WEIGHTED INEQUALITIES FOR PRODUCT FRACTIONAL INTEGRALS

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Abstract. We investigate one and two weight norm inequalities for product fractional integrals

\[ I_{m,n}^{\alpha,\beta} f(x,y) = \left( \iint_{\mathbb{R}^m \times \mathbb{R}^n} |x-u|^{\alpha-m} |y-t|^{\beta-n} f(u,t) \, du \, dt \right) \]

in \( \mathbb{R}^m \times \mathbb{R}^n \). We show that in the one weight case, most of the 1-parameter theory carries over to the 2-parameter setting - the one weight inequality

\[ \| I_{m,n}^{\alpha,\beta} f \|_{L^q} \leq A_p^{(\alpha,m), (\beta,n)} (w_p) \| f \|_{L^p}, \quad f \geq 0, \]

is equivalent to finiteness of the rectangle characteristic

\[ A_p^{(\alpha,\beta), (m,n)} (w) = \sup_{I,J} |I|^{\frac{\alpha - m}{p} - 1} |J|^{\frac{\beta - n}{q} - 1} \left( \iint_{I \times J} w^q \right)^{\frac{1}{q}} \left( \iint_{I \times J} w^{-p'} \right)^{\frac{1}{p'}}, \]

which is in turn equivalent to the diagonal and balanced equalities

\[ 0 < \frac{\alpha}{m} - \frac{\beta}{n} - \frac{1}{p} - \frac{1}{q}. \]

Moreover, the optimal power of the characteristic that bounds the norm is \( 2 + 2 \max \left\{ \frac{\alpha}{q}, \frac{\beta}{p'} \right\} \). However, in the two weight case, apart from the trivial case of product weights, the rectangle characteristic fails to control the operator norm of \( I_{\alpha,\beta} : L^p (v^p) \rightarrow L^q (w^q) \) in general.

On the other hand, in the half-balanced case \( \min \left\{ \frac{\alpha}{m}, \frac{\beta}{n} \right\} = \frac{1}{p} - \frac{1}{q} \), we prove that the rectangle characteristic is sufficient for the weighted norm inequality in the presence of a side condition - either \( w^q \) is in product \( A_1 \) or \( v^{-p'} \) is in product \( A_1 \).

Moreover, the Stein-Weiss extension of the classical Hardy - Littlewood - Sobolev inequality to power weights carries over to the 2-parameter setting with nonproduct power weights using a 'sandwiching' technique, providing our main positive result in two weight 2-parameter theory.

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1. INTRODUCTION

The theory of weighted norm inequalities for product operators, i.e., those operators commuting with a multiparameter family of dilations, has proved challenging since the pioneering work of Robert Fefferman [Fef] in the 1980’s involving covering lemmas for collections of rectangles. The purpose of the present paper is to settle some of the basic questions arising in the weighted theory for the special case of product fractional integrals, in particular the relationship between their norm inequalities and their associated rectangle characteristics. Our four main results can be split into two distinct parts, which are presented largely independent of each other:

1. In the first part of the paper, we consider the general two weight norm inequality $I_{m,n}^{\alpha,\beta}: L^p(v^p) \to L^q(w^q)$ for product fractional integrals $I_{m,n}^{\alpha,\beta}$ when the indices are subbalanced:

$$\min\left\{\frac{\alpha}{m}, \frac{\beta}{n}\right\} \geq \frac{1}{p} - \frac{1}{q} > 0.$$ 

We focus separately on the three subcases where the indices are balanced $\frac{\alpha}{m} = \frac{\beta}{n} = \frac{1}{p} - \frac{1}{q} > 0$, half subbalanced $\min\left\{\frac{\alpha}{m}, \frac{\beta}{n}\right\} = \frac{1}{p} - \frac{1}{q} > 0$ with $\frac{\alpha}{m} \neq \frac{\beta}{n}$, and finally strictly subbalanced, $\min\left\{\frac{\alpha}{m}, \frac{\beta}{n}\right\} > \frac{1}{p} - \frac{1}{q} > 0$. It turns out that the one weight theory in 1-parameter carries over to the product setting in the balanced case, some of the familiar two weight theory in 1-parameter carries over in the half balanced case, and finally, the rectangle characteristic is not sufficient for the norm inequality in the strictly subbalanced case, without assuming additional side conditions on the weights. More precisely, we prove:

(a) In the balanced case, the one weight inequality in the product setting is equivalent to finiteness of the product Muckenhoupt characteristic.

(b) In the half balanced case, we use the one weight inequality to show that the two weight inequality holds if in addition to the Muckenhoupt characteristic, we have one of the side conditions, $w^q \in A_1 \times A_1$ or $v^{-p} \in A_1 \times A_1$.

(c) In the strictly sub-balanced case, a simple construction shows that the rectangle characteristic is not sufficient.

2. In the second part of the paper, we give a sharp product version of the Stein-Weiss extension of the classical Hardy-Littlewood-Sobolev theorem for (nonproduct) power weights, which we establish by a (somewhat complicated) method of iteration, something traditionally thought unlikely.

Our positive results for one weight theory and two power weight theory are obtained using the tools of iteration, Minkowski’s inequality and a sandwiching argument. Now we begin to describe these matters in detail.

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1We thank H. Tanaka for pointing out an error in our counterexample with rectangle $A_1$ weights in the previous version of this paper, and also for bringing to our attention ”The $n$-linear embedding theorem for dyadic rectangles” by H. Tanaka and K. Yabuta, arXiv 1710.08059v1, which obtains boundedness of certain product fractional integrals with reverse doubling weights. The counterexample for the two-tailed characteristic in the previous version of this paper also contained an error.
Let $m, n \geq 1$. For indices $1 < p, q < \infty$ and $0 < \alpha < m$ and $0 < \beta < n$, we consider the weighted norm inequality
\[
\left\{ \int\int_{\mathbb{R}^m \times \mathbb{R}^n} I_{\alpha, \beta}^{m,n} f(x,y)^q w(x,y)^{\frac{q}{p}} \, dx \, dy \right\}^{\frac{1}{p}} \leq N_{p,q}^{(\alpha,\beta),(m,n)}(v,w) \left\{ \int\int_{\mathbb{R}^m \times \mathbb{R}^n} f(u,t)^p v(u,t)^{\frac{p}{q}} \, du \, dt \right\}^{\frac{1}{q}}
\]
for the product fractional integral
\[
I_{\alpha, \beta}^{m,n} f(x,y) = \int\int_{\mathbb{R}^m \times \mathbb{R}^n} |x-u|^{\alpha-m} |y-t|^{\beta-n} f(u,t) \, du \, dt, \quad (x,y) \in \mathbb{R}^m \times \mathbb{R}^n,
\]
and characterize when one weight and two weight inequalities are equivalent to the corresponding product fractional Muckenhoupt characteristic,
\[
A_{p,q}^{(\alpha,\beta),(m,n)}(v,w) = \sup_{I \subset \mathbb{R}^m, J \subset \mathbb{R}^n} |I|^{\frac{m-1}{p}} |J|^{\frac{n-1}{q}} \left( \int\int_{I \times J} u^q \, du \right)^{\frac{1}{q}} \left( \int\int_{I \times J} v^{-p'} \, dv \right)^{\frac{1}{p'}},
\]
and its two-tailed variant,
\[
\widehat{A}_{p,q}^{(m,n),(\alpha,\beta)}(v,w) = \sup_{I \times J \subset \mathbb{R}^m \times \mathbb{R}^n} |I|^{\frac{m-1}{p}} |J|^{\frac{n-1}{q}} \left( \int\int_{I \times J} (s_{I \times J} u)^q \, ds \right)^{\frac{1}{q}} \left( \int\int_{I \times J} (s_{I \times J} v^{-p'}) \, ds \right)^{\frac{1}{p'}}.
\]
where
\[
(1.3) \quad s_{I \times J}(x,y) = \left( 1 + \frac{|x-c_I|}{|I|^{\frac{1}{m}}} \right)^{\alpha-m} \left( 1 + \frac{|y-c_J|}{|J|^{\frac{1}{n}}} \right)^{\beta-n}.
\]
and that $p < q$; then the finiteness of the operator norm $N_{p,q}^{(m,n)}(w)$ is equivalent to finiteness of the characteristic $A_{p,q}^{(m,n)}(w)$, where as is conventional, we are suppressing redundant indices in the one weight diagonal balanced case. In addition, we characterize the optimal power of the characteristic that controls the operator norm. These one weight results are proved by an iteration strategy using Minkowski’s inequality that also yields two weight results for the special case of product weights.

For the general two weight case, we show that in the absence of any side conditions on the weight pair $(v,w)$, the operator norm $N_{p,q}^{(\alpha,\beta),(m,n)}(v,w)$ is never controlled by the two-tailed characteristic $\widehat{A}_{p,q}^{(m,n),(\alpha,\beta)}(v,w)$, not even the weak type operator norm of the much smaller dyadic fractional maximal function $M_{\alpha,\beta}^{dy}$ (see below for definitions). On the other hand, new two weight results can be obtained from known norm inequalities, such as the one weight and product weight results mentioned above, by the technique of ‘sandwiching’.

**Lemma 1.** If $\{ (V^i, W^i) \}_{i=1}^N$ is a sequence of weight pairs ‘sandwiched’ in a weight pair $(v,w)$, i.e.
\[
\frac{w(x,y)}{v(u,t)} \leq \sum_{i=1}^N W^i(x,y) \leq \sum_{i=1}^N V^i(u,t),
\]
then
\[
N_{p,q}^{(\alpha,\beta),(m,n)}(v,w) \leq \sum_{i=1}^N N_{p,q}^{(\alpha,\beta),(m,n)}(V^i, W^i).
\]

\[\text{In the case } N = 1, \text{ the hypothesis is implied by } w \leq W^1 \text{ and } V^1 \leq v, \text{ hence the terminology ‘sandwiched’}.\]
Proof. This follows immediately from setting \( g = fV \) in the identity,

\[
N^{(\alpha, \beta), (m, n)}_{p, q}(V, W) = \sup_{\|g\|_{L^p} \leq 1, \|h\|_{L^q} \leq 1} \int_{\mathbb{R}^m \times \mathbb{R}^n} h(x, y) W(x, y) |x - u|^\alpha - m |y - t|^\beta - n \frac{g(u, t)}{V(u, t)} dx dy dt.
\]

\( \square \)

We mention two simple examples of sandwiching, the first example sandwiching a one weight pair, and the second example sandwiching two product weight pairs:

1. In the case of diagonal and balanced indices \( \frac{1}{p} - \frac{1}{q} = \frac{\alpha}{m} = \frac{\beta}{n} \), if there is a one weight pair \((u, w)\) with \( A^{(\alpha, \beta), (m, n)}_{p, q}(u) < \infty \) sandwiched in \((v, w)\), then \( N^{(\alpha, \beta), (m, n)}_{p, q}(v, w) \lesssim A^{(\alpha, \beta), (m, n)}_{p, q}(u) < \infty \).

2. If \( w(x, y) = |(x, y)|^{-\gamma} \) and \( v(x, y) = |(x, y)|^\delta \) are power weights on \( \mathbb{R}^{m+n} \), then \( N^{(\alpha, \beta), (m, n)}_{p, q}(v, w) < \infty \) provided the indices satisfy product conditions corresponding to those of the Stein-Weiss theorem in 1-parameter (see below), that in turn generalize the classical Hardy-Littlewood-Sobolev inequality. In this example there are two weight pairs \{ \( (V, W), (V', W') \) \} depending on the indices and sandwiched in \((v, w)\) where each weight is an appropriate product power weight.

We begin by briefly recalling the 1-parameter weighted theory of fractional integrals.

2. 1-PARAMETER THEORY

Define \( \Omega^m_\alpha(x) = |x|^\alpha - m \) and set \( I^m g = \Omega^m_\alpha * g \). The following one weight theorem for fractional integrals is due to Muckenhoupt and Wheeden.

Theorem 1. Let \( 0 < \alpha < m \). Suppose \( 1 < p < q < \infty \) and

\[
(2.1) \quad \frac{1}{p} - \frac{1}{q} = \frac{\alpha}{m}.
\]

Let \( w(x) \) be a nonnegative weight on \( \mathbb{R}^m \). Then

\[
\left\{ \int_{\mathbb{R}^m} I^m_\alpha f(x)^q w(x)^q \, dx \right\}^{\frac{1}{q}} \leq N_{p,q}(w) \left\{ \int_{\mathbb{R}^m} f(x)^p w(x)^p \, dx \right\}^{\frac{1}{p}}
\]

for all \( f \geq 0 \) for \( N_{p,q}(w) < \infty \) if and only if

\[
(2.2) \quad A_{p,q}(w) = \sup_{\text{cubes } I \subset \mathbb{R}^m} \left( \frac{1}{|I|} \int_I w(x)^q \, dx \right)^{\frac{1}{q}} \left( \frac{1}{|I|} \int_I w(x)^{-p'} \, dx \right)^{\frac{1}{p'}} < \infty.
\]

In fact, assuming only that \( 1 < p, q < \infty \), the balanced condition \( (2.1) \) is necessary for the norm inequality \( (2.2) \). See Theorem 13 in the Appendix for this.

A two weight analogue for \( 1 < p \leq q < \infty \) was later obtained by Sawyer [Saw] that involved testing the norm inequality and its dual over indicators of cubes times \( w^q \) and \( v^{-p'} \) respectively, namely for all cubes \( Q \subset \mathbb{R}^m \),

\[
\left\{ \int_{\mathbb{R}^m} I^m_\alpha \left( 1_Q v^{-p'} \right) (x)^q w(x)^q \, dx \right\}^{\frac{1}{q}} \leq T^{\alpha, m}_{p,q}(v, w) |Q|^\frac{1}{p'},
\]

\[
\left\{ \int_{\mathbb{R}^m} I^m_\alpha \left( 1_Q w^q \right) (x)^p v(x)^{-p'} \, dx \right\}^{\frac{1}{p'}} \leq T^{\alpha, m}_{q', p'} \left( \frac{1}{|w|^q} \frac{1}{v^p} \right) |Q|^{\frac{1}{p'}}.
\]

Later yet, it was shown by Sawyer and Wheeden [SaWh], using an idea of Kokilashvili and Gabidzashvili [KoGa], that in the special case \( p < q \), the testing conditions could be replaced with a two-tailed two weight version of the \( A_{p,q} \) condition \( (2.2) \):

\[
(2.3) \quad \sup_{I \subset \mathbb{R}^m} \left| \frac{|I|}{|I|^{\frac{\alpha}{m}}} \left( \frac{1}{|I|} \int_I \left[ \hat{s}_I(x) w(x) \right]^q \, dx \right)^{\frac{1}{q}} \left( \frac{1}{|I|} \int_I \left[ \hat{s}_I(x) v(x)^{-1} \right]^p \, dx \right)^{\frac{1}{p}} \right| \equiv A^{\alpha, m}_{p,q}(v, w) < \infty,
\]

where the tail \( \hat{s}_I \) is given by

\[
\hat{s}_I(x) = \left( 1 + \frac{|x - c_I|}{|I|^{\frac{1}{m}}} \right)^{\alpha - m},
\]
and $c_I$ is the center of the cube $I$. Note that in Sawyer and Wheeden [SaWh], this condition was written in terms of the rescaled tail $s_I = |I|^{rac{m}{n}-1} s_I$, and it was also shown that the two-tailed condition $\tilde{A}_{p,q}$ condition \(2.3\) could be replaced by a pair of corresponding one-tailed conditions.

**Theorem 2.** Suppose $1 < p < q < \infty$ and $0 < \alpha < m$. Let $w(x)$ and $v(x)$ be a pair of nonnegative weights on $\mathbb{R}^m$. Then

\[
\left\{ \int_{\mathbb{R}^m} \left( \int_{\mathbb{R}^m} f(x)^q w(x)^q \, dx \right)^{\frac{1}{q}} \right\}^{\frac{1}{p}} \leq N_{p,q}^{\alpha,m}(v,w) \left\{ \int_{\mathbb{R}^m} \left( \int_{\mathbb{R}^m} f(x)^p v(x)^p \, dx \right)^{\frac{1}{p}} \right\}^{\frac{1}{q}}
\]

for all $f \geq 0$ if and only if the $\tilde{A}_{p,q}$ condition \(2.3\) holds, i.e. $\tilde{A}_{p,q}^{\alpha,m}(v,w) < \infty$. Moreover, the best constant $N_{p,q}^{\alpha,m}(w,v)$ in \(2.3\) is comparable to $\tilde{A}_{p,q}^{\alpha,m}(w,v)$.

The special case of power weights $|x|^{\gamma}$ had been considered much earlier, and culminated in the following 1958 theorem of Stein and Weiss [StWe2].

**Theorem 3.** Let $w_\gamma(x) = |x|^{-\gamma}$ and $v_\delta(x) = |x|^{\delta}$ be a pair of nonnegative power weights on $\mathbb{R}^m$ with $-\infty < \gamma, \delta < \infty$. Suppose $1 < p \leq q < \infty$ and $\alpha \in \mathbb{R}$ satisfy the strict constraint inequalities,

\[
0 < \alpha < m \quad \text{and} \quad \alpha + \gamma < m \quad \text{and} \quad \frac{q}{p} \delta < m,
\]

together with the inequality

\[
\gamma + \delta \geq 0,
\]

and the power weight equality,

\[
\frac{1}{p} - \frac{1}{q} = \frac{\alpha - (\gamma + \delta)}{m}.
\]

Then \(2.3\) holds, i.e.

\[
\left\{ \int_{\mathbb{R}^m} \left( \int_{\mathbb{R}^m} f(x)^q \, |x|^{-\gamma q} \, dx \right)^{\frac{1}{q}} \right\}^{\frac{1}{p}} \leq N_{p,q}^{\alpha,m}(w_\gamma,v_\delta) \left\{ \int_{\mathbb{R}^m} \left( \int_{\mathbb{R}^m} f(x)^p \, |x|^{-\gamma p} \, dx \right)^{\frac{1}{p}} \right\}^{\frac{1}{q}}.
\]

The previous theorem cannot be improved when the weights are restricted to power weights. Indeed, $\alpha > 0$ follows from the local integrability of the kernel, both $q\gamma < m$ and $p\gamma < m$ follow from the local integrability of $w^q$ and $v^{-p'}$, and then both $\gamma + \delta \geq 0$ and $\frac{q}{p} - \frac{1}{q} = \frac{\alpha - (\gamma + \delta)}{m}$ follow from the finiteness of the Muckenhoupt condition $A_{p,q}(v,w)$ using standard arguments (see e.g. the proof of Theorem 3 below). These conditions then yield

\[
\alpha = m \left( \frac{1}{p} - \frac{1}{q} \right) + \gamma + \delta < m \left( \frac{1}{p} - \frac{1}{q} \right) + \frac{m}{q} + \frac{m}{p'} = m.
\]

A routine calculation shows that the aforementioned conditions on the indices $\alpha, m, \gamma, \delta, p, q$ are precisely those for which the characteristic $A_{p,q}(w_\gamma,v_\delta)$ is finite. Finally, the necessity of the remaining condition $p \leq q$ is an easy consequence of Maz’ja’s characterization [Maz] of the Hardy inequality for $q < p$. See Theorem 14 in the Appendix for this. Altogether, this establishes the succinct conclusion that the power weight norm inequality holds if and only if $p \leq q$ and the characteristic is finite.

Next, in the one weight setting, we recall the solution to the ‘power of the characteristic’ problem for fractional integrals due to Lacey, Moen, Perez and Torres in [LaMoPeTo]. See the Appendix for a different proof of this 1-parameter theorem that reveals the origin of the number $1 + \max \left\{ \frac{q'}{q}, \frac{q}{p'} \right\}$ to be the optimal exponent in the inequality $\mathcal{A}_{p,q}(w) \leq A_{p,q}(w)^{1+\max \left\{ \frac{q'}{q}, \frac{q}{p'} \right\}}$ where $\mathcal{A}_{p,q}(w)$ is a one-tailed version of $A_{p,q}(w)$.

**Theorem 4.** Let $0 < \alpha < m$. Suppose $1 < p \leq q < \infty$ and

\[
\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{m}.
\]

Let $w(x)$ be a nonnegative weight on $\mathbb{R}^m$. Then

\[
\left\{ \int_{\mathbb{R}^m} \left( \int_{\mathbb{R}^m} f(x)^q \, w(x)^q \, dx \right)^{\frac{1}{q}} \right\}^{\frac{1}{p}} \leq N_{p,q}(w) \left\{ \int_{\mathbb{R}^m} \left( \int_{\mathbb{R}^m} f(x)^p \, w(x)^p \, dx \right)^{\frac{1}{p}} \right\}^{\frac{1}{q}}
\]
consider the two weight norm inequality for nonnegative functions

\[ \|M_{\alpha} f\|_{L^p(\sigma)} \leq C_{p,q,n} A_{p,q} (w) \left( \frac{\|w\|_p}{\|w\|_q} \right)^{1+\max\left\{ \frac{\alpha}{q}, \frac{\beta}{p} \right\}}. \]

The power \( 1 + \max\left\{ \frac{\alpha}{q}, \frac{\beta}{p} \right\} \) is sharp, even when \( w \) is restricted to power weights \( w(x) = |x|^\gamma \).

Finally we recall the extremely simple proof of the equivalence of the dyadic characteristic

\[ K_{p,q}^{\alpha,\beta}(\sigma, \omega) \equiv \sup_{Q \subset \mathbb{R}^n \text{dyadic}} |Q|^{\frac{\alpha}{q} - \frac{\beta}{p}} \left( \frac{|Q|_\sigma}{|Q|_\omega} \right)^{\frac{\beta}{p}}, \]

where the supremum is taken over dyadic cubes, and the weak type \((p, q)\) operator norm \( N_{p,q}^{\alpha,\beta}(\sigma, \omega) \) of the dyadic fractional maximal operator \( M_{\alpha} \) with respect to \((\sigma, \omega)\):

\[ N_{p,q}^{\alpha,\beta}(\sigma, \omega) \equiv \sup_{f \geq 0} \frac{\sup_{\lambda > 0} \lambda \left( \{ M_{\alpha} (f \sigma) > \lambda \} \right)^{\frac{p}{q}}}{\int f^p d\sigma}. \]

This proof illustrates the power of an effective covering lemma for weights, something that is sorely lacking in the 2-parameter setting, and accounts for much of the negative nature of our results (c.f. Example 2 below). Recall that the 1-parameter dyadic fractional maximal operator \( M_{\alpha} \) acts on a signed measure \( \mu \) in \( \mathbb{R}^n \) by

\[ M_{\alpha} \mu(x) = \sup_{Q \text{ dyadic}} |Q|^{\alpha-n} \int_Q d\mu. \]

**Lemma 2.** Let \((\sigma, \omega)\) be a locally finite weight pair in \( \mathbb{R}^n \), and let \( 1 < p \leq q < \infty \). Then \( M_{\alpha} : L^p(\sigma) \to L^{q,\infty}(\omega) \) if and only if \( A_{p,q}^{\alpha,\beta}(\sigma, \omega) < \infty \).

**Proof.** Fix \( \lambda > 0 \) and \( f \geq 0 \) bounded with compact support. Let \( \Omega_\lambda \equiv \{ x \in \mathbb{R}^n : M_{\alpha} f \sigma > \lambda \} \). Then

\[ \Omega_\lambda = \bigcup_k Q_k, \quad |Q_k|^{\alpha-n} \int_{Q_k} f d\sigma > \lambda, \]

and we have

\[ \left( \frac{\lambda |\Omega_\lambda|^{\frac{1}{q}}}{|Q_k|^{\frac{1}{q}}} \right)^p = \lambda^p \left( \sum_k |Q_k|^{\frac{p}{q}} \right) \leq \lambda^p \sum_k |Q_k|^{\frac{p}{q}} = \sum_k \lambda^p |Q_k|^{\frac{p}{q}} \]

\[ < \sum_k \left( |Q_k|^{\alpha-n} \int_{Q_k} f d\sigma \right)^p |Q_k|^{\frac{1}{q}} = \sum_k \left( |Q_k|^{\alpha-n} |Q_k|^{\frac{p}{q}} |Q_k|^{\frac{1}{q}} \right)^p |Q_k|^{1-p} \int_{Q_k} f d\sigma \]

\[ \leq \sum_k \left( A_{p,q}^{\alpha,\beta}(\sigma, \omega) \right)^p \int_{Q_k} f^p d\sigma \leq A_{p,q}^{\alpha,\beta}(\sigma, \omega)^p \| f \|_{L^p(\sigma)}^p, \]

which gives

\[ \| M_{\alpha} f \|_{L^{q,\infty}(\omega)} = \sup_{\lambda > 0} \lambda |\Omega_\lambda|^{\frac{1}{q}} \leq A_{p,q}^{\alpha,\beta}(\sigma, \omega) \| f \|_{L^p(\sigma)}, \]

for all \( f \geq 0 \) bounded with compact support.

The proof of the converse statement is standard, similar to but easier than that of Lemma 2 below. \( \square \)

### 3. 2-parameter theory

Define the product fractional integral \( I_{\alpha,\beta}^{m,n} \) on \( \mathbb{R}^m \times \mathbb{R}^n \) by the convolution formula

\[ I_{\alpha,\beta}^{m,n} f \equiv \Omega_{\alpha,\beta}^{m,n} * f, \]

where the convolution kernel \( \Omega_{\alpha,\beta}^{m,n} \) is a product function:

\[ \Omega_{\alpha,\beta}^{m,n}(x, y) = |x|^{\alpha-n} |y|^{\beta-n}, \quad (x, y) \in \mathbb{R}^m \times \mathbb{R}^n. \]

Let \( v(x, y) \) and \( w(x, y) \) be positive weights on \( \mathbb{R}^m \times \mathbb{R}^n \). For \( 1 < p, q < \infty \) and \( 0 < \alpha < m \), \( 0 < \beta < n \), we consider the two weight norm inequality for nonnegative functions \( f(x, y) \):

\[ \left\{ \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} I_{\alpha,\beta}^{m,n} f(x, y)^q w(x, y)^q \, dx \, dy \right\}^{\frac{1}{q}} \leq N_{p,q}^{(\alpha,\beta),(m,n)}(v, w) \left\{ \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} f(x, y)^p v(x, y)^p \, dx \, dy \right\}^{\frac{1}{p}}. \]
If we define absolutely continuous measures $\sigma, \omega$ by

$$d\sigma(x, y) = v(x, y)^{-p'} \, dx \, dy \quad \text{and} \quad d\omega(x, y) = w(x, y)^{q} \, dx \, dy,$$

then the two weight norm inequality (3.1) is equivalent to the norm inequality

$$\left\{ \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} I_{\alpha,\beta}^{m,n}(f \sigma)(x, y)^{\frac{q}{p} \left(\frac{1}{p} + \frac{1}{q} - \frac{1}{\sigma} - \frac{1}{\omega} \right)} \, dx \, dy \right\}^{\frac{1}{\frac{q}{p} \left(\frac{1}{p} + \frac{1}{q} - \frac{1}{\sigma} - \frac{1}{\omega} \right)}} \leq N^{(\alpha,\beta),(m,n)}_{p,q}(\sigma, \omega) \left\{ \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} f(x, y)^{p} \, d\sigma(x, y) \right\}^{\frac{1}{p}},$$

where now the measure $\sigma$ appears inside the argument of $I_{\alpha,\beta}^{m,n}$, namely in $I_{\alpha,\beta}^{m,n}(f \sigma)$. In this form, the norm inequality makes sense for arbitrary locally finite Borel measures $\sigma, \omega$ since for nonnegative $f \in L^p(\sigma)$, the function $f$ is measurable with respect to $\sigma$ and the integral $\int I_{\alpha,\beta}^{m,n}(x-u, y-t) f(u, t) \, d\sigma(u, t)$ exists. Note also that the best constants $A_{p,q}^{(\alpha,\beta),(m,n)}(v, w)$ and $N_{p,q}^{(\alpha,\beta),(m,n)}(\sigma, \omega)$ coincide under the standard identifications in (3.2), and this accounts for the use of blackboard bold font to differentiate the two best constants.

A necessary condition for (3.3) to hold is the finiteness of the corresponding product fractional characteristic $K^{(\alpha,\beta),(m,n)}_{p,q}(\sigma, \omega)$ of the weights,

$$K^{(\alpha,\beta),(m,n)}_{p,q}(\sigma, \omega) \equiv \sup_{I \subset \mathbb{R}^m, J \subset \mathbb{R}^n} \left| I \right|^{\frac{1}{p} - \frac{1}{\sigma}} \left| J \right|^{\frac{1}{q} - \frac{1}{\omega}} \left( \int_{I \times J} \frac{\left| I \times J \right|}{\left| I \right|^{\frac{1}{p}}} \right)^{\frac{1}{q}} \left( \int_{I \times J} \frac{\left| I \times J \right|}{\left| J \right|^{\frac{1}{q}}} \right)^{\frac{1}{p}},$$

as well as the finiteness of the larger two-tailed characteristic,

$$\hat{K}^{(\alpha,\beta),(m,n)}_{p,q}(\sigma, \omega) \equiv \sup_{I \times J \subset \mathbb{R}^m \times \mathbb{R}^n} \left| I \right|^{\frac{1}{p} - \frac{1}{\sigma}} \left| J \right|^{\frac{1}{q} - \frac{1}{\omega}} \left( \int_{I \times J} \frac{\left| I \times J \right|}{\left| I \right|^{\frac{1}{p}}} \right)^{\frac{1}{q}} \left( \int_{I \times J} \frac{\left| I \times J \right|}{\left| J \right|^{\frac{1}{q}}} \right)^{\frac{1}{p}},$$

where $\hat{s}_{I \times J}(x, y)$ is the ‘tail’ defined in (1.3) above. When (3.2) holds, we write $A_{p,q}^{(\alpha,\beta),(m,n)}(v, w) = K^{(\alpha,\beta),(m,n)}_{p,q}(\sigma, \omega)$ and $\hat{A}_{p,q}^{(\alpha,\beta),(m,n)}(v, w) = \hat{K}^{(\alpha,\beta),(m,n)}_{p,q}(\sigma, \omega)$, and in the one weight case $v = w$ we simply write $A^{(\alpha,\beta),(m,n)}_{p,q}(w)$ and $\hat{A}_{p,q}^{(\alpha,\beta),(m,n)}(w)$.

