A NOTE ON DOMAIN MONOTONICITY FOR THE NEUMANN EIGENVALUES OF THE LAPLACIAN

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Abstract. Given a convex domain and a convex subdomain we prove a variant of domain monotonicity for the Neumann eigenvalues of the Laplacian. As an application of our method we also obtain an upper bound for Neumann eigenvalues of the Laplacian of a convex domain.

1. Introduction

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with piecewise smooth boundary. For the Neumann eigenvalues \( 0 = \lambda_0^N(\Omega) < \lambda_1^N(\Omega) \leq \lambda_2^N(\Omega) \leq \cdots \leq \lambda_k^N(\Omega) \leq \cdots \) of the Laplacian on \( \Omega \) we prove a variant of domain monotonicity:

**Theorem 1.1.** There exists a universal constant \( C > 0 \) such that for any two bounded convex domains \( \Omega \subseteq \Omega' \) in \( \mathbb{R}^n \) with piecewise smooth boundaries the eigenvalues of their Neumann eigenvalues satisfy

\[
\lambda_k^N(\Omega') \leq Cn^2 \lambda_k^N(\Omega).
\]

The constant in the above theorem can be chosen as \( C = (92)^2 \).

The following example (for \( p = 1 \)) indicates the sharpness of the above inequality with respect to the order of \( n \).

**Example 1.1.** Let \( p \in [1, 2] \) and \( B_p^n \) be the \( n \)-dimensional \( \ell_p \)-ball centered at the origin. Suppose that \( r_{n,p} \) is the positive number such that \( \text{vol}(r_{n,p}B_p^n) = 1 \) and set \( \Omega' := r_{n,p}B_p^n \). Then \( r_{n,p} \sim n^{1/p} \) and \( \lambda_1^N(\Omega') \geq c \) for some absolute constant \( c > 0 \) ([23] Section 4 (2)). If the segment in \( \Omega' \) connecting the origin and \( (r_{n,p}, 0, 0, \cdots, 0) \) is approximated by a convex domain \( \Omega \) in \( \Omega' \) then \( \lambda_1^N(\Omega) \sim r_{n,p}^{-2} \sim n^{-2/p} \).

We remark that domain monotonicity for the Dirichlet eigenvalues \( \lambda_1^D(\Omega) \leq \lambda_2^D(\Omega) \leq \cdots \leq \lambda_k^D(\Omega) \leq \cdots \) is an easy consequence of the Courant minimax principle ([5]). Actually in this case we have

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\( \lambda_k^D(\Omega') \leq \lambda_k^D(\Omega) \) for any two bounded domains \( \Omega \subseteq \Omega' \). An example of a ball and its dumbbell like subdomain shows a sort of convexity cannot be avoided in the assumption of the above theorem.

In [10] the author proved that \( \lambda_k^N(\Omega') \leq C(n \log k)^2 \lambda_k^N(\Omega) \) for some universal constant \( C > 0 \) under the same assumption of the above theorem. In particular this implies

\[ \lambda_k^N(\Omega') \leq C(n \log k)^2 \lambda_k^N(\Omega). \]

Theorem [11] removes the log \( k \) factor and thus improves this inequality.

As a byproduct of our method we obtain an upper bound estimate of Neumann eigenvalues of the Laplacian concerning the Pólya conjecture. See Section 4.

2. Preliminaries

Let \( \Omega \) be a bounded domain in a Euclidean space with piecewise smooth boundary and \( \{\Omega_i\}_{i=0}^l \) be a finite partition of \( \Omega \) by subdomains; \( \Omega = \bigcup_i \Omega_i \) and \( \text{vol}(\Omega_i \cap \Omega_j) = 0 \) for different \( i \neq j \). The following proposition was due to Buser [3, 8.2.1 Theorem]. See [13] for an weaker form and also [12, Proposition 6.1] for generalization.

