ON COSILTING HEARTS OVER THE KRONECKER ALGEBRA

Alessandro Rapa

Abstract

This paper is about the hearts arising from torsion pairs of finite type in the category of modules over the Kronecker algebra. After a characterization of the simple objects in these hearts, we describe their atom spectrum and compute their Gabriel dimension.

1 Introduction

With the introduction of $\tau$-tilting theory for finite dimensional algebras by Adachi, Iyama and Reiten, in [1], and the proof of the bijection between support $\tau$-tilting modules and functorially finite torsion classes, a renewed focus has been established on the study of torsion pairs in the category of finite dimensional modules over a finite dimensional algebra. After that, in [20], the authors looked at the bigger picture by studying the collection of all the torsion pairs, not only the functorially finite ones, in the category of finite dimensional modules over a finite dimensional algebra, exploring its lattice theoretical properties.

For a finite dimensional algebra $\Lambda$, the torsion pairs in the category $\Lambda$-mod of finite dimensional $\Lambda$-modules are in bijection with the torsion pairs of finite type in the category $\Lambda$-Mod of all $\Lambda$-modules (see [18]). A torsion pair is of finite type if the torsionfree class is closed under direct limits. In this paper we direct our attention on the torsion pairs of finite type of $\Lambda$-Mod and on the hearts of the t-structures arising from them, describing properties that come from the characterization of their simple objects.

After this, we use the description of the simples to compute the Gabriel dimension.

The construction of a t-structure arising from a torsion pair goes back to Happel, Reiten and Smalø [23]. It is well known that the heart of a t-structure is always an abelian category and, in our specific case, since the considered torsion pairs are cogenerated by a cosilting $\Lambda$-module (cf. [15] and [39]).

First of all, we give a complete description of the simple objects in these hearts and their injective envelopes. In order to do so we use the notion of torsionfree, almost torsion object defined in [3] (together with the dual concept of torsion, almost torsionfree object) and introduced also in [12] under the name of minimal extending modules. Torsionfree, almost torsion objects, with respect to a given torsion pair, are torsionfree objects whose proper quotients are torsion (almost torsionfree objects are defined dually). In a sense, one can think about torsionfree, almost torsion objects as objects very close to the “border” of the torsion part.

After this, we use the description of the simples to compute the atom spectrum of the different hearts. The atom spectrum has been introduced by Kanda, in [24], for a general abelian category and it is a generalization of the prime spectrum for commutative rings. Accordingly, it has a structure of topological space and, for a Grothendieck category, it is strongly related to the spectrum of the indecomposable injective objects. The elements of the atom spectrum are called atoms and they are built as equivalence classes of monoform objects. For a monoform object $X$, the corresponding atom is denoted by $\overline{X}$.

The author acknowledges partial support by Fondazione Cariverona, program ”Ricerca Scientifica di Eccellenza 2018”, project ”Reducing complexity in algebra, logic, combinatorics - REDCOM”

Theorem (Theorem 7.5). Let $\Lambda$ be the Kronecker algebra (over a field $k$). Denote by $P_1$ and $Q_1$ the simple projective and the simple injective $\Lambda$-modules, respectively. For a torsion pair of finite type $\mathfrak{t}$ in $\Lambda$-Mod, we denote by $\mathcal{A}$ the heart of the t-structure arising from $\mathfrak{t}$.

If $\mathfrak{t}$ is cogenerated by a finite dimensional cosilting $\Lambda$-module, we get:

- $\text{ASpec}(\mathcal{A}) \cong \text{ASpec}(k$-$\text{Mod}) = \{k\}$, for the torsion pair cogenerated by $Q_1$.
- $\text{ASpec}(\mathcal{A}) \cong \text{ASpec}(\Lambda$-$\text{Mod}) = \{P_1, Q_1\}$, for all the other torsion pairs.
Let \( t \) be the family of homogeneous tubes in \( \Lambda\text{-Mod} \), indexed by the projective line \( \mathbb{P}^1_k \). Consider \( U \subseteq \mathbb{P}^1_k \) and denote by \( G \) the generic \( \Lambda \)-module. If \( t \) is generated by \( q \) or by a nonempty subset of \( t \) indexed by \( U \subseteq \mathbb{P}^1_k \), then:

\[
\text{ASpec}(\mathcal{A}) = \bigcup \left\{ \bigcup \{ S \mid S \text{ simple regular in a tube indexed by } U \} \cup \bigcup \{ S \mid S \text{ simple regular in a tube indexed by } \mathbb{P}^1_k \setminus U \} \right\}.
\]

In the last part of the paper we give a direct computation of the Gabriel dimension of the different hearts. The notion of Gabriel dimension has been introduced by Gabriel, in [21], under the name of Krull dimension, as a way to understand the complexity of a Grothendieck category using an iterated localization procedure. Whenever the Grothendieck category is a Gabriel category, i.e., a category with finite Gabriel dimension, the atom spectrum is in bijective correspondence with the spectrum of the indecomposable injective objects (see [36]). Our result is that all the hearts arising from the torsion pairs of finite type are Gabriel categories, giving an a posteriori evidence of the fact that the atom spectrum of every heart bijectively corresponds to the spectrum of the indecomposable injective objects in that heart.

## 2 Preliminaries

### 2.1 Torsion pairs and t-structures

Let \( \mathcal{G} \) be a Grothendieck category. Let \( \mathcal{M} \) be a class of objects in \( \mathcal{G} \) and let \( X \in \mathcal{G} \). We say that:

- \( X \) is generated by \( \mathcal{M} \), if \( X \) is a quotient object of coproducts of objects in \( \mathcal{M} \).
- \( X \) is cogenerated by \( \mathcal{M} \), if \( X \) is a subobject of products of objects in \( \mathcal{M} \).

We denote by:

- \( \text{Gen}\mathcal{M} \): the class of all objects in \( \mathcal{G} \) generated by \( \mathcal{M} \).
- \( \text{Cogen}\mathcal{M} \): the class of all objects in \( \mathcal{G} \) cogenerated by \( \mathcal{M} \).
- \( \text{Add}\mathcal{M} \): the class of objects in \( \mathcal{G} \) isomorphic to a direct summand of a direct product of objects in \( \mathcal{M} \).
- \( \text{Prod}\mathcal{M} \): the class of objects in \( \mathcal{G} \) isomorphic to a direct summand of a direct product of objects in \( \mathcal{M} \).

If \( \mathcal{M} = \{ M \} \) for \( M \in \mathcal{G} \), we write Gen\( M \), Cogen\( M \), Add\( M \) and Prod\( M \). All these classes are full subcategories of \( \mathcal{G} \). We say that:

- \( \mathcal{M} \) is generating for \( \mathcal{G} \) if \( \mathcal{G} = \text{Gen}\mathcal{M} \).
- \( \mathcal{M} \) is cogenerated for \( \mathcal{G} \) if \( \mathcal{G} = \text{Cogen}\mathcal{M} \).

**Definition 2.1.** A torsion pair is a pair \( t = (T, F) \), where \( T \) and \( F \) are two full subcategories of \( \mathcal{G} \), such that:

1. \( \text{Hom}_{\mathcal{G}}(T, F) = 0 \).
2. For any \( X \in \mathcal{G} \), there is a short exact sequence \( 0 \to T \to X \to F \to 0 \), where \( T \in T \) and \( F \in F \).

We call \( T \) (resp. \( F \)) the torsion class (resp. the torsionfree class). We say that a torsion pair \( t \) is:

- split: if every short exact sequence \( 0 \to T \to X \to F \to 0 \), with \( T \in T \) and \( F \in F \), splits.
- hereditary: if the torsion class \( T \) is closed under subobjects.
- of finite type: if the torsionfree class \( F \) is closed under direct limits.

Given a class of objects \( \mathcal{M} \subset \mathcal{G} \), we set \( \mathcal{M}^{+o} = \text{Ker Hom}_{\mathcal{G}}(\mathcal{M}, -) \) and \( \mathcal{M}^{+1} = \text{Ker Ext}^1_{\mathcal{G}}(\mathcal{M}, -) \). Dually, we define the classes \( ^{+o}\mathcal{M} \) and \( ^{+1}\mathcal{M} \). If \( \mathcal{M} = \{ M \} \) for \( M \in \mathcal{G} \), we write \( M^{+o} \), \( M^{+1} \), \( ^{+o}M \) and \( ^{+1}M \).

**Remark 2.2.** Fix a torsion pair \( t = (T, F) \), it follows from the definition that \( F = T^{+o} \) and \( T = F^{+1} \). In particular, \( T \) is closed under extensions, quotient objects and all coproducts that exist in \( \mathcal{G} \) and, dually, \( F \) is closed under extensions, subobjects and products.

Let \( \mathcal{M} \) be a class of objects in \( \mathcal{G} \) and \( t = (T, F) \) a torsion pair in \( \mathcal{G} \). We have that:

- \( t \) is generated by \( \mathcal{M} \) if \( F = M^{+o} \) (and \( T = ^{+o}(M^{+o}) \)).
- \( t \) is cogenerated by \( \mathcal{M} \) if \( T = ^{+o}M \) (and \( F = (^{+o}M)^{+o} \)).

If \( \mathcal{G} \) is a locally noetherian Grothendieck category, denote by \( \mathcal{G}_0 = \text{fp}(\mathcal{G}) \) the full subcategory of finitely presented objects. We have the following:

**Theorem 2.3.** [13 §4.4/17 Lemma 3.11] Let \( \mathcal{G} \) be a locally noetherian Grothendieck category. There is a bijective correspondence between as follows:

\[
\begin{align*}
\{ \text{torsion pairs of finite type in } \mathcal{G} \} & \longleftrightarrow \{ \text{torsion pairs in } \mathcal{G}_0 \} \\
(T, F) & \longleftrightarrow (T \cap \mathcal{G}_0, F \cap \mathcal{G}_0) \\
(\lim T_0, \lim F_0) & \longleftrightarrow \mathcal{G}_0(T_0, F_0)
\end{align*}
\]

Moreover, \((\lim T_0, \lim F_0)\) coincides with the torsion pair \((\text{Gen} T_0, \text{Cogen} T_0)\) generated by \( T_0 \).
Consider now a triangulated category \( \mathcal{D} \), with shift functor \( [1] \).

**Definition 2.4.** A pair of full subcategories of \( \mathcal{D} \), \((\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})\), is called a \( t \)-structure if it satisfies the properties below.

We use the following notation: \( \mathcal{D}^{\leq n} = \mathcal{D}^{\leq 0}[-n] \) and \( \mathcal{D}^{\geq n} = \mathcal{D}^{\geq 0}[-n] \).

1. \( \text{Hom}_\mathcal{D}(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1}) = 0 \),
2. \( \mathcal{D}^{\leq 0} \subseteq \mathcal{D}^{\leq 1} \) (and \( \mathcal{D}^{\geq 0} \supseteq \mathcal{D}^{\geq 1} \)),
3. For every object \( X \in \mathcal{D} \), there is a triangle \( A \to X \to B \to A[1] \), with \( A \in \mathcal{D}^{\leq 0} \) and \( B \in \mathcal{D}^{\geq 1} \).

For a \( t \)-structure \((\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})\), the full subcategory defined as:

\[
\mathcal{A} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}
\]

is called the heart of the \( t \)-structure. \( \mathcal{A} \) is always an abelian category and its abelian structure comes from the triangulated structure of \( \mathcal{D} \) (i.e., a short exact sequence \( 0 \to X \to Y \to Z \to 0 \) in \( \mathcal{A} \) is exact if and only if there is a triangle \( X \to Y \to Z \to X[1] \) in \( \mathcal{D} \) with \( X, Y, Z \in \mathcal{A} \)).

**Example 2.5.** Consider a Grothendieck category \( \mathcal{G} \) and a torsion pair \((\mathcal{Q}, \mathcal{C})\) in it. The full subcategories of \( \mathcal{D}^b(\mathcal{G})\):

\[
\mathcal{D}^{\leq 0} = \{ X \in \mathcal{D}^b(\mathcal{G}) \mid H^0(X) \in \mathcal{Q}, H^i(X) = 0 \ for \ i > 0 \},
\]

\[
\mathcal{D}^{\geq 0} = \{ X \in \mathcal{D}^b(\mathcal{G}) \mid H^{-1}(X) \in \mathcal{C}, H^i(X) = 0 \ for \ i < -1 \}
\]

form a \( t \)-structure \((\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})\), called the HRS-\( t \)-structure, see \[23\] Proposition 2.1. Its heart is the category:

\[
\mathcal{A} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0} = \{ X \in \mathcal{D}^b(\mathcal{G}) \mid H^0(X) \in \mathcal{Q}, H^{-1}(X) \in \mathcal{C}, H^i(X) = 0 \ for \ i \neq -1, 0 \}
\]

and it is called the HRS-heart. In the sequel, we will denote by \( \mathcal{A} = \mathcal{G}(\mathcal{Q}, \mathcal{C}) \) the HRS-heart of the HRS-\( t \)-structure arising from the torsion pair \((\mathcal{Q}, \mathcal{C})\) on the category \( \mathcal{G} \).

