Sparse recovery by non-convex optimization – instance optimality

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Abstract

In this note, we address the theoretical properties of \( \Delta_p \), a class of compressed sensing decoders that rely on \( \ell^p \) minimization with \( 0 < p < 1 \) to recover estimates of sparse and compressible signals from incomplete and inaccurate measurements. In particular, we extend the results of Candès, Romberg and Tao [4] and Wojtaszczyk [30] regarding the decoder \( \Delta_1 \), based on \( \ell^1 \) minimization, to \( \Delta_p \) with \( 0 < p < 1 \). Our results are two-fold. First, we show that under certain sufficient conditions that are weaker than the analogous sufficient conditions for \( \Delta_1 \) the decoders \( \Delta_p \) are robust to noise and stable in the sense that they are \((2, p)\) instance optimal for a large class of encoders. Second, we extend the results of Wojtaszczyk to show that, like \( \Delta_1 \), the decoders \( \Delta_p \) are \((2, 2)\) instance optimal in probability provided the measurement matrix is drawn from an appropriate distribution.

1 Introduction

The sparse recovery problem received a lot of attention lately, both because of its role in transform coding with redundant dictionaries (e.g., [9, 28, 29]), and

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perhaps more importantly because it inspired compressed sensing [3, 4, 13], a novel method of acquiring signals with certain properties more efficiently compared to the classical approach based on Nyquist-Shannon sampling theory. Define $\Sigma^N_S$ to be the set of all $S$-sparse vectors, i.e.,

$$
\Sigma^N_S := \{ x \in \mathbb{R}^N : |\text{supp}(x)| \leq S \},
$$

and define compressible vectors as vectors that can be well approximated in $\Sigma^N_S$. Let $\sigma_S(x)_{\ell^p}$ denote the best $S$-term approximation error of $x$ in $\ell^p$ (quasi-)norm where $p \geq 0$, i.e.,

$$
\sigma_S(x)_{\ell^p} := \min_{v \in \Sigma^N_S} \|x - v\|_p.
$$

Throughout the text, $A$ denotes an $M \times N$ real matrix where $M < N$. Let the associated encoder be the map $x \mapsto Ax$ (also denoted by $A$). The transform coding and compressed sensing problems mentioned above require the existence of decoders, say $\Delta : \mathbb{R}^M \mapsto \mathbb{R}^N$, with roughly the following properties:

(C1) $\Delta(Ax) = x$ whenever $x \in \Sigma^N_S$ with sufficiently small $S$.

(C2) $\|x - \Delta(Ax + e)\| \lesssim \|e\| + \sigma_S(x)_{\ell^p}$, where the norms are appropriately chosen. Here $e$ denotes measurement error, e.g., thermal and computational noise.

(C3) $\Delta(Ax)$ can be computed efficiently (in some sense).

Below, we denote the (in general noisy) encoding of $x$ by $b$, i.e.,

$$
b = Ax + e. \quad (1)
$$

In general, the problem of constructing decoders with properties (C1)-(C3) is non-trivial (even in the noise-free case) as $A$ is overcomplete, i.e., the linear system of $M$ equations in (1) is underdetermined, and thus, if consistent, it admits infinitely many solutions. In order for a decoder to satisfy (C1)-(C3), it must choose the “correct solution” among these infinitely many solutions. Under the assumption that the original signal $x$ is sparse, one can phrase the problem of finding the desired solution as an optimization problem where the objective is to maximize an appropriate “measure of sparsity” while simultaneously satisfying the constraints defined by (1). In the noise-free case, i.e., when $e = 0$ in (1), under certain conditions on the $M \times N$ matrix $A$, i.e., if $A$ is in general position, there is a decoder $\Delta_0$ which satisfies $\Delta_0(Ax) = x$ for all $x \in \Sigma^N_S$ whenever $S < M/2$, e.g., see [14]. This $\Delta_0$ can be explicitly computed via the optimization problem

$$
\Delta_0(b) := \arg \min_y \|y\|_0 \text{ subject to } b = Ay. \quad (2)
$$
Here $\|y\|_0$ denotes the number of non-zero entries of the vector $y$, equivalently its so-called $\ell^0$-norm. Clearly, the sparsity of $y$ is reflected by its $\ell^0$-norm.

1.1 Decoding by $\ell^1$ minimization

As mentioned above, $\Delta_0(Ax) = x$ exactly if $x$ is sufficiently sparse depending on the matrix $A$. However, the associated optimization problem is combinatorial in nature, thus its complexity grows quickly as $N$ becomes much larger than $M$. Naturally, one then seeks to modify the optimization problem so that it lends itself to solution methods that are more tractable than combinatorial search. In fact, in the noise-free setting, the decoder defined by $\ell^1$ minimization, given by

$$\Delta_1(b) := \arg \min_y \|y\|_1 \text{ subject to } Ay = b,$$  \hspace{1cm} (3)

recovery $x$ exactly if $x$ is sufficiently sparse and the matrix $A$ has certain properties (e.g., [4, 6, 9, 14, 15, 26]). In particular, it has been shown in [4] that if $x \in \Sigma^N_S$ and $A$ satisfies a certain restricted isometry property, e.g., $\delta_3S < 1/3$ or more generally $\delta_{(k+1)S} < \frac{k-1}{k+1}$ for some $k > 1$ such that $k \in \frac{1}{S} N$, then $\Delta_1(Ax) = x$ (in what follows, $N$ denotes the set of positive integers, i.e., $0 \notin \mathbb{N}$). Here $\delta_S$ are the $S$-restricted isometry constants of $A$, as introduced by Candès, Romberg and Tao (see, e.g., [4]), defined as the smallest constants satisfying

$$(1 - \delta_S)\|c\|^2_2 \leq \|Ac\|^2_2 \leq (1 + \delta_S)\|c\|^2_2$$  \hspace{1cm} (4)

for every $c \in \Sigma^N_S$. Throughout the paper, using the notation of [30], we say that a matrix satisfies RIP$(S, \delta)$ if $\delta_S < \delta$.

Checking whether a given matrix satisfies a certain RIP is computationally intensive, and becomes rapidly intractable as the size of the matrix increases. On the other hand, there are certain classes of random matrices which have favorable RIP. In fact, let $A$ be an $M \times N$ matrix the columns of which are independent, identically distributed (i.i.d.) random vectors with any sub-Gaussian distribution. It has been shown that $A$ satisfies RIP $(S, \delta)$ with any $0 < \delta < 1$ when

$$S \leq c_1 M/\log(N/M),$$  \hspace{1cm} (5)

with probability greater than $1 - 2e^{-c_2 M}$ (see, e.g., [1], [5], [6]), where $c_1$ and $c_2$ are positive constants that only depend on $\delta$ and on the actual distribution from which $A$ is drawn.

In addition to recovering sparse vectors from error-free observations, it is important that the decoder be robust to noise and stable with regards to the “compressibility” of $x$. In other words, we require that the reconstruction error scale well with the measurement error and with the “non-sparisity” of the sig-
nal (i.e., (C2) above). For matrices that satisfy RIP($(k+1)S, \delta$), with $\delta < \frac{k-1}{k+1}$ for some $k > 1$ such that $k \in \frac{1}{S} \mathbb{N}$, it has been shown in [4] that there exists a feasible decoder $\Delta_1^\epsilon$ for which the approximation error $\|\Delta_1^\epsilon(b) - x\|_2$ scales linearly with the measurement error $\|e\|_2 \leq \epsilon$ and with $\sigma_S(x)_1$. More specifically, define the decoder

$$\Delta_1^\epsilon(b) = \arg\min_y \|y\|_1 \text{ subject to } \|Ay - b\|_2 \leq \epsilon.$$  

(6)

The following theorem of Candès et al. in [4] provides error guarantees when $x$ is not sparse and when the observation is noisy.

**Theorem 1.1** [4] Fix $\epsilon \geq 0$, suppose that $x$ is arbitrary, and let $b = Ax + e$ where $\|e\|_2 \leq \epsilon$. If $A$ satisfies $\delta_{2S} + 3\delta_{4S} < 2$, then

$$\|\Delta_1^\epsilon(b) - x\|_2 \leq C_{1,S}\epsilon + C_{2,S} \frac{\sigma_S(x)_1}{\sqrt{S}}.$$  

(7)

For reasonable values of $\delta_{4S}$, the constants are well behaved; e.g., $C_{1,S} = 12.04$ and $C_{2,S} = 8.77$ for $\delta_{4S} = 1/5$.

**Remark 1.2** This means that given $b = Ax + e$, and $x$ is sufficiently sparse, $\Delta_1^\epsilon(b)$ recovers the underlying sparse signal within the noise level. Consequently the recovery is perfect if $\epsilon = 0$.

**Remark 1.3** By explicitly assuming $x$ to be sparse, Candès et. al. [4] proved a version of the above result with smaller constants, i.e., for $b = Ax + e$ with $x \in \Sigma_S^N$ and $\|e\|_2 \leq \epsilon$,

$$\|\Delta_1^\epsilon(b) - x\|_2 \leq C_S\epsilon,$$  

(8)

where $C_S < C_{1,S}$.

**Remark 1.4** Recently, Candès [2] showed that $\delta_{2S} < \sqrt{2} - 1$ is sufficient to guarantee robust and stable recovery in the sense of (7) with slightly better constants.

In the noise free case, i.e., when $\epsilon = 0$, the reconstruction error in Theorem 1.1 is bounded above by $\sigma_S(x)_1/\sqrt{S}$, see (7). This upper bound would sharpen if one could replace $\sigma_S(x)_1/\sqrt{S}$ with $\sigma_S(x)_2$ on the right hand side of (7) (note that $\sigma_S(x)_1$ can be large even if all the entries of the reconstruction error are small but nonzero; this follows from the fact that for any vector $y \in \mathbb{R}^N$, $\|y\|_2 \leq \|y\|_1 \leq \sqrt{N}\|y\|_2$, and consequently there are vectors $x \in \mathbb{R}^N$ for which $\sigma_S(x)_1/\sqrt{S} \gg \sigma_S(x)_2$, especially when $N$ is large). In [10] it was shown that the term $C_{2,S}\sigma_S(x)_1/\sqrt{S}$ on the right hand side of (7) cannot be replaced with $C\sigma_S(x)_2$ if one seeks the inequality to hold for all $x \in \mathbb{R}^N$ with a fixed matrix $A$, unless $M > cN$ for some constant $c$. This is unsatisfactory since the paradigm of compressed sensing relies on the ability of recovering sparse or compressible vectors $x$ from significantly fewer measurements than
the ambient dimension $N$.

