CONSTRUCTION OF POINCARÉ-TYPE SERIES BY GENERATING KERNELS

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Abstract. Let \( \Gamma \subset \text{PSL}_2(\mathbb{R}) \) be a Fuchsian group of the first kind having a fundamental domain with a finite hyperbolic area, and let \( \tilde{\Gamma} \) be its cover in \( \text{SL}_2(\mathbb{R}) \). Consider the space of twice continuously differentiable, square-integrable functions on the hyperbolic upper half-plane, which transform in a suitable way with respect to a multiplier system of weight \( k \in \mathbb{R} \) under the action of \( \tilde{\Gamma} \). The space of such functions admits the action of the hyperbolic Laplacian \( \Delta_k \) of weight \( k \). Following an approach of [JvPS16] (where \( k = 0 \)), we use the spectral expansion associated to \( \Delta_k \) to construct a wave distribution and then identify the conditions on its test functions under which it represents automorphic kernels and further gives rise to Poincaré-type series. An advantage of this method is that the resulting series may be naturally meromorphically continued to the whole complex plane. Additionally, we derive sup-norm bounds for the eigenfunctions in the discrete spectrum of \( \Delta_k \).

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1. Introduction

Let $\Gamma \subset \text{PSL}_2(\mathbb{R})$ be a Fuchsian group of the first kind having a fundamental domain $\mathcal{F}$ with a finite hyperbolic area. It acts on the complex upper half-plane $\mathbb{H}$ and the quotient space can be identified with the Riemann surface $M = \Gamma \backslash \mathbb{H}$.

We fix a real weight $k$, such that there exists a unitary multiplier system $\chi$ of weight $k$ on the cover $\tilde{\Gamma}$ of $\Gamma$ in $\text{SL}_2(\mathbb{R})$. Throughout the paper we will assume the weight $k$ and the multiplier system to be arbitrary but fixed.

The hyperbolic Laplacian of weight $k$ on $M$ is the operator

$$\Delta_k = -y^2(\partial_x^2 + \partial_y^2) + 2k iy\partial_x$$

acting on the space $D_k$ of all twice continuously differentiable, square-integrable functions on $\mathbb{H}$ which transform in a suitable way with respect to the weight $k$ unitary multiplier system $\chi$ on $\tilde{\Gamma}$ (a precise definition is given in Section 2).

The operator $\Delta_k$ is a special case of a differential operator investigated by Maass in [Maa52] (see also [Roe56]); in some papers (e.g. [Os90]), $\Delta_k$ is referred to as the Maass-Laplacian. It is the analogue of the non-Euclidean Laplacian for non-analytic automorphic forms on $\Gamma \backslash \mathbb{H}$ of weight $k$. Namely, the weighted Laplacian preserves the transformation behavior of functions from $D_k$ (cf. formula (7) below); it can be represented as a composition of differential operators of the first order (the lowering and raising Maass operators) mapping weight $k$ forms into forms of weight $k \pm 1$ (or $k \pm 2$, depending on the scaling taken); the kernel of $\Delta_k - k(1 - k)$ is isomorphic to the set of meromorphic differentials on $\mathcal{F}$ of weight $k$ with unitary multiplier system, see [Fi87, Section 1.4] and references therein.

The spectral theory of $\Delta_k$ was carefully developed in [Roe66, Roe67], [El73a, El73b] and [Pa75, Pa76a, Pa76b]. It was proved that the Hilbert space of all square-integrable functions on $\mathbb{H}$ which transform in a suitable way with respect to the weight $k$ unitary multiplier system on $\tilde{\Gamma}$ is a direct sum of countably many finite dimensional eigenspaces spanned by the eigenfunctions $\varphi_j$ associated to the discrete eigenvalues $\lambda_j$ of $\Delta_k$ and, in the case when the surface is non-compact, the Eisenstein series associated to each cusp and corresponding to the continuous spectrum.

In the seminal paper [Fay77], Fay computed the basic eigenfunction expansions (in both rectangular and hyperbolic polar coordinates) of the resolvent kernel for the operator $\Delta_k$ acting on the Hilbert space of real weight $k$ automorphic forms. He applied it to the construction of an automorphic “prime form” and automorphic functions with prescribed singularities. Following the ideas of Selberg [Se56], Hejhal developed the trace formula for $\Delta_k$ on the space $\Gamma \backslash \mathbb{H}$ and derived various applications of it, such as a distribution of pseudo-primes and the computation of the dimension of the space of classical cusp forms of weight $2k \in \mathbb{N}$.

1.1. Poincaré-type series. The resolvent kernel for $\Delta_k$ is the special case of Poincaré-type series, which is the series defined by summing over the group $\Gamma$ (or over its cover $\tilde{\Gamma}$) the weight $k$ point-pair invariant, multiplied by certain other factors depending on the element of $\Gamma$ (or $\tilde{\Gamma}$) in the index of the sum. Loosely speaking, the weight $k$ point-pair invariant is a function $k(z, w)$ of $z, w \in \mathbb{H}$ which is radial/spherical (meaning it depends only upon the hyperbolic distance between $z$ and $w$) and which transforms “nicely” with respect to the weight $k$ pseudo-action of $\tilde{\Gamma}$. In most applications (and so is the case with the resolvent kernel), the point-pair invariant is taken to depend upon one or two complex parameter(s) with large enough real part (to ensure the convergence of the series).
When the point-pair invariant is well-chosen, Poincaré-type series become very important objects of study. This is because their Fourier expansions (in different coordinates) together with a Laurent/Taylor series expansion in an additional complex variable $s$ (usually at $s = 1$ or $s = 0$) carry important information. For example, when the multiplier system is identity, the constant term in the Laurent series expansion of the resolvent kernel at $s = 1$ gives rise to a holomorphic automorphic “prime form”, see [Fay77, Theorem 2.3]. Such a form is an important object in the construction of automorphic forms with prescribed singularities. In case when $M$ is compact and the multiplier system is one-dimensional, the constant term in the Laurent series expansion of the resolvent kernel at $s = 1$ gives rise to a unique Prym differential with multipliers, see [Fay77, p. 163]. The resolvent kernel asymptotic is used more recently in [FJK19] with $k \in \frac{1}{2} \mathbb{N}$ to establish effective sup-norm bounds on average for weight $2k$ cusp forms for $\Gamma$.

In the series of papers [Br85, Br86a, Br86b], Bruggeman constructed families of Eisenstein and Poincaré series on the full modular group, which depend on a complex variable $s$ parameterizing the eigenvalue $1/4 - s^2$ of $\Delta_k$, and also depend continuously (in fact real-analytically) on the weight $k$ (which in this special case can be taken to belong to the interval $(0, 12)$). Bruggeman proved that, for $k \neq 0$, all square-integrable modular forms (of a certain type) occur in such families.

1.2. Our results. In this paper, we follow an approach of [JvPS16] and [CJS20] towards the construction of Poincaré-type series associated to the weighted Laplacian. It is similar to Bruggeman’s in the sense that we undertake the “operator” approach, looking at appropriate distributions. However, our starting point is the “wave distribution” (see Definition 16), a concept which does not appear in [Br85]–[Br86b].

We illustrate this approach in Section 6 by constructing a new Poincaré-type series $K_s(z, w)$, for $z, w \in \mathcal{F}$ and $\text{Re}(s) \gg 0$, which transforms “nicely” with respect to the weight $k$ multiplier system. We then obtain its meromorphic continuation with respect to the $s$ variable and deduce its representation in terms of the sum over the group of a certain point-pair invariant.

More precisely, starting with the spectral expansion theorem ([FS7, Theorem 1.6.4]), we construct the wave distribution associated to the weighted Laplacian. We prove in Proposition 17 that the wave distribution acts on a rather large space of test functions and that it can be represented as an integral operator with a certain kernel (Theorem 18). In Proposition 17, we also derive sufficient conditions on the test function so that the wave distribution acting on this test function produces an $L^2$-automorphic kernel.

To guarantee the absolute convergence of the aforementioned objects we need bounds for their discrete and, in case when the surface is non-compact, continuous spectra. It turned out that even though the spectral properties of $\Delta_k$ are well studied, both analytically and computationally (see [St08]), the properties of the eigenvalues associated to its discrete spectrum that are different from the minimal eigenvalue $|k|(1 - |k|)$ did not get much attention in the non-compact setting. This is probably because in the non-compact setting the discrete spectrum still remains very mysterious; for example it is not even known in general whether it is finite or infinite.

For that reason, in Section 4 we prove the sup-norm bound $\sup_{z \in \mathcal{F}} |\varphi_j(z)| \ll |\lambda_j|$ for the eigenfunctions associated to discrete eigenvalues $\lambda_j$ of $\Delta_k$, uniform in $j$. This result is of independent interest, because it is proved in a general setting of a possibly non-compact surface, real weight $k$ and vector-valued eigenfunctions $\varphi_j$. In the non-compact setting, we also derive the sup-norm bound for the growth of a certain weighted integral of the (vector-valued) Eisenstein series along the critical line (Proposition 12(b)).
The definition of the wave distribution enables one to construct automorphic kernels through the action of the wave distribution on suitably chosen test functions. In Section 6, a new $L^2$-automorphic kernel $K_s(z,w)$, called the basic automorphic kernel, is constructed through the action of the wave distribution on the test function $g_s(u) = \frac{\Gamma(s-1/2)}{\Gamma(s)} \cosh(u)^{(s-1/2)}$ for $\text{Re}(s) \gg 0$. The kernel $K_s(z,w)$ is called the basic kernel, because, as will be seen in [KKMMMS20], both the resolvent kernel and, consequently, the Eisenstein series, can be expressed in terms of this kernel and its translates in the $s$-variable. It is analogous to the basic automorphic kernel constructed in [CJS20] in the setting of smooth, compact, projective Kähler varieties.

Using the properties of the wave distribution, it is possible to construct Poincaré-type series that are not square-integrable by taking appropriate sums/integrals of the wave distribution (see e.g. [JvPS16, Section 7] in the special case of the multiplier system equal to 1). We leave this investigation to the subsequent paper [KKMMMS20].

This approach to the construction of Poincaré-type series has many advantages. Firstly, the construction depends only on the spectral properties of the Laplacian, and we believe it can be applied in more general settings, with the Laplacian replaced by the Casimir element (see a discussion in Section 2.2 below). Second, the problem of the meromorphic continuation of Poincaré series, which is usually attacked by means of Fourier series expansion and serious analytic considerations related to the coefficients in those series, is simplified. Namely, the meromorphic continuation essentially boils down to establishing a suitable functional relation between the Fourier transform of the test function at $s$ and at $s + \alpha$, for a suitable translation parameter $\alpha$ (see Lemma 19 below). For this reason, the scaling factor $\frac{\Gamma(s-1/2)}{\Gamma(s)}$ appears in the test function $g_s(u)$ above.

Moreover, this approach provides additional flexibility in the construction of Poincaré series, depending on the desired properties of the series, under the action of $\Delta_k$. Namely, assume that one is interested in the construction of Poincaré series $P_s(z,w)$ on $M$, such that $\Delta_k P_s(z,w)$ equals a certain function of $P_s(z,w)$. Then, representing $P_s(z,w)$ as the wave distribution acting on an unknown test function, this construction boils down to solving a second order differential equation satisfied by this test function, with some natural boundary conditions, such as e.g. decay to zero as $\text{Re}(s) \to \infty$. This task is not easy, but it may turn out to be easier than solving the partial differential equation that is to be satisfied by the point-pair invariant generating the series $P_s(z,w)$ as a sum over the group $\Gamma$ (or its cover $\tilde{\Gamma}$).

1.3. Outline of the paper. The paper is organized as follows: in Section 2 we introduce the basic notation, define the weighted Laplacian, the unitary multiplier system and the spaces of functions we are interested in and we recall the spectral expansion theorem. In Section 3 the construction of the geometric automorphic kernel is presented, following the approach undertaken in [He76] and [He83] and the pre-trace formula for the resolvent kernel is recalled from [Fi87]. Section 4 is devoted to proof of the non-trivial sup-norm bound for the eigenfunctions of the weighted Laplacian, a result necessary for the construction of the wave distribution associated to $\Delta_k$ in Section 5. Properties of the wave distribution are identified in Section 5 and applied to the construction of the basic automorphic kernel in Section 6.

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2. Preliminaries

2.1. Basic notation. Let $\Gamma \subset \text{PSL}_2(\mathbb{R})$ denote a Fuchsian group of the first kind. It acts by fractional linear transformations on the hyperbolic upper half-plane $\mathbb{H} := \{x + iy \mid x, y \in \mathbb{R}; y > 0\}$. We choose once and for all a connected fundamental domain $\mathcal{F} \subseteq \mathbb{H}$ for $\Gamma$. We further assume $\mathcal{F}$ (and therefore every fundamental domain) to have finite hyperbolic area. Then $M := \Gamma \backslash \mathbb{H}$ is a finite volume hyperbolic Riemann surface, which we allow to have elliptic fixed points and $c_\Gamma$ cusps. Locally, $M$ is identified with its universal cover $\mathbb{H}$, and each point on $M$ has a unique representative in $\mathcal{F}$. We rely on this identification of $M$ with $\mathcal{F}$ whenever a definition of a function on $M$ uses the choice of a representative in $\mathbb{H}$. This in particular applies to the kernel functions in this paper.

Let $\tilde{\Gamma}$ denote the cover of $\Gamma$ in $\text{SL}_2(\mathbb{R})$, i.e. the set of all matrices $\gamma \in \text{SL}_2(\mathbb{R})$ such that $[\pm \gamma] \in \Gamma$. Throughout this paper, assume that $\tilde{\Gamma}$ contains $-I$, where $I$ stands for the identity element of $\text{SL}_2(\mathbb{R})$.

Let $\mu_{\text{hyp}}$ denote the hyperbolic metric on $M$, which is compatible with the complex structure of $M$, and has constant negative curvature equal to minus one. The hyperbolic line element $ds^2_{\text{hyp}}$ is given by $ds^2_{\text{hyp}} := \frac{dx^2 + dy^2}{y^2}$. Denote the hyperbolic distance from $z \in \mathbb{H}$ to $w \in \mathbb{H}$ by $d_{\text{hyp}}(z, w)$. It satisfies the relation

$$\cosh(d_{\text{hyp}}(z, w)) = 1 + 2u(z, w),$$

where

$$u(z, w) := \frac{|z - w|^2}{4 \text{Im}(z) \text{Im}(w)}.$$

In the sequel, we will need the displacement function $\sigma(z, w)$, which is defined as

$$\sigma(z, w) := 1 + \frac{|z - w|^2}{4 \text{Im}(z) \text{Im}(w)} = \frac{|z - \overline{w}|^2}{4 \text{Im}(z) \text{Im}(w)}.$$

2.2. Weighted Laplacian. For any real $k$, denote by

$$\Delta_k = -y^2(\partial_x^2 + \partial_y^2) + 2k iy \partial_x$$

the hyperbolic Laplacian on $M$ of weight $k$, which will be applied to twice differentiable functions $f : \mathbb{H} \to \mathbb{C}$. H. Maass [Maa52] introduced in broader generality, for real numbers $\alpha$ and $\beta$, the differential operator

$$\Delta_{\alpha, \beta} = -y^2 (\partial_x^2 + \partial_y^2) + (\alpha - \beta) iy \partial_x - (\alpha + \beta) y \partial_y.$$

Specializing to $\alpha + \beta = 0$, we recover the classical Laplace-Beltrami operator on $\mathbb{H}$ of weight $\alpha - \beta$ (which is, among others, subject of Roelcke’s work [Roel56, Roel66, Roel67]).

