CONNES INTEGRATION FORMULA FOR THE NONCOMMUTATIVE PLANE

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ABSTRACT. Our aim is to prove the integration formula on the noncommutative (Moyal) plane in terms of singular traces à la Connes.

1. Introduction

Let $M$ be a compact Riemannian manifold. The following formula can be found in p. 34 in [1] and in Corollary 7.21 in [9].

\[
\text{Tr}_\omega(M_f(1-\Delta)^{-\frac{d}{2}}) = \int_M f \, d\text{vol}, \quad f \in C_\infty(M).
\]

Here, $M_f$ is the multiplication operator, $\Delta$ is the Hodge-Laplacian operator on $L_2(M, \text{vol})$ and $\text{Tr}_\omega$ is the Dixmier trace on the ideal $L_1, \infty$ (see Section 2). Also, Corollary 7.22 in [9] wrongly extends this result to $f \in L_1(M, \text{vol})$ (in fact, $f \in L_2(M, \text{vol})$ is the necessary and sufficient condition for this formula to hold; see [14] or the book [15] for detailed proofs).

According to [1], formula (1) “led Connes to introduce the Dixmier trace as the correct operator theoretical substitute for integration of infinitesimals of order one in non-commutative geometry.” It appears suitable to refer to (1) and similar results as the “Connes Integration Formula”.

Compactness of the (resolvent of the) Hodge-Dirac operator plays a crucial role in the proofs of Connes Integration Formula for unital spectral triples (see [1] and [9]). For non-unital spectral triples (including non-compact manifolds), the proofs become radically harder. Even the case of the simplest non-compact manifold $\mathbb{R}^d$ required a substantial effort and the first reasonable answer was very recently given in [11] (see the book [15] for detailed proofs).

In this paper, we investigate the validity of Connes Integration Formula for the noncommutative (Moyal) plane $\mathbb{R}^d_\theta$ (here, $\theta$ is a non-degenerate antisymmetric matrix). Earlier attempts in this direction can be found in [8] (see Proposition 4.17 there), [2] and [3]. We substantially strengthen corresponding results from these papers and present a completely different approach to Connes Integration Formula. The novelty of our approach is in the consistent use of Cwikel estimates for the noncommutative plane (obtained in a recent paper [12]) — see Section 2.

Our main result is the following theorem.

**Theorem 1.1.** If $x \in W^{d,1}(\mathbb{R}^d_\theta)$, then $x(1-\Delta)^{-\frac{d}{2}} \in L_1, \infty$ and

\[
\varphi(x(1-\Delta)^{-\frac{d}{2}}) = \tau_\theta(x)
\]

for every normalised continuous trace $\varphi$ on $L_1, \infty$. 

Here, \( W^{d,1}(\mathbb{R}^d_0) \) is a Sobolev space on \( \mathbb{R}^d_0 \) and \( \tau_0 \) is the faithful normal semifinite trace on \( L_{\infty}(\mathbb{R}^d_0) \).

Section 2 involves the preliminaries necessary to prove Theorem 1.1. In Section 3, we prove that

\[ \varphi(x(1 - \Delta)^{-\frac{d}{2}}) = c_\varphi \tau_0(x), \quad x \in W^{d,1}(\mathbb{R}^d_0), \]

for every normalised trace on \( L_{1,\infty} \). In Section 4, we construct one particular \( x \in W^{d,1}(\mathbb{R}^d_0) \) such that \( \varphi(x(1 - \Delta)^{-\frac{d}{2}}) \) does not depend on the choice of a normalised continuous trace \( \varphi \). The combination of these results yield Theorem 1.1.

2. Preliminaries

2.1. General notation. Fix throughout a separable infinite dimensional Hilbert space \( H \). We let \( \mathcal{L}(H) \) denote the algebra of all bounded operators on \( H \). For a compact operator \( T \) on \( H \), let \( \mu(k,T) \) denote the \( k \)-th largest singular value (these are the eigenvalues of \( |T| \)). The sequence \( \mu(T) = \{\mu(k,T)\}_{k \geq 0} \) is referred to as to the singular value sequence of the operator \( T \). The standard trace on \( \mathcal{L}(H) \) is denoted by \( \text{Tr} \).

Fix an orthonormal basis in \( H \) (the particular choice of a basis is inessential). We identify the algebra \( l_\infty \) of bounded sequences with the subalgebra of all diagonal operators with respect to the chosen basis. For a given sequence \( \alpha \in l_\infty \), we denote the corresponding diagonal operator by diag(\( \alpha \)).

2.2. Schatten ideals \( \mathcal{L}_p \) and \( \mathcal{L}_{p,\infty} \), \( p > 0 \). For every \( p > 0 \), we set

\[ \mathcal{L}_p = \{ T \in \mathcal{L}(H) : \text{Tr}(|T|^p) < \infty \}. \]

We set

\[ \|T\|_p = \left(\text{Tr}(|T|^p)\right)^{\frac{1}{p}}, \quad T \in \mathcal{L}_p. \]

For every \( p > 0 \), \( \| \cdot \|_p \) is a quasi-norm\(^1\) and \( (\mathcal{L}_p, \| \cdot \|_p) \) is a quasi-Banach space. For \( p \geq 1 \), \( \| \cdot \|_p \) is a norm. For \( p < 1 \), the space \( (\mathcal{L}_p, \| \cdot \|_p) \) is not Banach — that is, its quasi-norm is not equivalent to any norm.

