A mathematical theory of imperfect communication: 
Energy efficiency considerations in multi-level coding

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Abstract

A novel framework is presented for the analysis of multi-level coding that takes into account degrees of freedom attended and ignored by the different levels of analysis. It follows that for a multi-level coding system, skipped or incomplete error correction at many levels can save energy and provide equally good results to perfect correction. This is the case for both discrete and continuous cases. This has relevance to approximate computing, and also to deep learning networks, which can readily be construed as multiple levels of inadequate error correction reacting to some input signal, but which are typically considered beyond analysis by traditional information theoretical methods. The finding also has significance in natural systems, e.g. neuronal signaling, vision, and molecular genetics, which can be characterized as relying on multiple layers of inadequate error correction.

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1 Introduction

Many, if not most, forms of communication can be construed as using multiple levels of encoding and decoding. A note is typed into an email, which is encoded into a series of bytes, organized into packets of bits, and sent as current fluctuations to some other computer. On receipt they become bits, then bytes, then letters on the recipient’s screen. It is possible to discern similar arrangements in natural systems. For example, both phonological and visual processing have long been understood to be arranged in tiers of functionally similar

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syntactic operations feeding the processors of the next higher tier (Liberman and Prince, 1977; Goldsmith, 1979; Marr, 1982).

Since the establishment of communication theory with Shannon (1948), there has been little consideration of ensembles of levels as a functional unit and how levels relate to one another. There has been extensive examination of concatenated codes, where two encodings are combined into one (e.g. Dumer, 1998). This is related, but differs by assuming the intermediate level to be isolated from further input.

Shannon’s goal was the engineering of perfect reproduction of a sender’s message at the receiver, in the presence of noise. The same tools can be used to analyze the performance of complex multi-level systems when the reproduction may not be perfect. We proceed by examining the case of discrete noiseless transmission with an eye to expanding to multiple levels, then move on to an analysis of the energy use of a two-level system in the context of noise, and how that can be extended to an arbitrary number of levels. Finally, we show how the concepts introduced in these sections can be applied to the case of continuous transmission, with noise.

2 Discrete noiseless transmission

Consider some set of symbols $\mathcal{R} = \{r_1, r_2, r_3, \ldots\}$ each of which may be translated into one or more from $\mathcal{Q} = \{q_1, q_2, q_3, \ldots\}$ for transmission. On receipt, the original members of $\mathcal{R}$ are recreated from measurements of $\mathcal{Q}$ via a process reversing what came before. If $R$ is a message made of $r \in \mathcal{R}$ and $Q$ a message of $q \in \mathcal{Q}$ then together they can be arranged in a Markov chain:

$$R \rightarrow Q \rightarrow R'$$

We banish the passive voice and consider two agents to accomplish the encoding and decoding, respectively:

$$R \rightarrow \text{Agent 1} \rightarrow Q \rightarrow \text{Agent 2} \rightarrow R'$$ (1)

Assume for the moment that Agent 1 encodes each symbols of $\mathcal{R}$ into a two-letter “word” composed of symbols of $\mathcal{Q}$. Consider an input alphabet $\{A, B, C, D\}$, to be encoded into two-letter words from the alphabet $\{a, b\}$:

- $\vec{A} \rightarrow \vec{a}a$
- $\vec{B} \rightarrow \vec{a}b$
- $\vec{C} \rightarrow \vec{b}a$
- $\vec{D} \rightarrow \vec{b}b$

Because the positions of the two output letters are independently variable factors, each two-letter word can be represented as a point in a two-dimensional phase space:
Three-letter words could similarly be described in three dimensions, four-letter words with four dimensions, and so on. Indeed, one can regard the dimensions of the phase space as representing generalized “degrees of freedom,” each of which might represent letter position, or something else entirely, such as whether the symbol is printed in red or perhaps transmitted on an independent transmission channel.

One might classify agents into “aggregators” where the number of input degrees of freedom is greater than in their output and “distributors” where the input degrees are fewer than the output. The action of a distributor implies conditionality of its output on the degrees of freedom. For example, the probability of encountering some symbol might depend on whether one is considering the first or second symbol of a word. Conversely, the action of an aggregator is to assert conditionality on its inputs. A distributor feeding three output degrees of freedom to an aggregator expecting three inputs might perfectly agree with its partner:

\[
R \rightarrow \text{Agent 1} \rightarrow Q_1 \rightarrow \text{Agent 2} \rightarrow R'
\]

In such a case, Agent 2 simply reassembles what Agent 1 took apart. Perhaps Agent 1 encoded its input into a 3-symbol word and Agent 2 is simply decoding the codewords back into the source alphabet. Agent 1 asserts that its outputs are conditional on three degrees of freedom, and Agent 2 agrees that its inputs are also conditional on those same degrees.

Contrast that with another geometry, which accomplishes much the same thing, but over a more complicated network.

\[
R \rightarrow \text{Agent 1} \rightarrow Q_1 \rightarrow \text{Agent 1a} \rightarrow Q_1' \rightarrow \text{Agent 2} \rightarrow R'
\]

In this case, Agent 1 still asserts that its outputs are conditional on the degrees of freedom and Agent 2 asserts the same about its inputs, but the conditionality is irrelevant to the action of Agent 1a and the other agents in the middle. The Markov assumption implies these agents are free to treat the input they receive as coming from a stochastic source of uncorrelated symbols, one at a time.

Conditionality decreases information, so as a consequence, an analysis of the inputs and outputs of the agents in the middle would imply more bits of information than the analysis of Agent 1 would imply. Distributors, therefore, increase information locally, while aggregators decrease it.

