QUANTIZATION OF HITCHIN INTEGRABLE SYSTEM VIA POSITIVE
CHARACTERISTIC

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To David Kazhdan with admiration

Abstract. The main result of the seminal (unpublished) work of Beilinson-Drinfeld is the con-
struction of an automorphic sheaf corresponding to a local system which carries the additional
structure of an oper. This is achieved by quantizing the Hitchin intergrable system. In this note
we show (in the case of $G = GL(n)$) that this result admits a short proof based on positive
characteristic methods.

In the appendix we study the restriction of the $p$-curvature ($p$-Hitchin) map to the space of
opers, we show it is finite and flat by checking it is asymptotic to Frobenius at infinity. We
speculate on the relation of this result to a conjecture of Frenkel, Etingof and Kazhdan on the
spectrum of global critically twisted differential operators on $\text{Bun}_G$ acting on the space of $L^2$
sections of the bundle of half forms.

1. Introduction

Geometric Langlands duality predicts the existence of an automorphic $\mathcal{D}$-module $\mathcal{M}_L$ on $\text{Bun}_G$
attached to a (de Rham) $L^G$-local system $L$. Here $G$, $L^G$ are reductive groups dual in the sense
of Langlands, $\text{Bun}_G$ is the moduli stack of $G$-bundles on a complete smooth irreducible curve $C$
and the local system $L$ on $C$ with structure group $L^G$ is assumed to be irreducible (i.e., it does not
admit a reduction to a proper parabolic subgroup).

In their celebrated unpublished work [BD] Beilinson and Drinfeld explain that geometric Lang-
lands duality can be thought of as a quantization of a natural duality for the Hitchin integrable
systems associated to two Langlands dual groups $G$, $L^G$. Furthermore, they present a construction
of $\mathcal{M}_L$ for a local system $L$ which carries an additional structure of an oper, see [BDop] for an
introduction to this notion. Their construction uses local to global arguments, it relies heavily on
representation theory of affine Lie algebras at the critical level.

In this note we describe, for $G = GL_n$, a much shorter construction bypassing affine Lie algebra
representations, relying instead on reduction to positive characteristic.

More precisely, we use the (easy) construction of automorphic $\mathcal{D}$-modules $\mathcal{M}_L$ for a generic local
system $L$ on a curve over a field of positive characteristic [BB]. We establish the following property
of this construction playing a crucial role in the present work: the (critically twisted) $\mathcal{D}$-module
$\mathcal{M}_L$ is generated by a global section if and only if the local system $L$ carries the structure of an
oper. The proof of this property is closely related to ideas of [BD].

As a formal consequence of that property we deduce that the (partially defined) geometric
Langlands equivalence in positive characteristic of [BB] sends the free critically twisted $\mathcal{D}$-module
to the structure sheaf of space opers, see section 8.
This allows us to show that global sections of the sheaf of critically twisted differential operators (i.e., global sections of the corresponding free rank 1 twisted \(D\)-module) on \(\text{Bun}_G\) is a flat deformation of the ring of functions on the Hitchin base, by first doing it in positive characteristic and then deducing the general case by a standard argument. The construction of automorphic \(D\)-modules corresponding to opers follows from this in view of the observation from section \(\S\).

Let us mention that the theme started in [BB], where geometric Langlands duality was established for \(GL_n\) local systems with a smooth spectral curve has been developed in [Gr] where the case of not necessarily smooth spectral curves has been treated, in [CZ], [CZ1] dealing with \(G\) local systems for \(G \neq GL_n\) and in [Tr] where study of quantum geometric Langlands duality in that setting is initiated. However, the present note is the first work (to the authors’ knowledge) where this type of result is connected to the original setting of a characteristic zero base field.

The argument is based on an interesting property of the so called \(p\)-Hitchin map (\(p\)-curvature) map \(h_p\) restricted to the space of opers \(\text{Op}\) established in Appendix A. Roughly speaking, it says that the map \(h_p|\text{Op}\) is asymptotic to Frobenius at infinity (see Lemma A.3 for a precise formulation). This implies that \(h_p|\text{Op}\) is finite, flat of degree \(p^d\) where \(d = \dim(\text{Bun}_G)\) (Theorem A.1). In contrast with some other steps in the argument, this phenomenon is special to positive characteristic, it has not, to our knowledge, appeared in previous works in geometric Langlands duality (a somewhat related concept of a dormant oper defined by Mochizuki [Mo] has been studied in [JP], [W] etc). For the group \(G = GL(1)\) the map \(h_p|\text{Op}\) is described the Hasse-Witt matrix (cf. Remark A.10), so one can consider \(h_p|\text{Op}\) a noncommutative counterpart of the Hasse-Witt matrix.

We refer the reader to Remark A.11 for a discussion of the formal parallel between a resulting description of the spectrum of global twisted differential operators on \(\text{Bun}_G\) acting on half-forms and a conjecture of [EFK] describing the spectrum of such a ring of differential operators acting on the Hilbert space of \(L^2\) sections of half forms on the moduli space of bundles on a complex curve.

We finish the Introduction with a technical remark. Below we use a general construction, the derived category of asymptotic \(D\)-modules on stacks. Here by an asymptotic \(D\)-module on a smooth algebraic variety \(X\) we mean a sheaf of modules over the sheaf of rings \(D_h(X)\), the sheaf of Rees algebras corresponding to the sheaf of filtered algebras \(D(X)\). Thus \(D_h(X)\) is a flat sheaf of rings over polynomials in \(h\), such that \(D_h(X)/(h-1) = D(X), D_h(X)/h = \mathcal{O}(T^*X)\). We refer to [L] for a discussion of standard functors on the derived category of \(D_h\)-modules. The proof of Proposition 3 relies on an extension of this theory to smooth algebraic stacks over a field of an arbitrary characteristic which does not seem to be documented in the literature.

The proof of Lemma 4 uses rudimentary theory of derived stacks (the only derived stacks appearing here are derived fiber products of ordinary stacks), we use [H] as a general reference for their basic properties.

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2. Notations and statement of the main result

We mostly work over a field $k$ of characteristic different from 2, we fix a complete smooth geometrically connected curve $C$ over $k$ of genus at least 2. Let $G = GL_n$, $\text{Bun}$ will denote the moduli stack of rank $n$ vector bundles over $C$; let $\text{Bun}^d$ denote the component of parametrizing bundles of degree $d$.

Recall the stack $\text{Bun}$ with a map $\text{Bun} \to \text{Bun}$ which is a $\mathbb{G}_m$-gerbe, [BB] §4.6. The categories of coherent sheaves and $D$-modules on $\text{Bun}$ and $\text{Bun}$ are closely related, while $\text{Bun}$ has the advantage of being good in the sense of [BD]; we let $\text{Bun}^d \subset \text{Bun}$ be the image of $\text{Bun}^d$.

Let $D\text{-mod}_{\text{Bun}}$ be the category of twisted $D$-modules on $\text{Bun}$, where the class of the twisting equals half of the class corresponding to the canonical line bundle on $\text{Bun}$ and similarly for $\text{Bun}$. Notice that a square root of the canonical line bundle on $\text{Bun}$ is known to exist, thus this category is equivalent to the category of twisted $D$-modules on $\text{Bun}$, respectively $\text{Bun}$. Let $D_{\text{Bun}} \in D\text{-mod}_{\text{Bun}}$ denote the sheaf of twisted differential operators with the same twisting. (Notice that $D_{\text{Bun}}$ is not a sheaf of rings on $\text{Bun}$, see [BD, Sect. 1.1.3].) Let also $D_{\text{Bun}}$ be the derived category of $D$-modules on $\text{Bun}$.

Thus $D\text{-mod}_{\text{Bun}}$ is the heart of the natural $t$-structure on $D_{\text{Bun}}$. We also use similar notations with $\text{Bun}$ replaced by $\text{Bun}$.

Let $\text{Op}$ denote the space of marked opers, see [BDop] for a general introduction to this notion, see also the definition (in the version we use) below before Corollary 6.

The main result of this note is the following

**Theorem 1.** (a) For every $d \in \mathbb{Z}$ we have a canonical isomorphism $\Gamma(\text{Bun}^d, D_{\text{Bun}}) \cong \Gamma(\text{Op})$.

(b) For a point $x \in \text{Op}$ corresponding to a local system $\mathcal{L}_x$ the $D$-module $D_{\text{Bun}} \otimes_{\mathcal{O}_{\text{Op}}} \mathcal{L}_x$ is a Hecke eigenmodule with respect to the local system $\mathcal{L}_x$.

Our strategy is to first establish the Theorem when $k$ has prime characteristic using the result of [BB] and then formally deduce the characteristic zero case.

3. Hecke functor and filtrations

In this section we introduce a filtration on the image of the free $D$-module under the Hecke functor. This is done uniformly in all characteristics by a direct argument independent of [BB]; the idea is close in spirit to [BD] §5.5.

We now recall the definition of the Hecke functor corresponding to the tautological representation of $GL_n$.

Let $\mathcal{H}$ be the stack parametrizing inclusions of vector bundles of rank $n$, $\mathcal{E}_1 \hookrightarrow \mathcal{E}_2$, whose cokernel is of length one. We define $q_1, q_2 : \mathcal{H} \to \text{Bun}$ and $q_C : \mathcal{H} \to C$ by $q_1 : (\mathcal{E}_1 \subset \mathcal{E}_2) \mapsto \mathcal{E}_1$ and $q_C : (\mathcal{E}_1 \subset \mathcal{E}_2) \mapsto x$ where $x$ is determined by the short exact sequence $0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{O}_x \to 0$.

We also consider the projections $q_{i,C} = (q_i, q_{C}) : \mathcal{H} \to \text{Bun} \times C$ ($i = 1, 2$). Notice that both $q_{1,C}$ and $q_{2,C}$ are $\mathbb{P}^{n-1}$ bundles.

The following statement is standard.

**Lemma 2.** The relative tangent bundles $\mathcal{T}_1, \mathcal{T}_2$ for the maps $q_{1,C}, q_{2,C}$ admit a nondegenerate pairing $\mathcal{T}_1 \otimes \mathcal{T}_2 \to q_C^* \mathcal{O}_C = q_{C}^* \Omega_C$.

We let $D_C$ denote the sheaf of twisted differential operators on $C$ that act on sections of the line bundle $\mathcal{T}_C^\otimes n/2$; here for odd $n$ we use the choice of a square root of the canonical bundle $\Omega_C$. We let $D_C, D_{\text{Bun} \times C}$ denote the corresponding derived categories of twisted $D$-modules.

