Non-Abelian Gerbes from Strings on a Branched Space-Time

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Abstract

As superstring solitons that carry Neveu–Schwarz charge can be described in terms of gerbes, one expects non-Abelian gerbes to appear e.g. in the exotic six-dimensional world-volume theories of coinciding NS5 branes. We consider open bosonic strings on a space-time that is branched in such a way that the $B$-field is provided with the same Lie algebra structure as the world-volume gauge field on a D-brane. These considerations motivate a generalization of the cocycle conditions and the transformation rules of an Abelian gerbe in hypercohomology. The resulting system incorporates in a natural way the NS two-form, the RR gauge field, the Chan–Paton gauge field, the relevant gauge transformations and the holonomies associated to Wilson surface observables.
1 Introduction

One of the most intriguing systems in string theory is that of \( r \) overlapping NS5 branes [1] in the decoupling limit. The world-volume theory of an isolated NS5 brane in type IIA is given by a \( N = (0,2) \) supersymmetric theory in six-dimension, and it therefore involves self-dual antisymmetric tensorfields [2, 3]. For several NS5-branes the dominant low energy degrees of freedom are tensionless strings that arise in M-theory from M2-branes suspended between M5-branes when the M5-branes’ world-volumes coincide. These theories appear also in type IIB compactified on \( \mathbb{K}^3 \) [4, 5], in which formulation it becomes clear that the appearing tensionless strings are not fundamental strings and that they have \( ADE \) gauge symmetry. The heterotic description involves a small \( E_8 \) instanton [6] at the core of which gauge symmetry is enhanced. Hence, it is natural to look for a way to describe these systems in terms of a local quantum field theory that involves a non-Abelian self-dual two-form in six dimensions. However, such a theory does not exist [7] (cf. also [8]), and one needs an indirect or nonperturbative description. Wilson surfaces and loop equations in these systems have been studied in [9]. There is also a M(atrix) theory construction [10].

Another interesting system involves the \( N = (1,1) \) ‘new’ gauge theories of Witten in six-dimensions [11]. In type IIB they appear on a \( \mathbb{C}^2/\mathbb{Z}_r \) orbifold when nonperturbative closed string states appear from open strings that start and end on points in \( \mathbb{C}^2 \) that are identified by the \( \mathbb{Z}_r \) action. In low energies the dynamics should however reduce to the six-dimensional infrared-free \( N = (1,1) \) Yang–Mills theory.

The bulk origin of the anti-symmetric tensor field is the Neuveu–Schwarz two-form \( B \), the gauge field of fundamental string charge. There are many interesting phenomena connected to it, such as the appearance of noncommutative Yang–Mills theories in a constant background condensate [12, 13, 14]. If the curvature \( [H] \) of the \( B \) field is a torsion class in integral cohomology, D-brane charges can be classified [15] in a twisted version of K-theory [16], and the Chan–Paton gauge fields appear as connections on a module of a noncommutative algebra [17]. If the curvature is not a torsion class then classification in terms of K-theory fails. For general curvature \( H \), and thus bound states that involve nontrivial NS five-brane charge, the classification problem is still open.

The proper mathematical framework for treating these three-form fluxes seems to be that of gerbes [18]. In all generality gerbes are sheafs of groupoids
but they can be understood more concretely as collections of local principal bundles and their isomorphisms. As such they also include the modules of noncommutative spaces that appear in noncommutative Yang–Mills. Abelian gerbes allow for a geometric interpretation in terms of local line bundles, and of hypercohomology. The role of hypercohomology is to provide a differential geometric framework for studying gerbes. Physically this corresponds to finding the correct local degrees of freedom for field theory. However, this has been done until now only for Abelian gerbes. Non-Abelian gerbes do exist, actually the concept was originally introduced to formulate noncommutative cohomology. However, the description uses holonomies and isomorphisms, which physically corresponds to a Wilson loop, or surface, observables.

Recently it was shown that the local line bundles of Hitchin indeed appear in effective type IIA solutions in massive supergravity, when NS5 branes and D6-branes are involved. The same considerations also gave reason to suspect that gerbes should enter whenever NS charges are involved, including the world-sheet theories on D-branes. However, these theories involve non-Abelian bundles on the world-volumes, and the Abelian gerbes are clearly not suited for describing them. In this article our aim is to find a straightforward non-Abelian generalization of the Abelian hypercohomology underlying a general gerbe.

In order to do this we shall consider the quantum theory of strings on a branched space-time or, more concretely, multivalued $B$-fields. These models are adequate for describing strings both on an orbifold $\mathbb{C}^2/\mathbb{Z}_r$ in type IIB description, and on $r$ coinciding NS5 branes, when each one of them is carrying an independent $B$-field. This serves as a rough bosonic model for strings in both six dimensional $N = (0,2)$ and $N = (1,1)$ systems as we are not imposing self-duality constraints on the two-form. In fact, what follows does not depend on the dimensionality of the brane, either. These arguments are sufficient for establishing what fields and which symmetries to expect in a differential geometry description of gerbe. A simple non-Abelian generalization of the cocycle conditions and symmetry transformations of an Abelian gerbe then yields a strongly constrained system which fits well in the physical picture; it involves a collection of essentially Abelian RR fields, and a Chan–Paton and NS two-form, which will turn out to be non-Abelian, though in a somewhat restricted sense. The result is suitably Abelian as to allow us to use some field theory intuition in these systems, but display the underlying non-Abelian structure clearly enough, so that we see why and when we would
have to move from a local field theory description to a nonperturbative one, and to Wilson surfaces.

The plan of the paper is as follows: In the next section we shall consider string sigma-models and non-Abelian currents on a branched space-time. These results motivate in Section 3 a non-Abelian generalization of Abelian hypercohomology. In Section 4 we show how the mathematical framework fits together with what we know about superstring solitons, and their effective low energy theories.

2 World-sheet actions and currents

We shall start by considering the bosonic string sigma-model in a framework that naturally accommodates what we know about open string dynamics in the presence of several D-branes.

