Calculus of Variations

Principal spectral curves for Lane–Emden fully nonlinear type systems and applications

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Abstract
In this paper we exploit the phenomenon of two principal half eigenvalues in the context of fully nonlinear Lane–Emden type systems with possibly unbounded coefficients and weights. We show that this gives rise to the existence of two principal spectral curves on the plane. We develop an anti-maximum principle, which is a novelty even for Lane–Emden systems involving the Laplacian operator. As applications, we derive a maximum principle in small domains for these systems, as well as existence and uniqueness of positive solutions in the sublinear regime. Most of our results are new even in the scalar case, in particular for a class of Isaac’s operators with unbounded coefficients, whose $W^{2,q}$ regularity estimates we also prove.

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1 Introduction and main results

In this paper we study existence and uniqueness properties of the Dirichlet problem for partial differential fully nonlinear systems of Lane–Emden nature with weights, such as

\[
\begin{cases}
  F_1(x, u, Du, D^2u) + \lambda \tau_1(x)v^{q-1}v = f_1(x) & \text{in } \Omega \\
  F_2(x, v, Dv, D^2v) + \mu \tau_2(x)u^{p-1}u = f_2(x) & \text{in } \Omega \\n  u = v & = 0 \quad \text{on } \partial \Omega,
\end{cases}
\]

(1.1)

in the viscosity sense. Here \( \Omega \) is a \( C^{1,1} \) bounded domain in \( \mathbb{R}^N \) with \( N \geq 1, \lambda, \mu \in \mathbb{R} \) and \( p, q > 0 \) are constants, \( F_i \) is a uniformly elliptic fully nonlinear operator in nondivergence form, \( f_i \in L^q(\Omega) \) for some \( q > N, i = 1, 2 \), and the respective weights satisfy

\[
\tau_i \in L^q(\Omega) \quad \text{with} \quad \tau_i \geq 0 \quad \text{in } \Omega, \quad i = 1, 2, \quad |\text{supp } \tau_1 \cap \text{supp } \tau_2| > 0. \quad (1.2)
\]

Here \( \tau_i \geq 0 \) means that \( \tau_i \geq 0 \) a.e. in \( \Omega \) and \( \tau_i \not\equiv 0 \). When \( p = q = 1 \), \( \tau_1 = \tau_2 =: \tau \), \( F_1 = F_2 =: F \) and \( f_1 = f_2 =: f \), we recover the scalar case

\[
F(x, u, Du, D^2u) + \lambda \tau(x)u = f(x) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega, \quad (1.3)
\]

for which we also present new results.

Spectral properties of uniformly elliptic PDEs in nonvariational form have long been recognized since the seminal work [7]. Its fully nonlinear scalar theory in terms of viscosity solutions unveils the phenomenon of two half eigenvalues corresponding to both positive and negative eigenfunctions. The notion of principal eigenvalue associated to a positive eigenfunction for fully nonlinear operators, defined through the validity of the maximum principle (i.e. in light of [7]), dates back from [9]. Independently in [13] it appears the notion of two half principal eigenvalues, associated to positive and negative eigenfunctions, for each Pucci operator. The latter in turn complements earlier results on existence and simplicity of first positive eigenvalues obtained for Pucci operators in [23] (radial) and [39] (nonradial) through a generalized Krein–Rutman theorem. It was in [40] that a complete program in the spirit of [7] was developed for convex operators with bounded coefficients. The case of nonconvex operators (again with bounded coefficients) was analyzed in [4] under additional continuity restrictions on the data and on the operators.

Spectral properties for related cooperative systems with linear operators in nondivergence form \( F_i = L_i \) have been extensively investigated. We quote [46] for the case \( p = q = 1 \), and its extension in [10] to non-smooth domains. Moreover, the theory of [7] through the maximum principle was extended to linear systems in [14, Sections 13, 14], which in turn was extended to fully nonlinear systems and half eigenvalues in [41]. When more general power-like nonlinearities are taken into account, still for linear operators, still for linear operators, in [30] a spectral curve was constructed for (LE) when \( pq = 1 \); more recently, related comparison principles appear in [28]. Both [28, 30] deal with linear operators and unbounded coefficients and weights.

Our main goal here is to understand the phenomenon of two principal half eigenvalues induced by the fully nonlinear operators \( F_1, F_2 \) in light of [7, 40], for the Lane–Emden coupling prototype in the regime \( pq = 1 \), including nonconvex operators with possibly unbounded weights and coefficients. We show that the homogeneous version of (1.1), i.e.

\[
\begin{cases}
  F_1(x, u, Du, D^2u) + \lambda \tau_1(x)v^{q-1}v = 0 & \text{in } \Omega \\
  F_2(x, v, Dv, D^2v) + \mu \tau_2(x)u^{p-1}u = 0 & \text{in } \Omega \\n  u = v & = 0 \quad \text{on } \partial \Omega,
\end{cases}
\]

(LE)
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gives rise to the existence of two principal spectral curves to (LE) in the plane \((\lambda, \mu)\). We stress that principal eigenvalues are related to the solvability of (1.1), and to the validity of maximum principles, which we also study. Moreover, we develop an anti-maximum principle, which is a novelty even for Lane–Emden systems involving the Laplacian operator. All our results are valid also for a class of Isaac’s operators with unbounded coefficients (1.11) (see Example 1.14 ahead), and therefore are new and improve results in the literature even in the scalar case. In this context, we mention that in [35] it was started a spectral analysis involving a class of proper operators with unbounded drift and weight in the scalar case, but only in what concerns existence of eigenvalues. Here we complement that study, by giving a full characterization of the first scalar eigenvalues in terms of validity of maximum principles, solvability of the Dirichlet problem, and more generally the validity of Alexandrov-Bakelman-Pucci (ABP) inequality for nonproper and possible nonconvex operators, under improved assumptions. Observe that, once ABP is proved, for any solution \((u, v)\) we have \(uv > 0\) in \(\Omega\) whenever \(u\) is signed in \(\Omega\).

This problem brings about several applications. For instance, one may view the pair \((\lambda, \mu)\) as risk-sensitive averages of the weights \(\tau_1\) and \(\tau_2\), respectively, over the diffusions \(F_1, F_2\), see [2, 22]. Besides, it characterizes the range of solvability for equations with superlinear gradient growth, as well as existence and uniqueness of positive solutions for (LE) in the sublinear regime \(pq < 1\), which we also prove.

### 1.1 Assumptions on the operators

Next we list our hypotheses on the operators \(F_1\) and \(F_2\). We denote by \(\mathbb{S}^N\) the space of \(N \times N\)–symmetric matrices. Let us bear in mind the following general structural hypothesis on a fully nonlinear operator \(F : \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N \to \mathbb{R}\) given by

\[
\mathcal{L}^-(x, r - s, \xi - \eta, X - Y) \leq F(x, r, \xi, X) - F(x, s, \eta, Y) \leq \mathcal{L}^+(x, r - s, \xi - \eta, X - Y),
\]

for all \(X, Y \in \mathbb{S}^N, \eta, \xi \in \mathbb{R}^N, r, s \in \mathbb{R},\) and \(x \in \Omega\), where \(F(\cdot, 0, 0, 0) \equiv 0\), and

\[
\mathcal{L}^\pm(x, r, \xi, X) := \mathcal{L}^\pm_0(x, \xi, X) \pm \vartheta(|r|), \quad \text{for } \mathcal{L}^\pm_0(x, \xi, X) := \mathcal{M}^\pm(X) \pm \gamma(x)|\xi|,
\]

for \(\gamma, \vartheta \in L^0(\Omega), \vartheta > N,\) with \(\gamma \geq 0\) and \(\vartheta \geq 0\) a.e. in \(\Omega\). Also, \(\mathcal{M}^\pm = \mathcal{M}^\pm_{\alpha, \beta}\) are the Pucci’s extremal operators with ellipticity constants \(0 < \alpha \leq \beta\), see (2.1) ahead.

Note that (H1) corresponds to a uniform bound for all operators satisfying a prescribed ellipticity. In order to measure how far a particular fully nonlinear operator \(F\) is from a linear one, in the spirit of [4, 24, 40], we may construct from (H1) a more accurate structure:

\[
F_s(x, r - s, \xi - \eta, X - Y) \leq F(x, r, \xi, X) - F(x, s, \eta, Y) \leq F^*(x, r - s, \xi - \eta, X - Y),
\]

where

\[
F^*(x, r, \xi, X) := \sup_{r', \xi', X'} \{F(x, r + r', \xi + \xi', X + X') - F(x, r', \xi', X')\},
\]

\[
F_s(x, r, \xi, X) := \inf_{r', \xi', X'} \{F(x, r + r', \xi + \xi', X + X') - F(x, r', \xi', X')\},
\]

for all \(x \in \Omega, r \in \mathbb{R}, \xi \in \mathbb{R}^N, X \in \mathbb{S}^N\). Assume that \(F\) satisfies (H1). Then both \(F^*, F_s\) satisfy (H1) as well; \(F^*\) is convex and \(F_s\) is concave in \((r, \xi, X); F = F^*\) if and only if \(F\)
is convex, \( F = F_* \) if and only if \( F \) is concave. Also, the following identity holds
\[
F^*(x, r, \xi, X) = -F_*(x, -r, -\xi, -X);
\]
see [4, Proposition 4.2]. Finally, we have the ordering \( \mathcal{L}^- \leq F_* \leq F \leq F^* \leq \mathcal{L}^+ \).

**Definition 1.1** For a function \( w \) and an operator \( F \), we consider the following notations:

(a) \( F[w] := F(x, w, Dw, D^2w) \);

(b) \( \lambda^+_1(F(\vartheta)) \) is the principal weighted eigenvalue associated to a positive eigenfunction of the scalar Dirichlet problem \( F[u] + \lambda \vartheta u = 0 \) in \( \Omega \), \( u = 0 \) on \( \partial\Omega \), see Sect. 2.1;

(c) viscosity solutions are meant in the \( L^N \)-viscosity sense, see Sect. 2.3;

(d) we say that \( F \) enjoys \( W^{2,\varrho} \) regularity if any viscosity solution \( u \) of \( F[u] = f(x) \) in \( \Omega \) with \( f \in L^0(\Omega) \) belongs to \( W^{2,\varrho}_{\text{loc}}(\Omega) \), and in addition \( u \in W^{2,\varrho}(\Omega) \) if \( u = \psi \) on \( \partial\Omega \), \( \psi \in W^{2,\varrho}(\Omega) \).

Having in mind the existence of eigenvalues, another condition we ask on an operator \( F \) is that it satisfies a positive homogeneity of order one, namely
\[
F(x, tr, t\xi, tX) = t F(x, r, \xi, X) \quad \text{for all } t \geq 0, \quad \text{for any } X \in S^N, \xi \in \mathbb{R}^N, r \in \mathbb{R}, \text{ and } x \in \Omega.
\]

Moreover, we consider the following control of oscillation in the \( x \)-entry:
\[
\forall \theta_0 > 0, \quad \exists r_0 > 0 : \quad \|\beta_F(x, \cdot)\|_{L^\infty(B_r(x))} \leq \theta_0 r^{N/\varrho} \quad \text{for all } r \leq r_0, \quad x \in \overline{\Omega}, \quad (1.5)
\]
where \( \beta_F(x, y) := \sup_{X \in S^N} |F(x, 0, 0, X) - F(y, 0, 0, X)| \|X\|^{-1} \) for \( x, y \in \overline{\Omega} \). It holds for instance when \( F \) satisfies \( |F(x, 0, 0, X) - F(y, 0, 0, X)| \leq \omega(|x - y|) \|X\| \) for all \( x, y \in \overline{\Omega}, X \in S^N \).

Finally, we assume that the Dirichlet problems associated to \( F_* \) and \( F^* \) are uniquely solvable (see Sect. 3) in the scalar sense together with regularity of solutions in a suitable Sobolev sense. In other words, in terms of Definition 1.1 (b), (d), we ask that \( F_i \) satisfy the following
\[
\lambda^+_1(F^*(\vartheta)) > 0, \quad F, F^* \text{ satisfy (1.5),} \quad (H3)
\]
\[
F \text{ enjoys } W^{2,\varrho} \text{ regularity.} \quad (H4)
\]

Here and onward in the text, the drift \( \gamma \) and the zero order term \( \vartheta \) in (H1) may be unbounded, and this is an advantage of our paper over the usual literature [4, 40], even in the scalar case. We observe that, in the system, the structure of each \( F_i \) with respect to the zero order term in (H1) could be taken in terms of functions \( \varrho_i \), which gives the possibility of prescribing different weights in (H3), see also Remark 2.10. We avoid including so many indexes in order to make the presentation cleaner.

Existence and positiveness in (H3) are verified if \( F^* \) is a proper operator (nonincreasing in \( r \)) by [35], and we will see this extends for nonproper operators as well, check Lemma 3.1 ahead. Meanwhile, (H4) will hold true if \( F \) is a convex (or concave) operator in \( X \) and satisfy (1.5), see Lemma 3.5 (consequently, under (1.5), \( F^* \) fulfills (H4)). However, (H4) also covers some nonconvex operators, for instance the asymptotic recession profiles in [37] for which a \( W^{2,\varrho} \) theory is available. In particular, our results are valid for a class of Isaac’s operators which are sufficiently close to a Bellman operator that has good regularity-estimates, see Example 1.14. Indeed, in Sect. 8 we prove that they verify (H4) under (8.1), even in the presence of unbounded coefficients.
It is worthwhile to mention that hypothesis (H4) is not overly restrictive and in fact it is a natural condition when dealing with comparison principles for $L^p$-viscosity solutions, see [35, 42]. Instead in the universe of $C$-viscosity solutions it is possible to skip it by the price of asking stronger continuity assumptions on the coefficients which is not our intention here, see [4]. Fully nonlinear equations with measurable ingredients were introduced in [15] for which a more general $L^p$-viscosity notion of solution is required. It is a modern theory which still develops, and results in such direction with unbounded coefficients are rather involved and delicate, see [26, 35, 44].

In this paper we treat the optimality of scalar spectral properties of fully nonlinear operators, and we also exploit the differences arising in the case of systems; both on a scenario with possible unbounded drift and weights.

**Remark 1.2** Note that hypothesis (H3) allows us to treat nonproper operators. This is equivalent to ask $\lambda_1^+(F(\vartheta)) > 0$ when $F$ is a linear operator. For systems, this condition on the operators $F_1, F_2$ is somehow required in [28, 30] in terms of MP’s validity, see Definition 1.6. Instead, in [41, Theorem 1] the coupling proposed does not fall upon the nonlinearities but on the operators. On the other hand, the regularity assumption (H4) for $F_1, F_2$ seems to be optimal in the fully nonlinear case, with respect to the previous scalar works [4, 40], since there is no need to assume neither convexity nor continuity on the data.

**1.2 Statement of the main results**

Let us consider the space $E_\varrho = W^{2,\varrho}(\Omega) \cap C(\Omega)$. Our main results are in the sequel.

**Theorem 1.3** (Existence, simplicity, and asymptotics) Let $\Omega \subset \mathbb{R}^N$ be a bounded $C^{1,1}$ domain. Assume $pq = 1, \tau_1, \tau_2$ satisfy (1.2), and $F_1, F_2$ satisfy (H1)–(H4). Then there exist two spectral curves

$$\Lambda_1^\pm(\lambda) = (\lambda, \mu_1^\pm(\lambda)) \in \mathbb{R}^2, \text{ for all } \lambda > 0,$$

in the first quadrant, corresponding to signed eigenfunctions $\varphi_1^\pm, \psi_1^\pm \in E_\varrho$ such that both the pairs $\varphi_1^+, \psi_1^+ > 0$ and $\varphi_1^-, \psi_1^- < 0$ satisfy (LE) in the strong sense.

The eigenfunctions $(\varphi_1^+, \psi_1^+)$ and $(\varphi_1^-, \psi_1^-)$ are unique in the sense that any other eigenfunction $(u^\pm, v^\pm)$ corresponding to $\Lambda_1^\pm(\lambda)$ satisfies $u^\pm \equiv t\varphi_1^\pm$ and $v^\pm \equiv t^p\psi_1^\pm$ for a suitable $t \in \mathbb{R}^\pm$. Furthermore, if $(u, v)$ is a signed solution of (LE), then necessarily $(\lambda, \mu) \in \Lambda_1^\pm$.

Moreover, $\mu_1^\pm$ is continuous and strictly decreasing with $\lambda$, and the following asymptotic behavior holds

$$\mu_1^\pm(\lambda) \to \infty \text{ as } \lambda \to 0, \quad \mu_1^\pm(\lambda) \to 0 \text{ as } \lambda \to \infty. \quad (1.6)$$

**Remark 1.4** For the explicit shape of $\Lambda_1^\pm$, see (2.12) ahead.

In what follows we deal with geometric properties of the first spectral curves and their characterization. In what follows, for a parametrized curve $\Sigma = \Sigma(\lambda) = (\lambda, \varrho(\lambda))$, where $\lambda > 0$ and $\varrho(\lambda)$ continuous, we say that $\Sigma$ is above $\Lambda_1^\pm$ if $\mu_1^\pm(\lambda) < \varrho(\lambda)$ for every $\lambda > 0$.

**Theorem 1.5** (Local isolation) Assume $pq = 1, \tau_1, \tau_2$ satisfy (1.2), $F_1, F_2$ satisfy (H1)–(H4), and take $\Lambda_1^\pm$ from Theorem 1.3. Then there exists a curve $\Sigma = (\lambda, \varrho(\lambda))$, strictly above $\Lambda_1^\pm$, such that $\varrho$ is strictly decreasing, $\varrho(\lambda) \to \infty$ as $\lambda \to 0$, $\varrho(\lambda) \to 0$ as $\lambda \to \infty$, with the property that: if $(\lambda, \mu)$ is an eigenvalue of (LE) in the region

$$\{(\lambda, \mu) \in \mathbb{R}^2 : \lambda > 0, \ 0 < \mu < \varrho(\lambda)\},$$
then necessarily $(\lambda, \mu) \in \Lambda_1^+ \cup \Lambda_1^-$. In other words, in the first quadrant, below and slightly above $\Lambda_1^+$ there are no other eigenvalues of (LE).

Next we see how the region below each curve $\Lambda_1^+$ gives a complete characterization of the plane $\mathbb{R}^2$ in terms of maximum and minimum principles, and in terms of the solvability of the associated Dirichlet problem. This extends [28] to viscosity solutions, and plays the role of the condition $\lambda < \lambda_1^+$ in the scalar case.

**Definition 1.6** (MP and mP) We say that the maximum principle (MP) holds for (LE) if any viscosity subsolution of (LE), that is, any solution pair $u, v \in C(\bar{\Omega})$ of

\[
F_1[u] + \lambda \tau_1(x)|v|^{q-1}v \geq 0, \quad F_2[v] + \mu \tau_2(x)|u|^{p-1}u \geq 0 \quad \text{in} \ \Omega, \quad u, v \leq 0 \text{ on} \ \partial \Omega
\]

(1.7)

satisfies $u, v \leq 0$ in $\Omega$. Likewise, we say that the minimum principle (mP) holds for (LE) if $u, v \geq 0$ in $\Omega$ for any viscosity supersolution pair $u, v \in C(\bar{\Omega})$ of

\[
F_1[u] + \lambda \tau_1(x)|v|^{q-1}v \leq 0, \quad F_2[v] + \mu \tau_2(x)|u|^{p-1}u \leq 0 \quad \text{in} \ \Omega, \quad u, v \geq 0 \text{ on} \ \partial \Omega.
\]

(1.8)

Let $C_1^+$ be the open region in the first quadrant below $\Lambda_1^+$, and similarly for $C_1^-$ associated to $\Lambda_1^-$, namely:

\[
C_1^\pm := \{ (\lambda, \mu) \in \mathbb{R}^2 : \lambda > 0, \ 0 < \mu < \mu_1^\pm(\lambda) \}.
\]

(1.9)

**Theorem 1.7** (Characterization of $\Lambda_1^\pm$) Assume $pq = 1$, $\tau_1, \tau_2$ satisfy (1.2), $F_1, F_2$ satisfy (H1)–(H4), and let $\Lambda_1^\pm$ be as in Theorem 1.3. Then:

(i) $(\lambda, \mu) \in \overline{C_1^- \setminus \Lambda_1^+}$ if, and only if, MP holds for (LE);

(ii) $(\lambda, \mu) \in C_1^- \setminus \Lambda_1^+$ if, and only if, mP holds for (LE).

Let us now consider the Dirichlet problem (1.1) in the viscosity sense, for functions $f_1, f_2 \in L^6(\Omega)$, $q > N$, with $u, v \in C(\bar{\Omega})$.

**Theorem 1.8** (Solvability of the Dirichlet problem) Assume $F_1, F_2$ satisfy (H1)–(H4), $pq = 1$, $\tau_1, \tau_2$ satisfy (1.2), and let $f_1, f_2 \in L^6(\Omega)$.

(i) If $(\lambda, \mu) \in C_1^+ \cap C_1^-$, then (1.1) is solvable among viscosity solutions.

