A large deviation principle for block models

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Abstract

We initiate a study of large deviations for block model random graphs in the dense regime. Following [14], we establish an LDP for dense block models, viewed as random graphons. As an application of our result, we study upper tail large deviations for homomorphism densities of regular graphs. We identify the existence of a “symmetric” phase, where the graph, conditioned on the rare event, looks like a block model with the same block sizes as the generating graphon. In specific examples, we also identify the existence of a “symmetry breaking” regime, where the conditional structure is not a block model with compatible dimensions. This identifies a “reentrant phase transition” phenomenon for this problem—alogous to one established for Erdős–Rényi random graphs [13, 14]. Finally, extending the analysis of [33], we identify the precise boundary between the symmetry and symmetry breaking regime for homomorphism densities of regular graphs and the operator norm on Erdős–Rényi bipartite graphs.

Keywords: large deviation, block models, symmetry/symmetry-breaking, bipartite Erdős–Rényi graph.

Contents

1 Introduction 2
1.1 Graph limit theory: a brief review ........................................ 3
1.2 A large deviation principle for block models ........................... 5
1.3 LDP for graph parameters and the associated variational problem 6
1.4 The existence of a symmetric regime for $d$-regular graphs .......... 7
1.5 A non-symmetric regime in special cases .............................. 9
1.6 Bipartite Erdős–Rényi graphs: symmetry vs. symmetry breaking .10
1.7 History and related work ............................................. 11

2 Large deviation principle .............................................. 12
2.1 Preliminaries ............................................................. 12
2.2 Lower bound ............................................................ 16
2.3 Upper bound ............................................................ 18
2.4 Proof of Theorem 2 ...................................................... 24

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3 \( \phi_r \): Monotonicity, continuity, and examples

3.1 Preliminaries ......................................................... 26
3.2 Establishing the continuity of \( \phi_r \) ................................ 27
3.3 Properties of \( \phi_r \) at points of continuity ......................... 33

4 The symmetric regime in general block models

4.1 A key lemma for establishing the symmetric regime ................. 36
4.1.1 Proofs of supporting lemmas ..................................... 36
4.2 Symmetry with a unique minimizer for small \( \delta \) .................... 40
4.2.1 Proofs of supporting lemmas ..................................... 40
4.3 A symmetric regime for larger \( \delta \) ................................. 44

5 Symmetry breaking in special cases

5.1 Proof of Lemma 35 ...................................................... 45
5.2 Proof of Lemma 36 ...................................................... 46
5.2.1 Erdős–Rényi graphs with a planted independent set ............. 47
5.2.2 Erdős–Rényi graphs with a planted clique ....................... 47
5.2.3 Erdős–Rényi graphs with a planted clique and independent set . 48

6 Bipartite Erdős–Rényi graphs

6.1 Density of \( d \)-regular subgraphs .................................... 49
6.1.1 Proof for the symmetric regime .................................. 49
6.1.2 Proof for the non-symmetric regime .............................. 50
6.2 Largest eigenvalue ..................................................... 52

7 Open questions

8 Appendix

8.1 Weak topology LDP upper bound .................................... 54
8.2 Other useful results ................................................... 55
8.3 Behavior at \( t = t_{\text{max}} \) ............................................ 57
8.3.1 Proof of Theorem 5 .................................................. 57
8.3.2 Elaboration on Remark 4 .......................................... 58

1 Introduction

The study of large deviation problems on random graphs has a long and rich history in Probability and Combinatorics. Research in this area is motivated by the following fundamental question: \textit{What is the structure of a random graph, conditioned on a rare event?}

In a seminal paper, Chatterjee and Varadhan \[14\] formalized this question by combining the theory of graph limits \[31\] with classical Large Deviations theory \[19\], and established a Large Deviation Principle (LDP) for the Erdős–Rényi binomial random graph \( G(n, p) \). This is the simplest random graph model, constructed by adding edges independently among \( n \) vertices with probability \( p \). As an application of this LDP, Chatterjee and Varadhan \[14\] examined upper tail large deviations for regular subgraph counts. The homomorphism density \( t(H, G) \) of a graph \( H \) on \( v \) vertices measures the probability that \( H \) appears on \( v \) randomly chosen vertices of a graph \( G \) (see Definition 5). Let \( H \) be a \( d \)-regular graph, and for notational convenience, define the event \( \mathcal{E}_\delta = \{ t(H, G) > (1 + \delta)E[t(H, G)] \} \). Chatterjee and Varadhan \[14\] established the existence of \( 0 < \delta_{\text{min}}(H) < \delta_{\text{max}}(H) \) such that if \( \delta < \delta_{\text{min}}(H) \) or \( \delta > \delta_{\text{max}}(H) \), conditioned on \( \mathcal{E}_\delta \), \( G(n, p) \) “looks like” an Erdős–Rényi random graph, albeit with a higher edge density. They call this the “replica symmetric” phase. On the contrary, \[14\] also establishes that for \( p \) sufficiently small, there exists \( \delta \in [\delta_{\text{min}}(H), \delta_{\text{max}}(H)] \) such that conditioned on \( \mathcal{E}_\delta \), the graph is not distributed as an Erdős–Rényi random graph—this regime was termed as the “replica symmetry breaking” regime. Using the framework of \[14\], Lubetzky and Zhao \[33\] characterized the precise boundary between the symmetry and the symmetry-breaking regimes, in terms of \( \delta \) and \( p \). We defer an in-depth survey of large deviations on random graphs to Section 1.7.

