BETA-INTEGRALS AND FINITE ORTHOGONAL SYSTEMS OF WILSON POLYNOMIALS

NERETIN Yu.A.

In this paper, we derive the beta-integral

\[(0.1) \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \prod_{k=1}^{3} \frac{\Gamma(a_k + is)}{\Gamma(2is)\Gamma(b + is)} \right|^2 ds = \frac{\Gamma(b - a_1 - a_2 - a_3) \prod_{1 \leq k < l \leq 3} \Gamma(a_k + a_l)}{\prod_{k=1}^{3} \Gamma(b - a_k)}\]

(I could not find this integral in literature). We construct the system of orthogonal polynomials related to this integral and also systems related to the Askey integral (0.5) and the 5-\(H_5\)-Dougall formula (0.6). It turns out that all these systems are the Wilson polynomials (see [Wil], [AAR], [KS]) outside the domain of positivity of the classical weight.

0.1. Beta-integrals. The integral (0.1) is a representative of a large family of so-called beta-integrals. We mention only the beta-integrals that in either case will appear in this paper: the de Branges–Wilson integral [dB], [Wil] (see also [AAR], 3.6)

\[(0.2) \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \prod_{k=1}^{4} \frac{\Gamma(a_k + is)}{\Gamma(2is)} \right|^2 ds = \prod_{1 \leq k < l \leq 4} \Gamma(a_k + a_l),\]

the Second Barnes Lemma (see [AAR], Theorem 2.4.3)

\[(0.3) \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Gamma(a_1 - is)\Gamma(a_2 - is)\Gamma(b_1 + is)\Gamma(b_2 + is)\Gamma(b_3 + is)}{\Gamma(a_1 + a_2 + b_1 + b_2 + b_3 + is)} ds = \frac{\prod_{1 \leq k \leq 5} \Gamma(a_1 + b_k)\Gamma(a_2 + b_k)}{\prod_{1 \leq k < l \leq 3} \Gamma(a_1 + a_2 + b_k + b_l)},\]

the Nassrallah–Rahman integral (see [Ask2], [RS])

\[(0.4) \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \prod_{j=1}^{5} \frac{\Gamma(a_j + is)}{\Gamma(2is)\Gamma(\sum_{j=1}^{5} a_j + is)} \right|^2 ds = 2 \prod_{1 \leq k < l \leq 5} \Gamma(a_k + a_l) \prod_{k=1}^{5} \Gamma(a_1 + a_2 + a_3 + a_4 + a_5 - a_k),\]

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and the Askey integral (see [RS])

\[
(0.5) \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Gamma(1-2s)\Gamma(1+2s)}{\prod_{k=1}^{4} \Gamma(a_k+s)\Gamma(a_k-s)} ds = \frac{\Gamma(a_1 + a_2 + a_3 + a_4 - 3)}{\prod_{1 \leq k < l \leq 4} \Gamma(a_k + a_l - 1)}.
\]

In the last case, the integrand has poles on the integration contour at the points \(s = \pm n/2, n = 1, 2, \ldots\) We can understand this integral in two ways.

The first way. Fix \(\alpha\) such that 0 < \(\alpha\) < 1. Consider the integral from \(-M - \alpha\) to \(N + \alpha\), where \(M, N\) are positive integers. We understand this integral as the principal value near each pole. For \(\Re \sum a_j < 3\), the limit of this integral as \(M, N \to +\infty\) exists.

The second way. Consider the integral from \(-M - \alpha\) to \(N + \alpha\) over the contour that passes the poles from above and consider the limit as \(M, N \to +\infty\). The both variants coincide since the integrand is an even function (the corrections in the pairs \(\pm k/2\) of poles cancel).

More complete lists of beta-integrals are contained in [Ask2], [RS], there are also numerous \(q\)-analogs (see [AW], [Ask2], [RS]), discrete analogs and multivariate analogs (see [Gus1], [Gus2]) of beta-integrals.

Among discrete analogs, we need the Dougall formula

\[
(0.6) \quad \sum_{n=-\infty}^{\infty} \frac{\alpha + n}{\prod_{j=1}^{4} \Gamma(a_j + a_j + n)\Gamma(a_j - a_j - n)} = \frac{\sin 2\pi \alpha}{2\pi} \frac{\Gamma(a_1 + a_2 + a_3 + a_4 - 3)}{\prod_{1 \leq j < k \leq 4} \Gamma(a_j + a_k - 1)}.
\]

Remark. In formulas (0.1)–(0.4), we assume that the parameters \(a_j, b_j\) are real and positive. To be definite, consider integral (0.2). Represent it in the form

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\prod_{k=1}^{4} \Gamma(a_k + is)\Gamma(a_k - is)}{\Gamma(2is)\Gamma(-2is)} ds = \frac{\prod_{1 \leq k < l \leq 4} \Gamma(a_k + a_l)}{\Gamma(a_1 + a_2 + a_3 + a_4)}.
\]

By the analytic continuation, this equality is valid for \(\Re a_j > 0\). We can replace this restriction by a weaken variant \(a_k + a_l \neq 0, -1, -2, \ldots\) (for all the pairs \(k, l\)), but in this case we must change the contour of integration (see Subsection 0.4) or add the sum of residues in the right-hand side of the equation. We can understand all the integrals (0.1)–(0.4) and (2.2)–(2.5) in a similar way.

0.2. Nassrallah–Rahman integral. Our derivation of the integral (0.1) is very simple, but the integral itself can be obtained by degenerations and hypergeometric transformations from the following wonderful Nassrallah–Rahman integral

\[
(0.7) \quad \int_{-\infty}^{\infty} \left| \frac{\Gamma(a + is)\Gamma(b + is)\Gamma(c + is)\Gamma(d + is)\Gamma(f + is)\Gamma(g + is)}{\Gamma(2is)\Gamma(\lambda + is)\Gamma(\mu + is)} \right|^2 ds =
\]

\[
= \frac{4\pi^3 \sin(a + b + c + d)\pi \Gamma(a + b)\Gamma(a + c)\Gamma(a + d)\Gamma(1 + 2a)}{\sin(b + c)\pi \sin(b + d)\pi \sin(c + d)\pi}
\]

\[
\times \frac{\Gamma(a + f)\Gamma(f - a)\Gamma(g + a)\Gamma(g - a)}{\Gamma(1 + a - b)\Gamma(1 + a - c)\Gamma(1 + a - d)\Gamma(\lambda + a)\Gamma(\lambda - a)\Gamma(\mu + a)\Gamma(\mu - a)}
\]

\[
\times_{\alpha} F_{\beta} \left[ \begin{array}{c} 2a, 1 + a, a + b, a + c, a + d, a + f, a + g, 1 + a - \lambda, 1 + a - \mu \\ a, 1 + a - b, 1 + a - c, 1 + a - d, 1 + a - f, 1 + a - g, a + \lambda, a + \mu \end{array} : 1 \right]
\]
where \( \cdots \) denotes the sum of the similar summands, where \( a \) changes by \( b, c, d \).

