PEIERLS BRACKETS IN THEORETICAL PHYSICS

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Abstract. Peierls brackets are part of the space-time approach to quantum field theory, and provide a Poisson bracket which, being defined for pairs of observables which are group invariant, is group invariant by construction. It is therefore well suited for combining the use of Poisson brackets and the full diffeomorphism group in general relativity. The present paper provides an introduction to the topic, with applications to field theory and point Lagrangians.
1. Introduction

Although the Hamiltonian formalism provides a powerful tool for studying general relativity [1], its initial-value problem and the approach to canonical quantization [2], it suffers from severe drawbacks: the space + time split of \((M, g)\) disagrees with the aims of general relativity, and the space-time topology is taken to be \(\Sigma \times \mathbb{R}\), so that the full diffeomorphism group of \(M\) is lost [3,4].

However, as was shown by DeWitt in the sixties [5], it remains possible to use a Poisson-bracket formalism which preserves the full invariance properties of the original theory, by relying upon the work of Peierls [6]. In our paper, whose aims are pedagogical, we begin by describing the general framework, assuming that the reader has just been introduced to the DeWitt condensed notation [5]. Let us therefore consider, in field theory, disturbances which satisfy the homogeneous equation \(S_{\alpha ij} \delta \varphi^j = 0\), \(S\) being the classical action functional. On using the DeWitt super-condensed notation we write therefore

\[ S^2_2 \delta \varphi = 0. \] \hspace{1cm} (1.1)

Hereafter \(R^i_\alpha\) are the generators of infinitesimal gauge transformations, with associated

\[ R_{i\alpha} \equiv \gamma_{ij} R^j_\alpha \]

built from a local and symmetric matrix \(\gamma_{ij}\) which is taken to transform like \(S_{\alpha ij}\) under group transformations. We also consider

\[ R^\alpha_i \equiv \tilde{\gamma}^{\alpha\beta} R_{i\beta}, \]

where by hypothesis the matrix \(\tilde{\gamma}^{\alpha\beta}\) is local, non-singular, symmetric, and transforms according to the adjoint representation of the infinite-dimensional invariance group.

We expect to impose supplementary conditions on infinitesimal disturbances, chosen in the form \((R^t_i\) being the transpose of generators of infinitesimal gauge transformations)

\[ R^t_i \gamma \, \delta \varphi = 0. \] \hspace{1cm} (1.2)
Thus, on defining the operator

$$F \equiv S_2 + \gamma R \tilde{\gamma}^{-1} R^t \gamma,$$  \hspace{1cm} (1.3)

we look for $\delta \varphi$ solving the homogeneous equation

$$F \delta \varphi = 0,$$  \hspace{1cm} (1.4)

and we expect to determine $\delta \varphi$ throughout space-time if it and its derivatives are specified on any spacelike hypersurface $\Sigma$. Now on defining

$$\tilde{G} \equiv G^+ - G^-,$$  \hspace{1cm} (1.5)

we arrive at an integral formula for $\delta \varphi$, i.e.

$$\delta \varphi = \int_{\Sigma} \tilde{G} \hat{\mathcal{J}}^\mu \delta \varphi d\Sigma_\mu.$$  \hspace{1cm} (1.6)

The advanced and retarded Green functions $G^\pm$ are left inverses of $-F$:

$$G^\pm \tilde{F} = -\mathbb{I} \implies \tilde{G} \hat{\mathcal{F}} = 0.$$  \hspace{1cm} (1.7)

Furthermore, the form of $F$ and arbitrariness of Cauchy data $\hat{\mathcal{J}}^\mu \delta \varphi$ imply that $G^\pm$ are right inverses as well, i.e.

$$\hat{\mathcal{F}} G^\pm = -\mathbb{I} \implies \hat{\mathcal{F}} \tilde{G} = 0.$$  \hspace{1cm} (1.8)

If symmetry of $F$ is required, one also finds

$$(G^\pm)^t = G^\mp,$$  \hspace{1cm} (1.9a)

and hence

$$(\tilde{G})^t = -\tilde{G}.$$  \hspace{1cm} (1.10a)

When indices are used, the above properties read

$$G^{ij} = G^{-ji}, \ G^{-ij} = G^{+ji},$$  \hspace{1cm} (1.9b)
\[ \tilde{G}^{ij} = -\tilde{G}^{ji}, \quad (1.10b) \]

because
\[ G^{\pm ij} - \tilde{G}^{\mp ji} = G^{\pm ik}(F_{kl} - F_{lk})G^{\mp jl}, \quad (1.11) \]

and \( F_{kl} \) is symmetric. These properties show that, on defining \( \delta^\pm_A B \equiv \varepsilon B, G^{\pm ij} A, j \), one has, on relabelling dummy indices,
\[ \delta^\pm_A B = \varepsilon B, G^{\pm ji} A, i = \varepsilon A, G^{\mp ij} B, j = \delta^\mp_B A. \quad (1.12) \]