**Proposition 2.1 ([3]).** Under the above situation, we have

\[ \lambda_{l+1}^N(\Omega) \geq \min_{i=0,1, \ldots, l} \lambda_1^N(\Omega_i). \]

We use the following relation between diameter and the first positive Neumann eigenvalue of the Laplacian under the convexity assumption.

**Proposition 2.2 ([20, (1.2)]).** Let \( \Omega \) be a bounded convex domain in a Euclidean space. Then we have

\[ \lambda_1^N(\Omega) \geq \frac{\pi^2}{(\text{diam} \, \Omega)^2}. \]

From Proposition 2.1 we can obtain a lower bound of eigenvalues of the Laplacian once we give a partition. In the proof of Theorem 1.1 we use the Voronoi partition to get the lower bound of \( \lambda_k^N(\Omega) \).

Let \( X \) be a metric space and \( \{x_i\}_{i \in I} \) be a subset of \( X \). For each \( i \in I \) we define the **Voronoi cell** \( C_i \) associated with the point \( x_i \) as

\[ C_i := \{ x \in X \mid d(x, x_i) \leq d(x, x_j) \text{ for all } j \neq i \}. \]

Note that if \( X \) is a bounded convex domain \( \Omega \) in a Euclidean space then \( \{C_i\}_{i \in I} \) is a convex partition of \( \Omega \) (the boundaries \( \partial C_i \) may overlap each other). Observe also that if the balls \( \{B(x_i, r)\}_{i \in I} \) of radius \( r \) covers \( \Omega \) then \( C_i \subseteq B(x_i, r) \), and thus \( \text{diam} \, C_i \leq 2r \) for any \( i \in I \).
3. Proof of Theorem 1.1

The following lemma is a key to prove Theorem 1.1.

**Lemma 3.1.** Let $\Omega$ be a bounded convex domain in $\mathbb{R}^n$ with a piecewise smooth boundary. Given $r > 0$ suppose that $\{x_i\}_{i=0}^l$ is an $r$-separated set in $\Omega$. Then

$$r \leq \frac{45n}{\sqrt{\lambda_1^N(\Omega)}}.$$

To prove Lemma 3.1 we use the boundary concentration inequality established in [11, 12] and a variant of it. The boundary concentration inequality is an analogue of the exponential concentration inequality due to Gromov and Milman ([14]).

For a subset $A$ of a metric space $X$ and $r > 0$, $B_r(A)$ denotes the closed $r$-neighborhood of $A$.

**Lemma 3.2** (Boundary concentration inequality, [11, Proposition 2.1]). Let $\Omega$ be a (not necessarily convex) bounded domain in $\mathbb{R}^n$ with a piecewise smooth boundary and let $\mu$ be the uniform probability measure on $\Omega$. For any $r > 0$ we have

$$\mu(\Omega \setminus B_r(\partial \Omega)) \leq \exp \left(1 - \sqrt{n\lambda_1^D(\Omega)r}\right).$$

For a bounded domain $\Omega$ with a piecewise smooth boundary $\partial \Omega$ and a piecewise smooth open domain $U$ of $\partial \Omega$ we consider the following mixed eigenvalue problem:

$$\Delta \phi = -\lambda \phi \text{ on } \Omega, \phi = 0 \text{ on } \partial \Omega \setminus U, \text{ and } \frac{\partial \phi}{\partial \nu} = 0 \text{ on } U,$$

where $\nu$ is the outer unit normal. On this problem the eigenvalues consists of a discrete positive sequence ([5, Theorem 1 of Chapter I]). Let $\lambda_k^U(\Omega)$ be the $k$th eigenvalue of the problem.

The proof of the following lemma is the same as the proof of [11, Proposition 2.1] and we omit it.