We discuss several integrals (2.2)–(2.5) intermediate between (0.2), (0.1) and the Nassrallah–Rahman integral. All these integrals can be obtained from (0.7) by degenerations and hypergeometric transforms, however I never have seen the final formulas (2.2)–(2.5). Nevertheless they deserve to be written (especially the integral representation (2.2) for \( {}_3F_2(1) \)), furthermore, our calculations are very simple conceptually and technically.

0.3. Finite systems of orthogonal polynomials. We write explicitly the system of polynomials \( p_n(s^2) \) orthogonal with respect to the weight

\[
w_1(s) = \left| \frac{\prod_{k=1}^{3} \Gamma(a_k + is)}{\Gamma(2 is) \Gamma(b + is)} \right|^2 ds.
\]

Observe that this weight decreases as

\[
s^{-2(b-a_1-a_2-a_3)-1},
\]

hence only a finite number of the moments

\[
\int_0^\infty s^{2k}w_1(s)ds
\]

exists. Thus our system of orthogonal polynomials is finite by the definition.

We also construct the system of polynomials orthogonal with respect to the discrete weight

\[
w_2(s) = \sum c_n \delta_n,
\]

where \( \delta_n \) is the delta-function supported by the point \( n \), and \( c_n \) are the summands of the Dougall formula (0.6).

This weight also polynomially decreases. For some \( a_j \) and \( \alpha \) the weight is positive, in some cases it is sign indefinite. The latter is not very important for play formulas.

Finally, we construct a system of polynomials (again, this system is finite) associated with the weight \( w_3(s) \), where \( w_3(s) \) is the integrand in the Askey integral (0.5). This weight is not positive; moreover, it is not a measure, but we can integrate with respect to it.

It turns out that all the three systems consist of the Wilson polynomials. Furthermore, the systems related to the weights \( w_2(s) \) and \( w_3(s) \) coincide (up to a correction factor in the orthogonality relations).

In first time, finite systems of orthogonal polynomials (with respect to a continuous weight) were discovered by Romanovski [Rom] in 1929 (he constructed analogues of the Jacobi polynomials). Askey [Ask1] obtained the system of polynomials related to the Ramanujan beta-integral. Various finite families of orthogonal polynomials were considered in a series of papers of Lesky [Les1]–[Les4]. Some applications of such constructions are contained in [Pee], [BO].

0.4. Notation. \( (a)_k := \frac{\Gamma(a + k)}{\Gamma(a)} \) is the Pochhammer symbol.

\[
_qF_p \left[ \begin{array}{c} a_1, \ldots, a_q \\ b_1, \ldots, b_p \end{array} ; z \right] := \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_q)_k}{k! (b_1)_k \ldots (b_p)_k} z^k
\]
is the generalized hypergeometric function. Next,

\[ \Gamma \left[ \begin{array}{c} u_1, \ldots, u_l \\ v_1, \ldots, v_m \end{array} \right] := \frac{\Gamma(u_1) \ldots \Gamma(u_l)}{\Gamma(v_1) \ldots \Gamma(v_m)}. \]

Mellin–Barnes integrals are defined by

\[ \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma \left[ \begin{array}{c} a_1 + s, \ldots, a_m + s, b_1 - s, \ldots, b_n - s \\ c_1 - s, \ldots, c_k - s, d_1 + s, \ldots, d_l + s \end{array} \right] z^{-s} ds. \]

We suppose that the integration is given over the contour passing from \(-i\infty\) to \(+i\infty\) and separating the left series of the poles

\[ s = -a_1 - j, \ldots, -a_m - j, \quad \text{where} \quad j = 0, 1, 2, \ldots, \infty, \]

from the right series of the poles

\[ s = b_1 + j, \ldots, s = b_n + j. \]

We assume that none of poles from the left series coincides with a pole from the right series (this implies the existence of a contour). Using the standard rules, we can represent a Mellin–Barnes integral as a finite sum of hypergeometric functions with \(\Gamma\)-factors. This rules (for \(m + n \geq k + l\), bellow this is always satisfied) are contained in the books of Slater [Sla] and Marichev [Mar], see also [PBM].

§1. Preliminaries and preliminary calculations

1.1. Index hypergeometric transform. Fix \(a, b > 0\). Let \(f\) be a function on the half-line \(x \geq 0\). The index hypergeometric transform \(J_{b,c}\) is defined by the formula

\[ (1.1) \quad Jf(s) = \frac{1}{\Gamma(a + b)} \int_0^\infty f(x) _2F_1(a + is, a - is; a + b; -x)x^{a+b-1}(1 + x)^{a-b} dx. \]

The inverse transform is given by

\[ (1.2) \quad J^{-1}g(x) = \frac{1}{\pi \Gamma(a + b)} \int_0^\infty g(s) _2F_1(a + is, a - is; a + b; -x) \left| \frac{\Gamma(a + is)\Gamma(b + is)}{\Gamma(2is)} \right|^2 ds. \]

The transformation \(J\) is a unitary operator

\[ L^2 \left( \mathbb{R}^+, x^{a+b-1}(1 + x)^{a-b} \right) \rightarrow L^2 \left( \mathbb{R}^+, \left| \frac{\Gamma(a + is)\Gamma(b + is)}{\Gamma(2is)} \right|^2 \right) \]

The unitarity condition (the Plancherel formula) in the explicit form is

\[ (1.3) \quad \int_0^\infty f_1(x)f_2(x)x^{a+b-1}(1 + x)^{a-b} dx = \frac{1}{\pi} \int_0^\infty g_1(s)g_2(s) \left| \frac{\Gamma(a + is)\Gamma(b + is)}{\Gamma(2is)} \right|^2 ds \]

