Global Hölder estimates for hypoelliptic operators with drift on homogeneous groups

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Abstract. Let \( X_0, X_1, \ldots, X_q \) be left invariant real vector fields on the homogeneous group \( G \), satisfying Hörmander’s condition on \( \mathbb{R}^N \). Assume that \( X_1, \ldots, X_q \) are homogeneous of degree one and \( X_0 \) is homogeneous of degree two. In this paper we consider the following hypoelliptic operator with drift

\[
L = \sum_{i,j=1}^{q} a_{ij} X_i X_j + a_0 X_0,
\]

where \( (a_{ij}) \) is a \( q \times q \) positive constant matrix and \( a_0 \neq 0 \), and obtain Global Hölder estimates for \( L \) on \( G \) by establishing several estimates of singular integrals.

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1. INTRODUCTION

Let \( G \) be a homogeneous group and \( X_0, X_1, \ldots, X_q \) be left invariant real vector fields on \( \mathbb{R}^N \) \( (q < N) \). Assume that \( X_1, \ldots, X_q \) are homogeneous of degree one and \( X_0 \) is homogeneous of degree two, satisfying Hörmander’s condition

\[
\text{rank} \mathcal{L}(X_0, X_1, \ldots, X_q)(x) = N, x \in \mathbb{R}^N,
\]

where \( \mathcal{L}(X_0, X_1, \ldots, X_q) \) denotes the Lie algebra generated by \( X_0, X_1, \ldots, X_q \). In this paper we are interested in the following hypoelliptic operator with drift

\[
L = \sum_{i,j=1}^{q} a_{ij} X_i X_j + a_0 X_0. \tag{1.1}
\]

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where $a_0 \neq 0, (a_{ij})_{i,j=1}^q$ is a constant matrix satisfying
\begin{equation}
\mu^{-1} |\xi|^2 \leq \sum_{i,j=1}^q a_{ij} \xi_i \xi_j \leq \mu |\xi|^2, \xi \in \mathbb{R}^q,
\end{equation}
for a constant $\mu > 0$.

Many authors paid attention to the hypoelliptic operator. The outstanding result in [8] points out that Hörmander’s condition implies (actually, is equivalent to) the hypoellipticity of $L$ in (1.1). The existence of fundamental solutions for homogeneous hypoelliptic operators on nilpotent Lie groups was investigated by Folland in [6]. Bramanti and Brandolini in [2] proved the uniqueness of homogeneous fundamental solutions for $L$. Let us note that $L$ includes the classic Laplace operator and parabolic operator on Euclidean spaces. Another special case of $L$ is
\begin{equation}
L_1 = \sum_{i,j=1}^q a_{ij} \partial_{x_i x_j}^2 + \sum_{i,j=1}^n b_{ij} x_i \partial_{x_j} - \partial_t,
\end{equation}
where $(x,t) \in \mathbb{R}^{n+1}, X_0 = \sum_{i,j=1}^n b_{ij} x_i \partial_{x_j} - \partial_t$ and $X_i = \partial_{x_i}, i = 1, 2, \ldots, q, (a_{ij})_{i,j=1}^q$ is a positive matrix in $\mathbb{R}^q$, $(b_{ij})$ is a constant matrix with a suitable upper triangular structure. Note that $L_1$ belongs to a class of Kolmogorov-Fokker-Planck ultraparabolic operators. The operator $L_1$ appears in many research fields, for instance, in stochastic processes and kinetic models (see [3–5]), and in mathematical finance theory (see [1, 12]). After the previous study on $L_1$ in [9, 10], the authors of [7, 11, 13] established an invariant Harnack inequality for the non-negative solution of $L_1 u = 0$ by applying the mean value formula. With the theory of singular integral, Polidoro and Ragusa in [14] concluded some Morrey-type imbedding results and gave a local Hölder continuity of the solution.

