Gröbner–Shirshov bases for
Vinberg–Koszul–Gerstenhaber right-symmetric
algebras

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Abstract: In this paper, we establish the Composition-Diamond lemma for right-symmetric algebras. As an application, we give a Gröbner-Shirshov basis for universal enveloping right-symmetric algebra of a Lie algebra.

Key words: right-symmetric algebra, Gröbner-Shirshov basis, normal form

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1 Introduction

Gröbner and Gröbner–Shirshov bases theories were invented independently by A.I. Shirshov [55] for Lie algebras and H. Hironaka [33] and B. Buchberger [24, 25] for associative-commutative algebras.

Shirshov’s paper [55] based on his papers [54] (Gröbner–Shirshov bases theory for (anti) commutative algebras, the reduction algorithm for (anti-) commutative algebras) and [52] (Lyndon–Shirshov words which were defined some earlier in [45], but incidentally that was unknown for 25 years in Russia and these words were called Shirshov’s regular words, see, for example, [2, 3, 4, 8, 57, 46], see also [29]), Lyndon–Shirshov basis of a

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free Lie algebra (see also [28]). The latter Shirshov’s papers [51, 52] were based on his Thesis [49], A.G. Kurosh (adv), published in three papers [50] (on free Lie algebras: $K_r$-Lemma (Lazard–Shirshov elimination process), the subalgebra theorem (Shirshov–Witt theorem)), [51] (on free (anti-) commutative algebras: linear bases, the subalgebra theorems), and [53] (on free Lie algebras: a series of bases with well-ordering of basic Lie words such that $[w] = [[u][v]] > [v]$, see also [59]; the series is called now Hall sets or Hall–Shirshov bases). Shirshov’s Thesis, in turn, was in line with a Kurosh’s paper [40] (on free non-associative algebras: the subalgebra theorem). Also Shirshov’s paper [54] was in a sense of a continuation of a paper by A.I. Zhukov [61], a student of Kurosh (on free non-commutative algebras: decidability of the word problem for non-associative algebras). The difference from the Zhukov’s approach was that Zhukov did not use any linear ordering of non-associative words, but just the partial deg-ordering to compare two words by the degree (length).

It would be not a big exaggeration to say that Shirshov’s paper [55] was between line of the Kurosh’s program of study free algebras of different classes of non-associative algebras.

Shirshov’s paper [55] contained implicitly the Gröbner–Shirshov bases theory for associative algebras too because he constantly used that any Lie polynomial is at the same time a non-commutative polynomial. For example, the maximal term of a Lie polynomial is defined as its maximal word (in the deg-lex ordering) as a non-commutative polynomial, the definition of a Lie composition (Lie $S$-polynomial) of two Lie polynomials begins with the definition of their composition as non-commutative polynomials and follows by putting some special Lie brackets on it, and so on. The main Composition (-Diamond) lemma for associative polynomials is actually proved in the paper and we need only to “forget” about Lie brackets in the proof of this lemma for Lie polynomials, i.e. to change Lie polynomials to non-commutative (associative) polynomials ([55], Lemma 3). Explicitly Composition (-Diamond) lemma was formulated much later in papers L.A. Bokut [9] and G. Bergman [5].

Last years there were quite a few results on Gröbner–Shirshov bases for associative algebras, Lie (super) algebras and irreducible modules for them, Kac–Moody algebras, Coxeter groups, braid groups, quantum groups, conformal algebras, free inverse semigroups, Kurosh’s Ω-algebras, Loday’s dialgebras and Leibniz algebras, Rota-Baxter algebras, and so on, see, for example, books [22, 47], papers [47, 41, 19, 42, 43, 44, 34, 35, 23, 36, 10, 6, 37, 18, 16, 7], and surveys [17, 20, 21, 15, 11, 14]. Actually, conformal algebras, dialgebras, Rota-Baxter algebras are examples of Ω-algebras. For non-associative Ω-algebras, Composition-Diamond lemma was proved in [30]. The case of associative Ω-algebras (associative algebras with any set $\Omega$ of multi-linear operations) was treated in [13] with an application to free ($\lambda$-) Rota-Baxter algebras (the latter are associative algebras with linear operation $P(x)$ and the identity $P(x)P(y) = P(P(x)y) + P(xP(y)) + \lambda P(xy)$, where $\lambda$ is a fix element of a ground field, see, for example, [31]). Composition–Diamond lemma for dialgebras [12] has an application to the PBW theorem for universal enveloping dialgebras of Leibniz algebras (see [11]).