**Lemma 3.** For $1 < p, q < \infty$ and $0 < \alpha \leq m$, $0 < \beta \leq n$, we have

$$\hat{K}^{(\alpha,\beta),(m,n)}_{p,q}(\sigma, \omega) \leq N_{p,q}^{(\alpha,\beta),(m,n)}(\sigma, \omega).$$

**Proof.** To see this, we begin by noting that for any rectangle $I \times J$ we have

$$\left| I \right|^{\frac{1}{p}} |x-u| \leq \left( \left| I \right|^{\frac{1}{p}} + |x-c_I| \right) \left( \left| I \right|^{\frac{1}{p}} + |u-c_I| \right)$$

and $|y-t| \leq \left( \left| J \right|^{\frac{1}{q}} + |y-c_J| \right) \left( \left| J \right|^{\frac{1}{q}} + |t-c_J| \right),$

i.e.

$$|x-u| \leq \left| I \right|^{\frac{1}{p}} \left( 1 + \frac{|x-c_I|}{\left| I \right|^{\frac{1}{p}}} \right) \left( 1 + \frac{|u-c_I|}{\left| I \right|^{\frac{1}{p}}} \right)$$

and $|y-t| \leq \left| J \right|^{\frac{1}{q}} \left( 1 + \frac{|y-c_J|}{\left| J \right|^{\frac{1}{q}}} \right) \left( 1 + \frac{|t-c_J|}{\left| J \right|^{\frac{1}{q}}} \right),$

and hence

$$\left| x-u \right|^{\alpha-m} \left| y-t \right|^{\beta-n} \geq \left| I \right|^{\frac{1}{p} - 1} \left( 1 + \frac{|x-c_I|}{\left| I \right|^{\frac{1}{p}}} \right)^{\alpha-m} \left( 1 + \frac{|u-c_I|}{\left| I \right|^{\frac{1}{p}}} \right)^{\alpha-m} \left| J \right|^{\frac{1}{q} - 1} \left( 1 + \frac{|y-c_J|}{\left| J \right|^{\frac{1}{q}}} \right)^{\beta-n} \left( 1 + \frac{|t-c_J|}{\left| J \right|^{\frac{1}{q}}} \right)^{\beta-n} \hat{s}_{I \times J}(x, y) \hat{s}_{I \times J}(u, t).$$
Thus for $R > 0$ and $f_R(u, t) \equiv 1_{B(0, R) \times B(0, R)}(u, t) \tilde{s}_Q(u, t)^{p'}$, we have
\[
I_{\alpha, \beta}^{m, n}(f_R \sigma)(x, y) = \int_{B(0, R) \times B(0, R)} |x - u|^{-\alpha - n} |y - t|^{-\beta - n} \tilde{s}_{I \times J}(u, t)^{p'} \, d\sigma(u, t)
\geq \int_{B(0, R) \times B(0, R)} |J|^{\frac{p'}{p} - 1} |J|^{\frac{p'}{p} - 1} \tilde{s}_{I \times J}(x, y) \tilde{s}_{I \times J}(u, t) \tilde{s}_{I \times J}(u, t)^{p'} \, d\sigma(u, t)
= |I|^{\frac{p'}{p} - 1} |J|^{\frac{p'}{p} - 1} \tilde{s}_{I \times J}(x, y) \int_{B(0, R) \times B(0, R)} \tilde{s}_{I \times J}(u, t)^{p'} \, d\sigma(u, t).
\]
Substituting this into the norm inequality (3.3) gives
\[
|I|^{\frac{p'}{p} - 1} |J|^{\frac{p'}{p} - 1} \left( \int_{\mathbb{R}^m \times \mathbb{R}^n} \tilde{s}_{I \times J}(x, y)^q \, d\omega(x, y) \right)^{\frac{1}{q'}} \leq N_{p, q}^{(\alpha, \beta), (m, n)}(\sigma, \omega) \left( \int_{\mathbb{R}^m \times \mathbb{R}^n} f_R(u, t)^p \, d\sigma(u, t) \right)^{\frac{1}{p'}} \leq N_{p, q}^{(\alpha, \beta), (m, n)}(\sigma, \omega),
\]
and upon dividing through by \( \left( \int_{B(0, R) \times B(0, R)} \tilde{s}_{I \times J}(u, t)^{p'} \, d\sigma(u, t) \right)^{\frac{1}{p'}} \), we obtain
\[
|I|^{\frac{p'}{p} - 1} |J|^{\frac{p'}{p} - 1} \left( \int_{B(0, R) \times B(0, R)} \tilde{s}_{I \times J}(x, y)^q \, d\omega(x, y) \right)^{\frac{1}{q'}} \leq N_{p, q}^{(\alpha, \beta), (m, n)}(\sigma, \omega), \quad \text{for all } R > 0 \text{ and all rectangles } I \times J.
\]

Now take the supremum over all $R > 0$ and all rectangles $I \times J$ to get (3.10). \( \square \)

**Remark 1.** We have the ‘duality’ identities $N_{p, q}^{(\alpha, \beta), (m, n)}(\sigma, \omega) = N_{q', p'}^{(\alpha, \beta), (m, n)}(\omega, \sigma)$ and $\hat{N}_{p, q}^{(\alpha, \beta), (m, n)}(\sigma, \omega) = \hat{N}_{q', p'}^{(\alpha, \beta), (m, n)}(\omega, \sigma)$.

**Remark 2.** Since $\tilde{s}_{I \times J} \approx 1$ on $I \times J$, we have the inequality $N_{p, q}^{(\alpha, \beta), (m, n)}(\sigma, \omega) \lesssim \hat{N}_{q', p'}^{(\alpha, \beta), (m, n)}(\omega, \sigma)$. In particular, we see that in the case
\[
\frac{1}{p} - \frac{1}{q} - \frac{\alpha}{m} - 1 = 0,
\]
say $\frac{1}{p} - \frac{1}{q} - \frac{\alpha}{m} = \varepsilon > 0$, we have $\frac{\alpha}{m} - 1 = -\varepsilon - \frac{1}{p} - \frac{1}{q}$, and so
\[
|I \times J|^{-\varepsilon} \left( \int_{I \times J} d\sigma \right)^{\frac{1}{p'}} \left( \int_{I \times J} d\omega \right)^{\frac{1}{q'}} \lesssim \hat{N}_{p, q}^{(\alpha, \beta), (m, n)}(\sigma, \omega) \lesssim \hat{N}_{p, q}^{(\alpha, \beta), (m, n)}(\sigma, \omega),
\]
for all rectangles $I \times J$. Thus the finiteness of $\hat{N}_{p, q}^{(\alpha, \beta), (m, n)}(\sigma, \omega)$ implies that the measures $\sigma$ and $\omega$ are carried by disjoint sets. We shall not have much more to say regarding this case.
4. Statements of problems and theorems, and simple proofs

The 2-parameter questions we investigate in this paper are these.

1. Is the finiteness of the one weight characteristic $A_{p,q}(w)$ in (4.1) below sufficient for the one weight norm inequality (4.1) with $w = v$ when the indices are balanced, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{m} = \frac{\beta}{n}$, and if so what is the dependence of the operator norm $N_{p,q}(w)$ on the characteristic $A_{p,q}(w)$?

2. If the operator norm $N_{p,q}^{(\alpha,\beta),(m,n)}(\sigma, \omega)$ fails to be controlled by the characteristic $A_{p,q}^{(\alpha,\beta),(m,n)}(\sigma, \omega)$, what additional side conditions on the weights $\sigma, \omega$ are needed for finiteness of the characteristic $A_{p,q}^{(\alpha,\beta),(m,n)}(\sigma, \omega)$ to imply the norm inequality (4.3)?

4.1. A one weight theorem. The special ‘one weight’ case of (4.1), namely when $v = w$, is equivalent to finiteness of the product characteristic, and we can calculate the optimal power of the characteristic.

Theorem 5. Let $\alpha, \beta > 0$, and suppose $1 < p, q < \infty$. Let $w(x, y)$ be a nonnegative weight on $\mathbb{R}^m \times \mathbb{R}^n$. Then the norm inequality

$$\left\{ \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} t_{x, y}^{\alpha, \beta} f(x, y)^q w(x, y)^q \, dx \, dy \right\}^{\frac{1}{q}} \leq N_{p,q}(w) \left\{ \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} f(x, y)^p w(x, y)^p \, dx \, dy \right\}^{\frac{1}{p}}$$

holds for all $f \geq 0$ if and only if both

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{m} = \frac{\beta}{n},$$

and

$$A_{p,q}(w) \equiv \sup_{I \subset \mathbb{R}^m, J \subset \mathbb{R}^n} \left( \frac{1}{|I|} \int_I \int_J w(x, y) q \, dx \, dy \right)^{\frac{1}{q}} \left( \frac{1}{|J|} \int_J \int_J w(x, y)^{-p} \, dx \, dy \right)^{\frac{1}{p}} < \infty,$$

if and only if both (4.2) and

$$\sup_{y \in \mathbb{R}^n} \left\{ \sup_{I \subset \mathbb{R}^m} \left( \frac{1}{|I|} \int_I w^q(x) \, dx \right)^{\frac{1}{q}} \left( \frac{1}{|I|} \int_I w^{q'}(x)^{-p'} \, dx \right)^{\frac{1}{p'}} \right\} \leq A_{p,q}(w),$$

$$\sup_{x \in \mathbb{R}^m} \left\{ \sup_{J \subset \mathbb{R}^n} \left( \frac{1}{|J|} \int_J w^q(x) \, dy \right)^{\frac{1}{q}} \left( \frac{1}{|J|} \int_J w^{q'}(y)^{-p'} \, dy \right)^{\frac{1}{p'}} \right\} \leq A_{p,q}(w).$$

Moreover

$$N_{p,q}(w) \leq A_{p,q}(w)^{2 + 2\max\{\frac{\alpha}{m}, \frac{\beta}{n}\}}.$$

and the exponent $2 + 2\max\{\frac{\alpha}{m}, \frac{\beta}{n}\}$ is best possible.

There is a substitute for the case $\alpha = \beta = 0$ due to R. Fefferman [Fef see page 82], namely that the strong maximal function

$$Mf(x, y) \equiv \sup_{I, J} \frac{1}{|I| |J|} \int_I \int_J |f(x, y)| \, dx \, dy,$$

is bounded on the weighted space $L^p(w^p)$ if and only if $A_p(w) = A_{p,p}(w)$ is finite. This is proved in [Fef see page 82] as an application of a rectangle covering lemma, but can also be obtained using iteration, specifically from Theorem 11 below as $M$ is dominated by the iterated operators in (4.1). Another substitute for the case $\alpha = \beta = 0$ is the boundedness of the double Hilbert transform

$$Hf(x, y) \equiv \int_I \int_J \frac{1}{x - u} \frac{1}{y - t} f(u, t) \, du \, dt,$$

on $L^p(w^p)$ if and only if $A_p(w)$ is finite. This result, along with corresponding results for more general product Calderón-Zygmund operators, can be easily proved using Theorem 11 below, and are left for the reader.
4.1.1. Application to a two weight half-balanced norm inequality. Here is a two weight consequence of Theorem 6 when the indices satisfy the half-balanced condition.

**Theorem 6.** Suppose that $1 < p, q < \infty$ and $0 < \frac{\alpha}{m}, \frac{\beta}{n} < 1$ satisfy the half-balanced condition

\[
\frac{1}{p} - \frac{1}{q} = \min \left\{ \frac{\alpha}{m}, \frac{\beta}{n} \right\},
\]

If one of the weights $w^q$ or $v^{-p'}$ is in the product $A_1 \times A_1$ class, then the following norm inequality holds,

\[
\int_{\mathbb{R}^m} \int_{\mathbb{R}^n} I_{\alpha,m}^{m,n} f(x,y)^q w(x,y)^q \, dx \, dy \left( \frac{1}{q} \right) \leq C_{p,q} A_{p,q}^{(\alpha,\beta), (m,n)}(v,w) \left( \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} f(x,y)^p v(x,y)^p \, dx \, dy \right)^{\frac{1}{p}},
\]

where $C_{p,q}$ depends also on either $\|w^q\|_{A_1 \times A_1}$ or $\|v^{-p'}\|_{A_1 \times A_1}$.

This theorem shows that in the half-balanced case, the simple characteristic $A_{p,q}^{(\alpha,\beta), (m,n)}$ controls the two weight norm inequality \([3]\) under either of the side conditions (i) $w^q \in A_1 \times A_1$ or (ii) $v^{-p'} \in A_1 \times A_1$. This is in stark contrast to the strictly subbalanced case $\frac{1}{p} - \frac{1}{q} < \min \left\{ \frac{\alpha}{m}, \frac{\beta}{n} \right\}$ where not even both side conditions $v^{-p'}, w^q \in A_1 \times A_1$ are sufficient for control of the norm inequality by the two-tailed characteristic $A_{p,q}^{(\alpha,\beta), (m,n)}(v,w)$. On the other hand, we cannot replace the side condition that $w^q \in A_1 \times A_1$ or $v^{-p'} \in A_1 \times A_1$ in the above theorem with the smaller side condition $w^q \in A_{\infty} \times A_{\infty}$ or $v^{-p'} \in A_{\infty} \times A_{\infty}$, or even with $w^q \in A_q \times A_q$ or $v^{-p'} \in A_{p'} \times A_{p'}$, as evidenced by the following family of examples, whose properties we prove below.

**Example 1.** Let $1 < p < q < \infty$ and $\frac{\alpha}{m} = \frac{1}{p} - \frac{1}{q}$. Set

\[
v_1(y) = |y|^{-\frac{\alpha}{m}} \quad \text{and} \quad w_1(x) = (1 + |x|)^{-m} \quad \text{for} \quad x, y \in \mathbb{R}^m.
\]

Then $v_1^{-p'}(\mathbb{R}^m) \in A_{p'}$ and $A_{p,q}^{(\alpha,\beta), (m,n)}(v_1, w_1) < \infty = A_{p,q}^{(\alpha,\beta), (m,n)}(v_1, w_1)$. Now let $(v_2, w_2)$ be any weight pair in $\mathbb{R}^n$ satisfying both $v_2^{-p'} \in A_{p'}(\mathbb{R}^n)$ and $0 < A_{p,q}^{(\alpha,\beta), (m,n)}(v_2, w_2) < \infty$. Then with $v(y_1, y_2) = v_1(y_1)v_2(y_2)$ and $w(x_1, x_2) = w_1(x_1)w_2(x_2)$, we have $v^{-p'} \in (A_{p'} \times A_{p'})(\mathbb{R}^{m+n})$ and

\[
A_{p,q}^{(\alpha,\beta), (m,n)}(v,w) = A_{p,q}^{(\alpha,\beta), (m,n)}(v_1, w_1) A_{p,q}^{(\alpha,\beta), (m,n)}(v_2, w_2) < \infty,
\]

\[
A_{p,q}^{(\alpha,\beta), (m,n)}(v,w) = A_{p,q}^{(\alpha,\beta), (m,n)}(v_1, w_1) A_{p,q}^{(\alpha,\beta), (m,n)}(v_2, w_2) = \infty.
\]

In particular, the two weight norm inequality \([3]\) fails to hold for the weight pair $(v, w)$ despite the fact that $v_1^{-p'}$ belongs to the product $A_{p'} \times A_{p'}$ class, and $A_{p,q}^{(\alpha,\beta), (m,n)}(v,w)$ is finite.

4.2. Two weight theorems - counterexamples. Without any side conditions at all on the weights, the characteristic $A_{p,q}^{(\alpha,\beta), (m,n)}(\sigma, \omega)$ never controls the operator norm $N_{p,q}^{(\alpha,\beta), (m,n)}(\sigma, \omega)$ for the product fractional integral, and not even for the smaller product dyadic fractional maximal operator $M_{\alpha,\beta}^{\text{dy}}$ defined on a signed measure $\mu$ by

\[
M_{\alpha,\beta}^{\text{dy}} \mu (x,y) = \sup_{R = I \times J \text{ dyadic}} \left| I \right|^\frac{\alpha}{m} - 1 \left| J \right|^\frac{\beta}{n} - 1 \int \left| \int_{I \times J} d |\mu| \right|.
\]

(note that $M_{\alpha,\beta}^{\text{dy}} \mu \leq I_{\alpha,\beta}^{m,n} \mu$ when $\mu$ is positive). Compare this to Lemma 2 in the 1-parameter setting.

**Example 2.** Let $0 < \alpha, \beta < 1$ and $1 < p, q < \infty$. Given $0 < \rho < \infty$, define a weight pair $(\sigma, \omega_\rho)$ in the plane $\mathbb{R}^2$ by

\[
\sigma = \delta_{(0,0)} \quad \text{and} \quad \omega_\rho = \sum_{P \in \mathcal{P}} \delta_P \quad \text{where} \quad \mathcal{P} = \left\{ (2^k, 2^{-\rho k}) \right\}_{k=1}^\infty.
\]
If \( R = I \times J \) is a rectangle in the plane \( \mathbb{R} \times \mathbb{R} \) with sides parallel to the axes that contains \((0, 0)\) and satisfies \( R \cap \mathcal{P} = \{(2^k, 2^{-\rho k})\}_{k=L}^{L+N} \) for some \( L \geq 1 \) and \( N \geq 0 \), then it follows that \( \int_I \, d\omega_\rho \approx \sum_{k=L}^{L+N} 1 \approx N + 1 \), and
\[
|I|^\alpha |J|^\beta - 1 \left( \int_I \, d\omega_\rho \right)^{\frac{1}{\alpha}} \left( \int_J \, d\sigma \right)^{\frac{1}{\beta}} \lesssim (2^{L+N})^\alpha - 1 \left( 2^{-\rho(L+1)} \right)^{\beta - 1} (N + 1)^\frac{1}{\alpha}.
\]

If \( \rho \leq \frac{1-\alpha}{1-\beta} \), then this latter expression is \( 2^{\rho(1-\beta)} \) times \( 2^{-L[(1-\alpha) - \rho(1-\beta)]} 2^N(\alpha - 1)(N+1)^\frac{1}{\alpha} \), which is uniformly bounded, and hence \( \mathcal{A}_{p,q}^{(\alpha, \beta)}((0,0), \omega) \) is finite. On the other hand, the function \( f(x,y) \equiv 1 \) satisfies \( f \in L^p(\sigma) \), while the strong dyadic fractional maximal function satisfies
\[
\mathcal{M}_{\alpha, \beta}^d f \sigma(2^N, 2^{-\rho N}) \geq \left| (0,2^N) \right|^{\alpha - 1} \left| (0, 2^{-\rho N}) \right|^{\beta - 1} \left( \int_{[0,2^N] \times [0, 2^{-\rho N}]} f \, d\sigma \right) = 2^{N(\alpha - 1 - \rho(\beta - 1))} \geq 1,
\]
for all \( N \geq 1 \) provided \( p \geq \frac{1-\alpha}{1-\beta} \), so that the weak type operator norm of \( \mathcal{M}_{\alpha, \beta}^d f \) is infinite if \( \rho \geq \frac{1-\alpha}{1-\beta} \):
\[
\left\| \mathcal{M}_{\alpha, \beta}^d \right\|_{L^q(\omega_\rho)} = \sup_{\lambda > 0} \lambda \left\{ \left. f \in \mathbb{R}^n : \mathcal{M}_{\alpha, \beta}^d f \sigma > \lambda \right\} \right\|_{L^p(\sigma)} = \frac{1}{2} \left( \sum_{N=1}^{\infty} \left| (2^N, 2^{-\rho N}) \right|_{\omega_\rho} \right)^\frac{1}{\alpha} \approx \left( \sum_{N=1}^{\infty} \frac{1}{N} \right)^\frac{1}{\alpha} = \infty.
\]

**Remark 3.** It is easy to verify that when \( \sigma = \delta_{(0,0)} \), the operator norm \( \mathcal{N}_{p,q}^{(\alpha, \beta)}(\delta_{(0,0)}, \omega) \) is in fact equivalent to the two-tailed characteristic \( \widehat{A}_{\alpha, \beta}^{(\alpha, \beta)}(m_n) \) \( \delta_{(0,0)}, \omega \) for all measures \( \omega \). Indeed, as shown in the Appendix below, \( \mathcal{N}_{p,q}^{(\alpha, \beta)}(\delta_{(0,0)}, \omega) \) and \( \widehat{A}_{\alpha, \beta}^{(\alpha, \beta)}(m_n) \) \( \delta_{(0,0)}, \omega \) are each equivalent to
\[
\left\{ \left( \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} |x|^{\alpha - m} q |y|^{\beta - n} q \, d\omega(x,y) \right) \frac{1}{\sqrt{q}} \right\}.
\]

Thus the previous example simply produces a weight pair \((\sigma, \omega)\) for which \( \mathcal{N}_{p,q}^{(\alpha, \beta)}(\sigma, \omega) < \infty \) and \( \widehat{A}_{\alpha, \beta}^{(\alpha, \beta)}(m_n) (\sigma, \omega) = \infty \).

**4.3. Two weight theorems - power weights.** We will see below that in the special case that both weights are product weights, i.e. \( w(x,y) = w_1(x) w_2(y) \) and \( v(x,y) = v_1(x) v_2(y) \), then the one parameter theory carries over fairly easily to the multiparameter setting. Despite the negative nature of the previous theorem, the 1958 result of Stein and Weiss on power weights does carry over to the multiparameter setting using the sandwiching technique of Lemma 4 - where here the power weights are not product power weights (the theory for product power weights reduces trivially to that of the 1-parameter setting).

At this point it is instructive to observe that the kernel of the 1-parameter fractional integral \( I_{\alpha, \beta}^{m+n} \) is trivially dominated by the kernel of the 2-parameter fractional integral \( I_{\alpha, \beta}^{m,n} \), and hence the corresponding 1-parameter conditions in Theorem 5 are necessary for boundedness of \( I_{\alpha, \beta}^{m,n} \) - namely \( p \leq q \) and
\[
0 < \alpha + \beta < m + n \text{ and } \gamma q < m + n \text{ and } \delta p' < m + n,
\]
where the displayed conditions are equivalent to finiteness of the 1-parameter Muckenhoupt characteristic \( A_{p,q}^{\alpha, \beta, m+n} (v_3, w_\gamma) \). The boundedness of \( I_{\alpha, \beta}^{m,n} \) is instead given by finiteness of the rectangle Muckenhoupt characteristic \( A_{p, q}^{(\alpha, \beta)}(m_n) (v_3, w_\gamma) \). Here now is our extension of the classical Stein-Weiss theorem to the product setting.
Theorem 7. Let \( w_\gamma(x, y) = |(x, y)|^{\gamma} = (|x|^2 + |y|^2)^{\frac{\gamma}{2}} \) and \( v_\delta(x, y) = |(x, y)|^{\delta} = (|x|^2 + |y|^2)^{\frac{\delta}{2}} \) be a pair of nonnegative power weights on \( \mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n \) with \(-\infty < \gamma, \delta < \infty\). Let \( 1 < p, q < \infty \) and \(-\infty < \alpha, \beta < \infty\). Then the Muckenhoupt characteristic is finite, i.e. \( (4.11) \) holds, i.e.

\[
\left\{ \int_{\mathbb{R}^{m+n}} I_{\alpha,\beta}^{\gamma,q} \frac{x}{|x|^\gamma} \ dx \right\}^{\frac{1}{q}} \leq N_{(m,n)}^{(p,q)} \left\{ \int_{\mathbb{R}^{m+n}} f(x,y)^p |(x,y)|^{\delta p} \ dx \right\}^{\frac{1}{p}},
\]

for all \( f \geq 0 \)

(2) the indices \( p, q \) satisfy

\[
(4.11) \quad p \leq q,
\]

and the Muckenhoupt characteristic \( A_{(p,q)}^{(m,n)}(v_\delta, w_\gamma) \) is finite, i.e.

\[
(4.12) \quad \sup_{I \subset \mathbb{R}^{m+n}, J \subset \mathbb{R}^{n}} |I|^{\frac{1}{p} - 1} |J|^{\frac{1}{q} - 1} \left( \int_{I \times J} |(x,y)|^{\gamma} \ dx \right)^{\frac{1}{q}} \left( \int_{I \times J} |(u,t)|^{\delta p'} \ du \right)^{\frac{1}{p'}} < \infty,
\]

(3) the indices satisfy \((4.11)\) and

\[
(4.13) \quad \frac{1}{p} - \frac{1}{q} + \frac{\gamma + \delta}{m + n} = \frac{\alpha + \beta}{m + n},
\]

and

\[
(4.14) \quad \gamma + \delta \geq 0,
\]

and

\[
(4.15) \quad \beta - \frac{n}{p} < \delta \text{ and } \alpha - \frac{m}{p} < \delta \text{ when } \gamma \geq 0 \geq \delta,
\]

\[
(4.15) \quad \beta - \frac{n}{q'} < \gamma \text{ and } \alpha - \frac{m}{q'} < \gamma \text{ when } \delta \geq 0 \geq \gamma.
\]

In the absence of \((4.11)\), a characterization in terms of indices of the finiteness of the Muckenhoupt characteristic \( A_{(p,q)}^{(m,n)}(v_\delta, w_\gamma) \) is given by the following theorem, not used in this paper. For any real number \( t \) let \( t_\ast = \max \{t, 0\} \) and \( t_\ast = \max \{-t, 0\} \) be the positive and negative parts of \( t \). Note that \( t = t_\ast + t_\ast \). For the statement of the next theorem, we will use the notation \( 0_\ast \leq A \) to mean the inequality \( 0 < A \).

Theorem 8. Suppose that

\[
1 < p, q < \infty \text{ and } -\infty < \alpha, \beta < \infty,
\]

and \( w_\gamma(x, y) = |(x, y)|^{\gamma} \) and \( v_\delta(x, y) = |(x, y)|^{\delta} \) for \( (x, y) \in \mathbb{R}^m \times \mathbb{R}^n \) with \(-\infty < \gamma, \delta < \infty\). Let

\[
\Gamma \equiv \frac{1}{p} - \frac{1}{q} = \frac{1}{q'} - \frac{1}{p'},
\]

and set

\[
\Delta_{p,q}^{\gamma,\delta}(m) \equiv \left( \gamma - \frac{m}{q} \right)_+ + \left( \delta - \frac{m}{p} \right)_+,
\]

\[
\Delta_{p,q}^{\gamma,\delta}(n) \equiv \left( \gamma - \frac{n}{q} \right)_+ + \left( \delta - \frac{n}{p'} \right)_+.
\]

Then the Muckenhoupt characteristic is finite, i.e.

\[
A_{(p,q)}^{(m,n)}(v_\delta, w_\gamma) < \infty,
\]

if and only if the power weights \( w_\gamma^q \) and \( v_\delta^{-p'} \) are locally integrable,

\[
(4.16) \quad \gamma q < m + n \text{ and } \delta p' < m + n,
\]
and the following power weight equality and constraint inequalities for \( \alpha \) and \( \beta \) hold:

\[
\Gamma = \frac{\alpha + \beta - \gamma - \delta}{m + n},
\]

\[
\Gamma + \frac{\Delta_{p,q}^\delta (n)}{m} \leq \frac{\alpha}{m} \leq \Gamma + \frac{\Delta_{p,q}^\delta (m)}{m},
\]

\[
\Gamma + \frac{\Delta_{p,q}^\delta (m)}{n} \leq \frac{\beta}{n} \leq \Gamma + \frac{\Delta_{p,q}^\delta (n)}{n}.
\]

For the proof of Theorem 8 see the Appendix.

4.4. Two weight theorems - product weights. When both \( v(u,t) = v_1(u) v_2(t) \) and \( w(x,y) = w_1(x) w_2(y) \) are product weights, the one-parameter theory carries over fairly easily, and also for product measures \( \sigma = \sigma_1 \times \sigma_2 \) and \( \omega = \omega_1 \times \omega_2 \). Recall that

\[
\widehat{K}_{p,q}^{(\alpha,\beta),(m,n)} (\sigma, \omega) = \sup_{I \subset \mathbb{R}^m, \, J \subset \mathbb{R}^n} |I|^{\frac{\alpha}{m} - 1} |J|^{\frac{\beta}{n} - 1} \left( \int_{\mathbb{R}^m \times \mathbb{R}^n} s_{IJ}^q d\omega \right)^{\frac{1}{q}} \cdot \left( \int_{\mathbb{R}^m \times \mathbb{R}^n} \hat{s}^q_{IJ} d\sigma \right)^{\frac{1}{q}},
\]

and so for product measures \( \sigma = \sigma_1 \times \sigma_2 \) and \( \omega = \omega_1 \times \omega_2 \), we have

\[
\widehat{K}_{p,q}^{(\alpha,\beta),(m,n)} (\sigma, \omega) = \sup_{I \subset \mathbb{R}^m, \, J_1, J_2 \subset \mathbb{R}^n} |I|^{\frac{\alpha}{m} - 1} \left( \int_{J} s_{J_1}^q d\omega_1 \right)^{\frac{1}{q}} \cdot \left( \int_{J_2} \hat{s}_{J_1}^q d\sigma_1 \right)^{\frac{1}{q}} \cdot \sup_{J_1 \subset \mathbb{R}^n} |I|^{\frac{\beta}{n} - 1} \left( \int_{J} s_{J_2}^q d\omega_2 \right)^{\frac{1}{q}} \cdot \left( \int_{J_2} \hat{s}_{J_2}^q d\sigma_2 \right)^{\frac{1}{q}}.
\]

Theorem 9. Suppose that \( 1 < p, q < \infty \) and \( 0 < \frac{\alpha}{m}, \frac{\beta}{n} < 1 \). If both \( \sigma = \sigma_1 \times \sigma_2 \) and \( \omega = \omega_1 \times \omega_2 \) are product measures on \( \mathbb{R}^m \times \mathbb{R}^n \), then the norm inequality (3.3) is characterized by the two-tailed Muckenhoupt condition (\ref{eq:3.3}), i.e.

\[
\| I_{\alpha,\beta}^{v,w} (\sigma, \omega) \| \approx \widehat{K}_{p,q}^{(\alpha,\beta),(m,n)} (\sigma, \omega).
\]

As we will see later, the proof of this theorem follows immediately from the 1-parameter version, Theorem 2 together with the measure version of the iteration Theorem 12.

4.5. Two weight \( T1 \) or testing conditions. Recall that a weight \( u \) satisfies the product doubling condition if \( \| 2R \|_u \leq C \| R \|_u \) for all rectangles \( R \), and that this condition implies the weaker product reverse doubling condition. The theorem here shows that under certain side conditions on the weights \( \sigma \) and \( \omega \), the characteristic \( A_{p,q}^{(\alpha,\beta),(m,n)} (\sigma, \omega) \) controls the testing conditions for \( I_{\alpha,\beta}^{m,n} \).

