Phase factors of periodically driven two-level systems

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Abstract
Using a specially developed perturbative solution for periodically driven quantum systems, we show how to obtain phase factors for both a single qubit and for two interacting qubits. The method is easily implemented by numerical routines. Comparing to more traditional methods, it presents the advantage of being stable for long-time periods, therefore allowing the computation of phase factors with arbitrary precision. We furthermore explore the possibility of implementing a quantum phase gate using the perturbative solution. Our method allows the numerical computation of phase factors with arbitrary precision provided, of course, sufficient computational resources are at disposal.

Keywords: geometric phases, periodically driven quantum systems, perturbation theory for periodically driven qubits

1. Introduction
The theoretical computation of the behaviour of mechanical systems (classical or quantum) interacting with periodically or quasi-periodically time dependent external sources requires a very good control of their behaviour at large times. Naive solution methods can lead to intractable problems involving secular terms (polynomially growing terms depending on time) or small denominators (specially under quasi-periodic interactions). The usual numerical integration methods may also be unstable for large times and can induce uncontrolled errors. These problems have been first identified in Celestial Mechanics and are ubiquitous in physical systems under periodic or quasi-periodic time interactions. The analysis of these stability problems and their solutions is a wide area of research both in Physics and Applied Mathematics, and led to important developments, as the Poincaré-Lindstedt series and the KAM theory.

The main goal in the perturbative treatment of such systems is to express physically meaningful quantities in term of uniformly convergent series depending on time, that means, in terms of series that, when truncated, differ from the exact solution by at most a fixed small amount that does not increase with time. The computation of quantum phases is a relevant physical situation where such uniform, i.e., time stable approximations are required, since these phases are sensitive to errors that increase as the time goes to infinity.

In this work we discuss a particular situation involving quantum phases of a qubit where these computations can be carried out in a controlled way.

From the physical side, the study of geometric phases has attracted significant interest since it was shown that they could be used to process quantum information [1] and, due to its geometric properties, they present an inherent resilience to fluctuation errors in the control parameters. Geometric phases have been studied in circuit QED [2–4] and the experimental implementations of geometric phase in the context of quantum computation, sometimes referred to as geometric quantum computation (GQC), has been fruitful [5, 6]. Nevertheless, to obtain the expression for the geometric phase acquired by a qubit, many works implement the rotating wave approximation (RWA) [5, 7, 8]. As every approximation, the RWA has its realm of validity and applicability that has been extensively studied [9–13].

In this work, we consider the evolution of a qubit driven by periodic fields. Instead of the RWA, we use the solution of the Schrödinger equation obtained in [14, 15] (see also [16–19]) to compute the total, dynamical and geometric phase for system of one and two qubits. Since the solution used is uniformly convergent in time, the expressions for the phases are robust when long-time periods are considered. We first
2. Description of the model and methods

Let us start by considering a two-level system with the following Hamiltonian:

$$H_k(t) = \epsilon \sigma_3 - f(t) \sigma_k,$$  \hspace{1cm} (1)

where $\sigma_k, k = 1, 2, 3$, are the usual Pauli matrices, $\epsilon$ is a real constant and $f(t)$ is a periodic function of time with frequency $\omega > 0$. One of the possible physical systems it describes is a qubit with energy levels $\pm \epsilon$ (corresponding to the ‘free’ Hamiltonian $\epsilon \sigma_3$) coupled to an external time-dependent transversal magnetic field (corresponding to the ‘interaction’ Hamiltonian $-f(t) \sigma_1$). The later term induces transitions between the two eigenstates of the free system.

This is one of the most fundamental quantum systems and we are interested in considering the term $\epsilon \sigma_3$ as a perturbation, by considering $\epsilon$ small, with the interaction $f$ depending periodically on time.

In [14–18] a perturbative method inspired in the Poincaré-Lindstedt series was developed that provides uniformly converging expansions in time for the solution of Schrödinger’s equation $i \frac{d}{dt} \psi(t) = H_k(t) \psi(t)$ in the cases when $f$ is a periodic or a quasi-periodic function.

