HARMONIC SECTIONS OF TANGENT BUNDLES EQUIPPED WITH RIEMANNIAN $g$-NATURAL METRICS

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Abstract

Let $(M, g)$ be a Riemannian manifold. When $M$ is compact and the tangent bundle $TM$ is equipped with the Sasaki metric $g^s$, the only vector fields which define harmonic maps from $(M, g)$ to $(TM, g^s)$, are the parallel ones. The Sasaki metric, and other well known Riemannian metrics on $TM$, are particular examples of $g$-natural metrics. We equip $TM$ with an arbitrary Riemannian $g$-natural metric $G$, and investigate the harmonicity of a vector field $V$ of $M$, thought as a map from $(M, g)$ to $(TM, G)$. We then apply this study to the Reeb vector field and, in particular, to Hopf vector fields on odd-dimensional spheres.

1 Introduction

Let $(M, g)$ be an $n$-dimensional Riemannian manifold. Its tangent bundle $TM$, equipped with the so-called Sasaki metric $g^s$, has been extensively studied by several authors and in many different contexts.

In particular, given a compact Riemannian manifold $(M, g)$, Nouhaud [N] considered the problem of determining harmonic sections of $(TM, g^s)$, that is, vector fields $V \in \mathfrak{X}(M)$ which define harmonic maps from $(M, g)$ to $(TM, g^s)$. She found the expression of the energy associated to $V$ and proved that parallel vector fields are all and the ones harmonic sections. Ishihara [I] obtained independently the same result, giving also the explicit expression of the tension field associated to a vector field $V$.

Given a vector field $V$ over a compact Riemannian manifold, the energy associated to the map $V : (M, g) \rightarrow (TM, g^s)$ admits the following very simple expression [N], [W0]:

\[ E(V) = \frac{n}{2} \text{vol}(M) + \frac{1}{2} \int_M ||\nabla V||^2 dv_g, \]

which, up to a constant, also corresponds to the total bending of $V$ [W1].

More recently, Gil-Medrano [G1] proved that critical points of $E : \mathfrak{X}(M) \rightarrow \mathbb{R}$, that is, the energy functional restricted to vector fields, are again parallel vector fields. Moreover, in the same paper she also determined the tension field associated to a unit vector field $V : (M, \bar{g}) \rightarrow (T_1M, g^s)$, where $\bar{g}$ is a new Riemannian metric on $M$, and investigated the problem of determining when $V$ defines a harmonic map.

Investigating critical points of the energy associated to vector fields is an interesting purpose under different points of view. On the one hand, in many cases a distinguished vector field appears in a natural way, and it is worthwhile to see how the criticality of such a vector field is

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related to the geometry of the manifold. A well known example of this situation is given by the Reeb vector field $\xi$ of a contact metric manifold ([31], [32]). On the other hand, vector fields determining harmonic maps, provide new and interesting examples of harmonic maps having as target some Riemannian manifolds endowed of an highly non-trivial geometry. For more details and the state of the art for criticality of vector fields, we can refer to the survey ([22]).

The Sasaki metric $g^s$ has been the most investigated among all possible Riemannian metrics on $TM$. However, in many different contexts such metrics showed a very "rigid" behaviour. Moreover, $g^s$ represents only one possible choice inside a wide family of Riemannian metrics on $TM$, known as Riemannian $g$-natural metrics, which depend on several independent smooth functions from $\mathbb{R}^n$ to $\mathbb{R}$. As their name suggests, those metrics arise from a very "natural" construction starting from a Riemannian metric $g$ over $M$. The introduction of $g$-natural metrics moves from the classification of natural transformations of Riemannian metrics on manifolds to metrics on tangent bundles [KoMSl], or equivalently, the description of all first order natural operators $D : S^2_+T \sim (S^2T^+)T$, transforming Riemannian metrics on manifolds into metrics on their tangent bundles [KoMSl] (see also [A]). Riemannian $g$-natural metrics have been completely described in [AS2]. They depend on six smooth functions from $\mathbb{R}^n$ to $\mathbb{R}$, special choices of which give all the well known examples of Riemannian metrics on $TM$ as $g^s$ itself, the Cheeger-Gromoll metric $g_{GC}$ and the metrics investigated in [O] (cf. Remark [1]).

Both the rigidity of the Sasaki metric, and the fact mentioned above that several well known examples of Riemannian metrics on $TM$ are $g$-natural, make interesting to investigate criticality of a vector field $V$, when $g^s$ is replaced by an arbitrary Riemannian $g$-natural metric $G$. In particular, the following questions arise:

1) When $V : (M, g) \rightarrow (TM, G)$ defines a harmonic map?

2) When $V$ is a critical point for the energy $E$ restricted to vector fields?

The aim of this paper is to answer the questions above. Note that in the study of Question 1, we shall find new examples of harmonic maps from $M$ to $TM$, defined by non-parallel vector fields (as Reeb vector fields and Hopf vector fields). The paper is organized in the following way. In Section 2, we shall recall the definition and basic properties of $g$-natural metrics on $TM$. The energy associated to $V : (M, g) \rightarrow (TM, G)$ when $M$ is compact, is explicitly calculated in Section 3, while in Section 4 we shall calculate the tension field associated to $V$. In Section 5, we shall determine some families of Riemannian $g$-natural metrics for which, as for $g^s$, parallel vector fields are all and the ones defining harmonic maps. In Section 6, we shall consider vector fields which are critical points for $E : X(M) \rightarrow \mathbb{R}$, emphasizing the cases when this property is not equivalent to harmonicity of $V : (M, g) \rightarrow (TM, G)$. Finally, in Section 7 we shall apply our study to the case of the Reeb vector field $\xi$ of a contact metric manifold and, in particular, to Hopf vector fields on odd-dimensional spheres.

2 Basic formulae on $g$-natural metrics on tangent bundles

Let $(M, g)$ be an $n$-dimensional Riemannian manifold and $\nabla$ its Levi-Civita connection. At any point $(x, u)$ of its tangent bundle $TM$, the tangent space of $TM$ splits into the horizontal and vertical subspaces with respect to $\nabla$:

$$(TM)_{(x,u)} = \mathcal{H}_{(x,u)} \oplus \mathcal{V}_{(x,u)}.$$  

For any vector $X \in M_x$, there exists a unique vector $X^h \in \mathcal{H}_{(x,u)}$ (the horizontal lift of $X$ to $(x, u) \in TM$), such that $\pi \circ X^h = X$, where $\pi : TM \rightarrow M$ is the natural projection. The vertical lift of a vector $X \in M_x$ to $(x, u) \in TM$ is a vector $X^v \in \mathcal{V}_{(x,u)}$ such that $X^v(df) = Xf$, for all functions $f$ on $M$. Here we consider 1-forms $df$ on $M$ as functions on $TM$ (i.e., $(df)(x, u) = uf$).
The map $X \to X^h$ is an isomorphism between the vector spaces $M_z$ and $\mathcal{H}_{(x,u)}$. Similarly, the map $X \to X^v$ is an isomorphism between $M_x$ and $\mathcal{V}_{(x,u)}$. Each tangent vector $\vec{Z} \in (TM)_{(x,u)}$ can be written in the form $\vec{Z} = X^h + Y^v$, where $X, Y \in M_x$ are uniquely determined vectors. Horizontal and vertical lifts of vector fields on $M$ can be defined in an obvious way and are uniquely defined vector fields on $TM$.

We now write $F$ for the natural bundle with $FM = \pi^*(T^* \otimes T^*)M \to M$. Then, we have $Ff(X_x, g_x) = (Tf)_*, f_*X_x, (Tf)_*\otimes f_*g_x$ for all manifolds $M$, local diffeomorphisms $f$ of $M$, $X_x \in T_x M$ and $g_x \in (T^* \otimes T^*)_x M$. The sections of the canonical projection $FM \to M$ are called $F$-metrics in literature. So, if we denote by $\otimes$ the fibered product of fibered manifolds, then the $F$-metrics are mappings $TM \otimes TM \otimes TM \to \mathbb{R}$ which are linear in the second and the third argument.

For a given $F$-metric $\delta$ on $M$, there are three distinguished constructions of metrics on the tangent bundle $TM$ [KSe]:

(a) If $\delta$ is symmetric, then the Sasaki lift $\delta^s$ of $\delta$ is defined by

$$
\begin{align*}
\delta^s_{(x,u)}(X^h, Y^h) &= \delta(u; X, Y), \\
\delta^s_{(x,u)}(X^h, Y^v) &= 0, \\
\delta^s_{(x,u)}(X^v, Y^h) &= 0, \\
\delta^s_{(x,u)}(X^v, Y^v) &= \delta(u; X, Y),
\end{align*}
$$

for all $X, Y \in M_x$. When $\delta$ is non degenerate and positive definite, so is $\delta^s$.

(b) The horizontal lift $\delta^h$ of $\delta$ is a pseudo-Riemannian metric on $TM$, given by

$$
\begin{align*}
\delta^h_{(x,u)}(X^h, Y^h) &= 0, \\
\delta^h_{(x,u)}(X^h, Y^v) &= \delta(u; X, Y), \\
\delta^h_{(x,u)}(X^v, Y^h) &= \delta(u; X, Y), \\
\delta^h_{(x,u)}(X^v, Y^v) &= 0,
\end{align*}
$$

for all $X, Y \in M_x$. If $\delta$ is positive definite, then $\delta^s$ is of signature $(m, m)$.

(c) The vertical lift $\delta^v$ of $\delta$ is a degenerate metric on $TM$, given by

$$
\begin{align*}
\delta^v_{(x,u)}(X^h, Y^h) &= \delta(u; X, Y), \\
\delta^v_{(x,u)}(X^h, Y^v) &= 0, \\
\delta^v_{(x,u)}(X^v, Y^h) &= 0, \\
\delta^v_{(x,u)}(X^v, Y^v) &= 0,
\end{align*}
$$

for all $X, Y \in M_x$. The rank of $\delta^v$ is exactly that of $\delta$.

If $\delta = g$ is a Riemannian metric on $M$, then these three lifts of $\delta$ coincide with the three well-known classical lifts of the metric $g$ to $TM$.

The three lifts above of natural $F$-metrics generate the class of $g$-natural metrics on $TM$. The introduction of $g$-natural metrics moves from the description of all first order natural operators $D : S^2_T T^* \dashv (S^2 T^*)T$, transforming Riemannian metrics on manifolds into metrics on their tangent bundles, where $S^2_T T^*$ and $S^2 T^*$ denote the bundle functors of all Riemannian metrics and all symmetric $(0, 2)$-tensors over $n$-manifolds respectively. For more details about the concept of naturality and related notions, we can refer to [KoMSl].

Every section $G : TM \to (S^2 T^*)T$ $TM$ is called a (possibly degenerate) metric. Then there is a bijective correspondence between the triples of first order natural $F$-metrics $(\zeta_1, \zeta_2, \zeta_3)$ and first order natural (possibly degenerate) metrics $G$ on the tangent bundles given by (cf. [KSc]):

$$
G = \zeta_1 + \zeta_2^h + \zeta_3^v.
$$

Therefore, to find all first order natural operators $S^2_T T^* \dashv (S^2 T^*)T$ transforming Riemannian metrics on manifolds into metrics on their tangent bundles, it suffices to describe all first order natural $F$-metrics, i.e. first order natural operators $S^2_T T^* \dashv (T, F)$. In this sense, it is shown in [KSc] (see also [KoMSl] and [AS1]) that all first order natural $F$-metrics $\zeta$ in dimension $n > 1$ form a family parametrized by two arbitrary smooth functions $\alpha_0, \beta_0 : \mathbb{R}^+ \to \mathbb{R}$, where
\( \mathbb{R}^+ \) denotes the set of all nonnegative real numbers, in the following way: For every Riemannian manifold \((M, g)\) and tangent vectors \(u, X, Y \in M_x\)

\[
\zeta_{(M,g)}(u)(X,Y) = \alpha_0(g(u,u))g(X,Y) + \beta_0(g(u,u))g(u,X)g(u,Y).
\]

If \( n = 1 \), then the same assertion holds, but we can always choose \( \beta_0 = 0 \). In particular, all first order natural \( F \)-metrics are symmetric.

We shall call a metric \( G \) on \( TM \), coming from \( g \) by a first order natural operator \( S^2 T^* \to (S^2 T^*)^* \), a \( g \)-natural metric \( \text{AS2} \). All \( g \)-natural metrics on the tangent bundle of a Riemannian manifold \((M, g)\) are completely determined as follows:

**Proposition 1** (\( \text{AS2} \)). Let \((M, g)\) be a Riemannian manifold and \( G \) be a \( g \)-natural metric on \( TM \). Then there are six smooth functions \( \alpha_i, \beta_i : \mathbb{R}^+ \to \mathbb{R}, i = 1, 2, 3 \), such that for every \( u, X, Y \in M_x \), we have

\[
\begin{align*}
G_{(x,u)}(X^h, Y^h) &= (\alpha_1 + \alpha_3)(r^2)g_x(X,Y) + (\beta_1 + \beta_3)(r^2)g_x(X,u)g_x(Y,u), \\
G_{(x,u)}(X^h, Y^v) &= \alpha_2(r^2)g_x(X,Y) + \beta_2(r^2)g_x(X,u)g_x(Y,u), \\
G_{(x,u)}(X^v, Y^h) &= \alpha_2(r^2)g_x(X,Y) + \beta_2(r^2)g_x(X,u)g_x(Y,u), \\
G_{(x,u)}(X^v, Y^v) &= \alpha_1(r^2)g_x(X,Y) + \beta_1(r^2)g_x(X,u)g_x(Y,u),
\end{align*}
\]

where \( r^2 = g_x(u,u) \).