(ii) If $f_1, f_2 \leq 0$ a.e. and $(\lambda, \mu) \in C_1^+$, then (1.1) has a unique nonnegative solution pair in $E_\varepsilon$.

(iii) If $f_1, f_2 \geq 0$ a.e. and $(\lambda, \mu) \in C_1^-$, then (1.1) has a unique nonpositive solution pair in $E_\varepsilon$.

In other words, as long as we are in the region $C_1^+ \cap C_1^-$ we obtain the complete standard solvability of the Dirichlet problem (1.1). The solvability up to $C_1^+ \cup C_1^-$ may not hold in general (e.g. [40, Theorem 1.8]), in contrast to the case of linear eigenvalues in [28]. However, if the pair $f_1, f_2$ has the “good” sign, we do obtain solvability in this larger region.

**Remark 1.9** When $p = q = 1$, the principal spectral curve for linear operators and its properties also follow from the papers [10, 14, 46]. For instance, for each fixed $\lambda, \mu > 0$, there is $\lambda_1 = \lambda_1(\lambda, \mu) \in \mathbb{R}$, such that

\[
\begin{pmatrix}
\Delta u \\
\Delta v
\end{pmatrix}
+
\begin{pmatrix}
0 & \lambda \tau_1(x) \\
\mu \tau_2(x) & 0
\end{pmatrix}
\begin{pmatrix}
u \\
v
\end{pmatrix}
=
-\lambda_1
\begin{pmatrix}
u \\
v
\end{pmatrix}
\]
has a positive solution, with all the properties stated so far. One could prove the strict monotonicity of $\lambda_1$ in $(\lambda, \mu)$ following [7] so, for each $\lambda > 0$, there is $\mu(\lambda) > 0$ such that $\lambda_1(\lambda, \mu(\lambda)) = 0$. Therefore the principal curve is obtained, and if $(\lambda, \mu)$ is below the curve then $\lambda_1(\lambda, \mu) > 0$, so there is a maximum principle result. Exactly in the same way, still for $p = q = 1$, from [41] follow two principal spectral curves if the Laplacian is replaced by fully nonlinear operators with bounded coefficients, with the related properties of maximum principles and solvability of the Dirichlet problem.

In general, problems involving nonlinear Hamiltonian systems may be much more involved than their linear counterpart—for instance we mention the so called Lane–Emden conjecture, see [38, 45], a long standing open problem for which only partial results are known. In particular, the lack of Lipschitz regularity in one of the nonlinearities $u^p$ and $v^q$ when $pq = 1$ (but not necessarily $p = q = 1$) breaks down some arguments used in the references mentioned in Remark 1.9.

On the other hand, in comparison with spectral properties for linear operators developed in [30] and complemented in [28] for $pq = 1$, it is undeniable that replacing the linear by a fully nonlinear operator in there brings new difficulties that need to be overcome in the universe of viscosity solutions—for instance by considering sub and supersolutions that are merely continuous in maximum principle like results.

Finally, we highlight a last aspect of our work: it refers to a recent research line on equations driven by fully nonlinear operators with unbounded coefficients. Replacing continuity by merely measurability and integrability in a suitable Lebesgue space justifies the use of $L^p$-viscosity solutions instead of $C$-viscosity solutions in the first place, see [15]. When further the drift term is assumed to be unbounded, it is a theme that has gained some attention in the last years and requires different approaches, see for instance [12, 26, 35, 43, 44]. Here, moreover, we allow both the drift and the weights involving the first eigenvalues to be unbounded. Up to our knowledge, apart from the study initiated in [35], it is the first time spectral properties on fully nonlinear nonlinear operators with unbounded coefficients are addressed. We expect that our results can give some insights to the theory and be useful even for single equations.

**Remark 1.10** In particular, Theorem 1.8 applied to $F_1 = F_2 = \mathcal{L}^+$, $\tau_1 = \tau_2 = \tau$, $p = q = 1$ (scalar case) gives the optimal range of solvability in [42, Proposition 3.4]; see also Remark p.595 therein where the problem of spectral properties for these operators under unbounded coefficients was left open. See also our Theorem 3.4 in Sect. 3 for a priori bounds of ABP type to solutions produced by Theorem 1.8 in the scalar case, as well as necessary conditions which characterize their validity.

In what concerns the optimality of Theorem 1.8, we show that an anti-maximum principle occurs when we move a little bit above the region $\mathcal{C}_1^+ \cup \mathcal{C}_1^-$. This type of result is essential, for instance, in bifurcation and resonance phenomena [3, 21, 22]. A classical reference for it in the linear scalar case is [17] (see also [8]), while its fully nonlinear scalar counterpart can be found in [4, Theorem 2.5]. Here we extend [4] to fully nonlinear Lane–Emden systems as follows.

**Theorem 1.11** (Anti-maximum principle) Let $F_1, F_2$ satisfy (H1)–(H4), $pq = 1$, $\tau_1, \tau_2$ satisfy (1.2), and $f_i \in L^q(\Omega)$ with $f_i \not\equiv 0$, $i = 1, 2, q > N$. Then there exists a curve $\Gamma = (\lambda, \tilde{\gamma}(\lambda))$, depending on $f_1, f_2$, which is above $\Lambda_1^{\pm}$, where $\tilde{\gamma}$ is strictly decreasing, $\tilde{\gamma}(\lambda) \to 0$ as $\lambda \to \infty$, and $\tilde{\gamma}(\lambda) \to \infty$ as $\lambda \to 0$, such that

(i) if $f_1, f_2 \leq 0$ a.e., $\Lambda_1^+$ is below or coincides with $\Lambda_1^-$, and $(\lambda, \mu)$ is a pair between $\Lambda_1^-$ and $\Gamma$, then any solution pair $u, v \in C(\overline{\Omega})$ of (1.1) satisfies $u, v < 0$ in $\Omega$;
(ii) if \( f_1, f_2 \geq 0 \) a.e., \( \Lambda_1^- \) is below or coincides with \( \Lambda_1^+ \), and \( (\lambda, \mu) \) is a pair between \( \Lambda_1^+ \) and \( \Gamma \), then any solution pair \( u, v \in C(\overline{\Omega}) \) of (1.1) verifies \( u, v > 0 \) in \( \Omega \).

We highlight that Theorem 1.11 is new even in the case of the standard Lane–Emden system involving the Laplacian operator, i.e. when \( F_1 = F_2 = \Delta \). Up to our knowledge, this is the first result on anti-maximum principle regarding strongly coupled systems.

1.3 Examples and applications

We start by highlighting that the curves \( \Lambda_1^+ \) and \( \Lambda_1^- \) obtained in Theorem 1.3 can be different when the operator is not linear, as shows the following example.

**Example 1.12** In light of [13], one may consider the Fucik-like spectrum

\[
Lu + \lambda u^+ - \frac{\lambda}{\kappa} u^- = 0,
\]

associated to the linear operator \( Lu := \text{tr}(A(x)D^2u) + \gamma(x) \cdot Du \), where \( \kappa > 0 \) is fixed, which can be viewed as the spectrum of a nonlinear convex or concave operator given by

\[
F_1[u] := \max\{Lu, \kappa Lu\} = -\lambda u \quad \text{if} \quad \kappa \geq 1, \quad F_2[u] := \min\{Lu, \kappa Lu\} = -\lambda u \quad \text{if} \quad \kappa \leq 1.
\]

Now let us fix \( \kappa > 1 \) and \( p, q > 0 \) such that \( pq = 1 \). From [30], there exists a first positive eigenvalue-parameter \( \sigma \) (see Sect. 2.4) and an eigenfunction pair \( (\varphi, \psi) \) so that

\[
L\varphi + \sigma \psi^q = 0, \quad L\psi + \sigma \varphi^p = 0, \quad \varphi, \psi > 0 \quad \text{in} \quad \Omega, \quad \varphi, \psi = 0 \quad \text{on} \quad \partial\Omega. \tag{1.10}
\]

Then the pair \( \varphi_1^+ := \varphi, \psi_1^+ := t\psi \), with \( t > 0 \) to be chosen, solves

\[
F_1[\varphi_1^+] = \max\{L\varphi, \kappa L\varphi\} = -\sigma \psi^q = -\sigma t^{-q}(\varphi_1^+)^q,
\]

\[
F_2[\psi_1^+] = \min\{tL\psi, \kappa tL\psi\} = -\kappa t\sigma \varphi^p = -\kappa t\sigma (\varphi_1^+)^p,
\]

so it is a positive eigenfunction pair if one chooses \( t := \kappa^{-\frac{1}{q}} \). Moreover, this eigenpair is unique up to scaling by our Theorem 1.3, and we conclude \( \lambda_1^+(F_1, F_2) = \kappa^{\frac{1}{q}} \sigma \).

Analogously, the pair given by \( \varphi_1^- := -\varphi, \psi_1^- := -s\psi \) solves

\[
F_1[\varphi_1^-] = \max\{-L\varphi, -\kappa L\varphi\} = \kappa \sigma \varphi^q = -\kappa \sigma s^{-q} |\psi_1^-|^{-q-1}\psi_1^-,
\]

\[
F_2[\psi_1^-] = \min\{-sL\psi, -\kappa sL\psi\} = s \sigma \varphi^p = -s \sigma |\psi_1^-|^{-p-1}\psi_1^-,
\]

and becomes a negative eigenfunction pair when \( s := \kappa^{-\frac{1}{p}} \)—again unique up to scaling by Theorem 1.3, and \( \lambda_1^-(F_1, F_2) = \kappa^{\frac{1}{p}} \sigma \), compare with the scalar case in [4, Example 3.10].

Now, since \( \kappa > 1 \), then one always has \( \lambda_1^+(F_1, F_2) \neq \lambda_1^-(F_1, F_2) \), for \( q \neq 1 \), and so by scaling one recovers that the two parallel curves \( \Lambda_1^+ \) and \( \Lambda_1^- \) are different, see Sect. 2.4. Furthermore, \( \Lambda_1^+ \) stays below \( \Lambda_1^- \) if \( q < 1 \); while \( \Lambda_1^+ \) lies above \( \Lambda_1^- \) if \( q > 1 \).

On the other hand, it is also simple to verify that for \( \kappa > 1 \), \( \lambda_1^+(F_1, F_1) = \sigma < \kappa \sigma = \lambda_1^-(F_1, F_1) \) and \( \lambda_1^-(F_2, F_2) = \sigma < \kappa \sigma = \lambda_1^+(F_2, F_2) \), for all \( p, q > 0 \) with \( pq = 1 \).

More generally, one can also consider different operators \( L_1, L_2 \) in (1.10).

The next examples comprise important classes of fully nonlinear operators for which all our results apply, being novelties even in the scalar case in the context of unbounded drift and weight.
Example 1.13 Simple prototypes we may have in mind are extremal operators involving Pucci’s, for instance $F_1 = F_1^+ = \mathcal{L}^+$, $F_2 = (F_2)_+ = \mathcal{L}^-$, with $\vartheta = \ell \theta$, for some $\ell > 0$ and $\theta \in L^0(\Omega)$ satisfying $\vartheta \geq 0$ a.e. in $\Omega$. They obviously fulfill (H1), (H2), (1.5), and (H4). Moreover, recall $\lambda_1^+(\mathcal{L}_0^+(\theta)) > 0$ by [35] (see also our Proposition 2.9), and $\lambda_1^-(\mathcal{L}_0^-(\theta)) = \lambda_1^+(\mathcal{L}_0^-(\theta))$. Thus (H3) is verified for $F_1, F_2$ if one chooses $\ell < \lambda_1^+(\mathcal{L}_0^+(\theta))$ as in (3.3), see Lemma 3.1.

Example 1.14 A bit more sophisticated model case arising from control theory are Hamilton-Jacobi-Bellman-Isaac’s type operators [4, 22, 36, 40], with unbounded coefficients, such as

\[ F_1[w] = \sup_{s \in \mathbb{N}} \inf_{t \in \mathbb{N}} L_{s,t}(w), \quad F_2[w] = \inf_{s \in \mathbb{N}} \sup_{t \in \mathbb{N}} L_{s,t}(w), \quad \text{(1.11)} \]

wher $L_{s,t}$ for $s, t \in \mathbb{N}$ is a linear operator in the form

\[ L_{s,t}(w) = \text{tr}(A_{s,t}(x)D^2w) + \gamma_{s,t}(x) \cdot Dw + \ell \vartheta_{s,t}(x)w, \quad \ell < \lambda_1^+(\mathcal{L}_0^+(\vartheta)), \]

\[ |\gamma_{s,t}| \leq \gamma, \quad |\vartheta_{s,t}| \leq \vartheta, \quad \gamma, \vartheta \in L^0(\Omega), \quad \alpha I \leq A_{s,t} \leq \beta I, \]

\[ A_{s,t} \in C(\overline{\Omega}) \text{ uniformly in } s, t \in \mathbb{N}, \quad \text{(1.12)} \]

for $0 < \alpha \leq \beta$. For instance, when $L_{s,t} = L_{0,t}$ for all $s \in \mathbb{N}$ then $F_1$ and $F_2$ are called Bellman operators, which are concave and convex, respectively. In the general case, $F_1$ and $F_2$ are neither convex nor concave, and are called Isaac’s operators. Again, (H3) holds for $F_1, F_2$ under (1.12) as in (3.3), see Lemma 3.1; while we show (H4) for (1.11)–(1.12) under (8.1) in Sect. 8.

Results in the superlinear and sublinear regimes As a byproduct of our arguments, we complement a study on maximum principles in small domains for Lane–Emden systems. Besides being of independent interest [7], they play an important role in symmetry problems [6, 18], aside from spectral constructions when the domain is not smooth [40]. It also appeared in [32] as the main tool to derive a Unique Continuation Principle of radial fully nonlinear type in the case $pq \geq 1$. For $pq = 1$, an explicit form was previously/independently proved in [28, Theorem 1.3, Corollary 1.1], under a smallness hypothesis on the weights. Here we instead make it an alternative, by asking either the domain or the weights to be small. This in particular extends and unifies [35, Lemma 5.4] (for domains with small measure) and [42, the maximum principle in Proposition 3.4] (for operators with small weight) even in the scalar case. Furthermore, in what concerns the system, the result is valid not only for $pq = 1$ but also for $pq \geq 1$.

Theorem 1.15 Let $F_1, F_2$ satisfy (H1). Let $\tau_1, \tau_2 \in L^0(\Omega), \varrho > N, \ ps \geq 1$, and $\lambda, \mu \geq 0$. Then the following MP result holds.

(i) Assume (1.5), (H2) hold for $F_i^*$ with $\lambda_{i,1}^+(F_i^*(\vartheta)) > 0, i = 1, 2$. Then, there exists $\varepsilon_0 > 0$, depending on $N, \varrho, p, q, \alpha, \beta, \lambda, \mu, \|\gamma\|_{L^0(\Omega)}, \|\vartheta\|_{L^0(\Omega)}, \|\tau_1\|_{L^0(\Omega)}, \|\tau_2\|_{L^0(\Omega)}, \|u\|_{L^0(\Omega)}$, $\|v\|_{L^0(\Omega)}$, $\text{diam} \Omega$, and $\lambda_{1,1}^+(F_i^*(\vartheta))$, such that if

\[ \|\Omega\| \leq \varepsilon_0 \quad \text{or} \quad \min\{\|\tau_1\|_{L^0(\Omega)}, \|\tau_2\|_{L^0(\Omega)}\} \leq \varepsilon_0, \]

then any viscosity subsolution pair $u, v \in C(\overline{\Omega})$ of

\[ F_1[u] + \lambda \tau_1(x)|v|^{q-1}v \geq 0, \quad F_2[v] + \mu \tau_2(x)|u|^{p-1}u \geq 0 \quad \text{in } \Omega, \quad u, v \leq 0 \quad \text{on } \partial \Omega, \]

satisfies $u, v \leq 0 \quad \text{in } \Omega.
(ii) Assume (1.5), (H2) hold for \((F_i)_*\) with \(\lambda^+_i((F_i)_*(\vartheta)) > 0\), \(i = 1, 2\). Then, there exists \(\varepsilon_0 > 0\), depending on \(N, \varrho, p, q, \alpha, \beta, \lambda, \mu, \|y\|_{L^p(\Omega)}, \|\vartheta\|_{L^q(\Omega)}, \|\tau_1\|_{L^p(\Omega)}, \|\tau_2\|_{L^q(\Omega)}, \|u\|_{L^\infty(\Omega)}, \|v\|_{L^\infty(\Omega)}, \text{diam}(\Omega)\), and \(\lambda^+_i((F_i)_*(\vartheta))\), such that if

\[
\begin{align*}
&\text{either } |\Omega| \leq \varepsilon_0 \text{ or } \min\{\|\tau_1\|_{L^p(\Omega)}, \|\tau_2\|_{L^q(\Omega)}\} \leq \varepsilon_0, \\
&\text{then for any } u, v \in C(\Omega) \text{ viscosity supersolution of } \\
&F_1[u] + \lambda \tau_1(x)|v|^{q-1}v \leq 0, \quad F_2[v] + \mu \tau_2(x)|u|^{p-1}u \leq 0 \text{ in } \Omega, \quad u, v \geq 0 \text{ on } \partial\Omega,
\end{align*}
\]

one has \(u, v \geq 0\) in \(\Omega\).

If \(pq = 1\), then \(\varepsilon_0\) does not depend on a bound from above of the \(L^\infty\) norms of \(u, v\).

On the other hand, the eigenvalue problem furnishes unique solvability in the sublinear regime.

**Theorem 1.16** (Sublinear regime) Assume \(F_1, F_2\) verify (H1)–(H4). Let \(f_i \in L^0(\Omega)\) with \(f_i \leq 0\) a.e. in \(\Omega\), \(i = 1, 2\), and \(p, q > 0\) such that \(pq < 1\). Then the problem (1.1) is uniquely solvable among positive viscosity solutions for all \(\lambda, \mu > 0\).

Problems of this nature, for instance involving the Laplacian operator, have been studied in [19, Theorem 3] [30], and [11, Theorem 7.1]. We also mention that uniqueness results imply that solutions inherit all symmetries of the problem. For example, if the operator and its domain are radially symmetric, then so is the solution. Furthermore, uniqueness simplifies the dynamics of evolution problems, and in many cases provides global stability properties of equilibrium.

### 1.4 Structure of the paper

The rest of the paper is organized as follows. In Sect. 2 we recall some preliminary facts and definitions. In particular, we define the notion of principal eigenvalues for the system (LE).

Section 3 is devoted to the study of the scalar case, namely we prove how principal eigenvalues relate to maximum principles of ABP type, from which we obtain Theorem 1.15 as a consequence.

In Sect. 4 we prove some auxiliary results for systems, in particular sufficient conditions that imply uniqueness of solutions to systems (up to scaling), and a priori bounds for the first eigenvalue.

Section 5 addresses the main properties of the first eigenvalue problem when \(pq = 1\). It contains the proofs of Theorems 1.3, 1.5, 1.7 and 1.8 (i).

In Sect. 6 we deal with the anti-maximum principle, proving Theorem 1.11.

Section 7 is dedicated to solvability of the Dirichlet problem when the functions \(f_i\) have a sign. In this spirit we develop a unified proof for both Theorem 1.8 (ii–iii) and Theorem 1.16.

Finally, Sect. 8 is devoted to \(W^{2,\varrho}\) regularity of Isaac’s operators in Example 1.14.

### 2 Preliminaries

In this section we begin by recalling some of the different notions of viscosity solutions and their equivalence under regularity of the data. We then recall important results such as
the Alexandrov-Bakelman-Pucci (ABP for short) maximum principle, the strong maximum principle and Hopf’s lemma. In the second part we introduce the notion of principal eigenvalues and comment on the scaling properties of the system \((1.1)\). All these results are used throughout the paper.

### 2.1 Some known results

Let us start by recalling the definition of Pucci’s operators

\[
M^+_{\alpha, \beta}(X) := \sup_{\alpha I \leq \beta I} \text{tr}(AX), \quad M^-_{\alpha, \beta}(X) := \inf_{\alpha I \leq \beta I} \text{tr}(AX),
\]

and of viscosity and strong solutions in what follows.

**Definition 2.1** Let \( f \in L^\varepsilon_{\text{loc}}(\Omega) \) for some \( \varepsilon \geq N \), and \( F \) an operator satisfying \((\text{H1})\). We say that \( u \in C(\Omega) \) is an \( L^\varepsilon \)-viscosity subsolution (resp. supersolution) of \( F[u] = f(x) \) in \( \Omega \) if whenever \( \phi \in W^{2, \varepsilon}_{\text{loc}}(\Omega), \varepsilon > 0 \), and \( \mathcal{O} \subseteq \Omega \) open are such that

\[
F(x, u(x), D\phi(x), D^2\phi(x)) - f(x) \leq -\varepsilon \quad \text{if } x \in \mathcal{O},
\]

for \( \varepsilon > 0 \), then \( u - \phi \) cannot have a local maximum (minimum) in \( \mathcal{O} \). In this case we also say that \( u \) is a viscosity solution of the inequality \( F[u] \geq f(x) \) (resp. \( F[u] \leq f(x) \)) in \( \Omega \).

