NAVIER-STOKES LIMIT OF GLOBALLY HYPERBOLIC MOMENT EQUATIONS

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Abstract. This paper is concerned with the Navier-Stokes limit of a class of globally hyperbolic moment equations from the Boltzmann equation. We show that the Navier-Stokes equations can be formally derived from the hyperbolic moment equations for various different collision mechanisms. Furthermore, the formal limit is justified rigorously by using an energy method. It should be noted that the hyperbolic moment equations are in non-conservative form and do not have a convex entropy function, therefore some additional efforts are required in the justification.

1. Introduction. The Boltzmann equation [6], rooted in the Hamiltonian dynamics of a gas particle system, is a popular gas kinetic equation with applications ranging for rarefied gas flows to plasma physics [27, 44, 7, 1, 42, 14, 33, 40, 4]. It satisfies the conservation laws of fluid mass, momentum and energy, and also admits the H-theorem—a kinetic description of the second law of thermodynamics. Though the Boltzmann equation has a solid physical ground, it is computationally costly due to its high dimension and therefore is not directly usable in engineering applications. To remedy this, many simplifications or approximations have been proposed, such as the discrete velocity models [36], the moment closure systems [25, 34], and so on. These models are aimed at reducing the dimension of the Boltzmann equation while maintaining the main physical properties and have attracted much attention in recent years [50, 18].

This paper is concerned with the moment closure systems of the Boltzmann equation. The moment systems were initially proposed in Grad’s pioneer work [25] by expanding the phase density function in terms of the Hermite polynomials and projecting each term of the Boltzmann equation into the moment space. The resulting equations, the so-called moment systems, are first order partial differential equations with source terms, for which the hyperbolicity is important. However, Grad’s moment system is not globally hyperbolic, as explicitly shown in [37] and [9]. The hyperbolicity is necessary for a system of first-order PDEs to be well-posed [38] and the well-posedness reflects the observability of physics phenomenon. There has been a persistent effort to construct well-posed generalized hydrodynamic models by various regularizations [41, 5, 23]. It is only recent that Grad’s moment
system is intrinsically regularized to be globally hyperbolic [9, 8]. Particularly, a
general framework was proposed in [11, 21] to derive hyperbolic moment systems for
general kinetic equations. Numerical results in [13, 12] indicated that the hyperbolic
moment systems are reasonable and reliable approximations to the kinetic equation.

Since the moment systems are used to model (nonequilibrium) fluid flows, they
are naturally required to be compatible with the classical Euler or Navier-Stokes
equations. At this point, let us mention that the compatibility with the Euler equa-
tions was recently justified in [51] for the globally hyperbolic moment systems by
checking Yong’s structural stability condition proposed in [46] for hyperbolic relax-
ation systems. The structural stability condition to the moment systems is same
as H-theorem to the Boltzmann equation. It characterizes the dissipation property
of the moment systems. Particular, under this condition, it can be shown that
the solution of the moment systems will tend to the solution of the corresponding
equilibrium system, the Euler equations, as the Knudsen number goes to zero. Ad-
ditionally, under the stability condition, a formal asymptotic approximation of the
initial-layer solution to the nonlinear moment system can been constructed [47].

It is well-known that the Navier-Stokes equations can be formally derived from
the Boltzmann equation [15]. There are many works devoted to a rigorous proof
of the derivation [2, 19, 26, 24, 20]. In [24], a complete proof for the limit of the
Boltzmann equation to the incompressible Navier-Stokes equations was established
for certain classes of collision kernels. In [20], the authors presented a rigorous
derivation of the steady incompressible Navier-Stokes system from the Boltzmann
theory. However, the derivation from the moment systems to the Navier-Stokes
equations is a different issue and it seems that not many works have been done.
This paper presents a rigorous derivation of the Navier-Stokes equations from the
globally hyperbolic moment systems constructed in [9, 8, 11].

In this work, I proceed to study the Navier-Stokes limit of the hyperbolic moment
systems. First I formally derive the Navier-Stokes equations from the general glob-
ally hyperbolic moment systems with the Chapman-Enskog expansion. By doing
so, I obtain the coefficients of viscosity and thermal conductivity for general binary
collision models. Furthermore, I present a rigorous justification of the formal deriva-
tion for different collision mechanisms such as BGK, Shakhov and ES-BGK models,
as well as the binary collision model. Namely, the globally hyperbolic moment
systems are compatible with the Navier-Stokes equations as the Knudsen number
tends to zero.

Remark that the globally hyperbolic moment systems are not in conservative
form. It is not clear whether they admit a convex entropy function. Therefore,
the general results in [45] do not apply to the moment systems. In fact, the non-
conservative terms need to be treated carefully. This is the main point of the present
paper. Meanwhile, the authors of [10] formally derived the Navier-Stokes equations
only from the modified 13-moment system for Maxwell molecules. I extend this
result for moment systems of arbitrary order with BGK, Shakhov and ES-BGK models,
as well as the binary collision model.

The paper is organized as follows. In the next section, I present the globally hy-
perbolic moment systems for the Boltzmann equation. In Section 3, I show that the
Navier-Stokes equations can be derived formally from the globally hyperbolic mo-
ment systems by using the Chapman-Enskog expansion. The last section is devoted
to a proof of the compatibility with the Navier-Stokes equations. The Appendixes
A and B are devoted to reviewing the hyperbolicity and Yong’s structural stability
condition, respectively. The details of formal derivation is given in the Appendix C for the binary collision mechanism.

2. **Hyperbolic moment systems of the boltzmann equation.** The Boltzmann equation in the phase space \((x, \xi)\) and time \(t\) can be written as

\[
\partial_t f + \xi \cdot \nabla f = \frac{1}{\varepsilon} Q(f, f).
\]

Here \(f := f(x, \xi, t)\) is the mass density distribution function of the space-velocity-time \((x, \xi, t) \in \mathcal{R}^D \times \mathcal{R}^D \times [0, \infty)\) with \(D\) the dimensions of space, \(Q(f, f)\) denotes a collision term, and \(\varepsilon\) is the Knudsen number. Having the distribution function, we can compute the macroscopic density \(\rho\), flow velocity \(u\), temperature \(\theta\), pressure tensor \(p = (p_{ij})\) and the heat flux \(q\) as

\[
\rho = \int_{\mathcal{R}^D} f d\xi, \quad \rho u = \int_{\mathcal{R}^D} \xi f d\xi, \quad \rho(u^2 + D\theta) = \int_{\mathcal{R}^D} \xi^2 f d\xi,
\]

\[
p = \int_{\mathcal{R}^D} (\xi - u) \otimes (\xi - u) f d\xi, \quad q = \frac{1}{2} \int_{\mathcal{R}^D} |\xi - u|^2 (\xi - u) f d\xi
\]

where the symbol \(\otimes\) stands the tensor product. Note that the deviatoric pressure \(P\) and the stress tensor \(\sigma = (\sigma_{ij})_{D \times D}\) are defined as \(P = \frac{1}{D} \sum_{d=1}^D p_{dd}\) and \(\sigma_{ij} = p_{ij} - \delta_{ij}\) with \(\delta_{ij}\) the Kronecker delta.

In [25], Grad proposed a well-known method to derive moment equations from the Boltzmann equation. In the method, the distribution function \(f(x, \xi, t)\) is expanded in terms of multi-dimensional Hermite tensor polynomials as

\[
f(x, t, \xi) = \sum_{\alpha \in \mathbb{N}^D} f_{\alpha}(x, t) H_{\alpha}^{[u(x,t), \theta(x,t)]}(\xi) w_{\alpha}^{[u(x,t), \theta(x,t)]}(\xi).
\]

Here \(\alpha\) is the usual \(D\)-dimensional multi-index and \(H_{\alpha}^{[u, \theta]}(\xi)\) are the Hermite basis functions defined as

\[
H_{\alpha}^{[u, \theta]}(\xi) = \frac{1}{\omega^{[u, \theta]}(\xi)} \prod_{d=1}^D \frac{\partial^{\alpha_d}}{\partial \xi_{d}^{\alpha_d}} w^{[u, \theta]}(\xi), \quad w_{\alpha}^{[u, \theta]}(\xi) = \frac{f_{eq}(x, t, \xi)}{\rho(x, t)}
\]

with \(f_{eq}\) the local equilibrium

\[
f_{eq} = f_{eq}(x, t, \xi) = \frac{\rho(x, t)}{[2\pi \theta(x, t)]^{D/2}} \exp \left( -\frac{|\xi - u(x, t)|^2}{2\theta(x, t)} \right).
\]

Thanks to the orthogonality of the basis functions, the expansion coefficients can be expressed as [9]

\[
f_{\alpha} = \frac{\delta^{[\alpha]}_{\alpha!}}{\alpha!} \int_{\mathcal{R}^D} f H_{\alpha}^{[u, \theta]}(\xi) d\xi,
\]

where \(|\alpha| = \sum_{d=1}^D \alpha_d\) and \(\alpha! = \prod_{d=1}^D \alpha_d!\). In particular, the first few coefficients satisfy

\[
\begin{align*}
    f_0 &= \rho, \quad f_{e_i} = 0, \quad \sum_{d=1}^D f_{2e_d} = 0, \\
    \sigma_{ij} &= (1 + \delta_{ij}) f_{e_i + e_j}, \quad q_i = f_{3e_i} + \sum_{d=1}^D f_{e_i + 2e_d},
\end{align*}
\]

where \(e_i\) is the \(i\)-th column of the unit matrix \(I_D\).