For any two objects \( X, Z \in \mathcal{A} \) there are functorial isomorphisms \( \text{Ext}^i_{\mathcal{A}}(X, Z) \cong \text{Hom}_{\mathcal{D}^b(\mathcal{G})}(X, Z[i]) \), for \( i = 0, 1 \). Moreover, the pair \((\mathcal{C}[1], \mathcal{Q})\) is a torsion pair in \( \mathcal{A} = \mathcal{G}(\mathcal{Q}, \mathcal{C}) \), see \[23\] Corollary 1.2.2(b).

We recall some conditions on the torsion pair in \( \mathcal{G} \) affecting the HRS-heart \( \mathcal{A} = \mathcal{G}(\mathcal{Q}, \mathcal{C}) \).

**Theorem 2.6.**

(i) \[23\] Theorem 5.2] \( \mathcal{A} \) is hereditary if and only if the torsion pair \((\mathcal{Q}, \mathcal{C})\) is split and \( \text{pdim}_{\mathcal{A}} Q \leq 1 \), for any \( Q \in \mathcal{Q} \).

(ii) \[24\] Corollary 4.10] Suppose that either \( \mathcal{C} \) is generating or \( \mathcal{Q} \) is cogenerating for \( \mathcal{G} \). Then \( \mathcal{A} \) is a Grothendieck category if and only if \((\mathcal{Q}, \mathcal{C})\) is of finite type in \( \mathcal{G} \).

(iii) \[23\] Theorem 5.2] If \( \mathcal{G} \) is locally noetherian, then \( \mathcal{A} \) is a locally coherent Grothendieck category if and only if \((\mathcal{Q}, \mathcal{C})\) is of finite type.

### 2.2 Silting and cosilting modules

In this section we recall, mainly from \[7\] and \[14\], the notions of (co)silting module and (co)silting torsion pair. Let \( A \) be a ring. We denote by \( \text{Mod-}A \) (resp. \( \text{A-Mod} \)) the category of right (resp. left) \( A \)-modules and by \( \text{mod-}A \) (resp. \( \text{A-mod} \)) the category of finitely generated right (resp. left) \( A \)-modules.

Let \( \sigma: \mathcal{P} \to \mathcal{Q} \) be a map between two projective \( A \)-modules and consider:

\[
\mathcal{D}_\sigma = \{ X \in \text{Mod-}A \mid \text{Hom}_A(\sigma, X) \text{ surjective} \}
\]

**Definition 2.7.** An \( A \)-module \( T \) is called silting if it admits a projective presentation \( P \xrightarrow{\sigma} Q \to T \to 0 \) such that \( \text{Gen} T = \mathcal{D}_\sigma \). In this case, the torsion class \( \text{Gen} T \) is called silting class. \( T \) is called tilting if \( T^{\perp 1} = \text{Gen} T \).

Two silting modules \( T \) and \( T' \) are equivalent if they generate the same silting class or, equivalently, if \( \text{Add} T = \text{Add} T' \). An \( A \)-module \( T \) is tilting if and only if \( T \) is silting with respect to a projective presentation \( 0 \to P \xrightarrow{\sigma} Q \to T \to 0 \), with \( \sigma \) injective (see \[7\] Proposition 3.12).

**Remark 2.8.** \[7\] Remark 3.11] If \( T \) is a silting module, then \((\text{Gen} T, T^{\perp 1})\) is a torsion pair.

In a dual fashion, let \( \omega: E \to F \) be a map between two injective \( A \)-modules and consider:

\[
\mathcal{C}_\omega = \{ X \in \text{Mod-}A \mid \text{Hom}_A(\omega, X) \text{ surjective} \}
\]

**Definition 2.9.** An \( A \)-module \( C \) is called cosilting if it admits an injective copresentation \( 0 \to C \to E_0 \xrightarrow{\omega} E_1 \) such that \( \text{Cogen} C = \mathcal{C}_\omega \). In this case, the torsionfree class \( \text{Cogen} C \) is called cosilting class. \( C \) is called cotilting if \( \perp 1 C = \text{Cogen} C \).

Two cosilting modules \( C \) and \( C' \) are equivalent if they generate the same cosilting class or, equivalently, if \( \text{Prod} C = \text{Prod} C' \). An \( A \)-module \( C \) is cotilting if and only if \( C \) is cosilting with respect to an injective copresentation \( 0 \to C \to E_0 \xrightarrow{\omega} E_1 \) \( \to 0 \), with \( \omega \) surjective (see \[14\]).

**Remark 2.10.** \[7\] Remark 3.11] If \( C \) is a cotilting module, then \((\perp 1 C, \text{Cogen} C)\) is a torsion pair.

A class \( \mathcal{C} \) in \( \text{Mod-}A \) is called definable if it is closed under direct products, direct limits and pure submodules. The next proposition is a summary of some results proved in \[14\] and \[39\] (cf. also \[15\]).
Proposition 2.11. Let $A$ be a ring. Every cosilting $A$-module is pure-injective. Every cosilting class is a definable subcategory of $\text{Mod}-A$ and moreover the cosilting classes are precisely the definable torsionfree classes in $\text{Mod}-A$.

Remark 2.12. By the previous proposition we can infer that the cosilting torsion pairs $(+C, \text{Cogen } C)$ are precisely the torsion pairs of finite type.

2.3 Simple objects in the heart

Let $\mathcal{G} = A\text{-Mod}$, for a ring $A$. Consider a torsion pair $\mathfrak{t} = (\mathcal{Q}, \mathcal{C})$ in $\mathcal{G}$. The following statements are part of [4], an ongoing work by Angeleri Hügel, Herzog and Laking. We omit the proofs of the Propositions in this section and we refer to [4].

Definition 2.13. An $A$-module $Y$ is torsionfree, almost torsion if it satisfies:

(i) $Y \in \mathcal{C}$ and all proper quotient modules of $Y$ are in $\mathcal{Q}$.
(ii) For any short exact sequence $0 \to Y \to B \to C \to 0$ with $B \in \mathcal{C}$, then $C \in \mathcal{C}$.

We say that $Y$ is torsion, almost torsionfree if it satisfies the dual properties.

We have the following:

Lemma 2.14. [4] Let $X, X' \in \mathcal{G}$ be both torsionfree, almost torsion, or both torsion, almost torsionfree. If $\text{Hom}_\mathcal{G}(X, X') \neq 0$, then $X \cong X'$.

Let $\mathcal{A} = \mathcal{G}(\mathcal{Q}, \mathcal{C})$ be the HRS-heart of the torsion pair $(\mathcal{Q}, \mathcal{C})$. We can compute kernels and cokernels of morphisms in $\mathcal{A}$ via the following formulas:

Lemma 2.15. [4]

(1) Let $f : X \to Y$ be a morphism in $\mathcal{A} = \mathcal{G}(\mathcal{Q}, \mathcal{C})$, and let $Z$ be the cone of $f$ in $\mathcal{D}^b(\mathcal{G})$. Consider the canonical triangle, given by the truncation functors:

$$K = \tau_{\leq -1} Z \to Z \to \tau_{\geq 0} Z \to K[1]$$

where $\tau_{\leq -1} Z \in \mathcal{D}^{\leq -1}$ and $\tau_{\geq 0} Z \in \mathcal{D}^{\geq 0}$. Then:

$$\text{Ker}_\mathcal{A}(f) = K[-1] \quad \text{Coker}_\mathcal{A}(f) = \tau_{\geq 0} Z.$$

(2) Let $h : X \to Y$ be a morphism in $\mathcal{G}$ with $X, Y \in \mathcal{C}$. Then:

• $h[1] : X[1] \to Y[1]$ is a monomorphism in $\mathcal{A}$ if and only if $\text{Ker } h = 0$ and $\text{Coker } h \in \mathcal{C}$.
• $h[1] : X[1] \to Y[1]$ is an epimorphism in $\mathcal{A}$ if and only if $\text{Coker } h \in \mathcal{Q}$.

(3) Let $h : X \to Y$ be a morphism in $\mathcal{G}$ with $X, Y \in \mathcal{Q}$. Then:

• $h$ is a monomorphism in $\mathcal{A}$ if and only if $\text{Ker } h \in \mathcal{C}$.
• $h$ is an epimorphism in $\mathcal{A}$ if and only if $\text{Coker } h = 0$ and $\text{Ker } h \in \mathcal{Q}$.

The following is the main characterization Theorem for simple objects in the HRS-heart of an HRS-t-structure.

Theorem 2.16. [4] (cf. [28, Lemma 2.2]) The simple objects in $\mathcal{A}$ are precisely the objects $S$ of the form $S = Y[1]$ with $Y$ torsionfree, almost torsion, or $S = Q$ with $Q$ torsion, almost torsionfree.

Moreover, if the heart $\mathcal{A}$ is a Grothendieck category, it is possible to describe the injective envelopes of the objects in the heart by means of special covers and envelopes.

Proposition 2.17. [4] Let $\mathcal{G} = A\text{-Mod}$. Consider a torsion pair $\mathfrak{t} = (\mathcal{Q}, \mathcal{C})$ such that $\mathcal{A} = \mathcal{G}(\mathcal{Q}, \mathcal{C})$ is a Grothendieck category.

(1) Let $Y \in \mathcal{C}$, and let $0 \to Y \xrightarrow{f} B \to C \to 0$ be a special $\mathcal{C}^{\perp 1}$-envelope. Then $Y[1] \xrightarrow{f[1]} B[1]$ is an injective envelope of $Y[1]$ in $\mathcal{A}$.

(2) Let $Q \in \mathcal{Q}$, and let $0 \to B \xrightarrow{f} C \xrightarrow{g} Q \to 0$ be a special $\mathcal{C}$-cover. Then $Q \to B[1]$ is an injective envelope of $Q$ in $\mathcal{A}$.

3 The case of the Kronecker algebra

Let us now consider the Kronecker algebra $\Lambda$, i.e. the path algebra over a field $k$ of the quiver:

\[ \bullet \quad \bullet \quad \bullet \]

This is a tame hereditary algebra and the Auslander-Reiten quiver of $\Lambda$-mod is:

where $p$ is the preprojective component, whose modules are denoted by $P_i$, for $i \geq 1$, indexed in such a way that $\text{dim}_k \text{Hom}_\Lambda(P_i, P_{i+1}) = 2$. Dually, $q$ is the preinjective component, whose modules are denoted by $Q_i$, for $i \geq 1$,
indexed in such a way that \( \dim_k \text{Hom}_\Lambda(Q_{i+1}, Q_i) = 2 \) and \( t \) is a sincere stable and separating family of regular homogeneous tubes, \( t_x \), indexed by the projective line over \( k \), \( t = \bigcup_{x \in \text{Proj}_k} t_x \). We denote by \( S^\infty_x \) and \( S^{-\infty}_x \) the Pr"ufer and the adic module, respectively, corresponding to the simple regular \( t \)-homogeneous tubes, \( k \) indexed in such a way that \( \dim_k \) has infinite length over \( \Lambda \), but finite length over its endomorphism ring. Recall, from \([29][30]\), that \( \text{End}_G \) is a division ring.

