Abstract

In this paper, we showcase the interplay between discrete and continuous optimization in network-structured settings. We propose the first fully decentralized optimization method for a wide class of non-convex objective functions that possess a diminishing returns property. More specifically, given an arbitrary connected network and a global continuous submodular function, formed by a sum of local functions, we develop Decentralized Continuous Greedy (DCG), a message passing algorithm that converges to the tight $(1 - 1/e)$ approximation factor of the optimum global solution using only local computation and communication. We also provide strong convergence bounds as a function of network size and spectral characteristics of the underlying topology. Interestingly, DCG readily provides a simple recipe for decentralized discrete submodular maximization through the means of continuous relaxations. Formally, we demonstrate that by lifting the local discrete functions to continuous domains and using DCG as an interface we can develop a consensus algorithm that also achieves the tight $(1 - 1/e)$ approximation guarantee of the global discrete solution once a proper rounding scheme is applied.

This work was done while A. Mokhtari was a Research Fellow at the Simons Institute for the Theory of Computing.
1. Introduction

In recent years, we have reached unprecedented data volumes that are high dimensional and sit over (clouds of) networked machines. As a result, decentralized collection of these data sets along with accompanying distributed optimization methods are not only desirable but very often necessary [Boyd et al., 2011].

The focus of this paper is on decentralized optimization, the goal of which is to maximize/minimize a global objective function – distributed over a network of computing units – through local computation and communications among nodes. A canonical example in machine learning is fitting models using M-estimators where given a set of data points the parameters of the model are estimated through an empirical risk minimization [Vapnik, 1998]. Here, the global objective function is defined as an average of local loss functions associated with each data point. Such local loss functions can be convex (e.g., logistic regression, SVM, etc) or non-convex (e.g., non-linear square loss, robust regression, mixture of Gaussians, deep neural nets, etc) [Mei et al., 2016]. Due to the sheer volume of data points, these optimization tasks cannot be fulfilled on a single computing cluster node. Instead, we need to opt for decentralized solutions that can efficiently exploit dispersed (and often distant) computational resources linked through a tightly connected network. Furthermore, local computations should be light so that they can be done on single machines. In particular, when the data is high dimensional, extra care should be given to any optimization procedure that relies on projections over the feasibility domain.

In addition to large scale machine learning applications, decentralized optimization is a method of choice in many other domains such as Internet of Things (IoT) [Abu-Elkheir et al., 2013], remote sensing [Ma et al., 2015], multi-robot systems [Tanner and Kumar, 2005], and sensor networks [Rabbat and Nowak, 2004]. In such scenarios, individual entities can communicate over a network and interact with the environment by exchanging the data generated through sensing. At the same time they can react to events and trigger actions to control the physical world. These applications highlight another important aspect of decentralized optimization where private data is collected by different sensing units [Yang et al., 2017]. Here again, we aim to optimize a global objective function while avoiding to share the private data among computing units. Thus, by design, one cannot solve such private optimization problems in a centralized manner and should rely on decentralized solutions where local private computation is done where the data is collected.

Continuous submodular functions, a broad subclass of non-convex functions with diminishing returns property, have recently received considerable attention [Bach, 2015; Bian et al., 2017]. Due to their interesting structures that allow strong approximation guarantees [Mokhtari et al., 2017a; Bian et al., 2017], they have found various applications, including robust budget allocation [Staib and Jegelka, 2017; Soma et al., 2014], online resource allocation [Eghbali and Fazel, 2016], learning assignments [Golovin et al., 2014], as well as Adwords for e-commerce and advertising [Devanur and Jain, 2012; Mehta et al., 2007]. However, all the existing work suffer from centralized computing. Given that many information gathering, data summarization, and non-parametric learning problems are inherently related to large-scale submodular maximization, the demand for a fully decentralized solution is immediate. In this paper, we develop the first decentralized framework for both continuous and discrete submodular functions. Our contributions are as follows:

- **Continuous submodular maximization:** For any global objective function that is monotone and continuous DR-submodular and subject to any down-closed and bounded convex body, we develop Decentralized Continuous Greedy, a decentralized and projection-free algorithm that achieves the tight \((1 - 1/e - \epsilon)\) approximation guarantee in \(O(1/e^2)\) rounds of local communication.

- **Discrete submodular maximization:** For any global objective function that is monotone and submodular and subject to any matroid constraint, we develop a discrete variant of the Decentralized Continuous Greedy algorithm that achieves the tight \((1 - 1/e - \epsilon)\) approximation ratio in \(O(1/e^3)\) rounds of communication.
2. Related Work

Decentralized optimization is a challenging problem as nodes only have access to separate components of the global objective function, while they aim to collectively reach the global optimum point. Indeed, one naive approach to tackle this problem is to broadcast local objective functions to all the nodes in the network and then solve the problem locally. However, this scheme requires high communication overhead and disregards the privacy associated with the data of each node. An alternative approach is the master-slave setting (Bekkerman et al., 2011; Shamir et al., 2014; Zhang and Lin, 2015) where at each iteration, nodes use their local data to compute the information needed by the master node. Once the master node receives all the local information, it updates its decision and broadcasts the decision to all the nodes. Although this scheme protects the privacy of nodes it is not robust to machine failures and is prone to high overall communication time. In decentralized methods, these issues are overcome by removing the master node and considering each node as an independent unit that is allowed to exchange information with its neighbors.

Convex decentralized consensus optimization is a relatively mature area with a myriad of primal and dual algorithms (Bertsekas and Tsitsiklis, 1989). Among primal methods, decentralized (sub)gradient descent is perhaps the most well known algorithm which is a mix of local gradient descent and successive averaging (Nedic and Ozdaglar, 2009; Yuan et al., 2016). It also can be interpreted as a penalty method that encourages agreement among neighboring nodes. This latter interpretation has been exploited to solve the penalized objective function using accelerated gradient descent (Jakovetić et al., 2014; Qu and Li, 2017), Newton’s method (Mokhtari et al., 2017b; Bajovic et al., 2017), or quasi-Newton algorithms (Eisen et al., 2017). The methods that operate in the dual domain consider a constraint that enforces equality between nodes’ variables and solve the problem by ascending on the dual function to find optimal Lagrange multipliers. A short list of dual methods are the alternating directions method of multipliers (ADMM) (Schizas et al., 2008; Boyd et al., 2011), dual ascent algorithm (Rabbat et al., 2005), and augmented Lagrangian methods (Jakovetic et al., 2015; Chatzipanagiotis and Zavlanos, 2015; Mokhtari and Ribeiro, 2016). Recently, there have been many attempts to extend the tools in decentralized consensus optimization to the case that the objective function is non-convex (Di Lorenzo and Scutari, 2016; Sun et al., 2016; Hajinezhad et al., 2016; Tatarenko and Touri, 2017). However, such works are mainly concerned with reaching a stationary point and naturally cannot provide any optimality guarantee.

In this paper, our focus is to provide the first decentralized algorithms for both discrete and continuous submodular functions. Indeed, it is known that the centralized greedy approach of (Nemhauser et al., 1978), and its many variants (Feige et al., 2011; Buchbinder et al., 2015, 2014; Feldman et al., 2017; Mirzasoleiman et al., 2016), reach tight approximation guarantees in various scenarios. Since such algorithms are sequential in nature, they do not scale to massive datasets. To partially resolve this issue, MapReduce style algorithms, with a master-slave architecture, have been proposed (Mirzasoleiman et al., 2013; Kumar et al., 2015; da Ponte Barbosa et al., 2015; Mirrokni and Zadimoghaddam, 2015; Qu et al., 2015).

One can extend the notion of diminishing returns to continuous domains (Wolsey, 1982; Bach, 2015). Even though continuous submodular functions are not generally convex (nor concave) Hassani et al. (2017) showed that in the monotone setting and subject to a general bounded convex body constraint, stochastic gradient methods can achieve a 1/2 approximation guarantee. The approximation guarantee can be tightened to $(1 - 1/e)$ by using Frank-Wolfe (Bian et al., 2017) or stochastic Frank-Wolfe (Mokhtari et al., 2017a).

3. Notation and Background

In this section, we review the notation that we use throughout the paper. We then give the precise definition of submodularity in discrete and continuous domains.
Notation. Lowercase boldface $v$ denotes a vector and uppercase boldface $W$ a matrix. The $i$-th element of $v$ is written as $v_i$ and the element on the $i$-th row and $j$-th column of $W$ is denoted by $w_{i,j}$. We use $\|v\|$ to denote the Euclidean norm of vector $v$ and $\|W\|$ to denote the spectral norm of matrix $W$. The null space of matrix $W$ is denoted by null($W$). The inner product of vectors $x, y$ is indicated by $\langle x, y \rangle$, and the transpose of a vector $v$ or matrix $W$ are denoted by $v^\dagger$ and $W^\dagger$, respectively. The vector $1_n \in \mathbb{R}^n$ is the vector of all ones with $n$ components, and the vector $0_p \in \mathbb{R}^p$ is the vector of all zeros with $p$ components.

Submodularity. A set function $f : 2^V \rightarrow \mathbb{R}_+$, defined on the ground set $V$, is called submodular if for all $A, B \subseteq V$, we have

$$f(A) + f(B) \geq f(A \cap B) + f(A \cup B).$$

We often need to maximize submodular functions subject to a down-closed set family $I$. In particular, we say $I \subset 2^V$ is a matroid if 1) for any $A \subset B \subset V$, if $B \in I$, then $A \in I$ and 2) for any $A, B \in I$ if $|A| < |B|$, then there is an element $e \in B$ such that $A \cup \{e\} \in I$.

The notion of submodularity goes beyond the discrete domain (Wolsey, 1982; Vondrák, 2007; Bach, 2015). Consider a continuous function $F : \mathcal{X} \rightarrow \mathbb{R}_+$, where the set $\mathcal{X} \subseteq \mathbb{R}^p$ is of the form $\mathcal{X} = \prod_{i=1}^p \mathcal{X}_i$ and each $\mathcal{X}_i$ is a compact subset of $\mathbb{R}_+$. We call the continuous function $F$ submodular if for all $x, y \in \mathcal{X}$ we have

$$F(x) + F(y) \geq F(x \lor y) + F(x \land y), \tag{1}$$

where $x \lor y := \max(x, y)$ (component-wise) and $x \land y := \min(x, y)$ (component-wise). In this paper, our focus is on differentiable continuous submodular functions with two additional properties: monotonicity and diminishing returns. Formally, a submodular function $F$ is monotone if

$$x \leq y \implies F(x) \leq F(y), \tag{2}$$

for all $x, y \in \mathcal{X}$. Note that $x \leq y$ in (2) means that $x_i \leq y_i$ for all $i = 1, \ldots, p$. Furthermore, a differentiable submodular function $F$ is called DR-submodular (i.e., shows diminishing returns) if the gradients are antitone, namely, for all $x, y \in \mathcal{X}$ we have

$$x \leq y \implies \nabla F(x) \succeq \nabla F(y). \tag{3}$$

When the function $F$ is twice differentiable, submodularity implies that all cross-second-derivatives are non-positive (Bach, 2015), and DR-submodularity implies that all second-derivatives are non-positive (Bian et al., 2017). In this work, we consider the maximization of continuous submodular functions subject to down-closed convex bodies $C \subset \mathbb{R}_+^p$ defined as follows. For any two vectors $x, y \in \mathbb{R}_+^p$, where $x \leq y$, down-closedness means that if $y \in C$, then so is $x \in C$. Note that for a down-closed set we have $0_p \in C$.

4. Decentralized Submodular Maximization

In this section, we state the problem of decentralized submodular maximization in continuous and discrete settings.

Continuous Case. We consider a set of $n$ computing machines/sensors that communicate over a graph to maximize a global objective function. Each machine can be viewed as a node $i \in \mathcal{N} \equiv \{1, \ldots, n\}$. We further assume that the possible communication links among nodes are given by a bidirectional connected communication graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ where each node can only communicate with its neighbors in $\mathcal{G}$. We formally use $\mathcal{N}_i$ to denote node $i$’s neighbors. In our setting, we assume
that each node \( i \in \mathcal{N} \) has access to a local function \( F_i : \mathcal{X} \to \mathbb{R}_+ \). The nodes cooperate in order to maximize the aggregate monotone and continuous DR-submodular function \( F : \mathcal{X} \to \mathbb{R}_+ \) subject to a down-closed convex body \( C \subset \mathcal{X} \subset \mathbb{R}_n^p \), i.e.,

\[
\max_{x \in C} F(x) = \max_{x \in \mathcal{X}} \frac{1}{n} \sum_{i=1}^n F_i(x).
\] (4)

The goal is to design a message passing algorithm to solve (4) such that: (i) at each iteration \( t \), the nodes send their messages (and share their information) to their neighbors in \( \mathcal{G} \), and (ii) as \( t \) grows, all the nodes reach to a point \( x \in C \) that provides a (near-) optimal solution for (4).

**Discrete Case.** Let us now consider the discrete counterpart of problem (4). In this setting, each node \( i \in \mathcal{N} \) has access to a local set function \( f_i : 2^V \to \mathbb{R}_+ \). The nodes cooperate in maximizing the aggregate monotone submodular function \( f : 2^V \to \mathbb{R}_+ \) subject to a matroid constraint \( \mathcal{I} \), i.e.,

\[
\max_{S \in \mathcal{I}} f(S) = \max_{S \in \mathcal{I}} \frac{1}{n} \sum_{i=1}^n f_i(S).
\] (5)

Note that even in the centralized case, and under reasonable complexity-theoretic assumptions, the best approximation guarantee we can achieve for Problems (4) and (5) is \( (1 − 1/e) \) (Feige, 1998). In the following, we show that it is possible to achieve the same approximation guarantee in a decentralized setting.

5. Decentralized Continuous Greedy Method

In this section, we introduce the Decentralized Continuous Greedy (DCG) algorithm for solving Problem (4). Recall that in a decentralized setting, the nodes have to cooperate (i.e., send messages to their neighbors) in order to solve the global optimization problem. We will explain how such messages are designed and communicated in DCG. Each node \( i \) in the network keeps track of two local variables \( x_i, d_i \in \mathbb{R}^p \) which are iteratively updated at each round \( t \) using the information gathered from the neighboring nodes. The vector \( x_i \) is the local decision variable of node \( i \) at step \( t \) whose value we expect to eventually converge to the \( (1 − 1/e) \) fraction of the optimal solution of Problem (4). The vector \( d_i \) is the estimate of the gradient of the global objective function that node \( i \) keeps at step \( t \).

To properly incorporate the received information from their neighbors, nodes should assign non-negative weights to their neighbors. Define \( w_{ij} \geq 0 \) to be the weight that node \( i \) assigns to node \( j \). These weights indicate the effect of (variable or gradient) information nodes received from their neighbors in order to update their local (variable or gradient) information. Indeed, the weights \( w_{ij} \) must fulfill some requirements (later described in Assumption [A]), but they are design parameters of DCG and can be properly chosen by the nodes prior to the implementation of the algorithm.

The first step at each round \( t \) of DCG is updating the local gradient approximation vectors \( d_i \) using local and neighboring gradient information. In particular, node \( i \) computes its vector \( d_i \) according to the update rule

\[
d_i^t = (1 − \alpha) \sum_{j \in \mathcal{N}_i \cup \{i\}} w_{ij}d_j^{t-1} + \alpha \nabla F_i(x_i^t),
\] (6)

where \( \alpha \in [0, 1] \) is an averaging coefficient. Note that the sum \( \sum_{j \in \mathcal{N}_i \cup \{i\}} w_{ij}d_j^{t-1} \) in (6) is a weighted average of node \( i \)'s vector \( d_j^{t-1} \) and its neighbors \( d_j^{t-1} \), evaluated at step \( t-1 \). Hence, node \( i \) computes the vector \( d_i \) by evaluating a weighted average of its current local gradient \( \nabla F_i(x_i^t) \) and the local and neighboring gradient information at step \( t-1 \), i.e., \( \sum_{j \in \mathcal{N}_i \cup \{i\}} w_{ij}d_j^{t-1} \). Since the vector \( d_i \) is
evaluated by aggregating gradient information from neighboring nodes, it is reasonable to expect that $d_i^t$ becomes a proper approximation for the global objective function gradient $(1/n) \sum_{k=1}^n \nabla f_k(x_i^t)$ as time progresses. Note that to implement the update in (6) nodes should exchange their local vectors $d_i^t$ with their neighbors.

