Highly Eccentric Kozai Mechanism and GW Observation for Neutron Star Binaries

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The Kozai mechanism for a hierarchical triple system could reduce the merger time of inner eccentric binary emitting gravitational waves (GWs), and has been qualitatively explained with the secular theory that is derived by averaging short-term orbital revolutions. However, with the secular theory, the minimum value of the inner pericenter distance could be excessively limited by the averaging operation. Compared with traditional predictions, the actual evolution of an eccentric inner binary could be accompanied by (i) a higher characteristic frequency of the pulse-like GWs around its pericenter passages, and (ii) a larger residual eccentricity at its final inspiral phase. These findings would be important for GW astronomy with the forthcoming advanced detectors.

I. INTRODUCTION

Today, large-scale laser interferometers are under development to attain a world-wide network of second-generation GW detectors [1]. Their overall sensitivities will be improved by a factor of ∼ 10, with drastic noise reduction at the lower frequency regime down to ∼ 10Hz [1]. Accordingly, understandings of basic properties of potential astrophysical sources have become significant, more than ever.

One of the most promising targets of these detectors is inspiral of a neutron star binary (NSB), and in this paper we focus our attention to GW observation for NSBs. From identified samples in our Galaxy, NSBs are expected to have very small residual eccentricities (O(10^{-5})) around 10Hz [2, 3].

Meanwhile, it has been pointed out that the Kozai mechanism might play important roles for compact binary mergers [4-6]. This mechanism works for hierarchical triple systems, and oscillates pericenter distances of inner binaries, due to exchange of angular momenta between the inner/outer orbits [7]. This characteristic feature can be qualitatively understood with the secular theory for which, following a perturbative method in analytical mechanics, we effectively average out short-term fluctuations associated with both the inner/outer orbital revolutions [8, 9]. Since energy loss due to GW emission depends strongly on pericenter distance, the Kozai mechanism can largely reduce the merger time of an inner NSB of a triple system. This interesting possibility has been actively discussed mostly with the secular theory including the averaging operations [4-6] (see also [10]).

In this paper, we only discuss relativistic effects for hierarchical triples, but tidal effects around planets also depend strongly on orbital distance [11] (see also [12] for collisions of white dwarf binaries). For extra-solar planetary systems (e.g. Hot Jupiters [13, 14]), an investigation similar to this work would be worth considering.

II. SECULAR THEORY

We study evolution of a hierarchical triple system of point masses \( m_0, m_1 \) and \( m_2 \). We basically use the geometrical units with \( G = c = (m_0 + m_1 + m_2) = 1 \). The inner binary is composed by \( m_0 \) and \( m_1 \), and we denote its semimajor axis by \( a_1 \) and its instantaneous orbital separation by \( d_1 \). In the next section, we also introduce astrophysical units, considering \( m_0, m_1 \) as a NSB. For the outer third body \( m_2 \), we define its semimajor axis by \( a_2 \), relative to the mass center of the inner binary (total mass \( M_1 = m_0 + m_1 \)). Likewise, we use the labels \( j = 1 \) and \( 2 \) for the inner and outer orbital elements (e.g. \( e_1 \) for the inner eccentricity), and assume hierarchical orbital configurations with \( \alpha \equiv a_1/a_2 \ll 1 \).

First, we briefly discuss long-term secular evolution of the triple system in Newtonian dynamics, following the approach developed by von Zeipel [8]. By suitably using canonical transformations, we effectively average the short-term fluctuations associated with both the inner and outer mean anomalies \( l_1 \) and \( l_2 \) (the instantaneous angular positions of the inner/outer point masses [11]). The relevant Hamiltonian after the averaging operations, can be evaluated perturbatively with the expansion parameter \( \alpha \ll 1 \). The leading order (quadrupole) term
configuration is obtained after averaging the inner mean anomaly \( I_1 \) as

\[
H_{\text{qd}} = \mathcal{O}(\alpha^2)
\]

\[
H_{\text{qd}} = C_{\text{qd}} \left[ (2 + 3e_1^2)\left(1 - 3\theta^2\right) - 15e_1^2\left(1 - \theta^2\right) \cos 2\omega_1 \right]
\]

