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Real Paley-Wiener theorems for the Dunkl transform on $\mathbb{R}^d$

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Abstract

In this paper, we establish real Paley-Wiener theorems for the Dunkl transform on $\mathbb{R}^d$. More precisely, we characterize the functions in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ and in $L^2_k(\mathbb{R}^d)$ whose Dunkl transform has bounded, unbounded, convex and nonconvex support.

Key word: Dunkl transform on $\mathbb{R}^d$. Real Paley-Wiener theorems.
AMS subject classification: 42B10.

1 Introduction

In the last few years there has been a great interest to real Paley-Wiener theorems for certain integral transforms, see [15] for an overview references and details for this question.

In this paper we consider the Dunkl operators $T_j, j = 1, \ldots, d$, which are the differential-difference operators introduced by C.F.Dunkl in [5]. These operators are very important in pure Mathematics and in Physics. They provide a useful tool in the study of special functions with root systems (see [6].)

C.F.Dunkl in [7] (see also [8]) has studied a Fourier transform $\mathcal{F}_D$, called Dunkl transform defined for a regular function $f$ by

$$\forall x \in \mathbb{R}^d, \mathcal{F}_D f(x) = \int_{\mathbb{R}^d} K(-ix,y)f(y)\omega_k(y)dy,$$

where $K(-ix,y)$ represents the Dunkl kernel and $\omega_k$ a weight function.

The aim purpose of this paper is to prove real Paley-Wiener theorems on the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ and on $L^2_k(\mathbb{R}^d)$. More precisely we consider first the Paley-Wiener spaces
associated with the Dunkl operators:

\[
PW_k^2(\mathbb{R}^d) = \{ f \in \mathcal{E}(\mathbb{R}^d)/\forall n \in \mathbb{N}, \ \triangle_k^n f \in L_k^2(\mathbb{R}^d) \text{ and } R_{f_j} \triangle_k = \lim_{n \to \infty} ||\triangle_k f||_{k,2}^n < +\infty\}
\]

\[
PW_k(\mathbb{R}^d) = \{ f \in \mathcal{E}(\mathbb{R}^d)/\forall n, m \in \mathbb{N}, \ (1 + ||x||)^m \triangle_k^n f \in L_k^2(\mathbb{R}^d) \text{ and } R_{f_j} \triangle_k < +\infty\},
\]

where \( \mathcal{E}(\mathbb{R}^d) \) is the space of \( C^\infty \)-functions on \( \mathbb{R}^d \), \( \triangle_k = \sum_{j=1}^{d} T_j^2 \) the Dunkl-Laplacian operator, \( L_k^2(\mathbb{R}^d) \) the space of square integrable functions with respect to the measure \( \omega_k(x)dx \) and ||.||_{k,2} the norm of the space \( L_k^2(\mathbb{R}^d) \).

We establish that \( \mathcal{F}_D \) is a bijection from \( PW_k^2(\mathbb{R}^d) \) onto \( L_k^2(\mathbb{R}^d) \) (the space of functions in \( L_k^2(\mathbb{R}^d) \) with compact support), and from \( PW_k(\mathbb{R}^d) \) onto \( D(\mathbb{R}^d) \) (the space of \( C^\infty \)-functions on \( \mathbb{R}^d \) with compact support).

Next, we characterize the \( L_k^2(U) \)-functions by their Dunkl transform, where \( U \) is respectively a disc, a symmetric body, a nonconvex and an unbounded domain in \( \mathbb{R}^d \). These results are the real Paley-Wiener theorems for square integrable functions with respect to the measure \( \omega_k(x)dx \).

We generalize also a theorem of H.H.Bang [2] by characterizing the support of the Dunkl transform of functions in \( S(\mathbb{R}^d) \) by an \( L^p \) growth condition. More precisely these real Paley-Wiener theorems can be stated as follow:

- The Dunkl transform \( \mathcal{F}_D(f) \) of \( f \in S(\mathbb{R}^d) \) vanishes outside a polynomial domain \( U_P = \{ x \in \mathbb{R} : P(x) \leq 1 \} \), with \( P \) a non constant polynomial, if and only if
  \[
  \limsup_{n \to +\infty} ||P^n(iT)f||_{k,p} \leq 1, \ 1 \leq p \leq \infty,
  \]
  with \( T = (T_1, ..., T_d) \) and ||.||_{k,p} is the norm of the space \( L_k^p(\mathbb{R}^d) \) of \( p \)th integrable functions on \( \mathbb{R}^d \) with respect to the measure \( \omega_k(x)dx \).

- A function \( f \in S(\mathbb{R}^d) \) is the Dunkl transform of a function vanishing in some ball with radius \( r \) centered at the origin, if and only if
  \[
  \lim_{n \to -\infty} \sum_{m=0}^{\infty} \frac{(n \triangle_k)^m f}{m!} ||\frac{1}{k}\exp(-r^2), \ 1 \leq p \leq \infty.
  \]

This paper is arranged as follows:

In the second section we recall the main results about the harmonic analysis associated with the Dunkl operators.

The third section is devoted to study the functions such that the support of their Dunkl transform are compact, and to establish the real Paley-Wiener theorems for \( \mathcal{F}_D \) on the Schawrz space \( S(\mathbb{R}^d) \).

In the fourth section we characterize the functions in \( S(\mathbb{R}^d) \) such that their Dunkl transform vanishes outside a polynomial domain.

In the fifth section we give a necessary and sufficient condition for functions in \( L_k^2(\mathbb{R}^d) \) such that their Dunkl transform vanishes in a disc.

We study in the sixth section the functions such that their Dunkl transform satisfies
the symmetric body property, and we derive a real Paley-Wiener type theorem for these functions.

2 Harmonic analysis associated for the Dunkl operators.

In the first two subsections we collect some notations and results on Dunkl operators, the Dunkl kernel and the Dunkl intertwining operators (see [6],[7],[8]).

2.1 Reflection groups, root system and multiplicity functions

We consider $\mathbb{R}^d$ with the euclidean scalar product $<..>$ and $||x|| = \sqrt{\langle x,x \rangle}$. On $\mathbb{C}^d$, $||.||$ denotes also the standard Hermitian norm while for all $z = (z_1, ... , z_d)$, $w = (w_1, ... , w_d) \in \mathbb{C}^d$,

\[ <z,w> = \sum_{j=1}^{d} z_j \bar{w}_j. \]

For $\alpha \in \mathbb{R}^d \backslash \{0\}$, let $\sigma_{\alpha}$ be the reflection in the hyperplan $H_{\alpha} \subset \mathbb{R}^d$ orthogonal to $\alpha$, i.e.

\[ \sigma_{\alpha}(x) = x - 2\frac{\langle \alpha, x \rangle}{||\alpha||^2} \alpha. \] (1)

A finite set $R \subset \mathbb{R}^d \backslash \{0\}$ is called a root system if $R \cap \mathbb{R}.\alpha = \{\alpha, -\alpha\}$ and $\sigma_{\alpha}R = R$ for all $\alpha \in R$. For a given root system R the reflection $\sigma_{\alpha}, \alpha \in R$, generate a finite group $W \subset O(d)$, the reflection group associated with R. We denote by $|W|$ its cardinality. All reflections in W correspond to suitable pairs of roots. For a given $\beta \in \mathbb{R} \backslash \alpha \in R \cap H_{\alpha}$, we fix the positive subsystem $R_+ = \{\alpha \in R / \langle \alpha, \beta \rangle > 0\}$, then for each $\alpha \in R$, either $\alpha \in R_+$ or $-\alpha \in R_+$.

A function $k : R \rightarrow \mathcal{C}$ on a root system R is called a multiplicity function if it is invariant under the action of the associated reflection group W. If one regards $k$ as a function on the corresponding reflections, this means that $k$ is constant on the conjugacy classes of reflections in W. For abbreviation, we introduce the index

\[ \gamma = \gamma(k) = \sum_{\alpha \in R_+} k(\alpha). \] (2)

Moreover, $\omega_k$ denotes the weight function

\[ \omega_k(x) = \prod_{\alpha \in R_+} |\langle \alpha, x \rangle|^{2k(\alpha)}, \] (3)

which is $W-$invariant and homogeneous of degree $2\gamma$.