Using the gradient approximation vector $d_i^t$, each node $i$ evaluates its local ascent direction $v_i^t$ by solving the following linear program

$$v_i^t = \arg \max_{v \in C} \langle d_i^t, v \rangle.$$ \hspace{1cm} (7)

The update in (7) is also known as conditional gradient update. Ideally, in a conditional gradient method, we should choose the feasible direction $v \in C$ that maximizes the inner product by the full gradient vector $\frac{1}{n} \sum_{k=1}^n \nabla F_k(x_i^t)$. However, since in the decentralized setting the exact gradient $\frac{1}{n} \sum_{k=1}^n \nabla F_k(x_i^t)$ is not available at the $i$-th node, we replace it by its current approximation $d_i^t$ and hence we obtain the update rule (7).

After computing the local ascent directions $v_i^t$, the nodes update their local variables $x_i^t$ by averaging their local and neighboring iterates and ascend in the direction $v_i^t$ with stepsize $1/T$ where $T$ is the total number of iterations, i.e.,

$$x_i^{t+1} = \sum_{j \in \mathcal{N}_i \cup \{i\}} w_{ij} x_j^t + \frac{1}{T} v_i^t.$$ \hspace{1cm} (8)

The update rule (8) ensures that the neighboring iterates are not far from each other via the averaging term $\sum_{j \in \mathcal{N}_i \cup \{i\}} w_{ij} x_j^t$, while the iterates approach the optimal maximizer of the global objective function by ascending in the conditional gradient direction $v_i^t$. The update in (8) requires a round of local communication among neighbors to exchange their local variables $x_i^t$. The steps of the DCG method are summarized in Algorithm 1.

Indeed, the weights $w_{ij}$ that nodes assign to each other cannot be arbitrary. In the following, we formalize the conditions that they should satisfy (Yuan et al., 2016).

**Assumption 1** *The weights that nodes assign to each other are nonnegative, i.e., $w_{ij} \geq 0$ for all $i, j \in \mathcal{N}$, and if node $j$ is not a neighbor of node $i$ then the corresponding weight is zero, i.e., $w_{ij} = 0$ if $j \notin \mathcal{N}_i$. Further, the weight matrix $W \in \mathbb{R}^{n \times n}$ with entries $w_{ij}$ satisfies

$$W^\dagger = W, \quad W1_n = 1_n, \quad \null(I - W) = \text{span}(1_n).$$ \hspace{1cm} (9)*
The first condition in (9) ensures that the weights are symmetric, i.e., $w_{ij} = w_{ji}$. The second condition guarantees the weights that each node assigns to itself and its neighbors sum up to 1, i.e., $\sum_{j=1}^{n} w_{ij} = 1$ for all $i$. Note that the condition $W_1 = 1_n$ implies that $I - W$ is rank deficient. Hence, the last condition in (9) ensures that the rank of $I - W$ is exactly $n - 1$. Indeed, it is possible to optimally design the weight matrix $W$ to accelerate the averaging process as discussed in (Boyd et al., 2004), but this is not the focus of this paper. We should emphasize that $W$ is not a problem parameter, and we design it prior to running DCG.

Notice that the stepsize $1/T$ and the conditions in Assumption 1 on the weights $w_{ij}$ are needed to ensure that the local variables $x_t^i$ are in the feasible set $C$, as stated in the following proposition.

**Proposition 1** Consider the proposed DCG method outlined in Algorithm 1. If Assumption 1 holds and nodes start from $x_0^i = 0_p \in C$, then the local iterates $x_t^i$ are always in the feasible set $C$, i.e., $x_t^i \in C$ for all $i \in N$ and $t = 1, \ldots, T$.

**Proof:** Check Section 9.1 in the Appendix. ■

Let us now explain how DCG relates to and innovates beyond the existing work in submodular maximization as well as decentralized convex optimization. Note that in order to solve Problem (4) in a centralized fashion (i.e., when every node has access to all the local functions) we can use the continuous greedy algorithm (Vondrak, 2008), a variant of the conditional gradient method. However, in decentralized settings, nodes have only access to their local gradients, and therefore, continuous greedy is not implementable. Similar to the decentralized convex optimization, we can address this issue via local information aggregation. Our proposed DCG method incorporates the idea of choosing the ascent direction according to a conditional gradient update as is done in the continuous greedy algorithm (i.e., the update rule (7)), while it aggregates the global objective function information through local communications with neighboring nodes (i.e., the update rule (8)).

Unlike traditional consensus optimization methods that require exchanging nodes’ local variables only (Nedic and Ozdaglar 2009; Nedic et al. 2010), DCG also requires exchanging local gradient vectors to achieve a $(1 - 1/e)$ fraction of the optimal solution at each node (i.e., the update rule (6)). This major difference is due to the fact that in conditional gradient methods, unlike proximal gradient algorithms, the local gradients can not be used instead of the global gradient. In other words, in the update rule (7), we can not use the local gradients $\nabla F_i(x_t^i)$ in lieu of $d_t^i$. Indeed, there are settings for which such a replacement provides arbitrarily bad solutions. We formally characterize the convergence of DCG in Theorem 7.

### 5.1 Extension to the Discrete Setting

In this section we show how DCG can be used for maximizing a decentralized submodular set function $f$, namely Problem (5), through its continuous relaxation. Formally, in lieu of solving Problem (5), we can form the following decentralized continuous optimization problem

$$
\max_{x \in C} \frac{1}{n} \sum_{i=1}^{n} F_i(x),
$$

where $F_i$ is the multilinear extension of $f_i$ defined as

$$
F_i(x) = \sum_{S \subseteq V} f_i(S) \prod_{i \in S} x_i \prod_{j \notin S} (1 - x_j),
$$

and the down-closed convex set $C = \text{conv}\{1_I : I \in I\}$ is the matroid polytope. Note that the discrete and continuous optimization formulations lead to the same optimal value (Calinescu et al., 2011).

Based on the expression in (11), computing the full gradient $\nabla F_i$ at each node $i$ will require an exponential computation in terms of $|V|$, since the number of summands in (11) is $2^{|V|}$. As a result,
Gradient approximations. Therefore, the sequence of shown recently (Mokhtari et al., 2017a) that the averaging technique in (12) reduces the noise of the estimations of the local gradients and continuous versions of DCG are the same at each round. However, since we are using unbiased the continuous DCG method (Algorithm 1) is in Step 5 in which the exact local gradient $\nabla F_i(x^t_i)$ is replaced by the stochastic approximation $\nabla F_i(x^t_i)$ for cheap unbiased gradient estimator $\nabla g$ is replaced by the stochastic approximation as time progresses.

In the discrete setting, we will slightly modify the DCG algorithm and work with unbiased estimates of the gradient that can be computed in time $O(|V|)$ (see Appendix 9.7 for one such estimator). More precisely, in the discrete setting, each node $i \in \mathcal{N}$ updates three local variables $x^t_i, d^t_i, g^t_i \in \mathbb{R}^{|V|}$. The variables $x^t_i, d^t_i$ play the same role as in DCG and are updated using the messages received from the neighboring nodes. The variable $g^t_i$ at node $i$ is defined to approximate the local gradient $\nabla F_i(x^t_i)$. Consider the vector $\nabla F_i(x^t_i)$ as an unbiased estimator of the local gradient $\nabla F_i(x^t_i)$ at time $t$, and define the vector $g^t_i$ as the outcome of the recursion

$$g^t_i = (1 - \phi)g^{t-1}_i + \phi \nabla F_i(x^t_i), \quad (12)$$

where $\phi \in [0, 1]$ is the averaging parameter. We initialize all vectors as $g^0_i = 0 \in \mathbb{R}^{|V|}$. It was shown recently (Mokhtari et al., 2017a) that the averaging technique in (12) reduces the noise of the gradient approximations. Therefore, the sequence of $g^t_i$ approaches the true local gradient $\nabla F_i(x^t_i)$ as time progresses.

The steps of the Decentralized Continuous Greedy for the discrete setting is summarized in Algorithm 2. Note that the major difference between the Discrete DCG method (Algorithm 2) and the continuous DCG method (Algorithm 1) is in Step 5 in which the exact local gradient $\nabla F_i(x^t_i)$ is replaced by the stochastic approximation $g^t_i$ which only requires access to the computationally cheap unbiased gradient estimator $\nabla F_i(x^t_i)$. The communication complexity of both the discrete and continuous versions of DCG are the same at each round. However, since we are using unbiased estimations of the local gradients $\nabla F_i(x_i)$, the Discrete DCG takes more rounds to converge to a near-optimal solution compared to continuous DCG. We characterize the convergence of Discrete DCG in Theorem 8. Further, the implementation of Discrete DCG requires rounding the continuous solution to obtain a discrete solution for the original problem without any loss in terms of the objective function value. The provably lossless rounding schemes include the pipage rounding (Calinescu et al., 2011) and contention resolution (Chekuri et al., 2014).

6. Convergence Analysis

In this section, we study the convergence properties of DCG in both continuous and discrete settings. In this regard, we assume that the following conditions hold.

**Algorithm 2** Discrete DCG at node $i$

**Require:** $\alpha, \phi \in [0, 1]$ and weights $w_{ij}$ for $j \in \mathcal{N}_i \cup \{i\}$

1. Initialize local vectors as $x^0_i = d^0_i = g^0_i = 0$
2. Initialize neighbor’s vectors as $x^0_j = d^0_j = 0$ if $j \in \mathcal{N}_i$
3. for $t = 1, 2, \ldots, T$
4. Compute $g^t_i = (1 - \phi)g^{t-1}_i + \phi \nabla F_i(x^t_i)$
5. Compute $d^t_i = (1 - \alpha) \sum_{j \in \mathcal{N}_i \cup \{i\}} w_{ij} d^{t-1}_j + \alpha g^t_i$
6. Exchange $d^t_i$ with neighboring nodes $j \in \mathcal{N}_i$
7. Evaluate $v^t_i = \arg\max_{v \in \mathcal{C}} \langle d^t_i, v \rangle$
8. Update the variable $x^{t+1}_i = \sum_{j \in \mathcal{N}_i \cup \{i\}} w_{ij} x^t_j + \frac{1}{T} v^t_i$
9. Exchange $x^{t+1}_i$ with neighboring nodes $j \in \mathcal{N}_i$
10. end for
11. Apply proper rounding to obtain a solution for (5)

6. Convergence Analysis

In this section, we study the convergence properties of DCG in both continuous and discrete settings. In this regard, we assume that the following conditions hold.
Assumption 2  Euclidean distance of the elements in the set $C$ are uniformly bounded, i.e., for all $x, y \in C$ we have
\[ \|x - y\| \leq D. \] (13)

Assumption 3  The local objective functions $F_i(x)$ are monotone and DR-submodular. Further, their gradients are $L$-Lipschitz continuous over the set $X$, i.e., for all $x, y \in X$
\[ \|\nabla F_i(x) - \nabla F_i(y)\| \leq L\|x - y\|. \] (14)

Assumption 4  The norm of gradients $\|\nabla F_i(x)\|$ are bounded over the convex set $C$, i.e., for all $x \in C$, $i \in N$,
\[ \|\nabla F_i(x)\| \leq G. \] (15)

The condition in Assumption 2 guarantees that the diameter of the convex set $C$ is bounded. Assumption 3 is needed to ensure that the local objective functions $F_i$ are smooth. Finally, the condition in Assumption 4 enforces the gradients norm to be bounded over the convex set $C$. All these assumptions are customary and necessary in the analysis of decentralized algorithms. For more details, please check Section VII-B in Jakovetić et al. (2014).

We proceed to derive a constant factor approximation for DCG. Our main result is stated in Theorem 7. However, to better illustrate the main result, we first need to provide several definitions and technical lemmas. Let us begin by defining the average variables $\bar{x}^t$ as
\[ \bar{x}^t = \frac{1}{n} \sum_{i=1}^{n} x^t_i. \] (16)

In the following lemma, we establish an upper bound on the variation in the sequence of average variables $\{\bar{x}^t\}$.

Lemma 2  Consider the proposed DCG algorithm defined in Algorithm 1. Further, recall the definition of $\bar{x}^t$ in (16). If Assumptions 2 and 3 hold, then the difference between two consecutive average vectors is upper bounded by
\[ \|\bar{x}^{t+1} - \bar{x}^t\| \leq \frac{D}{T}. \] (17)

Proof:  Check Section 9.2 in the Appendix.

Recall that at every node $i$, the messages are mixed using the coefficients $w_{ij}$, i.e., the $i$-th row of the matrix $W$. It is thus not hard to see that the spectral properties of $W$ (e.g. the spectral gap) play an important role in the the speed of achieving consensus in decentralized methods.

Definition 3  Consider the eigenvalues of $W$ which can be sorted in a nonincreasing order as $1 = \lambda_1(W) \geq \lambda_2(W) \cdots \geq \lambda_n(W) > -1$. Define $\beta$ as the second largest magnitude of the eigenvalues of $W$, i.e.,
\[ \beta := \max\{|\lambda_2(W)|, |\lambda_n(W)|\}. \] (18)

As we will see, a mixing matrix $W$ with smaller $\beta$ has a larger spectral gap $1 - \beta$ which yields faster convergence (Boyd et al. 2004; Duchi et al. 2012). In the following lemma, we derive an upper bound on the sum of the distances between the local iterates $x^t_i$ and their average $\bar{x}^t$, where the bound is a function of the graph spectral gap $1 - \beta$, size of the network $n$, and the total number of iterations $T$. 


Lemma 4 Consider the proposed DCG algorithm defined in Algorithm 1. Further, recall the definition of $\bar{x}^t$ in (16). If Assumptions 1 and 2 hold, then for all $t \leq T$ we have

$$\left( \sum_{i=1}^{n} \| x^t_i - \bar{x}^t \|^2 \right)^{1/2} \leq \frac{\sqrt{nD}}{T(1-\beta)}. \tag{19}$$

Proof: Check Section 9.3 in the Appendix. ■

Let us now define $\bar{d}^t$ as the average of local gradient approximations $d^t_i$ at step $t$, i.e.,

$$\bar{d}^t = \frac{1}{n} \sum_{i=1}^{n} d^t_i. \tag{20}$$

We will show in the following that the vectors $d^t_i$ also become uniformly close to $\bar{d}^t$. 

Lemma 5 Consider the proposed DCG algorithm defined in Algorithm 1. If Assumptions 1 and 3 hold, then

$$\left( \sum_{i=1}^{n} \| d^t_i - \bar{d}^t \|^2 \right)^{1/2} \leq \frac{\alpha \sqrt{nG}}{1 - \beta(1-\alpha)}. \tag{21}$$

Proof: Check Section 9.4 in the Appendix. ■

Lemma 5 guarantees that the individual local gradient approximation vectors $d^t_i$ are close to the average vector $\bar{d}^t$ if the parameter $\alpha$ is small. To show that the gradient vectors $d^t_i$, generated by DCG, approximate the gradient of the global objective function, we further need to show that the average vector $\bar{d}^t$ approaches the global objective function gradient $\nabla F$. We prove this claim in the following lemma.

Lemma 6 Consider the proposed DCG algorithm defined in Algorithm 1. If Assumptions 1-4 hold, then

$$\| \bar{d}^t - \frac{1}{n} \sum_{i=1}^{n} \nabla F_i(\bar{x}^t) \| \leq (1 - \alpha)^t G + \left( \frac{(1-\alpha)LD}{\alpha T} + \frac{LD}{T(1-\beta)} \right). \tag{22}$$

Proof: Check Section 9.5 in the Appendix. ■

By combining Lemmas 5 and 6 and setting $\alpha = 1/\sqrt{T}$ we can conclude that the local gradient approximation vector $d^t_i$ of each node $i$ is within $O(1/\sqrt{T})$ distance of the global objective gradient $\nabla F(\bar{x}^t)$ evaluated at $\bar{x}^t$. We use this observation in the following theorem to show that the sequence of iterates generated by DCG achieves the tight $(1 - 1/e)$ approximation ratio of the optimum global solution.

