EXISTENCE OF SOLUTION AND ASYMPTOTIC BEHAVIOR
FOR A CLASS OF PARABOLIC EQUATIONS

ANDERSON L. A. DE ARAUJO
Universidade Federal de Viçosa, Departamento de Matemática
Avenida Peter Henry Rolfs, s/n
CEP 36570-900, Viçosa, MG, Brasil

MARCELO MONTENEGRO*
Universidade Estadual de Campinas, IMECC, Departamento de Matemática
Rua Sérgio Buarque de Holanda, 651
CEP 13083-859, Campinas, SP, Brasil

(Communicated by Jaeyoung Byeon)

Abstract. We prove existence and uniqueness of a positive solution for a class of quasilinear parabolic equations. We also show some maximum principles on the derivatives of the solution and study the asymptotic behavior of the solution near the maximal time of existence.

1. Introduction. The aim of this paper is to study the following equation

\begin{equation}
\begin{aligned}
u_t &= \frac{u_{xx}}{1 + (u_x)^2} - \frac{\lambda}{u^\alpha}, & & x \in (0, a), \quad t \in [0, T], \\
u_x(0, t) &= 0, u_x(a, t) = 0, & & t \in [0, T], \\
u(x, 0) &= u_0(x), & & x \in [0, a],
\end{aligned}
\end{equation}

(1.1)

where $0 < \alpha \leq 1$ and $\lambda > 0$ is a parameter.

We prove that there exists a unique positive solution in Theorem 2.1. We also study the asymptotic behavior. Let $T^* > 0$ be the maximal time of existence of the unique solution $u$. We show that $T^* < \infty$ and $u(x, t) \to 0$ as $t \to T^-$ at only one point $x \in [0, a]$, namely $x = 0$. For that matter we derive some maximum principles for $u_t$, $u_x$ and $u_{xt}$. As a byproduct we show a few estimates of $u$ near the vanishing time $T^*$. This is done in Lemma 2.2, Theorem 2.3 and Proposition 1. The phenomenon that $u$ tends to zero in finite time is known as quenching, pinching or necking. This is a reminiscent behavior to that occurring with the mean curvature flow evolving on a bone-shaped surface when it collapses. The general equation for a moving surface $S(t)$ along the mean curvature flow starts with $S(0)$, which is a compact $n$-dimensional surface without boundary and smoothly embedded in $\mathbb{R}^{n+1}$, $n \geq 1$, which is locally represented by a diffeomorphism $u_0 : \Omega \to \mathbb{R}^{n+1}$, where $\Omega \subset \mathbb{R}^n$ and $u_0(\Omega) \subset S(0)$. Let $H(., t)$ be the mean curvature of $S(t)$, for

2020 Mathematics Subject Classification. Primary: 35K10, 35B50; Secondary: 35K90, 35K67.

Key words and phrases. Parabolic equation, existence of solution, uniqueness, maximum principle, asymptotic behavior.

The authors have been supported by FAPESP and CNPq.

* Corresponding author.
each \( t > 0 \) the objective is to look for a family of maps \( u(\cdot, t) : \Omega \rightarrow \mathbb{R}^{n+1} \) with \( u(\Omega, t) \subset S(t) \) such that for some \( T > 0 \),

\[
u_t(x, t) = -H(x, t) \nu(x, t), \quad (x, t) \in \Omega \times (0, T)
\]

(1.2)

and

\[ u(x, 0) = u_0, \quad x \in \Omega. \]

If the normal derivative is equal to zero on the boundary

\[
\frac{\partial u}{\partial \nu} = 0,
\]

(1.4)

and if \( u \) is symmetric about one of the coordinate axis, denoted hereafter simply by \( x \), then (1.2)–(1.4) is rewritten as

\[
\begin{cases}
  u_t = \frac{u_{xx}}{1 + u_x^2} - \frac{n-1}{u}, & x \in (0, a), \ t \in [0, T], \\
u(x, 0) = 0, u_x(a, t) = 0, & t \in [0, T], \\
u(x, 0) = u_0(x), & x \in [0, a].
\end{cases}
\]

Equation (1.1) clearly reduces to (1.5) when \( \lambda = n-1 \) and \( \alpha = 1 \), see also [1, 10, 17].

Classical references focusing many aspects on evolution problems of curves and surfaces are [2, 4, 7, 8, 11, 12]. Prior results on evolution of cylindrically symmetric surfaces can be found in [1, 21]. If \( u_0(x) = c > 0 \) is constant, then particular solution of (1.5) is the shrinking cylinder \( u(x, t) = ((n - 1)(c^2 - 2t))^{1/2} \) which collapses to a line at time \( T = c^2/2 \). It was shown in [13, Section 5] that a solution of (1.5), if its curvature develops a singularity, then it collapses near the maximal time \( T \) like \( (T - t)^{1/2} \), if the initial surface \( u_0 \) has nonpositive curvature. Other singularity estimates have been obtained in [3, 10, 18]. Evolution of initially convex surfaces have been studied in [5, 14]. More recently, evolution of star-shaped surfaces were considered in [15, 20]. Neck pinching phenomena have been studied in [9]. In this way a result for (1.5) was proved in [17] says that under nonhomogeneous boundary conditions the solution has a finite time singularity at one of the extreme points of the interval \([0, a]\). In [6] the authors studied the periodic problem and if the initial condition \( u_0 \) has nonpositive mean curvature, then by [6, Theorem 2.3] the surface pinches in finite time only at \( 2k\pi \) with \( k \in \mathbb{Z} \).

We describe next the idea to solve (1.1) or the particular case (1.5). This is done in Lemma 3.1. We introduce a function \( \Phi \) which makes the equation (1.1) uniformly elliptic. An additional technical issue should be regard. The function \( 1/u^n \) is singular at \( u = 0 \), so that we replace it by a nonsingular function \( f \) and solve the problem

\[
\begin{cases}
  u_t = \Phi'(u_x)u_{xx} - \nu f(u), & x \in [0, a], \ t > 0, \\
u(x, 0) = 0, u_x(a, t) = 0, & t > 0, \\
u(x, 0) = u_0(x), & x \in [0, a].
\end{cases}
\]

