Noncommuting Flux Sectors in a Tabletop Experiment

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Abstract

We show how one can use superconductors and Josephson junctions to create a laboratory system which can explore the groundstates of the free electromagnetic field in a 3-manifold with torsion in its cohomology.
1. Introduction

Abelian gauge theories exhibit a curious uncertainty principle between the topological classes of electric and magnetic flux sectors. One version of this phenomenon arose in string theory \([1]\) and it has been thoroughly explored in \([2][3]\). The uncertainty principle even applies to ordinary 3 + 1 dimensional Maxwell theory, and hence it is natural to ask if one could devise an experiment to demonstrate it. This paper shows that such an experiment is indeed possible. Moreover, it is related to recent ideas for designing topologically protected qubits in quantum computation \([4]\).

The effect we wish to demonstrate arises when one considers Maxwell theory in spacetimes of the form \(Y \times \mathbb{R}\) where \(Y\) is an oriented 3-manifold with torsion in its integral cohomology group \(H^2(Y)\). \([2]\) In particular, the groundstates of the Maxwell theory will form an irreducible representation of the Heisenberg group

\[
\text{Heis}(\text{Tors}H^2(Y) \times \text{Tors}H^2(Y))
\]  

where the cocycle defining the Heisenberg group is defined by the link pairing. This would appear, at first sight, to be an extremely esoteric observation. Nevertheless, we will show that the basic phenomenon can in principle be experimentally observed in a tabletop experiment using only appropriate arrays of Josephson junctions.

In trying to devise an experiment that exhibits this phenomenon we are immediately confronted with a discouraging fact, which was pointed out to us by M. Freedman: For any region \(R \subset \mathbb{R}^3\) the cohomology groups \(H^2(R)\) and \(H_1(R)\) are torsion free. See the appendix for an explanation. We will show, however, that by combining superconductors with a new device \([5][4]\) based on Josephson junctions one can make identifications on the holonomies of the gauge field which (in the limit of low energies and large capacitance) mimic the identifications needed to define an abstract 3-manifold with torsion in its homology.

The new device may be described as a superconducting current mirror, herafter referred to as an SCM. It can be realized as a pair of capacitively coupled Josephson junction chains \([6]\), though the implementation has not yet been achieved experimentally. An ideal SCM is an electric circuit element with four superconducting leads whose energy in the absence of a magnetic field is given by \(E = f(\varphi_1 - \varphi_2 + \varphi_3 - \varphi_4)\), where \(\varphi_1, \varphi_2, \varphi_3, \varphi_4\)

\(^1\) Except in the appendix, all homology and cohomology groups in this paper will have coefficients in \(\mathbb{Z}\).
are the values of the superconducting phase on the leads and $f$ is a function with a global minimum at 0. It has been observed recently in [4] that the SCM can be turned into a topologically protected qubit by connecting the four leads diagonally, which is described by setting $\varphi_1 = \varphi_3$ and $\varphi_2 = \varphi_4$. Under these circumstances, the energy has two equal minima at $\varphi_1 - \varphi_2 = 0$ and $\varphi_1 - \varphi_2 = \pi$. In this paper we build on the same idea but interpret it differently. While the above description may be viewed as an “electrical engineering approach” where one thinks of an electric circuit in terms of currents (or superconducting phases), we suggest that the two-fold degenerate ground state can also be understood as a property of the electromagnetic field in the free space surrounding the superconductor.

We now discuss the general principles by which one can map superconducting circuits to properties of the groundstates of free Maxwell theory on three-manifolds $Y$. We are aiming to write an effective quantum mechanical system for the low energy degrees of freedom. Consider quantum Maxwell theory in spatial $\mathbb{R}^3$, but with a connected region $S$ filled with superconductor. This will be related to Maxwell theory on $\mathbb{R}^3/\sim$ where $\sim$ identifies $S$ to a single point $\mathcal{P}$. The reason is that $\vec{E} = \vec{B} = 0$ inside $S$, so that inside $S$ there is only a flat gauge field. Suppose, for the moment, that the bosons which condense in the superconductor have the elementary unit of charge. Then, by flux quantization, the holonomies

$$\exp(2\pi i \int_\gamma A)$$

around homotopically nontrivial cycles $\gamma \subset S$ must be trivial, and hence the gauge field in $S$ is trivial. Therefore, the gauge bundle with connection restricted to $S$ is trivial and $S$ can be identified to a point.