Let \( P_1 \) and \( Q_1 \) be the simple projective and the simple injective \( \Lambda \)-modules, respectively. Following \([8]\), for a nonempty subset \( U \subseteq \text{Proj}_k \), we denote by \( \Lambda_U \) the universal localization of \( \Lambda \) at the set of morphisms given by the projective resolutions of modules in the tubes indexed by elements in \( U \). The silting \( \Lambda \)-modules have been completely described in \([8]\) and all of them, except two, are tilting. By taking the duality with respect to the injective cogenerator, silting right \( \Lambda \)-modules turns out to be in bijective correspondence to cosilting left \( \Lambda \)-modules (see \([5][Corollary 3.7]\)). Therefore, we can summarize in the following table all the nonzero silting and cosilting \( \Lambda \)-modules (together with the correspondent cosilting classes):

| silting right \( \Lambda \)-module | cosilting left \( \Lambda \)-module | cosilting class |
|-----------------------------------|-----------------------------------|----------------|
| \( P_1 \)                         | \( Q_1 \)                         | Cogen(\( Q_1 \)) |
| \( Q_1 \)                         | \( P_1 \)                         | Cogen(\( P_1 \)) |
| \( (P_1 \oplus P_{i+1})_{i \geq 1} \) | \( (Q_1 \oplus Q_{i+1})_{i \geq 1} \) | Cogen(\( Q_i \)), \( i \geq 1 \) |
| \( (Q_{i+1} \oplus Q_i)_{i \geq 1} \) | \( (P_{i+1} \oplus P_i)_{i \geq 1} \) | Cogen(\( P_i \)), \( i \geq 1 \) |
| \( (\Lambda_U \oplus \Lambda_U / \Lambda)_{\emptyset \neq U \subseteq \text{Proj}_k} \) | \( \text{Cogen}(C_U) \), \( \emptyset \neq U \subseteq \text{Proj}_k \) | Cogen(\( W \)) = \( t^\infty \) |
| \( L_\Lambda \)                    | \( D(L_\Lambda) \)                  | \( \text{Cogen}(W) \) = \( t^\infty \) |

In this table \( L_\Lambda \) denotes the Lukas tilting module and the only cosilting, non cotilting, \( \Lambda \)-modules are \( P_1 \) and \( Q_1 \) (cf. \([13][Corollary 3.10]\)). Moreover, the cotilting module \( D(L_\Lambda) \) is equivalent to the Reiten-Ringel tilting-cotilting module \( \Lambda W = G \oplus \bigoplus_{x \in \text{Proj}_k} S^\infty_x \) and it corresponds to \( C_U \), for \( U = \emptyset \) (see \([29]\)).

### 3.1 Hearts arising from cosilting torsion pairs

Let \( \mathcal{G} = \Lambda\text{-Mod} \). We analyze more specifically the torsion pairs arising from the cosilting classes described above, giving also a description of the different HRS-hearts related to them. By \([32][Theorem 5.2]\), all the hearts we are going to describe are locally coherent categories.

- **For the simple cosilting module \( Q_1 \), the cosilting torsion pair is \( (Q, \mathcal{C}) = (t^0 Q_1, \text{Cogen}(Q_1)) \). Since \( t^0(Q_1) = \text{Gen}(P_1) \), we have \( (Q, \mathcal{C}) = (\text{Gen}(P_1), P_1) \) which is the torsion pair generated by the silting module \( P_1 \). Hence, by \([28][Corollary 4.7]\), the heart \( \mathcal{A} = \mathcal{G}(Q, \mathcal{C}) \) is equivalent to the category of modules over \( \text{End}(P_1) \), so \( \mathcal{A} \cong \Lambda\text{-Mod} \).

- **For the simple silting module \( P_1 \), the cosilting torsion pair \( (Q, \mathcal{C}) = (t^0 P_1, \text{Cogen}(P_1)) \). Since the class \( t^0 P_1 = \text{Gen}(P_2) = \text{Gen}(P_2 \oplus P_3) \) (cf. \([2][Example 6.9]\)), we have \( (Q, \mathcal{C}) = (\text{Gen}(P_2 \oplus P_3), (P_2 \oplus P_3)^{t^0}) \) which is the torsion pair generated by the silting module \( P_2 \oplus P_3 \). Hence, by \([28][Corollary 4.7]\), the heart \( \mathcal{A} = \mathcal{G}(Q, \mathcal{C}) \) is equivalent to the category of modules over \( \text{End}(P_2 \oplus P_3) \), so \( \mathcal{A} \cong \Lambda\text{-Mod} \).

- **The torsion pair cogenerated to the cotilting module \( Q_1 \oplus Q_2 \) is the trivial one \( (0, \Lambda\text{-Mod}) \), so the heart \( \mathcal{A} = \mathcal{G}(Q, \mathcal{C}) \) is \( \Lambda\text{-Mod} \) itself.

- **For \( i > 0 \), \( Q_{i+1} \), which is the direct limit of \( Q_1 \), \( \cdots \), \( Q_{i-1} \), \( \text{add}\{Q_k | k \geq i + 1\} \) \( \cup \) \( \text{add}\{P_k | k > i + 1\} \cup \text{t} \cup \text{q} \), which is generated by the tilting module \( Q_{i+1} \oplus Q_i \), so the heart \( \mathcal{A} = \mathcal{G}(Q, \mathcal{C}) \) is equivalent to \( \Lambda\text{-Mod} \).

- **The torsion pair cogenerated to the cotilting module \( P_1 \oplus P_{i+1}, \) for \( i \geq 1 \), which is the direct limit closure of the torsion pair \( \text{add}\{P_k \mid k > i + 1\} \cup \text{t} \cup \text{q} \), which is generated by the tilting module \( P_{i+1} \oplus P_i \), so the heart \( \mathcal{A} = \mathcal{G}(Q, \mathcal{C}) \) is equivalent to \( \Lambda\text{-Mod} \).
by $(\gen(P_{t+2}), P_{t+2}^{\perp_0})$, which is generated by the tilting module $P_{t+2} \oplus P_{t+3}$, so the heart $\mathcal{A} = \mathcal{G}(\mathcal{Q}, \mathcal{C})$ is equivalent to $\Lambda$-Mod.

- The cotilting module $C_U$, for $U = \emptyset$, is the so-called Reiten-Ringel tilting module $W$:
  \[
  W = G \oplus \bigoplus_{x \in P_k^1} S_x^\infty
  \]
  and it cogenerates the torsion pair $(\mathcal{Q}, \mathcal{C}) = (\perp^U W, \text{Cogen } W)$, which is generated by $q$. The heart $\mathcal{A} = \mathcal{G}(\mathcal{Q}, \mathcal{C})$ is equivalent to the locally noetherian Grothendieck category $\text{Qcoh} P_k^1$ of quasi-coherent sheaves over $P_k^1$ (see [6, Section 3.1]).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Fig2.png}
\caption{Auslander-Reiten quiver of the heart $\mathcal{A} = \mathcal{G}(\mathcal{Q}, \mathcal{C})$, for $U = \emptyset$.}
\end{figure}

- $C_U$, for $\emptyset \neq U \subsetneq P_k^1$, is the module
  \[
  C_U = G \oplus \prod_{x \in U} S_x^{-\infty} \oplus \bigoplus_{x \notin U} S_x^\infty
  \]
  and it cogenerates the torsion pair $(\mathcal{Q}_U, \mathcal{C}_U) = (\perp^U C_U, \text{Cogen } C_U)$, which is generated by the set $\bigcup_{x \in U} t_x \cup q$. This cotilting module is not $\Sigma$-pure-injective, therefore, by [26, Proposition 5.6], the heart $\mathcal{A}_U = \mathcal{G}(\mathcal{Q}_U, \mathcal{C}_U)$ is not locally noetherian.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Fig3.png}
\caption{Auslander-Reiten quiver of the heart $\mathcal{A}_U = \mathcal{G}(\mathcal{Q}_U, \mathcal{C}_U)$, for $U \subsetneq P_k^1$.}
\end{figure}

- If $U = P_k^1$, then $C_U$ is the module
  \[
  C_U = G \oplus \prod_{x \in P_k^1} S_x^{-\infty}
  \]
  and it cogenerates the torsion pair $(\text{Gen } t, \mathcal{F}) = (\perp^U C_U, \text{Cogen } C_U)$, which is generated by $t$. The cotilting module $C_U$ is not $\Sigma$-pure-injective, therefore, also in this case, via [26, Proposition 5.6], the heart $\mathcal{A}_U = \mathcal{G}(\text{Gen } t, \mathcal{F})$ is not a locally noetherian category.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Fig4.png}
\caption{Auslander-Reiten quiver of the heart $\mathcal{A}_U = \mathcal{G}(\text{Gen } t, \mathcal{F})$, for $U = P_k^1$.}
\end{figure}

**Remark 3.1.** Notice that there are inclusions $\mathcal{F} \subset \mathcal{C}_U \subset \mathcal{C}$ and $\mathcal{Q} \subset \mathcal{Q}_U \subset \text{Gen } t$.
Theorem 3.2. Let $\mathcal{U} \subseteq \mathbb{P}_k^1$. The complete list of simple objects in the heart $\mathcal{A}_U = \mathcal{G}(\mathcal{Q}_U, \mathcal{C}_U)$ is:

- $\{S_x \mid S_x$ simple regular in $\bigcup_{x \in \mathcal{U}} t_x\} \cup \{S_x[1] \mid S_x$ simple regular in $\bigcup_{x \in \mathcal{U}} t_x\}$, whenever $\mathcal{U} \neq \mathbb{P}_k^1$.
- $\{S_x \mid S_x$ simple regular $\} \cup \{G[1]\}$, whenever $\mathcal{U} = \mathbb{P}_k^1$.

Moreover, for $\mathcal{U} \subseteq \mathbb{P}_k^1$, we have that:

- The short exact sequence $0 \to S_x \to S_x^{-\infty}[1] \to S_x^{-\infty}[1] \to 0$ is a minimal injective coresolution in $\mathcal{A}_U$ for a simple regular module $S_x \in \bigcup_{x \in \mathcal{U}} t_x$.
- The short exact sequence $0 \to S_x[1] \to S_x^\infty[1] \to S_x^\infty[1] \to 0$ is a minimal injective coresolution in $\mathcal{A}_U$ for a simple regular module $S_x \in \bigcup_{x \in \mathcal{U}} t_x$.
- If $\mathcal{U} = \mathbb{P}_k^1$, then the object $G[1]$ is simple injective in $\mathcal{A}_U$.

Proof. To prove the claim, according to Theorem 2.16 we show that:

1. If $\mathcal{U} \neq \emptyset$, then $X \in \mathcal{Q}_U$ is almost torsionfree if and only if $X$ is simple regular in $\bigcup_{x \in \mathcal{U}} t_x$.
2. If $\mathcal{U} \neq \mathbb{P}_k^1$, then $X \in \mathcal{C}_U$ is almost torsion if and only if $X$ is simple regular in $\bigcup_{x \in \mathcal{U}} t_x$.
3. If $\mathcal{U} = \emptyset$, there are no torsion, almost torsionfree modules.
4. If $\mathcal{U} = \mathbb{P}_k^1$, then $X \in \mathcal{C}_U$ is almost torsion if and only if $X \cong G$.

First of all, observe that the indecomposable modules in $\mathcal{C}_U$ are the modules in $\mathfrak{p} \cup \bigcup_{x \in \mathcal{U}} t_x$ and the indecomposable modules in $\mathcal{Q}_U$ are the modules in the union $\bigcup_{x \in \mathcal{U}} t_x \cup \mathfrak{q}$.

1. Let $S_x$ be simple regular in $\bigcup_{x \in \mathcal{U}} t_x$. Then $S_x \in \mathcal{Q}_U$ is torsion, almost torsionfree:

   i. All proper subobjects of $S_x$ are preprojective, hence in $\mathcal{C}_U$.
   ii. Let $0 \to A \to B \to S_x \to 0$ be an exact sequence with $B \in \mathcal{Q}_U$. Consider the canonical exact sequence $0 \to A' \to \overline{A} \to 0$ with $A' \in \mathcal{Q}_U$ and $\overline{A} \in \mathcal{C}_U$, and assume that $\overline{A} \neq 0$. In the push-out diagram:

   $\begin{array}{ccc}
   0 & \to & A \\
   & & \downarrow \alpha \\
   0 & \to & \overline{A} \\
   \end{array}$

   the map $\alpha$ is surjective and thus $B' \in \mathcal{Q}_P$. Notice that $B'$ cannot have nonzero direct summands in $\mathfrak{q}$, because they would be submodules of Ker $g' \cong \overline{A} \in \mathcal{C}_U$. So we conclude that $B' \in \mathcal{C} \cap \text{Gen} \mathfrak{t}$ and therefore it is union of modules belonging to $\mathcal{t}$, cf. 29.3.4 and 3.5. Moreover, $B' \in \mathcal{Q}_U$, therefore we can say that it is union of modules belonging to $\bigcup_{x \in \mathcal{U}} t_x$. But then, since $\mathcal{C} \cap \text{Gen} \mathfrak{t}$ is an exact abelian subcategory of $\Lambda$-Mod, also Ker $g' \cong \overline{\mathcal{A}} \in \mathcal{C}_U$ must be union of modules belonging to $\bigcup_{x \in \mathcal{U}} t_x$, a contradiction. This proves that $A \in \mathcal{C}_U$.

