NORM DISCONTINUITY AND SPECTRAL PROPERTIES OF
ORNSTEIN-UHLENBECK SEMIGROUPS

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ABSTRACT. Let $E$ be a real Banach space. We study the Ornstein-Uhlenbeck semigroup $P = \{P(t)\}_{t \geq 0}$ associated with the Ornstein-Uhlenbeck operator

$$Lf(x) = \frac{1}{2} \text{Tr} Q D^2 f(x) + \langle Ax, Df(x) \rangle, \quad x \in E.$$ 

Here $Q \in \mathcal{L}(E^*, E)$ is a positive symmetric operator and $A$ is the generator of a $C_0$-semigroup $S = \{S(t)\}_{t \geq 0}$ on $E$. Under the assumption that $P$ admits an invariant measure $\mu_\infty$ we prove that if $S$ is eventually compact and the spectrum of its generator is nonempty, then

$$\|P(t) - P(s)\|_{\mathcal{L}(L^1(\mu_\infty))} = 2 \quad \text{for all } t, s \geq 0 \text{ with } t \neq s.$$ 

This result is new even when $E = \mathbb{R}^n$. We also study the behaviour of $P$ in the space $BUC(E)$. We show that if $A \neq 0$ there exists $t_0 > 0$ such that

$$\|P(t) - P(s)\|_{\mathcal{L}(BUC(E))} = 2 \quad \text{for all } 0 \leq t, s \leq t_0 \text{ with } t \neq s.$$ 

Moreover, under a nondegeneracy assumption or a strong Feller assumption, the following dichotomy holds: either

$$\|P(t) - P(s)\|_{\mathcal{L}(BUC(E))} = 2 \quad \text{for all } t, s \geq 0, \ t \neq s,$$

or $S$ is the direct sum of a nilpotent semigroup and a finite-dimensional periodic semigroup. Finally we investigate the spectrum of $L$ in the spaces $L^1(\mu_\infty)$ and $BUC(E)$.

1. Introduction and Preliminaries

In this paper we study certain properties of the Ornstein-Uhlenbeck semigroup in spaces of continuous functions and integrable functions. This semigroup is associated with the stochastic linear Cauchy problem

$$
\begin{cases}
    dU(t) = AU(t) \, dt + B \, dW_H(t), \\
    U(0) = x.
\end{cases}
$$

Here $A$ is assumed to be the infinitesimal generator of a $C_0$-semigroup $S = \{S(t)\}_{t \geq 0}$ on a real Banach space $E$, $B$ is a bounded operator from a real Hilbert space $H$ into $E$, $W_H = \{W_H(t)\}_{t \geq 0}$ is an $H$-cylindrical Brownian motion, and $x \in E$ is an initial value. As is well known, the above problem admits a unique weak solution if and only if for all $t \geq 0$ there exists a centred Gaussian Radon measure $\mu_t$ on $E$ whose covariance operator $Q_t \in \mathcal{L}(E^*, E)$ is given by

$$\langle Q_t x^*, y^* \rangle = \int_0^t \langle S(s)BB^*S^*(s)x^*, y^* \rangle \, ds, \quad x^*, y^* \in E^*,$$
where $E^*$ denotes the topological dual of $E$. Under this assumption the solution $U = \{U(t, x)\}_{t \geq 0}$ of (1.1) is given by the stochastic Itô integral

$$U(t, x) = S(t)x + \int_0^t S(t-s)B \, dW_H(s),$$

see [3, 9, 23]. For more information on Gaussian measures in infinite dimensions we refer to [2, 27].

The Ornstein-Uhlenbeck semigroup $P = \{P(t)\}_{t \geq 0}$ associated with $A$ and $B$ is defined on the space $C_b(E)$ of bounded real-valued continuous functions on $E$ by

$$(1.2) \quad P(t)f(x) := \mathbb{E}(f(U(t, x))) = \int_E f(S(t)x + y) \, d\mu_t(y), \quad x \in E, \ f \in C_b(E).$$

This semigroup leaves $BUC(E)$, the space of bounded real-valued uniformly continuous functions on $E$, invariant and has been studied by many authors [4, 8, 14, 15, 16, 25, 26]. It is well known that $P$ fails to be strongly continuous with respect to the supremum norm of $BUC(E)$ unless $A = 0$. Therefore it is natural to introduce the closed subspace $BUC^0(E)$ consisting of all functions on which $P$ acts in a strongly continuous way. This subspace is invariant under $P$, and the restriction $P^o$ of $P$ is strongly continuous on $BUC^0(E)$. It is well known that the behaviour of $P^o$ is quite pathological. For instance, in the setting of a Hilbert space $E$ it was shown in [25] that one has

$$(1.3) \quad \|P^o(t) - P^o(s)\|_{\mathcal{L}(BUC^0(E))} = 2$$

whenever $\mu_t \perp \mu_s$, i.e., the measures $\mu_t$ and $\mu_s$ are mutually singular. Here $\| \cdot \|_{\mathcal{L}(X)}$ denotes the uniform operator norm of the Banach space $\mathcal{L}(X)$ of all bounded linear operators on $X$. For the heat semigroup, which corresponds to the case $A = 0$, (1.3) was established earlier in [11]. In Section 2 we extend this result to Banach spaces and complement it by showing that (1.3) also holds whenever $S(t) \neq S(s)$. It follows that if $A \neq 0$, then there exists $t_0 > 0$ such that

$$(1.4) \quad \|P^o(t) - P^o(s)\|_{\mathcal{L}(BUC^0(E))} = 2 \quad \text{for all} \ 0 \leq s, t \leq t_0, \ t \neq s.$$ 

In particular, if $A \neq 0$, then $P^o$ always fails to be norm continuous on $BUC^0(E)$ for $t > 0$. In the converse direction we show that for fixed $t, s \geq 0$, (1.3) and $S(t) = S(s)$ imply $\mu_t \perp \mu_s$. These results are used to prove the following dichotomy: either

$$(1.5) \quad \|P^o(t) - P^o(s)\|_{\mathcal{L}(BUC^0(E))} = 2 \quad \text{for all} \ t, s \geq 0, \ t \neq s,$$

or $S$ is the direct sum of a nilpotent semigroup and a finite-dimensional periodic semigroup. Note that this result is new even when $E$ is finite-dimensional. The probabilistic interpretation of (1.4) and (1.5) is that $\text{sup}_{x \in E} \|\mu_{t,x} - \mu_{s,x}\|_{\text{var}} = 2$, for $t, s \geq 0$ with $t \neq s$, where $\mu_{t,x}$ denotes the law of the process $U(t, x)$ which solves (1.1), and $\| \cdot \|_{\text{var}}$ is the total variation norm.

Related to the problem of norm discontinuity is the problem of characterizing the spectrum of the generator $L_{P^o}$ of $P^o$. For finite-dimensional spaces $E$, it was shown in [17] that if the operator $Q := B \circ B^*$ is invertible and the spectrum $\sigma(A)$ of $A$ is contained in $\{\lambda \in \mathbb{C} : \text{Re} \lambda < 0\}$, then

$$(1.6) \quad \sigma(L_{P^o}) = \mathbb{C}^-,$$

where $\mathbb{C}^- := \{\lambda \in \mathbb{C} : \text{Re} \lambda \leq 0\}$, and every $\lambda \in \mathbb{C}$ with $\text{Re} \lambda < 0$ is an eigenvalue. By standard results from semigroup theory, (1.6) already implies that $P^o$ cannot be eventually norm continuous in $BUC^0(E)$. Below we obtain an extension of (1.6) to the case where $S$ is an eventually compact semigroup on a Banach space $E$.

