Spectral partitions on infinite graphs

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Statistical models on infinite graphs may exhibit inhomogeneous thermodynamic behaviour at macroscopic scales. This phenomenon is of geometrical origin and may be properly described in terms of spectral partitions into subgraphs with well defined spectral dimensions and spectral weights. These subgraphs are shown to be thermodynamically homogeneous and effectively decoupled.

I. INTRODUCTION

The study of model systems without translation invariance is an interesting and complex subject of modern statistical mechanics. A very general description of this situation is in terms of statistical models on graphs, that is on generic networks formed by sites, where dynamical variables reside, and links connecting pairwise sites whose variables are coupled. This is the direct extension of the typical setup valid for crystalline lattices, which are indeed very special, homogeneous graphs.

On the other hand, graphs are not in general homogeneous and the main question is how these inhomogeneities affect physical properties and give rise to relevant changes with respect to lattices. While small scale inhomogeneities will affect local properties, one expects that only large scale inhomogeneities are relevant for bulk thermodynamic properties. Most likely, the latter properties are those that show universal features which depend only on a few global parameters, just as in the case of lattices. The study of such universality requires consideration of infinite graphs (with certain natural restrictions given below), where the thermodynamic limit is taken.

The main relevant geometrical parameter affecting universal properties is the spectral dimension $\tilde{d}$ of an infinite graph $\mathcal{G}$. It generalizes the Euclidean dimension of lattices to arbitrary real values and is naturally defined from the infrared behaviour of the spectral density of the Laplacian operator on $\mathcal{G}$. An equivalent definition, the one adopted in this work, is in terms of average properties of random walks on $\mathcal{G}$ at large times, that is to say of the singularities of the Gaussian model on the same graph.

On the other hand, the spectral dimension of the whole graph $\mathcal{G}$, by itself turns out not to be sensitive to macroscopic inhomogeneities strong enough to give rise to true thermodynamic inhomogeneities. Indeed it may happen that distinct macroscopic parts of an infinite graph exhibit distinct thermodynamic behaviour. We shall show below that such parts can be characterized in terms of their own spectral dimension, possibly plus a spectral weight, resulting in an effective spectral partition of $\mathcal{G}$. The crucial point is that these parts form subgraphs which are thermodynamically independent, that is to say completely uncoupled as far as thermodynamic properties are concerned. In other words, inhomogeneous thermodynamic behaviour on the same infinite graph necessarily imply effective decoupling.

II. INFINITE GRAPHS: BASIC DEFINITIONS, MEASURE AND AVERAGES

A (unoriented) graph $\mathcal{G}$ is the ordered couple $(G, G_L)$ formed by a countable set $G$ of vertices (or sites, or nodes), that we shall generically indicate with small-case Latin letters, $i, j, k, \ldots$, and a set $G_L$ of unoriented links (or bonds) which connect pairwise the sites and are therefore naturally denoted by couples $(i, j) = (j, i)$. When the set $G$ is finite, $\mathcal{G}$ is a finite graph and we shall denote $N$ the number of vertices of $\mathcal{G}$. A subgraph $\mathcal{G}'$ of $\mathcal{G}$ is a graph such that $G' \subseteq G$ and $G'_L \subseteq G_L$. A subgraph is said to be complete if its has all the available links, that is if, given the subset of nodes $G'$, the subset of links $G'_L$ is the largest possible one.
A path in $\mathcal{G}$ is a sequence of consecutive links $\{(i,k)(k,h)\ldots(n,m)(m,j)\}$. A graph is said to be connected, if for any two points $i, j \in G$ there is always a path joining them. In the following we will consider only connected graphs.

The graph topology can be algebraically described by its adjacency matrix $A$ with elements

$$A_{ij} = \begin{cases} 1 & \text{if } (i, j) \in G_L \\ 0 & \text{if } (i, j) \notin G_L \end{cases} \quad (2.1)$$

The Laplacian matrix $L$ on the graph $\mathcal{G}$ has elements:

$$L_{ij} = z_i \delta_{ij} - A_{ij} \quad (2.2)$$

where $z_i = \sum_j A_{ij}$, the number of nearest neighbours of $i$, is called the coordination number (or degree) of site $i$. Here we will consider graphs with $z_{\text{max}} = \sup_i z_i < \infty$.

