Classical Nature of the Evolution of Dark Energy Density

Wei Yuan\textsuperscript{1} and Yu-xin Liu\textsuperscript{1,2,3}\textsuperscript{1}

\textsuperscript{1}Department of Physics, Peking University, Beijing 100871, China
\textsuperscript{2}The Key Laboratory of Heavy Ion Physics, Ministry of Education, Beijing 100871, China
\textsuperscript{3}Center of Theoretical Nuclear Physics, National Laboratory of Heavy Ion Accelerator, Lanzhou 730000, China
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By ignoring the local density fluctuations, we construct an uniform Higgs-field’s (inflaton’s) quantum theory with varying effective Planck constant \(\hbar_v(t) \propto R(t)^{-3}\) for the evolution of the dark energy density during the epoch after inflation. With presumable sufficient inflation in the very early period (time-scale is \(t_{inf}\)), so that \(\hbar_v \to 0\), the state of universe decomposes into some decoherent components, which could be the essential meaning of phase transition, and each of them could be well described by classical mechanics for an inharmonic oscillator in the corresponding potential-well with a viscous force. We find that the cosmological constant at present is \(\Lambda_{now} \approx 2.05 \times 10^{-3}\) eV, which is almost independent of the choice of potential for inflaton, and agrees excellently with the recent observations. In addition, we find that, during the cosmic epoch after inflation, the dark energy is almost conserved as well as the matter’s energy, therefore the “why now” problem can be avoided.

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Introduction. – Up to now, the dominant energy density which governs the evolution of the whole universe is still dark energy density \(\Omega\), which can also be understood in terms of Einstein’s cosmological constant \(\Lambda = \Omega^{1/4}\). However, the size of a small mass scale, \(\Lambda \sim 10^{-3}\) eV, has not yet been derived from a fundamental theory, and its nature has not been understood either.

On the other hand, in very early universe, the dark energy density is expected to maintain at extremely high level for a while to realize the well known inflation \(\text{2}\), finally, it rolls down the hill of the potential during the epoch after inflation \(\text{2}\). If one expects to understand the evolution of dark energy density and the acceleration of universe during epoch after the abrupt inflation, a detailed theory for the rolling-down process (perhaps for the inflation process itself) seems to be required. Such a theory should also avoid the familiar “why now” problem: why do we find ourself in such a epoch when the cosmological constant is near zero \(\text{2}\) and why do we live during an era when the energy densities in matter and dark energy are comparable \(\text{3}\).

In this paper, we establish a quantum theory of uniform scalar field for the evolution of dark energy, where the local density fluctuations are ignored, and it is expected to carry the leading order effects of the evolution of the dark energy density. In addition, our calculations will make no use of the conceptions such as, effective potential, statistical ensembles, finite temperature quantum field theory (QFT), which are defined to describe a static, equilibrium system, however, as we will see below, the real situation is non-equilibrium. Nevertheless, as we give up here the concept of statistical ensembles, it seems to rise another familiar problem: in a pure quantum picture, how can we define the events such as phase transition or spontaneous symmetry breaking? In traditional quantum measurement theory, events are related to the entanglement between the apparatus and the system which attracts our interests. Thus, we could not imagine any event that has emerged in a pure quantum evolution before we take an apparatus to observe it, because such imaginations will destroy the coherence between each component of the quantum state. We thought it is trustless to imagine the existence of some environments which will induce the event of phase transition to occur in the early universe. The crucial problem turns out to be whether we can find a way to realize the decoherence between each phase in a pure quantum process without any environment. To this problem, the irradiative arguments have been put forward (see for example Ref. \(\text{6}\)). In this paper, we propose that the inflation of early universe will indeed help us to realize the events of phase transition and the spontaneous symmetry breaking. Moreover, the followed evolution of dark energy density can be well handled by a classical theory.