Let us next assume that the limit $Q_\infty := \lim_{t \to \infty} Q_t$ exists in the weak operator topology of $\mathcal{L}(E^*, E)$ and that there exists a centred Gaussian Radon measure $\mu_\infty$ with covariance operator $Q_\infty$. A sufficient condition for this is that the Gaussian Radon measures $\mu_t$ exist and $S$ is uniformly exponentially stable; cf. [9, Chapter 9], [24]. The measure $\mu_\infty$ is invariant for $P$, in the sense that for all $f \in BUC(E)$ and $t \geq 0$,

$$\int_E P(t)f(x) \, d\mu_\infty(x) = \int_E f(x) \, d\mu_\infty(x).$$
By a standard argument, the semigroup \( P \) has a unique extension to a strongly continuous contraction semigroup on \( L^p(E, \mu_\infty) \) for all \( p \in [1, \infty) \). For \( p \in (1, \infty) \), the behaviour of this semigroup is well understood. We refer to [6, 7, 18, 19, 22], where the domain of the generator, its spectrum, and analyticity properties are characterized.

The behaviour of \( P \) in \( L^1(E, \mu_\infty) \) is much less well understood. For finite-dimensional spaces \( E \) it is shown in [18] that the \( L^1(E, \mu_\infty) \)-spectrum of its generator \( L_P \) equals \( \mathbb{C}^- \). To the best of our knowledge, it is an open problem whether this result extends to infinite dimensions. Furthermore no \( L^1 \)-analogue of (1.3) seems to be known. In Section 3 we will first show, for finite-dimensional spaces \( E \), that

\[
\| P(t) - P(s) \|_{L^1(E, \mu_\infty)} = 2
\]

whenever \( t > s \geq 0 \). Then we extend this result to infinite dimensions in the setting of eventually compact semigroups \( S \), and, extending a result for \( E = \mathbb{R}^d \) in [18], we prove that the spectrum of \( L_P \) equals \( \mathbb{C}^- \).

Our approach is based on a technique introduced by Davies and Simon [10] which may be described as follows. If \( B_1 \) and \( B_2 \) generate \( C_0 \)-semigroups of contractions \( T_1 \) and \( T_2 \) on a Banach space \( X \), then \( B_1 \) belongs to the limit class of \( B_2 \) if there exists a sequence of invertible isometries \( V_n : X \to X \) such that

\[
R(\lambda, B_1)x = \lim_{n \to \infty} V_n^{-1} R(\lambda, B_2)V_n x, \quad x \in E.
\]

Here \( R(\lambda, B_k) = (\lambda - B_k)^{-1}, \ k = 1, 2 \). This is equivalent to require that, for each \( t > 0 \),

\[
T_1 x = \lim_{n \to \infty} V_n^{-1} T_2 V_n x, \quad x \in E.
\]

In this situation one has

\[
\| T_2(t) - T_2(s) \|_{L^1(X)} \geq \| T_1(t) - T_1(s) \|_{L^1(X)}, \quad t, s \geq 0,
\]

and

\[
\| R(\lambda, B_2) \|_{L^1(X)} \geq \| R(\lambda, B_1) \|_{L^1(X)}, \quad \lambda \in \mathcal{g}_\infty(B_1) \cap \mathcal{g}_\infty(B_2),
\]

where \( \mathcal{g}_\infty(B_k) \) denotes the connected component of the resolvent set \( \mathcal{g}(B_k) \) containing \( +\infty, k = 1, 2 \). This technique is applied in the situation where \( B_2 \) is a suitable realization of the generator of \( P \) and \( B_1 \) is a realization of the generator of the drift semigroup \( R \) associated with \( A \). This semigroup is defined on \( C_b(E) \) by

\[
R(t)f(x) := f(S(t)x), \quad x \in E, \ f \in C_b(E).
\]

Throughout this paper, a Gaussian measure is a centred Gaussian Radon measure.

2. The Ornstein-Uhlenbeck semigroup in spaces of continuous functions

In this section we study various properties of the Ornstein-Uhlenbeck semigroup \( P \) and the drift semigroup \( R \) in the spaces \( C_0(E) \) and \( BUC(E) \). We denote by \( \| \cdot \| \) the supremum norm.

As semigroups on \( C_0(E) \), both \( P \) and \( R \) are strongly continuous with respect to the mixed topology. This topology is defined as the finest locally convex topology in \( C_0(E) \) which agrees on every norm bounded set with the topology of uniform convergence on compact sets; see [28, 29] for a detailed investigation of its properties. This topology is complete and may be used to define the infinitesimal generators \( L_P \) and \( L_R \) of \( P \) and \( R \) by taking, for \( T = P \) or \( R \),

\[
\mathcal{D}(L_T) := \left\{ f \in C_0(E) : \lim_{t \downarrow 0} \frac{1}{t} (T(t)f - f) \text{ exists} \right\},
\]

\[
L_T f := \lim_{t \downarrow 0} \frac{1}{t} (T(t)f - f), \quad f \in \mathcal{D}(L_T),
\]

where the limits are taken with respect to the mixed topology. We have \( f \in \mathcal{D}(L_T) \) if and only if the following two conditions hold:
(i) \( \lim \sup_{t \downarrow 0} \frac{1}{t} \|T(t)f - f\| < \infty \);

(ii) there exists a function \( g \in C_b(E) \) such that for all \( x \in E \),
\[
\lim_{t \downarrow 0} \frac{1}{t} (T(t)f(x) - f(x)) = g(x).
\]

In this situation, \( L_T f = g \).

On a suitable core of smooth cylindrical functions, \( L_P \) and \( L_R \) are given by
\[
L_P f(x) = \frac{1}{2} \text{Tr} Q D^2 f(x) + \langle Ax, Df(x) \rangle,
\]
\[
L_R f(x) = \langle Ax, Df(x) \rangle,
\]
where \( \text{Tr} \) denotes the trace and \( Q := BB^* \). We refer to [14, 15] for proofs and more details. Alternative approaches to diffusion semigroups in spaces of continuous functions may be found in [4, 16, 26].

Both \( P \) and \( R \) leave the closed subspace \( BUC(E) \) of \( C_b(E) \) invariant, but even on this smaller space both semigroups fail to be strongly continuous with respect to the supremum norm, unless \( A = 0 \). It is easy to see, cf. [8, Lemma 3.2], that the closed subspaces of \( BUC(E) \) on which \( P \) and \( R \) act in a strongly continuous way with respect to the supremum norm coincide. This common subspace will be denoted by \( BUC^\circ(E) \). Thus,
\[
BUC^\circ(E) = \{ f \in BUC(E) : \lim_{t \downarrow 0} \|P(t)f - f\| = 0 \} = \{ f \in BUC(E) : \lim_{t \downarrow 0} \|R(t)f - f\| = 0 \}.
\]

The restrictions of \( P \) and \( R \) to \( BUC^\circ(E) \), denoted by \( P^\circ \) and \( R^\circ \) respectively, are strongly continuous with respect to the supremum norm. Their generators \( L_{P^\circ} \) and \( L_{R^\circ} \) are characterized as follows; see [8, Proposition 3.5] for a related result.

