An inner full-state feedback control model for traffic flow system

Lidong Zhang¹*, Xiaowei Li² and Jian Wang³

¹School of transportation and logistics engineering, Shandong Jiaotong University, Jinan, Shandong, 250357, China
²School of rail transit, Shandong Jiaotong University, Jinan, Shandong, 250357, China
³School of automobile engineering, Shandong Jiaotong University, Jinan, Shandong, 250357, China

*Corresponding author’s e-mail: zhanglidong1102@126.com

Abstract. In order to discover the compound effect of traffic system’s state variables on its stability properties and inner mechanism, we presented a type of inner full-state feedback control strategy based on optimal velocity model of traffic flow system. We took the car’s velocity and headway as the inner feedback control input parameters and car’s acceleration as the system’s output parameter. With Lyapunov’s direct method, we deduced the new state space equation and its stability condition. Under its stable conditions, we studied the effect of full-state feedback control strategy on traffic flow and the jamming transition by use of numerical and analytical methods. The new system’s controllability and observability were also studied. Numerical simulations verified our analytical results’ correctness.

1. Introduction

Traffic jam is a heavy problem to prevent the city’s fast development because of quickly increasing vehicle population in our country. To release the traffic jam’s negative effect, many scholars have studied traffic system in the field of physics from diverse sides and they proposed many theoretical models, including macroscopic models, mesoscopic models and microscopic models. Among all these models, car-following model as a kind of typical microscopic model has attracted many scholars’ hot interest since it was proposed by Pipes in 1953[1]. There was a long blank period from Pipes’s model to find a new type one. Till 1995, Bando[2] proposed a new model named optimal velocity model(i.e.OVM), by Taylor’s expansion formula. The OVM is a milestone in traffic flow theory. From then on, car-following theory was intensively studied based on Bando’s OVM model by the scholars around the world. We summarized the representative achievements as follows. All the models could be classified into two kinds based on their time-domain characteristics. One kind is continuous model and the other one is discrete model. The traffic flow system changes their motion state continuously in continuous model, comparing to discrete traffic flow model. The typical continuous models include multi-anticipative effect car-following model by Lenz[3], car-following model with a next nearest neighbour interaction by Nagatani[4] and Sawada[5], full velocity difference car-following model[6], full velocity difference model with a nonlinear analysis method[7], an Intelligent Transportation System (ITS) environment model by Ge[8], car-following model with a consideration of infinite forward looking velocity difference effects by Li[9]. Also, some scholars improved the car-following model by considering both the driver’s attribution, the inter-vehicle’s communication and the bounded driver’s
rationality[10], the model with velocity and acceleration difference[11], and optimal velocity difference model[12], et.al. Yu calibrated the optimal velocity car-following model using field data[13,14]. Traffic flow system is also a kind of complex industrial system. Some scholars studied the traffic flow system from the viewpoint of control theory and control engineering. For example, they presented cascade compensation, proportional-differential compensation and speed feedback compensation to improve the performance of the car-following system[15–17]. Some scholar also extended the study method to modern control theory, such as the stability of the full velocity difference model with the use of Lyapunov’s method[18]. With the coming of autonomous cars era, a new kind model considering adjustable sensitivity, strength factor and mean expected velocity field size parameters has been put forward[19]. The discrete traffic flow model include coupled map car-following model[20]. In 2019, Zhu presented a systematic method to discretize the continuous car-following model into a difference equation, and obtained the impulse transfer function [21]. In a word, although the mechanism of the traffic flow theory in the OV traffic model has been revealed by several theoretical studies, to our knowledge, the problem how to prevent traffic jams has not yet been solved completely. So we present an inner full-state feedback control strategy to further study the jam mechanism of traffic flow system. Our paper is organized as follows. Section 2 explains the optimal velocity traffic flow state space model and its Lyapunov direct method stability analysis. Section 3, the inner full-state feedback control method is introduced for the prevention of traffic jam, this method analyses system controllability and observability properties. Section 4 gives the computer numerical simulations to prove our results obtained from theoretical analysis. Conclusions are presented in Section 5.

