Predictions for measuring the 21-cm multi-frequency angular power spectrum using SKA-Low

Rajesh Mondal, 1* Abinash Kumar Shaw 2, Ilian T. Iliev, 1 Somnath Bharadwaj, 2 Kanan K. Datta, 3 Suman Majumdar, 4,5 Anjan K. Sarkar 6 and Keri L. Dixon 7

1 Astronomy Centre, Department of Physics and Astronomy, University of Sussex, Brighton BN19QH, UK
2 Department of Physics & Centre for Theoretical Studies, Indian Institute of Technology Kharagpur, Kharagpur 721302, India
3 Department of Physics, Presidency University, 86/1 College Street, Kolkata 700073, India
4 Discipline of Astronomy, Astrophysics and Space Engineering, Indian Institute of Technology Indore, Simrol, Indore 453552, India
5 Department of Physics, Blackett Laboratory, Imperial College, London SW7 2AZ, UK
6 Astronomy & Astrophysics Group, Raman Research Institute, Bengaluru 560080, India
7 New York University Abu Dhabi, PO Box 129188, Saadiyat Island, Abu Dhabi, United Arab Emirates

ABSTRACT
The light-cone (LC) effect causes the mean as well as the statistical properties of the redshifted 21-cm signal $T_b(\hat{n}, \nu)$ to change with frequency $\nu$ (or cosmic time). Consequently, the statistical homogeneity (ergodicity) of the signal along the line of sight (LoS) direction is broken. This is a severe problem particularly during the Epoch of Reionization (EoR) when the mean neutral hydrogen fraction ($\bar{x}_{\text{HI}}$) changes rapidly as the universe evolves. This will also pose complications for large bandwidth observations. These effects imply that the 3D power spectrum $P(k)$ fails to quantify the entire second-order statistics of the signal as it assumes the signal to be ergodic and periodic along the LoS. As a proper alternative to $P(k)$, we use the multi-frequency angular power spectrum (MAPS) $C_\ell(\nu_1, \nu_2)$ which does not assume the signal to be ergodic and periodic along the LoS. Here, we study the prospects for measuring the EoR 21-cm MAPS using future observations with the upcoming SKA-Low. We find that the EoR 21-cm MAPS can be measured at a confidence level $\gtrsim 5\sigma$ at angular scales $\ell \sim 1300$ for total observation time $t_{\text{obs}} \gtrsim 128$ hrs across $\sim 44$ MHz observational bandwidth. These results are very relevant for the upcoming large bandwidth EoR experiments as previous predictions were all restricted to individually analyzing the signal over small frequency (or equivalently redshift) intervals.

Key words: cosmology: theory – observations – dark ages, reionization, first stars – diffuse radiation – large-scale structure of Universe – methods: statistical – technique: interferometric.

INTRODUCTION

The Epoch of Reionization (EoR) is one of the important periods in the evolutionary history of our Universe. During this epoch, the ionizing radiation from the first luminous sources in the Universe gradually ionizes the neutral Hydrogen (H_i) in the intergalactic medium (IGM). As more and more these sources form, the ionized (H II) regions grow and eventually overlap and fill almost the entire IGM. Our present knowledge about this epoch is very limited. The current measurements of the Thomson scattering optical depth (Planck Collaboration et al. 2016a,b), a measure of the line of sight (LoS) free electron opacity to cosmic microwave background (CMB) radiation in the IGM, suggest that the mean neutral fraction $\bar{x}_{\text{HI}}$ falls by $\sim 0.1$ at $z \sim 10$ from a completely neutral IGM. The second observation is the Gunn-Peterson optical depth of the high redshift quasar spectra (Becker et al. 2001; Fan et al. 2006, 2002; Becker et al. 2015). These measurements show an absorption trough at $z \lesssim 6$ which indicates that the IGM was neutral at 0.1% level by $z \sim 6$. The third and the most recent constraint comes from the measurements of the luminosity function and clustering properties of high-$z$ Lyman-α emitters (Konno et al. 2014; Santos et al. 2016; Zheng et al. 2017; Ota et al. 2017). These studies indicate a patchy distribution of H i and infer a sharp increase in $\bar{x}_{\text{HI}}$ at redshifts larger than $z \sim 7$. The findings of all these indirect observations provide an overall indication that the EoR probably extends over a redshift range $6 \lesssim z \lesssim 12$ (Robertson et al. 2013, 2015; Mitra et al. 2015, 2017; Dai et al. 2019). However, these indirect observations are not able to shed light on various fundamental issues, such as the exact duration and timing of the reionization, properties of the ionizing sources, the topology of H i at different cosmic times etc.

Observations of the redshifted 21-cm signal caused by the hyperfine transition of H i in the IGM is the most promising...
probes of the EoR (Scott & Rees 1990; Bharadwaj & Sethi 2001). There has been a considerable observational effort devoted to measuring the EoR 21-cm signal using the presently operating radio interferometers e.g. the GMRT1 (Paciga et al. 2013), LOFAR2 (van Haarlem et al. 2013; Yatawatta et al. 2013), the MWA3 (Jacobs et al. 2016), and PAPER4 (Parsons et al. 2014; Jacobs et al. 2013; Ali et al. 2015). The presently operating (first generation) radio interferometers are not sensitive enough to make tomographic images of the EoR 21-cm signal and can only make a statistical detection of the signal. Observing the EoR 21-cm signal is one of the major scientific goals of the upcoming radio telescopes e.g. SKA5 (Mellema et al. 2013; Koopmans et al. 2015) and HERA6 (DeBoer et al. 2017). These observations are very challenging due to the presence of foregrounds, system noise, and other calibration errors. Foregrounds are ~4–5 orders of magnitude stronger than the expected signal (Ali et al. 2008; Bernardi et al. 2009; Ghosh et al. 2012; Paciga et al. 2013), and modeling or removing them from the actual data is more complicated. However, in this work, we assume the idealistic scenario where foregrounds can be removed completely.

The upcoming SKA-Low will have 512 stations7 and each of them will be ~35 m in diameter. These stations will consist of several log-periodic dipole antennas. The telescope will also have ~20 deg² field of view, a compact core and 3 spiral arms which will extend up to ~60 km. SKA-Low will have enough sensitivity over a large range of frequencies (frequency band of 50 – 350 MHz) to image the EoR 21-cm signal (Mellema et al. 2015). Unlike the CMB, we can map the large-scale structure (LSS) of the universe in 3D using the redshifted 21-cm signal, with the third dimension being frequency (or cosmic time or redshift). However, one has to be very careful while quantifying the EoR 21-cm signal as the mean, as well as other statistical properties of the signal change with varying frequency or redshift due to the light-cone (LC) effect (Barkana & Loeb 2006; Datta et al. 2012; Mondal et al. 2018).