**Proposition 2.1.** We have
\[
\mathcal{D}(L_{P^\circ}) = \{ f \in \mathcal{D}(L_P) \cap BUC^\circ(E) : L_P f \in BUC^\circ(E) \},
\]
\[
\mathcal{D}(L_{R^\circ}) = \{ f \in \mathcal{D}(L_R) \cap BUC^\circ(E) : L_R f \in BUC^\circ(E) \}.
\]

**Proof.** Let \( T = P \) or \( R \).

The inclusion ‘\( \subseteq \)’ is clear. To prove the inclusion ‘\( \supseteq \)’ let \( f \in \mathcal{D}(L_T) \cap BUC^\circ(E) \) be such that \( L_T f \in BUC^\circ(E) \). Then,
\[
\lim_{t \downarrow 0} \sup_{x \in E} \frac{1}{t} |(T(t)f(x) - f(x)) - L_T f(x)| = \lim_{t \downarrow 0} \sup_{x \in E} \left| \frac{1}{t} \int_0^t T(s) L_T f(x) \, ds - L_T f(x) \right| = \lim_{t \downarrow 0} \frac{1}{t} \int_0^t T^\circ(s) L_T f(x) \, ds = 0,
\]
where the first identity is a consequence of the fact that \( T \) is strongly continuous with respect to the mixed topology. This proves that \( f \in \mathcal{D}(L_{T^\circ}) \). \( \square \)

We do not know whether \( \mathcal{D}(L_{P^\circ}) \) is always contained in \( \mathcal{D}(L_{R^\circ}) \).

The following simple observation, cf. the proof of [25, Lemma 2.3], will be useful.

**Lemma 2.2.** Let \( T = P \) or \( R \). For \( f \in BUC(E) \) and \( \delta > 0 \) define
\[
f_\delta(x) := \frac{1}{\delta} \int_0^\delta T(t)f(x) \, dt, \quad x \in E.
\]
Then \( f_\delta \in BUC^\circ(E) \). Moreover, \( \lim_{\delta \downarrow 0} f_\delta = f \) in the mixed topology inherited from \( C_b(E) \).
Proof. First note that \( t \mapsto T(t)f(x) \) is continuous for all \( x \in E \), and therefore the function \( f_\delta \) is well defined. It is clear that \( f_\delta \in BUC(E) \) and \( \|f_\delta\| \leq 1 \). For all \( x \in E \) and \( t \in (0, \delta) \) we have

\[
|T(t)f_\delta(x) - f_\delta(x)| = \frac{1}{\delta} \left| \int_0^\delta T(s)f(x) \, ds - \int_0^\delta T(s)f(x) \, ds \right| \leq \frac{2t}{\delta} \|f\|.
\]

Thus \( \|T(t)f_\delta - f_\delta\| \leq 2\delta^{-1}t\|f\| \), which shows that \( f_\delta \in BUC^\circ(E) \). The final statement is obvious.

Obviously, if \( S(t) = S(s) \) for certain \( t, s \geq 0 \), then \( R(t) = R(s) \). The following lemma describes what happens if \( S(t) \neq S(s) \).

Lemma 2.3. For all \( t, s \geq 0 \) such that \( S(t) \neq S(s) \) we have \( \|R^t(t) - R^s(s)\|_{\mathcal{L}(BUC^\circ(E))} = 2 \).

Proof. Fix \( t, s \geq 0 \) such that \( S(t) \neq S(s) \). We may assume that \( t > s \geq 0 \). Choose \( x_0^* \in E^* \) such that \( S^*(t)x_0^* \neq S^*(s)x_0^* \). Noting that \( S^*(s)x_0^* \neq 0 \) we pick \( x_0 \in E \) such that \( \langle x_0, S^*(t)x_0^* \rangle = 0 \) and \( \langle x_0, S^*(s)x_0^* \rangle = 0 \). The function \( f(x) := \cos \langle x, x_0^* \rangle \) defines an element of \( BUC(E) \) and we have

\[
\|R(t)f - R(s)f\| \geq |R(t)f(x_0) - R(s)f(x_0)| = 2.
\]

Given \( \varepsilon > 0 \) we choose \( \delta > 0 \) small enough such that

\[
|\langle R(t)f_\delta(x_0) - R(s)f_\delta(x_0), f_\delta(x_0) \rangle| = \|R(t)f_\delta(x_0) - R(s)f_\delta(x_0)\| \geq 2 - \varepsilon,
\]

where \( f_\delta \) is defined as in the previous lemma. Since \( f_\delta \in BUC^\circ(E) \), \( \|f_\delta\| \leq 1 \), and \( \|R^t(t)\| \leq 1 \), \( \|R^s(s)\| \leq 1 \), the lemma follows.

In combination with the technique described in Introduction we obtain a similar result for the Ornstein-Uhlenbeck semigroup:

Proposition 2.4. For all \( t, s \geq 0 \) such that \( S(t) \neq S(s) \) we have \( \|P^t(t) - P^s(s)\|_{\mathcal{L}(BUC^\circ(E))} = 2 \).

Proof. Define the invertible isometries \( V_n : BUC(E) \to BUC(E) \) by

\[
V_n f(x) = f(n^{-1}x), \quad x \in E, \quad f \in BUC(E).
\]

We will show that \( L_{P^\circ} \) belongs to the limit class of \( L_{P^\circ} \). To this end, for any \( f \in BUC(E) \) and \( x \in E \), one has

\[
|V_n^{-1}P(t)V_n f(x) - R(t)f(x)| \leq \int_E |f(S(t)x + n^{-1}y) - f(S(t)x)| \, d\mu_t(y) \leq \int_E \omega_f(n^{-1}y) \, d\mu_t(y),
\]

where \( \omega_f \) denotes the modulus of continuity of \( f \). Letting \( n \to \infty \), the last term tends to 0 by the dominated convergence theorem. Hence, for any \( f \in BUC(E) \),

\[
\lim_{n \to \infty} \|V_n^{-1}P(t)V_n f - R(t)f\| = 0.
\]

The result now follows from Lemma 2.3.

Corollary 2.5. If \( A \neq 0 \), then there exists \( t_0 > 0 \) such that \( \|P^t(t) - P^s(s)\|_{\mathcal{L}(BUC^\circ(E))} = 2 \) for all \( 0 \leq t, s \leq t_0, t \neq s \).

Proof. If such \( t_0 \) does not exist, there exist sequences \( s_n \downarrow 0 \) and \( t_n \downarrow 0 \) with \( s_n \leq t_n \) such that \( \|P^t(t_n) - P^s(s_n)\|_{\mathcal{L}(BUC^\circ(E))} < 2 \) for all \( n \). By Proposition 2.4, \( S(s_n) = S(t_n) \) for all \( n \). Fixing an element \( x \in D(A) \), for all \( n \) we obtain

\[
\int_{s_n}^{t_n} S(r)Ax \, dr = S(t_n)x - S(s_n)x = 0.
\]

Upon dividing both sides by \( t_n - s_n \) and passing to the limit \( n \to \infty \) we obtain \( Ax = 0 \). This being true for all \( x \in D(A) \) we conclude that \( A = 0 \).

\[\square\]
By a result of [25] the same conclusion holds for $A = 0$ if the range of $Q$ is infinite-dimensional; see also [11] where the special case of a Hilbert space $E$ was considered.

We proceed with a different sufficient condition for norm discontinuity which, for the case of a Hilbert spaces $E$, is implicitly contained in [25]. Two probability measures $\mu$ and $\nu$ are called disjoint, notation $\mu \perp \nu$, if there exist disjoint measurable sets $A$ and $B$ such that $\mu(A) = \nu(B) = 1$. The measures $\mu$ and $\nu$ are called equivalent, notation $\mu \sim \nu$, if they are mutually absolutely continuous, i.e., $\mu \ll \nu$ and $\nu \ll \mu$.

**Proposition 2.6.** For all $t, s \geq 0$ such that $\mu_t \perp \mu_s$ we have $\|P^\circ(t) - P^\circ(s)\|_{\mathcal{L}(\text{BUC}^\circ(E))} = 2$.

