FINITE EULER HIERARCHIES AND INTEGRABLE UNIVERSAL EQUATIONS

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Abstract

Recent work on Euler hierarchies of field theory Lagrangians iteratively constructed from their successive equations of motion is briefly reviewed. On the one hand, a certain triality structure is described, relating arbitrary field theories, classical topological field theories – whose classical solutions span topological classes of manifolds – and reparametrisation invariant theories – generalising ordinary string and membrane theories. On the other hand, finite Euler hierarchies are constructed for all three classes of theories. These hierarchies terminate with universal equations of motion, probably defining new integrable systems as they admit an infinity of Lagrangians. Speculations as to the possible relevance of these theories to quantum gravity are also suggested.

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1. Introduction

Using techniques of triangulations of Riemann surfaces and integrations over random matrices, recent developments in string theory have exposed\(^1\) profound connections between two dimensional quantum gravity coupled to a variety of matter fields, two dimensional topological gravity and classically integrable systems in two dimensions. On the other hand, it was the problem of quantising the latter classes of theories that also brought\(^2\) quantum groups and related algebraic structures into the arena of present day mathematical physics. Motivated by these results, some time ago we considered\(^3\) the two dimensional Bateman equation\(^4\) for a field \(\phi(x, y)\) function of variables \(x\) and \(y\):

\[
\phi_{xx} \phi_y^2 + \phi_{yy} \phi_x^2 - 2\phi_{xy} \phi_x \phi_y = 0 . \tag{1.1}
\]

This nonlinear equation possesses interesting properties, which are easily established from its determinant form

\[
\det \begin{pmatrix} 0 & \phi_x & \phi_y \\ \phi_x & \phi_{xx} & \phi_{xy} \\ \phi_y & \phi_{xy} & \phi_{yy} \end{pmatrix} = 0 . \tag{1.2}
\]

This makes it obvious that the equation is covariant under arbitrary \(GL(2)\) linear transformations of the base space coordinates \(x\) and \(y\), showing that such transformations map solutions into one another. More importantly however, the equation is also covariant under redefinitions of the field as \(\phi \to \varphi = F(\phi)\), with \(F(\phi)\) a totally arbitrary function. This property is one of general covariance in the target space parametrised by \(\phi\), which shall thus be referred to as such. Hence, under field redefinitions, solutions to (1.1) fall into topological classes of the target space. The Bateman equation is a simple example of a classical topological field theory\(^5\), namely a field theory whose space of classical solutions spans topological classes of manifolds. Classical topological field theories may be viewed as being halfway between ordinary field theories and quantum topological field theories\(^6\).

There is still another property of the Bateman equation reminiscent of quantum topological field theories. Namely, (1.1) follows\(^3\) from the massless Klein-Gordon equation in \(2 + 1\) dimensions

\[
\phi_{tt} = \phi_{xx} + \phi_{yy} , \tag{1.3a}
\]

by imposing the nonlinear constraint

\[
\phi_t^2 = \phi_x^2 + \phi_y^2 . \tag{1.3b}
\]

However, these are the equations of motion for the action

\[
\int dt \, dx \, dy \, \frac{1}{2}(1 + \lambda)(\phi^2_t - \phi^2_x - \phi^2_y) , \tag{1.4}
\]

