HODGE BUNDLES ON SMOOTH COMPACTIFICATIONS OF SIEGEL VARIETIES AND APPLICATIONS

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Siegel varieties are locally symmetric varieties. They are important and interesting in algebraic geometry and number theory. We construct a canonical Hodge bundle on a Siegel variety so that the holomorphic tangent bundle can be embedded into the Hodge bundle; we obtain that the canonical Bergman metric on a Siegel variety is same as the induced Hodge metric and we describe asymptotic behavior of this unique Kähler-Einstein metric explicitly; depending on these properties and the uniformitarian of Kähler-Einstein manifold, we study extensions of the tangent bundle over any smooth toroidal compactification (Theorem 1.4, Theorem 1.5 and Theorem 1.9 in Section 1). We apply these results of Hodge bundles, together with Siegel cusp modular forms, to study general type for Siegel varieties (Theorem 2.10 in Section 2).

1. Hodge bundles on Siegel varieties

1.1. Construction of Hodge bundles on Siegel varieties

1.2. Degeneration of canonical metrics on Siegel varieties

2. General type for Siegel varieties $A_{g,l}$

2.1. Structures of morphisms $\lambda_{n,m} : A_{g,n} \to A_{g,m}$ for all $m|n$

2.2. Siegel modular forms

2.3. A Criterion on varieties of general type

2.4. General type of Siegel varieties with suitable level structures

References

We begin by establishing the basic notation.

Notation.

Throughout this paper, $g$ is an integer more than two.

In this paper, we fix a real vector space $V_\mathbb{R}$ of dimensional $2g$ and fix a standard symplectic form $\psi = \begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix}$ on $V_\mathbb{R}$. For any non-degenerate skew-symmetric bilinear form $\tilde{\psi}$ on $V_\mathbb{R}$, it is known that there is an element $T \in \text{GL}(V_\mathbb{R})$ such that $^tT\tilde{\psi}T = \psi$. We also fix a symplectic basis $\{e_i\}_{1 \leq i \leq 2g}$ of the standard symplectic space $(V_\mathbb{R}, \psi)$ such that

$\psi(e_i, e_{g+i}) = -1$ for $1 \leq i \leq g$, and $\psi(e_i, e_j) = 0$ for $|j - i| \neq g$.

- Denote by $V_\mathbb{Z} := \bigoplus_{1 \leq i \leq 2g} \mathbb{Z} e_i$, then $V_\mathbb{R} = V_\mathbb{Z} \otimes \mathbb{R}$ and $V_\mathbb{Z}$ is a standard lattice in $V_\mathbb{R}$. In this paper, we fix the lattice $V_\mathbb{Z}$ and fix the rational space $V_\mathbb{Q} := V_\mathbb{Z} \otimes \mathbb{Q}$.
- For any $\mathbb{Z}$-algebra $\mathcal{R}$, we define $V_\mathcal{R} := V_\mathbb{Z} \otimes \mathcal{R}$ and we write

$\text{Sp}(g, \mathcal{R}) := \{ h \in \text{GL}(V_\mathcal{R}) | \psi(hu, hv) = \psi(u, v) \text{ for all } u, v \in V_\mathcal{R} \}$.

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Since $\det M = \pm 1$ for all $M \in \text{Sp}(g,\mathbb{Z})$, $\text{Sp}(g,\mathbb{Z})$ is a subgroup of $\text{Sp}(g,\mathbb{Q})$.

- Denote congruent groups by

\[(0.0.3) \quad \Gamma_g(1) := \text{Sp}(g,\mathbb{Z}), \quad \Gamma_g(n) := \{ \gamma \in \text{Sp}(g,\mathbb{Z}) \mid \gamma \equiv I_{2g} \mod n \} \forall n \geq 2.\]

Obviously, each $\Gamma_g(n)$ is a normal subgroup of $\text{Sp}(g,\mathbb{Z})$ with finite index. For convenience, we write $\Gamma_g$ for $\Gamma_g(1)$.

- A subgroup $\Gamma \subset \text{Sp}(g,\mathbb{Q})$ is said to be **arithmetic** if $\rho(\Gamma)$ is commensurable with $\rho(\text{Sp}(g,\mathbb{Q})) \cap \text{GL}(n,\mathbb{Z})$ for some embedding $\rho : \text{Sp}(g,\mathbb{Q}) \hookrightarrow \text{GL}(n,\mathbb{Q})$. By a result of Borel, a subgroup $\Gamma \subset \text{Sp}(g,\mathbb{Q})$ is arithmetic if and only if $\rho'(\Gamma)$ is commensurable with $\rho'(\text{Sp}(g,\mathbb{Q})) \cap \text{GL}(n',\mathbb{Z})$ for every embedding $\rho' : \text{Sp}(g,\mathbb{Q}) \hookrightarrow \text{GL}(n')$. We note that a subgroup $\Gamma \subset \text{Sp}(g,\mathbb{Z})$ is arithmetic if and only if $[\text{Sp}(g,\mathbb{Z}) : \Gamma] < \infty$.

- Let $k'$ be a subfield of $\mathbb{C}$. An automorphism $\alpha$ of a $k'$-vector space is defined to be **neat** (or **torsion free**) if its eigenvalues in $\mathbb{C}$ generate a torsion free subgroup of $\mathbb{C}$. An element $h \in \text{Sp}(g,\mathbb{Q})$ is said to be neat (or torsion free) if $\rho(h)$ is neat for one faithful representation $\rho$ of $\text{Sp}(g,\mathbb{Q})$. A subgroup $\Gamma \subset \text{Sp}(g,\mathbb{R})$ is neat if all elements of $\Gamma$ are torsion free. It is known that if $n \geq 3$ then $\Gamma_g(n)$ is a neat arithmetic subgroup of $\text{Sp}(g,\mathbb{Q})$.

The Siegel space $\mathfrak{H}_g$ of degree $g$ is a complex manifold defined to be the set of all symmetric matrices over $\mathbb{C}$ of degree $g$ whose imaginary parts are positive defined. The action of $\text{Sp}(g,\mathbb{R})$ on $\mathfrak{H}_g$ is defined as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \tau := \frac{A \tau + B}{C \tau + D}.$$ 

It is known that the simple real Lie group $\text{Sp}(g,\mathbb{R})$ acts transitively. A **Siegel variety** is defined to be $\mathcal{A}_{g,\Gamma} := \Gamma \backslash \mathfrak{H}_g$, where $\Gamma$ is an arithmetic subgroup of $\text{Sp}(g,\mathbb{Q})$.

- A Siegel variety $\mathcal{A}_{g,\Gamma}$ is a normal quasi-projective variety.

- Any neat arithmetic subgroup $\Gamma$ of $\text{Sp}(g,\mathbb{Q})$ acts freely on the Siegel Space $\mathfrak{H}_g$, then the induced $\mathcal{A}_{g,\Gamma}$ is a regular quasi-projective complex variety of dimension $g(g+1)/2$.

- A **Siegel variety of degree $g$ with level $n$** is defined to be

$$\mathcal{A}_{g,n} := \Gamma_g(n) \backslash \mathfrak{H}_g.$$ 

Thus, the Siegel varieties $\mathcal{A}_{g,n}$ ($n \geq 3$) are quasi-projective complex manifolds.

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1. Hodge bundles on Siegel varieties

Let $\Gamma$ be a neat arithmetic subgroup of $\text{Sp}(g, \mathbb{Q})$. Let $V_\mathbb{Q}$ be the fixed rational symplectic vector space as in the introduction of notations.

The Siegel space $\mathfrak{H}_g$ can be realized as a period domain parameterizing weight one polarized Hodge structures:

**Proposition** (Borel’s embedding cf. [Sat & Del73]). Define

$$\mathfrak{G}_g = \mathfrak{G}(V_\mathbb{R}, \psi) := \{ F^1 \in \text{Grass}(g, V_\mathbb{C}) \mid \psi(F^1, F^1) = 0, \sqrt{-1}\psi(F^1, F^1) > 0 \}.$$  

The map $\iota : \mathfrak{H}_g \xrightarrow{\cong} \mathfrak{G}_g \tau \mapsto F^1_\tau$ identifies the Siegel space $\mathfrak{H}_g$ with the period domain $\mathfrak{G}_g$, where $F^1_\tau$ is the subspace of $V_\mathbb{C}$ spanned by the column vectors of $\begin{pmatrix} \tau & I_g \end{pmatrix}$. Moreover, the map $h$ is biholomorphic.

There is a natural variation of Hodge structures on the Siegel space $\mathfrak{H}_g$:

**Corollary** (Cf. [Del79]). Gluing Hodge structures on $\mathfrak{H}_g$ altogether, the local system $\mathcal{V} := V_\mathbb{Q} \times \mathfrak{H}_g$ admits a homogeneous variation of polarized rational Hodge structures of weight one on $\mathfrak{H}_g$.

Let $o$ be an arbitrary fixed base point in $\mathcal{A}_g,\Gamma$. Since $\mathfrak{H}_g$ is simply connected, the fundamental group of $\mathcal{A}_g,\Gamma$ has $\pi_1(\mathcal{A}_g, o) \cong \Gamma$. Then, there is a natural local system $\mathcal{V}_g,\Gamma := V_\mathbb{Q} \times \Gamma \mathfrak{H}_g$ on $\mathcal{A}_g,\Gamma$ given by the fundamental representation $\rho : \pi_1(\mathcal{A}_g, o) \to \text{GSp}(V, \psi)(\mathbb{Q})$. Actually the $\mathbb{Q}$-local system $\mathcal{V}_g,\Gamma$ admits a variation of polarized rational Hodge structures of weight one on $\mathcal{A}_g,\Gamma := \Gamma \backslash \mathfrak{H}_g$ by using the arguments in Section 4 of [Zuc81]. More precisely, in our previous paper (cf. Proposition 1.8 in Section 1 of [YZ11]) we obtain:

1. The local system $\mathcal{V}_g,\Gamma$ admits a variation of polarized rational Hodge structures on $\mathcal{A}_\Gamma$, and the associated period map attached to this PVHS is

   $$(1.0.4) \quad \iota_{\Gamma} : \mathcal{A}_g,\Gamma \xrightarrow{\cong} \Gamma \backslash \mathfrak{G}_g.$$  

2. Let $\bar{\mathcal{A}}_g,\Gamma$ be an arbitrary smooth compactification of $\mathcal{A}_g,\Gamma$ with simple normal crossing divisor $D_\infty := \mathcal{A}_g,\Gamma \setminus \mathcal{A}_g,\Gamma$. Around the boundary divisor $D_\infty$, all local monodromies of any rational PVHS $\mathcal{V}$ on $\mathcal{A}_g,\Gamma$ are unipotent.

Now, we fix this rational PVHS $\mathcal{V}_g,\Gamma$ throughout this paper.

1.1. Construction of Hodge bundles on Siegel varieties. Most materials in this subsection are taken from [Sch73], [Sim90] and [Zuo00].

We note that $\mathbb{H} := \mathcal{V}_g,\Gamma \otimes \mathbb{C}$ is a flat complex vector bundle on the $\mathcal{A}_g,\Gamma$ with a flat connection $\mathbb{D}$. There is a filtration of $C^\infty$ vector bundles over $\mathcal{A}_g,\Gamma$ $0 = \mathbb{F}^2 \subset \mathbb{F}^1 \subset \mathbb{F}^0 = \mathbb{H}$, whose fibers at each point $\tau \in \mathcal{A}_g,\Gamma$ gives a Hodge filtration isomorphic to $F^*_\tau := (0 \subset F^1_\tau \subset V_\mathbb{C})$. The vector bundle $\mathbb{H}$ admits a positive Hermitian metric $H$ induced by the polarization $\psi$ of the Hodge structures as follows:

$$(1.0.5) \quad < u, v >_H := \psi(C_\tau u, v) \quad \forall u, v \in \mathbb{H}_\tau,$$

where each $C_\tau$ is the Weil operator on the $F^*_\tau$. We usually call this metric $H$ the Hodge metric on $\mathbb{H}$. Let $\mathbb{H}^{p,q} := \mathbb{H}^p \cap \overline{\mathbb{H}^q}$. The smooth decomposition $\mathbb{H} = \bigoplus \mathbb{H}^{p,q}$ is orthogonal with respect to the Hodge metric $H$.

Let $\mathbb{D}^{0,1}$ be the $(0, 1)$-part of the flat connection $\mathbb{D}$ and $\mathbb{D}^{1,0}$ the $(1, 0)$-part of $\mathbb{D}$. The $\mathbb{D}^{0,1}$ gives a holomorphic structure on $\mathbb{H}$, so that $\mathcal{H} := (\mathbb{H}, \mathbb{D}^{0,1})$ is the corresponding holomorphic bundle.
The $\mathbb{D}^{1,0}$ guarantees $\mathcal{H}$ has an integrable holomorphic connection $\mathbb{D}^{1,0} : \mathcal{H} \to \mathcal{H} \otimes \Omega^1_{A,g,\Gamma}$. It is known that all subbundles $\mathbb{F}^p$'s admit the holomorphic structure $\mathbb{D}^{0,1}$ naturally, so that we have the corresponding holomorphic subbundles $\mathbb{F}^p$. Moreover, we have the Griffiths transversality :

\begin{equation}
\mathbb{D}^{1,0} : \mathbb{F}^p \longrightarrow \mathbb{F}^{p-1} \otimes \Omega^1_{A,g,\Gamma}, \forall p.
\end{equation}

Define $E^{p,2-p} := \mathbb{F}^p / \mathbb{F}^{p+1}$ $\forall p$. We know that each holomorphic vector bundle $E^{p,q}$ is $C^\infty$-isomorphic to the vector bundle $\mathbb{H}^{p,q}$. We set $E := \text{Gr}(\mathcal{H}) = \bigoplus_p E^{p,2-p}$. The flat connection $\mathbb{D}$ on $\mathcal{H}$ actually induces a global holomorphic structure $\mathcal{D}$ on $E$ such that each $E^{p,q}$ is a holomorphic subbundle of $E$. We write :

\begin{equation}
E^{p,q} = (\mathbb{H}^{p,q}, \mathcal{D}), \quad \text{and} \quad E = \bigoplus \mathbb{H}^{p,q}, \mathcal{D}).
\end{equation}

The holomorphic vector bundles $E$ and $E^{p,q}$'s are endowed natural Hermitian metrics induced by $H$. For convenience, we still call these Hermitian metrics the Hodge metrics and still write these Hermitian metrics as $\overline{H}$.