The aim of the paper is to prove global Hölder estimates on the homogeneous group $G$ for $L$ by applying the properties of the fundamental solution for $L$ and several estimates of singular integrals on the homogeneous space. The method here is inspired by that used in [14]. Our results reflect the relations between the Morrey norms of $Lu$ and Hölder exponents for $u$ and $X_i u, i = 1, 2, \ldots, q$. In order to state our main results, we first introduce the definition of Morrey space.

**Definition 1.** For $p \in (1, \infty), \lambda \in [0, Q)$, the Morrey space on homogeneous group $G$ is defined by
\begin{equation}
L^{p,\lambda}(G) = \{ g \in L^p_{loc}(G) : \|g\|_{L^{p,\lambda}(G)} < \infty \},
\end{equation}
where
\begin{equation}
\|g\|_{L^{p,\lambda}(G)} = \left( \sup_{r>0, x \in G} \int_{B_r(x)} \frac{1}{r^\lambda} |g(y)|^p dy \right)^{1/p},
\end{equation}
$B_r(x)$ and $Q$ will be given in (2.1) and (2.2), respectively. Here $L^{p,0}(G) = L^p(G)$. 
The main results of this paper are as follows. For the case $\lambda \neq 0$, we have

**Theorem 1.**

1. If $1 < p < \frac{Q}{2}$, $Q - 2p < \lambda < Q - p$, then there exists a positive constant $c = c(p, \lambda)$ such that for every $u \in C_0^\infty(G)$ and any $x, z \in G, x \neq z$,

$$\frac{|u(x) - u(z)|}{\|z^{-1} \circ x\|^{\theta}} \leq c \|Lu\|_{L^{p, \lambda}(G)},$$

where $\theta = \frac{2p + \lambda - Q}{p}$.

2. If $1 < p < \frac{Q}{2}$, $Q - p < \lambda < Q$, then there exists a positive constant $c = c(p, \lambda)$ such that for every $u \in C_0^\infty(G)$ and any $x, z \in G, x \neq z$,

$$\frac{|X_i u(x) - X_i u(z)|}{\|z^{-1} \circ x\|^{\theta}} \leq c \|Lu\|_{L^{p, \lambda}(G)},$$

where $i = 1, \ldots, q$ and $\theta = \frac{p + \lambda - Q}{p}$.

For $\lambda = 0$, we have the following results, which restores the known result previously proved in [1].

**Remark 1.**

1. Assume $\frac{Q}{2} < p < Q$. Then there exists a positive constant $c = c(p)$ such that for every $u \in C_0^\infty(G)$ and any $x, z \in G, x \neq z$,

$$\frac{|u(x) - u(z)|}{\|z^{-1} \circ x\|^{\theta}} \leq c \|Lu\|_{L^p(G)},$$

where $\theta = \frac{2p - Q}{p}$.

2. Assume $p > Q$. Then there exists a positive constant $c = c(p)$ such that for every $u \in C_0^\infty(G)$ and any $x, z \in G, x \neq z$,

$$\frac{|X_i u(x) - X_i u(z)|}{\|z^{-1} \circ x\|^{\theta}} \leq c \|Lu\|_{L^p(G)},$$

where $i = 1, \ldots, q$ and $\theta = \frac{p - Q}{p}$.

The plan of the paper is as follows: in Section 2 we introduce some knowledge of homogeneous group and related lemmas. Estimates of two integral operators are proved. Section 3 is devoted to the proof of the main result.

## 2. Preliminary

Given a pair of mappings:

$$[(x, y) \mapsto x \circ y] : \mathbb{R}^N \times \mathbb{R}^N \mapsto \mathbb{R}^N, [x \mapsto x^{-1}] : \mathbb{R}^N \mapsto \mathbb{R}^N,$$
which are smooth, it follows that \( \mathbb{R}^N \) with these mappings forms a group, and the identity is the origin. If there exist \( 0 < \omega_1 \leq \omega_2 \leq \ldots \leq \omega_N \), such that the dilations

\[ D(\lambda) : (x_1, \ldots, x_N) \mapsto (\lambda^{\omega_1} x_1, \ldots, \lambda^{\omega_N} x_N), \lambda > 0, \]

are group automorphisms, then the space \( \mathbb{R}^N \) with this structure is called a homogeneous group and denoted by \( G \).

