Splitting of folded strings in AdS$_3$

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Abstract

In this paper we present semiclassical computations of the splitting of folded spinning strings in AdS$_3$, which may be of interest in the context of AdS/CFT duality. We start with a classical closed string and assume that it can split into two closed string fragments, if at a given time two points on it coincide in target space and their velocities agree. First we consider the case of the folded string with large spin. Assuming the formal large-spin approximation of the folded string solution in AdS$_3$, we can completely describe the process of splitting: compute the full set of charges and obtain the string solutions describing the evolution of the final states. We find that, in this limit, the world surface does not change in the process and the final states are described by the solutions of the same type as the initial string, i.e. the formal large-spin approximation of the folded string in AdS$_3$. Then we consider the general case — splitting of string given by the exact folded string solution. We find the expressions for the charges of the final fragments, the coordinate transformations diagonalizing them and, finally, their energies and spins. Due to the complexity of the initial string profile, we cannot find the solutions describing the evolution of the final fragments, but we can predict their qualitative behavior. We also generalize the results to include circular rotations and windings in $S^5$.

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1 Introduction

Decay properties of massive strings have been studied for a long time [1, 2, 3]. Classical string solutions have proved to be a useful tool for exploring the AdS/CFT correspondence in the sector of large charges [16, 17, 18, 19, 20, 21, 22, 23].

For flat Minkowski space splitting of semiclassical strings was analyzed in detail in [12, 13], for $R_t \times S^3$ space in [14, 15]. There is an obvious lack of results in AdS space, and the purpose of the present paper is to fill this gap. Following the conventional approach, we start with a classical closed string and assume that it can split into two fragments, if at a given time $\tau_0$ two points on it coincide in target space and their velocities agree. Closed string periodicity conditions are separately imposed on each of the two final pieces. Initial conditions are defined by the initial string at $\tau_0$. The relations between the energies and spins of the cut fragments — together with “conservation laws” of splitting $E(E_1, E_{II}, ...), S(S_1, S_{II}, ...)$, etc — are completely determined by the charge conservation. Thus they may be found (at least parametrically) for the initial string solution of arbitrary complexity.

Determining the evolution is much more complicated: one has to solve the string equations with the boundary conditions given by a part of the profile of the initial string. At the moment, this is possible only in the simplest cases.

The main purpose of this paper is to investigate splitting of folded spinning string in AdS$_3$ [17]

$$Y_0 + iY_5 = \text{dn}[\kappa \ell^{-1} \sigma, -\ell^2] e^{i\kappa \tau}, \quad Y_1 + iY_2 = \ell \, \text{sn}[\kappa \ell^{-1} \sigma, -\ell^2] e^{i\omega \tau},$$

$$\kappa = \frac{2}{\ell} \, K[-\ell^2], \quad \frac{w^2}{\kappa^2} = 1 + \frac{1}{\ell^2},$$

(1.1)

where $\text{sn}[z, m]$ and $\text{dn}[z, m]$ are Jacobi elliptic functions, $K[z]$ is the complete elliptic integral of the first kind. First we consider the limit of the folded string with large spin. Then solution (1.1) may be approximated by

$$Y_0 + iY_5 = \cosh(\kappa \sigma) e^{i\omega \tau}, \quad Y_1 + iY_2 = \sinh(\kappa \sigma) e^{i\omega \tau}, \quad \kappa = \omega \gg 1.$$  

(1.2)

In this simple case, we can completely describe the process of splitting: compute the full set of charges and find string solutions describing the evolution of the final states. It appeared that when such a string splits, the world surface does not change in the process and the final states are described by the solutions of the same type as (1.2):

$$Y_{1,II0} + iY_{1,II5} = \cosh(\kappa_{1,II} \sigma) e^{i\kappa_{1,II} \tau}, \quad Y_{1,II1} + iY_{1,II2} = \sinh(\kappa_{1,II} \sigma) e^{i\kappa_{1,II} \tau}, \quad \kappa_{1,II} = \kappa \frac{\pi + 2\sigma_0}{2\pi},$$

(1.3)

where $\sigma_0$ parameterizes the coordinate of the splitting point.

In the general case we find expressions for the charges of the final fragments, the coordinate transformations that diagonalize them and, at the end, their energies and spins as the functions of $\ell$ and $\sigma_0$ (in the coordinate system where no non-Cartan components present). These are

$$E_{1,II} = \frac{\sqrt{\lambda}}{2} \sqrt{(\kappa C_{1,II} + \omega S_{1,II})^2 - M_{1,II}^2 (\omega + \kappa)^2 + \frac{\sqrt{\lambda}}{2} \sqrt{(\kappa C_{1,II} - \omega S_{1,II})^2 - M_{1,II}^2 (\omega - \kappa)^2}},$$

$$S_{1,II} = \frac{\sqrt{\lambda}}{2} \sqrt{(\kappa C_{1,II} + \omega S_{1,II})^2 - M_{1,II}^2 (\omega + \kappa)^2} - \frac{\sqrt{\lambda}}{2} \sqrt{(\kappa C_{1,II} - \omega S_{1,II})^2 - M_{1,II}^2 (\omega - \kappa)^2}.$$  

(1.4)
Here

\[ \kappa C_{1,II} = \frac{1}{2} S_{\text{fold}} + \frac{\ell}{\pi} E [am[\kappa \ell^{-1} \sigma_0, -\ell^2], -\ell^2] , \]
\[ \omega S_{1,II} = \frac{1}{2} S_{\text{fold}} + \sqrt{1 + \ell^2} \left( -\frac{2}{\pi} \sigma_0 K[-\ell^2] + E [am[\kappa \ell^{-1} \sigma_0, -\ell^2], -\ell^2] \right) , \]
\[ M_{I,II} = \pm \frac{\ell^2}{\kappa \pi} \text{cn} [\kappa \ell^{-1} \sigma_0, -\ell^2] , \]

where \( E_{\text{fold}} = \sqrt{\lambda} S_{\text{fold}} \) and \( S_{\text{fold}} = \sqrt{\lambda} S_{\text{fold}} \) are the energy and spin of the folded string \([1,1]\); \( E[z] \) and \( \mathbb{E}[z,m] \) are the complete and incomplete elliptic integrals of the second kind, respectively, and \( \text{cn}[z,m] \) is a Jacobi elliptic function. These relations parametrically encode the conservation laws of splitting, namely \( E(E_1, E_{\text{II}}), S(S_1, S_{\text{II}}), \) etc.

Due to the complexity of the folded string profile \([1,1]\), we are unable to find the solutions describing the evolution of the final fragments explicitly. However, we can describe the evolution qualitatively. Let us examine the case of large but not infinitely large (as in \([1,2]\)) spin, with the cut occurring far enough from the string ends for \( \sigma_0 \) to satisfy \( \kappa (\pi/2 - \sigma_0) \gg 1 \). In this limit one expects the final pieces to have almost the standard folded shape \([1,1]\), disturbed by a kink moving along the string, similar to the one observed in flat Minkowski space \([12]\). The kink is a “correction” to the “leading” folded shape of the cut fragments, thus the angle of bending has to depend on the position of the kink. It may be substantial at the string ends but must be small close to the center.

The results obtained for the folded string in AdS\(_3\) generalizes to include circular rotations and windings in S\(^5\). We discuss such a generalization with the example of the string in AdS\(_3 \times \)S\(^3\).

The rest of the paper is organized as follows. In Section 2 we introduce notations and discuss a general approach to studying splitting of classical bosonic closed strings in AdS\(_5 \times \)S\(^5\). Section 3 is a review of the splitting of the folded strings in flat Minkowski space. Section 4 is dedicated to the splitting of Gubser–Klebanov–Polyakov folded strings in AdS\(_3\). The results obtained in AdS\(_3\) are generalized to include circular rotations and windings in S\(^5\) in Section 5.

## 2 Splitting of closed strings in AdS\(_5 \times \)S\(^5\). General formalism.

In this section we discuss a general approach to studying of splitting of classical closed bosonic strings in AdS\(_5 \times \)S\(^5\).