Theorem 10. Suppose that \( 1 < p < q < \infty \) and \( 0 < \frac{\alpha}{m}, \frac{\beta}{n} < 1 \) satisfy

\[
\frac{1}{p} - \frac{1}{q} < \min \left\{ \frac{\alpha}{m}, \frac{\beta}{n} \right\},
\]

and suppose that the locally finite positive Borel measures \( \sigma \) and \( \omega \) satisfy the product doubling condition and have product reverse doubling exponent \( (\varepsilon, \varepsilon') \) that satisfies

\[
1 - \frac{\alpha}{m} < \varepsilon < \frac{1 - \alpha}{q + \frac{2}{p}} \quad \text{and} \quad 1 - \frac{\beta}{n} < \varepsilon' < \frac{1 - \beta}{q + \frac{2}{p}}.
\]

Then if the two weight characteristic \( A_{p,q}^{(\alpha,\beta),(m,n)} (\sigma, \omega) \) is finite, the following testing (or \( T1 \)) conditions hold: for all rectangles \( R \subset \mathbb{R}^m \times \mathbb{R}^n \),

\[
\left\{ \int_{\mathbb{R}^m \times \mathbb{R}^n} I_{\alpha,\beta}^{m,n} (1_R \sigma) (x,y)^q \, d\omega (x,y) \right\}^{\frac{1}{q}} \leq C_{p,q} A_{p,q}^{(\alpha,\beta),(m,n)} (\sigma, \omega) \| R \|_\sigma^{\frac{1}{q}},
\]

\[
\left\{ \int_{\mathbb{R}^m \times \mathbb{R}^n} I_{\alpha,\beta}^{m,n} (1_R \omega) (x,y)^{q'} \, d\sigma (x,y) \right\}^{\frac{1}{q'}} \leq C_{p,q} A_{p,q}^{(\alpha,\beta),(m,n)} (\sigma, \omega) \| R \|_\omega^{\frac{1}{q'}}.
\]
Remark 4. The testing condition in the first line of (4.19) only requires the reverse doubling assumption on $\sigma$, while the testing condition in the second line only requires reverse doubling of $\omega$.

The proofs of our positive results in the one weight case involve the standard techniques of iteration, Lebesgue's differentiation theorem and Minkowski's inequality, and the proof of the product version of the Stein-Weiss two power weight extension involves the sandwiching technique as well, and finally the derivation of the testing conditions from the characteristic and reverse doubling assumptions on the weights requires a quasiorthogonality argument. We begin with the simpler one weight norm inequality, and the special case of the two weight inequality when the weights are product weights. Some of this material generalizes naturally from product fractional integrals to \textit{iterated operators} to which we now turn.

5. Iterated Operators

The product fractional integral $I_{\alpha, \beta}^{m,n} f = \Omega_{\alpha, \beta}^{m,n} * f$, where

$$\Omega_{\alpha, \beta}^{m,n} (x, y) = |x|^\alpha |y|^\beta - n, \quad (x, y) \in \mathbb{R}^m \times \mathbb{R}^n,$$

is an example of an \textit{iterated operator}. In order to precisely define what we mean by an iterated operator, we denote the collection of nonnegative measurable functions on $\mathbb{R}^n$ by

$$\mathcal{N}(\mathbb{R}^n) \equiv \{ g : \mathbb{R}^n \to [0, \infty] : g \text{ is Lebesgue measurable} \},$$

and we refer to a mapping $T : \mathcal{N}(\mathbb{R}^n) \to \mathcal{N}(\mathbb{R}^n)$ from $\mathcal{N}(\mathbb{R}^n)$ to itself as an \textit{operator} on $\mathcal{N}(\mathbb{R}^n)$, without any assumption of additional properties. If $T_1$ is an operator on $\mathcal{N}(\mathbb{R}^m)$, we define its \textit{product extension} to an operator $T_1 \otimes \delta_0$ on $\mathcal{N}(\mathbb{R}^m \times \mathbb{R}^n)$ by

$$(T_1 \otimes \delta_0) f (x, y) = T_1 f^y (x), \quad f \in \mathcal{N}(\mathbb{R}^m \times \mathbb{R}^n),$$

and similarly, if $T_2$ is an operator on $\mathcal{N}(\mathbb{R}^n)$ we define its \textit{product extension} to an operator $\delta_0 \otimes T_2$ on $\mathcal{N}(\mathbb{R}^m \times \mathbb{R}^n)$ by

$$(\delta_0 \otimes T_2) f (x, y) = T_2 f_x (y), \quad f \in \mathcal{N}(\mathbb{R}^m \times \mathbb{R}^n).$$

Definition 1. If $T_1$ is an operator on $\mathcal{N}(\mathbb{R}^m)$ and $T_2$ is an operator on $\mathcal{N}(\mathbb{R}^n)$, then the composition operator

$$T = (\delta_0 \otimes T_2) \circ (T_1 \otimes \delta_0)$$

on $\mathcal{N}(\mathbb{R}^m \times \mathbb{R}^n)$ is called an \textit{iterated operator}.

To see that $I_{\alpha, \beta}^{m,n}$ is an iterated operator, define $\Omega_{\alpha}^{m} (x) = |x|^\alpha - m$ and $\Omega_{\beta}^{n} (y) = |y|^\beta - n$ and set $I_{\gamma}^k g = \Omega_{\gamma}^k * g$. Then extend the operator $I_{\alpha}^{m}$ from $\mathbb{R}^m$ to the product space $\mathbb{R}^m \times \mathbb{R}^n$ by defining

$$(I_{\alpha}^{m} \otimes \delta_0) f (x, y) = [\Omega_{\alpha}^{m} \otimes \delta_0] * f (x, y)$$

$$= \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} \Omega_{\alpha}^{m} (x - u) \delta_0 (y - v) f (u, v) d u d v$$

$$= \int_{\mathbb{R}^m} \Omega_{\alpha}^{m} (x - u) f (u, y) d u$$

$$= I_{\alpha}^{m} f^y (x),$$

where $f^y (u) \equiv f (u, y)$, and similarly define

$$(\delta_0 \otimes I_{\beta}^{n}) f (x, y) = I_{\beta}^{n} f_x (y),$$
where \( f_x(v) \equiv f(x, v) \). Then from \( \Omega^{m,n}_{\alpha,\beta}(x, y) = \Omega^m_\alpha(x) \Omega^n_\beta(y) \) we have

\[
I^{m,n}_{\alpha,\beta} f(x, y) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Omega^{m,n}_{\alpha,\beta}(x - u, y - v) f(u, v) \, du \, dv
\]

\[
= \int_{\mathbb{R}^n} \Omega^n_\beta(y - v) \left\{ \int_{\mathbb{R}^m} \Omega^m_{\alpha,\beta}(x - u) f^v(u) \, du \right\} \, dv
\]

\[
= \int_{\mathbb{R}^n} \Omega^n_\beta(y - v) \{ I^m_{\alpha} f^v(x) \} \, dv
\]

\[
= \Omega^n_\beta \{ (I^m_{\alpha} \otimes \delta_0) f(x, \cdot) \} \, dv
\]

\[
= I^n_{\beta} \{ (I^m_{\alpha} \otimes \delta_0) f(x, \cdot) \} \, dv
\]

\[
= (\delta_0 \otimes I^n_{\beta}) \circ (I^m_{\alpha} \otimes \delta_0) f(x, y)
\]

Thus

\[
I^{m,n}_{\alpha,\beta} = (\delta_0 \otimes I^n_{\beta}) \circ (I^m_{\alpha} \otimes \delta_0)
\]

and similarly we have

\[
I^{m,n}_{\alpha,\beta} = (I^m_{\alpha} \otimes \delta_0) \circ (\delta_0 \otimes I^n_{\beta})
\]

which expresses \( I^{m,n}_{\alpha,\beta} \) as an iterated operator, namely the composition of two commuting operators \( I^m_{\alpha} \otimes \delta_0 \) and \( \delta_0 \otimes I^n_{\beta} \).

More generally, we can consider the operator

\[
T f(x, y) = (K * f)(x, y) = \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} K(x - u, y - v) f(u, v) \, du \, dv
\]

where \( K \) is a \textit{product kernel} on \( \mathbb{R}^m \times \mathbb{R}^n \),

\[
K(x, y) = K_1(x) K_2(y), \quad (x, y) \in \mathbb{R}^m \times \mathbb{R}^n,
\]

and obtain the factorizations

\[
T = (K_1 \otimes \delta_0) \circ (\delta_0 \otimes K_2) = (\delta_0 \otimes K_2) \circ (K_1 \otimes \delta_0)
\]

of \( T \) into iterated operators where

\[
(K_1 \otimes \delta_0) \ast g(x, y) = \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} K_1(x - u) \delta_0(y - v) g(u, v) \, du \, dv
\]

\[
= \int_{\mathbb{R}^n} K_1(x - u) g(u, y) \, du
\]

\[
= K_1 \ast g^u(x)
\]

and

\[
(\delta_0 \otimes K_2) \ast h(x, y) = K_2 \ast g_x(y)
\]

As a final example, let \( T_1 = M_{\mathbb{R}^m} \) be the Hardy-Littlewood maximal operator on \( \mathbb{R}^m \) and let \( T_2 = M_{\mathbb{R}^n} \) be the Hardy-Littlewood maximal operator on \( \mathbb{R}^n \). Then the iterated operator

\[
(5.1) \quad T = (\delta_0 \otimes T_2) \circ (T_1 \otimes \delta_0) \circ (\delta_0 \otimes M_{\mathbb{R}^m}) \circ (M_{\mathbb{R}^n} \otimes \delta_0)
\]

is usually denoted \( M_{\mathbb{R}^{m,n}}(M_{\mathbb{R}^{m,n}}) \), and the other iterated operator by \( M_{\mathbb{R}^{m,n}}(M_{\mathbb{R}^{m,n}}) \). They both dominate the strong maximal operator \( \mathcal{M} \) given in (4.6).
5.1. One weight inequalities for iterated operators. Define the iterated Lebesgue spaces $L_{m,n}^{p,q}$ on $\mathbb{R}^m \times \mathbb{R}^n$ by

$$\|F\|_{L_{m,n}^{p,q}} = \left\| \|F\|_{L^p(\mathbb{R}^m)} \right\|_{L^q(\mathbb{R}^n)} = \left\{ \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^m} F(x,y)^p \, dx \right\}^{\frac{q}{p}} \, dy \right\}^{\frac{1}{q}}.$$ 

In the proof of the next theorem we will use Minkowski’s inequality for nonnegative functions, which written out in full is

$$\left\{ \int_{\mathbb{R}^m} \left\{ \int_{\mathbb{R}^n} F(x,y)^p \, dx \right\}^{\frac{q}{p}} \, dy \right\}^{\frac{1}{q}} \leq \left\{ \int_{\mathbb{R}^m} \left\{ \int_{\mathbb{R}^n} F(x,y)^q \, dx \right\}^{\frac{1}{q}} \, dy \right\}^{\frac{1}{q}}.$$

**Theorem 11.** Let $1 \leq p \leq q \leq \infty$ and $T_1 : N(\mathbb{R}^m) \to N(\mathbb{R}^m)$ and $T_2 : N(\mathbb{R}^n) \to N(\mathbb{R}^n)$. Suppose that $v(x,y) \geq 0$ on $\mathbb{R}^m \times \mathbb{R}^n$ satisfies

$$(5.2) \quad \|T_1 g\|_{L^q((v y) v)} \leq C_1 \|g\|_{L^p((v y) v)} , \quad \text{for all } g \geq 0,$$

uniformly for $y \in \mathbb{R}^n$, and

$$(5.3) \quad \|T_2 h\|_{L^q((v y) v)} \leq C_2 \|h\|_{L^p((v y) v)} , \quad \text{for all } h \geq 0,$$

uniformly for $x \in \mathbb{R}^m$. Then the iterated operator $T = (\delta_0 \otimes T_2) \circ (T_1 \otimes \delta_0)$ satisfies

$$\|T f\|_{L^q(v y)} \leq C_1 C_2 \|f\|_{L^p(v y)} , \quad \text{for all } f \geq 0.$$ 

**Proof.** We have

$$\|T f\|_{L^q(v y)} = \left\{ \int_{\mathbb{R}^m} \left\{ \int_{\mathbb{R}^n} (\delta_0 \otimes T_2) \circ (T_1 \otimes \delta_0) f(x,y)^q v(x,y)^q \, dy \, dx \right\} \right\}^{\frac{1}{q}}$$

$$= \left\{ \int_{\mathbb{R}^m} \left\{ \int_{\mathbb{R}^n} \left| T_2 \left[ (T_1 \otimes \delta_0) f \right]\right| v_x (y)^q \, dy \right\} \right\}^{\frac{1}{q}}$$

$$= \left\{ \int_{\mathbb{R}^m} \left\| T_2 \left[ (T_1 \otimes \delta_0) f \right]\right\|_{L^q((v y) v)} \, dx \right\}^{\frac{1}{q}}$$

$$\leq C_2 \left\{ \int_{\mathbb{R}^m} \left\| (T_1 \otimes \delta_0) f \right\|_{L^p((v y) v)} \, dx \right\}^{\frac{1}{q}}$$

$$= C_2 \left\{ \int_{\mathbb{R}^m} \left\{ \int_{\mathbb{R}^n} (T_1 \otimes \delta_0) f(x,y)^p v(x,y)^p \, dy \right\} \right\}^{\frac{1}{q}} \, dx \right\}^{\frac{1}{q}},$$

where we have used $h = [(T_1 \otimes \delta_0) f]^q \geq 0$ in (5.3). Then by Minkowski’s inequality applied to the nonnegative function $F = (T_1 \otimes \delta_0) f(x,y) v(x,y)$, this is dominated by

$$C_2 \left\{ \int_{\mathbb{R}^m} \left\{ \int_{\mathbb{R}^n} (T_1 \otimes \delta_0) f(x,y)^q v(x,y)^q \, dx \right\} \right\}^{\frac{1}{q}} \, dy \right\}^{\frac{1}{q}}$$

$$= C_2 \left\{ \int_{\mathbb{R}^m} \left\{ \int_{\mathbb{R}^n} T_1 f^q (x)^q v^q (x)^q \, dx \right\} \right\}^{\frac{1}{q}} \, dy \right\}^{\frac{1}{q}}$$

$$= C_2 \left\{ \int_{\mathbb{R}^m} \|f\|^p_{L^p((v y) v)} \, dy \right\}^{\frac{1}{p}}$$

$$\leq C_2 C_1 \left\{ \int_{\mathbb{R}^n} \|f\|^p_{L^p((v y) v)} \, dy \right\}^{\frac{1}{p}} = C_2 C_1 \|f\|_{L^p(v y)} ,$$

where we have used $g = f^q \geq 0$ in (5.2). \[\square\]

The following porisms, or ‘corollaries of the proof’, of Theorem 11 will find application in proving Theorem 9 below.
Porism1: If we replace (5.3) with the more general two weight inequality

\[ \|T_2 h\|_{L^q(w_x)} \leq C_2 \|h\|_{L^p(v_y)}, \quad \text{for all } h \geq 0, \]

for some weight \( w(x, y) \geq 0 \) on \( \mathbb{R}^m \times \mathbb{R}^n \), then the iterated operator \( T = (\delta_0 \otimes T_2) \circ (T_1 \otimes \delta_0) \) satisfies the two weight inequality

\[ \|T f\|_{L^q(w_y)} \leq C_1 C_2 \|f\|_{L^p(w_x)}, \quad \text{for all } f \geq 0. \]

To prove this Porism, we modify the first display in the proof of Theorem [11] to this,

\[
\begin{align*}
\|T f\|_{L^q(w_y)} & = \left\{ \int_{\mathbb{R}^m} \left( \int_{\mathbb{R}^n} |(T_1 \otimes \delta_0) f|_x(y) \, w_x(y) \, dy \right)^q \, dx \right\}^{1/q} \\
& = \left\{ \int_{\mathbb{R}^m} \left( \int_{\mathbb{R}^n} |T_2 [(T_1 \otimes \delta_0) f]_x(y) \, w_x(y) \, dy \right)^q \, dx \right\}^{1/q} \\
& \leq C_2 \left\{ \int_{\mathbb{R}^m} \left( \int_{\mathbb{R}^n} (T_1 \otimes \delta_0) f(x,y)^p \, v(x,y) \, dy \right)^q \, dx \right\}^{1/q},
\end{align*}
\]

and then the remainder of the proof of Theorem [11] applies verbatim.

There is also the following symmetrical porism whose proof is left to the reader.

Porism2: If \( \tilde{T} = (T_1 \otimes \delta_0) \circ (\delta_0 \otimes T_2) \) is the composition of the two iterated operators in the reverse order, and if

\[ \|T_1 g\|_{L^q(w_y)} \leq C_1 \|g\|_{L^p(v_y)}, \quad \text{for all } g \geq 0, \]

uniformly for \( y \in \mathbb{R}^n \), and

\[ \|T_2 h\|_{L^q(v_x)} \leq C_2 \|h\|_{L^p(w_x)}, \quad \text{for all } h \geq 0, \]

uniformly for \( x \in \mathbb{R}^m \), then

\[ \|T f\|_{L^q(w_y)} \leq C_1 C_2 \|f\|_{L^p(v_x)}, \quad \text{for all } f \geq 0. \]

In the special case that

\[ \frac{1}{p} - \frac{1}{q} = \frac{\alpha}{m} = \frac{\beta}{n}, \]

we can exploit the equivalence of the product \( A_{p,q} \) condition (4.3) with the iterated \( A_{p,q} \) condition (4.3) to obtain a characterization of the one weight \( L^p \rightarrow L^q \) inequality for \( I_{\alpha,\beta}^{m,n} \). In fact, condition (5.3) is actually necessary for the product \( A_{p,q} \) condition (4.3) to be finite.

Definition 2. We set

\[ A_{p,q}^{(\alpha,\beta),(m,n)} (w) \equiv \sup_{I \subset \mathbb{R}^m, J \subset \mathbb{R}^n} \left| I \right|^{\frac{\alpha}{m} + \frac{\beta}{n} - \frac{1}{p}} \left| J \right|^{\frac{\alpha}{m} + \frac{\beta}{n} - \frac{1}{q}} \left( \frac{|I \times J|}{|I|} \right)^{\frac{1}{q}} \left( \frac{|I \times J|}{|I|} \right)^{\frac{1}{p}}. \]

Claim 1. If \( N_{p,q}^{(\alpha,\beta),(m,n)} (w) < \infty \) for some weight \( w \), then (5.3) holds.
Proof. Following the proof of Theorem 13 in the Appendix, we apply Hölder’s inequality with dual exponents $\frac{p'}{p'} + 1$ to obtain

$$1 = \left\{ \frac{1}{|I \times J|} \int_{I \times J} w^{\frac{p'}{p'} + 1} w^{-\frac{p'}{p'} + 1} \right\}^{\frac{p'}{p'} + 1}$$

Then we conclude that

$$|I|^\frac{\alpha}{m} + 1 - \frac{1}{p} |J|^\frac{\alpha}{m} - \frac{1}{p}$$

for all rectangles $I \times J$, which implies $\frac{\alpha}{m} = \frac{1}{p} - \frac{1}{q} = \frac{\beta}{p}$ as required. \hfill \Box

5.2. Proof of Theorem 5

Proof of Theorem 5. By Claim 1 the balanced and diagonal condition (5.5) holds. Thus we have the necessity of (4.3) follows from Lemma 3 above:

Then we conclude that

$$|I|^\frac{\alpha}{m} + 1 - \frac{1}{p} |J|^\frac{\alpha}{m} - \frac{1}{p}$$

By letting the cubes $J$ and $I$ shrink separately to points $y \in \mathbb{R}^n$ and $x \in \mathbb{R}^m$, we obtain (4.4). Then from the one weight theorem of Muckenhoupt and Wheeden above, Theorem 4 we conclude that both (5.2) and (5.3) hold with $T_1g = \Omega_m^\alpha * g$ and $T_2h = \Omega_p^\beta * h$. Then the conclusion of Theorem 11 proves the norm inequality (4.1).

Finally, the estimate (4.5) follows using the estimate of Lacey, Moen, Pérez and Torres LaMoPeTo, which gives

$$C_1 \lesssim A_{p,q}(w^y)^{q \max \left\{ 1, \frac{\alpha}{m} \right\}} \left( 1 - \frac{\alpha}{m} \right)$$

uniformly in $x$ and $y$, upon noting that the characteristic $A_{p,q}$ used in [LaMoPeTo] is the $q^{th}$ power of that defined there. Then using that

$$q \max \left\{ 1, \frac{p'}{q} \right\} \left( 1 - \frac{\alpha}{m} \right) = \max \left\{ q, p' \right\} \left( \frac{1}{q} + \frac{1}{p'} \right) = 1 + \max \left\{ \frac{p'}{q}, \frac{q}{p'} \right\},$$

we obtain (4.3). Sharpness of the exponent follows upon taking product power weights and product power functions and then arguing as in the previous work of Buckley [?, Buc] and Lacey, Moen, Pérez and Torres [LaMoPeTo]. \hfill \Box

5.3. Two weight inequalities for product weights. If in addition we consider product weights, then we can prove two weight versions of the theorem and corollary above using essentially the same proof strategy, namely iteration of one parameter operators.
**Theorem 12.** Let $1 \leq p \leq q \leq \infty$ and $T_1 : N^m (\mathbb{R}^m) \to N^m (\mathbb{R}^m)$ and $T_2 : N^n (\mathbb{R}^n) \to N^n (\mathbb{R}^n)$. Suppose that $w (x, y) = w_1 (x) w_2 (y)$ and $v (x, y) = v_1 (x) v_2 (y)$ are both product weights, and that the weight pairs $(w_1, v_1)$ and $(w_2, v_2)$ on $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively satisfy

$$\|T_1 g\|_{L^p(w_1)} \leq C_1 \|g\|_{L^p(v_1^*)} , \text{ for all } g \geq 0,$$

and

$$\|T_2 h\|_{L^p(w_2^*)} \leq C_2 \|h\|_{L^p(v_2^*)} , \text{ for all } h \geq 0.$$

Then the iterated operator $T = (\delta_0 \otimes T_2) \circ (T_1 \otimes \delta_0)$ satisfies

$$\|T f\|_{L^q(w)} \leq C_1 C_2 \|f\|_{L^p(v \otimes v)} , \text{ for all } f \geq 0.$$

**Proof.** We have

$$\|T f\|_{L^q(w)} = \left\{ \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} (\delta_0 \otimes T_2) \circ (T_1 \otimes \delta_0) f (x, y)^q w (x, y)^q \, dy \, dx \right\}^{\frac{1}{q}}$$

$$= \left\{ \int_{\mathbb{R}^m} \left[ \int_{\mathbb{R}^n} |T_2 [(T_1 \otimes \delta_0) f]_x (y)|^q w_1 (x)^q w_2 (y)^q \, dy \right] \, dx \right\}^{\frac{1}{q}}$$

$$= \left\{ \int_{\mathbb{R}^m} \|T_2 [(T_1 \otimes \delta_0) f]_x \|_{L^q(w_2^*)}^q w_1 (x)^q \, dx \right\}^{\frac{1}{q}}$$

$$\leq C_2 \left\{ \int_{\mathbb{R}^m} \|[(T_1 \otimes \delta_0) f]_x \|_{L^p(v_2^*)}^p w_1 (x)^q \, dx \right\}^{\frac{1}{p}}$$

$$= C_2 \left\{ \int_{\mathbb{R}^m} \left\{ \int_{\mathbb{R}^n} (T_1 \otimes \delta_0) f (x, y)^p v_2 (y)^p w_1 (x)^q \, dx \right\}^\frac{1}{p} dy \right\}^{\frac{1}{q}}$$

where we have used $h = [(T_1 \otimes \delta_0) f]_x \geq 0$ in (5.7). Then by Minkowski’s inequality applied to the nonnegative function $F = (T_1 \otimes \delta_0) f (x, y) v_2 (y) w_1 (x)$, this is dominated by

$$C_2 \left\{ \int_{\mathbb{R}^m} \left\{ \int_{\mathbb{R}^n} (T_1 \otimes \delta_0) f (x, y)^p v_2 (y)^p w_1 (x)^q \, dx \right\}^\frac{1}{p} dy \right\}^{\frac{1}{q}}$$

$$= C_2 \left\{ \int_{\mathbb{R}^m} \left\{ \int_{\mathbb{R}^n} T_1 f^v (x)^q v_2 (y)^q w_1 (x)^q \, dx \right\}^\frac{1}{q} v_2 (y)^p \, dy \right\}^{\frac{1}{p}}$$

$$= C_2 \left\{ \int_{\mathbb{R}^m} \|T_1 f^v \|_{L^q(v_1^*)}^p v_2 (y)^p \, dy \right\}^{\frac{1}{p}}$$

$$\leq C_2 C_1 \left\{ \int_{\mathbb{R}^m} \|f^v \|_{L^p(v_1^*)}^p v_2 (y)^p \, dy \right\}^{\frac{1}{p}} = C_2 C_1 \|f\|_{L^p(v \otimes v)} ,$$

where we have used $g = f^v \geq 0$ in (5.6).

**Corollary 1.** Suppose $1 < p < q < \infty$ and $\frac{1}{p} - \frac{1}{q} = \frac{\omega}{m} \neq \frac{\omega}{n}$. Let $w (x, y) = w_1 (x) w_2 (y)$ and $v (x, y) = v_1 (x) v_2 (y)$ be a pair of nonnegative product weights on $\mathbb{R}^m \times \mathbb{R}^n$. Then

$$\left\{ \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} f (x, y)^q w (x, y)^q \, dxdy \right\}^{\frac{1}{q}} \leq C_{p, q} (w, v) \left\{ \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} f (x, y)^p v (x, y)^p \, dxdy \right\}^{\frac{1}{p}}$$

for all $f \geq 0$ if and only if

$$\sup_{I \subset \mathbb{R}^m, J \subset \mathbb{R}^n} \left( \frac{1}{|I|} \right)^{\frac{1}{p}} \int_{\mathbb{R}^m \times \mathbb{R}^n} (\hat{S}_{I, J} w) (x, y)^q \, dxdy \right\}^{\frac{1}{q}}$$

$$\times \left( \frac{1}{|J|} \right)^{\frac{1}{p}} \int_{\mathbb{R}^m \times \mathbb{R}^n} (\hat{S}_{I, J} v^{-1}) (x, y)^p \, dxdy \right\}^{\frac{1}{p}} \equiv \tilde{A}_{p, q} (w, v) < \infty,$$
where \( \hat{s}_{I, J}(x, y) = \hat{s}_I(x) \hat{s}_J(y) = \left(1 + \frac{|x-y|}{|I|^\frac{1}{m}}\right)^{-m} \left(1 + \frac{|x-y|}{|J|^\frac{1}{n}}\right)^{-n}. \) Moreover,

\[ C_{p, q}(w, v) \approx \tilde{A}_{p, q}(w, v). \]

**Proof.** The necessity of (5.9) is again a standard exercise in adapting the one parameter argument in Sawyer and Wheeden to the setting of product weights.

Now we turn to the sufficiency of (5.9). Since our weights \( w \) and \( v \) are product weights, the double integrals on the left hand side of (5.9) each factor as a product of integrals over \( \mathbb{R}^m \) and \( \mathbb{R}^n \) separately, e.g.

\[
\left( \frac{1}{|I|} \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} \left(\hat{s}_{I, J}(w)(x, y)^q \cdot dy \right) dx \right)^\frac{1}{q} = \left( \frac{1}{|I|} \int_{\mathbb{R}^m} \left(\hat{s}_I w_1(x)^q \cdot dx \right)^{\frac{1}{q}} \left( \frac{1}{|J|} \int_{\mathbb{R}^n} \left(\hat{s}_J w_2(y)^q \cdot dy \right)^{\frac{1}{q}} \right)^{\frac{1}{q}}.
\]

As a consequence, the characteristic \( \tilde{A}_{p, q}(w, v) \) defined in (5.9) can be rewritten as

\[ \tilde{A}_{p, q}(w, v) = \hat{A}_{p, q}(w_1, v_1) \hat{A}_{p, q}(w_2, v_2). \]

From the two weight theorem of Sawyer and Wheeden above, Theorem 2, we conclude that (5.6) holds with \( T_1 g = \Omega_{1}^m \ast g \) and constant \( C_1 = C \hat{A}_{p, q}(w_1, v_1) \), and that (5.7) holds with \( T_2 h = \Omega_{2}^n \ast h \) and constant \( C_2 = C \hat{A}_{p, q}(w_2, v_2) \). This precise dependence on \( \hat{A}_{p, q}(w_2, v_2) \) is not explicitly stated in [2] but it is easily checked by tracking the constants in the proof given there. See also the detailed proof in the appendix below. Then the conclusion of the theorem proves the norm inequality (5.8), and also the equivalence (5.10), in view of (5.11).

**Remark 5.** If we restrict the function \( f \) in the norm inequality in the corollary above to be a product function \( f(x, y) = f_1(x) f_2(y) \), then \( \left(\Omega_{\alpha, \beta}^m \ast f \right)(x, y) \) is the product function \( \Omega_{\alpha}^m \ast f_1(x) \Omega_{\beta}^n \ast f_2(y) \), and there is a particularly trivial proof of the norm bound for such \( f \):

\[
\left\{ \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} \left(\Omega_{\alpha, \beta}^m \ast f \right)(x, y)^q \cdot w(x, y)^q \cdot dx \cdot dy \right\}^{\frac{1}{q}} = \left\{ \int_{\mathbb{R}^m} \Omega_{\alpha}^m \ast f_1(x)^q \cdot w_1(x)^q \cdot dx \right\}^{\frac{1}{q}} \left\{ \int_{\mathbb{R}^n} \Omega_{\beta}^n \ast f_2(y)^q \cdot w_2(y)^q \cdot dy \right\}^{\frac{1}{q}} \leq C_1 \left\{ \int_{\mathbb{R}^m} f_1(x)^p \cdot w_1(x)^p \cdot dx \right\}^{\frac{1}{p}} \cdot C_2 \left\{ \int_{\mathbb{R}^n} f_2(y)^p \cdot w_2(y)^p \cdot dy \right\}^{\frac{1}{p}},
\]

**Remark 6.** We have the following pointwise limit for \( w_1 \) above:

\[ \lim_{r \to 0} \left( \frac{1}{|I|} \int_{\mathbb{R}^m} \hat{s}_I w_1(x)^q \cdot dx \right)^{\frac{1}{q}} = C w_1(x_0), \quad \text{for a.e. } x_0 \in \mathbb{R}^m.
\]

Indeed, with \( I_0 = [-\frac{1}{2}, \frac{1}{2}] \), the function

\[ \hat{s}_{I_0}(x)^q = (1 + |x|)^{-m-\alpha q} = (1 + |x|)^{-m-\alpha p}, \quad x \in \mathbb{R}^m,
\]

is such that the family \( \left\{ \frac{1}{r^m} \hat{s}_{I_0} \left( \frac{x}{r} \right) \right\}_{r > 0} \) is an approximate identity on \( \mathbb{R}^m \), and thus (5.12) holds at every Lebesgue point of \( w_1^q \), since \( w_1^q \) obviously satisfies the growth condition,

\[ \int_{\mathbb{R}^m} (1 + |x|)^{-m-\frac{1}{p}} \cdot w_1(x)^q \cdot dx = \int_{\mathbb{R}^m} (1 + |x|)^{-q(m-\alpha)} \cdot w_1(x)^q \cdot dx \leq C < \infty,
\]

when \( v_1 \) is not the trivial weight identically infinity. Similar pointwise limits hold for the remaining three functions \( w_2, v_1 \) and \( v_2 \).
6. Proof of Theorem and Example

We begin with this easy lemma.