We introduce the Mehta-type constant

\[ c_k = (\int_{\mathbb{R}^d} \exp(-||x||^2)\omega_k(x) \, dx)^{-1}, \] (4)
Remark

For \( d = 1 \) and \( W = Z_2 \), the multiplicity function \( k \) is a single parameter denoted \( \gamma > 0 \) and we have
\[
\forall x \in \mathbb{R}, \omega_k(x) = |x|^{2\gamma}.
\]

2.2 Dunkl operators- The Dunkl kernel and the Dunkl intertwining operator

**Notations.** We denote by
- \( C(\mathbb{R}^d) \) \((\text{resp } C_c(\mathbb{R}^d)) \) the space of continuous functions on \( \mathbb{R}^d \) \((\text{resp. with compact support}) \).
- \( C^p(\mathbb{R}^d) \) \((\text{resp } C^p_c(\mathbb{R}^d)) \) the space of functions of class \( C^p \) on \( \mathbb{R}^d \) \((\text{resp. with compact support}) \).
- \( E(\mathbb{R}^d) \) the space of \( C^\infty \)-functions on \( \mathbb{R}^d \).
- \( C^\infty_0(\mathbb{R}^d) \) the space of \( C^\infty \)-functions on \( \mathbb{R}^d \) which vanish at the infinity.
- \( S(\mathbb{R}^d) \) the space of \( C^\infty \)-functions on \( \mathbb{R}^d \) which are rapidly decreasing as their derivatives.
- \( D(\mathbb{R}^d) \) the space of \( C^\infty \)-functions on \( \mathbb{R}^d \) which are of compact support.

We provide these spaces with the classical topology.

We consider also the following spaces
- \( E'(\mathbb{R}^d) \) the space of distributions on \( \mathbb{R}^d \) with compact support. It is the topological dual of \( E(\mathbb{R}^d) \).
- \( S'(\mathbb{R}^d) \) the space of tempered distributions on \( \mathbb{R}^d \). It is the topological dual of \( S(\mathbb{R}^d) \).

The Dunkl operators \( T_j, j = 1, \ldots, d \), on \( \mathbb{R}^d \) associated with the finite reflection group \( W \) and the multiplicity function \( k \) are given by
\[
T_j f(x) = \frac{\partial}{\partial x_j} f(x) + \sum_{\alpha \in \mathbb{R}_+} k(\alpha) \alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{<\alpha, x>}, \quad f \in C^1(\mathbb{R}^d). \tag{5}
\]

In the case \( k = 0 \), the \( T_j, j = 1, \ldots, d \), reduce to the corresponding partial derivatives. In this paper, we will assume throughout that \( k \geq 0 \) and \( \gamma > 0 \).

The Dunkl Laplacian \( \triangle_k \) on \( \mathbb{R}^d \) is defined by
\[
\triangle_k f = \sum_{j=1}^d T_j^2 f = \Delta f + 2 \sum_{\alpha \in \mathbb{R}_+} k_\alpha \delta_\alpha(f), \quad f \in C^2(\mathbb{R}^d), \tag{6}
\]
where \( \Delta = \sum_{j=1}^d \partial_j^2 \) the Laplacian on \( \mathbb{R}^d \) and
\[
\delta_\alpha(f)(x) = \frac{<\nabla f(x), \alpha>}{<\alpha, x>} - \frac{f(x) - f(\sigma_\alpha(x))}{<\alpha, x>^2},
\]
with $\nabla f$ the gradient of $f$.

For $f$ in $C^1_c(\mathbb{R}^d)$ and $g$ in $C^1(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} T_j f(x) g(x) \omega_k(x) \, dx = - \int_{\mathbb{R}^d} f(x) T_j g(x) \omega_k(x) \, dx, \quad j = 1, \ldots, d. \quad (7)$$

For $y \in \mathbb{R}^d$, the system

$$\begin{cases}
T_j u(x, y) = y_j u(x, y), & j = 1, \ldots, d, \\
u(0, y) = 1, & \text{for all } y \in \mathbb{R}^d.
\end{cases}$$

admits a unique analytic solution on $\mathbb{R}^d$, denoted by $K(x, y)$ and called Dunkl kernel. This kernel has a unique holomorphic extension to $\mathbb{C}^d \times \mathbb{C}^d$.

**Example.**

If $d = 1$ and $W = Z_2$, the Dunkl kernel is given by

$$K(z, w) = j_{\frac{1}{2}}(izw) + \frac{zw}{2\gamma + 1} j_{\frac{3}{2}}(izw), \quad z, w \in \mathbb{C}, \quad (8)$$

where for $\alpha \geq -\frac{1}{2}$, $j_\alpha$ is the normalized Bessel function of index $\alpha$ defined by

$$j_\alpha(z) = 2^\alpha \Gamma(\alpha + 1) \frac{J_\alpha(z)}{z^\alpha} = \Gamma(\alpha + 1) \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{z}{2})^{2n}}{n! \Gamma(\alpha + 1 + n)} \quad (9)$$

with $J_\alpha$ is the Bessel function of first kind and index $\alpha$.

The Dunkl kernel possesses the following properties

**Proposition 2.1.**

i) For all $z, w \in \mathbb{C}^d$ we have

$$K(z, w) = K(w, z) \quad ; K(z, 0) = 1 \quad \text{and} \quad K(\lambda z, w) = K(z, \lambda w), \quad \text{for all } \lambda \in \mathbb{C}. \quad (10)$$

ii) For all $\nu \in \mathbb{N}^d, x \in \mathbb{R}^d$ and $z \in \mathbb{C}^d$, we have

$$|D^\nu_x K(x, z)| \leq ||x||^{|\nu|} \exp(||x|| ||\text{Re} z||), \quad (11)$$

and for all $x, y \in \mathbb{R}^d$ :

$$|K(ix, y)| \leq 1, \quad (12)$$

with $D^\nu_x = \frac{\partial^\nu}{\partial x_1^{\nu_1} \ldots \partial x_d^{\nu_d}}$ and $|\nu| = \nu_1 + \ldots + \nu_d$.

iii) For all $x, y \in \mathbb{R}^d$ and $g \in W$ we have

$$K(-ix, y) = \overline{K(ix, y)}, \quad \text{and} \quad K(gx, gy) = K(x, y). \quad (13)$$

iv) The function $K(x, z)$ admits for all $x \in \mathbb{R}^d$ and $z \in \mathbb{C}^d$ the following Laplace type integral representation

$$K(x, z) = \int_{\mathbb{R}^d} e^{<y, z>} d\mu_x(y), \quad (14)$$

where $\mu_x$ is a probability measure on $\mathbb{R}^d$, with support in the closed ball $B(o, ||x||)$ of center $o$ and radius $||x||$. (See [11]).
The Dunkl intertwining operator $V_k$ is defined on $C(\mathbb{R}^d)$ by

$$
\forall x \in \mathbb{R}^d, \quad V_k f(x) = \int_{\mathbb{R}^d} f(y) d\mu_x(y),
$$

where $\mu_x$ is the measure given by the relation (14).

The operator $V_k$ satisfies the following properties

i) We have

$$
\forall x \in \mathbb{R}^d, \forall z \in \mathcal{O}^d, \quad K(x, z) = V_k(e^{<.,z>})(x).
$$

ii) The operator $V_k$ is a topological isomorphism from $E(\mathbb{R}^d)$ onto itself satisfying the transmutation relation

$$
\forall x \in \mathbb{R}^d, \quad T_j V_k(f)(x) = V_k(\frac{\partial}{\partial y_j} f)(x), \quad j = 1, ..., d, f \in E(\mathbb{R}^d).
$$

(16)

ii) For each $x \in \mathbb{R}^d$ there exists a unique distribution $\eta_x$ in $E'(\mathbb{R}^d)$ with support in the ball $B(o, ||x||)$, such that for all $f$ in $E(\mathbb{R}^d)$ we have

$$
V_k^{-1}f(x) = \langle \eta_x, f \rangle.
$$

(17)

(See [16]).

