Mass and Spin of Poincaré Gauge Theory

by

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Abstract

We discuss two expressions for the conserved quantities (energy momentum and angular momentum) of the Poincaré Gauge Theory. We show, that the variations of the Hamiltonians, of which the expressions are the respective boundary terms, are well defined, if we choose an appropriate phase space for asymptotic flat gravitating systems. Furthermore, we compare the expressions with others, known from the literature.

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1. Introduction

If one looks for expressions of energy for a gravitating system, a natural candidate for such an expression will be given by the Hamiltonian. In gravitational theories the Hamiltonian can be written in the form $H = dB + J \cong dB$, where $J$ vanishes for exact solutions. Consequently the energy of exact solutions may be defined by $E := \int_\Sigma H = \int_{\partial \Sigma} B$, where $\Sigma$ is a 3-dim. spacelike hypersurface. However the Hamiltonian is not completely fixed by the requirement of generating the correct field equations, it can be modified by adding a total divergence or equivalently a boundary term at spatial infinity. As pointed out by Regge and Teitelboim [1] in the case of General Relativity, one has to adjust the boundary term in such a way that the variation of the Hamiltonian is well defined; this means that no variations of the derivatives of the variables occur. But this argumentation of Regge and Teitelboim fixes only the integrals, not the integrands, and the whole discussion depends on the phase space chosen. Therefore some freedom in constructing energy expressions still exists.

In this paper, we will discuss two possible boundary terms for energy momentum and angular momentum of the Poincaré Gauge Theory (PGT). One of the expressions was given by Nester [2], the other one (see [3]) is a modification of it. Both expressions were tested in [4] with exact solutions, but a detailed discussion has not been given. We will show that the variations of the respective Hamiltonians are well defined, in the sense of Regge and Teitelboim, if we choose an appropriate phase space for asymptotic flat gravitating systems.

Suitable expressions for the conserved quantities of the PGT for asymptotic flat solutions were given earlier by Hayashi and Shirafuji [5] and by Blagojević and Vasič [6]. In their works they have to restrict themself to an asymptotic Cartesian basis. Also approaches were made for calculating conserved quantities of the PGT in asymptotic anti-de Sitter space times, see [7] and [8], but they didn’t proof to be successful. One advantage of the expressions discussed here is that they need no restriction to an asymptotic Cartesian basis and can be evaluated also in asymptotic anti-de Sitter space times.

First we will give a brief introduction into the framework of the PGT. Then we will calculate the fall off of asymptotic flat solutions of the PGT in order to be able to fix the phase space. In section 3 we will write down the Hamiltonian and the expressions we will deal with. The variation of the Hamiltonian and the argumentation of Regge and Teitelboim are worked out in section 4. In section 5 we show that the integrals of our expressions are indeed finite and conserved. Finally, in section 6, we will compare them with the work of Hayashi & Shirafuji [5] and Blagojević & Vasič [6].

Let us shortly recapitulate the underlying theory and fix the conventions. The PGT (see, for instance, [9,10]) is a gauge theory of gravity in which spacetime is represented by a 4-dimensional Riemann-Cartan manifold. The gauge potentials are the orthonormal basis 1-forms $\partial^\alpha$ and the connection 1-forms $\omega_{\alpha}^\beta$. The corresponding field strengths are the torsion $\Theta^\alpha = D \partial^\alpha := d \partial^\alpha + \omega^\alpha_\mu \wedge \partial^\mu$ and the curvature $\Omega^\alpha_{\beta} := d \omega^\alpha_\beta + \omega^\beta_\gamma \wedge \omega^\gamma_\alpha$. The sources of the gravitational fields are the 3-forms of material energy-momentum $\Sigma_\alpha$ and spin angular-momentum $\tau^\alpha_{\beta}$ which are variational derivatives of the material Lagrangian with respect to the gauge potentials. In order to have a local Poincaré
invariant Lagrangian for the gravitational field, the gravitational Lagrangian should be of the form

\[ L = L_G(\vartheta^\alpha, \Theta^\alpha, \Omega_{\alpha \beta}) + L_M(\vartheta^\alpha, \psi, D\psi) \quad . \] (1.1)

Variation with respect to the potentials yield the field equations

\[ DH_\alpha - \epsilon_\alpha = \Sigma_\alpha \quad \text{and} \quad DH_{\alpha \beta} - \epsilon_{\alpha \beta} = \tau_{\alpha \beta} \quad , \] (1.2)

where

\[ H_\alpha := - \frac{\partial L_G}{\partial d\vartheta^\alpha} = - \frac{\partial L_G}{\partial \Theta^\alpha} \quad , \quad H^\alpha_{\beta} := - \frac{\partial L_G}{\partial d\omega^\alpha_{\beta}} = - \frac{\partial L_G}{\partial \Omega^\alpha_{\beta}} \quad , \] (1.3)

and

\[ \epsilon_\alpha = e_\alpha L_G + (e_\alpha \Theta^\mu) \wedge H_\mu + (e_\alpha \Omega^\mu_{\nu}) \wedge H^\mu_{\nu} \quad , \quad \epsilon_{\alpha \beta} = \partial_{[\beta} \wedge H_{\alpha]} \quad . \] (1.4)

In this article we restrict ourself to Lagrangians, which are at most quadratic in the field strengths. This leads to

\[ L = \frac{\Lambda_{\cos}}{l^2} \eta - \frac{1}{2} \Theta^\alpha \wedge H_\alpha - \frac{1}{2} \Omega_{\alpha \beta} \wedge H^\alpha_{\beta} + \frac{a_0}{4l^2} \Omega^\alpha_{\beta} \wedge \eta^\alpha_{\beta} \quad . \] (1.5)

where \( \Lambda_{\cos} \) is the cosmological constant, \( l \) the Planck length, \( \eta \) the volume 4-form, \( \eta^\alpha_{\beta} = \ast(\vartheta^\alpha \wedge \vartheta^\beta) \), and \( \ast \) the Hodge star. Then the field momenta can be expressed in terms of the irreducible pieces of the field strength [11]:

\[ H_\alpha = - \frac{1}{l_0^2} \ast \left( \sum_{n=1}^{3} a_n (n) \Theta_\alpha \right) \quad , \quad H_{\alpha \beta} = - \frac{a_0}{2l_0^2} \eta_{\alpha \beta} - \frac{1}{\kappa} \ast \left( \sum_{n=1}^{6} b_n (n) \Omega_{\alpha \beta} \right) \quad , \] (1.6)

where \( \kappa \), \( a_i \), and \( b_i \) are coupling constants. As we are interested in asymptotic flat solutions, we set \( \Lambda_{\cos} = 0 \).

In this article we use Greek letters to denote anholonomic indices, and Latin letters for holonomic indices. The metric is given by \( g_{\alpha \beta} = diag(-1, 1, 1, 1) \).