One can also consider a generalization of the adjacency matrix, which corresponds to the ferromagnetic and uniformly bounded coupling $J_{ij}$, with $J_{ij} \neq 0 \iff A_{ij} = 1$ and $\sup J_{ij} < \infty$, $\inf J_{ij} > 0$. The elements of the generalized Laplacian matrix then read:

$$L_{ij} = J_i \delta_{ij} - J_{ij} \quad (2.3)$$

where $J_i = \sum_j J_{ij}$.

Every connected graph $\mathcal{G}$ is endowed with an intrinsic metric generated by the chemical distance $r_{i,j}$, which is defined as the number of links in the shortest path(s) connecting vertices $i$ and $j$.

Let us now consider thermodynamic averages on infinite graphs $\mathcal{G}$. The Van Hove sphere $S_{o,r} \subset \mathcal{G}$ of centre $o \in G$ and radius $r$ is the complete subgraph of $\mathcal{G}$ containing all $i \in G$ whose distance from $o$ is $\leq r$ and all the links of $\mathcal{G}$ joining them. We will call $N_{o,r}$ the number of vertices contained in $S_{o,r}$.

In the thermodynamic limit the average $[f]_G$ of a real–valued function $f$ on $G$ is:

$$[f]_G \equiv \lim_{r \to \infty} \frac{1}{N_{o,r}} \sum_{i \in \partial S_{o,r}} f_i \quad (2.4)$$

This average does not depend on the choice of the origin $o \in G$ provided $f$ is bounded from below and

$$\lim_{r \to \infty} \frac{|\partial S_{o,r}|}{N_{o,r}} = 0 \quad (2.5)$$

where $|\partial S_{o,r}|$ is the number of the vertices of the sphere $S_{o,r}$ connected with the rest of the graph $\mathcal{G}$. Here we shall restrict our attention to graphs with this property.

The measure $|A|$ of a subset $A \subset G$ is the average value $[\chi(A)]_G$ of its characteristic function $\chi_i(A)$ defined by $\chi_i(A) = 1$ if $i \in A$ and $\chi_i(A) = 0$ if $i \notin A$. The measure of a subset of links $G'_{L} \subseteq G_L$ is similarly given by:

$$|G'_{L}| \equiv \lim_{r \to \infty} \frac{N'_{L,r}}{N_{o,r}} \quad (2.6)$$

where $N'_{L,r}$ is the number of links of $G'_{L}$ contained in the sphere $S_{o,r}$. Any two nonzero–measure subsets $A$ and $B$ of $G$ are said to be equivalent if their symmetric difference has zero measure, that is $|A| = |B| = |A \cap B|$. For any given nonzero–measure subsets $A \subset G$ we shall denote $\{A\}$ its equivalence class. Then $A$ is said to be a representative of $\{A\}$. With the subgraph $\mathcal{G}'$ defined by the ordered double $(G', G'_L)$, we identify the measure of the subgraph as the measure $|G'|$ of its points.

Given a (nonzero–measure) subset $A \subset G$, we define the average on $A$ of any real–valued function $f$ on $G$ as:

$$[f]_A = [\chi(A)]_G f \quad (2.7)$$

By definition $[f]_A$ is a function only of the equivalence classes, that is $[f]_A = [f]_{\{A\}}$. Moreover, quite evidently $[f]_C = [f]_A + [f]_B$ whenever $C = A \cup B$ and $|A \cap B| = 0$.

Given a complete subgraph $\mathcal{M} = (M, M_L)$, we denote $\overline{\mathcal{M}}$ its complement in $\mathcal{G}$. This is formed by all points that do not belong to $M$ and by all links of $G_L$ which connect them. $\mathcal{M}$ is therefore a complete subgraph. We call the pair $(\mathcal{M}, \overline{\mathcal{M}})$ a partition of order two of $\mathcal{G}$ whenever both $M$ and its complement $\overline{M}$ are nonzero–measure subsets of $G$.