Consider D-branes \( Q_i \), where \( i = 1, \ldots, r \). Each of these branes carries a Chan–Paton vector potential \( A_{M,i} \), where \( M, N = 1, \ldots, D \) are space-time indices. In the sigma-model this condensate field is integrated over the components of the boundary of the world-sheet \( \Sigma \) on different D-branes, namely \( \partial \Sigma_i \). It is natural to think of the boundary as a vector in \( H^1(M) \otimes h \) where \( h \) is a vector space of dimension \( r \) with basis \( \{ e^i \} \). Similarly, \( A \) should be thought of as an element of \( \Omega^1(M, h^*) \). One particular realization of this is to take \( h \) to be the Cartan subalgebra of some Lie algebra \( \mathfrak{g} \) of rank \( r \).

Let us in particular consider the configuration where all of the D-branes \( Q_i \) lie on top of each other, and the boundaries \( \partial \Sigma_i \) coincide with \( \partial \hat{\Sigma} \). Then we can write

\[
\int_{\partial \Sigma} A = \sum_i \int_{\partial \Sigma_i} A_{M,i} \partial_\varphi X^M_i \, d\tau = \int_{\partial \hat{\Sigma}} (A_M, \partial_\varphi X^M) \, d\tau .
\]

where \( \varphi \) denotes \( \tau \) for parallel and \( \sigma \) for transverse coordinates to the D-brane, and \( (\ ,\ ) \) is the Killing form of the Lie algebra. The index of the boundary component became in this way formally an index of the at the moment diagonal coordinate matrix \( X^M \).

Let us next turn to the B-field, and the full world-sheet. Consider, for simplicity, a world-sheet that is composed of disjoint cylinders \( \Sigma_{ij} \) that connect the boundaries \( \partial \Sigma_i \) and \( \partial \Sigma_j \). As we already attached the vectors \( e^i \) and \( e^j \) to them it is natural to attach their difference \( e^i - e^j = \alpha \) to the interpolating world-sheet \( \Sigma_\alpha \), hence \( \partial \Sigma_\alpha = \partial \Sigma_i - \partial \Sigma_j = e^i - e^j = \alpha \). In...
particular in the case that the Lie group $g$ is just $A_r$ the vector $\alpha$ is one of its roots.

In string theory there is only one bulk $B$-field. However, it is natural to associate different pull-backs of this field $B_{MN,\alpha}$ for each component of the world-sheet $\Sigma_\alpha$. For disjoint cylinders we can write without any loss of generality

$$B_{MN,\alpha} = (B_{MN}, \alpha^\vee),$$

where $\alpha^\vee$ is the coroot. The pertinent world-sheet integrals can now be written in the form

$$\int_{\Sigma} B = \sum_{\alpha} \int_{\Sigma_\alpha} (B, \alpha^\vee).$$

### 2.1 Non-Abelian currents

Until now all Lie algebra has been used for keeping track of disconnected components of the world-sheet. Consider now a configuration where the cylinders fuse into one geometrical object. Let us take this limit in such a way that the components of the fields $B_{MN}$ and $A_M$ stay independent on each component cylinder; This means that these fields effectively live on different branches of space-time. For the vector fields this is the standard limit of coinciding D-branes. As to what concerns the $B_{MN}$ field, this kind of a situation could arise for instance at an orbifold such as $\mathbb{C}^2/\mathbb{Z}_r$ where the different components come from different fundamental domains of the $\mathbb{Z}_r$ action, though we do not have an explicit CFT construction for this.

Suppose further that we have two cylinders $\Sigma_\alpha$ and $\Sigma_\beta$ that overlap on one of their boundaries. In the limit we are taking both of the cylinders are forced to occupy the same part of space-time, and as they are connected, one is simply folding one on top of the other. However, as far as the $B$ field is concerned we could have equally well started with the combined cylinder $\Sigma_{\alpha+\beta}$. This means that even in the limit where one allows the $B$ field be independent on different world-sheets we have to impose a consistency condition

$$B_{\alpha} + B_{\beta} = B_{\alpha+\beta}.$$
Assuming (2) solves this condition, as anticipated.

The result of this analysis is hence that the string world-sheets on different branches of space-time and the \( r \) different boundaries of the cylinders can both be associated to the same Cartan subalgebra of a Lie algebra \( \mathfrak{g} = A_r \), and the connecting cylinders to the roots of the same algebra. This argument generalizes to world-sheets with an arbitrary number of boundary components. For instance, given the boundary \( \Sigma = e^1 + e^2 - e^3 = \mathbf{v} \), the correct \( B \)-field proportional to \( (B, \mathbf{v}) \). These world-sheets belong naturally to some representation of \( \mathfrak{g} \), with Dynkin labels \( \mathbf{v} \). Note also that we are not restricted to the unitary series, but modding by a suitable symmetry we get all of the simply laced Lie algebrae in the \( ADE \) series.

In order to learn how to describe these models in effective field theory we need a theory that can accommodate non-Abelian \( B \) fields. This question will occupy us for the rest of the paper. Define\(^4\) \( A_M = A_{M,i} H^i \) and \( B_{MN} = B_{MN,i} H^i \). It is also useful to introduce the non-Abelian line element

\[
\mathrm{d}X^M = \mathrm{d}\tau \partial_\varphi X^M_i H^i + \mathrm{d}z \partial X^M_\alpha E^\alpha + \mathrm{d}\bar{z} \partial X^M_{-\bar{\alpha}} E^{-\alpha}.
\]

Then the full non-Abelian world-sheet action can be succinctly summarized in

\[
S = \text{tr} \int_\Sigma (G_{MN} + B_{MN}) \, \mathrm{d}X^M \mathrm{d}X^N + \text{tr} \int_{\partial \Sigma} A_M \, \mathrm{d}X^M.
\]

We stress that the Lie algebra indices \( i \) for a boundary component and \( \alpha \) for a connecting world-sheet arose geometrically when one evaluated coordinate functions on different components of the world-sheet.