There is a slight ambiguity in the notation used: the operator $\Delta_{\alpha, \beta}$ with $\alpha - \beta = \alpha + \beta = k$ is also called the weighted Laplacian of weight $k$ in the literature. It is that one which is used in the theory of mock modular forms (see e.g. [BK18]).

Our choice of the weighted Laplacian is the specialization to weight $2k$ of the Laplace-Beltrami operator

$$\tilde{\Delta} = -y^2 (\partial_x^2 + \partial_y^2) + y \partial_x \partial_y,$$

which in turn equals the Casimir operator for $\text{SL}(2, \mathbb{R})$, up to a multiplicative constant. More precisely, the action of $\text{SL}(2, \mathbb{R})$ on $L^2(\tilde{\Gamma} \backslash \text{SL}(2, \mathbb{R}))$ by right translations comes along with an action of its Lie algebra on $C^\infty$-vectors, given by differential operators. The Casimir element generates the center of the universal enveloping Lie algebra and,
written with respect to the coordinates $\partial_x, \partial_y, \partial_\theta$, this operator coincides with the Laplace-Beltrami operator $\Delta$ above. By Schur’s lemma, the Casimir acts as a constant on any irreducible representation of $\text{SL}(2, \mathbb{R})$. In turn, any eigenfunction of the Casimir, respectively the Laplace-Beltrami, together with its $\text{SL}(2, \mathbb{R})$-translates generates an irreducible representation. On the other hand, when restricting to eigenfunctions of weight $2k$ for the maximal compact subgroup $\text{SO}(2)$ of $\text{SL}(2, \mathbb{R})$, the Casimir operator specializes to our choice of the weighted Laplacian $\Delta_k$. In turn, the isomorphism of $\overline{\Gamma \backslash \text{SL}(2, \mathbb{R})}/\text{SO}(2)$ with $\Gamma \backslash \mathbb{H}$ induces an isomorphism of the $\text{SO}(2)$-eigenfunctions on $\overline{\Gamma \backslash \text{SL}(2, \mathbb{R})}$ with automorphic forms of weight $2k$ on $\mathbb{H}$.

2.3. **Unitary multiplier system.** For every $\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma$ and every complex number $z$, define $j(\gamma, z) := cz + d$ and $J_{\gamma,k}(z) := \exp(2ik \arg j(\gamma, z))$.

**Definition 1.** A function $\mu : \mathbb{H}^2 \to \mathbb{C}^*$ satisfying the transformation property

$$\mu(\gamma z, \gamma w) = \mu(z, w) J_{\gamma,k}(z) J_{\gamma,k}(w)^{-1}$$

for all $\gamma \in \text{SL}_2(\mathbb{R})$ and all $z, w \in \mathbb{H}$ is called a **weight $k$ point-pair invariant**.

Note that, due to the fact that $\text{SL}_2(\mathbb{R})$ acts transitively on point-pairs of a fixed hyperbolic distance, a point-pair invariant of weight zero is just an ordinary point-pair invariant depending only on the hyperbolic distance of the point-pair $(z, w)$. Further, if $\mu$ is a weight $k$ point-pair invariant and $\Phi$ is a weight zero point-pair invariant, then $\mu \cdot \Phi$ is also a weight $k$ point-pair invariant. Furthermore, if $\nu$ is also a weight $k$ point-pair invariant, then $\mu/\nu$ is a point-pair invariant of weight zero.

**Lemma 2.** Decompose the real number $k = k_1 + k_2$ with $k_1 \in \mathbb{Z}$ and $k_2 \in (-\frac{1}{2}, \frac{1}{2}]$, and define $z^k = z^{k_1} \exp(k_2 \log z)$ for the principal branch of the complex logarithm $\log z$. The function $H_k : \mathbb{H}^2 \to \mathbb{C}^*$ given by

$$H_k(z, w) := \left( -\frac{|z - w|^2}{(z - w)^2} \right)^k = \left( \frac{z - w}{w - \overline{z}} \right)^k = \left( 1 - \frac{\zeta}{1 - \zeta^2} \right)^k, \quad \text{for} \quad \zeta = \frac{z - w}{z - \overline{w}},$$

is a weight $k$ point-pair invariant.

**Proof.** The function $r : \mathbb{H}^2 \to \mathbb{C}$ given by

$$r(z, w) = 1 - \frac{z - w}{z - \overline{w}} = \frac{2i \text{Im} w}{z - \overline{w}}$$

determines under $\text{SL}_2(\mathbb{R})$ as

$$r(\gamma z, \gamma w) = r(z, w) \left( \frac{cz + d}{cw + d} \right).$$

Note that, for all $z, w \in \mathbb{H}$, we have $z - \overline{w} \in \mathbb{H}$. In particular, $0 \leq \arg(z - \overline{w}) < \pi$, which implies that

$$-\frac{\pi}{2} < \arg(r) = \frac{\pi}{2} - \arg(z - \overline{w}) \leq \frac{\pi}{2}.$$

We claim that the argument of $r$ transforms as

$$\arg(r(\gamma z, \gamma w)) = \arg(r(z, w)) + \arg(cz + d) - \arg(cw + d).$$

To see this, notice that both $cz + d$ and $cw + d$ belong either to upper or lower complex half-plane. In particular, either $\arg(cz + d), \arg(cw + d) \in (0, \pi)$ or $\arg(cz + d), \arg(cw + d) \in \ldots$
\((-\pi, 0)\), and it follows that
\[
\arg \left( \frac{cz + d}{cw + d} \right) = \arg(cz + d) - \arg(cw + d).
\]
Consequently, since in \((4)\) both values of \(r\) have arguments in \((-\frac{\pi}{2}, \frac{\pi}{2})\) and since
\[
\arg(r(\gamma z, \gamma w)) = \arg(r(z, w)) + \arg \left( \frac{cz + d}{cw + d} \right) + 2\pi l
\]
for some \(l \in \mathbb{Z}\), we must have \(l = 0\). Since
\[
H_k(z, w) = H_k(r(z, w)) = \left( \frac{r}{|r|} \right)^{2k_1} \cdot \left( \frac{r}{|r|} \right)^{2k_2},
\]
the claim of the lemma follows trivially for \(k = k_1 \in \mathbb{Z}\), using the definition of \(J_{\gamma, k_1}\). For \(k = k_2 \in (-\frac{1}{2}, \frac{1}{2})\), it follows from the above by noticing that, for exponent \(2k_2 \in (-1, 1]\), the power is still given by multiplying the argument, \((e^{i \arg(z)})^{2k_2} = e^{2ik_2 \arg(z)}\). The lemma follows for arbitrary real \(k\) from our choice of the \(k\)-th power.

For every \(\gamma_1 = (a_1, a_2)\) and \(\gamma_2 = (b_1, b_2)\) \(\in\) \(SL_2(\mathbb{R})\), write \(\gamma_1\gamma_2 = (c_1, c_2)\). For every \(z \in \mathbb{H}\), we have
\[
a_3\gamma_2z + a_4 = \frac{c_3z + c_4}{b_3z + b_4}
\]
and therefore there exists an integer \(w(\gamma_1, \gamma_2) \in \{-1, 0, 1\}\), which is independent of \(z\), such that
\[
2\pi w(\gamma_1, \gamma_2) = \arg(a_3\gamma_2z + a_4) + \arg(b_3z + b_4) - \arg(c_3z + c_4).
\]
The function \(\omega_k(\gamma_1, \gamma_2) := \exp(4\pi i kw(\gamma_1, \gamma_2))\) is called a factor system of weight \(k\).

Let \((V, \langle \cdot, \cdot \rangle)\) be a \(d\)-dimensional unitary \(\mathbb{C}\)-vector space \((d < \infty)\), where the inner product \(\langle \cdot, \cdot \rangle\) is semi-linear in the first argument. Let \(U(V)\) denote the unitary group, i.e. the automorphisms \(u\) of \(V\) respecting the scalar product, \(\langle u(v), u(w) \rangle = \langle v, w \rangle\) for all \(v, w \in V\).

**Definition 3.** A (unitary) multiplier system of weight \(k\) on \(\tilde{\Gamma}\) is a function \(\chi : \tilde{\Gamma} \rightarrow U(V)\) which satisfies the properties:
\begin{enumerate}
  \item \(\chi(-I) = e^{-2\pi ik}\id_V\) and \(\chi(\gamma(\gamma_1, \gamma_2)) = \omega_k(\gamma_1, \gamma_2)\chi(\gamma_1)\chi(\gamma_2)\).
\end{enumerate}

If \(\tilde{\Gamma}\) contains parabolic elements, then there exists a unitary multiplier system on \(\tilde{\Gamma}\) for every weight \(k \in \mathbb{R}\); when \(\tilde{\Gamma}\) does not contain parabolic elements, a unitary multiplier system on \(\tilde{\Gamma}\) exists for certain rational values of weight \(k\), depending on the signature of the group \(\Gamma\), see [Fli87, Proposition 1.3.6]. From now on, we fix \(k \in \mathbb{R}\) such that there exists a unitary multiplier system \(\chi : \tilde{\Gamma} \rightarrow U(V)\) of weight \(k\) on \(\tilde{\Gamma}\), which we also fix.

**Lemma 4.** For every weight \(k\) point-pair invariant \(\mu\) such that the series
\[
S_{\Gamma, \mu}(z, w) := \sum_{\gamma \in \tilde{\Gamma}} \chi(\gamma)J_{\gamma, k}(w)\mu(z, \gamma w)
\]
is absolutely convergent for all \(z, w \in \mathbb{H}\), we have the identity
\[
S_{\Gamma, \mu}(\eta z, w)J_{\eta, k}(z)^{-1} = \chi(\eta)S_{\Gamma, \mu}(z, w)
\]
for all \(\eta \in \tilde{\Gamma}\).
Proof. We have to prove that
\[ \chi(\eta) \sum_{\gamma \in \tilde{\Gamma}} \chi(\gamma) J_{\gamma,k}(w) \mu(z, \gamma w) = \sum_{\gamma \in \tilde{\Gamma}} \chi(\gamma) J_{\gamma,k}(w) J_{\eta,k}(z)^{-1} \mu(\eta z, \gamma w) \]
for every \( \eta \) in \( \tilde{\Gamma} \). Setting \( \gamma' = \eta^{-1} \gamma \) and summing over \( \gamma' \) instead of \( \gamma \) by absolute convergence of the series, the above follows from the definitions of multiplier system and weight \( k \) point-pair invariant, combined with the implication of (5) that, for any \( w \in \mathbb{H} \) and any \( \eta, \gamma \in \tilde{\Gamma} \),
\[ \omega_k(\eta, \gamma) = J_{\eta,k}(\gamma w) J_{\gamma,k}(w) J_{\eta \gamma,k}(w)^{-1}. \]
\[ \square \]

2.4. Spectral expansion. For every \( \gamma \in \tilde{\Gamma} \), define the linear operator \(|[\gamma, k]|\) on the space of functions \( f : \mathbb{H} \rightarrow V \) by
\[ f|[\gamma, k](z) := f(\gamma z) J_{\gamma,k}(z)^{-1}. \]
It is important to notice that \( \Delta_k \) commutes with \(|[\gamma, k]|\), in other words
\[ \Delta_k (f|[\gamma, k]) = ([\Delta_k f]|[\gamma, k]|) \]
for every twice continuously differentiable function \( f : \mathbb{H} \rightarrow V \). It follows that, if \( f \) is such a function and it additionally satisfies
\[ (6) \quad f|[\gamma, k] = \chi(\gamma) f \]
for every \( \gamma \in \tilde{\Gamma} \), then
\[ (7) \quad ([\Delta_k f]|[\gamma, k]|) = \chi(\gamma) \Delta_k f. \]
Notice that if \( f_1, f_2 : \mathbb{H} \rightarrow V \) are functions satisfying (6) then \( \langle f_1, f_2 \rangle \) is a \( \tilde{\Gamma} \)-invariant, vector-valued function on \( \mathbb{H} \). Let \( \mathcal{F} \) denote an arbitrary fundamental domain of \( \Gamma \). Let \( \mathcal{H}_k \) denote the space of (equivalence classes of \( \mu_{\text{hyp}} \)-almost everywhere equal) \( \mu_{\text{hyp}} \)-measurable functions \( f : \mathbb{H} \rightarrow V \) which satisfy the properties:
\begin{enumerate}
  \item \( f|[\gamma, k](z) = \chi(\gamma) f(z) \) for all \( \gamma \in \tilde{\Gamma} \) and
  \item \( \|f\|^2 := \int_{\mathcal{F}} \langle f, f \rangle d\mu_{\text{hyp}} < \infty. \)
\end{enumerate}
It follows that \( \mathcal{H}_k \) is a Hilbert space when equipped with the scalar product
\[ \langle f, g \rangle := \int_{\mathcal{F}} \langle f, g \rangle d\mu_{\text{hyp}}. \]
For all \( f_1, f_2 \in \mathcal{H}_k \), the function \( \langle f_1, f_2 \rangle \) given by the scalar product on \( V \) determines an almost everywhere well-defined function on \( \mathbb{H} \). In particular, when \( V = \mathbb{C} \), \( \langle f, g \rangle = \bar{f} \cdot g \) and \( \langle f, g \rangle = \int_{\mathcal{F}} f(z) g(z) d\mu_{\text{hyp}}(z) \) is the usual \( L^2 \)-scalar product. From now on, the equivalence class of a function \( f : \mathbb{H} \rightarrow V \) under the equivalence relation \( \mu_{\text{hyp}} \)-almost everywhere equal will be denoted by \( f \) by abuse of notation. Moreover, identify \( V = \mathbb{C}^d \), which implies that
\[ (8) \quad \langle x, y \rangle = \sum_{j=1}^d x_j y_j \]
for every \( x = (x_1, \ldots, x_d)^t \) and \( y = (y_1, \ldots, y_d)^t \) in \( V \). Here, \( X^t \) denotes the transpose of a matrix \( X \). With these conventions, measurability, differentiability, integrability, etc. of any function \( f : \mathbb{H} \rightarrow V \) are defined component-wise.
The norm on $V$ corresponding to the scalar product $\langle \cdot , \cdot \rangle$ will be denoted by $| \cdot |_V$. We will at times apply the Hermitian inner product to $d \times d$ matrices, more precisely to $xy^\dagger$ for arbitrary $x, y \in V$. Denote the resulting norm by $| \cdot |_{d \times d}$ and note that

$$|xy^\dagger|_{d \times d} = |x||y|_V.$$ 

Let $D_k$ denote the set of all twice continuously differentiable functions $f \in \mathcal{H}_k$ such that $\Delta_k f \in \mathcal{H}_k$. The operator $\Delta_k : D_k \to \mathcal{H}_k$ is essentially self-adjoint [Fis87, Theorem 1.4.5]. Let $\tilde{\Delta}_k : \tilde{D}_k \to \mathcal{H}_k$ denote the unique maximal self-adjoint extension of $\Delta_k$ with $\tilde{D}_k$ as its domain.