The first step is to consider a rotation of $\pi/2$ around the $y$-axis, implemented by the rotation operator $R_y(\pi/2) = \exp(-i \pi \sigma_y / 4) = \frac{1}{\sqrt{2}} (1 - i \sigma_y)$. Schrödinger’s equation on this new rotated frame becomes

$$i \frac{d}{dt} \psi_2(t) = H_2(t) \psi_2(t),$$  \hspace{1cm} (2)

where

$$\psi_2(t) = R_y(\pi/2) \psi(t)$$  \hspace{1cm} (3)

and

$$H_2(t) = \epsilon \sigma_1 + f(t) \sigma_y.$$  \hspace{1cm} (4)

The Hamiltonian $H_2(t)$ has the advantage of being diagonal for the unperturbed case $\epsilon = 0$, when $H_2(t)$ reduces to $f(t) \sigma_y$. Hence, for the unperturbed case we have the explicit solution $\psi_2(t) = \exp\left(-i \int_0^t f(t') dt' \sigma_y\right)$ valid for all times.

As elaborated in [14–18], this facilitates the problem of finding a convergent perturbative solution for $\psi_2(t)$ for the perturbed case when $\epsilon \neq 0$ by means of an Ansatz similar to the Poincaré-Lindstedt series.

The method developed in [14] and [15] is valid for small $\epsilon$ and periodic $f$. The quasi-periodic case was analysed in [19]. It consists in writing a perturbative expansion in $\epsilon$ for the time evolution operator. This method has been proven to have the following advantages: the series expansion is uniformly convergent in time, the expression obtained for the time evolution operator is given in terms of a series and so are easily implementable in numerical calculations and they can be employed for any periodic function. The uniform convergence is of great importance, since it means the results lead to stable numerical calculations and therefore allows the study of long-time behaviour of the observable quantities of the system.

It was shown in [14] that the time evolution operator $U(t)$ for the system described by (4) can be written as

$$U(t) = \begin{pmatrix} R(t)(1 + i g_0 S(t)) & -i e R(t) S(t) \\ -i e R(t) S(t) & R(t)(1 - i g_0 S(t)) \end{pmatrix},$$  \hspace{1cm} (5)

where $R(t)$ and $S(t)$ are given by

$$R(t) = e^{-it\Omega} \sum_{m \in \mathbb{Z}} R_m e^{i m \omega t}$$  \hspace{1cm} (6)

and

$$S(t) = \sigma_0 + e^{2it\Omega} \sum_{m \in \mathbb{Z}} S_m e^{i m \omega t}.$$  \hspace{1cm} (7)

$R_m$ and $S_m$ are coefficients of the Fourier expansion of $R(t)$ and $S(t)$, respectively. Together with the Rabi frequency $\Omega$ and the constants $g_0$ and $\sigma_0$, they can all be obtained from rather complex but convergent power series expansions in $\epsilon$, involving the the Fourier coefficients of $f$ and its frequency $\omega$. See [15] as well as [14, 16–18] for explicit formulas and examples. Sometimes we will refer to the matrix elements of $U(t)$, for example, $U_{11}(t) = R(t)(1 + i g_0 S(t))$ and $U_{12}(t) = -i e R(t) S(t)$.

As done in [15], we implemented numerically the method developed there for a perturbation of the form

$$f(t) = F_0 + A \cos(\omega t),$$  \hspace{1cm} (8)

where $F_0$ is a real number, $A$ and $\omega$ is the amplitude and the frequency of the periodic perturbation, respectively.

Following the directions of the original paper, the method was applied to several values of $\omega$ and $\epsilon$, the former ranging from 1.0 to 10.0 and the later from 0.01 to 0.40. For all these values, the unitarity test was sufficiently satisfactory, since the error is bounded by $3 \times 10^{-3}$ in one specific case (for $\omega = 1.0$ and $\epsilon = 0.40$), but for most cases, is bounded by $10^{-5}$ or even $10^{-10}$.

