METRIC DUALITY BETWEEN POSITIVE DEFINITE KERNELS AND BOUNDARY PROCESSES

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Abstract. We study representations of positive definite kernels $K$ in a general setting, but with view to applications to harmonic analysis, to metric geometry, and to realizations of certain stochastic processes. Our initial results are stated for the most general given positive definite kernel, but are then subsequently specialized to the above mentioned applications. Given a positive definite kernel $K$ on $S \times S$ where $S$ is a fixed set, we first study families of factorizations of $K$. By a factorization (or representation) we mean a probability space $(B, \mu)$ and an associated stochastic process indexed by $S$ which has $K$ as its covariance kernel. For each realization we identify a co-isometric transform from $L^2(\mu)$ onto $\mathcal{H}(K)$, where $\mathcal{H}(K)$ denotes the reproducing kernel Hilbert space of $K$. In some cases, this entails a certain renormalization of $K$. Our emphasis is on such realizations which are minimal in a sense we make precise. By minimal we mean roughly that $B$ may be realized as a certain $K$-boundary of the given set $S$. We prove existence of minimal realizations in a general setting.

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1. Introduction

A variety of notions of “boundary” and boundary representation for general classes of positive definite kernels are established in [JT16]. It allows us to carry over results and notions from classical harmonic analysis on the disk to this wider context (see [JP98a, JP98b, Str98]).

More specifically, starting with a given positive definite (p.d.) kernel $K$ on $S \times S$, we introduce generalized boundaries for the set $S$ that carry $K$. It is a measure theoretic “boundary” in the form of a probability space, but it is not unique. The set of measure boundaries will be denoted $\mathcal{M}(K)$. Indeed, there exists such a

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generalized boundary probability space associated to any p.d. kernel. For example, as an element in \( M(K) \), we can take a “measure” boundary to be the Gaussian process having \( K \) as its covariance kernel. This exists by Kolmogorov’s consistency theorem.

The p.d. kernels include those defined on infinite discrete sets, for example sets of vertices in electrical networks, or discrete sets which arise from sampling operations performed on p.d. kernels in a continuous setting, and with the sampling then referring to suitable discrete subsets. See, e.g., [HJY11, JS13, ZS16].

The purpose of the present paper is to study a metric duality between (I) and (II) below, where

(I) \( K : S \times S \rightarrow \mathbb{C} \) is a given positive definite (p.d.) kernel defined on a fixed set \( S \), i.e., for \( \forall N \in \mathbb{N}, \forall \{ s_i \}_{i=1}^N, s_i \in S, \forall \{ \xi_i \}_{i=1}^N, \xi_i \in \mathbb{C} \), we have

\[
\sum_i \sum_j \xi_i \xi_j K(s_i, s_j) \geq 0; \quad \text{and} \quad (1.1)
\]

(II) measure space \((B, \mathcal{B}, \mu)\) where \( B \) is a set equipped with a \( \sigma \)-algebra \( \mathcal{B} \) of subsets, and \( \mu \) is a probability measure defined on \( \mathcal{B} \).

In particular, \( \mu \) satisfies \( \mu(\emptyset) = 0 \), \( \mu(B) = 1 \), \( \mu(F) \geq 0 \forall F \in \mathcal{B} \), and if \( \{ F_i \}_{i \in \mathbb{N}} \subset \mathcal{B}, F_i \cap F_j = \emptyset, i \neq j \) in \( \mathbb{N} \), then \( \mu(\bigcup_i F_i) = \sum_i \mu(F_i) \).

**Definition 1.1.** Let \( K \) be a p.d. kernel as in (I). We shall denote by \( \mathcal{H}(K) \) the corresponding reproducing kernel Hilbert space (RKHS), i.e., \( \mathcal{H}(K) \) is the Hilbert-completion of span \( \{ K_s := K(\cdot, s) ; s \in S \} \), with respect to the inner product

\[
\left\langle \sum_i \xi_i K_{s_i}, \sum_j \xi_j K_{s_j} \right\rangle_{\mathcal{H}(K)} := \sum_i \sum_j \xi_i \xi_j K(s_i, s_j).
\]

The following reproducing property holds:

\[
f(s) = \left\langle f, K(\cdot, s) \right\rangle_{\mathcal{H}(K)}, \forall s \in S, \forall f \in \mathcal{H}(K).
\]

**Definition 1.2.** Given \( K \) as in (I), and \((B, \mu)\) as in (II), we shall say that \((B, \mu) \in M(K)\) if there is a function \( k : S \rightarrow L^2(\mu) \) such that

\[
K(s, t) = \int_B k_s(x) \overline{k_t(x)} d\mu(x)
\]

holds for all \((s, t) \in S \times S\). We shall say that \((B, \mu)\) is tight (or minimal) iff the span of \( \{ k_s ; s \in S \} \) is dense in \( L^2(B, \mu) \).