In case when $\tilde{\Gamma}$ contains parabolic elements, let $\zeta_1, \ldots, \zeta_{cr}$ denote a complete system of representatives of the $\Gamma$-equivalence classes of cusps of $\Gamma$. Choose matrices $A_1, \ldots, A_{cr} \in \text{SL}_2(\mathbb{R})$, such that the stabilizers $\Gamma_{\zeta_j} := \{ \gamma \in \tilde{\Gamma} \mid \gamma \zeta_j = \zeta_j \}$ are generated by $-I$ and $T_j := A_j^{-1}(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) A_j$. Let $m_j$ denote the multiplicity of the eigenvalue $1$ of $\chi(T_j)$. For every $j \in \{1, \ldots, cr\}$, choose an orthonormal basis $\{v_{j1}, \ldots, v_{jd}\}$ of $V$ such that

$$\chi(T_j)v_{jl} = e^{2\pi i \beta_{jl}}v_{jl}, \text{ with } \begin{cases} \beta_{jl} = 0, & \text{if } 1 \leq l \leq m_j \\ \beta_{jl} \in (0, 1), & \text{if } m_j < l \leq d. \end{cases}$$

For every $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\text{Re}(s) > 1$, define the parabolic Eisenstein series of weight $k$ for the cusp $\zeta_j$, the multiplier system $\chi$ and the eigenvector $v_{jl}$ as the series

$$E_{jl}(z, s) := \frac{1}{2} \sum_{\gamma \in \Gamma_{\zeta_j} \backslash \tilde{\Gamma}} \omega_k(A_j \gamma)^{-1}\chi(\gamma)^{-1}v_{jl}J_{A_j \gamma, k}(z)^{-1}(\text{Im}(A_j \gamma z))^s.$$ (9)

This series converges uniformly absolutely in $(z, s) \in \mathbb{H} \times \{ s \in \mathbb{C} \mid \text{Re}(s) > 1 + \varepsilon \}$ for every $\varepsilon > 0$, hence it defines a $C^\infty$-function from $\mathbb{H} \times \{ s \in \mathbb{C} \mid \text{Re}(s) > 1 \}$ to $V$, which is holomorphic in $s$. It was shown in [Roe67] that, for every $s \in \mathbb{C}$ such that $\text{Re}(s) > 1$, the series $E_{jl}(\cdot, s)$ is an eigenfunction of $\tilde{\Delta}_k$, with eigenvalue $s(1-s)$:

$$\Delta_k E_{jl}(\cdot, s) = s(1-s)E_{jl}(\cdot, s).$$ (10)

Furthermore, for every fixed $z \in \mathbb{H}$, the series $E_{jl}(z, \cdot)$ can be extended to a meromorphic function on $\mathbb{C}$, which is denoted in the same way. This function has only simple poles in the half-plane $\{ s \in \mathbb{C} \mid \text{Re}(s) > 1/2 \}$, which all lie in the interval $(1/2, 1]$. It has no poles on the line $\{ s \in \mathbb{C} \mid \text{Re}(s) = 1/2 \}$, from which it follows that $E_{jl}$ is continuous on $\mathbb{H} \times \{ s \in \mathbb{C} \mid \text{Re}(s) = 1/2 \}$. The Eisenstein series $E_{jl}$ satisfies (10) in this domain. Recall the following theorem from [Fis87, pp. 37–38]:

**Theorem 5** (Spectral expansion). Every function $f \in \tilde{D}_k$ has an expansion of the following form:

$$f(z) = \sum_{n \geq 0} (\varphi_n, f) \varphi_n(z) + \frac{1}{4\pi} \sum_{j=1}^{cr} \sum_{l=1}^{m_j} \int_{-\infty}^{\infty} (E_{jl}(\cdot, 1/2 + it), f) E_{jl}(z, 1/2 + it) dt,$$

where $(\varphi_n)_{n \geq 0}$ is a countable orthonormal system of eigenfunctions of $\tilde{\Delta}_k : \tilde{D}_k \to \mathcal{H}_k$. The series $\sum_{n \geq 0} (\varphi_n, f) \varphi_n$ converges uniformly absolutely on compact subsets of $\mathbb{H}$. When $\Gamma$ is cocompact, the second sum on the right hand side of the above equation is identically zero.

**Remark 6.** In the sequel, whenever we apply the spectral expansion theorem to cocompact $\Gamma$, we will assume that the sum over parabolic elements is identically zero and we will not treat that case separately.
Let $|k|(1 - |k|) = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots$ denote the discrete eigenvalues corresponding to the orthonormal system $(\varphi_n)_{n \geq 0}$ and write
\begin{equation}
\lambda_n = 1/4 + t_n^2
\end{equation}
for every $n$ where $t_n = \sqrt{\lambda_n - 1/4}$ and $t_n \in (0, iA]$ when $\lambda_n < 1/4$; here, $A$ is defined as
\begin{equation}
A := \max \{1/2, |k| - 1/2\}
\end{equation}
and note that $|k|(1 - |k|) \geq 1/4 - A^2$. Every $\lambda_n$ occurs with finite multiplicity $\mu_n$ and the series $\sum_{n \geq 0} \lambda_n^{-2}$ converges [Fi87, Theorem 1.6.5].

3. The automorphic kernel

In this section, we recall the construction of automorphic forms for $\Gamma$ with multiplier system $\chi$, using point-pair kernel functions (i.e. kernel functions depending only upon the hyperbolic distance between the points).

3.1. Selberg Harish-Chandra transform. Following [He83, pp. 386–387], let $\Phi$ be a real-valued function defined on $[0, \infty)$, four times differentiable in this interval and such that $|\Phi(\ell)(t)| \ll (t + 4)^{-\delta - \ell}$, for $\ell = 0, 1, 2, 3, 4$ and for some $\delta > \max\{1, |k|\}$. To the weight $k$ point-pair invariant
\begin{equation}
k(z, w) := H_k(z, w) \cdot \Phi \left( \frac{|z - w|^2}{\text{Im}(z) \text{Im}(w)} \right),
\end{equation}
where $z, w \in \mathbb{H}$, we associate the automorphic kernel
\begin{equation}
K_\Gamma(z, w) := \frac{1}{2} \sum_{\gamma \in \Gamma} \chi(\gamma) J_{\gamma, k}(w) k(z, \gamma w),
\end{equation}
which takes values in the endomorphism ring $\text{End}(V)$. Note that $\Phi \left( \frac{|z - w|^2}{\text{Im}(z) \text{Im}(w)} \right)$ is a weight zero point-pair invariant. Due to the bounds on the derivatives of $\Phi$ and to Lemma 4, the automorphic kernel $K_\Gamma$ belongs to $\tilde{D}_k$ as a function of $z$.

The Selberg Harish-Chandra transform $h_\Phi$ of a function $\Phi$ satisfying the conditions stated above can be computed using the following three steps:

(i) compute
\begin{equation}
Q(y) = \int_{-\infty}^{\infty} \Phi(y + v^2) \left( \frac{v + 4 + iv}{v + 4 - iv} \right)^k dv
\end{equation}
for $y \geq 0$;

(ii) set $g(u) = \frac{Q(2(\cosh u - 1))}{\cosh(u - 1)}$;

(iii) the Selberg Harish-Chandra transform of $\Phi$ is the Fourier transform of $g$, i.e.
\begin{equation}
h_\Phi(r) = \int_{-\infty}^{\infty} g(u) e^{iru} du.
\end{equation}

The Selberg Harish-Chandra transform exists for complex numbers $r$ with suitably bounded imaginary part.

Remark 7. A slightly different, yet equivalent version of the Selberg Harish-Chandra transform of the point-pair invariant is given in [Fay77, Theorem 1.5]. In the cited text, the automorphic kernel constructed from the point-pair invariant is defined as
\begin{equation}
\tilde{K}_\Gamma(z, w) = \sum_{\gamma \in \Gamma} \chi(\gamma) \left( \frac{cw + d}{cw + d} \right)^k \left( \frac{z - \gamma w}{\gamma w - z} \right)^k g(\cosh(d_{\text{hyp}}(z, \gamma w))),
\end{equation}
where
under the assumption that \( g(u) \) is a continuous function of \( u > 1 \), with a majorant \( g_1(u) \in L^1 \cap L^2(1, \infty) \) satisfying the following condition: for any \( \delta > 0 \) there exists a constant \( m(\delta) > 0 \) such that, for all \( z, w \in \mathbb{H} \) with \( d_{\text{hyp}}(z, w) > \delta \),

\[
g_1(\cosh(d_{\text{hyp}}(z, w))) \leq m(\delta) \cdot \int_{d_{\text{hyp}}(\zeta, w) < \delta} g_1(\cosh(\text{hyp}(\zeta))) \, d\mu_{\text{hyp}}(\zeta).
\]

The Selberg Harish-Chandra transform \( h \) of the point-pair invariant function \( g \) is given by the formula

\[
h(r) = 2\pi \int_1^\infty g(\cosh(y)) \left( \frac{2}{\cosh y + 1} \right)^r F \left( r - k, r + k; 1; \frac{\cosh y - 1}{\cosh y + 1} \right) \, d(cosh(y)),
\]

where \( F(a, b; c; z) \) stands for the (Gauss) hypergeometric function.

In fact, equation (11) yields that \( K_H(z, w) = K_G(z, w) \), where \( K_G(z, w) \) is defined by (14) with the point-pair invariant function \( \Phi \) defined in (13) given by \( \Phi(x) = g(1 + \frac{x}{2}) \); in particular \( h = h_{\Phi} \).

For a function \( h : D \to \mathbb{C} \), where \( D \) is a subset of \( \mathbb{C} \), and a constant \( a > 0 \) define the following conditions:

(S1) \( h(r) \) is an even function.

(S2) \( h(r) \) is holomorphic in the strip \( |\text{Im}(r)| < a + \epsilon \) for some \( \epsilon > 0 \).

(S3) \( h(r) \ll (1 + |r|)^{-2-\delta} \) for some fixed \( \delta > 0 \) as \( |r| \to \infty \) in the set of definition of condition (S2).

Choosing \( a = A \) as in (12), the conditions (S1)-(S3) are actually the assumptions posed on the test function \( h \) in the trace formula [He83, Theorem 6.3]. The following proposition holds:

**Proposition 8** ([He83, Section 9.7]). Let \( A \) be defined as in (12) and \( \lambda_j = 1/4 + t_j^2 \) as in (11). Suppose that the Selberg Harish-Chandra transform \( h_\Phi \) exists and satisfies conditions (S1)-(S3) for \( a = A \). Then the automorphic kernel (14) admits a spectral expansion of the form

\[
K_G(z, w) = \sum_{\lambda_j \geq |k| \cdot (1-k)} h_\Phi(t_j) \varphi_j(z) \overline{\varphi_j}(w)^t
\]

(15)

\[
+ \frac{1}{4\pi} \sum_{j=1}^c \sum_{l=1}^{m_j} \int_{-\infty}^\infty h_\Phi(r) E_{jl}(z, 1/2 + ir) \overline{E_{jl}(w, 1/2 + ir)} \, dr,
\]

which converges absolutely and uniformly on compacta.

When \( \Gamma \) is cocompact, according to Remark 6, the second sum on the right hand side of (15) is identically zero.

The assumptions on the test function \( h \), which ensure the convergence of the series and the integral on the right-hand side of (15), can be relaxed. Namely, we will prove in Section 5 that, if the function \( h \) satisfies the conditions (S1),

(S2') \( h(r) \) is well-defined and even for \( r \in \mathbb{R} \cup [-ia, ia] \),

and (S3) in the set of definition of condition (S2') (that is, as \( |r| \to \infty \)), then the series and integrals on the right-hand side of (15) are well-defined and converge absolutely and uniformly on compacta. However, the assumptions (S1), (S2') and (S3) do not imply that the right-hand side of (15) represents a spectral expansion of some \( L^2 \)-automorphic kernel for \( a = A \).
3.2. Resolvent kernel and pre-trace formula. Let $\rho(\tilde{\Delta}_k)$ denote the resolvent set of $\tilde{\Delta}_k$, i.e. the set of all complex numbers $\lambda$ for which the linear operator $(\tilde{\Delta}_k - \lambda \mathrm{id}_{\mathcal{H}_k})^{-1} : \mathcal{H}_k \to \tilde{D}_k$ is bounded. According to [Fi87, pp. 25–27], the resolvent kernel associated to the operator $\tilde{\Delta}_k$ is the integral kernel of the operator $(\tilde{\Delta}_k - s(1-s))^{-1}$, defined for all $s \in \mathbb{C} \setminus \{k-n, -k-n \mid n = 0, 1, 2, \ldots\}$ with $\operatorname{Re}(s) > 1$ and $z, w \in \mathbb{H}$ such that $z \neq \gamma w$ for all $\gamma \in \Gamma$ as the automorphic kernel

$$
G_s(z, w) := \frac{1}{2} \sum_{\gamma \in \Gamma} \chi(\gamma) k_s(\sigma(z, \gamma w)) J_{\gamma,k}(w) H_k(z, \gamma w),
$$

with the point-pair invariant function

$$
k_s(\sigma) := \sigma^{-s} \frac{\Gamma(s-k) \Gamma(s+k)}{4\pi \Gamma(2s)} F(s+k, s-k; 2s; \frac{1}{\sigma}),
$$

where $\sigma := \sigma(z, w)$ is defined by (13) and $F(\alpha, \beta; \gamma; z)$ denotes the classical Gauss hypergeometric function.

The series on the right-hand side of (16) converges normally in the variables $z, w \in \mathbb{H}$ such that $z \neq \gamma w$ and $s \in \mathbb{C} \setminus \{k-n, -k-n \mid n = 0, 1, 2, \ldots\}$ with $\operatorname{Re}(s) > 1$ with respect to the operator norm in the ring of endomorphisms of $V$.