Two points raised by the above proposal require further discussion. First, in Nature the condensing bosons – the Cooper pairs – actually have twice the elementary charge, so the above argument leaves open the possibility that holonomies around noncontractible loops in $S$ can be $-1$. Let $\mathcal{L}$ be the line bundle corresponding to the representation with the elementary charge. Any superconducting circuit in $\mathbb{R}^3$ can be described by a globally defined $U(1)$ connection $A$ on $\mathcal{L}$. It is therefore completely defined by its fieldstrength, which vanishes in the superconducting region $S$. This holds for both $A$ as well as the connection $2A$ on $\mathcal{L} \otimes \mathcal{L}$. However, the latter has holonomy = 1 inside $S$. Now, we can unambiguously obtain the fieldstrength $F(A)$ from that of $F(2A) = 2F(A)$, and hence we

\[2\]

Our conventions for gauge fields are those of [2][3]. In particular, $A$ is normalized so that $F = dA$ locally, and $F$ has integral periods.
can thereby reconstruct the original connection $A$ from $2A$. The important point is that it is the connection $2A$ which has a nice description in terms of an effective field theory in the complementary vacuum region. Of course, field configurations $A$ with holonomy $-1$ around loops in $S$ do exist, but these correspond to higher energy states, and are not important to the low energy effective field theory. (In particular, the magnetic energy of a half flux quantum trapped in a superconducting ring is much greater than the ground state splitting in an SCM-based qubit.) For simplicity, in the following we continue to assume that the boson which condenses in the superconductor has the elementary unit of charge. By the above remarks we can always map the low energy states of this hypothetical system to the case where the condensing boson has twice the elementary charge, as it is in Nature.

The second point is that the region $\mathbb{R}^3/\sim$ is not necessarily a manifold. In formulating Maxwell theory directly on this space we must use a boundary condition. We assume that at $\mathcal{P}$, the point of identification, the fields are zero. Alternatively, we can work with $3$-manifolds with boundary $Y$ with superconducting boundary conditions on $\partial Y$, i.e., $A$ is trivial on $\partial Y$, in which case the uncertainty principle on topological sectors is determined by the link pairing (again a perfect pairing):

$$\text{Tors} H^2(Y, \partial Y) \times \text{Tors} H^2(Y) \to U(1).$$  \hfill (1.3)

(In general boundary conditions on a free Maxwell field in a spacetime $M$ are formulated by using the 2-form $\Omega = \int_{\partial M} \delta A \wedge * \delta F$ to define a symplectic form on the Hamiltonian reduction of fieldspace. A boundary condition is a Lagrangian subspace with respect to this form. We choose trivial connection on $\partial Y$ which entails the standard conditions $E_{\parallel} = 0$ and $B_{\perp} = 0$ at the boundary of the superconductor. If it were possible to condense magnetic monopoles in nature we could use the electromagnetic dual boundary condition, and in this case, the construction of qubits would be quite easy.)

Now let us discuss the effective quantum theory. First, in a region surrounded by superconductor as in fig. 1, the wavefunction is a function of the gauge-invariant variable $u_{12} := \varphi_1 - \varphi_2 + 2\pi \int_1^2 A$ where $\varphi_i$ is the phase of the superconducting condensate and the contour integral is along a short vertical path from region 1 to region 2. The variable $u_{12}$ is defined modulo $2\pi \mathbb{Z}$. 

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Fig. 1: Layer between two superconductors.

Fig. 2: A SCM with leads 1, 2, 3, 4. In this paper we treat the region inside the dotted lines as a black box.