The LC effect breaks the statistical homogeneity (ergodicity) along the LoS direction. Moreover, the main assumption that goes into the estimation of the power spectrum \(P(k)\) or equivalently into the 3D Fourier transform is that the signal is ergodic and periodic. As a consequence of this fundamental difference between the assumption for Fourier transform and the actual properties of the signal, the spherically averaged 3D power spectrum \(P(k)\) fails to quantify the entire second-order statistics of the signal (Mondal et al. 2018) and gives a rather biased estimation of the signal (Trott 2016). This is particularly severe during the EoR when the \(\delta_{\text{M}}\) changes rapidly as the reionization proceeds. This will also pose complications for broad bandwidth observations with SKA-Low (Mondal et al. 2019). The issue here is ‘how to quantify the statistics of the EoR 21-cm signal in the presence of the LC effect’. As a proper alternative to \(P(k)\), we use the multi-frequency angular power spectrum (MAPS) \(\mathcal{C}_{\ell}(v_1, v_2)\) (Datta et al. 2007; Mondal et al. 2018, 2019) which does not assume ergodicity and periodicity along the LoS. The only assumption is that the EoR 21-cm signal is statistically homogeneous and isotropic in different directions on the sky plane. The visibilities are the main observables in every radio-interferometric observations and the MAPS is directly associated with these visibility correlations. Therefore, it is relatively easy to estimate MAPS from the observations (Bharadwaj & Ali 2005; Ali et al. 2008; Ghosh et al. 2011).

Several studies have been made to quantify the sensitivity for measuring the EoR 21-cm power spectrum with different instruments (Morales 2005; McQuinn et al. 2006; Zaroubi et al. 2012; Beardsley et al. 2013; Pober et al. 2014; Ewall-Wice et al. 2016; Shaw et al. 2019). These predictions were restricted to individually analyzing over small redshift (or equivalently frequency) intervals where they have worked with the 3D power spectrum \(P(k)\). However, there is no such restriction for the MAPS and we can, in principle, consider the entire bandwidth for the analysis. Here, we have made the SNR predictions for measuring the EoR 21-cm MAPS using future observations with SKA-Low. We have considered a scenario where the observed EoR 21-cm MAPS is the sum of the EoR 21-cm MAPS and the system noise only, ignoring any contribution from the foregrounds to the observed signal. We have used numerical simulations to calculate the EoR 21-cm MAPS for our analysis.

The paper is structured as follows. In Section 2, we briefly describe the simulations used to generate the EoR 21-cm light-cones. Starting from the basic definition of the MAPS, we derive the expressions for the noise MAPS and MAPS error-covariance in Section 3. In Section 5, we report the results i.e. the estimated MAPS, MAPS error-covariance and SNR. Finally, in Section 6, we summarize our results and conclude. Throughout the paper, we have used the values of cosmological parameters \(\Omega_{m0} = 0.27, \Omega_{\Lambda0} = 0.73, \Omega_{b0}h^2 = 0.02156, h = 0.7, \sigma_8 = 0.8, \text{and } n_s = 0.9619\). These values are consistent with the latest results from WMAP (Komatsu et al. 2011) and Planck combined with other available constraints (Planck Collaboration et al. 2015, 2016a).

2 SIMULATING THE EOR 21-CM SIGNAL

2.1 The Simulation

The density fields and halo catalogues are obtained from a high-resolution, large-volume N-body PRACE4LOFAR simulation (Giri et al. 2019). This simulation was run using the CowperM code (Harnois-Deraps et al. 2013) and followed 6912⁶ particles in a comoving 500 h⁻¹Mpc ≈ 714 Mpc per side volume to enable reliable halo identification (with 25 particles or more) down to \(10^9 M_\odot\). The reionization process is simulated using the C²-RAY code (Mellema et al. 2006) on a 30⁰ grid with sources and density fields based on the N-body data following the method presented in Iliev et al. (2007) and Dixon et al. (2016). Specifically, for this work, we have used the data from the 500Mpc_r1.0e3Cs300 reionization simulation of Dixon et al. (2016). We refer the reader to Dixon et al. (2016) for details of the notation and setup, with only a brief summary provided here.

The density fields are calculated using SPH-like smoothing. The sources of ionization are associated with the resolved halos, complemented by the sub-grid model for the low-mass atomically-cooling halos \(10^6 < M_{halo} < 10^9 \) (Ahn et al. 2015). For a source with halo mass \(M\) and lifetime \(t_s\), we assign ionizing photon emissivity according to

\[
N_\gamma = g_\gamma \frac{M_{\odot}}{\mu_{mp}(10 \text{Myr})},
\]

where the efficiency \(g_\gamma\) combines the ionizing photon production

1 http://www.gmrt.ncra.tifr.res.in
2 http://www.lofar.org
3 http://www.mwatelescope.org
4 http://eor.berkeley.edu
5 http://www.skatelescope.org
6 http://reionization.org
7 SKA1_LowConfigurationCoordinates-1.pdf
efficiency of the stars per stellar atom, $N_s$, the star formation efficiency, $f_s$, and the escape fraction, $f_{\text{esc}}$:

$$g_\gamma = f_s f_{\text{esc}} N_s \left( \frac{10 \text{ Myr}}{t_s} \right).$$

(e.g. Haiman & Holder 2003; Iliev et al. 2012). The high-mass sources ($M > 10^9 M_\odot$) are assumed unaffected by the radiative feedback, while for local ionization the low-mass sources have a mass-dependent efficiency

$$g_\gamma \propto \left[ \frac{M}{9 \times 10^8 M_\odot} - \frac{1}{9} \right].$$

We have generated the coeval brightness temperature ($\delta T_b$) cubes at 125 different redshifts in the range $6 \leq z < 16$, and the resulting reionization history is shown in Figure 1.