3. Total, dynamical and geometric phases

We now show the calculations of the total, dynamical and geometric phases for the two-level system considered. The total phase of the system is simply given by

$$\phi_{\text{tot}}(t) = \arg \langle \psi(0), \psi(t) \rangle,$$  \hspace{1cm} (9)

and the dynamical phase $\alpha_{\text{dyn}}$ is given by

$$\alpha_{\text{dyn}}(t) = i \int_0^t \langle \psi(t'), \psi(t') \rangle dt'.$$  \hspace{1cm} (10)
where \( \psi(0) = \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) \) with \( |\alpha|^2 + |\beta|^2 = 1 \) and \( \psi(t) \) are the state vectors of the system at the initial instant of time and for an instant of time \( t \), respectively. The dot indicates derivation relative to time. The geometric phase \( \gamma_{geo} \) is simply the difference between the total and dynamical phases:

\[
\gamma_{geo}(t) = \phi_{tot}(t) - \alpha_{dyn}(t). \tag{11}
\]

We note that the phase factors are functions of time, since they are defined by the evolution of the state vector \( \psi(t) \).

When performing the following calculations, we shall consider the state vector correspondent to the rotated Hamiltonian given by (3). The resulting expressions become

\[
\phi_{tot}(t) = \arg \{\text{Re} \, U_{11}(t) + i(-2 \text{Re}(\bar{\alpha} \beta) \text{Im} \, U_{11}(t)) + 2 \text{Im}(\bar{\alpha} \beta) \text{Re} \, U_{12}(t) + (2|\alpha|^2 - 1) \text{Im} \, U_{12}(t))\}
\]

and

\[
\alpha_{dyn}(t) = |\alpha|^2 \left( -\text{Im} \int_0^t a_{11}(t')dt' + i \text{Re} \int_0^t a_{12}(t')dt' \right) - 2i \text{Re}(\bar{\alpha} \beta) \text{Re} \int_0^t a_{11}(t')dt' - 2i \text{Im}(\bar{\alpha} \beta) \text{Im} \int_0^t a_{12}(t')dt' + |\beta|^2 \left( -\text{Im} \int_0^t a_{11}(t')dt' - i \text{Re} \int_0^t a_{12}(t')dt' \right), \tag{13}
\]

where \( a_{11}(t) \) and \( a_{12}(t) \) are matrix elements of the product of \( U^\dagger(t) \) and \( \dot{U}(t) \):

\[
U^\dagger(t) \dot{U}(t) = \begin{pmatrix}
  a_{11}(t) & a_{12}(t) \\
  -\bar{a}_{12}(t) & \bar{a}_{11}(t)
\end{pmatrix} \tag{14}
\]

The expression for the dynamical phase involves integrations over time of the expansions. Although there are lots of integration routines, using them in the highly oscillatory functions often results in a large error due to the routine. Thus, the integrations were carried out analytically term by term in the Fourier expansions and then implemented numerically.

The previous expressions determine the total and dynamical phase for the system for any instant of time. Next, it is necessary to define the instant of time that is physically meaningful to the calculations of the phase acquired by the system. One could argue that the appropriate instant of time would be the ‘natural’ frequency of the system, characterised by the Rabi frequency \( \Omega \). But we must recollect the nature of the geometric phase, that is, the phase acquired over the course of the evolution of the system resulted from the geometrical properties of the parameter space of the Hamiltonian.