We will need some technical details: The connection \( \omega^\alpha_{\beta} = r^\alpha_{\beta} + K^\alpha_{\beta} \) splits into a purely Riemannian part \( r^\alpha_{\beta} \) and the contortion \( K^\alpha_{\beta} \). The purely Riemannian part of the curvature is denoted by \( R^\alpha_{\beta} \) (Riemann 2-form). The Lie derivative of a scalar valued form \( \Psi \) with respect to a vector field \( \xi \) is given by \( \ell_\xi \Psi := \xi \lfloor d\Psi + d(\xi \lceil \Psi) \). For tensor valued forms we have to use

\[ \mathcal{L}_v \Psi^\alpha_{\beta} := \ell_v \Psi^\alpha_{\beta} + \Psi^\mu_{\beta} l^\alpha_{\mu} - \Psi^\alpha_{\mu} l^\beta_{\mu} \quad , \quad l^\alpha_{\beta} := e_\alpha \lfloor \ell_\xi \vartheta^\beta \quad , \] (1.7)

which may be more generally written as

\[ \mathcal{L}_v \Psi = \ell_v \Psi - l^\alpha_{\beta} S^\alpha_{\beta} \Psi \quad , \] (1.8)

where \( S^\alpha_{\beta} \) is the generator of the Lorentz group in the respective representation.
2. The fall off of asymptotic flat PGT solutions

For our discussion of the Hamiltonian and its boundary term we have to choose the phase space of our system. Therefore we consider first the behaviour of asymptotically flat exact solutions of the PGT (compare [9],[12]).

We demand the solutions to be asymptotically flat and we use an asymptotically Cartesian coordinate system. Therefore we have a radial coordinate \( r \). For a function, we define the fall off by

\[
\lim_{r \to \infty} (r^n f) = \text{constant} \iff f = O_n
\]

e.t.c. We say a \( p \)-form \( \omega \) is \( O_n \), \( \omega = O_n \), iff all their components with respect to the asymptotically Cartesian basis are at least \( O_n \).

We start with the requirement

\[
e_i^\alpha \to \delta_i^\alpha + O_1 \quad \text{and} \quad e_i^{\alpha,j} \to O_2 \quad \text{and} \quad \omega_{i\alpha}^\beta \to O_1 \quad \text{and} \quad \omega_{i\alpha}^{\beta,j} \to O_2 \quad . \quad (2.1)
\]

where \( e_i^\alpha \) are the components of the basis 1-forms and \( \omega_{i\alpha}^\beta \) the components of the connection forms with respect to the asymptotically Cartesian holonomic basis: \( \vartheta^\alpha = e_i^\alpha dx^i \), \( \omega_{\alpha \beta} = \omega_{i\alpha}^{\beta} dx^i \). Of course, it follows that \( g_{ij} = o_{ij} + O_1 \), \( g_{ij,k} = O_2 \) (\( o_{ij} \) is the Minkowskian metric tensor). For the field strength we have \( \Theta^\alpha = O_1 \), \( d\Theta^\alpha = O_2 \), \( \Omega_{\alpha \beta} = O_2 \), \( d\Omega_{\alpha \beta} = O_3 \).

As we are interested in the asymptotics of isolated gravitating systems, we only consider the vacuum field equations. For our purpose it is useful to split the momenta into an “Einsteinian” part and the rest (1):

\[
H^E_\alpha := H_\alpha - a_0 H_\alpha \quad \text{where} \quad H_\alpha^E := - \frac{1}{2l^2} K^{\mu \nu} \wedge \eta_{\mu \nu \alpha} \quad \text{and} \quad (2.2)
\]

\[
H_{\alpha\beta}^E := - \frac{a_0}{2l^2} \eta_{\alpha\beta} + \overline{H}_{\alpha\beta} := \overline{H}_{\alpha\beta} - a_0 H^E_{\alpha\beta} \quad . \quad (2.3)
\]

Then \( \overline{H}_{\alpha\beta} \) is at least \( O_2 \). Because of \( D\overline{H}_{\alpha\beta}^E = \vartheta_{[\alpha} \wedge \overline{H}_{\beta]} \) we get from the second field equation \( D\overline{H}_{\alpha\beta} = \vartheta_{[\alpha} \wedge \overline{H}_{\beta]} \), which can be solved for \( \overline{H}_\alpha \):

\[
\overline{H}_\alpha = e^\mu [D\overline{H}_{\alpha\mu} + \frac{1}{4} \vartheta_{\alpha} \wedge (e^\nu] e^\mu] D\overline{H}_{\nu \mu}) \quad , \quad (2.4)
\]

and the irreducible decomposition of this equation gives

\[
(a_n + a_0 a_n)^{(n)} \Theta^\alpha = O_3 \quad , \quad (2.5)
\]

where \([a_1, a_2, a_3] = [-1, 2, \frac{1}{7}]\). Because of the identity

\[
D \overline{H}_\alpha^E - \frac{1}{2} (e_{\alpha}] \Theta^\beta) \wedge \overline{H}_\beta^E + \frac{1}{2} \Theta^\beta \wedge (e_{\alpha}] \overline{H}_\beta^E) \equiv \frac{1}{2l^2} (R^{\mu \nu} - \Omega^{\mu \nu}) \wedge \eta_{\mu \nu \alpha} \quad , \quad (2.6)
\]

(1) If we choose the paramater \( a_i, b_i \) such that \( H_{\alpha\beta} = \eta_{\alpha\beta}/(2l^2) \) and \( H_\alpha = 0 \), then we get just Einstein’s theory, whereas in the teleparallelism \( \Omega_{\alpha \beta} = 0 \) the choice \( H_\alpha = -K^{\mu \nu} \wedge \eta_{\mu \nu \alpha}/(2l^2) \) leads to the teleparallel equivalent of General Relativity.
(ηαβγ = γ(θα ∧ θβ ∧ θγ)) we get in the case that $\bar{H}_\alpha = 0$, for the left hand side of the first field equation (which we will call $F_\alpha$):

$$F_\alpha = \frac{a_0}{2l^2}(\Omega_{\mu}^\nu - R_{\mu}^\nu) \land \eta_{\nu \alpha} - \frac{a_0}{2l^2} \Omega_{\mu}^\nu \land \eta_{\nu \alpha} + \text{ squares of curvature}.$$ \hspace{1cm} (2.7)

Here the first term originates from the torsion part, and the last two terms from the curvature part of the field equation. If $\bar{H}_\alpha$ is not equal to zero, then it is of order $O_3$ (see (1.6,2.3)) and gives no new contribution to the last equation (up to $O_4$ terms). Hence, in any case we have

$$F_\alpha = O_4 - \frac{a_0}{2l^2} R_{\mu}^\nu \land \eta_{\nu \alpha}.$$ \hspace{1cm} (2.8)

If we require $\omega_{\alpha \beta} = O_{1+ \gamma}$ with $\gamma > 0$, then the term $O_4$ in the equation (2.7) above will change into $O_{4+ \gamma}$, and also $\bar{H}_\alpha$ will be of order $O_{3+ \gamma}$.

