Black hole entropy for the general area spectrum

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Abstract. We consider the possibility that the horizon area is expressed by the general area spectrum in loop quantum gravity when we leave off the semiclassical consideration. To check this idea, we calculate the number of degrees of freedom in spin-network states related to its area. We obtain that logarithm of this number is proportional to its area as in previous works where the simplified area formula has been used. Our result shows that we should be careful in justifying (or falsifying) the area spectrum if we respect to leave off the semiclassical consideration.

One of the most mysterious things about black holes is their entropy $S$ which is not proportional to their volume but to their horizon area $A$. This was first pointed out related to the first law of black hole thermodynamics. The famous relation $S = A/4$ has been established by the discovery of the Hawking radiation. Recently, its statistical origin has been discussed in the candidate theories of quantum gravity, such as string theory [1], or loop quantum gravity (LQG) [2], etc. It has been discussed that LQG can describe its statistical origin independent of black hole species because of its background independent formulation [3]. For this reason, we concentrate on LQG here.

Quantum states in LQG are described by spin-network [4], and basic ingredients of the spin-network are edges, which are lines labelled by spin $j (j = 0, 1/2, 1, 3/2, \cdots)$ reflecting the SU(2) nature of the gauge group, and vertices which are intersections between edges. For three edges having spin $j_1, j_2,$ and $j_3$ that merge at an arbitrary vertex, we have following conditions.

\[ j_1 + j_2 + j_3 \in \mathbb{N}, \]
\[ j_i \leq j_j + j_k, \quad (i, j, k \text{ different from each other.}) \] (1) (2)

These conditions guarantee the gauge invariance of the spin-network.

Using this formalism, general expressions for the spectrum of the area and the volume operators can be derived [5, 6]. For example, the area spectrum $A_j$ is

\[ A_j = 4\pi \gamma \sum_i \sqrt{2j_i^u(j_i^u + 1) + 2j_i^d(j_i^d + 1) - j_i^t(j_i^t + 1)} \] (3)

where $\gamma$ is the Barbero-Immirzi parameter related to an ambiguity in the choice of canonically conjugate variables [7]. The sum is added up all intersections between a surface and edges. Here, the indices $u, d,$ and $t$ mean edges upper side, down side, and tangential to the surface, respectively (We can determine which side is upper or down side arbitrarily).

In [2], it was proposed that black hole entropy is obtained by counting the number of degrees of freedom about $j$ when we fix the horizon area where a simplified area formula is used. This
simplified area formula is obtained by assuming that there are no tangential edges on black hole horizon, that is \( j_i^t = 0 \). We obtain \( j_i^u = j_i^d := j_i \) by using the condition (2). Then, we consider the degrees of freedom about \( j \) satisfying

\[
A_j = 8\pi\gamma \sum_i \sqrt{j_i(j_i+1)} = A. \tag{4}
\]

The standard procedure is to impose the Bekenstein-Hawking entropy-area law \( S = A/4 \) for macroscopic black holes in order to fix the value of \( \gamma \). Ashtekar et al. in [3] extended this idea using the isolated horizon framework (ABCK framework) [8]. Error in counting in this original work has been corrected in [9, 10].

These rely on semiclassical consideration since it is difficult to express black holes in full quantum gravity. Interesting idea to circumvent this difficulty is to define the horizon as a boundary which is accessible to the external observer at spatial infinity [11]. If we respect this viewpoint, it is valuable to examine (3) as a horizon area. We can also discuss this possibility as follows. Imagine to describe it by spin-network states. If we suppose that its boundary which is accessible to the external observer at spatial infinity [11]. If we respect

One of the criteria to check (3) as a horizon area is examining the number of degrees of freedom related to (3). We notice that both counting the horizon freedom in [3] and \( j \) freedom in [2] obtain the entropy proportional to the area (4). Based on this observation, we assume that counting \( j \) freedom can be a good indicator in judging the area spectrum (3) as a horizon area by investigating whether or not the entropy is proportional to the area (3).

1. Consideration of the general area spectrum

In the case for (4), it has been shown that isolated horizon conditions do not affect the number of states in the limit \( A \to \infty \). Based on this observation, we consider only degrees of freedom about its area (3) as a first step. Then in this case, we also denote number of states as \( N(A) \) which is defined as

\[
N(A) := \{ (j_1^u, j_1^d, j_1^t, \ldots, j_n^u, j_n^d, j_n^t) | j_i^u, j_i^d, j_i^t \in \mathbb{N}/2, \ j_i^t \neq (0, 0, 0) \}
\]

\( j_i^u, j_i^d, j_i^t \) should satisfy (1) and (2). Then, we perform counting as follows. First, we count the case \( j_i^u \in \mathbb{N}/2 \) to simplify the problem and denote the corresponding number of states as \( N(A)_{\text{even}} \). Then, we have \( j_i^u + j_i^d := n \in \mathbb{N} \) by (1). If we fix \( n \), we can classify the possible \( j_i^u, j_i^d, j_i^t \) as follows, which is one of the most important parts in this paper. First, we have \( (j_i^u, j_i^d) = \left( \frac{n}{2} \pm \frac{s}{2}, \frac{n}{2} \mp \frac{s}{2} \right) \) (double-sign corresponds) for \( 0 \leq s \leq n, s \in \mathbb{N}/2 \) to satisfy (2). Then, for each \( s \), possible value of \( j_i^t \) is \( j_i^t = s, s+1, \ldots, n \) to satisfy (2).

