Algebraic treatment of alpha-cluster nuclei

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Abstract. I discuss an algebraic treatment of alpha-cluster nuclei in the framework of the Algebraic Cluster Model. It is suggested to treat $^{12}\text{C}$ as an oblate top with $D_3h$ symmetry (equilateral triangle) and $^{16}\text{O}$ as a spherical top with tetrahedral $T_d$ symmetry (regular tetrahedron). The structure of the rotational bands is a consequence of the underlying symmetry, and can be used to help determine the geometrical configuration of the $\alpha$-particles.

1. Introduction
The structure of $^{12}\text{C}$ has been the subject of many studies, both experimental and theoretical, since the early days of nuclear physics [1]. Especially the (collective) nature of the $0^+$ Hoyle state at 7.65 MeV which is of crucial importance in stellar nucleosynthesis to explain the observed abundance of $^{12}\text{C}$, has presented a formidable challenge to nuclear structure calculations, such as cluster model [2], (no-core) shell model [3, 4] or \textit{ab initio} lattice calculations [5].

Recently, three different experimental collaborations reported the measurement for a low-lying $2^+$ state around 10 MeV with a width of approximately 0.8 MeV which was interpreted as a rotational excitation of the Hoyle state, thus providing the first evidence of a collective band structure built on top of the Hoyle state. This $2^+$ state was observed in the $^{12}\text{C}(\alpha, \alpha')$ reaction at an energy $9.75 \pm 0.15$ MeV and width $0.75 \pm 0.15$ MeV [6]. A combination of the $(\alpha, \alpha')$ and $(p, p')$ reactions showed evidence for a $2^+$ state at $E = 9.75 \pm 0.15$ MeV and $\Gamma = 0.75 \pm 0.15$ MeV [7]. Finally, experiments using the $^{12}\text{C}(\gamma, \alpha)^8\text{Be}$ reaction showed a state at $10.03 \pm 0.11$ MeV and $\Gamma = 0.80 \pm 0.13$ MeV [8].

In view of the new experimental information, in this contribution I revisit an algebraic treatment of alpha-cluster nuclei, called the Algebraic Cluster Model, and discuss two special solutions which are relevant to the alpha-cluster nuclei $^{12}\text{C}$ [9] and $^{16}\text{O}$ [10].

2. Algebraic Cluster Model
Algebraic models and spectrum generating algebras have played an important role in the study of both many-body and few-body systems. Generally speaking, in algebraic models energy eigenvalues and eigenvectors are obtained by diagonalizing a finite-dimensional matrix, rather than by solving a set of coupled differential equations in coordinate space. As examples, I mention the interacting boson model which has been very successful in the description of collective states in nuclei [11], and the vibron model [12] which was introduced to describe vibrational and rotational excitations in diatomic molecules. In spectroscopic studies these algebraic methods provide a powerful tool to study symmetries and selection rules, to classify the basis states, and to calculate matrix elements of physical observables.
The Algebraic Cluster Model (ACM) is an interacting boson model to describe the relative motion of the n clusters based on the spectrum generating algebra of $U(\nu + 1)$ where $\nu = 3(n-1)$ represents the number of relative spatial degrees of freedom (after removal of the center of mass). For the two-body problem the ACM reduces to the $U(4)$ vibron model [12] and for three-body clusters to the $U(7)$ model [9, 13, 14, 15].

The spatial degrees of freedom for n-body systems are taken as the relative Jacobi coordinates $\vec{\rho}_k$ and the center of mass $\vec{R}$

\[
\vec{\rho}_k = \frac{1}{\sqrt{k(k+1)}} \left( \sum_{i=1}^{k} \vec{r}_i - k\vec{r}_{k+1} \right), \quad k = 1, \ldots, n-1,
\]

\[
\vec{R} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \vec{r}_i,
\]

and their conjugate momenta. Here $\vec{r}_i$ are the coordinates of the $i$-th cluster ($i = 1, \ldots, n$). Instead of a formulation in terms of coordinates and momenta the method of bosonic quantization is used which consists of introducing a dipole boson with $L^P = 1^-$ ($b^\dagger_k$) for each independent relative Jacobi vector ($k = 1, \ldots, n-1$), and an auxiliary scalar boson with $L^P = 0^+$ ($s^\dagger$). The scalar boson does not represent an independent degree of freedom, but is added under the restriction that the total number of bosons $N = n_s + \sum_k n_k$ is conserved. This procedure leads to a compact spectrum generating algebra of $U(3n-2)$ whose model space is spanned by the symmetric irreducible representation $[N]$ which contains the oscillator shells with $n_b = \sum_k n_k = 0, 1, 2, \ldots, N$. The introduction of the scalar boson makes it possible to investigate the dynamics of $n-1$ vector degrees of freedom including situations in which there is a mixing of oscillator shells.