### 3. The Hamiltonian

We rewrite the Lagrangian in the following form (see [13])

$$L = -\Theta^\alpha \land H_\alpha - \Omega_{\alpha \beta} \land H_\alpha^\beta + D\psi \land P - \Lambda(\vartheta^\alpha, H_\alpha, H_\alpha^\beta, \psi, P).$$ \hspace{1cm} (3.1)

The field equations follow from the variational principle regarding the potentials $\vartheta^\alpha$, $\omega_{\alpha \beta}$, $\psi$ and the momenta $H_\alpha$, $H_\alpha^\beta$ and $P$ as independent. The potential $\Lambda$ is quadratic in $H_\alpha$ and $H_\alpha^\beta$ in such a way, as to reproduce relations equivalent to (1.2). The Lagrangian is invariant under diffeomorphisms and $SO(3,1)$ rotations. This invariance leads in the usual way to the Noether identities. We vary the Lagrangian,

$$\delta L = d[\delta \vartheta^\alpha \land H_\alpha - \delta\omega_{\alpha \beta} \land H_\alpha^\beta + \delta\psi \land P] + \delta\vartheta^\alpha \land \frac{\delta L}{\delta \vartheta^\alpha} + \delta\omega_{\alpha \beta} \land \frac{\delta L}{\delta \omega_{\alpha \beta}} +$$

$$\delta\psi \land \frac{\delta L}{\delta \psi} + \frac{\delta L}{\delta H_\alpha} \land \delta H_\alpha + \frac{\delta L}{\delta H_\alpha^\beta} \land \delta H_\alpha^\beta + \frac{\delta L}{\delta P} \land \delta P$$ \hspace{1cm} (3.2)

and deal only with the symmetry transformations Lorentz rotations and diffeomorphisms. Then we have for instance $\delta\vartheta^\alpha = \varepsilon^\alpha_{\beta \gamma} \vartheta^\beta - \ell_\xi \vartheta^\alpha$ or, $\delta\psi = \varepsilon^\beta_{\alpha \gamma} \psi - \ell_\xi \psi$ etc. $(S^\alpha_{\beta}$ are the generators of Lorentz rotations, $\varepsilon^\beta_{\alpha}$ arbitrary parameters, and $\xi$ is an arbitrary vector field, generating the diffeomorphism). Considering only diffeomorphism invariance, we will get eventually the first Noether identity (compare [10], [13]):

$$\langle \xi \mid \vartheta^\alpha \rangle \land D \frac{\delta L}{\delta \vartheta^\alpha} + (-1)^{p+1} \langle \xi \mid \psi \rangle \land D \frac{\delta L}{\delta \psi} + D \frac{\delta L}{\delta H_\alpha} \land \langle \xi \mid H_\alpha \rangle + D \frac{\delta L}{\delta H_\alpha^\beta} \land \langle \xi \mid H_\alpha^\beta \rangle +$$

$$(-1)^{p+1} D \frac{\delta L}{\delta P} \land \langle \xi \mid P \rangle = \langle \xi \mid D\vartheta^\alpha \rangle \land \frac{\delta L}{\delta \vartheta^\alpha} + \langle \xi \mid \Omega_{\alpha \beta} \rangle \land \frac{\delta L}{\delta \omega_{\alpha \beta}} + \langle \xi \mid D\psi \rangle \land \frac{\delta L}{\delta \psi} +$$

$$\frac{\delta L}{\delta H_\alpha} \land \langle \xi \mid DH_\alpha \rangle + \frac{\delta L}{\delta H_\alpha^\beta} \land \langle \xi \mid DH_\alpha^\beta \rangle + \frac{\delta L}{\delta P} \land \langle \xi \mid DP \rangle ,$$ \hspace{1cm} (3.3)
and we get the second Noether identity by using the Lorentz invariance of the Lagrangian,

\[ D \frac{\delta L}{\delta \omega_{\alpha \beta}} = -\nabla^{[\alpha} \frac{\delta L}{\delta \psi} \wedge S_{\beta]}^{\alpha \beta} \psi \wedge \frac{\delta L}{\delta H_{\alpha]}^{[\beta}} \wedge H_{\beta]} - \frac{\delta L}{\delta H_{\mu}^{[\alpha}} \wedge H_{\mu]}^{\alpha \mu}
\]

\[ + \frac{\delta L}{\delta H_{\mu}^{\alpha \alpha}} \wedge H_{\mu}^{\mu} + \frac{\delta L}{\delta P} \wedge PS_{\alpha \beta}. \]  

(3.4)

The invariance of the Lagrangian leads also to the Noether current. We can identify the Noether current 3-form from (3.2) as

\[ H = \xi \wedge N^{\alpha} - \delta \omega_{\alpha \beta} \wedge H_{\alpha \beta} + \delta \psi \wedge P. \]  

(3.5)

For a timelike vector field \( \xi \) and vanishing \( \epsilon \), the Noether current is just the canonical Hamiltonian of the theory, and therefore we will call in future \( H \) also the (generalized) Hamiltonian.

This Hamiltonian can be recast in the form:

\[ H \equiv dB + J := d \left[ \xi \wedge H_{\alpha} + \tilde{\epsilon}_{\alpha} \wedge H_{\alpha \beta} - (\xi \wedge P) \right] + (\xi \wedge \vartheta^{\alpha}) \wedge \frac{\delta L}{\delta \vartheta^{\alpha}} +
\]

\[ (\xi \wedge \psi) \wedge \tilde{\epsilon}_{\alpha} \wedge \frac{\delta L}{\delta \psi} \wedge \frac{\delta L}{\delta H_{\alpha}^{\beta}} \wedge (\xi \wedge H_{\alpha}) + \left( \frac{\delta L}{\delta H_{\alpha}^{\alpha \beta}} \wedge (\xi \wedge H_{\alpha \beta}) + (-1)^{p+1} \frac{\delta L}{\delta P} \wedge (\xi \wedge P) \right), \]  

(3.6)

where \( \tilde{\epsilon}_{\alpha} := \epsilon_{\alpha} + \xi \omega_{\alpha \beta}. \) Obviously, the Hamiltonian is weakly conserved (that is conserved for exact solutions), \( dH \cong 0. \)

For a space time symmetry (a Killing field \( \xi \)) it is known that the gravitational part of the Hamiltonian is conserved, even if the gravitational field equations are not fulfilled. With the help of the Noether identities, we calculate the derivative of the Hamiltonian (\( \tilde{\omega} \) is the transposed connection \(^{(2)} \), \( \tilde{\omega}_{\alpha \beta} := \omega_{\alpha \beta} + e_{\alpha} \Theta_{\beta} \)):

\[ dH = dJ = L_{\xi} \omega_{\alpha} \wedge \frac{\delta L}{\delta \omega_{\alpha}^{\beta}} + L_{\xi} \psi \wedge \frac{\delta L}{\delta \psi} + \left( \frac{\delta L}{\delta H_{\alpha}^{\alpha \beta}} \wedge L_{\xi} H_{\alpha}^{\alpha} \right) +
\]

\[ \frac{\delta L}{\delta H_{\alpha}^{\alpha \beta}} \wedge L_{\xi} H_{\alpha \beta} + \frac{\delta L}{\delta P} \wedge L_{\xi} P + d \left[ (\tilde{\epsilon}_{\alpha} - \tilde{D}_{\alpha} \epsilon_{\beta}) \wedge \frac{\delta L}{\delta \omega_{\alpha}^{\beta}} \right]. \]  

(3.7)

We see, that if \( \xi \) is a Killing field and the matter field equation is fulfilled, then the Hamiltonian is conserved, if we choose \( \epsilon_{\alpha} = l_{\alpha} \) (because of the identity \( l_{\alpha} = \xi \omega_{\alpha}^{\beta} \equiv \tilde{D}_{\alpha} \epsilon_{\beta} \)).

