Topological and Nontopological Edge States Induced by Qubit-Assisted Coupling Potentials

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In the usual Su–Schrieffer–Heeger (SSH) chain, the topology of the energy spectrum is divided into two categories in different parameter regions. Here, the topological and nontopological edge states induced by qubit-assisted coupling potentials in circuit quantum electrodynamics (QED) lattice modeled as a SSH chain are studied. It is found that, when the coupling potential added on only one end of the system raises to a certain extent, the strong coupling potential will induce a new topologically nontrivial phase accompanied by the appearance of a nontopological edge state, and the novel phase transition leads to the inversion of odd–even effect directly.

Furthermore, the topological phase transitions when two unbalanced coupling potentials are injected into both ends of the circuit QED lattice are studied, and it is found that the system exhibits three distinguishing phases with multiple flips of energy bands. These phases are significantly different from the previous phase induced via unilateral coupling potential due to the existence of a pair of nontopological edge states. The scheme provides a feasible and visible method to induce different topological and nontopological edge states through controlling the qubit-assisted coupling potentials in circuit QED lattice both in experiment and theory.

1. Introduction

The discovery of topological insulators[1,2] establishes a new path for the study of novel quantum phase transitions of the matters. The SSH model,[3] which is viewed as the simplest 1D tight-binding model with the staggered dimerized hopping bonds and possesses topologically trivial and nontrivial phases simultaneously, is widely investigated in recent years. In the context of usual SSH chain, various kinds of backgrounds have been investigated in detail, including topological phase transition,[4–10] topological properties and $PT$ symmetry effect,[11–15] multibody effect of Hubbard model[16] and other interactions,[17–20] topological edge states and invariants,[21–24] simulation of SSH chain,[25–28] etc.

Especially, the topological edge states and the nontopological bound states in SSH chains, based on various system, have also been widely investigated. For example, the topological Floquet edge states based on the $z$-axis periodically curved waveguides have been reported,[29] in which the coexistence of topological and nontopological edge states is analytically obtained. The chiral bound states in the photonic analog of the Su–Schrieffer–Heeger model have also been explored based on the topological waveguide quantum electrodynamics bath,[30] in which the bound state can give rise to exotic many-body phases. Further, the multiple topological edge states of the two interacting electrons based on the driven quantum dot arrays have also been investigated,[31] in which the driving allows to suppress all the undesired hopping processes and enhance the necessary ones to ensure the topological protection.

We note that these previous works are mainly based on the curvature design of the optical waveguides in the third dimension[29] or based on the time-dependent driving terms,[31] which might greatly or marginally increase the complexity of experiments. Here, inspired by the above previous works, we propose a scheme to investigate the topological edge states and the nontopological boundary states based on a 1D simple modulated superconducting circuit QED lattice. Superconducting circuit QED[32–35] lattice, recently, is being one of the most appealing and promising candidates for the quantum simulation of bosonic systems[36–41] with the fast-developing fields of micro–nanomanufacturing and materials processing technology. Compared with waveguides and atom systems, the circuit QED system has the following advantages: i) the electric dipole moments between the superconducting circuit qubits are much stronger, which enables the ultrastrong coupling possible; ii) it does not need ultralow temperature cooling and possesses longer interaction time between cavity field and qubits; iii) the energy level spacing of superconducting qubits is highly controllable in experiment, which can achieve the periodic change of circuit parameters; iv) the setup of the superconducting circuit...
QED system can be easily built under the present technology conditions. Many schemes have been put forward to study the topological properties of matters based on circuit QED lattice system. Mei et al. have proposed schemes to simulate and detect the photonic chern insulator\(^{[42]}\) and Weyl semimetal phase\(^{[43]}\) by periodically varying the coupling parameters based on a 1D circuit QED lattice. Li et al. have proposed a scheme to explore photonic topological insulator states in a circuit-QED lattice.\(^{[44]}\) Kapit et al.\(^{[45]}\) have studied the fractional quantum Hall states of light. Fitzpatrick et al.\(^{[46]}\) have reported the observation of a dissipative phase transition in a 1D circuit QED lattice.

