Extremal variety as the foundation of a cosmological quantum theory

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ABSTRACT

Dynamical systems of a new kind are described. These are based on the extremization of a non-local and non-additive quantity that we call the variety of a system. In these systems all dynamical quantities are relational, and particles have properties, and can be identified, only through the values of these relational quantities. The variety then measures how uniquely each of the elements of the system can be distinguished from the others in terms of the relational variables. Thus a system with extremal variety is one in which the parts are related to the whole in as distinct a way as possible.

We study several dynamical systems which are defined by setting the action of the system equal to its variety, and find that they have the following characteristics: 1) The dynamics is deterministic globally, but stochastic on the smallest scales. 2) At an intermediate scale structures emerge which are stable under the stochastic perturbations at smaller scales. 3) In near to external configurations one sees the emergence of both short ranged repulsive forces and long ranged attractive forces, 4) The dynamics is invariant under permutations of the labels of the particles. For these reasons it seems possible that extremal variety models could provide a foundation for a new kind of non-local hidden variables theory, which could be applicable in a cosmological context.

In addition, the mathematical definition of variety may provide a quantitative tool to study self-organizing systems, because it distinguishes highly structured, but asymmetric, configurations such as one finds in biological systems from both random configurations and configurations such as crystals which are highly ordered by virtue of having a large symmetry group.
1 Introduction

This paper is addressed to the question of how to construct a physical theory that could apply to the universe as a whole. It is motivated by the idea that the problem of constructing a theory that could apply to a single and entire universe differs fundamentally from the problem of constructing a theory to describe phenomena occurring in a small portion of the universe. There are several reasons for this belief; perhaps the primary one is that a theory of a portion of the universe can refer, in the definition of its kinematical quantities and interpretational schemes, to things that are outside the domain of the phenomena that are being described. Indeed, all of our successful physical theories make use of this opportunity. Some examples of these external elements are the fixed inertial frames of Galilean and Poincare invariant theories, the external measuring devices of quantum mechanics and the external time coordinate that is the basis of all non-general relativistic dynamics.

As long as the domain in which the theory is actually compared to experiment involves only a portion of the universe there is no problem with the use of these external elements. However, when we enlarge our ambition and attempt to construct a theory that could apply to the whole universe, we should not be surprised to discover difficulties arising because the familiar structures that previously were defined in terms of things external to the system now have nothing to which to refer. This is the origin of the criticisms of Newtonian mechanics known as Mach’s principle[1] and also of many of the difficulties, both interpretational and technical, in the study of quantum cosmology[2, 3].

Thus, the problem of constructing a cosmological theory is largely the problem of constructing a sensible theory that does not make any use of fixed, a priori, or background structures. Indeed, one of Einstein’s main motivations for constructing general relativity was Mach’s criticism of Newtonian dynamics exactly on this point[4], and he succeeded to the extent that the only full scale example we have of a theory which does not rely on external or fixed nondynamical structures is general relativity itself. Specifically, the diffeomorphism invariance of general relativity is, in a certain sense[5, 6], an exact indication of the lack of background structure in the theory. Thus, at this time, the problem of constructing a cosmological theory consists chiefly of the problem of merging general relativity with quantum theory, and one of the basic aspects of this problem is the problem of constructing quantum
field theories which are diffeomorphism invariant.

There are, broadly speaking, two ways in which this problem may be resolved. The first is by inventing a reformulation and reinterpretation of the quantization process—the process in which classical theories are turned into quantum theories that is suitable to diffeomorphism invariant theories such as general relativity and that could stand as a theory of a single universe. Several such programs are currently under active development, from which we have learned, and will continue to learn, a great deal that is useful. However, we find ourselves, for reasons we have argued elsewhere [2, 3, 5, 7, 8, 9, 10], skeptical as to whether any effort in this direction alone can be completely successful.

The second direction to search for a resolution to these problems is by the invention of a new theory, based on new principles, that would reduce, in the appropriate limits, to classical general relativity and quantum field theory. This paper is intended as a contribution to this second program.

The work we will describe here has been underway for a number of years, and involves projects undertaken both singly and jointly by the authors. In this work we have attempted to start, so to speak, from the other side, and to pose the question, what kind of mathematical structure could be appropriate for the description of the physics of an entire universe? We have found our efforts to answer this question converging from more than one direction on a particular kind of structure. This structure is a certain kind of non-local dynamical law which can be applied to a large number of different model systems. These dynamical systems are defined by means of a variational principle, which is based on a type of quantity that we call, generically, the variety of the system. The variety is a potential that is defined on the configuration space of the system, i.e. it is a function of the configuration of the system. The variety is a non-local and non-additive quantity, which can only be applied to a system as a whole. It measures, in a certain sense, how unique, one from another, the different parts of the system are.

We consider here two kinds of variational principles involving the variety. In the first, the variety alone is extremized, in the second, and more conventional, the variety is extremized in conjunction with a kinetic energy term.

We find that dynamical systems based on extremising the variety by itself have a number of general characteristics. 1) They are deterministic globally, but the behavior on the smallest scales is stochastic. This is because the
variational principle is not locally additive, so that global extrema are not local extrema. 2) The extremal configurations have structures on an intermediate scale that are stable under the stochastic perturbations at smaller scales. Given a notion of time (which we will discuss below) these structures evolve in ways that are slow on the scale of the fundamental time step. 3) In spite of the overall non-locality of the dynamics, which is responsible for the stochastic behavior at the smallest scales, in near to extremal configurations an approximate local dynamics emerges on these intermediate scales. Further, in some of the models we see the emergence of short ranged repulsive interactions and long ranged attractive forces. 4) The extremal configurations are, in a certain sense, the opposite of ordered systems in that they have as little symmetry as possible. 5) The action principle is always invariant under the group of permutations of the elements of the system, so that these are deterministic dynamical systems of intrinsically identical particles. (The precise meaning of this last statement will be clarified below.)

Because of the suggestive character of some of these results, as well as for other reasons that we will elaborate on shortly, we believe that it might ultimately be possible to use this kind of dynamical systems to describe a cosmological hidden variable theory. This would be a theory that would resolve together the problems of quantum cosmology and the interpretational problems of quantum mechanics in a theory that was at once a theory of cosmology and a theory of the deterministic—but highly non-local—dynamics behind quantum mechanics. However, we want to stress that we are not here proposing such a theory; having come to the conclusion that a new kind of dynamical structure would be needed in the construction of any such theory, we intend here only to study a possible candidate for such a dynamics. The systems that we analyze here are thus intended only as toy models.

Besides these long term aims, we also have the impression that extremal variety systems are interesting in their own right. They introduce kinds of structures that, so far as we know, have not hitherto been studied. Besides the possible application to physics, they may also be interesting for other fields such as information theory, the study of neural networks, the study of biological systems or the study of self-organizing systems in general. In this paper we will touch only briefly on this aspect of the subject, it will be taken up in detail in another place.

