LARGE TIME WELLPOSEDNESS TO THE 3-D CAPILLARY-GRAVITY WAVES IN THE LONG WAVE REGIME

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Abstract. In the regime of weakly transverse long waves, given long-wave initial data, we prove that the nondimensionalized water wave system in an infinite strip under influence of gravity and surface tension on the upper free interface has a unique solution on $[0, T/\varepsilon]$ for some $\varepsilon$ independent of constant $T$. We shall prove in the subsequent paper [22] that on the same time interval, these solutions can be accurately approximated by sums of solutions of two decoupled Kadomtsev-Petviashvili (KP) equations.

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1. Introduction

1.1. General setting. The aim of this paper is to prove that in the regime of weakly transverse long waves, given long-wave initial data, the nondimensionalized water wave system in an infinite strip under influence of gravity and surface tension on the upper free interface has a unique solution on $[0, T/\varepsilon]$ for some $\varepsilon$ independent of constant $T$. More precisely, we consider the irrotational flow of an incompressible, inviscid fluid in an infinite strip with impermeable bottom under the influence of gravity and surface tension on the upper free interface. In
this setting, we may assume that the free surface is described by the graph \( z = \zeta(t, x, y) \), and \( z = -d + b(x, y) \) (the constant \( d > 0 \)) describes the bottom of the infinite strip. Since the fluid is incompressible and irrotational, there exists a velocity potential \( \phi \) such that the velocity field is given by \( v = \nabla \phi \). Then one can reduce the motion of the fluid to a system in terms of the velocity potential \( \phi \) and \( \zeta \):

\[
\begin{aligned}
\partial_t^2 \phi + \partial_y^2 \phi + \partial_z^2 \phi &= 0, \\
\nabla \phi &= 0, \\
\partial_t \zeta + \nabla_h \phi \cdot \nabla_h \zeta &= \partial_z \phi, \\
\partial_t \phi + \frac{1}{2} |\nabla_h \phi|^2 + (\partial_z \phi)^2 + g \zeta &= \kappa \nabla_h \cdot \left( \frac{\nabla_h \zeta}{\sqrt{1 + |\nabla_h \zeta|^2}} \right), \quad z = \zeta,
\end{aligned}
\]

where \( \nabla_h = (\partial_x, \partial_y) \) and \( \partial_h \phi \) denotes the outward normal derivative at the bottom of the fluid region, \( g, \kappa > 0 \) denotes the gravitational force constant and the surface tension coefficient respectively.

It is well-known [9, 10, 35] that the water wave system (1.1) can be reduced to a system of two evolution equations coupling the parametrization of the free surface \( \zeta \) and the trace of the velocity potential \( \phi \) at the free surface. More precisely, let \( n_+ \) be the outward unit normal vector to the free surface,

\[
\psi(t, x_h) \overset{\text{def}}{=} \phi(t, x_h, \zeta(t, x_h)) \quad \text{with} \quad x_h \overset{\text{def}}{=} (x, y)
\]

and the (rescaled) Dirichlet-Neumann operator \( G(\zeta, b) \) (or simply \( G(\zeta) \))

\[
G(\zeta) \psi \overset{\text{def}}{=} \sqrt{1 + |\nabla_h \zeta|^2} \partial_n \phi |_{z=\zeta(t, x_h)}.
\]

Taking the trace of (1.1) on the free surface \( z = \zeta(t, x_h) \), the system (1.1) is equivalent to (see [9, 10, 35] for instance)

\[
\begin{aligned}
\partial_t \zeta - G(\zeta) \psi &= 0, \\
\partial_t \psi + g \zeta + \frac{1}{2} |\nabla_h \psi|^2 - \frac{|G(\zeta) \psi + \nabla_h \zeta \cdot \nabla_h \psi|^2}{2(1 + |\nabla_h \zeta|^2)} &= \kappa \nabla_h \cdot \left( \frac{\nabla_h \zeta}{\sqrt{1 + |\nabla_h \zeta|^2}} \right).
\end{aligned}
\]

Recently this subject of water wave problem has attracted the interest of lots of mathematicians. Concerning 2-D water wave system, when the surface tension is neglected and the motion of free surface is a small perturbation of still water, one could check Nalimov [23], Yoshihara [34] and Craig [8]. In general, the local wellposedness of 2-D full water wave problem was solved by Wu [30] and see also Ambrose and Masmoudi [4], where they firstly studied the 2-D irrotational water wave problem with nonzero surface tension and proved the local wellposedness of the problem, then they showed that as the surface tension goes to zero, the solutions of nonzero surface tension problem goes to solutions of the corresponding zero surface tension problem. (See similar result by the same authors for the 3-D problem in [5].) One may also check [32] for the most recent almost global wellposedness to the 2-D full water wave system without surface tension.

Concerning the 3-D water wave problem without surface tension, Wu [31] proved its local wellposedness under the assumptions that the fluid is irrotational and there is no self-intersection point on the initial surface. Lannes [17] considered the same problem in the case of finite depth under Eulerian coordinates. More recently, following Lannes [17]’s framework, Ming and Zhang [21] proved the local wellposedness of water wave system in an infinite strip and under the influence of surface tension on the free interface. Recently, Alazard, Burq and Zuily [1] studied the regularities to the local solutions of 3-D water wave system, Germain, Masmoudi and Shatah [11], Wu [33] independently proved the global wellposedness of the 3-D water system without surface tension. D. Lannes proved very recently a more general
well-posedness result on the two-fluid system with surface tension on the interface \[16\], and he also stated a stability criterion for these two-fluid interfaces and some applications.

When the initial vorticity does not equal zero, Iguchi, Tanaka and Tani \[12\] proved the local wellposedness of the free boundary problem for an incompressible ideal fluid in two space dimensions without surface tension. Similar result was proved by Ogawa and Tani \[24\] to the case with surface tension. And in \[25\], Ogawa and Tani generalized the wellposedness result in \[24\] to the case of finite depth. One may check \[7, 20, 29, 36\] for some recent study on the local wellposedness of free boundary problem of 3-D Euler equations under the Taylor sign condition on the initial interface.

1.2. Nondimensionalized water-wave system and main results. The complexity of the full water wave system led physicists and mathematicians to derive simpler sets of equations likely to describe the dynamics of \((1.1)\) in some specific physical regimes. Yet the mathematical analysis of these models on their relevance as approximate models for the water wave equations only began three decades ago.

In the particular regime of weakly transverse long waves, Craig \[8\] and Kano and Nishida \[15\] gave a first justification of the 1-D Boussinesq systems. However, the convergence result proved in \[15\] is given on a time scale which is too short to capture the nonlinear and dispersive effects for Boussinesq systems; the correct large time convergence result was later proved by Craig in \[8\]. In the 2-D case, assuming the large time wellposedness of the dimensionless water wave equations, Bona, Colin and Lannes \[6\] justified the Boussinesq approximation. Notice that at the first order, the Boussinesq systems reduce to two decoupled Korteweg-de Vries (KdV) equations in 1-D case and Kadomtsev-Petviashvili (KP) equations in 2-D case. Many papers addressed the problem of the validity of KdV model \([8, 14, 27, 28, 13]\). For the KP model, a first attempt was done in \[14\] for small and analytic initial data. But as in \[15\], the time scale considered is again unfortunately too small for the relevant dynamics. In \[19\], Lannes and Saut proved the KP limit by assuming a large time wellposedness theorem and a specific control of the solutions to the dimensionless full water wave system without surface tension.

In the fundamental paper \[2\], Alvarez-Samaniego and Lannes systematically justified various 3-D asymptotic models, including shallow-water equations, Boussinesq system, KP approximation, Green-Naghdi equations, Serre approximation and full-dispersion model for the water wave system without surface tension.

As is well-known, the proof of large-time wellposedness of dimensionless form of \((1.2)\) is the most delicate point in the justification of the related approximations. The purpose of our paper is to prove that: in the long wave regime, the evolution of long wave-length initial data to \((1.2)\) has a unique solution on \([0, T/\varepsilon]\) for some \(\varepsilon\) independent positive time \(T\). The main idea of our proof is similar as in our previous work \[21\] but more complicated. We use the similarity between the main part of Dirichlet-Neumann operator and the surface tension operator to construct the energy for the linearized system, and we also use a Nash-Moser iteration theorem to handle with the loss of derivatives in the energy estimates. We refer to \[26, 16\] for another way to prove the well-posedness without using Nash-Moser iteration by taking the sufficient amount of derivatives to the system.

Now we are going to introduce the specific regime we used in our paper. This regime of weakly transverse long waves can be specified in terms of relevant characters of the wave, namely, its typical amplitude \(\alpha\), the mean depth \(d\), the typical wavelength \(\lambda\) along the longitudinal direction (say, the \(x\) axis), and \(\lambda\sqrt{\varepsilon}\) the wavelength in \(y\) direction, \(B\) the amplitude
of the variations of the bottom topography, which satisfy
\begin{equation}
\varepsilon = \frac{a}{d} = \frac{d^2}{\lambda^2} = \frac{B}{d} \ll 1.
\end{equation}

The asymptotic study becomes more transparent when working with variables scaled in such a way that the dependent quantities and the initial data which appear in the initial value problem are all of order one. The relation \((\ref{1.3})\) which sets the KP regime here are connected with small parameters in the nondimensionalized equations of motion.

For simplicity, we take gravitational constant \(g = 1\) in \((\ref{1.2})\) and denote the dimensionless variables with a prime. We set
\begin{equation}
x = \lambda x', \quad y = \frac{\lambda}{\sqrt{\varepsilon}} y', \quad z = d z', \quad t = \frac{\lambda}{\sqrt{d}} t',
\end{equation}
\begin{equation}
\zeta = a \zeta', \quad \phi = \frac{a}{\sqrt{d}} \lambda \phi', \quad b = B b', \quad \psi = \frac{a}{\sqrt{d}} \lambda \psi'.
\end{equation}
Then we write the dimensionless form of \((\ref{1.1})\) as follows (by neglecting the prime)
\begin{equation}
\begin{aligned}
\varepsilon \partial_x^2 \phi + \varepsilon^2 \partial_y^2 \phi + \partial_z^2 \phi &= 0, \\
-\varepsilon \nabla_h^x (b \phi) \cdot \nabla_h^x \phi + \partial_t \phi &= 0, \\
\partial_t \zeta + \varepsilon \nabla_h^z \zeta \cdot \nabla_h^z \phi &= \frac{1}{\varepsilon} \partial_t \phi, \\
\partial_t \phi + \frac{1}{2} (\varepsilon|\nabla_h^z \phi|^2 + (\partial_z \phi)^2) + \zeta &= \varepsilon \partial_t \phi, \\
\partial_t \phi &= \frac{1}{2} \varepsilon \nabla_h^z \zeta / \sqrt{1 + \varepsilon^3 |\nabla_h^z \zeta|^2}, \quad z = \varepsilon \zeta
\end{aligned}
\end{equation}
where \(\nabla_h^z \equiv (\partial_x, \sqrt{\varepsilon} \partial_y)\). We define the scaled Dirichlet-Neumann operator \(G[\varepsilon \zeta]\) by
\begin{equation}
G[\varepsilon \zeta] \psi \equiv (-\varepsilon \nabla_h^z (\varepsilon \zeta) \cdot \nabla_h^z \Phi + \partial_z \Phi)|_{z = \varepsilon \zeta},
\end{equation}
with \(\Phi\) solving
\begin{equation}
\begin{aligned}
\partial_t^2 \Phi + \varepsilon \partial_x^2 \Phi + \varepsilon^2 \partial_y^2 \Phi &= 0, \\
\Phi|_{z = \varepsilon \zeta} &= \psi, \\
\partial_{n}^P \Phi|_{z = -1 + \varepsilon b} &= 0.
\end{aligned}
\end{equation}
Here \(\partial_{n}^P \Phi\) is the outward conormal derivative associated to the elliptic equation \((\ref{1.7})\), i.e.,
\begin{equation}
\partial_{n}^P \Phi|_{z = -1 + \varepsilon b} \equiv n \cdot P_0 \nabla \Phi|_{z = -1 + \varepsilon b}
\end{equation}
with \(P_0 = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\),
where \(\nabla \equiv (\nabla_h, \partial_z)\) and \(n\) stands for the outward unit normal vector to the bottom of the infinite strip \(\{(x, y, z)| -1 + \varepsilon b(x, y) < z < \varepsilon \zeta(t, x, y)\}\).

Then similar to \((\ref{1.2})\), the system \((\ref{1.5})\) becomes
\begin{equation}
\begin{aligned}
\partial_t \zeta - \frac{1}{\varepsilon} G[\varepsilon \zeta] \psi &= 0, \\
\partial_t \psi + \zeta + \frac{1}{2} |\nabla_h^z \psi|^2 - \frac{\varepsilon^2}{2} (1 + \varepsilon^3 |\nabla_h^z \zeta|^2) (1 + \varepsilon^3 |\nabla_h^z \zeta|^2) &= \alpha \varepsilon \nabla_h^z \cdot (\nabla_h^z \zeta / \sqrt{1 + \varepsilon^3 |\nabla_h^z \zeta|^2}) \\
\zeta|_{t=0} &= \zeta_0, \\
\psi|_{t=0} &= \psi_0.
\end{aligned}
\end{equation}
where \(\alpha = \kappa/d^2\) is the so-called Bond number.

The uniform energy estimates for the solutions to the linearized system of \((\ref{1.8})\) plays an essential role in the proof of the large time well-posedness for the nonlinear system. Compared with \([2]\), there is an additional term on the left hand side of the linearized system \((\ref{6.2})\) due to the appearance of surface tension term in \((\ref{1.3})\). Then the ordinary energy functional given in \([2]\) will not work for \((\ref{6.2})\), otherwise, there will be a loss of one order derivative in the energy estimates. The key point here is that we observed the resemblance between the principle part
of the Dirichlet-Neumann operator and the linearized surface tension operator, and based on this fact we constructed an effective energy functional to obtain the uniform energy estimate for the linearized system \([5,2]\). This new energy functional leads to the use of a parameterized Sobolev space and some complicated pseudo-differential operator estimates in the process of the energy estimates. With these preparations, we can use a modified version of Nash-Moser iteration theorem in \([3]\) to prove the large time existence of solutions to \((1.8)\).

Before presenting our main results, we introduce the following function space

**Definition 1.1.** We define the space \(X^s\) as
\[
X^s = \left\{ U = (\zeta, \psi)^T : \zeta \in H^{2s+1}(\mathbb{R}^2), \nabla_h \psi \in H^{2s-\frac{1}{2}}(\mathbb{R}^2)^2 \right\}
\]
endowed with the semi-norm
\[
|U|_{X^s} = \sqrt{\varepsilon}|\zeta|_{H^{2s+1}} + |\zeta|_{H^{2s}} + \sqrt{\varepsilon}|\nabla_h \zeta|_{H^s} + |\zeta|_{H^s} + |\Psi|_{H^{2s}} + |\Psi|_{H^s}
\]
for \(\Psi \equiv |D_h^s|/\left(1 + \sqrt{\varepsilon}|D_h^s|\right)^{\frac{3}{2}}, \left|D_h^s\right|\) the Fourier multiplier with the symbol \((\xi_1^2 + \varepsilon \xi_2^2)^{\frac{1}{2}}\), and \(H_x^s(\mathbb{R}^2)\) is the space of tempered distributions \(v\) so that
\[
(1.9) \quad |v|_{H_x^s} = \left|\left(1 + |D_h^s|^{\frac{1}{2}}\right)^2 v\right|_{L^2} < \infty.
\]

**Remark 1.1.** The scaled Sobolev space \(H_x^s\) is naturally connected with the equivalent form for the energy functional introduced in Section 6, which is crucial to obtain an uniform energy estimates for the linearized water-wave system.

Our result of this paper is as follows.

**Theorem 1.1.** Let the Bond number \(\alpha > 0\) and \(\alpha \neq \frac{1}{3}, s \geq m_0\) for some \(m_0 \in (9,10)\). Assume that there exist \(P > D > 0\) such that \(b \in H^{2s+2P+1}(\mathbb{R}^2)\) and bounded initial data \((\zeta_0^\varepsilon, \psi_0^\varepsilon) \in X^{s+P}\) satisfy
\[
\inf_{\mathbb{R}^2} \left(1 + \varepsilon \zeta_0^\varepsilon - \varepsilon b\right) > 0 \quad \text{uniformly for } \varepsilon \in (0,1).
\]

Then there exits \(T > 0\) such that \((1.8)\) has a unique family of solutions \((\zeta^\varepsilon, \psi^\varepsilon)^{0<\varepsilon<1}\) on \([0, T]\) with \((\zeta_0^\varepsilon, \psi_0^\varepsilon)^{0<\varepsilon<1}\), and \((\sqrt{\varepsilon} \partial_y \psi^\varepsilon)^{0<\varepsilon<1}\) being uniformly bounded in \(C([0,T/\varepsilon]; H^{s+D+\frac{1}{2}}(\mathbb{R}^2))\).

**Remark 1.2.** Let
\[
\zeta^{KP}_{\pm}(t,x,y) \equiv \frac{1}{\sqrt{2}} (\zeta_+ (\varepsilon t, x - t, y) - \zeta_- (\varepsilon t, x + t, y)),
\]
where \(\zeta_\pm (\tau, X, Y)\) solve the uncoupled KP equations
\[
(KP)^\pm \quad \partial_\tau \zeta_\pm \pm \frac{1}{2} \partial_X^2 \partial_Y^2 \zeta_\pm \pm \left(\frac{\alpha}{2} - 1\right) \partial_X^3 \zeta_\pm \pm \frac{3\sqrt{2}}{4} \zeta_\pm \partial_X \zeta_\pm = 0.
\]
We shall prove in \([22]\) that: in addition to the assumptions in Theorem 1.1, we assume moreover
\[
\lim_{\varepsilon \to 0} |\zeta_0^\varepsilon - \zeta_0|_{H^{s+D+\frac{1}{2}}} = 0 \quad \text{and} \quad \lim_{\varepsilon \to 0} |\partial_x \psi_0^\varepsilon - \partial_x \psi_0|_{H^{s+D-\frac{1}{2}}} = 0
\]
with \((\partial_x \psi_0, \zeta_0) \in \partial_x H^{s+D-\frac{1}{2}}(\mathbb{R}^2)\) and \((\partial_y^2 \partial_x \psi_0, \partial_y^2 \zeta_0) \in \partial^2 H^{s+D-\frac{3}{2}}(\mathbb{R}^2)\). Then \((KP)^\pm\) with initial data \((\partial_x \psi_0 \pm \zeta_0)/\sqrt{2}\) has a unique solution \(\zeta_\pm \in C([0,T/\varepsilon]; H^{s+D+\frac{1}{2}}(\mathbb{R}^2))\). Furthermore, there holds
\[
\lim_{\varepsilon \to 0} |\zeta^\varepsilon - \zeta^{KP}_{\varepsilon}|_{L^\infty([0,T/\varepsilon] \times \mathbb{R}^2)} = 0.
\]
In case when the Bond number \( \alpha = \frac{1}{d} \), the coefficients of the third order dispersion terms in \((KP)^\pm\) vanish and the resulting equations become illposed. These third order terms in \((KP)^\pm\) equations represent the leading order dispersive effects in the water-wave problem and their disappearance means that in this parameter regime the water waves are almost dispersionless. To model interesting behaviors and capture the dispersive nature of the water-wave problem for this parameter regime in our following paper \cite{22}, we need to modify the scaling in (1.14) firstly and then prove the large-time existence for the new water-wave system. More precisely, we set

\[
x = \lambda x', \quad y = \frac{\lambda}{\varepsilon} y', \quad z = d z', \quad t = \frac{\lambda}{\sqrt{d}} t',
\]

(1.10)

\[
\zeta = a \zeta', \quad \phi = \frac{a}{\sqrt{d}} \lambda \phi', \quad b = B b', \quad \psi = \frac{a}{\sqrt{d}} \lambda \psi',
\]

with

\[
\varepsilon = \sqrt{\frac{a}{d} - \frac{d^2}{\lambda^2}} = \sqrt{\frac{B}{d}}.
\]

Then similar to (1.5), we obtain the following dimensionless form of the original system (by neglecting the prime)

\[
\begin{eqnarray*}
\varepsilon \partial_x^2 \phi + \varepsilon^3 \partial_x^2 \phi + \partial_x^2 \phi &=& 0, \quad -1 + \varepsilon^2 b < z < \varepsilon^2 \zeta, \\
-\varepsilon \nabla_h^\varepsilon (\varepsilon^2 b) \cdot \nabla_h^\varepsilon \phi + \partial_z \phi &=& 0, \\
\partial_t \zeta + \varepsilon^2 \nabla_h^\varepsilon \cdot \nabla_h^\varepsilon \phi &=& \frac{1}{\varepsilon} \partial_z \phi, \\
\partial_t \phi + \frac{1}{2} (\varepsilon^2 |\nabla_h^\varepsilon \phi|^2 + \varepsilon^3 (\partial_z \phi)^2) + \zeta &=& \alpha \varepsilon \nabla_h^\varepsilon \cdot (\nabla_h^\varepsilon \zeta / \sqrt{1 + \varepsilon^5 |\nabla_h^\varepsilon \zeta|^2}), \\
\end{eqnarray*}
\]

(1.11)

where \( \tilde{\nabla}_h^\varepsilon \) \( \equiv (\partial_x, \varepsilon \partial_y) \) and \( \alpha = \kappa / d^2 \) is still the Bond number. We define a new scaled Dirichlet-Neumann operator \( \tilde{G}[^2 \zeta] \) by

\[
\tilde{G}[^2 \zeta] \psi := (-\varepsilon \nabla_h^\varepsilon (\varepsilon^2 \zeta) \cdot \tilde{\nabla}_h^\varepsilon \phi + \partial_z \phi) |_{z=\varepsilon^2 \zeta},
\]

with \( \phi \) solving

\[
\begin{eqnarray*}
\partial_x^2 \phi + \varepsilon \partial_x^2 \phi + \varepsilon^3 \partial_y^2 \phi &=& 0, \quad -1 + \varepsilon^2 b < z < \varepsilon^2 \zeta, \\
\partial_x \phi |_{z=\varepsilon^2 \zeta} = \psi, \quad \partial_x^3 \phi |_{z=-1+\varepsilon^2 b} = 0,
\end{eqnarray*}
\]

and

\[
\partial_x^3 \phi |_{z=-1+\varepsilon^2 b} \equiv \bar{P}_0 \nabla \phi |_{z=-1+\varepsilon^2 b} \quad \text{with} \quad \bar{P}_0 = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon^3 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

Let \( \psi(t, x_h) \equiv \phi |_{z=\varepsilon^2 \zeta} = \phi(t, x_h, \varepsilon^2 \zeta) \). Then similar to (1.8), the new dimensionless system of \((\phi, \zeta)\) can be reformulated as a system of \((\psi, \zeta)\):

\[
\begin{eqnarray*}
\partial_t \zeta - \frac{1}{\varepsilon} \tilde{G}[^2 \zeta] \psi &=& 0, \\
\partial_t \psi + \zeta + \varepsilon^2 |\tilde{\nabla}_h^\varepsilon \psi|^2 - \frac{1}{\varepsilon^3} \varepsilon (\tilde{G}[^2 \zeta] \psi + \varepsilon^2 \nabla_h^\varepsilon \zeta \cdot \nabla_h^\varepsilon \psi)^2 / (1 + \varepsilon^5 |\nabla_h^\varepsilon \zeta|^2) &=& \alpha \varepsilon \nabla_h^\varepsilon \cdot (\nabla_h^\varepsilon \zeta / \sqrt{1 + \varepsilon^5 |\nabla_h^\varepsilon \zeta|^2}), \\
\zeta |_{t=0} = \zeta_0, \quad \psi |_{t=0} = \psi_0.
\end{eqnarray*}
\]

(1.13)

To describe the function space for the initial data such that (1.13) has a unique solution on \([0, T/\varepsilon^2]\), we need to modify Definition 1.1 as below:
Definition 1.2. We define the space \( \tilde{X}^s \) as
\[
\tilde{X}^s \overset{\text{def}}{=} \left\{ U = (\zeta, \psi)^T : \zeta \in H^{2s+1}(\mathbb{R}^2), \nabla_h \psi \in H^{2s-\frac{1}{2}}(\mathbb{R}^2)^2 \right\}
\]
endowed with the semi-norm
\[
|U|_{\tilde{X}_+} \overset{\text{def}}{=} \sqrt{\epsilon} |\zeta|_{H^{2s+1}} + |\zeta|_{H^{2s}} + \sqrt{\epsilon} \left| \nabla^2_h \zeta \right|_{H^s} + |\zeta|_{H^s} + |\Psi \psi|_{H^{2s}} + |\tilde{\Psi} \psi|_{H^s}
\]
for a new regularizing Poisson operator \( \tilde{\Psi} \overset{\text{def}}{=} [\tilde{D}_h^\epsilon]/(1 + \sqrt{\epsilon} |\tilde{D}_h^\epsilon|)^\frac{1}{2} \) with \( \tilde{D}_h^\epsilon = \frac{1}{\epsilon} \nabla^2_h \), and \( H^s(\mathbb{R}^2) \) is the space of tempered distributions \( \psi \) so that
\[
(1.14) \quad |\psi|_{\tilde{H}_x^s} \overset{\text{def}}{=} \|(1 + |\tilde{D}_h^\epsilon|^2)^\frac{1}{2} \psi\|_{L^2} < \infty.
\]

Our second main result is as follows.

Theorem 1.2. (Degenerate case) Let \( \alpha = \frac{1}{3} + \epsilon \theta, \theta \geq 0 \) fixed and \( s \geq m_0 \) for some \( m_0 \in (9,10) \). Assume that there exists \( P > D > 0 \) such that for all \( b \in H^{2s+2P+1}(\mathbb{R}^2) \) and bounded initial data \((\zeta_0, \psi_0) \in \tilde{X}^{s+P} \) satisfying
\[
\inf_{\mathbb{R}^2} (1 + \epsilon^2 \zeta_0^2 - \epsilon^2 b) > 0 \quad \text{uniformly for} \quad \epsilon \in (0,1).
\]
Then there exits \( T > 0 \) such that \((1.13)\) has a unique family of solutions \((\zeta, \psi_\epsilon)_{t < \epsilon < 1}\) on \([0, T/\epsilon^2]\) which satisfy \((\zeta_\epsilon)_{0 < \epsilon < 1}, (\partial_x \psi_\epsilon)_{0 < \epsilon < 1}, \text{and} (\epsilon \partial_y \psi_\epsilon)_{0 < \epsilon < 1}\) are uniformly bounded in \( C([0,T/\epsilon^2]; H^{2s-D-\frac{1}{2}}(\mathbb{R}^2)) \).

Remark 1.3. In fact, these two theorems above are two particular results of a more general existence theorem. First of all, define as in [2] that
\[
\epsilon = \frac{a}{d}, \quad \mu = \frac{d^2}{\lambda^2}, \quad \beta = \frac{B}{d},
\]
and set the dimensionless variable (with prime) as below
\[
x = \lambda x', \quad y = \gamma y', \quad z = dz', \quad t = \frac{\lambda}{\sqrt{d}} t',
\]
\[
(1.15) \quad \zeta = a \zeta', \quad \phi = \frac{a}{\sqrt{d}} \lambda \phi', \quad b = B b', \quad \psi = \frac{a}{\sqrt{d}} \lambda \psi',
\]
One can derive a more general water-wave system of \((\psi, \zeta)\):
\[
(1.16) \quad \begin{cases}
\partial_t \zeta - \frac{1}{\mu} G[\epsilon \zeta] \psi = 0, \\
\partial_t \psi + \zeta + \frac{\epsilon \mu}{2} \nabla_h^2 \psi - \frac{1}{\epsilon \mu} G[\epsilon \zeta] \psi + \epsilon \nabla_h^2 \zeta \cdot \nabla_h^2 \psi)^2 / (1 + \epsilon^2 \mu \nabla_h^2 \zeta)^2 \\
= \alpha \mu \nabla_h^2 \cdot \left( \nabla_h^2 \zeta / \sqrt{1 + \epsilon^2 \mu \nabla_h^2 \zeta} \right), \\
\zeta|_{t=0} = \zeta_0', \quad \psi|_{t=0} = \psi_0',
\end{cases}
\]
with \( \psi(t, x_h) \overset{\text{def}}{=} \phi|_{z=\epsilon \zeta} = \phi(t, x_h, \epsilon \zeta), \quad \nabla_h^\epsilon \overset{\text{def}}{=} (\partial_x, \gamma \partial_y)^T \) and the nondimensionalized Dirichlet-Neumann operator \( G[\epsilon \zeta] \) defined by
\[
(1.17) \quad G[\epsilon \zeta] \psi := (-\mu \nabla_h^\epsilon (\epsilon \zeta) \cdot \nabla_h^\epsilon \phi + \partial_z \phi)|_{z=\epsilon \zeta},
\]
with \( \phi \) solving
\[
\begin{cases}
\partial_t^2 \phi + \mu \partial_z^2 \phi + \gamma^2 \mu \partial_y^2 \phi = 0, \quad -1 + \epsilon b < z < \epsilon \zeta, \\
\phi|_{z=\epsilon \zeta} = \psi, \quad \partial_z^1 \phi|_{z=-1+\epsilon b} = 0,
\end{cases}
\]
and
\[ \partial^n_x^0 \phi|_{z=-1+\epsilon b} \overset{\text{def}}{=} n \cdot P_0 \nabla \phi|_{z=-1+\epsilon b} \quad \text{with} \quad P_0 = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \gamma^2 \mu & 0 \\ 0 & 0 & 1 \end{pmatrix}. \]

We can have a large-time existence result similar to Theorem 1.1 for solutions to the general system (1.16) on time interval \([0, T]\) following the proof of Theorem 1.1. Then Theorem 1.1 and Theorem 1.2 are indeed two particular results of this result. In fact, one can take \( \epsilon = \mu = \epsilon \), \( \gamma = \sqrt{\epsilon} \) in system (1.16) to arrive at Theorem 1.1, and one can take \( \epsilon = \mu^2 = \epsilon^2 \), \( \gamma = \sqrt{\epsilon} = \epsilon \) to arrive at Theorem 1.2.

