SURFACE AREA AND OTHER MEASURES OF ELLIPSOIDS

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Abstract. We begin by studying the surface area of an ellipsoid in \( \mathbb{E}^n \) as the function of the lengths of the semi-axes. We write down an explicit formula as an integral over \( S^{n-1} \), use this formula to derive convexity properties of the surface area, to give sharp estimates for the surface area of a large-dimensional ellipsoid, to produce asymptotic formulas for the surface area and the isoperimetric ratio of an ellipsoid in large dimensions, and to give an expression for the surface in terms of the Lauricella hypergeometric function. We then write down general formulas for the volumes of projections of ellipsoids, and use them to extend the above-mentioned results to give explicit and approximate formulas for the higher integral mean curvatures of ellipsoids. Some of our results can be expressed as isoperimetric results for higher mean curvatures.

Introduction

We study the mean curvature integrals of an ellipsoid \( E \) in \( \mathbb{E}^n \) as functions of the lengths of its semiaxes – the 0-th mean curvature integral is simply the surface area of the ellipsoid \( E \). The goal is to study the properties of these integral mean curvatures as functions of the lengths of the (semi)axes of the ellipsoid. We derive explicit formulas, very good approximations, and asymptotic results. In addition, some of our results can be viewed as isoperimetric results.

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for ellipsoids, and we conjecture generalizations to hold for arbitrary convex bodies.

In detail, we first write down a formula (3) expressing the surface area of $E$ in terms of an integral of a simple function over the sphere $S^{n-1}$. This formula will be used to deduce a number of results:

(1) The ratio of the surface area to the volume of $E$ (call this ratio $R(E)$) is a norm on the vectors of inverse semi-axes. (Theorem 1).

(2) By a simple transformation (introduced for this purpose in [7], though doubtlessly known for quite some time) $R(E)$ can be expressed as a moment of a sum of independent Gaussian random variables; this transformation can be used to evaluate or estimate quite a number of related spherical integrals (see Section 2).

(3) Sharp bounds (15) on the ratio of $R(E)$ to the $L^2$ norm of the vectors of inverses of semi-axes are derived.

(4) We write down a very simple asymptotic formula (Theorem 11) for the surface area of an ellipsoid of a very large dimension with “not too different” axes. In particular, the formula holds if the ratio of the lengths of any two semiaxes is bounded by some fixed constant (Corollary 12).

(5) Finally, we give an identity expressing the surface area of $E$ as a linear combination of Lauricella hypergeometric functions.

We then go on to give similar explicit formulas for the higher mean curvature integrals of ellipsoids, by first computing the volumes of projections of ellipsoids onto subspaces (Sections 9 and 10) and then writing down a simple approximation (Theorem 20) for the $k$-th integral mean curvature of an ellipsoid. This estimate (for a fixed $k$) does not differ from the true value of the $k$-th mean curvature by more than a (dimension independent) constant factor. The worst possible functional dependence of our estimate on the dimension $n$ is $O(n^{1/4})$, which comes to pass when $k = n/2$. Our estimates on the error are sharp. Unfortunately, it seems difficult to derive a law of large numbers (as we describe above for the surface area – the 0-th integral mean curvature).

In Section 6 we comment on some historical antecedents of our work, and in Section 11 we interpret some of our inequalities as isoperimetric inequalities.
Notation. Let \((S, \mu)\) be a measure space with \(\mu(S) < \infty\). We will use the notation
\[
\int_S f(x) \, d\mu \overset{\text{def}}{=} \frac{1}{\mu(S)} \int_S f(x) \, d\mu.
\]
In addition, we shall denote the area of the unit sphere \(S^n\) by \(\omega_n\) and we shall denote the volume of the unit ball \(B^n\) by \(\kappa_n\).

1. Cauchy’s formula

Let \(K\) be a convex body in \(\mathbb{E}^n\). Let \(u \in S^{n-1}\) be a unit vector, and let us define \(V_u(K)\) to be the (unsigned) \(n-1\)-dimensional volume of the orthogonal projection of \(K\) in the direction \(U\). Cauchy’s formula (see [9, Chapter 13]) then states that
\[
V_{n-1}(\partial K) = \frac{n-1}{\omega_{n-2}} \int_{S^{n-1}} V_u(K) \, d\sigma = (n-1) \frac{\omega_{n-1}}{\omega_{n-2}} \int_{S^{n-1}} V_u(K) \, d\sigma,
\]
where \(d\sigma\) denotes the standard area element on the unit sphere.

In the case where \(K = E\) is an ellipsoid, given by
\[
E = \{ x \in \mathbb{E}^n \mid \sum_{i=1}^n q_i^2 x_i^2 \leq 1 \}
\]
the volume \(V_u\) of the projection is computed in Example 33 as a special case of more general projection results:
\[
V_u(E) = \kappa_{n-1} \sqrt{\frac{\sum_{i=1}^n u_i^2 q_i^2}{\prod_{i=1}^n q_i}}.
\]

Since
\[
V_u(E) = \frac{\kappa_n}{\prod_{i=1}^n q_i},
\]
we can rewrite Cauchy’s formula (1) for \(E\) in the form:
\[
\mathcal{R}(E) \overset{\text{def}}{=} \frac{V_{n-1}(\partial E)}{V_n(E)} = n \int_{S^{n-1}} \sqrt{\sum_{i=1}^n u_i^2 q_i^2} \, d\sigma,
\]
where \(\mathcal{R}(E)\) is the isoperimetric ratio of \(E\).

Theorem 1. The ratio \(\mathcal{R}(E)\) is a norm on the vectors \(q\) of lengths of semiaxes \((q = (q_1, \ldots, q_n))\).

