Azumaya algebras with involution and classical semisimple group schemes

S. Srimathy

ABSTRACT

Let $S$ be a non-empty scheme with $2$ invertible. In this paper we present a functor $F : AZ_{n'}^* \rightarrow GS_n^*$ where $AZ_{n'}^*$ and $GS_n^*$ are fibered categories over $Sch_S$ given respectively by degree-$n'$ Azumaya algebras with an involution of type $*$ and rank-$n$ adjoint group schemes of classical type $*$ with absolutely simple fibers. Here $n'$ is a function of $n$. We show that this functor is an equivalence of fibered categories using étale descent, thus giving a classification of adjoint (as well as simply connected) group schemes over $S$, generalizing the well known case when the base scheme is the spectrum of a field. In particular, this implies that every adjoint group scheme of classical type with absolutely simple fibers is isomorphic to the neutral component of the automorphism group scheme of a unique (up to isomorphism) Azumaya algebra with involution. We also show interesting applications of this classification such as specialization theorem for isomorphism classes of Azumaya algebras with involution over Henselian local rings, uniqueness of integral model for groups with good reduction over discrete valued fields and discuss its implications on the Grothendieck-Serre conjecture over certain domains.

1. Introduction

It is well known that the category of rank-$n$ (with some $n$ excepted) absolutely simple adjoint (or simply connected) algebraic groups of a given classical type over any field $F$ (char $F \neq 2$) is equivalent to the category of degree-$n'$ central simple algebras with involution of analogous type over $F$. Here $n'$ is a function of $n$. Moreover, the functor which gives this equivalence is obtained by taking a given central simple algebra with involution over $F$ to the identity component of its automorphism group (or its simply connected cover for the simply connected case). This result is originally due to Weil ([Wei60]) and the proof of this equivalence can also be found in [KMRT98] §26, Chapter VI]. This gives neat classification results for groups of classical types in terms of central simple algebras with involution which can be translated to the well understood language of sesquilinear forms over division algebras. This kind of classification is very useful for studying many properties of algebraic groups and the projective homogeneous varieties associated to them.

In this paper, we show a similar classification for adjoint (as well as simply connected) group schemes over an arbitrary scheme where is $2$ invertible. Since we are in the general case of arbitrary base scheme, we use the language of stacks and gerbes to prove that the fibered category of degree-$n'$ Azumaya algebras with an involution of a given type is equivalent to the fibered category of rank-$n$ classical adjoint group

2010 Mathematics Subject Classification 14L15, 16H05

Keywords: classical group schemes, Azumaya algebra with involution, classification of semisimple group schemes
schemes with absolutely simple fibers of the corresponding type via étale descent. As before, \( n' \) is a function of \( n \) with some \( n \) excepted. This implies that an adjoint group scheme of classical type (see Definition 3.2) with absolutely simple fibers is isomorphic to the neutral component of automorphism group scheme of a unique (up to isomorphism) Azumaya algebra with involution. We also give a few applications of this classification such as a specialization theorem for Azumaya algebras with involution and uniqueness of integral model for group schemes with good reduction. Another interesting corollary is that the Grothendieck-Serre conjecture on principal \( G \)-bundles holds whenever \( G \) is an adjoint group scheme with absolutely simple fibers over \( R \) where is \( R \) is a regular local ring containing a field of characteristic \( \neq 2 \) or \( R \) is a semilocal Bézout domain with \( 2 \) invertible.

2. Notations

Throughout this paper, \( S \) denotes a non-empty scheme with \( 1/2 \in O_S(S) \). The category of schemes over \( S \) will be denoted by \( \text{Sch}_S \). Given a scheme \( X \) over \( S \) and a point \( s \in S \), \( k(s) \), \( X_s \), \( X_\pi \) denote respectively the residue field, fiber and geometric fiber at \( s \). For a presheaf \( F \) over a scheme \( X \), \( F_x \) denotes its stalk at the point \( x \in X \). The ring of \( n \times n \) matrices is denoted by \( M_n \). The identity matrix of size \( n \) is denoted by \( I_n \). For a sheaf of algebra \( A \), \( A^{\text{op}} \) denotes the sheaf of opposite algebra given by \( U \to A(U)^{\text{op}} \). The reference [DG70a] is mentioned as SGA3.

3. Group schemes over an arbitrary scheme

In this section we recall the necessary results from the literature on group schemes over an arbitrary scheme. The main sources of reference are SGA3 and [Dem65].

A group scheme \( G \) over \( S \) is called reductive (resp. semi-simple, adjoint, simply connected) if \( G \) is affine, smooth over \( S \) and for every \( s \in S \), the geometric fiber \( G_\pi \) is a connected reductive (resp. semi-simple, adjoint, simply-connected) group ([Dem65, Def. 2.1.2] and [SGA3, Exp. XXII, Def. 4.3.3]).