Even though one cannot obtain bounds on the approximation error in terms of $\sigma_S(x)_{\ell_2}$ with constants that are uniform on $x$ (with a fixed matrix $A$), the situation is significantly better if we relax the uniformity requirement and seek for a version of (7) that holds “with high probability”. Indeed, it has been recently shown by Wojtaszczyk that for any specific $x$, $\sigma_S(x)_{\ell_2}$ can be placed in (7) in lieu of $\sigma_S(x)_{\ell_1}/\sqrt{S}$ (with different constants that are still independent of $x$) with high probability on the draw of $A$ if (i) $M > cS \log N$ and (ii) the entries $A$ is drawn i.i.d. from a Gaussian distribution or the columns of $A$ are drawn i.i.d. from the uniform distribution on the unit sphere in $\mathbb{R}^M$ [30]. In other words, the encoder $\Delta_1 = \Delta_1^0$ is “(2,2) instance optimal in probability” for encoders associated with such $A$, a property which was discussed in [10].

Following the notation of [30], we say that an encoder-decoder pair $(A, \Delta)$ is $(q, p)$ instance optimal of order $S$ with constant $C$ if

$$
\|\Delta(Ax) - x\|_q \leq C \frac{\sigma_S(x)_{\ell_p}}{S^{1/p-1/q}}
$$

holds for all $x \in \mathbb{R}^N$. Moreover, for random matrices $A_\omega$, $(A_\omega, \Delta)$ is said to be $(q, p)$ instance optimal in probability if for any $x$ (9) holds with high probability on the draw of $A_\omega$. Note that with this notation Theorem 1.1 implies that $(A, \Delta_1)$ is (2,1) instance optimal (set $\epsilon = 0$), provided $A$ satisfies the conditions of the theorem.

The preceding discussion makes it clear that $\Delta_1$ satisfies conditions (C1) and (C2), at least when $A$ is a sub-Gaussian random matrix and $S$ is sufficiently small. It only remains to note that decoding by $\Delta_1$ amounts to solving an $\ell_1$ minimization problem, and is thus tractable, i.e., we also have (C3). In fact, $\ell_1$ minimization problems as described above can be solved efficiently with solvers specifically designed for the sparse recovery scenarios (e.g. [27], [16], [11]).

1.2 Decoding by $\ell^p$ minimization

We have so far seen that with appropriate encoders, the decoders $\Delta_1^\epsilon$ provide robust and stable recovery for compressible signals even when the measurements are noisy [4], and that $(A_\omega, \Delta_1)$ is (2,2) instance optimal in probability [30] when $A_\omega$ is an appropriate random matrix. In particular, stability and robustness properties are conditioned on an appropriate RIP while the instance optimality property is dependent on the draw of the encoder matrix (which is typically called the measurement matrix) from an appropriate distribution, in addition to RIP.
Recall that the decoders $\Delta_1$ and $\Delta'_1$ were devised because their action can be computed by solving convex approximations to the combinatorial optimization problem (2) that is required to compute $\Delta_0$. The decoders defined by

$$
\Delta_p(b) := \arg\min_y \| y \|_p \text{ s.t. } Ay - b \leq \epsilon,
$$

and

$$
\Delta'_p(b) := \arg\min_y \| y \|_p \text{ s.t. } Ay = b,
$$

with $0 < p < 1$ are also approximations of $\Delta_0$, the actions of which are computed via non-convex optimization problems that can be solved, at least locally, still much faster than (2). It is natural to ask whether the decoders $\Delta_p$ and $\Delta'_p$ possess robustness, stability, and instance optimality properties similar to those of $\Delta_1$ and $\Delta'_1$, and whether these are obtained under weaker conditions on the measurement matrices than the analogous ones with $p = 1$.

Early work by Gribonval and co-authors [19–22] take some initial steps in answering these questions. In particular, they devise metrics that lead to sufficient conditions for uniqueness of $\Delta_1(b)$ to imply uniqueness of $\Delta_p(b)$ and specifically for having $\Delta_p(b) = \Delta_1(b) = x$. The authors also present stability conditions in terms of various norms that bound the error, and they conclude that the smaller the value of $p$ is, the more non-zero entries can be recovered by (11). These conditions, however, are hard to check explicitly and no class of deterministic or random matrices was shown to satisfy them at least with high probability. On the other hand, the authors provide lower bounds for their metrics in terms of generalized mutual coherence. Still, these conditions are pessimistic in the sense that they generally guarantee recovery of only very sparse vectors.

Recently, Chartrand showed that in the noise-free setting, a sufficiently sparse signal can be recovered perfectly with $\Delta_p$, where $0 < p < 1$, under less restrictive RIP requirements than those needed to guarantee perfect recovery with $\Delta_1$. The following theorem was proved in [7].

**Theorem 1.5** [7] Let $0 < p \leq 1$, and let $S \in \mathbb{N}$. Suppose that $x$ is $S$-sparse, and set $b = Ax$. If $A$ satisfies $\delta_{kS} + k^{\frac{2}{p}-1}k^{\frac{2}{p}-1} - 1$ for some $k > 1$ such that $k \in \frac{1}{S}\mathbb{N}$, then $\Delta_p(b) = x$.

Note that, for example, when $p = 0.5$ and $k = 3$, the above theorem only requires $\delta_{3S} + 27\delta_{4S} < 26$ to guarantee perfect recovery with $\Delta_{0.5}$, a less restrictive condition than the analogous one needed to guarantee perfect reconstruction with $\Delta_1$, i.e., $\delta_{3S} + 3\delta_{4S} < 2$. Moreover, in [8], Staneva and Chartrand study a modified RIP that is defined by replacing $\|Ac\|_2$ in (1) with $\|Ac\|_p$. They show that under this new definition of $\delta_S$, the same sufficient condition as in Theorem 1.5 guarantees perfect recovery. Steneva and Chartrand also show that if $A$ is an $M \times N$ Gaussian matrix, their sufficient condition is satisfied provided $M > C_1(p)S + pC_2(p)S \log(N/S)$, where $C_1(p)$ and $C_2(p)$ are
given explicitly in [8]. It is important to note is that \( pC_2(p) \) goes to zero as \( p \) goes to zero. In other words, the dependence on \( N \) of the required number of measurements \( M \) (that guarantees perfect recovery for all \( x \in \Sigma^N_S \)) disappears as \( p \) approaches 0. This result motivates a more detailed study to understand the properties of the decoders \( \Delta_p \) in terms of stability and robustness, which is the objective of this paper.

1.2.1 Algorithmic Issues

Clearly, recovery by \( \ell^p \) minimization poses a non-convex optimization problem with many local minimizers. It is encouraging that simulation results from recent papers, e.g., [7,25], strongly indicate that simple modifications to known approaches like iterated reweighted least squares algorithms and projected gradient algorithms yield \( x^* \) that are the global minimizers of the associated \( \ell^p \) minimization problem (or approximate the global optimizers very well). It is also encouraging to note that even though the results presented in this work and in others [7,19–22,25] assume that the global minimizer has been found, a significant set of these results, including all results in this paper, continue to hold if we could obtain a feasible point \( \tilde{x}^* \) which satisfies \( \| \tilde{x}^* \|_p \leq \| x \|_p \) (where \( x \) is the vector to be recovered). Nevertheless, it should be stated that to our knowledge, the modified algorithms mentioned above have only been shown to converge to local minima.

1.3 Paper Outline

In what follows, we present generalizations of the above results, giving stability and robustness guarantees for \( \ell^p \) minimization. In Section 2.1 we show that the decoders \( \Delta_p \) and \( \Delta_2^p \) are robust to noise and (2,p) instance optimal in the case of appropriate measurement matrices. For this section we rely and expand on our note [25]. In Section 2.3 we extend [30] and show that for the same range of dimensions as for decoding by \( \ell^1 \) minimization, i.e., when \( A_\omega \in \mathbb{R}^{M \times N} \) with \( M > cS \log(N) \), \((A_\omega, \Delta_p)\) is also (2,2) instance optimal in probability for \( 0 < p < 1 \), provided the measurement matrix \( A_\omega \) is drawn from an appropriate distribution. The generalization follows the proof of Wojtaszczyk in [30]; however it is non-trivial and requires a variant of a result by Gordon and Kalton [18] on the Banach-Mazur distance between a \( p \)-convex body and its convex hull. In Section 3 we present some numerical results, further illustrating the possible benefits of using \( \ell^p \) minimization and highlighting the behavior of the \( \Delta_p \) decoder in terms of stability and robustness. Finally, in Section 4 we present the proofs of the main theorems and corollaries.

While writing this paper, we became aware of the work of Foucart and Lai [17]
which also shows similar \((2, p)\) instance optimality results for \(0 < p < 1\) under different sufficient conditions. In essence, one could use the \((2, p)\)-results of Foucart and Lai to obtain \((2, 2)\) instance optimality in probability results similar to the ones we present in this paper, albeit with different constants. Since neither the sufficient conditions for \((2, p)\) instance optimality presented in [17] nor the ones in this paper are uniformly weaker, and since neither provide uniformly better constants, we simply use our estimates throughout.

2 Main Results

In this section, we present our theoretical results on the ability of \(\ell^p\) minimization to recover sparse and compressible signals in the presence of noise.

2.1 Sparse recovery with \(\Delta_p\): stability and robustness

We begin with a deterministic stability and robustness theorem for decoders \(\Delta_p\) and \(\Delta'_p\) when \(0 < p < 1\) that generalizes Theorem 1.1 of Candès et al. Note the associated sufficient conditions on the measurement matrix, given in (12) below, are weaker for smaller values of \(p\) than those that correspond to \(p = 1\). The results in this subsection were initially reported, in part, in [25].

In what follows, we say that a matrix \(A\) satisfies the property \(P(k, S, p)\) if it satisfies

\[
\delta_{kS} + k^{\frac{2}{p} - 1} \delta_{(k+1)S} < k^{\frac{2}{p} - 1} - 1, \tag{12}
\]

for \(S \in \mathbb{N}\) and \(k > 1\) such that \(k \in \frac{1}{S} \mathbb{N}\).

**Theorem 2.1 (General Case)** Let \(0 < p \leq 1\). Suppose that \(x\) is arbitrary and \(b = Ax + e\) where \(\|e\|_2 \leq \epsilon\). If \(A\) satisfies \(P(k, S, p)\), then

\[
\|\Delta'_p(b) - x\|_2^p \leq C_1 \epsilon^p + C_2 \sigma_S(x)^p_{\ell^p} \frac{\sigma_S(x)^p_{\ell^p}}{S^{1-p/2}}, \tag{13}
\]

where

\[
C_1 = 2^p \frac{1 + k^{p/2 - 1} (2/p - 1)^{-p/2}}{(1 - \delta_{(k+1)S})^{p/2} - (1 + \delta_{kS})^{p/2} k^{p/2 - 1}}, \quad \text{and} \quad \tag{14}
\]

\[
C_2 = 2 \left( \frac{2-p}{2-p} \right)^{p/2} \frac{k^{1-p/2}}{k^{1-p/2}} \left( 1 + \frac{(2/p - 1)\frac{2}{p} + k^{p/2 - 1}(1 + \delta_{kS})^{p/2}}{(1 - \delta_{(k+1)S})^{p/2} - (1 + \delta_{kS})^{p/2} k^{1-p/2}} \right). \tag{15}
\]

**Remark 2.2** By setting \(p = 1\) and \(k = 3\) in Theorem 2.1, we obtain Theorem 1.1 with precisely the same constants.
Remark 2.3 The constants in Theorem 2.1 are generally well behaved; e.g., $C_1 = 5.31$ and $C_2 = 4.31$ for $\delta_{4S} = 0.5$ and $p = 0.5$. Note for $\delta_{4S} = 0.5$ the sufficient condition (12) is not satisfied when $p = 1$, and thus Theorem 2.1 does not yield any upper bounds on $\|\Delta_1(b) - x\|_2$ in terms of $\sigma_S(x)_{1\ell}$.