The directions from which we have been led to the study of extremal variety dynamics include a search for a nonlocal dynamics that could under-
lie a hidden variables theory, a search for a formulation of dynamics that realizes the principles set by Leibniz in his criticisms of Newtonian dynamics and the search for a dynamics that could be applied to a combinatorial system of relations that would cause a condensation to a phase that could be described in terms of the geometry of a low dimensional spacetime. These motivations are recalled briefly in the next section, but are described in detail in the cited references. In this paper we concentrate on describing the models and the results of our studies of them.

In the following section we elaborate on the motivations from quantum gravity and the interpretational problems of quantum mechanics for attempting the construction of non-local and relational dynamical systems. In section 3 we give a general characterization of extremal variety dynamical systems, and introduce much of the language and mathematical structure that is used to describe them. Section 4 is devoted to examples of extremal variety systems constructed from graphs. These have been described before, so the discussion here is brief. Three new models of extremal variety systems, which are much easier to study than the graph theory models, are described in section 5. We close in section 6 by summarizing what we have learned from these systems and setting forth some conjectures and directions for future study.

2 Motivation: issues facing attempts to construct theories of a whole universe

When one approaches the problem of constructing a theory of cosmology, several new issues emerge which do not have to be faced in any less ambitious undertaking. As these issues form the motivation for the work presented here, we will begin by briefly sketching them.

1) As we stated in the introduction, all existing physical theories posit the existence of fixed background structures which are non-dynamical, but are necessary to fix both the forms of the equations and the interpretive rules that describe the evolution of the dynamical quantities that the theory describes. These background structures refer either explicitly or implicitly to things that exist in the universe but are outside of the domain of description
of the theory.

A theory of the whole universe must do away with these external background structures, for, in principle, nothing may be outside of the domain of description of such a theory. In terms of the question of the nature of space and time the task of doing away with background structures is intimately related to the absolute/relative debate, which has been one of the central themes in the development of dynamics from the Greeks to Einstein[15].

A good example of a theory in which some of the background structures in classical mechanics have been eliminated is the set of models developed by one of us in collaboration with Bruno Bertotti in the context of the classical dynamics of point particles[13, 5, 12]. In these models a dynamics is constructed in which the measurements of space and time are purely relational, so that only relative distances play a role, and external frames are not used. Here one sees two key features that emerge when one eliminates background structure in favor of explicitly dynamical structure. The first is that gauge symmetries arise, which indicate that certain mathematical structures, such as the coordinates of particles or time parameters, are auxiliary quantities and do not refer to anything that is physically meaningful. The diffeomorphism invariance of general relativity is, in fact, a gauge symmetry exactly of this kind. Its consequence is that spacetime points are not physically meaningful, so that no physically meaningful observables may refer to them.

The second feature that arises, in both these model systems and in general relativity (when cosmological boundary conditions are used), is that certain global constraints arise which are reflected in the local physics. Thus, in all of these cases, the total momentum, energy and angular momentum of the universe are constrained to be zero. This, is of course, the only logical possibility, as there is no external or background structure with respect to which these quantities could be defined. But we believe that it is also an indication of a deeper consequence of the elimination of background structures. In conventional physical theories the properties of individual particles, or of fields at individual points, are well defined independently of the existence of other particles or of the field values at other points, because they are defined with respect to the background structure. Thus, the basic entities of these theories can have a priori properties. In a theory without background structure, this option is not available; all properties of individual particles must be defined in terms of their relations with the other particles in the universe. This turns out to introduce new kinds of couplings between the local physics...
and the global state of the system that are not present in conventional background dependent physics. Mach’s principle, in which the global distribution of matter is posited to affect the local inertial frames, and through them the properties of individual particles, is one example of this kind of local/global coupling that comes from the elimination of background structure\cite{12, 13, 14}.

A related consequence is that, in a theory without background structure, there are no purely kinematical quantities. All observable, physically meaningful, quantities rely on dynamics for their definition, so that the conventional distinction between kinematics and dynamics breaks down. This is especially apparent in general relativity, as well as in particle mechanics with genuine time reparametrization gauge symmetry\cite{1}. In these cases the problem of determining the physical observables—those functions of the phase space variables that are invariant under the gauge transformations—is a dynamical problem, and it requires the solution of the equations of motion\cite{16, 22}.

We will see in the models to be described below this intertwining of local and non-local dynamics, and of dynamics and kinematics.

2) For the reason just cited, as well as for reasons described elsewhere\cite{11, 9, 17}, we have been interested for some time in the problem of whether alternatives to quantum mechanics can be constructed that maintain a realist ontology and epistemology. This leads to the problem of constructing hidden variables theories and, through the work of Bell\cite{18} and the related experimental developments\cite{19}, to the problem of constructing a non-local hidden variables theory. This is relevant to our project here, because the kind of non-locality that the experimental results seem to call for is so drastic as to make it seem likely that any non-local hidden variables theory must be a cosmological theory. According to the experiments, as well as to the quantum theory itself, non-local correlations are established between any two particles whenever they interact. These correlations are carried forever by the particles, so that at any time in the future the complete determination of the quantum state of a single electron requires a knowledge of the quantum states of all the particles it has since interacted with. If this is to be replicated by a hidden variables theory, the hidden variables which describe a single particle must tie it to all of the particles it interacted with in the past. Thus, the causal structure of such a hidden variables theory must be a

\footnote{By which we mean that there is no external time and the Hamiltonian constraint is quadratic in the canonical momenta\cite{10, 3}.}
representation of the history of the universe.

This circumstance leads naturally to the idea of relational hidden variables theories\[1, 2, 3]. In such a theory the hidden variables give, not a more detailed description of the state of an individual electron, but a more detailed description of the relations which exist between that electron and the other particles in the universe. The problem of constructing an acceptable hidden variables theory thus leads naturally to the same issues that the problem of constructing a cosmological theory without background structures lead to: how to construct a sensible dynamics in which all quantities are relational rather than absolute quantities which necessarily refer to a fixed background structure. Indeed, it was an attempt to elaborate such theories that first led to the formulation of the idea of extremal variety\[3].

3) In conventional physical theory, the description of a given phenomenon is divided into two parts: the specification of the laws and the specification of the initial conditions. As has been stressed before\[16, 22, 20], this separation is somewhat problematical when we come to the case of cosmology. First, as the universe only happens once, or at least as we will only get to observe one instance of a universe, it is not clear what empirical meaning this distinction has when applied to the universe as a whole. But, more importantly, what we really want and need from a cosmological theory is a theory of origins, a theory that will explain to us why the universe is the way it is. If we stick to a more or less conventional dynamical framework, then what we need is exactly a theory of initial conditions\[2].

This problem becomes especially acute if we recall that, as a result of the time reparametrization invariance of cosmological theories, and the resulting blurring of the distinction between kinematics and dynamics, there is no physical, diffeomorphism invariant, meaning to the notion of initial conditions in the cosmological context\[16, 24]. Both classically and quantum mechanically in a cosmological theory there is a space of states, but as there is no time, and no Hamiltonian, these states do not evolve\[3].