Formalism. – We start with the simplest model for a single scalar Higgs field \(\phi\). However, as we will see below, the results are almost independent of the choice of potential \(V(\phi)\) and can be easily generalized to the case with multiple Higgs fields. By excluding any local density fluctuation, the space-differential term can be deleted from the Lagrangian

\[
\mathcal{L}(\phi) = \frac{1}{2}\left(\frac{d\phi}{dt}\right)^2 + \frac{1}{2}\mu^2 \phi^2 - \frac{1}{4}\lambda \phi^4 - \frac{\mu^4}{4\lambda}, \quad (1)
\]

and the path integral formalism is

\[
\int [D\phi(t)]e^{i \int d^4x \mathcal{L}} = \int [D\phi(t)]e^{i \int d\hbar_v(t)^{-1}L(\phi, \frac{d\phi}{dt})}, \quad (2)
\]

with \(\hbar_v \equiv v^{-1}\) and \(v\) being the volume of the universe. Because the universe is expanding with respect to time, the effective Planck constant \(\hbar_v\) is time dependent. How-
ever, the system can still be quantized by the canonical method which is equivalent to the path integral approach. With such a canonical quantization, we obtain a Schrödinger-like equation with varying Planck constant

$$i\hbar_v(t)\frac{\partial}{\partial t}\Psi(\phi, t) = [-\frac{\hbar_v}{2}\frac{\partial^2}{\partial \phi^2} + V(\phi)]\Psi(\phi, t). \quad (3)$$

It should be noted that the mass dimension of $\hbar_v$ is 3, thus the mass dimension of Hamiltonian $\mathcal{H}$ is 4, then $\mathcal{H}$ represents actually the operator of energy density rather than energy. Since we only take the uniform scalar field configurations into account, the expectation value $\Omega(t) \equiv \langle \Psi(t)|\mathcal{H}|\Psi(t)\rangle$ is just the dark energy density. Considering the Hermite property of the Hamiltonian, we have the time-differential of $\Omega$ as

$$\frac{d\Omega(t)}{dt} = 2\hbar_v^{-1}\frac{d\hbar_v}{dt} T(t), \quad (4)$$

where $T = \langle \Psi(t)|\frac{\partial^2}{\partial \phi^2}|\Psi(t)\rangle$ is the average kinetic energy density. It has been well established that, after sufficient inflation, the space-time can be described by the de-Sitter metric
ds^2 = dt^2 - R(t)^2d\vec{x}^2, \quad (5)

where the first order Friedmann equation for $R(t)$ reads

$$R^{-1}\frac{dR}{dt} = \frac{8\pi G}{3}\Omega^{1/2}, \quad (6)$$

where $G$ is the gravitation constant. Considering the relation $\hbar_v = v^{-1} \propto R^{-3}$, we can rewrite the first order Friedmann equation as

$$\hbar_v^{-1}\frac{d\hbar_v}{dt} = -(24\pi G\Omega)^{1/2}. \quad (7)$$

Substituting Eq. (7) into Eq. (4), we get

$$\frac{d\Omega(t)}{dt} = -2(24\pi G\Omega)^{1/2} T(t). \quad (8)$$

On one hand, Eqs. (3) and (7) could describe a theoretically solvable quantum system. However, they are worthwhile only if the quantum fluctuations are indispensable, for instance, during the cosmic epoch of inflation. One can imagine that $\hbar_v$ is very large in that period, it forces us to make use of Eqs. (3) and (7) exactly (we have not yet investigated such process, inflation itself. Moreover, in such an epoch, de-Sitter metric is plausible). On the other hand, once inflation has lasted continuously for a while ($t_{inf}$) and quenched, Eqs. (7) and (8) imply that the effective Planck constant becomes very small and the energy density decreases to a level well below the potential density barrier. At that moment, one might have reasons to expect that the quantum-mechanical tunneling effects are strongly restrained by the smallness of $\hbar_v$ and $\Omega$, thus the wave function in the negative $\phi$ region will decohere with the wave function in the positive $\phi$ region. We emphasize that the decoherence of these two equivalent components is a reasonable signal of the event of phase transition (viz. spontaneous symmetry breaking) which occurs at the time $t_c \simeq t_{inf}$. The followed evolution of the localized wave packet is known as rolling-down. Guth had investigated a similar process (with a parameter in Eq. (1) being set as $\lambda = 0$), and pointed out that the evolution of such a quantum wave packet can be understood with a probability distribution which describes the classical trajectories rolling in the well. Actually, we do not need to worry about the problem of which trajectory we should choose, because these trajectories are different from each other only by a series of time-translations, and the typical translation’s scale is extremely small compared to nowadays cosmic age. The problem is essentially that, after a long time evolution, no observation can distinguish these different trajectories. Therefore, the evolution of each localized wave function in the corresponding well of the potential density surface can be well described by classical mechanics. Here, it means that, during the cosmic epoch after inflation, we can take the classical kinematics as the complement for Eq. (8), so that it is solvable. The dark energy density in each well can then be given as

