Fractal behaviours of networks induced on infinite tree structures by random walks

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Abstract. Tree graphs such as Cayley trees provide a stage to support the self-organization of fractal networks by the flow of walkers from the root vertex to the outermost shell of the tree graph. This network model is a typical example that demonstrates the ability of a random process on a network to generate fractality. However, the finite scale of the tree structure assumed in the model restricts the size of fractal networks. In this study, we removed the restriction on the size of the trees by introducing a lifetime $\tau$ (number of steps of random walks) of walkers. As a result, we successfully induced a size-independent fractal structure on a tree graph without a boundary. Our numerical results show that the mean number of offspring $n_b$ of the original tree structure determines the value of the fractal box dimension $d_b$ through the relation $d_b - 1 = (n_b - 1)^{-\theta}$. The lifetime $\tau$ controls the presence or absence of small-world and scale-free properties. The ideal fractal behaviour can be maintained by selecting an appropriate value of $\tau$. The numerical results contribute to the development of a systematic method for generating fractal small-world and scale-free networks while controlling the value of the fractal box dimension. Unlike other models that use recursive rules to generate self-similar structures, this model specifically produces small-world fractal networks with scale-free properties.

1. Introduction

Fractality is a universal property found in real networks, including the World Wide Web, social networks, and biological networks [1]. Recent studies on model networks have revealed that random processes occurring on networks could be a mechanism that induces the fractal properties of networks [2, 3]. It is known that recursive rules generating self-similar patterns produce fractal networks [4, 5]. However, mechanisms based on random phenomena should be more appropriate for the explanation of fractality, considering that most real-world networks appear to be self-organized by random events.

The addition of shortcut edges to the traces of random walkers on a finite tree graph [3] is an example of such random processes that can generate fractal networks. However, this model assumed a finite number of shells in the initial tree graph, in which walkers flow from the root vertex to the outermost shell of the tree graph. Owing to this assumption, a size effect exists in this model. For example, if the box-counting method [1] is applied to the resulting graph, the fractal relation $N_b \sim l_b^{-d_b}$ is limited to a range of $l_b$ smaller than the number of shells of the original tree, where $N_b$ is the minimum number of boxes required to tile the entire network, $l_b$ is the size of the boxes, and $d_b$ is the fractal box dimension. The mean path length $\langle l \rangle$ is also
bounded above by the number of shells. Furthermore, the fractal relation begins to decay as the graph spreads over a wide area of the finite tree graph.

The aim of this study is to remove the size effect on the fractality from the model and to study the properties of the resulting networks regardless of the network size. The newly introduced assumptions are as follows. First, random walkers are removed after executing a finite number of random steps (we call this the lifetime $\tau$). Second, each newly born random walker starts with a vertex randomly chosen from the vertices that previous walkers have reached at least once. These assumptions allow us to not assume a finite number of shells, which is needed to consider the flow of walkers from the root vertex to the outermost shell. In the current model, the mean number of offspring of the initial tree $n_b$ and the lifetime of walkers $\tau$ control the network properties, including the fractality, and scale-free and small-world properties. The knowledge obtained in this study would provide a systematic method for generating networks while controlling the small-world properties and the value of the fractal box dimension $d_b$.

This paper is organized as follows. The next section details the model. In section 3, we present the numerical results for fractal analysis using the box-counting and cluster-growing methods [6]. Subsection 3.1 clarifies the dependence of $d_b$ on $n_b$ and $\tau$. In subsection 3.2, we investigate the small-world property of the resulting graph using the cluster-growing method. In subsection 3.3, we summarise the numerical results for the power-law index describing the degree distribution along with other properties. The conclusion is presented in Section 4.

2. Model

We consider a tree graph without a boundary as a framework that supports the self-organization of a fractal network (Fig. 1(a)). The tree graph prepared as an initial state is generated by the following rule.