In our case, the parameter space is two-dimensional, with each dimension associated to the parameters \( A \) and \( \omega \) in (8). So, if we consider a cyclic evolution on the parameter space and a fixed amplitude \( A \) of the external field, the relevant instant of time is precisely

\[
t_w = \frac{2\pi}{\omega}. \tag{15}
\]

Therefore, the expressions (12), (13) and (11) for the respective total phase, dynamical phase and geometric phase of the system are taken at \( t_w \). Next, we present some results of our calculations for the phase factors of the system as graphical representations. In order to illustrate the method, we considered the initial state vector to be \( \psi(0) = \vert 0 \rangle \), that is, the state vector is initially aligned with the \( z \)-axis. The calculations were performed for values of \( \epsilon \) ranging from 0.01 to 0.40 with steps of 0.01; and values of \( \omega \) ranging from 1.0 to 10.0 with steps of 0.5.

As previously stated, the numerical implementation of the total phase was easily accomplished. We note that since the total phase is defined as an argument, there was no need to test if the numerical function had relevant imaginary parts due to built-in machine errors. Figure 1(a) shows the relation between the values of the total phase and the parameter \( \epsilon \). We can see that the absolute value of the total phase is proportional to the value of \( \epsilon \). According to the interpretation of (1) in which \( \epsilon \) is the energy gap between the two eigenstates of \( \sigma_3 \), we can say that the total phase is proportional to this gap. Moreover, we note that as the value of \( \omega \) increases, the rate in which the total phase increases with \( \epsilon \) decreases. In other words, the value of \( \omega \) modulates the curve \( \phi_{tot} \times \epsilon \). Figure 1(b) shows graphs of the total phase as a function of \( \omega \) with fixed values of \( \epsilon \). The same behaviour observed in figure 1(a) is present in figure 1(b), but in this case, the value
of $\epsilon$ modulates the curve $\phi_{\text{tot}} \times \omega$ in the following way: as $\epsilon$ increases, the curve gets more accentuated. It is also notable that for $\omega$ around 2.0, the absolute value of the total phase is maximised.

The numerical implementation of the dynamical phase is not as straightforward as that of the total phase, since it involves several integrations over time (equation (13)). These integrations, as we said before, were done analytically and then numerically implemented. The dynamical phase is expected to be real, but the expansions in our implementation are truncated, so we tested if the imaginary part of the dynamical phase had relevant contributions. The imaginary parts equal zero within the machine accuracy. The relation between the dynamical phase and the values of $\omega$ has a particular behaviour: for $\omega = 1.0, 1.5, 2.0, 2.5$ the curve $\alpha_{\text{dyn}} \times \epsilon$ resembles a parabola and for higher values the curve resembles a linear function. Figure 2(a) shows the dynamical phase as a function of $\epsilon$ for some fixed values of $\omega$. Figure 2(b) shows the curve $\alpha_{\text{dyn}} \times \omega$ for some fixed values of $\epsilon$. We can see that, similar to figure 1(b), $\epsilon$ seems to modulate the curve and there is a value of $\omega$ that maximises $\alpha_{\text{dyn}}$, but this value shifts according to the value of $\epsilon$.

A similar behaviour of the total phase is observed for the geometric phase in figures 3(a) and (b): the absolute value of the geometric phase increases as $\epsilon$ increases, the curve $\gamma_{\text{geo}} \times \omega$ is modulated by $\epsilon$ and it presents a value of $\omega$ that maximises the absolute value of the geometric phase.

We next consider two qubits with individual Hamiltonians given by (1). When considering that the two systems do not interact with each other, the phase factors obtained for the composite system are simply the algebraic sum of the individual phase factors. In order to explore how the phase factors of the composite system change when interactions are taken into account, we considered an interaction given by

$$H(t) = \kappa v(t) \sigma_3^{(a)} \otimes \sigma_3^{(b)},$$

where $\kappa$ is a real constant and $v(t)$ is a real function of time. The corresponding Hamiltonian in the rotated frame is given by

$$H_{\text{rot}}(t) = \tilde{R}_z(\pi/2)(\kappa v(t) \sigma_3^{(a)} \otimes \sigma_3^{(b)})\tilde{R}_z(\pi/2)^* = \kappa v(t) \sigma_1^{(a)} \otimes \sigma_1^{(b)},$$