Recall from [Fi87, Formula (2.1.4) on p. 46] the pre-trace formula that follows from the computation of the trace of the resolvent kernel $G_s(z, w)$:

**Lemma 9.** For all $s, t \in \mathbb{C} \setminus \{k-n, -k-n \mid n = 0, 1, 2, \ldots\}$ with $\operatorname{Re}(t), \operatorname{Re}(s) > 1$ and $z \in \mathbb{H}$, we have

$$
\sum_{n \geq 0} \left( \frac{1}{\lambda_n - \lambda} - \frac{1}{\lambda_n - \mu} \right) \left| \varphi_n(z) \right|^2_V + \frac{1}{4\pi} \sum_{j=1}^{m_1} \sum_{l=1}^{m_2} \int_{-\infty}^{\infty} \left( \frac{1}{1 + r^2 - \lambda} - \frac{1}{1 + r^2 - \mu} \right) 
\times \left| E_{jl}(z, \frac{1}{2} + ir) \right|^2_V dr
= -\frac{d}{4\pi} (\psi(s+k) + \psi(s-k) - \psi(t+k) - \psi(t-k))
+ \frac{1}{2} \sum_{\gamma \in \Gamma_{\{\pm 1\}}} \operatorname{Tr}(\chi(\gamma)) (k_s(\sigma(z, \gamma z)) - k_t(\sigma(z, \gamma z))) J_{\gamma,k}(z) H_k(z, \gamma z),
$$

where $\lambda := s(1-s)$, $\mu := t(1-t)$ and $\psi(x) := \frac{\Gamma'(x)}{\Gamma(x)}$ is the digamma function.

Moreover, by Dini’s theorem, all the sums and integrals in (17) converge uniformly for every $s, t$ as above and $z \in \mathbb{H}$.

When $\Gamma$ is cocompact, the sum over cusps on the left hand side of (17) is identically zero.

4. Sup-norm bounds for the eigenfunctions associated to discrete eigenvalues

In this section, we use (17) to derive the sup-norm bounds for the norm $|\varphi_n(z)|_V$, when $z \in \mathcal{F}$. Hence, among others, we need an upper bound for the absolute value of the difference $q_k(s; z, \gamma z) := k_s(\sigma(z, \gamma z)) - k_{s+1}(\sigma(z, \gamma z))$ analogous to the bound derived in [FJK19, Lemma 6.2]. The proof of [FJK19, Lemma 6.2] could be adopted to our setting when the weight $k$ is not a positive integer or a half-integer. However, we give a direct proof of a better bound, valid for all real weights $k$. 


Lemma 10. Let \( k \in \mathbb{R} \) and let \( s > |k| \) be a real number. Then
\[
|g_k(s; z, \gamma z)| \leq \frac{s}{2\pi(s^2 - k^2)} \sigma(z, \gamma z)^{-s}.
\]

Proof. From the definition of hypergeometric series in terms of the Pochhammer symbol \((a)_j := \Gamma(a + j)/\Gamma(a)\) and the identity \((a + 1)_j = (a)_{j+1}/a\), we obtain that
\[
k_{s+1}(\sigma(z, \gamma z)) = \sigma(z, \gamma z)^{-s} \frac{\Gamma(s-k)\Gamma(s+k)}{4\pi\Gamma(2s+1)} \sum_{j=0}^{\infty} \frac{(s+k)_{j+1}(s-k)_{j+1}(j+1)}{(j+1)!\Gamma(2s+1)_{j+1}} \sigma(z, \gamma z)^{-j+1},
\]
so that
\[
g_k(s; z, \gamma z) = \sigma(z, \gamma z)^{-s} \frac{\Gamma(s-k)\Gamma(s+k)}{4\pi\Gamma(2s)} \sum_{j=0}^{\infty} \frac{(s+k)_{j}(s-k)_{j}}{j!\Gamma(2s+1)_{j}} \sigma(z, \gamma z)^{-j}.
\]
Since \( \sigma(z, \gamma z) \geq 1 \), application of [GR07, Formula 9.122.1] gives
\[
\sum_{j=0}^{\infty} \frac{(s+k)_{j}(s-k)_{j}}{j!\Gamma(2s+1)_{j}} \sigma(z, \gamma z)^{-j} \leq \sum_{j=0}^{\infty} \frac{(s+k)_{j}(s-k)_{j}}{j!\Gamma(2s+1)_{j}} = \frac{\Gamma(2s+1)}{\Gamma(s-k+1)\Gamma(s+k+1)},
\]
which leads to
\[
g_k(s; z, \gamma z) \leq \frac{2s}{4\pi(s-k)(s+k)} \sigma(z, \gamma z)^{-s}.
\]
The proof is complete; note that we have omitted the absolute values because all expressions are positive, due to the fact that \( s > |k| \) is real. \( \square \)

Remark 11. In the case where \( s = k + \epsilon \), for some \( \epsilon \in (0,1) \) and some positive integer \( k \), the upper bound from (18) becomes \( \frac{k+\epsilon}{2\pi(2k+\epsilon)} \sigma(z, \gamma z)^{-s} \). This is obviously less than \( \frac{3}{2\pi\epsilon} \sigma(z, \gamma z)^{-s} \) for all positive integers \( k \), hence the bound \( [18] \) is better than the one obtained in [FJK19, Lemma 6.2] using the representation of the resolvent kernel as an integral transform of the heat kernel.

Next, we derive the sup-norm bound for the eigenfunctions of the weighted Laplacian associated to discrete eigenvalues \( \lambda_j, j \geq 0 \), and for the integral of the Eisenstein series, when \( M \) is non-compact. Throughout this section, identify the surface \( M \) with the fundamental domain \( \mathcal{F} \). Let \( Y > 1 \) be arbitrary and let \( \mathcal{F}_Y \) denote the neighbourhood of the cusp \( \zeta_j, j \in \{1, \ldots, c_\Gamma\} \), characterized by
\[
A_j^{-1} \mathcal{F}_Y = \{ z \in \mathbb{H} | -1/2 \leq \text{Re}(z) \leq 1/2, \text{Im}(z) \geq Y \},
\]
where \( A_j \) is the scaling matrix associated to the cusp \( \zeta_j \), for every \( j \in \{1, \ldots, c_\Gamma\} \). Denote by \( \mathcal{F}_Y \) the closure of the complement of \( \bigcup_{j=1}^{c_\Gamma} \mathcal{F}_j^Y \) with respect to \( \mathcal{F} \) (note that \( \mathcal{F} = \mathcal{F}_Y \) if \( \Gamma \) is cocompact).

We introduce the constant
\[
C(k; M, d) := \frac{d(|k| + 2)}{8\pi(|k| + 1)} + \left( \frac{|k| + 2}{|k| + 1} \right)^2 \frac{d}{2\text{vol}_{\text{hyp}}(\mathcal{F})} e^{\frac{2}{3}\text{diam}_{\text{hyp}}(\mathcal{F})},
\]
where \( \text{diam}_{\text{hyp}}(\mathcal{F}) \) denotes the hyperbolic diameter of the fundamental domain \( \mathcal{F} \). The constant \( C(k; M, d) \) clearly depends upon the surface and the multiplier system, but not on the eigenvalue. With this notation, the following proposition holds:
Proposition 12. (a) Let $\varphi_j(z)$ be the eigenfunction of the Laplacian $\Delta_k$ associated to the discrete eigenvalue $\lambda_j$. Then

$$\sup_{z \in \mathcal{F}} |\varphi_j(z)|_V \leq C(k, M, d) |\lambda_j|,$$

where the constant $C(k, M, d)$ depends on the surface and the multiplier system, but not on the eigenvalue. When $\lambda_j \geq 3 + |k|$, one can take

$$C(k, M, d) = (C(k, M, d)(|k| + 2))^{\frac{1}{2}}.$$

(b) In case when $\widetilde{\Gamma}$ contains parabolic elements, for any $j \in \{1, \ldots, c_\Gamma\}$ and $l \in \{1, \ldots, m_\Gamma\}$, the following bound for the parabolic Eisenstein series $\delta$ of weight $k$ for the cusp $c_\gamma$, the multiplier system $\chi$ and the eigenvector $v_jl$ holds:

$$\sup_{z \in \mathcal{F}} \int_{-\infty}^{\infty} \frac{1}{(\frac{1}{2} + r^2 + |k| + 2)^2} |E_{jl}(z, \frac{1}{2} + ir)|^2 \, dr \leq \frac{2\pi}{|k| + 2} C(k, M, d).$$

Proof. Take $s = |k| + 2$ and $t = |k| + 3$ in Lemma 3 (note that $s, t \notin \{k - n, -k - n \mid n = 0, 1, 2, \ldots\}$). Start with an upper bound for the right-hand side of (17). The sum of the values of digamma functions may be evaluated by applying the functional equation

$$\psi(z + 1) = \psi(z) + z^{-1}:$$

$$\frac{d}{4\pi} |\psi(s + k) + \psi(s - k) - \psi(t + k) - \psi(t - k)| = \frac{d(|k| + 2)}{8\pi(|k| + 1)}.$$

To bound the sum, use inequality (18). Recall that $|J_{\gamma, k}(z) H_k(z, \gamma z)| = 1$ for all $z$ and $\gamma$ and that $\chi$ is unitary, so that

$$\left| \frac{1}{2} \sum_{\gamma \in \Gamma \setminus \{\pm I\}} \text{Tr}(\chi(\gamma)) (k_s(\sigma(z, \gamma z)) - k_t(\sigma(z, \gamma z))) J_{\gamma, k}(z) H_k(z, \gamma z) \right|$$

$$\leq \sum_{\gamma \in \Gamma \setminus \{I\}} \frac{d(|k| + 2)}{8\pi(|k| + 1)} |\sigma(z, \gamma z)|^{-|(k| + 2)}.$$

Furthermore, applying [FKJK19] Lemma 3.7 with $\delta = |k| + 2$, we deduce that, for any $Y > 1$ and any $z \in \mathcal{F}_Y$,

$$\sum_{\gamma \in \Gamma \setminus \{I\}} \frac{d(|k| + 2)}{8\pi(|k| + 1)} |\sigma(z, \gamma z)|^{-|(k| + 2)} \leq \left( \frac{|k| + 2}{|k| + 1} \right)^2 \frac{d B_Y}{2},$$

where $B_Y = \exp(\frac{3}{2}\text{diam}_{\text{hyp}}(\mathcal{F}_Y)) \text{vol}_{\text{hyp}}(\mathcal{F}_Y)^{-1}$. Note that, for every $Y \geq 2$, $B_Y$ is bounded by $\exp(\frac{3}{2}\text{diam}_{\text{hyp}}(\mathcal{F})) \text{vol}_{\text{hyp}}(\mathcal{F})^{-1}$. Hence, for all $z \in \mathcal{F}_Y$ and $Y \geq 2$, the right-hand side of (17) is bounded from above by the constant $C(k, M, d)$ defined in (19).

Now, specialize the pre-trace formula (17) to either one summand or one integral on the left-hand side.

(a) Since there are only finitely many eigenvalues that are less than $3 + |k|$, it is sufficient to prove (20) for eigenvalues $\lambda_j \geq 3 + |k|$. Therefore, assume that $\lambda_j \geq 3 + |k|$. Our choice of $s$ and $t$ in Lemma 11 together with above computations and the assumption on $\lambda_j$, lead to the inequality

$$\sup_{z \in \mathcal{F}_Y} |\varphi_j(z)|_V^2 \leq C(k, M, d)(|k| + 2)\lambda_j^2,$$

which holds for all $Y \geq 2$. It remains to extend it to $z \in \mathcal{F}$. 

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Since all eigenfunctions \( \varphi_j \) are continuous on \( \overline{F} \) and the area of \( F \) is finite (with the area of the boundary equal to zero, since \( \Gamma \) is of the first kind), one deduces that 

\[
\sup_{z \in F} |\varphi_j(z)|_V = \lim_{p \to \infty} \mu(F)^{-1/p} \|\varphi_j(z)|_V \|_p = \lim_{p \to \infty} \left( \mu(F)^{-1} \int_F |\varphi_j(z)|_V^p d\mu_{\text{hyp}}(z) \right)^{1/p},
\]

where \( \| \cdot \|_p \) denotes the \( L^p \)-norm (see e.g. [Ch84, Formula (22) on p. 100] for the analogous statement related to eigenfunctions of the Laplacian).

Let \( \{Y_n\}_{n \geq 1} \) be an increasing sequence of real numbers bigger than 2, tending to infinity. For every \( p > 1 \), the monotone convergence theorem applied to the sequence \( |\varphi_j(z)|^p \cdot 1_{F_{Y_n}}(z) \), where \( 1_{F_{Y_n}}(z) \) denotes the characteristic function of the set \( F_{Y_n} \), yields that

\[
\int_F |\varphi_j(z)|_V^p d\mu_{\text{hyp}}(z) = \lim_{n \to \infty} \int_{F_{Y_n}} |\varphi_j(z)|_V^p d\mu_{\text{hyp}}(z) \leq C(k, M, d)^{p/2} (|k| + 2)^{p/2} |\lambda_j|^p \mu(F).
\]

Therefore,

\[
\sup_{z \in F} |\varphi_j(z)|_V \leq C(k, M, d)^{1/2} (|k| + 2)^{1/2} |\lambda_j|.
\]

(b) If \( M \) has cusps, fix \( j \in \{1, \ldots, c_r\} \) and \( l \in \{1, \ldots, m_j\} \). The above computations imply that, for \( s = |k| + 2 \) and for all \( Y \geq 2 \),

\[
\sup_{z \in F_Y} \int_{-\infty}^{\infty} \frac{1}{(\frac{1}{4} + r^2 + s^2)^2 - s^2} |E_{jl}(z, \frac{1}{2} + ir)|_V^2 dr \leq \frac{2\pi}{|k| + 2} C(k, M, d).
\]

Proceeding analogously as above, define the function

\[
G(z) := \int_{-\infty}^{\infty} \frac{1}{(\frac{1}{4} + r^2 + s^2)^2 - s^2} |E_{jl}(z, \frac{1}{2} + ir)|_V^2 dr,
\]

which is continuous and non-negative on \( \overline{F} \). Applying the monotone convergence theorem to the sequence \( G(z)^p \cdot 1_{F_{Y_n}}(z) \), together with the fact that the sup-norm is the limit of \( L^p \)-norms, and reasoning as in the proof of part (a), we deduce that

\[
\sup_{z \in F} \int_{-\infty}^{\infty} \frac{1}{(\frac{1}{4} + r^2 + s^2)^2 - s^2} |E_{jl}(z, \frac{1}{2} + ir)|_V^2 dr \leq \frac{2\pi}{|k| + 2} C(k, M, d).
\]

This completes the proof. \( \square \)

5. The wave distribution associated to the weighted Laplacian

5.1. The heat and Poisson kernel. In this section, we define the Poisson kernel for the weighted Laplacian \( \Delta_k \) via the heat kernel. For any \( t > 0 \) and \( \rho \geq 0 \), define the heat kernel

\[
K_{\text{heat}}(t; \rho) := \frac{\sqrt{2}e^{-\rho^2/4}}{(4\pi t)^{3/2}} \int_{\rho}^{\infty} \frac{re^{-r^2/4t}}{\sqrt{\cosh(r) - \cosh(\rho)}} T_{2k} \left( \frac{\cosh(r/2)}{\cosh(\rho/2)} \right) dr,
\]

where

\[
T_{2k}(x) = \frac{1}{2} \left[ (x + \sqrt{x^2 - 1})^{2k} + (x - \sqrt{x^2 - 1})^{2k} \right],
\]
for any real $k$. Here the $k$-th powers are chosen as in Lemma 2. Note that, for $k \in \mathbb{Z}$, the function $T_{2k}(x)$ coincides with the $2k$-th Chebyshev polynomial. The hyperbolic heat kernel on $\mathbb{H}$ is defined by

$$K_\mathbb{H}(t; z, w) := K_{\text{heat}}(t; d_{\text{hyp}}(z, w)) \quad (z, w \in \mathbb{H}).$$

For any $t > 0$ and $k \in \mathbb{R}$, the same argument as in [FJK16] p. 136 shows that the heat kernel $K_\mathbb{H}(t; \rho)$ is strictly monotonic decreasing with respect to $\rho > 0$. In the spirit of [Fay77] p. 157, the hyperbolic heat kernel on $M$ associated to $\Delta_k$ is defined as

$$(23) \quad K_{\text{hyp}}(t; z, w) := \frac{1}{2} \sum_{\gamma \in \Gamma} \chi(\gamma) \left( \frac{cw + d}{cw + d} \right)^k \left( \frac{z - \gamma w}{\gamma w - z} \right)^k K_\mathbb{H}(t; z, \gamma w) \quad (z, w \in \mathcal{F}).$$

**Lemma 13.** For any $k \in \mathbb{R}$ and $t > 0$, $K_{\text{hyp}}(t; z, w)$ converges absolutely and uniformly on any compact subset of $\mathcal{F} \times \mathcal{F}$.