In this paper, we are doing Gröbner-Shirshov bases theory for right-symmetric algebras (RS-algebras) (they are defined by the following identity $(x, y, z) = (x, z, y)$ for the associator $(x, y, z) = (xy)z - x(yz)$. Under this name, these algebras appeared in
the paper E.B. Vinberg ([60], 1960) (actually, he invented left-symmetric (LS-) algebras, \((x, y, z) = (y, x, z)\)). Independently they were introduced by J.-L. Koszul ([38], 1961) and M. Gerstenhaber ([32], 1963) (under the name pre-Lie algebras). As was pointed out by D. Burde [26], the same algebras first appeared in a paper by A. Cayley in 1896 (see [27]). Survey [26] contains detailed discussion of the origin, theory and applications of LSA in geometry and physics together with an extensive bibliography.

D. Segal [48] found a linear basis of a free LS-algebra and applied it for PBW type theorem for universal LS enveloping algebra of a Lie algebra. Vasil’eva and Mikhalev [58] found another proof of the former Segal’s result (and a more general result for LS superalgebras) using Composition-Diamond lemma from Zhukov’s paper [61]. Kazybaev, Makar-Limanov and Umirbaev [39] found some new properties of the Segal’s basic words and proved the Freiheitssatz and decidability of the word problem for one-relator RS-algebras. Actually, it is a generalization of well known Shirshov’s results for one-relator Lie algebras [54].

2 Composition-Diamond lemma for right-symmetric algebras

Let \(X = \{x_i | i \in I\}\) be a set, \(X^*\) the set of all associative words \(u\) in \(X\), \(X^{**}\) the set of all non-associative words \((u)\) in \(X\), and \(|(u)|\) the length of the word \((u)\).

Let \(I\) be a well ordered set. We order \(X^{**}\) by the induction on the lengths of the words \((u)\) and \((v)\) in \(X^{**}\):

(i) If \(|((u)(v))| = 2\), then \((u) = x_i > (v) = x_j\) if and only if \(i > j\).

(ii) If \(|((u)(v))| > 2\), then \((u) > (v)\) if and only if one of the following cases holds:

(a) \(|(u)| > |(v)|\).

(b) If \(|(u)| = |(v)|\), \((u) = ((u_1)(u_2))\) and \((v) = ((v_1)(v_2))\), then \((u_1) > (v_1)\) or \((u_2) > (v_2)\).

It is clear that the order \(<\) on \(X^{**}\) is well ordered. This order is called deg-lex (degree-lexicographical) order and we use this order throughout this paper.

We now cite the definition of good words (cf. [48]) by induction on length:

1) \(x_i\) is a good word for any \(x_i \in X\).

Suppose that we define good words of length \(\leq n\).

2) non-associative word \(((u)(w))\) is called a good word if

(a) both \((v)\) and \((w)\) are good words,

(b) if \((v) = ((v_1)(v_2))\), then \((v_2) \leq (w)\).

We denote \((u)\) by \([u]\), if \((u)\) is a good word.

Let \(W\) be the set of all good words in the alphabet \(X\) and \(RS\langle X \rangle\) the free right-symmetric algebra over a field \(k\) generated by \(X\). Then \(W\) forms a linear basis of the free right-symmetric algebra \(RS\langle X \rangle\), see [48].
Every nonzero element $f$ in $RS(X)$ can be uniquely represented as
$$f = \lambda_1[w_1] + \lambda_2[w_2] + \cdots + \lambda_n[w_n]$$
where $[w_i] \in W, 0 \neq \lambda_i \in k$ for all $i$ and $[w_1] > [w_2] > \cdots > [w_n]$. Denote by $\bar{f}$ the leading word $[w_1]$ of $f$. $f$ is called monic if the coefficient of $\bar{f}$ is 1.