Substituting the above expansion into the Boltzmann equation and matching the coefficients of the basis functions, we obtain a hierarchy of infinitely many governing
equations for \( f_\alpha \) with \( \alpha \in \mathbb{N}^D \). In [25], Grad chosen a positive integer \( M(\geq 3) \), discarded all the evolution equations for \( f_\alpha \) and set \( f_\alpha = 0 \) with \( |\alpha| > M \) to obtain the following \( M \)-th order moment system

\[
\frac{\partial f_\alpha}{\partial t} + \mathcal{L}_\alpha(F_{|\alpha|-3}, F_{|\alpha|-2}, F_{|\alpha|-1}, F_{|\alpha|}, F_{|\alpha|+1}) = Q_\alpha.
\]

Here the vector \( F_m \) is composed of all \( f_\alpha \) satisfying \( |\alpha| = m \) and

\[
\mathcal{L}_\alpha = \mathcal{L}_\alpha(F_{|\alpha|-3}, F_{|\alpha|-2}, F_{|\alpha|-1}, F_{|\alpha|}, F_{|\alpha|+1})
\]

\[=: \sum_{d=1}^{D} \left( \theta \frac{\partial f_{\alpha-e_d}}{\partial x_d} + u_d \frac{\partial f_\alpha}{\partial x_d} + (1 - \delta_{|\alpha|, M})(\alpha_d + 1) \frac{\partial f_{\alpha+e_d}}{\partial x_d} \right)
\]

\[+ \sum_{d=1}^{D} f_{\alpha-e_d} \frac{\partial u_d}{\partial t} + \sum_{k,d=1}^{D} \frac{\partial u_k}{\partial x_d} (\theta f_{\alpha-e_k-e_d} + u_d f_{\alpha-e_k} + (\alpha_d + 1)f_{\alpha-e_k+e_d})
\]

\[+ \frac{1}{2} \sum_{d=1}^{D} f_{\alpha-2e_d} \frac{\partial \theta}{\partial t} + \sum_{k,d=1}^{D} \frac{1}{2} \frac{\partial \theta}{\partial x_d} (\theta f_{\alpha-2e_k-e_d} + u_d f_{\alpha-2e_k} + (\alpha_d + 1)f_{\alpha-2e_k+e_d}),
\]

where \( |\alpha| \leq M \). Here \( (\cdot)_{\alpha} \) is taken as zero if any component of \( \alpha \) is negative and

\[
Q_\alpha = \frac{1}{\varepsilon} \frac{\theta_{|\alpha|}}{\alpha!} \int_{\mathbb{R}^D} Q(f,f) h^{[\alpha,\theta]}(\xi) d\xi
\]

with \( Q_0 = Q_{eq} = \sum_{d=1}^{D} Q_{2e_d} = 0 \). It is well-known that Grad’s moment system lacks of global hyperbolicity and it was found recently that it is not hyperbolic even around the local Maxwellian [37, 10].

In fixing the drawback of Grad’s moment systems, the authors in [9] proposed the following moment system

\[
\frac{\partial f_\alpha}{\partial t} + \bar{\mathcal{L}}_\alpha(F_{|\alpha|-3}, F_{|\alpha|-2}, F_{|\alpha|-1}, F_{|\alpha|}, F_{|\alpha|+1}) = Q_\alpha, \quad |\alpha| \leq M.
\]

Here \( \bar{\mathcal{L}}_\alpha \) is obtained by modifying \( \mathcal{L}_\alpha \) in the following fashion

\[
\bar{\mathcal{L}}_\alpha = \bar{\mathcal{L}}_\alpha(F_{|\alpha|-3}, F_{|\alpha|-2}, F_{|\alpha|-1}, F_{|\alpha|}, F_{|\alpha|+1})
\]

\[= \mathcal{L}_\alpha - \sum_{k,d=1}^{D} \frac{\partial u_k}{\partial x_d} \delta_{|\alpha|, M}(\alpha_d + 1)f_{\alpha-e_k+e_d} - \sum_{k,d=1}^{D} \frac{1}{2} \frac{\partial \theta}{\partial x_d} \delta_{|\alpha|, M}(\alpha_d + 1)f_{\alpha-2e_k+e_d}.
\]

These systems were shown in [17] to be globally hyperbolic and satisfy Yong’s stability condition [46, 47] with commonly used approximate collision terms including the binary collision model. For the convenience of the reader, the hyperbolicity [16] and Yong’s structural stability condition [46] are reviewed in Appendixes A and B.

We conclude this section with the source terms \( Q_\alpha \) for four different collision mechanisms:

1. The Bhatnagar-Gross-Krook(BGK) model [3]:

\[
\frac{1}{\varepsilon} Q(f,f) = \frac{1}{\tau} (f_{eq} - f)
\]

with \( \tau \) a relaxation time. For this model, the source term \( Q_\alpha \) is

\[
Q_\alpha^{BGK} = \begin{cases} 
0, & |\alpha| = 0, 1 \\
-\frac{\varepsilon}{\tau}, & |\alpha| \geq 2.
\end{cases}
\]
2. The Shakhov model \[39\] :

\[
\frac{1}{\varepsilon} Q(f, f) = \frac{1}{\tau} (f_s - f)
\]

with

\[
f_s = f^{eq} \left( 1 + \frac{(1 - Pr) q^T (\xi - u)}{(D + 2) \rho \theta^2} \left( \frac{|\xi - u|^2}{\theta} - (D + 2) \right) \right),
\]

and \(Pr\) the Prandtl number. For this model, the source term \(Q_\alpha\) is

\[
Q_{\alpha}^{\text{Shakhov}} = \begin{cases} 
0, & |\alpha| = 0, 1, \\
\frac{1 - Pr}{(D + 2) \rho} q_i - \frac{f_{e_i + 2 e_k}}{\tau}, & \alpha = e_i + 2 e_k, \ i, k = 1, \ldots, D, \\
-\frac{f_{\alpha}}{\tau}, & \text{otherwise}.
\end{cases}
\]

(5)

3. The ES-BGK model \[28\] :

\[
\frac{1}{\varepsilon} Q(f, f) = \frac{Pr}{\tau} (f_G - f)
\]

with

\[
f_G = \frac{\rho}{\sqrt{\det(2\pi \Lambda)}} \exp \left( -\frac{1}{2} (\xi - u)^T \Lambda^{-1} (\xi - u) \right).
\]

Here \(\Lambda = (\lambda_{ij}) \in \mathbb{R}^{D \times D}\) is a symmetric positive definite matrix with entries

\[
\lambda_{ij} = \begin{cases} 
\theta \delta_{ij}, & \alpha = 0, \\
0, & |\alpha| \text{ odd}, \\
\frac{1 - 1/Pr}{\rho} \sum_{d=1}^D \frac{\sigma_{ia} G_{a - e_i - e_d}}{\tau}, & |\alpha| \geq 2 \text{ and } \alpha_i > 0.
\end{cases}
\]

(6)

4. The binary collision model \[33\]. This model is commonly used to model the dilute gas and has the following quadratic form

\[
Q(f, f) = \int (f_1 f' - f f') g \, db d \xi_1.
\]

(7)

Here \(g = |\xi_1 - \xi|\) is the asymptotic relative velocity, \(b\) is the impact parameter and \(\epsilon\) is the azimuthal angle,

\[
f = f(x, t, \xi), \quad f_1 = f(x, t, \xi_1), \quad f' = f(x, t, \xi'), \quad f_1' = f(x, t, \xi_1'),
\]

where \(\xi\) and \(\xi_1\) are the velocities of two particles before collision, \(\xi'\) and \(\xi_1'\) are their velocities after collision. For the binary collision model, we have

\[
Q_{\alpha}^{\text{Binary}} = -C_0 \rho \sqrt{\theta} f_\alpha + \frac{C_1 \rho}{\tau} \sum_{1 < |\beta| \leq M, \beta \neq \alpha} f_\beta Q_{\beta,0}^{\alpha} (\theta)
\]

\[
+ \frac{C_2}{\tau} \sum_{1 < |\gamma|, |\beta| \leq M} f_\beta f_\gamma Q_{\beta,\gamma}^{\alpha} (\theta),
\]

(8)

where \(C_0 > 0\) and \(C_1, C_2\) are constants independent of \(\tau\). See Appendix C for details.
3. Formal derivation of the Navier-Stokes equations. In this section, we show that the Navier-Stokes equations can be derived formally from the hyperbolic moment equations (3) with the four different collision mechanisms.

For this purpose, we firstly rewrite the moment equations in (3). For $\alpha = 0$, we recall $f_0 = \rho$ and have the mass equation

$$\frac{\partial \rho}{\partial t} + \sum_{d=1}^{D} u_d \frac{\partial \rho}{\partial x_d} + \sum_{d=1}^{D} \rho \frac{\partial u_d}{\partial x_d} = 0.$$  

For $\alpha = e_d$, we refer to (2) and have the momentum equation

$$\rho \frac{\partial u_i}{\partial t} + \sum_{d=1}^{D} \rho u_d \frac{\partial u_i}{\partial x_d} + \frac{\partial (\rho \theta)}{\partial x_i} + \sum_{d=1}^{D} \frac{\partial \sigma_{id}}{\partial x_d} = 0.$$  

Moreover, summing up the equations in (3) for $\alpha = 2 e_i$ over $i$ yields the energy equation

$$\rho \frac{\partial \theta}{\partial t} + \sum_{d=1}^{D} \rho u_d \frac{\partial \theta}{\partial x_d} + 2 \frac{\partial}{\partial x_d} \left( \rho u_d \left( e + \frac{u^2}{2} \right) \right) + \sum_{d=1}^{D} \frac{\partial q_d}{\partial x_d} + \frac{\partial}{\partial x_d} \left( \sum_{k,d}^{D} \sigma_{kd} \frac{u_k}{u} \right) = 0.$$  

It is not difficult to rewrite the above equations into the following conservative form

$$\frac{\partial \rho}{\partial t} + \sum_{d=1}^{D} \frac{\partial (\rho u_d)}{\partial x_d} = 0,$$

$$\frac{\partial (\rho u_i)}{\partial t} + \sum_{d=1}^{D} \frac{\partial (\rho u_d u_i + \rho \theta \delta_{i,d})}{\partial x_d} + \frac{\partial \rho}{\partial x_i} \frac{\partial \theta}{\partial x_d} + \sum_{d=1}^{D} \frac{\partial \sigma_{id}}{\partial x_d} = 0,$$

$$\frac{\partial}{\partial t} \left( \rho \left( e + \frac{u^2}{2} \right) \right) + \sum_{d=1}^{D} \frac{\partial}{\partial x_d} \left( \rho u_d \left( e + \frac{u^2}{2} \right) + \rho \theta u_d \right) + \sum_{d=1}^{D} \frac{\partial q_d}{\partial x_d} + \frac{\partial}{\partial x_d} \left( \sum_{k,d}^{D} \sigma_{kd} \frac{u_k}{u} \right) = 0.$$  

(9)

Here $e = \frac{D}{2} \theta$ is the internal energy. This system of equations is not closed due to the stress tensor $\sigma_{ij}$ and heat flux $q_i$.