Using the Lemma it is easy to see that the sum of pull-backs under $q_1, q_2$ and $q_C$ of the above twisting classes equals the class of a line bundle (we will use a more precise information about this
natural correspondence up to twist by the line bundle $K$ derived stacks. The cotangent bundles and/or the relevant fiber products may have to be taken in the category of coherent sheaves on the cotangent bundles given by the natural correspondence; while the push-forward functor under a proper morphism $f$ is compatible with the natural (derived) functor from the category of coherent sheaves on the cotangent bundles equipped with a good filtration. Thus we can define the Hecke functor uniformly in all characteristics, a direct analogue of this definition for arbitrary reductive group $G$ works for Hecke functors corresponding to minuscule coweights only: doing it for more general weights requires intersection cohomology sheaves which do not admit a direct generalization to positive characteristic.

The functors $H$ and $H^\vee$ are adjoint in the following sense: there is a functorial isomorphism

\[ (1) \quad \text{Hom}_{D_{\text{Bun}} \times C}(M \boxtimes (T_C^{\otimes n/2} \otimes N), H(M')) \cong \text{Hom}_{D_{\text{Bun}} \times C}(H^\vee(M) \otimes p_C^! N, M' \boxtimes T_C^{\otimes n/2}) \]

for all $M, M' \in D_{\text{Bun}}$ and $N \in D_C\text{-mod}$, where $p_C$ is the projection $\text{Bun} \times C \to C$ and $D_C\text{-mod}$ is the category of untwisted $D$-modules on $C$ (here we use a choice of a square root of $T_C$). The above adjunction can also be characterized as follows. Let us consider the bifunctor $D_{\text{Bun}}^\text{op}\times D_{\text{Bun}} \to D(C)$, where $D(C)$ is the derived category of complexes of (untwisted) $D_C$-modules, given by $(M, M') \mapsto \text{Hom}_{C}(M, M') := p_C^! \text{Hom}_{D_{\text{Bun}} \times C}(\mathcal{O}_C, (M, M'))$.

The functor $\text{Hom}_{C}(\mathcal{O}_C, (M, M'))$ is characterized by the following adjunction:

\[ \text{Hom}_{D(C)}(N, \text{Hom}_{C}(M_1, M_2)) \cong \text{Hom}_{D_{\text{Bun}} \times C}(M_1 \otimes p_C^! N, M_2) \]

Using $\text{Hom}_{C}(\mathcal{O}_C, (M, M'))$, the isomorphism $(1)$ can be rewritten as follows:

\[ (2) \quad \text{Hom}_{C}(\mathcal{O}_C, (M \boxtimes T_C^{\otimes n/2}, H(M'))) \cong \text{Hom}_{C}(H^\vee(M), M' \boxtimes T_C^{\otimes n/2}) \]

Let $(D_C)^{\leq n}$ denote the term of the standard filtration by the order of a differential operator.

**Proposition 3.** The object $H(D_{\text{Bun}})$ lies in the abelian category $D\text{-mod}_{D_{\text{Bun}} \times C}$. Furthermore, the $D_{\text{Bun}} \times C$ module $H(D_{\text{Bun}})$ admits a canonical map $c: D_{\text{Bun}} \times C \to H(D_{\text{Bun}})$, such that the restriction of $c$ to $D_{\text{Bun}} \boxtimes (D_C)^{\leq n}$ is an isomorphism of quasicoherent sheaves.

**Proof.** The proof uses the category of “asymptotic” $D$-modules $\mathcal{D}_h$ (cf. [L]). Recall that for a smooth variety $X$ the sheaf of rings $\mathcal{D}_h(X)$ is obtained from the filtered sheaf of rings $\mathcal{D}(X)$ (differential operators on $X$) by the Rees construction. Thus $\mathcal{D}_h$ is a sheaf of graded rings on $X$ with a central section $h$, such that $\mathcal{D}/h$ is isomorphic to the sheaf $\mathcal{O}_T^h \times X$, while the localization $\mathcal{D}(h)$ is isomorphic to $\mathcal{D}(X)[h, h^{-1}]$, one then considers the (derived) category of sheaves of graded modules. A similar construction applies to twisted differential operators on stacks in the sense of [BD]. Notice that the subcategory of $h$-torsion free coherent asymptotic $D$-modules is equivalent to the category of coherent $D$-modules equipped with a good filtration.

The push-forward and pull-back functors are defined for “asymptotic” (twisted) $D$-modules in a way compatible with the natural (derived) functor from the category of $D_h$ modules to that of $D$-modules (quotient by $h - 1$), see [L]. Moreover, as shown in *loc. cit.* the pull-back functor is compatible under the specialization at $h = 0$ with the functor between the derived categories of coherent sheaves on the cotangent bundles given by the natural correspondence; while the push-forward functor under a proper morphism $f: X \to Y$ is compatible with the functor given by the natural correspondence up to twist by the line bundle $K_X \otimes K_Y^{-1}$.

This theory can be generalized to smooth algebraic stacks. Notice that in the stack case the cotangent bundles and/or the relevant fiber products may have to be taken in the category of derived stacks.
We now proceed to spell this out in the present case. Let Hitch = $T^*\text{Bun}$. Recall that Hitch is the stack parametrizing Higgs fields, i.e., pairs $(E, \phi)$ where $E$ is a rank $n$ bundle on $C$ and $\phi \in H^0(\text{End}(E) \otimes \Omega_C)$.

Let $\mathcal{H}_{\text{Hitch}}$ denote the Hitchin Hecke stack parametrizing triples $(E_1 \subset E_2, \phi)$ where $(E_1 \subset E_2) \in \mathcal{H}$ and the Higgs field $\phi : E_2 \to E_2 \otimes \Omega_C$ satisfies $\phi(E_1) \subset E_1 \otimes \Omega_C$.

We have $pr_1, pr_2 : \mathcal{H}_{\text{Hitch}} \to \text{Hitch}$, $pr_i : (E_1 \subset E_2, \phi) \mapsto (E_i, \phi|_{E_i})$ and $pr_C : \mathcal{H}_{\text{Hitch}} \to T^*C$ sending $(E_1 \subset E_2, \phi)$ to $(x, \xi)$ where $x$ and $\xi$ are determined by $\mathcal{O}_x \cong E_2/E_1$, $(\phi - \xi \otimes \text{id}_{E_2})(E_2) \subset E_1 \otimes \Omega_C$. Here $\mathcal{H}_{\text{Hitch}}$ is considered as a derived stack.

The free rank one $D$-module $D_{\text{Bun}} \in \mathcal{D}_{\text{modBun}}$ equipped with the standard filtration determines an object in the category of asymptotic $D$-modules on $\text{Bun}$ which we denote by $\widetilde{D}_{\text{Bun}}$. Applying the Hecke functor $H_{\text{asymp}} = (q_{2,C})_! q_1^*$ to this object (where $H_{\text{asymp}}$ denotes the Hecke functor on the category of asymptotic $D$-modules) we get an object $H(\widetilde{D}_{\text{Bun}})$.

Using the above compatibility of pull-back and push-forward functors with the specialization at $h = 0$ and Lemma 2 one checks that the corresponding coherent sheaf $H(\widetilde{D}_{\text{Bun}}) \otimes \mathcal{L}_{\mathcal{K}[h]}$ is given by:

$$H(\widetilde{D}_{\text{Bun}}) \otimes \mathcal{L}_{\mathcal{K}[h]} \cong (pr_2 \times pr_C)_! pr_1^!(\mathcal{O}_{\text{Hitch}}).$$

Let $\mathcal{C}_{\text{univ}} \subset \text{Hitch} \times T^*C$ be the universal spectral curve, i.e., $\mathcal{C}_{\text{univ}}$ parametrizes the data of $(E, \phi, x, \xi)$ where $x \in C$, $\xi \in T^*C|_x$ such that $\det(\phi|_x - \xi \cdot \text{id}) = 0$.

**Lemma 4.** We have $(pr_2 \times pr_C)_! pr_1^!(\mathcal{O}_{\text{Hitch}}) \cong \mathcal{O}_{\mathcal{C}_{\text{univ}}}$.

**Proof.** Consider subvarieties $S \subset \mathfrak{g}_n \times \mathbb{A}^1$ and $\tilde{S} \subset \mathfrak{gl}_n \times \mathbb{P}^{n-1} \times \mathbb{A}^1$ given by: $S = \{(x, t) \mid \det(x - t \cdot \text{id}) = 0\}$, $\tilde{S} = \{(x, t, \xi) \mid x|_t = t \cdot \text{id}\}$. Let $\pi : S \to \mathfrak{gl}_n \times \mathbb{A}^1$ be the natural projection. It is a standard fact that

$$\pi_*(\mathcal{O}_S) \cong \mathcal{O}_S.$$

Consider the natural "evaluation" map $\text{Hitch} \times T^*C \to (\mathfrak{gl}_n \times \mathbb{A}^1)/(GL_n \times \mathbb{G}_m)$ (where $GL_n$ acts on $\mathfrak{g}_n$ by the adjoint action and $\mathbb{G}_m$ acts on $\mathbb{A}^1$ by dilatations). Namely, the line bundle $\Omega_C$ defines a $\mathbb{G}_m$-torsor on $C$, and also we have a tautological $GL_n$-torsor on $\text{Bun} \times C$ corresponding to the universal rank $n$ vector bundle $\mathcal{E}_{\text{univ}}$ on $\text{Bun} \times C$. We pull back both torsors to $\text{Hitch} \times T^*C$ and take their fiber product, thus getting a $(GL_n \times \mathbb{G}_m)$-torsor $\mathcal{P}$ on Hitch $\times T^*C$. Further, we have canonical sections of the pullbacks of $\Omega_C$ to $T^*C$ and of $\text{End} \mathcal{E}_{\text{univ}} \otimes \text{pr}_C^*\mathcal{O}_C$ to Hitch $\times T^*C$. Using the tautological trivializations of the pullbacks of both bundles to $\mathcal{P}$, pulling back the canonical sections defines a $(GL_n \times \mathbb{G}_m)$-equivariant map $\mathcal{P} \to \mathfrak{gl}_n \times \mathbb{A}^1$ which, passing to the quotient stacks, gives our map $\text{Hitch} \times T^*C \to (\mathfrak{gl}_n \times \mathbb{A}^1)/(GL_n \times \mathbb{G}_m)$.

We have natural isomorphisms:

$$H_{\text{Hitch}} \cong (\mathcal{S}/(GL_n \times \mathbb{G}_m) \times (\mathfrak{gl}_n \times \mathbb{A}^1)/(GL_n \times \mathbb{G}_m)) (\text{Hitch} \times T^*C);$$

$$\mathcal{C}_{\text{univ}} \cong (\mathcal{S}/(GL_n \times \mathbb{G}_m) \times (\mathfrak{gl}_n \times \mathbb{A}^1)/(GL_n \times \mathbb{G}_m)) (\text{Hitch} \times T^*C).$$

Here both fiber products are understood to be derived, thus both formulas are isomorphisms of derived stacks.

Thus base change isomorphism applies, see [TT §3.1] and [TT Proposition 1.4].