$$\Omega = T + V(\phi) = \frac{1}{2}(\frac{d\phi}{dt})^2 + V(\phi). \quad (9)$$

Combining Eq. (9) with Eq. (8), one can easily obtain a solvable classical equation for $\phi(t)$

$$\frac{d^2\phi}{dt^2} = -\frac{dV(\phi)}{d\phi} - (24\pi G\Omega)^{1/2}\frac{d\phi}{dt}. \quad (10)$$

Eq. (10) is just the equation of the motion of a point mass evolving in the potential $V(\phi)$ in the presence of a viscous force $-(24\pi G\Omega)^{1/2}\frac{d\phi}{dt}$. Such a viscous force results naturally from the quantum effects and the gravity effects. It should be emphasized that Eq. (10) can be independently derived by combining the full Friedmann equations (the first order and the second order) with the classical mechanics of uniform Higgs field, which has been implied in the chaotic inflation theory. However, our derivation for Eq. (10), which has not made use of the second order Friedmann equation, indicates that the Schrödinger-like equation in Eq. (3) is not only naturally consistent with the second order Friedmann equation but also a proper substitution for Eq. (10) during the epoch of inflation. Here, we just need to note that Eq. (10) can only be used precisely for the epoch after inflation but it is not true for the inflation process itself.

Since the $\phi$ is expected to be near the bottom of the potential density surface $V(\phi)$ (here we choose $\phi \sim \frac{\phi_f}{\sqrt{\hbar_v}}$), we can then, at the lowest order, approximate the potential $V(\phi)$ to be

$$V(\phi_f) \approx \mu^2\phi_f^2. \quad (11)$$
with $\phi = \frac{T}{\omega} + \phi_f$, and the corresponding equation for $\phi_f(t)$ becomes

$$\frac{d^2 \phi_f}{dt^2} = -\omega^2 \phi_f - (24\pi G)^{1/2} \frac{1}{2} \left( \frac{d\phi_f}{dt} \right)^2 + \frac{1}{2} \omega^2 \phi_f^2 \frac{d^2 \phi_f}{dt^2},$$

(12)

with $\omega = \sqrt{2\mu}$ (we will verify the validity of this approximation later). It is obvious that there still exist difficulties in solving Eq. (12) exactly. Nevertheless, there is a reasonable way to obtain the evolution of $\Omega(t)$. If we ignore the viscous force at first, the system reduces to a harmonic oscillator, its time-period is $\frac{2\pi}{\omega}$. Then we turn on the viscous force, but do not change the behavior of the harmonic oscillation in one time-period. After some calculations, we obtain the negative work made by the viscous force in this period as

$$W_T = -\frac{2\pi}{\omega} (24\pi G)^{1/2} \Omega^{3/2}.$$  

(13)