In view of (9) and also for the notational convenience, we rewrite the equations in (3) as

$$\partial_t w + \sum_{j=1}^{D} \partial_{x_j} a_j(w, v) = 0,$$

$$\partial_t v + \sum_{j=1}^{D} b_j(w, v) \partial_{x_j} w + \sum_{j=1}^{D} c_j(w, v) \partial_{x_j} v = \frac{H(w, v)}{\tau}.$$  

(10)

Here $w = (\rho, \rho u, \rho e + \frac{u^2}{2})^T$ stands for the conservation variables and $v$ denotes the other moments in (3). Note that this non-conservative system is different from the conservative ones studied in [45]. From the equation (9) we can see that $a_j(w, v)$ has the following form

$$a_j(w, v) = a_j(w, 0) + a_{jv}(w, 0)v$$

with

$$v = (\sigma_{ij}, q_k, f_\alpha)^T.$$
for \(i, j, k = 1, \cdots, D, 3 \leq |\alpha| \leq M\) and \(\alpha \neq 3e_k\). We take
\[
G := \{(w, v) : \rho > 0, \quad \theta > 0\}
\]
as the domain for the moment system (10).

Next we write down the evolution equations for \(v\). From (3) we easily deduce that the equations for the stress tensor \(\sigma_{ij}\) and heat flux \(q_i = 2f_{3e_i} + \sum_{j=1}^{D} f_{e_j} e_j\) are
\[
\frac{\partial \sigma_{ij}}{\partial t} + \sum_{d=1}^{D} u_d \frac{\partial \sigma_{ij}}{\partial x_d} + \sum_{d=1}^{D} (\sigma_{id} + \rho \theta \delta_{id}) \frac{\partial u_i}{\partial x_d} + \sum_{d=1}^{D} (\sigma_{dj} + \rho \theta \delta_{dj}) \frac{\partial u_j}{\partial x_d} - \sum_{d=1}^{D} 2 \delta_{ij} \frac{\partial q_d}{\partial x_d}
\]
\[
- \sum_{d,k=1}^{D} 2 \delta_{ij} (\sigma_{kd} + \rho \theta \delta_{kd}) \frac{\partial u_d}{\partial x_k} + \sum_{d=1}^{D} (e_i + e_j + e_d) \frac{\partial f_{e_i+e_j+e_d}}{\partial x_d} = Q_{ij},
\]
\[
\frac{\partial q_i}{\partial t} + \sum_{d=1}^{D} u_d \frac{\partial q_i}{\partial x_d} + \sum_{d=1}^{D} q_d \frac{\partial u_i}{\partial x_d} + \sum_{d=1}^{D} \sum_{k,d=1}^{D} (e_d + e_k + e_i) f_{e_d+e_k+e_i} \frac{\partial u_k}{\partial x_d}
\]
\[
+ D + 2 \sum_{d=1}^{D} p_{id} \frac{\partial \theta}{\partial x_d} + \sum_{d=1}^{D} \sigma_{id} \frac{\partial \theta}{\partial x_d} + \sum_{d=1}^{D} \sigma_{id} \frac{\partial \theta}{\partial x_d} - \sum_{k,d=1}^{D} \frac{\sigma_{ik}}{\rho} \frac{\partial p_{ik}}{\partial x_d}
\]
\[
+ \sum_{k,d=1}^{D} (e_i + 2e_k + e_d) \frac{\partial f_{e_i+2e_k+e_d}}{\partial x_d} = Q_i.
\]

The evolution equations for other components of \(v\) are given in equation (3).

About the source terms \(H(w, v)\), we recall equations (4)-(8) and see that
\[
H(w, v) = -v
\]
for the BGK model;

\[
H(w, v) = \begin{cases} 
-\sigma_{ij}, & v = \sigma_{ij}, \\
-Pr q_i, & v = q_i,
\end{cases}
\]

\[
-Pr(f_\alpha - G_\alpha), & v = f_\alpha \text{ and } |\alpha| \text{ is even},
\]
\[
-Pr f_\alpha, & \text{otherwise},
\]

for the ES-BGK model;

\[
H(w, v) = \begin{cases} 
-\sigma_{ij}, & v = \sigma_{ij}, \\
-Pr q_i, & v = q_i,
\end{cases}
\]

\[
-f_{e_i+2e_k} + \frac{1}{D+2} q_i, & v = f_{e_i+2e_k}, \ k \neq i,
\]

\[
-f_{\alpha}, & \text{otherwise},
\]

for the Shakhov model; and

\[
H(w, v) = \begin{cases} 
-\rho \sqrt{\theta} \sigma_{ij} + C_1 \rho \sum \beta \sum_{\beta, \gamma} f_{\beta} Q_{\beta,0}^{e_i+e_j}(\theta) + C_2 \sum \beta \sum_{\beta, \gamma} f_{\beta} Q_{\beta,0}^{e_i+e_j}(\theta), & v = \sigma_{ij}, \\
-\frac{\rho \sqrt{\theta} q_i + C_1 \rho \sum \beta \sum_{\beta, \gamma} f_{\beta} Q_{\beta,0}^{e_i+e_j}(\theta) + C_2 \sum \beta \sum_{\beta, \gamma} f_{\beta} Q_{\beta,0}^{e_i+e_j}(\theta), & v = q_i,
\end{cases}
\]

\[
-C_0 \rho \sqrt{\theta} f_{\alpha} + C_1 \rho \sum \beta \sum_{\beta, \gamma} f_{\beta} Q_{\beta,0}^{e_i+e_j}(\theta) + C_2 \sum \beta \sum_{\beta, \gamma} f_{\beta} Q_{\beta,0}^{e_i+e_j}(\theta), & \text{otherwise},
\]

for the binary collision model. In (12), the summations are same as that in (8). The detailed derivation for the binary collision term is given in Appendix C. Here and below, \(C_0, C_1\) and \(C_2\) are generic constants which may change from line to line.

For \(H(w, v)\), we have the following lemma.
Lemma 3.1. For the BGK, ES-BGK, Shakhov and the binary collision models, \(H(w, v)\) satisfies
\[
H(w, 0) = 0, \quad H_w(w, 0) = 0 \quad (13)
\]
and \(H_v(w, 0)\) is invertible.

Proof. The statements are obviously true for the BGK and Shakhov models.

For the ES-BGK model, we only need to consider even \(|\alpha|\), since it is obviously true otherwise. In this case, \(H(w, v) = -Pr(f_\alpha - G_\alpha)\). From (6), we know that
\[
G_\alpha = \frac{1 - 1/Pr}{\rho} \sum_{i,d=1}^{D} \frac{1}{\alpha_i} \sigma_{id} G_{\alpha_{e_i} - e_{id}}, \quad G_{e_i + e_j} = \frac{1 - 1/Pr}{\alpha_i \rho} \sigma_{ij}.
\]
From the above recursive formula, we know that \(G_\alpha\) only involves \(\rho\) and \(\sigma_{ij}\). Moreover, it is straightforward to show that \(G_\alpha = 0 \) and \(\frac{dG}{d\rho} = 0\) when \(\sigma_{ij} = 0\). Thus, it follow that \(H(w, 0) = 0\) and \(H_w(w, 0) = 0\). In addition, it is also obvious that \(H_v(w, 0)\) is an invertible matrix since that \(Pr\) is positive.

For binary collision term, we know that the equations (10) satisfies the Yong's structural stability condition, which means that the Jacobian matrix of \(H(w, v)\) is similar with an invertible matrix [17]. And according to the equation (12), \(H(w, 0) = 0, H_w(w, 0) = 0\) are obviously true. Therefore, the matrix \(H_v(w, 0)\) is invertible.

With this lemma, we rewrite the \(v\)-equation in (10) as
\[
H(w, v) = \tau \left( \partial_v + \sum_{j=1}^{D} b_j(w, v) \partial_{x_j} w + \sum_{j=1}^{D} c_j(w, v) \partial_{x_j} v \right). \quad (14)
\]
This equation indicates that \(H(w, v) = O(\tau)\) for \(\tau\) small. On the basis of Lemma 3.1, we apply the implicit function theorem to \(H = H(w, v)\) to get \(v = v(w, H)\) for \(H\) sufficient small. Since \(v(w, 0) = 0\), we have \(v(w, H) = O(H) = O(\tau)\). Substituting \(v = O(\tau)\) into the equation (14), we obtain
\[
v = \tau L^{-1}(w) \sum_{j=1}^{D} b_j(w, 0) \partial_{x_j} w + O(\tau^2), \quad (15)
\]
where \(L(w) = H_v(w, 0)\).

For the moment equations (3) and (11), the equation (14) are
\[
\sigma_{ij} = -\mu \tau \left( \frac{\partial \sigma_{ij}}{\partial t} + \sum_{d=1}^{D} u_d \frac{\partial \sigma_{ij}}{\partial x_d} + \sum_{d=1}^{D} (\sigma_{id} + \rho \theta \delta_{id}) \frac{\partial u_j}{\partial x_d} + \sum_{d=1}^{D} (\sigma_{jd} + \rho \theta \delta_{jd}) \frac{\partial u_i}{\partial x_d} \right)
\]
\[
- \sum_{d=1}^{D} \frac{2}{D} \delta_{ij} \frac{\partial \bar{u}_d}{\partial x_d} - \sum_{d,k=1}^{D} \frac{2}{D} \delta_{ij} (\sigma_{kd} + \rho \theta \delta_{kd}) \frac{\partial u_d}{\partial x_k}
\]
\[
+ \sum_{d=1}^{D} (e_i + e_j + e_d)! \frac{\partial f_{e_i + e_j + e_d}}{\partial x_d} \right) + O(\tau^2),
\]
\[
q_i = -\kappa \tau \left( \frac{\partial q_i}{\partial t} + \sum_{d=1}^{D} u_d \frac{\partial q_i}{\partial x_d} + \sum_{d=1}^{D} q_i \frac{\partial u_d}{\partial x_d} + \sum_{d=1}^{D} q_d \frac{\partial u_i}{\partial x_d} + D + 2 \sum_{d=1}^{D} \frac{p_{id}}{2} \frac{\partial \theta}{\partial x_d} \right)
\]
\[
- \sum_{k,d=1}^{D} \frac{\sigma_{ik}}{\rho} \frac{\partial \bar{p}_{ik}}{\partial x_d} + \sum_{d=1}^{D} \theta \frac{\partial \sigma_{id}}{\partial x_d} \right) + \sum_{k,d=1}^{D} \frac{\sigma_{id}}{\rho} \frac{\partial \theta}{\partial x_d} \left( \frac{e_d + e_k + e_i}{\partial x_d} + \frac{e_k + e_i}{\partial x_d} \right)
\]
Moreover, equation (15) becomes

\[ E_{\alpha}(F_{|\alpha|-3}, F_{|\alpha|-2}, F_{|\alpha|-1}, F_{|\alpha|}, F_{|\alpha|+1}), \quad |\alpha| \geq 3, \alpha \neq 3e_i. \]

