Abstract: This paper considers risks of the investment portfolio, which consist of distributed mortgages and sold European call options. It is assumed that the stream of the credit payments could fall by a jump. The time of the jump is modeled by the exponential distribution. We suggest that the returns on stock are variance-gamma distributed. The value at risk, the expected shortfall and the entropic risk measure for this portfolio are calculated in closed forms. The obtained formulas exploit the values of generalized hypergeometric functions.

Keywords: variance-gamma distribution; credit risk; call option; exponential distribution; shortfall risk; generalized hyperbolic function

1. Introduction

As there are a lot of various types of trading financial instruments on financial markets, a number of different investment strategies can happen. It is supposed throughout this paper that the investor accumulates money by selling call options and issues loans on the sum. We discuss the risk of such a portfolio. It is connected with the possible fall of mortgage payments and the growth of the asset price. The time of negative jump of the payment rate is assumed to be exponentially distributed.

The exponential distribution is exploited widely for the modeling of credit risk. The area of use covers both the stock (see Bäuer 2015; Carr and Wu 2010) and interest rate (Eberlein et al. 2013; Grbac 2009; Madan 2014) frameworks. Let us notice that the hazard rate theory is an extension of the idea of default time modeling by the exponential distribution, see, for example, the book by Bielecki and Rutkowski (2002).

The variance-gamma distribution was first introduced in Madan and Seneta (1990) and Madan and Milne (1991), who discussed its symmetric version in the context of various problems of mathematical finance. Madan et al. (1998) defined the variance-gamma process as a time-changed Brownian motion with drift, or as a difference of two gamma processes, and supplied its properties. For a review of various investigations for the variance-gamma process and its properties, see also Seneta (2007).

There is much work that supports the use of the variance-gamma process in finance. Among others, let us mention the papers by Daal and Madan (2005), Finlay and Seneta (2006), Linders and Stassen (2016), Luciano et al. (2016), Luciano and Schoutens (2006), Moosbrucker (2006), Mozumder et al. (2015), Rathgeber et al. (2016), and Wallmeier and Diethelm (2012), where the variance-gamma distribution is confirmed as a very good model to make out the statistics. For approximations of processes by the variance-gamma one, see Eichelsbacher and Thäle (2015).

Various procedures can be exploited for the computing in the variance-gamma model. See, for example, Avramidis et al. (2003), Fu (2007) and the monograph by Korn et al. (2010) on Monte-Carlo methods, Carr and Madan (1999), and the review paper by Eberlein (2014) and Wang (2009) on Fourier transform methodologies. American-style options are discussed in Hirsa and Madan (2004) and
Almendral and Oosterlee (2007). Closed-form solutions are presented in Madan et al. (1998) and then developed in Ivanov (2018), Ivanov and Ano (2016) and Ivanov and Temnov (2016).

This paper continues the direction of Madan et al. (1998), giving analytical expressions that allow one to compute the value at risk, the expected shortfall risk and the entropic risk measure for the portfolio that is introduced above.

2. Model

It is assumed that at the start of investment, the banker has a portfolio with the value

\[ X_0 = xc, \]

where \( x > 0 \) is the number of sold European call options and \( c > 0 \) is the price of one option. The strategy of the investor is to hand out credits on this sum of money, that is, the banker has the capital

\[ X_1 = \int_0^1 R_t dt - x(S - K)^+, \]

at the end of investment, where \( K \geq 0 \). Normalizing with \( x \), one can see that it is enough to estimate risks of the return,

\[ X = \int_0^1 r_t dt - (S - K)^+, \]

where \( r_t = R_t/x \).

It is supposed that the stock return is variance-gamma distributed, that is,

\[ S = \mu + \theta \gamma + \sigma N_\gamma, \]

where \( \gamma \) has the gamma distribution with unit mean, that is, with the probability density function

\[ f(\gamma, x) = \frac{a^a}{\Gamma(a)} x^{a-1} e^{-ax}, \]

where \( \Gamma(u), u > 0 \), is the gamma function. Thus, there are the sold European call options on interest rate (for example, swaptions or caplets) in the Merton variance-gamma model, or even the options on stock if the stock tail is modeled directly by the variance-gamma distribution but not by its exponential.

The process \( r_t \) is presumed to stay constant \( r_t = r_0 \geq 0 \) if the credit return does not change over the investment period, and to become \( r_t = r_1 \geq 0, r_1 \leq r_0 \) if the return falls. It is assumed that the time \( \tau \) of the return jump is exponentially distributed with the intensity \( \lambda \). It is easy to see, then, that

\[ X = r_0 I_{\{\tau > 1\}} + (r_0 \tau + r_1 (1 - \tau)) I_{\{\tau \leq 1\}} - (S - K)^+. \]

Next, we set

\[ \text{sign}(u) = \begin{cases} 1 & \text{if } u > 0, \\ 0 & \text{if } u = 0, \\ -1 & \text{if } u < 0, \end{cases} \]

and use notations

\[ \Psi(u), u \in \mathbb{R}, \quad \mathcal{B}(u_1, u_2), u_1 > 0, u_2 > 0, \quad \mathcal{K}_{u_1}(u_2), u_1 \in \mathbb{R}, u_2 > 0 \]
for the normal distribution function, the beta function and the MacDonald function (the modified Bessel function of the second kind), respectively. The hypergeometric Gauss function is denoted as

$$F(u_1, u_2, u_3; u_4), \ u_1, u_2, u_3 \in \mathbb{R}, u_4 < 1.$$  

Also, the degenerate Appell functions (or the Humbert series), which is the double sum,

$$\Phi(u_1, u_2, u_3; u_4, u_5) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(u_1)_m (u_2)_n}{m! n!} u_4^m u_5^n,$$

with $u_1, u_2, u_3, u_5 \in \mathbb{R}$ and $|u_4| < 1$, and where $(u)_l$, $l \in \mathbb{N} \cup \{0\}$, the Pochhammer’s symbol, is exploited. For more information on the special mathematical functions above, see the monographs by Bateman and Erdélyi (1953), Whittaker and Watson (1990) and also Gradshteyn and Ryzhik (1980).

Also, we need to represent the following constants and functions for the formulating of results. Let

$$w = \frac{1}{2} \left( \frac{\sigma \lambda}{r_1 - r_0} \right)^2 + \frac{\lambda \theta}{r_1 - r_0}, \quad z = \frac{\sigma \lambda}{r_1 - r_0} + \frac{\theta}{\sigma},$$  

$$\bar{w} = \frac{(\zeta \sigma)^2}{2} + \zeta \theta, \quad \bar{z} = \zeta \sigma + \frac{\theta}{\sigma},$$  

$$\Omega_j = \frac{\Gamma(a + j + \frac{1}{2})}{\Gamma(a + j)} \left( \frac{B\left(\frac{1}{2}, a + j\right)}{\sqrt{2}} + \frac{\theta}{\sqrt{a}} \right)^2 F\left(a + j + 1, \frac{1}{2}, 2; -\frac{\theta^2}{2a\sigma^2}\right),$$  

$$\Psi = \frac{\Gamma\left(a + \frac{1}{2}\right)}{(a - w)^{\frac{1}{2}}} \left( \frac{B\left(\frac{1}{2}, a\right)}{\sqrt{2}} + \frac{z}{\sqrt{a - w}} \right)^2 F\left(a + 1, 1, 3 \frac{1}{2} 2; -\frac{z^2}{2(a - w)}\right),$$  

$$Y_j = \frac{|k_j|^{\mu - \frac{1}{2}}} {\sqrt{2} \sqrt{(a - w)^{\frac{1}{2}}} \left( B(a, 1) \left| k_j \right| K_{a + \frac{1}{2}} \left( \frac{\left| k_j \right|}{\sqrt{2}} \right) + k_j K_{a - \frac{1}{2}} \left( \frac{\left| k_j \right|}{\sqrt{2}} \right) \right) \times \Phi\left(a, 1 - a, a + 1; 1 + v, -k_j(1 + v)\right) - (1 + v)k_j B(a, 1 + 1) \times K_{a - \frac{1}{2}} \left( \frac{\left| k_j \right|}{\sqrt{2}} \right) \Phi\left(a + 1, 1 - a, a + 2; 1 + v, -k_j(1 + v)\right)}$$  

with $v = \frac{z}{\sqrt{2} + 2(a - w)}$ and $k_j = (\mu + u - \rho l - K) \sqrt{v^2 + 2(a - w)}$,

$$\Lambda_j = \frac{|s_j|^{\rho + 1} \frac{1}{2} e^{(1+q)u + l}} {a^{\rho + 1}} \left( B(a + l, 1) \left| s_j \right| K_{a + l + \frac{1}{2}} \left( \frac{|s_j|}{\sqrt{2}} \right) + s_j K_{a + l - \frac{1}{2}} \left( \frac{|s_j|}{\sqrt{2}} \right) \right) \Phi\left(a + l, 1 - a - l, a + l + 1; 1 + q, -s_j(1 + q)\right) - (1 + q) B(a + l + 1, 1) K_{a + l - \frac{1}{2}} \left( \frac{|s_j|}{\sqrt{2}} \right) \times \Phi\left(a + l + 1, 1 - a - l, a + l + 2; 1 + q, -s_j(1 + q)\right).$$
where \( q = \frac{\theta}{\sqrt{\theta^2 + 2n\sigma^2}} \), \( s_j = \frac{\theta(\mu - u - r - K)}{\sqrt{\theta^2 + 2n\sigma^2}} \),

\[
\Xi_j = \frac{e^{\theta(\mu - u - r - K)}/[\theta^2 + 2n\sigma^2]}{a^{\theta + 1/2} [z]^{\theta - 1}} \left( K_{a + 1/2} \left( \frac{|z|}{\theta} \right) \right) + \\
+ \text{sign}(\mu + u - r - K) K_{a - 1/2} \left( \frac{|z|}{\theta} \right),
\]

(11)

\[
\Psi = \frac{2^{\theta - 1/2}}{a} \left( 1/2 \right) \left( a + 1 \right) \left( 2/2(w - a) \right) \left( 1/2(w - a) - 2^\theta \right),
\]

(12)

\[
\Theta_j = \frac{(c - 1)^{\theta + 1/2} \sqrt{1/2}}{(w - a)^\theta} \left[ B(a, 1) \Phi \left( a, 1 - a, a + 1; \frac{1}{2}, l_j, 1 - c \right) \right. \\
\times \left( l_j K_{a + 1/2} \left( |l_j| \right) + l_j K_{a - 1/2} \left( |l_j| \right) \right) + (c - 1) l_j K_{a - 1/2} \left( |l_j| \right) B(a + 1, 1) \times \\
\left. \times \Phi \left( a + 1, 1, a - a + 2; \frac{1}{2}, l_j, 1 - c \right) \right],
\]

(13)

where \( c = -\frac{z}{\sqrt{2^\theta - 2(a - w)}} \) and \( l_j = (\mu + u - r - K) \sqrt{2^\theta - 2(a - w)} \),

\[
\Lambda_j = \frac{e^{\theta(\mu - u - r - K)}/[\theta^2 + 2n\sigma^2]}{a^{\theta + 1/2} [s]^{\theta - 1}} \left\{ B(a + l, 1) \left( s \right) K_{a + l + 1/2} \left( |s| \right) + \\
+ s K_{a + l - l/2} \left( |s| \right) \Phi \left( a + l, 1 - a, a + l + 1; \frac{1}{2}, -s(1 + q) \right) - \\
- (1 + q) s B(a + l + 1, 1) K_{a + l - l/2} \left( |s| \right) \times \\
\times \Phi \left( a + l + 1, 1 - a, a + l + 2; \frac{1}{2}, -s(1 + q) \right) \right\},
\]