Theorem 7 Consider the proposed DCG method outlined in Algorithm 1. Further, consider $x^*$ as the global maximizer of Problem (14). If Assumptions 1-4 hold and we set $\alpha = 1/\sqrt{T}$, for all nodes $j \in N$, the local variable $x^T_j$ obtained after $T$ iterations satisfies

$$F(x^*_j) \geq (1 - e^{-1})F(x^*) - \frac{LD^2 + GD}{T^{1/2}} - \frac{GD}{T^{1/2}(1-\beta)} - \frac{LD^2}{2T} - \frac{GD + LD^2}{T(1-\beta)}. \tag{23}$$

Proof: Check Section 9.6 in the Appendix. ■
Theorem 7 shows that the sequence of the local variables $x^t_j$, generated by DCG, is able to achieve the optimal approximation ratio $(1 - 1/e)$, while the error term vanishes at a sublinear rate of $O(1/T^{1/2})$, i.e.,

$$F(x^T_j) \geq (1 - 1/e)F(x^*) - O \left( \frac{1}{(1 - \beta)T^{1/2}} \right),$$

which implies that the iterate of each node reaches an objective value larger than $(1 - 1/e - \epsilon)OPT$ after $O(1/e^2)$ rounds of communication. It is worth mentioning that the result in Theorem 7 is consistent with classical results in decentralized optimization that the error term vanishes faster for the graphs with larger spectral gap $1 - \beta$. We proceed to study the convergence properties of Discrete DCG in Algorithm 2. To do so, we first assume that the variance of the stochastic gradients $\nabla F_i(x)$ used in Discrete DCG is bounded. We justify this assumption in Remark 9.

**Assumption 5** The variance of the unbiased estimators $\nabla \tilde{F}(x)$ is bounded above by $\sigma^2$ over the convex set $C$, i.e., for any $i \in \mathcal{N}$ and any vector $x \in C$ we can write

$$E \left[ \| \nabla \tilde{F}_i(x) - \nabla F_i(x) \|^2 \right] \leq \sigma^2,$$

where the expectation is with respect to the randomness of the unbiased estimator.

In the following theorem, we show that Discrete DCG achieves a $(1 - 1/e)$ approximation ratio for Problem 5.

**Theorem 8** Consider our proposed Discrete DCG algorithm outlined in Algorithm 2. Recall the definition of the multilinear extension function $F_i$ in (11). If Assumptions 7,8 hold and we set $\alpha = T^{-1/2}$ and $\phi = T^{-2/3}$, then for all nodes $j \in \mathcal{N}$ the local variables $x^T_j$ obtained after running Discrete DCG for $T$ iterations satisfy

$$E \left[ F(x^T_j) \right] \geq (1 - e^{-1})F(x^*) - \frac{GD + LD^2}{T(1 - \beta)} - \frac{LD^2}{2T} - \frac{\sqrt{6}LD^2}{T^{2/3}} - \frac{\sqrt{T^2LD^2}}{(1 - \beta)T^{2/3}} - \frac{D(\sigma^2 + G^2)^{1/2}}{T^{1/2}(1 - \beta)} - \frac{DG + LD^2}{T^{1/2}} - \frac{\sqrt{2}G^2 + 4DG}{T^{1/3}} \leq \frac{\sqrt{4ALD^2}}{(1 - \beta)T^{1/3}},$$

where $x^*$ is the global maximizer of Problem 10.

**Proof:** Check Section 9.8 in the Appendix. \hfill \blacksquare

Theorem 8 states that the sequence of the local variables $x^t_j$, generated by Discrete DCG, is able to achieve the optimal approximation ratio $(1 - 1/e)$ in expectation, while the error term vanishes at a sublinear rate of $O(1/T^{1/3})$, i.e.,

$$E \left[ F(x^T_j) \right] \geq (1 - e^{-1})F(x^*) - O \left( \frac{1}{(1 - \beta)T^{1/3}} \right).$$

Hence, the iterate of each node reaches an objective value larger than $(1 - 1/e - \epsilon)OPT$ after $O(1/e^3)$ rounds of communication.

**Remark 9** For any submodular set function $h : 2^V \to \mathbb{R}$ with associated multilinear extension $H$, it can be shown that its Lipschitz constant $L$ and the gradient norm $G$ are both bounded above by $m_f\sqrt{|V|}$, where $m_f$ is the maximum marginal value of $f$, i.e., $m_f = \max_{i \in V} f(\{i\})$ (see Hassani et al. 2017). Similarly, it can be shown that for the unbiased estimator in Appendix 9.7 we have $\sigma \leq m_f\sqrt{|V|}$. 

11
7. Numerical Experiments

We will consider a discrete setting for our experiments and use Algorithm 2 to find a decentralized solution. The main objective is to demonstrate how consensus is reached and how the global objective increases depending on the topology of the network and the parameters of the algorithm.

For our experiments, we have used the MovieLens data set. It consists of 1 million ratings (from 1 to 5) by $M = 6000$ users for $p = 4000$ movies. We consider a network of $n = 100$ nodes. The data has been distributed equally between the nodes of the network, i.e., the set of users has been partitioned into 100 equally-sized sets and each node in the network has access to only one chunk (partition) of the data. The global task is to find a set of $k$ movies that are most satisfactory to all the users (the precise formulation will appear shortly). However, as each of the nodes in the network has access to the data of a small portion of the users, the nodes have to cooperate (exchange information) in order to fulfill the global task.

We consider a well motivated objective function for the experiments. Let $r_{\ell,j}$ denote the rating of user $\ell$ for movie $j$ (if such a rating does not exist in the data we assign $r_{\ell,j}$ to 0). We associate to each user $\ell$ a “facility location” objective function $g_{\ell}(S) = \max_{j \in S} r_{\ell,j}$, where $S$ is any subset of the movies (i.e. the ground set $V$ is the set of the movies). Such a function shows how much user $\ell$ will be “satisfied” by a subset $S$ of the movies. Recall that each node $i$ in the network has access to the data of a (small) subset of users which we denote by $U_i$. The objective function associated with node $i$ is given by $f_i(S) = \sum_{\ell \in U_i} g_{\ell}(S)$. With such a choice of the local functions, our global task is hence to solve problem (5) when the matroid $I$ is the $k$-uniform matroid (a.k.a. the $k$-cardinality constraint).

We consider three different choices for the underlying communication graph between the 100 nodes: A line graph (which looks like a simple path from node 1 to node 100), an Erdos-Renyi random graph (with average degree 5), and a complete graph. The matrix $W$ is chosen as follows (based on each of the three graphs). If $(i,j)$ is an edge of the graph, we let $w_{i,j} = 1/(1+\max(d_i,d_j))$. If $(i,j)$
Figure 2: The average objective value is plotted as a function of the cardinality constraint $k$ for different choices of the communication graph as well as number of iterations $T$. Note that “ER” stands for the Erdos-Renyi graph with average degree 5, “Line” stands for the line graph and “Complete” is for the complete graph. We have run Algorithm 2 for $T = 50$ and $T = 1000$.

is not an edge and $i, j$ are distinct integers, we have $w_{i,j} = 0$. Finally we let $w_{i,i} = 1 - \sum_{j \in N} w_{i,j}$. It is not hard to show that the above choice for $W$ satisfies Assumption 1.

Figure 1 shows how consensus is reached w.r.t each of the three underlying networks. To measure consensus, we plot the (logarithm of) distance-to-average value $\frac{1}{n} \sum_{i=1}^{n} ||x_i^T - \bar{x}^T||$ as a function of the total number of iterations $T$ averaged over many trials (see (16) for the definition of $\bar{x}$). It is easy to see that the distance to average is small if and only if all the local decisions $x_i^T$ are close to the average decision $\bar{x}^T$. As expected, it takes much less time to reach consensus when the underlying graph is fully connected (i.e. complete graph). For the line graph, the convergence is very slow as this graph has the least degree of connectivity.

Figure 2 depicts the obtained objective value of Discrete DCG (Algorithm 2) for the three networks considered above. More precisely, we plot the value $\frac{1}{n} \sum_{i=1}^{n} f(x_i^T)$ obtained at the end of Algorithm 2 as a function of the cardinality constraint $k$. We also compare these values with the value obtained by the centralized greedy algorithm (i.e. the centralized solution). A few comments are in order. The performance of Algorithm 2 is close to the centralized solution when the underlying graph is the Erdos-Renyi (with average degree 5) graph or the complete graphs. This is because for both such graphs consensus is achieved from the early stages of the algorithm. By increasing $T$, we see that the performance becomes closer to the centralized solution. However, when the underlying graph is the line graph, then consensus will not be achieved unless the number of iterations is significantly increased. Consequently, for small number of iterations (e.g., $T \leq 1000$) the performance of the algorithm will not be close to the centralized solution. Indeed, this is not surprising as the line graph is poorly connected.

8. Conclusion

In this paper, we proposed the first fully decentralized optimization method for maximizing a monotone and continuous DR-submodular function where its components are available at the different
nodes of a connected graph. We developed Decentralized Continuous Greedy (DCG) that achieves a 
\((1 - 1/e - \epsilon)\) approximation guarantee with \(\mathcal{O}(1/\epsilon^2)\) local rounds of communication. We also showed 
that our continuous algorithm can be used to provide the first \((1 - 1/e)\) tight approximation guaran-
tee for maximizing a monotone submodular set function subject to a general matroid constraint in a 
decentralized fashion. In particular, we demonstrated that by lifting the local discrete functions to 
the continuous domain and using DCG as an interface, after \(\mathcal{O}(1/\epsilon^3)\) rounds of communication, each 
node achieves a tight \((1 - 1/e - \epsilon)\) fractional approximate solution. Such solutions can be efficiently 
rounded in order to obtain discrete solutions with the same approximation guarantee.

Acknowledgments

The work of A. Mokhtari was partially supported by the DIMACS/Simons Collaboration on Bridging 
Continuous and Discrete Optimization through NSF grant CCF-1740425. The work of A. Karbasi 
was supported by DARPA Young Faculty Award (D16AP00046).

9. Appendix

9.1 Proof of Proposition 1

Define \(x_{con} = [x_1; \ldots; x_n] \in \mathbb{R}^{np}\) and \(v_{con} = [v_1; \ldots; v_n] \in \mathbb{R}^{np}\) as the concatenation of the local 
variables and descent directions, respectively. Using these definitions and the update in (8) we can 
write

\[
x_{con}^{t+1} = (W \otimes I)x_{con}^t + \frac{1}{T}v_{con}^t,
\]

where \(W \otimes I \in \mathbb{R}^{np \times np}\) is the kronecker product of the matrices \(W \in \mathbb{R}^{n \times n}\) and \(I \in \mathbb{R}^{p \times p}\). If we set 
\(x_0^t = 0_p\) for all nodes \(i\), it follows that \(x_{con}^0 = 0_{np}\). Hence, by applying the update in (28) recursively 
we obtain that the iterate \(x_{con}^t\) is equal to

\[
x_{con}^t = \frac{1}{T} \sum_{s=0}^{t-1} (W \otimes I)^{t-1-s}v_{con}^s.
\]

We proceed by showing that if the local blocks of a vector \(v_{con} \in \mathbb{R}^{np}\) belong to the feasible set 
\(\mathcal{C}\), i.e., \(v_i \in \mathcal{C}\) for \(i = 1, \ldots, n\), then the local vectors of \(y_{con} = (W \otimes I)v_{con} \in \mathbb{R}^{np}\) also in the set \(\mathcal{C}\). 
Note that if the condition \(y_{con} = (W \otimes I)v_{con}\) holds, then the \(i\)-th block of \(y_{con} = [y_1; \ldots; y_n]\) can 
be written as

\[
y_i = \sum_{j=1}^n w_{ij}v_j.
\]

Since we assume that all \(\{v_j\}_{j=1}^n\) belong to the set \(\mathcal{C}\) and the set \(\mathcal{C}\) is convex, the weighted average 
of these vectors also is in the set \(\mathcal{C}\), i.e., \(y_i \in \mathcal{C}\). This argument indeed holds for all blocks \(y_i\) and 
therefore \(y_i \in \mathcal{C}\) for \(i = 1, \ldots, n\). This argument verifies that if we apply any power of the matrix 
\(W \otimes I\) to a vector \(v_{con} \in \mathbb{R}^{np}\) whose blocks belong to the set \(\mathcal{C}\), then the local components of the 
output vector also belong to the set \(\mathcal{C}\). Therefore, the local components of each of the terms 
\((W \otimes I)^{t-1-s}v_{con}\) in (29) belong to the set \(\mathcal{C}\). The fact that \(x_i\) which is the \(i\)-th block of the vector \(x_{con}\) is 
the average of \(T\) terms that are in the set \(\mathcal{C}\) \((x_{con})\) is the average of the vectors 
\((W \otimes I)^{t-1}v_{con}, \ldots, (W \otimes I)^0v_{con}\) with weights \(1/T\) and the vector \(0_{np}\) with weight \((T - t)/T)\), 
implies that \(x_i \in \mathcal{C}\). This result holds for all \(i \in \{1, \ldots, n\}\) and the proof is complete.
9.2 Proof of Lemma 2

By averaging both sides of the update in (8) over the nodes in the network and using the fact $w_{ij} = 0$ if $i$ and $j$ are not neighbors we can write

$$\frac{1}{n} \sum_{i=1}^{n} x_{i}^{t+1} = \frac{1}{n} \sum_{i=1}^{n} \sum_{j \in \mathcal{N}_{i} \cup \{i\}} w_{ij} x_{j}^{t} + \frac{1}{T} \frac{1}{n} \sum_{i=1}^{n} v_{i}^{t}$$

$$= \frac{1}{n} \sum_{j=1}^{n} \sum_{i=1}^{n} w_{ij} x_{j}^{t} + \frac{1}{T} \frac{1}{n} \sum_{i=1}^{n} v_{i}^{t}$$

$$= \frac{1}{n} \sum_{j=1}^{n} x_{j}^{t} \sum_{i=1}^{n} w_{ij} + \frac{1}{T} \frac{1}{n} \sum_{i=1}^{n} v_{i}^{t}$$

$$= \frac{1}{n} \sum_{j=1}^{n} x_{j}^{t} + \frac{1}{T} \frac{1}{n} \sum_{i=1}^{n} v_{i}^{t}, \quad (31)$$

where the last equality holds since $W^{T}1_{n} = 1_{n}$ (i.e. $W$ is a doubly stochastic matrix). By using the definition of the average iterate vector $\bar{x}^{t}$ and the result in (31) it follows that

$$\bar{x}^{t+1} = \bar{x}^{t} + \frac{1}{T} \frac{1}{n} \sum_{i=1}^{n} v_{i}^{t}. \quad (32)$$

Since $v_{i}^{t}$ belongs to the convex set $C$ its Euclidean norm is bounded by $\|v_{i}^{t}\| \leq D$ according to Assumption 2. This inequality and the expression in (32) yield

$$\|\bar{x}^{t+1} - \bar{x}^{t}\| \leq \frac{D}{T}, \quad (33)$$

and the claim in (17) follows.