Random graphs are simple stochastic models for large networks observed in myriad scientific applications, and in this context, it is often natural to study graph models with inhomogeneities or constraints. Large deviation phenomena are of natural interest in this general setting, although progress in this direction requires several new ideas. The study of large deviations for constrained random graphs has
been initiated in the recent literature—[18] studies large deviations for the uniform random graph with a given number of edges, while in [20], in joint work with Souvik Dhara, one of the authors studied large deviations for random graphs with given degrees. Finally, [7] focuses on large deviations for random regular graphs in the sparse regime. In contrast, large deviations for inhomogeneous random graphs is relatively unexplored (see [7] for some preliminary results on sparse graphs). This paper seeks to fill this gap, by initiating the study of large deviations for block model random graphs.

A block model on \( k \) blocks is specified by a set of values \( \{p_{ij}\}_{1 \leq i \leq j \leq k} \), \( p_{ij} \in [0,1] \). A graph on \( kn \) vertices is sampled from this model as follows: (i) collect the vertices into \( k \) groups of size \( n \), indexed by \( 1,\ldots,k \), and (ii) connect two vertices from groups \( i \) and \( j \) with probability \( p_{ij} \). (See Section 1.1 for a formal definition.) Our contributions in this article can be summarized as follows:

1. We adopt the framework of [14], and establish an LDP for block model random graphs, viewed as random graphons. The induced law of the random graph satisfies an LDP with speed \( n^2 \)—the rate function in this case is the lower semicontinuous envelope of an appropriate relative entropy functional (see formal statement as Theorem 1). Perhaps surprisingly, although the block model is quite similar to the Erdős–Rényi random graph, our derivation of the LDP requires going substantially beyond the ideas introduced in [14]. In particular, the derivation of the LDP in [14] relies heavily on the fact that an Erdős–Rényi random graph remains invariant in law under permutations of the vertices, a fact that is no longer true for general block models. To overcome this barrier, we rely on a two-step approach: (a) Using Szemerédi’s Regularity Lemma, we construct a Szemerédi net of block graphons and cover an event by a finite union of open balls centered on the elements of this net. Thus it suffices to characterize the limiting probability of each open ball. (b) To this end, we employ a “method of types”-style argument, similar to the classical proof of Sanov’s Theorem. A similar two-step strategy was employed earlier in [20] while deriving an LDP for random graphs with given degrees. A crucial technical difference between the two settings is that in [20], the graphon being sampled from was bounded away from zero and one, whereas our results include block models that take value zero or one. As an immediate application of this general result, we obtain an LDP for the Erdős–Rényi bipartite graph.

2. Our general LDP, in turn, directly implies an LDP for graph parameters continuous with respect to the cut topology (see Theorem 2), e.g. homomorphism density, largest eigenvalue, etc. For such graph parameters, the rate function is expressed as a variational problem on the space of graphons.

3. Next, we turn our attention to the variational problem for upper tail large deviations of regular subgraphs. In Theorem 3 we establish that close to the expected value, this problem exhibits a symmetric phase—where the variational problem admits a unique solution, which exhibits the same block structure as the base graphon. We also demonstrate that for large target values of the homomorphism density, the variational principle admits a unique symmetric solution.

4. In some specific block graphons, we exhibit the existence of a non-symmetric phase—where there does not exist a symmetric optimizer (see Section 1.5 for the specific examples). This establishes an analogue of the reentrant phase transition phenomenon, noted earlier for the upper tail problem on Erdős–Rényi random graphs [13] [14].

5. Finally, we turn to the bipartite Erdős–Rényi random graph in Section 1.6 and study the variational problems corresponding to the upper tails of regular subgraphs and largest eigenvalue. We extend the analysis of Lubetzky and Zhao [33] and determine the precise transition boundary between the symmetric and the symmetry-breaking regimes.

We present a brief review of the relevant facts from graph limit theory [8, 9, 32] and detail our main results in the rest of this section.

1.1 Graph limit theory: a brief review

In this section, we collect some facts from the theory of dense graph limits [8, 9, 32] which will be relevant for the subsequent discussion. We refer the interested reader to [31] for an in-depth survey of this area.
Define the function $k_n : [0, 1] \to [n]$ as
\[
k_n(x) = \begin{cases} 
1 & 0 \leq x \leq \frac{1}{n}, \\
\frac{i}{n} & \frac{i}{n} < x \leq \frac{i+1}{n}, \quad 1 < i \leq n. 
\end{cases}
\] (1)

**Definition 1** (Graphon). Let $W$ be the space of all measurable functions $f : [0, 1]^2 \to [0, 1]$ such that $f(x, y) = f(y, x)$ for all $(x, y) \in [0, 1]^2$. We call $f \in W$ a graphon.

**Definition 2** (Empirical Graphon). Let $G$ be a simple graph on $[n] = \{1, \ldots, n\}$. The empirical graphon $f^G : [0, 1]^2 \to [0, 1]$ is defined as follows
\[
f^G(x, y) = \begin{cases} 
1 & (k_n(x), k_n(y)) \text{ is an edge in } G, \\
0 & \text{otherwise.} 
\end{cases}
\]

Next, we recall the notions of the cut distance and cut metric.

**Definition 3** (Cut Distance). The cut distance between two graphons $f, g \in W$ is defined as
\[
d_{\square}(f, g) = \sup_{S, T \subseteq [0, 1]} \left| \int_{S \times T} (f(x, y) - g(x, y)) \, dx \, dy \right|,
\]
where $S, T$ are measurable subsets of $[0, 1]$.