(14)

with \( s = (\mu - K) \sqrt{2^\theta + 2(a - w)} \),

\[
\Upsilon = \frac{e^{\theta(\mu - u - r - K)}/[\theta^2 + 2n\sigma^2]}{a^{\theta + 1/2} [s]^{\theta - 1}} \left\{ B(a, 1) \left( k \right) K_{a + 1/2} \left( |k| \right) + k K_{a - 1/2} \left( |k| \right) \right. \\
\times \Phi \left( a + 1 - a, a + 1; \frac{1}{2}, -k(1 + v) \right) - \\
\left. (1 + v) k B(a + 1, 1) \times \\
K_{a - 1/2} \left( |k| \right) \Phi \left( a + 1, 1 - a, a + 2; \frac{1}{2}, -k(1 + v) \right) \right\},
\]

(15)

where \( k = (\mu - K) \sqrt{2^\theta + 2(a - w)} \),

\[
\Xi = \frac{e^{\theta(\mu - u - r - K)}/[\theta^2 + 2n\sigma^2]}{a^{\theta + 1/2} [s]^{\theta - 1}} \left( K_{a + 1/2} \left( \frac{|z|}{\theta} \right) \right) + \\
+ \text{sign}(\mu - K) K_{a - 1/2} \left( \frac{|z|}{\theta} \right),
\]

(16)
Theorem 1. Let the random variable \( X \) be defined by (1)–(4). Then the probability \( P(X \leq u) \) can be calculated with respect to the next ratios, where the constant and functions \( w, z, \Psi, Y, \Lambda, \Xi, \Theta, \Lambda, \bar{Y}, \bar{\Theta} \) are defined in (5) and (7)–(17).

**Ratio 1:** \( r_0 > r_1 \).

**Ratio 1.1:** \( u < r_1 \).

**Ratio 1.1.1:** \( a > w \).

\[
\Theta = \frac{(c-1)^{|a|}a^{1/2}d}{(a-\mu)^{2}} \left[ B(a,1) \Phi \left( a, 1-a, a+1; \frac{1-\xi}{2}, I(1-c) \right) \times 
\right. \\
\left. \times \left( |l| K_{a+1/2} (|l|) + IK_{a-1/2} (|l|) \right) + (c-1) IK_{a-1/2} (|l|) B(a+1,1) \times 
\right. \\
\left. \times \Phi \left( a+1, 1-a, a+2; \frac{1-\xi}{2}, I(1-c) \right) \right] 
\]

with \( l = (\mu - K) \sqrt{z^2 - 2(w-a)} \),

\[
\mathcal{A}_j = 2e^{\frac{\mu - (a-\mu)}{\sigma^2}} \left( \frac{|l|}{\sqrt{\sigma^2 + 2\sigma^2 a}} \right) \times 
\]

\[
\mathcal{A}_t = 2e^{\frac{\mu - (a-\mu)}{\sigma^2}} \left( \frac{|K - \mu|}{\sqrt{\sigma^2 + 2\sigma^2 a}} \right) \times 
\]

\[
\mathcal{M} = \frac{\Gamma \left( a + \frac{1}{2} \right)}{\Gamma \left( a + \frac{1}{2} \right)} \left( B \left( \frac{1}{2}, a \right) + \frac{\sqrt{2}}{\sqrt{a-w}} \left( a + \frac{1}{2} \right) \right) 
\]

and

\[
\bar{Y}_j = \frac{\tilde{k}_j^{1/2}(1+\tilde{\nu})^a}{(a-\tilde{u})^a} \left\{ B(a,1) \left( |\tilde{k}_j| K_{a+1/2} (|\tilde{k}_j|) + \tilde{k}_j K_{a-1/2} (|\tilde{k}_j|) \right) \times 
\right. \\
\left. \times \Phi \left( a, 1-a, a+1; \frac{1+\tilde{\nu}}{2}, -\tilde{k}_j (1+\tilde{\nu}) \right) - (1+\tilde{\nu}) \tilde{k}_j B(a+1,1) \times 
\right. \\
\left. \times K_{a-1/2} \left( |\tilde{k}_j| \right) \Phi \left( a+1, 1-a, a+2; \frac{1+\tilde{\nu}}{2}, -\tilde{k}_j (1+\tilde{\nu}) \right) \right\} 
\]

with \( \tilde{\nu} = \frac{\tilde{\nu}}{\sqrt{z^2 + 2(a-\tilde{u})}} \) and \( \tilde{k}_j = (\mu + u - r_i - K) \sqrt{z^2 + 2(a-\tilde{u})} \).

3. Results

First, let us discuss the basic monetary risk measure for an investment portfolio, that is, the value at risk (VaR). Properties and methods of computing of the VaR in various models are discussed in particular in the papers by Berkowitz et al. (2002), Chen and Tang (2005), Pritsker (1997) and Tsyurmasto et al. (2014). If the financial losses are modeled by a random variable \( X \) and \( 0 < \alpha < 1 \), the \( \alpha \)-VaR of \( X \), \( \text{VaR}_\alpha (X) \), is defined as the low \( \alpha \)-quantile of the distribution of \( X \) with minus sign, that is,

\[
\text{VaR}_\alpha (X) = -u_\alpha, \quad \text{where} \ u_\alpha = \inf \{ u \in \mathbb{R} : P(X \leq u) \geq \alpha \}. 
\]

Thereby, one needs to be able to compute the probability \( P(X \leq u) \) if they want to derive \( \text{VaR}_\alpha (X) \).
If $K + r_1 = \mu + u$, 

$$P(X \leq u) = \frac{a^a}{\Gamma(a) \sqrt{2\pi}} \left( \Lambda_{10} + e^{-\frac{\lambda(t_{1-u+K-u})}{0^{+1}}} (\Omega_0 - Y_1) \right).$$  \hspace{1cm} (22)$$

When $K + r_0 = \mu + u$, 

$$P(X \leq u) = \frac{a^a}{\Gamma(a) \sqrt{2\pi}} \left( \Lambda_{10} + e^{-\frac{\lambda(t_{1-u+K-u})}{0^{+1}}} (\Omega_0 - Y_1) \right).$$  \hspace{1cm} (23)$$

If $K + r_1 \neq \mu + u$ and $K + r_0 \neq \mu + u$, 

$$P(X \leq u) = \frac{a^a}{\Gamma(a) \sqrt{2\pi}} \left( \Lambda_{10} + e^{-\frac{\lambda(t_{1-u+K-u})}{0^{+1}}} (Y_0 - Y_1) \right).$$  \hspace{1cm} (24)$$

Ratio 1.1.2: $a = w$.

When $K + r_1 = \mu + u$, 

$$P(X \leq u) = \frac{a^a}{\Gamma(a) \sqrt{2\pi}} \left( \Lambda_{10} + e^{-\frac{\lambda(t_{1-u+K-u})}{0^{+1}}} (Y_0 - Y_1) \right).$$  \hspace{1cm} (25)$$

If $K + r_0 = \mu + u$, 

$$P(X \leq u) = \frac{a^a}{\Gamma(a) \sqrt{2\pi}} \left( \Lambda_{10} + e^{-\frac{\lambda(t_{1-u+K-u})}{0^{+1}}} (Y_0 - Y_1) \right).$$  \hspace{1cm} (26)$$

When $K + r_1 \neq \mu + u$ and $K + r_0 \neq \mu + u$, 

$$P(X \leq u) = \frac{a^a}{\Gamma(a) \sqrt{2\pi}} \left( \Lambda_{10} + e^{-\frac{\lambda(t_{1-u+K-u})}{0^{+1}}} (Y_0 - Y_1) \right).$$  \hspace{1cm} (27)$$

Ratio 1.1.3: $a < w$, $\theta \neq 0$.

If $K + r_1 = \mu + u$, 

$$P(X \leq u) = \frac{a^a}{\Gamma(a) \sqrt{2\pi}} \left( \Lambda_{10} + e^{-\frac{\lambda(t_{1-u+K-u})}{0^{+1}}} (Y_0 - Y_1) \right).$$  \hspace{1cm} (28)$$

When $K + r_0 = \mu + u$, 

$$P(X \leq u) = \frac{a^a}{\Gamma(a) \sqrt{2\pi}} \left( \Lambda_{10} + e^{-\frac{\lambda(t_{1-u+K-u})}{0^{+1}}} (Y_0 - Y_1) \right).$$  \hspace{1cm} (29)$$

If $K + r_1 \neq \mu + u$ and $K + r_0 \neq \mu + u$, 

$$P(X \leq u) = \frac{a^a}{\Gamma(a) \sqrt{2\pi}} \left( \Lambda_{10} + e^{-\frac{\lambda(t_{1-u+K-u})}{0^{+1}}} (Y_0 - Y_1) \right).$$  \hspace{1cm} (30)$$

Ratio 1.2: $r_1 \leq u < r_0$.

Ratio 1.2.1: $a > w$. 

Ratio 1.2.2: $a < w$.
When $K = \mu$,

$$P(X \leq u) = 1 - e^{-\frac{a}{\mu} + \frac{\lambda(\mu)}{\mu} \frac{1}{\Gamma(a) \sqrt{2\pi}}} (\mu_0 + \frac{\lambda(\mu)}{\mu \Gamma(a) \sqrt{2\pi}}).$$

If $K + r_0 = \mu + u$,

$$P(X \leq u) = 1 - e^{-\frac{a}{\mu} + \frac{\lambda(\mu)}{\mu} \frac{1}{\Gamma(a) \sqrt{2\pi}}} + \frac{a^2 e^{-\frac{a}{\mu}}}{\Gamma(a) \sqrt{2\pi}} \left( 2^{\frac{1}{2}} \Gamma(a + \frac{1}{2}) \right).$$

When $K \neq \mu$ and $K + r_0 \neq \mu + u$,

$$P(X \leq u) = 1 - e^{-\frac{a}{\mu} + \frac{\lambda(\mu)}{\mu} \frac{1}{\Gamma(a) \sqrt{2\pi}}} + \frac{a^2 e^{-\frac{a}{\mu}}}{\Gamma(a) \sqrt{2\pi}} \left( 2^{\frac{1}{2}} \Gamma(a + \frac{1}{2}) \right).$$

Ratio 1.2.2: $a = w$.

When $K = \mu$,

$$P(X \leq u) = 1 - e^{-\frac{a}{\mu} + \frac{\lambda(\mu)}{\mu} \frac{1}{\Gamma(a) \sqrt{2\pi}}} (\mu_0 + \frac{\lambda(\mu)}{\mu \Gamma(a) \sqrt{2\pi}}).$$

If $K + r_0 = \mu + u$,

$$P(X \leq u) = 1 - e^{-\frac{a}{\mu} + \frac{\lambda(\mu)}{\mu} \frac{1}{\Gamma(a) \sqrt{2\pi}}} + \frac{a^2 e^{-\frac{a}{\mu}}}{\Gamma(a) \sqrt{2\pi}} \left( 2^{\frac{1}{2}} \Gamma(a + \frac{1}{2}) \right).$$

When $K \neq \mu$ and $K + r_0 \neq \mu + u$,

$$P(X \leq u) = 1 - e^{-\frac{a}{\mu} + \frac{\lambda(\mu)}{\mu} \frac{1}{\Gamma(a) \sqrt{2\pi}}} + \frac{a^2 e^{-\frac{a}{\mu}}}{\Gamma(a) \sqrt{2\pi}} \left( 2^{\frac{1}{2}} \Gamma(a + \frac{1}{2}) \right).$$

Ratio 1.2.3: $a < w, \theta \neq 0$.

