Nonlinear Holomorphic Supersymmetry on Riemann Surfaces

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Abstract

We investigate the nonlinear holomorphic supersymmetry for quantum-mechanical systems on Riemann surfaces subjected to an external magnetic field. The realization is shown to be possible only for Riemann surfaces with constant curvature metrics. The cases of the sphere and Lobachevski plane are elaborated in detail. The partial algebraization of the spectrum of the corresponding Hamiltonians is proved by the reduction to one-dimensional quasi-exactly solvable $\mathfrak{sl}(2,\mathbb{R})$ families. It is found that these families possess the “duality” transformations, which form a discrete group of symmetries of the corresponding 1D potentials and partially relate the spectra of different 2D systems. The algebraic structure of the systems on the sphere and hyperbolic plane is explored in the context of the Onsager algebra associated with the nonlinear holomorphic supersymmetry. Inspired by this analysis, a general algebraic method for obtaining the covariant form of integrals of motion of the quantum systems in external fields is proposed.

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1 Introduction

In the pioneer paper [1], the usual linear supersymmetry [2] was generalized by employing the higher-derivative supercharges. The characteristic property of such a generalization is the polynomiality of the corresponding superalgebra in even integrals of motion of the system. This makes the polynomial (nonlinear) supersymmetry to be similar to the Yangian and finite W-algebras [3, 4, 5]. The supersymmetry of such a type was found in various physical models [6, 7, 8, 9, 10, 11]. This provides a solid background for physical interest in the nonlinear supersymmetry.

The nonlinear holomorphic supersymmetry (n-HSUSY) [12, 13, 14, 15] is a natural generalization of the usual linear [2] and higher-derivative (polynomial) [1, 9, 10] supersymmetries. Its construction was triggered by the observation of the quantum anomaly problem, which appears under attempt to quantize the classical one-dimensional system with nonlinear supersymmetry of arbitrary order \( n \in \mathbb{N}, n > 1 \) [9]. The important result in resolving this problem was obtained in Ref. [12], where we showed that the anomaly-free quantization is possible only for the peculiar class of the superpotentials, for which the corresponding quantum n-HSUSY turns out to be directly related to the quasi-exactly solvable (QES) systems [17, 18, 19]. Such one-dimensional quantum mechanical systems were also independently discussed within the framework of the so called Type A \( N \)-fold supersymmetry [20, 21, 22, 23, 24], where, in particular, the equivalence between the nonlinear supersymmetry and the \( \mathfrak{sl}(2, \mathbb{R}) \) scheme for the 1D QES systems was demonstrated [22].

On the other hand, nowadays, the two-dimensional dynamics attracts a considerable attention in the context of the both, linear [2, 25, 26, 27], and nonlinear [28, 29, 30], supersymmetries. The particularly interesting 2D system, related to the quantum Hall effect [31, 32, 33], is the non-relativistic charged spin-1/2 particle in a stationary magnetic field [34, 35, 25]. In Ref. [13], we generalized the n-HSUSY to the 2D case represented by the system of a charged spin-1/2 particle with gyromagnetic ratio \( 2n \) (or, spin-n/2 particle with gyromagnetic ratio 2) moving in the plane with magnetic field of a specific form. The most important result of such a generalization was the observation of some universal nonlinear algebraic relations underlying the n-HSUSY.

In the recent paper [15], the nonlinear relations of Ref. [13] were identified as the Dolan-Grady relations [36]. As a result, the construction of the n-HSUSY was reduced to the following universal (representation-independent) algebraic structure. Its supercharges have the form of (anti)holomorphic polynomials in the two mutually conjugate operators (generating elements), in whose terms all the components of the n-HSUSY construction are built. It is these generating elements that obey the Dolan-Grady relations and produce the associated infinite-dimensional Onsager algebra [37]. The knowledge of the universal algebraic construction essentially facilitated the search for the central charges of the n-HSUSY, and simplified the calculation of its nonlinear superalgebra. Having it, we proposed a generalization of the n-HSUSY in the form of nonlinear pseudo-supersymmetry [15], which is related to the \( PT \)-symmetric quantum mechanics [38, 39, 40].

The power of the algebraic formulation of the n-HSUSY [17] is that it does not refer to the nature of the space-time manifold which can be lattice, continuum or noncommutative space. In this sense the n-HSUSY is a direct algebraic generalization of the usual quantum-mechanical supersymmetry [2].
The present paper is devoted to investigation of the quantum mechanical problem of realization of the \( n \)-HSUSY on Riemann surfaces with an inhomogeneous magnetic field. Applying the algebraic construction of Ref. \[15\], we find a necessary condition for such a realization, and discuss the spectral problem in the two possible (non-trivial) cases of the sphere and Lobachevski plane subjected to an external axial magnetic field of a peculiar form. Reducing the \( n \)-HSUSY systems to one dimension, we find the new discrete symmetry of the 1D QES potentials, which induces the “duality” transformations between the associated distinct \( \mathfrak{sl}(2, \mathbb{R}) \) schemes. The analysis of the axial symmetry of the 2D \( n \)-HSUSY systems in terms of the intrinsic algebra generated by the covariant derivatives allows us to propose (beyond the context of supersymmetry) the general algebraic method for obtaining the covariant form of integrals of motion of quantum systems in a curved space in the presence of an external gauge field.

The paper is organized as follows. In Section 2 we discuss the realization of the \( n \)-HSUSY on Riemann surfaces with a magnetic field, and find the zero modes corresponding to the cases of the sphere and Lobachevski plane. In Section 3 we demonstrate that the spectral problem for these two cases can be partially solved by a reduction to one dimension. The obtained 1D systems admit distinct \( \mathfrak{sl}(2, \mathbb{R}) \) representations, that is reflected in existence of the discrete symmetry of the potentials. This symmetry is investigated in Section 4. In Section 5 we study the algebraic content of the systems on the sphere and hyperbolic plane in the context of the associated infinite-dimensional contracted Onsager algebra \[15\]. Section 6 develops the algebraic approach for finding the covariant form of the integrals of motion of the quantum systems in an external field with symmetry. In Section 7 we discuss the obtained results and specify some problems to be interesting for further consideration.

2 \( n \)-HSUSY and Riemann surfaces

2.1 \( n \)-HSUSY algebraic structure

The system with nonlinear holomorphic supersymmetry is described by the Hamiltonian \[15\]
\[
\mathcal{H}_n = \frac{1}{4} \{ \bar{Z}, Z \} + \frac{n}{4} [Z, \bar{Z}] \sigma_3. \tag{2.1}
\]
Here \( \sigma_3 \) is the diagonal Pauli matrix, \( n \in \mathbb{N} \), and the mutually conjugate operators \( Z \) and \( \bar{Z} \) obey the nonlinear Dolan-Grady relations
\[
[Z, [Z, [Z, \bar{Z}]]] = \omega^2 [Z, \bar{Z}], \quad [\bar{Z}, [\bar{Z}, [Z, \bar{Z}]]] = \bar{\omega}^2 [Z, \bar{Z}]. \tag{2.2}
\]
These relations guarantee the existence of the odd integrals of motion (supercharges), \( Q_n \) and \( \bar{Q}_n \), which are defined by the recurrent relations
\[
Q_n = \left( \bar{Z}^2 - \left( \frac{n-1}{2} \right)^2 \omega^2 \right) Q_{n-2}, \quad Q_0 = \sigma_+ , \quad Q_1 = Z \sigma_+, \tag{2.3}
\]
\[
\bar{Q}_n = \left( Z^2 - \left( \frac{n-1}{2} \right)^2 \bar{\omega}^2 \right) \bar{Q}_{n-2}, \quad \bar{Q}_0 = \sigma_- , \quad \bar{Q}_1 = \bar{Z} \sigma_- ,
\]
containing a constant parameter \( \omega \in \mathbb{C}, \bar{\omega} = \omega^*, \) and \( \sigma_\pm = \frac{1}{2} (\sigma_1 \pm i \sigma_2) \). The supercharges \( Q_n, \bar{Q}_n \) generate a nonlinear superalgebra of order \( n \) \[15\]. In Section 5 we shall return to the
discussion of the algebraic construction underlying the nonlinear holomorphic supersymmetry.

2.2 Fixing geometry

We are going to investigate the problem of realization of the $n$-HSUSY on a 2D surface with a nontrivial Riemann geometry. The metrics of any two-dimensional surface can be represented locally in the conformally flat form,

$$ds^2 = \rho(z, \bar{z}) dz d\bar{z} = g_{z\bar{z}} dz d\bar{z} + g_{zz} d\bar{z} dz,$$

where $z = x^1 + ix^2$, $\bar{z} = x^1 - ix^2$ are the isothermal coordinates. Given the isothermal coordinates on the neighbouring patches, the metrics on the patches are related by conformal transformations. Hence, we have

$$g_{z\bar{z}} = \frac{\rho}{2}, \quad g_{z\bar{z}} = \frac{2}{\rho}, \quad g_{zz} = g_{\bar{z}\bar{z}} = 0, \quad \sqrt{g} = \rho,$$

where $g = \det g_{\mu\nu}$.