The action for a bosonic string in AdS\(_5 \times \)S\(^5\) reads

\[ I_B = \frac{1}{2} T \int d\tau \int_0^{2\pi} d\sigma (L_{\text{AdS}} + L_S), \quad T = \frac{R^2}{2\pi \alpha'} = \frac{\sqrt{\lambda}}{2\pi}, \]

where

\[ L_{\text{AdS}} = -\partial_a Y_P \partial^a Y^P - \bar{\Lambda} (Y_P Y^P + 1), \quad L_S = -\partial_a X_M \partial^a X_M + \Lambda (X_M X_M - 1). \]

Here \( X_M, M = 1,\ldots, 6 \) and \( Y_P, P = 0,\ldots, 5 \) are embedding coordinates of \( R^6 \) with the Euclidean metric \( \delta_{MN} = (+1,+1,+1,+1,+1,+1) \) in \( L_S \) and of \( R^{2,4} \) with \( \eta_{PQ} = (-1,+1,+1,+1,+1,-1) \) in \( L_{\text{AdS}}, \) respectively \( (Y_P = \eta_{PQ} Y^Q) \). \( \Lambda \) and \( \bar{\Lambda} \) are the Lagrange multipliers imposing the two hypersurface conditions:

\[ \eta_{PQ} Y^P Y^Q = -1 , \quad X_M X_M = 1. \]
The action (2.1) is supplemented with the conformal gauge constraints
\[ Y_P \dot{Y}^P + Y'_P Y'^P + \dot{X}_M X_M + X'_M X'_M = 0, \quad \dot{Y}_P Y'^P + \dot{X}_M X'_M = 0 \] (2.4)
and the closed string periodicity conditions
\[ Y_P(\tau, \sigma + 2\pi) = Y_P(\tau, \sigma), \quad X_M(\tau, \sigma + 2\pi) = X_M(\tau, \sigma). \] (2.5)
The classical equations of motion following from (2.1) are
\[ \partial^\alpha \partial_\alpha Y_P - \tilde{\Lambda} Y_P = 0, \quad \tilde{\Lambda} = \partial^\alpha Y_P \partial_\alpha Y^P, \quad Y_P Y'^P = -1, \]
\[ \partial^\alpha \partial_\alpha X_M + \Lambda X_M = 0, \quad \Lambda = \partial^\alpha X_M \partial_\alpha X_M, \quad X_M X_M = 1. \] (2.6)
The action is invariant under the $SO(2,4)$ and $SO(6)$ rotations with correspondent conserved (on-shell) charges
\[ S_{PQ} = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} (Y_P \dot{Y}_Q - Y_Q \dot{Y}_P), \quad J_{MN} = \sqrt{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} (X_M \dot{X}_N - X_N \dot{X}_M). \] (2.7)
We will be working with “spinning” string solutions which have nonzero values of these charges.
It is useful to solve the constraints (2.3) by choosing an explicit parametrization of the embedding coordinates $Y_P$ and $X_M$, e.g.
\[ Y_{05} = Y_0 + iY_5 = \cosh \rho e^{it}, \]
\[ Y_{12} = Y_1 + iY_2 = \sinh \rho \cos \theta e^{i\phi_1}, \quad Y_{34} = Y_3 + iY_4 = \sinh \rho \sin \theta e^{i\phi_2}; \]
\[ X_{12} = X_1 + iX_2 = \sin \gamma \cos \psi e^{i\varphi_1}, \quad X_{34} = X_3 + iX_4 = \sin \gamma \sin \psi e^{i\varphi_2}, \]
\[ X_{56} = X_5 + iX_6 = \cos \gamma e^{i\varphi_3}. \] (2.8)
The corresponding metrics take the form
\[ ds^2_{AdS_5} = - \cosh^2 \rho \, dt^2 + \cosh^2 \rho \left( d\theta^2 + \cos^2 \theta \, d\phi_1^2 + \sin^2 \theta \, d\phi_2^2 \right) \] (2.10)
\[ ds^2_{S^5} = \cos^2 \gamma \, d\varphi_3^2 + d\gamma^2 + \sin^2 \gamma \left( d\psi^2 + \cos^2 \psi \, d\varphi_1^2 + \sin^2 \psi \, d\varphi_2^2 \right). \] (2.11)
The Cartan generators of $SO(2,4)$ corresponding to the three linear isometries of the $AdS_5$ metric are the translations in the AdS-time $t$ and two angles $\phi_1$ and $\phi_2$ :
\[ S_0 \equiv S_{05} \equiv E = \sqrt{\lambda} \mathcal{E}, \quad S_1 \equiv S_{12} = \sqrt{\lambda} \mathcal{S}_1, \quad S_2 \equiv S_{34} = \sqrt{\lambda} \mathcal{S}_2. \] (2.12)
The Cartan generators of $SO(6)$ corresponding to the three linear isometries of the $S^5$ metric are the translations in the three angles $\varphi_1$, $\varphi_2$ and $\varphi_3$ :
\[ J_1 \equiv J_{12} = \sqrt{\lambda} \mathcal{J}_1, \quad J_2 \equiv J_{34} = \sqrt{\lambda} \mathcal{J}_2, \quad J_3 \equiv J_{56} = \sqrt{\lambda} \mathcal{J}_3. \] (2.13)
Let us consider a string solution
\[ X_M = X_{inM}(\tau, \sigma), \quad Y_P = Y_{inP}(\tau, \sigma) \] (2.14)
The energy and spin $S_n$, $J_k$. We assume that if at a given $\tau_0$ two points on the string coincide in the target space
\[ X_{inM}(\tau_0, \sigma_1) = X_{inM}(\tau_0, \sigma_2) \quad Y_{inP}(\tau_0, \sigma_1) = Y_{inP}(\tau_0, \sigma_2) \quad (2.15) \]
and their velocities agree
\[ \dot{X}_{inM}(\tau_0, \sigma_1) = \dot{X}_{inM}(\tau_0, \sigma_2) \quad \dot{Y}_{inP}(\tau_0, \sigma_1) = \dot{Y}_{inP}(\tau_0, \sigma_2), \quad (2.16) \]
then the string can split into two pieces
\[ \text{fragment I} : \sigma \in (0, \sigma_1) \cup (\sigma_2, 2\pi) \]
\[ \text{fragment II} : \sigma \in (\sigma_1, \sigma_2). \quad (2.17) \]
The behavior of the cut fragments is governed by equations (2.4) and (2.6) with the boundary conditions defined by the initial string at the moment of splitting:
\[ X_{I,M}(\tau_0, \sigma) = X_{inM}(\tau_0, \sigma) \quad Y_{I,P}(\tau_0, \sigma) = Y_{inP}(\tau_0, \sigma) \quad \sigma \in (0, \sigma_1) \cup (\sigma_2, 2\pi); \quad (2.18) \]
\[ X_{II,M}(\tau_0, \sigma) = X_{inM}(\tau_0, \sigma) \quad Y_{II,P}(\tau_0, \sigma) = Y_{inP}(\tau_0, \sigma) \quad \sigma \in (\sigma_1, \sigma_2). \]
The closed string periodicity conditions are imposed on each fragment separately:
\[ X_{I,II,M}(\tau, \sigma) = X_{I,II,M}(\tau, \sigma + 2\pi_{II}) \quad Y_{I,II,P}(\tau, \sigma) = Y_{I,II,P}(\tau, \sigma + 2\pi_{II}), \quad \text{where} \quad 2\pi_1 = 2\pi - (\sigma_2 - \sigma_1) \quad 2\pi_{II} = \sigma_2 - \sigma_1. \quad (2.19) \]
Conditions (2.18) and (2.19) uniquely determine the final states. The relations between the energies ($E_{I,II}$) and spins ($S_{I,II}, J_{I,II}$) of the cut fragments — together with “conservation laws” of splitting $E(E_1, E_{II}, ...)$, $S(S_1, S_{II}, ...)$, etc — are completely determined by the charge conservation. Thus they may be found (at least parametrically) for the initial string solution of arbitrary complexity. Determining the evolution is much more complicated: one has to solve the string equations (2.4), (2.6) with the boundary conditions (2.18) and (2.19). At the moment, this is possible only in the simplest cases.