Let \( G \) be an adjoint (or simply connected) group scheme \( G \) over \( S \). The type (resp. rank) of \( G \) at \( s \in S \) is the type (resp. rank) of \( G_\pi \) ([SGA3, Exp. XXII, Def. 2.7]). The type and rank of \( G \) are locally constant functions over \( S \) ([SGA3, Exp. XXII, Prop.2.8] and [SGA3, Exp. XIX, Cor. 2.6]).

**Remark 3.1.** Any adjoint (resp. simply connected) group scheme over \( S \) is isomorphic to the Weil restriction \( R_{S'/S}(G') \) of an adjoint (resp. simply connected) group scheme \( G' \) with absolutely simple fibers over \( S' \) where \( S' \to S \) is a finite étale cover. Moreover, the pair \( (S', G') \) is uniquely determined upto a unique \( S \)-isomorphism (see [Con14a, Proposition 6.4.4 and Remark 6.4.5]). Therefore, classification of adjoint group schemes over \( S \) reduces to classification of adjoint group schemes with absolutely simple fibers.

**Definition 3.2.** Let \( G \) be an adjoint group scheme over \( S \) with absolutely simple fibers. We say that \( G \) is of type \( A \) (resp. \( B \), \( C \), \( D \)) and rank \( n \) if every fiber \( G_\pi \), \( s \in S \) is of type \( A \) (resp. \( B \), \( C \), \( D \)) and rank \( n \). In general, we say that \( G \) is of classical type if it is of type \( A \), \( B \), \( C \) or \( D \).

**Proposition 3.3.** Let \( G \) and \( H \) be adjoint group schemes over \( S \) with absolutely simple fibers of a given type and rank. Then locally for the étale topology on \( S \), \( G \) and \( H \) are isomorphic. In fact, any reductive group scheme with root datum \( \mathcal{R} \) is étale locally isomorphic to a unique Chevalley group over \( \text{spec } \mathbb{Z} \) with root datum \( \mathcal{R} \).

**Proof.** See [Dem65, § 5, Cor. 5.1.4.1 and Prop. 5.1.6].


AZUMAYA ALGEBRAS WITH INVOLUTION AND CLASSICAL SEMISIMPLE GROUP SCHEMES

Let $G$ be any group scheme of locally finite type over $S$. In [SGA3, Exp. VI\text{B}, Def 3.1], the notion of the neutral component of $G$ denoted by $G^0$ is defined by the functor

$$T \mapsto G^0(T) = \{u \in G(T) | \forall s \in S, u_s(T_s) \subset (G_s)^0\}$$

We refer the reader to [SGA3, § 3, Exp. VI\text{B}] for more details. It can be shown that the neutral component of $G$ is stable under base change. We restate here below.

**Proposition 3.4.** ([SGA3, Exp. VI\text{B}, Prop. 3.3]) Let $G$ be a group scheme over $S$. Then for any scheme $S' \to S$, we have

$$(G \times_S S')^0 = G^0 \times_S S'$$

i.e., the functor $G \to G^0$ commutes with base change.

We will be using the following result about $G^0$.

**Proposition 3.5.** ([Con14a, Prop 3.1.3]) Let $G$ be a smooth separated group scheme of finite presentation such that $G^0_s$ is reductive for all $s \in S$. Then $G^0$ is a reductive group scheme over $S$ that is open and closed in $G$.

For a group scheme $G$ over $S$, we associate the automorphism functor $\text{Aut}(G)$ on $\text{Sch}_S$ defined by

$$\text{Aut}(G) : S' \mapsto \text{Aut}_{S'-\text{grp}}(G_{S'})$$

**Proposition 3.6.** Let $G$ a semisimple group scheme over $S$. Then the functor $\text{Aut}(G)$ is represented by a smooth, affine scheme over $S$.

**Proof.** See [SGA 3, Exp. 24, Theorem 1.3(i) and Corollary 1.6].

**Remark 3.7.** A representable functor on $\text{Sch}_S$ is a sheaf for the fpqc topology ([Vis05, Theorem 2.55]). Therefore if $G$ is semisimple, by Proposition 3.6, $\text{Aut}(G)$ is a sheaf for the fpqc topology and hence for the étale topology on $\text{Sch}_S$.

4. Azumaya algebras with involution over an arbitrary scheme

Recall that an Azumaya algebra $A$ over $S$ is an $\mathcal{O}_S$-algebra that is locally free and finite type as an $\mathcal{O}_S$-module such that the canonical homomorphism

$$A \otimes A^{op} \to \text{End}_{\mathcal{O}_S-\text{mod}}(A)$$

$$a \otimes b \mapsto (x \mapsto a \cdot x \cdot b)$$

is an isomorphism. This implies that there is an étale covering $\{U_i \to S\}$ such that $A \otimes_{\mathcal{O}_S} \mathcal{O}_{U_i} \simeq \mathcal{M}_{n_i}(\mathcal{O}_{U_i})$ for some $n_i$. If $n_i = n$ for all $i$ (this happens for example when $S$ is connected), we call $A$ an Azumaya algebra of degree $n$ over $S$. Also recall that for an Azumaya algebra $A$ over $S$, $A_s \otimes k(s)$ is a central simple algebra over $k(s)$ for every $s \in S$ (see [Mil80] Chapter IV, § 2, Prop. 2.1)).