A strong subsolution (resp. supersolution) belongs to \( W^{2, \varepsilon}_{\text{loc}}(\Omega) \) and satisfies the inequality

\[
F[u] \geq f(x) \quad \text{(resp. } F[u] \leq f(x)),
\]

at almost every point \( x \in \Omega \). In each situation, a solution \( u \in C(\Omega) \) is meant to be both subsolution and supersolution.

We now comment on the equivalence of these definitions.

- The notions of \( L^\varepsilon \)-viscosity and strong solutions are equivalent whenever the solution belongs to the space \( W^{2, \varepsilon}_{\text{loc}}(\Omega) \), see [25, Theorem 3.1, Proposition 9.1].

- Under \((\text{H1})\), the concepts of \( L^\varepsilon \) and \( L^N \) viscosity solutions are also equivalent whenever \( \varepsilon > N \) and \( f \in L^\varepsilon_{\text{loc}}(\Omega) \), see [34, Proposition 2.9] (observe that \( L^\varepsilon_{\text{loc}}(\Omega) \subset L^N_{\text{loc}}(\Omega) \)).

Thus, given \( f \in L^\varepsilon_{\text{loc}}(\Omega) \), throughout the text, we say simply viscosity solution of \( F[u] = f \) to mean an \( L^N \)-viscosity solution. This in turn is equivalent to be a strong solution when hypothesis \((\text{H4})\) is in force.

**Remark 2.2** In order to unify the notation, we always assume \( f, \gamma \in L^\varepsilon \) by means of producing \( C^{1, \alpha} \) solutions under \((1.5)\) as in Proposition 2.8 (such strategy was also employed in [4] to treat nonconvex operators), despite sometimes this integrability can be relaxed to \( L^N \).

Next we recall the ABP maximum principle for proper operators with unbounded drift (for a proof see [26, Proposition 2.8]). Recall from \((1.4)\) the notation

\[
\mathcal{L}^\pm_0[u] := \mathcal{L}^\pm_0(x, Du, D^2u) = M^\pm(D^2u) \pm \gamma(x)|Du|,
\]

and

\[
\mathcal{L}^\pm_f[u] := \mathcal{L}^\pm_0[u] \pm f(x)|u|.
\]

**Proposition 2.3** (ABP) Let \( f \in L^N(\Omega), \gamma \in L^\varepsilon_{\text{loc}}(\Omega) \) for \( \varepsilon > N \), and \( u \in C(\Omega) \) be a viscosity solution of \( \mathcal{L}_0^+[u] \geq f(x) \) in \( \Omega^+ \) (resp. \( \mathcal{L}_0^-[u] \leq f(x) \) in \( \Omega^- \)), where \( \Omega^\pm = \Omega \cap \{\pm u > 0\} \).

Then

\[
\max_{\Omega^+} u \leq \max_{\partial \Omega} u^+ + C \|f^-\|_{L^N(\Omega)} \quad \text{(resp. } \min_{\Omega^-} u \geq \min_{\partial \Omega} (-u^-) - C \|f^+\|_{L^N(\Omega)}),
\]

(2.2)
for a universal constant $C = C(N, \alpha, \beta, \|\gamma\|_{L^{p}}, \text{diam}(\Omega)) > 0$, which is bounded if these quantities are bounded from above. We denote this constant by $C_{A}$.

**Remark 2.4** Recall that $F$ is called proper if $F(x, r, \xi, X) \leq F(x, s, \xi, X)$ for $r \geq s$. Therefore, the previous statement can be applied to proper operators, since for instance

$$\mathcal{L}^+_{\Omega} [u] \geq F(x, 0, Du, D^2u) \geq F(x, u, Du, D^2u) = F[u] \quad \text{in } \Omega^+.$$  

A consequence of ABP is the following result on the stability of viscosity solutions (see [42, Theorem 4], which is based on [15, Theorem 3.8]).

**Proposition 2.5** (Stability) Let $F', F_k$ be operators satisfying (H1). $f, f_k \in L^{\infty}(\Omega)$. Let $u_k \in C(\Omega)$ be a viscosity solution of $F_k[u_k] \geq f_k(x)$ in $\Omega$ (resp. $f(x)$) for all $k \in \mathbb{N}$. Suppose $u_k \to u$ in $L^{\infty}_{\text{loc}}(\Omega)$ as $k \to \infty$ and that, for each ball $B \subset \subset \Omega$ and $\varphi \in W^{2,\infty}(B)$, setting

$$g_k(x) := F_k(x, u_k, D\varphi, D^2\varphi) - f_k(x) \quad \text{and} \quad g(x) := F(x, u, D\varphi, D^2\varphi) - f(x),$$

we have $\|(g_k - g)\|_{L^{\infty}(B)} \to 0$ as $k \to \infty$, (resp. $\|(g_k - g)\|_{L^{\infty}(B)} \to 0$ as $k \to 0$). Then $u$ is a viscosity solution of $F[u] \geq f(x)$ (resp. $\leq f(x)$) in $\Omega$.

Next we recall some important results concerning the strong maximum principle and the Hopf lemma from [43]. We often refer to them simply by SMP and Hopf along the text.

**Proposition 2.6** (SMP) Let $\Omega$ be a $C^{1,1}$ domain and $u$ a viscosity solution of $\mathcal{L}^{-}[u] \leq 0$, $u \geq 0$ in $\Omega$, where $\gamma, \vartheta \in L^{\infty}_{\text{loc}}(\Omega)$. Then either $u > 0$ in $\Omega$ or $u \equiv 0$ in $\Omega$.

**Proposition 2.7** (Hopf) Let $\Omega$ be a $C^{1,1}$ domain and $u$ a viscosity solution of $\mathcal{L}^{-}[u] \leq 0$, $u > 0$ in $\Omega$, where $\gamma, \vartheta \in L^{\infty}_{\text{loc}}(\Omega)$. If $u(x_0) = 0$ for some $x_0 \in \partial\Omega$, then $\lim_{t \to 0^+} u(x_0 + t\nu)/t > 0$, where $\nu$ is the interior unit normal vector to $\partial\Omega$ at $x_0$.

In [43], Propositions 2.6 and 2.7 are proved for $\vartheta \equiv 0$, but the same proofs there work for any coercive operator, in particular for $\mathcal{L}^{-} = \mathcal{L}^{+} - \vartheta(x)u$ since $u \geq 0$; see also [44].

To conclude we recall $C^{1,\alpha}$ regularity estimates for equations with unbounded drift from [35]. Since Theorem 1 therein was stated for bounded zero order terms, we briefly explain how to deduce them also for merely $L^{\infty}$-integrable ones.

**Proposition 2.8** Assume $F$ satisfies (H1), $f \in L^{\infty}(\Omega)$, $q > N$, and $\Omega \subset \mathbb{R}^N$ is a bounded domain. Let $u$ be a viscosity solution of $F[u] = f(x)$ in $\Omega$. Then, there exists $\alpha \in (0, 1)$ and $\theta_0 = \theta_0(\alpha)$, depending on $q, \omega, \lambda, \Lambda, \|\gamma\|_{L^{\infty}(\Omega)}$, such that if (1.5) holds for all $r \leq \min[r_0, \text{dist}(x, \partial\Omega)]$, for some $r_0 > 0$ and for all $x \in \Omega$, this implies that $u \in C^{1,\alpha}_{\text{loc}}(\Omega)$ and for any subdomain $\Omega' \subset \subset \Omega$,

$$\|u\|_{C^{1,\alpha}(\Omega')} \leq C \left\{ \|u\|_{L^{\infty}(\Omega)} + \|f\|_{L^{p}(\Omega)} \right\}$$

where $C$ depends on $r_0, q, \omega, \lambda, \Lambda, \|\gamma\|_{L^{p}(\Omega)}$, $\|\vartheta\|_{L^{p}(\Omega)}$, $\text{dist}(\Omega')$, $\text{dist}(\partial\Omega)$.

If in addition, $\partial\Omega \in C^{1,1}$ and $u \in C(\Omega) \cap C^{1,\gamma}(\partial\Omega)$, then there exists $\alpha \in (0, \tau)$ and $\theta_0 = \theta_0(\alpha)$, depending on $q, \omega, \lambda, \Lambda, \|\gamma\|_{L^{p}(\Omega)}$, so that if (1.5) holds for some $r_0 > 0$ and for all $x \in \Omega$, this implies that $u \in C^{1,\alpha}(\Omega)$ and

$$\|u\|_{C^{1,\alpha}(\Omega')} \leq C \left\{ \|u\|_{L^{\infty}(\Omega)} + \|f\|_{L^{p}(\Omega)} + \|u\|_{C^{1,\gamma}(\partial\Omega)} \right\}$$

where $C$ depends on $r_0, n, p, \lambda, \Lambda, \alpha, \|\gamma\|_{L^{p}(\Omega)}$, $\|\vartheta\|_{L^{p}(\Omega)}$, $\text{dist}(\Omega)$, $\text{dist}(\partial\Omega)$. 

\[\square\]
Proof We first observe that our structural hypothesis (H1) takes into account a Lipschitz modulus of continuity as the zero order term. Moreover, when $\mu = 0$ in \cite[Theorem 1]{35}, one can perform a simpler rescaling of variable $W = N(0)$ as in \cite[Remark 3.4]{35}, but instead we use directly (1.5) (for $\beta_F$ in place of $\beta_F$ there) and set $\tilde{u}(x) := \frac{u(\sigma(x))}{W}$ in \cite[Claim 3.2]{35}. This allows us to achieve the regularity desired for $\vartheta \in L^p(\Omega)$, with the estimates depending on $\|\vartheta\|_{L^p(\Omega)}$.

2.2 Principal scalar eigenvalues for proper operators

In the scalar case, for an operator $F$ satisfying (H1), (H2), we set

$$\lambda_1^\pm(F(\vartheta)) = \sup \left\{\lambda \in \mathbb{R}, \Phi_\lambda^\pm \neq \emptyset\right\},$$

where

$$\Phi_\lambda^+ = \left\{\phi : \phi > 0 \text{ in } \Omega, \ F[\phi] + \lambda \vartheta(x)\phi \leq 0 \text{ in } \Omega\right\},$$

$$\Phi_\lambda^- = \left\{\phi : \phi < 0 \text{ in } \Omega, \ F[\phi] + \lambda \vartheta(x)\phi \geq 0 \text{ in } \Omega\right\}.$$

Our goal is to show that these suprema are achieved accordingly to Definition 1.1 (b), i.e. there exist eigenfunctions $u^\pm$ such that $-F[u] = \lambda_1^\pm u^\pm$ in $\Omega$—for instance when $F = F_i$, $i = 1, 2$.

The first step is to deduce it, in light of \cite{35}, for proper operators $F$ with unbounded drift and weight for which it holds (H1), (H2), (1.5), and (H4).

Proposition 2.9 Let $\Omega \subset \mathbb{R}^N$ be a bounded $C^{1,1}$ domain, $\tau \in L^\rho(\Omega)$, $\tau \geq 0$, $\rho > n$, where $F$ is a proper operator satisfying (H1), (H2), (1.5), and (H4), for $\gamma, \vartheta \in L^\rho(\Omega)$. Then $\lambda_1^\pm > 0$ and $F$ has two signed eigenfunctions $\varphi_1^\pm \in C^{1,\gamma}(\Omega)$ so that

$$F[\varphi_1^\pm] + \lambda_1^\pm \tau(x)\varphi_1^\pm = 0, \quad \pm \varphi_1^\pm > 0 \text{ in } \Omega, \quad \varphi_1^\pm = 0 \text{ on } \partial \Omega, \quad \max_{\Omega}(\pm \varphi_1^\pm) = 1.$$

Proof Let us first observe that Proposition 2.9 is already proved in \cite[Theorem 5.2]{35} when $\vartheta$ is bounded. We stress that $C^{1,\gamma}$ regularity estimates in Proposition 2.8 hold for unbounded $\vartheta$. On the other hand, we note that the solvability asked in \cite[Theorem 5.2]{35}, hypothesis (H) of Theorem 5.2 is now ensured due to (H4), since in this case unique solvability of the Dirichlet problem comes from \cite[Theorem 1 (i), (ii)]{42}. Moreover, the existence result on first eigenvalues does not require the drift nor the zero order term to be bounded. Indeed, the bound \cite[(5.8)]{35} is replaced by Lemma 5.7 there, with the blow-up argument comprising an unbounded zero order term as well; see ahead Step 2 in the proof of our Proposition 4.5, by taking $u = v$, $\tau_1 = \tau_2$.

Hence, for such $F$, the following ordering holds

$$\lambda_1^+(F^*(\vartheta)) = \lambda_1^-(F_*(\vartheta)) \leq \lambda_1^+(F(\vartheta)), \quad \lambda_1^-(F^*(\vartheta)) \leq \lambda_1^-(F(\vartheta)) = \lambda_1^+(F_*(\vartheta)),$$

(2.3)
since $F^*$ is convex and $F_*$ is concave, see \cite[Proposition 4.2]{4} and \cite[Lemma 1.1]{40}.

Remark 2.10 The following monotonicity property with respect to the weight holds:

if $\vartheta_1 \leq \vartheta_2$ a.e. in $\Omega$ then $\lambda_1^+(F(\vartheta_1)) \geq \lambda_1^+(F(\vartheta_2)).$

(2.4)

They are instrumental in risk-sensitive control and probabilistic arguments, see \cite{2, 20}.
2.3 Definition of principal eigenvalues for systems

Inspired by [7, 10, 40], we define the notion of principal eigenvalues for the system (LE) as follows:

$$\lambda_{1}^{\pm} = \lambda_{1}^{\pm}(F_1, F_2) = \lambda_{1}^{\pm}(F_1(\tau_1), F_2(\tau_2)) := \sup \{ \lambda \in \mathbb{R} : \Psi_{\lambda}^{\pm} \neq \emptyset \},$$

where

$$\Psi_{\lambda}^{+} = \{(\varphi, \psi) ; \varphi, \psi > 0 \text{ in } \Omega, \ F_1[\varphi] + \lambda \tau_1(x)\psi^q \leq 0, \ F_2[\psi] + \lambda \tau_2(x)\varphi^p \leq 0 \text{ in } \Omega \},$$

$$\Psi_{\lambda}^{-} = \{(\varphi, \psi) ; \varphi, \psi < 0 \text{ in } \Omega, \ F_1[\varphi] + \lambda \tau_1(x)\psi^q \geq 0, \ F_2[\psi] + \lambda \tau_2(x)\varphi^p \geq 0 \text{ in } \Omega \},$$

with inequalities holding in the $L^N$-viscosity sense, for functions $\varphi, \psi \in C(\overline{\Omega})$. When necessary, we will also highlight the dependence of $\lambda_{1}^{\pm}$ on $\Omega$. Observe that

$$\lambda_{1}^{\pm}(G_1, G_2) = \lambda_{1}^{\pm}(F_1, F_2), \text{ for } G_i(x, r, p, X) = -F_i(x, -r, -p, -X). \quad (2.5)$$

**Remark 2.11** By (2.3) and hypothesis (H3) on $F = F_i$, we have $\lambda_{1}^{\pm}(F_i(\vartheta)) > 0$, $i = 1, 2$. Then we infer that $\lambda_{1}^{\pm}(F_1, F_2) > 0$. Indeed, to fix the ideas let us consider the $\lambda_{1}^{+}$ case. Taking the positive eigenfunctions $\varphi_{1}^{+}$ and $\psi_{1}^{+}$ associated to $\lambda_{1}^{+}(F_1(\vartheta)) > 0$ and $\lambda_{1}^{+}(F_2(\vartheta)) > 0$ respectively, we have $(\varphi_{1}^{+}, \psi_{1}^{+}) \in \Psi_{0}^{+}$, from which the desired bound follows.

Let us also denote

$$m_1 = \min \{\lambda_{1}^{+}(F_1, F_2), \lambda_{1}^{-}(F_1, F_2)\}, \quad M_1 = \max \{\lambda_{1}^{+}(F_1, F_2), \lambda_{1}^{-}(F_1, F_2)\}. \quad (2.6)$$

2.4 Scaling and asymptotic behavior

In this section we show some equivalent forms of problem (LE) obtained by means of a suitable scaling, and how to build a spectral curve starting from a scalar-like eigenvalue. These observations are crucial in the proof of our main results, as often it will be convenient to take $\lambda = \mu$ in (LE).

Take $p, q > 0$ with $pq = 1$, and consider:

$$-F_1[u] = \lambda \tau_1(x)|v|^{q-1}v, \quad -F_2[v] = \lambda \tau_2(x)|u|^{p-1}u \text{ in } \Omega, \quad u = v = 0 \text{ on } \partial \Omega. \quad (2.7)$$

Let us check that to study this system for $\lambda > 0$ is, in a way, equivalent to studying (LE) for $(\lambda, \mu)$ in the first quadrant.

Assume we have an eigenfunction pair $(u_0, v_0)$ associated to an eigenvalue $\lambda_0 > 0$, i.e. $(u_0, v_0) \neq (0, 0)$ viscosity solution to

$$-F_1[u_0] = \lambda_0 \tau_1(x)|v_0|^{q-1}v_0, \quad -F_2[v_0] = \lambda_0 \tau_2(x)|u_0|^{p-1}u_0 \text{ in } \Omega, \quad u_0 = v_0 = 0 \text{ on } \partial \Omega. \quad (2.8)$$

for some $\lambda_0 > 0$. We infer that this implies the existence of a curve of eigenvalues of the form $(\lambda, \mu) \in \mathbb{R}^{+} \times \mathbb{R}^{+}$, with associated eigenfunctions $u, v$ such that

$$-F_1[u] = \lambda \tau_1(x)|v|^{q-1}v, \quad -F_2[v] = \mu \tau_2(x)|u|^{p-1}u \text{ in } \Omega, \quad u = v = 0 \text{ on } \partial \Omega. \quad (2.9)$$

(i.e., they solve (LE)). Indeed, given $\lambda > 0$, set

$$u = u_0, \quad v = \lambda_0^{\frac{p}{q-1}}v_0, \quad \text{and } \mu = \frac{\lambda_0^{\frac{1}{q-1}}}{\lambda^{\frac{1}{q}}}.$$ 

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By the homogeneity assumption (H2), we see that (2.9) is satisfied. In other words, given \( \lambda_0 > 0 \), (2.7) produces a spectral curve \( \Lambda_{\lambda_0} \) parametrized by

\[
\Lambda_{\lambda_0}(\lambda) = (\lambda, \mu(\lambda)), \quad \text{where} \quad \mu(\lambda) = \frac{\lambda^{p+1}}{\lambda^p}, \lambda > 0.
\]

(2.11)

Observe that \( \lambda \mapsto \mu(\lambda) \) is one-to-one, \( \mu(\lambda) \to 0 \) as \( \lambda \to \infty \), \( \mu(\lambda) \to \infty \) as \( \lambda \to 0 \). Moreover, \( \Lambda_{\lambda_0} \cap \Lambda_{\lambda'_0} \) if \( \lambda_0 \neq \lambda'_0 \), and \( \mathbb{R}^+ \times \mathbb{R}^+ = \bigcup_{\lambda_0} \Lambda_{\lambda_0} \).

From these comments one derives that, via the relation (2.10), to study (LE) or \( \text{(2.7)} \), and that statements in the Introduction can be equivalently written in terms of these scalings. Since in what follows we are going to consider almost exclusively the first quadrant of the plane \((\lambda, \mu)\) (with the exception of Theorem 1.7), we will equivalently write (LE) in the form (2.7).

In particular, ahead in Sect. 5.1 we will prove that the principal eigenvalues \( \lambda_1^\pm = \lambda_1^{\pm}(F_1, F_2) \), defined in the previous section, exist and are positive, being associated with positive solutions of

\[
-F_1[u] = \lambda_1^\pm \tau_1(x)|v|^{q-1}v, \quad -F_2[v] = \lambda_1^\pm \tau_2(x)|u|^{p-1}u \quad \text{in} \quad \Omega, \quad u = v = 0 \text{ on } \partial \Omega.
\]

From this one builds two spectral curves

\[
\Lambda_1^\pm(\lambda) = (\lambda, \mu_1^\pm(\lambda)), \quad \text{where} \quad \mu_1^\pm(\lambda) = \frac{(\lambda_1^\pm)^{p+1}}{\lambda^p}, \quad \text{for all} \quad \lambda > 0.
\]

(2.12)

Each \( \mu_1^\pm(\lambda) \) is strictly decreasing as a function of \( \lambda \), satisfying the asymptotic behavior (1.6) stated in Theorem 1.3. Moreover, in view of the proof of Theorem 1.7, observe that saying that \((\lambda, \mu)\) lies for instance below the curve \( \Lambda_1^+ \) and in the first quadrant is completely equivalent to saying that

\[
\mu < \frac{(\lambda_1^1)^{p+1}}{\lambda^p} \quad \Leftrightarrow \quad \lambda_0 := (\mu \lambda)^{\frac{1}{p+1}} < \lambda_1^+.
\]

Remark 2.12 Just to justify other scaling prototypes we might find in the literature, instead of (2.7) we could also have written either

\[
-F_1[u] = \tau_1(x)|v|^{q-1}v, \quad -F_2[v] = \lambda \tau_2(x)|u|^{p-1}u \quad \text{in} \quad \Omega, \quad u = v = 0 \text{ on } \partial \Omega;
\]

or

\[
-F_1[u] = \lambda \tau_1(x)|v|^{q-1}v, \quad -F_2[v] = \tau_2(x)|u|^{p-1}u \quad \text{in} \quad \Omega, \quad u = v = 0 \text{ on } \partial \Omega;
\]

which are both equivalent to (2.7) whenever we are in the first quadrant.