Next we consider the superconducting mirror (SCM) shown in fig. 2. The SCM adds a term to the Hamiltonian for the low energy modes given by $f(u_{14} - u_{23})$ where $f$ is a function of a periodic variable with a single minimum at 0. This is a result of [5]. For small devices we can consider the function to be $f(u_{12} + u_{34})$. Note that in a limit (such as the semiclassical limit) in which the potential function dominates the low energy quantum mechanics this imposes the constraint

$$u_{12} + u_{34} = 0 \pmod{2\pi\mathbb{Z}} \tag{1.5}$$

This is the origin of the name “superconducting mirror.” It reflects the fact that the currents $J_i = \frac{\partial}{\partial u_i} E$, where $E$ is the energy, are all equal in magnitude.

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on the ground state. Note too that we have used
\[ u_{14} - u_{23} - (u_{12} + u_{34}) = - \left( \int_1^2 A + \int_2^3 A + \int_3^4 A + \int_4^1 A \right) \] (1.6)
so, if the magnetic flux through the device is small we can neglect the right hand side. For definiteness we will sometimes take the potential term in the Hamiltonian to be:
\[ -J \cos(u_{12} + u_{34}) \] (1.7)

2. An Example

Consider the Klein bottle \( K \). We use this to form a twisted interval bundle
\[ I \to Y \to K \]
where the twisting cancels \( w_1(K) \) so that \( Y \) is orientable. The boundary \( \partial Y \) is a torus - the orientation double cover of \( K \). We are going to design a situation where Maxwell theory is effectively placed in the three manifold \( Y \).  
\[ \begin{align*}
\pi_1(K) &= \langle a, b | aba^{-1}b^{-1} = 1 \rangle \\
\text{abelianization} &= \mathbb{Z} \times \mathbb{Z}_2,
\end{align*} \]
Recall that \( \pi_1(K) = \langle a, b|aba^{-1}b = 1 \rangle \), so the abelianization is \( \mathbb{Z} \times \mathbb{Z}_2 \), and hence
\[ H^2(Y, \partial Y) \cong H_1(Y) \cong H_1(K) \cong \mathbb{Z} \times \mathbb{Z}_2 \]
has torsion.

\[ \text{Fig. 3: A sketch of the manifold } N. \]

\[ ^4 \text{As mentioned above, the theory really lives on } Y/\partial Y, \text{ or on } Y \text{ with superconducting boundary conditions on } \partial Y. \]
Of course, $K$ and $Y$ cannot be embedded into $\mathbb{R}^3$, but they can be immersed with only double points. The double points of the immersion $i(K)$ trace out a figure $X \times S^1$, where here $X$ is a literal $X$. Consider a thickening of $i(K)$, e.g., the region occupied by a model of $i(K)$ made with glass. The region occupied by the glass is a 3-manifold with boundary which we will denote $N$. A sketch of $N$ appears in fig. 3. The boundary $\partial N$ is the disjoint union of two tori $T^2$, and $N$ itself may be viewed as a cobordism from $T^2$ to $T^2$ obtained by cutting a small solid torus from within a small ball within a larger solid torus.

![Fig. 4: Region near the double point of immersed Klein bottle (dashed line). It has been thickened, and there is superconductor in the shaded region.](image)

Now imagine that $N$ is not filled with glass but with vacuum, and that the complementary region $\mathbb{R}^3 - N$ is filled with superconductor, both on the inner region and on the outer region. Consider the neighborhood of a double point as in fig. 4.

A priori there are 4 gauge invariant variables $u_{12}, u_{23}, u_{34}, u_{41}$ satisfying the constraint $\sum u_{i,i+1} = 0$. However, in the complement of $N$ a point in region 1 is continuously connected to a point in region 3, and similarly a point in region 2 to a point in region 4. Therefore, if the gauge field is zero there is only one independent variable, say $u = u_{12}$. The effective Hamiltonian is given by

$$H = \frac{Q^2}{2C},$$

where $Q = e^* N$ is the charge for $N$ Cooper pairs of charge $e^*$ and $C$ is an effective capacitance. The superconducting state is not a state of definite $N$ but is rather described by a wavefunction of the conjugate variable $u$ so that $N = -i \frac{\partial}{\partial u}$ so that

$$H = -\frac{(e^*)^2}{2C} \left( \frac{\partial}{\partial u} \right)^2$$

(2.1)
Since $u \sim u + 2\pi \mathbb{Z}$, the gauge invariant configuration space is a circle. There is no potential function, and hence there is a unique normalizable ground state $\Psi_{\text{ground}}(u) = \text{constant}.$