Involutions on central simple algebras are well studied in the literature ([KMRT98]). In a similar fashion one can also define involution on Azumaya algebras over the scheme $S$. This is discussed in detail in [KPS90] and [PS92] which we briefly recall now. A classification of involutions on Azumaya algebras in a more general setting can be found in [FW20, § 5].

An *involution of first kind* $\sigma$ on $A$ is an isomorphism of $\mathcal{O}_S$-algebras

$$\sigma : A \to A^{op}$$
such that $\sigma^\op \circ \sigma$ is the identity. The involution $\sigma$ on $A$ is said to be of orthogonal type (resp. symplectic type) if étale locally on $S$, it corresponds to a non-degenerate symmetric (resp. skew-symmetric) bilinear form on a locally free $\mathcal{O}_S$-module with values in a line bundle over $S$ (see [PS92 § 1.1]).

Now let $\pi : T \to S$ be a an étale covering of degree-2. Let $A$ be an Azumaya algebra over $T$. An involution $\sigma$ of second kind (a.k.a unitary type) on $A$ is an anti-automorphism on $A$ of order 2 which on $\mathcal{O}_T$, restricts to the non-trivial element of the Galois group of the covering. Locally for the étale topology on $S$ unitary involutions correspond to hermitian forms on locally free $\mathcal{O}_T$-modules ([PS92 § 1.2]).

The data of an Azumaya algebra $A$ with an involution $\sigma$ over $S$ is denoted by $(A, \sigma)$. A homomorphism between Azumaya algebras with involution is a homomorphism between the algebras which respects the involution structure.

**Definition 4.1.** In the case above where $A$ is an Azumaya algebra over a quadratic étale extension $T$ of $S$ and $\sigma$ is unitary, we will make a slight abuse of notation and call $(A, \sigma)$ an Azumaya algebra with unitary involution over $S$ even though the center of $A$ is not $\mathcal{O}_S$. This agrees with the corresponding notion of central simple algebras with unitary involution defined in [KMRT98 Chapter I, § 2.B].

**Remark 4.2.** If $(A, \sigma)$ is a degree-$n$ Azumaya algebra with involution over $S$ where $\sigma$ is of a given type, then for any $s \in S$, $(A_s \otimes k(s), \sigma_s \otimes k(s))$ is a degree-$n$ central simple algebra with involution of the same type over $k(s)$.

We now give an étale local description of Azumaya algebras with involution.

**Proposition 4.3.** Let $(A, \sigma)$ be a degree-$n$ Azumaya algebra with involution over $S$. Then locally for the étale topology on $S$, we have

1. $(A, \sigma) \cong (\mathcal{M}_n, tr)$ when $\sigma$ is of orthogonal type where $tr : A \to A^{tr}$ is the transpose involution.
2. $(A, \sigma) \cong (\mathcal{M}_2m, sp)$ when $\sigma$ is of symplectic type. Here $sp$ is the involution on $\mathcal{M}_{2m}$ given by $A \to J_m A^{tr} J_m^{-1}$ where $J_m = \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix}$ is the standard matrix associated to an alternating form.
3. $(A, \sigma) \cong (\mathcal{M}_n \times \mathcal{M}_n^{\op}, \epsilon)$ when $\sigma$ is of unitary type where $\epsilon : (A, B^{\op}) \to (B, A^{\op})$ is the exchange involution.

**Proof.** All of the above are well known if the base scheme $S$ is the spectrum of a field (see [KMRT98]). For the general case, proofs of (1) and (2) can be easily derived and can also be found in the literature. See for example, [PS92 § 1.1] and [Knu91] Chapter III, § 8.5. We could not find the proof of (3) anywhere in the literature, so we give a proof here.

In this case, $A$ an Azumaya algebra with center $\mathcal{O}_T$ where $T \to S$ is a degree-2 étale cover of $S$ with the non-trivial element in its Galois group denoted by $\tau$. Consider the degree-2 étale cover $p : T \times_T T \to T$ obtained via the étale base change $\pi : T \to S$. It follows from Galois theory of schemes that $(T \times_T T, \pi^* \tau) \cong (T \times T, ex)\) (see the proof of Theorem 5.10 in [Len]) where $ex : (x, y) \to (y, x)$. Now $p^* A$ is an Azumaya algebra over $T \amalg T$. Therefore the center $\mathcal{O}_T \times \mathcal{O}_T$ of $p^* A$ contains the idempotent $i = (1, 0)$ where $ex(i) = 1 - i$. Note that $B = i(p^* A)$ an Azumaya algebra over $T$ and we have an isomorphism of Azumaya algebras with unitary involution given by

$$(p^* A, p^* \sigma) \xrightarrow{\cong} (B \times B^{\op}, \epsilon) \quad a \mapsto (i \cdot a, (i \cdot (p^* \sigma(a)))^{\op})$$

By taking a suitable étale covering of $T$ which splits $B$, we obtain (3).
5. The Group scheme of Automorphisms of Azumaya algebras with involution

Let \((A, \sigma)\) be a degree-\(n\) Azumaya algebra with involution of any type over \(S\). Consider the functor

\[
\text{Aut}(A, \sigma) : \text{Sch}_S \rightarrow \text{Groups}
\]

\[
(U \rightarrow S) \mapsto \text{Aut}_{O_{U\rightarrow \text{alg}}}(i^*A, i^*\sigma)
\]

where \(\text{Aut}_{O_{U\rightarrow \text{alg}}}(i^*A, i^*\sigma)\) is the group of \(O_U\)-algebra automorphisms of \(i^*A\) compatible with the involution \(i^*\sigma\).