Let us choose the operators $Z$ and $\bar{Z}$ in the form of covariant derivatives,

$$Z = \nabla_z = D + \partial \log \rho \cdot \hat{S}, \quad \bar{Z} = -\nabla_{\bar{z}} = -g_{z\bar{z}} \bar{D},$$

where the operator $\hat{S}$ defines the helicity, or the conformal spin of a tensor $T^{(s)}$, $\hat{S} T^{(s)} = s T^{(s)}$, $s \in \mathbb{Z}$, while $D = \partial + i\hat{e} A$ and $\bar{D} = \bar{\partial} + i\hat{\bar{e}} \bar{A}$ are the $U(1)$ covariant derivatives with $A$, $\bar{A}$ to be the components of the vector potential corresponding to the case of stationary magnetic field. The operator $\hat{e}$ gives the electric charge of the tensor field. By definition, the derivative $\nabla_z$ maps the tensor fields of the helicity $s$ into those of the helicity $s - 1$, while $\nabla_{\bar{z}}$ transforms the tensors of the helicity $s$ into those of the helicity $s + 1$. The operators (2.5) are mutually conjugate with respect to the scalar product

$$\langle T^{(s)}, U^{(s')} \rangle = \int d^2 z \sqrt{g} (g_{z\bar{z}})^s T^{(s)} U^{(s')}$$

defined on the space of the tensor fields satisfying the appropriate boundary conditions.

In representation (2.5) we have

$$[Z, \bar{Z}] = B \hat{e} - \frac{1}{2} \mathcal{R} \hat{S},$$

where the scalar curvature of the Riemann surface $\mathcal{R}$ and the magnetic field $B$ are

$$\mathcal{R} = -\frac{4}{\rho} \partial \bar{\partial} \ln \rho, \quad B \hat{e} = -\frac{2}{\rho} [D, \bar{D}].$$

As a result, the nonlinear Dolan-Grady relations (2.2) take the form

$$\left((\nabla_z^2 B) - \omega^2 B\right) \hat{e} - \frac{1}{2} \left((\nabla_z^2 \mathcal{R}) - \omega^2 \mathcal{R}\right) \hat{S} - (\nabla_z \mathcal{R}) \nabla_z = 0,$$

$$\left((\nabla_{\bar{z}}^2 B) - \bar{\omega}^2 B\right) \hat{e} - \frac{1}{2} \left((\nabla_{\bar{z}}^2 \mathcal{R}) - \bar{\omega}^2 \mathcal{R}\right) \hat{S} + (\nabla_{\bar{z}} \mathcal{R}) \nabla_{\bar{z}} = 0.$$
The relations (2.9) have to be satisfied as the operator identities. Hence, the coefficients at the operators $\hat{e}, \hat{S}, \nabla_z$ and $\nabla\bar{z}$ have to vanish independently. The coefficients at the covariant derivatives vanish only when the scalar curvature is a constant,

$$\mathcal{R} = \text{const.} \quad (2.10)$$

With this condition, the coefficients at the operator $\hat{S}$ result in the equation $\omega^2 \mathcal{R} = 0$, and, hence, there are two possibilities: either $\omega = 0$, or $\mathcal{R} = 0$. The latter case corresponds to the plane geometry, and it was analysed in detail in Ref. [13]. Therefore, in what follows we shall discuss the former case.

For $\omega = 0$, there arise the contracted Dolan-Grady relations,

$$[Z, [Z, [Z, \bar{Z}]]] = 0, \quad [\bar{Z}, [\bar{Z}, [Z, \bar{Z}]]] = 0. \quad (2.11)$$

The nonlinear supersymmetry associated with the contracted Dolan-Grady relations (2.11) is generated by the supercharges of the form

$$Q_n = Z^n \sigma_+, \quad \bar{Q}_n = \bar{Z}^n \sigma_- \quad (2.12)$$

corresponding to the limit $\omega \to 0$ of the general case (2.3).

One notes that, in principle, the restriction (2.10) can be overcome if to endow the Riemann structure of the 2D surface with the torsion. Then, for $\omega \neq 0$, the Dolan-Grady relations (2.2) will result in a set of nonlinear equations on the scalar curvature and on the torsion. However, here we restrict ourselves by the torsion-free case with the relation (2.10) to be the necessary condition for existence of the $n$-HSUSY on a Riemann surface.

The general uniformization theorem [42] implies that, up to the holomorphic equivalence, there are just three distinct simply connected Riemann surfaces: a) the sphere $S^2$ with the conformal factor

$$\rho(z, \bar{z}) = \frac{1}{(1 + \beta^2 z \bar{z})^2}, \quad (2.13)$$

where $z \in \mathbb{C} \cup \{\infty\}$; b) the complex plane $\mathbb{C}$ with the flat metric, $\rho = 1$; c) the Lobachevski (hyperbolic) plane with the metric (2.4) defined on the disk $|z| < \beta^{-1}$ by

$$\rho(z, \bar{z}) = \frac{1}{(1 - \beta^2 z \bar{z})^2} \quad (2.14)$$

In Eqs. (2.13), (2.14), the positive parameter $\beta$ defines the value of the scalar curvature, $\mathcal{R} = \pm 8 \beta^2$, where the plus corresponds to the sphere, while the minus does to the hyperbolic plane. In the case of the sphere the parameter $\beta$ is related to its radius as $\beta^{-1} = 2R$.

In what follows we will mainly analyse the case of the sphere, while all the corresponding results for the Lobachevski plane can formally be obtained by the change $\beta \to i\beta$. At the same time, the construction of the $n$-HSUSY on the plane [13] can be reproduced from the cases corresponding to the surfaces (2.13) and (2.14) via the appropriate limit procedure $\beta \to 0$.
2.3 Fixing magnetic field

We have not analysed yet the conditions corresponding to the vanishing of the coefficients at $\hat{e}$. They produce the equations on the external magnetic field. In correspondence with Eq. (2.8), the magnetic field is given locally by the relation

$$Bdv = dA,$$

where $dv = (i\sqrt{g/2})dz \wedge d\bar{z}$ is the area (volume) element, and $A = Adz + \bar{A}d\bar{z}$. In correspondence with (2.9), for $\omega = 0$ we have the following equations on the magnetic field $B = -ig^{\bar{z}z}(\partial\bar{A} - \bar{\partial}A)$:

$$\partial \left( \frac{1}{\rho(z, \bar{z})} \partial B(z, \bar{z}) \right) = 0, \quad \bar{\partial} \left( \frac{1}{\rho(z, \bar{z})} \bar{\partial} B(z, \bar{z}) \right) = 0.$$

For the both cases (2.13) and (2.14), their solution can be represented in the form

$$B(z, \bar{z}) = c_1 \sqrt{\rho(z, \bar{z})} + c_0,$$

with $c_0, c_1 \in \mathbb{R}$. In the case of the sphere the (normalized for $2\pi$) total magnetic flux,

$$2\pi \Phi = \int_{S^2} Bdv = 2\pi \int_0^\infty B(r)\rho(r)rdr = \frac{\pi (2c_0 + c_1)}{2\beta^2},$$

is quantized, $\Phi \in \mathbb{Z}$. Here we have used the polar representation of the complex coordinates, $z = re^{i\phi}$, $\bar{z} = re^{-i\phi}$.

In the case of the hyperbolic plane, the magnetic flux is divergent, but the relation (2.16) (with the substitution $\beta^2 \to -\beta^2$) still can be used in the form $\Phi = -(2c_0 + c_1)/(4\beta^2)$ as a mere combination of the parameters which can acquire any real value.

In what follows, we fix the charge of the system to be $e = +1$.

2.4 Zero modes

Let us find the zero mode subspace of the $n$-HSUSY realized on the Riemann surfaces (2.13), (2.14) with magnetic field (2.15). To this end we note that the space of the states of the system (2.1) is a direct sum of the two Hilbert spaces, $F_- \oplus F_+$, formed by the states $\Psi^- \in F_-$ and $\Psi^+ \in F_+$, being the eigenstates of $\sigma_3$, $\sigma_3\Psi^\pm = \pm\Psi^\pm$. We will refer to the spaces $F_-$ and $F_+$, respectively, as to the “bosonic” and “fermionic” subspaces. We also assume that the “bosonic” wave functions are the tensors of helicity $s$. Then, in accordance with the structure of the supercharges (2.12), the “fermionic” wave functions will be the tensors of the helicity $s - n$.

The Hamiltonian acting on the “bosonic” wave functions can be rewritten in the complex coordinates as

$$H_n^{(s)} = -\frac{1}{2\rho} \left( \{\hat{D}, D\} + 2s\partial\log\rho \cdot \hat{D} \right) - \frac{n}{4}B - s(n + 1)\beta^2.$$
The upper index in the parentheses denotes the helicity of the tensor that the Hamiltonian acts on. In these notations the “fermionic” Hamiltonian is $H^{(s-n)}_n$.

