Equilibrium customer and socially optimal balking strategies in a constant retrial queue with multiple vacations and N-policy

Zhen Wang · Liwei Liu · Yiqiang Q. Zhao

Abstract
In this paper, equilibrium strategies and optimal balking strategies of customers in a constant retrial queue with multiple vacations and the N-policy under two information levels, respectively, are investigated. We assume that there is no waiting area in front of the server and an arriving customer is served immediately if the server is idle; otherwise (the server is either busy or on a vacation) it has to leave the system to join a virtual retrial orbit waiting for retrials according to the FCFS rules. After a service completion, if the system is not empty, the server becomes idle, available for serving the next customer, either a new arrival or a retried customer from the virtual retrial orbit; otherwise (if the system is empty), the server starts a vacation. Upon the completion of a vacation, the server is reactivated only if it finds at least N customers in the virtual orbit; otherwise, the server continues another vacation. We study this model at two levels of information, respectively. For each level of information, we obtain both equilibrium and optimal balking strategies of customers, and make corresponding numerical comparisons. Through Particle Swarm Optimization (PSO) algorithm, we explore the impact of parameters on the equilibrium and social optimal thresholds, and obtain the trend in changes, as a function of system parameters, for the optimal social welfare, which provides guiding significance for social planners. Finally, by comparing the social welfare under two information levels, we find that whether the
system information should be disclosed to customers depends on how to maintain the growth of social welfare.

**Keywords** Multiple vacations · Equilibrium strategies · Balking strategies · Particle swarm optimization algorithm · Information accuracy

**Mathematics Subject Classification** 60K25 · 91B50

### 1 Introduction

In many service and electronic commerce systems, there exists a new trend to study the behavior of customers in queuing models. In these models, customers can decide whether to join or balk, according to a natural tendency to maximize their personal utility. To this end, from the perspective of game theory, the decentralized behavior of customers in the queuing system has attracted extensive attentions in recent decades. Generally, queuing systems are divided into the observable case and the unobservable case depending on whether customers can obtain the information about the system upon arrival. The observable case was first studied by Naor (1969), who analyzed an $M/M/1$ queue model with a linear reward-cost structure, and obtained equilibrium and social optimal strategies. Subsequently, Naor’s study was extensively extended, see e.g. Edelson and Hilderbrand (1975), Johansen and Stidham (1980), Stidham (1985). Specifically, Edelson and Hilderbrand (1975) complemented the unobservable case to Naor’s model. Chen and Frank (2001) generalized the model of Naor’s, who assumed that customers and servers use the same discount rate to maximize their expected discount utility. Afterward, some authors studied equilibrium strategies in various invisible models with many different characteristics. The monograph of Hassin and Haviv (2003) summarized the main results of the subject under different levels of information.

The present paper aims to discuss equilibrium strategies and socially optimal balking strategies of customers in an $M/M/1$ constant retrial queue with multiple vacations and the $N$-policy. Customers’ retrials are a common phenomenon in service systems and enterprise engineering. For example, arriving calls to a call center will be connected immediately if service staff is available, otherwise customers may have to retry for service after a random time. With the development of information technology, modern call centers may provide some levels of information to callers, e.g., the current number of customers waiting for service and/or expected waiting time, among other possibilities. Server vacation is another useful concept in modeling for situations, in which optimization of resources and/or reduction of cost are/is required. For vacation models, due to technical and cost (or other) reasons, the server might not be able to obtain the information about the current system capacity during the vacation, or it is impossible for the server to immediately return to work when the number of customers reaches a predetermined threshold, or the number of customers in the system is small at the end of the server vacation so that the server is reluctant to return to work, or return to work at the normal service rate. In addition, too frequent startups and changeovers on operations could lead to severe server wear and overhead...
on cost, the $N$-policy is usually used to solve this predicament, such as warehouse management systems, order-based production systems and possible others. With an $N$-policy, when all customers are served, i.e., the system is empty, the server will shut off. Once the system has accumulated $N$ customers, the server will be activated again. Yadin and Naor (1963) first presented the $N$-strategy in the queuing system in 1963. Subsequently, several works (Balachandran 1973; Shanthikumar 1981; Doshi 1986) promoted this control policy through different models and methods. But none of these works with $N$ policy have studied strategic behavior. In recent years, some works (Guo and Hassin 2011; Guo and Li 2013; Sun et al. 2016; Wang et al. 2017) have analyzed the strategic behavior of the queuing system with $N$-policy, which has attracted more and more scholars to study queuing system with $N$-policy from an economic viewpoint. Therefore, a model combining customer retrials, server multiple vacations, and the $N$-policy is of practical interest and is our focus of this paper.

As for studies on equilibrium balking strategies of customers, Burnetas and Economou (2007) considered queueing models with setup times under several information levels; Economou and Kanta (2008) discussed balking strategies for an observable queue with breakdowns and repairs; Liu et al. (2012) explored an observable queue under single vacation policy; Ma et al. (2013) presented equilibrium balking behavior under a multiple vacation policy; Sun et al. (2016) investigated equilibrium strategies and optimal balking strategies for an unobservable queue with double adaptive working vacations. Customers’ equilibrium strategies for queue systems with retrial were also reported in the literature, for example, when balking is not allowed (Kulkarni 1983; Zhang et al. 2012), and when balking is allowed (Economou and Kanta 2011; Wang and Zhang 2013; Kumar et al. 2010). Regarding models implemented with the $N$-policy, Guo and Hassin (2011), Guo and Hassin (2012) investigated models at two information levels with homogeneous and heterogeneous customers, respectively; Guo and Li (2013) addressed the same issue for systems, which are partially observable, such as the system capacity is observable, or the state of system is observable. Wang et al. (2017) presented customers’ strategic behavior and the social optimal problem in a constant retrial queue with the $N$-policy. Sun et al. (2017) discussed equilibrium strategies and balking strategies with multiple vacations and the $N$-policy.

However, different from the previously mentioned literature on the $N$-policy, the present paper assumes that the system can be reactivated if and only if, upon the completion of a vacation, the server finds at least $N$ customers in the virtual orbit; otherwise, the server continues another vacation.

This paper studies equilibrium strategies and optimal balking strategies of customers in a queue with a constant retrial rate, multiple server vacations, and the $N$-policy under two information levels (the observable case and the unobservable case). In this system, there is no waiting area in front of the server and an arriving customer will be serviced immediately if the state of server is idle; otherwise (when the state of the server is busy or on vacation, it has to leave the system to join a virtual retrial orbit waiting for retries according to the FCFS rule. After the completion of each service, the server will take a vacation if the system is empty, or becomes idle if there is at least one customer in the orbit. The idle server will serve the next customer, either a new arrival or a retried customer, whichever comes earlier. The server
be reactivated, upon return from the vacation, if at least $N$ customers are presented in the virtual orbit; otherwise, the server will start another vacation.

To conclude our main contributions made in this paper, we emphasize that to our best knowledge, a model, which combines features of the retrial, multiple vacations, and the $N$-policy, has not been considered in the literature for the purpose of customers’ equilibrium and optimal balking strategies. Although our work combines the work of Wang et al. (2017), Sun et al. (2016), and others, our work deals with problems completely different from their work, or our work in the analysis of more complex or can summarize some of the existing work. For example, the works of Wang et al. (2017) assumes that arriving customers can only observe the server’s state. Whereas in our work, there are two types to present information level, i.e., observable case and unobservable case. Moreover, the framework and problem setting of Wang et al. (2017) are different from ours. Our work is more complex than the work of Sun et al. (2016) and can summarize their work. The first contribution is our problem construction and derivation of performance measures that are important for equilibrium analysis. Second, for each type of information level, we determine equilibrium strategies and optimal balking strategies of customers and social welfare. For the observable case, in order to ensure that the server can be reactivated, we derive the optimal balking threshold of customers in the vacation state, which must be greater than $N - 1$. Therefore, there are three different queuing cases for the observable case, and we study the corresponding stationary distributions for the three queuing cases, and obtain the equilibrium social welfare per time unit. For the unobservable case, we derive the positive equilibrium arrival rate and optimal arrival rate. Third, due to the complexity of equations involved, explicit expressions for the socially optimal balking thresholds and optimal social welfare are not available in general. Hence, we use the improved Particle swarm optimization (PSO) algorithm to solve the complex analytic characteristics, by which the numerical optimal solution $(n^*(1), n^*(2))$ and optimal social welfare $U_s(n^*(0), n^*(1))$ are obtained. By comparing the numerical results for the two different information levels, respectively, we conclude that the customers’ equilibrium behavior makes the system more congested than that under the socially optimal strategy, and whether the system information should be disclosed to customers depends on how to maintain the growth of the social welfare (i.e., potential demand arrivals). Obviously, in order to maximize social welfare, which factor determines the level of information disclosure and when to disclose system information to customers are also crucial for the server or social planner.

Motivated by cost control and information guidance, this work wants to present a practical queuing system suitable for cost control, avoiding frequent startup or damage of equipment, efficient office of the company, etc. More specifically, this work considers $N$-policy, multiple vacations, retrial behavior, and integrates two different information levels, which makes this work more thoughtful. On the other hand, our urgent motivation is to study this integrated model’s customer strategy behavior from the economic viewpoint based on a game-theoretic analysis. The research results provide management guidance for social planners. Two potential application scenarios of our model are given below.

The proposed model has a potential application in company document processing. A company often has many departments, and the general manager needs to examine
and approve each department’s documents within a certain period or go out to deal with other matters. Considering the working time and efficiency of the general manager, the assistant will collect the documents of each department and submit them to the general manager for approval in a suitable period or when a certain amount is accumulated. If there are newly arrived documents within the general manager’s approval period, these documents can be handled directly by the general manager. New documents arriving outside the general manager’s approval period can be listed on the assistant’s waiting list according to the first-come, first-served (FCFS) discipline or the messenger can give up the possibility of waiting. In this scenario, documents, assistant’s waiting list, seek services independently of other documents after some time, general manager, and a suitable period or a certain amount corresponding to the customers, the orbit, retrial policy, the server, and \( N \)-policy, respectively, in the queueing terminology.

As another application of our retrial model, we take an order-based production system as an example. In order to avoid frequent start-up of equipment and unprofitable production (considering the profit and cost of product production), the factory’s production line will start up only after signing the order quantity sufficient to cover all expenses. A production line of a factory is usually in three periods: vacation, idle and busy. The vacation period is generally used to sign orders sufficient to cover costs (substantial orders and minor orders); the idle period is used to prepare materials for production; the busy period is used to produce orders signed. If an idle production line’s surplus material can make a new arrival order, the production line can directly complete the order. New orders that arrive during the busy period can be joined the waiting list or abandon the factory contract. The new order is independent of other orders and is completed on the first-come, first-served (FCFS) discipline. In this scenario, orders, waiting list, the new order is independent of other orders and is completed on the FCFS principle, the production line of the factory, and order quantity sufficient to cover all expenses corresponding to the customers, the orbit, retrial policy, the server, and \( N \)-policy, respectively, in the queueing terminology.

The remaining sections are organized as follows: In Sect. 2, we describe the model in detail. We derive the corresponding stationary distributions for the three queueing cases, equilibrium thresholds, and social welfare per time unit in Sect. 3. Section 4 contributes to studies for the unobservable case, and we derive the equilibrium arrival rate and optimal arrival rate. Section 5 focuses on using numerical analysis to explore the theoretical findings in the previous sections and compare the observable and unobservable cases of this model. Section 6 presents discussions and possible further studies.

## 2 Model description

Consider a single-server retrial queueing system with a constant retrial rate, multiple server vacations, and with \( N \)-policy and exhaustive service. We assume that customers arrive to the system according to a Poisson process with rate \( \lambda \), served by a single server with exponential service rate \( \mu \). There is no waiting area in front of the server. An arriving customer will be serviced immediately if the server is idle and leave the system immediately upon the completion of the service; otherwise (the server is either busy or
on a vacation), it will join a virtual retrial orbit according to the first-come, first-served
discipline (FCFS). In practice, a customer in the orbit can be viewed as a customer
on the waiting list. After the completion of a service, the server becomes idle and
immediately searches for the customer from the top of the waiting list. The time of
the search is a random variable, exponentially distributed with rate $\theta$. In the search
process, if a new customer arrives, the search will be immediately interrupted and
the server will return to serving the arriving customer; otherwise (no arrivals during
the search process), the customer at the head of the waiting line will be served and
will leave the system upon the completion of its service. After all customers in the
system are served, or when the system becomes empty, the server will take a vacation
of exponential amount of time $V$ with rate $\xi$. During the vacation time, the server will
be not available to serve customers. Upon the completion of a vacation, the server will
continue to another (independent) vacation with the same parameter if there are fewer
than $N$ customers in the system; otherwise, the server will return from vacations (to
idle state) and immediately start the same search process as that for the case, described
above, when the server becomes idle from busy. This type of queue systems is referred
to as the M/M/1/MV queue. Inter-arrival times of customers, service times and the
times of retrials are assumed to be mutually independent.