with \( C_{\text{qd}} \equiv m_0m_1m_2\alpha^2/16M_1a_2(1 - e_1^2)^{3/2} \) and the argument of the inner pericenter \( \omega_1 \) \[11\]. Here we define \( \theta \equiv \cos I \) with the opening angle \( I \) between the inner/outer orbital angular momentum vectors (identical to the angle \( i \) in \[9\]). We denote the next order (octupole) term by \( H_{\text{oc}} = \mathcal{O}(\alpha^3) \) \[9\]. For our secular analysis of the inner binary, we keep up to this term for the gravitational perturbation externally induced by \( m_2 \).

But there exists a relation \( H_{\text{oc}} \propto (m_0 - m_1) \), resulting in \( H_{\text{oc}} = 0 \) for \( m_0 = m_1 \) \[9\]. Later we use this property to examine possible effects of the sub-leading terms.

Next, we mention general relativistic corrections to the system, using the post-Newtonian (PN) expansion. The lowest order (1PN) term \( H_{1\text{pn}} \) for our hierarchical configuration, the dissipation predominantly works for terms of \( \mathcal{O}(\alpha^3) \) \[9\]. Combining these with the conservative contributions, we have \( H_{\text{c}} \) for the secular evolution is given by

\[
H_{\text{c}} = H_{\text{qd}} + H_{\text{oc}} + H_{1\text{pn}},
\]

and the system is conservative (thus putting the subscript "c" above) \[9, 8\]. Using canonical equations and transformations of variables, we have e.g.

\[
\left( \frac{dw_1}{dt} \right)_c = 6C_{\text{qd}} \left( \frac{4\theta^2}{G_1} + \cdots \right) + \text{O.T.} + \frac{3}{a_1(1 - e_1^2)} \left( \frac{M_1}{a_1} \right)^{3/2}
\]

\[
\left( \frac{dw_1}{dt} \right)_c = 30C_{\text{qd}} \frac{e_1(1 - e_1^2)}{G_1} (1 - \theta^2) \sin 2\omega_1 + \text{O.T.},
\]

(\( da_1/\dot{d}t \)) = (\( da_2/\dot{d}t \)) = 0 and the scaling relations (\( de_2/\dot{d}t \)) = O(T) and (\( dw_2/\dot{d}t \)) = O(\( \alpha^3 \)). Here we defined \( G_1 = m_0m_1 \left[ a_1(1 - e_1^2)/M_1 \right]^{1/2} \) and put O.T. for terms of \( \mathcal{O}(\alpha^3) \) originating from \( H_{\text{oc}} \) \[9, 8\]. The total angular momentum is conserved with \( \sqrt{a_2(1 - e_2^2)}/\alpha \) for the magnitude of the outer one.

The triple system becomes dissipative at the 2.5PN order, due to emission of GWs. Given our hierarchical configuration, the dissipation predominantly works for the inner binary, and we include its effects only for \( a_1 \) and \( e_1 \), using standard formulae for isolated eccentric binaries \[13\]. Combining these with the conservative contributions, we can write down the final expressions for the secular evolution such as \( \dot{\omega}_1/\dot{d}t = (\dot{w}_1/\dot{d}t)_c \),

\[
\frac{da_1}{dt} = -\frac{64m_0m_1M_1}{5a_1^3(1 - e_1^2)^{7/2}} \left( 1 + \frac{73}{24} e_1^4 + \frac{37}{96} e_1^6 \right),
\]

\[
\frac{de_1}{dt} = -\frac{304m_0m_1M_1e_1}{15a_1^3(1 - e_1^2)^{7/2}} \left( 1 + \frac{121}{304} e_1^2 + \left( \frac{de_1}{dt} \right)_c \right),
\]

which have strong dependencies on \( 1 - e_1 \). We also have \( a_2 = \text{const} \). These secular equations have been widely used for analyzing long-term evolutions of relativistic hierarchical triple systems \[9, 8\].

### III. Numerical Results

In this section, we numerically discuss the Kozai mechanism for relativistic hierarchical triples, first using the secular equations and then directly integrating the PN equations for three-body systems. While a triple system has many parameters, we fix most of them to concisely explain our new findings.