In the gauge

$$A = -i \frac{\bar{z}}{8} (c_1 \rho + 4 \beta^2 \Phi \sqrt{\rho}) \quad \text{and} \quad \bar{A} = i \frac{z}{8} (c_1 \rho + 4 \beta^2 \Phi \sqrt{\rho}),$$

(2.18)

corresponding to the magnetic field (2.15), (2.16), the states of the form

$$\psi_n^{(-)}(z, \bar{z}) = \rho^{\frac{s}{2} - \frac{n}{2} + \frac{n-k}{2}} e^{s \beta \sqrt{\rho}} \sum_{k=0}^{n-1} \varphi_k^{(-)}(z) z^k,$$

(2.19)

$$\psi_n^{(+)}(z, \bar{z}) = \rho^{\frac{n}{2} - \frac{n-k}{2}} e^{-s \beta \sqrt{\rho}} \sum_{k=0}^{n-1} \varphi_k^{(+)}(z) z^k$$

are the zero modes of the supercharges $Q_n$ and $\bar{Q}_n$, respectively. In the case of the sphere, the condition of normalizability fixes the form of the functions $\varphi_k^{(+)}$, $\varphi_k^{(-)}$,

$$\varphi_k^{(-)}(\bar{z}) = \sum_{q=0}^{2(n-s\Phi-1)-k} \varphi_{kq}^{(-)} \bar{z}^q, \quad \varphi_k^{(+)}(\bar{z}) = \sum_{q=0}^{2(s\Phi-1)-k} \varphi_{kq}^{(+)} \bar{z}^q,$$

(2.20)

where we have introduced the notation

$$s_{\Phi} = s - \frac{1}{2} \Phi.$$

(2.21)

The physical sense of the parameter $s_{\Phi}$ will be found below under analysis of the contracted Onsager algebra associated with the system. The corresponding number of the normalizable zero modes in the “bosonic” and “fermionic” sectors is given by

$$N_B = \frac{1}{2} n (3n - 4s_{\Phi} - 1), \quad N_F = - \frac{1}{2} n (n - 4s_{\Phi} + 1).$$

Here we assume that $N_B, N_F \in \mathbb{N}$; for other values of the parameters $n, s$ and $\Phi$ the corresponding number of zero modes is implied to be equal to zero. One can verify that in the spheric case $N_B + N_F > 0$ for any $n \in \mathbb{N}$, and, therefore, the nonlinear supersymmetry of the corresponding system is always unbroken.

In the case of the hyperbolic plane, the normalizability is managed by the sign of the parameter $c_1$, and the corresponding space of zero modes is infinite-dimensional. Moreover on the hyperbolic plane, unlike the sphere case, the supercharges $Q_n$ and $\bar{Q}_n$ can not have zero modes simultaneously.

So, the nonlinear holomorphic supersymmetry of quantum-mechanical systems on 2D surfaces leads to the strong restrictions on configurations of the external geometry and magnetic field. Moreover, the systems with the $n$-HSUSY on the sphere are quasi-exactly solvable since the space of zero modes of the supercharges is finite-dimensional in this case. In the next section we will demonstrate that the same is also true for the systems on the hyperplane.
3 Reduction to $\mathfrak{sl}(2,\mathbb{R})$ quasi-exactly solvable families

In this section we consider the spectral problem for the “bosonic” Hamiltonian (2.17) and show that it can be reduced to a one-dimensional problem corresponding to $\mathfrak{sl}(2,\mathbb{R})$ quasi-exactly solvable families of the systems. The spectral problem in the “fermionic” sector can be analysed in a similar way.

First, we note that the functions of the form $z^m f(\bar{z}z)$, $m \in \mathbb{Z}$, constitute a subspace invariant with respect to the action of the Hamiltonian (2.17) in the gauge (2.18). The parameter $m$, as will be shown, is the quantum number corresponding to the integral (5.6) associated with the axial symmetry of the system. Departing from the form of zero modes (2.19), we search for the eigenfunctions of the form

$$\psi(z, \bar{z}) = z^m (\bar{z}z)^a \rho(\bar{z}z)^b e^{c \sqrt{\rho(\bar{z}z)}} \varphi(\bar{z}z).$$  \hspace{1cm} (3.1)

Now, we introduce the variable

$$y = \sqrt{\rho(\bar{z}z)}$$

with the domain $0 \leq y \leq 1$. The anzatz (3.1) reduces the two-dimensional Hamiltonian (2.17) to the one-dimensional operator

$$H_{\mathfrak{sl}(2,\mathbb{R})} = 2\beta^2 \left( T_0^2 - T_- T_+ \right) + \alpha_+ T_+ + \alpha_0 T_0 + \alpha_- T_- + \text{const}$$  \hspace{1cm} (3.2)

if the parameters $a$, $b$ and $c$ are chosen in the following way:

$$a = \frac{\sigma - 1}{2} m, \quad b = \frac{\nu}{2} s_\Phi - \frac{s}{2} + \frac{\sigma - \nu}{4} m, \quad c = \mu \frac{c_1}{8\beta^2}.  \hspace{1cm} (3.3)$$

The parameters $\mu$, $\nu$ and $\sigma$ acquire the values $\pm 1$, while the operators

$$T_+ = (N - 1)y - y^2 \frac{d}{dy}, \quad T_0 = y \frac{d}{dy} - \frac{N - 1}{2}, \quad T_- = \frac{d}{dy}$$  \hspace{1cm} (3.4)

generate the $\mathfrak{sl}(2,\mathbb{R})$ algebra with the Casimir operator fixed by the parameter $N$. When $N \in \mathbb{N}$, the realization (3.4) gives an irreducible $N$-dimensional representation on the space of polynomials of the degree up to $N - 1$. In this case the operator (3.2) corresponds to some QES system \[16, 17, 18\].

The corresponding parameters of the $\mathfrak{sl}(2,\mathbb{R})$ scheme, $N$, $\alpha_\pm$ and $\alpha_0$, read as

$$N = \mu n + \frac{\nu - \sigma}{2} m - (\mu + \nu)s_\Phi,$$

$$\alpha_+ = -\mu \frac{c_1}{2},$$

$$\alpha_0 = \beta^2 \left(2\mu n - (\nu - \sigma) m - (\mu - \nu) 2s_\Phi \right) - \mu \frac{c_1}{2},$$

$$\alpha_- = \beta^2 \left(1 - \mu n + \frac{3\nu + \sigma}{2} m + (\mu - 3\nu) s_\Phi \right).$$  \hspace{1cm} (3.5)

So, we have formally $2^3 = 8$ solutions parametrised by the triple $(\mu, \nu, \sigma)$. Of course, the solutions (3.3) realize a true quasi-exactly solvable system only if the condition $N \in \mathbb{N}$ is
satisfied. For example, the solution with the triple \((-1, +1, +1)\) does not correspond to any QES system since in this case \(N = -n\), while throughout the paper we assume \(n \in \mathbb{N}\). But as it will be shown further, this solution together with the other solutions reflects some discrete symmetry of the resulting one-dimensional system. We also note that under the change \((\mu, \nu, \sigma) \rightarrow (-\mu, -\nu, -\sigma)\) in (3.3), the parameter \(N\) changes its sign. This means that for any choice of the parameters \(n, m\) and \(s\), at most four solutions from (3.3) can realize the proper finite-dimensional \(\mathfrak{sl}(2, \mathbb{R})\) representations of the QES system.

From the 2D viewpoint, the conditions of regularity at zero and of finiteness at infinity\(^1\) for the functions (3.1), (3.3) lead, correspondingly, to the constraints
\[
\sigma m \geq 0, \quad \nu m \leq 2(\nu s - s). \tag{3.6}
\]
The condition of normalizability of the wave functions results, in turn, in the inequality
\[
\nu (2s - m) + 1 > 0. \tag{3.7}
\]
In all the relations (3.6), (3.7) we imply that the parameters \(m, s\) and \(\Phi\) accept integer values only.

Changing the variable,
\[
y = \cos^2 \beta x,
\]
with \(0 \leq x \leq \frac{\pi}{2\beta}\), and following the standard procedure of the \(\mathfrak{sl}(2, \mathbb{R})\) scheme for QES systems [10, 11, 12], one can transform the operator (3.3) to the one-dimensional Hamiltonian of the standard form,
\[
H = -\frac{1}{2} \frac{d^2}{dx^2} + V(x).
\]
All the solutions (3.3) lead to the very potential
\[
V(x) = -\frac{c_0^2}{32\beta^2} \cos^4 \beta x + \frac{c_1}{4} \left(2(s - n) + \frac{c_1}{8\beta^2}\right) \cos^2 \beta x + \frac{\beta^2}{8} \left(4m^2 - 1\right) \cot^2 \beta x + \frac{\beta^2}{8} \left(4(2s - m)^2 - 1\right) \tan^2 \beta x. \tag{3.8}
\]
This means that the potential (3.8) possesses some discrete symmetry, which relates the solutions (3.5). We discuss this symmetry in the next section.