Precise conditions on \( f \) and \( \Phi \) will be timely presented. We derive a bound \( 0 < m \leq u \leq M \) for the solution \( u \), and \( f(u) = 1/u^n \) in the interval \([m, M]\). Since \( \Phi(s) = \arctan(s) \) in some interval \([-\ell_0, \ell_0]\) and \( \arctan'(s) = 1/(1 + s^2) \), thus a solution of (1.6) turns out to be a solution (1.1), because there holds also an estimate like \(|u_x| \leq \ell_0 \). The initial datum \( u_0 > 0 \) belongs to \( H_{3+\eta}((0, a) \times (0, T)) \) and a solution is a function \( u \in H_{2+\eta}((0, a) \times (0, T)) \cap C^0([0, a] \times [0, T]) \) for some \( \gamma > 0 \). These spaces are revoked in Subsection 1.1.
We quote now the properties of the function $\Phi$. Let $\ell > 0$ be a positive constant such that
\[ \max_{x \in [0, a]} |u_0'(x)| \leq \ell \]  
and let $\Phi$ be the function defined by
\[ \Phi(s) = \int_0^s \phi_\ell(x)dx, \]  
where
\[ \phi_\ell(s) = \begin{cases} 
\frac{1}{1+s^2}, & |s| \leq \ell \\
p(|s|), & \ell < |s| \leq 2\ell \\
p(2\ell), & |s| > 2\ell 
\end{cases} \]
and $p : \mathbb{R} \rightarrow \mathbb{R}$ is the polynomial defined by
\[ p(s) = \frac{-3 + \ell^2}{2 \ell^2(3\ell^4 + \ell^6 + 1 + 3\ell^2)} s^4 - \frac{4 \ell^2 - 7}{3 (1 + 2\ell^2 + \ell^4)(1 + \ell^2)} s^3 + \frac{151\ell^4 - 11\ell^2 + 6}{6 (1 + \ell^2)^3}. \]  
Let
\[ g(s) = \frac{1}{1+s^2}. \]
Thus, in fact, $\phi_\ell$ is $C^2$, since the polynomial $p$ satisfies
\[ p(\ell) = g(\ell), \quad p'(\ell) = g'(\ell), \quad p''(\ell) = g''(\ell), \quad p'(2\ell) = p''(2\ell) = 0. \]
We use Maple computer assistance to show the values of $p(\ell)$, $p'(\ell)$ and $p''(\ell)$ in the Appendix. Other properties of the function $\Phi$ are
\[ \Phi : \mathbb{R} \rightarrow \mathbb{R} \text{ belongs to } C^3; \]
there exists a constant $\gamma > 0$ such that $\gamma \leq \Phi'(s) \leq 1$ for every $s \in \mathbb{R}$;
\[ \Phi'' \leq 0 \text{ in } (0, \infty); \]
there is a constant $B > 0$ such that $|\Phi''(s)| \leq B$ for every $s \in \mathbb{R}$.

The best constants $\gamma$ and $B$ are computed in the Appendix.

The definition of $f$ is as follows. Let
\[ \theta = \min_{x \in [0, s]} u_0(x) \text{ and } m = \frac{\theta}{4}, \]
Define
\[ f(z) = \begin{cases} 
q \left( \frac{m}{2} \right), & z < \frac{m}{2} \\
q(z), & \frac{m}{2} \leq z < m \\
\frac{1}{z^\alpha}, & z \geq m
\end{cases} \]
where $q : \mathbb{R} \rightarrow \mathbb{R}$ is the polynomial
\[ q(z) = \frac{\alpha (5 + \alpha)}{m^{\alpha+1}} z^4 - \frac{4 \alpha (2\alpha + 11)}{3 m^{\alpha+3}} z^3 + \frac{1}{2} \frac{\alpha (29 + 5\alpha)}{m^{\alpha+2}} z^2 - \frac{\alpha (6 + \alpha)}{m^{\alpha+1}} z + \frac{1}{6} \frac{(7\alpha + \alpha^2 + 6)}{m^{\alpha}}. \]  
and
\[ h(s) = \frac{1}{s^\alpha}. \]
The polynomial \( q \) satisfies

\[ q(m) = h(m), \quad q'(m) = h'(m), \quad q''(m) = h''(m), \quad q'(m/2) = q''(m/2) = 0. \tag{1.20} \]

Constants \( q(m/2), q(m), q'(m) \) and \( q''(m) \) will be computed in the Appendix.

**Remark 1.** Also, in the Appendix we show that the function \( f : \mathbb{R} \to \mathbb{R} \) is \( C^2 \) by (1.20). The following relations will be important in the course of the paper

\[ 0 < f(z) \leq \frac{1}{48m^2}(13\alpha + \alpha^2 + 48), \quad \forall z \in \mathbb{R} \tag{1.21} \]

and

\[ -\frac{2}{27} (343 + 147\alpha + 21\alpha^2 + \alpha^3)\alpha m^{-\alpha-1} \leq f'(z) \leq 0, \quad \forall z \in \mathbb{R}. \tag{1.22} \]

1.1. **Function spaces.** The conditions on the initial datum are described in the sequel. Throughout the paper \( \eta \) designates various constants \( 0 < \eta < 1 \) that may differ from place to place.

\[ u_0 \in H_{3+\eta}([0,a]), \quad u_0 > 0, \quad u'_0 \geq 0 \quad \text{and} \quad u''_0 \neq 0. \tag{1.23} \]

We need to prescribe the compatibility conditions

\[ u_0(0) = u'_0(a) = 0. \tag{1.24} \]

By a solution of (1.1), (1.5) or (1.6) we mean a function \( u : [0,a] \times [0, \Upsilon) \to \mathbb{R} \) for some time \( 0 < \Upsilon < \infty \) that belongs to

\[ H_{2+\eta}((0,a) \times (0, \Upsilon)) \cap C^0([0,a] \times [0, \Upsilon)) \]

which satisfies the problem. We denote by \( \Gamma_Y \) the parabolic boundary

\[ \Gamma_Y = ((0,a) \times \{0\}) \cup (\{0,a\} \times (0, \Upsilon)). \tag{1.25} \]

The norm of a point in \( [0,a] \times [0, \Upsilon] \) is denoted by \( |(x,t)| = max\{|x|,|t|^{1/2} \} \).

It is worth to define the following spaces when \( k \geq 1 \),

\[ C^{k,\frac{1}{2}}((0,a)\times(0, \Upsilon)) = \{ u : (0,a)\times(0, \Upsilon) \to \mathbb{R} : \exists D_x^i D_t^j u \in C^0((0,a)\times(0, \Upsilon)) \mid \forall i+2j \leq k \} \]

and

\[ H_{k+\eta}((0,a) \times (0, \Upsilon)) = \{ u \in C^{k,\frac{1}{2}}((0,a) \times (0, \Upsilon)) : |u|_{k+\eta} < \infty \}, \tag{1.26} \]

where \( \left\lfloor \frac{k}{2} \right\rfloor \) is the integer part of \( \frac{k}{2} \),

\[ |u|_{k+\eta} = \sum_{i+2j \leq k} \sup_{(x,t) \times (0, \Upsilon)} |D_x^i D_t^j u| + |u|_{k+\eta} + \langle u \rangle_{k+\eta} \]

with

\[ |u|_{k+\eta} = \sum_{i+2j = k} \sup_{(x,t) \neq (y,s)} \frac{|D_x^i D_t^j u(x,t) - D_x^i D_t^j u(y,s)|}{|(x,t) - (y,s)|^\eta} \]

and

\[ \langle u \rangle_{k+\eta} = \sum_{i+2j = k-1} \sup_{(x,t) \neq (y,s)} \frac{D_x^i D_t^j u(x,t) - D_x^i D_t^j u(y,s)}{|t-s|^\frac{1+\eta}{2}}. \]

The corresponding spaces of functions \( u(x) \) defined on the interval \( I = [0,a] \) are the following

\[ H_{2+\eta} = \{ u : |u|_{2+\eta}^I < \infty \}, \tag{1.27} \]

where,

\[ |u|_{2+\eta}^I = |u|_{1+\eta}^I + |u_{xx}|_{\eta}^I \]
with
\[ |u|_{1+\eta}^I = |u|_{\eta}^I + |u_x|_{\eta}^I \]
and
\[ |u|_{\eta}^I = \sup_I |u(x)| + \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^{\eta}}. \]
Analogously we define
\[ H_{3+\eta}(I) = \{ u : |u|_{3+\eta}^I < \infty \} \]
where
\[ |u|_{3+\eta}^I = |u|_{2+\eta}^I + |u_{xxx}|_{\eta}^I. \]

2. Statements of the main results. Recall (1.17), we are going to show that the existence time of a solution is at least
\[ T_0 = \frac{\theta}{4 \left( \lambda \left( \frac{\alpha}{\eta} \right)^\alpha + \frac{\eta}{\alpha} \right)}. \] (2.1)
We state next the existence and uniqueness of the solution of (1.1).

