Vacuum Polarization at Finite Temperature around a Magnetic Flux Cosmic String

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Abstract

We consider a general situation where a charged massive scalar field $\phi(x)$ at finite temperature interacts with a magnetic flux cosmic string. We determine a general expression for the Euclidean thermal

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Green’s function of the massive scalar field and a handy expression for a massless scalar field. With this result, we evaluate the thermal average $< \phi^{(2)}(x) >$ and the thermal average of the energy-momentum tensor of a nonconformal massless scalar field.

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1 Introduction

In the General Relativity framework, a straight, infinite, static string, lying on the $z$-axis, is described by a static metric with cylindrical symmetry

$$ds^2 = -dt^2 + dz^2 + d\rho^2 + B^2 \rho^2 d\varphi^2$$

(1)

where $\rho \geq 0$ and $0 \leq \varphi < 2\pi$ and the constant $B$ is related to the linear mass density $\mu$ as $B = 1 - 4\mu$ \footnote{We use here the system of units $G = \hbar = c = 1$.}. Spacetime (1) is locally but not globally flat \footnote{Spacetime (1) is locally but not globally flat}. The presence of the string leads to an azimuthal deficit angle $8\pi \mu$ and, as a result, this spacetime has a conical singularity.

One of the most remarkable features of spacetime (1) is the fact that fields are sensitive to its global structure and interesting effects can occur \footnote{Example:}. For
instance, in the context of Quantum Field Theory in curved spacetimes, the vacuum state of a quantum field is polarized due solely to the presence of the string in this geometry. This problem has been studied in many papers for various massless fields, including the case where the cosmic string carries an internal magnetic flux (for a review, see Dowker [3]).

Recently, Guimarães and Linet [4] have considered a more general situation where a charged massive scalar field interacts with a cosmic string carrying an internal magnetic flux $\Phi$. They have found the Euclidean Green’s function $G_E^{(\Phi)}(x, x_0, m)$ of the massive scalar field in a local form in which $G_E^{(\Phi)}(x, x_0, m)$ is the sum of the usual Euclidean Green’s function and a regular term $G_E^{(\Phi)}(x, x_0, m)$, for $x$ close enough to $x_0$. This form is very convenient because the renormalization is straightforward and consists of removing the usual Euclidean Green’s function from the expression of $G_E^{(\Phi)}(x, x_0, m)$. Thus, the calculus of the vacuum expectation values of the energy-momentum tensor consists in taking the coincidence limit $x = x_0$ of the regular term and its derivatives.

The phenomenon of vacuum polarization of a quantum field by a cosmic string carrying an internal magnetic flux can be understood as a realization in Cosmology of the current Aharonov-Bohm effect [5]. Indeed, a quantum
field placed in the exterior region of the string acquires an additional phase shift proportional to the magnetic flux even though there is no magnetic field outside the string [6]. Our purpose in this paper is to analyse the vacuum polarization of a nonconformal, charged massive scalar field at finite temperature by a magnetic flux cosmic string. Recently, Frolov et al. [7] considered the vacuum polarization of a nonconformal, massless scalar field at finite temperature on a cone. Our work differs from theirs in two important aspects. First, we consider a cosmic string which carries an internal magnetic flux. Second, we restrict our calculations to values of constant $B$ such that $B > 1/2$, while their calculations is for arbitrary values of $B$. However, it must be remarked that setting $B > 1/2$ represents no physical restriction because strings of cosmological interest are of order $\mu \sim 10^{-6}$.

To study the vacuum polarization phenomenon, we need first to compute the thermal Green’s function of the scalar field. Since it is more convenient to work in the Euclidean approach to quantum theory, we will, in practice, compute the Euclidean thermal Green’s function of the scalar field. This represents no complication because, spacetime (1) being globally static, the Euclidean metric is obtained straightforwardly by a Wick rotation $(t = -i\tau)$ in the coordinate $t$ of metric (1). The method used here is the same as in Linet’s
paper [8] which works in the framework of the Schwinger-DeWitt formalism [9]. Namely, the Euclidean thermal Green’s function $G^{(\Phi)}_{ET}(x, x_0, m)$ is calculated from its corresponding Euclidean thermal heat kernel $K^{(\Phi)}_{ET}(x, x_0, s)$ by an integral on the variable $s$. Moreover, since the metric we are dealing with is ultrastatic (static and $g_{00} = 1$), the Euclidean thermal heat kernel $K^{(\Phi)}_{ET}(x, x_0, s)$ can be derived from the Euclidean zero-temperature heat kernel $K^{(\Phi)}_E(x, x_0, s)$ [10]. Thus, our task is, first of all, to determine $K^{(\Phi)}_E(x, x_0, s)$. Since in the paper [4], the Euclidean zero-temperature Green’s function $G_E(x, x_0, m)$ was already determined, we can easily find $K_E(x, x_0, s)$ from it.

