On the Schrödinger–Lohe Hierarchy for Aggregation and Its Emergent Dynamics

Seung-Yeal Ha¹,² · Hansol Park³

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Abstract
The Lohe hierarchy is a hierarchy of finite-dimensional aggregation models consisting of the Kuramoto model, the complex Lohe sphere model, the Lohe matrix model and the Lohe tensor model. In contrast, the Schrödinger–Lohe model is the only known infinite-dimensional Lohe aggregation model in literature. In this paper, we provide an explicit connection between the Schrödinger–Lohe model and the complex Lohe sphere model, and then by exploiting this explicit relation, we construct infinite-dimensional liftings of the Lohe matrix and the Lohe tensor models. In this way, we establish the Schrödinger–Lohe hierarchy which corresponds to the infinite-dimensional extensions of the Lohe hierarchy. For the proposed hierarchy, we provide sufficient frameworks leading to the complete aggregation in terms of coupling strengths and initial configurations.

Keywords
Aggregation · Emergence · Kuramoto model · Lohe tensor model · Quantum synchronization · Tensors

Mathematics Subject Classification 82C10 · 82C22 · 35B37

1 Introduction

Collective behaviors often appear in classical and quantum many-body systems, e.g., aggregation of bacteria, herding of sheep, schooling of fish, synchronous firing of fireflies and array of Josephson junctions in semiconductors [1,2,14,24,34,36,38] etc. Despite of their ubiquity in our nature, model-based studies on the collective dynamics were first begun only...
in a half century ago by two pioneers, Winfree and Kuramoto [27,28,39,40]. Recently, due to applications in the control of drones, self-driving cars and sensor networks, research on the collective dynamics has received lots of attentions from diverse scientific disciplines such as applied mathematics, biology, control theory and statistical physics, etc.

In this paper, we are interested in the Kuramoto model [7,10,23] and its high-dimensional extensions such as the Lohe sphere model [6,8,25,32], the Lohe matrix model [22,29–31]. See [5,11–13,32,37,42] for other related models. Aforementioned aggregation models were further extended to the ensemble of Lohe tensors by the authors in [20] in which we call it as the Lohe tensor model which completes the Lohe hierarchy (LH) comprising of finite-dimensional aggregation models such as the Kuramoto model, the Lohe sphere (LS) model, the Lohe matrix (LM) model and the Lohe tensor (LT) models, whereas in an infinite-dimensional setting, the Schrödinger–Lohe (SL) model is the only known Lohe type aggregation model so far. In what follows, we address the following two questions:

- \( (Q1) \) What is the connection between the SL model and the finite-dimensional aggregation models in the LH?
- \( (Q2) \) If such connection exists, can we establish a Schrödinger–Lohe hierarchy (SLH) consisting of the infinite-dimensional analogs of the aggregation models in LH?

The main results of this paper are positive answers for the above two questions. First, we provide an explicit connection between the SL model and the LS model. If the solution to the SL model is expanded in terms of a basis consisting of suitable standing wave solutions, we show that coefficients satisfy the Lohe sphere model on \( (L^2 \cap L^\infty)(\mathbb{Z}_+) \). Second, we employ the idea of connecting the SL model and LS model to introduce infinite-dimensional analogs of the LM and LT models which will be coined as the SL matrix and SL tensor models. Since the details are rather messy, we will not go into the details here and we instead leave the detailed results in Sects. 4 and 5. In this manner, we establish the Schrödinger–Lohe hierarchy.

The rest of this paper is organized as follows. In Sect. 2, we review minimal materials on tensors which are enough to understand the rest of paper, and then briefly review the Lohe hierarchy consisting of the Kuramoto model, complex Lohe sphere model, the generalized Lohe matrix model and the Lohe tensor model, and study basic properties such as conservation law and solution splitting property of each model. In Sect. 3, we study a priori estimates on the SL model and propose a new SL type model with rotational couplings and then discuss its connection with the Kuramoto model. In Sect. 4, we present an explicit bridge between the Schrödinger Lohe model and the complex Lohe sphere model on \( (\ell^2 \cap \ell^\infty)(\mathbb{Z}_+) \), and then using this idea of an explicit bridge, we provide an extension of the generalized Lohe matrix model to a Schrödinger setting. In Sect. 5, we further propose a Schrödinger type extension of the Lohe tensor model and finally establish the Schrödinger–Lohe hierarchy. In some sense, unlike to the standard approaches such as classical and quantum BBGKY hierarchies in classical and quantum physics which is a top-down approach, whereas our approach is a bottom-up approach to go from low-rank models to high-rank models. Finally, Sect. 6 is devoted to a brief summary of main results and some remaining issues to be discussed in a future work.

### 1.1 Notation

For complex-valued functions \( \psi \) and \( \varphi \) in \( L^2(\mathbb{T}^d) \), we define the inner product and its associated norm as follows:
\[ \langle \psi | \varphi \rangle = \int_{\mathbb{T}^d} \overline{\psi}(x) \varphi(x) \, dx, \quad \| \psi \| := \sqrt{\langle \psi | \psi \rangle}. \]

## 2 Preliminaries

In this section, we briefly review basics material of tensors, tensor space and tensor contraction, and then we introduce the Lohé hierarchy and we review their basic properties such as conservation laws and solution splitting property.

### 2.1 Tensors and Tensor Contraction

A tensor denotes a multi-dimensional array of complex numbers with several indices. Thus, it can be viewed as a generalization of vector and matrix, and the rank of a tensor is the number of indices, i.e., a rank-\( m \) tensor of dimensions \( d_1 \times \cdots \times d_m \) is an element of \( \mathbb{C}^{d_1 \times \cdots \times d_m} \). Hence, a rank-\( m \) tensor \( T \in \mathbb{C}^{d_1 \times \cdots \times d_m} \) can also be identified as a multilinear map from \( \mathbb{C}^{d_1} \times \cdots \times \mathbb{C}^{d_m} \) to \( \mathbb{C} \). Complex numbers, complex vectors and complex matrices correspond to rank-0, 1 and 2 tensors, respectively.

For a rank-\( m \) tensor \( T \) and a multi-index \( \alpha_* = (\alpha_1, \ldots, \alpha_m) \in \{1, \ldots, d_1\} \times \cdots \times \{1, \ldots, d_m\} \), we denote the \( \alpha_* \)-th component of \( T \) by \( [T]_{\alpha_*} = [T]_{\alpha_1 \cdots \alpha_m} \), and we also denote \( \bar{T} \) by the rank-\( m \) tensor whose components are the complex conjugate of the elements of \( T \):

\[ [\bar{T}]_{\alpha_1 \cdots \alpha_m} = [\bar{T}]_{\alpha_1 \cdots \alpha_m}. \]

Finally, we set \( T_m(\mathbb{C}) := T_m(d_1, \ldots, d_m; \mathbb{C}) \) to be the set of all rank-\( m \) tensors with complex entries and the size \( d_1 \times \cdots \times d_m \). One of key basic operations in \( T_m(\mathbb{C}) \) is a tensor contraction which yields a low-rank tensor by contracting repeated variables in the expressions. Note that the inner product between rank-1 tensors and matrix product between rank-2 tensors can be defined as special cases of tensor contractions: for \( v, w \in T_1(d_1; \mathbb{C}) \) and \( A, B \in T_2(d_1, d_1; \mathbb{C}) \),

\[ \langle v, w \rangle := [\bar{v}]_\alpha [w]_\alpha, \quad [AB]_{\alpha \beta} := [A]_{\alpha \gamma} [B]_{\gamma \beta}, \]

where we used Einstein summation rule for repeated indices.

For a rank-\( m \) tensor \( T \in T_m(\mathbb{C}) \), we also set

\[ [T]_{\alpha_*} := [T]_{\alpha_1 \alpha_2 \cdots \alpha_m}, \quad [T]_{\alpha_0} := [T]_{\alpha_0 \alpha_2 \cdots \alpha_m}, \quad [T]_{\alpha_1} := [T]_{\alpha_1 \alpha_2 \cdots \alpha_m}. \]

Moreover, for a special rank-\( 2m \) tensor \( S \in T_{2m}(d_1, \ldots, d_m, d_1, \ldots, d_m; \mathbb{C}) \), one has

\[ [S]_{\alpha_* \beta_*} := [S]_{\alpha_1 \alpha_2 \cdots \alpha_m \beta_1 \beta_2 \cdots \beta_m}. \]

Next, we define Frobenius inner product, corresponding norm on \( T_m(\mathbb{C}) \), ensemble diameter as follows: for a tensor ensemble \( \{ T_i \} \subset T_m(\mathbb{C}) \),

\[ \langle T_i, T_j \rangle_F := [\bar{T}_i]_{\alpha_0} [T_j]_{\alpha_0}, \quad \| T_i \|_F^2 := \langle T_i, T_i \rangle_F \quad \text{and} \quad \mathcal{D}(T) := \max_{1 \leq i, j \leq N} \| T_i - T_j \|_F. \]

For an elementary introduction to tensors and elementary tensor operations, we refer to introductory articles [4,33].
2.2 The Lohe Hierarchy

In this subsection, we review the Lohe hierarchy consisting of finite-dimensional Lohe type aggregation models and their basic properties such as conservation laws and solution splitting property:

Lohe tensor model \(\rightarrow\) Lohe matrix model \(\rightarrow\) Lohe sphere model.

2.2.1 The Lohe Tensor Model

Let \(\{T_j\}\) be a homogeneous Lohe tensor flock whose dynamics is governed by the following Cauchy problem:

\[
\frac{dT_j}{dt} = [F]_{\alpha\beta_0} T_j, \\
T_j\big|_{t=0} = T_j^{\text{in}}, \quad j = 1, \ldots, N,
\]

where \(\alpha, \beta\) is a nonnegative coupling strength and \(F\) is a Skew–Hermitian rank-2m tensor in \(T_{2m}(d_1, \ldots, d_m, d_1, \ldots, d_m; \mathbb{C})\) with the following three properties: for \(n \in \mathbb{Z}_+\),

\[
[F_0^{\alpha\beta}]_{\alpha\beta} = \delta\alpha\beta, \quad [F]_{\alpha\beta} = [F]_{\beta\alpha}, \quad [F^T]_{\alpha\beta} = [F]_{\alpha\beta}, \quad [F]^T_{\alpha\beta} = [F]_{\alpha\beta},
\]

for all \(i \in \{0, 1\}^m\) with \(\kappa_i \neq 0\) and \(\delta\alpha\beta\) is defined as follows:

\[
\delta\alpha\beta = \begin{cases} 
1, & \alpha = \beta \quad \forall k = 1, 2, \ldots, m, \\
0, & \text{otherwise}.
\end{cases}
\]

Although (2.1) and (2.2) look so complicated, it admits a conservation law and solution splitting property. For this, we consider the Cauchy problem to the subsystem of (2.1) with zero free flow \(F \equiv 0\) and the initial data:

\[
\frac{dS_j}{dt} = \sum_{i} \kappa_i \left( [S_c]_{\alpha\beta} [\tilde{S}_j]_{\alpha\beta} [S_j]_{\alpha\beta(1-i_\alpha)} - [S_j]_{\alpha\beta} [\tilde{S}_c]_{\alpha\beta} [S_j]_{\alpha\beta(1-i_\alpha)} \right),
\]

\[
S_j\big|_{t=0} = T_j^{\text{in}}, \quad j = 1, 2, \ldots, N.
\]

**Proposition 2.1** \([18, 20]\) Let \(\{T_j\}\) and \(\{S_j\}\) be solutions to (2.1)–(2.3) with the same initial data \(\{T_j^{\text{in}}\}\), respectively. Then, the following assertions hold.

1. \(\|T_j\|_F\) is a conserved quantity:

\[
\|T_j(t)\|_F = \|T_j^{\text{in}}\|_F, \quad t \geq 0, \quad j = 1, \ldots, N.
\]

2. The Lohe tensor flow (2.2) can be represented as a composition of free flow and nonlinear flow:

\[
T_j(t) = e^{tF} S_j(t), \quad t \geq 0, \quad j = 1, \ldots, N,
\]
where \( e^{tF} \) is a matrix exponential defined by the following relation:

\[
[e^{tF}]_{\alpha \beta} = \sum_{n=0}^{\infty} \frac{t^n}{n!}[F^n]_{\alpha \beta}.
\]

### 2.2.2 The Lohe Matrix Model

Let \( \{A_j\} \) be a collection of complex \( d_1 \times d_2 \) Lohe matrices whose dynamics is governed by the following Cauchy problem [21]:

\[
\dot{A}_j = BA_j + \kappa_0(A_c A_j^\dagger A_j - A_j A_c^\dagger A_j) + \kappa_1(A_j A_j^\dagger A_c - A_c A_j^\dagger A_j), \quad t > 0,
\]

\[
A_j \bigg|_{t=0} = A_{jn}, \quad j = 1, 2, \ldots, N, \tag{2.4}
\]

where \( \kappa_0 \) and \( \kappa_1 \) are nonnegative coupling strengths, and \( \dagger \) denotes the Hermitian conjugate, \( A_c := \frac{1}{N} \sum_{k=1}^{N} A_k \) and the rank-4 tensor \( B \in T_4(d_1, d_2, d_1, d_2; C) \) satisfies

\[
[B]_{\alpha \beta \gamma \delta} = -[B]_{\gamma \delta \alpha \beta}, \quad 1 \leq \alpha, \gamma \leq d_1, \quad 1 \leq \beta, \delta \leq d_2, \quad j = 1, \ldots, N,
\]

\[
[e^{-Bt}]_{\alpha \beta \gamma \delta} [e^{Bt}]_{\gamma \delta \alpha \beta} = \delta_{\alpha \beta} \delta_{\gamma \delta}.
\]

Consider the corresponding nonlinear sub-system associated with (2.4):

\[
\dot{N}_j = \kappa_1(N_c N_j^\dagger N_j - N_j N_c^\dagger N_j) + \kappa_2(N_j N_j^\dagger N_c - N_j N_c^\dagger N_j), \quad t > 0,
\]

\[
N_j \bigg|_{t=0} = A_{jn}^i. \tag{2.6}
\]

Similar to Proposition 2.1, we have a conservation law and solution splitting property.

**Proposition 2.2** [19] Let \( \{A_j\} \) and \( \{N_j\} \) be solutions to (2.4)–(2.6), respectively. Then, the following assertions hold.

1. \( \|A_j\|_F \) is a conserved quantity:

\[
\|A_j(t)\|_F = \|A_{jn}\|_F, \quad t \geq 0, \quad j = 1, \ldots, N.
\]

2. The Lohe matrix flow can be represented as a composition of free flow and nonlinear flow.

\[
A_j(t) = e^{tB} N_j(t), \quad t \geq 0, \quad j = 1, \ldots, N,
\]

where \( e^{tB} \) is given as follows.

\[
[e^{tB}]_{\alpha \beta \gamma \delta} := \sum_{n=0}^{\infty} \frac{t^n}{n!}[B^n]_{\alpha \beta \gamma \delta}.
\]

### 2.2.3 The Lohe Hermitian Sphere Model

Let \( \{v_j\} \) be a collection of the complex vectors in \( \mathbb{C}^d \) whose dynamics is governed by the following Cauchy problem:
\begin{align}
\dot{v}_j &= \Omega v_j + \kappa_0 (v_c \langle v_j, v_j \rangle - \langle v_c, v_j \rangle) + \kappa_1 \left( \langle v_j, v_c \rangle - \langle v_c, v_j \rangle \right)v_j, \quad t \geq 0, \\
v_j \big|_{t=0} &= v_j^{in}, \quad j = 1, 2, \ldots, N,
\end{align}

(2.7)

where \( \kappa_0 \) and \( \kappa_1 \) are nonnegative coupling strengths, \( v_c := \frac{1}{N} \sum_{i=1}^{N} v_i \) and \( \Omega \) is \( d \times d \) skew-Hermitian with the property \( \Omega^\dagger = -\Omega \).

Note that for a real vector \( v_j = x_j \in \mathbb{R}^d \), the third term in the R.H.S. of (2.7) vanishes, and system (2.7) reduces to the complex Lohe sphere model in [6]:

\begin{align}
\dot{x}_j &= \Omega x_j + \kappa_0 \left( x_c \langle x_j, x_j \rangle - x_j \langle x_c, x_j \rangle \right).
\end{align}

Next, we consider the corresponding nonlinear subsystem associated with (2.7):

\begin{align}
\dot{w}_j &= \kappa_0 (w_c \langle w_j, w_j \rangle - w_j \langle w_c, w_j \rangle) + \kappa_1 \left( \langle w_j, w_c \rangle - \langle w_c, w_j \rangle \right)w_j, \quad t > 0, \\
w_j \big|_{t=0} &= w_j^{in}, \quad j = 1, \ldots, N.
\end{align}

(2.8)

Similar to Proposition 2.1, we have a conservation law and solution splitting property.

**Proposition 2.3** [19] Let \( \{v_j\} \) and \( \{w_j\} \) be solutions to (2.7) and (2.8) with the same initial data \( \{v_j^{in}\} \), respectively. Then, the following assertions hold.

1. \( \|v_j\| \) is a conserved quantity:
   \( \|v_j(t)\| = \|v_j^{in}\|, \quad t \geq 0, \quad j = 1, \ldots, N. \)

2. The Lohe sphere flow can be represented as a composition of free flow and nonlinear flow:
   \( v_j = e^{B_t} w_j, \quad j = 1, \ldots, N. \)

### 3 Schrödinger–Lohe Type Models

In this section, we first review the Schrödinger–Lohe model and its basic properties, and then we introduce a variant of the Schrödinger–Lohe model with rotational couplings and study its connection with the Kuramoto model.

