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Abstract. The Poisson σ-models are described, as well as the σ-models connected by the Poisson-Lie T-duality. Common algebraic structures are emphasized, as the models are investigated from the geometrical point of view. The possibility to introduce a duality transformation of the Poisson-Lie σ-models is discussed. A general method, which enables one to construct a Poisson-Lie sigma model using an arbitrary Lie bialgebra, is given. The correspondence to the the $R^2$ gravity action is shown, as well as other physically relevant examples of the Poisson-Lie sigma models.

1. Introduction
The effort to generalize the string T-duality to non-Abelian groups of symmetries resulted in the discovery of the Poisson-Lie T-duality, published in [1]. As the name suggests, a Poisson-Lie structure is hidden in the construction of the dualizable σ-models. Meanwhile, a class of topological field theories was investigated in [2, 3], which were built as σ-models on a Poisson manifold. These are of special interest, because of their relation to 2D-gravity and gauge theories.

Recently, a bulk-boundary duality transformation was proposed in [4] for Poisson-Lie σ-models calculated from the r-matrix – a solution of the Yang-Baxter equation. Motivated by this work, we build a Poisson-Lie structure that corresponds to an arbitrary Lie bialgebra, and investigate, whether the duality of the solutions of the Poisson σ-models can be introduced.

The first section of our paper deals with the basic geometrical aspects concerning the Poisson manifolds and the Poisson σ-model. Then we focus on a special class – the Poisson-Lie σ-models. In the second part we give the description of the Poisson-Lie T-duality, and discuss the obstructions to develop the duality transformation between Poisson σ-models.

2. Poisson σ-models
We summarize some properties of the Poisson manifolds. See [5] or [6] for a more detailed description. The Poisson manifold is a differentiable manifold equipped with a Poisson bracket, i.e. a skew-symmetric bilinear mapping $\{,\} : C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M)$ which verifies the Leibnitz rule $\{fg,h\} = f\{g,h\} + \{f,h\}g$ and the Jacobi identity $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$. We restrict our considerations to the finite dimensional case and represent the Poisson bracket by a bivector field $\Pi$, called the Poisson bivector. Let $x^i$ be the set of coordinate functions on $U \subset M$, then

$$\Pi = \Pi^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} \in \Lambda^2 TM,$$

$$\Pi^{ij} \in C^\infty(M), \quad \Pi^{ij}(x) = \{x^i, x^j\}.$$
Such a $\Pi$ has to be skew-symmetric and satisfy the condition following from the Jacobi identity. This can be written as

$$\Pi^{ij} = -\Pi^{ji}, \quad \Pi^{ij} \frac{\Pi^{kl}}{\partial x^j} + \Pi^{ij} \frac{\Pi^{kl}}{\partial x^j} + \Pi^{ij} \frac{\Pi^{lk}}{\partial x^j} = 0. \quad (3)$$

We do not need the Poisson bivector to be non-degenerate. The Poisson manifolds are not symplectic in general, and a class of nonconstant functions $f$ may exist, such that $\{f, g\} = 0 \ \forall g \in C^\infty(M)$. These are called the Casimir functions. To every smooth function on $M$ a Hamiltonian vector field can be assigned as

$$X_f(g) := \Pi(df, dg) = \{f, g\}, \quad X_f = \Pi^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}. \quad (4)$$

The Poisson bivector is invariant under the flow of the Hamiltonian vector fields, and as a consequence of Jacobi identity, these vector fields form an infinite-dimensional algebra. In the same manner, a map $P_x : T_x^* M \to T_x M$ can be defined as $P_x(\alpha) = \Pi(\alpha, \cdot)(x)$ which specifies subspaces $P_x(T_x^* M) \subset T_x M$ spanned by the values of the Hamiltonian vector fields. Although the rank of $\Pi$ is not constant on $M$ and the dimension of these subspaces varies, by virtue of the Jacobi identity the generalized distribution is integrable and we obtain a generalized symplectic foliation of the Poisson manifold. The Poisson structure can be consistently restricted to a leaf as non-degenerate, and the leaves become symplectic manifolds. This is summarized by Weinstein’s splitting theorem, which states that if for an $n$-dimensional Poisson manifold $\text{Rank}\Pi = 2s$ at some point $x$, then there exists an $(n-2s)$-dimensional submanifold $N$ of $M$, which is transversal to the space $P_x(T_x^* M)$, and local canonical coordinates $(x^1, \ldots, x^s, p_1, \ldots, p_s, z^1, \ldots, z^{n-2s})$, such that $\{z^i, z^j\}(x) = 0$, and

$$\{f, g\} = \sum_{i,j}^{n-2s} \frac{\partial f}{\partial z^i} \frac{\partial g}{\partial z^j} \{z^i, z^j\} + \sum_{i=1}^s \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial p_j} - \frac{\partial g}{\partial x^i} \frac{\partial f}{\partial p_j}. \quad (5)$$