**Remark 1.4.** Let
\[ \zeta^K_P(t, x, y) \overset{\text{def}}{=} \frac{1}{\sqrt{2}} \left( \zeta_+(\epsilon^2 t, x - t, y) - \zeta_-(\epsilon^2 t, x + t, y) \right), \]
where \( \zeta_{\pm}(\tau, X, y) \) solve the uncoupled fifth order KP equations
\[(KP^{5\text{th}}) \pm \partial_t \zeta_{\pm} + \frac{1}{2} \partial_x^{-1} \partial_x^2 \zeta_{\pm} \mp \frac{\theta}{2} \partial_x^3 \zeta_{\pm} \pm \frac{1}{90} \partial_x^5 \zeta_{\pm} + \frac{3\sqrt{2}}{4} \zeta_{\pm} \partial_x \zeta_{\pm} = 0.\]

We shall prove in [22] that: under the assumptions in Theorem 1.2, we assume moreover
\[ \lim_{\epsilon \to 0} |\zeta_\epsilon^K - \zeta_0|_{H^{s+D-\frac{1}{4}} \cap \partial_x H^{s+D-\frac{3}{4}}} = 0 \quad \text{and} \quad |\partial_x \psi_\epsilon^K - \partial_x \psi_0|_{H^{s+D-\frac{1}{4}}} = 0 \]
with \((\partial_x \psi_0, \zeta_0) \in H^{s+D-\frac{1}{4}}(\mathbb{R}^2) \cap \partial_x H^{s+D-\frac{3}{4}}(\mathbb{R}^2)\) and \(( \partial_x^3 \partial_x \psi_0, \partial_x^3 \zeta_0) \in \partial_x H^{s+D-6}(\mathbb{R}^2)\).

Then \((KP^{5\text{th}})_{\pm}\) with initial data \((\partial_x \psi_0 \pm \zeta_0)/\sqrt{2}\) has a unique solution \( \zeta_{\pm} \in C([0, T]; \mathcal{H}^{s+D-\frac{1}{4}}(\mathbb{R}^2))\). Furthermore, there holds
\[ \lim_{\epsilon \to 0} |\zeta_\epsilon^K - \zeta_{\epsilon KP}|_{L^\infty([0, T/e^2] \times \mathbb{R}^2)} = 0. \]

### 1.3. Scheme of the proof and organization of the paper and notations.

In Section 2, we shall present various product laws and commutator estimates in the scaled Sobolev spaces; We provide uniform estimates for the solutions of scaled Laplacian equations in the Section 3; While in Section 4, we modify some results from [18] on the calculus of pseudo-differential operators with rough symbols; We shall study the Dirichlet-Neumann operator in Section 5; With the preparation in the above sections, we shall prove large-time uniform estimates for the solutions of the linearized system of (1.8), which is the crucial step in the proof of the large-time wellposedness result for (1.8) in Section 7. In the appendix, we shall present a variance of Nash-Moser iteration Theorem in [3], which has been used in Section 7.

Let us complete this section by some notations, which we shall use throughout the paper. We shall always denote by \( C(\lambda_1, \lambda_2, \cdots) \) a generic positive constant which is a nondecreasing function of its variables, \( t_0 \) a fixed number in \((1, 2)\), and \( m_0 \overset{\text{def}}{=} t_0 + 8 \). We denote \( \nabla_h \overset{\text{def}}{=} (\partial_x, \partial_y) \), \( \nabla^e_h \overset{\text{def}}{=} (\partial_x, \sqrt{\epsilon} \partial_y) \), the scaled horizontal derivatives, \( D^e_h \overset{\text{def}}{=} \frac{1}{\epsilon} \nabla^e_h \), \( \nabla \overset{\text{def}}{=} (\nabla_h, \partial_z) \), \( \nabla^e \overset{\text{def}}{=} (\sqrt{\epsilon} \nabla^e_h, \partial_z) \) the scaled full derivative, \( \xi^e = (\xi_1, \sqrt{\epsilon} \xi_2) \in \mathbb{R}^2 \), and \( |D^e_h| \) the Fourier multiplier with the symbol \( |\xi^e| \). \( \Lambda \) and \( \Lambda_e \) are Fourier multiplier with the symbol \((1 + |\xi^e|^2)^{\frac{1}{2}}\) and \((1 + |\xi^e|^2)^{\frac{1}{2}}\) respectively. We denote \( | \cdot |_p \) the \( L^p(\mathbb{R}^2) \) norm, \( \| \cdot \|_p \) the \( L^p(S) \) norm with \( S = \mathbb{R}^2 \times [-1, 0] \), \( H^s(\mathbb{R}^2) \) the usual Sobolev spaces with the norm \( |f|_{H^s} \overset{\text{def}}{=} |\Lambda^s f|_2 \), \( |f|_{H^s_e} \overset{\text{def}}{=} |\Lambda_e^s f|_2 \), \( |f|_{H^s_{ee}} \overset{\text{def}}{=} |\Lambda_e^s \Lambda_e f|_2 \), \( |f|_{H^s_{e0}} \overset{\text{def}}{=} |\Lambda_e^s \Lambda_e^0 f|_2 \), \( |f|_{H^s_{0e}} \overset{\text{def}}{=} |\Lambda^s_e \Lambda_e^0 f|_2 \) and \( |f|_{H^s_{00}} \overset{\text{def}}{=} |\Lambda^s_e \Lambda_e^0 \Lambda_e^0 f|_2 \).
Proof. We use \( \partial_s H^s(\mathbb{R}^2) \) to refer to the space of all the distributions \( v \) such that there exists \( \tilde{v} \in H^s(\mathbb{R}^2) \) with \( \partial_s \tilde{v} = v \), and we write \( |v|_{\partial_s H^s} \overset{\text{def}}{=} |\tilde{v}|_{H^s} \). We define similarly for \( \partial_z^2 H^s(\mathbb{R}^2) \). Finally we shall always use the convention that

\[
A_s = B_s + \langle C_s \rangle_{s > s_0} = \left\{ \begin{array}{ll}
B_s & \text{if } s \leq s_0, \\
B_s + C_s & \text{if } s > s_0.
\end{array} \right.
\]

2. Preliminaries

Recall that \( |f|_{H^s} = |\Lambda^s f|_2 \) is the norm of the scaled Sobolev space \( H^s_\varepsilon(\mathbb{R}^2) \). It is easy to observe by a scaling argument that

\[ \text{Lemma 2.1. Let } \tau, s \geq 0. \text{ There exists an } \varepsilon \text{ independent constant } C \text{ such that} \]

(i) If \( f \in H^s(\mathbb{R}^2) \) and \( \frac{1}{p} = \frac{1}{2} - \frac{s}{2} \),

\[ |f|_p \leq C \varepsilon^{-\frac{\tau}{s}} |f|_{H^\tau_\varepsilon} \quad \text{for } 0 \leq s < 1, \quad \text{and} \quad |f|_\infty \leq C \varepsilon^{-\frac{1}{2}} |f|_{H^\varepsilon_\varepsilon} \quad \text{for } s > 1; \]

(ii) If \( f, g \in H^s \cap H^{t_0}(\mathbb{R}^2) \),

\[ |fg|_{H^\varepsilon_\varepsilon} \leq C \left( |f|_{H^\tau_\varepsilon} |g|_{H^\tau_\varepsilon} + |f|_\infty |g|_{H^{t_0}} \right); \]

(iii) If \( F \in C^\infty(\mathbb{R}) \) with \( F(0) = 0 \) and \( f \in H^s \cap L^\infty(\mathbb{R}^2) \),

\[ |F(f)|_{H^\varepsilon_\varepsilon} \leq C |f|_{H^\varepsilon_\varepsilon}; \]

(iv) If \( f \in H^{s+r} \cap H^{t_0+1}(\mathbb{R}^2) \) and \( g \in H^{s+r-1} \cap H^{t_0}(\mathbb{R}^2) \),

\[ \left| [\Lambda^s f] g \right|_{H^\varepsilon_\varepsilon} \leq C \left( |\nabla^\varepsilon f|_{H^{s+r-1}_\varepsilon} |g|_{H^\varepsilon_\varepsilon} + |\nabla^\varepsilon f|_{H^{t_0}} |g|_{H^{t_0} \varepsilon} \right); \]

\[ \left| [\Lambda^s f] g \right|_{H^\varepsilon_\varepsilon} \leq C \varepsilon^{-\frac{1}{2}} \left( |\nabla^\varepsilon f|_{H^{s+r-1}_\varepsilon} |g|_{H^{t_0+1}_\varepsilon} + |\nabla^\varepsilon f|_{H^{t_0}} |g|_{H^{t_0} \varepsilon} \right). \]

Remark 2.1. (2.4) still holds with \( \Lambda^s \) being replaced by \( |D^\varepsilon_h|^s \) for \( s \in 2\mathbb{N} \).

\[ \text{Lemma 2.2. Let } s \in \mathbb{R} \text{ and } \nabla u \in L^2([-1,0]; H^{s-1}(\mathbb{R}^2)) \text{ with } u(x_h,0) = 0. \text{ Then for any } j \in [-1,0], \text{ one has} \]

\[ |u|_{j} \leq C \varepsilon^{-\frac{j}{2}} \| \nabla^\varepsilon \Lambda^{s-j-1} u \|_2 \]

for the constant \( C \) independent of \( \varepsilon \).

Proof. Let \( \gamma \overset{\text{def}}{=} \sqrt{\varepsilon} \). It is easy to observe that

\[ \Lambda^s u(x_h, z) = \gamma^{-1} \Lambda^s u_\gamma(x, \frac{y}{\gamma}, z), \quad u_\gamma(x, y, z) = \gamma u(x, \gamma y, z), \]

we have

\[ |u|_{j} \leq \gamma^{-\frac{j}{2}} \| \Lambda^{s-j-1} u_\gamma \|_2 = \gamma^{-\frac{j}{2}} \| \Lambda^{s-j-1} u_\gamma \|_{H^2}. \]
Whereas $u(x_h,0) = 0$, applying Cauchy-Schwartz inequality gives
\[
(1 + |\xi|^2)^{\frac{1}{2}}|s^{-1}u_\gamma(\xi, z)|^2 \\
= - \int_0^1 (1 + |\xi|^2)^{\frac{1}{2}} s^{-1}u_\gamma(\xi, z) \partial_z s^{-1}u_\gamma(\xi, z) dz \\
\leq \left( \int_0^1 (1 + |\xi|^2)|s^{-1}u_\gamma(\xi, z)|^2 dz \right)^{\frac{1}{2}} \left( \int_0^1 |\partial_z s^{-1}u_\gamma(\xi, z)|^2 dz \right)^{\frac{1}{2}},
\]
which implies that
\[
|s^{-1}u_\gamma(\cdot, t)|^2_{H^s} \leq \|s^u_\gamma\| \|\partial_z s^{-1}u_\gamma\| = \gamma \|s^u_\gamma\| \|\partial_z s^{-1}u_\gamma\| \\
\leq \frac{1}{2} \gamma^2 \|s^u_\gamma\|^2 + \frac{1}{2} \|\partial_z s^{-1}u_\gamma\|^2,
\]
from which and (2.6), we deduce that
\[
\|u(\cdot, t)|_{H^s} = C \gamma^{-\frac{1}{2}} (\gamma \|s^u_\gamma\| + \|\partial_z s^{-1}u_\gamma\|) \leq C \gamma^{-\frac{1}{2}} \|\nabla s^{-1}u_\gamma\|.
\]
where in the last step, we used again the fact that $u(x_h,0) = 0$ such that $\|s^{-1}u\| \leq C \|\partial_z s^{-1}u\|$. □

We introduce the following scaled 2nd-order elliptic operator
\[
\varphi(x) \equiv \left| D_h^0 \right|^2 - \frac{e^{3}(\partial_x a)^2 D^2_x + 2e \partial_x a \partial_y a D_x D_y + e^2(\partial_y a)^2 D^2_y}{1 + e^{3}|\nabla h|^2},
\]
which is a part of the linearized operator for the nonlinear system corresponding to the surface tension term.

**Lemma 2.3.** Let $s \in [0, 1]$. Then for $k \in \mathbb{N}, s \geq 0$ and $f \in H^{2k+s} \cap H^{l_0}(\mathbb{R}^2)$, we have
\[
|\varphi(x)|^k f_{H^s} \leq M(a)(|f|_{H^{2k+s} + |f|_{H^0}}) |\nabla a|_{H^{2k+s}}; \\
|\varphi(x)|^k f_{H^s} \geq M(a)^{-1}|f|_{H^{2k+s} - M(a)(1 + |h|^0_{H^{2k+s} + |f|_{H^0}})}.
\]

Here and in what follows $M(a)$ always denotes a constant depending on $|a|_{H^{m_0}}$.

**Proof.** One can deduce this lemma from Lemma 3.5 and Lemma 3.6 in [21] by a scaling argument. For completeness, we shall present the outline of the proof here. Indeed for the first estimate, one only need to use Proposition 2.1 and an interpolation argument. Now we focus on the sketch of the proof for the second estimate.

We use an inductive argument on $k$. Let us first deal with the case when $k = 1$. Toward this, we write $\varphi(x)$ as
\[
\varphi(x) = \sum_{i,j=1,2} \left[ \delta_{ij} - (1 + e^3|\nabla h|^2)^{-1} e^3 \partial_i a \partial_j a \right] D_i^x D_j^x \\
\text{def} = \sum_{i,j=1,2} g_{ij}(e^2 |\nabla h|^2) D_i^x D_j^x \quad \text{for} \quad (D_1^x, D_2^x) = D_h^0.
Then we have

\[ (\partial_{c}(a)f, f) = - \sum_{i,j = 1,2} (g_{ij}(\epsilon^{2} \nabla_{h}^{x} a) \partial_{c}^{i} \partial_{c}^{j} f, f) \]

\[ = \sum_{i,j = 1,2} (g_{ij}(\epsilon^{2} \nabla_{h}^{x} a) \partial_{c}^{i} f, \partial_{c}^{j} f) + (\partial_{c} g_{ij}(\epsilon^{2} \nabla_{h}^{x} a) \partial_{c}^{i} f, f) \]

\[ \geq M(a)^{-1} |\nabla_{h}^{x} f|_{L^{2}}^{2} - M(a) |f|_{L^{2}} |\nabla_{h}^{x} f|_{L^{2}} \]

\[ \geq M(a)^{-1} |f|_{H^{2}}^{2} - M(a) |f|_{L^{2}}^{2}. \]

Whereas notice that

\[ |(\partial_{c}(a)\nabla_{h}^{x} f, \nabla_{h}^{x} f)| \leq |(\partial_{c}(a) f, \nabla_{h}^{x} f)| + \sum_{i,j = 1,2} |((\nabla_{h}^{x} g_{ij}) \partial_{c}^{i} f, \nabla_{h}^{x} f)| \]

\[ \leq |(\partial_{c}(a) f|_{L^{2}} + M(a)|f|_{H^{2}}^{2})| |\nabla_{h}^{x} f|_{H^{2}}^{2}. \]

As a consequence, we obtain

\[ M(a)^{-1} |\nabla_{h}^{x} f|_{H^{2}}^{2} \leq (\partial_{c}(a) \nabla_{h}^{x} f, \nabla_{h}^{x} f) + M(a) |\nabla_{h}^{x} f|_{L^{2}}^{2} \]

\[ \leq \frac{M(a)^{-1}}{2} |\nabla_{h}^{x} f|_{H^{2}}^{2} + M(a) |\partial_{c}(a)f|_{L^{2}}^{2} + M(a) |f|_{L^{2}}^{2}. \]

This ensures

\[ |\partial_{c}(a)f|_{L^{2}} \geq M(a)^{-1} |f|_{H^{2}} - M(a) |f|_{L^{2}}, \]

from which and Kato-Ponce type commutator estimate, we infer

\[ |\partial_{c}(a)f|_{H^{2}} \geq |\partial_{c}(a) \Lambda_{x}^{2} f|_{L^{2}} - |(\Lambda_{x}^{2} g_{ij}) \partial_{c}^{i} f|_{L^{2}} \]

\[ \geq M(a)^{-1} |f|_{H^{2}+s} - M(a) |f|_{H^{2}} - M(a) (|\nabla_{h}^{x} a|_{H^{2+s}} |f|_{H^{2+s}} + |f|_{H^{2+s}}), \]

that is

\[ |\partial_{c}(a)f|_{H^{2}} \geq M(a)^{-1} |f|_{H^{2+s}} - M(a) (1 + |\nabla_{h}^{x} a|_{H^{2+s}}) |f|_{H^{2+s}}. \]

Now we assume inductively that for \(1 \leq \ell \leq k - 1\)

\[ |\partial_{c}(a)^{\ell} f|_{H^{2}} \geq M_{1}(a)^{-1} |f|_{H^{2+\ell}} - M_{1}(a) (1 + |\nabla_{h}^{x} a|_{H^{2+\ell}}) |f|_{H^{2+\ell}}. \]

Then we deduce from the induction assumption that

\[ |\partial_{c}(a)^{k} f|_{H^{2}} = |\partial_{c}(a)^{k-1} \partial_{c}(a) f|_{H^{2}} \geq M(a)^{-1} |\partial_{c}(a) f|_{H^{2(k-1)+s}} \]

\[- M(a) (1 + |\nabla_{h}^{x} a|_{H^{2(k-1)+s}}) |\partial_{c}(a) f|_{H^{2(k-1)+s}}. \]

which together with an interpolation argument implies the second inequality of the lemma.

\[ \square \]

**Lemma 2.4.** Let \(k \in \mathbb{N}, s \geq 0\). Then for any \(f \in H^{2k+s} \cap H^{0} \cap H^{0+2}(\mathbb{R}^{2})\) and \(g \in H^{2k+s} \cap H^{0+1}(\mathbb{R}^{2})\), there hold

\[ |(\partial_{c}(a)^{k}, f)|_{H^{2}} \leq M(a) \left( |f|_{H^{0+2}} + |g|_{H^{2k+s+1}} + |f|_{H^{2k+s+1}} + |a|_{H^{2k+s+1}} |g|_{H^{0+2}} + |g|_{H^{0+2}} |f|_{H^{2k+s+1}} \right) \]

\[ |(\partial_{c}(a)^{k}, \nabla_{h}^{x} g)|_{H^{2}} \leq \epsilon^{2} M(a) \left( |\nabla_{h}^{x} g|_{H^{2k+s+1}} + |a|_{H^{2k+s+1}} |\nabla_{h}^{x} g|_{H^{0}} \right) \]

\[ |(\partial_{c}(a)^{k}, \nabla_{h}^{x} g)|_{H^{2}} \leq \epsilon^{2} M(a) \left( |\nabla_{h}^{x} g|_{H^{2k+s+1}} + |a|_{H^{2k+s+2}} |\nabla_{h}^{x} g|_{H^{0}} \right) \]
Proof. Firstly let’s focus on the proof for the first inequality. Indeed thanks to Lemma 2.1 and Sobolev inequality, it reduces to prove that

\[
\left|\partial_x (a)^k - |D_h^x|^2k, f\right|_{H^s} \leq M(a) \left( |f|_{H^{s_0+2}} |g|_{H^{2k+s-1}_t} + |f|_{H^{2k+s}} |g|_{H^{s_0}} + |a|_{H^{2k+s}} |f|_{H^{s_0+1}_t}\right) \equiv I_k(s, f, g).
\]

We shall use an inductive argument on \(k\) to prove (2.8). We first infer from Lemma 2.1 that

\[
\left|\partial_x (a) - |D_h^x|^2, f\right|_{H^s} \leq M(a) \left( |f|_{H^{s_0+1}_t} |g|_{H^{s_0}} + |f|_{H^{s_0+2}} |g|_{H^{s_0}_t}\right)_{s > t_0 - 1} \quad (2.9)
\]

This shows (2.8) for \(k = 1\). Now we assume that (2.8) holds for \(k \leq \ell - 1\). To prove the case of \(k = \ell\), we write

\[
\left[\partial_x (a) - |D_h^x|^{2\ell}, f\right] g = \partial_x (a) \left[\partial_x (a)^{\ell-1} - |D_h^x|^{2(\ell-1)}, f\right] g + \left(\partial_x (a) - |D_h^x|^2, f\right) \partial_x (a)^{\ell-1} g + \left[|D_h^x|^2, f\right] (\partial_x (a)^{\ell-1} - |D_h^x|^{2(\ell-1)}) g.
\]

We first get by applying Lemma 2.3 and the induction assumption that

\[
\left|\partial_x (a)\left[\partial_x (a)^{\ell-1} - |D_h^x|^{2(\ell-1)}, f\right] g\right|_{H^s} \leq M(a) \left( \left|\partial_x (a)^{\ell-1} - |D_h^x|^{2(\ell-1)}, f\right|_{H^{s_0+1}_t} + |a|_{H^{s_0+1}_t} \left|\partial_x (a)^{\ell-1} - |D_h^x|^{2(\ell-1)}, f\right|_{H^{s_0}} \right) \\
\leq I_\ell(s, f, g) + M(a) |a|_{H^{s_0+1}_t} \left( \left|f\right|_{H^{s_0+2}} \left|g\right|_{H^{2(\ell-1)+t_0+1}_t} + \left|f\right|_{H^{2(\ell-1)+t_0}_t} \left|g\right|_{H^{s_0}_t} \right) \\
\leq I_\ell(s, f, g),
\]

where in the last inequality we used the following interpolation inequalities

\[
|a|_{H^{s_0+1}_t} \leq |a|^{\frac{1-\theta}{H^{s_0+1}_t}} |a|^{\theta}_{H^{2s+\beta}_t}, \quad \theta = \frac{s - t_0 + 2}{s + 2\ell - t_0 - 1}, \\
|a|_{H^{2(\ell-1)+t_0}_t} \leq |a|^{\frac{1-\theta}{H^{s_0+1}_t}} |a|^{\theta}_{H^{2s+\beta}_t}, \quad \theta = \frac{2(\ell - 1) - 1}{s + 2\ell - t_0 - 1}, \\
|g|_{H^{2(\ell-1)+t_0-1}_t} \leq |g|^{\frac{1-\theta}{H^{s_0}_t}} |g|^{\theta}_{H^{2s+\beta+1}_t}, \quad \theta = \frac{2(\ell - 1) - 1}{s + 2\ell - t_0 - 1}, \\
|f|_{H^{2(\ell-1)+t_0}_t} \leq |f|^{\frac{1-\theta}{H^{s_0+1}_t}} |f|^{\theta}_{H^{2s+\beta+1}_t}, \quad \theta = \frac{2(\ell - 1) - 1}{s + 2\ell - t_0 - 1},
\]

such that for example,

\[
|a|_{H^{s_0+1}_t} |g|_{H^{2(\ell-1)+t_0-1}_t} \leq M(a) \left( |a|_{H^{2s+\beta+1}_t} |g|_{H^{s_0+1}_t} + |g|_{H^{2s+\beta+1}_t} \right).
\]

Similarly applying Lemma 2.1 ensures that

\[
\left|\partial_x (a) - |D_h^x|^2\right| \left[|D_h^x|^{2(\ell-1)}, f\right] g \left|_{H^s} \leq I_\ell(s, f, g).
\]
Thanks to (2.9), one has
\[
\left| \partial_x (a) - |D_h^s|^2, f \right| \partial_x (a)^{\ell-1} g |_{H^s_x} \\
\leq M(a) \left( |f| |H^{s+1}_x| |\partial_x (a)^{\ell-1} g |_{H^{s+1}_x} + (|f| |H^{s+2}_x| |\partial_x (a)^{\ell-1} g |_{H^{s+2}_x})_{s > t_0-1} \\
+ \left| |a| |H^{s+1}_x| f |H^{s+2}_x| |\partial_x (a)^{\ell-1} g |_{H^{s+2}_x})_{s > t_0} \right) ,
\]
while it follows from Lemma 2.3 and an interpolation argument that for \( s > t_0 - 1 \),
\[
|f| |H^{s+2}_x| |\partial_x (a)^{\ell-1} g |_{H^{s+2}_x} \leq M(a) |f| |H^{s+2}_x(\epsilon^{-1} + t_0 + |a| |H^{2(\epsilon^{-1} + t_0 + 1)}_x) |_{H^{s+2}_x} \\
\leq I_\epsilon(s, f, g),
\]
and for \( s > t_0 \),
\[
|a| |H^{s+1}_x| f |H^{s+2}_x| |\partial_x (a)^{\ell-1} g |_{H^{s+2}_x} \leq M(a) |a| |H^{s+1}_x| f |H^{s+2}_x| (|g| |H^{2(\epsilon^{-1} + t_0 + 1)}_x + |a| |H^{2(\epsilon^{-1} + t_0 + 1)}_x \leq I_\epsilon(s, f, g).
\]
As a consequence, we obtain
\[
\left| \partial_x (a) - |D_h^s|^2, f \right| \partial_x (a)^{\ell-1} g |_{H^s_x} \leq I_\epsilon(s, f, g).
\]
Similarly we can deduce from (2.4) and Lemma 2.3 that
\[
\left| |D_h^s|^2, f \right| \partial_x (a)^{\ell-1} | - |D_h^s|^2(\ell-1)))g |_{H^s_x} \leq I_\epsilon(s, f, g).
\]
This proves (2.8) for \( k = \ell \), which proves the first estimate of the lemma.

For the second inequality, one can use a similar inductive argument to prove it. And it’s almost the same for the third inequality by noticing that \( \partial_x (a) = f(\epsilon^3(\nabla^2_h a)^2)(D_h^s)^2(\text{formally}) \) and one has
\[
\left| (\partial_x (a), \nabla) |_{H^s_x} = \left| \left| f(\epsilon^3(\nabla^2_h a)^2), \nabla \right| (D_h^s)^2 g \right|_{H^s_x} \\
\leq \epsilon^2 M(a) | |D_h^s|^2 g |_{H^s_x} + |a| |H^{s+1}_x| |D_h^s|^2 g |_{L^2} \leq \epsilon^2 M(a) | |D_h^s|^2 g |_{H^{s+1}_x} + |a| |H^{s+1}_x| |D_h^s|^2 g |_{H^s_x}.
\]
This completes the proof of the lemma.

Let us conclude this section by recalling a result from [24] on the anisotropic Poisson regularization.

