The Mass of an Asymptotically Hyperbolic Manifold with a Non-compact Boundary

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Abstract. We define a mass-type invariant for asymptotically hyperbolic manifolds with a non-compact boundary which are modelled at infinity on the hyperbolic half-space and prove a sharp positive mass inequality in the spin case under suitable dominant energy conditions. As an application, we show that any such manifold which is Einstein and either has a totally geodesic boundary or is conformally compact and has a mean convex boundary is isometric to the hyperbolic half-space.

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1. Introduction

Given a non-compact Riemannian manifold \((M, g)\) of dimension \(n \geq 3\) arising as the (time-symmetric) initial data set for a solution \((\overline{M}, \overline{g})\) of Einstein field equations in dimension \(n + 1\), standard physical reasoning suggests the existence of a geometric invariant defined in terms of the asymptotic behavior of the underlying metric at spatial infinity. Roughly speaking, it is assumed that in the asymptotic region \((M, g)\) converges to some reference space \((N, b)\), which by its turn is required to propagate to a static solution (i.e. a solution displaying a time-like vector field whose orthogonal distribution is integrable), and the mass invariant, which is denoted by \(m_{(g, b)}\) and should be interpreted as the total energy of the isolated gravitational system modelled by \((\overline{M}, \overline{g})\), is designed so as to capture the coefficient of the leading term in the asymptotic expansion of \(g\) around \(b\). In particular, the important question arises as to

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whether, under a suitable dominant energy condition, the invariant in question satisfies the positive mass inequality

\[ m_{(g,b)} \geq 0, \]

with equality taking place if and only if \((M, g) = (N, b)\) isometrically.

The classical example is the asymptotically flat case, where the reference space is \((\mathbb{R}^n, \delta)\), the Euclidean space endowed with the standard flat metric \(\delta\). Here, \(m_{(g, \delta)}\) is the so-called ADM mass and it has been conjectured that the corresponding positive mass inequality holds true whenever the scalar curvature \(R_g\) of \(g\) is non-negative. After previous contributions by Schoen–Yau [38] if \(n \leq 7\) and by Witten and Bartnik [5, 41] in the spin case, the conjecture has at last been settled in independent contributions by Schoen–Yau [36] and Lohkamp [27].

Partly motivated by the so-called AdS/CFT correspondence, which in the Euclidean semi-classical limit highlights Einstein metrics with negative scalar curvature, recently there has been much interest in studying similar invariants for non-compact Riemannian manifolds whose geometry at infinity approaches some reference space with constant negative sectional curvature [2, 21]. A notable example occurs in case the model geometry is hyperbolic space \((\mathbb{H}^n, b)\). The novelty here is that the asymptotic invariant is not a number but instead a linear functional on the space \(\mathcal{N}_b\) of static potentials \(V: \mathbb{H}^n \to \mathbb{R}\) satisfying

\[ \nabla^2 b V = V b; \]

see [12, 13, 21, 32]. However, symmetry considerations allow us to extract a mass-like invariant (i.e. a real number) out of the given functional, so it makes sense to ask whether the inequality similar to (1.1) holds, with the corresponding rigidity statement characterizing the reference space. After preliminary contributions by Min–Oo [33], Anderson–Dahl [3] and Wang [40], the conjecture has been confirmed in case the underlying manifold is spin by Chruściel–Herzlich [12]. Elementary proofs of this result in special cases are available in [15, 17] (graphical manifolds) and [7] (small perturbations of the standard hyperbolic metric). We also refer to [28] for a treatment of the non-time-symmetric case. Regarding the not necessarily spin case, we should mention the works by Andersson–Cai–Galloway [4] and Chruściel–Delay [11], where the given manifold is assumed to be asymptotically hyperbolic in the conformally compact sense.

At least in the asymptotically flat case, the positive mass inequality (1.1) has applications that transcend its physical motivation. In particular, it has been crucially used in Schoen’s solution of the Yamabe problem [25, 37] and in the investigation of multiplicity and compactness issues for solutions to this same problem [9, 18]. The need to examine these questions for compact manifolds with boundary suggested the consideration of a mass-type invariant for asymptotically flat manifold with a non-compact boundary \(\Sigma\) modelled on the Euclidean half-space \(\mathbb{R}^n_+\). In [1] a positive mass inequality has been established for this invariant under the assumptions that the scalar curvature \(R_g\) and the mean curvature \(H_g\) along the boundary are both nonnegative.
and that the double of the underlying manifold satisfies the standard (i.e. boundaryless) mass inequality. In view of the recent progress due to Schoen–Yau and Lohkamp mentioned above, the positive mass theorem in [1] actually holds in full generality.

The purpose of this article is to devise a mass-type invariant which at the same time extends those considered in [1,12]. More precisely, here we take as reference space the hyperbolic half-space \((H^n_+, b)\), which is obtained by cutting the standard hyperbolic space \((H^n, b)\) along a totally geodesic hypersurface \(\partial H^n_+\). We make use of Witten’s spinorial approach to establish, for asymptotically hyperbolic spin manifolds \((M,g)\) with a non-compact boundary \(\Sigma\) and which are modeled at infinity on \((H^n_+, b, \partial H^n_+)\), a sharp positive mass inequality under suitable lower bounds on the scalar curvature \(R_g\) and the mean curvature \(H_g\) of \(\Sigma\); see Definition 2.3 and Theorems 5.2 and 5.4. The following rigidity statements are then consequences of our main results.

**Theorem 1.1.** Let \((M,g, \Sigma)\) be an asymptotically hyperbolic spin manifold with \(R_g \geq -n(n-1)\) and \(H_g \geq 0\). Assume further that \(g\) agrees with the reference hyperbolic metric \(b\) in a neighborhood of infinity. Then \((M,g, \Sigma) = (H^n_+, b, \partial H^n_+)\) isometrically.

**Theorem 1.2.** Let \((M,g, \Sigma)\) be an asymptotically hyperbolic spin manifold. Assume further that \(g\) is Einstein and that \(\Sigma\) is totally geodesic. Then \((M,g, \Sigma) = (H^n_+, b, \partial H^n_+)\) isometrically.

The Proof of Theorem 1.2 also makes use of an alternate definition of the mass in terms of the Einstein tensor on the interior and the Newton tensor of the boundary as described in Theorem 3.7. This has recently been established in [16] by adapting an argument first put forward by Herzlich [22] in the boundaryless case. It should be pointed out that the boundaryless version of Theorem 1.2 may be obtained by a similar argument. In this case we use the boundaryless version of the alternate definition (3.11), which already appears in [22], and the positive mass theorem in [12]. This actually provides a noteworthy extension of a celebrated rigidity result by Anderson and Dahl [3] originally proved in the more restrictive setting of conformal compactness; see also [21, Theorem 4.5]. More precisely, the following result, which may be thought of as a corollary of our proof of Theorem 1.2, holds.

**Theorem 1.3.** Let \((M,g)\) be a complete, boundaryless spin manifold which is asymptotically hyperbolic (in the sense of [12]) and assume further that \(g\) is Einstein. Then \((M,g) = (H^n, b)\) isometrically.

We now discuss another rigidity result stemming from our main theorems which is more directly related to the AdS/CFT correspondence mentioned above. Compared with Theorem 1.2, it allows us to substantially relax the assumption on the geometry of the non-compact boundary at the expense of requiring a more restrictive behavior at infinity, namely conformal compactness.

Let \(\overline{M}\) be a compact \(n\)-manifold carrying a \((n-2)\)-dimensional corner which can be written as \(S \cap \Sigma\), where \(S\) and \(\Sigma\) are smooth hypersurfaces of \(M\)
such that $\partial M = S \cup \Sigma$, with $S$ being connected. Let $g$ be a Riemannian metric on $M := \text{int } M \cup \Sigma$. We say that $(M, g)$ is \textit{conformally compact} if there exists a collar neighborhood $U \subset \tilde{M}$ of $S$ such that on $\text{int } U$ we may write $g = s^{-2}\tilde{g}$ with $\tilde{g}$ extending to a sufficiently regular metric on $U$ so that $S$ and $\Sigma$ meet orthogonally (with respect to $\tilde{g}$) along their common boundary $S \cap \Sigma$, where $s : U \to \mathbb{R}$ is a \textit{defining function} for $S$ in the sense that $s \geq 0$, $s^{-1}(0) = S$, $ds|_S \neq 0$ and $\nabla_{\tilde{g}} s$ is tangent to $\Sigma$ along $U \cap \Sigma$. The restriction $\tilde{g}|_S$ defines a metric which changes by a conformal factor if the defining function is changed. Thus, the conformal class $[\tilde{g}|_S]$ of $\tilde{g}|_S$ is well defined. We then say that the pair $(S, [\tilde{g}|_S])$ is the \textit{conformal infinity} of $(M, g)$.

If $|ds|_\tilde{g} = 1$ along $S$, then $(M, g)$ is \textit{weakly} asymptotically hyperbolic in the sense that its sectional curvature converges to $-1$ as one approaches $S$. In this case, if $h_0$ is a metric on $S$ representing the given conformal infinity, then there exists a unique defining function $t$ in $U$ so that

$$g = \sinh^{-2} t \left( dt^2 + h_t \right),$$

where $h_t$ is a $t$-dependent family of metrics on $S$ with $h_t|_{t=0} = h_0$.

\textbf{Remark 1.4.} Recall that (1.3) is established by means of a conformal deformation of the type $\tilde{g} = e^{2f} \tilde{g}$. Thus, if $\tilde{s} = e^f s$, then $t$ defined by $\tilde{s} = \sinh t$ is required to be the distance function to $S$ with respect to $\tilde{g}$. The condition $|dt|_{\tilde{g}} = 1$ turns out to be a first order PDE for $f$, namely

$$\partial_s f = \frac{s}{2} \left( e^{2f} - |\nabla_{\tilde{g}} f|^2_{\tilde{g}} \right) + \frac{1 - |\nabla_{\tilde{g}} s|^2_{\tilde{g}}}{2s},$$

for which $S$ is a non-characteristic hypersurface. Thus, (1.4) can be solved in $U$ with initial data $f = 0$ on $S$, so that (1.3) is retrieved by setting $\tilde{g} = dt^2 + h_t$, where $h_t$ is the restriction of $\tilde{g}$ to the level hypersurfaces of $t$; see [3,29] for further details. If $\xi$ is a normal unit vector field with respect to $\tilde{g}$ along $U \cap \Sigma$, then using that $\partial_s / \partial \xi = (\nabla_{\tilde{g}} s, \xi)_{\tilde{g}} = 0$ we easily check from (1.4) that $p = \partial f / \partial \xi$ satisfies

$$\partial_s p = s \left( e^{2f} p - (\nabla_{\tilde{g}} f)(p) \right),$$

and since $p = 0$ for $s = 0$ we see that $\partial f / \partial \xi = 0$ along $U \cap \Sigma$. Since

$$\nabla_{\tilde{g}} \tilde{s} = e^{-f} (s \nabla_{\tilde{g}} f + \nabla_{\tilde{g}} s),$$

this means that $\nabla_{\tilde{g}} t = \cosh^{-1} t \nabla_{\tilde{g}} \tilde{s}$ remains tangent to $\Sigma$ along $U \cap \Sigma$.