2.1. Poisson $\sigma$-model

The building blocks of the Poisson $\sigma$-model are a 2-dimensional orientable differentiable manifold $\Sigma$, referred to as the worldsheet, and an $n$-dimensional Poisson manifold $(M, \Pi)$, called the target space. We label the local coordinates on $\Sigma$ as $\sigma^\mu$, on $M$ as $x^i$.

The dynamical field of the model is a vector bundle map $(X, \tilde{A}) : T\Sigma \to T^* M$, composed of a base manifold map $X : \Sigma \to M$ and a total spaces map $\tilde{A} : T\Sigma \to T^* M$. Authors mostly prefer to express the action in local coordinates. To adopt this notation, we define $A : T\Sigma \to \Gamma(\Sigma, \Lambda^*(T^* M))$ as $\langle A, V \rangle(\sigma) := \tilde{A}(V_\sigma) \in T^\Sigma_{X(\sigma)} M$ and expand it as $A(\sigma) = A_i(\sigma)dx^i |_{X(\sigma)}$. The action of the Poisson $\sigma$-model, as originally appeared in [2], then reads

$$S_{X,A} = \int_{\Sigma} A_i \wedge dX^i + \frac{1}{2} \Pi^{ij}(X) A_i \wedge A_j, \quad (6)$$

where $dX^i = X^*(dx^i)$. In order not to bother with the boundary conditions, we impose the condition $\partial \Sigma = \emptyset$ on the worldsheet.

2.2. Solution of the Poisson $\sigma$-model

Field equations in the bulk follow from the action (6) as

$$dX^i + \Pi^{ij}(X) A_j = 0, \quad dA_i + \frac{1}{2} \frac{\partial \Pi^{jk}}{\partial x^i} (X) A_j \wedge A_k = 0. \quad (7)$$
The consistency of (7) is ensured by (3). These equations simplify significantly in the local canonical coordinates \((X^I, X^i), I = 1 \ldots n - 2s; i, j = 1 \ldots 2s\). According to (5), (7) reduces to
\[
dX^I = 0, \quad dA_I = 0, \quad dA_j = \Omega_{ij} dX^i.
\]
We see that \(X^I\) have to remain constant but arbitrary, \(X^i\) are undetermined, while their choice determines \(A_I\). Any of the \(A_I\) is equivalent to \(A_I + dh_I\). Solving the Equations of Motion of the Poisson \(\sigma\)-model is therefore equivalent to the problem of finding the local canonical coordinates. This is not easier at all, but at least it offers an alternative way of solving (7).

### 2.3. Physically relevant models

We have constructed the action of a Poisson \(\sigma\)-model and found its solution. To demonstrate an application of this construction, we first examine the action of a bosonic string.

\[
S_{str} = \frac{1}{4\pi \alpha'} \int_\Sigma d^2\sigma \sqrt{\det(h)} (G_{ij}(X)h^{\mu\nu} + e^{\mu\nu} B_{ij}(X)) \partial_\mu X^i \partial_\nu X^j,
\]
where \(h\) is a metric on \(\Sigma\), \(G\) is a metric on \(M\), \(B\) is a 2-form on \(M\). Introducing fields \(A_i\) as in (3), we may consider the first order action
\[
S_{str}^{(1)} = \int_\Sigma A_i \wedge dX^i + \pi \alpha' E^{ij}(X) A_i \wedge A_j + \frac{1}{2} \Pi^{ij}(X) A_i \wedge A_j,
\]
with Equations of Motion
\[
dX^i + \Pi^{ij}(X) A_j + E^{ij}(X) * A_j = 0.
\]
This model is classically equivalent to (9) if we insert (11) into (10), and set
\[
(G + B)^{-1} = E + \frac{1}{2\pi \alpha'} \Pi.
\]
The Poisson \(\sigma\)-model action is obtained from (10) through the Seiberg-Witten limit \(\alpha' \to 0\) while keeping \((G, B)\) fixed.