**Lemma 2.5.** Let \( \chi \in C_\infty^0(\mathbb{R}) \) with \( \chi(0) = 1 \), and define \( u^\dagger = \chi(z \sqrt{\epsilon}|D_h^s|)u \). Then for any \( s \in \mathbb{R} \), if \( u \in H^{s-\frac{1}{2}}(\mathbb{R}^2) \), we have
\[
c_1 \left| \frac{1}{(1 + \sqrt{\epsilon}|D_h^s|)^{\frac{3}{2}}} u \right|_{H^s_x} \leq \left| \left| \Lambda^s u^\dagger \right|_{2} \leq c_2 \left| \frac{1}{(1 + \sqrt{\epsilon}|D_h^s|)^{\frac{3}{2}}} u \right|_{H^s_x},
\]
and if \( u \in H^{s+\frac{1}{2}}(\mathbb{R}^2) \), we have
\[
c'_1 \sqrt{\epsilon} |\nabla u|_{H^s_x} \leq \left| \left| \Lambda^s \nabla^s u^\dagger \right|_{2} \leq c'_2 \sqrt{\epsilon} |\nabla u|_{H^s_x}.
\]
Here \( c_1, c_2, c'_1 \) and \( c'_2 \) are positive constants depending only on \( \chi \).
3. Elliptic estimates on the infinite strip

In this section, we consider the following boundary value problem on the infinite strip

\[
\begin{aligned}
\partial^2_{zz} \Phi + \varepsilon \partial_{z} \Phi + \varepsilon^2 \partial^2_{y} \Phi &= 0, & -1 + \varepsilon b(x_h) < z < \varepsilon \zeta(t, x_h), \\
\partial_{n} \Phi |_{z=\varepsilon \zeta} &= \psi, & \partial_{n} \Phi |_{z=-1+\varepsilon b} = 0,
\end{aligned}
\]

under the assumption that

\[
1 + \varepsilon \zeta - \varepsilon b \geq h_0 \quad \text{for some} \quad h_0 > 0.
\]

We denote by \( S \) a diffeomorphism from \( S = \mathbb{R}^2 \times [-1, 0] \) to the fluid domain \( \Omega = \mathbb{R}^2 \times [-1 + \varepsilon b(x_h), \varepsilon \zeta(x_h)] \) so that

\[
\begin{aligned}
S : (x_h, z) \in S \mapsto S(x_h, z) = (x_h, z + \sigma(x_h, z)) \in \Omega
\end{aligned}
\]

for \( \sigma(x_h, z) = -\varepsilon zb(x_h) + \varepsilon (z + 1)\zeta(x_h) \).

Using this diffeomorphism \( S \), the elliptic equation (3.1) can be equivalently formulated as an elliptic problem with variable coefficients on the flat strip so that

\[
\begin{aligned}
\nabla \cdot P^\varepsilon[\sigma] \nabla u = 0 & \quad \text{in} \quad S, \\
u |_{z=0} = \psi, & \quad \partial^P_n u |_{z=-1} = 0,
\end{aligned}
\]

where \( u = \Phi \circ S \) and \( \partial^P_n \) denotes the conormal derivative associated with \( P^\varepsilon[\sigma] \), i.e.,

\[
\partial^P_n u = -e_3 \cdot P^\varepsilon[\sigma] \nabla u |_{z=-1}.
\]

Here \( e_3 = (0, 0, 1)^T \). Moreover, we write

\[
\nabla \cdot P^\varepsilon[\sigma] \nabla = \nabla^\varepsilon \cdot (I + Q^\varepsilon[\sigma]) \nabla^\varepsilon
\]

with

\[
Q^\varepsilon[\sigma] = \begin{pmatrix}
\partial_{x} \sigma & 0 & -\sqrt{\varepsilon} \partial_{y} \sigma \\
0 & \partial_{y} \sigma & -\varepsilon \partial_{y} \sigma \\
-\sqrt{\varepsilon} \partial_{y} \sigma & -\varepsilon \partial_{y} \sigma & \frac{-\varepsilon \partial_{x} \sigma + \partial_{y} \sigma}{1 + \partial_{x} \sigma}
\end{pmatrix}.
\]

**Notation 3.1** Throughout this paper, we shall always denote

\[
M(\sigma) = \frac{1}{h_0} C(b|_{H^{m_0}}, |\zeta|_{H^{m_0}})
\]

to be a constant which is a nondecreasing function to all arguments.

To transform the Dirichlet boundary data \( \psi \) in (3.4) to be zero, we are led to consider the following elliptic problem

\[
\begin{aligned}
\nabla^\varepsilon \cdot (1 + Q^\varepsilon[\sigma]) \nabla^\varepsilon u = \nabla^\varepsilon \cdot g & \quad \text{in} \quad S, \\
u |_{z=0} = 0, & \quad \partial^P_n u |_{z=-1} = -e_3 \cdot g |_{z=-1},
\end{aligned}
\]

with

\[
\partial^P_n u |_{z=-1} = -e_3 \cdot (1 + Q^\varepsilon[\sigma]) \nabla^\varepsilon u |_{z=-1}.
\]

**Proposition 3.1.** Let \( s \geq 0 \), and \( \zeta, b \in H^{m_0} \cap H^{s+1}(\mathbb{R}^2) \) satisfy (3.2). Then for all \( g \in C([-1, 0]; H^s(\mathbb{R}^2)^3) \), (3.4) has a unique variational solution \( u \in H^1(S) \) so that

\[
\|
L^\varepsilon \nabla^\varepsilon u \|_2 \leq M(\sigma) \left( \|L^\varepsilon \nabla^\varepsilon g \|_2 + \langle \|L^\varepsilon \nabla^\varepsilon g \|_2, (\zeta, b) \rangle_{H^{s+1}_{\varepsilon} \times H^{s+1}_{\varepsilon}}, s > t_0 + 1 \right).
\]

**Proof.** Thanks to (2.1)-(2.4), one can deduce Proposition 3.1 by exactly following the same line as the proof of Proposition 2.4 in [2] (see also the proof of Proposition 3.2 below), and we omit the details here. □
Notation 3.2. For $u \in H^{\frac{7}{2}}(\mathbb{R}^2)$, we define $u^b$ as the solution of
\begin{align}
(3.7) \quad \begin{cases}
\nabla^\varepsilon \cdot (1 + Q^\varepsilon[\sigma]) \nabla^\varepsilon u^b = 0 & \text{in } S, \\
u^b|_{z=0} = u, \quad \partial_n u^b|_{z=-1} = 0.
\end{cases}
\end{align}

As an immediate corollary of Proposition 3.1, we obtain

Corollary 3.1. Let $s \geq 0$, and $\zeta, b \in H^{m0} \cap H^{s+1}(\mathbb{R}^2)$ satisfy (3.7). Then for any $u \in H^{s+\frac{1}{2}}(\mathbb{R}^2)$, one has
\[ \|\nabla^\varepsilon u^b\|_2 \leq \sqrt{\varepsilon} M(\sigma) \left( \|\Psi u|_{H^{s+1}_\varepsilon} + \langle\|\Psi u|_{H^{m0}_\varepsilon}\rangle_{H^{s+1}_\varepsilon}|_{s>t_0} \right). \]

Proof. Let $v \overset{\text{def}}{=} u^b - u^\dagger$ being given by Lemma 2.5. Then $v$ solves
\begin{align*}
(3.7) \quad \begin{cases}
\nabla^\varepsilon \cdot (1 + Q^\varepsilon[\sigma]) \nabla^\varepsilon v = -\nabla^\varepsilon \cdot (1 + Q^\varepsilon[\sigma]) \nabla^\varepsilon u^\dagger & \text{in } S, \\
u|_{z=0} = 0, \quad \partial_n^P u|_{z=-1} = e_3 \cdot (1 + Q^\varepsilon[\sigma]) \nabla^\varepsilon u^\dagger|_{z=-1}.
\end{cases}
\end{align*}

Thanks to (2.1)-(2.2), we get by applying Proposition 3.1 for $g = -(1 + Q^\varepsilon[\sigma]) \nabla^\varepsilon u^\dagger$ that
\[ \|\nabla^\varepsilon v\|_2 \leq M(\sigma) \left( \|\nabla^\varepsilon u^\dagger\|_2 + \langle\|\nabla^\varepsilon u^\dagger\|_2\rangle_{H^{s+1}_\varepsilon}|_{s>t_0} \right), \]
which together with Lemma 2.5 completes the proof of the corollary. 

Besides (3.6), we also need to deal with a more general elliptic problem as follows
\begin{align}
(3.8) \quad \begin{cases}
\nabla^\varepsilon \cdot (1 + Q^\varepsilon[\sigma]) \nabla^\varepsilon u = f & \text{in } S, \\
u|_{z=0} = 0, \quad \partial_n^P u|_{z=-1} = g.
\end{cases}
\end{align}

Proposition 3.2. Under the assumptions of Proposition 3.1 for any given $f \in L^2((-1, 0); H^{s-1}(\mathbb{R}^2))$ and $g \in H^s(\mathbb{R}^2)$, (3.8) has a variational solution $u \in H^1(S)$ so that
\[ \|\nabla^\varepsilon u\|_2 \leq M(\sigma) \left( e^{-\frac{1}{4} \|\nabla^\varepsilon f\|_2} + \langle e^{-\frac{1}{4} \|\nabla^\varepsilon f\|_2}\rangle_{H^{s+1}_\varepsilon}|_{s>t_0+1} \right), \]
\[ \|\nabla^\varepsilon u\|_2 \leq M(\sigma) \left( e^{-\frac{1}{4} \|\nabla^\varepsilon f\|_2} + \langle e^{-\frac{1}{4} \|\nabla^\varepsilon f\|_2}\rangle_{H^{s+1}_\varepsilon}|_{s>t_0+1} \right), \]
and if $s \geq 1$, we have
\[ \|\nabla^\varepsilon \partial_z u\|_2 \leq M(\sigma) \left( \|\nabla^\varepsilon f\|_2 + \sqrt{\varepsilon} |g|_{H^{s}_\varepsilon} + \langle \|\nabla^\varepsilon f\|_2 + \sqrt{\varepsilon} |g|_{H^{m_0}_\varepsilon}\rangle_{H^{s+1}_\varepsilon}|_{s>t_0+1} \right). \]

Proof. Since the existence part can be obtained by a standard argument, here we just present the detailed proof of the estimates. Indeed recall that what we mean by $u$ is a variational solution of (3.8): for any $\phi \in C^\infty_0((-1, 0) \times \mathbb{R}^2)$, there holds
\begin{align}
(3.9) \quad \int_S (1 + Q^\varepsilon[\sigma]) \nabla^\varepsilon u \cdot \nabla^\varepsilon \phi \, dx_h \, dz = - \int_S f \, \phi \, dx_h \, dz + \int_{\mathbb{R}^2} g \, \phi|_{z=-1} \, dx_h.
\end{align}

Taking $\phi = \Lambda^\varepsilon u$ in (3.9) results in
\[ \int_S (1 + Q^\varepsilon[\sigma]) \nabla^\varepsilon \Lambda^\varepsilon u \cdot \nabla^\varepsilon \Lambda^\varepsilon u \, dx_h \, dz \]
\begin{align}
(3.10) \quad \leq - \int_S (\Lambda^\varepsilon, Q^\varepsilon[\sigma]) \nabla^\varepsilon u \cdot \nabla^\varepsilon \Lambda^\varepsilon u + \Lambda^\varepsilon u + \Lambda^\varepsilon f \, \Lambda^\varepsilon u \, dx_h \, dz + \int_{\mathbb{R}^2} \Lambda^\varepsilon g \, \Lambda^\varepsilon u|_{z=-1} \, dx_h
\end{align}
\[ \leq \|\nabla^\varepsilon u\|_2 \|\nabla^\varepsilon \Lambda^\varepsilon u\|_2 + \|\nabla^\varepsilon f\|_2 \|\nabla^\varepsilon \Lambda^\varepsilon u\|_2 + \|\Lambda^\varepsilon g\|_2 \|\Lambda^\varepsilon u\|_{z=-1}. \]
Thanks to Proposition 2.3 in [2], we have
\[
\int_S (1 + Q^e[\sigma]) \nabla^e \Lambda^e_{\varepsilon} u \cdot \nabla^e \Lambda^e_{\varepsilon} u \, dx_h \, dz \geq M(\sigma)^{-1} \| \Lambda^e_{\varepsilon} \nabla^e u \|_2^2,
\]
While it follows from (2.4) that
\[
\| [\Lambda^e_{\varepsilon}, Q^e[\sigma]] \nabla^e u \|_2 \leq M(\sigma) \left( \| \Lambda^e_{\varepsilon-1} \nabla^e u \|_2 + \langle \| \Lambda^e_{\varepsilon} \nabla^e u \|_2, (\zeta, b) \rangle_{H^{s+1}_{\varepsilon}} \right),
\]
and thanks to \( u(x_h, 0) = 0 \), we have
\[
\sqrt{\varepsilon} \| \Lambda^e_{\varepsilon+1} u \|_2 \leq C \| \Lambda^e_{\varepsilon} \nabla^e u \|_2, \quad \| \Lambda^e_{\varepsilon} u \|_{z=\varepsilon} \leq \| \Lambda^e_{\varepsilon} \partial_z u \|_2.
\]
Plugging all the above estimates into (3.10) and using Young’s inequality yield that
\[
\| \Lambda^e_{\varepsilon} \nabla^e u \|_2 \leq M(\sigma) \left( \| \Lambda^e_{\varepsilon-1} \nabla^e u \|_2 + \langle \| \Lambda^e_{\varepsilon} \nabla^e u \|_2, (\zeta, b) \rangle_{H^{s+1}_{\varepsilon}} \right)
\]
\[
+ \varepsilon^{-\frac{1}{2}} \| \Lambda^e_{\varepsilon-1} f \|_2 + |g|_{H^s_{\varepsilon}}
\]
from which and an interpolation argument, we deduce that
\[
\| \Lambda^e_{\varepsilon} \nabla^e u \|_2 \leq M(\sigma) \left( \| \nabla^e u \|_2 + \langle \| \Lambda^e_{\varepsilon} \nabla^e u \|_2, (\zeta, b) \rangle_{H^{s+1}_{\varepsilon}} \right)
\]
\[
+ \varepsilon^{-\frac{1}{2}} \| \Lambda^e_{\varepsilon} \nabla^e u \|_2 + |g|_{H^s_{\varepsilon}}
\]
Whereas taking \( \phi = u \) in (3.3), we get
\[
\| \nabla^e u \|_2 \leq M(\sigma) \langle \| f \|, \| g \| \rangle_{H^s_{\varepsilon}}
\]
Consequently, we arrive at
\[
(3.11) \quad \| \Lambda^e_{\varepsilon} \nabla^e u \|_2 \leq M(\sigma) \left( \| \Lambda^e_{\varepsilon} \nabla^e u \|_2 + \langle \| \Lambda^e_{\varepsilon} \nabla^e u \|_2, (\zeta, b) \rangle_{H^{s+1}_{\varepsilon}} \right)
\]
\[
+ \varepsilon^{-\frac{1}{2}} \| \Lambda^e_{\varepsilon} \nabla^e u \|_2 + |g|_{H^s_{\varepsilon}}
\]
which implies the first inequality of Proposition 3.2

To prove the second inequality, we only need to replace the estimate for the boundary term. Indeed thanks to Lemma 2.2 one has
\[
\left| \int_{R^2} \Lambda^e_{\varepsilon} g^{\varepsilon} u \big|_{z=-1} \, dx_h \right| \leq |g|_{H^{s+\frac{1}{2}}_{\varepsilon}} \, |u|_{z=-1} \big|_{H^{s+\frac{1}{2}}_{\varepsilon}} \leq C \varepsilon^{-\frac{1}{2}} |g|_{H^{s+\frac{1}{2}}_{\varepsilon}} \, \| \Lambda^e_{\varepsilon} \nabla^e u \|_2,
\]
which along with the proof of (3.11) gives the second inequality of Proposition 3.2

Finally, we get by using the elliptic equation and (2.2) to obtain
\[
\| \Lambda^e_{\varepsilon} \nabla^e u \|_2 \leq M(\sigma) \left( \| \Lambda^e_{\varepsilon} \nabla^e u \|_2 + \sqrt{\varepsilon} \| \Lambda^e_{\varepsilon} \nabla^e u \|_2 \right)
\]
\[
+ \langle \| \Lambda^e_{\varepsilon} \nabla^e u \|_2 + \sqrt{\varepsilon} \| \Lambda^e_{\varepsilon} \nabla^e u \|_2, (\zeta, b) \rangle_{H^{s+1}_{\varepsilon}} \rangle_{s+1}
\]
This together with the first inequality implies the third inequality of the proposition. This finishes the proof of Proposition 3.2
4. Calculus of pseudo-differential operators with symbols of limited smoothness

In this section, we shall adapt some results from [18] on the calculus of pseudo-differential operators with symbols of limited smoothness to our setting here. More precisely, we shall consider symbols of the form

\[ \sigma(x_h, \xi) = \Sigma(v(x_h), \xi) \]

for \( v \in C^0(\mathbb{R}^2)^p \) with \( p \in \mathbb{N} \), and \( \Sigma \) is a function defined as follows (see [18]):

**Definition 4.1.** Let \( m \in \mathbb{N}_0, p \in \mathbb{N} \), and \( \Sigma \) be a function defined on \( \mathbb{R}_0^p \times \mathbb{R}_\xi^2 \). We say \( \Sigma(v, \xi) \in C^\infty(\mathbb{R}_0^p, \dot{\mathcal{M}}^m) \) if

1. \( \Sigma \in C^\infty(\mathbb{R}_0^p; C^m) \) and \( |\partial^\beta \Sigma(v, \xi)| \leq C_\beta |\xi|^{m-\beta} \) for any \( \xi \in \mathbb{R}^2 \), \( |\beta| \leq m \);
2. for any \( \alpha \in \mathbb{N}^p, \beta \in \mathbb{N}^2 \), there exists a non-decreasing function \( C_{\alpha, \beta}(\cdot) \) such that

\[
\sup_{|\xi| \geq \frac{1}{4}} |\xi|^{-m} (|\partial^\alpha \partial^\beta \Sigma|(v, \xi)) \leq C_{\alpha, \beta}(|\xi|).
\]

For given \( \Sigma, v \) and \( \varepsilon \in [0, 1] \), we consider pseudo-differential operators, \( \text{Op}_\varepsilon(\sigma) \), defined by

\[
\text{Op}_\varepsilon(\sigma)u(x_h) \overset{\text{def}}{=} (2\pi)^{-2} \int_{\mathbb{R}^2} e^{ix_h \cdot \xi} \sigma(x_h, \xi^{\varepsilon}) \hat{u}(\xi) d\xi, \quad \xi^{\varepsilon} = (\xi_1, \sqrt{\varepsilon} \xi_2).
\]

**Proposition 4.1.** Let \( m \in \mathbb{N}_0, p \in \mathbb{N} \). Then for given \( \Sigma \in C^\infty(\mathbb{R}_0^p, \dot{\mathcal{M}}^m) \), \( v \in H^{t_0}(\mathbb{R}^2) \) and \( \sigma(x_h, \xi) = \Sigma(v(x_h), \xi) \), one has

\[
|\text{Op}_\varepsilon(\sigma)u|_{H^s} \leq C(|v|_{\infty}) \left( |D^m_h u|_{H^s} + \left\langle \varepsilon^{-\frac{1}{4}} |v|_{H^{t_0}} |D^m_h u|_{H^s} \right\rangle \right) + |v|_{H^s} |D^m u|_{H^{t_0}} s > t_0,
\]

for all \( 0 \leq s \leq t_0 \).

**Proof.** Using the scaling argument, one can reduce the proof of Proposition 4.1 to the case when \( \varepsilon = 1 \). We first split \( u \) as the low and high frequency part so that

\[
u = u_{hf} + u_{lf}, \quad u_{lf} = \psi(D) u,
\]

where \( \psi \in C^\infty_0(\mathbb{R}^d) \) and \( \psi \equiv 1 \) near the origin. Let \( \sigma_0(\xi) \overset{\text{def}}{=} \Sigma(0, \xi) \), it is easy to observe that

\[
|\text{Op}(\sigma_0) u|_{H^s} \leq C|D|^m u|_{H^s}.
\]

Whence without loss of generality, we may assume that \( \sigma_0(\xi) = 0 \). While thanks to Corollary 30 of [18], we have

\[
|\text{Op}(\sigma) u_{hf}|_{H^s} \leq C(|v|_{\infty}) \left( |u_{hf}|_{H^{s+m}} + |v|_{H^{t_0}} |u_{hf}|_{H^{s+m}} + \left\langle |v|_{H^s} |u_{hf}|_{H^{t_0+m}} \right\rangle \right) s > t_0,
\]

\[
\leq C(|v|_{\infty}) \left( |D|^m u|_{H^s} + |v|_{H^{t_0}} |D|^m u|_{H^s} + \left\langle |v|_{H^s} |D|^m u|_{H^{t_0}} \right\rangle \right).
\]

On other hand, notice that

\[
|\text{Op}(\sigma) u_{lf}|_{H^s} = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{ix_h \cdot \xi} \sigma(x_h, \xi) \hat{u}(\xi) d\xi,
\]

and \( |e^{ix_h \cdot \xi} \sigma(x_h, \xi)|_{H^s} \leq C(\xi)^s |\sigma(\cdot, \xi)|_{H^s} \), which along with (2.3) ensures that

\[
|\text{Op}(\sigma) u_{lf}|_{H^s} \leq C \sup_{|\xi| \leq 1} (|\xi|^{-m} |\sigma(\cdot, \xi)|_{H^s}) \int_{|\xi| \leq 1} |\xi|^m |\hat{u}(\xi)| d\xi
\]

\[
\leq C(|v|_{\infty}) |v|_{H^s} |D|^m u|_{2}.
\]

This finishes the proof of Proposition 4.1. \( \square \)
To handle the composition and commutator between two pseudo-differential operators of limited-smooth symbols, we recall the following symbols for $n \in \mathbb{N}_0$:

$$
\sigma_1 \ast_n \sigma_2(x, \xi) \overset{\text{def}}{=} \sum_{|\alpha| \leq n} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \sigma_1(x, \xi) \partial_x^\alpha \sigma_2(x, \xi) \quad \text{and} \\
\{\sigma_1, \sigma_2\}_n(x, \xi) \overset{\text{def}}{=} \sigma_1 \ast_n \sigma_2(x, \xi) - \sigma_2 \ast_n \sigma_1(x, \xi).
$$

**Proposition 4.2.** Let $m_1, m_2, n \in \mathbb{N}_0$ with $m^+ = \max(m_1, m_2)$, $m^- = \min(m_1, m_2) \geq n$. Let $\sigma^j(x, \xi) = \Sigma^j(v^j(x, \xi))$ (j = 1, 2) with $p_j \in \mathbb{N}$, $\Sigma^j \in C^\infty(\mathbb{R}^{p_j}, \mathcal{M}^{m_j})$ and $v_j \in H^{1+a+m^+1}(\mathbb{R}^2)$. Then for any $0 \leq s \leq t_0 + 1$, there holds

$$
|\text{Op}_\varepsilon(\sigma^1) \circ \text{Op}_\varepsilon(\sigma^2) u - \text{Op}_\varepsilon(\sigma_1 \ast_n \sigma_2) u|_{H^s} \\
\leq C(|v|_{W^{n+1, \infty}}) \left\{ \frac{|D^m \sigma|_{H^{s+m-n-1}}}{H^s} |v|_{W^{n+1, \infty}} + \left( \varepsilon^{-\frac{1}{2}} |v|_{H^{s+m+1}} \right) \frac{|D^m \sigma|_2}{H^s} \right\} \\
+ \left( \varepsilon^{-\frac{1}{2}} |v|_{H^{s+m+1}} + \varepsilon^{-\frac{1}{2}} |v|_{H^{s+1}} \right) \frac{|D^m \sigma|_{H^s}}{H^{s+m+1}} \right\}.
$$

In particular, we have

$$
|\text{Op}_\varepsilon(\sigma^1), \text{Op}_\varepsilon(\sigma^2) u - \text{Op}_\varepsilon(\sigma_1 \ast_n \sigma_2) u|_{H^s} \\
\leq C(|v|_{W^{n+1, \infty}}) \left\{ \frac{|D^m \sigma|_{H^{s+m-n-1}}}{H^s} |v|_{W^{n+1, \infty}} + \left( \varepsilon^{-\frac{1}{2}} |v|_{H^{s+m+1}} \right) \frac{|D^m \sigma|_2}{H^s} \right\} \\
+ \left( \varepsilon^{-\frac{1}{2}} |v|_{H^{s+m+1}} + \varepsilon^{-\frac{1}{2}} |v|_{H^{s+1}} \right) \frac{|D^m \sigma|_{H^s}}{H^{s+m+1}} \right\}.
$$

Here $v = (v^1, v^2)$.

**Proof.** Again using a scaling argument, one can reduce the proof of Proposition 4.2 to the case when $\varepsilon = 1$. As in the proof of Proposition 4.1, we split $u$ into $u = u_{lf} + u_{hf}$ given by (4.11). Then we get by applying Theorem 7 and Theorem 8 in [18] that

$$
|\text{Op}(\sigma^1) \circ \text{Op}(\sigma^2) u_{lf} - \text{Op}(\sigma_1 \ast_n \sigma_2) u_{lf}|_{H^s} \\
\leq C(|v|_{W^{n+1, \infty}}) \left\{ |u_{lf}|_{H^{s+n+1-m-n-1}} |v|_{W^{n+1, \infty}} + \left( |v|_{H^{s+m+1}} \right) \frac{|D^m \sigma|_{H^{s+m+1}}} {H^s} \right\} \\
+ \left( |v|_{H^{s+m+1}} + |v|_{H^{s+1}} \right) \frac{|D^m \sigma|_{H^s}} {H^{s+m+1}} \right\}.
$$

Setting $\sigma_0^j(\xi) = \Sigma^j(0, \xi)$ and $\sigma_0^2(\xi) = \Sigma^j(0, \xi)$, we write

$$
\text{Op}(\sigma^1) \circ \text{Op}(\sigma^2) u - \text{Op}(\sigma_1 \ast_n \sigma_2) u = \text{Op}(\sigma^1) \circ \text{Op}(\sigma^2 - \sigma_0^j) - \text{Op}(\sigma_1 \ast_n (\sigma^2 - \sigma_0^j)) \\
+ \text{Op}(\sigma_1 - \sigma_0^j) \circ \text{Op}(\sigma_0^j) - \text{Op}((\sigma_1 - \sigma_0^j) \ast_n \sigma_0^2).
$$

It follows from the proof of Proposition 4.1 that

$$
|\text{Op}(\sigma_1 \ast_n (\sigma^2 - \sigma_0^j)) u_{lf}|_{H^s} \leq C(|v|_{W^{n, \infty}}) |D^{m_2-m-n} u_{lf}|_2,
$$

where $m_2 = \max(m_1, m_2)$.
and it is easy to observe that
\[
|\text{Op}(\sigma^1) \circ \text{Op}(\sigma^2 - \sigma_0^2)u_{lf}|_{H^s} \leq C(|v|_{\infty}) \left( |\text{Op}(\sigma^2 - \sigma_0^2)u_{lf}|_{H^{s+m_1}} + |v|_{H^s}|\text{Op}(\sigma^2 - \sigma_0^2)u_{lf}|_{H^{t_0+m_1}} \right)
\]
\[
\leq C(|v|_{\infty}) \left( |v|_{H^{s+m_1}}||D||m_2u_{lf}|_{2} + |v|_{H^s}|v|_{H^{t_0+m_1}}||D||m_2u_{lf}|_{2} \right),
\]
which implies that
\[
|\text{Op}(\sigma^1) \circ \text{Op}(\sigma^2)u_{lf} - \text{Op}(\sigma^1\sigma^2)u_{lf}|_{H^s} \leq C(|v|_{W^{n,\infty}})(1 + |v|_{H^{t_0+m_1}})|v|_{H^{s+m_1}}||D||m_2u_{lf}|_{2}.
\]
This along with (4.3) concludes the proof of Proposition 4.2. \hfill \square

**Proposition 4.3.** Let \( m_1, m_2, n \in \mathbb{N}_0 \) with \( m_1, m_2 \geq n \) and \( m^{-} \overset{\text{def}}{=} \min(m_1, m_2) \). Let \( \sigma^1(\xi) \in \mathcal{M}^{m_1} \), and \( \sigma^2(x_h, \xi) = \Sigma(v(x_h), \xi) \) with \( p \in \mathbb{N}, \Sigma \in C^\infty(\mathbb{R}^p, \mathcal{M}^{m_2}) \) and \( v \in H^{t_0+m_1}(\mathbb{R}^2) \). Then for any \( 0 \leq s \leq t_0 + 1 \), there holds
\[
|\sigma^1(D_h^s) \circ \text{Op}(\sigma^2)u|_{H^s} - \text{Op}(\sigma^1\sigma^2)u|_{H^s} \leq C(|v|_{W^{n+1,\infty}})(||D_h^{m_1+2}u||_{H_{-1}^{s+1+m_1}} + \epsilon^{-\frac{1}{4}}|v|_{H^{s+m_1}}(||D_h^{m_2}u||_{H_t^{t_0}} + ||D_h^{m_2}u||_{2})).
\]

**Proof.** Similar to the proof of Proposition 4.2, one first reduces the proof of this proposition to the case when \( \epsilon = 1 \). For the high frequency part, \( u_{hf} \) of \( u \), we use Corollary 39 in [2] so that
\[
|\sigma^1(D_h^s) \circ \text{Op}(\sigma^2)u_{hf} - \text{Op}(\sigma^1\sigma^2)u_{hf}|_{H^s} \leq C(|v|_{W^{n+1,\infty}})(||v|_{H^{s+m_1+m_2-n-1}}||v|_{W^{n+1,\infty}} + |v|_{H^{s+m_1}}||v|_{H^{t_0+m_2}}).
\]
The low frequency part, \( u_{lf} \) of \( u \) can be obtained by exactly the same line as the proof to Proposition 4.2. \hfill \square

5. The Dirichlet-Neumann Operator

The goal of this section is to study the Dirichlet-Neumann operator defined by (1.6), which will be the key ingredient used in the proof of the first part of Theorem 1.1. Firstly thanks to the argument at the beginning of section 3, we write (1.7) on \( \Omega = \mathbb{R}^2 \times [-1 + \epsilon b(x_h), \epsilon \zeta(t, x_h)] \) into a problem on the flat strip \( S \):
\[
\begin{aligned}
\begin{cases}
\nabla_{h,z} \cdot \mathcal{P}^\epsilon[\sigma]\nabla_{h,z} \psi^b = 0, & \text{in } S, \\
\psi|_{z=0} = 0 = \psi, & \partial^n_h \psi|_{z=-1} = 0.
\end{cases}
\end{aligned}
\]
where \( \psi^b = \Phi \circ S, \mathcal{P}^\epsilon[\sigma] = I + Q^\epsilon(\sigma), Q^\epsilon(\sigma) \) and \( S \) are given by (3.5) and (3.3) respectively. Then we can write the Dirichlet-Neumann operator as
\[
G[\epsilon \zeta] \psi = \partial^n_h \psi|_{z=0} = -e_3 \cdot \mathcal{P}^\epsilon[\sigma] \nabla \psi^b|_{z=0}.
\]

5.1. Some basic properties. For the convenience of the readers, we shall first recall some basic properties of Dirichlet-Neumann operator from [2].