   Conversely, if $X \in \mathcal{Q}_U$ is almost torsionfree, then $X \notin \mathcal{C}_U = (\bigcup_{x \in \mathcal{U}} t_x)^{\perp}$, so there is a simple regular module $S_x \in \bigcup_{x \in \mathcal{U}} t_x$ with a non-zero map $f : S_x \to X$. But then $S_x \cong X$ by Lemma 2.14.

2. We now turn to the case $\mathcal{U} \neq \mathbb{P}_k^1$, which is somehow dual to case (1), and pick a simple regular module $S_x \in \bigcup_{x \in \mathcal{U}} t_x$. First of all, observe that the generic module $G \in \mathcal{t}^{\perp} \subset \mathcal{C}_U$ is not almost torsion, indeed: consider the exact sequence $0 \to S_x^{-\infty} \to G^{(1)} \to S_x^{\infty} \to 0$, from 13 Lemma 2.4, for a set $I$. This sequence yields a nonzero map $f : G \to S_x^\infty$, defined as the composite $G \to G^{(1)} \to S_x^\infty$. It is clear that $\text{Im} f \in \mathcal{C}_U$, since $\mathcal{C}_U$ is closed under subobjects, and therefore $G$ has a proper quotient in $\mathcal{C}_U$. Then $S_x \in \mathcal{C}_U$ is torsionfree, almost torsion:

   i. All proper quotients of $S_x$ are in $\mathcal{Q}$.
   ii. Let $0 \to S_x \to B \to C \to 0$ be an exact sequence with $B \in \mathcal{C}_U$. Consider the canonical exact sequence $0 \to C' \to C \to 0$ with $C' \in \mathcal{Q}_U$ and $\overline{C} \in \mathcal{C}_U$ and assume $C' \neq 0$. In the pullback diagram:

   $\begin{array}{ccc}
   0 & \to & S_x \\
   & & \downarrow \alpha \\
   0 & \to & B \\
   \end{array}$

   $\alpha$ is injective and then $B' \in \mathcal{C}_U$. Moreover, $B' \in \text{Gen} \mathfrak{t}$ indeed, consider the following diagram:

   $\begin{array}{ccc}
   0 & \to & t(B') \\
   & & \downarrow \pi \\
   0 & \to & B' \\
   \end{array}$
So, let us assume that such $S_x$ does not exist. Then $X \in t^{1,0}$, and by [29, 6.6] there is a short exact sequence

$$0 \rightarrow X \xrightarrow{f} G^{(a)} \rightarrow Z \rightarrow 0$$

where $G^{(a)} \in C_{\mathcal{U}}$ and thus, by property (ii), also $Z$ belong to $C_{\mathcal{U}}$. Moreover, $X \notin p$, because every $P \in p$ is the first term of a short exact sequence $0 \rightarrow P \rightarrow P' \rightarrow S_x \rightarrow 0$ with $P' \in p \subseteq C_{\mathcal{U}}$ and a simple regular module $S_x \in \bigcup_{x \in t} t_x \subset \mathcal{Q}_{\mathcal{U}}$. Furthermore, $X$ is indecomposable, since if it is not it would have a proper quotient in $C_{\mathcal{U}}$. Moreover $X$ is also in $\perp p$, because if there is a nonzero map $X \rightarrow P$, with $P \in p$, then $X$ has a direct summand in $p$, but $X$ is indecomposable, so $X \notin p$, contradiction. It follows that $X \in \perp p$. In fact, any $0 \neq h : X \rightarrow S_x$ with $S_x$ simple regular would have to be a proper epimorphism with $S_x \in \mathcal{Q}_{\mathcal{U}}$. But then $\text{Ext}^1_{\Lambda}(Z,S_x) \cong D \text{Hom}(S_x,Z) = 0$, and $h$ would factor through $f$, contradicting the fact that $\text{Hom}_{\Lambda}(G,S_x) = 0$. So we conclude that $X$ belongs to $t^{1,0} \cap \perp t = \text{Add} G$ (see [29, 4]). But then $X \cong G$, which is impossible as we have observed above.

(3): Assume $\mathcal{U} = \varnothing$. Then $\mathcal{Q}_{\mathcal{U}} = \text{Add} q$, and every $Q \in q$ is the end-term of a short exact sequence $0 \rightarrow S_x \rightarrow Q' \rightarrow Q \rightarrow 0$, where $Q' \notin q$ and $S_x \notin q$ is simple regular, but $Q$ is almost torsionfree. Moreover, all simple regular modules are torsionfree, almost torsion by (2).

(4): It remains to check the case $\mathcal{U} = \mathbb{P}^1_k$. Then the torsion pair is $(\text{Gen} \ t, \mathcal{F})$, where $\mathcal{F} = t^{1,0}$ and $G$ is torsionfree, almost torsion. Indeed, $G \in F$, and

(i) If $g : G \rightarrow B$ is a proper epimorphism, and $0 \rightarrow B' \rightarrow B \rightarrow \overline{B} \rightarrow 0$ is the canonical exact sequence with $B' \in \text{Gen} \ t$ and $0 \neq \overline{B} \in \mathcal{F}$, then $\overline{B} \in \mathcal{F} \cap \perp t = \text{Add} G$. So $G \xrightarrow{g} B \rightarrow \overline{B}$ is a morphism over a simple artinian ring $Q$, which is Morita equivalent to $\text{End}_{\Lambda}(G)$ (see [19] and [10, 1.7 and 1.8] for the details). Thus, $G \xrightarrow{g} B \rightarrow \overline{B}$ a split monomorphism, which is a contradiction. Hence $B \in \text{Gen} \ t$.

(ii) If $0 \rightarrow G \rightarrow B \rightarrow C \rightarrow 0$ is an exact sequence with $B \in \mathcal{F}$, applying $\text{Hom}_{\Lambda}(S_x,-)$, with $S_x$ a simple regular module, we obtain an exact sequence:

$$\text{Hom}_{\Lambda}(S_x,B) \rightarrow \text{Hom}_{\Lambda}(S_x,C) \rightarrow \text{Ext}^1_{\Lambda}(S_x,G) \cong D \text{Hom}_{\Lambda}(G,S_x)$$

where the first and third term are zero, showing that $C \in \mathcal{F}$.

Conversely, if $X \in \mathcal{F}$ is almost torsion, then $X$ is cogenerated by $G$, hence $\text{Hom}_{\Lambda}(X,G) \neq 0$, and $X \cong G$ by Lemma 2.13.

Finally, to prove that the injective coresolutions have the stated form, we apply Proposition 2.17 first to the special $C_{\mathcal{U}}$-cover $0 \rightarrow S_x^\infty \rightarrow S_x^\infty \rightarrow S_x \rightarrow 0$ of $S_x \in \bigcup_{x \in t} t_x$ and then to the special $C_{\mathcal{U}}^{1,0}$-envelope $0 \rightarrow S_x \rightarrow S_x^\infty \rightarrow S_x^\infty \rightarrow 0$ of $S_x \in \bigcup_{x \in t} t_x$.

Remark 3.3. Recall from [35] Theorem 5.2 that $\mathcal{A}_{\mathcal{U}}$ is hereditary only if $(\mathcal{Q}_{\mathcal{U}}, \mathcal{U})$ is a split torsion pair. But if $\mathcal{U}$ is infinite and $S_x$ is a simple regular in the tube $t_x$, then by [31, Proposition 5] there is a non-split exact sequence $0 \rightarrow \bigoplus_{x \in t} S_x \rightarrow \prod_{x \in t} S_x \rightarrow G^{(a)} \rightarrow 0$ with $\bigoplus_{x \in t} S_x \in \mathcal{Q}_{\mathcal{U}}$ and $G^{(a)} \in t^{1,0} \subset C_{\mathcal{U}}$.

4 The Atom Spectrum

In this section we introduce the notion of atom spectrum for a Grothendieck category. This is a generalization of the prime spectrum for a commutative ring and, in a similar fashion, it is endowed with a topological space structure.

The notion of atom spectrum has been introduced by Kanda in [23, in the more general setting of abelian categories. Moreover, some properties will appear in a forthcoming work by Vámos and Virili, [36].

Definition 4.1. Let $\mathcal{G}$ be a Grothendieck category and let $X$ be an object of $\mathcal{G}$. $X$ is said to be monoform if, given any subobject $H$ of $X$ and a morphism $\varphi \in \text{Hom}_{\mathcal{G}}(H,X)$, $\varphi$ is non-zero if and only if it is a non-zero monomorphism.

Lemma 4.2. [36, Lemma 2.10] Let $X \in \mathcal{G}$. The following conditions are equivalent:

(i) $X$ is monoform.

(ii) for any non-zero subobject $H \subseteq X$, the unique object isomorphic to both a subobject of $X$ and to a subobject of $X/H$ is the zero object.

(iii) $X$ is uniform and, for any non-zero subobject $H \subseteq X$, the unique object isomorphic to both a subobject of $H$ and to a subobject of $X/H$ is the zero object.
Any simple object is monoform. We say that two monoform objects \(X\) and \(X'\) in \(\mathcal{G}\) are atom-equivalent if there exist a nonzero \(H \in \mathcal{G}\) such that \(X \supseteq H \subseteq X'\). Clearly, two non isomorphic simple objects are not atom-equivalent.

**Example 4.3.** Let \(\Lambda\) be the Kronecker algebra. As mentioned before, the two simple \(\Lambda\)-modules \(P_1\) and \(Q_1\) are monoform. On the contrary, the generic module \(G\) is not monoform, indeed we have a short exact sequence \(0 \to P_1 \to G \to \bigoplus S_n \to 0\) (see [20, Theorem 6.1]) and \(P_1\) is a submodule of any Prüfer module.

**Lemma 4.4.** [24, Lemma 5.8] [36, Lemma 2.13] Two monoform objects \(X\) and \(X'\) are atom-equivalent if and only if \(E(X) \cong E(X')\).

As proven in [24, Proposition 2.8], the atom-equivalence relation is an equivalence relation. We denote by \(\mathcal{X}\) the class of all monoform objects atom-equivalent to the monoform object \(X\), these equivalence classes are called atoms.

**Definition 4.5.** The atom spectrum of an abelian category \(\mathcal{G}\), denoted by \(\text{ASpec}(\mathcal{G})\), is the class of all atoms in \(\mathcal{G}\).

Recall that, if \(t = (\mathcal{T}, \mathcal{F})\) is a hereditary torsion pair in \(\mathcal{G}\), then \(\mathcal{T}\) is a localizing subcategory of \(\mathcal{G}\). Hence, we have the so-called localization sequence:

\[
\mathcal{T} \xrightarrow{\text{inc}} \mathcal{G} \xrightarrow{\mathcal{Q}} \mathcal{G}/\mathcal{T}
\]

where \(\text{inc}\) is the inclusion functor, \(\mathcal{T}\) is the left-exact torsion radical (see [33, §VI.3]), the Grothendieck category \(\mathcal{G}/\mathcal{T}\) is the localization of \(\mathcal{G}\) at \(\mathcal{T}\) and \(\mathcal{Q}\) and \(\mathcal{S}\) are the quotient functor and the section functor, respectively, of the localization.

**Definition 4.6.** Let \(t = (\mathcal{Q}, \mathcal{C})\) be a torsion pair in \(\mathcal{G}\). An object \(Y\) in \(\mathcal{G}\) is \(t\)-cocritical if \(Y \in \mathcal{C}\) and all proper quotients of \(Y\) are in \(\mathcal{Q}\).