In our geometrical units, we fix the masses at \( M_1 = 0.2, m_2 = 0.8 \), and the initial orbital parameters at \( \alpha_1 = 3.57 \times 10^4, a_2 = 60a_1 = 2.14 \times 10^7 = a_21 \) (i.e. initially \( \alpha = 1/60 \)), \( e_1 = 0.2 \) and \( e_2 = 0.6 \). We also set the initial angular variables at \( \omega_1 = \pi/2 \) and \( \Omega_1 = \omega_2 = 3 \) (\( \Omega_1 \): the longitude of the inner ascending node \[11\]). For our study, the remaining important parameter is the initial inclination \( I_1 \). We explore the regime \( I_1 \sim 90^\circ \) for which an inner binary can merge in a short time (also preferable for costly direct calculations).

For actual astrophysical system, we presume that the inner binary is a NSB with their total mass \( M_1 = 2.8M_\odot \). Then the initial axes correspond to \( a_1 = 0.05 \) AU and \( a_2 = a_21 = 3AU \). Below, instead of the direct time variable \( t \), we use the effective outer revolution cycles \( N_2 \equiv t/P_2 \) defined with the initial orbital period \( P_21 = 2\pi a_21^{3/2} \) (corresponding to 1.38yr). The primary GW frequency, of a quasi-circular inner binary becomes 10Hz (\( \sim \)lower end of the advanced detectors) at the critical separation \( a_1 = a_{1\text{cr}} \equiv 34.6 \).

Since observed NSBs have nearly equal masses (with relative difference of \( \lesssim 7\% \) \[2\]), we mainly set \( m_0 = m_1 = 0.1 \) in geometrical units. For an isolated binary with a semimajor axis \( a \equiv 0.05AU \) and masses \( m_0 = m_1 = 1.4M_\odot \), the merger time due to GW emission becomes \( 1.0 \times 10^{10}\text{yr} \) even for \( e = 0.7 \).

As mentioned earlier, the octupole term \( H_{\text{oc}} \) vanishes for \( m_0 = m_1 \). In order to safely estimate its potential effects, we also examine the case \( (m_0, m_1) = (0.11, 0.09) \).

#### A. Results with the Secular Theory

As an example for predictions of the secular theory, in Fig.1, we provide the inner semimajor axis \( a_1 \) and pericenter distance \( r_{p1} \equiv a_1(1 - e_1) \), as functions of the outer cycles \( N_2 \). Their ratio \( r_{p1}/a_1 \) is identical to \( (1 - e_1) \). The basic parameters for this calculation are given in the caption.

The inner binary merges at \( N_2m = 1209 \) that is considerably smaller than the cycles \( N_{2m} = O(10^{10-11}) \) for isolated binaries with moderate initial eccentricities \[3, 8\]. Due to the Kozai mechanism, the inner eccentricity \( e_1 \)
oscillates in the range $0.2 \lesssim e_1 \lesssim 0.9992$, and the minimum pericenter distances becomes $r_{p1} \simeq 300$.

When we switch off the radiation reaction and also drop the octupole and higher terms, we have conserved quantities in the secular theory, as mentioned after Eq.(5) (in particular $\sqrt{a_2(1 - e_2^2)}$). These conserved quantities actually allows us to set a lower limit $r_{p1} \simeq 300$ close to Fig.1 (see e.g. [2] for the role of the 1PN effect).

In Fig.1, the energy of the inner binary is radiated mostly around the close approaches $d_1 \sim 300$. As discussed in the literature [2, 6], the oscillation amplitude of $e_1$ decreases gradually due to the 1PN apsidal precession (the last term in Eq.(4)), and, at $N_2 \gtrsim 1000$, the inner elements evolve, as if an isolated binary. The binary becomes nearly circular at the final phase close to the merger. At the critical separation $a_1 = a_{1cr}$, the residual eccentricity becomes $e_{1cr} = 5.3 \times 10^{-3}$.