**Definition 4** (Cut Metric). For $f, g \in W$, the cut metric is defined as
\[
d_{\square}(f, g) = \inf_{\phi \in \mathcal{M}} d_{\square}(f, \phi) = \inf_{\phi \in \mathcal{M}} d_{\square}(f^\phi, \phi) = \inf_{\phi \in \mathcal{M}} d_{\square}(f^\phi, g),
\]
where $\mathcal{M}$ denotes the set of bijective, Lebesgue measure-preserving maps $\phi : [0, 1] \to [0, 1]$.

We will establish our large deviation principle in the natural quotient space associated with $\mathcal{M}$.

**Definition 5** (Homomorphism Density). Let $H = (V(H), E(H))$ be a simple graph, where the vertices are labeled as $[v] = \{1, \ldots, v\}$, where $v = |V(H)|$. Define the homomorphism density of $H$ in $f \in W$ as
\[
t(H, f) = \int_{[0, 1]^v} \prod_{(i, j) \in E(H)} f(x_i, x_j) \, dx_1 \ldots dx_v.
\]

Since $t(H, f) = t(H, g)$ whenever $f \sim g$, $t(H, \cdot)$ is well-defined on $\widetilde{W}$. With a slight abuse of notation, we use the same symbol for the function $t(H, \cdot) : \widetilde{W} \to [0, 1] : \tilde{f} \mapsto t(H, \tilde{f})$. As shown in [32], this function is continuous for any finite graph $H$.

In this article, we study large deviations for block model random graphs. To this end, we denote $\mathcal{B}^+$ as the set of block graphons where the width of the blocks are given by the values in the vector $\gamma$, which we assume to be rational. Let $\Delta_m = \{ \gamma \in [0, 1]^m : \sum_{i=1}^m \gamma_i = 1, \gamma_i \in \mathbb{Q}\}$ denote the rational points in the $(m-1)$-dimensional simplex.

**Definition 6**. Given $\gamma \in \Delta_m$, we define $I_1 = [0, \gamma_1]$ and
\[
I_j = \left[ \sum_{k=0}^{j-1} \gamma_k, \sum_{k=0}^{j} \gamma_k \right], \quad 1 < j \leq m.
\]
From these intervals, define the interval membership function
\[
k_\gamma(x) = \sum_{j=1}^m \mathbb{1}\{x \in I_j\}.
\] (2)
When $\gamma$ is clear from context, we write $k(x)$. Let $\mathcal{B}^\gamma$ be the set of graphons $f \in \mathcal{W}$ of the form

$$f(x, y) = p_k(x, k(y)),$$

where $p_{ij} = p_{ii} \in [0, 1]$. We call such a graphon an $m$-block graphon. When $\gamma$ is clear from context, we write $f \in \mathcal{B}^\gamma$ as $f = (p_{ij})_{i,j \in [m]}$. When $\gamma = (1/m, \ldots 1/m)$, we say $f \in \mathcal{B}^\gamma$ is a uniform size (or simply uniform) $m$-block graphon. Let

$$\mathcal{B}^{\gamma^*} = \{ f \in \mathcal{B}^\gamma : f \notin \mathcal{B}^\eta \text{ for all } \eta \in \Delta_{m-1} \}.$$ 

In other words, $\mathcal{B}^{\gamma^*}$ is the subset of graphons in $\mathcal{B}^\gamma$ that cannot be described by a smaller number of blocks. Let

$$\tilde{\mathcal{B}}^\gamma = \{ \tilde{f} \in \tilde{\mathcal{W}} : \delta_\square(\tilde{f}, g) = 0 \text{ for some } g \in \mathcal{B}^\gamma \}.$$ 

Finally we define the sampling distribution for dense block model random graphs. We recall that $\mathcal{A}$ denotes the Borel $\sigma$-algebra over the metric space $(\mathcal{W}, \delta_\square)$. 

**Definition 7** (Sampling from a block model). Let $W_0 = (p_{ij})_{i,j \in [k]}$ be a uniform $k$-block graphon. Let $\mathbb{P}_{kn,W_0}$ denote the probability distribution over $\mathcal{W}$ obtained by sampling from $W_0$ as follows. Construct a simple graph $G$ on $kn$ vertices with unique labels in $[kn]$. Independently, add an edge between vertex $i$ and vertex $j$ with probability $W_0(i/kn, j/kn) = p_{ij} (j/n)$. Return the empirical graphon $f_G$. Let $\tilde{\mathbb{P}}_{kn,W_0}$ denote the probability distribution induced on $\tilde{\mathcal{W}}$ by the measure $\mathbb{P}_{kn,W_0}$, i.e., $\tilde{\mathbb{P}}_{kn,W_0}(A) = \mathbb{P}_{kn,W_0}(\tilde{f}_G \in A)$ for all $A \in \tilde{\mathcal{A}}$.

**Remark 1.** Note that any graphon with rational-length blocks is a uniform $k$-block graphon for some $k$, and thus the above scheme can be used to sample from such graphons.

### 1.2 A large deviation principle for block models

First we define the relative entropy function, both pointwise and for entire graphons. These definitions will be used to define the rate function for the LDP. Throughout we use the conventions $0 \log 0 = 0$ and $0 \log(0/0) = 0$.

