An inhomogeneous solution of Einstein equations with viscous fluids

Z. Haba
Institute of Theoretical Physics, University of Wroclaw, 50-204 Wroclaw, Plac Maxa Borna 9, Poland

Assuming conformally flat metric we obtain inhomogeneous solutions of Einstein equations with the energy-momentum of a viscous fluid. We suggest that the viscous solution can be applied as a model of an expanding inhomogeneous dark energy.

I. INTRODUCTION

Einstein equations are studied with various assumptions on the energy-momentum of “matter”. The quantum form of the equations with the energy-momentum of the fields of the standard model cannot be realized because of the difficulties with the quantization of the gravitational field. We must use approximations for the rhs of Einstein equations. The most studied models involve a perfect fluid on the rhs of Einstein equations. The solutions can describe an expanding universe or compact objects like the static as well as expanding and collapsing stars [1][2] (for inhomogeneous solutions and their relevance in gravity see [3][4][5][6][7]). However, it is obvious that the assumption of a perfect fluid on the rhs of Einstein equations, usually applied in gravitational models, is an idealization. All physical fluids at high temperature have a non-zero viscosity. It is not simple to include viscosity in the solutions of Einstein equations. Concerning the homogeneous solutions, it is known that the FLWR form of the solution admits only the bulk viscosity. The effect of the bulk viscosity on the expansion of the universe has been discussed in [2][8][9][10]. The introduction of the bulk viscosity modifies the formula for the pressure which leads to interesting reformulations in the dynamics and thermodynamics depending on the change of the equation of state [9][10][11]. There are many examples of inhomogeneous solutions of Einstein equations with a perfect fluid on the rhs [3][4]. However, it seems that no examples are known of inhomogeneous solutions with a shear viscosity of the fluid. Recently the relativistic fluid equations have been studied in the heavy-ion physics (with a suggestion of a possible simulation of the Big Bang, see the review [12]). Some solutions of the hydrodynamics equations for the perfect fluid have been obtained, (see e.g., [13]). Little is known about solutions with viscosity (see however [14]). Another approach to such models is developed in refs.[15][16][17] where the Einstein tensor for various metrics is interpreted as an energy-momentum tensor of a fluid (including possibly a viscous fluid).

In this paper we assume that the metric is conformally flat. A large class of models can be expressed in a conformally flat form including the FLWR solutions [18] [19]. However, the FLWR metric admits only bulk viscosity of the fluid. We obtain conformally flat inhomogeneous solutions with a non-zero shear and bulk viscosity which satisfies the dark energy equation of state \( \rho = -p \). It is known that if the equation of state for the energy-momentum of a perfect fluid has the form \( \rho = -p \) then from the conservation law it follows that \( p = \text{const} \). In such a case the dark energy is just the cosmological constant. We obtain a solution in the form of an expanding viscous fluid. The expansion is at lower rate than the expansion of radiation and of the dust. The resulting viscous fluid could be a candidate for a dark energy dominating the energy-momentum at large time.

II. THE ENERGY-MOMENTUM TENSOR

We consider Einstein equations

\[ G_{\mu\nu} = 8\pi GT_{\mu\nu}, \]  

where \( G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \) is the Einstein tensor and \( T_{\mu\nu} \) is the energy-momentum. We set the velocity of light \( c=1 \) and the length will be measured in Planck units \( \sqrt{\frac{8\pi G}{\hbar}} \) (so coordinates will be dimensionless), hence we set \( \sqrt{8\pi G} = 1 \) from now on. In the case of a dust with a density \( \rho \) the energy-momentum \( T_{\mu\nu} = \rho v_\mu v_\nu \) is conserved \( \nabla_\mu T^{\mu\nu} = 0 \) (together with the current \( \nabla_\mu (\rho v^\mu) = 0 \)), where the relativistic velocity \( v_\mu \) satisfies the equation

\[ g^{\mu\nu} v_\mu v_\nu = 1. \]  

We can apply the Hamilton-Jacobi theory to express the velocity by the action \( S \), \( v_\mu = \frac{\partial S}{\partial \pi^\mu} \). Then, equation (2) reads

\[ g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} = 1 \]  

Eqs.(1) and (3) can be considered as a system of equations which determine the metric \( g_{\mu\nu} \) and the fluid velocities \( v_\mu \). We extend this scheme to the viscous energy-momentum

\[ T_{\mu\nu} = (\rho + p) v_\mu v_\nu - g_{\mu\nu} p - \eta (\nabla_\mu v_\nu + \nabla_\nu v_\mu) - \gamma g_{\mu\nu} g^{\alpha\beta} \nabla_\alpha v_\beta, \]

where \( p \) is the pressure. In physical fluids the shear \( \eta \) and bulk \( \gamma \) viscosities depend in an involved way on the

*Electronic address: zbigniew.haba@uwr.edu.pl
temperature and density of the fluids. In our solution the viscosities depend on the metric through its scale factor (the viscosities are decreasing with an expansion of the fluid).