This paper is organized as follows. In section II, we briefly review the Euclidean Green’s function at zero-temperature. Then, we derive its corresponding Euclidean heat kernel. In section III, we determine the Euclidean thermal heat kernel and its corresponding Green’s function. In section IV, by applying the results obtained in section III we give the thermal average $\langle \phi^2(x) \rangle$ in a general form. Then, we give both zero and high-temperatures limits of this general expression. In section V, we determine the thermal average of the energy-momentum tensor of a nonconformal massless scalar field. We provide the general expression and then make the zero and high-
temperatures limits. We compare our results in the case where the flux vanishes with the ones of Frolov et al. [7] when $B > 1/2$. In section VI, we add some concluding remarks.

2 Preliminaries

We consider the four-dimensional Riemannian metric described by

$$ds^2 = d\tau^2 + dz^2 + d\rho^2 + B^2 \rho^2 d\varphi^2,$$

(2)
in a coordinate system $(\tau, z, \rho, \varphi)$, with $\rho \geq 0$ and $0 \leq \varphi < 2\pi$, which is obtained from (1) by means of a Wick rotation ($t = -i\tau$) in the coordinate $t$.

As we already mentioned in the introduction, metric (2) has a global conical structure. Moreover, we will consider that the cosmic string, which is the source of geometry (2), carries an internal magnetic flux $\Phi$.

The Euclidean Green’s function of a charged massive scalar field $G^{(\gamma)}_E(x, x_0, m)$ obeys the covariant Laplace equation in the space described by metric (2)

$$\left[\frac{\partial^2}{\partial \tau^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{B^2 \rho^2} \frac{\partial^2}{\partial \varphi^2} - m^2\right]G^{(\gamma)}_E(x, x_0, m) = -\frac{1}{B} \delta^{(4)}(x, x_0)$$

(3)
and satisfies the following boundary conditions

\[ G_E^{(\gamma)}(\tau, z, \rho + 2\pi, \varphi) = e^{2i\pi\gamma}G_E^{(\gamma)}(\tau, z, \rho, \varphi) \]

\[ \frac{\partial}{\partial \varphi}G_E^{(\gamma)}(\tau, z, \rho, \varphi + 2\pi) = e^{2i\pi\gamma}\frac{\partial}{\partial \varphi}G_E^{(\gamma)}(\tau, z, \rho, \varphi), \quad (4) \]

where \( \gamma \) is the fractional part of \( \Phi/\Phi_0 \), \( \Phi_0 \) being the flux quantum \( 2\pi/e \), and lies in the interval \( 0 \leq \gamma < 1 \). We shall remark that the case \( \gamma = 0 \) represents the situation where the flux is vanishing and the case \( \gamma = 1/2 \) describes a twisted scalar field around the axis \( \rho = 0 \) [2]. We also require that \( G_E^{(\gamma)}(x, x_0, m) \) must vanish when \( x \) and \( x_0 \) are infinitely separated.

An Euclidean Green’s function which is solution to equation (3) and that satisfies requirements (4) was already obtained in the paper of ref. [4]. When \( B > 1/2 \), it is presented in a local form valid in the domain \( \frac{\pi}{B} - 2\pi < \varphi - \varphi_0 < 2\pi - \frac{\pi}{B} \) as a sum of the usual Green’s function in Euclidean space and a regular term, which is the net effect of the conical structure of (2). That is

\[ G_E^{(\gamma)}(x, x_0, m) = \frac{mK_1[mr_4]}{4\pi^2r_4} + G_E^{(\gamma)}(x, x_0, m), \quad (5) \]

