TROPICAL CYCLES AND CHOW POLYTOPES

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Abstract. The Chow polytope of an algebraic cycle in a torus depends only on its tropicalisation. Generalising this, we associate a Chow polytope to any abstract tropical variety in a tropicalised toric variety. Several significant polyhedra associated to tropical varieties are special cases of our Chow polytope.

The Chow polytope of a tropical variety \( X \) is given by a simple combinatorial construction: its normal subdivision is the Minkowski sum of \( X \) and a reflected skeleton of the fan of the ambient toric variety.

1. Introduction

Several well understood classes of tropical variety are known to correspond to certain regular subdivisions of polytopes, in a way that provides a bijection of combinatorial types.

1. Hypersurfaces in \( \mathbb{P}^{n-1} \) are set-theoretically cut out by principal prime ideals. If the base field has trivial valuation, then \( \text{Trop} \, V(f) \) is the fan of all cones of positive codimension in the normal fan to its Newton polytope \( \text{Newt}(f) \).

In the case of general valuation, the valuations of coefficients in \( f \) induce a regular subdivision of \( \text{Newt}(f) \), and \( \text{Trop} \, V(f) \) consists of the non-full-dimensional faces in the normal complex (in the sense of Section 2.2).

2. Linear spaces in \( \mathbb{P}^{n−1} = \mathbb{P}(k^n) \) are cut out by ideals generated by linear forms. To a linear space \( X \) of dimension \( n - d - 1 \) is associated a matroid \( M(X) \), whose bases are the sets \( I \in \binom{[n]}{d+1} \) such that the projection of \( X \) to the coordinate subspace \( \mathbb{K}\{e_i : i \notin I\} \) has full rank. If the base field has trivial valuation, then \( \text{Trop} \, X \) is a subfan (the Bergman fan [3]) of the normal fan to the matroid polytope

\[
\text{Poly}(M(X)) = \text{conv}\{\sum_{j \in J} e_j : J \text{ is a basis of } M(X)\}
\]

of \( M(X) \). In the case of general valuations, the valuations of the Plücker coordinates induce a regular subdivision of \( \text{Poly}(M(X)) \) into matroid polytopes, and \( \text{Trop} \, X \) consists of appropriate faces of the normal complex.

3. Zero-dimensional tropical varieties are simply point configurations. A zero-dimensional tropical variety \( X \) is associated to an arrangement \( \mathcal{H} \) of upside-down tropical hyperplanes with cone points at the points of \( X \): for instance, the tropical convex hull of the points of \( X \) is a union of closed regions in the polyhedral complex determined by \( \mathcal{H} \). The arrangement \( \mathcal{H} \) is dual to...
a fine mixed subdivision of a simplex, and \( X \) consists of the faces dual to little simplices in the normal complex of this subdivision.

The polytopes and subdivisions in this list are special cases of the *Chow polytope*, or subdivision of Chow polytopes, associated to any cycle \( X \) on \( \mathbb{P}^{n-1} \) as the weight polytope of the point representing \( X \) in the *Chow variety*, the parameter space of cycles. Although this is an entirely classical construction, in fact the Chow polytope subdivision of \( X \) depends only on the tropical variety \( \text{Trop} X \), and the construction can be extended to associate Chow polytope subdivisions to all tropical varieties in \( \mathbb{R}^{n-1} \).

This paper’s main theorem, Theorem 5.1, provides a simple tropical formula for this Chow polytope subdivision in terms of \( \text{Trop} X \), making use of a *stable Minkowski sum* operation on tropical cycles introduced in Section 3. The formula is similar to its classical analogue, and is even simpler in one salient respect, namely that there’s no need to invoke any sort of Grassmannian (Remark 5.3). The formula generalises to subvarieties of any projective toric variety.

There is however no general map in the reverse direction, from Chow polytope subdivision to tropical variety (that is, the bijection of combinatorial types in the opening examples is a special phenomenon). In Section 7 we present an example of two distinct tropical varieties with the same Chow polytope.

Finally, in Section 6 we use this machinery to at last record a proof of the fact that tropical linear spaces are exactly tropical varieties of degree 1.

2. Tropical setup

We begin with a few polyhedral notations and conventions. For \( \Pi \) a polyhedron in a real vector space \( V \) and \( u : V \to \mathbb{R} \) a linear functional, face \( u \Pi \) is the face of \( \Pi \) on which \( u \) is minimised, if such a face exists. For \( \Pi, P \) polyhedra, \( \Pi + P \) is the *Minkowski sum* \( \{ \pi + \rho : \pi \in \Pi, \rho \in P \} \), and we write \( -P = \{ -\rho : \rho \in P \} \) and \( \Pi - P = \Pi + (-P) \).

2.1. Tropical cycles. Let \( N_\mathbb{R} \) be a real vector space containing a distinguished full-dimensional lattice \( N \), so that \( N_\mathbb{R} = N \otimes \mathbb{R} \). This is all the structure necessary to define abstract tropical cycles in \( N_\mathbb{R} \), and this is the context in which we will work at first. However, we will often have the situation of Case 2.1.

**Case 2.1** (Projective tropical varieties). Let \( X \) be a classical subvariety of \( \mathbb{P}^{n-1} \) tropicalised with respect to the torus \( (\mathbb{K}^*)^n / \mathbb{K}^* \subseteq \mathbb{P}^{n-1} \), where the \( \mathbb{K}^* \) in the quotient embeds diagonally. Then \( X := \text{Trop} X \) is a tropical fan in \( N_\mathbb{R} = \mathbb{R}^n / (1, \ldots, 1) \), and \( N = \mathbb{Z}^n / (1, \ldots, 1) \) is the lattice of integer points within \( N_\mathbb{R} \). The dual vector space to \( N_\mathbb{R} \) is \( M_\mathbb{R} = (1, \ldots, 1)^\perp = (N_\mathbb{R})^\vee \) (sometimes it will be convenient to use a translate instead). This \( M_\mathbb{R} \) also carries its lattice \( M = M_\mathbb{R} \cap \mathbb{Z}^n = N^\vee \).

For maximal clarity we will write \( e_i \) for the image in \( N_\mathbb{R} \) of a basis element of \( \mathbb{R}^n \), and \( e^i \) for a basis element in the \( (\mathbb{R}^n)^* \) of which \( M_\mathbb{R} \) is a subspace. For \( J \subseteq [n] \), the notation \( e_J \) means \( \sum_{j \in J} e_j \), and \( e^J \) is analogously defined.

For a polyhedron \( \sigma \subseteq N_\mathbb{R} \), let \( \text{lin} \sigma \) be the translate of the affine hull of \( \sigma \) to the origin. We say that \( \sigma \) is *rational* if \( N_\sigma := N \cap \text{lin} \sigma \) is a lattice of rank \( \dim \sigma \).

The fundamental tropical objects we will be concerned with are abstract *tropical cycles* in \( N_\mathbb{R} \). See [2, Section 5] for a careful exposition of tropical cycles. Loosely,
a tropical cycle $X$ of dimension $k$ consists of the data of a rational polyhedral complex $\Sigma$ pure of dimension $k$, and for each facet $\sigma$ of $\Sigma$ an integer multiplicity $m_\sigma$ satisfying a balancing condition at codimension 1 faces, modulo identifications which ensure that the precise choice of polyhedral complex structure, among those with a given support, is unimportant. A tropical variety is an effective tropical cycle, one in which all multiplicities $m_\sigma$ are nonnegative.

We write $Z_k$ for the additive group of tropical cycles in $N\mathbb{R}$ of dimension $k$. We also write $Z = \bigoplus_k Z_k$, and use upper indices for codimension, $Z^k = Z_{\dim N\mathbb{R} - k}$. If $\Sigma$ is a polyhedral complex, then by $Z(\Sigma)$ (and variants with superscript or subscript) we denote the group of tropical cycles $X$ (of appropriate dimension) which can be given some polyhedral complex structure with underlying polyhedral complex $\Sigma$. Our notations $Z$ and $Z_k$ are compatible with [2], but we use $Z(\Sigma)$ differently (in [2] it refers merely to cycles contained as sets in $\Sigma$, a weaker condition).

If a tropical cycle $X$ can be given a polyhedral complex structure which is a fan over the origin, we call it a fan cycle. We prefer this word “fan”, as essentially in [12], over “constant-coefficient”, for brevity and for not suggesting tropicalisation; and over the “affine” of [2], since tropical affine space should refer to a particular partial compactification of $N\mathbb{R}$. We use notations based on the symbol $Z^\text{fan}$ for groups of tropical fan cycles.

In a few instances it will be technically convenient to work with objects which are like tropical cycles except that the balancing condition is not required. We call these unbalanced cycles and use notations based on the symbol $Z^\text{unbal}$. That is, $Z^\text{unbal}$ simply denotes the free Abelian group on the cones of $\Delta$. If $\sigma \subseteq N\mathbb{R}$ is a $k$-dimensional polyhedron, we write $[\sigma]$ for the unbalanced cycle $\sigma$ bearing multiplicity 1, and observe the convention $[\emptyset] = 0$. Then every tropical cycle can be written as an integer combination of various $[\sigma]$.

It is a central fact of tropical intersection theory that $Z^\text{fan}$ is a graded ring, with multiplication given by (stable) tropical intersection, which we introduce next, and grading given by codimension. The invocation of these notions in the toric context [10, Section 4] prefigured certain aspects of the tropical machinery:

**Theorem 2.2** (Fulton–Sturmfels). Given a complete fan $\Sigma$, $Z^\text{fan}(\Sigma)$ is the Chow cohomology ring of the toric variety associated to $\Sigma$.