We use the decomposition

\[ g^{\alpha \beta} = \epsilon^{\alpha \beta} + H^{\alpha \beta} \]

where

\[ H^{\alpha \beta} = g^{\alpha \beta} - \epsilon^{\alpha \beta} \]

in order to rewrite the energy-momentum (4) in the standard form of Landau-Lifshitz [20] and Weinberg [2][21]

\[ T^{\mu \nu} = (\rho + p_{c}) \epsilon^{\mu \nu} - p_{c} g^{\mu \nu} \]

\[ - \eta_{a} H^{\mu a} H^{\nu \beta} (\nabla_{\alpha} v_{\beta} + \nabla_{\beta} v_{\alpha} - \frac{2}{3} g_{a \beta} g^{\sigma \lambda} \nabla_{\sigma} v_{\lambda}) \]

\[ - \gamma_{a} H^{\mu \nu} g^{\alpha \beta} \nabla_{\alpha} v_{\beta} - \kappa_{a} (H^{\alpha \mu} \nu^{\nu} + H^{\nu \nu} \nu^{\nu}) Q_{\lambda} \]

where the heat current \( Q \) is

\[ Q_{\lambda} = \partial_{\lambda} T_{g} + T_{g} \epsilon^{\alpha \nu} \nabla_{\alpha} v_{\lambda} \]  

with the heat conductivity \( \kappa_{g} \) and temperature \( T_{g} \) satisfying the relations \( \partial_{\lambda} T_{g} = 0 \) and \( \kappa_{g} T_{g} = \eta_{g} \). So,

\[ \kappa_{g} Q_{\lambda} = \eta_{g} \epsilon^{\alpha \nu} \nabla_{\alpha} v_{\lambda} \]

The effective pressure and density are

\[ p_{e} = p - \frac{2}{3} \eta_{g} g^{\alpha \beta} \nabla_{\alpha} v_{\lambda} \]

\[ \rho_{e} = \rho - \gamma_{g} g^{\sigma \lambda} \nabla_{\sigma} v_{\lambda} \]  

We shall require for our solution in sec.4 that \( \rho + p = 0 \)

\[ \rho_{e} + p_{e} = \frac{2}{3} \eta_{g} \gamma_{g} g^{\sigma \lambda} \nabla_{\sigma} v_{\lambda} \]  

As is well-known [20][2] the form (5) of the energy-momentum with positive \( \eta, \kappa \) and \( \gamma \) ensures the increase of the entropy.

If eqs.(1) are to be non-contradictory we must have

\[ g^{\alpha \nu} \nabla_{\alpha} T_{\mu \nu} = 0. \]  

Eqs.(10) can be considered as differential equations (relativistic Navier-Stokes equations) relating \( p, p, \) and \( v \).

Eqs.(10) with the metric \( g_{\mu \nu} \) solving Einstein equations (1) follow from the Einstein equations. However, when \( v_{\alpha} \) satisfy (10) on a certain manifold with the metrics \( g_{\mu \nu} \) then only a subset of \( g_{\mu \nu}, v_{\alpha} \) will satisfy both eq.(10) and eq.(1).

### III. THE CONFORMALLY FLAT METRIC

We consider the conformally flat metric in four space-time dimensions

\[ ds^{2} = a(x)^{2}(d\tau^{2} - dx^{2}) = a(x)^{2} \eta^{\mu \nu} dx_{\mu} dx_{\nu}, \]  

where \( \eta^{\mu \nu} \) is the Minkowski metric. Then, the Einstein tensor can be expressed in the form [22][3][23]

\[ G_{\mu \nu} = (\tilde{\rho} + \tilde{p}) \tilde{u}_{\mu} \tilde{u}_{\nu} - \tilde{g}_{\mu \nu} + \Pi_{\mu \nu}, \]

where the fluid velocity is defined by

\[ \tilde{u}_{\mu} = \partial_{\mu} a (g^{\mu \nu} \partial_{\nu} a \partial_{\nu} a)^{-\frac{1}{2}}. \]