**Proposition 5.1.** Let \( \zeta, b \in H^{m_0}(\mathbb{R}^2) \) satisfy (3.2). Then we have
1. The Dirichlet-Neumann operator \( G[\epsilon \zeta] \) is self-adjoint:
\[
(u, G[\epsilon \zeta]v) = (v, G[\epsilon \zeta]u), \quad \forall u, v \in H^4(\mathbb{R}^2);
\]
2. For all \( u, v \in H^4(\mathbb{R}^2) \),
\[
|\langle u, G[\epsilon \zeta]v \rangle| \leq (u, G[\epsilon \zeta]u)^{\frac{1}{2}} (v, G[\epsilon \zeta]v)^{\frac{1}{2}};
\]
(3) For \( u \in H^{\frac{1}{2}}(\mathbb{R}^2) \),
\[
M(\sigma)^{-1} |\Psi u|^2 \leq (u, \frac{1}{\varepsilon} G[\varepsilon \zeta] u) \leq M(\sigma) |\Psi u|^2;
\]

(4) For \( \mathbf{v} \in H^{t_0+1}(\mathbb{R}^2)^2 \), \( u \in H^{\frac{1}{2}}(\mathbb{R}^2) \), and \( d_\varepsilon(\zeta) \) given by (2.7),
\[
|\mathbf{v} \cdot \nabla_\varepsilon u, \frac{1}{\varepsilon} G[\varepsilon \zeta] u| \leq M(\sigma) |\mathbf{v}|_{W^{1,\infty}} |\Psi u|^2 \quad \text{and}
\]
\[
|\left( d_\varepsilon(\zeta)^k, \mathbf{v} \cdot \nabla_\varepsilon h \right) u, \frac{1}{\varepsilon} G[\varepsilon \zeta] \left( d_\varepsilon(\zeta)^k, \mathbf{v} \cdot \nabla_\varepsilon h \right) u| \leq M(\sigma) |\mathbf{v}|_{H^{t_0+2}} \left( |\Psi u|^2_{H^{2k}} + |(\zeta, \mathbf{v})|_{H^{2k+2}} |\Psi u|^2_{H^{t_0+1}} \right).
\]

Proof. The second estimate in (4) can be deduced by following the proof of Proposition 3.7 (i) in [2], and all the other estimates can be found in [2]. \( \square \)

Proposition 5.2. Let \( s \geq t_0 \), and \( \zeta, b \in H^{s+\frac{3}{2}}(\mathbb{R}^2) \) satisfy (3.2). Then for any \( \psi \in H^{s+\frac{3}{2}} \), the mapping \( \zeta \mapsto G[\varepsilon \zeta] \psi \) is well-defined and differentiable in a neighborhood of \( \zeta \) in \( H^{s+\frac{3}{2}}(\mathbb{R}^2) \). Moreover, for any \( h \in H^{s+\frac{3}{2}}(\mathbb{R}^2) \), there holds
\[
d_\varepsilon G[\varepsilon \zeta] \psi \cdot h = -\varepsilon G[\varepsilon \zeta](hZ) - \varepsilon^2 \nabla_\varepsilon \cdot (h \mathbf{v}) \quad \text{with}
\]
\[
\mathbf{v} = \nabla_\varepsilon \psi - \varepsilon Z \nabla_\varepsilon \zeta \quad \text{and} \quad Z = \frac{1}{1 + \varepsilon^2 |\nabla_\varepsilon \zeta|^2} \left( G[\varepsilon \zeta] + \varepsilon^2 \nabla_\varepsilon \zeta \cdot \nabla_\varepsilon \psi \right).
\]

Proposition 5.3. Let \( s \geq t_0 \), and \( \zeta, b \in H^{t_0} \cap H^{s+1}(\mathbb{R}^2) \) satisfy (3.2). Then for any \( u \in H^{s+\frac{1}{2}}(\mathbb{R}^2) \), \( j \in \{0, 1, 2\} \) and \( h \in H^{t_0} \cap H^{s+1}(\mathbb{R}^2)^j \), one has
\[
\left| \frac{1}{\varepsilon} d_\varepsilon^j G[\varepsilon \zeta] u \cdot h \right|_{H^{s+\frac{1}{2}}} \leq \varepsilon^j M(\sigma) \left( |\Psi u|_{H^{s+1}} \prod_{k=1}^j |h_k|_{H^{t_0+1}} + |(\zeta, b)|_{H^{s+1}} \right)
\times |\Psi u|_{H^{t_0}} \prod_{k=1}^j |h_k|_{H^{t_0+1}} + |\Psi u|_{H^{t_0}} \sum_{k=1}^j |h_k|_{H^{s+1}} \prod_{l \neq k} |h_l|_{H^{t_0+1}} \right),
\]

Proof. We only present the proof for the case when \( j = 0 \), the other cases can be handled in a similar way (one may check the proof of Proposition 3.3 in [2]). Indeed for any \( v \in \mathcal{S}(\mathbb{R}^2) \), let \( u^b \) and \( u^\dagger \) be defined by (3.1) and (2.5) respectively. Then applying (2.2) and the fact that
\[
|\Lambda_{\varepsilon}^{-\frac{3}{2}} \nabla_\varepsilon u^\dagger|_2 \leq C |v|_2
\]
gives
\[
\left( \Lambda_{\varepsilon}^{s-\frac{3}{2}} G[\varepsilon \zeta] u, v \right) = \left( G[\varepsilon \zeta] u, \Lambda_{\varepsilon}^{s-\frac{1}{2}} v \right) = \int_S (1 + Q[\varepsilon \zeta]) \nabla_\varepsilon u^b \cdot \Lambda_{\varepsilon}^{s-\frac{1}{2}} \nabla_\varepsilon v^\dagger dx_h dz
\]
\[
= \int_S \Lambda_{\varepsilon}^s (1 + Q[\varepsilon \zeta]) \nabla_\varepsilon u^b \cdot \Lambda_{\varepsilon}^{-\frac{3}{2}} \nabla_\varepsilon v^\dagger dx_h dz
\]
\[
\leq C |\Lambda_{\varepsilon}^s (1 + Q[\varepsilon \zeta]) \nabla_\varepsilon u^b|_2 |v|_2
\]
\[
\leq M(\sigma) |v|_2 \left( |\Lambda_{\varepsilon}^s \nabla_\varepsilon u^b|_2 + |\Lambda_{t_0}^{t_0} \nabla_\varepsilon u^b|_2 |(\zeta, b)|_{H^{s+1}} \right),
\]
which along with Corollary 3.11 proves the proposition for the case \( j = 0 \). \( \square \)
Remark 5.1. We can also deduce from the proof of Proposition 3.3 in [2] that
\[
\left| \frac{1}{\sqrt{\varepsilon}} |A_{\varepsilon}^{m-\frac{1}{2}} d\xi G[\varepsilon \zeta] u \cdot h |_{H^s} \right| \leq \varepsilon^j M(\sigma) \left( |A_{\varepsilon}^{m} \mathcal{P} u |_{H^s} \prod_{k=1}^{j} |h_k|_{H^{t_0+m+1}} \right)
\]
+ \left| A_{\varepsilon}^{m+1}(\zeta, b) |_{H^s} |\mathcal{P} u |_{H^{t_0+m}} \prod_{k=1}^{j} |h_k|_{H^{t_0+m+1}} \right|
\]
+ \left| |\mathcal{P} u |_{H^{t_0+m}} \sum_{k=1}^{j} |A_{\varepsilon}^{m+1} h_k |_{H^s} \prod_{l \neq k} |h_l|_{H^{t_0+m+1}} \right|
\]
for \( m = 0, 1, 2, 3 \). This result is not sharp, but is enough for our applications in this paper.

Proposition 5.4. Let \( T > 0, b \in H^{m_0}(\mathbb{R}^2), \zeta \in C([0, T]; H^{m_0}(\mathbb{R}^2)) \) satisfy (5.2) for some \( h_0 \) independent of \( t \). Then for any \( u \in C^1([0, T]; H^{\frac{1}{2}}(\mathbb{R}^2)) \) and \( t \in [0, T] \), one has
\[
\left| \left( [\partial_t, G_\varepsilon] u(t), u(t) \right) \right| \leq \varepsilon M(\sigma(t)) |\nabla_h \partial_t \zeta|_{\infty} |\mathcal{P} u(t)|_2^2.
\]

5.2. The principle part of the DN operator. Recall that \( \sigma(t, x_h, z) = -\varepsilon b(x_h) + \varepsilon(1 + z)\zeta(t, x_h) \), we rewrite \( P^\varepsilon[\sigma] \) in (5.1) as
\[
P^\varepsilon[\sigma] = \begin{pmatrix}
P_1^\varepsilon & P_{d+1}^\varepsilon \\
(p^\varepsilon)^T & p_{d+1}^\varepsilon
\end{pmatrix},
\]
with
\[
P_1^\varepsilon = \varepsilon(1 + \varepsilon \zeta - \varepsilon b) \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix},
\]
\[
(p^\varepsilon)^T = -\varepsilon^2 \begin{pmatrix} -z \partial_z b + (z + 1) \partial_z \zeta \\ \varepsilon( -z \partial_y b + (z + 1) \partial_y \zeta) \end{pmatrix},
\]
\[
p_{d+1}^\varepsilon = \begin{pmatrix} 1 + \varepsilon^2(\varepsilon \partial_z b - \varepsilon(z + 1) \partial_z \zeta)^2 + \varepsilon^2(\varepsilon \partial_y b - \varepsilon(z + 1) \partial_y \zeta)^2 \\ 1 + \varepsilon^2(\varepsilon \partial_z b - \varepsilon(z + 1) \partial_z \zeta)^2 + \varepsilon^2(\varepsilon \partial_y b - \varepsilon(z + 1) \partial_y \zeta)^2 \end{pmatrix}.
\]

Then we have
\[
P^\varepsilon[\sigma] \equiv \nabla_{h,z} \cdot P^\varepsilon[\sigma] \nabla_{h,z} = p_{d+1} \partial_z^2 + (2p^\varepsilon \cdot \nabla_h + (\partial_z p_{d+1} + \nabla_h \cdot p^\varepsilon)) \partial_z + P_1^\varepsilon \Delta_h + ((\nabla_h \cdot P_1^\varepsilon) + \partial_z p^\varepsilon) \cdot \nabla_h.
\]

For simplicity, we shall neglect the subscript \( \varepsilon \) in what follows.

We now define the approximate operator to \( P^\varepsilon[\sigma] \) as follows
\[
P_{app}^{\varepsilon} \equiv p_{d+1} \partial_z - \eta_{-}(x_h, z, D_h^\varepsilon) \partial_z - \eta_{+}(x_h, z, D_h^\varepsilon)
\]
where \( \eta_{\pm}(x_h, z, D_h^\varepsilon) \) are pseudo-differential operators with symbols
\[
\eta_{\pm}(x_h, z, \xi^\varepsilon) = \frac{1}{p_{d+1}} \left( -i p \cdot \xi \pm \sqrt{p_{d+1} \xi \cdot P_1^\varepsilon - (p \cdot \xi)^2} \right)
\]
\[
\eta_{\pm}(x_h, z, \xi^\varepsilon) = \frac{1 + \partial_z \sigma}{1 + \varepsilon |\nabla_h^\varepsilon \sigma|^2} \left( (1 + \varepsilon |\nabla_h^\varepsilon \sigma|^2) \xi^\varepsilon \pm \sqrt{(1 + \varepsilon |\nabla_h^\varepsilon \sigma|^2)|\xi^\varepsilon|^2 - \varepsilon^2|\nabla_h^\varepsilon \sigma \cdot \xi^\varepsilon|^2} \right).
\]

It is easy to observe that there exists some constant \( c_+ > 0 \) so that
\[
\sqrt{\varepsilon} M(\sigma)|\xi^\varepsilon| \geq \text{Re}(\eta_{\pm}(x_h, z, \xi^\varepsilon)) \geq \sqrt{\varepsilon} c_+ |\xi^\varepsilon|,
\]
and there exists \( \Sigma_{\pm}(v, \xi) \in C^\infty(\mathbb{R}^3, \mathcal{M}) \) such that \( \eta_{\pm}(x_h, z, \xi^\varepsilon) = \sqrt{\varepsilon} \Sigma_{\pm}(\nabla^\varepsilon \sigma, \xi^\varepsilon) \).
As in [17], for $u \in \mathcal{S}(\mathbb{R}^2)$, we define the approximate solutions to (3.7) as

$$u_{\text{app}}^b(x_h, z) = \sigma_{\text{app}}(x_h, z, D^\varepsilon_h)u \quad \text{with}$$

$$\sigma_{\text{app}}(x_h, z, \xi) = \exp \left(-\int_z^0 \eta_+(x_h, s, \xi) ds\right),$$

(5.6)

and we define the symbol for the approximate Dirichlet-Neumann operator as

$$g(x_h, \xi^\varepsilon) \overset{\text{def}}{=} \sqrt{\varepsilon(1 + \varepsilon^4|\nabla^\varepsilon_h \xi|^2)|\xi|^2 - \varepsilon^4(\xi \cdot \nabla^\varepsilon_h \xi)^2}.$$  

(5.7)

Then it follows from [17] that

$$g(x_h, D^\varepsilon_h)\psi = \partial^P_n \psi_{\text{app}}|z=0.$$  

(5.8)

We’ll see that $g(x_h, D^\varepsilon_h)$ is the principle part of the D-N operator.

The goal of this subsection is to prove the following proposition which concerns the accuracy of the approximate Dirichlet-Neumann operator.

**Proposition 5.5.** Let $s \geq 0$, $u \in H^{s+\frac{1}{2}} \cap H^{0+\frac{1}{2}}(\mathbb{R}^2)$, and $u^b$ be defined by (3.7). Let

$$R[\varepsilon\zeta]u \overset{\text{def}}{=} G[\varepsilon\zeta]u - g(x_h, D^\varepsilon_h)u = \partial_n^P(u^b - u_{\text{app}})|z=0.$$  

Then we have

$$|R[\varepsilon\zeta]u|_{H^s_2} \leq \sqrt{\varepsilon} M(\sigma) \left(|\mathfrak{P}u|_{H^s_2} + |\mathfrak{P}u|_{H^{0}_2}|(\zeta, b)|_{H^{s+2}_2}\right).$$  

(5.9)

**Remark 5.2.** This estimate is not a standard one since the gain is of half instead of one derivative compared to similar estimates in [17, 21]. This is due to the need of $O(\sqrt{\varepsilon})$ term in the r.h.s. of (5.9). In fact, we refer to [16] for how to gain the full derivative without losing the $\sqrt{\varepsilon}$ in the r.h.s. of (5.2).

We start the proof of this proposition by the following lemma.

**Lemma 5.1.** Under the assumptions of Proposition 5.5, we have

$$\|\Lambda^\varepsilon_x \nabla^\varepsilon_h u_{\text{app}}^b\|_2 \leq M(\sigma) \left(|\mathfrak{P}u|_{H^s_2} + |\mathfrak{P}u|_{H^{0}_2}|(\zeta, b)|_{H^{s+2}_2}\right),$$

$$\|\Lambda^\varepsilon_x \partial_z u_{\text{app}}^b\|_2 \leq \sqrt{\varepsilon} M(\sigma) \left(|\mathfrak{P}u|_{H^s_2} + |\mathfrak{P}u|_{H^{0}_2}|(\zeta, b)|_{H^{s+2}_2}\right).$$

Proof. Thanks to (5.10), we write

$$u_{\text{app}}^b = \sigma_{\text{app}}(x_h, z, D^\varepsilon_h) \exp(-\frac{c_+}{2} \sqrt{\varepsilon z}|D^\varepsilon_h|) \exp(\frac{c_+}{2} \sqrt{\varepsilon z}|D^\varepsilon_h|)u$$

$$\overset{\text{def}}{=} \sigma_{\text{app}}(x_h, z, D^\varepsilon_h) \exp(\frac{c_+}{2} \sqrt{\varepsilon z}|D^\varepsilon_h|)u.$$  

(5.11)

Note by (5.5) that $\sigma_{\text{app}}(x_h, z, D^\varepsilon_h)$ is a pseudo-differential operator of order zero, and

$$\nabla^\varepsilon_h u_{\text{app}}^b = \left(-\int_z^0 (\nabla^\varepsilon_h \eta_+)(\cdot, s, \cdot) ds \sigma_{\text{app}}(x_h, z, D^\varepsilon_h) \right) (x_h, z, D^\varepsilon_h) + \sigma_{\text{app}}(x_h, z, D^\varepsilon_h) \nabla^\varepsilon_h u_{\text{app}}^b \right) \exp(\frac{c_+}{2} \sqrt{\varepsilon z}|D^\varepsilon_h|)u,$$
Lemma 5.2. Under the assumptions of Proposition 4.3, we denote \( u_t^\text{def} = u^b - u_{\text{app}}^b \). Then one has

\[
\| \Lambda_\varepsilon^s \nabla_{h_{\text{app}}^b} \|_2 \leq M(\varepsilon) \left( \| \Lambda_\varepsilon^s |D_{h_{\text{app}}}^b\| \exp \left( \frac{C_1}{2} \sqrt{\varepsilon} z |D_{h_{\text{app}}}^b| \right) u \|_2 + \| \Lambda_\varepsilon^s |D_{h_{\text{app}}}^b\| |(\zeta, b)_{H_{\varepsilon}^{l+2}} \right) \leq M(\varepsilon) \| |\Psi u|_{H_{\varepsilon}^l} + |\Psi u|_{H_{\varepsilon}^{l+1}} |(\zeta, b)_{H_{\varepsilon}^{l+2}} \right). \]

Proof. Thanks to the definition of \( u_t^b \), we find out that it solves

\[
\begin{align*}
\left\{ \begin{array}{l}
P_u^b = P(u^b - u_{\text{app}}^b) = - (P - \text{P}_{\text{app}}) u_{\text{app}}^b - \text{P}_{\text{app}} u_{\text{app}}^b \quad \text{in} \ S, \\
u_t^b|_{z=0} = 0, \quad \partial_n^p u_t^b|_{z=-1} = - \partial_n^p h_{\text{app}}^b|_{z=-1},
\end{array} \right.
\end{align*}
\]

where \( \text{P}_{\text{app}} \) is defined in (5.3). Then we get by applying the first inequality of Proposition 8.2 that

\[
\| \Lambda_\varepsilon^s \nabla_{h_{\text{app}}^b} \|_2 \leq M(\varepsilon) \left( \| \Lambda_\varepsilon^s |h_{\text{app}}^1 + h_{\text{app}}^2\|_2 + |\partial_n^p u_{\text{app}}^b|_{z=-1} |(\zeta, b)_{H_{\varepsilon}^{l+2}} \right) \]

Here

\[
\begin{align*}
h_{\text{app}}^1 &= (P - \text{P}_{\text{app}}) u_{\text{app}}^b \quad \text{and} \quad h_{\text{app}}^2 = \text{P}_{\text{app}} u_{\text{app}}^b.
\end{align*}
\]

In what follows, we shall estimate term by term the right hand side of (5.13).

**Step 1.** The estimate of \( h_{\text{app}}^1 \).

Let

\[
\begin{align*}
\tau_1(x, \varepsilon, z, D_{h_{\text{app}}}^b) &= \eta_- (x, \varepsilon, z, D_{h_{\text{app}}}^b) \circ \eta_+ (x, \varepsilon, z, D_{h_{\text{app}}}^b) - (\eta_- \eta_+) (x, \varepsilon, z, D_{h_{\text{app}}}^b), \\
\tau_2(x, \varepsilon, z, D_{h_{\text{app}}}^b) &= (\partial_z \eta_+) (x, \varepsilon, z, D_{h_{\text{app}}}^b).
\end{align*}
\]
Then we write

\[ \mathbf{P} - \mathbf{P}_{app} = -p_{d+1} \tau_1 + p_{d+1} \tau_2 + (\partial_z p_{d+1} + \nabla_h \cdot \mathbf{p}) \partial_z + (\nabla_h \cdot P_1 + \partial_z \mathbf{p}) \cdot \nabla_h. \]

While it is easy to observe that

\[ \nabla_h \cdot P_1 = -2 \left( \frac{\partial_z (\zeta - b)}{\epsilon \partial_y (\zeta - b)} \right) = -\partial_z \mathbf{p}, \]

\[ \nabla_h \cdot \mathbf{p} = \epsilon |D_h|^{3/2} \sigma = \epsilon^2 (-z|D_h|^{3/2} b + (z + 1)|D_h|^{3/2} \zeta), \]

\[ \partial_z p_{d+1} = 2 \epsilon^3 (-z\nabla_h b + (1 - z)\nabla_h \zeta) \cdot \nabla_h (\zeta - b) \]

one has \( \nabla_h \cdot P_1 + \partial_z \mathbf{p} = 0 \) and

\[ \partial_z p_{d+1} + \nabla_h \cdot \mathbf{p} = \epsilon^2 F(b, \nabla_h b, |D_h|^{3/2} b, \zeta, \nabla_h \zeta, |D_h|^{3/2} \zeta) \]

for some smooth function \( F \). Then it follows from (2.2)–(2.3) that

\[ \| \Lambda^s h_{app} \|_2 \leq M(\sigma) \left( \| \Lambda^s \tau_1(x_h, z, D_h^s) u_{app}^b \|_2 + \| \Lambda^s \tau_2(x_h, z, D_h^s) u_{app}^b \|_2 \right. \\ + \left. \epsilon^2 \| \Lambda^s \partial_z u_{app}^b \|_2 + \left( \| \Lambda^s \tau_1(x_h, z, D_h^s) u_{app}^b \|_2 + \| \Lambda^s \tau_2(x_h, z, D_h^s) u_{app}^b \|_2 \right) \right). \]

Applying Proposition 4.2 and Lemma 5.1 yields

\[ \| \Lambda^s \tau_1(x_h, z, D_h^s) u_{app}^b \|_2 \leq \epsilon^2 M(\sigma) \left( \| \Lambda^s \tau_1(x_h, z, D_h^s) u_{app}^b \|_2 + \| \Lambda^s \tau_2(x_h, z, D_h^s) u_{app}^b \|_2 \right) \]

\[ \leq \epsilon^2 M(\sigma) \left( \| \mathfrak{P} u \|_{H^s} + \| \mathfrak{P} u \|_{H^{s+1}} \right). \]

Similarly applying Proposition 4.1 and Lemma 5.1 gives

\[ \| \Lambda^s \tau_2(x_h, z, D_h^s) u_{app}^b \|_2 \leq \epsilon^2 M(\sigma) \left( \| \mathfrak{P} u \|_{H^s} + \| \mathfrak{P} u \|_{H^{s+1}} \right). \]

As a consequence, we obtain

\[ (5.15) \quad \| \Lambda^s h_{app} \|_2 \leq \epsilon^2 M(\sigma) \left( \| \mathfrak{P} u \|_{H^s} + \| \mathfrak{P} u \|_{H^{s+1}} \right). \]

**Step 2.** The estimate of \( h_{app}^2 \).

Thanks to (5.4), we write

\[ h_{app}^2 = \mathbf{P}_{app} u_{app}^b = p_{d+1}(\partial_z - \eta_-(x_h, z, D_h^s))\tau_3(x_h, z, D_h^s) \exp \left( \frac{c^+}{2} z \sqrt{\epsilon} |D_h^s| \right) u, \]

where \( \sigma_{app} \) is given by (5.11) and

\[ \tau_3(x_h, z, D_h^s) \overset{\text{def}}{=} \mathbf{O}_\epsilon(\eta_+ \sigma_{app}) - \mathbf{O}_\epsilon(\eta_+) \circ \sigma_{app}(x_h, z, D_h^s). \]

Applying Proposition 4.1 gives

\[ \| \Lambda^s h_{app}^2 \|_2 \leq M(\sigma) \left( \| \Lambda^s \nabla^e \tau_3(x_h, z, D_h^s) \exp \left( \frac{c^+}{2} z \sqrt{\epsilon} |D_h^s| \right) u \|_2 \right. \]

\[ + \left. \| \Lambda^s \nabla^e \tau_3(x_h, z, D_h^s) \exp \left( \frac{c^+}{2} z \sqrt{\epsilon} |D_h^s| \right) u \|_2 \right). \]

As in the proof of Proposition 4.1, we split \( u \) into the low frequency and high frequency parts so that \( u = u_{lf} + u_{hf} \) with \( u_{lf} = \psi(D_h^s) u \). Then we deduce from Proposition 4.2 and the
proof of Lemma 5.1 that
\[ \|\Lambda^+ \nabla \tau_3(x_h, z, D_h) \exp(\frac{c_+}{2} z \sqrt{\varepsilon}|D_h^\varepsilon|) u_{hf}\|_2 \leq \varepsilon^{\frac{7}{2}} M(\sigma) \left( \|\Lambda_\varepsilon^{t+1} \exp(\frac{c_+}{2} z \sqrt{\varepsilon}|D_h^\varepsilon|) u_{hf}\|_2 + \|\Lambda_\varepsilon^{t+1} \exp(\frac{c_+}{2} z \sqrt{\varepsilon}|D_h^\varepsilon|) u_{hf}\|_2(\zeta, b)|_{H^3}\right) \]
and similarly we have
\[ \leq \varepsilon^{\frac{7}{2}} M(\sigma)(|\mathcal{P} u|_{H^5} + |\mathcal{P} u|_{H^5}(\zeta, b)|_{H^3}). \]
and it follows from a similar procedure as that used in handling $u_{hf}$ in Proposition 4.2 that
\[ \|\Lambda^+ \nabla \tau_3(x_h, z, D_h) \exp(\frac{c_+}{2} z \sqrt{\varepsilon}|D_h^\varepsilon|) u_{hf}\|_2 \leq \varepsilon^{\frac{7}{2}} M(\sigma) \|D_h^\varepsilon\| \exp(\frac{c_+}{2} z \sqrt{\varepsilon}|D_h^\varepsilon|) u_{hf}\|_2(\zeta, b)|_{H^3} \]
\[ \leq \varepsilon^{\frac{7}{2}} M(\sigma)|\mathcal{P} u|_2(\zeta, b)|_{H^3}. \]
Whence we obtain
\[ (5.16) \quad \|\Lambda^+ h_{app}^2\|_2 \leq \varepsilon^{\frac{7}{2}} M(\sigma)(|\mathcal{P} u|_{H^5} + |\mathcal{P} u|_{H^5}(\zeta, b)|_{H^3}). \]

**Step 3.** The estimate of $\partial_n u_{app}$.

Noticing that
\[ \partial_n u_{app}(\zeta, -1) = \varepsilon^2 \Delta_h \nabla u_{app}(x_h, -1, D_h) u - p_{d+1}(\zeta, -1) \]
which together with (2.2.2) implies that
\[ |\partial_n u_{app}\|_{H^{3+1}} \leq M(\sigma) \left( \varepsilon^2 \|D_h^\varepsilon\| \|\sigma_{app}(x_h, -1, D_h) u\|_{H^{3+1}} \right) \]
\[ + |(|| \sigma_{app}(x_h, -1, D_h) u\|_{H^{3+1}} + \varepsilon^2 \|D_h^\varepsilon\| \|\sigma_{app}(x_h, -1, D_h) u\|_{H^{3+1}} \right) \]
\[ + |(|| \sigma_{app}(x_h, -1, D_h) u\|_{H^{3+1}} \|\zeta, b\|_{H^3}). \]
It is easy to observe from the proof of Lemma 5.1 that
\[ \sqrt{\varepsilon}|D_h^\varepsilon\| \sigma_{app}(x_h, -1, D_h u|_{H^{3+1}} \leq M(\sigma) \left( \|1 + \sqrt{\varepsilon}|D_h^\varepsilon|\|D_h^\varepsilon\| u|_{H^3} \right) \]
\[ + |(\|1 + \sqrt{\varepsilon}|D_h^\varepsilon|\|D_h^\varepsilon\| u|_{H^{3+1}} \|\zeta, b\|_{H^3}). \]
and similarly we have
\[ |(|| \sigma_{app}(x_h, -1, D_h) u\|_{H^{3+1}} \leq M(\sigma) \left( |\mathcal{P} u|_{H^5} + |\mathcal{P} u|_{H^{5}}(\zeta, b)|_{H^{3+1}} \right). \]
Therefore, we obtain that
\[ (5.17) \quad |\partial_n u_{app}\|_{H^{3+1}} \leq M(\sigma)(|\mathcal{P} u|_{H^5} + |\mathcal{P} u|_{H^{5}}(\zeta, b)|_{H^{3+1}}). \]
The above arguments also imply that
\[ (5.18) \quad |\partial_n u_{app}\|_{H^{3+1}} \leq \varepsilon^{\frac{7}{2}} M(\sigma)(|\mathcal{P} u|_{H^5} + |\mathcal{P} u|_{H^{5}}(\zeta, b)|_{H^{3+1}}). \]
Plugging (5.15)–(5.17) into (5.13) yields the first estimate of the lemma. The second inequality of the lemma can be deduced from the third inequality of Proposition 3.2 and (5.15)–(5.17). This finishes the proof of Lemma 5.2. □
With the above two lemmas, we can complete the proof of Proposition 5.5.

