A Characterization of the Prime Graphs of Solvable Groups

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Abstract

Let $\pi(G)$ denote the set of prime divisors of the order of a finite group $G$. The prime graph of $G$, denoted $\Gamma_G$, is the graph with vertex set $\pi(G)$ with edges $\{p,q\} \in E(\Gamma_G)$ if and only if there exists an element of order $pq$ in $G$. In this paper, we prove that a graph is isomorphic to the prime graph of a solvable group if and only if its complement is 3-colorable and triangle-free. We then introduce the idea of a minimal prime graph. We prove that there exists an infinite class of solvable groups whose prime graphs are minimal. We prove the $3k$-conjecture on prime divisors in element orders for solvable groups with minimal prime graphs, and we show that solvable groups whose prime graphs are minimal have Fitting length 3 or 4.

Keywords: prime graphs of finite groups, sets of element orders, solvable groups, Frobenius groups, 3-colorable graphs, triangle-free graphs, girth of a prime graph, $3k$ conjecture

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1. Introduction.

Prime graphs originated in the 1970s as a by-product of certain cohomological questions posed by K.W. Gruenberg. Shortly after their introduction, prime graphs became objects of interest in their own right, and since then numerous contributions have been made to the topic. The prime graphs of finite simple groups are well understood (see [9], [10], [17], [18], and [16]), as is the structure of groups with acyclic prime graphs (see [13]). Graph invariants such as diameter, spectrum, and degree sequence have also been extensively documented in [12], [15], and [14], respectively. There are still many unexplored questions about the way that graph theoretic properties influence group structure, and it is from this angle that our investigation proceeds.

There are three properties of solvable groups that are especially attractive to the study of prime graphs. We will show that these properties have graph theoretic equivalents, and that this enables us to extract structural information about a solvable group directly from its prime graph.

The first is a well known result by Hall in [4] that a finite group $G$ is solvable if and only if $G$ contains a Hall $\pi$-subgroup for every subset $\pi \subset \pi(G)$. Graph theoretically, this can be restated as the observation that $G$ is solvable if and only if every induced subgraph $\Gamma_G[\pi]$ is the prime graph of a Hall $\pi$-subgroup of $G$.

Of further use is the following proposition in [12], which we shall refer hereto as Lucido’s Three Primes Lemma.

**Lemma 1** (Lucido’s Three Primes Lemma). Let $G$ be a finite solvable group. If $p, q, r$ are distinct primes dividing $|G|$, then $G$ contains an element of order the product of two of these three primes.

Equivalently, if $G$ is solvable, $\Gamma_G$ cannot contain an independent set of size three. In other words, $\Gamma_G$ must be triangle-free.

The third property, established in [17], is that every solvable group with a disconnected prime graph must be either a Frobenius or 2-Frobenius group. In
particular, whenever an edge $pq$ is missing from the prime graph of a solvable group, the corresponding Hall $\{p, q\}$-subgroup $H_{pq}$ admits a fixed point free action between either the Sylow subgroups of $H_{pq}$ or their image in its Fitting quotient. Our characterization begins by defining an acyclic orientation of $\Gamma_G$ indicating the direction of this action for every edge $pq \in \Gamma_G$. We refer to this orientation as the Frobenius digraph of $G$, denoted $\Gamma_G$.

Studying small subgraphs of $\Gamma_G$ allows us to more accurately describe interactions between Hall subgroups. Doing so leads us to the primary result of this paper, which determines precisely which graphs may be realized as the prime graphs of solvable groups.

**Theorem 2.** An unlabeled graph $F$ is isomorphic to the prime graph of some solvable group if and only if its complement $\overline{F}$ is 3-colorable and triangle-free.

An immediate corollary of this characterization is that the prime graph of nearly every solvable group contains a clique of size three, aside from a few special cases, which we classify.

**Corollary 3.** The prime graph of any solvable group has girth 3 aside from the following exceptions: the 4-cycle, the 5-cycle, and the 7 unique forests that do not contain an independent set of size 3.

We devote the remainder of the paper to an extended application of the tools developed in the first section and the unique methods they provide. We introduce the class of prime graphs that are minimal with respect to the property that they are isomorphic to the prime graph of some solvable group. Next, we prove that any solvable group $G$ that has a minimal prime graph contains an element whose order is divisible by at least one third of the primes dividing the order of $G$, a special case of an earlier conjecture in [8]. We then discuss the structure of solvable groups with prime graphs isomorphic to the 5-cycle— which is minimal— and discover that these groups have Fitting length 3. The paper concludes with a proof that any solvable group with a minimal prime graph has Fitting length 3 or 4, and that this result is best possible.

**Theorem 4.** A solvable group with a minimal prime graph has Fitting length 3 or 4.

It is worth noting that this final theorem is reminiscent of Lucido’s work on solvable groups with prime graphs of diameter 3 in [12], which also have Fitting length 3 or 4.

Before we begin, let us briefly introduce the notational conventions to be used throughout this paper. Unless stated otherwise, all graphs will be assumed simple and unlabeled. In any graph isomorphism, both graphs will be considered unlabeled. When we say that an unlabeled graph $F$ is the prime graph of a
solvable group $G$, we mean that there is an isomorphism between $F$ and $\Gamma_G$. In an undirected graph $\Gamma$, we denote edges $\{p, q\} \in E(\Gamma)$ by $pq \in \Gamma$. For edges in a directed graph $\Gamma$, we write edges from $p$ to $q$ as either $pq \in \Gamma$ or $p \rightarrow q$ when unclear. Subgraphs of a graph $F$ induced by a subset $\pi \subseteq V(F)$ will be denoted $F[\pi]$. When we refer to cycles or paths in a directed graph, it is implicitly assumed that these cycles and paths are directed. We refer to paths on $n + 1$ vertices with $n$ edges as $n$-paths. Sometimes a 2-path on vertices $p, q, r$ will be written in line as $p \rightarrow q \rightarrow r$. Similarly, by $p \leftarrow q \rightarrow r$, we refer to the tree on vertices $p, q$ and $r$ with edges $q \rightarrow p$ and $q \rightarrow r$.

In an undirected graph $\Gamma$, the $k$-neighborhood of a vertex $v \in \Gamma$, which we will denote $N^k(v)$, is defined as the set of vertices $u \in \Gamma$ such that a path in $\Gamma$ connecting $u$ and $v$ exists, and such that the shortest such path has length $k$. In a directed graph $\Gamma$, a distinction is made between $k$-in- and $k$-out-neighborhoods of $v$. The $k$-in-neighborhood of $v$ is denoted $N^k_{\downarrow}(v)$ and consists of vertices $u \in \Gamma$ for which there exists a directed path in $\Gamma$ beginning with $u$ and ending with $v$ and such that the shortest such path has length $k$. The $k$-out-neighborhood of $v$, denoted $N^k_{\uparrow}(v)$, is defined similarly, with paths beginning with $v$ and ending in $u$.

We will sometimes write subgroups in the Fitting series of $G$ by $F_k(G)$ (or $F_k$ when there is no ambiguity), so that $F_1(G) = \text{Fit}(G)$, $F_2/F_1 = \text{Fit}(G/F_1)$, and so on. The Fitting length of $G$ will be denoted $\ell_F(G)$. For a prime $p$ dividing $|G|$, unless stated otherwise, we denote by $P$ an arbitrary Sylow $p$-subgroup of $G$, and all statements about $P$ apply to every Sylow $p$-subgroup of $G$. Finally, unless stated otherwise, for a set $\pi$ of primes dividing $|G|$, we denote by $H_\pi$ a Hall $\pi$-subgroup of $G$, unless $\pi$ consists of only two (or three, resp.) primes $p$ and $q$ (and $r$), in which case we write $H_{pq}$ (or $H_{pqr}$).

2. Characterization.

In this section, we characterize the prime graphs of solvable groups.

We would like to determine as much as possible about the way that Sylow subgroups of a solvable group $G$ interact given information from $\Gamma_G$. We know from Gruenberg-Kegel’s theorem [17, Thm. A] that any solvable group with a disconnected prime graph is Frobenius or 2-Frobenius. Therefore, we can pick out disconnected subgraphs of $\Gamma_G$ to find Hall subgroups that contain fixed point free action. We begin by defining 2-Frobenius groups, introducing some new terminology, and providing additional details regarding their structure.

Definition. A group $G$ is a 2-Frobenius group if $F_2$ and $G/F_1$ are Frobenius groups, where $F_1 = \text{Fit}(G)$ and $F_2/F_1 = \text{Fit}(G/F_1)$. We will often refer to the Frobenius kernel of $G/F_1$ as the upper kernel of $G$ and the Frobenius kernel of $F_2$ as the lower kernel of $G$. 

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Remark. The original definition of a 2-Frobenius group differs from the one above. The equivalence of the two relies on the nilpotence of Frobenius kernels, established for the special case of solvable groups by Higman[5, Thm. 4] in 1957, and in the general situation by J.G. Thompson in 1959.

We immediately see that in a 2-Frobenius group $G$, the primes dividing $[F_2 : F_1]$ are disjoint from those dividing $|F_1|$ or $|G : F_2|$. In fact, we will see they form a clique in $\Gamma_G$.

**Lemma 2.1.** Let $G$ be a 2-Frobenius group where $F_1 = \text{Fit}(G)$ and $F_2/F_1 = \text{Fit}(G/F_1)$. Then $G/F_2$ and $F_2/F_1$ are cyclic groups, $F_1$ is not a cyclic group, and the upper kernel of $G$ is a cyclic group of odd order.

**Proof.** A similar lemma appears in [19, Lemma. 2].

We notice that the simplest disconnected subgraphs of $\Gamma_G$ occur whenever an edge $pq \notin \Gamma_G$ as the induced subgraphs $\Gamma_G[p, q]$. This tells us that a Hall $\{p, q\}$-subgroup $H_{pq}$ is either Frobenius or 2-Frobenius. Thus it is convenient to distinguish the following type of 2-Frobenius groups.

**Definition.** If $G$ is a 2-Frobenius group for which there are primes $p$ and $q$ so that $G/F_2$ and $F_1$ are $p$-groups and $F_2/F_1$ is a $q$-group, we say that $G$ is a 2-Frobenius group of type $(p, q, p)$.

**Lemma 2.2.** If $H$ is a 2-Frobenius group of type $(p, q, p)$ for primes $p$ and $q$, then $F_2(H)$ has a complement in $H$, and the semidirect product of $\text{Fit}(H)$ with this complement is a Sylow $p$-subgroup of $H$.

**Proof.** This is a consequence of [19, Lemma. 2].

**Corollary 2.3.** Suppose $H$ is a 2-Frobenius group of type $(p, q, p)$ for primes $p$ and $q$. If $P$ is a Sylow $p$-subgroup of $H$, then $P$ is not a Frobenius complement. If $Q$ is a Sylow $q$-subgroup of $H$, then $Q$ is cyclic.

**Proof.** Observe that $Q$ is isomorphic to the upper Frobenius kernel of $H$, so $Q$ is cyclic by Lemma 2.1. By Lemma 2.2, $P$ is a nontrivial semidirect product. This implies that $P$ has more than one subgroup of order $p$. It is well known that Frobenius complements have unique subgroups of prime order. Therefore, $P$ cannot be a Frobenius complement.