Another example of the classical equivalence of the \(\sigma\)-model action is the \(SU(2)\) gauge theory. A Poisson bivector constructed as \(\Pi^{ij} = \sum_{k=1}^3 \epsilon^{ijk} X^k\) admits a Casimir function \(\sum_{k=1}^3 X^k X^k\). The structure constants of \(su(2)\) are \(c^{jk}_i = \epsilon^{ijk}\), and the action
\[
S_M^{(1)} = \int_\Sigma X^i (dA_i + \frac{1}{2} c^{jk}_i A_j \wedge A_k) - \frac{g}{2} \omega^i \sum_{k=1}^3 X^k X^k,
\]
where \(g\) is a coupling constant and \(\omega\) is a volume form on \(\Sigma\), gives the first order action of the Yang-Mills theory. Taking the limit \(g \to 0\) restores the Poisson \(\sigma\)-model action.

In these two examples, the geometrical structure was richer than the structure of the Poisson \(\sigma\)-model. To construct the Poisson \(\sigma\)-model we do not need any metric structure; all the initial data are the components of \(\Pi\), so we speak about a topological field theory.

### 2.4. Poisson-Lie group

A Certain class of the Poisson \(\sigma\)-models is of special interest due to its relation to the integrable models, and, as we shall see, to the models connected by the Poisson-Lie T-duality. If the Poisson manifold is a Lie group \(G\), we impose a condition of compatibility of the Poisson and the group structure to obtain a Poisson-Lie group. In particular, the multiplication map \(\cdot : G \times G \to G\) has to be the Poisson map. This results in the statement that \(\Pi\) has to be a multiplicative bivector field
\[
\Pi(g g') = (L_{g} \otimes L_{g'}) \Pi(g') + (R_{g'} \otimes R_{g'}) \Pi(g).
\]
We are interested in the construction of the multiplicative Poisson bivectors. Therefore, we will inspect the algebraic structures of the Poisson-Lie group.
2.5. Construction of the Poisson-Lie bivector

The crucial statement concerning Poisson-Lie group \((G, \Pi)\) is that there exists a unique Lie bialgebra structure \(\delta\) on \(g\), such that the bialgebra structure is given by the intrinsic derivative of \(\Pi\). Such a \((g, \delta)\) is called the tangent Lie bialgebra. Moreover, if \((g, \delta)\) is a finite dimensional Lie bialgebra, there exists a connected and simply connected Lie group \(G\) with the Lie algebra \(g\) and a Poisson-Lie group structure given by \(\Pi\), such that \((g, \delta)\) is the tangent Lie bialgebra to \((G, \Pi)\). See for example [4] for details.

To every bialgebra structure \((g, \delta)\), there exists a corresponding Manin triple \((\mathfrak{d}, g, \tilde{g})\), where \(\mathfrak{d}\) is a Lie algebra of dimension \(2n\) with a symmetric, non-degenerate, ad-invariant bilinear form \(\langle \ldots, \ldots \rangle_\mathfrak{d}\) and \(g, \tilde{g}\) are complementary \(n\)-dimensional subalgebras of \(\mathfrak{d}\), maximally isotropic w.r.t. \(\langle \ldots, \ldots \rangle_\mathfrak{d}\). The corresponding group structure is a Drinfeld double \(D \equiv (G, \tilde{G})\), a connected and simply connected Lie group with connected and simply connected subgroups \(G, \tilde{G}\), where the Lie algebras \((\mathfrak{d}, g, \tilde{g})\) form a Manin triple \((\mathfrak{d}, g, \tilde{g})\).

To construct the Poisson-Lie bivector of the desired properties on \(G\), we consider the matrix of the adjoint representation of \(G\) on the algebra \(g\)

\[
Ad_{g^{-1}} = \begin{pmatrix} a(g)^T & b(g)^T \\ 0 & d(g)^T \end{pmatrix}. \tag{15}
\]

To obtain \(\Pi\) in the basis of the right-invariant fields, we set

\[
\Pi(g) := -b(g).a^{-1}(g) = (b(g).a^{-1}(g))^T, \tag{16}
\]

which is transformed to the coordinate basis using the matrix of the components of the right-invariant fields.