Corollary 2.4 ((2, p) instance optimality) Let $0 < p \leq 1$. Suppose that $A$ satisfies $P(k, S, p)$. Then $(A, \Delta_p)$ is (2, p) instance optimal of order $S$ with constant $C_2^{1/p}$ where $C_2$ is as in (15).

Corollary 2.5 (sparse case) Let $0 < p \leq 1$. Suppose $x \in \Sigma^N_S$ and $b = Ax + e$ where $\|e\|_2 \leq \epsilon$. If $A$ satisfies $P(k, S, p)$, then

$$\|\Delta_p^\epsilon(b) - x\|_2 \leq (C_1)^{1/p} \epsilon,$$

where $C_1$ is as in (14).

Remark 2.6 Corollaries 2.4 and 2.5 follow from Theorem 2.1 by setting $\epsilon = 0$ and $\sigma_S(x)_{1\ell} = 0$, respectively. Furthermore, Corollary 2.5 can be proved independently of Theorem 2.1 leading to smaller constants. See [25] for the explicit values of these improved constants. Finally, note that setting $\epsilon = 0$ in Corollary 2.5, we obtain Theorem 1.5 as a corollary.

Remark 2.7 In [17], Foucart and Lai give different sufficient conditions for exact recovery than those we present. In particular, they show that if

$$\delta_{mS} < g(m) := \frac{4(\sqrt{2} - 1)(m/2)^{1/p-1/2}}{4(\sqrt{2} - 1)(m/2)^{1/p-1/2} + 2},$$

holds for some $m \geq 2, m \in \frac{1}{S} \mathbb{N}$, then $\Delta_p$ will recover signals in $\Sigma^N_S$ exactly. Note that the sufficient condition in this paper, i.e., (12), holds when

$$\delta_{mS} < f(m) := \frac{(m - 1)^{2/p-1} - 1}{(m - 1)^{2/p-1} + 1},$$

for some $m \geq 2, m \in \frac{1}{S} \mathbb{N}$. In Figure 1, we compare these different sufficient conditions as a function of $m$ for $p = 0.1, 0.5$, and 0.9 respectively. Figure 1 indicates that neither sufficient condition is weaker than the other for all values of $m$. In fact, we can deduce that (16) is weaker when $m$ is close to 2, while (17) is weaker when $m$ starts to grow larger. Since both conditions are only sufficient, if either one of them holds for an appropriate $m$, then $\Delta_p$ recovers all signals in $\Sigma^N_S$.

Remark 2.8 In [12], Davies and Gribonval showed that if one chooses $\delta_{2S} > \delta(p)$ (where $\delta(p)$ can be computed implicitly for $0 < p \leq 1$), then there exist matrices (matrices in $\mathbb{R}^{(N-1) \times N}$ that correspond to tight Parseval frames in $\mathbb{R}^{N-1}$) with the prescribed $\delta_{2S}$ for which $\Delta_p$ fails to recover signals in $\Sigma^N_S$.
Fig. 1. A comparison of the sufficient conditions on $\delta_{mS}$ in (17) and (16) as a function of $m$, for $p = 0.1$ (top), $p = 0.5$ (center) and $p = 0.9$ (bottom).

Note that this result does not contradict with the results that we present in this paper: we provide sufficient conditions (e.g., (12)) in terms of $\delta_{(k+1)S}$, where $k > 1$ and $kS \in \mathbb{N}$, that guarantee recovery by $\Delta_p$. These conditions are weaker than the corresponding conditions ensuring recovery by $\Delta_1$, which suggests that using $\Delta_p$ can be beneficial. Moreover, the numerical examples we provide in Section 3 indicate that by using $\Delta_p$, $0 < p < 1$, one can indeed recover signals in $\Sigma_2^N$, even when $\Delta_1$ fails to recover them (see Figure 2).

**Remark 2.9** In summary, Theorem 2.1 states that if (12) is satisfied then we can recover signals in $\Sigma_2^N$ stably by decoding with $\Delta_p$. It is worth mentioning that the sufficient conditions presented here reduce the gap between the conditions for exact recovery with $\Delta_0$ (i.e., $\delta_{2S} < 1$) and with $\Delta_1$, e.g., $\delta_{3S} < 1/3$. For example for $k = 2$ and $p = 0.5$, $\delta_{3S} < 7/9$ is sufficient. In the next subsection, we quantify this improvement.

### 2.2 The relationship between $S_1$ and $S_p$

Let $A$ be an $M \times N$ matrix and suppose $\delta_m$, $m \in \{1, \ldots, \lfloor M/2 \rfloor\}$ are its $m$-restricted isometry constants. Define $S_p$ for $A$ with $0 < p \leq 1$ as the largest
value of $S \in \mathbb{N}$ for which the slightly stronger version of (12) given by

$$\delta_{(k+1)S} < \frac{k^{\frac{p}{k}-1} - 1}{k^{\frac{p}{k}-1} + 1} \quad (18)$$

holds for some $k > 1$, $k \in \frac{1}{S}\mathbb{N}$. Consequently, by Theorem 2.1, $\Delta_p(Ax) = x$ for all $x \in \Sigma_{S_p}^N$. We now establish a relationship between $S_1$ and $S_p$.

**Proposition 2.10** Suppose, in the above described setting, there exists $S_1 \in \mathbb{N}$ and $k > 1$, $k \in \frac{1}{S_1}\mathbb{N}$ such that

$$\delta_{(k+1)S_1} < \frac{k - 1}{k + 1} \quad (19)$$

Then $\Delta_1$ recovers all $S_1$-sparse vectors, and $\Delta_p$ recovers all $S_p$ sparse vectors with

$$S_p = \left\lfloor \frac{k + 1}{k^{\frac{1}{p}-1} + 1} S_1 \right\rfloor.$$

**Remark 2.11** For example, if $\delta_{5S_1} < 3/5$ then using $\Delta_{\frac{5}{3}}$, we can recover all $S_{\frac{5}{3}}$-sparse vectors with $S_{\frac{5}{3}} = \left\lfloor \frac{5}{3}S_1 \right\rfloor$.

### 2.3 Instance optimality in probability and $\Delta_p$

In this section, we show that $(A_\omega, \Delta_p)$ is (2,2) instance optimal in probability when $A_\omega$ is an appropriate random matrix. Our approach is based on that of [30], which we summarize now. A matrix $A$ is said to possess the $LQ_1(\alpha)$ property if and only if

$$A(B_1^N) \supset \alpha B_2^M,$$

where $B_1^n$ denotes the $\ell^1$ unit ball in $\mathbb{R}^n$. In [30], Wojtaszczyk shows that random Gaussian matrices of size $M \times N$ as well as matrices whose columns are drawn uniformly from the sphere possess, with high probability, the $LQ_1(\alpha)$ property with $\alpha = \mu \sqrt{\log(N/M)}$. Noting that such matrices also satisfy RIP($((k+1)S, \delta)$ with $S < c \frac{M}{\log(N/M)}$, again with high probability, Wojtaszczyk proves that $\Delta_1$, for these matrices, is (2,2) instance optimal in probability of order $S$. Our strategy for generalizing this result to $\Delta_p$ with $0 < p < 1$ relies on a generalization of the $LQ_1$ property to an $LQ_p$ property. Specifically, we say that a matrix $A$ satisfies $LQ_p(\alpha)$ if and only if

$$A(B_p^N) \supset \alpha B_2^M.$$
We first show that a random matrix $A_\omega$, either Gaussian or uniform as mentioned above, satisfies the $LQ_p(\alpha)$ property with

$$\alpha = \frac{1}{C(p)} \left( \mu^2 \log \left( \frac{N/M}{\xi} \right) \right)^{(1/p-1/2)}.$$

Once we establish this property, the proof of instance optimality in probability for $\Delta_p$ proceeds largely unchanged from Wojtaszczyk’s proof with modifications to account only for the non-convexity of the $\ell^p$-quasinorm with $0 < p < 1$.

Next, we present our results on instance optimality of the $\Delta_p$ decoder, while deferring the proofs to Section 4. Throughout the rest of the paper, we focus on two classes of random matrices: $A_\omega$ denotes $M \times N$ matrices, the entries of which are drawn from a zero mean, normalized column-variance Gaussian distribution, i.e., $A_\omega = (a_{i,j})$ where $a_{i,j} \sim \mathcal{N}(0, 1/\sqrt{M})$; in this case, we say that $A_\omega$ is an $M \times N$ Gaussian random matrix. $\tilde{A}_\omega$, on the other hand, denotes $M \times N$ matrices, the columns of which are drawn uniformly from the sphere; in this case we say that $\tilde{A}_\omega$ is an $M \times N$ uniform random matrix. In each case, $(\Omega, P)$ denotes the associated probability space.

We start with a lemma (which generalizes an analogous result of [30]) that shows that the matrices $A_\omega$ and $\tilde{A}_\omega$ satisfy the $LQ_p(\alpha)$, $\alpha = 1/C(p)$, with probability $\geq 1 - e^{-cM}$, with $C(p)$ is a positive constant that depends only on $p$. (In particular, $C(1) = 1$ and see (30) for the explicit value of $C(p)$ when $0 < p < 1$). This statement is true also for $\tilde{A}_\omega$.

**Lemma 2.12** Let $0 < p \leq 1$, and let $A_\omega$ be an $M \times N$ Gaussian random matrix. For $0 < \mu < 1/\sqrt{2}$, suppose that $K_1 M (\log M)^{\xi} \leq N \leq e^{K_2 M}$ for some $\xi > (1 - 2\mu^2)^{-1}$ and some constants $K_1, K_2 > 0$. Then, there exists a constant $c = c(\mu, \xi, K_1, K_2) > 0$, independent of $p$, $M$, and $N$, and a set

$$\Omega_\mu = \left\{ \omega \in \Omega : A_\omega(B^N_p) \supset \frac{1}{C(p)} \left( \mu^2 \log \left( \frac{N/M}{\xi} \right) \right)^{1/p-1/2} B^M_2 \right\}$$

such that $P(\Omega_\mu) \geq 1 - e^{-cM}$.

In other words, $A_\omega$ satisfies the $LQ_p(\alpha)$, $\alpha = 1/C(p)$, with probability $\geq 1 - e^{-cM}$ on the draw of the matrix. Here $C(p)$ is a positive constant that depends only on $p$. (In particular, $C(1) = 1$ and see (30) for the explicit value of $C(p)$ when $0 < p < 1$). This statement is true also for $\tilde{A}_\omega$.