We introduce now the important concept of minimal distance $\overline{D}(A, B)$ between any pair $A, B$ of nonzero–measure subgraphs of $G$ such that $|A \cap B| = 0$. It is defined as

$$\overline{D}(A, B) = \min(n : |A \cap_n B| > 0) \quad (2.8)$$
where

\[ A \cap_n B = \{ i \in A : \text{dist}(i, B) = n \}, \quad \text{dist}(i, B) = \min_{j \in B} \text{r}_{i,j} \]  

(2.9)

For \( n = 0 \), \( \cap_n \) reduces to the usual intersection operator. Notice that, while in general the relation \( A \cap_n B \) is not symmetric in \( A, B \), the minimal distance is symmetric: \( D(A, B) = D(B, A) \). In fact, from the boundedness of \( z_i \), it can be shown by induction on \( n \) that

\[ |B \cap_n A| \geq (z_{\text{max}})^{-n} |A \cap_n B| \]  

(2.10)

so that

\[ |A \cap_n B| > 0 \implies |B \cap_n A| > 0 \]  

(2.11)

implying our assertion.

Consider now the minimal distance between the two members of a partition of order two. Suppose \( D(M, \bar{M}) = n > 1 \); then \( |M \cap_n \bar{M}| > 0 \Rightarrow |M \cap_{n-1} \bar{M}| > 0 \) from the boundedness of \( z_i \). This implies that if \( D(M, \bar{M}) \) is finite, then \( D(M, \bar{M}) = 1 \). In this case we may say that \( M \) and \( \bar{M} \) are densely interlaced, while in the opposite case that they are infinitely separated. From the definition of minimal distance, it follows that if two subgraphs \( A \) and \( B \) of \( G \) are infinitely separated, their common frontier \( \partial(A, B) \) (i.e. the links \( (i, j) \in G_L \) with \( i \in A \) and \( j \in B \)) is a zero–measure set. Then the two subgraphs can be disconnected by cutting such a zero–measure set of links. This relates the property of infinite separability to the simple separability property defined in [6]. Indeed, the two definitions coincide. We shall term separable partition a partition \( (M, \bar{M}) \) where \( M \) and \( \bar{M} \) are infinitely separated.

### III. The Gaussian Model: Infrared Behaviour and the Spectral Dimension

The Gaussian model on \( G \) is defined [2] by assigning a real–valued random variable \( \phi_i \) to each node \( i \in G \) and then prescribing the following probability measure

\[ d\mu_r[\phi] = \frac{1}{Z_r} \exp \left[ -\sum_{i,j \in S_{o,r}} \phi_i (L + m^2 \eta_{ij}) \phi_j \right] \prod_{i \in S_{o,r}} d\phi_i \]  

(3.1)

for the collection \( \phi = \{ \phi_i ; i \in S_{o,r} \} \). Here \( Z_r \) is the proper normalization factor, \( m > 0 \) is a free parameter and \( \eta \) is the diagonal matrix with elements \( \eta_{ij} = \eta_\delta_{ij} \) with the real numbers \( \eta_i \) positive definite and uniformly bounded throughout \( G \) (that is \( 0 < \eta_{\text{min}} \leq \eta_i \leq \eta_{\text{max}}, \forall i \in G \)).

The thermodynamic limit is achieved by letting \( r \to \infty \) and defines a Gaussian measure over the entire \( \phi = \{ \phi_i ; i \in G \} \) which does not depend on the centre of the Van Hove sphere \( \phi \). The covariance of this Gaussian process reads

\[ \langle \phi_i \phi_j \rangle \equiv C_{ij}(m^2) = (L + m^2 \eta_{ij})^{-1} \]  

(3.2)

and hence it satisfies by definition the Schwinger–Dyson (SD) equation

\[ (J_i + m^2 \eta_i)C_{ij}(m^2) - \sum_{k \in G} J_{ik} C_{kj}(m^2) = \delta_{ij} \]  

(3.3)

Setting

\[ C_{ij} = \frac{(1 - W)_{ij}^{-1}}{J_i + m^2 \eta_i}, \quad W_{ij} = \frac{J_{ij}}{J_j + m^2 \eta_j} \]  