\textbf{Definition 1.5.} Let $(M, g)$ be a weakly asymptotically hyperbolic manifold satisfying (1.3) as above. We say that $(M, g)$ is \textit{asymptotically hyperbolic} (in the conformally compact sense and with a non-compact boundary $\Sigma$) if its conformal infinity is $(S^n_{+1}, [h_0])$, where $h_0$ is a round metric on $S^n_{+1}$, the unit upper $(n - 1)$-hemisphere, and the following asymptotic expansion holds as $t \to 0$:

$$h_t = h_0 + \frac{t^n}{n!} h + k,$$

(1.5)
where $h$ and $k$ are symmetric 2-tensors on $S^{n-1}_+$ and the remainder term $k$ satisfies
\begin{equation}
|k| + |\nabla_{h_0} k| + |\nabla^2_{h_0} k| = o(t^{n+1}). \tag{1.6}
\end{equation}

It turns out that a manifold which is asymptotically hyperbolic in this sense is also asymptotically hyperbolic in the sense of Definition 2.3, so we may assign to it a mass-type invariant which extends Wang’s construction in the boundaryless case [40]. Thus, the rigidity statement of Theorem 5.4 yields another natural extension to our setting of the result by Andersson–Dahl [3,21] mentioned above in connection with Theorems 1.2 and 1.3.

**Theorem 1.6.** Let $(M, g, \Sigma)$ be a conformally compact, asymptotically hyperbolic spin manifold as above. Assume further that $g$ is Einstein and that the mean curvature of $\Sigma$ is everywhere nonnegative. Then $(M, g, \Sigma) = (\mathbb{H}^n_+, b, \partial \mathbb{H}^n_+)$ isometrically.

**Remark 1.7.** This rigidity result may be of some interest in connection with recent developments involving the construction of a holograph dual of a conformal field theory defined on a manifold with boundary, the so-called ADS/BCFT correspondence first put forward in [39]; see also [14,19,34] for further contributions. A mathematical treatment of results suggested in [34] appears in [30].

**Remark 1.8.** In recent years there have been considerable efforts by several authors in the direction of removing the spin assumption in the seminal rigidity result appearing in [3], the difficulty here coming from the fact that no suitable positive mass theorem has been proved in this generality; see for instance [10,26,35] and the references therein. In any case, one might ask whether analogous developments hold in the presence of a boundary as in Theorem 1.6 above. On the other hand, Theorems 1.2 and 1.3 show that conformal compactness is not really needed in order to obtain a rigidity statement as long as we remain in the spin category, where the pertinent positive mass inequality is available. This suggests a more ambitious goal, namely to investigate whether the corresponding rigidity persists for Einstein metrics (and totally geodesic boundaries) in the general asymptotically hyperbolic setup of Definition 2.3.

This paper is organized as follows. In Sect. 2 we define the relevant class of asymptotically hyperbolic manifolds with a non-compact boundary and in Sect. 3 we attach to each manifold in this class a mass functional whose geometric invariance is established. The proofs of the positive mass inequality for spin manifolds and its geometric consequences, including the rigidity statements above, are presented in Sect. 5. This uses some preparatory material regarding spinors on manifolds with boundary which is discussed in Sect. 4.

**2. Asymptotically Hyperbolic Manifolds**

Recall that the hyperboloid model for hyperbolic space in dimension $n$ is given by
\[ \mathbb{H}^n = \{ x \in \mathbb{R}^{1,n}; x_0 > 0, \langle x, x \rangle_L = -1 \}, \]
where $\mathbb{R}^{1,n}$ is the Minkowski space with the standard flat metric

$$\langle x, x \rangle_L = -x_0^2 + x_1^2 + \cdots + x_n^2.$$ 

The reference space we are interested in is $(\mathbb{H}_n^+, b, \partial \mathbb{H}_n^+)$, where $\mathbb{H}_n^+ = \{x \in \mathbb{H}^n; x_n \geq 0\}$ is the hyperbolic half-space endowed with the induced metric

$$b = \frac{dr^2}{1 + r^2} + r^2 h_0,$$

where $h_0$ stands for the canonical metric on the unit hemisphere $S_{n-1}^+$, and

$$r = \sqrt{x_1^2 + \cdots + x_n^2}.$$ 

Note that $\mathbb{H}_n^+$ carries a non-compact, totally geodesic boundary, namely $\partial \mathbb{H}_n^+ = \{x \in \mathbb{H}_n^+; x_n = 0\}$. Recall that the space of static potentials $\mathcal{N}_b$ on $\mathbb{H}^n$ is spanned by $V(0), V(1), \cdots, V(n)$, where $V(i) = x_i|_{\mathbb{H}^n}$. It is easy to check that

$$\frac{\partial V(i)}{\partial \eta} = 0, \quad i \neq n,$$ 

where $\eta$ is the outward unit normal to $\partial \mathbb{H}_n^+$. Thus, we are led to consider

$$\mathcal{N}_b^+ = \left\{ V \in \mathcal{N}_b; \frac{\partial V}{\partial \eta} = 0 \right\},$$

which is spanned by $V(0), V(1), \cdots, V(n-1)$. In particular, $V = O(r)$ as $r \to +\infty$ for any $V \in \mathcal{N}_b^+$.

**Remark 2.1.** The construction above can alternatively be carried out in the context of the so-called Poincaré model of hyperbolic geometry. Hence, we may consider the half $n$-disk $\mathbb{B}_n^+ = \{x' = (x_1, \ldots, x_n) \in \mathbb{R}^n; |x'| < 1, x_n \geq 0\}$ endowed with the metric

$$\hat{b} = \omega(x')^{-2} \delta, \quad \omega(x') = \frac{1 - |x'|^2}{2},$$

where $\delta$ is the standard Euclidean metric. The space $\mathcal{N}_b^+$ now is spanned by the functions

$$\hat{V}(0)(x') = \frac{1 + |x'|^2}{1 - |x'|^2}, \quad \hat{V}(1)(x') = \frac{2x_1}{1 - |x'|^2}, \cdots, \quad \hat{V}(n-1)(x') = \frac{2x_{n-1}}{1 - |x'|^2}.$$ 

Under the isometry between $(\mathbb{H}_n^+, b)$ and $(\mathbb{B}_n^+, \hat{b})$ given by stereographic projection “centered” at $(-1, 0, \ldots, 0)$, $\partial \mathbb{H}_n^+$ is mapped onto the unit $(n-1)$-disk $\partial \mathbb{B}_n^+$ defined by $x_n = 0$. Whenever convenience demands we will interchange freely between these models without further notice.
Remark 2.2. The linear space $\mathcal{N}^+_b$ can actually be thought of as a space of static potentials on $\mathbb{H}^n_+$ as follows. It is well known that a vacuum solution $(\bar{M}^{n+1}, \bar{g})$ of the Einstein field equations can be characterized as an extremizer of the Einstein–Hilbert functional
\[
\mathcal{E}[\bar{g}] = \int_{\bar{M}} (R_{\bar{g}} - 2\Lambda) d\bar{M},
\]
with $\Lambda \in \mathbb{R}$. In the case $\partial \bar{M} \neq \emptyset$, it is natural to consider instead the Gibbons–Hawking–York action
\[
\mathcal{E}[\bar{g}] = \int_{\bar{M}} (R_{\bar{g}} - 2\Lambda) d\bar{M} + \int_{\partial \bar{M}} (H_{\bar{g}} - \lambda) d\partial \bar{M}
\]
whose critical metrics satisfy the system
\[
\begin{cases}
\text{Ric}_{\bar{g}} - \frac{1}{2} R_{\bar{g}} \bar{g} + \Lambda \bar{g} = 0, & \text{in } \bar{M}, \\
\Pi_{\bar{g}} - H_{\bar{g}}|_{\partial \bar{M}} + \lambda \bar{g}|_{\partial \bar{M}} = 0, & \text{on } \partial \bar{M},
\end{cases}
\]
where $\Pi_{\bar{g}}$ is the boundary second fundamental form and $\lambda \in \mathbb{R}$. The first equation of (2.3) is the Einstein field equation for a vacuum spacetime with cosmological constant $\Lambda$, while the second one introduces the constant $\lambda$ which is related to the geometry of $\partial \bar{M}$. Recall that the spacetime $(\bar{M}, \bar{g})$ is said to be static if it carries a time-like Killing vector field $\xi$ whose orthogonal distribution is integrable or, equivalently, $\bar{g}$ can be written as a warped product $\bar{g} = -V^2 dt^2 + g$, where $g$ is a Riemannian metric on the spacelike slice $M^n$ and $V^2 = -\bar{g}(\xi, \xi)$. The leaves of this foliation are totally geodesic and isometric to each other. If $\partial M \neq \emptyset$, those leaves are orthogonal to $\partial \bar{M}$ which corresponds to $\xi$ being tangent to $\partial \bar{M}$. In this case, the system (2.3) is equivalent to requiring that $g$ and $V$ satisfy
\[
\begin{cases}
-VR_{ic}g + \nabla^2 g + \tilde{\Lambda} V g = 0, & \text{in } M, \\
\Pi_g - \tilde{\lambda} g|_{\partial M} = 0, & \text{on } \partial M, \\
\Delta g V = -\tilde{\Lambda} V, & \text{in } M, \\
\frac{\partial V}{\partial \eta} = \tilde{\lambda} V, & \text{on } \partial M,
\end{cases}
\]
where $\tilde{\Lambda} = \frac{2}{n-1} \Lambda$ and $\tilde{\lambda} = \frac{1}{n-1} \lambda$. Thus, by setting $\tilde{\Lambda} = -n$ and $\tilde{\lambda} = 0$, the elements $V \in \mathcal{N}^+_b$ are solutions of (2.4) when $(M, g) = (\mathbb{H}^n_+, b)$.

We now define the notion of an asymptotically hyperbolic manifold with a non-compact boundary having $(\mathbb{H}^n_+, b, \partial \mathbb{H}^n_+)$ as a model; this should be compared with the related boundaryless concept in [12]. Recall that $\mathbb{H}^n_+$ can be parameterized by polar coordinates $(r, \theta)$, where $r = \sqrt{x_1^2 + \cdots + x_n^2}$ and $\theta = (\theta_2, \ldots, \theta_n) \in \mathbb{S}^{n-1}$. For all $r_0 > 0$ large enough let us set $\mathbb{H}^n_{+,r_0} = \{ x \in \mathbb{H}^n_+ ; r(x) \geq r_0 \}$.

**Definition 2.3.** We say that $(M^n, g, \Sigma)$ is asymptotically hyperbolic (with a non-compact boundary $\Sigma$) if there exist $r_0 > 0$, a region $M_{ext} \subset M$ and a diffeomorphism $F : \mathbb{H}^n_+_{+,r_0} \to M_{ext}$ such that:
1. As \( r \to +\infty \), \( e := F^*g - b \) satisfies
\[
|e|_b + |\nabla e|_b + |\nabla^2 e|_b = O(r^{-\tau}), \quad \tau > \frac{n}{2};
\]
(2.5)

2. both \( \int_M r(R_g + n(n - 1))dM \) and \( \int_{\Sigma} rH_g d\Sigma \) are finite, where the asymptotical radial coordinate \( r \) has been smoothly extended to \( M \).

Remark 2.4. Although \( \Sigma \) may be disconnected, it follows from Definition 2.3 that \( \Sigma \) has exactly one non-compact component.

3. The Mass Functional and its Geometric Invariance

Here we define the mass functional for an asymptotically hyperbolic manifold and establish its geometric invariance. Given a chart at infinity \( F : \mathbb{H}^n_{+,r_0} \to M_{\text{ext}} \) as in Definition 2.3 we set, for \( r_0 < r < r' \), \( A_{r,r'} = \{ x \in \mathbb{H}^n_{+,r_0}; r \leq |x| \leq r' \} \), \( \Sigma_{r,r'} = \{ x \in \partial \mathbb{H}^n_{+,r_0}; r \leq |x| \leq r' \} \) and \( S_{r,r'}^{n-1} = \{ x \in \mathbb{H}^n_{+,r_0}; |x| = r \} \), so that
\[
\partial A_{r,r'} = S_{r,r'}^{n-1} \cup \Sigma_{r,r'} \cup S_{r,r'}^{n-1}.
\]

We represent by \( \mu \) the outward unit normal vector field to \( S_{r,r'}^{n-1} \) or \( S_{r,r'}^{n-1} \), computed with respect to the reference metric \( b \). Also, we consider \( S_{r,r'}^{n-2} = \partial S_{r,r'}^{n-1} \subset \partial \mathbb{H}^n_{+,r} \), endowed with its outward unit conormal field \( \vartheta \), again with respect to \( b \). We set \( e = g - b \), where we have written \( g = F^*g \) for simplicity of notation, and we define the 1-form
\[
\mathbb{U}(V,e) = V(\text{div}_b e - dtr_b e) - \nabla_b V \vartheta e + tr_b e \, dV,
\]
(3.1)
for a static potential \( V \in \mathcal{N}^+_b \).

