Bose-Einstein Condensate general relativistic stars

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We analyze the possibility that due to their superfluid properties some compact astrophysical objects may contain a significant part of their matter in the form of a Bose-Einstein condensate. To study the condensate we use the Gross-Pitaevskii equation, with arbitrary non-linearity. By introducing the Madelung representation of the wave function, we formulate the dynamics of the system in terms of the continuity equation and of the hydrodynamic Euler equations. The non-relativistic and Newtonian Bose-Einstein gravitational condensate can be described as a gas, whose density and pressure are related by a barotropic equation of state. In the case of a condensate with quartic non-linearity, the equation of state is polytropic with index one. In the framework of the Thomas-Fermi approximation the structure of the Newtonian gravitational condensate is described by the Lane-Emden equation, which can be exactly solved. The case of the rotating condensate is also discussed. General relativistic configurations with quartic non-linearity are studied numerically with both non-relativistic and relativistic equations of state, and the maximum mass of the stable configuration is determined. Condensates with particle masses of the order of two neutron masses (Cooper pair) and scattering length of the order of 10−0.3 × 1016 g/cm3 and minimum radii in the range of 10 − 20 km. In this way we obtain a large class of stable astrophysical objects, whose basic astrophysical parameters (mass and radius) sensitively depend on the mass of the condensed particle, and on the scattering length. We also propose that the recently observed neutron stars with masses in the range of 2 − 2.4M⊙ are Bose-Einstein Condensate stars.

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I. INTRODUCTION

At very low temperatures, particles in a dilute Bose gas can occupy the same quantum ground state, forming a Bose-Einstein (BEC) condensate, which appears as a sharp peak over a broader distribution in both coordinates and momentum space. The possibility to obtain quantum degenerate gases by a combination of laser and evaporative cooling techniques has opened several new lines of research, at the border of atomic, statistical and condensed matter physics (for recent reviews see [1, 2]).

To say that so many particles are in the same quantum state is equivalent in saying that these particles display the state coherence. That is, BEC is a particular case of coherence phenomena, related to the arising state coherence. As the gas is cooled, the condensation of a large fraction of the particle in a gas occurs via a phase transition, taking place when the wavelengths of individual particles overlap and behave identically. For the transition to take place, particles have to be strongly correlated with each other [1, 2].

For an ensemble of particles in thermodynamic equilibrium at temperature T, the thermal energy of a particle is given by kB(T, where kB is Boltzmann’s constant. For a particle of mass m, the thermal wavelength is \( \lambda_T = \sqrt{2\pi \hbar^2 / mk_B T} \). Particles become correlated with each other when their wavelengths overlap, that is, the thermal wavelength is greater than the mean inter-particles distance \( l, \lambda_T > l \). The average particle number \( n \) for \( N \) particles in a volume \( V \), \( n = N/V \), is related to the distance \( l \) through the relation \( nl^3 = 1 \). Hence the condition \( \lambda_T > l \) can be rewritten as \( n\lambda_T^3 > 1 \), which yields the inequality [3]

\[
T < \frac{2\pi \hbar^2}{mk_B n^{2/3}}.
\]

Hence a coherent state may develop if the particle density is high enough or the temperature is sufficiently low. An accurate description of the BEC for an ideal gas is based on the Bose-Einstein distribution \( f(p) = \{\exp[(\varepsilon_p - \mu)/k_BT] - 1\}^{-1} \), for particles with momentum \( p \), energy \( \varepsilon_p = p^2/2m \) and chemical potential \( \mu \). In the thermodynamic limit \( N \to \infty \), \( V \to \infty \), \( N/V \to \text{constant} \), the fraction of particles condensing to the state with \( p = 0 \) below the condensation temperature \( T_c \) is \( n_0 = 1 - (T/T_c)^{3/2} \), while \( n_0 = 0 \) above the condensation temperature. The condensation temperature is \( T_c = 2\pi \hbar^2 n^{2/3} / mk_B \zeta^{2/3} \), where \( \zeta = 2.612 \) [3]. The dynamical process of Bose-Einstein condensation in the canonical ensemble (fixed temperature T) has been studied in [4].
A non-ideal, weakly interacting Bose gas also displays Bose-Einstein condensation, though particles interactions deplete the condensate, so that at zero temperature the condensate fraction is smaller than unity, \( n_0 < 1 \). A system is called weakly interacting if the characteristic interaction radius \( r_{\text{int}} \) is much smaller than the mean inter-particles distance \( l \), \( r_{\text{int}} \ll l \). This inequality can be rewritten equivalently as \( n_{\text{int}}^3 \ll 1 \). If this condition holds, the system is called dilute [2].

Superfluid liquids, like \(^4\text{He} \), are far from being dilute. Nevertheless, one believes that the phenomenon of superfluidity is related with BEC. The experimental observations and the theoretical calculations estimate the condensate fraction for superfluid helium at \( T = 0 \) to be \( n_0 \approx 0.10 \). A strongly correlated pair of fermions can be treated approximately like a boson. This is why the arising superfluidity in \(^3\text{He} \) can be interpreted as the condensation of coupled fermions. Similarly, superconductivity may be described as the condensation of the Cooper pairs that are formed by the electrons or the holes [3].

An ideal system for the experimental observation of the BEC condensation is a dilute atomic Bose gas confined in a trap and cooled to very low temperatures. BEC were first observed in 1995 in dilute alkali gases such as vapors of rubidium and sodium. In these experiments, atoms were confined in magnetic traps, evaporatively cooled down to a fraction of a microkelvin, left to expand by switching off the magnetic trap, and subsequently imaged with optical methods. A sharp peak in the velocity distribution was observed below a critical temperature, indicating that condensation has occurred, with the alkali atoms condensed in the same ground state. Under the typical confining conditions of experimental settings, BECs are inhomogeneous, and hence condensates arise as a narrow peak not only in the momentum space but also in the coordinate space [4, 5].

If considering only two-body, mean field interactions, a dilute Bose-Einstein gas near zero temperature can be modelled using a cubic non-linear Schrödinger equation with an external potential, which is known as the Gross-Pitaevskii equation [2].

The possibility of the Bose-Einstein condensation has also been considered in nuclear and quark matter, in the framework of the analysis of the BCS-BEC crossover. At ultra-high density, matter is expected to form a degenerate Fermi gas of quarks in which the Cooper pairs of quarks condensate near the Fermi surface (color superconductor). If the attractive interaction is strong enough, at some critical temperature the fermions may condense into the bosonic zero mode, forming a Bose-Einstein quark condensate [6]. The basic concept of the BCS-BEC crossover is as follows: As long as the attractive interaction between fermions is weak, the system exhibits the superfluidity characterized by the energy gap in the BCS mechanism. On the other hand, if the attractive interaction is strong enough, the fermions first form bound molecules (bosons), then they start to condense into the bosonic zero mode at some critical temperature.

These two situations are smoothly connected without a phase transition [10].

One of the most striking features of the crossover is that the critical temperature in the BEC region is independent of the coupling for the attraction between fermions. This is because the increase of the coupling only affects the internal structure of the bosons, while the critical temperature is determined by the boson’s kinetic energy. Thus, the critical temperature reaches a ceiling for the large coupling as long as the binding effect on the boson mass can be neglected. Even in the nuclear matter where the interaction is relatively strong, the binding energy of the deuteron is much smaller than the nucleon mass. This fact allows us to work within a non-relativistic framework to describe such a crossover [10]. However, in relativistic systems where the binding energy cannot be neglected, there could be two crossovers in the relativistic fluids: one is the ordinary BCS-BEC crossover, where the critical temperature in the BEC region would not plateau because of the relativistic effect, and the second is the crossover from the BEC state to a relativistic state, the so-called relativistic BEC (RBEC), where the critical temperature increases to the order of the Fermi energy [10].

In isospin symmetric nuclear matter, neutron-proton (np) pairing undergoes a smooth transition leading from an assembly of np Cooper pairs at higher densities to a gas of Bose-condensed neutrons as the nucleon density is reduced to an extremely low value. This transition may be relevant to supernova matter or for the crust of neutron stars [11]. A mixture of interacting neutral and charged Bose condensates, which is supposed to be realized in the interior of neutron stars in the form of a coexistent neutron superfluid and protonic superconductor, was considered in [12].

The possibility of the existence of some Bose condensates in neutron stars was considered for a long time (see Glendenning [13] for a detailed discussion). The condensation of negatively charged mesons in neutron star matter is favored because such mesons would replace electrons with very high Fermi momenta. The in-medium properties of the \( K^- \) mesons may be such that they could condense in neutron matter as well. Bose-Einstein condensates of kaons/anti-kaons in compact objects were discussed recently [14, 15]. Pion as well as kaon condensates would have two important effects on neutron stars. Firstly, condensates soften the equation of state above the critical density for onset of condensation, which reduces the maximal possible neutron star mass. At the same time, however, the central stellar density increases, due to the softening. Secondly, meson condensates would lead to neutrino luminosities which are considerably enhanced over those of normal neutron star matter. This would speed up neutron star cooling considerably [13]. Another particle which may form a condensate is the H-dibaryon, a doubly strange six quark composite with spin and isospin zero, and baryon number two. In neutron star matter, which may contain a significant fraction of
A hyperons, the Λ’s could combine to form H-dibaryons. H-matter condensates may thus exist at the center of neutron stars [13]. Neutrino superfluidity, as suggested by Kapusta [16], may also lead to Bose-Einstein condensation [17].

Zero spin bosons, described by real or complex scalar fields, are the simplest particles which can be considered in the framework of quantum field theory and general relativity. Real scalar fields have equilibrium configurations that were discovered by Seidel and Suen [18] and are called oscillations. They are globally regular but are fully time dependent. As for their stability, they seem to be quite robust as far as numerical evolution is concerned [19]. The objects which can be formed by scalar fields have been investigated in detail by using mainly numerical tools [20]. Complex scalar fields can form stable equilibrium configurations called boson stars [21, 22], that are globally regular and whose energy density is time independent. The possibility that dark matter is in the form of a scalar field [23–25] or a Bose-Einstein condensate [26] has also been investigated extensively.

