The accessory parameter problem in positive characteristic

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Abstract

We study the existence of Fuchsian differential equations in positive characteristic with nilpotent $p$-curvature, and given local invariants. In the case of differential equations with logarithmic local monodromy, we determine the minimal possible degree of a polynomial solution.

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This paper deals with second order differential equations with regular singularities in characteristic $p > 0$. Our main interest is to characterize those differential equations with nilpotent (resp. nilpotent but nonzero) $p$-curvature. This problem is known as Dwork's accessory parameter problem. Differential equations with nilpotent $p$-curvature arise naturally in algebraic geometry. For example, Katz ([13]) showed that differential equations "coming from geometry", such as Picard–Fuchs differential equations, have nilpotent $p$-curvature. The nilpotence (resp. nonvanishing) of the $p$-curvature may be characterized in terms of the existence of polynomial solutions. The study of polynomial solution of differential equations in positive characteristic goes back to Dwork ([7], [8]) and Honda ([11]).

More recently, differential equations with nilpotent but nonzero $p$-curvature came up in Mochizuki’s work on $p$-adic uniformization ([14], [15]). Mochizuki develops the theory of indigenous bundles. On a curve of genus zero, these may be identified with differential equations with nilpotent but nonzero $p$-curvature ([6, §5]). To prove concrete existence results, the description of indigenous bundles as differential equations turns out to be more convenient.

Differential equation with nilpotent but nonzero $p$-curvature also arise in the theory of reduction to characteristic $p$ of Galois covers of curves. Solutions of differential equations arise in this context in the form of deformation data. In [6] one finds a correspondence between indigenous bundles and deformation data. (See also §2.2.)

This paper combines results from the work of Mochizuki on indigenous bundles with results on deformation data and techniques from the work of Honda and Dwork. It turns out that combing these techniques is very fruitful and allows to answer questions which are interesting from all three points of view.

We now give a more detailed description of the content of the paper.
We fix the number, \( r \), of singularities of the differential equation, together with a set \( \alpha = (\alpha_1, \ldots, \alpha_r) \) of local invariants, the local exponents. We study the stack \( \mathcal{N}_{0,r}(\alpha; n) \) of differential operators, \( L \), with local exponents \( \alpha \), nilpotent but nonzero \( p \)-curvature, and strength \( n \). The strength is a natural invariant of the differential operator introduced by Mochizuki ([14]) which is defined as the degree of the zero divisor of the \( p \)-curvature (§4). Essentially, it corresponds to the minimal degree of a polynomial solution of \( L \). We refer to §4 for a precise statement.

The main question we are interested in is to determine \( (\alpha, n) \) such that \( \mathcal{N}_{0,r}(\alpha; n) \) is nonempty, and to determine the dimension of the irreducible components of those \( \mathcal{N}_{0,r}(\alpha; n) \). This variant of Dwork’s accessory parameter problem we call the strong accessory parameter problem.

Our strongest results are in the case of logarithmic local monodromy (§4.4) which is the case on which the work of Mochizuki focusses. Mochizuki shows that if \( \mathcal{N}_{0,r}(\alpha; n) \) is nonempty, then every irreducible component has the maximal possible dimension \( r - 3 \). We determine all those \( n \) for which \( \mathcal{N}_{0,r}(\alpha; n) \) is nonempty. Equivalently, we determine which degrees occur as the minimal degree of a polynomial solution of a differential operator \( L \).

More precisely, we show the following.

**Theorem 4.10** Suppose that \( p > r - 2 \). Then the stack \( \mathcal{N}_{0,r}(\alpha = (0, \ldots, 0); n) \) is nonempty if and only if the strength, \( n \), is congruent to 0 mod 2\( p \) and satisfies \( 0 \leq n \leq (r - 2)(p - 1)/2 \).

The necessity of these conditions already follows from a result of Dwork ([8, Lemma 10.1]). To prove the theorem, we first apply a deformation technique due to Mochizuki to reduce ourselves to the case that \( r = n/p + 3 \). In this case, the minimal degree, \( d \), of a polynomial solution is less than \( p \). We then explicitly construct differential equations with a solution of this degree \( d \) (Proposition 4.12). Our method here is inspired by the work of Honda, Dwork and Beukers.

For general choice of the local exponents \( \alpha \) our results are less strong. We give a new necessary condition on \( n \) for \( \mathcal{N}_{0,r}(\alpha; n) \) to be nonempty (Lemma 4.3). Our main result in this case is a result on the dimension of the irreducible components of \( \mathcal{N}_{0,r}(\alpha; n) \) (Proposition 3.7). This is a (weaker) analog of the result which Mochizuki proved in the case of logarithmic local monodromy.

The key ingredient in the proof of Proposition 3.7 is the deformation theory of deformation data (§3), following Wewers ([20]). We use this as a replacement for Mochizuki’s results on deformation of indigenous bundles which are not available here. We also give some concrete examples which illustrates what to expect in the general case (§4.3).
1 Fuchsian differential equations in positive characteristic

The main goal of this section is to recall and reformulate some classical results on differential equations in positive characteristic. In §1.1 we recall some results of Dwork and Honda on polynomial solutions of differential equations. In §1.3 we recall the definition of the $p$-curvature and state some basic properties.

1.1 Algebraic solutions of Fuchsian differential equations

Let $X = \mathbb{P}^1_k$, and choose a parameter $x$ on $X$. Let $r \geq 3$, and suppose given pairwise distinct points $x_1, \ldots, x_r \in X$. We assume that $x_r = \infty$.

In this paper we consider Fuchsian differential operators

$$L = (\partial/\partial x)^2 + p_1(\partial/\partial x) + p_2$$

on $X$ with singularities in $x_1, \ldots, x_r = \infty$. Recall that this means that $p_1$ and $p_2$ are rational functions on $X$ which are regular outside the $x_i$ such that $\text{ord}_{x_i} p_i \geq i$ for all $j$. We call $L(u) = 0$ the corresponding differential equation. The differential operator $L_W = (\partial/\partial x) + p_1$ is called the Wronskian equation associated to $L$ (see [11, §1] for the relation of $L_W$ with the Wronskian.) All differential operators in this paper have order 2 and are supposed to be Fuchsian.

**Definition 1.1** Let $L$ be as in (1).

(a) We say that $L$ has nilpotent $p$-curvature if both $L$ and $L_W$ have a polynomial solution.

(b) Let $L$ be a differential operator with nilpotent $p$-curvature. We say that $L$ has nonzero $p$-curvature if the space of polynomial solutions of $L$ is 1-dimensional over $k[[x]]$.

We refer to [11] for a discussion of these notions. Nilpotent $p$-curvature is called “sufficiently many solutions in a weak sense” by Honda ([11]). To $L$ we may associate a flat vector bundle $(E, \nabla)$ of rank 2 on $X$. To this flat vector bundle is naturally associated the $p$-curvature (see for example [13]). In [11, Appendix] it is shown that the $p$-curvature of $E$ is nilpotent (resp. nonzero) if and only if $L$ satisfies the conditions of Definition 1.1. For a detailed discussion of the correspondence of $E$ and $L$ we refer to [6, §5]. In §1.3 we give a short introduction. For a definition of the $p$-curvature in terms of $L$ we refer to [7].

Since $L$ has regular singular points in $x_1, \ldots, x_r = \infty$, we may write

$$p_1 = \frac{P_1}{P_0}, \quad p_2 = \frac{P_2}{P_0} + \frac{P_3}{P_0^2},$$

where $P_0 = \prod_{i=1}^{r-1} (x - x_i)$. For $i = 1, \ldots, r - 1$, we define the local exponents $\{\alpha_i, \alpha'_i\} \in \mathbb{F}_p^\times$ of $L$ at $x_i$ by

$$\alpha_i + \alpha'_i = 1 - \frac{P_3(x_i)}{P_0(x_i)^2}, \quad \alpha_i \alpha'_i = \frac{P_3(x_i)}{P_0(x_i)^2}.$$
Similarly, for \( x = x_r = \infty \) we define the local exponents \( \{ \alpha_r, \alpha'_r \} \in \mathbb{F}_p^x \) of \( L \) at \( x_r \) by

\[
(3) \quad \alpha_r + \alpha'_r = \left( -1 + \frac{P_1(x)}{x^r - 2} \right) (x = \infty), \quad \alpha_r \alpha'_r = \left( \frac{P_2(x)}{x^r - 3} \right) (x = \infty).
\]

The local exponents are the eigenvalues of the local monodromy matrix of \( L \) at \( x = x_i \). They satisfy the Riemann relation

\[
(4) \quad \sum_{i=1}^{r} (\alpha_i + \alpha'_i) = r - 2.
\]

This follows by immediate verification (cf. [21]).

**Lemma 1.2**  
(a) Suppose that \( L \) has nilpotent \( p \)-curvature. Then the local exponents \( \alpha_i, \alpha'_i \) of \( L \) at \( x_i \) are in \( \mathbb{F}_p \).

(b) Let \( L \) be a differential operator as in (1) whose local exponents \( \{ \alpha_i, \alpha'_i \} \) are all elements of \( \mathbb{F}_p \). Suppose that \( L \) has a solution \( u \in k[x] \). Then \( L \) has nilpotent \( p \)-curvature.