Proof. The integrand in the formula (3) is a norm. \(\square\)
Corollary 2. There exist constants $c_{n,p}, C_{n,p}$, such that

$$c_{n,p} \|q\|_p \leq \mathcal{R}(q) \leq C_{n,p} \|q\|_p,$$

where $\|q\|_p$ is the $L^p$ norm of $q$.

Proof. Immediate (since all norms on a finite-dimensional Banach space are equivalent).

In the sequel we will find sharp bounds on the constants $c_{n,p}$ and $C_{n,p}$, but for the moment observe that if $a_i = 1 / q_i$, $i = 1, \ldots, n$, then

$$\|q\|_p = \prod_{i=1}^{n} q_i \sigma_{n-1}^{1/p}(\alpha_1^p, \ldots, \alpha_n^p),$$

where $\sigma_{n-1}$ is the $n-1$-st elementary symmetric function. In particular, for $p = 1$, Corollary 2 together with Eq. (4) gives the estimate of (8) (only for the 0-th mean curvature integral and with (for now) ineffective constants – the latter part will be remedied directly). To exploit the formula (3) fully, we will need a digression on computing spherical integrals.

2. Spherical integrals

In this section we will prove the following easy but very useful Theorem:

Theorem 3. Let $f(x_1, \ldots, x_n)$ be a homogeneous function on $\mathbb{E}^n$ of degree $d$ (in other words, $f(\lambda x_1, \ldots, \lambda x_n) = \lambda^d f(x_1, \ldots, x_n)$.) Then

$$\Gamma\left(\frac{n + d}{2}\right) \int_{S^{n-1}} f d\sigma = \Gamma\left(\frac{n}{2}\right) \mathbb{E}(f(X_1, \ldots, X_n)),$$

where $X_1, \ldots, X_n$ are independent random variables with probability density $e^{-x^2}$.

Proof. Let

$$E(f) = \int_{S^{n-1}} f(x) d\sigma,$$

and let $N(f)$ be defined as $\mathbb{E}(f(X_1, \ldots, X_n))$, where $X_i$ is a Gaussian random variable with mean 0 and variance 1/2, (so with probability density $n(x) = e^{-x^2}$) and $X_1, \ldots, X_n$ are independent. By definition,

$$N(f)(n) = c_n \int_{\mathbb{E}^n} \exp\left(-\sum_{i=1}^{n} x_i^2\right) f(x_1, \ldots, x_n) dx_1 \ldots dx_n,$$
where \( c_n \) is such that

\begin{equation}
(6) \quad c_n \int_{E^n} \exp\left(-\sum_{i=1}^{n} x_i^2\right) \, dx_1 \ldots dx_n = 1.
\end{equation}

We can rewrite the expression (5) for \( N(f) \) in polar coordinates as follows (using the homogeneity of \( f \)):

\begin{equation}
(7) \quad \text{Nmin}(n) = c_n \text{vol} S^{n-1} \int_0^\infty e^{-r^2 r^{n+d-1}} \, dr = c_n E(f) \int_0^\infty e^{-r^2 r^{n+d-1}} \, dr.
\end{equation}

Since, by the substitution \( u = r^2 \),

\[
\int_0^\infty e^{-r^2 r^{n+d-1}} \, dr = \frac{1}{2} \int_0^\infty e^{-u^{(n+d-2)/2}} \, du = \frac{1}{2} \Gamma\left(\frac{n+d}{2}\right).
\]

and Eq. (6) can be rewritten in polar coordinates as

\[
1 = c_n \text{vol} S^{n-1} \int_0^\infty r^{n-1} \, dr = \frac{c_n \text{vol} S^{n-1}}{2} \Gamma\left(\frac{n}{2}\right),
\]

we see that

\[
\Gamma\left(\frac{n+d}{2}\right) E(f) = \Gamma\left(\frac{n}{2}\right) N(f).
\]

Remark 4. In the sequel we will frequently be concerned with asymptotic results, so it is useful to state the following asymptotic formula (which follows immediately from Stirling’s formula):

\begin{equation}
(8) \quad \lim_{x \to \infty} \frac{\Gamma(x+y)}{\Gamma(x)(x+y)^y} = 1.
\end{equation}

It follows that for large \( n \) and fixed \( d \),

\begin{equation}
(9) \quad \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+d}{2}\right)} \sim \left(\frac{2}{n+d}\right)^{d/2}.
\end{equation}

3. An explicit formula for the surface area

The Theorem in the preceding section can be used to give explicit formulas for the surface area of an ellipsoid (this formula will not be used in the sequel, however). Specifically, in the book [4] there are formulas for the moments of of random variables which are quadratic
forms in Gaussian random variables. We know that for our ellipsoid \(E\),
\[
R(E) = n \int_{S^{n-1}} \sqrt{\sum_{i=1}^{n} u_i^2 q_i^2} \, d\sigma = n \frac{\Gamma \left( \frac{n}{2} \right)}{\Gamma \left( \frac{n+1}{2} \right)} \mathbb{E} \left( \sqrt{q_1^2 X_1 + \cdots + q_n^2 X_n} \right),
\]
where \(X_i\) is a Gaussian with variance \(1/2\). The expectation in the last expression is the \(1/2\)-th moment of the quadratic form in Gaussian random variables, and so the results of [4, p. 62] apply verbatim, so that we obtain:
(10)
\[
R(E) = n \frac{\Gamma \left( \frac{n}{2} \right)}{\Gamma \left( \frac{n+1}{2} \right) \Gamma \left( \frac{1}{2} \right)} \sqrt{\alpha} \int_0^\infty 1 \, \sum_{j=1}^{n} q_j^2 \left( \prod_{j=1}^{n} (1 - q_j^2 z) \right)^{-1/2} \, dz;
\]
note that \(\alpha\) in the above formula can be any positive number (as long as \(|1 - \alpha q_j^2| < 1\), for all \(j\).