On the other hand, we point out that we may reparametrize the curve (2.11) as

\[
\Lambda_{\lambda_0}(a) = (\lambda(a), \mu(a)), \quad \text{where} \quad \mu(a) = a\lambda(a), \quad \text{for} \quad a = \frac{\mu}{\lambda} = \frac{\lambda_0^{p+1}}{\lambda^p + 1}.
\]

This way one recovers the notation and asymptotic behavior from [30],

\[
\lambda(a) = \frac{\lambda_0}{a^{1/(p+1)}} \to 0 \text{ as } a \to +\infty, \quad \lambda(a) \to +\infty \text{ as } a \to 0,
\]

\[
\mu(a) = a\lambda = a\frac{p}{p+1}\lambda_0 \to +\infty \text{ as } a \to +\infty, \quad \mu(a) \to 0 \text{ as } a \to 0.
\]
3 The scalar case with unbounded coefficients

We first recall that, in the case of proper operators with unbounded drift and weights, existence of positive principal eigenvalues \( \lambda_1^\pm \) with associated eigenfunctions \( \varphi_1^\pm \) is proved in Proposition 2.9. In this section we start by extending the existence of eigenvalues for nonproper operators.

**Lemma 3.1** Set \( F_0[u] := F[u] - \partial(x)u \), where \( F \) satisfies (H1), (H2), (1.5), and (H4). Then the quantity defined by

\[
\lambda_1^+(F(\hat{\vartheta})) := \lambda_1^+(F_0(\hat{\vartheta})) - 1
\]

is the first eigenvalue associated to a positive eigenfunction of the scalar Dirichlet problem \( F[u] + \lambda \partial(x)u = 0 \) in \( \Omega \), \( u = 0 \) on \( \partial \Omega \). Moreover, if \( C_A \) is the ABP-constant in Proposition 2.3, then

\[
\lambda_1^+(F(\hat{\vartheta})) \geq \frac{1}{C_A \| \vartheta \|_{L^\infty(\Omega)}} - 1.
\]

In particular, assumption (H3) is verified whenever \( \| \vartheta \|_{L^\infty(\Omega)} < \frac{1}{C_A} \).

An analogous result holds for \( \lambda_1^+(F(\vartheta)) = \lambda_1^+(G(\hat{\vartheta})) \) by applying it to \( G \), see (2.5).

**Proof** Notice that \( F_0 \) is a proper operator, i.e. \( F_0(x, r, \xi, X) \leq F_0(x, s, \xi, X) \) for \( r \geq s \), since

\[
F_0(x, r, \xi, X) - F_0(x, s, \xi, X) \leq \partial(x)|r - s| - \partial(x)(r - s) = 2\partial(x)(r - s)^-
\]

for \( r, s \in \mathbb{R} \).

We evoke the existence and positivity of the first eigenvalue \( \lambda_1^+(F_0(\hat{\vartheta})) \) for the proper fully nonlinear operator \( F_0 \) with unbounded drift \( \gamma \) and weight \( \hat{\vartheta} \) from [35]. Hence, by the definition of scalar eigenvalue, one derives the first statement.

Now, if one writes \( \vartheta = \ell \hat{\vartheta} \), for \( \ell = \| \vartheta \|_{L^\infty(\Omega)} \) and \( \| \hat{\vartheta} \|_{L^\infty(\Omega)} = 1 \), then by definition of \( \lambda_1^+ \), one deduces \( \ell \lambda_1^+(F_0(\vartheta)) = \lambda_1^+(F_0(\hat{\vartheta})) \), and so

\[
\lambda_1^+(F(\vartheta)) > 0 \iff \ell < \lambda_1^+(F_0(\hat{\vartheta})).
\]

Let us check that (3.3) is verified if \( \| \vartheta \|_N \) is sufficiently small. We claim that \( \lambda_1^+(F_0(\vartheta)) \geq C > 0 \) uniformly in \( \vartheta \) whenever \( \| \vartheta \|_{L^\infty(\Omega)} \) is fixed. Indeed, since \( \lambda_0 = \lambda_1^+(F_0(\vartheta)) \) is well defined and positively attained by [35], then there exists a positive eigenfunction \( \varphi_0 \) related to \( \lambda_0 \) such that

\[
F_0[\varphi_0] = -\lambda_0 \theta(x) \varphi_0, \varphi_0 > 0 \text{ in } \Omega, \varphi_0 = 0 \text{ on } \partial \Omega.
\]

Since \( F_0 \) is a proper operator, then \( \mathcal{L}_0[\varphi_0] \geq -\lambda_0 \theta(x) \varphi_0 \) in \( \Omega^+ \) (see Remark 2.4). Therefore, ABP (Proposition 2.3) and \( \varphi_0 > 0 \) yield the existence of a universal constant \( C_A \) such that

\[
\sup_{\Omega} \varphi_0 \leq C_A \lambda_0 \sup_{\Omega} \varphi_0 \| \vartheta \|_{L^\infty(\Omega)}.
\]

In particular we achieve (3.3) by taking \( \vartheta = \hat{\vartheta} \) and \( \ell C_A < 1 \). Equivalently, for \( \vartheta = \hat{\vartheta} \) we derive (3.2) due to relation (3.1).

**Remark 3.2** The bound from below (3.2) extends and improves [42, Proposition 3.3] to the context of unbounded coefficients. In particular, with our spectral tools, the proof of [42, Theorem 1] can now be considerably shorten, see also Remark in p.595 there.
We now show how scalar principal eigenvalues are related to the validity of the Alexandrov-Bakelman-Pucci estimate in the scalar case, in the following sense.

**Definition 3.3** Let $F$ satisfy (H1). We say that ABP-MP (resp. ABP-mP) holds for $F$ in $\Omega$ if whenever $f \in L^N(\Omega)$ and $u \in C(\overline{\Omega})$ viscosity solution of $F[u] \geq f(x)$ (resp. $F[u] \leq f(x)$) in $\Omega$, then

$$\max_{\Omega} u \leq C_B \{ \max_{\partial \Omega} u^+ + \| f^- \|_{L^N(\Omega)} \} \quad \text{or} \quad \min_{\Omega} u \geq C_B \{ \min_{\partial \Omega} (-u^-) - \| f^+ \|_{L^N(\Omega)} \},$$

(3.5)

for some positive constant $C_B$ not depending on the norm of $u$.

**Theorem 3.4** Let $\Omega$ be a bounded $C^{1,1}$ domain. Assume (H1), (1.5) on $F$, (1.5), (H2) on $F^*$.

(i) If $\lambda_1^+(F^*(\vartheta)) = \lambda_1^-(F^*(\vartheta)) > 0$ then ABP-MP holds for $F$ in $\Omega$, with $C_B$ depending on $N, q, \alpha, \beta, \| \gamma \|_2, \| \vartheta \|_2, \text{diam}(\Omega), \lambda_1^+(F^*(\vartheta))$. On the other hand, if $F$ satisfies (H2), (H4), and $\lambda_1^+(F(\vartheta)) \leq 0$ then ABP-MP does not hold for $F$;

(ii) If $\lambda_1^+(F^*(\vartheta)) = \lambda_1^-(F^*(\vartheta)) > 0$ then ABP-mP holds for $F$ in $\Omega$, with $C_B$ depending on $N, q, \alpha, \beta, \| \gamma \|_2, \| \vartheta \|_2, \text{diam}(\Omega), \lambda_1^-(F^*(\vartheta))$. On the other side, if $F$ satisfies (H2), (H4), and $\lambda_1^-(F(\vartheta)) \leq 0$ then ABP-mP does not hold for $F$.

Note that, in order prove ABP-mP and ABP-MP, we do not need impose $F$ verifying (H2). This is good since, for instance, one may take $F^*$ to be $L^+$, which always satisfies (H2) even when $F$ does not. Of course notice that, if $F$ satisfies (H2), then this is also the case for $F^*$.

**Lemma 3.5** Let $\Omega \in C^{1,1}$ be a bounded domain. If $F$ is either a convex or concave operator in the X-entry, for which it holds (H1) and (1.5), then $F$ satisfies (H4). In particular, $\lambda_1^+(F(\vartheta))$ as in Lemma 3.1 is well defined for convex (or concave) operators satisfying only (H1), (H2), and (1.5).

**Proof** By [48, Theorem 5.3] we already know that $F$ enjoys $W^{2,2}$ interior regularity estimates. Thus it is enough to obtain the global statement. Let $u$ be a viscosity solution of $F[u] = f(x)$ in $\Omega$, where $f \in L^0(\Omega), u = \psi$ on $\partial \Omega$ for some $\psi \in W^{2,2}(\Omega)$. Then by the local regularity we know that $u$ is a strong solution. Thus, the $C^{1,a}$ global regularity in Proposition 2.8 and the proof of Nagumo’s lemma in [35, Lemma 4.4] imply the desired global regularity and estimates.

**Proof of Theorem 3.4** We only show item (i), since (ii) is analogous. Assume $u \in C(\overline{\Omega})$ is a viscosity solution of $F[u] \geq f(x)$ in $\Omega$. We first notice that, if $\lambda_1^+(F(\vartheta)) \leq 0$, then ABP-MP is not satisfied. Indeed, in this situation we obtain

$$F[\varphi_1^+] = -\lambda_1^+(\vartheta(x)) \varphi_1^+ \geq 0 \text{ in } \Omega,$$

with $\varphi_1^+ = 0$ on $\partial \Omega$, but $\varphi_1^+ > 0$ in $\Omega$. Consequently, ABP-MP does not hold in general.

Now set $\lambda_1^+ := \lambda_1^+(F^*(\vartheta)) > 0$. Let us show that this is a sufficient condition for ABP-MP.

Step 1) Let us check that, if there exists a solution $\psi \in W^{2,2}(\Omega) \cup C(\overline{\Omega})$ of $F^*[\psi] \leq 0$ in $\Omega$ such that $\psi > 0$ in $\Omega$, $\psi = 1$ on $\partial \Omega$, and $\psi \in [a, b]$ for some universal constants $0 < a < b$, then $F$ satisfies ABP-MP in $\Omega$, for a constant that depends also on $a = \inf_{\Omega} \psi$ and $b = \sup_{\Omega} \psi$.

Set $D := \| \psi \|_{C^1(\overline{\Omega})}$, Note that $v = \frac{u}{\psi}$ is a viscosity solution of $F[\psi][v] \geq f(x)$ in $\Omega$, where

$$F[\psi](x, r, \xi, X) := F(x, r \psi, r D \psi + \psi \xi, r D^2 \psi + \psi X + 2 D \psi \otimes \xi).$$

(3.6)
The operator $F_\psi$ satisfies (H1) with ellipticity constants $aa, b\beta$, with drift term $\gamma_\psi(x) = (2D + b)\gamma(x)$. Furthermore, $F_\psi$ is a proper operator: indeed, by applying (H1) for $F$, and (H2) for $F^\ast$, one finds

\[ F_\psi(x, r, \xi, X) - F_\psi(x, r, \eta, Y) \leq \mathcal{M}_{aa, b\beta}^+(X - Y) + \gamma_\psi(x)|\xi - \eta|, \]

\[ F_\psi(x, r, \xi, X) - F_\psi(x, s, \xi, X) \leq F^\ast[(r - s)\psi] = (r - s)F^\ast[\psi] \leq 0 \text{ for } r \geq s. \]

Now ABP for proper operators with unbounded coefficients (Proposition 2.3 and subsequent remark) produces the estimate (2.2) for $v = u/\psi$. In addition, $u \leq 0$ in the set where $v \leq 0$, and $u = v\psi \leq bv$ in the set where $v > 0$, and so $u$ satisfies (3.5).

Step 2) Now we prove that $\lambda_1^+ > 0$ yields the existence of a function $\psi$ as in Step 1.

Note that there exists a neighborhood of $\partial \Omega$ such that $\varphi^+_1$ attains its global maximum outside it. Moreover, as in [40, Lemma 4.5], by the Lipschitz estimate (see [43, Theorem 2.3] for a version of [40, Proposition 4.9] for unbounded coefficients) we may take this neighborhood $\mathcal{U}$ depending only on $N, \lambda, \Lambda, ||\gamma||_\rho, ||\partial \Omega||_N, \lambda_1^+$, i.e., uniform with respect to the class of equations we consider. Indeed, $F^\ast[\varphi^+_1] + \lambda_1^+\vartheta(x)\varphi^+_1 = 0$ in $\Omega$, and so $\varphi^+_1$ is a viscosity positive solution of $\mathcal{L}_0^+(\varphi^+_1) \geq -(1 + \lambda_1^+)\vartheta(x)\varphi^+_1$ in $\Omega$, $\varphi^+_1 = 0$ on $\partial \Omega$, and by [43, Theorem 2.3] we get

\[ 1 = \varphi^+_1(x_0) = \max_{\Omega} \varphi^+_1 \leq C(1 + \lambda_1^+)||\partial \Omega||_N \text{ dist}(x_0, \partial \Omega) \]

\[ \Rightarrow \text{dist}(x_0, \partial \Omega) \geq (C(1 + \lambda_1^+)||\partial \Omega||_N)^{-1}, \]

where $C$ is a universal positive constant depending only on $n, \varrho, \alpha, \beta, ||\gamma||_{L^\varrho(\Omega)}$, $\Omega$. We then take a compact set $K \subset (\mathbb{R}^N \setminus \mathcal{U})$ such that $\varphi^+_1$ attains its maximum equal to 1 in $K$ and

\[ |\Omega \setminus K| \leq \varepsilon := (2C_A ||\partial \Omega||_{L^\varrho(\Omega)})^{-\frac{N}{\varrho}}, \]

where $C_A$ is the constant in Proposition 2.3. Since $F^\ast$ satisfies (H1), by [42, Theorem 1.1(ii)] we may consider $w \in C(\bar{\Omega})$ a viscosity solution of the Dirichlet problem

\[ F^\ast(x, 0, 0, D^2w) + \gamma(x)|Dw| = f(x) \quad \text{in } \Omega, \quad w = 0 \text{ on } \partial \Omega, \quad (3.7) \]

where $f(x) = -2\vartheta(x)$ in $\Omega \setminus K$ and $f(x) = 0$ in $K$. Note that $w \in W^{2,\varrho}(\Omega)$ by Lemma 3.5.

Next, we apply ABP (Proposition 2.3) and Hölder’s inequality to find

\[ 0 < w \leq 2C_A||\vartheta||_{L^{N}(\Omega, K)} \leq 2C_A|\Omega \setminus K|^{1/N-1/\varrho}||\vartheta||_{L^{\varrho}(\Omega, K)} \]

\[ \leq 2C_A\varepsilon^{\frac{\varrho}{N\varrho}}||\vartheta||_{L^{\varrho}(\Omega)} = 1 \text{ in } \Omega. \]

Then $w$ also solves, in the strong sense,

\[ F^\ast[w + 1] \leq F^\ast(x, 0, 0, D^2w) + \gamma(x)|Dw| + \vartheta(x)(w + 1) = \vartheta(x)(w - 1) \leq 0 \text{ in } \Omega \setminus K. \]

Now we infer that Harnack inequality gives us $\varphi^+_1 \geq \eta$ on $K$, for some $\eta > 0$. In fact, since $\varphi^+_1$ is a positive solution of the inequalities $\mathcal{L}_0^+(\varphi^+_1) + (1 + \lambda_1^+)\vartheta(x)\varphi^+_1 \geq 0$ and $\mathcal{L}^-[\varphi^+_1] \leq 0$ in $\Omega$, this is a combination of the Local Maximum Principle for the nonproper operator $\mathcal{L}_0^+(1 + \lambda_1^+)\vartheta(x)\varphi^+_1$ with unbounded zero order term (which is obtained from [43, Theorem 2.5] through a Moser type argument, see details in [34, proof of Theorem 2.2]) followed by the Weak Harnack inequality for the proper operator $\mathcal{L}^- = \mathcal{L}_0^+ - \vartheta$ (since $\varphi^+_1 > 0$) with unbounded coercive term, see [44, Theorem 2.1]. This produces a positive constant $\eta$ depending on $n, \varrho, \alpha, \beta, \Omega, ||\gamma||_{L^\varrho(\Omega)}, ||\vartheta||_{L^{\varrho}(\Omega)}$ and $\lambda_1^+$. 

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Now we set \( A\lambda^+_1\eta = 2 \) and \( \psi = 1 + w + A\varphi^+_1 \). Thus \( 1 \leq \psi \leq 2 + A = b \) in \( \Omega \), \( \psi = 1 \) on \( \partial \Omega \), and \( \psi \) is a strong solution of
\[
F^*[\psi] \leq F^*[w + 1] + F^*[A\varphi^+_1] \leq f(x) + \vartheta(x)(w + 1) - A\lambda^+_1\vartheta(x)\varphi^+_1
\]
\[= h(x) \leq 0 \text{ in } \Omega, \]
since \( h(x) = \vartheta(x)(w + 1 - A\lambda^+_1\eta) \leq 0 \) in \( K \). Note that \( a = \inf_{\Omega} \psi = 1 \), and \( b = \sup_{\Omega} \psi \) depend only upon the same constants of ABP (see for instance [26]) since \( \psi = 1 \) on \( \partial \Omega \). In conclusion, \( F \) satisfies ABP-MP in \( \Omega \), which proves the theorem.

\[\blacksquare\]

**Remark 3.6** If \( F^* \) is a proper operator then Proposition 3.4 reduces to the usual ABP (Proposition 2.3). Moreover, in this case \( \psi \equiv 1 \) verifies the conditions in Step 1, since \( F^*[1] \leq F^*[0] = 0 \).

Let us now discuss some applications of Theorem 3.4. In what concerns the Dirichlet problem, the following solvability in the scalar case will be essential for the solvability of the system.

**Theorem 3.7** Let \( \Omega \subset C^{1,1} \) be a bounded domain. Assume (H1), (H3) on \( F \), and (H2) on \( F^* \). Let \( f \in L^q(\Omega), q > N \). Then there exists a viscosity solution \( u \in C^{1,\alpha}(\overline{\Omega}) \) of the problem
\[
F(x, u, Du, D^2u) = f(x) \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega. \tag{3.8}
\]
Further, if (3.8) possesses a strong solution \( u \in W^{2,q}_{\text{loc}}(\Omega) \), then \( u \) is the unique solution of (3.8) in the class of viscosity solutions. In particular, (3.8) is uniquely solvable under (H4).

**Proof** By Step 2 in the proof of Theorem 3.4 we know that there exists a function \( \psi \in W^{2,q}(\Omega) \) with \( \psi \geq 0 \) in \( \overline{\Omega} \) such that \( \psi \) solves \( F^*[\psi] \leq 0 \) in \( \Omega \) in the strong sense. Next, by Step 1 in that proof, one may define (3.6), from which we see that solving (3.8) is equivalent to solve \( F_\psi[v] = f(x) \) in \( \Omega \), where \( F_\psi \) is a proper operator satisfying (H1). In turn, the existence of a viscosity solution \( v \in C(\overline{\Omega}) \) to \( F_\psi[v] = f(x) \) comes from [42, Theorem 1(ii), case \( \mu = 0 \)]; and the uniqueness in the presence of a strong solution follows by [42, Theorem 1(iii), case \( \mu = 0 \)]. The regularity assertions are an immediate consequence of the \( C^{1,\alpha} \) results in Proposition 2.8.

As a nontrivial application of Theorem 3.4 to systems, we prove MP and mP for either domains with small measure or weights with small \( L^N \)-norm.

**Proof of Theorem 1.15** We only prove item (i), as (ii) is completely analogous. We start choosing \( A > 0 \) such that \( \lambda \mu^q C_B^{1+q} \leq A \), where \( C_B > 0 \) is the universal constant in Definition 3.3, which depends upon \( N, \alpha, \beta, \text{diam}(\Omega), \|\gamma\|_{L^q(\Omega)} \), in addition to \( \lambda^+_1(F^*_i(\vartheta)) \).