### 3 Splitting of folded strings in the flat space. A review.

In this section we review splitting of the folded strings in flat Minkowski space [12]. The solution for the folded strings in flat Minkowski space reads
\[ X_0 = \ell \tau, \quad X_1 = \rho \cos \phi = \ell \cos(\sigma) \cos(\tau), \quad X_2 = \rho \sin \phi = \ell \cos(\sigma) \sin(\tau). \quad (3.1) \]
The energy and spin
\[ E = \ell, \quad J = \frac{1}{2} \ell^2 \quad (3.2) \]
obeys the standard Regge relation $E^2 = 2J$.

Any two points on the string parameterized by $\sigma_1$ and $\sigma_2 = 2\pi - \sigma_1$ coincide in the target space and their velocities agree at any given time. Let us assume that at $\tau_0 = 0$ the string splits into two pieces. The cut occurs at $X_1 = \ell \cos(a\pi)$, $X_2 = 0$, i.e. $\sigma_{cut} = a\pi$ and $\sigma_{cut} = 2\pi - a\pi$ :
\[ \text{fragment I} : \sigma \in (0, a\pi) \cup (2\pi - a\pi, 2\pi) \quad 0 < a < \frac{1}{2}; \quad (3.3) \]
\[ \sigma = \pi \]
\[ \sigma = \pi/2 \]
\[ \sigma = 0, 2\pi \]
\[ \sigma = 3\pi/2 \]
\[ \sigma_{cut 1} = \pi a \]
\[ \sigma_{cut 2} = 2\pi - \pi a \]

Figure 1: Splitting of the folded string in the flat space.

Here without loss of generality \( 0 < a < \frac{1}{2} \), i.e. the fragment I is always “smaller” than the fragment II (see schematic plot in figure 1).

Quantum numbers of the fragment I are the energy \((\mathcal{E}_I)\), linear momentum \((P_{I} = \sqrt{\lambda P_{1i}})\) and angular momentum \((J_I)\):

\[
\mathcal{E}_I = P_{10} = 2 \int_0^{\pi a} \frac{d\sigma}{2\pi} \dot{X}_{01} = \ell a , \tag{3.4}
\]

\[
P_{11} = 0 , \tag{3.5}
\]

\[
P_{12} = 2 \int_0^{\pi a} \frac{d\sigma}{2\pi} \dot{X}_{12} = 2 \int_0^{\pi a} \frac{d\sigma}{2\pi} \cos(\sigma) = \frac{\ell \sin(\pi a)}{\pi} , \tag{3.6}
\]

\[
J_I = L_I + S_I = 2 \int_0^{\pi a} \frac{d\sigma}{2\pi} (X_{11} \dot{X}_{12} - \dot{X}_{11} X_{12}) = \ell^2 a \left( \frac{\sin(2\pi a)}{\pi a} - 1 \right) . \tag{3.7}
\]

Here the orbital momentum \((L_I = \sqrt{\lambda L_1})\) and spin \((S_I = \sqrt{\lambda S_1})\) are \(\hbar\)

\[
L_I = \ell^2 a \frac{\sin^2(\pi a)}{(\pi a)^2} , \quad S_I = \ell^2 a \left( \frac{\sin(2\pi a)}{\pi a} - \frac{\sin^2(\pi a)}{(\pi a)^2} - 1 \right) . \tag{3.8}
\]

The mass of the fragment I, i.e. its energy in the center-of-mass system read

\[
M_I^2 = \mathcal{E}_I^2 - P_{1}^2 = \ell^2 \left( a^2 - \frac{\sin^2(\pi a)}{\pi^2} \right) . \tag{3.9}
\]

Orbital momentum is defined as \(L_1 = X_{cm1} P_{12}\), where \(X_{cm1}\) is the coordinate of the center of mass of the string

\[
X_{cm1} = \frac{1}{\pi a} \int_0^{\pi a} d\sigma X_{11} = \frac{\ell \sin(\pi a)}{\pi a} . \]
The conserved charges for the fragment II may be found similarly:

\[ \mathcal{E}_II = \ell(1-a), \quad \mathcal{P}_II = -\frac{\ell \sin(\pi a)}{\pi}, \quad \mathcal{J}_II = \mathcal{L}_II + \mathcal{S}_II, \]

\[ M^2_{II} = \mathcal{E}^2_{II} - \mathcal{P}^2_{II} = \ell^2 \left( (1-a)^2 - \frac{\sin^2(\pi a)}{\pi^2} \right), \]  \hspace{1cm} (3.10)

\[ \mathcal{L}_II = \ell^2 (1-a) \frac{\sin^2(\pi a)}{(\pi(1-a))^2}, \quad \mathcal{S}_II = -\ell^2 (1-a) \left( 1 + \frac{\sin^2(\pi a)}{\pi(1-a)} + \frac{\sin(2\pi a)}{\pi(1-a)} \right). \]

The energy, linear momentum and angular momentum are conserved in the process of splitting:

\[ \mathcal{E}_I + \mathcal{E}_II = \mathcal{E}, \quad \mathcal{P}_{I2} + \mathcal{P}_{II2} = 0, \quad \mathcal{J}_I + \mathcal{J}_II = \mathcal{J}. \]  \hspace{1cm} (3.11)

The string solution describing the evolution of the final states may be found using the general solution for a closed bosonic string in flat Minkowski space. Imposing the boundary and periodicity conditions \([12]\) on it, one finds

\[ X_{I0} = 2\ell \tau, \]

\[ X_{I1} = \frac{\ell \sin(\pi a)}{\pi a} \left( 1 + 2 \sum_{n=1}^{\infty} (-1)^n \frac{\cos(n\tau)}{a} \cos\left(\frac{n\sigma}{a}\right) \right), \]

\[ X_{I2} = \frac{\ell \sin(\pi a)}{\pi a} \left( \tau + 2a \sum_{n=1}^{\infty} \frac{(-1)^n}{n(1-a)^2} \sin\left(\frac{2n\tau}{a}\right) \cos\left(\frac{2n\sigma}{a}\right) \right), \]  \hspace{1cm} (3.12)

where \(-\pi a < \sigma < \pi a\), and

\[ X_{II0} = 2\ell \tau, \]

\[ X_{II1} = -\frac{\ell \sin(\pi a)}{\pi(1-a)} \left( 1 + 2 \sum_{n=1}^{\infty} (-1)^n \frac{\cos\left(\frac{n\tau}{1-a}\right)}{\cos\left(\frac{n\sigma}{1-a}\right)} \right), \]

\[ X_{II2} = -\frac{\ell \sin(\pi a)}{\pi(1-a)} \left( \tau + 2(1-a) \sum_{n=1}^{\infty} \frac{(-1)^n}{n(1-a)^2} \sin\left(\frac{n\tau}{1-a}\right) \cos\left(\frac{n\sigma}{1-a}\right) \right), \]  \hspace{1cm} (3.13)

where \(-\pi(1-a) < \sigma < \pi(1-a)\).