**Theorem 2.1.** Let \( u_0 \) be as in (1.23)-(1.24), and \( T_0 > 0 \) be defined by (2.1). Then problem (1.1) possesses a unique positive solution \( u \in H_{3+\eta}((0,a) \times (0,T_0)) \cap C^0([0,a] \times [0,T_0]). \)

We need to establish some maximum principles to show that \( u \) tends to zero in finite time under suitable conditions. In the result below we need the following properties
\[ \frac{u''_0}{1 + (u_0')^2} - \frac{\lambda}{u_0^\alpha} \leq 0, \quad \forall x \in [0,a] \] (2.2)
and
\[ \left( \frac{u''_0}{1 + (u_0')^2} - \frac{\lambda}{u_0^\alpha} \right)' \geq 0, \quad \forall x \in [0,a]. \] (2.3)
Let \( T^* > 0 \) be the maximal time of existence of the positive unique solution \( u \) of (1.1).

**Lemma 2.2.** Let \( u \) be the solution of (1.1) under hypotheses of Theorem 2.1.
(a) If the condition (2.2) is satisfied, then \( u_t \leq 0, \quad \forall (x,t) \in [0,a] \times [0,T^*]. \)
(b) If the condition (1.23) is satisfied, then \( u_x \geq 0, \quad \forall (x,t) \in [0,a] \times [0,T^*]. \)
(c) If the conditions (1.23), (2.2) and (2.3) are satisfied, then \( u_{xt} \geq 0, \quad \forall (x,t) \in [0,a] \times [0,T^*]. \)

We show that \( T^* < \infty \) and that the solution tends to zero in finite time. As a consequence \( T_0 < T^* \), since as we shall see \( u(x,T_0) > 0 \) for every \( x \in [0,a] \).

**Theorem 2.3.** Let \( u \) be the solution of (1.1). Assume all hypotheses of Theorem 2.1 and Lemma 2.2 including the ones of items (a), (b) and (c). Then \( T^* < \infty \) and \( \lim_{t \to T^*^-} u(x,t) = 0 \) only at \( x = 0 \).

The following estimates complement the knowledge of the behavior of the solution of (1.1).
**Proposition 1.** Let $T^* > 0$ be the maximal time of existence of the positive unique solution $u$ of (1.1). Suppose all conditions of Theorem 2.3 and
\[ \frac{u_0''}{1 + (u_0')^2} - \frac{\lambda}{u_0^2} \neq 0. \] (2.4)

Then
(A) $K_1 \leq u(x, t)(T^* - t)^{-\frac{n}{n+1}}, \forall (x, t) \in [0, a] \times (0, T^*)$,
(B) $u(0, t)(T^* - t)^{-\frac{n}{n+1}} \leq K_2, \forall t \in (0, T^*)$,
(C) $-u_t(x, t)(T^* - t)^{-\frac{n}{n+1}} \leq K_3, \forall (x, t) \in [0, a] \times (0, T^*)$,

where $K_1, K_2, K_3$ are positive constants depending only on $\alpha$.

3. **Proof of Theorem 2.1.** We begin showing the existence of solution for problem (1.6).

**Lemma 3.1.** Let $\Phi$ and $f$ as defined in (1.8) and (1.18). Let $u_0$ be as in (1.23)-(1.24) and $T_0 > 0$ given by (2.1). Then (1.6) possesses a unique positive solution $u \in H^{3+\gamma}((0, a) \times (0, T_0)) \cap C^0([0, a] \times [0, T_0])$.

We prove Lemma 3.1 by using the the general theory of the fully nonlinear parabolic equations. To be complete, we paraphrase Lemma 14.11 and Theorems 8.3, 12.10 and 14.25 from [16].

**Lemma 3.2.** Let $F : (0, a) \times (0, T) \times \mathbb{R}^3 \to \mathbb{R}$ be a $C^1$ function such that
(i) there are constants $k > 0$ and $c > 0$ such that
\[ |F(x, t, z, p, 0)| \leq k|z|^2 + c \quad \forall (x, t) \in (0, a) \times (0, T), z \in \mathbb{R}, p \in \mathbb{R}; \]
(ii) there are constants $k' > 0$ and $c' > 0$ such that
\[ \text{sign}(z) F(x, t, z, p, 0) \leq k'|z| + c' \quad \forall (x, t) \in (0, a) \times (0, T), z \in \mathbb{R}, p \in \mathbb{R}; \]
(iii) there are constants $a_0 > 0$ and $a_1 > 0$ such that
\[ a_0 \leq F_r(x, t, z, p, r) \leq a_1 \quad \forall (x, t) \in (0, a) \times (0, T), (z, p, r) \in \mathbb{R}^3. \]
(iv) for every $K \geq 0$, there are constants $b_1 = b_1(K)$, $b_2 = b_2(K)$ and $0 < \Theta \leq 1$ such that
\[ |F(x, t, z, p, r) - F(y, s, w, q, r)| \leq \Theta \left[ |(x, t) - (y, s)| + |z - w| + |p - q| \right] \left[ b_1 + b_2 |r| \right] \]
holds for every $(x, t), (y, s) \in (0, a) \times (0, T)$ and $|z| + |w| + |p| + |q| \leq K$.

Then there is $T > 0$ such that problem
\[
\begin{cases}
-u_t + F(x, t, u, u_x, u_{xx}) = 0 & \text{in } (0, a) \times (0, T) \\
u_x(0, t) = u_x(a, t) = 0 & \text{in } (0, T) \\
u(0, 0) = u_0(x) & \text{in } [0, a]
\end{cases}
\] (3.1)
possesses a solution $u \in H^{3+\gamma}((0, a) \times (0, T)) \cap C^0([0, a] \times [0, T])$, provided that $u_0$ satisfies (1.23). Moreover, since $u_0 \in H^{3+\gamma}([0, a])$ and verifies (1.24), then $u \in H^{3+\gamma}((0, a) \times (0, T))$. And if $|u|_{H^{1+\gamma}}$ is bounded independently on time, then the solution exits for all $T > 0$.