(3.4)

one obtains the standard connection with the random walk (RW) over \( G \):

\[ (1 - W)_{ij}^{-1} = \sum_{t=0}^{\infty} (W^t)_{ij} = \sum_{\gamma : i \rightarrow j} W[\gamma] \]  

(3.5)

where the last sum runs over all paths from \( j \) to \( i \), each weighted by the product along the path of the one–step probabilities in \( W \):

\[ \gamma = (i, k_{t-1}, \ldots, k_2, k_1, j) \implies W[\gamma] = W_{ik_{t-1}} W_{k_{t-1}k_{t-2}} \cdots W_{k_{2}k_1} W_{kj} \]  

(3.6)
Notice that, as long as \( m > 0 \), we have \( \sum_i (W^t)_{ij} < 1 \) for any \( t \), namely the walker has a nonzero death probability. This implies that \( C_{ij} \) is a smooth functions of \( m^2 \) for \( m \geq \epsilon > 0 \). In the limit \( m \to 0 \) the walker never dies and the sum over paths in eq. (3.5) is dominated by the infinitely long paths which sample the large scale structure of the entire graph ("large scale" refers here to the metric induced by the chemical distance alone). This typically reflects itself into a singularity of \( C_{ij} \) at \( m = 0 \) whose nature does not depend on the detailed form of \( J_{ij} \) or \( \eta_i \), as long these stay uniformly positive and bounded.

Of particular importance is the leading singular infrared behaviour, as \( m^2 \to 0 \), of the average \([C(m^2)]_G \) of \( C_{ii}(m^2) \), which is a positive definite quantity, over all points \( i \) of the graph \( G \), which we may write in general as

\[
\text{Sing} \ [C(m^2)]_G \sim c(m^2)^{d/2-1} \tag{3.7}
\]

The parameter \( \tilde{d} \) is called the spectral dimension of the graph \( G \) and on regular lattices it coincides with the usual Euclidean dimension. Henceforth we shall call spectral weight the coefficient \( c \) in eq. (3.7). The name spectral dimension is related to the behaviour of the spectral density \( \rho(l) \) of low-lying eigenvalues of the Laplacian \( L \); indeed it can be shown [3] that \( \rho(l) \) scales as a power of \( l \) for \( l \to 0 \), that is \( \rho(l) \sim l^{d/2-1} \).

IV. LARGE SCALE INHOMOGENEITY: HOMOGENEITY CLASSES AND SPECTRAL CLASSES

In the study of statistical models one often has to deal with the average \([C(m^2)]_A \) of \( C_{ii}(m^2) \) over a generic positive measure subset \( A \subset G \) and in particular one has to consider the leading singular behaviour of \([C(m^2)]_A \) as \( m^2 \to 0 \). On regular lattices this singular behaviour is independent of \( A \) and it actually coincides with that obtained averaging over all points of \( G \):

\[
\text{Sing} \ [C(m^2)]_A = \text{Sing} \ [C(m^2)]_G , \quad \forall \ A \subset G , \quad |A| > 0 \tag{4.1}
\]

This property arises from the large scale homogeneity of regular lattices due to translation invariance. On graphs, where translation invariance is lost, this property can still hold if the inhomogeneity is limited to finite scales. More generally it may happen that inhomogeneity extends to large scales and the singular parts of eq. (4.1) are different on different subsets. However we will prove that such subsets must satisfy very strong topological constraints: a large–scale inhomogeneous graph always consists of homogeneous parts joined together by a zero–measure set of links. Therefore the splitting of infrared behaviour always corresponds to a macroscopically evident inhomogeneity of the graph.

In this section we will give a rigorous formulation of these statements through the following steps.

- Let us suppose that the graph \( G \) has indeed a large–scale inhomogeneity that manifests itself through the existence of at least one nonzero–measure subset \( A \subset G \) such that, as \( m^2 \to 0 \),

\[
\text{Sing} \ [C(m^2)]_A \sim c_A(m^2)^{d_A/2-1} \tag{4.2}
\]

with \( \tilde{d}_A \neq \tilde{d} \).