For a given \( 0 < p \leq \infty \), we let \( \mathcal{L}_{p,\infty} \) denote the principal ideal in \( \mathcal{L}(H) \) generated by the operator diag(\( \{(k + 1)^{-\frac{1}{p}}\}_{k \geq 0} \)). Equivalently,

\[ \mathcal{L}_{p,\infty} = \{ T \in \mathcal{L}(H) : \mu(k,T) = O((k + 1)^{-1/p}) \}. \]

We set

\[ \|T\|_{p,\infty} = \sup_{k \geq 0} (k + 1)^{1/p} \mu(k,T), \quad T \in \mathcal{L}_{p,\infty}. \]

For every \( p > 0 \), \( \| \cdot \|_{p,\infty} \) is a quasi-norm and \( (\mathcal{L}_{p,\infty}, \| \cdot \|_{p,\infty}) \) is a quasi-Banach space. For \( p > 1 \), \( \| \cdot \|_{p,\infty} \) is equivalent to a (unitarily invariant Banach) norm. For \( p \leq 1 \), the space \( (\mathcal{L}_{p,\infty}, \| \cdot \|_{p,\infty}) \) is not Banach — that is, its quasi-norm is not equivalent to any norm. In \([17]\), the Banach envelope of \( \mathcal{L}_{1,\infty} \) was thoroughly investigated.

\(^1\)A quasinorm satisfies the norm axioms, except that the triangle inequality is replaced by \( \|x + y\| \leq K(\|x\| + \|y\|) \) for some uniform constant \( K > 1 \).
2.3. Traces on $L_{1,\infty}$.

**Definition 2.1.** If $\mathcal{I}$ is an ideal in $L(H)$, then a unitarily invariant linear functional $\varphi: \mathcal{I} \to \mathbb{C}$ is said to be a trace.

Since $U^{-1}TU - T = [U^{-1}, TU]$ for all $T \in \mathcal{I}$ and for all unitaries $U \in L(H)$, and since the unitaries span $L(H)$, it follows that traces are precisely the linear functionals on $\mathcal{I}$ satisfying the condition

$$\varphi(TS) = \varphi(ST), \quad T \in \mathcal{I}, S \in L(H).$$

The latter may be reinterpreted as the vanishing of the linear functional $\varphi$ on the commutator subspace which is denoted $[\mathcal{I}, L(H)]$ and defined to be the linear span of all commutators $[T,S] : T \in \mathcal{I}, S \in L(H)$. It is shown in Lemma 5.2.2 in [15] that $\varphi(T_1) = \varphi(T_2)$ whenever $0 \leq T_1, T_2 \in \mathcal{I}$ are such that the singular value sequences $\mu(T_1)$ and $\mu(T_2)$ coincide.

For $p > 1$, the ideal $L_{p,\infty}$ does not admit a non-zero trace [7], while for $p = 1$, there exists a plethora of traces on $L_{1,\infty}$ (see e.g. [18] or [15]). A standard example of a trace on $L_{1,\infty}$ is a Dixmier trace introduced in [6] that we now explain.

**Definition 2.2.** Let $\omega$ be a free ultrafilter on $\mathbb{Z}_+$. The functional

$$\text{Tr}_\omega : A \to \lim_{n \to \omega} \frac{1}{\log(2+n)} \sum_{k=0}^n \mu(k, A), \quad 0 \leq A,$$

is finite and additive on the positive cone of $L_{1,\infty}$. Therefore, it extends to a trace on $L_{1,\infty}$. We call such traces Dixmier traces.

These traces clearly depend on the choice of the ultrafilter $\omega$ on $\mathbb{Z}_+$. Using a slightly different definition, this notion of trace was applied by Connes [4] in noncommutative geometry.

An extensive discussion of traces, and more recent developments in the theory, may be found in [15] including a discussion of the following facts. We refer the reader to an alternative approach to the theory of traces on $L_{1,\infty}$ suggested in [18] (based on the fundamental paper [16] by Pietsch).

1. All Dixmier traces on $L_{1,\infty}$ are positive.
2. All positive traces on $L_{1,\infty}$ are continuous in the quasi-norm topology.
3. There exist positive traces on $L_{1,\infty}$ which are not Dixmier traces (see [18]).
4. There exist traces on $L_{1,\infty}$ which fail to be continuous (see [15]).

**Definition 2.3.** We say that an operator $A \in L_{1,\infty}$ is measurable if $\varphi(A)$ does not depend on the choice of the continuous normalised trace $\varphi$ on $L_{1,\infty}$.

2.4. Noncommutative plane: algebra. Each assertion in this subsection is rigorously established in Section 6 in [12].

Our approach to the noncommutative plane is to introduce the von Neumann algebra generated by a strongly continuous family of unitary operators $\{U(t)\}_{t \in \mathbb{R}^d}$, $d \in \mathbb{N}$, satisfying the commutation relation

$$U(t+s) = \exp\left(-\frac{i}{2} (t, \theta s)\right) U(t)U(s), \quad t, s \in \mathbb{R}^d,$$

where $\theta$ is a fixed antisymmetric real $d \times d$ matrix. Namely, we set

$$\langle U(t)\xi\rangle(u) = e^{-\frac{i}{2}(t, \theta u)} \xi(u-t), \quad \xi \in L_2(\mathbb{R}^d), \quad u, t \in \mathbb{R}^d.$$


Definition 2.4. Let $d \in \mathbb{N}$ and let $\theta$ be a fixed non-degenerate\footnote{A non-degenerate antisymmetric matrix is automatically of even order.} antisymmetric real $d \times d$ matrix. The von Neumann subalgebra in $L(L_2(\mathbb{R}^d))$ generated by $\{U(t)\}_{t \in \mathbb{R}^d}$, introduced in \cite{8}, is called the noncommutative plane and denoted by $L_\infty(\mathbb{R}_d^d)$.

Example 2.5. If $d = 2$, then $L_\infty(\mathbb{R}_d^d)$ is generated by 2 unitary groups $t \to U_1(t)$, $t \to U_2(t)$, $t \in \mathbb{R}$ satisfying the condition

$$U_1(t_1)U_2(t_2) = e^{i\alpha t_1 t_2}U_2(t_2)U_1(t_1), \quad t_1, t_2 \in \mathbb{R}.$$  

Here, $U_1(t_1) = U((t_1,0))$ and $U_2(t_2) = U((0,t_2))$.