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\[2]as has been stressed especially by Hartle and Hawking\[20].

\[3]This does not mean that the formalism cannot describe phenomena that we would call evolution in time. But that time must be measured in terms of some dynamical degree of freedom of the theory, and the observables that describe evolution are really just describing invariant correlations between the degree of freedom chosen to be time and other degrees of freedom of the theory\[16, 21, 22]. It is, indeed, in the structure of such correlations that the dynamical content of the theory is recorded.
4) A question closely related to the question of the initial conditions is why the observed state of the universe seems so special. This circumstance, which has often been remarked upon, manifests itself in several, apparently independent, ways. It is worth recalling some of them here.

i) As has often been noted\cite{23, 24}, the overall configuration of the Universe is very special. It appears to have vanishing angular momentum, its density (in terms of fundamental units) is fantastically close to the critical density for a flat Friedman universe, and the microwave background is isotropic to at least $10^{-3}$. This is often interpreted to mean that the universe began in a very special state, for example, with the gravitational field completely unexcited\cite{23}.

ii) In spite of the apparent isotropy of the microwave background and observed galaxy distributions, the distributions of galaxies appear to be inhomogeneous out to the largest scales that have so far been probed\cite{25}. The most natural hypothesis that could be made concerning this structure is that it is the result of gravitational clumping that occurred since decoupling. Rather surprisingly, it seems to have been very difficult to make a theory based on this hypothesis work, even with the help of dark matter. Given that new observational discoveries, such as the large scale streaming, and the apparent periodic structure continue to be made, this is a subject that may remain open for some time.

iii) It appears that the values taken by a number of the fundamental constants of nature (for example, the 17 constants of the standard model of elementary particle physics) are special in the sense that were the values different by only a few percent the universe would be vastly different. For example, were the neutron-proton mass difference much bigger, there would be no stable nuclei, no stars, no nuclear physics or chemistry\cite{24}. Were it of the opposite sign, the primary component of the universe would be free neutrons. Several different examples of this circumstance are discussed in \cite{26, 27}. For our purposes it is worth remarking that in many of these cases it seems that the special values the constants take allow the universe to develop more structure than otherwise, for example, this is certainly the case with the neutron-proton mass difference.

Until recently, modern cosmology took the view that the large scale structure of the universe can be explained without making any specific hypotheses as to what is actually in the universe. This may have been plausible at a time when the simple model of a homogeneous and isotropic Friedman model
filled with "dust" was sufficient to account for all that was known. However as soon as one attempts to explain the real observed structure of the galaxies, or understand why the overall structure of the universe seems so fantastically special, such a view is clearly insufficient. In fact, most contemporary work on cosmology, such as the dark matter theories, inflation and cosmic string theories, rely on some hypotheses as to the content of the universe. If these do not succeed, then perhaps it will not seem too mad to entertain the idea that there are laws that govern the large scale structure of the universe and that these laws are intimately tied up, in the way we discussed above, with the laws that govern local physics.

3 Extremal variety as a dynamical principle

To summarize the argument of the previous section, we are in search of a new kind of dynamical theory which is characterized by the following conditions:

1) It is purely relational. All physical quantities describe relations between entities; conversely, none of the entities posited by the theory have any properties which are individually and independently defined; instead all properties are described in terms of the relations among the posited entities. Further, the use of fixed, background structure is to be as much as possible minimized.

2) It is non-local, because the experimental results show that the Bell inequalities are violated.

We will now describe a set of models which have these two features. In doing so, we will borrow some language from Leibniz’s *Monadology*\[28\], as we have found it to be the most appropriate language to use in this context. The extent to which these models do provide a mathematical realization of the ideas proposed in Leibniz’s writings will be discussed in another place\[29\].

We then posit a universe consisting of $N$ elements, which we will call monads. In a fundamental theory there should be no background space, so the monads do not have coordinates, or any other properties that refer to them individually. Instead, the state of the system is given in terms of relational quantities which are shared among monads. More particularly, each monad has a view, which describes its relation to the rest of the universe.

The unity of the universe finds its expression through the requirement that all the individual views are mutually consistent; this ensures that the
views can be regarded as all the different possible perspectives of a unique structure that is the configuration of the universe. Depending on the manner in which time is to be introduced into the scheme, such a configuration may represent either one instant of time or the entire history of the universe. The only a priori structure in the model is represented by the constraints which establish how the $N$ possible views corresponding to a given configuration do indeed fit together self-consistently to make such a configuration. The set of all such self-consistent views forms the configuration space, $\mathcal{S}$, of the universe.

The question then arises as to what kind of dynamics to posit for such a model. As there is, at this level, no spacetime, the normal dynamical quantities such as energy and momentum are inappropriate. Instead, we want to pick a dynamical principle that will push the system into configurations where the very special properties of the spacetime description will emerge. The usual dynamical quantities which are tied to spacetime symmetries—energy, momentum and angular momentum—will then be recovered as effective concepts.

An appropriate set of quantities which are candidates for the action of such a non-local, relational, system, are what we call *varieties*. A variety is a function of the views of the system which measures how different the different views of the system are one from another. To construct such a quantity we first introduce a symmetric array of *differences* $D_{ij}$. The $ij$’th is a function of the views $w_i$ and $w_j$ and is a measure of how different they are.

The variety can then be defined as

$$V = \sum_{i \neq j} D_{ij}$$  \hspace{1cm} (1)

As the simplest dynamical system we then posit that the state of the universe is one in which $V$ takes, at least locally, a maximal value, $N$ being held fixed.

On the face of it, such a principle will lead to the determination of one or several (if the extremal state is not unique) static configurations. However, this configuration could represent structure that corresponds to both space and time. Alternatively, time could be introduced explicitly by introducing

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4In the toy models the definition of $D_{ij}$ will also typically include a set of symmetry operations, so that what will be measured will be the difference in the views mod the symmetry operations. This has the effect of further diminishing the a priori structure of the model.
more conventional kinetic type terms into the variational principle. We shall return to this question later.

We will discuss later several alternative ways in which the variety of a system can be defined. Given either the above definition (1) or one of the alternatives, the variety can be understood to be, in a certain sense, a measure of the complexity or structure of the system. In addition, the condition of maximal variety can be understood as a condition of minimal symmetry, because if there is a symmetry operation, $S$, which is a permutation among the $N$ monads that leaves the views fixed, then there will be pairs of monads with no differences between their views.

We now go on to discuss four models in which this structure is realized. We want to emphasize that what we are describing are models, they are not proposals for a fundamental dynamics of the universe. They are designed to provide toys to investigate the kinds of mathematical phenomena that can occur in a dynamical system based on the principles stated above. We believe that some of the phenomena we find here are suggestive of various features of the real physical world and this, of course, encourages us. But for the moment this point is less important than the fact that these models manifest a rich array of behaviors and structures.

4 Extremal variety graphs

The first example we will give is what might be called a pure relational model, because there is no background space at all. The system is defined entirely in terms of a set of relational variables, one for every pair of monads. We will describe the simplest example of such a system, which is simply a graph.

