The lifetime of $B_c$-meson and some relevant problems

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Abstract

The lifetime of the $B_c$-meson is estimated with consistent considerations on all of the heavy mesons ($B^0, B^\pm, B_s, D^0, D^\pm D_s$) and the double heavy meson $B_c$. In the estimate, the framework, where the non-spectator effects for nonleptonic decays are taken into account properly, is adopted, and the parameters needed to be fixed are treated carefully and determined by fitting the available data. The bound-state effects in it are also considered. We find that in decays of the meson $B_c$, the QCD correction terms of the penguin diagrams and the main component terms $c_1O_1$, $c_2O_2$ of the effective interaction Lagrangian have direct interference that causes an enhancement about $3 \sim 4\%$ in the total width of the $B_c$ meson.

14.40.Nd, 14.40.Lb, 12.39.Hg, 12.38.Lg
I. INTRODUCTION

Recently the meson $B_c$ has been observed by CDF collaboration at Tevatron [1], that careful studies of the meson $B_c$ are motivated with the fresh reason.

It is known that $B_c$ is a meson of the ground state of a double heavy flavored quark-antiquark system (there is no light flavored quark as a valence being involved), and its decay must be either through the decays of each component (a heavy flavor), or through the annihilation of the two components (two heavy flavors), that is very different from $B$ and $D$ mesons. Of the decays, the contribution from each component ($\bar{b}$-quark or $c$-quark) to the total width happens to be comparable each other so in the future experiments it is accessible that to have precise measurements on each component. Namely with the meson of $B_c$ alone, one may investigate the two different heavy flavors simultaneously. Especially, certain decay mechanisms play a similar role in $D$-decays, $B$-decays and $B_c$-decays so some parameters appearing in $B_c$-decays should be the same as those in $B$-decays or in $D$-decays, therefore, when one estimates the $B_c$-decay one may use the experimental available data of $D$-decays and $B$-decays as input phenomenologically to determine them under the consistent considerations. Obviously in this way the estimate for $B_c$ lifetime should be comparatively reliable.

The meson $B_c$ certainly is an independent complement to $B$-mesons and $D$-mesons for studying the two heavy flavor $b$ and $c$ decays. Furthermore, if one carries on a comparatively study of the two heavy flavors, it has unique advantages. In this sense, $B_c$-meson will offer an extra interesting and unique laboratory for the heavy flavor decay studies.

There have been quite a lot of studies on the lifetime of $D$ and $B$ mesons and the meson $B_c$ as well [2, 3]. The reason in part is that for the lifetime it is comparatively ‘easy’. Due to the duality for quarks and hadrons:

$$\sum_{i,j} \langle q_i, \bar{g}_j \rangle \langle q_j, g_i \rangle = \sum_k |h_k \rangle \langle h_k|$$
where \( h_k, q_i \) and \( g_j \) denote hadrons, quarks and gluons respectively, the optical theorem may be set on the level of hadrons or the level of quark-gluons, an inclusive processes of hadrons can be turned onto a quark level instead. In general, the problem for evaluating a decay rate of a hadron is hard, because the relevant hadron matrix element cannot be handled reliably. The matrix element contains non-perturbative QCD effects, so one cannot compute them very satisfactorily based on an existent underlying-theory\(^1\). The optical theorem can be applied for evaluating lifetimes and certain inclusive processes, so the problem can be ‘solved’ in part: the non-perturbative effects are summed by the theorem, generally only in the initial state are still needed be handled. Thus in the studies of the lifetime and/or the inclusive process as well, one may focus the efforts mainly on the decay mechanisms.

For the estimate of the lifetimes and inclusive decays, the effective Lagrangian for weak decays with QCD corrections should be known \(^2\)\(^3\). With the effective Lagrangian for \( c \) and \( b \) decays, phenomenological analyses of \( D \) meson lifetimes and \( B \)-meson lifetimes have been made \(^4\). For all the heavy meson decays the contributions can be decomposed into three categories: the dominant one i.e. the direct decay of the heavy quark while the light quark remains as a spectator (this contribution is very sensitive to the heavy quark mass i.e. proportional to \( M_Q^5 \)); the non-spectator one from W-annihilation (or exchange) (WA or WE); and the one from the Pauli interference(PI) \(^5\). The parameters which are needed to be fixed are the quark masses, the matrix element \( \langle 0 | J_{\mu 5} | M_{(B_c, B, D)} \rangle \) relating to the decay constant, and the relevant non-factorizability parameters etc as well \(^6\). The range of all the parameters are known, but their precise values are not well calculable. The better way is to fix them phenomenologically by fitting data.

To have a better estimate of the lifetime of \( B_c \) than before, we will take a ‘consistent’

\(^1\)In principle the lattice gauge simulation may deal with the non-perturbative effects as well as one wishes, but in practice the computer ability now still is at quite sizable ‘distance’ to obtain sufficiently accurate results for calculating such hadron matrix elements.
view of the parameters appearing in estimating the lifetimes for all of the heavy mesons $D, D_s, B, B_s$ and those in estimating that for the meson $B_c$. Namely to estimate the lifetime of the meson $B_c$ with the parameters fixed by phenomenologically fitting the available data for the other heavy mesons. We also try to discuss some uncertainties of the estimate in the paper.

In literatures, the charm quark mass $m_c$ appearing in the estimate for $D$ and $B$ decays takes different values [8,15]. We think it is reasonable, because the mass appears in different situations: in the initial state for $D$-decays but in the final state for $B$-decays. In general, for the quark (antiquark) in the parent meson of a concerned decay mode, its mass should take its ‘pole’ value if the bound-state effects are ignored, whereas, for the masses of the product quarks (antiquarks) in the final state of an inclusive process, it is more reasonable to take relevant running masses and the running energy-scale should be the mass of the decaying quark (or mesons for WA and PI). Anyhow, this problem is somewhat subtle. In the earlier estimations for inclusive processes, the quark decays are considered only as if the quark is ‘free’. However, some authors have pointed out that the bound-state effects on the effective mass of the heavy quark should be taken into account. Namely the heavy quark effective mass, appearing in the formulation, should deviate from the pole value by an amount to correspond to the binding energy [16–18]. In our work, we also pay attention on the effects and use a parametrization which is a bit different from that of [17] to account for the bound-state effects on the mass of decaying quark (see the context below for details).

The relation between the pole mass and the running $\overline{MS}$ mass is used many times in our estimation, so we present it up to one loop level here precisely for convenience. It reads

\[ m = \bar{m}(\bar{m})(1 + \frac{4}{3} \frac{\alpha_s(\bar{m})}{\pi}), \]

where the running coupling constant at 1-loop level is

\[ \alpha_s = \frac{12\pi}{(33 - 2n_f)ln\frac{Q^2}{\Lambda^2}}, \]

with $\alpha_s(m_z^2) = 0.118$ [14]. The running mass runs as:
\[ \bar{m}(Q^2) = \frac{m}{(\frac{1}{2} \ln \frac{Q^2}{\Lambda^2})^{d_m}}, \quad (3) \]

where \( d_m = \frac{12}{33-2n_f} \).

For the lifetimes of \( B \) and \( D \) mesons, the contributions from penguin terms of the effective Lagrangian generally are not important \[4\] because of smallness of their coefficients \( c_3 \sim c_6 \). But as pointed out in \[19\], the penguin contributions to the charmless decays of \( B \)-mesons are not negligible. The reason is that for those modes the main contributions (since this parts are not zero, if we return back to the tree level, thus we will call them as ‘tree parts’ as in the most literature) suffer a cancelation \((c_1 + c_2/N_c)\) or \((c_2 + c_1/N_c)\), and the ‘tree part’ \( c_1 O_1 + c_2 O_2 \) does not contribute in addition, thus the penguin contributions become important.