where \(\lambda\) is a Lagrange multiplier for the constraint (1.3b). This action has the peculiarity of vanishing identically on the constraint surface, a property quite similar to that of topological field theories whose actions are typically given by surface terms\(^7\). Actually,
when represented as in (1.3), the Bateman equation is also reminiscent of the Nambu-Goto string\[8\]. Indeed, in the conformal gauge, the string equations of motion are
\[
\phi^\mu_{\tau\tau} = \phi^\mu_{\sigma\sigma},
\]
(1.4a)
whereas the constraints of reparametrisation invariance in the world-sheet are
\[
\phi^2_\tau + \phi^2_\sigma = 0 = \phi_\tau \phi_\sigma
\]
(1.4b)
(\phi^\mu are the Minkowski spacetime string coordinates while \(\tau\) and \(\sigma\) are the usual world-sheet variables). Except for the second condition in (1.4b), (1.4) is in direct correspondence with (1.3). Finally, given that the Nambu-Goto action for strings and membranes reads
\[
\mathcal{L}_{\text{Nambu-Goto}} = \left[\det (ij) N^t_i N_j^j\right]^{1/2},
\]
(1.5)
where \(N^\mu_j = \partial \phi^\mu / \partial x_j\), examples of classical topological field theories generalising the Bateman equation are simply obtained from the action
\[
\mathcal{L} = \left[\det (\mu \nu) NN^t\right]^{1/2},
\]
(1.6)
where the product of \(N^\mu_j\) with its transposed is taken in the reversed order. The equations of motion of (1.6) indeed transform covariantly under field redefinitions \(\phi^\mu \rightarrow \varphi^\mu = F^\mu(\phi^\nu)\) with \(F^\mu\) being arbitrary functions of all fields. In fact, as will be discussed later on, reparametrisation invariant field theories, of which Nambu-Goto actions are simple examples, and classical topological field theories of the type considered in this paper, of which (1.6) are also simple examples, are theories dual\[5\] to one another.

Integrability of the Bateman equation is made explicit by considering the variable \(u = \phi_x / \phi_y\), in terms of which the equation reduces to
\[
u_x = uu_y.
\]
(1.7)
Note how this choice of variable also makes the general covariance of (1.1) under arbitrary field redefinitions manifest, since (1.7) only involves the field \(u(x, y)\) which is invariant under field redefinitions of \(\phi(x, y)\). Integrability (à la Liouville) of the Bateman equation is also manifest from (1.7). Indeed, this equation is a simple reduction of the KdV equation in which the term linear in \(u_{yyy}\) is absent. The system thus possesses an infinity of conserved currents and charges, namely for every integer \(n\)
\[
\partial_x u^n = \partial_y \left[ \frac{n}{n+1} u^{n+1} \right],
\]
(1.8)
which are in involution for either of the two symplectic structures of the KdV hierarchy (the second structure thus also lacking the central extension term). In fact, all solutions to the Bateman equation may be defined implicitly, either by the constraint
\[
x F(\phi) + y G(\phi) = \text{constant},
\]
(1.9)
in the $\phi$-representation ($F(\phi), G(\phi)$ and “constant” being arbitrary), or by the constraint
\[ u = H(xu + y), \tag{1.10} \]
in the $u$-representation ($H(t)$ being arbitrary).

Finally, the Bateman equation also admits a variational formulation through an action principle. In fact, there exists an infinity of inequivalent (i.e. not differing by surface terms) local Lagrangian densities all leading to (1.1) as their equation of motion! This is quite remarkable indeed, since in the generic case only systems with a single degree of freedom admit an infinity of Lagrangian formulations\cite{9}, whereas we are dealing here with a field theory! The Lagrangian densities whose equation of motion is the Bateman equation are all weight one homogeneous functions of the first derivatives $\phi_x$ and $\phi_y$, namely $L(\lambda \phi_x, \lambda \phi_y) = \lambda L(\phi_x, \phi_y)$. Note that even though (1.1) is covariant under $GL(2)$ transformations and field redefinitions, these transformations need not be symmetries of any of these Lagrangians ($L$ always scales with a factor $F'(\phi)$ under field redefinitions $\phi \rightarrow F(\phi)$). The Bateman equation is thus an example of a system whose space of classical solutions admits more symmetries than its Lagrangian. In fact, the universality of the Bateman equation associated to this infinity of Lagrangian formulations is also at the origin of its integrability. In the general case of an arbitrary number of fields $\phi^a(x_i)$ ($a = 1, \cdots, D$; $i = 1, \cdots, d$) and variables $x_i$, the Euler-Lagrange equations of motion are given by the Euler operators
\[ E_a L = -\frac{\partial L}{\partial \phi^a} + \partial_i \left[ \frac{\partial L}{\partial \phi^a_i} - \partial_j \frac{\partial L}{\partial \phi^a_{ij}} + \cdots \right]. \tag{1.11} \]
Hence, for any Lagrangian whose dependence on the fields is only through their derivatives but not explicitly on the fields themselves – only Lagrangians of this type are considered in this work –, the equations of motion are simply conservation equations for a collection of currents. When the same equations admit an infinity of inequivalent Lagrangians, we then have in fact an infinity of currents and charges which are conserved for classical solutions (integrability à la Liouville still requires involution of all these charges for any sympletic structure(s) associated to Hamiltonian formulations of the same system. There are presumably an infinity of such structures for systems admitting an infinity of Lagrangians). In the case of the Bateman equation, this remark explains why all integer powers of $u$ define conserved charge densities. Indeed, we may always choose $L(\phi_x, \phi_y) = \phi_y F(\phi_x/\phi_y)$ with $F(u)$ arbitrary, so that
\[ \frac{\partial L}{\partial \phi_x} = F'(\phi_x/\phi_y), \quad \frac{\partial L}{\partial \phi_y} = F(\phi_x/\phi_y) - \frac{\phi_x}{\phi_y} F'(\phi_x/\phi_y). \tag{1.12} \]
Thinking now in terms of a power series expansion of $F(\phi_x/\phi_y)$ in the variable $u = \phi_x/\phi_y$, and using (1.12) in (1.11), one recovers indeed the conservation equation (1.8).

