Towards Bose-Einstein Condensation of Electron Pairs: Role of Schwinger Bosons

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Abstract

It can be shown that the bosonic degree of freedom of the tightly bound on-site electron pairs could be separated as Schwinger bosons. This is implemented by projecting the whole Hilbert space into the Hilbert subspace spanned by states of two kinds of Schwinger bosons (to be called binon and vacanon) subject to a constraint that these two kinds of bosonic quasiparticles cannot occupy the same site. We argue that a binon is actually a kind of quantum fluctuations of electron pairs, and a vacanon corresponds to a vacant state. These two bosonic quasiparticles may be responsible for the Bose-Einstein condensation (BEC) of the system associated with electron pairs. These concepts are also applied to the attractive Hubbard model with strong coupling, showing that it is quite useful. The relevance of the present arguments to the existing theories associated with the BEC of electron pairs is briefly commented.

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INTRODUCTION

The discovery of pseudogap phenomenon in the normal state of underdoped cuprate oxides renews the interest of exploiting the Bose-Einstein condensation (BEC) of electron pairs. One of theories attributes this unusual phenomenon to the BEC of the pre-formed singlet pairs under such an assumption that a smooth crossover from the BEC to the BCS regimes can emerge with increasing carrier concentration [1]. There are also other theories related to the pre-formed electron pairs (see, e.g. Ref. [2] for a review) or bipolarons [3] in cuprates. On the other hand, the evolution from the BCS theory to the BEC regime has been discussed long time ago by Eagles [4], Leggett [5], Nozières and Schmitt-Rink [6], and others (see, e.g. Ref. [7] for a review). A key ingredient is to modify the electron-phonon interacting potential by plugging a two-particle scattering length in the BCS reduced Hamiltonian so that a crossover parameter $\xi$ proportional to the inverse of the scattering length, is introduced, which results in the crossover from the BCS condensation ($\xi \to -\infty$) of the Cooper pairs to the BEC ($\xi \to +\infty$) of tightly bound electron pairs. The chemical potential is zero at the transition point. Note that the Cooper pairs are formed in the weak coupling limit in momentum space, while tightly bound electron pairs are formed in the strongly attractive coupling limit in real space.

With regard to the aforementioned theories, people intuitively believe that in the strong coupling limit the tightly bound electron pairs can be taken as bosons, each of which may be constituted by two fermions (e.g. electrons) through certain mechanism, and they can undergo the BEC below a certain temperature. However, whether or not the tightly bound electron pairs can really undergo the BEC directly, is an old but fundamental (and nontrivial) problem, particularly as we know that the bound electron pairs are not exactly bosons [8]. If not, what are the corresponding bosons in these theories? On the other hand, in the boson-fermion model for superconductivity [3] a localized boson (or a bipolaron) which was also regarded as a kind of tightly bound electron pair, was simply replaced by a hard-core boson. What is the underlying physics behind such a simple replacement? In this paper, we shall try to address these questions. Our result appears to show that the fluctuations of the tightly bound electron pairs play a central role. In the following, we shall first show why the tightly bound electron pairs could not undergo the BEC directly in momentum space using the standard argument, and then introduce two kinds of Schwinger bosons (binon and vacanon) and argue that a binon is a kind of fluctuations of the tightly bound electron pairs and a vacanon corresponds to a vacant state. These two bosonic quasiparticles may be responsible for the BEC of the system. These concepts are also applied to the attractive Hubbard model in the strong coupling limit, and it appears that this special system can be exploited on the basis of binons and vacanons. Finally, we shall discuss briefly the relevance of our argument to the existing theories concerning the BEC of electron pairs.

USEFUL NOTIONS

The operators of a tightly bound electron pair are defined in real space as usual in the following [9]:

$$b_i = c_{i\downarrow} c_{i\uparrow}, \quad b_i^\dagger = (b_i)^\dagger,$$

(1)
where \( c_{i\uparrow}(c_{i\downarrow}) \) is the annihilation operator of an electron with up(down)-spin at site \( i \) obeying the anticommutation relations. It is well-known that they satisfy the standard SU(2) algebra:

\[
[b_i, b_j^\dagger] = 2b_i^0 \delta_{ij}, \quad [b_i^\dagger, b_j^\dagger] = \pm b_i^\dagger \delta_{ij}, \quad \text{with} \quad b_i^0 = \frac{1}{2}(1 - n_i) \quad \text{and} \quad n_i = c_{i\uparrow}^\dagger c_{i\uparrow} + c_{i\downarrow}^\dagger c_{i\downarrow} \quad \text{the number operator of electrons at site} \quad i, \quad \text{as well as the bosonic property} \quad [b_i, b_j] = [b_i^\dagger, b_j^\dagger] = 0. \quad \text{Owing to Pauli’s exclusion principle, the tightly bound electron pairs exhibit the hard-core property, satisfying}
\]

\[
b_i^2 = b_i^{\dagger 2} = 0. \quad (2)
\]