For any $(w), (w_1) \in X^{**}$, denote by
$$(w)R_{(w_1)} = ((w)(w_1)).$$

The following results are actually proved in [39].

**Lemma 2.1** ([39]) In $X^{**}$, every good word $[w] \in W$ can be uniquely represented as
$$[w] = x_iR_{[w_1]}R_{[w_2]} \cdots R_{[w_n]}$$
where $[w_j] \in W$ for all $j$ and $[w_1] \leq [w_2] \leq \cdots \leq [w_n]$.

**Lemma 2.2** ([39]) Let $[u]$ and $[v]$ be arbitrary good words and assume that $[u] = x_iR_{[u_1]}R_{[u_2]} \cdots R_{[u_n]}$. Then, in $RS(X)$,
$$[u][v] = x_iR_{[u_1]} \cdots R_{[u_s]}R_{[v]}R_{[u_{s+1}]} \cdots R_{[u_n]}$$
where $[u_1] \leq \cdots \leq [u_s] \leq [v] < [u_{s+1}] \leq \cdots \leq [u_n]$ and $s \leq n$. By Lemma 2.2, we have

**Corollary 2.3** Let $[u], [v] \in W$ and $[u] = x_iR_{[u_1]} \cdots R_{[u_{m-1}]}R_{[u_m]} = [u'][u_m]$, where $[u'] = x_iR_{[u_1]} \cdots R_{[u_{m-1}]}$. If $[u_m] > [v]$, then, in $RS(X)$,
$$[u][v] = ((u')[v])[u_m]$$
and
$$[u'][[v][u_m]] < [u][v].$$

**Lemma 2.4** ([39]) Let $[u], [v],$ and $[w]$ be arbitrary good words. If $[u] < [v]$ then $\overline{[w][u]} < [w][v]$ and $[u][w] < [v][w]$. It follows that if $f, g \in RS(X)$, then $\bar{fg} = f\bar{g}$.

**Definition 2.5** Let $S \subset A$ be a set of monic polynomials, $s \in S$ and $(u) \in X^{**}$. We view $S$ as a new set of letters with $S \cap X = \emptyset$. We define $S$-word $(u)_s$ by induction:

(i) $(u)_s = s$ is an $S$-word of $S$-length 1.

(ii) If $(u)_s$ is an $S$-word with $S$-length $k$ and $[v]$ is a good word with length $l$, then
$$(u)_s[v] \text{ and } [v](u)_s$$
are $S$-words with $S$-length $k + l$.

**Definition 2.6** An $S$-word $(u)_s$ is called a normal $S$-word if $(u)_s = (asb)$ is a good word. We denote $(u)_s$ by $[u]_s$ if $(u)_s$ is a normal $S$-word. From Lemma 2.4 it follows that $\overline{[u]_s} = [u]_s$. 

4
Definition 2.7 Let \( f, g \in RS(X) \) be monic polynomials, \([w] \in W\), and \( a, b \in X^*\). Then there are two kinds of compositions.

(i) If \( \bar{f} = [agb] \), then
\[
(f, g)_{\bar{f}} = f - [agb]
\]
is called composition of inclusion.

(ii) If \( (\bar{f}[w]) \) is not good, then
\[
f \cdot [w]
\]
is called composition of right multiplication.

Let \( S \subset RS(X) \) be a given nonempty subset. The composition of inclusion \((f, g)_{\bar{f}}\) is said to be trivial modulo \( S \) if
\[
(f, g)_{\bar{f}} = \sum_i \alpha_i[a_is_ib_i]
\]
where each \( \alpha_i \in k \), \( a_i, b_i \in X^* \), \( s_i \in S \), \([a_is_ib_i]\) is normal \( S \)-word and \([a_is_ib_i] < \bar{f}\). If this is the case, then we write
\[
(f, g)_{\bar{f}} \equiv 0 \mod (S, \bar{f}).
\]

In general, for any normal word \([w]\) and \( p, q \in RS(X) \), we write
\[
p \equiv q \mod (S, [w])
\]
which means that
\[
p - q = \sum \alpha_i[a_is_ib_i], \text{ where each } \alpha_i \in k, a_i, b_i \in X^*, s_i \in S \text{ and } [a_is_ib_i] < [w].
\]