If $K = \mu$,

$$P(X \leq u) = 1 - e^{-\frac{a}{\mu} + \frac{\lambda(\mu)}{\mu} \frac{1}{\Gamma(a) \sqrt{2\pi}}} (\mu_0 + \Theta_0 - \Theta).$$

When $K + r_0 = \mu + u$,

$$P(X \leq u) = 1 - e^{-\frac{a}{\mu} + \frac{\lambda(\mu)}{\mu} \frac{1}{\Gamma(a) \sqrt{2\pi}}} + \frac{a^2 e^{-\frac{a}{\mu}}}{\Gamma(a) \sqrt{2\pi}} \left( 2^{\frac{1}{2}} \Gamma(a + \frac{1}{2}) \right).$$

If $K \neq \mu$ and $K + r_0 \neq \mu + u$,

$$P(X \leq u) = 1 - e^{-\frac{a}{\mu} + \frac{\lambda(\mu)}{\mu} \frac{1}{\Gamma(a) \sqrt{2\pi}}} + \frac{a^2 e^{-\frac{a}{\mu}}}{\Gamma(a) \sqrt{2\pi}} \left( 2^{\frac{1}{2}} \Gamma(a + \frac{1}{2}) \right).$$

Ratio 1.3: $u \geq r_0$. 
Then

\[ P(X \leq u) = 1. \]

**Ratio 2:** \( r_0 = r_1. \)

**Ratio 2.1:** \( u < r_0. \)

When \( K + r_0 = \mu + u, \)

\[ P(X \leq u) = \frac{a^a}{\Gamma(a) \sqrt{2\pi}} \mu_0. \]

If \( K + r_0 \neq \mu + u, \)

\[ P(X \leq u) = \frac{a^a}{\Gamma(a) \sqrt{2\pi}} \Lambda_{00}. \]

**Ratio 2.2:** \( u \geq r_0. \)

Then

\[ P(X \leq u) = 1. \]

Because VaR\(_\alpha\)(\(X\)) does not take into account the size of losses, the monetary measure ES\(_\alpha\)(\(X\)), the expected shortfall, which is defined as

\[ ES\alpha(X) = -\frac{1}{\alpha} \left[ \mathbb{E} I_{\{X \leq u\}_\alpha} + u_\alpha \left( \alpha - P(X \leq u_\alpha) \right) \right], \]

was recommended to financial institutions as a preferable alternative to the VaR in Basel III (2011). For continuous distributions, ES\(_\alpha\)(\(\cdot\)) coincides with the measure CVaR\(_\alpha\)(\(\cdot\)), the conditional value at risk, which is the mathematical expectation of “the tail” of the distribution:

\[ CVaR\alpha(X) = -\frac{1}{\alpha} \mathbb{E} I_{\{X \leq u\}_\alpha}. \]

The CVaR monetary measure was first introduced and studied for portfolio losses that are determined by the normal distribution in Rockafellar and Uryasev (2000). General properties of the CVaR were investigated in Rockafellar and Uryasev (2002). For methods of numerical estimation of the CVaR, see the works by Chernozhukov and Umantsev (2001) and Chun et al. (2012). For more recent studies, applications and investigations of the ES in various models, see in particular the works by Drapeau et al. (2014), Ivanov (2018), Kalinchenko et al. (2012) and Mafusalov and Uryasev (2016).

Obviously, it is required to obtain \( \mathbb{E} \left( X I_{\{X \leq u\}_\alpha} \right) \) for calculating both the expected shortfall and the conditional value at risk. Set

\[ D = \left( \frac{r_1 - r_0}{\lambda} - u \right) e^{\frac{\lambda(r_1 - r_0 + K - \mu)}{r_0 - r_1}}. \] (40)

\[ W_1 = \left( \frac{r_0 - r_1}{\lambda} + r_1 + K - \mu \right) \mu_0 - \frac{r_0 - r_1}{\lambda e^\lambda} \Lambda_{00} - \theta \Lambda_{11} - \sigma^2 \Lambda_{11}, \] (41)

\[ W_2 = \left( \frac{r_0 - r_1}{\lambda} + r_1 + K - \mu \right) \Lambda_{10} - \frac{r_0 - r_1}{\lambda e^\lambda} \mu_0 - \theta \Lambda_{11} - \sigma^2 \Lambda_{11}, \] (42)

\[ W_3 = \left( \frac{r_0 - r_1}{\lambda} + r_1 + K - \mu \right) \Lambda_{10} - \frac{r_0 - r_1}{\lambda e^\lambda} \Lambda_{00} - \theta \Lambda_{11} - \sigma^2 \Lambda_{11}. \] (43)
The next theorem is valid.

**Theorem 2.** If $X$ is defined by (1)–(4) and $u < r_1$, the expectation $E\left(X I_{\{X \leq u\}}\right)$ is computed with respect to the following ratios, where the constants and functions $w, z, U, W, Y, \Lambda, \Xi, \Psi, \Theta, \Lambda, W_1, W_2, W_3$ are defined in (5), (7)–(13), (18) and (40)–(43).

*Ratio 1:* $r_0 > r_1$.

*Ratio 1.1:* $a > w$.

If $r_1 + K = \mu + u$,

$$E\left(X I_{\{X \leq u\}}\right) = \frac{a^a}{\Gamma(a) \sqrt{2\pi}} \left(W_1 - W + Y_0\right). \quad (44)$$

When $r_0 + K = \mu + u$,

$$E\left(X I_{\{X \leq u\}}\right) = \frac{a^a}{\Gamma(a) \sqrt{2\pi}} \left(W_2 + \Theta\left(W - Y_1\right)\right). \quad (45)$$

If $r_1 + K \neq \mu + u$ and $r_0 + K \neq \mu + u$,

$$E\left(X I_{\{X \leq u\}}\right) = \frac{a^a}{\Gamma(a) \sqrt{2\pi}} \left(W_3 + \Theta\left(Y_0 - Y_1\right)\right). \quad (46)$$

*Ratio 1.2:* $a = w$.

When $r_1 + K = \mu + u$,

$$E\left(X I_{\{X \leq u\}}\right) = \frac{a^a}{\Gamma(a) \sqrt{2\pi}} \left(\frac{2^{a-\frac{1}{2}} \Gamma\left(a + \frac{1}{2}\right)}{a |z|^{2a}} + \Xi_0\right). \quad (47)$$

If $r_0 + K = \mu + u$,

$$E\left(X I_{\{X \leq u\}}\right) = \frac{a^a}{\Gamma(a) \sqrt{2\pi}} \left(W_2 + \Theta\left(\frac{2^{a-\frac{1}{2}} \Gamma\left(a + \frac{1}{2}\right)}{a |z|^{2a}} - \Xi_1\right)\right). \quad (48)$$

When $r_1 + K \neq \mu + u$ and $r_0 + K \neq \mu + u$,

$$E\left(X I_{\{X \leq u\}}\right) = \frac{a^a}{\Gamma(a) \sqrt{2\pi}} \left(W_3 + \Theta\left(\Xi_0 - \Xi_1\right)\right). \quad (49)$$

*Ratio 1.3:* $a < w$, $\theta \neq 0$.

If $r_1 + K = \mu + u$,

$$E\left(X I_{\{X \leq u\}}\right) = \frac{a^a}{\Gamma(a) \sqrt{2\pi}} \left(W_1 - W + \Theta_0\right). \quad (50)$$

When $r_0 + K = \mu + u$,

$$E\left(X I_{\{X \leq u\}}\right) = \frac{a^a}{\Gamma(a) \sqrt{2\pi}} \left(W_2 + \Theta\left(W - \Theta_1\right)\right). \quad (51)$$
If \( r_1 + K \neq \mu + u \) and \( r_0 + K \neq \mu + u \),

\[
E \left( X I_{\{X \leq u\}} \right) = \frac{a^a}{\Gamma(a) \sqrt{2\pi}} \left( \mathbb{W}_3 + \mathcal{D} (\Theta_0 - \Theta_1) \right). \tag{52}
\]

**Ratio 2:** \( r_0 = r_1 \).

When \( r_0 + K = \mu + u \),

\[
E \left( X I_{\{X \leq u\}} \right) = \frac{a^a}{\Gamma(a) \sqrt{2\pi}} \left( (r_0 + K - \mu) \Lambda_0 - \theta \Lambda_1 - \sigma \left( \frac{2\sigma^2}{\theta^2 + 2\sigma^2a} \right) \Gamma \left( a + \frac{1}{2} \right) \right).
\]

If \( r_0 + K \neq \mu + u \),

\[
E \left( X I_{\{X \leq u\}} \right) = \frac{a^a}{\Gamma(a) \sqrt{2\pi}} \left( (r_0 + K - \mu) \Lambda_0 - \theta \Lambda_1 - \sigma \mathcal{A}_0 \right). \tag{53}
\]

**Remark 1.** Theorem 2 gives us the value of \( E \left( X I_{\{X \leq u\}} \right) \) for \( u < r_1 \). If \( u \geq r_1 \), similar formulas can be derived as well. They will be dependent then also on the meanings of the functions \( \tilde{\Lambda}_l, \tilde{\Psi}, \tilde{\Theta}, \tilde{\Omega}, \tilde{\Xi} \) which are defined in (14)–(17) and (19).

Next, the entropic risk measure (see for details and the importance of this measure, for example, the monograph by Föllmer and Schied (2004) and the works by Barrieu and El Karoui (2005), Föllmer and Schied (2010), Ivanov and Temnov (2017) and Ivanov (2018)) for the tail of distribution \( X \) is defined as

\[
\text{ERM}_\zeta(X, u) = \frac{1}{\zeta} \log E \left( e^{-\zeta X I_{\{X \leq u\}}} \right), \quad \zeta > 0.
\]

Let

\[
\zeta \theta + \frac{\zeta^2 \sigma^2}{2} < a. \tag{55}
\]

The validity of this inequality is the necessary and sufficient condition for the finiteness of expectation

\[
E e^{\xi S} < \infty.
\]

Set

\[
C_1 = \frac{\zeta (r_0 - r_1) e^{-\lambda - \zeta(r_0 + K - \mu)}}{\lambda + \zeta (r_0 - r_1)}, \tag{54}
\]

\[
C_2 = \frac{\lambda e^{-\zeta(r_1 + K - \mu)}}{\lambda + \zeta (r_0 - r_1)}, \tag{55}
\]

\[
C_3 = \frac{\lambda e^{\frac{\lambda (\mu + K - r_1)}{r_1 - r_0} - \zeta u}}{\lambda + \zeta (r_0 - r_1)}. \tag{56}
\]

The next theorem allows to establish the value of the entropic risk measure for \( X \).
Theorem 3. Let $X$ be defined in (1)–(4). Assume that $u < r_1$ and (53) holds. Then the expectation $E \left( e^{-\xi X I_{\{X \leq u\}}} \right)$ can be computed with respect to the following ratios, where the constants and functions $w, z, w, \mathcal{Z}, \mathcal{W}, Y_j, \Xi_j, \Theta_j, \mathcal{W}, \bar{Y}_j$ are defined in (5) and (6), (8) and (9), (11)–(13) and (20) and (21).