The state of the system at time $t$ can be represented by a random vector
$\{(M(t), I(t))\}$, where $I(t)$ denotes the number of customers in the orbit, and $M(t)$
denotes the state of the server at time $t$:

$$M(t) = \begin{cases} 
0, & \text{on vacation;} \\
1, & \text{busy;} \\
2, & \text{idle.}
\end{cases}$$

Obviously, the stochastic process $\{(M(t), I(t))\}$ is a continuous-time Markov chain.
The corresponding transition rate diagram is shown in Fig. 1. Moreover, the observa-
able case means that arriving customers can observe all information about $M(t)$ and
$I(t)$, and the unobservable case implies that arriving customers can not observe any
information about $M(t)$ and $I(t)$.
The study of strategic customer behavior is important. In our case, we are interested in deciding whether an arriving customers would join or balk the system. Suppose that every customer receives the same reward of $R$ units for completing its service, which is used to quantify customer satisfaction or the added value of service. In addition, there is a waiting cost of $C$ units per time unit. The total waiting time for a customer is the continuous accumulation of the time when the customer reaches the system and until he/she leaves the system (including the service time). Customers are risk neutral and want to maximize their expected net benefit. Specifically, if a customer receives the reward of service is more than the expected waiting cost, then he/she will join the system. If a customer receives the reward of service equal to the expected waiting cost, the customer will be indifferent between joining and balking. Therefore, we only need to consider the reward with satisfying the following inequality:

$$R > \frac{C}{\mu}.$$  \hspace{1cm} (1)

The above inequality assures that all customers, who find the server being idle, always enter the system since his/her reward $R$ is more than the waiting cost during his/her expected service time $(1/\mu)$. We adopt a natural linear reward-cost structure, and $U$ is defined as the expected net benefit after the service completion, i.e., $U = R - CE[W]$, where $E[W]$ is mean sojourn time of the customer. Obviously, if the customer is balking, it will generate $U = 0$.

Under both levels of information levels, under the condition in (1), customers will be sure to enter the system if they find that the state of the system is idle upon arrivals. However, if an arriving customer finds a vacation or busy state, he/she has to decide whether to leave his/her contact details (enter retrial orbit) or leave for ever. We further assume that the arriving customers know the policy of the system, i.e., their decisions are irrevocable: the balking customers cannot retry and customers, who joined the system, cannot renege.

### 3 The observable case

As mentioned above, the observable case means that arriving customers can observe all information about $M(t)$ and $I(t)$. Obviously, the information about $I(t)$ (orbit length) is useful when the arriving customer finds the server being busy or on the vacation, but it is useless when the customer finds the server being idle upon arrivals, since in this case the customer will be served immediately regardless of the orbit length. Therefore, in all cases, the information about the system (both $M(t)$ and $I(t)$) is valuable for the arriving customer to make a better assessment on whether or not he/she should join the system. More specifically, if $M(t)$ is idle, the customer will join the system for sure regardless of $I(t)$; if $M(t)$ is busy or on vacation, according to the FCFS discipline, the arriving customer knows his/her position in the orbit, which can help him in deciding whether entering the system is preferable. In the observable case, define $W(i, n)$ to be the sojourn time of the marked customer, who joins the system at state $(i, n - 1) \ (i = 1, 2, 3)$. For studying optimal balking strategies, we need to
consider the expected (residual) net benefit of the marked customer, who is at the \(n\)th position in the orbit and the state of the server is \(i\), after he/she receives the service. We denote the equilibrium balking threshold of customers and the integrated strategy at state \(i\) by \(n_e(i)\) and \((n_e(0), n_e(1))\), respectively. In addition, we denote the socially optimal balking threshold of customers and the integrated strategy at state \(i\) by \(n^*(i)\) and \((n^*(0), n^*(1))\), respectively. To characterize \(n_e(0)\) and \(n_e(1)\), we first give the following theorem.

**Theorem 1** For the \(M/M/1\) constant retrial queue with multiple vacations and \(N\)-policy, when a marked customer is at \(n\)th position in the orbit and the state of server is \(i\) (\(i = 0, 1, 2\)), the mean (residual) sojourn time \(T(i, n)\) of the marked customer are given by, respectively,

\[
T(0, n) = \frac{1}{\xi} + n \cdot \frac{\lambda + \theta + \mu}{\mu \theta}, \quad n \geq N. 
\]

\[
T(1, n) = n \cdot \frac{\lambda + \theta + \mu}{\mu \theta} + \frac{1}{\mu}, \quad n = 0, 1 \ldots
\]

\[
T(2, n) = n \cdot \frac{\lambda + \theta + \mu}{\mu \theta}, \quad n = 1, 2 \ldots
\]

**Proof** Consider a marked customer arrived to the system, who found that the server is busy or on vacation. Clearly, the mean overall sojourn time of the marked customer is not affected by customers who arrive after the marked customer by finding the server being busy or on vacation, but it is affected by the customers, who enter the system after the marked customer by finding the server being idle, since in this case, by our imposed condition (1) they will join the system to receive the service immediately.

Since \(T(1, 0)\) represents the mean residual service time of the customer, who is receiving the service, we have

\[
T(1, 0) = \frac{1}{\mu}.
\]

For \(n \geq 1\), let \(m(n)\) be the probability of joining the virtual orbit for the arriving customer, who finds the server being busy and \(n\) customers being in the orbit. Then, based on a first step argument and noticing that the mean time to the next event is \(1/(\lambda m(n) + \mu)\), and the next event is an arrival or a service completion with probability \(\lambda m(n)/(\lambda m(n) + \mu)\) or \(\mu/(\lambda m(n) + \mu)\), respectively, we have

\[
T(1, n) = \frac{1}{\lambda m(n) + \mu} + \frac{\lambda m(n)}{\lambda m(n) + \mu} T(1, n) \\
+ \frac{\mu}{\lambda m(n) + \mu} T(2, n), \quad n = 1, 2, \ldots
\]

When the server state is idle, we can similarly get

\[
T(2, n) = \frac{1}{\lambda + \theta} + \frac{\lambda}{\lambda + \theta} T(1, n) + \frac{\theta}{\lambda + \theta} T(1, n - 1), \quad n = 1, 2 \ldots
\]
For \( i = 0 \), we only need expressions for \( n \geq N \) (see Remark 1), which is given by

\[
T(0, n) = \frac{1}{\xi} + T(2, n).
\] (8)

In terms of (7) and by solving (6) for \( T(1, n) \), we get

\[
T(1, n) = \frac{\lambda + \theta + \mu}{\mu \theta} + T(1, n - 1), \quad n = 1, 2, \ldots,
\] (9)

which leads to (3). Substituting (3) into (7) produces (4). Finally, substituting (4) into (8) gives (2), which completes the proof. \( \square \)

**Remark 1** In the above theorem, we did not provide the expression for \( T(0, n) \) when \( n < N \), since for our purpose, we only need the expression when the server can be reactivated, or \( n \geq N \).

### 3.1 Equilibrium

We first study the equilibrium balking behavior of customers in the observable case, i.e., the customers can observe both information of \( M(t) \) and \( I(t) \) at time \( t \). As mentioned above, the condition (1) ensures that the customers who find the server is idle always enter the system. The customers who find a vacation or busy state have to decide whether to leave their contact details (enter retrial orbit) or leave for ever. Therefore, we only need to consider that the system is in the state of vacation or busy upon the customers arrivals.

From Theorem 1, it is easy to know that the sojourn time \( W(0, n) \) of the marked customer satisfies the following equation when he/she encounters the system state \( (0, n) \):

\[
E[W(0, n)] = T(0, n) = \frac{1}{\xi} + n \cdot \frac{\lambda + \theta + \mu}{\mu \theta}, \quad n \geq N.
\] (10)

Define \( U_e(0, n) = R - CE[W(0, n)] \) to be the corresponding residual net benefit of the marked customer, and solve \( U_e(0, n) = 0 \) to get the equilibrium balking threshold:

\[
n_e(0) = \left\lfloor \frac{\mu \theta}{\lambda + \theta + \mu} \left( \frac{R}{C} - \frac{1}{\xi} \right) \right\rfloor,
\] (11)

where the floor function \( \lfloor x \rfloor \) is the largest integer smaller than \( x \). Similarly, when the server state is \( i = 1 \), we have

\[
E[W(1, n)] = T(1, n) = \frac{1}{\mu} + n \cdot \frac{\lambda + \theta + \mu}{\mu \theta}, \quad n = 0, 1, \ldots
\] (12)
Define $U_e(1, n) = R - C E[W(1, n)]$ to be the corresponding residual net benefit of the marking customer, and solve $U_e(1, n) = 0$ to get the balking threshold:

$$n_e(1) = \left\lfloor \frac{\mu \theta}{\lambda + \theta + \mu} \left( \frac{R}{C} - \frac{1}{\mu} \right) \right\rfloor. \quad (13)$$

Obviously, there are two possibilities: (i) if $\mu > \xi$, which implies $n_e(1) > n_e(0)$; and (ii) if $\xi > \mu$, which implies $n_e(0) > n_e(1)$. Moreover, in order to ensure that the server can always be reactivated, $n_e(0) > N - 1$ surely should be guaranteed. Hence, we need to discuss the stationary distribution of the system in three cases: Case 1: $N - 1 \leq n(0) < n(1)$; Case 2: $N - 1 \leq n(1) \leq n(0)$; and Case 3: $n(1) < N - 1 \leq n(0)$ for the observable case. Our focus in this section is to characterize the integrated balking threshold strategy $(n(0), n(1))$ for these three cases.

Case 1: For $N - 1 \leq n(0) < n(1)$, the corresponding transition rate diagram is showed in Fig. 2, and the state space of $\{(M(t), I(t))\}$ is given by:

$$\Omega_{ob1}^e = \{(0, n) : 0 \leq n \leq n(0) + 1\} \cup \{(1, n) : 0 \leq n \leq n(1) + 1\} \cup \{(2, n) : 1 \leq n \leq n(1) + 1\}. \quad (14)$$

Define the stationary distribution as

$$\pi_{i,n} = P\{M = i, I = n\} = \lim_{t \to \infty} P\{M(t) = i, I(t) = n\}, (i, n) \in \Omega_{ob1}^e, \quad i = 0, 1, 2.$$
\{(M(t), I(t))\} is given by \((14)\), and the stationary distribution \(\{\pi_{i,n} \mid (i, n) \in \Omega_{ob1}^c\}\) is given by:

\[
\pi_{0,n} = \begin{cases} 
\frac{\mu}{\lambda} \cdot \pi_{1,0}, & 0 \leq n \leq N - 1; \\
\frac{\mu}{\lambda} \cdot \left(\frac{\lambda}{\lambda + \xi}\right)^{n-N+1} \cdot \pi_{1,0}, & N \leq n \leq n(0); \\
\frac{\mu}{\xi} \cdot \left(\frac{\lambda}{\lambda + \xi}\right)^{n(0)-N+1} \cdot \pi_{1,0}, & n = n(0) + 1;
\end{cases}
\]

\[
\pi_{1,n} = \begin{cases} 
A_1 + A_2 \cdot F^n, & 0 \leq n \leq N - 1; \\
B_1 + B_2 \cdot F^n + D_1 \cdot \left(\frac{\lambda}{\lambda + \xi}\right)^n, & N \leq n \leq n(0); \\
B_1 + B_2 \cdot F^{n(0)+1} + D_1 \cdot \left(\frac{\lambda - F\xi}{\lambda + \xi}\right) \cdot \left(\frac{\lambda}{\lambda + \xi}\right)^{n(0)-1}, & n = n(0) + 1; \\
- (1 + \frac{\xi}{\theta}) \cdot \left(\frac{\lambda}{\lambda + \xi}\right)^{n(0)-N+1} \cdot \pi_{1,0}, & n = n(0) + 2; \\
B_1 + B_2 \cdot F^{n(0)+2} + D_1 \cdot \left(\frac{\lambda - F\xi - F^2\xi}{\lambda + \xi}ight) \cdot \left(\frac{\lambda}{\lambda + \xi}\right)^{n(0)-1}, & n = n(0) + 2; \\
E_2 \cdot F^n, & n(0) + 3 \leq n \leq n(1) + 1;
\end{cases}
\]

and

\[
\pi_{2,n} = \begin{cases} 
\frac{\mu}{\lambda + \theta} \cdot \pi_{1,n}, & 1 \leq n \leq N - 1; \\
\frac{\mu}{\lambda + \theta} \cdot \pi_{1,n} + \frac{\lambda}{\lambda + \xi} \cdot \pi_{0,n}, & N \leq n \leq n(0) + 1; \\
\frac{\mu}{\lambda + \theta} \cdot \pi_{1,n}, & n(0) + 2 \leq n \leq n(1) + 1;
\end{cases}
\]

where

\[
\begin{aligned}
A_1 &= \frac{\mu F}{\lambda (1 - F)} \cdot \pi_{1,0}, \\
A_2 &= \left(1 + \frac{\mu F}{\lambda (F - 1)}\right) \cdot \pi_{1,0}, \\
B_1 &= A_1 - D_1 \cdot \left(\frac{\lambda}{\lambda + \xi}\right)^{N-1} \cdot \frac{(\lambda - F\xi)}{(1 - F)(\lambda + \xi)}, \\
B_2 &= A_2 - D_1 \cdot \left(\frac{\lambda}{F(\lambda + \xi)}\right)^{N-1} \cdot \frac{\xi}{(1 - F)(\lambda + \xi)}, \\
F &= \frac{\lambda (\lambda + \theta)}{\theta \mu}, \\
D_1 &= \frac{\xi \mu (\lambda + \xi + \theta)(\lambda + \xi)^{N-1} \cdot \pi_{1,0}}{\lambda^{N-1} \left(\lambda (\lambda + \theta) + \theta \mu - \lambda \theta \mu - (\lambda + \theta)(\lambda + \xi)^2\right)},
\end{aligned}
\]

\[
E_2 = \frac{\pi_{1,n(0)+2}}{F^{n(0)+2}},
\]