- We then define \( M \subset G \) to be a maximally homogeneous (or more briefly maximal) subset with respect to \( \tilde{d}_A \) whenever

1. \( |M \cap A| > 0 \)
2. \( \text{Sing} \ [C(m^2)]_M \sim c_M(m^2)^{d_M/2-1} \), with \( \tilde{d}_M = \tilde{d}_A \).
3. For any nonzero–measure subset \( B \subset M \) we have \( \tilde{d}_B = \tilde{d}_M \).
4. There exists no \( B \supset M \) such that \( \tilde{d}_B = \tilde{d}_M \) and \( |B| > |M| \).

By this definition it follows that the set of all maximal subsets with respect to \( \tilde{d}_M \) coincides with the equivalence class \( \{ M \} \) and we will call it the homogeneity class of \( \tilde{d}_M \).

- Next we prove the

**Theorem 1:** The subgraphs \( M \) and its complement \( \bar{M} \) are infinitely separated, i.e. their minimal distance \( D(M, \bar{M}) \) is infinite and they define a separable partition of \( G \). Since this separability is induced by the spectral properties embodied by the spectral dimension, we call this a spectral partition (of order two) of \( G \).
Finally we consider a Gaussian model on the graph $\mathcal{M}$ showing that, from the infinite separability of $\mathcal{M}$ and $\mathcal{M}$ the spectral dimension of $\mathcal{M}$ is $\bar{d}$. Therefore, $\bar{d}_M$ is a property of the graph $\mathcal{M}$ and defines a spectral class. This chain of arguments may now be applied to $\mathcal{M}$, splitting off a new spectral class if $\mathcal{M}$ has a large scale inhomogeneity of the type given above. The process can be repeated until necessary, yielding a complete spectral partition of the original graph $\mathcal{G}$ into spectral classes.

Proof of Theorem 1:
Let us suppose ad absurdum that $\mathcal{D}(\mathcal{M}, \mathcal{M}) = 1$ and therefore that there exists a nonzero–measure subset $\mathcal{M}' \subset \mathcal{M}$ such that $\mathcal{D}(\mathcal{M}, \mathcal{M}') = 1$. From the maximality of $\mathcal{M}$ it follows that $\bar{d}_M \neq \bar{d}_{\mathcal{M}'}$. Let us consider the random walk representation (3.7) of $C_{ii}(m^2)$ with $i \in \mathcal{M}'$:

$$C_{ii}(m^2) = \frac{1}{J_i + m^2 \eta_i} \sum_{\gamma: \gamma \rightarrow i} W[\gamma]$$  \hspace{1cm} (4.3)

Next consider a site $k \in \mathcal{M}$ whose distance from $i$ is 1. This site exists from the hypothesis $\mathcal{D}(\mathcal{M}, \mathcal{M}') = 1$. Then, from the sum over paths in the left hand side of (4.3) let us retain only the paths containing $k$. Then, from the boundedness and positivity of $J_{ij}$ and $\eta_i$ one gets:

$$C_{ii}(m^2) \geq \frac{C_{kk}(m^2)}{J_{\text{max}} + m^2 \eta_{\text{max}}}$$  \hspace{1cm} (4.4)

Averaging over $\mathcal{M}$ and then over $\mathcal{M}'$ we get:

$$[C(m^2)]_{\mathcal{M}'} \geq K [C(m^2)]_{\mathcal{M}}$$ \hspace{1cm} (4.5)

where $K$ is a positive constant. Now, taking $m^2 \rightarrow 0$ and using the asymptotic expression for $[C(m^2)]$ given in (3.7) we obtain

$$(m^2)^{\bar{d}_{\mathcal{M}}/2-1} \geq K' (m^2)^{\bar{d}_{\mathcal{M}'}/2-1}$$  \hspace{1cm} (4.6)

Since this argument applies equally well with $\mathcal{M}$ and $\mathcal{M}'$ interchanged, one gets:

$$(m^2)^{\bar{d}_{\mathcal{M}'}/2-1} \geq K'' (m^2)^{\bar{d}_{\mathcal{M}}/2-1}$$  \hspace{1cm} (4.7)

which gives $\bar{d}_M = \bar{d}_{\mathcal{M}'}$ contradicting the hypothesis. Therefore $\mathcal{D}(\mathcal{M}, \mathcal{M}) = \infty$ and $\mathcal{M}$ and $\mathcal{M}'$ must be infinitely separated.