The following assertion is well-known. In [12], a spatial isomorphism is constructed.

Theorem 2.6. For every non-degenerate antisymmetric real matrix $\theta$, the algebra $L_\infty(\mathbb{R}_d^d)$ is isomorphic to $L(L_2(\mathbb{R}^d))$.

Having established the isomorphism between $r : L_\infty(\mathbb{R}_d^d) \to L(L_2(\mathbb{R}^d))$ we now equip $L_\infty(\mathbb{R}_d^d)$ with a faithful normal semifinite trace $\tau_\theta = \text{Tr} \circ r$.

We can now define $L_p$-spaces on $L_\infty(\mathbb{R}_d^d)$.

$$L_p(\mathbb{R}_d^d) = \{x \in L_\infty(\mathbb{R}_d^d) : \tau_\theta(|x|^p) < \infty\}.$$  

Lemma 2.7. An operator $x \in L_\infty(\mathbb{R}_d^d)$ is in $L_2(\mathbb{R}_d^d)$ if and only if

$$x = \text{Op}(f) \overset{\text{def}}{=} \frac{1}{(2\pi)^{d/4}} \int_{\mathbb{R}^d} f(s)U(s)ds$$  

for some unique $f \in L_2(\mathbb{R}^d)$ with $\|x\|_2 = \|f\|_2$.

Note that our picture is the Fourier dual of the one considered in [8]. More precisely, the paper [8] deals with operators of the form $\text{Op}(\mathcal{F}f)$, where $f$ is Schwartz (in [8], these operators are written simply as $f$).

2.5. Noncommutative plane: calculus. Each assertion in this subsection is rigorously established in Section 6 in [12].

Let $D_k$, $1 \leq k \leq d$ be multiplication operators on $L_2(\mathbb{R}^d)$

$$(D_k \xi)(t) = t_k \xi(t), \quad \xi \in L_2(\mathbb{R}^d).$$  

For brevity, we denote $\nabla = (D_1, \cdots, D_d)$. For every $1 \leq k \leq d$, we have

$$[D_k, U(s)] = sk U(s), \quad s \in \mathbb{R}^d.$$  

Moreover, we have

$$e^{i(t,\nabla)}U(s)e^{-i(t,\nabla)} = e^{i(t,s)}U(s), \quad s, t \in \mathbb{R}^d.$$  

If $[D_k, x] \in L(L_2(\mathbb{R}^d))$ for some $x \in L_\infty(\mathbb{R}_d^d)$, then $[D_k, x] \in L_\infty(\mathbb{R}_d^d)$. This crucial fact allows us to introduce mixed partial derivative $\partial^a x$ of $x \in L_\infty(\mathbb{R}_d^d)$.

\footnote{To be precise,}

$$x = \lim_{N \to \infty} \frac{1}{(2\pi)^{d/4}} \int_{[-N,N]^d} f(s)U(s)ds,$$

where the limit is taken in $L_2(\mathbb{R}_d^d)$. In what follows, we write the integral over $\mathbb{R}^d$ instead of the limit in order to lighten the notations.
Definition 2.8. Let $\alpha$ be a multiindex and let $x \in L_\infty(\mathbb{R}_d^d)$. If every repeated commutator $[D_{\alpha_1}, [D_{\alpha_1}, x]]$, $1 \leq j \leq n$, is a bounded operator on $L_2(\mathbb{R}^d)$, then the mixed partial derivative $\partial^\alpha x$ of $x$ is defined as

$$\partial^\alpha x = [D_{\alpha_1}, [D_{\alpha_2}, \cdots, [D_{\alpha_n}, x]].$$

In this case, we have that $\partial^\alpha x \in L_\infty(\mathbb{R}_d^d)$. As usual, $\partial^0 x = x$.

Therefore, we can introduce the Sobolev space $W^{m,p}(\mathbb{R}_d^d)$ associated with the noncommutative plane in the following way.

Definition 2.9. For $m \in \mathbb{Z}_+$ and $p \geq 1$, the space $W^{m,p}(\mathbb{R}_d^d)$ is the space of $x \in L_p(\mathbb{R}_d^d)$ such that every partial derivative of $x$ up to order $m$ is also in $L_p(\mathbb{R}_d^d)$. This space is equipped with the norm,

$$||x||_{W^{m,p}} = \sum_{|\alpha| \leq m} ||\partial^\alpha x||_p, \quad x \in W^{m,p}(\mathbb{R}_d^d).$$

The following assertion is one of the main results in [12].

Theorem 2.10. If $x \in W^{d,1}(\mathbb{R}_d^d)$, then

(a) $x(1 - \Delta)^{-\frac{d}{4}} \in L_1$ and

$$||x(1 - \Delta)^{-\frac{d}{4}}||_1 \leq c_d \|x\|_{W^{d,1}}.$$

(b) $x(1 - \Delta)^{-\frac{d}{4}} \in L_{1,\infty}$ and

$$||x(1 - \Delta)^{-\frac{d}{4}}||_{1,\infty} \leq c_d \|x\|_{W^{d,1}}.$$

3. Integration Formula Modulo a Constant Factor

For every $\phi \in L_\infty(\mathbb{R}^d)$, we define a bounded operator $T_\phi : L_2(\mathbb{R}_d^d) \rightarrow L_2(\mathbb{R}_d^d)$ by the formula

$$T_\phi : \int_{\mathbb{R}^d} f(s)U(s)ds \rightarrow \int_{\mathbb{R}^d} f(s)\phi(s)U(s)ds, \quad f \in L_2(\mathbb{R}^d).$$

Lemma 3.1. If $\phi$ is a Schwartz function, then $T_\phi : L_1(\mathbb{R}_d^d) \rightarrow L_1(\mathbb{R}_d^d)$.