The composition of right multiplication \( f \cdot [w] \) is said to be trivial modulo \( S \) if
\[
f \cdot [w] = \sum_i \alpha_i[a_is_ib_i]
\]
where each \( \alpha_i \in k \), \( a_i, b_i \in X^* \), \( s_i \in S \), \([a_is_ib_i]\) is normal \( S \)-word and \([a_is_ib_i] \leq f \cdot [w] \). If this is the case, then we write
\[
f \cdot [w] \equiv 0 \mod (S).
\]

Definition 2.8 Let \( S \subset RS(X) \) be a nonempty set of monic polynomials and the order < on \( X^{**} \) be defined as before. Then the set \( S \) is called a Gröbner-Shirshov basis in \( RS(X) \) if any composition in \( S \) is trivial modulo \( S \).

Lemma 2.9 Let \( S \subset RS(X) \) be a set of monic polynomials and \((u)_s\) an \( S \)-word. If any right multiplication composition in \( S \) is trivial modulo \( S \), then \((u)_s\) has a representation:
\[
(u)_s = \sum_i \alpha_i[u_i]_{s_i}
\]
where each \( \alpha_i \in k \), \([u_i]_{s_i}\) is a normal \( S \)-word and \([u_i]_{s_i} \leq (u)_s\).
Proof. We use induction on $(u)_s$. If $(u)_s = \bar{s}$, then $(u)_s = s$ and the result holds. Assume that $(u)_s > \bar{s}$. Then $(u)_s = (v)_s[w]$ or $(u)_s = [w](v)_s$. We consider only the case $(u)_s = (v)_s[w]$. The other case can be similarly proved.

By induction, we may assume that $(u)_s = [v]_s[w]$. If $[v]_s = s$, then the result holds clearly because each right multiplication composition in $S$ is trivial modulo $S$. Suppose that $[v]_s = [v_1]_s[v_2]$ or $[v]_s = [v_1][v_2]_s$. We consider only the case $[v]_s = [v_1]_s[v_2]$. The other case can be similarly proved. If $[v_2] \leq [w]$, then $(u)_s = [v]_s[w]$ is a normal $S$-word and we get the result. If $[v_2] > [w]$, then

$$(u)_s = ([v_1]_s[v_2])[w] = ([v_1]_s[w])[v_2] + [v_1]_s([v_2][w]) - [v_1]_s([w][v_2]).$$

By induction, $[v_1]_s[w] = \sum_{j} \beta_j[v]_j_s$, where $\beta_j \in k$, $[v]_j_s$ is a normal $S$-word, and $[v]_j_s \leq [v_1]_s[w]$. If $[v]_j_s = [v_1]_s[w]$, then $[v]_j_s[v_2]$ is a normal $S$-word since $[v]_j_s[v_2] = [v]_j_s[v_2] = ([v_1]_s[w])[v_2] = ([v_1]_s[v_2])[w] = (u)_s$ by Corollary 2.3. If $[v]_j_s < [v_1]_s[w]$, then $[v]_j_s[v_2] < ([v_1]_s[w])[v_2] = ([v_1]_s[w])[v_2] = (u)_s$. By Corollary 2.3 again,

$$[v_1]_s([v_2][w]), [v_1]_s([w][v_2]) < ([v_1]_s[w])[v_2] = (u)_s.$$

Now the result follows from the induction. \(\square\)

Lemma 2.10 Let $[a_1s_1b_1]$, $[a_2s_2b_2]$ be normal $S$-words. If $S$ is a Gröbner-Shirshov basis in $RS(X)$ and $[w] = [a_1s_1b_1] = [a_2s_2b_2]$, then

$$[a_1s_1b_1] \equiv [a_2s_2b_2] \mod (S, [w]).$$

Proof. We have $w = a_1s_1b_1 = a_2s_2b_2$ as associative words. There are two cases to consider.