Moreover, it can be observed that there exists an important property (i.e., an anticommutator):

\[
\{b_i, b_i^\dagger\} = 4(b_i^0)^2, \quad (3)
\]

implying that a tightly bound electron pair has fermionic characters. As the BEC takes place only in momentum space, we need to consider their Fourier transforms. The hermitian property \((b_i^\dagger = (b_i)^\dagger)\) and the consistency require that the Fourier transforms of the operators for the tightly bound electron pairs can only be defined through the following forms:

\[
b_i = \frac{1}{M} \sum_k e^{i\mathbf{k} \cdot \mathbf{r}_i} \tilde{b}_k, \quad \tilde{b}_k = \sum_i e^{-i\mathbf{k} \cdot \mathbf{r}_i} b_i
\]

\[
b_i^\dagger = (b_i)^\dagger = \frac{1}{M} \sum_k e^{-i\mathbf{k} \cdot \mathbf{r}_i} \tilde{b}_k^\dagger, \quad \tilde{b}_k^\dagger = \sum_i e^{i\mathbf{k} \cdot \mathbf{r}_i} b_i^\dagger
\]

\[
b_i^0 = \frac{1}{M} \sum_k e^{i\mathbf{k} \cdot \mathbf{r}_i} \tilde{b}_k^0, \quad \tilde{b}_k^0 = \sum_i e^{-i\mathbf{k} \cdot \mathbf{r}_i} b_i^0, \quad (4)
\]

where \( M \) is the number of lattice sites, and Kronecker’s delta function is defined by \( \delta_{ij} = \frac{1}{M} \sum_k e^{i\mathbf{k} \cdot (\mathbf{r}_i - \mathbf{r}_j)} \). It should be noted that the operator \( \tilde{b}_k^\dagger \) differs from the Cooper pair operator \( c_{-\mathbf{k} \uparrow} c_{\mathbf{k} \uparrow} \) (see below). The commutation relations of the Fourier transforms are of the form:

\[
[b_k, b_{k'}^\dagger] = 2\tilde{b}_k^0 \delta_{kk'}, \quad [\tilde{b}_k, b_{k'}^\dagger] = -\tilde{b}_{k+k'}, \quad [\tilde{b}_k^\dagger, b_{k'}^\dagger] = \tilde{b}_{k-k'}, \quad [\tilde{b}_k, b_{k'}^\dagger] = [\tilde{b}_k^\dagger, b_{k'}^\dagger] = 0. \quad \text{considering the Fourier transform of the electron operator,} \quad c_{i\sigma} = (1/\sqrt{M}) \sum_k \exp(i\mathbf{k} \cdot \mathbf{r}_i) c_{k\sigma} \quad (\sigma = \uparrow, \downarrow), \quad \text{one may find the relationship between operators} \quad \{\tilde{b}_k\} \quad \text{and electron operators:} \quad \tilde{b}_k = \sum_q c_{q+\frac{\tau}{2}} c_{-q+\frac{\tau}{2} \dagger}, \quad \tilde{b}_k^\dagger = (\tilde{b}_k)^\dagger, \quad \text{and} \quad \tilde{b}_k^0 = \frac{1}{2} \sum_q (\delta_{k,0} - c_{q+\frac{\tau}{2}} c_{-q+\frac{\tau}{2}} - c_{q+\frac{\tau}{2}}^\dagger c_{-q+\frac{\tau}{2}}^\dagger). \quad \text{Clearly, the tightly bound electron pair in momentum space consists of the superposition of two electrons with different momenta and is spin-singlet. Unlike the standard Cooper pair, the total momenta of the two pairing electrons here is nonzero. It is interesting to mention that the hard-core property like Eq. (3) no longer holds in momentum space, namely,} \quad \tilde{b}_k^2 = \sum_q (q'q \neq q) c_{q+\frac{\tau}{2}}^\dagger c_{-q+\frac{\tau}{2} \dagger} c_{q'+\frac{\tau}{2}} c_{-q'+\frac{\tau}{2}} \neq 0, \quad \text{and the anticommutator like Eq. (3)} \quad \text{will not remain true in momentum space, suggesting that the fermionic properties which exist in real space do not appear in momentum space. Therefore, in contrast to the Cooper pair} \quad c_{-\mathbf{k} \uparrow} c_{\mathbf{k} \uparrow} \quad \text{which possesses exactly the same commutation relations as the tightly bound electron pair in real space, the latter may purely show bosonic behavior in momentum space, which appears in turn to suggest that the BEC of the tightly bound electron pairs in momentum space be possible.}
TIGHTLY BOUND ELECTRON PAIRS IN MOMENTUM SPACE