Consider, then, a connected graph, $G$, on $N$ points, such as that shown in Figure 1, to be a model of a complete, self-contained, universe. The $N$ points are the monads, and the state of the system is given by the connections of the

\[\text{It seems to us that a graph is just about the simplest mathematical concept that one could take to represent a universe. For it enables one to express the idea that an interesting universe should contain genuinely distinct entities (this part is played by the vertices of the graph) that are nevertheless linked together in an indissoluble whole (the graph). Moreover, the parts of the universe (the vertices) only acquire genuine identity by virtue of their relations or connections to each other (the role of which is played by the lines of the graph). In a graph of the type considered here, all extraneous structure is pared away and we are left with what we believe is an irreducible model of a universe.}\]
graph. This can be described by the adjacency matrix, $A_{ij}$, which is defined so that, for $i \neq j$, $A_{ij} = 1$ if points $i$ and $j$ are connected and $A_{ij} = 0$ if they are not connected. We will also assume that the diagonal elements are zero, in accordance with the idea that monads have no a priori properties. The state space of the system is the space of all possible graphs on $N$ points. It has $2^{N(N-1)/2}$ elements, which grows very rapidly with $N$.

We now will introduce two ways in which the view of a point of a graph can be defined. The first definition of the view of a point is simply a list of other points that it is connected to. The space of views $V$ then consists of vectors in $\mathbb{R}^N$ whose components are either 0’s or 1’s. Thus, the $i$’th view, $\vec{w}_i$, is given by the $i$’th column of the adjacency matrix, so that

$$[w_i]^j = A_{ij}$$

This simplest definition might be called the first order view. (Of course, since the views are all derived from a single graph, they must satisfy a large number of consistency conditions.) Higher order views can be defined which measure the points that a point is connected to by more than one step of the lattice. For example, one definition of an $n$’th order view is,

$$[w^n_i]^j = (A^n)_{ij}$$

The differences $D_{ij}$ are then designed to measure how differently the $i$’th and $j$’th point are connected to the rest of the graph. The simplest definition is based on the first order views, it is

$$D_{ij} = |\vec{w}_i - \vec{w}_j|$$

where we use the Euclidean norm in $\mathbb{R}^N$.

The variety can then be defined according to (1).

The second definition of the view of a vertex is based on a suggestion by David Deutsch\(^\text{30}\). In this definition, one defines the concept of the $m$’th neighborhood of the $i$’th vertex. This is defined to be the subgraph $\mathcal{N}_i^m$ of $\mathcal{G}$ that consists of only the vertices within $m$ steps of $i$, and the connections

\(^6\)It is possible to consider generalizations in which there can be more than one connection, or more than one kind of connection between points, however we will discuss here only the simplest possible model, in which each relational variable is purely binary, the connection is either on or off.
between them. The original, i’th point, is a preferred, or marked, point on
the graph. We then define a notion of isomorphism between graphs with one
marked point: Two graphs are defined to be isomorphic if there is a one to
one and onto map that takes the vertices and lines of one into the vertices and
lines of the second, taking also the marked point to the marked point. We
then define $K_{ij}$ to be equal to the smallest $m$ such that $\mathcal{N}_i^m$ is not isomorphic
to $\mathcal{N}_j^m$. In this case it is appropriate to call $K_{ij}$ the relative indifference of
$i$ and $j$; for a large value of $K_{ij}$ means that $i$ and $j$ cannot be distinguished
readily by local comparisons. The variety is then defined in the following
way. For each vertex, $i$, we define its absolute indifference, $R_i$, to be the
maximum value of $K_{ij}$, for all $j \neq i$. This tells us how large neighborhoods
we must look at to distinguish the $i$’th point from all the others, just by the
topological structures of its neighborhoods. We then define the variety to be,

$$V' = \sum_i \frac{1}{R_i} \quad (5)$$

Thus, a graph has a maximal value of the variety if the sum of the $R_i$ is
minimized. This means that, overall, each point is distinguished from the
others in terms of its neighborhoods by using as small neighborhoods as
possible.

This second definition is, in a sense, more intrinsic than the first, because
it makes no use of any labeling of the graph. Using it, a vertex of a graph is
defined only by a description of its neighborhoods.

We believe that, at the most fundamental level, this is the way identity
should be conceived: all entities must be defined by their attributes—and
nothing else. Thus, one must not think of the vertices as little identical pre-
existing points waiting waiting to be connected up into a graph. Rather, the
vertices and lines, together with their intrinsic identities, come into existence
with the graph.

However, note that each of the definitions of variety we have just given
are invariant under a gauge symmetry, under which the vertices of the graph
are arbitrarily relabeled. One has therefore the option of regarding the ver-
tices as identical particles but requiring the dynamics to be invariant under
permutations among them. This establishes a certain similarity with the
symmetry principles of quantum mechanics that result from identity of par-
ticles; we shall see later that fermionic behavior is to a degree inherent in
maximal variety schemes.
A preliminary study of maximal variety graphs, using the first of these definitions, was performed several years ago, and described briefly in [3]. The basic difficulty is that the very large growth in the number of graphs with \(N\) means that statistical or Monte Carlo methods must be used to study graphs which are reasonably large. Using a Vax it was possible to study graphs consisting of up to about a thousand points. From this preliminary study a number of qualitative conclusions emerged:

i) Extremal variety graphs are very special; they have much higher variety than the average value of the ensemble of randomly generated graphs.

ii) Random trees have higher varieties than general random graphs.

iii) Graphs which are invariant under a symmetry operation cannot be extremal, each symmetry operation costs some variety. This suggests the point of view that extremal variety graphs are as asymmetric as possible.

Now, if a graph like this is to describe a universe, then the most important question is how space arises. What we mean by space, from a purely relational point of view, is that the sets of relative distances among a set of \(N\) particles is very highly structured, so that they can be represented by points in a low dimensional manifold. In the simplest case where that manifold is \(R^3\), this means that the \(N(N-1)/2\) relative distances are actually determined in terms of \(3N-6\) coordinates (We subtract 6 for the arbitrariness of the frame.) Furthermore, if we look at the distribution of the \(N(N-1)/2\) distances among \(N\) points in some \(R^m\), we see several characteristic features: 1) The distribution of distances \(d\) follows a power law, proportional to \(d^{m-1}\). 2) The ratio of the largest to the smallest distance is typically very large. To see how unusual it could be that a system of relations like a graph could approximate \(N\) points in an \(R^m\), with \(N\) much much larger than \(m\), one need only appreciate that in a random graph, such as one defined by some probability, \(p\), that any link is on, the distribution of distances is Gaussian, and the ratio of the largest to the smallest distance is, for almost all \(p\), a very slowly growing function of \(N\).

We can then ask, is there a dynamics that we can posit for graphs that will pick out the very unusual graphs whose distance relations (determined in some intrinsic way) could approximate those of a distribution of points in \(R^3\). It was, indeed, as a possible answer to this question that the idea of a variational principle based on the notion of variety was invented[3]. The intuitive idea is that a graph with a large variety will have a very spread out distribution of distances, so that the ratio of the largest to the shortest
distance should grow with $N$ much faster than in a random graph. While behavior of this kind was seen in the computer simulations that were done, it was not possible to make more stringent tests of the conjecture that an extremal variety graph will condense to a low dimensional space. To do this clearly requires large scale computer simulations, unless progress can be made in this direction by analytic means.