The above lemma for $p = 1$ can be found in [30]. As we will see in Section 4, the generalization of this result to $0 < p < 1$ is non-trivial and requires a result from [18], cf. [23], relating certain “distances” of $p$-convex bodies to their convex hulls. It is important to note that this lemma provides the machinery needed to prove the following theorem, which extends to $\Delta_p$, $0 < p < 1$, the analogous result of Wojtaszczyk [30] for $\Delta_1$. 

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In what follows, for a set $T \subseteq \{1, \ldots, N\}$, $T^c := \{1, \ldots, N\} \setminus T$; for $y \in \mathbb{R}^N$, $y_T$ denotes the vector with entries $y_T(j) = y(j)$ for all $j \in T$, and $y_T(j) = 0$ for $j \in T^c$.

**Theorem 2.13** Let $0 < p < 1$. Suppose that $A \in \mathbb{R}^{M \times N}$ satisfies RIP($S, \delta$) and $\text{LO}_p(\frac{1}{c_1(p)}(\mu^2/S)^{1/p-1/2})$ for some $\mu > 0$ and $C(p)$ as in (50). Let $\Delta$ be an arbitrary decoder. If $(A, \Delta)$ is $(\omega, \mu)$ instance optimal of order $S$ with constant $C_{2,\rho}$, then for any $x \in \mathbb{R}^N$ and $e \in \mathbb{R}^M$, all of the following hold.

1. $\|\Delta(Ax + e) - x\|_2 \leq C(\|e\|_2 + \frac{\sigma_S(x)e_2}{S^{1/p-1/2}})$
2. $\|\Delta(Ax) - x\|_2 \leq C(\|Ax\|_2 \| + \sigma_S(x)e_2)$
3. $\|\Delta(Ax + e) - x\|_2 \leq C(\|e\|_2 + \sigma_S(x)e_2 + \|Ax\|_2)$

Above, $T_0$ denotes the set of indices of the largest (in magnitude) $S$ coefficients of $x$; the constants (all denoted by $C$) depend on $\delta$, $\mu$, $p$, and $C_{2, \rho}$, but not on $M$ and $N$. For the explicit values of these constants see (35) and (39).

Finally, our main theorem on the instance optimality in probability of the $\Delta_p$ decoder follows.

**Theorem 2.14** Let $0 < p < 1$, and let $A_\omega$ be an $M \times N$ Gaussian random matrix. Suppose that $N \geq M[\log(M)]^2$. There exist constants $c_1, c_2, c_3 > 0$ such that for all $S \in \mathbb{N}$ with $S \leq c_1 M / \log(N/M)$, the following are true.

1. There exists $\Omega_1$ with $P(\Omega_1) \geq 1 - 3e^{-c_2 M}$ such that for all $\omega \in \Omega_1$

   $$\|\Delta_p(A_\omega(x) + e) - x\|_2 \leq C(\|e\|_2 + \frac{\sigma_S(x)e_2}{S^{1/p-1/2}}),$$

   for any $x \in \mathbb{R}^N$ and for any $e \in \mathbb{R}^M$.

2. For any $x \in \mathbb{R}^N$, there exists $\Omega_x$ with $P(\Omega_x) \geq 1 - 4e^{-c_3 M}$ such that for all $\omega \in \Omega_x$

   $$\|\Delta_p(A_\omega(x) + e) - x\|_2 \leq C(\|e\|_2 + \sigma_S(x)e_2),$$

   for any $e \in \mathbb{R}^M$.

The statement also holds for $\tilde{A}_\omega$, i.e., for random matrices the columns of which are drawn independently from a uniform distribution on the sphere.

**Remark 2.15** The constants above (both denoted by $C$) depend on the parameters of the particular $\text{LO}_p$ and RIP properties that the matrix satisfies, and are given explicitly in Section 4 see (38) and (41). The constants $c_1, c_2$, and $c_3$ depend only on $p$ and the distribution of the underlying random matrix (see the proof in Section 4.3) and are independent of $M$ and $N$. 

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Remark 2.16 Clearly, the statements do not make sense if the hypothesis of the theorem forces $S$ to be 0. In turn, for a given $(M, N)$ pair, it is possible that there is no positive integer $S$ for which the conclusions of Theorem 2.14 hold. In particular, to get a non-trivial statement, one needs $M > \frac{1}{c_1} \log(N/M)$.

Remark 2.17 Note the difference in the order of the quantifiers between conclusions (i) and (ii) of Theorem 2.14. Specifically, with statement (i), once the matrix is drawn from the “good” set $\Omega_1$, we obtain the error guarantee (20) for every $x$ and $e$. In other words, after the initial draw of a good matrix $A$, stability and robustness in the sense of (20) are ensured. On the other hand, statement (ii) concludes that associated with every $x$ is a “good” set $\Omega_x$ (possibly different for different $x$) such that if the matrix is drawn from $\Omega_x$, then stability and robustness in the sense of (21) are guaranteed. Thus, in (ii), for every $x$, a different matrix is drawn, and with high probability on that draw (21) holds.

Remark 2.18 The above theorem pertains to the decoders $\Delta_p$ which, like the analogous theorem for $\Delta_1$ presented in [30], requires no knowledge of the noise level. In other words, $\Delta_p$ provides estimates of sparse and compressible signals from limited and noisy observations without having to explicitly account for the noise in the decoding. This provides an improvement on Theorem 2.1 and a practical advantage when estimates of measurement noise levels are absent.

3 Numerical Experiments

In this section, we present some numerical experiments to highlight important aspects of sparse reconstruction by decoding using $\Delta_p$, $0 < p \leq 1$. First, we compare the sufficient conditions under which decoding with $\Delta_p$ guarantees perfect recovery of signals in $\Sigma_N^S$ for different values of $p$ and $S$. Next, we present numerical results illustrating the robustness and instance optimality of the $\Delta_p$ decoder. Here, we wish to observe the linear growth of the $\ell^2$ reconstruction error $\|\Delta_p(Ax + e) - x\|_2$, as a function of $\sigma_S(x)_{\ell^2}$ and of $\|e\|_2$.

To that end, we generate a $100 \times 300$ matrix $A$ whose columns are drawn from a Gaussian distribution and we estimate its RIP constants $\delta_S$ via Monte Carlo (MC) simulations. Under the assumption that the estimated constants are the correct ones (while in fact they are only lower bounds), Figure 2 (left) shows the regions where (12) guarantees recovery for different $(S, p)$-pairs. On the other hand, Figure 2 (right) shows the empirical recovery rates via $p$ quasinorm minimization: To obtain this figure, for every $S = 1, \ldots, 49$, we chose 50 different instances of $x \in \Sigma_N^{300}$ where non-zero coefficients of each were drawn i.i.d. from the standard Gaussian distribution. These vectors were encoded using the same measurement matrix $A$ as above. Since there is
no known algorithm that will yield the global minimizer of the optimization problem (11), we approximated the action of $\Delta_p$ by using a projected gradient algorithm on a sequence of smoothed versions of the $\ell^p$ minimization problem:

In (11), instead of minimizing the $\|y\|_p$, we minimized $\left(\sum_i (y_i^2 + \epsilon^2)^{p/2}\right)^{1/p}$ initially with a large $\epsilon$. We then used the corresponding solution as the starting point of the next subproblem obtained by decreasing the value of $\epsilon$ according to the rule $\epsilon_n = (0.99)\epsilon_{n-1}$. We continued reducing the value of $\epsilon$ and solving the corresponding subproblem until $\epsilon$ becomes very small. Note that this approach is similar to the one described in [7]. The empirical results show that $\Delta_p$ (in fact, the approximation of $\Delta_p$ as described above) is successful in a wider range of scenarios than those predicted by Theorem 2.1. This can be attributed to the fact that the conditions presented in this paper are only sufficient, or to the fact that in practice what is observed is not necessarily a manifestation of uniform recovery. Rather, the practical results could be interpreted as success of $\Delta_p$ with high probability on either $x$ or $A$.

Fig. 2. For a Gaussian matrix $A \in \mathbb{R}^{100 \times 300}$, whose $\delta_S$ values are estimated via MC simulations, we generate the theoretical (left) and practical (right) phase-diagrams for reconstruction via $\ell^p$ minimization.

Next, we generate scenarios that allude to the conclusions of Theorem 2.14. To that end, we generate a signal composed of $x_T \in \Sigma_{40}^{300}$, supported on an index set $T$, and a signal $z_{T^c}$ supported on $T^c$, where all the coefficients are drawn from the standard Gaussian distribution. We then normalize $x_T$ and $z_{T^c}$ so that $\|x_T\|_2 = \|z_{T^c}\|_2 = 1$ and generate $x = x_T + \lambda z_{T^c}$ with increasing values of $\lambda$ (starting from 0), thereby increasing $\sigma_{40}(x)\epsilon^2 \approx \lambda$. For this experiment, we choose our measurement matrix $A \in \mathbb{R}^{100 \times 300}$ by drawing its columns uniformly from the sphere. For each value of $\lambda$ we measure the reconstruction error $\|\Delta_p(Ax) - x\|_2$, and we repeat the process 10 times while randomizing the index set $T$ but preserving the coefficient values. We report the averaged results in Figure 3 (left) for different values of $p$. Similarly, we generate noisy observations $Ax_T + \lambda e$, of a sparse signal $x_T \in \Sigma_{40}^{300}$ where $\|x_T\|_2 = \|e\|_2 = 1$.
Fig. 3. Reconstruction error with compressible signals (left), noisy observations (right). Observe the almost linear growth of the error in compressible signals and for different values of $p$, highlighting the instance optimality of the decoders. The plots were generated by averaging the results of 10 experiments with the same matrix $A$ and randomized locations of the coefficients of $x$.

and we increase the noise level starting from $\lambda = 0$. Here, again, the non-zero entries of $x_T$ and all entries of $e$ were chosen i.i.d. from the standard Gaussian distribution and then the vectors were properly normalized. Next, we measure $\|\Delta_p(Ax_T + \lambda e) - x_T\|_2$ (for 10 realizations where we randomize $T$) and report the averaged results in Figure 3 (right) for different values of $p$. In both these experiments, we observe that the error increases roughly linearly as we increase $\lambda$, i.e., $\sigma_{40}(x)_2$ and the noise power, respectively. Moreover, when the signal is highly compressible or when the noise level is low, we observe that reconstruction using $\Delta_p$ with $0 < p < 1$ yields a lower approximation error than that with $p = 1$. It is also worth noting that for values of $p$ close to one, even in the case of sparse signals with no noise, the average reconstruction error is non-zero. This may be due to the fact that for such large $p$ the number of measurements is not sufficient for the recovery of signals with $S = 40$, further highlighting the benefits of using the decoder $\Delta_p$, with smaller values of $p$.