where the regular term is

\[ G_E^{(\gamma)}(x, x_0, m) = \frac{m}{8\pi^3B} \int_0^\infty \frac{K_1[mR_4(u)]}{R_4(u)}F_B^{(\gamma)}(u, \psi)du, \]
and $r_4$, $R_4(u)$ and $F_{B}^{(\gamma)}(u, \psi)$ are respectively

$$r_4 = [(\tau - \tau_0)^2 + (z - z_0)^2 + \rho^2 + \rho_0^2 - 2\rho\rho_0 \cos B(\varphi - \varphi_0)]^{1/2},$$

$$R_4(u) = [(\tau - \tau_0)^2 + (z - z_0)^2 + \rho^2 + \rho_0^2 + 2\rho\rho_0 \cosh u]^{1/2},$$

$$F_{B}^{(\gamma)}(u, \psi) = \frac{ie^{i(\psi - \pi/B)\gamma} \cosh[u(1 - \gamma)/B] - e^{-i(\psi + \pi/B)(1 - \gamma)} \cosh[u\gamma/B]}{\cosh u/B - \cos(\psi + \pi/B)} - \frac{ie^{i(\psi + \pi/B)\gamma} \cosh[u(1 - \gamma)/B] - e^{-i(\psi - \pi/B)(1 - \gamma)} \cosh[u\gamma/B]}{\cosh u/B - \cos(\psi - \pi/B)},$$

with $\psi \equiv \varphi - \varphi_0$.

Thus, at this point, we are able to apply the Schwinger-DeWitt formalism [9]. Namely,

$$G_{E}^{(\gamma)}(x, x_0, m) = \int_{0}^{\infty} K_{E}^{(\gamma)}(x, x_0, s)ds. \quad (7)$$

We remind that the heat kernel obeys the differential equation

$$(\frac{\partial}{\partial s} - \Box + m^2)K_{E}^{(\gamma)} = 0,$$

for $s > 0$ and satisfies the initial condition

$$K_{E}^{(\gamma)}(x, x_0, 0) = \frac{1}{B} \delta^{(4)}(x, x_0).$$

Taking into account the local form (5) of $G_{E}^{(\gamma)}(x, x_0, m)$, we can see that the corresponding heat kernel $K_{E}^{(\gamma)}(x, x_0, s)$ will be also represented as a sum of
the usual heat kernel in Euclidean space and a regular term. Besides, using the Laplace inverse transformation
\[
\sqrt{p} K_1 [d\sqrt{p}] = \int_0^\infty \frac{1}{4s^2} e^{-d^2/4s} e^{-ps} ds,
\]
and inserting in expression (7), with \( G_E^{(\gamma)}(x, x_0; m) \) given by (5) we deduce
\[
K_{E}^{(\gamma)}(x, x_0; s) = \frac{1}{16\pi^2 s^2} e^{-r^2/4s} e^{-m^2 s + K_{E}^{*\gamma}(x, x_0; s)},
\]
where the regular term is
\[
K_{E}^{*\gamma}(x, x_0; s) = \frac{1}{32\pi^3 B s^2} e^{-m^2 s} \int_0^\infty e^{-R^2(u)/4s} F_{B}^{(\gamma)}(u, \psi) du.
\]
for \( B > 1/2 \) and for \( x \) close enough to \( x_0 \).

Thus, we have determined an integral expression (8) the Euclidean (zero-temperature) heat kernel corresponding to the Euclidean (zero-temperature) Green’s function (5) of a charged, massive scalar field in the Riemannian metric (2).
3 The Euclidean Thermal Heat Kernel and its Corresponding Green’s Function

Just as a remainder of the paper, we recall that the thermal Green’s function of a charged scalar field $G^{(\gamma)}_{ET}(x, x_0; m)$ is a solution to equation (3), satisfies the boundary conditions (4), vanishes for spatial points $x^i$ and $x_0^i$ infinitely separated and moreover, it is periodic in the coordinate $\tau$ with a period given by $\beta = 1/kT$, $k$ and $T$ being respectively the Boltzmann constant and the temperature. To obtain $G^{(\gamma)}_{ET}(x, x_0, m)$ we will first derive the Euclidean thermal heat kernel $K^{(\gamma)}_{ET}(x, x_0; s)$, which can be determined straightforwardly from its zero-temperature limit $K^{(\gamma)}_E(x, x_0; s)$. Indeed, as shown in papers \[10\], for ultrastatic metrics such as (2), $K^{(\gamma)}_{ET}(x, x_0; s)$ can be factorized in the following way