From open string interactions between \( r \) coinciding D-branes we know\(^23, 30\) that the gauge fields \( A_M \) can be extended to the full Lie algebra \( \mathfrak{g} \), and that the scalars that appear as transversal coordinates take values in the same space. In (3) this means that we should allow \( A_M \) and hence \( \partial_\varphi X^M \) take arbitrary values in the full Lie algebra, and interpret

\[
\int_{\partial \Sigma} \delta(x - x(\tau)) \, \mathrm{d}X^M
\]

as the non-Abelian current\(^5\) carried by a particle moving along \( \partial \Sigma \).

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\(^4\)\(H^i\) are Cartan generators, \( \alpha \) the positive roots, and \( E_\alpha \) their generators [27].

\(^5\)Note that \( X^M \) denotes a sigma-model coordinate in all of the space-time directions; the physical transverse coordinates of the brane are included in \( A_M \).
Extending this procedure to the bulk fields (on the world-sheet) \( B_{MN} \) and \( X^M(z, \bar{z}) \) is tempting. This would formally mean that we introduce new degrees of freedom to the theory, namely the non-diagonal components of the \( B \)-field. These components couple to coordinate functions \( X^M \) on different world-sheets and would seem to correspond to strings propagating from one world-sheet – or sheet of space-time – to another.

### 2.2 Effective actions and symmetries

The action \((\ref{eq:effective-action})\) describes the coupling of a macroscopic string \( \hat{\Sigma} \) to the string condensates. From the effective field theory point of view it hence appears as a non-Abelian current. In order to address the dynamics of the full background fields \( A \) and \( B \) one produces the generating functional of their interactions from the path integral evaluated in the presence of this current. In the absence of the \( B \) field the functional is just the Wilson line \[31\]

\[
e^{-F[A_c,B_c]} = \left\langle \text{tr Pexp} - i \int_{\partial \hat{\Sigma}} A_c \right\rangle. \tag{8}
\]

It can be evaluated assuming that one is allowed to neglect derivative terms and commutators of the field strength, and the result is Tseytlin’s generalization of the DBI action \[32\].

This argument also tells us how to study non-Abelian \( B \)-fields in string theory. We like to simply insert the current \((\ref{eq:effective-action})\), and ask what the resulting generating function tells us about the dynamics of the field. As the microscopic description for non-Abelian \( B \) field is lacking we have to rely on indirect arguments, such as those that make use of the Wilson line above, or general underlying structures associated to the field, gerbes.

In order to find out what gauge symmetries we have, let us in particular assume first \( B = dC \), where \( C \) is a diagonal, but make no restrictions on \( A \). Then we can eliminate the \( B \) field, and confirm that the path integral \((8)\) is invariant under the transformations \( A + C \rightarrow k^{-1}(A + C + d)k \).

The coordinate system in which \( B \) is diagonal in isospin indices tells us what the geometrical direction in the isospin space should be – much like the coordinate system in which the gauge field of a D-particle is diagonal defines what we mean by asymptotic space-time. However, when \( A \) is generally non-Abelian, there should not be a particularly preferred choice of this diagonal direction because we can always change the basis in \((3)\). Hence, we may have an independent freedom to change \( B \) by an isospin rotation.
In all, we have found three, as it seems, independent symmetries of the
theory

(G1) The generalized NS symmetry for a local one-form $\eta \in \Omega^1(Q, h)$ where
$h \subset g$ is the Cartan subalgebra

$$A \rightarrow A - \eta \quad (9)$$

$$B \rightarrow B + d\eta \quad (10)$$

This symmetry relies heavily on the fact that $\eta$ is assumed diagonal
with respect to the basis (5).

(G2) The ordinary non-Abelian gauge symmetry $k : Q \rightarrow G$

$$A \rightarrow k^{-1}(A + dk) \quad (11)$$

$$B \rightarrow B \quad (12)$$

(G3) Provisionally, we include also the choice of the physical direction in
isospin space $h : Q \rightarrow G$

$$A \rightarrow A \quad (13)$$

$$B \rightarrow h^{-1}Bh \quad (14)$$

Note, however, that the non-Abelian DBI action is not invariant under
this symmetry unless $k = h$.

Next we shall try to combine these local symmetries and fields in a global
framework. For this we shall, however, have to find out how to describe a
non-Abelian gerbe in terms of differential geometry.

3 Hypercohomology

Let us consider a space-time manifold $X$ and a fixed open cover $\{U_\alpha\}$. The
isomorphism class of an Abelian one-gerbe with connective structure and
curving is given by a two-cocycle in the hypercohomology of the complex

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Neither the patches in the cover nor their intersections need to be contractible. Čech-
cohomology does not depend on the cover, if the cover is fine enough. Here we shall not
dwell on the dependence of the construction on the choice of cover. See, however, end of
Section 3.3.
\[ C^\infty \rightarrow \Omega^1 \rightarrow \Omega^2, \] A gerbe can hence be thought of as a two-cocycle in the hypercohomology \( H^2 \) of \( \check{\text{C}}ech\)-cocycles with de Rham forms on the coefficient sheaves. A representative of this class is then a closed two-cochain
\[
\omega = \left[ g_{\alpha\beta\gamma}^{[0]}, A_{\alpha\beta}^{[1]}, B_{\alpha}^{[2]} \right],
\] and any two representatives of the class are connected by a shift with an exact term, which will just turn out to be a gauge transformation. We shall give the cocycle conditions and the gauge transformation rules below.