**Proof:** Part (a) is proved in [11, Prop. 2.1]. If the local exponents of \( L \) are in \( \mathbb{F}_p \), then (6) implies that \( p_1 = Q'/Q \) for some polynomial \( Q \in k[x] \). Part (b) immediately follows from this observation (cf. [11, Cor. 1 to Prop. 2.3].) \( \square \)

**Lemma 1.3** Let \( u = u(x) \) be a polynomial solution of \( L(u) = 0 \). Then

(a) \( \deg(u) \equiv -\alpha_r \pmod{p} \) or \( \deg(u) \equiv -\alpha'_r \pmod{p} \),

(b) \( \text{ord}_{x_i}(u) \equiv \alpha_i \) or \( \text{ord}_{x_i}(u) \equiv \alpha'_i \pmod{p} \) for \( i \neq r \).

**Proof:** Suppose that \( x \) is a local parameter of \( X = \mathbb{P}^1_k \) at the singular point \( x_i \) for \( 1 \leq i \leq r \). Rewriting the differential equation in terms of \( x \) immediately yields the statement of the lemma; (a) corresponds to \( i = r \) and (b) corresponds to \( i \neq r \). \( \square \)

The following proposition is proved by Honda. It is a stronger version of Lemma 1.3.(b) in the case that \( L \) has zero \( p \)-curvature.

**Proposition 1.4 (Honda)** Let \( L \) be a differential operator whose \( p \)-curvature is zero. Let \( u_1, u_2 \in k[x] \) be two solutions of \( L \) which are independent over \( k[x]^p \). Suppose, moreover, that \( \delta := \deg(u_1) + \deg(u_2) \) is minimal.

(a) Then \( \deg(u_1) \) and \( \deg(u_2) \) are noncongruent to each other (modulo \( p \)).

In particular, \( \{ \deg(u_1), \deg(u_2) \} \equiv \{-\gamma_1, -\gamma_2\} \pmod{p} \) and \( \gamma_1 \neq \gamma_2 \pmod{p} \).

(b) Suppose that \( p \neq 2 \) and \( p \geq r - 2 \). Then

\[
r \leq \delta \leq (r - 1)p - (2r - 3).
\]
**Proof:** Part (a) is [11, Prop. 5.1]. Part (b) is [11, Theorem 7.c]

We conclude from Proposition 1.4 that if $L$ has nilpotent $p$-curvature then there exists a unique monic polynomial solution of minimal degree.

Let $L$ be a differential operator as above with nilpotent $p$-curvature. Let $u$ be a polynomial solution of minimal degree. Then $\alpha_i := \text{ord}_x u < p$ for $i \neq r$. We may suppose that $\alpha_i \equiv \alpha'_i \pmod{p}$ for $i \neq r$ (Lemma 1.3.b). For $i = r$, we suppose that $\text{deg}(u) \equiv -\alpha'_r \pmod{p}$. It follows that we may write

$$u = \prod_{i=1}^{r-1} (x - x_i)^{\alpha_i} v, \quad v \in k[x].$$

For $i = 1, \ldots, r$, we define integers $0 \leq t_i \leq p - 1$ by

$$(5) \quad t_i \equiv \alpha'_i - \alpha_i \pmod{p}.$$

The following proposition is an analog of Proposition 1.4.(b) in the case that the $p$-curvature is nonzero.

**Proposition 1.5 (Dwork)** Let $L$ be a differential operator with nilpotent but nonzero $p$-curvature. Let $u$ be a polynomial solution of minimal degree. There are exists a nonnegative integer $t$ such that

$$2 \text{deg}(v) + pt = (p - 1)(r - 2) - (\sum_{i=1}^{r} t_i).$$

**Proof:** This is [8, Lemma 10.1]

The following lemma is an easy consequence of Lemma 1.3. If $L$ is a differential operator with nilpotent $p$-curvature, the the lemma gives a sufficient criterion for the $p$-curvature of $L$ to be nonzero.

**Lemma 1.6** Let $L$ be a differential operator with nilpotent $p$-curvature. Suppose that $\alpha_i = \alpha'_i$, for some $i$. Then the $p$-curvature is nonzero.

**Proof:** Let $L$ be as in the statement of the lemma, and suppose that $\alpha + i = \alpha'_i$. In the case that $i = r$, the lemma follows from Lemma 1.3.(a). The general case is easily reduced to this case, by changing the coordinate $x$.

Beukers ([1]) proves the same result in the case of $r = 4$ singularities such that $\alpha_i = \alpha'_i = 0$ for all $i$.

### 1.2 Normalized differential operators

**Definition 1.7** Two differential equations $L_1$ and $L_2$ are called equivalent if they have the same set of singularities and there exists a rational function $v$ on $X$ such that $L_1(u) = 0$ if and only if $L_2(vu) = 0$ for all $u \in k((x))$. 
Let \( L_1 \) and \( L_2 \) be equivalent differential operators, and let \( \{ \alpha_i, \alpha'_i \} \) be the local exponents of \( L_1 \) at \( x_i \neq \infty \). There exist \( \mu_i \in \mathbb{F}_p \) such that the local exponents of \( L_2 \) at \( x_i \) are \( \{ \alpha_i - \mu_i, \alpha'_i - \mu_i \} \). Here \( \mu_i = \text{ord}_{x_i} v \), for \( v \) as in Definition 1.7.

**Definition 1.8** A differential equation \( L \) with nilpotent \( p \)-curvature is called normalized if its (unique) monic solution of minimal degree does not have zeros in the singular points \( x_i \). We denote the monic solution of minimal degree by \( u \), and put \( d = \deg(u) \).

The local exponents of a normalized differential operator are uniquely determined by \( \alpha := \{ \alpha_1, \ldots, \alpha_r \} \), due to the Riemann relation (4). Namely, the local exponents are \( (0, \alpha_i) \) at \( x = x_i \) for \( i \neq r \) and \( (-d, -d + \alpha_r) \) for \( x = x_r = \infty \).

Normalized differential operators satisfy

\[
\begin{align*}
    p_1 &= \sum_{i=1}^{r-1} \frac{1 - \alpha_i}{x - x_i}, & p_2 &= \frac{d(d - \alpha_r)x^{r-3} + \beta_{r-4}x^{r-4} + \cdots + \beta_0}{\prod_{i=1}^{r-1}(x - x_i)}. \\
\end{align*}
\]

Note that the properties 'nilpotent \( p \)-curvature' and 'nonzero \( p \)-curvature' only depends of the equivalence class of the differential operator. Moreover, every equivalence class of differential operators with nilpotent \( p \)-curvature contains a normalized one.

**1.3 The \( p \)-curvature** Let \( L = (\partial/\partial x)^2 + p_1(\partial/\partial x) + p_2 \) be a Fuchsian differential operator defined over \( k \) with nilpotent \( p \)-curvature. Let \( u \in k[x] \) be a polynomial solution of minimal degree. Lemma 1.2.(a) implies that there exists a polynomial \( Q \in k[x] \) such that \( p_1 = Q' / Q \). An explicit formula for \( Q \) is easily deduced from (6).

Let \( (L, u) \) be as above. Let \( D := \partial/\partial x \). We define a flat vector bundle \( \mathcal{E} \) on \( \mathbb{P}^1 \), as in the proof of [6, Proposition 5.3]. Namely, on \( \mathbb{A}^1 \) we let \( \mathcal{E} \) be the trivial bundle with basis \( e_1, e_2 \) and connection defined by

\[
\nabla(D) \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 & -p_2 \\ 1 & -p_1 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.
\]

It is shown in loc.cit. that \( \mathcal{E} \) extends to a flat vector bundle with regular singularities on \( \mathbb{P}^1 \).

Let \( T = (\Omega_{X/k}^+) \otimes -p \). The \( p \)-curvature is an \( \mathcal{O}_X \)-linear map

\[
\Psi_\mathcal{E} : T \rightarrow \text{End}_{\mathcal{O}_X}(\mathcal{E}).
\]

Since \( D^p = 0 \), the \( p \)-curvature of \( \mathcal{E} \) is defined by \( \Psi_\mathcal{E}(D^\otimes p) := \nabla(D)^p \) (cf. [6, §3.1]). It is well known that the \( p \)-curvature of \( L \) is nilpotent (resp. zero) if and only if the matrix of \( \Psi_\mathcal{E}(D^\otimes p) \) is nilpotent (resp. zero) ([11, Appendix]).
Proposition 1.9 The differential operator $L$ has nonvanishing $p$-curvature if and only if

$$ \left( \frac{\partial}{\partial x} \right)^{p-1} \frac{1}{Qu^2} \neq 0. $$

Proof: Suppose that (7) holds. The computation in the proof of [6, Proposition 5.3] implies that

$$ \Psi_{\mathcal{E}}(D \otimes p) \frac{e_1}{u} \neq 0. $$

We conclude that the $p$-curvature of $L$ is nonzero.

To prove the other implication, we suppose that $\Psi_{\mathcal{E}}(D \otimes p) \neq 0$. Let $\mathcal{M} \subset \mathcal{E}$ be the kernel of $\Psi_{\mathcal{E}} ([6, \S 3.1])$. Our assumption implies that this is a flat subbundle of $\mathcal{E}$ of rank 1. We may choose a horizontal section $\eta$ of $\mathcal{M}$ which generates $\mathcal{M}$ on a dense open subset of $\mathbb{P}^1_k$. Reversing the computation from [6, Proposition 5.3] implies that (7) holds. \hfill \Box

2 Deformation data

In this section, we prove a correspondence between deformation data and differential equations with nilpotent but nonzero $p$-curvature. This is a reformulation and adaptation to the present case of the result of [6]. For the convenience of the reader, we start by recalling the definition and basic properties of deformation data. We refer to [18, 19] for more details. A short introduction explaining how deformation data arise naturally in the theory of stable reduction can be found in [6, \S 4.2].

