac{1}{p_r})$. The integers $p_i > 1$ are such that $p_i$ divides $p_j$ for $i > j$. The value of $r \leq \left[\frac{d}{2}\right]$ will be referred to as the rank of the group $G$. An important subgroup is the center $C$ of $G$. It is generated by $(\Gamma_i' - 1)^{p_i}$ for $i = 1, \ldots, r$ and $\Gamma_d'\ldots\Gamma_2'$. For irreducible representations, all these matrices have to be a multiple of the identity $\tau_\mu I$, and the constants $\tau_\mu$ characterize the representation in question.

Now let us use the standard strategy of considering the maximal abelian subgroup $\mathcal{A}$. It is generated by $\Gamma_i', i = 1, \ldots, r$ (factoring out the center). Let $|\vec{\lambda}\rangle$ denote a vector which is a simultaneous eigenstate of these matrices with eigenvalues $\lambda_{i-1}$ (which can be written as an $r$ dimensional weight vector $\vec{\lambda}$). The matrices $\Gamma_{r+k-1}'$ now act on this state and transform it into another eigenstate with eigenvalue $(t_k \vec{\lambda})_i \equiv \lambda_{i-1} \exp\left\{\frac{2\pi i}{p_i} \delta_{ik}\right\}$. In this way, given one state $|\vec{\lambda}_0\rangle$ one generates $p_1 p_2 \ldots p_r$ different states $|\vec{\lambda}\rangle$ (Notice that $\lambda_{i-1} \neq 0$ since it is a representation of a group, and the matrices have to be invertible). Now we can normalize these eigenstates so that:

$$
\Gamma_{r+k-1}'|\vec{\lambda}\rangle = \chi_k t_k |\vec{\lambda}\rangle.
$$

(29)

The definition is consistent because the $\Gamma_{r+k-1}'$ with different $k$ commute. We have to insert $\chi_k$ because $\chi^p_k = \tau_{r+k-1}$ is a Casimir, which is not necessarily equal to 1.

Thus, we have explicitly constructed the irreducible representations of the twist group. The dimension of the representation is:

$$
N_0 = p_1 \times p_2 \times \ldots \times p_r.
$$

(30)

Henceforth, supposing that there is a solution in $N \times N$ matrices, then the representation is irreducible if and only if $N = N_0$. One can construct reducible solutions in all dimensions which are multiples of $N_0$.

The irreducible representation is unique modulo similarity transformations (Eq. 23) and multiplication of the matrices by a constant (Eq. 24).
Representations that are related by a similarity transformation are called equivalent. As we can see, equivalent representations are labelled by the Casimirs: \( \tau_{i-1} = \chi_{i-1}^\mu \), \( \tau_{r+i-1} = \chi_i^\mu \) (for \( i = 1 \ldots r \)) and \( \tau_\mu \) (for \( \mu = 2r, \ldots, d-1 \)). For unitary representations these Casimirs are phases. We are mainly interested in representations in terms of \( SU(N) \) matrices. In that case, the phases have a discrete set of values (\( \in Z_N \)) and the number of inequivalent irreducible representations is discrete. We will now calculate how many of them there are, since this will turn out to be relevant later on. We get:

\[
\# \text{ of irr. reps.} = \left( \frac{N_0}{p_1} \right)^2 \cdots \left( \frac{N_0}{p_r} \right)^2 \delta_0^{d-2r} = N_0^{d-2} .
\] (31)

This follows from counting the possible values of the Casimirs consistent with the requirement that the matrices belong to \( SU(N) \). Having studied the problem in general, let us now take a brief look at the 2, 3 and 4-dimensional cases, which are the interesting ones for us.

In 2 dimensions, we have 2 matrices \( \Gamma_0 \) and \( \Gamma_1 \) and a single twist tensor element \( n_{01} = \varphi_{01} N \). If \( q = \gcd(n_{01}, N) \) then there \( \exists \) a couple of integers \( n_1 \) and \( n_2 \) such that:

\[
n_1 N + n_2 n_{01} = q .
\] (32)

Now the canonical generators are \( \Gamma'_0 = \Gamma_0^{n_2} \), \( \Gamma'_1 = \Gamma_1 \), and \( p_1 = \frac{N}{q} \). The representation is irreducible if \( p_1 = N \) (\( \Rightarrow q = 1 \)). Our canonical basis amounts to \( \Gamma'_0 = Q_N \) and \( \Gamma'_1 = P_N^+ \) where:

\[
(P_N)_{kj} = z \delta_{j+k+1}, \\
(Q_N)_{kj} = z \delta_{j+k} \exp\left\{ \frac{2\pi i k}{N} \right\} ,
\] (33)

where indices are defined modulo \( N \), and \( z \) is chosen to ensure determinant equal to unity (\( z=1 \) for odd \( N \) and \( \exp\{\pi i \} \) for even \( N \)). Thus, the solution to our problem is \( \Gamma_0 = Q_N^{n_{01}} \) and \( \Gamma_1 = P_N^+ \), which is unique modulo similarity transformations (if \( N \) and \( n_{01} \) are coprime).

In three dimensions we have \( n_{ij} = \epsilon_{ijk} m_k \). Then the 3 canonical matrices are \( \Gamma'_1 = \Gamma(s^{(1)}) \), \( \Gamma'_2 = \Gamma(s^{(2)}) \) and \( \Gamma'_3 = \Gamma(m_0) \), where \( m_0 = \gcd(m, N) \). The vector \( s^{(1)} \) can be chosen among those that satisfy: \( \gcd(s^{(1)} \times m, N) = m_0 \). The other vector is such that \( (s^{(1)} \times m) \cdot s^{(2)} = m_0 \mod N \). One can easily see that \( p_1 = N/m_0 \) and the representation is irreducible for \( m_0 = 1 \). In this case the generator of the center of the group must be a multiple of the identity: \( \Gamma'_3 = z_3 I \). The inequivalent irreducible representations are labelled by \( z_3 \).
Finally, we arrive at the 4-dimensional case. We know that there are 2 subcases: rank 1 and rank 2 (rank 0 is trivial). The relevant quantities are \( p_1 \) and \( p_2 \) (for rank 2). In order to obtain these numbers it is not necessary to bring \( n_{\mu\nu} \) to the canonical form. They can be determined in terms of invariants of the transformations. We see that \( q_1 = \gcd(n_{\mu\nu}, N) \) is an invariant. By looking at the canonical form, one sees that:

\[
P_1 = \frac{N}{\gcd(n_{\mu\nu}, N)}.
\]

(34)

If we perform \( SL(4, \mathbb{Z}) \) transformations to \( n_{\mu\nu} \), there is another invariant:

\[
\kappa(n_{\mu\nu}) \equiv \frac{1}{4} n_{\mu\nu} \tilde{n}_{\mu\nu} = \vec{k} \cdot \vec{m} = Pf(n_{\mu\nu}),
\]

(35)

where \( \tilde{n}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} n_{\rho\sigma} \) and \( Pf(n_{\mu\nu}) \) stands for the Pfaffian of the antisymmetric matrix \( n_{\mu\nu} \). However, \( \kappa(n_{\mu\nu}) \) is non-invariant under the wider group of transformations \( GL(4, \mathbb{Z}) \), or under changes of \( n_{\mu\nu} \) by integer multiples of \( N \). By first bringing \( n_{\mu\nu} \) to a canonical form with \( SL(4, \mathbb{Z}) \) transformations and later applying the wider class of transformations, it is not hard to see that the other invariant \( p_2 \) can be obtained as follows:

\[
P_2 = \frac{N}{\gcd(\kappa(n_{\mu\nu}), q_1) \cdot N} \equiv \frac{N}{q_2}.
\]

(36)

We remind you that the necessary and sufficient condition for the existence of solutions is that \( N_0 \equiv p_1 p_2 \) divides \( N \). The representation is irreducible for \( N_0 = N \). The case of rank 1 corresponds to \( p_2 = 1 \Leftrightarrow q_2 = N \). One can summarise our results in the following statements:

- The necessary and sufficient condition for the existence of solutions is that \( \gcd(\kappa(n_{\mu\nu}), N) = N \). This case is called **Orthogonal twist**, since it corresponds to \( \vec{k} \cdot \vec{m} = 0 \mod N \). The opposite case is referred to as **Non-Orthogonal twist**.