These are the reciprocity relations, which express the idea that the retarded (resp. advanced) effect of \( A \) on \( B \) equals the advanced (resp. retarded) effect of \( B \) on \( A \). Another cornerstone of the formalism is a relation involving the Green function \( \hat{G} \) of the operator \( -\hat{F} \), having set \( R^k_{\beta} R^k_{\alpha} \equiv \hat{F}_{\beta\alpha} \); this is
\[ R \hat{G}^{\pm \gamma} = G^{\pm \gamma} R, \quad (1.13a) \]

which, on using indices in the condensed notation, reads
\[ R^i_{\alpha} \hat{G}^{\pm \alpha \beta} \tilde{\gamma}_{\beta \delta} = R^i_{\alpha} \hat{G}^{\pm \alpha \delta} = G^{\pm ij} \gamma_{jk} R^k_{\delta} = G^{\pm ij} R_{j\delta}. \quad (1.13b) \]

This holds because, for background fields satisfying the field equations, one finds that
\[ F_{ik} R^k_{\alpha} = R^i_{\beta} R_{k\beta} R^k_{\alpha} = R^i_{\beta} \hat{F}_{\beta\alpha}. \quad (1.14) \]

On multiplying this equation on the left by \( G^{\pm ji} \) and on the right by \( \hat{G}^{\pm \alpha \beta} \) one gets
\[ R^i_{\alpha} \hat{G}^{\pm \alpha \beta} = G^{\pm ji} R^i_{\beta}, \quad (1.15) \]

i.e. the desired formula (1.13b) is proved. Moreover, by virtue of (1.9b), the transposed equations
\[ \hat{G}^{\pm \alpha \beta} R^j_{\beta} = R^j_{\alpha} G^{\pm ij} \quad (1.16) \]
also hold. We are now in a position to define the Peierls bracket of any two observables $A$ and $B$. First, we consider the operation

$$D_A B \equiv \lim_{\varepsilon \to 0} \varepsilon^{-1} \delta^-_A B, \quad (1.17)$$

with $D_B A$ obtained by interchanging $A$ with $B$ in (1.17). The Peierls bracket of $A$ and $B$ is then defined by

$$(A, B) \equiv D_A B - D_B A = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[ \varepsilon A_1 G^+ B_1 - \varepsilon A_1 G^- B_1 \right] = A_1 \tilde{G} B_1 = A, i \tilde{G}^{ij} B_j, \quad (1.18)$$

where we have used (1.12) and (1.17) to obtain the last expression. Following DeWitt [7], it should be stressed that the Peierls bracket depends only on the behaviour of infinitesimal disturbances. This is not the same, however, as saying that quantum theory is a theory of infinitesimal disturbances of the underlying classical theory! This view would not take into account factor ordering problems in the evaluation of quantum commutators, nor the existence of non-classical phase effects. Nevertheless, DeWitt could show that quantum theory can be regarded as a theory of “finite but small” disturbances of the classical theory, and he stressed that the exact theory is indeed completely determined by the behaviour of infinitesimal disturbances.

In classical mechanics, following Peierls [6], we may arrive at the derivatives in (1.17) and (1.18) starting from the action functional $S \equiv \int L d\tau$ and considering the extremals of $S$ and those of $S + \lambda A$, where $\lambda$ is an infinitesimal parameter and $A$ any function of the path $\gamma$. Next we consider solutions of the modified equations as expansions in powers of $\lambda$, and hence the new set of solutions to first order reads

$$\gamma'(\tau) = \gamma(\tau) + \lambda D_A \gamma(\tau). \quad (1.19)$$

This modified solution is required to obey the condition that, in the distant past, it should be identical with the original one, i.e.

$$D_A \gamma(\tau) \to 0 \text{ as } \tau \to -\infty. \quad (1.20)$$
Similarly to the construction of the above “retarded” solution, we may define an “advanced” solution
\[ \gamma''(\tau) = \gamma(\tau) + \lambda D_A \gamma(\tau), \quad (1.21) \]
such that
\[ D_A \gamma(\tau) \to 0 \text{ as } \tau \to +\infty. \quad (1.22) \]

From these modified solutions one can now find \( D_A \gamma(\tau) \) along the solutions of the unmodified action and therefore, to first order, the changes in any other function \( B \) of the field variables, and these are denoted by \( D_A B \) and \( D_B A \), respectively.