**Lemma 4.7.** Let \(t = (\mathcal{T}, \mathcal{F})\) be a hereditary torsion pair and let \(\mathcal{Q}: \mathcal{G} \to \mathcal{G}/\mathcal{T}\) be the quotient functor. The following are equivalent for \(X \in \mathcal{G}\):

(i) \(X\) is \(t\)-cocritical  
(ii) \(\mathcal{Q}(X)\) is simple in \(\mathcal{G}/\mathcal{T}\) and \(X \in \mathcal{F}\)

**Proof.** (i) \(\Rightarrow\) (ii): Let \(Y\) be a nonzero subobject of \(\mathcal{Q}(X) \in \mathcal{G}/\mathcal{T}\). Then, applying the section functor \(\mathcal{S}: \mathcal{G}/\mathcal{T} \to \mathcal{G}\), we have \(\mathcal{S}(Y) \subseteq \mathcal{S}(\mathcal{Q}(X))\). Moreover, \(\mathcal{S}(\mathcal{Q}(X)) \subseteq E(X)\) (indeed, by [21, Proposition III.3.6], \(\mathcal{S}(\mathcal{Q}(X)) \subseteq \mathcal{S}(E(X)) \cong E(X)\)). \(X\) is essential in \(E(X)\), therefore we have that \(\mathcal{S}(Y) \neq 0\) if and only if \(\mathcal{S}(Y) \cap X \neq 0\), then, since \(X\) is \(t\)-cocritical, \(X/(\mathcal{S}(Y) \cap X) \in \mathcal{T}\). Applying the functor \(\mathcal{Q}\) to the short exact sequence:

\[
0 \to \mathcal{S}(Y) \cap X \to X \to X/(\mathcal{S}(Y) \cap X) \to 0
\]

we obtain that \(\mathcal{Q}(X) \cong \mathcal{Q}(\mathcal{S}(Y) \cap X) \subseteq \mathcal{Q}(\mathcal{S}(Y)) = Y \subseteq \mathcal{Q}(X)\). Therefore \(\mathcal{Y} \cong \mathcal{Q}(X)\).

(ii) \(\Rightarrow\) (i): Let \(Y\) be a nonzero subobject of \(X\), then \(Y \in \mathcal{F}\) and \(\mathcal{Q}(Y)\) is nonzero. Since \(\mathcal{Q}\) is an exact functor and \(\mathcal{Q}(X)\) is simple, \(\mathcal{Q}(Y) = \mathcal{Q}(X)\). Hence, applying \(\mathcal{Q}\) to the sequence \(0 \to Y \to X \to X/Y \to 0\), we get \(\mathcal{Q}(X/Y) = 0\) hence \(X/Y \in \mathcal{T}\).

**Proposition 4.8.** [36, Proposition 2.12] Let \(\mathcal{G}\) be a Grothendieck category and let \(X \in \mathcal{G}\). Consider the torsion pair \(t_X = (\mathcal{T}_X, \mathcal{F}_X)\) cogenerated by \(E(X)\). The followings are equivalent:

(i) \(X\) is monoform.
(ii) \(\mathcal{Q}_X(X)\) is the unique simple object in \(\mathcal{G}/\mathcal{T}_X\), up to isomorphisms (where \(\mathcal{Q}_X: \mathcal{G} \to \mathcal{G}/\mathcal{T}_X\) denotes the quotient functor relative to \(t_X\)).
(iii) there exists a hereditary torsion pair \(t = (\mathcal{T}, \mathcal{F})\) such that \(X \in \mathcal{F}\) and \(\mathcal{Q}(X)\) is simple in \(\mathcal{G}/\mathcal{T}\).

Moreover, if the above conditions are verified, a monoform object \(Y\) is atom-equivalent to \(X\) if and only if \(Y\) is isomorphic to a subobject of \(\mathcal{S}_X \mathcal{Q}_X(X)\) (where \(\mathcal{S}_X: \mathcal{G}/\mathcal{T}_X \to \mathcal{G}\) denotes the section functor relative to \(t_X\)).

**Remark 4.9.** Notice that, by Lemma 4.4, we can say that an object \(X\) is monoform if and only if there exists a hereditary torsion pair \(t = (\mathcal{T}, \mathcal{F})\) such that \(X\) is \(t\)-cocritical.

The atom spectrum of an abelian category may not be a set. On the other hand, we have that \(\text{ASpec}(\mathcal{G})\) is a set, whenever \(\mathcal{G}\) is a Grothendieck category. This is a consequence of the following lemma.

**Theorem 4.10.** [24, Theorem 5.9] [36, Lemma 2.13] Let \(\mathcal{G}\) be a Grothendieck category. There is a well-defined injective map of sets:

\[
\text{ASpec}(\mathcal{G}) \longrightarrow \text{Spec}(\mathcal{G})
\]

\[
\Xrightarrow{\text{inc}} \mathcal{G} \xrightarrow{\mathcal{Q}} \mathcal{G}/\mathcal{T} \xrightarrow{\mathcal{Q}} \mathcal{G}/\mathcal{T}
\]

If \(\mathcal{G}\) is a locally noetherian Grothendieck category, then this map is a well-defined bijection of sets.

**Example 4.11.** For the Kronecker algebra \(\Lambda\), the category \(\Lambda\)-Mod is a locally noetherian Grothendieck category. Therefore \(\text{ASpec}(\Lambda\text{-Mod}) = \text{Spec}(\Lambda)\) and the only two indecomposable injectives are the injective envelopes of the two simple \(\Lambda\)-modules, \(P_1\) and \(Q_1\), which are monoform. So we have \(\text{ASpec}(\Lambda\text{-Mod}) = (P_1, Q_1)\).
Topology and partial order on the atom spectrum

Let $G$ be a Grothendieck category. In [24], the author defines a topological space structure on $\mathrm{ASpec}(G)$ by means of the notion of atom support.

**Definition 4.12.** Let $M$ be an object of $G$. Define the *atom support* of $M$ as the set:

$$\mathrm{ASupp} M = \{ \alpha \in \mathrm{ASpec}(G) \mid \alpha = \overline{T} \text{ for a monoform subquotient } H \text{ of } M \}$$

Let $\Phi$ be a subset of $\mathrm{ASpec}(G)$. $\Phi$ is open if for any atom $\alpha \in \Phi$, there exists a monoform object $H \in G$ such that $\overline{T} = \alpha$ and $\mathrm{ASupp} H \subset \Phi$.

It is clear that, for a simple object $S$, $\mathrm{ASupp} S = \{ \overline{S} \}$. By [25] Proposition 3.2, the set of all open sets in $\mathrm{ASpec}(G)$ is given by the family $\{ \mathrm{ASupp} M \mid M \in G \}$. Moreover, if $G$ is locally noetherian, the family restricts to $\{ \mathrm{ASupp} M \mid M \text{ noetherian in } G \}$.

Open singletons in $\mathrm{ASpec}(G)$ are characterized by the following:

**Proposition 4.13.** [25] Proposition 3.7] Let $G$ be a locally noetherian Grothendieck category and let $\alpha \in \mathrm{ASpec}(G)$. The set $\{ \alpha \}$ is open if and only if there exists a simple object $S$ in $G$ such that $\overline{S} = \alpha$.

By [25] Proposition 3.5], the atom spectrum of $G$ is a $T_0$-space (or Kolmogorov space), i.e. a space in which, for any distinct points $x_1$ and $x_2$ in it, there exists an open set containing exactly one of them. It is well known that, if $X$ is a $T_0$-space, it is possible to define a partial order $\preceq$ on it, called specialization order, in the following way: for any $x, y \in X$, we define $x \preceq y$ if and only if $x \in \overline{\{y\}}$, where $\overline{\{y\}}$ is the topological closure of $\{y\}$ in $X$. Conversely, a partially ordered set $P$ can be seen as a topological space as follows: a subset $\Phi$ of $P$ is open if and only if for any $p, q \in P$ such that $p \preceq q$, $p \in \Phi$ implies $q \in \Phi$. These correspondences are mutually inverse.

$\mathrm{ASpec}(G)$ becomes a partially ordered set defining a specialization order $\preceq$ on it. We have the following:

**Proposition 4.14.** [25] Proposition 4.2] Let $G$ be a Grothendieck category and $\alpha, \beta \in \mathrm{ASpec}(G)$. Then the following are equivalent:

1. $\alpha \preceq \beta$, i.e. $\alpha \in \overline{\beta}$.
2. If $\Phi$ is an open subset of $\mathrm{ASpec}(G)$ such that $\alpha \in \Phi$, then $\beta \in \Phi$. In other words, $\beta$ belongs to the intersection of all the open subsets containing $\alpha$.
3. For any object $M$ in $G$ such that $\alpha \in \mathrm{ASupp} M$, we have $\beta \in \mathrm{ASupp} M$.
4. For any monoform object $H$ in $G$ such that $\overline{T} = \alpha$, we have $\beta \in \mathrm{ASupp} H$.

5 Atom spectrum of hearts in $\Lambda$-Mod

Recall that $\Lambda$ denotes the Kronecker algebra. We compute the atom spectra of the different HRS-hearts arising from the cotorsion pairs in $\Lambda$-Mod.

As a notation for the whole section, we denote by $\mathcal{U}$ the complement of $\mathcal{U}$ inside $\mathbb{P}^1_k$ and $G = \Lambda$-Mod.

First of all, notice that for the finite dimensional cosilting $\Lambda$-modules, the HRS-hearts are monoform, clearly not atom-equivalent.

As a notation for the whole section, we denote by $\overline{G}$ the set of the indecomposable objects in $\text{Prod}(\mathcal{C}_\mathcal{U})$. We focus on the torsion pair $\mathcal{T} = (\mathcal{Q}_\mathcal{U}, \mathcal{C}_\mathcal{U})$ generated by the set $\bigcup_{x \in \mathcal{U}} t_x \cup \mathfrak{q}$ and cogenerated by the cotilting module $\mathcal{C}_\mathcal{U}$.

By Theorem 4.4 there is an injection between $\mathrm{ASpec}(\mathcal{A}_\mathcal{U})$ and $\mathrm{Spec}(\mathcal{A}_\mathcal{U})$ which, by [17] Proposition 4.4, is equal to the set of the indecomposable objects in $\text{Prod}(\mathcal{C}_\mathcal{U})$. If $\mathcal{U} = \emptyset$, then, by the same Theorem, this injection is actually a bijection.

By Theorem 5.2, the simple objects in $\mathcal{A}_\mathcal{U}$ are:

$$\{ S_x \mid S_x \text{ simple regular in } \bigcup_{x \in \mathcal{U}} t_x \} \cup \{ S_x[1] \mid S_x[1] \text{ simple regular in } \bigcup_{x \in \mathcal{U}} t_x \}$$

and these are monoform, clearly not atom-equivalent.

Again by Theorem 5.2 the injective envelope of $S_x$, for $x \in \mathcal{U}$ is $S_x^{-\infty}[1]$, and the injective envelope of $S_x[1]$, for $x \in \mathcal{U}$ is $S_x^\infty[1]$. Recall that, as in [20] Section 8, we can decompose the subcategory $\mathcal{T} = \varprojlim t$ as a coproduct of categories $\mathcal{T}(x) = \varprojlim t_x$. We have:
Proposition 5.1. \(G[1]\) is a monomorphic object in \(\mathcal{A}_t = \mathcal{G}(\mathcal{Q}_t, \mathcal{C}_t)\).

Proof. By \([6]\) Corollary 5.8, in \(\mathcal{A}_t\) there is a hereditary torsion pair \(t_{\mathcal{U}} = (\mathcal{T}_t, F_{\mathcal{U}})\), where:

\[
\mathcal{T}_t = \bigcap_{x \in \mathcal{U}} \mathcal{T}(x)[1] \quad \text{and} \quad F_{\mathcal{U}} = \text{Cogen} \mathcal{C}_t[1].
\]

Clearly \(G[1] \in F_{\mathcal{U}}\). Let \(Z\) be a proper quotient of \(G[1]\). Hence \(Z \in \mathcal{C}_t[1]\), since \(\mathcal{C}_t[1]\) is a torsion class in the heart, meaning that \(Z = C[1]\) for some \(C \in \mathcal{C}_t\). Therefore, by Lemma 2.15 the proper epimorphism \(h[1] : G[1] \to C[1]\) in \(\mathcal{A}_t\) comes from a morphism in \(\text{A-Mod}\), \(h : G \to C\), with \(\text{Coker} h \in \mathcal{Q}_t\) and \(\text{Ker} h \neq 0\). The latter comes from the fact that the exact sequence \(0 \to \text{Ker} h \to G \xrightarrow{\bar{h}} \text{Im} h \to 0\) in \(\mathcal{C}_t\), therefore \(0 \to \text{Ker} h[1] \to G[1] \xrightarrow{\bar{h}[1]} \text{Im} h[1] \to 0\) is in \(\mathcal{C}_t[1]\) and \(h[1]\) is a proper epimorphism, so \(\text{Ker} h[1] \neq 0\).

