Properties of Convex Pentagonal Tiles for Periodic Tiling

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Abstract

A convex pentagonal tile is a convex pentagon that admits a monohedral tiling. We show that a convex pentagonal tile that admits a periodic tiling has a property in which the sum of three internal angles of the pentagon is equal to 360°.

Keywords: convex pentagon, tile, periodic tiling, monohedral tiling

1 Introduction

A tiling or tessellation of the plane is a collection of sets that are called tiles, which covers a plane without gaps and overlaps, except for the boundaries of the tiles. If all the tiles in a tiling are of the same size and shape, then the tiling is monohedral. Then, a polygon that admits a monohedral tiling is called a prototile of monohedral tiling, or simply, a polygonal tile. A tiling by convex polygons is edge-to-edge if any two convex polygons in a tiling are either disjoint or share one vertex or an entire edge in common [4,5,14,16–20]. A tiling of the plane is periodic if the tiling can be translated onto itself in two nonparallel directions [4,5,19].

In the classification problem of convex polygonal tiles, only the pentagonal case is open. At present, fifteen families of convex pentagonal tiles, each of them referred to as a “Type” (i.e., Type 1, Type 2, etc. up to Type 15), are known (see Figure 1) but it is not known whether this list is complete [2-14,17,20,22,23]. However, it has been proved that a convex pentagonal tile that can generate an edge-to-edge tiling belongs to at least one of the eight known types [1,4,16]. We are interested in the problem of convex pentagonal tiling (i.e., the complete list of Types of convex pentagonal tiles}, regardless of edge-to-edge and non-edge-to-edge tilings). In this paper, we prove the following.

Theorem 1. If a convex pentagon is a convex pentagonal tile that admits a periodic tiling, the pentagon has a property in which the sum of three internal angles of the pentagon is equal to 360°.

By using the notations A, B, C, D, and E for vertices of convex pentagon, the combinations in which the sum of three internal angles is equal to 360° can be expressed using the following equations:

\[ A + B + C = 360°, \quad B + C + D = 360°, \quad C + D + E = 360°, \quad D + E + A = 360°, \quad E + A + B = 360°, \]
\[ A + B + D = 360°, \quad B + C + E = 360°, \quad C + D + A = 360°, \quad D + E + B = 360°, \quad E + A + C = 360°, \]
\[ 2A + B = 360°, \quad 2A + C = 360°, \quad 2A + D = 360°, \quad 2A + E = 360°, \quad 2B + A = 360°, \quad 2B + C = 360°, \]

1 In May 2017, Michaël Rao declared that the complete list of Types of convex pentagonal tiles had been obtained (i.e., they have only the known 15 families), but it does not seem to be fixed as of October 2018 [10,23].

2 The classification of Types of convex pentagonal tiles is based on the essentially different properties of pentagons. The conditions of each Type express the essential properties. The classification problem of Types of convex pentagonal tiles and the classification problem of pentagonal tilings are quite different. The Types are not necessarily “disjoint,” that is, convex pentagonal tiles belonging to some Types also exist [14,18].
2B + D = 360°, 2B + E = 360°, 2C + A = 360°, 2C + B = 360°, 2C + D = 360°, 2C + E = 360°, 
2D + A = 360°, 2D + B = 360°, 2D + C = 360°, 2D + E = 360°, 2E + A = 360°, 2E + B = 360°, 
2E + C = 360°, 2E + D = 360°, 3A = 360°, 3B = 360°, 3C = 360°, 3D = 360°, 3E = 360°.

That is, a convex pentagonal tile that admits a periodic tiling has at least one of the above relationships.

2 Preparation

Definitions and terms of this section quote from [4].

Terms “vertices” and “edges” are used by both polygons and tilings. In order not to cause confusion, corners and sides are referred to instead of vertices and the edges of polygons, respectively. At a vertex of a polygonal tiling, corners of two or more polygons meet and the number of polygons meeting at the vertex is called the valence of the vertex, and is at least three (see Figure 2). Therefore, an edge-to-edge tiling by polygons is such that the corners and sides of the polygons in a tiling coincide with the vertices and edges of the tiling.

Two tiles are called adjacent if they have an edge in common, and then each is called an adjacent of the other. On the other hand, two tiles are called neighbors if their intersection is nonempty (see Figure 2).

There exist positive numbers $u$ and $U$ such that any tile contains a certain disk of radius $u$ and is contained in a certain disk of radius $U$ in which case we say the tiles in tiling are uniformly bounded.

A tiling $\mathcal{S}$ is called normal if it satisfies following conditions: (i) every tiles of $\mathcal{S}$ is a topological disk; (ii) the intersection of every two tiles of $\mathcal{S}$ is a connected set, that is, it does not consist of two (or more) distinct and disjoint parts; (iii) the tiles of $\mathcal{S}$ are uniformly bounded.