2.3 The Dunkl transform

Notations. We denote by $L^p_k(\mathbb{R}^d)$ the space of measurable functions on $\mathbb{R}^d$ such that

$$
||f||_{k,p} = \left( \int_{\mathbb{R}^d} |f(x)|^p \omega_k(x) \, dx \right)^{\frac{1}{p}} < +\infty, \quad if \ 1 \leq p < +\infty,
$$

$$
||f||_{k,\infty} = \text{ess sup}_{x \in \mathbb{R}^d} |f(x)| < +\infty.
$$

The Dunkl transform of a function $f$ in $D(\mathbb{R}^d)$ is given by

$$
\forall y \in \mathbb{R}^d, \quad \mathcal{F}_D(f)(y) = \int_{\mathbb{R}^d} f(x) K(-iy, x) \omega_k(x) \, dx.
$$

We give in the following some properties of this transform. (See [7][8]).

i) For all $f$ in $L^1_k(\mathbb{R}^d)$ we have

$$
||\mathcal{F}_D(f)||_{k,\infty} \leq ||f||_{k,1}.
$$

(19)

ii) For all $f$ in $\mathcal{S}(\mathbb{R}^d)$ we have

$$
\forall y \in \mathbb{R}^d, \quad \mathcal{F}_D(T_j f)(y) = iy_j \mathcal{F}_D(f)(y), \quad j = 1, ..., d.
$$

(20)

iii) For all $f$ in $L^1_k(\mathbb{R}^d)$ such that $\mathcal{F}_D(f)$ is in $L^1_k(\mathbb{R}^d)$, we have the inversion formula

$$
f(y) = \frac{c_k^2}{4^{\gamma + \frac{d}{2}}} \int_{\mathbb{R}^d} \mathcal{F}_D(f)(x) K(ix, y) \omega_k(x) \, dx, \quad a.e.
$$

(21)
Theorem 2.2. The Dunkl transform $\mathcal{F}_D$ is a topological isomorphism.

i) From $\mathcal{S}(\mathbb{R}^d)$ onto itself.

ii) From $D(\mathbb{R}^d)$ onto $H(\mathbb{Q}^d)$ (the space of entire functions on $\mathbb{Q}^d$, rapidly decreasing and of exponential type.)

The inverse transform $\mathcal{F}_D^{-1}$ is given by

$$\forall y \in \mathbb{R}^d, \quad \mathcal{F}_D^{-1}(f)(y) = \frac{c_k^2}{4^{\gamma + \frac{d}{2}}} \mathcal{F}_D(f)(-y), \quad f \in \mathcal{S}(\mathbb{R}^d).$$ (22)

Theorem 2.3. i) Plancherel formula for $\mathcal{F}_D$.

For all $f$ in $\mathcal{S}(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} |f(x)|^2 \omega_k(x) \, dx = \frac{c_k^2}{4^{\gamma + \frac{d}{2}}} \int_{\mathbb{R}^d} |\mathcal{F}_D(f)(\xi)|^2 \omega_k(\xi) \, d\xi.$$ (23)

ii) Plancherel theorem for $\mathcal{F}_D$.

The renormalized Dunkl transform $f \rightarrow 2^{-(\gamma + \frac{d}{2})} c_k \mathcal{F}_D(f)$ can be uniquely extended to an isometric isomorphism on $L^2_k(\mathbb{R}^d)$.

Proposition 2.4. Let $1 \leq p \leq 2$. The Dunkl transform $\mathcal{F}_D$ can be extended to a continuous mapping from $L^p_k(\mathbb{R}^d)$ into $L^q_k(\mathbb{R}^d)$, with $q$ the conjugate component of $p$.

Definition 2.5. i) The Dunkl transform of a distribution $\tau$ in $\mathcal{S}'(\mathbb{R}^d)$ is defined by

$$\langle \mathcal{F}_D(\tau), \phi \rangle = \langle \tau, \mathcal{F}_D(\phi) \rangle, \quad \phi \in \mathcal{S}(\mathbb{R}^d).$$

ii) We define the Dunkl transform of a distribution $\tau$ in $\mathcal{E}'(\mathbb{R}^d)$ by

$$\forall y \in \mathbb{R}^d, \quad \mathcal{F}_D(\tau)(y) = \langle \tau, K(-ix, y) \rangle.$$ (24)

Theorem 2.6. The Dunkl transform $\mathcal{F}_D$ is a topological isomorphism.

i) From $\mathcal{S}'(\mathbb{R}^d)$ onto itself.

ii) From $\mathcal{E}'(\mathbb{R}^d)$ onto $H(\mathbb{Q}^d)$ (the space of entire functions on $\mathbb{Q}^d$, slowly increasing and of exponential type.)

Let $\tau$ be in $\mathcal{S}'(\mathbb{R}^d)$. We define the distribution $T_j \tau$, $j = 1, ..., d$, by

$$\langle T_j \tau, \psi \rangle = -\langle \tau, T_j \psi \rangle, \quad \text{for all} \quad \psi \in \mathcal{S}(\mathbb{R}^d).$$

This distribution satisfies the following properties

$$\mathcal{F}_D(T_j \tau) = i y_j \mathcal{F}_D(\tau), \quad j = 1, ..., d. \quad \text{(24)}$$

$$\mathcal{F}_D(\triangle_k \tau) = -||y||^2 \mathcal{F}_D(\tau). \quad \text{(25)}$$

We consider $f$ in $L^2_k(\mathbb{R}^d)$. We define the distribution $T_f$ in $\mathcal{S}'(\mathbb{R}^d)$ by

$$\langle T_f, \varphi \rangle = \int_{\mathbb{R}^d} f(x) \varphi(x) \omega_k(x) \, dx, \quad \varphi \in \mathcal{S}(\mathbb{R}^d).$$

In the following $T_f$ will be denoted by $f$. 7
Proposition 2.7. Let \( f \) be in \( L^2_k(\mathbb{R}^d) \). Then we have

\[
\mathcal{F}_D(\triangle_k f) = -||x||^2 \mathcal{F}_D(f). \tag{26}
\]

Proof

For all \( \varphi \in \mathcal{S}(\mathbb{R}^d) \) we have

\[
\langle \triangle_k f, \varphi \rangle = \langle f, \triangle_k \varphi \rangle = \int_{\mathbb{R}^d} f(x) \triangle_k \varphi(x) \omega_k(x) dx.
\]

But

\[
\langle \mathcal{F}_D(\triangle_k f), \varphi \rangle = \langle \triangle_k f, \mathcal{F}_D(\varphi) \rangle = \langle f, \triangle_k \mathcal{F}_D(\varphi) \rangle
\]

\[
= \int_{\mathbb{R}^d} f(y) \mathcal{F}_D(-||x||^2 \varphi(.))(y) \omega_k(y) dy
\]

\[
= - \int_{\mathbb{R}^d} \mathcal{F}_D(f)(x) ||x||^2 \varphi(x) \omega_k(x) dx
\]

\[
= \langle -||x||^2 \mathcal{F}_D(f), \varphi \rangle.
\]

Thus

\[
\mathcal{F}_D(\triangle_k f) = -||x||^2 \mathcal{F}_D(f).
\]

Notations. We denote by

- \( L^2_{k,c}(\mathbb{R}^d) \) the space of functions in \( L^2_k(\mathbb{R}^d) \) with compact support.
- \( \mathcal{H}_{L^2_k}(\mathbb{G}^d) \) the space of entire functions \( f \) on \( \mathbb{G}^d \) of exponential type such that \( f_{|\mathbb{R}^d} \) belongs to \( L^2_k(\mathbb{R}^d) \).

Theorem 2.8. The Dunkl transform \( \mathcal{F}_D \) is bijective from \( L^2_{k,c}(\mathbb{R}^d) \) onto \( \mathcal{H}_{L^2_k}(\mathbb{G}^d) \).

Proof

i) We consider the function \( f \) on \( \mathbb{G}^d \) given by

\[
\forall z \in \mathbb{G}^d, f(z) = \int_{\mathbb{R}^d} g(x) K(-ix, z) \omega_k(x) dx, \tag{27}
\]

with \( g \in L^2_{k,c}(\mathbb{R}^d) \).

By derivation under the integral sign and by using the inequality (11), we deduce that the function \( f \) is entire on \( \mathbb{G}^d \) and of exponential type.