The result of \(\lbrace 16 \rbrace\) is the same as that of the Sklyanin bracket, where Poisson bivector is calculated using an \(r\)-matrix. However, the technique just described works for an arbitrary Lie bialgebra.

The group structure of the Poisson-Lie \(\sigma\)-model brings a lot of interesting properties. Nevertheless, we shall now focus on a slightly different \(\sigma\)-models, which surprisingly have much in common with those just described.

3. Poisson-Lie T-duality

Let us now examine \(\sigma\)-models given by the action that in lightcone coordinates \(x_{\pm} = \frac{1}{2}(\tau \pm \sigma)\) on the worldsheet reads

\[
S_F(\phi) = \int_{\Sigma} dx_+ dx_- \partial_- \phi^\mu F_{\mu\nu}(\phi) \partial_+ \phi^\nu. \tag{17}
\]

Here \(\Sigma \subset \mathbb{R}^2\), \(\phi : \mathbb{R}^2 \mapsto M, \phi \in C^\infty, \mathbb{R}^2\) is equipped with the Minkowski metric, \(M\) is equipped with a second order covariant tensor field \(F\) and \(\phi^\mu : \mathbb{R}^2 \mapsto \mathbb{R}\), \(\mu = 1, \ldots, \dim M\) is the composition of \(\phi\) and components of a coordinate map on a chart of \(M\). We consider \(\phi^\mu\) to be a row vector, so here and in the following all the matrices are in fact transposed. The equations of motion follow from \(\lbrace 17 \rbrace\) as

\[
\partial_+ \partial_- \phi^\mu + \Gamma^\mu_{\nu\lambda} \partial_- \phi^\nu \partial_+ \phi^\lambda = 0, \tag{18}
\]

with \(\Gamma^\mu_{\nu\lambda} = \frac{1}{2}G^{\mu\nu}(F_{\rho\lambda,\nu} + F_{\nu\rho,\lambda} - F_{\nu\lambda,\rho})\). We consider \(G\), the symmetric part of \(F\), to be the metric on \(M\), while the antisymmetric part \(B\) to be the torsion potential.

Suppose that there is a Lie group \(G\) acting freely and transitively on \(M\) and \(M\) can be identified with \(G\). For more general cases see \[7\]. If the tensor \(F\) satisfies

\[
(L_{\nu a} F)_{\mu\nu} = F_{\mu\lambda} v^\nu_c v^\lambda_b F_{\nu\lambda}, \tag{19}
\]
where $e_{ac}^b$ are the structure constants of a Lie algebra $\mathfrak{g}$ of some Lie group $\tilde{G}$, and $v_a^b(g)$ are the components of left-invariant vector fields, we say that $\mathcal{F}$ is $G$-Poisson-Lie symmetric with respect to $\tilde{G}$, or that the $\sigma$-model has generalized symmetries. Self-consistency of this condition gives a condition on $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ to form mutually dual bialgebras. We again meet the structure of a Drinfeld double. \cite{19} first appeared in \cite{1}. It is solved by a technique presented there, which we are going do describe and which relies on the Poisson-Lie structure. We will construct $\sigma$-models on subgroups $G$ and $\tilde{G}$ and show that there exists a duality transformation between them.

We can choose bases $T_a \in \mathfrak{g}$, $\tilde{T}^a \in \tilde{\mathfrak{g}}$, such that $(T_a, \tilde{T}^b)_0 = \delta_a^b$. Due to ad-invariance of $(\ldots)_0$, the structure constants of $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ specify the algebra of $\mathfrak{d}$.

\begin{equation}
[T_a, T_b] = e_{ab}^c T_c, \quad \tilde{[T}^a, \tilde{T}^b] = e_{ab}^c \tilde{T}^c, \quad [T_a, \tilde{T}^b] = \tilde{e}^b_a \tilde{T}^c + \tilde{e}^{bc} T_c.
\end{equation}

Consider a regular linear map $E : \mathfrak{g} \rightarrow \tilde{\mathfrak{g}}$. Graph of $E$ specifies two subspaces
\begin{equation}
\mathcal{E}^+ := \text{Span} \left[ (T_i + E_{ij} \tilde{T}^j)_{i \in \tilde{m}} \right], \quad \mathcal{E}^- := \text{Span} \left[ (T_i - E_{ij} \tilde{T}^j)_{i \in \tilde{m}} \right].
\end{equation}