Theorem 3.1. If \((M,g,\Sigma)\) is an asymptotically hyperbolic manifold, then the quantity
\[
\mathbb{m}_{(g,b,F)}(V) = \lim_{r \to +\infty} \left[ \int_{S_{r,r'}^{n-1}} \langle \mathbb{U}(V,e), \mu \rangle dS_{r,r'}^{n-1} - \int_{S_{r'}^{n-2}} V e(\eta, \vartheta) dS_{r'}^{n-2} \right]
\]
(3.2)
exists and is finite.

Proof. The argument is based on the Taylor expansion, as \( r \to +\infty \), of the scalar curvature \( R_g \) around the reference metric \( b \); see [21,32] for nice descriptions of the underlying strategy. We have
\[
R_g = -n(n - 1) + \dot{R}_b e + \rho_b(e),
\]
where
\[
\dot{R}_b e = \text{div}_b (\text{div}_b e - dtr_b e) + (n - 1)tr_b e
\]
is the linearization of the scalar curvature at \( b \) and \( \rho_b(e) = O(r^{-2\tau}) \) denotes terms which are at least quadratic in \( e = g - b \). The key point is to properly handle the linear term. Indeed, a well-known computation gives the fundamental identity
\[
V \dot{R}_b e = \langle \dot{R}^*_b V, e \rangle + \text{div}_b \mathbb{U}(V,e),
\]
where
\[ \hat{R}^*_b V = \nabla^2_b V - Vb \]
is the formal $L^2$ adjoint of $\hat{R}_b$. Since $V \in \mathcal{N}_b^+$ we thus get from (1.2),
\[ V(R_g + n(n - 1)) = \text{div}_b U(V, e) + \rho_b(V, e), \quad (3.3) \]
where $\rho_b(V, e) = V\rho_b(e) = O(r^{-2\tau+1})$ since $V = O(r)$.

We now perform the integration of (3.3) over the half-annular region $A := A_{r,r'} \subset \mathbb{H}^n_+$ and eventually explore the imposed boundary conditions, namely
\[ \Pi_b = 0, \quad \frac{\partial V}{\partial \eta} = 0, \quad (3.4) \]
where $\Pi_b$ is the second fundamental form of $\partial\mathbb{H}^n_+$ and $V \in \mathcal{N}_b^+$; see (2.1). Integration yields
\[
\int_A V(R_g + n(n - 1))dA_b = \int_{S_{r,r'}^{n-1}} \langle U(V, e), \mu \rangle dS_{r,r'}^{n-1} - \int_{S_{r,r'}^{n-1}} \langle U(V, e), \mu \rangle dS_{r,r'}^{n-1} \\
+ \int_{\Sigma_{r,r'}} V(\text{div}_b e - \text{dtr}_b e)(\eta) d\Sigma_\beta \\
- \int_{\Sigma_{r,r'}} (\nabla_b V \perp e)(\eta) d\Sigma_\beta + \int_A \rho_b(V, e) dA_b,
\]
where $\beta = b|_{\Sigma}$ and we used that $dV(\eta) = 0$ by (3.4). We now recall (see [31, equation (34)] for example) that the mean curvature varies as
\[ 2\hat{H}_b e = [\text{dtr}_b e - \text{div}_b e](\eta) - \text{div}_\beta X - \langle \Pi_b, e \rangle_\beta, \]
where $X$ is the vector field dual to the 1-form $e(\eta, \cdot)|_{\Sigma}$ and we still denote $e = e|_{\Sigma}$. Hence, upon expansion around the reference metric $\beta$ and taking into account that $\Pi_b = 0$ by (3.4),
\[
2 \int_{\Sigma_{r,r'}} VH_g d\Sigma_\beta = \int_{\Sigma_{r,r'}} V(\text{dtr}_b e - \text{div}_b e)(\eta) d\Sigma_\beta - \int_{\Sigma_{r,r'}} V \text{div}_\beta X d\Sigma_\beta \\
+ \int_{\Sigma_{r,r'}} \rho_\beta(V, e) d\Sigma_\beta,
\]
where $\rho_\beta(V, e) = O(r^{-2\tau+1})$. Thus, if
\[ A_{r,r'}(g) := \int_A V(R_g + n(n - 1))dA_b + 2 \int_{\Sigma_{r,r'}} VH_g d\Sigma_\beta, \]
then
\[ A_{r,r'}(g) = \int_{S_{r,r'}^{n-1}} \langle U(V, e), \mu \rangle dS_{r,r'}^{n-1} - \int_{S_{r,r'}^{n-1}} \langle U(V, e), \mu \rangle dS_{r,r'}^{n-1} \\
- \int_{\Sigma_{r,r'}} (\nabla_b V \perp e)(\eta) d\Sigma_\beta - \int_{\Sigma_{r,r'}} V \text{div}_\beta X d\Sigma_\beta \\
+ \int_A \rho_b(V, e) dA_b + \int_{\Sigma_{r,r'}} \rho_\beta(V, e) d\Sigma_\beta. \]
But notice that

\[-V \text{div}_\beta X = -\text{div}_\beta (VX) + \langle \nabla_b V, X \rangle \beta\]

\[= -\text{div}_\beta (VX) + e(\nabla_b V, \eta)\]

\[= -\text{div}_\beta (VX) + (\nabla_b V \uparrow e)(\eta),\]

and we end up with

\[A_{r,r'}(g) = \int_{S_{r',+}^{n-1}} \langle \mathbb{U}(V, e), \mu \rangle dS_{r',+}^{n-1} - \int_{S_{r,+}^{n-1}} \langle \mathbb{U}(V, e), \mu \rangle dS_{r,+}^{n-1}\]

\[-\int_{\Sigma_{r,r'}} \text{div}_\beta (VX) d\Sigma_\beta\]

\[+ \int_{A_{r,r'}} \rho_b(V, e) dA_b + \int_{\Sigma_{r,r'}} \rho_\beta(V, e) d\Sigma_\beta\]

\[= \left( \int_{S_{r',+}^{n-1}} \langle \mathbb{U}(V, e), \mu \rangle dS_{r',+}^{n-1} - \int_{S_{r,+}^{n-2}} Ve(\eta, \emptyset) dS_{r,+}^{n-2} \right)\]

\[-\left( \int_{S_{r,+}^{n-1}} \langle \mathbb{U}(V, e), \mu \rangle dS_{r,+}^{n-1} - \int_{S_{r,+}^{n-2}} Ve(\eta, \emptyset) dS_{r,+}^{n-2} \right)\]

\[+ \int_{\Sigma_{r,r'}} \rho_b(V, e) dA_b + \int_{\Sigma_{r,r'}} \rho_\beta(V, e) d\Sigma_\beta.\]

Now, the assumption \(\tau > n/2\) in (2.5) implies that the last two integrals vanish as \(r \to +\infty\). Also, since \(V = O(r)\) the integrability assumptions in Definition 2.3, 2., imply that \(A_{r,r'}(g) \to 0\) as \(r \to +\infty\) and the result follows. \(\square\)

We should think of (3.2) as defining a linear functional

\[m_{(g,b,F)} : \mathcal{N}_b^+ \to \mathbb{R}.\]

We note however that the decomposition \(g = b + e\) used above depends on the choice of an admissible chart at infinity (the diffeomorphism \(F\)), so we need to check that \(m_{(g,b,F)}\) behaves properly as we pass from one such chart to another. For this we need some preliminary results.

**Lemma 3.2.** If \(V \in \mathcal{N}_b^+\) and \(\zeta\) is a vector field, then

\[\mathbb{U}(V, L_\zeta b) = \text{div}_b \mathbb{V}(V, \zeta, b),\]

with the 2-form \(\mathbb{V}\) being explicitly given by

\[\mathbb{V}_{ik} = V(\zeta_{i;k} - \zeta_{k;i}) + 2(\zeta_k V_i - \zeta_i V_k),\]

where the semi-colon denotes covariant derivation with respect to \(b\).

**Proof.** Using \((L_\zeta b)_{ij} = \zeta_{i;j} + \zeta_{j;i}\) and (3.1) we see that

\[\mathbb{U}_i = I_i^{(1)} + I_i^{(2)} + I_i^{(3)},\]
where
\[ I_1^{(1)} = V(\zeta_{i;k}^k + \zeta_{i;k}^k - 2\zeta_{i;k}^k) \]
\[ = V(\zeta_{i;k}^k + \zeta_{i;k}^k - 2\zeta_{i;k}^k + 2\text{Ric}_{ik}^b\zeta^k) \]
\[ = V(\zeta_{i;i}^k - \zeta_{i;i}^k),k - 2(n - 1)V\zeta_i, \]
\[ I_1^{(2)} = -(\zeta_{i;k} + \zeta_{k;i})V^k \]
\[ = (\zeta_{i;k} - \zeta_{k;i})V^k - 2\zeta_{i;k}V^k \]
\[ = (\zeta_{i;k} - \zeta_{k;i})V^k - 2(\zeta_i V^k)_{;k} + 2\zeta_i \Delta b V, \]

and
\[ I_1^{(3)} = 2\zeta_{i;k}^k V_i \]
\[ = 2(\zeta_i^k V_i)_{;k} - 2\zeta_i^k (V^2 b)_{ik}. \]

Thus, \[ \mathbb{U}_i = (V(\zeta_{i;i}^k - \zeta_{i;i}^k) - 2\zeta_i V^k + 2\zeta_i^k V_i)_{;k} \]
\[ + 2(-)(n - 1)V\zeta_i - (V^2 b)_{ik}^k \zeta^k + \Delta b V\zeta_i \]
\[ = (V(\zeta_{i;i}^k - \zeta_{i;i}^k) - 2\zeta_i V^k + 2\zeta_i^k V_i)_{;k} \]
\[ + 2(V b - V^2 b)_{ik}^k \zeta^k, \]

and the last term drops out since \( V \in \mathcal{N}_b^+ \). Hence, \( \mathbb{U}_i = b^k V_{ij;k} \).

The next result shows that the reference space \((\mathbb{H}_+^n, b, \partial \mathbb{H}_n^+)\) is rigid at infinity in the appropriate sense.

**Lemma 3.3.** If \( F : \mathbb{H}_+^n \to \mathbb{H}_+^n \) is a diffeomorphism such that \( F^* b = b + O(r^{-\tau}) \) as \( r \to +\infty \), then there exists an isometry \( I \) of \((\mathbb{H}_+^n, b)\) which preserves \( \partial \mathbb{H}_n^+ \) and satisfies
\[ F = I + O(r^{-\tau}), \]
with a similar estimate holding for the first order derivatives.

**Proof.** This is established by straightforwardly adapting the reasoning in the proofs of the corresponding results in [12, 13]. Therefore, the argument is omitted. \( \square \)

Suppose now that we have two diffeomorphisms, say \( F_1, F_2 : \mathbb{H}_+^n, r_0 \to M_{\text{ext}}, \) defining charts at infinity as above and consider \( F = F_1^{-1} \circ F_2 : \mathbb{H}_+^n, r_0 \to \mathbb{H}_+^n, r_0 \). It is clear that \( F^* b = b + O(r^{-\tau}), \tau > n/2, \) so by the previous lemma, \( F = I + O(r^{-\tau}) \) for some isometry \( I \) preserving \( \partial \mathbb{H}_n^+ \). The next result establishes the geometric invariance of the mass functional appearing in Theorem 3.1.