Similarly, given \((B, \mu)\) as in (II), we shall say that a p.d. kernel \( K \) is in \( \mathcal{H}(\mu) \) if there is a stochastic process \( \{ k_s \}_{s \in S} \) satisfying (1.4).

**Remark 1.3.** In [JT15a], we showed that for all p.d. kernel \( K(s, t) \), \((s, t) \in S \times S\), we have \( M(K) \neq \emptyset \). See more examples below.

Given \( K \) as in (I) then the problem (1.4) always has a solution in a discrete (atomic) measure space relative to the counting measure. Nonetheless, in the interesting solutions \((B, \mathcal{B}, \mu)\) to (1.4) we aim to achieve \( B \) as a “boundary space” to the given set \( S \) from (I); see the details in Section 5 below.

**Definition 1.4.** We shall say that a Hilbert space \( \mathcal{H} \) is separable if there is an orthonormal basis (ONB) \( \{ \beta_n \}_{n \in \mathbb{N}} \) indexed by \( \mathbb{N} \) (or a set of cardinality \( \aleph_0 \)), i.e., we have

\[
\langle \beta_n, \beta_m \rangle_{\mathcal{H}} = \delta_{n,m}, \quad \text{and} \quad (1.5)
\]
\|f\|_H^2 = \sum_{n \in \mathbb{N}} |\langle f, \beta_n \rangle_H|^2, \quad \forall f \in \mathcal{H}. \quad (1.6)

If only (1.6) holds, we say that \( \{\beta_n\}_{n \in \mathbb{N}} \) is a Parseval frame. In both cases, vectors \( f \) in \( \mathcal{H} \) always have the representation

\[ f = \sum_{n \in \mathbb{N}} \langle f, \beta_n \rangle_H \beta_n \quad (1.7) \]

where (1.7) converges in the norm \( \|\cdot\|_H \) of \( \mathcal{H} \).

**Lemma 1.5.** Let \( K \) be given and assumed positive definite (p.d.) on \( S \times S \), where \( S \) is a set, see (I). Let \( \mathcal{H} = \mathcal{H}(K) \) be the corresponding reproducing kernel Hilbert space (RKHS), assumed separable; see Definition 1.1. Let \( \{\beta_n\}_{n \in \mathbb{N}} \) be a Parseval frame, and set

\[ k_s(n) := \beta_n(s) = \langle \beta_n, K(\cdot, s) \rangle_H; \quad (1.8) \]

see (1.3). Then the system (1.8) is a solution to (1.4), but with the measure space \( \mathbb{N} \), and with counting measure.

**Proof.** The existence of a Parseval frame \( \{\beta_n\}_{n \in \mathbb{N}} \) is assumed, so (1.5)-(1.6) hold for the Hilbert space \( \mathcal{H} := \mathcal{H}(K) \). Now, for all pairs \( (s, t) \in S \times S \), we have

\[
K(s, t) = \langle K(\cdot, s), K(\cdot, t) \rangle_H \quad \text{(by the reproducing property)}
\]

\[
= \sum_{n \in \mathbb{N}} \langle K(\cdot, s), \beta_n \rangle_H \overline{\langle \beta_n, K(\cdot, t) \rangle_H}
\]

\[
= \sum_{n \in \mathbb{N}} \beta_n(s) \overline{\beta_n(t)}
\]

\[
= \sum_{n \in \mathbb{N}} k_s(n) k_t(n)
\]

which is the desired conclusion. \( \Box \)

**Corollary 1.6.** Let \( K \) be given as in Lemma 1.5 above, and let \( \{\beta_n\}_{n \in \mathbb{N}} \) be a Parseval frame, set \( k_s(n) := \beta_n(s) \), see (1.8); then this is a minimal solution, i.e., \( \{k_s(\cdot)\} \) is dense in \( l^2(\mathbb{N}) \).