**Lemma 4.** Suppose that $1 < p, q < \infty$ and $0 < \frac{\alpha}{m}, \frac{\beta}{n} < 1$ satisfy the half-balanced condition

\[
\frac{1}{p} - \frac{1}{q} = \min \left\{ \frac{\alpha}{m}, \frac{\beta}{n} \right\}.
\]

If $u$ is a positive weight on $\mathbb{R}^m \times \mathbb{R}^n$, then

\[
A_{p,q}(u) \leq \min \left\{ \| u^q \|_{A_1 \times A_1}, \| u^{-p'} \|_{A_1 \times A_1} \right\}.
\]

**Proof of Lemma** Suppose that $u^{-p'} \in A_1 \times A_1$. Then

\[
\frac{1}{|I \times J|} \int_{I \times J} u^{-p'} \leq \| u^{-p'} \|_{A_1 \times A_1} \left( \inf_{I \times J} u^{-1} \right)
\]

for all rectangles $I \times J$, and so

\[
\left( \frac{1}{|I \times J|} \int_{I \times J} u(x,y)^q \, dxdy \right)^{\frac{1}{q}} \leq \left( \frac{1}{|I \times J|} \int_{I \times J} u(x,y)^{-p'} \, dxdy \right)^{\frac{1}{p'}}
\]

\[
\leq \left( \frac{1}{|I \times J|} \int_{I \times J} u(x,y)^q \, dxdy \right)^{\frac{1}{q}} \| u^{-p'} \|_{A_1 \times A_1} \left( \inf_{I \times J} u^{-1} \right) \leq \| u^{-p'} \|_{A_1 \times A_1}^{\frac{1}{p'}},
\]

which proves the second assertion for $u^{-p'}$. Similarly, if $u^q \in A_1 \times A_1$, then

\[
\left( \frac{1}{|I \times J|} \int_{I \times J} u(x,y)^q \, dxdy \right)^{\frac{1}{q}} \leq \left( \frac{1}{|I \times J|} \int_{I \times J} u(x,y)^{-p'} \, dxdy \right)^{\frac{1}{p'}}
\]

\[
\leq \| u^q \|_{A_1 \times A_1} \left( \inf_{I \times J} u \right) \left( \frac{1}{|I \times J|} \int_{I \times J} u(x,y)^{-p'} \, dxdy \right)^{\frac{1}{p'}} \leq \| u^q \|_{A_1 \times A_1}^{\frac{1}{p}},
\]

and this completes the proof of Lemma.

**Proof of Theorem** Assume now that $v^{-p'} \in A_1 \times A_1$ and $\frac{1}{p'} - \frac{1}{q} = \frac{\alpha}{m} < \frac{\beta}{n}$, the other cases being similar. Then we have

\[
A_{p,q}^{(\alpha,\beta),(m,n)}(v, w) \equiv \sup_{I \subseteq \mathbb{R}^m, J \subseteq \mathbb{R}^n} |J|^\frac{\beta}{n} \left( \frac{1}{|I|} \int_{I \times J} w^q \right)^{\frac{1}{q}} \left( \frac{1}{|I|} \int_{I \times J} v^{-p'} \right)^{\frac{1}{p'}}
\]

and if we let $I$ shrink to a point $x \in \mathbb{R}^m$, we obtain the 1-parameter conclusion that

\[
(w_x(y), v_x(y)) \in A_{p,q}^{\beta,n}, \quad \text{uniformly a.e. } x \in \mathbb{R}^m.
\]
Then Minkowski’s inequality with $p \leq q$ gives
\[
\left\{ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} I_{\alpha, \beta}^m f(x, y)^q w(x, y)^q \, dx \, dy \right\}^{\frac{1}{q}} \\
= \left\{ \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} f(x, y)^q \right] w(x, y)^q \, dx \right\}^{\frac{1}{q}} \\
\leq A_{p, q}^{\beta, n} \left\{ \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} f(x, y)^p \right] v(x, y)^p \, dx \right\}^{\frac{1}{p}} \\
\leq A_{p, q}^{\beta, n} (v, w) \left\{ \int_{\mathbb{R}^n} f(x, y)^p \, v(x, y)^p \, dx \right\}^{\frac{1}{p}},
\]
where in the final line we have applied the 1-parameter one weight inequality in Theorem 1 since $v \in A_{p, q}$ by Lemma 3. This completes the proof of Theorem 6.

6.2. Example 1. Now we verify the properties claimed in Example 1. First we note the following discretization of $\frac{1}{|\Omega|} \int_{\mathbb{R}^n} s_I^p \, d\sigma$:
\[
\sum_{k=0}^{\infty} 2^{k[(\alpha-m)p'+m]} \frac{1}{|2^k I|} \int_{2^k I} d\sigma \\
= \sum_{k=0}^{\infty} 2^{k[(\alpha-m)p'+m]} \frac{1}{|2^k I|} \int_{2^k I \setminus 2^{k-1} I} d\sigma = \sum_{\ell=0}^{\infty} \left( \sum_{k=\ell}^{\infty} 2^{k[(\alpha-m)p']} \right) \int_{2^\ell I \setminus 2^{\ell-1} I} d\sigma \\
\approx \sum_{\ell=0}^{\infty} 2^{\ell[(\alpha-m)p']} \int_{2^\ell I \setminus 2^{\ell-1} I} d\sigma (x) \approx \frac{1}{|I|} \int_{\mathbb{R}^n} \left( 1 + \frac{|x - c_I|}{|I|^{\frac{1}{m}}} \right)^{(\alpha-m)p'} d\sigma (x) = \frac{1}{|I|} \int_{\mathbb{R}^n} s_I^p \, d\sigma,
\]

since $\alpha < m$. Thus we have
\[
\mathcal{K}_{\sigma, \alpha, \beta, I}^{\alpha, m} \left( \frac{|\sigma|}{|I|} \right)^p = \left\{ |I| \left( \frac{1}{|I/|} \int_{I} d\omega \right)^{\frac{1}{p}} \left( \frac{1}{|I|} \int_{\mathbb{R}^n} s_I^p \, d\sigma \right)^{\frac{1}{p'}} \right\}^{\frac{1}{p'}} \\
\approx |I| \left( \frac{1}{|I|} \int_{I} d\omega \right)^{\frac{1}{p}} \sum_{k=0}^{\infty} 2^{k[(\alpha-m)p'+m]} \frac{1}{|2^k I|} \int_{2^k I} d\sigma (x) \\
= \sum_{k=0}^{\infty} 2^{-k[(\alpha-m)p']} |2^k I|^{\left( \frac{1}{|I|} \int_{I} d\omega \right)^{\frac{1}{p}}} \sum_{k=0}^{\infty} 2^{-k[(\alpha-m)p'+m]} \left( \frac{1}{|2^k I|} \int_{I} d\omega \right)^{\frac{1}{p'}} \left( \frac{1}{|2^k I|} \right)^{\frac{1}{p'}} \\
= \sum_{k=0}^{\infty} 2^{-k[(\alpha-m)p']} |2^k I|^{\left( \frac{1}{|I|} \int_{I} d\omega \right)^{\frac{1}{p}}} \sum_{k=0}^{\infty} 2^{-k[(\alpha-m)p'+m]} \left( \frac{1}{|2^k I|} \int_{I} d\omega \right)^{\frac{1}{p'}} \left( \frac{1}{|2^k I|} \right)^{\frac{1}{p'}} \\
= \sum_{k=0}^{\infty} \left\{ |2^k I|^{\left( \frac{1}{|I|} \int_{I} d\omega \right)^{\frac{1}{p}}} \left( \frac{1}{|2^k I|} \right)^{\frac{1}{p'}} \right\}^{\frac{1}{p'}}.
\]

Now recall that
\[
v(y) = |y|^{-\frac{m}{p-m}} \quad \text{and} \quad w(x) = (1 + |x|)^{-m},
\]
so that $d\omega (x) = w(x)^q \, dx = (1 + |x|)^{-mp'} \, dx$ and $d\sigma (y) = v(y)^{-p'} \, dy = |y|^{-\frac{mp'}{p}} \, dy$. Then for $Q = [-R, R]^m$ we have

$$\mathcal{K}_{p,q}^{\alpha,m} (\sigma, \omega) [Q] \approx |Q|^\frac{m}{m'} \left( \frac{1}{|Q|} \int_Q (1 + |x|)^{-mp'} \, dx \right)^{\frac{1}{p'}} \left( \frac{1}{|Q|} \int_Q |y|^\frac{mp'}{p} \, dy \right)^{\frac{1}{p'}}$$

$$\approx R^{\alpha - m \frac{m'}{p} - m \frac{m}{p'}} \left( \frac{1}{R} \int_0^R (1 + r)^{-m \frac{m'}{p} + m \frac{m}{p'}} \, dr \right)^{\frac{1}{p'}} \left( \frac{1}{R} \int_0^R \frac{r^m}{|y|^{\frac{mp'}{p}}} \, dr \right)^{\frac{1}{p'}}$$

$$\approx R^{\alpha - m \frac{m}{p} - \frac{m}{p'}} \min \{ Q, 1 \}^m R^{\frac{m}{p} + \frac{m'}{p'}} \cong R^{\alpha + \frac{m}{p} - \frac{m'}{p}} \min \{ Q, 1 \}^m \leq C < \infty$$

since $\alpha + \beta = \alpha + \frac{m}{q} - \frac{m}{p} = 0$. Now if $Q = z + [-R, R]^m$ with $R \leq \frac{1}{\| z \|}$, we have

$$\mathcal{K}_{p,q}^{\alpha,m} (\sigma, \omega) [Q] = |Q|^\frac{m}{m'} \left( \frac{1}{|Q|} \int_Q (1 + |x|)^{-mp'} \, dx \right)^{\frac{1}{p'}} \left( \frac{1}{|Q|} \int_Q |y|^\frac{mp'}{p} \, dy \right)^{\frac{1}{p'}}$$

$$\approx R^{\alpha - m \frac{m}{p} - m \frac{m}{p'}} \left( \frac{1}{R} \int_0^R (1 + r)^{-m \frac{m}{p} + m \frac{m}{p'}} \, dr \right)^{\frac{1}{p'}} \left( \frac{1}{R} \int_0^R \frac{r^m}{|y|^{\frac{mp'}{p}}} \, dr \right)^{\frac{1}{p'}}$$

$$\leq R^{\alpha - m \frac{m}{p} - \frac{m}{p'}} \min \{ Q, 1 \}^m \leq C < \infty$$

since $0 \leq \frac{m}{q} \leq m$. On the other hand, if $Q = z + [-R, R]^m$ with $R > \frac{1}{\| z \|}$, then since $d\sigma (y) = |y|^{-\frac{mp'}{p}} \, dy$ is doubling and $d\omega (x) = (1 + |x|)^{-mp'} \, dx$ is radially decreasing, we have

$$\mathcal{K}_{p,q}^{\alpha,m} (\sigma, \omega) [Q] = |Q|^\frac{m}{m'} \left( \frac{1}{|Q|} \int_Q (1 + |x|)^{-mp'} \, dx \right)^{\frac{1}{p'}} \left( \frac{1}{|Q|} \int_Q |y|^\frac{mp'}{p} \, dy \right)^{\frac{1}{p'}}$$

$$\approx |[-R, R]^m|^\frac{m}{m'} \left( \frac{1}{|Q|} \int_Q (1 + |x|)^{-mp'} \, dx \right)^{\frac{1}{p'}} \left( \frac{1}{|Q|} \int_Q \frac{|y|^\frac{mp'}{p}}{|x|^\frac{mp'}{p}} \, dx \right)^{\frac{1}{p'}}$$

$$\approx |[-R, R]^m|^\frac{m}{m'} \left( \frac{1}{|[-R, R]^m|} \int_{[-R, R]^m} (1 + |x|)^{-mp'} \, dx \right)^{\frac{1}{p'}} \left( \frac{1}{|[-R, R]^m|} \int_{[-R, R]^m} \frac{|y|^\frac{mp'}{p}}{|x|^\frac{mp'}{p}} \, dx \right)^{\frac{1}{p'}}$$

which by (6.1) is bounded by $C$.

On the other hand, with $I = [-1, 1]^m$, we have

$$|2^k I|^\frac{m}{m'} \left( \frac{1}{|2^k I|} \int_{2^k I} d\omega \right)^{\frac{1}{p'}} \left( \frac{1}{|2^k I|} \int_{2^k I} |2^k I| \right)^{\frac{1}{p'}}$$

$$\approx |2^k I|^\frac{m}{m'} \left( \frac{1}{|2^k I|} \int_{2^k I} d\sigma \right)^{\frac{1}{p'}} \left( \frac{1}{|2^k I|} \int_{2^k I} |2^k I| \right)^{\frac{1}{p'}}$$

$$\approx 2^{k[a - m\left(\frac{1}{p} - \frac{1}{p'}\right)]} 2^{-km \left[\frac{m}{q} + \frac{m}{p} - \frac{m}{p'}\right]} \left( \int_{2^k I} d\sigma \right)^{\frac{1}{p'}} = 2^{k[a - m]} \left( \int_{2^k I} d\sigma \right)^{\frac{1}{p'}}.$$
7. Proof of Theorem 7

Here we give the proof of our main positive two weight result, Theorem 7 beginning with the necessity of (4.11) and (4.12) for the norm inequality (4.10), i.e. the proof of (1) \(\implies\) (2). The necessity of (4.11) follows from the inequality
\[
|x - u|^{\alpha - m} |y - t|^{\beta - n} \geq |(x - u, y - t)|^{\alpha + \beta - (m + n)},
\]
which shows that the 1-parameter fractional integral \(I_{\alpha,\beta} f\) is dominated by the 2-parameter fractional integral \(I_{\alpha,\beta}^{m,n} f\) when \(f\) is nonnegative. Thus Theorem 14 shows that \(p \leq q\). Local integrability of the kernel is necessary for the norm inequality, and so \(\alpha, \beta > 0\). The necessity of finiteness of the two-tailed Muckenhoupt characteristic \(A_{p,q}(\alpha,\beta,\gamma_q)(v_\alpha, w_\beta)\) now follows from Proposition 3 at the end of the appendix, and then we use \(A_{p,q}(\alpha,\beta,\gamma_q)(v_\alpha, w_\beta) \leq A_{p,q}(\alpha,\beta,\gamma_q)(v_\alpha, w_\beta)\).

Now we turn to proving (2) \(\implies\) (3), i.e. that (4.11) and (4.12) imply the conditions (4.13), (4.14) and (4.15) on indices, namely

\[
(7.1)
\]
\[
\frac{1}{p} - \frac{1}{q} + \frac{\gamma + \delta}{m + n} = \frac{\alpha + \beta}{m + n},
\]
\[
\gamma + \delta \geq 0,
\]
\[
\beta - \frac{n}{p} < \delta\quad \text{and} \quad \alpha - \frac{m}{p} < \delta \quad \text{when} \quad \gamma \geq 0 \geq \delta,
\]
\[
\beta - \frac{n}{q} < \gamma\quad \text{and} \quad \alpha - \frac{m}{q} < \gamma \quad \text{when} \quad \delta \geq 0 \geq \gamma.
\]
So assume both (4.11) and (4.12). We begin with the necessity of the equality in the top line of (7.1). This follows immediately from the calculation,
\[
|t|^\frac{\beta}{\alpha} - 1 |t|^\frac{\delta}{\gamma} - 1 \left( \int_{I \times J} |(x, y)|^{-\gamma q} \, dx \, dy \right) \frac{1}{p} \left( \int_{I \times J} |(x, y)|^{-\delta p'} \, dx \, dy \right) \frac{1}{q}
\]
\[
eq a - m t^{\beta - n} t^{-\gamma + \frac{m + n}{p}} |I|^\frac{\beta}{\alpha} - 1 |J|^\frac{\delta}{\gamma} - 1 \left( \int_{I \times J} |(x, y)|^{-\gamma q} \, dx \, dy \right) \frac{1}{p} \left( \int_{I \times J} |(x, y)|^{-\delta p'} \, dx \, dy \right) \frac{1}{q}
\]
which implies \(\alpha - m + \beta - n - \gamma + \frac{m + n}{p} - \delta + \frac{m + n}{q} = 0\). Next we note that power weights have support equal to the entire Euclidean space, and so Remark 2 shows that
\[
\frac{\alpha}{m} - \frac{\beta}{n} \geq \frac{1}{p} - \frac{1}{q},
\]
and combining this with (4.13) gives the second line in (7.1). Finally, we turn to proving the necessity of the third and fourth lines in (7.1).

For this, we consider \(P \times Q\) to be centered at the origin. Define the truncated cubes \(P^\varepsilon = P \setminus \{|x| < \varepsilon\} \subset \mathbb{R}^m\) and \(Q^\varepsilon = Q \setminus \{|y| < \varepsilon\} \subset \mathbb{R}^n\) for some \(\varepsilon > 0\). Let \(0 < \varepsilon < 1\). We set \(|Q|^{\frac{1}{2}} = 1\) and \(|P|^{\frac{1}{2}} = \lambda\). Suppose that \(\frac{\alpha}{m} - \left(\frac{1}{p} - \frac{1}{q}\right) > 0\). Then we have
\[
(7.2)
\]
\[
\lim_{\lambda \to 0} \left| P \right|^{\frac{\alpha}{m} - \left(\frac{1}{p} - \frac{1}{q}\right)} |Q|^{\frac{\beta}{n} - \left(\frac{1}{p} - \frac{1}{q}\right)} \left\{ \frac{1}{|P||Q|} \int_{P \times Q^\varepsilon} \left( \int_{|x| + |y|} \gamma^q \, dx \, dy \right)^\frac{1}{q} \right\} \left\{ \frac{1}{|P||Q|} \int_{P \times Q^\varepsilon} \left( \int_{|x| + |y|} \delta^{\left(\frac{p}{r} - 1\right)} \, dx \, dy \right)^\frac{1}{r} \right\}
\]
\[
= \lim_{\lambda \to 0} \lambda^{\alpha - m \left(\frac{1}{p} - \frac{1}{q}\right)} \left\{ \frac{1}{|P||Q|} \int_{P \times Q^\varepsilon} \left( \int_{|x| + |y|} \gamma^q \, dx \, dy \right)^\frac{1}{q} \right\} \left\{ \frac{1}{|P||Q|} \int_{P \times Q^\varepsilon} \left( \int_{|x| + |y|} \delta^{\left(\frac{p}{r} - 1\right)} \, dx \, dy \right)^\frac{1}{r} \right\}
\]
\[
= \left( \lim_{\lambda \to 0} \lambda^{\alpha - m \left(\frac{1}{p} - \frac{1}{q}\right)} \right) \left\{ \int_{Q^\varepsilon} \left( \int_{|y|} \gamma^q \, dy \right)^\frac{1}{q} \right\} \left\{ \int_{Q^\varepsilon} \left( \int_{|y|} \delta^{\left(\frac{p}{r} - 1\right)} \, dy \right)^\frac{1}{r} \right\} = 0 \quad \text{for every} \quad \varepsilon > 0.
\]
Case 1: Suppose $\gamma > 0, \delta \leq 0$. Let $|Q|^{\frac{1}{p'}} = 1$ and $|P|^{\frac{1}{q'}} = \lambda$. Suppose $\alpha - m\left(\frac{\lambda}{p} - \frac{1}{q}\right) = 0$. Then we have

\begin{equation}
|P|^{\frac{1}{p'}} \left(\frac{1}{q'} - \frac{1}{q}\right) \left|Q\right|^{\frac{1}{q'} - \frac{1}{q'}} \left\{ \frac{1}{|P||Q|} \int_{P \times Q^c} \left(\frac{1}{|x| + |y|}\right)^{\gamma q} dxdy \right\}^{\frac{1}{q}} \left\{ \frac{1}{|P||Q|} \int_{P \times Q^c} \left(\frac{1}{|x| + |y|}\right)^{\delta \left(\frac{p}{p+1}\right)} dxdy \right\}^{\frac{p}{p+1}} \gtrsim \left\{ \int_{Q} \left(\frac{1}{|y|}\right)^{\gamma q} dy \right\}^{\frac{1}{q}} \left\{ \int_{|y| \leq 1} \left(\frac{1}{|y|}\right)^{\gamma q} dy \right\}^{\frac{1}{q}} \gtrsim \left\{ \int_{\lambda < |y| \leq 1} \left(\frac{1}{|y|}\right)^{\gamma q} dy \right\}^{\frac{1}{q}}.
\end{equation}

A direct computation shows

\begin{equation}
\int_{\lambda < |y| \leq 1} \left(\frac{1}{|y|}\right)^{\gamma q} dy \approx \ln \left(\frac{1 + \lambda}{2\lambda}\right) \text{ if } \gamma = \frac{n}{q}
\end{equation}

and

\begin{equation}
\int_{\lambda < |x| \leq 1} \left(\frac{1}{|x| + |y|}\right)^{\gamma q} dy \approx \left(\frac{1}{2\lambda}\right)^{\gamma q - \beta} - \left(\frac{1}{\lambda + 1}\right)^{\gamma q - \beta} \text{ if } \gamma > \frac{n}{q}.
\end{equation}

Using (7.3), (7.4) and (7.5), and letting $\lambda \to 0$, we obtain that

\begin{equation}
\gamma < \frac{n}{q}.
\end{equation}

On the other hand, suppose that $\alpha - m\left(\frac{\lambda}{p} - \frac{1}{q}\right) > 0$. Then we have

\begin{equation}
\left|P\right|^{\frac{1}{p'}} \left(\frac{1}{q'} - \frac{1}{q}\right) \left|Q\right|^{\frac{1}{q'} - \frac{1}{q'}} \left\{ \frac{1}{|P||Q|} \int_{P \times Q^c} \left(\frac{1}{|x| + |y|}\right)^{\gamma q} dxdy \right\}^{\frac{1}{q}} \left\{ \frac{1}{|P||Q|} \int_{P \times Q^c} \left(\frac{1}{|x| + |y|}\right)^{\delta \left(\frac{p}{p+1}\right)} dxdy \right\}^{\frac{p}{p+1}} \gtrsim \left\{ \int_{Q} \left(\frac{1}{|y|}\right)^{\gamma q} dy \right\}^{\frac{1}{q}} \left\{ \int_{|y| \leq \lambda} \left(\frac{1}{|y|}\right)^{\gamma q} dy \right\}^{\frac{1}{q}} \gtrsim \left\{ \int_{0 < |y| \leq \lambda} \left(\frac{1}{|y|}\right)^{\gamma q} dy \right\}^{\frac{1}{q}} \gtrsim \left(\frac{1}{\lambda}\right)^{\frac{1}{q}} ^{\gamma q - \gamma + m\left(\frac{\lambda}{p} - \frac{1}{q}\right)}.
\end{equation}

Recall the estimate in (7.2). Now we note that (7.7) converges to zero as $\lambda \to 0$. By putting (7.7) together with (7.6), we obtain

\begin{equation}
\gamma < \frac{n}{q} + \alpha - m\left(\frac{1}{p} - \frac{1}{q}\right).
\end{equation}

The formula in (4.13) implies that (7.8) is equivalent to

\begin{equation}
\beta - \frac{n}{p} < \delta.
\end{equation}

Switching the roles of $P$ and $Q$ in the argument above shows that

\begin{equation}
\alpha - m\left(\frac{1}{p} - \frac{1}{q}\right) < \delta.
\end{equation}
Using (7.11), (7.12) and (7.13), and letting 

$$\delta < n$$

together with (7.14), we have

$$\int_Q \left( \frac{1}{|y|} \right)^\gamma dy \approx \left( \frac{1}{\lambda + |y|} \right) \delta \left( \frac{1}{\pi r} \right)$$

A direct computation shows

$$\int_{\lambda < |y| \leq 1} \left( \frac{1}{\lambda + |y|} \right) \delta \left( \frac{1}{\pi r} \right) dy \approx \ln \left( \frac{1 + \lambda}{\lambda} \right)$$

if \( \delta = n \left( \frac{p - 1}{p} \right) \)

and

$$\int_{\lambda < |y| \leq 1} \left( \frac{1}{\lambda + |y|} \right) \delta \left( \frac{1}{\pi r} \right) dy \approx \left( \frac{1}{2\lambda} \right) \delta \left( \frac{1}{\pi r} \right) - \left( \frac{1}{\lambda + 1} \right) \delta \left( \frac{1}{\pi r} \right) - n \left( \frac{p - 1}{p} \right)$$

if \( \delta > n \left( \frac{p - 1}{p} \right) \).

Using (7.11), (7.12) and (7.13), and letting \( \lambda \to 0 \), we obtain

$$\delta < n \left( \frac{p - 1}{p} \right)$$

On the other hand, suppose that \( \alpha - m \left( \frac{1}{p} - \frac{1}{4} \right) > 0 \). Then we have

$$\int_Q \left( \frac{1}{|y|} \right)^\gamma dy \approx \left( \frac{1}{\lambda + |y|} \right) \delta \left( \frac{1}{\pi r} \right)$$

Recall the estimate in (7.22). Now we note that (7.15) converges to zero as \( \lambda \to 0 \). By putting (7.15) together with (7.14), we have

$$\delta < n \left( \frac{p - 1}{p} \right) + \alpha - m \left( \frac{1}{p} - \frac{1}{q} \right)$$

The formula in (4.13) implies that (7.16) is equivalent to

$$\beta - n \left( \frac{q - 1}{q} \right) < \gamma$$

Switching the roles of \( P \) and \( Q \) in the argument above shows that

$$\alpha - m \left( \frac{q - 1}{q} \right) < \gamma$$

This completes the proof that (7.1) is necessary for (4.11) and (4.12), and hence we have proved (2) \( \implies \) (3).

Now we turn to proving (3) \( \implies \) (1), i.e. that (4.11) and (7.1) are sufficient for the norm inequality (4.10). To this end we use Young’s inequality

$$a^{1-\theta}b^\theta \leq \sqrt{a^2 + b^2} \leq a + b,$$
valid for \( a, b \geq 0 \) and \( 0 \leq \theta \leq 1 \), in order to define weight pairs to which the sandwiching principle can be applied. The special cases \( \alpha = m \) and \( \beta = n \), along with some additional exceptional cases, will be treated at the end of the proof.

### 7.1. The nonexceptional cases.

We assume that \( \alpha < m \) and \( \beta < n \) and show here that \( N_{p,q}^{(\alpha,\beta),(m,n)}(v_\delta, w_\gamma) < \infty \) follows from (4.11), (4.13), (4.14) and (4.15) when both \( \gamma \geq 0 \) and \( \delta \geq 0 \), and also when either \( \gamma < 0 \) or \( \delta < 0 \).

**Case 1:** First we suppose that \( \gamma \geq 0 \) and \( \delta \geq 0 \). Define the weight pair

\[
(V(u,t), W(x,y)) = \left( |u|^{\frac{\delta}{m+n}}, |t|^{\frac{\delta}{m+n}}, \left( \frac{1}{|x|} \right)^{\frac{\gamma m}{m+n}}, \left( \frac{1}{|y|} \right)^{\frac{\gamma n}{m+n}} \right)
\]

where the indices \( \delta_1, \delta_2, \gamma_1, \gamma_2 \) satisfy

\[
(7.20) \quad \frac{\delta_1 m}{m+n} + \frac{\delta_2 n}{m+n} = \delta \quad \text{and} \quad \frac{\gamma_1 m}{m+n} + \frac{\gamma_2 n}{m+n} = \gamma,
\]

and

\[
(7.21) \quad \alpha - \left( \frac{\gamma_1 m}{m+n} + \frac{\delta_1 m}{m+n} \right) = \Gamma = \beta - \left( \frac{\gamma_2 n}{m+n} + \frac{\delta_2 n}{m+n} \right),
\]

i.e. \( \frac{\gamma_1 m}{m+n} + \frac{\delta_1 m}{m+n} = \alpha - m\Gamma \) and \( \frac{\gamma_2 n}{m+n} + \frac{\delta_2 n}{m+n} = \beta - n\Gamma \).

Solving for

\[
\Delta_1 \equiv \frac{\delta_1 m}{m+n} , \quad \Delta_2 \equiv \frac{\delta_2 n}{m+n} , \quad \Gamma_1 \equiv \frac{\gamma_1 m}{m+n} \gamma_1 , \quad \Gamma_2 \equiv \frac{\gamma_2 n}{m+n} ,
\]

we obtain the system

\[
(7.22) \quad \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
\Delta_1 \\
\Delta_2 \\
\Gamma_1 \\
\Gamma_2
\end{bmatrix} =
\begin{bmatrix}
\delta \\
\beta - n\Gamma \\
\gamma \\
\alpha - m\Gamma
\end{bmatrix},
\]

which in reduced row echelon form is

\[
\begin{bmatrix}
1 & 0 & 0 & -1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\Delta_1 \\
\Delta_2 \\
\Gamma_1 \\
\Gamma_2
\end{bmatrix} =
\begin{bmatrix}
\delta - \beta + n\Gamma \\
\beta - n\Gamma \\
\gamma \\
\alpha - m\Gamma - \delta + \beta - n\Gamma
\end{bmatrix}.
\]

The system is solvable since \( \alpha - m\Gamma - \delta + \beta - n\Gamma = 0 \) by the power weight equality in the first line of (4.17), and the general solution to the system (7.22) is thus given by

\[
\begin{bmatrix}
\Delta_1 \\
\Delta_2 \\
\Gamma_1 \\
\Gamma_2
\end{bmatrix} =
\begin{bmatrix}
\delta - \beta + n\Gamma \\
\beta - n\Gamma \\
\gamma \\
\alpha - m\Gamma - \delta + \beta - n\Gamma
\end{bmatrix} + \lambda
\begin{bmatrix}
-1 \\
1 \\
1 \\
-1
\end{bmatrix} = z_\lambda , \quad \lambda \in \mathbb{R}.
\]

Among these solution vectors \( z_\lambda \), we will find a vector satisfying all of the constraint inequalities needed below.

Now by Young’s inequality (7.19) and (7.20), we have

\[
\frac{w(u,t)}{v(u,t)} \leq \frac{W(x,y)}{V(u,t)},
\]

and so by the sandwiching principle, Lemma II we have

\[
(7.23) \quad N_{p,q}^{(\alpha,\beta),(m,n)}(v, w) \leq N_{p,q}^{(\alpha,\beta),(m,n)}(V, W).
\]
Moreover, the weights $V, W$ are product weights,

\[
V(u, t) = V_1(u) V_2(t) \quad \text{and} \quad W(x, y) = W_1(x) W_2(y);
\]

\[
V_1(u) = |u|^{\frac{\delta_1}{m}} \quad , \quad V_2(t) = |t|^{\frac{\delta_1}{n}},
\]

\[
W_1(x) = \left(\frac{1}{|x|}\right)^{\frac{\delta_1}{m+n}} \quad , \quad W_2(y) = \left(\frac{1}{|y|}\right)^{\frac{\delta_1}{m+n}},
\]

where the 1-parameter weight pairs $(V_1(u), W_1(x))$ and $(V_2(t), W_2(y))$ each satisfy the hypotheses of Theorem 3 on $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively for an appropriate choice of solution vector above.