On properties of this transform, see [FJK], [Koo1]–[Koo3], [Yak], [Ner]. The operator \(J\) is called also the Olevsky transform, the Jacobi transform, the Fourier–Jacobi
1.2. Gauss hypergeometric function. We use some simple properties of
the hypergeometric functions $2F_1$, in particular: the Gauss formula (see [HTF1],
(2.1.14), [AAR], Theorem 2.2.2)

$$2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad \text{Re} \,(c-a-b) > 0. \quad (1.4)$$

and the Bolza formula (see [HTF1] (2.1.22-23))

$$2F_1(a, b; c; -x) = (1 + x)^{-a}2F_1(a, c-b; c; \frac{x}{x+1}) = (1 + x)^{-a-b}2F_1(c-a, c-b; c; -x). \quad (1.5)$$

Recall that the function $2F_1(a, b; c; x)$ admits the analytical continuation to the
half-line $x < 0$. The asymptotics of the function $2F_1(a + is, a - is; a + b; -x)$ as
$x \to +\infty$ has the form

$$2F_1(a + is, a - is; a + b; -x) = \psi(x)x^{-\lambda},$$

where $\psi$ is a bounded function for $s \neq 0$ and $\psi(x)/\ln(x)$ is a bounded function for
$s = 0$ (for a more explicit expression, see [HTF1], (2.3.2.9)).

The asymptotics of the function $2F_1(a + is, a - is; a + b; -x)$ as $s \to +\infty$ has the form

$$2F_1(a + is, a - is; a + b; -x) = \lambda(s)s^{-a-b+1/2},$$

where $\lambda(s)$ is a bounded function (see the Watson formula [HTF1], (2.3.2.17)).

Recall three representations of the Gauss hypergeometric function as Barnes
integrals see [HTF1], (2.1.15), [Mar], §2.10, formulas 11(1), 19(1), 22(1), [AAR],
Theorems 2.4.1, 2.4.2, [PBM], 8.4.49(13), (20), (22); below $x > 0$

$$\frac{1}{2\pi i} \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \int_{-i\infty}^{i\infty} \frac{\Gamma(a-s)\Gamma(b-s)\Gamma(s)}{\Gamma(c-s)} x^{-s}ds = 2F_1(a, b; c; -x); \quad (1.7)$$

$$\frac{1}{2\pi i} \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)} \int_{-i\infty}^{i\infty} \frac{\Gamma(s)\Gamma(s+c-a+b)\Gamma(a-s)\Gamma(b-s)x^{-s}}{\Gamma(s+c-a)\Gamma(s+c-b)} ds = 2F_1(a, b; c; 1-x); \quad (1.8)$$

$$\frac{1}{2\pi i} \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(s+c-a)\Gamma(s+c-b)} \int_{-i\infty}^{i\infty} \frac{\Gamma(s)\Gamma(s+c-a-b)}{\Gamma(s+c-a)\Gamma(s+c-b)} x^{-s}ds =$$

$$= \left\{ \begin{array}{ll} (1-x)^{c-1}2F_1(a, b; c; 1-x); & x > 1, \\ 0; & x < 1 \end{array} \right. \quad (1.9)$$
1.3. Γ-function. We shall use the following integral representation of the beta-function
\[ \int_0^\infty \frac{x^{\alpha-1}}{(1 + x)^\rho} \, dx = B(\alpha, \rho - \alpha), \]
The substitution \( x/(x + 1) = z \) transforms this integral to the usual definition of the beta-function.

Asymptotics of the function \( \Gamma(z) \) is given by the Stirling formula
\[ \Gamma(z) = \sqrt{2\pi z} z^{-1/2} e^{-z} (1 + O(1/z)); \quad z \to \infty \quad \arg z < \pi - \varepsilon. \]

In particular (see [HTF1], (1.18.3), (1.18.6)),
\[ \Gamma(z + a) \Gamma(z + b) \sim z^{a-b}, \quad z \to \infty, \quad \arg z < \pi - \varepsilon, \]
\[ |\Gamma(a + is)| \sim \sqrt{2\pi |s|^{a-1/2}} e^{-|s|\pi/2}, \quad s \to +\infty. \]

1.4. Mellin transform. Consider a function \( f \) on the half-line \( x \geq 0 \). Its Mellin transform is defined by the formula
\[ f^\ast(s) = \int_0^\infty x^s f(x) \frac{dx}{x}. \]
The inversion formula is
\[ f(x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} f^\ast(s) x^{-s} ds. \]

The Mellin transform takes the convolution
\[ \int_0^\infty f_1(x/t)f_2(t) \frac{dt}{t} \]
to the product of the Mellin transforms \( f_1^\ast(s)f_2^\ast(s) \). For precise statements, see for instance [Mar].

Let \( a > 0 \). Obviously, the function \( f(ax) \) is mapped to \( a^{-s} f^\ast(s) \). Also, the function \( x^\alpha f(x) \) corresponds to \( f^\ast(s + \alpha) \).

1.5. We need some integrals of the form
\[ (1.10) \int_0^\infty \frac{x^{\alpha-1}}{(x + z)^\rho} F_1(p, q; r; -x) \, dx = \]
\[ = \frac{1}{2\pi i} z^{\alpha-\rho} \left[ \int_{-i\infty}^{i\infty} \frac{\Gamma \left[ \frac{r}{p, q, \rho} \right]}{\Gamma \left[ \frac{s + \alpha, \rho - s - \alpha, p + s, q + s, -s}{r + s} \right]} \right] z^s ds \]
The Mellin–Barnes integral in the right-hand side can be expressed as a linear combination of two \( 3F_2 \). As a quite standard (see [Mar]) pattern, we prove the identity (1.10); this statement also is quite standard (see [PBM], formula (2.21.1.16)). For this, represent \( x^\alpha (x + z)^{-\rho} \) as the inverse Mellin transform
\[ x^\alpha (x + z)^{-\rho} = \frac{z^{\alpha-\rho}}{2\pi i \Gamma(\rho)} \int_{-i\infty}^{i\infty} \Gamma(s + \alpha) \Gamma(\rho - s - \alpha) z^s x^{-s} ds \]
(for a verification, it is sufficient to evaluate the direct Mellin transform). Applying the Barnes integral (1.7) and the convolution formula for the Mellin transform, we obtain (1.10).

