Consensus vs Broadcast in Communication Networks with Arbitrary Mobile Omission Faults

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Abstract. We compare the solvability of the Consensus and Broadcast problems in synchronous communication networks in which the delivery of messages is not reliable. The failure model is the mobile omission faults model. During each round, some messages can be lost and the set of possible simultaneous losses is the same for each round. We investigate these problems for the first time for arbitrary sets of possible failures. Previously, these sets were defined by bounding the numbers of failures. In this setting, we present a new necessary condition for the solvability of Consensus that unifies previous impossibility results in this area. This condition is expressed using Broadcastability properties. As a very important application, we show that when the sets of omissions that can occur are defined by bounding the numbers of failures, counted in any way (locally, globally, etc.), then the Consensus problem is actually equivalent to the Broadcast problem.

1 Introduction

We consider synchronous communication networks in which some messages can be lost during each round. These omission faults can be permanent or not; a faulty link can become reliable again after an unpredictable number of rounds, and it can continue to alternate between being reliable and faulty in an unpredictable way. This model is more general than other models, such as component failure models, in which failures, once they appear somewhere, are located there permanently. The model that we use, called the mobile faults or dynamic faults model, was introduced in [SW89] and is discussed further in [SW07]. An important property of these systems is that the set of possible simultaneous omissions is the same for each round. In some sense, the system has no “memory” of the previous failures. Real systems often exhibit such memory-less behaviour.

In previous research, the sets of possible simultaneous omissions were defined by bounding the numbers of omissions. Recent work on this subject includes [SW07], in which omissions are counted globally, and [SWK09], in which the number of omissions is locally bounded. It has also been shown to be good for layered analysis [MR02]. In this paper we consider the most general case of such systems, i.e. systems in which the set of possible simultaneous omissions is arbitrary. This allows the modelling of any system in which omissions can happen transiently, in any arbitrary pattern, including systems in which the communications are not symmetric.

We investigate two fundamental problems of Distributed Computing in these networks: the Consensus problem and the Broadcast problem. While it has long been known that solvability of the Broadcast problem implies solvability of the Consensus problem, we prove here that these problems are actually equivalent (from both the solvability and complexity points of view) when the sets of possible omissions are defined by bounding the number of failures, for any possible way of counting them (locally, globally, any combination, etc.).

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1.1 The Consensus Problem

The Consensus problem is a very well studied problem in the area of Distributed Algorithms. It is defined as follows. Each node of the network starts with an initial value, and all nodes of the network have to agree on a common value, which is one of the initial values. Many versions of the problem concern the design of algorithms for systems that are unreliable.

The Consensus problem has been widely studied in the context of shared memory systems and message passing systems in which any node can communicate with any other node. Surprisingly, there have been few studies in the context of communication networks, where the communication graph is not a complete graph. In one of the first thorough studies [SW07], Santoro and Widmayer investigate some $k$–Majority Problems that are defined as follows. Each node starts with an initial value, and every node has to compute a final decided value such that there exists a value (from the set of initial values) that is decided by at least $k$ of the nodes. The Consensus problem (called the Unanimity problem in [SW07]) is the $n$–Majority problem where $n$ is the number of nodes in the network.

In their paper, Santoro and Widmayer give results about solving the Consensus problem in communication networks with various types of faults including omission faults. For simplicity, we focus here only on omission faults. We believe that our results can be quite easily extended to other fault models, using the methodology of [SW07].

1.2 The Broadcast Problem

Two of the most widely studied patterns of information propagation in communication networks are broadcasting and gossiping. A broadcast is the distribution of an initial value from one node of a network to every other node of the network. A gossip is a simultaneous broadcast from every node of the network. The Broadcast problem that we study in this paper is to find a node from which a broadcast can be successfully completed.

There are close relationships between broadcasting and gossiping, and the Consensus problem. Indeed, the Consensus problem can be solved by first gossiping and then applying a deterministic function at each node to the set of initial values. But a gossip is not actually necessary. If there exists a distinguished node $v_0$ in the network, then a Consensus algorithm can be easily derived from an algorithm that broadcasts from $v_0$. However the Broadcast problem and the Consensus problem are not equivalent, as will be made clearer in Section 3.

1.3 Our Contributions

In this paper, we investigate systems in which the pattern of omission failures is arbitrary. A set of simultaneous omissions is called a communication event. We characterize the solvability of the Broadcast and Consensus problems subject to an arbitrary family of possible communication events. A node from which it is possible to broadcast if the system is restricted to a given communication event is called a source for the communication event. We prove that the Broadcast problem is solvable if and only if there exists a common source for all communication events. To study the Consensus problem, we define an equivalence relation on a family of communication events based on the collective local observations of the events

\footnote{Note that some of the terminology that we use in this paper is different from the terminology of Santoro and Widmayer. We are investigating relationships between different areas of distributed algorithms, and some terminology (such as $k$–agreement) has different meanings in the different areas.}
by the sources. We prove in Theorem 4.10 that the Consensus problem is not solvable for a family of communication events if the Broadcast problem is not solvable for one class of the equivalence relation. For Consensus to be solvable, the sources of a given event must be able to collectively distinguish communication events with incompatible sources. We conjecture that this is actually a sufficient condition.