Given two rational polyhedra $\sigma$ and $\tau$, we define a multiplicity $\mu_{\sigma,\tau}$ arising from the lattice geometry, namely the index

$$\mu_{\sigma,\tau} = [N_{\sigma + \tau} : N_\sigma + N_\tau].$$

We define two variations where we require, respectively, transverse intersection and linear independence:

$$\mu^\bullet_{\sigma,\tau} = \begin{cases} 
\mu_{\sigma,\tau} & \text{if } \text{codim}(\sigma \cap \tau) = \text{codim } \sigma + \text{codim } \tau \\
0 & \text{otherwise},
\end{cases}$$

$$\mu^\boxplus_{\sigma,\tau} = \begin{cases} 
\mu_{\sigma,\tau} & \text{if } \text{dim}(\sigma + \tau) = \text{dim } \sigma + \text{dim } \tau \\
0 & \text{otherwise}.
\end{cases}$$

Alternatively, $\mu^\boxplus_{\sigma,\tau}$ is the absolute value of the determinant of a block matrix consisting of a block whose rows generate $N_\sigma$ as a $\mathbb{Z}$-module above a block whose rows generate $N_\tau$, in coordinates providing a basis for any $(\text{dim } \sigma + \text{dim } \tau)$-dimensional...
lattice containing \( N_{\sigma+\tau} \). Likewise \( \mu^\bullet_{\sigma,\tau} \) can be computed from generating sets for the dual lattices.

If \( \sigma \) and \( \tau \) are polytopes in \( N_\mathbb{R} \) which are either disjoint or intersect transversely in the relative interior of each, their stable tropical intersection is

\[
(2.1) \quad [\sigma] \cdot [\tau] = \mu^\bullet_{\sigma,\tau} [\sigma \cap \tau].
\]

If \( X = \sum_\sigma m_\sigma [\sigma] \) and \( Y = \sum_\tau n_\tau [\tau] \) are unbalanced cycles such that every pair of facets \( \sigma \) of \( X \) and \( \tau \) of \( Y \) satisfy this condition, then their stable tropical intersection is obtained by linear extension,

\[
(2.2) \quad X \cdot Y = \sum_{\sigma,\tau} m_\sigma n_\tau \cdot \mu^\bullet_{\sigma,\tau} [\sigma \cap \tau].
\]

If \( X \) and \( Y \) are tropical cycles, so is \( X \cdot Y \) (see \[4\]). For a point \( v \in N_\mathbb{R} \), let \([v] \boxplus Y\) denote the translation of \( Y \) by \( v \); this is a special case of a notation we introduce in Section \[3\]. If \( X \) and \( Y \) are rational tropical cycles with no restrictions, then for generic small displacements \( v \in N_\mathbb{R} \) the faces of \( X \) and \([v] \boxplus Y\) intersect suitably for equation \( (2.2) \) to be applied. In fact the facets of the intersection \( X \cdot ([v] \boxplus Y) \) vary continuously with \( v \), in a way that can be continuously extended to all \( v \). This is essentially the \textit{fan displacement rule} of \[10\], which ensures that \( X \cdot Y \) is always well-defined.

**Definition 2.3.** Given two tropical cycles \( X, Y \), their \textit{(stable) tropical intersection} is

\[
X \cdot Y = \lim_{v \to 0} X \cdot ([v] \boxplus Y).
\]

We introduce a few more operations on cycles. Firstly, there is a cross product defined in the expected fashion. Temporarily write \( Z(V) \) for the ring of tropical cycles defined in the vector space \( V \). Let \( (N_i)_\mathbb{R}, i = 1, 2 \), be two real vector spaces. Then there is a well-defined bilinear cross product map

\[
x : Z^{\text{unbal}}((N_1)_\mathbb{R}) \otimes Z^{\text{unbal}}((N_2)_\mathbb{R}) \to Z^{\text{unbal}}((N_1 \oplus N_2)_\mathbb{R})
\]

linearly extending \([\sigma] \times [\tau] = [\sigma \times \tau]\), and the exterior product of tropical cycles is a tropical cycle.

Let \( h : N \to N' \) be a linear map of lattices, inducing a map of real vector spaces which we will also denote \( h : N_\mathbb{R} \to N'_\mathbb{R} \) (an elementary case of a tropical morphism). Cycles can be pushed forward and pulled back along \( h \). These are special cases of notions defined in tropical intersection theory even in ambient tropical varieties other than \( \mathbb{R}^n \) (in the general case, one can push forward general cycles but only pull back complete intersections of Cartier divisors \[2\]).

Given a cycle \( Y = \sum_\sigma m_\sigma [\sigma] \) on \( N'_\mathbb{R} \), its pullback is defined in \[11\] as follows. This is shown in \[10\] Proposition 2.7) to agree with the pullback on Chow rings of toric varieties.

\[
h^*(Y) = \sum_{\sigma : \sigma \text{ meets im } h \text{ transversely}} m_\sigma [N_{h^{-1}(\sigma)} : h^{-1}(N'_\sigma)][h^{-1}(\sigma)]
\]

The pushforward is defined in \[12\] in the tropical context, and is shown to coincide with the cohomological pushforward in \[17\] Lemma 4.1. If \( X = \sum_\sigma m_\sigma [\sigma] \) is a cycle on \( N_\mathbb{R} \), its pushforward is

\[
h_*(X) = \sum_{\sigma : h|\sigma \text{ injective}} m_\sigma [N_{h(\sigma)} : h(N_\sigma)][h(\sigma)].
\]
In these two displays, the conditions on \( \sigma \) in the sum are equivalent to \( h^{-1}(\sigma) \) or \( h(\sigma) \), respectively, having the expected dimension. Pushforwards and pullbacks of tropical cycles are tropical cycles.

2.2. Normal complexes. Write \( M = N^\vee, M_\mathbb{R} = N_\mathbb{R}^\vee \) for the dual lattice and real vector space. Let \( \pi : M_\mathbb{R} \times \mathbb{R} \to M_\mathbb{R} \) be the projection to the first factor. A polytope \( \Pi \subseteq M_\mathbb{R} \times \mathbb{R} \) induces a regular subdivision \( \Sigma \) of \( \pi(\Pi) \). Our convention will be that regular subdivisions are determined by lower faces: so the faces of \( \Sigma \) are the projections \( \pi(\text{face}_{(u,1)}(\Pi)) \). We will also write \( \text{face}_u \Sigma \) to refer to this last face. In general, we will not consider regular subdivisions \( \Sigma \) by themselves but will also want to retain the data of \( \Pi \). More precisely, what is necessary is to have a well-defined normal complex; for this we need only \( \Sigma \) together with the data of the heights of the vertices of \( \Pi \) visible from underneath, equivalently the lower faces of \( \Pi \). (When we refer to “vertex heights” we shall always mean only the lower vertices.)

Definition 2.4. The (inner) normal complex \( N(\Sigma, \Pi) \) to the regular subdivision \( \Sigma \) induced by \( \Pi \) is the polyhedral subdivision of \( N_\mathbb{R} \) with a face

\[
\text{normal}(F) = \{ u \in N_\mathbb{R} : W \subseteq \text{face}_{(u,1)}(\Pi) \}
\]

for each face \( F = \text{conv}(\pi(W)) \) of \( \Sigma \).

We will allow ourselves to write \( N^1(\Sigma) \) for \( N^1(\Sigma, \Pi) \) when \( \Pi \) is clear from context. If \( \Pi \) is contained in \( M_\mathbb{R} \times \{0\} \), which we identify with \( M_\mathbb{R} \), then \( N(\Sigma, \Pi) \) is the normal fan of \( \Pi \).

We give multiplicities to the faces of the skeleton \( N^e(\Sigma, \Pi) \) of \( N(\Sigma, \Pi) \) so as to make it a cycle, which we also denote \( N^e(\Sigma, \Pi) \). To each face \( \text{normal}(F) \in N(\Sigma, \Pi) \) of codimension \( e \), we associate the multiplicity \( m_{\text{normal}(F)} = \text{vol} F \) where \( \text{vol} \) is the normalised lattice volume, i.e. the Euclidean volume on \( \text{lin} F \) rescaled so that any simplex whose edges incident to one vertex form a basis for \( \text{lin} F \) has volume 1. In fact \( N(\Sigma, \Pi) \) is a tropical cycle. In codimension 1 a converse holds as well.

Theorem 2.5. 
(a) For any rational regular subdivision \( \Sigma \) in \( M_\mathbb{R} \) induced by a polytope \( \Pi \) in \( M_\mathbb{R} \times \mathbb{R} \), the skeleton \( N^e(\Sigma, \Pi) \) is a tropical variety.
(b) For any tropical variety \( X \in Z^1(N_\mathbb{R}) \), there exists a rational polytope \( \Pi \) in \( M_\mathbb{R} \times \mathbb{R} \) and induced regular subdivision \( \Sigma \), unique up to translation and adding a constant to the vertex heights, such that \( X = N^1(\Sigma, \Pi) \).

Part (a) in the case of fans, i.e. \( \Pi \subseteq M_\mathbb{R} \times \{0\} \), is a foundational result in the polyhedral algebra \cite[Section 11]{22}. The statement for general tropical varieties follows since the normal complex of \( \Sigma \) is just the slice through the normal fan of \( \Pi \) at height 1, and this slicing preserves the balancing condition. Part (b) is also standard, and is a consequence of ray-shooting algorithms, the codimension 1 case of Theorem 4.9.

One more fact will be important when we move beyond \( \mathbb{P}^{n-1} \) as ambient variety. This is the content of \cite[Theorem 5.1]{22} cast tropically.

Theorem 2.6. Let \( \iota : N \to N' \) be an inclusion of lattices such that \( \iota N \) is saturated in \( N' \), and \( \iota^T \) the dual projection. For any polytope \( \Pi' \) in \( M'_\mathbb{R} \times \mathbb{R} \), let \( \Pi = (\iota^T \times \text{id})\Pi' \).
be its projection to $M_{R} \times R$, and let $\Sigma'$ and $\Sigma$ be the induced regular subdivisions. Then $N(\Sigma) = N(\Sigma') \cdot [\iota N]$.