**Ratio 1.** $r_0 > r_1$.

When $r_1 + K = \mu + u$,

$$E \left( e^{-\xi X I_{\{X \leq u\}}} \right) = \frac{a^\mu}{\Gamma(a) \sqrt{2\pi}} \left( C_1 \bar{Y}_0 + C_2 \mathcal{W} - C_3 (\mathcal{W} - Y_0) \right). \quad (57)$$

If $r_0 + K = \mu + u$,

$$E \left( e^{-\xi X I_{\{X \leq u\}}} \right) = \frac{a^\mu}{\Gamma(a) \sqrt{2\pi}} \left( C_1 \bar{Y}_0 + C_2 \bar{Y}_1 - C_3 (Y_1 - \mathcal{W}) \right). \quad (58)$$

When $r_1 + K \neq \mu + u$ and $r_0 + K \neq \mu + u$,

$$E \left( e^{-\xi X I_{\{X \leq u\}}} \right) = \frac{a^\mu}{\Gamma(a) \sqrt{2\pi}} \left( C_1 \bar{Y}_0 + C_2 \bar{Y}_1 - C_3 (Y_1 - Y_0) \right). \quad (59)$$

**Ratio 1.1.** $a > w$.

If $r_1 + K = \mu + u$,

$$E \left( e^{-\xi X I_{\{X \leq u\}}} \right) =$$

$$= \frac{a^\mu}{\Gamma(a) \sqrt{2\pi}} \left( C_1 \bar{Y}_0 + C_2 \mathcal{W} - C_3 \left( \frac{2^{a-1} \Gamma \left( a + \frac{1}{2} \right)}{a|z|^{2a}} - \Xi_0 \right) \right). \quad (60)$$

When $r_0 + K = \mu + u$,

$$E \left( e^{-\xi X I_{\{X \leq u\}}} \right) =$$

$$= \frac{a^\mu}{\Gamma(a) \sqrt{2\pi}} \left( C_1 \bar{W} + C_2 \bar{Y}_1 - C_3 \left( \Xi_1 - \frac{2^{a-1} \Gamma \left( a + \frac{1}{2} \right)}{a|z|^{2a}} \right) \right). \quad (61)$$

If $r_1 + K \neq \mu + u$ and $r_0 + K \neq \mu + u$,

$$E \left( e^{-\xi X I_{\{X \leq u\}}} \right) = \frac{a^\mu}{\Gamma(a) \sqrt{2\pi}} \left( C_1 \bar{Y}_0 + C_2 \bar{Y}_1 - C_3 (\Xi_1 - \Xi_0) \right). \quad (62)$$

**Ratio 1.2.** $a = w$.

If $r_1 + K = \mu + u$,

$$E \left( e^{-\xi X I_{\{X \leq u\}}} \right) =$$

$$= \frac{a^\mu}{\Gamma(a) \sqrt{2\pi}} \left( C_1 \bar{Y}_0 + C_2 \mathcal{W} - C_3 \left( \frac{2^{a-1} \Gamma \left( a + \frac{1}{2} \right)}{a|z|^{2a}} - \Xi_0 \right) \right). \quad (63)$$

When $r_0 + K = \mu + u$,

$$E \left( e^{-\xi X I_{\{X \leq u\}}} \right) =$$

$$= \frac{a^\mu}{\Gamma(a) \sqrt{2\pi}} \left( C_1 \bar{W} + C_2 \bar{Y}_1 - C_3 \left( \Xi_1 - \frac{2^{a-1} \Gamma \left( a + \frac{1}{2} \right)}{a|z|^{2a}} \right) \right). \quad (64)$$

If $r_0 + K \neq \mu + u$ and $r_0 + K \neq \mu + u$,

$$E \left( e^{-\xi X I_{\{X \leq u\}}} \right) =$$

$$= \frac{a^\mu}{\Gamma(a) \sqrt{2\pi}} \left( C_1 \bar{Y}_0 + C_2 \bar{Y}_1 - C_3 (\Xi_1 - \Xi_0) \right). \quad (65)$$
When \( r_1 + K \neq \mu + u \) and \( r_0 + K \neq \mu + u \),
\[
E \left( e^{-\zeta X} I_{\{X \leq u\}} \right) = \frac{a^a}{\Gamma(a) \sqrt{2\pi}} \left( C_1 \tilde{Y}_0 + C_2 \tilde{Y}_1 - C_3 (\Theta_1 - \Theta_0) \right). \tag{65}
\]

Ratio 2. \( r_0 = r_1 \).

If \( r_0 + K = \mu + u \),
\[
E \left( e^{-\zeta X} I_{\{X \leq u\}} \right) = \frac{a^a e^{(\mu - r_0 - K)} \zeta}{\Gamma(a) \sqrt{2\pi}} \tilde{Y}_0. \tag{66}
\]

When \( r_0 + K \neq \mu + u \),
\[
E \left( e^{-\zeta X} I_{\{X \leq u\}} \right) = \frac{a^a e^{(\mu - r_0 - K)} \zeta}{\Gamma(a) \sqrt{2\pi}} \tilde{Y}_0. \tag{67}
\]

The example below explains how the obtained results can be easily enough exploited by practitioners.

Example 1. If we discuss the calculation of downside risks at practice as an application of our results, it is usual to assume that \( u < 0 \) in Theorems 1–3. Since \( r_1 > 0 \), we have that the condition \( u < r_1 \) of Theorems 2 and 3 holds.

Also, it is natural to propose that the call option strike price \( K > 0 \). Hence a practitioner often has that \( K + r_0 > \mu + u \) and \( K + r_1 > \mu + u \). Therefore, if one wants to compute the expected shortfall of the considered portfolio, they need in fact only the formulas (46) and (52) instead of the result of Theorem 2. Similarly, the results of Theorems 1 and 3 are not too complicated either.

4. Conclusions

This paper introduces the formulas that allow one to calculate the VaR, the expected shortfall risk and the entropic risk measure for an investor who issues mortgages on the income earned from the sale of a package of European call options. It is assumed that the underlying stock return is variance-gamma distributed but the credit payment flow could drop down. The time of possible downside jump of credit payments is proposed to be distributed exponentially. The size of this jump is assumed to be constant and it is expected that the random size of the jump will be discussed in future investigations. Also, a number of jumps of the payment flow can be considered as well. Another idea is to model the stock return by the variance-gamma distribution with extra randomness in mean and volatility.

5. Proofs

Proof of Theorem 1. Since
\[
P(X \leq u) = E \left( E \left( I_{\{X \leq u\}} | S \right) \right),
\]
one could start from the computing of the function
\[
g(S) = E \left( I_{\{X \leq u\}} | S \right) = P \left( r_0 I_{\{\tau > 1\}} + (r_0 \tau + r_1 (1 - \tau)) I_{\{\tau \leq 1\}} \leq u + (S - K)^+ | S \right). \tag{68}
\]

Let \( r_0 = r_1 \). Then, obviously,
\[
g(S) = I_{\{r_0 \leq u + (S - K)^+ \}} \tag{69}
\]
and

\[ P(X \leq u) = E g(S) = I_{\{r_0 \leq u\}} + P(S \geq r_0 - u + K) I_{\{r_0 > u\}}. \]  

(70)

Assume that \( r_0 > r_1 \). Then, if

\[ r_0 \leq u, \]  

(71)

the function

\[ g(S) = 1. \]  

(72)

Let \( r_0 > r_1 \) and (71) be not satisfied. Then,

\[
g(S) = e^{-\lambda} I_{\{r_0 \leq u+(S-K)^+\}} + P \left( \tau \leq \min \left\{ \frac{u+(S-K)^+ - r_1}{r_0 - r_1}, 1 \right\} \bigg| S \right) =
\]

\[
= \left( 1 - e^{-\frac{\lambda(r_1 - (S-K)^+) - u}{r_0 - r_1}} \right) I_{\left\{ \frac{u+(S-K)^+ - r_1}{r_0 - r_1} \geq 1 \right\}} + e^{-\lambda} I_{\{r_0 \leq u+(S-K)^+\}}
\]

\[
= I_{\{r_0 \leq u+(S-K)^+\}} + \left( 1 - e^{-\frac{\lambda(r_1 - (S-K)^+) - u}{r_0 - r_1}} \right) I_{\left\{ \frac{u+(S-K)^+ - r_1}{r_0 - r_1} < 1 \right\}}.
\]  

(73)

It follows from (73) that

\[ g(S) = l_1(S) + l_2(S), \]  

(74)

where

\[ l_1(S) = I_{\{r_0 \leq u+(S-K)^+\}} \]

and

\[ l_2(S) = \left( 1 - e^{-\frac{\lambda(r_1 - (S-K)^+) - u}{r_0 - r_1}} \right) I_{\{r_1 \leq u+(S-K)^+ < r_0\}}. \]

Next, we have that

\[
E l_1(S) = E I_{\{S \geq K, r_0 \leq u + S - K\}} = P(S \geq \max \{K, K + r_0 - u\})
\]  

(75)

and

\[
E l_2(S) = \left( 1 - e^{-\frac{\lambda(r_1 - u)}{r_0 - r_1}} \right) E I_{\{S < K, r_1 \leq u < r_0\}} +
\]

\[ + E \left( 1 - e^{-\frac{\lambda(r_1 - S + K - u)}{r_0 - r_1}} \right) I_{\{S \leq K, r_1 \leq u + S - K < r_0\}} =
\]

\[
= \left( 1 - e^{-\frac{\lambda(r_1 - u)}{r_0 - r_1}} \right) P(S < K) I_{\{r_1 \leq u < r_0\}} +
\]

\[ + E \left( 1 - e^{-\frac{\lambda(r_1 - S + K - u)}{r_0 - r_1}} \right) I_{\{\max \{K, r_1 - u + K\} \leq S < r_0 - u + K\}}. \]  

(76)
Combining together (69)–(76), one can observe that

$$
P(X \leq u) = I_{\{r_0 \leq u\}} + P(S \geq r_0 - u + K)I_{\{r_0 > u\}} + \left(1 - e^{-\frac{\lambda(1-a)}{\eta}}\right)P(S < K)I_{\{r_0 > r_1, r_1 \leq u < r_0\}} + P(\max\{K, r_1 - u + K\} \leq S < r_0 - u + K)I_{\{r_0 > r_1, r_0 > u\}} - e^{-\frac{\lambda(1-a)}{\eta}}E\left(\frac{1}{\eta} e^{\frac{1}{2} \sigma^2 \gamma} \right),
$$

and therefore it is necessary to compute

$$
Ee^{AS}I_{\{S \geq C\}},
$$

where $A \in \{0, \frac{\lambda}{r_1 - r_0}\} \leq 0$ and $C \in \mathbb{R}$.