$$K^{(\gamma)}_{ET}(x, x_0; s) = \Theta_3\left(i\frac{\beta(\tau - \tau_0)}{4s} \mid i\frac{\beta^2}{4\pi s}\right)K^{(\gamma)}_E(x, x_0; s), \quad (9)$$

where $\Theta_3$ is the Theta function \[11\] defined as

$$\Theta_3(z \mid t) = \sum_{n=-\infty}^{\infty} e^{i\pi n^2 t + 2i\pi n z}.$$ 

Substituting (8) into (9), we have
\[ K^{(\gamma)}_{ET}(x, x_0; s) = \frac{1}{16\pi^2s^2} \Theta_3(i\frac{\beta(\tau - \tau_0)}{4s} \mid i\frac{\beta^2}{4\pi s})e^{-r^2/4s}e^{-m^2s} + K^{*\gamma}_{ET}(x, x_0; s), \] 

(10)

with

\[ K^{*\gamma}_{ET}(x, x_0; s) = \frac{1}{32\pi^3Bs^2} \Theta_3(i\frac{\beta(\tau - \tau_0)}{4s} \mid i\frac{\beta^2}{4\pi s})e^{-m^2s} \int_0^\infty e^{-R^2(u)/4s} F^{(\gamma)}_B(u, \psi) du. \]

Applying formula (7) we finally find the expression of the Euclidean thermal Green’s function for the massive scalar field

\[ G^{(\gamma)}_{ET}(x, x_0; m) = \frac{1}{16\pi^2} \int_0^\infty \Theta_3(i\frac{\beta(\tau - \tau_0)}{4s} \mid i\frac{\beta^2}{4\pi s})e^{-r^2/4s}e^{-m^2s/s^2} ds + G^{*\gamma}_{ET}(x, x_0; m), \] 

(11)

where the regular term is

\[ G^{*\gamma}_{ET}(x, x_0; m) = \frac{1}{32\pi^3B} \int_0^\infty \int_0^\infty \Theta_3(i\frac{\beta(\tau - \tau_0)}{4s} \mid i\frac{\beta^2}{4\pi s})e^{-R^2(u)/4s}e^{-m^2s/s^2} F^{(\gamma)}_B(u, \psi) dsdu. \]

Now, expression (11) cannot be integrated. A handy form will be provided in the case where the mass of the scalar field tends to zero. So that, we set \( m = 0 \) in expression (11) and we obtain
with the regular term

\begin{equation}
D_{ET}^{*(\gamma)}(x, x_0) = \frac{1}{32\pi B} \int_0^\infty \int_0^\infty \frac{1}{s^2} e^{-R_z(u)/4s} \Theta_3(i\beta(u) \Theta_3(i\beta - \tau_0) \mid i\beta^2 / 4\pi s) \int_0^\infty F_B^{(\gamma)}(u, \psi) dsdu.
\end{equation}

Following Linet’s procedure in paper [8], we change the variable \( s \) for \( t \) in the way

\[ t = \beta^2 \frac{1}{4\pi^2 s}, \]

and substitute in expression (12). We obtain

\begin{equation}
D_{ET}^{(\gamma)}(x, x_0) = \frac{1}{4\beta^2} \int_0^\infty \Theta_3(i\pi \kappa t \mid i\pi t) e^{-(d^2 + \kappa^2)t} dt
\end{equation}

\begin{equation}
+ \frac{1}{8\pi B \beta^2} \int_0^\infty \int_0^\infty \Theta_3(i\pi \kappa t \mid i\pi t) e^{-(D^2(u) + \kappa^2)t} F_B^{(\gamma)}(u, \psi) dsdu,
\end{equation}

where we define the functions \( d, D(u) \) and \( \kappa \) respectively as

\[ d = \frac{\pi}{B} [(z - z_0)^2 + \rho^2 + \rho_0^2 - 2\rho \rho_0 \cos B(\varphi - \varphi_0)]^{1/2}, \]

\[ D(u) = \frac{\pi}{B} [(z - z_0)^2 + \rho^2 + \rho_0^2 + 2\rho \rho_0 \cosh u]^{1/2}, \]