We have succeeded\cite{3,10,5} in generalising the above attractive properties of the Bateman equation to situations for which the numbers of fields and variables are arbitrary. In doing so, we also discovered some interesting new structures – of which the Bateman
equation is but the simplest illustration – described as follows. On the one hand, there exist transformations between arbitrary field theories (whose Lagrangians do not explicitly depend on the fields but only on their derivatives), reparametrisation invariant field theories and classical topological field theories. These transformations define a closed triality diagram relating all three types of theories and their classical solutions in a unique manner. These are the C- and R-maps and their inverse maps briefly described below. On the other hand, specific Euler hierarchies of field theories also play a distinguished rôle. Euler hierarchies[3,5] are hierarchies of Lagrangian field theories in which each theory is constructed in an arbitrary way out of the equations of motion of the theory at the previous level in the hierarchy. Namely, there exist finite Euler hierarchies, i.e. hierarchies terminating after a finite number of iterations of the procedure just described, and ending with equations of motion which are universal, i.e. which are independent of the arbitrariness inherent to the construction of the Euler hierarchy. Thus by construction, these equations admit an infinity of inequivalent Lagrangians. Moreover, since all Lagrangians of these finite Euler hierarchies depend on derivatives of fields only, for the reasons discussed above, this raises the strong possibility that these universal equations are actually new integrable systems in arbitrary dimensions. Finally, using these results and the aforementioned triality, one has a way of constructing dual finite Euler hierarchies of arbitrary field theories, of classical topological field theories and of reparametrisation invariant ones, all leading to universal equations of motion. In particular, in this manner one obtains new string and membrane theories with dynamics being described by universal equations of motion which are most certainly integrable, in contradistinction to Nambu-Goto membranes.

In this short note, it is not possible to present all the considerations relating to these results, for which the reader is referred back to the original work[3,10,5]. Here, we shall merely sketch the main conclusions. First, the close analogy between properties of Lagrangians for reparametrisation invariant field theories and for classical topological ones is presented in the next section. In the following two sections, the C- and R-maps are briefly described. In sect.5, the generic finite Euler hierarchy for one field is constructed, leading through the C- and R-maps to the associated two dual hierarchies, whose universal equations of motion are respectively a generalisation of the Bateman equation to an arbitrary number of dependent variables and a universal (d−1)-dimensional membrane in (d+1) dimensions. To conclude, we briefly comment on further results[3,5] concerning arbitrary numbers of fields, while also suggesting directions for further investigations.

2. Homogeneity, Reparametrisation Invariance and General Covariance

Generally, consider theories of $D$ fields $\phi^a$ depending on $d$ coordinates $x_i$, with Lagrangian densities $\mathcal{L}(\phi^a, \phi_i^a)$ functions of first and second derivatives of the fields only. Even though the results of this section remain valid for Lagrangians depending on derivatives of arbitrary higher order, the restriction to first and second derivatives only is sufficient for our purposes. The equations of motion for the fields $\phi^a$ are thus given by $\mathcal{E}_a \mathcal{L}[\phi^a] = 0$, where the Euler operators $\mathcal{E}_a$ are given in (1.11).