2. Mathematical properties of Peierls brackets

We are now aiming to prove that \((A, B)\) satisfies all properties of a Poisson bracket. The first two are indeed obvious:
\[ (A, B) = -(B, A), \quad (2.1) \]
\[ (A, B + C) = (A, B) + (A, C), \quad (2.2) \]
whereas the proof of the Jacobi identity is not obvious and is therefore presented in detail.

First, by repeated application of (1.18) one finds
\[ P(A, B, C) \equiv (A, (B, C)) + (B, (C, A)) + (C, (A, B)) \]
\[ = A_{i,\ell} \tilde{G}^{i\ell} \left( B_{j} \tilde{G}^{jk} C_{k} \right)_{,\ell} + B_{j} \tilde{G}^{i\ell} \left( C_{k} \tilde{G}^{ki} A_{i} \right)_{,\ell} + C_{k} \tilde{G}^{kl} \left( A_{j} \tilde{G}^{ij} B_{j} \right)_{,l} \]
\[ = A_{i,\ell} B_{j} C_{k} \left( \tilde{G}^{ij} \tilde{G}^{kl} + \tilde{G}^{ji} \tilde{G}^{ki} \right) + A_{i} B_{j l} C_{k} \left( \tilde{G}^{j k} \tilde{G}^{i\ell} + \tilde{G}^{k l} \tilde{G}^{i j} \right) \]
\[ + A_{i} B_{j} C_{k, l} \left( \tilde{G}^{i j} \tilde{G}^{k l} + \tilde{G}^{i l} \tilde{G}^{j k} \right) \]
\[ + A_{i} B_{j} C_{k} \left( \tilde{G}^{i l} \tilde{G}^{j k}_{,l} + \tilde{G}^{i j} \tilde{G}^{k l}_{,l} + \tilde{G}^{k l} \tilde{G}^{i j}_{,l} \right). \quad (2.3) \]

Now the antisymmetry property (1.10b), jointly with commutation of functional derivatives: \( T_{,il} = T_{,li} \) for all \( T = A, B, C \), implies that the first three terms on the last equality in (2.3) vanish. For example one finds
\[ A_{i,\ell} B_{j} C_{k} \left( \tilde{G}^{ij} \tilde{G}^{kl} + \tilde{G}^{ji} \tilde{G}^{ki} \right) = A_{i,\ell} B_{j} C_{k} \left( \tilde{G}^{ij} \tilde{G}^{kl} + \tilde{G}^{ji} \tilde{G}^{kl} \right) \]
\[ = -A_{i,\ell} B_{j} C_{k} \left( \tilde{G}^{ji} \tilde{G}^{ik} + \tilde{G}^{ij} \tilde{G}^{kl} \right) = 0, \quad (2.4) \]
and an entirely analogous procedure can be applied to the terms containing the second functional derivatives $B_{jl}$ and $C_{kl}$. The last term in (2.3) requires new calculations because it contains functional derivatives of $\tilde{G}^{ij}$. These can be dealt with after taking infinitesimal variations of the equation $F_{ik} G^{\pm kj} = -\delta_i^j$, so that

$$F \delta G^{\pm} = -(\delta F) G^{\pm},$$

and hence

$$G^{\pm} F \delta G^{\pm} = F G^{\pm} \delta G^{\pm} = -\delta G^{\pm} = -G^{\pm} (\delta F) G^{\pm},$$

i.e.

$$\delta G^{\pm} = G^{\pm} (\delta F) G^{\pm}. \tag{2.6b}$$

Thus, the desired functional derivatives of advanced and retarded Green functions read

$$G^{\pm ij}_{,c} = G^{\pm ia} F_{ab,c} G^{\pm bj} = G^{\pm ia} \left( S_{,ab} + R_{\alpha a} R_{\beta b}^{\alpha} \right)_{,c} G^{\pm bj}$$

$$= G^{\pm ia} S_{,abc} G^{\pm bj} + G^{\pm ia} R_{\alpha a,c} R_{\beta b}^{\alpha} G^{\pm bj} + G^{\pm ia} R_{\alpha a,b} R_{\beta c}^{\alpha} G^{\pm bj}. \tag{2.7}$$

In this formula the contractions $R_{\beta b}^{\alpha} G^{\pm bj}$ and $G^{\pm ja} R_{\alpha a}$ can be re-expressed with the help of Eqs. (1.15) and (1.16), and eventually one gets

$$G^{\pm ij}_{,c} = G^{\pm ia} S_{,abc} G^{\pm bj} + G^{\pm ia} R_{\alpha a,c} \tilde{G}^{\pm \alpha \beta} R_{\beta j}^{\alpha} + R_{\alpha \beta} \tilde{G}^{\pm \beta} R_{\beta b}^{\alpha} G^{\pm bj}. \tag{2.8}$$

