\(\omega^\omega\)-BASE AND INFINITE-DIMENSIONAL COMPACT SETS IN LOCALLY CONVEX SPACES

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Abstract. A locally convex space (lcs) \(E\) is said to have an \(\omega^\omega\)-base if \(E\) has a neighborhood base \(\{U_\alpha : \alpha \in \omega^\omega\}\) at zero such that \(U_\beta \subseteq U_\alpha\) for all \(\alpha \leq \beta\). The class of lcs with an \(\omega^\omega\)-base is large, among others contains all \((LM)\)-spaces (hence \((LF)\)-spaces), strong duals of distinguished Fréchet lcs (hence spaces of distributions \(D'(\Omega)\)). A remarkable result of Cascales-Orihuela states that every compact set in a lcs with an \(\omega^\omega\)-base is metrizable. Our main result shows that every uncountable-dimensional lcs with an \(\omega^\omega\)-base contains an infinite-dimensional metrizable compact subset. On the other hand, the countable-dimensional space \(\varphi\) endowed with the finest locally convex topology has an \(\omega^\omega\)-base but contains no infinite-dimensional compact subsets. It turns out that \(\varphi\) is a unique infinite-dimensional locally convex space which is a \(k_\mathbb{R}\)-space containing no infinite-dimensional compact subsets. Applications to spaces \(C_p(X)\) are provided.

1. Introduction

A topological space \(X\) is said to have a neighborhood \(\omega^\omega\)-base at a point \(x \in X\) if there exists a neighborhood base \((U_\alpha(x))_{\alpha \in \omega^\omega}\) at \(x\) such that \(U_\beta(x) \subseteq U_\alpha(x)\) for all \(\alpha \leq \beta\) in \(\omega^\omega\). We say that \(X\) has an \(\omega^\omega\)-base if it has a neighborhood \(\omega^\omega\)-base at each point of \(X\). Evidently, a topological group (particularly topological vector space (tvs)) has an \(\omega^\omega\)-base if it has a neighborhood \(\omega^\omega\)-base at the identity. The classical metrization theorem of Birkhoff and Kakutani states that a topological group \(G\) is metrizable if and only if \(G\) is first-countable. Then, as easily seen, if \((U_n)_{n \in \omega}\) is a neighborhood base at the identity of \(G\), then the family \(\{U_\alpha : \alpha \in \omega^\omega\}\) formed by sets \(U_\alpha = U_\alpha(0)\) forms an \(\omega^\omega\)-base (at the identity) for \(G\). Locally convex spaces (lcs) with an \(\omega^\omega\)-base are known in Functional Analysis since 2003 when Cascales, Kąkol, and Saxon [7] characterized quasi-barreled lcs with an \(\omega^\omega\)-base. In several papers (see [16] and the references therein) spaces with an \(\omega^\omega\)-base were studied under the name lcs with a \(\mathcal{G}\)-base, but here we prefer (as in [1]) to use the more self-suggesting terminology of \(\omega^\omega\)-bases.

In [8] Cascales and Orihuela proved that compact subsets of any lcs with an \(\omega^\omega\)-base are metrizable. This refers, among others, to each \((LM)\)-space, i.e. a countable inductive limit of metrizble lcs, since \((LM)\)-spaces have an \(\omega^\omega\)-base. Also the following metrization theorem holds together a number of topological conditions.

Theorem 1.1. [16] Corollary 15.5] For a barrelled lcs \(E\) with an \(\omega^\omega\)-base, the following conditions are equivalent.

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(1) $E$ is metrizable;
(2) $E$ is Fréchet-Urysohn;
(3) $E$ is Baire-like;
(4) $E$ does not contain a copy of $\varphi$, i.e. an $\aleph_0$-dimensional vector space endowed with the finest locally convex topology.

Hence every Baire lcs with an $\omega_\omega$-base is metrizable. The space $\varphi$ appearing in Theorem 1.1 has the following properties:

(1) $\varphi$ is the strong dual of the Fréchet-Schwartz space $\mathbb{R}^\omega$.
(2) All compact subsets in $\varphi$ are finite-dimensional.
(3) $\varphi$ is a complete bornological space, see [23], [21], [16].

Being motivated by above’s results, especially by a remarkable theorem of Cascales-Oruñuela mentioned above, one can ask for a possible large class of lcs $E$ for which every infinite-dimensional subspace of $E$ contains an infinite-dimensional compact (metrizable) subset. Surely, each metrizable lcs trivially fulfills this request. We prove however the following general

**Theorem 1.2.** Every uncountably-dimensional lcs $E$ with $\omega_\omega$-base contains an infinite-dimensional metrizable compact subset.

Theorem 1.2 will be proved in Section 4. An alternative proof will be presented in Section 5 as a consequence of Theorem 5.2.

The uncountable dimensionality of the space $E$ in Theorem 1.2 cannot be replaced by the infinite-dimensionality of $E$: the space $\varphi$ is infinite-dimensional, has an $\omega_\omega$-base and contains no infinite-dimensional compact subsets. However, $\varphi$ is a unique locally convex $k_\mathbb{R}$-space with this property. Recall [20] that a topological space $X$ is a $k_\mathbb{R}$-space if a function $f : X \to \mathbb{R}$ is continuous whenever for every compact subset $K \subseteq X$ the restriction $f \upharpoonright K$ is continuous. We prove the following

**Theorem 1.3.** A lcs $E$ is topologically isomorphic to the space $\varphi$ if and only if $E$ is a $k_\mathbb{R}$-space containing no infinite-dimensional compact subsets.

Theorem 1.3 implies that a lcs is topologically isomorphic to $\varphi$ if and only if it is homeomorphic to $\varphi$. This topological uniqueness property of the space $\varphi$ was first proved by the first author in [2].

The following characterization of the space $\varphi$ can be derived from Theorems 1.2 and 2.1. It shows that $\varphi$ is a unique bornological space for which the uncountable dimensionality in Theorem 1.2 cannot be weakened to infinite dimensionality.

**Theorem 1.4.** A lcs $E$ is topologically isomorphic to the space $\varphi$ if and only if $E$ is bornological, has an $\omega_\omega$-base and contains no infinite-dimensional compact subset.

Theorem 1.2 provides a large class of concrete (non-metrizable) lcs containing infinite-dimensional compact sets.