In this paper, resorting to circuit QED lattice lattice, we propose a scheme to study the topological and nontopological edge states as well as phase transitions induced by unilateral and unbalanced bilateral qubit-assisted coupling potentials. We find that the present circuit QED lattice system possesses topologically different edge states and experiences different phase transitions with the varying of the coupling potentials. When the unilateral coupling potential is added into the system, comparing with the usual SSH chain, the present system exhibits a new topological nontrivial phase accompanied with a nontopological edge state under strong unilateral coupling potential regime. The difference between the new topological and the usual SSH topological phase is that the new phase is topologically nontrivial in the whole hopping parameter region corresponding to an even number of lattice sites. When the two unbalanced bilateral coupling potentials are added into the system, the system has two nontopological edge states appearing in the energy spectrum under strong unbalanced bilateral coupling potentials limit. Accordingly, the energy bands of the system experiences multiple inversions with the change of the bilateral coupling potentials, and the new phases occur at the point of energy band inversion. We find that, in contrast to the usual SSH model, the present system possesses the nontopological and topological edge states simultaneously under strong unilateral and unbalanced bilateral coupling potential regimes.

The paper is organized as follows: In Section 2, we derive the effective Hamiltonian of the qubit-assisted circuit QED lattice system under periodic boundary condition. In Section 3, we study the effects of the unilateral and unbalanced bilateral coupling potentials. Finally, a conclusion is given in Section 4.

### 2. System and Hamiltonian

We consider a circuit QED lattice composed of \(N\) transmission line resonators, as depicted in Figure 1. In this lattice, each resonator module contains a two-level qubit \(q_n\) with coupling strength \(g_n\), and the two adjacent resonators are coupled via another two-level qubit \(Q_n\). Under periodic boundary condition (connecting two ends of the circuit QED lattice system via qubit \(Q_n\)), the system can be described by the Hamiltonian.

\[
H = H_0 + H_{\text{inter}} + H_{\text{disp}}
\]

where

\[
H_0 = \sum_n \left[ \omega_n a_n^\dagger a_n + \frac{\omega_0}{2} \sigma_{z,n} \right]
\]

\[
H_{\text{inter}} = \sum_n \left[ \Omega_n a_n e^{i\omega_0 t} + \Omega_n^* a_n^\dagger e^{-i\omega_0 t} \right]
\]

\[
H_{\text{disp}} = \sum_n \left[ G_n \left( \sigma_{+n} a_n + \sigma_{-n} a_n^\dagger \right) + G_{n+1} \left( \sigma_{+n+1} a_{n+1} + \sigma_{-n-1} a_{n-1}^\dagger \right) \right]
\]

Here, \(H_0\) represents the free energy of resonators and qubits, \(\omega_0\) \((\omega_0)\) is the energy level spacing of the two-level qubit \(q_n\) \((Q_n)\), \(\omega_0\) is the frequency of transmission line resonator \(a_n\), \(\sigma_{\pm} = (\sigma_{z} \pm i\sigma_{x})\) is Pauli z operator. \(H_{\text{inter}}\) represents that each resonator is driven by an external field with frequency \(\omega_0\) and strength \(\Omega_n\). \(H_{\text{disp}}\) describes the interaction between resonator and qubit \(q_n\) in the \(n\)th resonator module, in which \(g_n\) is the fixed basic coupling strength and \(g_n\) represents that the coupling strength in the \(n\)th resonator module can be modulated on the basis of \(g_n\). \(H_{\text{disp}}\) denotes the coupling between two adjacent resonators via qubit \(Q_n\) with the fixed and modulated coupling strength \(G_n\) and \(G_n\), and \(\sigma_{\pm} = (|e\rangle\langle h| + |h\rangle\langle e|)\) \((|e\rangle\langle h|, |g\rangle\langle e|)\) with \(|g\rangle\) and \(|e\rangle\) being the ground and excited states of a qubit.