This can also be expressed in terms of special functions. First, we need a definition:

**Definition 5.** Let \(a, b_1, \ldots, b_n, c, x_1, \ldots, x_n\) be complex numbers, with \(|x_i| < 1\), \(i = 1, \ldots, n\), \(\Re a > 0\), \(\Re (c - a) > 0\). We then define the Lauricella Hypergeometric Function \(F_D(a; b_1, \ldots, b_n; c; x_1, \ldots, x_n)\) as follows:

(11)
\[
F_D(a; b_1, \ldots, b_n; c; x_1, \ldots, x_n) = \frac{\Gamma(c)}{\Gamma(a) \Gamma(c - a)} \int_0^1 u^{a-1} (1 - u)^{c-a-1} \prod_{i=1}^{n} (1 - ux_i)^{-b_i} \, du.
\]
We also have the series expansion:

(12)
\[
F_D(a; b_1, \ldots, b_n; c; x_1, \ldots, x_n) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{(a)_{m_1+\cdots+m_n}}{c_{m_1+\cdots+m_n}} \prod_{i=1}^{n} (b_i)_{m_i} \prod_{i=1}^{n} x_i^{m_i} / m_i!;
\]
valid whenever \(|x_i| < 1\), \(\forall i\).

Now, we can write

(13)
\[
R(E) = n \frac{\Gamma^2 \left( \frac{n}{2} \right)}{\Gamma^2 \left( \frac{n+1}{2} \right)} \sqrt{\alpha} \times \sum_{j=1}^{n} q_j^2 F_D \left( 1/2; \eta_{ij}, \ldots, \eta_{nj}; n+1/2; 1 - \alpha q_j^2, \ldots, 1 - \alpha q_n^2 \right),
\]
where \( \eta_{ij} = 1/2 + \delta_{ij} \), and \( \alpha \) is a positive parameter satisfying \( |1 - \alpha q^2| < 1 \).

4. Laws of large numbers

Many of the results in this section will require the following basic lemmas.

**Lemma 6.** Let \( F_1, \ldots, F_n, \ldots \) be a sequence of probability distributions whose first moments converge to \( \mu \) and whose second moments converge to 0. then \( F_i \) converge to the Dirac delta function distribution centered on \( \mu \).

**Proof.** Follows immediately from Chebyshev’s inequality. \( \square \)

**Lemma 7.** Suppose the distributions \( F_1, \ldots, F_n, \ldots \) converge to the distribution \( F \), and the expectations of \( |x|^\alpha \) with respect to \( F_1, \ldots, F_n, \ldots \) are bounded. Then the expectation of \( |x|^\beta \), \( 0 \leq \beta < \alpha \) converges to the expectation of \( |x|^\beta \) with respect to \( F \).

**Proof.** See [2, pp. 251-252]. \( \square \)

**Theorem 8.** Let \( Y_1, \ldots, Y_n, \ldots \) be independent random variables with means \( 0 < \mu_1, \ldots, \mu_n, \ldots < \infty \) and variances \( \sigma^2_1, \ldots, \sigma^2_n, \ldots < \infty \) such that

\[
\lim_{n \to \infty} \frac{\sum_{i=1}^{n} \sigma_i^2}{\left( \sum_{i=1}^{n} \mu_i \right)^2} = 0.
\]

Then

\[
\lim_{n \to \infty} \mathbb{E} \left( \frac{Y_1 + \cdots + Y_n}{\sum_{i=1}^{n} \mu_i} \right)^\alpha = 1,
\]

for \( \alpha < 2 \).

**Proof.** Consider the variable

\[
Z_n = \frac{\sum_{i=1}^{n} Y_i}{\sum_{i=1}^{n} \mu_i}.
\]

It is not hard to compute that

\[
\sigma^2(Z_n) = \frac{\sum_{i=1}^{n} \sigma_i^2}{\left( \sum_{i=1}^{n} \mu_i \right)^2},
\]

while

\[
\mu(Z_n) = 1,
\]

so by assumption (14) and Lemma 6, \( Z_n \) converges in distribution to the delta function centered at 1. The conclusion of the Theorem then follows from Lemma 7. \( \square \)
Lemma 9. Let $X$ be normal with mean $0$ and variance $1/2$ (so probability density $e^{-x^2}/\sqrt{\pi}$.) Then

$$\mathbb{E}(|X|^p) = \frac{\Gamma\left(\frac{p+1}{2}\right)}{\sqrt{\pi}}.$$  

Proof.

$$\mathbb{E}(|X|^p) = \frac{2}{\sqrt{\pi}} \int_0^\infty x^p e^{-x^2} \, dx = \frac{1}{\sqrt{\pi}} \int_0^\infty u^{(p-1)/2} e^{-u} = \frac{\Gamma\left(\frac{p+1}{2}\right)}{\sqrt{\pi}}.$$

\qed

Theorem 10.

$$\int_{S^{n-1}} ||u||_p \, d\sigma \sim \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} \left( \frac{\Gamma\left(\frac{n+1}{2}\right)}{n \sqrt{\frac{1}{2} \sum_{i=1}^n q_i^2}} \right)^\frac{1}{p}.$$  

Proof. This follows immediately from the 1-homogeneity of the $L^p$ norm, the results of Section 2, Theorem 8, and Lemma 9. \qed