Proof. We claim that

$$T_\phi x = \int_{\mathbb{R}^d} (\mathcal{F}\phi)(u)U(-\theta^{-1} u)xU(\theta^{-1} u)du, \quad x \in L_2(\mathbb{R}_d^d).$$

Since both sides above define bounded operators on $L_2(\mathbb{R}_d^d)$ and since the set $\{\text{Op}(f) : f \text{ is Schwartz}\}$ is dense in $L_2(\mathbb{R}_d^d)$, it suffices to establish the claim for

$$x = \int_{\mathbb{R}^d} f(s)U(s)ds, \quad f \in S(\mathbb{R}^d).$$

Using the inverse Fourier transform, we write

$$\phi(s) = \int_{\mathbb{R}^d} (\mathcal{F}\phi)(u)e^{i(u,s)}du, \quad s \in \mathbb{R}^d.$$

Since both $f$ and $\mathcal{F}\phi$ are Schwartz functions, it follows that

$$T_\phi x = \int_{\mathbb{R}^d \times \mathbb{R}^d} f(s)(\mathcal{F}\phi)(u)e^{i(u,s)}U(s)du.$$

It follows from (2) that

$$e^{i(u,s)}U(s) = U(-\theta^{-1} u)U(s)U(\theta^{-1} u).$$
Therefore,
\[ T_\varphi x = \int_{\mathbb{R}^d} (\mathcal{F}\varphi)(u) \left( \int_{\mathbb{R}^d} f(s)U(-\theta^{-1}u)U(s)U(\theta^{-1}u)ds \right) du. \]
Using the definition of \( x \), we obtain
\[ \int_{\mathbb{R}^d} f(s)U(-\theta^{-1}u)U(s)U(\theta^{-1}u)ds = U(-\theta^{-1}u)xU(\theta^{-1}u). \]
This proves the claim.

Now, we prove the assertion of the lemma as follows.
\[ \| T_\varphi x \|_1 \leq \int_{\mathbb{R}^d} |(\mathcal{F}\varphi)(u)| \cdot \| U(-\theta^{-1}u)xU(\theta^{-1}u) \|_1 du = \| \mathcal{F}\varphi \|_1 \| x \|_1. \]

\[ \square \]

**Lemma 3.2.** For every \( x \in W^{d,1}(\mathbb{R}^d) \), the mapping
\[ t \mapsto U(-t)xU(t), \quad t \in \mathbb{R}^d, \]
is a continuous \( W^{d,1}(\mathbb{R}^d) \)-valued function. Moreover,
\[ \| U(-t)xU(t) \|_{W^{d,1}} = \| x \|_{W^{d,1}}. \]

**Proof.** It follows from Leibniz rule that
\[ [D_k, U(-t)xU(t)] = [D_k, U(-t)] \cdot xU(t) + U(-t) \cdot [D_k, x] \cdot U(t) + U(-t)x \cdot [D_k, U(t)] = -t_k U(-t)xU(t) + U(-t)[D_k, x]U(t) + t_k U(-t)xU(t) = U(-t)[D_k, x]U(t). \]
Iterating the latter inequality, we obtain
\[ \partial^\alpha(U(-t)xU(t)) = U(-t)\partial^\alpha(x)U(t). \]
Thus,
\[ \| U(-t)xU(t) \|_{W^{d,1}} = \sum_{|\alpha| \leq d} \| \partial^\alpha(U(-t)xU(t)) \|_1 = \sum_{|\alpha| \leq d} \| \partial^\alpha(x) \|_1 = \| x \|_{W^{d,1}}. \]

We now establish the continuity. For every \( y \in \mathcal{L}_1 \), the mapping
\[ t \mapsto V(-t)yV(t), \quad t \in \mathbb{R}^d, \]
is continuous in the \( \mathcal{L}_1 \)-norm whenever the mapping \( t \mapsto V(t) \) is strongly continuous. Recall that \( (L_\infty(\mathbb{R}^d), \tau_\theta) \) is *-isomorphic (so that trace is preserved) to \( (\mathcal{L}(L_2(\mathbb{R}^d)), \text{Tr}) \). Thus, the mapping
\[ t \mapsto U(-t)\partial^\alpha(x)U(t) = \partial^\alpha(U(-t)xU(t)) \]
is continuous in \( L_1 \)-norm. This completes the proof. \[ \square \]

**Lemma 3.3.** (a) If \( f \) is Schwartz, then \( \text{Op}(f) \in W^{d,1}(\mathbb{R}^d) \).
(b) The set \( \{ \text{Op}(f) : f \text{ is Schwartz} \} \) is dense in \( L_1(\mathbb{R}^d) \). In particular, \( W^{d,1}(\mathbb{R}^d) \) is dense in \( L_1(\mathbb{R}^d) \).

**Proof.** There exists a sequence \( \{ e_{kl} \}_{k,l \geq 0} \subset L_\infty(\mathbb{R}^d) \) such that
\begin{itemize}
  \item[(i)] \( e_{k1}, e_{k2} = \delta_{k1}, \delta_{k2} e_{k1}, e_{k2} \) and \( e_{kl}^* = e_{lk} \).
  \item[(ii)] \( \tau_\theta(e_{kk}) = 1 \).
  \item[(iii)] \( \sum_{k \geq 0} e_{kk} = 1 \) in strong operator topology.
  \item[(iv)] for every \( k, l \geq 0 \), there exists a Schwartz function \( f_{kl} \) such that \( e_{kl} = \text{Op}(f_{kl}) \).
\end{itemize}
The existence of such a sequence is established in Lemma 2.4 in [8] (see also additional references therein). A particular formula for $f_{kl}$ can be found on p. 618 in [8] in terms of Laguerre polynomials.