Now, let us consider the effect of inserting an SCM at a fixed angle $\theta \in S^1$ in the set of double points $X \times S^1$ of $i(K)$. According to [4] this adds a term (1.7) to the Hamiltonian. Because of the topology $u_{12} = u_{34} = u$. In terms of the circuit in fig. 2 we would be connecting leads 1 to 3 and leads 2 to 4. Thus the effective Hamiltonian is

$$H = -\frac{(e^*)^2}{2C} \left( \frac{\partial}{\partial u} \right)^2 - J \cos(2u) \quad (2.2)$$

For $JC \gg 1$, $J > 0$ the groundstates are well-approximated by states localized near $u = 0, \pi$, denoted $|0\rangle$ and $|\pi\rangle$. Thus the space of groundstates is effectively 2-dimensional.

As an aside, we note that the Schrödinger equation for this potential is the well-known Mathieu equation and can be “solved exactly.” The groundstate is of course unique and, for $JC \gg 1$, closely approximated by $\frac{1}{\sqrt{2}}(|0\rangle + |\pi\rangle)$. However, in the limit

$$q := -\frac{CJ}{(e^*)^2} \to -\infty \quad (2.3)$$

there are two low-lying states with energy eigenvalues

$$E_1 - E_0 \sim J|q|^{-1/4} e^{-4\sqrt{|q|}} \quad (2.4)$$

(we drop a numerical constant). This confirms and quantifies our expectation that, to exponential accuracy there is a two-dimensional space of degenerate groundstates.

We claim that this 2-dimensional space of approximate groundstates naturally forms the irreducible representation of the Heisenberg group $\text{Heis}(\mathbb{Z}_2 \times \mathbb{Z}_2)$. The Heisenberg operators corresponding to the clock and shift operators are

$$P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (2.5)$$

According to the principles discussed above, effectively the theory has been put on the space $Y/\partial Y$. In the Maxwell theory picture $P, Q$ correspond to measuring magnetic and electric fluxes, that is, the magnetic and electric first Chern class $c_1$, respectively. Let us discuss the physical implementation of these operations in the corresponding superconducting circuit, following [4]. In the analogous SCM as in fig. 2 the leads 1 and 3 are connected and the leads 2 and 4 are connected. The operation of $P$ corresponds to inserting a device between leads 1 and 4 to measure the phase $\exp(iu)$. The operator $Q$ is trickier.

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5 We are neglecting some physical effects, e.g. electron tunneling through the vacuum, which only give exponentially small contributions to the ground state splitting.
Fig. 5: A setup for the realization of the shift operator $Q$ or the measurement in the eigenbasis of $Q$.

The operator $Q$ as a unitary transformation of the quantum state corresponds to the adiabatic change of $\varphi_1 - \varphi_3$ from 0 to $2\pi$ or $-2\pi$. To realize this transformation, one needs to insert a capacitor on the wire connecting nodes 1 and 3, breaking the identification $u_{13} = 0$ (see fig. 5). Then the (classical) groundstate equation becomes

$$u_{13} + 2u_{32} = 0 \pmod{2\pi} \quad (2.6)$$

If we increase $u_{13}$ from 0 to $2\pi$ adiabatically, then $u_{32}$ shifts from 0 to $-\pi$. Therefore, after reconnecting the terminals, $u_{12}$ has shifted from 0 to $-\pi \cong +\pi$, and we have implemented the shift operator $Q$.

The above argument also shows that $Q = e^{2\pi in}$, where $n = -i \partial / \partial u_{13}$ is the operator of electric charge on either capacitor plate. By measuring this charge, one can perform a measurement in the eigenbasis of $Q$. The charge is related to the electric field in the capacitor and is therefore observable, though a practical implementation of the measurement might ultimately use a different principle. If the SCM is in state $\frac{1}{\sqrt{2}}(|u \cong 0\rangle + |u \cong \pi\rangle)$, then the charge takes on integer values, otherwise the charge is half-integer.