#### 3.1 The Schrödinger–Lohe Model

Let \( \{\psi_j\} \) be a collection of \( N \) complex-valued functions in \( C(\mathbb{R}_+; L^2(\mathbb{T}^d)) \) whose dynamics is governed by the following Cauchy problem:

\begin{align}
i \partial_t \psi_j &= \mathcal{H} \psi_j + \frac{i}{N} \sum_{k=1}^{N} (\psi_k \langle \psi_j | \psi_j \rangle - \langle \psi_k | \psi_j \rangle) \psi_j, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{T}^d, \\
\psi_j \big|_{t=0} &= \psi_j^{in}, \quad j = 1, \ldots, N,
\end{align}

(3.1)

where \( \mathcal{H}(x, p) = \mathcal{H}(x, -i\nabla_x) \) is a one-body hermitian Hamiltonian. The global existence of strong and smooth solutions to (3.1) was studied in [15] using the standard energy method and
asymptotic dynamics of (3.1) has been extensively discussed in [16,17]. Now, we consider the corresponding nonlinear flow for (3.1) with the same initial data:

\[
\begin{align*}
  i \partial_t \varphi_j &= \frac{ik}{N} \sum_{k=1}^{N} (\varphi_k \langle \varphi_j | \varphi_k \rangle - \langle \varphi_k | \varphi_j \rangle \varphi_k), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{T}^d, \\
  \varphi_j \bigg|_{t=0} &= \varphi_j^{iin}, \quad j = 1, \ldots, N.
\end{align*}
\]

(3.2)

Next, we present the conservation of \(L^2\)-norm and the solution splitting property.

**Proposition 3.1** [9] Let \(\{\psi_j\}\) and \(\{\varphi_j\}\) be global smooth solutions to (3.1) and (3.2), respectively. Then, the following assertions hold.

1. \(\|\psi_j\|\) is a conserved quantity:
   \[\|\psi_j(t)\| = \|\psi_j^{iin}\|, \quad t \geq 0, \quad j = 1, \ldots, N.\]

2. The Schrödinger–Lohe flow can be represented as a composition of free flow and nonlinear flow:
   \[\psi_j = e^{-iHt} \varphi_j, \quad j = 1, \ldots, N.\]

**Remark 3.1** The second property is called the “solution splitting property”.

### 3.2 The Schrödinger–Lohe Model with Rotational Couplings

Below, we introduce a variant of the SL model motivated. Recall that two coupling terms in the complex Lohe sphere model:

\[
\kappa_0 \left( v_c - \langle v_c, v_j \rangle v_j \right) + \kappa_1 \left( \langle v_j, v_c \rangle - \langle v_c, v_j \rangle \right) v_j.
\]

(3.3)

The coupling term involving with \(\kappa_0\) has the same structure as that of the SL model discussed in the previous subsection. Next, we propose a SL type model motivated by the coupling term involving with \(\kappa_1\) responsible for rotational motion.

Consider the Cauchy problem to the Schrödinger–Lohe model with rotational couplings:

\[
\begin{align*}
  i \partial_t \psi_j &= \mathcal{H} \psi_j + \frac{ik}{N} \sum_{k=1}^{N} ((\psi_j | \psi_k) - \langle \psi_k | \psi_j \rangle) \psi_j, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{T}^d, \\
  \psi_j \bigg|_{t=0} &= \psi_j^{iin}, \quad j = 1, \ldots, N.
\end{align*}
\]

(3.4)

Note that the coupling term can be rewritten using the average wave function \(\psi_c := \frac{1}{N} \sum_{j=1}^{N} \psi_j\):

\[\frac{ik}{N} \sum_{k=1}^{N} ((\psi_j | \psi_k) - \langle \psi_k | \psi_j \rangle) \psi_j = \langle \psi_j | \psi_c \rangle - \langle \psi_c | \psi_j \rangle.\]

This exactly coincides with the second term in (3.3). The global well-posedness of (3.4) can be treated similarly as in [15]. Thus, we focus on the a priori asymptotic dynamics for (3.4).

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Next, we study a connection between (3.4) and the Kuramoto model. For this, we consider the following setting:

$$\mathcal{H} = 0, \quad \psi_{in}^j(x) = e^{i\theta_{in}^j} \psi(x),$$

(3.5)

where $\psi$ is an $L^2$-function with $\|\psi\|_2 = 1$.

**Proposition 3.2** Suppose that the setting (3.5) holds, and let $\{\psi_j\}$ be a global smooth solution to (3.4). Then, one has

$$\begin{cases}
\psi_j(t, x) = e^{i\theta_{in}^j(t)} \psi(x), & (t, x) \in \mathbb{R}_+ \times \mathbb{T}^d, \\
\dot{\theta}_j = \frac{2\kappa}{N} \sum_{k=1}^{N} \sin(\theta_k - \theta_j), & j = 1, \ldots, N, \\
\theta_j \big|_{t=0} = \theta_{in}^j.
\end{cases}$$

(3.6)

**Proof** Note that $\psi_j$ satisfies

$$\begin{align*}
\partial_t \psi_j &= \frac{\kappa}{N} \sum_{k=1}^{N} ((\psi_j | \psi_k) - (\psi_k | \psi_j)) \psi_j = \kappa ((\psi_j | \psi_c) - (\psi_c | \psi_j)) \psi_j \\
&= 2\kappa i \cdot \text{Im}(\langle \psi_j | \psi_c \rangle) \psi_j,
\end{align*}$$

where we used $\langle \psi_c | \psi_j \rangle = \langle \psi_j | \psi_c \rangle$.

This yields

$$\psi_j(x, t) = \psi_{in}^j(x) e^{2\kappa i \int_0^t \text{Im}(\langle \psi_j | \psi_c \rangle)(s) ds} = e^{2\kappa i \int_0^t \text{Im}(\langle \psi_j | \psi_c \rangle)(s) ds} e^{i\theta_{in}^j} \psi(x).$$

Thus, it is reasonable to set the ansatz for $\psi_j$ as follows.

$$\psi_j(t, x) = e^{i\theta_{in}^j(t)} \psi(x).$$

(3.7)

This implies

$$\partial_t \psi_j = i\dot{\theta}_j \psi_j, \quad \langle \psi_j | \psi_k \rangle = e^{i(\theta_k - \theta_j)} \|\psi\|^2_{L^2} = e^{i(\theta_k - \theta_j)}.$$

We use the above calculation and the ansatz (3.7) into (3.6) to see

$$i\dot{\theta}_j \psi_j = \frac{\kappa}{N} \sum_{k=1}^{N} \left( e^{i(\theta_k - \theta_j)} - e^{i(\theta_j - \theta_k)} \right) \psi_j$$

which yields

$$\dot{\theta}_j = \frac{2\kappa}{N} \sum_{k=1}^{N} \sin(\theta_k - \theta_j).$$

\(\Box\)

Next, we show that system (3.5) admits a conservation law as the original S-L model.

**Lemma 3.1** Let $\{\psi_j\}$ be a global smooth solution to (3.4). Then, for $t > 0$ and $i, j = 1, \ldots, N$, one has

$$\frac{d}{dt} \langle \psi_i | \psi_j \rangle = \kappa ((\psi_c | \psi_i - \psi_j) - (\psi_i - \psi_j | \psi_c))(\psi_i | \psi_j), \quad \|\psi_j(t)\| = \|\psi_{in}^j\|.$$
Proof (i) We use (3.4) to get
\[
\frac{d}{dt}(\psi_i|\psi_j) = (\partial_t \psi_i|\psi_j) + (\psi_i|\partial_t \psi_j) = (-i\mathcal{H}\psi_i|\psi_j) + (\psi_i|\psi_j) - i\mathcal{H}(\psi_j|\psi_i)
\]
\[
+ \frac{\kappa}{N} \sum_{k=1}^N ((\psi_k|\psi_i) - (\psi_i|\psi_k) + (\psi_j|\psi_k) - (\psi_k|\psi_j))(\psi_i|\psi_j)
\]
(3.8)
(ii) We set \(i = j\) in (3.8) to see
\[
\frac{d}{dt}(\psi_j|\psi_j) = \frac{d}{dt} \|\psi_j\|^2 = 0.
\]
\[\Box\]

For a global smooth solution \(\{\psi_j\}\) and \(i_1, i_2, \ldots, i_m \in \{1, 2, \ldots, N\}\), we introduce a functional \(\mathcal{J}(\Psi)\):
\[
\mathcal{J}(\Psi) := \langle \psi_{i_1} | \psi_{i_2} \rangle \cdot \langle \psi_{i_2} | \psi_{i_3} \rangle \cdots \langle \psi_{i_m} | \psi_{i_1} \rangle.
\]
(3.9)

Proposition 3.3 Let \(\{\psi_i\}\) be a global smooth solution to (3.4). Then, we have two conservation laws:
\[
\frac{d}{dt}\mathcal{J}(\Psi) = 0 \quad \text{and} \quad \frac{d}{dt} |\langle \psi_i | \psi_j \rangle|^2 = 0, \quad t > 0.
\]

Proof (i) We use (3.8) and (3.9) to obtain
\[
\frac{d}{dt}(\langle \psi_{i_1} | \psi_{i_2} \rangle \cdot \langle \psi_{i_2} | \psi_{i_3} \rangle \cdots \langle \psi_{i_m} | \psi_{i_1} \rangle) = \langle \psi_{i_1} | \psi_{i_2} \rangle \cdot \langle \psi_{i_2} | \psi_{i_3} \rangle \cdots \langle \psi_{i_m} | \psi_{i_1} \rangle
\]
\[
\times \sum_{k=1}^N ((\psi_k|\psi_{i_1} - \psi_{i_2}) + (\psi_{i_2} - \psi_{i_3}) + \cdots + (\psi_{i_m} - \psi_{i_1}))
\]
\[
- (\langle \psi_{i_1} - \psi_{i_2} \rangle + (\psi_{i_2} - \psi_{i_3}) + \cdots + (\psi_{i_m} - \psi_{i_1})|\psi_k) = 0.
\]
(ii) We set \(m = 2, \quad i_1 = i, \quad i_2 = j\) to get the desired estimate. \[\Box\]

Next, we recall Barbalat’s lemma to be used crucially in the following sections.

Lemma 3.2 [3] (i) Suppose that a real-valued function \(f : [0, \infty) \rightarrow \mathbb{R}\) is uniformly continuous and it satisfies
\[
\lim_{t \to \infty} \int_0^t f(s)ds \quad \text{exists}.
\]
Then, \(f\) tends to zero as \(t \to \infty:\)
\[
\lim_{t \to \infty} f(t) = 0.
\]

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(ii) Suppose that a real-valued function \(f : [0, \infty) \to \mathbb{R}\) is continuously differentiable, and \(\lim_{t \to \infty} f(t) = f_\infty \in \mathbb{R}\). If \(f'\) is uniformly continuous, then

\[
\lim_{t \to \infty} f'(t) = 0.
\]

**Theorem 3.1** Let \(\{\psi_i\}\) be a global smooth solution to (3.4) with \(\kappa > 0\). Then, we have

(i) \[
\frac{d}{dt} \sum_{i,j=1}^{N} \|\psi_i - \psi_j\|_2^2 = -2N\kappa \sum_{i=1}^{N} |\langle \psi_c | \psi_i \rangle - \langle \psi_i | \psi_c \rangle|^2 \leq 0.
\]

(ii) \[
\lim_{t \to \infty} \sum_{i=1}^{N} |\langle \psi_c | \psi_i \rangle - \langle \psi_i | \psi_c \rangle|^2 = 0.
\]

**Proof** (i) We use the conservation of \(\|\psi_i\|_2 = 1\) to get

\[
\frac{d}{dt} \sum_{i,j=1}^{N} \|\psi_i - \psi_j\|_2^2 = -\frac{d}{dt} \sum_{i,j=1}^{N} \left( \langle \psi_i | \psi_j \rangle + \langle \psi_j | \psi_i \rangle \right)
= -\frac{\kappa}{N} \sum_{i,j,k=1}^{N} (\langle \psi_k | \psi_i \rangle - \langle \psi_i | \psi_j \rangle)(\langle \psi_i | \psi_j \rangle - \langle \psi_j | \psi_i \rangle)
= -\frac{2\kappa}{N} \sum_{i,j,k=1}^{N} (\langle \psi_k | \psi_i \rangle - \langle \psi_i | \psi_k \rangle)(\langle \psi_i | \psi_j \rangle - \langle \psi_j | \psi_i \rangle)
= -2N\kappa \sum_{i=1}^{N} (\langle \psi_c | \psi_i \rangle - \langle \psi_i | \psi_c \rangle)(\langle \psi_i | \psi_c \rangle - \langle \psi_c | \psi_i \rangle)
= -2N\kappa \sum_{i=1}^{N} |\langle \psi_c | \psi_i \rangle - \langle \psi_i | \psi_c \rangle|^2 \leq 0.
\]

(3.10)

(ii) It follows from the result of (i) and boundedness that \(\sum_{i,j=1}^{N} \|\psi_i - \psi_j\|_2^2\) converges as time goes infinity. Then, it follows from the boundedness of \(\frac{d\psi_i}{dt}\) that

\[
\frac{d^2}{dt^2} \sum_{i,j=1}^{N} \|\psi_i - \psi_j\|_2^2 = -2N\kappa \frac{d}{dt} \left( \sum_{i=1}^{N} (\langle \psi_c | \psi_i \rangle - \langle \psi_i | \psi_c \rangle)(\langle \psi_i | \psi_c \rangle - \langle \psi_c | \psi_i \rangle) \right).
\]

By Barbalat’s lemma, one has

\[
\lim_{t \to \infty} \frac{d}{dt} \sum_{i,j=1}^{N} \|\psi_i - \psi_j\|_2^2 = 0.
\]

This and (3.10) yield

\[
\lim_{t \to \infty} \sum_{i=1}^{N} |\langle \psi_c | \psi_i \rangle - \langle \psi_i | \psi_c \rangle|^2 = 0.
\]

\(\square\)
Remark 3.2 By the non-increasing property of the relative $L^2$-distances between wave functions, we can see that system (3.4) does not admit a periodic solution except equilibrium solutions.

Next, we study the solution splitting property of (3.4). Consider the corresponding nonlinear system:

$$i \partial_t \varphi_j = \frac{i \kappa}{N} \sum_{k=1}^{N} (\langle \varphi_j | \varphi_k \rangle - \langle \varphi_k | \varphi_j \rangle) \varphi_j, \ (t, x) \in \mathbb{R}_+ \times \mathbb{T}^d,$$

$$\varphi_j \bigg|_{t=0} = \psi_j^{in}, \ j = 1, \ldots, N. \tag{3.11}$$

Theorem 3.2 Let $\{\varphi_j\}$ and $\{\psi_j\}$ be global smooth solutions to (3.4) and (3.11), respectively. Then, one has

$$\psi_j(t, x) = e^{-\frac{i}{H}t} \varphi_j(t, x), \ j = 1, \ldots, N.$$

Proof It follows from (3.4) that

$$i \partial_t \psi_i = \mathcal{H} \psi_i + \frac{i \kappa}{N} \sum_{k=1}^{N} (\langle \psi_i | \psi_k \rangle - \langle \psi_k | \psi_i \rangle) \psi_i.$$

Then, we have

$$i \partial_t (e^{\frac{i}{H}t} \psi_i) = \frac{i \kappa}{N} \sum_{k=1}^{N} \left( (e^{\frac{i}{H}t} \psi_i) e^{\frac{i}{H}t} \psi_k - (e^{\frac{i}{H}t} \psi_k) e^{\frac{i}{H}t} \psi_i \right) e^{\frac{i}{H}t} \psi_i.$$

This yields the desired result. \qed

3.3 The Generalized S-L Model

In this subsection, we study emergent dynamics of the generalized S-L model:

$$i \partial_t \psi_j = \mathcal{H}_j \psi_j + \frac{i \kappa_0}{N} \sum_{k=1}^{N} \left( \psi_k (\psi_j | \psi_j) - \langle \psi_k | \psi_j \rangle \psi_j \right)$$

$$+ \frac{i \kappa_1}{N} \sum_{k=1}^{N} \left( (\psi_j | \psi_k) - \langle \psi_k | \psi_j \rangle \right) \psi_j, \ (t, x) \in \mathbb{R}_+ \times \mathbb{T}^d. \tag{3.12}$$

First, we recall the concept of functional derivative.

Definition 3.1 [41] Let $L^2(\mathbb{T}^d)$ be the complex vector space and $\mathcal{F} : L^2(\mathbb{T}^d) \to \mathbb{C}$ be a functional, and we write $\mathcal{F} = \mathcal{F}(\psi), \ \psi \in L^2(\mathbb{T}^d)$. Fix the point $\psi \in L^2(\mathbb{T}^d)$ and given $\varphi \in L^2(\mathbb{T}^d)$, we define

$$\frac{\delta \mathcal{F}(\psi)}{\delta \psi}(\varphi) := \lim_{h \to 0} \frac{\mathcal{F}(\psi + h \varphi) - \mathcal{F}(\psi)}{h}.$$
Now we introduce a potential $V(\Psi)$ with $\Psi = (\psi_1, \psi_2, \ldots, \psi_N)$ as follows:

$$V(\Psi) = -\frac{\kappa}{2N} \sum_{j,k=1}^{N} \langle \psi_j | \psi_k \rangle = -\frac{\kappa}{2N} \sum_{j,k=1}^{N} \int_{\mathbb{T}^d} \overline{\psi_j} \psi_k dx$$

$$= -\frac{\kappa}{2N} \sum_{j,k=1}^{N} \int_{\mathbb{T}^d} \left( \text{Re}(\psi_j) \text{Re}(\psi_k) + \text{Im}(\psi_j) \text{Im}(\psi_k) \right) dx$$

$$- \frac{\kappa i}{2N} \sum_{j,k=1}^{N} \int_{\mathbb{T}^d} \left( \text{Re}(\psi_j) \text{Im}(\psi_k) - \text{Re}(\psi_k) \text{Im}(\psi_j) \right) dx$$

$$= -\frac{\kappa}{2N} \sum_{j,k=1}^{N} \int_{\mathbb{T}^d} \left( \text{Re}(\psi_j) \text{Re}(\psi_k) + \text{Im}(\psi_j) \text{Im}(\psi_k) \right) dx.$$

Note that the potential $V$ is an analytic function in terms of their arguments $\psi_j$’s.