The potential (3.8) has the following behaviour near the point \(x = 0\):
\[
V(x) = \text{const} + \frac{4m^2 - 1}{8x^2} + \mathcal{O}(x^2).
\]
Therefore, in the case \(m = \pm \frac{1}{2}\), it is formally possible to extend the domain of definition of \(x\) to the symmetric interval \(-\frac{\pi}{2\beta} \leq x \leq \frac{\pi}{2\beta}\) (or to \(\mathbb{R}\) in the case of the hyperbolic plane, see below).

The case of the plane corresponds to the limit \(\beta \to 0\). Rescaling and shifting the parameters,
\[
c_0 \to c_0 + \frac{c_1}{\beta^2}, \quad c_1 \to -\frac{c_1}{\beta^2},
\]
\(^1\)Since we consider the sphere, it is necessary to require the regularity of all the wave functions for \(|z| \to \infty\).
in this limit we recover from Eq. (2.15) the magnetic field of the form

\[ B(r) = c_1 r^2 + c_0, \quad r^2 = \bar{z}z, \]

and find that the described reduction results in the one-dimensional QES potential

\[ V(x) = \frac{c_1^2}{32} x^6 + \frac{c_1 c_0}{8} x^4 + \frac{1}{8} \left( c_0^2 - 2c_1 (2n - m) \right) x^2 + \frac{4m^2 - 1}{8x^2} + \text{const}, \quad (3.9) \]

defined on the half-line \( 0 \leq x < \infty \). Hence, the proper limit procedure reproduces correctly the results derived earlier in Ref. [13].

The formulas corresponding to the case of the hyperbolic plane are obtained by the formal substitution \( \beta \to i\beta \) with the change of the domains of the variables \( y \) and \( x \) to \( 1 \leq y < \infty \) and \( 0 \leq x < \infty \). Under such transformations, the trigonometric potential \((3.8)\) is converted into the hyperbolic QES potential

\[ V(x) = \frac{c_1^2}{32\beta^2} \cosh^4 \beta x + \frac{c_1}{4} \left( 2(s_\Phi - n) - \frac{c_1}{8\beta^2} \right) \cosh^2 \beta x \]

\[ + \frac{\beta^2}{8} \left( 4m^2 - 1 \right) \coth^2 \beta x + \frac{\beta^2}{8} \left( 4 \left( 2s_\Phi - m \right)^2 - 1 \right) \tanh^2 \beta x. \quad (3.10) \]

The condition of regularity at zero is the same as for the sphere, \( \sigma m \geq 0 \), while the condition of finiteness at infinity has to be discarded. Besides, the normalizability of the functions \((3.1)\), \((3.3)\) is changed for \( \mu c_1 > 0 \) instead of \((3.7)\). Since the magnetic flux diverges in this case, the parameter combination \( \Phi \), specified at the very end of Section 2.3, can acquire any real value. Therefore, in general, only the solutions \((+1, -1, +1), (+1, -1, -1)\) and \((-1, +1, -1)\) can produce the proper finite-dimensional \( sl(2, \mathbb{R}) \) representations, while other values of the triple \((\mu, \nu, \sigma)\) serve only for integer values of the parameter \( \Phi \).

### 4 “Duality” transformations

In the previous section we have seen that the reduction of the two-dimensional Hamiltonian, associated with the \( n \)-HSUSY on the sphere or hyperbolic plane with magnetic field \((2.15)\), results in the one-dimensional quantum system with the potential \((3.8)\) or \((3.10)\), respectively. The resulting 1D Hamiltonians can be transformed to the QES operator \((3.2)\) with appearance of the eight different families of the \( sl(2, \mathbb{R}) \) schemes. Here we investigate the question what these different families of the \( sl(2, \mathbb{R}) \) schemes mean from the point of view of the corresponding one-dimensional quantum mechanical QES system.

First, we note that the potential \((3.3)\) is effectively four-parametric since the parameter \( \beta \) can be absorbed by rescaling \( x \to \beta^{-1} x, c_1 \to \beta^2 c_1, H \to \beta^{-2} H \), while \( s \) and \( \Phi \) enter only in the combination \((2.21)\). The parameters \( n, 2s_\Phi \) and \( m \) take integer values, and \( c_1 \) is real. These parameters enter into the potential \((3.3)\) in quadratic combinations. This motivates us to search for the linear transformations of the parameters, which leave the potential to be invariant. It is convenient to represent them in the matrix form

\[ Y' = GY, \quad (4.1) \]
with \( Y^T = (n, 2s_0, m, c_1) \). Let us introduce the following matrices:

\[
G_1 = \begin{pmatrix}
-1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}, \quad G_2 = \begin{pmatrix}
1 & -1 & 1 & 0 \\
0 & -1 & 2 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad G_3 = \begin{pmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & -2 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}. \tag{4.2}
\]

One can verify that the potential \( (3.8) \) stands invariant under the transformations \( (4.1) \) generated by the set of the mutually commuting matrices \( G \) generated by the set of the discrete transformations \( (4.1) \) with \( Y \). Different solutions \( (3.5) \) at the level of the \( \text{sl}_2 \) give some irreducible representation of the Abelian discrete group of reflections in three axes.

Another nontrivial example of such discrete transformations is given by the QES potential \( (3.9) \) related to the \( \text{sl}(2, \mathbb{R}) \)-operator

\[
H_{QES} = -2T_0T_+ + \beta_+T_+ + \beta_0T_0 + \beta_-T_-, \tag{4.5}
\]

where the \( \text{sl}(2, \mathbb{R}) \)-generators are given by Eq. \( (3.4) \) with \( y = x^2 \) \([16, 43]\). The complete set of the discrete transformations \( (4.1) \) with \( Y^T = (n, m, c_0, c_1) \) is given in this case by the matrices \( \mathbb{I}, -\mathbb{I}, \tilde{G}, -\tilde{G} \), where

\[
\tilde{G} = \begin{pmatrix}
1 & -1 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

In comparison with \( (3.8) \), the case \( (3.3) \) with \( n, m \in \mathbb{Z}, c_0, c_1 \in \mathbb{R} \), is more simple, and, hence, more illustrative. Let us discuss here the corresponding eigenfunctions, which can be found algebraically. One can represent them formally as

\[
\psi(x) = P_{N-1} \left( x^2 \right) x^\frac{1}{2} \exp \left( -\frac{\mu}{16} c_1 x^4 - \frac{\mu}{8} c_0 x^2 \right), \quad \text{with} \quad N = \mu n - \frac{\mu + \nu}{2} m, \tag{4.6}
\]

and \( \mu, \nu \) acquiring the values \( \pm 1 \). For the functions \( (4.6) \) to be normalized, one has to impose some restrictions on the parameters. There are four distinct cases, which can be marked by the pair \( (\mu, \nu) \):

\[
\begin{align*}
(+1, -1) & : N = n \in \mathbb{N}, \quad c_1 > 0, \quad m \in \mathbb{Z}_-; \\
(+1, +1) & : N = n - m \in \mathbb{N}, \quad c_1 > 0, \quad m \in \mathbb{Z}_+; \\
(-1, +1) & : N = -n \in \mathbb{N}, \quad c_1 < 0, \quad m \in \mathbb{Z}_+; \\
(-1, -1) & : N = m - n \in \mathbb{N}, \quad c_1 < 0, \quad m \in \mathbb{Z}_-.
\end{align*}
\]
The restrictions on $c_1$ arise from the requirement of the normalizability, while those on $m$ do from the requirement of the vanishing of the functions at $x = 0$, since we consider the spectral problem on the half-line, $0 \leq x < \infty$. In fact, the cases $(+1, -1)$, $(−1, +1)$ correspond to the conventional $\mathfrak{sl}(2, \mathbb{R})$ representation of the potential (3.9) (see, e.g., Refs. [17, 43]), while those $(+1, +1)$, $(−1, −1)$ were not discussed earlier. All the four cases are related by the discrete transformations as it is reflected on the diagram:

$(+1, −1) \xrightarrow{\tilde{G}} (+1, +1)$

$(−1, +1) \xrightarrow{\tilde{G}} (−1, −1)$

In principle, one can treat the system (3.9) from the more general point of view by assuming that all the parameters $n$, $m$, $c_0$ and $c_1$ are real. In this case the Hamiltonian can also be reduced to the form (4.5), but if there are no additional restrictions on the parameters, the $\mathfrak{sl}(2, \mathbb{R})$ generators realize an infinite-dimensional representation of the algebra. As a consequence, the 1D system with the potential (3.9) is not quasi-exactly solvable. Now, if any of the restrictions $\pm n \in \mathbb{N}$ or $\pm (n − m) \in \mathbb{N}$ is imposed, the 1D system under consideration becomes to be quasi-exactly solvable since with such a restriction the corresponding representation of the $\mathfrak{sl}(2, \mathbb{R})$ is finite-dimensional. For example, when $m = \frac{1}{2}$, the domain of definition of the potential (3.9) can be extended to the whole real axis. Let $c_1 > 0$, then the pair $(+1, −1)$ with $N = n \in \mathbb{N}$ gives even eigenfunctions (4.6), while the pair $(+1, +1)$ with $N = n − \frac{1}{2} \in \mathbb{N}$ provides the odd eigenfunctions. These two cases correspond to the two distinct forms of the QES $x^0$-potential [17].