As for \( B_c \) meson, the problem has not been investigated very carefully. For instance, in the earlier paper \[9\], the lifetime of \( B_c \) was estimated where the bound state effect was carefully handled in terms of the Bethe-Salpeter equation, but in the effective Lagrangian the penguin contributions were ignored. Recently, Beneke and Buchalla \[11\] also presented an evaluation of the \( B_c \) lifetime, where they also ignored the penguins’. For the spectator mechanism, the contribution from the penguin terms in \( B \) decays has been estimated by Bagan et al. \[7\], and their results show that at most it can give rise to a few thousandths of changes, so in general it can be neglected. However, for the WA and PI terms, the operators induced by the penguin diagrams are \( \sum_{i=3}^{6} c_i O_i \) which contain terms \((\bar{s}_i b_j ) (\bar{c}_j c_i) \) and \((\bar{s}_i b_j)(\bar{c}_j c_i)\), where \( i, j \) are color indices, so in some \( B_c \) decays they can interfere with the ‘tree part’

\[ L_{\text{eff}}^{(\text{tree})} = V_{cb} V_{cs}^{*} [c_1 \bar{c}_L \gamma_{\mu} b_L \bar{s}_L \gamma^{\mu} c_{L} + c_2 \bar{s}_L \gamma_{\mu} b_L \bar{c}_L \gamma^{\mu} c_{L}]. \]

Therefore, in \( B^- \) and \( D^- \) decays, the contributions relating to penguins can be proportional only to \( |c_i|^2 \), or \( |c_i^* c_j| (i, j = 3 \sim 6) \), so they are small \[4\], whereas, in some \( B_c \) decays there may exist \( |c_i^* c_1| \) and \( |c_i^* c_2| \) terms, which are not so small, and such interference may bring up a few percents of corrections in its lifetime. Indeed our numerical results show the fact that the interference can make a change in lifetime of \( B_c \) so large as \( 3 \sim 4\% \) of the total.
In the paper, including the penguin contributions and fixing the parameters with a ‘consistent’ view to fit the data of $B, B_s$ and $D, D_s$ decays, we re-estimate the lifetimes, the branching ratios of the semileptonic and pure leptonic decays of the meson $B_c$. We expect to gain more information on $B_c$ about QCD and decay mechanisms.

The paper is organized as follows: after the introduction, we present the formulation in Sec.II, and give the numerical results and the concerned phenomenological parameters in Sec.III, then put conclusions and discussions in the last section. For convenience, we collect some useful formulars in Appendix A.

II. FORMULATION

In this section we will describe the different mechanisms to the lifetimes for the mesons $D, B$ and $B_c$ etc., and present the formulas for later numerical calculations.

A. The spectator components and the contributions from $b$ or $c$ decays

With quark-hadron duality and the optical theorem, the ‘full’ inclusive decay width (the lifetime) of a heavy hadron $H_Q$ (containing a heavy quark $Q = b, c$) is related to the absorptive part of the forward matrix element of the transition operator $\hat{T}$.

$$\Gamma(H_Q \to X) = \frac{1}{m_{H_Q}} Im \int d^4x \langle H_Q | \hat{T} | H_Q \rangle = \frac{1}{2m_{H_Q}} \langle H_Q | \hat{\Gamma} | H_Q \rangle,$$

where

$$\hat{T} = T \{ i \mathcal{L}_{eff}(x), \mathcal{L}_{eff}(0) \},$$

and $\mathcal{L}_{eff}$ is the relevant effective weak Lagrangian which is responsible for the decay. For the concerned final state $X$ with designated quark-antiquark combination, up-to order $1/m_Q^3$ we have:
\[
\Gamma(H_Q \to X) = \frac{G_F^2 m_Q^2}{192 \pi^3} |V(C K M)|^2 \left\{ c^X_3 \langle H_Q | \bar{Q}Q | H_Q \rangle + c^X_5 \frac{\langle H_Q | \bar{Q}i \sigma \cdot GQ | H_Q \rangle}{m_Q^2} \right. \\
+ \sum_i c^X_{6,i} \frac{\langle H_Q | (\bar{Q} \Gamma_i q)(\bar{q} \Gamma_i Q) | H_Q \rangle}{m_Q^3} \bigg\} + O(1/m_Q^4). \quad (5)
\]

Here only the heavy quark \((b, c)\) quark) decays are concerned. In the spectator components of the decays, for the heavy meson decays, the light flavor in the heavy meson remains as a spectator; for the \(B_c\)-meson decays there are two possibilities: \(\bar{b}\) decays, while the \(c\)-quark remains as a spectator, and \(c\) decays, while the \(\bar{b}\)-quark remains as a spectator. In principle, in each spectator components there are two ‘further’ components the semileptonic one and the non-leptonic one:

\[
\Gamma(b \to c) = \sum_{l=e,\mu,\tau} \Gamma_{b \to c l \nu} + \sum_{q=u,d,s,c} \Gamma_{b \to c \bar{q}q} \quad (6)
\]

for \(b\)-decay;

\[
\Gamma(c \to s) = \sum_{l=e,\mu} \Gamma_{c \to s l \nu} + \sum_{q=u,d,s} \Gamma_{c \to s \bar{q}q} \quad (7)
\]

for \(c\)-decay. As for the concerned \(B_c\) meson, being a double heavy meson, its two components \(\bar{b}\)-quark and \(c\)-quark, each plays the decay role and the spectator role once in tern, so both of eqs.(6,7) as the spectator components make contributions to \(B_c\) decay.

The semileptonic and non-leptonic decay rates of \(b\)- quark up-to the order \(1/m_b^2\) are evaluated by many authors \([2,10,23]\). Since in our numerical computations we need to use their formulas, so we quote them from the references into Appendix A. For \(c \to s\), the formulation is similar and even simpler, we also include the useful formulas in Appendix A.

**B. The non-spectator components in D and B meson decays**

The non-spectator contributions are crucially important to the \(D\) inclusive decays. For instance, the PI contribution may explain the data why \(\tau_{D^\pm} \sim 2\tau_{D^0}\), but \(\tau_{B^\pm} \sim \tau_{B^0}\). Moreover, the penguin contributions in \(D(D_s)\) and \(B(B_s)\) decays are negligible as aforementioned and the bound state effects emerge. Whereas, all the non-factorization effects cannot be reliably well-determined yet.
With straightforward calculations, the precise operators for the non-spectator contributions may be obtained.

(a) For the $D(D_s)$ decays:

$$
\Gamma^{WE}(D^0) = -\Gamma_0\eta_{\text{spec}}\frac{m_D^2}{m_c^2}|V_{cs}|^2|V_{ud}|^2 + |V_{cd}|^2|V_{us}|^2(1-x_+)^2\left\{\left(\frac{N}{c_1} + 2c_1c_2 + Nc_2^2\right)\right\}
\times\left\{\left(1 + \frac{x_+}{2}\right)B_1 - (1 + 2x_+)B_2\right\} + 2c_1^2\left\{\left(1 + \frac{x_+}{2}\right)\xi_1 - (1 + 2x_+)\xi_2\right\}.
\Gamma^{WE}(D^-) = -\Gamma_0\eta_{\text{spec}}\frac{p_D^2}{m_c^2}|V_{cs}|^2|V_{ud}|^2\left\{\left(\frac{N}{c_1} + 2c_1c_2 + Nc_2^2\right)(B_1 - 1 - 2x_+)\right\} + 2c_2^2\left(\xi_1 - \xi_2\right).
\Gamma^{WA}(D_s^+) = -\Gamma_0\eta_{\text{spec}}\frac{m_D^2}{m_c^2}|V_{cs}|^2|V_{ud}|^2\left\{\left(\frac{N}{c_1} + 2c_1c_2 + Nc_2^2\right)(B_1 - 2B_2) + 2c_2^2\left(\xi_1 - \xi_2\right)\right\}
\Gamma^{PI}(D_s^+) = -\Gamma_0\eta_{\text{spec}}\frac{p_D^2}{m_c^2}|V_{cs}|^2|V_{ud}|^2\left\{\left(\frac{N}{c_1} + 2c_1c_2 + Nc_2^2\right)(1 - 2x_+)\right\} + 2c_2^2\left(\xi_1 - \xi_2\right).
\Gamma(D_s^+ \to \tau \nu) = \frac{G_F m_D^2 f_D^2}{8\pi} |V_{cs}|^2 (1 - \frac{m_D^2}{M_D^2})^2.
\end{equation}

where

$$
\Gamma_0 = \frac{G_F^2 m_c^2}{192\pi^3}, \eta_{\text{spec}} = 16\pi^2 f_D^2 \frac{m_D}{m_c^3},
\begin{align*}
x_+ &= \frac{m^2}{p^2_+}; p_+ = p_c + p_q, \\
x_- &= \frac{m^2}{p^2_-}; p_- = p_c - p_q.
\end{align*}