**Proof.** Immediate from the details in the proof of Lemma 1.5. In particular, if \( f \in \mathcal{H} = \mathcal{H}(K) \) satisfies \( f \perp k_\cdot(s) \), then \( \langle f, \beta_n(\cdot) \rangle = 0 \) for all \( n \in \mathbb{N} \), so by (1.6) we get

\[ \|f\|_H^2 = \sum_{n \in \mathbb{N}} |\langle f, \beta_n \rangle_H|^2 = 0. \]

\( \Box \)

**Discussion of the literature.** The theory of RKHS and their applications is vast, and below we only make a selection. Readers will be able to find more cited there. As for the general theory of RKHS in the pointwise category, we find useful [AD92, ABDD93, AD93, LMP09, PR16]. The applications include fractals (see e.g., [Aro43, AJSV13, BH14]); probability theory [Jr68, HE15, MSF+16, EMESO17, Sai16, CMPS17, PVPK17]; and application to learning theory [SZ04, SZ09a, SZ09b, JT15b]. For recent applications, we refer to [JT15b, JPT15, JPT16].
2. Properties of solutions to the factorization problem

Let $K : S \times S \to \mathbb{C}$ be a given p.d. kernel, as specified in (I) from Section 1 above. Solutions $(B, \mu, \{k_s\}_{s \in S})$ to the problem (1.4) are called factorizations.

**Proposition 2.1.** Let $K$ on $S \times S$ be given, and let $(B, \mu, \{k_s\}_{s \in S})$ be a solution to the factorization problem (1.4). Then the assignment

$$W(K(\cdot, s)) := k_s \in L^2(\mu)$$

extends by linearity to an isometry, denoted $W : \mathcal{H}(K) \to L^2(\mu)$, and its adjoint $V := W^* : L^2(\mu) \to \mathcal{H}(K)$ is the following transform

$$(Vf)(s) = \int_B f(x) \overline{k_s(x)} d\mu(x),$$

and we have

$$W^*W = VW = I_{\mathcal{H}(K)},$$

while $WW^* = WV$ is a projection in the Hilbert space $L^2(\mu)$. 

**Proof.** It is immediate from (1.4) that the operator $W$ from (2.1) is isometric $\mathcal{H}(K) \to L^2(\mu)$. For $f \in L^2(\mu)$, and $s \in S$, we have

$$\langle W(K(\cdot, s)), f \rangle_{L^2(\mu)} = \langle k_s(x) \overline{f(x)}, d\mu(x) \rangle = \langle K(\cdot, s), W^*f \rangle_{\mathcal{H}(K)} = \langle W^*f, s \rangle,$$

so the formula (2.2) follows, and we infer that

$$\left( s \mapsto (Vf)(s) = \int_B f(x) \overline{k_s(x)} d\mu(x) \right) \in \mathcal{H}(K).$$

We will show that for $\forall N \in \mathbb{N}, \forall \{s_i\}_{i=1}^N, s_i \in S, \{\xi_i\}_{i=1}^N, \xi_i \in \mathbb{C}$, we have

$$\left| \sum_{i=1}^N \xi_i (Vf)(s_i) \right|^2 \leq \|f\|_{L^2(\mu)}^2 \sum_{i=1}^N \sum_{j=1}^N \xi_i \overline{\xi_j} K(s_i, s_j),$$

and the desired conclusion (2.4) follows. We now show that (2.5) holds:

$$\text{LHS}(2.5) = \left| \int_B f(x) \sum_i \xi_i k_{s_i}(x) d\mu(x) \right|^2$$

$$\leq \int_B |f(x)|^2 d\mu(x) \int_B \left| \sum_i \xi_i k_{s_i}(x) \right|^2 d\mu(x) \quad \text{(by Schwarz in } L^2(\mu))$$

$$\quad = \|f\|_{L^2(\mu)}^2 \sum_i \sum_j \xi_i \overline{\xi_j} \int_B k_{s_i}(x) \overline{k_{s_j}(x)} d\mu(x)$$

$$\quad = \|f\|_{L^2(\mu)}^2 \sum_i \sum_j \xi_i \overline{\xi_j} K(s_i, s_j) = \text{RHS}(2.5),$$

and the proof is completed. □
3. The boundary space

Given \((K, S)\) as in (I), i.e., \(S\) is a set and \(K\) is a p.d. kernel on \(S \times S\), let \(\mathcal{M}(K)\) be the boundary space consisting of all measure spaces \((B, \mu)\) satisfying (1.4); see Definition 1.2.

In the discussion below, we shall introduce an order relation on \(\mathcal{M}(K)\). We show that there is always a minimal element in \(\mathcal{M}(K)\).