2. Traffic flow state space model

2.1. Model

At first, we analyse a vehicle group that consists of many vehicles running on a single road without overtaking behaviour. Let us consider the OV traffic model under close boundary conditions (Figure 1.). We define several notations used in the following context, including the position of the $n^{th}$ vehicle $x_n(t)$, its velocity $v_n(t)$, and its acceleration $a_n(t)$. They have the relation as in formula (1),

$$v_n(t) = a_n(t), \quad x_n(t) = v_n(t)$$

Modern control theory is a good tool to describe complex system. Now, we can plot the traffic flow system’s signal diagram as in Figure 2. Now we will deduce its state space description.

![Figure 1. OVM traffic flow car-following diagram under an open boundary](image)

![Figure 2. State space of controlled OVM traffic flow](image)

For above system described in Figure 2, its state space equation can be written as,
\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align*}
\]  
(2)

Here, the state variables describing the traffic flow system are \( x = \begin{bmatrix} v_n(t) \\ \Delta x_n(t) \end{bmatrix} \), \( u = v_{n+1}(t) \), \( y = v_n(t) \).

Based on Bando’s model, the \( n^{th} \) vehicle’s dynamics in the queue can be rewritten as in (3),

\[
\begin{align*}
\frac{dv_n(t)}{dt} &= \alpha [F(\Delta x_n(t)) - v_n(t)], \\
\frac{d(\Delta x_n(t))}{dt} &= v_{n+1}(t) - v_n(t),
\end{align*}
\]

where \( \Delta x_n(t) \) is the headway value between the leading \((n+1)^{th}\) car and following \( n^{th} \) vehicle, that is \( \Delta x_n(t) = x_{n+1}(t) - x_n(t) \). \( \alpha > 0 \) is the driver’s sensitivity factor. \( N \) is the total number of the vehicle in the queue. \( F(\Delta x_n(t)) \) is the OV function for \( n^{th} \) vehicle which is function of the headway distance \( \Delta x_n(t) \). Assuming that the system is steady, that is the vehicles run constantly with speed \( v_i \), then the system’s dynamical steady state is given in equation (4),

\[
[v_n^*(t), \Delta x_n^*(t)]^T = [v_0, F^{-1}(v_0)]^T
\]

(4)

This state shows that all the vehicles run orderly with velocity \( v_0 \) and headway value \( F^{-1}(v_0) \). Since system dynamics has the one-way direction, the stability of the entire system can be reduced to stability of steady state (3). So the linearized system of vehicle control system (3) around its steady state (4) is,

\[
\begin{align*}
\frac{d\tilde{v}_n(t)}{dt} &= -\alpha \phi \tilde{v}_n(t) - \alpha \phi \Delta x_n(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tilde{v}_{n+1}(t) \\
\frac{d\Delta x_n(t)}{dt} &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{v}_n(t) \\ \Delta x_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tilde{v}_{n+1}(t)
\end{align*}
\]

(5)

where \( \tilde{v}_n(t) := v_n(t) - v_0 \), \( \Delta x_n(t) := \Delta x_n(t) - F^{-1}(v_0) \), \( \phi \) is the slope of the OV function at \( \Delta x_n(t) = F^{-1}(v_0) \).

Therefore, we obtain \( A = \begin{bmatrix} -\alpha & \alpha \phi \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = [0] \).

Based on system modern control theory, the transfer function from input variable \( \tilde{v}_{n+1}(t) \) to output variable \( \tilde{v}_n(t) \) is described as,

\[
G(s) = C [sI - A]^{-1} B = \frac{\alpha \phi}{(s^2 + \alpha s + \alpha \phi)}
\]

(6)

2.2. Full state-feedback control

2.2.1. Controller setting

To design the state variable process controller, we firstly assume that all the system’s states are available for feedback. If we have access to the complete state \( x(t) \) for all \( t \), then the system’s inner control input \( u(t) \) is given by,

\[
u = -k x
\]

(7)
To determine the gain matrix $k$ is one of the main goal for the full-state feedback design procedure. With the system defined by the state variable model, $\dot{x} = Ax + Bu$, the control feedback is given by, $u = -k \dot{x}$.

\[
\begin{cases}
\frac{dv_\alpha(t)}{dt} = \alpha \{F(\Delta x_n(t)) - v_\alpha(t)\} + k_1 v_\alpha(t) + k_2 \Delta x_n(t), \\
\frac{d(\Delta x_n(t))}{dt} = v_{n+1}(t) - v_n(t),
\end{cases} 
\quad (n = 1, 2, \ldots, N) \tag{8}
\]