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1In fact, the base change isomorphism is stated in loc. cit. for derived product of (derived) schemes rather than derived stacks. The fiber product we presently consider is locally a quotient of a fiber product of schemes by an action of an affine algebraic group; moreover, it is a union of open (derived) substacks of this form. Thus the base change isomorphism in this case follows from loc. cit.
Thus the lemma follows from (3) and (4).

Remark 1. a) Using Koszul resolution one can write down an explicit sheaf of DG-algebras on $\text{Bun} \times C$, such that its derived category of sheaves of modules is identified with the derived coherent sheaves category of the derived fiber product. Thus one can work with these categories without invoking the general theory of derived stacks, see [R] where this approach is spelled out in another context.

b) We do not know if the derived fiber product in the last displayed formulas is essentially derived, i.e., if some of higher $\text{Tor}$'s between the structure sheaves of the two factors over the structure sheaf of the base are nonzero. If this is not the case the isomorphisms can be understood as isomorphisms of ordinary stacks.

We are now ready to finish the proof of the Proposition. Recall that an object $M$ in the derived category of asymptotic $\mathcal{D}$-modules on a stack $X$ such that the induced object $M = M \otimes^L_{k[x]} k \in D^b(\text{Coh}^{\mathbb{G}_m}(T^*X))$ lies in homological degree zero amounts to a $\mathcal{D}$-module with a good filtration whose associated graded is isomorphic to $M$. Thus comparing (3) with Lemma 4 we see that $H(D_{\text{Bun}})$ is a $\mathcal{D}$-module with a good filtration whose associated graded is isomorphic to $\mathcal{O}_{\mathbb{C}_{\text{univ}}}$. Since the latter coherent sheaf is cyclic, we see that $H(D)$ is a cyclic $\mathcal{D}$-module with a canonical generator. Since the sheaf of regular functions on Hitch $\times T^*C \to \text{Hitch} \times C$ maps isomorphically to $\mathcal{O}_{\mathbb{C}_{\text{univ}}}$, the Proposition follows.

Recall the ring of twisted differential operators $\mathcal{D}_C$ introduced after Lemma 2. By an oper we will understand an $\mathcal{O}$-coherent $\mathcal{D}_C$ module $\mathcal{O}$ of rank $n$ which has a good filtration whose associated graded is isomorphic to $\text{gr}(\mathcal{D}_C)_{<n}$. Choosing a theta-characteristic (i.e., a square root of the cotangent bundle) we can identify the category of $\mathcal{D}_C$-modules with the category of $\mathcal{D}$-modules and connect this with the standard definition of a (marked) oper. It is standard that opers in this sense are parametrized by a variety which we will denote $\text{Op}$.

Corollary 5. Assume that $M \in D^b(\mathcal{D}_{\text{mod}} \text{Bun})$ satisfies the Hecke eigenproperty with respect to a local system $\mathcal{L} \in \mathcal{D}_{\text{C-mod}}$. Assume\footnote{The first assumption holds automatically if $\text{char} \ k = 0$. If $\text{char} \ k = p > 0$, then $\text{deg}(\mathcal{L}) \equiv (1-g)n(n-1) \ (\text{mod} \ p)$, and $\text{deg}(\mathcal{L})$ is determined by the character by which $\mathbb{G}_m$ acts on the quasicoherent sheaf underlying $M$ (where we use that $\text{Bun}$ is a $\mathbb{G}_m$ gerbe over $\text{Bun}$). It is not hard to see that $\text{RHom}_{\mathcal{D}_{\text{mod}} \text{Bun}}(\mathcal{D}_{\text{Bun}}, M) = 0$ automatically unless $\text{deg}(\mathcal{L}) = (1-g)n(n-1)$.} that $\mathcal{L}$ has degree $(1-g)n(n-1)$ and does not admit an oper structure. Then

$$\text{RHom}_{\mathcal{D}_{\text{mod}} \text{Bun}}(\mathcal{D}_{\text{Bun}}, M) = 0.$$ $$\text{Proof.}$$

The proof will proceed by contradiction. Let $\mathcal{L}$ be the corresponding local system and set $V := \text{RHom}_{\mathcal{D}_{\text{mod}} \text{Bun}}(\mathcal{D}_{\text{Bun}}, M)$, thus $V \neq 0$ by assumption.

Consider $\text{pr}_{C}^\mathcal{O}(H(M))$, the sheaf direct image of the $\mathcal{D}$-module $H(M)$ under the projection $\text{pr}_C : \text{Bun}_n \times C \to C$. Then the Hecke eigen-property of $M$ shows that

$$\text{pr}_{C}^\mathcal{O}(H(M)) \cong V \otimes \mathcal{L}.$$
On the other hand, Proposition \[3\] and isomorphism \[2\] imply that the object in the derived category of quasicoherent sheaves \( pr_{C,*}(H(M)) \) satisfies:
\[
pr_{C,*}(H(M)) \cong R\text{Hom}_{/C}(\mathcal{D}_{\text{Bun}} \boxtimes \mathcal{T}_{C}^{\otimes n/2}, H(M)) \otimes \mathcal{T}_{C}^{\otimes n/2} \\
\cong R\text{Hom}_{/C}(H^{\vee}(\mathcal{D}_{\text{Bun}}), M \boxtimes \mathcal{T}_{C}^{\otimes n/2}) \otimes \mathcal{T}_{C}^{\otimes n/2} \\
\cong pr_{C,*}(R\text{Hom}(\mathcal{D}_{\text{Bun}} \boxtimes (\mathcal{D}_{C})_{<n}, M \boxtimes \mathcal{T}_{C}^{\otimes n/2})) \otimes \mathcal{T}_{C}^{\otimes n/2} \\
\cong \text{RHom}_{D\text{-mod}_{\text{un}}}(\mathcal{D}_{\text{Bun}}, M) \otimes \text{Hom}_{O_{C}}((\mathcal{D}_{C})_{<n}, \mathcal{T}_{C}^{\otimes n/2}) \otimes \mathcal{T}_{C}^{\otimes n/2} \\
\cong V \otimes (\mathcal{D}_{C})_{<n}
\]
where \( H^{\vee} \) is the adjoint Hecke functor \( H^{\vee}: D\text{-mod}_{\text{Bun}} \to D\text{-mod}_{\text{un}} \times C \).

Comparing the two displayed isomorphisms we see that \( \text{oblv}_{C}(\mathcal{L}) \) admits an injective map into \( (\mathcal{D}_{C})_{<n} \) as a coherent sheaf. Since an injective map between coherent sheaves on a curve having the same degree and the same generic rank has to be an isomorphism, we see that \( \mathcal{L} \) has an oper structure. 

\[\square\]

4. PROOF OF THE MAIN THEOREM IN THE CASE \( \text{char} \ k = p > 0 \)

It is easy to deduce the assertion of the theorem for \( k \) from the assertion for the algebraic closure of \( k \), so we assume for simplicity that \( k \) is algebraically closed.

Recall that Hitch = \( T^{*}\text{Bun} \) and \( \tilde{C}_{\text{univ}} \) is the universal spectral curve. Let \( h: \text{Hitch} \to B \) be the Hitchin map and \( \pi: \tilde{C}_{\text{univ}} \to B \) be the projection. Let \( B_{s} \supset B_{s} \) be the open subsets in the Hitchin base \( B \) parametrizing the points \( x \in B \) such that the fiber \( \pi^{-1}(x) \) is reduced, respectively, smooth.

In this section we assume that the base field \( k \) has prime characteristic \( p \). Then \( \mathcal{D}_{\text{Bun}} \) can be thought of as a sheaf over Hitchin(1), where the superscript denotes the Frobenius twist.

Let \( \text{Loc} \) denote the moduli stack of \( \mathcal{D}_{C} \)-modules which are locally free of rank \( n \) as an \( \mathcal{O} \)-module. Recall \[BB\] that we have the Frobenius-Hitchin map \( h: \text{Loc} \to B(1) \); for example, for \( x \in B_{s} \) the fiber of \( h \) over \( x \) is the abelian algebraic group \( \text{Pic}(\tilde{C}_{x}) \), while the fiber of \( \pi_{p} \) over \( x^{(1)} \in B(1) \) is the torsor over the abelian algebraic group \( \text{Pic}(\tilde{C}_{x})^{(1)} \) (here \( x^{(1)} \) denotes the image of \( x \) under Frobenius).

We will need the following result proven in the Appendix.

**Proposition 6.** The composition \( \text{Op} \to \text{Loc} \xrightarrow{\pi_{p}} B(1) \) is a flat finite map of degree \( p^{N} \), where \( N = \dim(\text{Bun}) \).

Set Hitch\( s = h^{-1}(B_{s}) \), Loc\( s = \pi_{p}^{-1}(B_{s}^{(1)}) \), Op\( s = \text{Op} \times_{B(1)} B_{s}^{(1)} \). \( \mathcal{D}_{\text{Bun}} = \mathcal{D}_{\text{Bun}} \mid_{\text{Hitch}}(1) \), where in the last expression we use the same notation \( \mathcal{D}_{\text{Bun}} \) for the object in \( \mathcal{D}_{\text{Bun}} \) and the corresponding sheaf on Hitchin(1).

Recall that the main result of \[BB\] is an equivalence\[3\]
\[
D^{b}((\mathcal{D}_{\text{Bun}})_{\text{-mod}_{\text{coh}}}) \cong D^{b}(\text{Coh}(\text{Loc}_{s}^{(1-g)n(n-1)})),
\]
where \( -\text{mod}_{\text{coh}} \) stands for the category of coherent sheaves of modules, and \( \text{Loc}_{s}^{(1-g)n(n-1)} \) stands for the component of Loc\( s \) classifying \( \mathcal{D}_{C} \)-modules (locally free of rank \( n \) over \( \mathcal{O}_{C} \) with smooth

\[3\]In fact, this is a version of the equivalence constructed in loc. cit.: there the stack Bun and ordinary \( D \)-modules are considered instead of \( \text{Bun} \) and twisted \( D \)-modules. It is not hard to deduce that version from the result of loc. cit. Note in particular that replacing Bun by \( \text{Bun} \) in the left-hand side corresponds to restricting to one of the connected components of Loc in the right-hand side.
$p$-spectral curve) whose underlying $\mathcal{O}_C$-module has degree $(1 - g)n(n - 1)$. We let $\Phi$ denote that equivalence.

The first step in the proof of the Theorem is the following

**Proposition 7.** We have $\Phi((\mathcal{D}_{\text{Bun}})_s) \cong (\text{Op}_s \to \text{Loc}_s)_*(\mathcal{O}_{\text{Op}_s})$.

**Proof.** It is easy to see that the tautological map from the space $\text{Op}$ of marked opers to $\text{Loc}$ is a composition of the map $\text{Op} \to \text{Op}/\mathbb{G}_m$ and a closed embedding $\text{Op}/\mathbb{G}_m \to \text{Loc}$, where we use the trivial action of $\mathbb{G}_m$ on $\text{Op}$. Thus the direct image of $\mathcal{O}_{\text{Op}}$ to $\text{Loc}$ decomposes as a direct sum indexed by characters of $\mathbb{G}_m$: we claim that the summand corresponding to the character $t \mapsto t^d$ is canonically isomorphic to $\Phi((\mathcal{D}_{\text{Bun}})_s))$.