Having the two examples, one can suppose that the existence of the discrete symmetry transformations is general for all the one-dimensional QES $\mathfrak{sl}(2, \mathbb{R})$ systems. Let us give some simple arguments in favour of this conjecture.

Changing appropriately the variable and realizing the similarity transformation, one can represent any QES $\mathfrak{sl}(2, \mathbb{R})$ Hamiltonian in the canonical form [10, 11, 12, 13]

$$H_{QES} = -P_4(y) \frac{d^2}{dy^2} + \left(\frac{N - 1}{2} P_4'(y) - P_2(y)\right) \frac{d}{dy} + \frac{N}{2} \left(P_2'(y) - \frac{N - 1}{6} P_4''(y)\right),$$

(4.7)

where $P_k(y)$ is a polynomial of the $k$th degree. The potential of the corresponding Schrödinger equation is given by

$$V(x) = \frac{N(N + 2)}{12} \left(\frac{3}{4} P_4'^2 - P_4''\right) + \frac{N + 1}{4} \left(2 P_2' - P_2 P_4' \frac{P_4'}{P_4}\right) + \frac{1}{4} \frac{P_2^2}{P_4} + \text{const},$$

(4.8)

where the right hand side is evaluated at $y = f^{-1}(x)$ with $f^{-1}$ being the inverse function of elliptic integral

$$f(y) = \int \frac{dy}{\sqrt{P_4(y)}}.$$

The canonical form (4.7) is not unique because of the existence of a “residual” symmetry, which allows us to fix essentially the polynomial $P_4$ [19]. Therefore, one can consider that
coefficients of the polynomial $P_2$ and the discrete parameter $N$ span the whole parametric space of the system. From (4.8) it follows that the parameters enter into the potential in quadratic combinations (see the explicit set of QES potentials, e.g., in Ref. [43]). Therefore, appealing to the cases (3.8) and (3.9), it seems plausible that the transformations of the form (4.1) should also exist in the general case (4.8). Of course, in general, for the transformed parameter $N$ to remain integer, one has to treat some parameters as to be also integer.

Let us stress that if one treats a 1D QES system in the framework of the corresponding 2D supersymmetric system, then the discrete transformations can intertwine the distinct sectors of the spectrum of the same 2D system, or the spectra of different 2D systems. This property as well as the relation $G^2 = 1$ allows us to treat (4.1) as some kind of “duality” transformations. For example, one can verify that the zero modes (2.19), (2.20) are spanned by the solutions (3.3) with the triples $(+1, -1, +1)$ and $(+1, -1, -1)$, which are related by the “duality” transformation generated by the $G_3$ from Eq. (4.2).

In conclusion of this section we note that in the context of QES systems, some similar duality transformations were discussed in Refs. [44, 45, 46]. However, the “duality” transformations discussed here are essentially different from the transformations of Refs. [44, 45, 46]. In the latter case the duality connects different 1D QES systems, while in the present case it relates different parts of the spectrum of one and the same 1D QES system. These parts correspond to different $\mathfrak{sl}(2, \mathbb{R})$ schemes of the given potential.

## 5 The contracted Onsager algebra

In this section, following Ref. [15], we analyse the algebraic structure underlying the 2D spherical and hyperbolic systems with the nonlinear holomorphic supersymmetry.

The operators $Z_0 \equiv Z$ and $\bar{Z}_0 \equiv \bar{Z}$ together with the contracted Dolan-Grady relations (2.11) recursively generate the infinite-dimensional contracted Onsager algebra:

\[
\begin{align*}
[Z_k, \bar{Z}_l] &= B_{k+l+1}, \\
[Z_k, B_l] &= Z_{k+l}, \\
[B_k, \bar{Z}_l] &= \bar{Z}_{k+l}, \\
[Z_k, Z_l] &= 0, \\
[Z_k, \bar{Z}_l] &= 0, \\
[B_k, B_l] &= 0,
\end{align*}
\]

(5.1)

where $k, l \in \mathbb{Z}_+$ and $B_0 = 0$ is implied. In general, the algebra (5.1) admits the infinite set of the commuting charges [15],

\[
\mathcal{J}_n^I = \frac{1}{2} \sum_{p=1}^{l} \left( \{ \tilde{Z}_{p-1}, Z_{l-p} \} - B_p B_{l-p} \right) + \frac{n}{2} B_l \sigma_3,
\]

(5.2)

which includes the Hamiltonian (2.1), $\mathcal{J}_n^I = 2 \mathcal{H}_n$.

The operators $Z, \bar{Z}$ in the representation (2.3) with the conformal factor (2.13) corresponding to the sphere generate the following finite-dimensional algebra which we call the intrinsic:

\[
\begin{align*}
[Z, \bar{Z}] &= G - 4\beta^2 S_\Phi, \\
[Z, G] &= -4\beta^2 \mathcal{D}, \\
[\bar{Z}, G] &= 4\beta^2 \mathcal{D}, \\
[Z, \mathcal{D}] &= G, \\
[\bar{Z}, \mathcal{D}] &= -G, \\
[S_\Phi, Z] &= -Z, \\
[S_\Phi, \bar{Z}] &= \bar{Z}, \\
[S_\Phi, \mathcal{D}] &= -\mathcal{D}, \\
[S_\Phi, \mathcal{D}] &= \mathcal{D}, \\
[Z, \mathcal{D}] &= \mathcal{D}, \\
[Z, \mathcal{D}] &= G, \\
[G, \mathcal{D}] &= [G, \mathcal{D}] = 0,
\end{align*}
\]

(5.3)
where
\[ S_\Phi = \hat{S} - \frac{1}{2} \Phi, \quad G = B(\bar{z}z) - 2\beta^2 \Phi. \] (5.4)

The operator \( G \) satisfies the obvious relation \( \iint_{S^2} G \, dv = 0 \), whereas the multiplicative operators \( \mathcal{D} \) and \( \bar{\mathcal{D}} \) are, in correspondence with Eq. (5.3), the tensor fields of helicity \(-1\) and \(+1\), respectively, and have the coordinate representation \( \mathcal{D} = \frac{1}{2} \rho(\bar{z}z) \), \( \bar{\mathcal{D}} = \frac{1}{2} c_1 \bar{z} \).

The operators \( \mathcal{D} \) and \( \bar{\mathcal{D}} \) are mutually adjoint with respect to the scalar product (2.6). The intrinsic algebra (5.3) and all the corresponding relations for the hyperbolic plane (2.14) can be reproduced by the formal change \( \beta \to i\beta \).

The intrinsic algebra (5.3) has the following two Casimir operators:
\[ C = G^2 + 8\beta^2 \bar{\mathcal{D}} \mathcal{D}, \quad \mathcal{W} = \bar{\mathcal{D}} Z + \bar{Z} \mathcal{D} - \mathcal{D} \bar{\mathcal{D}} - G(S_\Phi - 1). \] (5.5)

In the given representation, we have \( C = \frac{1}{4} c_1^2 \cdot I \), while the operator \( \mathcal{W} \) in the gauge (2.18) has the form
\[ \mathcal{W} = \frac{1}{2} c_1 (z \partial - \bar{z} \bar{\partial} - S_\Phi). \] (5.6)

From the coordinate representation (5.6), one can conclude that the operators \( \mathcal{D} \) and \( \bar{\mathcal{D}} \) represent the components of the Killing vector associated with the axial symmetry, and the Casimir operator \( \mathcal{W} \) is proportional to the generator of this symmetry. It is interesting to note that in terms of the algebra (5.3), the symmetry generator itself has the form \( J = \mathcal{W}/\sqrt{C} \), i.e. it defines “spin” of representations of the algebra (see below), and the eigenvalue (2.21) of the operator \( S_\Phi \) has a sense of “effective helicity” shifted by the quantized magnetic flux. Therefore, in the case of the odd values of the magnetic flux, we have here an example of the “boson-fermion” transmutation. We shall return to the discussion of this aspect of the system below.