Note that \((u, v)\) is a pair of viscosity solutions to
\[
-F^*_1[u] \leq \lambda \tau_1(x)(v^+)^q, \quad -F^*_2[v] \leq \mu \tau_2(x)(u^+)^p \text{ in } \Omega.
\]
Hence, Theorem 3.4 just proved, applied to the scalar equation for \( u \) and \( v \), yields
\[
\sup_{\Omega} u \leq C_B \lambda \|\tau_1\|_{L^q(\Omega)} \sup_{\Omega} (v^+)^q, \quad \sup_{\Omega} v \leq C_B \mu \|\tau_2\|_{L^p(\Omega)} \sup_{\Omega} (u^+)^p. \tag{3.9}
\]
If we had either \( u \leq 0 \) or \( v \leq 0 \) in \( \Omega \), then by (3.9) we would obtain \( u, v \leq 0 \) in \( \Omega \), and the proof is done. We then assume that both \( u \) and \( v \) assume their positive maxima in \( \Omega \). Then,
\[
\sup_{\Omega} (u^+) \leq C_B^{1+q} \lambda \mu^q \|\tau_1\|_{L^q(\Omega)} \|\tau_2\|_{L^q(\Omega)}^q \sup_{\Omega} (u^+)^{pq} \leq A \|\tau_1\|_{L^q(\Omega)} \|\tau_2\|_{L^q(\Omega)}^q \|u\|_{L^q(\Omega)}^{pq-1} \sup_{\Omega} (u^+).
\]
For MP with small weights \( \tau_1, \tau_2 \in L^N_+ (\Omega) \), we choose \( \varepsilon_0 > 0 \) with \( \| \tau_1 \|_{L^N (\Omega)} \| \tau_2 \|_{L^N (\Omega)}^q \leq \varepsilon_0 \) so that \( A \varepsilon_0 \| u \|_{\infty}^{\frac{pq}{2}} \leq 1/2 \). Then, \( u \leq 0 \) in \( \Omega \), and so does \( v \) by (3.9). Upon performing the above argument with \( \sup_{\Omega} (v^+) \), one can assume instead \( \| \tau_1 \|_{L^N (\Omega)} \| \tau_2 \|_{L^N (\Omega)} \leq \varepsilon_0 \).

On the other hand, for \( \tau_1, \tau_2 \in L^E_+ (\Omega) \), say \( \| \tau_1 \|_{L^E (\Omega)} \| \tau_2 \|_{L^E (\Omega)}^q \leq W \), we apply Holder inequality to obtain

\[
\sup_{\Omega} (u^+) \leq AW \| \Omega (\frac{1}{\varepsilon_0^q - \frac{1}{2}}) \| u \|_{\infty}^{\frac{pq}{2}} \sup_{\Omega} (u^+).
\]

Then we pick \( \varepsilon_0 > 0 \) with \( |\Omega| \leq \varepsilon_0 \) such that \( AW \varepsilon_0 (\frac{1}{\varepsilon_0^q - \frac{1}{2}}) \| u \|_{\infty}^{\frac{pq}{2}} \leq 1/2 \), from which we also derive \( u, v \leq 0 \) in \( \Omega \).

The argument for supersolutions is analogous, by considering the negative parts. In any case, observe that, if \( pq = 1 \), then \( \varepsilon_0 \) can be chosen independently of the \( L^\infty \)-norm of \( u \) and \( v \).

\( \square \)

4 Auxiliary results for systems

In this section we consider some fundamental results which appear throughout the text. We start with an instrumental proposition to our analysis of uniqueness results.

**Proposition 4.1** Let \( F_1, F_2 \) satisfy (H1), (H2), (H3). Let \( pq = 1, \lambda, \mu \geq 0, \) and \( (u_1, v_1), (u_2, v_2) \) in \( C (\overline{\Omega}) \) be viscosity solutions of

\[
\begin{align*}
F_1[u_1] + \lambda \tau_1 (x) |u_1|^{q-1} v_1 \leq 0 & \text{ in } \Omega \\
F_2[v_1] + \mu \tau_2 (x) |u_1|^{p-1} u_1 \leq 0 & \text{ in } \Omega \\
u_1, v_1 & > 0 \text{ in } \Omega
\end{align*}
\]

In addition, assume that

\[
\text{either } u_2 (x_0) > 0 \text{ or } v_2 (x_0) > 0, \text{ for some } x_0 \in \Omega; \quad (4.1)
\]

and that one of the pairs of solutions is in \( E_0 = W^{2, q} (\Omega) \cap C (\overline{\Omega}) \). Then \( u_1 \equiv tu_2 \) and \( v_1 \equiv t^p v_2 \) in \( \Omega \) for some \( t > 0 \).

Analogously, if \( pq = 1, \lambda, \mu \geq 0, \) and \( (u_1, v_1), (u_2, v_2) \) in \( C (\overline{\Omega}) \) satisfy

\[
\begin{align*}
F_1[u_1] + \lambda \tau_1 (x) |u_1|^{q-1} v_1 \leq 0 & \text{ in } \Omega \\
F_2[v_1] + \mu \tau_2 (x) |u_1|^{p-1} u_1 \leq 0 & \text{ in } \Omega \\
u_1, v_1 & < 0 \text{ in } \Omega
\end{align*}
\]

with either \( u_2 (x_0) < 0 \) or \( v_2 (x_0) < 0, \) for some \( x_0 \in \Omega; \) and that one of the pairs of solutions is in \( E_0 \). Then \( u_1 \equiv tu_2 \) and \( v_1 \equiv t^p v_2 \) in \( \Omega \) for some \( t > 0 \).

**Remark 4.2** The assumption (4.1) on the pair \( (u_2, v_2) \) means that the system (LE) does not satisfy the maximum principle for \( (\lambda, \mu) \).

**Proof** Let us prove the first statement, since the other one is carried out similarly.

We observe that (4.1) implies \( \sup_{\Omega} u_2 > 0 \) and \( \sup_{\Omega} v_2 > 0 \). Indeed, say \( u_2 (x_0) > 0 \), which yields \( \sup_{\Omega} u_2 > 0 \). If we had \( v_2 \leq 0 \) in \( \Omega \) then \( -F_1^* [u_2] \leq 0 \) in \( \Omega, u_2 \leq 0 \) on \( \partial \Omega \), so we would get \( u_2 \leq 0 \) by ABP-MP (since we assume \( F_1 \) satisfies (H3), the first eigenvalue of \( F_1^* \) is positive and so Theorem 3.4 can be applied). This yields a contradiction.
Observe that for each compact set $K \subset \Omega$, with $x_0 \in K$, there exists $s_K$ such that $u_1 > s_K u_2$, $v_1 > s_K^p v_2$ in $K$. This comes from the fact that min $u_1$, min $v_1$, max $u_2$, max $v_2$ are positive over $K$.

Next we claim that $u_1 \geq s u_2$, $v_1 \geq s^p v_2$ in a neighborhood of $\partial \Omega$ for some small $s > 0$.

It is enough to prove the first inequality; the second one is analogous. Notice that $u_1 - s u_2 \geq 0$ on $\partial \Omega$, for all $s > 0$. Fix $\hat{x} \in \partial \Omega$. If $u_1(\hat{x}) - s u_2(\hat{x}) > 0$, then by continuity of $u_1$ and $u_2$ up to the boundary there exists a neighborhood of $\hat{x}$, namely $\hat{B}$, such that $u_1 - s u_2 > 0$ in $\Omega \cap \hat{B}$. Assume then $u_1(\hat{x}) - s u_2(\hat{x}) = 0$ for some $s$. Thus $u_1(\hat{x}) = u_2(\hat{x}) = 0$. Let us look at the quantities

$$A_i = \lim_{t \to 0^+} \frac{u_i(\hat{x} + tv) - u_i(\hat{x})}{t}, \quad i = 1, 2,$$

where $v$ is the interior unit normal vector to $\partial \Omega$ at $\hat{x}$. Hopf’s lemma for viscosity solutions (Proposition 2.7) yields $A_1 > 0$. If we had $A_2 \leq 0$, then $A_1 - s A_2 > 0$ for all $s > 0$. Otherwise, if $A_2 > 0$ then we may pick some small $\hat{s} > 0$ such that $A_1 - \hat{s} A_2 > 0$. Recall that one of the solutions pair is in $E_o$, then one of the $A_i$’s is the normal derivative of $u_i$ at $\hat{x}$. Since $u_1(\hat{x}) - s u_2(\hat{x}) = 0$, this is enough to ensure that $u_1 - s u_2 > 0$ in $\Omega \cap \hat{B}$. A covering argument then concludes the claim.

Therefore one obtains the existence of some $s_0 > 0$ such that $u_1 \geq s_0 u_2$ and $v_1 \geq s_0^p v_2$ in $\Omega$, for all $s \leq s_0$. In particular, the following set is nonempty,

$$S = \{s > 0 : u_1 > s u_2, \quad v_1 > s^p v_2 \text{ in } \Omega\},$$

and the quantity $s_* = \sup S$ is well defined. We have $s_* < +\infty$ by (4.1).

Notice that $w := u_1 - s_* u_2 \geq 0$ and $z := v_1 - s_*^p v_2 \geq 0$ in $\Omega$. Moreover, $s_*^p |u_2|^{p-1} u_2 \leq u_1^p$ and $s_* |v_2|^{p-1} v_2 \leq v_1^q$ in $\Omega$. By using also (H1), (H2), and $\lambda, \mu \geq 0$, one sees that $w, z$ satisfy

$$(F_1)_* [w] \leq F_1[u_1] - s_* F_1[u_2] \leq -\lambda \tau_1(x) v_1^q + \lambda \tau_1(x) s_* |v_2|^{q-1} v_2 \leq 0,$$

$$(F_2)_* [z] \leq F_2[v_1] - s_*^p F_2[v_2] \leq -\mu \tau_2(x) u_1^p + \mu \tau_2(x) s_*^p |u_2|^{p-1} u_2 \leq 0,$$

(4.2) in the viscosity sense in $\Omega$, with $w, z \geq 0$ on $\partial \Omega$. Whence, by applying twice Theorem 3.4 and SMP for scalar equations we get either $w > 0$ or $w \equiv 0$ in $\Omega$; and either $z > 0$ or $z \equiv 0$ in $\Omega$.

Notice that $w > 0$ is equivalent to $z > 0$ by (4.2). In other words, one has either $w, z > 0$ or $w, z \equiv 0$. Under the latter we are done.

Suppose on the contrary that $w, z > 0$ in $\Omega$. Now we may reproduce the preceding argument with the pair $(u_2, v_2)$ replaced by $(w, z)$ in order to conclude the existence of some small $\varepsilon > 0$ such that $u_1 > (s_* + \varepsilon) u_2$ in $\Omega$. But this contradicts the definition of $s_*$ as the supremum of $S$. \hfill $\Box$

**Corollary 4.3** Let $F_1, F_2$ satisfy (H1), (H2), (H3). Let $pq = 1$, $\lambda \geq 0$, and let $(u_1, v_1)$, $(u_2, v_2)$ be viscosity solutions of

$$\begin{cases}
F_1[u_1] + \lambda \tau_1(x) |v_1|^{q-1} v_1 = 0 & \text{in } \Omega, \\
F_2[v_2] + \lambda \tau_2(x) |u_1|^{p-1} u_1 = 0 & \text{in } \Omega, \\
u_1, v_1 = 0 & \text{on } \partial \Omega,
\end{cases} \quad \begin{cases}
F_1[u_2] + \Lambda \tau_1(x) |v_2|^{q-1} v_2 = 0 & \text{in } \Omega, \\
F_2[v_2] + \Lambda \tau_2(x) |u_2|^{p-1} u_2 = 0 & \text{in } \Omega, \\
u_2, v_2 = 0 & \text{on } \partial \Omega,
\end{cases}$$

with one of the solutions pair in $E_o$. Suppose that either $u_i, v_i > 0$ in $\Omega$ or $u_i, v_i < 0$ in $\Omega$, $i = 1, 2$. Then $\lambda = \Lambda$, and $u_1 \equiv \tau u_2$, $v_1 \equiv \tau^p v_2$ in $\Omega$ for some $\tau > 0$.

We conclude the section presenting a priori bounds for the first eigenvalue.
Lemma 4.4  Let $F_1, F_2$ satisfy (H1), (H2), (H3) with $\tau_i(x) \geq \delta$ a.e. in $B_R$ for $i = 1, 2$, for some $B_R \subset \subset \Omega$, then

$$\lambda_1^\pm(F_1(\tau_1), F_2(\tau_2), \Omega) \leq \delta^{-1} \lambda_1^\pm(F_1(1), F_2(1), B_R).$$

Proof  We work on the $\lambda_1^+$ case, since for $\lambda_1^-$ it is just a question of replacing $F_1$ by $G_i(x, r, p, X) = -F_i(x, -r, -p, -X)$, recall (2.5). Observe that $\lambda_1^+(F_i, \Omega) \leq \lambda_1^+(F_i, B_R)$, by definition. Also, both quantities are nonnegative by Remark 2.11. Hence, given

$$A := \{ \lambda \in \mathcal{R}, \, \Psi_\lambda^+(\Omega) \neq \emptyset \}, \quad B := \{ \lambda \in \mathcal{R}, \, \Psi_\lambda^-(B_R) \neq \emptyset \},$$

it is enough to see that $A \cap \{ \lambda \geq 0 \} \subset B/\delta \cap \{ \lambda \geq 0 \}$, since

$$\lambda_1^+(F_1(\tau_1), F_2(\tau_2), \Omega) = \sup_{A} \lambda = \lambda_1^+(F_1(1), F_2(1), B_R) = \sup_{B} \lambda,$$

as settled before in Sect. 2.3. Let $\lambda \in A \cap \{ \lambda \geq 0 \}$, then there exist positive functions $\varphi, \psi \in C(\overline{\Omega})$ solving $F_i[\varphi] + \lambda \varphi(x)\psi^q \leq 0$, $F_2[\psi] + \lambda \tau_2(x)\varphi \leq 0$ in $\Omega$, in the viscosity sense. Hence, $(\varphi, \psi)$ is a positive viscosity solution of $F_1[\varphi] + \lambda \delta \psi^q \leq 0$, $F_2[\psi] + \lambda \delta \varphi \leq 0$ in $B_R$, and so $\delta \lambda \in B$. \hfill \Box

Proposition 4.5  Suppose $F_1, F_2$ verify (H1)--(H4), $pq = 1$, and $\tau_i \geq \delta > 0$, $\tau_2 \geq \delta > 0$ a.e. in $B_R \subset \subset \Omega$, for some $R \leq 1$. Let $\lambda > 0$ and $(u, v)$, with $uv > 0$ in $\Omega$, be a solution of

$$-F_1[u] = \lambda \varphi(x)|u|^{q-1} v, \quad -F_2[v] = \lambda \tau_2(x)|u|^{p-1} u \quad \text{in} \quad \Omega, \quad u = v = 0 \text{ on } \partial \Omega.$$

Then $\lambda \leq C$, for a positive constant $C$ depending on $N, \alpha, \beta, \Omega, \|\gamma\|_{L^0(\Omega)}, \|\vartheta\|_{L^0(\Omega)}, \delta,$ and $R$.

Proof  We use some constructions from [7, 30, 35]. By Lemma 4.4, it is enough to prove the result for $\tau_i \equiv 1$, $i = 1, 2$. Let us consider $u > 0$ and $v > 0$ in $\Omega$; for the case with $u < 0$ and $v < 0$ in $\Omega$ it is sufficient to apply the case with positive sign to $G_i$ in place of $F_i$, see (2.5).

Step 1) We first consider a bounded drift and zero order term $\gamma \in L^\infty(\Omega)$, namely $|\gamma(x)| \leq \gamma$ and $|\vartheta| \leq \vartheta$ for a.e. $x \in \Omega$. Let us prove in what follows that there exists $C > 0$ depending only on $N, \alpha, \beta, R, \gamma$, and $\vartheta$, such that

$$\lambda_1^\pm(F_1(1), F_2(1), B_R) \leq \frac{C}{R^2}.$$

Moreover, if $\gamma, \vartheta = 0$ then the constant does not depend on $R$, for all $R > 0$ such that $B_R \subset \subset \Omega$.

Note that the function $U(x) = (R^2 - |x|^2)^2$ is a positive strong solution of $F_1[U] + \frac{C_0}{R^2} U \geq 0$ in $B_R$, see [7]. Next we consider the strong solution of

$$F_2[V] + U^p = 0 \text{ in } B_R, \quad V = 0 \text{ on } \partial B_R,$$

given for instance in [40]. Here $V > 0$ in $B_R$ by Theorem 3.4 and SMP, since $\tau_2 > 0$ a.e. and $U > 0$ in $B_R$. Also, $\delta \vartheta V > 0$ on $\partial \Omega$ by Hopf, where $\nu$ is the interior unit normal. Now, since $pq = 1$, without loss of generality we may assume $q \leq 1$. Thus $U^{1/q} \in C^1(\overline{\Omega})$, and we can pick up some $a > 0$ large enough so that

$$U^{1/q} \leq a V \text{ in } B_R.$$

Therefore $U$ becomes a strong solution of $F_1[U] + \frac{a^2 C_0}{R^2} V^q \geq 0$ in $B_R$.\hfill \Box
Set $C = a^d C_0 > 1$. Suppose by contradiction that there exists some $\lambda > \frac{C}{R^2}$ such that $\varphi, \psi \in \Psi^+_1(B_R) \neq \emptyset$, i.e. let $(\varphi, \psi) \in C(B_R)$ be positive viscosity solutions of $F_1[\varphi] + \lambda \varphi^q \leq 0$ and $F_2[\psi] + \lambda \varphi^p \leq 0$ in $B_R$. They also solve $F_1[\varphi] + \frac{C}{R^2} \psi^q \leq 0$, $F_2[\psi] + \frac{C}{R^2} \varphi^p \leq 0$ in $B_R$. 

Now we apply Proposition 4.1 to obtain that $\varphi = tU$ and $\psi = t^p V$ in $B_R$ for some $t > 0$. However, this is not possible since $\varphi > 0$ on $\partial B_R \subset \Omega$, while $U = 0$ on $\partial B_R$.

Step 2) In general case we assume $\gamma, \vartheta \in L^\infty_+(\Omega)$. Let us show that there exists a universal constant such that $\lambda^{\frac{1}{N} - \frac{\gamma}{\vartheta}} \leq C(\|\gamma\|_{L^\infty(\Omega)} + \|\vartheta\|_{L^\infty(\Omega)})^2$.

We argue by contradiction. Suppose there exist sequences $\gamma_k, \vartheta_k \in L^\infty_+(\Omega)$ with $\|\gamma_k\|_{L^\infty(\Omega)} + \|\vartheta_k\|_{L^\infty(\Omega)} \leq C$, but $\|\gamma_k\|_{L^\infty(\Omega)} + \|\vartheta_k\|_{L^\infty(\Omega)} \to +\infty$, and let the respective eigenvalue problem

$$F_k^1[U_k] + \lambda_k v_k^q = 0, \quad F_k^2[V_k] + \lambda_k u_k^p = 0, \quad u_k, v_k > 0 \text{ in } B_R, \ u_k, v_k = 0 \text{ on } \partial B_R,$$

in the viscosity sense, such that $\lambda_k \to +\infty$ as $k \to \infty$, where $F_k^i$ is a fully nonlinear operator satisfying (H1)–(H4) for the respective $\gamma_k$ and $\vartheta_k$. Up to using the rescaling $(u, v) \mapsto (tu, t^p v)$ with $t = 1/\|u\|_\infty$, we may assume $\max_{\Omega} u_k = 1$, $\max_{\Omega} v_k = c_0$ for all $k \in \mathbb{N}$. Say $\max_{\Omega} u_k = u_k(x_k^0)$ for $x_k^0 \in \Omega$. Then, $x_k \to x_0 \in \Omega$ as $k \to +\infty$, up to a subsequence.