1.6. Corollaries from formula (1.10).

Lemma 1.1. a) The operator $J$ takes the function

$$(1 + x)^{-a-c} \to \frac{\Gamma(c+is)\Gamma(c-is)}{\Gamma(c+a)\Gamma(c+b)}.$$ 

b) The operator $J$ takes the function

$$\frac{(1 + x)^{b-a}}{(x + z)c+b} \to \frac{\Gamma(c+is)\Gamma(c-is)}{\Gamma(c+a)\Gamma(c+b)} \, _2F_1\left[ \begin{array}{c} c + is, c - is \\ c + a \\ \end{array} ; 1 - z \right].$$ 

c) The operator $J$ takes the function

$$x^{-u-a} \to \frac{\Gamma(-u+b)}{\Gamma(a+u)} \cdot \frac{\Gamma(u+is)\Gamma(u-is)}{\Gamma(b+is)\Gamma(b-is)}.$$ 

Proof. We must substitute these $f$ to formula (1.1). We can simply refer to the rows (2.21.1.4), (2.21.1.5), (2.21.1.15) of Prudnikov, Brychkov, Marichev, vol.3, or to the rows (7.512) of Ryzhik–Gradshteyn.

But it is better to explain what happened and why for some values of the parameters the integral (1.10) admits an explicit evaluation and for some values this is impossible.

1. For $\alpha = r$ in the right-hand side of (1.10), two $\Gamma$-factors in the integrand cancel. The remain is a Barnes integral of the type (1.8); this proves the statement b).

2. Substituting $z = 1$ to the statement a), we obtain the claim b).

3. Substitute $z = 1$ to (1.10). For $r = p + q + \rho$, the right-hand side can be evaluated by the Second Barnes Lemma (0.3). This gives c).

1.7. Integrals with products of hypergeometric functions. A direct application of the convolution formula for the Mellin transform gives

$$(1.11) \quad \int_0^\infty x^{a-1} \, _2F_1\left[ \begin{array}{c} p,q \\ r \\ \end{array} ; -\omega x \right] \, _2F_1\left[ \begin{array}{c} u,v \\ w \\ \end{array} ; -\tilde{\omega} x \right] \, dx = \frac{1}{2\pi i} \omega^{-\alpha} \Gamma \left[ \begin{array}{c} r, u, v, p, q \\ u, v, p, q \end{array} \right] \int_{-i\infty}^{i\infty} \left[ \begin{array}{c} \alpha + s, u + s, v + s, p - \alpha - s, q - \alpha - s, -s \\ r - \alpha - s, w + s \end{array} \right] (\omega^{-s}) \, ds.$$ 

Substitute $\alpha = w = r$ to this identity. Then four $\Gamma$-factors of the integrand in the right-hand side cancel. Applying the Barnes formula (1.8), we obtain

$$(1.12) \quad \int_0^\infty x^{r-1} \, _2F_1\left[ \begin{array}{c} p,q \\ r \\ \end{array} ; -\omega x \right] \, _2F_1\left[ \begin{array}{c} u,v \\ r \\ \end{array} ; -\tilde{\omega} x \right] \, dx = \omega^{-r+u}\tilde{\omega}^{-u} \Gamma \left[ \begin{array}{c} r, r, p - r + u, q - r + u, p - r + v, q - r + v \\ u, v, p, q + p + q + a + v - 2r \end{array} \right] \times \, _2F_1\left[ \begin{array}{c} p - r + u, q - r + u \\ p + q + u + v - 2r \end{array} ; 1 - \omega \right].$$
It is formula (2.21.9.7) from Tables [PBM] (where a prime in the second index of hypergeometric function in the right-hand side is lost).

We also need the integral

\[
\int_0^1 z^{\mu-1}(1-z)^{\nu-1} {}_2F_1 \left[ \begin{array}{c} \alpha, \beta \\ \nu \end{array} ; 1-z \right] {}_2F_1 \left[ \begin{array}{c} \phi, \psi \\ \xi \end{array} ; 1-z \right] dz =
\]

\[
\frac{1}{2\pi i} \Gamma \left[ \phi, \psi, \xi - \phi, \psi - \xi \right] \times \int_{-\infty}^{\infty} \Gamma \left[ \mu + s, \mu + \nu - \alpha - \beta + s, \phi + s, \psi + s, -s, \xi - \phi - \psi + s \right] ds.
\]

To deduce this formula, we represent \((1-z)^{\nu-1} {}_2F_1(\alpha, \beta; 1-z)\) as the integral (1.9), and represent \(_2F_1(\phi, \psi; \xi; 1-z)\) as the integral (1.8). It remains to apply the convolution formula.

1.9. Corollaries from the formula (1.11).

Lemma 1.2. a) The transform \(J\) takes

\[
_2F_1 \left[ \begin{array}{c} p + b, q + b \\ a + b \end{array} ; -\frac{x}{y} \right] (1+x)^{b-a}
\]

to

\[
\frac{y^{b-q}\Gamma(a+b)}{\Gamma(p+q)\Gamma(p+b)\Gamma(q+b)} \cdot \frac{\Gamma(p+is)\Gamma(p-is)\Gamma(q+is)\Gamma(q-is)}{\Gamma(a+is)\Gamma(a-is)} {}_2F_1 \left[ \begin{array}{c} p + is, p - is \\ p + q \end{array} ; 1 - y \right].
\]

b) The transform \(J\) takes \(_2F_1\left[ \begin{array}{c} p + b, q + b \\ a + b \end{array} ; -x \right] (1+x)^{b-a}\) to

\[
\frac{\Gamma(a+b)}{\Gamma(p+q)\Gamma(p+b)\Gamma(q+b)} \cdot \frac{\Gamma(p+is)\Gamma(p-is)\Gamma(q+is)\Gamma(q-is)}{\Gamma(a+is)\Gamma(a-is)}.
\]

c) The transform \(J\) takes \(_2F_1\left[ \begin{array}{c} a + c, a + d \\ a + b + c + d \end{array} ; -x \right] \) to

\[
\frac{\Gamma(a+b+c+d) \cdot \Gamma(c+is)\Gamma(c-is)\Gamma(d+is)\Gamma(d-is)}{\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)\Gamma(c+d)}.
\]

Proof. The claim a) is an immediate corollary of formula (1.11). The substitution \(y = 1\) to a) gives the statement b) (of course, it is more pleasant to substitute \(\omega = \tilde{\omega}\) directly to (1.11).