(9)

In the equations, the hadronic parameters are defined as follows:

$$
\frac{g^{\mu\nu}}{2m_D} < D_q|O^{\mu\nu}_q|D_q > = \frac{f_D^2 m_D}{8} B_1,
\frac{g^{\mu\nu}}{2m_D} < D_q|T^{\mu\nu}_q|D_q > = \frac{f_D^2 m_D}{8} \xi_1,
\frac{p^{\mu}p^{\nu}}{2m_D^2} < D_q|O^{\mu\nu}_q|D_q > = \frac{f_D^2 m_D}{8} B_2,
\frac{p^{\mu}p^{\nu}}{2m_D^2} < D_q|T^{\mu\nu}_q|D_q > = \frac{f_D^2 m_D}{8} \xi_2,
\end{equation}

(10)
where

\[
O_{\mu\nu}^a = \bar{c}\gamma_\mu Lq\bar{q}\gamma_\nu Lc, \\
T_{\mu\nu}^a = \bar{c}\gamma_\mu T^a Lq\bar{q}\gamma_\nu T^a Lc, 
\]

with \( T^a = \frac{\lambda^a}{2} \) and \( \lambda^a \) being the Gell-Mann Matrices.

(b) For the \( B(B_s) \) decays:

\[
\Gamma^{\text{WE}}(B_d^0) = -\Gamma_0 \eta_{\text{spec}} |V_{ud}|^2 (1 - z_+)^2 \left\{ \left( \frac{c_1^2}{N} + 2c_1c_2 + Nc_2^2 \right)[(1 + \frac{z_+}{2})B_1 - (1 + 2z_+)B_2] \\
+ 2c_1^2[(1 + \frac{z_+}{2})\epsilon_1 - (1 + 2z_+)\epsilon_2] \right\} \\
- \Gamma_0 \eta_{\text{spec}} |V_{cd}|^2 \sqrt{1 - 4z_+} \left\{ \left( \frac{c_1^2}{N} + 2c_1c_2 + Nc_2^2 \right)[(1 - z_+)B_1 - (1 + 2z_+)B_2] \\
+ 2c_1^2[(1 - z_+)\epsilon_1 - (1 + 2z_+)\epsilon_2] \right\}, \\
\Gamma^{\text{PI}}(B^-) = \Gamma_0 \eta_{\text{spec}} \frac{p^2}{m_B^2} (1 - z_-)^2 \left\{ (c_1^2 + c_2^2)(B_1 + 6\epsilon_1) + 6c_1c_2B_1 \right\}, \\
\Gamma^{\text{WE}}(B_s^0) = -\Gamma_0 \eta_{\text{spec}} |V_{us}|^2 (1 - z_+)^2 \left\{ \left( \frac{c_1^2}{N} + 2c_1c_2 + Nc_2^2 \right)[(1 + \frac{z_+}{2})B_1 - (1 + 2z_+)B_2] \\
+ 2c_1^2[(1 + \frac{z_+}{2})\epsilon_1 - (1 + 2z_+)\epsilon_2] \right\} \\
- \Gamma_0 \eta_{\text{spec}} |V_{cs}|^2 \sqrt{1 - 4z_+} \left\{ \left( \frac{c_1^2}{N} + 2c_1c_2 + Nc_2^2 \right)[(1 - z_+)B_1 - (1 + 2z_+)B_2] \\
+ 2c_1^2[(1 - z_+)\epsilon_1 - (1 + 2z_+)\epsilon_2] \right\}. 
\]

(12)

where

\[
\Gamma_0 = \frac{G_F^2 m_b^5}{192\pi^3} |V_{cb}|^2, \eta_{\text{spec}} = 16\pi^2 \frac{f_{B_s}^2 m_{B_s}^3}{m_b^2}, \\
z_+ = \frac{m_c}{m_{B_s}}, z_- = \frac{m_c}{p_-} = \frac{m_c^2}{(p_b - p_\mu)^2}. 
\]

(13)

Analogous to D meson, the parameters \( B_1, B_2, \epsilon_1 \) and \( \epsilon_2 \) are defined:

\[
\frac{g^{\mu\nu}}{2m_{B_q}} < B_q |O_{\mu\nu}^a| B_q > \equiv \frac{f_{B_q}^2 m_{B_q}}{8} B_1, \\
\frac{g^{\mu\nu}}{2m_{B_q}} < B_q |T_{\mu\nu}^a| B_q > \equiv \frac{f_{B_q}^2 m_{B_q}}{8} \epsilon_1, \\
\frac{p^{\mu}p^{\nu}}{2m_{B_q}^3} < B_q |O_{\mu\nu}^a| B_q > \equiv \frac{f_{B_q}^2 m_{B_q}}{8} B_2, \\
\frac{p^{\mu}p^{\nu}}{2m_{B_q}^3} < B_q |T_{\mu\nu}^a| B_q > \equiv \frac{f_{B_q}^2 m_{B_q}}{8} \epsilon_2. 
\]

(14)
where

\[ O_{\mu}^{q} = \bar{b}\gamma_{\mu}Lq\gamma_{\mu}Lb, \]
\[ T_{\mu}^{q} = \bar{b}\gamma_{\mu}T^{a}Lq\gamma_{\mu}T^{a}Lb. \]  

(15)

C. The non-spectator components in \( B_c \) decays

As pointed above, the spectator contribution to the \( B_c \) lifetime is a sum of that from \( \bar{b} \) and \( c \) individual decays as pointed above:

\[ \Gamma^{\text{spectator}} = \Gamma_{b}^{\text{spectator}} + \Gamma_{\bar{c}}^{\text{spectator}}, \]  

(16)

and \( \Gamma_{b}^{\text{spectator}} \) and \( \Gamma_{\bar{c}}^{\text{spectator}} \) are the same as they are in \( B \) and \( D \) decays and given in eqs.(6,7).

Now let us deal with the non-spectator contributions which are different from that in \( B \) and \( D \) decays.

To estimate the non-spectator components in the \( B_c \) decays, let us write the relevant effective Lagrangian precisely here:

\[ L_{c}^{\Delta C=1}(\mu = m_c) = -\frac{4G_F}{\sqrt{2}}V_{cs}V_{ud}^{*}\left\{ c_1(\mu)(\bar{s}\gamma_{\mu}Lc)(\bar{u}\gamma_{\mu}Ld) + c_2(\mu)(\bar{u}\gamma_{\mu}Lc)(\bar{s}\gamma_{\mu}Ld) \right\} + \text{h.c.}, \]  

(17)

and

\[ L_{c}^{\Delta B=1}(\mu = m_b) = -\frac{4G_F}{\sqrt{2}}\left\{ V_{cb}[V_{ud}^{*}(c_1(\mu)O_{1}^{u} + c_2(\mu)O_{2}^{u}) + V_{cs}^{*}(c_1(\mu)O_{1}^{c} + c_2(\mu)O_{2}^{c}) + \sum_{l=e,\tau,\mu} \bar{\ell}\gamma_{\mu}L\ell\gamma_{\mu}Lb + V_{cs}^{*}\sum_{i=3}^{6} c_i O_i] \right\} + \text{h.c.}, \]  

(18)

where the operators are

\[ O_{1}^{c} = \bar{s}\gamma_{\mu}Lc\bar{c}\gamma_{\mu}Lb, \]
\[ O_{1}^{u} = \bar{d}\gamma_{\mu}Lu\bar{c}\gamma_{\mu}Lb, \]
\[ O_{2}^{c} = \bar{s}\gamma_{\mu}Lc\bar{c}_{j}\gamma_{\mu}Lb, \]
\[ O_{2}^{u} = \bar{d}\gamma_{\mu}Lu\bar{c}_{j}\gamma_{\mu}Lb, \]
\[ O_3 = \bar{s} \gamma_\mu L b \bar{c} \gamma^\mu L c , \]
\[ O_4 = \bar{s}_i \gamma_\mu L b_j \bar{c}_j \gamma^\mu L c_i , \]
\[ O_5 = \bar{s} \gamma_\mu L b \bar{c} \gamma^\mu R c , \]
\[ O_6 = \bar{s}_i \gamma_\mu L b_j \bar{c}_j \gamma^\mu R c_i , \]

and \( c_i (i = 1, 2, \cdots) \), denoting the Wilson coefficients due to QCD corrections, will take the values as those in ref. [4]. Here we consider the non-spectator components in \( B_c \) decays by two steps. The first step is to compute the relevant operators up to the order \( O(1/m_Q^4) \) and then to evaluate the contributions precisely.