**Lemma 3.3.** Let $\Omega$ be a (not necessarily convex) bounded domain in $\mathbb{R}^n$ with a piecewise smooth boundary and $U$ be a piecewise smooth open domain of $\partial \Omega$. Let $\mu$ denotes the uniform probability measure on $\Omega$. For any $r > 0$ we have

$$\mu(\Omega \setminus B_r(\partial \Omega \setminus U)) \leq \exp \left(1 - \sqrt{n\lambda_1^U(\Omega)r}\right).$$

Since convex domains in $\mathbb{R}^n$ enjoy the CD($0$, $n$) condition in the sense of Lott-Villani-Sturm ([19] [24] [25]) and CD($0$, $n$) spaces satisfy the Bishop-Gromov volume comparison theorem ([25, Theorem 2.3])...
we obtain the following lemma. Here we give a direct proof so that this paper become self-contained.

**Lemma 3.4** (Bishop-Gromov inequality). Let $\Omega$ be a convex domain in $\mathbb{R}^n$. Then for any $x \in \Omega$ and any $R > r > 0$ we have

$$(3.1) \quad \frac{\operatorname{vol}(B(x, r) \cap \Omega)}{\operatorname{vol}(B(x, R) \cap \Omega)} \geq \left(\frac{r}{R}\right)^n.$$  

**Proof.** Recall that the Brunn-Minkowski inequality ([13]) states that for any two measurable subsets $A$, $B$ in $\mathbb{R}^n$ and $t \in [0, 1]$ we have

$$\operatorname{vol}((1-t)A + tB) \geq (1-t) \operatorname{vol}(A) + t \operatorname{vol}(B),$$  

(3.2)

where $$(1-t)A + tB := \{(1-t)x + ty \mid x \in A, y \in B\}.$$  

The inequality (3.1) follows by putting $A := \{x\}$, $B := B(x, R) \cap \Omega$, and $t := \frac{r}{R}$ in (3.2). This completes the proof. \(\Box\)

**Proof of Lemma 3.1.** Let $B_i := B(x_i, r/8) \cap \Omega$. For any positive number $r' < \frac{1}{8} \min_{i \neq j} d(B_i, B_j)$ we set $\tilde{B}_i := B_{r'}(B_i) \cap \Omega$. We also set $A := \bigcup_{i=0}^k \tilde{B}_i$. Then the (usual) domain monotonicity gives

$$\lambda_i^N(\Omega) \leq \max_{i=0,1,\ldots,p} \nu_1(\tilde{B}_i).$$

Putting $\nu_1(\tilde{B}_i) := \lambda_i^{\tilde{B}_i \cap \partial \Omega}(\tilde{B}_i)$ if $\tilde{B}_i \cap \partial \Omega \neq \emptyset$ and $\nu_1(\tilde{B}_i) := \lambda_i^{\tilde{B}_i}(\tilde{B}_i)$ if $\tilde{B}_i \cap \partial \Omega = \emptyset$ we then get

$$\lambda_i^N(\Omega) \leq \max_{i=0,1,\ldots,p} \nu_1(\tilde{B}_i).$$

Suppose that the maximum of the right-hand side is attained by $\nu_1(\tilde{B}_{i_0})$. Let $\mu_{i_0}$ be the uniform probability measure on $B_{i_0}$. The Bishop-Gromov inequality implies that

$$\mu_{i_0}(B_{i_0}) \geq \left(\frac{r}{r + r'}\right)^n \geq \frac{1}{5^n}.$$  

Thus Lemmas 3.2 and 3.3 imply that

$$\exp\left(1 - \sqrt{\nu_1(\tilde{B}_{i_0})}\right) \geq \begin{cases} \mu_{i_0}(\tilde{B}_{i_0} \setminus B_s(\partial \tilde{B}_{i_0} \setminus \partial \Omega)) & (\tilde{B}_{i_0} \cap \partial \Omega \neq \emptyset), \\ \mu_{i_0}(\tilde{B}_{i_0} \setminus B_s(\partial \tilde{B}_{i_0})) & (\tilde{B}_{i_0} \cap \partial \Omega = \emptyset). \end{cases}$$
This shows that if \(\exp(1 - \sqrt{\nu_1(\hat{B}_{i_0})} s) < 1/5^n\) then \(r' \leq s\). That is, as long as
\[
\frac{1}{\sqrt{\nu_1(\hat{B}_{i_0})}}(1 + n \log 5) < s,
\]
we have \(r' \leq s\). Therefore we get
\[
r' \leq \frac{1}{\sqrt{\nu_1(\hat{B}_{i_0})}}(1 + n \log 5) \leq \frac{4n}{\sqrt{\lambda_k^N(\Omega)}}.
\]
Since \(r'\) can be sufficiently close to \(\frac{1}{8} \min_{i \neq j} d(B_i, B_j)\) and \(\min_{i \neq j} d(B_i, B_j)\) is at least \(3r/4\) we obtain the lemma. ✷