The \( \check{\text{C}}ech\)-coboundary operator \( \delta \) acts by adding an index to \( h_{\alpha\beta} \), such that for instance
\[
\delta h_{\alpha\beta\gamma} = h_{\beta\gamma} h_{\alpha\gamma}^{-1} h_{\alpha\beta}.
\] The \( \check{\text{C}}ech\)-indices always remain antisymmetric, and \( \delta^2 = 0 \). The zero-forms are multiplicative and the higher de Rham forms additive. The coboundary operator of the complex introduced above is, when acting on an \( i \)-form,
\[
\mathcal{D} = d_{\text{deRham}} + (-1)^i \delta_{\check{\text{Cech}}}.
\] The statement that \( \omega \) be closed under this coboundary operator \( \mathcal{D}\omega = 0 \) gives the cocycle conditions
\[
g_{\beta\gamma\delta}^{-1} g_{\alpha\delta\beta}^{-1} g_{\alpha\beta\gamma}^{-1} = 1 \quad \text{on} \quad \mathcal{U}_{\alpha\beta\gamma\delta},
\]
\[
g_{\alpha\beta\gamma}^{-1} d g_{\alpha\beta\gamma} - A_{\beta\gamma} + A_{\alpha\gamma} - A_{\alpha\beta} = 0 \quad \text{on} \quad \mathcal{U}_{\alpha\beta\gamma},
\]
\[
d A_{\alpha\beta} + B_{\beta} - B_{\alpha} = 0 \quad \text{on} \quad \mathcal{U}_{\alpha\beta}.
\] Gauge symmetry arises from shifting the gerbe \( \omega \) by an exact term
\[
\omega' = \omega + \mathcal{D}\lambda,
\] where \( \lambda \) is a cochain in the lower complex \( C^\infty \rightarrow \Omega^1 \)
\[
\left[ h_{\alpha\beta}^{[0]}, \eta_{\alpha}^{[1]} \right].
\] The index in square brackets denotes the de Rham form-degree, and Greek alphabet is used to label the intersections of local patches where the object is defined. Thus, for instance, \( A_{\alpha\beta}^{[1]} \) is a one-form defined on every twofold intersection of local charts, namely \( \mathcal{U}_\alpha \cap \mathcal{U}_\beta = \mathcal{U}_{\alpha\beta} \).
This makes $g_{\alpha\beta\gamma}$ into an ordinary Čech-cocycle, $A_{\alpha\beta}$ into a connection of a line bundle defined on $U_{\alpha\beta}$, and $H = dB_\alpha$ into a globally defined three-form. Concretely,

\begin{align*}
g'_{\alpha\beta\gamma} &= g_{\alpha\beta\gamma} h_{\beta\gamma} h^{-1}_{\alpha\gamma} h_{\alpha\beta} \\
A'_{\alpha\beta} &= A_{\alpha\beta} + h^{-1}_{\alpha\beta} d h_{\alpha\beta} - \eta_\beta + \eta_\alpha \\
B'_\alpha &= B_\alpha + d\eta_\alpha.
\end{align*}

Similarly, $\Lambda$ becomes a one-cocycle, if one imposes the condition $D\Lambda = 0$. It is easy to see that $\Lambda$ is then a principal bundle with connection $\eta$ and transition functions $h$.

### 3.1 Non-Abelian gerbes

A straightforward attempt to simply add a Lie algebra (or group) index to $w$ fails as it seems to be rather difficult to define a non-Abelian generalization of the cocycle condition (18). Indeed, there is no non-Abelian Čech-theory, and hence a non-Abelian hypercohomology formulation is absent as well. Gerbes were originally invented for studying non-Abelian cohomology [18], and they have appeared e.g. in [20] as Dixmier–Dady sheaves of groupoids [19, 20]. Abelian gerbes carry a natural differential geometry, but this structure has not been extended to the non-Abelian case [18, 23]. For physics the differential complexes are rather essential as they correspond to physical fields. The formulation in terms of groupoids corresponds rather to a QFT formulation in terms of Wilson line and other holonomy operators [23].

The idea of the present construction comes from D-particle physics. There a number of classical particles moving in the target space is described by a diagonal matrix. In physical processes this coordinate matrix has to be asymptotically diagonal – in some basis – but the dynamics of the theory allow processes where also off-diagonal elements of the coordinate matrices are excited. Then the notion of local space-time vanishes and we are in the realm of noncommutative geometry, or stringy geometry. In particular, one could consider a process where the in-coming and out-going particles live in different Cartan subalgebrae of the pertinent group; then the definition of space-time seems to have changed under the process.

In what follows we shall study a non-Abelian generalization of the Abelian hypercohomology relevant for gerbes, which is in many respects still essentially Abelian. One circumvents the problem of defining non-Abelian Čech-
cohomology in formulae (18) and (23) by assuming that the system is actually Abelian on threefold intersections. One can still allow non-Abelian behaviour outside these by assumption isolated patches on the manifold. More concretely we choose for each threefold intersection \( \mathcal{U}_{\alpha\beta\gamma} \) a fixed torus \( T_{\alpha\beta\gamma} \) inside a Lie group \( G \), and assume that the Čech two-cocycles \( g_{\alpha\beta\gamma} \) as well as the restrictions of the gauge transformations \( h_{\alpha\beta} \) on any \( \mathcal{U}_{\alpha\beta\gamma} \) take values in the same fixed torus. Outside triple intersections we need not constrain these transformations. However, we shall have to impose further assumptions on other fields.

The obvious generalization of the one-form is now to make it a connection on a principal \( G \)-bundle on \( \mathcal{U}_{\alpha\beta} \). The cocycle condition (19) can then be accepted as is; for the fixed torus part the condition is then just a collection of Abelian equations, for the rest of the Lie algebra it reduces to a cocycle condition that does not involve \( g_{\alpha\beta\gamma} \). This restriction on the form of the one-form \( A_{\alpha\beta} \) only constrains the field on the triple intersections. Under the non-Abelian gauge transformations we have to impose further

\[
\delta(\text{Ad}(h_{\alpha\beta})A_{\alpha\beta}) = 0 .
\]

This condition can be solved by assuming that on three-fold intersections the one-forms take values in the Lie algebra of the fixed torus \( \mathfrak{t}_{\alpha\beta\gamma} = \text{Lie}(T_{\alpha\beta\gamma}) \). Finally, the two-form field should be made Lie algebra valued as well.