It follows from Corollary 3 that the complex $\Phi((\mathcal{D}_{\text{Bun}})_s))$ is supported on $\text{Op}_s/\mathbb{G}_m$.

From Proposition 6 we see that its support is finite over the base $\mathcal{B}^{(1)}$. Since Fourier-Mukai transform is exact on sheaves with finite support, sending such a sheaf of length $r$ to a vector bundle of rank $r$, we see that $\Phi((\mathcal{D}_{\text{Bun}})_s))$ is concentrated in homological degree zero; moreover, its pull-back to $\text{Op}_s$ is flat of rank $p^N$ as a module over $\mathcal{O}_{\mathcal{B}^{(1)}}$.

We claim that $\Phi((\mathcal{D}_{\text{Bun}})_s))$ is scheme theoretically supported on $\text{Op}_s/\mathbb{G}_m$.

First of all, $\Phi((\mathcal{D}_{\text{Bun}})_s))$ is torsion free as an $\mathcal{O}(\mathcal{B}^{(1)})$ module. Thus it suffices to check this claim over the generic point of $\mathcal{B}^{(1)}$.

To see this notice that a coherent sheaf on $\text{Loc}$ which is generically set theoretically but not scheme theoretically supported on $\mathcal{B}^{(1)}$ would have length greater than one at the generic point of $\text{Op}$. Then its direct image to $\mathcal{B}^{(1)}$ would have generic rank greater than $p^N$, which contradicts the second paragraph of the proof.

Now, flatness of $\Phi((\mathcal{D}_{\text{Bun}})_s))$ over $\mathcal{O}_{\mathcal{B}^{(1)}}$ implies it’s a Cohen-Macaulay module over $\mathcal{O}_{\text{Op}_s}$. Since $\text{Op}_s$ is smooth, it is actually a locally free module, since its degree over $\mathcal{O}_{\mathcal{B}^{(1)}}$ equals that of $\mathcal{O}_{\text{Op}_s}$, we conclude that the pull-back of $\Phi((\mathcal{D}_{\text{Bun}})_s))$ to $\text{Op}_s$ is a line bundle on $\text{Op}_s$. Since $\text{Op}_s$ is an open subvariety in $\text{Op}$ which is isomorphic to the affine space, every line bundle on $\text{Op}_s$ is trivial. Since $\mathbb{G}_m$ acts by the character $t \mapsto t^d$ on the $\Phi(\mathcal{F})$ for $\mathcal{F}$ supported on $\mathcal{B}^{(1)}$, we get the statement. \Box

The Proposition implies that for all $d \in \mathbb{Z}$

$$A_d := \Gamma(\mathcal{D}_{\text{Bun}})
\subset \Gamma((\mathcal{D}_{\text{Bun}})_s) \cong \text{End}((\mathcal{D}_{\text{Bun}})_s) \cong \Gamma(\mathcal{O}_{\text{Op}_s})$$

is a commutative algebra.

Fix some $d \in \mathbb{Z}$. Proposition allows one to construct a family of opers on $\text{C}$ parametrized by $\text{Spec}(A_d)$. The family can be described as a $\mathcal{D}_C$-module $\mathcal{F}_{\text{univ}}$, with an $A_d$ action and a filtration which is flat over $A_d$ and such that $\mathcal{F}_{\text{univ}} \otimes A_d k_x$ is an over every $x \in \text{Spec}(A_d)$ and is constructed as follows. The sheaf $\mathcal{F}_{\text{univ}}$ is the (sheaf theoretic) direct image to the second factor of the $\mathcal{D}_{\text{Bun}}$-module $H(\mathcal{D}_{\text{Bun}}^{d-1})$, equipped with the natural $A_d$ action and the filtration coming from Proposition 3. The $A_d$-action is defined by presenting $\mathcal{F}_{\text{univ}}$ as the pushforward to $C$ of the local Hom from $\mathcal{D}_{\text{Bun}}^{d} \otimes \mathcal{C}$ to $H(\mathcal{D}_{\text{Bun}}^{d-1})$ and using $A_d$-action on the first argument.

Actually the definition of this action can be generalized to a functor $\Upsilon: \mathcal{D}\text{-mod}_{\text{Bun}^{d} \times C} \to (A_d \otimes \mathcal{D}_C)\text{-mod}$ given by pushforward to $C$, and $\mathcal{F}_{\text{univ}} = \Upsilon(H(\mathcal{D}_{\text{Bun}}^{d-1}))$. Applying the functor $\Upsilon$ to the map $c$ from Proposition 3 we get a map $\Upsilon(c): \Upsilon((\mathcal{D}_{\text{Bun}}^{d} \times C)) = A_d \otimes \mathcal{D}_C \to \mathcal{F}_{\text{univ}}$ which restricts to an isomorphism of $A_d \otimes \mathcal{O}_\text{C}$-modules $A_d \otimes (\mathcal{D}_C)^{<n} \to \mathcal{F}_{\text{univ}}$. From this it is straightforward to see that $\mathcal{F}_{\text{univ}}$ defines an $A_d$-family of opers.

Thus we get a map $\Pi: \text{Spec}(A_d) \to \text{Op}$. We will show that it is an isomorphism.
It is easy to deduce from the Hecke eigen-property for the equivalence $\Phi$ that base-change of
$\Pi$ from $B^{(1)}$ to $B^{(1)}_{s}$ coincides with the (dual of) isomorphism $[3]$. From this we see that the composition

$$\mathcal{O}(\text{Op}) \xrightarrow{\Pi} A_{d} \subset \Gamma((\mathcal{D}_{\text{Bun}}^{d})_{s}) \xrightarrow{\text{prop.}[2]} \mathcal{O}(\text{Op}_{s})$$

is the natural inclusion. Thus $A_{d}$ is isomorphic to subalgebra of $\mathcal{O}(\text{Op}_{s})$ containing $\mathcal{O}(\text{Op})$. Since $\mathcal{O}$ is normal (it is isomorphic to an affine space: indeed, by Lemma $[A,5]$ part (1), $\mathcal{O}$ is affine and there is a filtration on $\mathcal{O}(\text{Op})$ with $gr \mathcal{O}(\text{Op}) \cong \mathcal{O}(B)$ which is a polynomial algebra; see also $[JP$, Proposition 3.2.3]), it would suffice to show that $A_{d}$ is finitely generated as a module over $\mathcal{O}(\text{Op})$.

We will in fact prove finite generation over a smaller algebra.

**Lemma 8.** $A_{d}$ is a finitely generated torsion free module over $\mathcal{O}(B^{(1)})$.

**Proof.** Consider the filtration on $\mathcal{D}_{\text{Bun}}$ and the induced one on $A_{d}$ by degree of differential operator. Then we have $\text{gr} \mathcal{D}_{\text{Bun}}^{d} = (T^{*}\text{Bun}^{d} \to \text{Bun}^{d}), \mathcal{O}_{T^{*}\text{Bun}^{d}}$. The induced filtration on $\mathcal{O}(B^{(1)}) \to A_{d}$ coincides with the one coming from the grading on $\mathcal{O}(B^{(1)})$ multiplied by $p$, and the associated graded map to this embedding is dual to $\text{Fr}_{B} \circ h$: $\text{Hitch} \to B^{(1)}$. It is known that all global functions on $T^{*}\text{Bun}^{d}$ are pullbacks from the Hitchin base, so we have $\Gamma(\text{gr} \mathcal{D}_{\text{Bun}}^{d}) = \mathcal{O}(B)$. On the other hand, there is an inclusion $\text{gr} A \hookrightarrow \Gamma(\text{gr} \mathcal{D}_{\text{Bun}}^{d}) = \mathcal{O}(B)$. Thus $\text{gr} A$ identifies with an $\mathcal{O}(B^{(1)})$-submodule in $\mathcal{O}(B)$, therefore it is finitely generated and torsion-free over $\mathcal{O}(B^{(1)})$. But then so is $A_{d}$, as desired. \hfill $\square$

As explained above, the lemma shows that the map $\Pi$ is an isomorphism, so that for any $d$ we have a canonical isomorphism

$$A_{d} \cong A := \mathcal{O}(\text{Op}).$$

It also follows that $\mathcal{F}_{\text{univ}}^{d}$ are identified for all $d \in \mathbb{Z}$, so we write $\mathcal{F}_{\text{univ}}$ for the sheaf isomorphic to all of them.

**Lemma 9.** The map $c: \mathcal{D}_{\text{Bun}^{d} \times C} = \mathcal{D}_{\text{Bun}^{d}} \boxtimes \mathcal{D}_{C} \to H(\mathcal{D}_{\text{Bun}^{d-1}})$ from Proposition 3 intertwines the two A-actions coming from the $A$-action on $\mathcal{D}_{\text{Bun}}$.

**Proof.** Since both the source and the target are torsion-free as modules over $\mathcal{O}(B^{(1)}) \subset Z(\mathcal{D}\text{-mod}_{\text{Bun}})$ (where $Z$ stands for the center of a category, i.e., the ring of endomorphisms of the identity functor), it is enough to show that the statement of the lemma holds after tensoring by $\mathcal{O}(\text{Op}_{s})$ over $\mathcal{O}(\text{Op})$. The localized map $c_{s}$ is a morphism in $(\mathcal{D}\text{-mod}_{\text{Bun}^{d} \times C})_{s}$, and we can apply $\Phi$ in the first factor. Then Proposition $[7]$ and Hecke eigen-property of $\Phi$ imply that $\Phi(c)$ is a map of $(\mathcal{O}\text{-Loc}_{s} \boxtimes \mathcal{D}_{C})$-modules scheme-theoretically supported on $\text{Op}_{s} / \mathbb{G}_{m} \times C \subset \text{Loc}_{s} \times C$, and that the elements of $A = \mathcal{O}(\text{Op})$ act by multiplication by the same functions on this support. \hfill $\square$

### 4.1. End of proof of Theorem 1 in the positive characteristic case.

Thus we proved the first part of the theorem. The second statement of the theorem follows from the construction of the morphism $\Pi$. Indeed, we need to construct an isomorphism

$$H(\mathcal{D}_{\text{Bun}}) \cong \mathcal{D}_{\text{Bun}} \boxtimes_{A} \mathcal{F}_{\text{univ}},$$

where we used the following notation: if $A$ is a $k$-algebra, $X, Y$ are $k$-schemes (or stacks) and $\mathcal{F}$, respectively $\mathcal{G}$, are quasi-coherent sheaves on $X$, resp. $Y$, with a right, resp. left, $A$-actions, then we define a quasi-coherent sheaf on $X \times Y$: $\mathcal{F} \boxtimes_{A} \mathcal{G} := (\mathcal{F} \boxtimes \mathcal{G}) \otimes_{A \boxtimes \mathcal{A}^{op}} A$, where the rightmost symbol $A$ refers to the regular $A$-bimodule.