The preceding lemmas provide a description of Hall $\{p, q\}$-subgroups for all nonedges $pq \notin \Gamma_G$, but what do these facts mean together? We are motivated to study the formation of edges in $\Gamma_G$ and watch for emergent properties in the group structure, suspecting the whole to be greater than the sum of its parts. To this end, we assign directions to the edges in $\Gamma_G$. 5
**Definition.** Define an orientation of $\Gamma_G$ for a finite solvable group $G$ as follows. For each edge $pq \in \Gamma_G$, a Hall $\{p, q\}$-subgroup $H_{pq}$ is either a Frobenius or 2-Frobenius group by [17, Thm. A]. If $H_{pq}$ is a Frobenius group with complement a Sylow $p$-subgroup and kernel a Sylow $q$-subgroup, we direct the edge $pq$ in $\Gamma_G$ so that $p \to q$. If $H_{pq}$ is a 2-Frobenius group of type $(p, q, p)$, we direct the edge $pq$ in $\Gamma_G$ by $p \to q$. We call this orientation the Frobenius digraph of $G$, denoted $\Gamma_G$.

**Remark.** We choose to direct the edges associated with 2-Frobenius groups in $\Gamma_G$ based on the “higher” Frobenius action so that the orientation is preserved when taking factor groups. This way, we are guaranteed that if $p \to q$ in $\Gamma_G$, the Frobenius kernel of either $H_{pq}$ or $H_{pq}/\text{Fit}(H_{pq})$ will be a Sylow $q$-subgroup. It is also possible to define $\Gamma_G$ so that edges corresponding to 2-Frobenius groups are oriented based on “lower” Frobenius action, that is, to direct $q \to p$ in $\Gamma_G$ if $H_{pq}$ is a 2-Frobenius group of type $(p, q, p)$.

When $r \to q$ and $q \to p$ in $\Gamma_G$, we notice that $\Gamma_G[\{p, q, r\}]$ is disconnected, so a Hall $\{p, q, r\}$-subgroup must be Frobenius or 2-Frobenius. We next define 2-Frobenius groups of type $(p, q, r)$, which we then show are closely related to such 2-paths $r \to q \to p$ in $\Gamma_G$.

**Definition.** Suppose that there exist distinct primes $p$, $q$, and $r$ so that $G = PQR$, where $P$, $Q$, and $R$ are Sylow $p$, $q$, and $r$-subgroups respectively, $PQ$ is a Frobenius group with kernel $P$, and $QR$ is either a 2-Frobenius group of type $(r, q, r)$ or a Frobenius group with Frobenius kernel $Q$. Then we say that $G$ is a 2-Frobenius group of type $(p, q, r)$.

Observe that if $G$ is a 2-Frobenius group of type $(p, q, r)$, then $\Gamma_G$ has the form $r \to q \to p$. We next show that the converse is true, that is, that subgraphs of $\Gamma_G$ of the form $r \to q \to p$ correspond to 2-Frobenius Hall subgroups of type $(p, q, r)$.

**Lemma 2.4.** If $r \to q \to p$ is a 2-path in $\Gamma_G$ for a solvable group $G$, then a Hall $\{p, q, r\}$-subgroup $H_{pqr}$ is 2-Frobenius of type $(p, q, r)$.

**Proof.** Let $H = H_{pqr}$. The prime graph of $H$ has two connected components $\{p, r\}$ and $\{q\}$, so $H$ is Frobenius or 2-Frobenius by [17, Thm. A]. Suppose first that $H$ is a Frobenius group with kernel $K$ and complement $C$. One of the connected components is the set of primes dividing $|C|$ and the other is the set of primes dividing $|K|$. Thus either $K$ or $C$ is a Sylow $q$-subgroup of $H$. Suppose $C$ is a Sylow $q$-subgroup of $H$. Since $K$ is nilpotent, the Sylow $r$-subgroup $R$ of $K$ is normal in $K$. However, then $C$ normalizes $R$ and $RC$ is a Frobenius group with kernel $R$, contradicting that $r \to q$ in $\Gamma_G$. On the other hand, if $K$ is the Sylow $q$-subgroup of $H$, we see that $HP$ is a Frobenius group with kernel $H$, contradicting $q \to p$ in $\Gamma_G$. Thus we conclude that $H$ is not Frobenius.
We now know that \( H \) is 2-Frobenius, so it remains to be shown that \( H \) is of type \( (p, q, r) \). Observe that either \( F_2 / F_1 \) is the Sylow \( q \)-subgroup of \( H / F_1 \) or \( F_1 \) and \( H / F_2 \) are both \( q \)-groups. Suppose \( F_1 \) and \( H / F_2 \) are \( q \)-groups. Let \( C \) be a complement to \( F_2 \) in \( H \), and let \( R \) be a Sylow \( r \)-subgroup of \( H \). Since \( G / F_1 \) is a Frobenius group, \( RF_1 \) is normal in \( G \). We see that \( CRF_1 \) is a Hall \( \{q, r\} \)-subgroup of \( H \) and is a 2-Frobenius group of type \( (q, r, q) \), and we have \( q \rightarrow r \) in \( \Gamma_G \), a contradiction. Thus, \( F_2 / F_1 \) is the Sylow \( q \)-subgroup of \( G \), which is cyclic by Lemma 2.1. We note that a Hall \( \{q, r\} \)-subgroup of \( H \) is either Frobenius or 2-Frobenius of type \( (r, q, r) \). Since \( q \rightarrow p \) in \( \Gamma_G \), we know that a Hall \( \{p, q\} \)-subgroup of \( H \) is either Frobenius or 2-Frobenius of type \( (p, q, p) \). In latter case, we know that the Sylow \( q \)-subgroup is not cyclic by Corollary 2.3 and this contradicts the fact we have seen that a Sylow \( q \)-subgroup is cyclic. Therefore, a Hall \( \{p, q\} \)-subgroup is a Frobenius group, and we conclude that \( H \) has type \( (p, q, r) \).

If we can describe the Sylow subgroups of 2-Frobenius groups of type \( (p, q, r) \), we can read off the structure of Sylow \( p \), \( q \), and \( r \)-subgroups of \( G \) whenever \( r \rightarrow q \rightarrow p \) in \( \Gamma_G \).

**Lemma 2.5.** Suppose that \( p, q \), and \( r \) are distinct primes and \( G = PQR \) is a 2-Frobenius group of type \( (p, q, r) \), where \( P \), \( Q \), and \( R \) are Sylow \( p \)-, \( q \)-, and \( r \)-subgroups, respectively. Then \( P \) is not cyclic, \( Q \) is cyclic, and \( R \) is not generalized quaternion.

**Proof.** We observe that \( Q \) is isomorphic to the upper kernel of \( G \). Thus \( Q \) is cyclic by Lemma 2.1. If \( r \) does not divide \( |F_1(G)| \), then \( R \) is isomorphic to a Frobenius complement of \( G / F_1 \), and therefore cyclic by Lemma 2.1. If \( r \) does divide \( |F_1(G)| \), then \( H_{qr} \) is 2-Frobenius, so \( R \) is not a Frobenius complement by Lemma 2.3. In both cases, we see that \( R \) is not generalized quaternion. By Lemma 2.2, we know that \( F_2(G) \) has a complement \( C \). Observe that \( C \) will normalize \( PQ \). It is not difficult to see that \( PQC \) is a 2-Frobenius group, and so, by Lemma 2.1, we see that \( P \) is not cyclic.

In fact, we can extend this result to gain even more information from 2-paths in \( \Gamma_G \).

**Corollary 2.6.** Let \( G \) be a solvable group. If \( p_1 \rightarrow p_2 \rightarrow p_3 \) is a 2-path in \( \Gamma_G \), then for every prime \( q \in N_1^1(p_3) \), a Hall \( \{q, r\} \)-subgroup \( H_{qr} \) is a Frobenius group for every prime \( r \in N_1^1(q) \).

**Proof.** Consider a Hall \( \{p_1, p_2, p_3\} \)-subgroup \( H_{p_1p_2p_3} \). We have by Lemma 2.4 that \( H_{p_1p_2p_3} \) is a 2-Frobenius group of type \( (p_3, p_2, p_1) \). We know that the Hall \( \{p_2, p_3\} \)-subgroup \( H_{p_2p_3} \) is a Frobenius group, and by Lemma 2.5, \( P_3 \) is not cyclic.
Figure 1: Setup of Corollary 2.6.

With this in mind, let \( q \in N_1^1(p_3) \) be arbitrary and consider a prime \( r \in N_1^1(q) \). Let \( H_{qr} \) be a Hall \( \{q,r\} \)-subgroup of \( G \). Note that the prime graph of \( H_{qr} \) is disconnected, so \( H_{qr} \) is either Frobenius or 2-Frobenius of type \((r,q,r)\). We show that it is Frobenius. We first show the result if \( r = p_3 \). If \( H_{qp_3} \) is a 2-Frobenius group of type \((q,p_3,q)\), then \( P_3 \) must be cyclic by Corollary 2.3, contradicting Lemma 2.5. Thus \( H_{qp_3} \) is a Frobenius group with complement \( Q \). For each remaining prime \( r \in N_1^1(q) \) other than \( p_3 \), we note that if \( H_{qr} \) is a 2-Frobenius group of type \((q,r,q)\), then \( Q \) cannot be a Frobenius complement by Corollary 2.3. However, this contradicts that \( Q \) is a Frobenius complement in \( H_{qp_3} \). Hence \( H_{qr} \) is Frobenius as well.

Our investigation of paths in \( \Gamma_G \) concludes with the following corollary, which strongly elucidates which structures may occur in \( \Gamma_G \). This theorem constitutes the primary argument of one direction of the characterization.

**Corollary 2.7.** The Frobenius digraph of a solvable group cannot contain a directed 3-path.

**Proof.** Suppose that \( p_1 \to p_2 \to p_3 \to p_4 \) is a 3-path in the Frobenius digraph of \( G \). We have that \( H_{p_1p_2p_3} \) is 2-Frobenius of type \((p_1,p_2,p_3)\) by Lemma 2.4. If \( P_2 \) is a Sylow \( p_2 \)-subgroup of \( G \), then \( P_2 \) is cyclic by Lemma 2.5. On the other hand, \( H = H_{p_2p_3p_4} \) is 2-Frobenius of type \((p_2,p_3,p_4)\) by Lemma 2.4. Applying Lemma 2.5, it follows that \( P_2 \) is not cyclic. This is a contradiction.

**Remark.** It is obvious that the Frobenius digraph is acyclic when we assume that each arrow corresponds to a Frobenius group, since the order of a Frobenius complement divides the order of its kernel minus one [6, Lem. 16.6]. That this remains true when we allow the possibility of 2-Frobenius groups is not as easy, but follows immediately from Corollary 2.7.

Conversely, we show that any digraph violating neither Corollary 2.7 nor Lucido’s Three Primes Lemma is isomorphic to the Frobenius digraph of some solvable group.