It is very simple to characterize Broadcastability (see Theorem 4.4), so we get very simple and efficient impossibility proofs for solving Consensus subject to arbitrary omission failures. These impossibility conditions are satisfied by the omission schemes of [SW07] and [SWK09]. This means that our results encompass all previous known results in the area.

Furthermore, we prove that under very general conditions, in particular when the possible simultaneous omissions are defined by bounding the number of omissions, for any way of counting omissions, there is actually only one equivalence class when the system is not broadcastable. An important application is that, the Consensus problem is exactly the same as the Broadcast problem for most omission fault models. Therefore, it is possible to deduce complexity results for the Consensus problem from complexity results about broadcasting with omissions.

1.4 Related Work

In [CBS09], the authors present a model that can describe benign faults. This model is called the “Heard-Of” model. It is a round-based model for an omission-prone environment in which the set of possible communication events is not necessarily the same for each round. However, they require a time-invariance property. As a special case, they present a characterization that shows that solving Consensus in this environment is equivalent to solving a Selection problem. They also present algorithms for some families of omission schemes. It is not possible to derive our simple characterizations from [CBS09].

In [SW07], the Consensus and related Agreement problems are investigated for networks in which there are at most $f$ omissions during any given round. It is proved that it is impossible to solve Consensus if $f$ is at least the minimum degree of the graph. A Consensus algorithm is presented for the case where $f$ is strictly smaller than the connectivity of the network. In [FG11], a reduction to the two process case is used to show that the connectivity of the graph is indeed the exact limit for consensus is such omission schemes. In this paper, we generalize these results, showing that exact limits for Consensus can be derived from exact limits for Broadcast.

While the above studies use a global failure metric, a local failure metric is investigated in [SWK09], distinguishing send and receive omissions. The authors describe which bounds allow Consensus to be solved, using a proof technique based on a Withholding Lemma. They claim that Consensus is solvable if and only if no node can withhold its information from some other part of the network. We will show that this is not true when the pattern of omissions can be arbitrary. We present (in Example 2.5) a system in which nodes can withhold information infinitely but Consensus is solvable.

Finally, it is worth noting that, although both [SW07] and [SWK09] are using the classic bivalency proof technique, it is not possible to derive any of the results of [SW07] or [FG11] (global bound on omissions) from the results of [SWK09] (local bound) as the omission schemes are not comparable. Our results consolidate these previous results. Furthermore, our approach is more general than these previous approaches and is more suitable for applications to new omission metrics.
2 Definitions and Notation

2.1 Communication Networks

We model a communication network by a digraph $G = (V, E)$ which does not have to be symmetric. If we are given an undirected graph $G$, we consider the corresponding symmetric digraph. We always assume that nodes have unique identities. Given a set of arcs $E$, we define $h(E) = \{ t \mid (s, t) \in E \}$, the set of nodes that are heads of arcs in $E$.

All sub-digraphs that we consider in this paper are spanning subgraphs. Since all spanning subgraphs of a digraph have the same set of nodes, we will use the same notation to refer to both the set of arcs of a sub-digraph and the sub-digraph with that set of arcs when the set of nodes is not ambiguous.

2.2 Omission Schemes

In this section, we introduce our model and the associated notation. Communication in our model is synchronous but not reliable, and communication is performed in rounds. Communication with omission faults is described by a spanning sub-graph of $G$ with the semantics that are specified in Section 2.4.

Throughout this paper, the underlying graph $G = (V, E)$ is fixed, and we define the set $\Sigma = \{(V, E') \mid E' \subseteq E\}$. This set represents all possible simultaneous communications given the underlying graph $G$.

**Definition 2.1.** An element of $\Sigma$ is called a communication event (or event for short). An omission scenario (or scenario for short) is an infinite sequence of communication events. An omission scheme over $G$ is a set of omission scenarios.

A natural way to describe communications is to consider $\Sigma$ to be an alphabet, with communication events as letters of the alphabet, and scenarios as infinite words. We will use standard concatenation notation when describing sequences. If $w$ and $w'$ are two sequences, then $ww'$ is the sequence that starts with the ordered sequence of events $w$ followed by the ordered sequence of events $w'$. This notation is extended to sets in an obvious way. The empty word is denoted $\varepsilon$. We will use the following standard notation to describe our communication schemes.

**Definition 2.2 ([PP04]).** Given $R \subseteq \Sigma$, $R^*$ is the set of all finite sequences of elements of $R$, and $R^\omega$ is the set of all infinite ones.

The set of all possible scenarios on $G$ is then $\Sigma^\omega$. A given word $w \in \Sigma^*$ is called a partial scenario and $|w|$ is the length of this partial scenario. An omission scheme is then a subset $S$ of $\Sigma^\omega$. A mobile omission scheme is a scheme that is equal to $R^\omega$ for some subset $R \subseteq \Sigma$.