Therefore the physical results presented above show that the possibility of the existence of a Bose-Einstein condensate inside compact astrophysical objects or the existence of stars formed entirely from a Bose-Einstein condensate cannot be excluded a priori. Such a possibility has in fact suggested recently. Wang [27] used the Gross-Pitaevskii equation, together with the associated energy functional and the Thomas-Fermi approximation, to study a cold star composed of a dilute Bose-Einstein condensate. For a static star, the exact solution for the density distribution was obtained. A number of perturbative solutions for the case of a slowly rotating star have also been derived. The effect of a scalar dark matter background on the equilibrium of degenerate stars was studied by Grifols [28], with a particular focus on white dwarfs, and the changes induced in their masses and radii.

A detailed analytical and numerical analysis of the Newtonian Bose-Einstein condensate systems was performed recently in [29] and [30], respectively. In [29] an approximate analytical expression of the mass-radius relation of a Newtonian self-gravitating Bose-Einstein condensate with short-range interactions, described by the Gross-Pitaevskii-Poisson system, was obtained. For repulsive short-range interactions (positive scattering lengths), configurations of arbitrary mass do exist, but their radius is always larger than a minimum value. For attractive short-range interactions (negative scattering lengths), equilibrium configurations only exist below a maximum mass. The equation of hydrostatic equilibrium describing the balance between the gravitational attraction and the pressure due to quantum effects and short-range interactions (scattering) was numerically solved in [30].

It is the purpose of the present paper to develop a general and systematic formalism for the study of gravitationally bounded Bose-Einstein condensates, in both Newtonian and general relativistic situations. Our approach is independent of the nature of the condensate. As a starting point we generalize the Gross-Pitaevskii equation by allowing an arbitrary form of the non-linearity. To obtain a transparent description of the physical properties of the BECs we introduce the hydrodynamical representation of the wave function, which allows the formulation of the dynamics of the condensate in terms of the continuity and hydrodynamic Euler equations. Hence the Bose-Einstein gravitational condensate can be described as a gas whose density and pressure are related by a barotropic equation of state. In the case of a condensate with quartic non-linearity, the equation of state of the condensate is given by a polytropic equation of state with polytropic index $n = 1$. In the framework of the Thomas-Fermi approximation, with the quantum potential neglected, the structure of the gravitational BEC is described by the Lane-Emden equation, which can be solved analytically. Hence the mass and the radius of the condensate can be easily obtained. The case of the rotating Newtonian condensate is also discussed, by using the generalized Lane-Emden equation.

By using the equation of state corresponding to the Bose-Einstein condensates with quartic non-linearity we consider the general relativistic properties of condensate stars, by numerically integrating the structure equations (the mass continuity and the Tolman-Oppenheimer-Volkoff equation) for a static configuration. In our general relativistic study we consider the cases of condensates described by both non-relativistic and relativistic equations of state, respectively. The maximum mass and the corresponding radius are obtained numerically. Bose-Einstein condensate stars with particle masses of the order of two neutron masses (Cooper pair) and scattering length of the order of $10^{-20}$ fm have maximum masses of the order of $2M\odot$, maximum central density of the order of $0.1 - 0.3 \times 10^{16}$ g/cm$^3$ and minimum radii in the range of $10 - 20$ km.

The present paper is organized as follows. The Gross-Pitaevskii equation is written down in Section II. The hydrodynamical representation for the study of the gravitationally bounded BECs is introduced in Section III. The static and slowly rotating Newtonian condensates are analyzed in Section IV. The maximum mass of Newtonian condensate stars is discussed in Section V. The properties of the general relativistic static condensates with quartic non-linearity are studied in Section VI for both non-relativistic and relativistic equations of state, respectively. The astrophysical implications of our results are considered in Section VII. We discuss and conclude our results in Section VIII.

II. THE GROSS-PITAEVERSKII EQUATION FOR THE BOSE-EINSTEIN CONDENSATE STARS

In a quantum system of $N$ interacting condensed bosons most of the bosons lie in the same single-particle...
where $\hat{\Psi}(\vec{r})$ and $\hat{\Psi}^+(\vec{r})$ are the boson field operators that annihilate and create a particle at the position $\vec{r}$, respectively, and $V(\vec{r} - \vec{r}')$ is the two-body interatomic potential. $V_{\text{rot}}(\vec{r})$ is the potential associated to the rotation of the condensate, and is given by

$$V_{\text{rot}}(\vec{r}) = f_{\text{rot}}(t) \frac{m \omega^2}{2} r^2,$$

where $\omega$ is the angular velocity of the condensate and $f_{\text{rot}}(t)$ a function which takes into account the possible time variation of the rotation potential. For a system consisting of a large number of particles, the calculation of the ground state of the system with the direct use of Eq. (2) is impracticable, due to the high computational cost.

Therefore the use of some approximate methods can lead to a significant simplification of the formalism. One such approach is the mean field description of the condensate, which is based on the idea of separating out the condensate contribution to the bosonic field operator. For a uniform gas in a volume $V$, BEC occurs in the single particle state $\Psi_0 = \sqrt{N/V}$, having zero momentum. The field operator can then be decomposed in the form $\hat{\Psi}(\vec{r}) = \sqrt{N/V} + \hat{\Psi}'(\vec{r})$. By treating the operator $\hat{\Psi}'(\vec{r})$ as a small perturbation, one can develop the first order theory for the excitations of the interacting Bose gases.

In the general case of a non-uniform and time-dependent configuration, the field operator in the Heisenberg representation is given by

$$\hat{\Psi}(\vec{r}, t) = \psi(\vec{r}, t) + \hat{\Psi}'(\vec{r}, t),$$

where $\psi(\vec{r}, t)$, called the condensate wave function, is the expectation value of the field operator, $\psi(\vec{r}, t) = \langle \hat{\Psi}(\vec{r}, t) \rangle$. It is a classical field and its absolute value fixes the number density of the condensate through $n(\vec{r}, t) = |\psi(\vec{r}, t)|^2$. The normalization condition is $N = \int n(\vec{r}, t) d^3 \vec{r}$, where $N$ is the total number of particles in the star.

The equation of motion for the condensate wave function is given by the Heisenberg equation corresponding to the many-body Hamiltonian given by Eq. (2),

$$i\hbar \frac{\partial}{\partial t} \hat{\Psi}(\vec{r}, t) = \left[ \hat{\Psi}(\vec{r}, t), \hat{\mathcal{H}} \right] =$$

$$\left[ -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{rot}}(\vec{r}) + V_{\text{ext}}(\vec{r}) + \int d\vec{r}' \hat{\Psi}^+(\vec{r}', t) V(\vec{r} - \vec{r}') \hat{\Psi}(\vec{r}', t) \right] \hat{\Psi}(\vec{r}, t).$$

Replacing $\hat{\Psi}(\vec{r}, t)$ by the condensate wave function $\psi$ gives the zeroth-order approximation to the Heisenberg equation. In the integral containing the particle-particle interaction $V(\vec{r} - \vec{r}')$ this replacement is in general a poor approximation for short distances. However, in a dilute and cold gas, only binary collisions at low energy are relevant and these collisions are characterized by a single parameter, the $s$-wave scattering length, independently of the details of the two-body potential. Therefore, one can replace $V(\vec{r} - \vec{r}')$ by an effective interaction $V(\vec{r} - \vec{r}') = \lambda \delta(\vec{r} - \vec{r}')$, where the coupling constant $\lambda$ is related to the scattering length $a$ through $\lambda = 4\pi\hbar^2a/m$. Hence, we assume that in a medium composed of scalar particles with non-zero mass, the range of Van der Waals-type scalar mediated interactions among nucleons becomes infinite, when the medium makes a transition to a Bose-Einstein condensed phase.

With the use of the effective potential the integral in the bracket of Eq. (5) gives $\lambda |\psi(\vec{r}, t)|^2$, and the resulting equation is the Schrödinger equation with a quartic nonlinear term. However, in order to obtain a more general description of the Bose-Einstein condensate stars, we shall assume an arbitrary non-linear term $g(|\psi(\vec{r}, t)|^2)$.

From a physical point of view these modifications can be understood as follows. The inter-particle interaction can be written as $V(\vec{r}) = u_0 \delta^d_a(\vec{r})$, where $u_0$ is the amplitude of the inter-particle repulsion and $\delta^d_a(\vec{r})$ denotes any well localized $d$-dimensional function that transforms into the mathematical Dirac delta distribution when the range of interactions $a \to 0$. Assume that the inter-particle interaction is so strong that each particle is localized within a cage formed by its neighbors. In the dilute limit $n d \ll 1$ the size of this cage can be estimated as $R \sim n^{-1/d}$ and the ground state energy per particle follows from the uncertainty principle as $\hbar^2/mR^2 \sim \hbar^2 n^{2/d}/m$. The ground state energy which would go into the energy functional is given by $\hbar^2 n(2d-4)/d \sim m u_0^2$. The strong interaction assumption is valid if the interaction energy per particle $u_0/R^2$ is much bigger than the ground state energy per particle, i.e. $u_0/R^2 \gg \hbar^2/mR^2$. The condition for the strong coupling limit can be written as $\hbar^2 n(2d-4)/d \sim m u_0^2 \ll 1$. As space dimensionality decreases, it becomes increasingly harder for the repulsive particles to avoid collisions. Thus, in general, the quartic non-linearity in the energy functional should be replaced by $|\psi|^{2+2d}/d$. The non-linearity may also be logarithmic in $|\psi|^2$. For $d = 1$ we have a $|\psi|^6$ interaction.