**Proof.** The proof of Lemma 3.1 is fractioned in several steps.

**Step one.** Existence of local solution in time.

Let
\[ F(x, t, z, p) = \Phi'(p)r - \lambda f(z), \] (3.2)
with \( \Phi \) and \( f \) given by (1.8) and (1.18), respectively. The local existence of a solution for problem (1.6) is performed by means of Lemma 3.2. We start verifying its hypotheses. Condition (i) is fulfilled since
\[
F(x, t, z, p, 0) = -\lambda f(z) \leq 0.
\]
The condition (ii) is verified in two steps, recall (1.14). If \( z > 0 \), then
\[
\text{sign}(z)F(x, t, z, p, 0) = -\lambda f(z) \leq 0.
\]
If \( z < 0 \), then
\[
\text{sign}(z)F(x, t, z, p, 0) = \lambda f(z) = \lambda q \left( \frac{m}{2} \right) \leq \lambda q \left( \frac{m}{2} \right) + |z|.
\]
As for condition (iii), notice that \( F_r(x, t, z, p, r) = \Phi'(p) \) we take \( a_0 = \gamma \) and \( a_1 = 1 \), recall (1.14). The condition (iv) is satisfied if one takes \( \Theta = 1, b_1 = \sup_{z \in \mathbb{R}} |\lambda f'(z)| < \infty \), by Remark 1, and \( b_2 = \sup_{s \in \mathbb{R}} |\Phi''(s)| \). Indeed,
\[
\left| F(x, t, z, p, r) - F(y, s, w, q, r) \right| \leq \left| (\Phi'(p) - \Phi'(q)) r \right| + |\lambda| \left| f(z) - f(w) \right|
\]
By the mean value theorem applied to \( \Phi' \) and \( f \) we obtain
\[
\left| F(x, t, z, p, r) - F(y, s, w, q, r) \right| \leq \sup_{z \in \mathbb{R}} |\Phi''(\zeta)||p-q||r| + \sup_{z \in \mathbb{R}} |\lambda f'(z)||z-w| \leq b_2|p-q||r| + b_1|z-w|
\]
and then
\[
\left| F(x, t, z, p, r) - F(y, s, w, q, r) \right| \leq (|(x,t)-(y,s)|+|z-w|+|p-q|)(b_1+b_2|r|).
\]
Thus there is a local solution until some time \( T \) of (1.6). By (1.24), it also follows the regularity of \( u \) in the space \( H_{3+\eta} \).

The uniqueness and extension of \( u \) until time \( T = T_0 \), with \( T_0 \) given by (2.1) will be completed after Lemmas 3.3 and 3.4.

**Step two.** Existence of a subsolution and a supersolution of (1.6).

**Lemma 3.3.** Recall \( \Phi \) and \( f \) as in (1.8) and (1.18). Let \( u_0 \) with properties (1.23)–(1.24) and \( T_0 > 0 \) defined by (2.1). Then problem (1.6) has a subsolution \( \underline{u} \) and a supersolution \( \overline{u} \) both defined in \([0,a] \times [0,T_0]\).

**Proof.** Recall (1.17) and define
\[
u(x, t) = -\frac{\theta}{2a^2} + \theta - \left( \lambda \left( \frac{4}{\theta} \right)^{\alpha} + \frac{\theta}{a^2} \right) t \quad \forall (x, t) \in [0,a] \times [0,T_0]. \tag{3.3}
\]
Hence
\[
\underline{u}(x, t) \geq \underline{u}(a, t) = -\frac{\theta}{2} + \theta - \left( \lambda \left( \frac{4}{\theta} \right)^{\alpha} + \frac{\theta}{a^2} \right) t \\
\geq \theta - \left( \lambda \left( \frac{4}{\theta} \right)^{\alpha} + \frac{\theta}{a^2} \right) T_0 = \theta = m \quad \forall (x, t) \in [0,a] \times [0,T_0] \tag{3.4}
\]
and
\[
\underline{u}_t - \underline{u}_{xx} = -\left( \lambda \left( \frac{4}{\theta} \right)^{\alpha} + \frac{\theta}{a^2} \right) + \frac{\theta}{a^2} \leq -\lambda \left( \frac{4}{\theta} \right)^{\alpha} \leq -\lambda f(u).
\]
Since $u_{xx}(x, t) = -\theta/a^2 \leq 0$ and $\Phi'(s) \leq 1$ we obtain

$$u_t \leq u_{xx} - \lambda f(u) \leq \Phi'(u_x)u_{xx} - \lambda f(u), \quad u_x(0, t) = 0,$$

by (3.3) and (3.4) we have

$$u_x(a, t) = -\theta \leq 0$$

and

$$u(x, 0) = -\frac{\theta x^2}{2a^2} + \theta \leq \theta u_0(x).$$

Therefore $u$ is a subsolution.

Let $\overline{u}(x, t) = M$ be the constant

$$M = 2 \max_{x \in [0,a]} u_0(x). \quad (3.5)$$

Recall that $f > 0$, see (1.18). Clearly

$$\overline{u}_t = 0 = \Phi'(\overline{u}_x)\overline{u}_{xx} - \lambda f(\overline{u}), \quad \overline{u}_x(0, t) = 0, \quad \overline{u}_x(a, t) = 0$$

and

$$\overline{u}(x, 0) = M \geq u_0(x).$$

Therefore $\overline{u}$ is a supersolution. \hfill \square

**Step three.** The subsolution and supersolution of (1.6) are ordered as $\underline{u} \leq \overline{u}$.

**Lemma 3.4.** Let $\Phi$ and $f$ as in (1.8) and (1.18), and suppose that $u_0$ satisfies (1.23)–(1.24). Let $\underline{u}, \overline{u} \in C^2(0, a) \times [0, T_0]) \cap C^0([0, a] \times [0, T_0])$ be a subsolution and a supersolution of (1.6), respectively, with $T_0 > 0$ defined by (2.1). Then

$$\underline{u} \leq \overline{u}, \quad \forall (x, t) \in [0, a] \times [0, T_0].$$

**Proof.** We have

$$\begin{cases}
\underline{u}_t \leq \Phi'(\underline{u}_x)\underline{u}_{xx} - \lambda f(\underline{u}), & (x, t) \in [0, a] \times [0, T_0], \\
\underline{u}_x(0, t) \leq 0, \underline{u}_x(a, t) \leq 0, & t \in [0, T_0], \\
\underline{u}(x, 0) \leq u_0(x), & x \in [0, a],
\end{cases} \quad (3.6)$$

and

$$\begin{cases}
\overline{u}_t \geq \Phi'(\overline{u}_x)\overline{u}_{xx} - \lambda f(\overline{u}), & (x, t) \in [0, a] \times [0, T_0], \\
\overline{u}_x(0, t) \geq 0, \overline{u}_x(a, t) \geq 0, & t \in [0, T_0], \\
\overline{u}(x, 0) \geq u_0(x), & x \in [0, a].
\end{cases} \quad (3.7)$$