5 Some simpler models of extremal variety

Because the graphs do not live in any background space, and in general are hard to imbed meaningfully in any space (this is, indeed, the point) it is often hard to see what is going on when one is working with a large graph. Because of this, and the general computational difficulties of working with large graphs, it is useful to define some toy models that involve points living in some fixed background structure, but still use a principle of extremal variety to generate structure. This allows the general properties of systems based on the idea of maximal variety to be studied without having to face the particularly difficult problem of how the very structured system of spatial relations among points in a low dimensional manifold is generated. More particularly, we would like to separate the study of how a variational principle can generate structure in general from the much more difficult question of how it can generate the particular kinds of structure that is required if a set of points and relations is going to condense in a way that they can be imbedded in a low dimensional space.

It is easy to invent such systems. We will discuss here three that have been studied in some detail analytically and numerically.

5.1 A first one dimensional model

Consider a one dimensional lattice with $P$ sites. The ends are identified, so that we have a circle. On these sites we distribute $N$ ($< P$) identical particles (See Figure 2). Given an appropriate definition of variety, we can then study the properties of distributions of the particles on the lattice that extremise it.

In such models it is appropriate to use the second definition of the variety we introduced in the previous section. To do this let us introduce an arbitrary
labeling \( i = 1, \ldots, N \) of the particles. A state of the system, which consists of the positions of these particles, will be called \( S \). More properly, the state consists of the positions of the particles mod the permutation group, as nothing will depend on the arbitrary labeling of the particles. For the moment, we shall allow distributions in which more than one particle is placed on a given site.

We want to define the variety in such a way that it depends only on the relations between the particles on the line. We then consider the \( m \)'th neighborhood, \( \mathcal{N}_i^m \), of the \( i \)'th particle to consist of the \( m \) sites to the left and the \( m \) sites to the right of the particle (see Figure 3). As in the case of the graphs, we need a definition of when two neighborhoods are isomorphic. To make the model more interesting, we define this in such a way that there is no intrinsic orientation of the neighborhoods on the lattice. Thus, two \( m \) neighborhoods, \( \mathcal{N}_i^m \) and \( \mathcal{N}_j^m \) are isomorphic if they are identical up to reflection around the central point. (See Figure 4).

As before, we then define the indifference, \( K_{ij} \), to be the smallest \( m \) such that \( \mathcal{N}_i^m \) is not isomorphic to \( \mathcal{N}_j^m \).

We may note that it may be the case for some particular state \( S \) that for two particles \( i \) and \( j \) their neighborhoods do not become distinct before \( m \) reaches its natural upper limit \( m^* \), which is \( m^* = (P - 1)/2 \) if \( P \) is odd and \( P/2 - 1 \) if \( P \) is even. This is always the case if particles \( i \) and \( j \) are placed on the same site, but it also occurs for many symmetric configurations of the particles with not more than one particle per site. (Incidentally, analogous possibilities also exist in the graph model.) We will thus call a configuration in which there exists, for all \( i \) and \( j \), an \( m \leq m^* \) such that the \( m \)'th neighborhood of \( i \) and \( j \) are distinct, Leibnizian configurations; the remaining configurations will be called non-Leibnizian.

We can deal with non-Leibnizian configurations in one of two ways. We can simply say they are not allowed. Alternatively, if for particles \( i \) and \( j \) no distinguishing neighborhoods with \( m \leq m^* \) exist, we can set the indifference \( K_{ij} = M \), where \( M \) is any number such that \( M > m^* \).

Now, for each particle, say the \( k \)'th one, we may define the relative indifference of \( k \), denoted \( R_k \) to be the maximum of the \( D_{ki} \), over all the \( i \). This is the order of the neighborhood it is necessary to go to distinguish the \( k \)'th

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7After Leibniz’s principle of the identity of indiscernibles.
particle from all the others. The variety is then defined as,

$$V = \frac{1}{N} \sum_{k} \frac{1}{R_k}$$  \hspace{1cm} (6)$$

A configuration with high variety is then one in which each particle is distinguished from each other one with the smallest neighborhoods.

Note that if non-Leibnizian configurations are allowed among the trial configurations the above definition of $K_{ij}$ for indistinguishable $i$ and $j$ will ensure, especially for large $P$, that the non-Leibnizian configurations have a variety much lower than the extremal configurations. In particular, this means that our dynamical principle will automatically exclude the placing of two particles on the same site (such configurations being automatically non-Leibnizian). As a consequence, the particles will exhibit fermionic type behavior, which will be achieved dynamically and not through imposition of a kinematical condition.

The above model was studied numerically \footnote{Using Lightspeed C on a Machintosh II.} by means of a program which generated states which are local extrema of $V$. The program begins by generating an initial state $S$ randomly. It then randomly picks one of the $N$ particles, say the $k$'th one, and moves it to a new unoccupied position, trying first one step to the left or right, then two steps, and so on. It tests to see if the move increases the variety or not, if it does it keeps it, if it does not, it goes on to try the next unoccupied spaces to the left or right. The program stops when there is no move for any of the particles that will increase the variety of the state. The resulting state, $S_{\text{ext}}$ is then a local extremum of $V$. Some local extremum of $V$ are shown in Figures 5-8 for several different $N$'s and $P$'s. From these results we may draw some conclusions:

1) The maximal variety states are highly structured. It is, indeed, easy to distinguish each point from all the others by inspection, by noting visually its neighborhood.

A maximal variety state is then very different from both a random state and a highly ordered state. We will discuss this distinction further below.

2) Even for these relatively small $N$ and $P$, there are several distinct local extrema. This tells us that $V$ must be a rather complicated function on the space of states. For example, in Figures 6-8 we see several different
local extrema for the same \( N \) and \( P \). These configurations are similar, in the qualitative appearance, but distinct.

For example for small \( N \) compared to \( P \) there are two or more independent groups, separated by quite a bit of space. The tendency of the particles to form groups is a result of the fact that if there is more than one lone particle, each separated by quite a bit of space, it will take a large \( m \) to distinguish them. This \( m \) can be decreased if at least one of them moves close to a group that can distinguish it. Thus, as long as there is more than one lone particle there is what can be thought of as a long range attractive force between lone particles and groups and among lone particles. (Indeed, with the form (6) of \( V \) the potential pulling lone particles to groups is proportional to the inverse distance!) In a solution the variety is extremized, and there cannot be a situation with more than one lone particle. However, there can be left out exactly one lone particle, as this will be easily distinguished from all the others. Indeed, from the results we see that in the extremal configurations there is often one, but never more than one, lone particle.

At the same time there must also be a repulsive short ranged interaction, because to extremize the variety the groups should all be distinct from each other. We cannot have too many identical groups of two or three, each with a lot of space around them.

The emergence of both long ranged attractive forces and short ranged repulsive forces are examples of how local dynamics can emerge from a global, non-local variational principle. We note, in particular, that the short ranged repulsive forces are very similar in their action to the degeneracy pressure in fermions that results from the kinematic Pauli principle. (We have already shown how the requirement of extremal variety effectively prevents the occurrence of configurations with more than one particle at a given site).