By Taylor’s expansion formula, the traffic flow control system’s linearized form around steady point is,

\[
\begin{bmatrix}
\frac{d\tilde{v}_\alpha(t)}{dt} \\
\frac{d\Delta x_n(t)}{dt}
\end{bmatrix} =
\begin{bmatrix}
k_1 - \alpha & \alpha \Phi + k_2 \\
-1 & 0
\end{bmatrix}
\begin{bmatrix}
\tilde{v}_\alpha(t) \\
\Delta x_n(t)
\end{bmatrix} +
\begin{bmatrix}
0 \\
1
\end{bmatrix}
\tilde{v}_{n+1}(t) \tag{9}
\]

where $\tilde{v}_\alpha(t) := v_\alpha(t) - v_0$, $\Delta x_n(t) := \Delta x_n(t) - F^{-1}(v_0)$, $\Phi$ is the slope of the OV function at $\Delta x_n(t) = F^{-1}(v_0)$,

\[
\Phi := \frac{dF(\Delta x)}{d(\Delta x)}|_{\Delta x=F^{-1}(v_0)} \tag{10}
\]

Let’s define $\bar{A} = \begin{bmatrix} k_1 - \alpha & \alpha \Phi + k_2 \\ -1 & 0 \end{bmatrix}$, $\bar{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\bar{C} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\bar{D} = 0$, $x = \begin{bmatrix} \tilde{v}_\alpha(t) \\ \Delta x_n(t) \end{bmatrix}$, and $u = \tilde{v}_{n+1}(t)$, $y = \tilde{v}_\alpha(t)$. $\mathcal{L}$ denotes the Laplace transform operator. $V_n(s) := \mathcal{L}[v_\alpha(t)]$, $V_{n+1}(s) := \mathcal{L}[v_{n+1}(t)]$.

The traffic flow system’s transfer function $G(s)$ is in equation (11),

\[
\tilde{G}(s) = C[SI-A]^tB \tag{11}
\]

Therefore, the relationship between the $(n+1)th$ vehicle velocity disturbance and the $n^{th}$ vehicle velocity disturbance is described by,

\[
V_n(s) = \tilde{G}(s)V_{n+1}(s) \tag{12}
\]

And the modified transfer function $\tilde{G}(s)$ as

\[
\tilde{G}(s) = \frac{\alpha \Phi + k_2}{d(s)} \tag{13}
\]
where the characteristics polynomial \( \overline{d}(s) \) is, 
\[ \overline{d}(s) := s^2 + (\alpha - k_1)s + (\alpha \Phi + k_2). \]

2.2.2. Linear stability analysis

Now let us apply the linear stability theory to analyse the traffic flow control system’s stable condition. Assuming that all vehicles move with the identical headway \( h \) and the optimal velocity \( V(h) \), and the relative speed \( \Delta v_n(t) \) is zero. We let \( x_n^{(0)}(t) \) represent the uniform steady state as in equation (14),
\[ x_n^{(0)}(t) = h n + V(h) t \quad \text{with} \quad h = L / N \] (14)

Also, \( N \) is the number of cars and \( h \) is the average headway.

Now, let \( y_n(t) \) be a small deviation from the steady-state flow \( x_n^{(0)}(t) \), and \( x_n(t) = x_n^{(0)}(t) + y_n(t) \). Then the linearized form of equation (9) is rewritten as,
\[
\begin{align*}
\frac{d^2 y_n(t)}{dt^2} &= \alpha (\Phi \Delta y_n(t) - \frac{dy_n(t)}{dt}) + k_1 \frac{dy_n(t)}{dt} + k_2 \Delta y_n(t), \\
\frac{d(\Delta y_n(t))}{dt} &= \frac{dy_{n+1}(t)}{dt} - \frac{dy_n(t)}{dt}, \\
&\quad (n = 1, 2, \ldots, N) \tag{15}
\end{align*}
\]

By expanding \( y_n(t) = \exp(ikt + z t) \), we derive the following \( z \) equation (16),
\[ z^2 + (\alpha - k_1)z - (\alpha \Phi + k_2)(e^z - 1) = 0 \] (16)
The solution of equation (16) is,
\[ z = z_1(ik) + z_2(ik)^2 + \cdots \] (17)

Accordingly, the first- and second-order terms of \( ik \) are,
\[ z_1 = \frac{\alpha \Phi + k_2}{\alpha - k_1}, \quad z_2 = \frac{\alpha \Phi + k_2}{2(\alpha - k_1)} - \frac{(\alpha \Phi + k_2)^2}{(\alpha - k_1)^3}, \quad \alpha \neq k_1 \] (18)

As we all know that if \( z_2 \) is negative, the uniformly steady-state flow becomes unstable for long-wavelength modes. And if \( z_2 \) is a positive, the uniform traffic flow is stable. Therefore, the neutral stability criteria for this steady state are summarized as in Lemma 2.