**Lemma 3.3** Let $V$ be a potential defined in (3.13). Then, the functional derivative of $\frac{\delta V}{\delta \psi_i}$ is explicitly given as follows. Then one has

$$\frac{\delta V(\Psi)}{\delta \psi_i}(\varphi) = -\kappa \langle \psi_i | \varphi \rangle, \quad \forall \ varphi \in L^2(\mathbb{T}^d), \quad i = 1, \ldots, N.$$

**Proof** For fixed $i \in \{1, \ldots, N\}$, we rewrite (3.13) as

$$V(\Psi) = -\frac{\kappa}{2N} \int_{\mathbb{T}^d} \left( (\text{Re}(\psi_i))^2 + (\text{Im}(\psi_i))^2 \right) dx$$

$$- \frac{\kappa}{2N} \sum_{k \neq i} \int_{\mathbb{T}^d} \left( \text{Re}(\psi_i) \text{Re}(\psi_k) + \text{Im}(\psi_i) \text{Im}(\psi_k) \right) dx$$

$$- \frac{\kappa}{2N} \sum_{j \neq i} \int_{\mathbb{T}^d} \left( \text{Re}(\psi_j) \text{Re}(\psi_i) + \text{Im}(\psi_j) \text{Im}(\psi_i) \right) dx$$

$$- \frac{\kappa}{2N} \sum_{k \neq i, j \neq i} \int_{\mathbb{T}^d} \left( \text{Re}(\psi_j) \text{Re}(\psi_k) + \text{Im}(\psi_j) \text{Im}(\psi_k) \right) dx.$$

We set

$$\Phi = (0, \ldots, 0, \varphi, 0, \ldots, 0).$$

first $(i - 1)$
Then, it follows from (3.14) that

\[ V(\Psi + h\Phi) = -\frac{\kappa}{2N} \int_{\mathbb{T}^d} \left( (\text{Re}(\psi_i + h\varphi))^2 + (\text{Im}(\psi_i + h\varphi))^2 \right) dx \]

\[ - \frac{\kappa}{2N} \sum_{k \neq i} \int_{\mathbb{T}^d} \left( \text{Re}(\psi_i + h\varphi)\text{Re}(\psi_k) + \text{Im}(\psi_i + h\varphi)\text{Im}(\psi_k) \right) dx \]

\[ - \frac{\kappa}{2N} \sum_{j \neq i} \int_{\mathbb{T}^d} \left( \text{Re}(\psi_j)\text{Re}(\psi_i + h\varphi) + \text{Im}(\psi_j)\text{Im}(\psi_i + h\varphi) \right) dx \]

\[ - \frac{\kappa}{2N} \sum_{k \neq i, j \neq i} \int_{\mathbb{T}^d} \left( \text{Re}(\psi_j)\text{Re}(\psi_k) + \text{Im}(\psi_j)\text{Im}(\psi_k) \right) dx \]

\[ \quad \quad = - \frac{\kappa h}{2N} \sum_k \int_{\mathbb{T}^d} \left( \text{Re}(\varphi)\text{Re}(\psi_k) + \text{Im}(\varphi)\text{Im}(\psi_k) \right) dx \]

\[ - \frac{\kappa h}{2N} \sum_j \int_{\mathbb{T}^d} \left( \text{Re}(\psi_j)\text{Re}(\varphi) + \text{Im}(\psi_j)\text{Im}(\varphi) \right) dx \]

\[ - \frac{\kappa h^2}{2N} \int_{\mathbb{T}^d} \left( (\text{Re}(\phi))^2 + (\text{Im}(\phi))^2 \right) dx \]

\[ + V(\Psi). \]

This yields

\[ \frac{\delta V(\Psi)}{\delta \psi_i}(\varphi) = -\frac{\kappa}{N} \sum_k \int_{\mathbb{T}^d} \left( \text{Re}(\psi_k) - i\text{Im}(\psi_k) \right) \cdot \left( \text{Re}(\varphi) + i\text{Im}(\varphi) \right) dx \]

\[ = -\frac{\kappa}{N} \sum_k \int_{\mathbb{T}^d} \overline{\psi_k} \varphi dx = -\kappa \int_{\mathbb{T}^d} \overline{\psi_c} \varphi dx = -\kappa (\psi_c | \varphi). \]

Next, we show that the SL model (4.8) with \( \mathcal{H}_j = 0, j = 1, \ldots, N \) can be rewritten as a gradient flow. For this, we rewrite (4.8) with \( H_j = 0 \) as an mean-field form:

\[ \partial_t \psi_j = \kappa_0 (\psi_j - (\psi_c | \psi_j) \psi_j) + \kappa_1 ((\psi_j | \psi_c) - (\psi_c | \psi_j)) \psi_j. \]  

(3.15)

**Proposition 3.4** (Gradient flow formulation) Suppose the coupling strengths \( \kappa_0 \) and \( \kappa_1 \) satisfy

\[ \kappa_1 = -\frac{1}{2} \kappa_0. \]  

(3.16)

Then, the SL model (3.15) is a gradient flow with analytical potential \( V \) in (3.13):

\[ \partial_t \psi_j = -\frac{\delta V(\Psi)}{\delta \psi_j}(\psi_j) \bigg|_{\mathcal{H}^d} \quad t > 0, \quad j = 1, \ldots, N. \]

**Proof** Recall that

\[ \mathcal{H}^d = \{ \psi \in L^2(\mathbb{T}^d) : \| \psi \| = 1 \}. \]
Then, we can define the projection from $L^2(\mathbb{R}^d)$ onto the tangent plane of $\mathbb{H}S^d$ at $\psi$ as follows:

$$\phi \bigg|_{T_{\psi} \mathbb{H}S^d} = \phi - \frac{1}{2}(\langle \phi, \psi \rangle + \langle \psi, \phi \rangle)\psi, \quad \forall \phi \in L^2(\mathbb{T}^d).$$

By the assumptions (3.16) and Lemma 3.3, system (3.15) can be rewritten as

$$\partial_t \psi_j = \kappa_0(\psi_c - \langle \psi_c | \psi_j \rangle \psi_j) - \frac{1}{2} \kappa_0((\psi_j | \psi_c) - \langle \psi_c | \psi_j \rangle) \psi_j$$

$$= \kappa_0 \left( \psi_c - \frac{1}{2} \left( (\psi_c | \psi_j) + \langle \psi_j | \psi_c \rangle \right) \psi_j \right)$$

$$= \kappa_0 \psi_c \bigg|_{T_{\psi_j} \mathbb{H}S^d} = -\frac{\delta V(\Psi)}{\delta \psi_j} \bigg|_{T_{\psi_j} \mathbb{H}S^d}.$$ 

As a direct application of Proposition 3.4, we have the convergence of the flow $e^{-Ht} \psi_j$ as $t \to \infty$.

**Corollary 3.1** Suppose coupling strengths $\kappa_0, \kappa_1$ satisfy

$$\kappa_1 = -\frac{1}{2} \kappa_0,$$

and let $\psi_j = \psi_j(t, x)$ be a solution to system (3.15) with initial data $\{\psi_j^{in}\}$. Then, one has

$$\lim_{t \to \infty} \|\partial_t \psi_j(t)\| = 0.$$

**Proof** We use the result of Proposition 3.4 to see

$$\frac{dV}{dt} = \sum_j \delta V(\Psi) \frac{\delta \psi_j}{\delta \psi_j} \partial_t \psi_j = -\sum_j \left( \frac{\delta V(\Psi)}{\delta \psi_j} \bigg|_{T_{\psi_j} \mathbb{H}S^d} \right) \left( \frac{\delta V(\Psi)}{\delta \psi_j} \bigg|_{T_{\psi_j} \mathbb{H}S^d} \right)$$

$$= -\sum_j \left\| \frac{\delta V}{\delta \psi_j} \bigg|_{T_{\psi_j} \mathbb{H}S^d} \right\|^2 = -\sum_j \|\partial_t \psi_j\|^2 \leq 0.$$

Similarly, we can obtain the boundedness of the second derivative $\frac{d^2V}{dt^2}$. Then we can apply Barbalat’s lemma (Lemma 3.2) to obtain

$$\lim_{t \to \infty} \|\partial_t \psi_j(t)\| = 0.$$

**4 The Schrödinger–Lohe Matrix Model**

In this section, we briefly discuss basic properties to the SL model and present the infinite-dimensional analog of the complex LS model.
4.1 A Bridge Between the SL and LS Models

For a given Hermitian Hamiltonian $\mathcal{H}(x, p)$, let $\{\phi_{\alpha_1}\}$ and $\{E_{\alpha_1}\}$ be an orthonormal basis consisting of eigenfunctions and their corresponding eigenvalues for $\mathcal{H}$:

$$\mathcal{H}\phi_{\alpha_1} = E_{\alpha_1}\phi_{\alpha_1}, \quad \alpha_1 = 1, 2, \ldots .$$

Then the standing wave solution $\Phi_{\alpha_1}(t, x) := e^{-iE_{\alpha_1}t}\phi_{\alpha_1}(x)$ satisfies the linear Schrödinger equation:

$$i\partial_t \Phi_{\alpha_1} = \mathcal{H}\Phi_{\alpha_1}, \quad \alpha_1 = 1, 2, \ldots ,$$

and we set $\psi_j$ to be a linear combination of $\{\Phi_{\alpha_1}\}_{\alpha_1}$ as follows:

$$\psi_j(t, x) = \sum_{\alpha_1} [v_j(t)]_{\alpha_1} \Phi_{\alpha_1}(t, x), \quad j = 1, \ldots , N. \tag{4.1}$$

Suppose that $\psi_j$ satisfies the SL model with $\|\psi_j\|_2 = 1$:

$$i\partial_t \psi_j = \mathcal{H}\psi_j + \frac{i\kappa}{N} \sum_{k=1}^{N} (\psi_k - \langle \psi_k | \psi_j \rangle \psi_j). \tag{4.2}$$

We use (4.1) to rewrite the L.H.S. of (4.2) to see

$$i\partial_t \psi_j = \sum_{\alpha_1} ([v_j]_{\alpha_1} i\partial_t \Phi_{\alpha_1} + [\dot{v}_j]_{\alpha_1} i\Phi_{\alpha_1}) = \sum_{\alpha_1} ([v_j]_{\alpha_1} \mathcal{H}\Phi_{\alpha_1} + [\dot{v}_j]_{\alpha_1} i\Phi_{\alpha_1})$$

$$= \mathcal{H}\psi_j + i \sum_{\alpha_1} [\dot{v}_j]_{\alpha_1} \Phi_{\alpha_1}. \tag{4.3}$$

Now, we equate (4.2) and (4.3) to get

$$\mathcal{H}\psi_j + i \sum_{\alpha_1} [\dot{v}_j]_{\alpha_1} \Phi_{\alpha_1} = \mathcal{H}\psi_j + \frac{i\kappa}{N} \sum_{k=1}^{N} (\psi_k - \langle \psi_k | \psi_j \rangle \psi_j)$$

$$= \mathcal{H}\psi_j + \frac{i\kappa}{N} \sum_{k=1}^{N} \sum_{\alpha_1} ([v_k]_{\alpha_1} - \langle \psi_k | \psi_j \rangle [v_j]_{\alpha_1}) \Phi_{\alpha_1}.$$ 

This yields

$$\sum_{\alpha_1} [\dot{v}_j]_{\alpha_1} \Phi_{\alpha_1} = \frac{\kappa}{N} \sum_{k=1}^{N} \sum_{\alpha_1} ([v_k]_{\alpha_1} - \langle \psi_k | \psi_j \rangle [v_j]_{\alpha_1}) \Phi_{\alpha_1}. \tag{4.4}$$

Since $\{\Phi_{\alpha_1}\}$ is an orthonormal basis, one has

$$\frac{d}{dt} [v_j]_{\alpha_1} = \frac{\kappa}{N} \sum_{k=1}^{N} ([v_k]_{\alpha_1} - \langle \psi_k | \psi_j \rangle [v_j]_{\alpha_1}), \quad j = 1, \ldots , N, \quad \alpha_1 = 1, 2, \ldots . \tag{4.4}$$

For each $j = 1, \ldots , N$, we define an infinite complex vector in $(\ell^\infty \cap \ell^2)(\mathbb{Z}_+)$:

$$v_j = ([v_j]_1, [v_j]_2, \ldots ).$$
We use the definition of $\langle \cdot | \cdot \rangle$ to get

$$\langle \psi_k | \psi_j \rangle = \sum_{\alpha_1, \beta_1} [v_k]_{\alpha_1} [v_j]_{\beta_1} = \sum_{\alpha_1, \beta_1} [\bar{v}_k]_{\alpha_1} [v_j]_{\beta_1} \langle \Phi_{\alpha_1} | \Phi_{\beta_1} \rangle$$

$$= \sum_{\alpha_1} [\bar{v}_k]_{\alpha_1} [v_j]_{\alpha_1} = \langle v_k | v_j \rangle. \quad \text{(4.5)}$$

Finally, we combine (4.4) and (4.5) to derive an infinite-dimensional counterpart for the complex Lohe sphere model on $(\ell^2 \cap \ell^\infty)(\mathbb{Z}_+)$:

$$\dot{v}_j = \frac{\kappa}{N} \sum_{k=1}^{N} (v_k - \langle v_k | v_j \rangle v_j), \quad j = 1, \ldots, N. \quad \text{(4.6)}$$

Before we close this subsection, consider a possibility whether we can deal with heterogeneous hamiltonians or not. In fact, the preceding argument connecting the SL model and infinite-dimensional LHS model can be further applied for the generalized SL model with heterogeneous hamiltonians. More precisely, we consider the situation in which $\mathcal{H}_1, \ldots, \mathcal{H}_N$ can be simultaneously diagonalizable. From the well-known fact from linear algebra, $\mathcal{H}_i$ are simultaneously diagonalizable if and only if $\mathcal{H}_i$ and $\mathcal{H}_j$ commute for all $i, j \in \{1, 2, \ldots, N\}$. That means $\mathcal{H}_i$ commute, we can follow similar preceding arguments to obtain the infinite vector form of Schrödinger–Lohe model.

Next, we will try to find specific conditions of $\{\mathcal{H}_i\}$ for

$$\mathcal{H}_i = -\Delta_x + V_i(x), \quad [\mathcal{H}_i, \mathcal{H}_j] = 0, \quad 1 \leq i, j \leq N.$$ 

Let $\phi \in C^2(\mathbb{T}^d)$ be a test function. Then by direct calculation, one has

$$[\mathcal{H}_i, \mathcal{H}_j]\phi = (-\Delta + V_i)(-\Delta + V_j)\phi - (-\Delta + V_j)(-\Delta + V_i)\phi$$

$$= (-\Delta + V_i)(-\Delta \phi + V_j\phi) - (-\Delta + V_j)(-\Delta \phi + V_i\phi)$$

$$= (-\Delta + V_i)(-\Delta \phi + V_j\phi) + V_i V_j \phi$$

$$= (\Delta^2 \phi - V_i \Delta \phi - \Delta (V_j \phi) + V_i V_j \phi)$$

$$= (V_j - V_i) \Delta \phi - \Delta (V_j - V_i) \phi$$

$$= \Delta (V_j - V_i) \phi - \nabla (V_j - V_i) \cdot \nabla \phi.$$

Hence, the relation $[\mathcal{H}_i, \mathcal{H}_j] = 0$ implies

$$\nabla (V_j - V_i) = 0, \quad 1 \leq i, j \leq N.$$

Again, this implies

$$V_i(x) = V(x) + v_i, \quad i \in \{1, 2, \ldots, N\}$$

for some common potential $V(x)$ and $v_i \in \mathbb{R}$. This case has been studied from [15–17].

### 4.2 The SLM Model

In the previous subsection, we showed that the SL model can be reduced to the extended complex LS model on $(\ell^2 \cap \ell^\infty)(\mathbb{Z}_+)$. Below, we propose a Schrodinger–Lohe type model which can be reduced to the LM model in Sect. 2.2, and study its emergent dynamics in a
priori setting by assuming a global well-posedness of a smooth solution.

Recall the complex LM model for $d_1 \times d_2$ complex matrix $A_j$:

$$
\dot{A}_j - BA_j = \kappa_0(A_c A_j^* A_j - A_j A_c^* A_j) + \kappa_1(A_j A_j^* A_c - A_c A_j^* A_j).
$$

(4.7)

free flow cubic mean-field interactions

In the sequel, we present the Schrödinger–Lohe type model which can be associated with the generalized Lohe matrix model (4.7).

Next, we introduce the Schrödinger–Lohe matrix model (SLM) for a homogeneous ensemble. First, we set

$$
\mathcal{H}(x_1, x_2, p_1, p_2) := \mathcal{H}_1(x_1, p_1) + \mathcal{H}_2(x_2, p_2), \quad (x_1, x_2) \in \mathbb{T}^d \times \mathbb{T}^d,
$$

where the one-body potential is assumed to be continuous for our emergent dynamics. However, for a global well-posedness of classical solutions based on energy method, we might need to assume high regularity of the potential.

For notational simplicity, we suppress $t$-dependence on $\Psi$ and use a handy notation for a partial inner product:

$$
\Psi_j(x_1, x_2) := \Psi_j(t, x_1, x_2), \quad t \geq 0, \quad (x_1, x_2) \in \mathbb{T}^d \times \mathbb{T}^d,
$$

and for $\Psi(x_1, x_2)$ and $\tilde{\Psi}(x_1, x_2)$, we set

$$
\langle \Psi(x_2^*) | \tilde{\Psi}(x_2) \rangle := \int_{\mathbb{T}^d} \tilde{\Psi}(x_1, x_2) \Psi(x_1^*, x_2) dx_1^*,
$$

$$
\langle \Psi(x_1^*) | \tilde{\Psi}(x_1) \rangle := \int_{\mathbb{T}^d} \tilde{\Psi}(x_1, x_2^*) \Psi(x_1^*, x_2^*) dx_2^*.
$$

(4.8)

Now, we propose the Schrödinger–Lohe matrix (SLM) model as follows: for $t > 0$ and $x_i, x_i^* \in \mathbb{T}^d$,

$$
\begin{aligned}
&i\partial_t \Psi_j(x_1, x_2) - \mathcal{H}_j \Psi_j(x_1, x_2) \\
&\quad = i \kappa_0 \int_{\mathbb{T}^d} \left( \Psi_j(x_1^*, x_2^*) \Psi_j(x_1^*, x_2) - \Psi_j(x_1, x_2^*) \Psi_j(x_1^*, x_2) \right) dx_1^* dx_2^* \\
&\quad + i \kappa_1 \int_{\mathbb{T}^d} \left( \Psi_j(x_1, x_2^*) \Psi_j(x_1^*, x_2^*) - \Psi_j(x_1^*, x_2^*) \Psi_j(x_1^*, x_2^*) \right) dx_1^* dx_2^*,
\end{aligned}
$$

$$
\Psi_j \bigg|_{t=0} = \Psi_j^{in}, \quad j = 1, \ldots, N,
$$

(4.9)

where $\Psi_c := \frac{1}{N} \sum_{k=1}^N \Psi_k$.