3. Generalization to other Heisenberg groups

3.1. An array of SCM’s
Let us consider a system of $n$ SCM’s which will be, roughly speaking, connected in series. The effective Hamiltonian in the $JC \gg 1$ limit sets

$$\varphi_1 - \varphi_2 + \varphi_3 - \varphi_4 = 0$$
$$\varphi_5 - \varphi_6 + \varphi_7 - \varphi_8 = 0$$
$$\varphi_9 - \varphi_{10} + \varphi_{11} - \varphi_{12} = 0$$

$$\ldots$$

$$\varphi_{4n-3} - \varphi_{4n-2} + \varphi_{4n-1} - \varphi_{4n} = 0$$

Now connect the wires so that $\varphi_1 = \varphi_3, \varphi_5 = \varphi_7, \ldots, \varphi_{4j+1} = \varphi_{4j+3}, \ldots, \varphi_{4n-3} = \varphi_{4n-1}$. We further connect wires so that

$$\varphi_1 = \varphi_6$$
$$\varphi_4 = \varphi_5 = \varphi_{10}$$
$$\varphi_8 = \varphi_9 = \varphi_{14}$$

$$\ldots$$

$$\varphi_{4n-4} = \varphi_{4n-3}$$
that is, \( \varphi_{4j} = \varphi_{4j+1} = \varphi_{4j+6} \) for \( 1 \leq j \leq n-2 \), and finally connect \( \varphi_{4n} = \varphi_2 \). See fig. 6 for the case \( n = 3 \).

With the above connections the groundstate equations become:

\[
\begin{pmatrix}
2 & -1 & 0 & \cdots & \cdots & 0 \\
-1 & 2 & -1 & \cdots & \cdots & 0 \\
0 & -1 & 2 & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 2 & -1 & 0 \\
0 & \cdots & \cdots & -1 & 2 & -1 \\
0 & \cdots & \cdots & 0 & -1 & 2 \\
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2 \\
u_3 \\
\vdots \\
u_n \\
\end{pmatrix} = 0
\]

(3.3)

where

\[
\begin{align*}
u_1 &= \varphi_1 - \varphi_2 \\
u_2 &= \varphi_5 - \varphi_2 \\
u_3 &= \varphi_9 - \varphi_2 \\
&\quad \vdots \\
u_n &= \varphi_{4n-3} - \varphi_2
\end{align*}
\]

(3.4)

The solution is \( u_j = ju_1 \) and

\[(n + 1)u_1 = 0 \pmod{2\pi\mathbb{Z}} \]

(3.5)

Incidentally, a model Hamiltonian analogous to (2.1) is

\[
H = -\frac{(e^*)^2}{2C} \sum_i \left( \frac{\partial}{\partial u_i} \right)^2 - \frac{1}{2} J \sum_{\alpha > 0, \text{simple}} (e^{i\alpha \cdot \Phi} + e^{-i\alpha \cdot \Phi})
\]

(3.6)

where \( \Phi = \sum \alpha_i u_i \) and \( \alpha_i \) are the simple roots of \( A_n \), and \( u_i \sim u_i + 2\pi \). This system is very closely related to the exactly soluble Toda system. However, as in the \( n = 1 \) case, for \( (CJ) >> (e^*h)^2 \) there are - to exponentially good accuracy - \( (n+1) \) degenerate groundstates. One natural basis is given by \( |r\rangle := |u_1 = \frac{r}{n+1} 2\pi, r = 0, \ldots, n. \)

We now claim that this set of groundstates can be regarded as the irreducible representation of \( \text{Heis} (\mathbb{Z}_{n+1} \times \mathbb{Z}_{n+1}) \), generalizing the example we studied previously. To justify this claim we need to explain how to implement the standard clock and shift operators.