Let $T(A_g,\Gamma)$ be the real tangent bundle of $A_g,\Gamma$. According to $\pm \sqrt{-1}$-eigenvalues of the complex structure $J$ on $T(A_g,\Gamma)$, there is a $C^\infty$-decomposition $T(A_g,\Gamma) \cong \mathbb{C} = T^{1,0}(A_g,\Gamma) \oplus T^{0,1}(A_g,\Gamma)$. The real tangent bundle $T(A_g,\Gamma)$ undertakes the holomorphic tangent bundle $T_{A,g,\Gamma} := (\Omega^1_{A,g,\Gamma})^\vee$ in sense that

\begin{equation}
T^{1,0}(A_g,\Gamma) \overset{\cong}{\longrightarrow} T_{A,g,\Gamma}, \quad T^{0,1}(A_g,\Gamma) \overset{\cong}{\longrightarrow} T_{A,g,\Gamma}.
\end{equation}

Let $(p, q)$ be a pair of integers. For any local holomorphic vector filed $\mathcal{X}$ of $T_{A,g,\Gamma}$, there is a local $\mathcal{O}_{A,g,\Gamma}$-linear morphism $\theta^{p,q}(\mathcal{X}) : E^{p,q} \to E^{p-1,q+1}$ by the Griffiths transversality [1.0.6]. Then we get an $\mathcal{O}_{A,g,\Gamma}$-linear morphism $\theta^{p,q} : E^{p,q} \to E^{p-1,q+1} \otimes \Omega^1_{A,g,\Gamma}$, and so we get the adjoint map $\theta_{H}^{p-1,q+1} : E^{p-1,q+1} \to E^{p,q} \otimes \Omega^1_{A,g,\Gamma}$ of $\theta^{p,q}$ given by $\langle \theta^{p,q}(s), t \rangle \equiv < s, \theta_{H}^{p-1,q+1}(t) >_{H}$, where $s$ (resp. $t$) is a local section of $E^{p,q}$ (resp. $E^{p-1,q+1}$). Clearly $\theta^{p,q}$ can be regarded as an $\mathcal{O}_{A,g,\Gamma}$-linear morphism. The **Higgs field** $\theta$ on $E$ is defined as follows :

\[ \theta = \bigoplus_{p,q} \theta^{p,q} : \bigoplus_{p,q} E^{p,q} \longrightarrow \bigoplus_{p,q} E^{p,q} \otimes \Omega^1_{A,g,\Gamma}. \]

Respectively, the adjoint morphism of $\theta$ is defined to be $\theta^{*}_H := \bigoplus_{p,q} \theta^{p,q*}_H$.

**Remark.** Let $A^1$ be the dual of the sheaf of $C^\infty$ germs of $T(A_g,\Gamma)$. Then there is a $C^\infty$ splitting $A^1 = A^{1,0} \oplus A^{0,1}$ where $A^{1,0}$ (resp. $A^{0,1}$) is the dual of the sheaf of $C^\infty$ germs of $T_{A,g,\Gamma}$ (resp. $\overline{T_{A,g,\Gamma}}$). We can extend $\theta$ and $\theta^{*}_H$ naturally as $C^\infty$ morphisms

\[ \theta : C^\infty(E) \longrightarrow C^\infty(E) \otimes A^{1,0}, \]

\[ \theta^{*}_H : C^\infty(E) \longrightarrow C^\infty(E) \otimes A^{0,1}, \]

where $C^\infty(E)$ is the sheaf of $C^\infty$ germs of $E$.

Let $\nabla_H$ be the unique Chern connection on $(E, H)$. Thus, the connection $\nabla_H$ is compatible with the Hodge metric, and its $(0,1)$-part has $\nabla_{H}^{0,1} = \overline{\mathcal{D}}$. Define $\partial := \nabla_{H}^{0,1}$. We immediately obtain $\partial^2 = \overline{\mathcal{D}}^2 = 0$, and get the Chern curvature form

\[ \Theta(E, H) := \nabla_H \circ \nabla_H = (\overline{\mathcal{D}}^2)^{1,1}. \]
Lemma 1.1. We have:

\[ \overline{\partial} (\theta) := \overline{\partial} \circ \theta + \theta \circ \overline{\partial} = 0, \]

\[ \partial (\theta^*_h) := \partial \circ \theta^*_h + \theta^*_h \circ \partial = 0. \]

Proof. One can find these two equalities in [Sim88] & [Sim90]. Here we give a direct proof.

The morphism \( \theta \) is naturally holomorphic by the definition, so that the first equality is automatically true. Now, we begin to prove the second equality.

It is sufficient to prove the equality at an arbitrary point \( p \). Let \((U, p)\) be a local coordinate neighborhood of \( p \). Let \( \{e_\alpha\} \) be a local holomorphic basis of \( E \). We then get a local holomorphic basis \( \{e^\alpha\} \) of \( E^\vee|_U := \text{Hom}(E|_U, \mathcal{O}_U) \) as follows: For each \( \alpha \), let \( e^\alpha \) be the dual of \( e_\alpha \), i.e.,

\[ e^\alpha \in E^\vee|_U \text{ such that } e^\alpha(e_\beta) = \begin{cases} 1, & \beta = \alpha; \\ 0, & \beta \neq \alpha. \end{cases} \]

We call the local holomorphic basis \( \{e^\alpha\} \) of \( E^\vee|_U \) as a local dual base of \( \{e_\alpha\} \). Let \( \{l_1, \cdots, l_m\} \) be a local holomorphic basis of \( \mathcal{T}_{A_{g,r}} \) and \( \{\phi_1, \cdots, \phi_m\} \subset \Omega^1_{A_{g,r}} \) its local holomorphic dual basis. Locally, we can write

\[ \theta = \sum_{i=1}^m A^i \phi_i, \quad \theta^*_h = \sum_i B^i \overline{\phi}_i \]

where \( A^i := A^i_{\beta} e_\alpha \otimes e^\beta \) and

\[ B^i := B^i_{\beta} e_\alpha \otimes e^\beta \text{ with } B^i_{\beta} := \sum_{\gamma, \delta} H_{\alpha \gamma} A^i_{\delta} \overline{H^{\delta \beta}}, H_{\alpha \gamma} := <e_\alpha, e_\gamma> \).

Form the first equality in the lemma, we get

\[ 0 = \overline{\partial} \theta = \sum_{i=1}^m \sum_{j=1}^n A^i_{j} \overline{\phi}_j \wedge \phi_i = \sum_{i=1}^m \sum_{j=1}^n A^i_{\beta} e_\alpha \otimes e^\beta \overline{\phi}_j \wedge \phi_i \]

where \( A^i_{j} \)’s for all \( j \) are covariant partial derivations of the tensor \( A^i \), and so we obtain

\[ A^i_{\beta j} = 0 \text{ on } U \forall i, j, \alpha, \beta. \]

On the other hand, we compute that

\[ \partial \theta^*_h = \sum_{i=1}^m \sum_{j=1}^n B^i_{j} \overline{\phi}_j \wedge \phi_i = \sum_{i=1}^m \sum_{j=1}^n B^i_{\beta j} e_\alpha \otimes e^\beta \overline{\phi}_j \wedge \phi_i, \]

where \( B^i_{j} \)’s for all \( j \) are covariant partial derivations of the tensor \( B^i \). It is well-known that one can contract the neighborhood \((U, p)\) sufficiently small to get a special holomorphic local basis \( \{e_\alpha\} \) of \( E \) over \( U \) such that \( H(p) = \text{Id}, \ dH(p) = 0 \) under the frame \( \{e_\alpha\} \). Then, at the point \( p \), we have:

\[ B^i_{\beta j}(p) = A^i_{\beta j}(p) = 0, \forall i, j, \alpha, \beta. \]

Thus \( \partial \theta^*_h = 0 \) at the point \( p \).

\[ \square \]

Corollary 1.2. Let \((E, H)\) be Hermitian vector bundle in \([1.0.7] \). We have:

\[ \Theta(E, H) = -(\theta \wedge \theta^*_h + \theta^*_H \wedge \theta), \]

\[ \theta \wedge \theta = -\overline{\partial}(\theta) = 0, \]

\[ \theta^*_H \wedge \theta^*_h = -\overline{\partial}(\theta^*_h) = 0. \]
Proof. It is known the flat connection $\mathcal{D}$ on $\mathbb{H}$ has the following decomposition
\[
\mathcal{D} = \nabla_H + \theta + \theta^*_H.
\]
Since $\mathcal{D}^2 = 0$, we can finish the proof by the lemma. □

Attached to the PVHS $\mathcal{V}_{g, \Gamma}$, we finally obtain the associated Hodge bundle $(E, \theta, H)$ on $A_{g, \Gamma}$, i.e., a holomorphic system
\[
(1.2.1) \quad (E = \oplus E^{p,q}, \theta = \oplus \theta^{p,q})
\]
with a Hermitian metric $H$ satisfying the following properties:
- $E^{p,q}$ are orthogonal to each other under the metric $H$;
- $\theta \wedge \theta = 0$;
- $\theta^{p,q} : E^{p,q} \rightarrow E^{p-1,q+1} \otimes \Omega^1_{A_{g, \Gamma}}$.

The dual local system $\mathcal{V}'_{g, \Gamma} = \mathcal{V}'_{\mathcal{Q}} \times_{\Gamma} \mathcal{V}_g$ admits a polarized rational VHS of weight $-1$ on $A_{g, \Gamma}$, its associated Hodge bundles is $(E^{p,q})'$ with
\[
E^{p,q} = (E^{p,q})' = E^{q,p},
\]
and the Higgs field $\theta^{end} : \text{End}(E) \rightarrow \text{End}(E) \otimes \Omega^1_{A_{g, \Gamma}}$ given by
\[
\theta^{end}(u \otimes v') = \theta(v) \otimes v' + u \otimes \theta(v').
\]
We notes that $\text{End}(E)$ has a holomorphic subbundle
\[
\text{End}(E)^{-1,1} = \bigoplus_{p+q=1} E^{p,q} \otimes E^{q-1,p+1},
\]
\[
= (E^{0,1})^{\otimes 2}.
\]

We still use $H$ to denote the induced Hermitian metric on $E^{\vee}$ and $\text{End}(E)$. Throughout this section, we now fix the Hermitian bundles $(E, H)$, $(\text{End}(E), H)$, and $(E^{p,q}, H)$s, $(\text{End}(E)^{p,q}, H)$s.

1.2. Degeneration of canonical metrics on Siegel varieties. Let $\tilde{A}_{g, \Gamma}$ be a smooth compactification of $A_{g, \Gamma}$ such that the divisor $D_{\infty} = \tilde{A}_{g, \Gamma} \setminus A_{g, \Gamma}$ is simple normal crossing. Since any local monodromy of $\mathcal{V}_{g, \Gamma}$ around $D_{\infty}$ is unipotent, the Hodge bundle $(E, \theta)$ has a Deligne’s canonical extension $(\tilde{E} = \oplus \tilde{E}^{p,q}, \tilde{\theta} = \oplus \tilde{\theta}^{p,q})$ with $\tilde{\theta}^{p,q} : \tilde{E}^{p,q} \rightarrow \tilde{E}^{p-1,q+1} \otimes \Omega^1_{\tilde{A}_{g, \Gamma}}(\log D_{\infty})$. Deligne’s extension of $(\text{End}(E), \theta^{end})$ is $(\text{End}(\tilde{E}), \tilde{\theta}^{end})$. The morphism $\tilde{\theta}^{1,0} : \tilde{E}^{1,0} \rightarrow \tilde{E}^{0,1} \otimes \Omega^1_{\tilde{A}_{g, \Gamma}}(\log D_{\infty})$ represents the global section $\tilde{\theta}^{1,0} \in H^0(\tilde{A}_{g, \Gamma}, \tilde{E}^{0,1} \otimes \Omega^1_{\tilde{A}_{g, \Gamma}}(\log D_{\infty}))$. Then, we obtain a sheaf morphism
\[
(1.2.2) \quad \rho : \mathcal{A}_{g, \Gamma}(- \log D_{\infty}) \rightarrow \tilde{E}^{0,1} \otimes \Omega^2_{\mathcal{A}_{g, \Gamma}}.
\]
Define the restriction map $\rho_0 := \rho|_{A_{g, \Gamma}}$. 

Lemma 1.3. The holomorphic tangent bundle $\mathcal{T}_{A_g,\Gamma}$ of $A_{g,\Gamma}$ is a holomorphic subbundle of $(E^{0,1})^{\otimes 2}$. Moreover, the morphism $\rho_0 : \mathcal{T}_{A_g,\Gamma} \hookrightarrow (E^{0,1})^{\otimes 2}$ is an inclusion of vector bundles.

Proof. We known that the vector bundles $E^{1,0}$, $E^{0,1}$, $\mathcal{O}_{A_g,\Gamma}$, $\Omega^1_{A_{g,\Gamma}}$ are all $\text{Sp}(g,\mathbb{R})$-homogenous, and the morphism $\theta^{1,0} : E^{1,0} \rightarrow E^{0,1} \otimes \Omega^1_{A_{g,\Gamma}}$ is a $\text{Sp}(g,\mathbb{R})$-equivariant morphism. Thus, the morphism $\rho_0 : \mathcal{T}_{A_g,\Gamma} \hookrightarrow (E^{0,1})^{\otimes 2}$ is $\text{Sp}(g,\mathbb{R})$-equivariant. We verify the inclusion at the base point $o \in A_{g,\Gamma}$: At point $o$, we have $E^{1,0}|_o = H^{1,0}, E^{0,1}|_o = H^{0,1}$ and $\mathcal{T}_{A_{g,\Gamma},o} \subset \text{Hom}(H^{1,0}, H^{0,1}) = (H^{0,1})^{\otimes 2}$ by Borel’s embedding. The construction of the Hodge bundle $(E, \theta)$ shows that the inclusion $\mathcal{T}_{A_{g,\Gamma},o} \subset \text{Hom}(H^{1,0}, H^{0,1}) = (H^{0,1})^{\otimes 2}$ is just the morphism $\rho_0$ at the point $o$. \hfill \square

We now introduce an induced $\text{Sp}(g,\mathbb{R})$-invariant positive Hermitian metric $H(H$\textit{odge metric}$)$ on $A_{g,\Gamma}$ by the following inclusion

$$\rho_0 : \mathcal{T}_{A_g,\Gamma} \hookrightarrow \text{End}(E)^{-1,1} \subset \text{End}(E).$$

Let $\{l_1, \ldots, l_m\}$ be a holomorphic basis of $\mathcal{T}_{A_g,\Gamma}$ on a local neighborhood $(U, z)$ of $A_{g,\Gamma}$, and $\{\phi_1, \ldots, \phi_m\}$ be the dual holomorphic basis of $\Omega^1_{A_{g,\Gamma}}$ over $U$. We define

$$(1.3.1) \quad H(l_i, \overline{l_j}) := <\rho_0(l_i), \rho_0(l_j)>_H.$$ 

Since $\rho_0$ can be linearly extended to a morphism of sheaves of $C^\infty$ germs as well as $\theta$ does, we then obtain a metric $H$ on $A_{g,\Gamma}$. The Kähler form of $H$ on $U$ can be written locally as

$$(1.3.2) \quad \omega_H = \sum_{i,j=1}^m H(l_i, \overline{l_j}) \phi_i \wedge \overline{\phi_j}.$$ 

Theorem 1.4. Let $\Gamma$ be a neat arithmetic subgroup of $\text{Sp}(g,\mathbb{Z})$.