1. Pauli interference (PI) operators

The Pauli interference (PI) operators \( \hat{\Gamma}_{\text{tree}}^{\text{PI}} \) and \( \hat{\Gamma}_{\text{penguin}}^{\text{PI}} \) which correspond to the non-leptonic decay induced by the tree part and penguin respectively are given by:

\[
\hat{\Gamma}_{\text{tree}}^{\text{PI}} = \frac{2G_F^2}{\pi} |V_{cb}|^2 |V_{cs}|^2 (1 - z_-)^2 p_-^2 \cdot \left\{ 2c_1c_2 \cdot \bar{b} \gamma_\mu L c^i \bar{c}^j \gamma^\mu L b^j + (c_1^2 + c_2^2) \bar{b} \gamma_\mu L c^i \bar{c}^j \gamma^\mu L b^j \right. \\
+ \left. (c_3^2 + c_4^2 + 2c_1c_3 + 2c_2c_4) \cdot \bar{b} \gamma_\mu L c^i \bar{c}^j \gamma^\mu L b^j \right\} \\
+ \frac{G_F^2}{3\pi} |V_{cb}|^2 |V_{cs}|^2 (1 - z_-)^2 \cdot \left\{ (1 - z_-) p_-^2 g_\mu^\nu + 2(1 + 2z_-) p_-^\mu p_-^\nu \right. \\
+ \left. \left\{ 2c_5c_6 \cdot \bar{b} \gamma_\mu L b^i \bar{c}^j \gamma_\nu R c^j + (c_5^2 + c_6^2) \cdot \bar{b} \gamma_\mu L b^i \bar{c}^j \gamma_\nu R c^j \right\} \\
- \frac{G_F^2}{\pi} |V_{cb}|^2 |V_{cs}|^2 (1 - z_-)^2 \bar{m}_c p^\alpha \cdot \left\{ [c_2c_6 + c_3c_6 + c_1c_5 + c_4c_5] \\
\cdot \left[ \bar{b} \gamma_\mu L c^i \bar{c}^j \gamma_\alpha \gamma_\mu L b^j + \bar{b} \gamma_\mu \gamma_\alpha R c^i \bar{c}^j \gamma^\mu L b^j \right] \\
+ [c_2c_5 + c_3c_5 + c_1c_6 + c_4c_6] \\
\cdot \left[ \bar{b} \gamma_\mu L c^i \bar{c}^j \gamma_\alpha \gamma_\mu L b^j + \bar{b} \gamma_\mu \gamma_\alpha R c^i \bar{c}^j \gamma^\mu L b^j \right] \right\} \right\} ,
\]

where

\[
z_- = \frac{\bar{m}_c^2}{p_-^2}, p_- = p_b - \bar{p}_\ell \cdot (21)
\]
2. Weak annihilation (WA) operators

The weak annihilation operators are $\hat{\Gamma}^\text{WA}_\text{tree}$, $\hat{\Gamma}^\text{WA}_\text{penguin}$, and $\hat{\Gamma}^\text{WA}(B_c \to \tau \nu_\tau)$ which correspond to the non-leptonic decay induced by the tree part, penguin and the pure leptonic (PL) decay respectively.\footnote{Due to helicity suppression, the decays $B_c \to l(e, \mu) + \nu$ are ignorable for the low order estimate of the lifetime, thus we do so here.}

\[
\hat{\Gamma}^\text{WA}_\text{tree} = -\frac{2G_F^2}{3\pi} |V_{cb}|^2 |V_{cs}|^2 (1 - z_+)^2 \cdot \left\{ (1 + \frac{z_+}{2})p_+^2 g^{\mu \nu} - (1 + 2z_+)p_+^\mu p_+^\nu \right\} \\
\hat{\Gamma}^\text{WA}_\text{penguin} = -\frac{2G_F^2}{3\pi} |V_{cb}|^2 |V_{cs}|^2 (1 - z_+)^2 \cdot \left\{ (1 + \frac{z_+}{2})p_+^2 g^{\mu \nu} - (1 + 2z_+)p_+^\mu p_+^\nu \right\} \\
\hat{\Gamma}^\text{WA}(B_c \to \tau \nu_\tau) = -\frac{2G_F^2}{3\pi} |V_{cb}|^2 (1 - z_\tau)^2 \cdot \left\{ (1 + \frac{z_\tau}{2})p_+^2 g^{\mu \nu} - (1 + 2z_\tau)p_+^\mu p_+^\nu \right\} \\
\]

where the parameters $p_+$, $z_+$ and $z_\tau$ are defined by

\[
p_+ = p_b + p_c, \\
z_+ = \frac{\bar{m}_c^2}{p_+^2} = \frac{\bar{m}_c^2}{M_{B_c}^2}, \\
z_\tau = \frac{m_\tau^2}{p_+^2} = \frac{m_\tau^2}{M_{B_c}^2}. \tag{26}
\]
3. The contributions from the non-spectator WA and PI to the lifetime for $B_c$ meson

With the optical theorem, substituting all the above operators $\hat{\Gamma}^W_A$, $\hat{\Gamma}^P_I$ into the relevant matrix element, we may estimate the non-spectator contributions to the lifetime of $B_c$ meson:

$$\Gamma = \frac{1}{2M_{B_c}} \langle Bc|\hat{\Gamma}|Bc\rangle,$$

where $\hat{\Gamma}$ denotes the relevant operators for PI and WA given in the above subsections.

According to eq. (27 to evaluate the lifetime, finally some hadronic matrix elements need to be determined, whereas, having nonpertubative nature, they cannot be determined by well-established theories so far. Let us discuss their determination here for a while.

First of all, some parameters, such as $B_1, B_2, \tilde{B}_1, \tilde{B}_2, \epsilon_1, \epsilon_2, \tilde{\epsilon}_1$ and $\tilde{\epsilon}_2$, appear in the corresponding estimates for $B$ and $D$ decays too. Precisely for $B_c$ decays, they are

$$\frac{1}{2M_{B_c}} \langle Bc|O_{V-A}^c|Bc\rangle = \frac{f_{B_c}^2 M_{B_c}}{8} B_1,$$

$$\frac{1}{2M_{B_c}} \langle Bc|O_{S-P}^c|Bc\rangle = \frac{f_{B_c}^2 M_{B_c}}{8} B_2,$$

$$\frac{1}{2M_{B_c}} \langle Bc|T_{V-A}^c|Bc\rangle = \frac{f_{B_c}^2 M_{B_c}}{8} \epsilon_1,$$

$$\frac{1}{2M_{B_c}} \langle Bc|T_{S-P}^c|Bc\rangle = \frac{f_{B_c}^2 M_{B_c}}{8} \epsilon_2,$$