Under the handy notation (4.8), system (4.9) becomes

$$
\begin{aligned}
&i\partial_t \Psi_j(x_1, x_2) - \mathcal{H}_j \Psi_j(x_1, x_2) \\
&\quad = i \kappa_0 \int_{\mathbb{T}^d} \left( \langle \Psi_j(x_2^*) | \Psi_j(x_2) \rangle \Psi_c(x_1, x_2^*) - \langle \Psi_c(x_2^*) | \Psi_j(x_2) \rangle \right) dx_2^* \\
&\quad + i \kappa_1 \int_{\mathbb{T}^d} \left( \langle \Psi_j(x_2^*) | \Psi_c(x_2) \rangle - \langle \Psi_c(x_2^*) | \Psi_j(x_2) \rangle \right) \Psi_j(x_1, x_2^*) dx_2^*,
\end{aligned}
$$

(4.10)

$$
\Psi_j \bigg|_{t=0} = \Psi_j^{in}, \quad j = 1, \ldots, N.
$$

Since a global well-posedness of (4.9) can be treated using a standard energy method as in [15] for the SL model in a suitable Sobolev space setting, we will focus on the emergent
dynamics in a priori setting. Notice that the R.H.S. of (4.7) and (4.9) are structurally the same.

**Proposition 4.1** Let \( \{ \Psi_j \} \) be a global smooth solution to (4.9) with the initial data \( \| \Psi_j^{in} \| = 1 \). Then \( L^2 \)-norm of \( \Psi_j \) is a conserved quantity:

\[
\frac{d}{dt} \| \Psi_j(t) \| = 0, \quad t > 0, \quad i = 1, \ldots, N.
\]

where

\[
\| \Psi_j(t) \| := \int_{\mathbb{T}^d} |\Psi_j(t, x_1, x_2)|^2 dx_2 dx_1, \quad t \geq 0.
\]

**Proof** By definition of \( \| \Psi_j \|^2 \), one has

\[
\frac{d}{dt} \| \Psi_j \|^2 = \int_{\mathbb{T}^d} (\partial_t \Psi_j(x_1, x_2))\overline{\Psi_j(x_1, x_2)}dx_1 dx_2 + \text{(c.c.)}, \quad (4.11)
\]

where (c.c.) denotes the complex conjugate of the first term.

This yields

\[
\int_{\mathbb{T}^d} (\partial_t \Psi_j(x_1, x_2))\overline{\Psi_j(x_1, x_2)}dx_1 dx_2 = -\int_{\mathbb{T}^d} i\mathcal{H} \Psi_j(x_1, x_2)\overline{\Psi_j(x_1, x_2)}dx_1 dx_2
\]

\[
+ \frac{K_0}{N} \sum_{k=1}^{N} \int_{\mathbb{T}^d} \left( \psi_k(x_1, x_2^*) \overline{\psi_j(x_1^*, x_2^*)} \psi_j(x_1^*, x_2) - \psi_j(x_1, x_2^*) \overline{\psi_j(x_1^*, x_2^*)} \psi_j(x_1^*, x_2) \right)
\]

\[
\times \overline{\psi_j(x_1, x_2)} dx_1^* dx_2^* dx_1 dx_2.
\]

\[
+ \frac{K_1}{N} \sum_{k=1}^{N} \int_{\mathbb{T}^d} \left( \psi_j(x_1, x_2^*) \overline{\psi_j(x_1^*, x_2^*)} \psi_k(x_1^*, x_2) - \psi_j(x_1, x_2^*) \overline{\psi_j(x_1^*, x_2^*)} \psi_j(x_1^*, x_2) \right)
\]

\[
\times \overline{\psi_j(x_1, x_2)} dx_1^* dx_2^* dx_1 dx_2.
\]

We use \( \mathcal{H}^* = \mathcal{H} \), (4.11) and (4.12) to get

\[
\int_{\mathbb{T}^d} (\partial_t \Psi_j(x_1, x_2))\overline{\Psi_j(x_1, x_2)}dx_1 dx_2 + \text{(c.c.)} = 0.
\]

This yields the desired estimate. \( \Box \)

Consider the Cauchy problem to the nonlinear system associated with (4.9):

\[
\begin{cases}
\begin{align*}
i \partial_t \varphi_j(x_1, x_2) &= i \kappa_0 \int_{\mathbb{T}^d} \left( \left( \overline{\varphi_j(x_2)} \varphi_j(x_2) \right) \varphi_c(x_1, x_2^*) - \left( \varphi_c(x_2^*) \overline{\varphi_j(x_2)} \right) \varphi_j(x_1, x_2^*) \right) dx_2^* \\
&+ i \kappa_1 \int_{\mathbb{T}^d} \left( \left( \overline{\varphi(x_2)} \varphi(x_2) \right) - \left( \varphi(x_2^*) \overline{\varphi_j(x_2)} \right) \varphi_j(x_1, x_2^*) \right) dx_2^* ,
\end{align*}
\end{cases}
\]

(4.13)

Now we will show the solution splitting property of the SLM model (4.9).

**Proposition 4.2** Let \( \{ \Psi_j \} \) and \( \{ \varphi_j \} \) be global smooth solutions to (4.9) and (4.13), respectively. Then, one has

\[
\Psi_j(t, x_1, x_2) = e^{-i\mathcal{H}t} \varphi_j(t, x_1, x_2), \quad j = 1, \ldots, N.
\]
**Proof** Since $\mathcal{H}_1(x_1, -i\nabla_{x_1})$ and $\mathcal{H}_2(x_2, -i\nabla_{x_2})$ commute, we have

$$e^{-i\mathcal{H}_1 t} = e^{-i\mathcal{H}_1 t} e^{-i\mathcal{H}_2 t} = e^{-i\mathcal{H}_2 t} e^{-i\mathcal{H}_1 t},$$

- Step B: For $t \in \mathbb{R}_+$, $x_i \in \mathbb{T}^d$, $j = 1, \ldots, N$, we set

$$\varphi_j(t, x_1, x_2) := e^{i\mathcal{H}_t} \Psi_j(t, x_1, x_2), \quad \text{or} \quad \Psi_j(t, x_1, x_2) = e^{-i\mathcal{H}_t} \varphi_j(t, x_1, x_2).$$

Suppose that $\Psi_j$ satisfies system (4.9). Then, it suffices to show that $\varphi_j$ satisfies (4.13). For this, we multiply $e^{i\mathcal{H}_t}$ to (4.9) and compare the L.H.S. and R.H.S of the resulting relation to derive (4.13) for $\varphi_j$.

- **Estimate of L.H.S.** By direct calculation, one has

$$e^{i\mathcal{H}_t} i\partial_t \Psi_j - e^{i\mathcal{H}_t} \mathcal{H} \Psi_j = i\partial_t (e^{i\mathcal{H}_t} \Phi_j) = i\partial_t \varphi_j. \quad (4.14)$$

- **Estimate of R.H.S.** Recall the R.H.S.:

$$i\kappa_0 e^{i\mathcal{H}_t} \int_{\mathbb{T}^d} \left( \left( \langle \Psi_j(x_1^2) | \Psi_j(x_2) \rangle \Psi_e(x_1, x_1^2) - \langle \Psi_e(x_1) | \Psi_j(x_2) \rangle \Psi_j(x_1, x_1^2) \right) dx_2^2 
+ i\kappa_1 e^{i\mathcal{H}_t} \int_{\mathbb{T}^d} \left( \left( \langle \Psi_j(x_1^2) | \Psi_e(x_2) \rangle \Psi_j(x_1, x_1^2) - \langle \Psi_e(x_2) | \Psi_j(x_2) \rangle \Psi_j(x_1, x_2^2) \right) dx_2^2 \right) \right). \quad (4.15)$$

Then, we use the isometry of $e^{i\mathcal{H}_t}$ to rewrite (4.15) as

$$i\kappa_0 \int_{\mathbb{T}^d} \left( \langle \varphi_j(x_1^2) | \varphi_j(x_2) \rangle \varphi_e(x_1, x_1^2) - \langle \varphi_e(x_1) | \varphi_j(x_2) \rangle \varphi_j(x_1, x_1^2) \right) dx_2^2 
+ i\kappa_1 \int_{\mathbb{T}^d} \left( \langle \varphi_j(x_1^2) | \varphi_e(x_2) \rangle \varphi_j(x_1, x_1^2) - \langle \varphi_e(x_2) | \varphi_j(x_2) \rangle \varphi_j(x_1, x_2^2) \right) dx_2^2. \quad (4.16)$$

Finally, we combine (4.14) and (4.16) to get the desired system (4.13). \hfill \square

### 4.3 Reduction to the LM Model

Next, we discuss how system (4.9) can be reduced to the generalized Lohe matrix model (2.4) using the same strategy in Sect. 4.1.

Suppose that $\mathcal{H}$ is additive without any interaction Hamiltonians:

$$\mathcal{H}(x_1, x_2, -i\nabla_{x_1}, -i\nabla_{x_2}) = \mathcal{H}_1(x_1, -i\nabla_{x_1}) + \mathcal{H}_2(x_2, -i\nabla_{x_2}).$$

It follows from Chapter 11 of [26] that we know that the eigenfunctions of the Hermitian operator $\mathcal{H}$ forms the orthonormal basis. Let $\{\phi_{a_1}^1(x_1)\}_{a_1=1}^\infty$ and $\{\phi_{a_2}^2(x_2)\}_{a_2=1}^\infty$ be two orthonormal basis consisting of eigenfunctions of $\mathcal{H}_i$:

$$\mathcal{H}_1 \phi_{a_1}^1 = E_{a_1}^1 \phi_{a_1}^1 \quad \text{and} \quad \mathcal{H}_2 \phi_{a_2}^2 = E_{a_2}^2 \phi_{a_2}^2.$$

Now, we introduce standing wave solutions $\Phi_{a_1}^1$ and $\Phi_{a_2}^2$ as follows:

$$\Phi_{a_1}^1(t, x_1) := e^{-iE_{a_1}^1 t} \phi_{a_1}^1(x_1) \quad \text{and} \quad \Phi_{a_2}^2(t, x_2) := e^{-iE_{a_2}^2 t} \phi_{a_2}^2(x_2).$$

Then, it is easy to see

$$i\partial_t \Phi_{a_1}^1 = E_{a_1}^1 \Phi_{a_1}^1, \quad i\partial_t \Phi_{a_2}^2 = E_{a_2}^2 \Phi_{a_2}^2,$$

$$\mathcal{H}_1 \Phi_{a_1}^1 = E_{a_1}^1 \Phi_{a_1}^1, \quad \mathcal{H}_2 \Phi_{a_2}^2 = E_{a_2}^2 \Phi_{a_2}^2. \quad (4.17)$$

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Due to (4.17), the tensor product
\[
(\Phi_{\alpha_1}^1 \otimes \Phi_{\alpha_2}^2)(t, x_1, x_2) := \Phi_{\alpha_1}^1(t, x_1) \Phi_{\alpha_2}^2(t, x_2)
\]
satisfies two-dimensional linear Schrödinger equation with the Hamiltonian \( \mathcal{H} \):
\[
i \partial_t (\Phi_{\alpha_1}^1 \otimes \Phi_{\alpha_2}^2) = (E_{\alpha_1}^1 + E_{\alpha_2}^2)(\Phi_{\alpha_1}^1 \otimes \Phi_{\alpha_2}^2) = \mathcal{H}(\Phi_{\alpha_1}^1 \otimes \Phi_{\alpha_2}^2). \tag{4.18}
\]
Now, we expand \( \Psi_j = \Psi_j(t, x_1, x_2) \) in terms of the basis \( \{\Phi_{\alpha_1}^1 \otimes \Phi_{\alpha_2}^2\}_{\alpha_1, \alpha_2} \):
\[
\Psi_j = \sum_{\alpha_1, \alpha_2} [A_j(t)]_{\alpha_1 \alpha_2} (\Phi_{\alpha_1}^1 \otimes \Phi_{\alpha_2}^2), \tag{4.19}
\]
Here we set \( A_j = ([A_j]_{\alpha \beta}) \) to be an infinite matrix (see a review paper [35] for theory of infinite matrices).

**Proposition 4.3** Let \( \{\Psi_j\} \) be a global smooth solution to (4.9), and let \( A_j = ([A_j]_{\alpha \beta}) \) be an infinite matrix whose elements is given as a coefficient in (4.19). Then, the matrix ensemble \( \{A_j\} \) satisfies the generalized Lohe matrix model on \( (l^\infty \cap l^2)(\mathbb{Z}_+^2) \):
\[
\dot{A}_j = \kappa_0(A_c A_j^* A_j - A_j A_c^* A_c) + \kappa_1(A_j A_j^* A_c - A_j A_c^* A_j), \quad j = 1, \ldots, N.
\]

**Proof** We differentiate (4.19) with respect to \( t \) and use (4.18) to derive
\[
i \partial_t \Psi_j = \sum_{\alpha_1, \alpha_2} \left[ [\dot{A}_j]_{\alpha_1 \alpha_2} i(\Phi_{\alpha_1}^1 \otimes \Phi_{\alpha_2}^2) + [A_j]_{\alpha_1 \alpha_2} i \partial_t (\Phi_{\alpha_1}^1 \otimes \Phi_{\alpha_2}^2) \right]
= \sum_{\alpha_1, \alpha_2} \left[ [\dot{A}_j]_{\alpha_1 \alpha_2} i(\Phi_{\alpha_1}^1 \otimes \Phi_{\alpha_2}^2) + [A_j]_{\alpha_1 \alpha_2} \mathcal{H}(\Phi_{\alpha_1}^1 \otimes \Phi_{\alpha_2}^2) \right]. \tag{4.20}
\]
We substitute (4.19) into (4.9) to find
\[
i \sum_{\alpha_1, \alpha_2} [\dot{A}_j]_{\alpha_1 \alpha_2} (\Phi_{\alpha_1}^1 \otimes \Phi_{\alpha_2}^2)(x_1, x_2)
= i \kappa_0 \int_{\mathbb{T}^d} \left( \left< \Psi_j(x_2^*) \Psi_j(x_2) \right> \Psi_c(x_1, x_2^*) - \left< \Psi_c(x_2^*) \Psi_j(x_2) \right> \Psi_j(x_1, x_2^*) \right) dx_2^*
=: I_{11}
+ i \kappa_1 \int_{\mathbb{T}^d} \left( \left< \Psi_j(x_2^*) \Psi_c(x_2) \right> \Psi_j(x_1, x_2^*) - \left< \Psi_c(x_2^*) \Psi_j(x_2) \right> \Psi_j(x_1, x_2^*) \right) dx_2^*. \tag{4.21}
\]
Below, we estimate \( I_{11} \) separately.
• Estimate of $I_{11}$ By direct estimate, one has

$$I_{11} = \int_{\mathbb{R}^d} \left( \left| \langle \Psi_j(x^*_1) \rangle \Psi_j(x^*_2) \Phi_c(x_1, x^*_2) \right| - \left| \langle \Phi_c(x^*_1) \rangle \Psi_j(x_1, x^*_2) \right| \right) dx^*_2$$

$$= \sum_{\alpha, \beta, \gamma, \delta, \epsilon, \eta} \int_{\mathbb{R}^d} \left( [A_c]_{\alpha \beta} [\tilde{A}_j]_{\gamma \delta} [A_j]_{\epsilon \eta} - [A_j]_{\alpha \beta} [\tilde{A}_c]_{\gamma \delta} [A_j]_{\epsilon \eta} \right) \times \Phi_{\alpha}^1(x_1) \Phi_{\beta}^2(x_2) \Phi_{\gamma}^1(x_1) \Phi_{\delta}^2(x_2) \Phi_{\epsilon}^1(x_2) \Phi_{\eta}^2(x_2) dx_1^* dx_2^*$$

$$= \sum_{\alpha, \beta, \gamma, \eta} \left( [A_c]_{\alpha \beta} [\tilde{A}_j]_{\gamma \eta} [A_j]_{\eta \gamma} - [A_j]_{\alpha \beta} [\tilde{A}_c]_{\gamma \eta} [A_j]_{\eta \gamma} \right) \Phi_{\alpha}^1(x_1) \Phi_{\beta}^2(x_2) \Phi_{\gamma}^1(x_1) \Phi_{\eta}^2(x_2) (x_1, x_2).$$

(4.22)

• Estimate of $I_{12}$ Similarly, one has

$$I_{12} = \sum_{\alpha, \eta} \left( [A_j]_{\alpha \beta} A_c - [A_j]_{\alpha \beta} A_c \right) \Phi_{\alpha}^1 \otimes \Phi_{\eta}^2 (x_1, x_2).$$

(4.23)

We combine (4.21), (4.22) and (4.23) to get

$$\sum_{\alpha_1, \alpha_2} [\dot{A}_j(t)]_{\alpha_1 \alpha_2} i(\Phi_{\alpha_1}^1 \otimes \Phi_{\alpha_2}^2)$$

$$= i \sum_{\alpha, \eta} \left( \kappa_0 [A_c A^*_j A_j - A_j A^*_c A_j]_{\alpha \eta} + \kappa_1 [A_c A^*_j A_j - A_j A^*_c A_j]_{\alpha \eta} \right) (\Phi_{\alpha}^1 \otimes \Phi_{\eta}^2).$$

Then, we use the orthonormality of \{\Phi_{\alpha_1}^1 \otimes \Phi_{\alpha_2}^2\} to see

$$[\dot{A}_j]_{\alpha \beta} = \kappa_0 [A_c A^*_j A_j - A_j A^*_c A_j]_{\alpha \beta} + \kappa_1 [A_c A^*_j A_j - A_j A^*_c A_j]_{\alpha \beta}.$$

This yields

$$\dot{A}_j = \kappa_0 (A_c A^*_j A_j - A_j A^*_c A_j) + \kappa_1 (A_j A^*_j A_c - A_j A^*_c A_j).$$