This $\Sigma$ is the image subdivision of $\Sigma'$ of [16]; this is the natural notion of projection for regular subdivisions with vertex heights.

3. Minkowski sums of cycles

Let $N$ be any lattice. For a tropical cycle $X = \sum m_{\sigma}[\sigma]$, we let $X_{\text{refl}} = \sum m_{\sigma}[-\sigma]$ denote its reflection about the origin. (This is the pushforward or pullback of $X$ along the linear isomorphism $x \mapsto -x$.)

Given two polyhedra $\sigma, \tau \subseteq N_{R}$, define the (stable) Minkowski sum

\[ [\sigma] \boxplus [\tau] = \mu_{\sigma, \tau}[\sigma + \tau]. \]

Compare [2.1]. If $X$ and $Y$ are cycles in $N_{R}$, then we can write their intersection and Minkowski sum in terms of their exterior product

\[ X \cdot Y = \iota^{*}(X \times Y). \]

\[ X \boxplus Y_{\text{refl}} = \phi_{\iota}(X \times Y) \]

Since pullback is well-defined and takes tropical cycles to tropical cycles, it follows immediately that there is a well-defined bilinear map $\boxplus : Z_{\text{bal}} \otimes Z_{\text{bal}} \to Z_{\text{bal}}$ extending (3.1), restricting to a bilinear map $\boxplus : Z \otimes Z \to Z$.

A notion of Minkowski sum for tropical varieties arose in [7] as the tropicalisation of the Hadamard product for classical varieties. The Minkowski sum of two tropical varieties in that paper’s sense can have dimension less than the expected dimension.

By contrast our bilinear operation $\boxplus$ should be regarded as a stable Minkowski sum for tropical cycles. It is additive in dimension, i.e. $Z_{d} \boxplus Z_{d'} \subseteq Z_{d+d'}$, just as stable tropical intersection is additive in codimension. The next lemma further relates intersection and Minkowski sum.

The balancing condition implies that for any tropical cycle $X$ in $N_{R}$ of dimension $\dim N_{R}$, $X(u)$ is constant for any $u \in N_{R}$ for which it’s defined. We shall denote this constant $\deg X$. Similarly, if $\dim X = 0$, then $X$ is a finite sum of points with multiplicities, and we will let $\deg X$ be the sum of these multiplicities. These are both special cases of Definition 3.3, to come.

Lemma 3.1. Let $X$ and $Y$ be tropical cycles on $N_{R}$, of complementary dimensions. Then

\[ \deg(X \cdot Y) = \deg(X \boxplus Y_{\text{refl}}). \]

Proof. Let $u \in N_{R}$ be generic. Let $\Sigma(X)$ and $\Sigma(Y)$ be polyhedral complex structures on $X$ and $Y$. The multiplicity of $X \boxplus Y_{\text{refl}}$ at a point $u \in N_{R}$ is

\[ (X \boxplus Y_{\text{refl}})(u) = \sum_{\sigma, \tau} \mu_{\sigma, \tau}, \]

summing over only those $\sigma \in \Sigma(X)$ and $\tau \in \Sigma(Y)_{\text{refl}}$ with $u \in \sigma + \tau$, i.e. with $\{u\} - \tau$ is $\Sigma(Y)$ empty. These $\{u\} - \tau$ are the cones of $\Sigma(Y)$, where $Y'$ =
Then by (2.2), \( \deg(X \cdot Y') \) is given by the very same expression (3.3) except with \( \mu^\bullet \) in place of \( \mu \); and by the fan displacement rule preceding Definition 2.3, \( \deg(X \cdot Y) = \deg(X \cdot Y') \). But \( \mu_{\sigma,\tau} = \mu_{\sigma,\tau} \) when \( \sigma \) and \( \tau \) are of complementary dimensions. \( \square \)

**Lemma 3.2.** Let \( X \) and \( Z \) be tropical cycles on \( N \), and \( Y \) a cycle which is a classical linear space through the origin, with \( X \subseteq Y \). Then

\[
X \oplus (Y \cdot Z) = Y \cdot (X \oplus Z).
\]

**Proof.** Replacing \( Z \) (and thus \( X \oplus Z \)) by a generic small translate, we may take the intersections to be set-theoretic intersections with lattice multiplicity. By linearity, we may assume \( X \) and \( Z \) are of the form \( [\sigma] \). Then this reduces to checking set-theoretic equality and checking equality of multiplicities, both of which are routine. \( \square \)

We specialise to Case 2.1. Let \( L \) be the fan of the ambient toric variety \( \mathbb{P}^n \), which is the normal fan in \( N \) to the standard simplex \( \text{conv}\{e^i\} \). The ray generators of \( L \) are \( e_i \in N \), and every proper subset of the rays span a face, which is simplicial. For \( J \subseteq [n] \) let \( C_J = \mathbb{R}_{\geq 0}\{e_j : j \in J\} \) be the face of \( L \) indexed by \( J \). Let \( L_k \) be the dimension \( k \) skeleton of \( L \) with multiplicities 1, that is, the canonical \( k \)-dimensional tropical fan linear space.

**Definition 3.3** ([2, Definition 9.13]). The degree of a tropical cycle \( X \in Z^c(N) \) is \( \deg X := \deg(X \cdot L_e) \).

The symbol \( \deg \) appearing on the right side is the special case defined just above for cycles of dimension 0. It is a consequence of the fan displacement rule that \( \deg X = \deg(X \cdot ([v] \oplus L_e)) \) for any \( v \in N \).

**Lemma 3.4.** Let \( X \in Z^c \). Then

\[
\deg(X \oplus L_{e-1}^\text{refl}) = e \deg X.
\]

**Proof.** By Lemma 3.1 we have

\[
\deg(X \oplus L_{e-1}^\text{refl}) = \deg((X \oplus L_{e-1}^\text{refl}) \cdot L_1)
= \deg(X \oplus L_{e-1}^\text{refl} \oplus L_1^\text{refl})
= \deg((L_{e-1}^\text{refl} \oplus L_1^\text{refl}) \cdot X^\text{refl})
= \deg((L_{e-1} \oplus L_1) \cdot X)
= \deg((eL_e) \cdot X)
= e \deg(X \cdot L_e)
= e \deg X. \square
\]

**Remark 3.5.** The classical projection formula of intersection theory is valid tropically [2, Proposition 7.7], and has an analogue for \( \oplus \). For a linear map of lattices \( h : N \rightarrow N' \) and cycles \( X \in Z(N) \) and \( Y \in Z(N') \), we have

\[
h_*(X \cdot h^*(Y)) = h_*(X) \cdot Y,
X \oplus h^*(Y) = h^*(h_*(X) \oplus Y).
\]

The facts in this section, as well as the duality given by polarisation in the algebra of cones which exchanges intersection and Minkowski sum, are all suggestive of the
existence of a duality between tropical stable intersection and stable Minkowski sum. However, we have not uncovered a better statement of such a duality than equations (3.2).

4. CHOW POLYTOPES

In this section we introduce Chow polytopes. There is little new content here: see [15], [13, ch. 4] and [8] for fuller treatments of this material, the first for the toric background, the second in the context of elimination theory, and the last especially from a computational standpoint. The assumptions of Case 2.1 will be in force for most of this section, and most of the rest of the paper.

Let $K$ be an algebraically closed field. Let $(K^*)^n$ be an algebraic torus acting via a linear representation on a vector space $V$, or equivalently on its projectivisation $\mathbb{P}(V)$. Suppose that the action of $(K^*)^n$ is diagonalisable, i.e. $V$ can be decomposed as a direct sum $V = \bigoplus V_i$ where $(K^*)^n$ acts on each $V_i$ by a character or weight $\chi^{w_i} : (K^*)^n \to K^*$. A character $\chi^{w_i}$ corresponds to a point $w_i$ in the character lattice of $(K^*)^n$, via $\chi^{w_i}(t) = t^{w_i}$. We shall always assume $V$ is finite-dimensional, except in a few instances where we explicitly waive this assumption for technical convenience.

Definition 4.1. Given a point $v \in V$ of the form $v = \sum_{k \in K} v_{i_k}$ with each $v_{i_k}$ nonzero, the weight polytope of $v$ is $\text{conv}\{w_k : k \in K\}$.

If $X \subseteq \mathbb{P}(V)$ is a $(K^*)^n$-equivariant subvariety, this defines the weight polytope of a point $x \in X$.

The Chow variety $\text{Gr}(d,n,r)$ of $\mathbb{P}^{n-1}$, introduced by Chow and van der Waerden in 1937 [6], is the parameter space of effective cycles of dimension $d-1$ and degree $r$ in $\mathbb{P}^{n-1}$. When we invoke homogeneous coordinates on $\mathbb{P}^{n-1}$ we will name them $x_1, \ldots, x_n$.

Example 4.2.

1. The variety $\text{Gr}(n-1,n,r)$ parametrising degree $r$ cycles of codimension 1 is $\mathbb{P}(K[x_1, \ldots, x_n]_r) \cong \mathbb{P}^{r+n-1}-1$. An irreducible cycle is represented by its defining polynomial.

2. The variety $\text{Gr}(d,n,1)$ parametrises degree 1 effective cycles, which must be irreducible and are therefore linear spaces. So $\text{Gr}(d,n,1)$ is simply the Grassmannian $\text{Gr}(d,n)$, motivating the notation.

The Chow variety $\text{Gr}(d,n,r)$ is projective. Indeed, we can present the coordinate ring of $\text{Gr}(n-d,n)$ in terms of (primal) Plücker coordinates, which we write as brackets:

$$\mathbb{K}[\text{Gr}(n-d,n)] = \mathbb{K}\left[\{J : J \in \binom{[n]}{n-d}\}\right]/(\text{Plücker relations}).$$

For our purposes the precise form of the Plücker relations will be unimportant. Then $\text{Gr}(d,n,r)$ has a classical embedding into the space $\mathbb{P}(\mathbb{K}[\text{Gr}(n-d,n)]_r)$ of homogeneous degree $r$ polynomials on $\text{Gr}(n-d,n)$ up to scalars, given by the Chow form [6]. We denote the Chow form of $X$ by $R_X$.