Proof of Theorem 1.1. Let \(R := 46n/\sqrt{\lambda_k^N(\Omega')}\). We take a maximal \(R\)-separated net \(\{x_i\}_{i=0}^l\) in \(\Omega'\). If \(l \geq k\) then by Lemma 3.1 we have
\[
\frac{46n}{\sqrt{\lambda_k^N(\Omega')}} = R \leq \frac{45n}{\sqrt{\lambda_k^N(\Omega')}} \leq \frac{45n}{\sqrt{\lambda_k^N(\Omega')}}.
\]
This is a contradiction. Hence \(l \leq k - 1\).

Let \(y_0, y_1, y_2, \cdots, y_l\) be maximal \(R\)-separated points in \(\Omega'\), where \(l \leq k - 1\). By the maximality we have \(\Omega' \subseteq \bigcup_{i=0}^l B(y_i, R)\). If \(\{\Omega'_i\}_{i=1}^l\) is the Voronoi partition associated with \(\{y_i\}\) then we have \(\text{diam } \Omega'_i \leq 2R\).

Setting \(\Omega_i := \Omega'_i \cap \Omega\) we get \(\Omega = \bigcup_{i=0}^l \Omega_i\) and \(\text{diam } \Omega_i \leq 2R\). Since each \(\Omega_i\) is convex, Proposition 2.2 gives \(\lambda_k^N(\Omega_i) \geq \pi^2/(2R)^2\). Applying Proposition 2.1 to the covering \(\{\Omega_i\}\) we obtain
\[
\lambda_k^N(\Omega) \geq \lambda_k^N(\Omega) \geq \pi^2/(2R)^2 \geq \lambda_k^N(\Omega')/(92n)^2,
\]
which yields the conclusion of the theorem. This completes the proof. □

4. An upper bound for Neumann eigenvalues

Let \(\Omega\) be a bounded domain with piecewise smooth boundary in a Riemannian manifold. We denote \(\lambda_k^N(\Omega)\) the \(k\)-th positive Neumann eigenvalue of the Laplacian on \(\Omega\), counted with multiplicities. Applying the method of the previous section we prove the following. Recall that \(\Omega\) is convex iff any minimizing geodesic connecting two points in \(\Omega\) is included in \(\Omega\).

Theorem 4.1. There is a universal constant \(C > 0\) satisfying the following. Let \(M\) be an \(n\)-dimensional complete Riemannian manifold
of nonnegative Ricci curvature and $\Omega$ be a bounded convex domain in $M$ with piecewise smooth boundary. Then we have

$$\lambda^N_k(\Omega) \leq C \left( \frac{k}{\omega_n \text{vol} \Omega} \right)^{\frac{2}{n}},$$

where $\omega_n$ is the volume of an $n$-dimensional Euclidean unit ball.

One can take $C = 10(46)^2$ in the above theorem.