On the other hand the relation (27) can also be written in the form

\[
\forall y \in \mathbb{R}^d, f(y) = \mathcal{F}_D(g)(y).
\]

Thus from Theorem 2.3 the function \( f_{|\mathbb{R}^d} \) belongs to \( L^2_k(\mathbb{R}^d) \). Thus \( f \in \mathcal{H}_{L^2_k}(\mathbb{G}^d) \).

ii) Reciprocally let \( \psi \) be in \( \mathcal{H}_{L^2_k}(\mathbb{G}^d) \). From Theorem 2.6 ii) there exists \( S \in \mathcal{E}'(\mathbb{R}^d) \) with support in the ball \( B(o, a) \) of center \( o \) and radius \( a \), such that

\[
\forall y \in \mathbb{R}^d, \psi(y) = \langle S_x, K(-ix, y) \rangle. \tag{28}
\]

On the other hand as \( \psi_{|\mathbb{R}^d} \) belongs to \( L^2_k(\mathbb{R}^d) \), then from Theorem 2.3 there exists \( h \in L^2_k(\mathbb{R}^d) \) such that

\[
\psi_{|\mathbb{R}^d} = \mathcal{F}_D(h). \tag{29}
\]
Thus from (28), for all \( \varphi \in D(\mathbb{R}^d) \) we have
\[
\int_{\mathbb{R}^d} \psi(y) \overline{F_D(\varphi)(y)\omega_k(y)} dy = \langle S_x, \int_{\mathbb{R}^d} K(-ix, y) \overline{F_D(\varphi)(y)\omega_k(y)} dy \rangle.
\]
Thus using (22) we deduce that
\[
\int_{\mathbb{R}^d} \psi(y) \overline{F_D(\varphi)(y)\omega_k(y)} dy = \frac{4^{\gamma + \frac{d}{2}}}{c_k^2} \langle S, \varphi \rangle. \tag{30}
\]
On the other hand (29) implies
\[
\int_{\mathbb{R}^d} \psi(y) \overline{F_D(\varphi)(y)\omega_k(y)} dy = \int_{\mathbb{R}^d} F_D(h)(y) \overline{F_D(\varphi)(y)\omega_k(y)} dy.
\]
But from Theorem 2.2 we deduce that
\[
\int_{\mathbb{R}^d} F_D(h)(y) \overline{F_D(\varphi)(y)\omega_k(y)} dy = \frac{4^{\gamma + \frac{d}{2}}}{c_k^2} \int_{\mathbb{R}^d} h(y) \overline{\varphi(y)\omega_k(y)} dy
= \frac{4^{\gamma + \frac{d}{2}}}{c_k^2} \langle T_{h\omega_k}, \varphi \rangle. \tag{31}
\]
Thus the relations (30),(31) imply
\[ S = T_{h\omega_k}. \]
This relation shows that the support \( h \) is compact. Then \( h \in L^2_{k,c}(\mathbb{R}^d) \)

### 2.4 The Dunkl translation operator and the Dunkl convolution product

**Definition 2.9.** Let \( y \in \mathbb{R}^d \). The Dunkl translation operator \( f \mapsto \tau_y f \) is defined on \( S(\mathbb{R}^d) \) by
\[
\forall x \in \mathbb{R}^d, \quad F_D(\tau_y f)(x) = K(-ix, y) F_D(f)(y). \tag{32}
\]

**Example**
Let \( t > 0 \), we have
\[
\forall x \in \mathbb{R}^d, \quad \tau_x(e^{-t\|\xi\|^2})(y) = \frac{M_k}{t^{\gamma + \frac{d}{2}}} K\left(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\right) e^{-\frac{\|x\|^2 + \|y\|^2}{4t}}, \tag{33}
\]
with \( M_k = (2^{\gamma + \frac{d}{2}}c_k)^{-1} \).

**Remark**
The operator \( \tau_y, y \in \mathbb{R}^d \), can also be defined on \( \mathcal{E}(\mathbb{R}^d) \) by
\[
\forall x \in \mathbb{R}^d, \quad \tau_y f(x) = (V_k)_x(V_k)_y[(V_k)^{-1}(f)(x + y)]. \tag{34}
\]
(See [17]).
At the moment an explicit formula for the Dunkl translation operator is known only
in the following two cases.

1st case: \( d = 1 \) and \( W = Z_2 \).

For all \( f \in C(\mathbb{R}) \) we have

\[
\forall x \in \mathbb{R}, \tau_y f(x) = \frac{1}{2} \int_{-1}^{1} f(\sqrt{x^2 + y^2 - 2xyt})(1 + \frac{x-y}{\sqrt{x^2 + y^2 - 2xyt}}) \Phi_k(t) dt
+ \frac{1}{2} \int_{-1}^{1} f(-\sqrt{x^2 + y^2 - 2xyt})(1 - \frac{x-y}{\sqrt{x^2 + y^2 - 2xyt}}) \Phi_k(t) dt,
\]
where
\[
\Phi_k(t) = \frac{\Gamma(k + \frac{1}{2})}{\sqrt{\pi} \Gamma(k)} (1 + t)^{(1 - t^2)^k}. \]

Moreover for all \( f \in L^p_k(\mathbb{R}), 1 \leq p \leq \infty \), we have

\[
\|\tau_y f\|_{k,p} \leq 3 \|f\|_{k,p}, \quad 1 \leq p \leq \infty. \]

(See [10][13]).

2nd case: For all \( f \in \mathcal{E}(\mathbb{R}^d) \) radial we have

\[
\forall x \in \mathbb{R}^d, \tau_y f(x) = V_k[f_0(\sqrt{||x||^2 + ||y||^2 + 2\langle x, y \rangle})](y),
\]
with \( f_0 \) the function on \([0, +\infty[\) given by \( f(x) = f_0(||x||) \).

Moreover for all \( f \in L^p_k(\mathbb{R}^d), 1 \leq p \leq \infty \), we have

\[
\|\tau_y f\|_{k,p} \leq \|f\|_{k,p}, \quad 1 \leq p \leq \infty. \]

(See [11][13]).

Using the Dunkl translation operator, we define the Dunkl convolution product of functions as follows (See [11][17]).

**Definition 2.10.** For \( f, g \) in \( D(\mathbb{R}^d) \), we define the Dunkl convolution product by

\[
\forall x \in \mathbb{R}^d, f * D g(x) = \int_{\mathbb{R}^d} \tau^x f(-y)g(y) d\omega_k(y). \tag{35}
\]

This convolution is commutative and associative and satisfies the following properties. (See [13]).

i) \( \mathcal{F}_D(f * D g) = \mathcal{F}_D(f) \mathcal{F}_D(g) \). \tag{36}

ii) Let \( 1 \leq p, q, r \leq +\infty \), such that \( \frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1 \). If \( f \) is in \( L^p_k(\mathbb{R}^d) \) radial and \( g \) an element of \( L^q_k(\mathbb{R}^d) \), then \( f * D g \) belongs to \( L^r_k(\mathbb{R}^d) \) and we have

\[
\|f * D g\|_{r,k} \leq \|f\|_{p,k} \|g\|_{q,k}. \tag{37}
\]

iii) Let \( d = 1 \) and \( W = Z_2 \). For all \( f \) in \( L^p_k(\mathbb{R}) \) and \( g \) an element of \( L^q_k(\mathbb{R}) \), the function \( f * D g \) belongs to \( L^r_k(\mathbb{R}) \) with \( \frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1 \) and we have

\[
\|f * D g\|_{r,k} \leq 3 \|f\|_{p,k} \|g\|_{q,k}. \tag{38}
\]
3 Functions with compact spectrum

First we recall that the spectrum of a function is the support of its Dunkl transform. We begin this section by the following definition.

Definition 3.1. i) We define the support of \( g \in L^2_k(\mathbb{R}^d) \) and we denote it by \( \text{supp} \ g \), the smallest closed set, outside which the function \( g \) vanishes almost everywhere.

ii) We denote by
\[
R_g = \sup_{\lambda \in \text{supp} \ g} ||\lambda||,
\]
the radius of the support of \( g \).

Remark

It is clear that \( R_g \) is finite if and only if, \( g \) has compact support.