Case 1. Suppose that $\bar{s}_1$ and $\bar{s}_2$ are disjoint, say, $|a_2| \geq |a_1| + |\bar{s}_1|$. Then, we can assume that

$$a_2 = a_1\bar{s}_1c \quad \text{and} \quad b_1 = c\bar{s}_2b_2$$

for some $c \in X^*$. Thus, $[w] = [a_1s_1c\bar{s}_2b_2]$. Now,

$$[a_1s_1b_1] - [a_2s_2b_2] = [a_1s_1c\bar{s}_2b_2] - [a_1s_1cs_2b_2] = [a_1s_1\bar{s}_2b_2] - (a_1s_1cs_2b_2) + (a_1s_1cs_2b_2) - [a_1s_1cs_2b_2] = (a_1s_1c(\bar{s}_2 - s_2)b_2) + (a_1(s_1 - \bar{s}_1)cs_2b_2).$$

Since $|\bar{s}_2 - s_2| < \bar{s}_2$ and $|s_1 - \bar{s}_1| < \bar{s}_1$, and by Lemmas 2.4, 2.9, we conclude that

$$[a_1s_1b_1] - [a_2s_2b_2] = \sum \alpha_i[u_is_i]$$

for some $\alpha_i \in k$, normal $S$-words $[u_is_i]$, such that $[u_is_i] < [w]$. Hence,

$$[a_1s_1b_1] \equiv [a_2s_2b_2] \mod (S, [w]).$$

Case 2. Suppose that $\bar{s}_1$ contains $\bar{s}_2$ as a subword. We assume that

$$\bar{s}_1 = [a\bar{s}_2b], \quad a_2 = a_1a \quad \text{and} \quad b_2 = b_1b_1,$$

that is, $[w] = [a_1[a\bar{s}_2b]b_1]$.
for the normal $S$-word $[as_2b]$. We have
\[
[a_1s_1b_1] - [a_2s_2b_2] = [a_1s_1b_1] - [a_1[as_2b]b_1] \\
= (a_1(s_1 - [as_2b])b_1) \\
= (a_1(s_1, s_2)s_1b_1).
\]

Since $S$ is a Gröbner-Shirshov basis, $(s_1, s_2)s_1 = \sum \alpha_i [c_i s_i d_i]$ for some $\alpha_i \in k$, normal S-words $[c_i s_i d_i]$ with each $[c_i s_i d_i] < s_1$. By Lemmas 2.4, 2.9, we have
\[
[a_1s_1b_1] - [a_2s_2b_2] = (a_1(s_1, s_2)s_1b_1) \\
= \sum \alpha_i (a_1[c_i s_i d_i]b_1) = \sum \beta_j [a_j s_j b_j]
\]
for some $\beta_j \in k$, normal S-words $[a_j s_j b_j]$ with each $[a_j s_j b_j] < [w] = [a_1s_1b_1]$.

Consequently, $[a_1s_1b_1] \equiv [a_2s_2b_2] \mod(S,[w])$. □

Lemma 2.11 Let $S \subset RS(X)$ be a set of monic polynomials and $Irr(S) = \{|u| \in W| [u] \neq [asb], a,b \in X^*, s \in S$ and $[asb]$ is a normal $S$-word}. Then for any $f \in RS(X)$,
\[
f = \sum_{|u_i| \leq \hat{f}} \alpha_i [u_i] + \sum_{[a_j s_j b_j] \leq \hat{f}} \beta_j [a_j s_j b_j],
\]
where each $\alpha_i, \beta_j \in k$, $[u_i] \in Irr(S)$ and $[a_j s_j b_j]$ is a normal S-word.

Proof. Let $f = \sum \alpha_i [u_i] \in RS(X)$ where $0 \neq \alpha_i \in k$ and $[u_1] > [u_2] > \cdots$. If $[u_1] \in Irr(S)$, then let $f_1 = f - \alpha_1 [u_1]$. If $[u_1] \notin Irr(S)$, then there exist some $s \in S$ and $a_1, b_1 \in X^*$ such that $\hat{f} = [a_1 s_1 b_1]$. Let $f_1 = f - \alpha_1 [a_1 s_1 b_1]$. In both cases, we have $\hat{f}_1 < \hat{f}$. Then the result follows by using induction on $\hat{f}$. □

Theorem 2.12 Let $S \subset RS(X)$ be a nonempty set of monic polynomials and the order $<$ be defined as before. Let $Id(S)$ be the ideal of $RS(X)$ generated by $S$. Then the following statements are equivalent:

(i) $S$ is a Gröbner-Shirshov basis in $RS(X)$.