We are now ready to write down an ansatz for the non-Abelian generalization of hypercohomology. Our strategy will be to try to find a non-Abelian generalization for the two-cochain \( \omega \), together with the corresponding new cocycle conditions and transformation rules. In addition to this two-cocycle it will turn out that it is actually necessary to also include a fixed one-cochain \( \psi \) together with its transformation rules. This one-cochain need not be closed. In the following, we consider then a fixed two-cocycle and a fixed one-cochain

\[
\begin{align*}
\omega &= \left[ g_{[0]}^{\alpha\beta\gamma}, A_{[1]}^{\alpha\beta}, B_{[2]}^{\alpha} \right] \\
\psi &= \left[ \phi_{[0]}^{\alpha\beta}, \chi_{[1]}^{\alpha} \right] ,
\end{align*}
\]

and the action of the two cochains

\[
\begin{align*}
\Lambda &= \left[ h_{[0]}^{\alpha\beta}, \eta_{[1]}^{\alpha} \right] \\
\kappa &= \left[ k_{[0]}^{\alpha} \right] .
\end{align*}
\]
on them. Here all of the one and two-forms take values in the Lie algebra \( g \), and the functions in the corresponding Lie group. As it will turn out, \( v \) describes a local principal bundle on each coordinate patch, and isomorphisms between bundles on different though intersecting patches. If the bundle were global, the cocycle conditions

\[
\phi_{\beta\gamma}\phi_{\gamma\alpha}\phi_{\alpha\beta} = 1 \\
\phi^{-1}_{\alpha\beta}(\chi_\alpha + d)\phi_{\alpha\beta} - \chi_\beta = 0
\]

would be satisfied everywhere. In the Abelian case the cocycle conditions can be succinctly stated by saying that \( v \) is closed, \( Dv = 0 \). The bundles \( v \) could fail to be a global bundle if the class \( g_{\alpha\beta\gamma} = \phi_{\beta\gamma}\phi_{\gamma\alpha}\phi_{\alpha\beta} \) were not trivial.

In this nested structure \( \kappa \) acts as gauge transformations on \( \lambda \) and \( v \), and both \( \kappa \) and \( \lambda \) act on \( w \). In particular, the action of \( \lambda \) on \( w \) is

\[
g'_{\alpha\beta\gamma} = g_{\alpha\beta\gamma}h_{\beta\gamma}h^{-1}_{\alpha\gamma}h_{\alpha\beta} \\
A'_{\alpha\beta} = h^{-1}_{\alpha\beta}(A_{\alpha\beta} - \eta_\beta + \eta_\alpha + d)h_{\alpha\beta} \\
B'_{\alpha} = B_\alpha + F(\eta_\alpha)
\]

where \( F(x) = dx + x \wedge x \). Having \( v \) at our disposal we could have defined \( B'_{\alpha} = B_\alpha + D(\chi_\alpha)\eta \) instead of (33), but this would lead to wrong transformation properties in (44) later on. The gauge transformations \( \kappa \) act according to

\[
h'_{\alpha\beta} = k^{-1}_{\alpha}\eta_{\alpha\beta}k_{\beta} \\
\eta'_{\alpha} = k^{-1}_{\alpha}\eta_{\alpha} + dk_{\alpha}
\]

on the local principal bundles. The action on \( w \) is both through

\[
B'_{\alpha} = k^{-1}_{\alpha}B_{\alpha}k_{\alpha}
\]

and through the action induced through \( \lambda \) in (37).

The highest object obeys the cocycle condition \( Dw = 0 \) namely,

\[
g_{\beta\gamma\delta}g^{-1}_{\alpha\gamma\delta}g_{\alpha\beta\delta}g^{-1}_{\alpha\beta\gamma} = 1 \quad \text{on} \quad U_{\alpha\beta\gamma\delta},
\]

\[
g^{-1}_{\alpha\beta\gamma}dg_{\alpha\beta\gamma} - A_{\beta\gamma} + A_{\alpha\gamma} - A_{\alpha\beta} = 0 \quad \text{on} \quad U_{\alpha\beta\gamma},
\]

\[
F(A_{\alpha\beta}) + B_{\beta} - B_{\alpha} = 0 \quad \text{on} \quad U_{\alpha\beta}.
\]
We call the collection of fields \( \underline{w} \) a non-Abelian one-gerbe if it satisfies these consistency conditions.

Where ever the “zero-gerbes” \( \underline{v} \) and \( \underline{\lambda} \) obey the cocycle conditions \( D\underline{v} = 0 \) or \( D\underline{\lambda} = 0 \) they are actually locally defined principal \( G \)-bundles. The former should not be assumed globally closed \( D\underline{v} \neq 0 \), as otherwise it would indeed extend to a global principal bundle, and the obstruction to this \( \underline{w} \) in which we are actually interested, should vanish. Also assuming \( \underline{\lambda} \) closed would imply that it act at least in the Abelian case trivially on \( \underline{w} \). Also \( \kappa \) has a geometrical interpretation: it is just the set of gauge transformations of \( \underline{\lambda} \).

We should take care that the cocycle conditions \( D\underline{w} = 0 \) are invariant under the action of \( \kappa \) and \( \lambda \). The first two conditions are still trivial, thanks to the assumption that all relevant fields collapse to tori on triple intersections, cf. (26). The last cocycle condition, however, gives a restriction on \( \underline{\lambda} \) and \( \kappa \).

In all generality

\[
0 = \left( \text{Ad}(h') - 1 \right) F(A) - \text{Ad}(h') \left( D(A) \delta \eta' - \delta \eta' \wedge \delta \eta' \right) + \delta F(\eta) + \delta \left( \text{Ad}(k) - 1 \right) (B + F(\eta)) ,
\]

(42)

where the prime denotes the action of \( \kappa \) on \( \underline{\lambda} \). In the next section, we shall simplify this condition. For this, however, we shall have to equip our construction with some more structure.

### 3.2 Consistency conditions

There is a way to restrict fields in order to make contact with the original hypercohomology. The idea is to restrict the covariant derivatives on the various principal bundles so that they commute with the Čech-coboundary operator\[^8\].

In what follows we derive the relevant commutative diagrams adding some geometrical assumptions. Note, however, that the rules of the previous chapter were not derived, but arose as a natural extension of Abelian structure. The analysis below serves hence as a justification for these definitions.