For simplicity, we set \(g_0 = G_0 = 1\) as the energy unit. In the interaction picture with respect to free energy, the interaction Hamiltonian in the dispersive regime is written as

\[
H_{\text{int}} = H'_0 + H'_{\text{inter}} + H'_{\text{disp}},
\]

with

\[
H'_0 = \sum_n \Delta_n a_n^\dagger a_n
\]

\[
H'_{\text{inter}} = \sum_n \left[ \frac{\Delta_n^2}{\Delta} \langle |e\rangle \langle h| \langle e| a_n^\dagger a_n - |g\rangle \langle h| a_n^\dagger a_n \rangle + \frac{\Delta_n^2}{\Delta} \langle |g\rangle \langle e| a_n^\dagger a_n - |e\rangle \langle h| a_n^\dagger a_n \rangle \right]
\]

\[
H'_{\text{disp}} = \sum_n \left[ \frac{2G_n^2}{\Delta} \langle |e\rangle \langle h| \langle e| a_n^\dagger a_n - |g\rangle \langle h| a_n^\dagger a_n \rangle + \frac{2G_n^2}{\Delta} \langle |g\rangle \langle e| a_n^\dagger a_n - |e\rangle \langle h| a_n^\dagger a_n \rangle \right]
\]
the qubits between resonators and qubits can be removed by preparing all
under open boundary condition, the total effective Hamiltonian
energy with respect to
\( q_n \) where

\[ H = H_0 + H_1 \]

(2)

where \( \Delta_e = \omega_e - \omega_d \) is the detuning of the resonator frequency,
\( \Delta_i = \omega_i - \omega_d \). \( \Delta_e = \omega_e - \omega_d \) represents the energy detuning
between the qubit \( q_i \) and \( q_d \). The couplings between resonators and qubits can be removed by preparing all
the qubits \( q_i \) (\( Q_i \)) in their ground states, thus the total effective Hamiltonian of the system can be expressed as

\[ H_{\text{eff}} = \sum_n (\Delta_e - \frac{\beta_n^2}{\Delta_e} - \frac{G_2^2}{\Delta_e}) a_n^{\dagger} a_n \]

(3)

where the first term denotes the onsite energy assisted by qubits \( q_n \) and \( Q_n \), and the second term denotes the coupling between two adjacent resonators assisted by qubit \( Q_n \). On the other hand, under open boundary condition, the total effective Hamiltonian can be written as

\[ H = \left( \Delta_e - \frac{\beta_n^2}{\Delta_e} - \frac{G_2^2}{\Delta_e} \right) a_n^{\dagger} a_n + \sum_{n=1}^{N-1} \left( \Delta_e - \frac{\beta_n^2}{\Delta_e} - \frac{G_2^2}{\Delta_e} \right) a_n^{\dagger} a_n \]

(4)

For simplicity, we make the following substitutions as

\[ -\frac{G_2 \cdot G_n \cdot \Delta e}{\Delta e} = t_1, \quad -\frac{G_2 \cdot G_n \cdot \Delta e}{\Delta e} = t_2, \quad -\frac{\beta_n^2}{\Delta_e} = V_1, \quad \text{and} \quad -\frac{\beta_n^2}{\Delta_e} = V_2. \]

We can always find a set of parameters to satisfy

\[ \Delta_e = \Delta_{e \text{odd}} = \frac{2t_1}{V_1}, \quad \Delta_{e \text{even}} = \frac{2t_2}{V_1}. \]

After resetting the zero point of energy with respect to \( \Delta_e \), the Hamiltonian becomes

\[ H = V_1 a_n^{\dagger} a_n + V_2 a_n^{\dagger} a_n + \sum_{n=1}^{N} t_1 (a_{n+1}^{\dagger} a_n + a_n^{\dagger} a_{n+1}) \]

(5)

where the first (second) term represents onsite modulation potential assisted by the qubits \( q_i \) (\( q_n \)) and \( Q_i \) (\( Q_n \)). Obviously, if the first two terms are removed, the Hamiltonian will be equivalent to a standard SSH model. In the following we study the effect of the onsite potentials induced by qubit-assisted coupling on the topology of the present system and reveal the role of the onsite modulation potentials in detail.

### 3. Topological Phases Induced by Coupling Potentials

Consider a generalized SSH model, in which the parameters are taken as \( t_1 = g_1 (1 + 0.5 \cos \theta) = 1 + 0.5 \cos \theta \) and \( t_2 = g_2 (1 - 0.5 \cos \theta) = 1 - 0.5 \cos \theta \). \( \theta \) represents a periodic parameter in the range of \( \theta \in [0, 2\pi] \). Experimentally, this parameter hopping between adjacent resonators can be realized via adiabatically controlling the coupling between resonator and qubit \( Q_n \). Just as discussed above, the Hamiltonian can be viewed as a modulated SSH model after dropping the first two terms, in which the Hamiltonian of the modulated SSH model in the momentum space can be written as

\[ H_{\text{eff}} = \frac{Z}{2 \pi} \sum_{\ell=\text{even}} \sigma_{\ell} H_{\ell} + \sigma_1^z H_{1} \]