It is easy to observe that

$$
E\left(e^{AS}I_{\{S \geq C\}}|\gamma\right) = e^{A(\mu + \theta\gamma)}E\left(e^{\lambda\sigma N_{\gamma}}I_{\{\sigma_{\gamma} \geq C-\mu - \theta\gamma\}}|\gamma\right) =
$$

$$
e^{A\mu + A(\sigma^2 A + 2\theta)\frac{\pi}{2}} \int_{C-\mu-(\theta + \sigma^2 A)\gamma}^{\infty} \frac{1}{\sigma \sqrt{2\pi\gamma}} e^{-\frac{x^2}{2\sigma^2 \gamma}} dx = e^{A\mu + A(\sigma^2 A + 2\theta)\frac{\pi}{2}} \gamma \Psi\left(\mu + \frac{C - \gamma(\sigma^2 A + \theta)}{\sigma \sqrt{\gamma}}\right) \Psi\left(\mu + \frac{C - \gamma(\sigma^2 A + \theta)}{\sigma \sqrt{\gamma}}\right).
$$

Keeping in mind (3), one could notice that it is required to calculate for the computing of (78) the integral

$$
I = \int_{0}^{\infty} x^a e^{-\left(b - \frac{\theta}{\sigma}\right)x} \Psi\left(h\sqrt{x} + \frac{p}{\sqrt{x}}\right) dx,
$$

where $a > -1$, $b > 0$, $p \in \mathbb{R}$ and

$$
(b, h) \in \left\{\left(0, \frac{\theta}{\sigma}\right), \left(\frac{\lambda}{2(r_1 - r_0)}, \frac{\lambda\sigma^2}{r_1 - r_0} + 2\theta, \frac{\lambda\sigma}{r_1 - r_0} + \frac{\theta}{\sigma}\right)\right\} = \left\{\left(0, \frac{\theta}{\sigma}\right), (\omega, z)\right\}.
$$

The integral (79) is the same as the one in the bottom of p. 207 of Ivanov and Ano (2016), which was partly computed for some values of the parameters therein. Further, we briefly summarize the results of Ivanov and Ano (2016) and calculate (79) for extra ratios between the parameters of the model.

Case 1. $a = b$.
Case 1.1. $h \geq 0$. Then, obviously, $I = \infty$.
Case 1.2. $h < 0$, $p = 0$. We have here that

$$
I = \int_{0}^{\infty} x^a \Psi\left(h\sqrt{x}\right) dx = \frac{h}{2(\alpha + 1)\sqrt{2\pi}} \int_{0}^{\infty} x^{a + \frac{1}{2}} e^{-\frac{x^2}{2}} dx = \frac{2^\alpha \Gamma\left(\alpha + \frac{3}{2}\right)}{(\alpha + 1)|h|^{2(\alpha + 1)}\sqrt{\pi}}.
$$
Case 1.3. $h < 0$, $p \neq 0$. Integrating by parts, we have in this case that

$$I = -\frac{1}{a+1} \int_0^\infty x^{a+1} d\Psi \left(h\sqrt{x} + \frac{p}{\sqrt{x}}\right) = \frac{1}{2(a+1)\sqrt{2\pi}} \int_0^\infty (px^{a-\frac{1}{2}} - h x^{a+\frac{1}{2}}) e^{-\frac{(hx+p)^2}{2}} dx.$$ 

The formula 3.471.9 from Gradshteyn and Ryzhik (1980) includes the identity

$$\int_0^\infty x^{a-1} e^{-\frac{a_1^2}{2}} - \frac{a_2 x}{\sqrt{2\pi}} dx = 2 \left(\frac{a_2}{a_3}\right)^{\frac{1}{2}} \Phi \left(2\sqrt{\frac{a_2}{a_3}}\right),$$  

(80)

where $a_1 \in \mathbb{R}$, $a_2 > 0$ and $a_3 > 0$. Using (80), we get that

$$I = \frac{e^{-hp}}{(a + 1)\sqrt{2\pi}} \left(\frac{B\left(\frac{1}{2}, a + 1\right)}{|a+\frac{1}{2}|^2} K_{a+\frac{1}{2}}(|hp|) - \frac{h/|p| a + \frac{1}{2}}{|a+\frac{1}{2}|^2} K_{a+\frac{1}{2}}(|ph|)\right) = \frac{e^{-hp}|p| a + \frac{1}{2}}{(a + 1)\sqrt{2\pi}|a+\frac{1}{2}|} \left(pK_{a+\frac{1}{2}}(|ph|) + |p|K_{a+\frac{1}{2}}(|ph|)\right).$$

Case 2. $a \neq b$, $p = 0$. Then,

$$I = \left| a - b \right|^{a+\frac{1}{2}} I,$$

where $\tilde{h} = \frac{h}{\sqrt{|a-b|}}$ and $\tilde{I} = \int_0^\infty x^a e^{-sign(a-b)x}\Psi \left(\tilde{h}\sqrt{x}\right) dx$.

Case 2.1. $b > a$, $\tilde{h} > -\sqrt{2}$. Then, $I = \infty$ with respect to Case 2.1 at p. 208 of Ivanov and Ano (2016).

Case 2.2. $b < a$. According to Case 2.2 at p. 208 of Ivanov and Ano (2016),

$$I = \frac{\Gamma(a+\frac{3}{2})}{\sqrt{2\pi}(a-b)^{a+\frac{1}{2}}} \left(\frac{B\left(\frac{1}{2}, a + 1\right)}{\sqrt{2}} + \frac{h}{\sqrt{a-b}} \Phi \left(\frac{3}{2}, \frac{1}{2}, \frac{3}{2}, \frac{2}{2}, \frac{h^2}{2(a-b)}\right)\right).$$

Case 2.3. $b > a$, $\tilde{h} < -\sqrt{2}$. We have that

$$\tilde{I} = \int_0^\infty x^a e^{\Psi \left(\tilde{h}\sqrt{x}\right)} dx = \int_0^\infty x^a e^x \left(\int_{-\infty}^{\frac{\tilde{h}}{\sqrt{2\pi}}} e^{\frac{u^2}{2}} du\right) dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\tilde{h}}{\sqrt{2\pi}}} e^{\frac{u^2}{2}} \left(\frac{\tilde{h}}{\sqrt{2\pi}}\right)^a du.\tag{81}$$

Using 3.382.4 from Gradshteyn and Ryzhik (1980), we get from (81) that

$$\tilde{I} = \frac{\Gamma(a+\frac{3}{2})}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\tilde{h}}{\sqrt{2\pi}}} \left(\nu^2 - 1\right)^{a-\frac{3}{2}} d\nu.$$ 

Set

$$u = \left(\frac{\nu^2}{2} - 1\right)^{-1}.$$
Then,
\[ v = -\sqrt{\frac{2}{u} + 2}, \quad dv = u^{-\frac{3}{2}}(2 + 2u)^{-\frac{1}{2}}du \]
and hence
\[
I = \frac{\Gamma\left(\alpha + \frac{3}{2}\right)}{2\sqrt{\pi}} \int_0^{\frac{h^2}{2}} u^a(1 + u)^{-\frac{1}{2}}du = \frac{2^{\frac{a}{2}}\Gamma\left(\alpha + \frac{3}{2}\right)}{(a + 1) \left(\frac{h^2}{2} - 2\right)^{a+1}} F\left(\frac{1}{2}, \alpha + 1, a + 2; \frac{2(b - a)}{2(b - a) - h^2}\right),
\]
where the last identity follows from 3.194.1 in Gradshteyn and Ryzhik (1980). Therefore,
\[
I = \frac{2^{\frac{a}{2}}\Gamma\left(\alpha + \frac{3}{2}\right)}{(a + 1) \left(\frac{h^2}{2} - 2\right)^{a+1}} \sqrt{\pi} F\left(\frac{1}{2}, \alpha + 1, a + 2; \frac{2(b - a)}{2(b - a) - h^2}\right).
\]

Case 3. \( a \neq b, p \neq 0 \). We have that
\[
I = \frac{1}{|a - b|^{\alpha+1}} I,
\]
where
\[
\hat{h} = \frac{h}{\sqrt{|a - b|}}, \quad \hat{p} = p\sqrt{|a - b|}
\]
and
\[
\hat{I} = \int_0^\infty x^a e^{-\text{sign}(a-b)xy} \left(\hat{h}\sqrt{x} + \frac{\hat{p}}{\sqrt{x}}\right) dx.
\]

Case 3.1. \( a < b, \hat{h} > -\sqrt{2} \). We conclude that \( \hat{I} = \infty \) by Case 3.1 at p. 210 of Ivanov and Ano (2016).

Case 3.2. \( a > b \). With respect to (21) of Ivanov and Ano (2016),
\[
I = \left|s\right|^{\alpha+\frac{1}{2}} e^{\left|\frac{1+q}{2}\right|^{a+1}} \left[ B(\alpha + 1, 1) \left( \left|s\right| K_{\alpha+\frac{1}{2}} \left( \left|s\right| \right) + sK_{\alpha+\frac{1}{2}} \left( \left|s\right| \right) \right) \right] \Phi \left( \alpha + 1, -\alpha, a + 2; \frac{1+q}{2}, -s(1+q) \right) - \\
(1+q)sB(\alpha + 2, 1) K_{\alpha+\frac{1}{2}} \left( \left|s\right| \right) \Phi \left( \alpha + 2, -\alpha, a + 3; \frac{1+q}{2}, -s(1+q) \right),
\]
where
\[
q = \frac{\hat{h}}{\sqrt{\hat{h}^2 + 2}} \quad \text{and} \quad s = \hat{p}\sqrt{\hat{h}^2 + 2}.
\]

Case 3.3. \( a < b, \hat{h} < -\sqrt{2} \). Let \( D(v) \) and \( H(v), v \leq V \in \mathbb{R} \) be two differentiable functions with \( D(V) = \hat{p}, H(V) = \hat{h} \) and \( \frac{D(-\infty)}{\sqrt{\hat{h}}} + H(-\infty)\sqrt{\hat{h}} = -\infty \). Since
\[
\psi \left( \frac{D(v)}{\sqrt{\hat{h}}} + H(v)\sqrt{\hat{h}} \right) = \int_{-\infty}^{\hat{p}} \psi' \left( \frac{D(u)}{\sqrt{\hat{h}}} + H(u)\sqrt{\hat{h}} \right) du = \\
= \int_{-\infty}^{\hat{p}} \psi' \left( \frac{D(u)}{\sqrt{\hat{h}}} + H(u)\sqrt{\hat{h}} \right) \left( \frac{D'(u)}{\sqrt{\hat{h}}} + H'(u)\sqrt{\hat{h}} \right) du,
\]
we get that
\[
\Psi \left( \frac{\hat{p}}{\sqrt{\lambda}} + \hat{h}\sqrt{x} \right) =
\int_{-\infty}^{V} \frac{1}{\sqrt{2\pi}} e^{-D(v)} H(v) - \frac{p^2_{(2)x}}{2} - \frac{p^2_{(2)x}}{2} \left( \frac{D'(v)}{\sqrt{x}} + H'(v) \sqrt{x} \right) dv.
\]

Set \( V = \hat{h} \) and
\[
H(v) = v, \quad D(v) = \frac{\hat{p}\sqrt{h^2 - 2}}{v^2 - 2}, \quad v \leq \hat{h}.
\]

Then,
\[
I = \frac{1}{\sqrt{2\pi}} \int_{0}^{h} x^\frac{1}{4} e^x \left[ \int_{-\infty}^{\hat{h}} e^{-\frac{\hat{p}\sqrt{h^2 - 2}}{\sqrt{v^2 - 2}} - \frac{\hat{p}^2(v^2 - 2)}{2(v^2 - 2)x} - \frac{v^2_2}{2x}} dv \right] dx.
\]