4.1. Asymptotics of $\mathbb{R}(E)$.

Theorem 11. Let $q_1, \ldots, q_n, \ldots$ be a sequence of positive numbers such that

$$\lim_{n \to \infty} \frac{\sum_{i=1}^n q_i^4}{\left(\sum_{i=1}^n q_i^2\right)^2} = 0.$$  

Let $E_n$ be the ellipsoid in $\mathbb{R}^n$ with semiaxes $a_1 = 1/q_1, \ldots, a_n = 1/q_n$. Then

$$\lim_{n \to \infty} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \frac{\mathbb{R}(E_n)}{n \sqrt{\frac{1}{2} \sum_{i=1}^n q_i^2}} = 1.$$  

Proof. The Theorem follows immediately from Theorem 8 and the results of Section 2. \qed

Corollary 12. Let $a_1, \ldots, a_n, \ldots$ be such that $0 < c_1 \leq a_i/a_j \leq c_2 < \infty$, for any $i, j$. Let $E_n$ be the ellipsoid with major semi-axes $a_1, \ldots, a_n$. Then

$$\lim_{n \to \infty} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \frac{\mathbb{R}(E_n)}{n \sqrt{\frac{1}{2} \sum_{i=1}^n \frac{1}{a_i^2}}} = 1.$$  

Proof. The quantities $q_1 = 1/a_1, \ldots, q_n = 1/a_n, \ldots$ clearly satisfy the hypotheses of Theorem 11. \qed
5. **General bounds on** $\mathbb{R}(E)$

We know that $\mathbb{R}(E)$ is a norm on the vector $\mathbf{q} = (q_1, \ldots, q_n)$ – let us agree to write

$$\|\mathbf{q}\|_\mathbb{R} \overset{\text{def}}{=} \frac{\mathbb{R}(E)}{n} = \int_{S^{n-1}} \sqrt[n]{\sum_{i=1}^{n} q_i^2 x_i^2} \, d\sigma,$$

where $\mathbf{q}$ is the vector of inverses of the major semi-axes of $E$.

We know that

$$c_n \|\mathbf{q}\| \leq \|\mathbf{q}\|_\mathbb{R} \leq C_n \|\mathbf{q}\|,$$

for some dimensional constants $c_n, C_n$. In this section we will give sharp estimates on the constants $c_n$ and $C_n$.

These estimates will depend on the following observation:

**Lemma 13.** Let $\alpha_1, \ldots, \alpha_n$ be nonnegative real numbers, and let

$$f(\alpha_1, \ldots, \alpha_n) = \int_{S^{n-1}} \sqrt[n]{\sum_{i=1}^{n} \alpha_i x_i^2} \, d\sigma.$$

Then $f(\alpha)$ is a concave function of the vector $\alpha = (\alpha_1, \ldots, \alpha_n)$.

**Proof.** The integrand is concave, since the square root is a concave function. The integral is thus also concave, as a sum of concave functions. \hfill \Box

**Lemma 14.** The ratio

$$\frac{\|\mathbf{q}\|_\mathbb{R}}{\|\mathbf{q}\|}$$

is maximized when all of the $q_i$ are equal; it is minimized when $q_2 = \cdots = q_n = 0$.

**Proof.** By homogeneity, we can assume that $\|\mathbf{q}\| = 1$. Now, let $\alpha_i = q_i^2$. Letting

$$f(\alpha) = \|\mathbf{q}\|_\mathbb{R},$$

we see that $f$ is a symmetric function, while Lemma 13 tells us that $f(\alpha)$ is a concave function. Since the set

$$S : \sum_{i=1}^{n} \alpha_i = 1$$

is convex, we know that the maximum of $f$ is attained at the point of maximum symmetry ($\alpha_i = 1/n$, for all $i$), and the minimum at an extreme point of $S$ – by symmetry any extreme point will do, for example $(1, 0, \ldots, 0)$. \hfill \Box
Corollary 15. The minimal value (previously denoted by $c_n$) of $\|q\|_R/\|q\|$ equals

$$\int_{S^n} |x_1| d\sigma = \frac{\Gamma \left( \frac{n}{2} \right)}{\sqrt{\pi} \Gamma \left( \frac{n+1}{2} \right)},$$

while the maximal value ($C_n$) of $\|q\|_R/\|q\|$ equals $1/\sqrt{n}$, or in other words,

(15) $$\frac{\Gamma \left( \frac{n}{2} \right)}{\sqrt{\pi} \Gamma \left( \frac{n+1}{2} \right)} \leq \|q\|_R/\|q\| \leq \frac{1}{\sqrt{n}}.$$

Remark 16. The left hand side of equation (15) is asymptotic to

$$\sqrt{\frac{2}{(n+1)\pi}},$$

(by (9)) so the ratio $C_n/c_n$ approaches $\sqrt{2/\pi} = 0.797$ as $n$ goes to infinity.

6. SOME HISTORICAL REMARKS

The perimeter of an ellipse has been studied since at least Fagnano (1716). The best approximation has been obtained by Ramanujan in 1914. The surface area of an ellipsoid was studied by Monge [5] and Legendre [3] by means of elliptic integrals. Monge also gave an approximate formula (as a series) which converges as long as the ellipsoid is not too round; Legendre gave a generally convergent series. Interesting estimates (also in dimension 3), especially for the mean curvature of the ellipsoid were given by G. Polya and G. Szegő in [6]. Almost none of the methods in the references cited above seem to extend to dimension higher than three. It would be interesting to extend the results and method of the current article to higher integral mean curvatures of ellipsoids, as studied in [8].