Proof. By assumption we have $\|\mu_t - \mu_s\|_{\text{var}} = 2$, where $\|\cdot\|_{\text{var}}$ denotes the total variation norm of a finite signed Radon measure. Identifying $\mu$ and $\mu_s$ with elements from the dual of $\text{BUC}^\circ(E)$, it will be enough to show that $\|\mu_t - \mu_s\|_{(\text{BUC}^\circ(E))'} = 2$. Indeed, once we know this, given $\varepsilon > 0$ we choose $g \in \text{BUC}^\circ(E)$ with $\|g\| = 1$ such that $|\langle g, \mu_t - \mu_s \rangle| \geq 2 - \varepsilon$ and observe that

$$
\|P^\circ(t) - P^\circ(s)\|_{\mathcal{L}(\text{BUC}^\circ(E))} \geq |\langle P^\circ(t)g(0) - P^\circ(s)g(0) \rangle| = |\langle g, \mu_t - \mu_s \rangle| \geq 2 - \varepsilon.
$$

Suppose $\nu$ is a finite signed Radon measure on $E$. Generalizing [25, Lemma 2.3], the proof will be finished by showing that

$$
\|\nu\|_{(\text{BUC}^\circ(E))'} = \|\nu\|_{\text{var}}.
$$

The inequality ‘$\leq$’ is clear. To check the inequality ‘$\geq$’, by the Jordan-Hahn decomposition it is enough to prove the assertion when $\nu$ is a Radon probability measure on $E$. By [1, Section 1.1], for any given $\varepsilon > 0$ there exists $f \in \text{BUC}(E)$ with $\|f\| \leq 1$ such that $\langle f, \nu \rangle \geq 1 - \varepsilon$. For $\delta > 0$ define $f_\delta \in \text{BUC}^\circ(E)$ as in Lemma 2.2. By inner regularity of $\nu$, the supremum of $\nu(K)$ with $K$ ranging over all compact subsets of $E$ equals 1. Hence to prove (2.1) it is enough to observe that by Lemma 2.2 we have $\lim_{\delta \downarrow 0} f_\delta = f$ uniformly on compact sets. $\square$

In the converse direction we have the following result.

**Proposition 2.7.** If $t, s \geq 0$ are such that $S(t) = S(s)$ and $\|P^\circ(t) - P^\circ(s)\|_{\mathcal{L}(\text{BUC}^\circ(E))} = 2$, then $\mu_t \perp \mu_s$.

Proof. Given $\varepsilon > 0$, there exist $f \in \text{BUC}^\circ(E)$ and $x \in E$ such that $|P^\circ(t)f(x) - P^\circ(s)f(x)| \geq 2 - \varepsilon$. Defining $g \in \text{BUC}^\circ(E)$ by $g(y) = f(S(t)x + y)$, this may be restated as

$$
\left| \int_E g(y) d\mu_t(y) - \int_E g(y) d\mu_s(y) \right| = \left| \int_E f(S(t)x + y) d\mu_t(y) - \int_E f(S(s)x + y) d\mu_s(y) \right| \geq 2 - \varepsilon.
$$

This shows that $\|\mu_t - \mu_s\|_{(\text{BUC}^\circ(E))'} \geq 2 - \varepsilon$. Since the choice of $\varepsilon > 0$ is arbitrary we obtain that

$$
2 \leq \|\mu_t - \mu_s\|_{(\text{BUC}^\circ(E))'} \leq \|\mu_t - \mu_s\|_{\text{var}} \leq 2,
$$

the second and third of these inequalities being obvious. Hence $\|\mu_t - \mu_s\|_{\text{var}} = 2$, which implies that $\mu_t \perp \mu_s$. $\square$

By putting these results together we have proved:

**Theorem 2.8.** For all $t, s \geq 0$ the following assertions are equivalent:

1. $\|P^\circ(t) - P^\circ(s)\|_{\mathcal{L}(\text{BUC}^\circ(E))} = 2$;
2. $S(t) \neq S(s)$ or $\mu_t \perp \mu_s$. 


It should be observed that neither $S(t) \neq S(s)$ implies $\mu_t \perp \mu_s$ or conversely. An example of a periodic semigroup with period 1 such that $\mu_t \perp \mu_s$ for all $t, s \geq 1$ is given in [21, Example 3.8]. On the other hand, if $\dim E < \infty$, then for any choice of $S$ and $B$ the measures $\mu_t$ and $\mu_s$ are mutually absolutely continuous for all $t, s \geq t_0$.

We continue with two examples which show that $\|P(t) - P(s)\|_{\mathcal{L}(\mathcal{BUC}(E))} < 2$ may occur for certain values of $t \neq s$.

Example 2.9 (Nilpotent $S$). Let $E = L^2(0, 1)$ and let $S$ be the nilpotent shift semigroup on $L^2(0, 1)$, see for instance [12, page 120]. Then $S(t) = S(s) = 0$ and $\mu_t = \mu_s = \mu_1$ for all $t, s \geq 1$. Hence, $P(t) = P(s)$ for all $t \geq s \geq 1$.

Example 2.10 (Periodic $S$ in finite dimensions). Let $H = E = \mathbb{R}^2$ and let $S$ be the rotation group on $\mathbb{R}^2$. Let $B := I$, the identity operator on $\mathbb{R}^2$. Since $S(t) = S(-t)$ for all $t \geq 0$, the covariance operator of $\mu_t$ is given by $Q_t = tI$. Hence $\mu_t$ is the Gaussian measure on $\mathbb{R}^2$ with variance $t$. For $k = 0, 1, 2, \ldots$ and $f \in \mathcal{BUC}(\mathbb{R}^2)$,

$$P^0(2k\pi)f(x) = \int_E f(S(2k\pi)x + y) d\mu_{2k\pi}(y) = \int_E f(x + y) d\mu_{2k\pi}(y).$$

For $j \geq 1$, $k \geq 1$, $j \neq k$, we have $S(2j\pi) = S(2k\pi)$ and $\mu_{2j\pi} \sim \mu_{2k\pi}$. Theorem 2.8 shows that $\|P^0(2j\pi) - P^0(2k\pi)\| < 2$.

We will show next that the above two examples are in some sense the only possible ones.

Recall that a Gaussian measure $\nu$ on $E$ is called nondegenerate if there exists no proper closed subspace $E_0$ of $E$ with $\nu(E_0) = 1$. It is easy to see that $\nu$ is nondegenerate if and only if its covariance operator has dense range.

For $t > 0$ fixed, $P$ is said to be strongly Feller at time $t$ if $P(t)f \in C_b(E)$ for all $f \in B_b(E)$. Here $B_b(E)$ denotes the space of real-valued bounded Borel functions on $E$. As is well known, $P$ is strongly Feller at time $t$ if and only if we have $S(t)E \subseteq H_{Q_t}$, where $H_{Q_t}$ is the reproducing kernel Hilbert space associated with $Q_t$; cf. [9, 21].

Theorem 2.11. Suppose $t > s \geq 0$ are such that $\|P^0(t) - P^0(s)\|_{\mathcal{L}(\mathcal{BUC}(E))} < 2$. Assume in addition that one of the following two assumptions is satisfied:

(i) $\mu_{t-s}$ is nondegenerate;
(ii) $P$ is strongly Feller at time $t - s$.

Then there exists a direct sum decomposition into $S$-invariant subspaces $E = E_0 \oplus E_1$, with $\dim E_1 < \infty$, such that $S$ is nilpotent on $E_0$ and periodic on $E_1$ with period $t - s$.

Proof. By Theorem 2.8, the assumption of the theorem implies that $S(t) = S(s)$ and $\mu_t \perp \mu_s$. By the Feldman-Hajek theorem [2, Theorem 2.7.2], $\mu_t \sim \mu_s$.