Remark 4.3. For $X$ irreducible, the Chow form $R_X$ is the defining polynomial of the locus of linear subspaces of $\mathbb{P}^{n-1}$ of dimension $n-d-1$ which intersect $X$. There is a single defining polynomial since $\text{Pic}(\text{Gr}(n-d,n)) = \mathbb{Z}$. 
The natural componentwise action $(\mathbb{K}^*)^n \acts \mathbb{K}^n$ induces an action $(\mathbb{K}^*)^n \acts S^*(\bigwedge^{n-d} \mathbb{K}^n)$. The ring $\mathbb{K}[\text{Gr}(n-d, n)]$ is a quotient of this symmetric algebra by the ideal of Plücker relations. This ideal is homogeneous in the weight grading, so the quotient inherits an $(\mathbb{K}^*)^n$-action. The Chow variety is an $(\mathbb{K}^*)^n$-equivariant subvariety of $\mathbb{P}(\mathbb{K}[\text{Gr}(n-d, n)])$, so we also get an action $(\mathbb{K}^*)^n \acts \text{Gr}(n-d, n)$. The weight spaces of $\mathbb{K}[\text{Gr}(n-d, n)]$ under the $(\mathbb{K}^*)^n$-action are spanned by monomials in the brackets $[J]$. The weight of a bracket monomial $\prod_i m_i x^j_i$. 

**Definition 4.4.** If $X$ is a cycle on $\mathbb{P}^{n-1}$ represented by the point $x$ of $\text{Gr}(d, n, r)$, the **Chow polytope** $\text{Chow}(X)$ of $X$ is the weight polytope of $x$.

**Example 4.5.**

1. The Chow form of a hypersurface $V(f)$ is simply its defining polynomial $f$ with the variables $x_k$ replaced by brackets $[k]$, so that the Chow polytope $\text{Chow}(V(f))$ is the Newton polytope $\text{Newt}(f)$.

2. The Chow form of a $(d-1)$-dimensional linear space $X$ is a linear form in the brackets, $\sum_J p_J[J]$, where the $p_J$ are the dual Plücker coordinates of $X$ for $J \in \binom{[n]}{n-d}$. Accordingly $\text{Chow}(X)$ is the polytope $\text{Poly}(M(X)^*)$ of the dual matroid. Note that this is simply the image of $\text{Poly}(M(X))$ under a reflection.

3. For $X = X_A$ an embedded toric variety in $\mathbb{P}^{n-1}$, the Chow polytope $\text{Chow}(X)$ is the secondary polytope associated to the vector configuration $A$ (Chapter 8.3). 

From a tropical perspective, the preceding setup has all pertained to the constant-coefficient case. Suppose now that the field $\mathbb{K}$ has a nontrivial valuation $\nu : \mathbb{K}^* \to \mathbb{Q}$, with residue field $k \leftarrow \mathbb{K}$. For instance we might take $\mathbb{K} = k\{t\}$ the field of Puiseux series over an algebraically closed field $k$, with the valuation $\nu : \mathbb{K}^* \to \mathbb{Q}$ by least degree of $t$. Let $X$ be a cycle on $\mathbb{P}^{n-1}$ with Chow form $R_X \in \mathbb{K}[\text{Gr}(n-d, n)]$. Let $\tau_1, \ldots, \tau_m \in \mathbb{K}$ be the coefficients of $R[X]$, so that $R[X]$ is defined over the subfield $k[\tau_1^{\pm 1}, \ldots, \tau_m^{\pm 1}] \subset \mathbb{K}$. The restriction of $\nu$ to this subfield is a discrete valuation, so we may assume that all the $\nu(\tau_i)$ are integers.

The torus $(\mathbb{K}^*)^n$ acts on $k[\text{Gr}(n-d, n)]$ just as before, and therefore acts on $k[\text{Gr}(n-d, n)][\tau_1^{\pm 1}, \ldots, \tau_m^{\pm 1}]$. Let $(\mathbb{K}^*)^n \times \mathbb{K}^* \acts k[\tau^{\pm 1}][\text{Gr}(n-d, n)]$ where the right factor acts on Laurent monomials in $\tau_1, \ldots, \tau_n$, with $\tau$ having weight $\sum_{i=1}^n a_i \nu(\tau_i)$. Let $\Pi$ be the weight polytope of the Chow form $R_X$ with respect to this action.

**Definition 4.6.** The **Chow subdivision** of a cycle $X$ on $\mathbb{P}^{n-1}$ over $(\mathbb{K}, \nu)$ is the regular subdivision $\text{Chow}_\nu(X)$ induced by $\Pi$.

The Chow subdivision is the non-constant-coefficient analogue of the Chow polytope, generalising the polytope subdivision of the opening examples. It appears as the secondary subdivision in Definition 5.5 of [16], but nothing is done with the definition in that work, and we believe this paper is the first study to investigate it in any detail. Observe that $\text{Chow}_\nu(X)$ is a subdivision of $\text{Chow}(X)$, and if $\nu$ is the trivial valuation, $\text{Chow}_\nu(X)$ is $\text{Chow}(X)$ unsubdivided. By $\mathcal{N}(\text{Chow}_\nu(X))$ we will always mean $\mathcal{N}(\text{Chow}_\nu(X), \Pi)$.

If $(u, v) : k^* \to (k^*)^n \times k^*$ is a one-parameter subgroup which as an element of $N \times \mathbb{Z}$ has negative last coordinate, then face$_u \text{Chow}_\nu(X) = \text{face}_{(u,v)} \Pi$ is bounded.
We observe that a bounded face $F = \operatorname{face}_u \operatorname{Chow}_\nu(X)$ of $\operatorname{Chow}_\nu(X)$ is the weight polytope of the toric degeneration $\lim_{t \to 0} u(t) \cdot X$. This follows from an unbounded generalisation of Proposition 1.3 of [15], which describes the toric degenerations of a point in terms of the faces of its weight polytope.

**Example 4.7.** Perhaps the simplest varieties not among our opening examples are conic curves in $\mathbb{P}^3$. Let $K = \mathbb{C}[[t]]$, and let $X \subseteq \mathbb{P}^3$ be the conic defined by the ideal

$$(tx - y + z - t^3 w, yz + tz^2 + t^2 yw - zw + (t^3 - t^7)w^2)$$
where \((x : y : z : w)\) are coordinates on \(\mathbb{P}^3\). The Chow form of \(X\) can be computed by the algorithm of [8, Section 3.1]. It is

\[
(2t^7 + t^6 + t^5 - t^4)[zw][yw] + (t^7 + t^6 - t^3)[yw]^2 + (2t^4 + t^3 + t^2 - 1)[zw][yz] + (-t^2 + 1)[yz][yz] + (-1)[yz][yz] + (2t^8 - 2t^4)[yw][zw] + (t^3 - t^2)[zw][zw] + (t^2 - t)[zw][zw] + (2t^3 - t)[zw][zw] + (t^3 - t^2)[zw][zw] + (t^2 - t)[zw][zw] + (2t^4 - t^2 + t)[zw][zw] + (-t^4 + t^3 - 2t)[yw][zw] + (2t^4 - t^3 - 2t)[yw][zw] + (-t^2 + t)[yw][zw] + (t^3 - t^2)[zw][zw] + (t^2 - 2t)[zw][zw] + (t^3 - t^2 + t)[zw][zw] + (t^2 - 2t)[zw][zw] + (2t^4 - t^3 + t)[zw][zw] + (t^3 - t^2 + t)[zw][zw] + (t^2 - 2t)[zw][zw] + (t^3 - t^2)[zw][zw] + (t^2 - t)[zw][zw].
\]

The Chow subdivision \(\text{Chow}_\nu(X)\) is the regular subdivision induced by the valuations of these coefficients. It is a 3-polytope subdivided into 5 pieces, depicted in Figure 1. The polytope \(\text{Chow}(X)\) of which it is a subdivision is an octahedron with two opposite corners truncated (it is not the whole octahedron, which is the generic Chow polytope for conics in \(\mathbb{P}^3\)).

Chow varieties and polytopes can also be defined for cycles on some more general spaces. For this we of course suspend the assumptions of Case 2.1. The groundwork for this construction is done in [19, Section I.3], and it’s also treated in [16].

**Case 4.8.** Let \(\iota : Y \subseteq \mathbb{P}^{n-1}\) be a projective toric variety with torus \(T\), included \(T\)-equivariantly in \(\mathbb{P}^{n-1}\). All our Chow constructions depend on \(\iota\), not merely \(Y\) alone. Let \(\Delta\) be the fan associated to \(Y\), and \(\mathbb{N}_R\) its underlying vector space, so that the fan structure defined on \(\mathbb{N}_R\) by its intersections with cones of the fan \(\mathcal{L}\) of \(\mathbb{P}^{n-1}\) is equal to \(\Delta\). The inclusion \(\iota\) corresponds to a linear inclusion \(\iota : \mathbb{N}_R \hookrightarrow \mathbb{R}/\langle 1, \ldots, 1 \rangle\), whose image we identify with \(\mathbb{N}_R\), turning tropical cycles in \(\mathbb{N}_R\) into tropical cycles in \(\mathbb{R}/\langle 1, \ldots, 1 \rangle\). These identifications are compatible with the corresponding classical ones.

Cycles in \(Y\) and \(Y\) inherit a degree via \(\iota\) and \(\iota\) respectively. For any given dimension \(d-1\) and degree \(r\), the Chow variety of dimension \(d-1\) degree \(r\) cycles for \(\iota\) is defined as the subvariety of \(\text{Gr}(d, n, r)\) whose points represent cycles in \(Y\).