Let us deduce c). Substituting \(f = {}_2F_1[...]\) to (1.1) and applying the Bolza transformation, we obtain

\[
\frac{1}{\Gamma(a+b)} \int_0^\infty x^{a+b-1} {}_2F_1 \left[ \begin{array}{c} b + d, b + c \\ a + b + c + d \end{array} ; -x \right] {}_2F_1 \left[ \begin{array}{c} a + is, a - is \\
 a + b \end{array} ; -x \right] dx.
\]

Furthermore, in the integrand in the right-hand side of (1.12), two \(\Gamma\)-factors cancel and we obtain

\[
\frac{1}{2\pi i} \Gamma \left[ \begin{array}{c} a + b + c + d \\ a + is, a - is, b + c, b + d \end{array} \right] \times 
\]

\[
\int_{-\infty}^{\infty} \Gamma \left[ \begin{array}{c} a + is + t, a - is + t, d - a - t, c - a - t, -t \\
 c + d - t \end{array} \right] dt.
\]
It remains to apply the Second Barnes Lemma (0.3).

§2. Degenerate cases of the Nassrallah–Rahman integral

In this section, we apply the Plancherel formula (1.3) for the index hypergeometric transform to various pairs of functions from Lemmas 1.1–1.2.

To be brief, we assume that the parameters $a, b, c, d, e, f, u, v$ are real positive.

2.1. De Branges–Wilson integral. Apply the Plancherel formula (1.3) to the pair of functions $(1 + x)^{-a-c}$ and $(1 + x)^{-a-d}$, see Lemma 1.1. In the left-hand side we have the beta-integral

\[
\int_0^\infty \frac{x^{a+b-1} dx}{(1 + x)^{a+b+c+d}}
\]

In the right-hand side, we obtain the de Branges–Wilson integral (up to $\Gamma$-factors). Evaluating (2.1), we obtain (0.2).

This proof of the de Branges–Wilson integral is known, see the work of Koornwinder [Koo2].

2.2. The integral (0.1). It is sufficient to apply the Plancherel formula to the pair of functions $x^{-u-a}$ and $x^{-v-a}$, see Lemma 1.1.c.

2.3. Apply the Plancherel formula (1.3) to the pair of functions

\[(1 + x)^{-a-e} \quad \text{and} \quad _2 F_1 \left[ \begin{array}{c} a + c, a + d \\ a + b + c + d \end{array}; -x \right].\]

In the left-hand side we have

\[
\int_0^\infty x^{a+b-1}(1 + x)^{-b-e} _2 F_1 \left[ \begin{array}{c} a + c, a + d \\ a + b + c + d \end{array}; -x \right] dx.
\]

We transform this by the Bolza formula and obtain

\[
\int_0^1 z^{a+b-1}(1 - z)^{c+e-1} _2 F_1 \left[ \begin{array}{c} a + c, b + c \\ a + b + c + d \end{array}; z \right] dz.
\]

This is a standard integral representation for $_3 F_2$ (see [AAR], (2.2.4)), and finally the left-hand side is

\[
\frac{\Gamma(a + b)\Gamma(c + e)}{\Gamma(a + b + c + e)} _3 F_2 \left[ \begin{array}{c} a + c, b + c, a + b \\ a + b + c + d, a + b + c + e \end{array}; 1 \right]
\]

Equating the left-hand and right-hand sides of the Plancherel formula, we obtain

\[
\frac{1}{\pi} \int_0^\infty \left| \frac{\Gamma(a + is)\Gamma(b + is)\Gamma(c + is)\Gamma(d + is)\Gamma(e + is)}{\Gamma(2is)} \right|^2 ds = \frac{\Gamma(a + b)\Gamma(a + c)\Gamma(a + d)\Gamma(a + e)\Gamma(b + d)\Gamma(b + e)\Gamma(c + d)\Gamma(c + e)}{\Gamma(a + b + c + d)\Gamma(a + b + c + e) \times \Gamma(a + b + c + d)\Gamma(a + b + c + e)} \times _3 F_2 \left[ \begin{array}{c} a + c, b + c, a + b \\ a + b + c + d, a + b + c + e \end{array}; 1 \right]
\]
Among the $\Gamma$-factors in the numerator, the factor $\Gamma(10N)$ is evaluated in Lemma 1.2b.

2.4. Now apply the Plancherel formula (1.3) to the pair of functions $f_i$ and $f_j$.

Remark. Obviously, the left-hand side of the identity is symmetric in $a, b, c, d, e$. Applying formula (1.11), we obtain

$$\int_0^\infty \left[ F(a+b+c+a+f.e) - F(b+c+b+f.e) \right] \frac{dz}{z} =$$

$$\int_0^\infty \left[ F(a+b+c+a+f.e) - F(b+c+b+f.e) \right] \frac{dz}{z}. $$

See Lemma 1.2c. In the left-hand side, we have $a+b+c+d+e=1$(1). The expression in the right-hand side can be represented as a linear combination of three functions $F_1, F_2, F_3$ (their index transforms are evaluated in Lemma 1.9).

2.5. Finally, we apply the Plancherel formula (1.3) to the pair of functions $f_i$ and $f_j$. We must evaluate the integral