First, let us consider Lagrangians with the following homogeneity property

\[
\mathcal{L}(R_i^j \phi_j^a, R_i^k R_j^l \phi_{kl}^a + T_{ij}^k \phi_k^a) = (\det R_i^j)^a \mathcal{L}(\phi_i^a, \phi_{ij}^a).
\]
Here, $R_i^j$ and $T_{ij}^k$ are arbitrary coefficients, and $\alpha$ is the weight of homogeneity of the Lagrangian. Using this property and the identities following from it by differentiation with respect to the parameters $R_i^j$ and $T_{ij}^k$, a straightforward calculation shows that the equations of motion obey the following relations

$$\phi_\alpha^i E_a L[\phi^a] = (\alpha - 1) \partial_i L(\phi_\alpha^i, \phi_\alpha^{ij}) .$$

(2.2)

Therefore, when $\alpha = 1$ there are $d$ identities among the $D$ equations of motion, leaving only $(D - d)$ independent equations of motion. In fact, the case $\alpha = 1$ corresponds to a reparametrisation invariant action. Any Lagrangian obeying (2.1) scales under reparametrisations with a factor which for $\alpha = 1$ precisely cancels the reparametrisation Jacobian for the integration measure $\prod_i dx_i$ over the coordinates. Hence in this case, the identities

$$\phi_\alpha^i E_a L[\phi^a] = 0 ,$$

(2.3)

are the Noether (or Ward) identities[11] due to reparametrisation invariance, leaving only $(D - d)$ independent equations of motion.

By analogy with (2.1), consider Lagrangians with the following homogeneity property

$$L(\phi_\alpha^i R_i^a, \phi_\alpha^{ij} R_{ij} + \phi_\alpha^{ib} T_j^a + \phi_\alpha^{ij} T_i^a) = (\det R_a^b)^\alpha L(\phi_\alpha^i, \phi_\alpha^{ij}) .$$

(2.4)

Here, $R_a^b$ and $T_{ij}^a$ are arbitrary coefficients, and $\alpha$ is the weight of homogeneity of the Lagrangian. Given this property, consider now the transformation of the equations of motion $E_a L[\phi^a]$ under arbitrary field redefinitions $\phi^a \to \varphi^a = F^a(\phi^b)$. Using the homogeneity property (2.4) and the identities following from it by differentiation with respect to $R_a^b$, $T_{ij}^a$, $\phi_\alpha^i$ and $\phi_\alpha^{ij}$, a straightforward calculation then shows that the equations of motion $E_a L[\varphi^a]$ for the transformed fields $\varphi^a$ are given in terms of those for the fields $\phi^a$ by

$$E_a L[\varphi^a] = (\det R_a^b)^\alpha (R^{-1})_a^b E_b L[\phi^a] + (R^{-1})_a^b \frac{\partial (\det R_a^b)^\alpha}{\partial \phi^b} (\alpha - 1) L(\phi_\alpha^i, \phi_\alpha^{ij}) ,$$

(2.5)

where $R_a^b = \partial F^b/\partial \phi^a$. Therefore, whenever the Lagrangian possesses the homogeneity property (2.4) with a weight $\alpha = 0$ or $\alpha = 1$, the equations of motion for the fields $\phi^a$ transform covariantly among themselves under arbitrary field redefinitions. In particular in the case of one field $D = 1$, the equation of motion is invariant when $\alpha = 1$. Hence, any Lagrangian $L(\phi_\alpha^a, \phi_\alpha^{ij})$ homogeneous in the sense of (2.4) with a weight $\alpha = 1$ or $\alpha = 0$ defines a classical topological field theory. In fact, the case $\alpha = 1$ plays a distinguished rôle, as will become clear in the next section when discussing the C-map.