In Fig.2, using the symbols on the solid lines, we show the duration $N_{2m}$ and the residual eccentricity $e_{1cr}$ at $a_1 = a_{1cr}$ for $I_1 \sim 90^\circ$. The results (circles) for $(m_0, m_1) = (0.1, 0.1)$ are similar to those (triangles) for $(m_0, m_1) = (0.11, 0.09)$. Therefore, for the present parameters, the octupole term plays a minor role, and the perturbative expansion itself is effective for the secular theory (see also [14, 16]).

### B. Direct Three-Body Calculations

Now we move to direct three-body calculations. We use PN equations of motions for spineless three-body systems, and handle the three particles equivalently. In addition to the conservative terms at the Newtonian, 1PN and 2PN orders (given e.g. in [17]), we included the dissipative 2.5PN terms by using Eq.(41) in [18]. Unless otherwise stated, we excluded the time consuming 2PN terms that would be briefly discussed later.

For numerical integration, we apply a fourth-order Runge-Kutta scheme with an adaptive time-step control [19]. We terminate our runs, when the inner semimajor axis decreases to $a_1 = a_{1cr}$ or when the instantaneous separation $d_1$ becomes less than $10M_1$. The later condition reflects our perturbative (PN) treatment of nonlinear gravity, but no run encountered this condition. For numerical evaluation of the orbital elements $a_j$ and $e_j$ ($j = 1, 2$), we use the consecutive maximum ($(1 + e_j)a_j$) and minimum ($(1 - e_j)a_j$) of the instantaneous orbital separations $d_j$.

For the direct calculations, we need to specify the initial mean anomalies $I_j$. Since three-body problem depends strongly on initial conditions, we randomly distribute the initial mean anomalies to examine statistical trends of evolutions. For each initial inclination $I_1$ and mass combination in Fig.2, we made 50 runs, and evaluated their median values and first/third quantiles of the durations $N_{2m}$ and the residual eccentricities $e_{1cr}$. For $m_0 = m_1 = 0.1$ and $I_1 = 90^\circ$, we additionally made 50 runs, including the 2PN terms, and obtained the median values $N_{2m} = 78.3$ and $e_{1cr} = 0.136$ that are close to the corresponding ones in Fig.2. Therefore, for our analyses, the 2PN effect would not be important.

We found that, in the direct calculations, the outer parameters $a_2$ and $e_2$ stay nearly at their initial values, in agreement with the secular theory. However, Fig.2 shows that the duration $N_{2m}$ and residual $e_{1cr}$ are totally different. [22]
To closely look at these discrepancies, we an illustrative sample among the 50 runs for $I_i = 91^\circ$ and $m_0 = m_1 = 0.1$. This run ended at $N_{2m} = 52.5$ with the residual $e_{1cr} = 0.313$ (close to the upper quantile in Fig.2). If we simply use the outer cycle $N_2$ (as in Fig.1), the semimajor axis $a_1$ comes to appear merely as a step function, and we cannot resolve its rapid final evolution. Therefore, for Fig.3, we plot, on a logarithm scale, the remaining cycles $\Delta N \equiv N_{2m} - N_2$ before the merger.

We can see that, up to $\Delta N = O(0.1)$, the axis $a_1$ is nearly a constant, but the pericenter distance $r_{p1}$ has a modulation period $\sim P_2$, the orbital period of the outer binary. This reflects the eccentric motion of the outer point mass $m_2$ characterized by $l_2$, rather than an effective ring in the secular theory. Temporally neglecting radiation reaction, we follow [12] and briefly discuss the impacts of this discreteness for evolution of the inner specific angular momentum vector $j_1$ (closely related to $e_1$ and $r_{p1}$ as $|j_1| = \sqrt{a_1(1-e_1^2)}$). Its variation $\Delta j_1$ due to $m_2$ in one inner orbital revolution, depends strongly on the exact position of $m_2$ and thus has a stochastic character (denoting its rms value by $\delta j_1$).

For $\delta j_1 \ll |j_1|$, the total variation of $j_1$ after a few outer orbital cycles could be close to that caused by the corresponding outer ring, and the orbital averaging could be efficient. However, for a highly eccentric case with $\delta j_1 \gg |j_1|$, the averaging method would break down, and consequently, the associated lower limit for $r_{p1}$ (mentioned in §III.A) would be no longer valid. In the direct three-body integral, the discreteness of $m_2$ is naturally included, and we have possibilities to realize $r_{p1}$ smaller than the limit obtained with the secular theory. While we temporally neglected radiation reaction for simplicity, we can expect similar differences for our dissipative systems.