**Definition 2.** We define a homogeneous norm \( \| \cdot \| \) in \( G \) by the following way: if for any \( x \in G, x \neq 0 \), it holds

\[ \|x\| = \rho \iff |D(1/\rho)x| = 1, \]

where \( |\cdot| \) denotes the Euclidean norm; also, let \( \|0\| = 0 \).

It is not difficult to derive that the homogeneous norm satisfies

1. \( \|D(\lambda)x\| = \lambda \|x\| \) for every \( x \in G, \lambda > 0 \);
2. there exists \( c(G) \geq 1 \), such that for every \( x, y \in G \),

\[ \|x^{-1}\| \leq c \|x\| \text{ and } \|x \circ y\| \leq c(\|x\| + \|y\|). \]

In view of the above properties, it is natural to define the quasidistance \( d : \)

\[ d(x, y) = \|y^{-1} \circ x\|. \]

The ball with respect to \( d \) is denoted by

\[ B(x; r) \equiv B_r(x) \equiv \{y \in G : d(x, y) < r\}. \quad (2.1) \]

Note \( B(0, r) = D(r)B(0, 1) \), therefore

\[ |B(x, r)| = r^Q |B(0, 1)|, x \in G, r > 0, \]

where

\[ Q = \omega_1 + \ldots + \omega_N. \quad (2.2) \]

We will call that \( Q \) is the homogeneous dimension of \( G \). In general, \( Q \geq 3 \).

**Definition 3.** A differential operators \( Y \) on \( G \) is said homogeneous of degree \( \beta (\beta > 0) \), if for every test function \( \varphi \),

\[ Y(\varphi(D(\lambda)x)) = \lambda^\beta (Y\varphi)(D(\lambda)x), \lambda > 0, x \in G; \]

A function \( f \) is called homogeneous of degree \( \alpha \), if

\[ f((D(\lambda)x)) = \lambda^\alpha f(x), \lambda > 0, x \in G. \]

**Remark 2.** Clearly, if \( Y \) is a differential operators of degree \( \beta \) and \( f \) is a function of homogeneous of degree \( \alpha \), then \( Yf \) is homogeneous of degree \( \alpha - \beta \).

**Lemma 1.** ([2]) The operator \( L \) possesses a unique fundamental solution \( \Gamma(\cdot) \), such that for every test function \( u \in C_0^\infty(G) \) and every \( x \in G \), it holds

1. \( \Gamma(\cdot) \in C^\infty(G\setminus\{0\}); \)
(2) \( \Gamma(\cdot) \) is homogeneous of degree \( 2 - Q \);
(3) \( u(x) = (Lu * \Gamma)(x) = \int_{\mathbb{R}^N} \Gamma(y^{-1} \circ x) Lu(y) dy \);
(4) \( X_i u(x) = \int_{\mathbb{R}^N} X_i \Gamma(y^{-1} \circ x) Lu(y) dy \).

Remark 3. If we set \( \Gamma_i = X_i \Gamma, i = 1, \ldots, q \), then it is obvious from Remark 2 that \( \Gamma_i(\cdot) \) is homogeneous of degree \( 1 - Q \).

Proposition 1. ([2]) Let \( f \in C^1(\mathbb{R}^N \setminus \{0\}) \) is a homogeneous function of degree \( \lambda < 1 \). Then there exist two constants \( c = c(G, f) > 0 \) and \( M = M(G) > 1 \), such that for any \( x, y \) satisfying \( \|x\| \geq M \|y\| \),
\[
| f(x \circ y) - f(x) | + | f(y \circ x) - f(x) | \leq c \|y\| \|x\|^{\lambda-1},
\]
where \( c = c(G, f) \sup_{z \in \Sigma_N} |\nabla f(z)|, \Sigma_N \) is the unit sphere of \( \mathbb{R}^N \).