**Definition 8** (Relative entropy). Define $I_{W_0} : \mathcal{W} \to \mathbb{R} \cup \{ \infty \}$ as

$$I_{W_0}(f) = \frac{1}{2} \int_{[0,1]^2} h_{W_0(x,y)}(f(x,y)) \, dx \, dy,$$

where $h_p(u)$ is the usual relative entropy,

$$h_p(u) = u \log \frac{u}{p} + (1-u) \log \frac{1-u}{1-p}.$$ 

Given $W_0$, let $\Omega = \{(x,y) : W_0(x,y) \in (0,1)\}$. Define

$$\Omega_\lambda = \{ f \in \mathcal{W} : \lambda(\{(x,y) : W_0(x,y) \neq W_0(x,y)\}) = 0 \}$$ (3)

and

$$\tilde{\Omega}_\lambda = \{ \tilde{f} \in \tilde{\mathcal{W}} : \delta_\square(\tilde{f}, g) = 0 \text{ for some } g \in \Omega_\lambda \},$$ (4)

where $\lambda(\cdot)$ is the Lebesgue measure on $[0,1]^2$. In other words, $\Omega_\lambda$ is the set of graphons that agree with $W_0$ wherever $W_0$ takes value 0 or 1, except possibly on a measure-zero set. Note that $\mathbb{P}_{kn,W_0}$ and $\tilde{\mathbb{P}}_{kn,W_0}$ are supported on $\Omega_\lambda$ and $\tilde{\Omega}_\lambda$ respectively. Lemma[3] states that $\tilde{\Omega}_\lambda$ is closed (and hence compact), and Proposition[4] states that $I_{W_0}$ is bounded on $\Omega_\lambda$, and infinite on $\mathcal{W} \setminus \Omega_\lambda$.

Note that Erdős–Rényi random graphs correspond to the constant base graphon $W_0 = p$—this model satisfies an LDP with speed $n^2$, and rate function $I_p$. However, in the general case, the function $I_{W_0}(\cdot)$ is not well-defined on the quotient space $\tilde{\mathcal{W}}$, and thus cannot be the rate function for our LDP. We introduce our candidate rate function $J_{W_0}$ on $\tilde{\mathcal{W}}$ as follows. To this end, we will use the symbols $B$ and $S$ to denote the closed balls in $\mathcal{W}$ and $\tilde{\mathcal{W}}$:

$$B(\tilde{f}, \varepsilon) = \{ g \in \mathcal{W} : \delta_\square(\tilde{f}, g) \leq \varepsilon \}$$

$$S(\tilde{f}, \varepsilon) = \{ \tilde{g} \in \tilde{\mathcal{W}} : \delta_\square(\tilde{f}, \tilde{g}) \leq \varepsilon \}.$$
Definition 9 (Rate function). The rate function is defined as

\[ J_{W_0}(\tilde{f}) = \begin{cases} \sup_{\delta > 0} \inf_{h \in B(\delta, 0)} I_{W_0}(h) & \tilde{f} \in \widehat{W}_\Omega \\ \infty & \tilde{f} \not\in \widehat{W}_\Omega. \end{cases} \]

In Section 2.1 we prove that \( J_{W_0} \) is lower semi-continuous on \( \widehat{W} \) (Lemma 4), and that it is bounded by some constant \( C(W_0) < \infty \) on \( \widehat{W}_\Omega \) (Proposition 1).

**Theorem 1.** For a block model \( W_0 \) with \( k \) uniform size blocks, the sequence \( \tilde{P}_{kn,W_0} \) obeys a large deviation principle in the space \( \widehat{W} \) (equipped with the cut metric \( \delta_\square \)) with rate function \( J_{W_0} \). Explicitly,

1. For any open set \( \tilde{U} \subseteq \tilde{W} \), \( \lim \inf_{n \to \infty} \frac{1}{(kn)^j} \log \tilde{P}_{kn,W_0}(\tilde{U}) \geq -\inf_{\tilde{f} \in \tilde{U}} J_{W_0}(\tilde{f}) \),

2. For any closed set \( \tilde{F} \subseteq \tilde{W} \), \( \lim \sup_{n \to \infty} \frac{1}{(kn)^j} \log \tilde{P}_{kn,W_0}(\tilde{F}) \leq -\inf_{\tilde{f} \in \tilde{F}} J_{W_0}(\tilde{f}) \).

(where we define the inf over the empty set to be \( \infty \).)

**Remark 2.** Note that any graphon with rational-length blocks is a uniform \( k \)-block graphon for some \( k \). Therefore, our result also describes large deviations events for any base graphon with rational-length blocks.

The proof of the LDP requires several new ideas, beyond those introduced in [14]. To explain the main additional difficulties, note that for the Erdős–Rényi random graph, \( W_0 \) is the constant graphon taking a value \( p \), and thus the cut-distance \( \delta_\square(W_0, f) \) to an arbitrary graphon \( f \in W \) is equal to the distance \( d_\square(W_0, f) \). Somewhat related, the relative entropy \( I_{W_0} \) is a well-defined rate function on equivalence classes \( f = \{ g : \delta_\square(f, g) = 0 \} \). Neither of these holds if \( W_0 \) is a block model with more than one block. To some extent, similar issues were faced in [20] in the context of large deviations for dense random graphs with given degrees. Our proof follows their general proof outline. However, the graphons \( W_0 \) considered in [20] are bounded away from zero and one, thus making the distinction between \( W \) and \( W_\Omega \) unnecessary. In contrast, the base graphon \( W_0 \) in our setting can have zero or one blocks—this creates many new analytic and probabilistic hurdles, and makes our analysis substantially more challenging.

### 1.3 LDP for graph parameters and the associated variational problem

In this section, we turn our attention to upper tail large deviations for continuous graph parameters.

**Definition 10.** A continuous graph parameter is a function \( \tau : \tilde{W} \to \mathbb{R} \) that is continuous with respect to \( \delta_\square \). We extend such a function \( \tau \) to \( W \) by setting \( \tau(f) = \tau(\tilde{f}) \), where as before, \( \tilde{f} \) is the equivalence class containing \( f \). We further write \( \tau(G) = \tau(f^G) \) for any graph \( G \). Finally, we set \( \tau_{\max}(\tilde{W}) = \max_{\tilde{f} \in \tilde{W}} \tau(\tilde{f}) \) and \( \tau_{\max}(\widehat{W}_\Omega) = \max_{\tilde{f} \in \widehat{W}_\Omega} \tau(\tilde{f}) \).

