Hybrid optimisation algorithm and its application for pattern synthesis of planar arrays

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Abstract: A hybrid approach to the synthesis of sparse arrays radiating shaped or focused beampattern is presented in this paper. Specifically, an iterative scheme is used where the prescribed pattern response in the mainlobe peak direction is cast as a multi-convex problem at each step that the non-convex lower bound constraint is relaxed while including a reweighted l1-norm minimisation based on the magnitudes of the elements. Numerical tests, referred to benchmark problems, show that the proposed method achieves improved radiation performance by using fewer number of radiating elements.

1 Introduction

Antenna measurements applications ranging from remote sensing and radar to modern wireless communication systems require the design of both linear arrays and planar arrays. This is the reason why the research interest of array pattern synthesis problem have recently drawn a lot of attention and interest. The purpose of pattern synthesis is to find the best solution of the array coefficients and the elements positions to generate desired radiation pattern. In [1–4], the iterative Fourier technique (IFT) for the synthesis of linear and planar arrays with uniform inter-element spacing is presented. This iterative algorithm makes use of the property that for a linear (planar) array with uniform spacing of the elements, an inverse Fourier transform relationship exists between the element excitations and the array factor (AF). By taking advantage of this property, the array element excitations can be derived from the prescribed AF through a direct Fourier transform.

The synthesis of sparse arrays with a minimum number of elements has several practical advantages such as the reduction of weight, cost, power consumption and the simplification of the BFN. Compared with uniformly spaced arrays, sparse arrays have the advantage of having peak sidelobe level and higher spatial resolution (i.e. narrow beamwidth) while using a reduced number of antennas. As expected, this also reduces the cost of whole antenna system. Some optimisation methods have been proposed to synthesise such arrays. One can mention spatial tapering designs [5], global optimisation algorithm (simulated annealing [6], genetic algorithm [7]), combinatorial approaches [8] and linear programming [9]. Recently, the extended forward-backward matrix pencil method has been applied to the multiple-pattern sparse array synthesis [10]. This method is indeed effective for reducing the number of elements.

With the recent advances in convex optimisation, another classic algorithm has lately been used to efficiently solve array synthesis problems. Specifically, in [11] and [12], centrosymmetric antenna layouts and conjugate symmetric beamforming weights are used so that the non-convex lower bound constraints on the beampattern can be convex. Thus, a mainlobe of an arbitrary beamwidth (ripple) can be obtained. The method takes definite advantage from the convexity of the problem with respect to conjugate symmetric excitations. Despite its excellent performances in terms of the minimum number of required radiating elements [11], the method allows to synthesise only real-valued powerpatterns. However, many applications require shaped beams with complex-valued responses.

After reviewing the advantages and disadvantages of the existing methods, a method to synthesise shaped beams with antenna arrays is described in this paper. The provided solutions exhibit key feature that allows to minimise the weight and complexity of the BFN and maximise the radiation performance of shaped power pattern. First, it shows and discusses techniques for the simplification of the array architecture, where the requirement of minimum number of antenna elements can be considered in the iterative optimisation problems. Second, differently from usual beam design methods, the proposed design can radiate beampatterns characterised by arbitrary beamwidth and response ripple through reformulating the non-convex lower bound constraints on the beampattern as an equivalent multi-convex optimisation problem with the addition of two auxiliary variables.

The paper is organised as follows: the optimisation problem is mathematically formulated where the design of reconfigurable sparse arrays generating shaped beams is presented in Section 2. A set of representative results is reported and discussed in Section 3. Some conclusions are drawn in Section 4.

2 Mathematical formulation

2.1 Antenna array

It uses the property that the AF of a planar array with elements arranged in a square grid at distance d along M columns and N rows, the array beampattern is defined as

\[ \text{AF}(u,v) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} W_{mn} e^{j(k/md + k/dv)}, \]

where \( u = \sin(\theta)\cos(\phi) \), \( v = \sin(\theta)\sin(\phi) \) are the observation direction, \( k = 2\pi/\lambda \) is the wavelength in freespace, \( \theta \) and \( \phi \) are the observation direction. Introducing the MN-dimensional steering and excitation vectors

\[ s = [1, e^{jkd_1}, \ldots, e^{jkd_{M-1}}, e^{jkd_{M-2d_1}}, \ldots, e^{jkd_{M-2(N-1)d_1}}, e^{jkd_{M-Nd_1}}, \ldots, e^{jkd_{M-Nd_{M-1}}}]^T, \]

and

\[ w = [W_{0,0}, W_{1,0}, \ldots, W_{M-2,0}, \ldots, W_{M-2,N-1}, W_{M-1,N-1}]^T. \]

The far field \( \text{AF}(u,v) \) radiated by the array is then

\[ \text{AF}(u,v) = s^H w, \]
where $(\cdot)^H$ denotes the Hermitian transposition. Two non-overlapping domains, $U_{ML}$ and $U_{SL}$ represent mainlobe and sidelobe regions. The power radiated by the array $|AF(u, v)|^2$ is

- as close as possible to a desired shape $d(u, v)$, in the shaped beam region $U_{ML}$;
- upper-bounded is defined by an envelope $\rho(u, v)$, in the sidelobe region $U_{SL}$.