From Proposition 1, it follows

Lemma 2. If \( K \in C^1(G \setminus \{0\}) \) is a homogeneous function of degree \( \alpha < 1 \) with respect to the group \( (D(\lambda))_{\lambda > 0} \), then there exist two constants \( c > 0 \) and \( M > 1 \), such that if \( \|x\| \geq M \|x^{-1} \circ z\| \), then
\[
|K(z) - K(x)| \leq c \|x^{-1} \circ z\| \|x\|^\alpha.
\]

By Lemma 1 and Lemma 2, we have immediately

Lemma 3. For every \( x, y, z \in G \), it holds
(1) there exists a constant \( c > 0 \), such that
\[
\Gamma(y^{-1} \circ x) \leq \frac{c}{\|y^{-1} \circ x\|^{Q-2}};
\]
\[
\Gamma_i(y^{-1} \circ x) \leq \frac{c}{\|y^{-1} \circ x\|^{Q-1}}.
\]
(2) there exist two constants \( c > 0 \) and \( M > 1 \), such that if \( \|y^{-1} \circ x\| \geq M \|x^{-1} \circ z\| \), then
\[
|\Gamma(y^{-1} \circ x) - \Gamma(y^{-1} \circ z)| \leq \frac{c}{\|y^{-1} \circ x\|^{Q-1}};
\]
\[
|\Gamma_i(y^{-1} \circ x) - \Gamma_i(y^{-1} \circ z)| \leq \frac{c}{\|y^{-1} \circ x\|^{Q}}.
\]

Now let us introduce two integral operators. For \( p \in (1, \infty) \) and \( \lambda \in [0, Q) \), fixed \( z \in G \) and \( \sigma > 0 \), we define for every \( g \in L^{p,\lambda}(G) \) that
\[
T_{ag}(x) = \int_{\|y^{-1} \circ x\| \geq \sigma \|z^{-1} \circ x\|} \frac{g(y)}{\|y^{-1} \circ x\|^{Q-\alpha}} dy, \alpha \in [0, Q);
\]
Lemma 4. If \( \lambda + p\alpha < Q \), then there exists \( c = c(p, \lambda, \alpha, \sigma) > 0 \), such that

\[
|T_\alpha g(x)| \leq c \| g \|_{L^{p, \lambda}(G)} \left\| z^{-1} \circ x \right\|^{\frac{\lambda + \alpha - Q}{\rho}} \cdot c \left( 2k^{-1} \sigma \left\| z^{-1} \circ x \right\| \right)^{\frac{\rho - 1}{\rho}} \int_{B_{2k^{-1} \sigma}} |g(y)| dy
\]

if \( \lambda + p\beta > Q \), then there exists \( c = c(p, \lambda, \beta, \sigma) > 0 \), such that

\[
|T^\beta g(x)| \leq c \| g \|_{L^{p, \lambda}(G)} \left\| z^{-1} \circ x \right\|^{\frac{\lambda + \beta - Q}{\rho}} \cdot c \left( 2k^{-1} \sigma \left\| z^{-1} \circ x \right\| \right)^{\frac{\rho - 1}{\rho}} \int_{B_{2k^{-1} \sigma}} |g(y)| dy
\]