Therefore the generalized Gross-Pitaevskii equation describing a gravitationally trapped Bose-Einstein con-
densate is given by
\[ i \hbar \frac{\partial}{\partial t} \psi(\vec{r}, t) = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{rot}}(\vec{r}) + V_{\text{ext}}(\vec{r}) + g' \left( |\psi(\vec{r}, t)|^2 \right) \right] \psi(\vec{r}, t), \]
where we denoted \( g' = dg/dn \).

As for \( V_{\text{ext}}(\vec{r}) \), we assume that it is the gravitational potential, \( V_{\text{ext}} = m\Phi \), and it satisfies the Poisson equation
\[ \nabla^2 \Phi = 4\pi G \rho, \]
where
\[ \rho = mn = m |\psi(\vec{r}, t)|^2, \]
is the mass density inside the Bose-Einstein condensate star.

### III. THE HYDRODYNAMICAL REPRESENTATION OF THE BOSE-EINSTEIN GRAVITATIONAL CONDENSATE

The physical properties of a Bose-Einstein condensate described by the generalized Gross-Pitaevskii equation given by Eq. (6) can be understood much more easily by using the so-called Madelung representation of the wave function \[1, 2\], which consists in writing \( \psi(\vec{r}, t) = \sqrt{n(\vec{r}, t)} \exp \left( \frac{i}{\hbar} S(\vec{r}, t) \right) \), where the function \( S(\vec{r}, t) \) has the dimension of an action.

By substituting the above expression of the wave function into Eq. (6) it decouples into a system of two differential equations for the real functions \( n \) and \( \vec{v} \), given by
\[ \frac{\partial n}{\partial t} + \nabla \cdot (m\vec{v}) = 0, \]
where we have introduced the quantum potential \( V_Q = -\frac{\hbar^2}{2m} \nabla^2 \sqrt{n}/\sqrt{n} \), and the velocity of the quantum fluid
\[ \vec{v} = \frac{\nabla S}{m}, \]
respectively. From its definition it follows that the velocity field is irrotational, satisfying the condition \( \nabla \times \vec{v} = 0 \).

The quantum potential \( V_Q \) has the property \[1\]
\[ n \nabla_i V_Q = \nabla_j \left( \frac{\hbar^2}{4m} n \nabla_i \nabla_j \ln n \right) = \nabla_j \sigma_{ij}^Q, \]
where \( \sigma_{ij}^Q = -(\hbar^2 n/4m) \nabla_i \nabla_j \ln n \) is the quantum stress tensor, which has the dimension of a pressure and is an intrinsically anisotropic quantum contribution to the equations of motion.

By taking into account that the flow is irrotational, the equations of motion of the gravitational ideal Bose-Einstein condensate take the form of the equation of continuity and of the hydrodynamic Euler equation,
\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0, \]
where we have denoted
\[ P \left( \frac{\rho}{m} \right) = g' \left( \frac{\rho}{m} \right) - \rho \left( \frac{\rho}{m} \right) \].

Therefore the Bose-Einstein gravitational condensate can be described as a gas whose density and pressure are related by a barotropic equation of state \[1\]. The explicit form of this equation depends on the form of the non-linearity term \( g \).

For a static ideal condensate, \( \vec{v} \equiv \vec{0} \). In this case, from Eq. (11), we obtain
\[ V_Q + V_{\text{rot}} + V_{\text{ext}} + g' = \text{constant} \]
Applying the operator \( \nabla^2 \) to both sides of Eq. (18) gives
\[ \nabla^2 \left( V_Q + V_{\text{rot}} + g' \right) + \nabla^2 V_{\text{ext}} = 0. \]

In the case of a condensate with a non-linearity of the form \( g(n) = k_0 m^2 n^2/2 \), where \( k_0 \) is a constant, and in the presence of a confining gravitational field \( V_{\text{ext}} = m\Phi \), it follows that the generalized potential \( V_{\text{gen}} = -V_Q - V_{\text{rot}} - k_0 m \rho \) satisfies the Poisson equation,
\[ \frac{1}{m} \nabla^2 V_{\text{gen}} = 4\pi G \rho. \]

If the quantum potential can be neglected, then from Eq. (19), by using the relations \( \nabla^2 V_{\text{rot}} = m\omega^2 \) and \( \nabla^2 g' = k_0 m \nabla^2 \rho \), it follows that the mass density of the condensate is described by a Helmholtz type equation, given by
\[ \nabla^2 \rho + \frac{4\pi G}{k_0} \rho + \omega^2 = 0. \]

### IV. STATIC AND SLOWLY ROTATING NEWTONIAN BOSE-EINSTEIN CONDENSATE STARS

When the number of particles in the gravitationally bounded Bose-Einstein condensate becomes large
enough, the quantum pressure term makes a significant contribution only near the boundary of the condensate. Hence it is much smaller than the non-linear interaction term. Thus the quantum stress term in the equation of motion of the condensate can be neglected. This is the Thomas-Fermi approximation, which has been extensively used for the study of the Bose-Einstein condensates [1]. As the number of particles in the condensate becomes infinite, the Thomas-Fermi approximation becomes exact [27, 29, 30]. This approximation also corresponds to the classical limit of the theory (it corresponds to neglecting all terms with powers of $\hbar$) or as the regime of strong repulsive interactions among particles. From a mathematical point of view the Thomas-Fermi approximation corresponds to neglecting all terms containing $\nabla \rho$ and $\nabla S$ in the equation of motion.

A. Static Bose-Einstein Condensates stars

In the case of a static Bose-Einstein condensate, all physical quantities are independent of time. Moreover, in the first approximation we also neglect the rotation of the star, taking $V_{rot} = 0$. Therefore the equations describing the static Bose-Einstein condensate in a gravitational field with potential $V_{ext} = m\Phi$ take the form

$$\nabla P = -\rho \nabla \Phi,$$

$$\nabla^2 \Phi = 4\pi G \rho.$$  

These equations must be integrated together with the equation of state $P = P(\rho)$, which follows from Eq. (17), and some appropriately chosen boundary conditions. By assuming that the non-linearity in the Gross-Pitaevskii equation is of the form

$$g(n) = \alpha n^\gamma,$$

where $\alpha$ and $\gamma$ are positive constants, it follows that the equation of state of the gravitational Bose-Einstein condensate is the polytropic equation of state,

$$P(\rho) = \alpha (\gamma - 1) n^\gamma = K \rho^\gamma,$$

where we denoted $K = \alpha (\gamma - 1) / m^\gamma$.

By representing $\gamma$ in the form $\gamma = 1 + 1/n$, where $n$ is the polytropic index, it follows that the structure of the static Bose-Einstein condensate star is described by the Lane-Emden equation,

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d\theta}{d\xi} \right) = -\theta^n,$$

where $\theta$ is a dimensionless variable defined via $\rho = \rho_c \theta^n$, $\xi$ is a dimensionless coordinate introduced via the transformation $r = [(n + 1)K \rho_c^{1/(n - 1)} / 4\pi G]^{1/2} \xi$ and $\rho_c$ is the central density of the condensate [32].

Hence the mass and the radius of the condensate are given by

$$R = \left[ \frac{(n + 1)K}{4\pi G} \right]^{1/2} \rho_c^{(1-n)/2n} \xi_1,$$  

and

$$M = 4\pi \left[ \frac{(n + 1)K}{4\pi G} \right]^{3/2} \rho_c^{(3-n)/2n} \xi_1^2 |\theta'(\xi_1)|,$$

respectively, where $\xi_1$ defines the zero-pressure and zero-density surface of the condensate: $\theta(\xi_1) = 0$ [32].

In the standard approach to the Bose-Einstein condensates, the non-linearity term $g$ is given by

$$g(n) = \frac{u_0}{2} |\psi|^{4} = \frac{u_0}{2} \rho^2,$$

where $u_0 = 4\pi \hbar^2 a / m$ [1]. The corresponding equation of state of the condensate is

$$P(\rho) = K \rho^2,$$

with

$$K = \frac{2\pi \hbar^2 a}{m^3} = 0.1856 \times 10^5 \left( \frac{a}{1 \text{ fm}} \right) \left( \frac{m}{2m_n} \right)^{-3},$$

where $m_n = 1.6749 \times 10^{-24}$ g is the mass of the neutron\(^1\).

Therefore, the equation of state of the Bose-Einstein condensate is a polytrope with index $n = 1$. In this case the solution of the Lane-Emden equation can be obtained in an analytical form, and the solution satisfying the boundary condition $\theta(0) = 1$ is [32]

$$\theta(\xi) = \frac{\sin \xi}{\xi}.$$  

The radius of the star is defined by the condition $\theta(\xi_1) = 0$, giving $\xi_1 = \pi$. Therefore the radius $R$ of the Bose-Einstein condensate is given by

$$R = \pi \sqrt{\frac{\hbar^2 a}{G m^3}} = 6.61 \left( \frac{a}{1 \text{ fm}} \right)^{1/2} \left( \frac{m}{2m_n} \right)^{-3/2} \text{ km}.$$  

For $m = 2m_n$ and $a = 1$ fm the radius of the condensate is $R \approx 7$ km. The radius of the gravitationally bounded Bose-Einstein condensate is independent on the central density and on the mass of the star, and depends only on the physical characteristics of the condensate.