\[ \kappa = \frac{\pi}{B} (\tau - \tau_0). \]
Let’s us turn our attention to the integral
\[ I(a, b) = \int_0^\infty \Theta_3(i\pi bt \mid i\pi t)e^{-(a^2+b^2)t}dt, \]
which appears in both terms of expression (13). This integral can be performed and gives
\[ I(a, b) = \frac{1}{2a}[\coth(a+ib) + \coth(a-ib)]. \tag{15} \]
Substituting this result in (13), we get the final form of the Euclidean thermal Green’s function for a massless scalar field
\[ D^{(\gamma)}_{ET}(x, x_0) = \frac{1}{4\pi\beta} \frac{\sinh(2\pi/\beta)d}{[\cosh(2\pi/\beta)d - \cos(2\pi/\beta)(\tau - \tau_0)]} \]
\[ + \frac{1}{8\pi^2 B\beta} \int_0^\infty \frac{\sinh[2\pi/\beta D(u)]F_B^{(\gamma)}(u, \psi)}{D(u)[\cosh 2\pi/\beta D(u) - \cos 2\pi/\beta(\tau - \tau_0)]}du, \tag{16} \]
for \( B > 1/2. \)

We must point out that, in the limite where the temperature goes to zero, the above result tends to the Euclidean Green’s function \( D^{(\gamma)}_E(x, x_0), \) already found in the paper of reference 4.
4 The Thermal Average $<\phi^2(x)>_\beta$

As an application of the results obtained in the previous section, we can evaluate the thermal average $<\phi^2(x)>_\beta$. The procedure consists of removing the usual (zero-temperature) Green’s function in the coincident limit $x = x_0$:

$$<\phi^2(x)>_\beta = \lim_{x \to x_0} [D_{ET}(x, x_0) - 1/4\pi^2 r_4^2].$$  \hspace{1cm} (17)

From (16) we obtain the general expression for the thermal average

$$<\phi^2(x)>_\beta = \frac{1}{12\beta^2} + \frac{1}{16\pi^2 B\beta\rho} \int_0^\infty \frac{\coth[\frac{2\pi}{\beta} \rho \cosh u/2]}{\cosh u/2} F_B^{(\gamma)}(u, 0) du, \hspace{1cm} (18)$$

where

$$F_B^{(\gamma)}(u, 0) = -2 \frac{\sin[\pi \gamma/B] \cosh[u(1 - \gamma)/B] + \sin[\pi(1 - \gamma)/B] \cosh[u\gamma/B]}{\cosh u/B - \cos \pi/B}.$$

We can recognize in expression (18) the usual thermal average in Euclidean space and the second term which is particular from the conical structure.

Unfortunately, we cannot integrate expression (18). Therefore, we will give some asymptotic limits of it. First, we consider the zero-temperature limit ($\beta \to \infty$), we have
\[< \phi^2(x) >_{\beta \to \infty} \sim \frac{\omega_2(\gamma)}{2\rho^2}, \quad (19)\]

where we define some quantities which will be useful later

\[\omega_2(\gamma) \equiv \frac{1}{16\pi^3 B} \int_0^\infty \frac{F_B^{(\gamma)}(u, 0)}{\cosh u/2} du,\]

\[\omega_4(\gamma) \equiv \frac{1}{32\pi^3 B} \int_0^\infty \frac{F_B^{(\gamma)}(u, 0)}{\cosh^2 u/2} du.\]

We remind that the constants \(\omega_2(\gamma)\) and \(\omega_4(\gamma)\) have already appeared in ref. \[4\] but their explicit expression have been found previously by Dowker \[12\]

\[\omega_2(\gamma) = -\frac{1}{8\pi^2 3} - \frac{1}{2B^2}[4(\gamma - \frac{1}{2})^2 - \frac{1}{3}],\]

\[\omega_4(\gamma) = -\frac{1}{720\pi^2} \{11 - \frac{15}{B^2} [4(\gamma - \frac{1}{2})^2 - \frac{1}{3}] \]
\[+ \frac{15}{8B^4} [16(\gamma - \frac{1}{2})^4 - 8(\gamma - \frac{1}{2})^2 + \frac{7}{15}]\},\]

for \(B > 1/2\). If we set \(\gamma = 0\) and \(\gamma = 1/2\) in expression (19) we get the same results as Smith \[2\].

