Hamiltonian models of multiphoton processes and four–photon squeezed states via nonlinear canonical transformations

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We introduce nonlinear canonical transformations that yield effective Hamiltonians of multiphoton down conversion processes, and we define the associated non–Gaussian multiphoton squeezed states as the coherent states of the multiphoton Hamiltonians. We study in detail the four–photon processes and the associated non–Gaussian four–photon squeezed states. The realization of squeezing, the behavior of the field statistics, and the structure of the phase space distributions show that these states realize a natural four–photon generalization of the two–photon squeezed states.

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Squeezed states [1] represent a remarkable improvement in interferometry, providing an example of clearly nonclassical states whose experimental realization is more readily accessible compared to that, e.g., of the number states. As is well known, squeezed states are minimum uncertainty states with unbalanced quantum noise on conjugate observables, tuned by the squeezing parameter. They are typically generated by two-photon parametric down conversion processes in media with second–order optical nonlinearity. In these systems, at variance with the case of one–photon processes, the modes of the electromagnetic field are not independently excited but instead coupled in pairs. As a consequence, the two–photon process is more adequately described by the field quadratures rather than by the single field mode operator (the annihilation operator) [2]. In recent years, different types of nonclassical states and generalizations of the squeezed states have been studied intensively in quantum optics as well as in atomic physics and in many other physical systems [3]. A challenging, yet so far elusive goal has been to find a multiphoton generalization of the effective two–photon Hamiltonian model for degenerate down conversion processes [4]. Multiphoton processes are in fact becoming of current interest in the study of the foundations of quantum mechanics. In particular, four–photon processes can be implemented by entangling pairs of two–photon squeezed states in a beam splitter [5], while high fidelity teleportation and the realization of EPR and GHZ states via four–photon processes have been recently demonstrated experimentally [6].

In this letter we introduce an effective Hamiltonian model of (degenerate) multiphoton down conversion processes and multiphoton squeezed states. The model is obtained by generalizing the canonical transformations originally exploited in [4] to define the two–photon squeezed states. The generalized transformations achieve the threefold goal of preserving the canonical commutation relations, of providing multiphoton down–conversion Hamiltonians, and of realizing squeezing.

We show that these transformations exist and are obtained by adding to the linear Bogoliubov transformation arbitrary but sufficiently regular nonlinear functions of one of the two field quadratures, i.e. the fundamental operators associated to two–photon processes [2]. Generalized canonical transformations yielding multiphoton processes have been first introduced in [6], with the nonlinearity placed on the first field quadrature \( \hat{X}_1 \), and the lowest order relevant case of four–photon processes has been discussed as well. However, the photon number distribution and the phase space distributions of the associated four–photon squeezed states differ radically from those of the two–photon squeezed states already for very small values of the strength of the nonlinearity (i.e. the strength of the multiphoton contributions).

Here we define a different class of four–photon squeezed states (FPSs). They are obtained by introducing a nonlinear canonical transformation with a quadratic nonlinearity placed on the second field quadrature \( \hat{X}_2 \), and by antisqueezing it. We show that these states, although obtained by a method similar to that exploited in [4], are physically very distinct from those with nonlinearity on \( \hat{X}_1 \), and are the proper four–photon analog of the two–photon squeezed states concerning the relevant physical aspects: down conversion Hamiltonian and squeezing, field statistics, and phase space distributions.