By virtue of the group invariance property satisfied by all physical observables, the second and third term on the right-hand side of Eq. (2.8) give vanishing contribution to (2.3). One is therefore left with the contributions involving third functional derivatives of the action. Bearing in mind that $S_{,abc} = S_{,acb} = S_{,bca} = \ldots$, one can relabel indices summed over, finding eventually (upon using (1.9b))

$$P(A,B,C) = A_i B_j C_k \left[ (G^{+ic} - G^{-ic})(G^{+ja} G^{+bk} - G^{-ja} G^{-bk}) \right.$$}

$$+ (G^{+jc} - G^{-jc})(G^{+ka} G^{+bi} - G^{-ka} G^{-bi}) \right.$$

$$+ (G^{+kc} - G^{-kc})(G^{+ia} G^{+bj} - G^{-ia} G^{-bj}) \right] S_{,abc}$$

$$= A_i B_j C_k \left[ (G^{+ia} - G^{-ia})(G^{+jb} G^{-kc} - G^{-jb} G^{+kc}) \right.$$}

$$+ (G^{+jb} - G^{-jb})(G^{+kc} G^{-ia} - G^{-kc} G^{+ia}) \right.$$}

$$+ (G^{+kc} - G^{-kc})(G^{+ia} G^{-jb} - G^{-ia} G^{+jb}) \right] S_{,abc} = 0. \tag{2.9}$$

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This sum vanishes because it involves six pairs of triple products of Green functions with opposite signs, i.e.

\[ G^{+ia}G^{-jb}G^{-kc}, \quad G^{-ia}G^{-jb}G^{+kc}, \quad G^{+jb}G^{+kc}G^{-ia}, \]
\[ G^{-jb}G^{-kc}G^{+ia}, \quad G^{+kc}G^{+ia}G^{-jb}, \quad G^{-kc}G^{-ia}G^{+jb}. \]

The Jacobi identity is therefore fulfilled. Moreover, the fourth fundamental property of Poisson brackets, i.e.

\[ (A, BC) = (A, B)C + B(A, C) \] (2.10)

is also satisfied, because

\[ (A, BC) = A, i \tilde{G}^{ik}(BC), k = A, i \tilde{G}^{ik}B, k C + BA, i \tilde{G}^{ik}C, k = (A, B)C + B(A, C). \] (2.11)

Thus, the Peierls bracket defined in (1.18) is indeed a Poisson bracket of physical observables. Equation (2.10) can be regarded as a compatibility condition of the Peierls bracket with the product of physical observables.

It should be stressed that the idea of Peierls [6] was to introduce a bracket related directly to the action principle without making any reference to the Hamiltonian. This implies that even classical mechanics should be considered as a “field theory” in a zero-dimensional space, having only the time dimension. This means that one deals with an infinite-dimensional space of paths \( \gamma : R \to Q \), therefore we are dealing with functional derivatives and distributions even in this situation where modern standard treatments rely upon \( C^\infty \) manifolds and smooth structures. Thus, the present treatment is hiding most technicalities involving infinite-dimensional manifolds. In finite dimensions on a smooth manifold, any bracket satisfying (2.2) and (2.10) is associated with first-order bidifferential operators [8,9]; in this proof it is important that the commutative and associative product \( BC \) is a local product. In any case these brackets at the classical level could be a starting point to define a \( * \)-product in the spirit of non-commutative geometry [10] or deformation quantization [11].
2.1 The most general Peierls bracket

The Peierls bracket is a group invariant by construction, being defined for pairs of observables which are group invariant, and is invariant under both infinitesimal and finite changes in the matrices $\gamma_{ij}$ and $\tilde{\gamma}_{\alpha\beta}$. DeWitt [5] went on to prove that, even if independent differential operators $P^\alpha_i$ and $Q^i_{i\alpha}$ are introduced such that

$$F_{ij} \equiv S_{ij} + P^\alpha_i Q^j_{\alpha}, \quad (2.12)$$

$$\hat{F}_{\alpha\beta} \equiv Q^i_{i\alpha} R^i_{\beta}, \quad (2.13)$$

$$F^\beta_{\alpha} \equiv R^i_{i\alpha} P^\beta_i, \quad (2.14)$$

are all non-singular, with unique advanced and retarded Green functions, the reciprocity theorem expressed by (1.12) still holds, and the resulting Peierls bracket is invariant under changes in the $P^\alpha_i$ and $Q^i_{i\alpha}$, by virtue of the identities

$$Q^i_{i\alpha} G^{\pm ij}_{\alpha} = G^{\pm \beta}_{\alpha} R^j_{\beta}, \quad (2.15)$$

$$G^{\pm ij}_{\alpha} P^j_{\beta} = R^i_{\alpha} \hat{G}^{\pm \alpha\beta}. \quad (2.16)$$