(6)

With the choice of \( V_1 = 0 \), the system possesses two degenerate topological zero energy states for \( \theta \in [0.5\pi, 1.5\pi] \), and it is topologically trivial for \( \theta \in [0, 0.5\pi] \cup [1.5\pi, 2\pi] \). A mild coupling potential added on the leftmost resonator splits the two original zero energy states from degeneracy when \( \theta \in [0.5\pi, 1.5\pi] \), as shown in the top panel of Figure 2a. And the separated edge state is localized near the leftmost (rightmost) site accompanying with the exponential attenuation (amplification) distributions at the odd (even) sites, as shown in the top panel of Figure 2b. We stress that the two separated gap states are topological states since they both are protected by the gap and possess the exponential distribution.

With the potential strength increasing continuously, we find that the left topological edge state (green line) gradually integrates into the bulk states in \( \theta \in [0.5\pi, 1.5\pi] \) accompanying with the appearances of a state (red line) in the gap and an isolated bulk state (blue line) on the top band in \( \theta \in [0.5\pi, 2\pi] \), as shown in the middle panel of Figure 2a. Especially, when the
coupling potential strength exceeds a certain value, the initial left topological edge state (green line) integrates into the bulk states completely in $\theta \in [0.5\pi, 1.5\pi]$, while a gap state (red line) and an isolated bulk state (blue line) appear in $\theta \in [0, 2\pi]$, as shown in the bottom panel of Figure 2a. We stress that the gap state (red line) and the isolated bulk state (blue line) are topologically different, since the gap state is protected by the gap and the isolated bulk state (blue line) is not protected by the gap. Thus, from the perspective of the topological protection, we treat the gap state as topological state and treat the isolated bulk state as nontopological state.

To further reveal the insightful topological difference, we plot the distributions of gap state (red line) and the isolated bulk state (blue line) when $\theta = 0.1\pi$, as shown in the middle panel of Figure 2b. Obviously, the blue isolated bulk state is always localized near the leftmost resonator with the bound state distribution, while the red gap state has the maximal distribution at the second resonator accompanying with the exponential attenuation distributions at even sites. Thus, together with the gap protection, the bounded characteristic at the first resonator of the blue isolated bulk state further verifies that the isolated bulk state is the nontopological edge state. Similarly, the exponential distribution near the second resonator of the red gap state for $\theta = 0.1\pi$ further implies that it is the topological edge state. In this way, we can identify the nontopological and topological edge states respectively. We stress that, the reason for the red topological edge state with the maximal distribution at the second resonator rather than the first resonator when $\theta = 0.1\pi$ can be explained as following. When the coupling potential strength $V_1$ far exceeds a certain strength, the leftmost resonator decouples from the rest of $N - 1$ resonators. The decoupling leads that the first resonator is bounded at the in situ corresponding to the nontopological edge state. Meanwhile, the decoupling also leads that a new topological left edge occurs at the second resonator, which interprets the reason that the red gap state has the maximal distribution at the second resonator accompanying with the exponential attenuation distributions at even sites. We also plot the distributions of the gap state (red line) and the isolated bulk state (blue line) when $\theta = \pi$, as shown in the bottom panel of Figure 2b. The results reveal that the system possesses a nontopological edge state and a topological right edge state simultaneously.

The more precise behavior of the phase transition for the system is given by the inverse participation ratio (IPR) of the nontopological and topological edge states with $\text{IPR} = \sum_{n=1}^{N} (|\langle \Psi_n | \Psi_n \rangle|^2)^{1/4}/(|\langle \Psi_n | \Psi_n \rangle|^2)$, in which $j$ represents the lattice index and $|\Psi_n \rangle$ is the corresponding nontopological and topological edge states. The numerical results are shown in Figure 2c. The different regions in Figure 2c represent three different phases of the system. In region I, the lattice system possesses topologically nontrivial edge states. On the contrary, the system is always topologically trivial in region II. A new novel phase transition occurs in region III, in which the system possesses topological and nontopological edge states simultaneously. Specially, we find that the phase boundary between region III and other regions approximately satisfies $V = \theta_i = 1 - 0.5 \cos \theta$. When $\theta = \pi$, the corresponding maximal value of the phase boundary is $V = 1.5$. Thus, in region III, the topological and nontopological edge states appear in the whole $\theta \in [0, 2\pi]$ at the same time when the coupling potential strength satisfies $V_1 > 1.5$.