We prove (b). Let $f$ be a Schwartz function. By Proposition 2.5 in [8], one can write $f$ as

$$f = \sum_{k,l \geq 0} c_{kl}f_{kl}, \quad \sum_{k,l \geq 0} |c_{kl}| < \infty.$$ 

Thus,

$$\text{Op}(f) = \sum_{k,l \geq 0} c_{kl}f_{kl},$$

where the series converges in $L_1$–norm. Thus, $\text{Op}(f) \in L_1(\mathbb{R}_d)$. Let $f_\alpha(t) = t^\alpha f(t)$, $t \in \mathbb{R}^d$. By [4], $\partial^\alpha(\text{Op}(f)) = \text{Op}(f_\alpha)$. Since $f_\alpha$ is also a Schwartz function, it follows that $\partial^\alpha(\text{Op}(f)) \in L_1(\mathbb{R}_d)$. This proves (b).

To prove (c), note that, for every $x \in L_1(\mathbb{R}_d)$,

$$\sum_{k,l \leq N} e_{kk}xe_{ll} = \left( \sum_{k \leq N} e_{kk} \right) x \left( \sum_{l \leq N} e_{ll} \right) \rightarrow x$$

in $L_1$–norm as $N \to \infty$. Note that $e_{kk}xe_{ll}$ is a scalar multiple of $e_{kl} = \text{Op}(f_{kl})$. Since a linear combination of Schwartz functions is again a Schwartz function, it follows that

$$\sum_{k,l \leq N} e_{kk}xe_{ll} \in \{ \text{Op}(f) : f \text{ is Schwartz} \} \subset W^{d,1}(\mathbb{R}_d).$$

This proves (c). \hfill \Box

**Lemma 3.4.** If $F$ is a continuous functional on $W^{d,1}(\mathbb{R}_d)$ such that

$$F(x) = F(U(-t)xU(t)), \quad x \in W^{d,1}(\mathbb{R}_d), \quad t \in \mathbb{R}^d,$$

then $F = \tau_\theta$ (up to a constant factor).

**Proof.** Let $T : W^{d,1}(\mathbb{R}_d) \to W^{d,1}(\mathbb{R}_d)$ be defined by setting

$$Tx = \int_{\mathbb{R}^d} U(-\theta^{-1}t)xU(\theta^{-1}t)e^{-\frac{1}{2}|t|^2} dt.$$ 

The integral is understood as a Bochner integral of a continuous $W^{d,1}(\mathbb{R}_d)$–valued function (the continuity and convergence of the integral follow from Lemma 3.2).

For every $x \in W^{d,1}(\mathbb{R}_d)$, we have

$$F(Tx) = \int_{\mathbb{R}^d} F(U(-\theta^{-1}t)xU(\theta^{-1}t))e^{-\frac{1}{2}|t|^2} dt = \int_{\mathbb{R}^d} F(x)e^{-\frac{1}{2}|t|^2} dt = (2\pi)^{\frac{d}{2}} F(x).$$

Thus,

$$F(x) = (2\pi)^{-\frac{d}{2}} F(Tx), \quad x \in W^{d,1}(\mathbb{R}_d).$$

We claim that $\|Tx\|_{W^{d,1}} \leq c_d \|x\|_1$ for every $x \in W^{d,1}(\mathbb{R}_d)$. To see this, let

$$x = \int_{\mathbb{R}^d} f(s)U(s)ds, \quad f \in L_2(\mathbb{R}^d).$$

If, in the proof of Lemma 3.1, we select $\phi(t) = e^{-\frac{1}{2}|t|^2}$, $t \in \mathbb{R}^d$, then the argument given there yields

$$Tx = \int_{\mathbb{R}^d} f(s)U(s)e^{-\frac{1}{2}|s|^2} ds.$$
By (1), we have
\[ \partial^\alpha(Tx) = \int_{\mathbb{R}^d} f(s) U(s) s^\alpha e^{-\frac{1}{2}|x|^2} ds. \]

Let \( \phi_\alpha(s) = s^\alpha e^{-\frac{1}{2}|s|^2}, \ s \in \mathbb{R}^d. \) We have that \( \partial^\alpha \circ T = T_{\phi_\alpha}. \) By Lemma 3.1
\( T_{\phi_\alpha} : L_1(\mathbb{R}^d_0) \to L_1(\mathbb{R}^d_0) \) is a bounded operator. This proves the claim.

For every \( x \in W^{d,1}(\mathbb{R}^d_0), \) we have
\[ |F(x)| = (2\pi)^{-\frac{d}{2}} |F(Tx)| \leq (2\pi)^{-\frac{d}{2}} \|F\|_{(W^{d,1})^*} \|Tx\|_{W^{d,1}} \leq c_d \|F\|_{(W^{d,1})^*} \|x\|_1. \]

Thus, a functional \( F \) on \( W^{d,1}(\mathbb{R}^d_0) \) is bounded in \( \| \cdot \|_1 \)-norm. By the Hahn-Banach Theorem, \( F \) extends to a bounded functional on \( L_1(\mathbb{R}^d_0). \) Hence, there exists \( y \in L_\infty(\mathbb{R}^d_0) \) such that
\[ F(x) = \tau_\theta(xy), \quad x \in W^{d,1}(\mathbb{R}^d_0). \]

Clearly,
\[ F(U(-t)xU(t)) = \tau_\theta(U(-t)xU(t)y) = \tau_\theta(xU(t)yU(-t)). \]

Comparing the last 2 equalities, we obtain
\[ \tau_\theta(xU(t)yU(-t)) = \tau_\theta(xy), \quad x \in W^{d,1}(\mathbb{R}^d_0). \]

Since \( W^{d,1}(\mathbb{R}^d_0) \) is dense in \( L_1(\mathbb{R}^d_0), \) it follows that \( y = U(t)yU(-t) \) for every \( t \in \mathbb{R}^d. \)

In other words, \( y \) commutes with every \( U(t) \) and, therefore, with every element in \( L_\infty(\mathbb{R}^d_0). \) Since \( L_\infty(\mathbb{R}^d_0) \) is a factor (see Theorem 2.6), it follows that \( y \) is a scalar operator. This completes the proof. \( \square \)