Proof. We follow the idea of Polidoro and Ragusa in [14]. If \( \lambda + p\alpha < Q \), then it obtains by decomposing the domain of integration and applying the Hölder inequality that

\[
|T_\alpha g(x)| \leq \sum_{k=1}^{\infty} \int_{B_{2k^{-1} \sigma}} |g(y)| dy \leq \sum_{k=1}^{\infty} \left( \frac{1}{2k^{-1} \sigma \left\| z^{-1} \circ x \right\|} \right)^{2k^{-1} \sigma \left\| z^{-1} \circ x \right\|} \left( \int_{B_{2k^{-1} \sigma}} |g(y)| dy \right)^{\frac{\rho}{\rho - 1}} \left\| g \right\|_{L^{p, \lambda}(G)} \left\| z^{-1} \circ x \right\|^{\frac{\lambda + \alpha - Q}{\rho}} \left( 2k^{-1} \sigma \left\| z^{-1} \circ x \right\| \right)^{\frac{\rho - 1}{\rho}} \sum_{k=1}^{\infty} \left( 2k^{-1} \sigma \left\| z^{-1} \circ x \right\| \right)^{\frac{\rho - 1}{\rho}}
\]

So (2.3) is proved, since the above series is convergent.

Similarly, if \( \lambda + p\beta > Q \), then

\[
|T^\beta g(x)| \leq \sum_{k=1}^{\infty} \int_{B_{2k^{-1} \sigma}} |g(y)| dy \leq \sum_{k=1}^{\infty} \left( \frac{1}{2k^{-1} \sigma \left\| z^{-1} \circ x \right\|} \right)^{2k^{-1} \sigma \left\| z^{-1} \circ x \right\|} \left( \int_{B_{2k^{-1} \sigma}} |g(y)| dy \right)^{\frac{\rho}{\rho - 1}} \left\| g \right\|_{L^{p, \lambda}(G)} \left\| z^{-1} \circ x \right\|^{\frac{\lambda + \beta - Q}{\rho}} \left( 2k^{-1} \sigma \left\| z^{-1} \circ x \right\| \right)^{\frac{\rho - 1}{\rho}} \sum_{k=1}^{\infty} \left( 2k^{-1} \sigma \left\| z^{-1} \circ x \right\| \right)^{\frac{\rho - 1}{\rho}}
\]
global Hölder estimates

\[ \leq \sum_{k=1}^{\infty} \left( \frac{1}{2^{-k} \sigma \|z^{-1} \circ x\|} \right)^{\beta} \left( \int_{B_{2^{-k} \sigma \|z^{-1} \circ x\|}(x)} |g(y)|^p dy \right)^{\frac{1}{p}} \]

\[ \left| B_{2^{-k} \sigma \|z^{-1} \circ x\|}(x) \right|^{\frac{p-1}{p}} \]

\[ \leq c \sum_{k=1}^{\infty} \left( \frac{1}{2^{-k} \sigma \|z^{-1} \circ x\|} \right)^{\beta} \left( \int_{B_{2^{-k} \sigma \|z^{-1} \circ x\|}(x)} |g(y)|^p dy \right)^{\frac{1}{p}} \]

\[ \left( 2^{-k} \sigma \|z^{-1} \circ x\| \right)^{\frac{(p-1)Q}{p}} \]

\[ \leq c \|g\|_{L^p(G)} \|z^{-1} \circ x\|^{\frac{p\bar{\alpha}+\lambda-Q}{p}} \sum_{k=1}^{\infty} \left( \frac{2^{Q-p\bar{\alpha}-\lambda}}{p} \right)^{k}. \]

This proves (2.4). \[\square\]

**Remark 4.** In particular, when \( \lambda = 0 \), we see that if \( p\alpha < Q \), then there exists a constant \( c = c(p, \alpha, \sigma) > 0 \), such that

\[ \left| Ta g(x) \right| \leq c \|g\|_{L^p(G)} \|z^{-1} \circ x\|^{\frac{p\alpha-Q}{p}} ; \quad (2.5) \]

if \( p\beta > Q \), then there exists a constant \( c = c(p, \beta, \sigma) > 0 \), such that

\[ \left| T^\beta g(x) \right| \leq c \|g\|_{L^p(G)} \|z^{-1} \circ x\|^{\frac{p\beta-Q}{p}} . \quad (2.6) \]