Let $H_Q$ be the reproducing kernel Hilbert space associated with $Q = BB^*$ and let $E_s$ denote the closure of the range of $S(s)$. Define $j : H_Q \to E_s$ by $j := S(s)B$ and $R \in \mathcal{L}(E_s^*, E_s)$ by $R := jj^* = S_s(s)Q_s^2(s)$, where $S_s(s)$ is the operator $S(s)$ regarded as an operator from $E$ to $E_s$. For $\tau > 0$ introduce the operators $R_\tau \in \mathcal{L}(E_s^*, E_s)$ by

$$R_\tau y^* := \int_0^\tau S(u)R S^*(u)y^* du, \quad y^* \in E_s^*,$$

where by abuse of notation we think of $S$ as a semigroup on $E_s$. Then $R_\tau$ is the covariance operator of the image measure $\nu_\tau = S_s(s)\mu_\tau$ on $E_s$, i.e., $R_\tau = S_s(s)Q_s S_s^*(s)$. Moreover,

$$\nu_s = S_s(s)\mu_s \sim S_s(s)\mu_t = \nu_t.$$

Clearly, (2.2)

$$S(t - s)|_{E_s} = I_{E_s}.$$  

By (2.2) and [21, Corollary 3.3], for $k \in \mathbb{N}$ such that $\bar{k}(t - s) \geq s$ we obtain

$$\nu_{\bar{k}(t - s)} = \nu_{s + \bar{k}(t - s) - s} \sim \nu_{t + \bar{k}(t - s) - s} = \nu_{(\bar{k} + 1)(t - s)}.$$
But by (2.3) we have $R_{(k+1)(t-s)} = (k+1)R_{t-s}$, and therefore the Feldman-Hajek theorem implies that the reproducing kernel Hilbert space $H_{R_{t-s}}$ associated with $R_{t-s}$ is finite-dimensional; cf. [2, Example 2.7.4].

We will show below that each of the conditions (i) and (ii) implies that the measure $\nu_{t-s}$ is nondegenerate. Once we know this, the proof can be finished as follows. Since $\nu_{t-s}$ is nondegenerate, the reproducing kernel Hilbert space $H_{R_{t-s}}$ is dense in $E_s$. It follows that $H_{R_{t-s}} = E_s$, which means that $E_s$ is finite-dimensional. Hence $E_s$ equals the range of $S(s) = S(t)$. By the semigroup property, $E_s$ equals also the range of $S(k(t-s))$, where the integer $k \geq 1$ is such that $s \leq k(t-s) < t$. In combination with (2.3) it follows that $S(k(t-s))$ is a projection in $E$. This proves the theorem, with $E_0 := \text{ker} S(k(t-s))$ and $E_1 := E_s = \text{range} S(k(t-s))$.

To finish the proof we show that both (i) and (ii) imply the nondegeneracy of the measure $\nu_{t-s}$.

First assume (i). It is immediate from the definition that the image of a nondegenerate Gaussian measure under a bounded operator with dense range is nondegenerate. Thus the nondegeneracy assumption on $\mu_{t-s}$ implies that $\nu_{t-s}$ is nondegenerate.

Next assume (ii). Write $H_{t-s} := H_{Q_{t-s}}$ for brevity and let $i_{t-s} : H_{t-s} \hookrightarrow E$ be the inclusion mapping. Recalling that $Q_{t-s} = i_{t-s} \circ i_{t-s}^*$, for all $u^* \in E_s^*$ and $x^* \in E^*$ such that $x^*|_{E_s} = u^*$ we have

$$\langle R_{t-s}u^*, u^* \rangle = \langle S_s(s)Q_{t-s}S_s^*(s)u^*, u^* \rangle = \langle S(s)Q_{t-s}S(s)x^*, x^* \rangle = \|i_{t-s}^*S^*(s)x^*\|^2_{H_{t-s}}.$$  

By the strong Feller property and a closed graph argument, $S(t-s)$ is bounded as an operator from $E$ to $H_{t-s}$. Denoting this operator by $T(t-s)$ we have $S(t-s) = i_{t-s} \circ T(t-s)$. Let $I : E_s \to E$ be the inclusion mapping. On $E_s$ we have $S(t) \circ I = I \circ S(t)$, where as before we abuse of notation by writing $S$ for the restriction of $S$ to $E_s$. Then, for all $x^* \in E^*$,

$$\|S^*(t)I^*x^*\| = \|I^*S^*(t-s)S^*(s)x^*\|$$

$$= \|I^*T^*(t-s)i_{t-s}^*S^*(s)x^*\| \leq \|T(t-s)I\|_{L^2(E_s, H_{t-s})} \|i_{t-s}^*S^*(s)x^*\|_{H_{t-s}}.$$  

Combining these things we obtain

$$\|T(t-s)I\|_{L^2(E_s, H_{t-s})} \|R_{t-s}I^*x^*, I^*x^*\| \geq \|S^*(t)I^*x^*\|^2 \geq c^2_t \|I^*x^*\|^2,$$

where the last estimate follows from the fact that $S$ is periodic on $E_s$. Since $I^*$ is a surjection from $E^*$ onto $E_s$, this gives that either $R_{t-s}$ is nondegenerate or $T(t-s)I = 0$. In the first case the proof is complete. If $T(t-s)I = 0$, then $S(t-s)I = 0$ as well, which means that $S(t-s) = 0$ on $E_s$. By periodicity this implies that $E_s = \{0\}$. This in turn implies that $S(s) = 0$, i.e., $S$ is nilpotent on $E$.

The nondegeneracy assumption on $\mu_{t-s}$ in (i) is fulfilled if $Q$ has dense range; this is proved in the same way as [13, Lemma 5.2].

**Corollary 2.12.** Let $\text{dim} E = \infty$, and assume that $S$ is analytic and condition (i) or (ii) is satisfied. Then for all $t > s \geq 0$ we have $\|P^o(t) - P^o(s)\|_{BUC(E)} = 2$.

**Proof.** An analytic $C_0$-semigroup on a nonzero Banach space cannot be nilpotent. Hence, Theorem 2.11 shows that if there exist $t > s \geq 0$ such that $\|P^o(t) - P^o(s)\|_{BUC(E)} < 2$, then $\text{dim} E < \infty$. □

Next we consider the case $A = 0$. In this situation one has $Q_t = tQ = tBB^*$, and since by our standing assumption these operators are Gaussian covariances, it follows that $Q$ is a Gaussian covariance. We denote the Gaussian measure on $E$ with covariance operator $Q$ by $\nu$. The semigroup $P$ is then the heat semigroup given by $P(t)f(x) = \int_E f(x + y) d\mu_t(y) = \int_E f(x + \sqrt{t}y) d\nu(y), \quad f \in \mathcal{C}_b(E)$.

The restriction of $P$ to $BUC(E)$ is strongly continuous with respect to the supremum norm. The infinitesimal generator $L_P$ of $P$ is given, on a suitable core of cylindrical functions, by $L_P f(x) = \frac{1}{2} \text{Tr} QD^2 f(x)$. 

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The following result was proved in [20] for the special case of an infinite-dimensional Hilbert space $E$. Our proof is essentially the same, the main difference being that the coordinate-free presentation simplifies matters somewhat. The spectrum and approximate point spectrum of $L_P$ in $BUC(E)$ are denoted by $\sigma(L_P)$ and $\sigma_n(L_P)$, respectively.