Proposition 3.2. Let \( g \in L^2_k(\mathbb{R}^d) \) such that for all \( n \in \mathbb{N} \), the function \( ||\lambda||^{2n}g(\lambda) \) belongs to \( L^2_k(\mathbb{R}^d) \). Then
\[
R_g = \lim_{n \to \infty} \left\{ \int_{\mathbb{R}^d} ||\lambda||^{4n} |g(\lambda)|^2 \omega_k(\lambda) d\lambda \right\}^{\frac{1}{4n}}.
\]  

Proof

We suppose that \( ||g||_{k,2} \neq 0 \), otherwise \( R_g = 0 \) and formula (39) is trivial. Assume now that \( g \) has compact support with \( R_g > 0 \). Then
\[
\left\{ \int_{\mathbb{R}^d} ||\lambda||^{4n} |g(\lambda)|^2 \omega_k(\lambda) d\lambda \right\}^{\frac{1}{4n}} \leq \left\{ \int_{||\lambda|| \leq R_g} |g(\lambda)|^2 \omega_k(\lambda) d\lambda \right\}^{\frac{1}{4n}} R_g.
\]
Thus we deduce that
\[
\limsup_{n \to \infty} \left\{ \int_{\mathbb{R}^d} ||\lambda||^{4n} |g(\lambda)|^2 \omega_k(\lambda) d\lambda \right\}^{\frac{1}{4n}} \leq \limsup_{n \to \infty} \left\{ \int_{||\lambda|| \leq R_g} |g(\lambda)|^2 \omega_k(\lambda) d\lambda \right\}^{\frac{1}{4n}} R_g = R_g.
\]
On the other hand, for any positive \( \varepsilon \) we have
\[
\int_{R_g - \varepsilon \leq ||\lambda|| \leq R_g} |g(\lambda)|^2 \omega_k(\lambda) d\lambda > 0.
\]
Hence
\[
\liminf_{n \to \infty} \left\{ \int_{\mathbb{R}^d} ||\lambda||^{4n} |g(\lambda)|^2 \omega_k(\lambda) d\lambda \right\}^{\frac{1}{4n}} \geq \liminf_{n \to \infty} \left\{ \int_{R_g - \varepsilon \leq ||\lambda|| \leq R_g} ||\lambda||^{4n} |g(\lambda)|^2 \omega_k(\lambda) d\lambda \right\}^{\frac{1}{4n}} \geq R_g - \varepsilon.
\]

Thus
\[
R_g = \lim_{n \to \infty} \left\{ \int_{\mathbb{R}^d} ||\lambda||^{4n} |g(\lambda)|^2 \omega_k(\lambda) d\lambda \right\}^{\frac{1}{4n}}.
\]
We prove now the assertion in the case where \( g \) has unbounded support. Indeed For any positive \( N \), we have
\[
\int_{||\lambda|| \geq N} |g(\lambda)|^2 \omega_k(\lambda) d\lambda > 0.
\]
Thus
\[
\liminf_{n \to \infty} \left\{ \int_{\mathbb{R}^d} ||\lambda||^{4n} |g(\lambda)|^2 \omega_k(\lambda) d\lambda \right\} \to \infty \geq N.
\]
This implies that
\[
\liminf_{n \to \infty} \left\{ \int_{\mathbb{R}^d} ||\lambda||^{4n} |g(\lambda)|^2 \omega_k(\lambda) d\lambda \right\} \to \infty.
\]

**Notations.** We denote by
- \( L^2_{k,R}(\mathbb{R}^d) := \{ g \in L^2_{k,c}(\mathbb{R}^d) / R_g = R \} \), for \( R \geq 0 \).
- \( D_R(\mathbb{R}^d) := \{ g \in D(\mathbb{R}^d) / R_g = R \} \), for \( R \geq 0 \).

**Definition 3.3.** We define the Paley-Wiener spaces \( PW^2_k(\mathbb{R}^d) \) and \( PW^2_{k,R}(\mathbb{R}^d) \) as follows
i) \( PW^2_k(\mathbb{R}^d) \) is the space of functions \( f \in \mathcal{E}(\mathbb{R}^d) \) satisfying
   a) \( \Delta_k^n f \in L^2(\mathbb{R}^d) \) for all \( n \in \mathbb{N} \).
   b) \( R_{\delta_k} := \lim_{n \to \infty} ||\Delta_k^n f||_{L^2} < \infty \).
ii) \( PW^2_{k,R}(\mathbb{R}^d) := \{ f \in PW^2_k(\mathbb{R}^d) / R_{\delta_k} = R \} \).

The real \( L^2 \)-Paley-Wiener theorem for the Dunkl transform can be formulated as follows

**Theorem 3.4.** The Dunkl transform \( \mathcal{F}_D \) is a bijection
i) from \( PW^2_{k,R}(\mathbb{R}^d) \) onto \( L^2_{k,R}(\mathbb{R}^d) \).
ii) from \( PW^2_k(\mathbb{R}^d) \) onto \( L^2_{k,c}(\mathbb{R}^d) \).

**Proof**

i) Let \( g \in PW^2_{k,R}(\mathbb{R}^d) \). Then from Proposition 2.7 the function \( \mathcal{F}_D(\Delta_k^n g)(\xi) = (-1)^n ||\xi||^{2n} \mathcal{F}_D(g)(\xi) \) belongs to \( L^2(\mathbb{R}^d) \) for all \( n \in \mathbb{N} \). On the other hand from Theorem 2.3 we deduce that
\[
\lim_{n \to \infty} \left\{ \int_{\mathbb{R}^d} ||\xi||^{4n} |\mathcal{F}_D(g)(\xi)|^2 \omega_k(\xi) d\xi \right\} \to \infty = R.
\]
Thus using Proposition 3.2 we conclude that \( \mathcal{F}_D(g) \) has compact support with \( R_{\mathcal{F}_D(g)} = R \).

Conversely let \( f \in L^2_{k,R}(\mathbb{R}^d) \). Then \( ||\xi||^nf(\xi) \in L^2_k(\mathbb{R}^d) \) for any \( n \in \mathbb{N} \), and \( \mathcal{F}_D^{-1} f \in D(\mathbb{R}^d) \). On the other hand from Theorem 2.3 we have
\[
\lim_{n \to \infty} \left\{ \int_{\mathbb{R}^d} |\Delta_k^n(\mathcal{F}_D^{-1} f)(x)|^2 \omega_k(x) dx \right\} \to \infty = R.
\]
Thus \( \mathcal{F}_D^{-1}(f) \in PW^2_{k,R}(\mathbb{R}^d) \).

ii) We deduce ii) from i).
Corollary 3.5. The Dunkl transform $\mathcal{F}_D$ is a bijection from $PW^2_k(\mathbb{R}^d)$ onto $\mathcal{H}_{L^2_k}(\mathbb{R}^d)$.  

Proof
We deduce the result from Theorem 3.4 ii) and Theorem 2.8.

Definition 3.6. i) The Paley-Wiener space $PW_k(\mathbb{R}^d)$ is the space of functions $f \in \mathcal{E}(\mathbb{R}^d)$ satisfying

a) $(1 + ||x||)^m \Delta_k^n \in L^2_k(\mathbb{R}^d)$ for all $n,m \in \mathbb{N}$.

b) $R^\Delta_k f := \lim_{n \to \infty} ||\Delta^n_k f||_{k,2}^{\frac{1}{p}} < \infty$.

ii) We have $PW_{k,R}(\mathbb{R}^d) := \{ f \in PW_k(\mathbb{R}^d) / R^\Delta_k f = R \}$, for $R \geq 0$.

Remark
We notice that the only difference between $PW^2_k(\mathbb{R}^d)$ and $PW_k(\mathbb{R}^d)$ is the extra requirement of polynomial decay to help ensure that $\mathcal{F}_D(f) \in \mathcal{E}(\mathbb{R}^d)$.

The real Paley-Wiener theorem for the Dunkl transform of functions in the preceding spaces is the following

Theorem 3.7. The Dunkl transform $\mathcal{F}_D$ is a bijection

i) from $PW_{k,R}(\mathbb{R}^d)$ onto $D_R(\mathbb{R}^d)$. 

ii) from $PW_k(\mathbb{R}^d)$ onto $D(\mathbb{R}^d)$.  

Proof
i) Let $g \in PW_{k,R}(\mathbb{R}^d) \subset PW^2_k(\mathbb{R}^d)$. Then $\mathcal{F}_D(g) \in \mathcal{E}(\mathbb{R}^d)$ since $g$ has polynomial decay, and by Theorem 3.4 the function $\mathcal{F}_D(g)$ has compact support with $R_{\mathcal{F}_D(g)} = R$.

Conversely, Let $f \in D_R(\mathbb{R}^d)$, then $\mathcal{F}_D^{-1}(f) \in \mathcal{S}(\mathbb{R}^d)$ and $\mathcal{F}_D^{-1}(f) \in PW^2_{k,R}(\mathbb{R}^d)$ by Theorem 3.4.

ii) We deduce the result from the i).

4 Dunkl transform of functions, with polynomial domain support

Let $P(x)$ be a non-constant polynomial.