\(^1\) In adopting this scaling, we have in mind the possibility that neutrons in the core of neutron stars form the equivalent of Cooper pairs and behave as bosons of mass $2m_n$. This means that we treat the core of neutron stars as a superfluid (see Section [VIII] for additional comments). However, our study may be valid in other circumstances so that we leave the mass $m$ unspecified.
The mass of the star is obtained as

$$M = 4\pi^2 \left( \frac{\hbar^2 a}{Gm^3} \right)^{3/2} \rho_c,$$

(34)

where we have used $|\theta'(\xi)| = 1/\pi$. As a function of the central density and of the coherent scattering length $a$, we obtain for the mass of the Bose-Einstein condensate star with quartic non-linearity the expression

$$M = 1.84 \left( \frac{\rho_c}{10^{16} \text{ g/cm}^3} \right) \left( \frac{a}{1 \text{ fm}} \right)^{3/2} \left( \frac{m}{2m_n} \right)^{-9/2} M_\odot.$$

(35)

For $m = 2m_n$, $a = 1 \text{ fm}$ and $\rho_c = 5 \times 10^{15} \text{ g/cm}^3$, the mass of the condensate is $M \approx 0.92 M_\odot$. However, this mass may be larger than the maximum mass allowed by general relativity (see below), and for a correct determination of the maximum mass of BEC stars we cannot ignore the effects induced by the space-time curvature.

The mass of the static condensate can be expressed in terms of the radius and central density by

$$M = \frac{4}{\pi} \rho_c R^3,$$

(36)

which shows that the mean density of the star $\bar{\rho} = 3M/4\pi R^3$ can be obtained from the central density of the condensate by the relation $\bar{\rho} = 3\rho_c/\pi^2$.

With respect to a scaling of the parameters $m$, $a$ and $\rho_c$ of the form $m \to \alpha_1 m$, $a \to \alpha_2 a$, $\rho_c \to \alpha_3 \rho_c$, the radius and the mass of the condensate have the following scaling properties:

$$R \to \alpha_1^{-3/2} \alpha_2^{1/2} R, \quad M \to \alpha_1^{-9/2} \alpha_2^{-3/2} \alpha_3 M.$$

(37)

**B. Slowly rotating Bose-Einstein Condensate stars**

The case of slowly rotating Bose-Einstein condensates can also be straightforwardly analyzed, by taking into account the fact that the condensate obeys a polytropic equation of state. The study of the slowly rotating polytropes was performed in detail in [32].

The Lane-Emden equation for a rotating Bose-Einstein condensate is

$$\frac{1}{\xi^2} \frac{\partial}{\partial \xi} \left( \xi^2 \frac{\partial \Theta}{\partial \xi} \right) + \frac{1}{\xi^2} \frac{\partial}{\partial \xi} \left[ (1 - \mu^2) \frac{\partial \Theta}{\partial \mu} \right] = -\Theta^n + \Omega,$$

(38)

where $\mu = \cos \theta$ and $\Omega = \omega^2/2\pi G \rho_c$. The volume $V_\omega$ and the mass $M_\omega$ of the condensate in slow rotation are given in the first order in $\Omega$ by

$$V_\omega = V_0 \left[ 1 + \frac{3\psi_0(\xi_1)}{\xi_1 |\theta'(\xi_1)|} \Omega \right],$$

(39)

and

$$M_\omega = M_0 \left[ 1 + \frac{\xi_1/3 - \psi_0(\xi_1)}{|\theta'(\xi_1)|} \Omega \right],$$

(40)

respectively. In these expressions, $M_0$ is given by Eq. [23] and $V_0 = (4/3)\pi R_0^3$ where $R_0$ is given by Eq. [27]. The values of the function $\psi_0$ are tabulated in [32]. Equations (39) and (40) represent the mass and volume relations for two stars with equal central density, one rotating with an angular velocity $\omega$ and the other non-rotating.

In the case of Bose-Einstein condensates with quartic non-linearity, corresponding to a polytropic index $n = 1$, the Lane-Emden equation can be integrated exactly (yielding $\Theta(\xi) = \sin \xi/\xi$ and $\psi_0(\xi) = 1 - \sin \xi/\xi$), giving for the volume $V_\omega$ and mass $M_\omega$ of the rotating condensate the following simple relations

$$V_\omega = V_0 (1 + 3\Omega),$$

(41)

$$M_\omega = M_0 \left[ 1 + \left( \frac{\pi^2}{3} - 1 \right) \Omega \right].$$

(42)

**V. MAXIMUM MASS OF THE STATIC NEWTONIAN BOSE-EINSTEIN CONDENSATE STARS: QUALITATIVE TREATMENT**

The numerical values of the basic parameters (mass and radius) of the condensed object sensitively depend on the mass $m$ of the particle, on the scattering length $a$, and on the central density $\rho_c$: $R = R(a, m)$, $M = M(a, m, \rho_c)$. Of course, in general, the values of the mass and radius of the gravitational condensate depend on the adopted model for the non-linearity.

The scattering length $a$ is defined as the zero-energy limit of the scattering amplitude $f$, and it can be related to the particle scattering cross section $\sigma$ by the relation $\sigma = 4\pi a^2$ [1]. On the other hand, the notion that particles like, for example, the quarks, retain their usual properties and interactions at the very high densities in the neutron stars may not be viable [13, 34]. In our calculations we use a “hard” core approximation of the potential. Therefore we accept that at high densities the “hard” core potential is in the QCD range of 1 fm and the allowed values of the scattering length $a$ may generally be in the interval $0.5 \text{ fm} \leq a < 1 - 2 \text{ fm}$, corresponding to a scattering cross section of around 1 mb.

The transition temperature to a Bose Einstein Condensate of dense matter can be written as

$$T_c = \frac{2\pi \hbar^2}{\zeta^{2/3} k_B m^{2/3}} \rho^{2/3}_{2/3} = 1.65 \times 10^{12} \times \left( \frac{\rho}{10^{16} \text{ g/cm}^3} \right)^{2/3} \left( \frac{m}{2m_n} \right)^{-5/3} \text{K.}$$

(43)

Neutron stars are born with interior temperatures of the order of $2 - 5 \times 10^{11}$ K, but they rapidly cool down via neutrino emission to temperatures of less than $10^{10}$ K within minutes. Also strange matter, pion condensates, $\lambda$ hyperons, $\delta$ isobars, or free quark matter might form under the initial thermal conditions prevailing in the very young neutron star. Hence a condensation process can
take place in the very early stages of stellar evolution. If the core is composed of only “ordinary” matter (neutrons, protons, and electrons), then when the temperature drops below about $10^9$ K all particles are degenerate. We expect that after a hundred years or so the core will become superfluid [12], and this may also favor the possibility of a Bose-Einstein Condensation through the BCS-BEC crossover.

Restriction on the maximum central density and maximum mass of the condensate quartic non-linearity can be obtained from the study of the speed of sound, defined as $c_s^2 = \partial P/\partial \rho$. With the use of Eq. (30) we obtain $c_s^2 = 2K\rho$. The causality condition implies $c_s \leq c$, where $c$ is the speed of light.

By introducing the dimensionless parameter $\kappa$, defined as

$$\kappa = \left( \frac{a}{1 \text{ fm}} \right)^{1/2} \left( \frac{m}{2m_n} \right)^{-3/2},$$

the causality condition gives the following upper bound for the central density of the condensate:

$$\rho_c \leq \frac{m^3 c^2}{4\pi ah^2} = 2.42 \times 10^{16} \kappa^{-2} \text{ g/cm}^3.$$  (45)

With the use of Eqs. (44) and (45) we obtain the following restriction on the maximum mass of the Bose-Einstein condensate with quartic non-linearity:

$$M \leq \pi \frac{\hbar c^2/\sqrt{a}}{(Gm)^{3/2}} = 4.46 \kappa M_\odot.$$  (46)

For $m = 2m_n$ we obtain the condition $\rho_c \leq [2.42/a(\text{fm})] \times 10^{16} \text{ g/cm}^3$. By taking into account that for this range of high densities a physically reasonable value for the scattering length is $a \approx 1$ fm, we obtain the restriction on the maximum mass of the Bose-Einstein condensate star from the causality condition as $M \leq 4.46 M_\odot$.

A stronger bound on the central density can be derived from the condition that the radius of the star $R$ must be greater than the Schwarzschild radius $R_S = 2GM/c^2$, $R \geq R_S$. For a Bose-Einstein condensate star $R_S$ can be expressed as a function of the central density and of the radius as $R_S = 8GR^3\rho_c/\pi c^2$. Then the condition of stability against gravitational collapse gives

$$\rho_c \leq \frac{m^3 c^2}{8\pi ah^2} = 1.21 \times 10^{16} \kappa^{-2} \text{ g/cm}^3.$$  (47)

a relation which for the condensate star with $m = 2m_n$ and $a = 1$ fm leads to the constraint $\rho_c \leq 1.21 \times 10^{16} \text{ g/cm}^3$. The constraint on the maximum mass for the stellar type Bose-Einstein condensate can be formulated as

$$M \leq \pi \frac{\hbar c^2/\sqrt{a}}{(Gm)^{3/2}} = 2.23 \kappa M_\odot.$$  (48)

With $a = 1$ fm and $m = 2m_n$ we obtain for the maximum mass of the Bose-Einstein condensate star the restriction $M \leq 2.23 M_\odot$.

For the $n = 1$ polytrope the radius of the star is independent on the central density. Generally, one may consider $a$ as a free parameter, which must be constrained by the physics of the nuclear interactions taking place in the system. However, due to the possible dependence of the free scattering length $a$ on the mass density, in the case of Bose-Einstein condensates there may be (indirect) dependence of the radius on the central density of the star.

Finally, we would like to point out that the estimates on the maximum mass obtained in the present Section are qualitative with respect to the numerical factors, and more precise values of the maximum mass of the BEC stars will be obtained in the next Section by using a full general relativistic approach.

VI. GENERAL RELATIVISTIC BOSE-EINSTEIN CONDENSATE STARS

In the previous Sections we have considered the gravitationally bounded Bose-Einstein condensate stars in the framework of Newtonian gravity. General relativistic effects may change the physical properties of compact objects in both a qualitative and quantitative way. For example, general relativity imposes a strict limit on the maximum mass of a stable compact astrophysical object, a feature which is missing for classical Newtonian stars. Therefore the study of the general relativistic Bose-Einstein condensates offers a better understanding of their physical properties. In the present Section, we study the properties of static general relativistic Bose-Einstein condensate stars.