7. HIGHER MEAN CURVATURES

In the sequel we will denote the surface area of the $n$-dimensional unit sphere by $\omega_n$, and the volume of the $n$-dimensional unit ball by $\kappa_n$. We recall that:
\[ \omega_n = \frac{2\pi^{(n+1)/2}}{\Gamma\left(\frac{n+1}{2}\right)}, \]
\[ \kappa_n = \frac{\omega_{n-1}}{n} = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}. \]

We will use \( M_k(K) \) to denote the integral \( k \)-th mean curvature of the boundary of a convex body \( K \). Recall that \( M_k(B^n(1)) = \omega_{n-1} \). \( B^n(R) \) is the unit ball of radius \( R \) in \( \mathbb{E}^n \). The following result can be found in Santaló’s book \[9\]:

\[ M_k^{(n)}(B^{n-k}) = \frac{\omega_k \omega_{n-k-2}}{(n-k-1)(n-k)}, \]

where \( M_k^{(n)}(K) \) denotes the \( k \)-th integral mean curvature of \( K \) viewed as a convex body in \( \mathbb{E}^n \).

We will also need the following:

**Theorem 17.** Let \( n - 1 > k > 1 \). Then

\[ \frac{M_k(B^n(R))}{M_k^{(n)}(B^{n-k-1}(R))} = 2(k-1)\pi^{3/2} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{k}{2}\right)\Gamma\left(\frac{n-k}{2}\right)}. \]

The proof will rely on Legendre’s duplication formula:

\[ \Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z) \Gamma(z + 1/2). \]
Proof of Theorem 17. Using the formulas for the surface area of the sphere and the $k$-th mean curvature of $B^n_k$ we write:

\[
\frac{M_k(B^n(R))}{M_k^{(n)}(B^{n-k-1}(R))} = (n-k-1) \left( \frac{n-1}{k} \right) \frac{\omega_{n-1}}{\omega_k \omega_{n-k-2}} = \frac{1}{2} (n-k-1) \left( \frac{n-1}{k} \right) \pi \frac{\Gamma \left( \frac{k+1}{2} \right) \Gamma \left( \frac{n-k-1}{2} \right)}{\Gamma \left( \frac{n-k}{2} \right)} =
\]

\[
\frac{1}{2} (n-k-1) \left( \frac{n-1}{k} \right) \pi \frac{\Gamma \left( \frac{k+1}{2} \right) \Gamma \left( \frac{n-k-1}{2} \right)}{\Gamma \left( \frac{n-k}{2} \right)} =
\]

\[
\frac{\Gamma \left( \frac{k+1}{2} \right) \Gamma \left( \frac{n-k-1}{2} \right)}{\Gamma \left( \frac{n-k}{2} \right)} =
\]

\[
\frac{\pi \sqrt{\pi}}{2} \frac{\Gamma \left( \frac{k+1}{2} \right) \Gamma \left( \frac{n-k-1}{2} \right)}{\Gamma \left( \frac{n-k}{2} \right) \Gamma \left( \frac{n-k-1}{2} \right)} =
\]

\[
\frac{(k-1) \pi}{4} \frac{\Gamma \left( \frac{k-1}{2} \right) \Gamma \left( \frac{n-k}{2} \right)}{\Gamma \left( \frac{n-k-1}{2} \right) \Gamma \left( \frac{n-k-1}{2} \right)}.
\]

We have used the assumption that $k > 1$ to factor out $k-1$ in the last line, and that $n-1 > k$ to factor out the $(n-k-1)$ in the first line.

Now we apply Legendre’s duplication formula (17) with $z = n/2$, $z = (k-1)/2$, and $z = (n-k-1)/2$, to get:

\[
\frac{\Gamma \left( \frac{n}{2} \right)}{\Gamma \left( \frac{n-1}{2} \right)} = \frac{2^{n-1}}{\sqrt{\pi}} \Gamma \left( \frac{n+1}{2} \right),
\]

\[
\frac{\Gamma \left( \frac{k-1}{2} \right)}{\Gamma \left( \frac{k-2}{2} \right)} = \frac{\sqrt{\pi}}{2^{k-2}} \frac{1}{\Gamma \left( \frac{k}{2} \right)},
\]

\[
\frac{\Gamma \left( \frac{n-k-1}{2} \right)}{\Gamma \left( \frac{n-k-1}{2} \right)} = \frac{\sqrt{\pi}}{2^{n-k-2}} \frac{1}{\Gamma \left( \frac{n-k}{2} \right)}.
\]

Substituting back into (18), we see that:

\[
\frac{M_k(B^n(R))}{M_k^{(n)}(B^{n-k-1}(R))} = 2(k-1) \pi^{3/2} \frac{\Gamma \left( \frac{n+1}{2} \right)}{\Gamma \left( \frac{k-1}{2} \right) \Gamma \left( \frac{n-k-1}{2} \right)}.
\]

□
Theorem 18. Let \( k > 1 \). Then
\[
\frac{M_k (B^n (R))}{M_k^{(n)} (B^{n-k-1} (R))} = \frac{k - 1}{\sqrt{k(k+1)}} \frac{\Gamma \left( \frac{n+1}{2} \right)}{\Gamma \left( \frac{n}{2} + 1 \right)} \frac{\Gamma \left( \frac{k+1}{2} \right)}{\Gamma \left( \frac{k}{2} \right)} \frac{\Gamma \left( \frac{n-k+1}{2} \right)}{\Gamma \left( \frac{n-k}{2} \right)} \sqrt{n/k+1} \sqrt{\frac{n}{k+1}}.
\]