\footnote{but according to Sergei Lukyanov, our system is is not integrable.}
defining the irreducible representation of $\text{Heis}(\mathbb{Z}_{n+1} \times \mathbb{Z}_{n+1})$. The clock operator is implemented by measuring the phase, say, of $u_1 = \varphi_{12}$—something which is experimentally quite feasible. The shift operator is performed in a way analogous to the case $n = 1$ (cf. fig. 5). First, we place a capacitor at point C in fig. 6 to break the relation $\varphi_1 = \varphi_3$. Next we adiabatically change the phase on the capacitor. The classical groundstate equations are modified so that the first equation in (3.3) is changed to

$$2u_1 - u_2 = \varphi_{13}$$

while the remaining equations in (3.3) are unchanged. These equations imply $u_i = \frac{n+1-i}{n+1} \varphi_{13}$ and in particular

$$(n + 1)u_1 - n\varphi_{13} = 0$$

so, increasing $\varphi_{13}$ from 0 to $2\pi$ adiabatically results in a phase shift of $u_1$ by $\frac{n}{n+1}2\pi$. After reconnecting leads 1 and 3, $u_1$ has shifted by $\frac{n}{n+1}2\pi \cong \frac{1}{n+1}2\pi$. We have thus implemented the (inverse of the) shift operator.

3.2. A corresponding 3-dimensional space

In this section we construct a three-dimensional space $Y$ which perfectly reproduces the identifications made in the above array of SCM’s.

![Diagram](image)

Fig. 7: The basic bordism $\kappa$ which can be concatenated in series.

Consider first an immersed bordism in $\mathbb{R}^2 \times I$ from two concentric circles to a single circle. Considering time evolution from the top to the bottom, at the top there are two concentric tubes. Then the inner tube passes through the outer tube as in the standard immersion of the Klein bottle. Beyond this point a slice $\mathbb{R}^2 \times \{t\}$ intersects the surface in
two nonintersecting nonconcentric circles. We now adjoin the standard 3-punctured sphere to give a bordism to a single circle. This bordism - which might be called the “Klein jug” is shown in fig. 7 and will be denoted by $\kappa$. If we cap off the bottom circle in $\kappa$ and let the two top concentric circles merge then we get the standard immersion of the Klein bottle.

Fig. 8: The case of $n = 3$. This corresponds to the circuit in fig. 6.

To produce the space in the $n > 1$ case we consider successive applications of the bordism $\kappa$. At the top we have $n + 1$ concentric circles. We apply $\kappa$ to the two innermost circles to obtain $n$ concentric circles and continue until there is a single circle at the bottom. We cap off the bottom circle and fuse the top $n + 1$ circles into a single circle. The case $n = 3$ is illustrated in fig. 8.

Labeling the regions $1, \ldots, 4n$ we find that the topology of this space precisely implements the identifications made above.

Fig. 9: Cell complex for the ideal space $Y$ for the case $n = 3$. 
The ideal space \( Y \) which has been immersed as above can be described as a 3-dimensional neighborhood of a 2-complex \( L \) (i.e. there is a deformation retraction of \( Y \) onto \( L \)). Therefore \( H^2(Y) \cong H^2(L) \). The 2-complex \( L \) can be thought of as an \( n+1 \)-punctured sphere with each of its boundary components identified in an orientation preserving fashion with a single circle. Accordingly, \( L \) has a cell decomposition consisting of two 0-cells \( p \) and \( q \), \( n+2 \) 1-cells \( x, y, a_1, \ldots, a_n \), and a single 2-cell \( c \). (See fig. 9.) The attaching maps for the 1- and 2-cells are given by

\[
\begin{align*}
\partial a_i &= q - p \\
\partial x &= q - p \\
\partial y &= p - q \\
\partial c &= y a_1 y a_2 \cdots a_n y x a_n^{-1} x \cdots x a_1^{-1} x
\end{align*}
\]

(3.9)

It follows that the 1-cycles are freely generated by \( x + y, a_1 + y, \ldots, a_n + y \), and the 1-boundaries are generated by \( (n+1)(x+y) \). So \( H_1(L) \) has rank \( n \) and \( \text{Tor}(H_1(L)) \cong \mathbb{Z}_{n+1} \). Also, there are no 2-cycles, so \( H_2(L) \cong 0 \). It now follows from the universal coefficient theorem that \( H^2(L) \cong \mathbb{Z}_{n+1} \). To see that the concatenated Klein jugs are indeed an immersion of \( Y \), pull out the handles of all the \( n \) jugs to obtain a sphere with \( (n+1) \) holes with its boundary circles identified.