The process to achieve desired features is to minimise the maximum distance $\epsilon$ between the targeted shape $d(u, v)$ and the far-field $|AF(u, v)|^2$. The constraints problem can be formulated as

$$\min_{\epsilon} \epsilon \begin{cases} \sup_{(u, v) \in U_{ML}} \left| H^2(u, v)w - d(u, v) \right| \leq \epsilon, \\ \left| H^2(u, v)w \right| \leq \rho(u, v), \text{ for } (u, v) \in U_{ML}. \end{cases}$$

(5)

the smallest shaped beam ripple $\epsilon$, the sidelobes are kept below the envelop $\rho(u, v)$.

### 2.2 Resolution method

The IFT plays the role of determining the excitations of the array elements to approximate a desired power pattern. The process of the Iterative FFT is formulated in detail as follows.

i. Start the synthesis using a set of the array coefficients with the uniform 1 for $M \times N$ aperture elements.
ii. The calculation of $AF$ using a $K \times K$ point 2-D discrete inverse FFT samples with $K > \text{maximum}(M, N)$.
iii. Match the mainlobe region of $AF$ to the mainlobe constrains and left unchanged the $AF$ values not violating the mainlobe thresholds like the $AF$ values required in the sidelobe region.
iv. Using a direct 2-D $K \times K$ points FFT to calculate an updated set of excitations, only the $M \times N$ samples associated with the array elements are retained.
v. The constraints for the resulting element excitations are enforced according to the following procedure.

a. In the case of centrosymmetry fashion, the patterns are obtained by subdividing the planar array into four symmetric quadrants (only the $(M \times N)/4$ samples belonging to the array are retained in each step of iterative algorithm) and imposing a quadrantal symmetry of the excitation amplitudes to the rest of the elements;

b. In case of an aperture with a non-rectangular shape, make the $M \times N - M_{\text{total}}$ ($M_{\text{total}}$ is the total number of element positions across the aperture) samples of $w_{\text{mn}}$ located outside the aperture equal to zero.

vi. Repeat iterative process 2–6 until all directions of the updated $AF$ satisfy prescribed beampattern requirements or the maximum number of iteration is reached.

In this paper, all synthesised tapers were obtained with the calculation of $AF$ for $K \times K$ far-field directions (set $K$ to 6001 in the following examples).

For the synthesis of reconfigurable planar arrays generating shaped beams, the mainlobe region $U_{ML}$ and sidelobe region $U_{SL}$ are sampled in $S_k = S(u, v)_k$ for $k = 1, \ldots, K$ and $S_q = S(u, v)_q$ for $q = 1, \ldots, Q$. It yields the real positive numbers $d_k = d(u, v)_k$ and $d_q = d(u, v)_q$. The problem (5) is thus approximated as

$$\min_{\epsilon} \epsilon \begin{cases} \max_{k = 1, \ldots, K} \left| H^2_k w - d_k \right| \leq \epsilon, \\ \left| H^2_k w \right| \leq \rho_{\epsilon}, \text{ for } q = 1, \ldots, Q. \end{cases}$$

(6)

The quadratic sidelobe constraints in (6) are standard convex programming routines can be easy to solved, and proposed mainlobe constraint can be rewritten as

$$\begin{align*}
\max_{k = 1, \ldots, K} \left| H^2_k w - d_k \right| & \leq \epsilon, \\
\left| H^2_k w \right| & \leq \rho_{\epsilon}, \text{ for } q = 1, \ldots, Q.
\end{align*}$$

(7)

or

$$\begin{align*}
\max_{l = 1, \ldots, K} \left| H^2_l w - d_l \right| & \leq \epsilon \quad \text{with } w_l = w_r. \\
\left| H^2_l w \right| & \leq \sqrt{\rho_{\epsilon}}, \text{ for } q = 1, \ldots, Q.
\end{align*}$$

(8)