This is proved as follows. The composition of $F_{ik}$ with the infinitesimal generators of gauge transformations yields

$$F_{ik} R^k_{\alpha} = P^\beta_i F^i_{\beta\alpha}, \quad (2.17)$$

and hence

$$G^{\pm ji}_{ik} R^k_{\alpha} = -R^j_{\alpha} = G^{\pm ji}_{i\alpha} P^\gamma_i F^j_{\gamma\alpha}, \quad (2.18)$$

which implies

$$R^j_{\alpha} G^{\pm \alpha\beta} = -G^{\pm ji}_{i\alpha} P^\gamma_i G^{\pm \alpha\beta} = G^{\pm ji}_{i\alpha} P^\beta_i, \quad (2.19)$$

i.e. Eq. (2.16) is obtained. Similarly,

$$R^i_{\alpha} F_{ij} = F^\beta_i Q^j_{\beta}, \quad (2.20)$$

and hence

$$G^{\pm \gamma}_{\alpha} R^i_{\gamma} F_{ij} = -Q^i_{j\alpha}, \quad (2.21)$$
which implies

\[ Q_{i\alpha} G^{\pm ij} = -G^{\pm}_{\alpha} \gamma R^{k}_{\gamma} F_{kIJ} G^{\pm ij} = G^{\pm}_{\alpha} \beta R^{j}_{\beta}, \tag{2.22} \]

i.e. Eq. (2.15) is obtained. Now we use the first line of Eq. (1.12) for \( \delta^{\pm}_{A} B \), jointly with

\[ G^{\pm ij} = G^{\mp ji} + G^{\pm k}(F_{kl} - F_{lk})G^{\mp jl}, \tag{2.23} \]

so that

\[ \delta^{\pm}_{A} B - \varepsilon B_{i} G^{\mp ji} A, j = \varepsilon B_{i} R^{i}_{\gamma} G^{\pm \alpha \gamma} Q_{\alpha} G^{\mp jl} A, j - \varepsilon B_{i} P^{\alpha}_{l} G^{\pm ik} Q_{k \alpha} G^{\mp jl} A, j. \tag{2.24} \]

Since \( B \) is an observable by hypothesis, the first term on the right-hand side of (2.24) vanishes. Moreover one finds, from (2.16)

\[ G^{\pm ik} P^{\alpha}_{l} Q_{k \alpha} G^{\mp jl} = G^{\pm il} R^{j}_{\beta} G^{\mp \beta \alpha} Q_{l \alpha}. \tag{2.25} \]

and hence also the second term on the right-hand side of (2.24) vanishes (\( A \) being an observable, for which \( R^{j}_{\beta} A, j = 0 \)), yielding eventually the reciprocity relation (1.12). Moreover, the invariance of the Peierls bracket under variations of \( P_{i\alpha} \) and \( Q_{i}^{\alpha} \) holds because

\[ \delta (\delta^{\pm}_{A} B) = \varepsilon B_{i} \delta G^{\pm ij} A, j = \varepsilon B_{i} G^{\pm ik} (\delta F_{kl}) G^{\pm lj} A, j \]

\[ = \varepsilon B_{i} G^{\pm ik} [(\delta P^{\alpha}_{k}) Q_{l \alpha} + P^{\alpha}_{k} (\delta Q_{l \alpha})] G^{\pm lj} A, j \]

\[ = \varepsilon B_{i} G^{\pm ik} (\delta P^{\alpha}_{k}) Q_{l \alpha} G^{\pm lj} A, j + \varepsilon B_{i} G^{\pm ik} P^{\alpha}_{k} (\delta Q_{l \alpha}) G^{\pm lj} A, j \]

\[ = \varepsilon B_{i} G^{\pm ik} (\delta P^{\alpha}_{k}) G^{\pm \beta} R^{j}_{\beta} A, j + \varepsilon B_{i} R^{i}_{\gamma} G^{\pm \gamma \alpha} (\delta Q_{l \alpha}) G^{\pm lj} A, j = 0, \tag{2.26} \]

where Eqs. (2.15) and (2.16) have been exploited once more.