For \( n = 1 \), the same holds with \( \beta_i = 0, i = 1, 2, 3 \).

**Notations 1.** In the sequel, we shall use the following notations:

- \( \phi_i(t) = \alpha_i(t) + t\beta_i(t) \)
- \( \alpha(t) = \alpha(t)(\alpha_1 + \alpha_3)(t) - \alpha_2(t) \)
- \( \phi(t) = \phi_1(t)(\phi_1 + \phi_3(t) - \phi_2(t) \)

for all \( t \in \mathbb{R}^+ \).

Riemannian \( g \)-natural metrics are characterized as follows:

**Proposition 2** (\( \text{AS2} \)). The necessary and sufficient conditions for a \( g \)-natural metric \( G \) on the tangent bundle of a Riemannian manifold \((M, g)\) to be Riemannian, are that the functions of Proposition 1 defining \( G \), satisfy the inequalities

\[
\begin{align*}
\alpha_1(t) &> 0, \\
\phi(t) &> 0,
\end{align*}
\]

for all \( t \in \mathbb{R}^+ \).

For \( n = 1 \), the system (2.3) reduces to \( \alpha_1(t) > 0 \) and \( \alpha(t) > 0 \), for all \( t \in \mathbb{R}^+ \).

**Convention 1.** a) In the sequel, when we consider an arbitrary Riemannian \( g \)-natural metric \( G \) on \( TM \), we implicitly suppose that it is defined by the functions \( \alpha_i, \beta_i : \mathbb{R}^+ \to \mathbb{R}, i = 1, 2, 3 \), given in Proposition 1 and satisfying (2.3).

b) Unless otherwise stated, all real functions \( \alpha_i, \beta_i, \phi_i, \alpha, \phi \) and their derivatives are evaluated at \( r^2 := g_x(u,u) \).

c) We shall denote respectively by \( R \) and \( Q \) the \textit{curvature tensor} and the \textit{Ricci operator} of a Riemannian manifold \((M, g)\). The tensor \( R \) is taken with the sign convention

\[
R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,
\]

for all vector fields \( X, Y, Z \) on \( M \).
Remark 1. In literature, there are some well known Riemannian metrics on the tangent bundle, which turn out to be special cases of Riemannian $g$-natural metrics (satisfying (2.3)). In particular:

- the Sasaki metric $g^s$ is obtained for
  \[
  (2.4) \quad \alpha_1(t) = 1, \quad \alpha_2(t) = \alpha_3(t) = \beta_1(t) = \beta_2(t) = \beta_3(t) = 0.
  \]
- the Cheeger-Gromoll metric $g_{GC}$ \cite{CG} is obtained when
  \[
  (2.5) \quad \alpha_2(t) = \beta_2(t) = 0, \quad \alpha_1(t) = \beta_1(t) = -\beta_3(t) = 1, \quad \alpha_3(t) = \frac{t}{1+t}.
  \]
- the two-parameters family of metrics investigated by Oproiu in \cite{O}, is obtained when there exist two smooth functions $v, w : \mathbb{R}^+ \rightarrow \mathbb{R}$, such that (see \cite{AS2})
  \[
  \begin{align*}
  (\alpha_1 + \alpha_3)(t) &= v(t/2), \quad (\beta_1 + \beta_3)(t) = w(t/2), \\
  \alpha_1(t) &= \frac{1}{v(t/2)}, \quad \beta_1(t) = -\frac{w(t/2)}{v(t/2)[v(t/2) + w(t/2)]}, \\
  \alpha_2(t) &= \beta_2(t) = 0.
  \end{align*}
  \]

Since $\alpha_2 = \beta_2 = 0$, all these metrics are examples of Riemannian $g$-natural metrics on $TM$, for which horizontal and vertical distributions are mutually orthogonal.

The Levi-Civita connection $\nabla$ of an arbitrary $g$-natural metric $G$ on $TM$, can be described as follows:

**Proposition 3** \cite{AS1}. Let $(M,g)$ be a Riemannian manifold, $\nabla$ its Levi-Civita connection and $R$ its curvature tensor. Let $G$ be a Riemannian $g$-natural metric on $TM$. Then the Levi-Civita connection $\nabla$ of $(TM,G)$ is characterized by

\[
(i)(\nabla_X Y^h)_{(x,u)} = (\nabla_X Y)^h_{(x,u)} + h\{A(u; X_x, Y_x)\} + v\{B(u; X_x, Y_x)\},
\]
\[
(ii)(\nabla_X Y^v)_{(x,u)} = (\nabla_X Y)^v_{(x,u)} + h\{C(u; X_x, Y_x)\} + v\{D(u; X_x, Y_x)\},
\]
\[
(iii)(\nabla_X Y^h)_{(x,u)} = h\{C(u; Y_x, X_x)\} + v\{D(u; Y_x, X_x)\},
\]
\[
(iv)(\nabla_X Y^v)_{(x,u)} = h\{E(u; X_x, Y_x)\} + v\{F(u; X_x, Y_x)\},
\]

for all vector fields $X, Y$ on $M$ and $(x, u) \in TM$. Here, $h\{\cdot\}$ and $v\{\cdot\}$ respectively denote the horizontal and vertical lifts of a vector tangent to $M$ and, for all $x \in M$ and vectors $u, X_x, Y_x$ tangent to $M$ at $x$, $A, B, C, D, E$ and $F$ are defined as follows:

\[
A(u; X_x, Y_x) = A_1[R_x(X_x, u)Y_x + R_x(Y_x, u)X_x] + A_2[g_x(Y, u)X_x + g_x(X, u)Y_x] + A_3g_x(R_x(X_x, u)Y_x, u) + A_4g_x(X_x, Y_x)u + A_5g_x(X_x, u)g_x(Y, u)u,
\]

where

\[
A_1 = \frac{-\alpha_3(\beta_1 + \beta_3)}{2\Phi},
\]
\[
A_2 = \frac{\alpha_2(\beta_1 + \beta_3)}{2\Phi},
\]
\[
A_3 = \frac{\alpha_3[\alpha_1(\beta_1 + \beta_3) - \phi_2\beta_3] + \alpha_2(\beta_1\alpha_3 - \beta_3\alpha_1]}{\alpha_3},
\]
\[
A_4 = \frac{\phi_2[\alpha_1 + \alpha_3]}{\alpha_3'},
\]
\[
A_5 = \frac{\alpha_2(\beta_1 + \beta_3) + (\beta_1 + \beta_3)[\alpha_2(\beta_2 - \phi_1(\beta_1 + \beta_3)] + (\alpha_1 + \alpha_3)(\alpha_1\beta_2 - \alpha_2\beta_1]}{\alpha_3}.
\]
\[ B(u; X_x, Y_x) = B_1 R_x(X_x, u)Y_x + B_2 R_x(X_x, Y_x)u + B_3 [g_x(Y_x, u)X_x + g_x(X_x, u)Y_x] + B_4 g_x(R_x(X_x, u)Y_x, u)u + B_5 g_x(X_x, Y_x)u + B_6 g_x(X_x, u)g_x(Y_x, u)u, \]

where

\[
\begin{align*}
B_1 &= \frac{\alpha^2}{\alpha}, \\
B_2 &= -\frac{\alpha_1(\alpha_1 + \alpha_3)}{2\alpha}, \\
B_3 &= -\frac{(\alpha_1 + \alpha_3)(\beta_1 + \beta_3)}{2\alpha}, \\
B_4 &= \frac{\alpha_2(\phi_2 + \phi_3(\beta_1 + \beta_3) + (\alpha_1 + \alpha_3)[\beta_1(\beta_1 - \beta_3)])}{\alpha_3}, \\
B_5 &= \frac{\phi_1 + \phi_3}{\alpha_3}, \\
B_6 &= \frac{\alpha(\phi_1 + \phi_3)(\beta_1 + \beta_3)^3 + (\beta_1 + \beta_3)[(\alpha_1 + \alpha_3)[\phi_1 + \phi_3(\beta_1 - \beta_3)] + \alpha_2(\beta_1 + \beta_3) - (\alpha_1 + \alpha_3)]}{\alpha_3}.
\end{align*}
\]

\[ C(u; X_x, Y_x) = C_1 R_x(Y_x, u)X_x + C_2 g_x(X_x, u)Y_x + C_3 g_x(Y_x, u)X_x + C_4 g_x(R_x(X_x, u)Y_x, u)u + C_5 g_x(X_x, Y_x)u + C_6 g_x(X_x, u)g_x(Y_x, u)u, \]

where

\[
\begin{align*}
C_1 &= -\frac{\alpha^2}{2\alpha}, \\
C_2 &= -\frac{\alpha_1(\beta_1 + \beta_3)}{2\alpha}, \\
C_3 &= \frac{\alpha_1(\alpha_1 + \alpha_3)}{2\alpha}, \\
C_4 &= \frac{\alpha_2(\phi_2(\beta_1 + \beta_3) + \phi_1(\beta_1 + \beta_3) - \phi_2 \beta_3)}{\alpha_3}, \\
C_5 &= \frac{\phi_1(\beta_1 + \beta_3)^2 + \phi_2(2\alpha_3 - \beta_2)}{\alpha_3}, \\
C_6 &= \frac{\alpha_3(\beta_1 + \beta_3)^3 + (\alpha_1(\beta_1 + \beta_3)[\phi_1 + \phi_3(\beta_1 - \beta_3)])}{\alpha_3} + \frac{\alpha_2(\alpha_3[\beta_1(\beta_1 + \beta_3)] + \alpha_2(\beta_1 + \beta_3) - (\alpha_1 + \alpha_3)]}{\alpha_3}.
\end{align*}
\]

\[ D(u; X_x, Y_x) = D_1 R_x(Y_x, u)X_x + D_2 g_x(X_x, u)Y_x + D_3 g_x(Y_x, u)X_x + D_4 g_x(R_x(X_x, u)Y_x, u)u + D_5 g_x(X_x, Y_x)u + D_6 g_x(X_x, u)g_x(Y_x, u)u, \]

where

\[
\begin{align*}
D_1 &= \frac{\alpha^2}{2\alpha}, \\
D_2 &= \frac{\alpha_2(\beta_1 + \beta_3)}{2\alpha}, \\
D_3 &= -\frac{\alpha_2(\alpha_1 + \alpha_3)}{2\alpha}, \\
D_4 &= \frac{\alpha_1(\alpha_1 + \alpha_3)[\alpha_1(\beta_1 - \alpha_2 \beta_3) + \alpha_2(\beta_2(\beta_1 + \beta_3) - \phi_1(\beta_1 + \beta_3)])}{2\alpha_3}, \\
D_5 &= \frac{\phi_1(\beta_1 + \beta_3) + \phi_3(2\alpha_3 - \beta_2)}{2\alpha_3}, \\
D_6 &= -\frac{\alpha_2(\beta_1 + \beta_3)^3 + (\alpha_1(\alpha_3[\beta_1(\beta_1 + \beta_3)]) + \alpha_2(\beta_2(\alpha_1 + \alpha_3) - \alpha_2(\beta_1 + \beta_3))] + \alpha_1(\beta_1 + \beta_3) - (\alpha_1 + \alpha_3)]}{2\alpha_3}.
\end{align*}
\]
$E(u; X_x, Y_x) = E_1[g_x(Y_x, u)X_x + g_x(X_x, u)Y_x] + E_2g_x(X_x, Y_x)u + E_3g_x(X_x, u)g_x(Y_x, u)u$

where

$$E_1 = \frac{\alpha_1(a'_1 + b'_1) - \alpha_2a'_1}{\alpha},$$
$$E_2 = \phi_1b_2 - \phi_2(\beta_1 - \alpha_1'),$$
$$E_3 = \frac{[\alpha(2\phi_1b'_1 - \phi_2b'_2) + 2\alpha' (\alpha_1 [\alpha_2(\beta_1 + \beta_2) - \beta_2(\alpha_1 + \alpha_3)] + \alpha_2(\beta_1 + \phi_3) - \beta_2\phi_2)]}{\alpha}\phi$$

(2.10)

$$F(u; X_x, Y_x) = F_1[g_x(Y_x, u)X_x + g_x(X_x, u)Y_x] + F_2g_x(X_x, Y_x)u + F_3g_x(X_x, u)g_x(Y_x, u)u$$

where

$$F_1 = -\frac{\alpha_2(a'_1 + b'_1) + (\alpha_1 + \alpha_3)\alpha'_1}{\alpha},$$
$$F_2 = \frac{(\phi_1 + \phi_3)(\beta_1 - \alpha_1') - \phi_2b_2}{\phi},$$
$$F_3 = \frac{[\alpha(\phi_1 + \phi_3)^2 - 2\phi_2b'_2 + 2\alpha' (\alpha_2(\beta_2(\alpha_1 + \alpha_3) - \beta_2(\beta_1 + \beta_2))] + (\alpha_1 + \alpha_3)\beta_2\phi_2 - \beta_1(\phi_1 + \phi_3)]}{\alpha}\phi$$

(2.11)

For $n = 1$, the same holds with $\beta_i = 0, i = 1, 2, 3$.