Here \( \mu = 1 \) for the BGK, Shakhov and ES-BGK models, while \( \mu = \frac{1}{\sqrt{\theta}} \) for the binary collision model, \( \kappa = 1 \) for the BGK model, \( \kappa = 1/Pr \) for the Shakhov model and ES-BGK model, and \( \kappa = \frac{3}{2} \mu \) for the binary collision model according to the equation (12). Moreover, equation (15) becomes

\[
\begin{align*}
\sigma_{ij} &= \tau \mu \rho \theta \left( \frac{\partial u_i}{\partial x_i} + \frac{\partial u_j}{\partial x_j} - \frac{2\delta_{ij}}{D} \sum_{d=1}^{D} \frac{\partial u_d}{\partial x_d} \right) + O(\tau^2), \\
q_i &= \tau \kappa \frac{D + 2}{2} \rho \theta \frac{\partial \theta}{\partial x_i} + O(\tau^2), \\
f_\alpha &= O(\tau), \quad |\alpha| \leq M
\end{align*}
\]

with \( i, j, k = 1, \ldots, D. \)

Dropping the high order term \( O(\tau^2) \) in the above expansions (15) and substituting the resultant results into the \( w \)-equation in (10), we arrive at the following second-order equations

\[
\partial_t w + \sum_{j=1}^{D} \partial_{x_j} a_{ij}(w, 0) = -\tau \sum_{j,k=1}^{D} \partial_{x_j} \left( a_{ij}(w, 0)L(w)^{-1} b_k(w, 0) \partial_{x_k} w \right). 
\tag{17}
\]

This procedure to derive (17) is the Chapman-Enskog expansion [32].

These are the classical Navier-Stokes equations.

\[
\frac{\partial \rho}{\partial t} + \sum_{d=1}^{D} \frac{\partial (\rho u_d)}{\partial x_d} = 0, \\
\frac{\partial (\rho u_i)}{\partial t} + \sum_{d=1}^{D} \frac{\partial (\rho u_d u_i + \rho \theta \delta_{id})}{\partial x_d} = \tau \sum_{j=1}^{D} \frac{\partial}{\partial x_j} \left( \mu \rho \theta \left( \frac{\partial u_i}{\partial x_i} + \frac{\partial u_j}{\partial x_j} - \frac{2\delta_{ij}}{D} \sum_{k=1}^{D} \frac{\partial u_k}{\partial x_k} \right) \right), \\
\frac{\partial}{\partial t} \left( \rho \left( e + \frac{u^2}{2} \right) \right) + \sum_{d=1}^{D} \frac{\partial}{\partial x_d} \left( \rho u_d (e + \frac{u^2}{2}) + \rho \theta u_d \right) = \tau \frac{D + 2}{2} \sum_{j=1}^{D} \frac{\partial}{\partial x_j} \left( \kappa \rho \theta \frac{\partial \theta}{\partial x_j} \right) + \tau \sum_{j,k=1}^{D} \frac{\partial}{\partial x_j} \left( \mu \rho \theta u_k \left( \frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j} - \frac{2}{D} \delta_{jk} \sum_{d=1}^{D} \frac{\partial u_d}{\partial x_d} \right) \right),
\tag{18}
\]

with viscosity coefficients \( \tau \mu \rho \theta \) and thermal conductivity \( \tau \kappa \frac{D + 2}{2} \rho \theta \). This procedure in deriving the Navier-Stokes equations is from [45].

We end up this section with the following:

**Remark 1.** Equations (16) can also be derived from ordered globally hyperbolic moment equations (OHME) proposed in [21], including the modified 13-moment system [10]. Indeed, except the modified 13-moment system, the OHME systems all contain the equations (11). For the modified 13-moment system, the equation for \( \sigma_{ij} \) is same as that in (11) and the equation for \( q_i \) differs from that in (11) only in the dependence on \( v \). Thus, the expressions for \( \sigma_{ij} \) and \( q_i \) in (16) remain unchanged for the OHME systems.
4. Rigorous justification. In this section, we rigorously justify the formal derivation of the Navier-Stokes equations from the hyperbolic moment systems. To this end, we construct an approximate solution to the system (3) or (10).

4.1. Approximate solutions. For (10), the approximate solution has the form
\[ w(x, t) \sim w_0(x, t) + \tau w_1(x, t), \quad v(x, t) \sim v_0(x, t) + \tau v_1(x, t). \]

Substituting this Ansatz into the equation in (10) and expanding each term into a power series of \( \tau \), we equate the coefficients of \( \tau^k \) with \( k \geq -1 \) to obtain
\[
\tau^{-1} : H(w_0, v_0) = 0,
\]
\[
\tau^0 : \partial_t w_0 + \sum_{j=1}^{D} \partial_x a_j(w_0, v_0) = 0,
\]
\[
\partial_t v_0 + \sum_{j=1}^{D} b_j(w_0, v_0) \partial_x j w_0 + \sum_{j=1}^{D} c_j(w_0, v_0) \partial_x j v_0 = H_w(w_0, v_0)w_1 + H_v(w_0, v_0)v_1,
\]
\[
\tau^1 : \partial_t w_1 + \sum_{j=1}^{D} \partial_x j a_j(w_0, v_0) v_1 + a_{jw}(w_0, v_0) w_1 + a_{jv}(w_0, v_0) v_1 = 0.
\]

By Lemma 3.1, we see from \( H(w_0, v_0) = 0 \) that \( v_0 = 0 \). Thanks to \( H_w(w, 0) = 0 \), the third equation above becomes
\[
\sum_{j=1}^{D} b_j(w_0, 0) \partial_x j w_0 = L(w_0)v_1.
\]

Thus, we deduce from the above equations that \( w_\tau := w_0 + \tau w_1 \) satisfies
\[
\partial_t w_\tau + \sum_{j=1}^{D} \partial_x j a_j(w_\tau, 0) = -\tau \sum_{j,k=1}^{D} \partial_x j \left( a_{jv}(w_\tau, 0)L(w_\tau)^{-1}b_k(w_\tau, 0)\partial_x k w_\tau \right) + R
\]
with
\[
R = \sum_{j=1}^{D} \partial_x j \left( a_j(w_\tau, 0) - a_j(w_0, 0) - \tau a_{jw}(w_0, 0)w_1 \right) - \tau \sum_{j,k=1}^{D} \partial_x j \left( a_{jv}(w_0, 0)L(w_0)^{-1}b_k(w_0, 0)\partial_x k w_0 - a_{jv}(w_\tau, 0)L(w_\tau)^{-1}b_k(w_\tau, 0)\partial_x k w_\tau \right)
\]

**Notation.** For vectors \( U, V \in R^k \) and matrix \( A \in R^{l \times k} \), our basic inner product and norm are
\[
\langle U, V \rangle = U^T V, \quad ||U|| = \langle U, U \rangle, \quad |A| = \max \{ |AU| : ||U|| = 1 \},
\]
where the superscript ‘\( T \)’ denotes the transpose of a vector or matrix. \( |U| \) denotes some norm of a vector or matrix. \( L_2 = L_2(R^D) \) is the space of square integrable (vector- or matrix-valued) functions on \( R^D \). For a non-negative integer \( s \), \( H_s = H_s(R^D) \) is defined as the space of functions whose distribution derivatives of order \( \leq s \) are all in \( L_2 \). We use \( ||U||_s \) to denote the standard norm of \( U \in H_s \), and \( ||U||_0 = ||U||_0 \). When \( A \) is a function of another variable \( t \) as well as \( x \), we write \( ||A(\cdot, t)||_s \) to recall that the norm is taken with respect to \( s \) while \( t \) is viewed as a parameter. In addition, we denote by \( C([0, T], X) \) the space of continuous functions on \([0, T]\) with values in a Banach space \( X \).
About this \( R = R(w_0, w_1, \tau) \), we have

**Lemma 4.1.** Let \( s > \frac{D}{2} + 1 \) be an integer. If \( w_0 \in C([0, T_*], H^{s+2}) \) and \( w_1 \in C([0, T_*], H^{s+2}) \), then \( R(w_0, w_1, \tau) \in C([0, T_*], H^s) \) and \( \| R \|_s = O(\tau^2) \).

This lemma indicates that \( w_* \) is an approximate solution to the moment system (10). It can be easily proved by using the following well-known calculus inequalities in Sobolev spaces (see, e.g., [46, 47]).

**Lemma 4.2.** Let \( s, s_1 \), and \( s_2 \) be three non-negative integers and \( s_0 = [D/2] + 1 \)

1. If \( s_3 = \min\{s_1, s_2, s_1 + s_2 - s_0\} \geq 0 \), then \( H^{s_1}H^{s_2} \subset H^{s_3} \);
2. Suppose \( s > s_0 + 1 \), \( A \in H^s \), and \( Q \in H^{s-1} \). Then for all multi-indices \( \alpha \) with \( \alpha \leq s \), \( \| [A, \partial_\alpha]Q \|_s \leq C_A\|A\|_s\|Q\|_{|\alpha|-1} \);
3. Suppose \( s > s_0 \), \( A \in C^1_0(G) \) and \( V \in H^1(R^d, G) \). Then \( A(V) \in H^s \) and

\[
\| A(V) \|_s \leq C_s\|A\|_s(1 + \|V\|_s^2). 
\]

The proof of this lemma can be found in [35].

### 4.2. Error analysis

Here we present a rigorous justification of the formal limit (17) or (18) derived in the previous section. The justification is similar to that in [45] but we have to deal with the non-conservative terms of the moment systems. In particular, now the symmetrizer is not block-diagonal and a convex entropy function is not assumed.