In the present application to $\alpha$-cluster nuclei the Hamiltonian has to be invariant under the permutation group $S_n$ for $n$ identical clusters. As a result, the most general one- and two-body Hamiltonian that describes the relative motion of a system of $n$ identical clusters, is a scalar under the permutation group $S_n$, is rotationally invariant, conserves parity as well as the total number of bosons $N$, is given by

\[
H = \epsilon_0 s^\dagger \tilde{s} - \epsilon_1 \sum_k b^\dagger_k \cdot \tilde{b}_k + u_0 s^\dagger s \tilde{s} \tilde{s} - u_1 \sum_k s^\dagger b^\dagger_k \cdot \tilde{b}_k \tilde{s} + v_0 \left[ \sum_k b^\dagger_k \cdot \tilde{b}_k \tilde{s} \tilde{s} + \text{h.c.} \right] + \sum_L \sum_{ijkl} v^{(L)}_{ijkl} [b^\dagger_i \times b^\dagger_j]^{(L)} \cdot [\tilde{b}_k \times \tilde{b}_l]^{(L)},
\]

with $\tilde{b}_{k,m} = (-1)^{-m} b_{k,-m}$ and $\tilde{s} = s$. The invariance under the permutation symmetry imposes further restrictions on the coefficients $v^{(L)}_{ijkl}$. By construction, the first five terms in Eq. (2) are invariant under the permutation group $S_n$.

In general, the eigenvalues and corresponding eigenvectors are obtained numerically by diagonalizing the Hamiltonian in an appropriate basis. By construction, the wave functions are characterized by the total number of bosons $N$, angular momentum and parity $L^P$, and their transformation property under $t$ the permutation group $S_n$. Since in the application to $\alpha$-cluster nuclei the excitation of the $\alpha$ particles themselves is not considered, the wave functions describing the relative motion have to be symmetric under $S_n$.

The ACM has a rich algebraic structure, which includes both continuous and discrete symmetries. It is of general interest to study limiting cases of the Hamiltonian in which the energy spectra can be obtained in closed form. The $S_n$ invariant ACM Hamiltonian of Eq. (2)
has two dynamic symmetries corresponding to the group chains

\[ U(3n - 2) \supset \begin{cases} U(3n - 3) \\ SO(3n - 2) \end{cases} \]  

(3)

These dynamic symmetries were studied in general for the \( n \)-body problem and were shown to correspond to the \((3n - 3)\)-dimensional (an)harmonic oscillator and a deformed oscillator, respectively \[10\].

In this contribution, I review two other limiting cases of \( S_n \) invariant Hamiltonians for which approximate solutions can be obtained in a semiclassical mean-field analysis and which could be of interest for the application to \( \alpha \)-cluster nuclei. For the case of three-body clusters I discuss the oblate symmetric top that corresponds to three particles located at the vertices of an equilateral triangle, and for four-body clusters the spherical top with four particles at the vertices of a regular tetrahedron.

3. Three-body clusters: oblate top

For the case of three identical clusters the spectrum generating algebra of the ACM is given by \( U(7) \). Here I consider a Hamiltonian which is special case of Eq. (2)

\[ H = \xi_1 (R^2 s^+ s^- - b_1^+ b_1^- - b_2^+ b_2^-) (R^2 \tilde{s} \tilde{s} - \tilde{b}_1 \cdot \tilde{b}_1 - \tilde{b}_2 \cdot \tilde{b}_2) + \xi_2 \left[(b_1^+ b_1^- - b_2^+ b_2^-) (\tilde{b}_1 \cdot \tilde{b}_1 - \tilde{b}_2 \cdot \tilde{b}_2) + 4 (b_1^+ b_2^- - b_2^+ b_1^-) (\tilde{b}_1 \cdot \tilde{b}_1)\right] + 2\kappa_1 (b_1^+ \tilde{b}_1 + b_2^+ b_2^- + \tilde{b}_2 \cdot b_2^-) (1) \cdot (b_1^+ \tilde{b}_1 + b_2^+ b_2^- + \tilde{b}_2 \cdot b_2^-) (1) + 3\kappa_2 (b_1^+ \tilde{b}_2 - b_2^+ \tilde{b}_1) (0) \cdot (b_1^+ \tilde{b}_1 - b_2^+ \tilde{b}_2)(0). \]  