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Theorem 2.8. For any 3-colorable, triangle-free (unlabeled) graph $F$, there exists a solvable group $G$ for which $F$ is isomorphic to the complement of the prime graph of $G$. Furthermore, there exists an acyclic orientation of $F$ that does not contain a directed 3-path, and given any such orientation $\overrightarrow{F}$, there exists a solvable group $G$ for which $\overrightarrow{F}$ is isomorphic to the Frobenius digraph of $G$.

Proof. We begin by showing that if $F$ is a 3-colorable, triangle free, then an acyclic orientation of $F$ exists that does not contain a 3-path. Take any 3-coloring of $F$ and arbitrarily label the vertices with numbers 1, 2, and 3 so that vertices of the same color have the same label. Direct the edges of $F$ from lower to higher numbered colors. By construction, the resulting orientation is acyclic and contains no 3-paths.

Now, let $\overrightarrow{F}$ be any acyclic orientation of $F$ that does not contain a directed 3-path. We now show that there is a solvable group $G$ whose Frobenius digraph is isomorphic to $\overrightarrow{F}$. Let $\mathcal{O}$ be the set of vertices in $\overrightarrow{F}$ with in-degree 0 and non-zero out-degree, $\mathcal{D}$ the set of vertices with both in- and out-degrees non-zero, and $\mathcal{I}$ the vertices with out-degree 0. (Here $\mathcal{O}$ reminds us of “outgoing” vertices, $\mathcal{D}$ reminds us of “double Frobenius” as by Lemma 2.4 vertices with this property imply the existence of a 2-Frobenius Hall subgroup, and $\mathcal{I}$ reminds us of “ingoing” vertices, including singleton vertices in $\mathcal{I}$.) Denote the number of vertices in each of these sets by $n_0, n_d$, and $n_i$, respectively.

Let $\mathcal{P} = \{p_j \in \mathbb{P} : j = 1, \ldots, n_o\}$ be a set of distinct primes and define $p = p_1p_2 \cdots p_{n_o}$. By Dirichlet’s theorem on arithmetic progressions, we can pick a set $\mathcal{Q} = \{q_k \in \mathbb{P} : k = 1, \ldots, n_d\}$ of distinct primes such that $q \equiv 1 \pmod{p}$ for every prime $q \in \mathcal{Q}$. Define a directed graph $\Lambda$ with vertex set $\mathcal{P} \cup \mathcal{Q}$ and edge set defined by the image of some fixed injective graph homomorphism $\Phi : \overrightarrow{F}[\mathcal{O} \cup \mathcal{D}] \to \Lambda$ mapping vertices in $\mathcal{O}$ to primes in $\mathcal{P}$ and vertices in $\mathcal{D}$ to primes in $\mathcal{Q}$. Let $T = C_{p_1} \times \cdots \times C_{p_{n_o}}$ and $U = C_{q_1} \times \cdots \times C_{q_{n_d}}$. Since $p_j \mid q_k - 1$ for each pair of primes $p_j \in \mathcal{P}, q_k \in \mathcal{Q}$, we can define a semidirect product $K = U \ltimes T$ by allowing $C_{p_j}$ to act fixed point freely on $C_{q_k}$ if $p_jq_k \in \overrightarrow{\Lambda}$ and trivially otherwise. It follows that each Hall $\{p_j, q_k\}$-subgroup of $K$ is a Frobenius group if $p_jq_k \in \overrightarrow{\Lambda}$ and a direct product otherwise. Note that $\overrightarrow{\Lambda}$ is the Frobenius digraph of $K$.

For each vertex $v \in \mathcal{I}$, let $\Phi_1(v)$ denote the set of primes in the image of $\overrightarrow{F}[\mathcal{N}_1^c(v)]$ under $\Phi$, with $\Phi_2(v)$ defined analogously. In the case that $\Phi_1(v) \neq \emptyset$, let $H_v$ be a Hall $\Phi_1(v)$-subgroup of $K$. Then $\text{Fit}(H_v)$ is a cyclic Hall $\Phi_1(v)$-subgroup of $K$. Let $\mathcal{R} = \{r_j \in \mathbb{P} : j = 1, \ldots, n_i\}$ be a set of distinct primes so that each $v_j \in \mathcal{I}$ is associated with a unique $r_j$. When $\Phi_1(v_j) = \emptyset$, define $R_{v_j} = C_{r_j}$. For the remaining vertices in $\mathcal{I}$, again by Dirichlet, we may insist that $r_j \equiv 1 \pmod{\text{Fit}(H_{v_j})}$. Then, by [7, Lem. 1.8], there exists a faithful irreducible $\mathbb{F}_{r_j}H_{v_j}$-module $R_{v_j}$ such that $\text{Fit}(H_{v_j})$ acts fixed point freely on $R_{v_j}$. Finally, define a direct product $J = R_{v_1} \times \cdots \times R_{v_{n_i}}$. Let $G = J \rtimes K$ be the
semidirect product where any subgroup \( C_s \leq K \) for \( s \in \mathcal{P} \cup \mathcal{Q} \) acts on \( R_{v_j} \) by the appropriate module action when \( s \in \Phi_1(v_j) \cup \Phi_2(v_j) \) and trivially otherwise. It follows that \( \overline{F} \) is isomorphic to the Frobenius digraph of \( G \).

Note that any group constructed in the method described above from graph with chromatic number 3 has Fitting length 3. We will return to this observation later in the paper during further examination of the connection between Fitting lengths and prime graphs.

As an immediate corollary to Theorem 2.8, we observe that the prime graphs of most solvable groups contain a 3-cycle, and in fact classify all exceptions.

Corollary 2.9. The prime graph of any solvable group has girth 3 aside from the following exceptions: the 4-cycle, the 5-cycle, and the 7 unique forests that do not contain an independent set of size 3.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{exceptions.png}
\caption{Exceptions to Corollary 2.9.}
\end{figure}

Proof. It is easily verifiable that each of the exceptional cases listed above have triangle free and 3-colorable complements, so by Theorem 2.8 each can be realized as the prime graph of some solvable group. Thus it remains to be shown that these exceptions are the only such prime graphs with girth not equal to 3.

Suppose that \( \Gamma \) is a connected non-cycle triangle-free graph on \( n \geq 5 \) vertices with independence number \( \alpha \), chromatic number \( \chi \), and maximum vertex degree \( \Delta \). By Brooks’ Theorem [11], we have \( \chi \leq \Delta \), whence

\[ \alpha \geq \frac{n}{\chi} \geq \frac{n}{\Delta} > \sqrt{n}. \]

Thus \( 2 < \sqrt{n} < \alpha \). It follows from Lucido’s Three Primes Lemma that \( \Gamma \) cannot be the prime graph of a solvable group. Likewise, any disconnected triangle-free graph on 5 or more vertices necessarily contains an independent set of size 3, as does any \( m \)-cycle for \( m \geq 6 \). One can easily verify by exhaustion that the 7 forests pictured above are exactly those forests on 4 or fewer vertices with independence number less than 3. This completes the proof.

Remark. A shrewd combinatorialist will recognize that the previous corollary admits an much simpler proof from Ramsey theory: \( r(3, 3) = 6 \) [3, Thm. 1].
We conclude the section with a characterization theorem, combining Corollary 2.7 and Theorem 2.8 into the following practical form.

**Theorem 2.10.** An unlabeled graph $F$ is isomorphic to the prime graph of some solvable group if and only if its complement $\overline{F}$ is 3-colorable and triangle-free.

**Proof.** If $F = \Gamma_G$ is isomorphic to the prime graph of some solvable group, then by Corollary 2.7, $\Gamma_G$ does not contain a 3-path. Thus by the Gallai-Roy theorem [1, Thm. 7.17], $\Gamma_G$ is 3-colorable. Also, $\overline{\Gamma_G}$ is triangle-free by Lucido’s Three Primes Lemma. The converse is given by Theorem 2.8. \qed

3. Minimal Prime Graphs.

For the remainder of the paper, we present an extended application of Theorem 2.10, which demonstrates how graph theoretic properties can influence the structure of solvable groups.

We first introduce a graph theoretic property we call *minimal*. Minimality was first observed as a property of the 5-cycle while studying the exceptions to Corollary 2.9. Solvable groups whose prime graphs are isomorphic to 5-cycles were discovered to have certain group theoretic properties; in particular, these groups have Fitting length 3. We prove this in Section 4. We anticipate that this bound generalizes as a result of minimality, and in fact we find in Section 5 that all solvable groups with minimal prime graphs have Fitting length 3 or 4. This result reminds us of Lucido’s similar conclusion in [13, Prop. 3] concerning solvable groups with prime graphs of diameter 3.

In this section, we outline some foundational results about minimality, culminating with the observation that a group with a minimal prime graph adheres to a conjectured bound on the number of prime divisors in the orders of its elements.

**Definition.** If $G$ is a finite solvable group and $\Gamma_G$ satisfies

(a) $|V(\Gamma_G)| > 1$,
(b) $\Gamma_G$ is connected,
(c) $\Gamma_G \setminus \{pq\}$ is not the prime graph of any solvable group for any $p, q \in \Gamma_G$,

then we say that $\Gamma_G$ is *minimal*.

One can easily verify that the 5-cycle is the smallest minimal prime graph. We now show that any minimal prime graph may be used to construct a new minimal prime graph of greater order.

**Proposition 3.1.** Any minimal prime graph may be used to construct a new minimal prime graph of greater order. In particular, the family of minimal prime graphs is infinite.
Proof. Suppose that $\Gamma'$ is a minimal prime graph and $v \in \Gamma'$. Let $\Gamma$ be the graph formed by adding a new vertex $u$ and a new edge $uv \in \Gamma$ such that $N^1(u) \setminus \{v\} = N^1(v) \setminus \{u\}$. (In other words, for any vertex $x \neq u$, there is an edge $xu \in \Gamma$ if and only if $xv \in \Gamma$.) We claim that $\Gamma$ is minimal. Since $N^1(u) \setminus \{v\} = N^1(v) \setminus \{u\}$, it follows that $\Gamma$ is 3-colorable and triangle-free, so $\Gamma$ is isomorphic to the prime graph of some solvable group. Thus it remains to show that $\Gamma \setminus \{uv\}$ is not the prime graph of a solvable group for any edge $uv \in \Gamma$.

Let $\Gamma^* = \Gamma \setminus \{v\}$, and note that $E(\Gamma) = E(\Gamma') \cup E(\Gamma^*) \cup \{uv\}$. Removing any edge $xy$ from $\Gamma'$ results in a graph whose complement contains a triangle or is not 3-colorable. In either case, since $\Gamma^* \setminus \{xy\}$ is an induced subgraph of $\Gamma \setminus \{xy\}$, we see that $\Gamma \setminus \{xy\}$ is not realizable as the prime graph of a solvable group. Noting that $\Gamma^*$ is isomorphic to $\Gamma'$ via the isomorphism mapping $u$ to $v$ and fixing all other vertices, we see that the same argument applies to edges $xy \in \Gamma^*$.