This special phase can be explained as odd–even effect inversion, which means that the present system possesses a topological energy level in the whole energy gap for the even number of lattice sites (the usual SSH model has a topological state in the whole energy gap for odd number of lattice sites), as shown in Figure 3. The odd–even effect inversion originates from the influence of a strong enough coupling potential $V_1$. As discussed above, the leftmost resonator will decouple from the rest of lattice chains under strong coupling potential regime, leading to the size of the rest of chains to be $N - 1$, which is odd for even $N$. The topological edge states appearing in the whole energy gap is widely applied to realizing robust quantum state transfer.\cite{47} Thus our scheme will provide a controllable technique to realize the quantum information processing using odd–even effect inversion by adjusting the coupling potential $V_1$ appropriately.
At the end of this part, we include some additional results. In all of the above cases, the value of $V_i$ is considered to be positive. It is worth emphasizing that the results are analogous for the case of the negative value of $V_i$. The difference is that the phase diagram related to negative $V_i$ is mirror symmetric on the line $V_i = 0$ compared to the case of positive $V_i$. And for the case of $V_i = 0$ and $V_2 \neq 0$, the final results are similar to the case that the coupling potential is added on the leftmost resonator.

3.2. Bilateral Unbalanced Coupling Potential Pattern

Without loss of generality, in the following we discuss the effects of the unbalanced coupling potentials added on the two ends of the system. Here we choose the two unbalanced potential strengths to be $V_i = V \cos \phi$ and $V_2 = V \sin \phi$, with $\phi$ being another periodic parameter varying in $\theta \in [0, 2\pi]$ and $V$ being a positive fixed value. Then the Hamiltonian of the system is given by

$$H = V \cos \phi a_{j=1}^\dagger a_j + V \sin \phi a_{j=N}^\dagger a_N$$

$$+ \sum_{j=odd} (1 + 0.5 \cos \theta)(a_{j=1}^\dagger a_j + a_{j=N}^\dagger a_{j+1})$$

$$+ \sum_{j=even} (1 - 0.5 \cos \theta)(a_{j=1}^\dagger a_j + a_{j=N}^\dagger a_{j+1})$$

(7)

One of the advantages for the choice of the coupling potential parameters is that the unbalanced bilateral coupling potential pattern can be transformed into the unilateral coupling potential pattern by choosing the parameter as $\phi = 0$ ($0.5\pi, \pi, 1.5\pi, 2\pi$). The system will display a more complex and fascinating phase transition for other values of $\phi$.

Under the weak coupling potential regime, the system always has two topological left and right edge states appearing in $\theta \in [0.5\pi, 1.5\pi]$ corresponding to all the values of $\phi$, as shown in Figure 4a,b. However, as shown in Figure 4c,d, the distributions of nontopological edge states are so slight that they are almost close to zero, which means that there is no nontopological edge states existing in the system. The numerical results obtained above are similar with the conclusions obtained in the unilateral coupling potential pattern under weak potential regime. The main difference between the unilateral and bilateral cases is the relative shifts of the positions of topological energy levels. More accurately, compared to the unilateral case, both of the topological energy levels will produce relative shifts at the same time in the bilateral case, as shown in Figure 4e–g. Specially, both of the separated topological edge states will become degenerate again corresponding to some specific values of $\phi$ (such as $\phi = 0.25\pi$ and $\phi = 1.25\pi$), as shown in Figure 4f.