(4)

For \( R^2 = 0 \), this Hamiltonian has \( U(7) \supset U(6) \) symmetry (anharmonic oscillator), whereas for \( R^2 = 1 \) and \( \xi_2 = 0 \) it has \( U(7) \supset SO(7) \) symmetry (deformed oscillator). The general case with \( R^2 \neq 0 \) and \( \xi_1, \xi_2 > 0 \) corresponds to a potential energy surface with the equilibrium shape of an equilateral triangle, in which the two Jacobi vectors \( \tilde{\rho}_1 \) and \( \tilde{\rho}_2 \) have equal length and are perpendicular \[9\]. The corresponding point group symmetry is \( D_{3h} \) whose subgroup \( D_3 \) is isomorphic to the permutation group \( S_3 \). Even though in this case the energy eigenvalues cannot be derived in closed form, an approximate expression for the energy levels can be obtained by making use of the method of intrinsic or coherent states (valid in the limit of large \( N \)) \[13, 15\]

\[ E(v_1, v_2, L, K) = \omega_1 (v_1 + \frac{1}{2}) + \omega_2 (v_2 + 1) + \kappa_1 L(L + 1) + \kappa_2 (K \pm 2l)^2, \]  

(5)

with frequencies \( \omega_1 = 4NR^2\xi_1 \) and \( \omega_2 = 4NR^2\xi_2/(1 + R^2) \). The quantum numbers have the following meaning: \( v_1, v_2 \) are vibrational quantum numbers: \( v_1 \) corresponds to a symmetric one-dimensional stretching vibration, and \( v_2 \) represents a two-dimensional bending vibration; \( l = v_2, v_2 - 2, \ldots, 1 \) or \( 0 \) is the vibrational angular momentum of the doubly degenerate vibration; \( L \) is the angular momentum, and \( K \) its projection on a body-fixed axis \[9\].

Fig. 1 shows the rotation-vibration spectrum of a triangular configuration with \( D_{3h} \) symmetry which is characterized by a series of rotational bands labeled by \( (v_1, v_2) \). For this case, the bands with \( (v_1, 0^0) \) can have angular momenta and parity \( L^P = 0^+, 2^+, 3^-, 4^\pm, \ldots \), whereas the angular momentum content of the doubly degenerate vibrations \( (v_1, 1^1) \) is given by \( L^P = 1^-, 2^+, 3^-, \ldots \), in agreement with Ref. \[16\]. The structure of the rotational bands can be considered as the fingerprint of the underlying \( D_{3h} \) symmetry of three identical structureless particles at the vertices of an equilateral triangle.
Figure 1. Rotation-vibration spectrum of an oblate top with $D_{3h}$ symmetry calculated using Eq. (5) with $\omega_1 = 7.0$, $\omega_2 = 9.0$, $\kappa_1 = 0.8$ and $\kappa_2 = 0.0$ MeV (only the levels with $E \leq 25$ MeV are shown). The levels are characterized by angular momentum and parity $L^P$, and the vibrational labels $(v_1, v_2)$. All states are symmetric under $S_3 \sim D_3$.

The nucleus $^{12}$C
The structure of $^{12}$C has been extensively investigated in $\alpha$-cluster models [1, 2, 17], the shell model from early calculations in the p-shell [18] to more recent studies with large shell mixing in the ab initio [3] or symmetry-adapted no-core shell model [4, 19], as well as ab initio calculations based on lattice effective field theory [5, 20].