**Proof.** Let $U$ be a compact subset of $\mathcal{F} \times \mathcal{F}$. Since $\chi$ is a unitary multiplier system and $\left| \left( \frac{cw + d}{cw + d} \right)^k \left( \frac{z - \gamma w}{\gamma w - z} \right)^k K_\mathbb{H}(t; z, \gamma w) \right| \leq \sqrt{d |K_\mathbb{H}(t; z, \gamma w)|}$, where $\cdot |_{\text{End}(V)}$ denotes hermitian norm on $\text{End}(V)$. Therefore, in order to prove the absolute and uniform convergence of $K_{\text{hyp}}(t; z, w)$ for any $t > 0$ and $(z, w) \in U$, we need to prove the convergence of the series

$$\sum_{\gamma \in \Gamma} K_\mathbb{H}(t; z, \gamma w)$$

in $\mathbb{C}$. Introduce the counting function

$$N(\rho; z, w) := \# \{ \gamma \in \Gamma \mid d_{\text{hyp}}(z, \gamma w) < \rho \},$$

which is defined for any $\rho > 0$ and $(z, w) \in U$. Then [PR09] gives a bound (uniformly for all $(z, w) \in U$) for the function $N(\rho; z, w)$, namely

$$N(\rho; z, w) = O_{\Gamma}(e^\rho),$$

where the implied constant depends only on $\Gamma$. By Stieltjes integral representation, we have

$$(25) \quad \sum_{\gamma \in \Gamma} K_\mathbb{H}(t; z, \gamma w) = \int_0^\infty K_{\text{heat}}(t; \rho) \ dN(\rho; z, w).$$

Using the fact that $K_{\text{heat}}(t; \rho)$ is a non-negative, continuous and monotonic decreasing function of $\rho$, write

$$(26) \quad \int_0^\infty K_{\text{heat}}(t; \rho) \ dN(\rho; z, w) = O \left( \int_0^\infty K_{\text{heat}}(t; \rho)e^\rho \ d\rho \right).$$

Following the idea of the proof of [FJK16] Proposition 3.3, we obtain that

$$(27) \quad K_{\text{heat}}(t; \rho) \leq e^{-\rho^2/(8t)} G_k(t),$$

where the function $G_k(t)$ is given by

$$G_k(t) := \frac{e^{-t/4}}{(4\pi t)^{3/2}} \int_0^\infty \frac{r e^{-r^2/(8t)}}{\sinh(r/2)} e^{kr} \ dr.$$
Combining (25), (26) and (27), we obtain that
\[
\sum_{\gamma \in \Gamma} K_{H}(t; z, \gamma w) = O_{F}(G_{k}(t)h(t)),
\]
with \( h(t) := \int_{0}^{\infty} e^{-\rho^{2}/(8t)} e^{\rho} d\rho \) and where the implied constant depends only on \( \tilde{\Gamma} \). Hence, the proof is complete. \( \square \)

Notice that using the notations, introduced in Section 2 of the paper, we can rewrite the heat kernel as
\[
K_{\text{hyp}}(t; z, w) = \frac{1}{2} \sum_{\gamma \in \Gamma} \chi(\gamma)J_{\gamma,k}(w)^{-1}H_{k}(z, \gamma w)^{-1}K_{H}(t; z, \gamma w) \quad (z, w \in \mathcal{F}).
\]

Now using the fact that \( H_{k} \) is a weight \( k \) point-pair invariant, \( \chi \) is a multiplier system and the relation
\[
J_{\eta,k}(\gamma z)J_{\gamma,k}(z) = \omega_{2k}(\eta, \gamma)J_{\eta,\gamma,k}(z), \quad (\eta, \gamma \in SL_{2}(\mathbb{R}), z \in \mathbb{H}),
\]
one can easily prove the following equations:
\[
K_{\text{hyp}}(t; \eta z, w) = K_{\text{hyp}}(t; z, w)J_{\eta,k}(z)^{-1}\chi(\eta)^{-1},
\]
\[
K_{\text{hyp}}(t; z, \eta w) = J_{\eta,k}(w)\chi(\eta)K_{\text{hyp}}(t; z, w),
\]
for \( t > 0, z, w \in \mathbb{H} \) and \( \eta \in \Gamma \). The hyperbolic heat kernel \( K_{\text{hyp}}(t; z, w) \) admits the spectral expansion
\[
K_{\text{hyp}}(t; z, w) = \sum_{\lambda_{j} \geq \lambda_{0}} e^{-\lambda_{j}t} \varphi_{j}(z)\varphi_{j}(w)^{t} + \frac{1}{4\pi} \sum_{j=1}^{\infty} \sum_{l=1}^{m_{j}} \int_{-\infty}^{\infty} e^{-(1/4+r^{2})\rho} E_{jl}(z, 1/2 + ir)\overline{E}_{jl}(w, 1/2 + ir)^{t} dr.
\]

Let \( U \subset \mathcal{F} \times \mathcal{F} \) be a compact subset and let \( t > 0 \). Using the sup-norm bound (20) for the eigenfunctions \( \varphi_{j} \) \( (j \geq 0) \) and applying the norm arising from the Hermitian inner product to a \( d \times d \) matrix in (8), we obtain that
\[
\left| \sum_{\lambda_{0} \leq \lambda_{j} < 1/4} e^{-\lambda_{j}t} \varphi_{j}(z)\varphi_{j}(w)^{t} + \sum_{\lambda_{j} \geq 1/4} e^{-\lambda_{j}t} \varphi_{j}(z)\varphi_{j}(w)^{t} \right|_{d \times d} \ll \sum_{\lambda_{j} \geq \lambda_{0}} \lambda_{j}^{-2},
\]
where \( \lambda_{0} = |k|(1-|k|) \) and the implied constant depends only on \( t \) and on the set \( U \), apart from \( k \) and \( d \) (this is possible because the number of eigenvalues in the interval \([\lambda_{0}, 1/4]\) is finite). The series \( \sum_{\lambda_{j} \geq 1/4} \lambda_{j}^{-2} \) is convergent ([P87], Theorem 1.6.5), and so is the series on the right-hand side of (28). Similarly, using Hölder’s inequality and Proposition 14(a) below, we deduce the absolute and uniform convergence of each integral on the right hand side of (28). Therefore, for any \( t > 0 \), the series and the integrals on the right hand side of (28) converge absolutely and uniformly on every compact subset of \( \mathcal{F} \times \mathcal{F} \).

From the integral representation of \( K_{H}(t; d_{\text{hyp}}(z, w)) \) and the spectral expansion (28), we deduce that \( K_{\text{hyp}}(t; z, w) \) satisfies the following estimates, stated component-wise:
\[
K_{\text{hyp}}(t; z, w) = O_{F,k}(t^{-3/2}e^{-d_{\text{hyp}}(z, w)/4t}) \quad \text{as } t \to 0,
\]
\[
K_{\text{hyp}}(t; z, w) = O_{F,k}(e^{-\lambda_{0}t}) \quad \text{as } t \to \infty.
\]
For every $Z \in \mathbb{C}$ with $\text{Re}(Z) \geq -|k|(1-|k|)$, $z, w \in \mathcal{F}$ and $u \in \mathbb{C}$ with $\text{Re}(u) \geq 0$, the translated by $-Z$ Poisson kernel $\mathcal{P}_{M,-Z}(u; z, w)$ is defined as

\begin{equation}
\mathcal{P}_{M,-Z}(u; z, w) := \frac{u}{\sqrt{4\pi}} \int_{0}^{\infty} K_{\text{hyp}}(t; z, w) e^{-Zt} e^{-u^2/4t} t^{-3/2} dt,
\end{equation}

where the integral is taken component-wise. This kernel is a fundamental solution of the associated differential operator $\Delta_k + Z - \partial_u^2$. Furthermore, using the spectral expansion of the heat kernel $K_{\text{hyp}}(t; z, w)$ and the identity (see [JLa03])

\[ e^{-a^2} = \frac{a}{\sqrt{4\pi}} \int_{0}^{\infty} e^{-at^2} dt, \quad a \geq 0, \quad a \in \mathbb{C}, \quad \text{Re}(a) \geq 0, \]

we have the following spectral expansion

\begin{align}
\mathcal{P}_{M,-Z}(u; z, w) &= \sum_{\lambda_0 \leq \lambda_j < 1/4} e^{-u\sqrt{\lambda_j + Z}} \varphi_j(z) \overline{\varphi_j(w)} t + \sum_{\lambda_j \geq 1/4} e^{-u\sqrt{\lambda_j + Z}} \varphi_j(z) \overline{\varphi_j(w)} t \\
&\quad + \frac{1}{4\pi} \sum_{j=1}^{c} \sum_{m=1}^{m_j} \int_{-\infty}^{\infty} e^{-u|m|} E_{jl}(z, 1/2 + ir) E_{jl}(w, 1/2 + ir)^t dr.
\end{align}

Following the steps of the proof of [JLa03 Theorem 5.2], using the estimates (29)–(30) for each component of the heat kernel $K_{\text{hyp}}(t; z, w)$ and the fact that

\[ K_{\text{hyp}}(t; z, w) - \sum_{\lambda_j \leq 1/4} e^{-\lambda_j t} = O(e^{-\lambda t}) \quad \text{as} \quad t \to \infty, \]

where $\lambda$ is the first eigenvalue of $\Delta_k$ bigger than $1/4$, one can deduce that, for $\text{Re}(u) > 0$ and $\text{Re}(u^2) > 0$, the Poisson kernel $\mathcal{P}_{M,-Z}(u; z, w)$ has an analytic continuation for each entry of the matrix to $Z = -1/4$. The continuation is given by

\begin{align}
\mathcal{P}_{M,1/4}(u; z, w) &= \sum_{\lambda_0 \leq \lambda_j < 1/4} e^{-u\sqrt{\lambda_j - 1/4}} \varphi_j(z) \overline{\varphi_j(w)} t + \sum_{\lambda_j \geq 1/4} e^{-u\lambda_j} \varphi_j(z) \overline{\varphi_j(w)} t \\
&\quad + \frac{1}{4\pi} \sum_{j=1}^{c} \sum_{m=1}^{m_j} \int_{-\infty}^{\infty} e^{-u|m|} E_{jl}(z, 1/2 + ir) E_{jl}(w, 1/2 + ir)^t dr,
\end{align}

where $t_j = \sqrt{\lambda_j - 1/4} \geq 0$, for $\lambda_j \geq 1/4$ and for $\lambda_j < 1/4$ we take the principal branch of $\sqrt{\lambda_j - 1/4}$.

When $\Gamma$ is cocompact, the sum over cusps (i.e. the last sum on the right hand side of (32)) is identically zero.

### 5.2. The wave distribution and its integral representation.

Let $L^1(\mathbb{R})$ denote the space of absolutely integrable functions on $\mathbb{R}$ and let $C_0^\infty(\mathbb{R})$ denote its subspace of all infinitely differentiable functions with compact support.

**Definition 14.** For any $a \geq 0$, denote by $L^1(\mathbb{R}, a)$ (resp. $S'(\mathbb{R}, a)$) the space of even functions $g$ in $L^1(\mathbb{R})$ (resp. in the Schwartz space on $\mathbb{R}$) such that $g(u) \exp(|u|a)$ is absolutely dominated by an integrable function on $\mathbb{R}$. Denote the Fourier transform of every $g \in L^1(\mathbb{R}, a)$ by

\begin{equation}
H(r, g) = \int_{-\infty}^{\infty} g(u) \exp(aru) \, du,
\end{equation}

with the domain extended to all $r \in \mathbb{C}$ for which it is well-defined.
Notice that, since \( g \) is assumed to be even,
\[
H(r, g) = 2 \int_0^\infty \cos(ur)g(u) \, du.
\]

The following result is a generalization of Lemma 3 in [JvPS16].

**Lemma 15.** Let \( n \geq 3 \) be an integer.

(a) Let \( g \in L^1(\mathbb{R}, a) \) be such that \( g^{(l)} \in L^1(\mathbb{R}) \) for \( 1 \leq l \leq n \), and \( \lim_{u \to \infty} g^{(l)}(u) = 0 \) for \( 0 \leq l \leq n - 1 \). Then the Fourier transform \( H(r, g) \) is well-defined for \( r \in \{ z \in \mathbb{C} \mid |\text{Im}(z)| \leq a \} \) and satisfies the conditions (S1), (S2'), and (S3) with \( \delta = n - 2 \).

(b) Let \( \eta > 0 \) and let \( g \in S'(\mathbb{R}, a + \eta) \) be such that \( g^{(j)}(u) \exp(|u|(a + \eta)) \) is absolutely bounded by some integrable function on \( \mathbb{R} \) for \( 1 \leq j \leq n - 1 \). Then the function \( H(r, g) \) satisfies the conditions (S1), (S2) for any \( 0 < \epsilon < \eta \), and (S3) with \( \delta = n - 2 \).

(c) If \( g \in S'(\mathbb{R}, a) \), then \( H(r, g) \) is a Schwartz function in \( r \in \mathbb{R} \).