The induced Hodge metric $H$ on the Siegel variety $A_{g,\Gamma} = \Gamma \backslash A_g$ is same as the canonical Bergman metric. Moreover, the Chern connection of $(\mathcal{T}_{A_{g,\Gamma}}, H)$ is compatible with the Levi-Civita connection of the Riemannian manifold $(A_{g,\Gamma}, H)$.

Proof. Notation as in the proof of the lemma[1.1]. Since the Hodge metric $H$ on $A_{g,\Gamma}$ is $\text{Sp}(g,\mathbb{R})$-invariant and $\text{Sp}(g,\mathbb{R})$ is a simple group, it is sufficient to show that $H$ is Kähler.

Let $p$ be an arbitrary point on $A_{g,\Gamma}$. Let $U$ be a suitable neighborhood of $p$ such that we can choose a local holomorphic coordinates $(z_1, \ldots, z_m)$ satisfying

$$\mathcal{T}_{A_{g,\Gamma}|U} = \text{span}\{\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_m}\}.$$ 

Let $\{e_\alpha\}$ be a local holomorphic basis of $E$ and $\{e^\alpha\}$ the local dual holomorphic basis of $E^\vee$. All calculation below are locally over $U$.

We write $\theta = \sum_{i=1}^k A^i dz_i$, where $A^i = \sum_{\alpha,\beta} A^i_{\alpha\beta} e_\alpha \otimes e^\beta \in \text{End}(E)$. The Kähler form is then

$$\omega_H := \sum_{i,j=1}^k H(l_i, \overline{l_j}) dz_i \wedge d\overline{z_j} = \sum_{i,j=1}^k <A^i, \overline{A^j}>_H dz_i \wedge d\overline{z_j}.$$ 

Thus, we have that

$$d\omega_H = \sum_{i,j} (d <A^i, \overline{A^j}>_H) \wedge dz_i \wedge d\overline{z_j}$$

$$= \sum_{i,j=1}^m (<\nabla_H A^i, \overline{A^j}>_H + <A^i, \nabla_H \overline{A^j}>_H) \wedge dz_i \wedge d\overline{z_j}.$$
where $\nabla_H$ is the Chern connection on $(\text{End}(E), H)$. For each $i = 1, \cdots, m$, we have:

$$\nabla_H A^i = \overline{\partial} A^i + \partial A^i = \partial A^i$$

$$= \sum_{k=1}^n A^{i,\alpha}_{j,k} e_\alpha \otimes e^\beta \phi_k.$$  

Since $\partial(\theta) = 0$ by the corollary \[1.2\] we have

$$A^{i,\delta}_{\alpha,j} = A^{j,\delta}_{\alpha,i} \forall i, j, \alpha, \delta.$$  

Contract the neighborhood $(U, p)$ sufficiently small, we can choose a special holomorphic basis $\{e_\alpha\} \text{ of } E$ over $U$ such that $H(p) = \text{Id}$, $dH(p) = 0$ under the frame $\{e_\alpha\}$. At the point $p$, we calculate

$$\begin{align*}
(d^{1,0} \omega_H)(p) &= \sum_{i,j,l} \sum_{\alpha,\beta} < A^{i,\alpha}_{j,l} e_\alpha \otimes e^\beta, A^{j,\beta}_{l,r} e_r \otimes e^\gamma >_H dz_l \wedge dz_i \wedge \overline{dz_j} \\
&= \sum_{j=1}^m \sum_{i,l=1}^n \overline{A^{i,\alpha}_{j,l}} A^{j,\alpha}_{i,l} dz_l \wedge dz_i \wedge \overline{dz_j} \\
&= \sum_{j=1}^m \sum_{i,l=1}^n \overline{A^{i,\alpha}_{j,l}} (\sum_{i,l=1}^n A^{i,\alpha}_{j,l} dz_l \wedge dz_i) \wedge \overline{dz_j} \\
&= 0.
\end{align*}$$  

Similarly, $d^{0,1} \omega_H = 0$ at the point $p$.

The rest is obvious. \hfill \Box

We still use $H$ to represent the dual metric of $\Omega^1_{A_{g,1}}$ induced by $(T_{A_{g,1}}, H)$. We write $\Theta(T_{A_{g,1}}, H)$ (resp. $\Theta(\Omega^1_{A_{g,1}}, H)$) as the Chern curvature form of the vector bundle $T_{A_{g,1}}$ (resp. $\Omega^1_{A_{g,1}}$). As the canonical Bergman metric on $A_{g,1}$ is Kähler-Einstein, there is $-1/2 \text{Trace}_H(\Theta(T_{A_{g,1}}, H)) = -\lambda \omega_H$, where $\lambda$ is a positive constant. Without lost of generality, we always assume $\lambda = 1$ for convenience.

**Theorem 1.5.** Let $\Gamma$ be a neat arithmetic subgroup of $\text{Sp}(g, \mathbb{Z})$. Let $\tilde{A}_{g,1}$ be an arbitrary smooth compactification (not necessary smooth toroidal compactification) of the Siegel variety $A_{g,1} := \Gamma \backslash S_g$ such that $D_\infty = \tilde{A}_{g,1} \backslash A_{g,1}$ is a simple normal crossing divisor. We have:

1. The canonical Bergman metric $H_{\text{can}}$ of $A_{g,1}$ has logarithmic degeneration along the boundary divisor $D_\infty$ in the sense of the following description:

Let $p$ a point in $\tilde{A}_{g,1}$ with a coordinate chart $(U, (z_1, \cdots, z_n)) (n = g(g + 1)/2)$ such that

$$U \cap A_{g,1} = \{(z_1, \cdots, z_l, \cdots, z_n) \mid 0 < |z_i| < r(i = 1, \cdots, l), |z_j| < r(j = l + 1, \cdots, n)\}.$$  

Let $\omega_{\text{can}}$ be the Kähler form of the canonical Bergman metric $H_{\text{can}}$. There holds

$$|\omega_{\text{can}}| \leq C \prod_{i=1}^l - \log |z_i|^M$$  

in the coordinate chart $\{(z_1, \cdots, z_l, \cdots, z_n) \mid 0 < |z_i| < r(i = 1, \cdots, l), |z_j| < r(j = l + 1, \cdots, m)\}$ of $p$ for a suitable $r > 0$, where $C, M$ are positive constants depending on $r$.

2. The Kähler form $\omega_{\text{can}}$ becomes a closed positive current $[\omega_{\text{can}}]$ on $\tilde{A}_{g,1}$. 

- 8 -
3. There is an equality \( c_1(\Omega^1_{\mathcal{A}_{g,\Gamma}}(D_{\infty})) = [\omega_{\text{can}}] \). The line bundle
\[
\omega_{\mathcal{A}_{g,\Gamma}}(D_{\infty}) = \bigwedge^{\dim \mathcal{A}_{g,\Gamma}} \Omega^1_{\mathcal{A}_{g,\Gamma}}(\log D_{\infty})
\]
is numerically effective on \( \mathcal{A}_{g,\Gamma} \).

4. The line bundle \( \omega_{\mathcal{A}_{g,\Gamma}}(D_{\infty}) \) is big on \( \mathcal{A}_{g,\Gamma} \).

Before proving the theorem 1.5, we review the theory of degeneration of Hodge metrics on any polarized variation of Hodge structures over a quasi Kähler manifold.

Let \( X \) be an open Kähler manifold of complex dimension \( m \). Let \( \overline{X} \) be one smooth compactification of \( X \) such that the boundary \( D := \overline{X} - X \) is a simple normal crossing divisor. Let \( j : X \xrightarrow{\sim} \overline{X} \) be the open embedding. Let \( V \) be an arbitrary polarized variation of real Hodge structures over \( X \) such that all monodromies around \( D \) are unipotent. Denote by \( V = V \otimes \mathcal{O}_X \).

Consequently, we have a Hodge filtration
\[
V = \mathcal{F}^0 \supset \mathcal{F}^1 \supset \cdots \supset \mathcal{F}^w \supset 0
\]
corresponding to the VHS \( V \), where \( w \) is the weight of the VHS \( V \).

Let \( (\Delta_1, z) \subset \overline{X} \) be a special coordinate neighborhood, i.e., a coordinate neighborhood isomorphic to the polycylinder \( \Delta^m (\Delta := \{ z \in \mathbb{C} \mid |z| \leq 1 \}) \) such that
\[
X \cap \Delta_1 \cong \{ z = (z_1, \cdots, z_l, \cdots, z_m) \in \Delta^m \mid z_1 \neq 0, \cdots, z_l \neq 0 \} = (\Delta^*)^l \times \Delta^{m-l}.
\]

We then have \( \Delta_1 \cap D_{\infty} \cong \{ (z_1, \cdots, z_l, \cdots, z_m) \mid z_1 \cdots z_l = 0 \} \).

For any real number \( 0 < \varepsilon < 1 \), let \( \Delta_\varepsilon \) be a scaling neighborhood of \( \overline{X} \), i.e.,
\[
\Delta_\varepsilon \subset \Delta_1 \text{ and } \Delta_\varepsilon \cong \{ (z_1, \cdots, z_l, \cdots, z_m) \in \Delta^m \mid |z_\alpha| \leq \varepsilon \text{ for } \alpha = 1, \cdots, l \}.
\]

Let \( \gamma_\alpha \) be a local monodromy around \( z_\alpha = 0 \) in \( \Delta_1 \) for \( \alpha = 1, \cdots, l \). Denote by
\[
N_\alpha = \log \gamma_\alpha := \sum_{j \geq 1} (-1)^{j+1} \frac{(\gamma_\alpha - 1)^j}{j}, \quad \forall \alpha,
\]
then each \( N_\alpha \) is nilpotent. Let \((v.\)\) be a flat multivalued basis of \( V \) over \( \Delta_1 \cap X \). The formula
\[
(v.)(z) := \exp\left(\frac{-1}{2\pi i} \sum_{\alpha=1}^l \log z_\alpha N_\alpha \right)(v.)(z)
\]
gives a single-valued basis of \( V \). Deligne’s canonical extension \( \overline{V} \) of \( V \) to \( \Delta_1 \) is generated by \((\overline{v}.\)\) (cf. [Sch73]). The construction of \( \overline{V} \) is independent of the choice of \( z_\alpha \)’s and \((v.\). For any holomorphic subbundle \( \mathcal{N} \) of \( V \), Deligne’s extension of \( \mathcal{N} \) is defined to be \( \overline{\mathcal{N}} := \overline{V} \cap j_* \mathcal{N} \). Then, we have extension of the filtration
\[
\overline{V} = \mathcal{F}^0 \supset \mathcal{F}^1 \supset \cdots \supset \mathcal{F}^w \supset 0,
\]
which is also a filtration of locally free sheaves.

Let \( N \) be a linear combination of \( N_\alpha \). Then \( N \) defines a weight flat filtration \( W_\bullet(N) \) of \( V_\mathbb{C} \) ([Del71], [Sch73]) by
\[
0 \subset \cdots \subset W_{i-1}(N) \subset W_i(N) \subset W_{i+1}(N) \subset \cdots \subset V_\mathbb{C}.
\]

Denote by \( W_\bullet^j := W_\bullet(\sum_{\alpha=1}^l N_\alpha) \) for \( j = 1, \cdots, l \). We can choose a multivalued flat multigrading
\[
V_\mathbb{C} = \bigoplus_{\beta_1, \cdots, \beta_l} V_{\beta_1, \cdots, \beta_l}
\]
such that

\[ \bigcap_{j=1}^{l} W_{\beta_j}^j = \sum_{k_j \leq \beta_j} \mathcal{V}_{k_1, \ldots, k_l}. \]

Let \( h \) be the Hodge metric on the PVHS \( \mathcal{V} \). In a special coordinate neighborhood \( \Delta_1 \), let \( v \) be a nonzero local multivalued flat section of \( \mathcal{V}_{k_1, \ldots, k_l} \), then \( (\tilde{v})(z) := \exp(-\frac{\sum_{a=1}^{l} \log z_{\alpha} N_{\alpha}}{2\pi i}) v(z) \) is a local single-valued section of \( \mathcal{V} \). There holds a norm estimate (Theorem 5.21 in \cite{CKS86})

\[ ||\tilde{v}(z)||_h \leq C'' \left( \prod_{\alpha=1}^{l} -\log |z_{\alpha}| \right)^{k_1/2} \left( \prod_{\alpha=1}^{l} -\log |z_{\alpha}| \right)^{k_2/2} \cdots \left( \prod_{\alpha=1}^{l} -\log |z_{\alpha}| \right)^{k_l/2} \]

on the region

\[ \Xi(N_1, \ldots, N_l) := \{(z_1, \ldots, z_l, \ldots, z_m) \in (\Delta^*)^l \times \Delta^{m-l} \mid |z_1| \leq |z_2| \leq \cdots \leq |z_l| \leq \varepsilon \} \]

for some small \( \varepsilon > 0 \), where \( C'' \) is a positive constant dependent on the ordering of \( \{N_1, N_2, \ldots, N_l\} \) and \( \varepsilon \). Since the number of the ordering of \( \{N_1, \ldots, N_l\} \) is finite, for any flat multivalued local section \( v \) of \( \mathcal{V} \) there exist positive constants \( C''(\varepsilon) \) and \( M'' \) such that

\[ (1.5.1) \quad ||\tilde{v}(z)||_h \leq C''(\varepsilon) \left( \prod_{\alpha=1}^{l} -\log |z_{\alpha}| \right)^{M''} \]

in the domain \( \{(z_1, \ldots, z_l, \ldots, z_m) \mid 0 < |z_i| < \varepsilon (i = 1, \ldots, l), |z_j| < \varepsilon (j = l + 1, \ldots, m)\} \).

Moreover, since the dual \( \mathcal{V}' \) is also a polarized real variation of Hodge structures, we then have that for any flat multivalued local section \( v \) of \( \mathcal{V} \) there holds

\[ (1.5.2) \quad C'_1 \left( \prod_{\alpha=1}^{l} -\log |z_{\alpha}| \right)^{-M} \leq ||\tilde{v}(z)||_h \leq C'_2 \left( \prod_{\alpha=1}^{l} -\log |z_{\alpha}| \right)^{M} \]

in the domain \( \{(z_1, \ldots, z_l, \ldots, z_m) \mid 0 < |z_i| < \varepsilon (i = 1, \ldots, l), |z_j| < \varepsilon (j = l + 1, \ldots, m)\} \) for some suitable \( \varepsilon > 0 \), where \( C'_1, C'_2 \) and \( M' \) are positive constants.

**Proposition 1.6.** Let \( X \) be an open Kähler manifold of dimension \( m \) and \( \overline{X} \) a smooth compactification of \( X \) such that the boundary \( D := \overline{X} - X \) is a simple normal crossing divisor. Let \( (\Delta_1, z = (z_1, \ldots, z_l, \ldots, z_m)) \subset \overline{X} \) be an arbitrary special coordinate neighborhood in which \( D \) is given by \( \prod_{i=1}^{l} z_i = 0 \).