In \(\mathcal{G}\) we have the sequence:

\[
0 \to \text{Ker} h \to G \xrightarrow{h} C \xrightarrow{\text{Coker} h} \text{Coker} h[1] \to 0
\]

Since \(G \in \mathcal{D} = \mathcal{A}_t\), which is a torsion class, and \(C \in \mathcal{C}_t\), which is a torsionfree class, we have that \(\text{Im} h \in \mathcal{C}_t \cap \mathcal{D}\), this means that \(\text{Im} h\) is of the form (see \([29]\) Theorem 6.4 and Section 8):

\[
\text{Im} h = G^{(\alpha)} \oplus \bigoplus_{x \in \mathcal{U}} S_x^{\infty(\beta_x)},
\]

for some cardinals \(\alpha, \beta_x\). Moreover:

\[
\text{Ker} h = \bigcap_{\alpha} \text{Ker} \pi_G \cap \bigcap_{x \in \mathcal{U}} \bigcap_{\beta_x} \text{Ker} \pi_x,
\]

where \(\pi_G\) and \(\pi_x\) are the corestrictions of the map \(\bar{h}\) to the different copies of \(G\) and \(S_x^{\infty}\) (for \(x \in \mathcal{U}\)) respectively.

Now, if \(G\) is a direct summand of \(\text{Im} h\), \(\pi_G\) is an isomorphism, \(\text{Ker} \pi_G = 0\) and \(\text{Ker} h = 0\), but this is a contradiction, since \(\text{Ker} h \neq 0\).

Therefore, \(\text{Im} h\) has no nonzero direct summands from \(\text{Add}(G)\), hence:

\[
\text{Im} h = \bigoplus_{x \in \mathcal{U}} S_x^{\infty(\beta_x)} \subseteq \bigcap_{x \in \mathcal{U}} \mathcal{T}(x) \subseteq \text{Gen} t.
\]

From the short exact sequence \(0 \to \text{Im} h \to C \to \text{Coker} h \to 0\) we have that \(C \in \text{Gen} t \cap \mathcal{C}_t\), therefore \(C \in \bigcap_{x \in \mathcal{U}} \mathcal{T}(x)\) and so \(C[1] = Z \in \mathcal{T}_t\).

Therefore, we have seen that any proper quotient of \(G[1]\) is in \(\mathcal{T}_t\), which means that \(G[1]\) is \(t_{\mathcal{U}}\)-cocrigid and so monomorphic by Proposition 1.3.

Remark 5.2. Notice that \(G[1]\) is not atom-equivalent to any simple object in \(\mathcal{A}_t\), since, if it is atom-equivalent to one of them then, by Lemma [14], their injective envelopes should be isomorphic, and this is a contradiction.

We can now describe the atom spectrum of \(\mathcal{A}_t\), as follows:

\[
\text{ASpec}(\mathcal{A}_t) = \overline{G[1]} \cup \bigcup_{x \in \mathcal{U}} \{S_x[1] \mid S_x \in \bigcup_{x \in \mathcal{U}} \text{t}_x\ \text{simple regular}\} \cup \{S_x^{\infty} \mid S_x \in \bigcup_{x \in \mathcal{U}} \text{t}_x\ \text{simple regular}\}.
\]

This shows that the injection between \(\text{ASpec}(\mathcal{A}_t)\) and \(\text{Spec}(\mathcal{A}_t)\) is actually a bijection also when \(\varnothing \neq \mathcal{U} \subseteq \mathcal{P}_k^1\), and the description of the atom spectrum is complete.

The partial order in \(\text{ASpec}(\mathcal{A}_t)\) is the following: the singletons \(\{S_x[1]\}\) and \(\{S_x\}\) are open by Proposition 1.3. Moreover, \(\overline{G[1]} \leq S_x[1]\), for any simple regular in \(\bigcup_{x \in \mathcal{U}} \text{t}_x\), indeed: let \(H\) be a monomorphic object in \(\mathcal{A}_t\) such that \(\overline{H} = \overline{G[1]}\), then \(H\) and \(G[1]\) have a common nonzero subobject \(Y\). For a simple regular module, we have the short exact sequence in \(\mathcal{A}_t\):

\[
0 \to S_x^{\infty} \to G[1] \to S_x^{\infty} \to 0
\]

Let \(Z\) be a pullback in the following diagram:

\[
0 \quad \rightarrow \quad \mathcal{Z} \quad \rightarrow \quad Y \quad \rightarrow \quad Y/\mathcal{Z} \rightarrow \mathcal{0}
\]

\[
0 \quad \rightarrow \quad S_x^{\infty} \quad \rightarrow \quad G[1] \quad \rightarrow \quad S_x^{\infty} \rightarrow \mathcal{0}
\]

where the last vertical arrow is a monomorphism by \([33]\) Proposition IV.5.1]. We have: \(Y/\mathcal{Z} \cong Y \rightarrow H\) and, since \(S_x^{\infty}\) is uniserial, \(S_x[1] \subseteq Y/\mathcal{Z} \subseteq S_x^{\infty}\). This means that \(S_x[1]\) in \(\text{ASupp} Y \subseteq \text{ASupp} H\) (see \([24]\) Propositions 3.3)).

By Proposition 1.4 (4), we reach the conclusion.

When \(\mathcal{U} \neq \varnothing\), let us suppose that \(\overline{G[1]} \leq S_x\), for \(S_x \in \bigcup_{x \in \mathcal{U}} \text{t}_x\), then, by Proposition 1.3, we have that \(S_x \in \text{ASupp} G[1]\), therefore \(S_x\) is atom equivalent to a subobject of a proper quotient object of \(G[1]\), but in Proposition 5.1 we have seen that any proper quotient of \(G[1]\) belongs to the hereditary torsion class \(\bigcap_{x \in \mathcal{U}} \mathcal{T}(x)[1]\), and so \(S_x\) has to be in there too. This is a contradiction.
5.2 Case \( \mathcal{U} = \mathbb{P}_k^1 \)

Consider now the torsion pair generated by \( t, t_\mathcal{U} = (\text{Gen} t, \mathcal{F}) \), in \( \mathcal{G} \), which is cogenerated by the cotilting module \( C_\mathcal{U} \). Also in this case we have an injective map between \( \text{ASpec}(\mathcal{A}_\mathcal{U}) \) and \( \text{Spec}(\mathcal{A}_\mathcal{U}) \), which corresponds, by [17, Proposition 4.4], to the set of indecomposable objects in \( \text{Prod}(C_\mathcal{U}[1]) \). From Theorem 5.2 we know that the simple objects in \( \mathcal{A}_\mathcal{U} \) are \( G[1] \) and \( S_x \), for any simple regular \( \Lambda \)-module in \( t \), and these are all monoform objects, clearly not atom-equivalent. Their injective envelopes are, respectively, \( G[1] \) and \( S_x \approx [1] \), for any \( x \in \mathbb{P}_k^1 \). Therefore we can conclude that the atom spectrum is:

\[
\text{ASpec}(\mathcal{A}_\mathcal{U}) = G[1] \cup \{ S_x | S_x \text{ simple regular } \Lambda \text{-module} \}.
\]

Therefore, \( \text{ASpec}(\mathcal{A}_\mathcal{U}) \) and \( \text{Spec}(\mathcal{A}_\mathcal{U}) \) are in bijection. Notice that any singleton \( \{ \alpha \} \), for \( \alpha \in \text{ASpec}(\mathcal{A}_\mathcal{U}) \), is open by Proposition [13, Proposition 1], meaning that the topology on \( \text{ASpec}(\mathcal{A}_\mathcal{U}) \) is discrete.

### 6 On Gabriel categories

Consider \( \mathfrak{t} = (\mathcal{T}, \mathcal{F}) \) a hereditary torsion pair in \( \mathcal{G} \) together with the localization sequence, as we have mentioned in Section 4:

\[
\begin{array}{ccc}
\mathcal{T} & \xrightarrow{\text{inc}} & \mathcal{G} \\
\downarrow \mathcal{T} & \searrow \mathcal{Q} & \swarrow \mathcal{S} \\
\mathcal{G}/\mathcal{T} & & \mathcal{G}/\mathcal{Q}
\end{array}
\]

For a set \( \mathcal{X} \) of objects in \( \mathcal{G} \), we denote by \( \langle \mathcal{X} \rangle_{\text{htor}} \) the smallest hereditary torsion class containing \( \mathcal{X} \).

**Definition 6.1.** The Gabriel filtration of \( \mathcal{G} \) is a transfinite chain of hereditary torsion classes of \( \mathcal{G} \)

\[
\mathcal{G}_{\alpha-1} \subseteq \mathcal{G}_0 \subseteq \mathcal{G}_1 \subseteq \cdots \subseteq \mathcal{G}_\alpha \subseteq \cdots
\]

where:

- \( \mathcal{G}_{\alpha-1} = \{0\} \)
- suppose that \( \alpha \) is an ordinal for which \( \mathcal{G}_\alpha \) has already been defined. Let \( Q_\alpha : \mathcal{G} \to \mathcal{G}/\mathcal{G}_\alpha \) be the quotient functor.

We define \( \mathcal{G}_{\alpha+1} \) as:

\[
\mathcal{G}_{\alpha+1} = \langle \mathcal{G}_\alpha \cup \{ X \in \mathcal{G} \mid Q_\alpha(X) \text{ is simple in } \mathcal{G}/\mathcal{G}_\alpha \} \rangle_{\text{htor}}
\]

- if \( \lambda \) is a limit ordinal, then:

\[
\mathcal{G}_\lambda = \bigcup_{\alpha < \lambda} \mathcal{G}_\alpha \quad \text{htor}
\]

Let \( \alpha \) be an ordinal. An object \( X \) in \( \mathcal{G} \) is said to be \( \alpha \)-torsion (resp. \( \alpha \)-torsionfree) if and only if \( X \in \mathcal{G}_\alpha \) (resp. \( X \in \mathcal{G}_{\alpha+\omega} \)).

**Remark 6.2.** The hereditary torsion pair \( \mathfrak{t}_\alpha = (\mathcal{G}_\alpha, \mathcal{G}_\alpha^{\perp_0}) \) induces the localization sequence:

\[
\begin{array}{ccc}
\mathcal{G}_\alpha & \xrightarrow{\text{inc}} & \mathcal{G} \\
\downarrow \mathcal{T}_\alpha & \searrow \mathcal{Q}_\alpha & \swarrow \mathcal{S}_\alpha \\
\mathcal{G}/\mathcal{T}_\alpha & & \mathcal{G}/\mathcal{Q}_\alpha
\end{array}
\]

and, for simplicity, a \( \mathfrak{t}_\alpha \)-cocritical object in \( \mathcal{G} \) will be called \( \alpha \)-cocritical.

Notice that, given the class \( \mathcal{G}_\alpha \) in the Gabriel filtration, we can define \( \mathcal{G}_{\alpha+1} \), using Lemma [17] as

\[
\mathcal{G}_{\alpha+1} = \langle \mathcal{G}_\alpha \cup \{ X \in \mathcal{G} \mid X \text{ is } \alpha \text{-cocritical} \} \rangle_{\text{htor}}.
\]

**Remark 6.3.** [17, Remark 2.12] Let \( G \) be a generator of the Grothendieck category \( \mathcal{G} \). It is known that \( \mathcal{G} \) has just a set of quotient objects. One can show that:

\[
\mathcal{G}_{\alpha+1} = \langle \mathcal{G}_\alpha \cup \{ X \in \mathcal{G} \mid X \text{ is a quotient of } G, Q_\alpha(X) \text{ is simple in } \mathcal{G}/\mathcal{G}_\alpha \} \rangle_{\text{htor}}
\]

This means that the Gabriel filtration eventually stabilizes, i.e. there is a cardinal \( \kappa \) such that \( \mathcal{G}_\alpha = \mathcal{G}_\kappa \) for all \( \alpha \geq \kappa \), just take \( \kappa = \sup \{ \alpha \mid \text{there is } H \subseteq G \text{ such that } Q_\alpha(G/H) \text{ is simple} \} \).

By virtue of the previous Remark, it is meaningful to consider the union \( \mathcal{G} = \bigcup_\alpha \mathcal{G}_\alpha \) of all the localizing subcategories in the Gabriel filtration.