Now, let us examine if the BEC of the noninteracting tightly bound electron pairs in momentum space can really occur. By noting the fact \( \sum_k \hat{b}^\dagger_k \hat{b}_k / M = \sum_i b_i b_i \), we define the number operator per volume, \( \hat{N}_k \), of the tightly bound electron pairs in momentum space as \( \hat{N}_k = \hat{b}^\dagger_k \hat{b}_k / M \). The eigenvalues of \( \hat{N}_k \) can be argued to be non-negative integers in the infinite volume of the system. This may be reached by observing the following facts. First, the eigenvalues of \( \hat{N}_k \) are non-negative, because for any eigenstate we can write \( \langle m | \hat{b}^\dagger_k \hat{b}_k | m \rangle = \sum_n \langle m | \hat{b}^\dagger_k | n \rangle \langle n | \hat{b}_k | m \rangle \geq 0 \) where \( \{|n\} \) is a complete orthonormal set of eigenstates. Second, we can construct the orthonormal set of eigenstates of \( \hat{N}_k \) as \( |n,k\rangle = [(M/n)!/(M!n!)]^{1/2} (\hat{b}^\dagger_k)^n |0\rangle \) satisfying \( \langle n',k' | n,k \rangle = \delta_{n,n'} \delta_{k,k'} \), where the Fock vacuum \( |0\rangle \) satisfies \( c_{k0} |0\rangle = 0 \). It is interesting to point out that the state \( |n,k\rangle \) possesses the on-site off-diagonal long-range order (ODLRO) in the two-particle reduced density matrix when \( n \) is on the order of \( M \). One may easily examine that \( \hat{N}_k |n,k\rangle = n(M - n + 1)/M |n,k\rangle \). Since \( n \leq M \) is integers, it is possible that, in the infinite volume (i.e., \( M \to \infty \)), the eigenvalues \( \hat{N}_k \) in the given \( k \) state can be taken as \( \hat{N}_k = 0, 1, 2, 3, \ldots, \infty \) with the degeneracy being 2. For the noninteracting system of the tightly bound electron pairs with the single-particle dispersion \( \epsilon_k \), it seems that one can get the average occupation number per volume \( \langle \hat{N}_k \rangle \) in the following way:

\[
\langle \hat{N}_k \rangle = \frac{2 \sum_{\hat{N}_k=0}^{\infty} e^{-\beta (\epsilon_k - \mu_b) \hat{N}_k} \hat{N}_k}{\sum_{\hat{N}_k=0}^{\infty} e^{-\beta (\epsilon_k - \mu_b) \hat{N}_k}} = \frac{2}{e^{\beta (\epsilon_k - \mu_b)} - 1},
\]

with \( \mu_b \) the chemical potential of the tightly bound electron pairs. If this formula is correct, the BEC will appear in this system, like the usual ideal Bose gas. Unfortunately, when one looks carefully at Eq. (5), one may find it incorrect. The reasoning is as follows. Equation (4) implies that the Hamiltonian for this noninteracting system is of the form \( (1/M) \sum_k (\epsilon_k - \mu_b) \hat{b}^\dagger_k \hat{b}_k \) which looks apparently like the conventional noninteracting Bose system. However, when considering the fact that the operator \( \hat{N}_k \) does not commute with the Hamiltonian of this form owing to \( [\hat{N}_k, \hat{N}_k'] \neq 0 \) for \( k \neq k' \), suggesting that the Hamiltonian cannot be diagonalized by the eigenstates of \( \hat{N}_k \), one may observe that the first equality of Eq. (4) is not justified. This could also be easily understood if we loosely treat the bound electron pairs as pseudospins and thereby the system as the XY model. Therefore, it might be reasonable to believe that the noninteracting tightly bound electron pairs in momentum space could not directly undergo the BEC in the usual (or strict) sense that the BEC is a phenomenon for the noninteracting Bose systems (4). On the other hand, as advocated by a number of people, there must be a crossover from the BCS theory to the BEC of the tightly bound electron pairs based on qualitative arguments associated with approximate calculations. Naively speaking, this is certainly sound, because if the attractive interactions between electrons are strong enough, the tightly bound electron pairs are formed, and the low-lying states of the system are thus occupied by electron pairs, which might have a macroscopic occupation at a certain quantum state due to the bosonic nature of these pairs. How to reconcile these facts? Actually, from our point of view the tightly bound electron pairs or Cooper pairs cannot undergo the BEC themselves directly, but their fluctuations probably can. These fluctuations can be constructed as genuine bosons through some kind of nonlinear transformations (10), in which Pauli’s exclusion principle seemingly no longer plays a role in the formalism. Consequently, owing to the genuinely bosonic statistical property of
these quasiparticles the BEC of the fluctuations appear to be possible. Emery and Kivelson have recently presented a nice example, similar in spirit to this argument, in which they assumed that the superconducting order parameter of a metal, with an amplitude and a phase, is complex, and emphasized the importance of phase fluctuations in superconductors with small superfluid density \[^{[11]}\]. Here we shall offer an alternative realisation.