9.3 Proof of Lemma 4

Recall the definitions $x_{con} = [x_{1}; \ldots; x_{n}] \in \mathbb{R}^{np}$ and $v_{con} = [v_{1}; \ldots; v_{n}] \in \mathbb{R}^{np}$ for the concatenation of the local variables and descent directions, respectively. These definitions along with the update in (8) lead to the expression

$$x_{con}^{t} = \frac{1}{T} \sum_{s=0}^{t-1} (W \otimes I)^{t-1-s} v_{con}^{s}. \quad (34)$$

If we premultiply both sides of (34) by the matrix $(\frac{1}{n}1_{n}^{\dagger} \otimes I)$ which is the kronecker product of the matrices $(1/n)(1_{n}1_{n}^{\dagger}) \in \mathbb{R}^{n \times n}$ and $I \in \mathbb{R}^{p \times p}$ we obtain

$$\left(\frac{1}{n}1_{n}^{\dagger} \otimes I\right) x_{con}^{t} = \frac{1}{T} \sum_{s=0}^{t-1} \left(\frac{1}{n}1_{n}^{\dagger} W^{t-1-s} \otimes I\right) v_{con}^{s}. \quad (35)$$

The left hand side of (35) can be simplified to

$$\left(\frac{1}{n}1_{n}^{\dagger} \otimes I\right) x_{con}^{t} = \bar{x}_{con}^{t}, \quad (36)$$
where \( \hat{x}_c^t = [\hat{x}^t_1; \ldots; \hat{x}^t_n] \) is the concatenation of \( n \) copies of the average vector \( \bar{x}^t \). Using the equality in \( 36 \) and the simplification \( \mathbf{1}_n\mathbf{1}_n^\dagger = \mathbf{1}_n \), we can rewrite \( 35 \) as

\[
\hat{x}_c^t = \frac{1}{T} \sum_{s=0}^{t-1} \left( \frac{1}{n} \mathbf{1}_n^\dagger \otimes \mathbf{1}_n \right) v^s_c. \tag{37}
\]

Using the expressions in \( 34 \) and \( 37 \) we can derive an upper bound on the difference \( \|x_c^t - \hat{x}_c^t\| \) as

\[
\|x_c^t - \hat{x}_c^t\| = \frac{1}{T} \left\| \sum_{s=0}^{t-1} \left( \left( \mathbf{W}^{t-s} - \frac{1}{n} \mathbf{1}_n^\dagger \right) \otimes \mathbf{1}_n \right) v^s_c \right\| \\
\leq \frac{1}{T} \sum_{s=0}^{t-1} \left\| \mathbf{W}^{t-s} - \frac{1}{n} \mathbf{1}_n^\dagger \right\| \|v^s_c\| \\
\leq \frac{\sqrt{nD}}{T} \sum_{s=0}^{t-1} \left\| \mathbf{W}^{t-s} - \frac{1}{n} \mathbf{1}_n^\dagger \right\|, \tag{38}
\]

where the first inequality follows from the Cauchy-Schwarz inequality and the fact that the norm of a matrix does not change if we kronecker it by the identity matrix, the second inequality holds since \( \|v^t_c\| \leq D \) and therefore \( \|v^t_c\| \leq \sqrt{nD} \). Note that the eigenvectors of the matrices \( \mathbf{W} \) and \( \mathbf{W}^{t-s} \) are the same for all \( s = 0, \ldots, t-1 \). Therefore, the largest eigenvalue of \( \mathbf{W}^{t-s} \) is 1 with eigenvector \( \mathbf{1}_n \) and its second largest magnitude of the eigenvalues is \( \beta^{t-1-s} \), where \( \beta \) is the second largest magnitude of the eigenvalues of \( \mathbf{W} \). Also, note that since \( \mathbf{W}^{t-s} \) has \( \mathbf{1}_n \) as one of its eigenvectors, then all the other eigenvectors of \( \mathbf{W} \) are orthogonal to \( \mathbf{1}_n \). Hence, we can bound the norm \( \|\mathbf{W}^{t-s} - (\mathbf{1}_n\mathbf{1}_n^\dagger)/(n)\| \) by \( \beta^{t-1-s} \). Applying this substitution into the right hand side of \( 38 \) yields

\[
\|x_c^t - \hat{x}_c^t\| \leq \frac{\sqrt{nD}}{T} \sum_{s=0}^{t-1} \beta^{t-1-s} \leq \frac{\sqrt{nD}}{T(1 - \beta)}. \tag{39}
\]

Since \( \|x_c^t - \hat{x}_c^t\|^2 = \sum_{i=1}^{n} \|x^t_i - \hat{x}_c^t\|^2 \), the claim in \( 19 \) follows.

### 9.4 Proof of Lemma \[5\]

Recall the definition of the vector \( x_c^t = [x_1; \ldots; x_n] \in \mathbb{R}^{np} \) as the concatenation of the local variables, and define \( d_c^t = [d_1; \ldots; d_n] \in \mathbb{R}^{np} \) as the concatenation of the local approximate gradients. Further, consider the function \( F_{con} : \mathcal{X}^n \to \mathbb{R} \) which is defined as \( F_{con}(x_c^t) = F_{con}(x_1, \ldots, x_n) := \sum_{i=1}^{n} F_i(x_i) \). According to these definitions and the update in \( 6 \), we can show that

\[
d_c^t = (1 - \alpha)(\mathbf{W} \otimes \mathbf{I})d_c^{t-1} + \alpha \nabla F_{con}(x_c^t), \tag{40}
\]

where \( \mathbf{W} \otimes \mathbf{I} \in \mathbb{R}^{np \times np} \) is the kronecker product of the matrices \( \mathbf{W} \in \mathbb{R}^{n \times n} \) and \( \mathbf{I} \in \mathbb{R}^{p \times p} \). Considering the initialization \( d_c^0 = \mathbf{0}_p \), applying the update in \( 40 \) recursively from step 1 to \( t \) leads to

\[
d_c^t = \alpha \sum_{s=1}^{t} ((1 - \alpha)\mathbf{W}^{t-s} \otimes \mathbf{I}) \nabla F_{con}(x_c^s), \tag{41}
\]

If we multiply both sides of \( 41 \) from left by the matrix \( \left( \frac{1}{n} \mathbf{1}_n^\dagger \otimes \mathbf{1}_n \right) \in \mathbb{R}^{np \times np} \) and use the properties of the weight matrix \( \mathbf{W} \), i.e., \( \mathbf{1}_n \mathbf{W}^{t-s} = \mathbf{1}_n^\dagger \), we obtain that

\[
d_c^t = \alpha \sum_{s=1}^{t} (1 - \alpha)^{t-s} \left( \frac{1}{n} \mathbf{1}_n^\dagger \otimes \mathbf{1}_n \right) \nabla F_{con}(x_c^s), \tag{42}
\]
Decentralized Submodular Maximization

where \( \bar{d}_t = [\bar{d}^t; \ldots; \bar{d}^t] \) is the concatenation of \( n \) copies of the average vector \( \bar{d}^t \). Hence, the difference \( \|d_t^{con} - \bar{d}_t^{con}\| \) can be upper bounded by

\[
\|d_t^{con} - \bar{d}_t^{con}\| = \alpha \left\| \sum_{s=1}^{t} (1 - \alpha)^{t-s} (W^{t-s} \otimes I) \nabla F_{con}(x_t^{con}) - \sum_{s=1}^{t} (1 - \alpha)^{t-s} \left[ \frac{1_n}{n} \otimes I \right] \nabla F_{con}(x_t^{con}) \right\|
\]

\[
= \alpha \left\| \sum_{s=1}^{t} (1 - \alpha)^{t-s} \left[ (W^{t-s} - \frac{1_n}{n}) \otimes I \right] \nabla F_{con}(x_t^{con}) \right\|
\]

\[
\leq \alpha \sqrt{nG} \sum_{s=1}^{t} (1 - \alpha)^{t-s} \beta^{t-s}
\]

\[
\leq \frac{\alpha \sqrt{nG}}{1 - \beta(1 - \alpha)},
\]  

(43)

where the first equality is implied by replacing \( d_t^{con} \) and \( \bar{d}_t^{con} \) with the expressions in (41) and (42), respectively, the second equality is achieved by regrouping the terms, the first inequality holds since \( \|\nabla F_i(x^t_i)\| \leq G \) and \( \|W^{t-s} - (1_n 1_n^T)/n\| \leq \beta^{t-s-1} \), and finally the last inequality is valid since \( \sum_{s=1}^{t} (1 - \alpha) \beta^{t-s} \leq \frac{1}{1 - (\beta(1 - \alpha))} \). Now considering the result in (43) and the expression \( \|d_t^{con} - \bar{d}_t^{con}\|^2 = \sum_{i=1}^{n} \|d_i^t - \bar{d}^t\|^2 \), the claim in (21) follows.

9.5 Proof of Lemma 6

Considering the update in (6), we can write the sum of local ascent directions \( d_i^t \) at step \( t \) as

\[
\sum_{i=1}^{n} d_i^t = (1 - \alpha) \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} d_{j}^{t-1} + \alpha \sum_{i=1}^{n} \nabla F_i(x^t_i)
\]

\[
= (1 - \alpha) \sum_{j=1}^{n} d_{j}^{t-1} \sum_{i=1}^{n} w_{ij} + \alpha \sum_{i=1}^{n} \nabla F_i(x^t_i)
\]

\[
= (1 - \alpha) \sum_{j=1}^{n} d_{j}^{t-1} + \alpha \sum_{i=1}^{n} \nabla F_i(x^t_i),
\]

(44)
where the last equality holds since \( \sum_{i=1}^{n} w_{ij} = 1 \) which is the consequence of \( \mathbf{W}^\dagger \mathbf{1}_n = \mathbf{1}_n \). Now, we use the expression in (44) to bound the difference \( \| \sum_{i=1}^{n} \mathbf{d}_i^t - \sum_{i=1}^{n} \nabla F_i(\bar{x}_i^t) \| \). Hence,

\[
\begin{align*}
&\left\| \sum_{i=1}^{n} \mathbf{d}_i^t - \sum_{i=1}^{n} \nabla F_i(\bar{x}_i^t) \right\| \\
= &\left\| (1 - \alpha) \sum_{j=1}^{n} \mathbf{d}_j^{t-1} + \alpha \sum_{i=1}^{n} \nabla F_i(\bar{x}_i^t) - \sum_{i=1}^{n} \nabla F_i(\bar{x}_i^t) \right\| \\
= &\left\| (1 - \alpha) \sum_{j=1}^{n} \mathbf{d}_j^{t-1} - (1 - \alpha) \sum_{i=1}^{n} \nabla F_i(\bar{x}_i^{t-1}) + (1 - \alpha) \sum_{i=1}^{n} \nabla F_i(\bar{x}_i^{t-1}) + \alpha \sum_{i=1}^{n} \nabla F_i(\bar{x}_i^t) - \sum_{i=1}^{n} \nabla F_i(\bar{x}_i^t) \right\| \\
= &\left\| (1 - \alpha) \left[ \sum_{j=1}^{n} \mathbf{d}_j^{t-1} - \sum_{i=1}^{n} \nabla F_i(\bar{x}_i^{t-1}) \right] + (1 - \alpha) \left[ \sum_{i=1}^{n} \nabla F_i(\bar{x}_i^{t-1}) - \sum_{i=1}^{n} \nabla F_i(\bar{x}_i^t) \right] \\
&\quad + \alpha \left[ \sum_{i=1}^{n} \nabla F_i(\bar{x}_i^t) - \nabla F_i(\bar{x}_i^t) \right] \right\| \\
\leq &\left\| (1 - \alpha) \left[ \sum_{j=1}^{n} \mathbf{d}_j^{t-1} - \sum_{i=1}^{n} \nabla F_i(\bar{x}_i^{t-1}) \right] + (1 - \alpha) \left[ \sum_{i=1}^{n} \nabla F_i(\bar{x}_i^{t-1}) - \sum_{i=1}^{n} \nabla F_i(\bar{x}_i^t) \right] \\
&\quad + \alpha \left[ \sum_{i=1}^{n} \nabla F_i(\bar{x}_i^t) - \nabla F_i(\bar{x}_i^t) \right] \right\| .
\end{align*}
\]

The first equality is the outcome of replacing \( \sum_{i=1}^{n} \mathbf{d}_i^t \) by the expression in (44), the second equality is obtained by adding and subtracting \( (1 - \alpha) \sum_{i=1}^{n} \nabla F_i(\bar{x}_i^{t-1}) \), in the third equality we regroup the terms, and the inequality follows from applying the triangle inequality twice. Applying the Cauchy–Schwarz inequality to the second and third summands in (45) and using the Lipschitz continuity of the gradients lead to

\[
\begin{align*}
&\left\| \sum_{i=1}^{n} \mathbf{d}_i^t - \sum_{i=1}^{n} \nabla F_i(\bar{x}_i^t) \right\| \\
\leq &\left\| (1 - \alpha) \left[ \sum_{j=1}^{n} \mathbf{d}_j^{t-1} - \sum_{i=1}^{n} \nabla F_i(\bar{x}_i^{t-1}) \right] + (1 - \alpha)L \sum_{i=1}^{n} \| \bar{x}_i^{t-1} - \bar{x}_i^t \| + \alpha L \sum_{i=1}^{n} \| \bar{x}_i^t - \bar{x}_i^t \| .
\end{align*}
\]

(46)

According to the result in Lemma 2, we can bound the \( \sum_{i=1}^{n} \| \bar{x}_i^{t+1} - \bar{x}_i^t \| \) by \( nD/T \). Further, the result in Lemma 4 shows that \( \left( \sum_{i=1}^{n} \| \bar{x}_i^t - \bar{x}_i^t \|^2 \right)^{1/2} \leq \frac{\sqrt{D}}{\sqrt{T}} \). Since by the Cauchy–Schwarz inequality it holds that \( \left( \sum_{i=1}^{n} \| \bar{x}_i^t - \bar{x}_i^t \|^2 \right)^{1/2} \geq \frac{1}{\sqrt{T}} \sum_{i=1}^{n} \| \bar{x}_i^t - \bar{x}_i^t \| \), it follows that \( \sum_{i=1}^{n} \| \bar{x}_i^t - \bar{x}_i^t \| \leq (nD)/(T(1 - \beta)) \). Applying these substitutions into (46) yields

\[
\begin{align*}
&\left\| \sum_{i=1}^{n} \mathbf{d}_i^t - \sum_{i=1}^{n} \nabla F_i(\bar{x}_i^t) \right\| \leq (1 - \alpha) \left\| \sum_{j=1}^{n} \mathbf{d}_j^{t-1} - \sum_{i=1}^{n} \nabla F_i(\bar{x}_i^{t-1}) \right\| + \frac{(1 - \alpha)LnD}{T} + \frac{\alpha LnD}{T(1 - \beta)} .
\end{align*}
\]

(47)
By multiplying both sides of (47) by $1/n$ and applying the resulted inequality recessively for $t$ steps we obtain

$$\left\| \frac{1}{n} \sum_{i=1}^{n} d_i^t - \frac{1}{n} \sum_{i=1}^{n} \nabla F_i(x^t) \right\| \leq (1 - \alpha)^t \left( \frac{1}{n} \sum_{j=1}^{n} d_j^0 - \frac{1}{n} \sum_{i=1}^{n} \nabla F_i(x^0) \right) + \alpha \frac{LD}{T(1-\beta)} \sum_{s=0}^{t-1} (1-\alpha)^s \sum_{i=1}^{n} \nabla F_i(x^s) + \frac{1}{n} \sum_{i=1}^{n} \nabla F_i(x^0) \right\| + \frac{(1-\alpha)LD}{\alpha T} + \frac{LD}{T(1-\beta)}$$

where the second inequality holds since $\sum_{j=1}^{n} d_j^0 = 0_p$ and $\sum_{s=0}^{t-1} (1-\alpha)^s \leq 1/\alpha$, and the last inequality follows from Assumption 4.