It remains to be shown that $\Gamma \setminus \{uv\}$ is not the prime graph of a solvable group. By minimality, a vertex $z \in \Gamma'$ such that $zv \not\in \Gamma'$, then $zu \not\in \Gamma$, so $zv, zu,$ and $uv$ form triangle in $\Gamma \setminus \{uv\}$. We conclude that $\Gamma$ is minimal.

By starting with the 5-cycle and repeatedly creating new graphs via the process described in the proof of Proposition 3.1, the reader may produce many examples of minimal prime graphs. It is important to note, however, that not all minimal prime graphs can be obtained in this way. One example is the Grötzsch graph with precisely one edge removed. (The reader should note that the Grötzsch graph is otherwise known as the Mycielski graph of order 4, or the triangle-free graph with chromatic number 4 with the smallest number of vertices, as shown in [2].)

Intuitively, groups with minimal prime graphs contain as many fixed-point-free actions as possible in a solvable group that is neither Frobenius nor 2-Frobenius, as their Frobenius digraphs are saturated with arrows. This rigid group structure causes their prime graphs to be somewhat well behaved. We see that if a group $G$ has a minimal prime graph, then for all subgroups $K \leq G$, we have $\Gamma_K = \Gamma_G$ if and only if $V(\Gamma_K) = V(\Gamma_G)$. Similarly, for all normal subgroups $N \unlhd G$, we have $\Gamma_{G/N} = \Gamma_G$ if and only if $V(\Gamma_{G/N}) = V(\Gamma_G)$.

**Lemma 3.2.** Suppose $G$ is a solvable group. If $\Gamma_G$ is minimal, then $\overline{\Gamma_G}$ is not 2-colorable.

**Proof.** Suppose that $\overline{\Gamma_G}$ is bipartite. Since minimal prime graphs are connected, there exists at least one non-edge between the color classes. Removing this edge from $\Gamma_G$ yields a graph with a bipartite, triangle-free complement, contradicting the minimality of $\Gamma_G$. □

For the remainder of the paper, we fix the following notation. Partition the vertices of $\overline{\Gamma_G}$ into three sets $O$, $D$, and $I$, where vertices in $O$ have zero
in-degree in $\Gamma_G$ (i.e., $p \in \mathcal{O}$ if and only if $N_1^+(p)$ is empty), vertices in $\mathcal{D}$ have non-zero in- and out-degrees (i.e., $p \in \mathcal{D}$ if and only if both $N_1^+(p)$ and $N_1^-(p)$ are nonempty), and vertices in $\mathcal{I}$ have zero out-degree (i.e. $p \in \mathcal{I}$ if and only if $N_1^-(p)$ is empty). The following lemma shows that these sets actually form a partition.

**Lemma 3.3.** Let $G$ be a solvable group. If $\Gamma_G$ is a minimal graph, then $\Gamma_G$ contains no singleton vertices.

**Proof.** Any singleton vertex in $\Gamma_G$ may be connected to any other vertex without creating a triangle or increasing the chromatic number of $\Gamma_G$, so a minimal prime graph contains no singleton vertices. \hfill \Box

It follows from Lemma 3.3 that $\mathcal{O}$, $\mathcal{D}$ and $\mathcal{I}$ are pairwise disjoint, and by Lemma 3.2 each must be nonempty. In particular, these sets provide a 3-coloring of $\Gamma_G$, and it is this 3-coloring we mean when we refer to a 3-coloring of $\Gamma_G$.

![Diagram](image.png)

**Figure 3:** An example of 3-coloring the complement of a minimal prime graph.

Next, we prove a technical lemma concerning the formation $q \leftarrow r \rightarrow p$ in $\Gamma_G$.

**Lemma 3.4.** Let $G$ be a solvable group and $p$, $q$, and $r$ be distinct primes dividing $|G|$. Suppose that a Hall $\{p, r\}$-subgroup of $G$ is a Frobenius group whose Frobenius kernel is a Sylow $p$-subgroup of $G$. Suppose additionally that $r \rightarrow q$ in $\Gamma_G$. Then a Hall $\{p, q, r\}$-subgroup of $G$ is a Frobenius group whose Frobenius kernel is a Hall $\{p, q\}$-subgroup of $G$. In particular, some Sylow $p$-subgroup and some Sylow $q$-subgroup of $G$ centralize each other.

**Proof.** Let $P$, $Q$, and $R$ be Sylow $p$-, $q$-, and $r$-subgroups, respectively, of $G$ so that $PQ$, $PR$, and $QR$ are subgroups. We know that $PR$ is a Frobenius group with Frobenius kernel $P$. Thus, $R$ is a Frobenius complement. Also, $QR$ is either a Frobenius group or a 2-Frobenius group of type $(r, q, r)$, but by Corollary 2.3,
it cannot be 2-Frobenius. Thus, \(QR\) is a Frobenius group with Frobenius kernel \(Q\). It is not difficult to see that this implies that \(PQR\) is a subgroup and in fact, it is a Frobenius group with Frobenius kernel \(PQ\). Since a Frobenius kernel is nilpotent, this implies that \(P\) and \(Q\) centralize each other. \(\square\)

Next, we define the sets \(\Pi = \{ p \in \Gamma_G : N_G^2(p) \neq \emptyset \}\). We must also introduce the binary octahedral group \(2O := \langle r, s, t \mid r^2 = s^3 = t^4 = rst \rangle\). \(2O\) is known under several guises. It has order 48, and it is the nonsplit extension of \(\text{SL}_2(3)\) by a cyclic group of order 2. In this paper, \(2O\) occurs as a Frobenius complement.

**Proposition 3.5.** Suppose \(G\) is a solvable group. If \(\Gamma_G\) is minimal, then \(\Pi \subseteq \mathcal{I}\) and \(H_{\Pi} \leq \text{Fit}(G)\).

**Proof.** We begin by noting that \(\Pi \subseteq \mathcal{I}\) by Corollary 2.7. It suffices to prove that a Sylow \(p\)-subgroup \(P\) is normal in \(G\) for every \(p \in \Pi\). We do this by showing that the normalizer of \(P\) contains a Sylow \(s\)-subgroup \(S\) for every prime \(s \in \pi(G)\). If \(s \rightarrow p\) in \(\Gamma_G\), then \(H_{sp}\) is Frobenius by Corollary 2.6, so the normalizer of \(P\) contains a Sylow \(s\)-subgroup of \(G\).

Suppose now that \(sp \in \Gamma_G\). By minimality, the removal of \(sp\) from \(\Gamma_G\) must create a triangle in \(\Gamma_G\) or increase the chromatic number of \(\Gamma_G\). Suppose \(s\) and \(p\) are in different color classes of \(\Gamma_G\). Then removing \(sp\) from \(\Gamma_G\) does not increase the chromatic number of \(\Gamma_G\), so there exists a prime \(t\) so that \(st, tp \in \Gamma_G\). In the case that \(s \in D\), we have that \(t \rightarrow s\) and \(t \rightarrow p\) in \(\Gamma_G\). By Corollary 2.6, we know that \(H_{pt}\) is a Frobenius group. Thus, we may apply Lemma 3.4 to see that some Sylow \(s\)-subgroup of \(G\) normalizes \(P\). If \(s \in \mathcal{O}\), then \(s \rightarrow t \rightarrow p\) in \(\Gamma_G\), so \(H_{stp}\) is a 2-Frobenius group of type \((p, t, s)\) by Lemma 2.4 and so, \(P\) is normalized by a Sylow \(s\)-subgroup.

Let \(s \in \mathcal{I}\). Suppose there exists a prime \(q \in N(p) \cap N(s)\). We know that \(H_{pq}\) is a Frobenius group by Corollary 2.6, so we may apply Lemma 3.4 to show that some Sylow \(s\)-subgroup of \(G\) normalizes \(P\). Assume now that \(N(p) \cap N(s) = \emptyset\). Then there exists a 2-path \(r \rightarrow q \rightarrow p\) in \(\Gamma_G\) for which \(qs \notin \Gamma_G\). Since \(q \in D\) and \(s \in \mathcal{I}\), there exists a prime \(t\) so that \(tq, ts \in \Gamma_G\) by minimality of \(\Gamma_G\). In particular, \(t \in N^2_G(s)\). By Corollary 2.7, \(t \rightarrow q \rightarrow p\) in \(\Gamma_G\). Let \(H = H_{stp}\). Then \(\Gamma_H\) consists of exactly the edges \(t \rightarrow q, q \rightarrow p, t \rightarrow s\). We conclude that \(\Gamma_H\) has diameter 3, so by [13, Prop. 3], either the Fitting length of \(H\) is 3 or the Fitting length of \(H\) is 4 and the binary octahedral group \(2O\) is a normal section of \(H\).

First, suppose that \(\ell_F(H) = 4\) and \(2O\) is a normal section of \(H\). Since \(K = H_{stp}\) is a 2-Frobenius group of type \((p, q, t)\) by Theorem 2.4, we know \(q \neq 2\) by Lemma 2.1. Let \(N\) and \(M\) be the normal subgroups of \(H\) so that \(M/N\) is isomorphic to \(2O\). It follows that \(G/N\) has a central subgroup of order 2. First we observe that if \(F_2(K) \leq N\), then we see that \(K/N\) is a cyclic Sylow \(t\)-subgroup of \(G/N\). On the other hand, if \(F_2(K)\) is not contained in \(K \cap N\),
then $K/(K \cap N) \cong KN/N$ is either a Frobenius group or a 2-Frobenius group. Since both Frobenius groups and 2-Frobenius groups have trivial center, we see that we 2 cannot divide $|K|$. We conclude that neither $t$ nor $p$ can be 2, which forces $s = 2$. Let $L$ be a Hall $\{2, t\}$-subgroup of $H$. We know that $L$ is either a Frobenius group whose Frobenius kernel is a 2-group or a 2-Frobenius group of type $(t, 2, t)$. Since a Sylow 2-subgroup is not cyclic it cannot be 2-Frobenius of type $(t, 2, t)$ by Lemma 2.3. On the other hand, if $L$ is a Frobenius group, then $LN/N$ cannot have a central subgroup of order 2. Therefore, we conclude that $s \neq 2$, and hence we cannot have that $\ell_F(H) = 4$.

It follows that $H$ has Fitting length 3. If $P$ is not normalized by $S$, then $P \not\subseteq \operatorname{Fit}(H)$. So we have $t \to q \to p$ in the Frobenius digraph of $H/\operatorname{Fit}(H)$. Thus $H/\operatorname{Fit}(H)$ contains a 2-Frobenius group of type $(p, q, t)$, which necessarily has Fitting length 3, a contradiction. This final contradiction shows that $S$ normalizes $P$ for any $s \in \Gamma_G$, so $P$ is normal in $G$. 

We now present the first structural consequence of minimality, exploiting the 3-colorability condition of Theorem 2.10.

**Definition.** Given a natural number $n$, denote by $\sigma(n)$ the number of prime divisors of $n$. Define $\sigma(G) = \max\{\sigma(o(g)) : g \in G\}$. Call a group $\sigma$-reduced if every prime $p$ appears in at most one chief factor of $G$.