**Proposition 2.13.** If $A = 0$ and $Q$ is not of finite rank, then $\sigma(L_P) = \sigma_n(L_P) = \mathbb{C}^\cdot$

**Proof.** Fix a sequence $(x_n^\ast)$ in $E^*$ such that $(B^*x_n^\ast)$ is an orthonormal sequence in $H$. Such a sequence exists since the range of $B^*$ is not finite-dimensional in $H$. For each $n \geq 1$ we consider the map $T_n : E \rightarrow \mathbb{R}^n$ defined by $T_n x := ((x, x_1^\ast), \ldots, (x, x_n^\ast))$.

The image measure of $\nu$ under $T_n$ equals $\gamma_n$, the standard Gaussian measure on $\mathbb{R}^n$.

Let $\Delta_n$ be the Laplace operator acting in $BUC(\mathbb{R}^n)$. Denoting the heat semigroup on $BUC(\mathbb{R}^n)$ generated by $\frac{1}{2}\Delta_n$ by $\{P_n(t)\}_{t \geq 0}$, for all $f \in BUC(\mathbb{R}^n)$ and $x \in E$ we have

$$P(t)f(T_n x) = \int_E f(T_n(x + \sqrt{t} \eta)) d\nu(\eta) = \int_{\mathbb{R}^n} f(T_n(x + \sqrt{t} \eta)) d\gamma_n(\eta) = P_n(t)f(T_n(x)).$$

From this it is immediate that $f \circ T_n \in D(L_P)$ whenever $f \in D(\Delta_n)$ and in this case,

$$L_P(f \circ T_n) = (\frac{1}{2}\Delta_n f) \circ T_n.$$

Fix $\lambda \in \mathbb{C}$ with $\text{Re}\lambda < 0$ and consider the functions $f_{n,\lambda}, g_{n,\lambda} \in BUC(\mathbb{R}^n)$ defined by

$$f_{n,\lambda}(\xi) = \exp\left(\frac{\lambda}{n}||\xi||^2\right) \quad \text{and} \quad g_{n,\lambda}(\xi) = \frac{-2\lambda||\xi||^2}{n^2}f_{n,\lambda}(\xi), \quad \xi \in \mathbb{R}^n.$$ We have $f_{n,\lambda} \in D(\Delta_n)$ and

$$(\lambda - \frac{1}{2}\Delta_n)f_{n,\lambda} = g_{n,\lambda}.$$ Hence $f_{n,\lambda} \circ T_n \in D(L_P)$ and

$$(\lambda - L_P)(f_{n,\lambda} \circ T_n) = g_{n,\lambda} \circ T_n.$$ Moreover,

$$\|f_{n,\lambda} \circ T_n\|_{BUC(E)} = \|f_{n,\lambda}\|_{BUC(\mathbb{R}^n)} = 1$$ and we compute

$$\|g_{n,\lambda} \circ T_n\|_{BUC(E)} = \|g_{n,\lambda}\|_{BUC(\mathbb{R}^n)} = \frac{2||\lambda||^2}{ne|\text{Re}\lambda|}.$$ This proves that the sequence $(f_{n,\lambda} \circ T_n)$ is an approximate eigenvector for $L_P$, with approximate eigenvalue $\lambda$. It follows that $\{\{\{Re\lambda < 0\}\} \subseteq \sigma_n(L_P)\}$. On the other hand, since $\{P(t)\}_{t \geq 0}$ is a contraction semigroup on $BUC(E)$, we have $\{\text{Re}\lambda > 0\} \subseteq \varrho(L_P)$, where $\varrho(L_P)$ denotes the resolvent set of $L_P$. Combining this, we see that $\sigma(L_P) = \mathbb{C}^\cdot$. Moreover, $i\mathbb{R} = \partial \sigma(L_P) \subseteq \sigma_n(L_P)$ by the general theory of semigroups, and therefore $\sigma(L_P) = \sigma_n(L_P) = \mathbb{C}^\cdot$. □

If $A = 0$ and $E = \mathbb{R}^d$, then $P$ is analytic and therefore $\sigma(L_P)$ is contained in a strict subsector in $\mathbb{C}^\cdot$. For $A \neq 0$, $Q$ invertible, and $E = \mathbb{R}^d$, it was shown in [17] that $\sigma(L_P) = \mathbb{C}^\cdot$ if $\sigma(A) \subseteq \{\lambda \in \mathbb{C} : \text{Re}\lambda < 0\}$ and that $\sigma(L_P) \supseteq \mathbb{C}^\cdot$ if $\sigma(A) \subseteq \{\lambda \in \mathbb{C} : \text{Re}\lambda > 0\}$, and that in both cases every $\lambda \in \mathbb{C}$ with $\text{Re}\lambda < 0$ is an eigenvalue. We have the following extension of this result to infinite dimensions:

**Theorem 2.14.** Assume that the operator $Q$ has dense range. Assume also that $S$ is eventually compact and that $\sigma(A)$ is nonempty.

1. If $\sigma(A) \subseteq \{\lambda \in \mathbb{C} : \text{Re}\lambda < 0\}$, then $\sigma(L_P) = \mathbb{C}^\cdot$ and every $\lambda \in \mathbb{C}$ with $\text{Re}\lambda < 0$ is an eigenvalue.

2. If $\sigma(A) \subseteq \{\lambda \in \mathbb{C} : \text{Re}\lambda > 0\}$, then $\sigma(L_P) \supseteq \mathbb{C}^\cdot$ and every $\lambda \in \mathbb{C}$ with $\text{Re}\lambda < 0$ is an eigenvalue.

The proof is based on the same Riesz projection argument as Theorem 3.7 below and is left to the reader.
3. The Ornstein-Uhlenbeck semigroup in spaces of integrable functions

Our approach to proving norm discontinuity of Ornstein-Uhlenbeck semigroups in $L^1$-spaces is based on the following observation.

Lemma 3.1. For all $t > s \geq 0$ with $S(t-s) \neq I$ there exist $x_0 \in E$ and $r > 0$ such that
\[ \{ x \in E : \| S(t)x - x_0 \| < r \} \cap \{ x \in E : \| S(s)x - x_0 \| < r \} = \emptyset. \]

Proof. Choose $x_0 \in E$ such that $S(t-s)x \neq x_0$. Let $M := \max\{ 1, \| S(t-s) \|_{\mathscr{L}(E)} \}$ and put
\[ r := \frac{1}{2M} \| S(t-s)x_0 - x_0 \|. \]
Suppose $x \in E$ is such that $\| S(s)x - x_0 \| < r$. We will prove that $\| S(t)x - x_0 \| \geq r$. By assumption there exists a vector $x_1 \in E$ with $\| x_1 \| < r$ such that $S(s)x = x_0 + x_1$. Then,
\[ \| S(t)x - x_0 \| = \| S(t-s)(x_0 + x_1) - x_0 \| \geq \| S(t-s)x_0 - x_0 \| - \| S(t-s)x_1 \| \geq 2Mr - Mr = Mr \geq r. \]

Until further notice we now specialize to the case where $E = \mathbb{R}^d$ and assume that $A$ is an $(d \times d)$-matrix with real coefficients. We write $S(t) = e^{tA}$. As before, $R$ indicates the drift semigroup given by (1.7). Let $C_c(\mathbb{R}^d)$ denote the space of continuous compactly supported functions on $\mathbb{R}^d$.