In this contribution, the spectrum of $^{12}$C is analyzed in the oblate top limit of the ACM which corresponds to a triangular configuration in which the three $\alpha$ particles are located at the vertices of an equilateral triangle (point group $D_{3h}$). The experimental spectrum of $^{12}$C is shown in Fig. 2, where it is compared with the exact energy spectrum of the oblate top Hamiltonian of Eq. (4) with $N = 10$ bosons [9]. The coefficients $\xi_1$, $\xi_2$, $\kappa_1$ and $\kappa_2$ were determined in a fit to the excitation energies of $^{12}$C (see Table 1). One can clearly identify in the experimental spectrum the states $0^+$, $2^+$, $3^-$, $4^+$ of the ground-state rotational band, and possibly the $4^-$ state at 13.35 MeV [21]. The $3^-$ state does not fall at the location expected for an oblate top, but at a somewhat higher excitation energy. Since the $3^-$ state has $K = 3$ the deviation from the simple rotational formula indicates large rotation-vibration interactions. The spectrum also shows an excited $0^+$ state at 7.65 MeV and a $1^-$ state at 10.84 MeV which could be interpreted as bandheads of the vibrational (stretching and doubly degenerate bending) excitations. In order to determine whether or not this is the case or rather those states represent other types of configurations, such as three $\alpha$ particles on a line [22] or a bent-arm configuration as suggested...
Figure 2. Comparison between the low-lying experimental spectrum of $^{12}$C and that calculated with the oblate top Hamiltonian of Eq. (4) with $N = 10$. The parameter values are given in Table 1. States with uncertain spin-parity assignment are in parentheses.

Recently in ab initio lattice calculations [5], one would have to identify the rotational sequences built on top of them which have a characteristic pattern for triangular configurations and another pattern for linear or bent configurations. In this respect, it is of special interest to investigate in more detail the nature of the $2^+$ state at 11.16 MeV and $2^-$ state at 11.83 MeV, which in the oblate top model could be interpreted as rotational excitations of the doubly degenerate vibration.

Recently, the structure of the Hoyle band was studied experimentally with the measurement of a new $2^+$ state with an energy in the range 9.75-10.03 MeV and a width of the order of 0.75-1.01 MeV [6, 7, 8], which was interpreted as a possible candidate for a rotational excitation of the $0^+$ Hoyle state. In Fig. 2, this new state is included by taking the average of the three measured energies. This assignment is supported by a recent analysis of the rms radii of the $0^+_2$ and $2^+_2$ states [23]. The radii were found to be quite similar thus providing evidence that these states belong to the same rotational band. In addition, there is experimental evidence for a state at 13.3 MeV which was assigned tentatively as a $4^+$ state belonging to the Hoyle band [24]. In order to distinguish between different geometrical configurations for the Hoyle band, such as linear, bent or triangular (see Fig. 3), the measurement of a possible $3^-$ Hoyle state is

Table 1. Coefficients used in the calculation of the energy spectrum and the transition form factors of $^{12}$C. The number of bosons is $N = 10$.

| Coefficient | Value 1 | Unit 1 | Value 2 | Unit 2 |
|-------------|---------|--------|---------|--------|
| $\xi_1$     | 0.1721  | MeV    | 0.7068  | MeV    |
| $\xi_2$     | 0.2745  | MeV    | 0.1276  | MeV    |
| $\beta$     | 1.74    | fm     | 0.52    | fm$^{-2}$ |
| $\alpha$    |         |        |         |        |

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Figure 3. Different α-cluster configurations for $^{12}$C: linear, bent and triangular with their respective point-group symmetries.

crucial, since its presence would indicate a triangular configuration, just as for the ground state band.

The moments of inertia of the ground band and the Hoyle band are very different. This leads to a somewhat larger radius for the Hoyle band than for the ground state band. Since for the oblate top, both the ground-state band and the Hoyle band arise from the same geometrical shape of an equilateral triangle, the radius of the Hoyle state can be estimated from the moments of inertia to be a factor $1^{1/5}$ larger than the rms radius of the ground state [9]. This value is a bit higher than, but essentially consistent with the one determined in the modified diffraction model [23].

For the oblate top, electromagnetic transition rates and form factors can be obtained in explicit form in the limit of large $N$. For example, the elastic form factor is given by

$$F(0^+_1 \rightarrow 0^+_1; q) = j_0(q\beta) e^{-q^2/4\alpha}.$$ (6)

The coefficient $\beta$ is determined from the first minimum in the elastic form factor, and subsequently the coefficient $\alpha$ is determined from the charge radius of $^{12}$C (see Table 1). Finally, since the energy spectrum is not very sensitive on the coefficient $R^2$ in the first term of the Hamiltonian of Eq. (4), its value is determined from the first minimum in the transition form factor to the Hoyle state $F(0^+_1 \rightarrow 0^+_2; q)$ [9].

Fig. 4 shows a comparison between experimental and theoretical form factors. The calculations give a reasonable description of the transition form factors to the rotational excitations of the ground state band, as well as the shape of the form factors leading to the $0^+$ Hoyle state, indicating that the $0^+_2$ state could indeed be the vibrational excitation of a three-alpha configuration with $D_{3h}$ symmetry albeit with a large mixing with other configurations. Finally, the transition form factor for the $2^+_2$ state shows a similar structure as the $0^+_2$ state.