Proof. First, we write
\[
\binom{n}{k+1} = \frac{\Gamma(n+1)}{\Gamma(k+2)\Gamma(n-k)}.
\]
and then, using Legendre’s duplication formula and the functional equation of the \( \Gamma \) function:
\[
\Gamma(n+1) = \frac{2^n}{\sqrt{\pi}} \Gamma \left( \frac{n}{2} + 1 \right),
\]
\[
\Gamma(k+2) = (k+1)k\Gamma(k) = (k+1)k\frac{2^{k-1}}{\sqrt{\pi}} \Gamma \left( \frac{k+1}{2} \right) \Gamma \left( \frac{k}{2} \right),
\]
\[
\Gamma(n-k) = \frac{2^{n-k-1}}{\sqrt{\pi}} \Gamma \left( \frac{n-k}{2} \right) \Gamma \left( \frac{n-k+1}{2} \right).
\]
The result follows by combining the above with the result of Theorem 17. \(\square\)

Remark 19. It is not hard to see that for a fixed \( k \), the right hand side of Eq. (19) approaches
\[
C(k) = \pi^{5/4} \frac{k - 1}{\sqrt{k(k+1)}} \sqrt{\frac{\Gamma((k+1)/2)}{\Gamma(k/2)}}
\]
as \( n \to \infty \). For a fixed \( m = n - k \) (but both \( k \) and \( n \) tending to \( \infty \)), the right hand side approaches
\[
D(m) = \pi^{5/4} \left( \frac{n-k+1}{2} \right)^{1/4}.
\]
Finally, if \( n, k, \) and \( n-k \) all approach infinity, the expression is asymptotic to
\[
B(n, k) = \pi^{5/4} \left( \frac{(k+1)(n-k+1)}{2(n+2)} \right)^{1/4}.
\]
It is not hard to see that for a given \( n \) \( B(n, k) \) is maximized when \( k = n/2 \), in which case
\[
B(n, n/2) = \left( \frac{n + 2}{8} \right)^{1/4}.
\]

8. Kubota’s formula

Cauchy’s formula expresses the surface area of a convex body \( K \) in terms of the average volume of the projections of \( K \) onto codimension 1 subspaces. Kubota’s Theorem (see \[9, Chapter 13\]) is a generalization, which expresses the \( k \)-th integral mean curvature in terms of the average volume of projections of \( K \) onto \( n - k - 1 \) dimensional subspaces:
\[
M_k(\partial K) = \frac{(n - r - 1)\omega_{n-1}}{\omega_{n-k-2}} \int_{G(n,n-k-1)} \text{vol}_{n-k-1}(P_x K) dx,
\]
where \( G(n, n - k - 1) \) is the Grassmannian of \( n - k - 1 \) dimensional linear subspaces of \( \mathbb{E}^n \), and \( P_x \) is the projection onto the subspace \( x \).

In the special case where \( K \) is an ellipsoid \( E \) with axes \( a_1, \ldots, a_n \), Theorem 30 gives us several explicit expressions for the integrand in Kubota’s formula. For the purposes of the next Theorem, Eq. (28b) will be the most useful.

**Theorem 20.** Let \( E \) be an ellipsoid with axes \( a_1, \ldots, a_n \). Let \( a_i \), for a multi-index \( i = (i_1, \ldots, i_{n-k-1}) \) be defined as:
\[
a_i = \prod_{l=1}^{n-k-1} a_{i_l},
\]
and let
\[
\mathcal{A} = \sqrt{\sum_{i} a_i^2},
\]
where the sum is taken over increasing multindices. Then
\[
M_k^{(n)}(B^n(1))\mathcal{A} \leq M_k(E) \leq \frac{M_k(B^n(1))}{\sqrt{\binom{n}{k+1}}} \mathcal{A}.
\]

**Proof.** The proof is identical to the proof of Lemma 14 except we use \( a_i \) as variables. With the normalization \( \mathcal{A} = 1 \) (allowed by homogeneity) we see that the maximal case corresponds to the ball of such a radius that \( a_i = 1/\sqrt{\binom{n}{n-k-1}} \), for any multi-index \( i \), and the minimum corresponds to \( a_1 = \ldots = a_{n-k-1} = 1 \), while \( a_{n-k} = \ldots = a_n = 0 \). □
The ratio of the right hand side of the inequality (21) to the left hand side is the subject of Theorem 18 and Remark 19. As commented in the Remark, the ratio is bounded for any fixed $k$, and in the worst case (for $k = n/2$), the ratio grows like $n^{1/4}$.

9. Some exterior algebra

Let $V$ be a vector space, and let $A$ be a linear transformation:

$$A \in \text{Hom}(V, V).$$

The exterior power $\bigwedge^k V$ is the vector space generated by multivectors of the form $v_1 \wedge \ldots \wedge v_k$, and so we define

$$\bigwedge^k A \in \text{Hom} \left( \bigwedge^k V, \bigwedge^k V \right)$$

by

$$\bigwedge^k A(v_1 \wedge \ldots \wedge v_k) = Av_1 \wedge \ldots \wedge Av_k.$$

From now on, we assume that $V$ is an $n$-dimensional Hilbert space. The vector space $\bigwedge^k V$ has a standard orthonormal basis: all multivectors of the form $e_{i_1} \wedge \ldots \wedge e_{i_k}$, where the $e_{i_l}$ are the standard orthonormal basis vectors in $V$, and $i_r \neq i_s$ for $r \neq s$. For notational convenience, we will henceforth denote such multi-indices by bold latin letters. In addition, if $i$ is a $k$-multindex, we define the $n-k$-multindex $\overline{i}$ by

$$e_1 \wedge e_{\overline{i}} = e_1 \wedge e_2 \ldots \wedge e_n.$$

Remark 21. Riemannian geometers would say that $e_{\overline{i}}$ is the image of $e_i$ by the Hodge $\ast$ operator.