But we cannot simply use the boundary term \( B \) of (3.6) as superpotential for conserved quantities. There exist mainly two obstacles:

One reason is that the variation principle, as used in PGT, doesn’t give a proper momentum (it leads just to \( \eta_{\alpha \beta} \)) for the linear (Hilbert-term) part of the Lagrangian. We can study the situation in the case of GR:

\(^{(2)}\) The name is only appropriate, if one chooses a holonomic basis.
From the Hilbert Lagrangian we can get, by adding a total divergence, the Lagrangian \( L' \), which is constructed out of squares of the connection,

\[
L_H = -\frac{1}{2l^2} R_{\alpha}{}^{\beta} \wedge \eta_{\beta} =: L' = -\frac{1}{2l^2} d(r_{\alpha}{}^{\beta} \wedge \eta_{\beta}) =: L' + dK. \tag{3.8}
\]

We choose \( L' \) as it does not contain second derivatives of the basis 1-form. The variation of this Lagrangian is

\[
\delta L' = \delta \vartheta^\mu \wedge \left[ M P_\mu + \frac{1}{2} d \vartheta_\alpha \wedge (\epsilon_\mu [ P^\alpha ) - \frac{1}{2} (\epsilon_\mu [ d \vartheta_\alpha ) \wedge P^\alpha \right] + d(\delta \vartheta^\mu \wedge P^\alpha) , \tag{3.9}
\]

the term in the brackets is the Einstein 3-form, and the momentum is given by \( P_\alpha = -1/(2l^2) r_{\mu} {}^\nu \wedge \eta^{\mu} {}^\nu_{\alpha} \), which was first introduced by Møller [16] and is a kind of anholonomic Freud potential. The Lagrangian may be now rewritten as \( L' \equiv 1/2 d \vartheta^\alpha \wedge P_\alpha \).

We split the variation into a Lie derivative \( \delta \xi = -\ell \xi \) and a rotational part, where the generators of Lorentz transformations are denoted by \( \varepsilon^\alpha {}^\beta \) \( ( \delta \varepsilon \vartheta^\alpha = \varepsilon^\alpha {}^\beta \vartheta^\beta ) \), \( \delta = \delta \xi + \delta \varepsilon \).

Rewriting (3.9) gives

\[
0 = \delta \vartheta^\alpha \wedge \frac{\delta L}{\delta \vartheta^\alpha} + d[\delta \vartheta^\alpha \wedge P^\alpha + \xi] L' + \delta \varepsilon K .
\]

We identify the term in the brackets as the (generalized Hamiltonian or) Noether current. For the variations we get

\[
\delta \vartheta^\alpha \wedge P^\alpha + \delta \xi K = \frac{M}{2l^2} D(\varepsilon^\beta {}^\alpha \wedge \eta_{\beta} ) \tag{3.10}
\]

and

\[
-\ell \xi \vartheta^\alpha \wedge P^\alpha + \xi] L' = -d(\varepsilon^\alpha \! P_\alpha) + \xi \wedge \frac{\delta L}{\delta \vartheta^\alpha} , \tag{3.11}
\]

therefore the superpotential for the Noether current is

\[
-\varepsilon^\alpha \! P_\alpha + \frac{1}{2l^2} \varepsilon^\beta {}^\alpha \wedge \eta_{\beta} . \tag{3.12}
\]

Apparently this term is not contained in \( B \) of (3.6).

The second reason is, that the boundary term \( B \) transforms not homogeneously. This restricts the range of application of this term to asymptotically Cartesian bases. If we want to improve \( B \) in this respect, we have to introduce an additional structure, a background field for instance. Moreover, in general spacetimes there is no Killing field at our disposal. But if we deal with spacetimes, which possess asymptotical symmetries, it is natural to use these asymptotical symmetries in order to fix the free parameters \( \xi \) and \( \varepsilon \). Therefore, we introduce a background spacetime \( b U_4 \) which is a copy of the asymptotic regions (spacelike infinity) of our physical spacetime \( U_4 \). This background space time will allow us also to construct covariant expressions for the boundary term, see below.

The geometric quantities \( \vartheta^\alpha \) and \( \omega^\alpha {}^\beta \) of the background space time should not exhibit
any dynamics. In order to deal with both the physical and the background quantities, we map the background space-time onto the physical space-time by some diffeomorphism \( f \) (we need only to identify the outer regions of the space times) \( f : \mathcal{U}_4 \rightarrow \mathcal{U}_4 \). This induces the forms \( \vartheta^\alpha := f^{-1*} b^\alpha \) and \( \omega^\alpha_\beta := f^{-1*} b^\alpha_\beta \) and vector fields \( \xi := f_* \xi \). For the diffeomorphism \( f \) we demand that \( g_{ij} = b_{ij} + O(1/r) \) (3.12)

for an appropriate coordinate system. The mapping can be constructed by identifying the coordinate systems \( x \) of \( \mathcal{U}_4 \) and \( y \) of \( \mathcal{U}_4 \), with \( f = y^{-1} \circ x \) if the coordinate systems fulfill (3.14) (see, for instance, [14] for a similar construction).

If we now vary the potentials and momenta of the physical spacetime, the quantities \( \vartheta^\alpha \) and \( \omega^\alpha_\beta \) remain of course fixed. Also \( \vartheta^\alpha \) and \( \omega^\alpha_\beta \) remain fixed, if the function \( f \) does not change. Notice that in this construction a change of \( \xi \), induced by variations of the potentials and momenta, can only occur, if the function \( f \) is affected by this variation. But in this case, also the quantities \( \vartheta^\alpha_\beta \) and \( \omega^\alpha_\beta \) have to change. We will not consider this possibility.

The Hamiltonian is not fixed by the conservation law or the field equations, we can always add a surface term, \( H = dB + J \rightarrow H_1 = dB_i + J \). This freedom we will use to improve \( B \) concerning the flaws mentioned above and make the variation of the (improved) Hamiltonian well defined in the sense of Regge and Teitelboim [1] (see below). As we are only interested in the behavior of the boundary terms in the asymptotic region of spacetime, we will henceforth neglect the matter fields, which are supposed to vanish in this region. Here we will show, that both of the following boundary terms

\[
B_1 = (\xi | \vartheta^\alpha) \wedge \Delta H_\alpha + \Delta \omega^\alpha_\beta \wedge (\xi | H^\alpha_\beta) + (\xi | \omega^\alpha_\beta) \wedge \Delta H^\alpha_\beta ,
\]

(where \( \Delta \omega^\alpha_\beta = \omega^\alpha_\beta - \vartheta^\alpha_\beta \) etc.) which was given by Nester [2] and (see [3])

\[
B_2 = (\xi | \vartheta^\alpha) \wedge \Delta H_\alpha + \Delta \omega^\alpha_\beta \wedge (\xi | H^\alpha_\beta) + \l^\circ D_\alpha \xi^\beta \wedge \Delta H^\alpha_\beta
\]

make the variations of the corresponding Hamiltonians well defined, if we choose suitable phase spaces for asymptotic flat solutions. The expressions were not deduced as Noether current of a suitable Lagrangian (like \( B \)), but the improvements were done by hand and we will justify this choice later (section 4). Observe that the expressions contains a term \( \omega^\alpha_\beta \xi | H^\alpha_\beta \), which equals the Møller potential in leading order. The expression \( \varepsilon^\alpha_\beta = e^\alpha_\beta \l^\circ D \xi^\beta \) which we choose in \( B_2 \), is a covariant generalization of the generators of rotations in Minkowski space and is in fact antisymmetric, since \( \xi \) is a killing vector of the background.