where $\tilde{R}_z(\pi/2) = R_z(\pi/2) \otimes R_z(\pi/2)$. $\epsilon_a$ and $\epsilon_b$ are the respective constants of the individual systems and $f_a(t)$ and $f_b(t)$ are the external fields applied to each subsystem. The
Hamiltonian of the composite system is

\[
H(t) = \begin{pmatrix}
    f_a(t) + f_b(t) & \epsilon_b & \epsilon_a & \kappa v(t) \\
    \epsilon_b & f_a(t) - f_b(t) & \kappa v(t) & \epsilon_a \\
    \epsilon_a & \kappa v(t) & -f_a(t) - f_b(t) & \epsilon_b \\
    \kappa v(t) & \epsilon_a(t) & \epsilon_b & -f_a(t) - f_b(t)
\end{pmatrix}
\]

In order to obtain the phase factors for the composite system, we consider the interaction picture. We will denote the state vector in this picture by \( \psi_I(t) \) and it relates to the state vector in the Schrödinger picture by the unitary transformation

\[
U(t) \psi(t) = \psi_I(t).
\]

where \( U(t) \) is the time evolution operator. In the interaction picture, the time evolution operator \( U_I(t) \) is given by the Dyson series

\[
U_I(t) = 1 + \sum_{n=1}^{\infty} (-i)^n \int_{t_0}^{t} V_I(t_1) dt_1 \cdots \int_{t_{n-1}}^{t} V_I(t_n) dt_n,
\]

Figure 4. Plots of the phase factors for the initial state \( |00\rangle \) as functions of the parameter \( \kappa \). The thick line represents the value of the phase factors for a system with non-interacting subsystems. The dashed line represents the interaction given by (28). We considered subsystems with \( \omega_a = 1.0, \omega_b = 2.0, \epsilon_a = \epsilon_b = 0.01 \) and \( t_0 = 0.5 \).

Figure 5. Plots of the phase factors for the initial state \( |00\rangle \) as functions of the instant of time of the interaction \( t_0 \). The thick line represents the value of the phase factors for a system with non-interacting subsystems. The dashed line represents the interaction given by (28). We considered subsystems with \( \omega_a = 1.0, \omega_b = 2.0, \epsilon_a = \epsilon_b = 0.01 \) and \( \kappa = 0.1 \). The time is measured in units of \( 2\pi/\omega \).
where $V_I(t)$ is the interaction Hamiltonian in the interaction picture given by

$$V_I(t) = \kappa V(t) \begin{pmatrix} V_{11}(t) & V_{12}(t) \\ V_{12}(t) & -V_{11}(t) \end{pmatrix}^{(a)} \otimes \begin{pmatrix} V_{11}(t) & V_{12}(t) \\ V_{12}(t) & -V_{11}(t) \end{pmatrix}^{(b)},$$

(21)

with

$$V_{11}(t) = -U_{11}(t) U_{12}(t) - U_{11}(t) U_{12}(t),$$

(22)

$$V_{12}(t) = U_{11}(t)^2 - U_{12}(t)^2.$$

(23)

The time evolution operator in the interaction picture given by the Dyson expansion in (20), considering the expression for the operator $V_I(t)$ in (21), is

$$U_I(t) = 1 - i \kappa \int_0^t (U_0^a(t') \sigma_1^{(a)} U_0(t')) dt' + O(\kappa^2).$$

(24)

We shall consider the Dyson expansion up to first order. The matrix form of the time evolution operator in the interaction picture, in first order, is given by

$$U_I(t) = 1 - i \kappa V^{(1)}(t),$$

(25)

We omitted the time-dependency of the expressions for $V_{11}(t)$ and $V_{12}(t)$ given by equations (22) and (23), respectively. The
operator \( V^{(1)}(t) \) will be useful for evaluating the expressions for the phase factors of the composite system. Also, we must note that \( V^{(1)}(t) \) is a self-adjoint operator, since \( \nu(t) \) is a real function of \( t \) and the matrix operator in the integrand on the right hand side of (25) is self-adjoint.