**Notation:** If a functor \(F\) on \(\text{Sch}_S\) is representable, let us denote the representing scheme by \(F\).

**Theorem 5.1.** The functor \(\text{Aut}(A, \sigma)\) is representable by a smooth, affine group scheme over \(S\).

**Proof.** Let \(S' \rightarrow S\) be an fpqc morphism. Recall that the category of affine \(S\)-schemes is equivalent to the category of affine \(S'\)-schemes with descent data. To see this use [BLR90 Theorem 4, Chapter 6] and the fact that for any scheme \(X\), the category of affine \(X\)-schemes is anti-equivalent to the category of quasi-coherent sheaves of \(O_X\)-algebras ([Sta20 Tag 01SS, Lemma 29.11.5] or see [Vis05 Theorem 4.33]). Moreover, the properties smooth and affine are fpqc local over the base (see [Sta20 Tag 02YJ] or [Gro65 Prop 2.7.1 and Prop 6.8.3]). Therefore by Proposition 4.3 it suffices to prove the theorem when \((A, \sigma)\) is split. So assume that \((A, \sigma)\) is split. Note that in this case the functor

\[
\text{Aut}(A) : (U \rightarrow S) \mapsto \text{Aut}_{O_{U\rightarrow \text{alg}}}(i^*A)
\]

is representable by a closed subscheme of the affine \(\mathbb{Z}\)-scheme \(\text{End}_{O_S\rightarrow \text{mod}}(A) \simeq M_r\) where \(r = \dim_{O_S}(A)\) (see [Mil80 §2, Chapter IV]). Module homomorphisms of \(A\) that respect the involution can be expressed as vanishing of polynomials and hence is representable by a closed subscheme of \(M_r\). The intersection of these two subschemes represents \(\text{Aut}(A, \sigma)\) and hence is an affine scheme over \(S\).

Now we prove that this affine scheme is smooth. For the unitary case, consider the functor

\[
\text{Aut}_{O_S\times O_S}(M_n \times M_n^{op}, \epsilon) : (U \rightarrow S) \mapsto \text{Aut}_{O_{U\times O_{U\rightarrow \text{alg}}}}(M_n(O_U) \times M_n(O_U)^{op}, \epsilon)
\]

\[
\simeq \text{Aut}_{O_{U\rightarrow \text{alg}}}(M_n(O_U))
\]

Hence \(\text{Aut}_{O_S\times O_S}(M_n \times M_n^{op}, \epsilon)\) is representable by a smooth affine \(\mathbb{Z}\)-scheme (see [Mil80 Chapter IV, §2] for affineness and [DG70 Ch. II, §5, 2.7] for smoothness). The functor \(\text{Aut}_{O_S}(O_S \times O_S)\) is representable by the finite group scheme \(\mathbb{Z}/2\mathbb{Z}\). Since representable functors on \(\text{Sch}_S\) are sheaves for the fpqc topology ([Vis05 Theorem 2.55]), we have the following exact sequence of sheaves over \(S\)

\[
1 \rightarrow \text{Aut}_{O_S\times O_S}(M_n \times M_n^{op}, \epsilon) \rightarrow \text{Aut}(M_n \times M_n^{op}, \epsilon) \rightarrow \text{Aut}_{O_S}(O_S \times O_S) \rightarrow 1
\]

(5.1)

Smoothness of \(\text{Aut}(M_n \times M_n^{op}, \epsilon)\) now follows from [SGA3, Exp. VIb, Prop. 9.2].

For the other cases, note that \(\text{Aut}(M_n, sp) \simeq Sp_n/Z\) and \(\text{Aut}(M_n, tr) \simeq O_n/Z\) (as fpqc quotients) where \(Sp_n, O_n\) denote respectively the symplectic group, orthogonal group and \(Z\) their respective centers. Required smoothness results for these group schemes follow from [DG70 Ch. II, §5, 2.7] and [SGA3, Exp. VIb, Prop. 9.2].

**Remark 5.2.** By [Vis05 Theorem 2.55] and Theorem 5.1, \(\text{Aut}(A, \sigma)\) is a sheaf for the fpqc topology.
VI, § 26]. This together with Theorem 5.1 and Proposition 3.5 shows that $S$-group scheme over $[KMRT98, Chapter VI, § 26]$. 