**Theorem 3.6.** For any solvable group $G$ for which $\Gamma_G$ is minimal,

$$\sigma(|G|) \leq 3\sigma(G).$$

**Proof.** It can be shown [6, Lem. 16.17] that $G$ contains at least one $\sigma$-reduced subgroup $\Sigma$ so that $\sigma(|\Sigma|) = \sigma(|G|)$. Since $\Sigma$ is a subgroup of $G$, we see that $pq \in \overline{\Gamma_G}$ for every $pq \in \Gamma_G$. Furthermore, since $\Gamma_G$ is minimal, every edge in $\Gamma_G$ must be present in $\overline{\Gamma_G}$. It follows that $\overline{\Gamma_G}$ isomorphic to $\Gamma_G$, and since the orientation of edges in $\overline{\Gamma_G}$ is closed under subgroups, we see in fact that $\overline{\Gamma_G}$ is isomorphic to $\Gamma_G$. Clearly $\sigma(\Sigma) \leq \sigma(G)$, so we may assume without loss of generality that $G$ is $\sigma$-reduced.

Because every prime appears in at most one chief factor of $G$, every Sylow subgroup of $G$ must be elementary abelian. Every vertex in $\mathcal{O}$ and $\mathcal{D}$ serves as a Frobenius complement in some Hall subgroup of $G$ or an appropriate quotient group, so it follows that every Sylow $p$-subgroup for $p \in \mathcal{O} \cup \mathcal{D}$ is cyclic of prime order. Then $H_{pq}$ is cyclic for any $p, q \in \mathcal{O} \cup \mathcal{D}$ for which $pq \in \Gamma_G$. It follows that $H_\mathcal{O}$ and $H_\mathcal{D}$ are nilpotent.

To prove that $H_\mathcal{I}$ is nilpotent, by Proposition 3.5 it suffices to show that $H_{ps}$ is nilpotent for any $p \in \mathcal{I}$ and $s \in \mathcal{I} \setminus \mathcal{I}$. We know that $N_{\mathcal{I}}(p) \cap \mathcal{I} = \emptyset$ and $N_{\mathcal{I}}(s) \cap \mathcal{D} = \emptyset$. Thus if we add the edge $ps$ to $\overline{\Gamma_G}$, $\mathcal{O}, \mathcal{D} \cup \{s\}, \mathcal{I} \cup \{p\}$ remains an admissible 3-coloring. From minimality, $p$ and $s$ must then share an in-neighbor
q ∈ O. Thus $H_{pq}$ is either a Frobenius kernel or an upper kernel in a $H_{qps}$, so $H_{ps}$ is nilpotent.

We conclude that $n := \max\{|O|, |D|, |I|\} \leq \sigma(G)$, and therefore

$$\sigma(|G|) \leq 3n \leq 3\sigma(G).$$

Remark. Theorem 3.6 is motivated by the conjecture in [8, Conj. 9] that, in fact, $\sigma(|G|) \leq 3\sigma(G)$ for every solvable group $G$. One may notice that minimality is only used in the above argument to prove that $I$ is nilpotent. Indeed, this approach works for any case where $n < \sigma(G) < |I|$, where $n = \max\{|O|, |D|\}$.

Actually, $n$ can be strengthened further to $n = \sum_k \max\{|O_k|, |D_k|\}$, where $k$ runs over the components of $\overline{\Gamma}_G[O \cap D]$, however in general the number of vertices in $I$ can be much greater in non-minimal prime graphs.

4. 5-cycles as Prime Graphs.

We continue our discussion of minimal prime graphs by returning to solvable groups with 5-cycle prime graphs. Intuitively, we think of these groups as a primordial model for groups with minimal prime graphs, expecting many of the group theoretic properties resulting from minimality to stem from those exhibited here. We develop this notion by showing that the 5-cycle is not only the minimal prime graph of smallest order, but also that every minimal prime graph contains a 5-cycle. This shows that groups with 5-cycle prime graphs occur as Hall subgroups in every solvable group with a minimal prime graph. From this observation, we are able to quickly derive an upper bound on the Fitting length of any solvable group with a minimal prime graph, though this bound will be improved in the next section. Most importantly, we prove that solvable groups with 5-cycle prime graphs have Fitting length exactly 3.

Lemma 4.1. Every minimal graph contains an induced 5-cycle.

Proof. Let $P$, $Q$, and $R$ be the color classes of any 3-coloring of $\overline{\Gamma}_G$. Because $\overline{\Gamma}_G$ is not 2-colorable, we can assume without loss of generality that for a prime $p \in P$, there exists some prime $q \in Q$ so that $pq \in \overline{\Gamma}_G$.

Suppose that a prime $r \in R$ exists such that $rp, rq \in \Gamma_G$. By minimality, removing the edge $rq$ will not yield the prime graph of a solvable group. Since $r$ and $q$ are in different color classes of $\overline{\Gamma}_G$, the graph $\overline{\Gamma}_G + rq$ admits the same 3-coloring as $\overline{\Gamma}_G$. Thus, a prime $p' \in P$ exists such that $p'r, p'q \in \overline{\Gamma}_G$. Similarly, since $rp \in \Gamma_G$, a prime $q' \in Q$ exists such that...

Figure 4: Cases for Lemma 4.1.
\(q'p, q'r \in \Gamma_G\), and so by Lucido's Three Primes Lemma, \(p'q' \in \Gamma_G\). It follows that \(\Gamma_G[\{p, p', q, q', r\}]\) is isomorphic to the 5-cycle.

Now suppose that no prime \(r \in R\) satisfies \(pr, qr \in \Gamma_G\). Again, since \(\Gamma_G\) is not 2-colorable, we can assume without loss of generality that there exists a prime \(r \in R\) so that \(pr \in \Gamma_G\), whence \(rq \in \Gamma_G\). Similarly, there exists a prime \(r' \in R\) so that \(qr' \in \Gamma_G\), whence \(r'p \in \Gamma_G\). Thus, by minimality there exists a prime \(p' \in P\) such that \(p'r, p'r' \in \Gamma_G\), whence \(p'q \in \Gamma_G\). It follows that \(\Gamma_G[\{p, p', q, r, r'\}]\) is isomorphic to the 5-cycle. \(\square\)

**Remark.** In a certain sense, the minimality criterion is a partial converse to Lucido’s Three Primes Lemma. Where this lemma asserts that \(qr, pq, pr\), Lucido’s Three Primes Lemma together with the minimality criterion assures the existence of a self-complementary object in every minimal graph.

Next, we make a simple observation connecting Lemma 4.1 to Proposition 3 of [13]. This allows us to easily derive an upper bound on the Fitting length of a group with a minimal prime graph.

**Corollary 4.2.** If \(G\) is a solvable group with a minimal prime graph, then
\[
3 \leq \ell_F(G) \leq 5.
\]

**Proof.** By Proposition 3.5, we have that \(H_\Pi \leq \text{Fit}(G)\). By Lemma 4.1, \(\Gamma_G\) contains at least one induced 5-cycle, each of which has contains at least one vertex in \(\Pi\). Let \(\Delta \subseteq \Pi\) be the smallest set of vertices such that \(\Gamma_G[\Delta]\) contains no induced 5-cycles. Then there exists a pentagon that loses exactly one vertex in \(\Gamma_G/\text{Fit}(\Gamma_G)\). Hence \(\Gamma_G/\text{Fit}(\Gamma_G)\) has diameter 3. By [13, Prop. 3], \(\ell_F(G/\Delta) \leq 4\), so \(\ell_F(G) \leq 5\). Every 2-Frobenius group has Fitting length 3, so since every minimal prime graph contains a 2-path, we obtain the lower bound \(3 \leq \ell_F(G)\). Thus \(3 \leq \ell_F(G) \leq 5\). \(\square\)

To show that solvable groups whose prime graph is isomorphic to a 5-cycle have Fitting length 3, we must first prove a general lemma.

**Lemma 4.3.** Let \(G\) be a solvable group such that \(\Gamma_G\) is minimal. Assume that \(H = H_{srq}\) is a Hall \(\{q, r, s\}\)-subgroup of \(G\) and that \(H_{sr}\) and \(H_{sq}\) are Hall \(\{r, s\}\)- and \(\{q, s\}\)-subgroups of \(H\), respectively.

(a) If \(s \rightarrow r \rightarrow q \in \Gamma_G\), then \(O_s(H_{srq}) = O_s(H_{sr})\).
(b) If \(q \leftarrow r \rightarrow s \in \Gamma_G\), then \(O_s(H_{srq}) = O_s(H_{sq}) = O_s(H_{sr})\).
(c) If \(s, q \in O\) and \(r \in N^1_{O}(s) \cap N^1_{O}(q)\). Then \(O_s(H_{srq}) = O_s(H_{sr})\).

**Proof.** Write \(F = \text{Fit}(H)\) and \(K = H_{sr}\). Note that in each case, \(\Gamma_H\) is disconnected, so \(H\) must be Frobenius or 2-Frobenius.
Proof. Let \( p_1, \ldots, p_5 \) be the prime divisors of \(|G|\). Because \( \Gamma_G \) is also isomorphic to the 5-cycle, we have by Corollary 2.7 that up to isomorphism there exists only one possible Frobenius digraph of \( G \). Without loss of generality, we label \( \Gamma_G \) as shown in Figure 5.

Observe that \( \Pi = \{p_4\} \), \( \mathcal{I} = \{p_4, p_5\} \), \( \mathcal{D} = \{p_3\} \), and \( \mathcal{O} = \{p_1, p_2\} \). In particular, \( H_{p_1p_3p_4} \) is 2-Frobenius by Lemma 2.4, so the Fitting length of \( G \) is at least 3. We apply Proposition 3.5 to see that \( P_4 \) is normal in \( G \). Notice that the primes dividing \(|\text{Fit}(G)|\) must be adjacent to \( p_4 \) and to each other.

We first suppose that \( H_{p_1p_3} \) is 2-Frobenius, so \( J = O_{p_1}(H_{p_1p_3}) \) is nontrivial. We show that \( J \) is normal in \( G \) by showing that the normalizer of \( J \) contains a Sylow subgroup for every prime dividing \(|G|\). It is obvious that \( J \) is normalized by a Sylow

![Figure 5: Frobenius digraph of G in Prop. 4.4.](image-url)
$p_1$- and $p_3$-subgroup of $G$. Let $H_{p_1p_3p_4}$ be a Hall $\{p_1, p_3, p_4\}$-subgroup containing $H_{p_1p_3}$. By Lemma 4.3(a), we see that $J = O_{p_1}(H_{p_1p_3p_4})$ and so $J$ is normalized by $P_4$. Let $H_{p_1p_3p_5}$ be a Hall $\{p_1, p_3, p_5\}$-subgroup containing $H_{p_1p_3}$ and $H_{p_1p_5}$. Applying Lemma 4.3(b), we see that $J = O_{p_1}(H_{p_1p_3p_5}) = O_{p_5}(H_{p_1p_5})$, and so $J$ is normalized by a Sylow $p_5$-subgroup of $G$. Finally, we use Lemma 4.3(c) to see that $J = O_{p_1}(H_{p_1p_2p_5})$, where $H_{p_1p_2p_5}$ is a Hall $\{p_1, p_2, p_5\}$-subgroup containing $H_{p_1p_5}$. We conclude that a Sylow $p_2$-subgroup of $G$ normalizes $J$. Thus we have shown that $J$ is normal in $G$. It is not difficult to see that $J = O_{p_1}(G)$. Any prime divisor of $|\text{Fit}(G)|$ must therefore be adjacent to $p_1$ in $\Gamma_G$, from which we conclude that $F = \text{Fit}(G) = J \times P_2$.