To see how the structure of \( V \) on the configuration space exhibits both these short ranged repulsive forces and long ranged attractive forces, we may consider a sampling of configurations through which the system approaches the extremum. This is indicated in Figures 9-11. In these Figures we show several numerical experiments in which we begin with a random configuration with \( N \) particles and watch how they move towards an extremal configuration. We see that, characteristically, the particles tend to move by sequences of steps which are occasionally punctuated by large jumps. The overall tendency is for the particles to pull together into groups, exactly as if they are moving under the influence of attractive forces.
It is interesting to note that if the new particle is added to a position far from any groups, and the original configuration already has a lone particle, both lone particles will move towards the nearest groups, which then absorb them. If, however, the new particle is added to a point in the midst of a group, it will likely have a small, but nonlocal and apparently random influence on many of the other particles as they shift around to find the configuration in which all are best distinguished.

We see that in both cases, the number and center of masses of the large groups are not changed by the perturbation of adding a new particle, even in the case that the small scale configuration of each group shifts quite a lot. This is an example of a further feature of the solutions which is of interest, namely at the intermediate scales of the groups, the structure of the solutions is stable against small perturbations. At the same time these perturbations cause changes at the smallest scale that are apparently random (in the sense that they have no local cause).

We can in addition say some things about the groups that the particles form themselves into. First, as we already remarked, each of these groups is different from all the others. This is necessary, of course, so that each member of each group is distinguished from all the others by as small neighborhoods as possible. If we look only at one particular group, consisting of a certain number of particles, we cannot predict exactly what form it will take. This illustrates a second feature of the solutions that we find very interesting, that the behavior of the system on small scales appears random if one looks for explanation in terms of a local law. At the same time, if one can look at the whole system one finds that the exact configuration is a solution to a non-local variational principle.

Finally, we see that because in an extremum there will be at most one group consisting of just one particle, the long range forces have disappeared, and there is no change in the variety if the overall distances between the groups change, as long as they remain large compared to the size of the groups. As we remarked above, in an extremum the long range interactions must disappear because every particle will be distinguished from every other one by small neighborhoods.

It is also interesting to look at the sequence of extrema which are produced by these experiments. These are shown in Figure 12. Each of these shows a sequence of extrema, each of which has been produced from the previous one by adding a monad at a random point and then finding a nearby extremum.
Thus, each line in these Figures corresponds to a line before a space in Figures 9-11.) We see again the combination of local stochasticity together with stability of structure on larger scales, as each of the groups grows, or new groups are started. This continues until the groups come close enough to each other that the short range interactions which attempt to keep the groups distinct begin to involve the particles in different groups, at which point they effectively merge.

We now turn to the discussion of a similar, but not identical, one dimensional model.

5.2 A second one dimensional model

In the first model we have put in a large amount of background structures in order to bring out in an easily visualized way how a variational principle based on variety produces structured configurations. However, as the basic conjectures which motivated the introduction of this kind of variational principle are based on the idea that we seek a theory with as little background structure as possible, it is interesting to ask to what extent we can eliminate the background structure in this model and still have a system which is easy to study. Of course, the basic structure we have put into the model is the one dimensional lattice and we do not want to eliminate this, or we are back with the much more difficult problem of how space arises. However, there are other aspects of background structure which are put into the model. For example, there are two kinds of sites on the lattice, occupied and unoccupied. In the model these are treated asymmetrically in the definition of the variety. Further, if we look at the solutions we see that there is no symmetry between the occupied and unoccupied sites.

Now, we might imagine a model in which this distinction between occupied and unoccupied sites is not put in at the beginning, so that the dynamics is completely symmetric under the exchange of the occupied for the unoccupied sites. It is then interesting to ask whether the solutions will break the symmetry. If this happens, then we will have a further example of the way in which a variational principle of the sort we have been discussing can introduce structure from an a-priori symmetric situation.

Our second one dimensional model is thus constructed so as to be symmetric in occupied and unoccupied sites. In this model we consider, again, a closed one-dimensional lattice with $P$ sites. On each slot we put either a
black ball or a white ball, each such configuration constitutes a state $S$. If $W$ is the number of white balls, and $B$ is the number of black balls, we have, because each site is filled, $W + B = P$. For each pair of distinct sites $i, j$ we form the relative indifference $K_{ij}$ as follows: We attempt to distinguish positions $i$ and $j$ in a way that is symmetric under both the interchange of the labels, white and black, and the exchange of left and right. For example, we might say that both neighbours of $i$ have the opposite colour to $i$, whereas one of $j$’s immediate neighbors has the same colour as $j$. Under such circumstances, $i$ and $j$ can be said to be 1-step distinguishable and we define $K_{ij}$ to be 1. However, it might be necessary to extend the comparison of the neighbourhoods of $i$ and $j$ to $m$ steps before the two positions can be distinguished; then $K_{ij} = m$. In making the comparison of the $m$’th neighbourhoods, we again use a definition in which two neighborhoods are identical if they are the same up to the exchange of black and white and left and right (See Figure 13).

Non-Leibnizian configurations can be treated as in the previous models. The complete set of relative indifferences $K_{ij}$ again forms a symmetric matrix (whose diagonal elements we set equal to zero by definition). The variety is then defined from this matrix. For convenience, in this model we work with the inverse of the variety, which we call the total indifference. This may be defined in two different ways: either by finding the maximum value of $K_{ij}$ in each row $i = 1, 2, \ldots, N$ and adding these $N$ largest indifferences to give a total indifference (this is analogous to the definition we used in the previous model) or by simply adding all the $N(N - 1)/2$ relative indifferences.

We have found in our calculations that the actual definition we used for the variety does not change the qualitative results, although the detailed form of the extremal solutions are, of course different. In our study of this second model we found it particularly useful to use this second definition, so that,

$$V'' = \frac{1}{\sum_{i<j} K_{ij}}$$

(7)

In our study of this second model, we were particularly interested in properties of the distribution of varieties over the states, such as the occurrence of both absolute and relative extrema. This particular definition is particularly well suited to this study, because the variety is the inverse of an integer. For example, with such a definition it follows trivially that there must be a
configuration of largest variety. Since the inverse of the variety (the absolute indifference) is a positive integer, there will, for each \( P \), be a least integer which is the inverse of the variety. Using, again, a Macintosh, we found it was not difficult to construct all possible configurations having a given number of slots up to \( P = 25 \) and study both the distribution of the variety over the ensemble and properties of both the absolute and relative extrema.

We now describe the results we obtained for all \( P \leq 25 \). First, for \( P < 7 \) there are no configurations in which every slot is distinguished from every one just by a description of its neighborhoods, mod the symmetries of the model. These are then all non-Leibnizian. For \( P = 7 \), there is only one Leibnizian configuration, which takes the form

\[
xx - x - - -
\]

(8)

Here, we have chosen arbitrarily that the \( x \)'s represent one color, say black, while the \( - \)'s represent the other. In this and the configurations below, we have also chosen a representation in which the largest string of \( - \)'s is to the right; because of the periodic boundary conditions and the left-right symmetry this is purely arbitrary, but it helps to compare results for different \( P \). Thus, the following configurations are identical descriptions of the state depicted in (8)

\[
- - x - xxx \quad x - xx - - -
\]

(9)

As \( P \) is increased, the number of distinct Leibnizian configurations increases quite rapidly, it is of the order of 1.2 million for \( P = 27 \). For each \( P \), up till 25, we found all configurations which maximized \( V \). These are shown in Figure (14).