3. The C-Map

To define the C-map, let us consider an arbitrary field theory Lagrangian $L(y_\alpha^a, y_{\alpha \beta}^a)$ dependent on the first and second derivatives of $p$ fields $y^a(x_\alpha)$ functions of $q$ coordinates $x_\alpha$. The associated action is thus

$$S[y^a] = \int \prod_\alpha dx_\alpha L(\frac{\partial y^a}{\partial x_\alpha}, \frac{\partial^2 y^a}{\partial x_\alpha \partial x_\beta}) .$$

(3.1)
Let us now introduce \( p \) additional coordinates \( \phi^a \), and extend the coordinate dependence of the fields \( y^a \) to also include a dependence on these new variables, \( \text{i.e.} \ y^a(x_\alpha, \phi^b) \). Nevertheless, the corresponding field theory is still described by the original Lagrangian \( \mathcal{L}(y^a_\alpha, y^a_{\alpha\beta}) \), with the action now given as in (3.1) with however an additional integration over the variables \( \phi^a \). In particular, the equations of motion for this new theory are identical to those of (3.1) since the Lagrangian is independent of any derivatives of the fields \( y^a \) with respect to the new variables \( \phi^a \). In other words, as far as the dynamical evolution of the fields \( y^a \) is concerned, the variables \( \phi^a \) are irrelevant.

The C-map is implemented by inverting the \( \phi^a \)-dependence of the fields \( y^a \), and by considering \( \phi^a \) as functions of \( x_i = (x_\alpha, y^a) \), \( \text{i.e.} \ \phi^a(x_i) \). This requires that the matrix of derivatives \( \partial y^a / \partial \phi^b \) be non singular, or equivalently that we have \( \det M_{ab} \neq 0 \) with \( M_{ab} = \partial \phi^b / \partial y^a \). By direct substitution in the Lagrangian for the original field theory \( y^a(x_\alpha, \phi^b) \) and its action, one then obtains a Lagrangian for a theory of \( D = p \) fields \( \phi^a(x_i) \) dependent on \( d = p + q \) independent variables \( x_i = (x_\alpha, y^a) \). In fact, as may be expected, the resulting theory is a classical topological field theory since, as was pointed out above, the \( \phi^a \)-dependence in the original theory is irrelevant, so that the equations of motion for the fields \( \phi^a \) in the new theory should be independent of the parametrisation used for these fields, namely their equations of motion should be generally covariant under arbitrary field transformations. Indeed, it is easy to check\(^5\) that the Lagrangian obtained through the C-map obeys the homogeneity property (2.4) with a weight \( \alpha = 1 \).

The C-map thus takes an arbitrary field theory of \( p \) fields in \( q \) dimensions into a classical topological field theory of \( D = p \) fields in \( d = p + q \) dimensions. However, if the original theory is itself already a classical topological field theory obeying (2.4) with \( \alpha = 1 \), the C-map only reproduces\(^5\) this original theory, in agreement with the fact that the inversion defining the C-map is essentially a redefinition of the fields \( y^a \). Moreover, it is also possible\(^5\) to define an inversion of the C-map, namely a transformation taking any classical topological field theory obeying (2.4) with \( \alpha = 1 \) and \( D = p, \ d = p + q \) into an arbitrary field theory with \( D = p \) and \( d = q \). In turn, the image of this latter theory under the C-map is again the original classical topological field theory. Hence, the C-map establishes a one-to-one correspondence between arbitrary field theories and classical topological field theories obeying (2.4) with \( \alpha = 1 \) and having fewer fields than coordinates.

4. The R-Map

To define the R-map, consider now an arbitrary field theory Lagrangian \( \mathcal{L}(\partial \phi^\mu / \partial z_i, \partial^2 \phi^\mu / \partial z_i \partial z_j) \) dependent on the first and second derivatives of \( p \) fields \( \phi^\mu(z_i) \) functions of \( q \) coordinates \( z_i \). The associated action is thus

\[
S[\phi^\mu] = \int \prod_i dz_i \mathcal{L}(\partial \phi^\mu / \partial z_i, \partial^2 \phi^\mu / \partial z_i \partial z_j). \tag{4.1}
\]

Let us now introduce \( q \) arbitrary functions \( x_i(z_j) \), and extend the field content of the theory to also include these extra degrees of freedom while still keeping the same Lagrangian and
action as in (4.1). Obviously, the equations of motion for $\phi^\mu$ remain as before, whereas those for the new fields $x_i$ are trivially satisfied since the Lagrangian is independent of these degrees of freedom.