We now argue that all the Sylow subgroups of $G/F$ are cyclic. Observe that $F = \text{Fit}(H_{p_1p_3p_4})$, and since $H_{p_1p_3p_4}$ is 2-Frobenius, we know by Lemma 2.1 that all the Sylow subgroups of $H_{p_1p_3p_4}/F$ are cyclic. This implies that the Sylow $p_1$- and $p_3$-subgroups of $G/F$ are cyclic. Since $J = O_{p_1}(H_{p_1p_3}) > 1$, we know that $H_{p_1p_3}$ is 2-Frobenius, so by Lemma 2.3 a Sylow $p_3$-subgroup of $G$ is cyclic. By Corollary 2.6, we see that $H_{p_2p_5}$ is a Frobenius group. The Sylow $p_5$-subgroup of $H_{p_2p_5}$ is cyclic, so a Sylow $p_2$-subgroup must be cyclic as well. Thus all the Sylow subgroups of $G/F$ are cyclic. It follows that $G/F$ has Fitting length at most 2, and so $G$ has Fitting length at most 3. For this case, this proves the theorem.

Next, suppose that $H_{p_1p_3}$ is a Frobenius group. We show that $P_3$ is normal in $G$. We know by Lemma 2.5 that $P_3$ is cyclic, so we have that $P_1$ is cyclic. We can apply Lemma 3.4 to see that $H_{p_1p_3p_5}$ is a Frobenius group whose Frobenius kernel is a Hall $\{p_3, p_5\}$-subgroup of $G$. This implies that the normalizer of $P_3$ contains Sylow $p_1$- and $p_3$-subgroups of $G$. We see that $p_2 \in N_G(p_4)$, so by Corollary 2.6, $H_{p_2p_4}$ is a Frobenius group, and by Lemma 3.4, there is a Hall $\{p_2, p_4, p_5\}$-subgroup of $G$ that is Frobenius with kernel $P_2P_3$. This implies that $P_2$ centralizes $P_3$, and the normalizer of $P_3$ contains a Sylow $p_2$-subgroup of $G$. We conclude that $P_3$ is normal in $G$. Notice that all the prime divisors of $|F|$ are adjacent to $p_5$ in $\Gamma_G$, so $F = P_4 \times P_5$.

If all Sylow subgroups of $G/F$ are cyclic, then $G/F$ has Fitting length at most 2, so $G$ has Fitting length at most 3. Thus we may assume that some Sylow subgroup of $G/F$ is not cyclic. Since we know the Sylow $p_1$- and $p_3$-subgroups are cyclic, it must be that a Sylow $p_2$-subgroup is not cyclic. We know that a Sylow $p_2$-subgroup $P_2$ is a Frobenius complement, and as it is not cyclic, we must have that $P_2$ is generalized quaternion. Note that since $H_{p_1p_3}$ is a Frobenius group, if $p_3 = 3$, then we must have $p_1 = 2$, and this cannot occur since $p_2 = 2$. Thus, we have that $p_3 \neq 3$.

Let $F_2/F = \text{Fit}(G/F)$. We know that $C_{G/F}(F_2/F) \leq F_2/F$, so we have $G/F_2 \leq \text{Aut}(F_2/F)$. If we assume that $F_2/F$ is cyclic, then $\text{Aut}(F_2/F)$ will be abelian, so $G/F_2$ will also be abelian. In this case, $G$ will have Fitting length at most 3. Thus, for the final step, we may assume that $F_2/F$ is not cyclic and

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work to obtain a contradiction.

Let $Q/F$ be the Sylow 2-subgroup of $F_2/F$, and let $D/F$ be the Hall 2-complement of $F_2/F$. We know that $F_2/F = Q/F \times D/F$ and that $D/F$ is cyclic. Thus, $G/F_2 \leq \text{Aut}(Q/F) \times \text{Aut}(D/F)$. We know that $\text{Aut}(D/F)$ is abelian. On the other hand, since $Q/F$ is a subgroup of a generalized quaternion group, and therefore cyclic or generalized quaternion. Thus $\text{Aut}(Q/F)$ is abelian unless $Q/F$ is isomorphic to the quaternion group $Q_8$ of order 8. In this case, $\text{Aut}(Q/F) \cong S_3$. We have seen that the conclusion of the theorem holds if $G/F_2$ is abelian, so we may assume that $Q/F$ is isomorphic to $Q_8$. If $C/F = C_{G/F}(Q/F)$, then $G/C \cong S_3$. In particular, $G/C$ has order 3, so $G'$ has a nontrivial Sylow 3-subgroup. Since $p_3 \neq 3$ and the only primes dividing $|G:F|$ are $p_1, p_2 = 2$, and $p_3$, this leaves us with $p_1 = 3$.

Notice $p_3$ does not divide $|G:C|$. Hence, $C$ contains a Sylow $p_3$-subgroup $P_3$ of $G$. Observe that $H_{p_1p_3} \cap F = 1$, so $H_{p_1p_3} \cong H_{p_1p_3}/F/F$. This implies that $p_1 = 3$ does not divide $|F_2/F|$, and thus, $D/F$ is a $p_3$-group. Notice that $P_3$ centralizes $Q/F$ since $P_3 \leq C$, and furthermore, since $D \leq P_3F$ and $P_3 \cong P_3/F/F$ is cyclic, we have since $P_3$ centralizes $D/F$. Thus $P_3$ centralizes $F_2/F$, so $P_3F = D$. Let $B/F = C_{G/F}(D/F)$. Since $D/F$ is cyclic, we know that $G/B$ is abelian. On the other hand, since $H_{p_1p_3}$ is a Frobenius group whose Frobenius kernel is $P_3$, we see that a Sylow $p_1$-subgroup of $B/F$ is trivial, and so $B$ has a trivial Sylow $p_1$-subgroup. Since $G/B$ is abelian, $G' \leq B$, and we conclude that $G'$ has a trivial Sylow $p_1$-subgroup. However, recalling that $p_1 = 3$, we saw earlier that $G'$ has a nontrivial Sylow $p_1$-subgroup. This contradiction completes the proof. □

5. Fitting Lengths and Minimal Prime Graphs.

In this section, we expand on the ideas used in the proof of Proposition 4.4 to show that all solvable groups with minimal prime graphs have Fitting length at most 4. The lower bound of $\ell_F(G) \geq 3$ found in Corollary 4.2 is certainly best possible, since by Lemma 3.2 there exists a 2-Frobenius Hall subgroup of type $(p,q,r)$ that has Fitting length 3. The upper bound, on the other hand, may be improved from $\ell_F(G) \leq 5$ to $\ell_F(G) \leq 4$ by further examining the graph theoretic properties implied by minimality.

We must first present several results that give more detail about the structure of a minimal prime graph. We begin by partitioning the sets $\mathcal{O}$ and $\mathcal{I}$ even further. As we will show in the following lemma, any $q \in \mathcal{O}$ is in contained in either $N^1_\uparrow(p)$ or $N^2_\uparrow(p)$ for any $p \in \mathcal{I}$. With this in mind, we define $\mathcal{O}_1(p) = N^1_\uparrow(p) \cap \mathcal{O}$, observing that $\mathcal{O} = \mathcal{O}_1(p) \cup N^2_\uparrow(p)$ is a disjoint union. We then define the following sets.

$$\mathcal{O}_1 = \cup_{p \in \Pi} \mathcal{O}_1(p) \quad \mathcal{O}_1^* = \bigcap_{p \in \Pi} \mathcal{O}_1(p) \quad \mathcal{O}_2 = \bigcap_{p \in \Pi} N^2_\uparrow(p) \quad \mathcal{O}_2^* = \cup_{p \in \Pi} N^2_\uparrow(p)$$
We observe that \( \mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2 = \mathcal{O}_1^* \cup \mathcal{O}_2^* \) are disjoint unions. Finally, set \( \Phi = \{ p \in \mathcal{I} \mid N_1^2(p) = \emptyset \} \). We obtain as a result of part (b) in the following lemma that \( \mathcal{I} = \Pi \cup \Phi \) is a disjoint union as well.

**Lemma 5.1.** Let \( G \) be a solvable group with a minimal prime graph.

(a) \( \mathcal{O} \subseteq N_1^1(p) \cup N_2^1(p) \) for any prime \( p \in \mathcal{I} \).

(b) \( \mathcal{O} \subseteq N_1^2(p) \) for any prime \( p \in \mathcal{I} \).

(c) If \( s \in \mathcal{O}_2 \), then \( \mathcal{D} \subseteq N_1^1(s) \).

**Proof.** Suppose that \( s \in \mathcal{O} \setminus N_1^1(p) \). Then \( \Gamma_G + \{ sp \} \) admits the same 3-coloring as \( \Gamma_G \), so by minimality \( \Gamma_G + \{ sp \} \) contains a triangle. It follows that there exists a prime \( d \in \mathcal{D} \) so that \( s \rightarrow d \rightarrow p \) in \( \Gamma_G \). This proves part (a). For part (b), we know that if \( p \in \Phi \), then \( N_2^2(p) = \emptyset \), so part (b) is an immediate consequence of part (a).

To prove part (c), suppose that \( sd \in \Gamma_G \) for some prime \( d \in \mathcal{D} \). Again by minimality, there exists a prime \( p \in \mathcal{I} \) such that \( s \rightarrow p \leftarrow d \). Since \( d \in N_1^1(p) \), we see that \( N_2^2(p) \) is nonempty, and thus, \( p \in \Pi \). However, by definition of \( \mathcal{O}_2 \), we obtain \( s \in N_2^2(p) \), contradicting Lucido’s Three Primes Lemma. Thus, \( s \rightarrow d \) in \( \Gamma_G \). \( \square \)

We now present an application of Lemmas 2.4 and 2.5 and Corollary 2.6, allowing us to characterize the Sylow subgroups for primes in \( \mathcal{D} \) and in \( \mathcal{O} \).

**Lemma 5.2.** Let \( G \) be a solvable group such that \( \Gamma_G \) is a minimal prime graph.

(a) If \( q \in \mathcal{D} \) and \( p \in N_1^1(q) \), then \( q \neq 2 \), a Sylow \( q \)-subgroup of \( G \) is cyclic, and a Hall \( \{ p, q \} \)-subgroup of \( G \) is a Frobenius group for which Sylow \( p \)-subgroup is the Frobenius kernel and a Sylow \( q \)-subgroup is a Frobenius complement.

(b) If \( s \in \mathcal{O}_1 \) and \( t \in N_1^1(s) \), then a Hall \( \{ s, t \} \)-subgroup of \( G \) is a Frobenius group whose Frobenius kernel is a Sylow \( t \)-subgroup of \( G \) and a Sylow \( s \)-subgroup of \( G \) is a Frobenius complement. In particular, a Sylow \( s \)-subgroup of \( G \) is either cyclic or generalized quaternion.