In this paper, we consider only mobile omission schemes. Note that we do not require $G$ to belong to $R$. A formal definition of an execution subject to a scenario will be given in Section 2.4. Intuitively, the $r$-th letter of a scenario will describe which communications are reliable during round $r$.

Finally, we recall some standard definitions for infinite words and languages over an alphabet $\Sigma$. Given $w = (a_1, a_2, \ldots) \in \Sigma^\omega$, a subword of $w$ is a (possibly infinite) sub-sequence $(a_{\sigma(1)}, a_{\sigma(1)}, \ldots)$, where $\sigma$ is a strictly increasing function. A word $u \in \Sigma^*$ is a prefix of $w \in \Sigma^*$ (resp. $w' \in \Sigma^\omega$) if there exists $v \in \Sigma^*$ (resp. $v' \in \Sigma^\omega$) such that $w = uv$ (resp. $w' = uv'$). Given $w \in \Sigma^\omega$ and $r \in \mathbb{N}$, $w|_r$ is the finite prefix of $w$ of length $r$. 

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Definition 2.3. Let \( w \in \Sigma^\omega \) and \( L \subset \Sigma^\omega \). Then \( \text{Pref}(w) = \{ u \in \Sigma^* | u \text{ is a prefix of } w \} \), and \( \text{Pref}(L) = \bigcup_{w \in L} \text{Pref}(w) \). A word \( w' \) is an extension of \( w \) in \( L \), if \( w w' \in L \).

2.3 Examples

We do not restrict our study to regular sets, however all omission schemes known to us are regular, including the following examples, so we will use the notation for regular sets. We present examples for systems with two processes but they can be easily extended to any arbitrary graph. The set \( \Sigma = \{ \circ \leftrightarrow \bullet, \circ \leftarrow \bullet, \circ \rightarrow \bullet, \circ \bullet \} \) is the set of directed graphs with two nodes \( \circ \) and \( \bullet \). The subgraphs in \( \Sigma \) describe what can happen during a given round with the following interpretation:

- \( \circ \leftrightarrow \bullet \): all messages that are sent are correctly received;
- \( \circ \leftarrow \bullet \): the message from process \( \circ \), if any, is not received;
- \( \circ \rightarrow \bullet \): the message from process \( \bullet \), if any, is not received;
- \( \circ \bullet \): no messages are received.

Example 2.4. The set \( \{ \circ \leftrightarrow \bullet \}^\omega \) corresponds to a reliable system. The set \( \mathcal{O}_1 = \{ \circ \leftrightarrow \bullet, \circ \leftarrow \bullet, \circ \rightarrow \bullet \}^\omega \) is well studied and corresponds to the situation in which there is at most one omission per round.

Example 2.5. The set \( \mathcal{H} = \{ \circ \leftarrow \bullet, \circ \rightarrow \bullet \}^\omega \) describes a system in which at most one message can be successfully received in any round, and if only one message is sent, it might not be received.

The examples above are examples of mobile omission schemes. The following is a typical example of a non-mobile omission scheme.

Example 2.6. Consider a system in which at most one of the processes can crash. From the communications point of view, this is equivalent to a system in which it is possible that no messages are transmitted by one of the processes after some arbitrary round. The associated omission scheme is the following:

\[ \mathcal{C}_1 = \{ \circ \leftrightarrow \bullet \}^\omega \cup \{ \circ \bullet \}^\omega (\{ \circ \rightarrow \bullet \}^\omega \cup \{ \circ \leftarrow \bullet \}^\omega) \].

2.4 Reliable Execution of a Distributed Algorithm Subject to Omissions

Given an omission scheme \( \mathcal{S} \), we define what is a successful execution of a given algorithm \( \mathcal{A} \) with a given initial configuration \( \iota \). Every process can execute the following communication primitives:

- \( \text{send}(v, msg) \) to send a message \( msg \) to an out-neighbour \( v \),
- \( \text{recv}(v) \) to receive a message from an in-neighbour \( v \).

An execution, or run, of an algorithm \( \mathcal{A} \) subject to scenario \( w \in \mathcal{S} \) is the following. Consider process \( u \) and one of its out-neighbours \( v \). During round \( r \in \mathbb{N} \), a message \( msg \) is sent from \( u \) to \( v \), according to algorithm \( \mathcal{A} \). The corresponding \( \text{recv}(v) \) will return \( msg \) only if \( E' \), the \( r \)-th letter of \( w \), is such that \( (u,v) \in E' \). Otherwise the returned value is \( \text{null} \). All messages sent in a round can only be received in the same round. After sending and receiving messages, all processes update their states according to \( \mathcal{A} \) and the messages they received. Given \( u \in \text{Pref}(w) \), let \( s^x(u) \) denote the state of process \( x \) at the end of the \( |u| \)-th round.
of algorithm $A$ subject to scenario $w$. The initial state of $x$ is $\iota(x) = s^x(\varepsilon)$. A configuration corresponds to the collection of local states at the end of a given round. An execution of $A$ subject to $w$ is the (possibly infinite) sequence of such message exchanges and corresponding configurations.