2.1 Definitions Let $k$ be an algebraically closed field of characteristic $p > 2$.

Definition 2.1 A deformation datum of type $(H, \chi)$ is a pair $(g, \omega)$, where $g : Z \to X = \mathbb{P}^1_k$ is a finite Galois cover of smooth projective curves and $\omega$ is a meromorphic differential form on $Z$ such that the following conditions hold.

(a) Let $H$ be the Galois group of $Z \to X$. Then

$$ \beta^* \omega = \chi(\beta) \cdot \omega, \quad \text{for all } \beta \in H. $$

Here $\chi : H \to \mathbb{F}_p^\times$ in an injective character.

(b) The differential form $\omega$ is logarithmic, i.e. of the form $\omega = df/f$, for some meromorphic function $f$ on $Z$.

Part (b) of Definition 2.1 may also be reformulated as: $\omega$ is fixed by the Cartier operator $\mathcal{C}$. Let $(g, \omega)$ be a deformation datum. For each closed point $x \in X$ we define the following invariants.

$$ m_x := |H_z|, \quad h_x := \text{ord}_x(\omega) + 1, \quad \sigma_x := h_x/m_x. $$

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Here $z \in Z$ is some point above $x$ and $H_z \subset H$ is the stabilizer of $z$. The invariant $\sigma_x$ is called the *ramification invariant* of the deformation datum at $x$. The following lemma gives some necessary conditions on the ramification invariant $\sigma_x$.

**Lemma 2.2** Let $(Z, \omega)$ be a deformation datum.

(a) For all $x \in X$, the ramification index $m_x$ divides $p - 1$. Moreover, $h_x$ and $m_x$ are relatively prime.

(b) If $h_x \neq 0$ then $\gcd(p, h_x) = 1$.

(c) For all but finitely many points $x \in X$ we have $\sigma_x = 1$.

(d) We have
\[
\sum_{x \in X} (\sigma_x - 1) = 2g - 2.
\]

Proof: This lemma is proved in [6, Lemma 4.3].

**Definition 2.3** Let $(Z, \omega)$ be a deformation datum on $X$.

(a) A point $x \in X$ is said to be a **critical point** of the deformation datum if $\sigma_x \neq 1$.

(b) A critical point $x \in X$ is **supersingular** if $\sigma_x = (p + 1)/(p - 1)$.

(c) A critical point $x \in X$ is **singular** if it is not a supersingular point and $\sigma_x \not\equiv 1 \pmod{p}$.

(d) A critical point $x \in X$ such that $\sigma_x \equiv 1 \pmod{p}$ is called a **spike**.

We refer to [6] for a motivation of the terminology. Lemma 2.2 (c) implies that a deformation datum has finitely many critical points.

**Definition 2.4** A **signature** is given by a finite set $M$ and a map
\[
\sigma : M \to \frac{1}{p - 1} \cdot \mathbb{Z}, \quad x \mapsto \sigma_x
\]
such that $\sigma_x \geq 0$ and $\sigma_x \neq 1, (p + 1)/(p - 1)$ for all $x \in M$ and such that the number
\[
d := \frac{p - 1}{2} \left( 2g - 2 - \sum_{x \in M} (\sigma_x - 1) \right)
\]
is a nonnegative integer. The **singularities** of $\sigma$ are the elements $x \in M$ with $\sigma_x \not\equiv 1 \pmod{p}$.

Given a deformation datum $(Z, \omega)$ on $X$, the invariants $\sigma_x$ defined in (8) give rise to a signature $\sigma$ (where $M$ is the set of points $x \in X$ with $\sigma_x \neq 1, (p + 1)/(p - 1)$). It follows from Lemma 2.2 (d) that the number $d$ defined above is the number of supersingular points.
We denote by $B$ the set of critical points of the deformation datum. Let $r$ be the number of singularities of the deformation datum. We say that the deformation datum $(Z, \omega)$ is trivial if $2g - 2 + r = 0$. In the rest of this paper, we exclude trivial deformation data. Without loss of generality, we may therefore suppose that $x_r = \infty$ is a singularity. Let $B' = \{ b \in B \mid x_b \neq \infty \}$.

We let $s - r$ be the number of spikes. We always enumerate the singularities (resp. spikes) of a deformation datum $(Z, \omega)$ as $x_1, \ldots, x_r$ (resp. $x_{r+1}, \ldots, x_s$) and write $\sigma_i := \sigma_{x_i}$ for $i = 1, \ldots, s$.

Define integers $0 \leq a_i < p - 1$ and $\nu_i \geq 0$ by

$$\sigma_i = \frac{a_i}{p-1} + \nu_i.$$  

Note that $x$ is supersingular if and only if $(a_x, \nu_x) = (2, 1)$.

Let $(Z, \omega)$ be a deformation datum of signature $(\sigma_i)$. Let $u \in k[x]$ be the monic polynomial whose zeros are exactly the supersingular points of the deformation datum (with multiplicity one).

For future reference we note that $Z$ is a connected component of the smooth projective curve defined by the Kummer equation

$$z^{p-1} = \prod_{i} (x - x_i)^{a_i} u^2,$$

cf. (9). Let $S \subset Z$ be the inverse image of the set of critical points $x_i \in X$ for which $\sigma_i = 0$. The definition of $\sigma_i$ implies that $S$ is the set of poles of $\omega$. Therefore $\omega$ is a section of the sheaf $\Omega^{\text{log}} := \Omega_{Z/k}(S)$ of differential 1-forms with simple poles in $S$. Definition 2.1.(a) implies therefore that $\omega \in H^0(Z, \Omega^{\text{log}})_\chi$.

Once $Z$ and $\chi$ are given, we may characterize the deformation data $(Z, \omega)$ as those section of $H^0(Z, \Omega^{\text{log}})_\chi$ which are fixed by the Cartier operator $C$. The following lemma is stated for completeness.

**Lemma 2.5** (a) The dimension of $H^0(Z, \Omega^{\text{log}})_\chi$ is $|B| - 1 - (\sum_i a_i)/(p-1)$.

(b) The differentials

$$\omega_j = \frac{x^{j-1}zdx}{\prod_{i \in B'} (x - x_i)}, \quad j = 1, \ldots, r - 1 - (\sum_i a_i)/(p-1),$$

form a basis of $H^0(Z, \Omega^{\text{log}})_\chi$.

**Proof:** This is proved like [2, Lemma 4.3].

**2.2 A correspondence** In this section we prove a correspondence between deformation data and solutions of Fuchsian differential equation in positive characteristic.

Let $k$ be an algebraically closed field of characteristic $p$. As before, we choose a parameter $x$ on $X = \mathbb{P}^1_k$. We write $D = \partial/\partial x$ and $f'$ for $D(f)$. 

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Suppose we are given a deformation datum \((Z, \omega)\) of type \((H, \chi)\) of signature \(\sigma\). As before we denote by \(x_1, \ldots, x_r = \infty\) the singular critical points of the deformation datum and by \(x_{r+1}, \ldots, x_s\) the spikes. After replacing \(k\) by a larger algebraically closed field, if necessary, we may suppose that all \(x_i\) are \(k\)-rational.

We let \(u \in k[x]\) be the monic polynomial with simple zeros in the supersingular points, and no other zeros. Put

\[
Q = \prod_{i=1}^{r-1} (x - x_i)^{1+a_i - \nu_i},
\]

where \(a_i\) and \(\nu_i\) are defined by (9), i.e.

\[
\sigma_i = \frac{a_i}{p-1} + \nu_i.
\]

The definition of \(\sigma_i\) implies that there exists an \(\epsilon \in k^\times\) such that

\[
\omega = \epsilon z \prod_{i \neq r} (x - x_i)^{\nu_i - 1} \, dx = \epsilon \frac{z}{Qu^2} \, dx.
\]

Here \(z\) is as in (10).

**Lemma 2.6** Let \(\omega\) be given by (12), and let \(u \in k[x]\) be the monic polynomial with simple zeros exactly in the supersingular points.

(a) The differential form \(\omega\) is logarithmic if and only if

\[
D^{p-1} \frac{1}{Qu^2} = -\epsilon^{p-1} \prod_{i=1}^{r} (x - x_i)^{p(\nu_i - 1)}.
\]

(b) The differential form \(\omega\) is exact if and only if

\[
D^{p-1} \frac{1}{Qu^2} = 0.
\]

(c) Suppose that \((Z, \omega)\) is a deformation datum. Then \(\text{Res}_x \frac{1}{Qu^2} = 0\), for \(x\) supersingular.

**Proof:** It is well known that \(\omega = F \, dx\) is logarithmic if and only if \(D^{p-1} F = -F^p\). (For an outline of the proof see [9, Exercise 9.6]). Therefore (11) implies that \(\omega\) is logarithmic if and only if

\[
D^{p-1} \frac{1}{Qu^2} = -\epsilon^{p-1} \prod_{i \neq r} (x - x_i)^{p(\nu_i - 1)}.
\]

This implies (a). It is well known that \(\omega\) is exact if and only if \(D^{p-1} F = z D^{p-1}(1/Qu^2) = 0\). This proves (b).
Let $\tau \in X$ be a supersingular point and write
\[
\frac{1}{Qu^2} = \sum_{n \geq -2} c_n (x - \tau)^n.
\]
Then (13) implies that
\[
D^{p-1} \frac{1}{Qu^2} = - \left[ \frac{c_{-1}}{(x - \tau)^p} + c_{p-1} + \cdots \right] = - \epsilon^{p-1} \prod_{i \neq r} (x - x_i)^{p(v_i - 1)}.
\]
Here $\epsilon \neq 0$ if and only if $\omega$ is logarithmic. We conclude that $c_{-1} = 0$, since $\tau \neq x_i$ for some $i \in \{1, \ldots, r\}$. This proves (b).