$$\int_0^\infty \Gamma(10N) \frac{dz}{z^2}. $$

(2.4) $\int_0^\infty \Gamma(10N) \frac{dz}{z^2} = \int_0^\infty \Gamma(10N) \frac{dz}{z^2}. $
Remark. The right-hand side can be written in the form
\[
\Gamma\left[\begin{array}{c}
u+v,p+q,p+b,q+b \\
a-v,u-v \\
\end{array}\right] \\
\times \left\{\Gamma\left[\begin{array}{c}
u+v,p+q,p+b,q+b \\
a-v,u-v \\
\end{array}\right] \times \right.
\begin{array}{c}
\frac{4F3}{4F3}\left[\begin{array}{c}
u+v,p+q,p+b,q+b \\
a-v,u-v \\
\end{array}\right] \times \\
\frac{4F3}{4F3}\left[\begin{array}{c}
u+v,p+q,p+b,q+b \\
a-v,u-v \\
\end{array}\right] \\
\end{array}
\right. \\
+ \Gamma\left[\begin{array}{c}
u+v,p+q,p+b,q+b \\
a-v,u-v \\
\end{array}\right] \times \\
\begin{array}{c}
\frac{4F3}{4F3}\left[\begin{array}{c}
u+v,p+q,p+b,q+b \\
a-v,u-v \\
\end{array}\right] \times \\
\frac{4F3}{4F3}\left[\begin{array}{c}
u+v,p+q,p+b,q+b \\
a-v,u-v \\
\end{array}\right] \\
\end{array}
\right.
\]
where \(m\) is a nonnegative integer, to this identity. Then two summands vanish (due \(\Gamma(-m)\) in the denominators), and we obtain \(4F3\)-Whipple transform (see [AAR], Theorem 3.3.3; it is mentioned below in Subsection 3.1).

\[2.6.\text{ Nassrallah–Rahman integral (0.4). If}
\]
\[a = b + u + v + p + q,
\]
then two \(\Gamma\)-factors in the right-hand side of (2.4) cancel. Applying the Second Barnes Lemma (0.3), we obtain (0.4).

\[2.7.\text{ Multiplying the both sides of (2.4) by } \Gamma^{-2}(b) \text{ and passing to the limit as } b \to +\infty, \text{ we obtain}
\]
\[2.5.\int_0^\infty \frac{\Gamma(p+is)\Gamma(q+is)\Gamma(u+is)\Gamma(v+is)}{\Gamma(2is)\Gamma(a+is)} \, ds =
\]
\[= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\Gamma(u+v,p+q)}{a-v,u-v} \, ds
\]
(in the right-hand side we have a linear combination of two functions \(3F2(1))\).

\[\text{§3. Finite systems of orthogonal polynomials.}
\]
\[3.1.\text{ Wilson polynomials. In the famous work [Wil], Wilson constructed the polynomials } p_n(s^2) \text{ orthogonal with respect to the weight}
\]
\[w(s) = \frac{1}{\pi} \left| \frac{\Gamma(a+is)\Gamma(b+is)\Gamma(c+is)\Gamma(d+is)}{\Gamma(2is)} \right|^2;
\]
they are defined by the formula
\[3.1.\quad p_n(a,b,c,d; s^2) =
\]
\[= (a+b)_n(a+c)_n(a+d)_n \quad 4F3\left[\begin{array}{c}
-n,n+a+b+c+d-1,a+is,a-is \\
\end{array}\right] ; 1
\]
The orthogonality relations have the form
\begin{equation}
\int_0^\infty p_n(a, b, c, d; s^2)p_m(a, b, c, d; s^2)w(s)ds = \frac{n!\Gamma(a + b + n)\Gamma(a + c + n)\Gamma(a + d + n)\Gamma(b + c + n)\Gamma(b + d + n)\Gamma(c + d + n)}{\Gamma(a + b + c + d + n)(a + b + c + d + 2n - 1)}\delta_{m,n}
\end{equation}

Evidently, the polynomials \( p_n(a, b, c, d; s^2) \) are symmetric with respect to \( b, c, d \). The symmetry in four indices \( a, b, c, d \) is equivalent to \( _4F_3 \)-Whipple transform mentioned above in Subsection 2.5.

3.2. Proof of orthogonality. Several proofs of orthogonality of the Wilson polynomials are known. Our purpose is to give a model proof that works in 3 cases of “exotic orthogonality” discussed below.

Lemma 3.1. Let \( V, W \) be \( N \)-dimensional linear spaces with bases \( e_j, f_j \) respectively. Define the sesquilinear form \( B(\cdot, \cdot) \) on \( V \times W \) by
\begin{equation}
B(e_k, f_l) = \frac{\Gamma(\nu + k + l)}{\Gamma(\mu + \nu + k + l)}
\end{equation}
where \( \mu, \nu \) are fixed. Then the system of vectors
\begin{align*}
R_n &= \sum_{j=1}^n \frac{(-n)_j(n + \mu + \nu - 1)_j}{(\nu)_j j!}e_j \\
T_n &= \sum_{j=1}^n \frac{(-n)_j(n + \mu + \nu - 1)_j}{(\nu)_j j!}f_j
\end{align*}
is biorthogonal, i.e., \( B(R_k, T_l) = 0 \) for \( k \neq l \). Moreover,
\begin{equation}
B(T_n, R_n) = \frac{n!\Gamma(\nu)\Gamma(\mu + n)}{\Gamma(\nu + n)\Gamma(n + \mu + \nu - 1)(\mu + \nu + 2n - 1)}.
\end{equation}

Proof. This lemma imitates the orthogonality relations for the Jacobi polynomials. Let \( V = W \) be the space of polynomials on the segment \([0, 1]\) with the scalar product
\begin{equation*}
<p, q> = \frac{1}{\Gamma(\mu)}\int_0^1 p(x)q(x)x^{\nu-1}(1 - x)^{\mu-1}dx.
\end{equation*}
Let \( e_n = f_n \) be the function \( x^n \). Then \( <e_k, f_l> \) coincides with (3.3). Next, \( R_n = T_n \) are the Jacobi polynomials \( P^{\mu-1, \nu-1}(2x - 1) \) in the standard notation (see [HTF2], 10.8).

Now let \( V = W \) be the space \( L^2 \) with respect to the Wilson weight \( w(s) \). Assume
\begin{align*}
e_k(s) &= \frac{(a + is)_k(a - is)_k}{\Gamma(a + c + k)\Gamma(a + d + k)}, \\
f_k(s) &= \frac{(b + is)_k(b - is)_k}{\Gamma(b + c + k)\Gamma(b + d + k)}.
\end{align*}
Then by (0.2),
\begin{align*}
\int_0^\infty e_k(s)f_m(s)w(s)ds &= \frac{1}{\pi \Gamma(a + c + k)\Gamma(a + d + k)\Gamma(b + c + m)\Gamma(b + d + m)} \\
&\times \int_0^\infty \left|\frac{\Gamma(a + k + is)\Gamma(b + m + is)\Gamma(c + is)\Gamma(d + is)}{\Gamma(2is)}\right|^2 ds = \\
&= \frac{\Gamma(c + d)\Gamma(a + b + k + m)}{\Gamma(a + b + c + d + k + m)}.
\end{align*}
We obtain the relations of the type (3.3) for scalar products. In notation of Lemma 3.1, we have
\[ R_n = \alpha_n p_n(a, b, c, d; s^2); \quad T_n = \beta_n p_n(b, a, c, d; s^2), \]
where \( \alpha_n, \beta_n \) are normalizing factors (we omit them to be brief). It remains to recall that the Wilson polynomials are symmetric with respect to \( a, b, c, d \). Hence \( R_n \) and \( T_n \) coincide up to a factor (precisely this place of the proof is surprising).