- The representation is irreducible, and hence unique modulo the transformations \( [24, 23] \) provided \( \gcd(N, n_{\mu\nu}, \kappa(n_{\mu\nu})/N) = 1 \). It must be one of the \( N^2 \) inequivalent irreducible representations.

- The representation has rank 2 provided \( \gcd(\frac{\kappa(n_{\mu\nu})}{q_1}, N) \neq N \)
Let us conclude this section by pointing to the relevant references where the study of twist eating solutions was addressed. The first solutions of this type were found in Ref. [15] and [16]. ’t Hooft addressed and solved the problem for the case of rank=1 in 4 dimensions [17]. This also covers the 2 and 3 dimensional cases, the latter studied previously in Ref. [16]. Rank 2 solutions, whose existence was established in Ref. [18], appear and become relevant in the Twisted-Eguchi-Kawai model [19, 20]. This triggered renewed interest in finding a complete solution to the problem. The final solution up to 4 dimensions was obtained in Refs. [20, 21]. Finally, the d-dimensional case was studied in Refs. [22, 23].

4.3 The algebra of twist-eaters

In this subsection we would like to consider the case of irreducible twists. Let us examine here what would be the equivalent of the Clifford algebra for our case. We consider the $\Gamma(n)$ matrices defined in Eq. 25. We will see in what follows that there are as many linearly independent such matrices as there are elements in the quotient group $G/C$. This group is abelian and isomorphic to $\mathbb{Z}_p^1 \times \mathbb{Z}_p^1 \times \ldots \times \mathbb{Z}_p^r \times \mathbb{Z}_p^r$. As a consequence of our study of the last subsection we know that the order of this group is $N^2$.

Since the representation is irreducible the matrices corresponding to an element of the center group $C$ are multiples of the identity. By virtue of the multiplication formula Eq. 26, the matrices corresponding to two elements belonging to the same class in $G/C$, differ by multiplication by a phase. Hence, at most there are as many linearly independent matrices as elements in $G/C$. Now let us prove the following result:

**Theorem 1** If for each class $a \in G/C$ we choose a representative:

$$\lambda(a) = e^{i\phi(a)} \Gamma(n(a)),$$

the corresponding matrices are linearly independent.

The proof is very similar to the corresponding one for the Clifford algebra. One makes use of the following result:

**Lemma 1** Let $A$ and $B$ be two invertible $N \times N$ matrices such that $A B = z B A$ with $z \neq 1$, then $z \in \mathbb{Z}_N$ and $Tr(A) = Tr(B) = 0$. 

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This lemma can be easily proven. By diagonalizing $A$, for example, one reaches the condition $(A_{ii} - z A_{jj}) B_{ij} = 0$. Now since $A$ is invertible $A_{ii} \neq 0$, and hence $B_{ii} = 0$. This shows that indeed the trace of $B$ is zero. Now, since $B$ is invertible, for any $i$ there must exist $i'$ such that $B_{ii'} \neq 0$. Hence, for this pair we must have $A_{ii} - z A_{i'i'} = 0$. If we start with $i'$ and repeat the argument, there must exist another $i''$ satisfying the same relation. Since the number of different eigenvalues of $A$ is finite, this implies that there is an integer $p$ such that $z^{-p} = 1$. Hence starting from one eigenvalue one generates a sequence of $p$ different ones. Now suppose $A$ has one eigenvalue $\tau$ with degeneracy $s$. Then as we have seen $z^{-1} \tau$ must also be an eigenvalue. In addition, its degeneracy must be $s$ as well. This is so because $B$ induces an homomorphism between the spaces of both eigenvalues, and due to the invertibility of $B$, it must be an isomorphism. In this way, from each eigenvalue one generates $ps$ vectors. Repeating the operation for an eigenvalue not contained in this set, we conclude $N = p (\sum s_i)$, and hence $p$ divides $N$. By diagonalising $B$ instead, one can show that the trace of $A$ is also zero. This completes the proof of the lemma. In fact, our lemma is very much related to the explicit construction of irreducible representations of $G$ done in the previous subsection.

As a consequence, the trace of all $\Gamma(n)$ matrices is zero, except for those which are elements of the center group. Now let us proceed to prove the linear independence, asserted in the Theorem. We will work by reductio ad absurdum. Consider that there exist a non-trivial linear combination of the $\lambda(a)$ matrices which is zero:

\[ \sum_{a \in G/C} c(a) \lambda(a) = 0 \quad . \tag{38} \]

Then, if we multiply both sides of the equation by $\Gamma^{-1}(n(a))$ and take the trace, we obtain $N \ e^{\phi(a)c(a)} = 0$. Since this is true for all $a$, all the coefficients must be zero, thus contradicting the non-triviality assumption. QED

Our previous theorem implies that the $\lambda(a)$ matrices define a basis of the complex vector space of all $N \times N$ matrices. The product of these matrices satisfies:

\[ \lambda(a) \lambda(b) = e^{d(a,b)} \lambda(a + b) \quad , \tag{39} \]

where, by virtue of Eq. 20, $d(a, b)$ is given by:

\[ d(a, b) = \phi(a) + \phi(b) - \phi(a + b) + \sum_{\mu > \nu} n_{\mu \nu} (a) n_{\nu}(b) \mod 2\pi \quad . \tag{40} \]
The phases \( d(a,b) \) depend upon the choice of representative \( (\phi(a), n(a)) \). However, the antisymmetric combination:

\[
< a, b > \equiv d(a, b) - d(b, a) = \sum_{\mu\nu} n_{\mu\nu} n_\mu(a) n_\nu(b) \mod 2\pi \tag{41}
\]
does not depend on the choice.

The real subspace of hermitian \( N \times N \) matrices is isomorphic to the Lie Algebra of \( U(N) \). This space can be obtained by linear combinations of the \( \lambda(a) \) matrices with coefficients satisfying certain relations. Using the freedom to make specific choices of the representatives \( (\phi(a), n(a)) \), one can simplify the form of the relations. For example, by choosing the matrices, such that \( \lambda(-a) = \lambda^+(a) \), the condition of hermiticity reads \( c(-a) = c^*(a) \). Hence, one can obtain the form of the structure constants of \( U(N) \) in the basis of \( \lambda(a) \) matrices, as follows:

\[
f_{abc} = -i \delta_{ca+b} (e^{id(a,b)} - e^{id(b,a)}) . \tag{42}
\]

Other choices of representatives can impose certain properties on the \( \lambda(a) \) matrices.

The algebra of twist-eaters was introduced in Refs. [24, 19] in 2 and 4 dimensions respectively. In those references, it was used as a basis of the Lie Algebra of \( SU(N) \). This will turn out to be useful later (See also Refs. [25, 26]).