where the relevant four-quark operators are

$$O_{V-A}^c = \bar{b}\gamma_\mu Lc\bar{c}\gamma^\mu Lb,$$

$$O_{S-P}^c = \bar{b}Lc\bar{c}Rb,$$

$$T_{V-A}^c = \bar{b}\gamma_\mu LT^a c\bar{c}\gamma^\mu LT^a b,$$
There are eight extra matrix elements corresponding to the newly emerged operators in the $B_c$ case. The ‘new’ matrix elements relate to the above parameters or new ones ($\epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6$) as follows:

\[
\begin{align*}
    \frac{1}{2M_{B_c}} \langle B_c | \bar{b}L \bar{c}cLb | B_c \rangle & \equiv -\frac{f_{Bc}^2 M_{B_c}}{8} \epsilon_3, \\
    \frac{1}{2M_{B_c}} \langle B_c | \bar{b}R \bar{c}cRb | B_c \rangle & \equiv -\frac{f_{Bc}^2 M_{B_c}}{8} \epsilon_4, \\
    \frac{1}{2M_{B_c}} \langle B_c | \bar{b}L \bar{c}c \gamma^\mu Lb | B_c \rangle & \equiv -\frac{f_{Bc}^2 M_{B_c}}{8} \epsilon_5, \\
    \frac{1}{2M_{B_c}} \langle B_c | \bar{b}R \bar{c}c \gamma^\mu Rb | B_c \rangle & \equiv -\frac{f_{Bc}^2 M_{B_c}}{8} \epsilon_6.
\end{align*}
\]

(29)

Generally speaking, we may assume that

\[
\tilde{B}_{1(2)} = B_{1(2)}; \quad \tilde{\epsilon}_{1(2)} = \epsilon_{1(2)},
\]

(30)

with symmetry consideration. As for the parameters $\epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6$, we would conjecture that $\epsilon_3, \epsilon_4 \simeq \epsilon_2$ and $\epsilon_5, \epsilon_6 \simeq \epsilon_1$ instead of precise computation.\footnote{The conjecture should be tested and proved later on. It should be considered as a working hypothesis.}
In the earlier literatures, usually $B_1 \approx B_2 \sim 1$ and $\epsilon_1 \sim -0.15$ from the lattice calculations and $\epsilon_2 = 0$. According to our numerical computations and trials to fit the data about the lifetimes of the heavy mesons $D^\pm, D^0, D_s, B^\pm, B^0$ and $B_s$ and their semileptonic decay branching ratios as well, we find that to adjust the values of them and the pole masses of $b$ and $c$ quarks, when the parameter $\epsilon_2 \neq 0$ etc, a better fit is obtained.

Now for the nonspectator component $P_I$, we have

$$
\Gamma_{\text{tree}}^{P_I} = \frac{G_F^2}{4\pi} f_{B_c}^2 M_{B_c} |V_{cb}|^2 |V_{cs}|^2 (1 - z_-)^2 p^2 \cdot \left\{ [2c_1 c_2 + \frac{1}{N} (c_1^2 + c_2^2)] B_1 + 2(c_1^2 + c_2^2) \epsilon_1 \right\} 
$$

(31)

$$
\Gamma_{\text{penguin}}^{P_I} = \frac{G_F^2}{4\pi} f_{B_c}^2 M_{B_c} |V_{cb}|^2 |V_{cs}|^2 (1 - z_-)^2 p^2 \cdot \left\{ [2c_2 c_4 + 2c_1 c_3 + 2c_3 c_4 + \frac{1}{N} (c_3^2 + c_4^2 + 2c_2 c_3 + 2c_1 c_4)] B_1 + 2(c_3^2 + c_4^2 + 2c_2 c_3 + 2c_1 c_4) \epsilon_1 \right\} 
$$

(32)

$$
-\frac{G_F^2}{4\pi} f_{B_c}^2 M_{B_c} |V_{cb}|^2 |V_{cs}|^2 (1 - z_-)^2 \cdot \left\{ [2c_5 c_6 + \frac{1}{N} (c_5^2 + c_6^2)] \frac{2 + z_-}{3} p^2 \tilde{B}_2 - \frac{1 + 2z_-}{6} (m_b^2 \tilde{B}_1 + m_c^2 \tilde{B}_1 - 4m_b m_c B_2 + 2m_b m_c B_1) \right\} + 2(c_5^2 + c_6^2) \left\{ \frac{2 + z_-}{3} p^2 \tilde{B}_2 - \frac{1 + 2z_-}{6} (m_b^2 \epsilon_1 + m_c^2 \epsilon_1 - 2m_b m_c (\epsilon_3 + \epsilon_4) + m_b m_c (\epsilon_5 + \epsilon_6)) \right\} 
$$

(33)

and for $WA$, we have

$$
\Gamma_{\text{tree}}^{WA} = -\frac{G_F^2}{12\pi} |V_{cb}|^2 |V_{cs}|^2 f_{B_c}^2 M_{B_c} (1 - z_+)^2 \cdot \left\{ [N c_1^2 + 2c_1 c_2 + \frac{c_2^2}{N}] \right\} 
$$

(34)

$$
\times \left\{ [(1 + \frac{z_+}{2}) M_{B_c}^2 B_1 - (1 + 2z_+) (m_b^2 B_2 + m_c^2 \tilde{B}_2 + 2m_b m_c B_2)] + 2c_2^2 [(1 + \frac{z_+}{2}) M_{B_c}^2 \epsilon_1 - (1 + 2z_+) (m_b^2 \epsilon_2 + m_c^2 \tilde{\epsilon}_2 + m_b m_c (\epsilon_3 + \epsilon_4))] \right\},
$$

$$
\Gamma_{B_s \rightarrow \tau \nu}^{WA} = -\frac{G_F^2}{12\pi} |V_{cb}|^2 |V_{cs}|^2 f_{B_c}^2 M_{B_c} (1 - z_+)^2 \cdot \left\{ (1 + \frac{z_+}{2}) M_{B_c}^2 B_1 \right\} 
$$

(35)

assumption. With the assumption eq.(30) in addition, all the assumptions here will cause an essential uncertainty for the estimates.
\[-(1 + 2z_\tau)(m_b B_2 + m_c \bar{B}_2 + 2mb m_c B_2)\}\right),

\[
\Gamma_{\text{penguin}}^{WA} = -\frac{G_F^2}{12\pi}|V_{cb}|^2|V_{cs}|^2 f_{B_c}^2 M_{B_c}(1 - z_+)^2 \cdot \left\{\left[\left(\frac{2c_2 + c_3}{N} + 2c_1 + c_4\right)(c_3 + Nc_4)\right]
\right.
\times\left(1 + \frac{z_+}{2}\right)M_{B_c}^2 B_1 - (1 + 2z_+)(m_b B_2 + m_c \bar{B}_2 + 2mb m_c B_2)

+ 2(2c_2 + c_3)c_3 \cdot \left(1 + \frac{z_+}{2}\right)\rho_+^2 \epsilon_1 - (1 + 2z_+)(m_b^2 \epsilon_2 + m_c^2 \bar{\epsilon}_2 + m_b m_c(\epsilon_3 + \epsilon_4))\right\}\right.