Summing the series up, we obtain

\[ X_{I,II \mu}(\sigma, \tau) = X_{I,II \mu}^+(\sigma^+) + X_{I,II \mu}^-(\sigma^-), \quad \sigma^\pm = \sigma \pm \tau, \]  \hspace{1cm} (3.14)

where

\[ X_{I^0}^{\pm} = \pm \frac{\ell}{2} a \sigma^\pm, \quad X_{I^1}^{\pm} = \frac{\ell}{2} C_I(\sigma^\pm), \quad X_{I^2}^{\pm} = \pm \frac{\ell}{2} \left[ \frac{\sin(a\pi)}{\pi} \sigma^\pm + S_I(\sigma^\pm) \right], \]  \hspace{1cm} (3.15)

\[ C_I(\xi) = \cos(a\xi), \quad S_I(\xi) = \sin(a\xi) - \frac{\sin(a\pi)}{\pi} \xi \quad \text{for } 0 \leq \xi < \pi, \]

\[ C_I(\xi) = \cos(a\xi - 2a\pi), \quad S_I(\xi) = \sin(a\xi - 2a\pi) - \frac{\sin(a\pi)}{\pi} (\xi - \pi) \quad \text{for } \pi \leq \xi < 2\pi \]

and

\[ X_{II^0}^{\pm} = \pm \frac{\ell}{2} (1-a) \sigma^\pm, \quad X_{II^1}^{\pm} = \frac{\ell}{2} C_{II}(\sigma^\pm), \quad X_{II^2}^{\pm} = \pm \frac{\ell}{2} \left[ -\frac{\sin(a\pi)}{\pi} \sigma^\pm + S_{II}(\sigma^\pm) \right], \]  \hspace{1cm} (3.16)

\[ C_{II}(\xi) = \cos((1-a)\xi + a\pi), \quad S_{II}(\xi) = \sin((1-a)\xi + a\pi) + \frac{\sin(a\pi)}{\pi} \xi \quad \text{for } 0 \leq \xi < 2\pi. \]
In the expressions (3.15) and (3.16), the world-sheet parameters are rescaled as

fragment I: \( \tau, \sigma \rightarrow a \tau, a \sigma \)

fragment II: \( \tau, \sigma \rightarrow (1 - a) \tau, (1 - a) \sigma \).  

(3.17)

The derivatives \( X_I', X_{II}' \), \( i = 1, 2 \) have discontinuities at the points of splitting, i.e. at \( \sigma^\pm = \pi \) for the fragment I and \( \sigma^\pm = 0 \) for the fragment II. These discontinuities show up as an angular bending on the folded shape of the strings moving along the strings as a function of \( \tau \) (for more details see the original paper [12]). Equations (2.6) are satisfied at each point on the string, in spite of the discontinuity. The \( \delta \)–functions arising from the second derivative \( \partial_{\sigma,\sigma} X_I, X_{II} \) cancel with those coming from \( \partial_{\tau,\tau} X_I, X_{II} \), due to the chiral properties of (3.14).

4 Splitting of folded strings in AdS$_3$.

In this section we discuss splitting of Gubser-Klebanov-Polyakov folded spinning strings in AdS$_3$.

4.1 Folded string in AdS$_3$

The folded string solution in the AdS$_3$ in the embedding coordinates read [17]

\[
Y_{05} = \cosh \rho \, e^{i \kappa \tau}, \quad Y_{12} = \sinh \rho \, e^{i \omega \tau},
\]

(4.1)

where

\[
\sinh \rho = \ell \, \text{sn}[\kappa \ell^{-1} \sigma, -\ell^2], \quad \cosh \rho = \text{dn}[\kappa \ell^{-1} \sigma, -\ell^2], \quad \frac{w^2}{\kappa^2} = 1 + \frac{1}{\ell^2}.
\]

(4.2)

Here \( \text{sn}[z, m] \) and \( \text{dn}[z, m] \) are the Jacobi elliptic functions, \( \ell \) defines the length of the string: \( \sinh \rho_{\text{max}} = \ell \).

Expressions (4.2) are valid on the interval \( 0 \leq \sigma < \frac{\pi}{2} \) only. To get the formal periodic solution on the interval \( 0 \leq \sigma < 2\pi \) one has to combine four stretches of (4.2):

\[
Y_{05} = \cosh \rho(\sigma)e^{i \kappa \tau}, \quad Y_{12} = \sinh \rho(\sigma)e^{i \omega \tau} \quad \text{for} \quad \sigma \in [0, \frac{\pi}{2})
\]

\[
Y_{05} = \cosh \rho(\pi - \sigma)e^{i \kappa \tau}, \quad Y_{12} = \sinh \rho(\pi - \sigma)e^{i \omega \tau} \quad \text{for} \quad \sigma \in \left[\frac{\pi}{2}, \pi\right)
\]

\[
Y_{05} = \cosh \rho(\pi - \sigma)e^{i \kappa \tau}, \quad Y_{12} = -\sinh \rho(\pi - \sigma)e^{i \omega \tau} \quad \text{for} \quad \sigma \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]
\]

\[
Y_{05} = \cosh \rho(2\pi - \sigma)e^{i \kappa \tau}, \quad Y_{12} = -\sinh \rho(2\pi - \sigma)e^{i \omega \tau} \quad \text{for} \quad \sigma \in \left[\frac{3\pi}{2}, 2\pi\right).
\]

(4.3)

and impose

\[
Y_P(\sigma + 2\pi) = Y_P(\sigma).
\]

(4.4)

The closed string periodicity conditions require

\[
\kappa = \frac{2}{\pi} \ell \, \mathbb{K}[-\ell^2].
\]

(4.5)

The energy and spin are

\[
\mathcal{E} = \frac{2}{\pi} \ell \, \mathbb{E}[-\ell^2], \quad \mathcal{S} = \frac{2}{\pi} \sqrt{1 + \ell^2} \left( \mathbb{E}[-\ell^2] - \mathbb{K}[-\ell^2] \right).
\]

(4.6)

Here \( \mathbb{K}[z] \) and \( \mathbb{E}[z] \) are the complete elliptic integrals of the first and second kinds, respectively.
The classical energy of the string in the limit of large spin is

\[ E \simeq S + \frac{\sqrt{\lambda}}{\pi} \ln \frac{S}{\sqrt{\lambda}} + \ldots, \quad \frac{S}{\sqrt{\lambda}} \gg 1. \]  

(4.7)

4.2 Large-spin limit. Formal \( \kappa = \omega \) approximation.

There is a useful simplification of the solution (4.2), when the spin of the folded string is large:

\[ \rho = \kappa \sigma, \quad \kappa = \omega \gg 1. \]  

(4.8)

This is a formal limit, as \( \kappa \rightarrow \omega \) implies \( \ell \rightarrow \infty \).

Expansion of the classical energy in large \( S \) is consistent with the one coming from (4.6) in the first two orders:

\[ E \simeq S + \frac{\sqrt{\lambda}}{\pi} \ln \frac{S}{\sqrt{\lambda}} + \ldots, \quad \frac{S}{\sqrt{\lambda}} \gg 1. \]  

(4.10)

Any two points on the string parameterized by \( \sigma_1 \) and \( \sigma_2 = \pi - \sigma_1 \) coincide in the target space and their velocities agree at any given time. Let us assume that at \( \tau_0 = 0 \) the string splits into two pieces. The cut occurs at \( \rho = \kappa \sigma_0 \), i.e. \( \sigma_{cut1} = \sigma_0 \) and \( \sigma_{cut2} = \pi - \sigma_0 \)

fragment I: \( \sigma \in (\sigma_0, \pi - \sigma_0) \), 
fragment II: \( \sigma \in (0, \sigma_0) \cup (\pi - \sigma_0, 2\pi) \), \( 0 < \sigma_0 < \frac{\pi}{2} \).  

(4.11)

Here without loss of generality \( 0 < \sigma_0 < \frac{\pi}{2} \), i.e. the fragment I is always smaller than the fragment II (see schematic plot in figure 2).

Approximation (4.8) is invalid close to the string ends, thus we have to demand

\[ \frac{\pi}{2} - \sigma_0 \gg \frac{1}{\kappa}. \]  

(4.12)

The charges \( (S_{I,H})_{PQ} \) of the cut fragments read

\[ (S_{I,H})_{05} = \frac{\kappa}{2\pi} \left( \frac{\pi}{2} + \sigma_0 \right) + \frac{\sinh(\kappa \pi) + \sinh(2\kappa \sigma_0)}{4\pi}, \] 
\[ (S_{I,H})_{12} = -\frac{\kappa}{2\pi} \left( \frac{\pi}{2} + \sigma_0 \right) + \frac{\sinh(\kappa \pi) + \sinh(2\kappa \sigma_0)}{4\pi}, \] 
\[ (S_{I,H})_{02} = -(S_{I,H})_{51} = \pm \frac{\cosh(\kappa \pi) - \cosh(2\kappa \sigma_0)}{4\pi}, \quad (S_{I,H})_{01} = 0, \quad (S_{I})_{52} = 0. \]  

(4.13)

They are conserved in the process of splitting

\[ E = S_{05} = (S_I)_{05} + (S_{II})_{05}, \quad S = (S_I)_{12} + (S_{II})_{12}, \]
\[ (S_{II})_{02} = (S_I)_{02} + (S_{II})_{02}, \quad (S)_{51} = (S_I)_{51} + (S_{II})_{51}. \]  

(4.14)

---

2 There is an elegant method to obtain expansion for \( E(S) \) in large or small \( S \) with arbitrary accuracy [24].

3 One has to be careful using (4.8) for computing charges. It is easy to see, that the absolute values of \( E \) and \( S \) in (4.13) approximately twice exceed those of (1.10) taken at equal \( \kappa \). This inconsistency comes from the fact, that approximation (1.8) is invalid close to the string ends [24], while the largest contribution to the charges comes exactly from them.
Spins of the fragments I and II given in (4.13) have non-Cartan components, as they are written in the “center-of-mass system” of the initial string (the coordinate system where $S_{PQ}$ of the string has Cartan components only). It is more natural to analyze the fragments in their own center-of-mass systems. Let us diagonalize $(S_{I,II})_{PQ}$.