**Remark 2.7.** With this definition of execution, the environment is independent of the actual behaviour of the algorithm, so communication failures do not depend upon whether or not messages are sent. This model is not suitable for modelling omissions caused by congestion. See [KKR03] for examples of threshold-based omission models.

**Definition 2.8.** A algorithm $A$ solves a problem $\mathcal{P}$ subject to omission scheme $S$ with initial configuration $\iota$, if, for any scenario $w \in S$, there exists $u \in \text{Pref}(w)$ such that the state $s^x(u)$ of each process $x \in V$ satisfies the specifications of $\mathcal{P}$ for initial configuration $\iota$. In such a case, $A$ is said to be $S$-reliable for $\mathcal{P}$.

**Definition 2.9.** If there exists an algorithm that solves a problem $\mathcal{P}$ subject to omission scheme $S$, then we say that $\mathcal{P}$ is $S$-solvable.

**Remark 2.10.** We emphasize that for an algorithm, “knowing” the omission scheme against which it runs is not the same as knowing whether or not a given message is actually received.

### 3 The Problems

#### 3.1 The Binary Consensus Problem

A set of synchronous processes wishes to agree about a binary value. This problem was first identified and formalized by Lamport, Shostak and Pease [PSL80]. Given a set of processes, a consensus protocol must satisfy the following properties for any combination of initial values [Lyn96]:

- **Termination**: every process decides some value;
- **Validity**: if all processes initially propose the same value $v$, then every process decides $v$;
- **Agreement**: if a process decides $v$, then every process decides $v$.

Consensus with these termination and decision requirements is more precisely referred to as **Uniform Consensus** (see [Ray02] for a discussion). Given a fault environment, the natural questions are: is Consensus solvable, and if it is solvable, what is the minimum number of rounds to solve it?

#### 3.2 The Broadcast Problem

Let $G = (V,E)$ be a graph. There is a broadcast algorithm from $u \in V$, if there exists an algorithm that can successfully transmit any value stored in $u$ to all nodes of $G$.

The **broadcast problem on graph** $G$ is to find a $u \in V$ and an algorithm $A$ such that $A$ is a broadcast algorithm from $u$. Given an omission scheme $S$ on $G$, $G$ is $S$-broadcastable if there exists a $u \in V$ such that there is an $S$-reliable broadcast algorithm $A$ from $u$. 

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3.3 First Reduction

The next proposition is quite well known but leads to very interesting questions.

**Proposition 3.1.** Let $G$ be a graph and $S$ an omission scheme for $G$. If $G$ is $S$-broadcastable, then Consensus is $S$-solvable on $G$.

*Proof.* If $G$ is $S$-broadcastable then there exists a node $u$ and an algorithm $A$ for broadcasting any value from $u$ subject to $S$. The consensus algorithm uses $A$ to broadcast the initial value of $u$, and then every node decides the value received from $u$. As the value that $u$ broadcasts is one of the initial values, the algorithm satisfies all three conditions, and is obviously $S$-reliable.

$\blacksquare$

We now present an example that shows that the converse is not always true.

**Example 3.2.** The omission scheme $H = \{\circ\leftarrow\bullet, \circ\rightarrow\bullet\}^\omega$ of Example 2.5 is an example of a system for which there is a Consensus algorithm but no Broadcast algorithm.

It is easy to see that it is not possible to broadcast from $\circ$ (resp. $\bullet$) subject to $H$ because $\circ\leftarrow\bullet^\omega$ (resp. $\circ\rightarrow\bullet^\omega$) is a possible scenario. However, the following one-round algorithm (the same for both processes) is an $H$-reliable Consensus algorithm:

- send the initial value;
- if a value is received, decide this value, otherwise decide the initial value.

This algorithm is correct, as exactly one process will receive a value, but it is not possible to know in advance whose value will be received.

We propose to study the following question: when is the solvability of Consensus equivalent to the solvability of Broadcast? That is, given a graph $G$, what are the mobile omission schemes $S$ on $G$ such that Consensus is $S$-solvable and $G$ is $S$-broadcastable. In the process of answering this question, we will give a simple characterization of the solvability of Broadcast and a necessary condition for the solvability of Consensus subject to mobile omission schemes.

4 Broadcastability

4.1 Flooding Algorithms

We start with a basic definition and lemma.

**Definition 4.1.** Consider a sub-digraph $H$ of $G$ and a node $u \in V$. A node $v \in V$ is reachable from $u$ in $H$ if there is a directed path from $u$ to $v$ in $H$. Node $u$ is a source for $H$ if every $v \in V$ is reachable from $u$ in $H$.

In a flooding algorithm, one node repeatedly sends a message to its neighbours, and each other node repeatedly forwards any message that it receives to its neighbours. The following useful lemma (from folklore) about synchronous flooding algorithms is easily extended to the omission context. Let $F^r_u$ denote a flooding algorithm that is originated by $u \in V$ and that halts after $r$ synchronous rounds.

**Lemma 4.2.** A node $u \in V$ is a source for $H$ if and only if for all $r \geq |V|$, $F^r_u$ is $H^\omega$-reliable for the Broadcast problem.
4.2 Characterizations of Broadcastability with Arbitrary Omissions

We have the following obvious but fundamental lemma. We say that a node is informed if it has received the value from the originator of a broadcast.