On the other hand, under the strong coupling potential regime, the system experiences multiple flips of energy bands with the varying of the parameter $\phi$. For example, in the case of $\phi = 0$ ($2\pi$), the system has one topological and one nontopological edge energy levels appearing in the whole $\theta \in [0, 2\pi]$. Accompanied with parameter $\phi$ raising from 0 to 0.25$\pi$, the $(N/2)$th green topological edge energy level integrates into the top energy band gradually in $\theta \in [0.5\pi, 1.5\pi]$ and moves toward the bottom band slowly in $\theta \in [0, 0.5\pi] \cup [1.5\pi, 2\pi]$, as shown in Figure 5a. Finally, the $(N/2)$th green topological edge energy level becomes degenerate with the $(N/2 - 1)$th red topological edge energy level which separates from bottom energy band in $\theta \in [0, 0.5\pi] \cup [1.5\pi, 2\pi]$. In the meanwhile, the $(N - 1)$th purple nontopological edge energy level separates from the top band gradually and the Nth blue nontopological edge energy level is close to the top band. Finally, the two nontopological edge energy levels become degenerate, as shown in Figure 5b. At the point $\phi = 0.25\pi$, a band inversion occurs for the energy spectrum, which means that the initial $(N/2)$th green topological edge energy level converts into the red topological edge energy level, and vice versa, as shown in Figure 5b. This band inversion means that the distributions of the corresponding topological edge states are reversed. The same phenomenon also occurs for both of the nontopological edge states. The arrows in Figure 5a–c show the inversion directly.

When the parameter $\phi$ changes from 0.25$\pi$ to 0.5$\pi$, the $(N/2)$th red topological energy level moves upward the upper energy band in $\theta \in [0, 0.5\pi] \cup [1.5\pi, 2\pi]$ and moves toward the bottom energy band in $\theta \in [0.5\pi, 1.5\pi]$. At the same time, the

![Figure 3. Odd–even effect inversion in strong coupling potential regime ($V_1 = 4g_0$). a) The energy spectrum of the system when the number of lattice site is $N = 100$. There exists a special energy level appearing in the whole energy gap. b) The energy spectrum of the system corresponding to $N = 99$. The energy gap only has the gap states in the part regions of the parameter $\theta$.](image-url)
(N/2 – 1)th green topological energy level is close to the bottom energy band and finally integrates into the bulk band in \( \theta \in [0, 0.5\pi] \cup [1.5\pi, 2\pi] \). The two degenerate nontopological edge energy levels move toward the opposite directions on the top of the upper energy band. And finally, the (N – 1)th blue nontopological energy level integrates into the upper band completely. The energy bands are reversed again at the point of \( \phi = 0.5\pi \), and the difference is that the (N/2)th red topological and the Nth purple nontopological energy levels do not inverse in the energy spectrum compared with the first band inversion, as shown in Figure 5d. More specifically, the (N/2 – 1)th green topological energy level flips into the (N/2 + 1)th bulk state, and the (N – 1)th blue nontopological energy level flips into the first bulk energy level of the bottom band.

With the parameter \( \phi \) raising from 0.5\( \pi \) to \( \pi \) sequentially, the (N/2)th red topological energy level integrates into the bottom band slowly and the (N/2 + 1)th green topological edge energy level separates from the upper band, as shown in Figure 5e. In the meanwhile, the Nth purple nontopological energy level is close to the upper band, and the blue nontopological energy level separates from the bottom band. Especially, the energy spectrum of the system becomes symmetrical on the zero energy surface when the parameter \( \phi \) is taken as \( \phi = 0.75\pi \), in which the system has a pair of topological and nontopological edge states respectively in \( \theta \in [0, 0.5\pi] \cup [1.5\pi, 2\pi] \) and only possesses a pair of nontopological edge states in \( \theta \in [0.5\pi, 1.5\pi] \), as shown in Figure 5f. The point of \( \phi = \pi \) is another critical point of energy band inversion, at which the (N/2 + 1)th green topological energy level remains unchanged and the (N/2)th red topological energy level flips into the (N/2 + 2)th bulk energy level, as shown in Figure 5g.h.

We emphasize that the process of energy band inversion is analogous to the case of \( \phi \in [0, \pi] \) when the parameter \( \phi \) belongs to \( [\pi, 2\pi] \). Now we give a brief summary. With \( \phi \) raising from 0 to 2\( \pi \), the number index of the green topological energy level experiences the following changes: \( N/2 \to N/2 – 1 \to N/2 + 1 \to N/2 + 2 \to N/2 \); the red topological energy level experiences the process of \( N/2 – 1 \to N/2 \to N/2 + 2 \to N/2 + 1 \to N/2 – 1 \); the purple nontopological energy state experiences the process of \( N – 1 \to N \to 2 \to N – 1 \); and the blue nontopological energy state experiences the process of \( N \to N – 1 \to 1 \to 2 \to N \).