+ \frac{G_F^2}{2\pi}|V_{cb}|^2|V_{cs}|^2 f_{B_c}^2 M_{B_c}^3 (1 - z_+)^2 \cdot \left\{\left[\left(\frac{c_2}{N} + 2c_5 c_6 + Nc_6^2\right)\bar{B}_2 + 2c_5^2 \tilde{\epsilon}_2\right]

- \frac{G_F^2}{4\pi}|V_{cb}|^2|V_{cs}|^2 f_{B_c}^2 M_{B_c} m_c(1 - z_+)^2 \cdot \left\{\left[\left(\frac{c_2 + c_3}{N} + c_1 + c_4\right)(c_5 + Nc_6)\right]

\times\left[2m_b B_2 + 2m_c \bar{B}_2\right] + 2(c_2 + c_3)c_5 \cdot \left[m_b(\epsilon_3 + \epsilon_4) + 2m_b \tilde{\epsilon}_2\right]\right\}\right.\]

D. The effective mass of the decaying heavy quark

The masses of the acting heavy quarks in a decay must be treated carefully although the bound-state effects make the problem complicated and obscure. It is commonly accepted that if the charm quark appears as a decay product, the mass should be its running one at the energy scale of the decaying quark or the meson, whereas, if it appears as the ‘parent(s)’ of the decay, the quark (antiquark) is not ”free”, but in a bound state, thus the pole mass should be taken and the bound-state effects on the mass must be taken into account too. Especially in spectator mechanism the decay possibility of the heavy quark is very sensitive to the value of its ‘adopted’ mass, hence what a value of the quark mass adopted in the estimate must pay special care. Narison [16] used the QCD sum rules to estimate the mass difference \(M_{b(c)}^{NR} - M_{b(c)}^{PT}\) where \(M_{PT}^2\) is the short-distance perturbative pole mass and \(M^{NR}\) is the long-distance QCD-related effective mass up-to two-loops. Whereas the authors of [17] attributed such effects into a factor which is multiplied to the decay width of the ”free” quark.

Here instead of deriving the modification factor with a relatively large uncertainty, we treat the problem phenomenologically i.e. by introducing a parametrization

\[
M_Q^{eff} = M_Q^{pole} - \Delta,
\]
where $\Delta$ manifests the bound-state effects, and it will be fixed phenomenologically. Note here that for each heavy meson there are three quantities: lifetime (total width), inclusive semileptonic branching ratio and pure leptonic branching ratio which may be used for phenomenological analyses, so the estimates here are still well-determined even when we introduce the parameter $\Delta$ here.

In the next section, we will discuss $\Delta$ and other related parameters more precisely.

With all the formulae derived above and the hadronic matrix elements, we can make numerical evaluation of the lifetime of $B_c$ straightforwardly.

III. NUMERICAL RESULTS

Since we carry out the estimate of the lifetime of $B_c$ with a ‘gloable’ comparison to all of the heavy and double heavy mesons, so the determination of all of the parameters by fitting the existence experimental data is ‘over-determined’ for our goal and has certain level tests. Therefore we evaluate the lifetimes, the semileptonic branching ratios and the pure leptonic branching ratios for all the mesons $D^\pm, D^0, D_s, B^\pm, B^0, B_s$ and $B_c$ in this section in turn and present the numerical results in this section.

A. For the heavy mesons $D$ and $B$

To evaluate the lifetimes of $D^0, D^\pm, D_s, B^0, B^\pm, B_s$ mesons and their branching ratios of the semileptonic decays, we use the formulae given in sections 2.1, 2.2 and the appendix. The parameter values are taken as follows. $|V_{cs}| = 0.974, |V_{ud}| = 0.975$, $\alpha_s(m_c) = 0.29$, $c_1(mw_c) = 1.30, c_2(m_c) = -0.57$, $B_1 = B_2 = 1, \epsilon_1 \simeq -0.05, \epsilon_2 = 0$, the decay constants of D mesons $f_D = 160$ MeV, $f_{D_s} = 190$ MeV. In the evaluation of the Pauli interference $PI$ contribution to $D$ decay width, we take the $p^2_\pi = (p_\pi - p_\ell)^2$ value as $0.5 M_D^2$ as done in ref. [24].
We take $m_b^{pole} = 5.02$ GeV, $m_c^{pole} = 1.88$ GeV \[26,27\]. By eq.(1), we have the running mass of the charm quark at various energy scales as

$$\overline{m}_c(m_c) = 1.67 \text{ GeV}, \quad \overline{m}_c(m_b) = 1.41 \text{ GeV}, \quad \overline{m}_c(m_{B_c}) = 1.37 \text{ GeV}.$$ 

By fitting data \[8\], we should have the quark masses as $m_s = 125$ MeV, $m_c^{eff} = 1.65$ GeV respectively. Then we obtain the D meson lifetimes: $\tau(D^0) = 0.419$ ps, $\tau(D^-) = 1.06$ ps, $\tau(D_s^-) = 0.446$ ps, and the branch ratio of the semileptonic decay of $D^0$ meson $B_{SL}(D^0) = 6.9\%$. Comparing to the experimental data: $\tau(D^0) = 0.415 \pm 0.04$ ps; $\tau(D^\pm) = 1.057 \pm 0.015$ ps; $\tau(D_s) = 0.467 \pm 0.017$ ps and $B_{SL}(D^0) = 6.75 \pm 0.29\%$, one can see the fit is quite well.

For the estimate of B meson lifetimes, we take the mass of the charm quark at final states $m_c$ to be running mass, namely it is different from the pole mass of charm quark in $D$-decays but the running mass at the energy scale $m_b$ i.e. $\overline{m}_c(m_b) = 1.41$ GeV. When calculating $PI$ contribution to the $B^-$ decay width, we take the value of $p_\perp^2 = (p_b - p_u)^2$ approximately to be $0.8 \, M_B^2$ \[24\]. For the other parameters in the numerical computations the values are adopted: $|V_{cb}| = 0.04$, $\alpha_s(m_b) = 0.20$, $c_1(m_b) = 1.150$, $c_2(m_b) = -0.313$ \[1\], $\alpha = 1.06$, $\beta = 1.32$ \[28\], $B1 = B2 = 1$ and $\epsilon_1 = -0.14, \epsilon_2 = -0.08$ \[27\]. The decay constants: $f_B = 200$ MeV and $f_{B_s} = 220$ MeV. Furthermore, taking the $b$ quark pole mass $m_b^{pole} = 5.02$ GeV and $m_b^{eff} = 4.89 \sim 4.91$ GeV, we obtain the results that

$$\tau(B^0) = 1.54\text{ps}, \quad \tau(B^-) = 1.74\text{ps}, \quad \tau(B_s^-) = 1.56\text{ps}, \quad B_{sl}(B^0) = 11.2\%, \quad \text{if } m_b^{eff} = 4.89\text{GeV} ;$$

$$\tau(B^0) = 1.52\text{ps}, \quad \tau(B^-) = 1.71\text{ps}, \quad \tau(B_s^-) = 1.54\text{ps}, \quad B_{sl}(B^0) = 11.2\%, \quad \text{if } m_b^{eff} = 4.90\text{GeV} ;$$

$$\tau(B^0) = 1.50\text{ps}, \quad \tau(B^-) = 1.68\text{ps}, \quad \tau(B_s^-) = 1.51\text{ps}, \quad B_{sl}(B^0) = 11.2\%, \quad \text{if } m_b^{eff} = 4.91\text{GeV} ,$$

where $B_{sl}$ indicates the branching ratio of the semileptonic decay. Comparing with the experimental data $\tau(B^0) = 1.56 \pm 0.04$ ps, $\tau(B^\pm) = 1.65 \pm 0.04$ ps, $\tau(B_s) = 1.54 \pm 0.07$ ps and $B_{SL}(B^0) = 10.5 \pm 0.008\%$ we can see the fit is quite good. According to the definition of $\Delta$, we have $\Delta_c \equiv m_c^{pole} - m_c^{eff} = 0.23$ GeV and $\Delta_b \equiv m_b^{pole} - m_b^{eff} = 0.11 \sim 0.13$ GeV, that is understandable.
With all the parameters obtained by fitting the lifetimes of $B^0, B^{±}, B_s, D^0, D^{±}, D_s$ and the branching ratios of the semileptonic decays of $B$ and $D$ mesons, we are proceeding to evaluate the lifetime of $B_c$-meson and its semileptonic decay rate.