**Definition 3.1.** Suppose \((B_i, \mathcal{B}_i, \mu_i) \in \mathcal{M}(K), i = 1, 2\). We say that
\[
(B_1, \mathcal{B}_1, \mu_1) \leq (B_2, \mathcal{B}_2, \mu_2)
\]
if \(\exists \varphi : B_2 \rightarrow B_1\), s.t.
\[
\mu_2 \circ \varphi^{-1} = \mu_1, \quad \text{and} \quad \varphi^{-1}(\mathcal{B}_1) = \mathcal{B}_2. \tag{3.3}
\]

**Lemma 3.2.** \(\mathcal{M}(K)\) has minimal elements.

**Proof.** If (3.2)-(3.3) hold, then
\[
L^2(B_1, \mu_1) \ni f \xrightarrow{W_{21}} f \circ \varphi \in L^2(B_2, \mu_2)
\]
is isometric, i.e.,
\[
\int_{B_2} |f \circ \varphi|^2 d\mu_2 = \int_{B_1} |f|^2 \, d\mu_1, \tag{3.4}
\]
and
\[
W_{B_2} = W_{21} W_{B_1} \text{ on } \mathcal{H}(K), \tag{3.5}
\]
i.e., the diagram commutes:

![Diagram]

We can then use Zorn’s lemma to prove that \(\forall K, \mathcal{M}(K)\) has minimal elements \((B, \mathcal{B}, \mu)\). (See the proof of Theorem 3.3 below.) But even if \((B, \mathcal{B}, \mu)\) is minimal, \(W_B : \mathcal{H}(K) \rightarrow L^2(\mu)\) may not be onto. \(\square\)

In the next result, we shall refer to the partial order “\(\leq\)” from (3.1) when considering minimal elements in \(\mathcal{M}(K)\). And, in referring to \(\mathcal{M}(K)\), we have in mind a fixed positive definite function \(K : S \times S \rightarrow \mathbb{C}\), specified at the outset; see (1.1).

**Theorem 3.3.** Let \((K, S)\) be a fixed positive definite kernel, and let \(\mathcal{M}(K)\) be the corresponding boundary space from Definition 1.2.

Then, for every \((X, \lambda) \in \mathcal{M}(K)\), there is a \((M, \nu) \in \mathcal{M}(K)\) such that
\[
(M, \nu) \leq (X, \lambda), \tag{3.6}
\]
and \((M, \nu)\) is minimal in the following sense: Suppose \((B, \mu) \in \mathcal{M}(K)\) and
\[
(B, \mu) \leq (M, \nu), \tag{3.7}
\]
then it follows that \((B, \mu) \simeq (M, \nu)\), i.e., we also have \((M, \nu) \leq (B, \mu)\).
Proof. We shall use Zorn’s lemma, and the argument from Lemma 3.2.

Let $L = \{(B, \mu)\}$ be a linearly ordered subset of $\mathcal{M}(K)$ s.t.
\[(B, \mu) \leq (X, \lambda), \quad \forall (B, \mu) \in L;\] and such that, for every pair $(B_1, \mu_1), i = 1, 2$, in $L$, one of the following two cases must hold:
\[(B_1, \mu_1) \leq (B_2, \mu_2), \quad \text{or} \quad (B_2, \mu_2) \leq (B_1, \mu_1).\] To apply Zorn’s lemma, we must show that there is a $(B_L, \mu_L) \in \mathcal{M}(K)$ such that
\[(B_L, \mu_L) \leq (B, \mu), \quad \forall (B, \mu) \in L.\] (3.10)

Now, using (3.8)-(3.9), we conclude that the measure spaces $\{(B, \mu)\}_L$ have an inductive limit, i.e., the existence of:
\[\mu_L := \text{ind limit}_{B \to B_L} \mu_B.\] (3.11)

In other words, we may apply Kolmogorov’s consistency (see, e.g., [PS75]) to the family $L$ of measure spaces in order to justify the inductive limit construction in (3.11).

We have proved that every linearly ordered subset $L$ (as specified) has a “lower bound” in the sense of (3.10). Hence Zorn’s lemma applies, and the desired conclusion follows, i.e., there is a pair $(M, \nu) \in \mathcal{M}(K)$ which satisfies the condition (3.7) from the theorem. □

4. Gaussian processes

By a theorem of Kolmogorov, every Hilbert space may be realized as a (Gaussian) reproducing kernel Hilbert space (RKHS), see e.g., [IM65, PS75, SNFBK10], and Theorem 4.2 below.

Remark 4.1.

