New criteria for Hunt’s hypothesis (H) of Lévy processes

Wei Sun
Department of Mathematics and Statistics
Concordia University
Montreal, H3G 1M8, Canada
wei.sun@concordia.ca

Abstract

In this paper, we present novel necessary and sufficient conditions for the validity of Hunt’s hypothesis (H) of Lévy processes. Based on these criteria, we obtain new examples of Lévy processes satisfying (H).

Keywords: Hunt’s hypothesis (H), Getoor’s conjecture, Lévy processes.

AMS Subject Classification: 60J45, 60G51.

1 Introduction

A Lévy process $X$ on $\mathbb{R}^n$ is said to satisfy Hunt’s hypothesis (H) if every semipolar set of $X$ is polar. More than forty years ago, Professor R.K. Getoor raised the problem that for which Lévy processes semipolar sets are always polar. He conjectured that essentially all Lévy processes satisfy (H). Getoor’s conjecture is the major open problem in the field of potential theory for Lévy processes (cf. e.g. [1, page 70]). In this paper, we will present new criteria for the validity of (H).

Let us start with a brief introduction to Hunt’s hypothesis (H). For simplicity, we consider here only (H) for Lévy processes; however, we should point out that (H) plays a crucial role in the potential theory of (dual) Markov processes. We refer the readers to [3, Chapter VI] for a systematic introduction to (H) for Markov processes.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X = (X_t)_{t \geq 0}$ be a Lévy process on $\mathbb{R}^n$ with Lévy-Khintchine exponent $\psi$, i.e.,

$$E[\exp\{i\langle z, X_t \rangle\}] = \exp\{-t\psi(z)\}, \quad z \in \mathbb{R}^n, \quad t \geq 0.$$
Hereafter $E$ denotes the expectation with respect to $P$ and $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product of $\mathbb{R}^n$. The classical Lévy-Khintchine formula tells us that

$$
\psi(z) = i \langle a, z \rangle + \frac{1}{2} \langle z, Q z \rangle + \int_{\mathbb{R}^n} \left(1 - e^{i\langle z, x \rangle} + i \langle z, x \rangle 1_{\{|x| < 1\}}\right) \mu(dx),
$$

where $a \in \mathbb{R}^n, Q$ is a symmetric nonnegative definite $n \times n$ matrix, and $\mu$ is a measure (called the Lévy measure) on $\mathbb{R}^n\setminus\{0\}$ satisfying $\int_{\mathbb{R}^n\setminus\{0\}} (1 \wedge |x|^2) \mu(dx) < \infty$. We use $\text{Re}(\psi)$ and $\text{Im}(\psi)$ to denote respectively the real and imaginary parts of $\psi$, and use $(a, Q, \mu)$ to denote $\psi$. For $x \in \mathbb{R}^n$, we denote by $P^x$ the law of $x + X$ under $P$. In particular, $P^0 = P$.

Denote by $\mathcal{B}^n$ the family of all nearly Borel measurable subsets of $\mathbb{R}^n$. For $D \subset E$, we denote the first hitting time of $D$ by

$$
\sigma_D := \inf\{t > 0 : X_t \in D\}.
$$

A set $D \subset E$ is called polar if there exists a set $C \in \mathcal{B}^n$ such that $D \subset C$ and $P^x(\sigma_C < \infty) = 0$ for every $x \in \mathbb{R}^n$. $D$ is called a thin set if there exists a set $C \in \mathcal{B}^n$ such that $D \subset C$ and $P^x(\sigma_C = 0) = 0$ for every $x \in \mathbb{R}^n$. $D$ is called semipolar if $D \subset \bigcup_{n=1}^{\infty} D_n$ for some thin sets $\{D_n\}_{n=1}^{\infty}$. $X$ is said to satisfy Hunt’s hypothesis (H) if every semipolar set of $X$ is polar.

To appreciate the importance of (H), we recall below some important principles of potential theory that are equivalent to (H). We refer the readers to [11, Proposition 1.1] for a summary of the proofs. For $\alpha > 0$, a finite $\alpha$-excessive function $f$ on $\mathbb{R}^n$ is called a regular potential provided that $E^x\{e^{-\alpha T_n}f(X_{T_n})\} \to E^x\{e^{-\alpha T}f(X_T)\}$ for $x \in E$ whenever $\{T_n\}$ is an increasing sequence of stopping times with limit $T$. Denote by $(U^\alpha)_{\alpha > 0}$ the resolvent operators for $X$.

- **Bounded maximum principle:** If $\nu$ is a finite measure with compact support $K$ such that $U^\alpha \nu$ is bounded, then $\sup\{U^\alpha \nu(x) : x \in E\} = \sup\{U^\alpha \nu(x) : x \in K\}$.

- **Bounded energy principle:** If $\nu$ is a finite measure with compact support such that $U^\alpha \nu$ is bounded, then $\nu$ does not charge semipolar sets.

- **Bounded regularity principle:** If $\nu$ is a finite measure with compact support such that $U^\alpha \nu$ is bounded, then $U^\alpha \nu$ is regular.

- **Bounded positivity principle:** If $\nu$ is a finite signed measure such that $U^\alpha \nu$ is bounded, then $\nu U^\alpha \nu \geq 0$, where $\nu U^\alpha \nu := \int_E U^\alpha \nu(x) \nu(dx)$.

Hunt’s hypothesis (H) is also equivalent to some other important properties of Markov processes. For example, Blumenthal and Getoor [11, Proposition (4.1)] and Glover [7, Theorem (2.2)] showed that (H) holds if and only if the fine and coarse topologies differ by polar sets; Fitzsimmons and Kanda [5] showed that (H) is equivalent to the dichotomy of capacity.
In spite of its importance, Hunt’s hypothesis (H) has been verified only in special situations. Some forty years ago, Getoor conjectured that essentially all Lévy processes satisfy (H). To motivate this paper, we recall below the results obtained so far for Getoor’s conjecture that are closely related to our results.

Blumenthal and Getoor [4] showed that all stable processes with index \( \alpha \in (0, 2) \) on the line satisfy (H). Kanda [12] and Forst [6] proved independently that (H) holds if \( X \) has bounded continuous transition densities (with respect to the Lebesgue measure \( dx \)) and the Lévy-Khintchine exponent \( \psi \) satisfies

\[
|\text{Im}(\psi)| \leq M(1 + \text{Re}(\psi))
\]

for some constant \( M > 0 \). Rao [14] gave a short proof of the Kanda-Forst theorem under the weaker condition that \( X \) has resolvent densities. In particular, for \( n \geq 1 \), all stable processes with index \( \alpha \neq 1 \) satisfy (H). Kanda [13] proved that (H) holds for stable processes on \( \mathbb{R}^n \) with index \( \alpha = 1 \) if we assume that the linear term vanishes. Glover and Rao [8] proved that \( \alpha \)-subordinates of general Hunt processes satisfy (H) (cf. Proposition 3.3 below). Rao [15] proved that if all 1-excessive functions of \( X \) are lower semicontinuous and

\[
|\text{Im}(\psi)| \leq (1 + \text{Re}(\psi))f(1 + \text{Re}(\psi)),
\]

where \( f \) is an increasing function on \([1, \infty)\) such that \( \int_N^\infty (\lambda f(\lambda))^{-1} d\lambda = \infty \) for every \( N \geq 1 \), then \( X \) satisfies (H).