Furthermore, we find that the cases of energy band inversion at the points of \( \phi = 0.25\pi \) (1.25\( \pi \)) and \( \phi = 0.5\pi \) (1.5\( \pi \), 2\( \pi \)) are significantly different, in which the former has two topological energy levels flipping at the same time and the latter only has one topological energy level being reversed. The reason is that, when the parameter satisfies \( \phi = 0.5\pi \) (1.5\( \pi \), 2\( \pi \)), the system simplifies to the unilateral coupling potential pattern, in which the left-most resonator decouples from the resonator chain due to the large unilateral coupling potential. While for \( \phi = 0.25 \) (1.25\( \pi \)), both the ends of the resonators decouple from the rest of lattice chains with \( N – 2 \) even number of lattice sites when the two strong balanced coupling potentials are added at both the ends of the system, and the physical structure in this case is completely different with the unilateral coupling potential pattern.

The multiple band inversions imply that the present system may possess several phases. To further clarify the results obtained above, we calculate the IPR for the corresponding nontopological and topological levels. The phase diagram of IPR is depicted in Figure 6. With the parameter \( \phi \) raising from 0 to \( \pi \), the system exhibits three different kinds of phases. In region 1, the system possesses a nontopological edge state and a topological edge state in the whole \( \theta \in [0, 2\pi] \). In region II, two topological edge states and two nontopological edge states appear in the energy spectrum of the system simultaneously. In region III, the system only has two nontopological edge states. We should stress that we only study the behavior of phase transition in the region of \( \phi \in [0, \pi] \), as for \( \phi \in [\pi, 2\pi] \), the results are homologous, thus no longer make discussion in detail.
3.3. Unbalanced Coupling Potential Added on Bulk

The advantage of the present circuit QED lattice system is that the coupling potential strength can be controlled at a single site level, meaning that the two coupling potentials $V \cos \phi$ and $V \sin \phi$ can also be added on arbitrary bulk sites by choosing the coupling parameters appropriately. When $\phi = 0$ and $N = 100$, we consider the coupling potential being injected into the $(N/2)$th bulk site. The corresponding energy spectrum and distributions of the states are shown in Figure 7a,b. One can see that there are two topological left (right) edge states appearing in the energy gap, which are localized at $j = 1(49)$ and $j = 51(100)$. In addition, there also exists a bound state localized at $j = 50$. The reason is that the existence of strong coupling potential separates the circuit QED lattice chain into two subchains at site $j = 50$. The two separated subchains exhibit a new topological right and left edge states, respectively. The numerical results are similar with the case of $N = 100$ when the parameters are chosen as $\phi = 0$ and $N = 101$, as shown in Figure 7c,d.

When $\phi = 0.25 \pi$ and $N = 100$, we consider that the two coupling potentials are added on the $(N/2)$th and the $(N/2 + 1)$th bulk sites, in which the system still has four topological edge states but has two bound states localized at $j = 50$ and $j = 52$ bulk sites, we find that there exists two pairs of topological states and two separated bound states in $\theta \in [0, 2\pi]$. In the meantime, a new localized state appears with being located at $j = 51$, as shown in Figure 8a,b. When the size of the lattice is set to $N = 101$ and the coupling potentials are added on $j = 50$ and $j = 52$ bulk sites, we find that there exists two pairs of topological states and two separated bound states in $\theta \in [0, 2\pi]$. The system is symmetrical on the zero energy level, and the system still has two pairs of topological edge states as well as two bound states in the whole gap, as shown in Figure 8e,f. The difference is that the localization of the isolated bulk site $j = 51$ becomes much more larger compared to the case of $\phi = 0.25 \pi$. We stress that the two strong coupling potentials can also be added on other bulk sites (such as $j = 37$ and $j = 68$ for $N = 101$), in which the system decouples from the three subchains with different lattice lengths. In this case, the system may possess more abundant phenomena.
3.4. Detections of Topological and Nontopological Edge States in Circuit QED Lattice System