Let $w = \overline{u} - \underline{u}$. By the mean value theorem

$$w_t = \Phi'(\overline{u}_x)\overline{u}_{xx} \Phi'(\underline{u}_x)\underline{u}_{xx} - \lambda (f(\overline{u}) - f(\underline{u}))
\begin{equation}
= \Phi'(\overline{u}_x)w_{xx} + \underline{u}_x\Phi'(\underline{u}_x)w_{xx} - \Phi'(u_x) - \lambda f(\overline{u}) - f(\underline{u})
\end{equation}
\begin{equation}
= \Phi'(\overline{u}_x)w_{xx} + \underline{u}_x\Phi'(\xi)w_{xx} - \lambda f'(u^*)w,
\end{equation}

where $\xi$ is between $\overline{u}_x$ and $\underline{u}_x$, and $u^*$ is between $\overline{u}$ and $\underline{u}$. Therefore

$$\begin{cases}
w_t - \Phi'(\overline{u}_x)w_{xx} - \underline{u}_x\Phi'(\xi)w_{xx} - \lambda f'(u^*)w \geq 0, & (x, t) \in [0, a] \times [0, T_0], \\
w_x(0, t) \geq 0, w_x(a, t) \geq 0, & t \in [0, T_0], \\
w(x, 0) \geq 0, & x \in [0, a].
\end{cases}$$

We are in position to apply the maximum principle [19, Lemma 2.1, p.54]. Hence $w \geq 0$ implying

$$\underline{u} \leq \overline{u}, \quad \forall (x, t) \in [0, a] \times [0, T_0]. \quad \square$$
Step four. Uniqueness and boundedness.

Proof. Bringing in the above information, if the solution \( u \) of (1.6) exists until time \( T_0 \), by Lemmas 3.3 and 3.4 we obtain uniqueness and
\[
0 < m \leq u(x, t) \leq u(x, t) \leq M, \quad \forall (x, t) \in [0, a] \times [0, T_0].
\]

We proceed to extend the solution until time \( T_0 \).

Step five. Estimate of \( u_x \).

Let \( \Gamma_T = ((0, a) \times \{0\}) \cup \{(0, a) \times [0, T]\} \) be the parabolic boundary of \([0, a] \times [0, T]\). Since \( u \) satisfies
\[
\frac{\partial u}{\partial t} = \Phi'(u)u_{xx} - \lambda f(u).
\]
Then, differentiating with respect to \( x \) we have
\[
\frac{\partial u}{\partial x} = \Phi'(u)u_{xx} - \lambda f'(u)u_x.
\]

Defining the operator
\[
Q(z) = -z_t + (\Phi'(z)z_x)_x - \lambda f'(u)z
\]
and denoting \( w = u_x \). Hence \( Qw = 0 \). And since \( w(x, t) \geq 0 \ \forall (x, t) \in \Gamma_T \) it follows from [16, Theorem 9.7] that \( w \geq 0 \) i.e.,
\[
u_x(x, t) \geq 0 \quad \forall (x, t) \in (0, a) \times [0, T).
\]

Now, we define the constant
\[
\vartheta = \max_{(y, s) \in \Gamma_T} \{u_x(y, s)\}.
\]
By the boundary and initial conditions we have
\[
\vartheta \leq \max_{x \in [0, a]} u_0'(x).
\]
\[
w(x, t) \leq \vartheta \leq \max_{x \in [0, a]} u_0'(x) \quad \forall (x, t) \in \Gamma_T.
\]
It follows from the maximum principle [16, Theorem 9.5] that
\[
\sup_{(0, a) \times [0, T]} w(x, t) \leq e^{(k+1)T} \sup_{\Gamma_T} w^+(x, t),
\]
where \( k = \lambda \sup_{z \in \mathbb{R}} |f'(z)| < \infty \), by Remark 1. Therefore,
\[
u_x(x, t) \leq e^{(k+1)T} \max_{x \in [0, a]} u_0'(x) \quad \forall (x, t) \in (0, a) \times [0, T),
\]
By (3.8), (3.9) and definition of \( T_0 \), we obtain
\[
0 \leq u_x(x, t) \leq K_0, \quad \forall (x, t) \in (0, a) \times [0, T),
\]
where \( K_0 = e^{(k+1)T_0} \max_{x \in [0, a]} u_0'(x) \). Constant \( K_0 \) does not depend on time, hence the boundedness of \( |u|_{L^{1+\alpha}} \) independently on time follows by [16, Theorem 12.10]. Thus the solution \( u \) exists until time \( T_0 \). The proof of Lemma 3.1 is complete.

Proof. To prove Theorem 2.1 we will put \( u_x \) in the range of \( \Phi' \). It follows from (3.10). The constant \( K_0 \) is independent on \( \ell > 0 \). Thus there exists a large enough \( \ell_0 > 0 \) such that \( K_0 \leq \ell_0 \). We conclude that
\[
|u_x(x, t)| < \ell_0, \quad \forall (x, t) \in [0, a] \times [0, T_0].
\]
The function \( \Phi \) is defined by (1.8)–(1.9), then \( \Phi'(u_x) = 1/(1 + u_x^2) \). Since \( m \leq u \leq M \), then \( u \) is also in the range of \( f \), so \( f(u) = 1/u^\alpha \). Therefore we conclude that problem (1.1) has a positive unique solution. \( \square \)
4. The behavior of the solution.

Proof. To prove Lemma 2.2, let \( u \) be the solution of (1.1). We continue to use \( \Phi \) to shorten notation.

We claim that the solution \( u \in H_{3+\eta} ((0, a) \times (0, T_0)) \) can be extended to \( u \in H_{3+\eta} ((0, a) \times (0, T^*) ) \) \( (T^* \) the maximum time of existence). Indeed, following with the same argument applied to prove of the Step five of Lemma 3.1, replacing \( T_0 \) by \( T^* \) and repeating the arguments to get (3.9) and (3.10), we obtain

\[
u_t(x,t) \leq \varepsilon^{(k+1)} T_0 \max_{x \in [0,a]} u_0'(x), \quad \forall (x,t) \in (0, a) \times [0, T_0),
\]

where \( k \) is as in (3.9). By (3.8), (3.9) and since \( T_0 < T^* \), we obtain

\[
0 \leq \nu_t(x,t) \leq K^*, \quad \forall (x,t) \in (0, a) \times [0, T_0),
\]

where \( K^* = \varepsilon^{(k+1)} T^* \max_{x \in [0,a]} u_0'(x) \). Constant \( K^* \) does not depend on time, hence the boundedness of \( |u|_{H^{3+\eta}} \) independently on time follows by [16, Theorem 12.10]. Thus the solution \( u \) exists until time \( T^* \), see conclusion of Lemma 3.2. Therefore, \( u \in H_{3+\eta} ((0, a) \times (0, T^*)), \) and in that follows we can apply the Maximum Principle [19, Lemma 2.1, p.54] in the time interval \( (0, T^*) \).

(a) Differentiating (1.1) with respect to \( t \) and using (2.2) we obtain

\[
\begin{align*}
\nu_{tt} &= \Phi'(u_0) \nu_{xx} + \Phi''(u_0) \nu_{x}^2 + \lambda \alpha u^{-\alpha-1} \rho \quad x \in [0,a], \quad t \in [0, T^*), \\
\nu_{tt}(0,t) &= \Phi'(u_0)(a) = 0, \quad t \in [0, T^*), \\
\nu_{t}(x,0) &= \Phi'(u_0)u_0' - \frac{\alpha}{u_0} \leq 0 \quad x \in [0,a].
\end{align*}
\]

Applying the the Maximum Principle [19, Lemma 2.1, p.54] for \( \nu_t \), we obtain \( \nu_t(x,t) \leq 0 \quad \forall (x,t) \in [0,a] \times [0, T^*) \).