## 5 Topology of gauge fields

In addition to twist, we must consider the ordinary topological charge (instanton number):

\[
Q = \frac{1}{16\pi^2} \int_{T^4} Tr(F_{\mu\nu}(x) \tilde{F}_{\mu\nu}(x)) d^4x . \tag{43}
\]

Using the fact that \( Tr(F_{\mu\nu}(x) \tilde{F}_{\mu\nu}(x)) \) is a pure divergence \( (\partial_\mu K^\mu) \), and integrating once, we get:

\[
Q = \sum_\mu \int \frac{d\sigma_\mu}{16 \pi^2} \Delta_\mu K^\mu , \tag{44}
\]

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where $\Delta_\mu$ is defined in Eq. [18], $d\sigma_\mu$ is the integral over the 3 dimensional face $x_\mu = 0$, and $K^\mu$ is given by:

$$K^\mu = 2 \epsilon_{\mu \nu \rho \sigma} Tr(A_\nu \partial_\rho A_\sigma - \frac{2}{3} i A_\nu A_\rho A_\sigma) .$$

The main ingredient that we need now, is how does $K^\mu$ change under a gauge transformation. We have:

$$K^\mu([\Omega] A) = K^\mu(A) - \frac{2}{3} \epsilon_{\mu \nu \rho \sigma} \partial_\rho Tr(\bar{A}_\nu A_\sigma) ,$$

with $\bar{A}_\nu = i \Omega^\nu \partial_\nu \Omega$. In our case $A_\nu(\tilde{\mu}) = [\Omega_\mu] A_\nu(0)$, and hence we get:

$$Q = -\frac{1}{24\pi^2} \epsilon_{\mu \nu \rho \sigma} \int d\sigma_\mu Tr(\bar{A}_\nu A_\sigma) + \frac{1}{8\pi^2} \epsilon_{\mu \nu \rho \sigma} \int d\sigma_\mu Tr(\Delta_\rho (\bar{A}_\nu A_\sigma)) ,$$

with $d\sigma_\mu$ the integral over the 2-dimensional surface $x_\mu = x_\rho = 0$, and $\bar{A}_\nu = i \Omega_\mu^+ \partial_\nu \Omega_\mu$. Now using $D_\rho \bar{A}_\nu = D_\rho (\bar{A}_\nu)$, with $D_\rho$ defined by

$$D_\rho \bar{A}_\nu = \Omega_\rho^+ (\Delta_\rho \bar{A}_\nu + \bar{A}_\nu \bar{A}_{\rho}^+ + \bar{A}_\nu \Omega_\rho^+) ,$$

and manipulating the previous equation, we arrive at:

$$Q = -\frac{1}{24\pi^2} \epsilon_{\mu \nu \rho \sigma} \int d\sigma_\mu Tr(\Omega_\nu^+ (\partial_\mu \Omega_{\mu}) \Omega_\nu^+ (\partial_\rho \Omega_{\mu}) \Omega_\nu^+ (\partial_\sigma \Omega_{\mu}))$$

$$-\frac{1}{8\pi^2} \epsilon_{\mu \nu \rho \sigma} \int d\sigma_\mu Tr(\Delta_\rho (\bar{A}_\nu A_\sigma)) ,$$

which is our final formula. As expected, only the twist matrices $\Omega_\mu$ enter the formula. Hence, given the twist matrices, one can compute the topological charge. Since for every twist we have found a particular set of twist matrices—the abelian ones—, we might compute the value of $Q$ for this class of solutions. By taking Eqs. [46][47] and substituting it into Eq. [18] we obtain:

$$Q = -\frac{\kappa(n_{\mu \nu})}{N} + \frac{1}{4} Tr(Q_{\mu \nu} \bar{Q}_{\mu \nu}) = -\frac{\kappa(n_{\mu \nu})}{N} + Tr(\bar{\epsilon} \beta) .$$

The second term is an integer. For example, had we taken the particular solution Eq. [19], we would have gotten $Q = \frac{\kappa(n_{\mu \nu})}{N}$. Actually, with a suitable choice of $\epsilon_i \equiv Q_{0i}$ and $\beta_i \equiv \frac{1}{2} Q_{ijk} \epsilon_{ijk}$, one can make the second term
of Eq. 49 take any possible integer value \( n \). A possible choice that does the job is \((i = 1, 2)\):

\[
\epsilon_i = \begin{pmatrix}
-k_i & 0 & \ldots & 0 \\
0 & 0 & \ddots & \\
\vdots & \ddots & \ddots & \\
0 & 0 & \ldots & 0
\end{pmatrix}, \quad \beta_i = \begin{pmatrix}
0 & 0 & \ldots & 0 \\
0 & -m_i & \ddots & \\
\vdots & \ddots & \ddots & \\
0 & 0 & \ldots & 0
\end{pmatrix}
\]

\( \epsilon_3 = \begin{pmatrix}
-k_3 - n & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & \ddots & \\
0 & 0 & n & \ddots & \\
\vdots & \ddots & \ddots & \ddots & \\
0 & 0 & \ldots & 0 & 0
\end{pmatrix}, \quad \beta_3 = \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 \\
0 & -m_3 - 1 & 0 & \ddots & \\
\vdots & \ddots & \ddots & \ddots & \\
0 & 0 & 1 & \ldots & \\
0 & \ldots & \ldots & \ldots & 0
\end{pmatrix}
\]

The previous formula makes sense for \( N \geq 3 \). For \( SU(2) \) and no-twist \((\vec{k} = \vec{m} = 0 \mod 2)\), there is an exception to the rule: Only even values of the topological charge are attainable in this case.

In summary, we have shown that in the presence of twist, the topological charge is given by:

\[
Q = -\frac{\kappa(n_{\mu\nu})}{N} + n, \quad n \in \mathbb{Z}.
\]

The previous formula was conjectured by ‘t Hooft [5, 6], and actually proven along similar lines to our presentation by Pierre van Baal [12]. A consequence one can derive from Eq. 51 is that for non-orthogonal twists \((\kappa(n_{\mu\nu}) \neq 0 \mod N)\) \( Q \) is not an integer. For orthogonal twists, one has twist-eating solutions, whose topological charge vanishes.

Now let us comment about the appropriate mathematical explanation of our results. The topological charge for an \( SU(N) \) group, is just minus the second chern number, which is an integer. In view of this, the previous result is perplexing. However, as mentioned previously, we have to look at our fields and bundles as being defined in \( SU(N)/\mathbb{Z}_N \). Hence, we should rather compute the previous expressions in the adjoint representation, which is a faithful representation of \( SU(N)/\mathbb{Z}_N \). To do so, we have to replace in Eq. 48 \( \lambda_a \) by \( \lambda_a^{(\text{adjoint})} \). The result gets multiplied by \( \frac{c_{A,F}}{c_{F,D}} = 2N \), where \( c_{A,F} \) and \( d_{A,F} \) are the quadratic Casimir and the dimension of the adjoint and fundamental representations. We get:

\[
Q^{(\text{adjoint})} = -2\kappa(n_{\mu\nu}) + 2Nn.
\]
which is an integer. The previous quantity is more appropriately referred to as the first Pontryagin index $p_1$. Indeed, mathematicians have proven that $p_1$ and twist provide a complete classification of the bundles [14].

6 from $U(N)$ to $SU(N)/\mathbb{Z}_N$

In this section, we will look at twist from a different point of view, by constructing a twisted $SU(N)$ gauge field as a projection of a $U(N)$ gauge field one[27]. Our starting point, hence, is a $U(N)$ gauge field on the torus:

$$A'_\mu(x) = A^\alpha_\mu(x) \lambda_\alpha.$$ 

The difference with respect to the $SU(N)$ field case, is the presence of an additional generator $\lambda_0 = \frac{1}{\sqrt{2N}}$. This field satisfies the boundary condition:

$$A'_\mu(x + \hat{\nu}) = [\Omega'_\nu(x)] A'_\mu(x) ,$$ (53)

where now the matrix $\Omega'_\nu(x)$ belongs to $U(N)$. The compatibility condition satisfied by these matrices is:

$$\Omega'_\rho(x + \hat{\nu}) \Omega'_\nu(x) = \Omega'_\nu(x + \hat{\rho}) \Omega'_\rho(x) .$$ (54)