Lemma 22. If $j \neq i$, then

$$e_j \wedge e_{\overline{i}} = 0.$$

Proof. One of the coordinates of $j$ must be the same as one of the coordinates of $\overline{i}$. □

In the sequel, we will use the following easy observation:

Lemma 23. Let $v \in \bigwedge^k V$, and let $i$ be a $k$-multindex. Then

$$\langle v, e_i \rangle = \frac{v \wedge e_{\overline{i}}}{e_1 \wedge \ldots \wedge e_n}.$$

Proof. Expand $v$ in coordinates; the result follows immediately from Lemma 22. □
Lemma 24 (Binet-Cauchy formula). Let $A, B \in \text{Hom}(V, V)$. Then,

$$\langle \bigwedge^k (AB)e_j, e_k \rangle = \frac{1}{k!} \sum_{\text{all } k\text{-multindices } i} \langle \bigwedge^k Ae_j, e_i \rangle \langle \bigwedge^k Be_i, e_k \rangle.$$ 

Proof. This is just the usual formula for matrix multiplication applied in the space $\bigwedge^k V$. □

We can use the results above to give some identities for projections:

9.1. On projections.

Theorem 25. Let $P$ and $Q$ be such that:

(1) $\text{rank } P = k$.
(2) $\text{rank } Q = n - k$.
(3) $P + Q = I$.

Then

(22) $$\langle \bigwedge^k P e_i, e_k \rangle = \langle \bigwedge^{n-k} Q e_i, e_k \rangle$$

Proof. 

$$\langle \bigwedge^k P e_i, e_k \rangle = \langle \bigwedge^k P e_i \wedge e_k \rangle = \langle \bigwedge^k P e_i \wedge \bigwedge^{n-k}(P+Q)e_k \rangle = \langle \bigwedge^k P e_i \wedge \bigwedge^{n-k} Q e_k \rangle = \langle \bigwedge^k (P+Q)e_i \wedge \bigwedge^{n-k} Q e_k \rangle = \langle \bigwedge^k e_i \wedge \bigwedge^{n-k} Q e_k \rangle = \langle \bigwedge^{n-k} Q e_i, e_k \rangle,$$

where we have used the observation that $\bigwedge^l P = 0$, whenever $l > \text{rank } P$. □
Suppose $W$ is a subspace of $V$, and let $w_1, \ldots, w_k$ be an orthonormal basis of $W$. Let $\Omega$ be the matrix whose columns are the vectors $(w_1, \ldots, w_k, 0, \ldots, 0)$ (padding $\Omega$ by zeros is not really necessary, but it will make the sequel slightly simpler notationally). We then have the following:

**Lemma 26.** Let $P$ be the orthogonal projection onto $W$. Then

$$P = \Omega \Omega^t.$$ 

**Proof.** The proof is by direct computation: we will show that $Q = \Omega \Omega^t$ is the sought-after projector. First, let $v$ be orthogonal to all of $W$. Then, it is clear that $\Omega^tv = 0$, and so $Qv = 0$. Now, consider $Qw_i$. First, $\Omega^tw_i = e_i$. Now, for any matrix $A$, $Ae_i$ is the $i$-th column of $A$. In particular, $\Omega e_i = w_i$, and so $Qw_i = w_i$, for all $i$. It follows that $Q$ is the sought-after projector. \qed

**Corollary 27.** Let $P$ and $\Omega$ be as above. Then

$$\left\langle \bigwedge^k P e_i, e_i \right\rangle = \left\langle \bigwedge^k \Omega e_i, e_i \right\rangle^2.$$ 

**Proof.** This is an immediate consequence of Lemma 26 above and Lemma 24. \qed

The following can be viewed as a generalization of the Pythagorean theorem:

**Theorem 28** (Generalized Pythagorean Theorem). Let $\Omega$ be as above. Then the sum of squares of $k \times k$ minors of $\Omega$ equals 1.

**Proof.** By examination of the characteristic polynomial of $P$, the product of the non-zero eigenvalues of $P$ equals the sum of the principal $k \times k$ minors, which, by corollary 27 equals the sum of squares of the $k \times k$ minors of $\Omega$. However, since $P$ is a projection, its non-zero eigenvalues are all equal to 1. \qed

10. HOW TO COMPUTE THE VOLUME OF A PROJECTED ELLIPSOID

First, consider a generalized ellipsoid $E(A)$ – the image of the unit ball in $\mathbb{R}^n$ under a linear transformation of rank $k$. We would like to know the $k$-dimensional volume of $E(A)$. The simplest situation is when

$$A_{ij} = \begin{cases} \lambda_i, & i = j, i \leq k, \\ 0, & \text{otherwise} \end{cases}$$

(23)
In this case,

\[ \text{vol}_k(E(A)) = \kappa_k \prod_{i=1}^{k} \lambda_i. \]

The general case is not much different: Any \( A \) of rank \( k \) can be written as \( U\Sigma V \), where \( V \in O(n) \), and \( U \) is in \( O(k) \subset O(k) \), while \( \Sigma \) is the diagonal matrix of type described in Eq. (23); the diagonal entries of \( \Sigma \) are the singular values of \( A \), which can be alternately described as the positive square roots of the (nonzero) eigenvalues of either \( A' A \) or \( A A' \). Let us state this as a theorem:

**Theorem 29.** Let \( E(A) \) be the image of the unit ball in \( \mathbb{R}^n \) under a transformation \( A \) of rank \( k \). Then

\[ \text{vol}_k(E(A)) = \kappa_k \prod_{i=1}^{k} \sigma_i, \]

where \( \sigma_1, \ldots, \sigma_k \) are the singular values of \( A \).