(b) Differentiating (1.1) with respect to \( x \) and using (1.23) one has

\[
\begin{align*}
\nu_{tt} &= \Phi'(u_0) \nu_{xx} + \Phi''(u_0) \nu_{x}^2 + \lambda \alpha u^{-\alpha-1} \rho \quad x \in [0,a], \quad t \in [0, T^*), \\
\nu_{tt}(0,t) &= \Phi'(u_0)(a) = 0, \quad t \in [0, T^*), \\
\nu_{t}(x,0) &= \nu_{0}'(x) \rho \geq 0 \quad x \in [0,a].
\end{align*}
\]

The Maximum Principle [19, Lemma 2.1, p.54] applied for \( \nu_x \) gives us \( \nu_x(x,t) \geq 0, \quad \forall (x,t) \in [0,a] \times [0, T^*) \).

(c) According to items (a) and (b) we have

\[
\nu_t(x,t) \leq 0 \text{ and } \nu_x(x,t) \geq 0 \quad \forall (x,t) \in [0,a] \times [0, T^*).
\]

Thus differentiating (4.2) with respect to \( t \) and using (2.3), we get

\[
\begin{align*}
\nu_{ttt} - \Phi'(u_0) \nu_{tt} - 2 \Phi''(u_0) \nu_{ttx} + \Phi'''(u_0) \rho_{xx}^2 + \lambda \alpha u^{-\alpha-1} + \Phi''(u_0) \nu_{x}^2 + \Phi'''(u_0) \nu_{xxx} \\
=- \lambda \alpha (a + 1) u^{-\alpha-2} u_t \rho_{x} \geq 0 \quad x \in [0,a], \quad t \in [0, T^*), \\
\nu_{tt}(0,t) &= \nu_{tt}(a) = 0, \quad t \in [0, T^*), \\
\nu_{tt}(x,0) &= \left( \Phi'(u_0) u_0'' - \frac{\alpha}{u_0} \right)' \geq 0, \quad x \in [0,a].
\end{align*}
\]

Hence, the Maximum Principle [19, Lemma 2.1, p.54] for \( \nu_{xx} \) implies that \( \nu_{xx}(x,t) \geq 0 \quad \forall (x,t) \in [0,a] \times [0, T^*). \) \( \Box \)
Proof. To prove Theorem 2.3, let $u$ be the solution of (1.1) defined on the maximal interval $[0, T^*)$. Define

$$H(t) = \int_0^a u(x, t)dx$$

for $t \in [0, T^*)$.

Observe that

$$H'(t) = \int_0^a u_t(x, t)dx = \int_0^a \left[ (\Phi(u_x))_x - \frac{\lambda}{u^\alpha} \right]dx$$

$$= \Phi(u_x(a, t)) - \Phi(u_x(0, t)) - \int_0^a \frac{\lambda}{u^\alpha}dx \leq -\frac{a\lambda}{M^\alpha},$$

where $M = 2\max_{x \in [0, a]} u_0(x)$. Since $\Phi$ is increasing (1.14) and $u$ is a decreasing function with respect to $t$ (see (a) of Lemma 2.2), we have

$$H'(t) \leq -\frac{a\lambda}{M^\alpha}.$$  \hfill (4.4)

Thus

$$H(t) \leq \Pi(t),$$

where $\Pi(t) = H(0) - \frac{a\lambda}{M}t$ for all $t \in [0, T^*)$.

The solution $u$ is bounded from above by $M$. Suppose by contradiction that $T^* = \infty$. Notice that $\lim_{t \to \infty} \Pi(t) = -\infty$.

If there is $\delta > 0$ such that

$$u(x, t) \geq \delta \quad \forall (x, t) \in [0, a] \times [0, \infty).$$

We get the contradiction

$$0 < a\delta \leq \Pi(t) \to -\infty \quad \text{as} \ t \to \infty.$$  \hfill (4.4)

If $u$ is not bounded away from zero, then $u(0, t) \to 0$ as $t \to +\infty$ because $u_t \leq 0$ (see (a) of Lemma 2.2). Again by (4.4) and as $u$ is an nondecreasing function with respect to $x$ (see (b) of Lemma 2.2) we obtain

$$0 \leq a \lim_{t \to +\infty} u(0, t) \leq \lim_{t \to +\infty} \int_0^a u(x, t)dx \leq \lim_{t \to +\infty} \Pi(t) = -\infty,$$

a contradiction. Therefore, $T^* < +\infty$ and the solution vanishes in finite time.

Notice that $u_{xt} \geq 0$ (see (c) of Lemma 2.2), so that the only vanishing point is $x = 0$. Indeed, for every fixed numbers $0 \leq x_1 \leq x_2 \leq a$ the function

$$z(t) = u(x_1, t) - u(x_2, t)$$

is nonincreasing in the interval $(0, T^*)$. Since $u_0(x) \geq 0$ and $u_0 \not\equiv 0$, we conclude that $z(0) \leq 0$ and $z(t_0) < 0$ for some $t_0 \in (0, T^*)$. Indeed, if $z(t) \geq 0$ for each $t \in (0, T^*)$, since $z(0) \leq 0$ and $z$ is nonincreasing in $(0, T^*)$, we conclude that $z(t) \leq 0$ for each $t \in (0, T^*)$. Hence, $z(t) \equiv 0$ in $(0, T^*)$. By continuity, $z(0) = 0$, that is, $0 = z(0) = u_0(x_1) - u_0(x_2)$, for all $x_1, x_2$, and $u_0$ is a constant, that contradicts $u_0 \not\equiv 0$. Therefore, $\lim_{t \to T^*} z(t) < 0$, which implies the assertion of the theorem. \hfill \Box

We proceed to obtain explicit estimates of the solution of problem (1.1) near the vanishing time $T^*$.

Proof. We will prove each item of Proposition 1. (A) Let $\eta^*$ be a small enough positive number such that $0 < \eta^* < T^*$.

We claim that $u_t(x, \eta^*) < 0$ in $[0, a - \eta^*]$. Indeed,
Define $J$ and arguments applied in the proof of Lemma 2.2-(δ) [p.54], we obtain the strict inequality

$$u_t(x,0) = 0 = \frac{u''_0}{1 + (u'_0)^2} - \frac{\lambda}{u'_0},$$

that contradicts (2.4). Therefore, $u_t(0,0) < 0$ and by continuity, there is $0 < \eta^*$ small enough such that $u_t(0, \eta^*) < 0$.

(2) $u_t(x, \eta^*) < 0$, for all $x \in (0, a)$. This conclusion follows with the same arguments applied in the proof of Lemma 2.2-(a), by application of [19, Lemma 2.1, p.54], we obtain the strict inequality

$$u_t(x,t) < 0 \quad \forall (x,t) \in (0, a) \times (0, T_0).$$

From (1) and (2) we conclude the claim.

Since $u_t(x, \eta^*) < 0$ and $u^\alpha(x, \eta^*) > 0$ in $[0, a - \eta^*]$ (see item (a) of Lemma 2.2).