### 2.2 Generating the light-cones

We have generated our light cones following the formalism presented in Datta et al. (2014) using the simulated coeval $\delta T_b$ cubes described in Section 2.1. We have generated two light-cones: LC1, centered at $z_c = 7.09$ (frequency $\nu_c = 175.58$ MHz) which corresponds to $\bar{\delta} H_1 = 0.50$ and LC2, centered at $z_c = 8.04$ ($\nu_c = 157.08$ MHz) which corresponds to $\bar{\delta} H_1 = 0.75$. The LC1 span the redshift range $6.15 \lesssim z \lesssim 8.25$, which corresponds to change in the mass-averaged $H_1$ fraction $\bar{\delta} H_1$ from end-to-end of the light cone, following the reionization history shown in Figure 1), is $\Delta \bar{\delta} H_1 \approx 0.79 - 0.02 = 0.77$, Whereas, LC2 span the range $6.92 \lesssim z \lesssim 9.40$, which corresponds to change in the $\bar{\delta} H_1$ is $\Delta \bar{\delta} H_1 \approx 0.90 - 0.42 = 0.48$. Note that the redshift ranges, channel widths and central frequencies assumed in the light-cones are only representative values and may change. We have chosen these to observe the behavior at two different stages of reionization history.

The right panels of Figure 2 and 3 show sections through the simulated LC 21-cm brightness temperature maps. As a comparison, the left panels of Figure 2 and 3 show the sections through coeval simulations at $z = 7.09$ and $z = 8.04$, respectively. The lower redshifts on the left side of the LC simulations correspond to the later stages of the evolution as compared to the higher redshifts shown on the right side. The ionized regions appear smaller in the LC simulations as compared to their coeval companion at the right side (early stage). Whereas, the ionized regions appear larger in the LC simulations as compared to their coeval case at later stages (left side).

### 3 THE MULTIFREQUENCY ANGULAR POWER SPECTRUM

The present study considers the question: ‘How to quantify the statistics of the redshifted EoR $H_1$ 21-cm signal $\delta T_b(K, \nu)$ when it is non-ergodic along the LoS (i.e. the signal varies significantly along the LoS)?’ (Trott 2016; Mondal et al. 2018, 2019). In the case of the evolving statistical properties of the signal within the observed volume, the 3D Fourier modes $k$ are not the correct choice of basis. Further, it assumes periodic boundary condition in all directions which is also not justified along the LoS. As a consequence, the power spectrum $P(k)$ is not optimal and gives a biased estimate of the true statistics (Trott 2016; Mondal et al. 2018). To avoid this issue, the previous power spectrum measurements are restricted to individually analyzing small redshift intervals (Morales 2005; McQuinn et al. 2006; Zaroubi et al. 2012; Datta et al. 2014; Pober et al. 2014; Ewall-Wice et al. 2016; Shaw et al. 2019).

The above-mentioned properties of the signal necessitate us to use the Multifrequency Angular Power Spectrum (MAPS) $G_f(v_1, v_2)$ which quantifies the entire second-order statistics of the EoR 21-cm signal (Mondal et al. 2018). It doesn’t assume the signal to be statistically homogeneous along the LoS. One can decompose

![Figure 1](image1.png)  
**Figure 1.** This shows the reionization history as a function of redshift that we have obtained from simulations.

![Figure 2](image2.png)  
**Figure 2.** This shows sections through the 3D 21-cm brightness temperature maps for the coeval (left) and LC (right) simulations. The boxes are centered at redshift 7.09 which corresponds to the comoving distance $r_c = 8865.64$ Mpc and $\delta H_1 = 0.50$.

![Figure 3](image3.png)  
**Figure 3.** Same as Figure 2 centered at redshift 8.04 which corresponds to the comoving distance $r_c = 9162.06$ Mpc and $\delta H_1 = 0.75$. 

In this work, we assume the plane of the sky is flat. Under the flat-sky approximation, we map the brightness temperature fluctuations $\delta T_b(x, y, z)$ from the Cartesian grid to a 3D rectangular grid in $(\theta, \nu)$ within our simulation box. We use $\theta_x = x/r$, $\theta_y = y/r$, and $\nu = z/r'$. We also keep the angular extent same at all frequency channels while performing this coordinate transformation.
\( \delta T_b(\mathbf{n}, \nu) \) into spherical harmonics \( Y^m_\ell(\mathbf{n}) \) as
\[
\delta T_b(\mathbf{n}, \nu) = \sum_{\ell,m} a_{\ell m}(\nu) Y^m_\ell(\mathbf{n}),
\]
and define the MAPS using
\[
C_\ell(v_1, v_2) = \langle a_{\ell m}(v_1) a_{\ell m}^*(v_2) \rangle \tag{5}
\]
The only assumption which goes into this definition is that the EoR 21-cm signal is statistically homogeneous and isotropic in different directions on the sky plane.

In this study, we have chosen to work in the flat-sky approximation where the redshifted 21-cm brightness temperature fluctuations can be expressed as \( \delta T_b(\theta, \nu) \). Here \( \theta \) denotes a 2D vector on the plane of the sky. Instead of \( \delta T_b(\theta, \nu) \), we use its 2D Fourier transform \( \hat{T}_b(U, \nu) \) where \( U \) is the Fourier conjugate of \( \theta \) described in the previous section. \( \hat{T}_b(U, \nu) \) is the primary observables measured in radio interferometric observations. Under the flat-sky approximation, we redefine the MAPS (eq. 5) as
\[
C_\ell(v_1, v_2) \equiv \hat{C}_\ell v_1(v_1, v_2) = \Omega^{-1} \langle \hat{T}_b(U, v_1) \hat{T}_b(U, v_2) \rangle ,
\]
where \( \Omega \) is the solid angle subtended by the transverse extent of the observation (or simulation) at the location of the observer and \( \ell \) is the corresponding angular multipole. The above definition of \( C_\ell(v_1, v_2) \) does not assume statistical homogeneity and periodicity along the LoS. However, note that if one imposes statistical homogeneity along the LoS, the MAPS \( C_\ell(v_1, v_2) \) is expected to depend only on the frequency separation \( \Delta \nu = |\nu_1 - \nu_2| \), i.e. \( C_\ell(v_1, v_2) \equiv C_\ell(\Delta \nu) \).