The following proposition is a light version of Theorem 1.1

**Proposition 3.5.** If \( x \in W^{d,1}(\mathbb{R}^d_0), \) then \( x(1 - \Delta)^{-\frac{d}{2}} \in L_{1,\infty} \) and
\[ \varphi(x(1 - \Delta)^{-\frac{d}{2}}) = c_\varphi \tau_\theta(x) \]
for every continuous trace on \( L_{1,\infty} \) and for some constant \( c_\varphi. \)

**Proof.** By Theorem 2.10 (3), the functional
\[ F : x \mapsto \varphi(x(1 - \Delta)^{-\frac{d}{2}}), \quad x \in W^{d,1}(\mathbb{R}^d_0), \]
is a well defined bounded linear functional on \( W^{d,1}(\mathbb{R}^d_0). \)

Since \( \varphi \) is unitarily invariant, it follows that
\[ \varphi(x(1 - \Delta)^{-\frac{d}{2}}) = \varphi(e^{i(t,\nabla)}x(1 - \Delta)^{-\frac{d}{2}} e^{-i(t,\nabla)}), \quad t \in \mathbb{R}^d. \]

By the Spectral Theorem, we have
\[ (1 - \Delta)^{-\frac{d}{2}} e^{-i(t,\nabla)} = e^{-i(t,\nabla)}(1 - \Delta)^{-\frac{d}{2}}, \]
and so
\[ \varphi(x(1 - \Delta)^{-\frac{d}{2}}) = \varphi(e^{i(t,\nabla)}xe^{-i(t,\nabla)}(1 - \Delta)^{-\frac{d}{2}}). \]

For every \( s \in \mathbb{R}^d, \) we have (see (3))
\[ e^{i(t,\nabla)} U(s)e^{-i(t,\nabla)} = e^{i(t,s)} U(s). \]

On the other hand, it follows from (2) that
\[ U(-\theta^{-1}t) U(s) U(\theta^{-1}t) = e^{i(t,s)} U(s). \]

Comparing preceding equalities, we arrive at
\[ e^{i(t,\nabla)} U(s)e^{-i(t,\nabla)} = U(-\theta^{-1}t) U(s) U(\theta^{-1}t). \]
It follows that
\[ e^{i(t,\nabla)}xe^{-i(t,\nabla)} = U(-\theta^{-1}t)xU(\theta^{-1}t), \quad x \in L_\infty(\mathbb{R}^d). \]

Combining the preceding paragraphs, we obtain
\[ \varphi(x(1-\Delta)^{-\frac{d}{2}}) = \varphi(U(-\theta^{-1}t)xU(\theta^{-1}t)(1-\Delta)^{-\frac{d}{2}}). \]

Applying Lemma 3.4 to our functional \( F \), we conclude the argument. \( \square \)

4. PROOF OF MEASURABILITY

**Lemma 4.1.** If \( K \in W^{2d+2,1}([0,1]^d \times [0,1]^d) \) and if \( T : L_2((0,1)^d) \to L_2((0,1)^d) \) is an integral operator with integral kernel \( K \), then \( T \in \mathcal{L}_1 \) and \( \|T\|_1 \leq c_d\|K\|_{W^{2d+2,1}} \).

**Proof.** Let \( K \in W^{2d+2,1}([-\pi,\pi]^d \times [-\pi,\pi]^d) \) be an extension of \( K \) such that
\[ \|K\|_{W^{2d+2,1}([-\pi,\pi]^d \times [-\pi,\pi]^d)} \leq c_d\|K\|_{W^{2d+2,1}([0,1]^d \times [0,1]^d)} \]
and such that \( K \) vanishes on and near the boundary. Thus, \( K \in W^{2d+2,1}(\mathbb{T}^d \times \mathbb{T}^d) \). Let \( S : L_2(\mathbb{T}^d) \to L_2(\mathbb{T}^d) \) be an integral operator with integral kernel \( K \). We have \( T = M_{\chi_{[0,1]^d}}SM_{\chi_{[0,1]^d}} \). Thus, \( \|T\|_1 \leq \|S\|_1 \).

Let us write Fourier series
\[ K(t,s) = \sum_{m_1,m_2 \in \mathbb{Z}^d} c_{m_1,m_2}e_{m_1}(t)e_{m_2}(s), \quad t, s \in \mathbb{T}^d. \]
Set
\[ S_{m_1,m_2}\xi = \langle \xi, e_{-m_2}\rangle e_{m_1}, \quad \xi \in L_2(\mathbb{T}^d). \]
It is an integral operator on \( L_2(\mathbb{T}^d) \) with the integral kernel \( (t,s) \to e_{m_1}(t)e_{m_2}(s) \).
Hence,
\[ S = \sum_{m_1,m_2 \in \mathbb{Z}^d} c_{m_1,m_2}S_{m_1,m_2}. \]
By triangle inequality, we have
\[ \|S\|_1 \leq \sum_{m_1,m_2 \in \mathbb{Z}^d} |c_{m_1,m_2}| \leq \sup_{m_1,m_2 \in \mathbb{Z}^d} (1 + |m_1|^2 + |m_2|^2)^{d+1} |c_{m_1,m_2}| \cdot \sum_{m_1,m_2 \in \mathbb{Z}^d} (1 + |m_1|^2 + |m_2|^2)^{-d-1}. \]
Observe that \( (1 + |m_1|^2 + |m_2|^2)^{d+1} |c_{m_1,m_2}| \) is the \((m_1,m_2)\)–th Fourier coefficient of the function \( (1 - \Delta_{\mathbb{T}^d})^{d+1}K \) (here, \( \Delta_{\mathbb{T}^d} \) is the Laplacian on the torus \( \mathbb{T}^d \)). Taking into account that Fourier coefficients do not exceed the \( L_1 \)–norm, we infer that
\[ (1 + |m_1|^2 + |m_2|^2)^{d+1} |c_{m_1,m_2}| \leq (2\pi)^{-2d} |(1 - \Delta_{\mathbb{T}^d})^{d+1}K|_1 \leq c_d\|K\|_{W^{2d+2,1}}. \]
Here, the last inequality follows from the definition of a Sobolev space. \( \square \)

In what follows, we consider the tensor product of 2 bounded operators on a Hilbert space \( H \) and a bounded operator on the Hilbert space \( H \otimes H \).