Since the ratio $W_T/\Omega = -\frac{2\pi}{\omega} (24\pi G)\Omega^{1/2} \propto \frac{\Omega^{1/2}}{\sqrt{\lambda \mu}}$, where $M_p = G^{-1/2}$ is the Planck scale of energy, is expected to be very tiny and the present observations indicate that such an expectation is true, we can really perform such a calculation in one period without changing the harmonic oscillation. Furthermore, during the epoch after inflation, $\frac{2\pi}{\omega}$ should be a very short time scale compared to the time scale of $\Omega(t)$'s evolution (as we will see self-consistently below), we can then take approximations $W_T \rightarrow d\Omega$ and $\frac{2\pi}{\omega} \rightarrow dt$. It turns out that Eq. (13) is a representation of the equation for the evolution of dark energy density, and can be written explicitly as

$$\frac{d\Omega}{dt} = -(24\pi G)^{1/2} \Omega^{3/2}.$$  

(14)

One can easily understand Eq. (14) by simply substituting $T = \Omega/2$ into Eq. (8), since it is the nature of quickly harmonic oscillation. Furthermore, it should be emphasized that the intrinsic parameters $\mu$ and $\lambda$ of the quantum field theory do not appear in Eq. (14), however, their effect is involved in the initial conditions, because Eq. (14) is valid only after some time $t_0 > t_{inf}$ which is the time for Eq. (11) to be valid, and $t_0$ depends definitely on $\mu$ and $\lambda$. Nevertheless, we take the initial conditions as $t_0$ and $\Omega_0$, with which we can easily find out the definite solution for Eq. (14) as

$$\Omega(t) = \frac{M_p^2}{6\pi t_{inf}^2 \Omega_0^2},$$

(15)

with $t_{inf}$ satisfying $\Omega_0 = \frac{M_p^2}{6\pi t_{inf}}$. On one hand, we need $\Omega_0$ to be small enough to ensure Eq. (11), which requires $\Omega_0 \ll \frac{\lambda}{\mu}$. On the other hand, we hope $t_{inf}$ is small enough compared to $t_{now} \approx 1.37 \times 10^{10}$ years (the age of universe with new accuracy from WMAP date) so that we can precisely make use of Eq. (15) in the epoch after inflation by simply ignoring $t_{inf}$, which requires $\Omega_0 \gg \frac{M_p^2}{6\pi t_{now}^2}$. Altogether, it requires

$$\frac{\mu^4}{\lambda} \gg \frac{M_p^2}{6\pi t_{now}^2} \approx 1.78 \times 10^{-47} \text{GeV}^4.$$  

(16)

We are glad to note that any reasonable choice for $\mu$ and $\lambda$ can perfectly satisfy such a condition. The last problem is that we have not found out a close formalism for $t_{inf}(\mu, \lambda)$ nor for $t_{now}(\mu, \lambda)$, it might need a full quantum treatment on Eqs. (8) and (7). Fortunately, whatever $t_{inf}(\mu, \lambda)$ should be, it can not change the solid fact that $t_{now} \gg t_0 > t_{inf}$. By rewriting $t_{inf} - t_0 = t_{eff}$, we have the varying cosmological constant as

$$\Lambda = \left[ \frac{M_p}{\sqrt{6\pi (t + t_{eff})}} \right]^{1/2}.$$  

(17)

When we apply Eq. (17) to the universe after inflation (actually, Eq. (17) can only be used in this case, since for very early universe, detailed treatment on the quantum nature of inflation is required, and the de-Sitter metric is plausible), $t_{eff} \ll t$, we can take $t_{eff}$ as zero. We obtain then the evolution of the cosmological constant with respect to time as shown in Fig. 1. With the age

![FIG. 1: The calculated evolution of Einstein’s cosmological constant with respect to time (with Planck scale of energy $M_p \approx 1.22 \times 10^{19} \text{GeV}$).](image)

of the present universe $t_{now} \approx 1.37 \times 10^{10}$ years and the Planck scale of energy $M_p = G^{-1/2} \approx 1.22 \times 10^{19} \text{GeV}$, we obtain the dark energy density or the Einstein’s cosmological constant at present as $\Omega_{now} \approx 1.78 \times 10^{-47}\text{GeV}^4$, or $\Lambda_{now} \approx 2.05 \times 10^{-3}$ eV. It is apparent that such a result agrees excellently with the recent SDSS and WMAP observation $\Lambda = 2.14 \pm 0.13 \times 10^{-3}$ eV. As for the behavior shown in Fig. 1 in the quantum inflating epoch, even though it can not be trusted precisely, it is still
consistent with the inflation picture which has been presumed in our calculations.