Since $B_R$ is a convex domain we know that $x_0 \in \Omega$. Let $2\rho = \text{dist}(x_0, \partial B_R) > 0$, so $x_k \in B_{2\rho}(x_0)$ for all $k \geq k_0$. Set $r_k = \lambda_k^{-1/2}$ and $U_k(x) = U_k(x_k^0 + r_k x)$, $V_k(x) = V_k(x_k^0 + r_k x)$. Thus, $(U_k, V_k)$ is a viscosity solution pair of

$$\tilde{F}_k^1[U_k] + V_k^q = 0, \quad \tilde{F}_k^2[V_k] + U_k^p = 0, \quad U_k, V_k > 0 \text{ in } \tilde{B}_k := B_{2\rho}(x_0),$$

where $\tilde{F}_k^i(x, r, p, X) = r_k^2 F_k^i(x_k^0 + r_k x, r, p/r_k, X/r_k^2)$ satisfies (H1)–(H4) for $\tilde{\gamma}_k$ and $\tilde{\vartheta}_k$, where $\tilde{\gamma}_k(x) = r_k^2 \gamma_k(x_k^0 + r_k x)$ and $\tilde{\vartheta}_k = r_k^2 \vartheta_k(x_k^0 + r_k x)$. Then one has

$$\|\tilde{\gamma}_k\|_{L^\infty} = r_k^{\frac{1}{N} - \frac{\gamma}{\vartheta}} \|\gamma_k\|_{L^\infty(\Omega)} \to 0 \text{ and } \|\tilde{\vartheta}_k\|_{L^\infty(\Omega)} = r_k^{\frac{2-\gamma}{\vartheta}} \|\vartheta_k\|_{L^\infty(\Omega)} \to 0 \text{ as } k \to \infty.$$

Also, $\sup_{\tilde{B}_k} U_k = U_k(0) = 1$ and $\sup_{\tilde{B}_k} V_k = V_k(0) \leq c_0$ for all $k \in \mathbb{N}$ with $B_{2\rho}(0) \subset \subset \tilde{B}_k$ for large $k$, for any fixed $R > 0$. By $C^{1,\alpha}$ regularity estimates (Proposition 2.8) we have $U_k, V_k \in C^{1,\alpha}_{\text{loc}}$ and

$$\|U_k\|_{C^{1,\alpha}(\overline{B_R}(0))}, \quad \|V_k\|_{C^{1,\alpha}(\overline{B_R}(0))} \leq C,$$

since the constant depends only on a bound from above on the $L^\infty$-norm of the coefficient $\gamma_k$, which is uniformly bounded. Hence, by compact embeddings, we have that there exists $U, V \in C^{1}(\overline{B_R}(0))$ such that $U_k \to U$, $V_k \to V$ as $k \to +\infty$, up to a subsequence. Doing the same for each ball $B_R(0)$, for every $R > 0$, it yields $U_k \to U$, $V_k \to V$ in $L^\infty_{\text{loc}}(\mathbb{R}^N)$. By applying a stability argument (Proposition 2.5) in each ball, one gets that $U, V$ is a viscosity solution of

$$J_1(x, D^2 U) + V^q = 0, \quad J_2(x, D^2 V) + U^p = 0 \text{ in } \mathbb{R}^N,$$

for some measurable operators $J_1$, $J_2$ satisfying (H1) with $\vartheta$ and $\gamma$ equal to zero. The operators $J_i$ are obtained by using Arzelà-Ascoli theorem, since $\tilde{F}_i$ are $(\alpha, \beta)$–uniformly elliptic, with zero and first order coefficients converging to zero. Further, $U(0) = 1$ implies $U > 0$ in $\mathbb{R}^N$ by SMP, and so $V > 0$ in $\mathbb{R}^N$. Thus, by Step 1,

$$1 \leq \lambda^+_i(J_1, J_2, B_{2\rho}) \leq \frac{C_0}{r^2} \text{ for all } r > 0,$$

where $C_0$ does not depend on $r$. One derives a contradiction by letting $r \to +\infty$. □
5 The first eigenvalue problem

In this section we investigate the main properties of the first eigenvalue problem. We stress once again that we can equivalently consider (LE), (2.7) if positive parameters are taken into account, recall Sect. 2.4. In particular, we recall that the existence of a principal eigenvalue for problem (2.7) implies the existence of a spectral curve for problem (LE). Moreover, \((\lambda, \mu)\) is below the curve \(\Lambda^+_0\) if and only if \(\lambda_0 < \lambda^+_0(F_1, F_2)\), where \(\lambda_0\) is defined through the identity
\[
\mu = \frac{\lambda^{p+1}_{0}}{p}. 
\]
Therefore, in what follows, we will always reduce ourselves to (2.7) by exploiting this scaling.

5.1 Existence and simplicity

We start by recalling a well known result from degree theory, see [5] for the proof.

**Proposition 5.1** Let \((\mathcal{E}, \| \cdot \|)\) be a Banach space. Let \(T : \mathbb{R}^+ \times \mathcal{E} \to \mathcal{E}\) be a completely continuous operator such that \(T(0, u) = 0\) for all \(u \in \mathcal{E}\); then there exists an unbounded, connected component \(C\) of \(\mathbb{R}^+ \times \mathcal{E}\) of solutions of \(u = T(\mu, u)\) and starting from \((0, 0)\).

We first show the following result.

**Proposition 5.2** Let \(F_1, F_2\) satisfy (H1)–(H4), \(pq = 1\), and assume (1.2). Then there exist positive numbers \(\sigma_1^\pm\), and signed functions \(\varphi_1^\pm, \psi_1^\pm\) in \(E_\partial = W^{2,q}(\Omega) \cap C(\overline{\Omega})\) such that
\[
\begin{align*}
F_1[\varphi_1^\pm] + \sigma_1^\pm \tau_1(x)(\varphi_1^\pm)^q &= 0 \quad \text{in } \Omega \\
F_2[\psi_1^\pm] + \sigma_1^\pm \tau_2(x)(\psi_1^\pm)^p &= 0 \quad \text{in } \Omega \\
\pm \varphi_1^\pm, \pm \psi_1^\pm &> 0 \quad \text{in } \Omega \\
\varphi_1^\pm &= \psi_1^\pm \quad \text{on } \partial \Omega,
\end{align*}
\]

**Proof** We prove the \(\sigma_1^+\) case; for \(\sigma_1^-\) is analogous by replacing the operator \(F_i\) by \(G_i\), see (2.5).

Step 1) Assume that \(\tau_i(x) > 0\) a.e. in \(\Omega\) for \(i = 1, 2\).

In this case we adapt Krein–Rutman’s theorem, exploiting ideas from [5, 39]. Let us consider the Banach space \(E = C_0^1(\overline{\Omega})\). Define \(F_i = -F_i^{-1} \circ \tau_i\) in \(E\), i.e.
\[
\mathcal{F}_i u = U \iff -F_i[U] = \tau_i(x)u \text{ in } \Omega, U = 0 \text{ on } \partial \Omega, i = 1, 2,
\]
in the viscosity sense. Thus \(\mathcal{F}_i : E \to E\) is well defined and completely continuous. Indeed, this comes from (H3), (H4) for \(F_i\), and Theorem 3.7.

Consider the closed positive cone \(K := \{ w \in E, w \geq 0 \}\). Note that \(K\) is solid, that is, it has nonempty interior. Then \(\mathcal{F}_i\) is strictly positive with respect to \(K\), in the sense that \(\mathcal{F}_i(K \setminus \{0\}) \subset \text{int}K\). This is due to ABP-MP, SMP, and Hopf for scalar equations (see Theorem 3.4, and Propositions 2.6, 2.7), since the weight \(\tau_i\) is strictly positive in \(\Omega\).

We first choose \(w_0 \in K \setminus \{0\}\). We can choose \(M > 0\) such that \(M \mathcal{F}_2(w_0) \geq w_0\), \(M \mathcal{F}_2(w_0^p) \geq w_0^p\) and \(M \mathcal{F}_1(w_0) \geq w_0\). In fact, this choice is possible because \(K\) is solid and each \(\mathcal{F}_i\) is a strongly positive operator with respect to \(K\). Fix \(\varepsilon > 0\) and define \(T_\varepsilon : \mathbb{R}^+ \times K \times K \to K \times K\) as
\[
T_\varepsilon(\mu, u, v) = (\mu \mathcal{F}_1(v^q) + \varepsilon \mu \mathcal{F}_2(w_0), \mu \mathcal{F}_2(u^p) + \varepsilon \mu \mathcal{F}_1(w_0)).
\]

By Proposition 5.1, there exists \(C_\varepsilon\), an unbounded connected component of solutions of \((u, v) = T_\varepsilon(\mu, u, v)\) which contains \((0, 0, 0)\). We claim that \(C_\varepsilon \subset [0, M] \times K \times K\). Indeed,
let \((\mu, u, v) \in C_\varepsilon\). In particular,

\[
u = \mu F_1(v^\varphi) + \varepsilon \mu F_2(w_0) \quad \text{and} \quad v = \mu F_2(u^\varphi) + \varepsilon \mu F_1(w_0).
\]

Then,

\[
u \geq \mu \varepsilon F_2(w_0) \geq \frac{\mu}{M} \varepsilon w_0,
\]

which implies that \(u^\varphi \geq \frac{\mu}{M} \varepsilon w_0^\varphi\). By applying \(F_2\), and using the comparison principle, see Theorem 3.4, we have

\[
F_2(u^\varphi) \geq \frac{\mu}{M} \varepsilon F_2(w_0^\varphi) \geq \frac{\mu}{M} \varepsilon \frac{1}{M} w_0^\varphi.
\]

Moreover, \(v \geq \mu F_2(u^\varphi) \geq \frac{\mu}{M} \varepsilon F_2(w_0^\varphi)\), whence \(v^\varphi \geq \frac{\mu}{M} \varepsilon \mu^\varphi \varepsilon w_0\) since \(pq = 1\). Now, applying \(F_1\),

\[
F_1(v^\varphi) \geq \frac{\mu}{M} \varepsilon F_1(w_0) \geq \frac{\mu}{M} \varepsilon \frac{1}{M} w_0,
\]

and therefore \(u \geq \mu F_1(v^\varphi) \geq \left(\frac{\mu}{M}\right)^{q+2} \varepsilon w_0\). Upon iteration, one gets

\[
u \geq \left(\frac{\mu}{M}\right)^k \varepsilon w_0 \quad \text{for all} \quad k \geq 2, \quad \text{where} \quad \alpha_k = k(q + 1) + 1.
\]

This gives us \(\mu \leq M\), and the claim is proved.

Therefore, there exists \((\mu_\varepsilon, u_\varepsilon, v_\varepsilon) \in C_\varepsilon\) with \(\|(u_\varepsilon, v_\varepsilon)\|_\infty = 1\). By compactness of \(F_1\) and \(F_2\), Theorem 3.4, and SMP, by taking \(\varepsilon \to 0\) we conclude that there exists \(\sigma_1 > 0\), in addition to \(u_1, v_1\) such that \(\|(u_1, v_1)\|_\infty = 1\),

\[
u_1 = \sigma_1 F_1(v_1^\varphi) \quad \text{and} \quad v_1 = \sigma_1 F_2(u_1^\varphi).
\]

The eigenfunctions belong to the interior of the positive cone \(K\), by the positivity of the operators \(F_i\) for \(i = 1, 2\). Then the result follows in the case of positive weights.

**Step 2)** In the general case we argue by approximation. Since \(\tau_1, \tau_2 > 0\) a.e. in a common set of positive measure – assumption (1.2)—then there exists some \(\delta > 0\) such that \(\{|\tau_1 \geq \delta\} \cap \{|\tau_2 \geq \delta\} > 0\). Say \(\tau_1, \tau_2 \geq \delta\) a.e. in some ball \(B_R \subset \subset \Omega, R \leq 1\). Take \(\varepsilon \in (0, 1)\) and define \(\tau_i^\varepsilon := \tau_i + \varepsilon > 0\) in \(\Omega\). By Step 1 we obtain the existence of \(\sigma_1^\varepsilon > 0\) and \(\varphi_i^\varepsilon, \psi_i^\varepsilon \in C^1(\overline{\Omega})\) such that

\[
F_1[\varphi_i^\varepsilon] + \sigma_1^\varepsilon \tau_i^\varepsilon F_1(x)(\varphi_i^\varepsilon)^q = 0, \quad F_2[\psi_i^\varepsilon] + \sigma_1^\varepsilon \tau_i^\varepsilon F_2(x)(\psi_i^\varepsilon)^p = 0 \quad \text{in} \ \Omega
\]

\[
\varphi_i^\varepsilon, \psi_i^\varepsilon > 0 \quad \text{in} \ \Omega, \quad \varphi_i^\varepsilon, \psi_i^\varepsilon = 0 \quad \text{on} \ \partial \Omega, \quad \max_{\Omega} \varphi_i^\varepsilon = 1, \quad \max_{\Omega} \psi_i^\varepsilon = 1.
\]

Then, by Proposition 4.5, \(0 < \sigma_1^\varepsilon \leq C_0\) for all \(\varepsilon \in (0, 1)\). So \(\sigma_1^\varepsilon \to \sigma_1 \in [0, C_0]\) up to a subsequence. Then, applying \(C^{1,\alpha}\) global regularity (Proposition 2.8), yields

\[
\|
\varphi_i^\varepsilon\|_{C^{1,\alpha}(\overline{\Omega})} \leq C \left\{ \|
\varphi_i^\varepsilon\|_{L^\infty(\Omega)} + \|\tau_i^\varepsilon\|_{L^\infty(\Omega)} \|\varphi_i^\varepsilon\|_{L^q(\Omega)} \right\} \leq C (\|
\tau_i\|_{L^p(\Omega)} + 1) \leq C,
\]

and analogously \(\|
\psi_i^\varepsilon\|_{C^{1,\alpha}(\overline{\Omega})} \leq C\). Hence the compact inclusion \(C^{1,\alpha}(\overline{\Omega}) \subset C^1(\overline{\Omega})\) yields \(\varphi_i^\varepsilon \to \varphi_1\) and \(\psi_i^\varepsilon \to \psi_1\) in \(C^1(\overline{\Omega})\), up to a subsequence, with \(\max_{\Omega} \varphi_1 = 1, \max_{\Omega} \psi_1 = 1, \varphi_1, \psi_1 \geq 0 \in \Omega, \) and \(\varphi_1, \psi_1 = 0 \) on \(\partial \Omega\). Since \(\tau_i^\varepsilon \to \tau_i\) in \(L^p(\Omega)\) as \(\varepsilon \to 0\), by stability of viscosity solutions we derive that \(\varphi_1, \psi_1\) is an \(L^N\)-viscosity solution pair of \(F_1[\varphi_1] + \sigma_1 \tau_1(x)\varphi_1^q = 0, F_2[\psi_1] + \sigma_1 \tau_2(x)\psi_1^p = 0\) in \(\Omega\), which allows us to apply \(C^{1,\alpha}\) regularity again to obtain that \(\varphi_1, \psi_1 \in C^{1,\alpha}(\overline{\Omega})\).

Next, by Theorem 3.4 and SMP, we have that \(\varphi_1 > 0\) in \(\Omega\); similarly \(\psi_1 > 0\). Moreover, we must have \(\sigma_1 > 0\). Indeed, \(\sigma_1 = 0\) produces \(F_1[\varphi_1] = 0\) in \(\Omega\) which in turn would give...
us \( \varphi_1 \leq 0 \) in \( \Omega \) by MP, since we are assuming \( \lambda_1^+ (F_1^+) > 0 \). Using the regularity property in (H4), \( \varphi_1 \) and \( \psi_1 \) turn out to be strong solutions.

**Proof of Theorem 1.3** Existence. We preliminarily notice that \( \lambda_1^+ \) is finite due to Proposition 4.5. Take \( \sigma_1^\pm \) as in Proposition 5.2. Let us prove that \( \sigma_1^\pm = \lambda_1^\pm \). By definition of \( \lambda_1^+ \), we have \( \sigma_1^+ \leq \lambda_1^+ \). Suppose there exists \( \varepsilon > 0 \) such that \( \sigma_1^+ < \lambda_1^+ - \varepsilon \). Then, by definition of \( \lambda_1^+ \), we can take \( \varphi, \psi > 0 \) such that

\[
F_1[\varphi] + (\lambda_1^+ - \varepsilon) \tau_1(x)\psi^q \leq 0, \quad F_2[\psi] + (\lambda_1^+ - \varepsilon) \tau_2(x)\varphi^p \leq 0.
\]

Since \( F_1[\varphi_1^+] + (\lambda_1^+ - \varepsilon) \tau_1(x)(\psi_1^+) \psi_1^q > 0 \) and \( F_2[\psi_1^+] + (\lambda_1^+ - \varepsilon) \tau_2(x)(\psi_1^+) \psi_1^p > 0 \), then by Proposition 4.1 (and up to suitable rescaling as in Sect. 2.4) we have \( \varphi = t\varphi_1^+ \) and \( \psi = t^p \psi_1^+ \) for a suitable \( t > 0 \), which is a contradiction. Similarly, we obtain \( \sigma_1^- = \lambda_1^- \).

**Simplicity.** Let us prove the case of \( \lambda_1^+ \), since for \( \lambda_1^- \) is analogous. Assume that \((u, v)\) is another eigenfunction corresponding to \( \lambda_1^+ \), with \( u \neq 0 \) or \( v \neq 0 \).

In the case that \( u \) or \( v \) attains a positive maximum in \( \Omega \), then we consider the positive eigenfunctions pair \((\varphi_1^+, \psi_1^+)\) of the operators \((F_1, F_2)\). Then we apply Proposition 4.1 to both pairs \((u, v)\) and \((\varphi_1^+, \psi_1^+)\) to obtain that \( u = t\varphi_1^+ \) and \( v = t^p \psi_1^+ \) in \( \Omega \) for a suitable \( t \in \mathbb{R} \).

If, in turn, \( u \leq 0 \) and \( v \leq 0 \) in \( \Omega \), then \( u < 0 \) and \( v < 0 \) in \( \Omega \) by the strong maximum principle for scalar equations. Notice that if one between \( u \) and \( v \) is \( \equiv 0 \), then also the other one has to be null. Then we consider the negative eigenfunctions pair \((\varphi_1^-, \psi_1^-)\) of the operators \((F_1, F_2)\). Therefore we use the fact that \( \lambda_1^- \) is the only positive eigenvalue corresponding to a negative eigenfunction. In other words, we apply Corollary 4.3 to obtain that \( \lambda_1^- = \lambda_1^+ \), as well as \( u \equiv t\varphi_1^- \) and \( v \equiv t^p \psi_1^- \) in \( \Omega \).

**5.2 Maximum principles**

**Proof of Theorem 1.7** We prove the statement regarding MP; the case mP is analogous.

Step 1) Let us first assume that \((\lambda, \mu) \in \mathbb{R}^+ \times \mathbb{R}^+\). Then, via the scaling (2.10), we reduce the problem to check if MP holds for

\[
-F_1[u] = \lambda_0 \tau_1(x)|v|^{q-1}v, \quad -F_2[v] = \lambda_0 \tau_2(x)|u|^{p-1}u \quad \text{in } \Omega, \quad u = v = 0 \text{ on } \partial \Omega.
\]

if and only if \( \lambda_0 < \lambda_1^+ \). We first notice that, if \( \lambda_0 \geq \lambda_1^+ \), then MP is not satisfied. Indeed, when \( \lambda_0 \geq \lambda_1^+ \) we get

\[
F_1[\varphi_1^+] + \lambda_0 \tau_1(x)(\psi_1^+) \psi_1^q = -\lambda_1^+ \tau_1(x)(\psi_1^+) \psi_1^q + \lambda_0 \tau_1(x)(\psi_1^+) \psi_1^q \geq 0,
\]

\[
F_2[\psi_1^+] + \lambda_0 \tau_2(x)(\varphi_1^+) \varphi_1^p \geq 0 \quad \text{in } \Omega,
\]

with \( \varphi_1^+, \psi_1^+ = 0 \) on \( \partial \Omega \), but \( \varphi_1^+ > 0, \psi_1^+ > 0 \) in \( \Omega \).

Now we prove that MP holds if \( 0 < \lambda_0 < \lambda_1^+ \). Let \((u, v)\) be a viscosity solution of

\[
F_1[u] + \lambda_0 \tau_1(x)|v|^{q-1}v \geq 0, \quad F_2[v] + \lambda_0 \tau_2(x)|u|^{p-1}u \geq 0 \quad \text{in } \Omega, \quad u, v \leq 0 \text{ on } \partial \Omega.
\]

Assume by contradiction that one between \( u, v \) has a positive maximum in \( \Omega \). Observe that \( \varphi_1^+, \psi_1^+ \) is a strong solution pair of

\[
F_1[\varphi_1^+] + \lambda_0 \tau_1(x)(\psi_1^+) \psi_1^q \leq 0, \quad F_2[\psi_1^+] + \lambda_0 \tau_2(x)(\varphi_1^+) \varphi_1^p \leq 0 \quad \text{in } \Omega, \quad \varphi_1^+, \psi_1^+ > 0 \quad \text{in } \Omega.
\]

Then Proposition 4.1 yields \( u = t\varphi_1^+, v = t^p \psi_1^+ \) in \( \Omega \), which contradicts \( \lambda_0 \neq \lambda_1^+ \). So \( u, v \leq 0 \) in \( \Omega \). 