**Lemma 4.2.** If \( T \in \mathcal{L}_{1,\infty} \) and \( S \in \mathcal{L}_1 \), then \( S \otimes T \in \mathcal{L}_{1,\infty} \) and
\[ \varphi(S \otimes T) = \text{Tr}(S) \cdot \varphi(T) \]
for every continuous trace \( \varphi \) on \( \mathcal{L}_{1,\infty} \).
Proof. Firstly, we show that $S \otimes T \in \mathcal{L}_{1,\infty}$. Let $z(t) = t^{-1}, t > 0$. By definition, we have $\mu(T) \leq \|T\|_{1,\infty}z$. The crucial fact that $\mu(S \otimes z) = \|S\|_1 z$ is proved on p. 211 in [3]. Thus,

$$\|S \otimes T\|_{1,\infty} = \|S \otimes \mu(T)\|_{1,\infty} \leq \|T\|_{1,\infty}\|S \otimes z\|_{1,\infty} = \|T\|_{1,\infty}\|S\|_1.$$ 

We now turn to the proof of (6). If $S$ is a rank one projection, then there is nothing to prove. If $S$ is a positive finite rank operator, then the assertion follows by linearity. If $S$ is an arbitrary finite rank operator, then the assertion again follows by linearity.

Let $S \in \mathcal{L}_1$ be arbitrary. Fix $\epsilon > 0$ and choose $S_1, S_2 \in \mathcal{L}_1$ such that $S = S_1 + S_2$, $S_1$ is finite rank and $\|S_2\|_1 \leq \epsilon$. Clearly,

$$\varphi(S \otimes T) - \text{Tr}(S) \cdot \varphi(T) =
\quad (\varphi(S_1 \otimes T) - \text{Tr}(S_1) \cdot \varphi(T)) + (\varphi(S_2 \otimes T) - \text{Tr}(S_2) \cdot \varphi(T)).$$

By the preceding paragraph, the summand in the first bracket vanishes. Thus,

$$\varphi(S \otimes T) - \text{Tr}(S) \cdot \varphi(T) = \varphi(S_2 \otimes T) - \text{Tr}(S_2) \cdot \varphi(T).$$

Hence,

$$|\varphi(S \otimes T) - \text{Tr}(S) \cdot \varphi(T)| \leq |\varphi(S_2 \otimes T)| + |\text{Tr}(S_2) \cdot \varphi(T)| \leq ||\varphi||_{\mathcal{L}_1,\infty} \cdot (\|S_2 \otimes T\|_{1,\infty} + \|\text{Tr}(S_2)\|_1)\|T\|_{1,\infty}.$$ 

By the norm estimate in the first paragraph and the assumption on $S_2$, we have

$$|\varphi(S \otimes T) - \text{Tr}(S) \cdot \varphi(T)| \leq 2\epsilon ||\varphi||_{\mathcal{L}_1,\infty} \|T\|_{1,\infty}.$$ 

Since $\epsilon > 0$ is arbitrarily small, the assertion follows. \hfill \Box

In the following lemma, we consider the direct sum of bounded operators on a Hilbert space $H$ as a bounded operator on a Hilbert space $\bigoplus_{m \geq 0} H$.

Lemma 4.3. If the operators $\{T_m\}_{m \geq 0}$ are pairwise orthogonal, i.e. $T_{m_1} T_{m_2} = T_{m_1} T_{m_2} = 0$ for $m_1 \neq m_2$, then $\sum_{m \geq 0} T_m$ is unitarily equivalent\footnote{To be pedantic, $\sum_{m \geq 0} T_m$ is unitarily equivalent to the direct sum $\bigoplus_{m \geq 0} T_m|_{r_m(H) \to r_m(H)}$, where $r_m$ is the projection defined in the proof of Lemma 4.3. Clearly, $T_m$ is unitarily equivalent to the direct sum $T_m|_{r_m(H) \to r_m(H)} \bigoplus \theta(1-r_m)(H) \to (1-r_m)(H)$. Thus, a direct sum $\bigoplus_{m \geq 0} T_m$ is unitarily equivalent to $\bigoplus_{m \geq 0} T_m \bigoplus 0$. In what follows, we ignore this subtle difference and write unitary equivalence as stated in Lemma 4.3.} to $\bigoplus_{m \geq 0} T_m$. Here, the sums are taken in the weak operator topology.

Proof. Let $p_1$ and $p_2$ be projections on $H$. Since $t \to t\frac{1}{t}$, $t > 0$, is an operator monotone function for every $n \geq 1$, it follows that

$$p_1 = p_1 \frac{1}{p_1} \leq (p_1 + p_2) \frac{1}{p_1 + p_2} \Rightarrow \text{supp}(p_1 + p_2).$$

Similarly, $p_2 \leq \text{supp}(p_1 + p_2)$ and, therefore,

$$p_1 \vee p_2 \leq \text{supp}(p_1 + p_2).$$

This simple fact can be also found in Proposition 2.5.14 in [10].