Indeed, to see this, note that the first weight pair $(V_1(u), W_1(x)) = (|u|^{\Delta_1}, |x|^{-\Gamma})$ on $\mathbb{R}^m$ satisfies the equality

\[
\frac{1}{p} - \frac{1}{q} = \frac{\alpha - (\Gamma_1 + \Delta_1)}{m}
\]

by (7.21). We also claim the inequalities

\[
q \Gamma_1 < m \quad \text{and} \quad p' \Delta_1 < m \quad \text{and} \quad \Gamma_1 + \Delta_1 \geq 0,
\]

for an appropriate family of solution vectors $z_\lambda$. The third inequality actually holds for all solution vectors $z_\lambda$ since

\[
\Gamma_1 + \Delta_1 = \gamma + \delta - \beta + n \Gamma \geq 0,
\]

by (4.17). Thus the equality (7.24), and the inequalities (7.25), all hold for those solution vectors $z_\lambda$ satisfying

\[
\Gamma_1 + \Delta_1 = \gamma + \delta - \beta + n \Gamma \geq 0.
\]

The second weight pair $(V_2(t), W_2(y)) = (|t|^{\Delta_2}, |y|^{-\Gamma_2})$ on $\mathbb{R}^n$ satisfies the equality

\[
\frac{1}{p} - \frac{1}{q} = \frac{\alpha - (\Gamma_2 + \Delta_2)}{n}
\]

by (7.21). We also claim the inequalities

\[
q \Gamma_2 < n \quad \text{and} \quad p' \Delta_2 < n \quad \text{and} \quad \Gamma_2 + \Delta_2 \geq 0,
\]

for an appropriate family of solution vectors $z_\lambda$. The third inequality actually holds for all solution vectors $z_\lambda$ since

\[
\Gamma_2 + \Delta_2 = \alpha + \beta - (m + n) \Gamma = \gamma + \delta \geq 0,
\]

by (4.17). Thus the equality (7.27), and the inequalities (7.28), all hold for those solution vectors $z_\lambda$ satisfying

\[
\Gamma_2 + \Delta_2 = \alpha + \beta - (m + n) \Gamma = \gamma + \delta \geq 0.
\]

In order to find $\lambda$ satisfying (7.26) and (7.29) simultaneously, we must establish the four strict inequalities in

\[
\max \left\{ \delta - \beta + n \Gamma - \frac{m}{p'}, -\frac{n}{q} \right\} < \min \left\{ \frac{m}{q} - \gamma, \frac{n}{p'} - \beta + n \Gamma \right\}.
\]
Now two of these four strict inequalities follow from the local integrability of the weights \(|(x,y)|^{-\gamma_q}\) and \(|(u,t)|^{-\gamma_{p'}}\) on \(\mathbb{R}^m \times \mathbb{R}^n\), namely
\[
\delta - \beta + n\Gamma - \frac{m}{p'} < \frac{n}{p'} - \beta + n\Gamma \quad \text{and} \quad -\frac{n}{q} < \frac{m}{p'} - \beta + n\Gamma;
\]
i.e. \(\delta < \frac{m+n}{p'}\) and \(\gamma < \frac{m+n}{q}\).

The other two strict inequalities follow from the assumptions that \(\alpha < m\) and \(\beta < n\), namely
\[
\delta - \beta + n\Gamma - \frac{m}{p'} < \frac{n}{p'} - \beta + n\Gamma \quad \text{and} \quad -\frac{n}{q} < \frac{m}{p'} - \beta + n\Gamma;
\]
i.e. \(\delta + \gamma < \beta - n\Gamma + m\left(\frac{1}{p'} + \frac{1}{q}\right)\) and \(\beta < n\left(\frac{1}{p'} + \frac{1}{q}\right) + n\Gamma;\)
i.e. \((m+n)\Gamma + (\delta + \gamma) < m + \beta\) and \(\beta < n;\)
i.e. \(\alpha + \beta < m + \beta\) and \(\beta < n,\)

where in the final line above we have used the power weight equality from the first line of (4.17).

Thus there does indeed exist a choice of \(\gamma \in \mathbb{R}\) so that the equalities (7.24), (7.27) and inequalities (7.25), (7.28) all hold. It now follows from Theorem [2] that
\[
N_{p,q}^{(\alpha,\beta),(m,n)} (V, W) \leq N_{p,q}^{(\alpha,m)} (V_1, W_1) N_{p,q}^{(\beta,n)} (V_2, W_2)
\]
and combined with (7.23) this yields
\[
N_{p,q}^{(\alpha,\beta),(m,n)} (v, w) < \infty,
\]
in the case \(\gamma \geq 0\) and \(\delta \geq 0.\)

**Case 2:** Next we suppose that \(\gamma < 0 < \delta\) and use the fourth line in (7.1). Let \(\rho \equiv \gamma + \delta \geq 0\) and \(\eta \equiv -\gamma > 0.\) Then by Young’s inequality (7.49), the weight pairs
\[
(V(u,t), W(x,y)) = \left( |u|^\frac{\rho}{\rho - \eta} |t|^\frac{\rho}{\rho + \eta}, |x|^\eta \right),
\]
\[
(V'(u,t), W'(x,y)) = \left( |u|^\frac{\rho}{\rho - \eta} |t|^\frac{\rho}{\rho + \eta}, |y|^\eta \right),
\]
where \(\rho_j + \eta_j = \rho + \eta = \delta > 0\) for \(j = 1, 2,\) satisfy
\[
\frac{w(x,y)}{v(u,t)} = \left( \frac{|x|^\eta + |y|^\eta}{|u|^\frac{\rho}{\rho - \eta} |t|^\frac{\rho}{\rho + \eta}} \right)^\frac{1}{2} = \left( \frac{|x|^\eta + |y|^\eta}{|u|^\frac{\rho}{\rho - \eta} |t|^\frac{\rho}{\rho + \eta}} \right)^\frac{1}{2} \left( \frac{|x|^\eta + |y|^\eta}{|u|^\frac{\rho}{\rho - \eta} |t|^\frac{\rho}{\rho + \eta}} \right)^\frac{1}{2} \left( \frac{|x|^\eta + |y|^\eta}{|u|^\frac{\rho}{\rho - \eta} |t|^\frac{\rho}{\rho + \eta}} \right)^\frac{1}{2} \left( \frac{|x|^\eta + |y|^\eta}{|u|^\frac{\rho}{\rho - \eta} |t|^\frac{\rho}{\rho + \eta}} \right)^\frac{1}{2}
\]
and so by the sandwiching principle, Lemma [11] we have
\[
N_{p,q}^{(\alpha,\beta),(m,n)} (v, w) \leq N_{p,q}^{(\alpha,\beta),(m,n)} (V, W) + N_{p,q}^{(\alpha,\beta),(m,n)} (V', W').
\]
Moreover, the weights \(V, V', W, W'\) are product weights,
\[
(V(u,t), V'(u,t)) = V_1(u) V_2(t) \quad \text{and} \quad V'(u,t) = V'_1(u) V'_2(t),
\]
\[
(W(x,y), W'(x,y)) = W_1(x) W_2(y) \quad \text{and} \quad W'(x,y) = W'_1(x) W'_2(y),
\]
where
\[
\begin{align*}
V_1(u) &= |u|^\frac{\rho}{\rho - \eta} |t|^\frac{\rho}{\rho + \eta}
V_2(t) &= |t|^\frac{\rho}{\rho + \eta}
V'_1(u) &= |u|^\frac{\rho}{\rho - \eta} |t|^\frac{\rho}{\rho + \eta}
V'_2(t) &= |t|^\frac{\rho}{\rho + \eta}
W_1(x) &= |x|^\eta
W_2(y) &= 1
W'_1(x) &= |x|^\eta
W'_2(y) &= 1
\end{align*}
\]
and where the 1-parameter weight pairs \((V_1 (u), W_1 (x)), (V'_1 (u), W'_1 (x))\) on \(\mathbb{R}^m\) and \((V_2 (t), W_2 (y)), (V'_2 (t), W'_2 (y))\) on \(\mathbb{R}^n\) each satisfy the hypotheses of Theorem 3 provided we choose \(\rho_1\) and \(\eta_1\) to satisfy

\[
\alpha - \left( - \eta + \frac{\rho_1 m}{m + n} \right) = \Gamma = \frac{\alpha + \beta - \rho}{m + n},
\]

\[
\beta - \left( - 0 + \frac{\rho_1 n}{m + n} \right) = \Gamma = \frac{\alpha + \beta - \rho}{m + n},
\]

\[\text{i.e.} \]

\[
\frac{\alpha - \rho_1}{m + n} + \frac{\eta - \eta_1}{m} = \frac{\alpha + \beta - \rho}{m + n} = \frac{\beta}{m + n};
\]

\[
\frac{\alpha}{m} + \frac{\eta - \eta_1}{m} = \frac{\alpha + \beta + \rho_1 - \rho}{m + n} = \frac{\beta}{m + n};
\]

\[
\frac{\beta}{n} - \frac{\alpha}{m} = \frac{\eta - \eta_1}{m} \quad \text{and} \quad \frac{\alpha + \beta}{m + n} = \frac{\beta}{m + n};
\]

\[
\frac{\eta - \eta_1}{m} = \frac{\beta}{n} - \frac{\alpha}{m} \quad \text{and} \quad \frac{\rho_1}{m + n} = \frac{\rho - (\alpha + \beta)}{m + n} + \frac{\beta}{n},
\]

so that

\[
\eta_1 = \alpha + \eta - \frac{m}{n} \beta \quad \text{and} \quad \rho_1 = \rho - (\alpha + \beta) + \frac{m + n}{n} \beta,
\]

and where \(\rho_2\) and \(\eta_2\) will be chosen below. Once we have established the appropriate hypotheses of Theorem 3 Theorem 12 will show that

\[N_{p,q}(\alpha, \beta, \rho, \eta) (v, w) < \infty,\]

in the case \(\gamma < 0 < \delta\).

To show that these four weight pairs satisfy the hypotheses of Theorem 3 we first note that

\[
\rho_1 = (m + n) \left[ \frac{(\gamma + \delta - (\alpha + \beta)}{m + n} + \frac{\beta}{n} \right] = (m + n) \left( \frac{\beta}{n} - \Gamma \right) \geq 0
\]

by the power weight equality in (4.17), and the third line in (4.17), and also note that

\[
\rho_1 + \eta_1 = (m + n) \left\{ \frac{\rho - (\alpha + \beta)}{m + n} + \frac{\beta}{n} \right\} + m \left\{ \frac{\alpha + \eta}{m} - \frac{\beta}{n} \right\}
\]

\[
= \rho - (\alpha + \beta) + \frac{m + n}{n} \beta + (\alpha + \eta) - \frac{m}{n} \beta = \rho + \eta.
\]

Next, we verify that the first weight pair \((V_1 (u), W_1 (x)) = \left( |u|^\frac{\rho_1 m}{m + n} + \eta_1, |x|^\eta \right)\) on \(\mathbb{R}^m\) satisfies the 1-parameter power weight equality

\[
\frac{1}{p} - \frac{1}{q} = \frac{\alpha - \left( - \eta + \frac{\rho_1 m}{m + n} + \eta_1 \right)}{m}
\]

as well as the 1-parameter constraint inequalities

\[
q (-\eta) < m \quad \text{and} \quad p' \left( \frac{\rho_1 m}{m + n} + \eta_1 \right) < m \quad \text{and} \quad -\eta + \frac{\rho_1 m}{m + n} + \eta_1 \geq 0.
\]

The equality (7.33) follows immediately from (7.31), the first line in (4.17), and (7.30). The first inequality in (7.34) is trivial, and the third inequality in (7.34) is

\[-\eta + \frac{\rho_1 m}{m + n} + \eta_1 \geq 0,
\]
which follows from (7.31):

\[(7.35) \quad -\eta + \frac{m}{m+n} \rho_1 + \eta_1 = -\eta + \frac{m}{m+n} \left[ \rho - (\alpha + \beta) + \frac{m+n}{n} \beta \right] + \alpha + \eta - \frac{m}{n} \beta \]

\[= -\eta + \frac{m}{m+n} (\rho - (\alpha + \beta)) + \frac{m}{n} \beta + \alpha + \eta - \frac{m}{n} \beta \]

\[= \frac{m}{m+n} (\rho - (\alpha + \beta)) + \alpha \]

\[= m \left( \frac{\rho - (\alpha + \beta)}{m+n} + \frac{\alpha}{m} \right) = m \left( \frac{\alpha}{m} - \Gamma \right) \geq 0. \]

The second inequality in (7.34) is

\[(7.36) \quad \frac{1}{m} \left( \frac{\rho_1 m}{m+n} + \eta_1 \right) < \frac{1}{p'}, \]

which, using \(\gamma = -\eta\) with the equality \(-\eta + \frac{m}{m+n} \rho_1 + \eta_1 = m \left( \frac{\alpha}{m} - \Gamma \right) = \alpha - m \Gamma\) just proved above in (7.35), is equivalent to

\[\frac{1}{m} \left( \frac{\alpha - m \Gamma + \eta}{m} \right) < \frac{1}{p'};\]

\[\iff \alpha + \eta - \frac{m}{n} \Gamma < \frac{1}{p'};\]

\[\iff \alpha < \Gamma + \frac{\gamma + \frac{1}{p'}}{m};\]

\[\iff \alpha - \frac{1}{q'} < \frac{\gamma}{m}. \]

Next we note that the weight pair \((V_2(t), W_2(y)) = \left( ||t|^{\frac{m}{m+n}}, 1 \right)\) on \(\mathbb{R}^n\) satisfies the 1-parameter power weight equality

\[\frac{1}{p} - \frac{1}{q} = \frac{\beta - (-0 + \frac{\rho_1 m}{m+n})}{n}, \]

and the 1-parameter constraint inequalities,

\[q(-0) < n \text{ and } p' \left( \frac{\rho_1 n}{m+n} \right) < n \text{ and } -0 + \frac{\rho_1 m}{m+n} \geq 0. \]

Indeed, for the equality we use (7.31), the first line in (4.17), and (7.30). The first of the constraint inequalities is trivial and the third constraint inequality follows from (7.32). The second of the constraint inequalities, namely \(p' \left( \frac{\rho_1 n}{m+n} \right) < n\), is equivalent to

\[\left( \frac{n}{m+n} \right) (m+n) \left( \frac{\beta}{n} - \Gamma \right) < \frac{n}{p'};\]

i.e.

\[\frac{\beta}{n} < \Gamma + \frac{1}{p'}. \]

However from the third line in (4.17) and the assumption that \(\gamma < 0\) we have

\[\frac{\beta}{n} \leq \Gamma + \gamma + \frac{\delta}{n} - \left( \frac{\delta}{p'} \right)_+ \]

\[= \Gamma + \frac{\gamma}{n} + \frac{1}{p'} + \frac{\delta - \frac{n}{p'}}{n} - \left( \frac{\delta}{n} - \frac{1}{p'} \right)_+ \]

\[= \Gamma + \frac{\gamma}{n} + \frac{1}{p'} - \left( \frac{\delta}{n} - \frac{1}{p'} \right)_- \]

\[\leq \Gamma + \frac{\gamma}{n} + \frac{1}{p'} < \Gamma + \frac{1}{p'}, \]
and this time there is no exceptional endpoint case. As indicated above, this completes the proof in the case
\( \gamma < 0 < \delta \).

The same arguments apply to the weight pair \((V', W')\), which we now sketch briefly. The weight pairs
\((V_2'(t), W_2'(y)) = \left( |t|^{\frac{\rho_2 m}{m+n} + \eta_2}, |y|^{\eta} \right)\) and \((V_1'(u), W_1'(x)) = \left( |u|^{\frac{\rho_1 m}{m+n}}, 1 \right)\) each satisfy the hypotheses of
Theorem 3 provided we choose \(\rho_2\) and \(\eta_2\) to satisfy

\[
\beta - \left( -\eta + \frac{\rho_2 m}{m+n} + \eta_2 \right) \quad = \quad \Gamma = \frac{\alpha + \beta - \rho}{m+n},
\]

\[
\alpha \left( -0 + \frac{\rho_2 m}{m+n} \right) \quad = \quad \Gamma = \frac{\alpha + \beta - \rho}{m+n},
\]

i.e.

\[
\frac{\alpha - \rho_2}{m} = \frac{\alpha + \beta - \rho}{m+n} - \frac{\rho_2}{n} + \frac{\eta - \eta_2}{n};
\]

\[
\frac{e}{m} = \frac{\alpha + \beta + \rho_2 - \rho}{m+n} = \frac{\beta}{n} + \frac{\eta - \eta_2}{n};
\]

\[
\frac{\alpha - \beta}{n} = \frac{\eta - \eta_2}{n} \quad \text{and} \quad \frac{\alpha + \beta}{m+n} = \frac{\alpha + \beta}{m+n} + \frac{\rho_2}{n};
\]

\[
\frac{\eta_2}{n} = \frac{\beta + \eta}{n} - \frac{\alpha}{m} \quad \text{and} \quad \frac{\rho_2}{n} = \frac{\rho - (\alpha + \beta)}{m+n} + \frac{\alpha}{m};
\]

so that

\[
\eta_2 = \beta + \eta - \frac{n}{m} \alpha \quad \text{and} \quad \rho_2 = \rho - (\alpha + \beta) + \frac{m+n}{m} \alpha.
\]

All of the constraint inequalities hold by arguments similar to those above, except of course in the
analogous exceptional endpoint case, and by way of example we treat just the second of the constraint
inequalities for the weight pair that gives rise to the exceptional endpoint case. The second constraint
inequality for the weight pair \((V_2'(t), W_2'(y)) = \left( |t|^{\frac{\rho_2 m}{m+n} + \eta_2}, |y|^{\eta} \right)\) on \(\mathbb{R}^n\) is

\[
\frac{1}{n} \left( \frac{\rho_2 m}{m+n} + \eta_2 \right) < \frac{1}{p'},
\]

Using \(\gamma = -\eta\) with the equality

\[
-\eta + \frac{n}{m+n} \rho_2 + \eta_2
\]

\[
= \quad -\eta + \frac{n}{m+n} \left[ \rho - (\alpha + \beta) + \frac{m+n}{m} \alpha \right] + \beta + \eta - \frac{n}{m} \alpha
\]

\[
= \quad -\eta + \frac{n}{m+n} \left( \rho - (\alpha + \beta) \right) + \frac{n}{m} \alpha + \beta + \eta - \frac{n}{m} \alpha
\]

\[
= \quad \frac{n}{m+n} \left( \rho - (\alpha + \beta) \right) + \beta
\]

\[
= \quad \frac{n}{m+n} \left( \rho - (\alpha + \beta) + \frac{\beta}{n} \right) = n \left( \frac{\beta}{n} - \Gamma \right),
\]

we see that (7.39) is equivalent to

\[
\frac{1}{n} \left( n \left( \frac{\beta}{n} - \Gamma \right) + \eta \right) < \frac{1}{p'}
\]

\[
\iff \quad \frac{\beta}{n} < \Gamma + \frac{1}{p'} + \frac{\gamma}{n}
\]

\[
\iff \quad \frac{\beta}{n} - \frac{1}{p'} < \frac{\gamma}{n}.
\]

Case 3: The case \(\delta < 0 < \gamma\) is handled similarly using the third line in (7.1).
7.2. The exceptional cases $\alpha = m$ or $\beta = n$. Here we consider the two cases where $\alpha = m$ or $\beta = n$. We first show that the power weight norm inequality (4.10) holds when $\alpha = m$ and $p \leq q$ and $A_{p,q}^{(\alpha,\beta),(m,n)}(v_\delta,w_\gamma) < \infty$. To see this, we note that from the first line in (9.29) that

\[
0 < \beta < n, \\
\frac{m}{q} < \gamma < \frac{m + n}{q}, \\
\frac{m}{p'} < \delta < \frac{m + n}{p'}.
\]

Then we compute that

\[
I_{m,\beta}^{m,n} f(x,y) = \int \int \mathbb{R}^n \times \mathbb{R}^n |y - t|^{\beta - n} f(u,t)\, du\, dt = \int \int |y - t|^{\beta - n} F(t)\, dt = I_{\beta}^{\gamma} F(y),
\]

where $F(t) \equiv \int \mathbb{R}^n f(u,t)\, du$. Thus we have

\[
\int \int \mathbb{R}^n \times \mathbb{R}^n |I_{m,\beta}^{m,n} f(x,y)|^q |(x,y)|^{-\gamma q} \, dx\, dy
\]

\[
= \int \mathbb{R}^n |I_{\beta}^{\gamma} F(y)|^q \left( \int \mathbb{R}^n |(x,y)|^{-\gamma q} \, dx \right) \, dy
\]

\[
\approx \int \mathbb{R}^n |I_{\beta}^{\gamma} F(y)|^q |y|^{m - \gamma q} \, dy,
\]

since

\[
\int \mathbb{R}^n |(x,y)|^{-\gamma q} \, dx \approx \int \mathbb{R}^n |(|x| + |y|)|^{-\gamma q} \, dx = |y|^{m - \gamma q} \int \mathbb{R}^n \left( \frac{|x|}{|y|} + 1 \right)^{-\gamma q} \, d \left( \frac{x}{|y|} \right) \approx |y|^{m - \gamma q}
\]

for $m - \gamma q < 0$. We also have

\[
\int \mathbb{R}^n |F(t)|^p |t|^{\left( \frac{\delta - \frac{m}{p'}}{\beta} \right) p} \, dt = \int \mathbb{R}^n \left| \int \mathbb{R}^n f(u,t)\, du \right|^p |t|^{\left( \frac{\delta - \frac{m}{p'}}{\beta} \right) p} \, dt
\]

\[
\leq \int \mathbb{R}^n \left\{ \int \mathbb{R}^n |f(u,t)|^p (|u| + |t|)^{\delta p'} \, du \right\} \left\{ \int \mathbb{R}^n (|u| + |t|)^{-\delta p'} \, du \right\}^{p-1} |t|^{\left( \frac{\delta - \frac{m}{p'}}{\beta} \right) p} \, dt
\]

\[
\approx \int \mathbb{R}^n \left\{ \int \mathbb{R}^n |f(u,t)|^p (|u| + |t|)^{\delta p'} \, du \right\} \left\{ |t|^{m - \delta p'} \right\}^{p-1} |t|^{\left( \frac{\delta - \frac{m}{p'}}{\beta} \right) p} \, dt
\]

\[
= \int \int \mathbb{R}^n \times \mathbb{R}^n |f(u,t)|^p (|u| + |t|)^{\delta p} \, du\, dt,
\]

since

\[
\int \mathbb{R}^n (|u| + |t|)^{-\delta p'} \, du \approx |t|^{m - \delta p'}
\]

for $m - \delta p' < 0$. Thus we conclude that (4.10) holds provided we have the 1-parameter power weight norm inequality

\[
\left( \int \mathbb{R}^n |I_{\beta}^{\gamma} F(y)|^q |y|^{-\left( \frac{\gamma - \frac{m}{q}}{\beta} \right) q} \, dy \right)^{\frac{1}{q}} \leq \left( \int \mathbb{R}^n |F(t)|^p |t|^{\left( \frac{\delta - \frac{m}{p'}}{\beta} \right) p} \, dt \right)^{\frac{1}{p'}}.
\]

But this inequality holds by Theorem 7.2 since the 1-parameter power weight equality holds,

\[
\beta - \left( \gamma - \frac{m}{q} \right) - \left( \delta - \frac{m}{p'} \right) = \beta - \gamma - \delta + m \left( \frac{1}{q} + \frac{1}{p'} \right)
\]

\[
= \alpha + \beta - \gamma - \delta + m \left( \frac{1}{q} + \frac{1}{p'} - 1 \right)
\]

\[
= (m + n) \Gamma - m \Gamma = n \Gamma,
\]
and each of the following four constraint inequalities holds,
\[ 0 < \beta < n, \]
\[ \left( \gamma - \frac{m}{q} \right) < \frac{n}{q}, \]
\[ \left( \delta - \frac{m}{p'} \right) < \frac{n}{p'}, \]
\[ \left( \gamma - \frac{m}{q} \right) + \left( \delta - \frac{m}{p'} \right) \geq 0. \]

A similar argument shows that (4.10) holds when \( \beta = n \) and \( p \leq q \) and \( A_{p,q}^{(\alpha, \beta), (m,n)}(v_\delta, w_\gamma) < \infty. \)

8. Proof of Theorem 10

Define the eccentricity \( \kappa(R) \) of a rectangle \( R = I \times J \) to be \( \kappa(R) = \ell(I) \ell(J). \) For \( j \in \mathbb{Z} \) define the conical operator \( \triangle_j I_{\alpha, \beta} \) acting on a measure \( \mu \) by
\[
\triangle_j I_{\alpha, \beta} \mu (x, y) \equiv \int \int_{S_j + (x, y)} \left( \frac{1}{|x - u|} \right)^{m-\alpha} \left( \frac{1}{|y - t|} \right)^{n-\beta} d\mu(u,t),
\]
where \( S_j = \left\{ (x, y) : |x| < 2^{-j+1}, |y| < 2^{-j+1} \right\} \) is a cone with aperture roughly \( 2^{-j} \) and slope roughly \( 2^{-j}. \)

Lemma 5. Suppose \( 1 < p < q < \infty, \) \( 0 < \alpha, \beta < 1 \) and that both \( \sigma \) and \( \omega \) are rectangle doubling, and that the reverse doubling exponent \( \varepsilon \) for \( \sigma \) satisfies
\[ 1 - \varepsilon < \frac{\alpha}{m} = \frac{\beta}{n}. \]

For \( \frac{1}{p} - \frac{1}{q} < \frac{\alpha}{m} = \frac{\beta}{n} \) we have
\[
(8.1) \quad \left( \int_{\mathbb{R}^m \times \mathbb{R}^n} |\triangle_j I_{\alpha, \beta} (1_R \sigma)(x, y)|^q d\omega(x, y) \right)^{\frac{1}{q}} \leq 2^{\varepsilon' j - k} \left( \int_R d\sigma \right)^{\frac{1}{p}},
\]
for all rectangles \( R \) with eccentricity \( \kappa(R) = 2^{-k}, \) and where \( \varepsilon' > 0 \) depends on \( p, q, \alpha, \beta, \varepsilon. \)

Proof. We will prove the special case when \( \ell(I) = 1 \) and \( \kappa(R) = 1, \) i.e. \( R \) is a square of side length 1. The general case is similar. We now place the origin so that \( R \) is the block \( B_0 = [1, 2]^m \times [1, 2]^n. \)

We first consider the region \( R_1 = \{(x, y) : |x| \leq \frac{1}{2} |y| \}. \) In this region the sum over \( j < 0 \) is easy. So we consider \( j > 0. \) To see that (8.1) holds with integration on the left restricted to \( R_1, \) we begin by noting that
\[
\triangle_j I_{\alpha, \beta} (1_R \sigma)(x, y) = \int \int_{R \setminus (S_j + (x, y))} \left( \frac{1}{|x - u|} \right)^{m-\alpha} \left( \frac{1}{|y - t|} \right)^{n-\beta} d\sigma(u,t)
= \sum_{r=1}^{2^j} \int \int_{R(r)} \left( \frac{1}{|x - u|} \right)^{m-\alpha} \left( \frac{1}{|y - t|} \right)^{n-\beta} d\sigma(u,t) 1_{R(r)}(x, y),
\]
where the tiles $R(r)$ are rectangles of size 1 by $2^{-j}$ and the tiles $R^*(r)$ are slightly enlarged reflections of the $R(r)$ centered on the $y$-axis. Now we continue by computing

$$
\int_\mathbb{R} |\triangle_j I_{\alpha, \beta} (1_{R\sigma}) (x, y)|^q \, d\omega (x, y)
$$

$$
= \int_\mathbb{R} \left\{ \sum_{r=1}^{2^j} \int_{R(r)} \left( \frac{1 - |x - u|}{|y - t|} \right)^{m-\alpha} \left( \frac{1}{|y - t|} \right)^{n-\beta} \, d\sigma (u, t) \right\}^q \, d\omega (x, y)
$$

$$
\approx \sum_{r=1}^{2^j} \int_{R^*(r)} \left\{ \int_{R(r)} \left( \frac{1 - |x - u|}{|y - t|} \right)^{m-\alpha} \left( \frac{1}{|y - t|} \right)^{n-\beta} \, d\sigma (u, t) \right\}^q \, d\omega (x, y),
$$

where matters have been reduced to the diagonal terms since $R^*(r) \cap R^*(s) = \emptyset$ unless $|r - s| \leq c_0$ (this follows easily from the fact that the tiles $R(r)$ are pairwise disjoint in $r$). Now we apply Hölder’s inequality to obtain

$$
(8.2) \quad \int_\mathbb{R} \sum_{r=1}^{2^j} \left\{ \int_{R(r)} \left( \frac{1 - |x - u|}{|y - t|} \right)^{m-\alpha} \left( \frac{1}{|y - t|} \right)^{n-\beta} \, d\sigma (u, t) \right\}^{\frac{q}{p}} \, w (x, y)^{\frac{q}{p}} \, dx \, dy
$$

$$
\leq \sum_{r=1}^{2^j} \left( \int_{R(r)} \, d\sigma \right)^{\frac{q}{p}} \left( \int_{R(r)} \left( \frac{1 - |x - u|}{|y - t|} \right)^{(m-\alpha)p'} \left( \frac{1}{|y - t|} \right)^{(n-\beta)p'} \, d\sigma (u, t) \right)^{\frac{q}{p}}
$$

$$
\leq \left( A_{p, q}^{\alpha, \beta} \right)^q \sum_{r=1}^{2^j} \left( \int_{R(r)} \, d\sigma \right)^{\frac{q}{p}} \left( \int_{R(r)} \left( \frac{1 - |x - u|}{|y - t|} \right)^{(m-\alpha)p} \left( \frac{1}{|y - t|} \right)^{(n-\beta)p} \, d\sigma (u, t) \right)^{\frac{q}{p}}
$$

$$
\leq \left( A_{p, q}^{\alpha, \beta} \right)^q \sum_{r=1}^{2^j} \left( \int_{R(r)} \, d\sigma \right)^{\frac{q}{p}} \left( \int_{R(r)} \left( \frac{1 - |x - u|}{|y - t|} \right)^{(m-\alpha)p} \left( \frac{1}{|y - t|} \right)^{(n-\beta)p} \, d\sigma (u, t) \right)^{\frac{q}{p}}
$$

Now we consider the region $R_2 \equiv \{(x, y) : |y| \leq |x| \leq 4 |y|\}$. Here we must perform an additional calculation involving the intersection of the tile $R(r)$ and the cone $S_j + (x, y)$:

$$
\int_\mathbb{R} \left\{ \sum_{r=1}^{2^j} \left( \int_{R(r)} \left( \frac{1 - |x - u|}{|y - t|} \right)^{m-\alpha} \left( \frac{1}{|y - t|} \right)^{n-\beta} \, d\sigma (u, t) \right) \right\} \, d\omega (x, y)
$$

$$
= \int_\mathbb{R} \left\{ \sum_{r=1}^{2^j} \left( \int_{R^*(r) \cap [S_j + (x, y)]} \left( \frac{1 - |x - u|}{|y - t|} \right)^{m-\alpha} \left( \frac{1}{|y - t|} \right)^{n-\beta} \, d\sigma (u, t) \right) \right\} \, d\omega (x, y),
$$

where now $R^*(r)$ can overlap $R(r)$ considerably. However, for each fixed $(x, y) \in R(r)$ we further decompose

$$
R(r) \cap [S_j + (x, y)] = \bigcup_{\ell} R_\ell (r)
$$

where the widths of the $R_\ell (r)$ form a geometric sequence that approaches 0 as $R_\ell (r)$ approaches $(x, y)$, and the dependence of $R_\ell (r)$ on $(x, y)$ is suppressed. Moreover, the tiles $R_\ell (r)$ are roughly rectangles of
dimension $2^{-\ell} \times 2^{-\ell-j}$, and so
\[
\int \int_{R(r) \cap [S_j + (x, y)]} \left( \frac{1}{|x-u|} \right)^{m-\alpha} \left( \frac{1}{|y-t|} \right)^{n-\beta} \, d\sigma(u, t) \lesssim \sum_{\ell=1}^{\infty} \int \int_{R_{\ell}(r)} \left( \frac{1}{2^{\ell-j}} \right)^{m-\alpha} \left( \frac{1}{2^{\ell-j}} \right)^{n-\beta} \, d\sigma(u, t).
\]

\[\approx 2^{j(n-\beta)} \sum_{\ell=1}^{\infty} 2^{j(m-\alpha+\beta)} |R_{\ell}(r)| \lesssim 2^{j(n-\beta)} \sum_{\ell=1}^{\infty} 2^{j(m-\alpha+\beta)} \left( 2^{-\varepsilon(m\ell+n+jn)} |R| \right)\]

\[\lesssim 2^{j(n-\beta-\varepsilon n)} \sum_{\ell=1}^{\infty} 2^{j(m-\alpha+\beta-(m+n)c)} |R| \approx 2^{jn(1-\frac{\beta}{m})} |R|_{\sigma}
\]

provided $1 - \varepsilon < \frac{\alpha}{m} = \frac{\beta}{n}$. Thus we have
\[
\int \int_{R_2} |\triangle_j I_{\alpha, \beta} (1_R \sigma)|^q \, d\omega
\]

\[= \int \int_{R_2} \left\{ \sum_{r=1}^{2^j} \sum_{\ell=1}^{N} \left[ \int \int_{R_{\ell}(r) \cap [S_j + (x, y)]} \left( \frac{1}{|x-u|} \right)^{m-\alpha} \left( \frac{1}{|y-t|} \right)^{n-\beta} \, d\sigma(u, t) \right] 1_{R_{\ell}(r)}(x, y) \right\}^q \, d\omega(x, y)
\]

\[\lesssim \int \int_{R_2} \left\{ \sum_{r=1}^{2^j} \sum_{\ell=1}^{N} \left[ 2^{jn(1-\frac{\alpha}{m}-\varepsilon)} |R|_{\sigma} \right] 1_{R_{\ell}(r)}(x, y) \right\}^q \, d\omega(x, y)
\]

\[\lesssim \sum_{r=1}^{2^j} \int \int_{R_2} \left\{ \sum_{\ell=1}^{N} \left[ 2^{jn(1-\frac{\alpha}{m}-\varepsilon)} |R|_{\sigma} \right] 1_{R_{\ell}(r)}(x, y) \right\}^q \, d\omega(x, y)
\]

since the $R_{\ell}(r)$ are essentially pairwise disjoint in both $r$ and $\ell$ (there is also decay in the kernel in the parameter $\ell$). Now we apply H"older’s inequality and continue as in (8.2) above.