Let \( \mathcal{V} \) be a polarized real VHS on \( X \) such that all local monodromies of \( \mathcal{V} \) around the simple normal crossing boundary divisor are unipotent, and \( h \) the Hodge metric on \( \mathcal{V} = \mathcal{V} \otimes \mathcal{O}_X \). Let \( \mathcal{N} \) be an arbitrary holomorphic subbundle of \( \mathcal{V} \) and \( \overline{\mathcal{N}} \) its Deligne's extension.

We have

\[ \Gamma(\Delta_\varepsilon, \overline{\mathcal{N}}) = \{ s \in \Gamma(\Delta_\varepsilon \cap X, \mathcal{N}) \mid ||s||_h \leq C \left( \sum_{\alpha=1}^{l} -\log |z_{\alpha}| \right)^M \} \]

for some constants \( C, M \), where \( \Delta_\varepsilon \) is a scaling neighborhood of \( \overline{X} \) for a sufficient small \( \varepsilon > 0 \).

**Proof.** Let \( (v) \) be a local flat multivalued basis of \( \mathcal{V} \) over \( \Delta_1 \cap X \), and so we have a local basis \( (\tilde{v})(z) = \exp(-\frac{1}{2\pi i} \sum_{\alpha=1}^{l} \log z_{\alpha} N_{\alpha}) v(z) \) of \( \overline{\mathcal{V}} \). Denote by \( h_{ij} = < \tilde{v}_i, v_j >_H \). According to the estimate \( (1.5.2) \) there are positive constants \( C, M \) such that

\[ (1.6.1) \quad |h_{ij}|, \quad \det(h_{ij}), \quad (\det(h_{ij}))^{-1} \leq C \left( \sum_{\alpha=1}^{l} -\log |z_{\alpha}| \right)^{2M} \]

in a suitable neighborhood \( \Delta_\varepsilon \). One can use Proposition 1.3 in \cite{Mum77} to finish the proof. 

\( \square \)
Proof of the theorem 1.5.

2. By the lemma 1.3 and the theorem 1.4, we can realize the holomorphic tangent bundle $T_{A_g,\Gamma}$ as a holomorphic subbundle of an Hodge bundle given by some PVHS, and The induced Hodge metric $H$ on the Siegel variety $A_{g,\Gamma} = \Gamma \backslash \mathfrak{H}_g$ is same as the canonical Bergman metric. We then finish the first statement by using the estimate 1.5.1.

3. By the statement (1), we have

$$\int_{\tilde{A}_{g,\Gamma}} |\omega_{\text{can}} \wedge \xi| < \infty$$

for any smooth $(2(\dim A_{g,\Gamma}) - 2)$-form $\xi$ on $\tilde{A}_{g,\Gamma}$, and so the form $\omega_{\text{can}}$ on $A_{g,\Gamma}$ defines a current $[\omega_{\text{can}}]$ on $\tilde{A}_{g,\Gamma}$ as follows:

$$< [\omega_{\text{can}}], \phi > = \int_{\tilde{A}_{g,\Gamma}} [\omega_{\text{can}}] \wedge \phi := \int_{A_{g,\Gamma}} \omega_{\text{can}} \wedge \phi,$$

where $\phi$ is a smooth $(2 \dim(A_{g,\Gamma}) - 2)$ form on $\tilde{A}_{g,\Gamma}$.

We begin to show that $d[\omega_{\text{can}}] = [d\omega_{\text{can}}] = 0$. Let $\xi$ be any smooth $(2 \dim(A_{g,\Gamma}) - 3)$-form on $\tilde{A}_{g,\Gamma}$. Let $T_\delta$ be a tube neighborhood of $D_{\infty,n} = \tilde{A}_{g,\Gamma} - A_{g,\Gamma}$ with radius $\delta$ and $M_\delta := \tilde{A}_{g,\Gamma} \setminus T_\delta$. Then $\partial T_\delta = -\partial M_\delta$. By definition, $< d[\omega_{\text{can}}], \xi > = -\int_{A_{g,\Gamma}} \omega_{\text{can}} \wedge d\xi$. On the other hand, we have that

$$0 = \int_{A_{g,\Gamma}} d\omega_{\text{can}} \wedge \xi = -\int_{A_{g,\Gamma}} \omega_{\text{can}} \wedge d\xi + \int_{A_{g,\Gamma}} d(\omega_{\text{can}} \wedge \xi) = -\int_{A_{g,\Gamma}} \omega_{\text{can}} \wedge d\xi + \lim_{\delta \to 0} \int_{M_\delta} d(\omega_{\text{can}} \wedge \xi)$$

(by Stoke’s theorem) $= -\int_{A_{g,\Gamma}} \omega_{\text{can}} \wedge d\xi + \lim_{\delta \to 0} \int_{\partial M_\delta} \omega_{\text{can}} \wedge \xi$

$$= -\int_{A_{g,\Gamma}} \omega_{\text{can}} \wedge d\xi.$$

Here we use that $\omega_H$ has Poincaré growth on $D_{\infty}$ to obtain

$$\lim_{\delta \to 0} \int_{\partial T_\delta} \omega_{\text{can}} \wedge \xi = 0.$$

4. Since $[\omega_{\text{can}}]$ is a positive closed current, it is a cohomology class on $\tilde{A}_{g,\Gamma}$ of type $(1,1)$. To prove that $[\omega_{\text{can}}]$ represents the first Chern class $c_1(\Omega^1_{\tilde{A}_{g,\Gamma}}(\log D_{\infty}))$, we only need to show the following equality

$$< [\omega_{\text{can}}], \eta > = c_1(\Omega^1_{\tilde{A}_{g,\Gamma}}(\log D_{\infty})), \eta >$$

for any closed smooth $(2 \dim(A_{g,\Gamma}) - 2)$-form $\eta$ on $\tilde{A}_{g,\Gamma}$. 
Let $\eta$ be an arbitrary closed smooth $(2 \dim(A_{g, \Gamma}) - 2)$-form on $\tilde{A}_{g, \Gamma}$. Let $\tilde{H}$ be an arbitrary Hermitian metric on the bundle $\Omega^1_{\tilde{A}_{g, \Gamma}}(\log D_{\infty})$. We have

$$<c_1(\Omega^1_{\tilde{A}_{g, \Gamma}}(\log D_{\infty})), \eta> := \int_{\tilde{A}_{g, \Gamma}} \text{Trace}_{\tilde{H}}(\Theta(\Omega^1_{\tilde{A}_{g, \Gamma}}(\log D_{\infty}), \tilde{H})) \wedge \eta$$

$$= \frac{-1}{2\pi \sqrt{-1}} \int_{\tilde{A}_{g, \Gamma}} \bar{\partial} \partial \log(\det \tilde{H}) \wedge \eta$$

$$= \frac{-1}{2\pi \sqrt{-1}} \int_{\tilde{A}_{g, \Gamma}} \bar{\partial} \partial \log(\det \tilde{H}) \wedge \eta$$

where $\Theta(\Omega^1_{\tilde{A}_{g, \Gamma}}(\log D_{\infty}), \tilde{H})$ is the Chern form of $(\Omega^1_{\tilde{A}_{g, \Gamma}}(\log D), \tilde{H})$, and

$$<[\omega_{can}], \eta> := \int_{A_{g, \Gamma}} \omega_{can} \wedge \eta$$

$$= \frac{-1}{2\pi \sqrt{-1}} \int_{A_{g, \Gamma}} \text{Trace}_{H_{can}} \Theta(\Omega^1_{A_{g, \Gamma}}, H_{can}) \wedge \eta$$

$$= \frac{-1}{2\pi \sqrt{-1}} \int_{A_{g, \Gamma}} \bar{\partial} \partial \log(\det H_{can}) \wedge \eta.$$

Thus, it is sufficient to show that

$$\lim_{\delta \to 0} \int_{M_\delta} \bar{\partial} \partial \log(\frac{\det H_{can}}{\det \tilde{H}}) \wedge \eta = 0.$$

We note that $\zeta := \partial \log \det H_{can} - \partial \log \det \tilde{H}$ is a global $(1, 0)$-form on $A_{g, \Gamma}$, we then get

$$\int_{M_\delta} \bar{\partial} \partial \log(\frac{\det H_{can}}{\det \tilde{H}}) \wedge \eta = \int \bar{\partial} \partial \zeta \wedge \eta$$

$$= - \int_{\partial T_\delta} \zeta \wedge \eta.$$

Using Proposition 5.22 in [CKS86], Kollár shows that the $(1, 0)$-form $\zeta$ near the boundary divisor $D_{\infty}$ is nearly bounded (cf. [Kol87]). Thus, we have

$$\lim_{\delta \to 0} \int_{\partial T_\delta} \zeta \wedge \eta = 0.$$

5. Since the metric connection form of any Hodge metric and its curvature form are both nearly bounded around the boundary divisor $D_{\infty}$ (cf. Proposition 5.7 [Kol87]), we have:

$$C_1(\Omega^1_{A_{g, \Gamma}}(\log D_{\infty}))^{\dim A_{g, \Gamma}} = (\frac{-1}{2\pi \sqrt{-1}})^{\dim A_{g, \Gamma}} \int_{A_{g, \Gamma}} \text{Trace}_H(\Theta(\Omega^1_{A_{g, \Gamma}}, H))^{\dim A_{g, \Gamma}} > 0.$$

Since $\omega_{A_{g, \Gamma}}(D_{\infty})$ is a numerically effective line bundle on $\tilde{A}_{g, \Gamma}$, we obtain that $\omega_{\tilde{A}_{g, \Gamma}}(D_{\infty})$ is a big line bundle by Siu’s numerical criterion in [Siu93].

\[\square\]

\textbf{Remark.} We should point out that the statement (4) is proven in [Mum77] by using some calculations depending on a smooth toroidal compactification $\tilde{A}_{g, \Gamma}$. Actually, the statement (4) can also be proven by Siu-Yau’s result on compactification (cf. Lemma 6 in [SY82]) or by Zuo’s result on positivity (cf. Theorem 0.1 in [Zuo00]).

Here is an interesting application of the estimates [1.5.1, 1.5.2] and the proposition [1.6]:
Lemma 1.7 (Moeller-Viehweg-Zuo [MVZ07]). Let $\Gamma$ be a neat arithmetic subgroup of $\text{Sp}(g, \mathbb{Z})$. Let $\mathcal{A}_{g,\Gamma}^\text{tor}$ be a smooth toroidal compactification of the Siegel variety $A_{g,\Gamma} := \Gamma \backslash \mathfrak{H}_g$ such that $D_\infty := \mathcal{A}_{g,\Gamma}^\text{tor} \setminus A_{g,\Gamma}$ is simple normal crossing. Let $(\mathcal{E}, \theta, h)$ be a homogenous Hodge bundle induced by PVHS on $A_{g,\Gamma}$.

Deligne’s canonical extension of the bundle $\mathcal{E}$ on $\mathcal{A}_{g,\Gamma}^\text{tor}$ coincides with the Mumford good extension(cf. [Mum77]) of $\mathcal{E}$ on $\mathcal{A}_{g,\Gamma}$ by the Bergman metric $h$.

Lemma 1.8. Let $\Gamma$ be a neat arithmetic subgroup of $\text{Sp}(g, \mathbb{Z})$. Let $\mathcal{A}_{g,\Gamma}^\text{tor}$ be a smooth toroidal compactification of the Siegel variety $A_{g,\Gamma} := \Gamma \backslash \mathfrak{H}_g$ such that $D_\infty := \mathcal{A}_{g,\Gamma}^\text{tor} \setminus A_{g,\Gamma}$ is simple normal crossing.

We have the following identifications

$$\mathcal{E}_{\mathcal{A}_{g,\Gamma}}^\text{tor}(-\log D_\infty) = \text{Sym}^2(E_{\mathcal{A}_{g,\Gamma}}^\text{tor}),$$

and

$$\omega_{\mathcal{A}_{g,\Gamma}}(D_\infty) = \dim C_{A_{g,\Gamma}} \Omega^1_{\mathcal{A}_{g,\Gamma}}(\log D_\infty) = (\det E_{\mathcal{A}_{g,\Gamma}}^\text{tor})^{g+1}.$$

Proof. We know that there is an inclusion $\mathcal{E}_{A_{g,\Gamma}} \hookrightarrow (E^\otimes 2)^{\otimes 2}$. Since the Higgs field has the property $\theta \wedge \theta = 0$, the holomorphic subbundle $\text{Sym}^2(E^{\otimes 1})$ of $(E^{\otimes 1})^{\otimes 2}$ must contain the bundle $\mathcal{E}_{A_{g,\Gamma}}$. According to rank $\mathcal{E}_{A_{g,\Gamma}} = \text{rank} \text{Sym}^2(E^{\otimes 1}) = g(g+1)/2$, we obtain $\mathcal{E}_{A_{g,\Gamma}} = \text{Sym}^2(E^{\otimes 1})$.

The holomorphic vector bundle $\text{Sym}^2(E_{\mathcal{A}_{g,\Gamma}}^\text{tor})$ on $\mathcal{A}_{g,\Gamma}^\text{tor}$ is Deligne’s extension of $\text{Sym}^2(E^{\otimes 1})$. Using the proposition 1.7 in the next subsection, $\text{Sym}^2(E_{\mathcal{A}_{g,\Gamma}}^\text{tor})$ is also the unique Mumford’s good extension of $\text{Sym}^2(E^{\otimes 1})$ by the Hodge metric $H$. Shown in Proposition 3.4 [Mum77], $\mathcal{E}_{\mathcal{A}_{g,\Gamma}}(-\log D_\infty)$ is the unique Mumford’s good extension of $\mathcal{E}_{A_{g,\Gamma}}$ by the metric $H$. Therefore,

$$\mathcal{E}_{\mathcal{A}_{g,\Gamma}}(-\log D_\infty) \cong \text{Sym}^2(E_{\mathcal{A}_{g,\Gamma}}^\text{tor}).$$

\[\square\]

Theorem 1.9. Let $\Gamma \subset \text{Sp}(g, \mathbb{Z})$ be a neat arithmetic subgroup. Let $\mathcal{A}_{g,\Gamma}^\text{tor}$ be a smooth toroidal compactification of the Siegel variety $A_{g,\Gamma} := \Gamma \backslash \mathfrak{H}_g$ such that $D_\infty := \mathcal{A}_{g,\Gamma}^\text{tor} \setminus A_{g,\Gamma}$ is a simple normal crossing divisor.

The logarithmic tangent bundle $\mathcal{T}_{\mathcal{A}_{g,\Gamma}}^\text{tor}(-\log D_\infty)$ is a stable vector bundle with respect to the canonical Bergman current $[\omega_{\text{can}}]$. 