3 The energy of a vector field $V : (M, g) \to (TM, G)$

We shall first discuss geometric properties of the map $V : (M, g) \to (TM, G)$ defined by a vector field $V \in \mathfrak{X}(M)$. It is well known that if $TM$ is equipped with the Sasaki metric $g^s$, then $V$ defines an isometry $V : (M, g) \to (TM, g^s)$, that is, it satisfies $V^*g^s = g$, if and only if $V$ is parallel. We now replace $g^s$ by an arbitrary Riemannian $g$-natural metric $G$. Since $V\cdot X = X^h + (\nabla_X V)^s$ for any vector field $X$, from (2.2) we obtain

$$(V^*G)(X, Y) = G(X^h + (\nabla_X V)^s, Y^h + (\nabla_Y V)^s)$$
$$= (\alpha_1 + \alpha_3)(r^2)g(X, Y) + (\beta_1 + \beta_3)(r^2)g(X, V)g(Y, V)$$
$$+ \alpha_2(r^2) [g(X, \nabla_Y V) + g(Y, \nabla_X V)]$$
$$+ \beta_2(r^2) [g(X, V)g(\nabla_Y V, V) + g(Y, V)g(\nabla_X V, V)]$$
$$+ \alpha_1(r^2)g(\nabla_X V, \nabla_Y V) + \beta_1(r^2)g(\nabla_X V, V)g(\nabla_Y V, V),$$

for all vector fields $X, Y$, where $r = ||V||$ is a smooth function from $M$ to $\mathbb{R}^+$. Note that by (3.1), in general $V^*G$ also depends on the length of $V$.

In particular, under the assumption $\beta_1 + \beta_3 = 0$, (determining a very large family of $g$-natural metrics, which includes $g^s$ and depends on five smooth functions $\alpha_1, \alpha_2, \alpha_3, \beta_1$ and $\beta_2$), from (3.1) we easily get the following

Proposition 4. Let $G$ be a Riemannian $g$-natural metric and $V \in \mathfrak{X}(M)$.

1) If $\beta_1 + \beta_3 = 0$, then

a) $\nabla V = 0$ implies that $V^*G$ is homothetic to $g$, with homothety factor $(\alpha_1 + \alpha_3)(\rho)$, where $\rho = ||V||^2$ is constant. In particular, $V$ is an isometry when in addition $(\alpha_1 + \alpha_3)(\rho) = 1$. 


b) If \( M \) is compact and \( V \) has constant length \( ||V|| = \sqrt{\rho} \), then

\[
V^*G = (\alpha_1 + \alpha_3)(\rho) g \iff \nabla V = 0.
\]

2) If \( \alpha_2 = \beta_1 = \beta_2 = \beta_3 = 0 \), then \( V \) is parallel if and only if \( V^*G = (\alpha_1 + \alpha_3)(r^2) g \).

\textbf{Proof.} 1): a): it follows at once from (3.1), rewritten when \( \beta_1 + \beta_3 = 0 \) and \( \nabla V = 0 \).

b): the "if" part follows from a). For the "only if" part, we consider a local orthonormal basis \( \{e_i\} \) on \( M \) and apply (3.1) to pairs \( (e_i, e_i) \), for all \( i=1,..,n \). Taking into account the fact that \( V^*G = (\alpha_1 + \alpha_3)(\rho) g \) and summing up over \( i \), we easily get

\[
2\alpha_2(\rho)\text{div}V + \alpha_1(\rho)||\nabla V||^2 = 0.
\]

Since \( M \) is compact, we can integrate (3.2) over \( M \) and we obtain

\[
\alpha_1(\rho) \int_M ||\nabla V||^2dv_g = 0,
\]

because \( \rho \) is constant and \( \int_M \text{div}Vdv_g = 0 \). By (2.3), \( \alpha_1 > 0 \). So, (3.3) yields \( ||\nabla V||^2 = 0 \), that is, \( V \) is parallel.

2): the "if" part follows directly from a). For the "only if" part, it is enough to rewrite (3.1) for \( \alpha_2 = \beta_1 = \beta_2 = \beta_3 = 0 \) and \( V^*G = (\alpha_1 + \alpha_3)(r^2) g \), and we get

\[
\alpha_1(r^2)g(\nabla_X V, \nabla_Y V) = 0
\]

for all vector fields \( X, Y \). Since \( \alpha_1 > 0 \), we then have \( \nabla V = 0 \) \( \Box \).

In order to provide some examples, note that, by (2.4) and (2.5), the Sasaki metric \( g^s \) and the Cheeger-Gromoll metric \( g_{CG} \) on \( TM \) satisfy conditions listed at points 2) and 1) of Proposition 2, respectively.

Next, let \( f : (M, g) \to (M', g') \) be a smooth map between Riemannian manifolds, with \( M \) compact. The energy of \( f \) is defined as the integral

\[
E(f) := \int_M e(f)dv_g
\]

where \( e(f) = \frac{1}{2}||f_*||^2 = \frac{1}{2}\text{tr}_g f^*g' \) is the so-called energy density of \( f \). With respect to a local orthonormal basis of vector fields \( \{e_1, ..., e_n\} \) on \( M \), it is possible to express the energy density as \( e(f) = \frac{1}{2} \sum_{i=1}^n g'(f_*e_i, f_*e_i) \). Critical points of the energy functional on \( C^\infty(M, M') \) are known as harmonic maps. They have been characterized in [ESa] as maps having vanishing tension field \( \tau(f) = \text{tr}\nabla df \). When \( (M, g) \) is a general Riemannian manifold (including the non-compact case), a map \( f : (M, g) \to (M', g') \) is said to be harmonic if \( \tau(f) = 0 \). For further details about the energy functional, we can refer to [EL1], [LV].

Let now \( (M, g) \) be a compact Riemannian manifold of dimension \( n \) and \( (TM, G) \) its tangent bundle, equipped with an arbitrary Riemannian \( g \)-natural metric \( G \). Each vector field \( V \in \mathfrak{X}(M) \) defines a smooth map \( V : (M, g) \to (TM, G), p \mapsto V_p \). By definition, the energy \( E(V) \) of \( V \) is the energy associated to the corresponding map \( V : (M, g) \to (TM, G) \). Therefore, \( E(V) = \int_M e(V)dv_g \), where the density function \( e(V) \) is given by

\[
e_p(V) = \frac{1}{2}||V_p||^2 = \frac{1}{2}\text{tr}_g(V^*G)_p = \frac{1}{2} \sum_{i=1}^n (V^*G)_p(e_i, e_i).
\]
\{e_1, \ldots, e_n\} being any local orthonormal basis of vector fields defined in a neighborhood of \(p\). Using formulae (i)-(iv) of Proposition 3, we then have

\[
e(V) = \frac{1}{2} \sum_{i=1}^{n} G_{V}(V_{e_{i}}, V_{e_{i}}) = \frac{1}{2} \sum_{i=1}^{n} G_{V}(e_{i}^{h}, e_{i}^{h})
\]

\[
= \frac{1}{2} \sum_{i=1}^{n} \{(\alpha_{1} + \alpha_{3})(r^{2})g(e_{i}, e_{i}) + (\beta_{1} + \beta_{3})(r^{2})g(e_{i}, V) + 2\alpha_{2}(r^{2})g(e_{i}, \nabla e_{i}, V) + 2\beta_{2}(r^{2})g(e_{i}, V)(\nabla e_{i}, V) + \alpha_{1}(r^{2})g(\nabla e_{i}, V, \nabla e_{i}, V) + \beta_{1}(r^{2})g(\nabla e_{i}, V, V)^{2} \}
\]

where \(r = ||V||\) and so,

\[
(3.5) \quad e(V) = \frac{1}{2} \left\{ n(\alpha_{1} + \alpha_{3})(r^{2}) + (\beta_{1} + \beta_{3})(r^{2})r^{2} + 2\alpha_{2}(r^{2})\text{div}(V)
\]

\[
+ 2\beta_{2}(r^{2})V(r^{2}) + \alpha_{1}(r^{2})||\nabla V||^{2} + \frac{1}{4} \beta_{1}(r^{2})||\text{grad} r^{2}||^{2} \right\}.
\]

We now assume that \(M\) is compact and we rewrite \(E(V) = \int_{M} e(V)dv_{g}\) for some special kinds of vector fields. More precisely, we consider vector fields of constant length and, as a special case, parallel vector fields.

For any constant \(\rho > 0\), we put

\[
\mathcal{X}^{\rho}(M) = \{V \in \mathcal{X}(M) : ||V||^{2} = \rho\}.
\]

So, if \(V \in \mathcal{X}^{\rho}(M)\), then \(V\) has constant length satisfying \(||V||^{2} = \rho\). By (3.5) and taking into account the definition of \(\phi_{i}\) given in Notations 1, we easily get that the energy of \(V\) is given by

\[
(3.6) \quad E(V) = \frac{1}{2}[n(\alpha_{1} + \alpha_{3}) + \phi_{1} + \phi_{3}](\rho) \cdot \text{vol}(M, g) + \frac{1}{2} \alpha_{1}(\rho) \cdot \int_{M} ||\nabla V||^{2}dv_{g}.
\]

Since \(\alpha_{1} > 0\), (3.6) implies that

\[
(3.7) \quad E(V) \geq \frac{1}{2}[(n - 1)(\alpha_{1} + \alpha_{3}) + \phi_{1} + \phi_{3}](\rho) \cdot \text{vol}(M, g) > 0,
\]

for all \(V \in \mathcal{X}^{\rho}(M)\). (The last inequality follows at once from Notations 1 and (2.3)). The equality holds in (3.7) if and only if \(V\) is parallel. Therefore, we have the following

**Theorem 1.** Let \((M, g)\) be a compact Riemannian manifold. Equipping TM with an arbitrary Riemannian \(g\)-natural metric \(G\), a vector field \(V \in \mathcal{X}^{\rho}(M)\) is an absolute minimum for the energy \(E : \mathcal{X}^{\rho}(M) \to \mathbb{R}\) restricted to \(\mathcal{X}^{\rho}(M)\) if and only if \(V\) is parallel.

In particular, from Proposition 3 and Theorem 1 it follows

**Corollary 1.** Let \((M, g)\) be a compact Riemannian manifold and \(V \in \mathcal{X}^{\rho}(M)\). With respect to a Riemannian \(g\)-natural metric \(G\) satisfying \(\beta_{1} + \beta_{3} = 0\), the following assertions are equivalent:

(i) \(V\) is an absolute minimum for the energy \(E : \mathcal{X}^{\rho}(M) \to \mathbb{R}\),

(ii) \(V\) is parallel,

(iii) \(V^{*}g = (\alpha_{1} + \alpha_{3})(\rho)g\) (that is, \(V : (M, g) \to (TM, G)\) is a homothetic immersion).
It is worth mentioning that Corollary 1 applies to both the Sasaki metric $g^s$ and the Cheeger-Gromoll metric $g_{CG}$ of $TM$.

Note that a parallel vector field $V$ necessarily has constant length. In fact, for all $X \in \mathfrak{X}(M)$ we have $2X(||V||^2) = g(\nabla_X V, V) = 0$. When $V$ is parallel, from (3.6) (or (3.5)) we have

$$E(V) = \frac{1}{2} \left[(n-1)(\alpha_1 + \alpha_3) + \phi_1 + \phi_3\right](\rho) \cdot \text{vol}(M, g),$$

where $||V||^2 = \rho$. By (3.3), a parallel vector field $V$ is a critical point for the energy restricted to the set

$$\mathfrak{X}_P(M) = \{V \in \mathfrak{X}(M) : \nabla V = 0\}$$

of all parallel vector fields, if and only if

$$[(n-1)(\alpha_1 + \alpha_3) + \phi_1 + \phi_3]'(\rho) = 0,$$

that is, $\rho = ||V||^2$ is a critical point of the function $[(n-1)(\alpha_1 + \alpha_3) + \phi_1 + \phi_3]$. As we shall see in the next Section, (3.9) is also a sufficient condition for a parallel vector field $V$ to define a harmonic map $V : (M, g) \to (TM, G)$.