The following proof is inspired by [1, Lemma 3]. A similar argument can be found in [12, Theorem 5]. A special case of the result can be found in [5, Proposition 3.2].

**Proposition 2.7** Let $(Z, \omega)$ be a deformation datum and $u$ be the monic polynomial with simple zeros exactly in the supersingular points.

(a) There exists a polynomial $P_2 = d(d + a_r)x^{r-2} + \beta_{r-3}x^{r-3} + \cdots + \beta_0 \in k[x]$ such that
\[
L_\omega(u) := u'' + \frac{Q'}{Q} u' + \frac{P_2}{\prod_{i=1}^{r-1}(x - x_i)} u = 0.
\]

(b) The differential operator $L_\omega$ defined in (a) is normalized. It has singularities at $x_1, \ldots, x_r = \infty$. Its local exponents at $x_i \neq \infty$ (resp. $x_r = \infty$) are $\{-\sigma_i; 0\}$ (resp. $\{-\deg(u); -\deg(u) + \sigma_r\}$ (mod $p$)).

(c) The $p$-curvature of $L$ is nilpotent but nonzero.

**Proof:** Let $\tau \in X$ be a supersingular point. Lemma 2.6.(c) implies that
\[
\text{Res}_{x = \tau} \frac{1}{Qu^2} = 0.
\]
The proof of [5, Proposition 3.2] also applies in our more general situation. We deduce that $u$ is a solution to a Fuchsian differential equation as in (a).

Since $p_2 := P_2 / \prod_{i=1}^{r-1}(x - x_i)$ has at most simple poles, it follows that $L_\omega$ is normalized. The singularities and local exponents are easily read off from the explicit expression for $L_\omega$. This proves (b). Part (c) follows from [6, Proposition 4.8].

The following proposition gives a converse to Proposition 2.7. It is a simplified version of [6, Proposition 5.3] which is stated in a different language.

**Proposition 2.8** Let $L = (\partial/\partial t)^2 + p_1(\partial/\partial t) + p_2$ be a normalized second order differential operator with regular singularities in $x_1, \ldots, x_r = \infty$ and local exponents $\{\alpha_i, 0\}$ (resp. $\{-d_i - d + \alpha_r\}$) at $x_i$ for $i \neq r$ (resp. $x_r$). Suppose that $L$ has nilpotent but nonzero $p$-curvature. Let $u$ be a polynomial solution of minimal degree. Then the pair $(L, u)$ is associated to a deformation datum $(Z, \omega)$ via the construction of Proposition 2.7.
Proof: Let \( L \) be as in the statement of the proposition, i.e. \( p_1 \) and \( p_2 \) are as in (6). We define
\[
Q = \prod_{i=1}^{r-1} (x - x_i)^{[1 - \alpha_i]}.
\]
Here \([a]\) denotes the unique integer satisfying \(0 \leq [a] < p\) and \([a] \equiv a \pmod{p}\).
We have \( p_1 = Q'Q \).

It follows from Proposition 1.9 that
\[
D^{p-1} \frac{1}{Qu^2} \neq 0.
\]
Denote by \( x_{r+1}, \ldots, x_s \) the points of \( X \), different from \( x_1, \ldots, x_r \), such that
\[
\text{ord}_{x_i} D^{p-1} \frac{1}{Qu^2} \neq 0.
\]
For \( i = 1, \ldots, s \) with \( i \neq r \) we define nonnegative integers \( \nu_i \) by
\[
\nu_i = \frac{1}{p} \left( \text{ord}_{x_i} D^{p-1} \frac{1}{Qu^2} \right) + 1.
\]
Moreover, we define integers \( 0 \leq a_i < p \) by \( a_i \equiv -\alpha_i + \nu_i \pmod{p} \), and put \( \sigma_i = a_i/(p-1) + \nu_i \). For \( i = r \) we define \( \sigma_r \) by the relation of Lemma 2.2.(d), and \( a_r, \nu_r \) by (9). Note that \( \sigma_i \equiv \alpha_i \pmod{p} \) if \( i \leq r \) and \( \sigma_i \equiv 1 \pmod{p} \) for \( i > r \).

After replacing \( k \) by a larger algebraically closed field, we may suppose that all points \( x_i \) are rational over \( k \). We let \( Z/k \) be the smooth projective curve defined by the Kummer equation (10). Lemma 2.6.(a) implies that there exists an \( \epsilon \in \kappa^\times \) such that the differential form \( \omega \) defined by (12) is logarithmic.

We claim that \( \sigma_r \geq 0 \). Indeed, if \( \sigma_r < 0 \) then the differential form \( \omega \) has a pole of order strictly larger than 1 at \( x = x_r = \infty \). But this is impossible since \( \omega \) is a logarithmic differential form. This implies that \((Z, \omega)\) defines a deformation datum. Its signature is \((\sigma_i)\). \( \square \)

Remark 2.9 Proposition 1.5 follows from Proposition 2.8. Namely, let \( L \) be a differential operator with nilpotent but nonzero \( p \)-curvature and let \( u \) be a polynomial solution of minimal degree. To prove Proposition 1.5, we may assume that \( L \) is normalized (§1.2). Proposition 2.8 implies that \((L, u)\) corresponds to a deformation datum \((Z, \omega)\). Therefore Lemma 2.2.(d) together with the estimates \( \sigma_i \geq t_i/(p-1) \) for \( i \leq r \) and \( \sigma_i > 1 \) for \( i > r \) implies that
\[
2 \deg(u) \leq (r-2)(p-1) - \sum_{i=1}^{r} t_i.
\]
3 Deformations of $\mathbf{\mu}_p$-torsors

In §4 we define the moduli space $\mathcal{N}_{0,r}(\alpha)$ of differential operators $L$ with nilpotent but nonzero $p$-curvature and local exponents $\alpha$. The results of §2.2 imply that $\mathcal{N}_{0,r}(\alpha)$ also parameterizes deformation data. We are interested in the dimension of the irreducible components of $\mathcal{N}_{0,r}(\alpha)$. Our main tool is a result of Wewers ([20]) on the deformation of deformation data. Below we see that this may be translated into the deformation of $\mathbf{\mu}_p$-torsors. In this section, we adapt some results of Wewers to our situation.

Let $(Z, \omega)$ a deformation datum of type $(H, \chi)$. We denote by $G$ the group scheme $\mathbf{\mu}_p \ltimes H$, as defined in [20, Section 4.1]. We associate to the deformation datum $(Z, \omega)$ a singular curve $Y$ together with an action of the group scheme $G$, as in [20, Construction 4.3]. Since $\omega$ is a logarithmic differential form, locally on $Z$ it may be written as

$$\omega = \frac{df}{f},$$

for some meromorphic functions $f$ on $Z$. Define $Y$, locally on $Z$, by the equation

$$y^p = f.$$  

Then $G$ obviously acts on $Y$ and the natural map $Y \to X$ is a $G$-torsor outside the branch points of $Z \to X$ ([20, Remark 4.6.i]). Moreover, [20, Remark 4.6.ii] implies that $Y$ is generically smooth.

Let $C_k$ be the category of local artinian $k$-algebras of equal characteristic $p$. A $G$-equivariant deformation of $Y$ over an object $A$ of $C_k$ is a flat $R$-scheme $Y_R$ together with an action of $G$ and an $G$-equivariant isomorphism $Y \cong Y_R \otimes_R k$.

We consider the deformation functor

$$R \mapsto \text{Def}(Y, G)(R)$$

which sends $R \in C_k$ to the set of isomorphism classes of $G$-equivariant deformations of $Y$ over $R$. Let

$$R \mapsto \text{Def}(X; x_i \mid i \in \mathbb{B})(R)$$

be the deformation functor which sends $R$ to the set of isomorphism classes of deformations of the pointed curve $(X; x_i \mid i \in \mathbb{B})$. We consider the points $x_i$ on $X$ to be ordered. Moreover, we consider the $x_i$ up to the action of $\text{PGL}_2(k)$, i.e. we suppose that $x_1 = 0, x_2 = 1, x_r = \infty$. We obtain a natural transformation

$$\text{Def}(Y, G) \longrightarrow \text{Def}(X; x_i \mid i \in \mathbb{B}).$$

Proposition 3.1 The deformation functor $\text{Def}(Y, G)$ is formally smooth.

Proof: We use the terminology of [20]. The proposition follows from [20, Theorem 4.8], if we show that $\text{Ext}^2_G(L,Y/k, \mathcal{O}_Y) = 0$.

In our situation, the integer $s = \dim_{\mathbb{Z}_p} V$ of [20] equals one. This implies that the sheaf $\mathcal{E}xt^1_G(L_Y/k, \mathcal{O}_Y)$ has support in isolated points (namely the critical points of the deformation datum). Since $H^1(X, \mathcal{E}xt^1_G(L_Y/k, \mathcal{O}_Y)) = 0$, it follows from [20, (43)] that

$$\text{Ext}^2_G(L_Y/k, \mathcal{O}_Y) = 0.$$
This implies that the deformation problem is formally smooth.