Rescaling appropriately the generators, it is always possible to reduce the constant \( 4\beta^2 \) to \( \epsilon = \pm 1 \), where ‘+’ corresponds to the sphere, while ‘−’ does to the hyperbolic plane. Then, one can redefine the generators of the algebra (5.3) as
\[ E_- = Z - \frac{1}{2} \mathcal{D}, \quad E_+ = \bar{Z} - \frac{1}{2} \bar{\mathcal{D}}, \quad E_0 = \epsilon S_\Phi, \] (5.7)
\[ T_- = \mathcal{D}, \quad T_+ = \bar{\mathcal{D}}, \quad T_0 = \epsilon G. \] (5.8)

From (5.3) it follows that the operators (5.8) span the Abelian algebra of translations, \( t(3) \), while the operators (5.7) together with (5.8) obey the commutation relations:
\[ [E_+, E_-] = \epsilon E_0, \quad [E_0, E_\pm] = \pm E_\pm, \quad [E_0, T_\pm] = \pm T_\pm, \quad [E_\pm, T_0] = \pm T_\pm, \quad [E_\pm, T_\mp] = \mp \epsilon T_0, \quad [E_-, T_+] = [E_+, T_-] = 0. \]

Therefore, one can conclude that in the case of the sphere the intrinsic algebra (5.3) is the algebra of the 3D Euclidean group of motions, \( \text{iso}(3) = t(3) \oplus_8 \text{so}(3) \), while in the case of the hyperbolic plane it is the Poincaré algebra, \( \text{iso}(2, 1) = t(3) \oplus_8 \text{so}(2, 1) \). In this context, the first Casimir operator in (5.5) is the “squared mass” operator, \( C = T_0^2 + 2\epsilon T_- T_+ \), while the second one is the Pauli-Lubanski “pseudo-scalar”, \( \mathcal{W} = \epsilon T_0(1 - E_0) + T_+ E_- + E_+ T_- \).
In terms of the algebra (5.3), the generators of the contracted Onsager algebra (5.1) read as

\[ Z_k = (-4\beta^2)^k (Z + kD), \]
\[ \bar{Z}_k = (-4\beta^2)^k (\bar{Z} + k\bar{D}), \]
\[ B_k = (-4\beta^2)^{k-1} (kG - 4\beta^2 S\Phi), \]

(5.9)

Hence, we have obtained the nontrivial realization of the contracted Onsager algebra in the sense that though the infinite set of generators (5.9) is represented linearly in terms of the finite number of the intrinsic algebra generators, nevertheless, \( Z_k, \bar{Z}_k \) and \( B_l \) do not turn into zero for \( k \geq 1, l \geq 2 \). In this context one notes that in the systems with nonlinear holomorphic supersymmetry investigated earlier [12, 13], only the non-contracted Onsager algebra case was realized nontrivially [15].

The operators \( Z_k, \bar{Z}_k \) with \( k \geq 1 \) also obey the contracted Dolan-Grady relations. Hence, one can try to use them to define a “new” physical system with the Hamiltonian of the same form (2.1). Since the operators \( D \) and \( \bar{D} \) are the tensors of helicity \(-1\) and \(+1\), the generating elements \( Z_k, \bar{Z}_k \) of the “new” system can be treated as a deformation of the original \( U(1) \)-connection. As a result, the “new” system constructed in such a way turns out to be equivalent to the “old” one up to a redefinition of the parameters \( c_0, c_1 \) and \( \beta \).

Among the central charges (5.2), besides the Hamiltonian \( H_n = \frac{1}{2} J_1^1 \), the operator \( J_2^2 \) is the only independent integral of motion, which can be represented in the form

\[ J_2^2 = -4\beta^2 (2J_1^1 + W) - \frac{1}{2} (4\beta^2)^2 S\Phi (S\Phi + n\sigma_3) - \frac{1}{2} C. \]

(5.10)

Hence, in the representation under consideration, the Hamiltonian \( H_n \) and the Casimir operator \( W \) form the set of independent central charges of the nonlinear superalgebra [15], which are the differential operators. We also note that the obvious integrals of motion, \( S \) and \( \sigma_3 \), of the supersymmetric system can be transformed into the central charges \( T_1 = 2\hat{S} + n\sigma_3 \) and \( T_2 = \hat{S}(\hat{S} + n\sigma_3) \), and in terms of them, the second structure in the integrals (5.10) is represented linearly,

\[ S\Phi (S\Phi + n\sigma_3) = \frac{1}{4} \Phi^2 - \frac{1}{2} \Phi T_1 + T_2. \]

All the commuting charges \( J_l^l \) (5.2) with \( l \geq 3 \) are not independent since they can be represented as:

\[ J_l^l = (-4\beta^2)^{l-2} \left( 4\beta^2 (l - 2) l J_1^1 + C_{2n}^l J_2^2 + \frac{1}{2} (4\beta^2)^2 C_{2n}^{l-1} S\Phi (S\Phi + n\sigma_3) - \frac{1}{2} C_3 \cdot C \right), \]

where \( C_{2n}^l = \frac{\delta_{l-n} \delta_{l-n}}{(l-n)!n!}, \ l, n \in \mathbb{N} \).

Thus, from the above algebraic analysis it follows that the operators \( W, T_1 \) and \( T_2 \) exhaust the list of independent nontrivial central charges of the \( n \)-HSUSY system described by the Hamiltonian \( H_n \) and by the supercharges (2.12). The given systems on the sphere and hyperbolic plane provide the first examples of the nontrivial realization of the contracted Onsager algebra in contrast with the cases of the plane [13] and 1D systems with \( n \)-HSUSY [12], where the realization of the algebra turns out to be trivial [13].
In conclusion of this section, let us return to the integral of motion $J = \mathcal{W}/\sqrt{C}$ constructed from the Casimir operators of the intrinsic algebra. Its coordinate form is

$$J = z\partial - \bar{z}\bar{\partial} - S_\Phi.$$  (5.11)

Since the $J$ is associated with the axial symmetry of the system, it has a sense of the total angular momentum operator. Then, the shifted helicity $S_\Phi$ (5.4) can be treated as the “effective spin”, controlling the phase of the quantum wave functions that they acquire under the $2\pi$-rotation of the system. In the spherical case, its eigenvalue $s_\Phi$ (2.21) is half-integer when the quantized magnetic flux $\Phi$ takes an odd value, i.e. we have here some kind of bose-fermi transmutation. On the other hand, in the case of the hyperbolic plane the magnetic flux is not quantized, the $s_\Phi$ can take arbitrary values, and one can say that the quantum states of the system are anyonic-like. However, such an interpretation has a weak point: the operator $J$ was obtained by us from the intrinsic algebra, whose $\text{iso}(3)$ (or $\text{iso}(2,1)$ in the hyperbolic case) generators $E_\pm, T_\pm, T_0$ are not integrals of motion, and, so, do not correspond to any symmetry of the system. Therefore, it seems that nothing prohibits to shift (5.11), say, by a constant term $f(c_0, c_1)$ such that $f(0, 0) = 0$. Below, however, we shall give an additional argument in favour of treating (5.11) as a total angular momentum operator.

6 Algebraic approach to analysis of symmetries in an external field

In the previous section we have established that the integral of motion corresponding to the axial symmetry of the system is the Casimir operator of the intrinsic algebra generated by the covariant derivatives (5.3). As a consequence, the operator $J = \mathcal{W}/\sqrt{C}$ automatically commutes with the Hamiltonian constructed in terms of covariant derivatives. In contrast with the generic case (2.15), the constant ($c_1 = 0$) magnetic field does not break the isometry of the sphere (hyperbolic plane). Therefore, the system defined by the Hamiltonian (2.1) with $B = \text{const}$ has to have three integrals of motion corresponding to the isometry of the background space. Here we analyse the symmetries of the system in the constant external magnetic field in the context of the corresponding intrinsic covariant algebra. This will help us to understand better the question of fixing the form of the total angular momentum operator (5.11). Moreover, our analysis will result in formulation of the general method (beyond the context of supersymmetry) of obtaining the explicitly covariant form of the integrals of motion.

Using the representation (2.17) of the “bosonic” Hamiltonian in the gauge (2.18) with $c_1 = 0$, one can find the coordinate form of the three integrals of motion,

$$J_+ = -\beta^2 z^2 \partial - \bar{\partial} + \left(2s - \frac{\Phi}{2}\right)\beta^2 z,$$

$$J_0 = z \partial - \bar{z}\bar{\partial} - s + \frac{\Phi}{2},$$

$$J_- = \partial + \beta^2 \bar{z}^2 \bar{\partial} - \frac{\Phi}{2} \beta^2 \bar{z},$$  (6.1)
where we assume that these operators act on the tensors of helicity $s$. Here, the constant term in the operator $J_0$ is completely fixed by the condition that the operators (6.1) realize a unitary representation of the non-Abelian algebra $su(2)$ being the isometry algebra of the sphere. The modification corresponding to the hyperbolic plane is reproduced, as always, by the formal change $\beta \to i\beta$, and in this case the integrals span the algebra $su(1, 1)$. The detailed discussion of different aspects of the systems on the sphere and hyperbolic plane in the constant magnetic field can be found in Refs. [33, 31, 33].

The integrals (6.1) are given in the non-covariant form. Nevertheless, we know that they have to be the scalar operators since we consider the spectral problem on the tensors of the definite rank, $s$, and the symmetry, generated by the integrals, corresponds to the degeneracy of the spectrum. The coordinate form (6.1) and the information about the type of the operators is enough to properly define their behaviour on the (co)tangent bundle, modifying appropriately the algebraic method used in the previous section, we develop a general method of finding the explicitly covariant form of the integrals.