It is now possible to decompose the gauge field into an $SU(N)$ part $A_\mu(x)$ and a $U(1)$ part, by considering the corresponding components of $A'_\mu(x)$. The boundary condition Eq. (53) somewhat mixes the two parts. However, we might factor the matrices $\Omega'_\nu(x)$ into an $SU(N)$ and a $U(1)$ part as follows:

$$\Omega'_\nu(x) = \exp\{i \omega_\nu(x)\} \Omega_\nu(x) ,$$ (55)

where $\Omega_\nu(x) \in SU(N)$ The decomposition is unique modulo multiplication of $\Omega_\nu(x)$ by an element of the center $z_\nu$ accompanied by the corresponding shift in $\omega_\nu(x)$. However, by continuity $z_\nu$ is space independent, and does not influence the following. Now, for the 2 different parts we have the following boundary conditions:

$$A_\mu(x + \hat{\nu}) = [\Omega_\nu(x)] A_\mu(x)$$ (56)

$$A'^0_\mu(x + \hat{\nu}) = A'^0_\mu(x) + \sqrt{2N} \partial_\mu \omega_\nu(x) .$$ (57)

The compatibility conditions read:

$$\Omega_\rho(x + \hat{\nu}) \Omega_\nu(x) = \exp\{2\pi i \frac{n_{\rho\nu}}{N}\} \Omega_\nu(x + \hat{\rho}) \Omega_\rho(x)$$ (58)

$$\Delta_\rho \omega_\nu(x) - \Delta_\nu \omega_\rho(x) = \frac{2\pi}{N} n_{\rho\nu} \mod 2\pi .$$ (59)

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Henceforth, in general, the resulting $SU(N)$ field has a non-trivial twist tensor $n_{\mu\nu}$. We can compute the field-strength tensor and decompose it into the $SU(N)$ and $U(1)$ part:

$$F'_{\mu\nu}(x) = F_{\mu\nu}(x) + F^0_{\mu\nu}(x) \lambda_0$$  \hspace{1cm} (60)$$

where $F^0_{\mu\nu}(x) = \partial_\mu A^0_{\nu}(x) - \partial_\nu A^0_{\mu}(x)$.

Now, we could use these results to relate the topological properties of the $U(N)$ field to those of $SU(N)$ gauge fields with twist. It is well-known from the theory of characteristic classes\cite{11} that the topological properties of these fields (actually of the bundles or transition matrices) are characterised by the Chern classes. In our case, we have the following topological invariants:

$$c_{\mu\nu} = \frac{1}{2\pi} \int dx_\mu \wedge dx_\nu \operatorname{Tr}(F'_{\mu\nu})$$ \hspace{1cm} (61)$$

$$k = \frac{1}{16\pi^2} \int_{T^4} \operatorname{Tr}(F'_{\mu\nu}(x) \tilde{F}'_{\mu\nu}(x)) d^4x$$ \hspace{1cm} (62)$$

The first set of invariants $c_{\mu\nu}$ is the integral of the first Chern class over the non-contractible 2-dimensional surfaces of the torus. These numbers are integers and hence do not depend on the actual surface chosen but only on the directions $\mu$ and $\nu$. The other quantity $k$ is again an integer and is obtained from the second Chern class. Now, we can evaluate these invariants in terms of our decomposed fields:

$$c_{\mu\nu} = \frac{N}{2\pi} (\Delta_\nu \omega_\mu(x) - \Delta_\mu \omega_\nu(x))$$ \hspace{1cm} (63)$$

$$k = Q + \frac{1}{4N} c_{\mu\nu} \hat{c}_{\mu\nu}$$ \hspace{1cm} (64)$$

where we have used the boundary conditions and compatibility equations. Notice, that using Eq. 59 one arrives at:

$$n_{\mu\nu} = c_{\mu\nu} \mod N$$ \hspace{1cm} (65)$$

which relates the twist tensor with the first Chern class of the original $U(N)$ field. Given this, Eq. 64 reproduces the expression of the topological charge $Q$ of an $SU(N)$ field with twist that we found before (Eq. 51).
7 Gauge invariant Quantities

The most important quantities of the theory are the Wilson loops. They are given by:

\[ W(\gamma) = Tr(U(\gamma : P \rightarrow P)) , \]  

where \( U(\gamma : P \rightarrow P) \) is the parallel transporter matrix along the closed loop \( \gamma \). Closed paths on the torus can be classified into the different homotopy classes, which form the group \( \pi_1(T^4) = \mathbb{Z}_N^4 \). An element \( w = (w_\mu) \) of this group contains all paths which wind \( w_\mu \) times around the \( \mu \)-th direction of the torus. The corresponding Wilson loops are called Polyakov loops or Polyakov lines. The simplest of these, are straight line Polyakov loops along the \( \mu \)-th direction:

\[ P_\mu(x) \equiv Tr(U_\mu(x)) = Tr(T \exp \left\{-i \int_0^{l_\mu} A_\mu(x) \, dx_\mu \right\} \Omega_\mu(x, x_\mu = 0)) . \]  

The occurrence of the twist matrix \( \Omega_\mu \) in the previous formula, is essential to guarantee gauge invariance. In general, one should insert a twist matrix every time that the loop \( \gamma \) traverses the boundary of the patch.

Polyakov lines are non-invariant under the \( \mathbb{Z}_N^4 \) discrete group of transformations (Eq. 14). If \( k \equiv (k_\mu) \) is an element of this group, the Polyakov loops transform \( W(\gamma) \rightarrow \exp \left\{2\pi i k \cdot w(\gamma)/N \right\} W(\gamma) \). It is also important to notice that Polyakov lines are non-periodic. For example, from its definition Eq. 67 one can easily deduce that:

\[ P_\mu(x + \hat{\nu}) = z_{\mu\nu} P_\mu(x) . \]  

8 Classical Solutions

This section will try to summarize what is known about solutions of the euclidean classical equations of motion of Yang-Mills fields on the torus. Classical solutions are extrema of the Yang-Mills (euclidean) action functional:

\[ S = \frac{1}{2} \int_{T^4} d^4 x \, Tr(F_{\mu\nu}(x) F_{\mu\nu}(x)) . \]  

They satisfy \( D_\mu F_{\mu\nu}(x) = 0 \). Some of the classical solutions are those which minimize the action within each twist and topological charge sector, and are
referred to as **minimum action solutions**. Using the Schwartz inequality one can derive the following bound for $S$:

$$S \geq 8\pi^2|Q|.$$  

(70)

The bound is saturated by self-dual and anti-self-dual solutions ($F_{\mu\nu}(x) = \pm \tilde{F}_{\mu\nu}(x)$), provided they exist. For non-orthogonal twists, the topological charge is never an integer, and henceforth the action can never be zero.

It is clear that given one classical solution, one can obtain others for the same twist, topological charge and action, by making a gauge transformation. They are said to be gauge-equivalent. Our purpose would be to find all gauge-inequivalent classical solutions. In the next sections we will review the different cases which have been studied in the literature.

### 8.1 Zero-action solutions

As mentioned previously, zero action solutions ($S = 0$) can only occur for orthogonal twists, which is the class of twists for which $\vec{k} \cdot \vec{m} = \kappa(n_{\mu\nu}) = 0 \mod N$. If we take a zero vector potential $A_\mu(x) = 0$, then this is only compatible with the boundary conditions Eq. 8 provided the twist matrices are constant. Indeed, this is the class of twist matrices $\Gamma_\mu$ which was studied previously under the name of twist eaters. It is now clear what is the origin of the name.

Having found zero-action solutions for all orthogonal twists, we turn now to the problem of determining how many gauge-inequivalent sets are there. Within the set of $A_\mu(x) = 0$ configurations, one is still free to make global gauge transformations, which will not change the value of the vector potential. However, these operations transform the twist matrices by a similarity transformation. Henceforth, the set of inequivalent twist-eaters labels the gauge-inequivalent configurations of this type.