Theorem 4.1. For any function $f \in \mathcal{S}(\mathbb{R}^d)$ the following relation holds

$$\lim_{n \to \infty} ||P(iT)^n f||_{k,p}^{\frac{1}{p}} = \sup_{y \in \text{supp} \mathcal{F}_D(f)} |P(y)|, \quad 1 \leq p \leq \infty,$$

(40)

with $T = (T_1, ..., T_d)$.

Proof
We consider $f \neq 0$ in $\mathcal{S}(\mathbb{R}^d)$. Set $q = \frac{p}{p-1}$ if $1 < p < \infty$ and $q = 1$ or $\infty$ if $p = \infty$ or $1$. 

The proof is divided in several steps. In the following three steps we suppose that

$$0 < \sup_{y \in \text{supp } F_D(f)} |P(y)| < \infty. \quad (41)$$

**First step:** In this step we shall prove that

$$\limsup_{n \to \infty} \|P(iT)^n f\|_{k,p}^{\frac{1}{p}} \leq \sup_{y \in \text{supp } F_D(f)} |P(y)|, 1 \leq p \leq \infty. \quad (42)$$

Let $2 \leq p < \infty$. Applying Proposition 2.4 we obtain

$$\|P(iT)^n f\|_{k,p} \leq C\|P(\xi)^n F_D(f)\|_{k,q}, \quad (43)$$

Thus

$$\limsup_{n \to \infty} \|P(iT)^n f\|_{k,p} \leq C\sup_{y \in \text{supp } F_D(f)} |P(y)|, \quad (44)$$

Suppose now that $1 \leq p < 2$. Hölder’s inequality gives

$$\|f\|_{k,p}^p = \int_{R^d} (1+\|x\|^2)^{-r^p}(1+\|x\|^2)^{r^p} f(x)|x|^{p}\omega_k(x)dx \leq \|(1+\|x\|^2)^{r^p} f\|_{k,2} \|(1+\|x\|^2)^{-r^p}\|_{k,2} \frac{2}{2-p}. \quad (45)$$

for $r > 2\gamma + d$.

Thus, from Proposition 2.7 we obtain

$$\|f\|_{k,p}^p \leq C\|(I-\triangle_k)^r [F_D(f)]\|_{k,2}. \quad (46)$$

Consequently for all $n \in \mathbb{N}$, we deduce that

$$\|P^n(iT)f\|_{k,p} \leq C\|\|(I-\triangle_k)^r [P^n(\xi)F_D(f)]\|_{k,2}. \quad (47)$$

On the other hand from Proposition 5.1 of [9] we have, the following relation:

For all $\mu \in \mathbb{N}^d \setminus \{0\}$ there exist: $t_p^0, t_p^1 \in [0,1], p = 1, \ldots, |\mu| - 1$, such that for all $u \in \mathcal{E}(\mathbb{R}^d)$ we have

$$T^\mu u(x) = D^\mu u(x) + \sum_{\alpha \in \mathbb{R}^+} \sum_{|\beta| = |\mu|} \sum_{p=1}^{|\mu|-1} Q_\mu(t_p^0, \ldots, t_p^0) D^\beta u(x - S_\mu(t_1^0, \ldots, t_p^0) < \alpha, x > \alpha)$$

$$+ \sum_{|\beta| = |\mu|} P_\mu(t_1^1, \ldots, t_1^1) D^\beta u(x - \tilde{S}_\mu(t_1^1, \ldots, t_1^1) < \alpha, x > \alpha), \quad (47)$$
where \( Q_p(t_1, ..., t_p), S_p(t_1, ..., t_p), p = 1, ..., \mu \) and \( P_p(t_1, ..., t_{|\mu|-1}), \tilde{S}_p(t_1, ..., t_{|\mu|-1}) \) are polynomials of degree at most \(|\mu|\), with respect to each variable.

From this relation and by induction one can show that

\[
|(I - \triangle_{\mu})^r [P^n_k(\xi) \mathcal{F}_D(f)(\xi)]|_{k, 2} \leq C n^{2r} \|P^{n-2r}(\xi) \varphi_n(\xi)\|_{k, 2}, \quad n > 2r, \tag{48}
\]

with \( \text{supp} \varphi \subset \text{supp} \mathcal{F}_D(f) \) and \( \|\varphi\|_{k, 2} \leq C_1 \), where \( C_1 \) is a constant independent of \( n \).

Hence, from the previous inequalities we deduce that

\[
\|P^n(iT)f\|_{k, p} \leq C \varphi n^{2r} \|P^{n-2r}(\xi) \varphi_n(\xi)\|_{k, 2} \leq C \varphi n^{2r} \sup_{y \in \text{supp} \mathcal{F}_D(f)} |P(y)|^{n-2r} |\varphi_n(\xi)|_{k, 2} \leq C \varphi C_1 n^{2r} \sup_{y \in \text{supp} \mathcal{F}_D(f)} |P(y)|^{n-2r}. \tag{49}
\]

Thus

\[
\limsup_{n \to \infty} \|P(iT)^n f\|_{k, p} \leq \sup_{y \in \text{supp} \mathcal{F}_D(f)} |P(y)|. \tag{50}
\]

- Let now \( p = \infty \). From the relation (22) We have

\[
\|f\|_{\infty, k} \leq \frac{c_k^2}{C_1} \|\mathcal{F}_D(f)\|_{k, 1}.
\]

On the other hand, from Cauchy-Schawrz’s inequality we obtain

\[
\|\mathcal{F}_D(f)\|_{k, 1} \leq C_0 \|(1 + \|\xi\|^2)^{\frac{2\nu + d}{2}} \mathcal{F}_D(f)(\xi)\|_{k, 2},
\]

where \( C_0 \) is a positive constant.

Combining the previous inequalities and replacing \( f \) by \( P(iT)^n f \), we deduce that there exists a positive constant \( C \) such that

\[
\|P(iT)^n f\|_{k, \infty} \leq C \|P^n(\xi)(1 + \|\xi\|^2)^{\frac{2\nu + d}{2}} \mathcal{F}_D(f)(\xi)\|_{k, 2}. \tag{51}
\]

Consequently,

\[
\limsup_{n \to \infty} \|P(iT)^n f\|_{k, \infty} \leq \sup_{y \in \text{supp}(1 + \|\xi\|^2)^{\frac{2\nu + d}{2}} \mathcal{F}_D(f)} |P(y)| = \sup_{y \in \text{supp} \mathcal{F}_D(f)} |P(y)|. \tag{52}
\]

Thus from (44), (50) and (52) we have

\[
\limsup_{n \to \infty} \|P(iT)^n f\|_{k, p} \leq \sup_{y \in \text{supp} \mathcal{F}_D(f)} |P(y)|, \quad 1 \leq p \leq \infty. \tag{53}
\]

**Second step:** In this step we want to prove that

\[
\lim_{n \to \infty} \|P(iT)^n f\|_{k, 2} = \sup_{y \in \text{supp} \mathcal{F}_D(f)} |P(y)|.
\]
For any \( \varepsilon \), \( 0 < \varepsilon < \sup_{y \in \text{supp} \mathcal{F}_D(f)} |P(y)| \), there exists a point \( x_0 \in \sup_{y \in \text{supp} \mathcal{F}_D(f)} |P(y)| \) such that
\[
|P(x_0)| > \sup_{y \in \text{supp} \mathcal{F}_D(f)} |P(y)| - \frac{\varepsilon}{2}
\]
As \( P \) is a continuous function, there exists a neighborhood \( U_{x_0} \) such that
\[
|P(x)| > \sup_{y \in \text{supp} \mathcal{F}_D(f)} |P(y)| - \varepsilon, \quad x \in U_{x_0}
\]
From Theorem 2.3 we deduce that
\[
\|P(iT)^n f\|_{k,2} = \frac{c^2_k}{4^{\gamma + \frac{\gamma}{2}} \|P(\xi)^n \mathcal{F}_D(f)\|_{k,2}}
\geq \frac{c^2_k}{4^{\gamma + \frac{\gamma}{2}} \|P(\xi)^n \mathcal{F}_D(f)1_{U_{x_0}}\|_{k,2}}
\]
where \( 1_{U_{x_0}} \) is the characteristic function of \( U_{x_0} \).
Thus
\[
\|P(iT)^n f\|_{k,2} \geq \frac{c^2_k}{4^{\gamma + \frac{\gamma}{2}}} \left( \sup_{y \in \text{supp} \mathcal{F}_D(f)} |P(y)| - \varepsilon \right)^n \|\mathcal{F}_D(f)1_{U_{x_0}}\|_{k,2}
\]
This inequality implies,
\[
\lim_{n \to \infty} \|P(iT)^n f\|_{k,2}^{\frac{1}{n}} \geq \left( \sup_{y \in \text{supp} \mathcal{F}_D(f)} |P(y)| - \varepsilon \right) \lim_{n \to \infty} \|\mathcal{F}_D(f)1_{U_{x_0}}\|_{k,2}^{\frac{1}{n}} = \sup_{y \in \text{supp} \mathcal{F}_D(f)} (|P(y)| - \varepsilon).
\]
But \( \varepsilon \) can be chosen arbitrarily small, thus from (53) and (54) the relation (40) follows for \( p = 2 \).