(c) If \( s \in \mathcal{O}_2^* \), then a Sylow \( s \)-subgroup of \( G \) is not generalized quaternion.

**Proof.** Suppose \( q \in \mathcal{D} \) and \( p \in N_1^1(q) \). There exists \( r \in \pi(G) \) so that \( r \rightarrow q \rightarrow p \) in \( \Gamma_G \). By Lemma 2.4, a Hall \( \{ p, q, r \} \)-subgroup of \( G \) is a 2-Frobenius group of type \( (r, q, p) \). In light of the definition of type \( (p, q, r) \) and Lemma 2.5, we see that \( q \neq 2 \) and a Sylow \( q \)-subgroup of \( G \) is cyclic, and a Hall \( \{ p, q \} \)-subgroup is a Frobenius group whose Frobenius kernel is a Sylow \( p \)-subgroup of \( G \). This proves part (a).

To prove part (b), we suppose \( s \in \mathcal{O}_1 \) and \( t \in N_1^1(s) \). Let \( p \) be a prime in \( \Pi \) so that \( s \in \mathcal{O}_1(p) \). There there exist primes \( q, r \in \pi(G) \) so that \( r \rightarrow q \rightarrow p \).
in \( \Gamma_G \). We now apply Corollary 2.6 to see that a Hall \( \{s, t\} \)-subgroup of \( G \) is a Frobenius group whose Frobenius kernel is a Sylow \( t \)-subgroup of \( G \) and where a Sylow \( s \)-subgroup is a Frobenius complement.

Finally, suppose \( s \in \mathcal{O}_2^* \). There exist primes \( p, q \in \pi(G) \) so that \( s \to q \to p \) in \( \Gamma_G \). This implies that a Hall \( \{p, q, s\} \)-subgroup is a 2-Frobenius of type \( (p, q, s) \). By Lemma 2.5, we conclude that a Sylow \( s \)-subgroup of \( G \) is not generalized quaternion. \( \square \)

With these results in mind, we are able to locate 2 as a vertex in \( \Gamma_G \) in the case that the Sylow 2-subgroups of \( G \) are generalized quaternion.

**Lemma 5.3.** Suppose \( G \) is a solvable group so that \( \Gamma_G \) is a minimal prime graph. If a Sylow 2-subgroup of \( G \) is generalized quaternion, then \( 2 \in \mathcal{O}_1^* \).

**Proof.** Let \( p \) be a prime in \( \mathcal{I} \). If \( p \in \mathcal{I} \), then the assertion is true by Lemma 2.5. We need to deal with the case where \( p \) is not in \( \mathcal{I} \). We know that if \( q \in \mathcal{O} \), then \( q \to p \) by Lemma 5.1(b). \( H_{qp} \) is either Frobenius or 2-Frobenius, and in either case, a Sylow \( p \)-subgroup is isomorphic to a Frobenius complement. Sylow subgroups for primes in \( \mathcal{D} \) are cyclic by Lemma 5.2(a). Generalized quaternion groups are not cyclic and may not be Frobenius kernels, so \( 2 \notin \mathcal{I} \cup \mathcal{D} \). By Lemma 5.2(c), Sylow subgroups for primes in \( \mathcal{O}_2^* \) are not generalized quaternion. Therefore, since \( \mathcal{O} = \mathcal{O}_1^* \cup \mathcal{O}_2^* \) is a disjoint union, we conclude that \( 2 \in \mathcal{O}_1^* \). \( \square \)

We can further characterize the Sylow subgroups associated with primes in \( \mathcal{O}_2 \) by separating those which are cyclic from those which are not. Let \( \mathcal{C} \) denote the primes in \( s \in \mathcal{O}_2 \) for which \( S \) is cyclic and set \( \mathcal{N} = \mathcal{O}_2 \setminus \mathcal{C} \). Note that when \( s \in \mathcal{O}_2 \) and \( t \in \mathcal{N}_1^*(s) \), we have by Lemma 2.2 that \( H_{st} \) is Frobenius if and only if \( s \in \mathcal{C} \).

Having established these refined partitions, we are ready to prove the final theorem.

**Theorem 5.4.** Let \( G \) be a solvable group such that \( \Gamma_G \) is minimal. Then \( \ell_F(G) \leq 4 \). Furthermore, if \( \ell_F(G) = 4 \), then \( 2O \) is a normal section of \( G \).

**Proof.** We claim that it suffices to find a nilpotent normal subgroup \( X \) such that all subgroups of \( G/X \) are cyclic or generalized quaternion. Assuming that such a subgroup \( X \) exists, take \( E/X = \text{Fit}(G/X) \), and note that all Sylow subgroups of \( E/X \) are also cyclic or generalized quaternion. Recall that any group that is cyclic or generalized quaternion has an abelian automorphism group, with the exception of \( Q_8 \), whose automorphism group is isomorphic to \( S_3 \). In either case, we see that the automorphism group of any Sylow subgroup of \( E/X \) has Fitting length at most 2. Since \( E/X \) contains its centralizer in \( G/X \), \( G/E \) is isomorphic to a subgroup of \( \text{Aut}(E/X) \). It is not difficult to see that \( \text{Aut}(E/X) \) is isomorphic to a direct
product of the automorphism groups of its Sylow subgroups, so \( \text{Aut}(E/X) \) has Fitting length at most 2. Therefore, \( G/E \) has Fitting length at most 2. Since \( E/X \) and \( X \) are both nilpotent, it follows that \( \ell_F(G) \leq 4 \).

We now prove our second assertion, that \( 2O \) is a normal section of \( G \) when \( \ell_F(G) \) is exactly 4. If this is the case, \( S_3 \) must be isomorphic to a subgroup of \( G/E \), and the Sylow 2-subgroup of \( E/X \) must be the quaternion group \( Q_8 \). Let \( D/X \) be the Hall 2-complement of \( E/X \), so that \( E/D \) is isomorphic to \( Q_8 \). Let \( C/D = C_{G/D}(E/D) \). Let \( T/D \) be a Sylow 2-subgroup of \( CE/D \). We know that \( T/D \) is a generalized quaternion since it is a nonabelian subgroup of a generalized quaternion group. By Dedekind's lemma, \( T = (C \cap T)E \), and this implies that \( T = E \). In particular, \( (C \cap E)/D \) is a Sylow 2-subgroup of \( C/D \). Let \( B/D \) be a 2-complement for \( C/D \). Since \( (C \cap E)/D \) is normal and has order 2, it is central. This implies that \( B/D \) is normal in \( C/D \), and hence characteristic. Hence \( B \) is normal in \( G \). We have \( B \cap E = D \), so \( BE/B \cong E/D \) is isomorphic to \( Q_8 \). Also, \( G/CE = G/BE \) is isomorphic to \( S_3 \). This implies that \( G/B \) is an extension of \( A_4 \) by \( Z_2 \). Since the Sylow 2-subgroup is a generalized quaternion, it cannot be a split extension. Thus, it is the nonsplit extension, and hence is isomorphic to \( 2O \). It is not difficult to show that \( G/X \) will have a normal subgroup isomorphic to \( 2O \).

We split the proof into two cases. First, we prove the theorem when \( N \) is empty, and second, we prove it when \( N \) is non-empty.

**Case: \( N = \emptyset \).**

We first prove the theorem under the assumption that \( N = \emptyset \). In this case, every prime divisor of \( |G : H_I| \) lies in \( D \cup C \cup O_1 \). By parts (a) and (b) of Lemma 5.2, Sylow subgroups corresponding to primes in \( D \cup O_1 \) are cyclic or generalized quaternion, and for \( c \in C \), every Sylow \( c \)-subgroup of \( G \) is cyclic by definition. Therefore, if \( H_I \) is normal in \( G \), every Sylow subgroup of \( G/H_I \) is cyclic or generalized quaternion, so by the first paragraph it suffices to show that \( H_I \) is nilpotent and normal in \( G \).

If \( \Phi = \emptyset \), then \( T = \Pi \), so we are done by Proposition 3.5. Therefore we suppose that \( \Phi \) is nonempty. It suffices to show that \( P \) is normal in \( G \) for every prime \( p \in \Phi \). We do this by showing that the normalizer of \( P \) contains a Sylow subgroup for every prime in \( \pi(G) \).

Suppose \( s \in O \). By Lemma 5.1(b), \( s \to p \) in \( \Gamma_G \). Thus, if \( H_{sp} \) is a Hall \( \{s, p\} \)-subgroup of \( G \) containing \( P \), then \( H_{sp} \) is either Frobenius or 2-Frobenius of type \( (s, p, s) \). Since Sylow subgroups for primes in \( O = C \cup O_1 \) are cyclic or generalized quaternion, \( H_{sp} \) cannot be 2-Frobenius by Lemma 2.2. Then \( H_{sp} \) is a Frobenius group, so \( P \) is normalized by some Sylow \( s \)-subgroup of \( G \).

Next, suppose \( q \in D \). By Lemma 5.1(c) there is an \( r \in O_2 = C \) so that \( r \to q \), and by Lemma 5.1(b), we have that \( r \to p \) in \( \Gamma_G \). We know that \( H_{rq} \) is either a Frobenius group or a 2-Frobenius group of type \( (r, q, r) \), and since \( R \) is cyclic, \( H_{rq} \) cannot be a 2-Frobenius group by Lemma 2.2. Thus, we may apply
Lemma 3.4 to see that \( H_{rp} \) is a Frobenius group whose Frobenius kernel is a Hall \( \{q, p\} \)-subgroup. In particular, \( P \) is centralized by some Sylow \( q \)-subgroup of \( G \).

Suppose now that \( t \in T \) and \( \mathcal{O}_1(t) \neq \emptyset \). Take \( r \in \mathcal{O}_1(t) \). By Lemma 5.1(b) we see that \( r \rightarrow p \). Applying Lemma 5.2(b), we see that \( H_{rp} \) is a Frobenius group. Thus, by a similar application of Lemma 3.4 to \( H_{rt} \), we see that \( P \) is centralized by some Sylow \( t \)-subgroup of \( G \).

It remains to check the primes in \( \Pi' = \{ q \in \mathcal{I} : \mathcal{O}_1(q) = \emptyset \} \). It is possible that this set is empty; if so, \( P \) is centralized by a Sylow \( t \)-subgroup for every prime \( t \in \Pi \), so \( P \) is normalized by every prime dividing \( |G| \), and we are finished. Otherwise, \( \mathcal{O}_1^* = \emptyset \).

In light of Lemma 5.3, a Sylow \( 2 \)-subgroup of \( G \) cannot be generalized quaternion, so a Sylow \( q \)-subgroup is cyclic for every prime \( q \in \mathcal{O}_1 \). Note that all prime divisors of \( |G : H_{\Pi}^*| \) lie in \( \mathcal{C} \cup \mathcal{O}_1 \cup \mathcal{D} \cup \mathcal{P} \). We continue to consider a prime \( p \in \Phi \).