4 The tension field associated to $V : (M, g) \to (TM, G)$

Let $(M, g)$ be a Riemannian manifold and $V \in \mathfrak{X}(M)$. The tension field associated to the map $V : (M, g) \to (TM, G)$, is defined as

$$\tau(V) : M \to V^{-1}(TTM)$$

$$p \mapsto \text{tr}(\nabla dV)_p.$$ 

Let $p$ be a point of $M$ and $\{e_1, ..., e_n\}$ a local orthonormal basis of vector fields, defined in a neighborhood of $p$. By (4.1), we have

$$\tau_p(V) = \sum_{i=1}^n (\nabla_{dV})(e_i, e_i)(p) = \sum_{i=1}^n \left(\nabla_{V^* e_i} V e_i - V^* (\nabla e_i e_i)\right)(p)$$

$$= \sum_{i=1}^n \left\{\tilde{\nabla}_{e_i} (\nabla e_i V)^v e_i^h + (\nabla e_i V)^v (\nabla e_i e_i)^v - (\nabla_{V^* e_i} V)^v\right\}(p)$$

$$= \sum_{i=1}^n \left\{\tilde{\nabla}_{e_i} e_i^h + \tilde{\nabla}_{V e_i} (\nabla e_i V)^v + \tilde{\nabla}_{(\nabla e_i V)^v} e_i^h + \tilde{\nabla}_{(\nabla V^v)^v} (\nabla e_i V)^v\right.$$ 
$$- (\nabla e_i e_i)^v - (\nabla_{V^* e_i} V)^v\right\}(p).$$
Hence, taking into account formulae of Proposition 3 for the Levi-Civita connection of an arbitrary Riemannian $g$-natural metric $G$ on $TM$, from (4.2) we easily get

\[
(4.3) \quad \tau_p(V) = \begin{cases} 
-2A_1 QV + 2C_1 \text{tr}[R(\nabla V, V)\cdot] + C_3 \sum_{i=1}^{n} e_i (r^2) e_i \\
+2C_2 Q V + E_1 \sum_{i=1}^{n} e_i (r^2) \nabla e_i V + \left[2A_2 - A_3 g(QV, V) + n A_4 \right] + A_5 r^2 + 2C_4 g(\text{tr}[R(\nabla V, V)\cdot], V) + 2C_5 \text{div} V + C_6 V(r^2) \\
+ E_2 ||\nabla V||^2 + \frac{1}{4} E_3 \sum_{i=1}^{n} [e_i (r^2)]^2 \right] V \end{cases} \]

where $r = ||V||$, $A_1, \ldots, F_1$ are evaluated at $r^2$ and $\bar{\Delta} V = - \text{tr} \nabla^2 V = - \sum_i (\nabla e_i \nabla e_i V - \nabla e_i e_i V)$ is the so-called rough Laplacian of $(M, g)$ calculated at $V$. Therefore, for the smooth map $V : (M, g) \to (TM, G)$ defined by a vector field $V \in \mathfrak{X}(M)$, by (4.3) we obtain the following

Theorem 2. Let $(M, g)$ be a compact Riemannian manifold. A vector field $V \in \mathfrak{X}(M)$ defines a harmonic map $V : (M, g) \to (TM, G)$ if and only if

\[
(4.4) \quad \tau_h(V) = -2A_1 QV + 2C_1 \text{tr}[R(\nabla V, V)\cdot] + C_3 \text{grad} r^2 + E_1 \nabla_{\text{grad} r^2} V \\
+2C_2 Q V + \left[2A_2 - A_3 g(QV, V) + n A_4 \right] + A_5 r^2 + 2C_4 g(\text{tr}[R(\nabla V, V)\cdot], V) \\
+2C_5 \text{div} V + C_6 V(r^2) + E_2 ||\nabla V||^2 + \frac{1}{4} E_3 ||\text{grad} r^2||^2 \right] V = 0
\]

and

\[
(4.5) \quad \tau_v(V) = -\bar{\Delta} V - B_1 QV + 2D_1 \text{tr}[R(\nabla V, V)\cdot] + D_3 \text{grad} r^2 + F_1 \nabla_{\text{grad} r^2} V \\
+2D_2 Q V + \left[2B_3 - B_4 g(QV, V) + n B_5 \right] + B_6 r^2 + 2D_4 g(\text{tr}[R(\nabla V, V)\cdot], V) \\
+2D_5 \text{div} V + D_6 V(r^2) + F_2 ||\nabla V||^2 + \frac{1}{4} F_3 ||\text{grad} r^2||^2 \right] V = 0,
\]

where, for all points $p \in M$, $\tau_h(V)(p)$ and $\tau_v(V)(p)$ denote the vectors tangent to $M$ at $p$, such that $\tau(V)_p = \{\tau_h(V)(p)\}^h + \{\tau_v(V)(p)\}^v$.

Remark 2. Since the condition $\tau(V) = 0$ has a tensorial character, as usual we can assume it as a definition of harmonic maps even when $M$ is not compact, and Theorem 2 extends at once to the non-compact case.
Remark 3. We now specify (4.1) and (4.5) for classical metrics on $TM$.

a) When $G = g^s$ is the Sasaki metric, we find the well known result: $V : (M, g) \to (TM, g^s)$ is a harmonic map if and only if

\begin{align}
(4.6) \quad \text{tr}[R(\nabla V, V)] &= 0 \quad \text{and} \\
(4.7) \quad \bar{\Delta} V &= 0.
\end{align}

b) When $G = g_{CG}$ is the Cheeger-Gromoll metric, then, using (2.5), (4.4) and (4.5), we easily get that $V : (M, g) \to (TM, g_{CG})$ is a harmonic map if and only if

\begin{align}
(4.8) \quad \text{tr}[R(\nabla V, V)] &= 0 \quad \text{and} \\
(4.9) \quad (1 + r^2) \bar{\Delta} V + \nabla_{\text{grad } r^2} V - \frac{1}{1 + r^2} \left[ (2 + r^2) ||\nabla V||^2 + \frac{1}{4} ||\text{grad } r^2||^2 \right] V &= 0.
\end{align}

Note that horizontal harmonicity of a vector field $V$, with respect to $g^s$ and $g_{CG}$, are expressed by the same condition.

We can now apply Theorem 2 to investigate relationships between harmonicity of maps defined by some special vector fields and properties of $g$-natural metrics.

a) Parallel vector fields

It is well known that the existence of a non-vanishing parallel vector field $V$ on a Riemannian manifold $(M, g)$ is equivalent to the local reducibility of $M$ as $\mathbb{R} \times M'$, equipped with the product metric, and $V$ is (locally) identified with a vector field tangent to the flat component $\mathbb{R}$ of the product. Rewriting (4.1) and (4.3) for a parallel vector field $V$, we have that $\tau(V) = 0$ (and so, $V$ defines a harmonic map from $(M, g)$ to $(TM, G)$) if and only if

\begin{align}
(4.10) \quad -2A_1(\rho)QV + [2A_2 - A_3 g(QV, V) + nA_4 + \rho A_5] (\rho) V &= 0 \\
(4.11) \quad -\bar{\Delta} V - B_1(\rho) QV + [2B_2 - B_4 g(QV, V) + nB_5 + \rho B_6] (\rho) V &= 0,
\end{align}

where $\sqrt{\rho} = ||V||$ is the constant length of $V$.

Since $V$ is tangent to the flat component $\mathbb{R}$ of the local decomposition $M = \mathbb{R} \times M'$, it annihilates the curvature. Moreover, $\bar{\Delta} V = 0$ for a parallel vector field. Therefore, (4.10) and (4.11) are equivalent to

\begin{align}
(4.12) \quad [2A_2 + nA_4 + \rho A_5] (\rho) = [2B_3 + nB_5 + \rho B_6] (\rho) = 0.
\end{align}

Using (2.6) and (2.7), we can easily conclude that (4.12) gives exactly (3.9), that is,

\begin{align}
[(n - 1)(\alpha_1 + \alpha_3) + \phi_1 + \phi_3]' (\rho) &= 0.
\end{align}

Therefore, we have the following

**Theorem 3.** A parallel vector field $V$ defines a harmonic map $V : (M, g) \to (TM, G)$ if and only if its constant length satisfies (3.9), that is, $\rho = ||V||^2$ is a critical point of the function

\begin{align}
(n - 1)(\alpha_1 + \alpha_3) + \phi_1 + \phi_3.
\end{align}

In particular:
Consider a vector field $\mathbf{V} \in \mathfrak{X}(M)$. Then, from (4.4), (4.5) we get at once the following

**Proposition 5.** A vector field $V \in \mathfrak{X}(M)$ satisfies $\tau(V) = 0$ (and so, it defines a harmonic map $V : (M, g) \rightarrow (TM, G)$) if and only if

$$
\Delta \nabla V = 0
$$

and

$$
\nabla \nabla V = 0.
$$

In the special case when $(M, g)$ has constant sectional curvature $k$, from Proposition 5 it follows

**Corollary 2.** Let $(M, g)$ be a Riemannian manifold of constant sectional curvature $k$. A vector field $V \in \mathfrak{X}(M)$ defines a harmonic map from $V : (M, g) \rightarrow (TM, G)$ if and only if

$$
2(k C_1 + C_2) (\nabla V) + \sum_{j=1}^{3} (\nabla V)^j = 0
$$

and

$$
\Delta \nabla V + 2 k \nabla V = 0.
$$

**Proof.** Since $(M, g)$ has constant sectional curvature $k$, its curvature tensor $R$ is given by

$$
R(X, Y) Z = k (g(Y, Z) X - g(X, Z) Y).
$$

By (4.19) it easily follows that $Q V = (n - 1) k V$ and $\nabla [R(\nabla V)] = -k (\nabla V) + k \nabla V$.

Using these formulae in (4.15) and (4.16), we respectively get (4.17) and (4.18).
connection of a \(g\)-natural metric \(G\) given in Proposition\(^3\)\(^{(4.15)}\) and \(\text{(4.16)}\) reduce respectively to

\[
(4.20) \quad C_1(\rho)\text{tr} [R(\nabla V, V)] + C_2(\rho)\nabla V + [C_4(\rho)g(\text{tr} [R(\nabla V, V)])/V] + C_5(\rho)\text{div} V = 0
\]

and

\[
(4.21) \quad -\Delta V + (2B_3(\rho) + nB_5(\rho) + \rho B_6(\rho) + F_2(\rho)/||\nabla V||^2) V = 0.
\]

In particular, \(\text{(4.21)}\) implies at once that \(\Delta V\) is collinear with \(V\). So, \(V\) is an eigenvector for the rough Laplacian \(\Delta\) and, since \(\sqrt{p} = ||V||\) is a constant, we have \(\Delta V = \frac{1}{p}||\nabla V||^2V\) and \(\text{(4.21)}\) implies

\[
(4.22) \quad \left( F_2(\rho) - \frac{1}{p} \right) ||\nabla V||^2 + (2B_3 + nB_5 + tB_6)(\rho) = 0.
\]

Again taking into account \(\alpha_2(\rho) = \beta_2(\rho) = 0, \quad (2.7)\) and \(\text{(2.11)}\), \(\text{(4.22)}\) may be easily rewritten as follows:

\[
(4.23) \quad \left( \frac{1}{\rho} \alpha_1 + \alpha_1' \right) (\rho)||\nabla V||^2 + \left[ (n - 1)(\alpha_1 + \alpha_3) + \phi_1 + \phi_3 \right]' (\rho) = 0.
\]

Because of \(\text{(4.23)}\), for different Riemannian \(g\)-natural metrics \(G\) some very different situations can occur about the harmonicity of the map \(V : (M, g) \to (TM, G)\) defined by \(V \in \mathcal{X}^p(M)\). The results are resumed in the following

Theorem 4. Let \((M, g)\) be a Riemannian manifold and \(G\) a Riemannian \(g\)-natural metric on \(TM\) satisfying \(\alpha_2(\rho) = \beta_2(\rho) = 0, \quad \rho > 0\). Then, a vector field \(V \in \mathcal{X}^p(M)\) defines a harmonic map \(V : (M, g) \to (TM, G)\) if and only if it satisfies \(\text{(4.20)}\) and \(\text{(4.21)}\). In particular:

(i) If

\[
(4.24) \quad \left( \frac{1}{\rho} \alpha_1 + \alpha_1' \right) (\rho) = \left[ (n - 1)(\alpha_1 + \alpha_3) + \phi_1 + \phi_3 \right]' (\rho) = 0,
\]

then \(V \in \mathcal{X}^p(M)\) defines a harmonic map \(V : (M, g) \to (TM, G)\) if and only if \(V\) is an eigenvector of \(\Delta\) and \(\text{(4.20)}\) holds.

(ii) If

\[
(4.25) \quad \left( \frac{1}{\rho} \alpha_1 + \alpha_1' \right) (\rho) \neq 0 = \left[ (n - 1)(\alpha_1 + \alpha_3) + \phi_1 + \phi_3 \right]' (\rho),
\]

then \(V \in \mathcal{X}^p(M)\) defines a harmonic map \(V : (M, g) \to (TM, G)\) if and only if \(V\) is parallel.