Hu and Sun [10] showed that if \( Q \) is non-degenerate then \( X \) satisfies (H); if \( Q \) is degenerate then, under the assumption that \( \mu(\mathbb{R}^n \setminus \sqrt{Q}\mathbb{R}^n) < \infty \), \( X \) satisfies (H) if and only if the equation

\[
\sqrt{Q}y = -a - \int_{\{x \in \mathbb{R}^n \setminus \sqrt{Q}\mathbb{R}^n : |x| < 1\}} x\mu(dx)
\]

has at least one solution \( y \in \mathbb{R}^n \). They also showed that if \( X \) is a subordinator satisfying (H) then its drift coefficient must be 0. Recently Hu, Sun and Zhang [11] gave a comparison result on Lévy processes which implies that big jumps have no effect on the validity of (H). Also, they gave a new necessary and sufficient condition for (H) (cf. Proposition 2.1 below) and obtained an extended Kanda-Forst-Rao theorem.

In the next section of this paper, we will present novel necessary and sufficient conditions for the validity of (H) for Lévy processes. It is a bit surprising to us that these criteria were not known until now. Based on these criteria, we will give in Section 3 new examples of Lévy processes satisfying (H).

2 New criteria for Hunt’s hypothesis (H)

Let \( X \) be a Lévy process on \( \mathbb{R}^n \) with Lévy-Khintchine exponent \( \psi \). From now on we assume that all 1-excessive functions are lower semicontinuous, equivalently,
X has resolvent densities. We refer the readers to [9, Theorem 2.1] for more characterizations of this weak assumption.

We define
\[ A := 1 + \text{Re}(\psi), \quad B := |1 + \psi|. \]

For a finite (positive) measure \( \nu \) on \( \mathbb{R}^n \), we denote
\[ \hat{\nu}(z) := \int_{\mathbb{R}^n} e^{i(z \cdot x)} \nu(dx). \]

\( \nu \) is said to have finite 1-energy if
\[ \int_{\mathbb{R}^n} \frac{A(z)}{B^2(z)} |\hat{\nu}(z)|^2 dz < \infty. \]  

Let \( \nu \) be a finite measure on \( \mathbb{R}^n \) of finite 1-energy. For \( \lambda > 0 \), define
\[ c(\lambda) := \int_{\mathbb{R}^n} \frac{\lambda}{\lambda^2 + B^2(z)} |\hat{\nu}(z)|^2 dz. \]

Then, we have (see [15, Theorem 1 and the proof of Theorem 2])
\[ \lim_{\lambda \to \infty} c(\lambda) = \lim_{\lambda \to \infty} \int_{\mathbb{R}^n} \frac{\lambda}{\lambda^2 + B^2(z)} |\hat{\nu}(z)|^2 1_{\{A(z) \leq \lambda\}} dz < \infty. \]  

The remarkable result of Rao ([15 Theorem 1]) tells us that whether \( X \) satisfies (H) depends on if the limit in (2.2) equals 0.

By [11, Theorems 4.3 and 5.1], we get the following proposition.

**Proposition 2.1.** Let \( f \) be an increasing function on \([1, \infty)\) such that \( \int_N^{\infty} (\lambda f(\lambda))^{-1} d\lambda = \infty \) for some \( N \geq 1 \). Then (H) holds for \( X \) if and only if
\[ \lim_{\lambda \to \infty} \int_{\{B(z) > A(z)f(A(z))\}} \frac{\lambda}{\lambda^2 + B^2(z)} |\hat{\nu}(z)|^2 dz = 0 \]  

for any finite measure \( \nu \) of finite 1-energy.

For \( \varsigma > 1 \), we define
\[ N_x^\varsigma := \varsigma^{(\varsigma x)} \text{ for } x \geq 0. \]

To simplify notations, we use \( N \) to denote \( N^\varsigma \) whenever there is no confusion caused. Throughout this paper, we use log to denote \( \log_e \).

**Theorem 2.2.** Let \( \nu \) be a finite measure on \( \mathbb{R}^n \) of finite 1-energy. Then, the following condition is fulfilled for any \( \delta > 0 \).

Condition \((C^\delta)\) : \[ \int_{\mathbb{R}^n} \frac{1}{B(z) \log(2 + B(z)) [\log \log(2 + B(z))]^{1+\delta}} |\hat{\nu}(z)|^2 dz < \infty. \]
Proof. We fix a $\delta > 0$ and let $\nu$ be an arbitrary finite measure on $\mathbb{R}^n$ of finite 1-energy. Denote

$$ F := \{ z \in \mathbb{R}^n : B(z) \geq 4A(z) \}. $$

To prove the theorem, it suffices to show that

$$ \int_F \frac{1}{B(z) \log(B(z)) [\log \log(B(z))]^{1+\delta}} |\nabla(z)|^2 dz < \infty. $$

Set $N_k = 2^{(2^k)}$ for $k \geq 1$. Define $f^\delta(\lambda) = 1$ when $1 \leq \lambda < 4$ and

$$ f^\delta(\lambda) = k^{1+\delta}(\log(N_k) - \log(N_{k-1})) \quad \text{when} \quad N_{k-1} \leq \lambda < N_k, \quad k \geq 2. $$

Then, we obtain by (2.2) that

$$ \int_1^\infty \frac{d\lambda}{\lambda f^\delta(\lambda)} \int_{\mathbb{R}^n} \lambda \frac{1}{A(z)} |\nabla(z)|^2 1_{\{A(z) \leq \lambda\}} dz = \int_{\mathbb{R}^n} |\nabla(z)|^2 dz \int_1^\infty \frac{d\lambda}{f^\delta(\lambda)(\lambda^2 + B^2(z))} = \int_{\mathbb{R}^n} \frac{A(z)}{B^2(z)} |\nabla(z)|^2 dz \int_1^\infty \frac{d\lambda}{\alpha(\lambda^2 + B^2(z))} \geq \int_{\mathbb{R}^n} \frac{A(z)}{B^2(z)} |\nabla(z)|^2 dz \sum_{k=2}^\infty \int_{\{1 \leq \lambda < \frac{N_k}{A(z)} \}} k^{1+\delta}(\log(N_k) - \log(N_{k-1}))(\eta^2 + \frac{B(z)}{A(z)})^2 \frac{d\eta}{\alpha(\lambda^2 + B^2(z))} = \int_{\mathbb{R}^n} \frac{1}{B(z)} |\nabla(z)|^2 dz \sum_{k=2}^\infty \arctan \frac{N_k}{B(z)} - \arctan \frac{N_k}{B(z)} \frac{\log(N_k) - \log(N_{k-1})}{k^{1+\delta}} \geq \sum_{k_0=2}^\infty \int_{\{N_{k_0-1} \leq B(z) < N_{k_0} \}} \frac{1}{B(z)} |\nabla(z)|^2 dz \sum_{k=2}^\infty \arctan \frac{N_k}{B(z)} - \arctan \frac{N_k}{B(z)} \frac{\log(N_k) - \log(N_{k-1})}{k^{1+\delta}} \geq \sum_{k_0=2}^\infty \int_{\{N_{k_0-1} \leq B(z) < N_{k_0} \}} \frac{1}{B(z)} |\nabla(z)|^2 dz \arctan \frac{N_k}{B(z)} - \arctan \frac{N_k}{B(z)} \frac{\log(N_k) - \log(N_{k_0-1})}{k_0^{1+\delta}} + \int_{\{N_{k_0-1} \leq B(z) < N_{k_0} \}} \arctan 2 - \arctan 1 \int_{\{z \in F : N_{k_0-1} \leq B(z) < N_{k_0} \}} 1 \frac{B(z)}{\log(N_k) - \log(N_{k_0-1})} $$

5
Therefore, Condition (C_δ) is fulfilled. \hfill \Box

Theorem 2.3. (i) $X$ satisfies (H) if the following condition holds:

**Condition (C^{log})**: For any finite measure $\nu$ on $\mathbb{R}^n$ of finite 1-energy, there exist a constant $\varsigma > 1$ and a sequence $\{y_k \uparrow \infty\}$ such that $y_1 > 1$ and

$$\sum_{k=1}^{\infty} \int \left\{ y_k \leq B(z) < (y_k)^\varsigma \right\} \frac{1}{B(z) \log B(z)} |\hat{\nu}(z)|^2 \, dz < \infty. \tag{2.4}$$

(ii) Suppose $X$ satisfies (H). Then, for any finite measure $\nu$ on $\mathbb{R}^n$ of finite 1-energy and any $\varsigma > 1$, there exists a sequence $\{y_k \uparrow \infty\}$ such that $y_1 > 1$ and (2.4) holds.