**Proof of Proposition 5.5.** Thanks to (5.9), for any \( v \in \mathcal{S}(\mathbb{R}^2) \), we get by applying Green's identity that

\[
\langle \Lambda_\varepsilon^s R[\varepsilon \zeta] u, v \rangle = \langle \partial^P_n u^b_r |_{z=0}, \Lambda_\varepsilon v \rangle \\
= -\langle \partial^P_n u^b_r |_{z=-1}, \Lambda_\varepsilon v |_{z=-1} \rangle + \int_S \left( P u^b_r \Lambda_\varepsilon \varepsilon v \varepsilon v_x + (1 + Q[\varepsilon]) \nabla^\varepsilon v_x u^b_r \cdot \nabla^\varepsilon \Lambda_\varepsilon \varepsilon v \varepsilon v_x \right) dx_h dz \\
= (\Lambda_\varepsilon^s \partial^P_n u^b_{app} |_{z=-1}, \chi(\sqrt{-\varepsilon} |D^\varepsilon_\varepsilon|) v) \\
- \int_S \left( \Lambda_\varepsilon^s P u^b_{app} v \varepsilon \varepsilon - \Lambda_\varepsilon^s (1 + Q[\varepsilon]) \nabla^\varepsilon v_x u^b_r \cdot \nabla^\varepsilon \varepsilon v \varepsilon v_x \right) dx_h dz.
\]

As \( \|v\|^2 \leq C\|v\|_2 \), applying Lemma 5.3 below ensures that

\[
\left| \langle \Lambda_\varepsilon^s R[\varepsilon \zeta] u, v \rangle \right| \leq C\|v\|_2 \left( \|\Lambda_\varepsilon^s \partial^P_n u^b_{app} |_{z=-1} \|_2 + \|\Lambda_\varepsilon^s P u^b_{app} \|_2 + \sqrt{\varepsilon} \|\Lambda_\varepsilon^s+1 (1 + Q[\varepsilon]) \nabla^\varepsilon v_x u^b_r \|_2 \right)
\]

which together with (5.15), (5.16), (5.18) and Lemma 5.2 implies that

\[
\left| \langle \Lambda_\varepsilon^s R[\varepsilon \zeta] u, v \rangle \right| \leq \sqrt{\varepsilon} M(\sigma) \|v\|_2 \left( \|P u\|_{H^2} + \|P u\|_{H^0} \|\zeta, b\|_{H^2} \right).
\]

This proves (5.10) by duality. \( \square \)

### 5.3. Commutator estimates.

In this subsection, we shall present several useful commutator estimates between the Dirichlet-Neumann operator and the elliptic operator \( \partial_\varepsilon(\zeta) \) defined by (2.7).

**Proposition 5.6.** Let \( k \in \mathbb{N} \), and \( \varepsilon, a, b \in H^{m_0} \cap H^{2k+2}(\mathbb{R}^2) \) satisfy (3.2). Then for any \( u \in \mathcal{H}^{l+\frac{3}{2}} \cap H^{2k+\frac{3}{2}}(\mathbb{R}^2) \), there holds

\[
(5.19) \left| \frac{1}{\varepsilon} G[\varepsilon \zeta], \partial_\varepsilon(\zeta)^k u \right|_2 \leq \varepsilon M(\sigma) \left( \|P u\|_{H^2} + \|P u\|_{H^0} \|\zeta, b\|_{H^{2k+2}} \right).
\]

**Proof.** Thanks to (3.5) and (5.2), for any \( v \in \mathcal{S}(\mathbb{R}^2) \), we get by applying Green’s identity that

\[
\langle [G[\varepsilon \zeta], \partial_\varepsilon(\zeta)^k] u, v \rangle = \langle G[\varepsilon \zeta] \partial_\varepsilon(\zeta)^k u, v \rangle - \langle \partial_\varepsilon(\zeta)^k G[\varepsilon \zeta] u, v \rangle \\
= \int_S \left\{ (1 + Q[\varepsilon]) \nabla^\varepsilon \partial_\varepsilon(\zeta)^k u \cdot \nabla^\varepsilon v + \partial_\varepsilon(\zeta)^k (1 + Q[\varepsilon]) \nabla^\varepsilon u \cdot \nabla^\varepsilon v \right\} dx_h dz \\
= \int_S \left\{ (1 + Q[\varepsilon]) \nabla^\varepsilon, \partial_\varepsilon(\zeta)^k u \cdot \nabla^\varepsilon v \right\} dx_h dz \\
+ (1 + Q[\varepsilon]) \nabla^\varepsilon \partial_\varepsilon(\zeta)^k u b \cdot \nabla^\varepsilon v \right\} dx_h dz \\
- \left[ \nabla^\varepsilon, \partial_\varepsilon(\zeta)^k \right] (1 + Q[\varepsilon]) \nabla^\varepsilon u b \cdot \nabla^\varepsilon v \right\} dx_h dz
\]

\[\def\A_1 = A_1 + A_2 + A_3.\]

To deal with \( A_1, A_2 \), we need the following lemma, which can be deduced from the proof of Lemma 3.1 in [2].

**Lemma 5.3.** For all \( f \in L^2(\mathbb{R}^2) \) and \( g \in H^1(\mathcal{S})^3 \), one has

\[
\left| \int_S \nabla^\varepsilon f \cdot g \, dx_h \, dz \right| \leq C \sqrt{\varepsilon} \|f\|_2 \|\Lambda_\varepsilon g\|_2.
\]
Applying Lemma 5.3 to $A_1$ gives
\[ |A_1| \leq C \sqrt{\varepsilon} |v|_2 |\Lambda_\varepsilon [(1 + Q[\sigma]) \nabla^\varepsilon, \partial_\varepsilon(\zeta) u^b]|_2, \]
but as
\[ [(1 + Q[\sigma]) \nabla^\varepsilon, \partial_\varepsilon(\zeta) u^b] = (1 + Q[\sigma]) [\nabla^\varepsilon, \partial_\varepsilon(\zeta) u^b] + [Q[\sigma], \partial_\varepsilon(\zeta) \nabla^\varepsilon u^b], \]
from which, Lemma 2.4 and Corollary 3.1, we deduce that
\[ |\Lambda_\varepsilon [(1 + Q[\sigma]) \nabla^\varepsilon, \partial_\varepsilon(\zeta) u^b]|_2 \leq \varepsilon M(\sigma) (|\Lambda_\varepsilon^2 \nabla^\varepsilon u^b|_2 + |\Lambda_\varepsilon^{t+1} \nabla^\varepsilon u^b|_2)((\zeta, b)|_{H^{t+2}}) \]
\[ \leq \varepsilon^{\frac{2}{3}} M(\sigma) (|\mathcal{P}u|_{H^{2k}} + |\mathcal{P}u|_{H^{t+1}}((\zeta, b)|_{H^{2k+2}}). \]
As a consequence, we obtain
\[ (5.21) \quad |A_1| \leq \varepsilon^2 M(\sigma) |v|_2 (|\mathcal{P}u|_{H^{2k}} + |\mathcal{P}u|_{H^{t+1}}((\zeta, b)|_{H^{2k+2}}). \]
Applying Lemma 5.3 again, we have
\[ |A_2| \leq \sqrt{\varepsilon M(\sigma)} |v|_2 |\Lambda_\varepsilon \nabla^\varepsilon((\partial_\varepsilon(\zeta) u^b) - \partial_\varepsilon(\zeta) u^b)|_2. \]
Thanks to (3.7), we find that $(\partial_\varepsilon(\zeta) u^b) - \partial_\varepsilon(\zeta) u^b$ solves
\[ \left\{ \begin{array}{l}
\nabla^\varepsilon \cdot (1 + Q[\sigma]) \nabla^\varepsilon ((\partial_\varepsilon(\zeta) u^b) - \partial_\varepsilon(\zeta) u^b) = g, \\
(\partial_\varepsilon(\zeta) u^b) - \partial_\varepsilon(\zeta) u^b|_{z=0} = 0, \\
\partial_n ((\partial_\varepsilon(\zeta) u^b) - \partial_\varepsilon(\zeta) u^b)|_{z=-1} = \varepsilon M(\sigma) \nabla^\varepsilon(\partial_\varepsilon(\zeta) u^b). 
\end{array} \right. \]
where
\[ g \overset{\text{def}}{=} - [\nabla^\varepsilon, \partial_\varepsilon(\zeta)] \cdot (1 + Q[\sigma]) \nabla^\varepsilon u^b - [\nabla^\varepsilon, \partial_\varepsilon(\zeta)] (1 + Q[\sigma]) \nabla^\varepsilon u^b. \]
Then we deduce from Proposition 3.1 and Proposition 3.2 that
\[ |\Lambda_\varepsilon \nabla^\varepsilon((\partial_\varepsilon(\zeta) u^b) - \partial_\varepsilon(\zeta) u^b)|_2 \leq M(\sigma) (|\Lambda_\varepsilon [(1 + Q[\sigma]) \nabla^\varepsilon, \partial_\varepsilon(\zeta) u^b]|_2 + \varepsilon^{-\frac{1}{2}} ||[\nabla^\varepsilon, \partial_\varepsilon(\zeta)] \cdot (1 + Q[\sigma]) \nabla^\varepsilon u^b||_2) \]
\[ \leq \varepsilon^{\frac{2}{3}} M(\sigma) (|\mathcal{P}u|_{H^{2k}} + |\mathcal{P}u|_{H^{t+1}}((\zeta, b)|_{H^{2k+2}}), \]
which gives
\[ (5.22) \quad |A_2| \leq \varepsilon^2 M(\sigma) |v|_2 (|\mathcal{P}u|_{H^{2k}} + |\mathcal{P}u|_{H^{t+1}}((\zeta, b)|_{H^{2k+2}}). \]
To deal with $A_3$, we apply Lemma 2.5, Lemma 2.4 and Corollary 3.1 to obtain
\[ |A_3| \leq c_2 ||[\nabla^\varepsilon, \partial_\varepsilon(\zeta)] \cdot ((1 + Q[\sigma]) \nabla^\varepsilon u^b)||_2 |v|_2 \]
\[ \leq \varepsilon^{\frac{2}{3}} M(\sigma) (|\mathcal{P}u|_{H^{2k}} + |\mathcal{P}u|_{H^{t+1}}((\zeta, b)|_{H^{2k+2}})|v|_2, \]
which along with (5.20), (5.22) concludes that
\[ |(G(\varepsilon \zeta), \partial_\varepsilon(\zeta) u, v)| \leq \varepsilon^2 M(\sigma) |v|_2 (|\mathcal{P}u|_{H^{2k}} + |\mathcal{P}u|_{H^{t+1}}((\zeta, b)|_{H^{2k+2}}), \]
and this implies (5.19). \hfill \square

**Remark 5.3.** It is easy to observe from the proof of Proposition 5.6 that
\[ |d_{k}^{m}m[A, \frac{1}{\varepsilon} G(\varepsilon \zeta)] u|_2 \leq \varepsilon M(\sigma) (|\Lambda_\varepsilon^{m+1} \mathcal{P}u|_{H^{t+1}} + |\mathcal{P}u|_{H^{t+2}} |\Lambda_\varepsilon^{3}(\zeta, b)|_{H^{t}}) \]
for $m = 0, 1$, which will be used later in the lower order energy estimate.

In order to deal with the energy estimate for the linearized system of (1.8), we need the following sharper commutator estimate.

---

**Remark 5.4.** It is easy to observe from the proof of Proposition 5.6 that
\[ |d_{k}^{m}m[A, \frac{1}{\varepsilon} G(\varepsilon \zeta)] u|_2 \leq \varepsilon M(\sigma) (|\Lambda_\varepsilon^{m+1} \mathcal{P}u|_{H^{t+1}} + |\mathcal{P}u|_{H^{t+2}} |\Lambda_\varepsilon^{3}(\zeta, b)|_{H^{t}}) \]
for $m = 0, 1$, which will be used later in the lower order energy estimate.
Theorem 5.1. Let \( s \geq 0, k \in \mathbb{N}, \) and \( \zeta, b \in H^{m_0} \cap H^{2k+s+3}(\mathbb{R}^2) \) satisfy (5.2). We denote \( \rho(\zeta) = (1 + \varepsilon^2|\nabla_\varepsilon^z \zeta|^2)^{1/2} \). Then for any \( u \in H^{2k+s-\frac{1}{2}} \cap H^{10+\frac{5}{2}}(\mathbb{R}^2) \), we have
\[
(5.23) \quad \left| \frac{1}{\varepsilon} \rho(\zeta)^{-1} G(\varepsilon \zeta), \delta_\varepsilon(\zeta)^b \right| u_{H^s_{\varepsilon \delta}} \leq \varepsilon M(\sigma) \left( |\mathcal{P} u|_{H^{2k+s-1}} + |\mathcal{P} u|_{H^{10+2}} |(\zeta, b)|_{H^{2k+s+3}} \right).
\]

Remark 5.4. Compared with (5.10), the commutator estimate (5.23) gains one more derivative. The key observation used to prove this theorem is that the symbol of the principle part of the operator \( \rho(\zeta)^{-1} G[\varepsilon \zeta] \) is the same as the square root of the symbol of \( \delta_\varepsilon(\zeta) \).

In what follows, we divide the proof of Theorem 5.1 into two parts. In the first part, we deal with the commutator estimate between the principle part of DN operator and \( \delta_\varepsilon(\zeta) \).

Proposition 5.7. Let \( s \geq 0, u \in H^{s+1} \cap H^{10+2}(\mathbb{R}^2) \), and \( g(x, \varepsilon \zeta) \) be determined by (5.7). Then under the assumptions of Theorem 5.1, one has
\[
(5.24) \quad \left| \frac{1}{\varepsilon} \rho(\zeta)^{-1} g(x, D^2_h \delta_\varepsilon(\zeta)), \delta_\varepsilon(\zeta)^b \right| u_{H^s_{\varepsilon \delta}} \leq \varepsilon^2 M(\sigma) \left( |\mathcal{P} u|_{H^{s+1}} + |\mathcal{P} u|_{H^{10+2}} |\zeta|_{H^{s+3}}(s_{t0+1}) \right).
\]

Proof. It is easy to observe from (4.2) and (5.16) that
\[
\{ \rho(\zeta)^{-1} g(x, \varepsilon \zeta), \delta_\varepsilon(\zeta)^b \} = 0,
\]
where \( \delta_\varepsilon(\zeta)^b \) denotes the symbol of the pseudo-differential operator with \( \delta_\varepsilon(\zeta) \) being defined in (2.7) so that \( \delta_\varepsilon(\zeta)^b = \varepsilon^2 (\rho(\zeta)^{-2}(\nabla_\varepsilon \zeta, \varepsilon \zeta)^2) \). Then we have
\[
\left[ \rho(\zeta)^{-1} g(x, D^2_h \delta_\varepsilon(\zeta)), \delta_\varepsilon(\zeta) \right] = \left[ \rho(\zeta)^{-1} g(x, D^2_h), \delta_\varepsilon(\zeta) \right] - \mathcal{O}_\varepsilon^2 \left\{ \rho(\zeta)^{-1} g(x, \varepsilon \zeta), \delta_\varepsilon(\zeta)^b \right\},
\]
from which and Proposition 4.2 we infer that
\[
\left| \rho(\zeta)^{-1} g(x, D^2_h \delta_\varepsilon(\zeta)), \delta_\varepsilon(\zeta) \right| u_{H^s_{\varepsilon \delta}} \leq \varepsilon^2 M(\sigma) \left( \varepsilon^2 \| D^2_h \| u_{H^s_{\varepsilon \delta}} + \varepsilon^2 \| \zeta \|_{H^{s+3}}(s_{t0+1}) \| D^2_h \| u_{H^{t0+1}} \right),
\]
which together with the fact that
\[
\| D^2_h \| u_{H^s_{\varepsilon \delta}} \leq \frac{\| D^2_h \| (1 + \varepsilon^{2} \| D^2_h \|)^{1/2} u_{H^s_{\varepsilon \delta}} \| u \|_{H^s_{\varepsilon \delta}} \leq \| \mathcal{P} u \|_{H^{s+1}}
\]
implies (5.24). \[\square\]

Next let us turn to the commutator estimate between \( R[\varepsilon \zeta] \) and \( \delta_\varepsilon(\zeta) \).

Lemma 5.4. Let \( s \geq 0, u \in H^{s+\frac{1}{2}} \cap H^{10+\frac{5}{2}}(\mathbb{R}^2) \). Let \( u^b \) be given by (5.7) and \( u^b \) be given by Lemma 5.2. We denote \( w^b_\varepsilon = \left( \delta_\varepsilon(\zeta)^b \right) u^b_\varepsilon - \left( \delta_\varepsilon(\zeta)^b \right) u^b_\varepsilon \). Then one has
\[
(5.25) \quad \| \Lambda^+ \varepsilon^2 \| w^b_\varepsilon \|_2 + \varepsilon^{2} \| \Lambda^+ \varepsilon^2 \| \delta_\varepsilon(\zeta)^b \|_2
\]
\[
\leq \varepsilon M(\sigma) \left( |\mathcal{P} u|_{H^{s+1}} + |\mathcal{P} u|_{H^{10}} |(\zeta, b)|_{H^{s+5}} \right) + \|\mathcal{P} u|_{H^{t0+1}} |(\zeta, b)|_{H^{s+5}}\|_{s_{t0+1}} + \|\mathcal{P} u|_{H^{t0+2}} |(\zeta, b)|_{H^{s+5}}\|_{s_{t0+1}} + \|\mathcal{P} u|_{H^{t0+3}} |(\zeta, b)|_{H^{s+5}}\|_{s_{t0+1}}.
\]

Proof. Thanks to (5.12), we have
\[
(5.26) \quad \mathcal{P} u^b_\varepsilon = \left[ \delta_\varepsilon(\zeta)^b, \mathcal{P} - \mathcal{P}_{\text{app}} \right] u^b_{\text{app}} + \left( \mathcal{P} - \mathcal{P}_{\text{app}} \right) \left( \delta_\varepsilon(\zeta)^b \right) u^b_{\text{app}}
\]
\[
- \left( \delta_\varepsilon(\zeta)^b \right) u^b_{\text{app}} + \left( \delta_\varepsilon(\zeta)^b \right) \mathcal{P}_{\text{app}} u^b_{\text{app}} - \mathcal{P}_{\text{app}} \left( \delta_\varepsilon(\zeta)^b \right) u^b_{\text{app}} + \left[ \delta_\varepsilon(\zeta)^b, \mathcal{P} \right] u^b_\varepsilon
\]
\[
\overset{\text{def}}{=} h_1 + h_2 + h_3 + h_4.
\]
together with the boundary conditions
\[ w^b_{\epsilon}|_{z=0} = 0, \quad \partial^P_{\tau} w^b_{\epsilon}|_{z=-1} = g_1 + g_2 \]
where
\[ g_1 \overset{\text{def}}{=} e_3 \cdot [(1 + Q[\sigma])\nabla^\varepsilon \partial_{\varepsilon}(\zeta)] u^b_{\epsilon}|_{z=-1}, \]
\[ g_2 \overset{\text{def}}{=} -e_3 \cdot [\partial_{\varepsilon}(\zeta), (1 + Q[\sigma])\nabla^\varepsilon \sigma_{app}(x_h, z, D^\varepsilon_h)] u|_{z=-1}. \]

In what follows, we just consider the case of \( s > t_0 + 1 \), the other cases can be handled in a similar way. In this case, we first get by applying Proposition 5.2 to (5.26) that
\[
\|\Lambda^{\varepsilon+1} \nabla^\varepsilon w^b_{\epsilon}\|_2 \leq M(\sigma) \left( \varepsilon^{-\frac{3}{4}} \|\Lambda^{\varepsilon}(h_1 + \cdots + h_4)\|_2 + \varepsilon^{-\frac{1}{4}} |g_1|_{H^{\varepsilon+\frac{1}{2}}_x} \right) \\
+ |g_2|_{H^{\varepsilon+1}_x} + \left( \varepsilon^{-\frac{3}{4}} \|\Lambda^{t_0-1}(h_1 + \cdots + h_4)\|_2 + |g_1 + g_2|_{H^{t_0}_x} \right) \left( |\zeta, b|_{H^{\varepsilon+2}_x} \right),
\]
which reduces the estimate of (5.25) to that of \( h_1, h_2, h_3, h_4 \) and \( g_1, g_2 \).

**Step 1.** The estimate of \( h_1 \).

Recall from the proof of Lemma 5.2 that
\[ \mathbf{P} - \mathbf{P}_{app} = -\partial_{x} p_{d+1} \tau_1 + p_{d+1} \tau_2 + (\partial_{x} p_{d+1} + \nabla_h \cdot \mathbf{p}) \partial_{z}, \]
so we write
\[ \left[ \partial_{\varepsilon}(\zeta), \mathbf{P} - \mathbf{P}_{app} \right] = \left[ \partial_{\varepsilon}(\zeta), \mathbf{P}_{d+1} \partial_1 \right] + \left[ \partial_{\varepsilon}(\zeta), p_{d+1} \tau_2 \right] + \left[ \partial_{\varepsilon}(\zeta), \partial_{x} p_{d+1} + \nabla_h \cdot \mathbf{p} \right] \partial_{z}. \]

To deal with \( \left[ \partial_{\varepsilon}(\zeta), \mathbf{P}_{d+1} \partial_1 \right] \), thanks to (5.14), we can split \( \left[ \partial_{\varepsilon}(\zeta), \tau_1 \right] \) as
\[ \left[ \partial_{\varepsilon}(\zeta), \tau_1 \right] = \left( \partial_{\varepsilon}(\zeta), \eta_- \right) - \text{Op}_{\varepsilon} \left( \partial_{\varepsilon}(\zeta), \eta_- \right) - \text{Op}_{\varepsilon} \left( \partial_{\varepsilon}(\zeta), \eta_+ \right) \]
\[ + \text{Op}_{\varepsilon} \left( \partial_{\varepsilon}(\zeta), \eta_- \right) + \text{Op}_{\varepsilon} \left( \partial_{\varepsilon}(\zeta), \eta_+ \right) + \left( \text{Op}_{\varepsilon} \left( \partial_{\varepsilon}(\zeta), \eta_- \right) - \text{Op}_{\varepsilon} \left( \partial_{\varepsilon}(\zeta), \eta_+ \right) \right) \]
\[ + \left( \text{Op}_{\varepsilon} \left( \partial_{\varepsilon}(\zeta), \eta_- \right) - \text{Op}_{\varepsilon} \left( \partial_{\varepsilon}(\zeta), \eta_+ \right) \right), \]
from which and Propositions 4.4.14.2 and Lemma 5.1, we deduce that
\[
\|\Lambda^{\varepsilon}_x \left[ \partial_{\varepsilon}(\zeta), \tau_1 \right] u^b_{app}\|_2 \leq \varepsilon^2 M(\sigma) \left( \|\Lambda^{\varepsilon+1}_x D^\varepsilon_h u^b_{app}\|_2 + \varepsilon^{-\frac{3}{4}} \|\zeta, b\|_{H^{\varepsilon+4}_x} \right) \\
+ \varepsilon^\frac{7}{8} M(\sigma) \left( \|\zeta, b\|_{H^{\varepsilon+4}_x} \right).
\]

This ensures
\[
\|\Lambda^{\varepsilon}_x \left[ \partial_{\varepsilon}(\zeta), p_{d+1} \tau_1 \right] u^b_{app}\|_2 \leq \|\Lambda^{\varepsilon}_x \left[ \partial_{\varepsilon}(\zeta), p_{d+1} \tau_1 \right] u^b_{app}\|_2 + \|\Lambda^{\varepsilon}_x p_{d+1} \left[ \partial_{\varepsilon}(\zeta), \tau_1 \right] u^b_{app}\|_2 \]
\[ \leq \varepsilon^\frac{7}{8} M(\sigma) \left( \|\zeta, b\|_{H^{\varepsilon+4}_x} \right). \]

Exactly following the same line, we obtain
\[
\|\Lambda^{\varepsilon}_x \left[ \partial_{\varepsilon}(\zeta), p_{d+1} \tau_2 \right] u^b_{app}\|_2 \leq \varepsilon^\frac{7}{8} M(\sigma) \left( \|\zeta, b\|_{H^{\varepsilon+4}_x} \right).
\]

Consequently, we arrive at
\[
\|\Lambda^{\varepsilon}_x h_1\|_2 \leq \varepsilon^\frac{7}{8} M(\sigma) \left( \|\zeta, b\|_{H^{\varepsilon+4}_x} \right).
\]

**Step 2.** The estimate of \( h_2 \).
Set
\[ w^b_{\text{app}} \overset{\text{def}}{=} \delta_{x}(\zeta)u^b_{\text{app}} - (\delta_{x}(\zeta))u^b_{\text{app}} = \left[ \delta_{x}(\zeta), \bar{\sigma}_{\text{app}}(x_h, z, D^e_{h}) \right] \exp \left( \frac{c}{2}z\sqrt{\varepsilon}|D^e_{h}| \right) u \]
\[ + \bar{\sigma}_{\text{app}}(x_h, z, D^e_{h}) \left[ \delta_{x}(\zeta), \exp \left( \frac{c}{2}z\sqrt{\varepsilon}|D^e_{h}| \right) \right] u, \]
with \( \bar{\sigma}_{\text{app}} \) being given by (5.11). Then we deduce from the proof of Propositions 4.1, 4.2 and Lemma 5.1 that
\[ \| \Lambda^e_{\varepsilon}(\nabla^e_{h}, \partial_z)w^b_{\text{app}} \|_2 \leq \varepsilon^{\frac{3}{2}} M(\sigma)(|\Psi u|_{H_{h}^{k+1}} + |\Psi u|_{H_{h}^{0+1}})(\zeta, b)|_{H_{h}^{k+1}}. \]
And we get by applying Propositions 4.1, 4.2 that
\[ \| \Lambda^e_{\varepsilon}h^2 \|_2 \leq \varepsilon^{\frac{3}{2}} M(\sigma)(|\Psi u|_{H_{h}^{k+1}} + |\Psi u|_{H_{h}^{0+1}})(\zeta, b)|_{H_{h}^{k+1}}. \]