For every $i \in B$, we let $\hat{Y}_i$ be the completion of $Y$ at $x_i$. Let $R \in \mathcal{C}_k$ and $Y_R$ be a $G$-equivariant deformation of $Y$. Write $(g_R : Z_R \rightarrow X_R, \omega_R)$ for the corresponding deformation datum. Let $i \in B$ and choose a point $z_i$ of $\tilde{Z}_0$ above $x_i \in X$. Let $H_i \subset H$ be the decomposition group of $z_i$. There exists a local parameter $t = t_i$ of $z_i$ on $Z_R$ and a character $\chi_i : H_i \rightarrow R^\times$ such that $O_{Z_R, z_i} = R[[t]]$ and $h^*t_i = \chi_i(h) \cdot t_i$ for all $h \in H_i$. We denote by $\tilde{Y}_{i,R}$ the completion of $Y_R$ at $x_i$; this is an equivariant deformation of $\hat{Y}_i$. We obtain a morphism

$$
\text{locgl} : \text{Def}(Y, G) -\rightarrow \prod_{i \in B} \text{Def}(\hat{Y}_i, G)
$$

called the local-global morphism ([20, Section 5.3]). The $G$-equivariant deformation $Y_R$ is called locally trivial if it lies in the kernel of the local global morphism. We denote by

$$
\text{Def}(Y, G)_{lt} \subset \text{Def}(Y, G)
$$

the subfunctor parameterizing locally trivial deformations; this is the image of the local-global morphism. We write

$$
\text{Def}(Y, G)_{lt} = \prod_{i \in B} \text{Def}(\hat{Y}_i, G)^\dagger.
$$

**Lemma 3.2** The tangent space to $\text{Def}(Y, G)_{lt}$ has dimension

$$
\frac{1}{p - 1} (\sum_{i \in B} a_i) - 1 = N.
$$

**Proof:** The tangent space to the deformation functor $\text{Def}(Y, G)_{lt}$ is

$$
H^1(X, \text{Hom}(\mathcal{L}_Y/k, \mathcal{O}_Y)) = H^1(X, M^H),
$$

[20, Proposition 4.10]. Here $M^H$ is defined in [20, Section 4.3]. In our situation it is the sheaf of derivations $D$ of $\mathcal{O}_X$ such that $D(\omega)$ is a regular function on $Z$. The proof of [20, Lemma 5.3] implies that $M^H$ is isomorphic to $((g_*) \mathcal{O}_Z)_X$. A local calculation shows that

$$
\text{deg}(M^H) = -\sum_{i \in B} \frac{a_i}{p - 1} = -N - 1.
$$

By the Riemann–Roch Theorem, the dimension of $H^1(X, M^H)$ equals $-1 + (\sum_{i \in B} a_i)/(p - 1)$. This proves the lemma. 

To compute the dimension of certain components of the moduli space of deformation data (§4), we need to modify the concept of locally trivial deformations, as defined in [20]. The reason for this is that we allow critical points with ramification invariant $\sigma_x > 2$. This happens, for example, for the spikes. In [20], Wewers focused on special deformation data, which satisfy $\sigma_x \leq 2$. 

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Let $x_i$ be a critical point of the deformation datum $(Z, \omega)$, and let $\sigma_i = a_i/(p-1) + \nu_i$ be its ramification invariant. In our situation, the local-global morphism (15) is formally smooth ([20, Remark 5.12]). Therefore the following proposition is proved like [20, Theorem 5.11.(i)].

**Proposition 3.3** The functor $\text{Def}(\hat{Y}_i, \mathcal{G})^\dagger$ admits a versal deformation over the ring

$$\hat{R}_i = W(k)[[t_{i,0}, \ldots, t_{i,\nu_i-1}]].$$

**Definition 3.4** Let $x_i$ be a critical point of the deformation datum and write $I_i = \{0 \leq j \leq \nu_i \mid j \equiv a_i \pmod{p}\}$. Put $\epsilon_i = |I_i|$. We define

$$\hat{R}_i^* = \hat{R}_i/(t_{i,j}, j \notin I_i).$$

We now give an interpretation of the rings $\hat{R}_i^*$. Let $v_i$ be a local parameter of a point $z_i$ of $Z$ above $x_i$. Put $n_i = (p-1)/\gcd(p-1, a_i)$. The definition of the integer $a_i$ implies that

$$(\varphi^*)^{a_i}v_i \equiv v_i^{b_i} \pmod{v_i^2},$$

where $b_i a_i/n_i \equiv 1 \pmod{(p-1)/n_i}$. Locally around $x_i$, we may choose a function $f$ such that $\hat{Y}_i^*$ is given by the Kummer equation $y^p = f$, where $f = 1 + v_i^{a_i+(p-1)\nu_i}/n_i$, since $\omega := df/f$ has a zero of order $(a_i + (p-1)\nu_i)/n_i - 1$ at $z_i$.

Let $\hat{Y}_i^*$ denote the restriction of the universal deformation of $\hat{Y}_i$ to $\text{Spec}(\hat{R}_i^*)$. Locally around $x_i$, the germ $\hat{Y}_i^*$ is given by a Kummer equation

$$y^p = F,$$

where $F = f + \sum_{j=0}^{\nu_i-1} t_{i,j}z^{(a_i+(p-1)j)/n_i}$,

since $H$ acts on $\hat{Y}_i$. The differential form corresponding to $\hat{Y}_i$ is given by $dF/F$.

We claim that $\text{Spec}(\hat{R}_i^*) \subset \text{Spec}(\hat{R}_i)$ is exactly the locus such that the restriction of $dF/F$ has a zero of order $(a_i + (p-1)\nu_i)/n_i - 1$ at $z_i$, i.e. such that the ramification invariant remains constant. Namely $\partial F/\partial v_i$ has a single zero in $v_i$ if and only if $t_{i,j} = 0$ for all $j$ such that $a_i - j \not\equiv 0 \pmod{p}$. This proves the claim.

Let $\text{Def}(Y, \mathcal{G}; \sigma) \subset \text{Def}(Y, \mathcal{G})$ be the deformation problem of $\mathcal{G}$-equivariant deformations with fixed signature. For every critical point of the deformation datum, we denote by $\text{Def}(\hat{Y}_i, \mathcal{G}; \sigma_i) \subset \text{Def}(\hat{Y}_i, \mathcal{G})$ the subfunctor parameterizing local deformations with given signature.

**Theorem 3.5**

(a) The deformation problem $\text{Def}(Y, \mathcal{G}; \sigma)$ is formally smooth.

(b) The functor $\text{Def}(\hat{Y}_i, \mathcal{G}; \sigma_i)$ admits a versal deformation over the ring $\hat{R}_i^*$.

(c) The dimension of $\text{Def}(Y, \mathcal{G}; \sigma)$ is

$$N + \sum_{i \in \mathbb{N}} \epsilon_i.$$

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Proof: Part (a) follows from Proposition 3.1. Part (b) follows from the above discussion.

We have already remarked that the local-global morphism (15) is formally smooth. This implies that the dimension of the deformation problem Def\((Y, G; \sigma)\) is equal to the dimension of its tangent space. It follows that

\[
\dim \text{Def}(Y, G; \sigma) = \dim \text{Def}(Y, G)_{\text{lt}} + \sum_{i \in B} (\dim \text{Def}(\hat{Y}_i, G) - \dim \text{Def}(\hat{Y}_i, G, \sigma_i))
\]

\[
= N + \sum_{i \in B} \epsilon_i.
\]

The last equality follows from Lemma 3.2 together with (a). This proves the theorem. □

The following lemma gives a bound on \(\epsilon_i\). We assume that \(p > r - 2\); this will be assumed in §4, as well.

Lemma 3.6 Assume that \(p > r - 2\).

(a) We have that \(\epsilon_i \in \{0, 1\}\).

(b) If \(x = x_i\) is supersingular, then \(\epsilon_i = 0\).

(c) If \(x = x_i\) is a spike, then \(\epsilon_i = 1\).

Proof: Lemma 2.2.(d) implies that

\[
\sum_{i=1}^{r} \sigma_i + \frac{2d}{p-1} + \sum_{i=r+1}^{s} (\sigma_i - 1) = r - 2.
\]

Here \(d\) is the number of supersingular points. This implies that if \(x = x_i\) is a singularity (i.e. \(1 \leq i \leq r\)), then \(\sigma_i \leq r - 2 < p\). We conclude that \(\epsilon_i \leq 1\). Since the supersingular points have ramification invariant \(\sigma = (p + 1)/(p - 1)\), it follows that they have \(\epsilon = 0\).

Now let \(x = x_i\) be a spike. Then \(\sigma_i \leq r - 2 + 1 = r - 1 \leq p\). Moreover, \(\sigma_i \equiv 1 \pmod{p}\). This implies that \(\nu_i \equiv 1 + a_i \pmod{p}\). Since \(\sigma_i \neq 1\), it follows that \(2 \leq \nu_i < p\). This implies that \(I_i = \{\nu_i - 1\}\) and hence that \(\epsilon_i = 1\). □

The following proposition characterizes the signatures \(\sigma\) for which the dimension of \(\text{Def}(Y, G; \sigma)\) is maximal.

Proposition 3.7 (a) Let \(\pi_\sigma : \text{Def}(Y, G; \sigma) \to \text{Def}(X; \{x_1, \ldots, x_r\})\) be the natural map which sends a deformation datum to its set of singularities. Then \(\pi_\sigma\) is finite.

(b) Suppose that \(p > r - 2\). Then the dimension of \(\text{Def}(Y, G; \sigma)\) equals \(r - 3 = \dim \text{Def}(X; \{x_1, \ldots, x_r\})\) if and only if

\[
\nu_i = \begin{cases} 
0 & \text{for } 1 \leq i \leq r, \\
2 & \text{for } r + 1 \leq i \leq s.
\end{cases}
\]
Proof: Part (a) follows from the main result of [7], by using the correspondence between deformation data and differential operators (§2.2). (Compare to Proposition 4.1.(a).)