\(\pi_{1,0}\) can be obtained by the normalization condition \(\sum_{(i,n) \in \Omega_{ob1}^c} \pi_{i,n} = 1\).
**Proof** From Fig. 2, the corresponding balance equations of the stationary distribution are given as follows,

\[ \lambda \pi_{0,0} = \mu \pi_{1,0}, \quad (23) \]

\[ \lambda \pi_{0,n} = \lambda \pi_{0,n-1}, \quad 1 \leq n \leq N - 1, \quad (24) \]

\[ (\lambda + \xi) \pi_{0,n} = \lambda \pi_{0,n-1}, \quad N \leq n \leq n(0), \quad (25) \]

\[ \xi \pi_{0,n(0)+1} = \lambda \pi_{0,n(0)}, \quad (26) \]

\[ (\lambda + \mu) \pi_{1,0} = \theta \pi_{2,1}, \quad (27) \]

\[ (\lambda + \mu) \pi_{1,n} = \lambda \pi_{1,n-1} + \lambda \pi_{2,n} + \theta \pi_{2,n+1}, \quad 1 \leq n \leq n(1), \quad (28) \]

\[ \mu \pi_{1,n(1)+1} = \lambda \pi_{1,n(1)} + \lambda \pi_{2,n(1)+1}, \quad (29) \]

\[ (\lambda + \theta) \pi_{2,n} = \mu \pi_{1,n}, \quad 1 \leq n \leq N - 1 \text{ and } n(0) + 2 \leq n \leq n(1) + 1, \quad (30) \]

\[ (\lambda + \theta) \pi_{2,n} = \mu \pi_{1,n} + \xi \pi_{0,n}, \quad N \leq n \leq n(0) + 1. \quad (31) \]

We first consider the stationary distribution \( \{ \pi_{0,n} | 0 \leq n \leq n(0) + 1 \} \). From (23) and (24), we can obtain

\[ \pi_{0,n} = \frac{\mu}{\lambda} \pi_{1,0}, \quad 0 \leq n \leq N - 1. \quad (32) \]

From (25) and (32),

\[ \pi_{0,n} = \frac{\lambda}{\lambda + \xi} \pi_{0,n-1} = \left( \frac{\lambda}{\lambda + \xi} \right)^{n-N+1} \pi_{0,N-1} = \left( \frac{\lambda}{\lambda + \xi} \right)^{n-N+1} \frac{\mu}{\lambda} \pi_{1,0} = \frac{\mu}{\lambda} \left( \frac{\lambda}{\lambda + \xi} \right)^{n-N+1} \pi_{1,0}, \quad N \leq n \leq n(0). \quad (33) \]

Based on (26) and (33), we can get

\[ \pi_{0,n(0)+1} = \frac{\lambda}{\xi} \left( \frac{\lambda}{\lambda + \xi} \right)^{n(0)-N+1} \frac{\mu}{\lambda} \pi_{1,0} = \frac{\mu}{\xi} \left( \frac{\lambda}{\lambda + \xi} \right)^{n(0)-N+1} \pi_{1,0}. \quad (34) \]

Therefore, we can get (15) from the above discussion.

Next, we consider the stationary distribution \( \{ \pi_{1,n} | 0 \leq n \leq N - 1 \} \). From (28) and (30), we can obtain

\[ (\lambda + \mu) \pi_{1,n} = \lambda \pi_{1,n-1} + \frac{\lambda \mu}{\lambda + \theta} \pi_{1,n} + \frac{\theta \mu}{\lambda + \theta} \pi_{1,n+1}, \quad 1 \leq n \leq N - 1. \quad (35) \]
The solution of (35) can be given by the following homogeneous linear difference equation:

$$\frac{\theta \mu}{\lambda + \theta} x_{n+1} - \left( \frac{\lambda + \theta \mu}{\lambda + \theta} \right) x_n + \lambda x_{n-1} = 0, \quad 1 \leq n \leq N - 1. \quad (36)$$

The characteristic equation corresponding to (36) is

$$\frac{\theta \mu}{\lambda + \theta} x^2 - \left( \frac{\lambda + \theta \mu}{\lambda + \theta} \right) x + \lambda = 0, \quad (37)$$

which has two roots: 1 and $F = \frac{\lambda(\lambda + \theta)}{\theta \mu}$. Let $x_n^h = A_1 + A_2 F^n$ be the general solution of (36), where $A_1$ and $A_2$ are the coefficients that need to be determined. From (27) and (30), we can obtain

$$\begin{cases} A_1 + A_2 = \pi_{1,0}, \\
(\lambda + \mu)(A_1 + A_2) = \theta \pi_{2,1} = \frac{\theta \mu}{\lambda + \theta} \pi_{1,1} = \frac{\theta \mu}{\lambda + \theta} (A_1 + A_2 F), \end{cases} \quad (38)$$

which yields

$$\begin{cases} A_1 = \frac{\mu F}{\lambda (1 - F)} \pi_{1,0}, \\
A_2 = \left( 1 + \frac{\mu F}{\lambda (1 - F)} \right) \pi_{1,0}. \end{cases} \quad (39)$$

Therefore,

$$\pi_{1,n} = A_1 + A_2 \cdot F^n, \quad 0 \leq n \leq N - 1, \quad (40)$$

where $A_1$ and $A_2$ are given by (39).

Now, let us continue to consider the stationary distribution $\{\pi_{1,n} \mid N \leq n \leq n(0)\}$.

From (28) and (31), we can obtain

$$\begin{align*}
\frac{\theta \mu}{\lambda + \theta} \pi_{1,n+1} - \left( \frac{\lambda + \theta \mu}{\lambda + \theta} \right) \pi_{1,n} + \lambda \pi_{1,n-1} \\
&= - \left( 1 + \frac{\theta}{\lambda + \xi} \right) \frac{\xi \mu}{\lambda + \theta} \left( \frac{\lambda}{\lambda + \xi} \right)^{n-N+1} \pi_{1,0}, \quad N \leq n \leq n(0). \quad (41)
\end{align*}$$

The solutions of (41) can be obtained through solving the following system of non-homogeneous linear difference equations:

$$\begin{align*}
\frac{\theta \mu}{\lambda + \theta} x_{n+1} - \left( \frac{\lambda + \theta \mu}{\lambda + \theta} \right) x_n + \lambda x_{n-1} \\
&= - \left( 1 + \frac{\theta}{\lambda + \xi} \right) \frac{\xi \mu}{\lambda + \theta} \left( \frac{\lambda}{\lambda + \xi} \right)^{n-N+1} \pi_{1,0}, \quad N \leq n \leq n(0), \quad (42)
\end{align*}$$
whose corresponding characteristic equation is given by

\[ \frac{\theta \mu}{\lambda + \theta} x^2 - \left( \frac{\theta \mu}{\lambda + \theta} \right) x + \lambda = - \left( \frac{1}{\lambda + \xi} \right) \frac{\xi \mu}{\lambda + \theta} \left( \frac{\lambda}{\lambda + \xi} \right)^{n-N+1} \pi_{1.0}. \]  

(43)

Define \( y^g_n = y^h_n + y^s_n \) as the general solution of (43), where \( y^h_n \) is the general solution of the homogeneous version of (43), which is \( y^h_n = B_1 + B_2 F^n \), and \( y^s_n \) is a specific solution of (43).

We consider a specific solution \( y^s_n = D_1 \left( \frac{\lambda}{\lambda + \xi} \right)^n \) of (43). Substituting it into (43), we can obtain

\[ D_1 = \frac{\xi \mu (\lambda + \xi + \theta)(\lambda + \xi)^{N-1} \cdot \pi_{1.0}}{\lambda^{N-1} \left( (\lambda(\lambda + \theta) + \theta \mu) - \lambda \theta \mu - (\lambda + \theta)(\lambda + \xi)^2 \right)}. \]  

(44)

Thus,

\[ y^g_n = B_1 + B_2 F^n + D_1 \left( \frac{\lambda}{\lambda + \xi} \right)^n, \quad N \leq n \leq n(0), \]  

(45)

where \( B_1 \) and \( B_2 \) are the coefficients that need to be determined. By considering (40), we get

\[
\begin{align*}
B_1 &= A_1 - D_1 \cdot \left( \frac{\lambda}{\lambda + \xi} \right)^{N-1} \cdot \frac{\lambda(1-F)-F\xi}{(1-F)(\lambda+\xi)}, \\
B_2 &= A_2 - D_1 \cdot \left( \frac{\lambda}{F(\lambda+\xi)} \right)^{N-1} \cdot \frac{\xi}{(1-F)(\lambda+\xi)}. 
\end{align*}
\]  

(46)

Therefore,

\[ \pi_{1,n} = B_1 + B_2 \cdot F^n + D_1 \cdot \left( \frac{\lambda}{\lambda + \xi} \right)^n, \quad N \leq n \leq n(0), \]  

(47)

where \( D_1, B_1 \) and \( B_2 \) are given by (44) and (46), respectively. Specially, based on (28), (31), (15) and (47), we can obtain the stationary distribution of \( \{\pi_{1,n(0)+1}\} \) as follows:

\[ \pi_{1,n(0)+1} = B_1 + B_2 \cdot F^{n(0)+1} \\
+ D_1 \cdot \left( \frac{\lambda - F\xi}{\lambda + \xi} \right) \cdot \left( \frac{\lambda}{\lambda + \xi} \right)^{n(0)-1} \\
- \left( 1 + \frac{\xi}{\theta} \right) \cdot \left( \frac{\lambda}{\lambda + \xi} \right)^{n(0)-N+1} \cdot \pi_{1.0}. \]  

(48)
Based on (28), (30), (31), (15) and (48), we can get the stationary distribution of \( \{ \pi_{1,n(0)+2} \} \) as follows:

\[
\pi_{1,n(0)+2} = B_1 + B_2 \cdot F^{n(0)+2} + D_1 \cdot \left( \frac{\lambda - F \xi - F^2 \xi}{\lambda + \xi} \right) \cdot \left( \frac{\lambda}{\lambda + \xi} \right)^{n(0)-1} - \frac{\lambda + \theta + \xi + F \theta + F \xi}{\theta} \cdot \left( \frac{\lambda}{\lambda + \xi} \right)^{n(0)-N+1} \cdot \pi_{1,0}.
\]

(49)

Continue our proof for the case of \( \{ \pi_{1,n} \} \) where \( n(0) + 3 \leq n \leq n(1) + 1 \). In this case, the general solution of (36) is \( z_n^h = E_1 + E_2 F^n \), where \( E_1 \) and \( E_2 \) are the coefficients that need to be determined. From (29), (30) and (49), we can obtain \( E_1 = 0 \) and

\[
E_2 = \frac{\pi_{1,n(0)+2}}{F^{n(0)+2}}.
\]

(50)

Therefore,

\[
\pi_{1,n} = E_2 \cdot F^n, \quad n(0) + 3 \leq n \leq n(1) + 1,
\]

(51)

which leads to (16).

Finally, we consider the stationary distribution of \( \{ \pi_{2,n} \} \) where \( 1 \leq n \leq n(1) + 1 \). From (16), (30) and (31), we can easily get (17).

In summary, (15), (16) and (17) are all related to \( \pi_{0,1} \), and we can get \( \pi_{0,1} \) by normalizing conditions \( \sum_{(i,n) \in \Omega_{ob1}^e} \pi_{i,n} = 1 \).

Based on Fig. 2 and Theorem 2, we know that the balking states of customers are \( (0, n(0) + 1) \) and \( (1, n(1) + 1) \). For the social optimization, which will be considered later, we define \( U^e_{ob1}(n(0), n(1)) \) to be the social welfare per time unit in Case 1: \( N - 1 \leq n(0) \leq n(1) \), or

\[
U^e_{ob1}(n(0), n(1)) = \lambda R (1 - \pi_{0,n(0)+1} - \pi_{1,n(1)+1}) - C \left( \sum_{n=0}^{n(0)+1} n \pi_{0,n} + \sum_{n=0}^{n(1)+1} n \pi_{1,n} + \sum_{n=1}^{n(1)+1} n \pi_{2,n} \right).
\]

(52)

Indeed, the first summand of (52) is the effective arrival rate at the system times the reward \( R \), while the second summand is the mean number of customers in the system. Obviously, the equilibrium social welfare is \( U^e_{ob1}(n_e(0), n_e(1)) \).