**BINONS AND VACANONS**

We introduce two kinds of bosons by which a tightly bound electron pair can be constituted. To this aim, we define

\[
b_i = f_i^\dagger a_i, \quad b_i^\dagger = a_i^\dagger f_i, \quad b_i^0 = \frac{1}{2}(n_i^f - n_i^a)
\]

where the operators \{\(f\)\} and \{\(a\)\} are of bosons, satisfying the standard commutation relations: \([f_i, f_j^\dagger] = [a_i, a_j^\dagger] = \delta_{ij}\) and \([f_i, f_j] = [f_i^\dagger, f_j^\dagger] = [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0\). In addition, they commute each other, namely \([\{f\}, \{a\}] = 0\). These bosons are hard-core bosons, obeying

\[
(n_i^a)^2 = n_i^a, \quad (n_i^f)^2 = n_i^f,
\]

with \(n_i^a(n_i^f) = f_i^\dagger f_i(a_i^\dagger a_i)\) the number operator of the respective bosons. Furthermore, we impose a constraint on the two bosons:

\[
n_i^a n_i^f = 0.
\]

This condition, arising from the consistency requirement with Eq.\(^\text{(3)}\), implies that the two bosons cannot occupy the same site. The underlying physics behind Eqs. \(^\text{(7)}\) and \(^\text{(8)}\) is still Pauli’s exclusion principle. One may check that with the above definitions all the conditions imposed on the tightly bound electron pair operators are indeed satisfied. We call the boson ”a” as *binon* and the boson ”f” as *vacanon*, which are actually two kinds of Schwinger bosons, because the transformation \(^\text{(3)}\) is seemingly similar to the form in the context of angular momentum discussed long time ago by Schwinger. It turns out that the number of the tightly bound electron pairs maps onto the following

\[
b_i^\dagger b_i \Rightarrow a_i^\dagger a_i.
\]

As a consequence, the total number of electrons, \(N_e = \sum_i n_i\), becomes \(N_e = M + N_a - N_f\), where \(N_a(f) = \sum_i n_i^{a(f)}\), is the total number of binons (vacanons). This reveals that the role of the tightly bound electron pairs played in the original electron system is now replaced by that of the new hard-core quasiparticles (i.e. binons and vacanons) in the new system. The fermionic property of the electron pairs is reflected by the conditions \(^\text{(7)}\) and \(^\text{(8)}\). As can be seen, this transformation is actually a projection procedure, which maps the whole Hilbert space \(\mathcal{V}\) spanned by all the states of the tightly bound electron pairs onto the Hilbert subspace \(\mathcal{V}_0\) spanned by those of the new bosons with the constraint that the two kinds of Schwinger bosons cannot occupy the same site. The projection operator \(\mathcal{P}\) can be defined such that \(\mathcal{P} : \mathcal{V} \Rightarrow \mathcal{V}_0\), and \(\mathcal{P} : b_i^\dagger b_i \Rightarrow a_i^\dagger a_i\) with the Hamiltonian \(\hat{H} \Rightarrow \hat{H}\). The transformation defined in Eq. \(^\text{(3)}\) just gives a possible manifestation of \(\mathcal{P}\). It is in some
sense very similar to the projection made in the context of the t-J model where the doubly-occupied sites are projected out. We could remark here that the Bose quasiparticles (binons and vacanons) in the system of the tightly bound electron pairs are somewhat in analogy with the magnons in magnetic systems.

The vacanon is an auxiliary Bose field by which the commutators for operators \{b_i\} are guaranteed. For the Fock vacuum \(|0\rangle\), we have \(b_i|0\rangle = 0\) and \(b_i^\dagger|0\rangle \neq 0\). By definition, we observe

\[
a_i|0\rangle = 0, \quad f_i|0\rangle \neq 0. \quad (10)
\]

This is the intrinsic difference between the two Schwinger bosons. The vacanon can be understood as a kind of empty or vacancy on the lattice in the sense that it cannot be destructed in vacuum [12]. Since a tightly bound electron pair carries spin zero and electric charge 2, namely, \(s_i^2 b_i^\dagger = 0\) and \(n_i b_i^\dagger = 2b_i^\dagger\), we have \(s_i^z a_i^\dagger = 0\) and \(n_i a_i^\dagger = 2a_i^\dagger\), where \(s_i^z\) is the z-component of the spin-1/2 operator, and use has been made of conditions (10). This shows that \(a\) \textit{binon also carries spin zero and electric charge 2}. It is not difficult to verify that a vacanon carries both charge and spin zero. In this way, the hopping of a tightly bound electron pair on a lattice from the site \(j\) to the site \(i\) is equivalent to the hopping of a binon from the site \(j\) to the site \(i\) while accompanying the hopping of a vacanon from the site \(i\) to the site \(j\), as depicted in Fig.1. This makes the separation of the genuinely charged hard-core bosonic degrees of freedom and the vacant states originally implied in the system of the tightly bound electron pairs. Note that the hopping occurs only between the occupied and unoccupied sites. Let us discuss briefly the physical meaning of the binon. It is readily checked that the following commutators hold: \([b_i, a_i] = [b_i, f_i^\dagger] = [b_i^\dagger, f_i] = [b_i^\dagger, a_i^\dagger] = 0\), \([b_i, f_i] = -a_i\), \([b_i, a_i^\dagger] = f_i^\dagger\), \([b_i^\dagger, a_i^\dagger] = -f_i\), \([b_i^\dagger, f_i^\dagger] = a_i^\dagger\). In momentum space, we have