Choosing two elements $V_{\pm} := \partial_{\pm l} l^{-1}$, the Equations of Motion on the double can be written as
\begin{equation}
V_{\pm} \in \mathcal{E}^\pm.
\end{equation}

Inserting $D \ni l = g \tilde{h}, g \in G, \tilde{h} \in \tilde{G}$ into \cite{22} we obtain
\begin{equation}
\begin{align*}
(\partial_+ g \partial_- g^{-1})_d &= (a^{-1}(g))_d e_{ac}^b [ -E_{ca}(g)(\partial_+ gg^{-1})^a ], \\
(\partial_- g \partial_+ g^{-1})_d &= (a^{-1}(g))_d e_{ac}^b [ E_{ac}(g)(\partial_- gg^{-1})^a ],
\end{align*}
\end{equation}
where $(\partial_+ gg^{-1})^a = \partial_+ \phi^a, e_{\mu}^a(g)$ and $e_a^\mu(g)$ are the components of the right-invariant Maurer-Cartan form $dgg^{-1}$. The dualizable $\sigma$-model which verifies \cite{19} is computed using
\begin{equation}
E(g) := (E(g)^{-1} + \Pi(g))^{-1}, \quad \Pi(g) := b(g) \cdot a(g)^{-1},
\end{equation}
where $a(g)$ and $b(g)$ are the submatrices of the adjoint representation of $G$ on $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$. Our $\sigma$-model action \cite{17} is calculated from \cite{22} as
\begin{equation}
S_{\mathcal{F}}(\phi) = \int_{\Sigma} dx_+ dx_- \left( E_{ab}(y, g)(\partial_- gg^{-1})^a (\partial_+ gg^{-1})^b \right).
\end{equation}

As we can see, the dualizable model is calculated using the same structures as the Poisson-Lie $\sigma$-model and we have the Poisson-Lie group structure. We can write the Equations of Motion \cite{18} as
\begin{equation}
\partial_+ (\partial_- \tilde{h} \tilde{h}^{-1})_{c} - \partial_- (\partial_+ \tilde{h} \tilde{h}^{-1})_{c} + \tilde{f}_{ab}^c (\partial_+ \tilde{h} \tilde{h}^{-1})_a (\partial_+ \tilde{h} \tilde{h}^{-1})_b = 0.
\end{equation}

If we switch $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$, we speak about the Poisson-Lie T-duality. But we can also decompose the algebra of the Drinfeld double into a new pair of subalgebras $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$ and speak about Poisson-Lie T-plurality. Expressing $\mathcal{E}^\pm$ in the $\tilde{T}, T$ bases, related to the original ones by
\begin{equation}
\begin{pmatrix} T \\ \tilde{T} \end{pmatrix} = \begin{pmatrix} \mathcal{P} & Q \\ \mathcal{R} & S \end{pmatrix} \begin{pmatrix} \tilde{T} \\ T \end{pmatrix},
\end{equation}
where $\mathcal{P}, Q, R, S$ are $\dim G \times \dim G$ matrices, and considering the action of $\tilde{G}$ on $\mathfrak{d}$, we calculate $\tilde{\mathcal{F}}$ for the $\sigma$-model on $\tilde{G}$ in a similar way as above by
\begin{equation}
\tilde{E} = (\mathcal{P} + E \cdot \mathcal{R})^{-1} \cdot (Q + E \cdot S),
\end{equation}
\begin{equation}
\tilde{E} = (\mathcal{P} + E \cdot S)^{-1}. \end{equation}
\[ \hat{E}(\hat{g}) = (\hat{E}^{-1} + \hat{\Pi}(\hat{g}))^{-1}, \quad \hat{\Pi}(\hat{g}) = \hat{b}(\hat{g}) \cdot \hat{a}(\hat{g})^{-1}, \tag{29} \]

and the fields \( \hat{e}(\hat{g}) \).

The Equations of Motion for both \( \sigma \)-models follow from (22), and the solutions are related by two possible decompositions of the element \( l \in D \)

\[ l = \hat{g} \hat{h} = \hat{g} \hat{h}. \tag{30} \]

The transformation between these two decompositions is the Poisson-Lie T-plurality, which we were trying to find.