3. Point Lagrangians

A basic question is whether the Peierls-bracket formalism is equivalent to the conventional canonical formalism when the latter exists. This is indeed the case, and the proof is given as follows in the case of point Lagrangians, relying upon the work in Ref. [5].
Let us consider a physical system possessing only a finite number of degrees of freedom, with Lagrangian $L$ depending on positions $q$ and velocities $v$. We use second-order formalism, *assuming* therefore that

$$v^i = \frac{d}{dt} q^i = \dot{q}^i. \quad (3.1)$$

We also assume that the equations defining canonical momenta $p_i$ as derivatives of $L$ with respect to $\dot{q}^i$ can be solved for $\dot{q}^i$ in terms of the $p_i$ and $q^i$, so that the Hessian matrix is non-singular. Our action functional is therefore

$$S = \int L(q, \dot{q}) dt,$$

whose second variation reads

$$\delta^2 S = \delta(\delta S) = \delta \int \left[ \frac{\delta L}{\delta q^i} \delta q^i + \frac{\delta L}{\delta \dot{q}^i} \delta \dot{q}^i \right]$$

$$= \int \left[ \frac{\delta^2 L}{\delta q^i \delta q^j} \delta q^i \delta q^j + \frac{\delta^2 L}{\delta q^i \delta \dot{q}^j} \delta q^i \delta \dot{q}^j + \frac{\delta^2 L}{\delta \dot{q}^i \delta q^j} \delta \dot{q}^i \delta q^j + \frac{\delta^2 L}{\delta \dot{q}^i \delta \dot{q}^j} \delta \dot{q}^i \delta \dot{q}^j \right]. \quad (3.2)$$

If we perform an integration by parts with respect to $\delta \dot{q}^i$, we obtain

$$\text{integrand of (3.2)} = \frac{\delta^2 L}{\delta q^i \delta q^j} \delta q^i \delta q^j - \frac{\delta^2 L}{\delta q^i \delta \dot{q}^j} \delta q^i \delta \dot{q}^j + \frac{d}{dt} \left[ \frac{\delta^2 L}{\delta \dot{q}^i \delta \dot{q}^j} \delta \dot{q}^i \delta \dot{q}^j \right]$$

$$- \frac{d}{dt} \left( \frac{\delta^2 L}{\delta q^i \delta \dot{q}^j} \right) \delta q^i \delta \dot{q}^j - \frac{\delta^2 L}{\delta \dot{q}^i \delta q^j} \delta \dot{q}^i \delta q^j + \frac{d}{dt}\left[ \frac{\delta^2 L}{\delta \dot{q}^i \delta \dot{q}^j} \delta \dot{q}^i \delta \dot{q}^j \right]$$

$$- \frac{d}{dt} \frac{\delta^2 L}{\delta q^i \delta \dot{q}^j} \delta q^i \delta \dot{q}^j - \frac{\delta^2 L}{\delta \dot{q}^i \delta \dot{q}^j} \delta \dot{q}^i \delta \dot{q}^j$$

$$= \left[ \frac{\delta^2 L}{\delta q^i \delta q^j} - \frac{d}{dt} \frac{\delta^2 L}{\delta q^i \delta \dot{q}^j} \right] \delta q^i \delta q^j$$

$$+ \left[ \frac{\delta^2 L}{\delta q^i \delta \dot{q}^j} - \frac{\delta^2 L}{\delta \dot{q}^i \delta q^j} - \frac{d}{dt} \frac{\delta^2 L}{\delta \dot{q}^i \delta \dot{q}^j} \right] \delta q^i \delta \dot{q}^j$$

$$+ \frac{d}{dt} \left[ \frac{\delta^2 L}{\delta \dot{q}^i \delta q^j} \delta \dot{q}^i \delta q^j + \frac{\delta^2 L}{\delta \dot{q}^i \delta \dot{q}^j} \delta \dot{q}^i \delta \dot{q}^j \right]. \quad (3.3)$$
The total derivative can be discarded in the previous equation, and hence we find

\[ \text{integrand of (3.2)} = A_{ij} \delta q^i \delta q^j' + B_{ij} \delta q^i \delta q^j' + C_{ij} \delta q^i \delta q^j', \]  

(3.4)

with obvious definition of \( A_{ij}, B_{ij}, \) and \( C_{ij} \) from the last three lines of (3.3). We can modify Eq. (3.4) using only un-primed indices by virtue of the following relations:

\[ \delta q^j' = \delta_j^j \left[ \frac{\partial \delta(t, t')}{\partial t} \delta q^j + \delta(t, t') \delta q^j \right], \]

(3.5)

\[ \delta q^j'' = \delta_j^j \left[ \frac{\partial^2 \delta(t, t')}{\partial t^2} \delta q^j + \frac{2}{\partial t} \delta(t, t') \delta q^j + \delta(t, t') \delta q^j \right]. \]