**Lemma 4.3.** Let \( u \in V, \tau \in \mathbb{N} \), and let Inform\((w)\) be the set of nodes informed by \( F_u^w \) under the partial execution subject to \( w \in \Sigma^* \). Then for any subword \( w' \) of \( w \), Inform\((w') \subseteq \text{Inform}(w)\).

**Theorem 4.4.** Let \( G \) be a graph and \( R \) a set of communication events for \( G \). Then \( G \) is \( R^\omega \)-broadcastable if and only if there exists \( u \in V \) that is a source for all \( H \in R \).

**Proof.** In the first direction, suppose that we have a broadcast algorithm from a given \( u \) that is \( R^\omega \)-reliable. Then an execution subject to \( H^\omega \) is successful for any \( H \in R \), so \( u \) is a source for \( H \) by Lemma 4.3.

In the other direction, choose the flooding algorithm \( F_u^{[R \times |V|]} \) to be the broadcast algorithm and consider a scenario \( w \in R^{[R \times |V|]} \). There is an event \( H \in R \) that appears at least \( |V| \) times in \( w \), hence \( H^{|V|} \) is a subword of \( w \). As \( u \) is a source for \( H \) by Lemma 4.3, Inform\((H^{|V|}) = V\). By Lemma 4.3 Inform\((w) = V\), and the flooding algorithm is \( R^\omega \)-reliable. \( \square \)

**Definition 4.5.** The set of sources of an event \( H \in \Sigma \) is \( B(H) = \{ u \in V \mid u \text{ is a source for } H \} \).

**Definition 4.6.** Let \( H_1, \ldots, H_q \in \Sigma \). Then the set \( \{H_1, \ldots, H_q\} \) is source-incompatible if

1. \( \forall 1 \leq i \leq q \), \( B(H_i) \neq \emptyset \),
2. \( \bigcap_{1 \leq i \leq q} B(H_i) = \emptyset \).

With these definitions we can restate the Broadcastability theorem (Theorem 4.4):

**Theorem 4.7.** Let \( G \) be a graph and \( R \) a set of communication events for \( G \). Then \( G \) is \( R^\omega \)-broadcastable if and only if every event in \( R \) has a source, and \( R \) is not source-incompatible.

4.3 A Converse Reduction

Given a subset \( X \subseteq V \) of vertices, and \( H \) an event we denote \( \text{In}_X(H) = \{ (v, u) \in H \mid u \in X \} \). Given \( R \) an omission scheme. We now define a more precise relation about indistinguishability.

**Definition 4.8.** Given three directed graphs \( G, H, K \in R \), we define the following relation denoted by \( G \alpha_K H \) if \( \text{In}_{B(K)}(G) = \text{In}_{B(K)}(H) \). The relation \( \alpha^* \) is the transitive closure of \( \alpha_K \) relations for any \( K \in R \).

We denote \( \beta \) the coarsest equivalence relation included in \( \alpha^* \) such that for all graphs \( G, H \)

(Closure Property) \( G \beta H \implies \exists H_0, \ldots, H_q \text{ and } K_1, \ldots, K_q \text{ such that} \)

- \( G = H_0, H = H_q \),
- \( \forall i \geq 1, H_i \beta G, K_i \beta G \),
- \( \forall i \geq 0, H_i \alpha_K H_{i+1} \).

The relation \( \alpha_K \) describes how some communication events are indistinguishable to the all the nodes of \( B(K) \). The relation \( \beta \) is well defined as the equality relation satisfies such a closure property. And for any two relations \( R_1 \) and \( R_2 \) that satisfy the property, we have \( R_1 \cup R_2 \) that satisfies the Closure.
Example 4.9. In $O_1$ from Example 2.4, there is only one equivalence class. Let’s see why. First, the sets of sources to consider are:
- $B(\circ \leftarrow \bullet) = \{\bullet\}$;
- $B(\circ \rightarrow \bullet) = \{\circ\}$;
- $B(\circ \leftrightarrow \bullet) = \{\circ, \bullet\}$.
We have $\circ \leftarrow \bullet \alpha \circ \leftrightarrow \bullet$ and $\circ \rightarrow \bullet \alpha \circ \leftrightarrow \bullet$. Therefore, all communication events are $\beta$–equivalent. In Example 2.5, $\beta$ has two equivalence classes, and every node can distinguish immediately which communication event happened.

As will be seen later in Section 6, the omission schemes in [SW07] and [SWK09], and more generally, all schemes that are defined by bounding the number of omissions in some way, have only one $\beta$–class when they are source-incompatible. Finally, we can now state the main theorem.

**Theorem 4.10.** Let $G$ a graph and $R$ a set of communication events for $G$. If Consensus is $R^\omega$–solvable then for every $\beta_R$–class $C$, $G$ is $C^\omega$–broadcastable.

5 Proof of Main Theorem

5.1 Events without Sources

First we consider the cases in which there are events without sources.