4. Four-body clusters: spherical top

As a second example, I discuss the spherical top with tetrahedral symmetry $T_d$ as a special case of the ACM for four identical clusters. The spectrum generating algebra is given by $U(10)$. Let us consider a $S_4$ invariant Hamiltonian of the form

$$H_4 = \xi_1 \left( R^2 s^4 s^\dagger - b_1^\dagger \cdot b_1^\dagger - b_2^\dagger \cdot b_2^\dagger - b_3^\dagger \cdot b_3^\dagger \right) \text{(h.c.)}$$

$$+ \xi_2 \left( -2\sqrt{2} b_1^\dagger \cdot b_3^\dagger + 2b_1^\dagger \cdot b_2^\dagger \right) \text{(h.c.)}$$

$$+ \left( -2\sqrt{2} b_2^\dagger \cdot b_3^\dagger + (b_1^\dagger \cdot b_1^\dagger - b_2^\dagger \cdot b_2^\dagger) \right) \text{(h.c.)}$$

$$+ \xi_3 \left( 2b_1^\dagger \cdot b_3^\dagger + 2\sqrt{2} b_1^\dagger \cdot b_1^\dagger \right) \text{(h.c.)}$$
Figure 4. Comparison between the experimental form factors $|\mathcal{F}(0^+_1 \rightarrow L^P_i; q)|^2$ of $^{12}$C for the final states $L^P_i = 0^+_1$ (top left), $2^+_1$ (top right), $0^+_2$ (bottom left) and $2^+_2$ (bottom right) and those obtained for the oblate top with $N = 10$. The parameter values are given in Table 1. The experimental data are taken from [25]-[30].

\begin{equation}
+(2b_2^\dagger \cdot b_0^\dagger + \sqrt{2} (b_1^\dagger \cdot b_1^\dagger - b_2^\dagger \cdot b_2^\dagger)) \text{ (h.c.)}
\end{equation}

\begin{equation}
+(b_1^\dagger \cdot b_1^\dagger + b_2^\dagger \cdot b_2^\dagger - 2b_3^\dagger \cdot b_3^\dagger) \text{ (h.c.)}
\end{equation}

\begin{equation}
+\kappa_1 \bar{L} \cdot \bar{L} + \kappa_2 (\bar{L} \cdot \bar{L} - \bar{I} \cdot \bar{I})^2,
\end{equation}

where $\bar{L}$ denotes the angular momentum in coordinate space and $\bar{I}$ the angular momentum in index space. For $R^2 = 0$, this Hamiltonian has $U(10) \supset U(9)$ symmetry (anharmonic oscillator), whereas for $R^2 = 1$ and $\xi_2 = \xi_3 = 0$ it has $U(10) \supset SO(10)$ symmetry (deformed oscillator). The general case with $R^2 \neq 0$ and $\xi_1, \xi_2, \xi_3 > 0$ corresponds to a geometric configuration in which the four clusters are located at the vertices of a regular tetrahedron: i.e. the three Jacobi vectors $\bar{\rho}_1$, $\bar{\rho}_2$ and $\bar{\rho}_3$ have equal length and are mutually perpendicular [10]. The corresponding point group symmetry is $T_d$ which is isomorphic to the permutation group $S_4$. Just as in the previous case of the oblate top, also for the Hamiltonian of Eq. (7) an approximate energy formula can be derived in a semiclassical mean-field analysis by making use of the methods of intrinsic or coherent states which is valid in the limit of large $N$. The resulting energy spectrum is that of
the vibrational and rotational excitations of a spherical top with tetrahedral symmetry

\[ E = \omega_1(v_1 + \frac{1}{2}) + \omega_2(v_2 + 1) + \omega_3(v_3 + \frac{3}{2}) + \kappa_1 L(L + 1), \]  

with frequencies \( \omega_1 = 4NR^2\xi_1, \omega_2 = 8NR^2\xi_2(1 + R^2) \) and \( \omega_3 = 8NR^2\xi_3/(1 + R^2) \). Here \( v_1 \) represents the vibrational quantum number for a symmetric stretching vibration, \( v_2 \) denotes a doubly degenerate \( E \) vibration, and \( v_3 \) a three-fold degenerate \( F \) vibration.