Suppose now that \( p > r - 2 \). We deduce from Theorem 3.5 and (17) that the dimension of \( \text{Def}(Y, \mathcal{G}; \sigma) \) equals \( r - 3 \) if and only if

\[
\sum_{i=1}^{r} (\nu_i - \epsilon_i) + \sum_{i=r+1}^{s} (\nu_i - \epsilon_i - 1) = 0.
\]

Let \( x = x_i \) be a singularity, i.e. \( 1 \leq i \leq r \), and suppose that \( \epsilon_i = 1 \). Definition 2.3 implies that \( \sigma_i \not\equiv 1 \mod p \). It follows therefore that \( \nu_i \geq 3 \), and hence that \( \nu_i - \epsilon_i > 0 \). If \( x = x_i \) is a spike, i.e. \( i > r \), we deduce from Lemma 3.6.(c) that \( \nu_i - \epsilon_i - 1 = \nu_i - 2 \). This proves the proposition.

\[\Box\]

4 The accessory parameter problem

This section contains the main results of the paper. In §4.1 we introduce the stacks \( N_{0,r}(\alpha) \) parameterizing differential equations with nilpotent but nonzero \( p \)-curvature. We also define the strength and prove a necessary condition for the substack \( N_{0,r}(\alpha; n) \) of operators of strength \( n \) to be nonempty (Lemma 4.3). In §4.2, we prove a result on the dimension of irreducible components of \( N_{0,r}(\alpha; n) \) for those \( (\alpha, n) \) for which this stack is nonempty, using the results of §3. In §4.4 we consider the case of logarithmic local monodromy.

4.1 Statement of the problem

Let \( r \geq 3 \) be an integer. We denote by \( \mathcal{M}_{0,r}/\mathbb{F}_p \) the stack parameterizing \( r \)-marked curves \((X; \{x_1, \ldots, x_r\})\) of genus zero. Fix a set \( \alpha = (\alpha_1, \ldots, \alpha) \) with \( \alpha_i \in \mathbb{F}_p \).

Dwork’s *accessory parameter problem* asks to determine the space of all normalized differential operators \( L \) with nilpotent \( p \)-curvature. This amounts to finding all \((x_i; \beta_i)\) such that the differential operator given by (6) is normalized and admits a polynomial solution. Dwork ([7]) proves that the algebraic space, \( V_N \), of such differential operators is a complete intersection. Moreover, Dwork shows that the natural projection of \( V_N \) on \( \mathcal{M}_{0,r} \) has degree \( p^{r-3} \) ([7, Corollary 4.2]).

In this section, we are interested in the moduli space of normalized differential operators with nilpotent but nonzero \( p \)-curvature, or, equivalently, the moduli space of deformation data.

Proposition 4.1  
(a) There exists a stack \( N_{0,r}(\alpha) \) parameterizing normalized differential operators with nilpotent, nonzero \( p \)-curvature with local exponents \( \alpha \).

(b) Let \( \pi : N_{0,r}(\alpha) \to \mathcal{M}_{0,r} \) be the natural projection which sends \( L \) to its set of singularities. Then \( \pi \) is finite.
(c) The degree of $\pi$ is less than or equal to $p^{r-3}$, with equality if and only if there do not exist normalized differential operators with local exponents $\alpha$ with zero $p$-curvature.

**Proof:** Mochizuki ([15, Chapter IV]) proves the existence of a stack parameterizing indigenous bundles with local exponents $\alpha$. It follows from [6, Theorem 4.11] and Proposition 2.8 that there is equivalence between indigenous bundles and normalized differential operators with nilpotent, nonzero $p$-curvature. Parts (b) and (c) follow, for example, from [7, Corollary 4.2]. □

Let $L$ be a normalized differential operator with nilpotent but nonzero $p$-curvature. Then Proposition 2.8 implies that $L$ defines a deformation datum $(Z, \omega)$. Therefore we may define the signature $\sigma = (\sigma_i)_{i \in \mathbb{R}}$, as in the proof of Proposition 2.8. In particular, we may define the spikes $x_{r+1}, \ldots, x_s$.

**Definition 4.2** Let $x$ be a critical point of the deformation datum corresponding to $L$. Define

$$n_x = \begin{cases} 
0 & \text{if } x \text{ is supersingular,} \\
(p-1)\sigma_x & \text{if } x \text{ is a singularity,} \\
(p-1)(\sigma_x - 1) & \text{if } x \text{ is a spike.}
\end{cases}$$

The number $n := \sum_x n_x$ is called the strength of $L$.

Definition 2.3 implies that $n_x$ is a nonnegative integer. We refer to [6] for an explanation of this notion; suitably defined, $n_x$ is the order of vanishing of the $p$-curvature at $x$. The proof of Proposition 2.8 implies that

$$n_x = \text{ord}_x \prod_{i=1}^{r-1} (x - x_i)^p D^{p-1} \frac{1}{Qu^2}.$$ 

This gives a concrete interpretation of $n_x$. The terminology ‘strength’ was introduced by Mochizuki ([15, Introduction, §1.2]). Note that the index $d$ in [15] denotes the strength and not the number of supersingular points. The following lemma states a few properties of $n$. The lemma follows immediately from (17).

**Lemma 4.3** Suppose that $n$ is the strength of a deformation datum. Then

(a) $n = (r - 2)(p - 1) - 2d$,

(b) $n$ is an even integer which is equivalent to $-\sum_{i=1}^{r} \alpha_i \pmod{p}$,

(c) $n \geq \sum_{\alpha_i \neq 0} (p - \alpha_i)$.

We denote by $N_{0,r}(\alpha; n)$ the substack of $N_{0,r}(\alpha)$ parameterizing normalized differential operators with nilpotent and nonzero $p$-curvature and strength $n$. The **strong accessory parameter problem** asks for the structure of $N_{0,r}(\alpha; n)$.
For example, it is natural to ask for which \( n \) the stack \( \mathcal{N}_{0,r}(\alpha; n) \) is nonempty or has maximal dimension (i.e. \( \dim \mathcal{N}_{0,r}(\alpha; n) = \dim \mathcal{M}_{0,r} = r - 3 \)). Also, one would like to know the degree of \( \mathcal{N}_{0,r}(\alpha; n) \rightarrow \mathcal{M}_{0,r} \), for those \( n \) for which \( \mathcal{N}_{0,r}[n] \) has maximal dimension.

**Remark 4.4** Lemma 4.3 gives a necessary condition for the nonemptiness of \( \mathcal{N}_{0,r}(\alpha; n) \). Namely for given \( \alpha \), a necessary condition for \( \mathcal{N}_{0,r}(\alpha; n) \) to be nonempty is that there exists a nonnegative integer \( n \) such that the (in)equalities of Lemma 4.3 are satisfied. We leave it to the reader to check that this is indeed a nontrivial condition.

As far as I know, the only results in this direction are for \( r \leq 4 \). These results are summarized in Example 4.5.

**Example 4.5** Suppose that \( \alpha_i = 0 \) for \( i = 1, \ldots, r \). This case we consider in more detail in §4.4.

(a) If \( r = 3 \) it follows from Proposition 1.4 that \( d < p \). Since \( d \equiv -1/2 \pmod{p} \), it follows that \( d = (p - 1)/2 \) and hence that \( \mathcal{N}_{0,3}(\alpha) = \mathcal{N}_{0,3}(\alpha; 0) \). In fact, this space consists of one point. Let \( r = 3 \) and suppose that \( x_1 = 0, x_2 = 1, x_3 = \infty \). Then the corresponding differential operator \( L \) is the Gauss hypergeometric differential operator ([6, Example 4.5]):

\[
L = \left( \frac{\partial}{\partial t} \right)^2 + \frac{2t - 1}{t(t - 1)} \left( \frac{\partial}{\partial t} \right) + \frac{1}{4t(t - 1)}.
\]

Its monic polynomial solution of minimal degree is the Hasse invariant:

\[
\Phi = \sum_{i=0}^{(p-1)/2} \left( \frac{p-1}{i} \right)^2 t^i.
\]

The corresponding deformation datum describes the stable reduction of the cover of modular curves \( X(2p) \rightarrow X(2) \) ([5]).

(b) Similarly, if \( r = 4 \) it follows from Proposition 1.5 or from [1] that \( d = p - 1 \), hence \( \mathcal{N}_{0,4}(\alpha) = \mathcal{N}_{0,4}(\alpha; 0) \). Lemma 1.6 and [7, Corollary 4.2] imply, moreover, that \( \mathcal{N}_{0,4}(\alpha) \) is nonempty. In [6, §6.2] it is shown that there exists a connected component which has degree one over \( \mathcal{M}_{0,4} \).

### 4.2 The dimension of the components of \( \mathcal{N}_{0,r}(\alpha; n) \)

Suppose that \( \mathcal{N} := \mathcal{N}_{0,r}(\alpha; n) \) is nonempty. In this section, we characterize the those irreducible components of \( \mathcal{N} \) which have maximal dimension \( r - 3 \). This is a direct consequence of the results of §3.

We fix local exponents \( \alpha = (\alpha_1, \ldots, \alpha_r) \) and a nonnegative integer \( n \equiv -\sum_{i=1}^{r} \alpha_i \pmod{p} \) such that \( d := [(r - 2)(p - 1) - n]/2 \) is a nonnegative integer and \( n \geq \sum_{\alpha_i \neq 0} (p - \alpha_i) \) (Lemma 4.3). Let \( s \) be defined by

\[
s - r = \frac{1}{p}[(r - 2)(p - 1) - 2d - \sum_{\alpha_i \neq 0} (p - \alpha_i)].
\]
It follows from the assumptions on \( n \) that \( s \) is a nonnegative integer.