### 2.1 Encoding degrees of freedom

If the symbols of an alphabet of length \(N\) can be ordered, they may be represented by a vector of \(N\) elements with a one in the position corresponding to that symbol and zeros everywhere else. For example, given an alphabet \(\mathcal{R}\) of \(N\) symbols, the first symbol would be represented as \(\vec{r}_1 = [1, 0, 0, \ldots]\), the second would be \(\vec{r}_2 = [0, 1, 0, \ldots]\), and so on. For a one-to-one transformation, it is simple to construct a transformation matrix where each
column represents the output encoding of a symbol of the input alphabet. Multiplication
by this matrix thus converts vector representations of the input alphabet symbols to vector
representations of the output.

Many transformations of interest involve aggregators or distributors, involving different
numbers of degrees of freedom for input and output. We can incorporate such transforma-
tions into our transformation matrix by defining a vector representation of a multi-
symbol word, appending the vector representations of each symbol to create a vector repre-
sentation of a multi-symbol word.

Consider the encoding above of the input alphabet \{\text{A, B, C, D}\}, into two-symbol words
of symbols from the alphabet \{\text{a, b}\}. If the encoding for a C is \text{ba}, one might have an input
\text{C} \equiv [0, 0, 1, 0] translate into an output \text{ba} = [0, 1, 1, 0] where the first two elements stand for
\text{b} = [0, 1] and the following two stand for \text{a} = [1, 0]. Here is a table of such an encoding:

\begin{align*}
\mathbf{\tilde{A}} & \equiv [1, 0, 0, 0] \rightarrow \mathbf{a}a \equiv [1, 0, 1, 0] \\
\mathbf{\tilde{B}} & \equiv [0, 1, 0, 0] \rightarrow \mathbf{a}b \equiv [1, 0, 0, 1] \\
\mathbf{\tilde{C}} & \equiv [0, 0, 1, 0] \rightarrow \mathbf{b}a \equiv [0, 1, 1, 0] \\
\mathbf{\tilde{D}} & \equiv [0, 0, 0, 1] \rightarrow \mathbf{b}b \equiv [0, 1, 0, 1]
\end{align*}  \tag{2}

It is straightforward to define a matrix \(T_{\mathcal{R}, \mathcal{D}}\), an agent can use to transform symbols of a
message \(R\) into \(Q\). We will use \(v\) for degrees of freedom and \(N_v\) for the average number of
degrees of freedom for a given transformation. Using \(v\) as a suffix will indicate a message
or an alphabet rendered to account for the degrees of freedom, so in this case, the agent
accepts a symbol of \(R\) and outputs two symbols of \(Q\):

\[ T_{\mathcal{R}, \mathcal{D}_v} \tilde{r} = \tilde{q}_v \] \tag{3}

The counterpart decoder to this agent must wait for a pair of symbols before putting out one
symbol of \(R\). The counterpart thus assumes that the symbols it receives one at a time are
organized into pairs.

A matrix like \(T_{\mathcal{R}, \mathcal{D}_v}\) will seldom be invertible, but in general it is possible to find an \(\tilde{r}\)
to solve the equation. Because the \(\tilde{q}_v\) was created by multiplication with \(T_{\mathcal{R}, \mathcal{D}_v}\), the rank of
an augmented matrix \([T_{\mathcal{R}, \mathcal{D}_v}, \tilde{q}_v]\) will be the same as the rank \(T_{\mathcal{R}, \mathcal{D}_v}\). By the Rouché-Capelli
theorem, equation (3) will have at least one solution, which is unique if the matrix is of full
rank.

The Moore-Penrose pseudoinverse can provide a solution for full rank matrices \cite{Penrose1955}, however the independence of the degrees of freedom militates against a transforma-
tion of sufficient size being of full rank. Consider a transformation where single symbols
of \(\mathcal{R}\) are transformed into pairs of symbols of \(\mathcal{D}\). This will be a matrix with \(N_v N_\mathcal{D} = 2 N_\mathcal{D}\)
rows. Because the degrees of freedom are independent, there are at most \(N_\mathcal{D}\) independent
rows. If there are more than \(N_\mathcal{D}\) columns, the matrix cannot be of full rank. A similar
demonstration is feasible with the rows of many-to-one transformations. For a transforma-
tion matrix with less than full rank the equation has infinite possible solutions, by the same
Rouché-Capelli theorem, if the columns span the problem space.

The addition of an objective function makes this into a linear programming optimization
problem, with a non-empty feasibility polytope. One might select an objective function
to minimize energy used, or conditional information, or some other quantity. One can
also restrict the possible solutions by subsetting the feasibility polytope. For example, the
transformation in (2) is uniquely decodable if the solutions are restricted to vectors with a
single one and zeros otherwise.

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2.2 Information

Many transformations of interest have redundancy, ambiguity, or degeneracy. DNA codons, for example, are degenerate, with many different codons indicating the same amino acid. Natural languages are frequently ambiguous and redundant, with homonyms and redundant information. For these cases, we can turn to a transition matrix, quite similar to the transformation matrix, where all the entries are conditional probabilities:

\[
A_{\mathcal{R}_{\mathcal{D}_v}} \equiv \begin{bmatrix}
P(q_1|v_1,r_1) & P(q_1|v_1,r_2) & P(q_1|v_1,r_3) & \ldots \\
P(q_2|v_1,r_1) & P(q_2|v_1,r_2) & P(q_2|v_1,r_3) & \ldots \\
\vdots & \vdots & \vdots & \ddots \\
P(q_1|v_2,r_1) & P(q_1|v_2,r_2) & P(q_1|v_2,r_3) & \ldots \\
P(q_2|v_2,r_1) & P(q_2|v_2,r_2) & P(q_2|v_2,r_3) & \ldots \\
\vdots & \vdots & \vdots & \ddots \\
P(q_1|v_3,r_1) & P(q_1|v_3,r_2) & P(q_1|v_3,r_3) & \ldots \\
P(q_2|v_3,r_1) & P(q_2|v_3,r_2) & P(q_2|v_3,r_3) & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

When multiplied by a symbol vector, this matrix will produce a probability distribution indicating the likely result. Further, if \(P(R)\) is the probability distribution of symbols in some message \(R\), then the distribution of the result, transformed into \(\mathcal{D}_v\) space, is \(A_{\mathcal{R}_{\mathcal{D}_v}}P(R) = P(Q|v)\). Like the transformation matrix, such a transition matrix is often singular, but solutions can be found using the same methods. Similarly, for the probability distribution of message \(R\) that has been successfully translated into \(Q\) and then to \(R'\), we can construct a matrix for a given solution \(P(R')\) which we will call \(A(R')^{-1}\) to show that it is the inverse of \(A_{\mathcal{R}_{\mathcal{D}_v}}\), when that matrix is invertible.