Figure 4 shows the scaled MAPS \( \Phi^2_\ell(v_1, v_2) = [\ell(\ell + 1)C_\ell(v_1, v_2)/2\pi] \) at four different values of \( \ell \) = 469, 768, 1257 and 2071 for the LC1 simulation which is centered at \( \bar{x}_H \approx 0.50 \). Figure 5 shows the same at \( \ell = 486, 796, 1304 \) and 2147 for the LC2 simulation which is centered at \( \bar{x}_H \approx 0.75 \). In this work, we focus mainly on the intermediate \( \ell \) modes as the detection of the signal will be difficult at large scales (\( \ell \lesssim 250 \)) due to the cosmic variance and at small scales (\( \ell \gtrsim 3500 \)) due to presence strong system noise. We see that the MAPS peaks when \( \nu_1 = \nu_2 \).\textsuperscript{1} along the diagonal line. The diagonal \( C_\ell(\nu) \) evolves considerably with the observed frequency \( \nu \). This is a direct consequence of the fact that the signal is non-ergodic along the frequency axis. We also find that the MAPS rapidly falls as the frequency separation \( |\nu_1 - \nu_2| \) increases and oscillates around zero for the larger frequency separation. Unlike the 3D power spectrum \( P(k) \), which captures only the information regarding the ergodic and periodic part of the signal, the MAPS \( C_\ell(v_1, v_2) \) contains the full information regarding the two-point statistics of the signal (Mondal et al. 2018). One can, in principle, use the entire information contained in \( C_\ell(v_1, v_2) \), i.e. all the diagonal and off-diagonal elements to better constrain the EoR. However, we focus mostly on the diagonal terms \( C_\ell(\nu) \). It will be difficult to detect the off-diagonal \( C_{\ell s} \), except for the small frequency separation \( |\nu_1 - \nu_2| \approx 1 \text{ MHz} \), due to poor signal to noise ratio. We refer the readers to section 5 for a detailed discussion on the detectability of the MAPS.

Figure 6 shows the diagonal components of the scaled MAPS \( \Phi^2_\ell(v, \nu) \) as a function of \( \nu \) for \( \ell \) values considered above for both simulations LC1 (black) and LC2 (red). It also shows the ergodic part of the signal \( \Phi^2_\ell(\nu, \nu) \) which is calculated at the central frequency \( \nu_c \) which is different for the LC1 and LC2 simulation. The 3D power spectrum \( P(k) \) misses out the part which is deviated from these horizontal dashed lines. We further see in Figure 6 that \( \Phi^2_\ell(\nu, \nu) \) peaks around a frequency corresponding to the global neutral fraction \( \bar{x}_H \approx 0.35 \) for both simulations. This is due to the presence of a significant number of large ionized bubbles at that stage of the EoR. The power spectrum at higher frequency decreases due to the rapid decline of the neutral fraction \( \bar{x}_H \). The characteristic size of ionized bubbles decreases at lower frequencies which causes the power spectrum to decrease. Similar results have been found in earlier studies (McQuinn et al. 2007; Lidz et al. 2008; Choudhury et al. 2009; Mesinger et al. 2011). We also notice that there is a ‘dip’ in the power spectrum \( \Phi^2_\ell(\nu, \nu) \) around a frequency corresponding to the global neutral fraction \( \bar{x}_H \approx 0.8 \) for all \( \ell \) modes for both simulations. During the early stages of reionization, the high-density regions get ionized first, and as a consequence, the large scale power decreases. This is reflected by the drop in the power across the four \( \ell \) panels when the neutral fraction is large. Later, as the reionization progresses further, the creation and growth of the ionized regions increase the power spectrum which peaks around \( \bar{x}_H \approx 0.35 \). Datta et al. (2014) have investigated the impact of the LC effect considering a similar reionization model and find a similar dip around \( \bar{x}_H \approx 0.8 \). The frequencies at which the minimum and maximum occur may change for different \( \ell \) values. However, we do not see any significant change in the locations of the maxima and minima for the \( \ell \) modes we consider.

4 OBSERVATIONAL CONSIDERATIONS

We now consider observations with a radio-interferometric array where the fundamental quantity is the visibility which is measured by each pair of antennas in the array. Considering any particular pair with \( d_0 \) being the antenna separation projected on the plane perpendicular to the LoS, the visibility measured at frequency \( \nu_1 \) and baseline \( U_n = d_0/\lambda_0 \) provides a direct estimate of \( \hat{T}_b(U_n, \nu_1) \) at the Fourier mode \( U_n \). Taking into account \( \delta T_b(U, \nu) \) the system noise contribution which is inherent in any radio-interferometric observation, the measured visibility actually provides us with \( \hat{T}_b(U_n, \nu_1) = \hat{T}_b(U_n, \nu_1) + \delta T_b(U_n, \nu_1) \), where we have assumed that the foregrounds have been completely removed and there are no calibration errors. The system noise at different baselines and frequency channels is uncorrelated. Using this in eq. (6) for the MAPS, we obtain
\[
C_\ell(v_1, v_2) = C_\ell(v_1, v_2) + \delta T_b(U_n, \nu_1) C_\ell(v_1, v_2),
\]
which can be estimated from the observed visibilities. Following the prescription in Bharadwaj et al. (2018), it is possible to avoid noise bias \( C_\ell(v_1, v_2) \) and obtain an unbiased estimate of \( C_\ell(v_1, v_2) \) from the measured visibilities. However, the noise contributions still persist in the error estimates and this cannot be avoided. In this work, we compute the error variance to predict the signal-to-noise ratio (SNR) of measuring MAPS using the upcoming SKA-Low. This also involves the estimation of system noise for which we use the telescope specifications of SKA-Low taken from the current proposed configuration document\textsuperscript{7}. Some important specifications\textsuperscript{8} which have been used in the computation of \( C_\ell(\nu, \nu) \) are tabulated in Table 1.

We consider observations tracking a field at declination DEC=\( -30^\circ \) for 8 hrs/night with 60 sec integration time following the formalism adopted by Shaw et al. (2019). We restrict our analysis to the baselines \( U \) corresponding to the antenna separations \( |d| < 19 \text{ km} \) as the baseline distribution falls off rapidly at larger \( |d| \) values.

\textsuperscript{7} The specifications assumed here may change in the final implementation of the telescope.

\textsuperscript{8}
Figure 4. This shows the MAPS $\Phi^2(\nu_1, \nu_2)$ at $\ell = 469, 768, 1257$ and 2071 (from left to right respectively) for the LC1 at $\nu_c = 175.58$ MHz.

Figure 5. Same as Figure 4 at $\ell = 486, 796, 1304$ and 2147 (from left to right respectively) for the LC2 at $\nu_c = 157.08$ MHz.