**Lemma 2. If the following condition is satisfied:**
\[ \Phi < \frac{(\alpha - k_1)^2 - 2k_2}{2\alpha} \] (19)

**then the traffic jam never occurs in the OV traffic model with inner state feedback control.**

Furthermore, we can see that when \( k_1 = k_2 = 0 \), then we get \( \Phi < \frac{\alpha}{2} \), which is the same as in Ref.[2].

3. Numerical simulations and experiments

3.1 Initial condition setting

Now let us assume that 100 (\( N = 100 \)) vehicles are travelling in a single lane, and none of them are passing the other vehicles, under the periodical boundary in Figure 2. The road length is \( L \) m.
\[ x_i(0) = 1.0 \text{m}; \quad x_i(0) = (i - 1)L / N \text{ m, for } i \neq 1; \quad \text{and} \quad v_i(0) = F_{v_i}(L / N) \] (20)

If the maximum velocity of the traffic queue is \( v_{\text{max}} \), and the safety distance for each two cascading vehicles is \( s_y \), then let \( \alpha = 0.85 \text{ } s^{-1} \), \( V_1 = 6.75 \text{ m/s} \), \( V_2 = 7.91 \text{ m/s} \), \( C_1 = 0.13 \text{ m}^{-1} \), \( C_2 = 1.57 \), and the optimal velocity function \( F(\Delta x_n(t)) \) is defined as in equation (21) as in Ref.[3],
\[ F(\Delta x_n(t)) = V_1 + V_2 \times \tanh(C_1 \Delta x_n(t) - C_2) \] (21)
3.2 Traffic flow stable and unstable state

Condition I: When $k_1 = 0.1$, $k_2 = 0.001$ and $L = 2400$, the system is stable. Subplot (a) in Figure 4 shows the 3-D velocity plot for all cars. It is easy to see that the system turns stable quickly once the disturbance disappears. We use the No 50th and 49th cars as an example for detailed analysis. Subplots (b) and (c) in Figure 5 show their acceleration and velocity tendencies, which shows good convergence. Subplot (d) in Figure 5 show the velocity of all cars at a given time such as $t=1000$, $t=2000$, $t=3000$ and $t=4000$. We can see that all cars converge with the elapse of time.

Condition II: When $k_1 = 0.1$, $k_2 = 0.001$ and $L = 1200$, the system is unstable. Subplot (a) in Figure 5. shows the 3-D velocity plot for all cars. It is easy to see that the system shows a chaos state once the disturbance disappears. We again use the No 50th and 49th cars as an example for detailed analysis. Subplots (b) and (c) in Figure 5. show their acceleration and velocity tendencies, which shows an oscillation state. Curves in subplot (d) of Figure 5. show the velocity of all cars at such given time as $t=1000$, $t=2000$, $t=3000$ and $t=4000$. We can see that all cars oscillate with the elapse of time.
4 Conclusions
The paper showed that the inner full-state feedback control method suppresses the traffic jam phenomenon in the OV traffic model. We derived a procedure to design the feedback gain $k$ such that the controlled traffic system can remain stable state. We have shown that it is possible to enhance the traffic current without jam by introducing the interaction of full-state. Furthermore, we use Lyapunov’s direct method to study the system’s stability, and to analyse the system’s controllability and observability. Our control scheme has following advantages: 1) The system’s controller fully considers the full state of the system including speed and headway; 2) Our control scheme is suitable for any size traffic flow system; 3) There is no need to change the model’s main parameters, which are also practical for real world traffic flow system. Furthermore, we concluded that the numerical simulations results are in good agreements with our theoretical results. In the future, we will examine how to use our results practical situations.

Acknowledgments
This work is partially supported by the National Natural Science Foundation of China (Grant No. 61773243), Science and Technology Project of Shandong Transportation Department (Grant No.2020B89-01). Also, we thank for the discussion with Prof. Heng Wei in University of Cincinnati U.S.A.

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