(i) Given a positive definite (p.d.) kernel $K$ on $S \times S$, there is then an associated mapping $E_S : S \to \{\text{Functions on } S\}$ given by
\[E_S(t) = K(t, \cdot),\] (4.1)
where the dot “$\cdot$” in (4.1) indicates the independent variable; so
\[S \ni s \longrightarrow K(t, s) \in \mathbb{C}.\]

(ii) We shall assume that $E_S$ is 1-1, i.e., if $s_1, s_2 \in S$, and $K(s_1, t) = K(s_2, t)$, \(\forall t \in S\), then it follows that $s_1 = s_2$. (This is not a strong limiting condition on $K$.)

(iii) We shall view the Cartesian product
\[B_S := \prod_S \mathbb{C} = \mathbb{C}^S\] (4.2)
as the set of all functions $S \to \mathbb{C}$.

It follows from assumption (ii) that $E_S : S \to B_S$ is an injection, i.e., with $E_S$, we may identify $S$ as a “subset” of $B_S$.

For $v \in S$, set $\pi_v : B_S \longrightarrow \mathbb{C},$
\[\pi_v(x) = x(v), \quad \forall x \in B_S;\] (4.3)
i.e., \(\pi_v\) is the coordinate mapping at \(v\). The topology on \(B_S\) shall be the product topology; and similarly the \(\sigma\)-algebra \(\mathcal{B}_S\) will be the the one generated by \(\{\pi_v\}_{v \in S}\); i.e., generated by the family of subsets

\[
\pi_v^{-1}(M), \ v \in S, \text{ and } M \subset \mathbb{C} \text{ a Borel set.} \tag{4.4}
\]

**Theorem 4.2** (Every p.d. kernel has a (non-minimal) Gaussian solution). Let \((S, K)\) be as specified in (1), then there is a Gaussian solution \((B, \mathcal{B}, \mu, \{k_s\}_{s \in S})\) to (1.4).

By a Gaussian solution we mean \((B, \mathcal{B}, \mu)\) is a probability space, and \(k : S \to L^2(\mu)\) has the following properties:

(a) Condition (1.4) holds. We shall write \(K(s, t) = \mathbb{E}(k_s \overline{k_t})\) where \(\mathbb{E}\) denotes the expectation with respect to \(\mu\);

(b) \(\mathbb{E}(k_s) = 0, \forall s \in S\);

(c) For every finite subset \(F \subset S\), the system of random variables \(\{k_s\}_{s \in F}\) is jointly Gaussian with covariance matrix \(M_F\) given by

\[
M_F(s, t) = K(s, t), \ \forall (s, t) \in F \times F. \tag{4.5}
\]

**Proof of Theorem 4.2 (sketch).** The result is essentially an application of the Kolmogorov extension principle (see e.g., [PS75]): Take

\[
B := \prod_{s \in S} \mathbb{C} = \text{all functions on } S, \tag{4.6}
\]

and set

\[
k_s(x) = x(s), \ \forall s \in S. \tag{4.7}
\]

Let \(F \subset S\) be a fixed subset, and let \(\mu_F\) be the Gaussian measure on \(\mathbb{C}^F\) which is specified by zero mean, and covariance matrix \(M_F\) as in (4.5). If \(M_F\) is invertible, then the density on \(\mathbb{C}^F\) computed w.r.t. Lebesgue measure on \(\mathbb{R}^{2|F|}\) is

\[
det(M_F)^{-|F|} \exp \left( -\frac{1}{2} \langle M_F^{-1}z_F, z_F \rangle_{\mu(F)} \right) \tag{4.8}
\]

where \(z_F\) denotes the point in \(l^2(F)\) with components \(z_j \in \mathbb{C}\) now indexed by \(j \in F\).

The system of measures \(\{\mu_F\}\) induced by all finite subsets of \(S\) then satisfies the Kolmogorov consistency equation: If \(F \subset F'\) are two finite subsets, then

\[
\mathbb{E}(\mu_{F'} | \mathbb{C}^F) = \mu_F \tag{4.9}
\]

where the notation in (4.9) refers to the conditional measure, and \(\mathbb{C}^F \to \mathbb{C}^{F'}\) via

\[
\mathbb{C}^{F'} = \mathbb{C}^F \times \mathbb{C}^{S \setminus F}. \tag{4.10}
\]

The existence of the desired probability measure \(\mu\) on \(\mathcal{B}\) now follows from Kolmogorov’s theorem, and we automatically get

\[
\mathbb{E}(\mu | \mathbb{C}^F) = \mu_F \tag{4.11}
\]

valid for all finite subsets \(F \subset S\). Now (4.11) refers to conditioning via \(B = \mathbb{C}^F \times \mathbb{C}^{S \setminus F}\).