3. Proofs of the main results

**Proof of Theorem 1.** (1) With the help of (3) in Lemma 1 and Lemma 3, we know that there exist constants \( c > 0 \) and \( M > 1 \) such that

\[ |u(x) - u(z)| = \left| \int_{\mathbb{R}^N} \Gamma(y^{-1} \circ x) - \Gamma(y^{-1} \circ z) Lu(y) dy \right| \]

\[ \leq \int_{\mathbb{R}^N} \left| \Gamma(y^{-1} \circ x) - \Gamma(y^{-1} \circ z) \right| |Lu(y)| dy \]

\[ \leq \int_{\|y^{-1} \circ x\| \geq M \|x^{-1} \circ z\|} \left| \Gamma(y^{-1} \circ x) - \Gamma(y^{-1} \circ z) \right| |Lu(y)| dy \]

\[ + \int_{\|y^{-1} \circ x\| < M \|x^{-1} \circ z\|} \left| \Gamma(y^{-1} \circ x) - \Gamma(y^{-1} \circ z) \right| |Lu(y)| dy \]

\[ \leq \int_{\|y^{-1} \circ x\| \geq M \|x^{-1} \circ z\|} \left| \Gamma(y^{-1} \circ x) - \Gamma(y^{-1} \circ z) \right| |Lu(y)| dy \]

\[ + \int_{\|y^{-1} \circ x\| < M \|x^{-1} \circ z\|} \left| \Gamma(y^{-1} \circ x) \right| |Lu(y)| dy \]
\[ + \int_{\|y^{-1} \circ x\| < \|x^{-1} \circ z\|} |F(y^{-1} \circ z)| |L(u(y)|dy \]

\[ \leq \int_{\|y^{-1} \circ x\| \geq \|x^{-1} \circ z\|} \frac{c \|x^{-1} \circ z\|}{\|y^{-1} \circ x\|^{Q-1}} |L(u)|dy \]

\[ + \int_{\|y^{-1} \circ x\| < \|x^{-1} \circ z\|} \frac{c}{\|y^{-1} \circ x\|^{Q-2}} |L(u)|dy \]

Noting that if \( \|y^{-1} \circ x\| \geq M \|x^{-1} \circ z\| \), then

\[ \|y^{-1} \circ x\| \geq M \|x^{-1} \circ z\| \geq \frac{M}{c} \|z^{-1} \circ x\| \]

if \( \|y^{-1} \circ x\| < M \|x^{-1} \circ z\| \), then

\[ \|y^{-1} \circ x\| < Mc \|z^{-1} \circ x\| \]

and

\[ \|y^{-1} \circ z\| \leq c (\|y^{-1} \circ x\| + \|x^{-1} \circ z\|) < c (M \|x^{-1} \circ z\| + \|x^{-1} \circ z\|) \]

\[ = c (1 + M) \|x^{-1} \circ z\| , \]

it follows

\[ |u(x) - u(z)| \leq \int_{\|y^{-1} \circ x\| < \|x^{-1} \circ z\|} \frac{c \|x^{-1} \circ z\|}{\|y^{-1} \circ x\|^{Q-1}} |L(u)|dy \]

\[ + \int_{\|y^{-1} \circ x\| \geq \|x^{-1} \circ z\|} \frac{c}{\|y^{-1} \circ x\|^{Q-2}} |L(u)|dy \]

\[ + \int_{\|y^{-1} \circ z\| < c(1 + M) \|x^{-1} \circ z\|} \frac{c}{\|y^{-1} \circ z\|^{Q-2}} |L(u)|dy \]

\[ = I_1 + I_2 + I_3. \]