Denote by \( U^* = (\tilde{w}^*, \tilde{v}^*)^T \) and \( w^*_\tau \) the solutions to systems (10) and (17), respectively. Our main result of this paper can be stated as

**Theorem 4.3.** Suppose \( s > [D/2] + 1 \) is an integer, initial data \( \tilde{U}(x, \tau) = (\tilde{w}(x, \tau), \tilde{v}(x, \tau))^T \) for the moment system (10) and \( \tilde{w}(x, \tau) \) for the formal limit (17) are in \( H^s \) for each \( \tau > 0 \), and satisfy

\[
\| \tilde{w}(x, \tau) - \tilde{w}(x, \tau) \|_s = O(\tau^2). 
\]

Then there exists \( \tau \)-independent positive constants \( T_* \) and \( K(T_*) \) such that both the systems (10) and (17) with the above initial data have unique solutions in \( C([0, T_*], H^s) \) satisfying

\[
\sup_{t \in [0, T_*]} \| (w^* - w^*_\tau)(\cdot, t) \|_s \leq K(T_*)\tau^2 
\]

for \( \tau \) sufficiently small.

To prove the above theorem, we know from [17] that the moment system (10) satisfies Young’s structural stability condition in [47]. Therefore, the singular perturbation theory developed therein applies. Particularly, the solution \( U^\tau = (w^\tau(x, t), v^\tau(x, t))^T \) to (10) with initial data \( \tilde{U}(x, \tau) = (\tilde{w}(x, \tau), \tilde{v}(x, \tau))^T \) exists in the time interval \([0, T_*]\), where the approximate solution \( U_\tau^1 = (w_0 + \tau w_1, v_0 + \tau v_1)^T \) can be constructed, and satisfies

\[
\sup_{t \in [0, T_*]} \| (U^\tau - U_\tau^1)(\cdot, t) \|_s \leq K\tau^2 
\]

with \( K \) a positive constant independent of \( \tau \).

In view of the known estimate in (21), it suffices to establish the following fact.
Theorem 4.4. Suppose $s > [D/2] + 1$ is an integer, $w_0(\cdot, 0)$, $w_1(\cdot, 0) \in H^{s+2}$, and $\hat{w}(x, \tau)$ is such that

$$\| w_0(x, 0) + \tau w_1(x, 0) - \hat{w}(x, \tau) \|_s = O(\tau^2).$$

Then the formal limit $(17)$ with initial data $\hat{w}$ has a unique solution $w^\tau_p \in C([0, T_*], H^s)$ satisfying

$$\sup_{t \in [0, T_*]} \| w^\tau_p(\cdot, t) - w_0(\cdot, t) - \tau w_1(\cdot, t) \|_s \leq K(T_*)\tau^2$$

for $\tau$ sufficiently small.

Proof. From the local existence theory [30] for the Navier-Stokes equations, the system $(17)$ has a unique solution $w^\tau_p = w^\tau_p(x, t)$ satisfying $w^\tau_p \in C([0, T], H^s)$. For $G_1 \subset G = \{(w, v) : \rho > 0, \theta > 0\}$, we define its maximum existence time as

$$T^\tau := \sup\{T > 0, w^\tau_p \in C([0, T], H^s), w^\tau_p(x, t) \in G_1\}.$$ 

It can be shown as in [48] that

$$T^\tau > T_*$$

provided that, as $\tau$ tends to zero, the following error estimates

$$\sup_{x,t} |w^\tau_p(x, t) - w_0(x, t) - \tau w_1(x, t)| = o(1),$$

$$\sup_t \| w^\tau_p(\cdot, t) - w_0(\cdot, t) - \tau w_1(\cdot, t) \|_s = O(1)$$

hold for $t \in [0, \min\{T_*, T^\tau\})$. Thus it remains to prove the error estimate with $T_*$ replaced by $\min\{T_*, T^\tau\}$.

To do this, we follow [31], and introduce the transformation

$$V = (\ln \frac{\rho}{\sqrt{\theta^\rho}} \frac{u^2}{2\theta} + \frac{5}{2} \frac{u}{\theta^\rho} - \frac{1}{\theta} )^T,$$

and rewrite the Navier-Stokes equations (17) into its normal form

$$V_t + \sum_{j=1}^D A^0(V)^{-1} A^j(V) V_{x_j} = \tau A^0(V)^{-1} \left( \sum_{j,k=1}^D (M^{jk}(V)\nu_{x_k}) x_j \right). \tag{22}$$

Here $A^0(V) = D_V w$ is a symmetric positive definite matrix, $A^j(V)$ are real symmetric matrices, $\sum_{j,k=1}^D M^{jk}(\zeta, \zeta_k)$ is symmetric positive definite for $\zeta \in R^D \setminus \{0\}$, and $\nu = (\frac{u}{\rho}, -\frac{1}{\theta})$. Likewise, we use the transformation above to rewrite the equations (19) as

$$V_{\tau t} + \sum_{j=1}^D A^0(V_\tau)^{-1} A^j(V_\tau) V_{\tau x_j} = \tau A^0(V_\tau)^{-1} \left( \sum_{j,k=1}^D (M^{jk}(V_\tau)\nu_{\tau x_k}) x_j \right) + A^0(V_\tau)^{-1} R.$$ 

Set

$$E = V^\tau - V_\tau = \begin{pmatrix} E^\nu \\ E^\theta \end{pmatrix}.$$
with $V^*$ a solution to (22). It is straightforward to verify that $E$ satisfies

$$
E_t + \sum_{j=1}^{D} A^0(V^*)^{-1} A^j(V^*) E_{x_j} = \tau A^0(V^*)^{-1} \left( \sum_{j,k=1}^{D} (M^{jk}(V^*) E_{x_j})_{x_k} \right) - A^0(V^*)^{-1} R
$$

$$
+ \sum_{j=1}^{D} \left( A^0(V^*)^{-1} A^j(V^*) - A^0(V^*)^{-1} A^j(V^*) \right) V_{x_j}
$$

$$
+ \tau \left( A^0(V^*)^{-1} - A^0(V^*)^{-1} \left( \sum_{j,k=1}^{D} (M^{jk}(V^*) \nu_{x_k})_{x_j} \right) \right)
$$

$$
+ \tau A^0(V^*)^{-1} \left( \sum_{j,k=1}^{D} \left( (M^{jk}(V^*) - M^{jk}(V^*)) \nu_{x_k} \right)_{x_j} \right).
$$

Applying $\partial_x^n$ to the last equation for multi-index $\alpha$ with $0 \leq |\alpha| \leq s$, we get

$$
E_{\alpha t} + \sum_{j=1}^{D} A^0(V^*)^{-1} A^j(V^*) E_{\alpha x_j}
$$

$$
= \tau A^0(V^*)^{-1} \left( \sum_{j,k=1}^{D} (M^{jk}(V^*) E_{\alpha x_k})_{x_j} \right) - \{ A^0(V^*)^{-1} R \}_\alpha +
$$

$$
\sum_{j=1}^{D} \left( [A^0(V^*)^{-1} A^j(V^*)]_{\alpha} E_{x_j} + \{ [A^0(V^*)^{-1} A^j(V^*) - A^0(V^*)^{-1} A^j(V^*)] V_{x_j} \}_\alpha \right)
$$

$$
- \tau A^0(V^*)^{-1} \left( \sum_{j,k=1}^{D} (M^{jk}(V^*) \partial_{\alpha} E_{x_k})_{x_j} \right)
$$

$$
- \tau [A^0(V^*)^{-1}, \partial_{\alpha}] \left( \sum_{j,k=1}^{D} (M^{jk}(V^*) E_{x_k})_{x_j} \right)
$$

$$
+ \tau \left( A^0(V^*)^{-1} - A^0(V^*)^{-1} \left( \sum_{j,k=1}^{D} (M^{jk}(V^*) \nu_{x_k})_{x_j} \right) \right)_\alpha
$$

$$
+ \tau A^0(V^*)^{-1} \left( \sum_{j,k=1}^{D} \left( (M^{jk}(V^*) - M^{jk}(V^*)) \nu_{x_k} \right)_{x_j} \right)_\alpha
$$

$$
\equiv G_1^\alpha + G_2^\alpha + G_3^\alpha + G_4^\alpha + G_5^\alpha + G_6^\alpha + G_7^\alpha.
$$

Here $G_i$ stands for the $i$-th line above. Recall that $A^0(V)$ is symmetric positive definite and $A^0(V)A^j(V)$ is symmetric. We multiply the last equation by $2 E^T_{\alpha} A^0(V^*)$ and integrate the resultant equation over $x \in \mathbb{R}^D$ to obtain

$$
\left( \int E^T_{\alpha} A^0(V^*) E_{\alpha} dx \right) = \int E^T_{\alpha} \left( A^0(V^*)_{t} + \sum_{j=1}^{D} (A^j(V^*))_{x_j} \right) E_{\alpha} dx
$$

$$
+ 2 \int E^T_{\alpha} A^0(V^*) (G_1^\alpha + G_2^\alpha + G_3^\alpha + G_4^\alpha + G_5^\alpha + G_6^\alpha + G_7^\alpha) dx.
$$

Next we estimate each term in the right-hand side of the last equation.
Here and below, $V^\tau = V^\tau(x,t)$ takes values in a compact subset. By using the equation of $V^\tau$ and the well-known embedding theorem, we have
\[
\int E_\alpha^T \left( A^0(V^\tau)_t + \sum_{j=1}^D (A^j(V^\tau))_{x_j} \right) d\alpha dx
\]
\[
\leq C \int |E_\alpha|^2 \left( |V^\tau_t| + \sum_{j=1}^D |V^\tau_{x_j}| \right) dx
\]
\[
\leq C \int |E_\alpha|^2 \left( \sum_{j=1}^D |V^\tau_{x_j}| + \tau |\nabla V^\tau|^2 + \tau |\nabla^2 V^\tau|^2 \right) dx
\]
\[
\leq C \left( \| V^\tau \|_s + \tau \| V^\tau \|_s^2 + \tau \| \nabla V^\tau \|_s^2 \right) \| E_\alpha \|^2 .
\]
Here and below, $C$ is a generic constant which may change from line to line. For $\int E_\alpha^T A^0(V^\tau)G_1^\alpha dx$, we integrate by part and use the Gårding’s inequality in [45] to obtain
\[
\int E_\alpha^T A^0(V^\tau)G_1^\alpha dx
\]
\[
= \tau \int E_\alpha^T A^0(V^\tau) A^0(V^\tau)^{-1} \left( \sum_{j,k=1}^D (M^j k(V^\tau)E^\nu_{\alpha x_k})_{x_j} \right) dx
\]
\[
- \int E_\alpha^T A^0(V^\tau) \{ A^0(V^\tau)^{-1} R \}_\alpha dx
\]
\[
= \tau \sum_{j,k=1}^D \int E^\nu_{\alpha x_j} M^j k(V^\tau)E^\nu_{\alpha x_k} dx
\]
\[
= \tau \sum_{j,k=1}^D \int E^\nu_{\alpha x_j} M^j k(V^\tau)E^\nu_{\alpha x_k} dx
\]
\[
\leq \tau c_0 \left( \| \nabla E^\nu_{\alpha x} \|^2 + C(1 + \| E \|_s^2) \right) \| E_\alpha \|^2 + C \tau^4
\]
with $c_0 > 0$ a constant. In the last step, we have also used Lemma 4.2 and the fact that $\| R \|_s = O(\tau^2)$.