**Corollary 1.5.** Every uncountable-dimensional subspace of an (LM)-space contains an infinite-dimensional compact set.
Let \( X \) be a Tychonoff space. By \( C_p(X) \) and \( C_k(X) \) we denote the space of continuous real-valued functions on \( X \) endowed with the pointwise and the compact-open topology, respectively. The problem of characterization of Tychonoff spaces \( X \) whose function spaces \( C_p(X) \) and \( C_k(X) \) admit an \( \omega^\omega \)-base is already solved. Indeed, by [16, Corollary 15.2] \( C_p(X) \) has an \( \omega^\omega \)-base if and only if \( X \) is countable. The space \( C_k(X) \) has an \( \omega^\omega \)-base if and only if \( X \) admits a fundamental compact resolution [11], for necessary definitions see below. Since every Čech-complete Lindelöf space \( X \) is a continuous image of a Polish space under a perfect map (and the latter space admits a fundamental compact resolution), the space \( C_p(X) \) has an \( \omega^\omega \)-base. So, we have another concrete application of Theorem 1.2.

Example 1.6. Let \( X \) be an infinite Čech-complete Lindelöf space. Then every uncountable-dimensional subspace of \( C_k(X) \) contains an infinite-dimensional metrizable compact set.

In Section 2 we show that all (bornological) lcs containing no infinite-dimensional compact subsets are bornologically (and topologically) isomorphic to a free lcs over discrete topological spaces. Consequently, in Sections 3 and 4 we study the free lcs \( L(\kappa) \) over infinite cardinals \( \kappa \), including \( L(\omega) = \varphi \). We introduce two concepts: the \((\kappa, \lambda)\)-tall bornology and the \((\kappa, \lambda)_p\)-equiconvergence, which will be used to obtain bornological and topological characterizations of \( L(\kappa) \). Both concepts apply to prove Theorem 1.2. To this end, we shall prove that each topological (vector) space with an \( \omega^\omega \)-base is \((\omega_1, \omega)\)-equiconvergent (and has \((\omega_1, \omega)\)-tall bornology). Another property implying the \((\omega_1, \omega)_p\)-equiconvergence is the existence of a countable cs\(^\bullet\)-network (see Theorem 1.2), which follows from the existence of an \( \omega^\omega \)-base according to Proposition 3.3. Linear counterparts of cs\(^\bullet\)-networks are radial networks introduced in Section 5 whose main result is Theorem 5.2 implying Theorem 1.2. Some applications of Theorem 1.2 to function spaces \( C_p(X) \) are provided in Section 6.

2. LOCALLY CONVEX SPACES CONTAINING NO INFINITE-DIMENSIONAL COMPACT SUBSETS

In this section we study lcs containing no infinite-dimensional compact subsets. We shall show that all such (bornological) spaces are bornologically (and topologically) isomorphic to free lcs over discrete topological spaces.

Recall that for a topological space \( X \) its free locally convex space is a lcs \( L(X) \) endowed with a continuous function \( \delta : X \to L(X) \) such that for any continuous function \( f : X \to E \) to a lcs \( E \) there exists a unique linear continuous map \( T : L(X) \to E \) such that \( T \circ \delta = f \). The set \( X \) forms a Hamel basis for \( L(X) \) and \( \delta \) is a topological embedding, see [22]; we also refer to [5] and [4] for several results and references concerning this concept; [5, Theorem 5.4] characterizes those \( X \) for which \( L(X) \) has an \( \omega^\omega \)-base.

Let \( E \) be a tvs. A subset \( B \subseteq E \) is called bounded if for every neighborhood \( U \subseteq E \) of zero there exists \( n \in \mathbb{N} \) such that \( B \subseteq nU \). The family of all bounded sets of \( E \) is called the bornology of \( E \). A linear operator \( f : E \to F \) between two tvs is called bounded if for any bounded set \( B \subseteq E \) its image \( f(B) \) is bounded in \( F \).

Two tvs \( E \) and \( F \) are

- **topologically isomorphic** if there exists a linear bijective function \( f : E \to F \) such that \( f \) and \( f^{-1} \) are continuous;
• bornologically isomorphic if there exists a linear bijective function $f : E \rightarrow F$ such that $f$ and $f^{-1}$ are bounded.

A lcs $E$ is called bornological if each bounded linear operator from $E$ to a lcs $F$ is continuous. A linear space $E$ is called $\kappa$-dimensional if $E$ has a Hamel basis of cardinality $\kappa$. In this case we write $\kappa = \dim(E)$.

A lcs $E$ is free if it carries the finest locally convex topology. In this case $E$ is topologically isomorphic to the free lcs $L(\kappa)$ over the cardinal $\kappa = \dim(E)$ endowed with the discrete topology.

The study around the free lcs $L(\omega) = \varphi$ has attracted specialists for a long time. For example, Nyikos observed [21] that each sequentially closed subset of $L(\omega)$ is closed although the sequential closure of a subset of $\varphi$ need not be closed. Consequently, $L(\omega)$ is a concrete “small” space without the Fréchet-Urysohn property. Applying the Baire category theorem one shows that $L(\omega)$ is not a Baire-like space (in sense of Saxon [23]) and a barrelled lcs $E$ is Baire-like if $E$ does not contain a copy of $L(\omega)$, see [23]. Although $L(\omega)$ is not Fréchet-Urysohn, it provides some extra properties since all vector subspaces in $L(\omega)$ are closed. In [17] we introduced the property for a lcs $E$ (under the name $C_{3^-}$) stating that the sequential closure of every linear subspace of $E$ is sequentially closed, and we proved [17, Corollary 6.4] that the only infinite-dimensional Montel (DF)-space with property $C_{3^-}$ is $L(\omega)$ (yielding a remarkable result of Bonet and Defant that the only infinite-dimensional Silva space with property $C_{3^-}$ is $L(\omega)$). This implies that barrelled (DF)-spaces and (LF)-spaces satisfying property $C_{3^-}$ are exactly of the form $M$, $L(\omega)$, or $M \times L(\omega)$ where $M$ is metrizable, [17, Theorems 6.11, 6.13].

The following simple theorem characterizes lcs containing no infinite-dimensional compact subsets.

**Theorem 2.1.** For a lcs $E$ the following conditions are equivalent:

1. Each compact subset of $E$ has finite topological dimension.
2. Each bounded linearly independent set in $E$ is finite.
3. $E$ is bornologically isomorphic to a free lcs.
4. $E$ is free.

If $E$ is bornological, then the conditions (1)–(3) are equivalent to

(4) $E$ is free.