We will construct the isomorphism as above for each component $\text{Bun}^{d}$ of $\text{Bun}$, so fix $d \in \mathbb{Z}$. Consider the functor $\mathcal{D}_{\text{Bun}^{d}} \boxtimes_{A} -: (A \boxtimes \mathcal{D}_{C})\text{-mod} \to \mathcal{D}\text{-mod}_{\text{Bun}^{d} \times C}$, which is the left adjoint to the
“d’th component” of the functor Υ used above. So we have a counit map \( a_M : \mathcal{D}_{\text{Bun}^d} \boxtimes \mathcal{A} \mathcal{Y}(M) \to M \) for any \( M \in \mathcal{D}_{\text{mod} \mathcal{Bun}^d \times C} \). We need to check that it is an isomorphism for \( M = H(\mathcal{D}_{\text{Bun}^{d-1}}) \). It is clear that if we apply the forgetful functor \( \mathcal{D}_{\text{mod} \mathcal{Bun}^d \times C} \to \mathcal{D}_{\text{mod} \mathcal{Bun}^{d-1}} \) to \( a_M \) then we will get the counit morphism for the similar adjunction between \( \mathcal{D}_{\text{mod} \mathcal{Bun}^d \times C} \) and \( (\mathcal{A} \otimes \mathcal{O}_C)\text{-mod} \). Now since locally over \( C \), \( H(\mathcal{D}_{\text{Bun}^{d-1}}) \) is isomorphic to \( (\mathcal{D}_{\mathcal{Bun}^d} \boxtimes \mathcal{O}_C)^{\otimes n} \) as a relative-D-module on \( \mathcal{Bun}^d \times C \) over \( C \), and our statement is local in \( C \), it suffices to check for \( M = \mathcal{D}_{\text{Bun}^d} \boxtimes \mathcal{O}_C \) where it is clear.

Now, to deduce Hecke eigen-property, we have to show that the constructed isomorphism commutes with the \( \mathcal{A} \)-action, where the action on \( H(\mathcal{D}_{\text{Bun}}) \) comes by transport of structure from the \( \mathcal{A} \)-action on \( \mathcal{D}_{\text{Bun}} \) and the action on \( \mathcal{D}_{\text{Bun}} \boxtimes \mathcal{A} \mathcal{F}_{\text{univ}} \) comes from either of the \( \mathcal{A} \)-actions contracted by the tensor product. For this we consider the commutative diagram

\[
\begin{array}{ccc}
\mathcal{D}_{\text{Bun}^d} \boxtimes \mathcal{D}_C & \to & \mathcal{D}_{\text{Bun}^d} \boxtimes \mathcal{A} \mathcal{Y}(\mathcal{D}_{\text{Bun}^d \times C}) \\
\mathcal{D}_{\text{Bun}^d} \boxtimes \mathcal{A} \mathcal{Y}(\mathcal{c}) & \sim & \mathcal{D}_{\text{Bun}^d} \boxtimes \mathcal{D}_C \\
\mathcal{D}_{\text{Bun}^d} \boxtimes \mathcal{A} \mathcal{F}_{\text{univ}} & \to & \mathcal{D}_{\text{Bun}^d} \boxtimes \mathcal{A} \mathcal{Y}(H(\mathcal{D}_{\text{Bun}^{d-1}})) \\
\mathcal{D}_{\text{Bun}^d} \boxtimes \mathcal{A} \mathcal{Y}(\mathcal{c}) & \sim & H(\mathcal{D}_{\text{Bun}^{d-1}}).
\end{array}
\]

By Lemma 9, the vertical arrows in this diagram commute with the \( \mathcal{A} \)-action. The top arrow commutes with the \( \mathcal{A} \)-action because of the naturality of \( a_M \). Hence the bottom arrow does, too, which is what we need. Note that \( \mathcal{D}_{\text{Bun}^d} \boxtimes \mathcal{A} \mathcal{Y}(\mathcal{D}_{\text{Bun}^d \times C}) \) a priori has three \( \mathcal{A} \)-actions, coming from the \( \mathcal{A} \)-actions on \( \mathcal{D}_{\text{Bun}^d} \), \( \mathcal{Y} \) and \( \mathcal{D}_{\text{Bun}^d \times C} \). The first two actions are contracted by the tensor product and hence give rise to the same action, while the last two actions agree because under the isomorphism \( \mathcal{Y}(\mathcal{D}_{\text{Bun}^d \times C}) \sim \mathcal{A} \otimes \mathcal{D}_C \) they go to the right, resp. left, actions on the first factor, and since \( \mathcal{A} \) is commutative, they are the same. Note that the first action is adapted to showing that the left vertical arrow commutes with \( \mathcal{A} \), and the last one is adapted to the horizontal map.

What we just proved implies the following in-families version of Hecke eigen-property. The \( \mathcal{A} \)-action on the \( \mathcal{D} \)-module \( \mathcal{D}_{\text{Bun}} \) allows to present it as a pushforward of a \( (\text{Spec} \mathcal{A} = \mathcal{O}_\text{Op}) \)-family \( \mathcal{M} \) of \( \mathcal{D} \)-modules. Applying to \( \mathcal{M} \) the relative version of the Hecke functor \( H \), we get a relative (twisted) \( \mathcal{D} \)-module on \( \text{Bun} \times \mathcal{O}_\text{Op} \times C \) over \( \mathcal{O}_\text{Op} \) which we denote by \( H_{\mathcal{O}_\text{Op}}(\mathcal{M}) \). Then it follows from what we proved that \( H_{\mathcal{O}_\text{Op}}(\mathcal{M}) \cong (\mathcal{M} \boxtimes \mathcal{O}_C) \otimes (\mathcal{O}_\text{Bun} \boxtimes \mathcal{F}_{\text{univ}}) \) where \( \mathcal{F}_{\text{univ}} \) is the Op-family of opers constructed from \( \mathcal{F}_{\text{univ}} \). Taking pullback to a closed point \( x \) of \( \mathcal{O}_\text{Op} \), we see that the derived specialization of the family \( \mathcal{M}, \mathcal{M}_x := L(\{x\} \times \text{Bun} \to \mathcal{O}_\text{Op} \times \text{Bun})^* \mathcal{M} = \mathcal{D}_{\text{Bun}} \boxtimes \mathcal{M}_x \), is a Hecke eigen-\( \mathcal{D} \)-module.

We want to show that \( \mathcal{M}_x \) is actually in the heart of \( D(\text{Bun}) \). In other words, we want to prove that \( \mathcal{D}_{\text{Bun}} \) is flat over \( A \). Since the \( \mathcal{A} \)-module \( \mathcal{D}_{\text{Bun}} \) is a (flat) deformation of \( \mathcal{O}_{\text{Hitch}} \) viewed as a module over \( \mathcal{O}(B) \), this follows from flatness of the Hitchin map.

\[ \square \]

\section{5. Proof of the main theorem in the case \( \text{char} \, k = 0 \)}

Considering the deformation of \( \mathcal{O}_{\mathcal{T}^* \text{Bun}^d} \) to \( \mathcal{D}_{\text{Bun}^d} \) we get an injection

\[ \text{gr} \, A_d \hookrightarrow \Gamma(\mathcal{O}_{\mathcal{T}^* \text{Bun}^d}) = \Gamma(\mathcal{O}_B). \]

Choose a finitely generated (over \( \mathbb{Z} \)) subring \( R \subset k \) and a smooth proper curve \( C_R \) over \( R \) whose base change to \( k \) is isomorphic to \( C \). We have a similar injection constructed from the moduli stack \( \text{Bun}_R \) for vector bundles over \( C_R \):

\[ \text{gr} \, A_{d,R} \hookrightarrow \Gamma(\mathcal{O}_{\mathcal{T}^* \text{Bun}^d_R}) = \Gamma(\mathcal{O}_{B_R}). \]
Here we use that the relative $D$-module $D_{\text{Bun}_{A_d}}$ is flat over $R$. This follows from the fact that $\text{Bun}_{A_d}$ is a family of good stacks in the sense of [BD, §1.1.1]: over a characteristic zero field this is proved in [BD, Prop.2.1.2], the general case is similar.

Since the right hand side is finitely generated as an $R$-module in every degree, equality of its elements can be checked by reduction modulo all maximal ideals of $R$. The same property therefore holds for $\text{gr} A_{A_d}$ and hence for $A_{A_d}$ itself. For every such maximal ideal $m$, the residue field $F := R/m$ is finite and we have an injective map $\iota_{d,R} : A_{A_d} \otimes_R F \hookrightarrow A_d,F$ whose target is commutative by the previous section. Hence so is the source and, since this holds for all $m$, we get that $A_d$ is commutative too.

Now the construction of the previous section yields a family of opers parametrized by $\text{Spec}(A_d)$, given by $\Upsilon(H(P_{\text{Bun}_{A_d}}))$ as before. Thus we get a map $\Pi : \text{Spec}(A_d) \to \text{Op}$ as explained in the previous section. Unraveling the definition, one can see that $\Pi = \Pi_R \otimes_R k$ and $\Pi^*_F = \iota_{d,R}^* (\Pi_R \otimes_R F)$. But $\Pi_F$ is an isomorphism by the previous section, and $\iota_{d,R}$ is injective, hence both $\iota_{d,R}$ and $\Pi_R \otimes_R F$ are actually isomorphisms.

Since the base change of $\Pi_R$ to all finite residue fields of $R$ are isomorphisms, we see that $\Pi_R$, and hence $\Pi = \Pi_R \otimes_R k$, is an isomorphism. This proves the first part of Theorem 1. One then proves an analogue of Lemma 9 in characteristic 0 by observing that it is enough to prove the statement for the reductions to finite residue fields of $R$. After that, the second part follows by the argument of the previous section.

**Appendix A. Hitchin map and opers in characteristic $p$ (by Roman Bezrukavnikov, Tsao-Hsien Chen, and Xinwen Zhu)**

This appendix is devoted to the proof of the following statement (Proposition 6 of the main text). Let $G$ be a connected reductive algebraic group over $k$. We assume that the characteristic $\text{char} k = p$ does not divide the order of the Weyl group $W$ of $G$.

**Theorem A.1.** Let $\text{Op}_G$ be the scheme of $G$-opers with marking (see §A.3). Then the composition

$$\pi_p : \text{Op}_G \to \text{Loc}_G \xrightarrow{h_p} B^{(1)}$$

is finite and faithfully flat of degree $p^{\dim B}$. Here $h_p$ is the $p$-Hitchin map.

**Remark A.2.** In the case $G = \text{PGL}_n$, the theorem above is a strengthening of a result of C. Pauly and K. Joshi [JP] who proved that the $p$-Hitchin map on the space of opers is finite.