The stated conditions (a)-(c) therefore follow, and the process \(\{k_s\}_{s \in S}\) in (4.7) has the desired properties. \(\square\)
**Definition 4.3.** Given $K$ on $S \times S$ p.d. as in (I) from Section 1. A solution to (4.1) (see (II)), $(B, \mathcal{R}, \mu, \{k_s\}_{s \in S})$, is said to be minimal if (Def.) the $L^2(\mu)$ closure of span $\{k_s ; s \in S\}$ is all of $L^2(\mu)$, i.e., span$L^2(\mu) \{k_s\}_{s \in S} = L^2(\mu)$.

**Remark 4.4.** It is known that the solution from Theorem 4.2 is generally not minimal; see e.g., [JR68, PS75, AD92, Lu08, AJSV13, Bre14, PS15, PR16].

Indeed, given $K$ on $S \times S$, p.d. as specified in (I), let $(B, \mathcal{R}, \mu, \{k_s\}_{s \in S})$ be the Gaussian solution from Theorem 4.2; then $L^2(B, \mu)$ is isomorphic to the symmetric Fock space $\mathcal{F}_S(\mathcal{H}_1)$ where $\mathcal{H}_1 = \text{span} \{k_s ; s \in S\}$.

**Example 4.5** (A p.d. kernel (the Szegő kernel) with minimal solutions). Let $D := \{z \in \mathbb{C} ; |z| < 1\}$, the open disk in the complex plane $\mathbb{C}$, and let
\[
K(z, w) = \frac{1}{1 - \overline{z}w}, \quad (z, w) \in \mathbb{D} \times \mathbb{D},
\]
(4.12) be the Szegő kernel (see e.g., [PR16]). Let $\mu$ be a singular measure on $T = \partial \mathbb{D} \cong [0, 1]$. We use the isomorphism $[0, 1] \cong T$, given by $[0, 1] \ni x \mapsto e(x) = e^{i2\pi x} \in T$. In this case, take
\[
k_z(x) = \frac{1}{1 - ze(x)}, \quad x \in [0, 1],
\]
(4.13) and suppose $f \in L^2([0, 1], \mu)$ satisfies
\[
\langle f, k_z \rangle_{L^2(\mu)} = 0, \ \forall z \in \mathbb{D}.
\]
Hence,
\[
\int_0^1 e(nx) f(x) d\mu(x) = 0, \ \forall n \in \mathbb{N}_0.
\]
(4.15) By the F. & M. Riesz theorem, we conclude that $fd\mu \ll dx$ holds, where $dx$ is standard Lebesgue measure. Since $fd\mu \perp dx$ by assumption, we conclude that $f = 0$ in $L^2(\mu)$.

**Theorem 4.6** ([HJW16]). Let $\mu$ be a singular probability measure on $[0, 1]$, and set
\[
b(z) := 1 - \int_0^1 \frac{d\mu(x)}{1 - ze(x)},
\]
(4.16) see (4.13), and
\[
K^{(b)}(z, w) = \frac{1 - b(z) \overline{b(w)}}{1 - \overline{z}w}, \quad (z, w) \in \mathbb{D} \times \mathbb{D},
\]
(4.17) see (4.12); then $\mu \in \mathcal{M}(K^{(b)})$; and it is a minimal solution, i.e.,
\[
k_z^{(b)} = \frac{1 - b(z) \overline{b(e(x))}}{1 - ze(x)}
\]
satisfies
\[
K^{b}(z, w) = \int_0^1 k_z^{(b)}(x) k_w^{(b)}(x) d\mu(x), \ (\text{see (1.4)})
\]
(4.18) and $\{k_z^{(b)}(x)\}_{z \in \mathbb{D}}$ spans a dense subspace in $L^2(\mu)$.

Moreover, $b$ is an inner function, i.e., $b \in H^\infty$, with boundary values $|b(e(x))| = 1 \ a.e. \ x$. For the RKHS of $K^{(b)}$ in (4.17), we have
\[
\mathcal{H}(K^{(b)}) = H^2 \ominus bH^2
\]
(4.19)
where $H^2$ is the standard Hardy space on $\mathbb{D}$.