**Proof.** (1) $\Rightarrow$ (2) Suppose that each compact subset of $E$ has finite topological dimension. Assuming that $E$ contains an infinite bounded linearly independent set, we can find a bounded linearly independent set $\{x_n\}_{n \in \omega}$ consisting of pairwise distinct points $x_n$. Then the sequence $(2^{-n}x_n)_{n \in \omega}$ converges to zero and

$$K = \bigcup_{n \in \omega} \left\{ \sum_{k=n}^{2n} t_k x_k : (t_k)_{k=n}^{2n} \in \prod_{k=n}^{2n} [0, 2^{-k}] \right\}$$

is an infinite-dimensional compact set in $E$, which contradicts our assumption.

(2) $\Rightarrow$ (3) Let $\tau$ be the finest locally convex topology on $E$. Then the identity map $(E, \tau) \rightarrow E$ is continuous and hence bounded. If each bounded linearly independent set in $E$ is finite, then each bounded set $B \subseteq E$ is contained in a finite-dimensional subspace of
$E$ and hence is bounded in the topology $\tau$. This means that the identity map $E \to (E, \tau)$ is bounded and hence $E$ is bornologically isomorphic to the free lcs $(E, \tau)$.

(3) $\Rightarrow$ (1) If $E$ is bornologically isomorphic to a free lcs $F$ then each bounded linearly independent set in $E$ is finite, since the free lcs $F$ has this property.

The implication (4) $\Rightarrow$ (3) is trivial. If $E$ is bornological then the implication (3) $\Rightarrow$ (4) follows from the continuity of bounded linear operators on bornological spaces.

The free lcs over discrete topological spaces are not unique lcs possessing no infinite-dimensional compact sets. A subset $B$ of a topological space $X$ is called functionally bounded if for any continuous real-valued function $f : X \to \mathbb{R}$ the set $f(B)$ is bounded.

**Proposition 2.2.** For a Tychonoff space $X$ the following conditions are equivalent:

1. each compact subset of the free lcs $L(X)$ has finite topological dimension;
2. each bounded linearly independent set in $L(X)$ is finite;
3. each functionally bounded subset of $X$ is finite.

**Proof.** The equivalence (1) $\Leftrightarrow$ (2) follows from the corresponding equivalence in Theorem 2.1. The implication (3) $\Rightarrow$ (1) follows from [6, Lemma 10.11.3], and (2) $\Rightarrow$ (3) follows from the observation that each functionally bounded set in a lcs is bounded.

### 3. Bornological and topological characterizations of the spaces $L(\kappa)$

In this section, given an infinite cardinal $\kappa$ we characterize the free lcs $L(\kappa)$ using some specific properties of the bornology and the topology of the space $L(\kappa)$.

Let $\kappa, \lambda$ be two cardinals. A lcs $E$ is defined to have $(\kappa, \lambda)$-tall bornology if every subset $A \subseteq E$ of cardinality $|A| = \kappa$ contains a bounded subset $B \subseteq A$ of cardinality $|B| = \lambda$.

**Theorem 3.1.** Let $\kappa$ be an infinite cardinal. For a lcs $E$ the following conditions are equivalent:

1. $E$ is bornologically isomorphic to the free lcs $L(\kappa)$;
2. each bounded linearly independent set in $E$ is finite and the bornology of $E$ is $(\kappa^+, \omega)$-tall but not $(\kappa, \omega)$-tall.

If $E$ is bornological, then the conditions (1)–(2) are equivalent to

3. $E$ is topologically isomorphic to $L(\kappa)$.

**Proof.** (1) $\Rightarrow$ (2): Assume that $E$ is bornologically isomorphic to $L(\kappa)$. Then $E$ has algebraic dimension $\kappa$ and each bounded linearly independent set in $E$ is finite (since this is true in $L(\kappa)$).

To see that the bornology of $E$ is $(\kappa^+, \omega)$-tall, take any set $K \subseteq E$ of cardinality $|K| = \kappa^+$. Since $E$ has algebraic dimension $\kappa$, there exists a cover $(B_\alpha)_{\alpha \in \kappa}$ of $E$ by $\kappa$ many compact sets. By the Pigeonhole Principle, there exists $\alpha \in \kappa$ such that $|K \cap B_\alpha| = \kappa^+$. This means that the bornology of $E$ is $(\kappa^+, \kappa^+)$-tall and hence $(\kappa^+, \omega)$-tall.

To see that the bornology of the space $E$ is not $(\kappa, \omega)$-tall, observe that the Hamel basis $\kappa$ of $L(\kappa)$ has the property that no infinite subset of $\kappa$ is bounded in $L(\kappa)$. Since $E$ is bornologically isomorphic to $L(\kappa)$, the image of $\kappa$ in $E$ is a subset of cardinality $\kappa$ containing no bounded infinite subsets and witnessing that $E$ is not $(\kappa, \omega)$-tall.
(2) $\Rightarrow$ (1): Assume that each bounded linearly independent set in $E$ is finite and the bornology of $E$ is $(\kappa^+, \omega)$-tall but not $(\kappa, \omega)$-tall. Let $B$ be a Hamel basis of $E$. We claim that $|B| = \kappa$. Assuming that $|B| > \kappa$, we conclude that $E$ is not $(\kappa^+, \omega)$-tall, which is a contradiction. Assuming that $|B| < \kappa$, we conclude that $E$ is the union of $< \kappa$ many bounded sets and hence is $(\kappa, \kappa)$-tall by the Pigeonhole Principle. But this contradicts our assumption. Therefore $|B| = \kappa$. Let $h : \kappa \to B$ be any bijection and $\bar{h} : L(\kappa) \to E$ be the unique extension of $h$ to a linear continuous operator. Since $B$ is a Hamel basis for $E$, the operator $\bar{h}$ is bijective. Since each bounded set in $E$ is contained in a finite-dimensional linear subspace, the operator $\bar{h}^{-1} : E \to L(\kappa)$ is bounded and hence $\bar{h} : L(\kappa) \to E$ is a bornological isomorphism.

If the space $E$ is bornological, then the equivalence $(1) \Leftrightarrow (3)$ follows from the bornological property of $E$ and $L(\kappa)$.

The $(\kappa, \omega)$-tallness of the bornology of a lcs $E$ has topological counterparts introduced in the following definition.