4.4 Emergent Dynamics

In this subsection, we introduce an order parameter and study emergent dynamics of (4.9). For a given ensemble \{\Psi_i = \Psi_i(x_1, x_2)\}, we set

$$\Psi_c := \frac{1}{N} \sum_{k=1}^{N} \Psi_k \quad \text{and} \quad R := \|\Psi_c\|.$$
Lemma 4.1 Let \( \{ \Psi_i \} \) be a global smooth solution to (4.9). Then, the order parameter \( R \) satisfies

(i) \( \frac{dR^2}{dt} = \frac{\kappa_0}{N} \sum_{i=1}^{N} \int_{\mathbb{T}^{2d}} \left| \int_{\mathbb{T}^{2d}} (\Psi_i(x_1, x_2) \Psi_i(x_1^*, x_2) - \bar{\Psi}_i(x_1, x_2) \bar{\Psi}_i(x_1^*, x_2)) dx_1 dx_2 \right|^2 dx_1 dx_1^* \)

\( + \frac{\kappa_1}{N} \sum_{i=1}^{N} \int_{\mathbb{T}^{2d}} \left| \int_{\mathbb{T}^{2d}} (\Psi_i(x_1, x_2) \Psi_i(x_1, x_2^*) - \bar{\Psi}_i(x_1, x_2) \bar{\Psi}_i(x_1, x_2^*)) dx_1 \right|^2 dx_2 dx_2^* \geq 0. \)

(ii) \( \frac{d}{dt} \sum_{i,j=1}^{N} \| \Psi_i - \Psi_j \|^2 \)

\( = -2\kappa_0 N \sum_{i=1}^{N} \int_{\mathbb{T}^{2d}} \int_{\mathbb{T}^{2d}} (\Psi_i(x_1, x_2) \Psi_i(x_1^*, x_2) - \bar{\Psi}_i(x_1, x_2) \bar{\Psi}_i(x_1^*, x_2)) dx_2 \right|^2 dx_1 dx_1^* \)

\( - 2\kappa_1 N \sum_{i=1}^{N} \int_{\mathbb{T}^{2d}} \int_{\mathbb{T}^{2d}} (\Psi_i(x_1, x_2) \Psi_i(x_1, x_2^*) - \bar{\Psi}_i(x_1, x_2) \bar{\Psi}_i(x_1, x_2^*)) dx_1 \right|^2 dx_2 dx_2^* \leq 0. \)

**Proof** (i) Note that

\[ \frac{d}{dt} \langle \Psi_i | \Psi_j \rangle = \langle \partial_t \Psi_i | \Psi_j \rangle + \langle \Psi_i | \partial_t \Psi_j \rangle. \]

Below, we estimate the second term \( \langle \Psi_i | \partial_t \Psi_j \rangle \). By direct calculation, one has

\[ \langle \Psi_i | \partial_t \Psi_j \rangle = \int_{\mathbb{T}^{2d}} \bar{\Psi}_i(x_1, x_2) \partial_t \Psi_j(x_1, x_2) dx_1 dx_2 = \int_{\mathbb{T}^{2d}} (-i) \bar{\Psi}_i(x_1, x_2) (\mathcal{H} \Psi_j)(x_1, x_2) dx_1 dx_2 \]

\( + \kappa_0 \int_{\mathbb{T}^{2d}} \bar{\Psi}_i(x_1, x_2) \psi_e(x_1, x_2^*) \psi_j(x_1^*, x_2^*) \psi_j(x_1^*, x_2^*) dx_1 dx_2 dx_1^* dx_2^* \)

\( - \kappa_0 \int_{\mathbb{T}^{2d}} \bar{\Psi}_i(x_1, x_2) \psi_e(x_1, x_2^*) \psi_j(x_1^*, x_2^*) \psi_j(x_1^*, x_2^*) dx_1 dx_2 dx_1^* dx_2^* \]

\( + \kappa_1 \int_{\mathbb{T}^{2d}} \bar{\Psi}_i(x_1, x_2) \psi_j(x_1, x_2^*) \psi_j(x_1^*, x_2^*) \psi_e(x_1^*, x_2) dx_1 dx_2 dx_1^* dx_2^* \)

\( - \kappa_1 \int_{\mathbb{T}^{2d}} \bar{\Psi}_i(x_1, x_2) \psi_j(x_1, x_2^*) \psi_j(x_1^*, x_2^*) \psi_e(x_1^*, x_2) dx_1 dx_2 dx_1^* dx_2^*. \)

The term \( T_{22}^{ij} \) will be cancelled with a similar term in \( \langle \partial_t \Psi_i | \Psi_j \rangle \) due to the Hermitian property of \( \mathcal{H} \). Then we have

\[ \frac{d}{dt} \sum_{i,j=1}^{N} \langle \Psi_i | \Psi_j \rangle = \kappa_0 \sum_{i,j=1}^{N} \left( T_{22}^{ij} + T_{22}^{ji} - T_{23}^{ij} - T_{23}^{ji} \right) + \kappa_1 \sum_{i,j=1}^{N} \left( T_{24}^{ij} + T_{24}^{ji} - T_{25}^{ij} - T_{25}^{ji} \right). \]
The other terms $\mathcal{I}_{2k}^{ij}$, $k = 2, \ldots, 5$ can be treated as follows.

\[
\sum_{i,j=1}^{N} \mathcal{I}_{22}^{ij} = N \sum_{i=1}^{N} \int_{\mathbb{T}^d} \overline{\Psi_c(x_1, x_2)} \psi_i(x_1, x_2) \overline{\psi_i(x_1^*, x_2^*)} \psi_i(x_1^*, x_2) dx_1 dx_2 dx_1^* dx_2^*,
\]
\[
\sum_{i,j=1}^{N} \mathcal{I}_{23}^{ij} = N \sum_{i=1}^{N} \int_{\mathbb{T}^d} \overline{\psi_i(x_1, x_2)} \overline{\psi_i(x_1^*, x_2^*)} \overline{\psi_i(x_1^*, x_2)} \psi_i(x_1^*, x_2) dx_1 dx_2 dx_1^* dx_2^*,
\]
\[
\sum_{i,j=1}^{N} \mathcal{I}_{24}^{ij} = N \sum_{i=1}^{N} \int_{\mathbb{T}^d} \overline{\psi_i(x_1, x_2)} \overline{\psi_i(x_1^*, x_2^*)} \overline{\psi_i(x_1^*, x_2)} \psi_i(x_1^*, x_2) dx_1 dx_2 dx_1^* dx_2^*,
\]
\[
\sum_{i,j=1}^{N} \mathcal{I}_{25}^{ij} = N \sum_{i=1}^{N} \int_{\mathbb{T}^d} \overline{\psi_i(x_1, x_2)} \psi_i(x_1, x_2^*) \overline{\psi_i(x_1^*, x_2^*)} \psi_i(x_1^*, x_2) dx_1 dx_2 dx_1^* dx_2^*.
\]

Note that

\[
\sum_{i,j=1}^{N} \left( \mathcal{I}_{22}^{ij} + \overline{\mathcal{I}_{22}^{ij}} - \mathcal{I}_{23}^{ij} - \overline{\mathcal{I}_{23}^{ij}} \right)
\]
\[
= -N \sum_{i=1}^{N} \int_{\mathbb{T}^d} \left( \overline{\psi_i(x_1, x_2)} \psi_i(x_1^*, x_2) - \overline{\psi_i(x_1, x_2)} \psi_i(x_1^*, x_2) \right) dx_1 dx_2 dx_1^* dx_2^* 
\]
\[
\times \left( \overline{\psi_i(x_1^*, x_2^*)} \psi_i(x_1^*, x_2^*) - \overline{\psi_i(x_1^*, x_2^*)} \psi_i(x_1^*, x_2^*) \right) dx_1 dx_2 dx_1^* dx_2^*
\]
\[
= -N \sum_{i=1}^{N} \int_{\mathbb{T}^d} \left( \int_{\mathbb{T}^d} \left( \overline{\psi_i(x_1, x_2)} \psi_i(x_1^*, x_2) - \overline{\psi_i(x_1, x_2)} \psi_i(x_1^*, x_2) \right) dx_2 \right) dx_1 dx_1^* dx_2^*
\]
\[
\times \left( \int_{\mathbb{T}^d} \left( \overline{\psi_i(x_1^*, x_2^*)} \psi_i(x_1^*, x_2^*) - \overline{\psi_i(x_1^*, x_2^*)} \psi_i(x_1^*, x_2^*) \right) dx_2^* \right) dx_2 dx_2^* dx_1^* 
\]
\[
= N \sum_{i=1}^{N} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \left( \overline{\psi_i(x_1, x_2)} \psi_i(x_1^*, x_2) - \overline{\psi_i(x_1, x_2)} \psi_i(x_1^*, x_2) \right) dx_2 \left( \overline{\psi_i(x_1^*, x_2^*)} \psi_i(x_1^*, x_2^*) - \overline{\psi_i(x_1^*, x_2^*)} \psi_i(x_1^*, x_2^*) \right) dx_2^* 
\]
\[
\geq 0.
\]

Similarly, we have

\[
\sum_{i,j=1}^{N} \left( \mathcal{I}_{24}^{ij} + \overline{\mathcal{I}_{24}^{ij}} - \mathcal{I}_{25}^{ij} - \overline{\mathcal{I}_{25}^{ij}} \right)
\]
\[
= N \sum_{i=1}^{N} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \left( \overline{\psi_i(x_1, x_2)} \psi_i(x_1^*, x_2) - \overline{\psi_i(x_1, x_2)} \psi_i(x_1^*, x_2) \right) dx_2 dx_2^* dx_1^* 
\]
\[
\geq 0.
\]

Finally we have

\[
\frac{d}{dt} \left( \sum_{i,j=1}^{N} \langle \psi_i | \psi_j \rangle \right)
\]
\[
= \kappa_0 N \sum_{i=1}^{N} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \left( \overline{\psi_i(x_1, x_2)} \psi_i(x_1^*, x_2) - \overline{\psi_i(x_1, x_2)} \psi_i(x_1^*, x_2) \right) dx_2 dx_2^* 
\]
\[
\int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \left( \overline{\psi_i(x_1, x_2)} \psi_i(x_1^*, x_2^*) - \overline{\psi_i(x_1, x_2)} \psi_i(x_1^*, x_2^*) \right) dx_2 dx_2^* dx_1^* 
\]
\[
+ \kappa_1 N \sum_{i=1}^{N} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \left( \overline{\psi_i(x_1, x_2)} \psi_i(x_1^*, x_2) - \overline{\psi_i(x_1, x_2)} \psi_i(x_1^*, x_2) \right) dx_2 dx_2^* dx_1^* 
\]
\[
\geq 0.
\]
(ii) By direct calculation, one has
\[
\frac{d}{dt} \left( \sum_{i,j=1}^{N} \| \Psi_i - \Psi_j \|^2 \right) = \frac{d}{dt} \left( \sum_{i,j=1}^{N} (2 - \langle \Psi_i | \Psi_j \rangle - \langle \Psi_j | \Psi_i \rangle) \right) = -2 \frac{d}{dt} \left( \sum_{i,j=1}^{N} \langle \Psi_i | \Psi_j \rangle \right)
\]
\[
= -2k_0 N \sum_{i=1}^{N} \int_{\mathbb{T}^{2d}} \int_{\mathbb{T}^d} (\Psi_i(x_1, x_2) \Psi_j(x_1^*, x_2) - \overline{\Psi_j(x_1, x_2)} \Psi_i(x_1^*, x_2)) dx_2 \, dx_1^* \quad (4.24)
\]
\[
= -2k_1 N \sum_{i=1}^{N} \int_{\mathbb{T}^{2d}} \int_{\mathbb{T}^d} (\Psi_i(x_1, x_2) \Psi_j(x_1^*, x_2) - \overline{\Psi_j(x_1, x_2)} \Psi_i(x_1^*, x_2)) dx_1 \, dx_2^* \leq 0.
\]

Next, we show that \( \frac{d^2}{dt^2} R^2 \) is uniformly bounded in \( t \) so that \( \frac{d}{dt} R^2 \) is uniformly continuous.

**Lemma 4.2** Let \( \{ \Psi_j \}_{j=1}^{N} \) be a global smooth solution of (4.9) satisfying a priori condition \( \| \mathcal{H} \Psi_j \| < C, \quad j = 1, \ldots, N \), for some positive constant \( C \). Then, the second derivative of \( R^2 = \| \Psi_c \|^2 \) is uniformly bounded in time. Furthermore, we have
\[
\lim_{t \to \infty} \frac{d}{dt} R(t)^2 = 0,
\]
where \( R \) is an order parameter.

**Proof** Note that
\[
\frac{d}{dt} R^2 = \frac{k_0}{N} \sum_{j=1}^{N} \int_{\mathbb{T}^{2d}} \left| \langle \Psi_c(x_1) | \Psi_j(x_1^*) \rangle - \langle \Psi_j(x_1) | \Psi_c(x_1^*) \rangle \right|^2 dx_1 dx_1^*
\]
\[
+ \frac{k_1}{N} \sum_{j=1}^{N} \int_{\mathbb{T}^{2d}} \left| \langle \Psi_c(x_2) | \Psi_j(x_2^*) \rangle - \langle \Psi_j(x_2) | \Psi_c(x_2^*) \rangle \right|^2 dx_2 dx_2^*.
\]

Then one has
\[
\frac{d^2}{dt^2} R^2 = \frac{k_0}{N} \sum_{j=1}^{N} \int_{\mathbb{T}^{2d}} \partial_t \left| \langle \Psi_c(x_1) | \Psi_j(x_1^*) \rangle - \langle \Psi_j(x_1) | \Psi_c(x_1^*) \rangle \right|^2 dx_1 dx_1^*
\]
\[
+ \frac{k_1}{N} \sum_{j=1}^{N} \int_{\mathbb{T}^{2d}} \partial_t \left| \langle \Psi_c(x_2) | \Psi_j(x_2^*) \rangle - \langle \Psi_j(x_2) | \Psi_c(x_2^*) \rangle \right|^2 dx_2 dx_2^*.
\]

Now we show the R.H.S. of the above relation is uniformly bounded. For this, note that
\[
\partial_t \left| \langle \Psi_c(x_1) | \Psi_j(x_1^*) \rangle - \langle \Psi_j(x_1) | \Psi_c(x_1^*) \rangle \right|^2
\]
\[
= \partial_t \left( \langle \Psi_c(x_1) | \Psi_j(x_1^*) \rangle - \langle \Psi_j(x_1) | \Psi_c(x_1^*) \rangle \right) \left( \langle \Psi_j(x_1) | \Psi_c(x_1^*) \rangle - \langle \Psi_j(x_1) | \Psi_c(x_1^*) \rangle \right)
\]
\[
= \frac{1}{N^2} \sum_{k=1}^{N} \partial_t \left( \langle \Psi_k(x_1) | \Psi_j(x_1^*) \rangle - \langle \Psi_j(x_1) | \Psi_k(x_1^*) \rangle \right) \left( \langle \Psi_j(x_1^*) | \Psi_k(x_1) \rangle - \langle \Psi_j(x_1^*) | \Psi_k(x_1) \rangle \right)
\]
\[
= \frac{1}{N^2} \sum_{k=1}^{N} \left( \langle \partial_t \Psi_k(x_1) | \Psi_j(x_1^*) \rangle - \langle \partial_t \Psi_k(x_1^*) | \Psi_j(x_1) \rangle \right) \left( \langle \Psi_j(x_1^*) | \Psi_k(x_1) \rangle - \langle \Psi_j(x_1^*) | \Psi_k(x_1) \rangle \right)
\]
+ \cdots.
\]

(4.24)
In what follows, we consider the first group appearing in the R.H.S. of (4.24). The other groups can be done similarly. To estimate efficiently, we take the following procedure. We first decompose $\partial_t \Psi_i$ into two parts:

$$
\partial_t \Psi_i = (\partial_t \Psi_i + i\mathcal{H} \Psi_i) - i\mathcal{H} \Psi_i.
$$

The first typical term of R.H.S. in (4.24) will be

$$
\langle \partial_t \Psi_k(x_1) | \Psi_j(x_1^*) \rangle \cdot (\langle \Psi_j(x_1^*) | \Psi_l(x_1) \rangle - \langle \Psi_l(x_1^*) | \Psi_j(x_1) \rangle) 
= \langle \partial_t \Psi_k(x_1) + i\mathcal{H} \Psi_k(x_1) | \Psi_j(x_1^*) \rangle \cdot (\langle \Psi_j(x_1^*) | \Psi_l(x_1) \rangle - \langle \Psi_l(x_1^*) | \Psi_j(x_1) \rangle)
=: \mathcal{J}_{11}
$$

in (4.25)

and

$$
-\langle \Psi_j(x_1) | \partial_t \Psi_k(x_1^*) \rangle \cdot (\langle \Psi_j(x_1^*) | \Psi_l(x_1) \rangle - \langle \Psi_l(x_1^*) | \Psi_j(x_1) \rangle) 
= - (\langle \Psi_j(x_1) | \partial_t \Psi_k(x_1^*) + i\mathcal{H} \Psi_k(x_1^*) \rangle \cdot (\langle \Psi_j(x_1^*) | \Psi_l(x_1) \rangle - \langle \Psi_l(x_1^*) | \Psi_j(x_1) \rangle))
=: \mathcal{J}_{12}
$$

and

$$
-\langle \Psi_j(x_1) | \partial_t \Psi_k(x_1^*) \rangle \cdot (\langle \Psi_j(x_1^*) | \Psi_l(x_1) \rangle - \langle \Psi_l(x_1^*) | \Psi_j(x_1) \rangle) 
= - (\langle \Psi_j(x_1) | \partial_t \Psi_k(x_1^*) + i\mathcal{H} \Psi_k(x_1^*) \rangle \cdot (\langle \Psi_j(x_1^*) | \Psi_l(x_1) \rangle - \langle \Psi_l(x_1^*) | \Psi_j(x_1) \rangle))
=: \mathcal{J}_{13}
$$

in (4.26)

• Case A (Boundedness of $\mathcal{J}_{11}$ and $\mathcal{J}_{13}$) We use the Cauchy-Schwarz inequality and $L^2$-conservation of $\Psi_i$ to find

$$
|\mathcal{J}_{11}| \leq \| \partial_t \Psi_k + i\mathcal{H} \Psi_k \| \cdot \| \Psi_j \| \cdot (\| \Psi_j \| \cdot \| \Psi_l \| + \| \Psi_l \| \cdot \| \Psi_j \|)
= \| \partial_t \Psi_k + i\mathcal{H} \Psi_k \| \cdot \| \Psi_j^{in} \| \cdot (\| \Psi_j^{in} \| \cdot \| \Psi_j^{in} \| + \| \Psi_j^{in} \| \cdot \| \Psi_j^{in} \|).
$$