**Proposition 5.1.** If there is an $H \in R$ that has no source, then Consensus is not $R^\omega$–solvable.

*Proof.* If $H$ has no source, then there exist two non-overlapping, non-empty subsets of nodes $U_0$ and $U_1$ such that there are no paths in $H$ from $V \setminus U_i$ to $U_i$, $i = 0, 1$.

We consider the three following initial configurations:
1. $I_0$: initial value is 0 at every node,
2. $I_1$: initial value is 1 at every node,
3. $I$: initial value is 0 if and only if the node belongs to $U_0$.

Under scenario $H^\omega$, $I$ is not distinguishable from $I_0$ (resp. $I_1$) for $U_0$ (resp. $U_1$). So subject to $H^\omega$, the algorithm decides 0 in $U_0$ and 1 in $U_1$, and this contradicts the Agreement property.

□

The proof of the main theorem uses an approach that is similar to the adjacency and continuity techniques of [SW07]. So, we will first prove these two properties. What should be noted is that the adjacency and continuity properties are mainly consequences of the fact that the scheme is a mobile scheme.

5.2 An Adjacency Property

**Lemma 5.2.** Let $H$ be a subgraph of $G$, and let $(s,t) \in H$. If $t \in B(H)$ then $s \in B(H)$.

**Proposition 5.3.** Let $H \in R$ and $w, w' \in R^*$ such that $|w| = |w'|$ and $s^p(w') = s^p(w)$ for all $p \in B(H)$. Then for all $k \in \mathbb{N}$ and all $p \in B(H)$, $s^p(wH^k) = s^p(w'H^k)$.

*Proof.* The proof relies upon Lemma 5.2 which implies that processes from $B(H)$ can only receive information from $B(H)$ under scenario $H^k$, for any $k \in \mathbb{N}$. □
5.3 A Continuity Property

Lemma 5.4. Let $H, H' \in \mathcal{R}$ such that $H_{\alpha_B(A)} H'$ for some $A \in \mathcal{R}$. Then for all $w \in \mathcal{R}^\omega$ and all $p \in B(A)$, $s^p(wH) = s^p(wH')$.

Proof. By definition of $\alpha_U$ relations, processes in $B(A)$ cannot distinguish $H$ from $H'$ meaning they are receiving the exact same messages from exactly the same nodes in both scenarios. Hence they end in the same states. $\square$

Proposition 5.5. Let $H, H' \in \mathcal{R}$ such that $H \beta H'$. Then for every $w \in \mathcal{R}^*$, there exist $H_1, \ldots, H_q$ in the $\beta$-class of $H$ and $H'$, and $A_0, \ldots, A_q \in \mathcal{R}$, such that for every $0 \leq i \leq q$ and every $p \in B(A_i)$, $s^p(wH_i) = s^p(wH_{i+1})$, where $H_0 = H$ and $H_{q+1} = H'$.

Proof. By Lemma 5.4 and definition of $\beta$. $\square$

5.4 End of Proof of Theorem 4.10

We will use a standard bivalency technique. We suppose that we have an algorithm that solves Consensus. A configuration is said to be 0-valent (resp. 1-valent) if all extensions decide 0 (resp. 1). A configuration is said to be bivalent subject to $L$ if there exists an extension in $L$ that decides 0 and another extension in $L$ that decides 1.

Lemma 5.6 (Restricted Initial Bivalent Configuration). If there exists a source-incompatible set $D$, then there exists an initial configuration that is bivalent subject to $D^\omega$.

Proof. Suppose that $\{H_1, \ldots, H_q\}$ is a source-incompatible set in $D$. There exist disjoint non-empty sets of nodes $M_1, \ldots, M_k$ such that $\forall i, \exists I \subseteq [1, k], B(H_i) = \bigcup_{j \in I} M_j$.

Consider $t_0$ (resp. $t_k$) in which all nodes of $\bigcup_{1 \leq j \leq k} M_j$ have initial value 0 (resp. 1). The initial configuration $t_0$ is indistinguishable from the configuration in which all nodes have initial value 0 for the nodes of $B(H)$ under scenario $H^\omega$, for every $i$. Hence $t_0$ is 0-valent. Similarly $t_k$ is 1-valent. We consider now the initial configurations $t_l, 1 \leq l \leq k - 1$ in which all nodes from $\bigcup_{1 \leq j \leq k-l} M_j$ have initial value 0, and all other nodes have initial value 1.

Suppose now that all initial configurations are univalent. Then there exists $1 \leq l \leq k$ such that $t_{l-1}$ is 0-valent and $t_l$ is 1-valent. As the set is source-incompatible, there must exist $i \in [1, q]$ such that $M_l \cap B(H_i) = \emptyset$. So, we can apply Proposition 5.3 to $H_i$. This means that all nodes in $B(H_i)$ decide the same value for both initial configurations, $t_{l-1}$ and $t_l$, under scenario $H^\omega$, and this is a contradiction. $\square$

Lemma 5.7 (Restricted Extension). Let $C$ be a $\beta$-class. Every bivalent configuration in $C^\omega$ has a succeeding bivalent configuration in $C^\omega$.