Formula (3.2) follows from (3.4).

3.3. Orthogonality relations associated with the integral (0.1). Consider the weight
\[ w_1(s) = \frac{1}{\pi} \left| \frac{\Gamma(p + is)\Gamma(u + is)\Gamma(v + is)}{\Gamma(2is)\Gamma(q + is)} \right|^2 \]
on the half-line \( s \geq 0 \).

Assume
\[ e_n(s) = \frac{\Gamma(-u - k + q)}{\Gamma(p + u + k)} (u + is)_k (u - is)_k; \]
\[ f_n(s) = \frac{\Gamma(-v - k + q)}{\Gamma(p + v + k)} (v + is)_k (v - is)_k. \]

Then
\[ \int_0^\infty e_k(s)f_l(s)w_1(s)\, ds = \frac{\Gamma(u + v + k + l)\Gamma(q - u - v - p - k - l)}{\Gamma(q - p)} = \]
\[ \frac{(-1)^{k+l}\pi}{\Gamma(q - p)\sin \pi(q - u - v - p)} \cdot \frac{\Gamma(u + v + k + l)}{\Gamma(1 - q + u + v + p + k + l)}. \]

Observe that the scalar products have the form (3.3). Thus we can repeat literally all the considerations of Subsection 3.2.

There exists a way that is even more simple.

**Lemma 3.2.** Let \( \ell \) be a linear functional on the space of even polynomials
\[ c_0 + c_1 s^2 + \cdots + c_N s^{2N} \]
and for
\[ h_k = (a + is)_k (a - is)_k \]
we have
\[ \ell(h_k) = C \cdot \frac{\Gamma(a + b + k)\Gamma(a + c + k)\Gamma(a + d + k)}{\Gamma(a + b + c + d + k)}. \]

Then for \( k + l \leq N \) the Wilson polynomials \( p_k, p_l \) satisfy
\[ \ell(p_k(a, b, c, d; \cdot))p_l(a, b, c, d; \cdot)) = \frac{C}{\Gamma(b + c)\Gamma(b + d)\Gamma(c + d)} \cdot \sigma_k(a, b, c, d)\delta_{k,l}, \]
where $\sigma_n(a, b, c, d)$ is the expression situated in the right-hand side of (3.2).

In other words, the Wilson polynomials form an orthogonal system with respect to the scalar product

\[ \langle f, g \rangle = \ell(fg) \]

in the space of polynomials.

**Proof.** Assume

\[ \ell(g) = \int_0^\infty g(s)w(s)ds, \]

where $w(s)$ is the Wilson weight. Then (3.7) is satisfied and our statement is a rephrasing of the orthogonality relations for the Wilson polynomials. \(\square\)

Now suppose

\[ \ell(g) = \int_0^\infty g(s)w_1(s)ds \]

and

\[ h_k(s) = (p + is)_k(p - is)_k. \]

We have

\[ \ell(h_k) = \frac{\Gamma(p + v + k)\Gamma(p + u + k)\Gamma(u + v)\Gamma(q - u - v - p - k)}{\Gamma(q - p - k)\Gamma(q - u)\Gamma(p - v)} = \]

\[ = \frac{\Gamma(u + v)\sin\pi(p - q)}{\Gamma(u - q)\Gamma(q - v)\sin\pi(u + v + p - q)} \cdot \frac{\Gamma(p + v + k)\Gamma(p + u + k)\Gamma(1 - q + p + k)}{\Gamma(1 - q + u + v + p + k)}. \]

Therefore, the system of the Wilson polynomials

\[ p_n(p, u, v, 1 - q; s^2); \quad 4n < q - p - u - v - 1 \]

is orthogonal with respect to the weight (3.6).

### 3.4. Orthogonality relations associated with Dougall formula.

Consider the weight

\[ w_2(t) = \sum_{-\infty}^{\infty} \frac{\alpha + t}{\prod_{j=1}^4 \Gamma(a_j + \alpha + t)\Gamma(a_j - \alpha - t)} \delta(t - n), \]

where $\delta(s - n)$ is the delta-function supported by the point $n$. For uniformity, substitute $t = is - \alpha$ and transform this expression to the form (3.8)

\[ w_2(s) := \prod_{j=1}^4 \frac{\sin\pi(a_j + \alpha)}{\pi} \cdot (is) \sum_{-\infty}^{\infty} \left\{ \prod_{j=1}^4 \Gamma(1 - a_j - is)\Gamma(1 - a_j + is) \cdot \delta(is - \alpha - n) \right\}. \]

To apply Lemma 3.2, suppose

\[ \ell(g) = \int g(s)w_2(s)ds \]

\[ h_k(s) = (1 - a_1 + is)_k(1 - a_1 - is)_k. \]
Then

\[ \ell(h_k) = \frac{\sin(2\pi \alpha)}{2\pi a_2 + a_3} \frac{\Gamma(a_2 + a_4) \Gamma(3 + a_4)}{\Gamma(a_1 + a_2 + a_3 + a_4 - k - 3)} \times \]
\[ = \frac{\sin(2\pi \alpha) \sin(a_1 + a_2) \sin(a_1 + a_3) \sin(a_1 + a_4)}{2\pi a_2 + a_3 \Gamma(a_2 + a_4) \Gamma(3 + a_4) \sin(a_1 + a_2 + a_3 + a_4)} \times \]
\[ \times \frac{\Gamma(-a_1 - a_2 + k + 2) \Gamma(-a_1 - a_3 + k + 2) \Gamma(-a_1 - a_4 + k + 2)}{\Gamma(4 - a_1 - a_2 - a_3 - a_4 + k)} \].