Now, we note that for a matrix \( M \) of rank \( k \), the product of the non-zero eigenvalues of \( M \) equals the sum of \( k \times k \) principal minors of \( M \) (this is immediate by examining the characteristic polynomial of \( M \). Thus Eq. (25) can be rewritten as:

\[ \text{vol}_k^2(E(A)) = \kappa_k \sum_{\text{principal submatrices } M \text{ of } A' A} \det M. \]

This last form is superior to Eq. (25), since it expresses the square of the volume as a polynomial in the entries of \( A \).

10.1. **How do we compute the volume of a projection of an ellipsoid?** Here we consider a special case: we take a non-degenerate ellipsoid \( E(\mathbb{I}) \), where, for simplicity, \( \mathbb{I} = \text{diag} a_1, \ldots, a_n \), and we would like to compute the volume of the projection of \( E(\mathbb{I}) \) onto a \( k \)-dimensional subspace \( W \) with the associated projector \( P \). In other words, we want to compute the volume of \( E(P\mathbb{I}) \). With the notation \( A = P\mathbb{I} \) we note that \( A' A = \mathbb{I} P \mathbb{I} \). To use the formula (26) we first note that if \( i \) is a multindex, then we have the following expression for the minors of \( A' A \):

\[ (\mathbb{I} P \mathbb{I})_i = a_i^2 P_i, \]
where, if $i = (i_1, \ldots, i_k)$, then

$$a_i = \prod_{l=1}^{k} a_{i_l}.$$  

We then have the following:

**Theorem 30** (Measure of ellipsoid projections). Let $E$ be the ellipsoid

$$E = \left\{ x \in \mathbb{R}^n \mid \sum_{i=1}^{n} \frac{x_i^2}{a_i^2} \leq 1 \right\}$$

Let $W$ be a subspace of $\mathbb{R}^n$, with an orthonormal basis $w_1, \ldots, w_k$, while $W^\perp$ has the orthonormal basis $w_{k+1}, \ldots, w_n$. Let $\Omega$ be the $n \times k$ matrix whose rows are the vectors $w_1, \ldots, w_k$, while $\Omega^\perp$ be the $n \times (n-k)$ matrix whose rows are $w_{k+1}, \ldots, w_n$. Let the projection onto $W$ be denoted by $P$, while the projection onto $W^\perp$ be denoted by $P^\perp$. Then, the $k$-dimensional volume $\text{vol}_k P(E)$ of the projection of $E$ onto $W$ can be expressed in any one of the following ways (all the sums below are taken over nondecreasing multindices $=(i_1, \ldots, i_k), \quad i_1 \leq i_2 \leq \ldots \leq i_k$):

\begin{align}
(28a) \quad \text{vol}_k P(E) &= \kappa_k \sqrt{\sum_{\text{nondecreasing } k\text{-multindices } i} P_i a_i^2}. \\
(28b) \quad \text{vol}_k P(E) &= \kappa_k \sqrt{\sum_{\text{nondecreasing } k\text{-multindices } i} \Omega_i^2 a_i^2}. \\
(28c) \quad \text{vol}_k P(E) &= \kappa_k \prod_{i=1}^{n} a_i \sqrt{\sum_{\text{nondecreasing } n-k\text{-multindices } i} \frac{P_i^\perp}{a_i^2}}. \\
(28d) \quad \text{vol}_k P(E) &= \omega_k \prod_{i=1}^{n} a_i \sqrt{\sum_{\text{nondecreasing } n-k\text{-multindices } i} \frac{(\Omega_i^\perp)^2}{a_i^2}}.
\end{align}

**Proof.** The expression (28a) follows immediately from Eq. (26). The expression (28b) follows for Eq. (28a) and Corollary 27. The expression (28c) follows from Eq. (28a) and Theorem 22. The expression (28d) follows from Eq. (28c) and Corollary 27. \[ \square \]

**Remark 31.** The last two expressions (28c) and (28d) in the above theorem are more useful when $k > n/2$; the forms (28b) and (28d) are useful when the subspaces are given by their generating vectors,
while (28a) and (28c) are more useful when the subspaces are given by their projectors.

**Example 32.** Suppose \( k = 1 \), so we are projecting on a subspace spanned by a (unit) vector \( \mathbf{v} = (v_1, \ldots, v_n) \). Then, the length of the projection of our ellipsoid is (according to Eq. (28b))

\[
\sum_{i=1}^{n} v_i^2 a_i^2.
\]

**Example 33.** Suppose \( k = n - 1 \), so we are projecting onto the orthogonal complement of the subspace subspace spanned by \( \mathbf{v} \) of the previous example. Then, the \( n - 1 \)-dimensional volume of the projection of our ellipsoid is (according to Eq. (28d)):

\[
\kappa_{n-1} \prod_{i=1}^{n} a_i \sum_{j=1}^{n} \frac{v_j^2}{a_j^2}.
\]

This formula was previously obtained (by completely different methods) by Connelly and Ostro in [1].

**11. Isoperimetric questions**

Theorem 20 can be expressed as follows: let \( \mathcal{E} \) be the set of ellipsoids such that the squares of the \( n - k - 1 \) dimensional volumes of the projections onto coordinate \( n - k - 1 \) dimensional subspaces equals 1 Then the largest value of \( M_k(E) \) for \( E \in \mathcal{E}(\mathcal{A}) \) is achieved by the \( n \)-dimensional ball (of radius \( \left( \frac{n}{n-k-1} \right)^{-1/(n-k-1)} \)) while the minimal value of \( M_k(E) \) is achieved by any \( n - k - 1 \) dimensional ellipsoid parallel to one of the coordinate subspaces.

It is natural to ask whether the above statement holds with the word “ellipsoid” replaced by the word “convex body” throughout. I believe that the answer is in the affirmative, but it is clear that the methods of this paper do not apply to this question in this generality.

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