\[
a_k^\dagger = \frac{1}{M} \sum_q (b_{q+\frac{1}{2}} f_{q+\frac{1}{2}} - f_{q-\frac{1}{2}} b_{q+\frac{1}{2}}), \quad (11)
\]

\[
f_k^\dagger = \frac{1}{M} \sum_q (b_{q-\frac{1}{2}} a_{q+\frac{1}{2}} - a_{q+\frac{1}{2}} b_{q-\frac{1}{2}}), \quad (12)
\]

where we have supposed that the Fourier transforms of the Schwinger bosons have the same form as electrons. Since the boson operator \(f_k^\dagger\) produces empty or vacant states, Eq. (11) shows that the binon is a kind of quantum fluctuations of the tightly bound electron pairs. This observation is quite consistent with the crude belief that the formation of the tightly bound electron pairs is a quantum fluctuation in the context of a continuum model and the attractive Hubbard model [7]. Here, we have mathematically presented a possible and explicit description for such fluctuations. By comparing the corresponding Fourier transforms we could obtain the relationship between binons, vacanons and electrons:

\[
\sum_q a_{q+k}^\dagger f_q = \sum_q c_{q+k}^\dagger c_{-q}^\dagger \quad \text{and} \quad \sum_q n_q = \sum_q [1 - (n_{q^+} + n_{-q+k})], \quad (9)
\]

where \(a_i^z = s_i^z a_i^\dagger (z = a, f)\). Equation (9) gives the condition \((1/M) \sum_k \hat{F}_k = 0\) with \(\hat{F}_k = \sum_{q, a, f} \delta_{q, a}^r a_{q+k}^\dagger f_{q+k}^\dagger f_q\). The vacanon is supposed to obey \((1/M) \sum_q f_{q+k}^\dagger f_q |0\rangle = \delta_{k,0} |0\rangle\). The total numbers of binons and vacanons satisfy \(N_a \in [0, N_e/2]\) and \(N_f \in [0, M - N_e/2]\). To this end, we could say that the bosons involved in the tightly bound electron pairs could be identified as binons and vacanons which are actually responsible for the BEC phenomenon in the system. Although the concept of binons and vacanons is developed on the basis of the noninteracting tightly
bound electron pairs, we anticipate that it would be applicable to interacting electrons, bipolarons, and other systems.

THE ATTRACTIVE HUBBARD MODEL WITH STRONG COUPLING

It is known that the attractive Hubbard model in the strong coupling limit bears the form [13]

\[ H = \sum_{i,j} J_{ij} b_i^\dagger b_j + \sum_{i,j} J_{ij} b_i^0 b_j^0 - \bar{\mu} \sum_i n_i + \text{const.} \]  \hspace{1cm} (13)

which is valid for arbitrary band filling, where \( \bar{\mu} = \mu + U/2 \) with \( \mu \) the chemical potential of electrons, can be viewed as the effective chemical potential, and \( J_{ij} = -2t_{ij}^2/|U| \). Hereafter we suppose \( J_{ii} = 0 \). In terms of the terminology of binons and vacanons, this Hamiltonian can be rewritten as

\[ H = \sum_{i,j} J_{ij} a_i^\dagger a_j + \frac{1}{4} \sum_{i,j} J_{ij} (n_i^a n_j^a + n_i^f n_j^f - 2n_i^a n_j^f) + \bar{\mu} (n_i^f - n_i^a) + \text{const.} \]  \hspace{1cm} (14)

Consequently, the attractive Hubbard model with strong coupling is mapped onto a coupled Bose system with binons and vacanons subject to the constraints mentioned before. The first term denotes the hopping process of binons and vacanons, as shown in Fig.1, while the second sum shows the interactions between binons and vacanons. In momentum space, the Hamiltonian reads

\[ H = \frac{1}{M} \sum_{k,q,q'} \epsilon_k a_{q+k}^\dagger a_{q'}^\dagger f_q f_{q'} - \frac{1}{4} \sum_k \epsilon_k (\rho_k^a - \rho_k^f)^2 + \bar{\mu} (n_k^f - n_k^a) + \text{const.} \]  \hspace{1cm} (15)

where \( \epsilon_k = -(2t^2/|U|) \sum_{\delta} \exp(i \mathbf{k} \cdot \delta) \) (\( \delta \) is the vector connecting the nearest-neighbor lattice sites if we assume \( t_{ij} = t \) for \( i,j \) being nearest neighbors and zero otherwise), and \( \rho_k^z = (1/M) \sum_{q} \psi_{q+k}^a \psi_{q+k}^f \). One may observe that the BEC phenomenon in the attractive Hubbard model with strong coupling can thus be explored on the basis of the above Hamiltonian in the restricted Hilbert subspace where the binons and vacanons cannot occupy the same site. Since this is a kind of interacting hard-core Bose gas, some approximations will be necessary. The details will be presented elsewhere.