Finally, in Figure 4, we plot the results of an experiment in which we generate signals $x \in \mathbb{R}^{200}$ with sorted coefficients $x(j)$ that decay according to some power law. In particular, for various values of $0 < q < 1$, we set $x(j) = cj^{-1/q}$ such that $\|x\|_2 = 1$. We then encode $x$ with 50 different $100 \times 200$ measurement matrices the columns of which were drawn from the uniform distribution on the sphere, and examine the approximations obtained by decoding with $\Delta_p$ for different values of $0 < p < 1$. The results indicate that values of $p \approx q$ provide the lowest reconstruction errors. Note that in Figure 4 we report the results in form of signal to noise ratios defined as

$$ SNR = 20 \log_{10} \left( \frac{\|x\|_2}{\|\Delta(Ax) - x\|_2} \right). $$
Fig. 4. Reconstruction signal to noise ratios (in dB) obtained by using $\Delta_p$ to recover signals whose sorted coefficients decay according to a power law $(x(j) = c j^{-1/q}, \|x\|_2 = 1)$ as a function of $q$ (left) and as a function of $p$ (right). The presented results are averages of 50 experiments performed with different matrices in $\mathbb{R}^{100 \times 200}$. Observe that for highly compressible signals, e.g., for $q = 0.4$, there is a 5 dB gain in using $p < 0.6$ as compared to $p = 1$. The performance advantage is about 2 dB for $q = 0.6$. As the signals become much less compressible, i.e., as we increase $q$ to 0.9 the performances are almost identical.

4 Proofs

4.1 Proof of Proposition 2.10

First, note that for any $A \in \mathbb{R}^{M \times N}$, $\delta_m$ is non-decreasing in $m$. Also, the map $k \mapsto \frac{k-1}{k+1}$ is increasing in $k$ for $k \geq 0$.

Set

$$L := (k+1)S_1, \quad \bar{\ell} = k^{\frac{p}{2-p}}, \quad \text{and} \quad \bar{S}_p = \frac{L}{\bar{\ell} + 1}. $$

Then

$$\delta_{(\bar{\ell}+1)\bar{S}_p} = \delta_{(k+1)S_1} < \frac{k-1}{k+1} + \frac{\bar{\ell}^{2-p}}{\bar{\ell} + 1}. $$

We now describe how to choose $\ell$ and $S_p$ such that $\ell \geq \bar{\ell}$, $S_p \in \mathbb{N}$, and $(\ell+1)S_p = L$ (this will be sufficient to complete the proof using the monotonicity observations above). First, note that this last equality is satisfied only if $(\ell, S_p)$ is in the set

$$\{ \left( \frac{n}{L-n}, L-n \right) : n = 1, \ldots, L-1 \}. $$

Let $n^*$ be such that

$$\frac{n^* - 1}{L - n^* + 1} \leq \frac{n^*}{L - n^*}. \quad (22)$$

To see that such an $n^*$ exists, recall that $\bar{\ell} = k^{\frac{p}{2-p}}$ where $0 < p < 1$. Also,
(k + 1)S_1 = L with $S_1 \in \mathbb{N}$, and $k > 1$. Consequently, $1 < \bar{\ell} < k \leq L - 1$, and $k \in \{ \frac{n}{L-n} : n = \left\lceil \frac{L}{2} \right\rceil , \ldots , L - 1 \}$. Thus, we know that we can find $n^*$ as above. Furthermore, $\frac{n^*}{L-n^*} > 1$. It follows from (22) that

$$L - n^* \leq \bar{S}_p < L - n^* + 1.$$  

We now choose

$$\ell = \frac{n^*}{L-n^*}, \quad \text{and} \quad S_p = \lfloor \bar{S}_p \rfloor = L - n^*.$$  

Then $(\ell + 1)S_p = L$, and $\ell \geq \bar{\ell}$. So, we conclude that for $\ell$ as above and

$$S_p = \lfloor \bar{S}_p \rfloor = \left\lfloor \frac{k + 1}{k^{2-p} + 1}S_1 \right\rfloor,$$

we have

$$\delta_{(\ell+1)S_p} \leq \frac{\bar{S}_p^{2-p} - 1}{\bar{S}_p^{2-p} + 1}.$$  

Consequently, the condition of Corollary 2.5 is satisfied and we have the desired conclusion.  

4.2 Proof of Theorem 2.1

We modify the proof of Candès et al. of the analogous result for the encoder $\Delta_1$ (Theorem 2 in [4]) to account for the non-convexity of the $\ell^p$ quasinorm. We give the full proof for completeness. We stick to the notation of [4] whenever possible.

Let $0 < p < 1$, $x \in \mathbb{R}^N$ be arbitrary, and define $x^* := \Delta^*_p(b)$ and $h := x^* - x$. Our goal is to obtain an upper bound on $\|h\|_2$ given that $\|Ah\|_2 \leq 2\epsilon$ (by definition of $\Delta^*_p$).

Below, for a set $T \subseteq \{1, \ldots , N\}$, $T^c := \{1, \ldots , N\} \setminus T$; for $y \in \mathbb{R}^N$, $y_T$ denotes the vector with entries $y_T(j) = y(j)$ for all $j \in T$, and $y_T(j) = 0$ for $j \in T^c$.

(1) We start by decomposing $h$ as a sum of sparse vectors with disjoint support. In particular, denote by $T_0$ the set of indices of the largest (in magnitude) $S$ coefficients of $x$ (here $S$ is to be determined later). Next, partition $T_0$ into sets $T_1, T_2, \ldots , |T_j| = L$ for $j \geq 1$ where $L \in \mathbb{N}$ (also to be determined later), such that $T_1$ is the set of indices of the $L$ largest (in magnitude) coefficients of $h_{T_0}$, $T_2$ is the set of indices of the second $L$ largest coefficients of $h_{T_0}$, and so on. Finally let $T_0 := T_0 \cup T_1$. We now obtain a lower bound for $\|Ah\|_2^p$ using the RIP constants of the matrix $A$. In particular, we have
\[ \|Ah\|_2^p = \|Ah_{T_0}\|_2^p + \sum_{j \geq 2} \|Ah_{T_j}\|_2^p \geq \|Ah_{T_0}\|_2^p - \sum_{j \geq 2} \|Ah_{T_j}\|_2^p \geq \left(1 - \delta_{L+|T_0|}\right)^{p/2}\|h_{T_0}\|_2^p - (1 + \delta_{L})^{p/2}\sum_{j \geq 2} \|h_{T_j}\|_2^p. \]  

(23)

Above, together with RIP, we used the fact that \(\|\cdot\|_2^p\) satisfies the triangle inequality for any \(0 < p < 1\). What now remains is to relate \(\|h_{T_0}\|_2^p\) and \(\sum_{j \geq 2} \|h_{T_j}\|_2^p\) to \(\|h\|_2\).

(II) Next, we aim to bound \(\sum_{j \geq 2} \|h_{T_j}\|_2^p\) from above in terms of \(\|h\|_2\). To that end, we proceed as in [4]. First, note that \(\|h_{T_{j+1}}(\ell)|^p \leq |h_{T_j}(\ell')|^p\) for all \(\ell \in T_{j+1}, \ell' \in T_j\), and thus \(\|h_{T_{j+1}}(\ell)|^p \leq \|h_{T_j}\|_p/L\). It follows that \(\|h_{T_{j+1}}\|_2^p \leq L^{1-\frac{2}{p}}\|h_{T_j}\|_2^p\), and consequently

\[ \|h_{T_j}\|_2^p \leq L^\frac{2}{p-1}\sum_{j \geq 1} \|h_{T_j}\|_p^p = L^\frac{2}{p-1}\|h_{T_0}\|_p^p. \]  

(24)

Next, note that, similar to the case when \(p = 1\) as shown in [4], the “error” \(h\) is concentrated on the “essential support” of \(x\) (in our case \(T_0\)). To quantify this claim, we repeat the analogous calculation in [4]: Note, first, that by definition of \(x^*\),

\[ \|x^*\|_p^p = \|x + h\|_p^p = \|x_{T_0} + h_{T_0}\|_p^p + \|x_{T_0} + h_{T_0}\|_p^p \leq \|x\|_p^p. \]

As \(\|\cdot\|_p^p\) satisfies the triangle inequality, we then have

\[ \|x_{T_0}\|_p^p - \|h_{T_0}\|_p^p + \|h_{T_0}\|_p^p - \|x_{T_0}\|_p^p \leq \|x\|_p^p. \]

Consequently,\n
\[ \|h_{T_0}\|_p^p \leq \|h_{T_0}\|_p^p + 2\|x_{T_0}\|_p^p. \]  

(25)

which, together with \((24)\), implies

\[ \sum_{j \geq 2} \|h_{T_j}\|_2^p \leq L^\frac{2}{p-1}\left(\|h_{T_0}\|_p^p + 2\|x_{T_0}\|_p^p\right) \leq \rho^{1-\frac{2}{p}}(\|h_{T_0}\|_p^p + 2\|T_0\|_p^p)^\frac{2}{p-1}\|x_{T_0}\|_p^p, \]  

(26)

where \(\rho := |T_0|/L\), and we used the fact that \(\|h_{T_0}\|_p^p \leq |T_0|^{1-\frac{2}{p}}\|h_{T_0}\|_2^p\) (which follows as \(\|\text{supp}(h_{T_0})\| = |T_0|\)). Using \((26)\) and \((23)\), we obtain

\[ \|Ah\|_2^p \geq C_{p,L,|T_0|}\|h_{T_0}\|_2^p - 2\rho^{1-\frac{2}{p}}|T_0|^{\frac{2}{p-1}}(1 + \delta_L)^{\frac{2}{p}}\|x_{T_0}\|_p^p, \]  

(27)

where

\[ C_{p,L,|T_0|} := (1 - \delta_{L+|T_0|})^{\frac{2}{p}} - (1 + \delta_L)^{\frac{2}{p}}\rho^{1-\frac{2}{p}}. \]  

(28)

At this point, using \(\|Ah\|_2 \leq 2\epsilon\), we obtain an upper bound on \(\|h_{T_0}\|_2\) given by

\[ \|h_{T_0}\|_2^p \leq \frac{1}{C_{p,L,|T_0|}} \left(2(2\epsilon)^p + 2\rho^{1-\frac{2}{p}}(1 + \delta_L)^{\frac{2}{p}}\|x_{T_0}\|_p^p/|T_0|^{1-\frac{2}{p}}\right), \]  

(29)
provided $C_{p,L,|T_0|} > 0$ (this will impose the condition given in (12) on the RIP constants of the underlying matrix $A$).