Figure 6. This shows the diagonal components of the scale-independent MAPS $\Phi^2(\nu, \nu)$ for LC1 (black) and LC2 (red). The LC1 and LC2 are respectively centered at frequency 175.58 MHz and 157.08 MHz (vertical dashed lines). We also show the ergodic component (mean) of MAPS $[\ell(\ell+1)C_\ell^2(\nu)/2\pi]$ (horizontal dashed lines). The $\bar{x}_{H_i}$ values corresponding to the frequencies are shown in the top x-axis.

Figure 7 shows the simulated SKA-Low baseline $U$ distribution ($\nu \nu$ coverage) at the two different central frequencies corresponding to LC1 and LC2 respectively. The signals at two different baselines $U$ separated by $< D/\lambda_i$ are correlated due to the overlap of the antenna beam pattern (Bharadwaj & Pandey 2003; Bharadwaj & Ali 2005). We grid the baselines $U_m$ with a grid of size $\Delta U_x = \Delta U_y = D/\lambda_i$ and count the number of measurements $\tau(U_g)$ that lie within a pixel centered at any grid point $U_g$.

We estimate the noise MAPS at the grid point $U_g$ following the calculation presented in White et al. (1999); Zaldarriaga et al.
Here the system temperature $T_{\text{sys}}$ is a sum of the sky temperature $T_{\text{sky}} = 60.2\pm 5.5$ K (Fixsen et al. 2011) and the receiver temperature $T_{\text{rec}}$. $N_{\ell}$ is the number of polarizations, $N_{i}$ is the number of observed nights, $\Delta t$ is the integration time, $a$ is the area of individual antenna in the array and $\hat{A}(U)$ is the Fourier transform of the primary beam of a station $\Lambda (\theta)$ which is approximated with a Gaussian $e^{-((\theta/a)^2)}$ (Choudhuri et al. 2014; Shaw et al. 2019). We express the total observation time using $t_{\text{obs}} = 8$ hrs $x N_{i}$ and this notation is used in the rest of the paper.

### 4.1 The binned weighted MAPS estimator

The simulated observations under consideration have $\sim 300 \times 300$ grid points on the $U$ plane and 313 frequency channels. This comes out to $\sim 14$ Million independent measurements of the MAPS which is computationally very expensive to deal with. Another problem is that the measurements at every individual grid point $U_{g}$ will be very noisy. To tackle these issues, we bin the $U$ space. We, however, lose the information at individual $U_{g}$ modes. This not only solves the computation problem but also increases the SNR of measurement within a bin. We use the binned weighted MAPS estimator $\tilde{C}_{\ell}^{t}(v_{1}, v_{2})$ which is the sum of the weighted brightness temperature fluctuation correlations between various grids within the bin. Exploiting the symmetry $\tilde{C}_{\ell}^{t}(v_{1}, v_{2}) = \tilde{C}_{\ell}^{t}(v_{2}, v_{1})$, the estimator $\hat{C}_{\ell}^{t}(v_{1}, v_{2})$ for the $i$-th bin is written as

$$\hat{C}_{\ell}^{t}(v_{1}, v_{2}) = \frac{1}{241} \sum_{U_{gi}} w(U_{g}, v_{1}) w(U_{g}, v_{2}) \times$$

$$\left[ T_{b2}^{2}(U_{g}, v_{1}) T_{b2}^{2}(U_{g}, v_{2}) \right] ,$$

(9)

where the sum $\sum_{U_{gi}}$ is over the $U_{g}$ grids within the $i$-th bin and $w(U_{g}, v)$ is the weight associated with the grid $U_{g}$ at frequency $v$. Here the angular multipole $\ell_{i} = 2\pi U_{i}$ (or $U_{i}$) is the weighted average of all $U_{gi}$ in the $i$-th bin. We have used equally spaced logarithmic binning and the bins here are semi-annuli of the width $\Delta U_{i} \times U_{i}$ (restricted to one half of the $U$ plane as the signal is real i.e. $T_{b2}^{2}(U_{g}, v_{1}) = T_{b2}^{2}(-U_{g}, v_{1})$).

The ensemble average of the estimator gives the bin-averaged MAPS

$$\langle \tilde{C}_{\ell}^{t}(v_{1}, v_{2}) \rangle = \tilde{C}_{\ell}^{t}(v_{1}, v_{2})$$

$$= \tilde{C}_{\ell}^{t}(v_{1}, v_{2}) + \delta_{\ell}^{K} \tilde{C}_{\ell}^{K}(v_{1}, v_{2}).$$

(10)

As mentioned earlier, it is possible to avoid the noise bias $\tilde{C}_{\ell}^{K}(v, v)$ (Bharadwaj et al. 2018) by subtracting out the contribution of the self-correlation of visibility from itself. This also leads to a loss of a part of the signal. However, this loss is extremely small ($< 0.01\%$) for long observations ($t_{\text{obs}} \sim 100$ hrs or larger) with 16 s integration time. It is therefore quite well justified to assume that we can obtain an unbiased estimate of $\tilde{C}_{\ell}^{t}(v_{1}, v_{2})$. In the subsequent analysis, we also do not consider any change in the weights along the frequency direction and express eq. 9 as

$$\tilde{C}_{\ell}^{t}(v_{1}, v_{2}) = \frac{1}{241} \sum_{U_{gi}} w(U_{g}, v_{1}) T_{b2}^{2}(U_{g}, v_{1}) T_{b2}^{2}(-U_{g}, v_{2})$$

$$+ T_{b2}^{2}(U_{g}, v_{2}) T_{b2}^{2}(-U_{g}, v_{1}) \right].$$

(11)

The weights $w(U_{g})$ are normalized such that $\sum_{U_{g}} w_{g} = 1$ where the sum runs over each grid point within a particular $U$ bin. As discussed later, the weights are selected in order to maximize the SNR of $\tilde{C}_{\ell}^{t}(v_{1}, v_{2})$ for each bin. This takes into account that the baselines $U_{g}$ do not uniformly sample the different grid points $U_{g}$ and consequently the ratio $\tilde{C}_{\ell}^{t}(v_{1}, v_{2})/\tilde{C}_{\ell}^{K}(v_{1}, v_{2})$ varies across the different grid points within a bin.