The first striking thing about these results is that for most \( P \) there is not a single global extremum of the variety. Instead there tend to be a number of distinct configurations which realize a single extremal value. (Because \( 1/V \) is an integer, this is possible.) The number of such configurations is not large; and does not seem to be increasing, for example, for \( P = 14 \) are nine, for \( p = 15 \) there are three, for \( P = 22 \) there are four. In addition, the extreme value of \( 1/V \) is often realized by configurations with different numbers of black and white balls. However, the occurrence of these multiple extrema is, perhaps, not puzzling in light of the rather large numbers around. For all \( P \) except the first few, the number of Leibnizian configurations is much greater.
than the maximal value of $1/V$, thus for each integer that $1/V$ takes there will be many configurations.

As in the case of the first model, the maximal variety configurations up to $P = 25$ all possess certain characteristic features. First, we see that the symmetry between black and white is spontaneously broken by the solutions, there is, in fact, no solution which maintains the symmetry of the dynamics. Indeed, the symmetry is broken in both the number and position; in all the extremal configurations with $P > 14$, significantly less than half the slots are of one color. Second, about one third of each extremal configuration is occupied by a uniform run of sites of all the same type. As represented above, these are the strings of zeros on the right of all configurations. We shall call this uniform run, perhaps wistfully, the space. In all cases, the space is bounded at one end by a single site of the opposite type followed by another site of the same type as the space. At the other end of the space, there are always two or three sites of the opposite type. After that the two types of site seem to alternate in a manner that is very hard to predict; without doing the exact calculation it seems to be impossible to say what will be found in the part of the configuration that is not the space.

It is interesting to examine the matrix of relative indifferences, for one of these configurations. This is shown, for one of the $P = 25$ configurations, in Figure 15.

It is perhaps worth commenting here on a characteristic feature of the maximal variety models that makes them very different from conventional dynamical systems. These are usually based on variational principles in which the action is a local additive quantity; as a consequence, any subsystem of a large system that is in an extremal state is itself also in an extremal state. This is not true in maximal variety schemes because of the extreme nonlocality of the variational principle. Thus, parts of the configurations shown in Figure (14) are very far from being in a state of maximal variety, even though the complete configurations are extremal. Indeed, what we have called the space is entirely devoid of variety. Intuitively, it is clear that interesting structure must have a background of less interesting structure to set it off. In maximal variety configurations both the background and the structure have been generated dynamically.

To conclude we see that the results of the study of this second model reinforce the results from the first model. We may then summarize the features of the extremal configurations of the one dimensional models as follows: 1)
On an intermediate scale there develop stable structures, while at the same time the small scale structure is apparently random. These structures consist of groups of one to four particles organized in a distinct way. 2) These structures are apparently stable under the small scale perturbations that arise when an additional particle is added. 3) The extremal configurations are intricately organized, but have no apparent symmetries. 4) We gain some information about the form of $V$ in neighborhoods of extremal configurations. In particular, we see evidence that lone particles feel long ranged attractive forces to groups and to other lone particles, as long as there is more than one lone particle in the configuration. We also see evidence of short ranged repulsive forces, which are necessary to keep the groups distinct. 5) Finally, the fact that rather similar structures arise in the two models, which use different definitions of both the kinematical state and the definition of the variety, suggests that these characteristics are rather general, and depend only weakly on the specifics of the definition of the variety that is being used. This reinforces the idea that these definitions do capture the intuitive idea of variety of views that motivated the mathematical definitions.

Before going on to the next model, we should remark that the two one dimensional models we have described here can be generalized to any higher dimension. This has not, so far, been done.

### 5.3 A two dimensional model

Another kind of model for maximal variety, also involving a distribution of points in a fixed space can be constructed as follows. Consider a scattering of $N$ points, representing our monads, in the two dimensional plane; their positions are given as usual by the points $\vec{x}_i$, $i = 1,...,N$. We can define a variational principle literally in terms of their views, i.e., in terms of what they "see" when they look around them. What the $i$'th monad sees is a distribution of points on the circle, which can be written $\rho(\theta)_i$. We then can define the indifference matrix, $K_{ij}$, between two monads to be a measure of the difference between their two distributions. As we do not want any preferred directions in the model, we can build into the definition a gauge symmetry in which each distribution can be rotated by an arbitrary angle, i.e.

$$\rho(\theta)_i \to \rho(\theta + \alpha_i)_i. \quad (10)$$
Since each $\alpha_i$ is independent, the gauge symmetry is $U(1)^N$. The variety may then be defined according to one of the two schemes discussed in section 4.

A model of this kind has been studied numerically by Nick Benton. In this model, which will be described in more detail in a separate publication\cite{31}, the circle is divided into octants and the distribution $\rho(\theta)_i$ is described by a histogram of the number of points that appear in each octant from the $i$'th point. The difference matrix is defined to be the minimum number of points by which the histograms differ, under the group of rotations by multiples of $\pi/4$. The variety is then defined by equation (6).

Some first results of this study are shown in figures (16)-(19). In Figure (16) we see a distribution of the varieties in an ensemble of random distributions of 50 points in the plane. In Figures (17)-(19) we see three configurations of very high variety. They are not actually extremals, but from Figure (16) we can see that they have large variety compared to the random ensemble (the small arrow to the right indicates where the configurations in Figures (17)-(19) are in the distribution.) We see again the characteristic features of extremal configurations that we found in one dimensional models. The distributions are highly asymmetric and highly structured, with different kinds of structure appearing on different scales. At the smallest scales, many but not all of the points are clumped in groups of two to four. These clumps are then distributed with respect to each other in a way that is apparently ordered on large scales; by the eye it appears that this one dimensional structure involves the clumps wanting to line up to form one dimensional structures. To what extent this one dimensional ordering of the clumps is real, or is an artifact of the eye’s tendency to put order in random distributions is presently under investigation.

\section{Remarks}

Before closing we would like to make three remarks concerning possible applications of the maximal variety models.

\subsection{Order and structure.}

It is clear intuitively that there is a large and qualitative difference between what we might call an ordered configuration and a structured configuration.
By an ordered configuration we mean a configuration of high symmetry, like a crystal. By a structured configuration we mean something like a living cell, a DNA molecule, or a large modern city in which all of the parts and their configurations are highly interrelated, but where there is no overall symmetry. It is clear that both types of configuration are far from random, highly ordered and highly structured configurations each have very small measure in random ensembles. Thus, each are characterized by low measures of entropy. However, it is also clear that ordered and structured configurations are very different from each other. A simple measure of the lack of randomness in a configuration, such as entropy, does not capture this difference. Both a crystal and a neural network have very low entropy.

This situation seems responsible for a certain amount of confusion, because while we have a strong intuitive feeling as to the difference between order and structure, between a piece of ice and a brain, it has not been clear how to make this intuition quantitative. It appears to us that variety, as we have defined it in this paper, does distinguish cleanly between order and structure, and could thus be useful for the study of the systems in which structure is generated.