Proof. Let $\mathcal{E}$ be Deligne’s canonical extension of the Hodge bundle $(E^{\otimes 2}, H)$, and $E_{\mathcal{A}_{g,\Gamma}}^{\text{tor}}$ be Deligne’s canonical extension of $E^{\otimes 1}$. By the lemma 1.3 and the lemma 1.8, the logarithmic tangent bundle $\mathcal{T}_{\mathcal{A}_{g,\Gamma}}^\text{tor}(-\log D_\infty) = \text{Sym}^2(E_{\mathcal{A}_{g,\Gamma}}^{\text{tor}})$, and so $\mathcal{T}_{\mathcal{A}_{g,\Gamma}}^\text{tor}(-\log D_\infty)$ is a holomorphic subbundle of $\mathcal{E} := E^{\otimes 2}$.

Let $\mathcal{G}$ be an arbitrary subbundle of $\mathcal{E}$ and $\tilde{H}$ an arbitrary Hermitian metric on $\mathcal{G}$. Let $\mathcal{G}_0 := \mathcal{G}|_{A_{g,\Gamma}}$. We know $\mathcal{G}$ is just Deligne canonical extension of $\mathcal{G}_0$. The degree of $\mathcal{G}$ with respect to the Bergman current is defined as

$$\deg_{[\omega_{\text{can}}]} \mathcal{G} := \langle c_1(\mathcal{G}), [\omega^{\text{dim} A_{g,\Gamma} - 1}] \rangle.$$
Let $\eta := \omega^{\dim A_{g,l}-1}$. Similar calculation as (3) of the theorem \[1.5\] we have that
\[
\deg [\omega_{\text{can}}] G = \int_{A_{g,l}} \text{Trace}_{\bar{H}}(\Theta(G, \bar{H})) \wedge [\eta] \\
= \frac{-1}{2\pi \sqrt{-1}} \int_{A_{g,l}} \partial \bar{\partial} \log(\det \bar{H}) \wedge [\eta] \\
= \frac{-1}{2\pi \sqrt{-1}} \int_{A_{g,l}} \partial \bar{\partial} \log(\det H) \wedge \eta \\
= \frac{-1}{2\pi \sqrt{-1}} \int_{A_{g,l}} \partial \bar{\partial} \log(\det H) \wedge \eta + \frac{1}{2\pi \sqrt{-1}} \int_{A_{g,l}} \partial \bar{\partial} \log(\det H \wedge \eta) \\
= \frac{-1}{2\pi \sqrt{-1}} \int_{A_{g,l}} \partial \bar{\partial} \log(\det H) \wedge \eta \\
= \int_{A_{g,l}} \text{Trace}_{H}(\Theta(G_0, H)) \wedge \omega^{\dim A_{g,l}-1}_{\text{can}}.
\]

Since the canonical Bergman metric is Kähler-Einstein, this essential property implies that the logarithmic tangent bundle $T^\text{tor}_{A_{g,l}}(-\log D_\infty)$ is a poly-stable vector bundle with respect to $[\omega]$.

On the other hand, $A_{g,l}$ is simple, then we obtain that the logarithmic tangent bundle $T^\text{tor}_{A_{g,l}}(-\log D_\infty)$ can not be decomposed into a direct sum by the argument in the third paragraph of Page 272 in \[Yau87\] and the argument of Page 478-478 in \[Yau93\].

\[\square\]

2. General type for Siegel varieties $A_{g,l}$

All definitions and notations related to toroidal compactifications of Siegel varieties can be found in \[AMRT, Chai, FC\] and \[YZ11\]. We do not recite these definitions and notations in this section again, and use them freely.

Let $\mathfrak{H}_0$ be the standard minimal cusp of the Siegel space $\mathfrak{H}_g$. Let $\Sigma_{\mathfrak{H}_0} := \{\sigma_{\mathfrak{H}_0}\}$ be a suitable $GL(g, \mathbb{Z})$-admissible polyhedral decomposition of $C(\mathfrak{H}_0)$ regular with respect to $Sp(g, \mathbb{Z})$ such that the induced symmetric $Sp(g, \mathbb{Z})$-admissible family $\{\Sigma_{\mathfrak{H}}\}_{\mathfrak{H}}$ of polyhedral decompositions is projective.

For any positive integer $l$, let $\overline{A}_{g,l}$ to be the symmetric toroidal compactification of the Siegel variety $A_{g,l} := \Gamma_g(l) \setminus \mathfrak{H}_g$ constructed by $\{\Sigma_{\mathfrak{H}}\}_{\mathfrak{H}}$, and let

$D_{\infty,l} := \overline{A}_{g,l} - A_{g,l}$

the boundary divisor. For convenience, we write $A_g$ for $A_{g,1}$.

2.1. Structures of morphisms $\overline{A}_{n,m} : \overline{A}_{g,n} \rightarrow \overline{A}_{g,m}$ for all $m|n$. For any positive integer $l$, we sketch a key-step in the construction of the symmetric compactification $\overline{A}_{g,l}$ as follows:

Let $\mathfrak{H}$ be an arbitrary cusp of depth $k$. $L_\mathfrak{H}(l) := \Gamma(l) \cap U^\mathfrak{H}(\mathbb{Q})$ is a full lattice in the vector space $U^\mathfrak{H}(\mathbb{C})$, and its dual is $M_\mathfrak{H}(l) := \text{Hom}_\mathbb{Z}(L_\mathfrak{H}(l), \mathbb{Z})$. Explicitly, let $\{\zeta_\alpha\}_{1}^{k(k+1)/2}$ be a lattice basis of $L_\mathfrak{H} := Sp(g, \mathbb{Z}) \cap U^\mathfrak{H}(\mathbb{Q})$ and $\{\delta_\alpha\}_{1}^{k(k+1)/2}$ the associated dual basis of $M_\mathfrak{H} := \text{Hom}_\mathbb{Z}(L_\mathfrak{H}, \mathbb{Z})$; then $\{\zeta'_\alpha := k\zeta_\alpha\}_{1}^{k(k+1)/2}$ is a lattice basis of $L_\mathfrak{H}(l)$, and $\{\delta'_\alpha := k\delta_\alpha\}_{1}^{k(k+1)/2}$ is the dual basis of $M_\mathfrak{H}(l)$. 
For any cone $\sigma \in \Sigma_3$, we get a toroidal variety $X_\sigma(l) := \text{Spec} \mathbb{C}[\sigma^\vee \cap M_3(l)]$; then we have

$$\Delta_{3,\sigma}(l) := \text{the interior of the closure of } \frac{D(\sigma)}{\Gamma(l) \cap U^3(Q)} \text{ in } X_\sigma(l) \times T_3(l) \frac{D(\delta)}{\Gamma(l) \cap U^3(Q)}$$

where $T_3(l) := \text{Spec} \mathbb{C}[M_3(l)]$ is a torus; gluing all $\Delta_{3,\sigma}(l)$ as $\sigma$ runs through $\Sigma_3$, we obtain an analytic variety $Z'_3(l)$ and an open morphism $\pi'_3(l) : Z'_3(l) \to \mathcal{A}_g(l)$. As in Section 2 of [YZII], we define

$$Z_3(l) := \frac{Z'_3(l)}{\Gamma(l) \cap N(3)/\Gamma(l) \cap U^3(\mathbb{R})}.$$ 

Let $n, m$ be two positive integers with $m|n$. We are going to construct a natural morphism $\lambda_{n,m} : \mathcal{A}_{g,n} \to \mathcal{A}_{g,m}$. Given a cusp $\mathfrak{c}$ and a cone $\sigma \in \Sigma_3$, the inclusion of the algebras $\mathbb{C}[\sigma^\vee \cap M_3(m)] \hookrightarrow \mathbb{C}[\sigma^\vee \cap M_3(n)]$ induces a finite surjective morphism $\lambda^\sigma : X_\sigma(n) \to X_\sigma(m)$. Therefore, we have an analytic surjective morphism

$$\lambda^\sigma : \Delta_{3,\sigma}(n) \to \Delta_{3,\sigma}(m),$$

such that any $\tau < \sigma$ there holds a commutative diagram

$$\begin{array}{ccc}
\Delta_{3,\tau}(n) & \overset{c}{\longrightarrow} & \Delta_{3,\tau}(n) \\
\lambda_{n,m}^\tau \downarrow & & \downarrow \lambda_{n,m}^\tau \\
\Delta_{3,\tau}(m) & \overset{c}{\longrightarrow} & \Delta_{3,\tau}(m)
\end{array}$$

and so we obtain a morphism $\lambda_{3}^\tau : Z'_3(n) \to Z'_3(m)$ by gluing all $\lambda_{n,m}^\sigma \forall \sigma \in \Sigma_3$. Since $\Gamma_g(n)$ is a normal subgroup of $\Gamma_g(m)$, the morphism $\lambda_{3}^\tau$ reduces to the morphism

$$\lambda_3 : Z_3(n) \to Z_3(m).$$

It can be verified straightforwardly that $\lambda_3$ is compatible with the morphisms $\Pi_{3\alpha,3\beta}$ and the action of $\Gamma$. Therefore, we have a global morphism

$$\lambda_{n,m} : \mathcal{A}_{g,n} \to \mathcal{A}_{g,m}.$$ 

Let $\sigma$ be an arbitrary topo-dimensional cone in $\Sigma_{3\alpha}$. Consider the inclusion

$$0 \to \mathbb{C}[\sigma^\vee \cap M_{3\alpha}(m)] \hookrightarrow \mathbb{Q}(\mathbb{C}[\sigma^\vee \cap M_{3\alpha}(n)])$$

where $\mathbb{Q}(\mathbb{C}[\sigma^\vee \cap M_{3\alpha}(n)])$ is the quotient field of the integral domain $\mathbb{C}[\sigma^\vee \cap M_{3\alpha}(n)]$. The algebra $\mathbb{C}[\sigma^\vee \cap M_{3\alpha}(n)]$ is indeed the integral closure of $\mathbb{C}[\sigma^\vee \cap M_{3\alpha}(m)]$ in $\mathbb{Q}(\mathbb{C}[\sigma^\vee \cap M_{3\alpha}(n)])$. Then, the compactification $\mathcal{A}_{g,n}$ is a normalization of the morphism $\mathcal{A}_{g,n} \to \mathcal{A}_{g,m}$ and so the morphism $\lambda_{n,m} : \mathcal{A}_{g,n} \to \mathcal{A}_{g,m}$ factors through the morphism $\lambda_{n,m} : \mathcal{A}_{g,n} \to \mathcal{A}_{g,m}$ (cf. [FC]). Thus, we obtain the following commutative diagram of morphisms

$$\begin{array}{ccc}
\mathcal{A}_{g,n} & \overset{\lambda_{n,m}}{\longrightarrow} & \mathcal{A}_{g,m} \\
\lambda_{n,m} \downarrow & & \downarrow \lambda_{n,m} \\
\mathcal{A}_{g,1} & \overset{\lambda_{n,1}}{\longrightarrow} & \mathcal{A}_{g,1}
\end{array}$$

**Lemma 2.1.** Let $n, m \geq 1$ be two positive integers with $m|n$. Let $\Sigma_{3\alpha} := \{ \sigma^{3\alpha}_m \}$ be a $\Gamma_{3\alpha}$ (or GL$(g, \mathbb{Z})$)-admissible polyhedral decomposition of $C(3\alpha)$ regular with respect to Sp$(g, \mathbb{Z})$.

Let $\mathcal{A}_{g,n}$ (resp. $\mathcal{A}_{g,m}$) be the symmetric toroidal compactification of $A_{g,n}$ (resp. $A_{g,m}$) constructed by $\Sigma_{3\alpha}$. The morphism $\lambda_{n,m} : \mathcal{A}_{g,n} \to \mathcal{A}_{g,m}$ has the following property:

$$\lambda_{n,m} D_{\infty,m} = \frac{n}{m} D_{\infty,n},$$
where $D_{\infty,m} := \overline{A}_{g,m} \setminus A_{g,m}$ and $D_{\infty,n} := \overline{A}_{g,n} \setminus A_{g,n}$.

Proof. By the construction of boundary divisors of Siegel varieties from edges of the fan $\Sigma_{\delta_0}$ in Theorem 2.22 of [Yau11], to study the relation between $D_{\infty,m}$ and $D_{\infty,n}$ is sufficient to study the morphism $\lambda^*_{\max} : \overline{\Delta}_{\delta_0,\sigma_{\max}}(n) \to \overline{\Delta}_{\delta_0,\sigma_{\max}}(m)$ for any top-dimensional cone $\sigma_{\max}$ in $\Sigma_{\delta_0}$.

We can choose a basis $\{\zeta_\alpha\}_{\alpha=1}^{g(g+1)/2}$ of $L_{\delta_0} := \text{Sp}(g,\mathbb{Z}) \cap U_{\delta_0}(\mathbb{Z})$ such that

$$\sigma_{\max} = \{ \sum_{\alpha=1}^{g(g+1)/2} \lambda_\alpha \zeta_\alpha \mid \lambda_\alpha \in \mathbb{R}_{\geq 0}, \ 1 \leq \alpha \leq g(g+1)/2 \}.$$ 

Let $\{\delta_\alpha\}_{\alpha=1}^{g(g+1)/2}$ be the dual basis of $\{\zeta_\alpha\}_{\alpha=1}^{g(g+1)/2}$. Then

$$\sigma^\vee_{\max} = \{ \sum_{\alpha=1}^{g(g+1)/2} \lambda_\alpha \delta_\alpha \mid \lambda_\alpha \in \mathbb{R}_{\geq 0}, \ 1 \leq \alpha \leq g(g+1)/2 \}.$$ 

Since the inclusion $0 \to \mathbb{C}[\sigma^\vee \cap M_{\delta_0}(m)] \to \mathbb{C}[\sigma^\vee \cap M_{\delta_0}(n)]$ is of the following type

$$0 \to \mathbb{C}[x_1, \ldots, x_i, \ldots, x_{g(g+1)/2}] \to \mathbb{C}[\sqrt{x_1}, \ldots, \sqrt{x_i}, \ldots, \sqrt{x_{g(g+1)/2}}],$$ 

we must have $\overline{\lambda}^*_{\alpha,m} D_{\infty,m} = \frac{n}{m} D_{\infty,n}$.

\[\square\]

2.2. Siegel modular forms. Some materials related to the Satake-Baily-Borel compactification of a Siegel variety are taken from [BB66]. We set

$$\overline{\mathcal{G}}_g := \{ F^1 \in \text{Grass}(g, V_\mathbb{C}) \mid \psi(F^1, F^1) = \sqrt{-1} \psi(F^1, F^\mathbb{T}) \geq 0 \},$$

$$\partial \mathcal{G}_g := \{ F^1 \in \overline{\mathcal{G}}_g \mid \sqrt{-1} \psi(F^1, F^\mathbb{T}) \geq 0, \psi(\cdot, \cdot) \text{ is degenerate on } F^1 \}.$$

A boundary component of the Siegel space $\mathcal{G}_g = \mathcal{G}(V, \psi)$ is a subset in $\partial \mathcal{G}_g$ given by

$$\mathfrak{F}(W_\mathbb{R}) := \{ F^1 \in \overline{\mathcal{G}}_g \mid F^1 \cap F^\mathbb{T} = W_\mathbb{R} \cap \mathbb{C} \text{ where } W_\mathbb{R} \text{ is an isotropic real subspace of } V_\mathbb{R} \}.$$

A boundary component $\mathfrak{F}(W)$ of the Siegel space $\mathcal{G}_g$ is rational(i.e., $\mathfrak{F}(W)$ is a cusp) if and only if $W = W_\mathbb{Q} \otimes \mathbb{R}$, where $W_\mathbb{Q}$ is an isotropic subspace of $(V_\mathbb{Q}, \psi)$.