(iii) If

\[
(4.26) \quad \left( \frac{1}{\rho} \alpha_1 + \alpha_1' \right) (\rho) = 0 \neq \left[ (n - 1)(\alpha_1 + \alpha_3) + \phi_1 + \phi_3 \right]' (\rho),
\]

there are not vector fields \(V \in \mathcal{X}^p(M)\) defining harmonic maps from \((M, g)\) to \((TM, G)\).

(iv) If

\[
(4.27) \quad \left( \frac{1}{\rho} \alpha_1 + \alpha_1' \right) (\rho) \neq 0 \neq \left[ (n - 1)(\alpha_1 + \alpha_3) + \phi_1 + \phi_3 \right]' (\rho),
\]

then \(V \in \mathcal{X}^p(M)\) defines a harmonic map \(V : (M, g) \to (TM, G)\) if and only if \(\text{(4.20)}\) holds, \(\Delta V\) is collinear to \(V\) and the length of \(\nabla V\) satisfies

\[
(4.28) \quad ||\nabla V||^2 = \frac{-\rho[(n - 1)(\alpha_1 + \alpha_3) + \phi_1 + \phi_3]' (\rho)}{(\alpha_1 + \rho \alpha_1') (\rho)}.
\]
In the case of the Cheeger-Gromoll metric $g_{CG}$, (2.23) easily implies \( \frac{1}{p} \alpha_1 + \alpha'_1 (\rho) \neq 0 \) and \( (n-1)(\alpha_1 + \alpha_3) + \phi_1 + \phi_3' (\rho) = 0 \). Therefore, when $TM$ is equipped with $g_{CG}$, the parallel ones are the only vector fields of constant length, defining harmonic maps. In particular, when $(M, g)$ has constant sectional curvature $k \neq 0$, then a vector field of constant length never defines a harmonic map $V : (M, g) \to (TM, g_{CG})$.

It is worthwhile to emphasize that, since a general Riemannian $g$-natural metric $G$ depends on six different smooth functions $\alpha_1$, $\alpha_2$, $\alpha_3$, $\beta_1$, $\beta_2$ and $\beta_3$ (satisfying inequalities (2.23)), in each of cases (i)-(iv) listed in Theorem 4 there are plenty of Riemannian $g$-natural metrics which furnish examples. We now illustrate some interesting cases:

**Example A:** Assume $(M, g)$ has constant sectional curvature $k$. For any $\varepsilon > 0$, there exists a family of Riemannian $g$-natural metrics $\{G_\varepsilon\}$, such that for all $\rho \geq \varepsilon$, $V \in \mathcal{X}^p(M)$ defines a harmonic map from $(M, g)$ to $(TM, G_\varepsilon)$ if and only if $\Delta V$ is collinear to $V$.

In fact, it suffices to consider the family of $g$-natural metrics $\{G_\varepsilon\}$ defined by the functions

\[
\begin{cases}
\alpha_1(t) = \lambda/t, & \text{(for } t \geq \varepsilon\), \text{ and prolonged smoothly and positively to } [0, \varepsilon]), \\
\alpha_2 = \beta_2 = 0, \\
\alpha_1 + \alpha_3 = \mu, \\
\beta_1 + \beta_3 = -k_0 \alpha_1, \\
\beta_1 \text{ arbitrary such that } \alpha_1(t) + t \beta_1(t) > 0 \text{ for all } t > 0,
\end{cases}
\]

(4.29)

where $\lambda > 0$ and $\mu > \sup(0, k\lambda)$. Formulae (4.29) ensure that each $G_\varepsilon$ is Riemannian and, for all $\rho \geq \varepsilon$, we are in case (i) of Theorem 4. Moreover, (4.20), equivalently (4.17), is satisfied. Note that whenever $\varepsilon \leq 1$, this case applies to Hopf vector fields of an odd-dimensional sphere.

**Example B:** For any $\delta > 0$, there exists a family of Riemannian $g$-natural metrics $\{G_\delta\}$, such that for all $\rho \geq \delta$, $V \in \mathcal{X}^p(M)$ never defines a harmonic map from $(M, g)$ to $(TM, G_\delta)$.

To show this, we consider the family of $g$-natural metrics $\{G_\delta\}$ described by

\[
\begin{cases}
\alpha_1(t) = \lambda/t, & \text{(for } t \geq \varepsilon\), \text{ and prolonged smoothly to } [0, \varepsilon]), \\
\alpha_2 = \beta_2 = 0, \\
\alpha_1 + \alpha_3 = \mu, \\
(\beta_1 + \beta_3)(t) = \eta/t^2 & \text{(for } t \geq \varepsilon\), \text{ and prolonged smoothly to } [0, \varepsilon)), \\
\beta_1 \text{ arbitrary such that } \alpha_1(t) + t \beta_1(t) > 0 \text{ for all } t > 0,
\end{cases}
\]

(4.30)

for some positive constants $\lambda, \eta$. Then, each $G_\delta$ is Riemannian and for all $\rho \geq \delta$, we are in case (iii) of Theorem 4. □

As concerns the meaning of condition (4.20), notice that, since $V \in \mathcal{X}^p(M)$, (4.20) implies

(4.31)

\[ (C_1 + \rho C_4)(\rho) g(\text{tr}[R(\nabla, V, V)], V) + \rho C_5(\rho) \text{div}V = 0. \]

When $M$ is compact, then $\int_M \text{div}V dv_g = 0$ and (4.31) reduces to

\[ (C_1 + \rho C_4)(\rho) \int_M g(\text{tr}[R(\nabla, V, V)], V) dv_g = 0, \]

which in particular is satisfied whenever

(4.32)

\[ \text{tr}[R(\nabla, V, V)] = 0. \]
Moreover, using formulae of Proposition 3, we can conclude that if \((\beta_1 + \beta_3)(\rho) = 0\) (and \(\alpha_2(\rho) = \beta_2(\rho) = 0\), then

\[
\begin{align*}
C_1(\rho) &= -\frac{\alpha_1^2}{\beta_3^2} > 0, \\
C_2(\rho) &= C_4(\rho) = C_5(\rho) = 0
\end{align*}
\]

and so, (4.20) reduces to (4.32). We now apply this information to the special case of Killing vector fields of constant length.

The early theory of harmonic unit vector fields developed by Gil-Medrano and other authors (see [G2] for a survey) shows that there are many interesting contexts in which non-parallel unit vector fields satisfying (4.32) appear.

Let \(V \in \mathfrak{X}(M)\) be a Killing vector field. As it is well-known, \(V\) satisfies (4.34)

\[
QV = \bar{\Delta}V.
\]

In the special case of an Einstein manifold \(M\), we have \(QV = \frac{S}{n}V\), \(S\) being the scalar curvature of \((M, g)\). Therefore, if \(V \in \mathfrak{X}^p(M)\), by (4.34) it then follows

\[
\bar{\Delta}V = ||\nabla V||^2 V = \frac{S}{n}V.
\]

Consider now any Riemannian \(g\)-natural metric \(G\) on \(TM\), satisfying \(\alpha_2(\rho) = \beta_2(\rho) = 0\) and

\[
(\tau \alpha_1)'(\rho) \frac{S}{n} = -\rho [(n-1)(\alpha_1 + \alpha_3) + \phi_1 + \phi_3]'(\rho).
\]

Because of (4.35) and (4.36), we can conclude that (4.21) and (4.22) (equivalently, (4.23)) are satisfied. Therefore, if \(\alpha_2(\rho) = \beta_2(\rho) = 0\), a Killing vector \(V \in \mathfrak{X}^p(M)\) defines a harmonic map \(V : (M, g) \to (TM, G)\) if and only if

\[
C_1(\rho) \text{tr}[R(\nabla V, V);] + C_4(\rho) g(\text{tr}[R(\nabla V, V);], V)V = 0.
\]

Assuming also \((\beta_1 + \beta_3)(\rho) = 0\), (4.33) holds and hence, (4.37) is equivalent to requiring that \(\text{tr}[R(\nabla V, V);] = 0\). So, we have at once the following

**Theorem 5.** Let \((M, g)\) be an Einstein manifold, \(V \in \mathfrak{X}^p(M)\) a Killing vector field and \(G\) a Riemannian \(g\)-natural metric on \(TM\), satisfying

\[
\begin{align*}
\alpha_2(\rho) &= \beta_2(\rho) = 0, \\
(\beta_1 + \beta_3)(\rho) &= 0, \\
(\tau \alpha_1)'(\rho) \frac{S}{n} &= -\rho [(n-1)(\alpha_1 + \alpha_3) + \phi_1 + \phi_3]'(\rho).
\end{align*}
\]

Then, \(V\) defines a harmonic map \(V : (M, g) \to (TM, G)\) if and only if \(\text{tr}[R(\nabla V, V);] = 0\).

### 5 Riemannian \(g\)-natural metrics having parallel vector fields as the only harmonic sections

Theorem shows under which assumptions on a Riemannian \(g\)-natural metric \(G\), a parallel vector field \(V\) defines a harmonic map \(V : (M, g) \to (TM, G)\). Obviously, this also includes the case of the Sasaki metric \(g^s\) on \(TM\). In fact, when \(M\) is compact, parallel vector fields are all and the ones defining harmonic maps from \((M, g)\) into \((TM, g^s)\) [1], [3]. We now consider the question whether this rigidity property is peculiar to the Sasaki metric, or there are other Riemannian \(g\)-natural metrics having the same property.
By Theorem 2, a vector field \( V \in \mathfrak{X}(M) \) defines a harmonic map \( V : (M, g) \to (TM, G) \) if and only if both (4.4) and (4.5) are satisfied. Moreover, (4.13) gives a necessary condition on these metrics, for the harmonicity of parallel vector fields. However, (4.13) is not sufficient in general to conclude that a vector field \( V \), satisfying (4.4) and (4.5), is parallel.

Looking for some special forms of equations (4.4) and (4.5), we determine two classes of Riemannian \( g \)-natural metric \( G \), for which harmonic sections are all and the ones parallel vector fields. The first class, which also includes the Sasaki metric as special case, was determined starting from formulae of Proposition 3 and using (5.1), it is easy to check

Proof. We first notice that (5.1) (or (5.2)) are Riemannian. Hence, (3.9) is satisfied, that is, \( \alpha \) and only if both (4.4) and (4.5) hold. Starting from formulae of Proposition 3 and using (5.1), it is easy to check

Theorem 6. Let \((M, g)\) be a compact Riemannian manifold and \( G \) be a Riemannian \( g \)-natural metric on \( TM \), satisfying one of the following sets of conditions:

either

\[
\begin{cases}
\alpha_2 = \beta_2 = 0, \\
\alpha_1 = \text{constant} > 0,
\end{cases}
\]

(5.1)

or

\[
\begin{cases}
\alpha_1 = \text{constant} > 0, \\
\alpha_2 = \text{constant} \neq 0, \\
\alpha = \alpha_1(\alpha_1 + \alpha_3) - \alpha_2^2 > 0, \\
\beta_1 = \beta_2 = 0, \\
\beta_3 > 0, \\
(n - 1)(\alpha_1 + \alpha_3) + \phi_1 + \phi_3 = \text{constant}.
\end{cases}
\]

(5.2)

Then, for any \( V \in \mathfrak{X}(M) \), \( V \) defines a harmonic map \( V : (M, g) \to (TM, G) \) if and only if \( V \) is parallel.

Proof. We first notice that (5.1) (or (5.2)) implies (2.3). So, \( g \)-natural metrics described by (5.1) (or (5.2)) are Riemannian.

Suppose now that (5.1) holds. If \( V \in \mathfrak{X}(M) \) is parallel, denote by \( \rho \) the constant value of \( ||V||^2 \). Because of Theorem 3, harmonicity of \( V : (M, g) \to (TM, G) \) is equivalent to (3.9). By (5.1), \( \alpha_1 + \alpha_3 \) is constant and \( \beta_1 + \beta_3 = 0 \). So,

\[
[(n - 1)(\alpha_1 + \alpha_3) + \phi_1 + \phi_3]' = [n(\alpha_1 + \alpha_3) + \beta_1 + \beta_3]' = 0.
\]

Hence, (3.9) is satisfied, that is, \( V \) defines a harmonic map into \((TM, G)\).