**Definition 3.2.** Let $\kappa, \lambda$ be cardinals. We say that a topological space $X$ is

- $(\kappa, \lambda)_p$-equiconvergent at a point $x \in X$ if for any indexed family $\{x_\alpha\}_{\alpha \in \kappa} \subseteq \{s \in X^\omega : \lim_{n \to \infty} s(n) = x\}$, there exists a subset $\Lambda \subseteq \kappa$ of cardinality $|\Lambda| = \lambda$ such that for every neighborhood $O_x \subseteq X$ of $x$ there exists $n \in \omega$ such that the set $\{\alpha \in \Lambda : x_\alpha(n) \notin O_x\}$ is finite;

- $(\kappa, \lambda)_k$-equiconvergent at a point $x \in X$ if for any indexed family $\{x_\alpha\}_{\alpha \in \kappa} \subseteq \{s \in X^\omega : \lim_{n \to \infty} s(n) = x\}$, there exists a subset $\Lambda \subseteq \kappa$ of cardinality $|\Lambda| = \lambda$ such that for every neighborhood $O_x \subseteq X$ of $x$ there exists $n \in \omega$ such that for every $m \geq n$ and $\alpha \in \Lambda$ we have $x_\alpha(m) \in O_x$;

- $(\kappa, \lambda)_p$-equiconvergent if $X$ is $(\kappa, \lambda)_p$-equiconvergent at every point $x \in X$;

- $(\kappa, \lambda)_k$-equiconvergent if $X$ is $(\kappa, \lambda)_k$-equiconvergent at every point $x \in X$.

It is easy to see that every $(\kappa, \lambda)_k$-equiconvergent space is $(\kappa, \lambda)_p$-equiconvergent. The following observation will be used below.

**Proposition 3.3.** If a lcs $E$ is $(\kappa, \lambda)_p$-equiconvergent, then its bornology is $(\kappa, \lambda)$-tall.

*Proof.* Given a subset $K \subseteq E$ of cardinality $|K| = \kappa$, for every $\alpha \in K$ consider the convergent sequence $x_\alpha \in X^\omega$ defined by $x_\alpha(n) = 2^{-n}\alpha$. Assuming that the lcs $E$ is $(\kappa, \lambda)_p$-equiconvergent, we can find a subset $L \subseteq K$ of cardinality $|L| = \lambda$ such that for every neighborhood of zero $U \subseteq E$ there exists $n \in \omega$ such that the set $\{\alpha \in L : 2^{-n}\alpha \notin U\}$ is finite. We claim that the set $L$ is bounded. Indeed, for every neighborhood $U \subseteq E$ of zero, we find a neighborhood $V \subseteq E$ of zero such that $[0, 1] \cdot V \subseteq U$. By our assumption, there exists $n \in \omega$ such that the set $F = \{\alpha \in K : 2^{-n}\alpha \notin V\}$ is finite. Find $m \geq n$ such that $2^{-m}\alpha \in U$ for every $\alpha \in F$. Then $2^{-m}L \subseteq 2^{-m}(L \setminus F) \cup 2^{-m}F \subseteq ([0, 1] \cdot V) \cup U = U$, and hence the set $L$ is bounded. \[\Box\]

Nevertheless, it seems that the following question remains open.

**Problem 3.4.** Assume that the bornology of a lcs $E$ is $(\omega_1, \omega)$-tall. Is it true that $E$ is $(\omega_1, \omega)_p$-equiconvergent?
Below we prove the following topological counterpart to Theorem 3.1.

**Theorem 3.5.** Let \( \kappa \) be an infinite cardinal. For a lcs \( E \) the following conditions are equivalent:

1. \( E \) is bornologically isomorphic to \( L(\kappa) \);
2. each compact subset of \( E \) has finite topological dimension, \( E \) is \((\kappa^+, \omega)_k\)-equiconvergent but not \((\kappa, \omega)_p\)-equiconvergent.
3. each compact subset of \( E \) has finite topological dimension, \( E \) is \((\kappa^+, \omega)_p\)-equiconvergent but not \((\kappa, \omega)_k\)-equiconvergent.

If \( E \) is bornological, then the conditions (1)-(3) are equivalent to

4. \( E \) is topologically isomorphic to \( L(\kappa) \).

**Proof.** (1) \( \Rightarrow \) (2): Assume that \( E \) is bornologically isomorphic to \( L(\kappa) \). By Theorems 3.1 each bounded linearly independent set in \( E \) is finite, and by Theorem 2.1 each compact subset of \( E \) is finite-dimensional. The linear space \( E \) has algebraic dimension \( \kappa \), being isomorphic to the linear space \( L(\kappa) \). Let \( B \) be a Hamel basis for the space \( E \).

To show that \( E \) is \((\kappa^+, \omega)_k\)-equiconvergent, fix an indexed family \( \{x_\alpha\}_{\alpha < \kappa^+} \subseteq \{s \in E^\omega : \lim_{n \to \infty} s(n) = 0\} \). Since bounded linearly independent sets in \( E \) are finite, for every \( \alpha \in \kappa^+ \) there exists a finite set \( F_\alpha \subseteq B \) such that the bounded set \( x_\alpha[\omega] \) is contained in the linear hull of \( F_\alpha \). Since \( |B| = \kappa < \kappa^+ \), by the Pigeonhole Principle, for some finite set \( F \subseteq B \) the set \( A = \{ \alpha \in \kappa^+ : F_\alpha = F \} \) is uncountable. Let \([F]\) be the linear hull of the finite set \( F \) in the linear space \( E \).

Consider the ordinal \( \omega + 1 = \omega \cup \{\omega\} \) endowed with the compact metrizable topology generated by the linear order. For every \( \alpha \in A \) let \( \bar{x}_\alpha : \omega + 1 \to [F] \) be the continuous function such that \( \bar{x}_\alpha[\omega] = x_\alpha \) and \( \bar{x}_\alpha(\omega) = 0 \). Let \( C_k(\omega + 1, [F]) \) be the space of continuous functions from \( \omega + 1 \) to \([F]\), endowed with the compact-open topology. Since \( A \) is uncountable and the space \( C_k(\omega + 1, [F]) \supseteq \{ \bar{x}_\alpha \}_{\alpha \in A} \) is Polish, there exists a sequence \( \{\alpha_n\}_{n \in \omega} \subseteq A \) of pairwise distinct ordinals such that the sequence \((\bar{x}_{\alpha_n})_{n \in \omega} \) converges to \( \bar{x}_\alpha \) in the function space \( C_k(\omega + 1, [F]) \). Then the set \( \Lambda = \{ \alpha_n \}_{n \in \omega} \subseteq \kappa^+ \) witnesses that \( E \) is \((\kappa^+, \omega)_k\)-equiconvergent to zero and by the topological homogeneity, \( E \) is \((\kappa^+, \omega)_k\)-equiconvergent. By Theorem 3.1 the bornology of the space \( E \) is not \((\kappa, \omega)_p\)-equiconvergent. By Proposition 3.3 the space \( E \) is not \((\kappa, \omega)_p\)-equiconvergent.