### 4.2 The error estimates

The EoR 21-cm signal is a highly non-Gaussian field (see e.g. Bharadwaj & Pandey 2005; Mondal et al. 2015; Yoshida et al. 2015; Majumdar et al. 2018). The non-Gaussian effects will play a significant contribution to the error estimates for the EoR 21-cm MAPS (Mondal et al. 2016, 2017; Shaw et al. 2019). However, we have not considered the non-Gaussian nature of the EoR 21-cm signal in this work and we have assumed the signal to be a Gaussian random field. The statistics of a Gaussian random field are completely specified by its second-order statistics (power spectrum) and all higher order statistics (bisppectrum, trispectrum etc.) are zero. Following the calculation presented in Appendix A, we write the MAPS error covariance as

$$X_{12,34}^{\ell_{i}} = \langle [\delta C_{\ell_{i}}^{t}(v_{1}, v_{2})][\delta C_{\ell_{i}}^{t}(v_{3}, v_{4})] \rangle$$

$$= \frac{1}{2} \sum_{U_{gi}} w_{g}^{2} \left[ C_{\ell_{i}}^{t}(v_{1}, v_{2}) C_{\ell_{i}}^{t}(v_{3}, v_{4}) \right]$$

$$+ C_{\ell_{i}}^{t}(v_{1}, v_{3}) C_{\ell_{i}}^{t}(v_{2}, v_{4}),$$

(12)
where the sum is over all the $U_{\ell i}$ grids within the $i$-th bin and $w_{\ell i} \equiv w(U_{\ell i})$. The variance in the measured $G_{\ell i}(v_1, v_2)$ is thus given by

$$X_{12,12}^2 = \langle \sigma_{\ell i}^2 \rangle = \langle (\delta C_{\ell i}^0(v_1, v_2))^2 \rangle \equiv \frac{1}{2} \sum_{U_{\ell i}} \left[ \frac{w_{g_{\ell i}}^2}{w_{\ell i}^2} \left[ \left( C_{\ell i}(v_1, v_1) + C_{\ell i}^N(v_1, v_1) \right) \left( C_{\ell i}(v_2, v_2) + C_{\ell i}^N(v_2, v_2) \right) + \left( C_{\ell i}(v_1, v_2) + C_{\ell i}^N(v_1, v_2) \right)^2 \right] \right]$$

The two terms in the right-hand side of eq. 14 are due to the cosmic variance and the system noise respectively. We require the EoR 21-cm MAPS $C_{\ell i}(v_1, v_2)$, the noise MAPS $C_{\ell i}^N(v_1, v_2)$ and appropriate weights $w_{\ell i}$ to estimate the errors (eqs. 13 and 14).

The variance by extremizing the SNR with respect to $w_{\ell i}$ with an assumption that the EoR 21-cm MAPS $C_{\ell i}(v_1, v_2)$ does not vary much within a $\ell$-bin and therefore $C_{\ell i}(v_1, v_2) = \tilde{C}_{\ell i}(v_1, v_2)$. Note that we consider the variation of the noise $C_{\ell i}^N(v_1, v_2)$ across the grid points within a bin. Considering two different frequency channels at $v_1$ and $v_2$ for a particular $\ell$-bin, we can then express the unnormalized weights in eq. 13 as

$$\hat{w}_{\ell i} = \left\{ (C_{\ell i}(v_1, v_1) + C_{\ell i}^N(v_1, v_1))(C_{\ell i}(v_2, v_2) + C_{\ell i}^N(v_2, v_2)) \right\}^{-1}$$

This implies that the grid points with higher noise have lower weights and contribute less to the estimator. The grid points which are unsampled during the observation (i.e. $\tau(U_{\ell i}) = 0$ and $C_{\ell i}^N(v,v) = 0$) have zero weights hence they do not contribute.

Using eqs. 13 and 15 we have the expression for the error variance

$$\langle \sigma_{\ell i}^2 \rangle = \frac{1}{2} \sum_{U_{\ell i}} \left\{ \frac{\tau(U_{\ell i})}{w_{\ell i}} \right\}^2$$

Now we discuss the behavior of the error variance $\langle \sigma_{\ell i}^2 \rangle$ (eq. 16) in two different scenarios. The MAPS error variance consists of the cosmic variance and the system noise $C_{\ell i}^0(v_1, v_2)$. We see from eq. 8 that the noise contribution drops off as $C_{\ell i}^N(v_1, v_2) \propto 1/t_{\text{obs}}$ with an increase in observation time. For small observation times, the estimated error variance is thus dominated by the large system noise and from eq. 16 we have

$$\langle \sigma_{\ell i}^2 \rangle \approx \frac{C_{\ell i}^0(v_1, v_2) + \delta_{\ell i}^{\text{K}(v_1, v_2)}}{2 \times \sum_{U_{\ell i}} \left\{ \tau(U_{\ell i}) \right\}^2} \times \left\{ \frac{8 \text{ hrs}}{t_{\text{obs}}} \right\}^2$$

In contrast, we have the other extreme $C_{\ell i}^0(v_1, v_2) \approx 0$ for very large observation times ($t_{\text{obs}} \to \infty$). In this case, the error variance approaches the cosmic variance (CV) limit and we have

$$\langle \sigma_{\ell i}^2 \rangle \approx \frac{\tilde{C}_{\ell i}(v_1, v_1)}{2N_{\ell i}}$$

where $N_{\ell i}$ is the number of sampled grid points in the $i$-th bin.

5 RESULTS

Figures 8 and 9 show the SNR for the diagonal elements of MAPS at the four representative $\ell$ values considered here for LC1 and LC2 respectively. For the moderate observation time $t_{\text{obs}} = 1024$ hrs, we see a correspondence of behavior between the SNR for MAPS and the signal (Figures 4 and 5). They both peak along the diagonal and fall rapidly away from the diagonal. The previous error estimates (Morales 2005; McQuinn et al. 2006; Zaroubi et al. 2012; Datta et al. 2014; Pober et al. 2014; Ewall-Wice et al. 2016; Shaw et al. 2019) are restricted to individually analyzing small frequency intervals centered at a particular frequency. However, we see that the error estimates, as well as the SNR values for MAPS, change with the frequency across the bandwidth. We shall discuss this in more detail in the following paragraph.