(ii) $f \in Id(S) \Rightarrow \hat{f} = [asb]$ for some $s \in S$ and $a,b \in X^*$, where $[asb]$ is a normal $S$-word.

(iii) $f \in Id(S) \Rightarrow f = \alpha_1[a_1 s_1 b_1] + \alpha_2[a_2 s_2 b_2] + \cdots$, where $\alpha_i \in k$, $[a_1 s_1 b_1] > [a_2 s_2 b_2] > \cdots$ and each $[a_i s_i b_i]$ is a normal $S$-word.

(iii) $Irr(S) = \{|u| \in W| [u] \neq [asb], a,b \in X^*, s \in S and [asb]$ is a normal $S$-word} is a linear basis of the algebra $RS(X)/Id(S)$.  

7
Proof. (i) ⇒ (ii). Let $S$ be a Gröbner-Shirshov basis and $0 \neq f \in Id(S)$. We can also assume, by Lemma 2.9 that

$$f = \sum_{i=1}^{n} \alpha_i [a_i s_i b_i]$$

where each $\alpha_i \in k$, $a_i, b_i \in X^*$, $s_i \in S$ and $[a_i s_i b_i]$ a normal $S$-word. Let

$$[w_1] = [a_1 s_1 b_1], \ [w_1] = [w_2] = \cdots = [w_l] \geq \cdots$$

We will use the induction on $l$ and $[w_1]$ to prove that $f = [a s b]$ for some $s \in S$ and $a, b \in X^*$. If $l = 1$, then $f = [a s b]$ and hence the result holds. Assume that $l \geq 2$. Then, by Lemma 2.10, we have

$$[a_1 s_1 b_1] \equiv [a_2 s_2 b_2] \ mod(S, [w_1]).$$

Thus, if $\alpha_1 + \alpha_2 \neq 0$ or $l > 2$, then the result holds. For the case that $\alpha_1 + \alpha_2 = 0$ and $l = 2$, we use the induction on $[w_1]$. Hence, the result follows.

(ii) ⇒ (ii)′. Assume that (ii) and $0 \neq f \in Id(S)$. Let $f = \alpha_1 f + \cdots$. Then, by (ii), $f = [a_1 s_1 b_1]$. Therefore,

$$f_1 = f - \alpha_1 [a_1 s_1 b_1], \ f_1 < f, \ f_1 \in Id(S).$$

Now, by using induction on $f$, we have (ii)′.

(ii)′ ⇒ (ii). This part is clear.

(iii) ⇒ (i). Suppose that $\sum \alpha_i [u_i] = 0$ in $RS\langle X \mid S \rangle$, where $\alpha_i \in k$, $[u_i] \in Irr(S)$. It means that $\sum \alpha_i [u_i] \in Id(S)$ in $RS\langle X \rangle$. Then all $\alpha_i$ must be equal to zero. Otherwise, we have $\sum \alpha_i [u_i] = [u_j] \in Irr(S)$ for some $j$ which contradicts (ii).

Now, for any $f \in RS\langle X \rangle$, by Lemma 2.11 we have

$$f = \sum_{[u_i] \in Id(S), \ [u_i] \leq f} \alpha_i [u_i] + \sum_{[a_j s_j b_j] \leq f} \beta_j [a_j s_j b_j].$$

Hence, (iii) follows.