Alternatively, one might take fixed twist matrices $\Omega_\mu(x) = \Gamma_\mu$, and consider different values of the vector potential. Since $S = 0 \Rightarrow F_{\mu\nu}(x) = 0$, there must exist $\Omega(x)$ such that:

$$A_\mu(x) = \imath \Omega(x) \partial_\mu \Omega^+(x).$$  

(71)

However, the vector potentials must satisfy the boundary condition Eq. 8, which in our case demands:

$$\Omega(x + \hat{\nu}) \partial_\mu \Omega^+(x + \hat{\nu}) = \Gamma_\nu \Omega(x) \partial_\mu \Omega^+(x) \Gamma^+_{\nu}.$$  

(72)
This can be shown to imply:

\[ \partial_\mu \tilde{\Gamma}_\nu(x) = 0 , \]  

where \( \tilde{\Gamma}_\nu = \Omega^+(x + \hat{\nu}) \Gamma_\nu \Omega(x) . \)  

(73)

(74)

Now the question is whether this is simply a gauge transformation of the previous \( (A_\mu(x) = 0) \) solution. If we perform a gauge transformation with \( \Omega^+(x) \), we indeed get \( A_\mu(x) \to 0 ; \Gamma_\mu \to \tilde{\Gamma}_\mu . \) The new twist matrices \( \tilde{\Gamma}_\mu \) are constant as a consequence of Eq. (73), and we are back to the previous situation. In summary, we have proven that:

- The class of gauge inequivalent zero-action solutions coincides with the class of inequivalent twist-eaters.

Let us now characterize the elements of this class in terms of gauge invariant quantities. If we evaluate all Wilson loops for the configuration with zero vector potential and twist matrices equal to \( \Gamma_\mu \), we get

\[ W(\gamma) = Tr(\Gamma_\mu \Gamma_\nu \cdots \Gamma_\sigma) = e^{i\alpha(\gamma)} Tr(\Gamma(\omega(\gamma))) , \]  

(75)

where \( s \) is the number of times the path \( \gamma \) crosses the edge of the hypercube \( \prod_{\mu}[0,l_\mu] \) and \( \alpha(\gamma) \) a path-dependent phase. Every time the curve \( \gamma \) crosses the edge of the hypercube along the \( \mu \)th direction one gets a factor of \( \Gamma_\mu^{\epsilon} \) in the previous expression, with \( \epsilon = \pm 1 \) depending on the sense of crossing. The final trace is only non-zero when \( \Gamma(\omega(\gamma)) \) is an element of the center \( C \) of the twist group. Henceforth, the Casimirs of the twist group \( G \) correspond to non-zero Polyakov lines.

We have set the mathematical problem properly: the space of gauge inequivalent zero-action solutions (or euclidean vacua), which is known as the euclidean vacuum valley, is the space of equivalence classes of N-dimensional representations of the twist group (In the mathematical literature, zero-action solutions are referred to as flat connections). In what follows, we will study the structure of this space for the different twists. The study naturally splits into the reducible and irreducible twist cases.

### 8.1.1 Irreducible twists

Under this name we include the values of \( n_{\mu\nu} \) and \( N \) for which we have an irreducible representation of the twist group. We recall that this occurs when \( p_1 \cdot p_2 = N \) for rank 2, or \( p_1 = N \) for rank 1. In either case we proved
previously that the number of inequivalent solutions is discrete. Indeed, we showed that the number of such solutions is $N^2$. There is always a Polyakov line which takes a different value in any two of these vacua. As a particular example, consider the twist $\vec{m} = (1, 1, 1); \vec{k} = (0, 0, 0)$. This has rank 1 and $p_1 = N$. The two independent Polyakov lines are $P_0 = Tr(U_0)/N$ and $P_{(1,1,1)} = Tr(U_1U_2U_3)/N$. Each one can take $N$ different values (the $N^{th}$ roots of unity).

Another example of rank 2 is $\vec{m} = (n, 0, 0); \vec{k} = (Nn, 0, 0)$ ($n$ divides $N$). In this case the independent Polyakov lines are: $Tr(U_2^{N/n})/N, Tr(U_3^{N/n})/N, Tr(U_0^n)/N$ and $Tr(U_1^n)/N$. The first 2 can take $n$ values and the last two $N/n$ values.

### 8.1.2 Reducible twists

This case is much more complicated than the previous one. We have $N = sN_0$, where $s$ is the number of irreducible components. One can bring the matrices to block diagonal form. In each $N_0 \times N_0$ box one has an irreducible representation of the twist group. Since unitary irreducible representations are unique modulo multiplication by a phase, we might write all the matrices in the form:

$$\Gamma_{\mu} = S_{\mu} \otimes \Gamma^{(N_0)}_{\mu},$$

(76)

where $\Gamma^{(N_0)}_{\mu}$ are the matrices of one irreducible representation, and $S_{\mu}$ are diagonal $SU(s)$ matrices.

The case that we will study first is the purely periodic one ($n_{\mu\nu} = 0$). This is a particular case of the general reducible case having $s = N$ and $\Gamma_{\mu} = S_{\mu}$. It might seem that the eigenvalues (elements) of $S_{\mu}$ label the different non-gauge equivalent solutions. In that case, the vacuum valley would be isomorphic with $S_d^{dN}$, where $d=4$ is the space-time dimension. However, this is not so, because there are different points of this manifold which are gauge equivalent. This occurs because there are similarity transformations which bring the $S_{\mu}$ back to diagonal form, namely those which exchange the order of the eigenvalues simultaneously in all d matrices. An additional complication comes from the requirement that the matrices have determinant equal to 1. This conditions fixes the value of the last eigenvalue in terms of the previous ones. Hence, we have the manifold $S_1^{d(N-1)}$. Now we may introduce the group of transformations $\mathcal{T}$ in this manifold corresponding to an exchange of the $N$ eigenvalues. This group is isomorphic to the permutation group of
N elements. Hence our vacuum valley is:

$$\mathcal{V} \equiv S_1^{d(N-1)}/\mathcal{T}.$$  \hspace{1cm} (77)

The disgusting feature of this space, is that it is an orbifold and not a manifold, corresponding to the fact that there are fixed points of the group of transformations $\mathcal{T}$. They occur when there are degenerate eigenvalues for all $S_\mu$ at the same time.

The presence of an infinite number of gauge-inequivalent vacua for periodic gauge fields on the torus was pointed out first in Ref. [28]. It was realized that there are special points in this space corresponding to the singular points. The name *toron* was used to refer to a gauge-equivalence class of vacua. In that reference, instead of setting $A_\mu(x) = 0$ and labelling the torons by the twist matrices, the latter were fixed to unity, and each toron was labelled by the value of $A_\mu(x)$. The transformations which leave the unit twist matrices invariant are the periodic gauge transformations. It is possible to choose one representative of each class of gauge-equivalent vector potentials, within the set of constant diagonal vector potentials. This is an alternative and equivalent description of the vacuum valley.

If we go to the generic irreducible case, our analysis and description remains valid after the substitution $N \rightarrow s$. Henceforth, the euclidean vacuum valley $\mathcal{V}$ is given by:

$$\mathcal{V} \equiv S_1^{d(s-1)}/\mathcal{T}_s,$$

where $\mathcal{T}_s$ is isomorphic to the permutation group of $s$ elements.