Performing the boost rotations independently for each string

\begin{align}
\begin{pmatrix}
\tilde{Y}_{I,II0} \\
\tilde{Y}_{I,II1}
\end{pmatrix} &= \begin{pmatrix}
\cosh \alpha_{I,II} & \sinh \alpha_{I,II} \\
\sinh \alpha_{I,II} & \cosh \alpha_{I,II}
\end{pmatrix}
\begin{pmatrix}
Y_{I,II0} \\
Y_{I,II1}
\end{pmatrix} \quad (4.15) \\
\begin{pmatrix}
\tilde{Y}_{I,II5} \\
\tilde{Y}_{I,II2}
\end{pmatrix} &= \begin{pmatrix}
\cosh \beta_{I,II} & \sinh \beta_{I,II} \\
\sinh \beta_{I,II} & \cosh \beta_{I,II}
\end{pmatrix}
\begin{pmatrix}
Y_{I,II5} \\
Y_{I,II2}
\end{pmatrix} \quad (4.16)
\end{align}

with the parameters ($\alpha_I, \beta_I$ for the fragment I and $\alpha_{II}, \beta_{II}$ for the fragment II)

\[ \alpha_{I,II} = \frac{\kappa}{2} \left( \frac{\pi}{2} \pm \sigma_0 \right), \quad (4.17) \]

we find the energies and spins of the cut fragments in their own center-of-mass systems:

\[ E_{I,II} \simeq \frac{\kappa}{2\pi} \left( \frac{\pi}{2} \mp \sigma_0 \right) + \frac{e^{\kappa(\frac{\pi}{2} \mp \sigma_0)}}{4\pi}, \quad S_{I,II} \simeq -\frac{\kappa}{2\pi} \left( \frac{\pi}{2} \pm \sigma_0 \right) + \frac{e^{\kappa(\frac{\pi}{2} \mp \sigma_0)}}{4\pi}. \quad (4.18) \]

These expressions coincide with (4.9) up to parameter definitions. The expansions of the classical energies $E_{I,II}(S_{I,II})$ in large spins obviously agree with (4.10):

\[ E_{I,II} \simeq S_{I,II} + \frac{\sqrt{\lambda}}{\pi} \ln \frac{S_{I,II}}{\sqrt{\lambda}} + \ldots, \quad \frac{S_{I,II}}{\sqrt{\lambda}} \gg 1. \quad (4.19) \]

Let us find the string solutions describing the evolution of the cut fragments.

---

4 Any rotation in the $Y_0 Y_5$, $Y_1 Y_2$, $Y_0 Y_2$ or $Y_2 Y_1$ would result in $(S_{I,II})_{01}$ and $(S_{I,II})_{52}$ gaining nonzero values. We are left only with boosts in $Y_0 Y_3$ and $Y_5 Y_2$ planes.

5 Making use of (4.12), we set $\sinh(\kappa(\frac{\pi}{2} \mp \sigma_0)) \sim \frac{1}{2} e^{\kappa(\frac{\pi}{2} \mp \sigma_0)}$. 

---

Figure 2: Splitting of the folded string in AdS$_3$. 

The evolution of the fragment I is governed by the string equations (2.4) and (2.6) with the initial conditions at \( \tau = 0 \) (written in the center-of-mass of the fragment):

\[
\begin{align*}
\tilde{Y}_{I0} &= \cosh \left( \kappa \left( \sigma - \frac{\pi}{4} - \frac{\sigma_0}{2} \right) \right), \quad \tilde{Y}_{I1} = \sinh \left( \kappa \left( \sigma - \frac{\pi}{4} - \frac{\sigma_0}{2} \right) \right), \\
\tilde{Y}_{I5} &= 0, \quad \tilde{Y}_{I2} = 0, \quad \frac{\partial}{\partial \tau} \tilde{Y}_{I0} = 0, \quad \frac{\partial}{\partial \tau} \tilde{Y}_{I1} = 0, \\
\frac{\partial}{\partial \tau} \tilde{Y}_{I5} &= \kappa \cosh \left( \kappa \left( \sigma - \frac{\pi}{4} - \frac{\sigma_0}{2} \right) \right), \quad \frac{\partial}{\partial \tau} \tilde{Y}_{I2} = \kappa \sinh \left( \kappa \left( \sigma - \frac{\pi}{4} - \frac{\sigma_0}{2} \right) \right),
\end{align*}
\]

for the interval \( 0 < \sigma < \frac{\pi}{2} \) and the same expressions with \( \sigma \to \pi - \sigma \) for the interval \( \frac{\pi}{2} < \sigma < \pi - \sigma_0 \).

After rescaling of the world-sheet parameters \( \sigma \) to \( \xi \) in such a way that \( 0 < \sigma < \frac{\pi}{2} \to -\frac{\pi}{2} < \xi < \frac{\pi}{2} \):

\[
\sigma = \frac{\pi - 2\sigma_0}{2\pi} \xi + \frac{\pi}{4} + \frac{\sigma_0}{2} \quad \text{and} \quad \tau = \frac{\pi - 2\sigma_0}{2\pi} \eta,
\]

we rewrite (4.20) in the following form

\[
\begin{align*}
\tilde{Y}_{I0} &= \cosh(\kappa_1 \xi), \quad \tilde{Y}_{I1} = \sinh(\kappa_1 \xi), \quad \tilde{Y}_{I5} = 0, \quad \tilde{Y}_{I2} = 0, \\
\frac{\partial}{\partial \eta} \tilde{Y}_{I0} = 0, \quad \frac{\partial}{\partial \eta} \tilde{Y}_{I1} = 0, \quad \frac{\partial}{\partial \eta} \tilde{Y}_{I5} &= \kappa_1 \cosh(\kappa_1 \xi), \quad \frac{\partial}{\partial \eta} \tilde{Y}_{I2} = \kappa_1 \sinh(\kappa_1 \xi), \\
\kappa_1 &= \kappa \left( \frac{\pi - 2\sigma_0}{2\pi} \right).
\end{align*}
\]

Such boundary conditions are satisfied by

\[
\tilde{Y}_{I05} = \cosh(\kappa_1 \xi) e^{i\kappa_1 \eta}, \quad \tilde{Y}_{I12} = \sinh(\kappa_1 \xi) e^{i\kappa_1 \eta}.
\]

That is the same as (4.8) up to parameter definitions.