Proof. Note first that Lemma 3.1 also holds in our nonlinear setting since the assumption of the nonnegativity of Ricci curvature and convexity of $\Omega$ imply the Bishop-Gromov inequality. Hence as in the proof of Theorem 1.1 at most $k$ balls of radius $46n/\sqrt{\lambda^N_k(\Omega)}$ covers $\Omega$. Since the Bishop inequality gives $\text{vol}(B(x, R) \cap \Omega) \leq w_n R^n$ for any $x \in \Omega$ and $R > 0$ (this follows directly from (3.1) by letting $r \to 0$) we have

$$\text{vol} \Omega \leq k \omega_n \left( \frac{46n}{\sqrt{\lambda^N_k(\Omega)}} \right)^{n}.$$

Thus $\Gamma(\frac{n}{2} + 1)^{\frac{2}{n}} \sim n$ shows (4.1). This completes the proof. $\square$

Let us review the previous known results relating with Theorem 4.1. Pólya conjectured that

$$\lambda^N_k(\Omega) \leq 4\pi^2 \left( \frac{k}{\omega_n \text{vol} \Omega} \right)^{\frac{2}{n}},$$

holds for any $k$ and any bounded domain $\Omega$ in $\mathbb{R}^n$ ([21, Chapter XIII]), that is, the principal term of the Weyl law provides a bound for the eigenvalues. Theorem 4.1 says that the conjecture is affirmative up to a multiplicative constant factor under the convexity assumption.

The conjecture is affirmative in the case of $k = 1, 2$. The case of $k = 1$ was proved by Weinberger removing some condition supposed by Szegő ([28, 20]). The case of $k = 2$ was solved affirmatively by Bucur and Henrot ([4]). For general $k$ Pólya ([22]) proved periodic tiling domains satisfies ([12]) and later Kellner ([13]) removed the periodic condition. Recently Filonov, Levitin, Polterovich and Sher showed the Pólya conjecture is affirmative for planar discs and planar sectors ([9]).

Using harmonic analysis Kröger ([17, Corollary 2]) proved that

$$\lambda^N_k(\Omega) \leq (2\pi)^2 \left( \frac{n + 2}{2} \right)^{\frac{2}{n}} \left( \frac{k}{\omega_n \text{vol} \Omega} \right)^{\frac{2}{n}} \sim \left( \frac{k}{\omega_n \text{vol} \Omega} \right)^{\frac{2}{n}}$$

holds for any (not necessarily convex) bounded domain $\Omega$ in $\mathbb{R}^n$ with piecewise smooth boundary, which is sharper than our estimate (4.1).
For a domain in a Riemannian manifold, Korevaar ([16], (0.3) Theorem) obtained much more general and deep result. For example from his result one can see that
\[
\lambda^N_k(\Omega) \leq c_n \left( \frac{k}{\omega_n \text{vol} \Omega} \right)^\frac{2}{n}
\]
holds for a (not necessarily convex) bounded domain \( \Omega \) in a manifold with non-negative Ricci curvature, where \( c_n > 0 \) is a numerical constant depending only on \( n \). He also got an estimate for a domain in a manifold with a lower Ricci curvature bound. The constant \( c_n \) comes from an upper bound of a number of balls of (some fixed) radius \( 5^{-1}R \) that covers an annulus \( B(x, 2R) \setminus B(x, R) \). Using the Bishop-Gromov inequality the upper bound can be estimated from above by an exponential in \( n \).

Colbois and Maerten ([7, Theorem 1.3]) showed that for each bounded domain \( \Omega \) in a complete Riemannian manifold with Ricci curvature bounded below by \(-(n - 1)a^2, a \geq 0\) we have
\[
\lambda^N_k(\Omega) \leq A_n a^2 + B_n \left( \frac{k}{\text{vol} \Omega} \right)^\frac{2}{n}
\]
for some numerical constants \( A_n, B_n > 0 \) depending only on \( n \). The constant \( B_n \) is depending on the covering number \( C(r) \) such that each ball of radius \( 4r \) in \( M \) may be covered by \( C(r) \) balls of some fixed radius \( r \). The covering number can be estimated from above by an exponential in \( n \) via the Bishop-Gromov inequality.

The same proof applies for an upper bound for eigenvalues of the Laplacian on closed Riemannian manifolds of nonnegative Ricci curvature. Since there is no boundary one can give a simpler proof of the corresponding statement of Lemma 3.1 as follows.