We have seen that the Sylow subgroups for primes in \( \mathcal{C} \cup \mathcal{O}_1 \cup \mathcal{D} \) normalize \( P \). If \( q \in \mathcal{P} \), then we know by Lemma 5.1(b) that \( \mathcal{O}_1(q) \) contains \( \mathcal{O}_1 \), so \( \mathcal{O}_1^* \) is not empty. We see from the previous paragraph that a Sylow \( q \)-subgroup of \( G \) normalizes \( P \). Hence \( PH_{\Pi}/H_{\Pi} \) is normal in \( G/H_{\Pi} \). As this is true for all the primes in \( \Phi \), it follows that \( H_{\Pi}/H_{\Pi}^* \) is normal in \( G/H_{\Pi} \) and nilpotent.

The prime divisors of \( |G : H_{\Pi}^*| \) must lie in \( \mathcal{C} \cup \mathcal{O}_1 \cup \mathcal{D} \), and we have seen that in this case the Sylow subgroups for primes in these three sets are cyclic. Thus all Sylow subgroups of \( G/H_{\Pi} \) are cyclic. We then have that \( G/H_{\Pi} \) has Fitting length at most 2. We claim that \( D/H_{\Pi} = \text{Fit}(G/H_{\Pi}) \) is a Hall \( \mathcal{D} \)-subgroup of \( G/H_{\Pi} \).

Consider a prime \( s \in \mathcal{C} \cup \mathcal{O}_1 \). Since \( \mathcal{O}_1^* = \emptyset \), there is a prime \( r \in \mathcal{D} \), so that \( s \rightarrow r \) in \( \Gamma_G^* \). Similarly, if \( r \in \mathcal{D} \), we can find a prime \( s \in \mathcal{C} \cup \mathcal{O}_1 \) so that \( s \rightarrow r \) in \( \Gamma_G^* \). In both cases, we have that \( H_{rs} \) is a Frobenius group. This implies that \( s \) does not divide \( |D : H_{\Pi}^*| \) and \( r \) does not divide \( |G : D| \), which proves the claim.

We have seen that, when \( p \in \Phi \), \( P \) is centralized by a Sylow \( d \)-subgroup for every \( d \in \mathcal{D} \). This implies that there is a Hall \( \mathcal{D} \)-subgroup \( H_{\mathcal{D}} \) of \( G \) that centralizes \( P \). Since this is true for every prime in \( \Phi \), we can find a Hall \( \Phi \)-subgroup \( H_{\Phi} \) of \( G \) that is centralized by \( H_{\mathcal{D}} \). We conclude that \( D/H_{\Pi}^* \cong H_{\Phi} \times H_{\mathcal{D}} \). We have seen that \( H_{\Phi} \cong H_{\Pi}/H_{\Pi}^* \) and \( H_{\mathcal{D}} \cong H/\Pi \) are nilpotent. It follows that \( D/H_{\Pi}^* \) is nilpotent. Since \( G/D \) and \( H_{\Pi}^* \) are both nilpotent, we conclude that \( \ell_F(G) \leq 3 \). This concludes the proof in the case that \( \mathcal{N} \) is empty.

Case: \( \mathcal{N} \neq \emptyset \).

We now suppose that \( \mathcal{N} \) is nonempty. Let \( L = H_{\Pi} \times \prod_{s \in \mathcal{N}} \mathcal{O}_s(G) \). Clearly, \( L \) is nilpotent and normal in \( G \). We now show that all the Sylow subgroups of \( G/L \) are cyclic or generalized quaternion. We then obtain the result by applying the first paragraph with \( X = L \). Observe that the primes dividing \( |G : L| \) lie in \( \mathcal{N} \cup \mathcal{C} \cup \mathcal{O}_1 \cup \mathcal{D} \cup \Phi \). We have seen that Sylow subgroups for primes in \( \mathcal{C} \cup \mathcal{O}_1 \cup \mathcal{D} \) must be cyclic or generalized quaternion. Thus, we need only consider primes in \( \Phi \cup \mathcal{N} \).

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Consider first a prime \( q \in \Phi \). By Lemma 5.1(b), we know that \( q \in N^1(s) \) for every prime \( s \in \mathcal{N} \). Fix a prime \( s \in \mathcal{N} \). Observe that \( s \in \mathcal{O}_2^* \), so a Sylow \( s \)-subgroup \( S \) is not generalized quaternion. Since \( S \) is also not cyclic, we conclude that \( S \) is not a Frobenius complement. We know that \( H_{qs} \) is either Frobenius or 2-Frobenius of type \( (s,q,s) \). Since \( S \) is not a Frobenius complement, we see that \( H_{qs} \) is not a Frobenius group. Thus, \( H_{qs} \) must be 2-Frobenius of type \( (s,q,s) \), and we may use Lemma 2.3 to see that a Sylow \( q \)-subgroup is cyclic. Hence, Sylow subgroups for every prime in \( \Phi \) are cyclic.

We now consider a prime \( s \in \mathcal{N} \). Since \( s \in \mathcal{O}_2 \), we know that there is a prime \( r \in \mathcal{D} \) so that \( s \rightarrow r \). Consider a Hall \( \{r,s\} \)-subgroup \( H_{rs} \) of \( G \). Let \( S \) be a Sylow \( s \)-subgroup contained in \( H_{rs} \), and let \( J = O_s(H_{rs}) \). By Lemma 5.3, \( S \) is not generalized quaternion. Since we also know that \( S \) is not cyclic, and \( H_{rs} \) is either Frobenius or 2-Frobenius, we deduce that \( H_{rs} \) is not Frobenius. Thus, \( H_{rs} \) is a 2-Frobenius group of type \( (s,r,s) \). By Lemma 2.1, \( S/J \) must be cyclic. If \( J = O_s(G) \), then \( S/L = J \), and so \( S/J \cong SL/L \). Since \( SL/L \) is a Sylow \( s \)-subgroup of \( G/L \), this implies that Sylow \( s \)-subgroups of \( G/L \) are cyclic. This implies that all of the Sylow subgroups of \( G/L \) are cyclic or generalized quaternion. Therefore we may once again obtain the result by applying the first paragraph with \( X = L \). Thus, the theorem will be proved once we prove that \( J = O_s(G) \).

We now prove that \( J = O_s(G) \). Observe first that \( O_s(G) \leq J \). We show that \( J \) is normal in \( G \) by showing that the normalizer of \( J \) contains a Sylow subgroup for every prime in \( \pi(G) \). Once we know \( J \) is normal in \( G \), then we will have \( J = O_s(G) \). By Lemma 5.1(b) and (c), if we have a prime \( q \in \mathcal{D} \cup \Phi \), then \( q \in N^1(s) \). Applying Lemma 4.3(b), we see that \( J = O_s(H_{sq}) \), where \( H_{sq} \) is a Hall \( \{s,q\} \)-subgroup containing \( J \). In particular, some Sylow \( q \)-subgroup normalizes \( O_s(G) \).

If we have a prime \( p \in \Pi \), then since \( s \in \mathcal{O}_2 \), we know that \( p \in N^2(s) \). Thus there exists a prime \( q \in \mathcal{D} \) so that \( s \rightarrow q \rightarrow p \in \Gamma^2 \). \( K \) be a Hall \( \{p,q,s\} \)-subgroup that contains \( H_{sq} \). Lemma 4.3(a) shows us that \( J = O_s(K) \). We have that \( J \) is normalized by the Sylow \( p \)-subgroup \( P \) of \( G \), which we recall is normal in \( G \), and is thus centralized by \( J \). Since \( K \) is a 2-Frobenius group, we see that \( C_K(P) = P \times J \). In particular, we deduce that \( J \) is a Sylow \( s \)-subgroup of \( C_K(P) \). Because this is true for every prime \( p \in \Pi \), we conclude that \( J \) is a Sylow \( s \)-subgroup of \( C_G(H_{rs}) \).

If \( t \in \mathcal{O} \) satisfies \( N^1(s) \cap N^1(t) \neq \emptyset \), then we may apply Lemma 4.3(c) to see that some Sylow \( t \)-subgroup normalizes \( J \). By Lemma 5.1(c), this occurs for all the primes in \( \mathcal{O}_2^* \). If \( \Phi \neq \emptyset \), then this holds for all primes \( t \in \mathcal{O} \) by Lemma 5.1(b). Notice that this implies that the normalizer of \( J \) contains Sylow subgroups for every prime dividing \( |G| \), and so we have that \( J \) is normal in \( G \). Thus, we may assume that \( \Phi \) is empty.
We have shown that \( J \leq C_G(H_\Pi) \). If \( J \) is normal in \( C_G(H_\Pi) \), then \( J \) is characteristic in \( C_G(H_\Pi) \) since \( J \) is a Sylow s-subgroup of \( C_G(H_\Pi) \). Because \( H_\Pi \) is normal in \( G \), it follows that \( C_G(H_\Pi) \) is normal in \( G \), and hence, \( J \) is normal in \( G \). Thus it remains to be proven that \( J \triangleleft C_G(H_\Pi) \). We now show that every prime dividing \( |C_G(H_\Pi)| \) is contained in \( \Pi \cup O_2 \). If \( t \in O_1 \cup D \), then there is a prime \( p \in \Pi \) so that \( p \in N^1(t) \). By Lemma 2.4 and Corollary 2.7, we know that \( H_{pt} \) is a Frobenius group, so \( t \) does not divide \( |C_G(P)| \). Since \( C_G(H_\Pi) \leq C_G(P) \), it follows that \( t \) does not divide \( |C_G(H_\Pi)| \). Since \( \Phi \) is empty, we can conclude that the only primes that divide \( |C_G(H_\Pi)| \) lie in \( \Pi \cup O_2 \). We have seen that the normalizer of \( J \) contains a Sylow subgroup of \( G \) for each prime in \( \Pi \cup O_2 \), and thus contains a Sylow subgroup of \( C_G(H_\Pi) \) for every prime divisor of \( |C_G(H_\Pi)| \). Thus \( C_G(H_\Pi) \) normalizes \( J \). This proves the theorem. \( \Box \)

We now demonstrate that the upper bound in Theorem 5.4 is in fact best possible by explicitly constructing a group of Fitting length 4 with a minimal prime graph. We will see that the Frobenius digraph of the resulting group may be obtained by linked vertex duplication of the 5-cycle. Let \( K = C_{11} \rtimes C_5 \) be a Frobenius group and take \( L = K \rtimes 2O \), where \( 2O \) is the binary octahedral group. Let \( V_1 \) be an absolutely irreducible \( \mathbb{F}_{23}[L] \)-module so that the fixed point space of the restriction of the module action to \( C_{11} \rtimes 2O \) is trivial. The smallest such \( V_1 \) has dimension 10. Let \( V_2 \) be an absolutely irreducible \( \mathbb{F}_{31}[L] \)-module so that \( C_5 \rtimes 2O \) acts fixed point freely on \( V_2 \). The smallest such \( V_2 \) has dimension 2. We have computationally verified that \((V_1 \times V_2) \rtimes L\) has Fitting length 4 and Frobenius digraph

\[
\begin{array}{c}
5 \\
11 \\
23 \\
3 \\
31
\end{array}
\]

which is minimal.

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