(III) To complete the proof, we will show that the error vector $h$ is concentrated on $T_{01}$. Denote by $h_{T_0}^c[m]$ the $m$th largest (in magnitude) coefficient of $h_{T_0}$ and observe that $|h_{T_0}^c[m]|^p \leq \|h_{T_0}^c\|^p/m$. As $h_{T_0}^c[m] = h_{T_0}^c[L + m]$, we then have

$$\|h_{T_0}^c\|^2 = \sum_{m \geq L+1} |h_{T_0}^c[m]|^2 \leq \sum_{m \geq L+1} \left( \frac{\|h_{T_0}^c\|^p}{m} \right)^{\frac{2}{p}} \leq \frac{\|h_{T_0}^c\|^2}{L^{\frac{2}{p}-1}(2/p - 1)}. \tag{30}$$

Here, the last inequality follows because for $0 < p < 1$

$$\sum_{m \geq L+1} m^{-\frac{2}{p}} \leq \int_L^\infty t^{-\frac{2}{p}} dt = \frac{1}{L^{\frac{2}{p}-1}(2/p - 1)}.$$

Finally, we use (29) and (30) to conclude

$$\|h\|^2 = \|h_{T_0}\|^2 + \|h_{T_0}^c\|^2 \leq \|h_{T_0}\|^2 + \left[ \frac{\|h_{T_0}\|^p + 2\|x_{T_0}^c\|^p}{L^{1-\frac{2}{p}(2/p - 1)^{-\frac{2}{p}}}} \right] \leq \left[ \left(1 + \rho^{1-\frac{2}{p}(2/p - 1)^{-\frac{2}{p}}} \right)\|h_{T_0}\|^2 + 2\rho^{1-\frac{2}{p}(2/p - 1)^{-\frac{2}{p}}} \|x_{T_0}^c\|^p \right] \leq \left(1 + \frac{1}{\rho} \right)\|h_{T_0}\|^2 \leq \left(1 + \frac{1}{\rho} \right)\|h\|_2 \tag{31}$$

Above, we used the fact that $\|h_{T_0}\|^p \leq |T_0|^{-\frac{2}{p}}\|h_{T_0}\|_2^p$, and that for any $a, b \geq 0$, and $\alpha \geq 1$, $a^\alpha + b^\alpha \leq (a+b)^\alpha$.

(IV) We now set $|T_0| = S$, $L = kS$ where $k$ and $S$ are chosen such that $C_{p,kS,S} > 0$ which is equivalent to having $k$, $S$, and $p$ satisfy (12). In this case, $\|x_{T_0}^c\|^p = \sigma_S(x)^p$, $\rho = 1/k$, and combining (29) and (31) yields

$$\|h\|^2 \leq C_1\rho + C_2\frac{\sigma_S(x)^p}{S^{1-\frac{2}{p}}} \tag{32}$$

where $C_1$ and $C_2$ are as in (14) and (15), respectively. \qed

4.3 Proof of Lemma 2.12

(1) The following result of Wojtaszczyk [30, Proposition 2.2] will be useful.

**Proposition 4.1** ([30]) Let $A_\omega$ be an $M \times N$ Gaussian random matrix, let $0 < \mu < 1/\sqrt{2}$, and suppose that $K_1 M (\log M) \xi \leq N \leq e^{CM}$ for some $\xi > (1 - 2\mu^2)^{-1}$ and some constants $K_1, K_2 > 0$. Then, there exists a constant
\[ c = c(\mu, \xi, K_1, K_2) > 0, \text{ independent of } M \text{ and } N, \text{ and a set} \]
\[ \Omega_\mu = \left\{ \omega : A_\omega(B_1^N) \supseteq \mu \sqrt{\frac{\log N/M}{M}} B_2^M \right\} \]

such that
\[ P(\Omega_\mu) \geq 1 - e^{-cM}. \]

The above statement is true also for \( \tilde{A}_\omega \).

We will also use the following adaptation of [18, Lemma 2] for which we will first introduce some notation. Define a body to be a compact set containing the origin as an interior point and star shaped with respect to the origin [23]. Below, we use \( \text{conv}(K) \) to denote the convex-hull of a body \( K \). For \( K \subseteq B \), we denote by \( d_1(K, B) \) the “distance” between \( K \) and \( B \) given by
\[ d_1(K, B) := \inf\{\lambda > 0 : K \subseteq B \subseteq \lambda K\} = \inf\{\lambda > 0 : \frac{1}{\lambda} B \subseteq K \subseteq B\}. \]

Finally, we call a body \( K \) \( p \)-convex if for any \( x, y \in K \), \( \lambda x + \mu y \in K \) whenever \( \lambda, \mu \in [0, 1] \) such that \( \lambda^p + \mu^p = 1 \).

**Lemma 4.2** Let \( 0 < p < 1 \), and let \( K \) be a \( p \)-convex body in \( \mathbb{R}^n \). If \( \text{conv}(K) \subseteq B_2^n \), then
\[ d_1(K, B_2^n) \leq C(p)d_1(\text{conv}(K), B_2^n)^{(2/p - 1)}, \]
where
\[ C(p) = \left(2^{1-p} + \frac{(1-p)2^{1-p/2}}{p}\right)^{\frac{2-p}{p}} \left(\frac{1}{(1-p)\log 2}\right)^{\frac{2-2p}{p}}. \]

We defer the proof of this lemma to the Appendix.

(II) Note that \( \tilde{A}_\omega(B_1^N) \subseteq B_2^M \). This follows because \( \|\tilde{A}_\omega\|_{1 \to 2} \), which is equal to the largest column norm of \( \tilde{A}_\omega \), is 1 by construction. Thus, for \( x \in B_1^N \),
\[ \|\tilde{A}_\omega(x)\|_2 \leq \|\tilde{A}_\omega\|_{1 \to 2}\|x\|_1 \leq 1, \]
that is, \( \tilde{A}_\omega(B_1^N) \subseteq B_2^M \), and so \( d_1(\tilde{A}_\omega(B_1^N), B_2^M) \) is well-defined. Next, by Proposition 4.1, we know that there exists \( \Omega_\mu \) with \( P(\Omega_\mu) \geq 1 - e^{-cM} \) such that for all \( \omega \in \Omega_\mu \),
\[ \tilde{A}_\omega(B_1^N) \supseteq \mu \sqrt{\frac{\log N/M}{M}} B_2^M \quad (33) \]
From this point on, let \( \omega \in \Omega_\mu \). Then
\[ B_2^M \supseteq \tilde{A}_\omega(B_1^N) \supseteq \mu \sqrt{\frac{\log N/M}{M}} B_2^M, \]
and consequently
\[ d_1(\tilde{A}_\omega(B_1^N), B_2^M) \leq \left(\mu \sqrt{\frac{\log N/M}{M}}\right)^{-1}. \] (34)

The next step is to note that \( \text{conv}(B_p^N) = B_1^N \) and consequently
\[ \text{conv} \left( \tilde{A}_\omega(B_p^N) \right) = \tilde{A}_\omega \left( \text{conv}(B_p^N) \right) = \tilde{A}_\omega(B_1^N). \]

We can now invoke Lemma 4.2 to conclude that
\[ d_1(\tilde{A}_\omega(B_p^N), B_2^M) \leq C(p) \left( \mu \sqrt{\frac{\log N/M}{M}} \right)^{-1/p} \]
\[ = C(p) d_1(\tilde{A}_\omega(B_1^N), B_2^M)^{-1/p}. \] (35)

Finally, by using (34), we find that
\[ d_1(\tilde{A}_\omega(B_p^N), B_2^M) \leq C(p) \left( \mu^2 \frac{\log N/M}{M} \right)^{1-1/p}, \] (36)
and consequently
\[ \tilde{A}_\omega(B_p^N) \supset \frac{1}{C(p)} \left( \mu^2 \frac{\log N/M}{M} \right)^{(1/p)-1/2} B_2^M. \] (37)

In other words, the matrix \( \tilde{A}_\omega \) has the LQ\(_p\)(\(\alpha\)) property with the desired value of \( \alpha \) for every \( \omega \in \Omega_\mu \) with \( P(\Omega_\mu) \geq 1 - e^{-cM} \). Here \( c \) is as specified in Proposition 4.1.

To see that the same is true for \( A_\omega \), note that there exists a set \( \Omega_0 \) with \( P(\Omega_0) > 1 - e^{-cM} \) such that for all \( \omega \in \Omega_0 \), \( \|A_j(\omega)\|_2 < 2 \), for every column \( A_j \) of \( A_\omega \) (this follows from RIP). Using this observation one can trace the above proof with minor modifications. \( \square \)

### 4.4 Proof of Theorem 2.13

We start with the following lemma, the proof of which for \( p < 1 \) follows with very little modification from the analogous proof of Lemma 3.1 in [30] and shall be omitted.

**Lemma 4.3** Let \( 0 < p < 1 \) and suppose that \( A \) satisfies RIP\((S, \delta)\) and LQ\(_p\)\( \left( \gamma_p/S^{1/p-1/2} \right) \) with \( \gamma_p := \mu^{2/p-1}/C(p) \). Then for every \( x \in \mathbb{R}^N \), there
exists $\tilde{x} \in \mathbb{R}^N$ such that

$$Ax = A\tilde{x}, \quad \|\tilde{x}\|_p \leq \frac{S^{1/p - 1/2}}{\gamma_p} \|Ax\|_2,$$

and $\|\tilde{x}\|_2 \leq C_3 \|Ax\|_2$.

Here, $C_3 = \frac{1}{\gamma_p} + \frac{\gamma_p(1-\delta)+1}{(1-\delta^2)\gamma_p}$. Note that $C_3$ depends only on $\mu$, $\delta$ and $p$.

We now proceed to prove Theorem 2.13. Our proof follows the steps of [30] and differs in the handling of the non-convexity of the $\ell^p$ quasinorms for $0 < p < 1$.

First, recall that $A$ satisfies RIP$(S, \delta)$ and LQ$_p(\gamma_p/S^{1/p-1/2})$, so by Lemma 4.3, there exists $z \in \mathbb{R}^N$ such that $Az = e$, $\|z\|_p \leq \frac{S^{1/p - 1/2}}{\gamma_p} \|e\|_2$, and $\|z\|_2 \leq C_3 \|e\|_2$. Now, $A(x + z) = Ax + e$, and $\Delta$ is $(2, p)$ instance optimal with constant $C_{2,p}$.

Thus,

$$\|\Delta(A(x) + e) - (x + z)\|_2 \leq C_{2,p} \frac{\sigma_S(x+z)_{\ell^p}}{S^{1/p-1/2}},$$

and consequently

$$\|\Delta(A(x) + e) - x\|_2 \leq \|z\|_2 + C_{2,p} \frac{\sigma_S(x+z)_{\ell^p}}{S^{1/p-1/2}}$$

$$\leq C_3 \|e\|_2 + 2^{1/p - 1} C_{2,p} \frac{\sigma_S(x)_{\ell^p}}{S^{1/p-1/2}} + \|z\|_2$$

$$\leq C_3 \|e\|_2 + 2^{1/p - 1} C_{2,p} \frac{\sigma_S(x)_{\ell^p}}{S^{1/p-1/2}} + 2^{1/p - 1} C_{2,p} \frac{\|e\|_2}{\gamma_p},$$

where in the third inequality we used the fact in any that $\ell^p$ quasinorm satisfies the inequality $\|a + b\|_p \leq 2^{\frac{1}{p} - 1} (\|a\|_p + \|b\|_p)$ for all $a, b \in \mathbb{R}^N$. So, we conclude

$$\|\Delta(A(x) + e) - x\|_2 \leq \left( C_3 + 2^{1/p - 1} C_{2,p} / \gamma_p \right) \|e\|_2 + 2^{1/p - 1} C_{2,p} \frac{\sigma_S(x)_{\ell^p}}{S^{1/p-1/2}}. \quad (38)$$

That is (i) holds with $C = C_3 + 2^{1/p - 1} C_{2,p} (1/\gamma_p + 1)$.