Case 2: For \( N - 1 \leq n(1) \leq n(0) \), the corresponding transition rate diagram is showed in Fig. 3, and the state space of \( \{(M(t), I(t))\} \) is given by

\[
\Omega_{ob2}^e = \{(0, n) : 0 \leq n \leq n(0) + 1 \} \cup \{(1, n) : 0 \leq n \leq n(0) + 1 \} \cup \{(2, n) : 1 \leq n \leq n(0) + 1 \}.
\]

(53)
The stationary distribution for this case (Case 2) is given in Theorem 3.

**Theorem 3** For the fully observable M/M/1 constant retrial queue with multiple vacations and the $N$-policy, if $N - 1 \leq n(1) \leq n(0)$, then the state space $\Omega_{ob}^N$ of $\{(M(t), I(t))\}$ is given by (53), and the stationary distribution $\{\pi_{i,n} \mid (i, n) \in \Omega_{ob}^N\}$ is given by:

$$\pi_{0,n} = \begin{cases} \frac{\mu \cdot A_1 + A_2 \cdot F^n}{\lambda + \xi}, & 0 \leq n \leq N - 1; \\ \frac{\mu \cdot B_1 + B_2 \cdot F^n}{\lambda + \xi} \cdot \pi_{1,0}, & N \leq n \leq n(0); \\ \frac{\mu \cdot C_1}{\lambda + \xi} \cdot \pi_{1,0}, & n = n(0) + 1; \end{cases}$$ (54)

$$\pi_{1,n} = \begin{cases} A_1 + A_2 \cdot F^n \cdot \left(\frac{\lambda}{\lambda + \xi}\right)^n, & 0 \leq n \leq N - 1; \\ B_1 + B_2 \cdot F^n \cdot D_1 \cdot \left(\frac{\lambda - F \xi}{\lambda + \xi}\right)^{n(1)-1}, & N \leq n \leq n(1); \\ B_1 + B_2 \cdot F^{n(1)+1} \cdot D_1 \cdot \left(\frac{\lambda - F \xi}{\lambda + \xi}\right)^{n(1)-1} \cdot \pi_{1,0}, & n = n(1) + 1; \\ -(\frac{\xi}{\lambda + \xi} + \frac{\xi}{\lambda + \xi}) \cdot \left(\frac{\lambda}{\lambda + \xi}\right)^{n(1)-1} \cdot \pi_{1,0}, & n = n(0) + 1; \\ (1 - F) \cdot B_1 + D_1 \left(\frac{\lambda}{\lambda + \xi} - F \left(\frac{\lambda}{\lambda + \xi}\right)^{n(1)-1} \cdot \psi(n) \cdot \pi_{1,0}, & n(1) + 2 \leq n \leq n(0) \end{cases}$$ (55)

and

$$\pi_{2,n} = \begin{cases} \frac{\mu \cdot \pi_{1,n}}{\lambda + \theta}, & 1 \leq n \leq N - 1; \\ \frac{\mu \cdot \pi_{1,n} + \psi(n)}{\lambda + \xi} \cdot \pi_{0,n}, & N \leq n \leq n(0) + 1; \end{cases}$$ (56)

where $A_i$ ($i = 1, 2$), $B_i$ ($i = 1, 2$), $F$ and $D_1$ are given by (18), (19), (20) and (21), respectively.

$$\psi(n) = \left(\frac{\lambda}{\lambda + \xi} + \frac{\lambda}{\lambda + \xi}\right)^{n-2} \cdot \left(1 + \frac{\lambda + \xi}{\theta} + \frac{\lambda + \xi}{\lambda + \xi} \cdot \left(\frac{\lambda}{\lambda + \xi}\right)^{n(1)+2-N}. \right.$$ (57)
\(\pi_{1,0}\) can be obtained by the normalization condition \(\sum_{(i,n)\in D_{ob2}} \pi_{i,n} = 1\).

**Proof** From Fig. 3, the corresponding balance equations of the stationary distribution are given as follows:

\[
\begin{align*}
\lambda \pi_{0,0} &= \mu \pi_{1,0}, & (58) \\
\lambda \pi_{0,n} &= \lambda \pi_{0,n-1}, & 1 \leq n \leq N - 1, & (59) \\
(\lambda + \xi) \pi_{0,n} &= \lambda \pi_{0,n-1}, & N \leq n \leq n(0), & (60) \\
\xi \pi_{0,n(0)+1} &= \lambda \pi_{0,n(0)}, & (61) \\
(\lambda + \mu) \pi_{1,0} &= \theta \pi_{2,1}, & (62) \\
(\lambda + \mu) \pi_{1,n} &= \lambda \pi_{1,n-1} + \lambda \pi_{2,n} + \theta \pi_{2,n+1}, & 1 \leq n \leq n(1), & (63) \\
\mu \pi_{1,n(1)+1} &= \lambda \pi_{1,n(1)} + \lambda \pi_{2,n(1)+1} + \theta \pi_{2,n(1)+2}, & (64) \\
\mu \pi_{1,n} &= \lambda \pi_{2,n} + \theta \pi_{2,n+1}, & n(1) + 2 \leq n \leq n(0), & (65) \\
\mu \pi_{1,n(0)+1} &= \lambda \pi_{2,n(0)+1}, & (66) \\
(\lambda + \theta) \pi_{2,n} &= \mu \pi_{1,n}, & 1 \leq n \leq N - 1, & (67) \\
(\lambda + \theta) \pi_{2,n} &= \mu \pi_{1,n} + \xi \pi_{0,n}, & N \leq n \leq n(0) + 1. & (68)
\end{align*}
\]

We first consider the stationary distribution \(\{\pi_{0,n} | 0 \leq n \leq n(0) + 1\}\). From (58)–(61), the discussion is similar to the discussion for (23)–(26), which leads to (54).

We next consider the stationary distribution \(\{\pi_{1,n} | 0 \leq n \leq N - 1\}\). From (62), (63) and (67), the discussion is similar to that for (40), and we can obtain

\[
\pi_{1,n} = A_1 + A_2 \cdot F^n, \ 0 \leq n \leq N - 1,
\]

where \(A_1\) and \(A_2\) are given by (39).

We now continue to consider the stationary distribution \(\{\pi_{1,n} | N \leq n \leq n(1)\}\). From (54), (63) and (68), the discussion is similar to that for (47), and we can obtain

\[
\pi_{1,n} = B_1 + B_2 \cdot F^n + D_1 \cdot \left(\frac{\lambda}{\lambda + \xi}\right)^n, \quad N \leq n \leq n(1),
\]

where \(D_1, B_1\) and \(B_2\) are given by (44) and (46), respectively. Specially, based on (54), (63) and (70), we can get the stationary distribution of \(\{\pi_{1,n(1)+1}\}\) as follows:

\[
\pi_{1,n(1)+1} = B_1 + B_2 \cdot F^{n(1)+1} + D_1 \cdot \left(\frac{\lambda - F \xi}{\lambda + \xi}\right) \cdot \left(\frac{\lambda}{\lambda + \xi}\right)^{n(1)-1} \\
- \left(\frac{\xi}{\lambda + \xi} + \frac{\xi}{\theta}\right) \cdot \left(\frac{\lambda}{\lambda + \xi}\right)^{n(1)-n+1} \cdot \pi_{1,0}. \quad (71)
\]
Based on (54), (64), (68) and (71), we can get the stationary distribution of \( \{\pi_{1,n(1)+2}\} \) as follows

\[
\pi_{1,n(1)+2} = (1 - F)B_1 + D_1 \left( \frac{\lambda}{\lambda + \xi} - F \right) \left( \frac{\lambda}{\lambda + \xi} \right)^{n(1)-1} \\
- \left( \frac{\xi}{\lambda} + \frac{\xi(\lambda + \xi)}{\lambda + \theta} + \frac{\xi}{\lambda + \xi} + \frac{\xi}{\theta} \right) \left( \frac{\lambda}{\lambda + \xi} \right)^{n(1)-N+2} \pi_{1,0}. \tag{72}
\]

Continue further to consider the stationary distribution of \( \{\pi_{1,n} | n(1)+3 \leq n \leq n(0)\} \). Based on (54), (65) and (68), we can obtain that

\[
\pi_{1,n} - \pi_{1,n-1} = - \left( \frac{\xi}{\lambda} + \frac{\xi}{\lambda + \xi} \right) \left( \frac{\lambda}{\lambda + \xi} \right)^{n-N} \pi_{1,0}. \tag{73}
\]

Recursively using (73), we can obtain that

\[
\pi_{1,n} = \left( \frac{\lambda}{\lambda + \xi} + \frac{\lambda}{\theta} \right) \left( \left( \frac{\lambda}{\lambda + \xi} \right)^{n-N} - \left( \frac{\lambda}{\lambda + \xi} \right)^{n(1)+2-N} \right) \\
+ \pi_{1,n(1)+2}, \quad n(1) + 2 \leq n \leq n(0). \tag{74}
\]

Thus,

\[
\pi_{1,n} = (1 - F)B_1 + D_1 \left( \frac{\lambda}{\lambda + \xi} - F \right) \left( \frac{\lambda}{\lambda + \xi} \right)^{n(1)-1} \\
+ \psi(n)\pi_{1,0}, \quad n(1) + 2 \leq n \leq n(0), \tag{75}
\]

where

\[
\psi(n) = \left( \frac{\lambda}{\lambda + \xi} + \frac{\lambda}{\theta} \right) \left( \frac{\lambda}{\lambda + \xi} \right)^{n-N} \\
- \left( 1 + \frac{\lambda + \xi}{\theta} + \frac{\xi(\lambda + \xi)}{\lambda + \theta} + \frac{\xi}{\lambda} + \frac{\xi}{\lambda + \xi} \right) \left( \frac{\lambda}{\lambda + \xi} \right)^{n(1)+2-N}. \tag{76}
\]

Specially, based on (54), (65) and (68), we can get the stationary distribution of \( \{\pi_{1,n(0)+1}\} \) as follows:

\[
\pi_{1,n(0)+1} = \pi_{1,n(0)} - \left( \frac{\xi}{\theta} \right) \left( \frac{\lambda}{\lambda + \xi} \right)^{n(0)-N+1} \pi_{1,0}. \tag{77}
\]

Therefore, taking into account all the above discussions, we can get (55).

Finally, we consider the stationary distribution of \( \{\pi_{2,n} | 1 \leq n \leq n(0) + 1\} \). From (55), (67) and (68), We can easily get (56).

In summary, (54), (55) and (56) are all related to \( \pi_{0,1} \), and we can get \( \pi_{0,1} \) by normalizing conditions \( \sum_{(i,n) \in \Omega_{ob2}} \pi_{i,n} = 1 \).
Based on Fig. 3 and Theorem 3, we know that the states at which customers will balk are \((0, n(0)+1)\) and \(\{1, n\} | n(1) + 1 \leq n \leq n(0)+1\). Denote \(U_{ob2}^e(n(0), n(1))\) to be the social welfare per time unit in Case 2, or \(N - 1 \leq n(1) \leq n(0)\). Then,

\[
U_{ob2}^e(n(0), n(1)) = \lambda R(1 - \pi_{0,n(0)+1}) - \sum_{n=n(1)+1}^{n(0)+1} \pi_{1,n}
- C(\sum_{n=0}^{n(0)+1} n\pi_{0,n} + \sum_{n=0}^{n(1)+1} n\pi_{1,n} + \sum_{n=1}^{n(0)+1} n\pi_{2,n}).
\] (78)

Obviously, the equilibrium social welfare is \(U_{ob2}^e(n_e(0), n_e(1))\).

Case 3: For \(n(1) < N - 1 \leq n(0)\), the corresponding transition rate diagram is showed in Fig. 4, and the state space of \(\{(M(t), I(t))\}\) is given by:

\[
\Omega_{ob3}^e = \{(0, n) : 0 \leq n \leq n(0) + 1\} \cup \{(1, n) : 0 \leq n \leq n(0) + 1\}
\cup\{(2, n) : 1 \leq n \leq n(0) + 1\}.
\] (79)

The stationary distribution for this case (Case 3) is given in Theorem 4.