**Definition 6.4.** For an object \( X \in \mathcal{G} \), we say that \( X \) has Gabriel dimension if there is a minimal ordinal \( \delta \) such that \( X \in \mathcal{G}_\delta \), and we write \( \text{Gdim}(X) = \delta \). If \( \mathcal{G} = \mathcal{G} \), we say that \( \mathcal{G} \) is a Gabriel category with Gabriel dimension \( \text{Gdim}(\mathcal{G}) = \kappa \), where \( \kappa \) is the smallest ordinal such that \( \mathcal{G}_\kappa = \mathcal{G} \).

**Proposition 6.5.** Every locally noetherian Grothendieck category is a Gabriel category.

**Proof.** Let \( \mathcal{G} \) be a locally noetherian Grothendieck category and consider its Gabriel filtration:

\[
\{0\} = \mathcal{G}_0 \subseteq \mathcal{G}_1 \subseteq \cdots \subseteq \mathcal{G}_\alpha \subseteq \cdots
\]
By Remark 5.5 this filtration stabilizes, i.e., there is a cardinal \( \kappa \) such that \( G_\alpha = G_\kappa \) for all \( \alpha \geq \kappa \). Let \( \mathcal{N} \) be the set of all the noetherian generators of \( G \). We prove that \( \mathcal{N} \subseteq G_\kappa \). Indeed: suppose that there is \( N \in \mathcal{N} \) such that \( N \notin G_\kappa \). Consider the set:

\[ \mathcal{I} = \{ X \subseteq N \mid N/X \notin G_\kappa \} \]

which is not empty since \( 0 \in \mathcal{I} \). Since \( N \) is noetherian, \( \mathcal{I} \) has a maximal element \( \bar{X} \). Therefore, for any proper subobject \( Y \) such that \( X \subseteq Y \subseteq N \), we have \( N/Y \notin G_\kappa \). Moreover, \( N/X \) is \( \kappa \)-torsionfree, indeed: if it is not, it has a nonzero \( \kappa \)-torsion part \( T_\kappa(N/X) \) such that \( (N/X)/T_\kappa(N/X) \) is \( \kappa \)-torsionfree, but all the proper quotients of \( N/X \) are \( \kappa \)-torsionfree, by the maximality of \( \bar{X} \). This means that \( N/X \) is \( \kappa \)-cocalcirical, indeed, it is \( \kappa \)-torsionfree and any proper quotient of \( N/X \) is in \( G_\kappa \). Hence, \( Q_\kappa(N/X) \) is simple in \( G/G_\kappa \) and then \( N/X \in G_{\kappa+1} = G_\kappa \). Contradiction. \( \mathcal{N} \subseteq G_\kappa \), therefore \( G_\kappa = G \).

Example 6.6. The Gabriel dimension of \( \Lambda \text{-Mod} \), where \( \Lambda \) is the Kronecker algebra, is 0. Indeed: \( G_0 = \{ (P_1, Q_1) \}_{\text{htor}} \), where \( P_1 \) and \( Q_1 \) are the simple \( \Lambda \)-modules, and there is a short exact sequence \( 0 \rightarrow P_1 \oplus P_1 \rightarrow P_2 \rightarrow Q_1 \rightarrow 0 \), since \( P_2 \) is the projective cover of \( Q_1 \). Therefore the module \( A_\Lambda \) is in \( G_0 \). Meaning that \( G_0 = \Lambda \text{-Mod} \).

7 Gabriel dimension of hearts in \( \Lambda \text{-Mod} \)

In this section, we prove that the different HRS-heart arising from the cosilting torsion pairs in \( \Lambda \text{-Mod} \) are Gabriel categories and we compute their Gabriel dimensions. For the finite dimensional cosilting \( \Lambda \)-modules, the computation is straightforward, indeed:

- For the simple injective \( Q_1 \), the heart \( \mathcal{A} \cong k \text{-Mod} \), therefore \( \text{Gdim}(\mathcal{A}) = 0 \), since it is a semisimple category.
- For all the other finite dimensional cosilting \( \Lambda \)-modules, the heart \( \mathcal{A} \cong \Lambda \text{-Mod} \), as seen in Example 6.6 therefore \( \text{Gdim}(\mathcal{A}) = 0 \), by Example 6.6

We are left with the hearts arising from the infinite dimensional cosilting modules \( C_\mathcal{U} \), for \( \mathcal{U} \subseteq \mathcal{P}_k^1 \).

7.1 Case \( \mathcal{U} = \emptyset \)

As we have seen in Section 5.1, \( \mathcal{A} \) is equivalent to \( \text{Qcoh}\mathcal{P}_k^1 \) and therefore a locally noetherian Grothendieck category. By Proposition 6.3, \( \mathcal{A} \) is a Gabriel category.

We build the Gabriel filtration step by step. Set \( G_{-1} = \{0\} \). We have, by definition:

\[ G_0 = \langle \{ X \in \mathcal{A} \mid X \text{ is simple in } \mathcal{A} \} \rangle_{\text{htor}} \]

therefore, using Theorem 3.2

\[ G_0 = \langle \{ Sx[1] \mid Sx \text{ simple regular } \Lambda \text{-module} \} \rangle_{\text{htor}}. \]

Following [5 Section 5.2], \( G_0 = \mathcal{T}[1] \), where \( \mathcal{T} = \lim\limits_{\rightarrow} t \) (see [29] Section 3.4), and the corresponding torsionfree class is \( G_0^{\perp_0} = \lim\limits_{\rightarrow}(Q \cup p)[1] \).

The next step is:

\[ G_1 = \langle G_0 \cup \{ X \in \mathcal{A} \mid Q_0(X) \text{ is simple in } \mathcal{A}/G_0 \rangle \rangle_{\text{htor}} \]

where \( Q_0 : \mathcal{A} \rightarrow \mathcal{A}/G_0 \) is the quotient functor. By Lemma 4.7, we have that the objects in \( \mathcal{A} \) which become simple objects in \( \mathcal{A}/G_0 \) are precisely the \( 0 \)-cocritical objects, i.e., the cocritical objects with respect to the torsion pair \( t_0 = (G_0, G_0^{\perp_0}) \). By Proposition 5.1 \( G[1] \) is monomorphism, hence it is \( 0 \)-cocritical via Remark 4.9. We claim the following:

Lemma 7.1. If \( X \) is a \( 0 \)-cocritical object in \( \mathcal{A} \), then \( Q_0(X) \cong Q_0(G[1]) \).

Proof. If \( X \in \mathcal{A} \) is a \( 0 \)-cocritical object, then, \( Q_0(X) \) is simple in \( \mathcal{A}/G_0 \) and moreover, by Remark 4.9 \( X \) is monomorphism in \( \mathcal{A} \).

Since we have a complete description of \( \text{ASpec}(A) \), we have that either \( X \in S_2[1] \) or \( X \in G[1] \): the first is not possible, because, if it is true, then \( X \) and \( S_2[1] \) have a common nonzero subobject, but \( S_2[1] \) is simple, then \( S_2[1] \subseteq X \), which is a contradiction since \( \text{Hom}_\mathcal{A}(S_2[1], X) = 0 \) (recall that \( S_2[1] \in G_0 \) and \( X \in G_0^{\perp_0} \)). Then \( X \in G[1] \), i.e., there is an object \( Y \in \mathcal{A} \) such that \( X \supseteq Y \subseteq G[1] \). This means that, in the quotient category \( \mathcal{A}/G_0 \), \( Q_0(Y) \subseteq Q_0(G[1]) \). But \( Q_0(X) \) is simple in \( \mathcal{A}/G_0 \) and, by Lemma 4.7, \( Q_0(G[1]) \) is simple too, therefore \( Q_0(X) \cong Q_0(G[1]) \).

We have \( G_1 = \langle G_0 \cup G[1] \rangle \rangle_{\text{htor}} \).

Theorem 7.2. If \( \mathcal{A} = G(Q, C) \), then \( \text{Gdim} \mathcal{A} = 1 \).

Proof. Consider the algebra \( \Lambda \) as a \( \Lambda \)-module. We have \( \Lambda \in p \subseteq \mathcal{F} = t^{\perp_0} \). By [29] Theorem 4.1, we have a short exact sequence: \( 0 \rightarrow \Lambda \rightarrow M \rightarrow M' \rightarrow 0 \), where \( M \in \text{Add}(G) \) and \( M' \) is a direct sum of Prüfer modules. Since, the sequence:

\[ 0 \rightarrow \Lambda \rightarrow G^{(\alpha)} \rightarrow M' \rightarrow 0 \]
lies entirely in \( \mathcal{C} \) and it gives rise to a short exact sequence in \( \mathcal{A} \), entirely lying in \( \mathcal{C}[1] \):

\[
0 \to \Lambda[1] \to G^{(\alpha)}[1] \to M[1] \to 0.
\]

Now, since \( G^{(\alpha)}[1] \in \mathcal{G}_1 \) and \( \mathcal{G}_1 \) is closed under subobjects, we have \( \Lambda[1] \in \mathcal{G}_1 \). For the same reason, the family \( \{ Z \mid Z \subseteq (\Lambda[1])^n, n \in \mathbb{N} \} \) is contained in \( \mathcal{G}_1 \) and this, by [10, Lemma 3.4], is a family of generators for \( \mathcal{A} \), therefore \( \mathcal{G}_1 = \mathcal{A} \). This means that the Gabriel filtration stops at \( \mathcal{G}_1 \), therefore \( \text{Gdim} \mathcal{A} = 1 \).

### 7.2 Case \( \emptyset \neq \mathcal{U} \subseteq \mathbb{P}^1_k \)

Set \( \mathcal{G}_{-1} = \{0\} \) and, by definition, we have:

\[
\mathcal{G}_0 = \{ \{X \in \mathcal{A}_U \mid X \text{ is simple in } \mathcal{A}_U \}\}_{\text{htor}}
\]

thus, by Theorem \[5.2\]

\[
\mathcal{G}_0 = \{ \{S_x \mid S_x \text{ simple regular in } \bigcup_{x \in \mathcal{U}} t_x \} \cup \{S_x[1] \mid S_x \text{ simple regular in } \bigcup_{x \in \mathcal{U}} t_x \}\}_{\text{htor}}.
\]

It is clear that all the modules in the ray of \( S_x \), for \( x \in \mathcal{U} \), are in \( \mathcal{G}_0 \), therefore the Prüfer modules \( S_x^{\infty} \), for \( x \in \mathcal{U} \), are in \( \mathcal{G}_0 \). The same argument can be applied to the ray starting at \( S_x[1] \), for \( x \in \mathcal{U} \), therefore \( S_x^{\infty}[1] \in \mathcal{G}_0 \), for \( x \in \mathcal{U} \).

The next torsion class in the Gabriel filtration is:

\[
\mathcal{G}_1 = \mathcal{G}_0 \cup \{ X \in \mathcal{A}_U \mid Q_0(X) \text{ is simple in } \mathcal{A}_U/\mathcal{G}_0 \}_{\text{htor}}
\]

where \( Q_0 : \mathcal{A}_U \to \mathcal{A}_U/\mathcal{G}_0 \) is the quotient functor. By Lemma \[4.1\] the simple objects in \( \mathcal{A}/\mathcal{G}_0 \) are precisely the cocritical objects with respect to the torsion pair \( \mathcal{t}_0 = (\mathcal{G}_0, \mathcal{G}_0^{\perp_0}) \), i.e. the 0-cocritical objects. Consider the torsion pair \( (\mathcal{T}_\mathcal{U}, \mathcal{F}_\mathcal{U}) \) defined in the proof of Proposition \[5.1\] where:

\[
\mathcal{T}_\mathcal{U} = \prod_{x \in \mathcal{U}} \mathcal{T}(x)[1] \quad \text{and} \quad \mathcal{F}_\mathcal{U} = \text{Cogen} \mathcal{C}_\mathcal{U}[1].
\]

It is clear that \( \mathcal{T}_\mathcal{U} \subseteq \mathcal{G}_0 \), hence \( \mathcal{G}[1] \) is 0-cocritical. Moreover, with the same argument as in Lemma \[4.1\] we can prove that if \( X \in \mathcal{A}_U \) is a 0-cocritical object, then \( Q_0(X) \cong Q_0(G[1]) \). In conclusion we have:

\[
\mathcal{G}_1 = (\mathcal{G}_0 \cup \mathcal{G}[1])_{\text{htor}}.
\]

**Theorem 7.3.** If \( \mathcal{A}_U = \mathcal{G}(Q_\mathcal{U}, \mathcal{C}_\mathcal{U}) \), then \( \text{Gdim} \mathcal{A}_U = 1 \).