The energy density is

\[ \tilde{\rho} = 3a^{-2} g^{\mu \nu} \partial_{\mu} a \partial_{\nu} a - g^{\mu \nu} \Pi_{\mu \nu}, \]

The pressure

\[ \tilde{p} = a^{-2} g^{\mu \nu} \partial_{\mu} a \partial_{\nu} a + g^{\mu \nu} \Pi_{\mu \nu}, \]

where

\[ \Pi_{\mu \nu} = -2a^{-1} \partial_{\mu} a \partial_{\nu} a. \]

Inserting the velocities (13) in eq.(12) we can also express \( G_{\mu \nu} \) in the form

\[ G_{\mu \nu} = 4a^{-2} \partial_{\mu} a \partial_{\nu} a - \tilde{p} g_{\mu \nu} + \Pi_{\mu \nu}, \]

Eqs.(12)-(16) suggest that \( \tilde{u}_{\mu} \sim \partial_{\mu} a \) can be a solution of Einstein equations (1). In the next section we show that this is really the case.

### IV. EINSTEIN EQUATIONS WITH A VISCOUS FLUID

In eq.(4) the covariant derivative is expressed by the Christoffel connection in conformally flat space [22](from now on the indices will be raised by means of the Minkowski metric)

\[ \Gamma_{\mu \nu}^{\lambda} = \delta_{\mu}^{\lambda} \partial_{\nu} a + \delta_{\nu}^{\lambda} \partial_{\mu} a - \eta_{\mu \nu} \partial^{\lambda} \ln a \]

It is easy to see that the equation

\[ G_{0 j} = T_{0 j} = -\eta_{g} (\nabla_{j} v_{0} + \nabla_{0} v_{j}) \]

is satisfied if the equation of state is

\[ \rho + p = 0 \]

the velocity

\[ v_{\mu} = \frac{\partial}{\partial x^{\mu}} a \]

and

\[ \eta_{g} = a^{-1}. \]

However, if the solution (21) of eq.(19) is to be the relativistic velocity then the normalization (2) must be satisfied i.e. \( g^{\mu \nu} \partial_{\mu} a \partial_{\nu} a = 1 \). We write \( a = \exp(\psi) \). Then,
on the basis of eqs.(2) and (21) \( \psi \) satisfies the Hamilton- Jacobi equation
\[ \eta^{\mu \nu} \partial_{\mu} \psi \partial_{\nu} \psi = 1. \] (23)
This is the Hamilton-Jacobi equation (as described at eq.(3)) for a free particle (with the unit mass) in the Minkowski space (its velocity is \( v_{\mu} = \partial_{\mu} \psi \)).

The remaining components of the energy-momentum tensor are
\[ T_{00} = -pa^2 - 2a^{-1} \nabla_0 v_0 - \gamma g(\nabla_0 v_0 - \nabla_k v_k), \] (24)
where
\[ \nabla_0 v_0 - \nabla_k v_k = \partial_0 v_0 - \partial_k v_k + 2a^{-1} v_0 \partial_0 a - 2a^{-1} v_k \partial_k a \] (25)
and
\[ T_{jk} = \delta_{jk} \left( pa^2 - 2(a^{-2} \partial_0 a v_0 - a^{-2} \partial_0 a v_0) - \gamma g(\partial_0 v_0 - \partial_j v_j + 2a^{-1} v_0 \partial_0 a - 2a^{-1} v_k \partial_k a) - a^{-1}(\partial_j v_k + \partial_k v_j - 2a^{-1} \partial_j a v_k - 2a^{-1} \partial_k a v_j) \right). \] (26)