4. Discussion

We have described the necessary elements of the Poisson-Lie \( \sigma \)-model and the Poisson-Lie T-duality. We see that the underlying algebraic structures are the same and an obvious question arises, whether we can develop some form of the duality transformation for the Poisson-Lie \( \sigma \)-model. The crucial concept of the duality is hidden in the possibility to express the Equations of Motion on the double in the form (22), (23) through the subspaces \( E^\pm \). The duality expresses these subspaces in different decompositions of the algebra of the double.

On the side of the Poisson-Lie \( \sigma \)-model, we have no \( E \) to specify the subspaces \( E^\pm \). Similarly, we can hardly restore (12) without having \( E \). We do not know yet, how to write the Equations of Motion on the double in such a way that it would be possible to pass from one model to the other one. Nevertheless, some form of the duality of Poisson-Lie \( \sigma \)-model was suggested by [4] as the bulk-boundary duality. Therefore we hope that an analogue of the T-duality transformation of the solutions can be developed as well.

Whether this attempt will be successful or not, we may inspect, if this effort is worthwhile. We demonstrate on an example that a physically relevant theory may be constructed on a Drinfeld double. Since the dual model is given purely by the algebraical data through (16), it can be easily computed later.

5. Example

To give an example, we choose the \( R^2 \)-gravity theory as in [2]. We shall use the fact that all the 6-dimensional Drinfeld doubles were classified in [8] and the intrinsic derivative of \( \Pi \) gives the algebra structure of the dual group. The Lagrangian reads

\[ \mathcal{L}_{R^2} = \frac{1}{4} \int d^2 x \sqrt{\text{det} \ g} \left( \frac{1}{4} R^2 + 1 \right), \tag{31} \]

where \( g \) is the 2-dimensional metric and \( R \) is the Ricci scalar of the associated Levi-Civita connection. Adopting the Einstein-Cartan variables \( (e^1, e^2) \) as the zweibein, and \( \omega \) as the connection 1-form, we have

\[ \mathcal{L}_{R^2} = \int_M X (de^1 - \omega \wedge e^2) + Y (de^2 + \omega \wedge e^1) + Z d\omega + \left( \frac{1}{4} - Z^2 \right) e^1 \wedge e^2. \tag{32} \]

Choosing \( X^i = (X, Y, Z), A_i = (e^1, e^2, \omega) \), we get the Poisson \( \sigma \)-model with

\[ \{X, Y\} = -Z^2 + \frac{1}{4}, \quad \{Y, Z\} = X, \quad \{Z, X\} = Y, \tag{33} \]

and local canonical coordinates

\[ \begin{align*}
\tilde{x} &= x^2 + y^2 - \frac{2}{3} z^3 + \frac{1}{2} z, \\
\tilde{y} &= \arctan \left( \frac{y}{x} \right), \\
\tilde{z} &= z.
\end{align*} \tag{34} \]
Is there a way we could build this structure on a subgroup $G$ of the Drinfeld double? From (14) we know that a multiplicative $\Pi$ vanishes at the unit element. We can choose $e = [0, 0, \pm \frac{1}{2}]$. The first choice induces the Lie algebra structure of Bianchi8, i.e. $(\mathfrak{sl}(2, \mathbb{R}))$, on $\tilde{\mathfrak{g}}$, and it can be built on the Abelian subgroup of the Drinfeld double $(1 | 8)$, where the numbers refer to the Bianchi classification. The latter choice induces the Lie algebra of Bianchi9 $(\mathfrak{so}(3))$. There are other possible Drinfeld doubles containing $(\mathfrak{so}(3))$ or $(\mathfrak{sl}(2, \mathbb{R}))$ than $(1 | 9)$ or $(1 | 8)$, but we are looking for an easy example, knowing that we have to find a coordinate transformation which will transform the computed $\Pi$ to (33). Such a coordinate transformation can be found; however, we noticed that much simpler calculations are needed considering doubles $(1 | 7_0)$, $(2i | 7_0)$ or $(2ii | 7_0)$, with $e = [0, 0, 0]$. Then

$$\{X, Y\} = 0, \quad \{Y, Z\} = X, \quad \{Z, X\} = Y,$$

and local canonical coordinates are

$$\tilde{x} = x^2 + y^2,$$
$$\tilde{y} = \arctan(\frac{y}{x}),$$
$$\tilde{z} = z.$$

The transformation to (33) can be done by the composition of (34) and the inverse of (36).

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