Let $D(r, M)$ be a closed circular disk of radius $r$, centered at any point $M$ of the plane. Let us place $D(r, M)$ on a tiling, and let $F_1$ and $F_2$ denote the set of tiles contained in $D(r, M)$ and the set of meeting boundary of $D(r, M)$ but not contained in $D(r, M)$, respectively. In addition, let $F_3$ denote the set of tiles surrounded by these in $F_2$ but not belonging to $F_2$. The set $F_1 \cup F_2 \cup F_3$ of tiles is called the patch $A(r, M)$ of tiles generated by $D(r, M)$.

For a given tiling $\mathcal{S}$, we denote by $v(r, M)$, $e(r, M)$, and $t(r, M)$ the numbers of vertices, edges, and tiles in $A(r, M)$, respectively. The tiling $\mathcal{S}$ is called balanced if it is normal and satisfies the following condition: the limits

$$v(\mathcal{S}) = \lim_{r \to \infty} \frac{v(r, M)}{t(r, M)} \quad \text{and} \quad e(\mathcal{S}) = \lim_{r \to \infty} \frac{e(r, M)}{t(r, M)}$$

exist. Note that $v(r, M) - e(r, M) + t(r, M) = 1$ is called Euler’s Theorem for Planar Maps.

**Statement 1** (Statement 3.3.13 in [4]). Every normal periodic tiling is balanced.

For a given tiling $\mathcal{S}$, we write $t_h(r, M)$ for the number of tiles with $h$ adjacents in $A(r, M)$, and $v_j(r, M)$ for the numbers of $j$-valent vertices in $A(r, M)$. Then the tiling $\mathcal{S}$ is called strongly balanced if it is normal and satisfies the following condition: all the limits

$$t_h(\mathcal{S}) = \lim_{r \to \infty} \frac{t_h(r, M)}{t(r, M)} \quad \text{and} \quad v_j(\mathcal{S}) = \lim_{r \to \infty} \frac{v_j(r, M)}{t(r, M)}$$

exist. Then,
Figure. 1: Convex pentagonal tiles of 15 families. Each of the convex pentagonal tiles is defined by some conditions between the lengths of the edges and the magnitudes of the angles, but some degrees of freedom remain. For example, a convex pentagonal tile belonging to Type 1 satisfies that the sum of three consecutive angles is equal to 360°. This condition for Type 1 is expressed as $A + B + C = 360°$ in this figure. The pentagonal tiles of Types 14 and 15 have one degree of freedom, that of size. For example, the value of $C$ of the pentagonal tile of Type 14 is $C = \cos^{-1}\left((3\sqrt{57} - 17)/16\right) \approx 1.2099 \text{ rad} \approx 70.32°$. The pale gray pentagons in each tiling indicate a fundamental region (the unit that can generate a periodic tiling by translation only).
The differences between corners and vertices, sides and edges, adjacents, and neighbors. The points \( A, B, C, E, F, \) and \( G \) are corners of the tile \( T \); but \( A, C, D, E, \) and \( G \) are vertices of the tiling (we note that the valence of vertices \( A \) and \( G \) is four, and the valence of vertices \( C, D, \) and \( E \) is three). The line segments \( AB, BC, CE, EF, FG, \) and \( GA \) are sides of \( T \), while \( AC, CD, DE, EG, \) and \( GA \) are edges of the tiling. The tiles \( T_1, T_3, T_4, T_5, \) and \( T_6 \) are adjacents (and neighbors) of \( T \), whereas tiles \( T_2 \) and \( T_7 \) are neighbors (but not adjacents) of \( T \).  

\[
\sum_{h \geq 3} t_h(\mathcal{I}) = 1 \quad \text{and} \quad v(\mathcal{I}) = \sum_{j \geq 3} v_j(\mathcal{I})
\]

hold. Therefore, every strongly balanced tiling is necessarily balanced.

**Statement 2** (Statement 3.4.8 in [4]). *Every normal periodic tiling is strongly balanced.*

**Statement 3** (Statement 3.5.13 in [4]). *For each strongly balanced tiling \( \mathcal{I} \) we have*

\[
\frac{1}{\sum_{j \geq 3} j \cdot w_j(\mathcal{I})} + \frac{1}{\sum_{h \geq 3} h \cdot t_h(\mathcal{I})} = \frac{1}{2}
\]

*where*

\[
w_j(\mathcal{I}) = \frac{v_j(\mathcal{I})}{v(\mathcal{I})}.
\]

Thus \( w_j(\mathcal{I}) \) can be interpreted as that fraction of the total number of vertices in \( \mathcal{I} \) which have valence \( j \), and \( \sum_{j \geq 3} j \cdot w_j(\mathcal{I}) \) is the *average* valence taken over all the vertices. Since \( \sum_{h \geq 3} t_h(\mathcal{I}) = 1 \) there is a similar interpretation of \( \sum_{h \geq 3} h \cdot t_h(\mathcal{I}) \): it is the *average* number of adjacents of the tiles, taken over all the tiles in \( \mathcal{I} \). Since the valence of the vertex is at least three,

\[
\sum_{j \geq 3} j \cdot w_j(\mathcal{I}) \geq 3.
\]
3 Proof of Theorem 1

Let $\mathfrak{S}_5^{ab}$ a strongly balanced tiling by convex pentagon. From [20], following propositions is known.