We will now construct the fibered categories of degree-$n$ Azumaya algebras with involution of a given type and the fibered category of rank-$n$ Azumaya algebras with involution of type $\sigma$ of $U$ over $S$. Moreover $Aut^0(A, \sigma)$ is of type $A$ if $\sigma$ is unitary, type $B$ if $\sigma$ is orthogonal and $n$ is odd, type $C$ if $\sigma$ is symplectic and type $D$ if $\sigma$ is orthogonal and $n$ is even.

**Proof.** By Lemma 7.1 below, for every $s \in S$ we have

$$Aut^0(A, \sigma)_s = (Aut(A, \sigma)_s)^0 \simeq Aut(A_{k(s)}, \sigma_{k(s)})^0 = Aut^0(A_{k(s)}, \sigma_{k(s)})$$

(5.2)

Now $Aut^0(A_{k(s)}, \sigma_{k(s)})$ is an absolutely simple adjoint algebraic group over $k(s)$ ([KMRT98, Chapter VI, § 26]). This together with Theorem 5.1 and Proposition 5.5 shows that $Aut^0(A, \sigma)$ is a reductive group scheme over $S$. The rest of the claim follows from (5.2), unravelling the definitions in §3 and [KMRT98 Chapter VI, §26].

**Remark 5.5.** In the above theorem, when $n = 2$, $Aut^0(A, \sigma)$ is not adjoint and when $n = 4$, the fibers of $Aut^0(A, \sigma)$ are not absolutely simple.

**Definition 5.6.** Based on Theorem 5.4 we say that the pair $(A, \sigma)$ is of type $\ast$ where $\ast$ is

- $A$ if $\sigma$ is unitary
- $B$ if $\sigma$ is orthogonal and the degree of $A$ is odd
- $C$ if $\sigma$ is symplectic
- $D$ if $\sigma$ is orthogonal and the degree of $A$ is even

6. The fibered categories $Az^n_\ast$ and $GS^n_\ast$

We will now construct the fibered categories of degree-$n$ Azumaya algebras with involution of a given type and the fibered category of rank-$n$ adjoint group schemes with absolutely simple fibers of a given type over $Sch_S$ and show that they are stacks (in fact gerbes) with respect to the étale topology. The classical reference for stacks and gerbes is [Gir71]. Other references are [Vis05], [Moe] and the Appendix in [DMT18].

**Definition 6.1.** Given a scheme $U$ in the category $Sch_S$, let $Az^n_\ast(U)$ denote the groupoid of degree-$n$ Azumaya algebras with involution of type $\ast$ (where $\ast$ is $A$, $B$, $C$ or $D$) over $U$ (morphisms are isomorphisms of Azumaya algebras over $U$ that respect the involution structures). For any morphism $f : V \to U$ in $Sch_S$ we have a pull-back functor

$$Az^n_\ast(U) \to Az^n_\ast(V)$$

$$(A, \sigma) \mapsto f^*(A, \sigma) := (f^*A, f^*\sigma)$$

which makes the assignment $U \to Az^n_\ast(U)$ a pseudo-functor on $Sch_S$ (see [Vis05], Chapter 3).

**Definition 6.2.** The fibered category of degree-$n$ Azumaya algebras with involution of type $\ast$ associated to the above pseudo-functor is denoted by $Az^n_\ast \to Sch_S$.

**Definition 6.3.** The fibered category $GS^n_\ast \to Sch_S$ of rank-$n$ adjoint group schemes with absolutely simple fibers of type $\ast$ (where $\ast$ is $A$, $B$, $C$ or $D$) is defined in a similar way where morphisms in every fiber $GS^n_\ast(U)$ are isomorphisms.
Proposition 6.4. The fibered categories \( Az^n_s \rightarrow \text{Sch}_S \) and \( GS^n_s \rightarrow \text{Sch}_S \) are stacks for the étale topology. In fact, they are gerbes.

Proof. By standard arguments from decent theory, it is easy to see that \( GS^n_s \rightarrow \text{Sch}_S \) is a stack for the étale topology. It is a gerbe by Proposition 3.3. For the case of \( Az^n_s \), we note from descent theory that quasi-coherent sheaves (as well as the morphisms) satisfy descent for the fqc topology ([BLR90, Chapter 6, Theorem 4]) and hence also for the étale topology. Quasi-coherent sheaves together with additional structure such as an algebra structure and involutions also descend (see for example [Vis05, § 4.2.2 and § 4.2.3] or [KO74, Chapter II, Theorem 3.4]). This shows that \( Az^n_s \rightarrow \text{Sch}_S \) is a stack for the étale topology. It is a gerbe by Proposition 4.3. \( \square \)

7. Equivalence of \( Az^n_s' \) and \( GS^n_s \)

Let \( Az^n_s \rightarrow \text{Sch}_S \) and \( GS^n_s \rightarrow \text{Sch}_S \) be the fibered categories defined in the previous section. In this section we describe a morphism between \( Az^n_s' \) and \( GS^n_s \) (where \( n' \) is determined by \( n \)) that will yield the required equivalence of fibered categories. We will need the following lemma.