5. Harmonic Analysis

In a general setting, positive definite (p.d.) kernels $K$ are defined on $S \times S$ where $S$ is a fixed set. In classical analysis such pairs $(K,S)$ have found uses in many problems in harmonic analysis, in complex analysis, in stochastic analysis, analysis on infinite graphs, and in PDE theory, the latter in the context of Green’s functions for elliptic operators. In the complex analysis setting, $S$ may be the disk $\mathbb{D}$, or the upper half-plane. For these applications, solutions typically entail consideration of boundaries, some in a natural geometric framework, and some more abstract. In some of the applications considered here, the notion of “boundary” is clear enough, for example for real or complex domains, but not for others. Take for example the case when $S$ may instead be the set of vertices in an infinite graph.

The problem considered in the present section is motivated by p.d. kernels arising naturally from classical frameworks, but our emphasis will be applications when there is not already a given, or a natural boundary available at the outset.

Example 5.1. Let $S := \mathbb{D}^k$ (the polydisk) with boundary $B := \mathbb{T}^k \simeq I^k$, and

$$K = \prod_{j=1}^{k} \left( \frac{1}{1 - z_j w_j} \right). \quad (5.1)$$

Recall the multi-index notation: $z = (z_1, \cdots, z_k)$, $w = (w_1, \cdots, w_k)$ in $\mathbb{D}^k$; and $z^n = z_1^{n_1} \cdots z_k^{n_k}$, $n = (n_1, \cdots, n_k) \in \mathbb{N}_0^k$.

**General setting.** Given $S$ a set, $K$ a p.d. kernel on $S$, $(B, \mathcal{B}, \mu)$ a probability space, and $K^* : S \times B \rightarrow \mathbb{C}$, assume that $\mu \in \mathcal{M}(K)$, $K \in \mathcal{H}(\mu)$ with reference to $S \leftarrow B$. That is,

$$K(z, w) = \int_B K^*_z(x) \overline{K^*_w(x)} d\mu(x), \ \forall (z, w) \in S \times S. \quad (5.2)$$

Set

$$E(K^*_z) = \int_B K^*_z(x) d\mu(x), \ \forall z \in S,$$

and

$$K^{ren}(z, w) = \frac{1}{E(K^*_z) E(K^*_w)} K(z, w), \ \forall (z, w) \in S \times S, \quad (5.3)$$

where “ren” := renormalization.

The kernel $K^{ren}$ in (5.3) is p.d., and we shall denote the corresponding RKHS $\mathcal{H}^{ren} = \mathcal{H}(K^{ren})$.

Set

$$(K^{ren}_z)^*(x) := \frac{K^*_z(x)}{E(K^*_z)}, \ \forall z \in S. \quad (5.4)$$

Lemma 5.2. We have

$$K^{ren}(z, w) = \int_B (K^{ren}_z)^*(x) \overline{(K^{ren}_w)^*(y)} d\mu(x), \quad (5.5)$$

on $S \times S$. 

Proof. Note that
\[
\text{RHS (5.5)} \overset{\text{by (5.4)}}{=} \frac{1}{\mathbb{E}(K_z^*) \mathbb{E}(K_w^*)} \int_B K_z^*(x) K_w^*(x) d\mu(x)
\]
by (5.2)
\[
\overset{\text{by (5.2)}}{=} \frac{K(z, w)}{\mathbb{E}(K_z^*) \mathbb{E}(K_w^*)} = K_{\text{ren}}(z, w) = \text{LHS (5.5)}.
\]
\[\square\]

Definition 5.3. Let \(S, B, \mu, K, K_{\text{ren}}, \) and \(K^*\) etc. be as above. Set
\[
K_{\text{ren}}^z \quad \overset{W_\mu}{\longrightarrow} \quad (K_{\text{ren}}^z)^*(x) = \left( \frac{K^*(x)}{\mathbb{E}(K_z^*)} \right).
\]
(5.6)
The assignment (5.6) extends by limit and closure to an isometry
\[
W_\mu : \mathcal{H}_{\text{ren}} \longrightarrow L^2(\mu),
\]
(5.7)
so that \(I_{\mathcal{H}_{\text{ren}}} = W_\mu^* W_\mu\).

Remark 5.4. (5.7) is immediate from (5.4)-(5.5). In general, \(W_\mu\) may not be onto; see below.

Lemma 5.5. The adjoint operator \(V_\mu := W_\mu^*\) of the isometry in (5.7) is a co-isometry, determined as follows:
\[
\mathcal{H}_{\text{ren}} \quad \overset{W_\mu}{\longrightarrow} \quad L^2(\mu) \quad \overset{V_\mu}{\longleftarrow} \quad \mathcal{H}_{\text{ren}}
\]
(5.8)
For \(f \in L^2(\mu), z \in S\), we have
\[
(V_\mu f)(z) = \int_B f(z) \left( \frac{K_z^*(x)}{\mathbb{E}(K_z^*)} \right) d\mu(x)
\]
\[= \frac{1}{\mathbb{E}(K_z^*)} \int_B f(x) K_z^*(x) d\mu(x).
\]
(5.9)
(We call \(V_\mu\) a normalized transform.)