Then, if the excitation vector $w_r$ or $w_l$ in (8) is alternatively determined, non-convex ripple constraint on the mainlobe beampattern can be transformed into convex one. By fixing $w_r$ for instance, the multidimensional $H^2_l w$ is constant, thus, the problem is mathematically formulated as

$$\begin{align*}
\min_{w_r} & \max_{l = 1, \ldots, K} \left| H^2_l w_r - d_l \right| \leq \epsilon, \\
& \left| H^2_l w_r \right| \leq \sqrt{\rho_{\epsilon}}, \text{ for } q = 1, \ldots, Q.
\end{align*}$$

(9)

The problem (9) is a convex problem with affine can be solved efficiently, to guarantee equivalence between the problem (9) and the original beampattern synthesis problem, $w_l$ must be equivalent to $w_r$ as written in (8), in this paper, a reasonable starting point $w_l$ for the algorithm is obtained by the above IFT method. Then, the procedure can be solved optimally by readily available software.

### 2.3 Discussion and remarks

This paper presents the novel hybrid algorithm combines IFT with quadratic programming procedure, some remarks on the proposed methods are:

i. The occurrence of a trap is a real problem which can be avoided only by a suitable choice of the starting point of the iteration. IFT method allows to maximise the radiation performance. Hence, the proposed design choose optimum solution $w_{\text{ml}}$ as reasonable starting point. Compared scheme preliminary introduced in [12], this start point closer to the globally optimal solution.

ii. As a scheme inspired from [12] is proposed, the step is an averaging and smoothing operation to ensure that asymptotically the difference between $w_r$ and $w_l$ vanishes.

In the case of centrosymmetry fashion, planar array are subdivided into four symmetric quadrants

$$\begin{align*}
[w_{0,0}, \ldots, w_{M/2-1,N/2-1}] & = [w_{M-1,N-1}, \ldots, w_{M-1,N}]. \\
[w_{M-1,0}, \ldots, w_{M-1,N}]. & = [w_{0,0}, \ldots, w_{M-1,N-1}, \ldots, w_{M-1,N}].
\end{align*}$$

(10)

The conjugate symmetric constraint are enforced on the array excitations, planar array are subdivided into four symmetric quadrants. Desired beam pattern is obtained by adding conjugate operator to the excitations of a couple of quadrants, and the problem to solve is then

$$\begin{align*}
[w_{0,0}, \ldots, w_{M/2-1,N/2-1}] & = [w_{M-1,N-1}, \ldots, w_{M-1,N}]. \\
[w_{M-1,0}, \ldots, w_{M-1,N}]. & = [w_{M-1,N-1}, \ldots, w_{M-1,N}].
\end{align*}$$

(11)

These limitations can be solved by just exploiting convex programming routines. In following example, all cases solved by the interior point method (IPM) [13].

### 3 Numerical results

#### 3.1 Flat-top beam synthesis with planar array

In the first test case, to analyse the method performance in dealing with flat-top beams, let the planar arrays composed of $11 \times 11$ isotropic sources arrangement in a centrosymmetric layout. The shaped beam regions $U_{ML}$ and sidelobe $U_{SL}$ regions are $(u^2 + v^2) \leq 0.2588^2$ and $0.3420^2 \leq (u^2 + v^2) \leq 1^2$, respectively. The desired pattern $d_k$ is equal to unity over $U_{SL}$, the synthesised beampattern.
Whereas the sidelobes are constrained to remain below $-17.34$ dB, the mainlobe response ripple $\varepsilon$ is 0.0849 dB.

### 3.2 Beampattern synthesis for planar arrays

In the second case, conjugate symmetric beamforming weights is used, and element failures is taken into account by setting their excitation values to zero. The synthesis method (7) is used to design a circular-shaped mainlobe with controlled sidelobe level. As in [12], the $\lambda/2$ spaced $11 \times 11$ array with the normalised embedded elements is shown in Fig. 2, accounting for 70% of those in the original array. The mainlobe region is $(u^2 + v^2) \leq 0.22$ with $r_{db} = 0.97$ dB in the shaped beam region. The sidelobe region is $(u^2 + v^2) \geq 0.4^2$, the minimal sidelobe level achieved is $-26.24$ dB. Compared with the experimental data from the literature [19], the sidelobe level of the shaped beam is 0.4 dB lower. Conjugate symmetric beamforming weights of excitation magnitudes and phases are given in Figs. 2 and 3. A final representative beampattern is shown in Figs. 4–6, the optimised array is shown in Fig. 7.

### 4 Conclusion

A hybrid algorithm for pattern synthesis with the minimum number of array elements is introduced in this paper. The proposed design providing two important contributions. First, there is no restriction regarding the type of array and pattern to be synthesised. Second, the method is easy to implement and there is no parameter to be tuned.
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