The implication (2) \( \Rightarrow \) (3) is trivial. To prove that (3) \( \Rightarrow \) (1), assume that each compact subset of \( E \) has finite topological dimension and \( E \) is \((\kappa^+, \omega)_p\)-equiconvergent but not \((\kappa, \omega)_k\)-equiconvergent. Let \( B \) be a Hamel basis in \( E \). By Theorem 2.1 the space \( E \) is bornologically isomorphic to \( L(|B|) \). Applying the (already proved) implication (1) \( \Rightarrow \) (2), we conclude that \( E \) is \(|B|^+, \omega)_k\)-equiconvergent, which implies that \(|B| \geq \kappa \) (as \( E \) is not \((\kappa, \omega)_k\)-equiconvergent). Assuming that \(|B| > \kappa \), we can see that the family \( \{x_b\}_{b \in B} \subseteq E^\omega \) of the sequences \( x_b(n) = 2^{-n}b \) witnesses that \( E \) is not \(|B|, \omega)_p\)-equiconvergent and hence not \((\kappa^+, \omega)_p\)-equiconvergent, which contradicts our assumption. So, \(|B| = \kappa \) and \( E \) is bornologically isomorphic to \( L(\kappa) \). If the space \( E \) is bornological, then the equivalence (1) \( \Leftrightarrow \) (4) follows from the bornological property of \( E \) and \( L(\kappa) \). \( \square \)
Observe that the purely topological properties (2), (3) in Theorem 3.5 characterize the free lcs $L(\kappa)$ up to bornological equivalence. We do not know whether the topological structure of the space $L(\kappa)$ determines this lcs uniquely up to a topological isomorphism.

**Problem 3.6.** Assume that a lcs $E$ is homeomorphic to the free lcs $L(\kappa)$ for some cardinal $\kappa$. Is $E$ topologically isomorphic to $L(\kappa)$?

By [2] the answer to this problem is affirmative for $\kappa = \omega$. This affirmative answer can also be derived from the following topological characterizations of the space $L(\omega) = \varphi$.

This characterization has been announced in the introduction as Theorem 1.3.

**Theorem 3.7.** A lcs $E$ is topologically isomorphic to the free lcs $L(\omega)$ if and only if $E$ is an infinite-dimensional $k_{\mathbb{R}}$-space containing no infinite-dimensional compact subset.

*Proof.* The “only if” part follows from known topological properties of the space $L(\omega) = \varphi$ mentioned in the introduction. To prove the “if” part, assume that a lcs $E$ is a $k_{\mathbb{R}}$-space and each compact subset of $E$ is finite-dimensional. Choose a Hamel basis $B$ in $E$ and consider the linear continuous operator $T : L(B) \to E$ such that $T(b) = b$ for each $b \in B$. Since $B$ is a Hamel basis, the operator $T$ is injective. We claim that the operator $T^{-1} : E \to L(B)$ is bounded. By Theorem 2.7 the linear hull of each compact subset $K \subseteq E$ is finite-dimensional, which implies that the restriction $T^{-1}|_K$ is continuous. Since $E$ is a $k_{\mathbb{R}}$-space, $T^{-1}$ is continuous and hence $T$ is a topological isomorphism. Then the free lcs $L(B)$ is a $k_{\mathbb{R}}$-space. Applying 15, we conclude that $B$ is countable and hence $E$ is topologically isomorphic to $L(\omega)$. \(\square\)

A Tychonoff space $X$ is called Ascoli if the canonical map $\delta : X \to C_k(C_k(X))$ assigning to each point $x \in X$ the Dirac functional $\delta_x : C_k(X) \to \mathbb{R}$, $\delta_x : f \mapsto f(x)$, is continuous. By 3, the class of Ascoli spaces includes all Tychonoff $k_{\mathbb{R}}$-spaces. By 15 a Tychonoff space $X$ is countable and discrete if and only if its free lcs $L(X)$ is Ascoli.

**Problem 3.8.** Assume that an infinite-dimensional lcs $E$ is Ascoli and contains no infinite-dimensional compact subsets. Is $E$ topologically isomorphic to the space $L(\omega)$?

4. **Equiconvergence of topological spaces and proof of Theorem 1.2**

In this section we establish two results related to equiconvergence in topological spaces.

**Theorem 4.1.** If a topological space $X$ admits an $\omega^\omega$-base at a point $x \in X$, then $X$ is $(\omega_1, \omega)_k$-equiconvergent at the point $x$.

*Proof.* Let $(U_f)_{f \in \omega^\omega}$ be an $\omega^\omega$-base at $x$. To show that $X$ is $(\omega_1, \omega)_k$-equiconvergent at $x$, fix an indexed family

\[
\{s_{\alpha} \}_{\alpha \in \omega_1} \subseteq \{ s \in X^\omega : \lim_{n \to \infty} s(n) = x \}
\]

of sequences that converge to $x$. For every $\alpha \in \omega_1$ consider the function $\mu_\alpha : \omega^\omega \to \omega$ assigning to each $f \in \omega^\omega$ the smallest number $n \in \omega$ such that $\{s_\alpha(m)\}_{m \geq n} \subseteq U_f$. It is easy to see that the function $\mu_\alpha : \omega^\omega \to \omega$ is monotone.

For every $n \in \omega$ and finite function $t \in \omega^n$, let $\omega^\omega_t = \{ f \in \omega^\omega : f | n = t \}$. By 4 Lemma 2.3.5, for every $f \in \omega^\omega$ there exists $n \in \omega$ such that $\mu_\alpha[\omega^\omega_f]_n$ is finite. Let $T_\alpha$ be the set of
all finite functions \( t \in \omega^{< \omega} = \bigcup_{n \in \omega} \omega^n \) such that \( \mu_\alpha[\omega^n] \) is finite but for any \( \tau \in \omega^{< \omega} \) with \( \tau \subset t \) the set \( \mu_\alpha[\omega^n] \) is infinite. It follows from [4 Lemma 2.3.5] that for every \( f \in \omega^\omega \) there exists a unique \( t_f \in T_\alpha \) such that \( t_f \subset f \).