Recalling Eq. (17) with approximation \( t_{\text{eff}} = 0 \), one can easily realize that a cosmological constant \( \Lambda = 2 \Lambda_{\text{now}} \) can only be observed at \( 1.03 \times 10^{10} \) years earlier, and \( \Lambda = \frac{4}{3} \Lambda_{\text{now}} \) can only be observed at \( 4.11 \times 10^{10} \) years later. It means that, in the cosmic epoch after inflation, the cosmological constant evolves very slowly. Moreover, by using the identity

\[
\frac{d(\Omega/\bar{h}_v)}{(\Omega/\bar{h}_v)dt} = -\frac{d\Omega}{dt} - h^{-1}_v \frac{dh_v}{dt},
\]

and Eqs. (7) and (14), we have

\[
\frac{d(\Omega_v)}{dt} = 0.
\]

It is presumable that, once the inflation has already ended, the probability of local materialization from false vacuum could be very small. Therefore, Eq. (19) means that the total dark energy is almost conserved just as the total energy included in matter (both luminous matter and dark matter) does. The above result indicates that, during the cosmic epoch after inflation, the ratio between the density of dark energy and the energy density of the matter, \( \frac{\Omega_{\text{DE}}}{\Omega_{\text{matt}}} \), is almost a constant (present observation shows \( \frac{\Omega_{\text{DE}}}{\Omega_{\text{matt}}} \approx \frac{1}{3} \)). Altogether, the “why now” problem is solved. However, it leaves a problem “why \( \frac{2}{3} \)” to be answered in the inflation epoch.

The presently obtained results indicate more interestingly is that, after the presumable sufficient inflation, the cosmology constant is almost independent of the intrinsic parameters of the quantum field theory. It means that one can choose any \( V(\phi) \) which could have arbitrary minimal positions, however, the inflation causes the components located in these valleys decohere from each other, and no matter where we live, the observed cosmological constant is the same. If there are \( N_\phi \) independent Higgs fields, which belong to different interior spaces, the cosmological constant should be \( N_\phi^{1/4} \times 2.05 \times 10^{-3} \) eV (note that energy density is additive quantity, but it is not true for cosmological constant). If the multiple Higgs fields belong to a single interior space, it should not change our result for the cosmological constant (\( 2.05 \times 10^{-3} \) eV). Precise observation for cosmological constant can then be taken as a signature to determine the \( N_\phi \) and test our theory itself. Our present results seem to support the case of \( N_\phi = 1 \) (Viz. all possible Higgs fields should belong to an interior space and hover around some valley of the potential density surface during the epoch after inflation).

Summary.– In this paper, we try to show that the relation between the epoch after inflation and the epoch during inflation is very subtle. During inflation, the quantum fluctuations are very strong, as Guth pointed out [7] that, one could expect to understand the production mechanism for essentially all matters, energy, entropy in such a process. Nevertheless, whatever the precise inflation picture should be, the rough properties of it seem to be sufficient to set the initial conditions for the precise interpretation of the slowly evolving dark energy density during the epoch after inflation. Thus, without adjusting any parameter, we give an evolution behavior of the dark energy density or the Einstein’s cosmological constant, and the results are agree excellently with the recent SDSS and WMAP observations. In addition, the “why now” problem is solved in our present approach. Thus we suppose that the present evolving cosmological constant is governed by a classical inharmonic oscillation and consistent with the inflation picture, but it is irrelevant to the details of quantum inflation itself. It provides then a classical nature for the evolution of the dark energy density after inflation.

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* corresponding author, yxliu@pku.edu.cn

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