**Lemma 4.2.** Let \( M \) be an \( n \)-dimensional closed Riemannian manifold of nonnegative Ricci curvature. Given \( r > 0 \) suppose that \( \{x_i\}_{i=0}^l \) is an \( r \)-separated set in \( M \). Then
\[
r \leq \frac{8n}{\sqrt{\lambda_l(M)}},
\]
where \( \lambda_l(M) \) is the \( l \)th nontrivial eigenvalue of the Laplacian on \( M \), counted with multiplicities.

**Proof.** Let \( B_i := B(x_i, r/2) \). Since \( B_i \cap B_j = \emptyset \) for distinct \( i, j \) the (usual) domain monotonicity yields
\[
\lambda_l(M) \leq \lambda^D_{l+1} \left( \bigcup_{i=0}^l B_i \right) \leq \max_{i=0, 1, \ldots, l} \lambda^D_i(B_i).
\]
By virtue of Cheng’s eigenvalue comparison theorem ([6, Theorem 1.1]) we have
\[ \lambda_i^D(B_i) \leq \lambda_1^D(\{|x| \leq r/2\}) = \left(\frac{r}{2}\right)^{-2} (j_{\frac{n}{2}-1,1})^2, \]
where \( j_{\frac{n}{2}-1,1} \) denotes the first positive zero of the Bessel function \( J_{\frac{n}{2}-1} \) ([5, Theorem 4 of Chapter II]). Since \( j_{\frac{n}{2}-1,1} \leq 2n \) ([27, P 486 (5)]) we thereby get
\[ \lambda_i(M) \leq \left(\frac{r}{2}\right)^{-2} (j_{\frac{n}{2}-1,1})^2 \leq 16r^{-2}n^2. \]
This completes the proof. \( \square \)

The same proof of Theorem 4.1 thus implies the following.

**Theorem 4.3.** There is a universal constant \( C > 0 \) satisfying the following. Let \( M \) be an \( n \)-dimensional closed Riemannian manifold of nonnegative Ricci curvature. Then we have
\[ \lambda_k(M) \leq C \left(\frac{k}{\omega_n \text{vol } M}\right)^{\frac{2}{n}}. \]

The above constant \( C \) can be taken as \( C = 640 \).

Buser proved in [2, Satz 7] (see also [1]) that
\[ \lambda_k(M) \leq \frac{(n-1)^2}{4} a^2 + c_n \left(\frac{k}{\text{vol } M}\right)^{\frac{2}{n}} \]
for a closed Riemannian manifold with Ricci curvature bounded below by \(-(n-1)a^2\), where \( c_n \sim n \). Li and Yau gave a similar and sharper estimate for a closed manifold with a lower Ricci curvature bound by using a covering and comparison method. In particular they gave an upper bound
\[ \lambda_k(M) \leq n(n+4)\omega_n^{\frac{4}{n}} \left(\frac{k+1}{\omega_n \text{vol } M}\right)^{\frac{2}{n}} \leq \left(\frac{k}{\omega_n \text{vol } M}\right)^{\frac{2}{n}} \]
for a closed manifold \( M \) with non-negative Ricci curvature ([18, Theorem 17]). Their estimate is the same with our estimate (4.3) in order. Our method looks somewhat simpler than their method.