\( \square \) Springer
Step 2) Let us now check that, if \((\lambda, \mu) \notin \mathbb{R}^+ \times \mathbb{R}^+,\) then MP does not hold. If \(\lambda = 0\) or \(\mu = 0,\) then we reduce ourselves to the scalar case, and the conclusion follows from [40] in the case of bounded drifts, while for unbounded ones this is a new contribution of this paper, see Sect. 3. The only thing which is left to prove is that if at least one between \(\lambda, \mu\) is negative, then the MP and the mP do not hold. Assume for instance \(\lambda < 0.\) Consider the eigenvalue problem

\[
F_1[\tilde{\psi}] + \lambda_0 \tau_1(x) \tilde{\psi}^q = 0, \quad (F_2)_s[\tilde{\psi}] + \lambda_0 \tau_2(x) \tilde{\psi}^p = 0 \text{ in } \Omega,
\]

where \(\tilde{\psi}, \tilde{\psi} > 0 \text{ in } \Omega, \tilde{\psi} = 0 \text{ on } \partial \Omega, \lambda_0 = \lambda_1^+(F_1, (F_2)_s).\) Let \(\tilde{\lambda} = 1/\delta^q, \tilde{\mu} = s\lambda.\) Hence, choosing a suitable \(s > 0,\) we have \(\tilde{\lambda} > 0\) and \(\tilde{\mu} < 0\) such that \(\tilde{\mu} \leq -\lambda_0.\)

Hence,

\[
\begin{aligned}
F_1[\tilde{\psi}] + \tilde{\lambda} \tau_1(x) |\tilde{\psi}|^{q-1} (-\tilde{\psi}) = F_1[\tilde{\psi}] - \tilde{\lambda} \tau_1(x) |\tilde{\psi}|^q = -\tau_1(x) (\lambda_0 + \tilde{\lambda}) |\tilde{\psi}|^q \leq 0, \\
F_2[-\tilde{\psi}] + \tilde{\mu} \tau_2(x) (\tilde{\psi})^p \leq - (F_2)_s[\tilde{\psi}] + \tilde{\mu} \tau_2(x) (\tilde{\psi})^p = \tau_2(x) (\lambda_0 + \tilde{\mu}) \varphi^p \leq 0.
\end{aligned}
\]

However, \(-\tilde{\psi} < 0 \text{ in } \Omega.\]

\[\square\]

5.3 The Dirichlet problem for \(\lambda < m_1\)

Now we focus on the nonhomogeneous Dirichlet Lane–Emden problem, proving item \((i)\) of Theorem 1.8. We postpone the proof of items \((ii)\) and \((iii)\) to Sect. 7 ahead, as they turn out to follow from an adaptation of the proof of Theorem 1.16. Recall \(m_1, M_1 \text{ from } (2.6).\) We consider functions \(f_i \in L^\infty(\Omega).\) Via the scaling \((2.10),\) we reduce our problem to the study of system

\[
\begin{aligned}
F_1[u] + \lambda \tau_1(x) |u|^{q-1} v = f_1(x) \quad &\text{in } \Omega, \\
F_2[v] + \lambda \tau_2(x) |u|^p |u|^{p-1} u = f_2(x) \quad &\text{in } \Omega, \\
u = v = 0 \quad &\text{on } \partial \Omega.
\end{aligned}
\]

and the condition \((\lambda, \mu) \in C_1^+ \cap C_1^- \text{ for } (LE) \text{ translates to } 0 < \lambda < m_1 \text{ for } (5.2).\)

**Proof of Theorem 1.8 (i)** We sketch the proof in light of [28]. One first obtains the following a priori bounds.

**Claim 1** Let \((u, v)\) be a viscosity solution of \((5.2).\) Then

\[\|u\|_\infty + \|v\|_\infty^p \leq C \{ \|f_1\|_{L^\infty(\Omega)} + \|f_2\|_{L^\infty(\Omega)} \}, \text{ for all } 0 < \lambda < m_1.\]

In order to see this, let us assume by contradiction that there exist solutions such that

\[\|u_k\|_\infty + \|v_k\|_\infty^p > k \{ \|f_1\|_\infty + \|f_2\|_\infty \}.\]

We normalize as in [28, Section 5] to get new functions \(\tilde{u}_k, \tilde{v}_k, \tilde{f}_1^k, \tilde{f}_2^k\) such that

\[\|\tilde{u}_k\|_\infty + \|\tilde{v}_k\|_\infty^p = 1, \quad \|\tilde{f}_1^k\|_\infty, \|\tilde{f}_2^k\|_\infty < 1/k.\]

The limit functions \(\tilde{u}, \tilde{v}\) solve \((LE)\) for \(\lambda = \mu,\) and they are nonzero due to our normalization. Hence, they are an eigenfunction, however \(\lambda < m_1,\) a contradiction. So Claim 1 is true.

Define \(H(t, u, v) = (U, V)\) as the viscosity solution to the problem

\[
\begin{aligned}
F_1[U] + \tau_1(x) t \lambda _\infty |v|^{q-1} v = tf_1(x) \quad &\text{in } \Omega, \\
F_2[V] + \tau_2(x) t \lambda |u|^p |u|^{p-1} u = tf_2(x) \quad &\text{in } \Omega, \\
U = V = 0 \quad &\text{on } \partial \Omega,
\end{aligned}
\]

In the following, we shall denote by \(U_\infty, V_\infty\) the unique solution of

\[
\begin{aligned}
F_1[U_\infty] + \tau_1(x) \lambda_\infty |V_\infty|^{q-1} V_\infty = 0 \quad &\text{in } \Omega, \\
F_2[V_\infty] + \tau_2(x) \lambda |U_\infty|^p |U_\infty|^{p-1} U_\infty = 0 \quad &\text{in } \Omega, \\
U_\infty = V_\infty = 0 \quad &\text{on } \partial \Omega.
\end{aligned}
\]
in the space \( C^1(\overline{\Omega}) \). By (H1)–(H4) for \( F_1, F_2 \), together with regularity-estimates \( C^{1,\alpha} \) theory of viscosity solutions, the map \( H \) is well defined and completely continuous. Observe that \( t\lambda < \min\{\lambda_1^-, \lambda_1^+\} \) for any \( t \in [0, 1] \). By using this and the estimates above, we have that any \((u, v)\) which satisfies \( H(t, u, v) = (u, v) \) is bounded by a constant not depending on \( t \). By degree fixed point theory we get a solution to our problem. \( \square \)

5.4 Local isolation

**Proof of Theorem 1.5** Once again, via the scaling (2.10), we reduce the problem to the study of (LE) with \( \lambda = \mu \), that is (2.7). Recall \( M_1, m_1 \) from (2.6). For \( 0 < \lambda < M_1 \), then the system (LE) with \( \mu = \lambda \) satisfies MP or mP depending on whether \( M_1 = \lambda_1^+ \) or \( \lambda_1^- \). Suppose that \( u, v \in C(\overline{\Omega}) \) is a nontrivial eigenfunction pair associated to the eigenvalue \( \lambda \), and let \( M_1 = \lambda_1^- \). If \( \lambda < \lambda_1^- \) then (LE) with \( \mu = \lambda \) satisfies mP. Therefore \( u, v \geq 0 \) in \( \Omega \), and \( u, v > 0 \) in \( \Omega \) by Hopf. By Proposition 4.1 one has \( \lambda = m_1 = \lambda_1^- \). Analogously we deduce that if \( M_1 = \lambda_1^+ \) then \( \lambda = \lambda_1^+ \). In particular, no eigenfunctions exist on the left of \( M_1 \), except from the one corresponding to \( m_1 \).

Let us show that there is a neighborhood on the right of \( M_1 \) where eigenfunctions do not exist. We assume w.l.g. that \( M_1 = \lambda_1^- \) (the other case is analogous). Let us take a sequence of positive eigenvalues \( \lambda_n = \lambda_1^- + \epsilon_n \) related to normalized eigenfunctions \((u_n, v_n)\) such that \( \epsilon_n \to 0 \). Then by stability of viscosity solutions (Proposition 2.5), \((u_n, v_n) \to (u, v)\) eigenfunction related to \( \lambda_1^- \). Then, by Proposition 4.1 one concludes \( u = t\varphi_1^- \) and \( v = t^p\psi_1^- \) for some \( t > 0 \). Note that by Krein–Rutman theorem \( \varphi_1 , \psi_1 \) belong to the interior of the cone of negative solutions. Then \( u_n, v_n < 0 \) for large \( n \). This implies by Proposition 4.1 that \( \lambda_n = \lambda_1^- \), which is a contradiction. \( \square \)

6 The anti-maximum principle

In this section we go along with the validity of the maximum principle, this time when it fails and takes the form of an anti-maximum principle. We move towards the proof of Theorem 1.11. Our approach relies on some arguments in [4, 24]. Once again, we reduce ourselves to one of the equivalent forms of (LE), and in particular we will always assume that the system is written in the form (2.7), see Sect. 2.4.

We start with an auxiliary nonexistence result. We assume \( F_i \) satisfies (H1)–(H3), \( i = 1, 2 \).

**Lemma 6.1** Let \( \tau_1, \tau_2 \in L^q(\Omega) \) for some \( q > N \), and \( f_i \in L^p(\Omega) \) such that \( f_i \neq 0 \).

(i) If \( \lambda \geq \lambda_1^+ \) and \( f_i \leq 0 \) then there is no nonnegative solution \( u, v \in C(\overline{\Omega}) \) of

\[
\begin{align*}
F_1[u] + \lambda \tau_1(x)|v|^{q-1}v & \leq f_1, & F_2[v] + \lambda \tau_2(x)|u|^{p-1}u & \leq f_2 & \text{in } \Omega, & u, v & \geq 0 \text{ on } \partial\Omega. 
\end{align*}
\]  

If in addition \( \lambda_1^+ \leq \lambda \leq \lambda_1^- \), the problem (6.1) does not possess a solution \( u, v \in C(\overline{\Omega}) \).

(ii) On the other hand, if \( \lambda \geq \lambda_1^- \) and \( f_i \geq 0 \) there is no nonpositive solution \( u, v \in C(\overline{\Omega}) \) of

\[
\begin{align*}
F_1[u] + \lambda \tau_1(x)|v|^{q-1}v & \geq f_1, & F_2[v] + \lambda \tau_2(x)|u|^{p-1}u & \geq f_2 & \text{in } \Omega, & u, v & \leq 0 \text{ on } \partial\Omega. 
\end{align*}
\]  

Moreover, if \( \lambda_1^- \leq \lambda \leq \lambda_1^+ \), the problem (6.2) does not possess a solution \( u, v \in C(\overline{\Omega}) \).
Proof We prove just (i), since the case (ii) is analogous. Assume by contradiction that there exists a solution pair \( u, v \geq 0 \) of (6.1) with \( \lambda \geq \lambda_1^+ \) and \( f_i \leq 0 \). Note that the case \( u, v \equiv 0 \) is not allowed by the hypothesis \( f_i \not\equiv 0 \). Thus, assume that \( u \not\equiv 0 \) or \( v \not\equiv 0 \). Thus by SMP we have \( u, v > 0 \) in \( \Omega \). The definition of \( \lambda_1^+ \) implies \( \lambda \leq \lambda_1^+ \), from which we deduce \( \lambda = \lambda_1^+ \). By Proposition 4.1 we obtain that \( u = t\varphi_1^+ \) and \( v = t\psi_1^+ \) for some \( t > 0 \). However, this leads to \( f_i \equiv 0, i = 1, 2 \), which contradicts the hypotheses.

Now assume in addition that \( \lambda \leq \lambda_1^+ \), and on the contrary that there exists a solution \( u, v \) of (6.1). By what we just proved, either \( u \) or \( v \) must be negative somewhere in \( \Omega \). Applying Proposition 4.1 we get that \( u \equiv t\varphi_1^− \) and \( v \equiv t\psi_1^− \) in \( \Omega \) for some \( t > 0 \). This yields \( \lambda = \lambda_1^− \), and so \( f_i \equiv 0, i = 1, 2 \), again a contradiction. \( \square \)

Proof of Theorem 1.11 We prove only (i); (ii) is similar. In order to get a contradiction, suppose that there exist values \( \lambda_k \) above \( \lambda_1^− \), and \( u_k, v_k \in C(\Omega) \) such that \( \lambda_k \to \lambda_1^− \), and \( u_k, v_k \in C(\overline{\Omega}) \) satisfying

\[
\begin{array}{ll}
F_1[u_k] + \lambda_k \tau_1(x)|v_k|^{q-1}v_k = f_1(x) & \text{in } \Omega \\
F_2[v_k] + \lambda_k \tau_2(x)|u_k|^{p-1}u_k = f_2(x) & \text{in } \Omega \\
u_k = v_k & \text{on } \partial \Omega
\end{array}
\]  

(6.3)

such that at least one between \( u_k \) or \( v_k \) is nonnegative somewhere in \( \Omega \). It then turns out that \( u_k \) or \( v_k \) is nonnegative somewhere for infinite \( k \)'s, say \( u_k \). Thus, take such \( x_k \in \Omega \) where \( u_k \) attains a nonnegative maximum at \( x_k \). In particular, for this subsequence one has

\[ u_k(x_k) \geq 0 \text{ and } Du_k(x_k) = 0. \]  

(6.4)

By taking a further subsequence we may assume \( x_k \to x_0 \) for some \( x_0 \in \overline{\Omega} \). We claim that

\[ \text{either } \sup_k \|u_k\|_{L^\infty(\Omega)} = +\infty \text{ or } \sup_k \|v_k\|_{L^\infty(\Omega)} = +\infty. \]  

(6.5)

Otherwise, if \( \|u_k\|_{L^\infty(\Omega)}, \|v_k\|_{L^\infty(\Omega)} \leq C \) for all \( k \), then by \( C^{1,\alpha} \) estimates and compact inclusion we obtain \( u, v \in C^1(\overline{\Omega}) \) such that \( u_k \to u \) and \( v_k \to v \) uniformly in \( \overline{\Omega} \). By stability of viscosity solutions we may pass to limits in (6.3), then \( u, v \) become solution of the problem

\[ F_1[u] + \lambda_1^− \tau_1(x)|v|^{q-1}v = f_1(x), \quad F_2[v] + \lambda_1^− \tau_2(x)|u|^{p-1}u = f_2(x) \text{ in } \Omega, \]

\[ u = v = 0 \text{ on } \partial \Omega. \]

But this contradicts Lemma 6.1 (i), whence (6.5) is proved.

Let us assume w.l.g. that \( \|u_k\|_\infty \to \infty \) (up to a subsequence), and let us define \( \theta_k = \|u_k\|_\infty \). We also consider the rescaling \( u_k = \theta_k U_k \) and \( v_k = \theta_k V_k \). Since the operators \( F_1, F_2 \) are positively 1-homogeneous,

\[ -F_1[U_k] = -\frac{F_1[u_k]}{\theta_k} = \frac{1}{\theta_k} (\lambda_k \tau_1(x)|v_k|^{q-1}v_k - f_1(x)) = \lambda_k \tau_1(x)|V_k|^{q-1}V_k - \frac{f_1(x)}{\theta_k}, \]

and

\[ -F_2[V_k] = -\frac{F_2[v_k]}{\theta_k} = \frac{1}{\theta_k} (\lambda_k \tau_2(x)|u_k|^{p-1}u_k - f_2(x)) = \lambda_k \tau_2(x)|U_k|^{p-1}U_k - \frac{f_2(x)}{\theta_k}, \]

with \( U_k = V_k = 0 \) on \( \partial \Omega \). Again by \( C^{1,\alpha} \) estimates, our construction, and taking a subsequence if necessary, we may assume \( U_k \to U, V_k \to V \) in \( C^1(\overline{\Omega}) \) for some \( U, V \in C^1(\overline{\Omega}) \). Notice that \( \|U_k\|_1 = 1 \), and hence the RHS in the last equality is uniformly bounded, therefore

\[ \square \]
by Theorem 3.4 $V_k$ is also bounded. Passing to limits via stability of viscosity solutions, we deduce that $U$, $V$ is a solution of

$$F_1[U] + \lambda_1^- \tau_1(x)|V|^{q-1}V = 0, \quad F_2[V] + \lambda_1^- \tau_2(x)|U|^{p-1}U = 0 \quad \text{in} \ \Omega,$$

$$U = V = 0 \text{ on } \partial \Omega.$$

If instead $\|v_k\|_\infty \to \infty$ we argue similarly, by defining $\theta_k = \|v_k\|_\infty^\theta$.

Anyway, our construction produces $\|U\|_\infty = 1$ or $\|V\|_\infty = 1$. Notice that if either $V \equiv 0$ or $U \equiv 0$ in $\Omega$, then $U \equiv V \equiv 0$ in $\Omega$ which produces a contradiction. W.l.g. suppose $\|U\|_\infty = 1$ and fix $x_1 \in \Omega$ such that $U(x_1) \neq 0$. By Proposition 4.1 we conclude that $U \equiv t \varphi$ and $V \equiv t^p \psi$ for some $t > 0$, where $\varphi = \varphi_1^+$, $\psi = \psi_1^+$ if $U(x_1) > 0$ (since $\lambda_1^- \geq \lambda_1^+$), while $\varphi = \varphi_1^-$, $\psi = \psi_1^-$ if $U(x_1) < 0$. Let us finish the proof by showing that both cases are not admissible.

First, if $U(x_1) < 0$, then $U, V < 0$ in $\Omega$. Using (6.4) we deduce that $U(x_0) = 0$, and $x_0 \in \partial \Omega$. However, $DU(x_0) \equiv 0$, in violation of Hopf lemma, and so $U(x_1) < 0$ fails to be true.

Finally, if $U(x_1) > 0$, then $U, V > 0$ in $\Omega$. Note that $U_k, V_k > 0$ in any given compact set $K \subset \Omega$, for $k$ suitably large. We claim that for $k$ sufficiently large we have $U_k, V_k \geq 0$ in $\Omega$. Indeed, we consider $K \subset \Omega$ such that $|\Omega \setminus K| < \varepsilon_0$, where $\varepsilon_0$ is the constant of Theorem 1.15. Thus $U_k, V_k \geq 0$ in $\partial(\Omega \setminus K)$, and so in $\Omega \setminus K$ by Theorem 1.15 since $\lambda_1^-(F_1^s(\vartheta)), \lambda_1^-(F_2^s(\vartheta)) > 0$ and $f_1, f_2 \leq 0$. Hence we derive a contradiction with Lemma 6.1, from where $U(x_1) > 0$ is also impossible. \hfill \Box

### 7 The signed Dirichlet problem

In this section we draw some attention to the Dirichlet problem (1.1), or equivalently to (2.7), see Sect. 2.4, in the case the functions $f_1$, $f_2$ have the “good” sign.

When $pq = 1$ we have seen that the Dirichlet problem (5.2) is solvable for any $\lambda < m_1$, with $m_1$ as in (2.6), independently of sign on $f_1$, $f_2$. Now we turn to the case $\lambda \in (m_1, M_1)$.

Moreover, a variation of such argument applies to show that the Dirichlet problem is uniquely solvable among positive viscosity solutions in the sublinear regime $pq < 1$. We start with the latter, by proving Theorem 1.16.

**Proof of Theorem 1.16** We borrow some ideas from [31, Section 5] and [30, Theorem 4.1], based on a Krasnoselskii [27] type argument.

**Step 1) Existence:** Let us consider the eigenvalue problem obtained in Theorem 1.3 for the operators $(F_1)_*$ and $(F_2)_*$, i.e. we take the strong solution pair $\varphi, \psi \in W^{2,p}_\text{loc}(\Omega)$ of

$$\begin{cases}
(F_1)_*[\varphi] + \lambda_1^+ \tau_1(x)\varphi^{1/p} = 0 & \text{in } \Omega \\
(F_2)_*[\psi] + \lambda_1^- \tau_2(x)\psi^{p} = 0 & \text{in } \Omega \\
\varphi, \psi > 0 & \text{in } \Omega \\
\varphi = \psi = 0 & \text{on } \partial \Omega.
\end{cases}
$$

(7.1)

Notice that the operators $(F_i)_*$ are homogeneous if $F_i$ are homogenous, namely satisfy (H2). Moreover, $\lambda_1^+ ((F_1)_*) = \lambda_1^- ((F_2)_*) = \lambda_1^+ ((F_1)_*) = \lambda_1^+ ((F_2)_*) > 0$, hence also (H3) is true. Finally, $(F_i)_*$ satisfy (H4) as they are concave operators with $(F_i)_*(\cdot, 0, 0, 0) = 0$, see Lemma 3.5.
For the sake of convenience we now look at the Dirichlet problem
\begin{align*}
F_1[u] + \tau_1(x)|v|^{q-1}v &= f_1(x) \quad \text{in } \Omega, \\
F_2[v] + \tau_2(x)|u|^{p-1}u &= f_2(x) \quad \text{in } \Omega, \\
u &= v = 0 \quad \text{on } \partial \Omega.
\end{align*}
(7.2)

Observe that (1.1) and (7.2) are equivalent up to scaling, since \(pq < 1\), see Sect. 2.4.

We first construct a subsolution pair \((u_0, v_0) = (\varepsilon \varphi, \varepsilon^k \psi)\) to (7.2), say for \(f_i \in L^0(\Omega)\) with \(f_i \leq 0\) a.e. in \(\Omega\). We use (7.1) and pick up some positive constants \(\varepsilon, k\) to be chosen. For this, we write in the a.e. sense,
\[-F_1[u_0] \leq -(F_1)_*[\varepsilon \varphi] = \lambda_1^+ \varepsilon \tau_1 \psi^{1/p} = \lambda_1^+ \tau_1(\varepsilon^p \psi)^{1/p} \leq \tau_1 \varepsilon^q v_0 - f_1(x),\]
and
\[-F_2[v_0] \leq -(F_2)_*[\varepsilon^k \psi] = \lambda_1^+ \varepsilon^k \tau_2 \varphi^p = \lambda_1^+ \tau_2(\varepsilon^k \varphi)^p \leq \tau_2 \varepsilon^p u_0 - f_2(x).\]

(7.3)

(7.4)
The choice of \(\varepsilon\) is made in order to have \(\lambda_1^+ \varepsilon^{1-kq} \leq \|\psi\|_{(p-1)/p}^{(pq-1)/p}\) in (7.3), for some \(1 - kq > 0\). In addition, for (7.4) we require \(\lambda_1^+ \varepsilon^{k-p} \leq 1\), with \(k > p\). Then, by diminishing \(\varepsilon\) if necessary, it is enough to choose \(k \in (p, 1/q)\).