**Proof.** As in the proof of Theorem \[7.2\] consider the regular \( \Lambda \)-module \( \Lambda \) and the short exact sequence: \( 0 \to \Lambda \to G^{(\alpha)} \to M' \to 0 \), where \( M' \) is a direct sum of Prüfer modules. Consider the canonical sequence given by the torsion pair \( (Q_\mathcal{U}, \mathcal{C}_\mathcal{U}) \) for \( M' \):

\[
0 \to t(M') \to M' \to M'/t(M') \to 0,
\]

where \( t(M') \in Q_\mathcal{U}, M'/t(M') \in \mathcal{C}_\mathcal{U} \) and \( Y \in \mathcal{C}_\mathcal{U} \) since \( G^{(\alpha)} \in \mathcal{C}_\mathcal{U} \). Moreover, since \( M' \) is a direct sum of copies of Prüfer modules, \( t(M') \) is a direct sum of Prüfer modules lying in \( \prod_{x \in \mathcal{U}} \mathcal{T}(x) \). So, the upper row becomes

\[
0 \to \Lambda \to Y \to \bigoplus_{x \in \mathcal{U}} S_x^{\infty(\beta_x)} \to 0
\]

and gives rise to a short exact sequence in \( \mathcal{A}_U \):

\[
0 \to \bigoplus_{x \in \mathcal{U}} S_x^{\infty(\beta_x)} \to \Lambda[1] \to Y[1] \to 0
\]

with \( \bigoplus_{x \in \mathcal{U}} S_x^{\infty(\beta_x)} \in \mathcal{G}_0 \). From the short exact sequence \( 0 \to Y \to G^{(\alpha)} \to M'/t(M') \to 0 \), which is entirely in \( \mathcal{C}_\mathcal{U} \), we obtain a short exact sequence in \( \mathcal{A}_U \):

\[
0 \to Y[1] \to G^{(\alpha)}[1] \to M'/t(M')[1] \to 0
\]
showing that \( Y[1] \in \mathcal{G}_1 \). Therefore, by the extension closure property of \( \mathcal{G}_1 \), \( \Lambda[1] \in \mathcal{G}_1 \). From \([10]\) Lemma 3.4 we know that the heart \( \mathcal{A}_U \) has a set of generators \( \{ Z \mid Z \subseteq (\Lambda[1])^n, n \in \mathbb{N} \} \), which is, therefore, entirely in \( \mathcal{G}_1 \) and so \( \mathcal{G}_1 = \mathcal{A}_U \), showing that \( \text{Gdim} \mathcal{A}_U = 1 \).

### 7.3 Case \( \mathcal{U} = \mathbb{P}_k^1 \)

By Theorem 5.2 the simple objects in \( \mathcal{A}_U \) are \( G[i] \) and \( S_x \), for \( x \in \mathbb{P}_k^1 \). Therefore, setting \( G^{-1} = \{ 0 \} \), we obtain:

\[
G_0 = \langle \{ S_x \mid S_x \text{ simple regular} \} \cup \{ G[1] \} \rangle_{\text{htor}}.
\]

It is clear that all the objects in the ray of \( S_x \), for any \( x \in \mathbb{P}_k^1 \), are in \( G_0 \), and hence all the Pr"{u}fer objects \( S_x^\infty \in G_0 \), for any \( x \in \mathbb{P}_k^1 \).

**Theorem 7.4.** If \( \mathcal{A}_U = \mathcal{G}(\text{Gen t}, \mathcal{F}) \), then \( \text{Gdim} \mathcal{A}_U = 0 \).

**Proof.** As for the previous case, we can show that the object \( \Lambda[1] \) is in \( G_0 \). Consider, as before, the short exact sequence:

\[
0 \to \Lambda \to G(\alpha) \to M' \to 0
\]

where \( M' \) is a direct sum of Pr"{u}fer modules. The first two terms of this short exact sequence are in \( \mathcal{F} \) and \( M' \in \text{Gen t} \), therefore, there is a short exact sequence in the heart \( \mathcal{A}_U \):

\[
0 \to M' \to \Lambda[1] \to G(\alpha)[1] \to 0.
\]

Since \( M' \) is a direct sum of Pr"{u}fer objects, \( M' \in G_0 \), hence \( \Lambda[1] \in G_0 \). This means that the set \( \{ Z \mid Z \subseteq (\Lambda[1])^n, n \in \mathbb{N} \} \) of generators of the heart given by \([10]\) Lemma 3.4 is entirely in \( G_0 \) showing that \( G_0 = \mathcal{A}_U \) and \( \text{Gdim} \mathcal{A}_U = 0 \).

We summarize all the results obtained in Sections 5 and 7 in the following Theorem.

**Theorem 7.5.** Let \( \mathcal{G} = \Lambda-\text{Mod} \), with \( \Lambda \) the Kronecker algebra. For any finite dimensional cosilting \( \Lambda \)-module, the heart \( \mathcal{A} \) of the t-structure arising from the corresponding cosilting torsion pair has \( \text{Gdim}(\mathcal{A}) = 0 \) and

\[
\text{ASpec}(\mathcal{A}) \cong \begin{cases} \{ \mathbb{K} \}, & \text{for the cosilting } \Lambda \text{-module } Q_1 \\ \{ P_1, Q_1 \}, & \text{otherwise} \end{cases}
\]

Consider \( \mathcal{U} \subseteq \mathbb{P}_k^1 \). Let \( C_U \) be the related infinite dimensional cosilting module, as in Table 4 and let \( \mathcal{A}_U \) be the heart of the t-structure arising from the corresponding cosilting torsion pair. We have the following:

- **If** \( \mathcal{U} \not\subseteq \mathbb{P}_k^1 \), **then** \( \text{Gdim} \mathcal{A}_U = 1 \) and

\[
\text{ASpec}(\mathcal{A}_U) = G[1] \cup \{ S_x \mid S_x \text{ simple regular} \} \cup \{ S_x | S_x \text{ simple regular} \} \cup \{ t_x \}.
\]

- **If** \( \mathcal{U} = \mathbb{P}_k^1 \), **then** \( \text{Gdim} \mathcal{A}_U = 0 \) and

\[
\text{ASpec}(\mathcal{A}_U) = G[1] \cup \{ S_x \mid S_x \text{ simple regular } \Lambda \text{-module} \}.
\]
References

[1] T. Adachi, O. Iyama, I. Reiten, \(\tau\)-tilting theory, Compos. Math. 150 (2014) 415-452.

[2] L. Angeleri Hügel, Infinite dimensional tilting theory, Advances in representation theory of algebras, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2013, pp. 1-37. MR 3220532

[3] L. Angeleri Hügel, On the abundance of silting modules, Contemporary Mathematics 716 (CONM/716)

[4] L. Angeleri Hügel, I. Herzog, R. Laking, ongoing work.

[5] L. Angeleri Hügel, M. Hrbeč, Silting modules over commutative rings, International Mathematics Research Notices 2017(13) (2017), 4131?4151.

[6] L. Angeleri Hügel, D. Kussin, Tilting and cotilting modules over concealed canonical algebras. Math. Z., (2016).

[7] L. Angeleri Hügel, F. Marks, J. Vitória, Silting modules, International Mathematics Research Notices 2016.4 (2016), 1251-1284.

[8] L. Angeleri Hügel, F. Marks, J. Vitória, Silting modules and ring epimorphisms, Adv. Math. 303 (2016), 1044-1076.

[9] L. Angeleri Hügel, D. Pospíšil, J. Šťovíček, J. Trlifaj, Tilting, cotilting, and spectra of commutative noetherian rings, Transactions of the Amer. Math. Soc. 366 (2014), pp.3487-3517.

[10] L. Angeleri Hügel, J. Sánchez, Tilting modules over tame hereditary algebras, J. Reine Angew. Math., 682 (2013), pp.1-48.

[11] A. A. Beilinson, J. Bernstein, P. Deligne, Faisceaux pervers, Astérisque 100 (1982).

[12] E. Barnard, A. Carroll, S. Zhu, Minimal inclusion of torsion classes, Algebraic Combinatorics, Volume 2 (2019) no. 5, p. 879-901.

[13] A. B. Buan, H. Krause, Cotilting modules over tame hereditary algebras, Pacific J. Math. 211 (2003), pp.41-59.

[14] S. Breaz, F. Pop, Cosilting modules, arXiv:1510.05098v2 (2017)

[15] S. Breaz, J. Žemlička, Torsion classes generated by silting modules, arXiv:1601.06655v3 (2017).

[16] R. Colpi, E. Gregorio, F. Mantese, On the heart of a faithful torsion theory, Journal of Algebra 307 (2007), pp.841-863.

[17] P. Čoupek, J. Šťovíček, Cotilting sheaves on noetherian schemes, arXiv:1707.01677v1 [math.AG] (2017)

[18] W. Crawley-Boevey, Locally finitely presented additive categories, Comm. Algebra 22 (1994), no. 5, pp.1641-1674.

[19] W. Crawley-Boevey, Regular modules for tame hereditary algebras, Proc. London Math. Soc. 62 (1991), pp.490-508.

[20] L. Demonet, O. Iyama, N. Reading, I. Reiten, H. Thomas, Lattice theory of torsion classes, arXiv:1711.01785v2.

[21] P. Gabriel, Des catégories abéliennes, Bull. soc. math. France, 90 (1962), pp.323-448.

[22] S. I. Gelfand, Yu. I. Manin, Methods of homological algebra, Springer-Verlag, 1996.

[23] D. Happel, I. Reiten, S. O. Smalø, Tilting in abelian categories and quasitilted algebras, Mem. Amer. Math. Soc. 120 (1996), viii+88.

[24] R. Kanda, Classifying Serre subcategories via atom spectrum, Adv. Math. 231 (2012), no. 3-4, pp.1572-1588.

[25] R. Kanda, Specialization orders on atom spectra of Grothendieck categories, J. Pure Appl. Algebra, 219 (2015), no. 11, pp.4907-4952.

[26] R. Laking, Purity in compactly generated derivators and t-structures with Grothendieck hearts, arXiv:1804.01364v1 (2018)

[27] C. E. Parra, M. Saorín, Direct limits in the heart of a t-structure: The case of a torsion pair, J. Pure Appl. Algebra 219 (2015), pp.4117-4143.

[28] C. Psaroudakis, J. Vitória, Realisation functors in tilting theory, Mathematische Zeitschrift 288.3-4 (2018): 965-1028.

[29] I. Reiten, C. M. Ringel, Infinite dimensional representations of canonical algebras, Canad. J. Math. 58 (2006), pp.180-224.
[30] C. M. Ringel, *Infinite dimensional representations of finite dimensional hereditary algebras*, Symposia Math. 23 (1979), pp.321-412.

[31] C.M. Ringel, *The Ziegler spectrum of a tame hereditary algebra*, Coll. Math. 76 (1998), pp.105-115.

[32] M. Saorín, *On locally coherent hearts*, Pacific J. Math. 287 (2017), n. 1, pp.199-221.

[33] B. Stenström, *Rings of quotients*. Springer-Verlag, New York, 1975.

[34] H. H. Storrer, *On Goldman's primary decomposition*, Lecture Notes in Mathematics (1972), vol. 246, pp.617-661.

[35] J. Šťovíček, O. Kerner, J. Trlifaj, *Tilting via torsion pairs and almost hereditary noetherian rings*, J. Pure Appl. Algebra 215 (2011), pp.2072-2085.

[36] P. Vámos, S. Virili, ongoing work.

[37] S. Virili, *Length functions of Grothendieck categories with applications to infinite group representations*, arXiv:1401.8306v1, (2014)

[38] J. Woolf, *Stability conditions, torsion theories and tilting*, J. London Math. Soc. 82 (2010), pp.663-682.

[39] P. Zhang, J. Wei, *Cosilting complexes and AIR-cotilting modules*, arXiv:1601.01385v1 (2016).

Dipartimento di Informatica - SETTORE DI MATEMATICA, UNIVERSITÀ DEGLI STUDI DI VERONA, STRADA LE GRAZIE, 15 - CA’ VIGNAL I, 37134, VERONA, ITALY

Email address: rapa.alessandro@gmail.com