Applying Lemma 4 \((\alpha = 1 \text{ and } \sigma = \frac{M}{c})\) and noting \(\lambda + p < Q\), there exists a constant \(c = c(p, \lambda, \sigma) > 0\) such that

\[ I_1 \leq c \|L_u\|_{L^p; \lambda(G)} \|z^{-1} \circ x\| \|z^{-1} \circ x\|^\frac{\beta + \lambda - Q}{p} = c \|L_u\|_{L^p; \lambda(G)} \|z^{-1} \circ x\|^\frac{2p + \lambda - Q}{p} ; \]

from Lemma 4 \((\beta = 2 \text{ and } \sigma = Mc; \beta = 2 \text{ and } \sigma = c(1 + M) \), respectively) and \(\lambda + 2p > Q\), it follows

\[ I_2 \leq c \|L_u\|_{L^p; \lambda(G)} \|z^{-1} \circ x\|^\frac{2p + \lambda - Q}{p} \]
and
\[ I_3 \leq c \| Lu \|_{L^p(G)} \| z^{-1} \circ x \|^{2p + 1 - \frac{Q}{p}}. \]

In conclusion, we deduce (1.3).

(2) We know from (4) in Lemma 1 and Lemma 3 that there exist two constants \( c > 0 \) and \( M > 1 \) such that
\[
|X_i u(x) - X_i u(z)| = \left| \int_{\mathbb{R}^N} \Gamma_i(y^{-1} \circ x) - \Gamma_i(y^{-1} \circ z) Lu(y) dy \right|
\leq \int_{\mathbb{R}^N} \left| \Gamma_i(y^{-1} \circ x) - \Gamma_i(y^{-1} \circ z) \right| |Lu(y)| dy
\leq \int_{\|y^{-1} \circ x\| \geq M \|x^{-1} \circ z\|} \left| \Gamma_i(y^{-1} \circ x) - \Gamma_i(y^{-1} \circ z) \right| |Lu(y)| dy
+ \int_{\|y^{-1} \circ x\| < M \|x^{-1} \circ z\|} \left| \Gamma_i(y^{-1} \circ x) - \Gamma_i(y^{-1} \circ z) \right| |Lu(y)| dy
\leq \int_{\|y^{-1} \circ x\| \geq M \|x^{-1} \circ z\|} \left| \Gamma_i(y^{-1} \circ x) - \Gamma_i(y^{-1} \circ z) \right| |Lu(y)| dy
+ \int_{\|y^{-1} \circ x\| < M \|x^{-1} \circ z\|} \frac{c}{\|y^{-1} \circ x\|^Q} |Lu(y)| dy
+ \int_{\|y^{-1} \circ x\| < M \|x^{-1} \circ z\|} \frac{c}{\|y^{-1} \circ z\|^Q} |Lu(y)| dy.
\]

Let us remark that if \( \|y^{-1} \circ x\| \geq M \|x^{-1} \circ z\| \), then
\[
\|y^{-1} \circ x\| \geq \frac{M}{c} \|x^{-1} \circ z\|;
\]
if \( \|y^{-1} \circ x\| < M \|x^{-1} \circ z\| \), then
\[
\|y^{-1} \circ x\| < M c \|x^{-1} \circ z\|
\]
and
\[
\|y^{-1} \circ z\| \leq c \left( \|y^{-1} \circ x\| + \|x^{-1} \circ z\| \right) < c \left( M \|x^{-1} \circ z\| + \|x^{-1} \circ z\| \right) = c (1 + M) \|x^{-1} \circ z\|.
\]
It implies

\[ |X_i u(x) - X_i u(z)| \leq \int \|y^{-1} \circ x\| \|x^{-1} \circ z\| \frac{c}{Q} |Lu(y)| dy \]

\[ + \int \|y^{-1} \circ x\| \leq M_c \|z^{-1} \circ x\| \frac{c}{Q-1} |Lu(y)| dy \]

\[ + \int \|y^{-1} \circ z\| \leq c(1+M) \|x^{-1} \circ z\| \frac{c}{Q-1} |Lu(y)| dy \]

\[ \leq I_4 + I_5 + I_6. \]