Let $p_m = \text{supp}(T_m)$ and $q_m = \text{supp}(T^*_m)$. It follows from the assumption that $p_{m_1} p_{m_2} = p_{m_1} q_{m_2} = q_{m_1} q_{m_2} = 0$, $m_1 \neq m_2$. Set $r_m = p_m \vee q_m$. We have

$$(p_{m_1} + q_{m_1})(p_{m_2} + q_{m_2}) = 0, \quad m_1 \neq m_2.$$ 

Thus,

$$\text{supp}(p_{m_1} + q_{m_1}) \cdot \text{supp}(p_{m_2} + q_{m_2}) = 0, \quad m_1 \neq m_2.$$
By the preceding paragraph, we have \( r_m r_m = 0, \) \( m_1 \neq m_2. \)
If \( T = \sum_{m \geq 0} T_m, \) then \( r_m T = T_m \) and \( T r_m = T_m \) for every \( m \geq 0. \) Thus, \( T = \bigoplus_{m \geq 0} T_m, \) where \( T_m \) acts on the Hilbert space \( r_m(H). \) \( \square \)

Let
\[
h(t) = (1 + \sum_{k=1}^{d} |t_k|^2)^{-\frac{d}{2}}, \quad t \in \mathbb{R}^d.
\]

The following proposition yields a special case of Theorem 13.

**Proposition 4.4.** If \( f \) is a Schwartz function supported on \([-1,1]^d\) and if \( f = \text{Op}(f) \), then \( x h(\nabla) \) is measurable.

**Proof.**

**Step 1:** We have that \( x h(\nabla) \) is an integral operator with the kernel
\[
K : (t,s) \to f(t-s) h(s) e^{\frac{1}{2} \langle s, \theta t \rangle}, \quad t, s \in \mathbb{R}^2.
\]
By assumption on \( f \), we have that
\[
f(s - t) = 0, \quad s \in m_1 + [0,1]^d, \quad t \in m_2 + [0,1]^2, \quad m_1 - m_2 \notin \{-1,0,1\}^d.\]

Thus,
\[
\text{op}(x h(\nabla)) = \sum_{\xi \in [-1,0,1]^d} \sum_{m \in \mathbb{Z}^d} T_{m,\xi} T_{m,\xi} = \sum_{m \in \mathbb{Z}^d} h(m) T_{m,\xi},
\]
where \( T_{m,\xi} \) is an integral operator whose integral kernel is given by the formula
\[
(t,s) \to f(t-s) e^{\frac{1}{2} \langle s, \theta t \rangle} \chi_{m_1 + [0,1]^2}(t) \chi_{m + [0,1]^4}(s), \quad t, s \in \mathbb{R}^d.
\]

**Step 2:**

We claim that \( T_{m,\xi} \) is measurable.
Note that the operators \( \{ T_{m,\xi} \}_{m \in \mathbb{Z}^d} \) are pairwise orthogonal. Therefore, we have (\( \sim \) denotes unitary equivalence)
\[
T_{m,\xi} \sim \bigoplus_{m \in \mathbb{Z}^d} (1 + |m|^2)^{-\frac{d}{2}} T_{m,\xi}.
\]

By definition, \( T_{m,\xi} : L^2(m + [-1,2]^d) \to L^2(m + [-1,2]^d). \) Define a unitary operator
\[
U_m : L^2([-1,2]^d) \to L^2([-1,2]^d)
\]
by setting
\[
(U_m \xi)(t) = e^{\frac{1}{2} \langle m, \theta t \rangle} \xi(t - m), \quad \xi \in L^2([-1,2]^d), \quad t \in m + [-1,2]^d.
\]

Define an operator \( S_{l_1} : L^2([-1,2]^d) \to L^2([-1,2]^d) \) to be an integral operator with the integral kernel
\[
(t,s) \to f(t-s) e^{\frac{1}{2} \langle s, \theta t \rangle} \chi_{l_1 + [0,1]^d}(t) \chi_{[0,1]^4}(s), \quad t, s \in [-1,2]^d.
\]

A direct computational argument shows that
\[
T_{m,\xi} = U_m S_{l_1} U^{-1}_m.
\]

\[\footnote{Indeed,}
\[
(U_m^{-1} \xi)(t) = e^{-\frac{1}{2} \langle m, \theta t \rangle} \xi(t + m), \quad \xi \in L^2(m + [-1,2]^d), \quad t \in [-1,2]^d.
\]

Thus,
\[
(S_{l_1} U^{-1}_m \xi)(t) = \chi_{l_1 + [0,1]^d}(t) \cdot \int_{[0,1]^d} f(t-s) e^{\frac{1}{2} \langle s, \theta (t+m) \rangle} \xi(s + m) ds.
\]
and Theorem 2.10 (a), we infer that \( \tau \) is a bounded function on \( \mathbb{R}^d \).

Let now \( x \in W^{d,1}(\mathbb{R}^d) \) be arbitrary. Since \( f_0 \) is a Schwartz function, it follows that

\[
\tau_0(x_0) = f_0(0) \neq 0.
\]

Without loss of generality, \( \tau_0(x_0) = 1 \). Let \( z = x - \tau_0(x)x_0 \in W^{d,1}(\mathbb{R}^d) \). Clearly, \( \tau_0(z) = 0 \). We have

\[
\varphi(x(1 - \Delta)^{-\frac{d}{2}}) = \varphi(z(1 - \Delta)^{-\frac{d}{2}}) + \tau_0(x) \cdot \varphi(x_0(1 - \Delta)^{-\frac{d}{2}}).
\]

By Proposition 4.4, the first summand vanishes. By the preceding paragraph, the second summand does not depend on \( \varphi \). This completes the proof. \( \square \)

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Thus,

\[
(U_mS_1U_m^{-1}\xi)(t) = \chi_{m+1+[0,1]^d} (t - m) \cdot \int_{[0,1]^d} e^{\frac{s}{2}(m,\theta t)} f(t - s - m) e^{\frac{s}{2}(s,\theta t)} \xi(s + m) ds =
\]

\[
= \chi_{m+t+1+[0,1]^d} (t) \cdot \int_{m+[0,1]^d} f(t - s) e^{\frac{s}{2}(s,\theta t)} \xi(s) ds.
\]
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