Nevertheless, by inverting the $z_i$ dependence of the theory, one obtains a new field theory with reparametrisation invariance. For this purpose, consider field configurations such that $\det M_{ij} \neq 0$ with $M_{ij} = \partial z_j / \partial x_i$. The $z_i$ dependence of the fields $x_i$ may then be inverted, leading to a field theory of $D = p + q$ fields $\phi^a = (\phi^\mu, z_i)$ functions of $d = q$ variables $x_i$, with an action obtained from (4.1) by direct substitution. The resulting field theory is reparametrisation invariant in the coordinates $x_i$. Clearly, this is to be expected since on the one hand, the choice for the functions $x_i(z_j)$ is totally arbitrary, and on the other hand, their equations of motion are trivial so that the transformed theory should indeed be independent of the choice of parametrisation in $x_i$. From a geometric point of view, the functions $\phi^\mu(z_i)$ define a $q$-dimensional subspace of the $(p + q)$-dimensional space spanned by the coordinates $\phi^a = (\phi^\mu, z_i)$. Introducing the fields $x_i(z_j)$ amounts to introducing an arbitrary intrinsic parametrisation of this subspace, without affecting its topological and geometrical properties as an embedded space. In other words, we are simply dealing with parametrised $(q - 1)$-dimensional membrane theories in $(p + q)$ dimensions.

The series of operations described above define the R-map, which thus takes any field theory of $p$ fields in $q$ dimensions into a reparametrisation invariant field theory of $D = p + q$ fields in $d = q$ dimensions. Again, it is straightforward to verify\textsuperscript{[5]} that the transformed Lagrangian obtained from (4.1) indeed satisfies the homogeneity property (2.1) with weight $\alpha = 1$. On the other hand, as is the case for the C-map, applying the R-map on a theory which is already reparametrisation invariant only reproduces\textsuperscript{[5]} the latter, the inversion defining the R-map being indeed a reparametrisation in the coordinates $z_i$. Similarly, it is also possible to define\textsuperscript{[5]} the inverse R-map taking any reparametrisation invariant theory of $D = p + q$ fields in $d = q$ dimensions into an arbitrary field theory with $D = p$, $d = q$, whose image under the R-map is again the original reparametrisation invariant theory. Hence, the R-map puts arbitrary field theories and reparametrisation invariant ones in one-to-one correspondence.

Composition of the C- and R-maps is obviously also possible. The following equivalences have already been mentioned:

$$C \circ C \sim C, \quad R \circ R \sim R.$$ (4.2)

Moreover, the successive application of the two maps leads\textsuperscript{[5]} to the further equivalences

$$C \circ R \sim C, \quad R \circ C \sim R.$$ (4.3)

Therefore, we have indeed the triality structure described in the introduction. Namely, given an arbitrary field theory (whose Lagrangian only depends on derivatives of fields), there exists a triplet of dual theories which is closed under the action of the C- and R-maps and their inverses. In addition to the original theory, the C-map produces a classical topological field theory obeying (2.4) with $\alpha = 1$ and fewer fields than coordinates,
whereas the R-map produces a reparametrisation invariant field theory with more fields than coordinates (whose number of independent equations is that of the original theory). Moreover, the C- and R-maps put the latter two theories in one-to-one correspondence.

5. The Generic Finite Hierarchy

Consider a collection $F_n(\phi_i)\ (n = 1, 2, \ldots)$ of arbitrary functions of the first derivatives of a field $\phi(x_i)$ dependent on $d$ coordinates $x_i$. The associated Euler hierarchy of Lagrangians is defined recursively by

$$L_n = F_n W_{n-1}, \quad W_0 = 1,$$

where the $W_n\ (n = 1, 2, \ldots)$ are the equations of motion

$$W_n = \mathcal{E} L_n,$$

with the Euler operator $\mathcal{E}$ defined in (1.11).