The infinite separability of $\mathcal{M}$ and $\mathcal{M}'$ implies that the two subgraphs can be disconnected by cutting a zero–measure set of links. This very peculiar property implies thermodynamic independence, that is the decoupling, in the thermodynamic limit, of a model defined on the whole graph $\mathcal{G}$ into two models defined independently on on $\mathcal{M}$ and $\mathcal{M}'$.

This applies in particular to the Gaussian model, so that the two averages of $C_{ii}(m^2)$ on $\mathcal{M}$ and $\mathcal{M}'$ are independent quantities, each satisfying a relation like eq. (4.5) with two distinct spectral dimensions. Most importantly, to any nonzero–measure subset of $\mathcal{M}$ there corresponds by construction the same spectral dimension $\bar{d}$ of $\mathcal{M}$. We can say then that $\bar{d}$ is a universal property of $\mathcal{M}$.

V. SPECTRAL WEIGHTS AND SUBCLASSES OF SPECTRAL CLASSES

In the singular behaviour of $[C(m^2)]$, inhomogeneity at large scale can appear also in the coefficient of the leading infrared part (3.7). However, following the same steps as the previous section, we will show that once again a splitting in the value of the coefficient corresponds to a macroscopic inhomogeneity of the graph and that a macroscopically homogeneous graph is indeed characterized by universal $\bar{d}$ and $c$. Actually in this case the proof is subtler and requires some further mathematical steps.

We first define the spectral subclasses of a given spectral class by looking at the spectral weight $c_A$, proceeding along steps similar to those followed above.
• Let us suppose that, for a given graph $G$ belonging to the spectral class characterized by $\bar{d}$, there exists at least one nonzero–measure subset $A \subset G$ such that, as $m^2 \to 0$,

$$\text{Sing} [C(m^2)]_A \sim c_A(m^2)^{d/2-1}$$  \hspace{1cm} (5.1)

with $c_A \neq c$, with $c$ given as in eq. (5.7).

• Then we say that a nonzero–measure subset $M \subset G$, which certainly is maximal w.r.t. $\bar{d}$, due to its universality, is maximal also w.r.t. $c$ whenever

1. $|M \cap A| > 0$
2. $\text{Sing} [C(m^2)]_M \sim c_M(m^2)^{d/2-1}$, with $c_M = c_A$.
3. For any nonzero–measure subset $B \subset M$ we have $c_B = c_M$.
4. There exists no $B \supset M$ such that $c_B = c_M$ and $|B| > |M|$.

By this definition it follows that the set of all maximal subsets with respect to $c_M$ coincides with the equivalence class $\{M\}$ and we will call it the homogeneity subclass of spectral weight $c_M$.

• We then prove the

**Theorem 2:** The subgraphs $M$ and its complement $\bar{M}$ are infinitely separated and define a spectral partition of $G$.

• Following the same steps as the previous section, we then consider a Gaussian model on the graph $M$ showing that, from the infinite separability of $M$ and $\bar{M}$, the coefficient of $\text{Sing} [C(m^2)]_M$ is $c_M$. Therefore we can say that $c_M$ is a universal property of the graph $M$ and defines a spectral subclass separated from the rest.