**A.1. Notations.** Let $C$ be a complete smooth curve over $k$. Let $G$ be a connected reductive algebraic group over $k$ of rank $l$. We denote by $\mathfrak{g}$ the Lie algebras of $G$. We fix a Borel subgroup $B_G \subset G$, and let $N$ be its unipotent radical and $T = B_G/N$. Let $Z(G)$ be the center of $G$. We denote by $G_{ad} = G/Z(G)$, $B_{ad} = B_G/Z(G)$ and $T_{ad} = T/Z(G)$. We denote the corresponding Lie algebras by $\mathfrak{b}$, $\mathfrak{n}$ and $\mathfrak{t}$.

**A.2. Hitchin map and $p$-Hitchin map.** In this subsection, we recall the definition of Hitchin and $p$-Hitchin map following [N] [CZ1] [BD].

**A.2.1. Hitchin map.** Let $k[\mathfrak{g}]$ and $k[\mathfrak{t}]$ be the algebras of polynomial functions on $\mathfrak{g}$ and $\mathfrak{t}$. By Chevalley’s theorem, we have an isomorphism $k[\mathfrak{g}]^G \simeq k[\mathfrak{t}]^W$. Moreover, $k[\mathfrak{t}]^W$ is isomorphic to a polynomial ring of $l$ variables $u_1, \ldots, u_l$ and each $u_i$ is homogeneous in degree $e_i$. Let $\mathfrak{e} : \text{Spec}(k[\mathfrak{t}]^W)$. Let

$$\chi : \mathfrak{g} \to \mathfrak{e}$$
be the map induced by \( k|c| \simeq k|g|^G \hookrightarrow k|g| \). This is \( G \times \mathbb{G}_m \)-equivariant map where \( G \) acts trivially on \( c \), and \( \mathbb{G}_m \) acts on \( c \) through the gradings on \( k|t|^W \). Let \( \mathcal{L} \) be an invertible sheaf on \( C \) and \( \mathcal{L}^x \) be the corresponding \( \mathbb{G}_m \)-torsor. Let \( \mathfrak{g}_\mathcal{L} = \mathfrak{g} \times \mathbb{G}_m \mathcal{L}^x \) and \( \mathfrak{c}_\mathcal{L} = \mathfrak{c} \times \mathbb{G}_m \mathcal{L}^x \) be the \( \mathbb{G}_m \)-twist of \( \mathfrak{g} \) and \( \mathfrak{c} \) with respect to the natural \( \mathbb{G}_m \)-action.

Let \( \text{Higgs}_{G, \mathcal{L}} = \text{Sect}(C, [\mathfrak{g}_\mathcal{L}/G]) \) be the stack of section of \([\mathfrak{g}_\mathcal{L}/G]\) over \( C \), i.e., for each \( k \)-scheme \( S \) the groupoid \( \text{Higgs}_{G, \mathcal{L}}(S) \) consists of maps over \( C \):

\[
\text{h}_{E, \phi} : C \times S \rightarrow [\mathfrak{g}_\mathcal{L}/G].
\]

Equivalently, \( \text{Higgs}_{G, \mathcal{L}}(S) \) consists of a pair \((E, \phi)\) (called a Higgs bundle), where \( E \) is a \( G \)-torsor over \( C \times S \) and \( \phi \) is an element in \( \Gamma(C \times S, \text{ad}(E) \otimes \mathcal{L}) \). If the group \( G \) is clear from the content, we simply write \( \text{Higgs}_G \) for \( \text{Higgs}_{G, \mathcal{L}} \).

Let \( B_\mathcal{L} = \text{Sect}(C, \mathfrak{c}_\mathcal{L}) \) be the scheme of sections of \( \mathfrak{c}_\mathcal{L} \) over \( C \), i.e., for each \( k \)-scheme \( S \), \( B_\mathcal{L}(S) \) is the set of sections over \( C \)

\[
b : C \times S \rightarrow \mathfrak{c}_\mathcal{L}.
\]

This is called the Hitchin base of \( G \).

The natural \( G \)-invariant projection \( \chi : \mathfrak{g} \rightarrow \mathfrak{c} \) induces a map

\[
[\chi_\mathcal{L}] : [\mathfrak{g}_\mathcal{L}/G] \rightarrow \mathfrak{c}_\mathcal{L},
\]

which in turn induces a natural map

\[
\text{h}_\mathcal{L} : \text{Higgs}_{G, \mathcal{L}} = \text{Sect}(C, [\mathfrak{g}_\mathcal{L}/G]) \rightarrow \text{Sect}(C, \mathfrak{c}_\mathcal{L}) = B_\mathcal{L}.
\]

We call \( h_\mathcal{L} : \text{Higgs}_{G, \mathcal{L}} \rightarrow B_\mathcal{L} \) the Hitchin map associated to \( \mathcal{L} \).

We are mostly interested in the case \( \mathcal{L} = \omega \). For simplicity, from now on we denote \( B = B_\omega \), \( \text{Higgs} = \text{Higgs}_\omega \) and \( h = h_\omega \), etc. We sometimes also write \( \text{Higgs}_G \) for \( \text{Higgs} \) to emphasize the group \( G \).

We fix a square root \( \kappa = \omega^{1/2} \) (called a theta characteristic of \( C \)). Recall that in this case, there is a section \( \epsilon_\kappa : B \rightarrow \text{Higgs} \) of \( h : \text{Higgs} \rightarrow B \), induced by the Kostant section \( k\text{os} : \mathfrak{c} \rightarrow \mathfrak{g} \).

Sometimes, we also call \( \epsilon_\kappa \) the Kostant section of the Hitchin fibration.

A.2.2. \( p \)-Hitchin map. Let \( \text{Loc}_G \) be the stack of (G-loc)-systems on \( C \), i.e., for every scheme \( S \) over \( k \), \( \text{Loc}_G(S) \) is the groupoid of all \( G \)-torsors \( E \) on \( C \times S \) together with a connection \( \nabla : T_{C \times S/S} \rightarrow \mathcal{T}_E \), here \( \mathcal{T}_E \) is the Lie algebroid of infinitesimal symmetry of \( E \). Recall the notion of \( p \)-curvature of a \( G \)-local system following [K] [Bo]: For any \((E, \nabla) \in \text{Loc}_G \) the \( p \)-curvature of \( \nabla \) is defined as

\[
\Psi(\nabla) : F^*T_{C} \rightarrow \text{ad}(E), \quad v \mapsto \nabla(v)^p - \nabla(v^p).
\]

We regard \( \Psi(\nabla) \) as an element \( \Psi(\nabla) \in \Gamma(C, \text{ad}(E) \otimes \omega^p) \) and call such a pair an \( F \)-Higgs field. The assignment \((E, \nabla) \rightarrow (E, \Psi(\nabla)) \) defines a map \( \Psi_G : \text{Loc}_G \rightarrow \text{Higgs}_{G, \omega^p} \). Combining this map with \( h_{\omega^p} \), we get a morphism from \( \text{Loc}_G \) to \( B_{\omega^p} \):

\[
\tilde{h}_p : \text{Loc}_G \rightarrow B_{\omega^p}.
\]

Observe that the pullback along \( F_C : C \rightarrow C^{(1)} \) induces a natural map \( F_{\omega^p} : B^{(1)} \rightarrow B_{\omega^p} \), where the superscript denotes the Frobenius twist. By [CZ1] Theorem 3.1, the \( p \)-curvature morphism \( \tilde{h}_p : \text{Loc}_G \rightarrow B_{\omega^p} \) factors through a unique morphism

\[
h_p : \text{Loc}_G \rightarrow B^{(1)}.
\]

We called this map the \( p \)-Hitchin map.

The construction of \( p \)-Hitchin map can be generalized to \( \lambda \)-connections. Recall that for any \( \lambda \in k \), a \( \lambda \)-connection on a \( G \)-torsor \( E \) is an \( \mathcal{O}_C \)-linear map \( \nabla_\lambda : T_C \rightarrow \mathcal{T}_E \) such that the composition
\(\sigma \circ \nabla_\lambda : T_G \to T_C\) is equal to \(\lambda \cdot \text{id}_{T_G}\), (where \(\sigma : \bar{\nabla}_E \to T_C\) is the natural projection). We denote by \(\text{Loc}_{G,\lambda}\) the stack of \(G\)-bundles on \(C\) with \(\lambda\)-connections. Then

\[
\text{Loc}_{G,1} = \text{Loc}_{G}, \quad \text{Loc}_{G,0} = \text{Higgs}_G.
\]

Let \((E, \nabla_\lambda) \in \text{Loc}_{G,\lambda}\). The \(p\)-curvature of \(\nabla_\lambda\) is defined as

\[
\Psi(\nabla_\lambda) : F^* T_C \to \text{ad}(E), \quad v \to \nabla_\lambda(v^p) - \lambda^{p-1}\nabla_\lambda(v^p).
\]

The map \(\text{Loc}_{G,\lambda} \to B_{\omega^p}, (E, \nabla_\lambda) \mapsto h_{\omega^p}(E, \Psi(\nabla_\lambda))\) factors through a unique map

\[
h_{p,\lambda} : \text{Loc}_{G,\lambda} \to B^{(1)},
\]

called the \(p\)-Hitchin map for \(\lambda\)-connections. It is clear that \(h_{p,1} = h_p\) and \(h_{p,0} = F \circ h\), where \(h : \text{Higgs} \to B\) is the usual Hitchin map and \(F : B \to B^{(1)}\) is the relative Frobenius of \(B\). From this perspective, the \(p\)-Hitchin map can be regarded as a deformation of the usual Hitchin map.

### A.3. Oper with marking

In this subsection, we recall the definition of opers with marking following [B]. There is a canonical decreasing Lie algebra filtration \(\{\mathfrak{g}^k\}\) of \(\mathfrak{g}\)

\[
\cdots \supset \mathfrak{g}^{-1} \supset \mathfrak{g}^0 \supset \mathfrak{g}^1 \supset \cdots
\]

such that \(\mathfrak{g}^0 = \mathfrak{b}, \mathfrak{g}^1 = \mathfrak{n}\) and for any \(i > 0\) (resp. \(< 0\)) weights of the action of \(t = \mathfrak{g}^0(\mathfrak{g})\) on \(\mathfrak{g}^i(\mathfrak{g})\) are sums of \(i\) simple positive (resp. \(-i\) simple negative) roots. In particular, we have \(\mathfrak{g}^{-1}(\mathfrak{g}) = \oplus \mathfrak{g}_\alpha\), where \(\alpha\) is a simple negative root and \(\mathfrak{g}_\alpha\) is the corresponding root space.