**Theorem 3.4.** Under the conditions above,
\[ m_{(g,b,F_1)}(V \circ I) = m_{(g,b,F_2)}(V), \text{ for all } V \in \mathcal{N}_b^+. \] (3.7)
Proof. Again we proceed as in [32]. We may assume that \( I \) is the identity, so that \( F = \exp \circ \zeta \), where \( \zeta = O(r^{-\tau}) \) is a vector field everywhere tangent to \( \Sigma \). Also, since the mass is defined asymptotically, we assume that \( M \) is diffeomorphic to \( \mathbb{H}^n_+ \) and identity \( M_{\text{ext}} \) with \( \mathbb{H}^n_+, r_0 \). For \( r > r_0 \), let \( S_{r-1} = \{ x \in \partial \mathbb{H}^n_+ : |x| \leq r \} \), so that \( S_{r-1} \cup S_{r-1}^n \) is the boundary of the compact region on \( \mathbb{H}^n_+ \) defined by \( |x| \leq r \). Now set

\[
e_1 = g_1 - b, \quad g_1 = F^*_1 \mu,
\]

and

\[
e_2 = F^*_2 \mu - b = F^*_1 \mu - b,
\]

so that

\[
E := e_2 - e_1 = F^* g_1 - g_1 = F^* (b + e_1) - (b + e_1) = L \zeta b + R_1,
\]

where \( R_1 = O(r^{-2\tau}) \) is a remainder term. It follows that

\[
\mathbb{U}(V, E) = \mathbb{U}(V, L \zeta b) + R_2,
\]

where \( R_2 = \mathbb{U}(V, R_1) = O(r^{-2\tau + 1}) \) vanishes after integration over \( S_{r-1} \) as \( r \to +\infty \). It follows that

\[
\lim_{r \to +\infty} \int_{S_{n+1}^{r-1}} \langle \mathbb{U}(V, E), \mu \rangle dS_{n+1}^{r-1} = \lim_{r \to +\infty} \int_{S_{n+1}^{r-1}} \langle \mathbb{U}(V, L \zeta b), \mu \rangle dS_{n+1}^{r-1}
\]

\[
= - \lim_{r \to +\infty} \int_{S_{n+1}^{r-1}} \langle \mathbb{U}(V, L \zeta b), \eta \rangle d\Sigma_r,
\]

where in the last step we used (3.5) to transfer the integral to \( S_{r-1}^{n-1} \). We shall compute the limit above by means of (3.6). Using an orthonormal \( b \)-frame \( \{ f_i \}_{i=1}^n \) so that \( \eta = -f_n = (0, 0, \ldots, -1) \) along the boundary,

\[
\langle \mathbb{U}(V, L \zeta b), \eta \rangle = b^{jk} V_{ij;k} \eta^j - \nabla_{n;k} \eta = -\nabla_{\eta \alpha \beta}, \quad \text{(sum over } 1 \leq \alpha \leq n - 1).\]

As \( \partial \mathbb{H}^n_+ \) is totally geodesic with respect to \( b \), this shows that

\[
\langle \mathbb{U}(V, L \zeta b), \eta \rangle \big|_{\Sigma_r} = \text{div}_\beta (\eta \cdot V).
\]

Thus,

\[
\int_{S_{n-1}^{r-1}} \langle \mathbb{U}(V, L \zeta b), \eta \rangle d\Sigma_r = \int_{S_{n-2}^{r-2}} \nabla (\eta, \theta) d\Sigma_s = - \int_{S_{n-2}^{r-2}} \nabla_{\eta \alpha \beta} dS_{n-2}^{r-2}.
\]

But

\[
\nabla_{\eta \alpha} = V (\zeta_{n;\alpha} - \zeta_{\alpha; n}) + 2(\zeta_\alpha V_n - \zeta_n V_\alpha) = -V_{\zeta_{\alpha; n}},
\]

where we used that \( V_n = 0 \) (due to (3.4)) and \( \zeta_{n;\alpha} = 0 = \zeta_n \) (since \( \zeta \) is tangent to \( \Sigma \)). It follows that

\[
\lim_{r \to +\infty} \int_{S_{n+1}^{n-1}} \langle \mathbb{U}(V, E), \mu \rangle dS_{n+1}^{n-1} = - \lim_{r \to +\infty} \int_{S_{n-2}^{n-2}} V_{\zeta_{\alpha; n}} \theta dS_{n-2}^{n-2}.
\]

(3.8)
To complete the argument, notice that
\[
- \int_{S_{r-2}^{n-2}} V E(\eta, \vartheta) dS_{r-2}^{n-2} = - \int_{S_{r-2}^{n-2}} V(\mathcal{L}_b \zeta)(\eta, \vartheta) dS_{r-2}^{n-2} \\
- \int_{S_{r-2}^{n-2}} V R_1(\eta, \vartheta) dS_{r-2}^{n-2},
\]
where the last integral vanishes at infinity. Finally, the remaining integral may be evaluated as
\[
- \int_{S_{r-2}^{n-2}} V(\mathcal{L}_b \zeta)(\eta, \vartheta) dS_{r-2}^{n-2} = \int_{S_{r-2}^{n-2}} V(\zeta_{\alpha; n} + \zeta_{n; \alpha}) \vartheta \alpha dS_{r-2}^{n-2},
\]
which clearly cancels out the contribution coming from the right-hand side of (3.8). We have thus shown that
\[
\lim_{r \to +\infty} \left[ \int_{S_{r+1}^{n-1}} \langle U(V, E), \mu \rangle dS_{r+1}^{n-1} - \int_{S_{r-2}^{n-2}} V E(\eta, \vartheta) dS_{r-2}^{n-2} \right] = 0,
\]
which finishes the proof. \(\square\)

Remark 3.5. Recalling that \(\eta = -f_n\) we have
\[
e(\eta, \vartheta) = -e(f_n, \vartheta) = -e_{\alpha n} \vartheta \alpha.
\]
Thus, the mass functional in (3.2) may be expressed in terms of \(e_{ik} = e(f_i, f_k)\) as
\[
m_{(g, b, F)}(V) = \lim_{r \to +\infty} \int_{S_{r+1}^{n-1}} (V (e_{i;k} - e_{k;i}) - e_{ik}V_k + e_{k}^i V_i) \mu^i dS_{r+1}^{n-1} \\
+ \lim_{r \to +\infty} \int_{S_{r-2}^{n-2}} V e_{\alpha n} \vartheta \alpha dS_{r-2}^{n-2}. \tag{3.9}
\]
In particular, if we express the metrics in Fermi coordinates around the boundary we get \(e_{\alpha n} = 0\), so in these new coordinates the last integral in the right-hand side of (3.9) does not contribute to the mass. We will use this choice of asymptotic coordinates and the ensuing simplified expression for the mass in the Proof of Theorem 5.2. It is not hard to check that under this change of coordinates (3.7) holds true with \(I\) being the identity isometry, so the mass functional remains the same.

To properly appreciate the relevance of Theorem 3.4, we note that the isometry group \(O^+(n - 1, 1)\) of the reference space \((\mathbb{H}_+^n, b, \partial \mathbb{H}_+^n)\), which is formed by those isometries of \(\mathbb{H}^n\) preserving \(\partial \mathbb{H}_+^n\), acts naturally on \(\mathcal{N}_b^+\), which is generated by \(\{V(0), V(1), \cdots, V(n-1)\}\), in such a way that the Lorentzian metric
\[
\langle \langle z, w \rangle \rangle = z_0w_0 - z_1w_1 - \cdots - z_{n-1}w_{n-1} \tag{3.10}
\]
is preserved. Here, we regard \(\{V(a)\}_{a=0}^{n-1}\) as an orthonormal basis and endow \(\mathcal{N}_b^+\) with a time orientation by declaring that \(V(0)\) is future directed. Thus, if for any admissible chart at infinity \(F\) we set
\[
P_a^{[F]} = m_{(g, b, F)}(V(a)), \quad 0 \leq a \leq n - 1,
\]
then Theorem 3.4 guarantees that $\langle\langle P^F, P^F \rangle\rangle$, the past/future pointing nature and the causal character of $P^F$ are chart independent indeed. Combined with the standard physical reasoning, this suggests the following conjecture.

**Conjecture 3.6** (Positive mass). Let $(M, g, \Sigma)$ be an asymptotically hyperbolic manifold with $R_g \geq -n(n-1)$ and $H_g \geq 0$. Then for any admissible chart $F$ the vector $P^F$ is time-like future directed, unless it vanishes and $(M, g, \Sigma)$ is isometric to $(\mathbb{H}^n_+, b, \partial \mathbb{H}^n_+)$. 

The “positive mass” terminology is justified by the fact that whenever the conjecture holds true, we may define the numerical invariant

$$m_{(g, b)} = \sqrt{\langle\langle P^F, P^F \rangle\rangle},$$

which happens to be independent of the chosen chart. This may be regarded as the total mass of the isolated gravitational system whose (time-symmetric) initial data set is $(M, g)$. Notice that in this case we always have $m_{(g, b)} > 0$ unless $(M, b, \Sigma)$ is isometric to $(\mathbb{H}^n_+, b, \partial \mathbb{H}^n_+)$. As remarked in the Introduction, our main results (Theorems 5.2 and 5.4) confirm the conjecture in case $M$ is spin.

For later reference we observe the existence of an alternate version of the asymptotic definition for the mass functional (3.2) in terms of the Einstein tensor $G_g = \text{Ric}_g - \frac{R_g}{2} g$ in the interior and the Newton tensor $J_g = \Pi_g - H_g g$, where $\Pi_g$ is the second fundamental form along the boundary, which has been recently established in [16]. More precisely, in terms of the modified Einstein tensor $\hat{G}_g = G_g - \frac{(n-1)(n-2)}{2} g$ we have the following result.

**Theorem 3.7** [16, Theorem 4.2]. For each $a = 0, 1, \cdots, n-1$ there exists a conformal vector field $X_a$ on $\mathbb{H}^n_+$ which is everywhere tangent to $\partial \mathbb{H}^n_+$ and which for any asymptotically hyperbolic manifold $(M, g, \Sigma)$ and admissible chart $F$ satisfies

$$m_{(g, b, F)}(V(a)) = d_n \lim_{r \to +\infty} \left[ \int_{\tilde{S}_{r,+}^{n-1}} \hat{G}_g(F_* X_a, \mu_g) d\tilde{S}_{r,+}^{n-1} + \int_{\tilde{S}_{r,-}^{n-2}} J_g(F_* X_a, \vartheta_g) d\tilde{S}_{r,-}^{n-2} \right],$$

(3.11)

where $d_n > 0$ is a dimensional constant, $\mu_g$ the outward unit normal to $\tilde{S}_{r,+}^{n-1} := F(S_{r,+}^{n-1}) \subset M$ and similarly for $\vartheta_g$. 

Remark 3.8. In general we may write
\[ \hat{G}_g = \tilde{G}_g + \frac{2-n}{2n} (R_g + n(n-1))g, \]
where
\[ \tilde{G}_g = \text{Ric}_g - \frac{R_g}{n} g \]
is the traceless Ricci tensor. In particular, if \( g \) is Einstein with \( \text{Ric}_g = -(n-1)g \), so that \( R_g = -n(n-1) \), then \( \tilde{G}_g = 0 \) and hence \( \hat{G}_g = 0 \) as well.

4. Spinors on Manifolds with Boundary

In this section we review the results in the theory of spinors on \( n \)-manifolds carrying a (possibly non-compact) boundary \( \Sigma \) which are needed in the rest of the paper. Our presentation uses the setup introduced in [23,24], and we refer to these works for further details on the material presented here.