Note that by the compactness of \( \tilde{W} \) and \( \widehat{W}_\Omega \), the maxima in the above expressions are actually maxima and not suprema.

**Definition 11.** Let \( \tau \) be a continuous graph parameter. For \( W_0 \in W \) and \( t \leq \tau_{\max}(\tilde{W}) \) we set

\[ \phi_{\tau}(W_0, t) = \min \{ J_{W_0}(\tilde{f}) : \tilde{f} \in \tilde{W}, \tau(\tilde{f}) \geq t \}. \tag{5} \]

For \( t > \tau_{\max}(\widehat{W}_\Omega) \), we set \( \phi_{\tau}(W_0, t) = \infty \).

Note that continuity of \( \tau \) and compactness of \( (\tilde{W}, \delta_\square) \) imply that \( \{ \tilde{f} \in \tilde{W} : \tau(\tilde{f}) \geq t \} \) is compact. Since the lower semi-continuous function \( J_{W_0} \) (Lemma 3) attains its minimum on any compact set, it follows that \( \phi_{\tau}(W_0, t) \) is well defined. Note also that \( \{ \tilde{f} \in \tilde{W} : \tau(\tilde{f}) \geq t \} \) has non-empty intersection with \( \widehat{W}_\Omega \) if \( t \leq t_{\max} := \tau_{\max}(\widehat{W}_\Omega) \), in which case \( \phi_{\tau}(W_0, t) \leq C(W_0) \), and that and \( \{ \tilde{f} \in \tilde{W} : \tau(\tilde{f}) \geq t \} \cap \widehat{W}_\Omega = \emptyset \) and \( \phi_{\tau}(W_0, t) = \infty \) if \( t > t_{\max} \). So in particular, \( \phi_{\tau}(W_0, t) \) is discontinuous at \( t = t_{\max} \). In addition, \( \phi_{\tau}(W_0, t) = 0 \) if \( t \leq \tau(W_0) \), and \( \phi_{\tau}(W_0, t) > 0 \) on \( (\tau(W_0), t_{\max}) \). To see this, observe that if \( t \leq \tau(W_0) \), then \( \{ \tilde{f} \in \tilde{W} : \tau(\tilde{f}) \geq t \} \) contains the equivalence class \( W_0 \), and thus \( \phi_{\tau}(W_0, t) = 0 \). On the other hand, \( J_{W_0}(f) = 0 \) if \( t \geq \tau(W_0) \), and \( \phi_{\tau}(W_0, t) > 0 \) for \( t \in (\tau(W_0), t_{\max}) \).

Our next result establishes \( \phi_{\tau} \) as the rate function for the upper tail large deviation of the graph parameter \( \tau \). Moreover, this result proves that conditioned on the rare event, the random graph concentrates on the minimizers of (5). This result is a direct adaptation of [33] Theorem 2.7 to general \( k \)-block graphons \( W_0 \).
Theorem 2. Let $W_0$ be a uniform $k$-block graphon. Let $\tau$ be a continuous graph parameter, $t \leq \tau_{\text{max}}(\hat{W})$, and let $G_{kn}$ be the graph on $kn$ vertices sampled from $W_0$ according to the probability distribution $\mathbb{P}_{kn,W_0}$. Recall $\phi_t(W_0,t)$ from (5), and assume that $\phi_t(W_0,\cdot)$ is continuous at $t$. Then

$$\lim_{n \to \infty} \frac{1}{(kn)^2} \log \mathbb{P}_{kn,W_0}(\tau(G_{kn}) \geq t) = -\phi_t(W_0,t).$$

Set $F^*$ to be the set of minimizers of (5). Then $F^*$ is a non-empty compact subset of $\hat{W}$. If $t > \tau_{\text{max}}$, then $\mathbb{P}_{kn,W_0}(\tau(G_{kn}) \geq t) = 0$, and if $t < \tau_{\text{max}}$, then for $n$ sufficiently large and each $\varepsilon > 0$, there exists $C \geq C(\tau, \varepsilon, W_0, t) > 0$ such that

$$\mathbb{P}_{kn,W_0}\left(\delta_{G(G_{kn},F^*)} < \varepsilon \bigg| \tau(G_{kn}) \geq t\right) \geq 1 - e^{-Cn^2}.$$

In particular, if $\hat{F}^* = \{\hat{f}^*\}$ for some $\hat{f}^* \in \hat{W}$, then as $n \to \infty$, the conditional distribution of $\hat{G}^{G_{kn}}$ given the event $\tau(G_{kn}) \geq t$ converges to the point mass at $\hat{f}^*$.

**Remark 3.** Note that, in general, Theorem 2 holds only at the continuity points of $\phi_t(W_0, t)$. Remark 7 explains that $\phi_t$ has at most countably many points of discontinuity when $\tau$ is a continuous graph parameter. Moreover, we establish (see Lemma 16) that $\phi_t$ is continuous on $\mathbb{R} \setminus \{\tau_{\text{max}}\}$ if $\tau$ satisfies the “sufficient increase property” (Definition 11). In turn, Lemma 14 establishes that homomorphism densities $t(H, \cdot)$ have the sufficient increase property for all finite graphs $H$ and all step functions $W_0$, and Lemma 15 establishes that the operator norm has the sufficient increase property for a specific family of graphons $W_0$, namely those which generate bipartite Erdős–Rényi graphs.