By Theorem 2.6, the transpose \(\iota^T\) projects the simplex conv \(\{e^i : i \in \{n\}\}\) onto a polytope \(Q\) with \(\mathcal{N}(Q) = \Delta\); this is the polytope associated to the ample divisor \(\iota^*\mathcal{O}(1)\). We define the Chow polytope and subdivision using the same projection. For \(X \subseteq Y\) a cycle, we define \(\text{Chow}_\iota(X) = \iota^T \text{Chow}(X)\). Similarly, if \(\Pi \subseteq \mathbb{M}_R \times \mathbb{R}\) is the polytope determining the regular subdivision \(\text{Chow}_\nu(X)\), then we define \(\text{Chow}_{\iota,\nu}(X)\) to be the regular subdivision of \((\iota^T \times \text{id}_\mathbb{R})(\Pi)\).

Returning to \(\mathbb{P}^{n-1}\) as ambient variety, Theorem 2.2 of [9] provides a procedure that determines the polytope \(\text{Chow}(X)\) given a fan tropical variety \(X = \text{Trop} X\). That procedure is the constant-coefficient case of the next theorem, Theorem 4.9, which can be interpreted as justifying our definition of the Chow subdivision. Theorem 4.9 determines \(\text{Chow}_\nu(X)\) for \(X = \text{Trop} X\) not necessarily a fan, by identifying the regions of the complement of \(\mathcal{N}(\text{Chow}_\nu(X))\) and the vertex of \(\text{Chow}_\nu(X)\) each of these regions is dual to.

**Theorem 4.9.** Let \(\dim X = d - 1\). Let \(u \in \mathbb{N}_R\) be a linear functional such that face\(_u\) \(\text{Chow}_\nu(X)\) is a vertex of \(\text{Chow}_\nu(X)\). Then

\[
in_u R_X = \prod_{J \in \binom{\{n\}}{n-d}} [J]^{\text{deg}([u + C_J] \cdot X)},
\]
Figure 2. Identifying a vertex of a Chow subdivision by Theorem 4.9. Coordinates of vertices of the curve are given in $\mathbb{R}^4/(1,1,1,1)$.

\[ \text{vertex}_{u} \operatorname{Chow}(X) = \sum_{J \in \binom{[n]}{\dim X}} \deg([u + C_{J}] \cdot X) e^J. \]

Recall that $C_{J} = \mathbb{R}_{\geq 0}\{e_{j} : j \in J\}$. The condition that $\text{face}_{u} \operatorname{Chow}(X)$ be a vertex is the genericity condition necessary for the set-theoretic intersection $(u + C_{J}) \cap X$ to be a finite set of points.

The constant-coefficient case of Theorem 4.9 is known as ray-shooting, and the general case as orthant-shooting, since the positions of the vertices of $\operatorname{Chow}(X)$ are read off from intersection numbers of $X$ and orthants $C_{J}$ shot from the point $u$.

**Example 4.10.** Let $X$ be the conic curve of Example 4.7. The black curve in Figure 2 is $X = \text{Trop} X$. Arbitrarily choosing the cone point of the red tropical plane to be $u \in N_{\mathbb{R}}$, we see that there are two intersection points among the various $[u + C_{J}] \cdot X$, the two points marked as black dots. Each has multiplicity 1, and they occur one each for $J = \{1, 3\}$ and $J = \{2, 3\}$. Accordingly $e^{\{1,3\}} + e^{\{2,3\}} = (1, 1, 2, 0)$ is the corresponding vertex of $\operatorname{Chow}(X)$ (compare Figure 1).

Theorem 4.9 is proved in the literature, in a few pieces. The second assertion, orthant-shooting in the narrow sense, for arbitrary valued fields is Theorem 10.1 of [16]. The first assertion, describing initial forms in the Chow form, is essentially Theorem 2.6 of [15]. This is stated in the trivial valuation case but of course extends to arbitrary valuations with our machinery of regular subdivisions in one dimension higher. The connection of that result with orthant shooting is as outlined in Section 5.4 of [26].
For arbitrary ambient varieties, the second statement of Theorem 4.9 takes the following form. (The analogue of the first statement is in [10 Section 10].) If $N_\mathbb{R}$ is a linear subspace of $N_\mathbb{R}^d$, and $X$ and $Y$ are tropical cycles in $N_\mathbb{R}$, by $X \cdot_{N_\mathbb{R}} Y$ we mean the stable tropical intersection taken in $N_\mathbb{R}$, i.e. where the displacement in the fan displacement rule is restricted to $N_\mathbb{R}$.

**Corollary 4.11.** With setup as in Case 4.8, let $\dim X = d - 1$ let $u \in N_\mathbb{R}$ be such that face $u$ Chow$_{t_0}(X)$ is a vertex of Chow$_{t_0}(X)$. Then

$$\text{vertex}_u \text{Chow}_{t_0}(X) = \sum_F \deg \left( \left[ u + \text{normal}_{N_\mathbb{R}}(F) \right] \cdot_{N_\mathbb{R}} X \right) m(F)$$

where $F$ runs over faces of $Q$ of dimension $\dim X$, and $m(F) = d \int_F x \, dx \in \mathbb{R}^n$.

**Proof.** By definition Chow$_{t_0}(X)$ is the image of Chow$_{t_0}(X)$ under $t^T$. For any tropical cycles $X \subseteq Y$ and $Z$, we have that $(Z \cdot Y) \cdot X = Z \cdot X$ (in treatments such as [2], which develop tropical cycles as zero loci of collections of rational functions and intersection as restriction of rational functions, this is immediate). Using this in (1.1) gives

$$\text{vertex}_u \text{Chow}_{t_0}(X) = \sum_{J \in \binom{[n]}{d-1}} \deg \left( \left[ u + C_J \right] \cdot [N_\mathbb{R}] \cdot_{N_\mathbb{R}} X \right) t^T(e^J)$$

By Theorem 2.6 the sum of these $[C_J] \cdot [N_\mathbb{R}]$ is $N_d - 1 Q$, with the natural tropical weights. A cone of $N_d - 1 Q$ may arise from multiple cones $C_J$. For each dimension $d - 1$ face $F$ of $Q$, consider the images of those vertices of $\text{conv}\{e_i \in [n]\}$ mapped onto it by $t^T$. Take a triangulation of $F$ using these images, coherent in the sense of [22] Section 4, and suppose the simplices used are the $S_j := \text{conv}\{t^Te^i : i \in J\}$ for $J \in T(F) \subseteq \binom{[n]}{d-1}$. Then we have

$$\text{vertex}_u \text{Chow}_{t_0}(X) = \sum_F \deg \left( \left[ u + \text{normal}_{N_\mathbb{R}}(F) \right] \cdot_{N_\mathbb{R}} X \right) \text{vol}(F) \sum_{J \in T(F)} t^T(e^J).$$

But for each $J$ we have $\int_{S_j} x \, dx = \text{vol}(S_j) t^T(e^J)/(d)$. Summing this integral over all the simplices in $F$ yields (4.2). \qed

5. From tropical variety to Chow polytope

Henceforth $d \leq n$ will be a fixed integer, and $X$ will be a $(d - 1)$-dimensional subvariety of the ambient toric variety, which is mostly $\mathbb{P}^{n-1}$.

As explained in [15], the torus $(\mathbb{K}^*)^n$ acts on the Hilbert scheme Hilb($\mathbb{P}^{n-1}$) in the fashion induced from its action on $\mathbb{P}^{n-1}$, and the map Hilb($\mathbb{P}^{n-1}$) $\rightarrow$ Gr($d, n, r$) sending each ideal to the corresponding cycle is $(\mathbb{K}^*)^n$-equivariant. This implies that deformations in Hilb($\mathbb{P}^{n-1}$) determine those in Gr($d, n, r$): if $u, u' \in N_\mathbb{R}$ are such that $\text{in}_u \mathcal{I}(X) = \text{in}_{u'} \mathcal{I}(X)$, where $\mathcal{I}$ denotes the defining ideal, then also $\text{in}_u R_X = \text{in}_{u'} R_X$. Accordingly each initial ideal of $\mathcal{I}(X)$ determines a face of Chow($X$), so that the Gröbner fan of $X$ is a refinement of the normal fan of Chow($X$).

The standard construction of the tropical variety $X$ via initial ideals [23 Theorem 2.6] shows that $X$ is a subfan of the Gröbner fan. But in fact $X$ is a subfan of the
coarser fan $\mathcal{N}(\text{Chow}(X))$, since the normal cone of a face $\text{face}_u \text{Chow}(X)$ appears in $X$ if and only if $X$ meets the maximal torus $(\mathbb{K}^*)^n / \mathbb{K}^* \subseteq \mathbb{P}^{n-1}$, and whether this happens is determined by the cycle associated to $X$. The analogue of this holds in the non-fan case as well. This reflects the principle that the information encoded in the Hilbert scheme but not in the Chow variety pertains essentially to nonreduced structure, while tropical varieties have no notion of embedded components and only multiplicities standing in for full-dimensional non-reduced structure.

The machinery of Section 3 allows us to give a lean combinatorial characterisation of the Chow subdivision in terms of Theorem 4.9.

**Main theorem 5.1.** For projective tropical varieties, we have

$$\mathcal{N}^1(\text{Chow}_\nu(X)) = X \oplus \mathcal{L}_{n-d-1} \text{refl}.\!
$$

In general, with the notation of Case 4.8

$$\mathcal{N}^1(\text{Chow}_\nu(X)) = X \oplus \mathcal{N}^d(Q) \text{refl}.\!
$$

To reiterate: Let $X$ be a $(d-1)$-cycle in a projective tropical variety $Y$, and let $X = \text{Trop } Y$. Then the codimension 1 part of the normal subdivision to the Chow subdivision of $X$ is the stable Minkowski sum of $X$ and the reflection of the codimension $d$ skeleton of the fan of $Y$ (with its natural weights under the embedding). In the projective case, the second summand is the reflected linear space $\mathcal{L}_{n-d-1} \text{refl}$. By Theorem 2.5(b), this uniquely determines $\text{Chow}_\nu(X)$ in terms of $X$, up to translation and adding a constant to the vertex heights.