\[\text{Fig. 10: The only kinds of singular points of immersions we need.}\]

In fact, any oriented, connected 3-manifold with non-empty boundary can be immersed in \( \mathbb{R}^3 \) as a neighborhood of a 2-complex with only double and triple points. Let \( M \) be such a 3-manifold and let \( f : M \to \mathbb{R}^3 \) be a constant map (all of \( M \) sent to a single point in \( \mathbb{R}^3 \)). Since the tangent bundle of \( M \) is trivializable \([7]\), \( f \) can be covered by a rank 3 bundle map \( f' : TM \to T\mathbb{R}^3 \). It now follows from the Smale-Hirsch immersion theorem \([7]\) that \( f' \) can be deformed through rank 3 bundle maps to the tangent map of an immersion \( g : M \to \mathbb{R}^3 \). (Note that here we use the fact that \( M \) has non-empty boundary,
since otherwise the hypotheses of the immersion theorem would not be satisfied.) Choose a deformation retraction of $M$ onto a 2-complex $L \subset M$. (This is equivalent to choosing a handle decomposition of $M$ which contains no 3-handles.) Deform $g$ so that its restriction to $L$ is a general position map with only double and triple points.\footnote{See fig. 10. Using the deformation retraction, we can further deform $g$ so that $g(M)$ is a small neighborhood of $g(L)$.}

We have explained above how to incorporate double points. Triple points do not require any special treatment: it is sufficient to include one SCM for each arc of double points.

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**Appendix A. Properties of torsion cohomology.**

In this appendix we construct the perfect pairing

$$\omega : \text{Tors}H^2(Y; \mathbb{Z}) \times \text{Tors}H^2(Y, \partial Y; \mathbb{Z}) \to \mathbb{R}/\mathbb{Z}. \quad (A.1)$$

for any orientable 3-manifold $Y$ with boundary. We also prove the following

**Theorem:** If $Y$ is an embedded 3-manifold in $\mathbb{R}^3$ then

$$\text{Tors}(H^2(Y; \mathbb{Z})) = \text{Tors}(H^2(Y; \partial Y, \mathbb{Z})) = 0. \quad (A.2)$$

\footnote{Once we have immersed the 3-manifold, we know that any small neighborhood in $L$ is embedded – there are no local singularities. Standard results on transversality allow us to assume, after a small perturbation, that the dimension of the intersection of the $i$-skeleton of $L$ and the $j$-skeleton of $L$ has dimension $i+j-3$. Similarly, the dimension of the triple intersection of the $i$, $j$ and $k$-skeleta of $L$ has dimension $i+j+k-6$. The only possibilities are (a) $2+2-3 = 1$, $2+1-3 = 0$ (we can assume that the 1-skeleton of $L$ coincides with the non-manifold points of $L$), and (c) $2+2+2-6 = 0$.}
In fact, such an embedded manifold cannot even have a finite fundamental group, as explained to us by Michael Freedman.

Let $Y$ be an orientable 3-manifold; without loss of generality we may assume that $Y$ is connected. Recall from the coefficient sequence

$$\cdots \to H^1(Y; \mathbb{R}) \to H^1(Y; \mathbb{R}/\mathbb{Z}) \xrightarrow{\beta} H^2(Y; \mathbb{Z}) \to H^2(Y; \mathbb{R}) \to \cdots \quad (A.3)$$

where $\beta$ is the Bockstein map that we can identify $\text{Tors}(H^2(Y; \mathbb{Z}))$ with the image of $\beta$. (Indeed, $\text{Im}\beta$ is finite since $\beta$ is a continuous map from a compact group to a discrete group. Conversely, any torsion element in $H^2(Y; \mathbb{Z})$ vanishes if we extend the coefficient group to $\mathbb{R}$.) The same is true for the relative cohomology $H^2(Y, \partial Y; \mathbb{Z})$.