Lemma 7.1. Let \((A, \sigma)\) be an Azumaya algebra with involution over a scheme \( X \). The assignment

\[
(A, \sigma) \mapsto \text{Aut}^0(A, \sigma)
\]

respects pull-backs i.e., for \( f : Y \rightarrow X \) we have a canonical isomorphism

\[
f^*(\text{Aut}^0(A, \sigma)) \simeq \text{Aut}^0(f^*(A, \sigma))
\]

Proof. We note that \( f^*(\text{Aut}(A, \sigma)) = \text{Aut}(f^*(A, \sigma)) \). Hence by the Yoneda lemma, there is a canonical isomorphism \( f^*(\text{Aut}(A, \sigma)) \simeq \text{Aut}(f^*(A, \sigma)) \). This together with Proposition 3.4 proves the claim. \( \square \)

Theorem 7.2. The functor

\[
\text{Aut}^0 : Az^n_s' \rightarrow GS^n_s
\]

\[
((A, \sigma) \rightarrow (B, \tau)) \mapsto (\phi \rightarrow i \circ \phi \circ i^{-1})
\]

is an equivalence of fibered categories where

- \( n > 1 \) and \( n' = n + 1 \) if \( * \) is of type \( A \)
- \( n' = 2n + 1 \) if \( * \) is of type \( B \)
- \( n' = 2n \) if \( * \) is of type \( C \)
- \( n > 2, n \neq 4 \) and \( n' = 2n \) if \( * \) is of type \( D \)

Proof. The functor \( \text{Aut}^0 \) defines a morphism of fibered categories by Lemma 7.1. Since \( Az^n_s' \rightarrow \text{Sch}_S \) and \( GS^n_s \rightarrow \text{Sch}_S \) are gerbes by Proposition 6.4 to show that \( \text{Aut}^0 \) is an equivalence it suffices to show that for any object \((A, \sigma)\) in \( Az^n_s' \) (say the split object), \( \text{Aut}^0 \) induces isomorphism of sheaves \( \text{Aut}(A, \sigma) \) and \( \text{Aut}(\text{Aut}^0(A, \sigma)) \) (see [Gir71, Chapter IV, § 3.1] or [Moe]). Now by Theorem 5.1 and Proposition 5.6 both \( \text{Aut}(A, \sigma) \) and \( \text{Aut}(\text{Aut}^0(A, \sigma)) \) are represented by smooth affine group schemes over \( S \) denoted by \( \text{Aut}(A, \sigma) \) and \( \text{Aut}(\text{Aut}^0(A, \sigma)) \) respectively. So it suffices to check that \( \text{Aut}^0 \) induces isomorphism at every fiber ([Sta20, Tag 0393], [Sta20, Tag 025G]). Again by the proof of Lemma 7.1 we see that for every \( s \in S \), \( \text{Aut}^0 \) induces morphism of schemes over \( k(s) \)

\[
\text{Aut}^0_{k(s)} : \text{Aut}(A_{k(s)}, \sigma_{k(s)}) \rightarrow \text{Aut}(\text{Aut}^0(A_{k(s)}, \sigma_{k(s)}))
\]
Now we are in the case of fields and the fact that $\text{Aut}^0_{k(s)}$ is an isomorphism follows from [KMRT98, Chapter VI, § 26].

**Remark 7.3.** In the case of $n = 1$ for type $A$, the above equivalence holds if $A^2_2$ is defined to be the fibered category of Azumaya algebras of degree 2 ([KMRT98, page 366]). Similarly, in the case of $n = 2$ for type $D$, since the corresponding root system is not irreducible, one has to remove the phrase “with absolutely simple fibers” in the definition of $\text{GS}_2^D$ for the above equivalence to hold. The proofs are similar.

**Remark 7.4.** The classification of simply connected group schemes with absolutely simple fibers over $S$ is similar to the adjoint case. The functor in this case is given by

$$\overline{\text{Aut}}^0 : (A, \sigma) \rightarrow \text{Aut}^0(A, \sigma)$$

where for a semisimple group scheme $G$ over $S$, $\overline{G}$ denotes its unique simply connected cover (see [Con14a, Exercise 6.5.2]).

**Remark 7.5.** Recall that when the base scheme is the spectrum of a field of characteristic 2, the groups $\text{Aut}^0(A, \sigma)$ are not smooth when $\sigma$ is orthogonal ([KMRT98, page 347]). Hence when 2 is not invertible in $S$, things look a little different for types $B$ and $D$. One could possibly work fppf locally instead of etale locally for these types to get a similar classification as in the case of fields ([Con14b, Lemma C.2.1]). We did not want to deal with these subtle technicalities in this paper that will take away the nice picture when 2 is invertible.

### 8. Applications

In this section we show some interesting consequences of Theorem 7.2.

For a ring $R$, let $P_{n'}(R, \ast)$ denote the isomorphism classes of degree-$n'$ Azumaya algebras with involution of type $\ast$ over $R$.