The cocycle condition (11) implies that \( \delta B \) is a covariantly constant section of the bundle where \( A \) is the connection. This means that under \( \underline{\lambda} \) for \( \eta = 0 \) it transforms according to \( \delta B'_{\alpha \beta} = \text{Ad}(k^{-1}_\alpha h_{\alpha \beta} k_\beta) B_{\alpha \beta} \). On the other

\[^8\] The formula (26) is actually already an example of this: it just states that \( \delta \text{Ad}(h) = \text{Ad}(g) \delta \). But the RHS is trivial because the fields are on a torus.
hand, we had already fixed $B$’s transformation properties in (38). Hence $\text{Ad}(h')\delta = \delta\text{Ad}(k)$, which means that the diagram

$$
\begin{array}{c}
\Omega^{[2]}_\alpha \xrightarrow{\text{Ad}(k)} \Omega^{[2]}_\alpha \\
\downarrow \delta \quad \quad \downarrow \delta
\end{array}
$$

should commute. This assumption relates the gauge transformations on twofold intersections so that $k_\alpha = k_\beta = h_{\alpha\beta}$.

In the Abelian case we found the globally defined three-form $H = dB_\alpha$ useful for distinguishing different gerbes. In the present situation we can build a covariant three-form under $\kappa$ from $B_\alpha$ on $U_\alpha$ by setting

$$H_\alpha = D(\chi_\alpha)B_\alpha . \tag{44}$$

The identity $D(A)\delta B = \delta D(\chi)B$ would imply on twofold intersections that $H_\alpha$ extend to a section of the local bundle associated to $D(\chi_\alpha)$. If this local bundle extends into a global one, $H$ extends to its section. This compatibility constraint is natural in the sense that it is just the covariantization of the observation that the exterior derivative and the Čech-coboundary operators commute in the diagram

$$
\begin{array}{c}
\Omega^{[2]}_\alpha \xrightarrow{D(\chi)} \Omega^{[3]}_\alpha \\
\downarrow \delta \quad \quad \downarrow \delta
\end{array}
$$

However, it is a restriction on $A_{\alpha\beta}$ and $\chi_\alpha$. The commutativity of the above diagram translates into the condition

$$[A, \delta B] = \delta [\chi, B] . \tag{46}$$

We shall give later an explicit example.

If instead of acting on $B$, we consider the action on the one-forms $\eta$, the result would be that the one-forms commute with each other $[\eta, \eta] = 0$ and with $A$, namely $[\eta, A] = 0$. We shall be lead to this result presently, though through another route. The same argument puts the one-forms $\chi$ on the same torus on double intersections.
Setting $h = k = 1$ in (42) yields
\[ F(A - \delta \eta) = F(A) - \delta F(\eta) \, . \] (47)

We shall impose this formula as a restriction on $A$ and $\delta \eta$. This leads to $\delta F(\eta) = d \delta F(\eta)$ corresponding to the commutative diagram
\[
\begin{array}{ccc}
\Omega^{[1]}_{\alpha} & \xrightarrow{F(\eta)} & \Omega^{[2]}_{\alpha} \\
\downarrow{\delta} & & \downarrow{\delta} \\
\Omega^{[1]}_{\alpha\beta} & \xrightarrow{d} & \Omega^{[2]}_{\alpha\beta}
\end{array}
\] (48)

One can verify that as was expected on general grounds, $[A, \delta \eta] = [\eta, \delta \eta] = 0$. Now (42) is identically satisfied. We shall have to ensure that the condition (45) is consistent with the transformation rules. This is not automatic, but we have to make yet a forth restriction
\[ \delta D(\chi) F(\eta) = 0 \, , \] (49)

or, equivalently, $\delta [\chi, F(\eta)] = 0$.

In summary, we have had to assume the commutativity of diagrams (43) and (45), and that conditions (47) and (49) hold. All of these conditions are geometrical, and fit nicely together with Abelian hypercohomology.

**A solution**

In order to see how these assumptions affect the differential forms it is useful to find concrete examples that satisfy them. The geometrical picture that arises from these considerations restricts the various fields in the following way:

**(C1)** The connections $\chi_{\alpha} \in \Omega^{[1]}(U_{\alpha}, g)$ define locally a subspace $\ker \text{ad}(\chi_{\alpha}) \in \Omega^*(U_{\alpha}, g)$. We can now choose the forms $B_{\alpha}$ so that their restrictions on $U_{\alpha\beta}$ belong there. Outside the double intersection there is no restriction.

**(C2)** Having hence fixed $\delta B$ on each $U_{\alpha\beta}$ we have actually also fixed $F(A) = -\delta B$. Because $[A, dA] = 0$ the connection $A$ and $\delta B$ should get their values in the same Cartan subalgebra.
(C3) On triple intersections $\mathcal{U}_{\alpha\beta\gamma}$ this Cartan subalgebra should be a part of the algebra $\mathfrak{t}_{\alpha\beta\gamma}$ of a fixed torus $T_{\alpha\beta\gamma}$.

(C4) The Čech 2-cocycle $g$ is built out of the transition functions $\phi$ of the local bundle $\mathfrak{g}$ according to

$$g_{\alpha\beta\gamma} = \phi_{\alpha\beta} \phi_{\beta\gamma} \phi_{\gamma\alpha}.$$  \hspace{1cm} (50)

On $\mathcal{U}_{\alpha\beta\gamma}$ $g$ is constrained to lie in the fixed torus $T_{\alpha\beta\gamma}$. The torus could vary as one moves over the triple intersection.

Our construction is hence essentially Abelian on triple and double intersections. The non-Abelianity of the construction lies outside the double intersections, and in the way in which these various locally Abelian constructions are related to each other. This means that the restrictions appear rather as boundary conditions. The crucially non-Abelian objects are the transition functions $\phi$, the one-forms $\chi$ and the two-forms $B$.

To see how this works, consider, for instance, how $\ker \text{ad}(\chi_{\alpha})$ viewed as a collection of sections of the local principal bundle $\mathfrak{g}$ transforms on a three-fold intersection $\mathcal{U}_{\alpha\beta\gamma}$ as we transport it around: The transition functions of the local bundle $\mathfrak{g}$ combine under this tour into $g$ as defined in (50). This holonomy does not need to be trivial as we did not assume that the local bundles combine into a global one. This far we did not impose much structure on $\mathfrak{g}$. Let us now suppose further that $g_{\alpha\beta\gamma}$ happens to be an element of $T_{\alpha\beta\gamma}$ in order to satisfy condition C4 above. In addition, assume that on four fold intersections the $g$ are compatible in the sense that $\delta g = 0$. Despite the notation, (54) does not imply that $g$ would be exact (or even closed) as a collection of Abelian Čech cocycles, because the transition functions $\phi$ are not in general Abelian.