**Theorem 4** For the fully observable M/M/1 constant retrial queue with multiple vacations and the N-policy, if \(n(1) < N - 1 \leq n(0)\), then the state space \(\Omega_{ob3}^e\) of \(\{(M(t), I(t))\}\) is given by (79), and the stationary distribution \(\{\pi_{i,n} | (i, n) \in \Omega_{ob3}^e\}\) is given by:
respectively,

\[
\begin{align*}
\pi_{0,n} &= \begin{cases}
\frac{\mu}{\lambda} \cdot \pi_{1,0}, & 0 \leq n \leq N - 1; \\
\frac{\mu}{\lambda} \cdot \left( \frac{\lambda}{\lambda + \xi} \right)^{n-N+1} \cdot \pi_{1,0}, & N \leq n \leq n(0); \\
\frac{\mu}{\xi} \cdot \left( \frac{\lambda}{\lambda + \xi} \right)^{n(0)-N+1} \cdot \pi_{1,0}, & n = n(0) + 1;
\end{cases}
\end{align*}
\]

\begin{equation}
(80)
\end{equation}

\[
\begin{align*}
\pi_{1,n} &= \begin{cases}
A_1 + A_2 \cdot F^n, & 0 \leq n \leq N - 1; \\
B_1 + B_2 \cdot F^n + D_1 \cdot \left( \frac{\lambda}{\lambda + \xi} \right)^n, & N \leq n \leq n(1); \\
B_1 + B_2 \cdot F^{n(1)+1} + D_1 \cdot \left( \frac{\lambda - F\xi}{\lambda + \xi} \right) \cdot \left( \frac{\lambda}{\lambda + \xi} \right)^{n(1)-1} - \frac{\xi}{\lambda + \xi} \cdot \pi_{1,0}, & n = n(1) + 1; \\
(1 - F)B_1 + D_1 \left( \frac{\lambda - F\xi}{\lambda + \xi} \right) \cdot \left( \frac{\lambda}{\lambda + \xi} \right)^{n(0)-N+1} + \psi(n) \cdot \pi_{1,0}, & n(1) + 2 \leq n \leq n(0).
\end{cases}
\end{align*}
\]

\begin{equation}
(81)
\end{equation}

and

\[
\begin{align*}
\pi_{2,n} &= \begin{cases}
\frac{\mu}{\lambda + \theta} \cdot \pi_{1,n}, & 1 \leq n \leq N - 1; \\
\frac{\mu}{\lambda + \theta} \cdot \pi_{1,n} + \frac{\xi}{\lambda + \xi} \cdot \pi_{0,n}, & N \leq n \leq n(0) + 1;
\end{cases}
\end{align*}
\]

\begin{equation}
(82)
\end{equation}

where \(A_i (i = 1, 2), B_i (i = 1, 2), F \) and \(D_1\) are given by (18), (19), (20) and (21), respectively.

\[
\psi(n) = \left( \frac{\lambda}{\lambda + \xi} + \frac{\xi}{\lambda} \right) \left( \frac{\lambda}{\lambda + \xi} \right)^{n-N} - \left( 1 + \frac{\lambda + \xi}{\theta} \right) \cdot \left( \frac{\xi}{\lambda + \xi} \right)^{n(1)+2-N}.
\]

\begin{equation}
(83)
\end{equation}

\(\pi_{1,0}\) can be obtained by the normalization condition \(\sum_{(i,n) \in \Omega_{ob3}^e} \pi_{i,n} = 1\).

From Fig. 4, we know that Case 3 and Case 2 have the same equilibrium equations. Hence, the stationary distribution of Case 3 is the same as that of Case 2, and therefore we omit the proof of Theorem 4. Based on Fig. 4 and Theorem 4, we can obtain that the states, at which customers will balk are \((0, n(0) + 1)\) and \((1, n) | n(1) + 1 \leq n \leq n(0) + 1\). Denote the social welfare per time unit in Case 3 by \(U_{ob3}^e(n(0), n(1))\). We then have

\[
U_{ob3}^e(n(0), n(1)) = \lambda R (1 - \pi_{0,n(0)+1} - \sum_{n=n(1)+1}^{n(0)+1} \pi_{1,n})
\]

\[
- C \left( \sum_{n=0}^{n(0)+1} n\pi_{0,n} + \sum_{n=0}^{n(1)+1} n\pi_{1,n} + \sum_{n=1}^{n(0)+1} n\pi_{2,n} \right).
\]

\begin{equation}
(84)
\end{equation}

Obviously, the equilibrium social welfare is \(U_{ob3}^e(n_e(0), n_e(1))\).
3.2 Social optimization

Summarize the above discussion, we define $U_s(n(0), n(1))$ as the social welfare per time unit, thus

$$U_s(n(0), n(1)) = \begin{cases} U_{ob1}(n(0), n(1)) & \text{if } N - 1 \leq n(0) \leq n(1); \\ U_{ob2}(n(0), n(1)) & \text{if } N - 1 \leq n(1) \leq n(0); \\ U_e(n(0), n(1)) & \text{if } n(1) < N - 1 \leq n(0). \end{cases} \tag{85}$$

From (85), we know that $U_s(n(0), n(1))$ is a piecewise function, in which specific explicit expression depends on $n(0), n(1),$ and $N.$ Social welfare is the sum of the expected net pay-off of all customers. Therefore, the socially optimal balking threshold considers all customers’ whole and achieves the maximum social welfare, which is different from the equilibrium balking threshold. We denote the socially optimal balking threshold of customers and the integrated strategy at state $i$ by $n^*(i) (i = 1, 2)$ and $(n^*(0), n^*(1))$, respectively. Socially optimal social welfare considers that the social welfare per time unit is maximized at the socially optimal balking thresholds $n^*(0)$ and $n^*(1).$ However, as discussed above, due to the complexity of the social welfare function (85), its analytic solution seems extremely difficult to obtain for the general case. Hence, we try to find the numerical optimal solution $(n^*(0), n^*(1))$ by some numerical algorithm. After exploring, we find that the Particle swarm optimization (PSO) algorithm is very suitable for finding our work’s numerical optimal solution. Although the PSO algorithm, Ant colony algorithm, and Fish swarm algorithm are all swarm intelligence optimization algorithms, the PSO algorithm needs less analyticity of the objective function. Besides, the genetic algorithms share information with each other, and the movement of the entire population is to move more evenly to the optimal area. The PSO algorithm belongs to one-way information flow, and the whole search and update process follows the current optimal solution. Therefore, in general, the convergence speed of PSO algorithms is faster. In Sect. 5, we briefly summarize the essential points of PSO algorithm, and find the numerical optimal solution $(n^*(0), n^*(1))$ of $\max \{U_s(n(0), n(1))\}$ through PSO algorithm, i.e., we define $U_s(n^*(0), n^*(1))$ as the socially optimal social welfare. So we can get $U_s(n^*(0), n^*(1)) = \max \{U_s(n(0), n(1))\}$ through PSO algorithm.

4 The unobservable case

In the observable case, we assume that the arriving customers can observe all information about $M(t)$ and $I(t).$ Now, in the unobservable case, we assume that the arriving customers cannot observe any information about $M(t)$ or $I(t),$ and all customers are indistinguishable. Then we assume that the arriving customers can choose to join the system with probability $q$ ($0 \leq q \leq 1$) or leave the system with complementary probability $\tilde{q} = 1 - q.$ So, the effective arrival rate is $\tilde{\lambda} = \lambda q,$ the equilibrium mixed strategy of the customers is denoted by the equilibrium arrival rate $\tilde{\lambda}_e = \lambda q_e$ ($q_e$ is equilibrium joining probability of the customers), and the socially optimal mixed
strategy is denoted by the optimal arrival rate \( \lambda^* = \lambda q^* \), where \( q^* \) is optimal joining probability of the customers. The corresponding transition rate diagram is showed in Fig. 5.

### 4.1 Equilibrium

In the unobservable case, the arriving customers can neither observe the state of the server \( M(t) \) nor the number of other customers \( I(t) \) in the orbit. In order to obtain the equilibrium arrival rate \( \lambda_e \) in this case, the stationary distribution needs to be determined first. From Fig. 5, it is easy to know that the process \( \{M(t), I(t)\} \) is a quasi-birth-and-death (QBD) process with

\[
\Omega_{un} = \{(0, n) : n \geq 0\} \cup \{(1, n) : n \geq 0\} \cup \{(2, n) : n \geq 1\}.
\]

If \( \rho = \lambda_e/\mu < 1 \), \((M, I)\) is defined as the stationary limit of the process \( \{M(t), I(t)\} \). The stationary distribution is denoted by:

\[
\pi = (\pi_{0,0}, \pi_{1,0}, \pi_1, \pi_2, \ldots, \pi_n, \ldots),
\]

where

\[
\pi_n = (\pi_{0,n}, \pi_{1,n}, \pi_{2,n}), \quad n \geq 1,
\]

with the definition of

\[
\pi_{i,n} = P\{M = i, I = n\} = \lim_{t \to \infty} P\{M(t) = i, I(t) = n\},
\]

\((i, n) \in \Omega_{un} \) for \( i = 0, 1, 2 \).
Let $Q$ be defined as the infinitesimal generator of the process. Then, the stationary probability vector $\pi$ can be solved through the equations $\pi Q = 0$:

\[
\begin{align*}
\bar{\lambda} \pi_{0,0} &= \mu \pi_{1,0}; \\
\bar{\lambda} \pi_{0,n} &= \bar{\lambda} \pi_{0,n-1}, \quad 1 \leq n \leq N - 1; \\
(\bar{\lambda} + \xi) \pi_{0,n} &= \bar{\lambda} \pi_{0,n-1}, \quad n \geq N; \\
(\bar{\lambda} + \mu) \pi_{1,0} &= \theta \pi_{2,1}; \\
(\bar{\lambda} + \mu) \pi_{1,n} &= \bar{\lambda} \pi_{1,n-1} + \bar{\lambda} \pi_{2,n} + \theta \pi_{2,n+1}, \quad n \geq 1; \\
(\bar{\lambda} + \theta) \pi_{2,n} &= \xi \pi_{0,n} + \mu \pi_{1,n}, \quad n \geq N.
\end{align*}
\]

From Fig. 5 and ordering the states in the state space $\Omega_{un}$ lexicographically, we can write the infinite generator for the Markov process $\{(M(t), I(t))\}$ as a block-partitioned form as follows:

\[
Q = \begin{pmatrix}
A_{0,0} & A_{0,1} \\
A_{1,0} & A_{1} \\
A_{2} & A_{1} & A_{0} \\
& \ddots & \ddots & \ddots \\
A_{2} & A_{1} & A_{0} \\
A_{2} & B_{1} & A_{0} \\
& \ddots & \ddots & \ddots
\end{pmatrix},
\]

where

\[
\begin{align*}
A_{0,0} &= \begin{pmatrix} -\bar{\lambda} & 0 \\
\mu & -(\bar{\lambda} + \mu) \end{pmatrix}, & A_{0,1} &= \begin{pmatrix} \bar{\lambda} & 0 \\
0 & -\bar{\lambda} \end{pmatrix}, & A_{1,0} &= \begin{pmatrix} 0 & 0 \\
0 & \theta \end{pmatrix}, \\
A_{0} &= \begin{pmatrix} \bar{\lambda} & 0 & 0 \\
0 & \bar{\lambda} & 0 \\
0 & 0 & 0 \end{pmatrix}, & A_{1} &= \begin{pmatrix} -\bar{\lambda} & 0 & 0 \\
0 & -(\bar{\lambda} + \mu) & \mu \\
0 & \bar{\lambda} & -(\bar{\lambda} + \theta) \end{pmatrix}, & A_{2} &= \begin{pmatrix} 0 & 0 & 0 \\
0 & 0 & \theta \end{pmatrix}, \\
B_{1} &= \begin{pmatrix} -(\bar{\lambda} + \xi) & 0 & \xi \\
0 & -(\bar{\lambda} + \mu) & \mu \\
0 & \bar{\lambda} & -(\bar{\lambda} + \theta) \end{pmatrix}.
\end{align*}
\]

**Theorem 5** For the unobservable case, the $M/M/1$ constant retrial queue with multiple vacations and the $N$-policy, given the arrival rate $\bar{\lambda}$, the stationary distribution $\{\pi_{i,n} | (i,n) \in \Omega_{un}\}$ is given by

\[
\pi_{i,n} = \begin{pmatrix} \pi_{0,0} & \pi_{0,1} & \cdots & \pi_{0,n} \\
\pi_{1,0} & \pi_{1,1} & \cdots & \pi_{1,n} \\
\vdots & \vdots & \ddots & \vdots \\
\pi_{N-1,0} & \pi_{N-1,1} & \cdots & \pi_{N-1,n} \\
\pi_{N,0} & \pi_{N,1} & \cdots & \pi_{N,n} \end{pmatrix},
\]

subject to the steady-state equations

\[
\begin{align*}
\bar{\lambda} \pi_{0,0} &= \mu \pi_{1,0}; \\
\bar{\lambda} \pi_{0,n} &= \bar{\lambda} \pi_{0,n-1}, \quad 1 \leq n \leq N - 1; \\
(\bar{\lambda} + \xi) \pi_{0,n} &= \bar{\lambda} \pi_{0,n-1}, \quad n \geq N; \\
(\bar{\lambda} + \mu) \pi_{1,0} &= \theta \pi_{2,1}; \\
(\bar{\lambda} + \mu) \pi_{1,n} &= \bar{\lambda} \pi_{1,n-1} + \bar{\lambda} \pi_{2,n} + \theta \pi_{2,n+1}, \quad n \geq 1; \\
(\bar{\lambda} + \theta) \pi_{2,n} &= \xi \pi_{0,n} + \mu \pi_{1,n}, \quad n \geq N.
\end{align*}
\]
\[ \pi_{0,n} = \begin{cases} \frac{\mu}{\lambda} \pi_{1,0}, & 0 \leq n \leq N - 1; \\ \frac{\mu}{\lambda + \xi} \pi_{1,0}, & n = N; \\ r_{11} \pi_{0,N}, & n \geq N; \end{cases} \]  
(94)

\[ \pi_{1,n} = \begin{cases} \frac{A_1 + A_2}{\bar{\lambda}(\lambda + \theta)} \pi_{1,0} + \frac{\lambda}{\bar{\lambda}(\lambda + \theta)} \xi \pi_{1,0}, & 0 \leq n \leq N - 1; \\ r_{12} \pi_{0,N} + r_{22} \pi_{1,N}, & n = N; \\ r_{12} \pi_{0,N} + r_{22} \pi_{1,N}, & n \geq N; \end{cases} \]  
(95)

and

\[ \pi_{2,n} = \begin{cases} \frac{\mu}{\lambda + \theta} \pi_{1,n}, & 1 \leq n \leq N - 1; \\ \frac{\mu}{\lambda + \theta} \pi_{0,N} + \frac{\mu}{\lambda + \theta} \pi_{1,N}, & n = N; \\ r_{13} \pi_{0,N} + r_{23} \pi_{1,N}, & n \geq N; \end{cases} \]  
(96)

where \( \bar{F} = \frac{\bar{\lambda}(\lambda + \theta)}{\lambda \mu} \).