Let us consider a single mode \( \hat{a} \) of the electromagnetic field. Canonical transformations involving higher powers of \( \hat{a} \) and \( \hat{a}^\dagger \) can be defined, although not directly in terms of the field modes, but rather of the field quadratures. There are only two possible one–mode generalized canonical transformations, and they read:

\[
\hat{b}_i = \mu a + i\nu \hat{a}^\dagger + \gamma F(\hat{X}_i), \quad i = 1, 2.
\]

Here, \( a, a^\dagger \) denote the one–mode fundamental canonical variables \( \langle [a, a^\dagger] = 1 \rangle \), \( \hat{b}_i \) denotes the transformed mode, \( F \) is a sufficiently regular, Hermitian function of one of the field quadratures \( \hat{X}_1 = (a + a^\dagger)/\sqrt{2}, \hat{X}_2 = -(a - a^\dagger)/\sqrt{2} \), and \( \mu, \nu, \gamma \) are complex coefficients, where \( \gamma \) is the “coupling” parameter measuring the strength of the nonlinearity. One transformation is generated by \( F(\hat{X}_1) \), the other by \( F(\hat{X}_2) \). Accordingly, the canonical constraint \( [\hat{b}_i, \hat{b}_j^\dagger] = 1 \) on the transformed modes yields two different sets of conditions on the coefficients of the transformations. The first condition is common to the two sets; it is the standard Bogoliubov constraint \( |\mu|^2 - |\nu|^2 = 1 \). The second condition, which remarkably does not depend on the form of the operatorial function \( F \), is either \( Re(\mu \gamma^* - \nu^* \gamma) = 0 \), or \( Im(\mu \gamma^* - \nu^* \gamma) = 0 \), depending on whether the nonlinearity is placed on \( \hat{X}_1 \) or on \( \hat{X}_2 \),
respectively. Here $Re(\cdot)$, $Im(\cdot)$ denote the real and the imaginary parts. The two transformations (I) yield two different one–mode multiphoton Hamiltonians $H_F^i = b_i^\dagger b_i + 1/2$ and $H_F^{II} = b_1^\dagger b_2 + 1/2$. It is instructive to write them in terms of the field quadratures. Let us denote by $X_i$ the quadrature argument of the nonlinear function $F$, and by $X_j$ the remaining quadrature ($i, j = 1, 2$, $i \neq j$). Without loss of generality we adopt the parameterization $\mu = \cosh r$, $\nu = \sinh r e^{2\phi}$, and we choose $\phi = 0$. We may then write:

$$H_F = \frac{e^{2r_i}}{2} X_i^2 + \frac{e^{-2r_i}}{2} \{X_i + \sqrt{2} \gamma_i e^{r_i} F(X_i)\}^2,$$

(2)

where $r_i = r_i$ and $r_2 = r_i$. The coefficient $\gamma_i = Im(\gamma)$, as in this case the canonical constraints imply $\gamma$ imaginary, while $\gamma_2 = \gamma$, as in this case the canonical constraints imply $\gamma$ real.

Finally, $H_F = H_F^i$ if $i = 1$, and $H_F = H_F^{II}$ if $i = 2$. It is well known that, for the two–photon squeezed state (TPSS), the instance $r > 0$ implies squeezing in $X_1$ as well as antisqueezing in $X_2$, due to the constraint of minimum Heisenberg uncertainty. In the canonical multiphoton extension, this is no longer true. In fact, we see from Eq. (3) that the nonlinear function $X_1$ is added to the other quadrature $X_2$. This shows that to the same TPSS obtained either by squeezing one quadrature or by antisqueezing the other quadrature, there correspond two different possible, physically distinct multiphoton squeezed states. Squeezing on $X_1$ is implied by the choice $r > 0$ and nonlinearity $F$ on $X_1$, while antisqueezing on $X_2$ is implied still by $r > 0$ but nonlinearity $F$ moved on $X_2$. The two distinct obtained by these two different nonlinear transformations reduce to the same TPSS with squeezing on $X_1$ (and thus antisqueezing on $X_2$) as the parameter $\gamma$ goes to zero. The Hamiltonian $H_F$ can be written down completely in terms of the fundamental modes $a, a^\dagger$ once a particular form of $F(X_i)$ is selected. The choice $F(X_i) = (X_i)^\alpha$, $i = 1, 2$, provides generic $2n$–photon Hamiltonians $H_{2n}$. The transformation generated by $F = (X_2)^2$ leads to a four–photon Hamiltonian $H_4^{II} = b_1^\dagger b_2 + 1/2$ that reads:

$$H_4^{II} = (\cosh 2r + 3\gamma^2) a^\dagger a + \sinh 2r + 3 4 \gamma^2 + \frac{1}{2}$$

$$+ \frac{1}{2} \gamma e^r (a^\dagger + a) + \frac{1}{2} (\sinh 2r - 3\gamma^2) (a^2 + a^\dagger)^2$$

$$+ \frac{1}{2} \gamma e^r (a^2 a + a^\dagger a^\dagger) - \frac{1}{2} \gamma e^r (a^3 + a^2)$$

$$+ \frac{3}{2} \gamma^2 a^2 a^\dagger a^\dagger a^\dagger - \frac{3}{2} \gamma^2 (a^2 a + a^\dagger a^\dagger) + \frac{1}{4} \gamma^2 (a^4 + a^\dagger a^\dagger).$$

(3)

We see that $H_4^{II}$ contains explicitly two–, three–, and four–photon degenerate parametric down conversion terms. The multiphoton squeezed states are the eigenvectors $\{|\beta\rangle_{\gamma,F}\}$ of the transformed annihilation operator $b$ (here $b$ stands generically for either $b_1$ or $b_2$), with eigenvalue $\beta$, and are the coherent states with respect to the vacuum state $|0\rangle_{\gamma,F}$ of the generic multiphoton Hamiltonian: $|\beta\rangle_{\gamma,F} = D(\beta)|0\rangle_{\gamma,F}$, where $D(\beta) = \exp(\beta b^\dagger - \beta^* b)$ denotes the Glauber unitary displacement operator. The vectors $\{|\beta\rangle_{\gamma,F}\}$ form an overcomplete set and resolve the identity: $1/\pi \int d^2\beta |\beta\rangle_{\gamma,F} \langle \beta| = I$. If, as for the standard squeezed states, we choose $\beta = \mu a + \nu a^\dagger$, where $\alpha = \omega_1 + \omega_2$ is the coherent amplitude, and again $\phi = 0$, the canonical transformations (I) are implemented by the unitary operators $U_i = \exp[\alpha e^{r_i} \gamma_i G(X_i)]D(\alpha)S(r)$. Here $S(r) = \exp[r(a^\dagger - a)^2]$ denotes the single–mode squeezing operator for $\phi = 0$, and $G(x) = \int_0^\infty F(y) dy$. The states $|\beta\rangle_{\gamma,F}$ are not minimum uncertainty states, but they approximate the minimum uncertainty on the characteristic time scale $\tilde{r}$ provided by the time dependence $r(t) = \tilde{r} t$ of the squeezing parameter (I). In the representation in which the quadrature argument of the function $F$ is diagonal, the eigenvalue equation for the states $|\beta\rangle_{\gamma,F}$ is a simple first order linear differential equation for the wave functions $\Psi^F_\beta(x_i) = \langle x_i | \beta \rangle_{\gamma,F}$, $i = 1, 2$. Integration yields

$$\Psi^F_\beta(x_i) = (\pi \sigma_i)^{-1/4} \exp \left[ -\frac{(x_i - x_i^{(0)})^2}{2 \sigma_i} \right]$$

$$\times \exp \left\{ i [c_i x_i + \sqrt{2} \gamma_i r_i G(x_i)] \right\},$$

(4)