Let \( \delta_\alpha(f) = \max \mu_\alpha[\omega^n] \geq \mu_\alpha(f) \). It is clear that the function \( \delta_\alpha : \omega^\omega \to \omega \) is continuous and hence \( \delta_\alpha \) is an element of the space \( C_p(\omega^\omega, \omega) \) of continuous functions from \( \omega^\omega \) to \( \omega \). Here we endow \( \omega^\omega \) with the product topology. The function space \( C_p(\omega^\omega, \omega) \) is endowed with the topology of pointwise convergence. By Michael’s Proposition 10.4 in [19], the space \( C_p(\omega^\omega, \omega) \) has a countable network.

Consider the function \( \delta : \omega_1 \to C_p(\omega^\omega, \omega), \delta : \alpha \mapsto \delta_\alpha \), and observe that \( \delta_\alpha(f) \geq \mu_\alpha(f) \) for any \( \alpha \in \omega_1 \) and \( f \in \omega^\omega \).

Since the space \( C_p(\omega^\omega, \omega) \) has countable network, there exists a sequence \( \{\alpha_n\}_{n \in \omega} \subseteq \omega_1 \) of pairwise distinct ordinals such that the sequence \( (\delta_{\alpha_n})_{n \in \omega} \) converges to \( \delta_\alpha \) in the function space \( C_p(\omega^\omega, \omega) \). We claim that the sequence \( (x_{\alpha_n})_{n \in \omega} \) witnesses that \( X \) is \((\omega_1, \omega)_\tau\)-equiconvergent at \( x \). Given any open neighborhood \( O_x \subseteq X \) of \( x \), find \( f \in \omega^\omega \) such that \( U_f \subseteq O_x \). Since the sequence \( (x_{\alpha_n}(n))_{n \in \omega} \) converges to \( x \), there exists \( m \in \omega \) such that \( \{x_{\alpha_n}(n)\}_{n \geq m} \subseteq U_f \). Since the sequence \( (\delta_{\alpha_n})_{n \in \omega} \) converges to \( \delta_\alpha \) in \( C_p(\omega^\omega, \omega) \) we can replace \( m \) by a larger number and additionally assume that \( \delta_{\alpha_n}(f) = \delta_\alpha(f) \) for all \( n \geq m \). Choose a number \( l \geq \delta_{\alpha_0}(f) \) such that for every \( n < m \) and \( k \geq l \) we have \( x_{\alpha_n}(k) \in O_x \).

On the other hand, for every \( n \geq m \) and \( k \geq l \) we have \( k \geq l \geq \delta_{\alpha_0}(f) = \delta_{\alpha_n}(f) \geq \mu_{\alpha_n}(f) \) and hence \( x_{\alpha_n}(k) \in U_f \subseteq O_x \). \( \square \)

Another property implying the \((\omega_1, \omega)_\tau\)-equiconvergence is the existence of a countable \( \text{cs}^\ast \)-network. First we introduce the necessary definitions.

Let \( x \) be a point of a topological space \( X \). We say that a sequence \( \{x_n\}_{n \in \omega} \subseteq X \) accumulates at \( x \) if for each neighborhood \( U \subseteq X \) of \( x \) the set \( \{n \in \omega : x_n \in U\} \) is infinite.

A family \( N \) of subsets of \( X \) is defined to be

- an \( \text{s}^\ast \)-network at \( x \) if for any neighborhood \( O_x \subseteq X \) of \( x \) and any sequence \( \{x_n\}_{n \in \omega} \subseteq X \) that accumulates at \( x \) there exists \( N \in N \) such that \( N \subset O_x \) and the set \( \{n \in \omega : x_n \in N\} \) is infinite;
- a \( \text{cs}^\ast \)-network at \( x \) if for any neighborhood \( O_x \subseteq X \) of \( x \) and any sequence \( \{x_n\}_{n \in \omega} \subseteq X \) that converges to \( x \) there exists \( N \in N \) such that \( N \subset O_x \) and the set \( \{n \in \omega : x_n \in N\} \) is infinite;
- a \( \text{cs}^\ast \)-network at \( x \) if for any neighborhood \( O_x \subseteq X \) of \( x \) and any sequence \( \{x_n\}_{n \in \omega} \subseteq X \) that converges to \( x \) there exists \( N \in N \) such that \( N \subset O_x \) and \( N \) contains some point \( x_n \);
- a network at \( x \) if for any neighborhood \( O_x \subseteq X \) the union \( \bigcup \{N \in N : N \subseteq O_x\} \) is a neighborhood of \( x \);

It is clear that for any family \( N \) of subsets of a topological space \( X \) and any \( x \in X \) we have the following implications.

\[
\begin{align*}
(N \text{ is an } \text{s}^\ast\text{-network at } x) & \quad \iff \quad (N \text{ is a network at } x) \\
(N \text{ is a } \text{cs}^\ast\text{-network at } x) & \quad \implies \quad (N \text{ is a } \text{cs}^\ast\text{-network at } x)
\end{align*}
\]
**Theorem 4.2.** If a topological space $X$ has a countable $cs^*$-network at a point $x \in X$, then $X$ is $(\omega_1, \omega)_p$-equiconvergent at $x$.

**Proof.** Let $N$ be a countable $cs^*$-network at $x$ and

\[ \{x_\alpha\}_{\alpha \in \omega_1} \subseteq \{s \in X^\omega : \lim_{n \to \infty} s(n) = x\}. \]

Endow the ordinal $\omega + 1 = \omega \cup \{\omega\}$ with the discrete topology. For every $\alpha \in \omega_1$ consider the function $\delta_\alpha : N \to \omega + 1$ assigning to each $N \in \mathbb{N}$ the smallest number $n \in \omega$ such that $x_\alpha(n) \in N$ if such number $n$ exists, and $\omega$ if $x_\alpha \notin N$ for all $n \in \omega$. Since $(\omega + 1)^N$ is a metrizable separable space, the uncountable set

\[ \{\delta_\alpha\}_{\alpha \in \omega_1} \subseteq (\omega + 1)^N \]

contains a non-trivial convergent sequence. Consequently, we can find a sequence $(\alpha_n)_{n \in \omega}$ of pairwise distinct countable ordinals such that the sequence $(\delta_{\alpha_n})_{n \in \omega}$ converges to $\delta_{\alpha_0}$ in the Polish space $(\omega + 1)^N$. We claim that the sequence $(x_{\alpha_n})_{n \in \omega}$ witnesses that the space $X$ is $(\omega_1, \omega)_p$-equiconvergent. Fix any neighborhood $U \subseteq X$ of zero.