We consider now the high-temperature limit \((\beta \to 0)\). We have

\[< \phi^2(x) >_{\beta \to 0} \sim \frac{1}{12\beta^2} + \frac{M^{(\gamma)}}{\beta \rho}, \quad (20)\]
where we define the following constants \( M^{(\gamma)} \) and \( N^{(\gamma)} \), which will be useful in further applications

\[
M^{(\gamma)} \equiv \frac{1}{16\pi^2 B} \int_0^\infty \frac{F_B^{(\gamma)}(u,0)}{\cosh u/2} du,
\]

and

\[
N^{(\gamma)} \equiv \frac{1}{32\pi^2 B} \int_0^\infty \frac{F_B^{(\gamma)}(u,0)}{\cosh^3 u/2} du.
\]

(21)

We obtain an explicit expression of (20) for first order in the angle deficit.

For \( \gamma \) defined in the interval \( 0 < \gamma < 1 \) and in the limit where \( B \to 1 \), we get

\[
< \phi^2(x) >_{\beta \to 0} \sim \frac{1}{12\beta^2} + \frac{(1-B)}{8\pi\beta\rho} [\gamma \Gamma(\frac{5}{2} - \gamma)\Gamma(\frac{1}{2}) + \Gamma(\frac{3}{2} + \gamma)\Gamma(\frac{3}{2} - \gamma)],
\]

(22)

\( \Gamma(z) \) being the Gamma function \[11\]. In the case where \( \gamma \to 0 \), we get the same result as Linet \[8\], the constants \( M^{(\gamma)} \) and \( N^{(\gamma)} \) reducing to \( M \) and \( N \) defined in the same paper. When \( \gamma = 1/2 \), we find the same as Rogatko \[13\].

5 The Thermal Average \( < T^\mu_\nu(x) >_\beta \)

The renormalized energy-momentum tensor of a nonconformal, charged, massless scalar field can be derived from its Green’s functions and derivatives in
the following way

\[ < T^\mu_\nu(x) >_\beta = \lim_{x \to x_0} D^\mu_\nu[D^{(\gamma)}_{ET}(x, x_0) - 1/4\pi^2 r^2_4], \]  

(23)

where \( D^\mu_\nu \) is a differential operator given by

\[ D^\mu_\nu = 2[(1 - 2\xi) \nabla_\nu \nabla^\mu_0 - 2\xi \nabla^\mu_\nu + (2\xi - 1/2)\delta^\mu_\nu \nabla_\alpha \nabla^\alpha_0], \]

\( \nabla_\nu(\nabla^\nu_0) \) corresponding to the covariant derivative with respect to the coordinate \( x(x_0) \) and \( \xi \) being the coupling parameter.

As in the case of the Green’s functions and thermal average \( < \phi^2(x) >_\beta \), \( < T^\mu_\nu(x) >_\beta \) will be represented as a sum of the thermal radiation in flat spacetime and a term which corresponds to the polarization effect due to the conical singularity. Thus, we have

\[ < T^\mu_\nu(x) >_\beta = \frac{\pi^2}{45\beta^4} diag(-3, 1, 1, 1) + < T^{\nu\mu}(x) >_\beta \]  

(24)

The second term of (24) is

\[ < T^{\nu\mu}(x) >_\beta = D^\mu_\nu D^{(\gamma)}_{ET}(x, x). \]  

(25)

The calculus are combersome but straightforward. We present directly our results. From (25) we get
\[
<T^x(x) >_\beta = I_1 + (2\xi - 1/2)[2I_2 + I_3 + I_4]
\]  
(26)

\[
<T^z(x) >_\beta = I_5 + I_6 + (2\xi - 1/2)[2I_2 + I_3 + I_4]
\]  
(27)

\[
<T^\rho(x) >_\beta = I_5 + I_6 - 4\xi[I_3 + I_4]
\]  
(28)

\[
<T^\phi(x) >_\beta = -2[I_5 + I_6 + \frac{1}{2}I_1] + 2\xi[I_2 + I_3 + I_4]
\]  
(29)