We introduce an operator \(\overline{H}\) on a matrix of probabilities \(A_{ij}\), that calculates the sum of \(y = -x \log x\) on the columns of a matrix:

\[
\overline{H}(A) \equiv \left[ -\sum_i P_{i1} \log P_{i1}, -\sum_i P_{i2} \log P_{i2}, -\sum_i P_{i3} \log P_{i3}, \ldots \right]
\]

One can verify that if \(A\) transforms one probability distribution into another, \(AP(X) = P(Y)\), then the inner product of \(\overline{H}(A)\) with the probability distribution \(P(X)\) gives the information in message \(Y\), given \(X\): \(\overline{H}(A) \cdot P(X) = H(Y|X)\). Applying the operator to a transition matrix, we have \(\overline{H}(A_{\mathcal{R}_{\mathcal{D}_v}}) \cdot P(R) = H(Q|v,R)\). The original transformation matrix \(T_{\mathcal{R}_{\mathcal{D}_v}}\) is simply a transition matrix where all the probabilities are one or zero, so \(\overline{H}(T_{\mathcal{R}_{\mathcal{D}_v}}) = 0\) and \(H(Q|v,R) = 0\) for transformations it describes.

Consider the equation \(A_{\mathcal{R}_{\mathcal{D}_v}}P(R) = P(Q|v)\). If the agent making the transformation is an aggregator, and if \(A_{\mathcal{R}_{\mathcal{D}_v}}\) is of full rank, then the matrix has a left inverse. If the matrix is not of full rank or if the agent is a distributor, one can solve the equation using linear programming. In this case, a solution-specific inverse can be constructed. For example, if a solution is found with the simplex method, the slack variable coefficients provide an inverse to that matrix to create a solution-specific inverse.

We define \(A(X_0)^{-1}\) to be a matrix that satisfies \(A(X_0)^{-1}X_0 = Y_0\) for some specific \(X_0\) that is the solution to the equation \(AX = Y_0\). If the original matrix is left-invertible, this is
the left inverse, and is a general solution. If not, this is the inverse constructed via the basis of the simplex solution and is specific only to $X_0$.

Going a step further, the information in $R'$ conditional on $R$ can be calculated from the matrix $A_{\mathcal{R},\mathcal{Q}}(R)^{-1}$:

$$H(A_{\mathcal{R},\mathcal{Q}}(R)^{-1}A_{\mathcal{R},\mathcal{Q}}) \cdot P(R) = H(R'|R)$$

(4)

If $A_{\mathcal{R},\mathcal{Q}}$ is left-invertible, then the product with $A_{\mathcal{R},\mathcal{Q}}(R)^{-1}$ is an identity matrix and $H(R'|R)$ is zero. Otherwise, the conditional information may be non-zero.

For the Markov chain described in (1), Agent 2 must decode what Agent 1 has encoded. Without the matrix, there is no equation to solve, so having an accurate copy of the encoding matrix or the functional equivalent of its inverse is a necessary, if not a sufficient, condition for an accurate reproduction of the original message. The picture of data flow might be adjusted to look like this:

$$R \rightarrow \text{Agent 1} \rightarrow Q \rightarrow \text{Agent 2} \rightarrow R'$$

(5)

The transmission problem is thus not merely moving the result of $A_{\mathcal{R},\mathcal{Q}}r_i$ from Agent 1 to Agent 2, but reproducing the $A_{\mathcal{R},\mathcal{Q}}$ originally used by Agent 1 so that Agent 2 has an equation to solve. There are a variety of ways to solve this transmission problem. Cryptographic communication often requires distribution of code books, or dated one-time pads. Computer communication has standards committees such as the ISO, ANSI, and IETF, whose role is to ensure that the encoder and decoder execute precisely inverse operations for the various communication standards. For some pair of transition matrix and solution, the magnitude of the $H(A_{\mathcal{R},\mathcal{Q}}(R)^{-1}A_{\mathcal{R},\mathcal{Q}})$ vector is a convenient measurement of the conditional information potentially introduced by the encoding and decoding, and thus a measure of the effective fidelity of the transmission $A_{\mathcal{R},\mathcal{Q}} \rightarrow A'_{\mathcal{R},\mathcal{Q}}$.

### 2.3 Multiple levels

Consider two layers of transformation: some set of symbols $\mathcal{S} = \{s_1, s_2, s_3, \ldots\}$ each of which can be translated into one or more from another set $\mathcal{R} = \{r_1, r_2, r_3, \ldots\}$ which can then be translated into one or more from $\mathcal{D} = \{d_1, d_2, d_3, \ldots\}$. One might imagine letters ($\mathcal{S}$) being decomposed into bits ($\mathcal{R}$) and then a series of voltage measurements ($\mathcal{D}$), for example, or a musical note name being decomposed into a frequency and then into a series of air pressure pulses, or any number of other such progressions. On the other side of the transmission, the original members of $\mathcal{S}$ are recreated from measurements of $\mathcal{D}$ via the reverse process. Again, we postulate independent agents to transform the messages. Each agent receives a series of input symbols from its predecessor and transforms it into a series of output symbols, with no awareness of the other agents apart from the series of symbols themselves, making this a Markov chain:

$$S \rightarrow \text{Agent 1} \rightarrow R \rightarrow \text{Agent 2} \rightarrow Q \rightarrow \text{Agent 3} \rightarrow R' \rightarrow \text{Agent 4} \rightarrow S'$$

(6)

If $\mathcal{R}$ has four symbols, with three degrees of freedom, it can describe an alphabet of as many as $4^3 = 64$ symbols without ambiguity. If $\mathcal{S}$ has only 20 symbols, we can describe the transformation $\mathcal{S} \rightarrow \mathcal{R}$ with a matrix of 20 columns (one for each input symbol) and
twelve rows (one for each output symbol and for each degree of freedom). If Agent 2 transforms its input symbols into three-bit binary numbers (two symbols, three degrees of freedom), it can use a six-row by four-column transformation matrix on each input symbol.