Figure 1. A typical network assuming $k_{\text{root}} = 2, n_b = 1.5$, and $\tau = 50$. (a) A tree graph prepared as an initial state (only vertices from the root vertex to the 9-th shell out of an infinite number of shells are illustrated.). (b) Subgraph $G_t$ (filled circles and bold lines) after the three walkers started. The number of vertices and edges in $G_{t=150}$ are $V_{t=150} = 26$ and $E_{t=150} = 59$, respectively. Edges added by different walkers are shown in different colours.

The vertex degree of the root vertex of the tree graph is $k_{\text{root}}$. The $k_{\text{root}}$ vertices directly connected to the root vertex are called the first shell. The number of offspring $n_b$ of each vertex in the first shell is probabilistically determined as:

$$n_b = \begin{cases} k_{\text{root}} & \text{(with probability } p) \\ k_{\text{root}} - 1 & \text{(with probability } 1 - p) \end{cases} \tag{1}$$

Consequently, the mean number of offspring of one vertex is $\bar{n}_b = k_{\text{root}}p + (k_{\text{root}} - 1)(1 - p) = k_{\text{root}} - 1 + p$. The set of offspring of vertices in the first shell is called the second shell. Similarly,
the vertices in the $l$-th shell are born from vertices in the $(l-1)$-th shell [Fig. 1(a)]. For simplicity, hereinafter, we denote $\bar{n}_b$ as $n_b$.

The tree graph provides vertices that can potentially be joined to an evolving graph $G_t$ that grows with discrete time $t (t = 0, 1, 2, \cdots)$. Graph $G_t$ [Fig. 1(b)] is constructed by adding shortcut edges to the tree graph according to movements of random walkers, as follows:

(i) A random walker is placed at the root vertex in the tree graph at time $t = 0$. At this stage, $G_{t=0}$ is only the root vertex with no edges.

(ii) The walker moves randomly from the current vertex to one of the neighbouring vertices. If the current vertex $v_t$ and the vertex where the walker was two steps previously $v_{t-2}$ are not joined, a shortcut edge is created between $v_t$ and $v_{t-2}$. Otherwise, no edges are added to the graph at time $t$. (Note that the walker is assumed to be able to move not only along the edges in the original tree graph but also along the edges created by the previous movement of walkers. The creation of a loop was also prohibited.)

(iii) Add 1 to $t$, and $G_t$ is renewed to the graph including the vertices that one of the walkers reached at least once, and edges that link these vertices (including edges added by movements of random walkers).

(iv) Only if $t$ is an integer multiple of the walker’s lifetime $\tau$, the walker is removed, and immediately, a new walker is born at a randomly selected vertex in $G_t$.

(v) The next step is (ii).

This model is different from the previous model in Ref.[3] in that $G_t$ develops in an infinite tree structure, and that the walker starts with a randomly selected vertex in $G_t$ and is removed after executing random walking of $\tau$ steps. Moreover, this model is similar to the model presented in Ref.[7] in the case when $\tau = \infty$. It should be noted that, according to Ref.[7], an edge condensation appears when $n_b < 1.5$ and $\tau = \infty$, along with the power-law form of the degree distribution, whereas the scale-free property disappears when $n_b > 1.5$ and $\tau = \infty$. (The value $n_b = 1.5$ corresponds to the mean distance 2 between bifurcation points.) This knowledge should help us to interpret the results for the present work.

3. Numerical results

3.1. Results for box-counting method

As expected from a previous study [3], the box-counting method identifies the fractal property of the model network (Fig. 2). However, in contrast to the results of [3], Figs. 2(a) and (b) show that the fractal relation holds for almost all ranges of $l_b$. In other words, the size effect is eliminated by using an infinite tree structure. It should also be observed in Figs. 2(a) and (b) that the fractal dimension $d_b$ is almost independent of $\tau$. Therefore, the results for $\tau = \infty$ provide the most information about the dependency of the box fractal dimension $d_b$ on the mean bifurcation number of the tree graph $n_b$ (Fig. 2(c)).