4.1. The Integral Lichnerowicz Formula on Spin Manifolds with Boundary

We assume that the given manifold \((M,g)\) is spin and fix once and for all a spin structure on \( TM \). We denote by \( SM \) the associated spin bundle and by \( \nabla \) both the Levi-Civita connection of \( TM \) and its lift to \( SM \). Also, \( c : TM \times SM \rightarrow SM \) is the associated Clifford product, so that the corresponding Dirac operator is
\[ D = \sum_{i=1}^{n} c(e_i) \nabla e_i, \]
where \( \{e_i\}_{i=1}^{n} \) is any orthonormal frame. Sometimes, when we wish to emphasize the dependence of \( c \) on \( g \) we append a superscript and write \( c = c^g \) instead (and similarly for the other geometric invariants associated to the given spin structure).

The need to consider chirality boundary conditions for spinors along the boundary leads us to implement a procedure introduced in [24] which allows us to treat the even and odd dimensional cases simultaneously. Given a spin manifold \( M \) as above we set \( \mathcal{E}M = SM \) if \( n \) is even and \( \mathcal{E}M = SM \oplus SM \) if \( n \) is odd. In this latter case, \( \mathcal{E}M \) becomes a Dirac bundle if we define, for a section
\[ \Psi = \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) \in \Gamma(\mathcal{E}M), \quad \psi_i \in \Gamma(SM), \]
the Clifford product and connection by
\[ c^\mathcal{E}(X)\Psi = \left( \begin{array}{c} c(X)\psi_1 \\ -c(X)\psi_2 \end{array} \right), \quad \nabla^\mathcal{E}_X \Psi = \left( \begin{array}{c} \nabla_X \psi_1 \\ \nabla_X \psi_2 \end{array} \right), \quad X \in \Gamma(TM). \]
As usual, the corresponding Dirac operator is
\[ D^\mathcal{E} \Psi = \sum_{i=1}^{n} c^\mathcal{E}(e_i) \nabla^\mathcal{E}_{e_i} \Psi = \left( \begin{array}{c} D\psi_1 \\ -D\psi_2 \end{array} \right). \]
Finally, in order to unify the notation we set $c^E = c$, $\nabla^E = \nabla$ and $D^E = D$ if $n$ is even whenever convenient.

We now define the *Killing connections* on $\mathcal{E}M$ by

$$\tilde{\nabla}^E_{\pm} = \nabla^E_{\pm} \pm \frac{i}{2} c^E(X).$$

The corresponding *Killing-Dirac operators* are defined in the usual way, namely

$$\tilde{D}^E_{\pm} = \sum_{i=1}^{n} c^E(e_i) \tilde{\nabla}^E_{e_i}^{\pm},$$

so that

$$\tilde{D}^E_{\pm} = D^E \mp \frac{n}{2}. \tag{4.1}$$

Given $\Psi \in \Gamma(\mathcal{E}M)$ we set

$$\tilde{\Theta}^\pm_\Psi(X) = -\langle \tilde{\nabla}^E_{\pm}(X)\Psi, \Psi \rangle, \quad X \in \Gamma(TM),$$

where

$$\tilde{\nabla}^E_{\pm}(X) = -\langle \tilde{\nabla}^E_{\pm} + c^E(X)\tilde{D}^E_{\pm} \rangle. \tag{4.2}$$

Using the standard Lichnerowicz formula we easily compute that

$$\text{div}_g \tilde{\Theta}^\pm_\Psi = |\tilde{\nabla}^E_{\pm}\Psi|^2 - |\tilde{D}^E_{\pm}\Psi|^2 + \frac{R_g + n(n-1)}{4} |\Psi|^2,$$

so if $\Omega \subset M$ is a compact domain with a nonempty boundary $\partial\Omega$, which we assume endowed with its inward pointing unit normal $\nu$, then integration by parts yields the integral version of the fundamental Lichnerowicz formula, namely

$$\int_{\Omega} \left( |\tilde{\nabla}^E_{\pm}\Psi|^2 - |\tilde{D}^E_{\pm}\Psi|^2 + \frac{R_g + n(n-1)}{4} |\Psi|^2 \right) dM$$

$$= \text{Re} \int_{\partial\Omega} \langle \tilde{\nabla}^E_{\pm}(\nu)\Psi, \Psi \rangle d\Sigma. \tag{4.3}$$

A key step in our argument is to rewrite the right-hand side of (4.3) along the portion of $\partial\Omega$ lying on $\Sigma$ in terms of the corresponding shape operator. Indeed, notice that $\Sigma$ carries the bundle $\mathcal{E}M|\Sigma$, obtained by restricting $\mathcal{E}M$ to $\Sigma$. This becomes a Dirac bundle if endowed with the Clifford product

$$c^{E,\tau}(X)\Psi = c^E(X) c^E(\nu)\Psi,$$

and the connection

$$\nabla^{E,\tau}_{X}\Psi = \nabla_{X}\Psi - \frac{1}{2} c^{E,\tau}(AX)\Psi, \tag{4.4}$$

where $A = -\nabla\nu$ is the shape operator of $\Sigma$, so the corresponding Dirac operator $D^{E,\tau} : \Gamma(\mathcal{E}M|\Sigma) \to \Gamma(\mathcal{E}M|\Sigma)$ is

$$D^{E,\tau}\Psi = \sum_{j=1}^{n-1} c^{E,\tau}(f_j) \nabla^{E,\tau}_{f_j}\Psi,$$
where \( \{ f_j \}_{j=1}^{n-1} \) is a local orthonormal tangent frame along \( \Sigma \). Choosing the frame so that \( A_g f_j = \kappa_j f_j \), where \( \kappa_j \) are the principal curvatures of \( \Sigma \), we have
\[
D_{E^\Sigma} \Psi = -c^E(\nu) D \Psi + \frac{H_g}{2} \Psi,
\]
where \( H_g = \kappa_1 + \cdots + \kappa_{n-1} \) is the mean curvature and
\[
D \Psi = \sum_{j=1}^{n-1} c^E(f_j) \nabla_{f_j}^E \Psi = c^E(\nu) \left( D_{E^\Sigma} \Psi - \frac{H_g}{2} \Psi \right).
\]
Since \( D = D^E - c^E(\nu) \nabla_{\nu}^E \), we obtain
\[
D_{E^\Sigma} \Psi = \frac{H_g}{2} \Psi - (\nabla_{\nu}^E + c^E(\nu) D^E) \Psi,
\]
which combined with (4.3) yields the following important result.

**Proposition 4.1.** Under the conditions above,
\[
\int_{\Omega} \left( |\tilde{\nabla}_{E^\Sigma}^E \pm \Psi|^2 - |\tilde{D}_{E^\Sigma}^E \pm \Psi|^2 + \frac{R_g + n(n-1)}{4} |\Psi|^2 \right) d\Omega
\]
\[
= \int_{\partial \Omega \cap \Sigma} \left( \langle \tilde{D}_{E^\Sigma}^E \pm \Psi, \Psi \rangle - \frac{H_g}{2} |\Psi|^2 \right) d\Sigma
\]
\[
+ \text{Re} \int_{\partial \Omega \cap \text{int} M} \left( \tilde{W}_{E^\Sigma}^E(\nu) \Psi, \Psi \right) d\partial \Omega,
\]
(4.5)
where
\[
\tilde{D}_{E^\Sigma}^E \pm = D_{E^\Sigma} \pm + \frac{n-1}{2} i c^E(\nu) : \Gamma(S\Sigma) \rightarrow \Gamma(S\Sigma).
\] (4.6)

**Remark 4.2.** It turns out that the extrinsic Dirac bundle \( (E|\Sigma, c_{E^\Sigma}, \nabla_{E^\Sigma}) \) can be naturally identified with certain Dirac bundles constructed out of the intrinsic induced spin bundle \( (S\Sigma, c^\gamma, \nabla^\gamma) \), where \( \gamma = g|\Sigma \) is the induced metric along \( \Sigma \). Thus,
\[
(E|\Sigma, c_{E^\Sigma}, \nabla_{E^\Sigma}) \cong \begin{cases} 
(S\Sigma \oplus S\Sigma, c^\gamma \oplus -c^\gamma, \nabla^\gamma \oplus \nabla^\gamma) & n \text{ even} \\
(S\Sigma \oplus S\Sigma, c^\gamma \oplus c^\gamma, \nabla^\gamma \oplus \nabla^\gamma) & n \text{ odd}.
\end{cases}
\]

Similar remarks hold for the corresponding Dirac operators. We refer to [24, Subsection 2.2] for a detailed discussion on these issues.

### 4.2. Chirality Boundary Conditions

To further simplify (4.5) we must impose suitable boundary conditions on \( \Psi \).

**Definition 4.3.** A chirality operator on a spin manifold \( (M, g) \) is a (point-wise) selfadjoint involution \( Q : \Gamma(E) \rightarrow \Gamma(E) \) which is parallel and anti-commutes with Clifford multiplication by tangent vectors.
If $n$ is even, it is well known that Clifford multiplication by the complex volume element provides a chirality operator. Using the formalism above, we easily see that in case $n$ is odd $Q : \Gamma(\mathcal{E} M) \to \Gamma(\mathcal{E} M)$,

$$Q \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix}$$

also defines a chirality operator. Thus, in either case we define the corresponding boundary chirality operator $Q = Q c^\mathcal{E} (\nu) : \Gamma(\mathcal{E} M|_\Sigma) \to \Gamma(\mathcal{E} M|_\Sigma)$. Clearly, $Q$ is a self-adjoint involution as well, so we may consider the projections

$$P_Q^{(\pm)} = \frac{1}{2} (\text{Id}_{\mathcal{E} M|_\Sigma} \pm Q) : \Gamma(\mathcal{E} M|_\Sigma) \to \Gamma(V^{\mathcal{E},(\pm)})$$

onto the $\pm 1$-eigenbundles $V^{\mathcal{E},(\pm)}$ of $Q$. Thus, $\Psi \in \Gamma(V^{\mathcal{E},(\pm)})$ if and only if $Q \Psi = \pm \Psi$. In the following we use the qualification standard to refer to any of these chirality structures on a spin manifold $M$.

**Remark 4.4.** If $n$ is odd, then $\Psi \in \Gamma(V^{\mathcal{E},(\pm)})$ if and only if

$$\Psi = \left( \begin{array}{c} \psi \\ \pm c(\nu) \psi \end{array} \right), \quad \psi \in \Gamma(SM|_\Sigma). \quad (4.7)$$

For any $\Psi \in \Gamma(\mathcal{E} M)$ we set $\Psi^{(\pm)} = P_{Q}^{(\pm)} \Psi \in \Gamma(V^{\mathcal{E},(\pm)})$, so that

$$\Psi = \Psi^{(+)} + \Psi^{(-)},$$

an orthogonal decomposition. Since $D^{\mathcal{E},\tau} c^\mathcal{E}(\nu) = -c^\mathcal{E}(\nu) D^{\mathcal{E},\tau}$ we have $D^{\mathcal{E},\tau} P_{Q}^{(\pm)} = P_{Q}^{(\mp)} D^{\mathcal{E},\tau}$, and hence

$$\langle D^{\mathcal{E},\tau} \Psi, \Psi \rangle = \langle D^{\mathcal{E},\tau} \Psi^{(+)}, \Psi^{(-)} \rangle + \langle D^{\mathcal{E},\tau} \Psi^{(-)}, \Psi^{(+)} \rangle. \quad (4.8)$$

**Definition 4.5.** If $\mathcal{E} M$ is endowed with the standard chirality operator $Q$ as above, then we say that $\Psi \in \Gamma(\mathcal{E} M)$ satisfies a chirality boundary condition if any of the identities $\Psi^{(\pm)} = 0$ holds along $\Sigma$.