B. For the double heavy meson $B_c$

The spectator component contribution to the $B_c$ lifetime is a sum of the individual $\bar{b}$ and $c$ quark decays, while leaving the other one as a spectator. When evaluating this contribution, $m_\bar{b}$ is its pole value at $p_\bar{b}^2 = m_\bar{b}^2$, and $m_c$ also the pole value $m_c(m_c)$. Whereas for the non-spectator contributions, i.e. the WA and PI pieces, the corresponding energy scale for the running charm-quark mass in the final state is taken as $M_{B_c}$. Now let us take the relevant parameters for $B_c$ as follows: $M_{B_c} = 6.25$ GeV, $M_{B_c}^* = 6.33$ GeV, $B_1 = B_2 = 1, \epsilon_1 = -0.14, \epsilon_2 = -0.08$. For the decay constant, we adopt Eichten and Quigg’s one $f_{B_c} = 500$ MeV [21] and the lattice one $f_{B_c} = 440$ MeV [22] respectively. Furthermore in the calculation of the $PI$ contribution, the quantity $p_\tau^2 = (p_\bar{b} - p_c)^2 \simeq 2m_\bar{b}^2 + 2m_c^2 - M_{B_c}^2$ is taken approximately. With the parameters described above, we obtain the numerical results and tabulate them in Table 1.

| $f_{B_c}$   | $\tau_{B_c}$ | $\Gamma_{\text{pen.}}$ | $\Gamma^{b\rightarrow c}$ | $\Gamma^{c\rightarrow s}$ | $\Gamma^{WA}$ | $\Gamma^{PI}$ | $\Gamma(\tau\nu)$ | $B_{SL}$ |
|------------|--------------|-------------------------|--------------------------|--------------------------|--------------|--------------|-----------------|---------|
| 440 MeV    | 0.362 (ps)   | 3.4%                    | 22.8%                    | 70.9%                    | 13.4%        | -7.1%        | 0.078 ps$^{-1}$ | 8.7%   |
| 500 MeV    | 0.357 (ps)   | 4.3%                    | 22.4%                    | 69.7%                    | 16.9%        | -9.0%        | 0.100 ps$^{-1}$ | 8.4%   |

In the table $f_{B_c}$ denotes the decay constant; $\tau_{B_c}$: the lifetime of $B_c$; $\Gamma_{\text{pen.}}$: the contribution from the interference between the penguin and ‘tree’ terms; $\Gamma(\tau\nu)$: the width of the pure leptonic decay ($\tau$ chennal only but almost equal to the total) and the $B_{SL}$: the branching ratio of the semileptonic decay of the meson $B_c$.

Since both the ‘parents’ $\bar{b}$ and $c$ quarks reside in a bound state ($B_c$ meson), the problem how to choose the value of the masses $m_\bar{b}$ and $m_c$ emerge as in the cases of the heavy mesons
D and B etc, but when taking all the parameters fixed by fitting data as the above, we obtain the results presented in table 1.

Let us discuss the bound-state effects on the \( \bar{b} \) and \( c \) quark-masses in \( B_c \) meson more precisely. Because \( B_c \) includes two heavy quarks i.e. is a double heavy meson, the bound-state effects might be greater than in the heavy mesons \( B, D \). We think that the values \( m_{c}^{\text{eff}} \) and \( m_{b}^{\text{eff}} \) might be smaller than \( m_{c}^{\text{eff}} = 1.65 \) GeV and \( m_{b}^{\text{eff}} = 4.9 \) GeV that we obtained in \( B \) and \( D \) decays. Phenomenologically, if in \( B_c \) meson, \( m_{c}^{\text{eff}}(B_c) = 1.55 \) GeV, \( m_{b}^{\text{eff}}(B_c) = 4.85 \) GeV, we obtain \( \tau(B_c) \approx 0.47 \) ps, which occasionally is closer to the center value of the \( B_c \) lifetime measured recently \[1\]. In this case, \( \Delta_c = 0.33 \) GeV and \( \Delta_b = 0.17 \) GeV. Because the rates of direct \( \bar{b} \) and \( c \) decays, which dominate the lifetime of \( B_c \) meson, are proportional to \( (M_{Q}^{\text{eff}})^5 \), the results are so sensitive to the effective masses.

### IV. CONCLUSION AND DISCUSSION

In this work, we estimate the lifetime of \( B_c \) in the ‘unique’ theoretical framework where the nonspectator effects are taken into account properly and the necessary parameters are determined by fitting the data of the heavy mesons \( B^0, B^{\pm}, B_s, D^0, D^{\pm}, D_s \) on the lifetimes and the branching ratios of the inclusive semileptonic decays as input.

Not only the uncertainties in the estimate are discussed but also the physical parameters appearing in the estimation are fixed in reasonable regions.

Even though not all of the parameters are fixed by fitting the available data for the heavy mesons \( B^0, B^{\pm}, B_s, D^0, D^{\pm}, D_s \), in order to carry out the estimate we make some reasonable assumptions or conjectures. In fact, in terms of the lattice calculation, QCD sum rules and other approaches, the parameters, such as \( B_1 \) and \( B_2 \) (the factors in the hadronic matrix elements, the values manifest the deviation to the vacuum saturation), may be determined at certain accurate level. The deviation from model-dependent calculation is small, and can be neglected in practice. Some of the parameters, such as \( \epsilon_1 \) and \( \epsilon_2 \), being realized to
relate to the non-factorization effects [33], may be calculated at the \(D\)-meson scale and the \(B\)-meson scale respectively in terms of the QCD sum rules [8,28,29]. It is known that the numerical values obtained by the QCD sum rules may have errors about 10 \(\sim\) 15\%, but they still can be used as inputs to the phenomenological calculations without causing too large errors. In this work we also carefully consider the quark masses and take into account of the bound-state effects. The consistency of our numerical results of \(B\) and \(D\) mesons confirms the validity of the parameter regions.

The earlier estimations on the lifetimes of \(B\) and \(D\) mesons and the semileptonic decay rates obviously deviate from the data. Luke, Savage and Wise [3] pointed out that in decay \(c \to X\bar{e}\nu_e\), the contribution of \(\alpha_s^2\) order is of the same magnitude as that of \(O(\alpha_s)\) and this higher order correction suppresses the semileptonic decay rate of \(D\)-meson. Taking into account this fact, we obtain numerical results of lifetimes of \(D\)-mesons and their semileptonic decay rate, and find that they are satisfactorily consistent with data. Whereas, for \(B\)-meson decays, the \(\alpha_s^2\) order correction, as well as the \(O(\alpha_s)\) correction, are smaller. With these corrections concerned, the results for \(B\)-mesons are also consistent with data within the experiment tolerance region. All these imply that the parameters taken as the above are reasonable.

When evaluating the \(B_c\) lifetime and its inclusive semileptonic decay rate, some new aspects must be taken into account. First there are several new operators in the effective Lagrangian playing roles. Their appearance is due to non-negligible \(m_c\), whereas in \(B\) and \(D\) cases, the light quark mass \(m_q\) is ignored with quite high accuracy. Correspondingly, several new hadronic matrix elements are induced by these operators. Some of them are also proportional to \(B_1\) and \(B_2\), which exist in the expressions for \(B\) and \(D\) meson decays, as long as the factorization theorem and the vacuum saturation hold. But in the non-factorization contributions, new parameters appear and \(\epsilon_3 \sim \epsilon_6\) is assumed. In this work, we have taken a naive symmetry consideration to let \(\epsilon_{3,4} \simeq \epsilon_2\) and \(\epsilon_{5,6} \simeq \epsilon_1\).

As discussed in the introduction, in the \(B_c\) case, the interference between the penguin and tree terms is not negligible. Namely, the penguin contribution to \(B_c\) lifetime is much
more important than to B and D decays. Our results confirm this allegation and we have found the contribution from the interference can be as large as $3 \sim 4\%$ of the total width. This fraction is measurable in accurate measurements. Since direct measurement of penguin diagrams would be interesting, these sizable value can be encouraging for future experiments.