### 8.2 Abelian solutions

The first class of solutions which were known for orthogonal and non-orthogonal twists are abelian. Consider first the abelian twist matrices given in Eqs. [16, 20, 21]. If $F_{\mu\nu}(x)$ and $A_\mu(x)$ are chosen to commute with these matrices, the vector potential should satisfy the following boundary condition:

$$\Delta_\nu A_\mu(x) = \partial_\mu \omega_\nu = \frac{\pi}{N} \frac{T_{\nu\mu}}{l_\mu}.$$  \hspace{1cm} (79)

A solution of this equation is given by:

$$A_\mu(x) = -\frac{\omega_\mu}{l_\mu} = -\frac{\pi}{N} \frac{x_\nu}{l_\mu l_\nu} T_{\mu\nu}$$  \hspace{1cm} (80)

$$F_{\mu\nu}(x) = \frac{2\pi}{N} \frac{T_{\mu\nu}}{l_\mu l_\nu}$$  \hspace{1cm} (81)
This is indeed a classical solution \((D_{\mu}F_{\mu\nu}(x) = \partial_{\mu}F_{\mu\nu}(x) = 0)\). We can compute for this configuration the value of the action and the topological charge:

\[
S = \frac{2\pi^2}{N^2} l_0 l_1 l_2 l_3 \text{Tr} \left( \frac{T_{\mu\nu}}{l_{\mu\nu}} \right)^2 = \frac{4\pi^2}{N^2} \sum_{i=1}^{3} [q_i \text{Tr}(E_i^2) + \frac{1}{q_i} \text{Tr}(B_i^2)] \tag{82}
\]

\[
Q = \frac{1}{4N^2} \text{Tr}(T_{\mu\nu}T_{\mu\nu}) = \frac{1}{N^2} \text{Tr}(\vec{E}\vec{B}) \tag{83}
\]

where \(T_{0i} = \mathcal{E}_i, T_{ij} = \epsilon_{ijk}B_k\) and

\[
q_i = \frac{l_0 l_1 l_2 l_3}{l_0^2 l_i^2}. \tag{84}
\]

As expected, the value of the topological charge is the one which follows from the twist matrices. Therefore, there are solutions for all values of the twist and the topological charge (excepting the odd integers for SU(2) with twist \(\vec{k} = \vec{m} = 0 \mod 2\)). The corresponding value of the action follows from Eq. 82. Here we will look in bigger detail to the self-dual (or anti-self-dual) solutions, which when existing have the minimum possible action within each twist and topological charge sector. The self-duality condition implies \(q_i \mathcal{E}_i = B_i\). As a consequence, the \(q_i\) have to be rational numbers:

\[
q_i = \frac{p_i}{p_i'}, \quad \text{with } p_i, p_i' \in \mathbb{Z}^+, \tag{85}
\]

and \(p_i\) and \(p_i'\) coprime. Henceforth, we might express \(\mathcal{E}_i = p_i' K_i\) and \(B_i = p_i K_i\), where \(K_i\) are traceless, integer, diagonal matrices. One gets:

\[
Q = \frac{1}{N^2} \sum_{i=1}^{3} p_ip_i' \text{Tr}(K_i^2) . \tag{86}
\]

We still have to put in the condition Eq. 24. Since there exist integers \(a_i\) and \(b_i\) such that \(a_i p_i + b_i p_i' = 1\) (\(i\) fixed), we may solve for \(K_i\):

\[
K_i = s_i + N S_i , \tag{87}
\]

where \(S_i\) are integer, diagonal matrices satisfying \(\text{Tr}(S_i) = -s_i\), and \(0 \leq s_i < N\) is such that \(k_i = p_i' s_i \mod N\) and \(m_i = p_i s_i \mod N\). The topological charge becomes:

\[
Q = \frac{3}{2} \sum_{i=1}^{3} p_i p_i' \left( -\frac{s_i^2}{N} + \text{Tr}(S_i^2) \right) \geq 0 . \tag{88}
\]
The first term is up to an integer equal to $-\kappa(n_{\mu \nu})$. For fixed size and twist the minimum value of the topological charge is $Q = \sum_i p_i' p_i s_i (N - s_i)/N$. If we are given the twist $(0 \leq k_i, m_i < N)$, it is always possible to find a given size of the torus for which there is an abelian self-dual solution.

We might summarize our results in the following statements:

- There are abelian solutions for all sizes, twists and topological charges, except for $SU(2)$, no-twist and odd $Q$.

- Self-dual ($Q \neq 0$) solutions occur for all twists and for some torus sizes, such that the ratios of the lengths in each direction are square roots of rational numbers: $l_i/l_0 = \sqrt{q_1 q_2 q_3}/q_i$. The minimum value of the topological charge for this configuration is $Q = \sum_i p_i' p_i s_i (N - s_i)/N$, where $p_i$ and $p_i'$ are defined in Eq. 85 and $s_i = a_i m_i + b_i k_i$ (with $a_i p_i + b_i p_i' = 1$).

- The only case in which these abelian solutions are minimum action solutions within each twist sector, is for $SU(2)$ and $\vec{k} = \vec{m} = (1, 0, 0)$ ($p_i = p_i' = 1$). By a misfortunate coincidence, this configuration was called toron by the authors of Ref. [29] —the same name used before for a different object—.

The anti-self-dual solutions have negative values of the topological charge and can be obtained from the previous ones by time reversal or parity.

### 8.3 Non-abelian constant field-strength solutions

'\t Hooft [17] discovered that a wider class of constant commuting $F_{\mu \nu}(x)$ solutions are compatible with the twisted boundary conditions. To find them, let us work the other way round and start with the expression of the vector potential and of the field strength tensor Eqs. 80-81. We will try to find out what are the possible twist matrices $\Omega_\mu(x)$ in this case. They must commute with $T_{\mu \nu}$. From the boundary conditions for $A_\mu(x)$ (8) one gets:

$$i \Omega_\nu(x) \partial_\mu \Omega^+_\nu(x) = \Delta_\nu A_\mu(x) = \frac{\pi}{T_{\mu \nu}} \frac{T_{\nu \mu}}{l_\mu} .$$  

The general solution of the previous equation is given by:

$$\Omega_\mu(x) = \exp\{i \frac{\pi}{N} \frac{x_\mu}{l_\mu} T_{\nu \mu} \} \Gamma_\mu ,$$

27
where $\Gamma_{\mu}$ are constant $SU(N)$ matrices which commute with $T_{\mu\nu}$. These matrices satisfy:

$$\Gamma_{\mu} \Gamma_{\nu} \Gamma_{\mu}^{+} \Gamma_{\nu}^{+} = \exp \{-2\pi i Q'_{\mu\nu} \} . \quad (91)$$

The form of $Q'_{\mu\nu}$ follows from our previous study of twist eaters. If there are $N_1$ coinciding eigenvalues in $T_{\mu\nu}$, we might set $\Gamma_{\mu}$ within this subspace to be one of the twist eating solutions. Hence, in this subspace, one has that the eigenvalues of $Q'_{\mu\nu}$ are given by $q_{\mu\nu}^{(1)}/N_1$, with $q_{\mu\nu}^{(1)}$ an orthogonal twist tensor in $SU(N_1)$. Now imposing the twist condition Eq. (10) we get:

$$T_{\mu\nu} = n_{\mu\nu} I + N Q'_{\mu\nu} . \quad (92)$$

The difference with respect to the purely abelian case lies in the expression of $T_{\mu\nu}$ (versus Eq. 21), since $Q'_{\mu\nu}$ is not an integer matrix. With this difference, expressions 82-83 are valid in this case as well. If we focus on self-dual solutions, the main difference with the previous case is the form of $K_i$, in which Eq. 87 is replaced by:

$$K_i = s_i + N \frac{s_i^{(a)}}{N} \delta_{\alpha} . \quad (93)$$

What we have done, is to separate the $N$ eigenvalues into sets of $N_\alpha$ coinciding ones, so that $\sum_\alpha N_\alpha = N$. The matrix $\delta_{\alpha}$ is the projector onto the subspace $\alpha$, and $s_i^{(a)}$ are integers such that $\sum_\alpha s_i^{(a)} = -s_i$. The formula for the topological charge is now:

$$Q = \sum_{i=1}^{3} p_i p_i' \left(-\frac{s_i^2}{N} + \sum_\alpha \frac{(s_i^{a})^2}{N_\alpha} \right) \geq 0 . \quad (94)$$

The abelian formula Eq. 88 is a particular case of this one for $N_\alpha = 1 \forall \alpha = 1 \ldots N$. The condition that the twist in the $\alpha$ subspace be orthogonal, amounts simply to the fact that the contribution of this space to the previous formula be an integer ($\sum_i p_i p_i' \frac{(s_i^{(a)})^2}{N_\alpha} \in \mathbb{Z}$).