Let

$$\delta_1 = \min_{x \in [0, a - \eta^*]} \{-u_t(x, \eta^*)u^\alpha(x, \eta^*)\}$$

and

$$\delta_2 = \min_{t \in [\eta^*, T^*]} \{-u_t(a - \eta^*, t)u^\alpha(a - \eta^*, t)\}.$$

Define $J : [0, a - \eta^*] \times [\eta^*, T^*] \to \mathbb{R}$ by

$$J(x,t) = u_t(x,t) + \lambda\delta u^{-\alpha}(x,t), \quad (4.5)$$

where $\delta = \min\{\delta_1, \delta_2, 1\}$. Differentiating (4.5) we obtain

$$J_t - \Phi'(u_x)J_{xx} = \frac{\Phi''(u_x)}{\Phi'(u_x)}(1 - \delta)u_{xx} - \Phi'(u_x)(\alpha + 1)\delta u^{-\alpha - 2}u_x^2. \quad (4.6)$$

Since $0 < \delta < 1$, $\Phi' > 0$ in $\mathbb{R}$ (1.14), $\Phi'' \leq 0$ in $[0, +\infty)$ (1.15), $u_x \geq 0$ in $[0, a] \times [0, T^*)$ (see item (b) of Lemma 2.2), and since $u_{xx} \geq 0$ in $[0, a] \times [0, T^*)$ (see item (c) of Lemma 2.2), we obtain

$$J_t - \Phi'(u_x)J_{xx} - \left(\frac{\Phi''(u_x)}{\Phi'(u_x)}u_{xx} + \lambda\alpha u^{-\alpha - 1}\right)J \leq 0, \quad \forall (x,t) \in [0, a - \eta^*] \times [\eta^*, T^*]. \quad (4.7)$$

Since $\delta \leq \frac{\delta_1}{\lambda}$ and $\delta \leq \frac{\delta_2}{\lambda}$ we have

$$J(0,t) \leq 0, \quad \forall t \in [\eta^*, T^*), \quad (4.8)$$

$$J(a - \eta^*, t) \leq 0, \quad \forall t \in [\eta^*, T^*), \quad (4.9)$$

and

$$J(x, \eta^*) \leq 0, \quad \forall t \in [0, a - \eta^*]. \quad (4.10)$$

Keeping in mind (4.7)-(4.10) we apply the Maximum Principle [19, Lemma 2.1, p.54] for $J$. Thus, we obtain

$$J(x,t) \leq 0, \quad \forall (x,t) \in [0, a - \eta^*] \times [\eta^*, T^*).$$

Hence

$$u_t(x,t)u^\alpha(x,t) \leq -\lambda\delta, \quad \forall (x,t) \in [0, a - \eta^*] \times [\eta^*, T^*). \quad (4.11)$$
Integrating (4.11) from \( t \) to \( T^* \) we obtain the following estimate
\[
u(x, t)^{n+1} \geq \lambda \delta (\alpha + 1)(T^* - t), \quad \forall (x, t) \in [0, a - \eta^*] \times [\eta^*, T^*).\]
Therefore,
\[
u(0, t)(T^* - t)^{-\frac{1}{n+1}} \geq (\lambda \delta (\alpha + 1))^{\frac{1}{n+1}}, \quad \forall t \in [\eta^*, T^*).
\]
Since the function \( \nu(0, t)(T^* - t)^{-\frac{1}{n+1}} \) is continuous and positive in [0, \( \eta^* \)]. Taking
\[
K_1 = \min \left\{ \min_{t \in [0, \eta^*]} \{ \nu(0, t)(T^* - t)^{-\frac{1}{n+1}}, (\lambda \delta (\alpha + 1))^{\frac{1}{n+1}} \} \right\}
\]
we obtain
\[
u(0, t)(T - t)^{-\frac{1}{n+1}} \geq K_1, \quad \forall t \in [0, T^*)
\]
and the conclusion follows from item (b) of Lemma 2.2.

(B) For every \( t \in [0, T^*) \), the minimum value of \( \nu(x, t) \) is attained at \( x = 0 \) (see item (b) of Lemma 2.2), then \( u_{xx}(0, t) > 0 \). Indeed, by item (b) of Lemma 2.2, \( u_x(x, t) \geq 0 \), \( \forall (x, t) \in [0, a] \times [0, T^*) \). Hence,
\[
u(0, t) \leq u(x, t), \forall (x, t) \in [0, a] \times [0, T^*),
\]
that is, \( x = 0 \) is the minimum point of \( u(x, t) \), for all \( t \in [0, T^*) \), in particular \( u_x(0, t) = 0 \). Therefore, \( u_{xx}(0, t) > 0 \).

By equation (1.1) and since \( \Phi \) is increasing (1.14), we have
\[
u_t(0, t) \geq -u^{-\alpha}(0, t).
\]
Integrating (4.12) from \( t \) to \( T^* \) we obtain \( K_2 = (1 + \alpha)^{\frac{1}{n+1}} \) such that
\[
u(0, t)(T^* - t)^{-\frac{1}{n+1}} \leq K_2, \quad \forall t \in [0, T^*).
\]
(C) Since \( u_{xx} \geq 0 \) in \([0, a] \times [0, T^*)\) by the item (c) of Lemma 2.2 and (4.12) we obtain
\[
u_t(x, t) \geq \nu_t(0, t) \geq -u^{-\alpha}(0, t) \quad \text{in} \quad [0, a] \times [0, T^*).
\]
Using item (A) of the present Proposition 1, we conclude that there is a constant \( K_3 = K^{-\alpha} \) such that
\[
u_t(x, t) \geq -u^{-\alpha}(0, t) \geq -K_3(T - t)^{-\frac{1}{n+1}}, \quad \forall (x, t) \in [0, a] \times [0, T^*). \]

5. Appendix. We will compute with Maple assistance some expressions appearing from (1.7) to (1.22).

5.1. Derivatives up to the second order of \( p \) and \( q \).

\[
p'(x) = \frac{2(-3 + \ell^2)}{\ell^2(3\ell^4 + \ell^0 + 1 + 3\ell^2)x^3} - \frac{3\ell^2 - 7}{(1 + 2\ell^2 + \ell^4)(1 + \ell^2)x^2} + \frac{2(12\ell^2 - 20)}{(1 + 2\ell^2 + \ell^4)(1 + \ell^2)x} - \frac{6(-3 + \ell^2)}{(1 + 2\ell^2 + \ell^4)(1 + \ell^2)},
\]
\[
p''(x) = \frac{2(2) - 20}{\ell^2(3\ell^4 + \ell^0 + 1 + 3\ell^2)x^2} - \frac{3\ell^2 - 7}{(1 + 2\ell^2 + \ell^4)(1 + \ell^2)x} + \frac{2(12\ell^2 - 20)}{(1 + 2\ell^2 + \ell^4)(1 + \ell^2)x} - \frac{6(-3 + \ell^2)}{(1 + 2\ell^2 + \ell^4)(1 + \ell^2)}.
\]
\[
g'(x) = -\frac{2x}{(1 + x^2)^2}, \quad g''(x) = \frac{2(3\ell^2 - 1)}{(1 + x^2)^3}.
\]
\[
q'(x) = \frac{4\alpha(5 + \alpha)}{m^{\alpha+4}}x^3 - \frac{4\alpha(2\alpha + 11)}{m^{\alpha+3}}x^2 + \frac{\alpha(29 + 5\alpha)}{m^{\alpha+2}}x - \frac{\alpha(6 + \alpha)}{m^{\alpha+1}}.
\]
\[ q''(x) = \frac{12\alpha(5 + \alpha)}{m^{\alpha+4}} x^2 - 8 \frac{\alpha(2\alpha + 11)}{m^{\alpha+3}} x + \frac{\alpha(29 + 5\alpha)}{m^{\alpha+2}}. \]