Define $\mathcal{H}_g^* := \bigcup_{\text{cusp } \mathfrak{F}} \mathfrak{F}$. The set $\mathcal{H}_g^*$ is stable under the action of $\text{Sp}(g, \mathbb{Q})$. Actually, the set $\mathcal{H}_g^*$ is a disjoint union of locally closed $\text{Sp}(g, \mathbb{Q})$-orbits $\mathcal{H}_g^* = \mathcal{O}_0 \cup \mathcal{O}_1 \cup \cdots \cup \mathcal{O}_g$, and each orbit $\mathcal{O}_r$ is a set of disjoint union rational boundary components with same rank, i.e.,

$$\mathcal{O}_{g-r} := \bigcup_{\text{dim } W = r} \mathfrak{F}(W).$$

In particular, $\mathcal{O}_g = \mathfrak{F}(\{0\}) = \mathcal{H}_g$. Let $\Gamma$ be an arbitrary arithmetic subgroup of $\text{Sp}(g, \mathbb{Q})$. Let $A_{g, \Gamma}^* := \Gamma \setminus \mathcal{H}_g^*$ be the Satake-Baily-Borel compactification of $A_{g, \Gamma}$. It is known that the analytic variety $A_{g, \Gamma}^*$ has an algebraic structure as a normal projective complex variety.

Let $l$ be an arbitrary positive integer. It is known that there is $\mathfrak{F}(W) \cong \mathcal{H}_{g-r}$ for any cusp with $\text{dim } W = r$. Therefore, the quotient $\Gamma_g(l) \setminus \mathcal{O}_{g-r}$ is a disjoint union of $[\text{Sp}(g, \mathbb{Z}) : \Gamma_g(l)]$ locally closed subsets, and each disjoint component of $\Gamma_g(l) \setminus \mathcal{O}_r$ is canonically isomorphic to the Siegel variety $A_{g-r,l}$. The Satake-Baily-Borel compactification $A_{g,l}^*$ of the Siegel variety $A_{g,l}$ is

$$A_{g,l}^* := \Gamma_g(l) \setminus \mathcal{H}_g^* = (\Gamma_g(l) \setminus \mathcal{O}_0) \cup (\Gamma_g(l) \setminus \mathcal{O}_1) \cup \cdots \cup (\Gamma_g(l) \setminus \mathcal{O}_g).$$
In particular, we have $A^*_g = A_g \cup A_{g-1} \cup \cdots \cup A_0$. The construction of Satake-Baily-Borel compactifications shows there is a natural morphism

$$\lambda^*_{n,m} : A^*_g \rightarrow A^*_{g,m}$$

for any two positive integers $m, n$ with $m|n$.

**Definition 2.2** (Cf. [Fre]). Let $k \geq 1, n \geq 1, g \geq 2$ be integers. A complex-valued function on $\mathfrak{H}_g$ is called a **Siegel modular form of weight** $k$, **degree** $g$ and **level** $n$ if the following conditions are satisfied:

- $f : \mathfrak{H}_g \rightarrow \mathbb{C}$ is a holomorphic function;
- $f(\tau) = (f|\gamma)(\tau) := \det(C\tau + D)^{-k} f(\gamma(\tau)) \quad \forall \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g(n)$.

Denote by

$$M_k(\Gamma_g(n)) := \{\text{Siegel modular forms of weight } k, \text{ degree } g \text{ and level } n\}.$$

Recall some important properties of Siegel modular forms (Cf. [Fre]): Let $f \in M_k(\Gamma_g(n))(g \geq 2)$. The Siegel modular form $f$ has an expansion of the form

$$f(\tau) = \sum_{2A \in \text{Sym}_g(\mathbb{Z}), A \geq 0} c(A) \exp(\frac{-1}{n} \sqrt{\text{Tr}(A\tau)})$$

where $c(A)$ are constant coefficients. The series converges absolutely on $\mathfrak{H}_g$ and uniformly on each subset of $\mathfrak{H}_g$ of the form $W^\epsilon_g = \{X + \sqrt{-1}Y \in \mathfrak{H}_g \mid Y \geq \epsilon I_g\}$ with $\epsilon > 0$. In particular, $f$ is bounded on each subsets. All coefficients $c(A)$ for $2A \in \text{Sym}_g(\mathbb{Z})$ and $A \geq 0$ satisfy

$$c(\v VAV \v) = (\det(V))^k \exp(-\frac{1}{n} \sqrt{-1} \text{Tr}(AVU))c(A)$$

for any $M \in \Gamma_g(n)$ of the form $M = M(V, U) = \begin{pmatrix} V^{-1} & U \\ 0 & tV \end{pmatrix}$. The series is called the Fourier expansion of $f$, and any $c(A)$ with $2A \in \text{Sym}_g(\mathbb{Z})$ and $A \geq 0$ is a Fourier coefficient of $f$.

Let $\mathfrak{H}$ be an arbitrary positive integer. Define $\tau_t = \begin{pmatrix} \tau' & 0 \\ 0 & \sqrt{-1}t \end{pmatrix}$ in $\mathfrak{H}_g$ with $\tau' \in \mathfrak{H}_{g-1}$, $t > 0$.

By Proposition 1.3 in Section 1 [YZ11], $\lim_{t \to \infty} \tau_t$ corresponds to the following element in $\partial \mathfrak{C}_g$

$$F_{\tau', \infty} := \text{the subspace of } V_C \text{ spanned by the column vectors of } \begin{pmatrix} \tau' & 0 \\ 0 & 1 \\ I_{g-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

Let $W_g$ be the one dimension isotropic real subspace of $V_\mathbb{R}$ generated by $e_g$. $F_{\tau', \infty}$ is actually in the cusp $\mathfrak{C}(W_g)$. The properties and guarantee that any modular $f \in M_k(\Gamma_g(n))$ can extend to be a holomorphic function on

$$\mathfrak{H}_g \cup \bigcup_{\text{Cusp } \mathfrak{C} \text{ with } \mathfrak{C}(W_g) \subseteq \mathfrak{C}} \mathfrak{C},$$

we then have

$$f(F_{\tau', \infty}) = \lim_{t \to \infty} f\left( \begin{pmatrix} \tau' & 0 \\ 0 & \sqrt{-1}t \end{pmatrix} \right) := \Phi_n(f)(\tau')$$

$$= \sum_{2A' \in \text{Sym}_{g-1}(\mathbb{Z}), A' \geq 0} c(\begin{pmatrix} A' & 0 \\ 0 & 0 \end{pmatrix}) \exp(\sqrt{-1} \pi \text{Tr}(A' \tau')).$$
Therefore, for \( g \geq 2 \) we can define the **Siegel operators** \( \Phi_n : M_k(\Gamma_g(n)) \rightarrow M_k(\Gamma_{g-1}(n)) \) by sending \( f \) to \( \Phi_n(f) \) for all positive integer \( n \). For any integers \( n \geq 1, k \geq 1, g \geq 2 \),

\[
S_k(\Gamma_g(n)) := \{ f \in M_k(\Gamma_g(n)) \mid \Phi_n(f|_{\gamma}) = 0 \text{ for } \forall \gamma \in \Gamma_g \}.
\]

is a set of all **Siegel cusp forms** of weight \( k \), degree \( g \) and level \( n \).

The Siegel space \( \mathfrak{H}_g \) has a global holomorphic coordinate system \( \tau \). Define a standard Euclidean form \( d\nu \) on \( \mathfrak{H}_g \) to be \( d\nu_\tau := \Lambda_{1 \leq i < j \leq g} d\tau_{ij} \) for \( \tau = (\tau_{ij})_{1 \leq i < j \leq g} \in \mathfrak{H}_g \). There is

\[
d\nu_M(\tau) = \det(C\tau + D)^{-(g+1)} d\nu_\tau \quad \text{for } M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(g, \mathbb{R}).
\]

Let \( \omega_{\mathfrak{H}_g} \) be the canonical line bundle on \( \mathfrak{H}_g \). For any form \( \varphi = f_\varphi \bigwedge_{1 \leq i < j \leq g} d\tau_{ij} \) in \( \Gamma(\mathfrak{H}_g, \omega_{\mathfrak{H}_g}^{\otimes k}) \), there is an associated smooth positive \((g+1)/2 \)-form

\[
(\varphi \wedge \overline{\varphi})^{1/k} := |f_\varphi|^{2/k} \bigwedge_{1 \leq i < j \leq g} \frac{\sqrt{-1}}{2\pi} d\tau_{ij} \wedge d\overline{\tau_{ij}}.
\]

**Lemma 2.3.** Let \( n \geq 3, k \geq 1, g \geq 2 \) be integers. Let \( f \in M_{k(g+1)}(\Gamma_g(n)) \) be a modular form. With respect to the correspondence

\[
M_{k(g+1)}(\Gamma_g(n)) \xrightarrow{\cong} \Gamma(\mathfrak{H}_g, \omega_{\mathfrak{H}_g}^{\otimes k}) \Gamma_g(n) := \{ s \in \Gamma(\mathfrak{H}_g, \omega_{\mathfrak{H}_g}^{\otimes k}) \mid s \text{ is } \Gamma_g(n)-\text{invariant} \}
\]

the following two conditions are equivalent:

a) \( f \in S_{k(g+1)}(\Gamma_g(n)) \);

b) the holomorphic form \( \varphi_f \) vanishes on all cusps of \( \mathfrak{H}_g \).

Moreover, if \( n \geq 3 \) then (a) or (b) is equivalent to the following

(c) \( \int_{A_{g,n}} (\varphi_f \wedge \overline{\varphi_f})^{1/k} < \infty \).

**Proof.** Let \( W_g \) be the one dimension isotropic real subspace of \( V_\mathbb{R} \) generated by \( e_g \).

"(a) \iff (b)": We only show the case of \( n = 1 \), the others are similar. Suppose \( f \) is a cusp form. Then, \( f \) vanishes on the cusp \( \mathfrak{F}(W_g) \), and so \( f \) vanishes on any cusp \( \mathfrak{F} \) with \( \mathfrak{F}(W_g) \subset \mathfrak{F} \). Since \( \varphi_f \) is a \( \Gamma_g \)-invariant form, \( \varphi \) vanishes on all proper cusps of \( \mathfrak{H}_g \). The converse part is obvious.

Assume that \( n \geq 3 \). We begin to show that "(a) \iff (c)"

We note that \( \int_{A_{g,n}} dV_B < \infty \) where

\[
dV_B(\tau) = \det(\text{Im}(\tau))^{-(g+1)} \bigwedge_{1 \leq i < j \leq g} \frac{\sqrt{-1}}{2\pi} d\tau_{ij} \wedge d\overline{\tau_{ij}},
\]

is the volume form of the \( \Gamma_g \)-invariant Bergman metric on \( \mathfrak{H}_g \).

- Suppose \( f \) is a cusp form. Then,

\[
f\left( \begin{pmatrix} \tau' & 0 \\ 0 & \sqrt{-1}y \end{pmatrix} \right) = O(\exp(-\frac{\pi}{2}y)) \quad \text{for } y >> 0,
\]

and so \( \int_{A_{g,n}} (\varphi_f^2 \wedge \overline{\varphi_f})^{1/k} < \infty \).
• Suppose \( f \) is not a cusp. Then, there is a \( \tau'_0 \in \mathcal{S}_{g-1} \) such that \( \Phi_n(f)(\tau'_0) \neq 0 \). Thus, there is a sufficiently small neighborhood \( U_{f',\infty} \) in \( \mathcal{S}_g \) such that \( |f(Z)| > c > 0 \) on \( U_{f',\infty} \) for some positive constant \( c \). Therefore,

\[
\int_{A_{g,n}} (\varphi f \wedge \overline{\varphi f})^{1/k} \geq \int_{\Gamma_0(U_{f',\infty})} (\varphi f \wedge \overline{\varphi f})^{1/k} = \infty.
\]

\[\square\]

**Proposition 2.4.** Let \( n \geq 3, k \geq 1, g \geq 2 \) be integers. Let \( \overline{A}_{g,n} \) be an arbitrary projective smooth toroidal compactification of \( A_{g,n} \) with simple normal crossing boundary divisor \( D_{\infty,n} := \overline{A}_{g,n} \setminus A_{g,n} \).

Then, we have:

\[
\Gamma(\overline{A}_{g,n}, \omega_{\overline{A}_{g,n}}(D_{\infty,n})^\otimes k) \cong \Gamma(A_{g,n}, \omega_{A_{g,n}}^k) \cong M_{k(g+1)}(\Gamma_g(n)),
\]

\[
\Gamma(\overline{A}_{g,n}, \omega_{\overline{A}_{g,n}}(D_{\infty,n})^\otimes k \otimes \omega_{\overline{A}_{g,n}}) \cong S_{k(g+1)}(\Gamma_g(n)).
\]

where \( \omega_{\overline{A}_{g,n}} \) is the canonical line bundle on \( \overline{A}_{g,n} \) and \( \omega_{A_{g,n}} \) is the canonical line bundle on \( A_{g,n} \).

Proof. With [AMRT], Mumford shows in [Mum77] that the canonical line bundle \( \omega_{A_{g,n}} \) extends to an ample line bundle \( L_{g,n} \) on \( A_{g,n}^* \) and that the canonical morphism \( \overline{\pi}_{g,n} : \overline{A}_{g,n} \to A_{g,n}^* \) is proper with \( \overline{\pi}_{g,n}^*(L_{g,n}) = \omega_{\overline{A}_{g,n}}(D_{\infty,n}) \).