For the fragment II we get similar result

\[
\tilde{Y}_{II05} = \cosh(\kappa_{II} \xi) e^{i\kappa_{II} \eta}, \quad \tilde{Y}_{II12} = \sinh(\kappa_{II} \xi) e^{i\kappa_{II} \eta}, \quad \kappa_{II} = \kappa \left( 1 + 2\sigma_0 \right). \quad (4.24)
\]

Making use of (4.9) and (4.18) the following conservation laws of the splitting may be derived:

\[
E^{1-2/\xi} = 4\pi \xi_{I}^{1-2/\xi} \xi_{II}^{1-2/\xi}, \quad S^{1+2/\xi} = 4\pi S_{I}^{1+2/\xi} S_{II}^{1+2/\xi}. \quad (4.25)
\]

The boost parameters (4.17) may be expressed as

\[
\alpha_{I,II} = \beta_{I,II} \simeq \pm \ln \frac{E^{1-2/\xi}}{E_{I,II}^{1-2/\xi}} \simeq \pm \ln \frac{S^{1+2/\xi}}{S_{I,II}^{1+2/\xi}}. \quad (4.26)
\]

Given (4.8), (4.23), (4.24) and (4.17) we see that when the initial string described by the formal \( \kappa = \omega \) limit of the folded string (4.8) splits into two pieces the world surface does not change and the fragments are described by the solutions of the same type as the parent string.

\[\text{Footnote: Here we used the relations}
\]

\[E = \frac{\kappa}{2\pi} + \frac{1}{4\pi} e^{\kappa \pi} \Rightarrow \ln E = \kappa \pi - \ln 4\pi + 2\pi \kappa e^{-\kappa \pi} \Rightarrow \kappa \pi = \ln E + \ln 4\pi - \frac{2}{E} \ln E.\]
It is interesting to point out that (4.8) is not just a formal approximation of the folded string profile (in the limit $\kappa = \omega \rightarrow \infty$), but a true solution of the string equations (2.4), (2.6) (with arbitrary values of $\kappa = \omega$). Strings of this type have a peculiar property. They may be divided into an arbitrary number of fragments each of which is an independent solution of the same type as (4.8), simply boosted from its center-of-mass. However, its stretches may not be consistently glued to form a closed string. Such glued string would have jumps of the first derivatives at the string ends of fragments each of which is an independent solution of the same type as (4.8), simply boosted from its center-of-mass.

In this section we discuss the most general case of splitting of folded strings in AdS$_3$. Starting with the folded string solution in its exact form (4.11), (4.12), (4.13) and, following the approach of section 4.2, we assume that the string splits into two fragments (I and II) defined in (4.11) and in figure 2.

The folded string solution in its exact form (4.1), (4.2), (4.5) and, following the approach of section 4.3 String with an arbitrary spin. The general case

In this section we discuss the most general case of splitting of folded strings in AdS$_3$. Starting with the folded string solution in its exact form (4.11), (4.12), (4.13) and, following the approach of section 4.2, we assume that the string splits into two fragments (I and II) defined in (4.11) and in figure 2. Their charges ($S_{I,II})_{PQ}$ read

\[
\begin{align*}
(S_{I,II})_{05} &= \kappa C_{I,II} = \frac{\ell}{\pi} \left( \mathbb{E}[-\ell^2] \mp \mathbb{E} \left[ \text{am} \left[ \kappa \ell^{-1} \sigma_0, -\ell^2 \right] \right] \right), \\
(S_{I,II})_{12} &= \omega S_{I,II} = \frac{-\sqrt{1 + \ell^2} (\pi + 2\sigma_0) K[-\ell^2] + \sqrt{1 + \ell^2} (\mathbb{E}[-\ell^2] \mp \mathbb{E} \left[ \text{am} \left[ \kappa \ell^{-1} \sigma_0, -\ell^2 \right] \right])}{\pi^2}, \\
(S_{I,II})_{02} &= \omega M_{I,II} = \pm \frac{1}{\pi} \sqrt{1 + \ell^2} \, \text{cn} \left[ \kappa \ell^{-1} \sigma_0, -\ell^2 \right], \\
(S_{I,II})_{15} &= \kappa M_{I,II} = \pm \frac{1}{\pi} \ell \, \text{cn} \left[ \kappa \ell^{-1} \sigma_0, -\ell^2 \right], \\
(S_{I,II})_{01} &= (S_{I,II})_{52} = 0,
\end{align*}
\]

where $\mathbb{E}[z, m]$ is the incomplete elliptic integral of the second kind. We want to find a coordinate systems where the non-Cartan components of the spins vanish and find the energies and spins of the final fragments. ($S_{I,II})_{PQ}$ may be diagonalized by boosts in the $Y_0 Y_1$ and $Y_2 Y_3$ planes (4.15), (4.16) with parameters

\[
\begin{align*}
\sinh(\alpha_{I,II} + \beta_{I,II}) &= -\frac{M_{I,II}(\omega + \kappa)}{\sqrt{(\kappa C_{I,II} + \omega S_{I,II})^2 - M_{I,II}^2(\omega + \kappa)^2}}, \\
\sinh(\alpha_{I,II} - \beta_{I,II}) &= \frac{M_{I,II}(\omega - \kappa)}{\sqrt{(\kappa C_{I,II} - \omega S_{I,II})^2 - M_{I,II}^2(\omega - \kappa)^2}}.
\end{align*}
\]

where

\[
\begin{align*}
C_{I,II} &= \frac{\ell}{\kappa \pi} \left( \mathbb{E}[-\ell^2] \mp \mathbb{E} \left[ \text{am} \left[ \kappa \ell^{-1} \sigma_0, -\ell^2 \right] \right] \right), \\
S_{II} &= -\frac{\pi + 2\sigma_0}{2} + C_{I,II}, \\
M_{I,II} &= \pm \frac{\ell}{\kappa \pi} \, \text{cn} \left[ \kappa \ell^{-1} \sigma_0, -\ell^2 \right], \\
\kappa &= \frac{2}{\pi} \ell \, \mathbb{K}[-\ell^2].
\end{align*}
\]

\footnote{In the flat space this inconsisteny is avoided due to the chiral properties of the solutions for the final fragments (see above).}

\footnote{Vanishing of the non-Cartan components of the spins implies}

\[
\begin{align*}
M_{I,II}(\kappa + \omega) \cosh(\alpha_{I,II} + \beta_{I,II}) + (\kappa C_{I,II} + \omega S_{I,II}) \sinh(\alpha_{I,II} + \beta_{I,II}) &= 0, \\
M_{I,II}(\kappa - \omega) \cosh(\alpha_{I,II} - \beta_{I,II}) + (\kappa C_{I,II} - \omega S_{I,II}) \sinh(\alpha_{I,II} - \beta_{I,II}) &= 0.
\end{align*}
\]

That leads to (4.28).
Then the energies and spins of the cut fragments read
\[
\mathcal{E}_{I,II} = \frac{1}{2} \sqrt{(\kappa C_{I,II} + \omega S_{I,II})^2 - M_{I,II}^2 (\omega + \kappa)^2} + \frac{1}{2} \sqrt{(\kappa C_{I,II} - \omega S_{I,II})^2 - M_{I,II}^2 (\omega - \kappa)^2},
\]
\[
\mathcal{S}_{I,II} = \frac{1}{2} \sqrt{(\kappa C_{I,II} + \omega S_{I,II})^2 - M_{I,II}^2 (\omega + \kappa)^2} - \frac{1}{2} \sqrt{(\kappa C_{I,II} - \omega S_{I,II})^2 - M_{I,II}^2 (\omega - \kappa)^2}.
\]

These relations parametrically encode the conservation laws of splitting, e.g. \(E(E_1, E_{II}), S(S_1, S_{II})\), etc.

The evolution of the fragments I and II is governed by the string equations (2.4) and (2.6) with the boundary conditions given by the initial string (4.1), (4.2), (4.5) on the intervals (4.11) at \(\tau_0 = 0\). Due to the complexity of the folded string profile (1.2), we are unable to find solutions to these equations. However, we could describe the evolution qualitatively based on the result of section 4.2 and section 3 in the limit of large — but not infinitely large (as in (1.8)) — spin, so long as the cut occurs far enough from the string ends for \(\sigma_0\) to satisfy \(\kappa(\pi/2 - \sigma_0) \gg 1\). In this case, one should expect the final pieces to have almost the standard folded shape (4.1), (4.2), (4.5), which is disturbed by a kink moving along the string, similar to that observed in flat Minkowski space, see section 3 and [12]. The kink is a “correction” to the “leading” folded shape of the cut fragments, thus the angle of bending has to depend on the position of the kink. It may be substantial at the string ends but must be small close to the center.

5 Splitting of strings in \(\text{AdS}_3 \times S^5\).

In this section we generalize the results for the splitting of the folded string in \(\text{AdS}_3\) to \(\text{AdS}_3 \times S^5\), including into consideration circular rotations and windings in \(S^5\).