Variety does this because configurations of high symmetry have very low variety. On the other hand, configurations with high variety appear to be highly organized in the sense that cities and biological systems are—the parts are distinct, but in ways that are highly related to each other. Further, a certain amount of small scale randomness coexists with an organization that can only be perceived by studying the system globally. On the other hand, random configurations come in the middle. If we consider Figure (16), which shows the distributions of varieties in a random ensemble we see a Gaussian distribution which is peaked at an intermediate value of the variety. Highly ordered systems are far to the left of the random distribution; they have very small varieties. Highly structured systems are far to the right; they have large varieties.

To summarize this point, varieties, as we have defined them here, are thus a collection of quantities that distinguish between the three kinds of configurations, ordered, random and structured. As such they could take us a step nearer to the goal, which has hitherto proved so elusive, of finding a rigorous formulation of the intuition that brains are different from crystals, as symphonies are different from a busy signal.

In this connection, we should like to mention Chaitin’s interesting at-
tempt [?] to capture mathematically the essence of structured complexity by means of an algorithmic version of Shannon’s concept of mutual information. Without going into details, we merely mention that the most important part of Chaitin’s approach is the identification within a complete structure of substructures that are isomorphic to each other. This is also central to our approach, since the isomorphic (and non-isomorphic) neighborhoods that we use to define variety are substructures in Chaitin’s sense. Thus, at the level of kinematic characterization, mutual information and variety represent two different ways of measuring structured complexity of a complete system in terms of isomorphisms between its parts. Chaitin’s approach is more general, but ours may be attractive in the prospect it offers for establishing organic connections between algorithmic complexity and conventional dynamics. The fact is that identity and identification, prime concerns of epistemology and information theory, have dynamical potential.

6.2 Two ideas about time

The basic theme of this paper is that maximal variety models might provide the mathematical foundations for new kinds of laws of physics that are intrinsically cosmological. Apart from the motivations described in the first two sections, we would like to keep away from any detailed speculations about how such laws are to be constructed. It is far too soon for such speculation, a great deal more needs to be known about these kinds of dynamical systems before such ideas can be seriously entertained. However, one aspect of the problem should be addressed: how is time to fit into such a dynamical scheme? We would like to suggest here two possible ways that might be further explored.

Variety as potential energy

As we have already noted, it is rather natural to think of variety in the systems we have described as a kind of non-local potential energy function. We have seen that while in random configurations the variety is a completely
non-local function, in near to extremal variety configurations we see evidence of both long ranged attractive forces, which hold the groups together, and short ranged forces, which insure that they are distinct from each other. We have noted the interesting similarity between this latter behavior and the consequences of the kinematic Pauli principle for fermions. It would then be very interesting to study a dynamical system, say for a system of particles in $R^n$, generated by a conventional form of a variational principle,

$$S = \int dt(T + V)$$  \hspace{1cm} (11)

where $V$ is the variety of the system and $T$ is a conventional kinetic energy.

There is a further direction for the development of such models, which is natural in view of their motivations in terms of the elimination of background structures. This is to combine the notion of extremal variety with the Machian dynamical systems developed by one of us in collaboration with Bertotti\textsuperscript{[12, 13, 5]}. In this case the variational principle would be of the form,

$$S = \int dt \sqrt{T'} \sqrt{V}$$  \hspace{1cm} (12)

where $V$ is again a measure of the variety and $T'$ is now a kinetic energy measured in terms of the intrinsic derivative\textsuperscript{[5]}. The product form now guarantees that the dynamics is invariant under time reparamaterization; the intrinsic derivative is defined independently of any background inertial frame. This form also lends itself very naturally to a situation where the individual particles are intrinsically identical, so that the label of the particles can be permuted by a time dependent gauge transformation.

We are presently studying such systems. We conjecture that there are systems of this form which can serve as non-local hidden variable theories for quantum mechanics, along the lines of the theories proposed by one of us in\textsuperscript{[11, 9]}. Further developments in this direction will be reported if they turn out to be fruitful.

**Time as creation**

A more radical notion of how time could be described in terms of extremal variety dynamics is the following. Let us suppose that at each time the universe is in an extremal variety configuration for $N$ monads, and that time
corresponds to the successive addition of monads to the system. In Figure (12) we see the result of stacking successive local extrema, each generated from the previous one by the addition of one particle. We see that independent groups appear which evolve in a way which seems slow and, to a good approximation, deterministic, on an intermediate scale, while the evolution at the smallest structure is stochastic. Indeed, this is exactly the kind of behaviour that we observe in the world around us, in which the gross structure seems to evolve in accordance with deterministic laws of classical physics, while the microscopic structure seems to obey probabilistic quantum laws.

7 Conclusions

In this paper we have explored the consequences of an argument, the main points of which are 1) A theory of the whole universe must differ substantially in its basic kinematical and dynamical structure from the theories that have so far been constructed in physics. 2) Such a theory must be relational; all kinematical quantities must refer to relations between the fundamental entities; no particle, or field, should have any a priori properties that are determined independently of whether or not anything else exists in the universe. 3) Dynamical systems in which the variables are relational are not very familiar to us; they are intrinsically non-local and may be non-additive, and thus cannot be expected to behave like the usual local systems we are familiar with.

Motivated by these considerations, we have invented and explored a new kind of relational dynamical system, which is based on the notion of extremal variety. These systems are based on an action principle, but the quantity which is extremized is proportional to a measure of the differences between the way each pair of entities is related to the rest of the system. This quantity is, as we have tried to argue, a measure of how structured the system is.

In the four model systems we have studied so far we find several common features that, at a rather general level, encourage our belief that dynamical systems based on extremal variety deserve more study as possible new approaches to combining quantum theory with a theory of space, time and cosmology. These features are: 1) The dynamics is deterministic globally and non-locally, but stochastic on the smallest scales. 2) Symmetry under exchanges of the labels of the particles is built in from the beginning, so
that the particles which are described are intrinsically identical. They are
distinguished only by the values of the relational dynamical variables. Both
of these are features we would like a hidden variables theory for quantum
mechanics to have. In addition, there is a hint that fermionic behavior can
be recovered directly from the basic variational principle and may not need
to be imposed a priori as a kinematic condition. 3) In extremal and near to
extremal configurations the particles tend to clump in groups that are sta-
ble under perturbations and (under at least one notion of time—the second
described above) evolve slowly and smoothly on the fundamental time scale.
While too much cannot be claimed from the results we have so far, let us say
only that this behavior is reminiscent of how we would like a classical world
to emerge from a fundamental theory. 4) By looking at near to extremal
configurations, as well as how the systems approach local extrema, we can
learn some general properties of the variety, as a function on the configura-
tion space. If we consider \( V \) as analogous to a potential energy, then we see
that in near to extremal configurations it is characterized by the presence of
long ranged attractive forces and short ranged repulsive forces. Again, it is
premature to claim too much from this, but it is suggestive.

We would like to close by stressing that all of the results we have reached
are preliminary; they are the result of simple computer simulations done
on simple model systems. Much more sophisticated studies need to be done,
both numerically and analytically, before we can have a clear idea of whether
or not extremal variety systems are the right answer to the problems posed
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