The total phase factor for the composite system is

\[
\phi_{tot}(t) = \arg \langle \psi_2(0), \psi_2(t) \rangle
\]

\[
= \arg \{ \langle \psi^{(a)}_2(0), U_0(t) \psi^{(a)}_2(0) \rangle \langle \psi^{(b)}_2(0), U_0(t) \psi^{(b)}_2(0) \rangle \} - i\kappa \langle \psi_2(0), U(t) V^{(1)}(t) \psi_2(0) \rangle \}
\]

(26)

Note that for \( \kappa = 0 \) the expression above reduces itself to the total phase of two non-interacting qubits.

Using (10) for the dynamical phase and the expansion in \( \kappa \) for the time evolution operator in the interaction picture, we have

\[
\alpha_{dyn}(t) = i \int_0^t \langle \psi_2(t'), \psi_2(t') \rangle \, dt'
\]

\[
= i \int_0^t \langle \psi_2(0), U^\ast(t') \tilde{U}(t') \psi_2(0) \rangle \, dt'
\]

\[
+ \kappa \int_0^t \langle \psi_2(0), U^\ast(t') U(t') V^{(1)}(t') \psi_2(0) \rangle \, dt'
\]

\[
+ \kappa \int_0^t \langle \psi_2(0), U^\ast(t') \tilde{U}(t') V^{(1)}(t') \psi_2(0) \rangle \, dt'
\]

\[
- \kappa \int_0^t \langle \psi_2(0), V^{(1)}(t') \psi_2(0) \rangle \, dt' + \mathcal{O}(\kappa^2),
\]

since \( U(t) \) is unitary, the identity \( U^\ast(t) \tilde{U}(t) = -\tilde{U}^\ast(t) U(t) \) holds and the third term on the right hand side of the expression above can be rewritten as the complex conjugate of the second term. Hence, the dynamical phase up to first order in \( \kappa \) is given by

\[
\alpha_{dyn}(t) = \alpha_{dyn}^{(0)}(t)
\]

\[
+ 2\kappa \operatorname{Re} \int_0^t \langle \psi_2(0), U^\ast(t') \tilde{U}(t') V^{(1)}(t') \psi_2(0) \rangle \, dt'
\]

\[
+ \kappa \int_0^t \langle \psi_2(0), V^{(1)}(t') \psi_2(0) \rangle \, dt' + \mathcal{O}(\kappa^2),
\]

(27)

where the \( \alpha_{dyn}^{(0)}(t) \) is exactly the expression for the dynamical phase for two non-interacting qubits. Also, the third term on the right hand side of (27) is the integral over time of the expectation value of the self-adjoint operator \( V^{(1)}(t) \). Therefore, this term is also real and so is the expression for the dynamical phase. The geometric phase for the composite system is still given by the difference between the total phase and the dynamical phase.

Now, let us consider the case in which the interaction is given by

\[
v(t) = \delta(t - t_0),
\]

where \( t_0 \) is any instant of time. The time evolution operator in the interaction picture, according to (24) and (25), is

\[
U(t) = I_4 - i\kappa \int_0^t \left( V^{(1)}(t') V^{(1)}(t') - V^{(1)}(t') V^{(1)}(t') \right) dt'
\]

\[
+ i \kappa \int_0^t \left( V^{(1)}(t') U(t') \psi_2(0) \right) \, dt' + \mathcal{O}(\kappa^2),
\]

(29)

where the time dependency of \( V_{11}(t) \) and \( V_{12}(t) \) are respectively given by (22) and (23). The time dependency in the second term on the right hand side was omitted, but we assume that \( 0 < t_0 < t \) and so, both \( V_{11}(t) \) and \( V_{12}(t) \) are calculated for \( t_0 \), as is indicated by the subscript on the matrix on the right hand side of (29).