**Corollary 8.1.** Let $R$ be a Henselian local ring with residue field $k$. Assume $\text{char } k \neq 2$. Then the restriction map

$$P_{n'}(R, \ast) \rightarrow P_{n'}(k, \ast)$$

is bijective for every $(n', \ast)$ listed in Theorem 7.2.

**Proof.** This follows from Lemma 7.1, Theorem 7.2 and [SGA3, Exp. XXIV, Prop. 1.21].

**Remark 8.2.** The above corollary is a generalization of a similar statement about isomorphism classes of Azumaya algebras. See Theorem 6.1 in [Gro95].

We will recall the concept of *good reduction* for algebraic groups defined in [CRR19]. Let $k$ be a discrete valued field with valuation $v$. Let $k_v$, $\mathcal{O}_v$ and $k(v)$ denote respectively the completion of $k$, the valuation ring of $k_v$ and the residue field. An absolutely almost simple linear algebraic group $G$ over $k$ is said to have *good reduction at $v$* if there exists a reductive group scheme $\mathcal{G}$ over $\mathcal{O}_v$ such that $\mathcal{G} \otimes_{\mathcal{O}_v} k_v \simeq G \otimes_k k_v$. In this case, let us call $\mathcal{G}$ an $\mathcal{O}_v$-*model* for $G$. Studying good reduction of $G$ has many applications such as computing the genus of algebraic group and proving Hasse principles. We refer the reader to [CRR19] for more details.

**Corollary 8.3.** Let $G$ be a rank-$n$ absolutely simple adjoint (or simply connected) algebraic group of classical type ($n \neq 4$ if $G$ is of type $D$) over a discrete valued field $k$ with valuation $v$. Assume that
the characteristic of the residue field $k^{(v)}$ is different from 2. Suppose $G$ has good reduction at $v$. Then any two $O_v$-models of $G$ are isomorphic. In other words, an $O_v$-model of $G$ if it exists is unique up to isomorphism.

Proof. This follows from Lemma[7.1] Theorem[7.2] and [BVG14 Theorem 3.7].

Remark 8.4. The classification in Theorem[7.2] can be used to translate the notion of good reduction of absolutely simple adjoint (or simply connected) classical algebraic groups to the notion of good reduction of the underlying sesquilinear forms over division algebras. See for example [Sri20, Remark 4.3].

Let $R$ be a regular local ring and let $G$ be a reductive groups scheme over $R$. A conjecture of Grothendieck and Serre ([Gro58 pp. 26-27, Remark 3], [Gro68 Remark 1.11(a)] and [Ser58 pp. 31, Remark]) states that rationally trivial principal $G$-homogeneous spaces are trivial i.e., the kernel of the canonical map

$$H^1(R, G) \to H^1(K, G)$$

where $K$ is the fraction field of $R$, is trivial. While the conjecture is still open to be proved in complete generality, the proofs for various cases of $R$ and $G$ have been established since Nisnevich’s thesis ([Nis82]). If $R$ is a regular local ring containing a field of characteristic $\neq 2$ or if $R$ is a semilocal Bézout domain with 2 invertible, I.Panin ([Pan03 Theorem 1.1]) and S.Beke ([BVG14 Theorem 3.7]) respectively have proved that any two Azumaya algebras with involutions over $R$ that are rationally isomorphic (i.e., isomorphic over the fraction field of $R$) are already isomorphic. This implies that for the above cases of $R$, the kernel of the map in (8.1) is trivial when $G \simeq \text{Aut}(A, \sigma)$, the automorphism scheme of an Azumaya algebra with involution over $R$. Since any rank-$n$ adjoint group scheme of classical type ($n \neq 4$ if the group is of type $D$) over $R$ with absolutely simple fibers is isomorphic to $\text{Aut}^0(A, \sigma)$ for some $(A, \sigma)$ by Theorem[7.2] we conclude:

Corollary 8.5. Let $R$ be a integral domain with 2 invertible that satisfies the property that any Azumaya algebra with involution over $R$ that is split over the fraction field of $R$ is already split. Then the Grothendieck-Serre conjecture is true for any rank-$n$ adjoint group scheme of classical type ($n \neq 4$ if the group is of type $D$) with absolutely simple fibers over $R$. In particular, this happens when $R$ is a regular local ring containing a field of characteristic $\neq 2$ or when $R$ is a semilocal Bézout domain with 2 invertible.

9. Acknowledgements

The author acknowledges the support of the DAE, Government of India, under Project Identification No. RTI4001. She thanks Jonathan Wise for pointing to an important reference and also thanks the anonymous referee for their useful comments and suggestions.

References

BLR90  Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. Néron models, volume 21 of Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Springer-Verlag, Berlin, 1990.

BVG14  Sofie Beke and Jan Van Geel. An isomorphism problem for Azumaya algebras with involution over semilocal Bézout domains. Algebr. Represent. Theory, 17(6):1635–1655, 2014.

Con14a  Brian Conrad. Reductive group schemes. In Autour des schémas en groupes. Vol. I, volume 42/43 of Panor. Synthèses, pages 93–444. Soc. Math. France, Paris, 2014.
S. SRIMATHY

Con14b Brian Conrad. Reductive group schemes. 2014.