The output from Agent 1, like its input, is merely a series of symbols. If, for example, it transforms single symbols of $S$ into triplets of $R$, Agent 2 will receive three times as many symbols as Agent 1. The fact that these groupings of three have meaning to Agent 1 is irrelevant to the operation of Agent 2, who is free to treat its input as a stochastic source of symbols conditional on nothing. Agent 4 assumes conditionality on the same degrees of freedom Agent 1 uses in its output and thus receives the message as intended. This is a simple consequence of the Markov property; each agent’s knowledge of the system is limited to its direct inputs. Conditionality requires knowledge of—or assumptions about—the state beyond those direct inputs.

Assume Agent 1 and Agent 2 are distributors and all the transformations involved have no ambiguity or degeneracy. Agent 1 uses $A_{S\mathcal{R}v}$ to convert symbols of message $S$ into $R_v$ and sees a message with $H(S)$ bits per symbol transform into one with $H(R|v)$ bits per symbol. Because this is a Markov chain, Agent 2 may treat the stream of symbols from Agent 1 as a stochastic source $R$ to which it can apply $A_{RQv}$. By respecting the order of processing, Agent 2 preserves the degrees of freedom so important to Agent 1 (and Agent 4) but they are irrelevant to its own operation. Were we to ask Agent 1 how many bits of information were in its input compared to its output, it would say they were both $N_SH(S) = N_RH(R|v)$. Were we to pose the same question to Agent 2, it will reply there are $N_RH(R)$ bits, a larger number than Agent 1 measures, because, from its perspective, the information is not conditional:

\[(\text{Output of Agent 1}) \quad H(R|v) \leq H(R) \quad \text{(Input of Agent 2)} \quad (7)\]

Entropy is not subjective, but one need not include degrees of freedom in an analysis if the symbols under consideration are independent of those degrees of freedom.

For Agent 1 and Agent 2 as distributors that appear to see an increase in information, and Agent 3 and Agent 4 are aggregators who see it decrease, we would have:

\[
\begin{align*}
N_SH(S) &= N_RH(R|v) \leq N_RH(R) = N_QH(Q|v) \leq N_QH(Q) \\
&\quad \text{Agent 1} \quad \text{Agent 2} \\
N_QH(Q) \geq N_QH(Q|v) = N_RH(R) \geq N_RH(R|v) = N_SH(S) \\
&\quad \text{Agent 3} \quad \text{Agent 4} \quad (8)
\end{align*}
\]

Information appears to be increasing as we move along the encodings, and decreasing as the message is decoded. From a global perspective, this is an illusion created by agents ignoring important features of their input, but this is appropriate in service of insight into the function of those agents, and we turn to that in the next section.
3 Discrete noisy transmission

Consider a similar arrangement to (6) in a noisy channel, where accuracy demands implementation of some system of error-checking. Such an arrangement is a more elaborate Markov chain, with separate symbol conversion and error correction steps. Expanding the steps $Q \rightarrow R' \rightarrow S'$ we now have a longer chain, where $Q$ is converted to $R'_t$ (the $t$ is for “tentative”), by solving the transition matrix $A_{\mathcal{RQ}^\nu}$. Additional data $\varepsilon_{R_t}$, received through some correction channel, is used to reduce uncertainty in $R'_t$, producing $R'_t$.

$$Q \rightarrow R'_t \rightarrow R' \rightarrow S'_t \rightarrow S'$$

The error-correction information, $\varepsilon_{R_t}$, could have arrived in the same channel as $R$, for example as parity bits, checksums, the extra bits added for a Hamming code, or some more elaborate forward error-correction (FEC) system still uninvented. It might also have been developed from other observations, experience, or prior arrangement. It could also arrive, as with the transformation information, through some completely different channel.

Consider the energy consumption of the two error correction steps $R_t \rightarrow R$ and $S_t \rightarrow S$. (We omit the $'$ to ease visual clutter; here we only address one side of the chain in (6).) For some error in the transmission of a symbol $q$, the error might be corrected at the first step, as it becomes a contribution to some $r$ or the second, as that $r$ contributes to some $s$. Error correction involves some way to compare observations with expectations. The energy spent will consist of two components: one to compare the symbols in a message with the expectations, and another to correct those seeming in need of repair. We attempt to set a lower bound to this energy.

For comparing symbols against expectations, we assume an efficient error-checking mechanism can do no better than an energy cost proportional to the total number of bits in the message. A message of twice the length costs twice the energy to review. Since there is a minimum energy cost to erasing information \cite{Landauer, Bennett}, there is also an energy cost to correcting a symbol. This could be as simple as the energy needed to erase a bit or as expensive as a request for retransmission, depending on context. We therefore model our second component with another linear function, dependent on both the number of errors and the efficacy of identifying them.

Let $K_R$ be an estimate of the per-symbol energy cost of assessing what the observations $R$ should be. We define a noise level (proportion of symbols transmitted incorrectly), $0 \leq f < 1$, an efficacy function (the proportion of errors actually found as a function of the energy spent finding them), $0 \leq z(K_R) < 1$, and a per-bit cost of repair, $L$, to write an expression for the work done in error correction during $Q \rightarrow R_v$:

$$E \geq K_R I(Q;R_v) + fL_R I(Q;R_v)z(K_R)$$ (10)

If the error rate ($f$) is 10% and the efficacy of the error detection ($z(K_R)$) is 80%, then 8% of the symbols in $R$ will need repair. Spending more energy per symbol in detection will bring $z$ closer to one and therefore require more energy for correction. We model the efficacy as a function of $K_R$ whose value is zero when $K_R$ is small and increases monotonically toward one. An inverse exponential captures the intuition that there is a point of diminishing returns, beyond which it costs significant amounts of energy to detect an
increasingly small number of errors, but the monotonic nature of the function is the only important assumption.