**Proposition 4.6.** If $\Psi$ satisfies a chirality boundary condition, then

$$\langle c^\mathcal{E}(\nu)\Psi, \Psi \rangle = 0, \quad (4.9)$$

and

$$\text{Re} \int_{\partial \Omega \cap \text{int} M} \left( \tilde{\nabla} c^\mathcal{E}(\nu) \Psi, \Psi \right) d\partial \Omega$$

$$= \int_{\Omega} \left( |\tilde{\nabla} c^\mathcal{E}(\nu) \Psi|^2 + |\tilde{\nabla} c^\mathcal{E}(\nu) \Psi|^2 + \frac{R_g + n(n-1)}{4} |\Psi|^2 \right) d\Omega$$

$$+ \int_{\partial \Omega \cap \Sigma} \frac{H_g}{2} |\Psi|^2 d\Sigma. \quad (4.10)$$

**Proof.** If $Q \Psi = \pm \Psi$ we have

$$\langle c^\mathcal{E}(\nu)\Psi, \Psi \rangle = \pm \langle c^\mathcal{E}(\nu)Q c^\mathcal{E}(\nu)\Psi, \Psi \rangle = \mp \langle Q c^\mathcal{E}(\nu)\Psi, c^\mathcal{E}(\nu)\Psi \rangle = \mp \langle c^\mathcal{E}(\nu)\Psi, Q c^\mathcal{E}(\nu)\Psi \rangle = -\langle c^\mathcal{E}(\nu)\Psi, \Psi \rangle,$$
which proves (4.9). On the other hand, from (4.8) we have $\langle D^\varepsilon \cdot \tau \Psi, \Psi \rangle = 0$. Thus, from (4.6) we get $\langle \tilde{D}^\varepsilon \cdot \tau \pm \Psi, \Psi \rangle = 0$, which together with (4.5) proves (4.10).

Finally, we remark that the projections $P^Q(\pm)$ define nice elliptic boundary conditions for the Dirac operator $\tilde{D}^\varepsilon$ considered above, as the following result shows.

**Proposition 4.7.** If $(M, g, \Sigma)$ is asymptotically hyperbolic as above with $R_g \geq -n(n-1)$ and $H_g \geq 0$, then for any $\Phi \in \Gamma(\mathcal{E}M)$ such that $\tilde{\nabla}^\varepsilon \cdot \Phi \in L^2(\mathcal{E}M)$ there exists a unique $\Xi \in L^2_1(\mathcal{E}M)$ solving the boundary value problem

\[
\begin{aligned}
\left\{ \begin{array}{l}
\tilde{D}^\varepsilon + \Xi = -\tilde{D}^\varepsilon \cdot \Phi \text{ in } M, \\
\Xi(\pm) = 0 \text{ on } \Sigma.
\end{array} \right.
\end{aligned}
\]

**Proof.** This is a direct consequence of the methods leading to [20, Corollary 4.19], so we merely sketch the argument. Firstly, since our asymptotically hyperbolic manifold $(M, g, \Sigma)$ clearly has bounded geometry, the trace theory developed in [20, Section 3.1] applies to our setting. In particular, the Green’s formula

\[
\int_M \left( \langle \tilde{D}^\varepsilon \cdot \Phi, \phi' \rangle - \langle \phi, \tilde{D}^\varepsilon \cdot \Phi' \rangle \right) \, dM = \int_\Sigma \langle \eta \cdot \phi, \phi' \rangle \, d\Sigma
\]

(4.11)

holds for any $\phi, \phi' \in L^2_1(\mathcal{E}M)$; see [20, Lemma 3.10] where this is proved for the standard Dirac operator $D^\varepsilon$ but in view of (4.1) it also holds true for $\tilde{D}^\varepsilon$. Thus,

\[
\tilde{D}^\varepsilon : \text{dom } \tilde{D}^\varepsilon \subset L^2_1(\mathcal{E}M) \to L^2(\mathcal{E}M),
\]

where the domain $\text{dom } \tilde{D}^\varepsilon$ is determined by the given chirality boundary condition, is a self-adjoint operator as that condition forces the vanishing of the right-hand side of (4.11). On the other hand, again from the trace theory we have

\[
\int_M \left( \langle \tilde{\nabla}^\varepsilon \cdot \Phi, \phi' \rangle - \langle \phi, \tilde{D}^\varepsilon \cdot \Phi' \rangle \right) \, dM + \int_\Sigma \frac{H_g}{2} |\phi|^2 \, d\Sigma = 0,
\]

(4.12)

for any $\phi \in \text{dom } \tilde{D}^\varepsilon$. Indeed, this follows from (4.10) if we take $\Omega = M_r$, the region in $M$ enclosed by $\Sigma_r \cup S^{n-1}_{r,+}$, and observe that the left-hand side, which involves an integration over $S^{n-1}_{r,+}$, vanishes as $r \to +\infty$ since $\phi \in L^2_1(\mathcal{E}M)$. In particular, if $\phi \in \text{ker } \tilde{D}^\varepsilon$, then $\phi$ is a Killing spinor as in Definition 4.8. However, it is well known that any such nontrivial object varies exponentially along any geodesic segment and therefore cannot lie in $\text{Dom}(\tilde{D}^\varepsilon)$. Hence, $\text{ker } \tilde{D}^\varepsilon = \{0\}$. Also, it is well known that a weighted Poincaré inequality, with the weight function being a positive constant in the asymptotic region, holds true in the asymptotically hyperbolic setting (the proof in [6, Theorem 9.10] works fine in the presence of a noncompact boundary as well). Thus, a standard argument ensures that $\tilde{D}^\varepsilon$ is coercive at infinity (with respect to the given boundary condition); see [20, Definition 4.17]. Thus, [20, Corollary 4.19]
applies. Since the compatibility condition in [20, equation (12)] with $\rho = 0$ is trivially satisfied because \( \ker \tilde{D}^{E,+} = \{0\} \), we may employ this corollary with $\psi = -\tilde{D}^{E,+}\Phi \in L^2(\mathcal{E}M)$. Here we use that the assumption $\tilde{\nabla}^{E,+}\Phi \in L^2(\mathcal{E}M)$ clearly implies that $\tilde{D}^{E,+}\Phi \in L^2(\mathcal{E}M)$ as well. In this way we obtain a spinor $\Xi(\pm)$ as in the statement of the proposition which is determined up to elements of $\ker \tilde{D}^{E,+} = \{0\}$, hence unique. \qed

4.3. Killing Spinors

We start by adapting a well-known definition.

**Definition 4.8.** We say that $\Phi \in \Gamma(\mathcal{E}M)$ is an imaginary Killing section to the number $\pm i/2$ if it is parallel with respect to $\tilde{\nabla}^{E,+}$, that is, $\nabla^{E}_{X}\Phi \pm \frac{i}{2} c^{E}(X)\Phi = 0$, $X \in \Gamma(TM)$. The space of all such sections is denoted by $\mathcal{K}^{g,\pm}(\mathcal{E}M)$. More generally, $\Phi$ is Killing-harmonic if it satisfies any of the equations $\tilde{D}^{E,+}\Phi = 0$.

**Remark 4.9.** If $n$ is odd, then

$$
\Phi = \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix}
$$

is imaginary Killing to the number $i/2$ if and only if $\phi_{\pm} \in \Gamma(\mathcal{S}M)$ is imaginary Killing to the number $\pm i/2$. Thus,

$$
\mathcal{K}^{g,\pm}(\mathcal{E}M) = \begin{cases} 
\mathcal{K}^{g,\pm}(\mathcal{S}M) & n \text{ even} \\
\mathcal{K}^{g,\pm}(\mathcal{S}M) \oplus \mathcal{K}^{g,-}(\mathcal{S}M) & n \text{ odd}
\end{cases}
$$

where $\mathcal{K}^{g,\pm}(\mathcal{S}M)$ is the space of imaginary Killing spinors to the number $\pm i/2$.

**Example 4.10.** The conformal relationship between $(\mathbb{B}^{n}_{+}, \tilde{b})$ and $(\mathbb{B}^{n}_{+}, \delta)$ described in (2.2) allows us to canonically identify the corresponding spinor bundles $\mathbb{S}^{n}_{+}\tilde{b}$ and $\mathbb{S}^{n}_{+}\delta$, so that $\phi \in \Gamma(\mathbb{S}^{n}_{+}\delta)$ corresponds to a certain $\overline{\phi} \in \Gamma(\mathbb{S}^{n}_{+}\tilde{b})$. Under this identification, if $u \in \Gamma(\mathbb{S}^{n}_{+}\delta)$ is a constant (i.e. $\nabla^{\delta}$-parallel) spinor, then the prescription

$$
\phi_{u,\pm}(x) := \omega(x)^{-1/2}(1 \pm i c^{\delta}(x))u \in \Gamma(\mathbb{S}^{n}_{+}\tilde{b})
$$

exhausts the space $\mathcal{K}^{b,\pm}(\mathbb{S}^{n}_{+})$ [8]. Here, $\delta$ refers to the Euclidean metric. In particular, if $n$ is odd, then by Remark 4.9,

$$
\Phi_{u,v} := \begin{pmatrix} \phi_{u,+} \\ \phi_{v,-} \end{pmatrix} \in \mathcal{K}^{b,+}(\mathbb{E}^{n}_{+}),
$$

for any pair of constant spinors $u, v \in \Gamma(\mathbb{S}^{n}_{+}\delta)$.

In general, if $(M, g, \Sigma)$ is asymptotically hyperbolic, we may consider the space

$$
\mathcal{K}^{g,\pm,\pm}(\mathcal{E}M) = \{ \Phi \in \mathcal{K}^{g,\pm}(\mathcal{E}M); Q^{g}\Phi = \pm \Phi \}$$
of all imaginary Killing sections to the number \( \pm \mathbf{i}/2 \) satisfying the corresponding chirality boundary condition. This space can be explicitly described for \( \mathbb{H}^n_+ \). In view of Example 4.10 above, it is convenient here to consider the half-disk model \( \mathbb{B}^n_+ \) in Remark 2.1. If \( Q^b \) is the standard chirality operator on \( (\mathbb{B}^n_+, \tilde{b}) \) and \( Q^\delta \) is the corresponding boundary chirality operator, then these data naturally induce corresponding operators \( Q^\delta \) and \( Q^\delta \) on \( (\mathbb{B}^n_+, \delta) \). Now let \( K^{\delta, (+)} \) be the space of all constant spinors \( u \in \Gamma(\mathbb{S}^n_+, \delta) \) that satisfy \( Q^\delta u = \pm u \).

**Proposition 4.11.** If \( n \) is even the prescription \( u \mapsto \phi_{u, \pm} \) in (4.13) defines isomorphisms \( K^{\delta, (+)} \simeq K^{b, +, (+)}(\mathbb{S}^n_+) \) and \( K^{\delta, (-)} \simeq K^{b, +, (-)}(\mathbb{S}^n_+) \).

**Proof.** Let \( \nu_b = \omega^{-1} \nu_\delta \) be the hyperbolic unit normal along \( \partial \mathbb{B}^n_+ \), where \( \nu_\delta = \partial_n \) is the Euclidean inward unit normal. Notice that \( e^\delta (\nu^\delta) e^\delta (x) = -e^\delta (x) e^\delta (\nu^\delta) \) if \( x \in \partial \mathbb{B}^n_+ \). Thus, if \( u \in K^{\delta, (+)} \),

\[
(\hat{Q}^b \phi_{u, \pm})(x) = (Q^b \hat{c}^b (\nu^b) \phi_{u, \pm})(x) = \omega(x)^{-1/2} Q^b (\nu^b) (\sqrt{-1} \mp c^\delta (x) u) = \omega(x)^{-1/2} \left( Q^b c^\delta (\nu^\delta) u \pm Q^b c^\delta (\nu^\delta) c^\delta (x) u \right) = \omega(x)^{-1/2} \left( \sqrt{-1} \mp c^\delta (x) Q^b c^\delta (\nu^\delta) u \right) = \omega(x)^{-1/2} \left( \sqrt{-1} \pm c^\delta (x) u \right) = \phi_{u, \pm}(x).
\]

Similarly, if \( u \in K^{\delta, (-)} \) we compute that \( (\hat{Q}^b \phi_{u, \pm})(x) = -\phi_{u, \pm}(x) \). \( \square \)

This leads to the following result, which confirms that \( \mathbb{H}^n_+ \) carries the maximal number of linearly independent such sections.