9.6 Proof of Theorem 7

Recall the definition of $\bar{x}^t = \frac{1}{n} \sum_{i=1}^{n} x_i^t$ as the average of local variables at step $t$. Since the gradients of the global objective function are $L$-Lipschitz we can write

$$\frac{1}{n} \sum_{i=1}^{n} F_i(\bar{x}^{t+1}) - \frac{1}{n} \sum_{i=1}^{n} F_i(\bar{x}^t) \geq \frac{1}{n} \sum_{i=1}^{n} \nabla F_i(\bar{x}^t), \bar{x}^{t+1} - \bar{x}^t - \frac{L}{2} \| \bar{x}^{t+1} - \bar{x}^t \|^2 \geq \frac{1}{n} \sum_{i=1}^{n} \nabla F_i(\bar{x}^t), \bar{x}^{t+1} - \bar{x}^t - \frac{L}{2T^2} \left( \frac{1}{n} \sum_{i=1}^{n} x_i^t \right)$$

where the equality holds due to the expression in [32]. Note that the term $\| (1/n) \sum_{i=1}^{n} v_i^t \|^2$ can be upper bounded by $D^2$ according to Assumption 2, since $(1/n) \sum_{i=1}^{n} v_i^t \in C$. Apply this substitution into (49) and add and subtract $(1/nT) \sum_{i=1}^{n} d_i^t$ to obtain

$$\frac{1}{n} \sum_{i=1}^{n} F_i(\bar{x}^{t+1}) - \frac{1}{n} \sum_{i=1}^{n} F_i(\bar{x}^t) \geq \frac{1}{T} \left( \frac{1}{n} \sum_{i=1}^{n} d_i^t, \frac{1}{n} \sum_{i=1}^{n} v_i^t \right) + \frac{1}{T} \left( \frac{1}{n} \sum_{i=1}^{n} \nabla F_i(\bar{x}^t), \frac{1}{n} \sum_{i=1}^{n} d_i^t, \frac{1}{n} \sum_{i=1}^{n} v_i^t \right) - \frac{LD^2}{2T^2}.$$ 

Now by rewriting the inner product $\langle \sum_{i=1}^{n} d_i^t, \sum_{i=1}^{n} v_i^t \rangle$ as $\sum_{i=1}^{n} \langle \sum_{j=1}^{n} d_j^t, v_j^t \rangle = \sum_{j=1}^{n} \langle \sum_{i=1}^{n} d_i^t, v_j^t \rangle$, we can rewrite the right hand side of (50) as

$$\frac{1}{n} \sum_{i=1}^{n} F_i(\bar{x}^{t+1}) - \frac{1}{n} \sum_{i=1}^{n} F_i(\bar{x}^t) \geq \frac{1}{nT} \sum_{j=1}^{n} \langle \sum_{i=1}^{n} d_i^t, v_j^t \rangle + \frac{1}{nT} \left( \frac{1}{n} \sum_{i=1}^{n} \nabla F_i(\bar{x}^t), \frac{1}{n} \sum_{i=1}^{n} d_i^t, \frac{1}{n} \sum_{i=1}^{n} v_i^t \right) - \frac{LD^2}{2T^2}$$

$$= \frac{1}{nT} \sum_{j=1}^{n} \langle d_j^t, v_j^t \rangle + \frac{1}{nT} \sum_{j=1}^{n} \langle \left( \frac{1}{n} \sum_{i=1}^{n} d_i^t - d_j^t \right), v_j^t \rangle + \frac{1}{T} \sum_{i=1}^{n} \left\| \nabla F_i(\bar{x}^t) - \frac{1}{n} \sum_{i=1}^{n} d_i^t, \frac{1}{n} \sum_{i=1}^{n} v_i^t \right\| - \frac{LD^2}{2T^2}.$$ 

(51)
Note that in the last step we added and subtracted \((1/nT) \sum_{j=1}^{n} (d_j^t, x^*)\). Now according to the update in \([7]\) we can write, \(\langle d_j^t, v_j^t \rangle = \max_{v \in \mathcal{C}} \langle d_j^t, v \rangle \geq \langle d_j^t, x^* \rangle\). Hence, we can replace \(\langle d_j^t, v_j^t \rangle\) by its lower bound \(\langle d_j^t, x^* \rangle\) to obtain

\[
\frac{1}{n} \sum_{i=1}^{n} F_i(\tilde{x}^{t+1}) - \frac{1}{n} \sum_{i=1}^{n} F_i(\tilde{x}^t) \geq \frac{1}{nT} \sum_{j=1}^{n} \langle (\frac{1}{n} \sum_{i=1}^{n} d_i^t - d_j^t), v_j^t \rangle + \frac{1}{T} \left( \frac{1}{n} \sum_{i=1}^{n} \nabla F_i(\tilde{x}^t) - \frac{1}{n} \sum_{i=1}^{n} d_i^t, \frac{1}{n} \sum_{i=1}^{n} v_i^t \right) - \frac{LD^2}{2T^2}.
\]

Adding and subtracting \(\frac{1}{nT} \sum_{j=1}^{n} (\sum_{i=1}^{n} d_i^t, x^*)\) and regrouping the terms lead to

\[
\frac{1}{n} \sum_{i=1}^{n} F_i(\tilde{x}^{t+1}) - \frac{1}{n} \sum_{i=1}^{n} F_i(\tilde{x}^t) \geq \frac{1}{nT} \sum_{j=1}^{n} \langle (\frac{1}{n} \sum_{i=1}^{n} d_i^t - \frac{1}{n} \sum_{i=1}^{n} d_i^t, x^*) \rangle + \frac{1}{nT} \sum_{j=1}^{n} \langle (\frac{1}{n} \sum_{i=1}^{n} d_i^t - \frac{1}{n} \sum_{i=1}^{n} d_i^t, x^*) \rangle + \frac{1}{T} \left( \frac{1}{n} \sum_{i=1}^{n} \nabla F_i(\tilde{x}^t) - \frac{1}{n} \sum_{i=1}^{n} d_i^t, \frac{1}{n} \sum_{i=1}^{n} v_i^t \right) - \frac{LD^2}{2T^2}.
\]

Further add and subtract the expression \(\frac{1}{nT} \sum_{j=1}^{n} (\sum_{i=1}^{n} \nabla F_i(\tilde{x}^t), x^*)\) and combine the terms to obtain

\[
\frac{1}{n} \sum_{i=1}^{n} F_i(\tilde{x}^{t+1}) - \frac{1}{n} \sum_{i=1}^{n} F_i(\tilde{x}^t) \geq \frac{1}{nT} \sum_{j=1}^{n} \langle (\frac{1}{n} \sum_{i=1}^{n} d_i^t - \frac{1}{n} \sum_{i=1}^{n} d_i^t), x^* \rangle + \frac{1}{nT} \sum_{j=1}^{n} \langle (\frac{1}{n} \sum_{i=1}^{n} d_i^t - \frac{1}{n} \sum_{i=1}^{n} d_i^t), x^* \rangle + \frac{1}{T} \left( \frac{1}{n} \sum_{i=1}^{n} \nabla F_i(\tilde{x}^t) - \frac{1}{n} \sum_{i=1}^{n} d_i^t, \frac{1}{n} \sum_{i=1}^{n} v_i^t \right) - \frac{LD^2}{2T^2}.
\]

The monotonicity of the average function \((1/n) \sum_{i=1}^{n} F_i(x)\) and its concavity along positive directions imply that \(\langle (1/n) \sum_{i=1}^{n} \nabla F_i(\tilde{x}^t), x^* \rangle \geq (1/n) \sum_{i=1}^{n} F_i(x^*) - (1/n) \sum_{i=1}^{n} F_i(\tilde{x}^t)\). By applying this substitution into \((54)\) and using the Cauchy-Schwarz inequality we obtain

\[
\frac{1}{n} \sum_{i=1}^{n} F_i(\tilde{x}^{t+1}) - \frac{1}{n} \sum_{i=1}^{n} F_i(\tilde{x}^t) \geq \frac{1}{nT} \left[ \sum_{i=1}^{n} F_i(x^*) - \sum_{i=1}^{n} F_i(\tilde{x}^t) \right] - \frac{1}{nT} \sum_{j=1}^{n} \left\| \frac{1}{n} \sum_{i=1}^{n} d_i^t - d_j^t \right\| \left\| v_j^t - x^* \right\| - \frac{1}{nT} \left\| \frac{1}{n} \sum_{i=1}^{n} d_i^t - \sum_{i=1}^{n} \nabla F_i(\tilde{x}^t) \right\| \left\| x^* - \frac{1}{n} \sum_{i=1}^{n} v_i^t \right\| - \frac{LD^2}{2T^2}.
\]

Now we proceed to derive lower bounds for the negative terms on the right hand side of \((55)\). Note that all \(v_i^t\) for \(i = 1, \ldots, n\) belong to the convex set \(\mathcal{C}\) and therefore the average vector \(\frac{1}{n} \sum_{i=1}^{n} v_i^t\) is
The result in Lemma 5 implies that \( \| \mathbf{x}^* - \frac{1}{n} \sum_{i=1}^{n} \mathbf{v}_i \| \) by \( D \) according to Assumption 2. Indeed, the norm \( \| \mathbf{v}_j - \mathbf{x}^* \| \) is also upper bounded by \( D \) and hence we can write

\[
\frac{1}{n} \sum_{i=1}^{n} F_i(\bar{x}^{t+1}) - \frac{1}{n} \sum_{i=1}^{n} F_i(\bar{x}^t) \\
\geq \frac{1}{nT} \left[ \sum_{i=1}^{n} F_i(\mathbf{x}^*) - \sum_{i=1}^{n} F_i(\bar{x}^t) \right] - \frac{D}{nT} \left\| \sum_{i=1}^{n} d_i - \sum_{i=1}^{n} \nabla F_i(\bar{x}^t) \right\| - \frac{D}{nT} \sum_{j=1}^{n} \left\| \frac{1}{n} \sum_{i=1}^{n} d_i - d_j \right\| - \frac{LD^2}{2T^2}.
\]

The result in Lemma 5 implies that \( \sum_{i=1}^{n} \| d_i - \bar{d}^t \|^2 \) \( \leq \frac{\alpha G}{1 - \beta(1 - \alpha)} \). Note that based on the Cauchy–Schwarz inequality it holds that \( \sum_{i=1}^{n} \| d_i - \bar{d}^t \|^2 \geq \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \| d_i - \bar{d}^t \| \), and hence, \( \sum_{i=1}^{n} \| d_i - \bar{d}^t \| \leq \frac{\alpha G}{1 - \beta(1 - \alpha)} \). Using this result and recalling the definition \( \bar{d}^t := \frac{1}{n} \sum_{i=1}^{n} d_i^t \), we obtain that

\[
\frac{1}{n} \sum_{j=1}^{n} \left\| \frac{1}{n} \sum_{i=1}^{n} d_i - d_j \right\| \leq \frac{\alpha G}{1 - \beta(1 - \alpha)}.
\]

Replace the term \( \frac{1}{n} \sum_{j=1}^{n} \left\| \frac{1}{n} \sum_{i=1}^{n} d_i - d_j \right\| \) in (56) by its upper bound in (57) and use the result in Lemma 6 to replace \( \frac{1}{T} \sum_{i=1}^{n} \| d_i - \nabla F_i(\bar{x}^t) \|^2 \) by its upper bound in (22). Applying these substitutions yields

\[
\frac{1}{n} \sum_{i=1}^{n} F_i(\bar{x}^{t+1}) - \frac{1}{n} \sum_{i=1}^{n} F_i(\bar{x}^t) \geq \frac{1}{T} \left[ \frac{1}{n} \sum_{i=1}^{n} F_i(\mathbf{x}^*) - \frac{1}{n} \sum_{i=1}^{n} F_i(\bar{x}^t) \right] - \frac{(1 - \alpha)G D}{T} \frac{LD^2}{(1 - \beta)T^2} - \frac{(1 - \alpha)LD^2}{\alpha T^2}.
\]

Set \( \alpha = 1/\sqrt{T} \) and regroup the terms to obtain

\[
\frac{1}{n} \sum_{i=1}^{n} F_i(\mathbf{x}^*) - \frac{1}{n} \sum_{i=1}^{n} F_i(\bar{x}^{t+1}) \leq \left( 1 - \frac{1}{T} \right) \left[ \frac{1}{n} \sum_{i=1}^{n} F_i(\mathbf{x}^*) - \frac{1}{n} \sum_{i=1}^{n} F_i(\bar{x}^t) \right] + \frac{(1 - (1/\sqrt{T}))^t G D}{T} + \frac{LD^2}{T^{3/2}} + \frac{LD^2}{(1 - \beta)T^2} + \frac{GD}{(1 - \beta)T^{3/2}} + \frac{LD^2}{2T^2}.
\]

By applying the inequality in (59) recursively for \( t = 0, \ldots, T - 1 \) we obtain

\[
\frac{1}{n} \sum_{i=1}^{n} F_i(\mathbf{x}^*) - \frac{1}{n} \sum_{i=1}^{n} F_i(\bar{x}^t) \leq \left( 1 - \frac{1}{T} \right)^T \left[ \frac{1}{n} \sum_{i=1}^{n} F_i(\mathbf{x}^*) - \frac{1}{n} \sum_{i=1}^{n} F_i(\bar{x}^0) \right] + \sum_{t=0}^{T-1} \left( 1 - \frac{1}{\sqrt{T}} \right)^t G D + \sum_{t=0}^{T-1} \frac{LD^2}{T^{3/2}} + \sum_{t=0}^{T-1} \frac{LD^2}{(1 - \beta)T^2} + \sum_{t=0}^{T-1} \frac{GD}{(1 - \beta)T^{3/2}} + \sum_{t=0}^{T-1} \frac{LD^2}{2T^2}.
\]
By using the inequality $\sum_{t=0}^{T-1}(1 - 1/\sqrt{T})^t \leq \sqrt{T}$ and simplifying the terms on the right hand side, we obtain that to the expression

$$
\frac{1}{n} \sum_{i=1}^{n} F_i(x^*) - \frac{1}{n} \sum_{i=1}^{n} F_i(x^T) - \frac{1}{e} \left[ \frac{1}{n} \sum_{i=1}^{n} F_i(x^*) - \frac{1}{n} \sum_{i=1}^{n} F_i(x^0) \right] + \frac{GD}{T^{1/2}} + \frac{LD^2}{T^{1/2}} + \frac{LD^2}{(1-\beta)T} + \frac{GD}{(1-\beta)T^{1/2}} + \frac{LD^2}{2T}
$$

where to derive the first inequality we used $(1 - 1/T)^T \leq 1/e$. Note that we set $x_i^0 = 0_p$ for all $i \in \mathcal{N}$ and therefore $x^0 = 0_p$. Since we assume that $F_i(0_p) \geq 0$ for all $i \in \mathcal{N}$, it implies that $\frac{1}{n} \sum_{i=1}^{n} F_i(0_p) = \frac{1}{n} \sum_{i=1}^{n} F_i(0_p) \geq 0$ and the expression in (61) can be simplified to

$$
\frac{1}{n} \sum_{i=1}^{n} F_i(x^T) \geq (1 - e^{-1}) \frac{1}{n} \sum_{i=1}^{n} F_i(x^*) - \frac{LD^2 + GD(1 + (1 - \beta)^{-1})}{T^{1/2}} - \frac{LD^2(0.5 + (1 - \beta)^{-1})}{T}.
$$

(62)

Also, since the norm of local gradients is uniformly bounded by $G$, the local functions $F_i$ are $G$-Lipschitz. This observation implies that

$$
\left| \frac{1}{n} \sum_{i=1}^{n} F_i(x^T) - \frac{1}{n} \sum_{i=1}^{n} F_i(x^T) \right| \leq \frac{G}{n} \sum_{i=1}^{n} \|x_i^T - x_i^0\| \leq \frac{GD}{T(1 - \beta)},
$$

where the second inequality holds by using the result in Lemma 3 and the Cauchy-Schwartz inequality. Therefore, by combining the results in (62) and (63) we obtain that for all $j = \mathcal{N}$

$$
\frac{1}{n} \sum_{i=1}^{n} F_i(x_j^T) \geq (1 - e^{-1}) \frac{1}{n} \sum_{i=1}^{n} F_i(x^*) - \frac{LD^2 + GD(1 + (1 - \beta)^{-1})}{T^{1/2}} - \frac{GD(1 - \beta)^{-1} + LD^2(0.5 + (1 - \beta)^{-1})}{T},
$$

(64)

and the claim in (23) follows.