**Step 3.** The estimate of \( h_3 \).
It is easy to observe from (5.6) that \( h_3 = [\delta_{x}(\zeta), P_{\text{app}} \circ \sigma_{\text{app}}] u \), since
\[ P_{\text{app}} \circ \sigma_{\text{app}} = p_{d+1}((\partial_z - \eta_-(x_h, z, D^e_{h}))\tau(x_h, z, D^e_{h}) \exp \left( \frac{c}{2}z\sqrt{\varepsilon}|D^e_{h}| \right), \]
\[ \tau(x_h, z, D^e_{h}) = \text{Op}_e(\eta_+\bar{\sigma}_{\text{app}}) - \eta_+(x_h, z, D^e_{h}) \circ \sigma_{\text{app}}(x_h, z, D^e_{h}), \]
we write
\[ h_3 = [\delta_{x}(\zeta), p_{d+1}((\partial_z - \text{Op}_e(\eta_-))\tau(x, z, D^e_{h}) \exp \left( \frac{c}{2}z\sqrt{\varepsilon}|D^e_{h}| \right)) u \]
\[ - p_{d+1} [\delta_{x}(\zeta), \text{Op}_e(\eta_-)] \tau(x_h, z, D^e_{h}) \exp \left( \frac{c}{2}z\sqrt{\varepsilon}|D^e_{h}| \right) u \]
\[ + p_{d+1} ((\partial_z - \text{Op}_e(\eta_-))[\delta_{x}(\zeta), \tau(x_h, z, D^e_{h})] \exp \left( \frac{c}{2}z\sqrt{\varepsilon}|D^e_{h}| \right) u \]
\[ + p_{d+1} (\partial_z - \text{Op}_e(\eta_-))\tau(x_h, z, D^e_{h}) [\delta_{x}(\zeta), \exp \left( \frac{c}{2}z\sqrt{\varepsilon}|D^e_{h}| \right) u. \]
Applying Propositions 4.1, 4.2 ensures that
\[ \| \Lambda^e_{\varepsilon}h_3 \|_2 \leq \varepsilon^{\frac{3}{2}} M(\sigma)(|\Psi u|_{H_{h}^{k+1}} + |\Psi u|_{H_{h}^{0+1}})(\zeta, b)|_{H_{h}^{k+1}}. \]

**Step 4.** The estimate of \( h_4 \).
Notice that
\[ h_4 = [\delta_{x}(\zeta), \nabla^e \cdot (1 + Q[\sigma])\nabla^e] u^b \]
\[ = [\delta_{x}(\zeta), \nabla^e \cdot (1 + Q[\sigma])] \nabla^e u^b + \nabla^e \cdot (1 + Q[\sigma]) [\delta_{x}(\zeta), \nabla^e] u^b, \]
which together with Lemma 2.4 and Lemma 5.2 implies that
\[ \| \Lambda^e_{\varepsilon}h_4 \|_2 \leq \varepsilon^2 M(\sigma)(\| \Lambda^{k+2}_{\varepsilon}\nabla^e u^b_v \|_2 + |(\zeta, b)|_{H_{h}^{k+1}}\| \Lambda^{k+1}_{\varepsilon}\nabla^e u^b_v \|_2) \]
\[ + \varepsilon M(\sigma)(\| \Lambda^{k+1}_{\varepsilon}\nabla^e u^b_v \|_2 + \| \Lambda^{k+1}_{\varepsilon}(\nabla^e)^2 u^b_v \|_2 + |(\zeta, b)|_{H_{h}^{k+1}}\| \Lambda^{k+1}_{\varepsilon}\nabla^e u^b_v \|_2) \]
\[ \leq \varepsilon^{\frac{3}{2}} M(\sigma)(|\Psi u|_{H_{h}^{k+1}} + |\Psi u|_{H_{h}^{0+1}})(\zeta, b)|_{H_{h}^{k+1}}. \]

**Step 5.** The estimate of \( g_1 \) and \( g_2 \).
We first get by applying Lemma 2.2 and Lemma 4.1 that
\[ |g_1|_{H^{s+\frac{1}{2}}_\varepsilon} \leq \varepsilon^{-\frac{1}{2}} \left\| \Lambda^{s}_\varepsilon \nabla \varepsilon \left[ (1 + Q[\sigma]) \nabla \varepsilon, \partial_\varepsilon (\zeta) \right] u_{r}^b \right\|_2 \]
\[ \leq \varepsilon^{\frac{1}{4}} M(\sigma) \left( \sqrt{\varepsilon} \left\| \Lambda^{s+2}_\varepsilon \nabla \varepsilon u_{r}^b \right\|_2 + \left\| \Lambda^{s+1}_\varepsilon \nabla \partial_\varepsilon u_{r}^b \right\|_2 \right) \]
\[ + \left( \sqrt{\varepsilon} \left\| \Lambda^{s+2}_\varepsilon \nabla \varepsilon u_{r}^b \right\|_2 + \left\| \Lambda^{s+1}_\varepsilon \nabla \partial_\varepsilon u_{r}^b \right\|_2 \right) |(\zeta, b)|_{H^{s+3}_\varepsilon}, \]
from which and Lemma 5.2, we infer that
\[ |g_1|_{H^{s+\frac{1}{2}}_\varepsilon} \leq \varepsilon^{\frac{1}{4}} M(\sigma) (|\Psi u|_{H^{s+1}_\varepsilon} + |\Psi u|_{H^{0+1}_\varepsilon} |(\zeta, b)|_{H^{s+3}_\varepsilon}). \]

To deal with \( g_2 \), we first rewrite it as
\[ g_2 = -\varepsilon \cdot [\partial_\varepsilon (\zeta), (1 + Q[\sigma]) \nabla \varepsilon] \sigma_{\text{app}}(x, z, D_\varepsilon^b) u|_{z=-1} \]
\[ -\varepsilon \cdot (1 + Q[\sigma]) \nabla \varepsilon [\partial_\varepsilon (\zeta), \sigma_{\text{app}}(x, z, D_\varepsilon^b)] u|_{z=-1} \]
\[ \overset{\text{def}}{=} g_{21} + g_{22}. \]

It follows from Lemma 2.4 and the proof of Lemma 5.1 that
\[ |g_{21}|_{H^{s+1}_\varepsilon} \leq \varepsilon M(\sigma) \left( |\nabla \varepsilon \sigma_{\text{app}}(x, z, D_\varepsilon^b) u|_{z=-1} \right)_{H^{s+2}_\varepsilon} \]
\[ + \left( |\nabla \varepsilon \sigma_{\text{app}}(x, z, D_\varepsilon^b) u|_{z=-1} \right)_{H^{0+1}_\varepsilon} |(\zeta, b)|_{H^{s+4}_\varepsilon} \]
\[ \leq \varepsilon M(\sigma) \left( |\Psi u|_{H^{s+1}_\varepsilon} + |\Psi u|_{H^{0+1}_\varepsilon} |(\zeta, b)|_{H^{s+4}_\varepsilon} \right), \]
and similarly, one has
\[ |g_{22}|_{H^{s+1}_\varepsilon} \leq \varepsilon M(\sigma) \left( |\Psi u|_{H^{s+1}_\varepsilon} + |\Psi u|_{H^{0+1}_\varepsilon} |(\zeta, b)|_{H^{s+4}_\varepsilon} \right). \]

This gives
\[ |g_{22}|_{H^{s+1}_\varepsilon} \leq \varepsilon M(\sigma) \left( |\Psi u|_{H^{s+1}_\varepsilon} + |\Psi u|_{H^{0+1}_\varepsilon} |(\zeta, b)|_{H^{s+4}_\varepsilon} \right). \]

Plugging (5.28)-(5.32) and (5.33) into (5.27) results in
\[ |\Lambda^{s+1}_\varepsilon \nabla \varepsilon u_{r}^b|_2 \leq \varepsilon M(\sigma) \left( |\Psi u|_{H^{s+1}_\varepsilon} + |\Psi u|_{H^{0+1}_\varepsilon} |(\zeta, b)|_{H^{s+4}_\varepsilon} \right). \]

On the other hand, it follows from the third inequality of Proposition 3.2 that
\[ |\Lambda^{s}_\varepsilon \nabla \partial_\varepsilon u_{r}^b|_2 \leq M(\sigma) \left( |\Lambda^{s}_\varepsilon (h_1 + \cdots + h_4)|_2 + \varepsilon^{\frac{1}{4}} |g_{21}|_{H^{s+\frac{1}{2}}_\varepsilon} + \sqrt{\varepsilon} |g_{22}|_{H^{s+\frac{1}{2}}_\varepsilon} \right) \]
\[ + \left( |\Lambda^{s}_\varepsilon (h_1 + \cdots + h_4)|_2 + \sqrt{\varepsilon} |g_{21} + g_{22}|_{H^{0+1}_\varepsilon} |(\zeta, b)|_{H^{s+2}_\varepsilon} \right) \]
\[ \leq \varepsilon^{\frac{1}{4}} M(\sigma) \left( |\Psi u|_{H^{s+1}_\varepsilon} + |\Psi u|_{H^{0+1}_\varepsilon} |(\zeta, b)|_{H^{s+4}_\varepsilon} \right). \]

This completes the proof of Lemma 5.4.
Proposition 5.8. Let \( s \geq 0 \) and \( u \in H^{s+\frac{3}{2}} \cap H^{t_0+\frac{5}{2}}(\mathbb{R}^2) \). Then under the assumptions of theorem 5.7 we have

\[
\left| \left[ \frac{1}{\varepsilon} R[\varepsilon \xi, \partial_\varepsilon(\zeta)] u \right] \right|_{H^s} \leq \sqrt{\varepsilon} M(\sigma) \left| \left[ \mathcal{P} u \right] \right|_{H^{s+1}} + \left| \left[ \mathcal{P} u \right] \right|_{H^{t_0}} \left| (\zeta, b) \right|_{H^{s+5}} + \left( \left| \left[ \mathcal{P} u \right] \right|_{H^{t_0+1}} \right)_{s>t_0} + \left( \left| \left[ \mathcal{P} u \right] \right|_{H^{t_0+2}} \right)_{s>t_0+1}.
\]

Proof. Thanks to (5.9), for any \( v \in \mathcal{S}(\mathbb{R}^2) \), we get by applying Green’s identity that

\[
\left( \Lambda_\varepsilon^s [R[\varepsilon \xi, \partial_\varepsilon(\zeta)] u, v \right) = \int_S \left\{ \nabla^\varepsilon \cdot \left( 1 + Q[\sigma] \right) \nabla^\varepsilon (\partial_\varepsilon(\zeta) u) \Lambda_\varepsilon^s v^t + (1 + Q[\sigma]) \nabla^\varepsilon (\partial_\varepsilon(\zeta) u) \Lambda_\varepsilon^s v^t \right. \\
- \nabla^\varepsilon \cdot (\partial_\varepsilon(\zeta) (1 + Q[\sigma]) \nabla^\varepsilon u^b) \Lambda_\varepsilon^s v^t - \partial_\varepsilon(\zeta) (1 + Q[\sigma]) \nabla^\varepsilon u^b \cdot \nabla^\varepsilon \Lambda_\varepsilon^s v^t \right\} dx_h dz,
\]

which ensures

\[
\left( \Lambda_\varepsilon^s [R[\varepsilon \xi, \partial_\varepsilon(\zeta)] u, v \right) = \int_S \left\{ \nabla^\varepsilon \cdot \left( 1 + Q[\sigma] \right) \nabla^\varepsilon ((\partial_\varepsilon(\zeta) u)^b) \\
- \partial_\varepsilon(\zeta) (1 + Q[\sigma]) \nabla^\varepsilon u^b \Lambda_\varepsilon^s v^t \\
- \partial_\varepsilon(\zeta) (1 + Q[\sigma]) \nabla^\varepsilon u^b \Lambda_\varepsilon^s v^t \right\} dx_h dz
\]

\[
\overset{\text{def}}{=} B_1 + B_2 + B_3 + B_4.
\]

Here \( v^t = (1 + z) \chi(z \sqrt{\varepsilon}|D^2_u|) v \) with \( \chi \) being given by Lemma 2.5.

Again we only consider the case of \( s > t_0 + 1 \), the other cases can be handled in a similar way. First of all, we get by applying Lemma 5.3 that

\[
|B_3| \leq C \sqrt{\varepsilon} |v|_2 \left| \Lambda_\varepsilon^{s+1} \left[ (1 + Q[\sigma]) \nabla^\varepsilon, \partial_\varepsilon(\zeta) \right] u^b \right|_2 + C |v|_2 \left| \Lambda_\varepsilon^{s} \left[ (1 + Q[\sigma]) \nabla^\varepsilon, \partial_\varepsilon(\zeta) \right] u^b \right|_2.
\]

Besides, Lemma 2.4 and the first inequality of Lemma 5.2 ensures that

\[
\left| \Lambda_\varepsilon^{s+1} \left[ (1 + Q[\sigma]) \nabla^\varepsilon, \partial_\varepsilon(\zeta) \right] u^b \right|_2 \leq \varepsilon M(\sigma) \left| \left| \Lambda_\varepsilon^{s+2} \nabla^\varepsilon u^b \right|_2 \right| + \left| (\zeta, b) \right|_{H^{t_0+4}} \left| \Lambda_\varepsilon^{t_0+1} \nabla^\varepsilon u^b \right|_2
\]

\[
\leq \varepsilon M(\sigma) \left| \left| \mathcal{P} u \right| \right|_{H^{s+1}} + \left| \left[ \mathcal{P} u \right] \right|_{H^{t_0}} \left| (\zeta, b) \right|_{H^{s+4}}
\]

and Lemma 2.4 and the second inequality of Lemma 5.2 implies that

\[
\left| \Lambda_\varepsilon^{s} \left[ (1 + Q[\sigma]) \nabla^\varepsilon, \partial_\varepsilon(\zeta) \right] u^b \right|_2 \leq \varepsilon M(\sigma) \left| \left| \Lambda_\varepsilon^{s+1} \nabla^\varepsilon u^b \right|_2 \right| + \left| (\zeta, b) \right|_{H^{t_0+3}} \left| \Lambda_\varepsilon^{t_0+1} \nabla^\varepsilon u^b \right|_2
\]

\[
\leq \varepsilon \frac{2}{M(\sigma)} \left| \left| \mathcal{P} u \right| \right|_{H^{s+1}} + \left| \left[ \mathcal{P} u \right] \right|_{H^{t_0+1}} \left| (\zeta, b) \right|_{H^{s+3}}.
\]

Here we used the fact that \( \left| \Lambda_\varepsilon^{s+1} \nabla^\varepsilon u^b \right|_2 \leq \left| \Lambda_\varepsilon^{s+1} \nabla^\varepsilon \partial_\varepsilon u^b \right|_2 \) due to \( u^b|_{z=0} = 0 \). So we arrive at

(5.34) \[
|B_3| \leq \varepsilon \frac{2}{M(\sigma)} |v|_2 \left| \left| \mathcal{P} u \right| \right|_{H^{s+1}} + \left| \left[ \mathcal{P} u \right] \right|_{H^{t_0+1}} \left| (\zeta, b) \right|_{H^{s+3}}.
\]

Notice that \( B_2 \) can be rewritten as

\[
B_2 = - \int_S \left\{ \sqrt{\varepsilon} \nabla^\varepsilon \cdot (\partial_\varepsilon(\zeta), (1 + Q[\sigma]) \nabla^\varepsilon u^b \Lambda_\varepsilon^s v^t + e_3 \cdot \partial_\varepsilon(\zeta, \partial_\varepsilon Q[\sigma] \nabla^\varepsilon u^b \Lambda_\varepsilon^s v^t + e_3 \cdot \partial_\varepsilon(\zeta, (1 + Q[\sigma]) \nabla^\varepsilon \partial_\varepsilon u^b \Lambda_\varepsilon^s v^t \right\} dx_h dz,
\]
which together with Lemma 2.4 and Lemma 5.2 yields that

\begin{equation}
|B_2| \leq \varepsilon^2 M(\sigma)|v|_2(\|\mathcal{P}u\|_{H^{\ell+1}} + |\mathcal{P}u|_{H^{\ell_0+1}}|\zeta|_{H^{\ell_0+4}}).
\end{equation}

Finally we have by Lemma 5.3 and Lemma 5.4 that

\begin{equation}
|B_1| + |B_4| \leq M(\sigma)|v|_2\left(\sqrt{\varepsilon}\|\Lambda_\varepsilon^{\ell_0+1}\nabla_\varepsilon w_\delta^b\|_2 + \|\Lambda_\varepsilon\nabla_\varepsilon (w_\delta^b + \partial_\varepsilon w_\delta^b)\|_2(\zeta, b)|_{H^{\ell_0+2}}\right)
\leq \varepsilon^4 M(\sigma)|v|_2(\|\mathcal{P}u\|_{H^{\ell_0+1}} + |\mathcal{P}u|_{H^{\ell_0+2}}|\zeta, b|_{H^{\ell_0+5}}).
\end{equation}

Summing up (5.34)-(5.36), we conclude Proposition 5.8 by the duality. \hfill \Box

**Remark 5.5.** It follows from the proof of Proposition 5.8 that

\[ |\left[\frac{1}{\varepsilon} R[\varepsilon\zeta], \partial_\varepsilon(\zeta) - |D_\varepsilon^k|^2\right] u|_{H^{\ell_0}} \leq \sqrt{\varepsilon} M(\sigma)\left(\|\mathcal{P}u\|_{H^{\ell_0+1}} + |\mathcal{P}u|_{H^{\ell_0+2}}|\zeta|_{H^{\ell_0+5}}\right) + \langle|\mathcal{P}u|_{H^{\ell_0+1}}|\zeta, b|_{H^{\ell_0+5}}\rangle_{s>t_0} + \langle|\mathcal{P}u|_{H^{\ell_0+2}}|\zeta, b|_{H^{\ell_0+5}}\rangle_{s>t_0+1} \}
\]

Thanks to Lemma 5.5 and the proof of Proposition 5.8, we also obtain that

**Proposition 5.9.** Let \( s \geq 0, k \in \mathbb{N} \), and \( u \in H^{2k+s+\frac{3}{2}} \cap H^{\ell_0+\frac{3}{2}}(\mathbb{R}^2) \). Then under the assumptions of theorem 5.7 one has

\[ |\left[\frac{1}{\varepsilon} R[\varepsilon\zeta], |D_\varepsilon^k|^2\right] u|_{H^{\ell_0}} \leq \sqrt{\varepsilon} M(\sigma)\left(\|\mathcal{P}u\|_{H^{2k+s-1}} + |\mathcal{P}u|_{H^{\ell_0+2}}|\zeta|_{H^{2k+s+3}}\right) \]

Now we are in a position to prove Theorem 5.1.

**Proof of Theorem 5.1.** Thanks to (5.9), we write

\begin{equation}
[\partial_\varepsilon(\zeta)^k, \frac{1}{\varepsilon}\rho(\zeta)^{-1} G[\varepsilon\zeta]] u = [\partial_\varepsilon(\zeta)^k, \frac{1}{\varepsilon}\rho(\zeta)^{-1} g(x_h, D_\varepsilon^k)] u
+ \left[\partial_\varepsilon(\zeta)^k, \frac{1}{\varepsilon}\rho(\zeta)^{-1} R[\varepsilon\zeta]\right] u \overset{\text{def}}{=} I_k + II_k.
\end{equation}

We shall prove by an induction argument on \( k \) that

\begin{equation}
|I_k|_{H^{\ell_0}} \leq \sqrt{\varepsilon} M(\sigma)\left(\|\mathcal{P}u\|_{H^{2k+s-1}} + |\mathcal{P}u|_{H^{\ell_0+2}}|\zeta|_{H^{2k+s+3}}\right).
\end{equation}

The case of \( k = 1 \) is a direct consequence of Proposition 5.7. We assume that (5.38) holds for \( k \leq \ell - 1 \). We turn now to prove the case of \( k = \ell \). We write

\[ I_k = \partial_\varepsilon(\zeta)^{\ell-1} \left[\frac{1}{\varepsilon}\rho(\zeta)^{-1} g(x_h, D_\varepsilon^k)\right] u + \left[\partial_\varepsilon(\zeta) - \frac{1}{\varepsilon}\rho(\zeta)^{-1} g(x_h, D_\varepsilon^k)\right] \partial_\varepsilon(\zeta)^{\ell-1} u.
\]

Using Lemma 2.5 and the induction assumption, we have by an interpolation argument that

\[ |\partial_\varepsilon(\zeta)^{\ell-1} \left[\frac{1}{\varepsilon}\rho(\zeta)^{-1} g(x_h, D_\varepsilon^k)\right] u|_{H^{\ell_0}} \leq M(\sigma)\left(||\partial_\varepsilon(\zeta)^{\ell-1} \left[\frac{1}{\varepsilon}\rho(\zeta)^{-1} g(x, D_\varepsilon^k)\right] u|_{H^{\ell_0+2}}
+ ||\partial_\varepsilon(\zeta)^{\ell-1} \left[\frac{1}{\varepsilon}\rho(\zeta)^{-1} g(x_h, D_\varepsilon^k)\right] u|_{H^{\ell_0+2}}|\zeta|_{H^{\ell_0+3}}\right)
\leq \sqrt{\varepsilon} M(\sigma)\left(\|\mathcal{P}u|_{H^{2k+s-1}} + |\mathcal{P}u|_{H^{\ell_0+2}}|\zeta|_{H^{2k+s+3}}\right),\]
and applying Proposition 5.7 and Lemma 2.3 gives
\[
|\left[ \frac{1}{e} \rho(\zeta) g(x_h, D_h^\varepsilon) \right] \partial_e(\zeta) \psi |_{H^2_h} \\
\leq \varepsilon^2 M(\sigma) \left( |\Psi \partial_e(\zeta)^{-1} u|_{H^{s+1}_h} + |\partial_e(\zeta)^{-1} u|_{H^{s+1}_h} \right) + \left| |\Psi \partial_e(\zeta)^{-1} u|_{H^{s+1}_h} \right|
\]
This proves (5.38) for \( k = \ell \).

Noticing that \( II_k \) can be rewritten as
\[
II_k = \left[ \frac{1}{e} \rho(\zeta) - 1 \right] R[\varepsilon \zeta] u + \rho(\zeta) - 1 \left[ \frac{1}{e} R[\varepsilon \zeta] \right] u,
\]
applying Lemma 2.4 and Proposition 5.5 gives
\[
\left| \partial_e(\zeta)^k \right|_{H^2_h} \leq \varepsilon^2 M(\sigma) \left( |\Psi u|_{H^{2s+1}} + |\Psi u|_{H^{s+1}} \right).
\]
Similarly, we can prove by an induction argument that
\[
\left| \frac{1}{e} \rho(\zeta)^{-1} \right| \partial_e(\zeta)^k - |D_h^\varepsilon | u \right|_{H^2_h} \\
\leq \varepsilon M(\sigma) \left( |\Psi u|_{H^{2s+1}} + |\Psi u|_{H^{s+1}} \right),
\]
which together with Proposition 5.9 implies that \( II_k \) also satisfies (5.38). In fact, the case of \( k = 1 \) comes from Remark 5.5.

6. LARGE TIME EXISTENCE FOR THE LINEARIZED SYSTEM

6.1. The linearized system. We first reformulate the original system (1.8) as
\[
\partial_t U + LU + \varepsilon A[U] = 0
\]
where
\[
U = (\zeta, \psi)^T, \quad A[U] = (A_1[U], A_2[U])^T, \quad L = \left( \begin{array}{cc} 0 & -\frac{1}{e} \varepsilon G[0] \\ 1 & 0 \end{array} \right),
\]
and
\[
A_1[U] \equiv -\frac{1}{e^2} (G[\varepsilon \zeta] \psi - G[0] \psi),
\]
\[
A_2[U] \equiv \frac{1}{2} |\nabla_h \psi|^2 - \left( \frac{e^2}{2} \frac{\varepsilon \zeta |\varphi| + \varepsilon^2 \nabla_h \zeta \cdot \nabla_h \psi}{2(1 + \varepsilon^2 |\nabla_h \zeta|^2)} \right)^2 - \alpha \nabla_h \cdot \left( \frac{\nabla_h \zeta}{1 + \varepsilon^2 |\nabla_h \zeta|^2} \right).
\]

Motivated by (17), we shall use Nash-Moser iteration Theorem to prove the large time existence of solutions to (6.1). Toward this, a key step will be to study the linearized system of (6.1). Indeed, we shall linearize the system (6.1) around an admissible reference state in the following sense:

Definition 6.1. Let \( T > 0 \). We say that \( \overline{U} = (\overline{\zeta}, \overline{\psi})^T \) is an admissible reference state to (6.1) on \([0, \frac{T}{e}]\) if there exists \( h_0 > 0 \) such that
\[
1 + \varepsilon \zeta - \varepsilon b \geq h_0 \quad \text{uniformly on} \quad \left[0, \frac{T}{e}\right] \times \mathbb{R}^2.
\]
Given an admissible reference state \( U = (\zeta, \psi)^T \), one can calculate by Proposition 5.2 (see also [2], [21]) that the linearized operator of the system (6.1) equals
\[
\mathcal{L} = \partial_t + \mathcal{L} + \varepsilon d_{\mathcal{L}} A
\]
\[
= \partial_t + \left( G[\varepsilon \zeta] : \nabla \psi + \varepsilon \nabla \psi \cdot \nabla \psi - \frac{1}{\varepsilon} G[\varepsilon \zeta] \right),
\]
where
\[
A \equiv \nabla \psi \cdot \left[ \frac{\nabla \zeta}{1 + \varepsilon^2 |\nabla \zeta|^2} - \frac{\varepsilon^2 \nabla \zeta \cdot \nabla \psi}{(1 + \varepsilon^2 |\nabla \zeta|^2)^{\frac{3}{2}}} \right],
\]
\[
\psi \equiv \nabla \zeta - \varepsilon Z \nabla \zeta, \quad Z \equiv \frac{1}{1 + \varepsilon^2 |\nabla \zeta|^2} \left( \varepsilon \nabla \zeta \psi + \varepsilon^2 \nabla \zeta \cdot \nabla \psi \right).
\]
This gives the linearized system of (6.1) as follows
\[
(6.2) \quad \mathcal{L} U = \varepsilon G, \quad U|_{t=0} = U^0.
\]
To solve (6.2), as in [17, 21] we introduce a new variable \( V \equiv (\zeta, \psi - \varepsilon \zeta) \) such that the principal symbol of the transformed linearized operator is trigonalized. Indeed the system (6.2) can be equivalently written as
\[
(6.3) \quad \nabla V = \varepsilon \psi, \quad V|_{t=0} = V^0,
\]
where
\[
\nabla \psi \equiv \partial_t + \left( \varepsilon \nabla \psi \cdot \nabla \psi - \frac{1}{\varepsilon} G[\varepsilon \zeta] \right),
\]
and
\[
a = 1 + \varepsilon (\varepsilon \psi \cdot \nabla Z + \partial_t Z), \quad H = (G_1, G_2 - \varepsilon \partial_t G_1).
\]

6.2. **Large time existence.** In this subsection, we shall prove the large time existence of smooth enough solution for the linearized system (6.2) and establish the uniform estimates for thus obtained solutions on \([0, T] \). Toward this, we first introduce the following definition.

**Definition 6.2.** Let \( s \in \mathbb{R} \) and \( T > 0 \).

1. We define the space \( X^s \) as
\[
X^s \equiv \left\{ U = (\zeta, \psi)^T : \zeta \in H^{2s+1}(\mathbb{R}^2), \psi \in H^{2s+\frac{3}{2}}(\mathbb{R}^2) \right\}
\]
endowed with the norm
\[
|U|_{X^s} \equiv \sqrt{\varepsilon} |\zeta|_{H^{2s+1}} + |\zeta|_{H^{2s+1}} + \sqrt{\varepsilon} |\nabla \zeta|_{H^{s}} + |\zeta|_{H^{s}} + |\nabla \psi|_{H^{2s}} + |\nabla \psi|_{H^{s}} + \varepsilon |\psi|_{H^{s-1}}.
\]
And \( X^s \equiv C(\mathbb{R}, X^s) \) endowed with its canonical norm.