We now consider the homogeneous continuum case at half-filling. The Bose field operators can be introduced through the following definitions:

\[ a_i^\dagger = \int d\mathbf{r} \delta(\mathbf{r} - \mathbf{r}_i) \psi_a^\dagger(\mathbf{r}), \quad \psi_a^\dagger(\mathbf{r}) = \sum_i \delta(\mathbf{r} - \mathbf{r}_i) a_i^\dagger. \]

\[ f_i^\dagger = \int d\mathbf{r} \delta(\mathbf{r} - \mathbf{r}_i) \psi_f^\dagger(\mathbf{r}), \quad \psi_f^\dagger(\mathbf{r}) = \sum_i \delta(\mathbf{r} - \mathbf{r}_i) f_i^\dagger, \]  \hspace{1cm} (16)

where \( [\psi_{a,f}(\mathbf{r}), \psi_{a,f}(\mathbf{r}')] = \delta(\mathbf{r} - \mathbf{r}'), \quad [\psi_{a,f}(\mathbf{r}), \psi_{a,f}(\mathbf{r}')] = [\psi_a(\mathbf{r}), \psi_f(\mathbf{r}')] = 0 \) and \( \{\psi_{a,f}(\mathbf{r}), \psi_{a,f}(\mathbf{r})\}^2 = \psi_{a,f}^\dagger(\mathbf{r}) \psi_{a,f}(\mathbf{r}) \). It follows from Eq. (1) that \( \psi_a^\dagger(\mathbf{r}) \psi_a(\mathbf{r}) \psi_f^\dagger(\mathbf{r}) \psi_f(\mathbf{r}) = 0 \). The Hamiltonian becomes
\[ H = \int d\mathbf{r} d\mathbf{r'} \psi_\alpha^\dagger(\mathbf{r}) \psi_\alpha(\mathbf{r'}) J(\mathbf{r} - \mathbf{r'}) \psi_f^\dagger(\mathbf{r'}) \psi_f(\mathbf{r}) + \frac{1}{4} \int d\mathbf{r} d\mathbf{r'} [\psi_\alpha^\dagger(\mathbf{r}) \psi_\alpha(\mathbf{r}) J(\mathbf{r} - \mathbf{r'}) \psi_\alpha^\dagger(\mathbf{r'}) \psi_\alpha(\mathbf{r}) + \psi_f^\dagger(\mathbf{r}) \psi_f(\mathbf{r}) J(\mathbf{r} - \mathbf{r'}) \psi_f^\dagger(\mathbf{r'}) \psi_f(\mathbf{r})] + \mu \int d\mathbf{r} \psi_f^\dagger(\mathbf{r}) \psi_f(\mathbf{r}) + \text{const.}, \]

where \( J(\mathbf{r} - \mathbf{r'}) = \sum_{ij} \delta(\mathbf{r} - \mathbf{r}_i) \delta(\mathbf{r'} - \mathbf{r}_j) J_{ij} \). From the equations of motion for the field operators \( \psi_\alpha(\mathbf{r}, t) \) and \( \psi_f(\mathbf{r}, t) \) we obtain

\[ i\hbar \frac{\partial \Psi(\mathbf{r}, t)}{\partial t} = h_{\text{eff}}(\mathbf{r}, t) \Psi(\mathbf{r}, t) \]

with

\[ \Psi(\mathbf{r}, t) = \begin{pmatrix} \psi_\alpha(\mathbf{r}, t) \\ \psi_f(\mathbf{r}, t) \end{pmatrix}, \]

\[ h_{\text{eff}}(\mathbf{r}, t) = \begin{pmatrix} \zeta(\mathbf{r}, t) - \bar{\mu} & \xi(\mathbf{r}, t) \\ -\bar{\zeta}(\mathbf{r}, t) & \bar{\xi}(\mathbf{r}, t) \end{pmatrix}, \]

\[ \xi(\mathbf{r}, t) = \int d\mathbf{r'} \psi_\alpha^\dagger(\mathbf{r'}, t) J(\mathbf{r} - \mathbf{r'}) \psi_\alpha(\mathbf{r'}, t), \]

\[ \zeta(\mathbf{r}, t) = \frac{1}{2} \int d\mathbf{r'} [\psi_\alpha^\dagger(\mathbf{r'}, t) J(\mathbf{r} - \mathbf{r'}) \psi_\alpha(\mathbf{r'}, t) - \psi_f^\dagger(\mathbf{r'}, t) J(\mathbf{r} - \mathbf{r'}) \psi_f(\mathbf{r'}, t)], \]

where we have noticed that \( J(\mathbf{r} - \mathbf{r'}) \) is a real function. One may observe that Eq. \((18)\) is nothing but the Schrödinger equation with the effective, time-dependent local Hamiltonian \( h_{\text{eff}}(\mathbf{r}, t) \), which implies that \( [\Psi(\mathbf{r}), H] = h_{\text{eff}}(\mathbf{r}) \Psi(\mathbf{r}) \). The formal solution of \( \Psi(\mathbf{r}, t) \) can be written as

\[ \Psi(\mathbf{r}, t) = \Psi(\mathbf{r}, t_0) e^{-i \int_{t_0}^t h_{\text{eff}}(\mathbf{r}, t') dt'}. \]