(3.6)

After substituting into Eq. (3.4), we find eventually

\[ \text{integrand of (3.2)} = \left[ \left( A_{ij} \delta(t, t') + B_{ij} \frac{\partial \delta(t, t')}{\partial t} + C_{ij} \frac{\partial^2 \delta(t, t')}{\partial t^2} \right) \delta q^j \right. \]

\[ + \left. \left( B_{ij} \delta(t, t') + 2C_{ij} \frac{\partial \delta(t, t')}{\partial t} \right) \delta q^j + C_{ij} \delta(t, t') \delta q^j \right] \delta q^i. \]

(3.7)

The result (3.8) yields the desired formula for the second functional derivative of the action, i.e.

\[ S_{ij'} = A_{ij} \delta(t, t') + B_{ij} \frac{\partial \delta(t, t')}{\partial t} + C_{ij} \frac{\partial^2 \delta(t, t')}{\partial t^2} \delta(t, t'). \]

(3.9)

Since \( S_{ik} G^{\pm kj} = -\delta_k^j \), the equation for the Green function reads

\[ A_{ik} G^{\pm kj} + B_{ik} \delta^{kj} + C_{ik} \delta^{kj} = -\delta_k^j. \]

(3.10)

Now it is crucial that \( C_{ij} \equiv -\frac{\partial^2 L}{\partial q^i \partial q^j} \) should have an inverse according to our assumptions (when this is not the case we are forced to use the constraint approach \([12,13]\) of Dirac and Bergmann). This makes it possible to write the solution of Eq. (3.10) as \( |t - t'| \to 0 \) in the form

\[ G^{\pm ij'} = -\theta(t', t)(t' - t) C^{-1} \delta_i^j + O(t - t')^2, \]

(3.11)
\[ G^{-ij'} = -\theta(t, t')(t - t')C^{-1i'j'} + O(t - t')^2, \]  
(3.12)

so that the “super-commutator function” (1.5) is given by

\[ \tilde{G}^{ij} = (t - t')C^{-1i'j'} + O(t - t')^2. \]  
(3.13)

By construction, \( \tilde{G}^{ij} \) solves the homogeneous equation

\[ A_{ik}\tilde{G}^{kj'} + B_{ik}\dot{\tilde{G}}^{kj'} + C_{ik}\ddot{\tilde{G}}^{kj'} = 0, \]  
(3.14)

and by inserting (3.13) into (3.14) one finds

\[ B_{i'k'}C^{-1k'j'} + C_{i'k'}\frac{\partial^2}{\partial t^2}\tilde{G}^{k'j'} = O(t - t'), \]  
(3.15)

which is solved by (cf. (3.13))

\[ \tilde{G}^{ij'} = (t - t')C^{-1i'j'} - \frac{1}{2}(t - t')^2C^{-1i'k'}B_{k'l'}C^{-1l'j'} + O(t - t')^3. \]  
(3.16)

Now we need the Peierls brackets \((q^i, q^j), (q^i, p_j)\) and \((p_i, p_j)\). The first task is easy, because

\[ (q^i(t), q^j(t)) = \lim_{t'\to t}(q^i(t), q^j(t')) \]
\[ = \lim_{t'\to t} \frac{\partial q^i}{\partial q^k(t)} \frac{\partial q^j}{\partial q^l(t')} = \lim_{t'\to t} \delta^i_k \delta^j_l \]
\[ = \lim_{t'\to t} \tilde{G}^{ij'} = 0, \]  
(3.17)

by virtue of (3.16). The remaining brackets require an intermediate step, i.e. evaluation of the Peierls brackets \((q^i, \dot{q}^i)\) and \((q^i, \ddot{q}^i)\), since for example

\[ (p_i, p_j) = \left( \frac{\partial L}{\partial q^i}, \frac{\partial L}{\partial \dot{q}^j} \right) \]
\[ = \frac{\partial^2 L}{\partial q^i \partial q^k}(q^k, \dot{q}^l) \frac{\partial^2 L}{\partial q^i \partial \dot{q}^j} + \frac{\partial^2 L}{\partial q^i \partial q^k}(q^k, \dot{q}^l) \frac{\partial^2 L}{\partial q^i \partial \dot{q}^j} \]
\[ + \frac{\partial^2 L}{\partial q^i \partial q^k}(q^k, \ddot{q}^l) \frac{\partial^2 L}{\partial q^i \partial \dot{q}^j} + \frac{\partial^2 L}{\partial q^i \partial q^k}(q^k, \ddot{q}^l) \frac{\partial^2 L}{\partial q^i \partial \dot{q}^j}. \]  
(3.18)
Indeed one finds

\[
(q^i, \dot{q}^j) = -(\dot{q}^j, q^i) = - \lim_{t' \to t} \dot{\tilde{G}}_{ji}' = -C^{-1ji} = -C^{-1ij},
\]