Consider the second level, the observations \( S \) that depend on \( R \), and the energy of error correction there. Assume all errors are corrected at one level or the other: if \( z(K_R) \) of the errors are fixed at \( R \), then \( 1 - z(K_R) \) are corrected at \( S \). We also assume for the moment that the transmission from \( R \) to \( S \) introduces no new errors. We can add to equation 10 and write an equation for the work done in error correction at the two levels, as a function of the energy invested in error correction at \( R \):

\[
E \geq K_R I(Q;R_v) + f L_R I(Q;R_v) z(K_R) + K_S I(R;S_v) + f L_S I(R;S_v) (1 - z(K_R)) \tag{11}
\]

We define a ratio \( \alpha \equiv I(R;S_v)/I(Q;R_v) \) to compare the number of bits of information in one set of observations with the number of bits in its successor set, recalling the point that the different levels may involve different degrees of freedom. Simplifying:

\[
\frac{E}{I(Q;R_v)} \geq K_R + f L_R z(K_R) + K_S \alpha + f L_S \alpha (1 - z(K_R)) \tag{12}
\]

Differentiating with respect to \( K_R \) and setting to zero to minimize:

\[
0 = 1 + f(L_R - \alpha L_S) \frac{dz}{dK_R} + \alpha \frac{dK_S}{dK_R} \tag{13}
\]

For \( \alpha < 1 \) and \( L_R/L_S > \alpha \), since \( z \) is a monotonically increasing function of \( K_R \) and \( K_S \) is independent of \( K_R \), then one of the terms on the right side of the equation is always positive and the other is zero. Thus there is no solution to this equation and therefore no positive value of \( K_R \) that will cost less energy than \( K_R = 0 \). For these cases, it will always save energy to skip error correction at \( R \) in favor of \( S \).

Note that \( dz/dK_R \) is close to zero at high levels of efficacy. For \( \alpha > 1 \) and \( L_R \) and \( L_S \) of comparable size, the second term in equation 13 is thus small unless \( \alpha \) is large. If the agent at \( S \) perceives much more information than the one at \( R \), then it might be efficient to do complete error correction at \( R \). Otherwise, so long as \( K_S \) is independent of \( K_R \), it is likely that a solution will occur at efficacy levels substantially lower than 100% at \( R \).

It is plausible that \( K_S \) might exhibit a dependence on \( K_R \), in which case there may be a non-zero solution to equation 13. Perhaps a certain amount of energy spent checking for errors at \( R \) would mean spending less at \( S \) to achieve the same result. We model this as the sum of \( b_S \), a component independent of \( K_R \), and another component that is a function of \( K_R \).

This function starts at some level \( k_S(0) \), the energy spent if no correction is done at \( R \), and declines to reach or approach zero for large values of \( K_R \):

\[
K_S = b_S + k_S(K_R) \tag{14}
\]

If the \( k_S(K_R) \) decreases from \( k_S(0) \) to zero, then for some or all of its domain, its derivative must be negative. Substituting into equation 13 and moving to the other side of the equation, the \( b_S \) term will disappear in the differentiation, leaving:

\[
-\frac{dk_S}{dK_R} = \frac{1}{\alpha} + f(L_R - \alpha L_S) \frac{dz}{dK_R} \tag{15}
\]

There are too many unknowns in this equation to say much about it, but some observations are possible. For example, for values of \( \alpha \) close to one, there may be a solution if the
derivative of $k_S$ is close to minus one, indicating that it might not matter whether correction happens at $R$ or $S$, which seems intuitively sensible. Further, if the cost of repair is substantially higher at $S$ than $R$ ($L_R/L_S < \alpha$), there may be a substantial range of $K_R$ values in which to find a minimum.

For values of $\alpha$ much smaller than one, if the value of $k_S$ declines abruptly at any point as $K_R$ increases, the left side of equation 15 will be large and make it more likely that a plausible selection of parameters would provide a solution to the equation, where a non-zero value for $K_R$ would minimize energy use. For example, this could be the case if noise above a certain level precluded efficient decoding at $S$ entirely and required a request for retransmission. Alternatively, if $k_S$ has only a gentle dependence on $K_R$, a solution would be less likely for $\alpha < 1$.

Assume there is a non-zero solution to equation 15 when $L_R/L_S > \alpha$. Because the derivative of $z(K_R)$ is always positive, it would occur where:

\[-\frac{dk_S}{dK_R} > \frac{1}{\alpha}\]  

By contrast, the condition of complete error correction at $R$ would have this derivative at or approaching zero. In other words, where there is a solution to equation 15 the correction at $R$ would be considered inadequate in isolation.

We have assumed no noise in the $R \rightarrow S$ step. Were we to reverse that assumption, the correction system at $S$ would still have to check all the bits, though it would be more expensive to correct the larger number of incorrect bits. In other words, noise would simply add a term to the right side of equation 11 proportional to $I(R;S)$. This quantity, and the noise in transmission from $R$ to $S$ would have no dependence on $K_R$ and so the term would disappear in the differentiation step. Noise may also reduce the value of $\alpha$, making it less likely to be worth doing error correction at $R$. In the case of noisy transmission where $K_S$ is dependent on $K_R$, noise will appear to reduce the efficacy of the correction at $R$, leading to a lower $dz/dK_R$, and making it more likely that there is a non-zero solution to equation 15.