So in particular, we know that for these graph parameters, the conclusions of Theorem 2 hold for all $t$, with the possible exception of $t = \tau_{\text{max}}$. As we will see in Theorem 5, for the graph parameter $\tau = t(H, \cdot)$, where $H$ is a finite $d$-regular graph, they also hold at $t = \tau_{\text{max}}$, in spite of the fact that $\phi_t(W_0, t)$ is not continuous at this point.

Theorem 2 establishes that typical behavior under the upper tail large deviation event is governed by the solutions of the variational problem (5). This directly motivates our subsequent investigations into the properties of this problem.

**Definition 12 (Symmetric Regime).** Let $W_0 \in B^{\gamma,*}$, and let $\tau$ be a continuous graph parameter. We say that $t \leq \tau_{\text{max}}$ is in the symmetric regime for $W_0$ and $\tau$ if all minimizers $\hat{g}$ of

$$\min_{f \in W} \{J_{W_0}(\hat{f}) : \tau(\hat{f}) \geq t\},$$

satisfy $\hat{g} \in B^{\gamma,*}$. We call the symmetric solution unique if a unique element of $\hat{W}_\Omega$ minimizes (5).

Theorem 2 implies that in the symmetric regime, the conditional distribution of the random graph concentrates on a set of graphons with block structure agreeing with $W_0$. In addition, when there is a unique symmetric solution, the graph concentrates on the point mass corresponding to this solution. Our subsequent results explore the existence of a symmetric regime for specific graph parameters.

Next we specialize to the graph parameter defined by $d$-regular subgraph densities, i.e., to the graph parameter $\tau : \hat{f} \mapsto t(H, \hat{f})$ for a $d$-regular graph $H$. In Section 1.3, we first show that for $\delta$ sufficiently small, $t = (1 + \delta)t(H, W_0)$ is in the symmetric regime of $W_0$ and this graph parameter, and that there is a unique solution to the variational problem. Then we show that when $\delta$ is sufficiently close to the maximum homomorphism density, $t$ is also in the symmetric regime. In Section 1.3, we study examples of two-block graphons $W_0$ that have a non-symmetric regime—this exhibits that in these examples, these two symmetric regimes are separated by a non-symmetric regime, establishing a “reentrant” phase transition phenomenon for large deviations in stochastic block models, analogous to the one established in [13, 14] for large deviations in Erdős–Rényi random graphs.

### 1.4 The existence of a symmetric regime for $d$-regular graphs

The next theorem establishes the existence of a unique symmetric regime for $\delta$ sufficiently small.

**Theorem 3.** Let $H$ be a $d$-regular graph, let $W_0 \in B^{\gamma,*}$, and let $t_{\text{max}} = \max_{f \in W_0} t(H, f)$. If $t(H, W_0) < t_{\text{max}}$, then there exists $\delta > 0$ sufficiently small such that for all $t \in [t(H, W_0), (1 + \delta)t(H, W_0))$, $t$ is in the symmetric regime for $W_0$ and $t(H, \cdot)$. Further, the symmetric optimizer is unique.

Next, we explore the variational problem near the maximum homomorphism density, and establish the existence of a symmetric regime in this setting. Note that the maximum homomorphism density of a fixed subgraph $H$ in a random graph drawn from $\mathbb{P}_{kn,W_0}$ is $\max_{f \in W_0} t(H, f)$. 7
Theorem 4. Let $H$ be a $d$-regular graph, let $W_0 \in B^{\gamma^+}$ and let $t_{\text{max}} = \max_{f \in W_0} t(H, f)$. If $t(H,W_0) < t_{\text{max}}$, then exists $\eta > 0$ such that for all $t \in (\eta, t_{\text{max}})$, $t$ is in the symmetric regime for $W_0$ and $t(H, \cdot)$. Further, the symmetric optimizer is unique.

Theorems 3 and 4 establish the existence of a symmetric regime for the homomorphism density of regular graphs. This is challenging due to the intractability of the rate function $J_{W_0}(\cdot)$, and is one of the main technical contributions of this paper. To this end, our first contribution is to establish that

$$\min\{J_{W_0}(\hat{f}) : \hat{f} \in W, \tau(\hat{f}) \geq t\} = \inf\{I_{W_0}(f) : f \in W, \tau(f) \geq t\}$$

(7)

under mild assumptions on the graph parameter $\tau$, which are satisfied for homomorphism densities and the operator norm (Lemma 20). This insight facilitates our subsequent analysis, and allows us to work with the relatively entropy functional $I_{W_0}$, instead of the complicated rate function $J_{W_0}$.

Even with this simplification, our proof is quite involved. To exhibit the existence of a symmetric phase, we will establish that for certain ranges of $t$ (depending on $W_0$), any minimizer of (6) is in $B^\gamma$. To this end, we will establish that if $\hat{f}$ is a minimizer of (6), there exists a sequence of block constant graphons $\{f_n : n \geq 1\} \subseteq B^\gamma$ such that $\delta \in (\hat{f}, f_n) \to 0$. This will imply that $\hat{f} \in B^\gamma$, as $B^\gamma$ is closed in $(\tilde{W}, \delta_t)$. We refer the reader to Section 4 for details on the construction of this sequence $\{f_n : n \geq 1\}$.