For all $f \in C_c(\mathbb{R}^d)$ we have
\[ \int_{\mathbb{R}^d} |R(t)f(x)| \, dx = \frac{1}{|\det(S(t))|} \int_{\mathbb{R}^d} |f(S(t)x)| \, dx = e^{-t \text{Tr}A} \int_{\mathbb{R}^d} |f(y)| \, dy. \]
It follows that the restrictions of $R(t)$ to $C_c(\mathbb{R}^d)$ extend to bounded operators on $L^1(\mathbb{R}^d)$ of norm $\| R(t) \|_{L^1(\mathbb{R}^d)} = e^{-t \text{Tr}A}$. Since also $\lim_{t \downarrow 0} \| R(t)f - f \|_{L^1(\mathbb{R}^d)} = 0$ for all $f \in C_c(\mathbb{R}^d)$ it follows that $R$ has a unique extension to a $C_0$-semigroup on $L^1(\mathbb{R}^d)$. The space $C_c(\mathbb{R}^d)$ is a core for its generator $L_R$ and we have
\[ L_R f(x) = \langle Ax, Df(x) \rangle, \quad x \in \mathbb{R}^d, \quad f \in C_c(\mathbb{R}^d). \]

Proposition 3.2. For all $t > s \geq 0$ with $S(t) \neq S(s)$ we have $\| e^{t \text{Tr}A} R(t) - e^{s \text{Tr}A} R(s) \|_{\mathscr{L}(L^1(\mathbb{R}^d))} = 2$.

Proof. Let $x_0 \in \mathbb{R}^d$ and $r > 0$ be as in Lemma 3.1. By Lemma 3.1,
\[ \| e^{t \text{Tr}A} R(t)1_{\{ \| x-x_0 \| < r \}} - e^{s \text{Tr}A} R(s)1_{\{ \| x-x_0 \| < r \}} \|
\leq\| e^{t \text{Tr}A}1_{\{ \| S(t)x-x_0 \| < r \}} - e^{s \text{Tr}A}1_{\| S(s)x-x_0 \| < r \}} \|
=\| e^{t \text{Tr}A}1_{\{ \| S(t)x-x_0 \| < r \}} - e^{s \text{Tr}A}1_{\| S(s)x-x_0 \| < r \}} \|
=\| e^{t \text{Tr}A}1_{\| R(t)1_{\{ \| x-x_0 \| < r \}} \| + e^{s \text{Tr}A}1_{\| R(s)1_{\| x-x_0 \| < r \}} \| = 2\| 1_{\{ \| x-x_0 \| < r \}} \|, \]
where in the last step we used (3.1). It follows that $\| e^{t \text{Tr}A} R(t) - e^{s \text{Tr}A} R(s) \| \geq 2$. Since by (3.1) we also have $e^{-\text{Tr}A} \| R(\tau) \| \leq 1$ for all $\tau \geq 0$, the proposition is proved.

Our next aim is to extend the Ornstein-Uhlenbeck semigroup $P$ to $L^1(\mathbb{R}^d)$ as well. For all $f \in C_c(\mathbb{R}^d)$ we have
\[ \int_{\mathbb{R}^d} |P(t)f(x)| \, dx \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(S(t)x + y)| \, dx \, d\mu(y) 
= e^{-t \text{Tr}A} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(\xi)| \, d\mu(y) = e^{-t \text{Tr}A} \int_{\mathbb{R}^d} |f(\xi)| \, d\xi \]
with equality for nonnegative functions $f$. It follows that the restrictions of the operators $P(t)$ to $C_c(\mathbb{R}^d)$ extend to bounded operators on $L^1(\mathbb{R}^d)$ of norm $\| P(t) \|_{\mathscr{L}(L^1(\mathbb{R}^d))} = e^{-t \text{Tr}A}$. Since also $\lim_{t \downarrow 0} \| P(t) - f \|_{L^1(\mathbb{R}^d)} = 0$ for all $f \in C_c(\mathbb{R}^d)$ it follows that the restriction of $P$ to $C_c(\mathbb{R}^d)$ has
a unique extension to a $C_0$-semigroup on $L^1(\mathbb{R}^d)$, which is still given by formula (1.2). The space $C^2_c(\mathbb{R}^d)$ is a core for its generator $L_P$ and we have

$$Lf(x) = \frac{1}{2} \text{Tr } QD^2f(x) + \langle Ax, Df(x) \rangle, \quad x \in \mathbb{R}^d, \ f \in C^2_c(\mathbb{R}^d).$$

**Theorem 3.3.** For all $t > s \geq 0$ with $S(t) \neq S(s)$ we have $\|e^{-t\text{Tr } A}P(t) - e^{-s\text{Tr } A}P(s)\|_{\mathcal{L}(L^1(\mathbb{R}^d))} = 2$.

**Proof.** For $n = 1, 2, \ldots$ let $V_n : L^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ denote the invertible isometry

$$V_n f(x) = n^{-d}f(n^{-1}x), \quad x \in \mathbb{R}^d, \ f \in L^1(\mathbb{R}^d).$$

As in the proof of Proposition 2.4 we see that $L_R$ belongs to the limit class of $L_P$. Hence by Proposition 3.2 and [10, Proposition 12], applied to the operators $L_P - \text{Tr } A$ and $L_R - \text{Tr } A$,

$$\|e^{-t\text{Tr } A}P(t) - e^{-s\text{Tr } A}P(s)\|_{\mathcal{L}(L^1(\mathbb{R}^d))} \geq \|e^{-t\text{Tr } A}R(t) - e^{-s\text{Tr } A}R(s)\|_{\mathcal{L}(L^1(\mathbb{R}^d))} \geq 2.$$ 

Since by (3.2) we also have $e^{\text{Tr } A}\|P(\tau)\| \leq 1$ for all $\tau \geq 0$, the theorem is proved.

Alternatively this theorem may be derived from Proposition 2.4 via the duality argument of [17, Lemma 3.6].

After these preparations we come to the main results of this section, which give conditions for norm discontinuity of $P$ in the space $L^1(E, \mu_\infty)$, where $\mu_\infty$ is the invariant measure for $P$ discussed in Section 1. Note that in finite dimensions, its existence is guaranteed under the mere assumption that the limit $Q_\infty := \lim_{t \to \infty} Q_t$ exists in $\mathcal{L}(\mathbb{R}^d)$. This will be assumed in the next result, in which $P$ denotes Ornstein-Uhlenbeck semigroup on $L^1(\mathbb{R}^d, \mu_\infty)$ and $L_P$ its generator. Since we are dealing with the finite-dimensional case, a sufficient condition for the existence of $Q_\infty$ is that $\sigma(A) \subseteq \{\text{Re } \lambda < 0\}$.

**Theorem 3.4.** Assume that the limit $Q_\infty := \lim_{t \to \infty} Q_t$ exists in $\mathcal{L}(\mathbb{R}^d)$ and let $\mu_\infty$ be the Gaussian measure on $\mathbb{R}^d$ with covariance matrix $Q_\infty$. Then for all $t, s \geq 0$ with $t \neq s$ we have

$$\|P(t) - P(s)\|_{\mathcal{L}(L^1(\mathbb{R}^d, \mu_\infty))} = 2.$$ 

**Proof.** As is well known [6, Proposition 1], the range of $Q_\infty$ is invariant under the action of $S$ and therefore we may assume without loss of generality that $\mu_\infty$ is nondegenerate. Moreover, the existence of $\mu_\infty$ implies that $S(t) \neq S(s)$ for all $t, s \geq 0$ with $t \neq s$, since otherwise the improper integral defining $Q_\infty$ will diverge.