Region $R_3 \equiv \{(x, y) : |x| \leq 4|y|\}$ is handled symmetrically to Region $R_1$, and this completes the proof of Lemma 5.

Now we can easily obtain the testing condition.

**Corollary 2.** Suppose $1 < p < q < \infty$, $0 < \alpha, \beta < 1$ and that both $\sigma$ and $\omega$ are rectangle doubling, and that the reverse doubling exponent $\varepsilon$ for $\sigma$ satisfies

\[1 - \varepsilon < \frac{\alpha}{m} = \frac{\beta}{n}.
\]

For $\frac{1}{p} - \frac{1}{q} < \frac{\alpha}{m} = \frac{\beta}{n}$ we have the testing condition

\[
\left( \int \int_{\mathbb{R}^m \times \mathbb{R}^n} |I_{\alpha, \beta} (1_R \sigma)|^q \, d\omega \right)^\frac{1}{q} \lesssim \ell_{p, q}^{(\alpha, \beta), (m, n)} (\sigma, \omega) \left( \int \int_{R} \, d\sigma \right)^\frac{1}{p}, \quad \text{for all } R = I \times J,
\]

as well as the dual testing condition in which the roles of the measures $\sigma$ and $\omega$ are reversed, and the exponents $p, q$ are replaced with $q', p'$ respectively.
Theorem 13. \[ \left( \int_{\mathbb{R}^m \times \mathbb{R}^n} |I_{\alpha,\beta} (1_{R\sigma})|^q \, d\omega \right)^{1/q} \leq \left( \int_{\mathbb{R}^m \times \mathbb{R}^n} \left( \sum_{\tau \in \mathbb{Z}} |I_{\alpha,\beta} (1_{R\sigma})|^q \right) \, d\omega \right)^{1/q} \]

\[ \leq \sum_{\tau \in \mathbb{Z}} \left( \int_{\mathbb{R}^m \times \mathbb{R}^n} |I_{\alpha,\beta} (1_{R\sigma})|^q \, d\omega \right)^{1/q} \]

\[ \leq \sum_{\tau \in \mathbb{Z}} 2^{-\epsilon |j-k|} \left( \int_{R} \, d\sigma \right)^{1/p} = C_{\epsilon} \left( \int_{R} \, d\sigma \right)^{1/p}. \]

\[ \] \thefootnote

9. Appendix

9.1. Sharpness in the Muckenhoupt-Wheeden theorem. Here we give a sharp form of the Muckenhoupt-Wheeden Theorem \footnote{5}

Theorem 13. Let \(0 < \alpha < m\), \(1 < p, q < \infty\), and let \(w(x)\) be a nonnegative weight on \(\mathbb{R}^m\). Then

\[ \left\{ \int_{\mathbb{R}^m} I_{\alpha}^m f(x)^q \ w(x)^q \, dx \right\}^{1/q} \leq N_{\alpha,m}^{\alpha,m}(w) \left\{ \int_{\mathbb{R}^m} f(x)^p \ w(x)^p \, dx \right\}^{1/p} \]

for all \(f \geq 0\) and \(N_{\alpha,m}^{\alpha,m}(w) < \infty\) if and only if both

\[ (9.2) \quad \frac{1}{p} - \frac{1}{q} = \frac{\alpha}{m}, \]

and

\[ A_{p,q}(w) \equiv \sup_{\text{cubes } I \subset \mathbb{R}^m} \left( \frac{1}{|I|} \int_I w(x)^q \, dx \right)^{1/q} \left( \frac{1}{|I|} \int_I w(x)^{-p'} \, dx \right)^{1/p'} < \infty. \]

Proof. Given Theorem \footnote{1} it remains only to prove that \((9.2)\) is necessary for \((9.1)\). To see this, we apply Hölder’s inequality with dual exponents \(\frac{\alpha+1}{p'}\) and \(p' + 1\) to obtain

\[ 1 = \left\{ \frac{1}{|I|} \int_I w^{\frac{\alpha+1}{p'}} w^{-\frac{\alpha+1}{p'}} \right\}^{\frac{\alpha+1}{p'}} \leq \left\{ \left( \frac{1}{|I|} \int_I w \right)^{\frac{\alpha+1}{p'}} \left( \frac{1}{|I|} \int_I w^{-p'} \right)^{\frac{\alpha+1}{p'}} \right\}^{\frac{\alpha+1}{p'}} = \left( \frac{1}{|I|} \int_I w \right)^{\frac{\alpha+1}{p'}} \left( \frac{1}{|I|} \int_I w^{-p'} \right)^{\frac{\alpha+1}{p'}}. \]

Then we conclude that

\[ |I|^{\frac{m-1}{m} + \frac{1}{p}} \leq |I|^{\frac{m-1}{m} + \frac{1}{p'}} \left( \frac{1}{|I|} \int_I w^q \right)^{\frac{1}{q}} \left( \frac{1}{|I|} \int_I w^{-p'} \right)^{\frac{1}{p'}} \leq A_{p,q}(w) \leq N_{\alpha,m}^{\alpha,m}(w) \]

for all cubes \(I\), which implies \(\frac{\alpha}{m} - 1 + \frac{1}{q} + \frac{1}{p'} = 0\) as required. \]
9.2. Sharpness in the Stein-Weiss theorem. Here we give a sharp form of the Stein-Weiss Theorem.  

**Theorem 14.** Let 

$$-\infty < \alpha, \beta, \gamma, \delta < \infty,$$

$$1 < p, q < \infty,$$

$$m \in \mathbb{N}.$$ 

Then the power weighted norm inequality 

$$\left\{ \int_{\mathbb{R}^m} I_\alpha^m f(x)^q |x|^{-\gamma q} \, dx \right\}^{\frac{1}{q}} \leq N_{p,q}(w, v) \left\{ \int_{\mathbb{R}^m} f(x)^p |x|^\delta p \, dx \right\}^{\frac{1}{p}},$$

holds if and only if the power weight equality 

$$\frac{1}{p} - \frac{1}{q} = \frac{\alpha - (\gamma + \delta)}{m}$$

holds along with the constraint inequalities, 

(9.5) \(0 < \alpha < m,\)

(9.6) \(p \leq q,\)

(9.7) \(q \gamma < m,\)

(9.8) \(p' \delta < m,\)

and 

(9.9) \(\gamma + \delta \geq 0.\)

**Proof.** The sufficiency of these conditions for (9.3) is the classical theorem of Stein and Weiss, so we turn to proving their necessity. The required local integrability of the power weights \(|x|^{-\gamma q}\) and \(|x|^{-\delta p'}\) shows that (9.7) and (9.8) hold. The kernel \(|x - y|^{\alpha - m}\) of the convolution operator must be locally integrable on \(\mathbb{R}^m\) and this implies that \(0 < \alpha.\) Using this, we can now use the argument in the proof of Proposition 2 to prove that finiteness of the Muckenhoupt characteristic \(A_{p,q}(w, v)\) is necessary, and this in turn implies both (9.9) and the power weight equality (9.4) just as in the proof of Theorem 8 above for the 2-parameter case.

From (9.4), (9.5) and (9.9) we now obtain 

$$\alpha = m \left( \frac{1}{p} - \frac{1}{q} \right) + (\gamma + \delta) < m \left( \frac{1}{p} - \frac{1}{q} \right) + \left( \frac{m}{q} + \frac{m}{p'} \right) = m,$$

which completes the proof that (9.5) holds.

Finally we turn to proving (9.6). Let \(f(y) = f(s),\) \(s = |y|,\) be a radial function on \(\mathbb{R}^m.\) Then \(I_\alpha f(x) = I_\alpha f(r), r = |x|,\) is also radial and 

$$I_\alpha f(x) = \int_{\mathbb{R}^m} |x - y|^{\alpha - m} f(y) \, dy \gtrsim \int_{|y| \leq |x|} |x|^{\alpha - m} f(y) \, dy = r^{\alpha - m} \int_0^r f(s) s^{m-1} \, ds.$$ 

Now suppose, in order to derive a contradiction, that (9.3) holds. Then we have 

$$\left\{ \int_0^\infty \left( \int_0^r f(s) s^{m-1} \, ds \right)^q \, r^{(\alpha - m)q - \gamma q} \, r^{n-1} \, dr \right\}^{\frac{1}{q}} \lesssim \left\{ \int_{\mathbb{R}^m} I_\alpha^m f(x)^q |x|^{-\gamma q} \, dx \right\}^{\frac{1}{q}} \leq N_{p,q}(w, v) \left\{ \int_{\mathbb{R}^m} f(x)^p |x|^\delta p \, dx \right\}^{\frac{1}{p}} = N_{p,q}(w, v) \left\{ \int_0^\infty f(s) s^{\delta p + m-1} \, ds \right\}^{\frac{1}{p}}$$

for all \(f \geq 0.\) With \(g(s) \equiv f(s) s^{m-1},\) this last inequality can be rewritten as 

$$\left\{ \int_0^\infty \left( \int_0^r g(s) \, ds \right)^q \, v(r) \, dr \right\}^{\frac{1}{q}} \lesssim N_{p,q}(w, v) \left\{ \int_0^\infty g(s) u(s) \, ds \right\}^{\frac{1}{p}}.$$
for all \( g \geq 0 \), and where the weights are given by \( v(r) = r^{(\alpha - m)q} - \gamma + n - 1 \) and \( u(s) = s^{\delta p} \). By a result of Maz'ja [Maz], the two weight Hardy inequality (9.10) with \( q < p \) holds if and only if

\[
\int_0^\infty \left( \left( \int_0^r u^{1-p'} \right)^{\frac{1}{p'}} \left( \int_r^\infty v \right)^{\frac{1}{p}} \right)^{\rho} v(r) \, dr < \infty,
\]

where \( \frac{1}{\rho} = \frac{1}{q} - \frac{1}{p'} > 0 \). But since the weights \( u \) and \( v \) are power functions, the integrand above is also a power function, and hence cannot belong to any Lebesgue space \( L^\rho(0, \infty) \), thus providing the required contradiction. \( \square \)

9.3. Optimal powers of Muckenhoupt characteristics. Recall the one parameter two-tailed characteristic \( \hat{A}_{p,q} \) given above by

\[
\hat{A}_{p,q}(v, w) \equiv \sup_{Q \subset \mathbb{R}^n} \left( \frac{1}{|Q|} \int_{\mathbb{R}^n} \left| \hat{s}_Q(x) w(x) \right|^q \, dx \right)^{\frac{1}{q}} \left( \frac{1}{|Q|} \int_{\mathbb{R}^n} \left| \hat{s}_Q(x) v(x)^{-1} \right|^{p'} \, dx \right)^{\frac{1}{p'}}
\]

where

\[
\hat{s}_Q(x) = \left(1 + \frac{|x - c_Q|}{|Q|^{\frac{1}{n}}} \right)^{-\alpha - n}, \quad c_Q \text{ is the center of } Q,
\]

and

\[
\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n},
\]

i.e.

\[
\frac{1}{q} + \frac{1}{p'} = 1 - \frac{\alpha}{n}.
\]

From Theorem 2 we know that the characteristic \( \hat{A}_{p,q}(w, v) \) is finite if and only if the following norm inequality for the fractional integral \( I_{\alpha}^v \) holds:

\[
\left\{ \int_{\mathbb{R}^n} I_{\alpha}^v f(x)^q \, w(x)^{\frac{1}{n}} \, dx \right\}^{\frac{1}{q}} \leq C_{p,q}(v, w) \left\{ \int_{\mathbb{R}^n} f(x)^p \, v(x)^{\frac{1}{n}} \, dx \right\}^{\frac{1}{p}}.
\]

Moreover, it is claimed there that

\[
C_{p,q}(v, w) \approx \hat{A}_{p,q}(v, w),
\]

and this equivalence can be verified by carefully tracking the constants in the proof of Theorem 1 in [SaWh]. We also have the same consequence for the one-tailed characteristic.

Porism: The same arguments as used in the proof of Theorem 1 in [SaWh] also prove that

\[
C_{\alpha,\beta}(v, w) \approx \overline{A}_{p,q}(v, w),
\]

where \( \overline{A}_{p,q} \) is the one-tailed characteristic,

\[
\overline{A}_{p,q}(v, w) = \sup_{Q \subset \mathbb{R}^n} \left( \frac{1}{|Q|} \int_{\mathbb{R}^n} \hat{s}_Q(x)^{\frac{1}{n}} \hat{s}_Q^{p'} w^{\frac{1}{n}} \, dx \right)^{\frac{1}{p'}} + \sup_{Q \subset \mathbb{R}^n} \left( \frac{1}{|Q|} \int_{\mathbb{R}^n} \hat{s}_Q^{q} w^{\frac{1}{n}} \right)^{\frac{1}{q}} \left( \frac{1}{|Q|} \int_{\mathbb{R}^n} v^{-p'} \, dx \right)^{\frac{1}{p'}}.
\]

9.3.1. Comparison with the inequality of Lacey, Moen, Pérez and Torres. Here we give a simple and instructive proof of the ‘A\(_1\) conjecture’ in the setting of fractional integrals. Recall that from [LaMoPeTo] we have the estimate

\[
C_{p,q}(w) \lesssim A_{p,q}(w)^{1 + \max\left\{ \frac{q}{1+q}, \frac{1}{p'} \right\}},
\]

and if we restrict the two weight result above to the case \( w = v \), we have the equivalence

\[
C_{\alpha,\beta}(w, w) \approx \hat{A}_{p,q}(w, w).
\]

Thus the estimate (9.13) is equivalent to

\[
\hat{A}_{p,q}(w) \lesssim A_{p,q}(w)^{\rho},
\]

\[
\hat{A}_{p,q}(w) \lesssim A_{p,q}(w)^{\rho}.
\]
and also equivalent to

\[
\|p,q\| \lesssim A_{p,q}(w)^{\rho},
\]

where \(\rho \equiv 1 + \max \left\{ \frac{p}{q}, \frac{p}{q'} \right\} \). Written out in full, one half of inequality (9.13) is

\[
\sup_{Q \subset \mathbb{R}^n} \left( \frac{1}{|Q|} \int_{\mathbb{R}^n} w_{Q}^q \right)^{\frac{1}{q}} \left( \frac{1}{|Q|} \int_{Q} w^{-p'} \right)^{\frac{1}{p'}}
\]

\[
\lesssim \left\{ \sup_{Q \subset \mathbb{R}^n} \left( \frac{1}{|Q|} \int_{Q} w^q \right)^{\frac{1}{q}} \left( \frac{1}{|Q|} \int_{Q} w^{-p'} \right)^{\frac{1}{p'}} \right\}^{\rho}.
\]

**Claim 2.** The inequality (9.15) holds directly, without any reference to norm inequalities at all.

**Proof.** Take the \(q\)th power of the left hand side of (9.10) and fix a cube \(Q\) which comes close to achieving the supremum over all cubes. Then we write

\[
\left( \frac{1}{|Q|} \int_{Q} w^q \right)^{\frac{1}{q}} \left( \frac{1}{|Q|} \int_{Q} w^{-p'} \right)^{\frac{1}{p'}} \lesssim \left( \frac{1}{|Q|} \int_{Q} w^{-p'} \right)^{\frac{1}{p'}} \sum_{k=0}^{\infty} 2^{k(q(a-n)q + n)} \frac{1}{|2^k Q|} \int_{2^k Q} w^q
\]

\[
\lesssim \left( \frac{1}{|Q|} \int_{Q} w^{-p'} \right)^{\frac{1}{p'}} \sum_{k=0}^{\infty} 2^{k(a-n)q + n} A_{p,q}(w)^q \left( \frac{1}{|2^k Q|} \int_{2^k Q} w^{-p'} \right)^{\frac{1}{p'}}
\]

\[
= A_{p,q}(w)^q \sum_{k=0}^{\infty} \left( \frac{|Q|}{|2^k Q|} \right)^{\frac{q}{2}} 2^{-kq} \lesssim A_{p,q}(w)^q \sum_{k=0}^{\infty} (2^{-k\delta})^\frac{q}{p'}
\]

\[
\lesssim A_{p,q}(w)^q \frac{1}{1 - 2^{-\delta \frac{q}{p'}}} \approx A_{p,q}(w)^q \frac{1}{\delta},
\]

where \(\delta\) is the reverse doubling exponent for the \(A_p\) weight \(w^{-p'}\),

\[|Q|_{w^{-p'}} \leq C 2^{-k\delta} |2^k Q|_{w^{-p'}} \text{ for all cubes } Q.\]

Now we claim that the reverse doubling exponents \(\delta = \delta \left( w^{-p'} \right) \) and \(\delta \left( w^q \right) \) for the weights \(w^{-p'}\) and \(w^q\) satisfy

\[
\frac{1}{\delta \left( w^{-p'} \right)} \leq C_{p,q,n} A_{p,q}(w)^{p'} \text{ and } \frac{1}{\delta \left( w^q \right)} \leq C_{p,q,n} A_{p,q}(w)^q.
\]

Indeed, we have for all \(f \geq 0\) and any \(0 < \varepsilon < 1\),

\[
\left( \frac{1}{|Q|} \int_{Q} f \right) \left( \frac{1}{|Q|} \int_{Q} w^q \right)^{\frac{1}{q}}
\]

\[
= \left( \frac{1}{|Q|} \int_{Q} (f^{\varepsilon} w) (f^{1-\varepsilon}) (w^{-1}) \right) \left( \frac{1}{|Q|} \int_{Q} w^q \right)^{\frac{1}{q}}
\]

\[
\leq \left( \frac{1}{|Q|} \int_{Q} (f^{\varepsilon} w)^q \right)^{\frac{1}{q}} \left( \frac{1}{|Q|} \int_{Q} (f^{1-\varepsilon})^{\frac{q}{p'}} \right)^{\frac{1}{p'}} \left( \frac{1}{|Q|} \int_{Q} w^{-p'} \right)^{\frac{1}{p'}} \left( \frac{1}{|Q|} \int_{Q} w^q \right)^{\frac{1}{q}}
\]

\[
= \left( \frac{1}{|Q|} \int_{Q} w^q \right)^{\frac{1}{q}} \left( \frac{1}{|Q|} \int_{Q} f^{\varepsilon q} w^{q} \right)^{\frac{1}{p'}} \left( \frac{1}{|Q|} \int_{Q} f^{(1-\varepsilon)q} w^{q} \right)^{\frac{1}{p'}} \left( \frac{1}{|Q|} \int_{Q} f^{(1-\varepsilon)\frac{q}{p'}} \right)^{\frac{1}{p'}}
\]

\[
\leq A_{p,q}(w) \left( \frac{1}{|Q|} \int_{Q} f^{\varepsilon q} w^q \right)^{\frac{1}{p'}} \left( \frac{1}{|Q|} \int_{Q} f^{(1-\varepsilon)\frac{q}{p'}} \right)^{\frac{1}{p'}}.
\]
and now plugging in \( f = 1_{\hat{s}Q} \) we get

\[
\left( \frac{|\hat{s}Q|}{|Q|} \right) \left( \int_Q w^q \right)^{\frac{1}{p}} \leq A_{p,q}(w) \left( \int_{\hat{s}Q} w^q \right)^{\frac{1}{p}} \left( \frac{|\hat{s}Q|}{|Q|} \right)^{\frac{1}{p} - \frac{1}{q}};
\]

\[
\frac{|Q|_{w^q}}{|\hat{s}Q|_{w^q}} \leq 3^{n \left( 1 + \frac{1}{p} \right)} A_{p,q}(w)^q.
\]

With \( \ell(Q) \) denoting the side length of \( Q \), we have

\[
|3Q|_{w^q} \leq \sum_{\alpha \in \{-1,0,1\}^n \setminus \{0,\ldots,0\}} |3(Q + \ell(Q)e_1)|_{w^q}
\]

\[
\leq \sum_{\alpha \in \{-1,0,1\}^n \setminus \{0,\ldots,0\}} 3^n \left( 1 + \frac{2}{p} \right) A_{p,q}(w)^q |Q + \ell(Q)e_1|_{w^q}
\]

\[
= 3^n \left( 1 + \frac{2}{p} \right) A_{p,q}(w)^q |3Q \setminus Q|_{w^q},
\]

and so

\[
\frac{|Q|_{w^q}}{|3Q|_{w^q}} = \frac{|3Q|_{w^q} - |3Q \setminus Q|_{w^q}}{|3Q|_{w^q}} \leq 1 - \frac{1}{3^n \left( 1 + \frac{2}{p} \right) A_{p,q}(w)^q} \equiv \gamma \in \left( 1 - 3^{-n \left( 1 + \frac{2}{p} \right)}, 1 \right).
\]

Iterating, we get

\[
\frac{|\frac{1}{3^n} Q|_{w^q}}{|\frac{1}{3^n} Q|_{w^q}} = \frac{|\frac{1}{3^n} Q|_{w^q} - |\frac{1}{3^n} Q \setminus \frac{1}{3^n} Q|_{w^q}}{|\frac{1}{3^n} Q|_{w^q}} \leq \gamma^k,
\]

from which we obtain

\[
\frac{|\frac{1}{3^n} Q|_{w^q}}{|\frac{1}{3^n} Q|_{w^q}} \leq C_{\gamma^{\ln \frac{1}{m^n} k}} = C2^{\ln \frac{1}{m^n} k} = C2^{-k \delta},
\]

where

\[
\delta = \delta(w^q) = \frac{\ln \frac{1}{q}}{\ln 3} = \frac{1}{\ln 3} \ln \frac{1}{3^n \left( 1 + \frac{2}{p} \right) A_{p,q}(w)^q} \approx \frac{1}{A_{p,q}(w)^q}.
\]

This proves the second assertion in (9.18), and the proof of the first assertion is similar.

Thus from (9.17) we obtain

\[
\overline{A}_{p,q}(w) = \sup_Q \left( \frac{1}{|Q|} \int_{\mathbb{R}^n} \hat{s}_Q w^q \right)^{\frac{1}{p}} \left( \frac{1}{|Q|} \int_Q w^{-p'} \right)^{\frac{1}{p'}} \leq C_{p,q,n} A_{p,q}(w) \left( \frac{1}{\delta(w^{-p'})} \right)^{\frac{1}{p'}} \leq C_{p,q,n} A_{p,q}(w)^{1 + \frac{\omega'}{q'}}.
\]

Similarly we obtain that the expression

\[
\sup_{Q \subset \mathbb{R}^n} \left( \frac{1}{|Q|} \int_{\mathbb{R}^n} w^q \right)^{\frac{1}{p}} \left( \frac{1}{|Q|} \int_Q \hat{s}_Q w^{-1}^{p'} \right)^{\frac{1}{p'}} \leq C_{q',p',n} A_{q',p'}(w^{-1})^{1 + \frac{\omega'}{q'}} = C_{q',p',n} A_{p,q}(w)^{1 + \frac{\omega'}{q'}}.
\]

This completes the proof of the claim.

Thus we have given in the Claim above a simple direct proof of the following theorem using the proof of Theorem 1 in [SaWh].
Theorem 15. (Lacey, Moen, Pérez and Torres) With $p, q, n$ as above we have

$$C_{p,q}(w) \leq C_{p,q,n} A_{p,q}(w)^{1 + \max\left\{ \frac{m}{p}, \frac{n}{q} \right\}}.$$  

9.4. The product fractional integral, the Dirac mass and a modified example. Here we consider both the two weight norm inequality \eqref{1-(3.3)} and the two-tailed characteristic \eqref{4.5} in the special case when $\sigma = \delta_{(0,0)}$, and show that they are equivalent in this case. When $\sigma = \delta_{(0,0)}$ the two weight norm inequality \eqref{1-(3.3)} yields

$$\mathbb{N}_{\alpha,\beta,\mu,\nu}^{(m,n)}(\delta_{(0,0)}, \omega) = \left\{ \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} \mathbb{I}_{\alpha,\beta}^{(m,n)} \delta_{(0,0)}(x,y)^{q} \, d\omega(x,y) \right\}^{\frac{1}{q}},$$

and since $\tilde{s}_{I \times J}(x,y) \equiv \left( 1 + \frac{|x-c_{I}|}{|I|^\frac{1}{m}} \right)^{\alpha-m} \left( 1 + \frac{|y-c_{J}|}{|J|^\frac{1}{n}} \right)^{\beta-n}$, \eqref{4.5} yields

$$\tilde{\mathbb{N}}_{\alpha,\beta,\mu,\nu}^{(m,n)}(\delta_{(0,0)}, \omega) \equiv \sup_{I \times J \subset \mathbb{R}^m \times \mathbb{R}^n} \left[ \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} \left( 1 + \frac{|x-c_{I}|}{|I|^\frac{1}{m}} \right)^{(\alpha-m)q} \left( 1 + \frac{|y-c_{J}|}{|J|^\frac{1}{n}} \right)^{(\beta-n)q} \, d\omega(x,y) \right]^{\frac{1}{q}} \equiv \sup_{I \times J \subset \mathbb{R}^m \times \mathbb{R}^n} \left( \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} \left( |I|^\frac{1}{m} + |x-c_{I}| \right)^{(\alpha-m)q} \left( 1 + \frac{|c_{I}|}{|I|^\frac{1}{m}} \right)^{(\beta-n)q} \, d\omega(x,y) \right) \equiv \mathbb{N}_{\alpha,\beta,\mu,\nu}^{(m,n)}(\delta_{(0,0)}, \omega).$$

If $I \times J$ has center $(c_{I}, c_{J}) = (0, 0)$, then this supremum is equal to or greater than

$$\left( \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} |x|^{(\alpha-m)q} |y|^{(\beta-n)q} \, d\omega(x,y) \right)^{\frac{1}{q}} = \mathbb{N}_{\alpha,\beta,\mu,\nu}^{(m,n)}(\delta_{(0,0)}, \omega).$$

Thus from Lemma 3 we conclude that

$$\mathbb{N}_{\alpha,\beta,\mu,\nu}^{(m,n)}(\delta_{(0,0)}, \omega) \approx \tilde{\mathbb{N}}_{\alpha,\beta,\mu,\nu}^{(m,n)}(\delta_{(0,0)}, \omega).$$

9.5. Reverse doubling and Muckenhoupt type conditions. Recall the reverse doubling condition with exponent $\epsilon > 0$, and rectangle reverse doubling condition with exponent pair $(\epsilon^1, \epsilon^2)$.

Definition 3. We say that a measure $\mu$ on $\mathbb{R}^m$ satisfies the reverse doubling condition $(R_{\epsilon^1} D_{\epsilon^2})$ with exponent $\epsilon > 0$ if

$$|sI|_{\mu} \leq C_{\epsilon^m} |I|_{\mu}, \quad 0 < s < 1,$$

for some constant $C$; and we say that a measure $\mu$ on $\mathbb{R}^m \times \mathbb{R}^n$ satisfies the rectangle reverse doubling condition $(R_{\epsilon^1} R_{\epsilon^2} D_{\epsilon^2})$ with exponent pair $(\epsilon^1, \epsilon^2)$ if

$$|sI \times tJ|_{\mu} \leq C_{\epsilon^m \epsilon^2} |I \times J|_{\mu}, \quad 0 < s, t < 1,$$

for some constant $C$. If $\epsilon = \epsilon^1 = \epsilon^2$, then we say simply that $\epsilon$ is the reverse doubling exponent for $\mu$.