Let \( B = \{1, \ldots, d + s\} \). We now define a signature \( \sigma = \sigma_{\alpha} = (\sigma_i)_{i \in B} \) as follows. For \( 1 \leq i \leq r \), we put

\[
\sigma_i = \begin{cases} 
  0 & \text{if } \alpha_i = 0, \\
  (p - \alpha_i)/(p - 1) & \text{otherwise}.
\end{cases}
\]

For \( r + 1 \leq i \leq s \), we put \( \sigma_i = (2p - 1)/(p - 1) \). For \( i > s \), we put \( \sigma_i = (p + 1)/(p - 1) \). Note that \( \sigma \) has \( s - r \) spikes and \( d \) supersingular points.

**Proposition 4.6** Suppose that \( p > r - 2 \) and that \( N_{0,r}(\alpha; n) \) is nonempty. Let \( N \) be an irreducible component of \( N_{0,r}(\alpha; n) \) of dimension \( r - 3 \). Then there exists a point \( L \in N \) whose signature is \( \sigma_{\alpha} \).

**Proof:** Let \( N \) be in the statement of the proposition, and suppose that \( \dim N = r - 3 \). Let \( L \) be the differential operator corresponding to the generic point of \( N \). Proposition 3.7.(b) implies that the signature of \( L \) satisfies \( \sigma_i < 1 \) for \( 1 \leq i \leq r \) and \( \sigma_i = (2p - 1)/(p - 1) \) if \( x_i \) is a spike. This implies that the signature of \( L \) is \( \sigma_{\alpha} \). \( \square \)

### 4.3 Examples

In this section, we illustrate the results of the previous section with some examples.

Let \( p = 7 \) and \( r = 4 \). We choose \( \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 5 \) and \( x_1 = 0, x_2 = 1, x_3 = \lambda, x_4 = \infty \), where we assume that \( \lambda \) is transcendental over \( \mathbb{F}_p \). In other words, we want to determine components of \( N_{0,4}(\alpha) \) with dimension \( r - 3 = 1 \). Put \( k = \mathbb{F}_p(\lambda) \). We consider differential operators \( L \) with singularities in \( (x_i) \) and local exponents \( \alpha = (\alpha_i) \), i.e. \( L \) is given by (6). However, we do not assume that \( L \) is normalized (§1.2). Let \( \gamma_1, \gamma_2 \) be the local exponents at \( \infty \). Therefore it follows from Lemma 1.3 that if \( u \in k[x] \) is a solution of \( L \) the \( \deg(u) \equiv -\gamma_1 \equiv 2, -\gamma_2 \equiv 4 \pmod{p} \). (Assuming that \( L \) is normalized would exclude one of these possibilities.)

Set \( u = \sum_{i \geq 0} u_i x^i \). One checks that \( L(u) = 0 \) if and only if the coefficients of \( u \) satisfy the recursion

\[
\lambda A_i u_{i+1} = (C_i - \beta) u_i - B_i u_{i-1},
\]

where

\[
A_i = (i - 1)(i + 1 - \alpha_1), \quad B_i = (i + \gamma_1 - 1)(i + \gamma_2 - 1),
\]

\[
C_i = i^2 (1 + \lambda) + i(\lambda(1 - \alpha_1 - \alpha_2) + 1 - \alpha_1 - \alpha_3).
\]

Propositions 1.4 and 1.5 imply that if \( L \) has nilpotent \( p \)-curvature then \( L \) has a solution of degree less than \( p \).

One deduces that \( L \) has a solution of degree 2 if and only if the accessory parameter \( \beta \) satisfies \( \beta \in \{0, -1, -\lambda\} \). Lemma 1.3.(b) implies in each of these cases
that the solution, \( u \), of degree 2 does not have zeros in \( x = 0, 1, \lambda \). Moreover, one checks that
\[
\left( \frac{\partial}{\partial x} \right)^{p-1} \frac{1}{x^3(x-1)^3(x-\lambda)^3u^2} \neq 0,
\]
as rational function of \( \lambda \). Therefore it follows from Proposition 1.9 that the \( p \)-curvature of \( L \) is nonzero for \( \beta \in \{0, -1, -\lambda\} \). (One could also deduce this from the recursion for the coefficients of \( u \).) Lemmas 2.2.(d) and 4.3 imply that the signature of \( (L, u) \) is \( \sigma = (\sigma_i) \) with \( \sigma_i = 2/(p-1) = 1/3 \) for \( i = 1, \ldots, 4 \). In particular, we have no spikes and \( n = 2(p-1) - 2d = 8 \).

We conclude that \( N_{0,4}(\alpha; 8) \subset N_{0,4}(\alpha) \) is dense. The degree of \( \pi : N_{0,4}(\alpha) \to M_{0,4} \) is 3. More precisely, \( N_{0,4}(\alpha; 8) \) consists of three irreducible components which each have degree 1 over \( M_{0,4} \).

Similarly, one shows that \( L \) has a solution, \( u \), of degree 4 if and only if
\[
5\beta^2 + \beta(1 + \lambda) + \lambda^2 + 3\lambda + 1 = 0.
\]
As before, it follows that \( u \) does not have zeros in \( x = 0, 1, \lambda \). One computes that the residues of \( 1/(x^3(x-1)^3(x-\lambda)^3u^2) \) in \( x = 0, 1, \infty \) are zero. This implies that
\[
\left( \frac{\partial}{\partial x} \right)^{p-1} \frac{1}{x^3(x-1)^3(x-\lambda)^3u^2} = 0
\]
(Lemma 2.6.(c)). It follows that the \( p \)-curvature of \( L \) is zero in this case. In fact, one computes that \( L \) also has a solution of degree \( 2 + p = 9 \).

As a second example, we consider \( p = 13, r = 4 \) and \( \alpha = (11, 11, 11, 10) \). Let \( x_1 = 0, x_2 = 1, x_3 = \lambda, x_4 = \infty \), with \( \lambda \) transcendental over \( \overline{\mathbb{F}}_p \), as in the previous example. We let \( L \) be given by (6), i.e. \( L \) is a general differential operator with local exponents \( \alpha \). Since we do not assume that \( L \) is normalized, it follows that the degree of a polynomial solution of \( L \) is congruent to 1, 4 (mod \( p \)).

Arguing as in the previous example, we find that \( L \) has a solution of degree 1 if and only if
\[
\beta^2 + 7(\lambda + 1)\beta + \lambda = 0.
\]
Moreover, for \( \beta \) satisfying (18), the monic polynomial solution, \( u \), of \( L \) does not have a zero in \( x = 0, 1, \lambda \). One computes that
\[
\left( \frac{\partial}{\partial x} \right)^{p-1} \frac{1}{x^3(x-1)^3(x-\lambda)^3u^2}
\]
has exactly one zero, which has order \( p \). Moreover, this zero is not in \( x = 0, 1, \lambda \), since \( \lambda \) is transcendental. This also follows from Proposition 4.6. In particular, the \( p \)-curvature of \( L \) is nonzero. We conclude that \( L \) has one spike, and hence that the signature is \((2/(p-1), 2/(p-1), 2/(p-1), 3/(p-1), 3(p-1)/(p-1)) \). It follows that the strength is \( n = 22 \).

We conclude that \( N_{0,4}(\alpha; 22) \subset N_{0,4}(\alpha) \) is dense and that the degree of \( \pi : N_{0,4}(\alpha) \to M_{0,4} \) is 2.
Similarly, one checks that $L$ has a polynomial solution of degree 4 if and only if
\begin{equation}
\lambda^3 + (2\beta + 9)\lambda^2 + (9 + 8\beta^2 + 4\beta)\lambda + 2\beta^3 + 8\beta^2 + 2\beta + 1 = 0.
\end{equation}

One checks that, for $\beta$ satisfying (19), the $p$-curvature of $L$ is nonvanishing, and that $L$ does not have a spike. The corresponding signature is therefore $\sigma = (2/(p-1), 2/(p-1), 2/(p-1), 10/(p-1))$. Hence the strength is $n = 16$. The differential operator is not normalized: following the convention of §1.2 we have that $\alpha' := (p - 2, p - 2, p - 2, 3)$ in this case.

We conclude that $N_{0,4}(\alpha'; 16) \subset N_{0,4}(\alpha')$ is dense and that the degree of $\pi : N_{0,4}(\alpha') \to M_{0,4}$ is 3.

**4.4 The case of logarithmic local monodromy**

In this section, we consider the strong accessory parameter problem in the case that all local monodromy matrices are nilpotent.

**Definition 4.7** Let $L$ be a normalized differential operator with nilpotent $p$- curvature, and let $\alpha = (\alpha_i)_{i=1}^r$ be its local exponents. We say that $L$ has logarithmic local monodromy if $\alpha_i = 0$ for $i = 1, \ldots, r$.

Let $d$ be a nonnegative integer congruent to $1 - r/2$ (mod $p$) such that $n := (p - 1)(r - 2) - 2d$ is nonnegative. In this section, we drop the local exponents from the notation, and write $N_{0, r}[n]$ for the stack parameterizing differential operators with nilpotent $p$- curvature and logarithmic local monodromy and strength $n$. Lemma 1.6 implies that the $p$-curvature of a differential operator is always nonzero, therefore this notation agrees with the notation in the previous section.