\[ \begin{align*}
A_1 &= \frac{\mu F}{\bar{\lambda}(1 - F)} \pi_{1,0}; \\
A_2 &= (1 + \frac{\mu F}{\bar{\lambda}(F - 1)}) \pi_{1,0}; \\
r_{11} &= \left( \frac{\bar{\lambda} + \xi}{\bar{\lambda} + \theta} \right)^n; \\
r_{12} &= \frac{\bar{\lambda}(\bar{\lambda} + \theta)}{\lambda \mu} \pi_{0,N} + \frac{\lambda + \theta}{\lambda \mu} \pi_{1,N}; \\
r_{13} &= \frac{\bar{\lambda}(\bar{\lambda} + \theta)}{\lambda \mu} \pi_{0,N} + \frac{\lambda + \theta}{\lambda \mu} \pi_{1,N}; \\
r_{22} &= \frac{\mu (\bar{\lambda} + \theta)}{\lambda + \theta}; \\
r_{23} &= \frac{\mu (\bar{\lambda} + \theta)}{\lambda + \theta};
\end{align*} \]  
(97)

and

\[ \pi_{1,0} = \frac{\bar{\lambda} \xi (\theta \mu - \bar{\lambda}^2 - \bar{\lambda} \theta)}{\mu^2 (\bar{\lambda} + \theta)(\bar{\lambda} + N \xi)}. \]  
(99)

**Proof** In order to obtain the stationary distribution of the system, we first need to obtain the rate matrix \( R \), which is the minimum non-negative solution of the following matrix quadratic equation:

\[ R^2 A_2 + RB_1 + A_0 = 0. \]  
(100)
By detailed calculations, we get the minimum non-negative solution of $R$ as follows:

$$R = \begin{pmatrix} \frac{\lambda + \xi}{\lambda + \eta} & \frac{\lambda^2 (\lambda + \theta + \xi)}{\theta \mu (\lambda + \eta)} & \frac{\lambda}{\theta} \\ 0 & \frac{\lambda (\lambda + \theta)}{\theta \mu} & \frac{\lambda}{\theta} \\ 0 & 0 & 0 \end{pmatrix}. \quad (101)$$

Using the matrix-geometric solution (see Neuts 1981), we have:

$$\pi_n = \pi_N R^{n-N}, \quad n \geq N, \quad (102)$$

and $(\pi_{0,0}, \pi_{1,0}, \pi_1, \pi_2 \ldots \pi_{N-1}, \pi_N)$ satisfies the following equation:

$$(\pi_{0,0}, \pi_{1,0}, \pi_1, \pi_2 \ldots \pi_{N-1}, \pi_N) B[R] = 0, \quad (103)$$

where

$$B[R] = \begin{pmatrix} A_{0,0} & A_{0,1} \\ A_{1,0} & A_1 & A_0 \\ A_2 & A_1 & A_0 \\ \vdots & \ddots & \ddots \\ A_2 & A_1 & A_0 \\ A_2 & RA_2 + B_1 \end{pmatrix}. \quad (104)$$

Substituting (104) into (103), we can get:

$$\begin{align*}
\tilde{\lambda} \pi_{0,0} &= \mu \pi_{1,0}; \\
\tilde{\lambda} \pi_{0,n} &= \tilde{\lambda} \pi_{0,n-1}; \quad 1 \leq n \leq N - 1; \\
(\tilde{\lambda} + \xi) \pi_{1,0,N} &= \tilde{\lambda} \pi_{0,N-1}; \\
(\tilde{\lambda} + \mu) \pi_{1,0} &= \theta \pi_{2,1}; \\
(\tilde{\lambda} + \mu) \pi_{1,n} &= \tilde{\lambda} \pi_{1,n-1} + \tilde{\lambda} \pi_{2,n} + \theta \pi_{2,n+1}, \quad 1 \leq n \leq N - 1; \\
\mu \pi_{1,N} &= \tilde{\lambda} \pi_{1,N-1} + \tilde{\lambda} \pi_{0,N} + \tilde{\lambda} \pi_{2,N}; \\
(\tilde{\lambda} + \theta) \pi_{2,n} &= \mu \pi_{1,n}, \quad 1 \leq n \leq N - 1; \\
(\tilde{\lambda} + \theta) \pi_{2,N} &= \xi \pi_{0,N} + \mu \pi_{1,N}. 
\end{align*} \quad (105)$$

By calculating (105), we can get

$$\pi_{0,n} = \begin{cases} \frac{\mu}{\tilde{\lambda}} \pi_{1,0}, & 0 \leq n \leq N - 1; \\
\frac{\mu}{\lambda + \xi} \pi_{1,0}, & n = N; \end{cases} \quad (106)$$

$$\pi_{1,n} = \begin{cases} \frac{A_1 + A_2 \cdot F^{'0}}{\theta (\tilde{\lambda} + \theta)} (A_1 + A_2 \cdot F^{N-1}) + \frac{\tilde{\lambda}(\tilde{\lambda} + \theta)}{\theta (\lambda + \xi)} \pi_{1,0}, & 0 \leq n \leq N - 1; \\
\frac{\tilde{\lambda}(\tilde{\lambda} + \theta)}{\theta (\lambda + \xi)} \pi_{1,0}, & n = N; \end{cases} \quad (107)$$
and

\[
\pi_{2,n} = \begin{cases} 
\frac{\mu}{\lambda + \theta} \pi_{1,n}, & 0 \leq n \leq N - 1; \\
\frac{\mu}{\lambda + \theta} \pi_{0,N} + \frac{\mu}{\lambda + \theta} \pi_{1,N}, & n = N;
\end{cases}
\]

where \( F = \frac{\bar{\lambda}(\lambda + \theta)}{\mu} \),

\[
\begin{align*}
A_1 &= \frac{\mu F}{\bar{\lambda}(1-F)} \pi_{1,0}, \\
A_2 &= (1 + \frac{\mu F}{\bar{\lambda}(F-1)}) \pi_{1,0}.
\end{align*}
\]

From (106)–(108), \( \pi_N = (\pi_{0,N}, \pi_{1,N}, \pi_{2,N}) \) can be obtained. By (101), \( R^{n-N} \) can be obtained as follows:

\[
R^{n-N} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\
0 & r_{22} & r_{23} \\
0 & 0 & 0 \end{pmatrix},
\]

where \( r_{11}, r_{12}, r_{13}, r_{22} \) and \( r_{23} \) see (98). Now, from (102), we can get:

\[
\begin{align*}
\pi_{0,n} &= r_{11} \pi_{0,N}, & n \geq N; \\
\pi_{1,n} &= r_{12} \pi_{0,N} + r_{22} \pi_{1,N}, & n \geq N; \\
\pi_{2,n} &= r_{13} \pi_{0,N} + r_{23} \pi_{1,N}, & n \geq N.
\end{align*}
\]

Hence, considering the above discussion, (94)–(96) can be obtained, and \( \pi_{1,0} \) can be calculated by the following normalization condition:

\[
\pi_{0,0} + \pi_{1,0} + \sum_{n=1}^{N-1} \pi_{n} e + \pi_{N} (I - R)^{-1} e = 1,
\]

where the expression for \( \pi_{1,0} \) is given in (99).

In Theorem 5, the stationary distribution under unobservable case was obtained by using the matric-analytic method, based on which we can get the mean queue length \( E[L(\bar{\lambda})] \) for the unobservable case, given by:

\[
E[L(\bar{\lambda})] = \sum_{n=1}^{\infty} n (\pi_{0,n} + \pi_{1,n} + \pi_{2,n}) \\
\quad = \frac{\bar{\lambda}}{\xi} + \frac{\bar{\lambda}}{\theta(\mu - \bar{\lambda})} + \frac{\theta \xi}{(\bar{\lambda} + \theta)(\bar{\lambda} + \xi)} + \frac{\theta(\mu - \xi)}{\mu^2(\bar{\lambda} + \xi)} + \frac{N(N-1)\xi}{\mu^2(\bar{\lambda} + N\xi)}.
\]
By using Little’s law for the whole system, we can get mean sojourn time $E[W(\bar{\lambda})]$:

$$
E[W(\bar{\lambda})] = \frac{E[L(\bar{\lambda})]}{\bar{\lambda}} = \frac{1}{\bar{\lambda}} + \frac{1}{\theta(\mu - \bar{\lambda})} + \frac{\theta\xi}{\bar{\lambda}(\lambda + \theta)(\lambda + \xi)} + \frac{\theta(\mu - \xi)}{\bar{\lambda}\mu^2(\lambda + \xi)} + \frac{N(N - 1)\xi}{\bar{\lambda}\mu^2(\lambda + N\xi)}.
$$

(114)

We obtain the equilibrium arrival rate and socially optimal arrival rate by the following theorems.

**Theorem 6** For the unobservable M/M/1 constant retrial queue with multiple vacations and the N-policy, we have the following conclusions on the equilibrium arrival rate:

(i) If $R < CE[W(\lambda_1)]$, it has no positive equilibrium arrival rate;

(ii) If $R = CE[W(\lambda_1)]$ (if and only if $\lambda_1 \leq \lambda$), it has one positive equilibrium arrival rate $\lambda_e = \lambda_1$;

(iii) $\lambda_e \in \{\lambda_2, \lambda_3\}$, if $R > CE[W(\lambda_1)]$ and $\lambda_3 \leq \lambda$;

$$
\lambda_e \in \{\lambda_2, \lambda\}, \quad \text{if } R > CE[W(\lambda_1)] \text{ and } \lambda_2 < \lambda < \lambda_3;
$$

$$
\lambda_e = \lambda, \quad \text{if } R > CE[W(\lambda_1)] \text{ and } \lambda = \lambda_2;
$$

no positive equilibrium rate, if $R > CE[W(\lambda_1)]$ and $\lambda_2 > \lambda$;

(115)

where $\lambda_1$ is the unique positive solution of $E[W'(\lambda)] = 0$, and $\lambda_2, \lambda_3$ (with $0 \leq \lambda_2 \leq \lambda_3$) are the positive solutions of $R - CE[W(\bar{\lambda})] = 0$.

**Proof** From (114), we can obtain the expected net benefit $U_e(\bar{\lambda})$ of the marked customer:

$$
U_e(\bar{\lambda}) = R - CE[W(\bar{\lambda})]
$$

$$
= R - C \left( \frac{1}{\bar{\lambda}} + \frac{1}{\theta(\mu - \bar{\lambda})} + \frac{\theta\xi}{\bar{\lambda}(\lambda + \theta)(\lambda + \xi)} + \frac{\theta(\mu - \xi)}{\bar{\lambda}\mu^2(\lambda + \xi)} + \frac{N(N - 1)\xi}{\bar{\lambda}\mu^2(\lambda + N\xi)} \right).
$$

(116)
Since the second-order derivative of $E[W(\bar{\lambda})]$ in $\bar{\lambda}$ is given by:

$$E[W''(\bar{\lambda})] = \frac{2}{\theta(\mu - \bar{\lambda})^3} + 2\theta\xi \left( \frac{1}{\bar{\lambda}^3(\bar{\lambda} + \theta)(\bar{\lambda} + \xi)} + \frac{1}{\bar{\lambda}(\bar{\lambda} + \theta)^3(\bar{\lambda} + \xi)} \right)$$

$$+ \frac{1}{\bar{\lambda}^2(\bar{\lambda} + \theta)^2(\bar{\lambda} + \xi)} + \frac{1}{\bar{\lambda}^2(\bar{\lambda} + \theta)^2(\bar{\lambda} + \xi)}$$

$$+ 2\theta(\mu - \xi) \left( \frac{1}{\bar{\lambda}^2(\bar{\lambda} + N\xi)} + \frac{1}{\bar{\lambda}^2(\bar{\lambda} + N\xi)} + \frac{1}{\bar{\lambda}^2(\bar{\lambda} + N\xi)} \right).$$

(117)

If $\bar{\rho} = \frac{\bar{\lambda}}{\mu} < 1$, then $E[W''(\bar{\lambda})]$ is positive. In this case, $E[W(\bar{\lambda})]$ is strictly convex of $\bar{\lambda}$. If we denote the positive solution of equation $E[W'(\bar{\lambda})] = 0$ by $\bar{\lambda}_1$, and the positive solutions of $U_e(\bar{\lambda}) = R - CE[W(\bar{\lambda})] = 0$ by $\bar{\lambda}_2$ and $\bar{\lambda}_3$ ($0 \leq \bar{\lambda}_2 \leq \bar{\lambda}_3$), then we have the following results:

1. When $R < CE[W(\bar{\lambda}_1)]$, i.e., $U_e(\bar{\lambda}_1) < 0$, $U_e(\bar{\lambda})$ is negative for every $\bar{\lambda}$. Therefore, it has no positive equilibrium arrival rate, which leads to (i).