Since $N$ is an $cs^*$-network, there exists $N \in \mathbb{N}$ and $n \in \omega$ such that $x_n \in N \subseteq U$. Hence

\[ d := \delta_{\alpha_0}(N) \leq n. \]

Since the sequence $(\delta_{\alpha_n})_{n \in \omega}$ converges to $\delta_{\alpha_0}$, there exists $l \in \omega$ such that

\[ \delta_{\alpha_k}(N) = \delta_{\alpha_0}(N) = d \]

for all $k \geq l$. Then for every $k \geq l$ we have $x_{\alpha_k}(d) \in N \subseteq U$. \hfill \Box

The following proposition (connecting $\omega^\omega$-bases with networks) is a corollary of Theorem 6.4.1 in [4].

**Proposition 4.3.** If $(U_\alpha)_{\alpha \in \omega^\omega}$ is an $\omega^\omega$-base at a point $x$ of a topological space $X$, then $(\bigcap_{\beta \in \tau_\alpha} U_\beta)^{\omega < \omega}$ is a countable $s^*$-network at $x$. Here $\tau_\alpha = \{\beta \in \omega^\omega : \alpha \subset \beta\}$ for any $\alpha \in \omega^\omega = \bigcup_{n \in \omega} \omega^n$.

As a consequence of the results presented above about the $(\kappa, \lambda)_p$-equiconvergence and the $(\kappa, \lambda)$-tall bornology for a lcs $E$, we propose the following proof of Theorem 1.2.

**Proof of Theorem 4.2**. If a lcs $E$ has an $\omega^\omega$-base, then by Theorem 1.1 the space $E$ is $(\omega_1, \omega)_k$-equiconvergent and hence $(\omega_1, \omega)_p$-equiconvergent. The $(\omega_1, \omega)_p$-equiconvergence of $E$ also follows from Proposition 4.3 and Theorem 4.2. Next, by Proposition 3.3 the space $E$ has $(\omega_1, \omega)$-tall bornology, which means that each uncountable set in $E$ contains an infinite bounded set. If $E$ has an uncountable Hamel basis $H$, then $H$ contains an infinite bounded linearly independent set, and by Theorem 2.1 the space $E$ contains an infinite-dimensional compact set. \hfill \Box

5. **Radial networks and another proof of Theorem 1.2**

A family $N$ of subsets of a linear topological space $E$ is called a radial network if for every neighborhood of zero $U \subseteq E$ and every every $x \in E$ there exist a set $N \in N$ and a nonzero real number $\varepsilon$ such that $\varepsilon \cdot x \in N \subseteq U$.

The following theorem is a “linear” modification of Theorem 4.2.
**Theorem 5.1.** If a lcsc $E$ has a countable radial network, then each uncountable subset in $E$ contains an infinite bounded subset.

*Proof.* Let $N$ be a countable radial network in $E$, and let $A$ be an uncountable set in $E$. Endow the ordinal $\omega + 1 = \omega \cup \{\omega\}$ with the discrete topology.

For every $\alpha \in A$ consider the function $\delta_\alpha : N \to \omega + 1$ assigning to each $N \in N$ the ordinal $\delta_\alpha(N) = \min \{ n \in \omega + 1 : 2^{-n} \cdot \alpha \in [-1, 1] \cdot N \}$. Here we assume that $2^{-\omega} = 0$.

Since $(\omega + 1)^N$ is a metrizable separable space, the uncountable set $\{\delta_\alpha\}_{\alpha \in A} \subseteq (\omega + 1)^N$ contains a non-trivial convergent sequence. Consequently, we can find a sequence $\{\alpha_n\}_{n \in \omega} \subseteq A$ of pairwise distinct points of $A$ such that the sequence $(\delta_{\alpha_n})_{n \in \omega}$ converges to $\delta_{\alpha_0}$ in the Polish space $(\omega + 1)^N$.

We claim that the set $\{\alpha_n\}_{n \in \omega}$ is bounded in $X$. Fix any neighborhood $U \subseteq X$ of zero. Since $N$ is a radial network, there exist a set $N \in N$ and a nonzero real number $\varepsilon$ such that $\varepsilon \cdot \alpha_0 \in N \subseteq U$. Then $d := \delta_{\alpha_0}(N) \in \omega$. Since the sequence $(\delta_{\alpha_n})_{n \in \omega}$ converges to $\delta_{\alpha_0}$, there exists $l \in \omega$ such that $\delta_{\alpha_k}(N) = \delta_{\alpha_0}(N)$ for all $k \geq l$. Then for every $k \geq l$ we have $2^{-d} \cdot \alpha_k \in [-1, 1] \cdot N \subseteq [-1, 1] \cdot U$ and hence $\{\alpha_k\}_{k \geq l} \subseteq [-2^d, 2^d] \cdot U$, which implies that the family $(\alpha_n)_{n \in \omega}$ is bounded in $X$. \hfill \Box

The implication $(1) \Rightarrow (7)$ in the following theorem provides an alternative proof of Theorem 1.2, announced in the introduction.

**Theorem 5.2.** For a lcsc $E$ consider the following properties:

1. $E$ has an $\omega^\omega$-base;
2. $E$ has a countable $s^*$-network at zero;
3. $E$ has a countable $cs^*$-network at zero;
4. $E$ has a countable $cs^\bullet$-network at zero;
5. $E$ has a countable radial network at zero;
6. each uncountable set in $E$ contains an infinite bounded subset;
7. $E$ contains an infinite-dimensional compact set.

Then $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6)$. If $E$ has uncountable Hamel basis, then $(6) \Rightarrow (7)$.

*Proof.* The implication $(1) \Rightarrow (2)$ follows from Proposition 4.3. The implications $(2) \Rightarrow (3) \Rightarrow (4)$ are trivial and $(4) \Rightarrow (5)$ follows from the observation that every $cs^\bullet$-network at zero in the space $E$ is a radial network for $E$. The implication $(5) \Rightarrow (6)$ is proved by Theorem 5.1.

If $E$ has an uncountable Hamel basis $H$, then by $(6)$, there exists an infinite bounded set $B \subseteq H$. By Theorem 2.1 the space $E$ contains an infinite-dimensional compact set. \hfill \Box

**Problem 5.3.** Is there an lcsc $E$ that has a countable radial network but does not have a countable $cs^\bullet$-network at zero?
6. Applications to spaces $C_p(X)$

A family $\{B_\alpha : \alpha \in \omega^\omega\}$ of bounded (compact) sets covering a lcs $E$ is called a bounded (compact) resolution if $B_\alpha \subseteq B_\beta$ for each $\alpha \leq \beta$. If additionally every bounded (compact) subset of $E$ is contained in some $B_\alpha$, we call the family $\{B_\alpha : \alpha \in \omega^\omega\}$ a fundamental bounded (compact) resolution of $E$.