**Proof. Assertion (i):**

Let $\nu$ be an arbitrary finite measure on $\mathbb{R}^n$ of finite 1-energy. We choose a constant $\varsigma > 1$ and a sequence $\{y_k \uparrow \infty\}$ described as in Condition (C^{log}). We assume without loss of generality that $y_1 > \varsigma$ and $(y_k)^\varsigma < y_{k+1}$ for $k \in \mathbb{N}$. Set

$$x_k = \log(\varsigma \log y_k) \quad \text{for} \quad k \in \mathbb{N}.$$  

Then $x_k + 1 < x_{k+1}$ for each $k \in \mathbb{N}$.

By (2.1), (2.2) and the dominated convergence theorem, we find that

$$\lim_{\lambda \to \infty} \int \left\{ B(z) \geq N_{x_k} \right\} \frac{\lambda}{\lambda^2 + B^2(z)} |\hat{\nu}(z)|^2 \, dz = \lim_{\lambda \to \infty} c(\lambda) < \infty. \tag{2.5}$$

We will show below that $\lim_{\lambda \to \infty} \int \left\{ B(z) \geq N_{x_k} \right\} \frac{\lambda}{\lambda^2 + B^2(z)} |\hat{\nu}(z)|^2 \, dz = 0$.

For $\lambda > 1$, we define

$$g(\lambda) := \sum_{k=1}^{\infty} \frac{1_{\{N_{s_k+1} \leq \lambda < N_{s_k+4}\}}}{\log \lambda}.$$
Then, we have
\[
\int_{N_{x_1}}^{\infty} \frac{g(\lambda)}{\lambda} \int_{\{B(z) \geq N_{x_1}\}} \frac{\lambda}{\lambda^2 + B^2(z)} |\hat{\varphi}(z)|^2 dz
= \int_{\{B(z) \geq N_{x_1}\}} |\hat{\varphi}(z)|^2 dz \int_{N_{x_1}}^{\infty} \frac{g(\lambda) d\lambda}{\lambda^2 + B^2(z)}
= \int_{\{B(z) \geq N_{x_1}\}} |\hat{\varphi}(z)|^2 dz \int_{N_{x_1}}^{\infty} \frac{g(A(z) \eta) A(z) d\eta}{A^2(z) \eta^2 + B^2(z)}
= \int_{\{B(z) \geq N_{x_1}\}} \frac{A(z)}{B^2(z)} |\hat{\varphi}(z)|^2 dz \int_{N_{x_1}}^{\infty} \frac{g(A(z) \eta) \left(\frac{B(z)}{A(z)}\right)^2 d\eta}{\eta^2 + \left(\frac{B(z)}{A(z)}\right)^2}
\leq \int_{\{B(z) \geq N_{x_1}\}} \frac{A(z)}{B^2(z)} |\hat{\varphi}(z)|^2 dz \sum_{k=1}^{\infty} \int_{\left\{\frac{N_{x_k} + \frac{3}{2}}{B(z)} \leq \eta < \frac{N_{x_k} + \frac{5}{2}}{A(z)}\right\}} \frac{\left(\frac{B(z)}{A(z)}\right)^2 d\eta}{\log \left(N_{x_k+\frac{1}{2}}\right)}
= \int_{\{B(z) \geq N_{x_1}\}} \frac{1}{B(z)} |\hat{\varphi}(z)|^2 dz \sum_{k=1}^{\infty} \arctan \frac{N_{x_k + \frac{3}{2}}}{B(z)} - \arctan \frac{N_{x_k + \frac{5}{2}}}{B(z)}
+ \sum_{l=1}^{\infty} \int_{\{N_{x_l+1} \leq B(z) < N_{x_{l+1}}\}} \frac{1}{B(z)} |\hat{\varphi}(z)|^2 dz \sum_{k=1}^{\infty} \arctan \frac{N_{x_k + \frac{3}{2}}}{B(z)} - \arctan \frac{N_{x_k + \frac{5}{2}}}{B(z)}
\sum_{k=1}^{\infty} \arctan \frac{N_{x_k + \frac{3}{2}}}{B(z)} - \arctan \frac{N_{x_k + \frac{5}{2}}}{B(z)}
= \sum_{l=1}^{\infty} \int_{\{N_{x_l} \leq B(z) < N_{x_{l+1}}\}} \frac{1}{B(z)} |\hat{\varphi}(z)|^2 dz \left\{ \sum_{k=1}^{l-1} \arctan \frac{N_{x_k + \frac{3}{2}}}{B(z)} - \arctan \frac{N_{x_k + \frac{5}{2}}}{B(z)} \right\}
+ \sum_{k=1}^{\infty} \arctan \frac{N_{x_k + \frac{3}{2}}}{B(z)} - \arctan \frac{N_{x_k + \frac{5}{2}}}{B(z)}
+ \sum_{l=1}^{\infty} \int_{\{N_{x_l+1} \leq B(z) < N_{x_{l+1}}\}} \frac{1}{B(z)} |\hat{\varphi}(z)|^2 dz \left\{ \sum_{k=1}^{l-1} \arctan \frac{N_{x_k + \frac{3}{2}}}{B(z)} - \arctan \frac{N_{x_k + \frac{5}{2}}}{B(z)} \right\}
+ \sum_{k=1}^{\infty} \arctan \frac{N_{x_k + \frac{3}{2}}}{B(z)} - \arctan \frac{N_{x_k + \frac{5}{2}}}{B(z)}
\leq \sum_{l=1}^{\infty} \int_{\{N_{x_l} \leq B(z) < N_{x_{l+1}}\}} \frac{1}{B(z)} |\hat{\varphi}(z)|^2 dz \left\{ \frac{N_{x_{l-1} + \frac{3}{2}}}{B(z)} \sum_{k=1}^{l-1} \frac{1}{\log \left(N_{x_k + \frac{1}{2}}\right)}
+ \frac{\pi}{2 \log \left(N_{x_l + \frac{1}{2}}\right)} \sum_{k=1}^{\infty} \frac{B(z)}{N_{x_k + \frac{1}{2}}} \right\}
\[
\begin{align*}
+ \sum_{l=1}^{\infty} \int_{\{N(x_l) + 1 \leq B(z) < N(x_{l+1})\}} \frac{1}{B(z)} |\hat{\nu}(z)|^2 dz \\
&\quad \left\{ \frac{N(x_l) + \frac{3}{2}}{B(z)} \sum_{k=1}^{l} \frac{1}{\log(N_{x_k} + \frac{1}{2})} \right\} \\
&\quad + \sum_{k=l+1}^{\infty} \frac{B(z)}{N_{x_k} + \frac{1}{2}} \log(N_{x_k} + \frac{1}{2}) \right\}
\end{align*}
\]

\[
\leq \sum_{l=1}^{\infty} \int_{\{N(x_l) \leq B(z) < N(x_{l+1})\}} \frac{1}{B(z)} |\hat{\nu}(z)|^2 dz \left( \frac{1}{\zeta^{x_l + \frac{1}{2}}(\log \zeta)(1 - \frac{1}{x})(B(z))^{1 - \zeta^{-\frac{1}{2}}} + \frac{\pi \zeta^\frac{1}{2}}{2 \log B(z)} + \frac{1}{\zeta^{x_l + \frac{1}{2}}(\log \zeta)(1 - \frac{1}{x})(B(z))^{1 - \zeta^{-\frac{1}{2}}}} \right)
\]