Let us consider the algebra generated by the operators $Z$ and $\bar{Z}$ in the case of the constant magnetic field,

\[ [Z, \bar{Z}] = -4\beta^2 S_\Phi, \quad [S_\Phi, Z] = -Z, \quad [S_\Phi, \bar{Z}] = \bar{Z}. \tag{6.2} \]

Here the operator $S_\Phi$ is formally the same as for the algebra (5.3), and $\beta$ is assumed to be pure imaginary in the case of the hyperbolic plane. One can note that this algebra is exactly the isometry algebra of the background space. As we will see later, this is a general (local) property of the spaces of constant sectional curvature when external fields are absent or constant.

Unlike the case of the inhomogeneous magnetic field (2.15), in the algebra (6.2) generated by the covariant derivatives only one additional element appears. Obviously, we cannot construct the integrals (6.1) only in terms of the operators $Z$, $\bar{Z}$ and $S_\Phi$. In the inhomogeneous case the components of the Killing vector corresponding to the axial symmetry were the necessary constituents of the covariant construction of the integral $\mathcal{W}$ (5.3). For the homogeneous magnetic field one needs to use the three Killing vectors. The form of their components in the coordinate system we are working in is the following:

\[ V_0 = \frac{1}{2} \rho(\bar{z}z)\bar{z}, \quad \bar{V}_0 = z, \quad V_1 = \frac{1}{2} \rho(\bar{z}z)\beta^2 z^2, \quad \bar{V}_1 = \beta^2 z^2, \quad V_2 = \frac{1}{2} \rho(\bar{z}z), \quad \bar{V}_2 = 1. \]

Under the change of the coordinates the $V_i$ ($\bar{V}_i$), $i = 0, 1, 2$, are transformed as the tensors of helicity $-1$ (+1). We will treat $V_i$ and $\bar{V}_i$ as multiplication operators for the given quantum system. In this context, the operators $V_i$ and $\bar{V}_i$ are mutually adjoint with respect to the scalar product (2.6). Together with the covariant derivatives, $V_i$ and $\bar{V}_i$ generate the following algebra:

\[
\begin{align*}
[Z, V_0] &= G_0, & [\bar{Z}, V_0] &= -G_0, & [Z, G_0] &= -4\beta^2 V_0, & [\bar{Z}, G_0] &= 4\beta^2 \bar{V}_0, \\
[Z, \bar{V}_1] &= G_1, & [\bar{Z}, V_1] &= -G_1, & [Z, G_1] &= 4\beta^2 V_2, & [\bar{Z}, G_1] &= 4\beta^2 \bar{V}_1, \\
[Z, V_2] &= -\bar{G}_1, & [\bar{Z}, V_2] &= G_1, & [Z, G_2] &= -4\beta^2 \bar{V}_1, & [\bar{Z}, G_2] &= -4\beta^2 V_2, \quad (6.3) \\
[S_\Phi, V_i] &= -V_i, & [S_\Phi, \bar{V}_i] &= \bar{V}_i, \\
\end{align*}
\]

\[
\begin{align*}
[Z, \bar{V}_i] &= [\bar{Z}, V_i] = [S_\Phi, G_\alpha] = [S_\Phi, \bar{G}_\alpha] = 0, \\
\end{align*}
\]
where \( i = 0, 1, 2, \alpha = 0, 1 \), and in the chosen coordinate system, \( \bar{G}_0 = G_0, G_1 \) and \( \bar{G}_1 \) are the scalar multiplicative operators of the form

\[
\begin{align*}
G_0 &= 2\sqrt{\rho(\bar{z}z)} - 1, \\
G_1 &= 2\beta^2 z \sqrt{\rho(\bar{z}z)}, \\
\bar{G}_1 &= 2\beta^2 \bar{z} \sqrt{\rho(\bar{z}z)}.
\end{align*}
\]

Since \( V_i, \bar{V}_i, G_\alpha \) and \( \bar{G}_\alpha \) are the multiplicative operators, they form an Abelian subalgebra. Moreover, from the commutation relations (6.2), (6.3) it follows that this Abelian subalgebra is an ideal. Finally, one can conclude that the algebra given by the commutation relations (6.2), (6.3) is isomorphic to the algebra isom, where \( t_-(3) = \text{span}\{V_1, \bar{G}_1, V_2\} \), \( t_0(3) = \text{span}\{V_0, G_0, \bar{V}_0\} \), \( t_+(3) = \text{span}\{V_2, G_1, \bar{V}_1\} \), and isom is the isometry algebra of the sphere or of the hyperbolic plane. The Abelian subalgebras \( t_\pm(3) \) are mutually conjugate with respect to the scalar product (2.4). The algebra \( g \) has seven Casimir operators constructed from the generators \( t_+(3) \), \( t_0(3) \). They are trivial in the given representation and their particular form is not important for the further discussion.

The Hamiltonian (2.1) is composed of the covariant derivatives, which belong to the algebra isom. Therefore, the Casimir operators of the subalgebras \( t_+ (3) \oplus_s \text{isom} \) and \( t_0(3) \oplus_s \text{isom} \), each of which is isomorphic to \( \text{iso}(3) \) (sphere) or to \( \text{iso}(2, 1) \) (hyperbolic plane),

\[
\begin{align*}
J_+ &= \bar{Z} V_2 - \bar{V}_1 Z + (S_\phi - 1) G_1, \\
J_0 &= \bar{V}_0 Z + \bar{Z} V_0 - (S_\phi - 1) G_0, \\
J_- &= V_2 Z - \bar{Z} V_1 + (S_\phi - 1) \bar{G}_1,
\end{align*}
\]

(6.4) commute with the Hamiltonian (2.1). These scalar operators are explicitly covariant with respect to the \( U(1) \) and (co)tangent bundles and in the gauge (2.18) their coordinate form coincide with the operators (1.1). Thus, we have found the covariant form of the integrals of motion for the given system (with \( B = \text{const} \)) in the pure algebraic manner.

It is worth noting that in the given representation the multiplicative operators obey some additional relations of a polynomial form. We will discuss only the most interesting of them. For example, one can verify that the multiplicative operators satisfy the constraints

\[
2\beta^2 V_0 - G_1 V_1 - \bar{G}_1 V_2 = 0, \quad (1 - G_0)V_2 - G_1 V_0 = 0, \quad (1 + G_0)V_1 - \bar{G}_1 V_0 = 0,
\]

and their conjugate. These constraints mean that the Killing vectors are not independent in some sense. The commutation of these relations with the covariant derivatives leads to another set of constraints:

\[
\begin{align*}
G_1 &= 4\beta^2 (\bar{V}_0 V_2 + \bar{V}_1 V_0), \\
\bar{G}_1 &= 4\beta^2 (\bar{V}_0 V_1 + \bar{V}_2 V_0), \\
G_0 &= 2 (\bar{V}_2 V_2 - \bar{V}_1 V_1).
\end{align*}
\]

Obviously, one can use these relations to exclude the operators \( G_0, G_1 \) and \( \bar{G}_1 \) from the algebra (6.2), (6.3). Then, one can say that the operators \( Z, \bar{Z}, V_i \) and \( \bar{V}_i \) generate some quadratic algebra. Therefore, the quantum system in the homogeneous magnetic field represents a physical system with the intrinsic nonlinear algebra.

In the case of an inhomogeneous magnetic field with the axial symmetry, the components of only one of the Killing vectors, say, \( V_0 \) and \( \bar{V}_0 \), can be considered as the proper multiplicative operators of the corresponding quantum system. Thus, for the axial magnetic field of a general form only the operators \( Z, \bar{Z}, V_0 \) and \( \bar{V}_0 \) can be considered as generating elements of
the algebra, intrinsic to the given quantum system. In general case, the covariant derivatives \( Z \) and \( \bar{Z} \) generate an infinite-dimensional subalgebra. Nevertheless, the class of systems with a finite-dimensional generated algebra should be large enough. Note, that the appearance of such a finite-dimensional algebra implies that the magnetic field obeys some conditions (like those appearing from the Dolan-Grady relations). The algebraic approach for obtaining the corresponding integrals of motion can be applied in these cases as well.

In addition to the gauge interaction, one can include into this scheme the interaction with a scalar potential, which, in general, should be treated as an independent generating element of the intrinsic algebra. As a toy example, one can include into the Hamiltonian (2.1) the interaction with the scalar potential of the form \( U = U(G_0, V_0 V_0) \). Obviously, such a potential does not change the intrinsic algebra \( \mathfrak{g} \) (6.3), but it breaks the symmetry of the quantum system to the axial subgroup generated by the operator \( J_0 \).