A. Static general relativistic BEC stars

For a static spherically symmetric star, the interior line element is given by

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\mu(r)}dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2).$$  (49)

The structure equations describing a general relativistic compact star are the mass continuity equation and the Tolman-Oppenheimer-Volkoff (TOV) equation of standard general relativity, and they are given by [12]

$$\frac{dM}{dr} = 4\pi \rho r^2,$$  (50)

$$\frac{dP(r)}{dr} = -G \frac{\left( \rho + P/c^2 \right) (4\pi P r^3/c^2 + M)}{r^2 [1 - 2GM(r)/c^2r]}. $$  (51)

These equations extend the classical condition of hydrostatic equilibrium for a self-gravitating gas to the context
of general relativity. We have written the energy density as \( \epsilon = \rho c^2 \). The system of equations \([50], [51]\) must be closed by choosing the equation of state for the thermodynamic pressure of the matter inside the star,

\[
P = P(\rho).
\]

At the center of the star, the mass must satisfy the boundary condition

\[
M(0) = 0.
\]

For the thermodynamic pressure \( P \) we assume that it vanishes on the surface, \( P(R) = 0 \).

The exterior of the Bose-Einstein condensate star is characterized by the Schwarzschild metric, describing the vacuum outside the star, and given by \([13]\)

\[
(\epsilon^\nu)^{\text{ext}} = (\epsilon^{-\nu})^{\text{ext}} = 1 - \frac{2GM}{c^2 r}, \quad r \geq R.
\]

The interior solution must match with the exterior solution on the vacuum boundary of the star.

B. Maximum mass of non-relativistic BECs with short-range interaction: \( n = 1 \) polytropes

We assume that, in general relativity, the BEC can still be described by the non-relativistic equation of state

\[
P = K \rho^2, \quad \text{with} \quad K = \frac{2\pi a \hbar^2}{m^3},
\]

(55)

corresponding to a polytropic equation of state, with polytropic index \( n = 1 \). The theory of polytropic fluid spheres in general relativity has been developed by Tooper \([32]\), and we shall use his formalism and notations. Therefore, we set

\[
\rho = \rho_c \theta, \quad p = K \rho_c^2 \theta^2, \quad \sigma = \frac{K \rho_c}{c^2},
\]

(56)

\[
r = \frac{\xi}{A}, \quad M(r) = \frac{4\pi \rho_c}{A^3} v(\xi), \quad A = \left( \frac{2\pi G}{K} \right)^{1/2},
\]

(57)

where \( \rho_c \) is the central density. In terms of these variables, the TOV equation and the mass continuity equation become

\[
\frac{d\theta}{d\xi} = -\frac{(1 + \sigma \theta)(v + 3\xi \theta^2)}{\xi^2(1 - 4\sigma v / \xi)},
\]

(58)

\[
\frac{dv}{d\xi} = \theta \xi^2.
\]

(59)

For a given value of the density parameter \( \sigma \), they have to be solved with the initial condition \( \theta(0) = 1 \) and \( v(0) = 0 \). Since \( v \sim \xi^3 \) as \( \xi \to 0 \), it is clear that \( \theta'(0) = 0 \). On the other hand, the density vanishes at the first zero \( \xi_1 \) of \( \theta \):

\[
\theta(\xi_1) = 0.
\]

(60)

This determines the boundary of the sphere. In the non-relativistic limit \( \sigma \to 0 \), the system \([68]-[70]\) reduces to the Lane-Emden equation \([26]\) with \( n = 1 \).

From the foregoing relations, we find that the radius, the mass and the central density of the configuration are given by

\[
R = \xi_1 R_\star, \quad M = 2\sigma v(\xi_1) M_\star, \quad \rho_c = \sigma \rho_\star,
\]

(61)

where the scaling parameters \( R_\star, M_\star \) and \( \rho_\star \) can be expressed in terms of the fundamental constants and the parameter \( \kappa \) as

\[
R_\star = \left( \frac{a \hbar^2}{Gm^3} \right)^{1/2} = 2.106 \kappa \text{ km},
\]

(62)

\[
M_\star = \frac{\hbar^2 \sqrt{\pi}}{(Gm)^{3/2}} = 1.420 \kappa M_\odot,
\]

(63)

\[
\rho_\star = \frac{m^3 c^2}{2\pi a \hbar^2} = 4.846 \times 10^{16} \kappa^{-2} \text{ g/cm}^3.
\]

(64)

We note that the expression of the scaled radius \( R_\star \) is the same as in the Newtonian regime (in particular it is independent on \( c \)), while the scaling of the mass and of the density are due to relativistic effects. By varying \( \sigma \) from 0 to \( +\infty \), we obtain the series of equilibria in the form \( M(\rho_c), R(\rho_c) \) and \( M(R) \).

The velocity of sound is \( c_\sigma = p'(\rho) = 2K \rho \). The condition that the velocity of sound at the center of the configuration (where it achieves its largest value) is smaller than the velocity of light can be expressed as \( 2K \rho_c \leq c^2 \), or equivalently as \( \sigma \leq \sigma_s \) with

\[
\sigma_s = \frac{1}{2}.
\]

(65)

The values of \( \xi_1 \) and \( v(\xi_1) \) at this point have been tabulated by Tooper (and confirmed by our numerical study):

\[
\xi_1 = 1.801, \quad v(\xi_1) = 0.4981.
\]

(66)

The corresponding values of radius, mass and central density are

\[
R_\star = 1.801 \left( \frac{a \hbar^2}{Gm^3} \right)^{1/2} = 3.790 \kappa \text{ km},
\]

(67)

\[
M_\star = 0.498 \frac{\hbar^2 \sqrt{a}}{(Gm)^{3/2}} = 0.707 \kappa M_\odot,
\]

(68)

\[
(\rho_c)_s = \frac{m^3 c^2}{4\pi a \hbar^2} = 2.423 \times 10^{16} \kappa^{-2} \text{ g/cm}^3.
\]

(69)
However, it is not granted that the criterion $\sigma > \sigma_s$ is equivalent to the condition of dynamical instability. The principle of causality is a necessary, but not a sufficient, condition of stability\(^2\). The condition of dynamical instability corresponds to the turning point of mass $dM = 0$ and there is no reason why this should be equivalent to $c_s = c$. In fact, our numerical study demonstrates that this is not the case. We find that the maximum mass does not exactly correspond to the point where the velocity of sound becomes equal to the velocity of light. In the series of equilibria (parameterized by the central density $\sigma$), the instability occurs sooner than predicted by the criterion \(\sigma > \sigma_s\). We find indeed that instability (corresponding to the mass peak) occurs for $\sigma \geq \sigma_c$ with

$$\sigma_c = 0.42.$$  \(\text{(70)}\)

The values of $\xi_1$ and $v(\xi_1)$ at this point are

$$\xi_1 = 1.888, \quad v(\xi_1) = 0.5954.$$  \(\text{(71)}\)

The corresponding values of radius, mass and central density are

$$R_{\text{min}} = 1.888 \left( \frac{ah^2}{Gm^3} \right)^{1/2} = 3.974 \kappa \text{ km},$$  \(\text{(72)}\)

$$M_{\text{max}} = 0.5001 \frac{hc}{(Gm)^{3/2}} = 0.710 \kappa M_\odot,$$  \(\text{(73)}\)

$$(\rho_c)_{\text{max}} = 0.42 \frac{m^2 a^2}{2\pi hl^2} = 2.035 \times 10^{16} \kappa^{-2} \text{ g/cm}^3.$$  \(\text{(74)}\)

We also note that the radius of a BEC star is necessarily smaller than

$$R_{\text{max}} = \sqrt{\frac{h^2 a}{Gm^3}} = 6.61 \kappa \text{ km},$$  \(\text{(75)}\)

corresponding to the Newtonian limit ($\sigma \rightarrow 0$). Therefore, its value $3.974 \kappa \leq R(\text{km}) \leq 6.61 \kappa$ is very much constrained.

The dimensionless curves giving the mass-central density, radius-central density, mass-radius relations and some density profiles are plotted in Figs. [1][2]. In Fig. [6] we present the mass-radius relation for $a = 1$ fm and different values of $m$.

### C. Maximum mass of relativistic BECs with short-range interaction

The previous treatment is approximate because we use the equation of state \[65\] obtained in the non-relativistic regime (i.e. from the Gross-Pitaevskii equation) but solve the TOV equation expressing the condition of hydrostatic equilibrium in general relativity. A fully relativistic approach based on the Klein-Gordon-Einstein system has been developed by Colpi \textit{et al.} \[22\]. In order to make the...
FIG. 3: Dimensionless radius-central density relation of a relativistic BEC with short-range interactions modeled by a \( n = 1 \) polytrope.

FIG. 4: Dimensionless mass-radius relation of a relativistic BEC with short-range interactions modeled by a \( n = 1 \) polytrope. There exists a maximum mass \( M_{\text{max}}/M_\ast = 0.5001 \) and a minimum radius \( R_{\text{min}}/R_\ast = 1.888 \) corresponding to a maximum central density \( \rho_\ast/\rho_\ast = 0.42\rho_\ast \). There also exists a maximum radius \( R_{\text{max}}/R_\ast = \pi \) corresponding to the Newtonian limit \( \sigma \to 0 \).

FIG. 5: Dimensionless density profiles corresponding to \( \sigma = 0 \) (Newtonian), \( \sigma = \sigma_c = 0.42 \) (maximum mass) and \( \sigma = \sigma_s = 1/2 \) (where \( c_s = c \)).