Example 6.1. Let $E$ be a metrizable lcs with a decreasing countable base $(U_n)_{n \in \omega}$ of absolutely convex neighbourhoods of zero. For $\alpha = (n_k)_{k \in \omega}$ put $B_\alpha = \bigcap_{k \in \omega} n_k U_k$ and observe that $\{B_\alpha : \alpha \in \omega^\omega\}$ is a fundamental bounded resolution in $E$.

A Tychonoff space $X$ is called pseudocompact if each continuous real-valued function on $X$ is bounded.

The first part of the following (motivating) result has been proved in [18]; since this is not published yet, we add a short proof.

Proposition 6.2. For a Tychonoff space $X$ the following assertions are equivalent:

1. The space $C_k(X)$ is covered by a sequence of bounded sets.
2. The space $C_p(X)$ is covered by a sequence of bounded sets.
3. $X$ is pseudocompact.

Moreover, the following assertions are equivalent:

4. $C_p(X)$ is covered by a sequence of bounded sets but is not covered by a sequence of functionally bounded sets.
5. $X$ is pseudocompact and contains a countable subset which is not closed in $X$ or not $C^*$-embedded in $X$.

Proof. (1) $\Rightarrow$ (2) is clear. (2) $\Rightarrow$ (3): Assume $C_p(X)$ is covered by a sequence of bounded sets but $X$ is not pseudocompact. Then $C_p(X)$ contains a complemented copy of $\mathbb{R}^\omega$, see [1]. But $\mathbb{R}^\omega$ cannot be covered by a sequence of bounded sets, otherwise would be $\sigma$-compact.

(3) $\Rightarrow$ (1): If $X$ is pseudocompact, then for every $n \in \mathbb{N}$ the set $B_n = \{f \in C(X) : \sup_{x \in X} |f(x)| \leq n\}$ is bounded in $C_k(X)$ and $\bigcup_{n \in \mathbb{N}} B_n = C_k(X)$.

The equivalence (4) $\iff$ (5) follows from [24, Problem 399]: $C_p(X)$ is covered by a sequence of functionally bounded subsets of $C_p(X)$ if and only if $X$ is pseudocompact and every countable subset of $X$ is closed and $C^*$-embedded in $X$. □

Example 6.3. $C_p([0, \omega_1))$ is covered by a sequence of bounded sets but is not covered by a sequence of functionally bounded sets.

By [10], $C_p(X)$ has a bounded resolution if and only if there exists a $K$-analytic space $L$ such that $C_p(X) \subseteq L \subseteq \mathbb{R}^X$. The problem when $C_p(X)$ has a fundamental bounded resolution is easier. As a simple application of Theorem [1,2] we prove the following

Proposition 6.4. For a Tychonoff space $X$ consider the following assertions:

1. $C_p(X)$ admits a fundamental bounded resolution $\{B_\alpha : \alpha \in \omega^\omega\}$.
2. $X$ is countable.
3. $\mathbb{R}^X = \bigcup_{\alpha \in \omega^\omega} \overline{B_\alpha}^{\mathbb{R}^X}$ for a fundamental bounded resolution $\{B_\alpha : \alpha \in \omega^\omega\}$ in $C_p(X)$. 

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(4) The strong (topological) dual $L_\beta(X)$ of $C_p(X)$ is a cosmic space, i.e. a continuous image of a metrizable separable space.

(5) $C_p(X)$ is a large subspace of $\mathbb{R}^X$, i.e. for every mapping $f \in \mathbb{R}^X$ there is a bounded set $B \subseteq C_p(X)$ such that $f \in \overline{B}^{\mathbb{R}^X}$.

Then (1) $\iff$ (2) $\iff$ (3) $\iff$ (4) $\implies$ (5) but (5) $\implies$ (2) fails even for compact spaces $X$.

The implication (1) $\implies$ (2) was recently proved by Ferrando, Gabriyelyan and Kąkol [9] (with the help of cs*-networks). We will derive this implication from Theorem 1.2.

Proof. (1) $\implies$ (2): If $C_p(X)$ has a fundamental bounded resolution $\{B_\alpha : \alpha \in \omega^\omega\}$, then the sets $U_\alpha = \{\xi \in L_\beta(X) : \sup_{f \in B_\alpha} |\xi(f)| \leq 1\}$ form an $\omega^\omega$-base in $L_\beta(X)$. By [14], every bounded set in $L_\beta(X)$ is finite-dimensional. Applying Theorem 1.2 we conclude that the Hamel basis $X$ of the lcs $L_\beta(X)$ is countable. (2) $\implies$ (1) is clear. (2) $\implies$ (3)$\wedge$ (5): Since $C_p(X)$ is dense in the metrizable space $\mathbb{R}^X$, the claims hold. (2) $\implies$ (4): If $X$ is countable, then $L_\beta(X)$ has a fundamental sequence of compact sets covering $L_\beta(X)$ and [19, Proposition 7.7] implies that $L_\beta(X)$ is an $\aleph_0$-space, hence cosmic. (4) $\implies$ (2): If $L_\beta(X)$ is cosmic, then it is separable, and [12, Corollary 2.5] shows that $X$ is countable. (5) $\not\implies$ (2): $C_p(X)$ over every Eberlein scattered, compact $X$ satisfies (5), see [13].

Item (5) in Proposition 6.4 is strictly connected with the following result.

**Theorem 6.5.** ([13], [12]) For a Tychonoff space $X$, the following conditions are equivalent:

(i) $C_p(X)$ is distinguished, i.e. the strong dual $L_\beta(X)$ of the space $C_p(X)$ is bornological.

(ii) The strong dual $L_\beta(X)$ of the space $C_p(X)$ is a Montel space.

(iii) $C_p(X)$ is a large subspace of $\mathbb{R}^X$.

(iv) The strong dual $L_\beta(X)$ of the space $C_p(X)$ carries the finest locally convex topology.

The following is a linear counterpart to item (4) in Proposition 6.4.

**Remark 6.6.** A Tychonoff space $X$ is finite if and only if $L_\beta(X)$ is a continuous linear image of a metrizable lcs.

Indeed, if $X$ is finite, nothing is left to prove. Conversely, assume that $L_\beta(X)$ is a continuous linear image of a metrizable lcs $E$ (by a one-to-one map). But $L_\beta(X)$ has only finite-dimensional bounded sets and $E$ fails this property. Hence $X$ is finite.
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