Let \(E\) be a \(B_G\)-torsor on \(C\) and \(E_G\) (resp. \(E_T\)) be the induced \(G\)-torsor (resp. \(T\)-torsor) on \(C\). In this subsection, we denote by \(\mathfrak{b}_E\) and \(\mathfrak{g}_{E_G} = \mathfrak{g}_E\) the associated adjoint bundles. Let \(\bar{T}_E\) and \(\bar{T}_{E_G}\) be the Lie algebroids of infinitesimal symmetries of \(E\) and \(E_G\). There is a natural embedding \(\bar{T}_E \to \bar{T}_{E_G}\) and we have a canonical isomorphism

\[
\bar{T}_{E_G}/\bar{T}_E \simeq (\mathfrak{g}/\mathfrak{b})_E =: E \times^B (\mathfrak{g}/\mathfrak{b}).
\]

For any connection \(\nabla\) on \(E_G\), we denote by \(\bar{\nabla}\) the composition

\[
\bar{\nabla} : T_C \xrightarrow{\nabla} \bar{T}_{E_G} \to \bar{T}_{E_G}/\bar{T}_E \simeq (\mathfrak{g}/\mathfrak{b})_E.
\]

**Definition A.3.** We fix a square root \(\kappa = \omega^{1/2}\) of the canonical bundle \(\omega\). A \(G\)-oper on \(C\) with marking is a triple \((E, \nabla, \phi)\) where \(E\) is a \(B_G\)-torsor on \(C\), \(\nabla\) is a connection on \(E_G\), and \(\phi : E_T \simeq \omega^{1/2} \times^G \mathbb{A}^p T\) is an isomorphism of \(T\)-torsor (we call \(\phi\) the marking), such that

1. The image of \(\bar{\nabla}\) lands in \((\mathfrak{g}^{-1}/\mathfrak{b})_E \subset (\mathfrak{g}/\mathfrak{b})_E\).
2. The composition

\[
T_C \xrightarrow{\bar{\nabla}} (\mathfrak{g}^{-1}/\mathfrak{b})_E \xrightarrow{pr_\alpha} (\mathfrak{g}_\alpha)_E
\]

is an isomorphism for every simple negative root \(\alpha\). Here

\[
pr_\alpha : (\mathfrak{g}^{-1}/\mathfrak{b})_E = \oplus (\mathfrak{g}_\beta)_E \to (\mathfrak{g}_\alpha)_E
\]

is the natural projection.
3. The condition (2) implies \(\bar{\nabla}\) induces an isomorphism

\[
E_T \times^T (\mathfrak{g}^1/\mathfrak{g}^2) \simeq \oplus_{i=1}^r (\mathfrak{g}_{\alpha_i})_E \xrightarrow{\hat{\phi}} (\omega^{1/2} \times^G \mathbb{A}^p T) \times^T (\mathfrak{g}^1/\mathfrak{g}^2).
\]

We require the marking \(\phi\) to be compatible with \(\hat{\phi}\).
We denote by $\text{Op}_G$ the scheme of $G$-opers with marking on $C$. Notice that if we drop condition (3) and the data of $\phi$ in the above definition, then we obtain the definition of $G$-opers in [BD]. As shown in loc. cit., a $G$-oper has $Z(G)$ as its automorphism group, the additional condition (3) eliminate these automorphisms (cf. [B Proposition 2.1]).

**Remark A.4.** When $G$ is of adjoint type, there exits a unique trivialization of $E$ such that the condition (3) is automatic. In general, the conditions (1) and (2) do not imply existence of $\phi$, hence we are limiting our collection of opers compared to [BD].

**Example A.5.** Consider the case $G = GL_n$. Then an oper with marking can be described in terms of vector bundles as follows: it consists of the data $(E, \{E_i\}_{i=1,\ldots,n}, \nabla, \phi)$ where $E$ is a rank $n$ vector bundle on $C$, $E_1 \subset E_2 \subset \cdots \subset E_n = E$ is a complete flag, $\nabla$ is a connection on $E$, and $\phi : E_1 \simeq \omega^{(n-1)/2}$ is an isomorphism, such that

1. $\nabla(E_i) \subset E_{i+1} \otimes \omega$.
2. For each $i$, the induced morphism $\text{gr}(E) \overset{\text{gr}(\nabla)}{\rightarrow} \text{gr}(E) \otimes \omega$ is an isomorphism.

**A.3.1. Affine space structure on $\text{Op}_G$.** We follow closely the presentation in [BD, 3.1.7-3.1.9]. We first consider the case $G = SL_2$. Let $(E, \nabla, \phi) \in \text{Op}_{SL_2}$. The isomorphism $\text{gr}(\nabla)$ takes the form $(n)_E \simeq E \times E \simeq E_T \times (g^1/g^2) \simeq \omega.$ Translating $\nabla$ by a section of $\omega^2 \simeq (n)_E \otimes \omega$, we get a new oper with marking and a direct computation shows that the resulting $\Gamma(C, \omega^2)$-action makes $\text{Op}_{SL_2}$ a $\Gamma(C, \omega^2)$-torsor.

Let $G$ be general reductive group. We pick a principal embedding $\iota : sl_2 \hookrightarrow g$ and a homomorphism $\iota : SL_2 \rightarrow G$ with $\text{dim}i = t$. Let us write $B_0 = T_0N_0$ for the standard Borel subgroup of $SL_2$ and $b_0 = t_0 + n_0$ the Borel subalgebra of $sl_2$. Without loss of generality we can assume $i(B_0) \subset B$. Let $\{e_0, h_0, f_0\}$ be the standard basis of the principal $sl_2 \subset g$ and let $V = g^{e_0}$. The grading on $g = \oplus_i g_i$ induces a grading $V = \oplus_i V_i$ on $V$ and we consider the $\rho^+(G_m)$-action on $V$ given by $t \cdot v_i = t^{i+1}v_i$ for $v \in V_i$. We set $V_0 = V \times G_m.$$\rho^+(G_m)$ $\omega$. Then the embedding $n_0 = \mathbb{C}e_0 \rightarrow V$ gives rise to an embedding

\begin{equation}
\omega^2 \simeq (n_0 \times T_0 \omega^{1/2}) \otimes \omega \rightarrow (V \times T_0, \omega^{1/2}) \otimes \omega \simeq V \times G_m.$$\rho^+(G_m)$ $\omega = V_0.$

Let $(E_0, \nabla_0, \phi_0) \in \text{Op}_{SL_2}$. The push-forward $i(E_0, \nabla_0, \phi_0) = (E, \nabla, \phi)$ along the principal homomorphism $\iota : SL_2 \rightarrow G$ is a $G$-oper with marking. Moreover we have an embedding

\begin{equation}
V_0 \simeq (V \times T_0, \omega^{1/2}) \otimes \omega \simeq (V \times B_0, E_0) \otimes \omega \rightarrow (B \times B E) \otimes \omega = (b)_E \otimes \omega.
\end{equation}

Thus translating $\nabla$ by a section of $v \in \Gamma(C, V_0) \subset \Gamma(C, (b)_E \otimes \omega)$ defines a new $G$-oper with marking denoted by $\nabla + v$. Consider the following $\Gamma(C, V_0)$-torsor

\[ \text{Op}_{SL_2} \times \Gamma(C, \omega^2) \Gamma(C, V_0), \]

where $\Gamma(C, \omega^2)$ acts on $\Gamma(C, V_0)$ via the map (A.2).

**Lemma A.6.** The principal $\iota : SL_2 \rightarrow G$ gives rise to an isomorphism

\[ \text{Op}_{SL_2} \times \Gamma(C, \omega^2) \Gamma(C, V_0) \simeq \text{Op}_G \quad (\nabla_0, v) \rightarrow \nabla + v. \]

**Proof.** It suffices to check the assertion locally. So it is enough to show that for any $(E, \nabla, \phi) \in \text{Op}_G$, there exists a unique trivialization of $E$ such that the connection has the form $\nabla = dx \otimes (f_0 + v)$.
where \( v : C \to V \). This follows from Kostant’s theorem, see [BDop] Section 3.4 or [B] Lemma 3.5].

A.3.2. Filtration on \( \mathcal{O}(\text{Op}_G) \). One defines a \((G, \lambda)\)-oper with marking as before by replacing connection \( \nabla \) by \( \lambda \)-connection \( \nabla_\lambda \). We denote by \( \text{Op}_{G,\lambda} \) the scheme of \((G, \lambda)\)-opers with marking. Clearly we have \( \text{Op}_{G,1} = \text{Op}_G \).

All \((G, \lambda)\)-opers with marking form a scheme equipped with a morphism \( q : \widetilde{\text{Op}_G} \to \mathbb{A}^1 \), such that the fiber of \( \text{Op}_G \) over \( \lambda \in \mathbb{A}^1(k) \) is \( \text{Op}_{G,\lambda} \). Moreover, there a \( \mathbb{G}_m \)-action on \( \text{Op}_G \), given by \((E, \nabla) \mapsto (E, t\nabla)\) and the morphism \( q \) is \( \mathbb{G}_m \)-equivariant. In addition, if we let \( \text{Bun}_N^\kappa \) denote the moduli of \( B_G \)-bundles \( E \) on \( C \) (say \( C \) projective) equipped with an isomorphism \( \phi : E_T \simeq \omega^{1/2} \times \mathbb{G}_m^2 \mathfrak{T} \) as in Definition [A.3], then there is a natural morphism \( f : \widetilde{\text{Op}_G} \to \text{Bun}_N^\kappa \). When \( G = \text{SL}_2 \), there are canonical isomorphisms \( \text{Bun}_N^\kappa \simeq H^1(C, \omega) \simeq \mathbb{A}^1 \) under which \( f \) is identified with \( q \).

By the same discussions as in Lemma [A.6] and the discussions before that, there is an action of \( \Gamma(C, \omega^2) \) on \( \text{Op}_{\text{SL}_2} \), and the map \( f \) (and \( q \)) is \( \Gamma(C, \omega^2) \)-equivariant. In addition, there is a canonical isomorphism

\[
\text{Op}_{\text{SL}_2} \times^{\Gamma(C, \omega^2)} \Gamma(C, V_\omega) \simeq \widetilde{\text{Op}_G}, \quad (\nabla_\lambda, v) \mapsto \iota_v(\nabla_\lambda) + v.
\]

**Lemma A.7.** The map \( q : \widetilde{\text{Op}_G} \to \mathbb{A}^1 \) is flat.

**Proof.** We will actually prove that the morphism is smooth. By the above isomorphism, it is enough to prove the statement for \( G = \text{SL}_2 \).

Recall that a morphism \( f : X \to Y \) of finite type smooth schemes over an algebraically closed field \( k \) is smooth if and only if for every closed point \( x \) of \( X \), the induced map of tangent spaces \( df : T_x X \to T_{f(x)} Y \) is surjective. Now we compute the tangent map in our situation. Since the map is \( \mathbb{G}_m \)-equivariant and has smooth fibers over \( \lambda \neq 0 \) by Lemma [A.6] it is smooth over \( \lambda \neq 0 \). Thus it is enough to compute it at points of \( \text{Op}_{\text{SL}_2} \) lying over \( \lambda = 0 \). Let \( (E_0, \nabla_0) \) be such a point (so \( \nabla_0 \) is a Higgs field). Then a standard deformation theory argument shows that the tangent space of \( \text{Op}_{\text{SL}_2} \) at this point is \( H^1(C, \omega) \oplus H^0(C, \omega^2) \) and the differential \( dq \) is identified with the projection to the first factor \( H^1(C, \omega) = k \). The lemma is proved.