The conserved quantities (total momentum and angular momentum) of asymptotic flat solutions are now calculated by integrating the surface term over a 2-sphere with radius \( R \) and take the limes \( R \rightarrow \infty \):

\[
Q(\xi) := \lim_{R \rightarrow \infty} \int_{S^2} B(\xi)
\]

(3.16)
By choosing the vector field $\xi$ to be one of the Killing-fields of the Minkowski-space, one get the corresponding conserved quantity. This calculations were done in [4] for both expressions with asymptotic flat and asymptotic constant curvature solutions of PGT. The results were, for the tested solutions, the same as for the corresponding solutions of General Relativity.

4. The variation of the Hamiltonian

The variation of the Hamiltonian has the general pattern:

$$\delta \int_{\Sigma} H = \int_{\Sigma} \left( \delta \vartheta^\alpha \wedge a_\alpha + \delta \omega^\alpha_\beta \wedge b^\alpha_\beta + c^\alpha \wedge \delta H^\alpha_\alpha + d^\alpha_\beta \wedge \delta H^\alpha_\beta + dX \right), \quad (4.1)$$

As pointed out by Regge and Teitelboim [1], we have to make sure that $\int_{\Sigma} dX$ vanishes, otherwise the field equations are not expressible as variational derivatives of the Hamiltonian with respect to the potentials and momenta. To reach this goal, we are free to add a boundary term to the Hamiltonian.

Nester set $\varepsilon^\alpha_\beta = 0$ and obtained the following term from the variation of his Hamiltonian $H_1 = dB_1 + J$:

$$X_1 = \delta(\xi | \vartheta^\alpha) \wedge \Delta H^\alpha_\alpha - (\delta \vartheta^\alpha) \wedge (\xi | H^\alpha_\alpha) + \Delta \omega^\alpha_\beta \wedge \delta(\xi | H^\alpha_\beta) + \delta(\xi | \omega^\beta_\beta) \wedge \Delta H^\alpha_\beta. \quad (4.2)$$

The variation of the Hamiltonian $H_2 = dB_2 + J$ yields:

$$X_2 = \delta(\xi | \vartheta^\alpha) \wedge \Delta H^\alpha_\alpha - (\delta \vartheta^\alpha) \wedge (\xi | H^\alpha_\alpha) + \Delta \omega^\alpha_\beta \wedge \delta(\xi | H^\alpha_\beta) + - (\xi | \omega^\beta_\beta) \wedge \delta H^\alpha_\beta + \delta(e_\alpha | \bar{\omega}^\beta_\beta) \wedge \Delta H^\alpha_\beta. \quad (4.3)$$

In (4.3) we set $\delta(e_\alpha | \bar{\omega}^\beta_\beta) = 0$, because we do not vary the background quantities. For completeness we also write down the variation of the canonical Hamiltonian (3.6):

$$X = \delta(\xi | \vartheta^\alpha) \wedge H^\alpha_\alpha - \delta \vartheta^\alpha \wedge (\xi | H^\alpha_\alpha) + \delta(\xi | \omega^\beta_\beta) \wedge H^\alpha_\beta - \delta \omega^\beta_\beta \wedge (\xi | H^\alpha_\beta) + \delta(e_\alpha | \bar{\omega}^\beta_\beta) \wedge \Delta H^\alpha_\beta. \quad (4.4)$$

These formulas are for no use, if we do not have a phase space given. Here we are interested in asymptotically flat solutions. For the potentials and momenta we do not demand that they fulfill the field equations, but we require that they possess the same asymptotic fall off as asymptotically flat solutions of the field equations as worked out in Sec. 2. For the background we choose the Minkowski space and a Cartesian basis. Then we have vanishing $\bar{\omega}^\alpha_\alpha$, $\bar{\omega}^\beta_\beta$, and the field momentum of rotation is reduced to $\bar{H}^\alpha_\beta = -(a_0/2l^2) \bar{\eta}^\alpha_\beta$. For the potentials we require the fall off as given in (2.1) and for the momenta as well as for their variations we require $H_\alpha = O_3$, $H^\alpha_\beta = O_2$. 
Now we can start with the variation of the Hamiltonian for asymptotic flat spacetimes as specified above. We begin with the variation of the canonical Hamiltonian and first consider only translations.

\[ X = \xi \left[ \delta \partial^\alpha \wedge H_\alpha + \delta \omega_\alpha^\beta \wedge H_\alpha^\beta \right]. \]  

(4.5)

The integral \( \int_\Sigma dX = \int_{\partial \Sigma} X \) will vanish, if \( X \) fall off faster than \( r^{-2} \). Because the variations of the potentials are independent, each of the terms should fall off faster than \( O_2 \) in order to make the variation of the Hamiltonian well defined. But solutions will in general not have this fall off. Moreover, the Hamiltonian will not give a reasonable energy momentum-expression in the case of GR. Therefore we turn to the Hamiltonian 2. To the canonical Hamiltonian (3.6) we add the following surface term:

\[ -d \left[ \xi \left( \omega_\alpha^\beta \wedge H_\alpha^\beta \right) \right]. \]  

(4.6)

The improved Hamiltonian is then given by

\[ \tilde{H} = d \left[ (\xi \partial^\alpha) \wedge H_\alpha + \omega_\alpha^\beta \wedge \xi H_\alpha^\beta \right] + J, \]  

(4.7)

which is just the Hamiltonian 2 for the case that \( \xi \) is a translational Killing field of the background Minkowski space time, because in this case \( \tilde{D}_\alpha \xi^\beta \) vanishes. The variation of this Hamiltonian leads to a boundary term

\[ X_2 = \xi \left[ \delta \partial^\alpha \wedge H_\alpha - \omega_\alpha^\beta \wedge \delta H_\alpha^\beta \right]. \]  

(4.8)

If we now use the relations (2.2,3), we obtain

\[ X_2 = \xi \left[ \delta \partial^\alpha \wedge \tilde{H}_\alpha + \frac{a_0}{2l^2} \delta \partial^\mu \wedge r_\alpha^\beta \wedge \eta^\alpha_{\beta \mu} - \omega_\alpha^\beta \wedge \delta \tilde{H}_\alpha^\beta \right] = \xi O_3. \]  

(4.9)

Therefore the variation of the generator of the translations is well-defined.

For spacelike rotations, the Killing field of the background is

\[ \xi^i = x_N \delta_M^i - x_M \delta_N^i, \quad M,N = 1,2,3 \quad \text{fixed}. \]  

(4.10)

Now \( \varepsilon^\beta_\alpha \) no longer vanishes. As \( \tilde{D}_\alpha \xi^\beta \) is evaluated on the background, its variation vanishes and the variation of the Hamiltonian 2 is also given by (4.9).