Then, in the same way, we can count the case \( j_i^t \in n + \frac{1}{2}, (n = 0, 1, 2, \ldots) \) and denote the corresponding number of states as \( N(A)_{\text{odd}} \). In summary, number of states \( N(A) \) can be written as

\[
N(A) = \sum_{n=1}^{\infty} \left[ \sum_{s=1}^{n} \sum_{t=s}^{n} 2\theta(A - 4\pi\gamma)N(A - x(n, s, t)) + \sum_{t=0}^{n} \theta(A - 4\pi\gamma)N(A - x(n, s = 0, t)) \right] + \sum_{n=0}^{\infty} \sum_{s=0}^{n} \sum_{t=s}^{n} 2\theta(A - 2\pi\gamma\sqrt{3})N(A - y(n, s, t)). \tag{5}
\]
For $A \to \infty$, by assuming the relation $N(A) = Ce^{4\gamma M A}$, where $C$ is a constant and substituting to the recursion relation (5), we obtain the beautiful formula as a generalization of the case (4) as,

$$1 = \sum_{n=1}^{\infty} \left[ \sum_{s=1}^{n} \sum_{t=0}^{s} 2\theta(A - 4\pi \gamma) \exp(-\gamma_M x(n,s,t)/4\gamma) + \sum_{t=0}^{n} \theta(A - 4\pi \gamma) \right] \times \exp(-\gamma_M x(n,s=0,t)/4\gamma) + \sum_{n=0}^{\infty} \sum_{s=0}^{n} \sum_{t=s}^{n} 2\theta(A - 2\pi \gamma \sqrt{3}) \exp(-\gamma_M y(n,s,t)/4\gamma).$$  

(6)

If we require $S = A/4$, we have $\gamma = \gamma_M = 0.7274\cdots$. This means that even if we use (3) as the horizon spectrum, we can reproduce the entropy formula $S = A/4$ by adjusting the Barbero-Immirzi parameter. This is nontrivial and is our main result in this paper.

2. Conclusion and Discussion

In this paper, we checked the possibility of the general area formula as a horizon area by counting the $j$ freedom. It is important that its entropy is proportional to its area. Then, it is natural to ask what the area formula should be in describing the horizon area. There are many possibilities examining the area spectrum. For example, we have not yet established the black hole thermodynamics in LQG which is one of the most important topics to be investigated. There is an idea that black hole evaporation process should also be described by using the general area formula [12]. Therefore, whether we can establish the generalized second law of black hole thermodynamics might be one of the criteria in judging which area formula is appropriate. For this purpose, black hole states in full quantum gravity is required since the exact counting is required. Though we do not take care of the topology of the horizon, discussing the difference caused by the topology is important as considered for the simplified area formula [13]. The covariant entropy bound [14] is also important which has been discussed in the LQG context recently [15]. Confirming LQG in many independent methods would be the holy grail of the theory.

References

[1] Strominger A and Vafa C 1996 Phys. Lett. B 379 99
Maldacena J M and Strominger A 1996 Phys. Rev. Lett. 77 428
[2] Rovelli C 1996 Phys. Rev. Lett. 77 3288
[3] Ashtekar A, Baez J, Corichi A and Krasnov K 1998 Phys. Rev. Lett. 80 904
Ashtekar A, Baez J and Krasnov K 2000 Adv. Theor. Math. Phys. 4 1
[4] Rovelli C and Smolin L 1995 Phys. Rev. D 52 5743
[5] Rovelli C and Smolin L 1995 Nucl. Phys. B 442 593; Erratum, ibid. 456 753
[6] Ashtekar A and Lewandowski J 1997 Class. Quantum Grav. 14 A55
[7] Barbero J F 1995 Phys. Rev. D 51 5507
Immirzi G 1997 Nucl. Phys. Proc. Suppl. B 57 65
[8] Ashtekar A, Corichi A and Krasnov K 1999 Adv. Theor. Math. Phys. 3 419
[9] Domagala M and Lewandowski J 2004 Class. Quantum Grav. 21 5233
[10] Meissner K A 2004 Class. Quantum Grav. 21 5245
[11] Krasnov K and Rovelli C Preprint gr-qc/0905.4916
[12] Ansari M H 2007 Nucl. Phys. B 783 179; ibid. 795 635
[13] Kloster S, Brannlund J and DeBenedictis A 2008 Class. Quantum Grav. 25 065008
[14] Bousso R 1999 J. High Energy Phys. JHEP07(1999)004
[15] Ashtekar A and Wilson-Ewing E 2008 Phys. Rev. D 78 064047