FIG. 6: Mass-radius dependence for general relativistic Bose-Einstein condensates with quartic non-linearity for \( a = 1 \) fm and different values of the mass \( m \). From top to bottom: \( m = m_n \), \( m = 1.25m_n \), \( m = 1.5m_n \), \( m = 1.75m_n \) and \( m = 2m_n \). For all configurations \( \rho_c \geq \rho_n \), where \( \rho_n = 2.026 \times 10^{14} \text{ g/cm}^3 \) is the nuclear density, and the causality condition \( c_s \leq c \) is satisfied.

The correspondence between BECs with short-range interactions described by the Gross-Pitaevskii equation and scalar fields with a \( \frac{1}{4} \lambda |\phi|^4 \) interaction described by the Klein-Gordon equation, we set \[ \lambda \equiv \frac{a \lambda_c}{8\pi} \approx \frac{a m c}{\hbar}, \] (76)

where \( \lambda_c = \hbar/mc \) is the Compton wavelength of the bosons. The equation of state \[ P = K \rho^2, \quad \text{with} \quad K = \frac{\lambda h^3}{4m^4c^3}, \] (77)

This returns the equation of state obtained by Arbey et al. \[ \lambda = \frac{a}{8\pi} \left( \frac{m}{2m_n} \right)^2 \] (78)

relativistic regime showing that the relation between \( a \) and \( \lambda \) given by Eq. (76) is correct. We note that the parameter \( \lambda \) can be expressed as

\[ \frac{\lambda}{8\pi} = \frac{a}{1 \text{ fm}} \left( \frac{m}{2m_n} \right). \] (78)

We can then use \( (\lambda, m) \) instead of \( (a, m) \) as independent physical variables. Finally, the dimensionless parameter \( \kappa \) can be written

\[ \kappa = 0.324 \left( \frac{\lambda}{8\pi} \right)^{1/2} \left( \frac{2m_n}{m} \right)^2. \] (79)

We can now express the results in terms of \( \lambda \). The
scaling of the maximum mass is given by
\[ M_\text{max} = \frac{h c^2}{(G m)^{3/2}} = \sqrt{\frac{\lambda}{8 \pi}} \frac{1}{m^2} (\frac{h c}{G})^{3/2} = \sqrt{\frac{\lambda M_\text{P}^3}{8 \pi \ h}}, \tag{80} \]
where \( M_\text{P} = (hc/G)^{1/2} \) is the Planck mass. This is the scaling of the maximum mass obtained by Colpi et al. \[22\] for a self-interacting scalar field. For \( \lambda \sim 1 \), it is of the order of the Chandrasekhar mass \( \sim M_\text{P}^3/m^2 \). On the other hand, the scaling of the minimum radius is given by
\[ R_\text{min} = \left( \frac{a h^2}{G m^3} \right)^{1/2} = \sqrt{\frac{\lambda h^3}{8 \pi G c}} \frac{1}{m^2} \frac{1}{\hbar} = \sqrt{\frac{\lambda M_\text{P}^3}{8 \pi \ m \ h}}. \tag{81} \]
This is the scaling of the minimum radius given by Arbey et al. \[23\] for a self-interacting scalar field. Finally, the scaling of the maximum density is given by
\[ \rho_\text{max} = \frac{m^3 c^2}{2 \pi \ h \ a^2} = \frac{4 m^4 c^3}{\lambda h^3}. \tag{82} \]
Now, the maximum mass obtained by Colpi et al. \[22\] in the fully relativistic regime is
\[ M_\text{max} = 0.22 \frac{\lambda M_\text{P}^3}{4 \pi m^2} = 0.22 \sqrt{2} M_\ast = 0.31 M_\ast, \tag{83} \]
which is smaller than our previous estimate \( M_\text{max} = 0.5001 M_\ast \) based on a non-relativistic equation of state. Therefore, relativistic effects tend to reduce the maximum mass.

Colpi et al. \[22\] showed that, in the Thomas-Fermi limit, the scalar field becomes equivalent to a fluid with an equation of state
\[ P = \frac{c^4}{36 K} \left[ \left( 1 + \frac{12 K}{c^2} \rho \right)^{1/2} - 1 \right]^2, \tag{84} \]
where \( K \) is given by Eq. \[74\]. For \( \rho \to 0 \) (low or moderate densities), we recover the polytropic equation of state \( p = K \rho^2 \) corresponding to a non-relativistic BEC with short-range interactions. For \( \rho \to +\infty \) (extremely high densities), we obtain the ultra-relativistic equation of state \( p = \rho c^2/3 \), similar to the one describing the core of neutron stars modeled by the ideal Fermi gas \[37–39\]. We know that a linear equation of state \( p = q \rho c^2 \) yields damped oscillations of the mass-central density relation, and a spiral structure of the mass-radius relation \[39\], similarly to the isothermal equation of state in Newtonian gravity \[40\]. Therefore, our BEC model will exhibit this behaviour, just like standard neutron stars. However, our BEC model differs from standard neutron star models in that, at low or moderate densities, \( p = K \rho^2 \) with \( K = 2 \pi \ h a^2/m^3 \) instead of \( p = K' \rho^{5/3} \) with \( K' = (1/5)(3/8 \pi)^{2/3} \ h / m^{8/3} \). This implies, in particular, the existence of a maximum radius given by Eq. \[74\], corresponding to the Newtonian limit.

Substituting the equation of state \[84\] in the TOV equations, using
\[ P'(\rho) = \frac{1}{3} c^2 \left[ 1 - \frac{1}{\sqrt{1 + 12 K \rho / c^2}} \right], \tag{85} \]
and introducing the same notations as before, we obtain
\[ \frac{d\theta}{d\xi} = -\frac{6 \frac{\xi}{\theta} (\sqrt{1 + 12 \sigma \theta} - 1)^2 + \sigma \theta}{\xi^2 (1 - 4 \sigma v/\xi)(1 - 1/\sqrt{1 + 12 \sigma})}, \tag{86} \]
\[ \frac{dv}{d\xi} = \theta \xi^2, \tag{87} \]
instead of Eqs. \[58\]–\[59\]. If we expand the square roots for \( \sigma \ll 1 \), we recover Eqs. \[58\]–\[59\]. However, this is not a uniform expansion and the two equations \[58\]–\[59\] and \[58\]–\[59\] are in fact different even for small values of \( \sigma \) (of course, they both reduce to the Lane-Emden equation \[20\] for \( \sigma = 0 \)).

The velocity of sound at the center of the configuration is
\[ (c_s^2)_0 = \frac{1}{3} c^2 \left( 1 - \frac{1}{\sqrt{1 + 12 \sigma}} \right), \tag{88} \]
and we always have \((c_s)_0 < c\). The series of equilibria becomes unstable after the first mass peak. We find that instability occurs for \( \sigma \geq \sigma_c' \) with
\[ \sigma_c' = 0.398. \tag{89} \]
The values of \( \xi_1 \) and \( v(\xi_1) \) at this point are
\[ \xi_1 = 1.923, \quad v(\xi_1) = 0.3865 \tag{90} \]
The corresponding values of the radius, mass and central
density are

\[ R'_{\text{min}} = 1.923 \left( \frac{ah^2}{Gm^3} \right)^{1/2} = 4.047 \kappa \text{ km}, \] (91)

\[ M'_{\text{max}} = 0.307 \frac{hc^2\sqrt{\pi}}{(Gm)^{3/2}} = 0.436 \kappa M_\odot, \] (92)

and

\[ \rho'_{\text{max}} = 0.398 \frac{m^3c^2}{2\pi ah^2} = 1.929 \times 10^{16} \kappa^{-2} \text{ g/cm}^3, \] (93)

respectively. The maximum mass \( M_{\text{max}} = 0.307 M_\odot \) is very close to the one [see Eq. (83)] found by Colpi et al. [22] by solving the Klein-Gordon-Einstein equations. This shows the accuracy of the hydrodynamical approach in the TF limit. The dimensionless curves giving the mass-central density relation, radius-central density, mass-radius relations and some density profiles are plotted in Figs. 7-9.

**Remark:** At \( T = 0 \), the first law of thermodynamics takes the form

\[ d\rho = \frac{P/\epsilon^2 + \rho}{n} dn, \] (94)

where \( nm \) is the rest-mass density. Integrating this relation with the equation of state [33], we can obtain the relation \( n(\rho) \). For \( \rho \to 0 \) (non-relativistic limit), we get \( \rho = nm \), leading to \( P \sim K\rho^2 = K(nm)^2 \) corresponding to a polytrope \( n = 1 \). For \( \rho \to +\infty \) (ultra-relativistic limit), we get \( \rho \propto (nm)^{4/3} \), leading to \( P \sim \rho c^2/3 \propto (nm)^{4/3} \) corresponding to a polytrope \( n = 3 \) like for an ultra-relativistic Fermi gas at \( T = 0 \) (standard neutron star). These results are consistent with those obtained by Goodman [24]. The proper number of particles is

\[ N = \int_0^R n(r) \left[ 1 - \frac{2GM(r)}{c^2r} \right]^{-1/2} 4\pi r^2 \, dr. \] (95)

It can be shown that a general relativistic, spherically symmetric, gaseous star at \( T = 0 \) is dynamically stable with respect to the Einstein equations if, and only if, it is a maximum of \( N(\rho) \) at fixed mass \( M(\rho) = M \) (see, e.g., [43, 44]). The first order variations \( \delta N - \alpha \delta M = 0 \), where
α is a Lagrange multiplier, yield the TOV equations determining steady state solutions. The ensemble of these solutions forms the series of equilibria. Then, using the Poincaré theorem [44], one can conclude that the series of equilibria becomes unstable at the first mass peak and that a new mode of instability appears at each turning point of mass in the series of equilibria (see [44] for an alternative derivation of these results). At these points, we have δN = δM = 0 so that the curve N(M) presents cusps at each point where M(ρc) reaches an extremal value. An illustration of this behavior is given in Figure 5 of [43]. These results of dynamical stability for general relativistic stars are similar to results of dynamical and thermodynamical stability for Newtonian self-gravitating systems [36, 40, 45].