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References

[1] P. Buser, A note on the isoperimetric constant. Ann. Sci. Ecole Norm. Sup. (4) 15(2), 213–230 (1982).
[2] P. Buser, Beispiele für $\lambda_1$ aus kompakten Mannigfaltigkeiten. Math. Z. 165, 107–133 (1979).
[3] P. Buser, Geometry and spectra of compact Riemann surfaces. Reprint of the 1992 edition. Modern Birkhauser Classics. Birkhauser Boston, Ltd., Boston, MA, 2010.
[4] D. Bucur and A. Henrot, Maximization of the second non-trivial Neumann eigenvalue, Acta Math. 222:2 (2019), 337–361.
[5] I. Chavel, Eigenvalues in Riemannian geometry, Pure and Applied Mathematics, 115. Academic Press, Inc., Orlando, FL, 1984.
[6] S. Y. Cheng, Eigenvalue comparison theorems and its geometric applications. Math. Z. 143 (1975), no. 3, 289–297.
[7] B. Colbois and D. Maerten, Eigenvalues estimate for the Neumann problem of a bounded domain. J. Geom. Anal. 18 (2008), no. 4, 1022–1032.
[8] R. J. Gardner, The Brunn-Minkowski inequality. Bull. Amer. Math. Soc. (N.S.) 39 (2002), no. 3, 355–405.
[9] N. Filonov, M. Levitin, I. Polterovich, and D. A. Sher, Pólya’s conjecture for Euclidean balls, to appear in Invent. Math.
[10] K. Funano, Applications of the ‘ham sandwich theorem’ to eigenvalues of the Laplacian. Anal. Geom. Metr. Spaces 4 (2016), no. 1, 317–325.
[11] K. Funano and Y. Sakurai, Concentration of eigenfunctions of the Laplacian on a closed Riemannian manifold. Proc. Amer. Math. Soc. 147 (2019), no. 7, 3155–3164.
[12] K. Funano and Y. Sakurai, Upper bounds for higher-order Poincaré constants. Trans. Amer. Math. Soc. 373 (2020), no. 6, 4415–4436.
[13] M. Gromov, Metric structures for Riemannian and non-Riemannian spaces, Based on the 1981 French original, With appendices by M. Katz, P. Pansu and S. Semmes. Translated from the French by Sean Michael Bates. Progress in Mathematics, 152. Birkhauser Boston, Inc., Boston, MA, 1999.
[14] M. Gromov and V. D. Milman, A topological application of the isoperimetric inequality. Amer. J. Math. 105 (1983), no. 4, 843–854.
[15] R. Kelhner, On a theorem of Polya, Amer. Math. Monthly 73:8 (1966), 856–858.
[16] N. Korevaar, Upper bounds for eigenvalues of conformal metrics. J. Differ. Geom. 37(1), 73–93 (1993).
[17] P. Kröger, Upper bounds for the Neumann eigenvalues on a bounded domain in Euclidean space. J. Funct. Anal. 106(2), 353–357 (1992).
[18] P. Li and S.-T. Yau, Estimates of eigenvalues of a compact Riemannian manifold. Proc. Sympos. Pure Math. 36, 205–239 (1980).
[19] J. Lott and C. Villani, Ricci curvature for metric-measure spaces via optimal transport. Ann. of Math. (2) 169 (2009), no. 3, 903–991.
[20] L. E. Payne and H. F. Weinberger. An optimal poincaré inequality for convex domains. Arch. Rational Mech. Anal., 5:286—292, 1960.
[21] G. Pólya, Mathematics and plausible reasoning, Oxford University Press, London, 1954.
[22] G. Pólya, *On the eigenvalues of vibrating membranes*, Proc. London Math. Soc. **11** (1961), 419-433
[23] S. Sodin, *An isoperimetric inequality on the $\ell_p$ balls*. (English, French summary) Ann. Inst. Henri Poincare Probab. Stat. **44** (2008), no. 2, 362–373.
[24] K-T. Sturm, *On the geometry of metric measure spaces. I*. Acta Math. **196** (2006), no. 1, 65–131.
[25] K-T. Sturm, *On the geometry of metric measure spaces. II*. Acta Math. **196** (2006), no. 1, 133–177.
[26] G. Szegő, *Inequalities for certain eigenvalues of a membrane of given area*, J. Rational Mech. Anal. **3** (1954), 343–356.
[27] G. N. Watson, *A treatise on the theory of Bessel functions*. (English summary) Reprint of the second (1944) edition. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1995.
[28] H. F. Weinberger, *An isoperimetric inequality for the $N$-dimensional free membrane problem*, J. Rational Mech. Anal. **5** (1956), 633–636.

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