The fundamental result is the following identity$^{[5,3]}$:

$$W_n = \frac{1}{(d-n)!} \epsilon_{i_1 \cdots i_d} \epsilon_{j_1 \cdots j_d} M_{i_1k_1}^{(1)} \cdots M_{i_nk_n}^{(n)} \phi_{k_1j_1} \cdots \phi_{k_nj_n} \delta_{i_{n+1}j_{n+1}} \cdots \delta_{i_dj_d},$$

where

$$M_{ij}^{(p)} = \frac{\partial^2 F_p}{\partial \phi_i \partial \phi_j}, \quad p = 1, 2, \cdots, n.$$

Consequently, $W_n$ is symmetric in the arguments $M^{(p)}\ (p = 1, 2, \cdots, n)$. Namely, the order in which the multiplicative factors $F_n(\phi_i)$ are introduced in the hierarchy is irrelevant. Moreover, the dependence of $W_n$ on these functions and on the second derivatives $\phi_{ij}$ separates$^{[5]}$. In particular, at level $\ell = d$ we obtain

$$W_d = \epsilon_{i_1 \cdots i_d} \epsilon_{j_1 \cdots j_d} M_{i_1j_1}^{(1)} \cdots M_{i_dj_d}^{(d)} \det \phi_{ij},$$

showing that the dependence on the functions $F_n(\phi_i)\ (n = 1, 2, \cdots, d)$ factorizes at this level, leading to the following universal equation of motion for the Lagrangian $L_d$ independently of the factors $F_n(\phi_i)\ (n = 1, 2, \cdots, d)$

$$\det \phi_{ij} = 0.$$

Since the choice for the functions $F_n(\phi_i)\ (n = 1, 2, \cdots, d)$ is totally arbitrary, (5.6) is indeed an example of an equation of motion admitting an infinite number of inequivalent Lagrangians. Moreover, since $W_d$ is always a surface term – being the equation of motion for a Lagrangian without an explicit dependence on the field $\phi$ –, one also concludes that there is an infinite number of inequivalent conserved currents for the universal equation (5.6), suggesting its possible integrability. Indeed for $d = 2$, (5.6) is a particular reduction
of Plebanski’s equation\cite{Plebanski} for self-dual gravity in four dimensions, a system known to be integrable.

The result (5.6) also implies that the hierarchy terminates at that level. Indeed, given any Lagrangian of the form

$$\mathcal{L}(\phi_i, \phi_{ij}) = F(\phi_i) \det \phi_{ij}, \quad (5.7)$$

as is $\mathcal{L}_{d+1}$, it is easily shown that its equation of motion $\mathcal{E}\mathcal{L}$ vanishes identically. Hence, the universal equation (5.6) is also the last non trivial equation of motion for the hierarchy. The recursive construction in (5.1) and (5.2) always leads to $W_n = 0$ for $n \geq d + 1$. This concludes the construction of the generic finite Euler hierarchy leading to the universal equation (5.6).

Using the C- and R-maps, it is now possible to construct two more dual finite Euler hierarchies also leading to universal equations. The C-map leads to a hierarchy of classical topological field theories for one field all obeying (2.4) with $\alpha = 1$. The R-map leads to a hierarchy of reparametrisation invariant field theories with one more field than independent coordinates, namely string and membrane theories. In fact, the former hierarchy is constructed as in (5.1) and (5.2) with the only further restriction that the functions $F_n(\phi_i)$ are now homogeneous and of weight $\alpha = 1$ but are otherwise arbitrary. Consequently, all equations of motion $W_n$ are invariant under redefinitions of the field $\phi$ (see sect.2). However, this hierarchy of classical topological field theories terminates at level $\ell = d - 1$ rather than $\ell = d$ as is the case for the generic hierarchy, with nevertheless the universal equation of motion

$$\det \begin{pmatrix} 0 & \phi_j \\ \phi_i & \phi_{ij} \end{pmatrix} = 0, \quad (5.8)$$

up to an overall factor encapsulating all the dependence on the functions $F_n(\phi_i)$ much as in (5.5). Clearly, (5.8) is a generalisation of the Bateman equation (1.1) to higher dimensions, the latter equation corresponding to $d = 2$ and thus to a hierarchy terminating at level $\ell = 1$ for any choice of weight one homogeneous function as initial Lagrangian, in agreement with the claims of the introduction.