**Proposition 1.** \[ 3 \leq \sum_{j \geq 3} j \cdot w_j(\mathfrak{S}_5^{ab}) \leq \frac{10}{3}. \]

From Proposition 1 if a convex pentagonal tile can generate $\mathfrak{S}_5^{ab}$, then the pentagon must be able to form a vertex of valence three. Here, a vertex in which two or more corners are concentrated on the side is called a pseudo-vertex. That is, the vertices $C$, $D$, and $E$ in Figure 2 are the pseudo-vertices of valence three.

Let $G_0$ be a convex pentagon that does not in which the sum of the three internal angles is equal to 360°. If $G_0$ is a convex pentagonal tile that can generate $\mathfrak{S}_5^{ab}$, we first consider that $G_0$ has only one corner at 90°. The reason for this is as follows: From Proposition 1 if $G_0$ is a convex pentagonal tile that can generate $\mathfrak{S}_5^{ab}$, $G_0$ must form a 3-valent vertex, and it is the 3-valent pseudo-vertex. Given that the total of the internal angles of any convex pentagon is 540°, when two corners at the 3-valent pseudo-vertex are different (for example, a pair of corners $A$ and $B$, a pair of corners $A$ and $C$, and so on), the convex pentagon must have a property in which the sum of three internal angles is equal to 360°. Therefore, if $G_0$ can form a 3-valent pseudo-vertex, we consider that $G_0$ has only one corner at 90°.

Hereafter, it is assumed that the corner $A$ of $G_0$ is 90°. If $G_0$ with $A = 90°$ generates $\mathfrak{S}_5^{ab}$, then the corner $A$ belongs to the 3-valent pseudo-vertex and the corners $B$, $C$, $D$, and $E$ belong to the vertices whose valence is four or more. (Of course, the corner $A$ may also have the properties of forming other vertices with valences of four or more. However, from Proposition 1, the corner $A$ must form a 3-valent pseudo-vertex.) If the corner $A$ belongs to the 3-valent pseudo-vertex and the corners $B$, $C$, $D$, and $E$ belong to the 4-valent vertices in $\mathfrak{S}_5^{ab}$ by using $G_0$ with $A = 90°$, we consider a model in which the ratio between the 3-valent pseudo-vertices and the 4-valent vertices is 1 : 2. (It is because that, when the corners $A$ of two pieces form one 3-valent pseudo-vertex, each corners $B$, $C$, $D$, and $E$ should appear two times at two 4-valent vertices [21].) For tiling of such models, the average valence of the vertices is given as follow:

\[ \sum_{j \geq 3} j \cdot w_j(\mathfrak{S}_5^{ab}) = \frac{3 + 2 \times 4}{1 + 2} = \frac{11}{3}. \]

This result contradicts Proposition 1. It is obvious that the average valence of the vertices will be even larger if the valence of the vertices to which the corners $B$, $C$, $D$, and $E$ belong is further increased. Therefore, there is no $G_0$ as a convex pentagonal tile that can generate $\mathfrak{S}_5^{ab}$. Then, from Statement 2 there is no $G_0$ as a convex pentagonal tile that can generate a periodic tiling.

Thus, a convex pentagonal tile that can generate a periodic tiling (i.e., a convex pentagonal tile that admits a periodic tiling) has a property in which the sum of three internal angles of the pentagon is equal to 360°.

4 Conclusions

We know for a fact that the convex pentagonal tiles belonging to families of Types 1–15 admit at least one periodic tiling. On the other hand, there is no assurance that all convex
pentagonal tiles admit at least one periodic tiling. In the solution to convex pentagonal tiling, it is necessary to consider whether there is a convex polygonal tile that admits infinitely many tilings of the plane, none of which is periodic (i.e., whether there is a convex pentagonal tile which is an aperiodic prototile) [19,20]. Currently, if a convex pentagonal tile that is an aperiodic prototile exists, it is unknown whether the pentagon must have a property in which the sum of three internal angles is equal to 360°.

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\[\text{A set of prototiles is called aperiodic if congruent copies of the prototiles admit infinitely many tilings of the plane, none of which are periodic. A tiling that has no periodicity is called nonperiodic. On the other hand, a tiling by aperiodic prototiles is called aperiodic. Note that, although an aperiodic tiling is a nonperiodic tiling, a nonperiodic tiling is not necessarily an aperiodic tiling.} \]
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