**Corollary 4.12.** We have

\[
\dim_{\mathbb{C}} K^{b, +, (+)}(\mathbb{E}^n_+) = 2^{k-1},
\]

if \( n = 2k \) or \( n = 2k + 1 \). As a consequence,

\[
\dim_{\mathbb{C}} K^{b, +}(\mathbb{E}^n_+) = 2^k = \text{rank } \mathbb{S}^n_+.
\]

**Proof.** The even case is obvious since \( \text{rank } \mathbb{S}^n_+ = 2^k \) equals the dimension of the space of constant spinors. If \( n \) is odd note that if \( \Psi_{u,v} \) in (4.14) is of the form (4.7), then

\[
v - c^\delta (x)v = \pm e^\delta (\nu^\delta)(u + c^\delta (x) u), \quad x \in \partial \mathbb{B}^n_+.
\]

By taking \( x = 0 \) yields \( v = \pm e^\delta (\nu^\delta) u \), so the entries in (4.14) depend on the same constant spinor. \( \square \)
5. The Positive Mass Theorem and its Consequences

In this section we present the proofs of our main results, namely Theorems 5.2 and 5.4. We then explain how they imply Theorems 1.1, 1.2 and 1.6 in the Introduction.

We consider an asymptotically hyperbolic manifold \((M, g, \Sigma)\) in the sense of Definition 2.3. We fix a chart at infinity \(F : (\mathbb{H}_+^n, r_0) \rightarrow (M_{\text{ext}}, g)\). In fact, since \(m_{(g, b, F)}\) is an asymptotic invariant, we may assume that \(F\) is a global diffeomorphism between \(\mathbb{H}_+^n\) and \(M\). In any case this allows us to construct a gauge map \(G\) acting on tangent vectors so that

\[
\langle GX, GY \rangle_g = \langle X, Y \rangle_b, \quad \langle GX, Y \rangle_g = \langle X, GY \rangle_g.
\]

and such that, in the asymptotic region,

\[
G = I - \frac{1}{2} \mathcal{H} + \mathcal{R}, \quad \mathcal{H} = O(r^{-\gamma}), \quad \mathcal{R} = O(r^{-2\tau}).
\]

In terms of an orthonormal \(b\)-frame \(\{f_i\}_{i=1}^n\) in Definition 2.3, this last requirement means that \(e_i = Gf_i\) given by

\[
e_i = f_i - \frac{1}{2} \mathcal{H}f_i + \mathcal{R}f_i
\]

is an orthonormal \(g\)-frame. Notice that \(\mathcal{H} = I - G^2 + O(r^{-2\tau})\), so that

\[
e(X, Y) = \langle \mathcal{H}X, Y \rangle_g + O(r^{-2\tau}),
\]

whenever \(X\) and \(Y\) are uniformly bounded vector fields.

The gauge map \(G\) induces an identification between the bundles \(\mathcal{E}_{\mathbb{H}_+^n, r_0}\) and \(\mathcal{E}M_{\text{ext}}\) endowed with the metric structures coming from \(b\) and \(g\), respectively. Thus, if \(\varphi\) is a cut-off function on \(M\) with \(\varphi = 1\) on \(M_{\text{ext}}\) and \(\Phi \in \Gamma(\mathcal{E}\mathbb{H}_+^n)\), then \(\Phi_* := \varphi G\Phi \in \Gamma(\mathcal{E}M)\) and the map \(\Phi \mapsto \Phi_*\) is a (fiber-wise) isometry in a neighborhood of infinity. We apply this construction to \(\Phi \in K^{b,+,(\pm)}(\mathbb{H}_+^n)\) an imaginary Killing section satisfying a chirality boundary-condition; see Example 4.10 and Corollary 4.12. Recalling (3.10), we set

\[
\mathcal{C}_b^+ = \{ V \in \mathcal{N}_b^+, \langle \langle V, V \rangle \rangle = 0, \langle \langle V, V(0) \rangle \rangle \geq 0 \}
\]

to be the future-pointing null cone.

**Proposition 5.1.** For any \(\Phi \in K^{b,+,(\pm)}(\mathbb{H}_+^n)\) we have that \(V_\Phi := \langle \Phi, \Phi \rangle \in \mathcal{N}_b^+\), and it satisfies \(\langle \langle V_\Phi, V_\Phi \rangle \rangle \geq 0\) and \(\langle \langle V_\Phi, V(0) \rangle \rangle \geq 0\). Moreover, every \(V \in \mathcal{C}_b^+\) can be written as \(V = V_\Phi\) for some \(\Phi \in K^{b,+,(\pm)}(\mathbb{H}_+^n)\). In particular, \(|\Phi|_b = O(r^{1/2})\).

**Proof.** If \(n\) is even so that \(\Phi = \phi_{u,+}\), direct computations starting from (4.13) show that for \(x' \in \mathbb{H}_+^n\),

\[
V_\Phi(x') = |u|^2 V(0)(x') + \sum_{i=1}^n \langle c^\delta(\partial x'_i) u, u \rangle V(i)(x')
\]

and \(\langle \langle V_\Phi, V_\Phi \rangle \rangle = |u|^4 + \sum_{i=1}^n \langle c^\delta(\partial x'_i) u, u \rangle^2\). However, the same calculation as that leading to (4.9) shows that \(\langle c^\delta(\partial x'_n) u, u \rangle = 0\). As shown in [8, Theorem 1], \(\langle \langle V_\Phi, V_\Phi \rangle \rangle\) is a nonnegative constant and of course the case \(\langle \langle V_\Phi, V_\Phi \rangle \rangle = 0\)
corresponds to \( V_{\Phi} \in C_b^+ \). Notice that the same conclusion holds if we had taken \( \Phi = \phi_{u,-} \) instead. If \( n \) is odd, (4.14) gives

\[
\langle \Phi, \Phi \rangle = \langle \phi_{u,+}, \phi_{u,+} \rangle + \langle \phi_{u,-}, \phi_{v,-} \rangle,
\]

which also proves the first assertions in those dimensions. Finally, the last assertion follows from the fact that \( V = O(r) \).

We now take a Killing section \( \Phi \in \mathcal{K}^{b,+,(\pm)}(\mathcal{E}\mathbb{H}^n_b) \) so that \( V_{\Phi} \in \mathcal{N}^b_+ \) as in Proposition 5.1. We may then extend the transplanted section \( \Phi_* \) to the whole of \( M \) so that the given chirality boundary condition is satisfied along \( \Sigma \). Also, a well-known computation shows that

\[
|\tilde{\nabla}^{E,+} \Phi_*|_g \leq C \left( |G - I|_b + |\nabla^b G|_b \right) |\Phi|_b = O(r^{-\tau + 1/2}),
\]

so that \( \tilde{\nabla}^{E,+} \Phi_* \in L^2 \) and we may apply Proposition 4.7 to obtain \( \Xi \in L^2(\mathcal{E}M) \) such that \( \tilde{D}^{E,+} \Xi = -\tilde{\nabla}^{E,+} \Phi_* \) and \( \Xi(\pm) = 0 \) along \( \Sigma \). Thus,

\[
\Psi_{\Phi} := \Phi_* + \Xi
\]

is Killing harmonic \( (\tilde{D}^{E,+} \Psi_{\Phi} = 0) \) and \( \Psi_{\Phi}(\pm) = 0 \) along \( \Sigma \). Moreover, by taking into account the identification between \( M \) and \( \mathbb{H}^n_b \) given by \( F \), we see that \( \Psi_{\Phi} \) asymptotes \( \Phi \) at infinity in the sense that \( \Psi_{\Phi} - \Phi \in L^2(M) \). We now state our first main result, which provides a Herzlich–Chruściel–Witten-type formula for the mass functional.

**Theorem 5.2.** With the notation above,

\[
\frac{1}{4} m_{(g,b,F)}(V_{\Phi}) = \int_M \left( |\tilde{\nabla}^{E,+} \Psi_{\Phi}|^2 + \frac{R_g + n(n - 1)}{4} |\Psi_{\Phi}|^2 \right) dM
\]

\[
+ \frac{1}{2} \int_\Sigma H_g |\Psi_{\Phi}|^2 d\Sigma,
\]

(5.5)

for any \( \Phi \in \mathcal{K}^{b,+,(\pm)}(\mathcal{E}\mathbb{H}^n_b) \).

For the proof we may assume that the chart at infinity is chosen as in Remark 3.5. After using (4.10) with \( \Omega = M_r \), the region in \( M \) bounded by \( \Sigma_r \cup S_{r,+}^{n-1} \), and \( \Psi = \Psi_{\Phi} \) we get

\[
\text{Re} \int_{S_{r,+}^{n-1}} \left\langle \tilde{\nabla}_\mu^{E,+} \Psi_{\Phi}, \Psi_{\Phi} \right\rangle dS_{r,+}^{n-1}
\]

\[
= \int_{M_r} \left( |\tilde{\nabla}^{E,+} \Psi_{\Phi}|^2 + \frac{R_g + n(n - 1)}{4} |\Psi_{\Phi}|^2 \right) dM_r
\]

\[
+ \frac{1}{2} \int_{\Sigma_r} H_g |\Psi_{\Phi}|^2 d\Sigma,
\]

and using that the second term in the right-hand side of (3.9) does not contribute to the mass, we need to check that

\[
\lim_{r \to +\infty} \text{Re} \int_{S_{r,+}^{n-1}} \left\langle \tilde{\nabla}_\mu^{E,+} \Psi_{\Phi}, \Psi_{\Phi} \right\rangle dS_{r,+}^{n-1}
\]

\[
= \lim_{r \to +\infty} \frac{1}{4} \int_{S_{r,+}^{n-1}} \langle U(V_{\Phi}, e), \mu \rangle dS_{r,+}^{n-1}.
\]

(5.6)
In fact, after splitting the integrand on the left-hand side by means of the decomposition $\Psi_\Phi = \Phi_* + \Xi$, we see that algebraic cancellations and the decaying properties of $\nabla^\Xi_\mu + \Phi_*$ and $\Xi$ imply that
\[
\lim_{r \to +\infty} \Re \int_{S_{r,+}^{n-1}} \langle \nabla^\Xi_\mu + \Psi_\Phi, \Psi_\Phi \rangle \, dS_{r,+}^{n-1} = \lim_{r \to +\infty} \Re \int_{S_{r,+}^{n-1}} \langle \nabla^\Xi_\mu + \Xi, \Phi_* \rangle \, dS_{r,+}^{n-1},
\]
so we shall focus on the last integrand.

To proceed we follow [25] and introduce the $(n-2)$-form
\[
\varepsilon = \langle (\ell_\alpha, \ell_m) \Phi_*, \Xi \rangle \, \ell_l \cdot \ell_m \, dM,
\]
where for simplicity we denote the Clifford multiplication by a dot.