Indeed, in Fig.3, at the turning point $\Delta N \sim 4 \times 10^{-2}$, the quantity $1 - e_1$ takes a minimum value, corresponding to $r_{p1} = 38$ (much smaller than Fig.1). Then the inner binary evolves almost independently of the outer body $m_2$ with rapidly decreasing $a_1$ from $a_1 = 3.0 \times 10^5$ but nearly conserving $r_{p1}$ for a while.

For an orbit with $1 - e_1 \ll 1$, GW emission is dominated at the pericenter passages, and the radiated energy there is given as $\delta E \sim -85\pi^2 m_0 m_1^2 M_1^{1/2} / (12 \sqrt{2} \pi_1^{7/2})$, depending strongly on $r_{p1}$ [21]. At the turning point in Fig.3, this amounts to a fraction

$$Y \sim 0.19 \left( \frac{a_1}{3.0 \times 10^5} \right)^{-7/2} \left( \frac{r_{p1}}{38} \right)$$

of the inner orbital energy $-m_0 m_1 / 2a_1$.

Meanwhile, for the secular theory, we can simply estimate the local minimum of $1 - e_1$ from Eq.(7) (with $de_1/dt = 0$) [3]. This is determined by the balance between the two effects, the dissipative radiation reaction working only around $d_1 = O(r_{p1}) \ll a_1$ and the tidal effect (by $m_2$) operating mainly during $d_1 = O(a_1)$. Neglecting the octupole terms, we obtain the minimum pericenter distance $r_{p1,\text{min}} = a_1 (1 - e_1)_{\text{min}}$ as

$$r_{p1,\text{min}} \sim 60 \left( \frac{a_2/a_1}{71} \right) \left( \frac{a_1}{3.0 \times 10^5} \right)^{1/6} \left( \frac{X}{1} \right)^{-1/3}$$

with the factor $X \equiv \sin^2 l / \sin 2a_1 \leq 1$. Thus, even with the highly conservative setting $X = 1$, the distance $r_{p1} = 38$ at the turning point in Fig.3 is not allowed in the secular theory, and the radiated fraction becomes at most $Y = 0.02$, in contrast to Eq.(8). Eq.(9) has been used in previous studies, with additionally evaluating $X$ [6]. But, along with the insufficient treatment of the discreteness effect (mentioned earlier), the two temporally separated effects are directly compared in Eq.(9) without resolving the inner orbital phase. Roughly speaking, even at $d_1 > r_{p1}$, a nearly radial inner orbit could be prohibited by the radiation reaction that intrinsically has no effect there.

**IV. DISCUSSIONS**

Finally, we comment on the implications of our results for GW astronomy. In Fig.3, after the turning point, the inner binary emits pulse-like GWs around the pericenter passages [21]. This waveform has a characteristic frequency $(M_1/r_{p1}^3)^{1/2} / \pi \sim 10$Hz that is $\sim 30$ times higher than the counterpart in Fig.1. While Figs.1 and 3 are given for a specific set of parameters, this shift would be encouraging for ground-based GW observation, given the formidable noise walls below $\sim 10$Hz [1].

Fig.2 shows that we could have larger residual eccentricities $e_{1cr}$ and also shorter merger times than the estimations by the secular theory. These differences are closely related to the decrease of the pericenter distances, and suggest a higher merger rate of NSBs in star clusters, as discussed earlier. For a quasi-circular binary, the residual eccentricity could be probed through the associated phase modulation of inspiral GWs [3].
NSB detectable with advanced detectors at SNR~15, the resolution of the residual value \(e_{1cr}\) (at 10Hz) would be \(\Delta e_{1cr} \simeq 0.01\). Interestingly, this is just between the two predictions in Fig.2 and we might discriminate the origins of NSB mergers with the upcoming GW detectors.

Acknowledgments

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[22] We have varied the initial parameter \(\alpha\) up to 1/20 (with fixing \(a_2 = a_{2i}\)) and our main results remain true.