As concerns the converse, if \( V \) defines a harmonic map into \((TM, G)\), then by Theorem 2, (4.4) and (4.5) hold. Starting from formulae of Proposition 5 and using (5.1), it is easy to check that (4.5) becomes

\[
-\bar{\Delta}V + \left[ \frac{\beta_1(r^2)}{\phi_1(r^2)} ||\nabla V||^2 + \frac{\beta_2(r^2)}{4\phi_1(r^2)} ||\text{grad} r^2||^2 \right] V = 0,
\]

(5.3)
where \( r^2 = ||V||^2 \). We take the scalar product of (5.3) by \( V \) and integrate over \( M \). Since
\[
\int_M g(\Delta V, V) dv_g = \int_M ||\nabla V||^2 dv_g,
\]
taking into account the definition of \( \phi_1 \), we get
\[
(5.4) \quad \int_M \frac{\alpha_1(r^2)}{\phi_1(r^2)} ||\nabla V||^2 dv_g - \int_M \frac{\beta'_1(r^2)}{4\phi_1(r^2)} ||\text{grad} \ r^2|| dv_g = 0.
\]
By (2.3) it follows \( \alpha_1, \phi_1 > 0 \). Moreover, by (5.1), \( \beta'_1 \leq 0 \). Therefore, \( 5.4 \) implies that \( V \) is parallel.

Next, assume \( 5.2 \) holds. Then \( 3.9 \) is satisfied and so, a parallel vector field is harmonic. Conversely, let \( V \in \mathfrak{X}(M) \) define a harmonic map into \((TM, G)\). By Theorem 2, (4.4) and (4.5) hold. Moreover, (5.2) implies that there exists a constant \( k \neq 0 \) (more explicitly, \( k = -\alpha_1/\alpha_2 \)), such that
\[
(2A_1, C_1, C_3, E_1, C_2, A_2, A_3, A_4, A_5, C_4, C_5, C_6, E_2, E_3) = k(B_1, D_1, D_3, F_1, D_2, B_3, B_1, B_5, B_6, D_4, D_5, D_6, F_2, F_3).
\]
We then divide (4.5) by \( k \) and subtract (4.5) by (4.4), and we obtain \( \Delta V = 0 \). Hence, \( 0 = \int_M g(\Delta V, V) dv_g = \int_M ||\nabla V||^2 dv_g \) and so, \( V \) is parallel \( \Box \)

Note that (5.1) determines a family of Riemannian \( g \)-natural metrics, depending on two real parameters \( \alpha_1 \) and \( \alpha_3 \) and a smooth function \( \beta_1 : \mathbb{R}^+ \to \mathbb{R} \) (satisfying some inequalities). Inside this class, the Sasaki metric \( g^s \) is the special case determined by \( \alpha_1 = 1 \) and \( \alpha_3 = \beta_1 = 0 \).

On the other hand, (5.2) also determines a family of Riemannian \( g \)-natural metrics, depending on two real parameters \( \alpha_1 \) and \( \alpha_2 \), and a smooth function \( \alpha_3 : \mathbb{R}^+ \to \mathbb{R} \), satisfying some inequalities. In fact, using the definitions of \( \phi_1 \), the last equation of (5.2) permits to write down \( \beta_3 \) in function of \( \alpha_1, \alpha_2 \) and \( \alpha_3 \). Obviously, this class does not contain the Sasaki metric, since in (5.2) we must have \( \alpha_2 \neq 0 \).

6 Critical points for the energy restricted to vector fields

Let \( (M, g) \) be a compact Riemannian manifold. We want to investigate conditions under which the map \( V : (M, g) \to (TM, G) \) associated to a vector field \( V \in \mathfrak{X}(M) \), is a critical point for the energy functional \( E : \mathfrak{X}(M) \to \mathbb{R} \), that is, only considering variations among maps defined by vector fields. Gil-Medrano [41] proved that, equipping \( TM \) with the Sasaki metric \( g^s \), \( V : (M, g) \to (TM, g^s) \) is a critical point for the energy functional \( E : \mathfrak{X}(M) \to \mathbb{R} \) if and only if \( V \) is parallel.

Consider now a vector field \( V \in \mathfrak{X}(M) \), and a smooth variation \( \{V_t\} \subset \mathfrak{X}(M) \) of \( V \), with \( |t| < \varepsilon \) and \( V_0 = V \). Note that \( \pi \circ V_t = \text{id}_M \) for all \( t \), where \( \pi : TM \to M \) is the natural projection and \( \text{id}_M \) the identity on \( M \). Therefore, the \textit{variational vector field} \( \tilde{W} \) associated to the variation satisfies
\[
\tilde{W}_p = \frac{\partial V_t(p)}{\partial t} \bigg|_{t = 0} \in V_{t_0} TM,
\]
for all \( p \in M \) and so, \( V \) is a critical point for \( E : \mathfrak{X}(M) \to \mathbb{R} \) if and only if
\[
0 = E'(0) = \frac{dE_t}{dt} \bigg|_{t = 0} = -\int_M G_{V_{t_0}} \left( \tau(V)_{t_0}, \tilde{W}_{t_0} \right) dv_g,
\]
for all variation \( \{V_t\} \subset \mathfrak{X}(M) \) of \( V \). Note that, as was already remarked in [41], for any vertical vector field \( W^v \), section of the bundle \( V^{-1}TTM \) of vector fields along \( V \), there exists a variation
\{V_i\} \subset \mathfrak{X}(M) of \ V, such that \ W^v = \frac{\partial V_i}{\partial t} \bigg|_{t=0}. So, by (6.1) it follows that \ V is a critical point for \ E: \mathfrak{X}(M) \to \mathbb{R} if and only if

\begin{equation}
(6.2) \quad \int_M G_{V_p} (\tau(V)_p, W^v_p) \, dv_g = 0,
\end{equation}

for all vector fields \ W \in \mathfrak{X}(M). Taking into account Proposition\[1\] we easily find that (6.2) is equivalent to

\begin{equation}
(6.3) \quad \int_M g (\alpha_2 \tau_h(V) + \beta_2 g(\tau_h(V), V)V + \alpha_1 \tau_v(V) + \beta_1 g(\tau_v(V), V)V, W) \, dv_g = 0,
\end{equation}

where \( \tau_h(V)(p), \tau_v(V)(p) \) denote the vectors tangent to \ M at \( p, \) such that \( \tau(V)_p = \{\tau_h(V)(p)\}_h + \{\tau_v(V)(p)\}_v, \) for all \( p \in M. \) Since (6.3) must hold for any vector field \( W \in \mathfrak{X}(M), \) it is equivalent to requiring that

\begin{equation}
(6.4) \quad T(V) := \alpha_2 \tau_h(V) + \beta_2 g(\tau_h(V), V)V + \alpha_1 \tau_v(V) + \beta_1 g(\tau_v(V), V)V = 0.
\end{equation}

Note that (6.4) expresses the \textit{vanishing of the projection of the tension field} \( \tau(V) \) \textit{into the vertical distribution}, with respect to an arbitrary Riemannian \( g \)-natural metric \( G. \)

Clearly, if \( V: (M, g) \to (TM, G) \) is a harmonic map, in particular \( V \) is a critical point for \( E: \mathfrak{X}(M) \to \mathbb{R}. \) This is also expressed by formula (6.4). In fact, if \( V: (M, g) \to (TM, G) \) is a harmonic map, then Theorem\[2\] implies that \( \tau_h(V) = \tau_v(V) = 0 \) and so, (6.4) holds. In general, the converse does not hold. To emphasize this, we consider the special situation when \( \alpha_2 = \beta_2 = 0. \) Under this assumption, \( T(V) = 0 \) is equivalent to requiring that \( \tau_v(V) = 0. \) In fact, taking into account \( \alpha_2 = \beta_2 = 0, \) if \( \tau_v(V) = 0 \) we have at once \( T(V) = 0. \) Conversely, if \( T(V) = 0, \) from \( \alpha_2 = \beta_2 = 0 \) it follows that (6.4) reduces to

\begin{equation}
(6.5) \quad \alpha_1 \tau_v(V) + \beta_1 g(\tau_v(V), V)V = 0.
\end{equation}

Taking the scalar product of both sides of (6.5) by \( V, \) since \( \phi_1 = \alpha_1 + r^2 \beta_1, \) where \( r^2 = ||V||^2, \) we get

\begin{equation}
(6.6) \quad \phi_1 g(\tau_v(V), V) = 0.
\end{equation}

By (2.3), \( \phi_1 > 0. \) Therefore, (6.6) gives \( g(\tau_v(V), V) = 0 \) and (6.5) reduces to \( \alpha_1 \tau_v(V) = 0. \) Again (2.3) gives \( \alpha_1 > 0 \) and so, \( \tau_v(V) = 0. \) In this way, by Theorem\[2\] we obtain at once the following

\textbf{Theorem 7.} Let \( (M, g) \) be a compact Riemannian manifold and \( G \) any Riemannian \( g \)-natural metric on \( TM \) with \( \alpha_2 = \beta_2 = 0. \) A vector field \( V \) on \( M \) defines a harmonic map \( V: (M, g) \to (TM, G) \) if and only if the following conditions hold:

i) \( \tau_h(V) = 0, \) and

ii) \( V \) is a critical point for \( E: \mathfrak{X}(M) \to \mathbb{R}, \) that is, \( T(V) = 0. \)

Coming back to the general case, we recall that formulae (4.4), (4.5) describe the tension field associated to \( V: (M, g) \to (TM, G), \) for an arbitrary Riemannian \( g \)-natural metric \( G. \) Using (4.4), (4.5) in (6.4) and taking into account (2.6), (2.11), some long but standard calculations lead to the following characterization:
Theorem 8. Let $(M, g)$ be a compact Riemannian manifold. A vector field $V \in \mathfrak{X}(M)$ is a critical point for $E : \mathfrak{X}(M) \to \mathbb{R}$ if and only if $T(V) = 0$, where

\begin{equation}
T(V) = -\alpha_1 \bar{\Delta}V + \left(\alpha'_2 - \frac{\beta_2}{2}\right) \mathrm{grad} r^2 + \alpha'_1 \mathrm{grad}_r V
- \left\{[(n-1)(\alpha_1 + \alpha_3) + \phi_1 + \phi_3]' + (2\alpha'_2 - \beta_2)\mathrm{div}V + (\alpha'_1 - \beta_1)||\nabla V||^2
\right. \\
- \beta_1 g(\bar{\Delta}V, V) - (\phi_2 C_6 + \phi_1 D_6 + \beta_2 C_2 + \beta_1 D_2 + \beta_2 C_3 + \beta_1 D_3) V(r^2)
- \frac{1}{4} (\phi_2 E_3 + \phi_1 F_3 + 2\beta_2 E_1 + 2\beta_1 F_1) ||\mathrm{grad} r^2||^2\right\} V,
\end{equation}

and all functions are evaluated at $r^2 = ||V||^2$.

Since the critical point condition $T(V) = 0$ has a tensorial character, it also makes sense when $(M, g)$ is not compact. For a general Riemannian manifold $(M, g)$, if a vector field $V$ satisfies $T(V) = 0$, we call it a $\mathfrak{X}$-harmonic vector field.

Remark 4. Specifying (6.7) for the Sasaki and Cheeger-Gromoll metrics of $TM$, we have the following results:

- if $G = g^s$, then (2.24) implies the well known formula $T(V) = -\bar{\Delta}V$.
- if $G = g_{CG}$, then applying (2.5) we easily obtain

\begin{equation}
T(V) = -\frac{1}{1 + r^2} \bar{\Delta}V - \frac{1}{(1 + r^2)^2} \mathrm{grad}_r V
- \frac{1}{1 + r^2} \left\{ -g(\bar{\Delta}V, V) + \frac{2 + r^2}{4(1 + r^2)} ||\nabla V||^2 - \frac{1}{4(1 + r^2)} ||\mathrm{grad} r^2||^2 \right\} V.
\end{equation}

We now determine $\mathfrak{X}$-harmonic vector fields, under some special assumptions either on the vector fields themselves or on the Riemannian $g$-natural metric $G$.

1) Parallel vector fields.
Suppose $V \in \mathfrak{X}(M)$ is a parallel vector field. Then, $\nabla V = 0$, $\bar{\Delta}V = 0$ and $||V||^2 = \rho$ is a constant. Thus, (6.7) reduces to

\begin{equation}
T(V) = -[(n-1)(\alpha_1 + \alpha_3) + \phi_1 + \phi_3]'(\rho) V.
\end{equation}

Hence, $T(V) = 0$ coincides with the necessary and sufficient condition we found in Theorem 3 for the harmonicity of $V : (M, g) \to (TM, G)$, and in Section 3 for critical points of the energy $E$ restricted to parallel vector fields. Therefore, we get the following

Theorem 9. Let $(M, g)$ be a Riemannian manifold and $G$ any Riemannian $g$-natural metric on $TM$. For a parallel vector field $V$ on $M$, the following statements are equivalent:

(a) $V : (M, g) \to (TM, G)$ is a harmonic map;

(b) $V$ is $\mathfrak{X}$-harmonic;

(c) $V$ is a critical point for $E$ in the set $\mathfrak{X}_P(M)$ of all parallel vector fields on $M$;

(d) $\rho = ||V||^2$ is a critical point for the function $[(n-1)(\alpha_1 + \alpha_3) + \phi_1 + \phi_3]$.