### 9.7 How to Construct an Unbiased Estimator of the Gradient in Multilinear Extensions

In this section, we provide an unbiased estimator for the gradient of a multilinear extension. We thus consider an arbitrary submodular set function $h : 2^V \rightarrow \mathbb{R}$ with multilinear $H$. Our goal is to provide an unbiased estimator for $\nabla H(x)$. We have $H(x) = \sum_{S \subseteq V} \prod_{i \in S} x_i \prod_{j \in S} (1 - x_j) h(S)$. Now, it can easily be shown that

$$
\frac{\partial H}{\partial x_i} = H(x; x_i \leftarrow 1) - H(x; x_i \leftarrow 0).
$$

where for example by $\langle x; x_i \leftarrow 1 \rangle$ we mean a vector which has value 1 on its $i$-th coordinate and is equal to $x$ elsewhere. To create an unbiased estimator for $\frac{\partial H}{\partial x_i}$ at a point $x$ we can simply sample a set $S$ by including each element in it independently with probability $x_i$ and use $h(S \cup \{i\}) - h(S \setminus \{i\})$ as an unbiased estimator for the $i$-th partial derivative. We can sample one single set $S$ and use the above trick for all the coordinates. This involves $n$ function computations for $h$. Having a mini-batch size $B$ we can repeat this procedure $B$ times and then average.
Note that since every element of the unbiased estimator is of the form \( h(S \cup \{i\}) - h(S \setminus \{i\}) \) for some chosen set \( S \), then due to submodularity of the function \( h \) every element of the unbiased estimator is bounded above by the maximum marginal value of \( h \) (i.e., \( \max_{i \in V} h(\{i\}) \)). As a result, the norm of the unbiased estimator (of the gradient of \( H \)) is bounded above by \( \sqrt{|V|} \max_{i \in V} h(\{i\}) \).

9.8 Proof of Theorem 8

The steps of the proof are similar to the one for Theorem 1. In particular, for the Discrete DCG method we can also show that the expressions in (49)-(56) hold and we can write

\[
\frac{1}{n} \sum_{i=1}^{n} F_i(x^{t+1}) - \frac{1}{n} \sum_{i=1}^{n} F_i(x^t) \geq \frac{1}{nT} \left[ \sum_{i=1}^{n} F_i(x^*) - \sum_{i=1}^{n} F_i(x^t) \right] - \frac{D}{nT} \sum_{j=1}^{n} \left\| \frac{1}{n} \sum_{i=1}^{n} d_i^t - d_j^t \right\| - \frac{LD^2}{2T^2}.
\]

Now we proceed to derive upper bounds for the norms on the right hand side of (65). To derive these bounds we use the results in Lemmata 2 and 3 which also hold for the Discrete DCG algorithm.

We first derive an upper bound for the sum \( \sum_{j=1}^{n} \left\| \frac{1}{n} \sum_{i=1}^{n} d_i^t - d_j^t \right\| \) in (65). To achieve this goal the following lemma is needed.

**Lemma 10** Consider the proposed Discrete DCG method defined in Algorithm 3. If Assumptions 4 and 5 hold, then for all \( i \in N \) and \( t \geq 0 \) the expected squared norm \( \mathbb{E} \left[ \| g_i^t \|^2 \right] \) is bounded above by

\[
\mathbb{E} \left[ \| g_i^t \|^2 \right] \leq K^2, 
\]

where \( K^2 = \sigma^2 + G^2 \).

**Proof:** Considering the condition in Assumption 5 on the variance of stochastic gradients, we can define \( K^2 := \sigma^2 + G^2 \) as an upper bound on the expected norm of stochastic gradients, i.e., for all \( x \in C \) and \( i \in N \)

\[
\mathbb{E} \left[ \| \nabla F_i(x^t) \|^2 \right] \leq K^2. 
\]

Now we use an induction argument to show that the expected norm \( \mathbb{E} \left[ \| g_i^t \|^2 \right] \leq K^2 \). Since the iterates are initialized at \( g_i^0 = 0 \), the update in (12) implies that \( \mathbb{E} \left[ \| g_i^t \|^2 | x^t \right] = \phi^2 \mathbb{E} \left[ \| \nabla F_i(x^t) \|^2 | x^t \right] \leq \phi^2 K^2 \leq K^2 \). Since \( \mathbb{E} \left[ \mathbb{E} \left[ \| g_i^t \|^2 | x^t \right] \right] = \mathbb{E} \left[ \| g_i^t \|^2 \right] \) it follows that \( \mathbb{E} \left[ \| g_i^t \|^2 \right] \leq K^2 \). Now we proceed to show that if \( \mathbb{E} \left[ \| g_i^{t-1} \|^2 \right] \leq K^2 \) then \( \mathbb{E} \left[ \| g_i^t \|^2 \right] \leq K^2 \).

Recall the update of \( g_i^t \) in (12). By computing the squared norm of both sides and using the Cauchy-Schwartz inequality we obtain that

\[
\| g_i^t \|^2 \leq (1 - \phi)^2 \| g_i^{t-1} \|^2 + \phi^2 \| \nabla F_i(x^t) \|^2 + 2\phi(1 - \phi)\| g_i^{t-1} \| \| \nabla F_i(x^t) \|.
\]

Compute the expectation with respect to the random variable corresponding to the stochastic gradient \( \nabla F_i(x^t) \) to obtain

\[
\mathbb{E} \left[ \| g_i^t \|^2 | x^t \right] \leq (1 - \phi)^2 \| g_i^{t-1} \|^2 + \phi^2 \mathbb{E} \left[ \| \nabla F_i(x^t) \|^2 | x^t \right] + 2\phi(1 - \phi)\| g_i^{t-1} \| \mathbb{E} \left[ \| \nabla F_i(x^t) \| | x^t \right].
\]

Note that according to Jensen’s inequality \( \mathbb{E} \left[ \| \nabla F_i(x^t) \|^2 \right] \leq K^2 \) implies that \( \mathbb{E} \left[ \| \nabla F_i(x^t) \| \right] \leq K \). Replacing these bounds into (69) yields

\[
\mathbb{E} \left[ \| g_i^t \|^2 | x^t \right] \leq (1 - \phi)^2 \| g_i^{t-1} \|^2 + \phi^2 K^2 + 2K\phi(1 - \phi)\| g_i^{t-1} \|.
\]
Now by computing the expectation of both sides with respect to all sources of randomness from \( t = 0 \) and using the simplification \( \mathbb{E} \left[ \mathbb{E} \left[ \| g_i^t \|^2 \mid x_i^t \right] \right] = \mathbb{E} \left[ \| g_i^t \|^2 \right] \) we can write
\[
\mathbb{E} \left[ \| g_i^t \|^2 \right] \leq (1 - \phi)^2 \mathbb{E} \left[ \| g_i^{t-1} \|^2 \right] + \phi^2 K^2 + 2K\phi(1 - \phi)\mathbb{E} \left[ \| g_i^{t-1} \| \right] \\
\leq (1 - \phi)^2 K^2 + \phi^2 K^2 + 2K\phi(1 - \phi)K \\
= K^2, \tag{71}
\]
and the claim in (66) follows by induction.

We use the result in Lemma 10 to find an upper bound for the sum \( (1/n) \sum_{j=1}^n \| \hat{d}_i^t - d_j^t \| \) on the right hand side of (65).

**Lemma 11** Consider the proposed Discrete DCG method defined in Algorithm 2. If Assumptions 1, 2, and 3 hold, then for all \( i \in \mathcal{N} \) and \( t \geq 0 \) we have
\[
\mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^n \| d_i^t - \bar{d}_i^t \| \right] \leq \frac{\alpha K}{1 - \beta(1 - \alpha)}, \tag{72}
\]
where \( K = (\sigma^2 + G^2)^{1/2} \).

**Proof:** Define the vector \( g_{con}^t = [g_1^t; \ldots; g_n^t] \) as the concatenation of the local vectors \( g_i^t \) at time \( t \). Further, recall the definitions of the vectors \( x_{con} = [x_1; \ldots; x_n] \in \mathbb{R}^{np} \) and \( d_{con} = [d_1; \ldots; d_n] \in \mathbb{R}^{np} \) as the concatenation of the local variables and local approximate gradients, respectively, and the definition of \( \bar{d}_{con}^t = [\bar{d}_1^t; \ldots; \bar{d}_n^t] \) as the concatenation of \( n \) copies of the average vector \( \bar{d}_i^t \). By following the steps of the proof for Lemma 5 it can be shown that
\[
\| \bar{d}_{con}^t - \bar{d}_i^t \| = \left\| \alpha \sum_{s=1}^t (1 - \alpha)^{t-s} \left[ (W^{t-s} - \frac{1}{n} \mathbb{1}^n) \otimes I \right] g_{con}^t \right\| \\
\leq \alpha \sum_{s=1}^t (1 - \alpha)^{t-s} \left\| (W^{t-s} - \frac{1}{n} \mathbb{1}^n) \otimes I \right\| \| g_{con}^t \| \\
\leq \alpha \sum_{s=1}^t (1 - \alpha)^{t-s} \| g_{con}^t \|. \tag{73}
\]
By computing the expected value of both sides and using the result in (66) we obtain that
\[
\mathbb{E} \left[ \| \bar{d}_{con}^t - \bar{d}_i^t \| \right] \leq \alpha \sqrt{nK} \sum_{s=1}^t (1 - \alpha)^{t-s} \beta^{t-s} \\
\leq \alpha \sqrt{nK} \frac{1}{1 - \beta(1 - \alpha)}, \tag{74}
\]
where to derive the first inequality we use the fact that
\[
\mathbb{E} \left[ \| g_{con}^t \| \right] \leq \left( \mathbb{E} \left[ \| g_{con}^t \|^2 \right] \right)^{1/2} = \left( \mathbb{E} \left[ \sum_{i=1}^n \| g_i^t \|^2 \right] \right)^{1/2} = \left( \sum_{i=1}^n \mathbb{E} \left[ \| g_i^t \|^2 \right] \right)^{1/2} \leq \sqrt{nK}. \tag{75}
\]
By combining the result in (74) with the inequality
\[
\frac{1}{n} \sum_{i=1}^n \| \bar{d}_i^t - \bar{d}_i^t \| \leq \frac{1}{\sqrt{n}} \left[ \sum_{i=1}^n \| d_i^t - \bar{d}_i^t \| \right]^{1/2} = \frac{1}{\sqrt{n}} \| \bar{d}_{con}^t - \bar{d}_i^t \|, \tag{76}
\]
the claim in (72) follows.

The result in Lemma [11] shows $\frac{1}{n} \sum_{i=1}^{n} \|d_i^t - d_i^t\|$ is bounded above by $(\alpha K)/(1 - \beta (1 - \alpha))$ in expectation. To bound the second sum in (65), which is $\sum_{i=1}^{n} \|d_i^t - \nabla F_i(x^t)\|$, we first introduce the following lemma, which was presented in [Mokhtari et al. 2017a] in a slightly different form.

**Lemma 12** Consider the proposed Discrete DCG method defined in Algorithm 2. If Assumptions 2 and 3 hold and we set $\phi = 1/T^{2/3}$, then for all $i \in \mathcal{N}$ and $t \geq 0$ we have

$$
\mathbb{E} \left[ \sum_{i=1}^{n} \|\nabla F_i(x_i) - g_i^t\|^2 \right] \leq \left( 1 - \frac{1}{2T^{2/3}} \right)^t nG^2 + \frac{6nL^2D^2C}{T^{4/3}} + \frac{2n\sigma^2 + 12nL^2D^2C}{T^{2/3}},
$$  

(77)

where $C := 1 + (2/(1 - \beta)^2)$.

**Proof:** Use the update $g_i^t := (1 - \phi)g_i^{t-1} + \phi \nabla \tilde{F}(x_i)$ to write the squared norm $\|\nabla F_i(x_i) - g_i^t\|^2$ as

$$
\|\nabla F_i(x_i) - g_i^t\|^2 = \|\nabla F_i(x_i) - (1 - \phi)d_{i-1} - \phi \nabla \tilde{F}_i(x_i)\|^2.
$$  

(78)

Add and subtract the term $(1 - \phi)\nabla F_i(x_i)$ to the right hand side of (78) and regroup the terms to obtain

$$
\|\nabla F_i(x_i) - g_i^t\|^2 = (1 - \phi)(\nabla F_i(x_i) - \nabla F_i(x_i^{t-1})) + (1 - \phi)(\nabla F_i(x_i^{t-1}) - g_i^{t-1})^2.
$$  

(79)

Define $\mathcal{F}_t$ as a sigma algebra that measures the history of the system up until time $t$. Expanding the square and computing the conditional expectation $\mathbb{E}[\cdot | \mathcal{F}_t]$ of the resulted expression yield

$$
\mathbb{E} \left[ \|\nabla F_i(x_i) - g_i^t\|^2 | \mathcal{F}_t \right] = \phi^2 \mathbb{E} \left[ \|\nabla F_i(x_i) - \nabla \tilde{F}_i(x_i)\|^2 | \mathcal{F}_t \right] + (1 - \phi)^2 \|\nabla F_i(x_i^{t-1}) - g_i^{t-1}\|^2 + (1 - \phi)2\|\nabla F_i(x_i^{t-1}) - \nabla F_i(x_i^{t-1})\| \|\nabla F_i(x_i^{t-1}) - g_i^{t-1}\|,
$$  

(80)

where we have used the fact $\mathbb{E} \left[ \nabla \tilde{F}_i(x_i) | \mathcal{F}_t \right] = \nabla F_i(x_i)$. The term $\mathbb{E} \left[ \|\nabla F_i(x_i^{t-1}) - \nabla \tilde{F}_i(x_i^{t-1})\|^2 | \mathcal{F}_t \right]$ can be bounded above by $\sigma^2$ according to Assumption 5. Based on Assumption 3, we can also show that the squared norm $\|\nabla F_i(x_i^{t-1}) - \nabla F_i(x_i^{t-1})\|^2$ is upper bounded by $L^2\|x_i - x_i^{t-1}\|^2$. Moreover, the inner product $2\langle \nabla F_i(x_i^{t-1}) - \nabla F_i(x_i^{t-1}) - d_{i-1} \rangle$ can be upper bounded by $2\|\nabla F_i(x_i^{t-1}) - d_{i-1}\|^2 + (1/\zeta)L^2\|x_i - x_i^{t-1}\|^2$ using Young’s inequality (i.e., $2\langle a, b \rangle \leq \zeta \|a\|^2 + \|b\|^2/\beta$ for any $a, b \in \mathbb{R}^n$ and $\zeta > 0$) and the condition in Assumption 3 where $\zeta > 0$ is a free scalar. Applying these substitutions into (80) leads to

$$
\mathbb{E} \left[ \|\nabla F_i(x_i^{t-1}) - g_i^{t-1}\|^2 | \mathcal{F}_t \right] \leq \phi^2 \sigma^2 + (1 - \phi)^2(1 + \zeta^{-1})L^2\|x_i - x_i^{t-1}\|^2 + (1 - \phi)^2(1 + \zeta)\|\nabla F_i(x_i^{t-1}) - g_i^{t-1}\|^2.
$$  

(81)

By setting $\zeta = \phi/2$ we can replace $(1 - \phi)^2(1 + \zeta^{-1})$ and $(1 - \phi)^2(1 + \zeta)$ by their upper bounds $(1 + 2\phi^{-1})$ and $(1 - \phi/2)$, respectively. Applying these substitutions and summing up both sides of the resulted inequality for $i = 1, \ldots, n$ lead to

$$
\mathbb{E} \left[ \sum_{i=1}^{n} \|\nabla F_i(x_i^{t-1}) - g_i^{t-1}\|^2 | \mathcal{F}_t \right] \leq n\phi^2 \sigma^2 + L^2(1 + 2\phi^{-1})\sum_{i=1}^{n} \|x_i - x_i^{t-1}\|^2 + \left( 1 - \frac{\phi}{2} \right) \sum_{i=1}^{n} \|\nabla F_i(x_i^{t-1}) - g_i^{t-1}\|^2.
$$  