It should be noted that, in Fig. 2(a), when $\tau$ is very large, two different fractal box dimensions can be detected: a local fractal box dimension $d_b$ for small $l_b$ and a long-range box dimension $d_{b{\text{long}}}$ for large $l_b$. In many cases, the value of $d_{b{\text{long}}}$ is close to 1. The appearance of long-range fractality is owing to the gradual movement of the walker from the root vertex. According to a previous study [7], the drift of the walker from the root vertex to the outer shells can be found when the bifurcation number $n_b$ is larger than 1.5 and $\tau = \infty$. Conversely, when $n_b$ is very close to or smaller than 1.5, the long-range fractality disappears (Fig. 2(b)).

The long-range fractality can also be eliminated by reducing the lifetime $\tau$ of the walkers, even if $n_b > 1.5$. This is because randomly selected vertices at which newly born walkers start are potentially near the root vertex. Fig. 2(a) shows that only the local fractal dimension $d_b$ remains, when the lifetime $\tau$ decreases. However, if $\tau$ is too short, a cutoff appears, above which
the fractal relation decays. This result indicates the existence of an appropriate lifetime which supports an ideal fractal relation.

Fig. 2(c) indicates a dependence of the local fractal box dimension \( d_b \) on \( n_b \), which suggests a relation \( d_b - 1 = (n_b - 1)^{-0.78} \). This functional form explains the asymptotic behaviour of \( d_b \) for large \( n_b \), where the local box dimension \( d_b \) is indistinguishable from the long-range box dimension 1. However, note that the functional form \( d_b - 1 = (n_b - 1)^{-0.78} \) is not valid when \( n_b \) is smaller than approximately 1.3. The fractal relation becomes obscure as \( n_b \) approaches 1, owing to the strong edge condensation.

3.2. Results for cluster-growing method

Fig. 3 shows typical results for the box-counting and the cluster-growing methods for different values of \( \tau (n_b = 1.6) \). If a finite tree structure was used, the mean path length \( \langle l \rangle \) and diameter \( D \) of \( G_t \) were bound above by the finite number of shells of the tree [3]. In the present model, owing to the usage of infinite number of shells, the size-effect is eliminated. As a result, the results for the cluster-growing method can be interpreted in accordance with that in previous studies on growth models of fractal networks [2, 8].

When the graph is not small-world, the fractal box and cluster dimensions \( (d_b \) and \( d_c \)) are equivalent. Such a result is observed when \( \tau = \infty \) in Figs. 3(a) and (b). Under this condition, as mentioned in the previous subsection, both local and long-range fractal dimensions were observed. Figs. 3(a) and (b) show that the box and the cluster dimensions take similar values in both contexts of local and long-range.

When the graph is small-world in the context of the mean shortest path length, the long-range dimension disappears, and the value of the cluster dimension \( d_c \) must vary with the number of vertices in \( G_t \). To reconcile the fractal relation \( (M_c) \sim l_c^{d_c} \), with the small-world property \( V_t \sim \exp (\langle l \rangle / \beta) \) (Fig.3(c)). (The \( V_t \)-dependence of \( d_c \) is given by \( d_c \simeq \log (V / \langle k \rangle) / \log (\log V) \), which is deduced from the fact that \( d_c \) is the inclination of a linear line in a double logarithmic chart, \( \log V - \log (\langle k \rangle) / (\log \langle l \rangle - \log 1) \), and the small-world property \( \langle l \rangle \propto \log V \) [2, 8].) When \( n_b < 1.5 \), the size-dependent \( d_c \) can be observed regardless of the value of \( \tau \), because the size-dependent \( d_c \) is specific to small-world growth graphs.
respectively; and in (d), curves with randomly chosen vertex in at most \( \langle l \rangle \) and \( V_t \) versus \( \tau \) relation in the box-counting method (Fig. 3(a) in case \( \tau = 0 \)), the difference between \( d_b \) and \( d_D \). However, if \( \tau \) is sufficiently reduced to make a clear appearance of the cutoff of the fractal relation in the box-counting method (Fig. 3(a) in case \( \tau = 100 \)), the difference between \( d_b \) and \( d_D \) becomes noticeable (compare Fig. 3(a) in case \( \tau = 100 \) with (d)).