Note that unlike doubling measures, nontrivial reverse doubling measures can vanish on open subsets. In particular, Cantor measures are typically reverse doubling, while never doubling. If both weights are reverse doubling, then the two-tailed, one-tailed, and no-tailed Muckenhoupt conditions are all equivalent.
Lemma 6. Suppose that $\sigma$ and $\omega$ are reverse rectangle doubling measures on $\mathbb{R}^n$, and that

$$0 < \alpha < m, \ 0 < \beta < n, \ 1 < p, q < \infty.$$ 

Then we have the following equivalence:

$$\mathcal{K}_{p,q}^{(\alpha, \beta), (m,n)} (\sigma, \omega) \approx \mathcal{K}_{p,q}^{(\alpha, \beta), (m,n)} (\sigma, \omega) \approx \mathcal{K}_{p,q}^{(\alpha, \beta), (m,n)} (\sigma, \omega).$$

Proof. We first consider the one-parameter equivalence

$$\mathcal{K}_{p,q}^{(\alpha, \beta)} (\sigma, \omega) \approx \mathcal{K}_{p,q}^{(\alpha, \beta)} (\sigma, \omega) \approx \mathcal{K}_{p,q}^{(\alpha, \beta)} (\sigma, \omega),$$

since the extension to two parameters will follow the same proof with obvious modifications. Since $\mathcal{K}_{p,q}^{(\alpha, \beta)} (\sigma, \omega) \leq \mathcal{K}_{p,q}^{(\alpha, \beta)} (\sigma, \omega)$, it suffices to prove

$$\mathcal{K}_{p,q}^{(\alpha, \beta)} (\sigma, \omega) \leq \mathcal{K}_{p,q}^{(\alpha, \beta)} (\sigma, \omega),$$

and we begin with the second inequality $\mathcal{K}_{p,q}^{(\alpha, \beta)} (\sigma, \omega) \leq \mathcal{K}_{p,q}^{(\alpha, \beta)} (\sigma, \omega)$ in (9.20).

So fix a cube $I$ which comes close to achieving the supremum in $\mathcal{K}_{p,q}^{(\alpha, \beta)} (\sigma, \omega)$ over all cubes, and suppose the tail $\tilde{s}_I$ occurs in the factor for $\omega$. We have

$$\sum_{k=0}^{\infty} 2^k (\alpha-m)|q+m| \frac{1}{|2^k I|} \int_{2^k I} d\omega (x)$$

$$= \sum_{k=0}^{\infty} 2^k (\alpha-m)|q+m| \frac{1}{|2^k I|} \sum_{\ell=0}^{k} \int_{2^\ell I \setminus 2^{\ell-1} I} d\omega (x) = \sum_{\ell=0}^{k} \left( \sum_{k=0}^{\infty} 2^k (\alpha-m)|q+m| \right) \int_{2^\ell I \setminus 2^{\ell-1} I} d\omega (x)$$

$$\approx \sum_{\ell=0}^{k} 2^\ell (\alpha-m)|q+m| \int_{2^\ell I \setminus 2^{\ell-1} I} d\omega (x) \approx \frac{1}{|I|} \int_{\mathbb{R}^m} \left( 1 + \frac{|x-c|}{|I|^{\frac{1}{m}}} \right)^{(\alpha-m)|q+m|} d\omega (x) = \frac{1}{|I|} \int_{\mathbb{R}^m} \tilde{s}_I^\ell d\omega,$$

since $\alpha < m$. Then we write

$$\mathcal{K}_{p,q}^{(\alpha, \beta)} (\sigma, \omega) \approx \left\{ \left| I \right|^\alpha \left( \int_{\mathbb{R}^m} \tilde{s}_I^\ell d\omega \right)^{\frac{1}{\alpha}} \right\}^q$$

$$\approx \left| I \right|^\alpha \frac{1}{|I|} \sum_{k=0}^{\infty} 2^k (\alpha-m)|q+m| \left( \frac{1}{|2^k I|} \int_{2^k I} d\omega \right) \left( \frac{1}{|I|} \int_{I} d\sigma \right)^\frac{1}{\alpha}$$

$$\lesssim \left| I \right|^\alpha \frac{1}{|I|} \sum_{k=0}^{\infty} 2^k (\alpha-m)|q+m| \left( \frac{1}{|2^k I|} \int_{2^k I} d\omega \right) \left( \frac{1}{|I|} \int_{I} C^{2-k\delta} |2^k I|^{\frac{1}{\sigma}} \right)$$

$$= \sum_{k=0}^{\infty} 2^k (\alpha-m)|q+m| \left( \frac{1}{|2^k I|} \int_{2^k I} d\omega \right) \left( \frac{1}{|I|} \int_{I} C^{2-k\delta} |2^k I|^{\frac{1}{\sigma}} \right)$$

$$= \sum_{k=0}^{\infty} C^{2-k\delta} 2^k (\alpha-m)|q+m| \left( \frac{1}{|2^k I|} \int_{2^k I} d\omega \right) \left( \frac{1}{|I|} \int_{I} C^{2-k\delta} |2^k I|^{\frac{1}{\sigma}} \right)$$

where $\varepsilon > 0$ is the reverse doubling exponent for the weight $\sigma$,

$$|Q|_\sigma \leq C^{2-k\delta} |2^k Q|_\sigma$$

for all cubes $Q$. We then dominate the term in braces by $\mathcal{K}_{p,q}^{(\alpha, \beta)} (\sigma, \omega)$ to obtain that

$$\mathcal{K}_{p,q}^{(\alpha, \beta)} (\sigma, \omega)^q \lesssim \mathcal{K}_{p,q}^{(\alpha, \beta)} (\sigma, \omega)^q \sum_{k=0}^{\infty} 2^{-k\delta} \lesssim \mathcal{K}_{p,q}^{(\alpha, \beta)} (\sigma, \omega)^q,$$

since $-k (\alpha-m)|q| + k [(\alpha-m)|q| + m] + km \frac{1}{\sigma} = 0$, and this completes the proof of the second inequality in (9.20). Note that if the cube $I$ which comes close to achieving the supremum in $\mathcal{K}_{p,q}^{(\alpha, \beta)} (\sigma, \omega)$ over all cubes has the tail $\tilde{s}_I$ occurring in the factor for $\sigma$, then the above argument applies using reverse doubling for the weight $\omega$. 

Now we consider the first inequality in (9.20). Here we can use reverse doubling for either $\sigma$ or $\omega$, and we will use a decomposition that uses reverse doubling for $\omega$ with exponent $\varepsilon > 0$, i.e. $|Q|_\omega \leq C 2^{-km\varepsilon} |2^k Q|_\omega$ for all cubes $Q$. This then has the analogous consequence for the $\omega$-integrals with tails:

$$\int_{R^m} \tilde{s}_I^q d\omega \approx \sum_{k=0}^\infty 2^k [(\alpha-m)q] \left( \int_{2^k I} d\omega \right) \leq \sum_{k=0}^\infty 2^k [(\alpha-m)q] \left( C 2^{-\ell \varepsilon m} \int_{2^k I} d\omega \right) \approx C 2^{-\ell \varepsilon m} \int_{R^m} \tilde{s}_I^q d\omega .$$

Now let $I$ be a cube which comes close to achieving the supremum in $\tilde{K}_{p,q}^{\alpha,m}(\sigma,\omega)$ over all cubes. Then we have

$$\tilde{K}_{p,q}^{\alpha,m}(\sigma,\omega)^{p'} \approx \left\{ |I|^{\frac{\ell - 1}{p'}} \left( \frac{1}{|I|} \int_{R^m} \tilde{s}_I^q d\sigma \right) \right\}^{p'} \approx |I|^{\frac{\ell - 1}{p'}} \left( \frac{1}{|I|} \int_{R^m} \tilde{s}_I^q d\sigma \right) \sum_{\ell=0}^\infty 2^\ell [(\alpha-m)p' + m] \left( \frac{1}{|2^\ell I|} \int_{2^\ell I} d\sigma \right) \approx |I|^{\frac{\ell - 1}{p'}} \left( \frac{1}{|2^\ell I|} \int_{2^\ell I} d\sigma \right) \sum_{\ell=0}^\infty C 2^{-\ell \varepsilon m} 2^{-\ell (\alpha - m \Gamma)} \tilde{s}_I^q \left( \frac{1}{|2^\ell I|} \int_{2^\ell I} d\sigma \right) .$$

We then dominate the term in braces by $\tilde{K}_{p,q}^{\alpha,m}(\sigma,\omega)^{p'}$ to obtain that

$$\tilde{K}_{p,q}^{\alpha,m}(\sigma,\omega)^{p'} \lesssim \tilde{K}_{p,q}^{\alpha,m}(\sigma,\omega)^{p'} \sum_{\ell=0}^\infty 2^{-\ell \varepsilon m} \lesssim \tilde{K}_{p,q}^{\alpha,m}(\sigma,\omega)^{p'},$$

since $-\ell (\alpha - m \Gamma) p' + \ell [(\alpha-m)p' + m] + \ell m \frac{q}{q'} = 0$, and this completes the proof of the first inequality in (9.20), which in turn completes the proof of the one parameter equivalence (9.19).

In order to prove the two parameter equivalence in Lemma 6 we begin with

$$\sum_{k,j=0}^\infty 2^k [(\alpha-m)q + j(\beta-n)q + n] \frac{1}{|2^k I \times 2^j J|} \int_{2^k I \times 2^j J} d\omega (x) \approx \frac{1}{|I \times J|} \int_{R^m \times R^n} \tilde{s}_I^q d\omega ,$$

and proceed with the two parameter analogue of (9.21), followed by the straightforward modifications of the remaining arguments. This completes our proof of Lemma 6. □

9.6. Proof of Theorem 8. Let $I = \prod_{i=1}^m [a_i, a_i + s]$ and $J = \prod_{j=1}^n [b_j, b_j + t]$ be cubes in $\mathbb{R}^m_+$ and $\mathbb{R}^n_+$ with side lengths $s > 0$ and $t > 0$ respectively. Then the local characteristic $A_{p,q}^{(\alpha,\beta),(m,n)}(v,w) [I,J]$ is given by

$$A_{p,q}^{(\alpha,\beta),(m,n)}(v,w) [I,J] = s^{\alpha-m} t^{\beta-n} \left( \int_{[a,a+s]^m} \int_{[b,b+t]^n} |(x,y)|^{-\gamma q} dx dy \right)^{\frac{1}{\gamma q}} \left( \int_{[a,a+s]^m} \int_{[b,b+t]^n} |(x,y)|^{-\delta p'} dx dy \right)^{\frac{1}{\delta p'}} = s^{\alpha-m} t^{\beta-n} T_{p,q}^{m,n} (-\gamma q) \frac{1}{\gamma q} + T_{p,q}^{m,n} (-\delta p') \frac{1}{\delta p'} ,$$
Recall that the corresponding sans serif local characteristic is given by

\[ \text{by the supremum of the sans serif local characteristics} \]

\[ A \leq \int_{[a_i, a_i + s]} \int_{[b_j, b_j + t]} \left| (x, y) \right|^\gamma dx dy = \int_{[a_i, a_i + s]} \int_{[b_j, b_j + t]} \left( |x|^2 + |y|^2 \right)^{\frac{\gamma}{2}} dx dy \]

\[ \approx \int_{[a_i, a_i + s]} \int_{[b_j, b_j + t]} \left( x_1 + \ldots + x_m + y_1 + \ldots + y_n \right)^\gamma dx dy \equiv I^{m,n}_{(a,b),(s,t)} (\eta), \]

for \((a, b) \in \mathbb{R}_+^m \times \mathbb{R}_+^n\). Two immediate necessary conditions for the finiteness of \(A^{(\alpha, \beta), (m,n)}_{p,q} (v, w)\) are the local integrability \(1.16\) of the weights \(w^\alpha\) and \(v^{-p'}\), namely \(\gamma q < m + n\) and \(\delta p' < m + n\). Define the sans serif local characteristic \(A^{(\alpha, \beta), (m,n)}_{p,q} (v, w) [I, J]\)

\[ A^{(\alpha, \beta), (m,n)}_{p,q} (v, w) [I, J] \equiv s^{m,n}_{(a_i, b_j); (s,t)} (-\gamma q) \left[ \left| \frac{m,n}{(a,b); (s,t)} (\alpha, \beta) \right| (-\delta p') \right]^{\frac{\gamma}{p'}}, \]

\[ \text{for } (a, b) \in \mathbb{R}_+^m \times \mathbb{R}_+^n. \]

Symmetry considerations show that the characteristic \(A^{(\alpha, \beta), (m,n)}_{p,q} (v, w)\) satisfies

\[ A^{(\alpha, \beta), (m,n)}_{p,q} (v, w) \equiv \sup_{I \times J \subset \mathbb{R}_+^m \times \mathbb{R}_+^n} A^{(\alpha, \beta), (m,n)}_{p,q} (v, w) [I, J] \approx \sup_{I \times J \subset \mathbb{R}_+^m \times \mathbb{R}_+^n} A^{(\alpha, \beta), (m,n)}_{p,q} (v, w) [I, J]. \]

Suppose that \(I = \prod_{i=1}^{m} [a_i, a_i + s] \subset \mathbb{R}_+^m\) and \(J = \prod_{j=1}^{n} [b_j, b_j + t] \subset \mathbb{R}_+^n\) are as above. If we slide (translate) the rectangle \(I \times J\) to a new position

\[ I' \times J' = \prod_{i=1}^{m} [a_i', a_i' + s] \times \prod_{j=1}^{n} [b_j', b_j' + t] \subset \mathbb{R}_+^m \times \mathbb{R}_+^n \]

in which their centers

\[ c_{I \times J} = (a, b), c_{I' \times J'} = (a', b') \in \mathbb{R}_+^m \times \mathbb{R}_+^n \]

lie on the same simplex

\[ S (\nu) \equiv \{(x, y) \in \mathbb{R}_+^m \times \mathbb{R}_+^n : x_1 + \ldots + x_m + y_1 + \ldots + y_n = \nu\}, \quad \nu > 0, \]

then the sans serif local characteristics are unchanged, i.e.

\[ A^{(\alpha, \beta), (m,n)}_{p,q} (v, w) [I, J] = A^{(\alpha, \beta), (m,n)}_{p,q} (v, w) [I', J']. \]

Thus it suffices to consider only rectangles \(I \times J \subset \mathbb{R}_+^m \times \mathbb{R}_+^n\) having the special forms

\[ I = [0, s]^{k-1} \times [a_k, a_k + s] \times [0, s]^{m-k} \text{ and } J = [0, t]^{n}, \]

and

\[ I = [0, s]^m \text{ and } J = [0, t]^{\ell-1} \times [b_{\ell}, b_{\ell} + t] \times [0, t]^{n-\ell}, \]

for some \(1 \leq k \leq m\) and \(1 \leq \ell \leq n\). We have thus shown that the characteristic \(A_{p,q}^{(\alpha, \beta), (m,n)} (v, w)\) is controlled by the supremum of the sans serif local characteristics \(A_{p,q}^{(\alpha, \beta), (m,n)} (v, w) [I, J]\) taken over rectangles \(I \times J\) of the special forms given in (9.23) and (9.24):

\[ A^{(\alpha, \beta), (m,n)}_{p,q} (v, w) \approx \sup_{I \times J \text{ as in } (9.23) \text{ or } (9.24)} A^{(\alpha, \beta), (m,n)}_{p,q} (v, w) [I, J]. \]

Let us further consider a rectangle \(I \times J\) of the form (9.23), and without loss of generality, we may take \(k = 1\) so that

\[ I = [a, a + s] \times [0, s]^{m-1} \text{ and } J = [0, t]^n. \]

Recall that the corresponding sans serif local characteristic is given by

\[ A^{(\alpha, \beta), (m,n)}_{p,q} (v, w) [I, J] = s^{\alpha-m} t^{\beta-n} \left[ \frac{m,n}{(a,0,\ldots,0); (s,t)} (-\gamma q) \right]^{\frac{\gamma}{p'}} \left[ \frac{m,n}{(a,0,\ldots,0); (s,t)} (-\delta p') \right]^{\frac{\gamma}{p'}}, \]
where \( 0 = (0, \ldots, 0) \in \mathbb{R}^n \) and

\[
I_{\{(0,0,\ldots,0),0\}}^{m,n}(s,t)(\eta) = \int_{x \in [a,a+s] \times [0,s]^{m-1}} \left( \int_{y \in [0,t]^n} (x_1 + \ldots + x_m + y_1 + \ldots + y_n)^\eta dy \right) dx.
\]

(1) First we note that if \( s \geq a > 0 \), then

\[
I_{\{(0,0,\ldots,0),0\}}^{m,n}(s,t)(\eta) \approx I_{\{(0,0,\ldots,0),0\}}^{m,n}(\eta), \quad \eta \geq 0,
\]

and

\[
I_{\{(0,0,\ldots,0),0\}}^{m,n}(s,t)(\eta) \lesssim I_{\{(0,0,\ldots,0),0\}}^{m,n}(\eta), \quad - (m+n) < \eta < 0,
\]

as is easily seen using the doubling and monotonicity properties of the locally integrable weight \((x_1 + \ldots + x_m + y_1 + \ldots + y_n)^\eta\). We conclude using (4.10) that

\[
(9.25) \quad s^{\alpha-m} t^{\beta-n} \left[ I_{\{(0,0,\ldots,0),0\}}^{m,n}(s,t) (-\gamma q) \right]^\frac{1}{\eta} \left[ I_{\{(0,0,\ldots,0),0\}}^{m,n}(\eta) (-\delta p') \right]^{\frac{m+n}{\eta}},
\]

holds when \( s \geq a > 0 \).

(2) Now consider the case \( 0 < s < a \leq t \). Then by doubling and monotonicity we have

\[
I_{\{(0,0,\ldots,0),0\}}^{m,n}(s,t)(\eta) \lesssim I_{\{(0,0,\ldots,0),0\}}^{m,n}(\eta), \quad - (m+n) < \eta < \infty,
\]

and so we conclude again using (4.10) that (9.25) also holds when \( 0 < s < a < t \).

(3) Now consider the remaining case \( 0 < s, t \leq a \). Since the weight \((x_1 + \ldots + x_m + y_1 + \ldots + y_n)^\eta\) is roughly equal to the constant \(a^q\) on the cube \([a,2a] \times [0,a]^{m-1} \times [0,a]^n\), we have

\[
A_{p,q}^{(\alpha,\beta),(m,n)}(v,w) \left[ [a,a+s] \times [0,s]^{m-1} \times [0,t]^n \right]
= s^{\alpha-m} t^{\beta-n} \left[ I_{\{(0,0,\ldots,0),0\}}^{m,n}(s,t) (-\gamma q) \right]^\frac{1}{\eta} \left[ I_{\{(0,0,\ldots,0),0\}}^{m,n}(\eta) (-\delta p') \right]^{\frac{m+n}{\eta}},
\]

\[
= s^{\alpha-m} t^{\beta-n} \left[ I_{\{(0,0,\ldots,0),0\}}^{m,n}(s,t) (-\gamma q) \right]^\frac{1}{\eta} \left[ I_{\{(0,0,\ldots,0),0\}}^{m,n}(\eta) (-\delta p') \right]^{\frac{m+n}{\eta}},
\]

\[
\approx s^{\alpha-m} t^{\beta-n} \Gamma a^{-(\gamma+\delta)},
\]

for \( 0 < s, t < a \). We note for future reference that

\[
A_{p,q}^{(\alpha,\beta),(m,n)}(v,w) \left[ [a,a+s] \times [0,s]^{m-1} \times [0,t]^n \right] \approx A_{p,q}^{(\alpha,\beta),(m,n)}(v,w) \left[ [a,a+s]^m \times [a,a+t]^n \right].
\]

Choosing \( s = t = a \) in (3) we see that \( a^{(\alpha+\beta)-(\gamma+\delta)-(m+n)\Gamma} \) must be bounded for all \( a > 0 \), which is equivalent to the balanced condition

\[
(9.26) \quad \Gamma = \frac{(\alpha + \beta) - (\gamma + \delta)}{m+n}.
\]

Now we also have that both \( s^{\alpha-m\Gamma} \) and \( t^{\beta-n\Gamma} \) are bounded for \( 0 < s, t \leq a \), which is equivalent to

\[
\frac{\alpha}{m} \geq \frac{\beta}{n} \geq \Gamma.
\]

If we fix \( t = 1 \) and let \( s = a \to \infty \), we obtain that the boundedness of \( a^{\alpha-m\Gamma} a^{-(\gamma+\delta)} \) for \( a \) large implies

\[
\frac{\alpha}{m} \leq \Gamma + \frac{\gamma + \delta}{m}.
\]

Similarly we have

\[
\frac{\beta}{n} \leq \Gamma + \frac{\gamma + \delta}{n},
\]

from which we conclude the diagonal inequality

\[
(9.27) \quad \left\{ \begin{array}{l}
\Gamma \leq \frac{\alpha}{m} \leq \Gamma + \frac{\gamma + \delta}{m} \\
\Gamma \leq \frac{\beta}{n} \leq \Gamma + \frac{\gamma + \delta}{n}
\end{array} \right.
\].
It is now an easy matter to verify that the supremum taken over \(0 < s, t \leq a\) of the sans serif characteristics arising in (3), is finite if and only if both the power weight equality (9.26) and diagonal inequalities (9.27)

\[
\sup_{0 < s, t \leq a} A_{\alpha, \beta, (m, n)}(v, w) \left[ [0, a + s] \times [0, a + t] \right] < \infty
\]

\[
\iff \sup_{0 < s, t \leq a} s^{\alpha-m} t^{\beta-n} \left[ a - (\gamma + \delta) \right] < \infty
\]

\[
\iff \begin{cases} \\
\quad \Gamma = \frac{(\alpha + \beta) - (\gamma + \delta)}{m+n} \\
\quad \Gamma \leq \frac{\alpha}{m} \leq \Gamma + \frac{\gamma + \delta}{m+n} \\
\quad \Gamma \leq \frac{\beta}{n} \leq \Gamma + \frac{\gamma + \delta}{m+n} \\
\end{cases}
\]

If we combine this with the conclusions of (1) and (2), along with a similar analysis of the case when \(I \times J\) has the form in (9.24), we obtain the following proposition.

**Proposition 1.** Suppose that \(\alpha, \beta > 0\), \(1 < p, q < \infty\), and \(w(x, y) = |(x, y)|^{-\gamma}\) and \(v(x, y) = |(x, y)|^{\delta}\) with \(-\infty < \gamma, \delta < \infty\) that satisfy (4.10). Then

\[
A_{\alpha, \beta, (m, n)}(v, w) \\
\approx \sup_{0 < s, t \leq \infty} A_{\alpha, \beta, (m, n)}(v, w) \left[ [0, s] \times [0, t] \right] \\
+ \sup_{0 < s, t \leq \infty} A_{\alpha, \beta, (m, n)}(v, w) \left[ [a, a + s] \times [a, a + t] \right],
\]

where the second summand is finite if and only if

\[
\begin{cases} \\
\quad \Gamma = \frac{(\alpha + \beta) - (\gamma + \delta)}{m+n} \\
\quad \Gamma \leq \frac{\alpha}{m} \leq \Gamma + \frac{\gamma + \delta}{m+n} \\
\quad \Gamma \leq \frac{\beta}{n} \leq \Gamma + \frac{\gamma + \delta}{m+n} \\
\end{cases}
\]

9.6.1. **Rectangles at the origin.** It now remains to determine under what conditions on the indices the first summand is finite, i.e. when

\[
\sup_{0 < s, t \leq \infty} A_{\alpha, \beta, (m, n)}(v, w) \left[ [0, s] \times [0, t] \right] < \infty.
\]

For this we start with

\[
A_{\alpha, \beta, (m, n)}(v, w) \left[ [0, s] \times [0, t] \right] \\
= s^{\alpha-m} t^{\beta-n} \left[ \int_{0}^{m,n} \left( \frac{m,n}{\{0,0\}: (s,t)} (\sigma) \int_{0}^{m,n} \left( \frac{m,n}{\{0,0\}: (s,t)} (\tau) \right) d\sigma d\tau \right) d\sigma d\tau \right.
\]

where if \(0 < s \leq t < \infty\),

\[
\int_{0}^{m,n} \left( \frac{m,n}{\{0,0\}: (s,t)} (\eta) \right) \\
= \int_{0}^{s} \int_{0}^{t} \left( \sigma + \tau \right)^{\gamma} d\sigma d\tau \\
\approx \int_{0}^{s} \left( \sigma + \tau \right)^{\gamma} d\sigma \\
= \int_{0}^{s} \left( \sigma + \tau \right)^{\gamma} d\sigma.
\]

We note that

\[
\int_{0}^{x} \left( \rho + 1 \right)^{\gamma} d\rho \approx \begin{cases} \\
\lambda^{m} \quad \text{if} \quad 0 < \lambda \leq 1 \\
\lambda^{m+\eta} \quad \text{if} \quad 1 \leq \lambda < \infty \quad \text{and} \quad m + \eta \neq 0 \\
1 + \ln \lambda \quad \text{if} \quad 1 \leq \lambda < \infty \quad \text{and} \quad m + \eta = 0
\end{cases}
\]
Now suppose that \( 0 < t \leq s < \infty \). Then we have with \( x = x_+ - x_- \) that

\[
\mathcal{I}_{s,t}^{m,n}(\eta) \approx \int_0^t \left\{ \int_0^\infty (\sigma + \tau)^{\eta \sigma^{m-1}} d\sigma \right\} \tau^{n-1} d\tau \\
= \int_0^t \left\{ \int_{\sigma=0}^\infty (\frac{s}{\tau} + 1)^{\eta \left( \frac{\sigma}{\tau} \right)^{m-1}} d \left( \frac{\sigma}{\tau} \right) \right\} \tau^{\eta + m + n - 1} d\tau \\
= \int_0^t \left\{ \int_{\sigma'=0}^\infty (\sigma' + 1)^{\eta (\sigma')^{m-1}} d\sigma' \right\} \tau^{\eta + m + n - 1} d\tau \\
\approx \int_0^t \left( \frac{s}{\tau} \right)^{(m+n)+\eta} \tau^{\eta + m + n - 1} d\tau = s^{(m+n)+\eta} t^{n-(m+n)-},
\]

provided \( m + \eta \neq 0 \) and \( m + \eta > -n \), and

\[
\mathcal{I}_{s,t}^{m,n}(-m) \approx \int_0^t \left\{ \int_{\sigma'=0}^\infty (\sigma' + 1)^{-m (\sigma')^{m-1}} d\sigma' \right\} \tau^{n-1} d\tau \approx \int_0^t \left( \ln \frac{s}{\tau} \right) \tau^{n-1} d\tau \\
= \frac{1}{n} t^\nu \ln s - \int_0^t \left( \ln \tau \right) \tau^{\nu-1} d\tau = \frac{1}{n} t^\nu \ln s + \frac{1}{n^\nu} t^\nu = \frac{1}{n} \left( \frac{1}{n} + \ln \frac{s}{\tau} \right),
\]

since \( \int x^{n-1} \ln x \, dx = \frac{1}{n} x^n \ln x - \frac{1}{n^2} x^n \). It will be convenient to simply understand that

\[
s^{0+} t^{-0-} = \ln \frac{s}{t} \text{ when } 0 < t \leq s < \infty.
\]

Similarly, if \( 0 < s \leq t < \infty \), then we have \( \mathcal{I}_{s,t}^{m,n}(\eta) \approx s^{m-(n+\eta)+} t^{(n+\eta)+} \) provided \( n + \eta > -m \), with the analogous understanding that \( s^{\nu-} t^{\nu+} = \ln \frac{s}{t} \) and so altogether we obtain

\[
\mathcal{I}_{s,t}^{m,n}(\eta) \approx \begin{cases} 
    s^{m-(n+\eta)+} t^{(n+\eta)+} & \text{if } 0 < s \leq t < \infty, \\
    s^{(m+n)+} t^{n-(m+n)+} & \text{if } 0 < t \leq s < \infty,
\end{cases} \quad m + n + \eta > 0.
\]

Thus if \( 0 < s \leq t < \infty \) we have, using \( m + n - \gamma > 0 \) and \( m + n - \delta \beta' > 0 \), that

\[
A^{(a,\beta),(m,n)}_{p,q}(0,1) [I,J] = s^{a-m} t^{\beta-n} I^{m,n}_{s,t} (-\gamma q)^\frac{n}{p} I^{m,n}_{s,t} (-\delta \beta')^\frac{n}{p} \\
= s^{a-m} t^{\beta-n} \left( s^{m-(n-\gamma)q} t^{(n-\gamma)q} \right)^\frac{n}{p} \left( s^{m-(n-\delta \beta')q} t^{(n-\delta \beta')q} \right)^\frac{n}{p} \\
= s^{a-m} \left( s^{m-(n-\gamma)q} t^{(n-\gamma)q} \right)^\frac{n}{p} \left( s^{m-(n-\delta \beta')q} t^{(n-\delta \beta')q} \right)^\frac{n}{p} \\
= s^{a-m} \left( s^{m+(n-\gamma)q} t^{(n-\gamma)q} \right)^\frac{n}{p} \left( s^{m-(n-\delta \beta')q} t^{(n-\delta \beta')q} \right)^\frac{n}{p} \\
= s^{a-m} \left( s^{m+(n-\gamma)q} t^{(n-\gamma)q} \right)^\frac{n}{p} \left( s^{m-(n-\delta \beta')q} t^{(n-\delta \beta')q} \right)^\frac{n}{p} \\
= s^{a-m} \left( s^{m+(n-\gamma)q} t^{(n-\gamma)q} \right)^\frac{n}{p} \left( s^{m-(n-\delta \beta')q} t^{(n-\delta \beta')q} \right)^\frac{n}{p} \\
= \left( s^{m+(n-\gamma)q} t^{(n-\gamma)q} \right)^\frac{n}{p} \left( s^{m-(n-\delta \beta')q} t^{(n-\delta \beta')q} \right)^\frac{n}{p},
\]

and similarly if \( 0 < t \leq s < \infty \), we have

\[
A^{(a,\beta),(m,n)}_{p,q}(1,0) [I,J] = s^{a-m} t^{\beta-n} I^{m,n}_{s,t} (-\gamma q)^\frac{n}{p} I^{m,n}_{s,t} (-\delta \beta')^\frac{n}{p} \\
= s^{a-m} t^{\beta-n} \left( s^{m-(n-\gamma)q} t^{(n-\gamma)q} \right)^\frac{n}{p} \left( s^{m-(n-\delta \beta')q} t^{(n-\delta \beta')q} \right)^\frac{n}{p} \\
= s^{a-m} \left( s^{m-(n-\gamma)q} t^{(n-\gamma)q} \right)^\frac{n}{p} \left( s^{m-(n-\delta \beta')q} t^{(n-\delta \beta')q} \right)^\frac{n}{p} \\
= s^{a-m} \left( s^{m+(n-\gamma)q} t^{(n-\gamma)q} \right)^\frac{n}{p} \left( s^{m-(n-\delta \beta')q} t^{(n-\delta \beta')q} \right)^\frac{n}{p} \\
= s^{a-m} \left( s^{m+(n-\gamma)q} t^{(n-\gamma)q} \right)^\frac{n}{p} \left( s^{m-(n-\delta \beta')q} t^{(n-\delta \beta')q} \right)^\frac{n}{p} \\
= \left( s^{m+(n-\gamma)q} t^{(n-\gamma)q} \right)^\frac{n}{p} \left( s^{m-(n-\delta \beta')q} t^{(n-\delta \beta')q} \right)^\frac{n}{p},
\]

with the understanding that

\[
(9.28) \quad s^{0+} t^{-0-} = \ln \frac{s}{t} \text{ when } 0 < t \leq s < \infty, \\
s^{-0-} t^{0+} = \ln \frac{t}{s} \text{ when } 0 < s \leq t < \infty.
\]

In order to efficiently calculate the remaining conditions, we assume for the moment that \( m \leq n \), and remove this restriction at the end.
Now we consider separately the (at most) five cases determined by $\delta$ (since the open interval $\left( \frac{m}{p}, \frac{n}{q} \right)$ is empty if $m = n$):

\[
\delta \in \left( -\infty, \frac{m}{p} \right), \quad \delta = \frac{m}{p}, \quad \delta \in \left( \frac{m}{p'}, \frac{n}{q} \right), \quad \delta = \frac{n}{q}, \quad \delta \in \left( \frac{n}{q}, \infty \right).
\]

**Case A:** $\delta < \frac{m}{p}$. Then for $0 < s \leq t < \infty$, we have

\[
A^{(\alpha,\beta),(m,n)}_{p,q}(v,w)[I,J] = s^{\alpha - m + \frac{m}{q} - \frac{m}{p'}} t^{\beta - n + \frac{n}{q} + \frac{n}{p'}} + s^{\alpha - m - \frac{m}{q} + \frac{m}{p'}} t^{\beta - n - \frac{n}{q} - \frac{n}{p'}} - \delta,
\]

and for $0 < t \leq s < \infty$, we have

\[
A^{(\alpha,\beta),(m,n)}_{p,q}(v,w)[I,J] = s^{\alpha - m + \frac{m}{q} - \frac{m}{p'}} t^{\beta - n + \frac{n}{q} + \frac{n}{p'}} - \delta.
\]

Now we consider separately within Case A the (at most) five subcases determined by $\gamma$ (since the open interval $\left( \frac{m}{q}, \frac{n}{q} \right)$ is empty if $m = n$):

\[
\gamma \in \left( -\infty, \frac{m}{q} \right), \quad \gamma = \frac{m}{q}, \quad \gamma \in \left( \frac{m}{q}, \frac{n}{q} \right), \quad \gamma = \frac{n}{q}, \quad \gamma \in \left( \frac{n}{q}, \infty \right).
\]

**Subcase A1:** $\gamma \in \left( -\infty, \frac{m}{q} \right)$. In this subcase $\frac{m}{q} - \gamma > 0$ and $\frac{n}{q} - \gamma > 0$, and so for $0 < s \leq t < \infty$, we have

\[
A^{(\alpha,\beta),(m,n)}_{p,q}(v,w)[I,J] = s^{\alpha - m + \frac{m}{q} + \frac{m}{p'}} t^{\beta - n + \frac{n}{q} - \frac{n}{p'}} - \delta,
\]

and for $0 < t \leq s < \infty$, we have

\[
A^{(\alpha,\beta),(m,n)}_{p,q}(v,w)[I,J] = s^{\alpha - m - \frac{m}{q} + \frac{m}{p'}} t^{\beta - n + \frac{n}{q} + \frac{n}{p'}} - \delta.
\]

Thus in the presence of the power weight equality (9.26), the boundedness of these two local characteristics $A^{(\alpha,\beta),(m,n)}_{p,q}(v,w)$ in the indicated ranges is equivalent to the four conditions:

\[
\alpha - m + \frac{m}{q} + \frac{m}{p'} \geq 0,
\]

\[
\alpha - m + \frac{m}{q} - \frac{m}{p'} - \delta \leq 0,
\]

\[
\beta - n + \frac{n}{q} + \frac{n}{p'} \geq 0,
\]

\[
\beta - n + \frac{n}{q} - \frac{n}{p'} - \delta \leq 0,
\]

i.e.

\[
\Gamma \leq \frac{\alpha}{m} \leq \Gamma + \frac{\gamma + \delta}{m},
\]

\[
\Gamma \leq \frac{\beta}{n} \leq \Gamma + \frac{\gamma + \delta}{n},
\]

which are the second and third lines in (4.17) in this case.