Next, for each \(n \geq 0\) we define recursively \((u_{n+1}, v_{n+1})\) as the unique strong solution of
\[-F_1[u_{n+1}] = \tau_1(x)v_n^q - f_1(x), \quad -F_2[v_{n+1}] = \tau_2(x)u_n^q - f_2(x) \quad \text{in } \Omega, \quad u_{n+1} = v_{n+1} = 0 \quad \text{on } \partial \Omega.
\]
Note that by \((H3)\) and ABP-mP for scalar equations (Theorem 3.4, since \(\lambda_1^-(F_i^*) \geq \lambda_1^+(F_i^*) > 0\)) one has \(u_{n+1}, v_{n+1} \geq 0\) in \(\Omega\), for all \(n \geq 0\).

Now we infer that the sequences \((u_n)\) and \((v_n)\) are monotone nondecreasing. This is accomplished via a monotone iterations technique. Indeed,
\[
F_1^*[u_n - u_{n+1}] \geq -F_1[u_{n+1}] + F_1[u_n] = \tau_1(x)(v_n^q - v_{n+1}^q),
\]
\[
F_2^*[v_n - v_{n+1}] \geq \tau_2(x)(u_n^q - u_{n+1}^q),
\]
for all \(n \geq 1\). For \(n = 0\) one has
\[
F_1^*[u_0 - u_1] \geq -F_1[u_1] + F_1[u_0] \geq \tau_1(x)(v_0^q - v_0^q) = 0,
\]
\[
F_2^*[v_0 - v_1] \geq \tau_2(x)(u_0^q - u_0^q) = 0.
\]
This implies \(u_1 \geq u_0\) and \(v_1 \geq v_0\) in \(\Omega\), by \((H3)\) and ABP-MP for scalar equations (Theorem 3.4), as desired.

Next we claim that \((u_n, v_n)\) is bounded in \(L^\infty \times L^\infty\). To see this we use the blow up method. Suppose \(\alpha_n = \|u_n\|_\infty \to \infty\) in order to get a contradiction. We know that \((\alpha_n)\) is a nondecreasing sequence. Hence we define the rescaled pair
\[
u_n = \alpha_n U_n \quad \text{and} \quad v_n = \alpha_n^Q V_n, \quad \text{where } Q = \frac{p + 1}{q + 1}.
\]

Properties of \(F_1^*, F_2^*\) and \((H1)\) for \(F_1, F_2\) give us
\[
-F_1^*[U_{n+1}] \leq -F_1[u_{n+1}] = \frac{\tau_1(x)v_n^q}{\alpha_n} - \frac{f_1(x)}{\alpha_n} \leq \tau_1(x)\alpha_n^{\frac{p-1}{q+1}} V_n^q - f_1(x),
\]
and analogously,

\[-F_2^*[V_{n+1}] \leq -\frac{F_2[v_{n+1}]}{\alpha_{n+1}^Q} = \frac{\tau_2(x)u_n^P}{\alpha_{n+1}^Q} - \frac{f_2(x)}{\alpha_{n+1}^Q} \leq \tau_2(x)\alpha_n^{p+1} U_n^P - f_2^e(x),\]

where \( f_1^e(x) = f_1(x)\alpha_n^{-1} \) and \( f_2^e(x) = f_2(x)\alpha_n^{-Q} \) for all \( n \geq 1 \). Since the last RHS converges to zero in \( L^q \), then by (H3) and ABP-MP we obtain that \( V_{n+1} \to 0 \). So, again ABP-MP for the first equation yields \( u_{n+1} \to 0 \), which derives a contradiction.

Therefore, since the sequences \( u_n^P \) and \( v_n^P \) are uniformly bounded from above and from below, by \( C^2 \) regularity estimates \( u_n, v_n \in C^2(\Omega) \) with \( \|u_n\|_{C^2(\Omega)}, \|v_n\|_{C^2(\Omega)} \leq C \). Thus compact inclusion and a standard stability argument of viscosity solutions lead to solution pair \( u, v \) of (LE). Here \( u, v > 0 \) in \( \Omega \) by the uniformly bound from below via \( u_0, v_0 \).

Step 2) Uniqueness:

Let \((u_1, v_1)\) and \((u_2, v_2)\) be two positive pairs of viscosity solutions to (LE). Since \( f_i \leq 0 \) we can use Hopf lemma to conclude \( \partial_i u_1 < 0 \) and \( \partial_i v_1 < 0 \) on \( \partial \Omega, i = 1, 2 \), where \( v \) is the exterior unit normal. Thus we may define (the nonempty set)

\[ S = \{ s > 0 : u_1 > s^{p+1} u_2, v_1 > s^{q+1} v_2 \text{ in } \Omega \}, \quad s_* = \sup S. \]

Here \( s_* < +\infty \) since \( u_2 \) and \( v_2 \) are positive. So, up to exchanging the roles of \((u_1, v_1)\) and \((u_2, v_2)\) if necessary, say that \( s_* \leq 1 \). Let us look at the nonnegative functions \( w, z \) given by

\[ w := u_1 - s_*^{p+1} u_2 \quad \text{and} \quad z := v_1 - s_*^{q+1} v_2 \quad \text{in } \Omega, \]

which satisfy in \( \Omega \), in the viscosity sense,

\[-(F_1)s[w] \geq -(F_1)[u_1] + s_*^{p+1} F_1[u_2] = \tau_1(x) v_1^q - f_1(x) - \tau_1(x) s_*^{p+1} v_2^q + s_*^{p+1} f_1(x) \geq \tau_1(x) v_1^q \left( 1 - s_*^{p+q} \right) - f_1(x) \left( 1 - s_*^{p+q} \right) \geq 0, \]

and similarly \(- (F_2)s[z] \geq 0\). Whence SMP for scalar equations and the strongly coupling of the Lane–Emden type system imply either \( w, z > 0 \) or \( w, z \equiv 0 \) in \( \Omega \).

Suppose on the contrary that \( w, z > 0 \) in \( \Omega \). Now we may repeat the preceding argument with the pair \((u_2, v_2)\) replaced by \((w, z)\) in order to conclude the existence of some small \( \varepsilon > 0 \) such that \( u_1 > (s_* + \varepsilon) u_2 \) in \( \Omega \). But this contradicts the definition of \( s_* \) as the supremum of \( S \).

Therefore \( w, z \equiv 0 \) in \( \Omega \). To finish we infer that \( s_* = 1 \); otherwise the strict inequalities above and SMP would be in force to produce the positivity of \( w \) and \( z \). So one concludes \( u_1 \equiv u_2 \) and \( v_1 \equiv v_2 \) in \( \Omega \), as desired. \( \square \)

Next we return to the regime \( pq = 1 \).

Proof of Theorem 1.8 (ii), (iii) We prove (ii); the case (iii) is similar. Since we have \( pq = 1 \), and \( f_i \equiv 0 \) is an eigenvalue problem which we have already studied, is enough to consider either \( f_1 \not\equiv 0 \) or \( f_2 \not\equiv 0 \). Therefore, we are going to obtain positive solutions, by SMP and strong coupling of the system. In particular, the uniqueness can be carried out as in Step 2 of the proof of Theorem 1.16. Let us show the existence assertion in (ii).

Step 1) Continuous and compactly supported \( f_i \), with uniformly positive weights.

Say \( \tau_i \geq a_i > 0 \) a.e. in \( \Omega \), and \( f_i \geq -b_i \) in \( K_i = \text{supp}(f_i) \subset \subset \Omega \), for some \( b_i > 0 \), \( i = 1, 2 \). In this case we choose a large constant \( A > 0 \) such that

\[ b_i \leq A a_i (\lambda_i^+ - \lambda)(\psi_i^+)^q \quad \text{a.e. in } K_1, b_2 \leq A^p a_2 (\lambda_2^+ - \lambda)(\varphi_i^+)^p \quad \text{a.e. in } K_2. \]
Then the pairs $u^* = A\phi_1^+, v^* = A^{1/q} \psi_1^+$ and $u_\ast = 0, v_\ast = 0$ satisfy

$$F_1[u_*] + \lambda \tau_1(x)|u_*|^{q-1}u_* \geq f_1(x) \geq F_1[u^*] + \lambda \tau_1(x)|v^*|^{q-1}v^* \quad \text{a.e. in } \Omega,$$

$$F_2[v_*] + \lambda \tau_2(x)|u_*|^{p-1}u_* \geq f_2(x) \geq F_2[v^*] + \lambda \tau_2(x)|u^*|^{p-1}u^* \quad \text{a.e. in } \Omega,$$

with $u^* = u_\ast = 0$ and $v^* = v_\ast = 0$ on $\partial \Omega$. Thus we apply the same monotone iterations technique as in Step 1 in the proof of Theorem 1.16, from the supersolution case instead of the subsolution one. This time is even a bit simpler since the supersolution already gives us a uniform bound from above on the uniform norm of the iterated solutions produced by the method.

Step 2) General nonnegative $f_i \in L^0(\Omega)$, but still uniformly positive weights.

We take sequences $(f_1^k), (f_2^k) \in L^0(\Omega)$ of continuous nonpositive functions with compact support in $\Omega$ such that $f_1^k \to f_1, f_2^k \to f_2$ in $L^0(\Omega)$. By Step 1, let $u_k, v_k \geq 0$ solving

$$F_1[u_k] + \lambda \tau_1(x) v_k^q = f_1^k(x), \quad F_2[v_k] + \lambda \tau_2(x) u_k^p = f_2^k(x) \quad \text{in } \Omega,$$

with $u_k = v_k = 0$ on $\partial \Omega$. We infer that

$$\|u_k\|_{L^\infty(\Omega)}, \|v_k\|_{L^\infty(\Omega)} \leq C \quad \text{for all } k. \quad (7.5)$$

Otherwise, assume for instance that $\theta_k = \|u_k\|_{L^\infty(\Omega)} \to \infty$ as $k \to \infty$, and set $u_k = \theta_k U_k$ and $v_k = \theta_k^p V_k$. Notice that the pair $(U_k, V_k)$ satisfies the equation

$$F_1[U_k] + \lambda \tau_1(x) V_k^q = f_1^k(x), \quad F_2[V_k] + \lambda \tau_2(x) U_k^p = f_2^k(x) \quad \text{in } \Omega,$$

and $\|U_k\|_\infty = 1$ for all $k$. Hence $\|V_k\|_\infty \leq C$ by Theorem 3.4 for scalar equations. Thus, by $C^{1,\alpha}$ estimates one gets that $\|U_k\|_{C^{1,\alpha}(\Omega)}, \|V_k\|_{C^{1,\alpha}(\Omega)} \leq C$. Extracting a subsequence if necessary we may assume $U_k \to U$ and $V_k \to V$ in $C^1(\overline{\Omega})$. Passing to limits, through stability of viscosity solutions we end up with $U, V \geq 0$ in $\Omega$ satisfying

$$F_1[U] + \lambda \tau_1(x) V^q = 0, \quad F_2[V] + \lambda \tau_2(x) U^p = 0 \quad \text{in } \Omega,$$

with $U, V = 0$ on $\partial \Omega$ and $\|U\|_\infty = 1$. Since $F_1[U] \leq 0$, by SMP for scalar equations we have $U > 0$ in $\Omega$. Whence $F_2[V] = -\lambda \tau_2(x) U^p \leq 0$, from which also $V > 0$ in $\Omega$. By Corollary 4.3 one derives $U = t \phi_1^+$ and $V = t^q \psi^q$ in $\Omega$, which contradicts the fact that $U, V$ satisfy

$$F_1[U] + \lambda_+ \tau_1(x) V^q \geq 0, \quad F_2[V] + \lambda_+ \tau_2(x) U^p \geq 0 \quad \text{in } \Omega,$$

since $\lambda < \lambda_+$. If instead $\|v_k\|_\infty \to \infty$ as $k \to \infty$ the argument is analogous, by taking $\theta_k = \|v_k\|_\infty^q$. Therefore one proves (7.5). Thus, again by $C^{1,\alpha}$ estimates, compact inclusion, and stability of viscosity solutions one finds a nonnegative limit solution pair $u, v \in C^{1,\alpha}(\overline{\Omega})$ of (1.1).

Step 3) General weights $\tau_1, \tau_2 \in L^q(\Omega)$ and general nonpositive $f \in L^0(\Omega)$.

This case is very similar to the preceding step, by arguing via uniform bounds and stability arguments. It is enough to pick up sequences of uniformly positive weights $\tau_1^k, \tau_2^k$ such that $\tau_1^k \to \tau_1$ and $\tau_2^k \to \tau_2$ in $L^q(\Omega)$. By Step 2, we then take $u_k, v_k \geq 0$ solving

$$F_1[u_k] + \lambda \tau_1^k(x) v_k^q = f_1(x), \quad F_2[v_k] + \lambda \tau_2^k(x) u_k^p = f_2(x) \quad \text{in } \Omega,$$

with $u_k = v_k = 0$ on $\partial \Omega$. Again one produces (7.5) by arguing exactly as in Step 2. Thus, $C^{1,\alpha}$ estimates, compact inclusion, and stability of viscosity solutions conclude the proof. □
8 An application: Isaac’s operators

In this section we prove that Isaac’s operators in the form (1.11) are examples for which all our results are new even in the scalar case. In order to do so, it is enough to prove that they satisfy (H4). We recall that the $W^{2,q}$ regularity results in [35] were extended in [48, Theorem 5.2] to the context of unbounded coefficients and superlinear gradient growth, but only for convex or concave operators. It is known that the same proof there works if the corresponding pure second order operator $F(x, 0, 0, X)$ enjoys $C^{1,1}$ regularity estimates, see [49, Remark 4.4] for instance.

The pure Isaac’s operator associated to (1.11) is again in the form (1.11) with $\gamma_{s,t}, \vartheta_{s,t} \equiv 0$ for all $s, t$. It is worth mentioning that $C^{1,1}$ regularity estimates do not hold in general for these operators, see [33]. On the other hand, in [29] the authors weakened the $C^{1,1}$ hypothesis to a $W^{2,p}$ one. More recently in [36], regularity is proved for Isaac’s operators with bounded drifts. Here we extend the preceding results to unbounded weights. This is of independent interest in view of applications [21, 22] to more general models driven by unbounded data.

As in [36], we assume that there exists $\bar{A}_t$ satisfying:

$$|A_s,t(x) - \bar{A}_t(x)| \leq \epsilon_1 \quad \text{uniformly in } x, s, t$$  \hspace{1cm} (8.1)

where $\epsilon_1$ is the number of condition $A_2$ in [36], with $\bar{A}_t \in C(\bar{\Omega})$ uniformly in $t \in \mathbb{N}$. In this case the corresponding homogeneous Belmann operators generated by $\bar{A}_t$ have $W^{2,q}$ regularity estimates for $q > q > n$, see [16].

**Proposition 8.1** ($W^{2,q}$ regularity estimates for Isaac’s operators) Let $\Omega \subset \mathbb{R}^N$ be a bounded $C^{1,1}$ domain, $f \in L^q(\Omega), q > N$, and assume (8.1). Then any viscosity solution $u \in C(\Omega)$ of

$$I[u] = f(x) \in \Omega, \quad \text{where } I[u] = \inf_{s \in \mathbb{R}} \sup_{t \in \mathbb{R}} L_{s,t} or I[u] = \sup_{s \in \mathbb{R}} \inf_{t \in \mathbb{R}} L_{s,t}$$

where $L_{s,t}$ satisfies (1.12), belongs to $W^{2,q}_{\text{loc}}(\Omega)$ and satisfies the estimate

$$\|u\|_{W^{2,q}(\Omega)} \leq C \{ \|u\|_{L^\infty(\Omega)} + \|f\|_{L^p(\Omega)} \}, \quad \text{for all } \Omega' \subset \subset \Omega,$$

where $C$ depends only on $N, q, \alpha, \beta, \|\gamma\|_{L^q(\Omega)}, \|\vartheta\|_{L^q(\Omega)}, \Omega'$, dist($\Omega', \partial \Omega$), and diam($\Omega$).

If in addition $u = \psi$ on $\partial \Omega$, for some $\psi \in W^{2,q}(\Omega)$, then $u \in W^{2,q}(\Omega)$ and

$$\|u\|_{W^{2,q}(\Omega)} \leq C \{ \|u\|_{L^\infty(\Omega)} + \|f\|_{L^p(\Omega)} + \|\psi\|_{W^{2,q}(\Omega)} \},$$

where $C$ depends only on $N, q, \alpha, \beta, \|\gamma\|_{L^q(\Omega)}, \|\vartheta\|_{L^q(\Omega)}, \partial \Omega$, and diam($\Omega$).

**Proof** To fix the ideas we consider the first expression for $I$; the second case is identical. Let $\gamma_k, \vartheta_k \in L^\infty(\Omega), f_k \in C(\Omega)$, be such that $f_k \to f$, $\gamma_k \to \gamma$ and $\vartheta_k \to \vartheta$ in $L^q(\Omega)$, with $\gamma_k \leq \gamma, \vartheta_k \leq \vartheta$, and $|f_k| \leq |f|$. Up to a subsequence, one can choose $\ell_k = \ell - \epsilon_k > 0$ for some $\epsilon_k \in (0, \ell), \epsilon_k \to 0$, such that

$$\ell_k < \lambda^+_k(L^+_k(\vartheta_k, \Omega), \text{ where } L^+_k[w] = M^+(D^2 w) + \gamma_k(x) |D w|$$

and $\ell$ as in (1.12). Note that $u \in C^{1,\alpha_0}_{\text{loc}}(\Omega)$ by Proposition 2.8. Let $u_k \in C^{1,\alpha_1}(\bar{B}_0)$ be a viscosity solution of

$$I_k[u_k] = f_k(x) \text{ in } B_0, \quad u_k = u \text{ on } \partial B_0,$$ \hspace{1cm} (8.2)

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given by Proposition 3.8, where $B_\rho$ is centered at $x_0 \in \Omega$, and
\[
I_k[w] := \inf_{s \in \mathbb{R}} \sup_{t \in \mathbb{R}} L_{s,t}[w], \quad \text{where } L_{s,t}[w] = \text{tr}(A_{s,t}(x)D^2 w) + \gamma_{s,t}(x)|Dw| + \ell_k \vartheta_{s,t}(x)w,
\]
with $A_{s,t}$ as in (1.12), but now $\left| \gamma_{s,t} \right| \leq \lambda_k$ and $|\vartheta_{s,t}| \leq \vartheta_k$ for all $s, t \in \mathbb{R}$.

Note that [47, Corollary 1.6] implies that $u_k$ is a viscosity solution of
\[
\inf_{s \in \mathbb{R}} \sup_{t \in \mathbb{R}} \text{tr}(A_{s,t}(x)D^2 u_k) = g_k(x), \quad \text{where } |g_k(x)| \leq |f| + \gamma(x)|Du_k| + \vartheta(x)|u_k| \in L^0(B_\rho).
\]

By [36, Theorem 1.1] one has $u_k \in W^2,0_{\text{loc}}(B_\rho)$, see also [1]. Now by the second part of Proposition 3.8, we know that $u_k$ is the unique viscosity solution of (8.2).

By the generalized Nagumo’s lemma in [35, Lemma 4.4] one gets
\[
\|u_k\|_{W^2,0(B_\rho)} \leq C \{\|u_k\|_{L^\infty(B_\rho)} + \|f\|_{L^0(B_\rho)}\}, \quad \text{for all } r < \rho, \quad (8.3)
\]
where $C_k$ remains bounded, since $\gamma_k$ and $\vartheta_k$ are bounded in $L^0(B_\rho)$. Moreover, since
\[
\ell_k < \lambda^+_1(L^+_k(\vartheta_k), \Omega) \leq \lambda^+_1(L^+_k(\vartheta_k), B_\rho) \leq \lambda^+_1(L^+_k(\vartheta_k), B_\rho),
\]
then one may apply ABP-MP and ABP-mP in Theorem 3.4 to obtain $\|u_k\|_{L^\infty(B_\rho)} \leq \|u\|_{L^\infty(B_\rho)} + C \|f\|_{L^0(B_\rho)}$, again the constant does not depend on $k$. This and (8.3) yield $\|u_k\|_{W^2,0(B_\rho)} \leq C$. Hence there exists $v \in C^1(\overline{B}_r)$ such that $u_k \to v$ in $C^1(\overline{B}_r)$, for all $r < \rho$.

Note that $\|u_k\|_{C^1(\overline{B}_\rho)} \leq C$ by global $C^{1,\alpha_1}$ estimates in Proposition 2.8. Thus, Proposition 2.5 implies that $v$ is a viscosity solution of
\[
I[v] = f(x) \text{ in } B_\rho, \quad v = u \text{ on } \partial B_\rho, \quad (8.4)
\]
Now, since $W^2,0(B_r)$ is reflexive, there exists $\tilde{v} \in W^2,0(B_r)$ such that $u_k$ converges weakly to $\tilde{v}$. By uniqueness of the limit, $\tilde{v} = v$ a.e. in $B_r$, and so $v \in W^2,0(B_r)$, for all $r < \rho$.

Now, since $u$ is already a viscosity solution of (8.4), by using again the uniqueness assertion in Proposition 3.8 one gets that $u \in W^2,0_{\text{loc}}(B_\rho)$. Since the ball $B_\rho$ is arbitrary, and in each ball the solution is unique, a covering argument produces $u \in W^2,0_{\text{loc}}(\Omega)$.

In the case of global regularity the argument is simpler, by taking $\Omega$ instead of $B_\rho$ in (8.2). \hfill \Box

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