Let us consider the string solution having the folded shape in the \(\text{AdS}_3\) and the circular one with windings in \(S^3\) :

\[
Y_{05} = \cosh \rho \ e^{i \kappa \tau}, \quad Y_{12} = \sinh \rho \ e^{i \omega \tau}, \quad X_{12} = a e^{i (\nu \tau + m \sigma)}, \quad X_{34} = b e^{i (\nu \tau - m \sigma)}, \quad a^2 + b^2 = 1, \quad m \in \mathbb{N},
\]

where

\[
\sinh \rho = \tilde{\ell} \ \text{sn} [\tilde{\tau} \tilde{\ell}^{-1} \sigma, -\tilde{\ell}^2], \quad \tilde{\kappa} = \frac{2}{\pi} \tilde{\ell} \ \mathbb{K} [-\tilde{\ell}^2],
\]

\[
\tilde{\omega}^2 \rho^2 = 1 + \frac{1}{\tilde{\ell}^2}, \quad \tilde{\omega}^2 = \omega^2 - (\nu^2 + m^2), \quad \tilde{\kappa}^2 = \kappa^2 - (\nu^2 + m^2).
\]

Comparing that with (4.1), (4.2) and (4.5), we see that the only result of accounting for the \(S^3\) part is redefinition of \(\kappa, \omega \rightarrow \tilde{\kappa}, \tilde{\omega}\). That is also true if one adds other spins and windings in \(S^5\).

Combining together four stretches of (5.1), each of which is valid on the interval \(0 \leq \sigma < \frac{\pi}{2}\), we obtain a periodic solution on the interval \(0 \leq \sigma < 2\pi\). Its classical energy and spins read

\[
\mathcal{E}_J = \frac{2}{\pi} \frac{\kappa}{\tilde{\kappa}} \tilde{\ell} \ \mathbb{E} [-\tilde{\ell}^2] = \frac{4}{\pi^2} \tilde{\ell}^2 + \frac{m^2 + (J_1 + J_2)^2}{\mathbb{K}^2 [-\tilde{\ell}^2]} \mathbb{E} [-\tilde{\ell}^2],
\]

\[
\mathcal{S}_J = \frac{2}{\pi} \frac{\omega}{\tilde{\omega}} \ \sqrt{1 + \tilde{\ell}^2} \ \mathbb{E} [-\tilde{\ell}^2] - \mathbb{K} [-\tilde{\ell}^2] = \frac{4}{\pi^2} (1 + \tilde{\ell}^2) + \frac{m^2 + (J_1 + J_2)^2}{\mathbb{K}^2 [-\tilde{\ell}^2]} \ (\mathbb{E} [-\tilde{\ell}^2] - \mathbb{K} [-\tilde{\ell}^2]),
\]

\[
J_1 = a^2 \nu, \quad J_2 = b^2 \nu, \quad \nu = J_1 + J_2.
\]

(5.3)
Following the approach of section [3], first, we consider the limit of the string with large spin in AdS. Then the AdS-part of the solution (5.2) may be approximated by
\[
\rho = \sqrt{\kappa^2 - (\nu^2 + m^2)} \sigma = \tilde{\kappa} \sigma, \quad \kappa = \omega, \quad \tilde{\kappa} \gg 1. \tag{5.4}
\]
This is a formal limit as \( \kappa \to \omega \) implies \( \tilde{\ell} \to \infty \).

The energy and AdS-spin of the string read
\[
E_J = S_{00} \simeq \frac{\kappa}{2\pi} + \frac{\kappa}{4\pi \tilde{\kappa}} e^{\tilde{\kappa} \pi}, \quad S_J = S_{12} \simeq -\frac{\kappa}{2\pi} + \frac{\kappa}{4\pi \tilde{\kappa}} e^{\tilde{\kappa} \pi}. \tag{5.5}
\]
Spins in \( S^3 \) are unaffected by the limit.

Two points on the string parameterized by \( \sigma_1 \) and \( \sigma_2 \) coincide in the target space and their velocities agree, if \( \sigma_1 = \sigma_0, \sigma_2 = \pi - \sigma_0 \) and
\[
\sigma_0 = \left( \frac{1}{2} - \frac{n}{m} \right) \pi, \quad n \in \mathbb{N}, \quad \text{if} \ m \neq 0 \tag{5.6}
\]
or for arbitrary \( \sigma_0 \) if \( m = 0 \). The string is not folded in AdS

Approximation (5.4) is invalid close to the string ends, thus we have to demand
\[
\frac{\pi}{2} - \sigma_0 \gg \frac{1}{\tilde{\kappa}} \tag{5.7}
\]
for the coordinates of the cut \( (\sigma_{\text{cut}}^1 = \sigma_0 \text{ and } \sigma_{\text{cut}}^2 = \pi - \sigma_0) \).

The charges \( (S_{IJ})_{PQ} \) of the cut fragments read
\[
(S_{I1})_{05} = \frac{\kappa}{2\pi} \left( \frac{\pi}{2} \mp \sigma_0 \right) + \frac{\kappa}{4\pi \tilde{\kappa}} \sinh(\tilde{\kappa} \pi) + \sinh(2\tilde{\kappa} \sigma_0), \\
(S_{I1})_{12} = -\frac{\kappa}{2\pi} \left( \frac{\pi}{2} \mp \sigma_0 \right) + \frac{\kappa}{4\pi \tilde{\kappa}} \sinh(\tilde{\kappa} \pi) - \sinh(2\tilde{\kappa} \sigma_0), \\
(S_{I2})_{02} = -(S_{I1})_{51} \pm \frac{\kappa}{\tilde{\kappa}} \cosh(\tilde{\kappa} \pi) - \cosh(2\tilde{\kappa} \sigma_0), \quad (S_{I1})_{01} = 0, \quad (S_{I2})_{52} = 0, \\
(J_{I1})_1 = a^2 \frac{\nu}{2\pi} (\pi \mp 2\sigma_0), \quad (J_{I1})_2 = b^2 \frac{\nu}{2\pi} (\pi \mp 2\sigma_0).
\tag{5.8}
\]
They are conserved in the process of splitting
\[
E_J = (S_{I1})_{05} + (S_{I2})_{05}, \quad S_J = (S_{I1})_{12} + (S_{I2})_{12}, \\
S_{02} = (S_{I1})_{02} + (S_{I2})_{02}, \quad S_{51} = (S_{I1})_{51} + (S_{I2})_{51}, \\
(J_{I1})_1 = (J_{I1})_1 + (J_{I2})_1, \quad (J_{I2})_2 = (J_{I1})_2 + (J_{I2})_2. \tag{5.9}
\]

It is natural to transform (5.8) to the center-of-mass systems of the final strings and explicitly find their energies and spins. \( (S_{I1})_{PQ} \) may be diagonalized by boosts in the \( Y_0Y_1 \) and \( Y_3Y_2 \) planes (4.15), (4.16) with parameters
\[
\alpha_{IJ} = \beta_{IJ} = \mp \frac{\tilde{\kappa}}{2} \left( \frac{\pi}{2} \pm \sigma_0 \right). \tag{5.10}
\]
We obtain the energies and AdS-spins of the fragments in the form
\[
E_{IJ} = \frac{\kappa}{2\pi} \left( \frac{\pi}{2} \pm \sigma_0 \right) + \frac{\kappa}{4\pi \tilde{\kappa}} e^{\tilde{\kappa} \left( \frac{\pi}{2} \pm \sigma_0 \right)}, \quad S_{IJ} = -\frac{\kappa}{2\pi} \left( \frac{\pi}{2} \mp \sigma_0 \right) + \frac{\kappa}{4\pi \tilde{\kappa}} e^{\tilde{\kappa} \left( \frac{\pi}{2} \mp \sigma_0 \right)}. \tag{5.11}
\]