In the subsequent results, we focus on the diagonal elements $v_1 = v_2$ of MAPS. Figures 10 and 11 show $\Phi^s(v, v)$ and the corresponding $5\sigma$ r.m.s. error estimates for LC1 and LC2 respectively. In the following analysis, we have considered four different observation times $t_{\text{obs}} = 128$ hrs, 1024 hrs, 10000 hrs, and 50000 hrs. We also show $\Phi$ which corresponds to $t_{\text{obs}} \to \infty$ where the system noise approaches zero. As discussed above, the cosmic variance and the system noise contribute to the total error budget (eq. 14). Considering the behavior of r.m.s. error at large angular scales, we see that the r.m.s. error is not much affected even if $t_{\text{obs}}$ is increased. Whereas the r.m.s. error decreases as $t_{\text{obs}}$ is increased at small angular scales. This confirms the fact that the cosmic variance dominates the total error at small $\ell$ and the system noise contribution dominates at large $\ell$. We also see that the r.m.s. error increases with decreasing frequency across the bandwidth of our simulations. This is due to the fact that the system noise contribution increases (eq. 8) with decreasing frequency. Considering Figure 10, we see that for any feasible $t_{\text{obs}}$, a $5\sigma$ detection the MAPS will not be possible at $\ell \leq 496$. The condition improves at $\ell \geq 796$ where SFA will be able to measure the MAPS at $> 5\sigma$ confidence over ~ 25 MHz frequency band for $t_{\text{obs}} \geq 128$ hrs, $\ell = 1257$ is a better scenario among the four $\ell$ values where $5\sigma$ detection will be possible roughly across the entire observational bandwidth for $t_{\text{obs}} \geq 128$ hrs. Whereas, the frequency band allowed for $> 5\sigma$ detection reduces at $\ell = 2071$ due to system noise domination. We find the behavior in Figure 11 is similar to that in LC1. The optimal angular multipole for detection, among the four $\ell$ values, is $\ell = 1304$ in Figure 11. The difference here is that the MAPS signal peaks at one end of the band as compared to LC1 where the signal peaks around the center of the frequency band.
plotted here. This is because the power spectrum in this simulation is maximum at the highest frequency unlike the LC1 simulation where the power spectrum peaks at some intermediate frequency.

6 DISCUSSION AND CONCLUSIONS

Several observational efforts are underway to detect the EoR 21-cm power spectrum $P(k)$ using the presently operating radio-interferometers across the globe. One of the key science goals of the future telescope SKA-Low is to measure the spherically averaged 3D EoR power spectrum $P(k)$. The definition of the spherically averaged 3D power spectrum makes use of the assumption that the signal is ergodic and periodic in all three spatial directions. However, the LC effect breaks both ergodicity and periodicity along the LoS of the observer. The problem is particularly severe during EoR where the mean H I fraction $\bar{\delta}_{\text{HI}}$ changes rapidly with redshift and this affects large bandwidth observations with different telescopes (Mondal et al. 2019). The spherically averaged 3D power spectrum $P(k)$ can no longer, therefore, be regarded as the correct estimator to quantify the second order statistics of the EoR 21-cm signal (Mondal et al. 2018), and any estimation using this may lead to a biased estimate for the statistics of the signal (Trott 2016). As an alternative to $P(k)$, we have used the MAPS $C_{\ell}(v_1, v_2)$ to quantify...
the two-point statistics of the EoR 21-cm signal. This does not assume the signal to be ergodic and periodic along the LoS. The only assumption here is that the signal is statistically homogeneous and isotropic in different directions on the observing plane of the sky. In this work, we have made predictions of the SNR for measuring the EoR 21-cm MAPS using future telescopes SKA-Low.

The sensitivity of any instrument to the measurement of the EoR 21-cm MAPS is limited by the errors, a part of which is inherent to the signal itself (cosmic variance), and the other part arises due to the system noise (external contamination). The EoR 21-cm signal is expected to be a highly non-Gaussian field (Bharadwaj & Pandey 2005; Mondal et al. 2015; Majumdar et al. 2018). The effects of this inherent non-Gaussianity play a significant role in the error estimates of the two-point correlation functions of the signal (Mondal et al. 2016, 2017; Shaw et al. 2019). However, in this work, we have assumed the signal is a Gaussian random field. We have used a 3D radiative transfer code C$^2$-RAY to generate the EoR 21-cm LC signal and incorporated observational effects like the array baseline distribution to predict the prospects of observing the bin-averaged MAPS using SKA-Low. We have considered two observations LC1, centered at $v_c = 175.58$ MHz which corresponds to $\bar{x}_{\text{H}_1} \approx 0.50$, and LC2, centered at $v_c = 157.08$ MHz corresponds to $\bar{x}_{\text{H}_1} \approx 0.75$. We
have also presented a detailed theoretical framework to quantify and interpret the error estimates for the MAPS incorporating the system noise. For moderate observation times, we have seen that the r.m.s. error $\sigma$ scales as $1/t_{\text{obs}}$, and consequently we have $\text{SNR} \propto t_{\text{obs}}$. In this case, we have found similar behavior between the signal and the SNR for MAPS (Figures 4, 5, 8 and 9). They both peak along the diagonal $v_1 = v_2$ and fall rapidly away from the diagonal. For further analysis, we have focused on the diagonal elements of the MAPS. We have found that the error predictions for MAPS are not much affected by the choice of $t_{\text{obs}}$ at large angular scales. This is due to the fact that cosmic variance dominates the total error budget at small $\ell$. We have also found that the r.m.s. error decreases as $t_{\text{obs}}$ is increased at small angular scales. This is because the system noise dominates the total error at large $\ell$. We have found that a $5\sigma$ detection of MAPS will not be possible with SKA-Low at $\ell < 496$ for LC1, and at $\ell < 486$ for LC2. Although, we have found that the SKA will be able to measure the MAPS at $\gtrsim 5\sigma$ confidence roughly across the $\sim 44$ MHz observational bandwidth at $\ell \sim 1300$ with $t_{\text{obs}} \gtrsim 128$ hrs. Whereas, the frequency band allowed for $\gtrsim 5\sigma$ detection reduces at higher values of $\ell$ due to system noise domination. We have noted that the r.m.s. error increases with decreasing frequency across the bandwidth of our simulations. This is due to the fact that the system noise contribution increases (eq. 8) with decreasing frequency. We have extended our analysis to study how the SNR for the diagonal elements of 21-cm MAPS changes with $\ell$ (Figures 12 and 13). Note that the system noise contribution, at a fixed $\ell$, decreases with as $t_{\text{obs}}$ increases but the cosmic variance remains unchanged. However, the cosmic variance dominates the total error at larger values of $\ell$ for a fixed $t_{\text{obs}}$. This interplay between the system noise and cosmic variance (as a function of $t_{\text{obs}}$) causes the peak of the SNR for MAPS to shifts toward larger values of $\ell$ as $t_{\text{obs}}$ is increased (Figures 12 and 13).