Up to first order in \( \kappa \), the time evolution operator in (29) is constant in time. Thus, the third term of the expression for the dynamical phase in (27), that involves the time derivative of \( V^{(1)}(t) \), is null. We implemented in our code routines that calculate the phase factors for the interaction given by (28).

To investigate the relation between the phase factors and the constant \( \kappa \), we considered a system composed of two commensurable subsystems with fixed \( \omega_{21}, \omega_{2b}, \epsilon_a \) and \( \epsilon_b \), a fixed \( \delta \) that characterises the delta interaction and we varied \( \kappa \) from 0 to 0.2, with steps of 0.01. Considering this set of parameters, the code calculates the phase factors for each of the computational basis states \( |00\rangle, |01\rangle, |10\rangle \) and \( |11\rangle \). Figure 4 shows the results for the initial state \( |00\rangle \) and \( \omega_{21} = 1.0, \omega_{2b} = 2.0, \epsilon_a = \epsilon_b = 0.01 \) and \( t_0 = 0.5 \). The results are similar for others sets of parameters. We note that since our approximation of the Dyson expansion (equation (20)) is only up to first order, the dependency of the phase factors on \( \kappa \) is linear. The parameter \( \kappa \) is not, as one could imagine, a parameter of the control space of the system. It simply modulates the interaction between the subsystems and can be thought of as an structural constant.

Figure 5 shows the dependency of the phase factors on the instant of time \( t_0 \) of the interaction for the initial state \( |00\rangle \). The presented relation between the phase factors and the constant \( \kappa \), we considered a system composed of two of the two non-interacting qubits.

4. Further results

Using the results obtained so far for two qubits, we may investigate once again the appropriate instant of time to calculate the phase factors. Following the same prerogative, that the instant to be considered corresponds to the time interval in which the system undergoes a cyclic evolution, we consider the probability of transition for the composite system:

\[
P(t) = |\langle \psi(t), U(t) \psi(0) \rangle|^2.
\]
figure 6 shows $P(t)$ as a function of time. We observe that the system returns to its initial state after a time $T_0 \approx 456\tau$, where $\tau = 2\pi/\omega$. $T_1$ is also obtained through $T_1 = 2\pi/\Omega$, where $\Omega$ is the Rabi frequency and is calculated numerically. We considered a system with $\omega_a = 1.0$, $\omega_b = 2.0$, $\epsilon_a = \epsilon_b = 0.01$. The constants that determine the interaction are $\kappa = 0.1$ and $\theta = 0.5 = 0.16\tau$. For this values, the correspondent Rabi frequency is $\Omega = 0.0022$, resulting in $T_0 \approx 456\tau$, as observed in figure 6.

Once we determined the period that the system takes to return to its initial state ($T_0$), we can calculate the total phase factor of the composite system.

Table 1 shows values of the total phase for a set of $\omega_a$ and $\omega_b$ values. We note that when $\omega_a = \omega_b$, we can write the following transformation:

$$B(\phi) = \begin{pmatrix} e^{i\phi} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{-i\phi} \end{pmatrix},$$

(30)

where $\phi$ is the total phase associated with the basis state $|00\rangle$. This transformation implements a conditional evolution of the basis states, we can say that (30) is a conditional phase gate in the sense that the state of one system influences the state of the other, although it does present the usual symmetric form of controlled phase shift gates. This gate is not purely geometrical, since the total phase factor involves both the dynamical and geometric phases. When $\omega_a = \omega_b$, the transformation on the basis state can no longer be represented by (30), as can be seen in table 1.

5. Conclusions

The main contribution of this work is the implementation of the method developed in [14] and [15] to obtain phase factors for a qubit and two interacting and non-interacting qubits. Since this method presents a solution stable for long-time periods, the resulting phase factors also present this property.