\(^9\text{Making use of (5.7), we set } \sinh(\tilde{\kappa} \left( \frac{\pi}{2} \pm \sigma_0 \right)) \to \frac{1}{2} e^{\tilde{\kappa} \left( \frac{\pi}{2} \pm \sigma_0 \right)}.\)
The evolution of the fragments (in the own center-of-mass system for each fragment) is described by

\[
\begin{align*}
(\bar{Y}_{1,II})_{05} &= \cosh(\tilde{\kappa}_{1,II} \xi)e^{i\kappa_{1,II} \eta} \\
(\bar{X}_{1,II})_{12} &= a e^{i(\kappa_{1,II} \eta + m_{1,II} \xi)} \\
(\bar{Y}_{1,II})_{12} &= \sinh(\tilde{\kappa}_{1,II} \xi)e^{i\kappa_{1,II} \eta} \\
(\bar{X}_{1,II})_{34} &= b e^{i(\kappa_{1,II} \eta - m_{1,II} \xi)},
\end{align*}
\]  

(5.12)

where

\[
\kappa_{1,II} = \frac{\pi_0 + 2\sigma_0}{2\pi}, \quad \tilde{\kappa}_{1,II} = \frac{\pi_0 + 2\sigma_0}{2\pi}, \quad \nu_{1,II} = \frac{\pi + 2\sigma_0}{2\pi}, \quad m_{1,II} = m \frac{\pi + 2\sigma_0}{2\pi}
\]  

(5.13)

and \(\sigma_0\) satisfy (5.6) if \(m \neq 0\). Note, that while the AdS part of (5.12) is just a large-spin approximation, the solution for the \(S^3\) part is exact.

Given (5.4), (5.12) and (5.10) we see that, when the initial string, described by the formal \(\kappa = \omega\) limit of the string (5.4) in AdS\(_3\times S^3\), splits into two pieces the world surface does not change and the fragments are described by the solutions of the same type as the parent string.

In the general case, starting from the exact solution in AdS\(_3\times S^3\) in the form (5.11), (5.2), we obtain the following expressions for the charges of the cut fragments (in the center-of-mass of the initial string):

\[
\begin{align*}
(S_{1,II})_{05} &= \frac{1}{2} \mathcal{E}_J \mp \sqrt{\frac{1}{\pi^2} \mathcal{J}^2 + \frac{\mathcal{J}^2}{4\mathcal{K}^2[-\mathcal{L}^2]} E \left[\text{am}[(\kappa\tilde{\ell}^{-1}\sigma_0, -\tilde{\ell}^2), -\tilde{\ell}^2]\right],} \\
(S_{1,II})_{12} &= \frac{1}{2} \mathcal{S}_J \mp \sqrt{\frac{1}{\pi^2} \mathcal{J}^2 + \frac{\mathcal{J}^2}{4\mathcal{K}^2[-\mathcal{L}^2]} \left(\mathcal{E} \left[\text{am}[(\kappa\tilde{\ell}^{-1}\sigma_0, -\tilde{\ell}^2), -\tilde{\ell}^2]\right] - \frac{2}{\pi} \sigma_0 \mathcal{K}[-\mathcal{L}^2]\right),} \\
(S_{1,II})_{02} &= \pm \omega \mathcal{K} \mathcal{L} \mathcal{J}_{\kappa\sigma} \tilde{\ell}^2 \mathcal{K}^{-1} \mathcal{L}^{-1} \mathcal{J}_{\kappa\sigma} \tilde{\ell}^2 \mathcal{K} \mathcal{L} \mathcal{J}_{\kappa\sigma}, \\
(S_{1,II})_{15} &= \pm \mathcal{K} \mathcal{L} \mathcal{J}_{\kappa\sigma} \tilde{\ell}^2 \mathcal{K}^{-1} \mathcal{L}^{-1} \mathcal{J}_{\kappa\sigma} \tilde{\ell}^2, \\
(J_{1,II})_{1} &= a^2 \frac{\nu}{2\pi} (\pi \mp 2\sigma_0), \\
(J_{1,II})_{2} &= b^2 \frac{\nu}{2\pi} (\pi \mp 2\sigma_0),
\end{align*}
\]  

(5.14)

where \(\mathcal{E}_J\) and \(\mathcal{S}_J\) are defined in (5.3). That may be transformed to the center-of-mass systems of the final states by the boosts in the \(Y_0 Y_1\) and \(Y_5 Y_2\) planes (4.14), (4.16) with parameters

\[
\sinh(\alpha_{1,II}^J + \beta_{1,II}^J) = -\frac{\tilde{M}_{1,II}(\omega + \kappa)}{\sqrt{(\kappa \tilde{C}_{1,II} + \omega \tilde{S}_{1,II})^2 - \tilde{M}_{1,II}^2(\omega + \kappa)^2}},
\]

\[
\sinh(\alpha_{1,II}^J - \beta_{1,II}^J) = \frac{\tilde{M}_{1,II}(\omega - \kappa)}{\sqrt{(\kappa \tilde{C}_{1,II} - \omega \tilde{S}_{1,II})^2 - \tilde{M}_{1,II}^2(\omega - \kappa)^2}},
\]  

(5.15)

where \(\tilde{C}_{1,II}, \tilde{S}_{1,II}\) and \(\tilde{M}_{1,II}\) are given by (4.29) with \(\ell\) replaced for \(\tilde{\ell}\).

The general expressions for the energies and spins of the fragments read

\[
\begin{align*}
(\mathcal{E}_{1,II})_1 &= \frac{1}{2} \sqrt{(\kappa \tilde{C}_{1,II} + \omega \tilde{S}_{1,II})^2 - \tilde{M}_{1,II}^2(\omega + \kappa)^2} + \frac{1}{2} \sqrt{(\kappa \tilde{C}_{1,II} - \omega \tilde{S}_{1,II})^2 - \tilde{M}_{1,II}^2(\omega - \kappa)^2}, \\
(\mathcal{S}_{1,II})_1 &= \frac{1}{2} \sqrt{(\kappa \tilde{C}_{1,II} + \omega \tilde{S}_{1,II})^2 - \tilde{M}_{1,II}^2(\omega + \kappa)^2} - \frac{1}{2} \sqrt{(\kappa \tilde{C}_{1,II} - \omega \tilde{S}_{1,II})^2 - \tilde{M}_{1,II}^2(\omega - \kappa)^2}, \\
(J_{1,II})_1 &= a^2 \frac{\nu}{2\pi} (\pi \mp 2\sigma_0), \\
(J_{1,II})_2 &= b^2 \frac{\nu}{2\pi} (\pi \mp 2\sigma_0).
\end{align*}
\]

These relations parametrically encode the conservation laws of splitting.
The evolution of the fragments I and II is governed by the string equations (2.4) and (2.6) with the boundary conditions given by the initial string (5.1), (5.2) on the intervals (4.11) at $\tau_0 = 0$ with $\sigma_0$ satisfying (5.6). The solutions describing the profiles of the fragments consist of AdS- and $S^3$-parts. The expressions for the $S^3$-parts presented in (5.12), but we are unable to find the exact expressions for the AdS-parts, due to the complexity of (5.2). Up to parameter definitions, the AdS-parts coincide with the solutions describing the splitted fragments of the folded string in pure AdS$_3$. This is based on the fact that the only result of accounting for the $S^5$ is redefinition of $\kappa,\omega \to \tilde{k},\tilde{w}$ and discretizing of $\sigma_0$, if any.

Concluding remarks

In this paper we have investigated splitting of folded spinning strings in AdS$_3$ and its generalization to include circular rotations and windings in $S^5$. We computed the energies and spins of products of splitting and showed that in the case of splitting of strings with large AdS-spins (which is of greatest interest in the context of AdS/CFT duality) the cut fragments are described by the solutions very similar to the initial string. The complexity of the exact folded string profile prevents us from finding the evolution of the final fragments by solving the string equations with boundary conditions given by the initial string. However, one hopes that this might be reachable “indirectly” by applying the finite gap technique (see [27, 28] for reviews). The profiles of the cut fragments are known at the moment of splitting, thus we can find the full set of the conserved charges for them, including the higher ones. This uniquely determines the algebraic surface which, being explicitly constructed, would allow the determination of the string profiles. Implementation of such an approach is promising, but quite complicated. It requires detailed investigation.

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