Next, consider the cup-products

$$H^1(Y; \mathbb{R}/\mathbb{Z}) \times H^2(Y, \partial Y; \mathbb{Z}) \to H^3(Y, \partial Y; \mathbb{R}/\mathbb{Z}) \cong \mathbb{R}/\mathbb{Z} \quad (A.4)$$

$$H^2(Y; \mathbb{Z}) \times H^1(Y, \partial Y; \mathbb{R}/\mathbb{Z}) \to H^3(Y, \partial Y; \mathbb{R}/\mathbb{Z}) \cong \mathbb{R}/\mathbb{Z} \quad (A.5)$$

By Poincaré duality, $H^2(Y, \partial Y; \mathbb{Z}) \cong H_1(Y)$ and $H^2(Y; \mathbb{Z}) \cong H_1(Y, \partial Y)$. The above maps are just the standard pairings between (the absolute or relative) homology and cohomology in dimension 1. They can also be interpreted as the homomorphisms $H^1(\cdots; \mathbb{R}/\mathbb{Z}) \to \hom(H_1(\cdots), \mathbb{R}/\mathbb{Z})$ from the universal coefficient sequence, which are actually isomorphisms since $\text{Ext}^1(G, \mathbb{R}/\mathbb{Z}) = 0$ for any $G$. Thus we have perfect pairings between the compact group $H^1(Y; \mathbb{R}/\mathbb{Z})$ and the discrete group $H^2(Y, \partial Y; \mathbb{Z})$, and also between $H^1(Y, \partial Y; \mathbb{R}/\mathbb{Z})$ and $H^2(Y; \mathbb{Z})$ (Pontryagin duality).

The perfect pairing (A.1) can be obtained from the cup-products (A.4) and (A.5) if we set

$$\omega(\beta h, \beta h') \overset{\text{def}}{=} h \cup \beta h' = \beta h \cup h', \quad (A.6)$$

where $h \in H^1(Y; \mathbb{R}/\mathbb{Z})$ and $h' \in H^1(Y, \partial Y; \mathbb{R}/\mathbb{Z})$. Both defining expressions are necessary to make sure that $\omega$ depends only on $\beta h, \beta h'$ rather than $h, h'$. But we need to demonstrate that the two definitions are equivalent. To this end, let us represent the cohomology classes $h, h'$ by simplicial cochains $c, c'$ with real coefficients. Then $\beta h$ and $\beta h'$ are represented by integral cocycles $dc$ and $dc'$, respectively. Passing to cohomology classes in the equation

$$d(c \cup c') = dc \cup c' - c \cup dc'$$
and taking the quotient over $\mathbb{Z}$, we get

$$0 = \beta h \cup h' - h \cup \beta h'.$$

Now, if $Y$ is an embedded 3-fold in $S^3$ then its complement $X$ is also an embedded 3-fold and $X \cup Y = S^3$, while $X \cap Y = \partial X = \partial Y$. From the exact sequence for the pair $(S^3, X)$ we learn

$$H^1(X; G) \cong H^2(S^3, X; G) \cong H^2(Y, \partial Y; G) \quad (A.7)$$

for any coefficient group $G$. Also we have

$$H^0(X; G)/G \cong H^1(S^3, X; G) \cong H^1(Y, \partial Y; G) \quad (A.8)$$

where the quotient by $G$ is by the 0-cycles which are the same constant on all the components of $X$.

Now we consider the commutative square

$$
\begin{array}{ccc}
H^0(X; \mathbb{R}/\mathbb{Z})/(\mathbb{R}/\mathbb{Z}) & \rightarrow & H^1(X; \mathbb{Z}) \\
\cong \downarrow & & \cong \downarrow \\
H^1(Y, \partial Y; \mathbb{R}/\mathbb{Z}) & \rightarrow & H^2(Y, \partial Y; \mathbb{Z})
\end{array}
$$

(A.9)

However, the image of the Bockstein map $H^0(X; \mathbb{R}/\mathbb{Z})/(\mathbb{R}/\mathbb{Z}) \rightarrow H^1(X; \mathbb{Z})$ must be zero since the domain is the direct product of copies of $\mathbb{R}/\mathbb{Z}$ and the image is a discrete group. Thus $\text{Tors}(H^2(Y, \partial Y; \mathbb{Z}))$ is zero and by the perfect pairing $\text{Tors}(H^2(Y; \mathbb{Z}))$ is also zero.
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