The R-map applied to the generic hierarchy (5.1) produces reparametrisation invariant theories of $D = d + 1$ fields $\phi^a(x_i)$ of $d$ variables $x_i$. It is convenient to define the Jacobians

$$J_a = (-1)^d \epsilon_{ab_1...b_d} \phi_{b_1}^{b_1} \cdots \phi_{b_d}^{b_d} . \quad (5.9)$$

Consider then a collection $F_n(\phi_i^a) \ (n = 1, 2, \ldots)$ of arbitrary functions with the homogeneity property

$$F_n(R_i^j \phi_i^a) = (\det R_i^j) F_n(\phi_i^a) . \quad (5.10)$$

The associated finite Euler hierarchy is then obtained from the iterative procedure

$$\mathcal{L}_n = F_n W_{n-1} , \quad W_0 = 1 , \quad (5.11)$$

where, due to reparametrisation invariance, the equations of motion always factorize\cite{Guarino, Faddeev} with the Jacobians $J_a$ so that for any fixed $a$, here not summed over,

$$W_n = \frac{1}{J_a} \mathcal{E}_a \mathcal{L}_n . \quad (5.12)$$
This hierarchy of reparametrisation invariant field theories thus terminates at level $\ell = d$ with an equation which, up to an overall factor containing all the dependence on the functions $F_n$, factorizes into the universal equation of motion (implicit summation over $a$ is again understood)

$$\det_{(ij)}(\phi_{ij}^a J_a) = 0.$$  

(5.13)

In fact, due to the Noether identities (2.3), there is only one independent equation of motion at each level of the hierarchy (essentially $W_n$ in (5.12)), which at level $\ell = d$ leads to (5.13). The universal equation of this hierarchy thus describes the dynamics of new $(d-1)$-dimensional membranes in a $(d+1)$-dimensional space which are most probably integrable systems.

6. Concluding Remarks

It is possible to extend the previous results to other finite Euler hierarchies leading to universal equations of motion in the case of arbitrary theories with more than one field. In turn, through the C- and R-maps, one obtains new finite Euler hierarchies of classical topological and reparametrisation invariant field theories, with universal equations, for which the difference between the numbers of fields and coordinates is arbitrary. The idea$^5$ is simply to consider the generic hierarchy with the single field $\phi$ being the sum of all fields $\phi^a$, i.e. $\phi = \phi^a \lambda^a$, and then expand all equations in the parameters $\lambda^a$. This leads to further generalisations of the Bateman equation in terms of classical topological field theories, as well as their dual reparametrisation invariant counterparts.

However, this does not answer the question whether there might still exist other independent finite Euler hierarchies with universal equations, for example with Grassmann odd degrees of freedom or supersymmetrisations of the ones above. Another possibility might be the extension of the Euler operator to some kind of noncovariant exterior derivative on field space, since for Lagrangians independent of the fields themselves the Euler operator is actually nilpotent$^{[3,13]}$, i.e. $E^2 = 0$, a property essential for the existence of the generic hierarchy above. On the other hand, would all field theories admitting an infinity of Lagrangians be the universal equations of some universal finite Euler hierarchy?

The whole issue of integrability of the equations constructed here remains to be solved. This question has been answered$^5$ in the affirmative in the simplest cases, but otherwise only general classes of solutions have been obtained$^5$ so far. Obviously, this problem is of interest since the present equations would probably define new integrable systems in arbitrary dimensions and as is well known, it is the study of classical and quantum integrability of two dimensional systems that led to the development of new mathematical techniques and the discovery of new algebraic structures such as quantum groups$^2$.

Finally, motivated by more physical considerations, the problem of quantising the equations constructed here is also of interest. On the one hand for classical topological field theories, an understanding of their quantum states could turn out to be of relevance to quantum gravity. As opposed to quantum topological field theories where it is hoped$^6$ that the geometrical structure of spacetime would result from some phase transition, in the context of classical topological field theories one may envisage the possibility that
spacetime geometry is actually a quantum phenomenon appearing only once these theories of spacetime topology are quantised! On the other hand, since ordinary critical strings always describe at least gravitational interactions, the BRST quantisation\cite{11} and physical spectrum of the new universal string and membrane equations appearing in our work is certainly a problem worth investigating as well, having in mind again possible applications to theories of gravity.

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