The lifetime of $B_c$-meson is estimated as $0.37 \sim 0.367$ ps for $f_{B_c} \sim 440 \sim 500$ MeV, which are smaller than the central value of the measurement $\tau_{B_c} = 0.46^{+0.18}_{-0.16}$ (stat) $\pm 0.063$ (syst) ps \cite{1}. Our earlier estimation of $\tau_{B_c}$ \cite{9} was 0.4 ps. Anisimov et al. estimated $\tau_{B_c}$ in the light-front constituent quark model and obtained $\tau_{B_c} = 0.59 \pm 0.06$ ps, which is larger than the measured value. In our estimation, we use the values of $M_{B_c}$ and $M_{B_c}^*$ as 6.25 GeV and 6.33 GeV \cite{2}, whereas the measurement is $M_{B_c} \sim 6.40 \pm 0.39$ (stat) $\pm 0.13$ (syst) GeV. When the bound-state effects on $\bar{b}$ and $c$ masses are reasonably taken into account, we can have $\tau_{B_c} \sim 0.47$ ps, which is very close to the present experimental center value of the $B_c$-meson lifetime. As noted, the change of $f_{B_c}$ itself only does not influence the result much, e.g. as $f_{B_c}$ changes from 440 MeV to 500 MeV, $\tau_{B_c}$ varies 1% only. Whereas, a large error in mass of the decay parents would result in an estimate error of about 4%, therefore, the more accurate lifetime and mass of $B_c$ meson will test the framework adopted here the deeper.

The more accurate experimental measurements will shed fresh lights on the framework adopted here, which involving the effective heavy flavour theory and the duality between quark states and hadronic states

$$\sum_{i,j} |q_i, g_j\rangle\langle q_i, g_j| = \sum_k |h_k\rangle\langle h_k|$$

etc. It will help to clarify many uncertainties.

**Acknowledgements**

This work was supported in part by the National Natural Science Foundation of China. One of the author (C.-H. Chang) would like to thank TH.-Division of CERN for their warm hospitality, since this paper is accomplished its final composition during his short visit of
CERN.
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Appendix A

The semileptonic and non-leptonic decay rates of $b$ quark through order $1/m_Q^2$ are given as following [2,25].

\[ \Gamma_{SL}(H_b) = \Gamma_0^{(b)} \cdot \eta(x_c, x_l, 0) \cdot \left[ I_0(x_c, 0, 0) \langle H_b | \bar{b}b | H_b \rangle - \frac{2 \langle \mu_G^2 \rangle H_b}{m_b^2} I_1(x_c, 0, 0) \right] ; \]

\[ \Gamma_{NL}(H_b) = \Gamma_0^{(b)} \cdot N \cdot \left\{ \left( c_1^2 + c_2^2 + \frac{2c_1 c_2}{N} \right) \cdot \left[ \alpha I_0(x_c, 0, 0) + \beta I_1(x_c, x_c, 0) \right] \langle H_b | \bar{b}b | H_b \rangle - \frac{2 \langle \mu_G^2 \rangle H_b}{m_b^2} (I_1(x_c, 0, 0) + I_1(x_c, x_c, 0)) \right\} - \frac{8 \langle \mu_G^2 \rangle H_b}{m_b^2} \frac{2c_1 c_2}{N} \cdot \left[ I_2(x_c, 0, 0) + I_2(x_c, x_c, 0) \right] \right\}, \tag{38} \]

where

\[ \Gamma_0^{(b)} = \frac{G_F^2 m_b^5 |V_{cb}|^2}{192\pi^3} ; \tag{39} \]

and the following notation has been used: $I_0$, $I_1$ and $I_2$ are phase-space factors, namely

\[ I_0(x, 0, 0) = (1 - x^2)(1 - 8x + x^2) - 12x^2 \log x, \]

\[ I_1(x, 0, 0) = \frac{1}{2} (2 - x \frac{d}{dx}) I_0(x, 0, 0), \]

\[ I_2(x, 0, 0) = (1 - x)^3, \]

\[ I_0(x, x, 0) = v(1 - 14x - 2x^2 - 12x^3) + 24x^2(1 - x^2) \log \frac{1 + v}{1 - v}, \]

\[ I_1(x, x, 0) = \frac{1}{2} (2 - x \frac{d}{dx}) I_0(x, x, 0), \]

\[ I_2(x, x, 0) = v(1 + \frac{x}{2} + 3x^2) - 3x(1 - 2x^2) \log \frac{1 + v}{1 - v}, \]

\[ x_c = (\bar{m}_c/m_b)^2, \quad v = \sqrt{1 - 4x}, \tag{40} \]

with $I_{0,1,2}(x, x, 0)$ describing the $b \rightarrow c\bar{c}s$ transitions.

And for $\eta(x_c, x_l, 0)$, which is the QCD radiative correction to the semileptonic decay rate. Its general analytic expression is given in [30]. The special case $\eta(x, 0, 0)$ is given in [31] and it can be approximated numerically by [32,25].
\[ \eta(x, 0, 0) \simeq 1 - \frac{2\alpha_s}{3\pi} \left[ (\pi^2 \frac{31}{4}) (1 - \sqrt{x})^2 + \frac{3}{2} \right]. \quad (41) \]

For the decay \( b \to c\tau\nu \), according to [7] we roughly have

\[ \Gamma(b \to c\tau\nu) \sim 0.25 \Gamma(b \to ce\nu). \quad (42) \]

The expressions are simpler for \( c \to s \):

\[
\Gamma_{SL}(H_c) = \Gamma_0^{(c)} \cdot \eta(x_s, x_t, 0) \left[ I_0(x_s, 0, 0) \langle H_c|c\bar{c}|H_c \rangle - \frac{2\langle \mu^2_G \rangle_{H_c}}{m_c^2} I_1(x_s, 0, 0) \right],
\]

\[
\Gamma_{NL}(H_c) = \Gamma_0^{(c)} \cdot N \left\{ (c_1^2 + c_2^2 + \frac{2c_1 c_2}{N}) \times [\alpha I_0(x_s, 0, 0) \langle H_c|c\bar{c}|H_c \rangle - \frac{2\langle \mu^2_G \rangle_{H_c}}{m_c^2} I_1(x_s, 0, 0)] - 8 \frac{\langle \mu^2_G \rangle_{H_c} 2c_1 c_2}{N} \cdot I_2(x_s, 0, 0) \right\},
\]

where

\[ \Gamma_0^{(c)} \equiv \frac{G_F^2 m_e^5}{192\pi^3} |V_{cs}|^2, \quad x_s = \frac{\bar{m}_s^2}{m_c^2} \quad (45) \]

and for the correction \( \eta(x_s, x_t, 0) \) in the c-decay case, we adopt a numerical expression from [5]. It reads

\[ \eta_{SL} = 1 - 2.08 \left( \frac{\alpha_s(m_c)}{\pi} \right) - 22.7 \left( \frac{\alpha_s(m_c)}{\pi} \right)^2. \quad (46) \]

For the dimension-three operator \( \bar{Q}Q \), the expectation value can be expressed at follows:

\[ \langle H_Q|\bar{Q}Q|H_Q \rangle = 1 - \frac{\langle (P_Q)^2 \rangle_{H_Q}}{2m_Q^2} + \frac{\langle \mu^2_G \rangle_{H_Q}}{2m_Q^2} + \mathcal{O}(1/m_Q^3); \quad (47) \]

where \( \langle (P_Q)^2 \rangle \equiv \langle H_Q|Q(iD)^2Q|H_Q \rangle \) denotes the average kinetic energy of the quark Q moving inside the hadron and \( \langle \mu^2_G \rangle_{H_Q} \equiv \langle H_Q|\bar{Q}_2^2 \sigma \cdot GQ|H_Q \rangle \).

Based on refs. [2,10] the kinetic terms take the values respectively as follows:

\[
\frac{\langle (P_b)^2 \rangle_B}{m_b^2} \simeq 0.016, \quad \frac{\langle (P_c)^2 \rangle_D}{m_c^2} \simeq 0.21, \quad \frac{\langle (P_b)^2 \rangle_B}{m_b^2} \simeq 0.04, \quad \frac{\langle (P_c)^2 \rangle_B}{m_c^2} \simeq 0.4. \quad (48)
\]

For the chromomagnetic operator one finds \( \langle \mu^2_G \rangle_{P_Q} \simeq \frac{3}{2} m_Q (M_{V_Q} - M_{P_Q}) \), where \( P_Q \) and \( V_Q \) denote the pseudoscalar and vector mesons, respectively.