Let $L$ correspond to a point of $N_{0, r}[n]$, and let $u$ be a polynomial solution of $L$ of minimal degree, $x_1, \ldots, x_r$ the singularities of $L$. Then $(L, u)$ corresponds to a deformation datum $(Z, \omega)$ (Proposition 2.8). Therefore we may use the terminology of §2. Let $x_{r+1}, \ldots, x_s$ be the spikes of $(Z, \omega)$ and let $\sigma_i = \alpha_i/(p - 1) + \nu_i$ be the ramification invariant of $x_i$ for $1 \leq i \leq s$ (compare to the proof of Proposition 2.8.) It follows from [6, Proposition 3.6.(i)] that $\sigma_i = 0$ for $i = 1, \ldots, r$.

The following theorem is proved by Mochizuki ([15, Introduction, Theorem 1.2]), by using a deformation argument. It is stronger than Proposition 4.6. Namely, Theorem 4.8 implies that every differential operator may be deformed to a differential operator of the same strength such that all spikes have the minimal possible ramification invariant, namely $\sigma = (2p - 1)/(p - 1)$. One expects this to hold for arbitrary local exponents, as well.

**Theorem 4.8 (Mochizuki)** Suppose that $N_{0, r}[n]$ is nonempty. Then all irreducible components of $N_{0, r}[n]$ have dimension $r - 3$.

The following lemma is a more precise version of Lemma 4.3 in the case of logarithmic local monodromy.
Lemma 4.9 Let \( r \geq 3 \) and \( n \geq 0 \) be integers such that \( \mathcal{N}_{0,r}[n] \) is nonempty.

(a) Then \( n \equiv 0 \pmod{2p} \).

(b) We have that \( 0 \leq n \leq (p-1)(r-2) \). Moreover, \( r \geq n/p + 3 \).

Proof: The statement that \( n \equiv 0 \pmod{p} \) follows from Definition 2.3.(d).

Lemma 2.2.(d) implies that \( 2d + n = (r-2)(p-1) \), since \( \sigma_i = 0 \) for \( i = 1, \ldots, r \).

It follows that \( n \) is even.

The bounds on \( n \) follow from Proposition 1.5. The inequality for \( r \) follows from (a).

To goal of the rest of this section is to prove the following theorem. This theorem follows immediately from Propositions 4.11 and 4.12.

Theorem 4.10 Suppose that \( p > r-2 \). Let \( 0 \leq n \leq (p-1)(r-2) \) be congruent to 0 \( \pmod{2p} \). Then \( \mathcal{N}_{0,r}[n] \) is nonempty.

Proposition 4.11 Suppose that \( \mathcal{N}_{0,r}[n] \) is nonempty. Then \( \mathcal{N}_{0,r+1}[n] \) is nonempty, as well.

Proof: Suppose that \( \mathcal{N}_{0,r}[n] \) is nonempty, and let \((L, u)\) correspond to a point of \( \mathcal{N}_{0,r}[n] \), i.e. \( u \) is a polynomial solution of \( L \) of minimal degree \( d \). The integer \( d \) satisfies \( d = [(r-2)(p-1)-n]/2 \leq (p-1)(r-2)/2 \). As usual, we denote by \( x_1, \ldots, x_r = \infty \) the singularities of \( L \) and by \( x_{r+1}, \ldots, x_s \) the spikes.

Proposition 2.8 implies that the pair \((L, u)\) corresponds to a deformation datum \((Z, \omega)\). The main result of [6] implies that this deformation datum corresponds to an indigenous bundle \( \mathcal{E}_1 \) on \((X_1 := \mathbb{P}^1; x_i)\).

Let \( \mathcal{E}_2 \) be the indigenous bundle corresponding to the deformation datum of Example 4.5.(a); this is the indigenous bundle corresponding to Gauss’ hypergeometric differential equation. The indigenous bundle \( \mathcal{E}_2 \) lives on the marked curve \((X_2; \tau_1 = 0, \tau_2 = 1, \tau_3 = \infty)\) and has no spikes, i.e. \( n = 0 \).

We define a stably marked curve \( X \) by identifying the point \( x_r = \infty \) on \( X_1 \) with the point \( \tau_3 = \infty \) on \( X_2 \). Mochizuki ([14, §I.2, page 1008]) shows that the indigenous bundles \( \mathcal{E}_1 \) define an indigenous bundle \( \mathcal{E} \) on \( X \). We refer to [14] for the precise definition of an indigenous bundle on a stably marked curve. The bundle \( \mathcal{E} \) is what Mochizuki calls an indigenous bundle of restrictable type. It is shown in [14, Proposition 2.11, §I.2] that \((X, \mathcal{E})\) deforms to an indigenous bundle \( \mathcal{E} \) on a smooth stably marked curve \((X; \tilde{x})\) which is a deformation of the stably marked curve \( X \). In particular, the bundle \( \mathcal{E} \) (or equivalently, the corresponding pair \((L, \tilde{u})\)) has \( r + 3 - 2 = r + 1 \) singular points and is spiked of strength \( n \). This proves that \( \mathcal{N}_{0,r+1}[n] \) is nonempty.

One could give an alternative proof of Proposition 4.11 without using the results of Mochizuki by using the ideas of [4, Section 2.5].

Proposition 4.12 Suppose that \( p > r-2 \). Let \( 0 \leq n \leq (p-1)(r-2) \) be an integer with \( n \equiv 0 \pmod{2p} \), and let \( r = n/p + 3 \). Then \( \mathcal{N}_{0,r}[n] \) is nonempty.
Proof: Note that the assumption that \( r = n/p + 3 \) implies that \( r \) is odd. Example 4.5 implies that the proposition holds for \( r = 3 \), therefore it is no restriction to suppose that \( r \geq 5 \), or, equivalently, \( n \geq 2p \). The degree, \( d \), of a minimal polynomial solution is now \( d = (p - r + 2)/2 \).

Let \( \zeta \in k \) be a primitive \((r - 2)\)th root of unity. It exists since we assumed that \( p > r - 2 \). We define \( x_1 = 0, x_r = \infty \). Moreover, for \( i = 0, \ldots, r - 3 \), we put \( x_{i+1} = \zeta^i \). We consider a general differential operator \( L \) with logarithmic local monodromy at \( x_1, \ldots, x_r \). By (6) we have that

\[
L = (x^{r-1} - x) \frac{\partial}{\partial x} + ((r - 1)x^{r-2} - 1) \frac{\partial}{\partial x} + (d^2 x^{r-3} + \beta_{r-4} x^{r-4} + \cdots + \beta_0).
\]

We want to determine \( \beta_i \) such that \( L \) has a polynomial solution, \( u_i \), of degree \( d \). Write \( u = \sum_{i \geq 0} u_i x^i \) with \( u_0 = 1 \). Then setting \( L(u) = 0 \) yields the following recursion for the coefficients:

\[
u_{i-r+3} \left( i - \frac{r-4}{2} \right)^2 + \sum_{j=0}^{r-4} u_{i-j} \beta_j = u_{i+1}(i + 1)^2. \tag{20}
\]

This determines the coefficients \( u_1, \ldots, u_{r-1} \) uniquely, in terms of the \( \beta_i \), since for \( 0 \leq i \leq p - 2 \) we have that \( i + 1 \not\equiv 0 \) (mod \( p \)).

To show the existence of a polynomial solution of degree \( d \) of \( L \), we have to show that we may choose \( \beta_i \in k \) such that \( u_{d+1} = \cdots u_{d+r-3} = 0 \). The recursion (20) then implies that we may assume that \( u_i = 0 \) for all \( i > d \).

One easily deduces from (20) that the total degree of \( u_i \) in the variable \( \beta_j \) is \( i \), and that

\[
u_i = \epsilon_i \beta_0^i + \text{terms of strictly lower degree, for some } \epsilon_i \in \mathbb{F}_p^\times. \tag{21}
\]

We now homogenize the equations \( u_{d+1} = 0, \ldots, u_{d+r-3} = 0 \), introducing a new variable \( \gamma \), and consider our equations as equations on \( \mathbb{F}_p^{r-3} \). Proposition 4.1.(b) implies that the solutions space of these equations is zero dimensional. Bezout’s Theorem implies that the total number of solutions (counted with multiplicity) is \( (d + 1)(d + 2) \cdots (d + r - 3) \). It remains to show that the number of solutions with \( \gamma = 0 \) is strictly less than this number.

We now compute the solutions with \( \gamma = 0 \). From the particular form of the equations, we deduce that if \( \gamma = 0 \) then also \( \beta_0 = 0 \). We now eliminate the variable \( \beta_0 \) from the equations \( u_{d+2} = 0, \ldots, u_{d+r-3} = 0 \). We first replace \( u_{d+i} = 0 \) by \( \mathcal{E}_{d+i} := u_{d+i} - \beta_0^{-1} u_{d+1} \mu_i = 0 \), where \( \mu_i = \epsilon_{d+i}/\epsilon_{d+1} \in \mathbb{F}_p^\times \). Then \( \mathcal{E}_{d+i} \) is divisible by \( \gamma \). Dividing out by a power of \( \gamma \) and substituting \( \beta_0 = 0 \), we obtain therefore for \( i > 1 \) a new equation of degree strictly smaller that \( d + i \). Since \( r - 3 > 1 \), we conclude that not all solutions of our system of equations are on the hyperplane at \( \infty \). This shows that the differential equation has a solution \( u \) of degree less than or equal to \( d \). Since \( u \) is nontrivial and \( d \) is the smallest possible degree of a solution, we conclude that \( \deg(u) = d \). This proves the proposition. \( \square \)
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