CRR19 Vladimir I. Chernousov, Andrei S. Rapinchuk, and Igor A. Rapinchuk. Spinor groups with good reduction. *Compos. Math.*, 155(3):484–527, 2019.

Dem65 Michel Demazure. Schémas en groupes réductifs. *Bull. Soc. Math. France*, 93:369–413, 1965.

DG70a M. Demazure and A. Grothendieck. *Schémas en groupes (SGA 3), I, II, III.* Lecture Notes in Math 151, 152, 153. Springer-Verlag, New York, 1970.

DG70b Michel Demazure and Pierre Gabriel. *Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs.* Masson & Cie, Éditeur, Paris; North-Holland Publishing Co., Amsterdam, 1970. Avec un appendice et Corps de classes local par Michiel Hazewinkel.

DM18 P. Deligne and J.S. Milne. Tannakian categories. *https://www.jmilne.org/math/xnotes/tc2018.pdf*, 2018.

FW20 Uriya A. First and Ben Williams. Involutions of Azumaya algebras. *Doc. Math.*, 25:527–633, 2020.

Gir71 Jean Giraud. *Cohomologie non abélienne.* Springer-Verlag, Berlin-New York, 1971. Die Grundlehren der mathematischen Wissenschaften, Band 179.

Gro58 Alexander Grothendieck. Torsion homologique et sections rationnelles. *Séminaire Claude Chevalley*, 3, 1958. talk:5.

Gro65 A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II. *Inst. Hautes Études Sci. Publ. Math.*, (24):231, 1965.

Gro68 Alexander Grothendieck. Le groupe de Brauer. II. Théorie cohomologique. In *Dix exposés sur la cohomologie des schémas*, volume 3 of *Adv. Stud. Pure Math.*, pages 67–87. North-Holland, Amsterdam, 1968.

Gro95 Alexander Grothendieck. Le groupe de Brauer. I. Algèbres d’Azumaya et interprétations diverses [MR0244269 (39 #5586a)]. In *Séminaire Bourbaki, Vol. 9*, pages Exp. No. 290, 199–219. Soc. Math. France, Paris, 1995.

KMRT98 Max-Albert Knus, Alexander Merkurjev, Markus Rost, and Jean-Pierre Tignol. *The book of involutions*, volume 44 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1998. With a preface in French by J. Tits.

Knu91 Max-Albert Knus. *Quadratic and Hermitian forms over rings*, volume 294 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1991. With a foreword by I. Bertuccioni.

KO74 Max-Albert Knus and Manuel Ojanguren. *Théorie de la descente et algèbres d’Azumaya*. Lecture Notes in Mathematics, Vol. 389. Springer-Verlag, Berlin-New York, 1974.

KPS90 M.-A. Knus, R. Parimala, and V. Srinivas. Azumaya algebras with involutions. *J. Algebra*, 130(1):65–82, 1990.

Len H. L. Lenstra. Galois theory for schemes. *Preprint (Univ. Leiden 2008), https://websites.math.leidenuniv.nl/algebra/GSchemes.pdf*.

Mil80 James S. Milne. *Étale cohomology*, volume 33 of *Princeton Mathematical Series*. Princeton University Press, Princeton, N.J., 1980.

Moe I. Moerdijk. Introduction to the language of stacks and gerbes. *Preprint, https://arxiv.org/abs/math/0212266*.

Nis82 Yevsey A. Nisnevich. *Étale cohomology and arithmetic of semisimple groups*. ProQuest LLC, Ann Arbor, MI, 1982. Thesis (Ph.D.)—Harvard University.

Pan03 Ivan Panin. Purity for similarity factors. *https://arxiv.org/pdf/math/0309054.pdf*, 2003.

PS92 R. Parimala and V. Srinivas. Analogues of the Brauer group for algebras with involution. *Duke Math. J.*, 66(2):207–237, 1992.

Ser58 Jean-Pierre Serre. Espaces fibrés algébriques. *Séminaire Claude Chevalley*, 3, 1958. talk:1.

Sri20 Srimathy Srinivasan. A finiteness theorem for special unitary groups of quaternionic skew-hermitian forms with good reduction. *Doc. Math.*, 25:1171–1194, 2020.
AZUMAYA ALGEBRAS WITH INVOLUTION AND CLASSICAL SEMISIMPLE GROUP SCHEMES

Sta20 The Stacks project authors. The stacks project. [https://stacks.math.columbia.edu], 2020.

Vis05 Angelo Vistoli. Grothendieck topologies, fibered categories and descent theory. In Fundamental algebraic geometry, volume 123 of Math. Surveys Monogr., pages 1–104. Amer. Math. Soc., Providence, RI, 2005.

Wei60 André Weil. Algebras with involutions and the classical groups. J. Indian Math. Soc. (N.S.), 24:589–623 (1961), 1960.

S. Srimathy
School of Mathematics, Tata Institute of Fundamental Research, Mumbai, India, email: srimathy@math.tifr.res.in