Note that even if the graph has the small-world property, the value of \( d_b \) remains stable. This is because the diameter \( D \) retains the non-small-world behaviour \( D \sim V^{1/d_D} \), in contrast to the mean path length \( \langle l \rangle \). Figs.3(a) and (c) show the consistency between the value of \( d_b \) and \( d_D \).

3.3. Degree distributions

Fig. 4 shows typical results for degree distribution. In general, as the mean vertex degree decreases, the power-law exponent \( \gamma \), describing the degree distribution as \( P(k) \sim k^{-\gamma} \), tends to increase. In the proposed model, an increase in \( n_b \) leads to an increase in \( \gamma \), because the wide spread of walkers owing to large \( n_b \) results in a rapid increase in the number of vertices compared to the increase in the number of edges. In particular, when \( n_b > 1.5 \) and \( \tau = \infty \), the degree distribution exhibits an exponential decay corresponding to \( \gamma = \infty \) (Fig. 4). Fig. 4 shows that the power-law form can be supported by reducing the value of \( \tau \), even when \( n_b > 1.5 \). Fig. 5 summarizes the numerical results for \( \gamma \) for various values of \( n_b \) and \( \tau \) along with the presence or absence of the small-world property.
Figure 4. Degree distributions (degree $k$ versus the number of vertices with degree $k$, $V(k)$) when $n_b = 1.6$ for $\tau = \infty$ ($\square$), $\tau = 400$ ($\bigcirc$), and $\tau = 100$ ($\triangle$). The fitted lines are the power-law form $V(k) \sim k^{-\gamma}$ with $\gamma = 2.4$.

Figure 5. Diagram representing the change of $\gamma$ for various $n_b$ and $\tau$. Circles ($\bigcirc$) and squares ($\square$) represent small-world (SW) and non-small-world fractal networks, respectively. Different states are vaguely (not exactly) separated by dashed curves.

4. Conclusion
In this study, we extend the model of a fractal network based on random walks from a finite tree structure to an infinitely tree structure by introducing a lifetime $\tau$ of walkers. As a result, we succeeded in eliminating the size effect on the fractal relations and the small-world property. The extension of the model allows a comprehensive investigation of the properties of model networks. The results are as follows.

The average number of offspring of the initial tree $n_b$ determines the value of the fractal box dimension $d_b$, whereas the lifetime of walkers $\tau$ does not significantly affect the value of $d_b$. Numerical calculations specify a formula, $d_b - 1 = (n_b - 1)^{-0.78}$. When $n_b > 1.5$ and $\tau$ is large, a long-range box dimension $d_b^{\text{long}}$ ($\approx 1$) is observed along with $d_b$. The appearance of long-range fractality is owing to the gradual movement of the walker from the root vertex. Therefore, long-range fractality can be eliminated by reducing $\tau$. As $\tau$ decreases, the graph changes from a non-small-world non-scale-free graph to a small-world scale-free graph. The ideal fractal behaviour can be supported by selecting an appropriate value of $\tau$. When $n_b < 1.5$, the current model produces fractal, small-world, and scale-free properties regardless of the value of $\tau$. However, the fractal relation tends to be obscure, as $n_b$ approaches 1. The disappearance of fractality was owing to strong edge condensation.

Our study enables us to systematically generate fractal small-world scale-free networks while controlling the value of $d_b$. In this model, the small-world property and fractality are inseparable when realising realistic values of $d_b$ ($\geq 2$). The small-world property urges the cluster dimension $d_c$ to increase with the graph size while keeping $d_b$ constant. This noticeable feature cannot be described using static fractal models. Conversely, the current model cannot describe non-small-world fractal networks with realistic values of $d_b$. Further extension of the model, such as probabilistic edge creation, would describe network properties, which could not be achieved with the current model.

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