In the spherical case, when the magnetic field is switched off, the generators (6.1), forming the \( \mathfrak{su}(2) \) algebra, realize the representations with integer weight only since we start form the wave functions to be tensor fields, i.e. initially spin of the system is integer. When the magnetic field is switched on, in accordance with the coordinate form of \( J_0 \) (6.1), the spin of the quantum system effectively undergoes the shift by \( \Phi/2 \). Hence, if the flux parameter \( \Phi \) is odd, the spin of the system is half-integer. This phenomenon is well-known, e.g., in the context of the charged system immersed into the field of a magnetic monopole [47]. At the same time, the 2D quantum system on the sphere subjected to the homogeneous magnetic field can be treated as a 3D system in the field of magnetic monopole reduced to the spherical geometry. In the case of the hyperbolic plane the real parameter \( \Phi \) is not restricted, and the generators (5.1) forming the noncompact \( \mathfrak{su}(1, 1) \) algebra realize a representation with arbitrary real weight [48] defined by the shifted helicity operator \( S_\Phi \).

The magnetic field of the form (2.15) possesses only the axial symmetry associated with the Killing vector \( V_0, \bar{V}_0 \). It means that the components of the other Killing vectors cannot be considered as a proper generating elements any more. Moreover, in the case (2.15) the multiplicative operators \( V_0, \bar{V}_0 \) are not independent generating elements since the intrinsic algebra (5.3) is generated merely by the covariant derivatives. Indeed, the operators \( \mathcal{D} \) and \( \bar{\mathcal{D}} \) in the algebra (5.3) are proportional, correspondingly, to \( V_0 \) and \( \bar{V}_0 \). The integral \( W (5.5) \) associated with the axial symmetry, being the Casimir of the algebra \( t(3) \oplus s \text{isom} \), is similar to the integrals (6.1). In this context, one can say that the presence of the inhomogeneous part in the magnetic field (2.15) breaks the algebra \( \mathfrak{g} \) (6.2), (6.3) to its subalgebra \( t_0(3) \oplus s \text{isom} \), and from the three Casimir operators (5.1) only the operator \( J_0 \) survives in the form of the integral (5.11). Such a relation between the cases with inhomogeneous axially symmetric magnetic field and homogeneous magnetic field supports our treatment of the operator (5.11) as the 2D total angular momentum operator of the system (2.1), (2.5), (2.15).

The above algebraic approach can be generalized to the case of curved spaces of higher dimensionality. To argue this, let us discuss a quantum system in a \( D \)-dimensional Riemannian space with a constant sectional curvature [43], the wave functions of which are covariant tensors. Then the Riemann tensor has the form

\[
R_{\mu\nu}^{\alpha\beta} = \frac{\mathcal{R}}{D(D-1)} \left( \delta_\alpha^\alpha \delta_\beta^\beta - \delta_\mu^\alpha \delta_\nu^\beta \right),
\]

where the constant \( \mathcal{R} \) is the scalar curvature. Obviously, such a space is a symmetric one.
and its isometry algebra is maximal: \( \mathfrak{so}(D+1) (\mathcal{R} > 0) \), \( \mathfrak{so}(D) (\mathcal{R} = 0) \) or \( \mathfrak{so}(D, 1) (\mathcal{R} < 0) \).

The covariant derivatives with the Riemann connection generate the corresponding isometry algebra. Indeed, their commutator can be represented in the form

\[
[\nabla_\mu, \nabla_\nu] = \frac{1}{2} R_{\mu\nu}^{\alpha\beta} M_{\alpha\beta} = \frac{\mathcal{R}}{D(D - 1)} M_{\mu\nu},
\]

where the operators \( M_{\mu\nu} \) are given in a matrix representation defined by the fields which the covariant derivatives act on. For example, the action of the operators on a contravariant vector field reads as \( M_{\mu\nu} \circ V^\alpha = (\delta_\mu^\alpha g_{\nu\gamma} - \delta_\nu^\alpha g_{\mu\gamma}) V^\gamma \), while for a spinorial field the corresponding matrix is proportional to \([\gamma_\mu, \gamma_\nu]\), where \( \gamma_\mu \) are the \( D \)-dimensional \( \gamma \)-matrices. The generalization to the fields of arbitrary type is straightforward. It is not difficult to verify that the covariant derivatives and the operators \( M_{\mu\nu} \) form the algebra isomorphic to the isometry algebra of the background space, i.e. the covariant derivatives are the generating elements of the isometry algebra. The number of Killing vector fields \( V^a = V^a_\mu(x) \partial_\mu \) forming the isometry algebra is equal to \( D(D + 1)/2 \). Let us suppose that the interaction with an external gauge field or with a scalar potential breaks this algebra to a subalgebra spanned by a subset of Killing vectors \( V_i, i = 1, \ldots, l < D(D + 1)/2 \). Treating the components \( V_i^\mu \) and the scalar potential (if it is present) as multiplicative operators of the given quantum system, one can consider them together with the covariant derivatives as generating elements of the intrinsic algebra of the system. The corresponding Hamiltonian is supposed to be composed of the covariant derivatives and of the scalar potential. Evidently, if the resulting algebra is finite-dimensional, then the covariant form of the corresponding integrals of motion can be found in closed terms.

7 Discussion and outlook

Let us summarize briefly the obtained results and discuss some open problems that deserve further attention.

We have discussed the application of the general algebraic scheme of the nonlinear holomorphic supersymmetry to the quantum mechanical systems with nontrivial 2D Riemann geometry in the presence of external magnetic field. The analysis of Dolan-Grady relations (2.2), underlying the construction [15], shows that

- The nonlinear holomorphic supersymmetry can be realized on Riemann surfaces with constant curvature only.

Moreover, for non-vanishing curvature the \( n \)-HSUSY can be realized solely in the case of the contracted Dolan-Grady relations (2.11). We have investigated the cases of the sphere and Lobachevski plane, which together with the zero curvature case of the plane analysed by us earlier [13], exhaust the simply connected Riemann surfaces. We demonstrated that for the \( n \)-HSUSY system on the sphere and hyperbolic plane a part of the spectrum can be found algebraically by reducing the corresponding 2D Hamiltonian to one-dimensional families of the QES Hamiltonians described by the \( \mathfrak{sl}(2, \mathbb{R}) \) scheme.

We have found that the one-dimensional QES potential corresponding to the sphere admits a discrete group of transformations of the parameters of the potential. The 1D
potential corresponding to the case of hyperbolic plane admits a similar discrete group of transformations if some parameter is restricted to be integer. In particular, this means that

- The discrete transformations (4.1)–(4.4) partially relate distinct sectors of the spectrum of the same 2D system, or the spectra of different 2D systems.

Due to the second relation in (4.4), the discrete transformations are a kind of duality transformations. We argued that, in general, other QES  sl(2, R) potentials also can admit such a symmetry group after discretization of some parameters. This was illustrated by the example of the QES potential (3.3). We are going to investigate in detail the revealed discrete symmetry of QES potentials elsewhere.

The analysis of the algebraic content of the model on the sphere and hyperbolic plane shows that all the observables of the systems can be represented in terms of the generators of the iso(3) (sphere) or iso(2, 1) (hyperbolic plane) intrinsic algebras. All the higher generators of the contracted Onsager algebra associated with the relations (2.11) are linearly dependent (see Eq. (5.9)) but non-vanishing. Hence,

- The supersymmetric 2D systems on the sphere and hyperbolic plane realize the non-trivial representations of the contracted Onsager algebra.

Using the representation (5.9), we have found that besides the Hamiltonian, the infinite set of commuting charges (5.2) contains, in fact, only one non-trivial independent integral of motion, which is a differential operator being, simultaneously, the Pauli-Lubanski “pseudo-scalar” \( \mathcal{W} \) (5.3) of the corresponding iso(3) or iso(2, 1) algebra. It plays the role of the 2D total angular momentum (5.1), and its noncoordinate part has a sense of the effective spin of the system. The value of the effective spin \( S_\phi \) (5.4), being a helicity shifted by the magnetic flux parameter, takes integer or half-integer values on the sphere, and corresponds to the anyonic-like quantum states on the hyperbolic plane.

The algebraic analysis allowed us to find the covariant form of the integral of motion associated with the axial symmetry. Following the same pattern we have found the covariant form of all the integrals in the case of the homogeneous magnetic field. On the basis of these explicit examples, we have proposed

- A general algebraic method to find the covariant form of integrals of motion of a quantum system in a curved space in the presence of an external gauge field.

The integrals obtained in this way have transparent commutation properties. We hope that this approach will be helpful for a wide class of quantum systems with symmetry.

Nowadays, a considerable attention is paid to a generalization of the 2D systems on Riemann surfaces with a constant magnetic field \[ B \neq \text{const} \] to the case of noncommutative surfaces. Such quantum systems are exactly solvable in the both cases of commutative and noncommutative spaces. The supersymmetric systems considered in this paper can be treated as a generalization of the constant magnetic field case to the case of the nontrivial field \( B \neq \text{const} \) accompanied by the transition from exactly solvable systems to quasi-exactly solvable ones. Therefore, the interesting problem is to generalize the 2D realizations of the nonlinear holomorphic supersymmetry to the cases of the noncommutative plane, sphere and hyperbolic plane. Such a generalization will be presented elsewhere.
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