It can be seen that the non-diagonal parts of \( G_{jk} \) (17) and \( T_{jk} \) (26) coincide if the velocity is determined by eq.(21) and \( \eta_{\mu \nu} = a^{-1} \) (eq.(22)). There remain the diagonal (\( \delta_{jk} \)) parts of eqs.(26) and (17). They are equal if
\[ p = (-3a\gamma + 1) - (a\gamma + 2)\partial^{\mu} \partial_{\mu} \psi \exp(-2\psi). \] (27)
The 00-equation as derived from eqs.(24) and (17) is satisfied if
\[ -p = (1 + 2a\gamma a^{-2} \eta^{\mu \nu} \partial_\mu a \partial_\nu a + (2 + a\gamma) a^{-1} \eta^{\mu \nu} \partial_\mu \partial_\nu a \] (28)
It coincides with eq.(27) (which followed from the spatial diagonal part of eq.(4)). So Einstein equations determine \( p \) and \( v_{\mu} \) in terms of \( \psi \) satisfying the Hamilton-Jacobi equation (23). The relativistic Navier-Stokes equations for \( v_{\mu} \) follow from eq.(10). These equations can be obtained simply by differentiation of eq.(28) and an insertion of \( \partial_\mu a = v_{\mu} \) because such a velocity is the solution of these equations.

The energy and pressure are functions of the solution \( \psi \) of eq.(23). A construction of the general solution in terms of characteristics is discussed in mathematical literature [24]. The solution can also be obtained by a calculation of the action for the relativistic Hamiltonian \( \sqrt{1 + \nu^2} \) [25]. Let us consider some special cases. \( \psi(x) = q_{\mu} x^{\mu} \) with \( q_{\mu} q_{\nu} = 1 \) is the solution of eq.(23). The next example is \( \psi(x) = \sqrt{x^2} \) where \( x^2 = x_\mu x^\mu \). It is the solution of eq.(23) for time-like \( x \). However, let us note that in these cases the viscosity term in eq.(17) (as well as eq.(2))can be written in the form
\[ \Pi_{\mu \nu} = (\delta \rho + \delta p) v_{\mu} v_{\nu} - \delta p g_{\mu \nu}. \]
with certain \( \delta \rho \) and \( \delta p \). Hence, the viscosity terms only modify the definition of \( \rho \) and \( p \). In the first case \( \nabla_\nu v_\mu + \nabla_\mu v_\nu \approx q_\mu q_\nu \approx v_\mu v_\nu \) (the perfect fluids of ref.[23]). In the second case \( \nabla_\nu v_\mu + \nabla_\mu v_\nu \approx v_\mu v_\nu + \eta_{\mu \nu} \delta p \).

A non-trivial viscosity results from the solution
\[ \psi = \sqrt{C_2 - \tau - \tau \sqrt{C_1 - \tau + r}}, \]
where \( r = |x| \). It solves eq.(23) if \( \tau < C_1 + r \) and \( \tau < C_2 - r \). We can get a solution for a large time \( \tau > C_1 + r \) and \( \tau > -C_2 + r \) if we write \( \psi \) in the form
\[ \psi = \sqrt{-C_2 + \tau + r \sqrt{C_1 + \tau - r}}, \] (29)
If \( C_1 = C_2 = 0 \) then we return to the solution \( \sqrt{x^2} \).

We obtain a local solution of eq.(23) depending on all variables if we treat eq.(23) as the Hamilton-Jacobi equation for the Hamiltonian \( \sqrt{1 + \nu^2} \). Through the separation of variables in cylindrical coordinates \( (r, \phi, z) \) we obtain
\[ \psi = -E\tau + L\phi + f(r) + \gamma z \] (30)
with the constant angular velocity \( v_{\phi} = L \) and radial velocity
\[ v_r = \frac{df}{dr} = \sqrt{E^2 - 1 - \gamma^2 - \frac{L^2}{r^2}}. \]
Hence,
\[ f(r) = L\sqrt{E^2 - 1 - \gamma^2 - \frac{L^2}{r^2}} + L \arctan \sqrt{\frac{1}{E^2 - 1 - \gamma^2 - \frac{L^2}{r^2}}}. \]
\( \psi \) of eq.(30) solves eq.(23) if \( r^2 \geq L^2(E^2 - 1 - \gamma^2)^{-1} \geq 0 \). This local solution of eq.(23) does not define \( a(\psi) = \exp(\psi) \) for all \( 0 \leq \phi \leq 2\pi \) because \( \psi(\phi + 2\pi) \neq \psi(\phi) \). However, the local coordinates define the Christoffel symbols (18) depending on \( \partial_\psi \psi \) which are functions only on \( r \). As a consequence the Riemann tensor depends solely on \( r \) and is well-defined if \( r^2 \geq L^2(E^2 - 1 - \gamma^2)^{-1} \geq 0 \). \( v_r = \partial_r \psi \) of eq.(30) solve Einstein equations (1) (which depend only on derivatives of \( \psi \)). We expect that there exists another system of neighborhoods with coordinates which extend the cylindrical coordinates \( (r, \phi, z) \). However, the range of small \( r \) seems to be an essential singularity of the solution (like \( r = 0 \) of the Schwarzschild solution). The function (30) describes the solution of the Hamilton-Jacobi equation corresponding to a relativistic particle moving in the Minkowski space with a forbidden region of large \( L^2 r^{-2} \) [26].