Theorem 5.1 should be taken as providing the extension of the notion of Chow polytope (via its normal fan) to tropical varieties.

**Definition 5.2.** Let the *Chow map* $\text{ch}$ for projective space be the map taking a tropical cycle $X$ of dimension $d$ to its (tropical) *Chow hypersurface*, the cycle $\text{ch}(X) = X \oplus \mathcal{L}_{n-d-1} \text{refl}$.

More generally, for an ambient projective toric variety $\iota : Y \to \mathbb{P}^{n-1}$, let the Chow map $\text{ch}_\iota$ be given by $\text{ch}_\iota(X) = X \oplus \mathcal{N}^d(Q) \text{refl}$. The dimension of $\text{ch}(X)$ is $(d-1) + (n-d-1) = n-2$, so its codimension is 1. Indeed $\text{ch}$ is a linear map $Z_{d-1} \to Z^{n-1}$. Likewise $\text{ch}_\iota$ is a linear map $Z_{d-1} \to Z_{\dim Y-1}$.

**Remark 5.3.** In the projective case, the support of $\text{ch}(X)$ is precisely the set of points $u \in N_\mathbb{R}$ such that a tropical $(n-d-1)$-plane centered at $u$ meets $X$. This is very reminiscent of the classical construction of the Chow form in Remark 4.3 which uses classical $(n-d-1)$-planes meeting $X$. The most significant difference between the two constructions is that the classical *Chow hypersurface* lies in $\text{Gr}(n-d,n)$, where it is the zero locus of the Chow form $R_X$. By contrast our tropical Chow hypersurface $\text{ch}(X)$ lies in the tropical torus $(\mathbb{K}^*)^n / \mathbb{K}^*$, in the same space as $X$. One might think of this as reflecting the presence in tropical projective geometry of a single canonical nondegenerate linear space $\mathcal{L}_e$ of each dimension, something with no classical analogue.

Following the classical construction more closely, one could associate to $X$ a hypersurface $Y$ in $\text{Trop } \text{Gr}(n-d,n)$, namely the tropicalisation of the ideal generated by $R_X$ and the Plücker relations. The torus action $(\mathbb{K}^*)^n / \mathbb{K}^* \times \text{Gr}(n-d,n)$ tropicalises to an action of $N_\mathbb{R}$ on $\text{Trop } \text{Gr}(n-d,n)$ by translation, i.e. an $(n-1)$-dimensional lineality space. Denote by $N_\mathbb{R} + 0$ the orbit of the origin in $\text{Trop } \text{Gr}(n-
Proof of Theorem 5.1. We begin in the projective Case 2.1. Given a regular subdivision $T$ of lattice polytopes in $M$ induced by $\Pi$, its support function $V_T: u \mapsto \text{face}_u T$ is a piecewise linear function whose domains of linearity are $\mathcal{N}(T, \Pi)$. We can view $V_T$ as an element of $(\mathbb{Z}^{\text{unbal}})^0 \otimes M$.

We take a linear map $\delta : (\mathbb{Z}^{\text{unbal}})^0 \otimes M \to (\mathbb{Z}^{\text{unbal}})^1$ such that $\delta(V_T) = \mathcal{N}^1(T, \Pi) \in Z^1$ for any regular subdivision $T$. The restriction of $\delta$ to the linear span of all support functions is a canonical map $\delta'$, which has been constructed as the map from Cartier divisors supported on $\mathcal{N}(T, \Pi)$ to Weil divisors on $\mathcal{N}(T, \Pi)$ in the framework of [2], or as the map from piecewise polynomials to Minkowski weights given by equivariant localisation in [18]. Roughly, $\delta'(V)$ is the codimension 1 tropical cycle whose multiplicity at a facet $\tau$ records the difference of the values taken by $V$ on either side of $\tau$. We can take $\delta$ as any linear map extending $\delta'$ such that $\delta(V)$ still only depends on $V$ locally; our only purpose in making this extension is to allow formal manipulations using unbalanced cycles.

Let $V = V_{\text{Chow},v}(X)$, and write $X = \sum_{\sigma \in \Sigma} m_\sigma [\sigma]$. Expanding (4.1) in terms of this sum, the value of $V$ at $u \in N_\mathbb{R}$ is

$$V = \sum_{\sigma \in \Sigma} m_\sigma \sum_{J \in (n \setminus d)} \deg([\sigma] \cdot [u + C_J]) e^J.$$

The intersection $[\sigma] \cdot [u + C_J]$ is zero if $u \notin \sigma - C_J$, and if $u \in \sigma - C_J$ it is one point with multiplicity $\mu_{\sigma, C_J}$. So

$$V = \sum_{\sigma \in \Sigma} m_\sigma \sum_{J \in (n \setminus d)} \mu_{\sigma, C_J} [\sigma - C_J] \otimes e^J.$$

Let $V_\sigma$ be the inner sum here, so that $V = \sum_{\sigma \in \Sigma} m_\sigma V_\sigma$. Then

$$\delta(V_\sigma) = \sum_{J \in (n \setminus d)} \sum_{\tau \text{ a facet of } \sigma - C_J} \delta([\tau] \otimes e^J).$$

Here, if $\tau$ is a facet of form $\sigma' - C_J$ for $\sigma'$ a facet of $\sigma$, then $e^j \in \mathbb{R} \tau$ so $\delta([\tau] \otimes e^j) = 0$ and the $\tau$ term vanishes. Otherwise $\tau$ has the form $\sigma - C_{J'}$ where $J' = J \setminus \{j\}$ for some $j \in J$. Regrouping the sum by $J'$ gives

$$\delta(V_\sigma) = \sum_{J' \in (n \setminus d_1)} \left( \sum_{J \in (n \setminus J')} \mu_{\sigma, C_{J' \cup \{j\}}} [\sigma - C_{J'}] \otimes e^j \right).$$

This should be compared to the fact that the Chow form of a cycle $X$ in $\text{Gr}(d, n, r)$ is of degree $r = \deg X$ in $\mathbb{K}([\text{Gr}(n - d, n)]$, and this ring is generated by brackets in $n - d = \text{codim} X$ letters.
where again we have omitted the terms \( \delta([\sigma - C_{J'}] \otimes e^I) = 0 \). Now, if \( j \notin J' \) then

\[
\mu_{\sigma,C_{J' \cup (j)}} = \mu_{\sigma,C_{J'}} \\
= [N_{\sigma + C_{J' \cup (j)}} : N_{\sigma + N_{C_{J' \cup (j)}}} ] \\
= [N : N_{\sigma} + N_{C_{J'}} + Ze_j] \\
= [N : N_{\sigma} + C_{J'} + Ze_j][N_{\sigma} + C_{J'} + Ze_j : N_{\sigma} + N_{C_{J'}} + Ze_i] \\
= [N : N_{\sigma} + C_{J'} + Ze_j][N_{\sigma} + C_{J'} : N_{\sigma} + N_{C_{J'}}] \\
= (e_j,p)\mu_{\sigma,C_{J'}}^\bigoplus
\]

where \( p \) is the first nonzero lattice point in the appropriate direction on a line in \( M_\mathbb{R} \) normal to \( \sigma + C_{J'} \). Then the components of \( p \) are the minors of a matrix of lattice generators for \( \sigma + C_{J'} \) by Cramer’s rule, and the last equality is a row expansion of the determinant computing \( \mu_{\sigma,C_{J'}}^\bigoplus \). If \( j \in J' \) then \( \mu_{\sigma,C_{J' \cup (j)}}^\bigoplus (e_j,p) = 0 = (e_j,p)\mu_{\sigma,C_{J'}}^\bigoplus \) also. So it’s innocuous to let the inner sum in (5.1) run over all \( j \in [n] \), and we get

\[
\delta(V_{\sigma}) = \sum_{J' \in \binom{[n] \setminus \{j\}}{d-1}} \left( \sum_{j \in [n]} \mu_{\sigma,C_{J'}}^\bigoplus (e_j,p)\delta([\sigma - C_{J'}] \otimes e^j) \right) \\
= \sum_{J' \in \binom{[n]}{d-1}} \mu_{\sigma,C_{J'}}^\bigoplus \delta([\sigma - C_{J'}] \otimes p) \\
= \sum_{J' \in \binom{[n]}{d-1}} \mu_{\sigma,C_{J'}} [\sigma - C_{J'}] \\
= ([\sigma] \bigoplus \mathcal{L}_{n-d-1}^{\refl}).
\]

We conclude that

\[
\mathcal{N}^1(\text{Chow}_\nu(X)) = \delta(V) = \sum_{\sigma} m_\sigma ([\sigma] \bigoplus \mathcal{L}_{n-d-1}^{\refl}) = X \bigoplus \mathcal{L}_{n-d-1}^{\refl}.
\]

Finally we handle the case of arbitrary ambient variety. We have that \( \mathcal{L}_{n-d-1} \) is the codimension \( d \) skeleton of the simplex \( S := \text{conv}\{e^i : i \in [n]\} \). Then

\[
\mathcal{N}^1(\text{Chow}_{\nu,J}(X)) = \mathcal{N}^1(i^T \text{Chow}_\nu(X)) \\
= Y \cdot \mathcal{N}^1(\text{Chow}_\nu(X)) \quad \text{by Theorem 2.6} \\
= Y \cdot (X \bigoplus \mathcal{N}^d(S)) \\
= X \bigoplus (Y \cdot \mathcal{N}^d(S)) \quad \text{by Lemma 3.2} \\
= X \bigoplus \mathcal{N}^d(Q) \quad \text{by Theorem 2.6} \quad \square
\]

6. Linear Spaces

A matroid subdivision (of rank \( r \)) is a regular subdivision of a matroid polytope (of rank \( r \)) all of whose facets are matroid polytopes, i.e. polytopes of the form \( \text{Poly}(M) \) defined in (1.1). The hypersimplex \( \Delta(r,n) \) is the polytope \( \text{conv}\{e^J : J \in \binom{[n]}{r}\} \). The vertices of a rank \( r \) matroid polytope are a subset of those of \( \Delta(r,n) \). We have the following polytopal characterisation of matroid polytopes due to Gelfand, Goresky, MacPherson, and Serganova.
Theorem 6.1. A polytope $\Pi \subseteq \mathbb{R}^n$ is a matroid polytope if and only if $\Pi \subseteq [0,1]^n$ and each edge of $\Pi$ is a parallel translate of $e^i - e^j$ for some $i, j$.