Proof. Consider a bivalent configuration obtained after a partial execution subject to $w \in C^\ast$. By way of contradiction, suppose that all succeeding configurations in $C^\omega$ are univalent. Then there exist succeeding configurations $wH$ and $wH'$ that are respectively 0-valent and 1-valent, as $w$ is bivalent.

By Proposition 5.5 there exist $H_1, \ldots, H_q$ in $C$ and $A_0, \ldots, A_q \in \mathcal{R}$ such that $s^p(wH_i) = s^p(wH_{i+1})$ for every $0 \leq i \leq q$ and every $p \in B(A_i)$, where $H_0 = H$ and $H_{q+1} = H'$. By hypothesis, all succeeding configurations $wH_i$ are univalent. As $H\alpha_{B(A_0)}H_1$, we get that
processes in \( B(A_0) \) are in the same state after \( H \) and after \( H_1 \). Hence, by Proposition 5.3, they are also in the same state after \( H A_i^k \) and after \( H_1 A_i^k \), so they decide the same value and \( wH_1 \) is 0–valent. We can repeat this for any \( 1 \leq i \leq q \). Hence \( wH' \) is also 0–valent, a contradiction. \( \square \)

We can now finish the proof with the standard bivalency arguments. Suppose that we have a source-incompatible set in the same \( \beta \)-class \( C \). Also suppose that there exists an \( R^\omega \)-reliable Consensus algorithm for \( G \). By Lemma 5.6, there exists an initial configuration that is bivalent in \( C^\omega \). From Lemma 5.7, we deduce that the algorithm does not satisfy the Termination property for Consensus on some execution subject to \( C^\omega \subset R^\omega \), which is a contradiction. Using Proposition 5.1 and Theorem 4.7, we conclude the proof of Theorem 4.10.

### 6 Solvability of Consensus vs Broadcast

In this section, we prove that the Consensus and Broadcast problems are equivalent for the large family of omission schemes that are defined over convex sets of events. This has very important consequences as checking Broadcastability is quite simple (see Theorem 4.4).

**Definition 6.1.** *A set of communication events* \( R \) *is convex if, for every* \( H, H' \in R \) *and every* \( a \in H', H \cup \{a\} \in R \).

Basically, this definition says that a convex set of communication events \( R \) is closed under the operation of adding a reliable communication event \( a \) from one event \( H' \) to another event \( H \). This is an important subfamily because sets of events that are defined by bounding the number of omissions, for any way of counting them, are convex. Stated differently, adding links to an event \( H \) with a bounded number of omissions cannot result in an event with more omissions. The convexity property does not depend upon the way that omissions are counted.

**Theorem 6.2.** *Let* \( R \subset \Sigma \) *be a convex set of communication events over a graph* \( G \). *Then Consensus is* \( R^\omega \)-*solvable if and only if* \( G \) *is* \( R^\omega \)-*Broadcastable.*

**Proof.** By Theorem 4.10, we only have to show that there is no source-incompatible set in \( R \). We will show that if there is such a set \( \{H_1, \ldots, H_q\} \), then there is only one \( \beta_R \) class.

There exist disjoint, non-empty sets of nodes \( M_1, \ldots, M_k \) such that \( \forall i, \exists I \subset [1, k], B(H_i) = \bigcup_{j \in I} M_j \). The \( M_j \) are “generators” for the sets of sources. We use \( M_J \) to denote \( \bigcup_{j \in J} M_j \) for any \( J \subset [1, k] \). Note that, as the intersection of the \( H_i \) is empty (Definition 4.6), for each \( j \in [1, k] \), there exists \( i_j \) such that \( M_j \cap B(H_{i_j}) = \emptyset \).

Now, consider \( H_1 \neq H_2 \in R \). We will show that \( H_1 \beta (H_1 \cup H_2) \). Using the decomposition into \( M_j \)s, there exist three mutually disjoint (possibly empty) subsets \( J_1, J_2, J_2 \) of \([1, k] \), such that \( B(H_1) = M_{J_1 \cup J} \) and \( B(H_2) = M_{J_2 \cup J} \).

Let \( H_1' = H_1 \cup \{(s, t) \in H_2 \mid t \in M_{J_1}\} \). As the intersection of \( B(H_2) \) with \( M_{J_1} \) is empty, we have \( H_1 \alpha_{B(H_2)} H_1' \).

Similarly, letting \( H_2'' = H_1' \cup \{(s, t) \in H_1 \mid t \in M_{J_2}\} \), we have \( H_1' \alpha_{B(H_1)} H_2'' \).

To obtain \( H_1 \cup H_2 \), we need to add to \( H_2'' \) the arcs with heads in \( B(H_1) \cap B(H_2) = M_J \). Let \( J = \{j_1, \ldots, j_k\} \) and \( K_k = H_1'' \bigcup_{l \leq k} M_{J_l} \).