**Proof of Theorem 2:**

To prove this theorem we first need the following lemma:

**Lemma:** Within a given spectral subclass, for any subset $A$ of the subclass, the asymptotic form of $[C(m^2)]_A$ is invariant under pre–averaging over any normalized point distribution with nonzero–measure support. In other words, if we define

$$[C(m^2)]_{A,\alpha} = \frac{\alpha C(m^2)_A}{\alpha_A}$$ \hspace{1cm} (5.2)

where $\alpha_i > 0$ on a subset of $A$ with nonzero measure, then again

$$\text{Sing} [C(m^2)]_{A,\alpha} \sim c_A(m^2)^{d/2-1}$$  \hspace{1cm} (5.3)

with no dependence at all for $c_A$ and $d$ on the distribution $\alpha = \{\alpha_i; i \in A\}$. The proof of this statement is elementary: we define the quantities

$$f_i = (m^2)^{-d/2+1}C_{ii}(m^2) - c_A$$  \hspace{1cm} (5.4)

Then, by construction, for any $\epsilon > 0$ there exist a $\delta > 0$ such that we have $|[f]_A| < \epsilon$ as soon as $m^2 < \delta$. Hence we also have

$$|[\alpha f]_A| < \left(\sup_{i \in A} \alpha_i\right) |[f]_A| < \left(\sup_{i \in A} \alpha_i\right) \epsilon$$  \hspace{1cm} (5.5)

which immediately implies our assertion.

Now we can prove Theorem 2:

Let us suppose ad absurdum that $\overline{D(M, \bar{M})} = 1$ and therefore that it exists a nonzero–measure subset $\bar{M}' \subset \bar{M}$ such that $\overline{D(M, \bar{M}')} = 1$. From the maximality of $\bar{M}$ it follows that $c_M \neq c_{\bar{M}'}$.

The following proof is given only for $d < 4$, owing to brevity and physical requirements. Indeed a real structure has necessarily a dimension $d \leq 3$; moreover, from a purely theoretical point of view, the class of models we have in mind, with site variables and link interactions, typically have 4 as an upper critical dimension for the scaling behaviour.
Let us consider first the case of a spectral class where \(C(m^2)\) diverges when \(m^2 \to 0\), that is such that \(\bar{d} < 2\). The Schwinger-Dyson equation for \(C_{ii}[m^2]\) reads:

\[
(J_i + m^2 \eta_i)C_{ii}(m^2) - \sum_{k \in \mathcal{G}} J_{ik}C_{ki}(m^2) = 1
\]

(5.6)

Averaging equation (5.6) over \(M\), we obtain the relation

\[
[J C]_M + m^2 [\eta C]_M - [J \cdot C]_M = |M|
\]

(5.7)

where \((J C)_i \equiv J_i C_{ii}\), \((\eta C)_i \equiv \eta_i C_{ii}\) and \((J \cdot C)_i = \sum_k J_{ik} C_{ki}\). We then divide by \([J C]_M\) and let \(m^2 \to 0\). Due to the divergence of \([J C]_M\) we have that, for any \(\epsilon > 0\) there exists a \(\delta > 0\) such that, as soon as \(m < \delta\),

\[
1 - \epsilon \leq \frac{[J C]_M}{[J C]_M}
\]

(5.8)

Next we set

\[
J_{\bar{M}' i} = \sum_{k \in \bar{M}'} J_{ik}, \quad (J \cdot C)_{\bar{M}' i} = \sum_{k \in \bar{M}'} J_{ik} C_{ki}
\]

(5.9)

and use the positivity of \(C_{ii} - C_{ik}\) to push the above inequality to

\[
1 - \epsilon \leq 1 - \frac{[J_{\bar{M}' i} C]_M}{[J C]_M} + \frac{[(J \cdot C)_{\bar{M}' i}]_M}{[J C]_M}
\]

(5.10)

which yields

\[
\lim_{m^2 \to 0} \frac{[J_{\bar{M}' i} C]_M}{[J_{\bar{M}' i} C]_M} = 1
\]

(5.11)

Owing to the symmetry of \(D(M, \bar{M}')\), we may repeat the above steps with \(M\) and \(\bar{M}'\) interchanged. Since the symmetry of \(J_{ij}\) and \(C_{ij}\) implies \([J \cdot C]_{\bar{M}' i} = [(J \cdot C)_{\bar{M}'}]_{\bar{M}'}\), we finally obtain

\[
\lim_{m^2 \to 0} \frac{[J_{\bar{M}' i} C]_M}{[J C]_{\bar{M}' i} C]_M} = 1
\]

(5.12)