Definition 6.2. Given a regular matroid subdivision $\Sigma$, its Bergman complex $\mathcal{B}(\Sigma)$ and co-Bergman complex $\mathcal{B}^*(\Sigma)$ are subcomplexes of $\mathcal{N}(\Sigma)$. The face of $\mathcal{N}(\Sigma)$ normal to $F \in \Sigma$ is a face of $\mathcal{B}(\Sigma)$ if and only if $F$ is the polytope of a loop-free matroid; it is a face of $\mathcal{B}^*(\Sigma)$ if and only if $F$ is the polytope of a coloop-free matroid.

We make $\mathcal{B}(\Sigma)$ and $\mathcal{B}^*(\Sigma)$ into tropical varieties by giving each facet multiplicity 1.

The Bergman fan, the fan case of the Bergman complex, was introduced in [3] (where an object named the “Bergman complex” different to ours also appears). Bergman complexes are much used in tropical geometry, on account of the following standard definition, appearing for instance in [24].

Definition 6.3. A tropical linear space is the Bergman complex of a regular matroid subdivision.

In the context of Chow polytopes it is the co-Bergman complex rather than the Bergman complex that arises naturally, on account of the duality mentioned in Example 4.5(2). Observe that the co-Bergman complex of a matroid subdivision is a reflection of the Bergman complex of the dual matroid subdivision; in particular any Bergman complex is a co-Bergman complex and vice versa.

Since there is a good notion of tropical degree (Definition 3.3), the following alternative definition seems natural.

Definition 6.4. A tropical linear space is a tropical variety of degree 1.

Theorem 6.5. Definitions 6.3 and 6.4 are equivalent.

The equivalence in Theorem 6.5 was noted by Mikhalkin, Sturmfels, and Ziegler and recorded in [14], but no proof was provided. One implication, that Bergman complexes of matroids have degree 1, follows from Proposition 3.1 of [24], which implies that the tropical stable intersection of a $(d-1)$-dimensional Bergman complex of a matroid subdivision with $\mathcal{L}_{n-d}$ (the Bergman complex of a uniform matroid) is a 0-dimensional Bergman complex, i.e. a point with multiplicity 1. Thus it remains to prove that degree 1 tropical varieties are (co-)Bergman complexes. In fact, let $X \subseteq \mathbb{N}$ be a degree 1 tropical variety of dimension $d-1$. We will show

1. The regular subdivision $\Sigma$ such that $ch(X) = \mathcal{N}(\Sigma)$ is dual to a matroid subdivision of rank $n - d$.
2. We have $X = \mathcal{B}^*(\Sigma)$.

Tropical varieties have an analogue of Bézout’s theorem. See for instance Theorem 9.16 of [2], which however only proves equality under genericity assumptions, not the inequality below. We will only need the theorem in the case that the varieties being intersected have degree 1.

Theorem 6.6 (Tropical Bézout’s theorem). Let $X$ and $Y$ be tropical varieties of complementary dimensions. We have $\deg(X \cdot Y) \leq \deg X \deg Y$, and equality is attained if $X$ and $Y$ are of sufficiently generic combinatorial type.
Lemma 6.7. If a tropical variety $X$ of degree 1 contains a ray in direction $-e_i$ for $i \in [n]$, then $-e_i$ is contained in the lineality space of $X$.

Proof. Consider the set

$$Y = \{u \in N_\mathbb{R} : u - ae_i \in X \text{ for } a \gg 0\}.$$ 

By assumption on $X$, $Y$ is nonempty. This $Y$ is the underlying set of a polyhedral complex; make it into a cycle by giving each facet multiplicity 1. In fact, $Y$ is a tropical variety, as any face $\tau$ of $Y$ corresponds to a face $\sigma$ of $X$ such that $\tau = \sigma + \mathbb{R}e_i$, and so $Y$ inherits balancing from $X$. Also $\dim Y = \dim X = d - 1$.

Since $Y$ is effective, some translate and therefore any translate of $L_{n-d-1}$ intersects $Y$ stably in at least one point.

Suppose $X$ had a facet $\sigma$ whose linear span didn’t contain $-e_i$. Then there is some translate $[u] \boxplus L_{n-d-1}$ which intersects relint $\sigma$, with the intersection lying on a face $u + C_J$ of $[u] \boxplus L_{n-d-1}$ for $i \in J$. Given this translate, any other translate $[u - ae_i] \boxplus L_{n-d-1}$ with $a \geq 0$ will intersect $X$ transversely in the same point of relint $\sigma$. For a sufficiently large, one of the points of $Y \cdot ([u - ae_i] \boxplus L_{n-d-1})$ lies in $X$, providing a second intersection point of $X$ and $[u - ae_i] \boxplus L_{n-d-1}$. By Bézout’s theorem this contradicts the assumption that $\deg X = 1$.

Proof of Theorem 6.5. To (1). Suppose $l \subseteq N_\mathbb{R}$ is a classical line in any direction $e_J$, $J \subseteq [n]$. By Lemma 6.1 and Theorem 6.6 we have

$$\deg(ch(X) \cdot [l]) = \deg((X \boxplus L_{(n-d-1)}} \cdot [l]) = \deg((L_{(n-d-1)} \boxplus [l] \cdot X) \leq 1$$

because $L_{(n-d-1)} \boxplus [l]$ is a degree 1 tropical variety. Since intersection multiplicities are positive, if $l$ intersects a facet $\sigma$ of $ch(X)$ then the multiplicity of the intersection is $\mu_{\sigma,l} = 1$.

Let $\sigma$ be a facet of $ch(X)$, and $l$ a line in direction $e_J$ intersecting it. Then $\mu_{\sigma,l} = \langle m, e_J \rangle$ where $m \in M_\mathbb{R}$ is the difference of the endpoints of the edge of $\Sigma$ dual to $\sigma$. Then $m$ is the product of a primitive normal vector to $\sigma$ and the multiplicity $m_\sigma$. The positive components of $m$ cannot have sum $k \geq 2$, or else, for a suitable choice of $J$, we would achieve $\mu_{\sigma,l} = \langle m, e_J \rangle = k$. Since $m$ is nonzero and normal to $(1, \ldots, 1)$ we must have $m = e_i - e_j$ for some $i \neq j \in [n]$. It follows that each edge of $\Sigma$ is a parallel translate of some $e_i - e_j$.

Furthermore, let $l \subseteq N_\mathbb{R}$ be a line in direction $e_i$, for $i \in [n]$. The vertices of $\Sigma$ attained as face$_u \Sigma$ for some $u \in l$ are in bijection with the connected components of the complement of $ch(X)$. So there are at most two of these vertices, and if there are two, say $m_0$ and $m_1$, we have $\langle m_1 - m_0, e_i \rangle = 1$. But among the vertices face$_u \Sigma$ for $u \in l$ are vertices $m$ minimising and maximising the pairing $\langle m, e_i \rangle$. Therefore, the projection of $\Sigma$ to the $i$th coordinate axis has length either 0 or 1.

For the remainder of the proof we fix a particular translation representative of $\Sigma$, namely the one whose projection onto the $i$th coordinate axis is either the point $\{0\}$ or the interval $[0, 1]$ for each $i \in [n]$. For this particular $\Sigma$, Theorem 6.1 implies that $\Sigma$ is a matroid subdivision.

Let $r$ be the rank of the matroid subdivision $\Sigma$. Let $e_J$ be one vertex of $\Sigma$, so that $|J| \in {\binom{n}{r}}$, and let $u$ be a linear form with face$_u \Sigma = e_J$. Then, for any $i \in [n] \setminus J$ and any $a > 0$, we have face$_{u + ae_i} \Sigma = e_j$, since $e_j \in$ face$_{e_i} \Sigma$. On the other hand, for any $i \in J$ and sufficiently large $a \gg 0$, we have face$_{u + ae_i} \Sigma \neq e_j$. 


and indeed face$_{u + ae_i} \Sigma$ will contain some vertex $e^\rho$ with $i \not\in J'$, whose existence is assured by our choice of translation representative for $\Sigma$. It follows that a ray $[u] \oplus [R_{\geq 0}(e_1)]$ of $[u] \oplus L_1$ intersects $ch(X)$ if and only if $i \in J$. Each intersection must have multiplicity 1, so

$$\deg(ch(X)) = \deg(ch(X) \cap ([u] \oplus L_1)) = |J| = r.$$ But by Proposition 3.4 we have that $\deg(ch(X)) = n - d$, so $r = n - d$ as claimed.