where

\[
I_1 \equiv \frac{1}{4B\beta^3} \int_0^\infty \frac{\coth[2\nu \rho \cosh u/2]}{\sinh^2[2\nu \rho \cosh u/2]} \cosh u/2 F^{(\gamma)}_{B}(u, 0) du
\]

\[
I_2 \equiv \frac{1}{4B\beta^3} \int_0^\infty \frac{\cosh[2\nu \rho \cosh u/2]}{\sinh^2[2\nu \rho \cosh u/2]} \cosh u/2 F^{(\gamma)}_{B}(u, 0) du
\]

\[
I_3 \equiv \frac{1}{8\pi B\beta^2 \rho^2} \int_0^\infty \frac{1}{\sinh^2[2\nu \rho \cosh u/2]} F^{(\gamma)}_{B}(u, 0) du
\]

\[
I_4 \equiv \frac{1}{16\pi^2 B\beta^3 \rho^3} \int_0^\infty \frac{\coth[2\nu \rho \cosh u/2]}{\cosh u/2} F^{(\gamma)}_{B}(u, 0) du
\]

\[
I_5 \equiv \frac{1}{16\pi^2 B\beta^2 \rho^2} \int_0^\infty \frac{1}{\sinh^2[2\nu \rho \cosh u/2] \cosh^2 u/2} F^{(\gamma)}_{B}(u, 0) du
\]

\[
I_6 \equiv \frac{1}{32\pi^2 B\beta \rho^3} \int_0^\infty \frac{\coth[2\nu \rho \cosh u/2]}{\cosh^3 u/2} F^{(\gamma)}_{B}(u, 0) du
\]

For the zero-temperature limit \((\beta \rightarrow \infty)\), we obtain

\[
<T^\rho(x) >_{\beta \rightarrow \infty} \sim [\omega_4(\gamma) - \frac{1}{3} \omega_2(\gamma)] \frac{1}{\rho^4} \text{diag}(1, 1, 1, -3)
\]

\[
+4(\xi - \frac{1}{6}) \omega_2(\gamma) \frac{1}{\rho^4} \text{diag}(1, 1, -1/2, 3/2),
\]

(30)
which is the same form presented in ref. [4].

In the high-temperature limit ($\beta \to 0$), we have

\begin{align*}
<T_{\tau}(x) >_{\beta \to 0} & \sim -\frac{\pi^2}{15 \beta^4} + \frac{(2\xi - 1/2) M^{(\gamma)}}{\beta \rho^3} \tag{31} \\
<T_{z}(x) >_{\beta \to 0} & \sim -\frac{\pi^2}{45 \beta^4} + \frac{N^{(\gamma)}}{\beta \rho^3} + \frac{(2\xi - 1/2) M^{(\gamma)}}{\beta \rho^3} \tag{32} \\
<T_{\rho}(x) >_{\beta \to 0} & \sim -\frac{\pi^2}{45 \beta^4} + \frac{N^{(\gamma)}}{\beta \rho^3} - \frac{2\xi M^{(\gamma)}}{\beta \rho^3} \tag{33} \\
<T_{\phi}(x) >_{\beta \to 0} & \sim -\frac{\pi^2}{45 \beta^4} - \frac{2N^{(\gamma)}}{\beta \rho^3} + \frac{4\xi M^{(\gamma)}}{\beta \rho^3} \tag{34}
\end{align*}

If we set $\gamma = 0$ in (31-34), our results are in agreement with Frolov et al. when $B > 1/2$ in their calculations.

6 Conclusion

In this paper we have considered the vacuum polarization effect on a charged scalar field by a magnetic flux cosmic string at finite temperature. We provided the general expression for the Euclidean thermal heat kernel and its corresponding Green’s function for a massive scalar field, and a handy form for a massless scalar field. The latter expression enabled us to evaluate the average of some physical quantities in a thermal bath. We calculated the
general expression of the thermal average of \(< \phi^2(x) >_\beta\) and, then, found both zero and high-temperature limits of it. We have also computed the thermal average of the energy-momentum tensor of a nonconformal, charged massless scalar field in a general form. We provided the high-temperature limit to \(< T_{\mu\nu}(x) >_\beta\). These formula generalize the work of Frolov et al. for the case where there exists an internal magnetic flux to the cosmic string but it is limited to \(B > 1/2\).

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