Thus we obtain that the system of the Wilson polynomials

\[ p_n(1 - a_1, 1 - a_2, 1 - a_3, 1 - a_4; s^2); \quad 4n < a_1 + a_2 + a_3 + a_4 - 3 \]
is orthogonal with respect to the Dougall weight (3.8).

3.5. Orthogonality relations associated with the Askey integral (0.5).

Now consider the Askey weight, i.e., the weight that is the integrand in (0.5)

\[ \frac{\Gamma(1 - 2s) \Gamma(1 + 2s)}{\prod_{j=1}^{4} \Gamma(a_j + s) \Gamma(a_j - s)} \]

In notation of Lemma 3.2,

\[ \ell(g) = \int_{-\infty}^{\infty} g(s) w_3(s) ds, \]

\[ h_k(s) = (1 - a_1 + s)_k (1 - a_1 - s)_k. \]

Then

\[ \ell(h_k) = \frac{1}{\Gamma(a_2 + a_3) \Gamma(a_2 + a_4) \Gamma(a_3 + a_4)} \times \]
\[ \times \frac{\Gamma(a_1 + a_2 + a_3 + a_4 - k - 3)}{\Gamma(a_1 + a_2 - k - 1) \Gamma(a_1 + a_3 - k - 1) \Gamma(a_1 + a_4 - k - 1)} \]

and we obtain the expression coinciding with (3.9) up to a constant factor. Hence, we obtain the same system (3.10) of orthogonal polynomials.

§4. Examples of index integrals

Index integrals are by themselves fairly known subject, see, for instance [Sla]. In this section, we present several amusing index integrals extending the integrals (0.1), (0.2), (2.2)–(2.5).

4.1. The index transform of the function \( (1 + x)^{b-\alpha}(x + y + 1)^{-c-b} \) was evaluated in Lemma 1.1. Using the inversion formula, we get

\[ \int_{0}^{\infty} \left[ \frac{\Gamma(a + is) \Gamma(b + is) \Gamma(c + is)}{\Gamma(2is)} \right]^2 \frac{\Gamma(c + is)}{\Gamma(2is)} \times \]
\[ = \frac{\Gamma(a + b) \Gamma(a + c) \Gamma(b + c)}{(1 + x + y)^{c+b}}. \]
4.2. Writing out the Plancherel formula for the functions \((1+x)^{b-a}(x+y+1)^{-c-b}\) and \((1+x)^{-a-d}\), we obtain (see de Branges [DB], Theorem 13)

\[
\int_0^\infty \left| \frac{\Gamma(a + is)\Gamma(b + is)\Gamma(c + is)\Gamma(d + is)}{\Gamma(2is)} \right|^2 \binom{c + is, c - is}{a + c} (d + is) \, ds = \\
\pi \Gamma(a + b)\Gamma(a + c)\Gamma(a + d)\Gamma(b + c)\Gamma(b + d)\Gamma(c + d) \binom{b + c, c + d}{a + b + c + d} (d + is) \, ds = \\
\frac{\pi \Gamma(a + b)\Gamma(a + c)\Gamma(a + d)\Gamma(b + c)\Gamma(b + d)\Gamma(c + d)}{\Gamma(a + b + c + d)} \binom{b + c, c + d}{a + b + c + d} (d + is) \, ds =
\]

This formula coincides (up to a permutation of letters) with the inversion formula for \(\binom{a + c, a + d}{a + b + c + d} (d + is)\), see Lemma 1.2.c (this implies the Second Barnes Lemma (0.3); evidently, it is not the most simple its proof).

4.3. Applying the Plancherel formula to the pair of functions \((1+x)^{b-a}(1+x+y)^{-b-c}\) and \((1+x)^{-a-d}(1+x+z)^{-b-d}\), we obtain

\[
\frac{1}{\pi} \int_0^\infty \left| \frac{\Gamma(a + is)\Gamma(b + is)\Gamma(c + is)\Gamma(d + is)}{\Gamma(2is)} \right|^2 \times \binom{c - is, c + is}{a + c} (d + is) \, ds = \\
\frac{\pi \Gamma(a + b)\Gamma(a + c)\Gamma(a + d)\Gamma(b + c)\Gamma(b + d)\Gamma(c + d)}{\Gamma(a + b + c + d)} \binom{b + c, c + d}{a + b + c + d} (d + is) \, ds =
\]

In the right-hand side we have one of integral representations of the Appel function \(F_1\). Substituting \(z = y\), we obtain

\[
\frac{1}{\pi} \int_0^\infty \left| \frac{\Gamma(a + is)\Gamma(b + is)\Gamma(c + is)\Gamma(d + is)}{\Gamma(2is)} \right|^2 \times \binom{c - is, c + is}{a + c} (d + is) \, ds = \\
\pi \Gamma(a + b)\Gamma(a + c)\Gamma(b + c)\Gamma(b + d)\Gamma(c + d) \times \binom{b + c, c + d}{a + b + c + d} (d + is) \, ds =
\]

4.4. Applying the Plancherel formula to the pair of functions \((1+x)^{b-a}(1+x+y)^{-c-b}\) and \(\binom{a + c, a + d}{a + b + c + d} (d + is)\), we get

\[
\frac{1}{\pi} \int_0^\infty \left| \frac{\Gamma(a + is)\Gamma(b + is)\Gamma(c + is)\Gamma(d + is)\Gamma(e + is)}{\Gamma(2is)} \right|^2 \binom{a + is, a - is}{a + c} \, ds = \\
\frac{1}{2\pi i} (1 + y)^{a - c} \Gamma(b + c)\Gamma(b + d)\Gamma(c + d) \times \\
\int_{-\infty}^{\infty} \Gamma \left[ \frac{a + b + s, a + c + s, a + d + s, e - a - s, -s}{a + b + c + d + s} \right] (1 + y)^s \, ds
\]

(it is a sum of two functions \(3F_2\)).

4.5. It is easy to extend this list. Even our Lemmas 1.1–1.2 were not completely utilized.
BETA-INTEGRALS AND FINITE ORTHOGONAL SYSTEMS OF WILSON POLYNOMIALS

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Math.Physics Group
Institute of Theoretical and Experimantal Physics
Bol’shaya Cheremushkinskaya, 25
Moscow 117259
Russia
e-mail neretin@main.mccme.rssi.ru