Then, \( \mathcal{O}_{A_{g,n}^*} = (\overline{\pi}_{g,n})_* \mathcal{O}_{\overline{A}_{g,n}} \) and so \((\overline{\pi}_{g,n})_* \omega_{\overline{A}_{g,n}}(D_{\infty,n})^\otimes k = (\overline{\pi}_{g,n})_* (\overline{\pi}_{g,n})^* L^\otimes k \cong L^\otimes k \). Thus,

\[
\Gamma(A_{g,n}^*, L_{g,n}^\otimes k) \cong \Gamma(A_{g,n}^*, (\overline{\pi}_{g,n})_* \omega_{\overline{A}_{g,n}}(D_{\infty,n})^\otimes k) \cong \Gamma(\overline{A}_{g,n}, \omega_{\overline{A}_{g,n}}(D_{\infty,n})^\otimes k).
\]

Let \( j : A_{g,n} \to A_{g,n}^* \) be open embedding. Since \( A_{g,n} \) is normal and \( \text{codim}(A_{g,n}^* \setminus A_{g,n}) = g \geq 2 \), we then have \( j_* \omega_{A_{g,n}}^k = L^\otimes k \). Thus, \( \Gamma(A_{g,n}^*, L^\otimes k) \cong \Gamma(A_{g,n}, \omega_{A_{g,n}}^k) \). That \( \Gamma(A_{g,n}, \omega_{A_{g,n}}^k) \cong M_{k(g+1)}(\Gamma_g(n)) \) is obvious.

By the lemma [2,3] and the lemma [2,4] we have

\[
\{ s \in \Gamma(A_{g,n}, \omega_{A_{g,n}}^k) \mid \int_{A_{g,n}} (s \wedge \overline{s})^{1/k} < \infty \}
\]

Shown in Theorem 2.1 of [Sak77], there holds

\[
\{ s \in \Gamma(A_{g,n}, \omega_{A_{g,n}}^k) \mid \int_{A_{g,n}} (s \wedge \overline{s})^{1/k} < \infty \} \cong \Gamma(\overline{A}_{g,n}, \omega_{\overline{A}_{g,n}}(D_{\infty,n})^\otimes k \otimes \omega_{\overline{A}_{g,n}}).
\]

\[\square\]

**Remarks.** Consider the short sequence

\[
0 \to \omega_{\overline{A}_{g,n}}(D_{\infty,n})^\otimes k \otimes \omega_{\overline{A}_{g,n}} \to \omega_{\overline{A}_{g,n}}(D_{\infty,n})^\otimes k \to \omega_{\overline{A}_{g,n}}(D_{\infty,n})^\otimes k|_{D_{\infty,n}} \to 0,
\]

we have:

a) That \( s \in \Gamma(\overline{A}_{g,n}, \omega_{\overline{A}_{g,n}}(D_{\infty,n})^\otimes k) \cong M_{k(g+1)}(\Gamma_g(n)) \) is a cusp form if and only if \( s|_{D_{\infty,n}} = 0 \).

Certainly, this result can also obtained by regarding the projective smooth toroidal compactification \( \overline{A}_{g,n} \) as the normalization of the blowing-up of \( A_{g,n}^* \), along the ideal sheaf \( \mathcal{I} \) supported on the subscheme \( A_{g,n}^* \setminus A_{g,n} \) (cf. Chap IV [AMRT]).

b) Since the line bundle \( \omega_{\overline{A}_{g,n}}(D_{\infty,n})^\otimes k \) is big, there is a finite positive number \( N_0 \) such that

\[
\dim \mathbb{C} S_{k(g+1)}(\Gamma_g(n)) = \dim \mathbb{C} \Gamma(\overline{A}_{g,n}, \omega_{\overline{A}_{g,n}}(D_{\infty,n})^\otimes k \otimes \omega_{\overline{A}_{g,n}}) > 0
\]

for any integer \( k \geq N_0 \).
2.3. A Criterion on varieties of general type. Let $X$ be a complex manifold which is a Zariski open set of a compact complex manifold $\overline{X}$ such that such that the boundary $D := \overline{X} - X$ is divisor with at most simple normal crossing. Let $L$ be a holomorphic line bundle on $\overline{X}$. For any positive integer $m$, let $\phi_{mL}$ be a meromorphic map define by a basis of $H^0(\overline{X}, mL)$. The $L$-dimension of $\overline{X}$ is defined to be

$$\kappa(L, \overline{X}) := \left\{ \begin{array}{ll} \max_{m \in N(L, X)} \{ \dim_{\mathbb{C}}(\phi_{mL}(\overline{X})) \}, & \text{if } N(L, \overline{X}) \neq \emptyset; \\
-\infty, & \text{if } N(L, \overline{X}) = \emptyset, \end{array} \right.$$ 

where $N(L, \overline{X}) := \{ m > 0 \mid \dim_{\mathbb{C}} H^0(\overline{X}, mL) > 0 \}$. We call $\kappa(X) := \kappa(O_X(K_X), \overline{X})$ Kodaira dimension of $X$, and $\kappa(X) := \kappa(O_X(K_X + D), \overline{X})$ logarithmic Kodaira dimension of $X$. $X$ is said to be of general type (resp. logarithmic general type) if $\kappa(X)$ (resp. $\kappa(X)$) equals to $\dim X$. All definitions above are independent of the choice of smooth compactification of $X$ (cf. [Iitaka77]).

By the theorem 1.5, we have:

**Corollary 2.5** (Mumford cf. [Mum77]). Let $g \geq 1, n \geq 3$ be two integers. The Siegel variety $A_{g, n}$ is of logarithmic general type.

In this subsection, we study that how to use the method of cyclic covering to get a variety of general-type from a variety of logarithmic general type.

**Lemma 2.6.** Let $(\overline{X}, D)$ be a compact complex manifold with a simple normal crossing divisor $D$. Let $m, l$ be two arbitrary positive integers, we then have an isomorphism

$$H^0(\overline{X}, O_{\overline{X}}(mK_{\overline{X}} + lD)) \cong \left\{ \begin{array}{ll} \text{m-ple n-form on } \overline{X} \text{ with at most } \text{l-ple poles along } D \end{array} \right\}. \tag{2.6.1}$$

**Proof.** We have a system of coordinates charts $\{(U_\alpha, (z^\alpha_1, \cdots, z^\alpha_n))\}_\alpha$ on $\overline{X}$ satisfying $\overline{X} = \bigcup_{\alpha} U_\alpha$. Let $\sigma$ be a holomorphic section of $O_{\overline{X}}(D)$ defining $D$. We can write $\sigma = \{ \sigma_\alpha \}_\alpha$ such that $(\sigma_\alpha) = D \cap U_\alpha$ with the rule $\sigma_\alpha = \delta_\alpha \sigma_\beta \forall \alpha, \beta$, where every $\delta_\alpha$ is a transition function of the line bundle $O_{\overline{X}}(D)$.

Let $\varphi \in H^0(\overline{X}, O(mK_{\overline{X}} + lD))$ be a global holomorphic section. We write $\varphi = \{ \varphi_\alpha \}_\alpha$ such that

$$\varphi_\alpha = k_\alpha^{m\beta} \delta_{\alpha\beta} \varphi_\beta \text{ on } U_\alpha \cap U_\beta,$$

where every $k_{\alpha\beta} = \text{det}(\partial^{\beta}_{\beta \alpha})$ is a transition function of the canonical line bundle $O_X(K_X)$. Then, we obtain the corresponding $m$-ple $n$-form $\omega = \{ \omega_\alpha \}_\alpha$ on $\overline{X}$ as follows:

$$\omega_\alpha := \frac{\varphi_\alpha}{\sigma_\alpha^l} (dz^\alpha_1 \wedge \cdots \wedge dz^\alpha_n)^m \text{ in } U_\alpha.$$ 

Conversely, let $\omega$ be a $m$-ple $n$-form on $\overline{X}$ with at most $l$-ple poles along $D$. Since $U_\alpha \cap D = \{ z^\alpha_i = 0 \}$, we have

$$\omega_\alpha := \omega|_{U_\alpha} = \frac{f_\alpha (dz^\alpha_1 \wedge \cdots \wedge dz^\alpha_n)^m}{(z^\alpha_i)^{s_i} \cdots (z^\alpha_i)^{s_l}} \text{ on } U_\alpha,$$

where every $s_i$ in an positive integer with $s_i \leq l$. Then, we can write $\omega = \{ \omega_\alpha \}_\alpha$ where

$$\omega_\alpha := \omega|_{U_\alpha} = \frac{\varphi_\alpha (dz^\alpha_1 \wedge \cdots \wedge dz^\alpha_n)^m}{\sigma_\alpha^l} \text{ on } U_\alpha.$$

Since $\omega_\alpha = \omega_\beta$ on $U_\alpha \cap U_\beta$, then $\varphi = \{ \varphi_\alpha \}_\alpha$ defines a global section in $H^0(\overline{X}, O(mK_{\overline{X}} + lD))$ with $\text{div}(\varphi) \sim mK_{\overline{X}} + lD$. \qed
Lemma 2.7 (Kawamata cf. [EV] & [Kawa81]). Let \( X \) be a \( n \)-dimensional quasi-projective nonsingular variety and let \( D = \sum_{i=1}^{r} D_i \) be a simple normal crossing divisor on \( X \). Let \( d_1, \ldots, d_r \) be positive integers. There exists a quasi-projective nonsingular variety \( Z \) and a finite surjective morphism \( \gamma : Z \to X \) such that

i. \( \gamma^* D_i = N_j (\gamma^* D_i)_{red} \) for \( i = 1, \ldots, r \);

ii. \( \gamma^* D \) is a simple normal crossing divisor.

Theorem 2.8. Let \( X \) be a complex non-singular quasi-projective variety of logarithmic general type. There is a nonsingular quasi-projective variety \( Y \) of general type with a finite surjective morphism \( f : Y \to X \).

Proof. Let \( n = \dim_{\mathbb{C}} X \). Let \( \overline{X} \) be a projective smooth compactification of \( X \) with a simple normal crossing boundary divisor \( B \). Since \( \kappa(Y) := \kappa(K_{\overline{X}} + B, \overline{X}) = n \), it is shown by Sakai in Proposition 2.2 of [Sak77] that for some integer \( N > 0 \) there are meromorphic differentials

\[
\eta_0, \ldots, \eta_n \in H^0(\overline{X}, \mathcal{O}_{\overline{X}}(N K_{\overline{X}} + (N - 1)B))
\]

such that \{\( \eta_i/\eta_0, \ldots, \eta_n/\eta_0 \)\} is a transcendence base of the function field \( \mathbb{C}(X) \).

Let \( d \) be an integer more than \( N \). We use Kawamata’s covering trick by setting \( d = N_1 = N_2 = \cdots \) in the lemma 2.7 to get a projective manifold \( \overline{X}_d \) and a finite surjective morphism \( f : \overline{X}_d \to \overline{X} \) such that \( f^*(B) = dD_d \), where \( D_d \) is a simple normal crossing divisor on \( \overline{X}_d \).

We begin to show that \( \overline{X}_d \) is of general type.

For any \( p \in \overline{X}_d \), we choose a local system of regular coordinates \( (z_1, \ldots, z_n) \) in the polycylindrical neighborhood \( U_p := \{ |z_i| \leq r_p, \ldots, |z_n| \leq r_p \} \) of \( p \) such that if \( p \in D_d \) then the equation \( z_1 \cdots z_s = 0 \) defines \( D_d \) around \( p \). For \( q = \pi_d(p) \), we choose a local system of regular coordinates \( (w_1, \ldots, w_n) \) in the polycylindrical neighborhood \( W_q := \{ |w_1| \leq r_q, \ldots, |w_n| \leq r_q \} \) of \( q \) such that if \( q \in B \) then the equation \( w_1 \cdots w_t = 0 \) defines \( B \) around \( q \).

By definition, we have \((J)^{-1}(B) \subset D_d \) and \( f^* w_i = \prod_j z_j^{n_{ij}} \epsilon_i \) with \( n_{ij} \geq 0 \), where \( \epsilon_i \) is an unit around \( p \). Thus we have \( f^* dw_i = \sum_j n_{ij} z^j + dz^j \) with \( dz^j \in \Omega^1(\log D_d) \) around the point \( p \).

Let \( \omega \in \{ \eta_0, \ldots, \eta_n \} \) be an element. Since

\[
H^0(\overline{X}, \mathcal{O}_{\overline{X}}(N K_{\overline{X}} + (N - 1)B)) \cong \left\{ \begin{array}{l}
\text{N-ple n-form on } \overline{X} \text{ with} \\
\text{at most } (N - 1) \text{-ple poles along } B
\end{array} \right\},
\]

we can write

\[
\omega = g(w) \frac{(dw_1 \wedge \cdots dw_n)^N}{w_1^{s_1} \cdots w_t^{s_t}} \text{ on } W_q
\]

where \( g(w) \) is a holomorphic function on \( W_q \) and \( s_1, \ldots, s_t \) are integers in \([0, N - 1]\). Around the point \( q \), we then have

\[
\omega = h(w)(\prod_{i=1}^{t} w_i) \frac{(dw_1 \wedge \cdots dw_n)^N}{w_1 \cdots w_t} \text{ on } W_q
\]

where \( h(w) \) is a holomorphic function on \( W_q \). Since \( f^*(B) = dD_d \), we get that

\[
f^* (\prod_{i=1}^{t} w_i) = (\prod_{j=1}^{s} z^j)^d \cdot \varepsilon \text{ around } p
\]

where \( \varepsilon \) is a unit around \( p \), and we get that

\[
f^*(\omega) = k(z)(\prod_{j=1}^{s} z^j)^d \frac{(dz_1 \wedge \cdots dz_n)^N}{z_1 \cdots z_s} \text{ around } p
\]
where \( k(z) \) is a holomorphic function around \( p \). Thus each \( f^*(\eta_i) \) is regular on \( \overline{Y_d} \). The lemma 2.6 says that all \( f^*(\eta_1), \ldots, f^*(\eta_n) \) are in \( H^0(\overline{X}, O_{\overline{X}}(NK_{\overline{X}})) \). Therefore, \( \{f^*(\eta_1), \ldots, f^*(\eta_n)\} \) is a transcendence base of the function field \( \mathbb{C}(\overline{Y_d}) \) and \( \overline{Y_d} \) is of general type. 

\[ \square \]

2.4. **General type of Siegel varieties with suitable level structures.** From Kawamata’s covering lemma [2.7] and the theorem [2.8] we immediately have:

**Corollary 2.9** (Mumford-Tai’s Theorem cf. Chap. IV. [AMRT] and [Mum77]). Let \( g \geq 1, l \geq 3 \) be two integers. There is a positive integer \( N(g,l) \) such that the Siegel variety \( A_{g,kl} \) is of general type for any integer \( k > N(g,l) \).

Now we describe a relation between the existence of nontrivial cusp forms and the type of manifolds: The existence of a nontrivial Siegel cusp form implies the general type of Siegel varieties with certain level structure. Actually, the spaces of Siegel cusp forms supply the following effective version of Mumford-Tai’s theorem:

**Theorem 2.10.** Let \( l \) be an arbitrary positive integer and let \( g \geq 2 \) be an integer. If

\[
N(g,l) := \min\{k \in \mathbb{Z}_{>0} \mid \dim_c S_k(g+1)(\Gamma_g(l)) > 0\}
\]

is a finite number then the Siegel variety \( A_{g,Nl} \) is of general type for any integer \( N \geq \max\{\frac{3}{4}, N(g,l)\} \).