where $\sigma_i = e^{-2r_i}$, $x_i^{(0)} = \sqrt{2} \alpha_1$, $x_i^{(0)} = -\sqrt{2} \alpha_2$, $c_1 = \sqrt{2} \alpha_2$, $c_2 = -\sqrt{2} \alpha_1$. The multiphoton squeezed states (II) are non–Gaussian because of the non–quadratic term $G(x_i)$ appearing in the phase of the wave function. However, the probability density $|\Psi^F_\beta(x_i)|^2$ is Gaussian in the representation in which the quadrature argument of the nonlinear function $F$ is diagonal. The probability density displays squeezing in this quadrature by the usual factor $e^{2r_i}$. This fact further shows the necessary link between squeezing and nonlinearity for the multiphoton states. From now on we specialize to the two possible four–photon cases, respectively associated to $F = (X_1)^2$ and to $F = (X_2)^2$. We work explicitly in the representation in which the first quadrature $X_1$ is diagonal. In this representation we can compare congruently the two different four–photon squeezed states (FPSSs), namely $\Psi^4_\beta(x)$ associated to $F = (X_1)^2$, and $\Psi^4_{\beta^*}(x)$ associated to $F = (X_2)^2$:

$$\Psi^4_\beta(x) = (\pi e^{-2r_i})^{-1/4} \exp \left[ -\frac{e^{2r}}{2} (x - \sqrt{2} \alpha_1)^2 \right]$$

$$\times \exp \left\{ i \sqrt{2} \left[ \alpha_2 x + e^{-r_i} \sqrt{\pi} m(\gamma)^{3/2} \right] \right\},$$

(5)

$$\Psi^4_{\beta^*}(x) = N^{-1/2} \exp(kx) Ai \left[ \frac{lx + m}{l^{2/3}} \right].$$

(6)

In Eq. (I) $N$ is the normalization factor, $Ai[y]$ denotes the Airy function that goes to zero as $y \to \infty$, $l = e^{r_i}/2 \gamma_i$, $m = e^{-2r_i}/16 \gamma^2 - \beta/\gamma_i$, $k = e^{-r_i}/4 \gamma_i$. The states (I) and (II) are both non–Gaussian. However, while $|\Psi^4_\beta(x)|$ is Gaussian, $|\Psi^4_{\beta^*}(x)|^2$ is not. Its behavior is shown in Fig. 1.
We now analyze the statistical properties of the state \( \Psi^I \).

In Fig. 2 we plot the photon number distribution \( P(n) \) of the TPSS (\( \gamma = 0 \)) and of the two FPSSs \( \Psi^I_4 \) and \( \Psi^{II}_4 \).

![Figure 2](image2.png)

**FIG. 2.** The photon number distribution \( P(n) \) at \( \beta = 3\sqrt{2} \) and \( r = 0.8 \) for the FPSS \( \Psi^I_4 \) with \( \gamma = 0.1 \) (dot–dashed line); for the same state with \( \gamma = 0.5 \) (doubledot–dashed line); for the FPSS \( \Psi^{II}_4 \) with \( \gamma = 0.05 \) (dashed line); and for the TPSS (full line).

We see from Fig. 2 that the form of \( P(n) \) for the state \( \Psi^I_4 \) is similar to that of the TPSS and is very stable as a function of the nonlinear coupling \( \gamma \). The only significant difference is that, for larger values of the coupling the peak of \( P(n) \) moves to the left, due to the growing influence of \( \gamma \) on the mean number of photons. We see instead that \( P(n) \) for the state \( \Psi^{II}_4 \) is unstable and is strongly deformed already for very small values of the nonlinear coupling.

The profoundly different behavior of \( P(n) \) for the states \( \Psi^I_4 \) and \( \Psi^{II}_4 \) is deeply rooted in the structure of their respective quasiprobability distributions in phase space. In Fig. 3 we plot the Wigner function for the state \( \Psi^I_4 \), while in Fig. 4 we plot it for the state \( \Psi^{II}_4 \). The nonclassical nature of the state \( \Psi^{II}_4 \) is more pronounced. Its Wigner function is deformed and attains also negative values. However, it is neither rotated nor translated with respect to that of the TPSS, at variance with the Wigner function of the state \( \Psi^I_4 \) which is both translated and rotated with respect to that of the TPSS. This difference affects crucially the properties of phase space interference. The latter is responsible for the oscillations of \( P(n) \) for the TPSS \( \Psi^I_4 \).