**Third step:** In this step we shall prove that
\[
\liminf_{n \to \infty} \|P(iT)^n f\|_{k,p}^{\frac{1}{n}} \geq \sup_{y \in \text{supp} \mathcal{F}_D(f)} |P(y)|, \quad 1 \leq p \leq \infty.
\]
Since \( f \in \mathcal{S}(\mathbb{R}^d) \), the iteration of the relation (7) implies the relation
\[
\int_{\mathbb{R}^d} P^n(-iT)f(x) P^n(iT)f(x) \omega_k(x) dx = \int_{\mathbb{R}^d} f(x) P^{2n}(iT)f(x) \omega_k(x) dx.
\]
Hence, by Hölder’s inequality,
\[
\|P^n(iT)f\|_{k,2}^{2} \leq \|f\|_{k,q} \|P^{2n}(iT)f\|_{k,p}.
\]
Consequently
\[
\lim_{n \to \infty} \|P^n(iT)f\|_{k,2}^{\frac{1}{n}} \leq (\lim_{n \to \infty} \|f\|_{k,q}^{\frac{1}{n}}) \liminf_{n \to \infty} \|P^{2n}(iT)f\|_{k,p}^{\frac{1}{n}} = \liminf_{n \to \infty} \|P^{2n}(iT)f\|_{k,p}^{\frac{1}{n}}.
\]
Applying now the relation (40) with \( p = 2 \), we conclude that
\[
\sup_{y \in \text{supp} \mathcal{F}_D(f)} |P(y)| = \lim_{n \to \infty} \|P^n(iT)f\|_{k,2}^{\frac{1}{n}} \leq \liminf_{n \to \infty} \|P^{2n}(iT)f\|_{k,p}^{\frac{1}{n}}.
\]
We replace in formula (56) the function $f$ by $P(iT)f$ and we obtain
\[ \|P^{n+1}(iT)f\|_{k,2} \leq \|P(iT)f\|_{k,q} \|P^{2n+1}(iT)f\|_{k,p}. \] (59)
Thus
\[ \sup_{y \in \text{supp} \mathcal{F}_D(f)} |P(y)| = \lim_{n \to \infty} \|P^{n+1}(iT)f\|_{k,2}^{1/2} \leq \liminf_{n \to \infty} \|P^{2n+1}(iT)f\|_{k,p}^{1/(2n+1)}. \] (60)
Using (58) and (60) we deduce that
\[ \sup_{y \in \text{supp} \mathcal{F}_D(f)} |P(y)| \leq \liminf_{n \to \infty} \|P^n(iT)f\|_{k,p}^{1/n}. \] (61)
Then formulas (61) and (53) give (40). Thus we have proved the theorem under the condition (41).

**Fourth step:** Suppose now $\sup_{y \in \text{supp} \mathcal{F}_D(f)} |P(y)| = +\infty$. Then for any $N > 0$ there exists a point $x_0 \in \text{supp} \mathcal{F}_D(f)$ such that $|P(x_0)| \geq 2N$. Since $P$ is a continuous function there exists a neighborhood $U_{x_0}$ of $x_0$ on which $|P(x)| > N$. Similarly that the previous calculation of second step we obtain
\[ \liminf_{n \to \infty} \|P^n(iT)f\|_{k,2}^{1/n} \geq \frac{\gamma^2}{4^n + 2} \liminf_{n \to \infty} \|P^n(\xi)\mathcal{F}_D(f)1_{U_{x_0}}\|_{k,2}^{1/n}, \]
\[ \geq N \liminf_{n \to \infty} \|f1_{U_{x_0}}\|_{k,2}^{1/n} = N. \]
We choose $N$ large, we obtain
\[ \lim_{n \to \infty} \|P^n(iT)f\|_{k,2}^{1/n} = \infty. \]
Finally if $\sup_{y \in \text{supp} \mathcal{F}_D(f)} |P(y)| = 0$ the identity (40) is clear for $p = 2$.
Hence the proof of the theorem is finished.

**Definition 4.2.** Let $P$ be a non-constant polynomial and $U_P = \{x \in \mathbb{R}^d, |P(x)| \leq 1\}$. The set $U_P$ is called a polynomial domain in $\mathbb{R}^d$.

**Remark**
A disc is a polynomial domain. A polynomial domain may be unbounded and non-convex, for example $U = \{x \in \mathbb{R}^d, |x_1 \ldots x_d| \leq 1\}$.

We have the following result.

**Corollary 4.3.** Let $f \in \mathcal{S}(\mathbb{R}^d)$. The Dunkl transform $\mathcal{F}_D(f)$ vanishes outside a polynomial domain $U_P$, if and only if,
\[ \limsup_{n \to \infty} \|P^n(iT)f\|_{k,p}^{1/n} = 1, \quad 1 \leq p \leq \infty. \] (62)

**Remark**
(i) If we take $P(y) = -|y|^2$, then $P(iT) = \Delta_k$, and Theorem 4.1 and Corollary 4.3 characterize functions such that the support of their Dunkl transform is a ball.
(ii) Theorem 4.1 and Corollary 4.3 generalize also the result obtained in [3].
5 Dunkl transform of functions vanishing on a Ball

The following theorem gives the radius of the large disc on which the Dunkl transform of functions in $L^2_k(\mathbb{R}^d)$ vanishes every where.

**Theorem 5.1.** Let $f \in L^2_k(\mathbb{R}^d)$. We consider the sequence

$$f_n(x) = E_n *_D f(x), \ x \in \mathbb{R}^d, \ n \in \mathbb{N} \setminus \{0\}.$$  \hfill (63)

where

$$E_n(y) = \frac{c_k}{(4n)^{\gamma + \frac{d}{2}}} e^{-\frac{||y||^2}{4n}}$$

Then

$$\lim_{n \to \infty} \sqrt{-\frac{1}{n} \ln ||f_n||_{k,2}} = \lambda_{F_D(f)},$$  \hfill (64)

where

$$\lambda_{F_D(f)} = \inf \{||\xi||, \ \xi \in \text{supp} F_D(f)\}.$$  \hfill (65)

**Remark**

The function $E_n$ is the Gauss kernel associated with Dunkl operators. From [11] p. 2424, we have

$$\forall x \in \mathbb{R}^d, \ F_D(E_n)(x) = e^{-n||x||^2}.$$  \hfill (66)

**Proof of Theorem 5.1**

First we remark that from (37) the function $f_n$ is well defined. We assume that $||f||_{k,2} > 0$, otherwise the result is trivial. To prove (64) it is sufficient to verify the equivalent identity

$$\lim_{n \to \infty} \frac{1}{n} \ln ||f_n||_{k,2} = \lambda_{F_D(f)}.$$  \hfill (67)

Using (66) and (37) we deduce that the Dunkl transform of $f_n(x)$ is $\exp(-\lambda_{F_D(f)}^2 F_D(f)(\xi))$. Then by applying Theorem 2.3 we obtain

$$||f_n||_{k,2} = \frac{c_k}{2^{\gamma + \frac{d}{2}}} ||\exp(-n||\xi||^2)F_D(f)(\xi)||_{k,2}$$

$$= \frac{c_k}{2^{\gamma + \frac{d}{2}}} ||f||_{k,2} \left\{ \int_{\text{supp} F_D(f)} \exp(-2n||\xi||^2) \frac{|F_D(f)(\xi)|^2}{||f||_{k,2}^2} \omega_k(\xi) d\xi \right\}^{\frac{1}{2}}.$$  \hfill (68)

On the other hand it is known that if $m$ is the Lebesque measure on $\mathbb{R}^d$ and $U$ a subset of $\mathbb{R}^d$ such that $m(U) = 1$, then for all $\phi$ in the Lebesgue space $L^p(U, dm)$, $1 \leq p \leq +\infty$, we have

$$\lim_{p \to +\infty} ||\phi||_{L^p(U; dm)} = ||\phi||_{L^\infty(U; dm)}.$$  \hfill (69)

By applying formula (69) with

$$U = \text{supp} F_D(f), \ \phi = \exp(-||\xi||^2), \ p = 2n, \ \text{and} \ dm(\xi) = \frac{|F_D(f)(\xi)|^2}{||f||_{k,2}^2} \omega_k(\xi) d\xi,$$
and using the fact that \( \lim_{n \to +\infty} \left( \frac{\epsilon_n}{2^{n+\frac{d}{2}}} \right)^{\frac{1}{n}} = 1. \)

We obtain
\[
\lim_{n \to \infty} \| f_n \|_{k,2} = \sup_{\xi \in \text{supp}\mathcal{F}(f)} \exp(-\|\xi\|^2) = \exp(-\lambda_{\mathcal{F}_D(f)}^2).
\] (70)

Which is the relation (67).