At this stage we apply the lemma given above with \(\alpha\) identified with \(J_{\bar{M}}\) or \(J_M\), namely

\[
[J_{\bar{M}' i} C]_M \sim c_M [J_{\bar{M}'}]_M (m^2)^{d/2-1}, \quad [J M C]_{\bar{M}'} \sim c_{\bar{M}'} [J M]_{\bar{M}'} (m^2)^{d/2-1}
\]

(5.13)

But \([J_{\bar{M}' i}]_M = [J M]_{\bar{M}'}\) so that eq. (5.12) implies \(c_M = c_{\bar{M}'}\), contradicting our initial hypothesis that \(D(M, \bar{M}) = 1\) with \(\bar{M}\) maximal. Hence necessarily \(D(M, \bar{M}) = \infty\), proving our assertion.

Let us now consider a spectral class where \(C(m^2)\) does not diverge in the limit \(m^2 \to 0\) while its first derivative with respect to \(m^2\), \(C'(m^2)\), diverges in the same limit. This is the case of a spectral class characterized by a spectral dimension \(2 < \bar{d} < 4\), where:

\[
[C'(m^2)]_{M,\alpha} = \frac{\alpha C'(m^2)_{\alpha}}{[\alpha]_M} \sim -(\bar{d}/2 - 1) c_M (m^2)^{\bar{d}/2-2}, \quad m^2 \to 0
\]

(5.14)

Taking the first derivative with respect to \(m^2\) in the Schwinger-Dyson equation (5.1), we obtain:

\[
\eta_i C_{ii}(m^2) + m^2 \eta_i C'_{ii}(m^2) = \sum_{k \in \mathcal{G}} J_{ik} [C'_{ki}(m^2) - C'_{ki}(m^2)]
\]

(5.15)

which can be averaged over \(M\) giving

\[
[\eta C]_M + m^2 [\eta C']_M = [J \cdot C']_M - [J \cdot C']_M
\]

(5.16)
Together with eq. (5.14), this implies
\[
\lim_{m^2 \to 0} (m^2)^{2-\tilde{d}/2} ((J \cdot C')_M - [J C']_M) = 0^+
\] (5.17)
that is, for any \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that, as soon as \( m^2 < \delta \)
\[
0 < \xi (\xi - 1) < \epsilon
\] (5.18)
with \( \xi = (m^2)^{2-\tilde{d}/2} \). This can be rewritten as
\[
0 < (J \cdot C')_M - [J M C']_M < \xi^{-1} \epsilon
\] (5.19)
Now, since \( C'_{ij} \equiv -\sum_k \eta_k C_k C_{kj} \) are the elements of a negative semi–definite matrix, one has that
\[
(J \cdot C')_M - [J M C']_M > 0.
\]
Therefore
\[
0 \leq |(J \cdot C')_M - [J M C']_M| < \xi^{-1} \epsilon
\] (5.20)
Again owing to the symmetry of \( D(M, \bar{M}') \), the previous steps can be repeated with \( M \) and \( \bar{M} \) interchanged, leading to:
\[
0 \leq |(J \cdot C')_{\bar{M}} - [J M C']_{\bar{M}}| < \xi^{-1} \epsilon
\] (5.21)
Since \( [(J \cdot C')_{\bar{M}}]_M = [(J \cdot C')_M]_{\bar{M}} \), these two relations imply:
\[
0 \leq |(J \cdot C')_{\bar{M}} - [J M C']_{\bar{M}}| < \xi^{-1} \epsilon
\] (5.22)
Eq. (5.14) entails in the limit \( m^2 \to 0 \):
\[
(J \cdot C')_M \sim -(\tilde{d}/2 - 1) c_M [J M C']_M \xi^{-1}, \quad |J M C'_{\bar{M}}| \sim -(\tilde{d}/2 - 1) c_{\bar{M}} [J M]_{\bar{M}} \xi^{-1}
\] (5.23)
so that, since \( [J M]_{\bar{M}} = [J M]_{\bar{M}} \) from (5.22) one obtains \( c_M = c_{\bar{M}} \), which contradicts our hypothesis \( D(M, \bar{M}') = 1 \) and therefore proves our assertion \( D(M, M) = \infty \).

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