Next, we move to the issue of uniqueness. For the regimes of $t$ (depending on $W_0$) covered in Theorems 3 and 4 for any symmetric optimizer $\hat{f}$, we establish that $J_{W_0}(\hat{f}) = I_{W_0}(g)$ for some $g \in B^\gamma$ with $\hat{g} = \hat{f}$. It follows that

$$J_{W_0}(\hat{f}) = \min\{I_{W_0}(h) : h \in B^\gamma, \tau(H, h) \geq t\}.$$

Thus the uniqueness of the above minimum implies the uniqueness of the symmetric optimizer. To establish this uniqueness, we first show that the minimizer must satisfy the constraint with equality. Note that the problem of minimizing $I_{W_0}(h)$ subject to $\tau(H, h) = t$ on $B^\gamma$ is a finite dimensional optimization problem with a convex objective and a single polynomial equality constraint. We convert this constraint into an implicit equation for one of the finite dimensional coordinates in terms of the others, and then show that the existence of two distinct minimizers can be used to construct a function $h$ which has lower relative entropy than the supposed minimizers, giving a contradiction.

Combined with Theorem 2, these two theorems characterize the “typical" structure of the graph, conditioned on an upper tail large deviation event for the graph parameter $\tau = \tau(H, \cdot)$ in the vicinity of the endpoints of $[t(H, W_0), t_{\text{max}}]$. However, as stated, Theorem 2 applies only for $t \in [t(H, W_0), t_{\text{max}}]$. It is natural to wonder what happens when $t = t_{\text{max}}$. This is the content of the next theorem.

To state it, we recall the notation $W_0 = (p_{ab})_{a,b \in [m]}$ for a graphon with blocks $I_a \times I_b$, $a,b \in [m]$, and define a block $I_a \times I_b$ to be relevant if $p_{ab} > 0$ and $t(H, W_0)$ strictly decreases if $p_{ab}$ is lowered. Note that by definition, all blocks where $p_{ab} = 0$ are not relevant, while the blocks where $p_{ab} = 1$ may or may not be relevant.

Theorem 5. Let $W_0$ be a uniform $k$ block graphon. Let $H$ be a finite $d$-regular graph, let $t_{\text{max}} = \max\{t(H, \hat{f}) : \hat{f} \in W_0\}$, and let $f_{\text{max}}$ be the step function which is equal to 1 on all relevant blocks, and equal to $W_0$ on all irrelevant blocks. Then $f_{\text{max}}$ is the unique minimizer of (6) at $t_{\text{max}}$. Moreover, for any $\delta > 0$, there exists a constant $C > 0$ such that

$$P_{kn,W_0} \left( \delta \in (\hat{f}_{G_{kn}}, f_{\text{max}}) < \delta \mid t(H, G_{kn}) \geq t_{\text{max}} \right) \geq 1 - \exp\left(-Cn^2\right),$$

implying that as $n \to \infty$, the conditional distribution of $\hat{f}_{G_{kn}}$ given the event $\tau(G_{kn}) \geq t_{\text{max}}$ converges to the point mass at $f_{\text{max}}$.

Remark 4. If $\tau(\hat{f}) = \tau_{\text{max}}(W_0)$ has a unique solution $\hat{f}_{\text{max}}$, it is immediately clear that

$$P_{kn,W_0} \left( \delta \in (\hat{f}_{G_{kn}}, \hat{f}_{\text{max}}) = 0 \mid \tau(\hat{f}_{G_{kn}}) \geq \tau_{\text{max}}(W_0) \right) = 1,$$

implying that the conditional distribution of $\hat{f}_{G_{kn}}$ given the event $\{\tau(\hat{f}_{G_{kn}}) \geq \tau_{\text{max}}(W_0)\}$ is the point mass at $\hat{f}_{\text{max}}$. The equation $\tau(\hat{f}) = \tau_{\text{max}}(W_0)$ has a unique solution, for example, if $\tau = t(H, \cdot)$, where $H$ is a finite d-regular graph, and all blocks of $W_0$ that are subsets of $\Omega$ are relevant. In this case $f_{\text{max}} = 1_{W_0 > 0}$. This also holds if $W_0$ is a bipartite graphon with two blocks and $\tau(\hat{f}) = \|f\|_{op}$ (see Theorem 3), again with $f_{\text{max}} = 1_{W_0 > 0}$. See the Appendix for additional details.

The proof of Theorem 5 is relatively straightforward given the proofs of Theorems 2 to 4 and is deferred to the Appendix.
1.5 A non-symmetric regime in special cases

Next, we establish the existence of a non-symmetric regime in some specific families of two-block graphons. Let

\[ f_{p,q,r}(x, y) = \begin{cases} 
  p & \text{if } (x, y) \in [0, \gamma]^2 \\
  r & \text{if } (x, y) \in (\gamma, 1]^2 \\
  q & \text{otherwise.}
\]  

We show the existence of a non-symmetric regime for base graphons of the form \( f_{0,p,0}' \), \( f_{1,p,0}' \), \( f_{1,p,0} \) when \( p \) is sufficiently small. The first model corresponds to an Erdős–Rényi random graph with a planted independent set, while the second example covers Erdős–Rényi graphs with a planted clique. Finally, the third graphon leads to a bipartite Erdős–Rényi random graph with a planted clique in one of the partitions.

**Theorem 6.** Let \( 0 < \gamma < 1 \), let \( H \) be a d-regular graph, and assume that

1. \( W_p = f_{0,p,0}' \) and \( 0 < t < t(H, f_{0,0,1}) \), or
2. \( W_p = f_{1,p,0}' \) and \( t(H, f_{1,0,0}) < t < 1 \), or
3. \( W_p = f_{1,p,0} \) and \( t(H, f_{1,0,0}) < t < t(H, f_{1,1,0}) \).