Theorem 9 includes as special cases both the Sasaki metric $g^s$ and the Cheeger-Gromoll metric $g_{CG}$ on $TM$, for which (d) is trivially satisfied and so, all parallel vector fields define harmonic maps.
2) Vector fields of constant length.
Considering a vector field \( V \in \mathcal{X}^p(M) \), by \( (6.7) \) we have that \( T(V) = 0 \) if and only if

\[
\alpha_1(\rho) \Delta V + \{ \beta_1(\rho) g(\Delta V, V) + [(n-1)(\alpha_1 + \alpha_3) + \phi_1 + \phi_3]'(\rho) \\
+ (2\alpha'_2 - \beta_2)(\rho) \operatorname{div} V + (\alpha'_1 - \beta_1)(\rho) \||\nabla V||^2 \} \ V = 0.
\]

By \( (6.8) \) it follows at once that \( \Delta V \) is collinear to \( V \). Therefore, since \( V \) has constant length \( ||V|| = \sqrt{\beta} \), we have \( \Delta V = \frac{1}{2} ||\nabla V||^2 V \) and from \( (6.8) \) we get

\[
(6.9) \quad \left( \frac{1}{\rho} \alpha_1 + \alpha'_1 \right)(\rho) ||\nabla V||^2 + (2\alpha'_2 - \beta_2)(\rho) \operatorname{div} V + [(n-1)(\alpha_1 + \alpha_3) + \phi_1 + \phi_3]'(\rho) = 0.
\]

Thus, Theorem 3 implies the following

**Theorem 10.** Let \( (M, g) \) be a Riemannian manifold and \( G \) any Riemannian \( g \)-natural metric on \( TM \). A vector field \( V \in \mathcal{X}^p(M) \) is \( \mathcal{X} \)-harmonic if and only if \( \Delta V \) is collinear to \( V \) and \( (6.9) \) holds.

For the Sasaki metric \( g^s \), an arbitrary vector field \( V \) is \( \mathcal{X} \)-harmonic if and only if \( \nabla V = 0 \). For the Cheeger-Gromoll metric \( g_{CG} \), by Theorem 10 we have the following

**Corollary 3.** Let \( (M, g) \) be a Riemannian manifold and equip \( TM \) with the Cheeger-Gromoll metric \( g_{CG} \). A vector field \( V \in \mathcal{X}^p(M) \) is \( \mathcal{X} \)-harmonic if and only if it is parallel (and so, if and only if \( V : (M, g) \to (TM, g_{CG}) \) is harmonic).

**Proof.** If \( V \in \mathcal{X}(M) \) is parallel, then the conclusion follows from Theorem 3. Conversely, assume \( V \in \mathcal{X}^p(M) \). Using \( (2.25) \), \( (6.9) \) gives at once \( \nabla V = 0 \). □

Equation \( (6.10) \) remains quite difficult to solve in full generality. For this reason, we consider the special case when \( \alpha_2(\rho) = \beta_2(\rho) = 0 \). Under this assumption, \( (6.9) \) becomes

\[
(6.10) \quad 2\alpha'_2(\rho) \operatorname{div} V + \left( \frac{1}{\rho} \alpha_1 + \alpha'_1 \right)(\rho) ||\nabla V||^2 + [(n-1)(\alpha_1 + \alpha_3) + \phi_1 + \phi_3]'(\rho) = 0.
\]

In particular, if \( \alpha'_2(\rho) = 0 \), then \( (6.10) \) gives exactly \( (4.23) \) which, together with the collinearity of \( \Delta V \) and \( V \), is equivalent to \( (4.21) \). Therefore, calculations above, together with Theorem 3 lead at once to the following

**Proposition 6.** Let \( (M, g) \) be a Riemannian manifold and \( V \in \mathcal{X}^p(M) \). For any Riemannian \( g \)-natural metric \( G \) on \( TM \), satisfying \( \alpha_2(\rho) = \alpha'_2(\rho) = \beta_2(\rho) = 0 \),

1. \( V \) is \( \mathcal{X} \)-harmonic if and only if \( (4.21) \) holds.
2. \( V \) defines a harmonic map \( V : (M, g) \to (TM, G) \) if and only if it is \( \mathcal{X} \)-harmonic and satisfies \( (4.21) \).

In particular, \( \mathcal{X} \)-harmonic vector fields do not necessarily define harmonic maps.

Taking into account formulae \( (4.21) \) determining the Riemannian \( g \)-natural metrics given in Example A, from Proposition 3 we obtain the following

**Corollary 4.** Let \( (M, g) \) be a Riemannian manifold of constant sectional curvature \( k \). For any \( \varepsilon > 0 \), there exists a family of Riemannian \( g \)-natural metrics \( \{ G_\varepsilon \} \), such that for all \( \rho \geq \varepsilon \), \( V \in \mathcal{X}^p(M) \) defines a harmonic map from \( (M, g) \) to \( (TM, G_\varepsilon) \) if and only if it is \( \mathcal{X} \)-harmonic.
7 Harmonicity of the Reeb vector field

We now apply the previous study to the case of some classic vector fields, namely, Reeb vector fields and Hopf vector fields, and we start by recalling some basic definitions and properties about contact metric manifolds.

Given a smooth manifold $M$ of odd dimension $n = 2m + 1$, a contact structure $(\eta, \varphi, \xi)$ over $M$ is composed by a global 1-form $\eta$ (the contact form) such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on $M$, a global vector field $\xi$ (the Reeb or characteristic vector field) and a global tensor $\varphi$, of type $(1,1)$, such that

\begin{equation}
\eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta\varphi = 0, \quad \varphi^2 = -I + \eta \otimes \xi. \tag{7.1}
\end{equation}

A Riemannian metric $g$ is said to be associated to the contact structure $(\eta, \varphi, \xi)$, if it satisfies

\begin{equation}
\eta = g(\xi, \cdot), \quad d\eta = g(\cdot, \varphi\cdot), \quad g(\cdot, \varphi\cdot) = -g(\varphi\cdot, \cdot). \tag{7.2}
\end{equation}

We refer to $(M, \eta, \xi)$ or to $(M, \eta, g, \xi, \varphi)$ as a contact metric manifold. As it is well known, the Reeb vector field $\xi$ plays a very important role in describing the geometry of a contact metric manifold. By (7.1) and (7.2) it follows at once that $\xi$ is a unit vector field on $(M, g)$, that is, $\xi \in \mathfrak{X}^1(M)$.

As it is well-known, the Reeb vector field $\xi$ satisfies

\begin{equation}
\nabla \xi = -\varphi - ph, \quad \nabla\xi = 0, \quad \text{div} \xi = 0, \tag{7.3}
\end{equation}

where $h = \frac{1}{2}\mathcal{L}_\xi \varphi$ is the Lie derivative of $\varphi$, and

\begin{equation}
||\nabla \xi||^2 = 2m + tr h^2 = 4m = g(Q\xi, \xi). \tag{7.4}
\end{equation}

Moreover, as it was proved in [P2],

\begin{equation}
\tilde{\Delta} \xi = 4m\xi - Q\xi. \tag{7.5}
\end{equation}

For further details, references and information about contact metric manifolds, we refer to [B].

In [P2], the third author introduced and studied $H$-contact spaces, that is, contact metric manifolds $(M, \eta, g, \xi, \varphi)$ whose Reeb vector field $\xi$ is a critical point for the energy functional $E$ restricted to the space $\mathfrak{X}^1(M)$ of all unit vector fields on $(M, g)$, considered as smooth maps from $(M, g)$ into the unit tangent sphere bundle $T^1M$, equipped with the Riemannian metric induced on $T^1M$ by the Sasaki metric $g^s$ of $TM$. As it was proved in [P2], $(M, \eta, g, \xi, \varphi)$ is $H$-contact if and only if $\xi$ is an eigenvector of the Ricci operator. (As a consequence, the class of $H$-contact manifolds is very large, since $\eta$-Einstein spaces, $K$-contact spaces, $(k, \mu)$-spaces and strongly locally $\phi$-symmetric spaces are all $H$-contact.)

We now use (7.3)-(7.5) to rewrite (4.15) and (4.16) for $\xi$. By Proposition 5 we then get the following

**Proposition 7.** Let $(M, \eta, g, \xi, \varphi)$ be a contact metric manifold and $G$ an arbitrary Riemannian $g$-natural metric on $TM$. The Reeb vector field $\xi$ defines a harmonic map $\xi : (M, g) \rightarrow (TM, G)$ if and only if

\begin{equation}
-2A_1(1)Q\xi + 2C_1(1)\text{tr}[R(\nabla\xi, \xi)] + \left[2A_2(1) + (2m + 1)A_4(1) + A_5(1)ight.
\end{equation}

\begin{equation}
+4mE_2(1) + 2C_4(1)g(\text{tr}[R(\nabla\xi, \xi)], \xi) - [A_3(1) + E_2(1)]g(Q\xi, \xi)] \xi = 0 \tag{7.6}
\end{equation}

and

\begin{equation}
[1 - B_1(1)]Q\xi + 2D_1(1)\text{tr}[R(\nabla\xi, \xi)] + \left[-4m + 2B_3(1) + (2m + 1)B_5(1)
\end{equation}

\begin{equation}
+ B_6(1) + 4mF_2(1) + 2D_4(1)g(\text{tr}[R(\nabla\xi, \xi)], \xi) - [B_4(1) + F_2(1)]g(Q\xi, \xi) \right] \xi = 0. \tag{7.7}
\end{equation}
Since \( C_1 = -\frac{\alpha_2^2}{\alpha_3} \neq 0 \), we can use (7.6) to write \( \text{tr}[R(\nabla,\xi,\xi)] \) as a linear combination of \( Q\xi \) and \( \xi \), and we get

\[
\text{tr}[R(\nabla,\xi,\xi)] = \frac{1}{2C_1(1)} \left\{ 2A_1(1)Q\xi - \left[ 2A_2(1) + (2m + 1)A_4(1) + A_5(1) + 4mE_2(1) + 2C_4(1)g(\text{tr}[R(\nabla,\xi,\xi)] - [A_3(1) + E_2(1)]g(Q\xi,\xi) \right] \xi \right\}.
\]

Replacing (7.8) in (7.7), we obtain

\[
\begin{align*}
(7.9) & \quad \left[1 - B_1(1) + \frac{2A_1(1)D_4(1)}{C_1(1)}\right] Q\xi + \left\{ \left[ -4m + 2B_3(1) + (2m + 1)B_5(1) + B_6(1) \right.ight. \\
& \quad + 4mF_2(1) + 2D_4(1)g(\text{tr}[R(\nabla,\xi,\xi)] - [B_3(1) + F_2(1)]g(Q\xi,\xi) \\
& \quad - D_1(1) \left[ 2A_2(1) + (2m + 1)A_4(1) + A_5(1) + 4mE_2(1) + 2C_4(1)g(\text{tr}[R(\nabla,\xi,\xi)] - [A_3(1) + E_2(1)]g(Q\xi,\xi) \right] \xi \\
& \quad - [A_3(1) + E_2(1)]g(Q\xi,\xi) \right\} \xi = 0.
\end{align*}
\]

Note that, by (2.6)-(2.9), we easily see that

\[
(7.10) \quad C_1(1) \text{tr}[R(\nabla,\xi,\xi)] + C_4(1)g(\text{tr}[R(\nabla,\xi,\xi)] - [A_3(1) + E_2(1)]g(Q\xi,\xi) \xi = 0
\]

and

\[
(7.11) \quad Q\xi = [4m - 2B_3(1) - (2m + 1)B_5(1) - B_6(1) - 4mF_2(1) + F_2(1)g(Q\xi,\xi)] \xi,
\]

respectively. (7.11) means that \( \xi \) is a Ricci eigenvector, that is, \( M \) is \( H \)-contact. Moreover, by (7.11), the corresponding Ricci eigenvalue \( g(Q\xi,\xi) \) depends on functions which determine the metric \( G \). On the other hand, by (7.10) we have \( g(Q\xi,\xi) = 2m - \text{tr}h^2 \) and so, (7.11) is equivalent to requiring that \( Q\xi \) is collinear to \( \xi \) and

\[
(7.12) \quad [F_2(1) - 1]h^2 = -[2B_3(1) + (2m + 1)B_5(1) + B_6(1) + 2mF_2(1) - 2m]j.
\]

Notice that since \( \xi \) is a unit vector, (7.12) also follows from (7.24). Taking into account formulae (2.6)-(2.11), we can write coefficients of (7.12) explicitly in function of \( \alpha_i, \beta_i \). Thus, (7.12) becomes

\[
(7.13) \quad (\text{tr}h^2 + 2m)(\alpha_1 + \alpha_4')(1) + [2m(\alpha_1 + \alpha_3) + \phi_1 + \phi_4'](1) = 0.
\]

As concerns (7.10), note that taking the scalar product of (7.10) by \( \xi \) and by an arbitrary vector field \( X \) orthogonal to \( \xi \), we obtain

\[
(7.14) \quad \begin{cases} [C_1(1) + C_4(1)]g(\text{tr}[R(\nabla,\xi,\xi)]) - [C_1(1)g(\text{tr}[R(\nabla,\xi,\xi)]X = 0 \end{cases} \quad \text{for all } X \perp \xi.
\]
As we already noticed, $C_1 = \frac{\alpha_1^2}{\alpha_2} \neq 0$. Moreover, since $\alpha_2(1) = \beta_2(1) = 0$, by (2.3) and the definition of $\phi$, $\alpha$ we easily get

$$C_1(1) + C_2(1) = -\frac{\alpha_1}{2(\alpha_1 + \alpha_3)} \neq 0.$$ 

Because of (7.14), (7.10) is equivalent to requiring $\text{tr}[R(\nabla, \xi, \xi)] = 0$. So, we have the following

**Theorem 12.** Let $(M, \eta, g, \xi, \varphi)$ be a contact metric manifold and $G$ any Riemannian $g$-natural metric on $TM$, satisfying $\alpha_2(1) = \beta_2(1) = 0$. Then $\xi$ defines a harmonic map $\xi : (M, g) \to (TM, G)$ if and only if $\xi$ is $H$-contact, (7.13) holds and $\text{tr}[R(\nabla, \xi, \xi)] = 0$.