**Subcase A2:** $\gamma = \frac{m}{q}$ and $m < n$. In this subcase $\frac{m}{q} - \gamma = 0$ and $\frac{n}{q} - \gamma > 0$, and so for $0 < s \leq t < \infty$, we have

\[
A^{(\alpha,\beta),(m,n)}_{p,q}(v,w)[I,J] = s^{\alpha - m + \frac{m}{q} - \frac{m}{p'}} t^{\beta - n + \frac{n}{q} + \frac{n}{p'}} - \delta,
\]

and for $0 < t \leq s < \infty$, we have from (9.28) that

\[
A^{(\alpha,\beta),(m,n)}_{p,q}(v,w)[I,J] = s^{\alpha - m - \frac{m}{q} - \delta} t^{\beta - n - \frac{n}{q} - \frac{n}{p'} \ln \frac{s}{t}}.
\]
Thus the boundedness of these two local characteristics $A_{p,q}^{(α,β),(m,n)}(v,w)$ in the indicated ranges is equivalent to the four conditions:

\[\alpha - m + \frac{m}{q} + \frac{m}{p'} ≥ 0,\]
\[\alpha - m + \frac{m}{p'} - \delta < 0,\]
\[β - n + \frac{n}{q} + \frac{n}{p'} > 0,\]
\[β - n + \frac{n}{q} - γ + \frac{n}{p'} - \delta ≤ 0,\]

i.e.

\[Γ ≤ \frac{α}{m} < Γ + \frac{γ + δ}{m},\]
\[Γ < \frac{β}{n} ≤ Γ + \frac{γ + δ}{n},\]

which are the second and third lines in (4.17) in this subcase.

**Subcase A3:** $γ ∈ \left(\frac{m}{q}, \frac{n}{q}\right)$. In this subcase $\frac{m}{q} - γ < 0$ and $\frac{n}{q} - γ > 0$, and so for $0 < s ≤ t < ∞$, we have

\[A_{p,q}^{(α,β),(m,n)}(v,w)[I,J] = s^{α-m+\frac{m}{q}+\frac{m}{p'}} t^{β-n+\frac{n}{q}+\frac{n}{p'}-\delta};\]

and for $0 < t ≤ s < ∞$, we have

\[A_{p,q}^{(α,β),(m,n)}(v,w)[I,J] = s^{α-m+\frac{m}{p'-}\delta} t^{β-n+\frac{n}{q}+\frac{n}{p'}-\gamma+\frac{γ}{p'}}.\]

Thus the boundedness of these two local characteristics $A_{p,q}^{(α,β),(m,n)}(v,w)$ in the indicated ranges is equivalent to the four conditions:

\[\alpha - m + \frac{m}{q} + \frac{m}{p'} ≥ 0,\]
\[\alpha - m + \frac{m}{p'} - \delta ≤ 0,\]
\[β - n + \frac{n}{q} + \frac{n}{p'} - γ + \frac{n}{p'} ≥ 0,\]
\[β - n + \frac{n}{q} - γ + \frac{n}{p'} - δ ≤ 0,\]

i.e.

\[Γ ≤ \frac{α}{m} ≤ Γ + \frac{γ + δ}{m} - \frac{γ - \frac{m}{q}}{m},\]
\[Γ + \frac{γ - \frac{m}{q}}{n} ≤ \frac{β}{n} ≤ Γ + \frac{γ + δ}{n},\]

which are the second and third lines in (4.17) in this subcase.

**Subcase A4:** $γ = \frac{n}{q}$ and $m < n$. In this subcase $\frac{m}{q} - γ < 0$ and $\frac{n}{q} - γ = 0$, and so for $0 < s ≤ t < ∞$, we have from (4.28) that

\[A_{p,q}^{(α,β),(m,n)}(v,w)[I,J] = s^{α-m+\frac{m}{q}+\frac{m}{p'}} t^{β-n+\frac{n}{q}+\frac{n}{p'}-\gamma+\frac{γ}{p'}} \ln \frac{t}{s};\]

and for $0 < t ≤ s < ∞$, we have

\[A_{p,q}^{(α,β),(m,n)}(v,w)[I,J] = s^{α-m+\frac{m}{p'}-\delta} t^{β-n+\frac{n}{q}+\frac{n}{p'}-γ+\frac{γ}{p'}}.\]
Thus the boundedness of these two local characteristics $A_{p,q}^{(\alpha,\beta),(m,n)}(v,w)$ in the indicated ranges is equivalent to the four conditions:

\[
\begin{align*}
\alpha - m + \frac{m}{q} + \frac{m}{p'} & > 0, \\
\alpha - m + \frac{m}{p'} - \delta & \leq 0, \\
\beta - n + \frac{n}{q} - \gamma + \frac{n}{p'} & \geq 0, \\
\beta - n + \frac{n}{p'} - \delta & < 0,
\end{align*}
\]

i.e.

\[
\begin{align*}
\Gamma & < \frac{\alpha}{m} \leq \Gamma + \frac{m}{m} + \delta, \\
\Gamma + \frac{\gamma - \frac{m}{m}}{n} & \leq \frac{\beta}{n} < \Gamma + \frac{\gamma + \delta}{n},
\end{align*}
\]

which are the second and third lines in (4.17) in this subcase.

**Subcase A5:** $\gamma \in \left(\frac{n}{q}, \infty\right)$. In this subcase $\frac{m}{q} - \gamma < 0$ and $\frac{n}{q} - \gamma < 0$, and so for $0 < s \leq t < \infty$, we have

\[
A_{p,q}^{(\alpha,\beta),(m,n)}(v,w)[I,J] = s^{\alpha - m + \frac{m}{q} - \gamma + \frac{m}{p'}} t^{\beta - n + \frac{n}{p'} - \delta},
\]

and for $0 < t \leq s < \infty$, we have

\[
A_{p,q}^{(\alpha,\beta),(m,n)}(v,w)[I,J] = s^{\alpha - m + \frac{m}{q} - \gamma + \frac{m}{p'}} t^{\beta - n + \frac{n}{p'} - \delta} \ln \frac{t}{s}.
\]

Thus the boundedness of these two local characteristics $A_{p,q}^{(\alpha,\beta),(m,n)}(v,w)$ in the indicated ranges is equivalent to the four conditions:

\[
\begin{align*}
\alpha - m + \frac{m}{q} + \frac{n}{q} - \gamma + \frac{n}{p'} & \geq 0, \\
\alpha - m + \frac{m}{p'} - \delta & \leq 0, \\
\beta - n + \frac{n}{q} - \gamma + \frac{n}{p'} & \geq 0, \\
\beta - n + \frac{n}{p'} - \delta & \leq 0,
\end{align*}
\]

i.e.

\[
\begin{align*}
\Gamma + \frac{\gamma - \frac{n}{q}}{m} & \leq \frac{\alpha}{m} \leq \Gamma + \frac{\gamma + \frac{n}{q}}{m}, \\
\Gamma + \frac{\gamma - \frac{n}{q}}{n} & \leq \frac{\beta}{n} \leq \Gamma + \frac{\gamma + \frac{n}{q}}{n},
\end{align*}
\]

which are the second and third lines in (4.17) in this subcase.

**Subcase A6:** $\gamma = \frac{n}{q}$ and $m = n$. In this case $\frac{m}{q} - \gamma = 0$ and $\frac{n}{q} - \gamma = 0$, and so for $0 < s \leq t < \infty$, we have from (9.28) that

\[
A_{p,q}^{(\alpha,\beta),(m,n)}(v,w)[I,J] = s^{\alpha - m + \frac{m}{q} - \gamma + \frac{m}{p'}} t^{\beta - n + \frac{n}{p'} - \delta} \ln \frac{t}{s}.
\]

and for $0 < t \leq s < \infty$, we have

\[
A_{p,q}^{(\alpha,\beta),(m,n)}(v,w)[I,J] = s^{\alpha - m + \frac{m}{q} - \gamma + \frac{m}{p'}} t^{\beta - n + \frac{n}{p'} - \delta} \ln \frac{s}{t}.
\]
Thus the boundedness of these two local characteristics $A_{p,q}^{(\alpha,\beta),(m,n)}(v,w)$ in the indicated ranges is equivalent to the four conditions:

\[
\begin{align*}
\alpha - m + \frac{m}{q} + \frac{m}{p'} > 0, \\
\alpha - m + \frac{m}{p'} - \delta < 0, \\
\beta - n + \frac{n}{q} + \frac{n}{p'} > 0, \\
\beta - n + \frac{n}{p'} - \delta < 0,
\end{align*}
\]

i.e.

\[
\begin{align*}
\Gamma < \frac{\alpha}{m} < \Gamma + \frac{\gamma + \delta}{m}, \\
\Gamma < \frac{\beta}{n} < \Gamma + \frac{\gamma + \delta}{n},
\end{align*}
\]

which are the second and third lines in (3.17) in this case.

Now we turn to the next major case.

**Case B**: $\delta = \frac{m}{p'}$. Then for $0 < s \leq t < \infty$, we have

\[
A_{p,q}^{(\alpha,\beta),(m,n)}(v,w)[I,J] = s^{\alpha-m+\frac{m}{q}+\frac{m}{p'} + \frac{\gamma}{p} - \beta - n + \frac{n}{q} - \frac{n}{p'} - \delta},
\]

and for $0 < t \leq s < \infty$, we have from (9.28) that

\[
A_{p,q}^{(\alpha,\beta),(m,n)}(v,w)[I,J] = s^{\alpha-m+\frac{m}{q}+\frac{m}{p'} - \beta - n + \frac{n}{q} + \frac{n}{p'} - \delta} \ln \frac{s}{t}.
\]

Now we consider separately within Case B the (at most) five subcases determined by $\gamma$ (since the open interval $\left(\frac{m}{q}, \frac{n}{q}\right)$ is empty if $m = n$):

**Subcase B1**: $\gamma \in \left(-\infty, \frac{m}{q}\right)$. In this subcase $\frac{m}{q} - \gamma > 0$ and $\frac{n}{q} - \gamma > 0$, and so for $0 < s \leq t < \infty$, we have

\[
A_{p,q}^{(\alpha,\beta),(m,n)}(v,w)[I,J] = s^{\alpha-m+\frac{m}{q}+\frac{m}{p'} + \frac{\gamma}{p} - \beta - n + \frac{n}{q} + \frac{n}{p'} - \delta},
\]

and for $0 < t \leq s < \infty$, we have

\[
A_{p,q}^{(\alpha,\beta),(m,n)}(v,w)[I,J] = s^{\alpha-m+\frac{m}{q}+\frac{m}{p'} - \beta - n + \frac{n}{q} + \frac{n}{p'} + \frac{\gamma}{p} - \delta} \ln \frac{s}{t}.
\]

Thus in the presence of the power weight equality (9.26), the boundedness of these two local characteristics $A_{p,q}^{(\alpha,\beta),(m,n)}(v,w)$ in the indicated ranges is equivalent to the four conditions:

\[
\begin{align*}
\alpha - m + \frac{m}{q} + \frac{m}{p'} &\geq 0, \\
\alpha - m + \frac{m}{p'} - \gamma + \frac{m}{p'} - \delta &< 0, \\
\beta - n + \frac{n}{q} + \frac{n}{p'} &> 0, \\
\beta - n + \frac{n}{p'} - \gamma + \frac{n}{p'} - \delta &\leq 0,
\end{align*}
\]

i.e.

\[
\begin{align*}
\Gamma &\leq \frac{\alpha}{m} < \Gamma + \frac{\gamma + \delta}{m}, \\
\Gamma &< \frac{\beta}{n} \leq \Gamma + \frac{\gamma + \delta}{n},
\end{align*}
\]

which are the second and third lines in (3.17) in this subcase.
Case B2: \( \gamma = \frac{m}{q} \). In this case \( \frac{m}{q} - \gamma = 0 \) and \( \frac{n}{q} - \gamma > 0 \), and so for \( 0 < s \leq t < \infty \), we have
\[
A_{p,q}^{(\alpha,\beta),(m,n)}(v,w)[I,J] = s^{\alpha-m+\frac{m}{q}+\frac{n}{q}+\gamma} t^{\beta-n+\frac{m}{p}+\gamma+\frac{n}{q}+\delta},
\]
and for \( 0 < t \leq s < \infty \), we have
\[
A_{p,q}^{(\alpha,\beta),(m,n)}(v,w)[I,J] = s^{\alpha-m+\frac{m}{q}+\gamma} t^{\beta-n+\frac{m}{p}+\delta} \ln \frac{s}{t}.
\]
This coincides with Subcase A2 above and so is equivalent to the four conditions
\[
\Gamma \leq \frac{\alpha}{m} < \frac{\gamma+\delta}{m},
\]
\[
\Gamma < \frac{\beta}{n} \leq \frac{\gamma+\delta}{n},
\]
which are the second and third lines in (4.17) in this subcase.

The remaining subcases B3, B4, B5 and B6 are handled in similar fashion.

Case C: \( \frac{m}{q} < \delta < \frac{n}{q} \). This is handled in similar fashion.

Case D: \( \delta = \frac{n}{p} \). This is handled in similar fashion.

Case E: \( \frac{m}{p} < \delta < \infty \). The subcases E1, E2, E3, E4 and E6, are handled in similar fashion, and we end with the remaining and final subcase.

Subcase E5: \( \frac{m}{p} < \gamma < \infty \). In this case we have both \( \frac{m}{q} - \delta < 0 \) and \( \frac{n}{q} - \gamma < 0 \) and so if \( 0 < s \leq t < \infty \), we have
\[
A_{p,q}^{(\alpha,\beta),(m,n)}(v,w)[I,J] = s^{\alpha-m+\frac{m}{q}+\frac{n}{q}+\gamma} t^{\beta-n+\frac{m}{p}+\gamma+\frac{n}{q}+\delta},
\]
and similarly if \( 0 < t \leq s < \infty \), we have
\[
A_{p,q}^{(\alpha,\beta),(m,n)}(v,w)[I,J] = s^{\alpha-m} t^{\beta-n+\frac{m}{q}+\gamma} \ln \frac{s}{t}.
\]
Thus in the presence of the power weight equality (9.26), the boundedness of these two local characteristics \( A_{p,q}^{(\alpha,\beta),(m,n)}(v,w)[I,J] \) in the indicated ranges is equivalent to the four conditions:
\[
\alpha - m + \frac{m}{q} + \frac{n}{q} - \gamma + \frac{m}{p} + \frac{n}{p} - \delta \geq 0,
\]
\[
\beta - n + \frac{n}{q} - \frac{m}{q} - \gamma + \frac{m}{p} + \frac{n}{p} - \delta \geq 0,
\]
i.e.
\[
\Gamma - \frac{n}{m} \left( \frac{1}{q} + \frac{1}{p'} \right) + \frac{\gamma+\delta}{m} \leq \frac{\alpha}{m} \leq 1,
\]
\[
\Gamma - \frac{m}{n} \left( \frac{1}{q} + \frac{1}{p'} \right) + \frac{\gamma+\delta}{n} \leq \frac{\beta}{n} \leq 1.
\]
which are the second and third lines in (4.17) in this subcase. Indeed, in this subcase, the second and third lines in (4.17) are
\[
\Gamma + \frac{\Delta_{p,q}^{\gamma,\delta}(m)}{m} \leq \frac{\alpha}{m} \leq \Gamma + \frac{\gamma+\delta}{m} - \frac{\Delta_{p,q}^{\gamma,\delta}(m)}{m},
\]
\[
\Gamma + \frac{\Delta_{p,q}^{\gamma,\delta}(m)}{n} \leq \frac{\beta}{n} \leq \Gamma + \frac{\gamma+\delta}{n} - \frac{\Delta_{p,q}^{\gamma,\delta}(n)}{n},
\]
where
\[
\Delta_{p,q}^{\gamma,\delta}(m) \equiv \left( \gamma - \frac{m}{q} \right) + \left( \delta - \frac{m}{p} \right),
\]
\[
\Delta_{p,q}^{\gamma,\delta}(n) \equiv \left( \gamma - \frac{n}{q} \right) + \left( \delta - \frac{n}{p} \right).
\]
This is equivalent to
\[ \Gamma + \frac{\gamma - \frac{n}{q} + \delta - \frac{n}{p'}}{m} \leq \frac{\alpha}{m} \leq \Gamma + \frac{\gamma + \delta - \frac{n}{q}}{m}, \]
\[ \Gamma + \frac{\gamma - \frac{m}{q} + \delta - \frac{m}{p'}}{n} \leq \frac{\beta}{n} \leq \Gamma + \frac{\gamma + \delta - \frac{n}{q}}{n}, \]
i.e.
\[ \Gamma - \frac{n}{m} \left( \frac{1}{q} + \frac{1}{p'} \right) + \frac{\gamma + \delta}{m} \leq \frac{\alpha}{m} \leq \Gamma + \frac{1}{q} + \frac{1}{p'} = 1, \]
\[ \Gamma - \frac{m}{n} \left( \frac{1}{q} + \frac{1}{p'} \right) + \frac{\gamma + \delta}{n} \leq \frac{\beta}{n} \leq \Gamma + \frac{1}{q} + \frac{1}{p'} = 1. \]

Finally we note that these families of inequalities for \( \alpha, \beta \) remain the same when \( m \geq n \). This concludes the proof of Theorem 8.

**Corollary 3.** If \( A_{p,q}^{(\alpha,\beta),(m,n)}(v_\delta,w_\gamma) < \infty \), then the following inequalities hold:
\[ \alpha \leq m \text{ and } \beta \leq n, \]
\[ \frac{\alpha}{m} \leq \min \left\{ \frac{\gamma}{m} + \frac{1}{q'}, \frac{\delta}{m} + \frac{1}{p'} \right\}, \]
\[ \frac{\beta}{n} \leq \min \left\{ \frac{\gamma}{n} + \frac{1}{q'}, \frac{\delta}{n} + \frac{1}{p'} \right\}. \]

In addition, we have the corresponding strict inequalities in the following cases:
\[ \frac{\alpha}{m} < 1 \text{ when either } \frac{\gamma}{m} - \frac{1}{q} \leq 0 \text{ or } \frac{\delta}{m} - \frac{1}{p'} \leq 0, \]
\[ \frac{\beta}{n} < 1 \text{ when either } \frac{\gamma}{n} - \frac{1}{q} \leq 0 \text{ or } \frac{\delta}{n} - \frac{1}{p'} \leq 0, \]
\[ \frac{\alpha}{m} < \frac{\gamma}{m} + \frac{1}{q'} \text{ when either } \frac{\gamma}{m} - \frac{1}{q} \geq 0 \text{ or } \frac{\delta}{m} - \frac{1}{p'} \leq 0, \]
\[ \frac{\alpha}{m} < \frac{\delta}{m} + \frac{1}{p'} \text{ when either } \frac{\gamma}{m} - \frac{1}{q} \leq 0 \text{ or } \frac{\delta}{m} - \frac{1}{p'} \geq 0, \]
\[ \frac{\beta}{n} < \frac{\gamma}{n} + \frac{1}{q'} \text{ when either } \frac{\gamma}{n} - \frac{1}{q} \geq 0 \text{ or } \frac{\delta}{n} - \frac{1}{p'} \leq 0, \]
\[ \frac{\beta}{n} < \frac{\delta}{n} + \frac{1}{p'} \text{ when either } \frac{\gamma}{n} - \frac{1}{q} \leq 0 \text{ or } \frac{\delta}{n} - \frac{1}{p'} \geq 0. \]

In particular, these strict inequalities for \( \frac{\alpha}{m} \) and \( \frac{\beta}{n} \) all hold if both \( \gamma \leq \frac{\min(m,n)}{q} \) and \( \delta \leq \frac{\min(m,n)}{p'} \) hold (c.f. (4.16) which shows that \( \gamma < \frac{m+n}{q} \) and \( \delta < \frac{m+n}{p'} \) are required by finiteness of \( A_{p,q}^{(\alpha,\beta),(m,n)}(v_\delta,w_\gamma) \)).

**Proof.** We compute
\[ \frac{\alpha}{m} \leq \Gamma + \frac{\gamma + \delta}{m} - \frac{\Delta_{p,q}^\gamma \delta(m)}{m} \]
\[ = \frac{1}{p} - \frac{1}{q} + \frac{\gamma + \delta}{m} \left\{ \left( \frac{\gamma}{m} - \frac{1}{q} \right)_+ + \left( \frac{\delta}{m} - \frac{1}{p'} \right)_+ \right\} \]
\[ = 1 - \left\{ \left( \frac{\gamma}{m} - \frac{1}{q} \right)_- + \left( \frac{\delta}{m} - \frac{1}{p'} \right)_- \right\}_+, \]
and similarly
\[ \frac{\beta}{n} \leq 1 - \left\{ \left( \frac{\gamma}{n} - \frac{1}{q} \right)_- + \left( \frac{\delta}{n} - \frac{1}{p'} \right)_- \right\}_+. \]
Using \(-x_- = \min \{0, x\}\) we also compute that

\[
\frac{\alpha}{m} \leq 1 - \left\{ \left( \frac{\gamma}{m} - \frac{1}{q'} \right)_- + \left( \frac{\delta}{m} - \frac{1}{p'} \right)_- \right\}
\]

\[
\leq \min \left\{ 1 - \left( \frac{\gamma}{m} - \frac{1}{q'} \right)_-, 1 - \left( \frac{\delta}{m} - \frac{1}{p'} \right)_- \right\}
\]

\[
\leq \min \left\{ 1 + \frac{\gamma}{m} - \frac{1}{q}, 1 + \frac{\delta}{m} - \frac{1}{p'} \right\}
\]

\[
= \min \left\{ \frac{\gamma}{m} + \frac{1}{q'}, \frac{\delta}{m} + \frac{1}{p'} \right\},
\]

and moreover that

\[
\frac{\alpha}{m} < \frac{\gamma}{m} + \frac{1}{q'} \quad \text{when either} \quad \frac{\delta}{m} - \frac{1}{p'} \leq 0 \quad \text{or} \quad \frac{\gamma}{m} - \frac{1}{q} \geq 0,
\]

\[
\frac{\alpha}{m} < \frac{\delta}{m} + \frac{1}{p} \quad \text{when either} \quad \frac{\gamma}{m} - \frac{1}{q} \leq 0 \quad \text{or} \quad \frac{\delta}{m} - \frac{1}{p'} \geq 0.
\]

Similarly we have

\[
\frac{\beta}{n} \leq \min \left\{ \frac{\gamma}{n} + \frac{1}{q'}, \frac{\delta}{n} + \frac{1}{p} \right\},
\]

and moreover that

\[
\frac{\beta}{n} < \frac{\gamma}{n} + \frac{1}{q'} \quad \text{when either} \quad \frac{\delta}{n} - \frac{1}{p} \leq 0 \quad \text{or} \quad \frac{\gamma}{n} - \frac{1}{q} \geq 0,
\]

\[
\frac{\beta}{n} < \frac{\delta}{n} + \frac{1}{p} \quad \text{when either} \quad \frac{\gamma}{n} - \frac{1}{q} \leq 0 \quad \text{or} \quad \frac{\delta}{n} - \frac{1}{p} \geq 0.
\]

□

We end the appendix with a variant of Lemma 3 in which the restrictions \(0 < \alpha \leq m\) and \(0 < \beta \leq n\) are no longer needed if one of the weights is a power weight.

**Proposition 2.** Suppose at least one of the weights \(\sigma, \omega\) is a power weight on \(\mathbb{R}^{m+n}\). Then for \(1 < p, q < \infty\) and \(\alpha, \beta > 0\), we have

\[
\tilde{K}_{p,q}^{(\alpha,\beta),(m,n)}(\sigma, \omega) \leq C \, K^{(\alpha,\beta),(m,n)}_{p,q}(\sigma, \omega),
\]

where \(C\) is a positive constant depending on \(m, n, \alpha, \beta, p, q\) and the power weight.

By duality we may suppose without loss of generality that \(\sigma\) is a locally integrable power weight. Indeed, modifying slightly the proof of Lemma 3 we have with

\[
\mathcal{E}_{x,y} (I, J) \equiv \left\{ (u, t) \in \mathbb{R}^m \times \mathbb{R}^n : |x - u| \geq |J|^{\frac{1}{m}} \quad \text{and} \quad |y - t| \geq |J|^{\frac{1}{n}} \right\},
\]

that

\[
|x - u|^{a - m} \quad |y - t|^{b - n} \geq 1_{\mathcal{E}_{x,y}(I, J)} ((u, t)) \quad |J|^{\frac{1}{m} - 1} \quad |J|^{\frac{1}{n} - 1} \quad \mathcal{S}_{I \times J} (x, y) \quad \mathcal{S}_{I \times J} (u, t),
\]

and so for \(R > 0\) and \(f_{R}(u, t) \equiv 1_{B(0,R) \times B(0,R)} (u, t) \mathcal{S}_{Q} (u, t)^{\nu - 1}\), we have

\[
I_{a,b}^{m,n} (f_{R})(x, y) = \int \int \left| x - u \right|^{a - m} \left| y - t \right|^{b - n} \mathcal{S}_{Q} (u, t)^{\nu - 1} d\sigma (u, t)
\]

\[
\geq \int \int 1_{\mathcal{E}_{x,y}(I, J)} ((u, t)) \quad |J|^{\frac{1}{m} - 1} \quad |J|^{\frac{1}{n} - 1} \quad \mathcal{S}_{I \times J} (x, y) \quad \mathcal{S}_{I \times J} (u, t) \quad \mathcal{S}_{Q} (u, t)^{\nu - 1} d\sigma (u, t)
\]

\[
= |J|^{\frac{1}{m} - 1} \quad |J|^{\frac{1}{n} - 1} \quad \mathcal{S}_{I \times J} (x, y) \quad \int \int 1_{\mathcal{E}_{x,y}(I, J)} ((u, t)) \quad \mathcal{S}_{I \times J} (u, t)^{\nu} d\sigma (u, t).
\]
It is at this point that we use our assumption that $\sigma(u, t) = |(u, t)|^\rho$, $\rho > -(m + n)$, is a locally integrable power weight in order to conclude that

$$\int_{B(0,R) \times B(0,R)} 1_{E_{x,y}(I,J)} ((u, t)) \, \hat{s}_{I \times J} (u, t)^p' \, d\sigma(u, t) \geq c_{m,n,\alpha,\beta,p,q,\rho} \int_{B(0,R) \times B(0,R)} \hat{s}_{I \times J} (u, t)^p' \, d\sigma(u, t)$$

holds for $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$ with a constant $c_{m,n,\alpha,\beta,p,q,\rho}$ independent of $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$. Now we continue with the proof of Lemma 3 as given earlier to conclude that

$$\hat{A}^{(\alpha,\beta),(m,n)}(\sigma, \omega) \leq \left( \frac{1}{c_{m,n,\alpha,\beta,p,q,\rho}} \right)^{\frac{1}{p}} \| A^{(\alpha,\beta),(m,n)}(\sigma, \omega) \|_{p,q}.$$ 

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