**Remark.** The proposition 2.3 guarantees that \( N(g,l) \) is a finite integer if \( g \geq 2 \) and \( l \geq 3 \). There are also many examples from number theory showing that \( N(g,1) \) is finite for some low degree \( g \).

**Example 2.11.** By the following list of examples of level one cusp forms for low degree \( g \), the Siegel varieties \( A_{g,n} \) below are of general type:

(i) \( A_{2,n} \) for \( g = 2 \) and \( n \geq 10 \), (ii) \( A_{3,n} \) for \( g = 3 \) and \( n \geq 9 \), (iii) \( A_{4,n} \) for \( g = 4 \) and \( n \geq 8 \).

- **Case** \( g = 2 \): Igusa shows in [Igu64] that there is a cusp form \( \chi_{10,2} \) of weight 10 with development

\[
\chi_{10} \left( \begin{array}{cc} \tau_1 & z \\ z & \tau_2 \end{array} \right) = (\exp(2\pi \sqrt{-1}\tau_1) \exp(2\pi \sqrt{-1}\tau_2) + \cdots)(pz)^2 + \cdots
\]

which vanishes along the "diagonal" \( z = 0 \) with multiplicity 2. So the zero divisor of \( \chi_{10,2} \) in \( A_2 \) is the divisor of abelian surfaces that are products of elliptic curves with multiplicity 2. Thus, there is a cusp form \( \vartheta_2 := \chi_{10,2}^3 \in S_{10(2+1)}(\Gamma_2) \).

- **Case** \( g = 3 \): Tsuyumine shows in [Tsu86] that the ring of classical modular forms \( \oplus M_k(\Gamma_3) \) is generated by 34 elements, and there is a cusp form \( \chi_{18,3} \) of weight 18, namely the product of the 36 even theta constants \( \theta[e] \). The zero divisor of \( \chi_{18,3} \) on \( A_3 \) is the closure of the hyperelliptic locus. Thus, there is a cusp form \( \vartheta_3 := \chi_{18,3}^2 \in S_{9(3+1)}(\Gamma_3) \).

- **Case** \( g = 4 \): Igusa shows in [Igu81] that up to isometry there is only one isomorphism class of even unimodular positive definite quadratic forms in 8 variables, namely \( E_8 \). In 16 variables there are exactly two such classes, \( E_8 \oplus E_8 \) and \( E_{16} \). To each of these quadratic forms in 16 variables we can associate a Siegel modular form on \( \Gamma_4 \) by means of a theta series: \( \theta_{E_8 \oplus E_8} \) and \( \theta_{E_{16}} \). The difference \( \chi_{8,4} := \theta_{E_8 \oplus E_8} - \theta_{E_{16}} \) is a cusp form of weight 8. The zero divisor of \( \chi_{8,4} \) on \( A_4 \) is the closure of the locus of Jacobians of Riemann surfaces of genus 4 in \( A_4 \) (cf. [Igu81] and [Poo96]). Thus, there is a cusp form \( \vartheta_4 := \chi_{8,4}^2 \in S_{8(4+1)}(\Gamma_4) \).

However these examples for low genus Siegel varieties of general type are not optimal. Actually, one has general type for \( g = 2, n \geq 4; g = 3, n \geq 3; g = 4, 5, 6, n \geq 2; g \geq 7, n \geq 1 \). The case \( A_{g,1} \) is still open, all other cases of low genus Siegel varieties are known to be rational or unirational. Except for the case \( A_{g,1} \), Hulek has completed the problem of general type for low
Moreover, the map \( \pi : W_j = QZ_{j-1}(W_{j-1}) \to W_{j-1} \mid 1 \leq j \leq l \) (\( QZ(X) \to X \) means the monoidal transform of \( X \) with the center \( Z \)) and reduced divisor \( D_j \) on \( W_j \) for \( 1 \leq j \leq l \) such that

- \( W_0 = W, D_0 = C \),
- \( D_j = \pi_j^{-1}(D_{j-1}) \) (Here \( \pi_j^{-1}(D_{j-1}) \) is defined to be \( \pi_j^*(D_{j-1})_{\text{red}} \)),
- \( Z_j \) is a nonsingular closed subvariety contained in \( D_j \),
- \( W^* := W_l \) is a nonsingular variety and \( D_\infty := D_l \) is a simple normal crossing divisor on \( W^* \).

Moreover, the map \( \pi = \pi_1 \circ \cdots \circ \pi_1 \) is a proper birational morphism from \( W^* \) to \( W \) such that the restriction morphism \( \pi|_{W^*\setminus D_\infty} : W^* \setminus D_\infty \to W \setminus C \) is an isomorphism.

**Proof of Theorem 2.12**. We fix a \( \Gamma_{g,n}(\text{or } \text{GL}(g, \mathbb{Z})) \)-admissible polyhedral decomposition \( \Sigma_{g_0} \) of \( C(\mathbb{A}) \) regular with respect to \( \text{Sp}(g, \mathbb{Z}) \) such that the induced symmetric \( \text{Sp}(g, \mathbb{Z}) \)-admissible family \( \{\Sigma_\mathfrak{g}\}_{\mathfrak{g}} \) of polyhedral decompositions is projective.

Let \( N \geq \max \{\frac{g}{2}, N(g, l)\} \) be an integer and define \( n := Nl \). Since we fixed the decomposition \( \Sigma_{g_0} \), we have the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{A}_{g,n} & \xrightarrow{\lambda_{n,l}} & \mathcal{A}_{g,l} \\
\pi_{g,n} \downarrow & & \downarrow \pi_{g,l} \\
\mathcal{A}^*_{g,n} & \xrightarrow{\lambda^*_{n,l}} & \mathcal{A}^*_{g,l}
\end{array}
\]

The lemma 2.1 shows that there is

\( \lambda_{n,l}D_{\infty,l} = ND_{\infty,n} \)

where \( D_{\infty,l} := \mathcal{A}_{g,l} \setminus \mathcal{A}_{g,l} \) and \( D_{\infty,n} := \mathcal{A}_{g,n} \setminus \mathcal{A}_{g,n} \).

The components of the boundary divisor \( D_{\infty,n} = \mathcal{A}_{g,n} \setminus \mathcal{A}_{g,n} \) may have self-intersections. However, Hironaka’s results on resolution of singularities show that there exists a smooth compactification \( \overline{\mathcal{A}}_{g,n} \) of \( \mathcal{A}_{g,n} \) and a proper birational morphism

\( \nu_n : \overline{\mathcal{A}}_{g,n} \to \mathcal{A}_{g,n} \)

such that

- \( \overline{D}_{\infty,n} = \nu_n^*(D_{\infty,n})_{\text{red}} \) is a simple normal crossing divisor, \( \overline{\mathcal{A}}_{g,n} - \overline{D}_{\infty,n} = \mathcal{A}_{g,n} \), and the restricted morphism \( \nu_n|_{\mathcal{A}_{g,n}} \) is the identity morphism;
- furthermore, write \( D_{\infty,n} = \sum_{i=1}^l D_i \) and let \( \tilde{D}_i \) be the strict transform of \( D_i \), then

\[
\nu_n^*(D_{\infty,n}) = \sum_{i=1}^l \tilde{D}_i + E, \quad \text{and} \quad \overline{D}_{\infty,n} = \sum_{i=1}^l \tilde{D}_i + E_{\text{red}}
\]
we have \( \Gamma(\mathcal{A}_{g,n}, \omega_{\mathcal{A}_{g,n}}^k) \) where \( d\nu \) is the standard Euclidean form on the Siegel space \( \mathfrak{H}_g \). By the lemma 2.3 we have \( f_{A,g,n}(\theta_f \wedge \overline{\theta_f})^{1/k} < \infty \). Theorem 2.1 in [Sak77] says that \( \theta_f \) defines a \( k \)-ple \( \frac{g(g+1)}{2} \)-form on \( \mathcal{A}_{g,n} \) with at most \((k-1)\)-ple poles along \( \overline{D}_{\infty,n} \), i.e.,

\[ \vartheta_g 
\in H^0(\mathcal{A}_{g,n}, \omega_{\mathcal{A}_{g,n}}^{\otimes k}_0 \otimes \mathcal{O}_{\mathcal{A}_{g,n}}((k_0 - 1)\overline{D}_{\infty,n})). \]

Let \( D_g \) be the Zariski closure in \( \mathcal{A}_{g,n} \) of the zero divisor of \( f \) on \( \mathcal{A}_{g,n} \), and let \( m_f \) be the vanishing order of \( f \) at the cusp \( \mathfrak{S}(W_g) \), where \( W_g \) is the one dimensional isotropic real subspace of \( V_2 \) generated by the vector \( e_g \). Since \( \mathcal{A}_{g,n} \) can be regarded as the normalization of the blowing-up of \( \mathcal{A}_{g,n}^* \) along the ideal sheaf \( \mathcal{J} \) supported on the subscheme \( \mathcal{A}_{g,n}^* \setminus \mathcal{A}_{g,n} \), we have

\[ \text{div}(\vartheta_g) = Nm_f \vartheta_{g}^{*}(D_{\infty,n}) + D_g = Nm_f \overline{D}_{\infty,n} + Nm_f(E - E_{\text{red}}) + D_g. \]

Thus, we get

\[ \omega_{\mathcal{A}_{g,n}}^{\otimes k_0} \otimes \mathcal{O}_{\mathcal{A}_{g,n}}((k_0 - 1)\overline{D}_{\infty,n}) = \mathcal{O}_{\mathcal{A}_{g,n}}(\text{div}(\vartheta_g)) \]

\[ = \mathcal{O}_{\mathcal{A}_{g,n}}(Nm_f \overline{D}_{\infty,n} + Nm_f(E - E_{\text{red}}) + D_g), \]

and

\[ \omega_{\mathcal{A}_{g,n}}^{\otimes k_0} = \mathcal{O}_{\mathcal{A}_{g,n}}((Nm_f - k_0 + 1)\overline{D}_{\infty,n} + Nm_f(E - E_{\text{red}}) + D_g). \]

Since \( D_{\infty,n}, (E - E_{\text{red}}) \) and \( D_g \) are all effective divisors on \( \mathcal{A}_{g,n} \), we have that

\[ \omega_{\mathcal{A}_{g,n}}^{\otimes k_0} \mathcal{O}_{\mathcal{A}_{g,n}}((h(\overline{D}_{\infty,n} + (E - E_{\text{red}}) + D_g)) \subset \omega_{\mathcal{A}_{g,n}}^{\otimes h k_0} \text{ for } \forall h > Nm_f. \]

Therefore \( \omega_{\mathcal{A}_{g,n}}^{\otimes k_0} \) becomes a big line bundle on \( \mathcal{A}_{g,n} \) by the corollary 2.5.

There is extensive work on the relationship between the existence of special modular forms and the geometry of moduli spaces of abelian varieties. The principal of using the existence of low weight cusp forms to study the Kodaira dimension of moduli spaces of polarized abelian varieties is first used in [GS96] and [Grit95], our theorem 2.10 provides a different version of this principal. The principal is also efficient for studying moduli spaces of K3 surfaces and moduli spaces of irreducible symplectic manifolds(cf. [Kon93] & [GHS11]); Gritsenko,Hulek and Sankaran have proven that the moduli spaces of polarized K3 surfaces are general type(cf. [GHS07]).

**REFERENCES**

[AMRT] Ash, A.; Mumford, D.; Rapoport, M.; Tai, Y.: *Smooth compactifications of locally symmetric varieties. Second edition. With the collaboration of Peter Scholze*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2010.

[BBS66] Baily, W.L. & Borel, A.: *Compactification of arithmetic quotients of bounded symmetric domains*. Ann. of Math. 84(1966) 442528.

[Chai] Chai, C.-L.: *Siegel moduli schemes and their compactifications over C*. Arithmetic geometry(Storrs, Conn., 1984), 231–251, Springer, New York, 1986.

[CK87] Cattani,E.; Kaplan, A.: *Degenerating variations of Hodge structure*. Actes du Colloque de Thorie de Hodge (Luminy, 1987). Astriques No. 179-180 (1989), 9, 67–96.

[CKS86] Cattani,E.; Kaplan, A. & Schmid, W.: *Degeneration of Hodge structures*. Ann. Math., 38, (1986), 457-535.
Schmid, W.: Variation of Hodge Structure: The singularities of the period mapping. Invent. math. 22 (1973) 211–319

Simpson, C.: Constructing variations of Hodge Structure using Yang-Mills theory and applications to uniformization. Journal of the AMS 1 (1988) 867-918

Simpson, C.: Harmonic bundles on noncompact curves. Journal of the AMS 3 (1990) 713–770

Siu, Y.-T. & Yau, S.-T.: Compactification of negatively curved complete Kähler manifolds of finite volume. Seminar on Differential Geometry, pp. 363–380, Ann. of Math. Stud., 102, Princeton Univ. Press, Princeton, N.J., 1982.

Siu, Y.-T.: An effective Matsusaka big theorem. Ann. Inst. Fourier (Grenoble) 43 (1993), no. 5, 1387-1405.

Tai, Y.: On the Kodaira dimension of the moduli space of abelian varieties. Invent. Math. 68 (1982), no. 3, 425–439.

Tian, G.; Yau, S.-T.: Existence of Kähler-Einstein metrics on complete Kähler manifolds and their applications to algebraic geometry. Mathematical aspects of string theory (San Diego, Calif., 1986), 574–628, Adv. Ser. Math. Phys., 1, World Sci. Publishing, Singapore, 1987.

Tsuyumine, S.: On Siegel modular forms of degree three. Amer. J. Math. 108 (1986), no. 4, 755–862.

van der Geer, G.: Siegel modular forms and their applications. (English summary) The 1-2-3 of modular forms, 181–245, Universitext, Springer, Berlin, 2008.

Wang, W.: On the smooth compactification of Siegel spaces. J. Differential Geom. 38 (1993), no. 2, 351–386.

Yau, S.-T.: On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I, Comm. Pure Appl. Math. 31 (1978), 339-411.

Yau, S.-T.: Uniformization of geometric structures. The mathematical heritage of Hermann Weyl (Durham, NC, 1987), 265–274, Proc. Sympos. Pure Math., 48, Amer. Math. Soc., Providence, RI, 1988.

Yau, S.-T.: A splitting theorem and an algebraic geometric characterization of locally Hermitian symmetric spaces. Comm. Anal. Geom. 1 (1993), no. 3–4, 473–486.

Yau, S.-T.; Zhang, Y.: The Geometry on Smooth Toroidal Compactifications of Siegel varieties. (2011), preprint [arXiv:1201.3785]

Zucker, S.: Locally homogeneous variations of Hodge structure. Enseign. Math. (2) 27 (1981), no. 3–4, 243-276 (1982).

Zuo, K.: On the negativity of kernels of Kodaira-Spencer maps on Hodge bundles and applications. Asian J. of Math. 4 (2000) 279-302.

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