The particular case studied explicitly by 't Hooft is that of 2 subspaces $N = N_1 + N_2$. We can work out explicitly the topological charge for this case, and we get:

$$Q = \sum_{i=1}^{3} p_i p_i' \frac{(s_i^1 N - s_i N_1)^2}{N_1 N_2 N} \quad (95)$$

28
In this case, as shown by 't Hooft [17], one can get solutions which have the minimum action within each twist sector. By taking $p_i = N_1$, $p'_i = N_2$, $s_{1,2} = s_{1,2}^1 = 0$, and $s_3^1 N - s_3 N_1 = 1$, one gets $Q = 1/N$. Other choices are easily extracted from Eq. 95.

8.4 General non-abelian minimum action solutions

Concerning non-abelian solutions in arbitrary sizes and topological charges not much is known analytically. Mathematicians have given existence proofs of some of these configurations [30]. However, a good deal is known about the form and properties of these solutions by numerical methods. We will in what follows give a brief description of these results without entering too much into the technical details of the method employed. For that we refer the reader to the original papers [31, 32, 33].

8.4.1 $SU(2)$

The numerical work in this case has concentrated in studying the minimum action solutions for the most spatially-symmetric choice of twist $\vec{m} = \vec{k} = (1, 1, 1)$ and size $(l_1 = l_2 = l_3 = l_s)$. In this case, $\kappa(n_{\mu\nu}) = \frac{1}{2}$ and the action of all configurations must be larger than $4\pi^2$. The numerical method used by the authors of Ref. [31] is based on the lattice gauge theory formulation of Wilson [1]. This is a gauge-invariant discretization of the Yang-Mills theory. As mentioned previously, putting the system on a box with periodic boundary conditions amounts to the discretization of a toroidal base space. Furthermore, it is possible to give in this case a lattice counterpart of the twisted boundary conditions [13, 19]. Although the lattice formulation destroys some of the topological features of the continuum theory, this is not the case for twist. Then, the point is to find by numerical methods the configuration which minimizes a discretized version of the continuum action-functional. The authors have used different discretized versions, but most of their results are based on Wilson’s discretized form. By employing a so-called classically improved lattice action, one makes the discretization errors smaller, but unfortunately the numerical techniques become more time consuming. In order to obtain the minimum lattice action configuration, the authors of Ref. [31, 32] used a local minimization method known as cooling [34, 35]. Finally, they extracted discretized estimates of the relevant quantities and studied their convergence towards the continuum limit as the
Figure 1: We plot the value of the lattice action $S$ divided by $4\pi^2$ as a function of the square lattice spacing $a^2$, for the SU(2) configuration which minimizes the action with twist $\vec{m} = \vec{k} = (1, 1, 1)$. The solid line is a linear fit to the data.

The lattices are hypercubic and with sizes which were always symmetric in space ($N_s^3 \times N_t$), where $N_s$ and $N_t$ are the number of points in the spatial and temporal directions. If one fixes the continuum unit of length by setting $l_s = 1$, then the lattice spacing is given by $a = \frac{1}{N_s}$.

The simplest quantities to study are global ones, such as the total action and topological charge. Fig. 1 shows the value of the Wilson action for different values of $a$ and $N_t/N_s \rightarrow l_t/l_s \approx 2$. The data approaches $4\pi^2$ linearly in $1/a^2$. The slope depends on the lattice action used, but the extrapolation can give very precise results. From the data of Fig. 1 one gets an extrapolated continuum action of $4\pi^2$ with a precision of 1 part in $10^5$, which is consistent with the size of the errors from other sources. At the
same time, the topological charge tends to \( Q = \frac{1}{2} \) with equivalent precision. This is a strong indication that one is indeed approaching a self-dual classical configuration. This is confirmed by studying self-duality directly (See Refs. [31, 32, 36] for details). The other quantities studied also show that the lattice minimum configuration approaches a continuum one. The accuracy of the approximation depends on the sizes, the quantity studied, the choice of discretized action and the choice of discretized estimate of the different continuum quantities involved. In any case, for the values studied, the numerical differences were at most of a few percent. In what follows, we will omit all reference to the method and concentrate on the results, taken as valid for the continuum minimum action configurations.

The next quantity to study is the energy profile of the classical solution, defined as:

\[
E(t) = \int d^3x \text{Tr}(\vec{E}^2(t, \vec{x}) + \vec{B}^2(t, \vec{x})).
\]  

(96)

For \( l_t/l_s > 1 \) this function has a maximum at a given time value, which will be called the time-center of the configuration, and chosen to correspond to \( t = 0 \). The main features shown by the energy-profile are the following:

- The curve approaches a well defined one when \( l_t/l_s \to \infty \).
- The curve is symmetrical about the time-center \( (t = 0) \).
- For large values of \( |t| \), \( E(t) \) goes to 0 exponentially.
- The curves for finite \( l_t/l_s > 1 \) differ from the infinite \( l_t/l_s \) one mostly at the tails. Actually, the shape of the energy profile is well described by a periodization of the \( l_t/l_s \to \infty \) curve.
- The same curve is obtained within errors if one takes twice the electric part (or magnetic part) of \( E(t) \), in agreement with self-duality.
- To show the quantitative aspect of the function, we show in Fig. 2, the shape of a function which fits the numerical data for the \( l_t/l_s \to \infty \) curve, with a precision of the width of the line. The function has only 4 parameters and its form is

\[
F(t) = \frac{1}{0.0113 + 0.00323 \cosh(8.515t) + 0.0459t^2}.
\]
Figure 2: We show the energy profile of the SU(2) minimal configuration with $\vec{m} = \vec{k} = (1, 1, 1)$.
Notice that the coefficient of the exponential decay is roughly equal to \( 2\pi \sqrt{2} \approx 8.88 \), which is the minimum curvature of the classical potential.

A qualitative conclusion is that the \( l_t/l_s \to \infty \) solution can be considered an \textit{instanton-like} solution, which tunnels between 2 classical zero-energy states. The exponential decay at \( \infty \) tells us that the system is not scale-invariant, being broken by the spatial size of the torus. We will refer to this object as \( Q = \frac{1}{2} \) \textit{instanton}.

In Refs. \[31, 32, 36\] the authors investigate in some detail the form of several physical quantities for this configuration. The results are obtained for a ratio \( N_t/N_s = l_t/l_s \approx 2 \) or higher, which except at the tails is already very close to \( \infty \). The field tensor \( F_{\mu\nu}(x) \), the straightline Polyakov loops and the vector potential itself are studied. The latter is obtained in the \( A_0 = 0 ; A_i(t = -\infty) = 0 \) gauge. The main results obtained are:

- The configuration action density has a maximum at a given point in space and time (center of the \( Q = \frac{1}{2} \) instanton). The size of the object (Full width at half maximum) is a large fraction of the torus spatial size.

- The configuration is invariant under spatial rotations belonging to the cubic group. This is the group which is left invariant by the boundary conditions.

- The solution is highly non-abelian. At the vicinity of the \( Q = \frac{1}{2} \) instanton center, the solution behaves similarly to the ordinary BPST instanton. Hence, \( E_1, E_2 \) and \( E_3 \) are orthogonal and of equal modulus in gauge space. In this sense it looks completely opposite to the constant field strength solutions (abelian and non-abelian), for which these quantities are parallel in Lie Algebra space.

- Both the electric field(= magnetic field) and vector potential components can be nicely fitted with a few Fourier coefficients.

- The space of solutions of this type \( (Q = \frac{1}{2} \) for this twist) depends on 4 parameters, which can be taken as the coordinates of the center of the configuration. This, as will be commented later, is what the index theorem predicts for this case. In addition, there is a discrete degeneracy, about which the index theorem can say nothing. Indeed, there are actually 8 gauge inequivalent families of instantons. They
differ, for example, by the sign of the straight Polyakov loops passing through the $Q = \frac{1}{2}$ instanton center.