\[
+ \sum_{l=1}^{\infty} \int_{\{N(x_l) + 1 \leq B(z) < N(x_{l+1})\}} \frac{1}{B(z)} |\hat{\nu}(z)|^2 dz \left( \frac{1}{\zeta^{x_l + \frac{1}{2}}(\log \zeta)(1 - \frac{1}{x})(B(z))^{1 - \zeta^{-\frac{1}{2}}} + \frac{\pi \zeta^\frac{1}{2}}{2 \log B(z)} + \frac{1}{\zeta^{x_l + \frac{1}{2}}(\log \zeta)(1 - \frac{1}{x})(B(z))^{1 - \zeta^{-\frac{1}{2}}}} \right)
\]

\[
\leq D_1 \int_{\mathbb{R}^n} \frac{B(z) \log(2 + B(z)) \log \log(2 + B(z))^{1 + \delta} |\hat{\nu}(z)|^2 dz}{B(z) \log B(z)} \quad \text{and Theorem 2.2 that}
\]

\[
\lim_{\lambda \to \infty} \int_{\{B(z) \geq N_{x_k}\}} \frac{\lambda}{\lambda^2 + B^2(z)} |\hat{\nu}(z)|^2 dz = 0.
\]

The proof of Assertion (i) is complete by Proposition 2.1.

Assertion (ii):

Suppose that $X$ satisfies (H). Let $\nu$ be an arbitrary finite measure on $\mathbb{R}^n$ of finite 1-energy and $\zeta > 1$ be a constant. By [11, Theorem 5.1], $\lim_{\lambda \to \infty} c(\lambda) = 0$. We choose an increasing sequence of positive numbers \{\lambda_k\} such that

\[
c(\lambda) \leq \frac{1}{2^k}, \quad \text{if } \lambda \geq x_k.
\]

We assume without loss of generality that $x_k + 1 < x_{k+1}$, $k \in \mathbb{N}$. Set

\[
y_k = N_{x_k}^\nu \quad \text{for } k \in \mathbb{N}.
\]

Denote

\[
F := \{ z \in \mathbb{R}^n : B(z) \geq 2A(z) \}.
\]
To prove (2.4), it suffices to show that

\[
\sum_{k=1}^{\infty} \int_{\{z \in F: N_{x_k} \leq B(z) < N_{x_k+1}\}} \frac{1}{B(z) \log B(z)} |\hat{v}(z)|^2 dz < \infty.
\]

For \(\lambda > 1\), we define \(f(\lambda) = \log \lambda\). Set

\[
\Lambda := \bigcup_{k=1}^{\infty} \{ \lambda : N_{x_k} \leq \lambda < N_{x_k+1}\}.
\]

By (2.7), we get

\[
\begin{align*}
\sum_{k=1}^{\infty} \int_{\Lambda \cap \{A(z) \leq \lambda\}} \frac{1}{\lambda^2 + B^2(z)} |\hat{v}(z)|^2 1_{\{A(z) \leq \lambda\}} dz & \\
= \int_{R^n} |\hat{v}(z)|^2 dz \int_{\Lambda \cap \{A(z) \leq \lambda\}} \frac{1}{\lambda^2 + B^2(z)} d\lambda & \\
= \int_{R^n} |\hat{v}(z)|^2 dz \int_{\Lambda \cap \{A(z) \leq \lambda\}} \frac{1}{\lambda^2 + B^2(z)} d\lambda & \\
= \int_{R^n} |\hat{v}(z)|^2 dz \int_{\Lambda \cap \{A(z) \leq \lambda\}} \frac{1}{\lambda^2 + B^2(z)} d\lambda & \\
\geq \int_{R^n} |\hat{v}(z)|^2 dz \sum_{k=1}^{\infty} \int_{\{1 \leq \lambda < N_{x_k+1}\}} \frac{1}{\lambda^2 + B^2(z)} d\lambda & \\
= \int_{R^n} |\hat{v}(z)|^2 dz \sum_{k=1}^{\infty} \int_{\{1 \leq \lambda < N_{x_k+1}\}} \frac{1}{\lambda^2 + B^2(z)} d\lambda & \\
\geq \sum_{k=1}^{\infty} \int_{\{z \in F: N_{x_k} \leq B(z) < N_{x_k+1}\}} \frac{1}{B(z)} |\hat{v}(z)|^2 dz (\arctan \frac{N_{x_k+1}}{B(z)} - \arctan (\frac{N_{x_k}}{B(z)})) & \\
= \sum_{k=1}^{\infty} \int_{\{z \in F: N_{x_k} \leq B(z) < N_{x_k+1}\}} \frac{1}{B(z)} |\hat{v}(z)|^2 dz (\arctan \frac{N_{x_k+1}}{B(z)} - \arctan (\frac{N_{x_k}}{B(z)})) & \\
+ \int_{\{z \in F: N_{x_k+1} \leq B(z) < N_{x_k+1}\}} \frac{1}{B(z)} |\hat{v}(z)|^2 dz (\arctan 2 - \arctan \frac{1}{2}) & \\
\geq \sum_{k=1}^{\infty} \int_{\{z \in F: N_{x_k} \leq B(z) < N_{x_k+1}\}} \frac{1}{B(z)} |\hat{v}(z)|^2 dz (\arctan 2 - \arctan \frac{1}{2}) & \\
+ \int_{\{z \in F: N_{x_k+1} \leq B(z) < N_{x_k+1}\}} \frac{1}{B(z)} |\hat{v}(z)|^2 dz (\arctan 1 - \arctan \frac{1}{2}) & \\
\geq \sum_{k=1}^{\infty} \int_{\{z \in F: N_{x_k} \leq B(z) < N_{x_k+1}\}} \frac{1}{B(z)} |\hat{v}(z)|^2 dz (\arctan 1 - \arctan \frac{1}{2}) & \\
\end{align*}
\]
The proof of Assertion (ii) is complete.

**Theorem 2.4.** (i) \(X\) satisfies (H) if the following condition holds:

**Condition \((C_{\log \log})\):** For any finite measure \(\nu\) on \(\mathbb{R}^n\) of finite 1-energy, there exist a constant \(\varsigma > 1\) and a sequence of positive numbers \(\{x_k\}\) such that \(N_{\varsigma x_1} > e, x_k + 1 < x_{k+1}, k \in \mathbb{N}, \sum_{k=1}^{\infty} \frac{1}{x_k} = \infty,\) and

\[
\sum_{k=1}^{\infty} \int_{\{N_{\varsigma x_k} \leq B(z) < N_{\varsigma x_{k+1}}\}} \frac{1}{B(z) \log B(z) \log \log B(z)} |\hat{\nu}(z)|^2 \, dz < \infty. \tag{2.8}
\]

(ii) Suppose \(X\) satisfies (H). Then, for any finite measure \(\nu\) on \(\mathbb{R}^n\) of finite 1-energy and any \(\varsigma > 1\), there exists a sequence of positive numbers \(\{x_k\}\) such that \(N_{\varsigma x_1} > e, x_k + 1 < x_{k+1}, k \in \mathbb{N}, \sum_{k=1}^{\infty} \frac{1}{x_k} = \infty,\) and (2.8) holds.

The proof of Theorem 2.4 is similar to, but more delicate than that of Theorem 2.3. We put it in the Appendix.