Proof. Immediate from the definitions. Indeed, for \(\forall f \in L^2(\mu), z \in S\), we have
\[
\langle V_\mu f, K_{\text{ren}}^z \rangle_{\mathcal{H}_{\text{ren}}} = \int_B f(x) \frac{K_z^*(x)}{\mathbb{E}(K_z^*)} d\mu(x) = \langle f, W_\mu K_{\text{ren}}^z \rangle_{L^2(\mu)}.
\]
The result follows, \(V^* = W_\mu, W_\mu^* = V_\mu\). \[\square\]

Corollary 5.6. The co-isometry \(V_\mu : L^2(\mu) \longrightarrow \mathcal{H}_{\text{ren}}\) is defined on all of \(L^2(\mu)\) if and only if
\[
\text{span} \{ K_z^*(\cdot) : z \in S \} = L^2(\mu),
\]
(5.10)
where the LHS of (5.10) denotes the \(L^2(\mu)\)-closed span of \(\{ K_z^*(\cdot) : z \in S \}\).

Proof. Recall that \(\text{ran}(W_\mu)^\perp = \ker(V_\mu)\), so \(\ker(V_\mu) = 0 \iff (5.10)\) holds. Note that when (5.10) holds then we get a unitary isomorphism \(V_\mu = W_\mu^*\). See (5.8). \[\square\]

Return to the polydisk in Example 5.1.
Lemma 5.7. Equation (5.10) holds in this special case, i.e.,

\[ \text{span}\{K^*_z(\cdot)\}_{x \in \mathbb{D}^k} = L^2(\mu) \]

\[ \Downarrow \]

\[ \text{span}\{e_{n_1}(x_1)e_{n_2}(x_2) \cdots e_{n_k}(x_k) : n \in \mathbb{N}_0^k\} \text{ is dense in } L^2(I^k, \mu). \tag{5.11} \]

Proof. (5.11) follows from the orthogonality relation in \( L^2(\mu) \),

\[ \{K^*_z(\cdot)\}_{z \in \mathbb{D}^k} \perp \{e_n(\cdot)\}_{n \in \mathbb{N}_0^k} \tag{5.12} \]

where \( \perp \) refers to the \( L^2(\mu) \)-inner product. In details, we have

\[ f \perp K^*_z, \forall z \in \mathbb{D}^k, \]

\[ \Downarrow \]

\[ \mathbb{D}^k \ni z \mapsto \langle K^*_z, f \rangle_\mu \equiv 0, \]

\[ \Downarrow \]

\[ \left( \frac{\partial}{\partial z} \right)^n \langle K^*_z, f \rangle_\mu = 0, \forall n \in \mathbb{N}_0^k, \]

\[ \Downarrow \]

\[ \langle e_n, f \rangle_\mu = 0, \forall n \in \mathbb{N}_0^k, \]

which is the desired conclusion. \( \square \)

In the case of polydisk, we know that \( 1/E(K^*_z) \) is a multiplier in \( \mathcal{H}(K) \), and so

\[ \mathcal{H}^{ren} \hookrightarrow \mathcal{H} \tag{5.13} \]

where (5.13) means containment of RKHSs, i.e.,

\[ \mathcal{H}^{ren} = \mathcal{H}(K^{ren}) \text{ is the RKHS of } K^{ren}; \text{ see (5.3)-(5.5).} \]

\[ \mathcal{H} = \mathcal{H}(K) \text{ is the RKHS of the Szegő kernel (5.1).} \]

Example 5.8. For \( k = 1 \), \( 1/E(K^*_z) = 1 - b(z) \), where \( b \) is the function corresponding to \( \mu \) via Herglotz

\[ \Re \left\{ \frac{1 + b(z)}{1 - b(z)} \right\} = P_z[\mu] = \int_0^1 \frac{1 - |z|^2}{|e(x) - z|^2} d\mu(x) \tag{5.14} \]

i.e., \( P_z[\mu] \) is the Poisson-kernel integral.

Remark 5.9. In the general case discussed above, we may not have that \( S \ni z \mapsto 1/E(K^*_z) \) is a multiplier in \( \mathcal{H}(K). \) (See (5.2)-(5.3) for the definitions.) It may not even be so for all kernels on \( \mathbb{D}^k. \)

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