It turns out that the definition (54) of $g$ produces the right three-index object even if the transition functions $\phi$ are just general isomorphisms between principal bundles on different charts. Then the cocycle condition has to be modified, however, and there does not seem to exist at present representations of the underlying sheaves of grupoids in terms of differential geometry on them, or hypercohomology.

For concreteness, let us consider a toy model on a triple intersection $\mathcal{U}_{123}$, with the group $G = SU(2)$. Suppose $\chi_1 = x(\sigma^1 - \sigma^2)$, $\chi_2 = x(\sigma^1 + \sigma^2)$, and $\chi_3 = -x(\sigma^1 + \sigma^2)$. Then the transition functions can be taken constants $\phi_{12} = i\sigma^1$, $\phi_{23} = i\sigma^3$, and $\phi_{31} = i\sigma^2$. The resulting holonomy is $g_{ijk} = -\varepsilon_{ijk}$.  

16
The differences in the $B$ field are
\[ \delta B_{12} = (B_2 - B_1) \sigma^1 + (B_2 + B_1) \sigma^2, \]
\[ \delta B_{23} = -(B_3 + B_2) (\sigma^1 + \sigma^2), \]
and \[ \delta B_{31} = (B_1 + B_3) \sigma^1 + (-B_1 + B_3) \sigma^2. \]
In order to find the connections $A$ we have to assume $B_1 = 0$. Then we can choose one-forms $A_{ij} = \varepsilon_{ijk} A_k (\sigma^1 + \sigma^2)$. This also fixes the embedding of the torus $T_{ijk} \subset SU(2)$.

**Transformations**

Given in the above sense consistent data $w$, $v$, let us now see what symmetries $\Lambda$, $\kappa$ are left.

(S1) On each patch $\kappa$ acts as the gauge symmetries of the bundle $v$.

(S2) On double intersections $U_{\alpha\beta}$ these transformations are fixed to coincide with the corresponding transformations of the gerbe $k_{\alpha} = h_{\alpha\beta}$.

(S3) On triple intersections also the transformations of the gerbe $h$ are fixed to respect the tori $T_{\alpha\beta\gamma}$.

(S4) The translations $\eta$ and their differences $\delta \eta$ belong on $U_{\alpha\beta}$ to a Cartan subalgebra $\ker \text{ad}(\chi_{\alpha}) \in \Omega^*(U_{\alpha\beta}, g)$.

In the previous toy model example the gauge transformations (S1) and (S2) make $\chi$ and $A$ into ordinary connections on the respective coordinate patches. The remaining shift symmetry $\eta$ can be found just as the $B_i$ were found above, except that now $[\eta, \delta \eta] = 0$. It follows, $\eta_1 = 0$, $\eta_2 = a(\sigma^1 + \sigma^2)$, and $\eta_3 = b(\sigma^1 + \sigma^2)$. It acts then on the triple intersections just as in the Abelian case, along the fixed torus. Assuming $A_2 = 0$ we can also choose $\eta_1 = a A_3 (\sigma^1 - \sigma^2)$ and $\eta_2 = \eta_3 = 0$.

### 3.3 Geometrical Interpretation

The non-Abelian gerbe $w$ found in the previous section provides a tool to study the set of local, non-Abelian principal bundles $v$. The local symmetry of the bundle is frozen on double intersections so that gauge transformations on both charts are identical. This transformation then acts also on the gerbe $w$. The two-form $B$ can be assumed to commute with the connection $\chi$ on two-fold intersections. $B$ provides us an Abelian connection $A$ on double intersections. The translations $\eta$ are Abelian on triple intersections as well, and act on $A$ in the same way as in the Abelian case.
The gerbe $\mathcal{w}$ would then look exactly like rank $G$ copies of Abelian gerbes, were it not for the fact that $B$ is generally Lie algebra valued outside double intersections, and that $h$ can mix the diagonal elements of $A$ on double intersections. The crucial non-Abelianity resides in the principal bundles $\mathcal{v}$, and $\mathcal{w}$ should be seen as an *almost* Abelian obstruction for extending $\mathcal{v}$ into a global bundle.

If the local bundles in $\mathcal{v}$ are trivial, then its sections can be conjugated to the Cartan subalgebras fixed on various double intersections. The transition functions $\phi$ do still not have to be trivial, but they act as isomorphisms between these tori. In particular, $g_{\alpha\beta\gamma}$ is an automorphism of the torus associated to $\chi_\alpha$, and maps the torus back to itself thus permuting the diagonal elements. In this way, the gerbe can be used to describe a *braid*.

**Limitations**

Čech-cohomology does not depend on the choice of cover if the cover is fine enough [34]. However, in our discussions the cover is very particular. For instance, if there were enough three-fold intersections to cover the whole space the whole construction would collapse to $r$ copies of Abelian gerbes. For our considerations it is, however, quite sufficient to know that there does exist a cover independent formulation of non-Abelian gerbes [18, 20] that describes obstructions to extend local bundles into global ones. We choose one of these configurations together with a cover that is as simple as possible but still carries the interesting information. In other words we smooth the system as much as possible, and try to push the obstructions to trivializing it into as small and isolated neighbourhoods as possible.

We have found a very particular differential geometry description of these objects as well. In that it seems to give an extension of Abelian hypercohomology the restrictions on the fields are under control. One should ask, however, whether the parametric tori $T_{\alpha\beta\gamma}$ are actually necessary data. The observation that $g_{\alpha\beta\gamma}$ can become non-Abelian in general gives hope that there actually might be a formulation where this data becomes superfluous. However, from the physical point of view the tori seem to be necessary, much as the physical Cartan subalgebrae in D-particle scattering, as we shall see presently.
4 String backgrounds and gerbes

In this section we make contact with the string theory considerations of Section 2.1, where we coupled the NS two-form fields and the Chan–Paton vector fields to a non-Abelian current carried by a world-sheet. Three different symmetries acted on these fields (G1), (G2), and (G3) in the notation of Section 2.2. We also have an essentially Abelian gauge field from the RR sector, and a gauge symmetry associated to it.