(3.19)

\[
(q^i, \dot{q}^j) = \lim_{t' \to t} \frac{\partial^2 \tilde{G}^{ij'}}{\partial t' \partial t'} = C^{-1ik} \left( B_{kl} - \frac{dC_{kl}}{dt} \right) C^{-1lj},
\]

(3.20)

and hence

\[
(q^i, p_j) = C^{-1ik} C_{kj} = \delta_j^i,
\]

(3.21)

\[
(p_i, p_j) = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} - \frac{\partial^2 L}{\partial q^i \partial \dot{q}^j} + B_{ij} - \frac{dC_{ij}}{dt} = 0.
\]

(3.22)

Thus, we fully recover the canonical commutation relations (3.17), (3.21) and (3.22), with Lagrangian sub-spaces corresponding to (3.17) and (3.22). The familiar Hamilton equations can also be recovered [5].

4. Concluding remarks

In agreement with the pedagogical aims of the Londrina school, we have presented a concise introduction to Peierls brackets in theoretical physics. For this purpose, we find it useful to supplement the previous discussion with the following correspondence of structures:

(i) Finite-dimensional manifold \( M \) in classical mechanics vs. infinite-dimensional manifold \( \Phi \) of field configurations.

(ii) Local coordinates \( \{\xi^i\} \) on \( M \) vs. field configurations \( \{\varphi^i\} \) on \( \Phi \).

(iii) Poisson bracket \( \{\xi^i, \xi^j\} = \omega^{ij} \), with \( \omega^{ij} \) invertible, vs. the Peierls bracket

\[
(\varphi^i, \varphi^j) = \varphi^i_k \tilde{G}^{kl} \varphi^j_l = \delta^i_k \tilde{G}^{kl} \delta^j_l = \tilde{G}^{ij}.
\]

(4.1)

(iv) Inverse matrix \( \omega_{ij} \) such that \( \omega_{ij} \omega^{jk} = \delta^k_i \), with associated symplectic form \( \omega \equiv \frac{1}{2} \omega_{ij} d\xi^i \wedge d\xi^j \), vs. inverse of \( \tilde{G}^{ij} \) built as

\[
\tilde{G}^{ik} \gamma_{im} \gamma_{kl} = \tilde{G}_{ml},
\]

(4.2)
for which $\tilde{G}_{m}^{d}k = \delta_{m}^{k}$. 

(v) Symplectic manifold $(M, \omega)$ vs. $(\Phi, \tilde{G})$.

(vi) Functions $f$ on $M$, i.e. $f \in F(M)$, vs. observables $A(\varphi)$ on the set $O(\Phi)$ of all observables on $\Phi$.

(vii) Poisson bracket in local coordinates:

$$\{f, h\} = \frac{\partial f}{\partial \xi^{i}}\omega^{ij}\frac{\partial h}{\partial \xi^{j}} = f_{,i}\omega^{ij}h_{,j}$$

vs. the Peierls bracket

$$(A, B) = A_{,i}\tilde{G}^{ij}B_{,j} = \int dx \int dy \frac{\delta A}{\delta \varphi^{i}(x)}\tilde{G}^{ij}(x, y)\frac{\delta B}{\delta \varphi^{j}(y)}. \quad (4.3)$$

(viii) Differential of $f$, i.e. $df = \frac{\partial f}{\partial \xi}d\xi = f_{,i}d\xi^{i}$, vs. variation of the functional $A(\varphi)$:

$$\delta A = \int \frac{\delta A}{\delta \varphi^{i}}d\varphi^{i}dx = A_{,i}\delta \varphi^{i}. \quad (4.4)$$

(ix) Poisson bracket $\{ , \} : F(M) \times F(M) \to F(M)$ vs. Peierls bracket

$$( , ) : O(\Phi) \times O(\Phi) \to O(\Phi).$$

Current applications of Peierls brackets deal with string theory [14,15], path integration and decoherence [16], supersymmetric proof of the index theorem [17], classical dynamical systems involving parafermionic and parabosonic dynamical variables [18], while for recent literature on covariant approaches to a canonical formulation of field theories we refer the reader to the work in Refs. [19-24].

In the infinite-dimensional setting which, strictly, applies also to classical mechanics, as we stressed at the end of section 2, we hope to elucidate the relation between a covariant description of dynamics as obtained from the kernel of the symplectic form, and a parametrized description of dynamics as obtained from any Poisson bracket, including the Peierls bracket.
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