In conclusion, our study indicates that the EoR 21-cm MAPS, which is directly related to the correlations between the visibilities measured in radio-interferometric observation, can be measured at a confidence level of $5\sigma$ or more at angular multipole $\ell \sim 1300$ for $t_{\text{obs}} \gtrsim 128$ hrs across $\sim 44$ MHz observational bandwidth using SKA-Low. The framework presented in this paper is general and can be applied to any radio-interferometer given the array baseline distribution. In this analysis, we have ignored foregrounds which will possibly restrict the $\ell$ ranges that can be used for measuring the EoR 21-cm MAPS. We plan to address this issue in our future work.

ACKNOWLEDGEMENTS
This work was supported by the Science and Technology Facilities Council [grant numbers ST/F002858/1 and ST/I000976/1] and the Southeast Physics Network (SEPNet). We acknowledge that the results in this paper have been achieved using the PRACE Research Infrastructure resources Curie based at the Très Grand Centre de Calcul (TGCC) operated by CEA near Paris, France and Marenostrum based in the Barcelona Supercomputing Center, Spain. Time on these resources was awarded by PRACE under PRACE4IOLOFAR grants 2012061089 and 2014102339 as well as under the Multi-scale Reionization grants 2014102281 and 2015122822. The authors gratefully acknowledge the Gauss Centre for Supercomputing e.V. (www.gauss-centre.eu) for funding this project by providing computing time through the John von Neumann Institute for Computing (NIC) on the GCS Supercomputer JUWELS at Jülich Supercomputing Centre (JSC). Some of the numerical computations were done on the Apollo cluster at The University of Sussex.

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MNRAS 000, 1–11 (2019)
APPENDIX A: MAPS ERROR COVARIANCE

The error covariance of the MAPS measured at the i-th and the j-th bins can be written as

\[ \Sigma_{ij} = \langle \hat{C}_i^2 (v_1, v_2) - \hat{C}_j^2 (v_1, v_2) \rangle \langle \hat{C}_j^2 (v_3, v_4) - \hat{C}_i^2 (v_3, v_4) \rangle \]

\[ = \langle \hat{C}_i^2 (v_1, v_2) \hat{C}_j^2 (v_3, v_4) \rangle - \langle \hat{C}_i^2 (v_1, v_2) \hat{C}_i^2 (v_3, v_4) \rangle - \langle \hat{C}_j^2 (v_1, v_2) \hat{C}_j^2 (v_3, v_4) \rangle + \langle \hat{C}_i^2 (v_1, v_2) \hat{C}_j^2 (v_3, v_4) \rangle. \]

(A1)

Using \( \hat{T}_{bb} (U, v) = \hat{T}_{bb} (U, v) + \hat{T}_{bb}^{N} (U, v) \) and eq. 7, the first ensemble average in eq. (A2) can be arranged as

\[ \langle \hat{T}_{bb}^{N} (U, v) \hat{T}_{bb}^{N} (U', v') \hat{T}_{bb}^{N} (U, v') \hat{T}_{bb}^{N} (U', v) \rangle = \Omega^2 \langle C(v_1, v_2) C(v_3, v_4) + \delta^{K}_{11} C(v_1, v_2) C(v_3, v_4) \rangle + \delta^{K}_{11} \langle C(v_1, v_2) C(v_3, v_4) + \delta^{K}_{11} C(v_1, v_2) C(v_3, v_4) \rangle \]

\[ + \delta^{K}_{11} \langle C(v_1, v_2) C(v_3, v_4) + \delta^{K}_{11} C(v_1, v_2) C(v_3, v_4) \rangle + \delta^{K}_{11} \langle C(v_1, v_2) C(v_3, v_4) + \delta^{K}_{11} C(v_1, v_2) C(v_3, v_4) \rangle \]

\[ + \delta^{K}_{11} \langle C(v_1, v_2) C(v_3, v_4) + \delta^{K}_{11} C(v_1, v_2) C(v_3, v_4) \rangle + \delta^{K}_{11} \langle C(v_1, v_2) C(v_3, v_4) + \delta^{K}_{11} C(v_1, v_2) C(v_3, v_4) \rangle \]

\[ + \delta^{K}_{11} \langle C(v_1, v_2) C(v_3, v_4) + \delta^{K}_{11} C(v_1, v_2) C(v_3, v_4) \rangle. \]

(A3)

where \( \delta^{K}_{11} \) are variables that are zero if \( U = U', \) and one if \( U = U', \) 0 otherwise. Here, we have considered that \( \hat{T}_{bb} (U, v) \) correlates at same baselines \( U \) and \( \hat{T}_{bb}^{N} (U, v) \) correlates at same \( U \) and frequency \( v. \) The MAPS estimations are restricted to the upper half of the baseline distribution. Hence, \( \delta^{K}_{11} \) reduces to

\[ \langle \hat{T}_{bb}^{N} (U, v) \hat{T}_{bb}^{N} (U', v') \hat{T}_{bb}^{N} (U, v') \hat{T}_{bb}^{N} (U', v) \rangle = \Omega^2 \langle C(v_1, v_2) C(v_3, v_4) \rangle + \delta^{K}_{11} \langle C(v_1, v_2) C(v_3, v_4) \rangle. \]

(A4)

Similarly, one can write down the other three ensemble averages of eq. (A2) by permuting the frequency indices in eq. (A4). Combining eq. (A1) and (A4) with the other ensemble averages, we write the error covariance in compact form

\[ X_{12,34}^{C} = \frac{1}{2} \sum_{U} \sum_{U'} w(U) w(U') \{ C_{i}^{k} (v_1, v_3) C_{j}^{k} (v_2, v_4) + C_{i}^{k} (v_2, v_3) C_{j}^{k} (v_1, v_4) \}. \]

(A5)

Finally, exploiting the Kronecker's delta, we obtain

\[ X_{12,34}^{C} = \frac{1}{2} \sum_{U} \sum_{U'} w^2 (C_{i}^{k} (v_1, v_3) C_{j}^{k} (v_2, v_4) + C_{j}^{k} (v_1, v_4) C_{i}^{k} (v_2, v_3)). \]

(A6)