We can write \( X = \xi |Y \) (for \( Y = Y^\alpha \eta_\alpha \)), and we are only interested in the projection of this integrand on the 2-sphere \( t=\text{const}, \ r=\text{const} \). This projection can be written as \( \xi^\alpha Y^0 dS_\alpha \sim \xi^\alpha Y^0 x_\alpha d\theta d\phi = 0 \). Therefore the variation of the generator of the rotations is well defined.

For the boosts we need stronger restrictions (compare [6]). We demand

\[ \omega_\alpha^\beta = O_{1+\gamma} \quad \text{where } \gamma > 0. \]  

(4.11)

Then the last term of the rhs of (4.9) is of order \( O_{3+\gamma} \), and, because of

\[ \xi^i = x_N \delta_0^i - x_0 \delta_N^i, \]  

(4.12)
we have
\[ X_2 = x_N \partial_t \left[ \frac{a_0}{2l^2} \delta \partial_x^\mu \wedge r_{x}^{\alpha} \wedge \eta^{\alpha}_{\beta \mu} \right] + O_{2+\gamma} \]  
(4.13)
Furthermore we have to require parity conditions (compare [1],[5],[6]),
\[ e_i^\alpha = \delta_i^\alpha + \frac{a_i^\alpha(n)}{r} + O_{(1+\zeta)} \quad \zeta > 0 \quad , \quad a_i^\alpha(n) = a_i^\alpha(-n) \]  
(4.14)
Then the Levi Civita connection is odd in leading order \((r_{x}^{\alpha} = \mu^{\alpha} + O_{2+\zeta})\), \(\mu^{\alpha} = O_2\) and \(\mu^{\alpha}(n) = -\mu^{\alpha}(-n))\). We find
\[ x_N \partial_t \left[ \frac{a_0}{2l^2} \delta \partial_x^\mu \wedge r_{x}^{\alpha} \wedge \eta^{\alpha}_{\beta \mu} \right] = \chi^a dS_a \]  
(4.15)
where \(\chi^a dS_a = -2x_N \delta e_i^\mu r_{x}^{\alpha} (e_i^{[\beta} e_j^{\gamma]} \mu e^{\alpha \alpha} + e_i^{[\beta} e_j^{\gamma]} e^{\alpha \beta} + e_i^{[\alpha} e_j^{\beta]} e^{\alpha \mu}) r_{0a}. \) We recognize that in leading order (that is \(O_2\)) \(\chi^a\) is an even function. Because \(dS_a\) is odd, the integral over a 2-sphere will vanish. Therefore the variations of the boost generators are well defined.

The variation of Nester’s Hamiltonian gives a boundary term
\[ X_1 = \xi \left[ \delta \partial_x^\alpha \wedge H_\alpha \right] + \delta (\xi \sqrt{\omega_\alpha^{\beta}} \wedge \sqrt{\Delta H_\alpha^{\beta}} + \sqrt{\omega_\alpha^{\beta}} \wedge \sqrt{\mu_\alpha^{\beta}}) \]  
(4.16)
which can be rewritten into
\[ X_1 = \xi \left[ \delta \partial_x^\alpha \wedge \bar{H}_\alpha \right] + \delta (\xi \sqrt{\omega_\alpha^{\beta}} \wedge \sqrt{\Delta H_\alpha^{\beta}} - \frac{a_0}{2l^2} \xi \left[ \delta \partial_x^\alpha \wedge r_{x}^{\alpha} \wedge \eta^{\alpha}_{\beta \gamma} \right] \\
+ \frac{a_0}{2l^2} \delta \partial_x^\gamma \wedge (\xi \sqrt{\omega_\alpha^{\beta}} \wedge \sqrt{\eta^{\alpha}_{\beta \gamma}} - \frac{a_0}{2l^2} \delta (\xi \sqrt{\omega_\alpha^{\beta}} \wedge \sqrt{\Delta \eta^{\alpha}_{\beta \gamma}} + (\xi \sqrt{\omega_\alpha^{\beta}} \wedge \bar{H}_\alpha^{\beta}) \right] \]  
(4.17)
We see that in the case of translations, \(X_1\) will fall off faster than \(1/r^2\). In the case of rotations, we have to impose the parity conditions
\[ e_i^\alpha = \delta_i^\alpha + \frac{a_i^\alpha}{r} + O_2 \]  
, with \(a_i^\alpha\) even,  
(4.18)
and the stronger fall off of the connection (4.11). Beside the stronger fall off condition, we have also to notice that \(B_2\) – contrary to \(B_1\) – transforms inhomogeneously under Lorentz transformations. Therefore the whole discussion of \(B_2\) is basis dependent.

5. Conservation and finiteness

Now we turn to the conservation and finiteness of the integrals. We now require that the variables fulfill the field equations. For this purpose we first observe that
\[ \omega_\alpha^{\beta} \wedge (\xi \sqrt{H_\alpha^{\beta}}) = a_0 \bar{H}_\mu \xi^\mu - \frac{a_0}{2l^2} \xi^\mu r_{x}^{\alpha} \wedge \eta^{\alpha}_{\beta \mu} + \omega_\alpha^{\beta} \wedge (\xi \sqrt{\bar{H}_\alpha^{\beta}}) \]  
(5.1)
where from it follows that

\[ \xi^\gamma H_\gamma + \omega_\alpha^\beta \wedge (\xi^\gamma H_\alpha^\beta) = -\frac{a_0}{2l^2} \xi^\mu r_\alpha^\beta \wedge \eta^\alpha_\beta \mu + \xi^\gamma H_\gamma + \omega_\alpha^\beta \wedge (\xi^\gamma \tilde{H}_\alpha^\beta) = 
\]

\[ -\frac{a_0}{2l^2} \xi^\mu r_\alpha^\beta \wedge \eta^\alpha_\beta \mu + \xi O_3 \quad . \quad (5.2) \]

For translations the boundary term \( B_2 \) gives just

\[ Q(\xi) = -\frac{a_0}{2l^2} \int_{S^2} r_\alpha^\beta \wedge \xi \eta^\alpha_\beta \quad . \quad (5.3) \]

We have \( r_\alpha^\beta \wedge d\eta^\alpha_\beta \mu = O_4 \), and for exact solutions of the (PGT) field equations we find \( (dr_\alpha^\beta) \wedge \eta^\alpha_\beta \mu = -(r_\nu^\beta \wedge r_\alpha^\nu) \wedge \eta^\alpha_\beta \mu + O_4 = O_4 \) and therefore also \( d(r_\alpha^\beta \wedge \xi \eta^\alpha_\beta) = O_4 (\xi^\mu = constant + O_4) \). Consequently their 4-momentum is conserved.