To summarize, for $\alpha < 1$, if $K_R$ and $K_S$ are independent and $L_R$ and $L_S$ of comparable size, it saves energy to skip error correction at $R$. Since work, as in equation 10, has a time component, it can save time on systems that are not energy-constrained. If $K_S$ is dependent on $K_R$, then investing energy at $R$ is efficient only if the decline in the energy necessary at $S$ is steep. Finally, even when it is efficient to correct at $R$, it is unlikely to be worth correcting 100%.

Consider further two or more aggregators in a row. If each agent asserts conditionality on the output of the previous agent, then as the message travels along the chain, the number of bits in the message appears to decrease to each successive agent. At the same time, however, if the dimensionality of the phase space increases, the valid code points will become ever more distant from one another in their respective phase spaces. An error that results in an invalid code point equidistant from two valid points, or a valid but incorrect code point, might require comparison with an external standard to be corrected. An invalid code point in a sparse phase space is less likely to be ambiguous, and selection of the nearest neighbor will be straightforward. Thus not only does the decreasing number of bits imply a reduction in the cost of error checking, but the per-bit cost may be reduced due to the sparsity of the valid code points. In terms of equation 12, in many cases we can expect not only $\alpha < 1$, but also $K_S < K_R$, making it even more likely that correction at $R$ will not be an efficient use of energy.
The findings for a two-level system can easily be extended to an arbitrary number of levels by considering multiple levels as one. Consider the sequence $Q \to R \to S \to T$. We can regard $S$ and $T$ as a single level while considering whether to do error correction at that level or at $R$. Once decided, we can further decide how much energy to invest in $S$ or $T$.

## 4 Continuous noisy transmission

Consider a system of communication agents as before, but where the messages consist of continuously varying quantities, such as a varying voltage or current, or a pulse of varying width or frequency. If we posit an expected probability density function for an incoming or outgoing signal, the discrepancy between the expected and actual behavior of the signal can be an effective measure of the information carried.

The inputs and outputs of some agent can each be represented by a point in two distinct phase spaces at some instant in time. The probability density function in multiple dimensions is a scalar field in this space, possibly time-varying. The discrepancy with some series of actual measurements can be calculated with a path integral.

An input signal $\vec{q}(t)$ to some agent is a moving point in an $\nu_q$-dimensional phase space, and the space has an associated probability density field $p(\vec{q})$. Let $\vec{r}(t)$, $\nu_r$, and $p(\vec{r})$ be similarly defined for the output of the agent. Both input and output are functions of time, so we can define a relative entropy as an integral along the two paths through their respective phase spaces:

$$D(p(\vec{q})\|p(\vec{r})) = \int_{t_0}^{t_1} p(\vec{q}(t)) \log \left( \frac{p(\vec{q}(t))}{p(\vec{r}(t))} \right) \, dt$$

This definition further allows calculation of the mutual information between the input and output signals, expressed as bits over some interval of time.

$$I(\vec{q};\vec{r}) = D(p(\vec{q},\vec{r})\|p(\vec{q})p(\vec{r}))$$

Consider an agent that outputs a signal $\vec{r}$ by applying a continuous function to two independent multi-dimensional inputs, $\vec{q}_1$ and $\vec{q}_2$ and appending the outputs. The independence of the inputs implies that $p(\vec{q}_1,\vec{r}) = p(\vec{r}) = p(\vec{q}_1)p(\vec{q}_2)$, so the relative entropy of input $\vec{q}_1$ and output $\vec{r}$ is the expected value of the entropy of $\vec{q}_1$ over some time interval:

$$I(\vec{q}_1;\vec{r}) = D(p(\vec{q}_1,\vec{r})\|p(\vec{q}_1)p(\vec{r}))$$

$$= \int_{t_0}^{t_1} p(\vec{q}_1,\vec{r}) \log \left( \frac{p(\vec{q}_1,\vec{r})}{p(\vec{q}_1)p(\vec{r})} \right) \, dt$$

$$= \int_{t_0}^{t_1} p(\vec{q}_1)p(\vec{q}_2) \log \left( \frac{p(\vec{q}_1)p(\vec{q}_2)}{p(\vec{q}_1)p(\vec{q}_1)p(\vec{q}_2)} \right) \, dt$$

$$= -\int_{t_0}^{t_1} p(\vec{q}_1)p(\vec{q}_2) \log p(\vec{q}_1) \, dt$$

$$= \mathbb{E} \left[ -\int_{t_0}^{t_1} p(\vec{q}_1) \log p(\vec{q}_1) \, dt \right] = \mathbb{E}[H(\vec{q}_1)]$$

One can easily generalize this result to an arbitrary number of independent input vectors.
\( \vec{q} \in Q \) such that \( p(\vec{r}) = \prod_{\vec{q} \in Q} p(\vec{q}) \) and write an expression for the overall entropy:

\[
H(\vec{r}) = -\int_{t_0}^{t_1} p(\vec{r}) \log p(\vec{r}) \, dt
\]

\[
= -\int_{t_0}^{t_1} \prod_{\vec{q} \in Q} p(\vec{q}) \log \prod_{\vec{q} \in Q} p(\vec{q}) \, dt
\]

\[
= \sum_{\vec{q} \in Q} E[H(\vec{q})]
\]

\[
= \sum_{\vec{q} \in Q} I(\vec{q}; \vec{r})
\] (20)

The entropy of the output conditional on some set of the inputs is simply the overall entropy minus the mutual information of the output with those inputs: \( H(\vec{r}|\vec{q}_i) = H(\vec{r}) - I(\vec{q}_i; \vec{r}) \). Therefore, from equation 20, the information in the output conditional on some set of inputs is the mutual information of the output and all the other inputs.

With these definitions, the correction of noise in transmission between agents can be addressed as in the discrete case developed in Section 3. Using the definition of mutual entropy in equation 18, that analysis works as well for the continuous case as for the discrete. It is therefore possible to predict that in the case of multiple continuous aggregators in a row, it will usually save energy to delay error correction, just as in the discrete case.