Next, we prove parts (ii) and (iii) of Theorem 2.13. As in the analogous proof of [30], Theorem 2.13 (ii) can be seen as a special case of Theorem 2.13 (iii), with $e = 0$. We therefore turn to proving (iii). Once again, by Lemma 4.3 there exists $v$ and $z$ in $\mathbb{R}^N$ such that the following hold.

$$Av = e; \quad \|v\|_p \leq \frac{S^{1/p - 1/2}}{\gamma_p} \|e\|_2, \quad \|v\|_2 \leq C_3 \|e\|_2,$$

and

$$Az = Ax_{T_0}; \quad \|z\|_p \leq \frac{S^{1/p - 1/2}}{\gamma_p} \|Ax_{T_0}\|_2, \quad \|z\|_2 \leq C_3 \|Ax_{T_0}\|_2.$$

Here $T_0$ is the set of indices of the largest (in magnitude) $S$ coefficients of $x$, and $T_0^c$ and $x_{T_0^c}$ are as in the proof of Theorem 2.1.
Similar to the previous part we can see that \( A(x_{T_0} + z + v) = Ax + e \) and by the hypothesis of \((2, p)\) instance optimality of \(\Delta\), we have

\[
\|\Delta(Ax + e) - (x_{T_0} + z + v)\|_2 \leq C_{2,p} \frac{\sigma_S(x_{T_0} + z + v)_{\ell,p}}{S^{1/p-1/2}}.
\]

Consequently observing that \( x_{T_0} = x - x_{T_0}^c \) and using the triangle inequality,

\[
\|\Delta(A(x + e) - x\|_2 \leq \|x_{T_0} - z - v\|_2 + C_{2,p} \frac{\sigma_S(x_{T_0} + z + v)_{\ell,p}}{S^{1/p-1/2}}
\]

\[
\leq \|x_{T_0} - z - v\|_2 + 2^{1/p-1}(C_{2,p}) \left( \frac{\|z\|_p + \|v\|_p}{S^{1/p-1/2}} \right)
\]

\[
\leq \sigma_S(x)\tilde{c}_2 + \|z\|_2 + \|v\|_2 + 2^{1/p-1}C_{2,p} \left( \frac{\|x_{T_0}\|_2}{\gamma_p} + \frac{\|e\|_2}{\gamma_p} \right)
\]

\[
\leq \sigma_S(x)\tilde{c}_2 + \left( C_3 + 2^{1/p-1}C_{2,p}\gamma_p \right) (\|e\|_2 + \|x_{T_0}\|_2). \quad (39)
\]

That is (iii) holds with \( C = 1 + C_3 + 2^{1/p-1}C_{2,p}\gamma_p \). By setting \( e = 0 \), one can see that this is the same constant associated with (ii). This concludes the proof of this theorem. \( \square \)

4.5 Proof of Theorem 2.14

First, we show that \( (A_\omega, \Delta_\rho) \) is \((2, p)\) instance optimal of order \( S \) for an appropriate range of \( S \) with high probability. One of the fundamental results in compressed sensing theory states that for any \( \delta \in (0, 1) \), there exists \( \tilde{c}_1, \tilde{c}_2 > 0 \) and \( \Omega_{\text{RIP}} \) with \( P(\Omega_{\text{RIP}}) \geq 1 - 2e^{-\tilde{c}_2 M} \), all depending only on \( \delta \), such that \( A_\omega, \omega \in \Omega_{\text{RIP}} \), satisfies RIP(\( \ell, \delta \)) for any \( \ell \leq \tilde{c}_1 \frac{M}{\log(N/M)} \). See, e.g., [6], [1], for the proof of this statement as well as for the explicit values of the constants. Now, choose \( \delta \in (0, 1) \) such that \( \delta < \frac{2^{2/p-1-1}}{2^{2/p-1+1}} \). Then, with \( \tilde{c}_1, \tilde{c}_2, \) and \( \Omega_{\text{RIP}} \) as above, for every \( \omega \in \Omega_{\text{RIP}} \) and for every \( S < \frac{\tilde{c}_1}{3 \log(N/M)} M \), the RIP constants of \( A_\omega \) satisfy (13) (and hence (12)), with \( k = 2 \). Thus, by Corollary 2.4 \((A_\omega, \Delta_\rho)\) is instance optimal of order \( S \) with constant \( C_{2,p}^{1/p} \) as in (15).

Now, set \( S_1 = c_1 \frac{M}{\log(N/M)} \) with \( c_1 \leq \tilde{c}_1/3 \) such that \( S_1 \in \mathbb{N} \) (note that such a \( c_1 \) exists if \( M \) and \( N \) are sufficiently large). By the hypothesis of the theorem, \( M \) and \( N \) satisfy the hypothesis of the Lemma 2.12 with \( \xi = 2, K_1 = 1 \), some \( 0 \leq \mu < 1/2 \), and an appropriate \( K_2 \) (determined by \( \tilde{c}_1 \) above). Because

\[
\left( \mu^2 \frac{\log(N/M)}{M} \right)^{1/p-1/2} = \left( \mu^2 \frac{c_1}{S_1} \right)^{1/p-1/2}
\]

by Lemma 2.12 there exists \( \Omega_{\mu}, P(\Omega_{\mu}) \geq 1 - e^{-cM} \) such that for every \( \omega \in \Omega_{\mu} \),
\( A_\omega \) satisfies \( LQ_p \left( \frac{\gamma_p(\mu)}{S_1^{1/p-1/2}} \right) \) where \( \gamma_p(\mu) := \frac{\mu^{1/p-1/2} \mu^{2/p-1}}{C(\mu)} \). Consequently, set \( \Omega_1 := \Omega_{RIP} \cap \Omega_\mu \). Then, \( P(\Omega_1) \geq 1 - 2e^{-c_2 M} - e^{-c M} \geq 1 - 3e^{-c_2 M} \), for \( c_2 = \min\{\bar{c}_2, c\} \). Note that \( c_2 \) depends on \( c \), which is now a universal constant, and \( \bar{c}_2 \), which depends only on the distribution of \( A_\omega \) (and in particular its concentration of measure properties, see [1]). Now, if \( \omega \in \Omega_1 \), \( A_\omega \) satisfies \( RIP(3S_1, \delta) \), thus \( RIP(S_1, \delta) \), as well as \( LQ_p \left( \frac{\gamma_p}{S_1^{1/p-1/2}} \right) \). Therefore we can apply part (i) of Theorem 2.13 to get the first part of this theorem, i.e.,

\[
\| \Delta(A_\omega(x) + e) - x \|_2 \leq C \left( \| e \|_2 + \frac{\sigma_{S_1}(x)_{e^p}}{S_1^{1/p-1/2}} \right).
\]

Here \( C \) is as in (38) with \( C_{2,p} = C_2^{1/p} \). To finish the proof of part (i), note that for \( S \leq S_1 \), \( \sigma_{S_1}(x)_{e^p} \leq \sigma_S(x)_{e^p} \) and \( S^{1/p-1/2} \leq S_1^{1/p-1/2} \).

To prove part (ii), first define \( T_0 \) as the support of the \( S_1 \) largest coefficients (in magnitude) of \( x \) and \( T_0^c = \{1, \ldots, N\} \setminus T_0 \). Now, note that for any \( x \) there exists a set \( \Omega_x \) with \( P(\Omega_x) \geq 1 - e^{-\bar{c} M} \) for some universal constant \( \bar{c} > 0 \), such that for all \( \omega \in \Omega_x \), \( \| A_\omega x_{T_0^c} \|_2 \leq 2\| x_{T_0^c} \|_2 = 2\sigma_{S_1}(x)_{e^2} \) (this follows from the concentration of measure property of Gaussian matrices, see, e.g., [1]). Define \( \Omega_x := \Omega_x \cap \Omega_1 \). Thus, \( P(\Omega_x) \geq 1 - 3e^{-c_2 M} - e^{-\tilde{c} M} \geq 1 - 4e^{-c_3 M} \) where \( c_3 = \min\{c_2, \bar{c}\} \). Note that the dependencies of \( c_3 \) are identical to those of \( c_2 \) discussed above. Recall that for \( \omega \in \Omega_1 \), \( A_\omega \) satisfies both \( RIP(S_1, \delta) \) and \( LQ_p \left( \frac{\gamma_p}{(S_1)^{1/p-1/2}} \right) \). We can now apply part (iii) of Theorem 2.13 to obtain for \( \omega \in \Omega_x \)

\[
\| \Delta(A_\omega(x) + e) - x \|_2 \leq C (3\sigma_{S_1}(x)_{e^2} + \| e \|_2).
\]

Above, the constant \( C \) is as in (39). Once again, note that for \( S \leq S_1 \), \( \sigma_{S_1}(x)_{e^2} \leq \sigma_S(x)_{e^2} \) to finish the proof for any \( S \leq S_1 \). \( \square \)

5 Appendix: Proof of Lemma 4.2

In this section we provide the proof of Lemma 4.2 for the sake of completeness and also because we explicitly calculate the optimal constants involved. Let us first introduce some notation used in [18] and [23].

For a body \( K \subset \mathbb{R}^n \), define its gauge functional by \( \| x \|_K := \inf \{ t > 0 : x \in tK \} \), and let \( T_q(K), \) \( q \in (1, 2] \), be the smallest constant \( C \) such that

\[
\forall m \in \mathbb{N}, \ x_1, \ldots, x_m \in K \quad \inf_{\epsilon_i = \pm 1} \left\{ \| \sum_{i=1}^m \epsilon_i x_i \|_K \right\} \leq C m^{1/q}.
\]
Given a $p$-convex body $K$ and a positive integer $r$, define

$$\alpha_r = \alpha_r(K) := \sup \{ \frac{\| \sum_{i=1}^{r} x_i \|_K}{r} : x_i \in K, i \leq r \}.$$ 

Note that $\alpha_r \leq r^{-1+1/p}$.

Finally, conforming with the notation used in [18] and [23], we define $\delta_K := d_1(K, \text{conv}(K))$. Note that this should not cause confusion as we do not refer to the RIP constants throughout the rest of the paper. It can be shown by a result of [24] that $\delta_K = \sup_r \alpha_r(K)$, cf. [18, Lemma 1] for a proof.

We will need the following propositions.