2. Let \( X^s \) be determined by Definition [17]. The semi-normed space \( (Y^s_T, |.|_{Y^s_T}) \) is defined as
\[
Y^s_T \equiv \left\{ U \in X^s : \|U\|_{X^s} \right\}.
\]

3. For any \( (G, U^0) \in X^s_T \times X^s \), we denote
\[
\mathcal{I}^s(t, U^0, G) \equiv |U^0|_{X^s}^2 + \varepsilon \int_0^t \sup_{|x| \leq r} |G(t')|^2_{X^s} dt'.
\]
Proposition 6.1. Let $k \in \mathbb{N}$, $2k \geq m_0$, $T > 0$ and $b \in H^{2k+5}(\mathbb{R}^2)$. Assume that $U = (\zeta, \psi)^T \in Y^{k}_T$ is an admissible reference state on $[0, T]$, for some $h_0 > 0$. Then for any $(G, U^0) \in X^k_T \times X^k$, (6.2) has a unique solution $U \in X^k_T$ such that for all $t \in [0, T]$,

$$|U(t)|_{X^k}^2 \leq C \left( \mathcal{T}^k(t, U^0, G) + |\Lambda^s_{\frac{5}{2}} U|_{Y^k_T}^2 \mathcal{T}^5(t, U^0, G) \right),$$

where $C = C(T, \frac{1}{h_0}, \alpha, \alpha^{-1}, \varepsilon, |b|_{H^{2k+5}}, |U|_{Y^{m_0}})$.

The proof of this proposition relies on the study of the trigo nalized linearized system (6.3), which we admit for the time being.

Proposition 6.2. Under the assumptions in Proposition 6.1, for any given $(H, V^0) \in X^k_T \times X^k$, (6.3) has a unique solution $V \in X^k_T$ such that for all $t \in [0, T]$,

$$|V(t)|_{X^k}^2 \leq C \left( \mathcal{T}^k(t, V^0, H) + |\Lambda^s_{\frac{5}{2}} U|_{Y^k_T}^2 \mathcal{T}^5(t, V^0, H) \right),$$

with $C$ being the same as in Proposition 6.1.

Proof of Proposition 6.1. Recalling that $U = (V_1, V_2 + \varepsilon Z V_1)$ and $H = (G_1, G_2 - \varepsilon Z G_1)$, it follows from Proposition 5.3 that

$$|U(t)|_{X^k}^2 \leq C \left( |V(t)|_{X^k}^2 + |\Lambda^s_{\frac{5}{2}} U|_{Y^k_T}^2 |V(t)|_{X^{m_0+1}}^2 \right) \leq C \left( \mathcal{T}^k(t, V^0, H) + |\Lambda^s_{\frac{5}{2}} U|_{Y^k_T}^2 \mathcal{T}^5(t, V^0, H) \right) \leq C \left( \mathcal{T}^k(t, U^0, G) + |\Lambda^s_{\frac{5}{2}} U|_{Y^k_T}^2 \mathcal{T}^5(t, U^0, G) \right).$$

\[ \square \]

In what follows, we shall use the energy method to prove Proposition 6.2. Notice that

$$S = \begin{pmatrix} -\alpha \varepsilon A & 0 \\ \frac{1}{\varepsilon} G[\varepsilon \zeta] & 0 \end{pmatrix}$$

is a symmetrizer of $\mathcal{M}$, so that a natural energy functional for the system (6.3) is given by

$$E^s(V)^{2 \equiv} (\Lambda^s V, S \Lambda^s V) = (\Lambda^s V_1, -\alpha \varepsilon A \Lambda^s V_1) + (\Lambda^s V_2, \frac{1}{\varepsilon} G[\varepsilon \zeta] V_2).$$

We shall see below that the estimate of $E^s(V)^2$ will lead to deal with the following commutators

$$\left( [\Lambda^s, A] V_1, G[\varepsilon \zeta] \Lambda^s V_2 \right) \quad \text{and} \quad \left( [\Lambda^s, G[\varepsilon \zeta] V_2, A \Lambda^s V_1] \right).$$

Since $[\Lambda^s, A]$ is a pseudo-differential operator of order $s + 1$ and $[\Lambda^s, G[\varepsilon \zeta]]$ is a pseudo-differential operator of order $s$, the above two commutators are dominated by (for example) $|V_1|_{H^{s+\frac{3}{2}}} |V_2|_{H^{s+\frac{1}{2}}}$, which can not dominated by the energy $E^s(V)$ due to

$$E^s(V) \sim |V_1|_{H^{s+1}} + |V_2|_{H^{s+\frac{1}{2}}} \quad \text{(essentially)}.$$

That is, we’ll lose one half derivative in the process of energy estimate if we choose to use this kind of symmetrizer. To overcome this difficulty, motivated by [21] we introduce a new energy functional $E^k(V)$ defined by

$$E^k(V)^{2 \equiv} \varepsilon^k(V)^2 + \varepsilon^k(V)^2$$
with $\mathcal{E}^k(V)$ and $\mathcal{E}_h(V)$ given by

$$
\mathcal{E}^k(V)^2 \overset{\text{def}}{=} (\Lambda^k V_1, (1 - \alpha \varepsilon A) \Lambda^k V_1) + (\Lambda^k V_2, \frac{1}{\varepsilon} G[\varepsilon \zeta] \Lambda^k V_2) + \varepsilon^2 (\Lambda^{k-1} V_2, \Lambda^{k-1} V_2),
$$
$$
\mathcal{E}_h(V)^2 \overset{\text{def}}{=} (\partial_\epsilon (\zeta) \rho^{-1} V_1, \rho (1 - \alpha \varepsilon A) \rho \partial_\epsilon (\zeta) \rho^{-1} V_1) + (\partial_\epsilon (\zeta) \rho V_2, \frac{1}{\varepsilon} G[\varepsilon \zeta] \partial_\epsilon (\zeta) \rho V_2)
$$

for $\partial_\epsilon (\zeta)$ given by (2.7) and $\rho = \rho(\zeta) = (1 + \varepsilon^3 |\nabla_0^k \zeta|^2)^{\frac{1}{2}}$. Let us also introduce

(6.6) \hspace{1cm} E^k(V)^2 \overset{\text{def}}{=} E^k(V)^2 + E_h(V)^2

for

$$
E^k(V)^2 \overset{\text{def}}{=} \varepsilon |\nabla_0^k V_1|^2_H + |V_1|^2_H + |\mathcal{P} V_2|^2_{H^k} + \varepsilon^2 |V_2|^2_{H^{k-1}},
$$
$$
E_h(V)^2 \overset{\text{def}}{=} \varepsilon |V_1|^2_{H^{2k+1}} + |V_1|^2_{H^{2k}} + |\mathcal{P} V_2|^2_{H^{2k}}.
$$

We have the following relation between $\mathcal{E}^k(V)$ and $E^k(V)$:

**Lemma 6.1.** Let $\mathcal{E}^k(V)$ and $E^k(V)$ be given by (6.5) and (6.6), respectively. We have

$$
M(\sigma)^{-1} E^k(V)^2 \leq \mathcal{E}^k(V)^2 + \varepsilon M(\sigma) (|V_1|^2_{H^0} + |\mathcal{P} V_2|^2_{H^0}) |\zeta|^2_{H^{2k+1}},
$$
$$
\mathcal{E}^k(V)^2 \leq M(\sigma) E^k(V)^2 + \varepsilon M(\sigma) (|V_1|^2_{H^0} + |\mathcal{P} V_2|^2_{H^0}) |\zeta|^2_{H^{2k+1}},
$$

$$
M(\sigma)^{-1} E^k(V)^2 \leq \mathcal{E}^k(V)^2 \leq M(\sigma) E^k(V)^2.
$$

**Proof.** The first inequality can be deduced from Proposition 5.1 Lemma 2.3 and Lemma 2.4 while the remaining ones are obvious.

Now let us turn to the proof of Proposition 6.2.

**Proof of Proposition 6.2.** With the *a priori* estimate (6.4), it is classical to prove the existence part of the proposition. So we only present the detailed proof of (6.4) for smooth enough solutions of (6.3).

**Step 1.** High order energy estimate.

Recall that $\partial_\epsilon (\zeta)$ is given by (2.7) and $\rho = \rho(\zeta) = (1 + \varepsilon^3 |\nabla_0^k \zeta|^2)^{\frac{1}{2}}$. Let $\lambda > 0$ to be determined later, and we denote $G \overset{\text{def}}{=} G[\varepsilon \zeta]$. Then applying a standard energy estimate to (6.3) yields

$$
\varepsilon^{\lambda} \eta \frac{d}{dt} \left( e^{-\varepsilon^\lambda \mathcal{E}^k(V(t))} \right) = -\varepsilon \lambda \mathcal{E}^k(V(t))^2 + 2 \partial_\epsilon (\zeta)^k \partial_t (\rho^{-1} V_1) + \rho (1 - \alpha \varepsilon A) \rho \partial_\epsilon (\zeta) \rho^{-1} V_1 + 2 \partial_\epsilon (\zeta)^k \partial_t V_2, \frac{1}{\varepsilon} G \partial_\epsilon (\zeta) \rho V_2) + 2 \partial_\epsilon (\zeta)^k \rho^{-1} V_1, \rho (1 - \alpha \varepsilon A) \rho \partial_\epsilon (\zeta) \rho^{-1} V_1 + 2 \partial_\epsilon (\zeta)^k \rho V_2, \frac{1}{\varepsilon} G \partial_\epsilon (\zeta) \rho V_2) + 2 \partial_\epsilon (\zeta)^k \rho^{-1} V_1, \rho (1 - \alpha \varepsilon A) \rho \partial_\epsilon (\zeta) \rho^{-1} V_1 + 2 \partial_\epsilon (\zeta)^k \rho V_2, \frac{1}{\varepsilon} G \partial_\epsilon (\zeta) \rho V_2)
$$

(6.7)

$$
\overset{\text{def}}{=} -\varepsilon \lambda \mathcal{E}^k(V)^2 + 2 D_1 + 2 D_2 + D_3.
$$
• The estimate of $D_1$

$$D_1 = (\partial_x \zeta^k \rho^{-1} \partial_t V_1, \rho(1 - \alpha \varepsilon A) \rho \partial_x \zeta^k \rho^{-1} V_1)$$

$$+ (\partial_x \zeta^k (\partial_t \rho^{-1}) V_1, \rho(1 - \alpha \varepsilon A) \rho \partial_x \zeta^k \rho^{-1} V_1) \overset{\text{def}}{=} D_1^1 + D_1^2.$$

Applying Lemma 2.3 gives

$$|D_1^2| \leq \mathcal{C}\left( |\partial_x \zeta^k (\partial_t \rho^{-1}) V_1|_2 |\partial_x \zeta^k \rho^{-1} V_1|_2 + \varepsilon |\partial_x \zeta^k (\partial_t \rho^{-1}) V_1|_{H^1} |\partial_x \zeta^k \rho^{-1} V_1|_{H^1} \right)$$

$$\leq \varepsilon^2 \mathcal{C}\left( |V_1|_{H^2}^2 + \varepsilon |V_1|_{H^2}^2 + |V_1|_{H^0}^2 |(\zeta, \partial_\zeta)|_{H^2}^2 \right).$$

While thanks to (6.3), we write

$$D_1^1 = (\partial_x \zeta^k \rho^{-1} \frac{1}{\varepsilon} \rho \partial_x \zeta^k \rho^{-1} V_1)$$

$$- (\partial_x \zeta^k \rho^{-1} \varepsilon \nabla_0^k \cdot (\mathbf{V}_1), \rho(1 - \alpha \varepsilon A) \rho \partial_x \zeta^k \rho^{-1} V_1)$$

$$+ (\partial_x \zeta^k \rho^{-1} \varepsilon H_1, \rho(1 - \alpha \varepsilon A) \rho \partial_x \zeta^k \rho^{-1} V_1) \overset{\text{def}}{=} D_1^{11} + D_1^{12} + D_1^{13}.$$

Let

$$\nabla_0^k = \frac{1}{2} (\nabla_0^k \cdot (\mathbf{V}_f) + \mathbf{V}_f \cdot \nabla_0^k f),$$

we write

$$D_1^{12} = - (\partial_x \zeta^k \rho^{-1} \varepsilon \nabla_0^k \cdot \mathbf{V}_1, \rho(1 - \alpha \varepsilon A) \rho \partial_x \zeta^k \rho^{-1} V_1)$$

$$- \frac{1}{2} (\partial_x \zeta^k \rho^{-1} \varepsilon (\nabla_0^k \cdot \mathbf{V}_f) V_1, \rho(1 - \alpha \varepsilon A) \rho \partial_x \zeta^k \rho^{-1} V_1)$$

$$= - \varepsilon \left( |\partial_x \zeta^k \rho^{-1}, \nabla_0^k \mathbf{V}_1|_2 \rho(1 - \alpha \varepsilon A) \rho \partial_x \zeta^k \rho^{-1} V_1 \right)$$

$$- \frac{1}{2} \varepsilon (\partial_x \zeta^k \rho^{-1} V_1, \rho(1 - \alpha \varepsilon A) \rho \partial_x \zeta^k \rho^{-1} V_1)$$

$$- \frac{1}{2} \varepsilon (\partial_x \zeta^k \rho^{-1} (\nabla_0^k \cdot \mathbf{V}_f) V_1, \rho(1 - \alpha \varepsilon A) \rho \partial_x \zeta^k \rho^{-1} V_1),$$

which together with Lemma 2.3, Lemma 2.4, and Proposition 5.3 implies that

$$|D_1^1| \leq \varepsilon \mathcal{C}\left( |V_1|_{H^2}^2 + \varepsilon |V_1|_{H^2}^2 + |V_1|_{H^0}^2 |(\zeta, \partial_\zeta)|_{H^2}^2 \right).$$

And similarly, one has

$$|D_1^3| \leq \varepsilon \mathcal{C}\left( |H_1|_{H^2}^2 + \varepsilon |H_1|_{H^2}^2 + |V_1|_{H^2}^2 + \varepsilon |V_1|_{H^0}^2 |(\zeta, \partial_\zeta)|_{H^2}^2 \right).$$

Here we used the fact that $|\mathbf{V}|_{H^2} \leq C |(\zeta, \partial_\zeta)|_{H^2}^2$. Consequently, we obtain

$$D_1 = D_1^1 + \mathcal{R}_1$$

with $D_1^1$ given by (6.8) and $\mathcal{R}_1$ satisfying

$$|\mathcal{R}_1| \leq \varepsilon \mathcal{C}\left( |H_1|_{H^2}^2 + \varepsilon |H_1|_{H^2}^2 + |V_1|_{H^2}^2 + \varepsilon |V_1|_{H^2}^2 \right)$$

$$+ (|H_1|_{H^0}^2 + |V_1|_{H^0}^2) |(\zeta, \partial_\zeta)|_{H^2}^2.$$

• The estimate of $D_2$
Again thanks to (6.3), we write

\begin{equation}
D_2 = - \left( \partial_\varepsilon (\zeta)^k (a - \alpha \varepsilon A)V_1, \frac{1}{\varepsilon} G \partial_\varepsilon (\zeta)^k V_2 \right) - \left( \partial_\varepsilon (\zeta)^k \varepsilon \cdot \nabla_\varepsilon V_2, \frac{1}{\varepsilon} G \partial_\varepsilon (\zeta)^k V_2 \right) \\
+ \left( \partial_\varepsilon (\zeta)^k \varepsilon H_2, \frac{1}{\varepsilon} G \partial_\varepsilon (\zeta)^k V_2 \right) \overset{\text{def}}{=} D_2^1 + D_2^3.
\end{equation}

Applying Lemma 2.3 Lemma 2.4 Proposition 5.1 and Proposition 5.3 yields

\begin{equation}
D_2^3 = - \varepsilon \left( \partial_\varepsilon (\zeta)^k, \varepsilon \cdot \nabla_\varepsilon V_2 \right) - \varepsilon \left( \partial_\varepsilon (\zeta)^k \varepsilon \cdot \nabla_\varepsilon (\partial_\varepsilon (\zeta)^k V_2), \frac{1}{\varepsilon} G \partial_\varepsilon (\zeta)^k V_2 \right) \\
\leq \varepsilon \mathcal{C} \left( |\mathcal{P} V_2|^2_{H^{2k}} + |\mathcal{P} V_2|^2_{H_{t_0+1}} \right) |(\zeta, \mathcal{P} \psi)|^2_{H^2_{t_0+3}};
\end{equation}

and similarly one has

\begin{equation}
|D_2^3| \leq \varepsilon \mathcal{C} \left( |\mathcal{P} H_2|^2_{H^{2k}} + |\mathcal{P} V_2|^2_{H^{2k}} \right) \left( |\mathcal{P} H_2|^2_{H^1} + |\mathcal{P} V_2|^2_{H^1} \right) |(\zeta, \mathcal{P} \psi)|^2_{H^2_{t_0+3}}.
\end{equation}

which leads to

\begin{equation}
D_2 = D_2^1 + \mathcal{R}_2,
\end{equation}

with \(D_2^1\) given by (6.11) and \(\mathcal{R}_2\) satisfying

\begin{equation}
|\mathcal{R}_2| \leq \varepsilon \mathcal{C} \left( |\mathcal{P} H_2|^2_{H^{2k}} + |\mathcal{P} V_2|^2_{H^{2k}} \right) \left( |\mathcal{P} H_2|^2_{H^1} + |\mathcal{P} V_2|^2_{H^1} \right) |(\zeta, \mathcal{P} \psi)|^2_{H^2_{t_0+3}}.
\end{equation}

**The estimate of \(D_1 + D_2\)**

Thanks to (6.3), we rewrite \(D_{11}^1\) as

\begin{equation}
D_{11}^1 = \left( \partial_\varepsilon (\zeta)^k \rho^{-1} \frac{1}{\varepsilon} G V_2, \rho^2 \partial_\varepsilon (\zeta)^k \rho^{-1} V_1 \right) - \left( \partial_\varepsilon (\zeta)^k \rho^{-1} \frac{1}{\varepsilon} G V_2, 2 \varepsilon \rho A \partial_\varepsilon (\zeta)^k \rho^{-1} V_1 \right) \\
= \left( \partial_\varepsilon (\zeta)^k \frac{1}{\varepsilon} G V_2, \partial_\varepsilon (\zeta)^k V_1 \right) + \left( \partial_\varepsilon (\zeta)^k \rho^{-1} \frac{1}{\varepsilon} G V_2, \rho \partial_\varepsilon (\zeta)^k \rho^{-1} V_1 \right) \\
+ \left( \rho \partial_\varepsilon (\zeta)^k \rho^{-1} \frac{1}{\varepsilon} G V_2, \partial_\varepsilon (\zeta)^k V_1 \right) - \alpha \varepsilon \left( \partial_\varepsilon (\zeta)^k \rho^{-1} \frac{1}{\varepsilon} G V_2, \rho \partial_\varepsilon (\zeta)^k \rho^{-1} V_1 \right) \\
- \alpha \varepsilon \left( \partial_\varepsilon (\zeta)^k \rho^{-1} \frac{1}{\varepsilon} G V_2, \rho A \partial_\varepsilon (\zeta)^k \rho^{-1} V_1 \right) \\
\overset{\text{def}}{=} \mathcal{G}_1 + \mathcal{G}_2 + \mathcal{G}_3 + \mathcal{G}_4 + \mathcal{G}_5.
\end{equation}

It follows from Lemma 2.3 2.4 Proposition 5.1 and Proposition 5.3 that

\begin{equation}
|\mathcal{G}_2| + |\mathcal{G}_3| \leq \varepsilon \mathcal{C} \left( |V_1|^2_{H^{2k}} + |\mathcal{P} V_2|^2_{H^{2k}} + \left( |V_1|^2_{H^1} + |\mathcal{P} V_2|^2_{H^1} \right) |(\zeta, \mathcal{P} \psi)|^2_{H^2_{t_0+1}} \right).
\end{equation}

Since \(A(\rho f)\) can be written as

\begin{equation}
A(\rho f) = - \partial_\varepsilon (\zeta) f + h_A \partial_\varepsilon (\zeta) \varepsilon h_A \partial_\varepsilon (\zeta) \varepsilon f + h_A' f
\end{equation}

for some smooth function \(h_A, h_A'\) depending on \(\varepsilon \zeta, \varepsilon^2 \zeta, \varepsilon^3 \zeta\), this implies that

\begin{equation}
[A(\rho), \partial_\varepsilon (\zeta)^k] = \left[ h_A \partial_\varepsilon (\zeta)^k, \partial_\varepsilon (\zeta)^k \right] + [h_A', \partial_\varepsilon (\zeta)^k].
\end{equation}

Therefore, one has

\begin{equation}
|\mathcal{G}_5| \leq \varepsilon \mathcal{C} \left( |V_1|^2_{H^{2k+1}} + |\mathcal{P} V_2|^2_{H^{2k}} + \left( |V_1|^2_{H^1} + |\mathcal{P} V_2|^2_{H^1} \right) |(\zeta, \mathcal{P} \psi)|^2_{H^2_{t_0+1}} \right).
\end{equation}

As a consequence, we obtain

\begin{equation}
D_{11}^1 = \mathcal{G}_1 + \mathcal{G}_4 + \mathcal{R}_3
\end{equation}
with $\mathcal{G}_1$, $\mathcal{G}_4$ given by (6.13) and $\mathcal{R}_3$ satisfying

$$\mathcal{R}_3 \leq \varepsilon C \left( |V_1|^2_{H^2_k} + \varepsilon |V_1|^2_{H^{2k+1}} + |\mathfrak{P} V_2|^2_{H^2_k} + |V_1|^2_{H^{2k+2}} + |\mathfrak{P} V_2|^2_{H^{2k+2}} \right).$$

On the other hand, let $a = 1 + \varepsilon h$ with $h \equiv \varepsilon \cdot \nabla^e \zeta + \partial_t \zeta$. Thanks to (6.11), we have

$$D_2^1 = - \left( \varepsilon \zeta^k V_1, \varepsilon \zeta^k V_1 \right)_{L^2(\Omega)} - \left( \varepsilon \zeta^k V_1, \varepsilon \zeta^k V_2 \right)_{L^2(\Omega)} + \left( \varepsilon \zeta^k \varepsilon \zeta^k V_1, \varepsilon \zeta^k \varepsilon \zeta^k V_2 \right)_{L^2(\Omega)} + \left( \varepsilon \zeta^k \varepsilon \zeta^k V_1, \varepsilon \zeta^k \varepsilon \zeta^k V_2 \right)_{L^2(\Omega)} = \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3 + \mathcal{H}_4 + \mathcal{H}_5.$$

Applying Lemma 2.3 and Proposition 5.6 gives

$$\mathcal{H}_4 \leq C \left| \varepsilon \zeta \mathfrak{P} \psi \right|_{H^2_k}^2 \left( |V_1|^2_{H^2_k} + |\mathfrak{P} V_2|^2_{H^2_k} + |V_1|^2_{H^{2k+2}} + |\mathfrak{P} V_2|^2_{H^{2k+2}} \right),$$

and Theorem 5.1 ensures that

$$\mathcal{H}_5 \leq C \left| \varepsilon \zeta \mathfrak{P} \psi \right|_{H^2_k} \left| \varepsilon \zeta \mathfrak{P} \psi \right|_{H^2_k} \left( |V_1|^2_{H^2_k} + |\mathfrak{P} V_2|^2_{H^2_k} + |V_1|^2_{H^{2k+2}} + |\mathfrak{P} V_2|^2_{H^{2k+2}} \right).$$

While it follows from Proposition 5.2 and Proposition 5.3 that

$$|b|_{H^{k+2}} \leq \sqrt{\varepsilon} \left( |\varepsilon \zeta \mathfrak{P} \psi|_{H^{k+2}} + \left| \partial_t \zeta \partial_t \mathfrak{P} \psi \right|_{H^{k+2}} \right),$$

from which and Proposition 5.1, Lemma 2.3 and Lemma 2.4 we infer that

$$\mathcal{H}_3 \leq C \left| \varepsilon \zeta \mathfrak{P} \psi \right|_{H^2_k} \left| \varepsilon \zeta \mathfrak{P} \psi \right|_{H^2_k} \left( |V_1|^2_{H^2_k} + |\mathfrak{P} V_2|^2_{H^2_k} + |V_1|^2_{H^{k+2}} + |\mathfrak{P} V_2|^2_{H^{k+2}} \right).$$

Consequently, we obtain

$$(6.16) \quad D_2^1 = \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{R}_4$$

with $\mathcal{H}_1$, $\mathcal{H}_2$ given by (6.15) and $\mathcal{R}_4$ satisfying

$$\mathcal{R}_4 \leq \varepsilon C \left( |V_1|^2_{H^2_k} + \varepsilon |V_1|^2_{H^{2k+1}} + |\mathfrak{P} V_2|^2_{H^2_k} + |V_1|^2_{H^{2k+2}} + |\mathfrak{P} V_2|^2_{H^{2k+2}} \right).$$

Summing up (6.10), (6.12), (6.14) and (6.16) yields that

$$(6.17) \quad |D_1 + D_2| = |\mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3 + \mathcal{R}_4| \leq \varepsilon C \left( |V_1|^2_{H^2_k} + \varepsilon |V_1|^2_{H^{2k+1}} + |\mathfrak{P} V_2|^2_{H^2_k} + |V_1|^2_{H^{2k+2}} + \varepsilon |H_1|^2_{H^2_k} + \varepsilon |H_1|^2_{H^{2k+1}} + |\mathfrak{P} H_2|^2_{H^2_k} + |V_1|^2_{H^{2k+2}} + |\mathfrak{P} V_2|^2_{H^{2k+2}} + |H_1|^2_{H^2_k} + |H_1|^2_{H^{2k+1}} \right).$$

- The estimate of $D_3$
The estimate of $D_3$ is much easier. For example, we get by applying Proposition 5.4 to the second term in $D_3$ that

$$\left| \langle d \cdot (\xi)^k \rangle V_2, [\partial_t, \frac{1}{\varepsilon} \xi^k \partial_t \xi^k V_2]\right| \leq C \left| \mathcal{P} \partial_t \xi^k V_2 \right|^2 \leq C \left( |\mathcal{P} V_2|^2_{L^2} + |\mathcal{P} V_2|^2_{H^{2k+2}} \right).$$

Following exactly the same line, we can obtain similar estimates for the other terms in $D_3$. This gives

$$|D_3| \leq C \left( |V_1|_{L^2}^2 + |V_1|_{H^{2k+1}}^2 + |\mathcal{P} V_2|^2_{L^2} + (|V_1|_{H^{2k+1}}^2 + |\mathcal{P} V_2|^2_{H^{2k+1}}) \right).$$

Plugging (6.17) and (6.18) into (6.14) results in

$$e^{\varepsilon \lambda t} \frac{d}{dt} \left( e^{-\varepsilon \lambda t} \epsilon^k (V) \right)^2 = -\varepsilon \lambda \epsilon^k (V)^2 + 2\left( \Lambda^k \partial_t V_1, (1 - \alpha \varepsilon \mathbf{A}) \Lambda^k V_1 \right) + 2\left( \Lambda^k \partial_t V_2, \frac{1}{\varepsilon} \xi \Lambda^k V_2 \right) + 2\epsilon^k H_0 \left( \partial_t \xi \mathbf{A} \right)$$

$$+ 2\epsilon^k (H_1, \mathbf{A} V_2) \left( \partial_t \xi \mathbf{A} \right) + 2\epsilon^k (H_2, \mathbf{A} V_2) \left( \partial_t \xi \mathbf{A} \right)$$

$$\leq -\varepsilon \lambda \epsilon^k (V)^2 + 2 E_1 + 2 E_2 + E_3 + 2 E_4.$$

First of all, it is easy to show that

$$|E_3| \leq C \left( \varepsilon |\nabla^2 h| V_1 |V_2|^2_{L^2} + |\mathcal{P} V_2|^2_{H^2} \right).$$