It turns out that the Hamiltonian \((17)\) can be equivalently rewritten as

\[ H = \frac{1}{2} \int d\mathbf{r} \Psi^\dagger(\mathbf{r}) (h_{\text{eff}}(\mathbf{r}) - \bar{\mu}) \Psi(\mathbf{r}) + \text{const.}, \]

where use has been made of \( J(0) = 0 \) and \( J(\mathbf{r} - \mathbf{r'}) = J(\mathbf{r'} - \mathbf{r}) \). It should be noted that \( \Psi(\mathbf{r}) \) itself does not fulfill the bosonic commutation relations but its elements do. If one makes proper approximations on \( h_{\text{eff}}(\mathbf{r}) \), one could get the Gross-Pitaevskii (GP)-like equation \((14)\) which is known to be capable of giving a better description on the condensate in dilute Bose gas near zero temperature. In this sense, Eq. \((18)\) can be regarded as a generalized GP equation for the system under interest. Like the Bogoliubov-de Gennes equation in superconductors, we may here linearize the Hamiltonian by replacing \( \xi(\mathbf{r}) \) and \( \zeta(\mathbf{r}) \) by their averages \( \bar{\xi}(\mathbf{r}) \equiv \langle \xi(\mathbf{r}) \rangle \) and \( \bar{\zeta}(\mathbf{r}) \equiv \langle \zeta(\mathbf{r}) \rangle \), respectively. Then, we can seek for the stationary solution of interest of the form \( h_{\text{eff}}(\mathbf{r}) \Psi(\mathbf{r}) \equiv E(\mathbf{r}) \Psi(\mathbf{r}) \), with

\[ E(\mathbf{r}) = \sqrt{(\bar{\zeta}(\mathbf{r}) - \bar{\mu})^2 + |\bar{\xi}(\mathbf{r})|^2}. \]

Within this linearizing approximation, the attractive Hubbard model in the strong coupling limit is mapped onto the system of noninteracting binons and vacans described by Eq.
For the \( N_a \)-binon state of the form
\[
|\phi(k)\rangle = \frac{1}{\sqrt{N^a_a}}[a^\dagger(k)]^{N^a_a}|0\rangle,
\]
(26)
where \( a^\dagger(k) \) is the Fourier transform of \( \psi^\dagger_a(r) \), one may verify that \( |\phi(k)\rangle \) is an eigenstate of the linearized Hamiltonian \( H \). Recall that
\[
\langle \phi(k)|\phi(k')\rangle = \delta_{kk'} \int dk |\phi(k)\rangle \langle \phi(k)| = 1.
\]
On the other hand, we can prove that
\[
\langle \phi(k)|\psi^\dagger_a(r)\psi_a(r')|\phi(k)\rangle = \frac{N_a}{M}e^{-ik\cdot(r-r')}.
\]
(27)
If \( N_a \) is of the order \( O(M) \), then \( \lim_{|r-r'|\to\infty}\langle \phi(k)|\psi^\dagger_a(r)\psi_a(r')|\phi(k)\rangle = O(1) \neq 0 \), implying the existence of ODLRO in the one-particle reduced density matrix, \( \rho_1 \), of the binons in the state \( |\phi(k)\rangle \). While the existence of ODLRO in \( \rho_1 \) in Bose systems implies the macroscopic occupation of bosons at a certain quantum state (i.e., the BEC), as discussed clearly by Penrose and Onsager [15], it suggests that the existence of the BEC of binons in the attractive Hubbard model in the strong coupling limit be possible. Consequently, we could understand qualitatively the whole crossover behavior from BCS to BEC in the attractive Hubbard model with varying the coupling constant as follows. In the weak coupling limit, the system should be in the BCS regime characterizing the formation of Cooper pairs; when the coupling is smoothly increasing to intermediate values, the system goes into the crossover regime in which Cooper pairs are getting broken and the charge fluctuations somehow become dominant; when the coupling becomes very strong, the system enters into the BEC regime in which the binons and vacanons play an important role. This understanding is qualitatively consistent with the results obtained in Ref. [7]. To this end, we have shown that the bosonic quasiparticles, binons and vacanons, play an important role in the tightly bound electron systems, which may be in fact responsible for the BEC of the system. Certainly, more analytical and numerical works are needed to confirm this observation.

CONCLUDING REMARKS

We have shown in this paper that the bosonic degree of freedom of the tightly bound on-site electron pairs could be separated as Schwinger bosons which are two Bose quasiparticles named as binon and vacanon. This can be implemented by projecting the whole Hilbert space into the Hilbert subspace spanned by states of binons and vacanons subject to a constraint that they cannot occupy the same site. We argue that a binon, carrying the same charge and spin as the tightly bound on-site electron pairs, is actually a kind of quantum fluctuations of electron pairs, and a vacanon, carrying both charge and spin zero, corresponds to a vacant state. These two bosonic quasiparticles may be responsible for the BEC of the system consisting of the tightly bound electron pairs. These concepts were also applied to the attractive Hubbard model with strong coupling in continuum case, showing that the binon-vacanon picture for describing the BEC phenomenon in this special system is useful. We expect that our arguments could be applicable to the correlated electron systems.