Whereas the angular momentum \( L \) is an exact symmetry of \( H_4 \) of Eq. (7), the angular momentum in index space \( I \) in general does not commute with the Hamiltonian. Only if \( \xi_2 = \xi_3 \) in Eq. (7), does \( I \) become a good quantum number. The rotational excitations of the ground state vibrational band of \( H_4 \) with \( (v_1, v_2, v_3) = (0, 0, 0) \) are characterized by \( L = I \). This property is a consequence from the fact that the operator \( \hat{L} \cdot \hat{L} - \hat{I} \cdot \hat{I} \) annihilates the coherent (or intrinsic) state corresponding to the rigid equilibrium shape of a regular tetrahedron.

Fig. 5 shows the structure of the spectrum of a spherical top with tetrahedral symmetry according to the approximate energy formula of Eq. (8). The energy spectrum consists of a series of rotational bands labeled by \( (v_1, v_2, v_3) \). It is assumed that the spin of the identical clusters is zero, as is relevant for the description of the \(^{16}O\) nucleus as a cluster of four \( \alpha \) particles. As a consequence, all states in Fig. 5 are symmetric under \( S_4 \). For this case, the bands with \( (v_1, 0, 0) \) can have angular momenta and parity \( L^P = 0^+, 3^−, 4^+, 6^±, \ldots \), whereas the angular momentum content of the doubly degenerate \( E \) vibration \( (v_1, 1, 0) \) is given by \( L^P = 2^±, 4^±, 5^±, \ldots \), and for the triply degenerate \( F \) vibration \( (v_1, 0, 1) \) by \( L^P = 1^−, 2^+, 3^±, 4^±, \ldots \), in agreement with Ref. [16]. A more detailed analysis of \(^{16}O\) in the ACM is forthcoming.

5. Summary and conclusions
In this contribution, I presented an algebraic treatment of \( \alpha \)-cluster nuclei in terms of the Algebraic Cluster Model which is based on the algebraic quantization of the relative Jacobi variables. The ensuing \( U(3n - 2) \) spectrum generating algebra for a system on \( n \) clusters incorporates all vibrational and rotational degrees of freedom from the outset. The permutation symmetry of the \( \alpha \)-clusters is taken into account in an exact manner. Two cases were discussed in more detail.

First it was suggested to describe the nucleus \(^{12}C\) as a system of three \( \alpha \)-particles located at the vertices of an equilateral triangle (oblate top). The spectrum of the ground state band of a triangular configuration is characterized by the rotational sequence: \( 0^+ \), \( 2^+ \), \( 3^− \), \( 4^± \), \ldots \), as is indeed observed in \(^{12}C\). A linear configuration would not have negative parity states, while a shell-model configuration would not have the \( 3^− \) state as a member of the rotational band but rather as an octupole vibration, \( i.e. \) it would not form a rotational sequence with the \( 0^+ \), \( 2^+ \), \( 4^± \) states. For this reason it is very important to look for the \( 3^− \) state of the Hoyle band in order to determine whether the geometrical configuration of the \( \alpha \)-particles for the Hoyle band is linear, bent or triangular.

As a second example, I showed some results for four identical clusters located at the vertices of a regular tetrahedron (spherical top) with possible applications for the nucleus \(^{16}O\) as a system of four \( \alpha \)-particles. As a consequence of the tetrahedral symmetry of this configuration, the ground state rotational band is given by the sequence \( 0^+, 3^−, 4^+, 6^±, \ldots \), just as is observed in the nucleus \(^{16}O\). The absence of a \( 2^+ \) state is due to the tetrahedral symmetry.

It is important to stress that these rotational sequences can be considered as the fingerprints of the underlying point-group symmetry of the system. If a physical system consists of three identical particles at the vertices of an equilateral triangle, then its spectrum must be as in Fig. 1, or for a system composed of four identical particles at the vertices of a regular tetrahedron, the spectrum must be as in Fig. 5.
In contrast with molecules, the vibrational and rotational frequencies in nuclei are comparable which implies large rotation-vibration interactions. The finite value of the number of bosons $N$ implies that the vibrational bands are no longer decoupled, but instead show an appreciable mixing between them.

Finally, it is important to note that the ACM provides a general framework to study the full rotational and vibrational structure of cluster systems which is not restricted to the case of identical particles discussed in this contribution. It can be applied to other situations as well, such as nonidentical particles [15] or other geometric configurations, rigid as well as non-rigid.

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