2. When $R = CE[W(\bar{\lambda}_1)]$, i.e., $U_e(\bar{\lambda}_1) = 0$, $U_e(\bar{\lambda})$ is negative for every $\bar{\lambda} \neq \bar{\lambda}_1$. Therefore, it has one positive equilibrium arrival rate $\bar{\lambda}_e = \bar{\lambda}_1$ (if and only if $\bar{\lambda}_1 \leq \lambda$), which leads to (ii).

3. When $R > CE[W(\bar{\lambda}_1)]$, i.e., $U_e(\bar{\lambda}_1) > 0$. If $\bar{\lambda}_3 \leq \lambda$, it has two positive equilibrium arrival rates $\bar{\lambda}_e \in \{\bar{\lambda}_2, \bar{\lambda}_3\}$, which leads to the first part of (115). If $\bar{\lambda}_2 < \lambda < \bar{\lambda}_3$, it has two positive equilibrium arrival rates $\bar{\lambda}_e \in \{\bar{\lambda}_2, \lambda\}$, which leads to the second part of (115). If $\lambda = \bar{\lambda}_2$, it has a unique positive equilibrium arrival rate $\bar{\lambda}_e = \bar{\lambda}_2$, which leads to the third part of (115). If $\bar{\lambda}_2 > \lambda$, obviously, it has no positive equilibrium arrival rate, which leads to the fourth part of (115).

\[\square\]

### 4.2 Social optimization

In this section, we discuss the optimal balking behavior of customers in the unobservable case in terms of the following Theorem.

**Theorem 7** For the unobservable M/M/1 constant retrial queue with multiple vacations and the N-policy, the socially optimal mixed strategy is given by

$$\bar{\lambda}^* = \begin{cases} \bar{\lambda}_1^*, & \text{if } \bar{\lambda}_1^* \leq \lambda; \\ \lambda, & \text{if } \bar{\lambda}_1^* > \lambda; \end{cases} \quad (118)$$

where $\bar{\lambda}_1^* (\bar{\lambda}_1^* > 0)$ is the solution of $U_S'(\bar{\lambda}) = 0$, and $U_S(\bar{\lambda}) = \bar{\lambda}R - CE[L(\bar{\lambda})]$.
Proof From (113), we can get the social welfare per time unite $U_s(\bar{\lambda})$ as follows:

$$U_s(\bar{\lambda}) = \bar{\lambda}R - CE[L(\bar{\lambda})]$$

$$= \bar{\lambda}R - C\left(\frac{\bar{\lambda}}{\xi} + \frac{\bar{\lambda}}{\theta(\mu - \bar{\lambda})}\right) + \frac{\theta\xi}{(\bar{\lambda} + \theta)(\bar{\lambda} + \xi)} + \frac{\theta(\mu - \bar{\lambda})}{\mu^2(\bar{\lambda} + \xi)} + \frac{N(N - 1)\xi}{\mu^2(\bar{\lambda} + N\xi)}.$$

(119)

The second-order derivative of $U_s(\bar{\lambda})$ in $\bar{\lambda}$ is given by:

$$U_s''(\bar{\lambda}) = -\frac{2C}{\theta(\mu - \bar{\lambda})^2} - \frac{2C\bar{\lambda}}{\theta(\mu - \bar{\lambda})^3} - \frac{2C\theta\xi}{(\bar{\lambda} + \theta)(\bar{\lambda} + \xi)^3} - \frac{2C\theta\xi}{(\bar{\lambda} + \theta)^2(\bar{\lambda} + \xi)^2} - \frac{2C\theta\xi}{(\bar{\lambda} + \theta)^3(\bar{\lambda} + \xi)^2}.$$

(120)

If $\rho = \frac{\bar{\lambda}}{\mu} < 1$, then $U_s''(\bar{\lambda}) < 0$. So, $U_s(\bar{\lambda})$ is strictly concave of $\bar{\lambda}$. If we denote the unique positive solution of equation $U_s'(\bar{\lambda}) = 0$ by $\bar{\lambda}_1^*$ ($\bar{\lambda}_1^* > 0$), then we have the following results:

(1) When $\bar{\lambda}_1^* \leq \lambda$, obviously, the socially optimal mixed strategy $\bar{\lambda}^*$ of customers is unique, which is $\bar{\lambda}_1^*$.

(2) When $\bar{\lambda}_1^* > \lambda$, the socially optimal mixed strategy $\bar{\lambda}^*$ of customers is unique, which is $\lambda$. $\square$

**Remark 2** When $\xi = 0$, i.e., the multiple vacations in our model become the classical single vacation, our model is similar to the model of Wang et al. (2017) with the assumption that only the server’s state is observable. But our work is different from the work of Wang et al. (2017). Specifically in these aspects: (1) The level of information presented is different; the work of Wang et al. (2017) assumes that arriving customers can only observe the server’s state. Whereas in our work, there are two types to present information level, i.e., the observable case is that the arriving customers can observe the server’s state and the number of customers in orbit; the unobservable case is that the server’s state and the number of customers in orbit are unobservable to the arriving customers. (2) The effective arrival rate of potential customers is different; the customers’ joining probabilities upon arrivals are conditional on the server’s state in the work of Wang et al. (2017), i.e., $\lambda_i = \lambda q_i$. Whereas in our work’s unobservable case, the effective arrival rate of customers is $\lambda q$. (3) The two works have different structures. It is clear that the two works deal with different problems. For example, in the work of Wang et al. (2017), the stationary distribution is obtained by using the partial generating functions. Whereas in our work, the stationary distribution of two information levels is obtained by solving the balance equations and the matrix-geometric solution, respectively. Although the work of Wang et al. (2017) is one of the motivations for our work, the two works studied different problems. The work of Wang et al. (2017) introduces the PSO algorithm to deal with the continuous nonlinear optimal problem of queuing game, which has a great inspiration for the numerical experiment.
part of our work. Moreover, our work explores the effect of information disclosure on equilibrium social welfare and optimal social welfare in numerical experiments.

**Remark 3** If the constant retrial is not considered in our model, then our model is similar to the work of Sun et al. (2016). However, the analysis of the multiple-vacations queuing system with customer retrial is more complex. For example, the study of the two models reflects the difference from the starting point, in the work of Sun et al. (2016), it’s easy to get the customers’ equilibrium balking threshold \( n_e(1) \) (i.e., equation (3.2)) and \( n_e(0) \) (i.e., equation (3.4)). Whereas in our work, due to the complexity of customer retrial in multiple-vacations queuing system, we need to use Theorem 1 to obtain \( n_e(1) \) and \( n_e(0) \). Moreover, it is is easy to get inequality relationship \( n_e(1) > n_e(0) \) in the work of Sun et al. (2016), and in order to ensure that the server can always be reactivated, \( n_e(0) > N - 1 \) surely should be guaranteed. Hence, the inequality relationship of the equilibrium balking threshold produced by the work of Sun et al. (2016) is \( n_e(1) > n_e(0) \geq N - 1 \). Whereas in our work, since the size of \( n_e(0) \) and \( n_e(1) \) depends on the values of \( \mu \) and \( \xi \), which leads to (i) if \( \mu > \xi \), which implies \( n_e(1) > n_e(0) \); and (ii) if \( \xi > \mu \), which implies \( n_e(0) > n_e(1) \). Hence, the inequality relationship of the equilibrium balking threshold produced by our work is \( n_e(1) > n_e(0) \geq N - 1 \), or \( n_e(0) \geq n_e(1) \geq N - 1 \), or \( n_e(0) \geq N - 1 > n_e(1) \). Moreover, our work also gives the reason for choosing the PSO algorithm in numerical experiments and gives the PSO algorithm’s key program.

### 5 Numerical results

In this section, we explore the previous theoretical results through numerical experiments. One should note that, due to the complexity of the equation (85), explicit expressions for the socially optimal balking threshold \( n^*(i) \), \( (i = 0, 1) \) and the optimal social welfare are not available in general. Hence, we use Particle swarm optimization (PSO) algorithm to numerically solve complex analytic characteristics in this section. The numerical optimal solution \( (n^*(0), n^*(1)) \) of \( \max_{(n(0), n(1))} U_s(n(0), n(1)) \) and optimal social welfare \( U_s(n^*(0), n^*(1)) \) can be obtained by PSO algorithm. PSO algorithm was introduced by Eberhart and Kennedy (1995) to solve continuous nonlinear optimization problems, and it has been widely used to solve optimal global solutions since it does not require many constraints and objective functions. The key procedure of applying the PSO algorithm to find the optimal solution (searching for socially optimal balking threshold \( n^*(i) \) \( (i = 0, 1) \)) is illustrated in Algorithm 1, where the velocity \( V_{id} \) and position \( X_{id} \) are generally provided by:

\[
V_{id} = \omega * V_{id} + c_1 * rand() * (pBest_{id} - X_{id}) + c_2 * rand() * (gBest_{id} - X_{id}),
\]

(121)

and

\[
X_{id} = X_{id} + V_{id},
\]

(122)
where \( i \) is the number of particles, \( d \) is the dimension, \( \text{rand()} \) is a random number in \((0, 1)\), \( c_1 \) and \( c_2 \) are the learning factor and \( \omega \) is the inertia factor.

\[ n^* = n^*(0), n^*(1) \]

**Algorithm 1** Searching for socially optimal balking threshold \( n^*(0) \) and \( n^*(1) \)

**Input:** \( R, C, \lambda, \mu, \xi, \theta, N \);

**Output:** \( n^*(0), n^*(1) \);

1: for each particle \( i \)
2: Initializing velocity \( V_{id} \) and position \( X_{id} \) for each particle \( i \)
3: Evaluating particle \( i \) and setting \( p_{Best_i} = X_{id} \)
4: end for
5: \( g_{Best} = \min \{ p_{Best_i} \} \)
6: while not stop
7: for \( i = 1 \) to \( M \)
8: Updating the velocity and position of particle \( i \)
9: Evaluating particle \( i \)
10: if \( \text{fit}(X_{id}) < \text{fit}(p_{Best_i}) \)
11: \( p_{Best_i} = X_{id} \)
12: if \( \text{fit}(p_{Best_i}) < \text{fit}(g_{Best_i}) \)
13: \( g_{Best_i} = p_{Best_i} \)
14: end for
15: end while
16: print \( g_{Best_i} \)
17: end

**5.1 Numerical results for the observable case**

Based on a large number of numerical experiments with a series of parameter choices, we conclude that key qualitative properties are independent of the choice of param-
Fig. 7 Socially optimal thresholds \((n^*(0), n^*(1))\) with respect to \(\xi\) when \(R = 15, C = 1, \lambda = 5, \mu = 3, \theta = 5, N = 7\)

To illustrate these properties, we present some exemplary results below. First, we explore the trend in changes for the socially optimal thresholds \((n^*(0), n^*(1))\) with respect to \(N\) and \(\xi\) in Figs. 6 and 7, respectively. They illustrate the following phenomena.

1. From Fig. 6, we can observe that \(n^*(0)\) and \(n^*(1)\) increase with \(N\), which illustrates that the social planner wants customers to actively join the system with the growth of \(N\).
2. In Fig. 6, it is clear \(n^*(0) \leq n^*(1)\) when \(N \leq 12, n^*(0) > n^*(1)\) when \(N > 12\). The reason for this is that \(U_s(n^*(0), n^*(1)) = U_{ob1}(n(0), n(1))\) when \(N \leq 12\), and \(U_s(n^*(0), n^*(1)) = U_{ob3}(n(0), n(1))\) when \(N > 12\). Specially, \(n^*(0) = N - 1\) when \(N > 7\). It indicates that \(n^*(0)\) is the minimum threshold to ensure server activity. Therefore, when the social planner sets a larger \(N\) value, the corresponding \(n^*(0)\) will be generated.
3. From Fig. 7, we can observe that both \(n^*(0)\) and \(n^*(1)\) decrease with \(\xi\), which illustrates that customers’ selfishness does not match the wishes of the social planner. Especially, \(n^*(0) = N - 1\) when \(\xi > 1.5\), which illustrates that the social planner does not want to accumulate many customers during the vocation.