(iii) ⇒ (i). Applying Lemma 2.11 to a composition of elements of $S$, we get by (iii) that any composition is trivial because any composition belongs to $Id(S)$. So $S$ is a Gröbner-Shirshov basis. □

3 Gröbner-Shirshov basis for universal enveloping right-symmetric algebra of a Lie algebra

The universal enveloping right-symmetric algebra of a Lie algebra is defined in the paper [48]. In this section, we give a Gröbner-Shirshov basis for such an algebra.
Theorem 3.1 Let \((\mathcal{L}, [,])\) be a Lie algebra with a well ordered basis \(\{e_i | i \in I\}\). Let

\[ [e_i, e_j] = \sum_m \alpha_{ij}^m e_m \]

where \(\alpha_{ij}^m \in k\). We denote \(\sum_m \alpha_{ij}^m e_m\) by \(\{e_i e_j\}\). Let

\[ U(\mathcal{L}) = RS(\{e_i\}_I | e_i e_j - e_j e_i = \{e_i e_j\}, i, j \in I) \]

be the universal enveloping right-symmetric algebra of \(\mathcal{L}\). Let

\[ S = \{f_{ij} = e_i e_j - e_j e_i - \{e_i e_j\}, i, j \in I \text{ and } i > j\}. \]

Then

(i) \(S\) is a Gröbner-Shirshov basis in \(RS(X)\) where \(X = \{e_i\}_I\).

(ii) \((\mathcal{L})\) \(\mathcal{L}\) can be embedded into the “universal enveloping right-symmetric algebra” \(U(\mathcal{L})\) as a vector space.

Proof. (i). It is clear that \(\overline{f_{ij}} = e_i e_j \ (i > j)\). So, there exists a unique kind of composition \(f_{ij} e_k \ (i > j > k)\). Then, in \(RS(X)\), we have

\[
\begin{align*}
f_{ij} e_k &= f_{ik} e_j + f_{jk} e_i - e_i f_{jk} + e_j f_{ik} - e_k f_{ij} - \sum_m \alpha_{jk}^m f_{im} - \sum_m \alpha_{ij}^m f_{km} - \sum_m \alpha_{ik}^m f_{mj} \\
&= (e_i e_j) e_k - (e_j e_i) e_k - (e_i e_k) e_j - (e_i e_k) e_i + (e_k e_i) e_j + (e_k e_i) e_i \\
&+ f_{jk} e_i - e_i f_{jk} + e_j f_{ik} - e_k f_{ij} - \sum_m \alpha_{jk}^m f_{im} - \sum_m \alpha_{ij}^m f_{km} - \sum_m \alpha_{ik}^m f_{mj} \\
&= (e_i e_k) e_j + (e_j e_k) e_i - e_i (e_k e_j) - e_j (e_k e_i) + e_j (e_k e_i) - e_j (e_i e_k) - e_k (e_i e_j) + (e_k e_j) e_i \\
&+ e_k (e_j e_i) + (e_i e_k) e_j + f_{jk} e_i - e_i f_{jk} + e_j f_{ik} - e_k f_{ij} \\
&- \sum_m \alpha_{jk}^m f_{im} - \sum_m \alpha_{ij}^m f_{km} - \sum_m \alpha_{ik}^m f_{mj} \\
&= - (e_i e_k) e_i + (e_k e_i) e_i + e_i (e_k e_j) - e_i (e_k e_i) + e_j (e_k e_i) - e_j (e_i e_k) - e_k (e_i e_j) + (e_k e_j) e_i \\
&- e_i e_j) e_k + (e_i e_k) e_j + f_{jk} e_i - e_i f_{jk} + e_j f_{ik} - e_k f_{ij} - \sum_m \alpha_{jk}^m f_{im} - \sum_m \alpha_{ij}^m f_{km} - \sum_m \alpha_{ik}^m f_{mj} \\
&= - \{e_j e_k\} e_i + e_i \{e_j e_k\} - e_j \{e_i e_k\} + e_k \{e_i e_j\} - \{e_i e_j\} e_k + \{e_i e_k\} e_j \\
&- \sum_m \alpha_{jk}^m f_{im} - \sum_m \alpha_{ij}^m f_{km} - \sum_m \alpha_{ik}^m f_{mj} \\
&= - \{e_k e_i\} e_j - \{e_i e_j\} e_k - \{e_j e_k\} e_i \\
&= 0 \text{ (by Jacobi identity).}
\end{align*}
\]

By invoking \(f_{mn} = -f_{nm}\), we have \(f_{ij} e_k \equiv 0 \ mod(S)\). Therefore, \(S\) is a Gröbner-Shirshov basis for \(U(\mathcal{L})\).

(ii). It follows from Theorem 2.12 \(\square\)
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