(82)
Now we proceed to derive an upper bound for the sum $\sum_{i=1}^{n} \|x_i^t - x_i^{t-1}\|^2$. Note that using the Cauchy-Schwartz inequality and the results in Lemmata 2 and 4 we can show that
\[
\sum_{i=1}^{n} \|x_i^t - x_i^{t-1}\|^2 \leq \sum_{i=1}^{n} \left( 3 \|x_i^t - \bar{x}^t\|^2 + 3 \|\dot{x}^t - x_i^{t-1}\|^2 + 3 \|x_i^{t-1} - x_i^{t-2}\|^2 \right)
\leq \frac{3nD^2}{T^2(1-\beta)^2} + \frac{3nD^2}{T^2} + \frac{3nD^2}{T^2(1-\beta)^2}
= \frac{3nD^2}{T^2} \left(1 + \frac{2}{(1-\beta)^2}\right).
\] (83)

Replace the sum $\sum_{i=1}^{n} \|x_i^t - x_i^{t-1}\|^2$ in (82) by its upper bound in (83) and compute the expectation with respect to $F_0$ to obtain
\[
\mathbb{E}\left[ \sum_{i=1}^{n} \|\nabla F_i(x_i^t) - g_i^t\|^2 \right]
\leq \left(1 - \frac{\phi}{2}\right) \mathbb{E}\left[ \sum_{i=1}^{n} \|\nabla F_i(x_i^{t-1}) - g_i^{t-1}\|^2 \right] + n\phi^2\sigma^2 + (1 + 2\phi^{-1}) \frac{3nL^2D^2}{T^2} \left(1 + \frac{2}{(1-\beta)^2}\right).
\] (84)

Set $\phi = T^{-2/3}$ to obtain
\[
\mathbb{E}\left[ \sum_{i=1}^{n} \|\nabla F_i(x_i^t) - g_i^t\|^2 \right]
\leq \left(1 - \frac{1}{2T^{2/3}}\right) \mathbb{E}\left[ \sum_{i=1}^{n} \|\nabla F_i(x_i^{t-1}) - g_i^{t-1}\|^2 \right] + \frac{n\sigma^2}{T^{4/3}} + \frac{3nL^2D^2C}{T^2} + \frac{6nL^2D^2C}{T^{4/3}},
\] (85)

where $C := \left(1 + \frac{2}{(1-\beta)^2}\right)$. Applying the expression in (85) recursively leads to
\[
\mathbb{E}\left[ \sum_{i=1}^{n} \|\nabla F_i(x_i^t) - g_i^t\|^2 \right]
\leq \left(1 - \frac{1}{2T^{2/3}}\right)^t \sum_{i=1}^{n} \|\nabla F_i(x_i^t) - d_0\|^2 + \left(\frac{n\sigma^2}{T^{4/3}} + \frac{3nL^2D^2C}{T^2} + \frac{6nL^2D^2C}{T^{4/3}}\right) \sum_{s=0}^{t-1} \left(1 - \frac{1}{2T^{2/3}}\right)^s
\leq \left(1 - \frac{1}{2T^{2/3}}\right)^t \sum_{i=1}^{n} \|\nabla F_i(x_i^t) - d_0\|^2 + \frac{2n\sigma^2}{T^{2/3}} + \frac{6nL^2D^2C}{T^{4/3}} + \frac{12nL^2D^2C}{T^{2/3}}
\leq \left(1 - \frac{1}{2T^{2/3}}\right)^t \frac{nG^2}{T^{2/3}} + \frac{2n\sigma^2}{T^{2/3}} + \frac{6nL^2D^2C}{T^{4/3}} + \frac{12nL^2D^2C}{T^{2/3}},
\] (86)

and the claim in (77) follows.

We use the result in Lemma 12 to derive an upper bound for $\|\frac{1}{n} \sum_{i=1}^{n} d_i^t - \frac{1}{n} \sum_{i=1}^{n} \nabla F_i(\bar{x}^t)\|$ in expectation.

Lemma 13 Consider the proposed Discrete DCG method defined in Algorithm 2. If Assumptions 2 and 3 hold and we set $\alpha = 1/\sqrt{T}$ and $\phi = 1/T^{2/3}$, then for all $i \in \mathcal{N}$ and $t \geq 0$ we have
\[
\mathbb{E}\left[ \left\| \frac{1}{n} \sum_{i=1}^{n} d_i^t - \frac{1}{n} \sum_{i=1}^{n} \nabla F_i(\bar{x}^t) \right\| \right] \leq G \left(1 - \frac{1}{T^{1/2}}\right)^t + G \left(1 - \frac{1}{2T^{2/3}}\right)^{t/2} + \frac{LD}{T^{1/2}} + \frac{LD}{T^{1/2}} + \frac{\sqrt{\beta}LDC^{1/2}}{T^{2/3}} + \frac{\sqrt{\beta}G}{T^{1/2}},
\] (87)
where $C := 1 + (2/(1 - \beta)^2)$.

**Proof:** The steps of this proof are similar to the ones in the proof of Lemma 6. It can be shown that

\[
\left\| \sum_{i=1}^{n} d_i^t - \sum_{i=1}^{n} \nabla F_i(\tilde{x}) \right\| \\
= \left\| (1 - \alpha) \sum_{j=1}^{n} d_j^{t-1} + \alpha \sum_{i=1}^{n} g_i - \sum_{i=1}^{n} \nabla F_i(\tilde{x}) \right\| \\
= \left\| (1 - \alpha) \sum_{j=1}^{n} d_j^{t-1} - (1 - \alpha) \sum_{i=1}^{n} \nabla F_i(\tilde{x}^{-1}) + (1 - \alpha) \sum_{i=1}^{n} \nabla F_i(\tilde{x}^{t-1}) + \alpha \sum_{i=1}^{n} g_i - \sum_{i=1}^{n} \nabla F_i(\tilde{x}) \right\| \\
\leq \left\| (1 - \alpha) \sum_{j=1}^{n} d_j^{t-1} - \sum_{i=1}^{n} \nabla F_i(\tilde{x}^{-1}) \right\| + (1 - \alpha) \left\| \sum_{i=1}^{n} \nabla F_i(\tilde{x}^{t-1}) - \sum_{i=1}^{n} \nabla F_i(\tilde{x}) \right\| + \alpha \left\| \sum_{i=1}^{n} g_i - \sum_{i=1}^{n} \nabla F_i(\tilde{x}) \right\| \\
\leq \left\| (1 - \alpha) \sum_{j=1}^{n} d_j^{t-1} - \sum_{i=1}^{n} \nabla F_i(\tilde{x}^{-1}) \right\| + (1 - \alpha) \left\| \sum_{i=1}^{n} \nabla F_i(\tilde{x}^{t-1}) - \sum_{i=1}^{n} \nabla F_i(\tilde{x}) \right\| + \alpha \left\| \sum_{i=1}^{n} g_i - \sum_{i=1}^{n} \nabla F_i(\tilde{x}) \right\| \\
+ \alpha \left\| \sum_{i=1}^{n} \nabla F_i(\tilde{x}) - \sum_{i=1}^{n} \nabla F_i(\tilde{x}) \right\|. \\
\tag{88}
\]

The first equality is the outcome of replacing $\sum_{i=1}^{n} d_i^t$ by the expression in (44), the second equality is obtained by adding and subtracting $(1 - \alpha) \sum_{i=1}^{n} \nabla F_i(\tilde{x}^{-1})$, in the third equality we regroup the terms, and the inequality follows from applying the triangle inequality twice. Applying the Cauchy–Schwarz inequality to the second and third summands in (45) and using the Lipschitz continuity of the gradients lead to

\[
\left\| \sum_{i=1}^{n} d_i^t - \sum_{i=1}^{n} \nabla F_i(\tilde{x}) \right\| \\
\leq \left(1 - \alpha\right) \left\| \sum_{j=1}^{n} d_j^{t-1} - \sum_{i=1}^{n} \nabla F_i(\tilde{x}^{-1}) \right\| + (1 - \alpha) L \sum_{i=1}^{n} \left\| \tilde{x}^{-1} - \tilde{x} \right\| \\
+ \alpha L \sum_{i=1}^{n} \left\| x_i^t - \tilde{x} \right\| + \alpha \left\| \sum_{i=1}^{n} \nabla F_i(\tilde{x}) \right\| \\
\leq \left(1 - \alpha\right) \left\| \sum_{j=1}^{n} d_j^{t-1} - \sum_{i=1}^{n} \nabla F_i(\tilde{x}^{-1}) \right\| + \frac{(1 - \alpha) L \ln D}{T} \\
+ \alpha \frac{\ln D}{T \left(1 - \beta\right)} + \alpha \sum_{i=1}^{n} \left\| g_i^t - \nabla F_i(x_i^t) \right\|, \\
\tag{89}
\]

where the last inequality follows from Lemmata 2 and 3. Using the inequality

\[
\frac{1}{\sqrt{n}} \mathbb{E} \left[ \sum_{i=1}^{n} \left\| g_i^t - \nabla F_i(x_i^t) \right\| \right] \leq \mathbb{E} \left[ \left( \sum_{i=1}^{n} \left\| g_i^t - \nabla F_i(x_i^t) \right\| \right)^2 \right]^{1/2} \leq \left( \mathbb{E} \left[ \sum_{i=1}^{n} \left\| g_i^t - \nabla F_i(x_i^t) \right\|^2 \right] \right)^{1/2},
\tag{90}
\]

The first inequality is the outcome of replacing $\sum_{i=1}^{n} d_i^t$ by the expression in (44), the second equality is obtained by applying the Cauchy–Schwarz inequality to the second and third summands in (45) and using the Lipschitz continuity of the gradients lead to

\[
\left\| \sum_{i=1}^{n} d_i^t - \sum_{i=1}^{n} \nabla F_i(\tilde{x}) \right\| \\
\leq \left(1 - \alpha\right) \left\| \sum_{j=1}^{n} d_j^{t-1} - \sum_{i=1}^{n} \nabla F_i(\tilde{x}^{-1}) \right\| + (1 - \alpha) L \sum_{i=1}^{n} \left\| \tilde{x}^{-1} - \tilde{x} \right\| \\
+ \alpha L \sum_{i=1}^{n} \left\| x_i^t - \tilde{x} \right\| + \alpha \left\| \sum_{i=1}^{n} \nabla F_i(\tilde{x}) \right\| \\
\leq \left(1 - \alpha\right) \left\| \sum_{j=1}^{n} d_j^{t-1} - \sum_{i=1}^{n} \nabla F_i(\tilde{x}^{-1}) \right\| + \frac{(1 - \alpha) L \ln D}{T} \\
+ \alpha \frac{\ln D}{T \left(1 - \beta\right)} + \alpha \sum_{i=1}^{n} \left\| g_i^t - \nabla F_i(x_i^t) \right\|, \\
\tag{89}
\]

where the last inequality follows from Lemmata 2 and 3. Using the inequality

\[
\frac{1}{\sqrt{n}} \mathbb{E} \left[ \sum_{i=1}^{n} \left\| g_i^t - \nabla F_i(x_i^t) \right\| \right] \leq \mathbb{E} \left[ \left( \sum_{i=1}^{n} \left\| g_i^t - \nabla F_i(x_i^t) \right\| \right)^2 \right]^{1/2} \leq \left( \mathbb{E} \left[ \sum_{i=1}^{n} \left\| g_i^t - \nabla F_i(x_i^t) \right\|^2 \right] \right)^{1/2},
\tag{90}
\]
and the result in Lemma 13 we obtain that

\[
\mathbb{E} \left[ \sum_{i=1}^{n} \| g_i^t - \nabla F_i(x_i^t) \| \right] \leq \sqrt{n} \left[ \left( 1 - \frac{1}{2T^{2/3}} \right)^t nG^2 + \frac{2n\sigma^2}{T^{2/3}} + \frac{6nL^2 D^2 C}{T^{3/3}} + \frac{12nL^2 D^2 C}{T^{2/3}} \right]^{1/2}
\]

\[
\leq nG \left( 1 - \frac{1}{2T^{2/3}} \right)^{t/2} \sqrt{2n\sigma} + \frac{\sqrt{6nLDC^{1/2}}}{T^{2/3}} + \frac{\sqrt{12nLDC^{1/2}}}{T^{1/3}},
\]  

(91)

where the second inequality holds since \( \sum_i a_i^2 \leq (\sum_i a_i)^2 \) for \( a_i \geq 0 \). Compute the expected value of both sides of (89) and replace \( \mathbb{E} \left[ \sum_{i=1}^{n} \| g_i^t - \nabla F_i(x_i^t) \| \right] \) by its upper bound in (91) to obtain

\[
\mathbb{E} \left[ \left\| \sum_{i=1}^{n} d_i^t - \sum_{i=1}^{n} \nabla F_i(x_i^t) \right\| \right] \leq (1 - \alpha) \mathbb{E} \left[ \left\| \sum_{j=1}^{n} d_j^{t-1} - \sum_{i=1}^{n} \nabla F_i(x_i^{t-1}) \right\| \right] + \frac{(1 - \alpha)LnD}{T} + \frac{\alpha LnD}{T(1 - \beta)}
\]

\[
+ \alpha nG \left( 1 - \frac{1}{2T^{2/3}} \right)^{t/2} + \frac{\sqrt{6nLDC^{1/2}}}{T^{2/3}} + \frac{\sqrt{12nLDC^{1/2}}}{T^{1/3}}.
\]  

(92)

By multiplying both sides of (91) by \( 1/n \) and applying the resulted inequality recessively for \( t \) steps we obtain

\[
\mathbb{E} \left[ \left\| \frac{1}{n} \sum_{i=1}^{n} d_i^t - \frac{1}{n} \sum_{i=1}^{n} \nabla F_i(x_i^t) \right\| \right]
\]

\[
\leq (1 - \alpha)^t \left\| \frac{1}{n} \sum_{j=1}^{n} d_j^{t-1} - \frac{1}{n} \sum_{i=1}^{n} \nabla F_i(x_i^{t-1}) \right\|
\]

\[
+ \left[ \frac{(1 - \alpha)LD}{T} + \frac{\alpha LD}{T(1 - \beta)} + \alpha G \left( 1 - \frac{1}{2T^{2/3}} \right)^{t/2} + \frac{\sqrt{6nLDC^{1/2}}}{T^{2/3}} + \frac{\sqrt{12nLDC^{1/2}}}{T^{1/3}} \right] \sum_{s=0}^{t-1} (1 - \alpha)^s
\]

\[
\leq (1 - \alpha)^t \frac{1}{n} \sum_{i=1}^{n} \| \nabla F_i(x_i^0) \| + \frac{(1 - \alpha)LD}{\alpha T} + \frac{LD}{T(1 - \beta)} + G \left( 1 - \frac{1}{2T^{2/3}} \right)^{t/2}
\]

\[
+ \frac{\sqrt{6LDC^{1/2}}}{T^{2/3}} + \frac{\sqrt{2\sigma}}{T^{1/3}},
\]

(93)

which yields the claim in (87).