A function \( f \in L^2_k(\mathbb{R}^d) \) is the Dunkl transform of a function vanishing in a neighborhood of the origin, if and only if, \( \lambda_{\mathcal{F}_D(f)} > 0 \), or equivalently, if and only if the limit (67) is less than 1. Thus we have proved the following result.

**Corollary 5.2.** The condition
\[
\lim_{n \to \infty} \| f_n \|_{k,2}^{\frac{1}{n}} < 1,
\] (71)

is necessary and sufficient for a function \( f \in L^2_k(\mathbb{R}^d) \) to have its Dunkl transform vanishing in a neighborhood of the origin.

**Remark**

From Theorem 3.3 and Corollary 5.2 it follows that the support of the Dunkl transform of a function in \( L^2_k(\mathbb{R}^d) \) is in the tore \( \lambda_{\mathcal{F}_D(f)} \leq \|\xi\| \leq R_{\mathcal{F}_D(f)} \), if and only if,
\[
\lambda_{\mathcal{F}_D(f)} \leq \lim_{n \to \infty} \sqrt{-\frac{1}{n} \ln \| f_n \|_{k,2}} \leq \lim_{n \to \infty} \| \Delta_k f \|_{k,2}^{\frac{1}{n}} \leq R_{\mathcal{F}_D(f)}.
\] (72)

**Theorem 5.3.** For any function \( f \in S(\mathbb{R}^d) \) the following relation holds
\[
\lim_{n \to \infty} \left| \sum_{m=0}^{\infty} \frac{(n \Delta_k)^m f}{m!} \right|_{k,p}^{\frac{1}{n}} = \exp(-\lambda_{\mathcal{F}_D(f)}^2), \quad 1 \leq p \leq \infty.
\] (73)

In particular, a function \( f \in S(\mathbb{R}^d) \) is the Dunkl transform of a function in \( S(\mathbb{R}^d) \) vanishing in the ball \( B(o,r) \) of center \( o \) and radius \( r \), if and only if we have
\[
\lim_{n \to \infty} \left| \sum_{m=0}^{\infty} \frac{(n \Delta_k)^m f}{m!} \right|_{k,p} \leq \exp(-r^2), \quad 1 \leq p \leq \infty.
\] (74)

**Proof**

A similar proof to that of Theorem 4.1, gives the result.

6 Dunkl transform of functions, vanishing outside a symmetric body

A subset \( K \) of \( \mathbb{R}^d \) is called a symmetric body if \(-x \in K \) for all \( x \in K \). The set \( K^* := \{ y \in \mathbb{R}^d, \langle x, y \rangle \leq 1 \text{ for all } x \in K \} \) is called the polar set of \( K \). We state now the following another real Paley-Wiener theorem.
Theorem 6.1. A function \( f \in \mathcal{E}(\mathbb{R}^d) \) is the Dunkl transform of a function in \( L_k^2(\mathbb{R}^d) \) vanishing outside a symmetric body \( K \), if and only if, \( T^\mu f \) belongs to \( L_k^2(\mathbb{R}^d) \) for all \( \mu = (\mu_1, \ldots, \mu_d) \in \mathbb{N}^d \), and for all \( n \in \mathbb{N} \) we have
\[
\sup_{a \in K^*} ||(a, T)^n f||_{k,2} \leq ||f||_{k,2}, \tag{75}
\]
where \( T = (T_1, \ldots, T_d) \).

Proof
Let \( f \in \mathcal{E}(\mathbb{R}^d) \) assume \( f \neq 0 \), otherwise the result is clear. We suppose that \( \mathcal{F}_D(f) \) which belongs in \( L_k^2(\mathbb{R}^d) \) vanishes outside a symmetric body \( K \). Then \( f \) is infinitely differentiable and belongs to \( L_k^2(\mathbb{R}^d) \) together with \( T^\mu f \) for all \( \mu = (\mu_1, \ldots, \mu_d) \in \mathbb{N}^d \). As the Dunkl transform of \( (i(\langle a, \xi \rangle)^n \mathcal{F}_D(f)(-\xi) = (\langle a, T \rangle)^n f \), then by applying Theorem 2.3, we obtain
\[
||(a, T)^n f||_{k,2} = \frac{c_k}{2^{\gamma-d/2} ||\langle a, \xi \rangle||_{\mathcal{F}_D(f)(\xi)}||_{k,2}.
\]
As \( K \) satisfies the symmetric property, we deduce that \( |\langle a, \xi \rangle| \leq 1 \) for all \( \xi \in K \) and \( a \in K^* \). Hence
\[
||(a, \xi)^n \mathcal{F}_D(f)(\cdot)||_{k,2}^2 = \int_K ||(a, \xi)^n \mathcal{F}_D(f)(\xi)|^2 \omega_k(\xi) d\xi
\leq \int_K |\mathcal{F}_D(f)(\xi)|^2 \omega_k(\xi) d\xi = \frac{4^{\gamma+d}}{c_k^2} ||f||_{k,2}^2.
\]
Thus
\[
\sup_{a \in K^*} ||(a, T)^n f||_{k,2} \leq ||f||_{k,2}.
\]

Conversely, we assume that the inequality (75) is valid for all \( n \in \mathbb{N} \). Since \( T^\mu f \in L_k^2(\mathbb{R}^d) \) for all \( \mu = (\mu_1, \ldots, \mu_d) \in \mathbb{N}^d \). Thus from Proposition 2.7 and Theorem 2.3 and the inequality (75) we obtain for all \( n \in \mathbb{N} \):
\[
\sup_{a \in K^*} ||(a, T)^n f||_{k,2} \leq \frac{2^{\gamma+d}}{c_k^2} ||f||_{k,2} \tag{77}
\]
Let \( \xi_0 \notin K \), that means there exists \( a_0 \in K^* \) such that \( \langle \xi_0, a \rangle > 1 \). Then there is a neighborhood \( U_{\xi_0} \) of \( \xi_0 \) with the property \( \langle \xi, a \rangle > \frac{1 + \langle \xi_0, a \rangle}{2} > 1 \), for all \( \xi \in U_{\xi_0} \). Thus for all \( n \in \mathbb{N} \):
\[
\frac{2^{\gamma+d}}{c_k} ||f||_{k,2} \geq \sup_{a \in K^*} ||(a, \xi)^n \mathcal{F}_D(f)(\xi)||_{k,2} \geq (\int_{U_{\xi_0}} ||(a, \xi)^n \mathcal{F}_D(f)(\xi)|^2 \omega_k(\xi) d\xi)^{\frac{1}{n}}
\geq \left( \frac{1 + \langle \xi_0, a \rangle}{2} \right)^n (\int_{U_{\xi_0}} |\mathcal{F}_D(f)(\xi)|^2 \omega_k(\xi) d\xi)^{\frac{1}{n}}.
\]
Since $(1 + \langle \xi_0, a \rangle)^n$ approaches $\infty$ as $n \to \infty$, (78) holds only if

$$\int_{U_{\mathcal{K}_0}} |\mathcal{F}_D(f)(\xi)|^2 \omega_k(\xi) d\xi = 0,$$

this implies that $\xi_0$ does not belongs to the support of $\mathcal{F}_D(f)$. Hence $\mathcal{F}_D(f) \subset K$, and Theorem 6.1 is proved.

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