On the geometrical side there is associated to the gerbe a local principal bundle $\nu = [\phi, \chi]$. This should be identified with the Chan–Paton bundle on a D-brane. The obstruction to extend this bundle is $g$. The gauge symmetry (G3) is then just the action of $\nu$.

On two-fold intersections we have the essentially Abelian gauge field $A$. This should be the RR gauge field for the D-particle, or the D6-brane. The Chan–Paton gauge transformations were correlated to the RR gauge transformations $h$ on these two-fold intersections. If the action of these gauge transformations is not Abelian it seems that an isospin rotation on the Chan–Paton sector induces a redefinition of which Cartan subalgebra the RR fields live.

We already noticed that the gauge transformations $h$ in $\lambda = [h, \eta]$ connect the Chan–Paton transformations and the RR gauge transformations. Also the transformations generated by $\eta$ play an important role. As they shift the $B$-field by the curvature $F(\eta)$ cf. (35) they are the natural generalization of the NS symmetry (G1). The $\eta$ transformation also acts on the RR field in the way NS transformation does. As was pointed out in [26] the Abelian version of the cocycle condition (41) guarantees that the right gauge invariant field strength is the same as in massive IIA supergravity, namely

$$\mathcal{F}^{[2]} = F(A_{\alpha\beta}) + B_\beta$$

(51)

It then readily follows that the two-form $B$ in $\nu$ is the NS two-form.

The NS gauge invariant combination in open string theory $B + F(A_{CP})$ appears here as well, but in the form $\mathcal{F} = B_\alpha + F(\chi_\alpha)$. Its curvature is $H = D(\chi)\mathcal{F}$, as it should be, but $\lambda$ does not seem to implement the NS symmetry (G1) correctly. Fortunately, all of the previous calculations on double intersections remain unchanged even if we extend the action of $\lambda$ onto $\nu$. Then we have to assume again $[\chi, \eta] = [\eta, \eta] = 0$, which makes $\eta$ into an effectively Abelian connection so that $F(\chi - \eta) = F(\chi) - F(\eta)$, and $\mathcal{F}$ is again invariant.
However, this would be pushing the NS symmetry too far. Though NS symmetry is present for Abelian currents coupled to the world-sheet – which we have again correctly reproduced above in hypercohomology – it is not there for non-Abelian currents, as it heavily relies on the Abelianity of $F(A_{CP})$. One should therefore think of the NS-symmetry ($G1$) rather as a freedom to redefine the connection $\chi$ by shifting it with suitably Abelian form. Our construction therefore necessitates a nontrivial non-Abelian extension of the NS-symmetry. There is exactly the same interplay between Abelian and non-Abelian currents in the effective supergravity Lagrangians and the above generalized hypercohomology.

Let us finally consider conserved charges. The local bundles $v$ are classified by the Chern class $\text{Ch}(F(\chi_\alpha))$. The bundles in $w$ also have nontrivial first Chern class $\text{ch}_1(F(A_{\alpha\beta}))$. The invariant quantity associated to $B$ is $\text{tr} \, H = \text{tr} \, D(\chi)B$. Consider its integral over a sphere $S^3$ that is divided into two discs $U_\alpha, U_\beta$, whose boundaries $S^2$ coincide. Then

$$Q_{NS} = \int_{S^3} \text{tr} \, H = \int_{S^2} \text{tr} \, (B_\alpha - B_\beta) = \int_{S^2} \text{tr} \, F(A_{\alpha\beta}) . \quad (52)$$

Thus NS charge is non-trivial, if $\text{tr}F(A_{\alpha\beta})$ has monopole number, i.e. there are D6-branes $[26]$. The NS charge is well defined under the $\eta$ shifts as well, because

$$\int_{S^2} \text{tr} \, \delta F(\eta_{\alpha\beta}) = \int_{S^2} \text{tr} \, d \delta \eta_{\alpha\beta} = 0 . \quad (53)$$

For fixed bases of RR fields these formulae yield charges that do not depend on $\eta$ or the choice of homology cycles, even if the traces are dropped in $(52)$.

5 Conclusions

We started by studying a branched cover of space-time and showed how the NS two-form fields are made to carry the same Lie algebra indices that the Chan–Paton gauge fields have. Much in the same way that the latter fields are promoted to non-Abelian Lie algebra fields in the case of a stack of coinciding D-branes, we argued that there should appear additional light degrees of freedom from strings that connect D-branes on different branches of the space-time. A DBI action argument was also used to indicate which symmetries there should be present.
The curvature of the NS $B$ field appears on the level of effective supergravity as the characteristic class of a gerbe. In order to set the stage for addressing dynamical issues concerning this non-Abelian $B$ field it is therefore necessary to generalize the Abelian hypercohomology construction. This we did, and the resulting structure incorporates strikingly well, and in particular without introducing unphysical degrees of freedom, all the relevant supergravity fields and symmetries.

This construction sheds light on the difficulties encountered in trying to describe perturbatively for instance the exotic $N = (0,2)$ theories in six dimensions. In the case of non-Abelian Yang–Mills the right object to study in supergravity seems to be the Wilson line, i.e. the holonomies of the principal bundle. It seems therefore that the right strategy to attack the dynamical problem here should be, analogously, to understand the holonomies of the gerbe using the techniques developed here. These very same holonomies arise also in guaranteeing that the string world-sheet measure is anomaly free. For instance the analysis in \cite{33} was concerned in essentially defining the the holonomy of an Abelian gerbe.

**Acknowledgements:** I thank L. Bonora, R. Iengo, and F. Thompson for useful discussions, and in particular P. Tran-Ngoc-Bich for collaboration in the early stages of this project. This work was supported in part by the European Union TMR program CT960045.

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