**Proof.** (a) For all \( r \in \{ z \in \mathbb{C} \mid |\text{Im}(z)| \leq a \} \), \( u \in \mathbb{R} \) and \( g \in L^1(\mathbb{R}, a) \), \( |g(u)\exp(iru)| \leq |g(u)|e^{|u|a} \) is dominated by an integrable function on \( \mathbb{R} \), and thus \( H(r, g) \) is well-defined. It is also even with respect to \( r \). Furthermore, for every \( r \in \mathbb{R} \), using the assumptions on the decay of \( g^{(l)} \) for \( 0 \leq l \leq n - 1 \) and the fact that \( g^{(2j+1)}(0) = 0 \) (since \( g \) is even), we obtain that
\[
\frac{1}{2}H(r, g) = \left[ \frac{1}{r} \sin(ur)g(u) \right]_0^\infty - \int_0^\infty \frac{1}{r} \sin(ur)g'(u) du
\]
\[
= \left[ \frac{1}{r^2} \cos(ur)g'(u) \right]_0^\infty - \int_0^\infty \frac{1}{r^2} \cos(ur)g''(u) du
\]
\[
= \ldots
\]
\[
= \left[ (-1)^{1+n/2} \frac{1}{r^{n-1}} \sin(ur)g^{(n-2)}(u) \right]_0^\infty + (-1)^{n/2} \int_0^\infty \frac{1}{r^{n-1}} \sin(ur)g^{(n-1)}(u) du
\]
\[
= \left[ (-1)^{1+n/2} \frac{1}{r^n} \cos(ur)g^{(n-1)}(u) \right]_0^\infty + (-1)^{n/2} \int_0^\infty \frac{1}{r^n} \cos(ur)g^{(n)}(u) du,
\]
when \( n \) is even. For odd \( n \), we obtain a similar series of equations, terminating at
\[
\pm \left[ \frac{1}{r^n} \sin(ur)g^{(n-1)}(u) \right]_0^\infty \pm \int_0^\infty \frac{1}{r^n} \sin(ur)g^{(n)}(u) du.
\]
Hence, using the definition of \( H(r, g) \) and the integrability conditions, it follows that
\[
\frac{(1 + |r|)^n}{2} |H(r, g)| \leq c \cdot \sum_{l=0}^{n} \int_0^\infty |g^{(l)}(u)| du \ll 1,
\]
for some constant \( c \). This proves that \( H(r, g) \) satisfies the condition (S3) for \( r \in \mathbb{R} \) with \( \delta = n - 2 \).

(b) By assumption, there is an integrable function \( G(u) \) dominating \( g(u) \exp(|u|(a + \eta)) \) absolutely. In turn,
\[
|g(u) \cos(ur)| \leq G(u) \exp(-|\eta - \epsilon| |u|)
\]
is uniformly bounded in the strip \( |\text{Im}(r)| \leq a + \epsilon \) for \( 0 < \epsilon < \eta \). Hence, the integral defining \( H(r, g) \) converges absolutely and uniformly on any compact set contained in such a strip, and thus defines a holomorphic function on the open strip \( \{ r \in \mathbb{C} \mid |\text{Im}(r)| < a + \eta \} \). In particular, conditions (S1) and (S2) are satisfied. Similarly, for \( j = 1, \ldots, n-1 \), the
functions \(g^{(j)}(u)\cos(\eta u)\), as well as \(g^{(j)}(u)\sin(\eta u)\), are bounded absolutely and uniformly in \(r\) by some integrable functions \(G_j(u)\exp(-(\eta - \epsilon)|u|)\). Recalling the computation involving partial integration from part (a), we obtain (S3) as well.

c) If \(g \in S'([\mathbb{R},a])\), then its Fourier transform \(H(r,g)\) is a Schwartz function in the variable \(r \in \mathbb{R}\). 

We now define the wave distribution.

**Definition 16** (Wave distribution). Let \(z, w \in \mathcal{F}\). For every \(g \in C_0^\infty(\mathbb{R})\), the wave distribution \(\mathcal{W}_{M,k,\chi}(z, w)\) applied to \(g\) is defined as

\[
\mathcal{W}_{M,k,\chi}(z, w)(g) := \sum_{\lambda_j \geq |k|(1-|k|)} H(t_j, g)\varphi_j(z|\varphi_j(w)|^t
\]

\[
+ \frac{1}{4\pi} \sum_{j=1}^\infty \sum_{l=1}^{m_j} \int_{-\infty}^\infty H(r, g)E_{jl}(z, 1/2 + ir)\overline{E_{jl}(w, 1/2 + ir)}dr,
\]

where \(\sqrt{\lambda_j - 1/4} = t_j \geq 0\) for \(\lambda_j \geq 1/4\) and \(t_j \in (0, iA]\) when \(\lambda_j < \frac{1}{4}\), where \(A\) is defined in (12).

**Proposition 17.** Let \(z, w \in \mathcal{F}\).

(a) For every \(g\) as in Lemma 13 (a) with \(a = A\) and \(n = 4\), the wave distribution \(\mathcal{W}_{M,k,\chi}(z, w)\) is well-defined.

(b) Let \(g \in S'([\mathbb{R}, A])\) satisfy the conditions of Lemma 13 (b) with \(n = 4\). Then \(\mathcal{W}_{M,k,\chi}(z, w)(g)\) represents the automorphic kernel \(K_{1/2}(z, w) = K_{1/2}(z, w)\) for the inverse Selberg Harish-Chandra transform \(\Phi\) of \(H(\cdot, g)\).

**Proof.** (a) By Lemma 13, the function \(H(r, g)\) is well-defined for all \(r \in \mathbb{C}\) with \(|\text{Im}(r)| \leq A\), which implies that the finite sum

\[
\sum_{|k|(1-|k|) \leq \lambda_j < \frac{1}{4}} H(t_j, g)\varphi_j(z|\varphi_j(w)|^t
\]

converges (recall that \(\lim_{j \to \infty} \lambda_j = \infty\)). Furthermore, if \(\lambda_j \geq \frac{1}{4}\), then \(t_j \in \mathbb{R}\) and thus \(H(t_j, g) \ll (1 + |t_j|)^{-4} \text{ as } j \to \infty\).

Fix \(z, w \in \mathcal{F}\) and observe that \(H(t_j, g)\varphi_j(z|\varphi_j(w)|^t \in \mathbb{C}^{d \times d}\). By applying the norm \(\cdot|_{d \times d}\) and Hölder’s inequality, we obtain that

\[
\sum_{\lambda_j \geq \frac{1}{4}} |H(t_j, g)\varphi_j(z|\varphi_j(w)|^t|_{d \times d} = \sum_{\lambda_j \geq \frac{1}{4}} |H(t_j, g)||\varphi_j(z)||\varphi_j(w)|^t
\]

\[
\ll \left( \sum_{\lambda_j \geq \frac{1}{4}} |H(t_j, g)||\varphi_j(z)||^2 \right)^{1/2} \left( \sum_{\lambda_j \geq \frac{1}{4}} |H(t_j, g)||\varphi_j(w)||^2 \right)^{1/2}.
\]

Note that, due to the estimate on \(|H(t_j, g)|\), each of the factors on the right-hand side can be compared with the sum occurring in the pre-trace formula (17). Use (17) with \(s = |k| + 2\) and \(t = |k| + 3\), as in the proof of Proposition 12 to obtain the following.
bound:
\[ \sum_{\lambda_j \geq 4} |H(t_j, g)||\varphi_j(z)|^2_E \ll \sum_{\lambda_j \geq 4} \frac{1}{(1 + |t_j|)^4} |\varphi_j(z)|^2_E \]
\[ \ll \sum_{\lambda_j \geq 4} t_j^4 + t_j^2 \left( \frac{1}{2} + 2s^2 \right) + \frac{1}{16} + \frac{1}{2} s^2 + s^2(s^2 - 1) |\varphi_j(z)|^2_E \]
\[ \leq C(k, M, d) \frac{1}{2(|k| + 2)}. \]

Convergence of the integral (uniform on compact subsets of \( \mathcal{F} \times \mathcal{F} \)) can be proved in a similar way, completing the proof.

(b) Because the properties of \( g \) imposed by assumption imply those claimed in part (a), the wave distribution is well-defined by the spectral expansion with coefficients \( H(r, g) \). The claim follows from Proposition 8 once we establish that \( H(r, g) \) belongs to the image of the Selberg Harish-Chandra transform. Hence, we have to invert steps (i)–(iii) of page 10. The inverse of \( H(r, g) \) under Fourier transformation (iii) is trivially \( g \). Since \( g \) is an even \( C^\infty \)-function, the inverse \( Q : \mathbb{R}^+ \to \mathbb{C} \) of \( g \) under (ii) exists and it is given by
\[ Q(y) = g \left( 2 \log \left( \frac{1}{2} \sqrt{y + 4} + \frac{1}{2} \sqrt{y} \right) \right). \]
It belongs to \( C^\infty(\mathbb{R}^+) \). Since \( g(u) \exp((A + \eta)u) \to 0 \) for \( u \to \infty \), we obtain that
\[ Q(y) \ll \exp \left( -(A + \eta)2 \log \left( \frac{1}{2} \sqrt{y + 4} + \frac{1}{2} \sqrt{y} \right) \right), \]
i.e.
\[ Q(y) \ll (y + 4)^{-(A + \eta)}. \]
Similarly, for its first derivative
\[ Q'(y) = g' \left( 2 \log \left( \frac{1}{2} \sqrt{y + 4} + \frac{1}{2} \sqrt{y} \right) \right) \cdot \frac{1}{\sqrt{y(y + 4)}}, \]
we find that
\[ Q'(y) \ll (y + 4)^{-(A + \eta) - 1}, \]
and for its second one
\[ Q''(y) = g'' \left( 2 \log \left( \frac{1}{2} \sqrt{y + 4} + \frac{1}{2} \sqrt{y} \right) \right) \cdot \frac{1}{\sqrt{y(y + 4)}} \]
\[ - g' \left( 2 \log \left( \frac{1}{2} \sqrt{y + 4} + \frac{1}{2} \sqrt{y} \right) \right) \cdot \frac{y + 2}{(y(y + 4))^{3/2}} \]
we obtain the estimate
\[ Q''(y) \ll (y + 4)^{-(A + \eta) - 2}. \]
By [He76, pp. 455–457], the inverse of \( Q \) under (i) is given by
\[ \Phi(x) = -\frac{1}{\pi} \int_{-\infty}^{\infty} Q'(x + t^2) \left( \frac{\sqrt{x + 4 + t^2} - t}{\sqrt{x + 4 + t^2} + t} \right)^k \ dt, \]
where \( \Phi \in C^1(\mathbb{R}^+) \) satisfies
\[ |\Phi(x)| \ll (x + 4)^{-\alpha} \quad \text{and} \quad |\Phi'(x)| \ll (x + 4)^{-\alpha - 1}, \quad 21 \]
for some \( \alpha > \max\{1, |k|\} \). We have to show that the integral in (38) is \( C^1 \) and satisfies the two conditions (37).

Let \( \beta = A + \eta + 1 \). The bound on \( Q' \) together with its differentiability allows us to conclude that the first condition of (37) holds, once we prove that

\[
\frac{-1}{\pi} \int_0^\infty (x+t^2)^{-\beta} \left[ \left( \frac{\sqrt{x+4+t^2}-t}{\sqrt{x+4+t^2}+t} \right)^k + \left( \frac{\sqrt{x+4+t^2}-t}{\sqrt{x+4+t^2}+t} \right)^{-k} \right] dt \ll (x+4)^{-(\beta-1/2)}.
\]

Let \( x_1 = x + 4 \) and introduce the change of variables \( y = \frac{\sqrt{x_1+t^2}-t}{\sqrt{x_1+t^2}+t} \) in the integral on the left-hand side of (38). Using

\[
t = \frac{\sqrt{x_1}(1-y)}{2\sqrt{y}}, \quad x_1 + t^2 = \frac{x_1(1+y)^2}{4y}, \quad dt = -\frac{\sqrt{x_1}(1+y)}{4y^2} dy,
\]

the integral becomes

\[
\frac{-1}{\pi} 4^{\beta-1} x_1^{-(\beta-1/2)} \int_0^1 (1+y)^{-2\beta+1} (y^{\beta+k-3/2} + y^{\beta-k-3/2}) dy,
\]

which is finite if and only if \( \beta - |k| - 3/2 < -1 \). This inequality in turn holds, due to our choice of \( A \) and \( \eta > 0 \). This proves (38) and the first part of (37).

Next, we prove that \( \Phi(x) \) is \( C^1 \) and that the second bound of (37) holds. In order to prove that \( \Phi(x) \) is \( C^1 \), it is sufficient to show that the integrand in (36) is differentiable in \( x \) and that the derivative of the integrand is bounded by some integrable function. If so, then

\[
\Phi'(x) = \frac{-1}{\pi} \int_x^\infty \frac{d}{dx} \left( Q'(x+t^2) \left( \frac{\sqrt{x+4+t^2}-t}{\sqrt{x+4+t^2}+t} \right)^k \right) dt.
\]

Differentiability of the integrand with respect to \( x \) is obvious, so it remains to prove that

\[
\frac{-1}{\pi} \int_{-\infty}^x \frac{d}{dx} \left( Q'(x+t^2) \left( \frac{\sqrt{x+4+t^2}-t}{\sqrt{x+4+t^2}+t} \right)^k \right) dt \ll (x+4)^{-(\beta+1/2)}.
\]

Analogously to the above, starting with the bound for \( Q'' \), we immediately deduce that

\[
\frac{-1}{\pi} \int_{-\infty}^\infty Q''(x+t^2) \left( \frac{\sqrt{x+4+t^2}-t}{\sqrt{x+4+t^2}+t} \right)^k dt \ll (x+4)^{-(\beta+1/2)}.
\]

Therefore, to prove (39) and complete the proof of part (b), it suffices to show that

\[
\frac{-1}{\pi} \int_0^\infty Q'(x+t^2) \frac{d}{dx} \left[ \left( \frac{\sqrt{x+4+t^2}-t}{\sqrt{x+4+t^2}+t} \right)^k + \left( \frac{\sqrt{x+4+t^2}-t}{\sqrt{x+4+t^2}+t} \right)^{-k} \right] dt \ll (x+4)^{-(\beta+1/2)}.
\]

This can be done analogously to the proof of the first bound in (37), i.e. take \( x_1 = x + 4 \) and change variables to \( y = \frac{\sqrt{x_1+t^2}-t}{\sqrt{x_1+t^2}+t} \). Using the bound for \( Q' \), it follows that the above
Proof. For every matrix-valued function \( \zeta \) with respect to \( \beta \) and set \( \theta = \sqrt{\lambda_j - 1/4} \) for every \( j \geq 0 \) is given by the principal branch of the square root. Recall that, if \( \lambda_j < 1/4 \), then \( t_j \in (0, iA] \), otherwise \( t_j \geq 0 \). Also, in case when \( \Gamma \) is cocompact, the second sum on the right hand side of (40) is identically zero.