For the remaining terms, we use Lemma 4.2 (the calculus inequalities in Sobolev spaces) to obtain
\[
\int E_\alpha^T A^0(V^\tau)G_2^\alpha dx
\]
\[
= \int \left( A^0(V^\tau)^{-1} A^j(V^\tau), \partial_\alpha \right) E_{x_j}
\]
\[
+ \left\{ \left( A^0(V^\tau)^{-1} A^j(V^\tau) - A^0(V^\tau)^{-1} A^j(V^\tau) \right)_{x_j} \right\}_\alpha \right) dx
\]
\[
\leq C \| E_\alpha \| \sum_{j=1}^D \left( \| A^0(V^\tau)^{-1} A^j(V^\tau) \|_s \right) \| E_{x_j} \|_{\alpha-lin}
\]
\[
+ \left( \| A^0(V^\tau)^{-1} A^j(V^\tau) \|_{\alpha} \right)
\]
\[
\leq C \| E_\alpha \| \left( \| A^0(V^\tau)^{-1} A^j(V^\tau) \|_s \right) \| E \|_{\alpha} + \| E \|_{\alpha} \left( 1 + \| V^\tau \|_s \right)
\]
\[
\leq C \| E \|_{\alpha} (1 + \| E \|_s).
\[
\int E_\alpha^T A^0(V^\tau)G_0^3 \, dx \\
= -\tau \int E_\alpha^T A^0(V^\tau)A^0(V^\tau)^{-1} \left( \sum_{j,k=1}^D ([M_{jk}(V^\tau)\nu_{\tau x_k})_{x_j}]^0 \right) \, dx \\
= \sum_{j,k=1}^D E_{\alpha x_j}^T [M_{jk}(V^\tau)\nu_{\tau}]_{x_k} \, dx \\
\leq C\tau \sum_{j,k=1}^D \| \nabla E_{\alpha x_j}^T \| || M_{jk}(V^\tau) ||_s \| \nabla E_{\nu}^T \|_{|\alpha|-1} \\
\leq C\tau \| \nabla E_{\nu}^T \| (1+\| E \|_s^2) \| E_{\nu}^T \|_{|\alpha|}.
\]

Similarly, we have
\[
\int E_\alpha^T A^0(V^\tau)G_0^3 \, dx \\
= -\tau \int E_\alpha^T A^0(V^\tau)A^0(V^\tau)^{-1} \left( \sum_{j,k=1}^D ([M_{jk}(V^\tau)E_{x_k})_{x_j}]^0 \right) \, dx \\
\leq C\tau \sum_{j,k=1}^D E_{\alpha x_j}^T [A^0(V^\tau)^{-1},\nu_{\tau}]_{x_k} [M_{jk}(V^\tau)E_{x_k})_{x_j}] \, dx \\
\leq C\tau \sum_{j,k=1}^D \| E_{\nu}^T \| || A^0(V^\tau)^{-1} ||_s \| (M_{jk}(V^\tau)E_{x_k})_{x_j} \|_{|\alpha|-1} \\
\leq C\tau \| E_{\nu}^T \| (1+\| V^\tau \|_s^2) \| M_{jk}(V^\tau) \|_s \| \nabla E_{\nu}^T \|_{|\alpha|} \\
\leq C\tau \| E_{\nu}^T \| (1+\| E \|_s^2) \| \nabla E_{\nu}^T \|_{|\alpha|}
\]

and
\[
\int E_\alpha^T A^0(V^\tau)G_0^3 \, dx \\
= \tau \int E_\alpha^T A^0(V^\tau) \left( (A^0(V^\tau)^{-1} - A^0(V^\tau)^{-1}) \left( \sum_{j,k=1}^D (M_{jk}(V^\tau)\nu_{\tau x_k})_{x_j} \right) \right) \, dx \\
\leq C\tau \int E_{\alpha}^T \left( (A^0(V^\tau)^{-1} - A^0(V^\tau)^{-1}) \left( \sum_{j,k=1}^D (M_{jk}(V^\tau)\nu_{\tau x_k})_{x_j} \right) \right) \, dx \\
\leq C\tau \| E_{\alpha} \| || A^0(V^\tau)^{-1} - A^0(V^\tau)^{-1} ||_s \| \sum_{j,k=1}^D (M_{jk}(V^\tau)\nu_{\tau x_k})_{x_j} \|_{|\alpha|-1} \\
\leq C\tau \| E \|_{|\alpha|}^2 (1+\| E \|_s^2).
\]

The term \( \int E_\alpha^T A^0(V^\tau)G_0^3 \, dx \) needs to be treated carefully:
\[
\int E_\alpha^T A^0(V^\tau)G_0^3 \, dx \\
= \tau \int E_\alpha^T A^0(V^\tau)A^0(V^\tau)^{-1} \left( \sum_{j,k=1}^D \left( (M_{jk}(V^\tau) - M_{jk}(V^\tau)\nu_{\tau x_k})_{x_j} \right) \right) \, dx \\
= \tau \int E_{\alpha}^T \left( \sum_{j,k=1}^D \left( (M_{jk}(V^\tau) - M_{jk}(V^\tau)\nu_{\tau x_k})_{x_j} \right) \right) \, dx.
\]
If $\alpha = 0$, this can be simply treated as

$$\tau \int E^\alpha_x \left( \sum_{j,k=1}^D \left( (M^{jk}(V^\tau) - M^{jk}(V_\tau)) \nu_{\tau x_k} \right)_{x_j} \right) \, \alpha \, dx$$

$$= - \tau \int E^\alpha_x \left( \sum_{j,k=1}^D \left( (M^{jk}(V^\tau) - M^{jk}(V_\tau)) \nu_{\tau x_k} \right) \, dx \right.$$

$$\leq C\tau \| \nabla E^\alpha_x \| \| E \|_\alpha.$$ 

Otherwise, $\alpha$ has a positive component, say the first one. Then we have

$$\tau \int E^\alpha_x \left( \sum_{j,k=1}^D \left( (M^{jk}(V^\tau) - M^{jk}(V_\tau)) \nu_{\tau x_k} \right)_{x_j} \right) \, \alpha \, dx$$

$$= - \tau \int E^\alpha_x \left( \sum_{j,k=1}^D \left( (M^{jk}(V^\tau) - M^{jk}(V_\tau)) \nu_{\tau x_k} \right) \, dx \right.$$

$$\leq C\tau \sum_{j,k=1}^D \| \nabla E^\alpha_x \| \| (M^{jk}(V^\tau) - M^{jk}(V_\tau)) \nu_{\tau x_k} \|_\alpha$$

$$\leq C\tau \sum_{j,k=1}^D \| \nabla E^\alpha_x \| \| M^{jk}(V^\tau) - M^{jk}(V_\tau) \|_\alpha$$

$$\leq C\tau \| \nabla E^\alpha_x \| \| E \|^2_\alpha (1 + \| E \|^*_\alpha). \quad (30)$$

In summary, we have

$$\int E^T_\alpha A^0(V^\tau)G^0_6 \, dx \leq C\tau \| \nabla E^\alpha_x \| \| E \| + C\tau \| \nabla E^\alpha_x \| \| E \|^2_\alpha (1 + \| E \|^*_\alpha). \quad (30)$$

The last term can be estimated as

$$\int E^T_\alpha A^0(V^\tau)G^0_7 \, dx$$

$$= \tau \int E^T_\alpha A^0(V^\tau)A^0(V^\tau)^{-1}, \partial_\alpha \left( \sum_{j,k=1}^D \left( (M^{jk}(V^\tau) - M^{jk}(V_\tau)) \nu_{\tau x_k} \right)_{x_j} \right) \, dx$$

$$\leq C\tau \int E^T_\alpha A^0(V^\tau)^{-1}, \partial_\alpha \left( \sum_{j,k=1}^D \left( (M^{jk}(V^\tau) - M^{jk}(V_\tau)) \nu_{\tau x_k} \right)_{x_j} \right) \, dx$$

$$\leq C\tau \| E_\alpha \| \| A^0(V^\tau)^{-1} \| \| \left( (M^{jk}(V^\tau) - M^{jk}(V_\tau)) \nu_{\tau x_k} \right)_{x_j} \|_{\alpha-1}$$

$$\leq C\tau \| E_\alpha \| (1 + \| V^\tau \|^*_\alpha) \| (M^{jk}(V^\tau) - M^{jk}(V_\tau)) \nu_{\tau x_k} \|_{\alpha}$$

$$\leq C\tau \| E_\alpha \| (1 + \| V^\tau \|^*_\alpha) \| (M^{jk}(V^\tau) - M^{jk}(V_\tau)) \|_{\alpha}$$

$$\leq C\tau \| E_\alpha \| (1 + \| V^\tau \|^*_\alpha) \| E \|_{\alpha} (1 + \| V^\tau \|^*_\alpha)$$

$$\leq C\tau \| E \|^2_\alpha (1 + \| E \|^*_\alpha). \quad (31)$$
Substituting the estimates in (24)–(31) into (23) yields
\[
\int_0^T E_\alpha^T A_\alpha(V^T) E_\alpha dx dt \leq C(\| V^T \|_s + \| V^T \|^2 + \| \nabla V^T \|_s) \| E_\alpha \|^2 \\
- 2\tau C_0 \| \nabla E_\alpha^\nu \|^2 + C(1 + \| E \|^2) \| E_\alpha \|^2 + C \tau^4 \\
+ C \| E \|^2_{\|\cdot\|_{\alpha}} (1 + \| E \|^2_{\|\cdot\|_{\alpha}}) + C \tau \| \nabla E_\alpha^\nu \| (1 + \| E \|^2_{\|\cdot\|_{\alpha}}) \| E^\nu \|_{\|\cdot\|_{\alpha}} \\
+ C \tau \| E_\alpha^\nu \| (1 + \| E \|^2_{\|\cdot\|_{\alpha}}) \| \nabla E^\nu \|_{\|\cdot\|_{\alpha}} + C \tau \| E \|^2_{\|\cdot\|_{\alpha}} (1 + \| E \|^2_{\|\cdot\|_{\alpha}}) \\
\leq C \| E \|^2_{\|\cdot\|_{s}} (1 + \| E \|^4 + 2\tau C_0 \| \nabla E_\alpha^\nu \|^2 + C \tau^4) \\
+ C \tau \| \nabla E^\nu \| (1 + \| E \|^4) dt + CT \tau^4 + \| E(0) \|_s.
\]