We define

**Condition \((C_0)\):** For any finite measure \(\nu\) on \(\mathbb{R}^n\) of finite 1-energy,

\[
\int_{\mathbb{R}^n} \frac{1}{B(z) \log(2 + B(z)) \log \log(2 + B(z))} |\hat{\nu}(z)|^2 \, dz < \infty,
\]

and

**Condition \((C_{B/A})\):** There exists a constant \(C > 0\) such that

\[
B(z) \leq CA(z) \log(2 + B(z)) \log \log(2 + B(z)), \quad \forall z \in \mathbb{R}^n.
\]

**Corollary 2.5.** **Condition \((C_{B/A})\) \Rightarrow Condition \((C_0)\) \Rightarrow (H).**

**Proof.** This is a direct consequence of Theorem 2.4.

### 3 Some consequences and examples

Theorems 2.3 and 2.4 provide novel necessary and sufficient conditions for the validity of Hunt’s hypothesis (H) for Lévy processes. Different from the classical Kanda-Forst condition (1.1) and Rao’s condition (1.2), our Conditions \((C_{\log})\) and
(C\textsuperscript{log log}) require only that Im(ψ) is partially well-controlled by 1 + Re(ψ). These weaker conditions are fulfilled by more general Lévy processes and reveal the more essential reason for the validity of (H).

In this section, we will give new examples of Lévy processes satisfying (H). The purpose of these preliminary examples is to exhibit the strength of the criteria obtained in Section 2. Further examples should be expected. In fact, based on Theorems 2.3 and 2.4, there are two ways to further explore Getoor’s conjecture. One the one hand, we can construct more general examples of Lévy processes satisfying Condition (C\textsuperscript{log}), or Condition (C\textsuperscript{log log}), and hence satisfy (H). One the other hand, we can consider if there exists a (maybe pathological) Lévy process which does not satisfy Condition (C\textsuperscript{log}) and hence does not satisfy (H).

First, we give the following consequence of Theorem 2.3.

**Proposition 3.1.** X satisfies (H) if the following conditions hold:

(i) \(\frac{1}{c}|z|^{\alpha_1} \leq A(z) \leq B(z) \leq c|z|^{\alpha_2}\) for \(|z| \geq 1\), where \(0 < \alpha_1 < \alpha_2 \leq 2\) and \(c > 1\) are constants.

(ii) There exist \(\varsigma > 1\), \(\kappa > 0\), and a sequence \(\{z_k\}\) such that \(z_1 > 1\), \(c^{\frac{\varsigma}{\alpha_1}} z_k^{\alpha_2} \alpha_1 < z_{k+1}\), \(k \in \mathbb{N}\), and

\[B(z) \leq \kappa A(z) \log(B(z)), \quad \text{for } z_k \leq |z| < c^{\frac{\varsigma}{\alpha_1}} z_k^{\alpha_2} \alpha_1, \quad k \in \mathbb{N}.\]

**Proof.** Suppose Conditions (i) and (ii) hold. Let \(\nu\) be a finite measure on \(\mathbb{R}^n\) of finite 1-energy. Set

\[y_k = cz_k^{\alpha_2}, \quad k \in \mathbb{N}.\]

Then,

\[
\sum_{k=1}^{\infty} \int_{\{y_k \leq B(z) < (y_k)^{\varsigma}\}} \frac{1}{B(z) \log B(z)} |\hat{\nu}(z)|^2 dz \\
\leq \sum_{k=1}^{\infty} \int_{\{z_k \leq |z| < c^{\frac{\varsigma}{\alpha_1}} z_k^{\alpha_2}\}} \frac{\kappa A(z)}{B^2(z)} |\hat{\nu}(z)|^2 dz + \int_{\{z: |z| < 1, B(z) \geq c\}} \frac{1}{B(z) \log B(z)} |\hat{\nu}(z)|^2 dz \\
\leq \int_{\mathbb{R}^n} \frac{\kappa A(z)}{B^2(z)} |\hat{\nu}(z)|^2 dz + \frac{\nu(\mathbb{R}^n)^2}{c \log c} \int_{\{|z| < 1\}} dz \\
< \infty.
\]

Hence Condition (C\textsuperscript{log}) is fulfilled and the proof is complete by Theorem 2.3(i).

**Remark 3.2.** Blumenthal and Getoor introduced in \[2\] different indexes for Lévy processes on \(\mathbb{R}^n\). In particular, they defined

\[\beta = \inf \left\{ \alpha \geq 0 : \frac{|\psi(z)|}{|z|^{\alpha}} \rightarrow 0 \quad \text{as} \quad |z| \rightarrow \infty \right\},\]

and

\[\beta'' = \sup \left\{ \alpha \geq 0 : \frac{\text{Re} \psi(z)}{|z|^{\alpha}} \rightarrow \infty \quad \text{as} \quad |z| \rightarrow \infty \right\}.
\]
It is well known that all stable processes with index $\alpha \in (0, 2]$ satisfy (H) (cf. Section 1 for the references). The main unknown case of Getoor’s conjecture is whether Hunt’s hypothesis (H) holds for Lévy processes with indexes $\beta'' < \beta$. Our Proposition 3.1 provides a sufficient condition for the validity of (H) in this case.

Note that in Condition (ii) of Proposition 3.1, $z_{k+1}$ may be chosen to be much bigger than $z_k$ for each $k \in \mathbb{N}$. Hence Proposition 3.1 can be used to construct a large class of Lévy processes with indexes $\beta'' < \beta$ and satisfying (H). We will give a new class of subordinators satisfying (H) in Example 3.4 below.

To the best of our knowledge, which subordinators satisfy (H) is unknown in general. To appreciate the importance of this problem, we recall below the remarkable result obtained by Glover and Rao.

**Proposition 3.3.** (Glover and Rao [8]) Let $(X_t)_{t \geq 0}$ be a standard Markov process on a locally compact space with a countable base and $(T_t)_{t \geq 0}$ be an independent subordinator satisfying Hunt’s hypothesis (H). Then $(X_{T_t})_{t \geq 0}$ satisfies (H).

**Example 3.4.** Let $0 < \alpha_1 < \alpha_2 < 1$, $c_1 > 1$, $\varsigma > 1$ and $\kappa_1 > 0$. Denote

$$c := c_1 \left( \frac{2}{\alpha_2} + \frac{1}{1 - \alpha_2} + 8 \right).$$

We choose a sequence $\{z_k\}$ satisfying $z_1 > 1$, $c^\varsigma z_k^{\alpha_1} < z_{k+1}$, $k \in \mathbb{N}$.

Let $X$ be a pure-jump subordinator with Lévy measure $\mu(dx) := \rho(x)dx$. Suppose $\rho$ satisfies the following conditions:

(i) $\frac{1}{c_1 x^{1+\alpha_1}} \leq \rho(x) \leq \frac{c_1}{x^{1+\alpha_2}}$ for $0 < x \leq 1$; and $\rho(x) = 0$, otherwise.

(ii) $\rho(x) \geq \frac{\kappa_1}{x^{1+\alpha_2}}$ for $\frac{1}{c_1 x^{1+\alpha_1}} \frac{1}{z_k^{\alpha_1}} \leq x < \frac{1}{z_k}$, $k \in \mathbb{N}$.

Then $X$ satisfies (H).