As an example, an auto-pilot incorporates two levels of continuous servo feedback: one for correcting errors in positioning the rudder and the other for correcting the direction of the aircraft or boat. Each level accepts multiple inputs and controls one output, so they are aggregators. This analysis thus suggests that one can increase efficiency by only correcting errors at the level of the aircraft direction. More precision in controlling the rudder position could be considered a waste of energy in this context, though obviously safety concerns also play a vital role in the design of such systems.

For distributors, minimizing energy use might indicate a modest error correction effort, but likely at some level of efficacy that could be characterized as inadequate in isolation. The overall navigation system that controls a flight path might be a distributor, communicating a direction to the autopilot as well as controlling orientation and height with ailerons and elevators. Since the ground-based navigation at the destination airport—a second level of correction—will eventually specify the exact runway location once the aircraft approaches, the heading specified by the navigation system need only be precise enough to get close to the destination, so error correction must achieve at least that level of precision, but no more.

### 5 Discussion

Energy use in computing has become an increasingly important issue, powered by two converging but independent forces. The first is the advent of tremendously effective, but tremendously compute-intensive machine learning applications and the second the advancing demands for both performance and battery life in mobile devices. In an architecture of multiple layers of analysis, it is clear from the analysis presented in Section 3 that it is often inefficient to insist on complete error correction at any individual level. One can go further to say, for example, that for a series of aggregators it may be a waste of time to bother with error correction at all, at least until the final layer.
This can have important implications for machine learning implementations since nodes in a neural network are aggregators, accepting multiple inputs and producing a single output. Such networks consist of multiple layers of such nodes, so one might predict that error correction—and thus precision of calculation—in neural networks is not important. Google’s experience with implementation of its TensorFlow computing software and hardware illustrates the point. In that case, the ever-increasing electricity usage of their translation software led Google to use quantization and low-precision libraries in the implementation of the software (Abadi et al., 2015) and to develop approximate hardware Tensor Processing Units (Jouppi et al., 2017). Energy savings and performance enhancement resulted with no loss of accuracy in the ultimate results. More generally, advances in approximate computing are motivated by a desire to save energy and time, though where and when to relax the demand for computational accuracy remains somewhat ad hoc (e.g. Xu et al., 2016).

Analyses of error-handling in mobile device communications, where energy consumption is among the important engineering constraints have some similarities. In a case where the per-bit cost of repair is much higher at one level than another, Zorzi and Rao (1999) finds that optimizing energy use depends on the error rate, and can involve skimping on the lower level. Raisinghani and Iyer (2004) surveys a number of investigations of strategies to improve reception by foregoing some error checking, but makes no general analysis of those findings beyond calling for network communication technologies able to make such inter-layer trade-offs.

Resource constraints are an important source of evolutionary selection pressures. Brains, for example, are expensive organs to support (Robin, 1973; Aiello and Wheeler, 1995), so strategies to minimize this energy use are important to an organism’s fitness. As a consequence, it is unsurprising to find natural systems using multiple levels of analysis and apparently inadequate error correction. There is empirical support for both: Clark (2013) reviews a great deal of support for multiple levels in cognition, but there is also evidence for the inadequacy of error correction in natural systems where such systems have been identified. For example, the behavior of retinal cells is often not adequate to disambiguate luminance values (Purves et al., 2004) and memory cues can aid phonological segmentation, but may still not be adequate to eliminate uncertainty (Gow and Zoll, 2002). Reproduction of DNA is a similar case, where one finds multiple levels of repair implemented in a cell (Fleck and Nielsen, 2004; Fijalkowska et al., 2012; Ganai and Johansson, 2016). However, the error correction in some levels can be artificially improved, implying that the natural state at those levels could be considered inadequate in isolation (e.g. Sivaramakrishnan et al., 2017; Ye et al., 2018).

The mechanism by which a natural system reproduces the encoding information at the point of decoding, as in (5), is also a source of interest, as important to the quality of transmission as the message itself. Engineered systems of communication have standards committees to ensure that senders and receivers are compatible. The mechanism by which this compatibility is created in natural systems deserves attention. Some epigenetic effects, such as the concentration and variety of non-coding RNA present in a cell (Collins et al., 2011) and methylation (Zemach et al., 2010) are related to the decoding in such a fashion.

Another important implication is that energy savings can arise through addition of a new level of analysis. A new aggregator, with a low $\alpha$, can reduce the number of bits necessary for analysis and thus be a more energy-efficient way to correct transmission errors than improving error correction directly. As a consequence, it is perhaps unlikely that in many
natural systems, the communication levels of the receiver correspond exactly to those of the sender. This, along with the adaptive possibilities of the transmission of encoding information, may have particular relevance to both the phylogenetic and ontogenetic development of natural systems.

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**References**

Abadi, M., Agarwal, A., Barham, P., Brevdo, E., Chen, Z., Citro, C., Corrado, G. S., Davis, A., Dean, J., Devin, M., Ghemawat, S., Goodfellow, I., Harp, A., Irving, G., Isard, M., Jia, Y., Jozefowicz, R., Kaiser, L., Kudlur, M., Levenberg, J., Mané, D., Monga, R., Moore, S., Murray, D., Olah, C., Schuster, M., Shlens, J., Steiner, B., Sutskever, I., Talwar, K., Tucker, P., Vanhoucke, V., Vasudevan, V., Viégas, F., Vinyals, O., Warden, P., Wattenberg, M., Wicke, M., Yu, Y., and Zheng, X. (2015). TensorFlow: Large-scale machine learning on heterogeneous systems. Software available from tensorflow.org.

Aiello, L. C. and Wheeler, P. (1995). The expensive-tissue hypothesis: The brain and the digestive system in human and primate evolution. *Current Anthropology*, 36(2):199–221.

Bennett, C. H. (2003). Notes on Landauer’s principle, reversible computation, and Maxwell’s Demon. *arXiv.org*, 2. [http://arxiv.org/abs/physics/0210005](http://arxiv.org/abs/physics/0210005).