Now we can complete the proof of Theorem 8 using the results in Lemmata 11 and 13 as well as the expression in (65). Replace the terms on the right hand side of (65) by their upper bounds in
Lemma 11 and 13 to obtain
\[
E \left[ \frac{1}{n} \sum_{i=1}^{n} F_i(\mathbf{x}^{t+1}) - \frac{1}{n} \sum_{i=1}^{n} F_i(\mathbf{x}^t) \right] \geq E \left[ \frac{1}{nT} \sum_{i=1}^{n} F_i(\mathbf{x}^t) - \frac{1}{n} \sum_{i=1}^{n} F_i(\mathbf{x}^t) \right] - \left( 1 - \frac{1}{T^{1/2}} \right)^t \frac{D \Gamma}{T} \frac{LD^2}{\alpha T^2} - \frac{LD^2}{T^2(1-\beta)} - \frac{\sqrt{\sigma} + \sqrt{\sigma^2 LD^2 C^{1/2}}}{T^{4/3}} - \frac{D(\sigma^2 + G^2)^{1/2}}{T^{3/2}(1-\beta(1-\alpha))} - \frac{LD^2}{2T^2}.
\]

Regrouping the terms implies that
\[
E \left[ \frac{1}{n} \sum_{i=1}^{n} F_i(\mathbf{x}^t) - \frac{1}{n} \sum_{i=1}^{n} F_i(\mathbf{x}^{t+1}) \right] \leq \left( 1 - \frac{1}{T} \right)^T E \left[ \frac{1}{n} \sum_{i=1}^{n} F_i(\mathbf{x}^t) - \frac{1}{n} \sum_{i=1}^{n} F_i(\mathbf{x}^0) \right] + \left( 1 - \frac{1}{T^{1/2}} \right)^t \frac{D \Gamma}{T} \frac{LD^2}{\alpha T^2} + \frac{LD^2}{T^{2}(1-\beta)} + \frac{\sqrt{\sigma} + \sqrt{\sigma^2 LD^2 C^{1/2}}}{T^{1/3}} + \frac{D(\sigma^2 + G^2)^{1/2}}{T^{3/2}(1-\beta(1-\alpha))} + \frac{LD^2}{2T^2}.
\]

Now apply the expression in (95) for \( t = 0, \ldots, T - 1 \) to obtain
\[
E \left[ \frac{1}{n} \sum_{i=1}^{n} F_i(\mathbf{x}^t) - \frac{1}{n} \sum_{i=1}^{n} F_i(\mathbf{x}^T) \right] \leq \left( 1 - \frac{1}{T} \right)^T E \left[ \frac{1}{n} \sum_{i=1}^{n} F_i(\mathbf{x}^0) - \frac{1}{n} \sum_{i=1}^{n} F_i(\mathbf{x}^0) \right] + \left( 1 - \frac{1}{T^{1/2}} \right)^t \frac{D \Gamma}{T} \frac{LD^2}{\alpha T^2} + \frac{LD^2}{T^{2}(1-\beta)} + \frac{\sqrt{\sigma} + \sqrt{\sigma^2 LD^2 C^{1/2}}}{T^{1/3}} + \frac{D(\sigma^2 + G^2)^{1/2}}{T^{3/2}(1-\beta(1-\alpha))} + \frac{LD^2}{2T^2} + \frac{4DG}{T^{1/3}} + \frac{\sqrt{\sigma} + \sqrt{\sigma^2 LD^2 C^{1/2}}}{T^{1/3}},
\]

where in the last inequality we use the inequalities
\[
\sum_{t=0}^{T} \left( 1 - \frac{1}{T^{1/2}} \right)^t \leq \frac{1}{1 - (1 - \frac{1}{2T^{1/2}})^2} \leq 4T^{2/3}
\]

and
\[
\sum_{t=0}^{T} \left( 1 - \frac{1}{T^{1/2}} \right)^t \leq T^{1/2}.
\]

Regrouping the terms and using the inequality \((1-1/T)^T \leq 1/e\) lead to
\[
E \left[ \frac{1}{n} \sum_{i=1}^{n} F_i(\mathbf{x}^T) \right] \geq (1 - e^{-1}) \frac{1}{n} \sum_{i=1}^{n} F_i(\mathbf{x}^t) - \frac{LD^2}{T^{1/2}} + \frac{LD^2}{T^{2/3}(1-\beta)} - \frac{\sqrt{\sigma} + \sqrt{\sigma^2 LD^2 C^{1/2}}}{T^{1/3}}.
\]
Now using the argument in (63), we can show that the result in (97) implies that for all \( j = \mathcal{N} \) it holds

\[
E \left[ \frac{1}{n} \sum_{i=1}^{n} F_i(x_j^T) \right] \geq (1 - e^{-1}) \frac{1}{n} \sum_{i=1}^{n} F_i(x^*) - \frac{LD^2}{T^{1/2}} - \frac{GD + LD^2}{T(1 - \beta)} - \frac{\sqrt{6}LD^2C^{1/2}}{T^{2/3}} - \frac{D(\sigma^2 + G^2)^{1/2}}{T^{1/2}} - \frac{LD^2}{2T} - \frac{DG}{T^{1/2}} - \frac{4DG}{T^{1/3}} - \frac{\sqrt{2}\sigma + \sqrt{12}LD^2C^{1/2}}{T^{1/3}}.
\]

Since \( C := 1 + \frac{2}{(1-\beta)^2} \), it can be shown that \( C^{1/2} = (1 + \frac{2}{(1-\beta)^2})^{1/2} \leq 1 + \frac{\sqrt{2}}{1-\beta} \). Applying this upper bound into (98) yields the claim in (26).

References

Mervat Abu-Elkheir, Mohammad Hayajneh, and Najah Abu Ali. Data management for the internet of things: Design primitives and solution. Sensors, 13(11):15582–15612, 2013.

F. Bach. Submodular functions: from discrete to continuous domains. arXiv preprint arXiv:1511.00394, 2015.

Dragana Bajovic, Dusan Jakovetic, Natasa Krejic, and Natasa Krklec Jerinkic. Newton-like method with diagonal correction for distributed optimization. SIAM Journal on Optimization, 27(2): 1171–1203, 2017.

Ron Bekkerman, Mikhail Bilenko, and John Langford. Scaling up machine learning: Parallel and distributed approaches. Cambridge University Press, 2011.

Dimitri P Bertsekas and John N Tsitsiklis. Parallel and distributed computation: numerical methods, volume 23. Prentice hall Englewood Cliffs, NJ, 1989.

Andrew An Bian, Baharan Mirzasoleiman, Joachim M. Buhmann, and Andreas Krause. Guaranteed non-convex optimization: Submodular maximization over continuous domains. In Proceedings of the 20th International Conference on Artificial Intelligence, pages 111–120, 2017.

Stephen Boyd, Persi Diaconis, and Lin Xiao. Fastest mixing markov chain on a graph. SIAM review, 46(4):667–689, 2004.

Stephen Boyd, Neal Parikh, Eric Chu, Borja Peleato, and Jonathan Eckstein. Distributed optimization and statistical learning via the alternating direction method of multipliers. Foundations and Trends® in Machine Learning, 3(1):1–122, 2011.

Niv Buchbinder, Moran Feldman, Joseph Naor, and Roy Schwartz. Submodular maximization with cardinality constraints. In Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2014, Portland, Oregon, USA, January 5-7, 2014, pages 1433–1452, 2014.

Niv Buchbinder, Moran Feldman, Joseph Naor, and Roy Schwartz. A tight linear time (1/2)-approximation for unconstrained submodular maximization. SIAM Journal on Computing, 44(5): 1384–1402, 2015.

Gruia Calinescu, Chandra Chekuri, Martin Pál, and Jan Vondrák. Maximizing a monotone submodular function subject to a matroid constraint. SIAM Journal on Computing, 40(6):1740–1766, 2011.
Nikolaos Chatzipanagiotis and Michael M Zavlanos. On the convergence rate of a distributed augmented Lagrangian optimization algorithm. In American Control Conference, ACC 2015, Chicago, IL, USA, July 1-3, 2015, pages 541–546, 2015.

Chandra Chekuri, Jan Vondrak, and Rico Zenklusen. Submodular function maximization via the multilinear relaxation and contention resolution schemes. SIAM J. Comput., 43(6):1831–1879, 2014.

Rafael da Ponte Barbosa, Alina Ene, Huy L. Nguyen, and Justin Ward. The power of randomization: Distributed submodular maximization on massive datasets. In Proceedings of the 32nd International Conference on Machine Learning, pages 1236–1244, 2015.

Nikhil R Devanur and Kamal Jain. Online matching with concave returns. In Proceedings of the 44th Symposium on Theory of Computing Conference, pages 137–144, 2012.

Paolo Di Lorenzo and Gesualdo Scutari. Next: In-network nonconvex optimization. IEEE Transactions on Signal and Information Processing over Networks, 2(2):120–136, 2016.

John C Duchi, Alekh Agarwal, and Martin J Wainwright. Dual averaging for distributed optimization: Convergence analysis and network scaling. IEEE Transactions on Automatic control, 57(3):592–606, 2012.

Reza Eghbali and Maryam Fazel. Designing smoothing functions for improved worst-case competitive ratio in online optimization. In Advances in Neural Information Processing Systems 29, pages 3279–3287, 2016.

Mark Eisen, Aryan Mokhtari, and Alejandro Ribeiro. Decentralized quasi-Newton methods. IEEE Transactions on Signal Processing, 65(10):2613–2628, 2017.

Uriel Feige. A threshold of ln n for approximating set cover. Journal of the ACM, 45(4):634–652, 1998.

Uriel Feige, Vahab S Mirrokni, and Jan Vondrak. Maximizing non-monotone submodular functions. SIAM Journal on Computing, 40(4):1133–1153, 2011.

Moran Feldman, Christopher Harshaw, and Amin Karbasi. Greed is good: Near-optimal submodular maximization via greedy optimization. arXiv preprint arXiv:1704.01652, 2017.

Daniel Golovin, Andreas Krause, and Matthew Streeter. Online submodular maximization under a matroid constraint with application to learning assignments. arXiv preprint arXiv:1407.1082, 2014.

Davood Hajinezhad, Mingyi Hong, Tuo Zhao, and Zhaoran Wang. NESTT: A nonconvex primal-dual splitting method for distributed and stochastic optimization. In Advances in Neural Information Processing Systems 29, pages 3207–3215, 2016.

S. Hamed Hassani, Mahdi Soltanolkotabi, and Amin Karbasi. Gradient methods for submodular maximization. In Advances in Neural Information Processing Systems 30, pages 5843–5853, 2017.

Dušan Jakovetić, Joao Xavier, and José MF Moura. Fast distributed gradient methods. IEEE Transactions on Automatic Control, 59(5):1131–1146, 2014.

Dusan Jakovetic, Jose MF Moura, and Joao Xavier. Linear convergence rate of a class of distributed augmented Lagrangian algorithms. IEEE Transactions on Automatic Control, 60(4):922–936, 2015.
Ravi Kumar, Benjamin Moseley, Sergei Vassilvitskii, and Andrea Vattani. Fast greedy algorithms in mapreduce and streaming. *TOPC*, 2(3):14:1–14:22, 2015.

Yan Ma, Haiping Wu, Lizhe Wang, Bormin Huang, Rajiv Ranjan, Albert Zomaya, and Wei Jie. Remote sensing big data computing: Challenges and opportunities. *Future Generation Computer Systems*, 51:47–60, 2015.

Aranyak Mehta, Amin Saberi, Umesh Vazirani, and Vijay Vazirani. Adwords and generalized online matching. *Journal of the ACM*, 54(5):22, 2007.

Song Mei, Yu Bai, and Andrea Montanari. The landscape of empirical risk for non-convex losses. *arXiv preprint arXiv:1607.06534*, 2016.

Vahab S. Mirrokni and Morteza Zadimoghaddam. Randomized composable core-sets for distributed submodular maximization. In *Proceedings of the Forty-Seventh Annual ACM on Symposium on Theory of Computing*, pages 153–205, 2015.

Baharan Mirzasoleiman, Amin Karbasi, Rik Sarkar, and Andreas Krause. Distributed submodular maximization: Identifying representative elements in massive data. In *Advances in Neural Information Processing Systems 26*, pages 2049–2057, 2013.

Baharan Mirzasoleiman, Ashwinkumar Badanidiyuru, and Amin Karbasi. Fast constrained submodular maximization: Personalized data summarization. In *Proceedings of the 33rd International Conference on Machine Learning*, pages 1358–1367, 2016.

Aryan Mokhtari and Alejandro Ribeiro. DSA: Decentralized double stochastic averaging gradient algorithm. *Journal of Machine Learning Research*, 17(61):1–35, 2016.

Aryan Mokhtari, Hamed Hassani, and Amin Karbasi. Conditional gradient method for stochastic submodular maximization: Closing the gap. *arXiv preprint arXiv:1711.01660*, 2017a.

Aryan Mokhtari, Qing Ling, and Alejandro Ribeiro. Network Newton distributed optimization methods. *IEEE Trans. on Signal Process.*, 65(1):146–161, 2017b.

Angelia Nedic and Asuman Ozdaglar. Distributed subgradient methods for multi-agent optimization. *IEEE Transactions on Automatic Control*, 54(1):48–61, 2009.

Angelia Nedic, Asuman Ozdaglar, and Pablo A Parrilo. Constrained consensus and optimization in multi-agent networks. *IEEE Transactions on Automatic Control*, 55(4):922–938, 2010.

George L Nemhauser, Laurence A Wolsey, and Marshall L Fisher. An analysis of approximations for maximizing submodular set functions–I. *Mathematical Programming*, 14(1):265–294, 1978.

Guannan Qu and Na Li. Accelerated distributed Nesterov gradient descent. *arXiv preprint arXiv:1705.07176*, 2017.

Guannan Qu, Dave Brown, and Na Li. Distributed greedy algorithm for satellite assignment problem with submodular utility function. *IFAC-PapersOnLine*, 48(22):258–263, 2015.

Michael Rabbat and Robert Nowak. Distributed optimization in sensor networks. In *Proceedings of the 3rd international symposium on Information processing in sensor networks*, pages 20–27. ACM, 2004.

Michael G Rabbat, Robert D Nowak, James Bucklew, et al. Generalized consensus computation in networked systems with erasure links. In *IEEE 6th Workshop on Signal Processing Advances in Wireless Communications*, pages 1088–1092. IEEE, 2005.
Ioannis D. Schizas, Alejandro Ribeiro, and Georgios B. Giannakis. Consensus in ad hoc WSNs with noisy links – Part I: Distributed estimation of deterministic signals. *IEEE Trans. Signal Processing*, 56(1):350–364, 2008.

Ohad Shamir, Nathan Srebro, and Tong Zhang. Communication-efficient distributed optimization using an approximate Newton-type method. In *Proceedings of the 31th International Conference on Machine Learning*, pages 1000–1008, 2014.

Tasuku Soma, Naonori Kakimura, Kazuhiro Inaba, and Ken Kawarabayashi. Optimal budget allocation: Theoretical guarantee and efficient algorithm. In *Proceedings of the 31th International Conference on Machine Learning*, pages 351–359, 2014.

Matthew Staib and Stefanie Jegelka. Robust budget allocation via continuous submodular functions. In *Proceedings of the 34th International Conference on Machine Learning*, pages 3230–3240, 2017.

Ying Sun, Gesualdo Scutari, and Daniel Palomar. Distributed nonconvex multiagent optimization over time-varying networks. In *50th Asilomar Conference on Signals, Systems and Computers*, pages 788–794, 2016.

Herbert G. Tanner and Amit Kumar. Towards decentralization of multi-robot navigation functions. In *Proceedings of the 2005 IEEE International Conference on Robotics and Automation*, pages 4132–4137, 2005.

Tatiana Tatarenko and Behrouz Touri. Non-convex distributed optimization. *IEEE Transactions on Automatic Control*, 62(8):3744–3757, 2017.

Vladimir Vapnik. *Statistical learning theory*. Wiley, 1998. ISBN 978-0-471-03003-4.

Jan Vondrák. Submodularity in combinatorial optimization. 2007.

Jan Vondrák. Optimal approximation for the submodular welfare problem in the value oracle model. In *Proceedings of the 40th Annual ACM Symposium on Theory of Computing*, pages 67–74, 2008.

Laurence A Wolsey. An analysis of the greedy algorithm for the submodular set covering problem. *Combinatorica*, 2(4):385–393, 1982.

Yuchen Yang, Longfei Wu, Guisheng Yin, Lijie Li, and Hongbin Zhao. A survey on security and privacy issues in internet-of-things. *IEEE Internet of Things Journal*, 4(5):1250–1258, 2017.

Kun Yuan, Qing Ling, and Wotao Yin. On the convergence of decentralized gradient descent. *SIAM Journal on Optimization*, 26(3):1835–1854, 2016.

Yuchen Zhang and Xiao Lin. DiSCO: Distributed optimization for self-concordant empirical loss. In *Proceedings of the 32nd International Conference on Machine Learning*, pages 362–370, 2015.