In fact, for $z \in \mathbb{R}$ with $|z| \geq 1$, we have

$$A(z) = 1 + \text{Re}\psi(z)$$
$$= 1 + \int_0^\infty (1 - \cos(zx)) \mu(dx)$$
$$\geq \int_0^{1/|z|} (1 - \cos(zx)) \frac{1}{c_1 x^{1+\alpha_1}} dx$$
$$\geq \int_0^{1/|z|} \frac{|z|^2 x^2}{4c_1 x^{1+\alpha_1}} dx$$
$$\geq \frac{1}{8c_1} |z|^{\alpha_1},$$

and

$$B(z) = |1 + \psi(z)|$$
\[ A(z) + |\text{Im}\psi(z)| \leq 1 + \int_0^\infty (1 - \cos(zx))\mu(dx) + \int_0^\infty |\sin(zx)|\mu(dx) \leq 1 + \int_{\frac{1}{|z|^2}}^\infty \left( \frac{|z|^2 x^2}{2} + |z|x \right) \frac{c_1}{x^{1+\alpha_2}} dx + 2 \int_{\frac{1}{|z|^2}}^1 \frac{c_1}{x^{1+\alpha_2}} dx \leq c_1 \left( \frac{2}{\alpha_2} + \frac{1}{1 - \alpha_2} + \frac{1}{2} \right) |z|^{\alpha_2}. \]

For \( z_k \leq |z| < c^{\frac{\alpha_1}{\alpha_2}} z_k^{\alpha_1} \), \( k \in \mathbb{N} \), we have

\[ A(z) \geq \int_{\frac{1}{|z|^2}}^\infty (1 - \cos(zx)) \frac{\kappa_1}{x^{1+\alpha_2}} dx \geq \int_{\frac{1}{|z|^2}}^1 \kappa_1 |z|^2 x^2 \frac{4}{4x^{1+\alpha_2}} dx \geq \frac{\kappa_1}{16} |z|^{\alpha_2}. \]

Hence Conditions (i) and (ii) of Proposition 3.1 are fulfilled and therefore \( X \) satisfies (H).

By Corollary 2.5, we obtain the following proposition.

**Proposition 3.5.** Let \( X \) be a Lévy process on \( \mathbb{R} \). Suppose that

\[ \liminf_{|z| \to \infty} \frac{|\psi(z)|}{|z|(\log \log |z|)^\delta} > 0 \]  \( \text{for some constant} \ \delta > 0. \) Then \( X \) satisfies (H).

**Example 3.6.** Let \( X \) be a Lévy process on \( \mathbb{R} \) with Lévy-Khintchine exponent \((a, Q, \mu)\). Suppose that there exist constants \( \delta > 0 \) and \( c > 0 \) such that

\[ d\mu \geq \frac{c[\log(-\log |x|)]^\delta}{x^2} dx \quad \text{on} \quad \left\{ x \in \mathbb{R} : 0 < |x| < \frac{1}{e} \right\}. \]

Similar to (3.1), we can show that (3.2) holds. Therefore, \( X \) satisfies (H) by Proposition 3.5. Note that in this example it does not matter if \( a \) or \( Q \) equals 0.

Finally, we give a result on the validity of Hunt’s hypothesis (H) for perturbed Lévy processes.

**Proposition 3.7.** Let \( X_1 \) and \( X_2 \) be two independent Lévy processes on \( \mathbb{R}^n \) with respective Lévy-Khintchine exponents \( \psi_1 \) and \( \psi_2 \). Suppose that \( X_2 \) satisfies (H) and there exists a constant \( c > 0 \) such that

\[ |\psi_1| \leq c(1 + \text{Re}(\psi_2)). \]  \( \text{for some constant} \ \delta > 0. \) Then, \( X_1 + X_2 \) satisfies (H).
Proof. Denote by \( \psi \) the Lévy-Khintchine exponent of \( X := X_1 + X_2 \). Then \( A(z) = 1 + \text{Re} \psi_1(z) + \text{Re} \psi_2(z) \) and \( B(z) = |1 + \psi_1(z) + \psi_2(z)| \) for \( z \in \mathbb{R}^n \). Let \( \nu \) be a finite measure of finite 1-energy. We assume without loss of generality that \( c > 1 \) and define \( f \equiv 10c \). We will show that (2.3) holds and hence \( X \) satisfies (H) by Proposition 2.1.

Note that (3.3) implies that
\[
(\text{Im} \psi_1(z))^2 \leq c^2 A^2(z).
\] (3.4)

Suppose \( B(z) > A(z)f(A(z)) \). Then,
\[
(\text{Im} \psi_2(z) + \text{Im} \psi_1(z))^2 > (100c^2 - 1) A^2(z).
\] (3.5)

By (3.4) and (3.5), we get
\[
(\text{Im} \psi_2(z) + \text{Im} \psi_1(z))^2 > \left(100 - \frac{1}{c^2}\right)(\text{Im} \psi_1(z))^2,
\]
which implies that
\[
(\text{Im} \psi_2(z))^2 + (\text{Im} \psi_1(z))^2 > \left(50 - \frac{1}{2c^2}\right)(\text{Im} \psi_1(z))^2
\]
and hence
\[
(\text{Im} \psi_2(z))^2 > \left(49 - \frac{1}{2c^2}\right)(\text{Im} \psi_1(z))^2.
\] (3.6)

By (3.6), we get
\[
(\text{Im} \psi_2(z) + \text{Im} \psi_1(z))^2 \geq \frac{(\text{Im} \psi_2(z))^2}{4} - (\text{Im} \psi_1(z))^2
\geq \left(\frac{1}{4} - \frac{1}{49 - \frac{1}{2c^2}}\right) \text{Im} \psi_2(z))^2.
\] (3.7)

By (3.7), we get
\[
\int_{\{B(z) > A(z)f(A(z))\}} \frac{\lambda}{\lambda^2 + (\text{Im} \psi(z))^2} |\hat{\nu}(z)|^2 dz \leq c' \int_{\mathbb{R}^n} \frac{\lambda}{\lambda^2 + (\text{Im} \psi_2(z))^2} |\hat{\nu}(z)|^2 dz
\] (3.8)

for some constant \( c' > 0 \). By (3.3) and the assumption that \( \nu \) has finite 1-energy, we have
\[
\int_{\mathbb{R}^n} \frac{1 + \text{Re} \psi_2(z)}{|1 + \psi_2(z)|^2} |\hat{\nu}(z)|^2 dz < \infty.
\] (3.9)

Therefore, the limit in (2.3) equals 0 by (3.8), (3.9) and the assumption that \( X_2 \) satisfies (H). The proof is complete. \( \blacksquare \)
Remark 3.8. In Proposition 3.7, if we assume that $X_2$ satisfies the Kanda-Forst condition (1.1) or Rao’s condition (1.2), then the result is obtained by the corresponding theorems of Kanda-Forst and Rao. But our Proposition 3.7 can be applied to general Lévy processes $X_2$ satisfying (H), e.g., the $X_2$ satisfying Conditions (i) and (ii) of Proposition 3.7.

Example 3.9. Let $X_1$ and $X_2$ be two independent Lévy processes on $\mathbb{R}^n$ with respective Lévy-Khintchine exponents $\psi_1$ and $\psi_2$. Suppose that $\text{Re}\, \psi_2(z) \geq c_1 |z|^\alpha$ for $|z| \geq 1$, where $0 < \alpha \leq 2$, $c_1 > 0$ are constants, and $X_2$ satisfies (H) (for example, $X_2$ is a subordinator as given in Example 3.4). Suppose $|\psi_1(z)| \leq c_2 |z|^\alpha$ for $|z| \geq 1$, where $c_2 > 0$ is a constant. Then, $X_1 + X_2$ satisfies (H) by Proposition 3.7.