Let $b$ be the density of $\mu_\infty$ with respect to the Lebesgue measure; this density exists since $\mu_\infty$ is assumed to be nondegenerate. Proceeding as in [18] we consider the invertible isometry $V : L^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d, \mu_\infty)$ given by $f \mapsto b^{-1}f$ and define $\tilde{P}(t) : L^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ by

$$\tilde{P}(t) = V^{-1} \circ P(t) \circ V, \quad t \geq 0.$$ 

Then $\tilde{P} = \{\tilde{P}(t)\}_{t \geq 0}$ is a $C_0$-semigroup on $L^1(\mathbb{R}^d)$ and by the computations in [18, Theorem 5.1] its generator $\tilde{L}$ is given by

$$\tilde{L}f(x) = \frac{1}{2} \text{Tr } QD^2f(x) + \langle \tilde{A}x, Df(x) \rangle + kf(x), \quad x \in \mathbb{R}^d, \ f \in C^2_c(\mathbb{R}^d),$$

where

$$\tilde{A} = -Q_\infty A^*Q_\infty^{-1}, \quad k = -\text{Tr } A = -\text{Tr } \tilde{A}.$$ 

The result now follows from Theorem 3.3 applied to $\tilde{L} - k$.

Returning to the setting of an arbitrary real Banach space $E$, we have the following extension of Theorem 3.4.

**Theorem 3.5.** Assume that the weak operator limit $Q_\infty := \lim_{t \to \infty} Q_t$ exists in $\mathcal{L}(E^*, E)$ and that it is the covariance operator of a Gaussian measure $\mu_\infty$ on $E$. Let $S$ be an eventually compact $C_0$-semigroup on $E$, and assume that its generator $A$ has nonempty spectrum. Then for all $t, s \geq 0$ with $t \neq s$ we have

$$\|P(t) - P(s)\|_{\mathcal{L}(L^1(E, \mu_\infty))} = 2.$$
Proof. Replacing $E$ by the closure of the reproducing kernel space associated with $Q_\infty$, which is invariant under $S$ by [6, Proposition 1], see also [21], we may assume without loss in generality that $\mu_\infty$ is nondegenerate.

Since $\sigma(A) \neq \emptyset$ we may fix some $\lambda_0 \in \sigma(A)$. Note that $\lambda_0$ is an isolated point in $\sigma(A)$. Let $\pi_0 : E \to E$ be the Riesz projection onto $E_0$, the finite dimensional subspace of $E$ generated by all generalized eigenvectors associated to $\lambda_0$, cf. [12, Corollary 3.2, page 330]. The projection $\pi_0$ commutes with the operators $S(t)$. Let $S_0$ denote the restriction of $S$ to $E_0$, with generator $A_0 \in \mathcal{L}(E_0)$, and define $Q_0 \in \mathcal{L}(E_0^*, E_0)$ by $Q_0 := \pi_0 Q \pi_0^*$. Here we think of $\pi_0$ as an operator from $E$ onto $E_0$. For $0 \leq t \leq \infty$ the covariance operator $Q_{0,t}$ associated with the image measure $\mu_{0,t} = \pi_0 \mu_t$ on $E_0$ is given by

$$Q_{0,t} x_0^* = \int_0^t S_0(s)Q_0 S_0^*(s)x_0^* ds, \quad x_0^* \in E_0^*.$$ 

Since $Q_{0,\infty}$ is nondegenerate and $\sigma(A_0) = \{\lambda_0\}$ we have $\text{Re} \lambda_0 < 0$.

For all $f \in L^1(E, \mu_{0,\infty})$, the function $f_0(x) := f(\pi_0 x)$, $x \in E$, belongs to $L^1(E, \mu_{\infty})$ and we have

$$\int_E |f_0(x)| d\mu_\infty(x) = \int_{E_0} |f(\xi)| d\mu_{0,\infty}(\xi).$$

Let $P_0$ be the corresponding Ornstein-Uhlenbeck semigroup on $E_0$, i.e.,

$$P_0(t)f(x_0) = \int_{E_0} f(S_0(t)x_0 + \xi) d\mu_{0,t}(\xi), \quad t \geq 0, \quad x_0 \in E_0, \quad f \in L^1(E_0, \mu_{0,\infty}).$$

With these notations,

$$\tag{3.6} (P_0(t)f)(\pi_0 x) = P(t)f_0(x).$$

Now let $t > s \geq 0$ be such that (3.4) holds. Then, by virtue of (3.5) and (3.6),

$$\|P(t) - P(s)\|_{\mathcal{L}(L^1(E, \mu_{\infty}))} \geq \sup_{\|f\|_{L^1(E_0, \mu_{0,\infty})} \leq 1} \|P(t)f_0 - P(s)f_0\|_{L^1(E, \mu_{\infty})}$$

$$= \sup_{\|f\|_{L^1(E_0, \mu_{0,\infty})} \leq 1} \|P_0(t)f - P_0(s)f\|_{L^1(E_0, \mu_{0,\infty})} = 2,$$

where the last step follows from the previous theorem. Since $P$ is a contraction semigroup in $L^1(E, \mu_{\infty})$, the equality (3.4) follows.

Our final result concerns the spectrum of $L_P$. The following description of $\sigma(L_P)$ in $L^1(\mathbb{R}^d, \mu_\infty)$ was shown in [18], where it was derived from the characterization of $\sigma(L_P)$ for $L^1(\mathbb{R}^d)$, see [17].

**Theorem 3.6.** Assume that the limit $Q_\infty := \lim_{t \to \infty} Q_t$ exists in $\mathcal{L}(\mathbb{R}^d)$ and let $\mu_\infty$ be the Gaussian measure on $\mathbb{R}^d$ with covariance matrix $Q_\infty$. The spectrum of $L_P$ in $L^1(\mathbb{R}^d, \mu_\infty)$ equals $\mathbb{C}^-$, and every $\lambda \in \mathbb{C}$ with $\text{Re} \lambda < 0$ is an eigenvalue of $L_P$.

In setting of a real Banach space $E$ we obtain the following extension:

**Theorem 3.7.** Assume that the weak operator limit $Q_\infty := \lim_{t \to \infty} Q_t$ exists in $\mathcal{L}(E^*, E)$ and that it is the covariance operator of a Gaussian measure $\mu_\infty$ on $E$. If $S$ is eventually compact and $\sigma(A) \neq \emptyset$, the spectrum of $L_P$ in $L^1(E, \mu_\infty)$ equals $\mathbb{C}^-$, and every $\lambda \in \mathbb{C}$ with $\text{Re} \lambda < 0$ is an eigenvalue of $L_P$.

**Proof.** We may assume that $\mu_\infty$ is nondegenerate. Fix $\lambda_0 \in \sigma(A)$. Using the notations of the proof of Theorem 3.5, let $P_{0,t}$ denote the generator of the semigroup $P_0$ on $L^1(E_0, \mu_{0,\infty})$. Theorem 3.6 implies
that $\sigma(L_{P\lambda}) = \mathbb{C}^-$ and that every $\lambda \in \mathbb{C}$ with $\text{Re}\ \lambda < 0$ is an eigenvalue of $L_{P\lambda}$. Let $f \in L^1(E_0, \mu_{0,\infty})$ be an associated eigenvector. Then $f_0(x) := f(\pi_0 x)$ defines a function $f \in L^1(E, \mu_\infty)$ satisfying

$$P(t)f_0(x) = P_0(t)f(\pi_0 x) = e^{\lambda t}f(\pi_0 x) = e^{\lambda t}f_0(x).$$

Hence, $P(t)f_0 = e^{\lambda t}f_0$, and $f_0$ is an eigenvector for $L_P$ with eigenvalue $\lambda$.

After the completion of this paper, the authors received the preprint [5] by Chojnowska-Michalik. She proves a related extension of Theorem 3.6: if the part of $A$ in the reproducing kernel Hilbert space of $\mu_{\infty}$ has an eigenvalue $\lambda$ with $\text{Re}\ \lambda < 0$, then $\sigma(L_P) = \mathbb{C}^-$. 

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