![Figure 3](image3.png)

**FIG. 3.** The Wigner quasiprobability distribution \( W(x_1, x_2) \) of the FPSS \( \Psi^I_4 \) with \( \beta = 3\sqrt{2} \), \( r = 0.8 \), and \( \gamma = 0.14 \).

The oscillations are preserved in a wide range of values of \( \gamma \) for the state \( \Psi^I_4 \), exactly because its Wigner function is neither translated nor rotated with respect to that of the TPSS, while they quickly disappear for the state \( \Psi^{II}_4 \), whose Wigner function is both translated and rotated with respect to that of the TPSS already at small values of \( \gamma \). In order to complete the analysis of the statistical properties of the states \( \Psi^I_4 \) and \( \Psi^{II}_4 \) we move to study their correlation functions. In Fig. 5 we plot the normalized second–order correlation functions \( g^{(2)}(0) \) of the TPSS (\( \gamma = 0 \)) and of the states \( \Psi^I_4 \) and \( \Psi^{II}_4 \), as a function of \( r \). We see that the state \( \Psi^{II}_4 \) follows a behavior similar to that of the TPSS, being exactly Poissonian at \( r = 0 \), slightly more sub–Poissonian for \( r < 0.9 \), and super–Poissonian for larger values of \( r \). The value of saturation of \( g^{(2)}(0) \) at asymptotically large values of \( r \) is an increasing function of \( \gamma \). All these characteristics remain stable as \( \gamma \) is varied. At variance with this behavior, the state \( \Psi^I_4 \) is always super–Poissonian, even at \( r = 0 \), and for small values of \( \gamma \).
FIG. 5. The second–order correlation $g^{(2)}(0)$ at $\beta = 3\sqrt{2}$ as a function of $r$ for the four–photon squeezed state $\Psi^I_4$ with $\gamma = 0.1$ (doubledot–dashed line); for the FPSS $\Psi^{II}_4$ with $\gamma = 0.1$ (dashed line); and for the TPSS (full line).

It is of particular importance to study the fourth–order correlation function in order to establish the degree of probability of simultaneous four–photon detection (bunching) [5]. In Fig. 6 we plot the normalized fourth–order correlation functions $g^{(4)}(0)$ of the TPSS ($\gamma = 0$) and of the states $\Psi^I_4$ and $\Psi^{II}_4$, as a function of $r$. Also in this case the FPSS $\Psi^{II}_4$ follows a behavior similar to that of the TPSS. It favors four–photon anti–bunching for values of $r < 0.8$. For larger values of $r$ it strongly favors four–photon bunching, and the value of saturation of $g^{(4)}(0)$ grows with $\gamma$. The state $\Psi^I_4$ instead always favors four–photon bunching even at $r = 0$ and for small values of $\gamma$.

FIG. 6. The fourth–order correlation $g^{(4)}(0)$ at $\beta = 3\sqrt{2}$ as a function of $r$ for the FPSS $\Psi^{II}_4$ with $\gamma = 0.1$ (doubledot–dashed line); for the FPSS $\Psi^I_4$ with $\gamma = 0.1$ (dashed line); and for the TPSS (full line). Inset: the same graph in the interval $[0, 1]$.

In summary, we have introduced an effective Hamiltonian model for multiphoton down conversion processes via a nonlinear generalization of the Bogoliubov transformation. This generalization allows to define two different classes of non–Gaussian multiphoton squeezed states. We have studied in detail the four–photon case, and we have shown that the four–photon squeezed states associated to quadratic nonlinearity in the second field quadrature and to antiqueseezing in the same quadrature are the natural four–photon extension of the two–photon squeezed states with respect to the Hamiltonian structure, the field statistics, and the Wigner quasiprobability distribution. We have considered the instance of degenerate multiphoton parametric down conversions. Work is in progress to address the description of nondegenerate, two– and multimode multiphoton processes.

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