4 Appendix

Proof of Theorem 2.4

Assertion (i):

Let $\nu$ be an arbitrary finite measure on $\mathbb{R}^n$ of finite 1-energy. We choose a constant $\varsigma$ and a sequence $\{x_k\}$ described as in Condition (C$\log\log$). We assume without loss of generality that $x_1 > 2$. We will show below that

$$\lim_{\lambda \to \infty} \int \frac{N_{x_k+\frac{1}{2}} \leq \lambda < N_{x_k+\frac{3}{2}}}{x_k \log \lambda} g(\lambda) d\lambda = 0.$$ 

Then, we have

$$\int_{N_{x_1}}^{\infty} \frac{g(\lambda) d\lambda}{\lambda} \int_{\{B(z) \geq N_{x_1}\}} \frac{\lambda}{\lambda^2 + B^2(z)} |\hat{\nu}(z)|^2 dz$$

$$= \int_{\{B(z) \geq N_{x_1}\}} |\hat{\nu}(z)|^2 dz \int_{N_{x_1}}^{\infty} \frac{g(\lambda) d\lambda}{\lambda^2 + B^2(z)}$$

$$= \int_{\{B(z) \geq N_{x_1}\}} A(z) \frac{B^2(z)}{B^2(z)} |\hat{\nu}(z)|^2 dz \int_{\{B(z) \geq N_{x_1}\}} A(z) \frac{B(z) \eta}{A(z)} \frac{B(z) \eta}{A(z)} d\eta$$

$$\leq \int_{\{B(z) \geq N_{x_1}\}} A(z) \frac{B^2(z)}{B^2(z)} |\hat{\nu}(z)|^2 dz \sum_{k=1}^{\infty} \int_{\frac{N_{x_k+\frac{1}{2}}}{A(z)} \leq \eta < \frac{N_{x_k+\frac{3}{2}}}{A(z)}} \frac{1}{x_k \log \left(\frac{N_{x_k+\frac{3}{2}}}{B(z)}\right)} - \frac{1}{x_k \log \left(\frac{N_{x_k+\frac{1}{2}}}{B(z)}\right)} dx_k \log \left(\frac{N_{x_k+\frac{3}{2}}}{B(z)}\right)$$

$$= \int_{\{B(z) \geq N_{x_1}\}} \frac{1}{B(z)} |\hat{\nu}(z)|^2 dz \sum_{k=1}^{\infty} \frac{\arctan \left(\frac{N_{x_k+\frac{3}{2}}}{B(z)}\right) - \arctan \left(\frac{N_{x_k+\frac{1}{2}}}{B(z)}\right)}{x_k \log \left(\frac{N_{x_k+\frac{3}{2}}}{B(z)}\right)}$$

15
\[
\begin{align*}
&= \sum_{l=1}^{\infty} \int_{\{N_{x_l} \leq B(z) < N_{x_{l+1}}\}} \frac{1}{B(z)} |\hat{\varphi}(z)|^2 dz \sum_{k=1}^{\infty} \frac{\arctan \frac{N_{x_k} + \frac{1}{2}}{B(z)} - \arctan \frac{N_{x_k} + \frac{1}{2}}{x_k \log(N_{x_k} + \frac{1}{2})}}{x_k \log(N_{x_k} + \frac{1}{2})} \\
&+ \sum_{l=1}^{\infty} \int_{\{N_{x_{l+1}} \leq B(z) < N_{x_{l+1}}\}} \frac{1}{B(z)} |\hat{\varphi}(z)|^2 dz \sum_{k=1}^{\infty} \frac{\arctan \frac{N_{x_k} + \frac{1}{2}}{B(z)} - \arctan \frac{N_{x_k} + \frac{1}{2}}{x_k \log(N_{x_k} + \frac{1}{2})}}{x_k \log(N_{x_k} + \frac{1}{2})} \\
&= \sum_{l=1}^{\infty} \int_{\{N_{x_l} \leq B(z) < N_{x_{l+1}}\}} \frac{1}{B(z)} |\hat{\varphi}(z)|^2 dz \left\{ \sum_{k=1}^{l-1} \frac{\arctan \frac{N_{x_k} + \frac{1}{2}}{B(z)} - \arctan \frac{N_{x_k} + \frac{1}{2}}{x_k \log(N_{x_k} + \frac{1}{2})}}{x_k \log(N_{x_k} + \frac{1}{2})} \\
&+ \sum_{k=l}^{\infty} \arctan \frac{N_{x_k} + \frac{1}{2}}{B(z)} - \arctan \frac{N_{x_k} + \frac{1}{2}}{x_k \log(N_{x_k} + \frac{1}{2})} \right\} \\
&+ \sum_{l=1}^{\infty} \int_{\{N_{x_{l+1}} \leq B(z) < N_{x_{l+1}}\}} \frac{1}{B(z)} |\hat{\varphi}(z)|^2 dz \left\{ \sum_{k=1}^{l-1} \frac{1}{x_k \log(N_{x_k} + \frac{1}{2})} \right\} \\
&+ \frac{\pi}{2 \log(N_{x_l} + \frac{1}{2})} \sum_{k=l}^{\infty} \frac{B(z)}{N_{x_k} + \frac{1}{2}} \right\} \\
&\leq \sum_{l=1}^{\infty} \int_{\{N_{x_l} \leq B(z) < N_{x_{l+1}}\}} \frac{1}{B(z)} |\hat{\varphi}(z)|^2 dz \left( \frac{1}{\varsigma^{x_l + \frac{1}{2}}(\log \varsigma)(1 - \frac{1}{\varsigma})(B(z))^{1 - \varsigma^{-\frac{1}{2}}} \\
&+ \frac{\pi}{\log B(z)[\log \log (B(z))] + \frac{1}{\varsigma^{x_l + \frac{1}{2}}(\log \varsigma)(1 - \frac{1}{\varsigma})(B(z))^{\frac{1}{2} - 1}} \\
&+ \frac{1}{\varsigma^{x_l + \frac{1}{2}}(\log \varsigma)(1 - \frac{1}{\varsigma})(B(z))^{\frac{1}{2} - 1}} \right) \\
&+ \sum_{l=1}^{\infty} \int_{\{N_{x_{l+1}} \leq B(z) < N_{x_{l+1}}\}} \frac{1}{B(z)} |\hat{\varphi}(z)|^2 dz \left( \frac{1}{\varsigma^{x_l + \frac{1}{2}}(\log \varsigma)(1 - \frac{1}{\varsigma})(B(z))^{1 - \varsigma^{-\frac{1}{2}}} \\
&+ \frac{1}{\varsigma^{x_l + \frac{1}{2}}(\log \varsigma)(1 - \frac{1}{\varsigma})(B(z))^{\frac{1}{2} - 1}} \right)
\end{align*}
\]
\begin{equation}
\leq D_1 \int_{\mathbb{R}^n} \frac{1}{B(z) \log(2 + B(z)) \log \log(2 + B(z))^{1+\delta}} |\hat{\nu}(z)|^2 dz \\
+ \pi (\log \varsigma) \frac{\zeta}{2} \sum_{l=1}^{\infty} \int_{\{N_{x_1} \leq B(z) < N_{x_1} + 1\}} \frac{1}{B(z) \log B(z) \log \log(B(z))} |\hat{\nu}(z)|^2 dz \\
< \infty, \tag{4.1}
\end{equation}

where \( D_1 > 0 \) and \( \delta > 0 \) are constants depending only on \( \varsigma \). The fact that \( \frac{\pi}{2} - \arctan x \leq \frac{1}{x} \) for \( x > 0 \) has been used above to get the second inequality.

Since \( \int_{N_{x_1}}^{\infty} \frac{g(\lambda) d\lambda}{\lambda} = \infty \), we obtain by (2.5), (2.8), (4.1) and Theorem 2.2 that

\begin{equation}
\lim_{\lambda \to \infty} \int_{\{B(z) \geq N_{x_1}\}} \frac{\lambda}{\lambda^2 + B^2(z)} |\hat{\nu}(z)|^2 dz = 0. \tag{4.2}
\end{equation}

The proof of Assertion (i) is complete by Proposition 2.1.