We have the forgetful map \( \text{Op}_{G,\lambda} \to \text{LocSys}_{G,\lambda}, \quad (E, \nabla_\lambda, \phi) \mapsto (E_G, \nabla_\lambda) \) and at \( \lambda = 0 \)

\[
(A.4) \quad B \simeq \text{Op}_{G,0} \to \text{Locg}, 0 = \text{Higgs}_G
\]

is the Kostant section \( \epsilon_\kappa \) induced by \( \kappa = \omega^{1/2} \).

The \( p \)-Hitchin map for \( \lambda \)-connections gives

\[
\tilde{\pi}_p : \widetilde{\text{Op}_G} \to B^{(1)} \times \mathbb{A}^1, \quad (E, \nabla_\lambda, \phi) \mapsto (h_{p,\lambda}(E_G, \nabla_\lambda), \lambda).
\]

The map \( \tilde{\pi}_p \) is compatible with the natural morphisms to \( \mathbb{A}^1 \) from both sides (namely the morphism \( q \) on the left and the second projection on the right), and is \( \mathbb{G}_m \)-equivariant where \( \mathbb{G}_m \) acts diagonally on \( B^{(1)} \times \mathbb{A}^1 \). We denote by \( \pi_{p,\lambda} : \text{Op}_{G,\lambda} \to B^{(1)} \) the base change of \( \tilde{\pi}_p \) to \( \lambda \in \mathbb{A}^1(k) \). When \( \lambda = 1 \), we get a map

\[
(A.5) \quad \pi_p := \pi_{p,1} : \text{Op}_G \to \text{Locg}, \quad h_{p,\lambda} \circ B^{(1)},
\]

and \( (A.4) \) implies \( \pi_{p,0} : B = \text{Op}_{G,0} \to B^{(1)} \) is the relative Frobenius morphism \( F : B \to B^{(1)} \).

\[\text{In loc. cit. the authors assume } G \text{ is adjoint, but since Kostant’s theorem holds for general reductive groups, the same proof applies to the general case.}\]
We recall the well-known equivalence of categories between $k$-algebras equipped with an exhaustive filtration and graded flat $k[\lambda]$-algebras (with $\deg \lambda = 1$). Namely, the equivalence sends a $k$-algebra $A$ equipped with an increasing filtration $F_\bullet A$ such that $k \subset F_0 A$, $\cup_i F_i A = A$, to the graded flat $k[\lambda]$-algebra $R = \oplus_i F_i A \lambda^i$. The quasi-inverse functor is given by sending $R = \oplus_i R_i$ to $A = R/(\lambda - 1)R$ with $F_i A = \Im(R_i \to A)$. Note that under this equivalence, $R/\lambda R$ can be identified with the associated graded of $A$. Now the discussion above imply the following lemma.

**Lemma A.8.** Let $\pi_p^* : \mathcal{O}(B^{(1)}) \to \mathcal{O}(\text{Op}_G)$ be the map of ring of functions corresponding to $\pi_p : \text{Op}_G \to B^{(1)}$. Then there are filtrations on $\mathcal{O}(\text{Op}_G)$ and $\mathcal{O}(B^{(1)})$ such that

1. The associated graded $\text{gr}(\mathcal{O}(\text{Op}_G)) \simeq \mathcal{O}(B)$ and $\text{gr}(\mathcal{O}(B^{(1)})) \simeq \mathcal{O}(B^{(1)})$.
2. $\pi_p^*$ is compatible with the filtrations.
3. The induced morphism $\text{gr}(\pi_p^*) : \mathcal{O}(B^{(1)}) \to \mathcal{O}(B)$ is the relative Frobenius map.

We need one more property of the above filtration on $\text{Op}_G$.

**Lemma A.9.** The exhaustive filtration on $\text{Op}_G$ is separated. In fact, $F_{-1} \mathcal{O}(\text{Op}_G) = 0$.

**Proof.** We claim that the degree $-1$ part of $\mathcal{O}(\widetilde{\text{Op}_G})$ is zero. First, we know that the $\mathbb{G}_m$-action on $\text{Op}_{G,0}$ is just by dilation of the vector space. So $\text{gr} \mathcal{O}(\text{Op}_G)$ is concentrated in non-negative degrees. It follows that

$$\cdots \xrightarrow{\lambda} \mathcal{O}(\widetilde{\text{Op}_G})^{-2} \xrightarrow{\lambda} \mathcal{O}(\widetilde{\text{Op}_G})^{-1} \xrightarrow{\lambda} \mathcal{O}(\widetilde{\text{Op}_G})^0 \xrightarrow{\lambda} \mathcal{O}(\widetilde{\text{Op}_G})^1 \xrightarrow{\lambda} \cdots .$$

We let $\mathcal{O}(\widetilde{\text{Op}_G})^{\text{deg}} \subset \mathcal{O}(\widetilde{\text{Op}_G})$ be the graded subspace with $\mathcal{O}(\widetilde{\text{Op}_G})^{\text{deg},m} = \lambda^m \mathcal{O}(\widetilde{\text{Op}_G})^{-n}$ for $n > 0$. Then $\mathcal{O}(\widetilde{\text{Op}_G})^{\text{deg}}$ is a graded ideal, defining a flat $\mathbb{G}_m$-invariant closed subscheme $\text{Op}_G^{\text{deg}} \subset \widetilde{\text{Op}_G}$ satisfying $\widetilde{\text{Op}_G}^{\text{nondeg}}|_0 = \widetilde{\text{Op}_G}|_0$. As $\widetilde{\text{Op}_G}|_1 = \text{Op}_G$ is smooth irreducible and $\dim(\widetilde{\text{Op}_G}|_1) = \dim(\widetilde{\text{Op}_G}|_0)$, we must have $\widetilde{\text{Op}_G}^{\text{nondeg}} = \widetilde{\text{Op}_G}$. It follows that $\mathcal{O}(\widetilde{\text{Op}_G})^{-1} = 0$. \square

**A.4. Proof of Theorem A.1.** Let $\pi_p : \text{Op}_G \to B^{(1)}$ be the map in (A.3). We first show that $\pi_p$ is finite and surjective. Thus we need to show that $\pi_p^* : \mathcal{O}(B^{(1)}) \to \mathcal{O}(\text{Op}_G)$ is injective and $\mathcal{O}(\text{Op}_G)$ is finitely generated as an $\mathcal{O}(B^{(1)})$-module. Since both rings $\mathcal{O}(\text{Op}_G)$ and $\mathcal{O}(B^{(1)})$ are filtered and $\pi_p^*$ is compatible with the filtrations, it is enough to show that the associated graded map $\text{gr}(\pi_p^*) : \text{gr}(\mathcal{O}(B^{(1)})) \to \text{gr}(\mathcal{O}(\text{Op}_G))$ is injective and $\text{gr}(\mathcal{O}(\text{Op}_G))$ is a finitely generated $\text{gr}(\mathcal{O}(B^{(1)}))$-module. But this is clear since by the lemma above $\text{gr}(\pi_p^*)$ is the Frobenius map. Now $\pi_p$ is a finite map between Op$_G$ and B$^{(1)}$, which are smooth of the same dimension, and therefore it is flat. In addition, as the relative Frobenius map $B \to B^{(1)}$ is of degree $p^{\dim B}$, so is $\pi_p$.

**Remark A.10.** Lemma[A.8] shows that the map $\pi_p : \text{Op}_G \to B^{(1)}$ is a deformation of the Frobenius morphism $Fr : B \to B^{(1)}$. In the special case when $G = GL(1)$ it is not hard to see that one can identify Op$_{GL(1)}$ with $B = \Gamma(C, \omega)$ so that the morphism $\pi_p$ is identified with $Fr - \mathcal{E}$ where $\mathcal{E} : \Gamma(C, \omega) \to \Gamma(C, \omega)^{(1)}$ is the map induced by Cartier isomorphism; it is closely related to the Hasse-Witt matrix of $C$. In particular, $\pi_p$ is purely inseparable if and only if $C$ is supersingular. It would be interesting to obtain a similar explicit description of the map $\pi_p$ for nonabelian $G$. 


**Remark A.11.** There is a parallel between analytic constructions involving a complex algebraic variety $X_C$ and algebraic constructions involving an algebraic variety $X_k$ over a field $k$ of characteristic $p > 0$. Thus the analytic exponential function is analogous to the Artin-Schreier polynomial $x^p - x$ and the Stone–von Neumann Theorem is parallel to the Azumaya property of crystalline differential operators: the former asserts that unique representation of a Heisenberg group on a Hilbert space is realized as $L^2(\mathbb{R}^n)$, while the latter implies that unique irreducible representation of the ring of differential operators on $\mathbb{A}^n$ with a fixed central character is realized as the space of functions on the Frobenius neighborhood of a point $O(\text{Fr}_N(x))$, $x \in \mathbb{A}^n$, see [BMR, Remark 2.2.4(3)].

Another parallel appearing in the literature is between the nonabelian Hodge theory for local systems on a complex curve and the (partially defined) Cartier transform [OV]: both construction produce a Higgs field starting from a local system on the curve. Notice that the support of the Cartier transform (when defined) of a $D$-module coincides with the support of the $D$-module as a module over the $p$-center.

In [EFK] one finds a conjectural description of the spectrum $S_C$ of the ring of global twisted differential operators on $\text{Bun}_G$ for a complex curve acting on the space of $L^2$ sections of $\Omega^{1/2}_{\text{Bun}}$. The above observation connecting $L^2(\mathbb{R}^n)$ to $O(\text{Fr}_N(x))$ suggests that $S_C$ is analogous to $S_k$, the spectrum of the action of the ring $\Gamma(D_{\text{Bun}})$ acting on the space of sections of $\Gamma(\Omega^{1/2}_{\text{Bun}} \mathcal{F}r_N(x))$, $x \in \text{Bun}$. Notice that this module has zero $p$-curvature, so Theorem A.1.1 implies that $S_k$ is a subset in $\pi^{-1}_p(0)$, the set of opers with zero $p$-curvature (dormant opers in the terminology of [Mo]). The analogy described in the previous paragraph suggests that the corresponding characteristic zero object should be the set of opers which under the nonabelian Hodge theory corresponds to a Higgs field $(E, \phi)$ with $\phi = 0$. A local system corresponding to such a Higgs field under the nonabelian Hodge theory has unitary monodromy. On the other hand, [EFK, Conjecture 1.11] asserts that $S_C$ is a subset in the set of opers with real monodromy. Thus the above heuristics agrees with the Conjecture of [EFK] up to the change of the real form of the complex group $G_C$.

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