5.2. Computation of some constants.

\[ p(\ell) = \frac{1}{1 + \ell^2}, \quad p'(\ell) = \frac{-2\ell}{(1 + \ell^2)^2}, \quad p''(\ell) = \frac{2(3\ell^2 - 1)}{1 + 3\ell^2 + 3\ell^4 + \ell^6}. \]

\[ q\left(\frac{m}{2}\right) = \frac{1}{48m^\alpha}(13\alpha + \alpha^2 + 48), \quad q(m) = \frac{1}{m^\alpha}, \quad q'(m) = \frac{-\alpha}{m^{\alpha+1}}, \quad q''(m) = \frac{\alpha(\alpha + 1)}{m^{\alpha+2}}. \]

\[ \gamma = \frac{3\ell^4 + 5\ell^2 + 6}{6(1 + \ell^2)^3}, \quad B = \frac{3}{8}\sqrt{3}. \]

5.3. Properties of \( f \) and \( f' \). We will show (1.21) exhibited in Remark 1. Clearly, \( f \) is \( C^2 \) by the previous two subsections. The critical points of \( f \) are given by

\[ z_1 = \frac{m}{2} \quad \text{and} \quad z_2 = \frac{(6 + \alpha)m}{5 + \alpha}. \]

Notice that \( m < z_2 \) and since \( f(z) \) restricted to the interval \([m/2, m]\) is \( q(z) \), the only critical point of \( f(z) \) in \([m/2, m]\) is \( z_1 \). Since

\[ f(z_1) = q\left(\frac{m}{2}\right) = \frac{1}{48m^\alpha}(13\alpha + \alpha^2 + 48) > \frac{1}{m^\alpha} = f(m) \]

we conclude that \( m \) is a minimum point of \( f \) and \( m/2 \) is a maximum point of \( f \) in the interval \((-\infty, m]\). In particular (1.21) follows.

We will verify the other estimate (1.22) in Remark 1. Indeed, the critical points of \( f' \) are

\[ y_1 = \frac{m}{2} \quad \text{and} \quad y_2 = \frac{129 + 5\alpha}{6} \frac{m}{5 + \alpha}. \]

We get by standard computations that

\[ \frac{m}{2} < \frac{129 + 5\alpha}{6} \frac{m}{5 + \alpha} < m. \]

Also,

\[ f'\left(\frac{m}{2}\right) = 0 \]

and

\[ f'\left(\frac{129 + 5\alpha}{6} \frac{m}{5 + \alpha}\right) = -\frac{2}{27} \frac{(343 + 147\alpha + 21\alpha^2 + \alpha^3)\alpha m^{-\alpha-1}}{(5 + \alpha)^2} < -\frac{\alpha}{m^{\alpha+1}} = f'(m). \]

We obtain that \( y_2 \) is the global minimum point of \( f' \) and \( m/2 \) is a global maximum point of \( f' \). Since the maximum of \( f' \) is attained in the interval \([m/2, m]\), we get immediately \( f'(z) \leq f'(m/2) = 0 \), concluding (1.22).

Acknowledgments. We would like to thank the referee for many valuable suggestions.
REFERENCES

[1] S. Altschuler, S. B. Angenent and Y. Giga, Mean curvature flow through singularities for surfaces of rotation, *J. Geom. Anal.*, 5 (1995), 293–358.

[2] S. Angenent, Parabolic equations for curves on surfaces: part I. Curves with $p$-integrable curvature, *Ann. Math.*, 132 (1990), 451–483.

[3] M. Athanassenas, Behaviour of singularities of the rotationally symmetric, volume–preserving mean curvature flow, *Calc. Var. PDE*, 17 (2003), 1–16.

[4] K. A. Brakke, The motion of a surface by its mean curvature, Princeton University Press, 2015.

[5] J. Escher and G. Simonett, The volume preserving mean curvature flow near spheres, *Proc. AMS*, 126 (1998), 2789–2796.

[6] J. Escher and B. V. Matioc, Neck pinching for periodic mean curvature flows, *Analysis*, 30 (2010), 253–260.

[7] L. C. Evans and J. Spruck, Motion of level sets by mean curvature I, *J. Differ. Geom.*, 33 (1991), 635–681.

[8] M. E. Gage and R. S. Hamilton, The heat equation shrinking convex plane curves, *J. Diff. Geom.*, 23 (1986), 69–96.

[9] Z. Gang and I. M. Sigal, Neck pinching dynamics under mean curvature flow, *J. Geom. Anal.*, 19 (2009), 36–80.

[10] Y. Giga, Y. Seki and N. Umeda, Mean curvature flow, closes open ends of noncompact surfaces of rotation, *Comm. Part. Diff. Eq.*, 34 (2009), 1508–1529.

[11] M. A. Grayson, The shape of afigure eight under the curve shortening flow, *Invent. Math.*, 90 (1989), 177–180.

[12] G. Huisken, Nonparametric mean curvature evolution with boundary conditions, *J. Differ. Equ.*, 77 (1989), 369–378.

[13] G. Huisken, Asymptotic behaviour for singularities of the mean curvature flow, *J. Differ. Geom.*, 31 (1990), 285–299.

[14] G. Huisken and C. Sinestrari, Convexity estimates for mean curvature flow and singularities of mean convex surfaces, *Acta Math.*, 183 (1999), 45–70.

[15] I. Kim and D. Kwon, On mean curvature flow with forcing, *Commun. Partial Differ. Equ.*, 45 (2020), 414–455.

[16] G. M. Lieberman, *Second Order Parabolic Differential Equations*, World Scientific, 2005.

[17] B. V. Matioc, value problems for rotationally symmetric mean curvature flows, *Arch. Math.*, 89 (2007), 365–372.

[18] J. A. McCoy, F. Y. Y. Mofarreh and G. H. Williams, Fully nonlinear curvature flow of axially symmetric hypersurfaces with boundary conditions, *Ann. Mat. Pura Appl.*, 193 (2014), 1443–1455.

[19] C. Y. Pao, *Nonlinear Parabolic and Elliptic Equations*, Plenum, 1992.

[20] K. Smoczyk, Starshaped hypersurfaces and the mean curvature flow, *Manuscr. Math.*, 95 (1998), 225–236.

[21] H. M. Soner and P. E. Souganidis, Singularities and uniqueness of cylindrically symmetric surfaces moving by mean curvature, *Commun. Partial Differ. Equ.*, 18 (1993), 859–894.

Received July 2020; revised December 2020.

E-mail address: anderson.araujo@ufv.br
E-mail address: msm@ime.unicamp.br