**Lemma 5.3.** We have
\[
\lim_{r \to +\infty} \int_{S_{r}^{n-2}} \varepsilon = 0.
\]

**Proof.** It follows from (5.3) that $\ell_l \cdot \ell_j = f_i \cdot f_j + O(r^{-\tau})$, so if we again take $f$ as in Remark 3.5 and use that $f_n \cdot f_\alpha \cdot dM = dS_{r}^{n-2}$ we have that, restricted to the boundary,
\[
\varepsilon = -4 \langle \ell_\alpha \cdot \ell_n \cdot \Phi_*, \Xi \rangle dS_{r}^{n-2} + \langle O(r^{-\tau}) \cdot \Phi_*, \Xi \rangle dS_{r}^{n-2}.
\]
Using that $dS_{r}^{n-2} = O(r^{n-2})$, $|\Phi_*| = O(r^{1/2})$ and the decaying properties of $\Xi$, we see that the last term integrates to zero at infinity. On the other hand, recalling that $\ell_n = -\nu$ and both $\Phi_*$ and $\Xi$ satisfy the chirality boundary conditions along $\Sigma$, the remaining integrand equals
\[
4 \langle \ell_\alpha \cdot \nu \cdot \Phi_*, \Xi \rangle = 4 \langle \ell_\alpha \cdot \nu \cdot (\pm Q
\nu \cdot \Phi_*), \pm Q \nu \cdot \Xi \rangle = -4 \langle Q \ell_\alpha \cdot \Phi_*, Q \nu \cdot \Xi \rangle = -4 \langle \ell_\alpha \cdot \Phi_*, \nu \cdot \Xi \rangle,
\]
so that
\[
\langle \ell_\alpha \cdot \nu \cdot \Phi_*, \Xi \rangle = \langle \nu \cdot \ell_\alpha \cdot \Phi_*, \Xi \rangle. \tag{5.7}
\]
From this and Clifford relations we get
\[
\langle \ell_\alpha \cdot \nu \cdot \Phi_*, \Xi \rangle = \frac{1}{2} \langle (\ell_\alpha \cdot \nu + \nu \cdot \ell_\alpha) \Phi_*, \Xi \rangle = -\langle \ell_\alpha, \nu \rangle \beta \langle \Phi_*, \Xi \rangle = 0,
\]
which completes the proof. \hfill \Box

A straightforward computation gives
\[
d\varepsilon = 4 \left( \nabla^\Xi_\mu + \ell_l \Phi_*, \Xi \right) - \langle \Phi_*, \nabla^\Xi_\mu + \ell_l \Xi \rangle \, \ell_l \cdot dM,
\]
where by (4.2),
\[
\nabla^\Xi_\mu + \ell_l = -(\xi_{lm} + \ell_l \cdot \ell_m) \nabla^\Xi_\mu + \ell_m = -\frac{1}{2} [\ell_l, \ell_m] \nabla^\Xi_\mu + \ell_m. \tag{5.8}
\]
Hence,

$$\int_{S^{n-1}} (\tilde{\nabla} \epsilon_1 + (\epsilon_1) \Phi_*) \Xi \epsilon_1 \downarrow dM - \int_{S^{n-1}} (\tilde{\nabla} \epsilon_1 + (\epsilon_1) \Xi \Phi_*) \epsilon_1 \downarrow dM = \frac{1}{4} \int_{S^{n-2}} \epsilon,$$

where we used that $S^{n-2} = \partial S^{n-1}$. Thus,

$$\text{Re} \int_{S^{n-1}} \langle \tilde{\nabla} \epsilon_1 + \Xi, \Phi_* \rangle \epsilon_1 \downarrow dM = - \text{Re} \int_{S^{n-1}} \langle (\tilde{\nabla} \epsilon_1 + (\epsilon_1) \cdot \tilde{D} \epsilon_1 \Phi_*) \Xi, \Phi_* \rangle \epsilon_1 \downarrow dM$$

$$= - \text{Re} \int_{S^{n-1}} \langle \tilde{\nabla} \epsilon_1 + (\epsilon_1) \Phi_*, \Xi \rangle \epsilon_1 \downarrow dM + \frac{1}{4} \text{Re} \int_{S^{n-2}} \epsilon$$

$$= - \text{Re} \int_{S^{n-1}} \langle \epsilon_1 \cdot \tilde{D} \epsilon_1 \Xi, \Phi_* \rangle \epsilon_1 \downarrow dM$$

$$= - \text{Re} \int_{S^{n-1}} \langle \tilde{\nabla} \epsilon_1 + (\epsilon_1) \Phi_*, \Xi \rangle \epsilon_1 \downarrow dM + \frac{1}{4} \text{Re} \int_{S^{n-2}} \epsilon$$

$$+ \text{Re} \int_{S^{n-1}} \langle \epsilon_1 \cdot \tilde{D} \epsilon_1 \Phi_*, \Phi_* \rangle \epsilon_1 \downarrow dM,$$

where we used that $\Psi_\Phi$ is Killing harmonic in the last step. Again due to the decay properties and (5.8), the first integral in the right-hand side above vanishes at infinity, so recalling that $\epsilon_i = f_i + O(r^{-\tau})$ we finally obtain

$$\lim_{r \to +\infty} \text{Re} \int_{S^{n-1}} \langle \tilde{\nabla} \mu + \Xi, \Phi_* \rangle dS^{n-1} = \lim_{r \to +\infty} \text{Re} \int_{S^{n-1}} \langle \mu \cdot \tilde{D} \epsilon_1 \Phi_*, \Phi_* \rangle dS^{n-1},$$

where we used Lemma 5.3 to get rid of the integral involving $\epsilon$. Now, a standard computation as in [12, Section 4] shows that

$$\lim_{r \to +\infty} \text{Re} \int_{S^{n-1}} \langle \mu \cdot \tilde{D} \epsilon_1 \Phi_*, \Phi_* \rangle dS^{n-1} = \lim_{r \to +\infty} \frac{1}{4} \int_{S^{n-1}} \langle \mathcal{U}(V_\Phi, \epsilon), \mu \rangle dS^{n-1},$$

which proves (5.6) and completes the Proof of Theorem 5.2.

We now explain how the mass formula (5.5) implies Conjecture 3.6 in case $M$ is spin. More precisely, we have the following result.

**Theorem 5.4.** Let $(M, g, \Sigma)$ be an asymptotically hyperbolic spin manifold with $R_g \geq -n(n-1)$ and $H_g \geq 0$. Then for any admissible chart $F$ the mass vector $\mathcal{P}^{[F]}$ is time-like future directed, unless it vanishes and $(M, g, \Sigma)$ is isometric to $(\mathbb{H}^n_+, b, \partial \mathbb{H}^n_+)$. 

**Proof.** From (5.5) and the assumptions $R_g \geq -n(n-1)$ and $H_g \geq 0$ we see that

$$\langle \langle \mathcal{P}^{[F]}, V \rangle \rangle \geq 0$$

(5.9)
for any $V = V_\Phi$ with $\Phi \in K^{b,+,(\pm)}(\mathcal{E}\mathbb{H}_n^+)$.

In particular, (5.9) holds for any $V \in C^+_b$ since, by Proposition 5.1, any such $V$ can be written as $V = V_\Phi$ for some $\Phi \in K^{b,+,(\pm)}(\mathcal{E}\mathbb{H}_n^+)$.

Hence, $\mathcal{P}[F]$ is time-like and future directed unless there exists some $V = V_\Phi \neq 0$ so that the equality holds in (5.9).

This last possibility implies by (5.5) that there exists a non-trivial Killing section $\tilde{\Psi} = \Psi_\Phi$ on $M$ satisfying the corresponding chirality boundary condition along $\Sigma$. In particular, $g$ is Einstein with $\text{Ric}_g = -(n-1)g$ so that $\tilde{G}_g = 0$ as in Remark 3.8. On the other hand, for any $X \in \Gamma(T\Sigma)$ we have, upon derivation of $Q\Psi = \pm \Psi$,

$$Qc^E(\nabla_X \nu)\Psi = -Qc^E(\nu)\tilde{\nabla}_X^E\Psi \pm \tilde{\nabla}_X^E\Psi = 0.$$  

Thus,

$$\langle (\nabla_X \nu) \mid \Psi \mid^2 = 2\langle \nabla_{\nabla_X \nu} \Psi, \Psi \rangle = -i\langle c^E(\nabla_X \nu)\Psi, \Psi \rangle = 0,$$

and since, for a nonzero $\Psi$, $|\Psi|^2$ is known to vary exponentially along any geodesic segment, this yields a contradiction unless $\nabla_X \nu = 0$, that is, $\Sigma$ is totally geodesic. Thus, by our alternate definition (3.11) of the mass functional we see that $\mathcal{P}[F]$ vanishes and hence the equality in (5.9) holds for any $V = V_\Phi$ with $\Phi \in K^{b,+,(\pm)}(\mathcal{E}\mathbb{H}_n^+)$. By Corollary 4.12 and (5.5), this means that $(M, g)$ carries as many Killing sections to the number $i/2$ (i.e. parallel sections for the connection $\tilde{\nabla}^E$+) as the reference space $(\mathbb{H}_n^+, b, \partial \mathbb{H}_n^+)$, which implies that $g$ is locally hyperbolic. Moreover, by (4.4), Remark 4.2 and Corollary 4.12, the restrictions of these Killing sections to $\Sigma$ generate a space of imaginary Killing spinors on $\Sigma$ with maximal dimension. In particular, this implies that $\Sigma$ has no compact components and so it is connected by Remark 2.4. Also, a direct application of Gauss formula shows that $R_\gamma + (n-1)(n-2) = 0$ so $(\Sigma, \gamma)$ is asymptotically hyperbolic as a boundaryless manifold (in the sense of [12]). A well-known result by H. Baum [8] implies that $(\Sigma, \gamma)$ is isometric to $(\mathbb{H}^{n-1}, \beta)$. Now, we double the manifold $(M, g, \Sigma)$ along $\Sigma$ obtaining a locally hyperbolic manifold $(\tilde{M}, \tilde{g})$ which is asymptotically hyperbolic as a boundaryless manifold. Standard topological arguments show that $(\tilde{M}, \tilde{g})$ is isometric to $(\mathbb{H}^n, b)$ and finally $(M, g, \Sigma)$ is isometric to $(\mathbb{H}_n^+, b, \partial \mathbb{H}^n_+)$.  

Clearly, Theorem 1.1 is an immediate consequence of Theorem 5.4. Also, Theorem 1.2 follows promptly from (3.11), Remark 3.8 and Theorem 5.4, and similarly for Theorem 1.3, which follows from the corresponding boundaryless statements.

We next present the Proof of Theorem 1.6. Since $g$ is Einstein, if $n = 3$ the result follows from Theorem 1.1. If $n \geq 4$ the well-known computation in [3] shows that the contribution for the mass functional coming from integration over the asymptotic hemisphere $S_\gamma^{n-1}$ vanishes. Thus, by (3.9) and the fact that $t \approx r^{-1}$, it remains to check that the asymptotic integral over $S^{n-2} = \partial S_\gamma^{n-1}$ vanishes as well, that is,

$$\lim_{t \to 0} \int_{S_t^{n-2}} V e_{1n} dS_t^{n-2} = 0,$$
where $V \in \mathcal{N}^+_b$ and we have chosen the frame $\mathfrak{f}$ so that $f_1 = \sinh t \partial_t = \vartheta$ and $f_n = -\eta$. Now recall that we can set up a gauge map $\mathcal{G}$ in the asymptotic region so that

$$\mathcal{G} f_i = f_i - \frac{1}{2} \mathcal{H} f_i + \mathcal{R} f_i,$$

is an orthonormal $g$-frame, where $\mathcal{H} = O(t^{n-2})$ and $\mathcal{R} = o(t^{n-1})$ by (1.6). But notice that (1.3) clearly leads to $\mathcal{G} f_1 = f_1$ so we actually have $\mathcal{H} f_1 = o(t^{n-1})$. Thus, by the analogue of (5.4),

$$e_{1n} = \langle \mathcal{H} f_1, f_n \rangle_g + o(t^{n-1}) = o(t^{n-1}).$$

Since $V = O(t^{-1})$ and $dS_t^{n-2} = O(t^{2-n})$,

$$\int_{S_t^{n-2}} V e_{1n} dS_t^{n-2} = o(1),$$

which proves the claim and finishes the Proof of Theorem 1.6.

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