**VII. ASTROPHYSICAL IMPLICATIONS**

One of the most important results in general relativistic astrophysics is the existence of a maximum mass of the neutron stars [47]. Ultra-dense compact objects may have a stable equilibrium configuration until their mass M is equal to the maximum mass Mmax. By integrating the mass continuity equation and the hydrostatic equilibrium equation for a star made of free, non-interacting, neutron gas, Oppenheimer and Volkoff [37] have shown that the maximum equilibrium mass is M_{OV} = 0.7M_\odot, with a corresponding radius of the order of R_{OV} = 9.6 km, and a central density of the order of \( \rho_c = 5 \times 10^{15} \text{ g/cm}^3 \). Using a variational method in which the equation of state was constrained to have sub-luminal sound velocity and to be stable against microscopic collapse, Rhodes and Ruffini [40], proved that, in the regions where it is uncertain, the equation of state that produces the maximum neutron star mass is the one for which the sound speed is equal to the speed of light, i.e. \( c^2 = \rho c^2 \). As a result, they found a maximum neutron star mass M_{max} \approx 3.2M_\odot, assuming uncertainty in the equation of state above a fiducial density \( \rho_0 \approx 4.6 \times 10^{14} \text{ g/cm}^3 \). More realistic models that take into account the composition of the star and the interaction between neutrons led to values of the maximum mass of the neutron stars in the range 1.5 – 3.2M_\odot [47]

The main reason for the lack of a better theoretical value of the maximum mass of the neutron star is the poor knowledge of the equation of state of hadronic matter at high densities.

With the use of Eqs. [62]-[64] it follows that the scaled mass, radius and central density satisfy the relations

\[
\frac{R_s}{M_s} = \frac{G}{c^2}, \quad \rho_s M_s^2 = \frac{c^6}{2\pi G^3}, \quad \rho_s R_s^2 = \frac{c^2}{2\pi G}.
\]

With the use of Eqs. [61], [70], [71] and [96], we obtain the following radius-mass, central density-mass, and central density-radius relations for the maximally stable non-relativistic BEC configuration,

\[
R_{\text{min}} = \frac{\xi_1}{2\sigma_c v(\xi_1)} \frac{G M_{\text{max}}}{c^2} = 5.599 \frac{M_{\text{max}}}{M_\odot} \text{ km},
\]

\[
(\rho_c)_{\text{max}} = 4\sigma_c^2 v^2(\xi_1) \frac{c^6}{2\pi G^3 M_{\text{max}}^2} = 1.026 \times 10^{16} \left( \frac{M_\odot}{M_{\text{max}}} \right)^2 \frac{\text{g}}{\text{cm}^3},
\]

\[
(\rho_c)_{\text{max}} = \sigma_c^2 \frac{c^6}{2\pi G R_{\text{min}}^2} = 3.215 \times 10^{15} \left( \frac{10 \text{ km}}{R_{\text{min}}} \right)^2 \frac{\text{g}}{\text{cm}^3}.
\]

For the relativistic BEC star, using Eqs. [89] and [90], we find

\[
R_{\text{min}} = 9.271 \frac{M_{\text{max}}}{M_\odot} \text{ km},
\]

\[
(\rho_c)_{\text{max}} = 3.682 \times 10^{15} \left( \frac{M_\odot}{M_{\text{max}}} \right)^2 \frac{\text{g}}{\text{cm}^3},
\]

\[
(\rho_c)_{\text{max}} = 3.160 \times 10^{15} \left( \frac{10 \text{ km}}{R_{\text{min}}} \right)^2 \frac{\text{g}}{\text{cm}^3}.
\]

In the non-relativistic case, the mass-radius ratio of the star can be expressed as

\[
\frac{2GM_{\text{max}}}{c^2 R_{\text{min}}} = \frac{4\sigma_c v(\xi_1)}{\xi_1} = 0.529,
\]

while, for the relativistic case, we obtain

\[
\frac{2GM_{\text{max}}}{c^2 R_{\text{min}}} = 0.319.
\]
A classical result by Buchdahl [48] shows that for static solutions of the spherically symmetric Einstein-matter systems, the total mass $M$ and the area radius $R$ of the boundary of the body obey the relation $2GM/c^2R \leq 8/9 = 0.888$, the equality sign corresponding to constant density stars. For BEC stars, Eqs. (103) and (104) obviously satisfy the Buchdahl inequality for the mass-radius ratio.

We emphasize that the radius $R_{\text{min}}$ and the central density $(\rho_c)_{\text{max}}$ given by Eqs. (99)-(102) only depend on the mass $M_{\text{max}}$. In particular, they do not explicitly depend on the two physical parameters of the model, the scattering length $a$, and the particle mass $m$. On the other hand, the maximum mass $M_{\text{max}}$ depends on these two parameters only through their ratio $a/m^3$, or equivalently, through the parameter $\kappa$, and it can be obtained from the relations

$$M_{\text{max}} = 0.71 \kappa M_\odot, \quad M'_{\text{max}} = 0.4368 \kappa M_\odot,$$  

respectively. Hence, all physical parameters of the model are determined by the mass $M_{\text{max}}$ of the star, which can be obtained from observations. Therefore, in the present model we have only one free parameter $\kappa = (a/\text{fm})^{1/2} (2m_n/m)^{3/2}$. With respect to a scaling of the scattering length and of the particle mass of the form

$$a/\text{fm} \rightarrow \beta_1 (a/\text{fm}), \quad m/2m_n \rightarrow \beta_2 (m/2m_n), \quad (106)$$

where $\beta_1, \beta_2$ are constants, the parameter $\kappa$ scales as

$$\kappa \rightarrow \beta_1^{1/2} \beta_2^{-3/2} \kappa.$$  

Since $\kappa = 0.324 (\lambda/8\pi)^{1/2} (2m_n/m)^2$, we equivalently conclude that the maximum mass $M_{\text{max}}$ depends on the two parameters $(\lambda, m)$ only through their ratio $\lambda/m^4$.

By assuming that the mass of the star is $M_{\text{max}} = 2M_\odot$, in the non-relativistic case we obtain for the parameters of the star $(\rho_c)_{\text{max}} = 0.256 \times 10^{16} \text{ g/cm}^3$ and $R_{\text{min}} = 11.2$ km, respectively, independently on the values of $a$ and $m$. On the other hand in this case $\kappa = 2.816$. If we take $m = 2m_n$, this corresponds to a scattering length $a = 7.93$ fm, and a coupling constant $\lambda = 1.90 \times 10^3$. For the relativistic equation of state, we find $(\rho_c)_{\text{max}} = 0.091 \times 10^{16} \text{ g/cm}^3$ and $R_{\text{min}} = 18.54$ km. For the parameter $\kappa$ we obtain $\kappa = 4.578$. If we take $m = 2m_n$, this corresponds to a scattering length $a = 21.0$ fm and a coupling constant $\lambda = 5.02 \times 10^3$. Therefore if we assume values of $\kappa$ of the order of $\kappa \approx 3$ in the non-relativistic regime, and $\kappa \approx 5$ in the relativistic regime, we obtain stellar objects with physical parameters in the range $M \sim 2M_\odot$, $R \sim 10 \sim 20$ km, and $\rho_c \sim 0.3 \sim 0.1 \times 10^{16} \text{ g/cm}^3$, respectively. The only free parameter in the model, $\kappa$, uniquely determines the mass $M_{\text{max}}$ of the star (or conversely).

It may be of interest to make a connection with the results of Oppenheimer & Volkoff [51]. In their model, the mass, the radius, and the central density of the critical configuration are

$$M_{\text{OV}} = 0.376 \left( \frac{\hbar c}{G} \right)^{3/2} \frac{1}{m_n^2} = 0.7 M_\odot,$$  

$$R_{\text{OV}} = 9.36 \frac{GM_{\text{OV}}}{c^2} = 9.6 \text{ km},$$

$$\left(\rho_c\right)_{\text{OV}} = 3.92 \times 10^{-3} \frac{\epsilon^6}{G^2 M_{\text{OV}}^2} = 5 \times 10^{15} \text{ g/cm}^3.$$  

In our model, introducing the parameter $\lambda$, the maximum mass is given by

$$M_{\text{max}} = 2\sigma_c v(\xi_1) \sqrt{\frac{\lambda}{8\pi}} \left( \frac{\hbar c}{G} \right)^{3/2} \frac{1}{m^2}. \quad (111)$$

If we write the boson mass as $m = km_n$, we obtain the maximum mass in the form

$$M_{\text{max}} = C\sqrt{\frac{\lambda}{k^2}} M_{\text{OV}},$$  

where $C = 0.265$ for a non-relativistic equation of state and $C = 0.163$ for a relativistic equation of state. Using the previous relations, we can then easily relate $R_{\text{min}}$ and $(\rho_c)_{\text{max}}$ to $R_{\text{OV}}$ and $(\rho_c)_{\text{OV}}$. Equation (112) clearly shows that, with respect to the standard Oppenheimer-Volkoff model, we have an additional parameter $\lambda$ (the strength of the self-interaction), which gives the possibility of obtaining higher values for the maximum mass. Using Eq. (79) which becomes $\kappa = 0.258\sqrt{\lambda}/k^2$, we can rewrite Eq. (112) as

$$M_{\text{max}} = 3.87 C\kappa M_{\text{OV}}.$$

Presently, there is conclusive observational evidence from pulsar studies for the existence of neutron stars with masses significantly greater than $1.5M_\odot$ [49]. By using the Shapiro time delay to measure the inclination, the mass of PSR J1614-223048 was recently determined to be $1.97 \pm 0.04M_\odot$ [50]. Moreover, a number of X-ray binaries seem to contain high-mass neutron stars: about $1.9M_\odot$ in the case of Vela X-1 and $2.4M_\odot$ in the case of 4U 1700-377 [10]. Even more intriguing is the case of the black widow pulsar B1957+20, with a best mass estimate of about $2.4M_\odot$ [51]. This system has both pulsar timing and optical light curve information. B1957+20 is located in an eclipsing binary system, consisting of the $1.6$ ms pulsar in a nearly circular $9.17$ h period orbit, and an extremely low mass companion, $M_e \approx 0.03M_\odot$. It is believed that irradiation of the companion by the pulsar strongly heats its cosmic environment to the point of ablation, leading to a comet-like tail, and a large cloud of plasma. The plasma cloud is responsible for the eclipsing. The pulsar is literally consuming its companion, hence the name black widow. The mass of the companion star has been reduced to a small fraction of its original.
mass. On the other hand, a measured mass of 2.4\(M_\odot\) would be incompatible with hybrid star models containing significant proportions of exotic matter in the form of hyperons, some forms of Bose condensates, or quark matter.