To (2). Fix some polyhedral complex structure on $X$. Given any $u \in N_R$ in the support of $ch(X)$, its multiplicity is $ch(X)(u) = 1$, and therefore by positivity there is a unique choice of a facet $\tau$ of $X$ and $J \subset \binom{[n]}{n-d-1}$ such that $u \in X - C_J$. Write $J = J(u)$. On the other hand, $\Sigma$ has a canonical coarsest possible polyhedral complex structure, on account of being a normal complex. We claim that $J(u)$ is constant for $u$ in the relative interior of each facet $\sigma$ of $\Sigma$, and thus we can write $J(\sigma) := J(u)$. Suppose not. Consider the common boundary $\rho$ of two adjacent regions $\sigma_1, \sigma_2$ of $\sigma$ on which $J(u)$ is constant. Suppose $\sigma_1 \subset \tau - C_{J_1}$. We have $\rho \subset \tau - C_{K}$ for $K \subset \binom{[n]}{n-d-2}$. There is a facet of $\Sigma$ of form $\sigma_j \subset \tau - C_{K_{J_k}}$ incident to $\rho$ for each $k \in [n] \setminus K$ such that $e_k$ is not contained in the affine hull of $\tau$. Since $\dim \tau = d - 1$, and any $d$ of the $e_k$ are independent in $N_R$, there exist at most $d - 1$ indices $k \in [n]$ such that $e_k$ is not contained in the affine hull of $\tau$, and hence at least

$$|[n] \setminus K| - (d - 1) = 3$$

indices $k \in [n]$ yielding facets of $\Sigma$. In particular $\sigma_1$ and $\sigma_2$ cannot be the only $(d - 1)$-dimensional regions in $\Sigma$ incident to $\rho$, and this implies $\sigma$ cannot be a facet of $\Sigma$, contradiction.

Now, every facet $\sigma$ of $ch(X)$ is normal to an edge of $\Sigma$, say $E_\sigma = \text{conv}\{e^K + e^j, e^K + e^k\}$ for $K \subset \binom{[n]}{n-d-1}$. Since $\Sigma \subset \Delta(n - d, n)$, $\sigma$ must contain a translate of the normal cone to $E_\sigma$ in $\mathbb{R}^n(\Delta(n - d, n))$, namely

$$\text{normal}(E_\sigma) = \{u \in N_R : u_j = u_k, u_i \leq u_j \text{ for } i \in K, u_i \geq u_j \text{ for } i \not\in K \cup \{j, k\}\}.$$ In particular $\sigma$ contains exactly $n - d - 1$ rays in directions $-e_i$, those with $i \in K$.

Let $R$ be the set of directions $-e_1, \ldots, -e_n$. Suppose for the moment that $X$ contains no lineality space in any direction $-e_i$. We have that $\sigma \subset X \oplus [-C_{J(\sigma)}]$. By Lemma 6.7 $X$ contains no rays in directions in $R$, so we must have that $J(\sigma) = K$ and $-C_{J(\sigma)}$ contains a ray in direction $-e_i$ for all $i \in K$. Now consider any face $\rho$ of $\sigma$ containing no rays in directions in $R$. Then we claim $\rho \subset X$. If this weren’t so, then there would be another face $\sigma'$ parallel to $\sigma$ and with $J(\sigma) = J(\sigma')$. But the edge $E_\sigma$ is determined by $J(\sigma) = K$ and the normal direction to $\sigma$, so $E_\sigma = E_{\sigma'}$, implying $\sigma = \sigma'$. On the other hand, the relative interior of any face of $\sigma$ containing a ray in direction $R$ is disjoint from $X$, since if $u$ is a point in such a face there exists $v \in -C_{J(\sigma)} \setminus \{0\}$ such that $u - v \in X$. So $X$ consists exactly of the faces of $ch(X)$ containing no ray in a direction in $R$.

If $X$ has a lineality space containing those $-e_j$ with $j \in J$, then let $X'$ be the pullback of $X$ along a linear projection with kernel span$\{-e_j : j \in J\}$. Then we can repeat the last argument using $X'$, and we get that $X$ consists exactly of the faces of $ch(X)$ containing no ray in a direction in $R \setminus \{-e_j : j \in J\}$.

Now, a face $\text{normal}(F)$ of $N(\Sigma)$ contains a ray in direction $-e_i$ if and only if the linear functional $\langle m, -e_i \rangle$ is constant on $m \in F$ and equal to its maximum.
for \( m \in \Sigma \). The projection of \( F \) to the \( i \)th coordinate axis is either \{0\}, \{1\}, or \([0, 1]\), so \( \text{normal}(F) \) contains a ray in direction \(-e_i\) if and only if the projection of \( F \) is \{1\}, or the projection of \( F \) and of \( \Sigma \) are both \{0\}. Projections taking \( \Sigma \) to \{0\} correspond to lineality directions in \( X \), so we have that \( X \) consists exactly of the faces of \( \text{ch}(X) \) which don’t project to \{1\} along any coordinate axis. These are exactly the coloop-free faces. \( \square \)

7. The kernel of the Chow map

In this section we will show that the Chow map \( \text{ch} : Z_{d-1} \to Z^1 \) has a nontrivial kernel. This implies that there exist distinct tropical varieties with the same Chow polynomial: \( Y \) and \( X + Y \) will be a pair of such varieties for any nonzero \( X \in \ker \text{ch} \), choosing \( Y \) to be any effective tropical cycle such that \( X + Y \) is also effective (for instance, let \( Y \) be a sum of classical linear spaces containing the facets of \( X \) that have negative multiplicity). Thus Chow subdivisions do not lie in a combinatorial bijection with general tropical varieties, as was the case for our opening examples.

There are a few special cases in which \( \text{ch} \) is injective. In the case \( d = n - 1 \) of hypersurfaces, \( \text{ch} \) is the identity. In the case \( d = 1 \), in which \( X \) is a point set with multiplicity, \( \text{ch}(X) \) is a sum of reflected tropical hyperplanes with multiplicity, from which \( X \) is easily recoverable. Furthermore, Conjecture 7.2 below would imply restrictions on the rays in any one-dimensional tropical fan cycle in \( \ker \text{ch} \), and one can check that no cycle with these restrictions lies in \( \ker \text{ch} \).

Example 7.1 provides an explicit tropical fan cycle in \( \ker \text{ch} \) in the least case, \((d, n) = (3, 5)\), not among those just mentioned. First we introduce the fan on which the example depends, which seems to be of critical importance to the behaviour of \( \ker \text{ch} \) in general.

Let \( A_n \subseteq \mathbb{R}^{n-1} \) be the fan in \( N_{\mathbb{R}} \) consisting of the cones \( \mathbb{R}_{\geq 0}\{e_{J_1}, \ldots, e_{J_l}\} \) for all chains of subsets
\[
\emptyset \subsetneq J_1 \subsetneq \cdots \subsetneq J_l \subsetneq [n].
\]
This fan \( A_n \) makes many appearances in combinatorics. It is the normal fan of the permutahedron, and by Theorem 6.1 also the common refinement of all normal fans of matroid polytopes. Its face poset is the order poset of the boolean lattice. Moreover, its codimension 1 skeleton is supported on the union of the hyperplanes \( \{x_i = x_j : i \neq j \in [n]\} \) of the type A reflection arrangement, i.e. the braid arrangement.

As in Section 2.4 the ring \( Z^\text{fan}(A_n) \) is the Chow cohomology ring of the toric variety associated to \( \Sigma \). This toric variety is the closure of the torus orbit of a generic point in the complete flag variety (which, to say it differently, is \( \mathbb{P}^{n-1} \) blown up along all the coordinate subspaces). The cohomology of this variety has been studied by Stembridge 25. We have that \( \dim Z^\text{fan}(A_n) = n! \), and \( \dim(Z^\text{fan})^k(A_n) \) is the Eulerian number \( E(n, k) \), the number of permutations of \([n]\) with \( k \) descents.

For any cone \( \sigma = \mathbb{R}_{\geq 0}\{e_{J_1}, \ldots, e_{J_l}\} \) of \( A_n \), and any orthant \( \sigma_{J_1}^\text{refl} = \mathbb{R}_{\geq 0}\{-e_j : j \in J'\} \), the Minkowski sum \( \sigma + \sigma_{J_1}^\text{refl} \) is again a union of cones of \( A_n \). Therefore \( \text{ch}(Z^\text{fan}(A_n)) \subseteq (Z^\text{fan})^1(A_n) \) always, and we find nontrivial elements of \( \ker \text{ch} \) whenever the dimension of \( Z^\text{fan}(A_n) \) exceeds that of \( (Z^\text{fan})^1(A_n) \), i.e. when \( E(n, n-d) > E(n, 1) \), equivalently when \( 2 < d < n - 1 \).
Example 7.1. For \((d, n) = (3, 5)\), we have \(E(5, 5 - 3) = 66 > 26 = E(5, 1)\), and the kernel of \(ch\) restricted to \(Z^\text{fan}_2(\mathcal{A}_5)\) is 40-dimensional. Two tropical varieties in \(Z^\text{fan}_2(\mathcal{A}_5)\) within \(N_R = \mathbb{R}^4\) with equal Chow hypersurfaces are depicted in Figure 3. As one often does, we have dropped one dimension in the drawing by actually drawing the intersections of these 2-dimensional tropical fans with a sphere centered at the origin in \(\mathbb{R}^4\), which are graphs in \(\mathbb{R}^3\). The difference of these varieties is an actual element of \(\ker ch\), involving the six labelled rays other than 123, which form an octahedron.

The property of \(\mathcal{A}_n\) that this example exploits appears to be essentially unique: this is part (a) of the next conjecture. This property, together with experimentation with fan varieties of low degree in low ambient dimension, also suggests part (b).

Conjecture 7.2.

(a) Let \(\Sigma\) be a complete fan such that the stable Minkowski sum of any cone of \(\Sigma\) and any ray \(\mathbb{R}_{\geq 0}(-e_i)\) is a sum of cones of \(\Sigma\). Then \(\mathcal{A}_n\) is a refinement of \(\Sigma\).

(b) The kernel of the restriction of \(ch\) to fan varieties is generated by elements of \(Z^\text{fan}(\mathcal{A}_n)\).

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