**Assertion (ii):**

Suppose \( X \) satisfies (H). Let \( \nu \) be an arbitrary finite measure on \( \mathbb{R}^n \) of finite 1-energy and \( \varsigma > 1 \) be a constant. We choose a sequence of increasing natural numbers \( \{p_k\} \) satisfying

\begin{equation}
c(\lambda) \leq \frac{1}{2^k}, \text{ if } \lambda \geq p_k. \tag{4.3}
\end{equation}

For \( m = 1, 2, \ldots \), we set \( x_{m,1} = p_m + 2 \) and choose \( k_m \) such that

\begin{equation}
1 \leq \frac{1}{p_m + 2} + \frac{1}{p_m + 4} + \cdots + \frac{1}{p_m + 2k_m} \leq 2. \tag{4.4}
\end{equation}

Define \( x_{m,l_m} = p_m + 2l_m \) for \( 1 \leq l_m \leq k_m \). We can require without loss of generality that

\begin{equation}
x_{1,1} > 1 - \frac{2 \log \log \varsigma}{\log \varsigma}, \tag{4.5}
\end{equation}

and

\begin{equation}
N_{x_{1,1}}^\varsigma > e \text{ and } x_{m,k_m} < p_{m+1}, \quad m = 1, 2, \ldots
\end{equation}

Denote

\[
F := \{ z \in \mathbb{R}^n : B(z) \geq 2A(z) \}.
\]

We will show below that

\[
\sum_{m=1}^{\infty} \sum_{l_m=1}^{k_m} \int_{\{z \in F : N_{x_{m,l_m}} \leq B(z) < N_{x_{m,l_m} + 1}\}} \frac{1}{B(z) \log B(z) \log \log(B(z))} |\hat{\nu}(z)|^2 dz < \infty.
\]

We define

\[
f(\lambda) = k(\log(N_k) - \log(N_{k-1})), \text{ when } N_{k-1} \leq \lambda < N_k, \quad k \geq 2.
\]

Set

\[
\Lambda := \bigcup_{m=1}^{\infty} \bigcup_{l_m=1}^{k_m} \{ \lambda : N_{x_{m,l_m}} \leq \lambda < N_{x_{m,l_m} + 1}\}.
\]
By (4.2) and (4.3), we get

\[
\begin{align*}
\infty & > \int_{\Lambda} \frac{d\lambda}{\lambda f(\lambda)} \int_{\mathbb{R}^n} \frac{\lambda}{\lambda^2 + B^2(z)} |\hat{v}(z)|^2 1_{\{A(z) \leq \lambda\}} dz \\
& = \int_{\mathbb{R}^n} |\hat{v}(z)|^2 dz \int_{\Lambda \cap \{A(z) \leq \lambda\}} \frac{d\lambda}{f(\lambda)(\lambda^2 + B^2(z))} \\
& = \int_{\mathbb{R}^n} |\hat{v}(z)|^2 dz \int_{\Lambda \cap \{A(z) \leq \lambda\}} \frac{A(z) d\eta}{f(A(z)\eta)(A^2(\eta)^2 + B^2(\eta))} \\
& = \int_{\mathbb{R}^n} \frac{A(z)}{B^2(z)} |\hat{v}(z)|^2 dz \int_{\Lambda \cap \{1 \leq \eta\}} \frac{(B(z))^2 d\eta}{f(A(z))(\eta^2 + (B(z)/A(z))^2)} \\
& = \int_{\mathbb{R}^n} \frac{1}{B(z)} |\hat{v}(z)|^2 dz \sum_{m=1}^{\infty} \sum_{l_m=1}^{k_m} \left( \sum_{m=1}^{\infty} \sum_{l_m=1}^{k_m} \left( x_{m,l_m} + 1 \right) (\log(N_{x_{m,l_m}+1}) - \log(N_{x_{m,l_m}})) \right)
\end{align*}
\]
\[
\arctan 1 - \arctan \frac{1}{2} \left( \frac{1}{x_{m,l} + 1} \left( \log(N_{x_{m,l} + 1}) - \log(N_{x_{m,l}}) \right) \right) \\
\geq \sum_{m=1}^{\infty} \sum_{l=1}^{k_m} \int_{\{z \in F : N_{x_{m,l}} \leq B(z) < N_{x_{m,l} + 1}\}} \frac{1}{B(z)} |\hat{\nu}(z)|^2 dz
\]
\[
= \frac{(x_{m,l} + 1)(\log(N_{x_{m,l} + 1}) - \log(N_{x_{m,l}}))}{2} \left( \arctan 1 - \arctan \frac{1}{2} \right) \\
\geq \frac{(\log \varsigma)(\arctan 1 - \arctan \frac{1}{2})}{2\varsigma} \sum_{m=1}^{\infty} \sum_{l=1}^{k_m} \int_{\{z \in F : N_{x_{m,l}} \leq B(z) < N_{x_{m,l} + 1}\}} \frac{1}{B(z) \log B(z)[\log \log(B(z))]} |\hat{\nu}(z)|^2 dz,
\]
(4.5)

where (4.4) has been used to obtain the last inequality.

Set \( \{x_k\} = \{x_{m,l} : 1 \leq l \leq k_m\} \). Then, we obtain by (4.5) that
\[
\sum_{k=1}^{\infty} \int_{\{z \in F : N_{x_k} \leq B(z) < N_{x_k + 1}\}} \frac{1}{B(z) \log B(z)[\log \log(B(z))]} |\hat{\nu}(z)|^2 dz < \infty.
\]

Therefore the proof of Assertion (ii) is complete by noting (2.1). \( \square \)

Acknowledgments

We acknowledge the support of NSERC and thank Dr. Z.C. Hu for comment on Proposition 3.7.

References

[1] Bertoin J. (1996) \textit{Lévy Processes}. Cambridge Univ. Press.

[2] Blumenthal R.M. and Getoor R.K. (1961) Sample functions of stochastic processes with stationary independent increments. J. Math. Mech. 10, 493-516

[3] Blumenthal R.M. and Getoor R.K. (1968) \textit{Markov Processes and Potential Theory}. Academic Press, New York and London

[4] Blumenthal R.M. and Getoor R.K. (1970) Dual processes and potential theory. Proc. 12th Biennial Seminar of the Canadian Math. Congress, 137-156

[5] Fitzsimmons P.J. and Kanda M. (1992) On Choquet’s dichotomy of capacity for Markov processes. Ann. Probab. 20, 342-349
[6] Forst G. (1975) The definition of energy in non-symmetric translation invariant Dirichlet spaces. Math. Ann. 216, 165-172

[7] Glover J. (1983) Topics in energy and potential theory. Seminar on Stochastic Processes, 1982, Birkhäuser, 195-202

[8] Glover J. and Rao M. (1986) Hunt’s hypothesis (H) and Getoor’s conjecture. Ann. Probab. 14, 1085-1087

[9] Hawkes J. (1979) Potential theory of Lévy processes. Proc. London Math. Soc. 38, 335-352

[10] Hu Z.C. and Sun W. (2012) Hunt’s hypothesis (H) and Getoor’s conjecture for Lévy processes. Stoch. Proc. Appl. 122, 2319-2328

[11] Hu Z.C., Sun W. and Zhang J. (2014) New results on Hunt’s hypothesis (H) for Lévy processes. Preprint, http://arxiv.org/abs/1210.2016

[12] Kanda M. (1976) Two theorems on capacity for Markov processes with stationary independent increments. Z. Wahrsch. verw. Gebiete 35, 159-165

[13] Kanda M. (1978) Characterisation of semipolar sets for processes with stationary independent increments. Z. Wahrsch. verw. Gebiete 42, 141-154

[14] Rao M. (1977) On a result of M. Kanda. Z. Wahrsch. verw. Gebiete 41, 35-37

[15] Rao M. (1988) Hunt’s hypothesis for Lévy processes. Proc. Amer. Math. Soc. 104, 621-624