Similarly, one has

$$
|\mathcal{J}_{13}| \leq \| \partial_t \Psi_k + i\mathcal{H} \Psi_k \| \cdot \| \Psi_j^{in} \| \cdot (\| \Psi_j^{in} \| \cdot \| \Psi_j^{in} \| + \| \Psi_j^{in} \| \cdot \| \Psi_j^{in} \|).
$$
Thus, we need to estimate the first factor $\|\partial_t \Psi_k + i\mathcal{H}\Psi_k\|$.

$$\|\partial_t \Psi_i + i\mathcal{H}\Psi_i\|$$

$$\leq \kappa_0 \left\| \int_{T^d} \left( \Psi_c(x_1, x_2) \bar{\Psi}_j(x_1^*, x_2^*) \Psi_j(x_1^*, x_2) \right. \right.$$

$$\left. - \Psi_j(x_1, x_2) \bar{\Psi}_c(x_1^*, x_2^*) \Psi_j(x_1^*, x_2) \right) dx_1^* dx_2^* \bigg\|$$

$$+ \kappa_1 \left\| \int_{T^d} \left( \Psi_j(x_1, x_2) \bar{\Psi}_c(x_1^*, x_2^*) \Psi_j(x_1^*, x_2) \right. \right.$$

$$\left. - \Psi_j(x_1, x_2) \bar{\Psi}_c(x_1^*, x_2^*) \Psi_j(x_1^*, x_2) \right) dx_1^* dx_2^* \bigg\|$$

$$\leq \kappa_0 \left( \left\| \int_{T^d} \Psi_c(x_1, x_2) \bar{\Psi}_j(x_1^*, x_2^*) \Psi_j(x_1^*, x_2) dx_1^* dx_2^* \right\|$$

$$+ \left\| \int_{T^d} \Psi_j(x_1, x_2) \bar{\Psi}_c(x_1^*, x_2^*) \Psi_j(x_1^*, x_2) dx_1^* dx_2^* \right\| \right)$$

$$+ \kappa_1 \left( \left\| \int_{T^d} \Psi_j(x_1, x_2) \bar{\Psi}_c(x_1^*, x_2^*) \Psi_j(x_1^*, x_2) dx_1^* dx_2^* \right\|$$

$$+ \left\| \int_{T^d} \Psi_j(x_1, x_2) \bar{\Psi}_c(x_1^*, x_2^*) \Psi_j(x_1^*, x_2) dx_1^* dx_2^* \right\| \right).$$

(4.27)

Next, we estimate only the first term in the R.H.S. of (4.27) as follows. For notational simplicity, we set

$$d\Sigma = dx_1 dx_2, \quad d\Sigma^* = dx_1^* dx_2^*, \quad d\Sigma^{**} = dx_1^{**} dx_2^{**}.$$  

Then, we use Hölder’s inequality to see

$$\left\| \int \Psi_i(x_1, x_2^*) \bar{\Psi}_j(x_1^*, x_2^*) \Psi_k(x_1^*, x_2) d\Sigma^* \right\|^4$$

$$= \left| \int \Psi_i(x_1, x_2^*) \bar{\Psi}_j(x_1^*, x_2^*) \Psi_k(x_1^*, x_2) \Psi_l(x_1^*, x_2) \bar{\Psi}_m(x_1^*, x_2) \bar{\Psi}_n(x_1^*, x_2) d\Sigma d\Sigma^* d\Sigma^{**} \right|^2$$

$$= \left| \int \Psi_i(x_1, x_2^*) \Psi_k(x_1^*, x_2) \Psi_j(x_1^*, x_2^*) \Psi_l(x_1^*, x_2) \Psi_m(x_1^*, x_2) \Psi_n(x_1^*, x_2) d\Sigma d\Sigma^* d\Sigma^{**} \right|^2$$

$$= \int |\Psi_i(x_1, x_2^*) \Psi_k(x_1^*, x_2) \Psi_j(x_1^*, x_2^*)|^2 d\Sigma d\Sigma^* d\Sigma^{**}$$

$$\times \int |\Psi_j(x_1^*, x_2^*) \Psi_i(x_1, x_2^*) \Psi_k(x_1^*, x_2)|^2 d\Sigma d\Sigma^* d\Sigma^{**}$$

$$= \|\Psi_i\|^4 \cdot \|\Psi_j\|^4 \cdot \|\Psi_k\|^2.$$  

(4.28)

Hence, the first term in the R.H.S. of (4.27) is bounded:

$$\left\| \int \Psi_i(x_1, x_2^*) \bar{\Psi}_j(x_1^*, x_2^*) \Psi_k(x_1^*, x_2) dx_1^* dx_2^* \right\| \leq \|\Psi_i\| \cdot \|\Psi_j\| \cdot \|\Psi_k\| < \infty.$$  

(4.29)

The other terms can be treated similarly. Hence, we have a uniform boundedness of $\|\partial_t \Psi_i + i\mathcal{H}\Psi_i\|_2$. We can also perform the same estimate for $\mathcal{J}_{13}$ to derive a uniform boundedness.
Case B (Boundedness of $\mathcal{J}_{12} + \mathcal{J}_{14}$) Recall the a-priori condition:

$$\|\mathcal{H}\Psi_i\| < C$$

for some positive constant $C$.

Then, one typical term in $\mathcal{J}_{12}$ can be estimated as follows.

$$\int_{T_{2d}} (-i\mathcal{H}\Psi_k(x_1)|\Psi_j(x_1^*)\rangle \langle \Psi_j(x_1)|\Psi_l(x_1^*)\rangle) dx_1 dx_1^*$$

$$\leq \int_{T_{4d}} \|\mathcal{H}\Psi_k(x_1, x_2)\Psi_j(x_1^*, x_2)\Psi_j(x_1, x_2^*)\Psi_l(x_1^*, x_2^*)\| dx_1 dx_1^* dx_2 dx_2^*$$

$$\leq \|\mathcal{H}\Psi_k\| \cdot \|\Psi_j\|^2 \cdot \|\Psi_l\|.$$

The other terms can be estimated similarly. Hence, we have the boundedness of $\mathcal{J}_{12} + \mathcal{J}_{14}$.

Now, we combine all the estimates in Case A and Case B this results and (4.24) to get the uniform boundedness of

$$\int_{T_{2d}} \partial_t \left( \langle \Psi_c(x_1)|\Psi_j(x_1^*)\rangle - \langle \Psi_j(x_1)|\Psi_c(x_1^*)\rangle \right)^2 dx_1 dx_1^*.$$

This yields the uniform boundedness of $\frac{d^2}{dt^2} R^2$, i.e., we have the uniform continuity of $\frac{dR^2}{dt}$. Finally we apply Barbalat’s lemma to derive the desired estimate:

$$\lim_{t \to \infty} \frac{dR^2}{dt} = 0.$$

\[\square\]

Remark 4.1 Below, we provide some comment on the a priori condition:

$$\|\mathcal{H}\Psi_i\| < C.$$

This is equivalent to

$$\|\mathcal{H}\Psi_i\|^2 = \langle \mathcal{H}\Psi_i, \mathcal{H}\Psi_i \rangle = \langle \Psi_i, \mathcal{H}^2\Psi_i \rangle = \langle \mathcal{H}^2 \rangle < \infty,$$

This means the expectation value of square of energy is uniformly bounded which is quite natural from physics’ viewpoint. For example, if hamiltonian consists of the smooth bounded potential $V$, the a priori condition is trivially true.

Theorem 4.1 Let $\{\Psi_i\}$ be a global smooth solution to (4.9) with a-priori condition $\|\mathcal{H}\Psi_j\| < C$ for some positive constant $C$ and the initial data satisfying $R^{in} > 0$. Then we have

(i) $R(t) \geq R^{in}, \quad t > 0$ and $\lim_{t \to \infty} \frac{d}{dt} R(t)^2 = 0.$

(ii) $\lim_{t \to \infty} \int_{T_{2d}} \left( \langle \Psi_c(x_1)|\Psi_j(x_1^*)\rangle - \langle \Psi_j(x_1)|\Psi_c(x_1^*)\rangle \right)^2 dx_1 dx_1^* = 0.$

(iii) $\lim_{t \to \infty} \int_{T_{2d}} \left( \langle \Psi_c(x_2)|\Psi_j(x_2^*)\rangle - \langle \Psi_j(x_2)|\Psi_c(x_2^*)\rangle \right)^2 dx_2 dx_2^* = 0.$

Proof It follows from Lemma 4.1 (i) that

$$\frac{dR^2}{dt} = \frac{\kappa_0}{N} \sum_{i=1}^{N} \int_{T_{2d}} \left( \langle \Psi_c(x_1)|\Psi_j(x_1^*)\rangle - \langle \Psi_j(x_1)|\Psi_c(x_1^*)\rangle \right)^2 dx_1 dx_1^*$$

$$+ \frac{\kappa_1}{N} \sum_{i=1}^{N} \int_{T_{2d}} \left( \langle \Psi_c(x_2)|\Psi_j(x_2^*)\rangle - \langle \Psi_j(x_2)|\Psi_c(x_2^*)\rangle \right)^2 dx_2 dx_2^* \geq 0.$$
Thus, one has
\[ R^2(t) \geq |R(t)|^2, \quad \text{i.e.,} \quad R(t) \geq R(t), \quad t > 0. \]
Since \( R \) is non-decreasing and bounded by 1, there exists \( R^\infty \in [R(t], 1] \) such that
\[ \lim_{t \to \infty} R(t) = R^\infty. \]
Also from uniform continuous of \( \frac{d}{dt} R^2 \) in Lemma 4.2, we have
\[ \lim_{t \to \infty} \frac{d}{dt} R(t)^2 = 0. \]
This implies that
\[ \frac{\kappa_0}{N} \sum_{i=1}^{N} \int_{\mathbb{R}^d} \left| \langle \Psi_i(x_1) | \Psi_j(x_1^+) \rangle - \langle \Psi_j(x_1) | \Psi_i(x_1^+) \rangle \right|^2 dx_1 dx_1^+ 
+ \frac{\kappa_1}{N} \sum_{i=1}^{N} \int_{\mathbb{R}^d} \left| \langle \Psi_i(x_2) | \Psi_j(x_2^+) \rangle - \langle \Psi_j(x_2) | \Psi_i(x_2^+) \rangle \right|^2 dx_2 dx_2^+ \to 0. \]
Since each term of L.H.S of above limit is non-negative, we have
\[ \lim_{t \to \infty} \int_{\mathbb{R}^d} \left| \langle \Psi_i(x_1) | \Psi_j(x_1^+) \rangle - \langle \Psi_j(x_1) | \Psi_i(x_1^+) \rangle \right|^2 dx_1 dx_1^+ = 0 \]
and
\[ \lim_{t \to \infty} \int_{\mathbb{R}^d} \left| \langle \Psi_i(x_2) | \Psi_j(x_2^+) \rangle - \langle \Psi_j(x_2) | \Psi_i(x_2^+) \rangle \right|^2 dx_2 dx_2^+ = 0 \]
for all \( j \in \{1, 2, \ldots, N\} \). \( \square \)

Next, we briefly discuss a global existence of smooth solutions to the SLM model (4.9) with heterogeneous Hamiltonians as follows:
\[ \mathcal{H}_j(x_1, x_2, -i\nabla x_1, -i\nabla x_2) = -\Delta x_1 - \Delta x_2 + V_{j1}(x_1) + V_{j2}(x_2). \]
In this case, system (4.9) can be expressed as follows:
\[ i\partial_t \Psi_j + (\Delta x_1 + \Delta x_2) \Psi_j = (V_{j1}(x_1) + V_{j2}(x_2)) \Psi_j \]
\[ + \frac{i\kappa}{N} \sum_{k=1}^{N} \int_{\mathbb{R}^d} \left( \Psi_k(x_1, x_2^+) \Psi_j(x_1^+, x_2^+) \Psi_j(x_1^+, x_2) - \Psi_j(x_1, x_2^+) \Psi_k(x_1^+, x_2^+) \Psi_j(x_1^+, x_2) \right) dx_1^+ dx_2^+. \]
(4.30)
By the same analysis in (4.28), one has
\[ \left\| \int \Psi_1(x_1, x_2^+) \Psi_2(x_1^+, x_2^+) \Psi_3(x_1^+, x_2) dx_1 dx_1^+ \right\|_{L^2} \leq \|\Psi_1\|_{L^2} \cdot \|\Psi_2\|_{L^2} \cdot \|\Psi_3\|_{L^2}. \]
(4.31)
Similar to (4.31), one has
\[ \left\| \int \Psi_1(x_1, x_2^+) \Psi_2(x_1^+, x_2^+) \Psi_3(x_1^+, x_2) dx_1 dx_1^+ \right\|_{L^2} \leq C \left( \|\Psi_1\|_{L^2} \cdot \|\Psi_2\|_{L^2} \cdot \|\Psi_3\|_{L^2} \right). \]
(4.32)
For $H^m$-solutions $\Psi = \{\Psi_i\}_{i=1}^N$ and $\tilde{\Psi} = \{\tilde{\Psi}_i\}_{i=1}^N$ to (4.30) in $\Omega := \mathbb{T}^d \times [0, T]$, we set

$$\mathcal{E}(T) := \max_{1 \leq j \leq N} \sup_{0 \leq t \leq T} \|\Psi_j(t, \cdot)\|_{H^m}, \quad \Delta(T) := \max_{1 \leq j \leq N} \sup_{0 \leq t \leq T} \|\Psi_j(t, \cdot) - \tilde{\Psi}_j(t, \cdot)\|_{H^m}.$$

**Lemma 4.3** Suppose one-body potentials $V_{j1}$ and $V_{j2}$ for $j = 1, 2, \ldots, N$ satisfy the following regularity:

$$\sum_{k=0}^m \left( \|\nabla^k V_{j1}\|_{L^\infty} + \|\nabla^k V_{j2}\|_{L^\infty} \right) \leq C_m < \infty,$$

for a positive integer $m$, and let $\Psi \in \mathcal{C}([0, T^*_1); H^m(\Omega))$ be a smooth solution to (4.30). Then, there exists a small positive constant $T^*_1$ depending only on $T(0)$ such for $T < T^*_1$,

$$\mathcal{E}(T) \leq 2\mathcal{E}(0).$$

**Proof** We basically use the same argument in Appendix A in [15]. For a desired estimate, we consider the following two cases.

- **Case A** ($m = 1$) We use (4.32) and energy estimate to obtain

$$\|\Psi_j(T)\|_{H^1} \lesssim \|\Psi_j^{in}\|_{H^1} + C \int_0^T \left( \|V_{j1} + V_{j2}\| \Psi_j \|_{H^1} + \frac{\kappa}{N} \sum_{k=1}^N (\|\Psi_k\|_{H^1} \|\Psi_j\|_{L^2}) \right) dt$$

$$\leq \|\Psi_j^{in}\|_{H^1} + 4\kappa CT \left[ \mathcal{E}(T) \left( \|V_{j1}\|_{L^\infty} + \|V_{j2}\|_{L^\infty} + \|\nabla V_{j1}\|_{L^\infty} + \|\nabla V_{j2}\|_{L^\infty} \right) \right].$$

(4.33)

- **Case B** ($2 \leq |\alpha_1| + |\alpha_2| \leq m$) Note that

$$i \partial_t \frac{\partial^{\alpha_1}}{\partial x_1} \frac{\partial^{\alpha_2}}{\partial x_2} \Psi_j + \Delta \frac{\partial^{\alpha_1}}{\partial x_1} \frac{\partial^{\alpha_2}}{\partial x_2} \Psi_j = \frac{\partial^{\alpha_1}}{\partial x_1} \frac{\partial^{\alpha_2}}{\partial x_2} (V_i(x) \Psi_i)$$

$$+ \frac{i k}{N} \sum_{k=1}^N \frac{\partial^{\alpha_1}}{\partial x_1} \frac{\partial^{\alpha_2}}{\partial x_2} \int \left( \Psi_k(x_1, x_2) \Psi_j(x_1, x_2) \Psi_j(x_1, x_2) \right) dx_1 dx_2.$$  

(4.34)

Similar to Case A, we perform energy estimates for (4.34):

$$\|\frac{\partial^{\alpha_1}}{\partial x_1} \frac{\partial^{\alpha_2}}{\partial x_2} \Psi_j(t)\|_{L^2} \leq \|\frac{\partial^{\alpha_1}}{\partial x_1} \frac{\partial^{\alpha_2}}{\partial x_2} \Psi_j^{in}\|_{L^2} + 4CT \left[ \mathcal{E}(T) \sum_{k=0}^m \|\nabla^k (V_{j1} + V_{j2})\|_{L^\infty} + \mathcal{E}(T)^3 \right].$$

(4.35)

Finally we combine estimates (4.33) and (4.34) to obtain

$$\|\Psi_j(t)\|_{H^m} \leq \|\Psi_j^{in}\|_{H^m} + 4CT \mathcal{E}(T) \left[ \sum_{k=0}^m \|\nabla^k (V_{j1} + V_{j2})\|_{L^\infty} + \mathcal{E}(T)^2 \right].$$

This yields

$$\mathcal{E}(T) \leq \mathcal{E}(0) + 4CT \mathcal{E}(T) \left[ \sum_{k=0}^m \|\nabla^k (V_{j1} + V_{j2})\|_{L^\infty} + \mathcal{E}(T)^2 \right].$$
By choosing sufficiently small $T \ll 1$, we obtain
\[ \mathcal{E}(T) \leq 2\mathcal{E}(0). \]
\[\square\]

From the same argument as in Lemma 4.3, we can obtain following lemma.