Following the same reasoning as in the proof of Proposition 17, part (a) (i.e., using the Hölder inequality, the estimate (11), comparing with the pre-trace formula (17) and
using the bounds obtained in Proposition 12, it is clear that for \( \text{Re}(\zeta) \geq 0 \) the series

\[
\sum_{\lambda_j \geq \lambda_0} w(\zeta, t_j)\varphi_j(z)\overline{\varphi_j(w)}^t
\]

converges absolutely and uniformly on \( \mathcal{F} \times \mathcal{F} \). The same holds for the integral

\[
\int_{-\infty}^{\infty} w(\zeta, |r|)E_{j,l}(z, 1/2 + ir)\overline{E_{j,l}(w, 1/2 + ir)}^t dr,
\]

for any pair \((j, l)\) with \(1 \leq j \leq c_r\) and \(1 \leq l \leq m_j\), when \(\Gamma\) is non-compact.

Therefore, for every arbitrary fixed \(\zeta \in \mathbb{C}\) with \(\text{Re}(\zeta) \geq 0\), \(\widetilde{W}(\zeta; z, w)\) is a well-defined matrix-valued function which converges absolutely and uniformly on \( \mathcal{F} \times \mathcal{F} \). Moreover, any of the first 4 derivatives with respect to \(\zeta\) of \(\widetilde{W}(\zeta; z, w)\) converges uniformly and absolutely, provided \(\text{Re}(\zeta) > 0\). Therefore, term by term differentiation is valid. By differentiating component-wise four times and using the fact that \(\frac{d}{d\zeta}w(\zeta, t) = e^{-\zeta t}\), we obtain that

\[
\frac{d^4}{d\zeta^4} \widetilde{W}(\zeta; z, w) = \sum_{\lambda_0 \leq \lambda_j < 1/4} e^{-\zeta \sqrt{\lambda_j - 1/4}} \varphi_j(z)\overline{\varphi_j(w)}^t + \sum_{\lambda_j \geq 1/4} e^{-\zeta \lambda_j} \varphi_j(z)\overline{\varphi_j(w)}^t
\]

\[
+ \frac{1}{4\pi} \sum_{j=1}^{c_r} \sum_{l=1}^{m_j} \int_{-\infty}^{\infty} e^{-\zeta |r|}E_{j,l}(z, 1/2 + ir)\overline{E_{j,l}(w, 1/2 + ir)}^t dr
\]

\[
= \mathcal{P}_{M,1/4}(\zeta; z, w),
\]

where \(\mathcal{P}_{M,1/4}\) is defined in (31). For \(\text{Re}(\zeta) > 0\), define

\[
\mathcal{P}^{(k)}(\zeta; z, w) = \begin{cases} 
\mathcal{P}_{M,1/4}(\zeta; z, w), & \text{if } k = 0 \\
\int_0^\zeta \mathcal{P}^{(k-1)}(\xi; z, w) d\xi, & \text{if } k \geq 1.
\end{cases}
\]

In the above definition, the integral is taken component-wise over a ray contained in the upper half-plane \(\text{Re}(\zeta) > 0\). With this definition, we have

\[
\mathcal{P}^{(k)}(\zeta; z, w) = \widetilde{W}(\zeta; z, w) + q(\zeta; z, w),
\]

where \(q(\zeta; z, w)\) is a \(d \times d\) matrix-valued function consisting of degree 3 polynomials in \(\zeta\), with coefficients depending on \(z\) and \(w\) at each component. For \(z \neq w\) and \(\zeta \to 0\), the function \(\mathcal{P}^{(0)}(\zeta; z, w)\) has a limit; therefore,

\[
\mathcal{P}^{(k)}(\zeta; z, w) = O(\zeta^k) \quad \text{as} \quad \zeta \to 0.
\]

For every \(u \in \mathbb{R}^+\), define

\[
W(u; z, w) = \frac{1}{2i} \left[ \left( \widetilde{W}(iu; z, w) + q(iu; z, w) \right) + \left( \widetilde{W}(-iu; z, w) + q(-iu; z, w) \right) \right].
\]

We claim that the function \(W(u; z, w)\) satisfies all the required conditions given in the statement. Using the spectral expansion (32) of the Poisson kernel \(\mathcal{P}_{M,1/4}(\zeta; z, w)\) and integrating it four times, we obtain the property (a). Assertion (b) follows using the bound (15) in (43). For a given \(g\) as in the statement, we can derive assertion (c) using integration by parts four times on the right-hand side of (10). \(\square\)
6. The basic automorphic kernel

In this section, we study two automorphic kernels, namely the basic and the geometric automorphic kernels. After defining the basic automorphic kernel \( K_\nu(z, w) \) for any \( z, w \in \mathcal{F} \) in an appropriate complex half \( s \)-plane in terms of the wave distribution applied to a test function, we prove that it has a meromorphic continuation to the whole complex \( s \)-plane. Then we introduce the geometric automorphic kernel \( \tilde{K}_\nu(z, w) \) and show that \( K_\nu(z, w) = \tilde{K}_\nu(z, w) \) for \( \text{Re}(s) > \max\{1, |k|\} \), thus also obtaining the meromorphic continuation of \( \tilde{K}_\nu(z, w) \) to the whole complex \( s \)-plane.

6.1. Construction and meromorphic continuation of the basic automorphic kernel. For \( z, w \in \mathcal{F} \) and \( s \in \mathbb{C} \) with \( \text{Re}(s) > \max\{1, |k|\} \), we define the basic automorphic kernel \( K_\nu(z, w) \) by

\[
K_\nu(z, w) := \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \mathcal{W}_{M, k, \chi}(z, w) \left( \cosh(u)^{-\left(s - \frac{1}{2}\right)} \right).
\]

Here, \( \mathcal{W}_{M, k, \chi}(z, w) \) is the wave distribution and it is applied to the test function

\[
g_\nu(u) = \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \cosh(u)^{-\left(s - \frac{1}{2}\right)}.
\]

Notice that \( g_\nu(u) \) satisfies the conditions in Lemma 15(a) with \( a = A \) and \( n = 4 \) where \( A \) is defined in (15). Thus, \( K_\nu(z, w) \) is well-defined by Proposition 14(a).

Lemma 19. For all \( s \in \mathbb{C} \) with \( \text{Re}(s) > \max\{1, |k|\} \), \( n \in \mathbb{N} \), and \( r \in \mathbb{R} \cup [-A_i, A_i] \), the Fourier transform \( H(r, \cdot) \) (see (43)) of \( g_\nu \), given by (48), satisfies the functional equation

\[
H(r, g_\nu) = \frac{2^{-2n} \Gamma(2s) n}{(\frac{s}{2} - \frac{1}{4} - ir)^n (\frac{s}{2} - \frac{1}{4} + ir)^n} H(r, g_{\nu + 2n}),
\]

where \((.)_n\) denotes the Pochhammer symbol.

Proof. Let \( n \in \mathbb{N} \) and \( s \in \mathbb{C} \) with \( \text{Re}(s) > \max\{1, |k|\} \). Definitions (34) and (48) imply that

\[
H(r, g_\nu) = \frac{2\Gamma(s - \frac{1}{2})}{\Gamma(s)} \int_0^\infty \cos(ur) \cosh(u)^{-(s-1/2)} du.
\]

When \( r \in \mathbb{R} \setminus \{0\} \) or \( r \in [-A_i, A_i] \setminus \{0\} \), or \( r = 0 \), we have (see [GR07, Formulas 3.985.1, 3.512.1 and 3.512.2, respectively])

\[
\int_0^\infty \cos(ur) \cosh(u)^{-\nu} du = \frac{2^{\nu-2}}{\Gamma(\nu)} \frac{\Gamma\left(\nu - ir\right)\Gamma\left(\nu + ir\right)}{\Gamma(s + n)} \frac{\Gamma\left(\nu + 2n\right)}{\Gamma(s)},
\]

where \( \text{Re}(\nu) > A \). Hence, for \( r \in \mathbb{R} \cup [-A_i, A_i] \), using (50) with \( \nu = s - 1/2, \nu = s - 1/2 + 2n \) and the Pochhammer symbol \((.)_n\), we obtain the identity (19). \( \square \)

The functional equation (49) enables us to deduce the meromorphic continuation of the kernel \( K_\nu(z, w) \) to the whole complex \( s \)-plane.

Theorem 20. For any \( z, w \in \mathcal{F} \), the basic automorphic kernel \( K_\nu(z, w) \) admits a meromorphic continuation to the whole complex \( s \)-plane. The possible poles of the function \( \Gamma(s - 1/2)^{-1} \Gamma(s) \) are located at the points \( s = 1/2 \pm i\lambda_j - 2n \), where \( n \in \mathbb{N} \) and \( \lambda_j = 1/4 \pm i\frac{\pi}{2} \) is a discrete eigenvalue of \( \Delta_k \). When \( \nu \) is non-compact, possible poles of \( K_\nu(z, w) \) are also located at the points \( s = 1 - \rho - 2n \), where \( n \in \mathbb{N} \) and \( \rho \in (1/2, 1] \) is a pole of the parabolic Eisenstein series \( E_{ij}(z, s) \), and the points \( s = \rho - 2n \), where \( n \in \mathbb{N} \) and \( \rho \) is a pole of \( E_{ij}(z, s) \) with \( \text{Re}(\rho) < 1/2 \).
Proof. The proof we present here follows closely the proof of [JvPS16, Theorem 10]. We assume $M$ is non-compact; in case of cocompact $\Gamma$, the sums over cusps below are identically zero and there are no poles stemming from poles of the Eisenstein series.

Let $B := \max\{1, |k|\}$. First we prove that $K_s(z, w)$ has a meromorphic continuation to the half-plane $\Re(s) > B - 2n$ for any $n \in \mathbb{N}$. For $s \in \mathbb{C}$ with $\Re(s) > B$ we use the wave representation (52) of $K_s(z, w)$ to obtain

$$K_s(z, w) = \sum_{\lambda_j \geq k(1-|k|)} H(t_j, g_s) \varphi_j(z) \overline{\varphi_j(w)}^t$$

(51)

$$+ \frac{1}{4\pi} \sum_{j=1}^{c_r} \sum_{l=1}^{m_j} \int_{-\infty}^{\infty} H(r, g_s) E_{jl}(z, 1/2 + ir) \overline{E_{jl}(w, 1/2 + ir)}^t dr,$$

where $\lambda_j = 1/4 + t_j^2$.

Letting $h_n(r, s) := (s - 1/4 - \frac{\iota r}{2}) (s - 1/4 + \frac{\iota r}{2})$ and using formula (49) in (51), we get

$$\frac{2^{2n} \Gamma(s)}{\Gamma(s + 2n)} K_s(z, w) = \sum_{\lambda_j \geq k(1-|k|)} \frac{H(t_j, g_{s+2n})}{h_n(t_j, s)} \varphi_j(z) \overline{\varphi_j(w)}^t$$

(52)

$$+ \frac{1}{4\pi} \sum_{j=1}^{c_r} \sum_{l=1}^{m_j} \int_{-\infty}^{\infty} \frac{H(r, g_{s+2n})}{h_n(r, s)} E_{jl}(z, 1/2 + ir) \overline{E_{jl}(w, 1/2 + ir)}^t dr.$$

It can be easily seen that $(a)_n = \prod_{j=0}^{n-1} (a + j)$. This implies that $h_n(r, s) \sim r^{2n}$ as $r \to \infty$. Hence, the series in (52) arising from the discrete spectrum is locally absolutely and uniformly convergent as a function of $s$ for $\Re(s) > B - 2n$ away from the poles of $h_n(r, s)^{-1}$, i.e. away from the zeros of $h_n(r, s)$. Using $(a)_n = \prod_{j=0}^{n-1} (a + j)$, we calculate the zeros of $h_n(r, s)$ for $\Re(s) > B - 2n$, which occur at the points $s = 1/2 \pm it_j - 2m$ for $m = 0, \ldots, n - 1$.

Next, we prove the meromorphic continuation of the integral coming from the continuous spectrum in (52). First, substitute $r \mapsto 1/2 + ir$ so that the integral is now over the vertical line whose real part is $1/2$ and observe that as a function of $s \in \mathbb{C}$ the integral, denote it by $I_{1/2, j}(s)$, is holomorphic for $s \in \mathbb{C}$ with $\Re(s) > B - 2n$ satisfying $\Re(s) \neq 1/2 - 2m$, where $m = 0, \ldots, n - 1$. In order to get the meromorphic continuation of this function across the lines $\Re(s) = 1/2 - 2m$, we will use the same method applied in the proofs of [JKvP10, Theorem 2] and [JvPS16, Theorem 10] or in [vP10].

As a first step, let $m = 0$ and choose $\epsilon > 0$ sufficiently small to guarantee that $E_{jl}(z, s)$ has no poles in the strip $1/2 - \epsilon < \Re(s) < 1/2 + \epsilon$. For $s \in \mathbb{C}$ with $1/2 < \Re(s) < 1/2 + \epsilon$, we apply the residue theorem to the function $I_{1/2, j}(s)$ to obtain the meromorphic continuation $I_{1/2, j}^{(1)}(s)$ of it in the strip $1/2 - \epsilon < \Re(s) < 1/2 + \epsilon$. Then, assuming $1/2 - \epsilon < \Re(s) < 1/2$ and using the residue theorem again, we get the meromorphic continuation $I_{1/2, j}^{(2)}(s)$ of the integral $I_{1/2, j}^{(1)}(s)$ to the strip $-3/2 < \Re(s) < 1/2$. Finally, adding the formulas coming from the applications of the residue theorem we obtain the meromorphic continuation of the integral $I_{1/2, j}(s)$ to the strip $-3/2 < \Re(s) \leq 1/2$. Similarly, we can get the meromorphic continuation of the integral $I_{1/2, j}(s)$ to the strip $-3/2 - 2m < \Re(s) \leq 1/2 - 2m$ for $m = 1, \ldots, n - 1$ by repeating this two-step process.

The poles that arise in this process are at $s = 1 - \rho - 2m$, where $\rho$ is a pole of the Eisenstein series $E_{jl}(z, s)$ belonging to the line segment $(1/2, 1]$, and at $s = \rho - 2m$, where $\rho$ is a pole of the Eisenstein series $E_{jl}(z, s)$ such that $\Re(s) < 1/2$, and $m = 0, \ldots, n - 1$. 26
This completes the proof of the meromorphic continuation of $K_s(z, w)$ to the whole $s$-plane, as $n \in \mathbb{N}$ was chosen arbitrarily. \hfill \Box

6.2. The geometric automorphic kernel. For $\text{Re}(s)$ sufficiently large and for any two points $z, w \in \mathcal{F}$, define the geometric automorphic kernel by

$$K_s(z, w) := \frac{\Gamma(s-k)\Gamma(s+k)}{2\pi\Gamma(s)^2} \sum_{\gamma \in \Gamma} \chi(\gamma) \cosh(d_{\text{hyp}}(z, \gamma w))^{-s}$$

$$\times F(-k, k; s; (1 + \cosh(d_{\text{hyp}}(z, \gamma w)))^{-1})J_{\gamma,k}(w)H_k(z, \gamma w),$$

where $F(-k, k; s; (1 + \cosh(d_{\text{hyp}}(z, \gamma w)))^{-1})$ stands for the (Gauss) hypergeometric function.