### 8.4.2 $SU(N)$

In Ref. [33, 37] a similar study is performed for other SU(N) groups. Again, the study focuses in a spatially symmetric situation: $l_1 = l_2 = l_3 = l_s$ and $\vec{n} = (1, 1, 1)$. Different groups are studied from N=3 to N=25. The reason for studying several groups will be more clear in the next section. Actually, there are two cases which are particularly interesting: $SU(3)$, because it is the physically relevant case for QCD, and $SU(\infty)$, because there is a hope that the solution simplifies in that limit and hence a higher chance of achieving an analytical solution, which is one of the goals of the numerical study. The temporal twist is chosen in several ways, but such that $|\kappa(n_{\mu\nu})| = 1$. Let us concentrate in this paragraph on the results for $N$ equal or close to 3. Essentially, the conclusions are similar to the SU(2) case, except that now the cubic symmetry is replaced by the cyclic group $x \rightarrow y \rightarrow z \rightarrow x$. Another change is that at the $Q = \frac{1}{N}$ instanton center, the three spatial components of the electric field are no longer orthogonal. In this sense, as $N$ grows the configuration looks more abelian at the center. The shape of the energy profile does not depend on the temporal twist (with $|\kappa(n_{\mu\nu})| = 1$) for $l_t/l_s$ large enough. The form of the energy profile does not change too much from one group to another, provided we scale the time variable by $1/N$ and profile itself by $N^2$. The resulting curves $N^2 E_N(t/N)$ are shown in Fig. 3 for various groups.

### 8.4.3 $SU(\infty)$

The large $N$ limit is particularly interesting, since ‘t Hooft [14] noticed, the configurations which have an action going like $1/N$ (unlike instantons), are not suppressed as $N \rightarrow \infty$ in the path integral. Furthermore, as explained previously there is a bigger hope to be able to obtain an analytical solution in this case. Unfortunately, the approach towards $N \rightarrow \infty$ in some quantities is fairly slow. Exploring larger groups numerically is limited by computer memory and time. This slow tendency shows up, for example, in the behaviour of the energy profile. For $N \approx 3$, as mentioned in the previous paragraph, its behaviour suggests that there could be a well-defined limit of the function $N^2 E_N(t/N)$ as $N \rightarrow \infty$. However, although numerically quite
Figure 3: Comparison of the energy profiles for different SU(N) groups. The profiles are scaled as shown in the figure $N^2 \mathcal{E}_N(t/N)$.

![Figure 3](image-url)
similar, there is no clear way to extrapolate to $SU(\infty)$. Actually for groups of the order of $N = 19$ the profile shows a double maximum structure, instead of the single maximum one.

There are, nevertheless, several features which quite consistently seem to become more exact as $N \to \infty$. One of the most remarkable is that gauge invariant quantities tend to be constant in space $[33]$. The final conclusions of this analysis are currently under study (Ref. [37]).

### 8.5 Other results on classical solutions

In this section, we would like to comment about the result obtained by P. van Baal and P. Braam [38] concerning the non-existence of $Q = 1$ instanton solutions on the torus without twist (See also [39]). A recent less technical discussion of this result is done in Ref. [40]. The result is based on Nahm’s transformation [41, 42]. This transformation takes an $U(N)$ self-dual gauge field $A_\mu(x)$ on the torus with topological charge $Q$, and gives rise to a new self-dual $U(Q)$ gauge field living in the dual torus, and whose topological charge is $N$. If we start with an $SU(N)$ gauge field, the transform is an $SU(Q)$ gauge field. As a consequence, it is clear that there cannot exist an $SU(N)$ self-dual gauge field with unit topological charge, since its Nahm’s transform would make no sense.

For completeness, let us sketch the definition of the Nahm’s transform for the gauge potential $A_\mu(x)$. First, one introduces a whole family of gauge inequivalent self-dual gauge fields as follows:

$$A_\mu(x, z) = A_\mu(x) + 2\pi z_\mu I ,$$  \hspace{1cm} (97)

where $0 \leq z_\mu < 1/l_\mu$ are real numbers. They label the points of the dual torus of size $\frac{1}{l_0} \times \ldots \times \frac{1}{l_3}$. This is so, because there exist a gauge transformation that maps $A_\mu(x, z_\nu)$ into $A_\mu(x, z_\nu + 1/l_\nu)$. The next step is to consider the Dirac operators corresponding to this family of gauge potentials. It is crucial that these operators have exactly $Q$ positive chirallity ($\psi^{(i)}(x, z)$ for $i = 1 \ldots Q$) and no negative chirallity solutions (the difference is fixed by the index theorem). Then the Nahm’s transform is given by:

$$\hat{A}^{ij}_\mu(z) = i \int dx \psi^{(i)}(x, z) \frac{\partial}{\partial z_\mu} \psi^{(j)}(x, z) .$$  \hspace{1cm} (98)

The proof that $\hat{A}^{ij}_\mu(z)$ is self-dual, follows very closely the ADHM construction [43, 44].
It is tempting to think that the Nahm’s transform could allow the construction of new self-dual solutions, starting from known ones. However, it is known that the Nahm’s transform of a constant field strength solution is another constant field strength solution. An explicit example is given in Ref. [40].

9 Fluctuations around classical solutions

In this section we will very briefly comment a few things about fluctuations around classical solutions. The first point we want to mention concerns zero modes. If we have a classical solution and we consider a small perturbation around it, the variation of the action is to first order equal to zero. To second order, we have a quadratic form given by an operator whose spectrum is the spectrum of fluctuations. For a minimum action solution all eigenvalues are positive or zero. The corresponding zero-eigenvalue eigenvectors are called zero-modes. If the classical solution depends on several parameters, the variation with respect to each parameter gives rise to a zero-mode. Hence, there are at least as many independent zero modes as parameters of which the solution depends. For a gauge theory, there are always zero-modes present, associated with gauge transformations of the solution. Hence, what one is really interested in, is in those zero-modes, giving rise to parameters, which are not associated to gauge transformations. The space of these parameters is what is called the moduli space. The number of independent zero-modes not associated with gauge transformations gives its dimensionality.

For the case of self-dual solutions, it was found some time ago [45, 46], that the Atiyah-Singer index theorem can tell us the dimensionality of the moduli space $d$, even if we do not know the analytic expression of the solutions. For ordinary instantons this is well-known: $SU(2)$ self-dual solutions on the sphere are known to depend on $d = 8|Q| - 3$ parameters ($Q$ is the topological charge). This was known before the ADHM method gave us a recipe to construct them. However, the result depends both on the gauge group and on the base-manifold (See Refs [47, 48]). In particular, for the torus $T_4$ the result is $d = 4c_A(G)|Q|$, where $c_A(G)$ is the quadratic Casimir in the adjoint representation of the group G. Hence, $SU(N)$ self-dual solutions depend on $d = 4N|Q|$ parameters.

As a consequence, of the previous formula, the $Q = \frac{1}{N}$ instantons which appear in twisted $SU(N)$ theories, depend on 4 parameters. These param-
eters are associated to the translational zero-modes in 4 dimensional space. There is no zero-mode associated with scale invariance, as for BPST instantons, since this symmetry is broken by the torus size. Actually, the form of $d$ is consistent with the interpretation of the space of self-dual solutions, as a liquid of $Q = \frac{1}{N}$ instantons done in Ref. [10].

An analysis of the fluctuation spectrum around the constant field-strength solutions of subsection 7.2 has been carried by P. van Baal [49]. He concluded that the only solutions which are stable are the self-dual (or anti-self-dual) ones. The eigenfunctions are given in terms of generalized Riemann theta functions.

10 Concluding remarks

In the previous pages, we have reviewed some of the most important results obtained for classical Yang-Mills fields on the torus, many of which turn out to be relevant when one studies the quantum mechanical system. The second part of my lectures at Peñiscola, covered this aspect. However, the written-up version of this part is not yet ready and will appear later. Unfortunately, a good deal of our motivation for plunging into the study of the classical system lies precisely in the second part. We hope this will not discourage the interested reader.

To conclude, I will just mention that some topics have been left out due to lack of space and time. For example, we have not included the results obtained about sphalerons in recent years [50, 51]. In any case, I hope that the present review will turn to be useful for those interested in studying the subject.
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