Integrating this inequality over \( t \in [0, T] \) with \( T < \min\{T_*, T^*\} \) and summing up the resultant inequality over all \( \alpha \) satisfying \( |\alpha| \leq M \), we have
\[
\| E(T) \|_s^2 + \tau C_0 \int_0^T \| \nabla E^\nu \|_{s}^2 dt \\
\leq C \int_0^T \| E \|_s^2 (1 + \| E \|_{s}^{4+2}) dt + CT \tau^4 + \| E(0) \|_s.
\]

Here we used the fact that \( \| E(0) \|_s = O(\tau^2) \). Then we apply Gronwall’s inequality and obtain
\[
\| E(T) \|_s^2 \leq CT \tau^4 \exp \left[ \int_0^T C(1 + \| E \|_{s}^{4+2}) dt \right] \equiv \phi(T). \tag{32}
\]

Furthermore, observe that
\[
\phi(T) = C\phi(T)(1 + \| E \|_{s}^{4+2}) \leq C\phi(T)(1 + \phi(T)^{2+1}).
\]

Applying the nonlinear Gronwall-type inequality \([47]\) the last inequality gives
\[
\| E(T) \|_s^2 \leq \phi(T) \leq e^{CT}.
\]

This completes the proof of Theorem 4.3. \( \square \)

5. **Conclusion.** In this work, we study the Navier-Stokes limit of the globally hyperbolic moment systems in \([9, 8]\). Formally we derive the Navier-Stokes equations from the globally hyperbolic moment systems with the Chapman-Enskog expansion. By doing so, we obtain the coefficients of viscosity and thermal conductivity for generally binary collision model. Furthermore, we rigorously justify the formal derivation for the globally hyperbolic moment systems accompanying the BGK, Shakhov and ES-BGK models, as well as the binary collision model. Namely, the globally hyperbolic moment systems are compatible with the Navier-Stokes equations as the Knudsen number tends to zero. Notice that the globally hyperbolic moment systems are different from the problem studied in \([45]\), since the present moment systems are non-conservative and may not have a convex entropy function.

This work shows the importance of Yong’s structural stability condition in analyzing the compatibility of the globally hyperbolic moment systems. In the future, we plan to check whether or not the other moment systems \([41, 22, 29]\) satisfy
the stability condition and analyze the compatibility with the corresponding classical systems. Furthermore, it would be interesting to investigate the compatibility numerically.

**Appendix A. Hyperbolic.** Let us consider a system of partial differential equations

\[ U_t + \sum_{j=1}^{D} A_j(U)U_{x_j} = Q(U), \quad (33) \]

where \( U, A_j(U) \) and \( Q(U) \) are the \( n \times n \)-matrix and \( n \)-vector smooth function of \( U \in G \) with state space \( G \) open and convex. Here, the subscripts \( t \) and \( x_j \) refer to the partial derivatives with respect to \( t \) and \( x_j \).

**Definition A.1.** A systems (33) is called a globally hyperbolic system if for all \( U \in G \), the coefficient matrix \( A \) satisfies

\[ \sup_{\xi \in \mathbb{R}^d} \| \exp(i \sum_{j=1}^{d} \xi_j A_j) \| < +\infty \]

with \( i \) the imaginary unit.

**Appendix B. Yong’s structural stability condition.** Consider a system of partial differential equations

\[ U_t + \sum_{j=1}^{D} A_j(U)U_{x_j} = Q(U), \]

where \( U, A_j(U) \) and \( Q(U) \) are the \( n \times n \)-matrix and \( n \)-vector smooth function of \( U \in G \) with state space \( G \) open and convex. Here, the subscripts \( t \) and \( x_j \) refer to the partial derivatives with respect to \( t \) and \( x_j \).

Yong developed a singular perturbation theory for nonlinear first-order hyperbolic system with stiff source term in several space variables, and proposed the structural stability condition. Under the structural stability condition, a formal asymptotic approximation of the initial-layer solution to the nonlinear problem are constructed. Moreover, with some regularity assumption on the solution, the existence of classical solutions is guaranteed in uniform time interval.

Define the equilibrium manifold

\[ E := \{ U \in G : Q(U) = 0 \}, \]

The stability condition reads

(i) There is an invertible \( n \times n \) matrix \( P(U) \) and an invertible \( r \times r \) matrix \( S(U) \), defined on the equilibrium manifold \( E \), such that

\[ P(U)Q(U) = \begin{bmatrix} 0 & 0 \\ 0 & S(U) \end{bmatrix} P(U), \quad \text{for} \quad U \in E, \]

(ii) There is a symmetric positive definite matrix \( A_0(U) \) such that

\[ A_0(U)A_j(U) = A_j(U)^T A_0(U), \quad \text{for} \quad U \in G, \]

(iii) The hyperbolic part and the source term are coupled in the sense

\[ A_0(U)Q(U) + Q(U)^T A_0(U) \leq -P^T(U) \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} P(U), \quad \text{for} \quad U \in E, \]
Here $I$ is the unit matrix of order $r$ and $Q_U = \frac{\partial Q}{\partial \theta}$.

As shown in [49], this set of conditions has been tacitly respected by many well-developed physical theories. Condition (i) is classical for initial value problems of the system of ordinary differential equations (ODE, spatially homogeneous systems), while (ii) means the symmetrizable hyperbolicity of the PDE system. Condition (iii) characterizes a kind of coupling between the ODE and PDE parts. Recently, this structural stability condition is shown in [17] to be proper for certain moment closure systems. On the other hand, this set of conditions implies the existence and stability of the zero relaxation limit of the corresponding initial value problems [47].

Appendix C. The binary collision model. In this appendix, I derive the expressions of $Q_\alpha$, which appears in (8) and (12). Based on this expression, we can obtain the viscosity coefficient $\mu = 1/\rho \sqrt{\theta}$ and thermal conductivity coefficient $\kappa = 3\mu/2$ for the binary collision model. First of all, it follows from the expansion (1) of the distribution that

$$Q_\alpha = \frac{1}{\varepsilon} \alpha_1! \int Q(f, f) H_\alpha^{[0, \theta]}(\xi) d\xi$$

$$= \frac{1}{\varepsilon} \alpha_1! \int \left( f(x, t, \xi') f(x, t, \xi) - f(x, t, \xi) f(x, t, \xi') \right) H_\alpha^{[0, \theta]}(\xi) \xi_1 - \xi |b| d\xi_1 d\xi$$

$$= \frac{1}{\varepsilon} \sum_{\beta, \gamma} f_\beta(x, t) f_\gamma(x, t) \frac{\theta^{[\alpha]}}{\alpha!}$$

$$\int \left( H_\beta^{[0, \theta]}(\xi_1) H_\gamma^{[0, \theta]}(\xi') H_\alpha^{[0, \theta]}(\xi) \right) - H_\beta^{[0, \theta]}(\xi_1) H_\gamma^{[0, \theta]}(\xi') H_\alpha^{[0, \theta]}(\xi)$$

$$= \frac{1}{\varepsilon} \sum_{\beta, \gamma} f_\beta f_\gamma Q_{\beta, \gamma}^{\alpha}.$$

Notice that $Q_{\beta, \gamma}^{\alpha}$ is symmetric with respect to $\beta$ and $\gamma$.

Thus, it remains to compute $Q_{\beta, \gamma}^{\alpha}$. To do this, we introduce $\eta = \frac{\xi-u}{\sqrt{\theta}}$ and rewrite

$$H_\alpha^{[0, \theta]}(\xi) = \frac{1}{w^{[0, \theta]}(\xi)} \prod_{d=1}^D \frac{\partial^{\alpha_d}}{\partial \xi^{\alpha_d}} w^{[0, \theta]}(\xi) = \frac{1}{\exp(-\frac{|\xi-u|^2}{2\theta})} \prod_{d=1}^D \frac{\partial^{\alpha_d}}{\partial \xi^{\alpha_d}} \exp(-\frac{|\eta|^2}{2})$$

$$= \sqrt{\theta}^{-\alpha} H^{[0, 1]}(\eta),$$

$$|\xi_1 - \xi| = \sqrt{\theta}|\eta_1 - \eta|,$$

$$w^{[0, \theta]}(\xi) = \frac{1}{2|\theta|^{D/2}} \exp(-\frac{|\xi-u|^2}{2\theta}) = \frac{1}{\theta^{D/2}} w^{[0, 1]}(\eta).$$

With these relations and the conservations of momentum and energy, it holds that

$$\eta_1' + \eta' = \eta_1 + \eta \quad |\eta_1'|^2 + |\eta'|^2 = |\eta_1|^2 + |\eta|^2.$$

$Q_{\beta, \gamma}^{\alpha}$ can be rewritten as

$$Q_{\beta, \gamma}^{\alpha}(\theta) = \sqrt{\theta}^{\alpha - |\beta| - |\gamma|} Q_{\beta, \gamma}^{\alpha}.$$
with
\[
\tilde{Q}_{\alpha,\gamma}^\rho = \frac{1}{\alpha!} \int (\mathcal{H}_\alpha^{[0,1]}(\eta_1) \mathcal{H}_\gamma^{[0,1]}(\eta') - \mathcal{H}_\alpha^{[0,1]}(\eta_1) \mathcal{H}_\gamma^{[0,1]}(\eta)) w^{[0,1]}(\eta) w^{[0,1]}(\eta_1) \eta - \eta| b b d \varepsilon d \eta_1 d \eta.
\]

Note that the \( \tilde{Q}_{\alpha,\gamma}^\rho \)'s are constants depending only of \( \alpha, \beta \) and \( \gamma \).