Note that it is possible to extend the above definition to $z, w \in \mathbb{H}$. Then, for any fixed $w \in \mathbb{H}$, the function $K_s(z, w)$ can be viewed as a map from $\mathbb{H}$ to End($V$) (which can be identified with $C^{d \times d}$). The following proposition shows that (for sufficiently large $\text{Re}(s)$) for any fixed $w \in \mathbb{H}$, the columns of the $d \times d$ matrix $\tilde{K}_s(z, w)$ belong to the space $\mathcal{H}_k$ (see Section 2.4), when viewed as maps from $\mathbb{H}$ to $V$.

**Proposition 21.** (a) The series in formula (53) converges normally with respect to the operator norm in the ring End($V$) of endomorphisms of $V$ in the variables $(z, w; s)$, with $s$ in the half-plane $\text{Re}(s) > 1$ and $z, w \in \mathcal{F}$. It defines a holomorphic function of $s$ in the half-plane $\text{Re}(s) > 1$. The convergence is uniform when $z, w \in \mathcal{F}$ are restricted to any compact subset of $\mathcal{F}$.

(b) The kernel $K_s(z, w)$ is a meromorphic function of $s$ in the half-plane $\text{Re}(s) > 1$, possessing simple poles in this half-plane only when $|k| > 1$. When $|k| > 1$, the simple poles are located at $s = |k| - n$, for integers $n \in [0, |k| - 1]$.

(c) For each $w \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\text{Re}(s) > \max\{1, |k|\}$ each column of the matrix $\tilde{K}_s(\cdot, w)$ defines a function in $\mathcal{H}_k$.

**Proof.** (a) For any $\gamma \in \tilde{\Gamma}$, the hypergeometric function $F(-k, k; s; (1 + \cosh(d_{\text{hyp}}(z, \gamma w)))^{-1})$ is well-defined for all $s \in \mathbb{C}$, due to the fact that

$$0 < (1 + \cosh(d_{\text{hyp}}(z, w)))^{-1} \leq \frac{1}{2},$$

for any two points $z, w \in \mathcal{F}$ (equality being attained when $z = w$). Moreover, for all $s$ with $\text{Re}(s) > 1$, the function $F(-k, k; s; (1 + \cosh(d_{\text{hyp}}(z, \gamma w)))^{-1})$ is holomorphic, since it is the sum of uniformly convergent holomorphic functions. For all $s \in \mathbb{C}$ such that $\text{Re}(s) > 1$, it is uniformly bounded by $F(-k, k; 1, \frac{1}{2})$.

Since $\chi$ is a unitary multiplier system and $|J_{2;k}(w)| = |H_k(z, w)| = 1$, when $d = 1$, the series appearing in the definition of the kernel $\tilde{K}_s(z, w)$ is dominated (uniformly in $s$, for $\text{Re}(s) > 1$) by the series

$$\sum_{\gamma \in \Gamma} \cosh(d_{\text{hyp}}(z, \gamma w))^{-\text{Re}(s)},$$

which converges in the half-plane $\text{Re}(s) > 1$ (see e.g. [vP10, Lemma 3.3.4]). The convergence is uniform when $z, w \in \mathcal{F}$ are restricted to any compact subset of $\mathcal{F}$.

When $d > 1$, in order to prove the normal convergence of the series in (53), it is sufficient to notice that $\chi$ can be identified with a unitary $d \times d$ matrix, with matrix norm induced from the Hilbert space norm obviously equal to $\sqrt{d}$. The normal convergence follows again from the convergence of the series (54) for $\text{Re}(s) > 1$, which is uniform when $z, w \in \mathcal{F}$ are restricted to any compact subset of $\mathcal{F}$. This proves part (a) of the Proposition.
(b) From part (a) it follows that the sum over \( \tilde{\Gamma} \) on the right-hand side of (53) is a holomorphic function in \( s \), for \( \text{Re}(s) > 1 \). Therefore the poles of \( \tilde{K}_s(z, w) \) in the half-plane \( \text{Re}(s) > 1 \) stem only from possible poles of the factor \( \Gamma(s - k)\Gamma(s + k)/\Gamma(s)^2 \). This factor can have poles in the half-plane \( \text{Re}(s) > 1 \) only when \( |k| > 1 \), and they are located at \( s = |k| - n \), for integers \( n \in [0, |k| - 1] \).

(c) To prove the last part, use [He83] formula (6.11) on p. 387 and note that \( L_2(\Gamma\backslash \mathbb{H}, m, \mathcal{W}) = \mathcal{H}_k \) in Hejhal’s notation. Therefore, it suffices to show that the function

\[
\Phi_s(4u) := \sqrt{\frac{2}{\pi}} \frac{\Gamma(s - \frac{1}{2})\Gamma(s + \frac{1}{2})}{\Gamma(s)} (1 + 2u)^{-s} F(-k, k; s; \frac{1}{2(1+u)}),
\]

where \( u = u(z, w) \) is given by (2), satisfies [He83] Assumption 6.1 on p. 387.

It is clear from the definition of \( F(-k, k; s; \frac{1}{2(1+u)}) \) that it is four times differentiable in \( u \geq 0 \) and its derivatives are uniformly bounded by \( F(-k, k; 1, \frac{1}{2}) \). The function \( F_s(u) = (1 + 2u)^{-s} \) is also four times differentiable as a function of \( u \) for any fixed \( s \in \mathbb{C} \) with \( \text{Re}(s) > \max\{1, |k|\} \) and satisfies the bound

\[
|F_s^{(j)}(u)| \ll (1 + u)^{-j - \text{Re}(s)},
\]

for \( j = 0, 1, 2, 3, 4 \), where the implied constant is independent of \( u \). It follows that \( \Phi_s(t) \) is four times differentiable as a function of the real parameter \( t \geq 0 \). Furthermore, for every \( s \in \mathbb{C} \) such that \( \text{Re}(s) > \max\{1, |k|\} \), the estimate

\[
|\Phi_s^{(j)}(t)| \ll (4 + t)^{-j - \text{Re}(s)}
\]

holds for \( j = 0, 1, 2, 3, 4 \) and \( t \geq 0 \). Thus, the proof is complete.

According to Proposition 21, for any fixed complex number \( s \) with \( \text{Re}(s) > \max\{1, |k|\} \), the kernel \( \tilde{K}_s(z, w) \) can be viewed as a map from \( F \times F \) to \( \text{End}(V) \). In the following proposition we prove that for all such \( s \), automorphic kernels \( \tilde{K}_s(z, w) \) and \( K_s(z, w) \) are equal on \( F \times F \).

**Proposition 22.** For all \( s \in \mathbb{C} \) with \( \text{Re}(s) > \max\{1, |k|\} \) and for all \( z, w \in F \),

\[
\tilde{K}_s(z, w) = K_s(z, w).
\]

**Proof.** In view of Proposition 21(c), it suffices to show that, for \( \text{Re}(s) > \max\{1, |k|\} \), the functions \( \tilde{K}_s(z, w) \) and \( K_s(z, w) \) have the same coefficients in the spectral expansion, which amounts to showing that the function \( \Phi_s(4u) \) defined in (55) is the inverse Selberg Harish-Chandra transform of the Fourier transform \( h_s \) of the function

\[
g_s(u) = \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} (\cosh u)^{-(s - \frac{1}{2})} = Q_s(e^u + e^{-u} - 2).
\]

Based on [He83] Formula (6.6) on p. 386, this is equivalent to showing that

\[
\Phi_s(4u) = \frac{\Gamma(s + \frac{1}{2})}{\pi \Gamma(s)} 2^{s-\frac{1}{2}} \int_{-\infty}^{\infty} \left(4u + t^2 + 2\right)^{-(s + \frac{1}{2})} \left(\frac{\sqrt{4u + 4 + t^2} - t}{\sqrt{4u + 4 + t^2} + t}\right)^k \, dt.
\]

Substitute \( y = 4u \) and denote by \( I(y) \) the integral on the right-hand side of (57). It follows immediately that

\[
I(y) = \int_0^\infty \left(\alpha + t^2 - 2\right)^{-(s + \frac{1}{2})} \left[\left(\frac{\sqrt{\alpha + t^2} - t}{\sqrt{\alpha + t^2} + t}\right)^k + \left(\frac{\sqrt{\alpha + t^2} - t}{\sqrt{\alpha + t^2} + t}\right)^{-k}\right] \, dt,
\]

where \( \alpha = \frac{y}{4} \).
where $\alpha = y + 4$. Introducing a new variable $x = \sqrt{\frac{\alpha+1}{2}}$, we obtain that

$$I(\alpha - 4) = \alpha^{-s}4^{-\frac{1}{2}} \int_0^1 (x^2 + 2x(1 - \frac{4}{\alpha}) + 1)^{-(s+\frac{1}{2})}((x^{s+k} + x^{s-k}) + (x^{s-k} + x^{s+k+1}))dx.$$ 

Substituting $x = 1/x$ in the two integrals containing exponents $x^{s-k}$ and $x^{s+k}$ yields the following simplified expression:

$$I(y) = \alpha^{-s}4^{-\frac{1}{2}} \int_0^\infty \frac{x^{s+k} + x^{s-k}}{(x^2 + 2x\frac{u}{u+1} + 1)^{s+\frac{1}{2}}}dx.$$ 

Next, write $I(y) = I(4u) = I_+(4u) + I_-(4u)$, where

$$I_\pm(4u) = \frac{1}{2}(u+1)^{-s} \int_0^\infty \frac{x^{s+k}}{(x^2 + 2x\frac{u}{u+1} + 1)^{s+\frac{1}{2}}}dx.$$ 

The integral on the right-hand side of the above equation appears in Formula 8.714.2 of [GR07] for the integral representation of the Legendre function, with $\cos(\varphi) = u/(u+1) \in (0,1)$, $\mu = s$ and $\nu = \pm k$. For $\Re(s) > \max\{1, |k|\}$, the conditions $\Re(\mu \pm \nu) > 0$ are fulfilled, hence

$$I_\pm(4u) = \frac{1}{2}(u+1)^{-s} \int_0^\infty \frac{x^{s+k}}{(x^2 + 2x\frac{u}{u+1} + 1)^{s+\frac{1}{2}}}dx$$

$$= \frac{2^{s-1}\Gamma(s+1)\Gamma(s \pm k + 1)\Gamma(s \mp k)}{\Gamma(2s+1)(1+2u)^{s/2}} \int_{\mu\pm\nu} P_{\mu\pm\nu} \left( \frac{u}{u+1} \right),$$

where $P_{\nu}$ stands for the Legendre function. Inserting the above expression for $I_\pm(4u)$ into Equation (57) after applying the doubling formula $\Gamma'(2s+1) = 2^{2s}\pi^{-\frac{1}{2}}\Gamma(s + \frac{1}{2})\Gamma(s+1)$ for the gamma function, we obtain that

$$\Phi_s(4u) = \frac{2^{-\frac{3}{2}}\Gamma(s+k)\Gamma(s-k)}{\sqrt{\pi}} \left((s+k)P_{s-k}^{-s} \left( \frac{u}{u+1} \right) + (s-k)P_{s+k}^{-s} \left( \frac{u}{u+1} \right)\right)(1+2u)^{-\frac{s}{2}}.$$ 

In order to prove Equation (57), it is left to show that

$$(s+k)P_{s-k}^{-s} \left( \frac{u}{u+1} \right) + (s-k)P_{s+k}^{-s} \left( \frac{u}{u+1} \right) = \frac{2}{\Gamma(s)}(1+2u)^{-\frac{s}{2}} F \left( \begin{array}{c} -k, \pm k + 1; s+1; \frac{1}{2(1+u)} \end{array} \right).$$

Apply Formula 8.704, with $x = u/(u+1) \in (0,1)$, $\mu = -s$ and $\nu = \pm k$; in order to express the Legendre function in terms of the hypergeometric function:

$$P_{s-k}^{-s} \left( \frac{u}{u+1} \right) = \frac{(1+2u)^{-\frac{s}{2}}}{\Gamma(s)} F \left( \begin{array}{c} -k, \pm k + 1; s+1; \frac{1}{2(1+u)} \end{array} \right).$$

Therefore, proof of Equation (57) reduces to proving that

$$\frac{1}{s} \left((s+k)F \left( -k, k+1; s+1; \frac{1}{2(1+u)} \right) + (s-k)F \left( k, -k+1; s+1; \frac{1}{2(1+u)} \right)\right)$$

$$= 2F \left( -k, k; s; \frac{1}{2(1+u)} \right).$$

The above identity follows immediately from the definition of the hypergeometric function and the property $(a+1)_j = (a)_j \frac{a+j}{a}$ of the Pochhammer symbol $(a)_j = \Gamma(a+j)/\Gamma(a)$, for all non-negative integers $j$, applied with $a = k$ and $a = -k$. \qed
Remark 23. When $k = 0$ and $\chi$ is the identity, the hypergeometric series $F\left(-k; k; s; \frac{1}{2(1+\chi)}\right)$ is identically equal to one, hence the series $\tilde{K}_s(z, w)$ coincides with the automorphic kernel $K_s(z, w)$ defined in [JvPS16 Formula (19)], up to the constant $\frac{1}{\sqrt{\pi}}$.

By the uniqueness of meromorphic continuation, combining the above proposition with Theorem 20 we arrive at the following corollary:

**Corollary 24.** For any $z, w \in \mathcal{F}$, the geometric kernel $\tilde{K}_s(z, w)$ admits a meromorphic continuation to the whole complex $s$-plane. The possible poles of the function $\Gamma(s)\Gamma(s - 1/2)^{-1}\tilde{K}_s(z, w)$ are located at the points $s = 1/2 \pm it_j - 2n$, where $n \in \mathbb{N}$ and $\lambda_j = 1/4 + t_j^2$ is a discrete eigenvalue of $\Delta_k$. In case when $M$ is non-compact, possible poles of $\tilde{K}_s(z, w)$ are also located at the points $s = 1 - \rho - 2n$, where $n \in \mathbb{N}$ and $\rho \in (1/2, 1]$ is a pole of the parabolic Eisenstein series $E_{\rho}(z, s)$, and at the points $s = \rho - 2n$, where $n \in \mathbb{N}$ and $\rho$ is a pole of $E_{\rho}(z, s)$ with $\text{Re}(\rho) < 1/2$.

**Remark 25.** Corollary 24 illustrates the strength of the approach to constructing Poincaré series using generating kernels. Namely, in order to deduce the meromorphic continuation of the automorphic kernel $\tilde{K}_s(z, w)$ from its geometric definition, one would have to consider some type of Fourier expansion (e.g. an expansion in rectangular or spherical coordinates at a certain point) and investigate certain properties of the coefficients in the expansion (e.g. uniform boundedness, analyticity, etc.). This is a heavy task, which we overcome by considering the wave distribution acting on the function $g_s$ defined in (50).

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