**Proposition 5.1 (sub-additivity of $\| \cdot \|_K^p$)** For the gauge functional $\| \cdot \|_K^p$ associated with a $p$-convex body $K \in \mathbb{R}^n$, the following inequality holds for any $x, y \in \mathbb{R}^n$.

$$\| x + y \|_K^p \leq \| x \|_K^p + \| y \|_K^p.$$  \hspace{1cm} (42)

**PROOF.** Let $r = \| x \|_K$ and $u = \| y \|_K$. If at least one of $r$ and $u$ is zero, then (42) holds trivially. (Note that, as $K$ is a body, $\| x \|_K = 0$ if and only if $x = 0$.) So, we may assume that both $r$ and $u$ are strictly positive. Since $K$ is compact, it follows that $x/r \in K$ and $y/u \in K$. Furthermore, $K$ is $p$-convex, i.e., for all $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$, we have $\alpha^{1/p}x/r + \beta^{1/p}y/u \in K$.

In particular, choose $\alpha = \frac{r^p}{r^p + u^p}$ and $\beta = \frac{u^p}{r^p + u^p}$. This gives $\frac{x + y}{(r^p + u^p)^{1/p}} \in K$.

Consequently, by the definition of the gauge functional $\| \cdot \|_K^p$, we have $\| \frac{x + y}{(r^p + u^p)^{1/p}} \|_K^p \leq 1$ and $\| \frac{x + y}{(r^p + u^p)^{1/p}} \|_K^p = 1$ and $\| x + y \|_K^p \leq r^p + u^p = \| x \|_K^p + \| y \|_K^p$. \hspace{1cm} \(\blacksquare\)

**Proposition 5.2** $T_2(B_2^n) = 1$.

**PROOF.** Note that $\| \cdot \|_{B_2^n} = \| \cdot \|_2$, and thus, by definition, $T_2(B_2^n)$ is the smallest constant $C$ such that for every positive integer $m$ and for every choice of points $x_1, \ldots, x_m \in B_2$,

$$\inf_{\epsilon_i = \pm 1} \left\{ \| \sum_{i=1}^{m} \epsilon_i x_i \|_2 \right\} \leq C \sqrt{m}. \hspace{1cm} (43)$$

For $m \leq n$, we can choose $\{x_1, \ldots, x_m\}$ to be orthonormal. Consequently,

$$\| \sum_{i=1}^{m} \epsilon_i x_i \|_2^2 = \sum_{i=1}^{m} \epsilon_i^2 = m,$$

and thus, $T_2 = T_2(B_2^n) \geq 1$. On the other hand, let $m$ be an arbitrary positive
integer, and suppose that \( \{x_1, \ldots, x_m\} \subset B^n_2 \). Then, it is easy to show that there exists a choice of signs \( \epsilon_i, i = 1, \ldots, m \) such that

\[
\inf_{\epsilon_i = \pm 1} \left\{ \left\| \sum_{i=1}^m \epsilon_i x_i \right\|_2 \right\} \leq \sqrt{m}.
\]

Indeed, we will show this by induction. First, note that \( \|\epsilon_1 x_1\|_2 = \|x_1\|_2 \leq 1 \). Next, assume that there exists \( \epsilon_1, \ldots, \epsilon_{k-1} \) such that

\[
\left\| \sum_{i=1}^{k-1} \epsilon_i x_i \right\|_2 \leq \sqrt{k-1}.
\]

Then (using parallelogram law),

\[
\min\{\left\| \sum_{i=1}^{k-1} \epsilon_i x_i + x_k \right\|_2, \left\| \sum_{i=1}^{k-1} \epsilon_i x_i - x_k \right\|_2 \} \leq \left\| \sum_{i=1}^{k-1} \epsilon_i x_i \right\|_2 + \left\| x_k \right\|_2 \leq k.
\]

Choosing \( \epsilon_k \) accordingly, we get

\[
\left\| \sum_{i=1}^k \epsilon_i x_i \right\|_2 \leq k,
\]

which implies that \( T_2 \leq 1 \). Using the fact that \( T_2 \geq 1 \) which we showed above, we conclude that \( T_2 = 1 \). \( \Box \)

**Proof of Lemma 4.2**

We now present a proof of the more general form of Lemma 4.2 as stated in [18] and [23] (albeit for the Banach-Mazur distance in place of \( d_1 \)). The proof is essentially as in [18], cf. [23], which in fact also works with the distance \( d_1 \) to establish an upper bound on the Banach-Mazur distance between a \( p \)-convex body and a symmetric body.

**Lemma 5.3** Let \( 0 < p < 1, q \in (1,2] \), and let \( K \) be a \( p \)-convex body. Suppose that \( B \) is a symmetric body with respect to the origin such that \( \text{conv}(K) \subset B \). Then

\[
d_1(K, B) \leq C_{p,q}[T_q(B)]^{\phi-1}[d_1(\text{conv}(K), B)]^\phi,
\]

where \( \phi = \frac{1/p-1/q}{1-1/q} \).

**Proof.** Note that \( K \subset \text{conv}(K) \subset B \), and therefore \( d_1(K, B) \) is well-defined. Let \( d = d_1(K, B) \) and \( T = T_q(B) \). Thus, \( (1/d)B \subset K \subset B \). Let \( m \) be a positive integer and let \( x_i, i \in \{1, 2, \ldots, 2^m\} \) be a collection of points in \( K \). Then, \( x_i \in B \) and by the definition of \( T \), there is a choice of signs \( \epsilon_i \)
so that \( \| \sum_{i=1}^{2^m} \epsilon_i x_i \|_B \leq T 2^{m/q} \). Since \( B \) is symmetric, we can assume that \( D = \{ i : \epsilon_i = 1 \} \) has \( |D| > 2^{m-1} \). Now we can write

\[
\begin{align*}
\| \sum_{i=1}^{2^m} x_i \|_K^p &= \| \sum_{i \in D} \epsilon_i x_i + 2 \sum_{i \notin D} x_i \|_K^p \\
&\leq d^p T^p 2^{mp/q} + 2^{mp} \alpha^{p}_{2m-1},
\end{align*}
\]

where the first inequality uses the sub-additivity of \( \| \cdot \|_K \) and the fact that \( (1/d)B \subset K \). Thus by taking the supremum in (44) over all possible \( x_i \)'s and dividing by \( 2^{mp} \), we obtain, for any \( m \),

\[
\alpha^{p}_{2m} \leq d^p T^p 2^{mp/q} + \alpha^{p}_{2m-1}.
\]

By applying this inequality for \( m-1, m-2, ..., k \), we obtain the following inequality for any \( k \leq m \)

\[
\alpha^{p}_{2m} \leq d^p T^p \sum_{i=k+1}^{\infty} 2^{-ip(1-1/q)} + \alpha^{p}_{2k} \leq d^p T^p 2^{-k p (1-1/q)} \frac{2}{p(1-1/q) \log 2} + 2^{k(1-p)}. \quad (45)
\]

Since \( \delta_K = \sup_r \alpha_r \), we now want to minimize the right hand side in (45) by choosing \( k \) appropriately. To that end, define

\[
f(k) := 2^{k(1-p)} + \frac{(dT)^p}{p(1-1/q) \log 2} 2^{-k p (1-1/q)p} \]

and

\[
A := \frac{(dT)^p}{p(1-1/q) \log 2}.
\]

Since \( \alpha^{p}_{2m} \leq f(k) \) for any \( k \in \{ 1, ..., m-1 \} \), the best bound on \( \alpha^{p}_{2m} \) is essentially given by \( f(k^*) \), where \( f'(k^*) = 0 \). However, since \( k^* \) is not necessarily an integer (which we require), we will instead use \( f(k^*+1) \geq f([k^*]) \geq f(k^*) \) as a bound. Thus, we solve \( f'(k^*) = 0 \) to obtain \( k^* = \frac{1}{p/q} \log 2 \left( \frac{Ap(1-1/q)}{1-p} \right) \). By evaluating \( f(k) \) at \( k^* + 1 \), we obtain \( \alpha_{2m} \leq (f(k^* + 1))^{1/p} \) for every \( m \geq k^* + 1 \). In other words, for every \( m \geq k^* + 1 \), we have

\[
\alpha_{2m} \leq (dT) \frac{1-p}{1-p/q} \left( 2^{1-p} + 2^{-p(1-1/q)} \frac{1-p}{p-p/q} \right) \left( \frac{1}{(1-p) \log 2} \right)^{1/p-1}. \quad \text{(46)}
\]

On the other hand, if \( m \leq k^* \), then \( \alpha^{p}_{2m} \leq 2^m 2^{m(1-p)} \leq 2^{k^*+1}(1-p) \). However, this last bound is one of the summands in the right hand side of (45) with \( k = k^* + 1 \) (which we provide a bound for in (46)). Consequently (46) holds for all \( m \). In particular, it holds for the value of \( m \) which achieves the supremum
of $\alpha_{2^n}$. Since $\delta_K = \sup_r \alpha_r$, we obtain

$$\delta_K \leq (dT)^{(1-p)/(1-p/q)} \left( 2^{1-p} + 2^{-p(1-1/q)} \frac{1 - p}{p(1 - 1/q)} \right)^{1/p} \left( \frac{1}{(1 - p) \log 2} \right)^{1/p}.$$  

(47)

**Remark 5.4** In the previous step we utilize the fact that in the derivations above we can replace every $2^m$ and $2^k$ with $m$ and $k$ respectively, thus every $m$ and $k$ with $\log_2 m$ and $\log_2 k$ without changing (46). This allows us to pass from the bound on $\alpha_{2^m}$ to $\delta_K = \sup_r \alpha_r$ without any problems.

Recalling the definitions of $d_1(\text{conv}(K), B)$ and $\delta_K$, note the following inclusions:

$$\frac{1}{\delta_K d_1(\text{conv}(K, B))} B \subset \frac{1}{\delta_K} \text{conv}(K) \subset K \subset \text{conv}(K) \subset B.$$  

(48)

Consequently $\frac{1}{\delta_K d_1(\text{conv}(K, B))} B \subset K \subset B$ and the inequality

$$d_1(K, B) = d \leq \delta_K d_1(\text{conv}(K), B)$$  

(49)

follows from the definition of $d_1(K, B)$. Combining (49) and (47) we complete the proof with

$$C_{p,q} = \left( 2^{1-p} + 2^{-p(1-1/q)} \frac{1 - p}{p(1 - 1/q)} \right)^{1-p/q} \left( \frac{1}{p^{(1-1/q)}} \right)^{-1/p^q} \left( \frac{1}{(1 - p) \log 2} \right)^{1/p^q}.$$  

$\square$

Finally, we choose above $B = B_n^2$ and $q = 2$, recall that $T = T_2(B_n^2) = 1$ (see Proposition 5.2), and obtain Lemma 4.2 as a corollary with

$$C(p) = \left( 2^{1-p} + \frac{(1 - p)2^{1-p/2}}{p} \right)^{2^{1-p}/p^2} \left( \frac{1}{(1 - p) \log 2} \right)^{2^{-2p}/p^2}.$$  

(50)

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