V. NON-SINGULAR SOLUTION WITH THE VISCOUS FLUID

We consider a special case of eq.(30) \( (L = \gamma = 0, \) spherical coordinates with \( \theta = \frac{\pi}{2} \) and \( r = \sqrt{x^2 + y^2 + z^2} \).
A spherically invariant solution of eq. (23) which makes sense for arbitrary \((\tau, r)\) can be expressed as

\[
\psi = \cosh(\alpha)\tau + \sinh(\alpha)r
\]

with an arbitrary real \(\alpha\).

For a better physical interpretation let us change co-
ordinates \((\tau, r) \to (t, R)\)

\[
t = \exp(\tau \cosh(\alpha) + r \sinh(\alpha)) = a
\]

\[
R = \tau \cosh(\alpha) + r \frac{\cosh^2(\alpha)}{\sinh(\alpha)}.
\]

Then, the metric is

\[
ds^2 = dt^2 - t^2 \frac{\cosh^2(\alpha)}{\sinh^2(\alpha)} dr^2 - t^2 \sinh^2(\alpha) (R - \ln(t))^2 (d\theta^2 + \sin^2 \theta d\phi^2).
\]

We can imbed this universe in the Minkowski space-time
\((\tilde{\tau}, \tilde{x})\) with \(\tilde{x} \geq |\tilde{x}|\) introducing the coordinates

\[
\tilde{\tau} = t \cosh(R \coth(\alpha)),
\]

\[
\tilde{x} = t \sinh(R \coth(\alpha)).
\]

Then

\[
ds^2 = d\tilde{\tau}^2 - d\tilde{x}^2 - \frac{1}{\tilde{\tau}}(d\tilde{\tau} - \tilde{x})^2 \sinh^2(\alpha) \ln^2 D(d\theta^2 + \sin^2 \theta d\phi^2),
\]

where

\[
D = (\tilde{\tau} - \tilde{x})^{1+q}(\tilde{\tau} + \tilde{x})^{1-q} = (\tilde{\tau}^2 - \tilde{x}^2) \exp(2q\tilde{\eta}),
\]

where we introduced the space-time rapidity \(\tilde{\eta}\) (often applied in a description of heavy ion-collisions [12])

\[
\tilde{\eta} = \frac{1}{2} \ln \left( (\tilde{\tau} - \tilde{x})(\tilde{\tau} + \tilde{x})^{-1} \right)
\]

and

\[
q = \tanh(\alpha).
\]

In these coordinates the radial velocity (the remaining components of the velocity are zero) has the simple form

\[
v_r = \sqrt{\tilde{\tau}^2 - \tilde{x}^2} \sinh(\alpha).
\]

From eq. (27) the energy density of the fluid is

\[
\rho = (\tilde{\tau}^2 - \tilde{x}^2)^{-1} \left( 3(1 + a\gamma_g) + 4(2 + a\gamma_g)(\ln(D))^{-1} \right).
\]

The density of the gravitational momentum (only the component with radial index is different from zero) is

\[
T_0^r = a^{-4}(4\partial_0 a\partial_r a - 2a\partial_0 \partial_r a)
\]

\[
= (\tilde{\tau}^2 - \tilde{x}^2)^{-1} 4 \cosh(\alpha) \sinh(\alpha) \simeq v_r^{-2}.
\]

VI. SUMMARY

We have derived a solution of Einstein equations with a viscous fluid which is different from the well-known homogeneous solutions with a perfect fluid. This kind of matter could be a constituent of the models of the universe or could exist in the form of (expanding) galaxies. The fluid density is decreasing like \(a^{-2}\) (if we assume that the bulk viscosity behaves as \(\gamma_g \simeq a^{-1}\) like the shear viscosity) with logarithmic corrections increasing the decay in comparison to the coasting cosmology [27]. A contribution of such a fluid to the total energy density (consisting of radiation, dark matter and "baryons") becomes relevant for a large time. It can be applied in models attempting to explain the coincidence problem.
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