We close this paper by discussing briefly the relevance of our argument to the existing theories concerning the BEC of electron pairs.
(1) The crossover picture [1] for high-temperature superconducting cuprates is based on such an observation that the superconducting transition temperature is proportional to the superfluid density in the underdoped regime which is supposed to be attributed to the BEC of pre-formed singlet pairs (or bosons). If this picture is sound, the dispersion relation of such bosons should depend linearly on momentum in two dimensions (see Appendix). The pre-formed bosons can be interpreted as the binons introduced in the present paper, and the superfluid density could be interpreted as $n_a$. In terms of this hypothesis, the London penetration depth obeys $\lambda^2(0)/\lambda^2(T) = 1 - (T/T_c)$, which somewhat deviates quantitatively from the standard BCS result for a $d_{x^2-y^2}$-wave superconductor [16]: $\lambda^2(0)/\lambda^2(T) \approx 1 - 0.65(T/T_c)$.

(2) In the crossover theory from the BCS to the BEC regimes in electron systems, the associated bosons could not be the tightly bound electron pairs. Instead, they may be understood as a kind of quantum fluctuations, e.g., the binons and vacanons specified in this paper.

(3) In the boson-fermion theory of superconductivity [3], the small bipolarons were considered as hard core bosons on a lattice, and were simply replaced by Bose operators in the formalism. Now, we have offered a reasonable explanation for such a replacement, namely, the hard core bosons can be reinterpreted as the binons and vacanons introduced in this paper [17].

(4) In Schafroth’s superconducting theory [18] which is being considered as a mechanism of bipolaronic theory of superconductivity [19], superconductivity arises from the BEC of charged ideal Bose gas. Based on our result, those charged bosons cannot be directly the tightly bound electron pairs, while they can be interpreted as the Schwinger bosons (e.g. binons).

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APPENDIX

In this appendix, for reader’s convenience we write down a general formula of the BEC transition temperature for an isotropic system of noninteracting bosonic quasiparticles with dispersion $\epsilon_p = \gamma |p|^\alpha$ where $p$ is the momentum, $\alpha$ and $\gamma$ are positive constants which are chosen in terms of the dimension of energy by [20]

$$T_c = \frac{(2^{1-\frac{1}{2\alpha}}\frac{\pi}{2}h)^{\alpha\gamma}}{k_B} \frac{\Gamma(d/2)n_\alpha}{\Gamma(d/2)\zeta(d/2)},$$

where $n = \langle \hat{N} \rangle / M$ is the density of the bosonic quasiparticles, $d$ is the dimensionality, $\Gamma(x)$ is the gamma function, and $\zeta(x) = \sum_{n=0}^{\infty} n^{-x}$ ($x > 1$) is the Riemann zeta function. The
condensed fraction of bosons, $n_0$, can be expressed by $n_0 = n [1 - (T/T_c)^{d/\alpha}]$. The Bernoulli equation becomes $PV = \frac{\alpha}{d} E$ with $E$ the internal energy. The specific heat for $T < T_c$ can be obtained by

$$c_v = \frac{(\frac{d}{\alpha} + 1)k_B}{2^{d-1} \pi^{d/2} \hbar^d \alpha} \frac{\Gamma(\frac{d}{\alpha} + 1) \zeta(\frac{d}{\alpha} + 1)}{\Gamma(\frac{d}{\gamma})} \left(\frac{k_BT}{\alpha}\right)^{\frac{d}{\alpha}}.$$

For $T > T_c$, one must consider the effect of the temperature-dependence of the chemical potential, $\mu = \mu(T)$, but keep the density fixed. At high temperatures, $c_v \to \frac{d}{\alpha} k_B n.$
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[8] In some literature the tightly bound local pairs were considered as a weakly nonideal Bose gas under the assumption of the very dilute limit. We believe that the approximation is not so good even for this limit, and especially, the approximations make the physical meaning of the bosons involved in the system ambiguous.

[9] Without loss of the generality, we here confine ourselves to consider only the on-site electron pairs. The intersite pairs, owing to the complexity, are left for studying elsewhere.

[10] Note that the Holstein-Primakoff transformation developed in the theory of spin waves in magnets cannot be used for this purpose, as noted long time ago in C. Kittel, Quantum Theory of Solids, (John Wiley, New York, 1963), though the tightly bound electron pairs can be regarded as the pseudospins.

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[17] At the last stage in preparing the present paper, we became aware of a very recent paper by B.K. Chakraverty, J. Ranninger and D. Feinberg [Phys. Rev. Lett. 81, 433 (1998)] in which two questions regarding the BEC of electron pairs raised in the context of questioning the possibility of bipolaronic superconductivity in oxide cuprates: (i) Is
a BEC of electron pairs at least in principle conceivable in a crystalline solid? and (ii) If so, could the possible origin for those electron pairs be of bipolaronic nature? While they argue them by invoking some experimental results, we have here offered alternative answers for these two nontrivial questions.

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FIG. 1. The hopping process of a tightly bound electron pair is associated with two excitations (i.e. binon and vacanon) with opposite hopping processes.