**Lemma 4.4** There exists a small positive constant $T^*_{2}$ depending only on $T(0)$ and $\mathcal{T}(0)$ such that for solutions $\Psi$ and $\tilde{\Psi}$ in $C([0, T^*_{2}); H^{m}(\Omega))$ to (4.30) with initial data $\Psi$ and $\tilde{\Psi}$, respectively and $T < T^*_{2}$,
\[ \Delta(T) \leq 2\Delta(0). \]

**Proof** Note that $\Psi_{t}$ and $\tilde{\Psi}_{t}$ satisfy
\[ i\partial_{t}\Psi_{j} + (\Delta_{x_{1}} + \Delta_{x_{2}})\Psi_{j} = (V_{j1}(x_{1}) + V_{j2}(x_{2}))\Psi_{j} \]
\[ + \frac{i\kappa}{N} \sum_{k=1}^{N} \int_{\mathbb{T}^{2d}} \left( \Psi_{k}(x_{1}, x_{2}^{*})\Psi_{j}(x_{1}^{*}, x_{2}^{*})\Psi_{j}(x_{1}^{*}, x_{2}) \right) dx_{1}^{*}dx_{2}^{*}, \]
(4.36)
and
\[ i\partial_{t}\tilde{\Psi}_{j} + (\Delta_{x_{1}} + \Delta_{x_{2}})\tilde{\Psi}_{j} = (V_{j1}(x_{1}) + V_{j2}(x_{2}))\tilde{\Psi}_{j} \]
\[ + \frac{i\kappa}{N} \sum_{k=1}^{N} \int_{\mathbb{T}^{2d}} \left( \tilde{\Psi}_{k}(x_{1}, x_{2}^{*})\tilde{\Psi}_{j}(x_{1}^{*}, x_{2}^{*})\tilde{\Psi}_{j}(x_{1}^{*}, x_{2}) \right) dx_{1}^{*}dx_{2}^{*}. \]
(4.37)

Then, it follows from (4.36) and (4.37) that
\[ i\partial_{t}(\Psi_{j} - \tilde{\Psi}_{j}) + (\Delta_{x_{1}} + \Delta_{x_{2}})(\Psi_{j} - \tilde{\Psi}_{j}) = (V_{j1}(x_{1}) + V_{j2}(x_{2}))(\Psi_{j} - \tilde{\Psi}_{j}) \]
\[ + \frac{i\kappa}{N} \sum_{k=1}^{N} \int_{\mathbb{T}^{2d}} \left[ (\Psi_{k} - \tilde{\Psi}_{k})(x_{1}, x_{2}^{*})\Psi_{j}(x_{1}^{*}, x_{2}^{*})\Psi_{j}(x_{1}^{*}, x_{2}) \right. \]
\[ + \tilde{\Psi}_{k}(x_{1}, x_{2}^{*})(\Psi_{j} - \tilde{\Psi}_{j})(x_{1}^{*}, x_{2}^{*})\Psi_{j}(x_{1}^{*}, x_{2}) \]
\[ + \tilde{\Psi}_{k}(x_{1}, x_{2}^{*})\tilde{\Psi}_{j}(x_{1}^{*}, x_{2}^{*})(\Psi_{j} - \tilde{\Psi}_{j})(x_{1}^{*}, x_{2}) \right] dx_{1}^{*}dx_{2}^{*}. \]
(4.38)
Since the estimate is almost the same as in Lemma 4.3, we omit its details and get the desired result. \[\square\]

Based on Lemmas 4.3 and 4.4, we are ready to present a local existence of $H^{m}$-solution in the following theorem.
Theorem 4.2 (Local existence) For a positive integer \( m \in \mathbb{Z}_+ \), suppose that initial data \( \Psi_i^{in} \in H^m(\mathbb{T}^d) \). Then, there exists a positive constant \( T \in (0, \infty) \) such that the initial value problem (4.30) has a unique local solution \( \psi_i \in C([0, T); H^m(\mathbb{T}^d)) \).

Proof The proof can be done using the standard successive approximation, Lemmas 4.3 and 4.4 to derive a unique solvability of the local \( H^m \)-solutions to (4.30). We omit its details. \( \square \)

Remark 4.2 By Sobolev’s embedding theorem, it is easy to see that for \( m > 2 + \frac{d}{2} \), then \( H^m \) solution is a classical solution.

Finally, we are ready to prove a global existence of classical solution.

Theorem 4.3 (Global existence) Suppose that the initial data \( \Psi_i^{in} \in H^m(\mathbb{T}^d) \). Then, for any \( T \in (0, \infty) \), the Cauchy problem for (4.30) has a unique global solution \( \{\Psi_i\} \) such that

\[
\Psi_i \in C([0, T), H^m(\mathbb{T}^d)) \cap C^1([0, T), H^{m-2}(\mathbb{T}^d)).
\]

Proof Since the global existence is also standard as in [15], we omit its details. \( \square \)

5 The Schrödinger–Lohe Tensor(SLT) Model

In this section, we introduce the highest object located in the vertex of the infinite-dimensional Schrödinger–Lohe hierarchy, namely “the Schrödinger–Lohe tensor model”. The construction is pretty much the same as the construction of the Schrödinger–Lohe matrix model in spirit in previous section. First, we choose a standing wave solution for each linear Schrödinger equation with Hamiltonian \( H_\alpha \) and then, take a tensor product on \((\mathbb{T}^d)^m = \mathbb{T}^d \times \cdots \times \mathbb{T}^d\), and then express our solution \( \Psi(t, x_1, \ldots, x_m) \) as a linear combination of \( \Phi^{1}_{\alpha_1} \otimes \Phi^{2}_{\alpha_2} \otimes \cdots \otimes \Phi^{m}_{\alpha_m} \) with time-dependent coefficients. Finally, we design our Schrödinger–Lohe tesnor model suitably so that it can be reduced into the Lohe tensor model as in the Schrödinger–Lohe matrix model.

5.1 Construction of SLT Model

Let \( x = (x_1, \ldots, x_m) \in \mathbb{T}^{md} \) and \( \mathcal{H} \) be an interaction free Hamiltonian:

\[
\mathcal{H}(x_1, \ldots, x_m, -i\nabla x_1, \ldots, -i\nabla x_m) = \sum_{i=1}^{N} \mathcal{H}_i(x_i, -i\nabla x_i).
\]

For each \( j = 1, \ldots, m \), we set \( \{\Phi^j_{\alpha_j}(t, x_j)\} \) be an orthonormal family of standing wave solutions to the linear Schrödinger equation associated with the Hamiltonian \( \mathcal{H}_j \). Then, the set \( \{\Phi^1_{\alpha_1} \otimes \Phi^2_{\alpha_2} \otimes \cdots \otimes \Phi^m_{\alpha_m}\} \) is an orthonormal family of basis, and for a wave function \( \Psi_i = \Psi_i(t, x_1, \ldots, x_m) \), we set

\[
\Psi_i(t, x) = \sum_{\alpha} [T_i(t)]_{\alpha_1} (\Phi^1_{\alpha_1} \otimes \Phi^2_{\alpha_2} \otimes \cdots \otimes \Phi^m_{\alpha_m})(t, x), \quad t \geq 0, \ x \in \mathbb{T}^{md}, \quad (5.1)
\]

where \( \alpha = (\alpha_1, \ldots, \alpha_m) \) and

\[
(\Phi^1_{\alpha_1} \otimes \Phi^2_{\alpha_2} \otimes \cdots \otimes \Phi^m_{\alpha_m})(t, x_1, \ldots, x_m) := \Phi^1_{\alpha_1}(t, x_1) \cdot \Phi^2_{\alpha_2}(t, x_2) \cdots \Phi^m_{\alpha_m}(t, x_m).
\]
For notational simplicity, we also introduce several handy notation.

\[
\Psi(x_{10}, \ldots, x_{m0}) = \Psi(x_{0}), \quad \Psi(x_{11}, \ldots, x_{m1}) = \Psi(x_{1}),
\]

\[
\Psi(x_{1i}, \ldots, x_{mim}) = \Psi(x_{imi}), \quad \Psi(x_{1(1-i)}, \ldots, x_{m(1-i)m}) = \Psi(x_{(1-i)}).
\]

\[
dx_{10} \cdots dx_{m0} = dx_{0}, \quad dx_{11} \cdots dx_{m1} = dx_{1},
\]

\[
dx_{1i} \cdots dx_{im} = dx_{imi}, \quad dx_{1(1-i)} \cdots dx_{m(1-i)m} = dx_{(1-i)}.
\]

Now, consider the Cauchy problem to the SLT model:

\[
\begin{aligned}
\left\{ \begin{array}{l}
\imath \partial_t \Psi_j(x_{0}) - H(\Psi_j(x_{0})) \\
= \imath \sum_{i_s \in [0,1]^m} \kappa_{i_s} \int_{\mathbb{R}^md} \left( \Psi_c(x_{imi}) \Psi_j(x_{imi}) - \Psi_j(x_{imi}) \Psi_c(x_{imi}) \right) dx_{imi}, \\
\Psi_j \bigg|_{t=0} = \Psi_j^0.
\end{array} \right.
\]

(5.2)

The global existence of (5.2) can be done as in Theorems 4.2 and 4.3. Note that the Schrödinger–Lohe model is a special case of Schrödinger–Lohe tensor model with \( m = 1 \), and Schrödinger–Lohe matrix model is a special case of Schrödinger–Lohe tensor model with \( m = 2 \). If we consider the case \( m = 0 \) of the Schrödinger–Lohe model, then \( \Psi_t \) is no longer a function of space, i.e., \( \Psi_t \) is a complex-valued function defined on time domain. Hence, we can easily derive the Kuramoto model from the Lohe tensor model.

**Proposition 5.1** Let \( \{\Psi_j\} \) be a global smooth solution to (5.2). Then \( L^2 \)-norm of \( \Psi_t \) is a conserved quantity.

\[
\frac{d}{dt} \| \Psi_i(t) \|^2 = 0, \quad t > 0, \quad i = 1, \ldots, N.
\]

**Proof** By direct calculation, one has

\[
\frac{d}{dt} \langle \Psi_i | \Psi_i \rangle = \langle \Psi_i | \imath \partial_t \Psi_i \rangle + (c.c.) = \langle \Psi_i | -iH \Psi_i(x_0) \rangle + \sum_{i_s \in [0,1]^m} \kappa_{i_s} \int_{\mathbb{R}^md} \left( \Psi_c(x_{imi}) \Psi_i(x_{imi}) - \Psi_i(x_{imi}) \Psi_c(x_{imi}) \right) dx_{imi} + (c.c.)
\]

\[
= \sum_{i_s \in [0,1]^m} \kappa_{i_s} \left( \langle \Psi_i | \int_{\mathbb{R}^md} \left( \Psi_c(x_{imi}) \Psi_i(x_{imi}) - \Psi_i(x_{imi}) \Psi_c(x_{imi}) \right) dx_{imi} \rangle + (c.c.) \right)
\]

\[
= \sum_{i_s \in [0,1]^m} \kappa_{i_s} \int_{\mathbb{R}^md} \left( \Psi_i(x_{0}) \Psi_c(x_{imi}) \Psi_i(x_{imi}) - \Psi_i(x_{0}) \Psi_c(x_{imi}) \Psi_i(x_{imi}) \right) dx_{imi} + (c.c.)
\]

\[
= \int_{\mathbb{R}^md} \left( \Psi_i(x_{0}) \Psi_c(x_{imi}) \Psi_i(x_{imi}) - \Psi_i(x_{0}) \Psi_c(x_{imi}) \Psi_i(x_{imi}) \right) dx_{imi} + (c.c.)
\]

\[
= \int_{\mathbb{R}^md} \left( \Psi_i(x_{0}) \Psi_c(x_{imi}) \Psi_i(x_{imi}) - \Psi_i(x_{0}) \Psi_c(x_{imi}) \Psi_i(x_{imi}) \right) dx_{imi} + (c.c.)
\]

Since \( x_{0} \) and \( x_{1} \) are dummy variables, we can exchange

\[
x_{1} \leftrightarrow x_{imi}, \quad x_{0} \leftrightarrow x_{(1-i)mi}.
\]

Hence, we can obtain

\[
\mathcal{I}_{31} = -\mathcal{I}_{32}
\]

to get the desired estimate \( \frac{d}{dt} \langle \Psi_i | \Psi_i \rangle = 0. \)
Consider the Cauchy problem to the nonlinear system associated with (5.2):

\[
\begin{aligned}
&\left\{ \begin{array}{l}
i\partial_t \phi_i(t, x_{a0}) \\
= i \sum_{i_a \in (0, 1)^m} \kappa_i \int_{\mathbb{T}^md} \left( \varphi_c(x_{a1}) \bar{\phi}_i(x_{1-i_a}) - \phi_i(x_{a1}) \bar{\varphi}_c(x_{1-i_a}) \right) dx_{a1}, \\
\phi_i \bigg|_{t=0} = \Psi_i^{in}.
\end{array} \right.
\end{aligned}
\]

Next, we derive the solution splitting property of (5.3).

**Proposition 5.2** Suppose that the one-body Hamiltonian $\mathcal{H}$ is additive:

\[
\mathcal{H}(x_1, -i\nabla x_1, \ldots, x_m, -i\nabla x_m) = \sum_{j=1}^m \mathcal{H}_j(x_j, -i\nabla x_j),
\]

and let $\{\Psi_i\}$ and $\{\psi_i\}$ be global smooth solutions to (5.2) and (5.3), respectively. Then, one has

\[
\Psi_j(t, x_1, x_2) = e^{-i\mathcal{H}(t)} \psi_j(t, x_1, x_2).
\]

**Proof** As in the proof of Proposition 4.2, we have

\[
\mathcal{H} = \sum_{k=1}^N \mathcal{H}_k, \quad \mathcal{H}_i \circ \mathcal{H}_j = \mathcal{H}_j \circ \mathcal{H}_i, \quad \forall i, j \in \{1, \ldots, N\}.
\]

These relations imply

\[
e^{-i\mathcal{H}(x_1, x_2, \ldots, x_m)t} = e^{-i\mathcal{H}(a_1, x_{a1})t} \cdot e^{-i\mathcal{H}(a_2, x_{a2})t} \cdots \cdot e^{-i\mathcal{H}(a_m, x_{a_m})t},
\]

where $\{a_1, a_2, \ldots, a_m\} = \{1, 2, \ldots, m\}$. Therefore, one has

\[
e^{-i\mathcal{H}(x_{a1})t} \cdot e^{i\mathcal{H}(x_1)t} \cdot e^{-i\mathcal{H}(x_{a1-1})t}.
\]

It follows from (5.2) that

\[
i \partial_t \Psi_i(t, x_{a0}) = \mathcal{H} \Psi_i(t, x_{a0})
\]

\[
+ i \sum_{i_a \in (0, 1)^m} \kappa_i \int_{\mathbb{T}^md} \left( \varphi_c(x_{a1}) \bar{\psi}_i(x_{1-i_a}) - \psi_i(x_{a1}) \bar{\varphi}_c(x_{1-i_a}) \right) dx_{a1}.
\]

Since we have

\[
i \partial_t \psi_i(t, x_{a0}) = \partial_t (e^{i\mathcal{H}(t)} \psi_i(t, x_{a0}))
\]

\[
= \sum_{i_a \in (0, 1)^m} \kappa_i e^{i\mathcal{H}(t)} \int_{\mathbb{T}^md} \left( \varphi_c(x_{a1}) \bar{\psi}_i(x_{1-i_a}) - \psi_i(x_{a1}) \bar{\varphi}_c(x_{1-i_a}) \right) dx_{a1}
\]

\[
= \sum_{i_a \in (0, 1)^m} \kappa_i \int_{\mathbb{T}^md} \left( e^{i\mathcal{H}(x_{a1})t} \varphi_c(x_{a1}) e^{i\mathcal{H}(t)} \psi_i(x_{1-i_a}) - \psi_i(x_{a1}) e^{i\mathcal{H}(x_{a1})t} \bar{\varphi}_c(x_{1-i_a}) \bar{\psi}_i(x_{1-i_a}) \right) dx_{a1}
\]

\[
+ \sum_{i_a \in (0, 1)^m} \kappa_i \int_{\mathbb{T}^md} \left( e^{i\mathcal{H}(x_{a1})t} \psi_i(x_{a1}) e^{i\mathcal{H}(t)} \bar{\varphi}_c(x_{1-i_a}) - \bar{\psi}_i(x_{a1}) e^{i\mathcal{H}(x_{a1})t} \bar{\varphi}_c(x_{1-i_a}) \bar{\psi}_i(x_{1-i_a}) \right) dx_{a1}.
\]
5.2 Reduction to the LT Model

In what follows, we present a reduction of the SLM to the Lohe tensor model in [19]. The basic idea is the same as in Sects. 4.1 and 4.3 for rank-1 and rank-2 tensors.