Let us consider the double integrals
\[
I_1 = \int_{0}^{\infty} \int_{-\infty}^{h} \left( x^{\frac{1}{4}} e^x \frac{p^2_{(2)x}}{2} \frac{\hat{p}^2(v^2 - 2)}{2(v^2 - 2)x} - \frac{v^2_2}{2x} \right) dv dx
\]
and
\[
I_2 = \int_{0}^{\infty} \int_{-\infty}^{\hat{h}} \left( \frac{p^2_{(2)x}}{2} \frac{\hat{p}^2(v^2 - 2)}{2(v^2 - 2)x} - \frac{v^2_2}{2x} \right) dv dx.
\]

We can notice that both \( I_1 \) and \( I_2 \) are integrals of constant sign functions. Therefore, if the iterated integrals
\[
\tilde{I}_1 = \int_{-\infty}^{\hat{h}} \left( \int_{0}^{\infty} \left( x^{\frac{1}{4}} e^x \frac{p^2_{(2)x}}{2} \frac{\hat{p}^2(v^2 - 2)}{2(v^2 - 2)x} - \frac{v^2_2}{2x} \right) dx \right) dv
\]
and
\[
\tilde{I}_2 = \int_{-\infty}^{\hat{h}} \left( \int_{0}^{\infty} \left( \frac{p^2_{(2)x}}{2} \frac{\hat{p}^2(v^2 - 2)}{2(v^2 - 2)x} - \frac{v^2_2}{2x} \right) dx \right) dv
\]
exist, then \( I_1 = \tilde{I}_1, I_2 = \tilde{I}_2 \), and the Fubini’s theorem can be exploited to \( I_1 \) and \( I_2 \), that is,
\[
I = \frac{1}{\sqrt{2\pi}} \left( \tilde{I}_1 - \tilde{I}_2 \right). \tag{83}
\]

Set \( s = \hat{p}\sqrt{h^2 - 2} \). Then we have, according to 3.471.9 from Gradshteyn and Ryzhik (1980), that
\[
\tilde{f}_1 = 2|s|^{\alpha + \frac{1}{2}} K_{\alpha + \frac{1}{2}}(|s|) \int_{-\infty}^{\hat{h}} (u^2 - 2)^{-\alpha - \frac{1}{2}} e^{-\frac{u^2}{2}} du
\]
and
\[
\hat{f}_2 = 2s|s|^{\alpha+\frac{1}{2}}K_{\alpha+\frac{1}{2}}(|s|) \int_{-\infty}^{\hat{h}} v \left(v^2 - 2\right)^{-\alpha-2} e^{-\frac{v^2}{2}} dv.
\]

Next, we make a change of variables
\[
v \to y, \quad y = -\frac{v}{\sqrt{v^2 - 2}}.
\]

Then,
\[
v = -\frac{y\sqrt{2}}{\sqrt{y^2 - 1}}, \quad v^2 - 2 = \frac{2}{y^2 - 1}, \quad dv = \frac{\sqrt{2}}{(y^2 - 1)^{\frac{3}{2}}} dy
\]
and we get that
\[
\hat{f}_1 = 2^{-\alpha}|s|^{\alpha+\frac{1}{2}}K_{\alpha+\frac{1}{2}}(|s|) \int_{1}^{\hat{h}} \left(2y^2 - 1\right)^{\alpha} e^{y^2} dy
\]
and
\[
\hat{f}_2 = -2^{-\alpha}s|s|^{\alpha+\frac{1}{2}}K_{\alpha+\frac{1}{2}}(|s|) \int_{1}^{\hat{h}} y \left(2y^2 - 1\right)^{\alpha} e^{y^2} dy.
\]

Set
\[
q = \frac{|\hat{h}|}{\sqrt{h^2 - 2}} \quad \text{and} \quad u = \frac{y - 1}{q - 1}.
\]

Then,
\[
\hat{f}_1 = 2^{-\alpha}(q - 1)|s|^{\alpha+\frac{1}{2}} e^s K_{\alpha+\frac{1}{2}}(|s|) \int_{0}^{1} ((q - 1)2u^2 + 2(q - 1)u)^{\alpha} e^{s(q-1)u} du
\]
\[
= (q - 1)^{\alpha+1}|s|^{\alpha+\frac{1}{2}} e^s K_{\alpha+\frac{1}{2}}(|s|) \int_{0}^{1} u^{\alpha} \left(\frac{(q-1)u}{2} + 1\right)^{\alpha} e^{s(q-1)u} du \tag{84}
\]
and
\[
\hat{f}_2 =
\]
\[
= -2^{-\alpha}(q - 1)s|s|^{\alpha+\frac{1}{2}} e^s K_{\alpha+\frac{1}{2}}(|s|) \left( \int_{0}^{1} ((q - 1)2u^2 + 2(q - 1)u)^{\alpha} e^{s(q-1)u} du \right)
\]
\[
+ (q - 1) \int_{0}^{1} u ((q - 1)2u^2 + 2(q - 1)u)^{\alpha} e^{s(q-1)u} du = \tag{85}
\]
\[
= - (q - 1)^{\alpha+1}|s|^{\alpha+\frac{1}{2}} e^s K_{\alpha+\frac{1}{2}}(|s|) \left( \int_{0}^{1} u^{\alpha} \left(\frac{(q-1)u}{2} + 1\right)^{\alpha} e^{s(q-1)u} du + \right)
\]
\[
+ (q - 1) \int_{0}^{1} u^{\alpha+1} \left(\frac{(q-1)u}{2} + 1\right)^{\alpha} e^{s(q-1)u} du.
\]
Applying 3.385 of Gradshteyn and Ryzhik (1980) to (84) and (85) and keeping in mind (83), we establish now that
\[
I = \frac{(q - 1)^{\alpha + 1} s^{\alpha + 1/2} e^{s}}{(b - a)^{\alpha + 1} \sqrt{2\pi}} \left[ B(\alpha + 1, 1) \Phi \left( \alpha + 1, -\alpha, \alpha + 2; \frac{1 - q}{2}, s(1 - q) \right) \times \right.
\]
\[
\times \left( |s| K_{\alpha + 1/2}(|s|) + s K_{\alpha + 1}(|s|) \right) + (q - 1) s K_{\alpha + 1/2}(|s|) B(\alpha + 2, 1) \times \right.
\]
\[
\times \Phi \left( \alpha + 2, -\alpha, \alpha + 3; \frac{1 - q}{2}, s(1 - q) \right) \right].
\]

Now, one might observe that Ratio 1.1 follows from (77), since then
\[
P(X \leq u) = P(S \geq K + r_0 - u) +
\]
\[
+ P(r_1 - u + K \leq S \leq r_0 - u + K) -
\]
\[
e^{-\frac{\lambda}{\sigma^2}} \mathbb{E} \left( e^{-\frac{\lambda S}{\sigma^2}} I_{r_1 - u + K \leq S \leq r_0 - u + K} \right) =
\]
\[
P(S \geq K + r_1 - u) -
\]
\[
e^{-\frac{\lambda}{\sigma^2}} \mathbb{E} \left( e^{-\frac{\lambda S}{\sigma^2}} I_{r_1 - u + K \leq S \leq r_0 - u + K} \right)
\]
and hence (22) results from Cases 2.2, 3.2, (23) comes after Case 3.2, 2.2, (24) issues from Case 3.2.

Further, one can notice that if \( w \geq a \), then
\[
\theta < \frac{\lambda \sigma^2}{2(r_0 - r_1)}
\]
and hence \( z < 0 \). Also, when \( w > a \), then
\[
z^2 - 2w = \frac{\theta^2}{\sigma^2} > 0,
\]
and since \( z < 0 \), we have that \( z < -\sqrt{2w} \). Therefore, one could derive (25) from Cases 2.2, 1.3, 1.2, (26) from Cases 3.2, 1.2, 1.3, (27) from Cases 3.2, 1.3, (28) from Cases 2.2, 3.3, 2.3, (29) from Cases 3.2, 2.3, 3.3, (30) from Cases 3.2, 3.3.

Next, Ratio 1.2 follows from (77), since then
\[
P(X \leq u) = P(S < K) \left( 1 - e^{\frac{\lambda(r_1 - u)}{\sigma^2}} \right) + P(S \geq K + r_0 - u) +
\]
\[
+ P(K \leq S < r_0 - u + K) - e^{\frac{\lambda(r_1 - u)}{\sigma^2}} \mathbb{E} \left( e^{\frac{\lambda K}{\sigma^2}} I_{K \leq S < r_0 - u + K} \right) =
\]
\[
= 1 - e^{\frac{\lambda(r_1 - u)}{\sigma^2}} \left( P(S < K) + e^{\frac{\lambda r_1}{\sigma^2}} \mathbb{E} \left( e^{\frac{\lambda K}{\sigma^2}} I_{K \leq S < r_0 - u + K} \right) \right) =
\]
\[
= 1 - e^{\frac{\lambda(r_1 - u)}{\sigma^2}} + e^{\frac{\lambda r_1}{\sigma^2}} \left( P(S < K) - e^{\frac{\lambda r_1}{\sigma^2}} \mathbb{E} \left( e^{\frac{\lambda K}{\sigma^2}} I_{K \leq S < r_0 - u + K} \right) \right),
\]
and we get (31) from Cases 2.2, 3.2, (32) from Cases 3.2, 2.2, (33) from Case 3.2, (34) from Cases 2.2, 1.3, 1.2, (35) from Cases 3.2, 1.2, 1.3, (36) from Cases 3.2, 1.3, (33) from Case 3.2, (37) from Cases 2.2, 3.3, 2.3, (38) from Cases 3.2, 2.3, 3.3, (39) from Cases 3.2, 3.3 and Ratio 1.3 immediately from (77).

Finally, Ratio 2.1 comes after (77) and Cases 2.2, 3.2. Ratio 2.2 follows straightly from (77).
Proof of Theorem 2. One can see that

$$E \left( XI_{\{X \leq u\}} \right) = E \hat{g}(S),$$

where

$$\hat{g}(S) = E \left( XI_{\{X \leq u\}} \right) = (r_0 - (S - K)^+) E \left( I_{\{\tau > 1, r_0 \leq u + (S - K)^+\}} \right) +$$

$$+ (r_1 - (S - K)^+) E \left( I_{\{\tau \leq 1, (r_0 - r_1) \tau \leq u + (S - K)^+-r_1\}} \right) S$$

$$+(r_0 - r_1) E \left( \tau I_{\{\tau \leq 1, (r_0 - r_1) \tau \leq u + (S - K)^+-r_1\}} \right) S =$$

$$= r_0 e^{-\lambda} I_{\{r_0 \leq u + (S - K)^+\}} + r_1 E \left( I_{\{\tau \leq 1, (r_0 - r_1) \tau \leq u + (S - K)^+-r_1\}} \right) S -$$

$$- (S - K)^+ g(S) + (r_0 - r_1) E \left( \tau I_{\{\tau \leq 1, (r_0 - r_1) \tau \leq u + (S - K)^+-r_1\}} \right),$$

where $g(S)$ is defined in (68).