For each \( l \in J \), there exists \( i_l \) such that \( M_l \cap B(H_{i_l}) = \emptyset \). Therefore, for all \( k \), \( K_{k-1} \alpha_{B(H_{i_k})} K_k \). So finally, we obtain \( H_1 \beta (H_1 \cup H_2) \). Similarly, \( H_2 \beta (H_1 \cup H_2) \), and \( H_1 \beta H_2 \). \( \square \)
Let $O_f(G)$ denote the set of communication events with at most $f$ omissions from the underlying graph $G$. An upper bound on $f$ for solvability of Consensus subject to $O_f(G)^\omega$ was given in [SW07], and it was proved to be tight with an ad hoc technique in [FG11]. We can now state this result as an immediate corollary of Theorem 6.2.

**Corollary 6.3.** Let $f \in \mathbb{N}$. Consensus is solvable subject to $O_f(G)^\omega$ if and only if $f < c(G)$, where $c(G)$ is the connectivity of the graph $G$.

The equivalence of Consensus and Broadcast includes the number of rounds to solve them.

**Proposition 6.4.** Let $R \subseteq \Sigma$ be a convex set of events. If Consensus is $R^\omega$-solvable then it is solvable with exactly the same number of rounds as Broadcast subject to $R^\omega$.

**Proof.** We only have to show that Consensus cannot be solved in fewer rounds than Broadcast. Due to space limitations, we only present a sketch of the proof. We use the same bivalency technique as in Section 5. So, suppose that Consensus is solvable in $r_c$ rounds while Broadcast needs more than $r_c$ rounds for any originator.

First, we show that there must be a bivalent initial configuration. Let $V = \{v_1, \ldots, v_n\}$ and let $\iota_l$ be the initial configuration in which $v_i$ has initial value 0 if $i \leq l$. If all initial configurations $\iota_l$, $1 \leq l \leq n$ are univalent, then there exists $l$ such that $\iota_{l-1}$ is 0-valent and $\iota_l$ is 1-valent. As a Broadcast from $v_l$ needs strictly more than $r_c$ rounds, there exists a vertex $v$ that does not receive the value from $v_l$, so no executions of length $r_c$ of the Consensus algorithm from initial configurations $\iota_{l-1}$ and $\iota_l$ can be distinguished by $v$. Consequently $v$ will decide the same value for both initial configurations, a contradiction.

Now we show that if all extensions of a bivalent configuration are univalent, then the Consensus algorithm needs more than $r_c$ rounds to conclude. Indeed, if we have an extension, starting with communication event $H_0$, that is 0-valent, and another extension, starting with communication event $H_1$, that is 1-valent, we can repeat the above technique by adding arcs to $H_0$ to obtain $H_0 \cup H_1$. The addition of arcs can be done by grouping them according to their heads. If only one node has a different state for two events, then it would need more than $r_c$ rounds to inform all other nodes. $\square$

There are many results concerning the Broadcast problem in special families of networks. General graphs are studied in [CDP94] and hypergraphs are studied in [MV98]. An optimal algorithm for the family of hypercubes in given in [DV99]. For a hypercube of dimension $n$, if at most $n - 1$ messages are lost during each round, then Broadcast can be solved in $n + 2$ rounds, compared to $n$ rounds when there are no omission faults. In [DV04], the precise impact on Broadcast of the actual number of faults is given. Based on the results in [DV04], we get the following bounds for Consensus.

**Corollary 6.5.** In hypercubes of dimension $n$, if the global number of omissions is at most $f$ per round, then
1. if $f \geq n$, then Consensus is not solvable,
2. if $f = n - 1$, Consensus is solvable in exactly $n + 2$ rounds,
3. if $f = n - 2$, Consensus is solvable in exactly $n + 1$ rounds,
4. if $f < n - 2$, Consensus is solvable in exactly $n$ rounds.

The following example shows that there are mobile schemes that are broadcastable but for which Consensus is solvable is fewer rounds than Broadcast.
Example 6.6. Let $R = \{H_1, H_2\}$ where $H_1$ (resp. $H_2$) is given by Fig. 1 (resp. Fig. 2). One can see that $d$ needs two rounds to broadcast in $H_1$, and $c$ needs two rounds to broadcast in $H_2$. Nodes $a$ and $b$ need more than two rounds. However there is a Consensus algorithm that finishes in one round. Notice that every node can detect which of the communication events actually happened, so the Consensus algorithm in which every node decides the value from $c$ if $H_1$ happened and the value from $d$ if $H_2$ happened uses only one round.

7 Conclusions

We have presented a new necessary condition for solving Consensus on communication networks subject to arbitrary mobile omission faults. We conjecture that this condition is actually sufficient, therefore leading to a complete characterization of the solvability of Consensus in environments with arbitrary mobile omissions. For a large class of environments that includes any environment defined by bounding the number of omissions during any round, for any way of counting omissions, we proved that the Consensus problem is actually equivalent to the Broadcast problem. We also gave examples (Ex. 2.5 and Ex. 6.6) showing how Consensus can differ from Broadcast for some environments.

Finally, by factoring out the broadcastability properties required to solve Consensus, we think that it is possible to extend this work to other kinds of failures, such as byzantine communication faults.

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