However, the mass and radius of the 2 - 2.4\(M_\odot\) neutron stars perfectly fit the expected properties of a Bose-Einstein condensate star. For \(\kappa \sim 3\), the mass of a typical general relativistic Bose-Einstein condensate star is of the order of two solar masses, with a radius of around 11 km. Therefore, we propose that the recently observed 2 - 2.4\(M_\odot\) mass neutron stars could be typical Bose-Einstein condensate stars.

A last comment may be in order. If we apply the same model (self-gravitating BEC with short-range interactions) to dark matter \({20}\), and use the Newtonian approximation (which is valid in this context), the radius of a dark matter halo is given by Eq. \({35}\), which can be rewritten

\[
R = 1.746 \times 10^{-2} \left( \frac{a}{1 \text{ fm}} \right)^{1/2} \left( \frac{m}{1 \text{ eV}/c^2} \right)^{-3/2} \text{kpc}. \tag{114}\]

Again, we note that the radius \(R\) determines the ratio \(a/m^3\) or \(\lambda/m^4\). Estimating the radius of dark matter halos by \(R = 10\) kpc, we obtain \(m^3/a = 3.049 \times 10^{-6} \text{(eV/c}^2)^3/\text{fm}\) and \(m^4/\lambda = 23.94 \text{(eV/c}^2)^4\). \(\star\)

VIII. DISCUSSIONS AND FINAL REMARKS

In the present paper, we have proposed that the core of neutron stars is a superfluid in which the neutrons form equivalent of Cooper pairs, so that they act as bosons of mass \(2m_\pi\). Therefore, once the Bose-Einstein condensation takes place, the neutron star should be modeled as a self-gravitating BEC star (boson star). In our approach, we also assume that the bosons have a self-interaction, described by a scattering length \(a\). The basic properties of the gravitationally bounded Bose-Einstein condensates have been obtained in both Newtonian and general relativistic regimes. To obtain the physical characteristics of the system, we have used the Madelung representation in which condensates can be modeled by using the hydrodynamic Euler equations describing a gas whose density and pressure are related by a barotropic equation of state. For the study of the Bose-Einstein condensate we have adopted the Thomas-Fermi approximation, which is valid if the total number of particles \(N\) obeys the condition \(N \gg R/\pi a\). A condition which can be reformulated, with the use of the mass density of the condensate, as \(n \gg 3\pi/4\pi^2 R^2 a\). This restriction is obviously satisfied by condensates with densities of the same order as the nuclear density.

In the physically most interesting case, corresponding to a quartic non-linearity term in the energy functional, the equation of state of the Bose-Einstein condensate is that of a polytropic with polytropic index \(n = 1\). In this case, the radius and the mass of the Newtonian stellar condensate can be obtained in an exact form. In a Bose-Einstein condensed neutron star the mass \(m\) of the particle does not need to coincide with the neutron mass. For the mass of the condensed particle we have used an effective value of the order of \(m^* = 2m_\pi\). This value is justified by the high densities in the neutron star cores, where the process of Bose-Einstein condensation is most likely to occur via the formation of Cooper pairs. However, we have also explicitly presented the numerical values of the basic physical parameters of the stars for other values of the mass.

General relativistic effects impose strong constraints on the maximum mass. In the framework of the general relativistic approach one must numerically integrate the structure equations of the star. In this way, we obtain a large class of stable astrophysical objects, whose basic parameters (mass and radius) depend on the particle mass \(m\) and scattering length \(a\). Since the values of \(a\) and \(m\) are not well-known, this offers the possibility to obtain a maximum mass for neutron stars that is larger than the Oppenheimer-Volkoff limit of \(0.7M_\odot\), and may be compatible with recent observational determinations of the masses of some neutron stars. This is possible because we have two new parameters in our model, the boson mass \(m\) and the scattering length \(a\), which give additional freedom (although these parameters should be ultimately determined by fundamental physics). We have found that the maximum mass \(M_{\text{max}}(\kappa)\) of the condensate star, given by Eqs. \(\text{(73)}\) or \(\text{(92)}\), depends in fact on a single parameter \(\kappa(a, m)\) which is proportional to the ratio \(a/m^3\) (or, equivalently, to the ratio \(\lambda/m^4\)). Since the radius \(R_{\text{min}}\) and the density \(\rho_{\text{max}}\) depend only on \(M_{\text{max}},\) all the physical properties of the BEC stars are determined by the parameter \(\kappa\). Condensates with particle masses of the order of two neutron masses and scattering length of the order of \(10 - 20\) fm (corresponding to \(\kappa \sim 3 - 5\)) have maximum masses of the order of \(2M_\odot\), minimum radii in the range of \(10 - 20\) km and maximum central density \(\rho_c \sim 0.3 - 0.110^6\text{ g/cm}^3\) in the non-relativistic and relativistic regimes, respectively. On the other hand, for \(a = 1\) fm, the maximum mass of the condensate varies between \(0.4 - 0.7M_\odot\) for \(m = m_\pi\) (corresponding to \(\kappa = 1\)), between \(1 - 2M_\odot\) for \(m = m_n\) (corresponding to \(\kappa = 2.8\)), and between \(10 - 16M_\odot\) for \(m = m_n/4\) (corresponding to \(\kappa = 22\)), a value which may correspond, for example, to the effective (density dependent) kaon mass \(m_{K}^*\) in the interior of neutron stars. Kaon condensation may provide an important example of Bose-Einstein type stellar condensate. \(\star\) Attraction from nuclear matter could bring down the mass of the kaon to an effective value of \(m_{K}^* \approx 200\text{ MeV/c}^2 \approx m_n/9.38\text{ MeV/c}^2\). From Eq. \(\text{(63)}\) it follows that Newtonian kaon condensates with kaon effective mass of the order of \(m_{K}^* = m_n/10\) could have masses as high as \(10^6M_\odot\), and radii of the order of 600 km [see Eq. \(\text{(63)}\) \(\star\). Thus, Bose-Einstein condensate stars, formed from small mass particles, may represent vi-
able candidates for the super-massive “black holes” that reside at the galactic centers. However, general relativistic effects strongly restrict the value of the maximum mass of super-massive BEC stars. From Eq. (44) it follows that for $m = m_\Lambda = m_n/10$ we have $\kappa = 89.44$. With the use of Eq. (73) we obtain a maximum mass of $M = 63.50 M_\odot$ for the kaon condensate star, with a radius $R = 355 \text{ km}$ [see Eq. (72)]. On the other hand, smaller mass condensed stars can have significantly higher maximum relativistic masses. Hence the Bose-Einstein condensation process in the early universe may have provided the seeds from which super-massive black holes were eventually formed through accretion of interstellar matter.

Presently, the mass of the neutron stars can be determined very accurately, and many of them have masses in the range of $2 - 2.4$ solar masses, which are very difficult to explain by the standard neutron matter models, including those with exotic matter like quarks. However, these mass values could be very easily explained by our model if neutron stars can be modeled as BEC stars.

Bose-Einstein condensate stars could have a normal matter crust, since we expect that the condensation cannot take place at densities smaller than the nuclear density or quark deconfinement density. The presence of the thin crust increases the mass and the radius of the condensate star by a factor of 10% or 17%, respectively. Therefore, the presence of a neutron crust does not modify significantly the basic physical properties of the star. Distinguishing between Bose-Einstein condensate stars and “standard” neutron stars or other type of condensate or quark stars could be an extremely difficult observational task. Similarly to the case of quark stars, we suggest that high energy radiation processes from the surface of the condensate may provide some distinctive features allowing a clear differentiation of these different types of stellar objects.

In a very general approach, one may assume that the masses $m$ of the particles forming the stellar type condensate are anisotropic, and they should be described by a mass tensor $m_{ij}$. Such anisotropic masses are known from condensed matter physics where they are encountered in effective mass calculations for electrons immersed in a band structure, in the case of excitons (electron-hole couples held together by the Coulomb attraction) and in BEC for semiconductors. The doping structure of the semiconductor and its anisotropies would give place to an effective mass matrix for the paraexcitons (singlet excitons) at least in the low momentum approximation. A different value for the effective mass $m$ may considerably increase (or decrease) the total mass of the condensate.

A rotating Bose-Einstein condensate may exhibit a very complex internal structure and dynamics, mainly due to the presence of vortex lattices. The vortex lattices may evolve kinetically, with each vortex following the streamline of a quadrupolar flow. The quadrupolar distortions can lead to a disordering of the vortex lattice, and to an instability due to inter-particle collisions, finite temperature effects or to the quadrupolar distortions induced by the external potential. On the other hand, due to the high neutrino emissivity, which is significantly enhanced due to the condensation, kaon condensate stars are very dark objects. Hence their observational detection may prove to be an extremely difficult task. The possible astrophysical/observational relevance of these processes will be considered in a future publication.

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