**Remark 5.** a) We recall that a unit vector field $U$ defines a harmonic map $U : (M, g) \to (T^1M, g^*)$ if and only if $\Delta U$ is collinear to $U$ and $\text{tr}[R(\nabla U, U)] = 0$ (see [HY1]). For a Riemannian $g$-natural metric $G$ on $TM$, satisfying $\alpha_2(1) = \beta_2(1) = 0$ and (7.13), Theorem 12 gives the following interesting fact: $\xi : (M, g) \to (TM, G)$ is a harmonic map if and only if $\xi : (M, g) \to (T^1M, g^*)$ is a harmonic map.

b) When $(\alpha_1 + \alpha'_1)(1) = 0$, formula (7.13) reduces to (3.9). On the other hand, if $(\alpha_1 + \alpha'_1)(1) \neq 0$, then (7.13) implies that $\text{tr}h^2$ is constant. All homogeneous contact metric manifolds provide examples of contact metric spaces for which $\text{tr}h^2$ is constant.

A $K$-contact space is a contact metric manifold $(M, \eta, g, \xi, \varphi)$ satisfying $h = 0$. As it was remarked in [P2], a $K$-contact space is necessarily $H$-contact. For a $K$-contact space, (7.13) clearly reduces to

$$2m(\alpha_1 + \alpha'_1)(1) + [2m(\alpha_1 + \alpha_3) + \phi_1 + \phi_3]'(1) = 0.$$ 

As it is well-known, Sasakian manifolds are $K$-contact, while the converse only holds in dimension three. Assume now $(M, \eta, g, \xi, \varphi)$ is Sasakian and consider a $\varphi$-basis on $M$, that is, an orthonormal basis of vector fields $\{e_1, \ldots, e_m, \varphi e_1, \ldots, \varphi e_m, \xi\}$. Taking into account (7.1), the first equation in (7.3) and the first Bianchi identity, we can see that the Reeb vector field $\xi$ satisfies

$$-\text{tr}[R(\nabla, \xi, \xi)] = \text{tr}[R((\varphi + \varphi h), \xi)] = \sum_{i=1}^{m} [R(\varphi e_i, \xi)e_i + R(\varphi^2 e_i, \xi)\varphi e_i] = \sum_{i=1}^{m} [R(\varphi e_i, \xi)e_i - R(e_i, \xi)\varphi e_i] = -\sum_{i=1}^{m} R(e_i, \varphi e_i)\xi = 0,$$

since on a Sasakian manifold, $R(X, Y)\xi = 0$ for all $X, Y$ orthogonal to $\xi$ [13]. Hence, Theorem 12 implies the following

**Theorem 13.** Let $(M, \eta, g, \xi, \varphi)$ be a Sasakian manifold, $\dim M = 2m + 1$ and $G$ any Riemannian $g$-natural metric on $TM$, satisfying $\alpha_2(1) = \beta_2(1) = 0$. Then, $\xi$ defines a harmonic map $\xi : (M, g) \to (TM, G)$ if and only if (7.15) holds.

Next, we shall investigate under which conditions the Reeb vector field is $X$-harmonic. Since $\xi$ is a unit vector field, it is $X$-harmonic if and only if (6.8) holds. Moreover, taking into account (7.3) and (7.5), (6.8) becomes

$$\alpha_1(1)Q\xi = \{4m(\alpha_1 + \alpha'_1)(1) + [2m(\alpha_1 + \alpha_3) + (\phi_1 + \phi_3)]'(1) - \alpha'_1(1)g(Q\xi, \xi)\} \xi.$$ 

Since $\alpha_1 > 0$, (7.16) gives that $\xi$ is a Ricci eigenvector. Using this fact and (7.3), (7.16) reduces to (7.13). Hence, from Theorem 12 we obtain the following
Theorem 14. Let \((M, \eta, g, \xi, \varphi)\) be a contact metric manifold and \(G\) an arbitrary Riemannian \(g\)-natural metric on \(TM\). If \(\xi\) is \(X\)-harmonic, then \(M\) is \(H\)-contact. Conversely, if \(M\) is \(H\)-contact, then \(\xi\) is \(X\)-harmonic if and only if (7.13) holds.

Remark 6. Note that (7.13) is not fulfilled neither by the Sasaki metric nor by the Cheeger-Gromoll metric on \(TM\), as it is easy follows from (2.4) and (2.5), respectively. So, when \((M, \eta, g, \xi, \varphi)\) is an arbitrary contact metric manifold and \(TM\) is equipped with either \(g^*\) or \(g_{CG}\), then the Reeb vector field \(\xi\) is never \(X\)-harmonic. In particular, in such cases, \(\xi\) never defines a harmonic map.

On the other hand, it is easy to exhibit examples of Riemannian \(g\)-natural metrics, satisfying (7.13). For example, (7.13) holds for all Riemannian \(g\)-natural metrics belonging to the two-parameters family satisfying

\[
\begin{align*}
\alpha_1(t) &= k_1 e^{-t}, \\
\alpha_3(t) &= k_2 - \alpha_1(t), \\
\alpha_2 &= \beta_1 = \beta_2 = \beta_3 = 0,
\end{align*}
\]

where \(k_1, k_2\) are positive constants.

We now apply Theorem 14 to special classes of contact metric manifolds, namely, \(K\)-contact and \((k, \mu)\)-spaces. If we assume \((M, \eta, g, \xi, \varphi)\) is \(K\)-contact, then \(Q\xi = 2m\xi\) and (7.16) becomes (7.15).

Next, we recall that a contact metric manifold \((M, \eta, g, \xi, \varphi)\) for which \(\xi\) belongs to the \((k, \mu)\)-nullity distribution, that is,

\[
R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY),
\]

where \(\kappa, \mu\) are constants, is called a \((k, \mu)\)-space. Such class of spaces extends that of Sasakian manifolds. The constant \(\kappa\) satisfies \(\kappa \leq 1\); if \(\kappa = 1\), then \(\mu = 0\) and \(M\) is Sasakian ([13], Theorem 7.7). Moreover, \((k, \mu)\)-spaces are examples of strongly pseudo-convex CR manifolds ([13], Theorem 7.6), and non-Sasakian \((k, \mu)\)-spaces are examples of locally \(\phi\)-symmetric spaces ([13], p. 118). From (7.17), one gets \(Q\xi = 2m\kappa\xi\) and so, (7.16) becomes

\[
2m(2 - \kappa)(\alpha_1 + \alpha_1')(1) + [2m(\alpha_1 + \alpha_3) + \phi + \phi_3]'(1) = 0.
\]

Then, by Theorem 14 we have the following

Theorem 15. Let \((M, \eta, g, \xi, \varphi)\) be a contact metric manifold and \(G\) any Riemannian \(g\)-natural metric on \(TM\).

(i) If \(M\) is \(K\)-contact, then \(\xi\) is \(X\)-harmonic if and only if (7.15) holds.

(j) If \(M\) is a \((k, \mu)\)-space, then \(\xi\) is \(X\)-harmonic if and only if (7.18) holds.

We now recall that Hopf vector fields on the unit sphere \(S^{2m+1}\), equipped with its canonical metric \(g_0\), are all the ones Killing unit vector fields on \(S^{2m+1}\) [12]. Moreover, a Hopf vector field \(\xi\) can always be considered as the Reeb vector field of a suitable Sasakian structure \((S^{2m+1}, \bar{\eta}, \bar{g_0}, \bar{\xi}, \bar{\varphi})\), where \(\bar{\eta} = g_0(\cdot, \xi)\) and \(\bar{\varphi} = -\nabla\bar{\xi}\). Taking into account Theorems 13 and 15 above, we have

Corollary 5. For all Riemannian \(g\)-natural metrics on \(TS^{2m+1}\), satisfying \(\alpha_2(1) = \beta_2(1) = 0\), a Hopf vector field \(\xi\) defines a harmonic map \(\bar{\xi} : (S^{2m+1}, g_0) \rightarrow (TS^{2m+1}, G)\) if and only if (7.15) holds.

Corollary 6. For all Riemannian \(g\)-natural metrics on \(TS^{2m+1}\), a Hopf vector field \(\xi\) is \(X\)-harmonic if and only if (7.15) holds.
References

[A] K.M.T. Abbassi, Note on the classification Theorems of $g$-natural metrics on the tangent bundle of a Riemannian manifold $(M, g)$, *Comment. Math. Univ. Carolinae.*, 45 (4) (2004), 591–596.

[AS1] K.M.T. Abbassi and M. Sarih, On natural metrics on tangent bundles of Riemannian manifolds, *Arch. Math. (Brno)*, 41 (2005), 71–92.

[AS2] K.M.T. Abbassi and M. Sarih, On some hereditary properties of Riemannian $g$-natural metrics on tangent bundles of Riemannian manifolds, *Diff. Geometry and Appl.*, (1) 22 (2005), 19–47.

[B] D.E. Blair, *Riemannian geometry of contact and symplectic manifolds*, Progress in Math. 203, Birkhäuser, 2002.

[CGr] J. Cheeger and D. Gromoll, On the structure of complete manifolds of non negative curvature, *Annals of Math.* 96 (1972), 413–443.

[EL1] J.Eells and L.Lemaire, A report on harmonic maps, *Bull. London Math. Soc.* 10 (1978), 1–68.

[EL2] J.Eells and L.Lemaire, Another report on harmonic maps, *Bull. London Math. Soc.* 20 (1988), 385–524.

[ESa] J.Eells and J.H. Sampson, Harmonic mappings of Riemannian manifolds, *Amer. Math. J.* 86 (1964), 109–160.

[G1] O Gil-Medrano, Relationship between volume and energy of vector fields, *Diff. Geom. Appl.* 15 (2001), 137–152.

[G2] O Gil-Medrano, Unit vector fields that are critical points of the volume and of energy: characterization and examples, in: Complex, contact and symmetric manifolds, Progress in Math., Birkhäuser 234 (2005), 165–186.

[I] T. Ishihara, Harmonic sections of tangent bundles, *J. Math. Tokushima Univ.* 13 (1979), 23–27.

[HYi] S.D. Han and J.W. Yim, Unit vector fields on spheres which are harmonic maps, *Math. Z.* 227 (1998), 83–92.

[KoMSl] I. Kolár, P.W. Michor and J. Slovák, *Natural operations in differential geometry*, Springer-Verlag, Berlin, 1993.

[KSe] O. Kowalski and M. Sekizawa, Natural transformations of Riemannian metrics on manifolds to metrics on tangent bundles—a classification, *Bull. Tokyo Gakugei Univ.* (4) 40 (1988), 1–29.

[N] O. Nouhaud, Applications harmoniques d’une variété Riemannienne dans son fibré tangent, *C.R. Acad. Sci. Paris* 284 (1977), 815–818.

[O] V. Oproiu, A Kähler Einstein structure on the tangent bundle of a space form, *Int. J. Math. Sci.* 25 (2001), 183–195.

[P1] D. Perrone, Harmonic characteristic vector fields on contact metric three-manifolds, *Bull. Austral. Math. Soc.* 67 (2003), 305–315.
[P2] D. Perrone, Contact metric manifolds whose characteristic vector field is a harmonic vector field, *Diff. Geom. Appl.* 20 (2004), 367–378.

[U] H. Urakawa, *Calculus of Variations and Harmonic Maps*, Transl. Math. Monographs Amer. Math. Soc. 132, 1993.

[W1] G. Wiegmink, Total bending of vector fields on Riemannian manifolds, *Math. Ann.* 303 (1995), 325–344.

[W2] G. Wiegmink, Total bending of vector fields on the sphere $S^3$, *Diff. Geom. and Appl.* 6 (1996), 219–236.

[Wo] C.M. Wood, On the energy of a unit vector field, *Geometriae Dedicata* 64 (1997), 319–330.

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