If $r_0 = r_1$, then it comes after (69) and (86) that

$$\hat{g}(S) = (r_0 - (S - K)^+) I_{\{r_0 \leq u + (S - K)^+\}} =$$

$$(r_0 - S + K) I_{\{S > K, r_0 \leq u + S - K\}}.$$

Hence one can observe, since $u < r_1$, that

$$E \left( XI_{\{X \leq u\}} \right) = (r_0 + K) P \{ S \geq r_0 + K - u \} - E \left( SI_{\{S \geq r_0 + K - u\}} \right).$$

When $r_0 > r_1$, then it follows from (86) that

$$\hat{g}(S) = r_0 e^{-\lambda} I_{\{r_0 \leq u + (S - K)^+\}} - (S - K)^+ g(S) +$$

$$+ r_1 I_{\{u + (S - K)^+ - r_1 \leq 0\}} E \left( \tau I_{\{\tau \leq \min\left\{ \frac{r_0 + (S - K)^+ - r_1}{0 - r_1} \right\} \}} \right) +$$

$$+ (r_0 - r_1) I_{\{u + (S - K)^+ - r_1 \leq 0\}} E \tau I_{\{\tau \leq \min\left\{ \frac{r_0 + (S - K)^+ - r_1}{0 - r_1} \right\} \}}.$$

Since

$$\int_0^A \lambda x e^{-\lambda x} dx = - \int_0^A x d e^{-\lambda x} = -A e^{-\lambda A} + \int_0^A e^{-\lambda x} dx =$$

$$= \frac{1}{A} - \left( \frac{1}{A} + A \right) e^{-\lambda A},$$

we have from (88) that

$$\hat{g}(S) = r_0 e^{-\lambda} I_{\{r_0 \leq u + (S - K)^+\}} - (S - K)^+ g(S) +$$

$$+ r_1 I_{\{u + (S - K)^+ \geq r_0\}} \left( 1 - e^{-\lambda} \right) +$$

$$+ r_1 I_{\{r_1 \leq u + (S - K)^+ \leq r_0\}} \left( 1 - e^{-\frac{\lambda (r_1 - u + (S - K)^+)}{r_0 - r_1}} \right) +$$

$$+ (r_0 - r_1) I_{\{u + (S - K)^+ \geq r_0\}} \left( \frac{1}{A} - \left( \frac{1}{A} + 1 \right) e^{-\lambda} \right) +$$

$$+ (r_0 - r_1) I_{\{r_1 \leq u + (S - K)^+ \leq r_0\}} \times$$

$$\times \left( \frac{1}{A} - \left( \frac{1}{A} + \frac{u + (S - K)^+ - r_1}{r_0 - r_1} \right) e^{-\frac{\lambda (r_1 - u + (S - K)^+)}{r_0 - r_1}} \right).$$
Substituting \( g(S) \) from (74) into (89), one can observe that

\[
\hat{g}(S) = \left( r_0 e^{-\lambda} + r_1 \left( 1 - e^{-\lambda} \right) + (r_0 - r_1) \left( \frac{1}{\lambda} - \left( \frac{1}{\lambda} + 1 \right) e^{-\lambda} \right) - \\
- (S - K)^+ I_{\{u+(S-K)^\geq 0\}} + \left( r_1 \left( 1 - e^{-\frac{\lambda (r_1 - u - (S - K)^+) S}{\lambda - 0 - r_1}} \right) + \right.
\]

\[
+ (r_0 - r_1) \left( \frac{1}{\lambda} + \frac{u + (S - K)^+ - r_1}{r_0 - r_1} \right) e^{-\frac{\lambda (r_1 - u - (S - K)^+)}{\lambda - 0 - r_1}} \right) -
\]

\[
- (S - K)^+ \left( 1 - e^{-\frac{\lambda (r_1 - u - (S - K)^+)}{\lambda - 0 - r_1}} \right) \right) I_{\{r_1 \leq u + (S - K)^+ < r_0\}}
\]

and therefore

\[
\hat{g}(S) = \left( \frac{r_0 - r_1}{\lambda} \left( 1 - e^{-\lambda} \right) + r_1 - (S - K)^+ \right) I_{\{u+(S-K)^\geq 0\}} +
\]

\[
+ \left( r_1 + \frac{r_0 - r_1}{\lambda} + \left( \frac{r_1 - r_0}{\lambda} - u \right) e^{-\frac{\lambda (r_1 - u - (S - K)^+)}{\lambda - 0 - r_1}} - (S - K)^+ \right) \times
\]

\[
I_{\{r_1 \leq u + (S - K)^+ < r_0\}}.
\]

Hence,

\[
\hat{g}(S) = \left( \frac{r_0 - r_1}{\lambda} \left( 1 - e^{-\lambda} \right) + r_1 - S + K \right) I_{\{S \geq r_0 + K - u\}} +
\]

\[
+ \left( r_1 + \frac{r_0 - r_1}{\lambda} + \left( \frac{r_1 - r_0}{\lambda} - u \right) e^{-\frac{\lambda (r_1 - u - S + K)}{\lambda - 0 - r_1}} - S + K \right) \times
\]

\[
I_{\{r_1 - u + K \leq S < r_0 - u + K\}}.
\]

Because of \( E \left( XI_{\{X \leq u\}} \right) = E \hat{g}(S) \), one can see that

\[
E \left( XI_{\{X \leq u\}} \right) = \left( \frac{r_0 - r_1}{\lambda} \left( 1 - e^{-\lambda} \right) + r_1 + K \right) P \{ S \geq r_0 + K - u \} -
\]

\[
- \left( \frac{r_1 - r_0}{\lambda} - u \right) \times
\]

\[
E \left( SI_{\{S \geq r_0 + K - u\}} \right) - \left( \frac{r_1 - r_0}{\lambda} - u \right) \times
\]

\[
e^{-\frac{\lambda (r_1 - u + K)}{\lambda - 0 - r_1}} E \left( \frac{\lambda S}{\lambda - 0 - r_1} I_{\{r_1 - u + K \leq S < r_0 - u + K\}} \right) +
\]

\[
+ \left( r_1 + \frac{r_0 - r_1}{\lambda} + K \right) P \{ r_1 - u + K \leq S < r_0 - u + K \}
\]

and therefore

\[
E \left( XI_{\{X \leq u\}} \right) = \left( r_1 + \frac{r_0 - r_1}{\lambda} + K \right) P \{ S \geq r_1 - u + K \} +
\]

\[
+ \frac{r_1 - r_0}{\lambda e^\lambda} P \{ S \geq r_0 - u + K \} - \left( \frac{r_1 - r_0}{\lambda} - u \right) \times
\]

\[
e^{-\frac{\lambda (r_1 - u + K)}{\lambda - 0 - r_1}} E \left( \frac{\lambda S}{\lambda - 0 - r_1} I_{\{r_1 - u + K \leq S < r_0 - u + K\}} \right).
\]
Next, one can see that for \( C \in \mathbb{R}_+ \),
\[
E \left( S_{I \{ S \geq C \}} \mid \gamma \right) = (\mu + \theta \gamma) P(S \geq C \mid \gamma) + \int_{C - \mu - \theta \gamma}^{\gamma} \frac{x}{\lambda} e^{-\frac{x^2}{2\sigma^2}} \, dx =
\]
\[
= (\mu + \theta \gamma) P(S \geq C \mid \gamma) + \frac{\sigma \sqrt{\pi}}{\sqrt{2\pi}} e^{-\frac{(C - \mu - \theta \gamma)^2}{2\sigma^2}}.
\]

Let \( C \neq \mu \). Then with respect to (80),
\[
E \left( S_{I \{ S \geq C \}} \right) = \mu P(S \geq C) + \frac{\theta a^d}{\Gamma(a)} \int_0^\infty x^{d-\frac{1}{2}} e^{-\frac{\theta^2 x}{\sigma^2}} \, dx + \frac{\sigma a^d}{\Gamma(a) \sqrt{\pi}} \int_0^\infty x^{d-\frac{1}{2}} e^{-\frac{\theta^2 x}{\sigma^2}} \, dx
\]
\[
\quad + \frac{\sigma a^d}{\Gamma(a) \sqrt{\pi}} \int_0^\infty x^{d-\frac{1}{2}} e^{-\frac{\theta^2 x}{\sigma^2}} \, dx + \left( C - \mu \right) \frac{\sqrt{\pi \sigma^2}}{\sqrt{\theta^2 + 2\sigma^2}} \Gamma \left( a + \frac{1}{2} \right).
\]

If \( C = \mu \), then obviously
\[
E \left( S_{I \{ S \geq C \}} \right) = \mu P(S > C) + \frac{\theta a^d}{\Gamma(a)} \int_0^\infty x^{d-\frac{1}{2}} e^{-\frac{\theta^2 x}{\sigma^2}} \, dx + \frac{\sigma a^d}{\Gamma(a) \sqrt{\pi}} \int_0^\infty x^{d-\frac{1}{2}} e^{-\frac{\theta^2 x}{\sigma^2}} \, dx + \left( C - \mu \right) \frac{\sqrt{\pi \sigma^2}}{\sqrt{\theta^2 + 2\sigma^2}} \Gamma \left( a + \frac{1}{2} \right).
\]

Thus, we get (44) from Cases 2.2, 3.2, (45) from Case 2.2, (46) from Case 2.2, (47) from Cases 2.2, 3.2, 1.2, 1.3, (45) from Cases 3.2, 2.2, 1.3, 1.2, (46) from Cases 3.2, 1.3, (50) from Cases 2.2, 3.2, 2.3, 3.3, (51) from Cases 3.2, 2.2, 3.3, 2.3, (52) from Cases 3.2, 3.3.

Finally, Ratio 2 follows from (87) and Cases 2.2, 3.2. \( \square \)

**Proof of Theorem 3.** If \( r_0 = r_1 \), then
\[
E \left( e^{-\xi X} I_{\{X \leq u\}} \right) = e^{-\xi x} E \left( e^{\xi (S-K)^+} \mid I_{\{S-K \geq r_0 - u\}} \right) =
\]
\[
\quad = e^{-\xi (r_0 + K)} E \left( e^{\xi S} \mid I_{\{S \geq r_0 + K - u\}} \right). \tag{90}
\]

Let \( r_0 > r_1 \). Then,
\[
E \left( e^{-\xi X} I_{\{X \leq u\}} \right) = e^{\xi (S-K)^+} I_{\{S-K \geq r_0 - u\}} P(\tau > 1) +
\]
\[
\quad + e^{\xi (S-K)^+} I_{\{S \geq r_0 + K - u\}} + e^{\xi (S-K)} \times
\]
\[
\quad \times I_{\{S \geq r_1 + K - u\}} E \left( e^{\xi (r_1 - r_0) \tau} \mid I_{\{\tau \leq \min \{1, \frac{S-K}{u_{r_0 - r_1}}\}} \right) \right) | S \right).
\]
Therefore,
\[
E\left( e^{-\xi T} I_{\{X \leq u\}} | S \right) = e^{-\lambda - \xi (r_0 + K) + \xi S} I_{\{S \geq r_0 + K - u\}} + e^\xi (S - K - r_1) \times \\
\times I_{\{S \geq r_0 + K - u\}} E\left( e^{\xi (r_1 - r_0)^T} I_{\{r \leq 1\}} \right) + e^\xi (S - K - r_1) \times \\
\times I_{\{r_1 + K - u \leq S < r_0 + K - u\}} E\left( e^{\xi (r_1 - r_0)^T} I_{\{r \leq 1 - \frac{u}{S - r_0 - r_1}\}} \right) | S
\]
(91)

in this case.