Recall that \( f_0 = \rho \) and \( f_{e_i} = 0 \). Then \( Q_\alpha \) can be rewritten as
\[
Q_\alpha = \frac{1}{\varepsilon} \sum_{\beta, \gamma} f_\beta f_\gamma Q_{\beta,\gamma}^\rho
= \frac{2}{\varepsilon} \sum_{1 < \beta \leq M} f_\beta \sqrt{\theta} \tilde{Q}_{\beta,0}^\rho + \frac{1}{\varepsilon} \sum_{1 < |\gamma| \leq M} f_\beta f_\gamma \sqrt{\theta} \tilde{Q}_{\beta,\gamma}^\rho
= - C_0 \rho \sqrt{\theta} f_\alpha + \frac{2}{\varepsilon} \sum_{1 < \beta \leq M} \sqrt{\theta} \tilde{Q}_{\beta,0}^\rho
+ \frac{1}{\varepsilon} \sum_{1 < |\gamma| \leq M} \sqrt{\theta} \tilde{Q}_{\beta,\gamma}^\rho.
\]

Here we use the fact that \( \tilde{Q}_{0,0}^\rho = 0 \), since \( \mathcal{H}_0^{[0,1]} = 1 \). A tedious calculation shows that \( C_0 = -\tilde{Q}_{0,0}^\rho \) are positive constants \([43]\). Here and below, \( C_0 \) is a generic constant which may change from line to line.

Next, we calculate the source term of the stress tensor \( \sigma_{ij} \) and heat flux \( q_i \) to obtain the coefficients of viscosity and thermal conductivity for the binary collision model. To do this, we look at these \( \tilde{Q}_{\alpha,\gamma}^\rho \) with \( \gamma = 0 \) more carefully. By using the identities
\[
\eta = \frac{\eta_1 + \eta}{2} - \frac{\eta_1 - \eta}{2}, \quad \eta_1 = \frac{\eta_1 + \eta}{2} + \frac{\eta_1 - \eta}{2},
\]
\[
\eta' = \frac{\eta_1' + \eta'}{2} - \frac{\eta_1' - \eta'}{2}, \quad \eta_1' = \frac{\eta_1' + \eta'}{2} + \frac{\eta_1' - \eta'}{2},
\]
\( \tilde{Q}_{\beta,0}^\rho \) can be computed as
\[
\tilde{Q}_{\beta,0}^\rho = \frac{1}{\alpha!} \int (\mathcal{H}_\beta^{[0,1]}(\eta_1) - \mathcal{H}_\beta^{[0,1]}(\eta_1)) w^{[0,1]}(\eta) w^{[0,1]}(\eta_1) | b b d| d \eta_1 d \eta
= \frac{1}{\alpha!} \int (\mathcal{H}_\beta^{[0,1]}(G + \frac{1}{2} g') - \mathcal{H}_\beta^{[0,1]}(G + \frac{1}{2} g)) w^{[0,1]}(\sqrt{2} G) w^{[0,1]}(\frac{g}{2})
\]
\[
\mathcal{H}_\alpha^{[0,1]}(G + \frac{1}{2} g)| b b d| d G \Omega \left( 2, \frac{1}{2} \right).
\]
Here \( g = \eta_1 - \eta \) and \( G = (\eta_1 + \eta)/2 \).

For \( Q_{ij} = (1 + \delta_{ij}) Q_{e_i + e_j,0} \), we consider the term \( \tilde{Q}_{e_i + e_j,0}^{e_i + e_j} \). With the help of the expansion of \( \mathcal{H}_\alpha^{[0,1]} \) in \([33]\), a tedious calculation shows that
\[
\tilde{Q}_{e_i + e_j,0}^{e_i + e_j} = \frac{1}{(e_i + e_j)!} \int (\mathcal{H}_\alpha^{[0,1]}(G + \frac{1}{2} g') - \mathcal{H}_\alpha^{[0,1]}(G + \frac{1}{2} g)) w^{[0,1]}(\sqrt{2} G)
\]
\[
\Omega \left( 2, \frac{1}{2} \right)
= \frac{1}{(e_i + e_j)!} 8 \sqrt{\pi} \left( \frac{2}{15} \delta_{ij} \delta_{kl} - \frac{1}{5} (\delta_{ij} \delta_{jk} + \delta_{ik} \delta_{jl}) \right) \Omega(2, 2).
\]
Here $\Omega(2, 2) = \int_0^\infty \int_0^\infty e^{-\gamma^2} \gamma (1 - \cos^2 \chi(b, \gamma)) \, db \, d\gamma$.

Therefore, combining the equation (34), we can obtain the source term of the stress tensor

$$Q_{ij} = (1 + \delta_{ij}) \frac{2\rho \sqrt{\theta}}{\varepsilon} \sum_{k, l} f_{e_k + e_l} \tilde{Q}_{e_k + e_l, 0} + (1 + \delta_{ij}) \frac{2\rho}{\varepsilon} \sqrt{\theta} \sum_{2 < \beta \leq M} f_\beta \tilde{Q}_{\beta, 0}^{e_i + e_j}$$

$$+ (1 + \delta_{ij}) \frac{1}{\varepsilon} \sum_{2 < \gamma, |\beta| \leq M} f_\beta f_\gamma \sqrt{\theta} \sum_{2 < \beta \leq M} f_\beta \tilde{Q}_{\beta, 0}^{e_i + e_j}$$

$$= -\frac{16 \rho \sqrt{\theta}}{5} \Omega(2, 2) \sigma_{ij} + (1 + \delta_{ij}) \frac{2\rho}{\varepsilon} \sqrt{\theta} \sum_{2 < \beta \leq M} f_\beta \tilde{Q}_{\beta, 0}^{e_i + e_j}$$

$$+ (1 + \delta_{ij}) \frac{1}{\varepsilon} \sum_{2 < \gamma, |\beta| \leq M} f_\beta f_\gamma \sqrt{\theta} \sum_{2 < \beta \leq M} f_\beta \tilde{Q}_{\beta, 0}^{e_i + e_j}$$

$$= \frac{\rho \sqrt{\theta}}{\tau} \sigma_{ij} + \frac{C_1}{\tau} \rho \sum_{2 < \beta \leq M} \sqrt{\theta} \sum_{2 < \beta \leq M} f_\beta \tilde{Q}_{\beta, 0}^{e_i + e_j}$$

$$+ \frac{C_2}{\tau} \sum_{2 < \gamma, |\beta| \leq M} \sqrt{\theta} \sum_{2 < \beta \leq M} f_\beta f_\gamma \tilde{Q}_{\beta, 0}^{e_i + e_j}.$$ (35)

Here $\tau = \frac{5\varepsilon}{16(1+\Omega(2, 2))}$ and $C_1, C_2$ are constant. Similarly, we can acquire the source term of the heat flux.

$$Q_i = Q_{3e_i} + \sum_{k=1}^{3} Q_{e_i + 2e_k}$$

$$= -\frac{2}{3} \frac{16 \rho \sqrt{\theta}}{5} \Omega(2, 2) q_i + \frac{2\rho}{\varepsilon} \sum_{1 < \beta \leq M} \sqrt{\theta} \sum_{1 < \beta \leq M} f_\beta (\tilde{Q}_{3e_i}^{e_i} + \sum_{k=1}^{3} \tilde{Q}_{e_i + 2e_k}^{e_i})$$

$$+ \frac{1}{\varepsilon} \sum_{1 < \gamma, |\beta| \leq M} \sqrt{\theta} \sum_{1 < \beta \leq M} f_\beta f_\gamma (\tilde{Q}_{3e_i}^{e_i} + \sum_{k=1}^{3} \tilde{Q}_{e_i + 2e_k}^{e_i})$$

$$= -\frac{2\rho \sqrt{\theta}}{3\tau} q_i + \frac{C_1}{\tau} \rho \sum_{1 < \beta \leq M} \sqrt{\theta} \sum_{1 < \beta \leq M} f_\beta \tilde{Q}_{\beta, 0}^{q_i} + \frac{C_2}{\tau} \sum_{1 < \gamma, |\beta| \leq M} \sqrt{\theta} \sum_{1 < \beta \leq M} f_\beta f_\gamma \tilde{Q}_{\beta, 0}^{q_i}.$$ (36)

with $\tilde{Q}_{\beta, \gamma}^{a_i} = \tilde{Q}_{\beta, \gamma}^{a_i} + \sum_{k=1}^{3} \tilde{Q}_{e_i + 2e_k}^{a_i}$. The calculation only need calculus knowledge, so we omit the details. Furthermore

$$Q_\alpha = \frac{1}{\varepsilon} \sum_{\beta, \gamma} f_\beta f_\gamma \tilde{Q}_{\beta, \gamma}^{a_i}$$

$$= \frac{2\rho}{\varepsilon} \sum_{1 < |\beta| \leq M} f_\beta \sqrt{\theta} \tilde{Q}_{\beta, 0}^{a_i} + \frac{1}{\varepsilon} \sum_{1 < |\gamma|, |\beta| \leq M} f_\beta f_\gamma \sqrt{\theta} \tilde{Q}_{\beta, 0}^{a_i}.$$ (37)

$$= -C_0 \frac{\rho \sqrt{\theta}}{\tau} f_\alpha + \frac{C_1}{\tau} \rho \sum_{1 < |\beta| \leq M} f_\beta \tilde{Q}_{\beta, 0}^{a_i}(\theta) + \frac{C_2}{\tau} \sum_{1 < |\gamma|, |\beta| \leq M} f_\beta f_\gamma \tilde{Q}_{\beta, \gamma}^{a_i}(\theta).$$
Consequently, with the help of (35), (36) and (37), we obtain the coefficients of viscosity $\tau \sqrt{\theta}$ and thermal conductivity $\frac{15}{4} \tau \sqrt{\theta}$ in (16). This indicates the correct Prandtl number $Pr = \frac{2}{3}$ for all monatomic gases.

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