- The estimate of $E_1$

Thanks to (6.3), we write

$$E_1 = -\left( \Lambda^k \varepsilon \nabla^k \cdot (\mathbf{v} V_1), (1 - \alpha \varepsilon \mathbf{A}) \Lambda^k V_1 \right) + \left( \Lambda^k \frac{1}{\varepsilon} \xi \Lambda^k V_2, (1 - \alpha \varepsilon \mathbf{A}) \Lambda^k V_1 \right)$$

$$+ \left( \Lambda^k \varepsilon \mathbf{h} \cdot (\mathbf{v} V_1), (1 - \alpha \varepsilon \mathbf{A}) \Lambda^k V_1 \right) \overset{\text{def}}{=} E_1^1 + E_1^2 + E_1^3.$$
and
\[
[A^k, \nabla^\varepsilon_h \mathbf{v}] = [A^k, \mathbf{v}] \cdot \nabla^\varepsilon_h + \frac{1}{2} [A^k, \nabla^\varepsilon_h \cdot \mathbf{v}],
\]
so it follows from Lemma 2.4 that
\[
|E^{11}_i| \leq \varepsilon C(\|A^k, \nabla^\varepsilon_h V_i\|_2 + \varepsilon^{\frac{1}{2}} \|\nabla^\varepsilon_h [A^k, \nabla^\varepsilon_h V_i]\|_2) (|V_i|_{H^k} + \varepsilon^{\frac{1}{2}} |\nabla^\varepsilon_h V_i|_{H^k})
\]
\[
\leq \varepsilon C(|V_i|_{H^k} + \varepsilon^{\frac{1}{2}} |\nabla^\varepsilon_h V_i|_{H^k} + |V_i|_{H^{t_0+1}} |\Lambda^2 \mathbf{v}|_{H^k}) (|V_i|_{H^k} + \varepsilon^{\frac{1}{2}} |\nabla^\varepsilon_h V_i|_{H^k})
\]
\[
\leq \varepsilon C \left( |V_i|_{H^k}^2 + \varepsilon |\nabla^\varepsilon_h V_i|_{H^k}^2 + |V_i|_{H^{t_0+1}}^2 (|\Lambda^3 \mathbf{w}|_{H^k}^2 + |\Lambda^2 |_{H^k}^2) \right).
\]
Similarly, we have
\[
|E^{12}_i + E^{13}_i| \leq \varepsilon C \left( |V_i|_{H^k}^2 + \varepsilon |\nabla^\varepsilon_h V_i|_{H^k}^2 + |V_i|_{H^{t_0+1}}^2 (|\Lambda^3 \mathbf{w}|_{H^k}^2 + |\Lambda^2 |_{H^k}^2) \right).
\]
To handle $E^{2}_1$, we write
\[
E^{2}_1 = \left( \frac{1}{\varepsilon} G^k \mathbf{V}_2, (1 - \alpha \varepsilon \mathbf{A}) A^k \mathbf{V}_i \right) + \left( [A^k, \frac{1}{\varepsilon} G] \mathbf{V}_2, (1 - \alpha \varepsilon \mathbf{A}) A^k \mathbf{V}_i \right) \overset{\text{def}}{=} E^{21}_i + E^{22}_i.
\]
As \[
|\mathbf{A}^2 \mathbf{w}|_{H^{k-1}} = |(\Lambda^2 \mathbf{w}) \varepsilon \mathbf{A} \mathbf{A}^k \mathbf{V}_i|_{2} \leq C \left( |\Lambda^2 \mathbf{w}|_{H^k} + |\Lambda^k \mathbf{w}|_{H^k} \right),
\]
which along with Remark 5.3 ensures that
\[
|E^{22}_i| \leq C \left( \|A^k, \frac{1}{\varepsilon} G\|_{2} \mathbf{V}_2 \|_2 + \varepsilon^{\frac{1}{2}} |D^\varepsilon_h \left[ A^k, \frac{1}{\varepsilon} G\right] \mathbf{V}_2\|_2 + |V_i|_{H^k} + \varepsilon^{\frac{1}{2}} |\nabla^\varepsilon_h V_i|_{H^k} \right)
\]
\[
\leq C \left( |V_i|_{H^k}^2 + \varepsilon |\nabla^\varepsilon_h V_i|_{H^k}^2 + |\mathbf{V}_2|_{H^k}^2 + |\mathbf{V}_2|_{H^{t_0+2}}^2 + |\Lambda^2 \mathbf{w}|_{H^k}^2 \right).
\]
As a consequence, we obtain
\begin{align*}
(6.22) \quad E_1 &= \left( \frac{1}{\varepsilon} G^k \mathbf{V}_2, (1 - \alpha \varepsilon \mathbf{A}) A^k \mathbf{V}_i \right) + \mathcal{R}_1
\end{align*}
with $\mathcal{R}_1$ satisfying
\[
\|\mathcal{R}_1\| \leq C \left( |V_i|_{H^k}^2 + |\nabla^\varepsilon_h V_i|_{H^k}^2 + |\mathbf{V}_2|_{H^k}^2 + |\mathbf{V}_2|_{H^{t_0+2}}^2 + |V_i|_{H^{t_0+1}}^2 |\Lambda^4 \mathbf{w}|_{H^k}^2 \right).
\]

- **The estimate of $E_2$**
  Thanks to (6.33), we write
\[
E_2 = - (\Lambda^k (\mathbf{A} - \alpha \varepsilon \mathbf{A}) V_i, \frac{1}{\varepsilon} G^k \mathbf{V}_2) - (\Lambda^k \varepsilon \mathbf{A} \cdot \nabla^\varepsilon_h \mathbf{v}, \frac{1}{\varepsilon} G^k \mathbf{V}_2)
\]
\[
+ (\Lambda^k \varepsilon H^k, \frac{1}{\varepsilon} G^k \mathbf{V}_2) \overset{\text{def}}{=} E^{12}_2 + E^{13}_2.
\]
Applying Proposition 5.1 gives
\[
|E^{2}_2| \leq \varepsilon C \|\Lambda^2 \mathbf{w}|_{H^k} \mathbf{V}_2|_{H^k}.
\]
While notice that
\[
E^{12}_2 = -(1 - \alpha \varepsilon \mathbf{A}) A^k \mathbf{V}_i, \frac{1}{\varepsilon} G^k \mathbf{V}_2) - ([A^k, 1 - \alpha \varepsilon \mathbf{A}] \mathbf{V}_i, \frac{1}{\varepsilon} G^k \mathbf{V}_2)
\]
\[
- (\Lambda^k \varepsilon \mathbf{A} V_i, \frac{1}{\varepsilon} G^k \mathbf{V}_2) \overset{\text{def}}{=} E^{11}_2 + E^{12}_2 + E^{13}_2.
\]
It follows from Lemma 2.4, Proposition 5.1 and an interpolation argument that
\[ |E_2^{12}| \leq C \|D_p^b\| [\Lambda^k, \alpha \varepsilon \Lambda] V_1 \sigma |\mathcal{V}\mathcal{W}|_{H^k} \]
\[ \leq \varepsilon C \left( |\mathcal{V}\mathcal{W}|_{H^k}^2 + \varepsilon |V_1|_{H^k}^2 + |\nabla^\varepsilon \cdot V_1|_{H^k}^2 + |V_1|_{H^{k+1}}^2 + |V_1|_{H^{k+2}}^2 + |V_1|_{H^{k+3}}^2 |\Lambda^3 \mathcal{W}|_{H^k}^2 \right). \]
Similarly, one has
\[ |E_2^{13}| = \varepsilon C \left( |\mathcal{V}\mathcal{W}|_{H^k}^2 + \varepsilon |\nabla^\varepsilon \cdot V_1|_{H^k}^2 + |V_1|_{H^k}^2 \right) \]
\[ + |V_1|_{H^{k+1}}^2 |\Lambda^3 \mathcal{W}|_{H^k}^2 + |\Lambda^3 (\partial_t \mathcal{W})^2|_{H^k} \right). \]
Noting that
\[ E_2 = -\varepsilon \left( [\Lambda^k, \mathcal{W}] \cdot \nabla^\varepsilon \cdot V_2, 1 - \alpha \varepsilon \Lambda \right) V_1 + \mathcal{W}_2 \]
from which, Proposition 5.1 and Lemma 2.4, we infer that
\[ |E_2^2| \leq \varepsilon C \left( |\mathcal{V}\mathcal{W}|_{H^k}^2 + |\mathcal{V}\mathcal{W}|_{H^k}^2 + |\mathcal{V}\mathcal{W}|_{H^{k+1}}^2 + |\Lambda^3 \mathcal{W}|_{H^k}^2 \right). \]
Therefore, we obtain
\[ (6.23) \quad E_2 = -\left( \frac{1}{\varepsilon} \varepsilon \Lambda^k V_1, (1 - \alpha \varepsilon \Lambda) \Lambda^k V_1 \right) + \mathcal{W}_2 \]
with \( \mathcal{W}_2 \) satisfying
\[ |\mathcal{W}_2| \leq C \left( |V_1|_{H^k}^2 + |\nabla^\varepsilon \cdot V_1|_{H^k}^2 + |V_1|_{H^{k+1}}^2 + |\mathcal{V}\mathcal{W}|_{H^k}^2 + |\mathcal{V}\mathcal{W}|_{H^{k+1}}^2 \right) \]
\[ + |\Lambda^3 \mathcal{W}|_{H^k}^2 + |\Lambda^2 (\partial_t \mathcal{W})^2|_{H^k} \right). \]

**The estimate of \( E_4 \)**
Again thanks to (6.3), we write
\[ E_4 = -\varepsilon^2 \left( \Lambda^{k-1} \cdot \nabla^\varepsilon \cdot V_1, \Lambda^{k-1} V_2 \right) - \varepsilon^2 \left( \Lambda^{k-1} \cdot \nabla^\varepsilon \cdot V_2, \Lambda^{k-1} V_1 \right) \]
\[ + \varepsilon^3 \left( \Lambda^{k-1} H_2, \Lambda^{k-1} V_2 \right) \equiv E_1^4 + E_2^4 + E_3^4. \]
It is easy to observe that
\[ |E_1^4| \leq C \varepsilon \left( (V_1)^2_{H^k} + |V_1|_{H^{k+1}}^2 + |V_1|_{H^{k+2}}^2 + |V_1|_{H^{k+3}}^2 |\Lambda^3 \mathcal{W}|_{H^k}^2 \right) \]
\[ + |\Lambda^2 (\partial_t \mathcal{W})^2|_{H^k} \right). \]
\[ |E_2^3| \leq C \varepsilon^2 |H_2|^2_{H^{k+1}} + |V_2|^2_{H^{k+1}} \right). \]
And one gets by using integration by parts that
\[ E_3^4 = -\varepsilon^3 \left( [\Lambda^{k-1}, \mathcal{W}] \cdot \nabla^\varepsilon \cdot V_2, \Lambda^{k-1} V_2 \right) + \frac{1}{2} \varepsilon^3 \left( (\nabla^\varepsilon \cdot \mathcal{W}) \Lambda^{k-1} V_2, \Lambda^{k-1} V_1 \right), \]
which together with Lemma 2.3 implies that
\[ |E_1^4| \leq C \varepsilon^2 |V_2|^2_{H^{k+1}} + |V_2|^2_{H^{k+1}} |\Lambda^3 \mathcal{W}|_{H^k}^2 \right). \]
Then we arrive at
\[ (6.24) \quad |E_4| \leq C \left( |V_1|_{H^k}^2 + |V_1|_{H^{k+1}}^2 + |V_1|_{H^{k+2}}^2 + |V_1|_{H^{k+3}}^2 |\Lambda^3 \mathcal{W}|_{H^k}^2 \right) \]
\[ + |V_1|_{H^{k+1}}^2 |\Lambda^3 \mathcal{W}|_{H^k}^2 + |\Lambda^2 (\partial_t \mathcal{W})^2|_{H^k} \right). \]
Plugging (6.21)-(6.24) into (6.20) results in
\[ e^{\lambda t} \frac{d}{dt} \left( e^{-\lambda t} \mathcal{E}^k(V(t))^2 \right) \leq -\varepsilon \lambda \mathcal{E}^k(V)^2 + \varepsilon \mathcal{C} \left( |V_1|_{H^k}^2 + |V_1|_{H^{2k+1}}^2 + |\nabla^k V_1|_{H^k}^2 + |\mathcal{P} V_2|_{H^k}^2 + |\mathcal{P} V_2|_{H^{k-1}}^2 + |H_1|_{H^k}^2 + |\mathcal{P} H_2|_{H^k}^2 + |\nabla^k H_1|_{H^k}^2 \right) \]
\[ + \varepsilon^2 |H_2|_{H_{k-1}}^2 + (|\mathcal{P}(V_2, H_2)|_{H^{k+2}}^2 + |(V_1, H_1)|_{H^{k+3}}^2) \]
\[ \times (|\lambda^4 \mathcal{E}^k(\zeta, \mathcal{P} \psi)|_{H^k}^2 + |\lambda^3 (\partial_k \zeta, \partial_t \mathcal{P} \psi)|_{H^k}^2). \]

Step 3. Full energy estimates.
Combining (6.19) with (6.25), we get by applying Lemma 6.1 that
\[ e^{\lambda t} \frac{d}{dt} \left( e^{-\lambda t} \mathcal{E}^k(V(t))^2 \right) \leq -\varepsilon \lambda \mathcal{E}^k(V)^2 + \varepsilon \mathcal{C} \mathcal{E}^k(V)^2 \]
\[ + \varepsilon \mathcal{C} \mathcal{E}^k(H)^2 + \varepsilon \mathcal{C} (\mathcal{E}^{t_0+3}(V)^2 + \mathcal{E}^{t_0+3}(H)^2) |\lambda^5(\zeta, \psi)|_{Y^k}^2. \]
Taking \( \lambda = \mathcal{C} \) in the above inequality and applying Lemma 6.1 again yields
\[ \mathcal{E}^k(V(t))^2 \leq \mathcal{C} \mathcal{T}^k(t, V_0, H) + \mathcal{C} (\mathcal{T}^{t_0}(t, V_0, H) + \varepsilon \mathcal{E}^{t_0}(V(t))^2) |\lambda^5(\zeta, \psi)|_{Y^k}^2 \]
\[ + \varepsilon \mathcal{C} \int_0^t \left( \mathcal{E}^{t_0+3}(V)^2 + \mathcal{E}^{t_0+3}(H)^2 \right) d\tau |\lambda^5(\zeta, \psi)|_{Y^k}^2. \]
On the other hand, it follows from (6.25) that
\[ \mathcal{E}^k(V(t))^2 \leq \mathcal{C} \mathcal{T}^k(t, V_0, H) + \mathcal{C} \varepsilon \int_0^t \mathcal{E}^k(V(\tau))^2 d\tau \]
\[ + \mathcal{C} \varepsilon \int_0^t \left( \mathcal{E}^{t_0+3}(V)^2 + \mathcal{E}^{t_0+3}(H)^2 \right) d\tau |\lambda^5(\zeta, \psi)|_{Y^k}^2. \]
After taking \( k = t_0 + 3 \) in the above inequality and applying Gronwall’s inequality, we plug the resulting inequality into (6.26) (where \( k = 5 \)) to yield that
\[ \mathcal{E}^5(V(t))^2 \leq \mathcal{C} \mathcal{T}^5(t, V_0, H), \]
which together with (6.26) implies that
\[ \mathcal{E}^k(V(t))^2 \leq \mathcal{C} \left( \mathcal{T}^k(t, V_0, H) + \mathcal{T}^5(t, V_0, H) |\lambda^5(\zeta, \psi)|_{Y^k}^2 \right). \]
This completes the proof of Proposition 6.2. \( \square \)

7. Large Time Existence for the Nondimensionalized Water-Wave System

The goal of this section is to use a modified Nash-Moser iteration theorem in the Appendix and the uniform estimates obtained for the linearized system (6.2) to solve the water-wave system (1.3) on \([0, \frac{T}{\varepsilon}]\). As noticed in Remark 1.3, there is no need to prove Theorem 1.2 here.

We start the proof of Theorem 1.1 with the following lemma:

Lemma 7.1. For all \( U_0 \in \mathcal{X}^s \), we denote \( S^\varepsilon(t) \) the solution operator to the linear system
\[
\begin{align*}
\partial_t V + \frac{1}{\varepsilon} \mathcal{L} V &= 0, \\
V|_{t=0} &= U_0.
\end{align*}
\]
Then for all \( T > 0 \), \( S^\varepsilon(\cdot)U_0 \) is well-defined in \( C([0, T]; \mathcal{X}^s) \). Moreover, for all \( t \in [0, T] \), there holds
\[ |S^\varepsilon(t)U_0|_{\mathcal{X}^s} \leq C(T, \frac{1}{h_0}, |b|_{H^{2+3}}) |U_0|_{\mathcal{X}^s}. \]
Proof. Indeed (7.2) can be deduced from the proof of Proposition (6.2) in the particular case when \( \varepsilon = (0, 0) \). \( \square \)

With \( V \overset{\text{def}}{=} S^c(t)U^0 \) thus defined, we shall seek for a solution \( U \) of (1.3) under the form \( U = V + W \), which is equivalent to solve the following system of \( W \):

\[
\begin{cases}
\partial_t W + \frac{1}{\varepsilon} \mathcal{L} W + \mathcal{F}[t, W] = h, \\
W|_{t=0} = (0, 0)^T,
\end{cases}
\]

(7.3)

where \( \mathcal{F}[t, W] \overset{\text{def}}{=} \mathcal{A}[V + W] - \mathcal{A}[V] \) and \( h \overset{\text{def}}{=} -\mathcal{A}[V] \).

Lemma 7.2. Let \( T > 0 \) and \( s \geq m_0 \). Then we have

1) The mapping \( \mathcal{L} : X^{s+1} \rightarrow X^s \) is well-defined and continuous, and the family of linear solution operators \( (S^c(t))_{0 < \varepsilon < 1} \) is uniformly bounded in \( C([-T, T]; \mathcal{L}(X^s, X^s)) \);

2) For \( 0 \leq j \leq 2 \), \( W \in X^{s+2}(\mathbb{R}^2) \) and \( (W_1, \ldots, W_j) \in X^{s+2}(\mathbb{R}^2)^j \),

\[
\sup_{t \in [0, T]} |d^j_W \mathcal{F}[t, W](W_1, \ldots, W_j)|_{X^s} \leq C(s, T, |W|_{X^{m_0}})
\]

\[
\times \left( \sum_{k=1}^{j} |A^2 W_k|_{X^s} \prod_{l \neq k} |W_l|_{X^{t_0+1}} + |A^2 W|_{X^s} \prod_{k=1}^{j} |W_k|_{X^{t_0+1}} \right);
\]

3) For \( 0 \leq j \leq 2 \), \( W \in X^s(\mathbb{R}^2) \) and \( (W_1, \ldots, W_j) \in X^s(\mathbb{R}^2)^j \),

\[
\sup_{t \in [0, T]} |A^2 d^j_W \mathcal{F}[t, W](W_1, \ldots, W_j)|_{X^s} \leq C(s, T, |W|_{X^{m_0}})
\]

\[
\times \left( \sum_{k=1}^{j} |W_k|_{X^s} \prod_{l \neq k} |W_l|_{X^{t_0+1}} + |W|_{X^s} \prod_{k=1}^{j} |W_k|_{X^{t_0+1}} \right)
\]

This lemma is a direct consequence of Proposition (5.3) and Remark (5.1).

Proof of Theorem 1.1. With the above preparations, this proof is much similar to that of Theorem 5.1 in [2], so we only sketch its proof here. Indeed rescaling the system (6.1) by using a new time variable \( t' = \varepsilon t \), we only need to show that there exists a \( T > 0 \) independent of \( \varepsilon \) so that the following system has a unique solution on \([0, T] :\)

\[
\begin{cases}
\partial_t U + \frac{1}{\varepsilon} \mathcal{L} U + \mathcal{A}[U] = 0, \\
U|_{t=0} = U^0.
\end{cases}
\]

(7.4)

As shown above, the solution of (7.4) can be equivalently decomposed into the sum of solution of (7.1) and solution of (7.3), so the proof of this theorem relies on the well-posedness of the nonlinear system (7.3). Here we use the Nash-Moser theorem (8.1) to solve it. Lemma 7.2 ensures the first two assumptions of Theorem (8.1) in the Appendix, and the third assumption of Theorem (8.1) follows from Proposition (6.1). Then applying Theorem (8.1) completes the proof of the theorem. \( \square \)

8. Appendix. A Nash-Moser iteration theorem

In order to solve the full water-wave system (1.8), here we present a variant of Nash Moser iteration theorem in [3]. As far as one can see, we present energy estimates with both scaled Sobolev spaces and standard Sobolev spaces. One will find out easily that the Banach space \( X^s \) in our paper doesn’t satisfy the definition of a ‘Banach scale’ in [3], and that’s the reason why a modified Nash-Moser based on [3] is needed in our paper.
We shall focus on the well-posedness of the singular evolution equations of the form

\[
\begin{align*}
\partial_t u^\varepsilon + \frac{1}{\varepsilon} \mathcal{L}^\varepsilon(t) u^\varepsilon + \mathcal{F}^\varepsilon[t, u^\varepsilon] &= h^\varepsilon, \\
\frac{d u^\varepsilon}{dt} &\big|_{t=0} = u_0^\varepsilon,
\end{align*}
\]

where \( \varepsilon \in (0, \varepsilon_0) \) is a small parameter, \( \mathcal{L}^\varepsilon(t) \) is a linear operator, while \( \mathcal{F}^\varepsilon[t, \cdot] \) is a nonlinear mapping. First of all, we introduce a family of Banach scale \( (X^s, | \cdot |_{X^s})_{s \in \mathbb{R}} \) in the following sense:

**Definition 8.1.** We say that a family of Banach spaces \( (X^s, | \cdot |_{X^s})_{s \in \mathbb{R}} \) are Banach scale if

1. For all \( s \leq s' \), one has \( X^{s'} \subset X^s \) and \( | \cdot |_{X^s} \leq | \cdot |_{X^{s'}} \);
2. There exits a family of smoothing operator \( S_\theta (\theta \geq 1) \) satisfying \( S_{2\theta} S_\theta = S_\theta \) and
   \[
   \forall s < s', \quad |(1 - S_\theta) u|_{X^s} \leq C_{s,s'} \theta^{s-s'} |u|_{X^{s'}};
   \]
3. There exists a linear positive operator \( \Lambda \) such that for \( m \geq 0 \),
   \[
   |S_\theta \Lambda^m u|_{X^s} \leq C\theta^m |u|_{X^s} \quad \text{and} \quad |\Lambda^m u|_{X^s} \leq C |u|_{X^{s+m}};
   \]
4. The norms satisfy a convexity property
   \[
   \forall s \leq s'' \leq s', \quad |u|_{X^{s''}} \leq C_{s,s',s''} |u|_{X^s}^\mu |u|_{X^{s'}}^{1-\mu},
   \]
   where \( \mu \) is determined by \( \mu s + (1-\mu)s' = s'' \).

**Notations.** If \( X_1 \) and \( X_2 \) be two Banach spaces, we denote by \( \mathcal{L}(X_1, X_2) \) the set of all continuous mappings from \( X_1 \) to \( X_2 \); If \( X \) is a Banach space and \( T > 0 \), \( X_T \) stands for \( C([0,T]; X) \) with the norm \( | \cdot |_{X_T} \); For \( \mathcal{F} \in C([0,T]; C^j(X_1, X_2)) \), we denote by \( d_j^u \mathcal{F} \) the \( j \)-th order derivatives of the mapping \( u \mapsto \mathcal{F}[\cdot, u] \).

**Assumption 8.1.** There exist \( T > 0, s_0 \in \mathbb{R} \) and \( m \geq 0 \) such that

1. For all \( s \geq s_0 \), one has \( (\mathcal{L}^\varepsilon(\cdot))_{0 < \varepsilon < \varepsilon_0} \) is bounded in \( C([0,T]; \mathcal{L}(X^{s+m}, X^s)) \);
2. For all \( g \in X^s \), the evolution operator \( (U^\varepsilon(\cdot))_{0 < \varepsilon < \varepsilon_0} \) defined by
   \[
   U^\varepsilon(t) g \overset{\text{def}}{=} u^\varepsilon(t), \quad \text{where} \quad \partial_t u^\varepsilon + \frac{1}{\varepsilon} \mathcal{L}^\varepsilon(t) u^\varepsilon = 0, \quad u^\varepsilon|_{t=0} = g
   \]
   is bounded in \( C([-T,T]; \mathcal{L}(X^s, X^s)) \) for \( s \geq s_0 \).

**Assumption 8.2.** There exist \( T > 0, s_0 \in \mathbb{R} \) and \( m \geq 0 \) such that for all \( s \geq s_0 \) and \( 0 \leq j \leq 2 \), one has \( \mathcal{F}^\varepsilon \in C([0,T]; C^j(X^{s+m}, X^s)) \) and for all \( u, v_1, \ldots, v_j \in X^{s+m} \),

\[
\sup_{[0,T]} \left| d_j^u \mathcal{F}[t, u](v_1, \ldots, v_j) \right|_{X^s} \leq C(s, T, |u|_{X^{s_0+m}})
\]

\[
\times \left( \sum_{k=1}^{j} |\Lambda^m v_k|_{X^s} \prod_{l \neq k} |v_l|_{X^{s_0+m}} + |\Lambda^m u|_{X^s} \prod_{k=1}^{j} |v_k|_{X^{s_0+m}} \right).
\]

Moreover for all \( u, v_1, \ldots, v_j \in X^s \cap X^{s_0+m} \),

\[
\sup_{[0,T]} \left| \Lambda^{-m} d_j^u \mathcal{F}[t, u](v_1, \ldots, v_j) \right|_{X^s} \leq C(s, T, |u|_{X^{s_0+m}})
\]

\[
\times \left( \sum_{k=1}^{j} |v_k|_{X^s} \prod_{l \neq k} |v_l|_{X^{s_0+m}} + |u|_{X^s} \prod_{k=1}^{j} |v_k|_{X^{s_0+m}} \right).
\]
In order to state the third assumption, we need to introduce some functional spaces as follows
\[
E_m^s \overset{\text{def}}{=} \cap_{i=0} \mathcal{C}^i([0, T]; X^s), \quad |u|_{E_m^s} \overset{\text{def}}{=} |u|_{X^s} + |\Lambda^{-m} \partial_t u|_{X^s}.
\]
\[
F_m^s \overset{\text{def}}{=} \mathcal{C}([0, T]; X^s) \times X^{s+m}, \quad |(f, g)|_{F_m^s} \overset{\text{def}}{=} |f|_{X^s} + |g|_{X^{s+m}},
\]
\[
\mathcal{J}_m^s(t, f, g) \overset{\text{def}}{=} |g|_{X^{s+m}} + \int_0^t \sup_{0 \leq t' \leq t'} |f(t'')|_{X^s} dt'.
\]

**Assumption 8.3.** There exists \( d_1 \geq 0 \) such that for all \( s \geq s_0 + m, \ u^\varepsilon \in E_m^{s+d_1}, \) and \((f^\varepsilon, g^\varepsilon) \in F_m^s, \) the IVP
\[
\partial_t v^\varepsilon + \frac{1}{\varepsilon} \mathcal{L} \varepsilon(t) v^\varepsilon + d_F F^\varepsilon[t, u^\varepsilon] v^\varepsilon = f^\varepsilon, \quad v^\varepsilon|_{t=0} = g^\varepsilon
\]
admits a unique solution \( v^\varepsilon \in \mathcal{C}([0, T]; X^s) \) for all \( \varepsilon \in (0, \varepsilon_0) \) and
\[
|v^\varepsilon|_{X^s} \leq C(\varepsilon_0, s, T, |u^\varepsilon|_{E_m^{s_0+m+d_1}}) \left( \mathcal{J}_m^s(t, f^\varepsilon, g^\varepsilon) + |\Lambda^{d_1} u^\varepsilon|_{E_m^s}^2 \mathcal{J}_m^{s_0+m}(t, f^\varepsilon, g^\varepsilon) \right).
\]

In what follows, we shall always denote
\[
D_1 = d_1 + m, \quad q = D - m, \quad \text{and} \quad P_{min} = D_1 + \frac{D}{q} \left( \sqrt{D_1 + \sqrt{2(D_1 + q)}} \right)^2.
\]

Then Nash-Moser iteration theorem is stated as follows.

**Theorem 8.1.** Let \( T > 0, \ s_0, \ m, \ d_1 \) be such that Assumptions 8.1-8.3 are satisfied. Let \( D > D_1, \ P > P_{min}, \ s \geq s_0 + m, \) and let \((h^\varepsilon, u^\varepsilon)_{0<\varepsilon<\varepsilon_0}\) be bounded in \( F_m^{s+P} \). Then there exist \( 0 < T' \leq T \) such that \((h^\varepsilon, u^\varepsilon)_{0<\varepsilon<\varepsilon_0}\) has a unique family of solutions \( \{u^\varepsilon\}_{0<\varepsilon<\varepsilon_0} \) which are uniformly bounded in \( \mathcal{C}([0, T']; X^{s+D}) \).

The proof of Theorem 8.1 essentially follows the framework of [3], and we omit the detailed proof here.

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