Next, we explore the trend in changes for \(n^*(i)\) and \(n_e(i)\) \((i = 0, 1)\) with respect to \(\theta\) and \(\mu\) in Figs. 8 and 9, respectively. They reveal the following phenomena.

1. From Figs. 8 and 9, we can observe that both \(n^*(i)\) and \(n_e(i)\) \((i = 0, 1)\) increase with respect to \(\theta\) and \(\mu\), respectively. It is obvious that the growth rate of \(n_e(i)\) \((i = 0, 1)\) is much faster than the growth rate of \(n^*(i)\) \((i = 0, 1)\).
2. \(n_e(i) > n^*(i)\) \((i = 0, 1)\) always holds as shown in Figs. 8 and 9, which illustrates that the customers’ individual behavior under the stable equilibrium can lead to system congestion more seriously.
Finally, Fig. 10 shows that the relationship between the optimal social welfare $U_s(n^*(0), n^*(1))$ and $N$, and the relationship between the optimal social welfare $U_s(n^*(0), n^*(1))$ and $\xi$. It reveals the following phenomena.

1. In Fig. 10a, the optimal social welfare $U_s(n^*(0), n^*(1))$ decreases with $N$. The reason is that when $N$ becomes larger, more customers will be accumulated, which leads to more waiting costs.

2. In Fig. 10b, the optimal social welfare $U_s(n^*(0), n^*(1))$ increases with $\xi$. When $\xi$ becomes bigger, it speeds up the operation of the system, which can then produce the social welfare more effectively. Moreover, when $\xi$ increases to a certain value, the social welfare reaches its maximum and remains stable afterwards.
5.2 Numerical results for the unobservable case

In the unobservable case, $\lambda_2$ is unstable from the equilibrium point of view. Thus, we only study the stable one with $\lambda_1$, $\lambda_3$ or $\lambda$, in the following numerical results. First, we explore the impact of the parameters $N$, $\xi$, $\theta$ and $\mu$ on the equilibrium arrival rate $\lambda_e = \lambda q_e$ and the optimal arrival rate $\lambda^* = \lambda q^*$ in Fig. 11, respectively. They illustrate the following phenomena.

1. From Fig. 11, we can always see that $\lambda_e$ (i.e. $\lambda q_e$) $\geq \lambda^*$ (i.e. $\lambda q^*$) with $N$, $\xi$, $\theta$ and $\mu$, respectively.
2. In Fig. 11a, $\lambda_e$ (i.e. $\lambda q_e$) decreases with $N$, whereas $\lambda^*$ (i.e. $\lambda q^*$) increases with $N$. The reason for this is that a larger value of $N$ reduces the enthusiasm of customers to join the system. However, a larger value of $N$ makes the social planner encourages more customers to enter the system.
3. From Fig. 11b–d, we can observe that the interests of customers and the social planner are coincident with each other with respect to $\xi$, $\theta$ and $\mu$. This is because that shortening the vacation time can reduce waiting costs, and increasing the retrial rate and service rate can accelerate switchover of the system back from vacations.

Moreover, Fig. 12 shows that $\lambda_e$ (i.e. $\lambda q_e$) $\geq \lambda^*$ (i.e. $\lambda q^*$) as a function of $\lambda$. It is obvious that the value of $N$ has no significance on $\lambda_e$ and $\lambda^*$ when $\lambda > \lambda_1$, but $\lambda_1$ increases with $N$. Thus, a higher value of $N$ will scare away customers unless $\lambda$ is big enough.

If all customers follow the stable equilibrium mixed strategy $\lambda_e$, then their equilibrium social welfare per time unit $U_s(\lambda_e)$ can be achieved. Similarly, the optimal social welfare per time unit $U_s(\lambda^*)$ can also be obtained. Figure 13 shows the trend in changes for the optimal social welfare per time unit $U_s(\lambda^*)$ with respect to $N$ and $\xi$, respectively. Obviously, the case shown in Fig. 13 is similar to that shown in Fig. 10.
Fig. 11 Equilibrium and optimal arrival rates with respect to $N$, $\xi$, $\theta$ and $\mu$, respectively. 

- (a) $R = 10$, $C = 1$, $\lambda = 3$, $\mu = 3$, $\xi = 0.5$, $\theta = 5$; 
- (b) $R = 20$, $C = 1$, $\lambda = 5$, $\mu = 5$, $\theta = 3$, $N = 6$; 
- (c) $R = 10$, $C = 1$, $\lambda = 5$, $\mu = 5$, $\xi = 3$, $N = 6$; 
- (d) $R = 10$, $C = 1$, $\lambda = 3$, $\xi = 3$, $\theta = 5$, $N = 3$.

Fig. 12 Equilibrium and optimal arrival rates for unobservable case with $R = 10$, $C = 1$, $\mu = 3$, $\xi = 2$, $\theta = 5$. 

- (a) $N = 3$; 
- (b) $N = 25$. 

5.3 The role of the information level on the equilibrium social welfare and optimal social welfare

An essential problem in the strategic customer queuing model is the level of information that the social planner should provide to customers. Figure 14 explores the trend in changes for the equilibrium social welfare of customers and optimal social welfare of customers under two levels of information. The properties shown here are presentative since the conclusions made are based on comprehensive numerical experiments with a broad choice of system parameters.

1. In Fig. 14a, it shows that the equilibrium strategy of customers is balking when $\lambda$ is small, since smaller $\lambda$ does not intend to activate the system. In this case, both $U_s(n_e(0), n_e(1))$ and $U_s(\lambda q_e)$ (i.e. $U_s(\lambda q_e)$) have a zero-increase-decrease trend with $\lambda$. However, $U_s(\lambda) \geq U_s(n_e(0), n_e(1))$ when $\lambda < \lambda_e$, since the arrival rate is small and the number of customers in the system is small. In this case, hiding the system information from customers helps to increase the number of customers entering the system, thereby increasing social welfare. Similarly, $U_s(\lambda) < U_s(n_e(0), n_e(1))$ when $\lambda > \lambda_e$, which shows that disclosing the system information can help reduce the system congestion, thus reducing waiting costs.

2. In Fig. 14b, it shows that the socially optimal mixed strategy of customers is balking when $\lambda$ is small. Both $U_s(n^*(0), n^*(1))$ and $U_s(\lambda^*)$ (i.e. $U_s(\lambda^*)$) keep growth, $U_s(\lambda^*)$ eventually becomes a constant. Similar to Fig. 14a, b also shows that $U_s(\lambda^*) \geq U_s(n^*(0), n^*(1))$ when $\lambda < \lambda^*$, $U_s(\lambda^*) < U_s(n^*(0), n^*(1))$ when $\lambda > \lambda^*$. This also shows that the information level of the system has a serious impact on the social welfare. Therefore, the social planner should choose the strategy consistent with the system designer to achieve social optimum.
Fig. 14  a Comparison of equilibrium social welfare under two information levels when $\mu = 3$, $\xi = 0.2$, $\theta = 5$, $R = 20$, $C = 1$, $N = 6$; b comparison of optimal social welfare under two information levels when $\mu = 3$, $\xi = 0.2$, $\theta = 5$, $R = 20$, $C = 1$, $N = 6$

6 Conclusions and further research

In this paper, we studied equilibrium strategies and optimal balking strategies of customers in a constant retrial queue with multiple vacations and the $N$-policy under two information levels (observable case and unobservable case), respectively. We determined equilibrium strategies and optimal balking strategies of customers and social welfare for each type of information level. For the observable case, in order to ensure that the server can be reactivated, we obtained that the optimal balking threshold of customers in the vacation state must be greater than $N - 1$. Therefore, there are three different queuing cases for the observable case, and we obtained the corresponding stationary distributions for the three queuing cases and determined the equilibrium social welfare per time unit. We got the positive equilibrium arrival rate and optimal arrival rate for the unobservable case, which are unique. In Sect. 5, we explored the previous theoretical results through numerical experiments. However, due to the complexity of the involved equations, explicit expressions for the socially optimal balking thresholds and optimal social welfare are not available in general. Hence, we use Particle Swarm Optimization (PSO) algorithm to solve the complex analytic characteristics. The numerical optimal solution $(n^*(1), n^*(2))$ and optimal social welfare $U_s(n^*(0), n^*(1))$ are obtained by PSO algorithm. By comparing the numerical results of the two information levels, we obtained that the customers’ behavior under the stable equilibrium makes the system more congested than that under the socially optimal one, and whether the system information should be disclosed to customers depends on how to maintain the growth of the social welfare (i.e., potential demand arrivals). Obviously, to maximize social welfare, which factor determines the level of information disclosure and when to disclose the system information to customers is also crucial for the server or social planner. Fortunately, this paper achieved this goal. In the future, it is necessary for us to consider the almost observable case of this model, i.e., the state of the server can be observed, but the number of customers in orbit cannot be observed.
Acknowledgements  We are grateful to the anonymous reviewers and editors for their constructive comments and feedback that help us to improve the presentation and quality of this manuscript. This work was supported in part by The National Natural Science Foundation of China (No. 61773014), the Research Fund for the Postgraduate Research and Practice Innovation Program of Jiangsu Province (No. KYCX20_0240), and the Natural Sciences and Engineering Research Council of Canada (No. 315660).

Availability of data and materials  Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Declaration

Conflict of interest  None of the authors have any competing interests in the manuscript.

References

Balachandran K (1973) Control policies for a single server system. Manag Sci 19(9):1013–1018
Burnetas A, Economou A (2007) Equilibrium customer strategies in a single server Markovian queue with setup times. Queueing Syst 56(3–4):213–228
Chen H, Frank MZ (2001) State dependent pricing with a queue. IIE Trans 33(10):847–860
Doshi B (1986) Queueing systems with vacations—a survey. Queueing Syst 1(1):29–66
Eberhart R, Kennedy J (1995) A new optimizer using particle swarm theory. In: Proceedings of the sixth international symposium on micro machine and human science MHS’95, pp. 39–43
Economou A, Kanta S (2008) Equilibrium balking strategies in the observable single-server queue with breakdowns and repairs. Oper Res Lett 36(6):696–699
Economou A, Kanta S (2011) Equilibrium customer strategies and social-profit maximization in the single-server constant retrial queue. Nav Res Logist 58(2):107–122
Edelson NM, Hilderbrand DK (1975) Congestion tolls for Poisson queueing processes. Econometrica 43(1):81–92
Guo P, Hassin R (2011) Strategic behavior and social optimization in Markovian vacation queues. Oper Res 59(4):986–997
Guo P, Hassin R (2012) Strategic behavior and social optimization in Markovian vacation queues: the case of heterogeneous customers. Eur J Oper Res 222(2):278–286
Guo P, Li Q (2013) Strategic behavior and social optimization in partially-observable Markovian vacation queues. Oper Res Lett 41(3):277–284
Hassin R, Haviv M (2003) To queue or not to queue: equilibrium behavior in queueing systems. Kluwer Academic Publishers, Boston
Johansen SG, Stidham S (1980) Control of arrivals to a stochastic input–output system. Adv Appl Probab 12(4):972–999
Kulkarni VG (1983) A game theoretic model for two types of customers competing for service. Oper Res Lett 2(3):119–122
Kumar BK, Vijayalakshmi G, Krishnamoorthy A, Basha SS (2010) A single server feedback retrial queue with collisions. Comput Oper Res 37(7):1247–1255
Liu W, Ma Y, Li J (2012) Equilibrium threshold strategies in observable queueing systems under single vacation policy. Appl Math Model 36(12):6186–6202
Ma Y, Liu W, Li J (2013) Equilibrium balking behavior in the Geo/Geo/1 queueing system with multiple vacations. Appl Math Model 37(6):3861–3878
Naor P (1969) The regulation of queue size by levying tolls. Econometrica 37(1):15–24
Neuts MF (1981) Matrix-geometric solutions in stochastic models. Johns Hopkins University Press, Baltimore
Shanthikumar J (1981) Optimal control of an M/G/1 priority queue via N-control. Am J Math Manag Sci 1(3):191–212
Stidham S (1985) Optimal control of admission to a queueing system. IEEE Trans Autom Control 30(8):705–713
Sun W, Li S, Cheng-Guo E (2016) Equilibrium and optimal balking strategies of customers in Markovian queues with multiple vacations and N-policy. Appl Math Model 40(1):284–301
Sun W, Li S, Tian N (2017) Equilibrium and optimal balking strategies of customers in unobservable queues with double adaptive working vacations. Qual Technol Quant Manag 14(1):94–113
Wang J, Zhang F (2013) Strategic joining in M/M/1 retrial queues. Eur J Oper Res 230(1):76–87
Wang J, Zhang X, Huang P (2017) Strategic behavior and social optimization in a constant retrial queue with the N-policy. Eur J Oper Res 256(3):841–849
Yadin M, Naor P (1963) Queueing systems with a removable service station. J Oper Res Soc 14(4):393–405
Zhang F, Wang J, Liu B (2012) On the optimal and equilibrium retrial rates in an unreliable retrial queue with vacations. J Ind Manag Optim 8(4):861–875

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.