Since
\[
E\left( e^{\lambda T} I_{\{r \leq C\}} \right) = \int_0^C \lambda e^{(A - \lambda)x} dx = \frac{\lambda}{A - \lambda} \left( e^{(A - \lambda)C} - 1 \right),
\]
we have from (91) that if \( r_0 > r_1 \), then
\[
E\left( e^{-\xi X} I_{\{X \leq u\}} | S \right) = e^{-\lambda - \xi (r_0 + K) + \xi S} I_{\{S \geq r_0 + K - u\}} + \\
\times \frac{\lambda e^{-\xi (r_1 + K - S)}}{A + \xi (r_0 - r_1)} \left( 1 - e^{-\lambda - \xi (r_0 - r_1)} \right) I_{\{S \geq r_0 + K - u\}} + \\
\times \frac{\lambda e^{-\xi (r_1 + K - S)}}{A + \xi (r_0 - r_1)} \left( 1 - e^{-\frac{(\lambda + \xi (r_0 - r_1) (u + S - K - r_1))}{r_1 - r_0}} \right) I_{\{r_1 + K - u \leq S < r_0 + K - u\}}
\]

and
\[
E\left( e^{-\xi X} I_{\{X \leq u\}} \right) = \frac{\xi (r_0 - r_1) e^{-\lambda - \xi (r_0 + K - S)}}{A + \xi (r_0 - r_1)} I_{\{S \geq r_0 + K - u\}} + \\
\times \frac{\lambda e^{-\xi (r_1 + K - S)}}{A + \xi (r_0 - r_1)} I_{\{S \geq r_1 + K - u\}} - \frac{\lambda e^{-\xi (r_1 + K - S)}}{A + \xi (r_0 - r_1)} \times \\
\times e^{-\frac{(\lambda + \xi (r_0 - r_1) (u + S - K - r_1))}{r_1 - r_0}} I_{\{r_1 + K - u \leq S < r_0 + K - u\}}.
\]

Hence
\[
\begin{align*}
E\left( e^{-\xi X} I_{\{X \leq u\}} \right) &= \frac{\xi (r_0 - r_1) e^{-\lambda - \xi (r_0 + K - S)}}{A + \xi (r_0 - r_1)} E\left( e^{\xi S} I_{\{S \geq r_0 + K - u\}} \right) + \\
&\times E\left( e^{\xi S} I_{\{S \geq r_0 + K - u\}} \right) \left( \frac{\lambda (u - K - r)}{A + \xi (r_0 - r_1)} \right) \\
&\times E\left( e^{\xi (r_1 - r_0)} I_{\{r_1 + K - u \leq S < r_0 + K - u\}} \right)
\end{align*}
\]
(92)

if \( r_0 > r_1 \).

Thereby, we get (57) from Cases 3.2, 2.2, (58) from Cases 2.2, 3.2, (59) from Case 3.2, (60) from Cases 3.2, 2.2, 1.3, (61) from Cases 2.2, 3.2, 1.2, (62) from Cases 3.2, 1.3, (63) from Cases 3.2, 2.2, 3.3, (64) from Cases 2.2, 3.2, 2.3, (65) from Cases 3.2, 3.3, (66) from Case 2.2 and (67) from Case 3.2. □

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References

Almendral, Ariel, and Cornelis W. Oosterlee. 2007. On American Options Under the Variance Gamma Process. Applied Mathematical Finance 14: 131–52. [CrossRef]

Avramidis, Athanasios N., Pierre L’Ecuyer, and Pierre-Alexandre Tremblay. 2003. Efficient simulation of gamma and variance-gamma processes. Paper presented at 2003 Winter Simulation Conference, New Orleans, LA, USA, 7–10 December, pp. 319–26.
Barrieu, Pauline, and Nicole El Karoui. 2005. Inf-convolution of risk measures and optimal risk transfer. *Finance and Stochastics* 9: 269–98. [CrossRef]

Bäurer, Patrick. 2015. Credit and Liquidity Risk in Lévy Asset Price Model. Ph.D. thesis, University of Freiburg, Breisgau, Germany, 165p.

Berkowitz, J., M. Pritsker, M. Gibson, and H. Zhou. 2002. How accurate are value-at-risk models at commercial banks. *The Journal of Finance* 57: 1093–111. [CrossRef]

Carr, Peter, and Liuren Wu. 2010. Stock options and credit default swaps: A joint framework for valuation and estimation. *Journal of Financial Econometrics* 8: 409–49. [CrossRef]

Chen, Song Xi, and Cheng Yong Tang. 2005. Nonparametric inference of value-at-risk for dependent financial returns. *Journal of Financial Econometrics* 3: 227–55. [CrossRef]

Eberlein, Ernst. 2014. Fourier based valuation methods in mathematical finance. In *Quantitative Energy Finance*. Edited by Fred Espen Benth, Valery A. Kholodnyi and Peter Laurence. New York: Springer, pp. 85–114.

Eberlein, Ernst, Zorana Grbac, and Thorsten Schmidt. 2013. Discrete tenor models for credit risky portfolios driven by time-inhomogeneous Lévy processes. *SIAM Journal on Financial Mathematics* 4: 616–49. [CrossRef]

Eichelsbacher, Peter, and Christoph Thäle. 2015. Malliavin-Stein method for variance-gamma approximation on Wiener space. *Electronic Journal of Probability* 20: 1–28. [CrossRef]

Finlay, Richard, and Eugene Seneta. 2006. Stationary-increment Student and variance-gamma processes. *Journal of Applied Probability* 43: 441–53. [CrossRef]

Fu, Michael C. 2007. Variance-Gamma and Monte-Carlo. In *Advances in Mathematical Finance*. Edited by Michael C. Fu, Robert A. Jarrow, Ju-Yi Yen and Robert J. Elliott. Boston: Birkhauser, pp. 21–34.

Föllmer, Hans, and Alexander Schied. 2004. *Stochastic Finance: An Introduction in Discrete Time*, 2nd revised and extended ed. Berlin: Walter de Gruyter & Co.

Föllmer, Hans, and Alexander Schied. 2010. Convex and Coherent Risk Measures. In *Encyclopedia of Quantitative Finance*. Edited by R. Cont. Hoboken: John Wiley & Sons, pp. 1200–04.

Bateman, Harry, and Arthur Erdelyi. 1953. *Higher Transcendental Functions*. New York: McGraw-Hill, Vol. I.

Drapeau, Samuel, Michael Kupper, and Antonis Papapantoleon. 2014. A Fourier approach to the computation of CVaR and optimized certainty equivalents. *Journal of Risk* 16: 3–29. [CrossRef]

Gradshteyn, Izrail Solomonovich, and Iosif Moiseevich Ryzhik. 1980. *Table of Integrals, Series and Products*. New York: Academic Press.

Grbac, Zorana. 2009. Credit Risk in Lévy Libor Modeling: Rating Based Approach. Ph.D. thesis, University of Freiburg, Freiburg im Breisgau, Germany, 172p.

Hirs, Ali, and Dilip B. Madan. 2004. Pricing American options under variance gamma. *Journal of Computational Finance* 7: 63–80. [CrossRef]

Ivanov, Roman V. 2018. On risk measuring in the variance-gamma model. *Statistics and Risk Modeling* 35: 23–33. [CrossRef]

Ivanov, Roman V., and Katsunori Ano. 2016. On exact pricing of FX options in multivariate time-changed Lévy models. *Review of Derivatives Research* 19: 201–16. [CrossRef]

Kalinenko, Konstantin, Stanislav Uryasev, and R. Tyrrell Rockafellar. 2012. Calibrating Risk Preferences with Generalized CAPM Based on Mixed CVaR Deviation. *Journal of Risk* 15: 45–70. [CrossRef]
Korn, Ralf, Elke Korn, and Gerald Kroisandt. 2010. Monte Carlo Methods and Models in Finance and Insurance. Boca Raton: Chapman and Hall.

Linders, Daniël, and Ben Stassen. 2016. The multivariate Variance Gamma model: basket option pricing and calibration. Quantitative Finance 16: 555–72. [CrossRef]

Luciano, Elisa, Marina Marena, and Patrizia Semeraro. 2016. Dependence calibration and portfolio fit with factor-based subordinators. Quantitative Finance 16: 1037–52. [CrossRef]

Luciano, Elisa, and Wim Schoutens. 2006. A multivariate jump-driven financial asset model. Quantitative Finance 6: 385–402. [CrossRef]

Madan, Dilip B. 2014. Modeling and monitoring risk acceptability in markets: The case of the credit default swap market. Journal of Banking Finance 47: 63–73. [CrossRef]

Madan, Dilip B., Peter P. Carr, and Eric C. Chang. 1998. The Variance Gamma Process and Option Pricing. European Finance Review 2: 79–105. [CrossRef]

Madan, Dilip B., and Frank Milne. 1991. Option pricing with VG martingale components. Mathematical Finance 1: 39–55. [CrossRef]

Madan, Dilip B., and Eugene Seneta. 1990. The Variance Gamma (V.G.) Model for Share Market Returns. Journal of Business 63: 511–24. [CrossRef]

Mafusalov, Alexander, and Stan Uryasev. 2016. CVaR (Superquantile) Norm: Stochastic Case. European Journal of Operational Research 249: 200–8. [CrossRef]

Moosbrucker, Thomas. 2006. Explaining the correlation smile using Variance Gamma distributions. The Journal of Fixed Income 16: 71–87. [CrossRef]

Mozumder, Sharif, Ghulam Sorwar, and Kevin Dowd. 2015. Revisiting variance gamma pricing: An application to S&P500 index options. International Journal of Financial Engineering 2: 1550022.

Pritsker, Matthew. 1997. Evaluating value at risk methodologies. Journal of Financial Services Research 12: 201–42. [CrossRef]

Rathgeber, Andreas W., Johannes Stadler, and Stefan Stöckl. 2016. Modeling share returns–An empirical study on the Variance Gamma model. Journal of Economics and Finance 40: 653–82. [CrossRef]

Rockafellar, R. Tyrrell, and Stanislav Uryasev. 2000. Optimization of conditional value-at-risk. Journal of Risk 2: 21–41. [CrossRef]

Rockafellar, R. Tyrrell, and Stanislav Uryasev. 2002. Conditional value-at-risk for general loss distribution. Journal of Banking and Finance 26: 1443–71. [CrossRef]

Tsyrumasto, Peter, Michael Zabarankin, and Stan Uryasev. 2014. Value-at-risk Support Vector Machine: Stability to Outliers. Journal of Combinatorial Optimization 28: 218–32. [CrossRef]

Seneta, Eugene. 2007. The Early Years of the Variance–Gamma Process. In Advances in Mathematical Finance. Edited by Michael C. Fu, Robert A. Jarrow, Ju-Yi Yen and Robert J. Elliott. Boston: Birkhauser, pp. 3–19.

Wallmeier, Martin, and Martin Diethelm. 2012. Multivariate downside risk: normal versus variance gamma. Journal of Futures Markets 32: 431–58. [CrossRef]

Wang, Jun. 2009. The Multivariate Variance Gamma Process and Applications in Multi-Asset Option Pricing. Ph.D. thesis, University of Maryland, College Park, MD, USA

Whittaker, Edmund Taylor, and George Neville Watson. 1990. A Course in Modern Analysis, 4th ed. Cambridge: Cambridge University Press.

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