Applying Lemma 4 (\( \alpha = 0 \) and \( \sigma = \frac{M}{c} \)) and \( \lambda < Q \), there exists a constant \( c = c(p, \lambda, \sigma) > 0 \) such that

\[ I_4 \leq c \|Lu\|_{L^{p, \lambda}(G)} \|x^{-1} \circ x\| \frac{\lambda - \Omega}{p} = c \|Lu\|_{L^{p, \lambda}(G)} \|x^{-1} \circ x\| \frac{p + \lambda - \Omega}{p}; \]

from Lemma 4 (\( \beta = 1 \) and \( \sigma = M_c \); \( \beta = 1 \) and \( \sigma = c(1+M) \), respectively) and \( \lambda + p > Q \), it gets

\[ I_5 \leq c \|Lu\|_{L^{p, \lambda}(G)} \|x^{-1} \circ x\| \frac{p + \lambda - \Omega}{p}. \]

and

\[ I_6 \leq c \|Lu\|_{L^{p, \lambda}(G)} \|x^{-1} \circ x\| \frac{p + \lambda - \Omega}{p}. \]

In conclusion we reach to (1.4).

REFERENCES

[1] E. Barucci, S. Polidoro, and V. Vespri, “Some results on partial differential equations and Asian options,” Math. Models Methods Appl. Sci., vol. 11, no. 3, pp. 475–497, 2001.
[2] M. Bramanti and L. Brandolini, “L^p estimates for uniformly hypoelliptic operators with discontinuous coefficients on homogeneous groups,” Rend. Sem. Mat. Univ. Pol. Torino, vol. 58, pp. 389–433, 2000.
[3] S. Chandrasekhar, “Stochastic problems in physics and astronomy,” Rev. Mod. Phys., vol. 15, pp. 1–89, 1943.
[4] S. Chapman and T. G. Cowling, The mathematical theory of nonuniform gases, 3rd ed. Cambridge: Cambridge University Press, 1990.
[5] J. J. Duderstadt and W. R. Martin, Transport theory, ser. A Wiley-Interscience Publication. New York: John Wiley & Sons, 1979.
[6] G. B. Folland, “Subelliptic estimates and function spaces on nilpotent Lie groups,” Ark. Mat., vol. 13, pp. 161–207, 1975.
[7] N. Garofalo and E. Lanconelli, “Level sets of the fundamental solution and Harnack inequality for degenerate equations of Kolmogorov type,” Trans. Am. Math. Soc., vol. 321, no. 2, pp. 775–792, 1990.
[8] L. Hörmander, “Hypoelliptic second order differential equations,” Acta Math., vol. 119, pp. 147–171, 1967.
[9] L. P. Kuptsov, “Fundamental solutions for a class of second-order elliptic-parabolic equations,” English Transl. Differential Equations, vol. 8, pp. 1269–1278, 1972.
[10] L. P. Kuptsov, “Mean value theorem and a maximum principle for Kolmogorov’s equation,” *English Transl. Math. Notes*, vol. 15, pp. 280–286, 1974.

[11] E. Lanconelli and S. Polidoro, “On a class of hypoelliptic evolution operators,” *Rend. Sem. Mat. Univ. Pol. Torino*, vol. 52, pp. 29–63, 1994.

[12] A. Pascucci, “Hölder regularity for a Kolmogorov equation,” *Trans. Am. Math. Soc.*, vol. 355, no. 3, pp. 901–924, 2003.

[13] A. Pascucci and S. Polidoro, “On the Harnack inequality for a class of hypoelliptic evolution equations,” *Trans. Am. Math. Soc.*, vol. 356, no. 11, pp. 4383–4394, 2004.

[14] S. Polidoro and M. A. Ragusa, “Sobolev-Morrey spaces related to an ultraparabolic equation,” *Manuscr. Math.*, vol. 96, no. 3, pp. 371–392, 1998.

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