Now we consider the rotations. For exact solutions it is

\[ H_2 = d \left[ \xi^\alpha H_\alpha^\beta + \omega_\alpha^\beta \wedge (\xi^\gamma H_\alpha^\beta) + \varepsilon^\beta_\alpha H^\alpha_\beta \right] = 
\]

\[ -\frac{a_0}{2l^2} d \left[ \xi^\mu r_\alpha^\beta \wedge \eta^\alpha_\beta \mu + \varepsilon^\beta_\alpha \eta^\alpha_\beta \right] + d \left( \xi^\mu O_3 + \varepsilon^\beta_\alpha O_2 \right) \quad . \quad (5.4) \]

Apparently the integration over the last term gives a finite result. The first term is a kind of anholonomic version of the Landau-Lifshitz expression, the latter one reads in the language of the exterior calculus,

\[ M^{ik} = \int_S (x^i_{\ell L} e^k_{\ell L} - x^k_{\ell L} e^i_{\ell L}) = \int_{S^2} (x^i_{\ell L} U^k - x^k_{\ell L} U^i) + \frac{1}{2l^2} \int_{S^2} \sqrt{-g} \eta^{ik} \quad . \quad (5.5) \]

Here

\[ \ell L U^i = \frac{1}{2l^2} \sqrt{-g} (m^m \wedge n^m i) \quad , \quad \ell L^i = d_{\ell L} U^i \quad (5.6) \]

are the superpotential and the energy complex of Landau-Lifshitz [15] respectively. With the help of the parity conditions (4.14) and the fall off (4.11), the expression (5.4) can be shown to be finite:

For rotations, the Killing field is given by (4.10), where now \( \alpha, \beta \in \{0, 1, 2, 3\} \). Consequently we have \( \partial_i \xi^j = \sigma_i \delta^j_M - \sigma_M \delta^j_i = \varepsilon^j_i \). Therefore \( d\xi^\beta = \varepsilon^\beta_\alpha \wedge \partial^\alpha + O_1^\beta \) (because \( \varepsilon^\beta_\alpha \) is evaluated with respect to the background basis, which differs from the physical one by \( O_1^\beta \)) where \( O_n^\alpha (O_n^\beta) \) means \( X + O_{n+\zeta} \) and \( X \) is a odd (even) term of order \( O_n \) (3).

Thus we have \( d\xi^\mu \wedge r_\alpha^\beta \wedge \eta^\alpha_\beta \mu = -\varepsilon^\beta_\alpha d\eta^\alpha_\beta + O_1^\beta \wedge r_\alpha^\beta \wedge \eta^\alpha_\beta \mu + O_{1+\zeta} \wedge r_\alpha^\beta \wedge \eta^\alpha_\beta \mu \), and the integration of this term will cancel the integration of \( d(\varepsilon^\beta_\alpha \wedge \eta^\alpha_\beta) \), because of \( d\varepsilon = 0 \) and \( r_\alpha^\beta = O_2^\alpha \). Now it remains to show that the integration over \( \xi^\mu d(r_\alpha^\beta \wedge \eta^\alpha_\beta) \)

(3) The term \( \tilde{D}_\alpha \xi^\beta \wedge \tilde{H}_\alpha^\beta \) gives no contribution to the integral, as it is easily seen, if we use the theorem of Stokes and \( d(\tilde{D}_\alpha \xi^\beta \wedge \tilde{H}_\alpha^\beta) = \tilde{D}_\alpha \xi^\beta \wedge d \tilde{H}_\alpha^\beta/(2l^2) = 0 \). But the term \( \tilde{H}_\alpha^\beta \) is important in the case of asymptotic anti-de Sitter spacetimes, for instance.
yields a finite value. Because of $G_\alpha = O_{1+\gamma}$, we have $dr_\alpha^\beta \wedge \eta^\alpha_{\beta\mu} = O_4^\mu$. Furthermore, we get $r_\alpha^\beta \wedge d\eta^\alpha_{\beta\mu} = O_4^\mu$. Thus, $\xi^\mu d(r_\alpha^\beta \wedge \eta^\alpha_{\beta\mu}) = O_3^\beta$, and the integral over this term is finite. Observe, that the only troublesome terms in (5.4) was the Einstein part. The term $(\xi^\mu O_3 + \varepsilon O_2)$ behaves well, and, imposing (4.11), will make the integral over this term vanish.

Now we have
\[ \frac{d}{dt} \int_\Sigma H = \int_\Sigma \xi_d H = \int_S \partial_t \right. \]

because of $dH = 0$.

Therefore the charge is conserved, if $\partial_t \right. \]$ falls off faster than $r^{-2}$, and this is the case for $H_2$ as shown above.

To get finite quantities for the boundary term $B_1$, which are also conserved, we have to impose stronger restrictions. We require that $\omega$ fulfills (4.11), and from (3.14) and (5.2) we find, that the only non-vanishing term in the boundary integral is
\[ -\frac{a_0}{2l^2} \left[ r_\alpha^\beta \wedge (\xi] \eta^\alpha_{\beta} + (\xi] \omega^\alpha_{\beta} \right] \wedge \Delta \eta^\alpha_{\beta} ] . \] (5.8)

Because the torsion, contained in the connection, is independent and cannot be cancelled out, we have to demand $\Theta_\alpha = O_{2+\gamma}$ (where $\gamma > 0$ is necessary for conservation) or parity conditions on the torsion. For the other term recall that $d(r_\alpha^\beta \wedge \eta^\alpha_{\beta\mu}) = O_4^\mu$, thus $d(\xi^\mu r_\alpha^\beta \wedge \eta^\alpha_{\beta\mu}) = d\xi^\mu \wedge (r_\alpha^\beta \wedge \eta^\alpha_{\beta\mu}) + \xi^\mu d(r_\alpha^\beta \wedge \eta^\alpha_{\beta\mu})$, where the last term is of order $\xi O_4^\mu$. Thus we have only to estimate $(d\xi^\mu) \wedge (r_\alpha^\beta \wedge \eta^\alpha_{\beta\mu})$. For $B_2$ this term was canceled in leading order by $d(\varepsilon \beta^\alpha \wedge \eta^\alpha_{\beta})$. Here we have to demand stronger parity conditions for the tetrads. With
\[ \omega_{i\alpha}^\beta = O_{2+\gamma} \quad \text{and} \quad e_i^\alpha = \delta_i^\alpha + \frac{a_i^\alpha}{r} + O_{2+\gamma} \quad \text{with} \ a_i^\alpha \ \text{even}, \] (5.9)

the boundary term $B_1$ will give finite and conserved quantities (where $\gamma > 0$ is only needed to show the conservation of the quantities).