In circuit QED lattice system, the bosonic photons in the resonators can occupy one particular eigenmode at the same time, which means that the topological and nontopological edge states can be directly detected in present photonic lattice system. According to the ref. [42], the detection of the edge and the bulk states can be realized via the input–output formalism. More specifically, the detection of the edge states can be realized by externally driving the resonators with the eigenenergy of the lattice. In this way, the distribution characteristics of the edge states can be revealed via the expectation value of the cavity fields. Under the strong unilateral coupling potential pattern in Figure 3a, the bosonic photons gather in the first resonator when the leftmost resonator is driven by the external laser whose driving frequency is tuned to the nontopological edge state energy, as shown in Figure 9a. The distribution signal is consistent with the characteristic of the left nontopological edge state. On the contrary, the distribution of the photons will possess maximal value at the second resonator when the same leftmost resonator is driven by the external laser whose driving frequency is tuned to the topological edge state energy, as shown in Figure 9b. Experimentally, we can realize the detection of the nontopological and topological edge states utilizing these peak signals of the bosonic photon distributions.

4. Conclusions

In conclusion, we have proposed a scheme for the investigation of topological and nontopological edge states induced by qubit-assisted unilateral and bilateral coupling potentials based on 1D circuit QED lattice system. When the coupling potential is added on only the leftmost resonator, the system experiences a phase transition with the varying of the coupling potential. Under the strong unilateral coupling potential regime, a new nontopological edge state appears at the leftmost resonator due to the decoupling between the first resonator and the rest of the resonators. In the meanwhile, the energy spectrum of the system has a topological energy level appearing in the whole gap for the even number of lattice sites, which is the so called odd–even effect inversion. We also focus on the situation of two unbalanced coupling potentials being added on both the ends of the system. We find that the energy spectrum of the system experiences the first band inversion when the bilateral coupling potentials are taken as identical values, and it experiences the second inversion when the bilateral coupling potentials become unilateral potential. These energy band inversions lead that the system possesses three different phases. Furthermore, we also study the situation of the coupling potentials added on the bulk of the system and in this case the system is separated into different subchains due to the decoupling from the bulk sites. Utilizing the advantage of the controllability at a single site level in circuit QED lattice, our
Figure 8. Energy spectra and distributions of the edge states when two coupling potentials are added on the bulk sites under large potential regime ($V = 2.5g_0$). a) Energy spectrum of the system when two coupling potentials are added on $j = 50$ and $j = 51$ bulk sites, $\phi = 0.25\pi$ and $N = 100$. b) The distribution of the edge states corresponding to (a). When the large double coupling potentials are added on bulk sites $j = 50$ and $j = 51$, the large coupling strengths make the sites $j = 50$ and $j = 51$ decouple from the chain. The decoupling generates a new topological right edge $j = 49$ and a new topological left edge $j = 52$. Thus, the system has topological left (right) edge states located at sites of $j = 49$ and $j = 52$, and has two nontopological bound state located at $j = 50$ and $j = 51$. c) Energy spectrum of the system when two coupling potentials are added on $j = 50$ and $j = 52$ bulk sites, $\phi = 0.25\pi$, and $N = 101$. d) The distributions of the edge states corresponding to (c). When the large double coupling potentials are added on bulk sites $j = 50$ and $j = 52$, the large coupling strengths make the sites $j = 50$ and $j = 52$ decouple from the chain. The decoupling generates a new topological right edge $j = 49$ and a new topological left edge $j = 53$ accompanying with a new isolated site $j = 51$. Thus, the system has topological left (right) edge states located at sites of $j = 49$ and $j = 53$, and has three bound state located at $j = 50$, $j = 51$, and $j = 52$. e) Energy spectrum of the system when two coupling potentials are added on $j = 50$ and $j = 52$ bulk sites, $\phi = 0.75\pi$ and $N = 101$. f) The distributions of the edge states corresponding to (e). The distributions of the localized states are similar with the case in (d).

Figure 9. The average photon number distribution of the nontopological and topological edge states in the case of strong unilateral coupling potential. a) The system possesses the maximal average photon number distribution at the first resonator when the first resonator is driven by external laser whose driving frequency is tuned to the nontopological edge state energy. b) The system possesses the maximal average photon number distribution at the second resonator when the first resonator is driven by external laser whose driving frequency is tuned to the topological edge state energy.
scheme provides a detectable method to study new topological phase transitions of SSH model both in experiment and theory.

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Conflict of Interest
The authors declare no conflict of interest.

Keywords
Circuit QEDs, nontopological edge states, SSH models, topological phase transition of SSH model both in experiment and theory.

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