The implementation of a quantum gate, when RWA is considered is a conditional phase gate in the sense that the state of one system influences the state of the other, although it does present the usual symmetric form of controlled phase shift gates. This gate is not purely geometrical, since the total phase factor involves both the dynamical and geometric phases. When $\omega_a = \omega_b$, the transformation on the basis state can no longer be represented by (30), as can be seen in table 1.

References

[1] Zanardi P and Rasetti M 1999 Holonomic quantum computation Phys. Lett. A 264 94–9
[2] Leek P J, Fink J M, Blais A, Bianchetti R, G’oppel M, Cambetta J M, Schuster D I, Frunzio L, Schoelkopf R J and Wallraff A 2008 Observation of Berry’s phase in a solid-state qubit Science 318 1889
[3] Berger S, Pechal M, Pugnett S, Jr A A A, Steffen L, Fedorov A, Wallraff A and Filipp S 2012 Geometric phases in superconducting qubits beyond the two-level-approximation Phys. Rev. B 85 220502
[4] Pechal M, Berger S, Jr A A A, Fink J M, Mlynek J A, Steffen L, Wallraff A and Filipp S 2012 Geometric phase and nonadiabatic effects in an electronic harmonic oscillator Phys. Rev. Lett. 108 170401
[5] Jones J A, Vedral V, Ekert A and Castagnoli G 2000 Geometric quantum computation using nuclear magnetic resonance Nature 403 869–71
[6] Anandan J, Christian J and Wanelik K 1997 Geometric phases in physics Am. J. Phys. 65 180
[7] Ekert A, Ericsson M, Hayden P, Inamori H, Jones J A, Oi D K L and Vedral V 2000 Geometric quantum computation J. Mod. Opt. 47 2501–13
[8] Sjöqvist E, Pati A K, Ekert A, Anandan J S, Ericsson M, Oi D K L and Vedral V 2000 Geometric phases for mixed states in interferometry Phys. Rev. Lett. 85 2845
[9] Bloch F and Siegert A 1940 Magnetic resonance for nonrotating fields Phys. Rev. 57 522–7
[10] Stevenson A F 1940 On the theory of the magnetic resonance method of determining nuclear moments Phys. Rev. 58 1061–7
[11] Bonacci D 2003 Rabi spectra—a simple tool for analyzing the limitations of RWA in modelling of the selective population transfer in many-level quantum systems arXiv:quant-ph/0309126
[12] Frasca M 2003 A modern review of the two-level approximation Ann. Phys. 306 193–208
[13] Spiegelberg J and Sjöqvist E 2013 Validity of the rotating-wave approximation in nonadiabatic holonomic quantum computation Phys. Rev. A 88 054301
[14] Barata J C A 2001 Converging perturbative solutions of the Schrödinger equation for a two-level system with a hamiltonian depending periodically on time Annales Henri Poincaré 2 963–1005
[15] Barata J C A and Cortez D A 2002 Time evolution of two-level systems driven by periodic fields Phys. Lett. A 301 350–60
[16] Barata J C A and Wreszinski W F 2000 Strong-coupling theory of two-level atoms in periodic fields Phys. Rev. Lett. 84 2112–5

[17] Barata J C A 2000 On formal quasi-periodic solutions of the Schrödinger equation for a two-level system with a Hamiltonian depending quasi-periodically on time Rev. Math. Phys. 12 25–64

[18] Barata J C A and Cortez D A 2003 Perturbative analysis of dynamical localisation J. Math. Phys. 44 1937–60

[19] Guido Gentile D A C and Barata J C A 2005 Stability for quasi-periodically perturbed Hill’s equations Commun. Math. Phys. 260 403–43

[20] Carollo A, Fuentes-Garidi I, França Santos M and Vedral V 2003 Geometric phase in open systems Phys. Rev. Lett. 90 160402

[21] Peixoto de Faria J G, Toledo Piza A F R and Nemes M C 2003 Phases of quantum states in completely positive non-unitary evolutions Europhysics Lett. 62 782–8