Bradley, S., Hax, A., and Magnanti, T. (1977). *Applied Mathematical Programming*. Addison-Wesley Publishing Company.

Clark, A. (2013). Whatever next? predictive brains, situated agents, and the future of cognitive science. *Behavioral and Brain Sciences*, 36(3):181–253.

Collins, L. J., Schönfeld, B., and Chen, X. S. (2011). The epigenetics of non-coding RNA. In Tollefsbol, T., editor, *Handbook of epigenetics: the new molecular and medical genetics*, chapter 4, pages 49–61. Academic, London.

Dumer, I. I. (1998). *Concatenated codes and their multilevel generalizations*, chapter 23, pages 1911–1988. Volume 2 of [Huffman et al. (1998)](http://www.fourmilab.ch/hypertext/entf4/html/).

Fijalkowska, I. J., Schaaper, R. M., and Jonczyk, P. (2012). DNA replication fidelity in *Escherichia coli*: a multi-DNA polymerase affair. *FEMS Microbiology Reviews*, 36(6):1105–1121.

Fleck, O. and Nielsen, O. (2004). Dna repair. *Journal of Cell Science*, 117(4):515–517.

Ganai, R. and Johansson, E. (2016). DNA Replication: A Matter of Fidelity. *Molecular Cell*, 62(5):745–755.
Goldsmith, J. A. (1979). *Autosegmental Phonology*. Outstanding Dissertations in Linguistics. Garland Publishing, New York.

Gow, D. W. and Zoll, C. (2002). The role of feature parsing in speech processing and phonology. In Csirmaz, A., editor, *Phonological Answers and Their Corresponding Questions*, volume 42 of *MIT Working Papers on Linguistics*, pages 55–68. MIT, Cambridge, MA.

Huffman, W. C., Brualdi, R. A., and Pless, V. S. (1998). *Handbook of Coding Theory*. Elsevier Science Inc., New York, NY, USA.

Jouppi, N. P., Young, C., Patil, N., Patterson, D., Agrawal, G., Bajwa, R., Bates, S., Bhatia, S., Boden, N., Borchers, A., Boyle, R., Cantin, P.-I., Chao, C., Clark, C., Coriell, J., Daley, M., Dau, M., Dean, J., Gelb, B., Ghaemmaghami, T. V., Gottipati, R., Gulland, W., Hagmann, R., Ho, C. R., Hogberg, D., Hu, J., Hundt, R., Hurt, D., Ibarz, J., Jaffey, A., Jaworski, A., Kaplan, A., Khaitan, H., Killebrew, D., Koch, A., Kumar, N., Lacy, S., Laudon, J., Law, J., Le, D., Leary, C., Liu, Z., Lucke, K., Lundin, A., MacKean, G., Maggiore, A., Mahony, M., Miller, K., Nagarajan, R., Narayanaswami, R., Ni, R., Nix, K., Norrie, T., Omernick, M., Penukonda, N., Phelps, A., Ross, J., Ross, M., Salek, A., Samadiani, E., Severn, C., Sizikov, G., Snelham, M., Souter, J., Steinberg, D., Swing, A., Tan, M., Thorson, G., Tian, B., Toma, H., Tuttle, E., Vasudevan, V., Walter, R., Wang, W., Wilcox, E., and Yoon, D. H. (2017). In-datacenter performance analysis of a Tensor Processing Unit. In *ISCA ’17: Proceedings of the 44th Annual International Symposium on Computer Architecture*, pages 1–12, New York, NY, USA. ACM.

Landauer, R. (1961). Irreversibility and heat generation in the computing process. *IBM Journal of Research and Development*, 5(3):183–191.

Liberman, M. and Prince, A. (1977). On stress and linguistic rhythm. *Linguistic Inquiry*, 8:249–336.

Marr, D. (1982). *Vision: A computational investigation into human representation and processing of visual information*. W.H. Freeman, San Francisco, CA.

Penrose, R. (1955). A generalized inverse for matrices. *Mathematical Proceedings of the Cambridge Philosophical Society*, 51(3):406â¬â§413.

Purves, D., Williams, S. M., Nundy, S., and Lotto, R. B. (2004). Perceiving the intensity of light. *Psychological Review*, 111(1):142–158.

Raisinghani, V. T. and Iyer, S. (2004). Cross-layer design optimizations in wireless protocol stacks. *Computer Communications*, 27(8):720–724.

Robin, E. D. (1973). The evolutionary advantages of being stupid. *Perspectives in Biology and Medicine*, 16(3):369–380.

Shannon, C. E. (1948). A mathematical theory of communication. *Bell System Technical Journal*, 27:379–423, 623–656. Reprinted in *Sloane and Wyner (1993)*.

Sivaramakrishnan, P., Sepúlveda, L. A., Halliday, J. A., Liu, J., Núñez, M. A. B., Golding, I., Rosenberg, S. M., and Herman, C. (2017). The transcription fidelity factor GreA impedes DNA break repair. *Nature*, 550:214–218.
Sloane, N. and Wyner, A. D., editors (1993). *Claude Elwood Shannon: Collected Papers*, New York. IEEE, IEEE Press.

Xu, Q., Mytkowicz, T., and Kim, N. S. (2016). Approximate computing: A survey. *IEEE Design Test*, 33(1):8–22.

Ye, L., Wang, C., Hong, L., Sun, N., Chen, D., Chen, S., and Han, F. (2018). Programmable DNA repair with CRISPRa/i enhanced homology-directed repair efficiency with a single Cas9. *Cell Discovery*, 4(1).

Zemach, A., McDaniel, I. E., Silva, P., and Zilberman, D. (2010). Genome-wide evolutionary analysis of eukaryotic DNA methylation. *Science*, 328(5980):916–919.

Zorzi, M. and Rao, R. R. (1999). Perspectives on the impact of error statistics on protocols for wireless networks. *IEEE Personal Communication*, 6(5):32–41.