Let \( \{\phi^i_{\alpha i}(x_i)\}_{\alpha i=1}^{\infty} \) be an orthonormal system consisting of eigenfunctions of \( \mathcal{H}_i \):

\[
\mathcal{H}_i \phi^i_{\alpha i} = E^i_{\alpha i} \phi^i_{\alpha i}, \quad i = 1, \ldots, N.
\]

Now, we introduce standing wave solution \( \Phi^i_{\alpha i} \) as follows:

\[
\Phi^i_{\alpha i}(t, x_i) := e^{-iE^i_{\alpha i}} t \phi^i_{\alpha i}(x_i), \quad i = 1, \ldots, N.
\]

Then, it is easy to see check

\[
i \partial_t (\Phi^1_{\alpha 1} \otimes \Phi^2_{\alpha 2} \otimes \cdots \otimes \Phi^m_{\alpha m}) = \mathcal{H}(\Phi^1_{\alpha 1} \otimes \Phi^2_{\alpha 2} \otimes \cdots \otimes \Phi^m_{\alpha m}).
\]

Now, we expand \( \Psi_j \) in terms of the basis \( \{\Phi^1_{\alpha 1} \otimes \Phi^2_{\alpha 2} \otimes \cdots \otimes \Phi^m_{\alpha m}\}_{\alpha} \):

\[
\Psi_j = \sum_{\alpha} [T_j(t)]_{\alpha} (\Phi^1_{\alpha 1} \otimes \Phi^2_{\alpha 2} \otimes \cdots \otimes \Phi^m_{\alpha m}). \tag{5.4}
\]

**Proposition 5.3** Let \( \{\Psi_j\} \) be a global smooth solution to (5.2). Then, the coefficient \( [T_j]_{\alpha} \) satisfies the Lohe tensor model:

\[
\frac{d}{dt} [T_j]_{\alpha} = \sum_{\alpha_i \in [0,1]^m} \kappa_{\alpha_i} \left( [T_c]_{\alpha i} [\tilde{T}_j]_{\alpha_1} [T_j]_{\alpha \setminus \{i\}} - [T_j]_{\alpha i} [\tilde{T}_c]_{\alpha_1} [T_j]_{\alpha \setminus \{i\}} \right).
\]

**Proof** We substitute (5.4) into the L.H.S. of (5.2) to get

\[
i \partial_t \Psi_j(t, x_0) = \sum_{\alpha} \left( \left[ \tilde{T}_j \right]_{\alpha} \Phi^1_{\alpha 1}(t, x_0) \Phi^2_{\alpha 2}(t, x_0) \cdots \Phi^m_{\alpha m}(t, x_0) \right.
\]

\[
+ \left[ T_j \right]_{\alpha} \partial_t \Phi^1_{\alpha 1}(t, x_0) \Phi^2_{\alpha 2}(t, x_0) \cdots \Phi^m_{\alpha m}(t, x_0) \right)
\]

\[
= \sum_{\alpha} \left( \left[ \tilde{T}_j \right]_{\alpha} \Phi^1_{\alpha 1}(t, x_0) \Phi^2_{\alpha 2}(t, x_0) \cdots \Phi^m_{\alpha m}(t, x_0) \right.
\]

\[
+ \sum_{\alpha} \left[ T_j \right]_{\alpha} \mathcal{H}(\Phi^1_{\alpha 1}(t, x_0) \Phi^2_{\alpha 2}(t, x_0) \cdots \Phi^m_{\alpha m}(t, x_0)) \right)
\]

\[
= i \left[ \tilde{T}_j \right]_{\alpha} \Phi^1_{\alpha 1}(t, x_0) \Phi^2_{\alpha 2}(t, x_0) \cdots \Phi^m_{\alpha m}(t, x_0) + \mathcal{H}(\Psi_j(t, x_0)).
\]

Now, we equate (5.2) and (5.5) to get

\[
\left[ \tilde{T}_j \right]_{\alpha_1 \cdots \alpha_m} \Phi^1_{\alpha 1}(t, x_0) \Phi^2_{\alpha 2}(t, x_0) \cdots \Phi^m_{\alpha m}(t, x_0)
\]

\[
= \sum_{\alpha_i \in [0,1]^m} \kappa_{\alpha_i} \left( \int_{T_m} \left( \psi_j(x_{\alpha_i}) \psi_j(x_{\alpha \setminus \{i\}}) - \psi_j(x_{\alpha_i}) \psi_j(x_{\alpha \setminus \{i\}}) \right) dx_{\alpha_i} \right).
\]

This yields

\[
\left[ \tilde{T}_j \right]_{\alpha} = \sum_{\alpha_i \in [0,1]^m} \kappa_{\alpha_i} \int_{T_m} \left( \psi_j(x_{\alpha_i}) \psi_j(x_{\alpha \setminus \{i\}}) - \psi_j(x_{\alpha_i}) \psi_j(x_{\alpha \setminus \{i\}}) \right)
\]

\[
\times \Phi^1_{\alpha 1}(t, x_0) \Phi^2_{\alpha 2}(t, x_0) \cdots \Phi^m_{\alpha m}(t, x_0) dx_{\alpha_i} dx_{\alpha_0} dx_{\alpha_0}.
\]
On the other hand, we use the relation (5.6) and the orthogonality of \( \{ \Phi_1^{(1)}(x_{10}) \otimes \cdots \otimes \Phi_m^{(m)}(x_{m0}) \}_{\alpha_1} \) to get
\[
\int_{\mathbb{T}^{2md}} \Psi_c(x_{a1}) \Phi_j(x_{a1}) \Phi_j(x_{a(1-i_a)}) \Phi_1^{(1)}(t, x_{10}) \Phi_2^{(2)}(t, x_{20}) \cdots \Phi_m^{(m)}(t, x_{m0}) dx_a dx_0 = [T_c]_{\alpha_1} [\tilde{T}_j]_{\alpha_1} [T_j]_{\alpha(1-i_a)}.
\]

By (5.6) and (5.7), one has the Lohe tensor model.

### Lemma 5.1

Let \( \{ \Psi_j \} \) be a global smooth solution to (5.2). Then we have
\[
\frac{d R^2}{dt} = \sum_{i_a \in [0,1]^m} \frac{\kappa_{i_a}}{N} \sum_{i=1}^N \int_{D^2} \left| \int_{D^2} (\Psi_c(x_{a0}) \Psi_j(x_{a(1-i_a)}) - \Psi_j(x_{a0}) \Psi_c(x_{a(1-i_a)})) \right| dx_0 dx A_1.
\]

**Proof** It follows from (5.2) and (5.8) that
\[
\frac{d}{dt} \Psi_c(x_{a0}) = -i \mathcal{H} \Psi_c(x_{a0}) + \sum_{i_a} \frac{\kappa_{i_a}}{N} \int_{D^2} \sum_{i=1}^N (\Psi_c(x_{a1}) \Psi_j(x_{a1}) \Psi_j(x_{a(1-i_a)}))
\]
\[
- \Psi_j(x_{a1}) \Psi_c(x_{a1}) \Psi_j(x_{a(1-i_a)}) dx_{a1}.
\]

This yields
\[
\langle \Psi_c | \partial_t \Psi_c \rangle = \langle \Psi_c | -i \mathcal{H} \Psi_c \rangle + \sum_{i_a} \frac{\kappa_{i_a}}{N} \sum_{i=1}^N \int_{D^2} \Psi_c(x_{a0})
\]
\[
\times \left( \Psi_c(x_{a1}) \Psi_j(x_{a1}) \Psi_j(x_{a(1-i_a)}) - \Psi_j(x_{a1}) \Psi_c(x_{a1}) \Psi_j(x_{a(1-i_a)}) \right) dx_{a1} dx_{a0}.
\]
Finally, one has
\[
\frac{d}{dt}(\Psi_c|\Psi_c) = (\Psi_c|\partial_t \Psi_c) + (c.c.)
\]
\[
= \sum_{i_0}^N \sum_{i=1}^{N} \int \mathcal{D}^2 \left( \Psi_c(x_{i_0}) \Psi_f(x_{i_1})(x_{i_0}) \Psi_f(x_{i_0(1-\i_0)}) - \Psi_f(x_{i_0}) \Psi_f(x_{i_0(1-\i_0)}) \right) dx_{i_1} dx_{i_0}
\]
\[
= I_4 + (c.c.).
\]

Now we simplify the term \( I_4 + \bar{I}_4 \) as follows.
\[
I_4 + \bar{I}_4 = \int \mathcal{D}^2 \left( \Psi_c(x_{i_0}) \Psi_f(x_{i_1}) \Psi_f(x_{i_0(1-\i_0)}) - \Psi_f(x_{i_0}) \Psi_f(x_{i_0(1-\i_0)}) \right) dx_{i_1} dx_{i_0}
\]
\[
= \int \mathcal{D}^2 \left( \Psi_c(x_{i_0}) \Psi_f(x_{i_0(1-\i_0)}) - \Psi_f(x_{i_0}) \Psi_f(x_{i_0(1-\i_0)}) \right) dx_{i_1} dx_{i_0}.
\]

Since \( x_{i_0} \) and \( x_{i_1} \) are dummy variables, we can interchange the variables in third term and forth term in R.H.S. of above equality:
\[
x_{i_0} \leftrightarrow x_{i_0(1-\i_0)} \quad \text{and} \quad x_{i_1} \leftrightarrow x_{i_0(1-\i_0)}
\]
to get
\[
I_4 + \bar{I}_4 = \int \mathcal{D}^2 \left( \Psi_c(x_{i_0}) \Psi_f(x_{i_0(1-\i_0)}) - \Psi_f(x_{i_0}) \Psi_f(x_{i_0(1-\i_0)}) \right) dx_{i_1} dx_{i_0}.
\]

This yields
\[
\int \mathcal{D}^2 \left( \Psi_c(x_{i_0}) \Psi_f(x_{i_0(1-\i_0)}) - \Psi_f(x_{i_0}) \Psi_f(x_{i_0(1-\i_0)}) \right) dx_{i_1} dx_{i_0}
\]
\[
= \left| \int \mathcal{D}_A dx \left( \Psi_c(x_{i_0}) \Psi_f(x_{i_0(1-\i_0)}) - \Psi_f(x_{i_0}) \Psi_f(x_{i_0(1-\i_0)}) \right) dx_{A0} dx_{A1} \right|^2 dx_{A0} dx_{A1}.
\]

Finally, one has
\[
\frac{dR^2}{dt} = \sum_{i_0}^N \sum_{i_1}^{N} \int \mathcal{D}_A dx \left( \Psi_c(x_{i_0}) \Psi_f(x_{i_0(1-\i_0)}) - \Psi_f(x_{i_0}) \Psi_f(x_{i_0(1-\i_0)}) \right) dx_{A0} dx_{A1}.
\]

\[\square\]

**Lemma 5.2** Let \( \{\Psi_j\} \) be a global smooth solution to (5.2) with a-priori condition \( ||\Psi|| < C \) for some positive constant \( C \). Then for each \( i_0 \) with \( k_{i_0} > 0 \) and \( j = 1, \ldots, N \), one has
\[
\lim_{t \to \infty} \int \mathcal{D}_A dx \left( \Psi_c(x_{i_0}) \Psi_j(x_{i_0(1-\i_0)}) - \Psi_j(x_{i_0}) \Psi_c(x_{i_0(1-\i_0)}) \right) dx_{A0} dx_{A1} = 0.
\]
**Remark 5.1**

Again, by Barbalat’s lemma, we have following theorem.

\[ \lim_{t \to \infty} \frac{d}{dt} \langle \Psi_c | \Psi_c \rangle = 0. \]

Then we have

\[
\frac{d}{dt} \langle \Psi_c | \Psi_c \rangle = \sum_{i_*} \frac{\kappa_{i_*}}{N} \sum_{\ell = 1}^N \int_{D_A^2} \left| \int_{D_B} (\frac{\Psi_c(x_0)}{x_0}) (\Psi_i(x_{(1-i_*)}) - \Psi_i(x_0)) (\Psi_e(x_{(1-i_*)})) dx_B \right|^2 dx_A dx_A.
\]

that we can easily obtain the boundedness of second derivative of \( \Psi_i \) for all \( i = 1, 2, \ldots, N \).

**Theorem 5.1**

Suppose the coupling strengths satisfy

\[ \kappa_{i_*} \geq 0, \quad \text{for } i_* \neq (0, \ldots, 0), \quad \kappa_{00-0} > 0. \]

and let \( \{ \Psi_i \} \) be a global smooth solution to (5.2) with a-priori condition \( \| \mathcal{H} \Psi_j \| < C \) for some positive constant \( C \). Then, either complete aggregation or bi-polar state occurs asymptotically.

**Proof** Since \( \kappa_{00-0} > 0 \), without loss of generality, we may set

\[ i_* \neq (0, 0, \ldots, 0) \text{ in (5.9)}. \]

Then, we have the term involving with \( \kappa_{00-0} \):

\[
\int_{D^2} \left| \left( \frac{\Psi_c(x_0)}{x_0} \right) (\Psi_i(y_0) - \Psi_i(x_0)) (\Psi_i(y_0) - \Psi_i(x_0)) \right|^2 dx_A dy_A
\]

\[
= \int_{D^2} \left( \frac{\Psi_c(x_0)}{x_0} \right) (\Psi_i(y_0) - \Psi_i(x_0)) (\Psi_i(y_0) - \Psi_i(x_0)) dx_A dy_A
\]

\[
= 2 \| \Psi_c \|^2 \cdot \| \Psi_i \|^2 - 2 \left( \langle \Psi_c, \Psi_i \rangle \right)^2 + 4 \text{Im}(\langle \Psi_e, \Psi_i \rangle)^2.
\]

This yields

\[
\lim_{t \to \infty} \left( \| \Psi_c \|^2 \cdot \| \Psi_i \|^2 - 2 \left( \langle \Psi_c, \Psi_i \rangle \right)^2 \right) = 0, \quad \lim_{t \to \infty} \text{Im}(\langle \Psi_c, \Psi_i \rangle)^2 = 0.
\]
So we have
\[
\|\Psi_c - (\Psi_c, \Psi_i)\|_2^2 = \|\Psi_c\|_2^2 + |\langle \Psi_c, \Psi_i \rangle|^2 - \langle \Psi_c, \Psi_c \rangle^2
\]
\[
= \|\Psi_c\|_2^2 \cdot \|\Psi_i\|_2^2 - |\langle \Psi_c, \Psi_i \rangle|^2 - \langle \Psi_i, \Psi_i \rangle^2 = \|\Psi_c\|_2^2 \cdot \|\Psi_i\|_2^2 - 2\text{Im}(\langle \Psi_c, \Psi_i \rangle)^2 \rightarrow 0,
\]
as time goes to infinity. Hence, we know that there exists complex scalar function \(\lambda_i(t)\) such that
\[
\|\Psi_c(t) - \lambda_i(t)\Psi_i(t)\|_2^2 \rightarrow 0. \tag{5.10}
\]
So we know
\[
\|\Psi_c - \langle \Psi_c, \Psi_i \rangle / \Psi_i\|_2^2 = \|\Psi_c - \lambda_i\Psi_i - \langle \Psi_c, \Psi_i \rangle / \Psi_i\|_2^2 \geq \|\Psi_c - \lambda_i\Psi_i - \langle \Psi_c, \lambda_i\Psi_i \rangle / \Psi_i\|_2 - \|\lambda_i - \overline{\lambda}_i\|_2.
\]
On the other hand, it follows from (5.10) that
\[
\lim_{t \to \infty} \|\lambda_i\Psi_i - \overline{\lambda}_i\Psi_i\|_2 = 0.
\]
That means we can set \(\lambda_i\) be real number. Also from \(\|\Psi_c\|_2 = |\lambda_i|\) and \(\|\Psi_c\|_2\) is nondecreasing, we can set
\[
|\lambda_i(t)| \geq \|\Psi_c(0)\|_2.
\]
From the triangle inequality
\[
\|\lambda_1\Psi_1 - \lambda_i\Psi_i\|_2 = \Psi_1\|(\Psi_c - \lambda_i\Psi_i) - (\Psi_c - \lambda_1\Psi_1)\|_2 \leq \|\Psi_c - \lambda_i\Psi_i\|_2 + \|\Psi_c - \lambda_1\Psi_1\|_2.
\]
If we set \(a_i = \lambda_i / \lambda_1\), we have
\[
\Psi_i - a_i\Psi_1 \rightarrow 0.
\]
Since \(|a_i| = 1\) and \(a_i\) are real numbers,
\[
a_i = \pm 1.
\]
This completes the proof. \(\square\)

**Lemma 5.3** Let \(\{\Psi_i\}\) be an ensemble with
\[
\Psi_1 = \cdots = \Psi_n = \Psi^\infty, \quad \Psi_{n+1} = \cdots = \Psi_N = -\Psi^\infty,
\]
where \(0 \leq n \leq N/2\) and \(\|\Psi^\infty\|_2 = 1\). Then we have
\[
\|\Psi_c\|_2 = \left|1 - \frac{2n}{N}\right|
\]
**Proof** By direct calculation, one has
\[
\Psi_c = \left(\frac{N - 2n}{N}\right)\Psi^\infty.
\]
Thus, we have
\[
\|\Psi_c\|_2 = 1 - \frac{2n}{N}.
\]
\(\square\)

**Remark 5.2** If \(\|\Psi_c(0)\|_F > 1 - \frac{2}{N}\), then a bi-polar state is impossible.
From above remark, we have following theorem.

**Theorem 5.2** Suppose that the Hamiltonian, coupling strengths and initial data satisfy

\[ H = 0, \quad \kappa_i \geq 0, \quad \forall i \neq (0,0,\ldots,0), \quad \kappa_{00,\ldots,0} > 0, \quad \|\Psi(0)\|_2 > 1 - \frac{2}{N}, \]

and let \( \{\Psi_i\} \) be a global smooth solution to (5.2). Then the complete aggregation occurs asymptotically.

**Proof** We use (5.1) and assumption to see

\[ R(t) \geq R(0) > 1 - \frac{2}{N}, \quad t \geq 0. \tag{5.11} \]

Suppose that a bi-polar state emerges: for some \( n \leq \lfloor N/2 \rfloor \), one has

\[
\lim_{t \to \infty} \|\Psi_j(t) - \Psi\|_2 = 0, \quad 1 \leq j \leq n, \\
\lim_{t \to \infty} \|\Psi_j(t) - (-\Psi)\|_2 = 0, \quad n + 1 \leq j \leq N.
\]

Then, it follows from Lemma 5.3 that

\[ \lim_{t \to \infty} R(t) = 1 - \frac{2n}{N}, \]

which is clearly contradictory to (5.11). Hence, we have the complete state aggregation. □

### 6 Conclusion

In this paper, we have proposed an infinite-dimensional Schrödinger–Lohe hierarchy consisting of the Schrödinger–Lohe model, the Schrödinger–Lohe matrix model and Schrödinger–Lohe tensor model. In a series of recent papers, the authors established the Lohe hierarchy consisting of the Kuramoto model, the Lohe sphere model, the Lohe matrix model and the Lohe tensor model. Prior to this work, the relation between the Schrödinger–Lohe model and the Lohe matrix model was a kind of mystery that remained unsolved in last ten years.

In this work, we have shown that the infinite-dimensional analog of the complex Lohe sphere model appears as a coefficient system of the Schrödinger–Lohe model. Thanks to this explicit connection between the complex Lohe sphere model and the Schrödinger–Lohe model, we establish an infinite-dimensional Schrödinger–Lohe hierarchy (see the diagram below):

```
Complex Lohe sphere ➔ Generalized Lohe Matrix ➔ Lohe Tensor
                      Quantum lifting
                                    Quantum lifting
Schrödinger Lohe ➔ Schrödinger Lohe Matrix ➔ Schrödinger Lohe Tensor
                      Quantum lifting
```

There are many unresolved issues related to this work. For example, in this paper, we considered the homogeneous ensemble with the same free flow. Thus, analysis on the emergent dynamics of heterogeneous ensemble is still far from complete, for example, we do not have a good analysis on the complete aggregation of the Schrödinger–Lohe model except a weak result on the practical aggregation. These issues will be left for a future work.
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