6. Relationship to other expressions for conserved quantities of PGT

We compare the potential $B_2$ with expressions as given in [5] and [6]. The first investigations about conserved quantities in PGT were done by Hayashi and Shirafuji [5]. In a Lagrangian approach, they started with a generator like (3.6) and substituted for the vector field $\xi$ just the Killing fields of Minkowski spacetime in a Cartesian basis. That is,
\[ \int_S H = e^i \int_{\partial S} P_i + b_{ij} \int_{\partial S} L^ij + d_{\alpha\beta} \int_{\partial S} S^\alpha_{\beta} , \] (6.1)

where momentum, angular momentum, and spin are given by
\[ P_i = \left( -\frac{\partial L_{HS}}{\partial d\alpha^\mu} \right) e_i^\alpha + \omega_{i\mu}^\nu \left( -\frac{\partial L_{HS}}{\partial d\omega^\alpha_{\beta}} \right) , \] (6.2)
\[ L^{ij} = x_i \left[ \epsilon_j \alpha \left( -\partial L_{HS} \partial \theta^\alpha \right) + \omega_j \beta \left( -\partial L_{HS} \partial \omega_\alpha \beta \right) \right] , \quad (6.3) \]

and \[ S^{\alpha \beta} = H^{\alpha \beta} + \frac{a_0}{2l^2} \delta^{\alpha \beta}. \quad (6.4) \]

In order to get reasonable results, they had to write the Hilbert part of the curvature in terms of the torsion and a divergence. The reason is already discussed in chapter 3: As the first part of (6.7) is not contained in the canonical Hamiltonian, Hayashi and Shirafuji wrote the Einstein-Cartan term for the Lagrangian (1.5) in the following way

\[-\frac{a_0}{2l^2} \Omega^{\alpha \beta} \wedge \eta^{\alpha \beta} = -\frac{a_0}{2l^2} \Omega^{\alpha \beta} \wedge \omega_\mu \wedge \eta^{\alpha \beta} - \frac{a_0}{2l^2} \Omega^{\alpha \beta} \wedge \delta^{\alpha \beta} - \frac{a_0}{2l^2} \partial (\omega_\alpha \wedge \eta^{\alpha \beta}) . \quad (6.5)\]

They discarded the exact term: \[ L_{HS} = L + a_0/(2l^2) d(\omega_\alpha \wedge \eta^{\alpha \beta}), \] where \( L \) is given in (1.5). Then the translational field momenta \((-\partial L_{HS} / \partial \theta^\alpha)\) picks up an additional term \(-a_0/(2l^2) \omega_\mu \wedge \eta^{\mu \nu \alpha} \). Therefore, the field momenta of Hayashi and Shirafuji are given by \((-\partial L_{HS} / \partial \theta^\alpha) = H^{\alpha \beta} - \frac{a_0}{2l^2} \omega_\mu \wedge \eta^{\mu \nu \alpha} \) and \(-L_{HS} / \partial \omega_\alpha \beta = H^{\alpha \beta} \).

Energy momentum and angular momentum are singled out by an appropriate choice of the parameters \( c, b, d \) in (6.1), where \( b \) and \( d \) has to be coupled by \( b_{ij} = \delta_i^\alpha \delta_j^\beta d_{\alpha \beta} \). The integrated quantities of Hayashi and Shirafuji then coincide with ours, provided the condition (4.11) is fulfilled.

To compare the expression (3.10) with the work of Blagojević and Vasilić [6], we evaluate our expression in the framework of the Ricci calculus. The boundary term is then given by \( (1/2) H^{\alpha \beta jk} \eta_{jk} := eH_\alpha \) etc., \( e := \det (e_i^\alpha) \)

\[ eB_2 = \frac{1}{2} \left[ \xi^\alpha H^j \beta \right] + \omega_{i \alpha} \xi^i H^\alpha \beta j^k + 2 \omega_{i \alpha} \xi^i H^\alpha \beta |^k |i + \epsilon^{\alpha \beta} \Delta H^\alpha \beta j^k \eta_{jk} \quad (6.6) \]

where \( \epsilon^{\alpha \beta} \) is given by \( D_\alpha \xi^\beta \). For the energy \( (\xi = \partial_0, \epsilon^{\alpha \beta} = 0) \) we obtain

\[ B_2 = \frac{1}{e} \left[ H_0^0 - \omega_{\alpha \beta} H^\alpha \beta b^a \right] dS_a \quad , \quad (6.7) \]

where \( B \) means the projection onto the spacelike hypersurface. For the linear momentum we get \( (\xi = \partial_c, \epsilon^{\alpha \beta} = 0) \)

\[ B_2 = \frac{1}{e} \left[ H_0^0 - 2 \delta^{\alpha}_{|b|} \omega_{\alpha \beta} H^\alpha \beta b^a \right] dS_a \quad , \quad (6.8) \]

for the angular momentum \( (\xi^i \) given in (4.10) \( \epsilon^{\alpha \beta} = g_{\alpha M} \delta^2_N - g_{N\alpha} \delta^2_M \))

\[ B_2 = \frac{1}{e} \left[ 2x_0 N H_{M0}^a + 2 \Delta H_{\alpha N}^0 a + 2x_0 N H_{M0}^0 \beta H^\alpha \beta b^a \right] dS_a \quad , \quad (6.9) \]

(observe, that we have for spatial rotations \( \xi^a dS_a = 0 \)), and for the boost \( (\xi^i \) given in (4.12) \( \epsilon^{\alpha \beta} = g_{\alpha N} \delta^2_0 - g_{0\alpha} \delta^2_N \))

\[ B_2 = \frac{1}{e} \left[ 2x_0 N H_0^0 a + 2 \Delta H_{N0}^0 a - 2x_0 \delta_{\alpha N} \omega_{\alpha \beta} H^\alpha \beta b^a - \omega_{\alpha \beta} H^\alpha \beta b^a \right] dS_a \quad . \quad (6.10) \]
Comparing the results with [6], we have to substitute $\mathcal{H}_{ij}^{\alpha \beta}$ by twice of the momentum of Blagojević & Vasić because their definition (of the generators of the Lorentz group and therefore) of the rotational momentum differ by an factor 2 from our definitions. Notice also that the leading term of the momentum is the Einstein part. Finally recall that $\mathcal{H}_{\alpha \beta}$ gives no contribution to the integrals, as shown in footnote (3). Then we can see that all of our integrated expressions coincide with the ones of Blagojević & Vasić (for the comparision of the integrals we have only to require (2.1)).

At the end we want to make a comment on the expressions of [7] and [8]. In this articles are the field momenta $H_\alpha$ and $H_{\alpha \beta}$ used as the integrands for total momentum and angular momentum, whereas the authors considered also asymptotic anti de Sitter spacetimes. As the field momenta bear indices, one has to choose carefully the basis system (which is a tedious task for complicated configurations) in order to get reasonable results. This was succesfully done in [7] for a Schwarzschild – anti-de-Sitter solution with torsion. But already the application of this method on a Kerr – anti-de-Sitter solution with torsion leads to an infinite angular momentum as $H_{\alpha \beta}$ is proportional to the curvature, and therefore does not have a suitable fall off. Moreover, as it is obvious, these quantities do not give reasonable values for solutions of the ECSK-theory or of GR.

7. Conclusion

We have discussed the behavior of the two expressions $B_1$, $B_2$ of eqs. 3.14,15 for conserved quantities (eq. 3.16) of PGT in asymptotic flat spacetimes. We have seen that the variation of the accompanying Hamiltonians are well defined for appropopriate phase spaces. The respective phase spaces of the two expressions differ slightly, whereas the approporriate phase space of $B_2$ (see eqs. 4.11,18) is larger than the one of $B_2$ (see eq. 5.9). Finally we have seen that the expression $B_2$, for appropriate boundary conditions, coincides with those of Hayashi & Shirafuji and Blagojević & Vasić. In [4] both expressions were tested with asymptotic flat and asymptotic constant curvature solutions of PGT and gave, for the tested solutions, the same results as the corresponding solutions of General Relativity. The advantage of the expressions is that they are not restricted on an asymptotically Cartesian basis and that they can be also used in asymptotic anti-de-Sitter spacetimes. Moreover it gives one compact expression for all of the ten conserved quantities.

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