Then there exists \( p_0 > 0 \) (whose value depends on which of the three cases we are considering) such that if \( p < p_0 \),

\[ \min\{I_{W_p}(\tilde{g}) : t(H, \tilde{g}) \geq t\} < \min\{I_{W_p}(\tilde{g}) : \tilde{g} \in \tilde{B}^\gamma, t(H, \tilde{g}) \geq t\}. \]

These statements imply that for \( p \) small enough, the optimizer of the variational problem is non-symmetric.

In Proposition 7, we show that \( \{\tilde{g} \in \tilde{B}^\gamma : t(H, \tilde{g}) \geq t\} \) is compact, which justifies the minimum on the right hand side in Theorem 6.

To establish this result, we recall that \( I_{W_0} \) is significantly more tractable than the rate function \( J_{W_0} \). Our first step (see Lemma 35) is to show that if \( W_0 \) is a graphon of the form \( f_{z_1,0,0}' \) or \( f_{z_2,0,0}' \) where \( z_1, z_2 \in \{0, 1\} \), then

\[ \min\{J_{W_0}(\tilde{f}) : \tilde{f} \in B^{(\gamma, 1-\gamma)}, \tau(f) \geq t\} = \min\{I_{W_0}(f) : f \in B^{(\gamma, 1-\gamma)} \cup B^{(1-\gamma, \gamma)}, \tau(f) \geq t\}. \]  

Next we show that for graphons \( W_p \) of the form \( W_p = f_{0,p,0}' \), \( W_p = f_{1,p,0}' \), \( W_p = f_{1,p,0} \) there exists \( p_0 > 0 \) such that if \( p < p_0 \),

\[ \inf\{I_{W_p}(f) : t(H, f) \geq t\} < \min\{I_{W_p}(f) : f \in B^{(\gamma, 1-\gamma)} \cup B^{(1-\gamma, \gamma)}, t(H, f) \geq t\} \]

for some range of \( t \) (Lemma 36). We establish this by constructing explicit graphons with lower entropy than that of all graphons in \( B^{(\gamma, 1-\gamma)} \cup B^{(1-\gamma, \gamma)} \). Together with 7 and 5, 9 implies the desired conclusion.
1.6 Bipartite Erdős–Rényi graphs: symmetry vs. symmetry breaking

Lubetzky and Zhao [33] characterize the symmetric regimes for $d$-regular subgraph counts and the largest eigenvalue in the Erdős–Rényi model. We extend these results to bipartite Erdős–Rényi random graphs. Let $f_p^\gamma$ denote the graphon $f_{0,p,0}^\gamma$, i.e.,

$$f_p^\gamma(x, y) = \begin{cases} 0 & (x, y) \in [0, \gamma]^2 \cup (\gamma, 1)^2 \\ p & \text{otherwise.} \end{cases}$$

![Figure 2: Illustration of the graphon $f_p^\gamma$.](image)

For $W_0$ of the form $f_p^\gamma$, the following theorem completely characterizes the symmetric and non-symmetric regime for $t(H, \cdot)$, where $H$ is a regular graph.

**Theorem 7.** Fix $0 < p < 1$ and $H$ a $d$-regular graph with $d \geq 1$. Let $W_0 = f_p^\gamma$. Let $r \in [p, 1]$ and define $t_r^\gamma = t(H, f_p^\gamma)$.

1. If $(r^d, h_p(r))$ lies on the convex minorant of $x \mapsto h_p(x^{1/d})$, then $t_r^\gamma$ is in the symmetric regime for $W_0$ and $t(H, \cdot)$. Moreover, $f_p^\gamma$ is the unique symmetric solution.

2. If $(r^d, h_p(r))$ does not lie on the convex minorant of $x \mapsto h_p(x^{1/d})$, then $t_r^\gamma$ is not in the symmetric regime of $W_0$ and $t(H, \cdot)$.

**Remark 5.** The symmetric regime for subgraph counts in Erdős–Rényi graphs [33] takes a similar form, with $t_r^\gamma$ replaced by $t(H, r)$, where $r$ denotes the constant graphon with value $r$. We can recover this result by setting $\gamma = 0$.

Finally, we characterize the symmetric regime for the largest eigenvalue. Similar to Erdős–Rényi graphs, the boundary for the symmetric regime for the largest eigenvalue coincides with that of the density of two-regular graphs.

**Definition 13.** For a graphon $f \in \mathcal{W}$, define the Hilbert–Schmidt kernel operator $T_f$ on $L^2([0, 1])$ by

$$(T_f u)(x) = \int_0^1 f(x, y)u(y)dy$$

for any $u \in L^2([0, 1])$. The operator norm is given by

$$\|f\|_{\text{op}} = \min\{c \geq 0 : \|T_f u\|_2 \leq c\|u\|_2 \text{ for all } u \in L^2([0, 1])\}.$$

**Lemma 1 ([33], Lemma 3.6).** The function $\|\cdot\|_{\text{op}}$ is a continuous extension of the normalized graph spectral norm, i.e., $\lambda_1(G)/n$ for a graph $G$ on $n$ vertices, to $(\mathcal{W}, \delta)$. Moreover, $\|\cdot\|_{\text{op}}$ is a continuous graph parameter.

**Theorem 8.** Fix $0 < p < 1$, and let $W_0 = f_p^\gamma$. Let $r \in [p, 1]$ and define $t_r^\gamma = \|f_p^\gamma\|_{\text{op}}$.

1. If $(r^2, h_p(r))$ lies on the convex minorant of $x \mapsto h_p(x^{1/2})$, then $t_r^\gamma$ is in the symmetric regime for $W_0$ and $\|\cdot\|_{\text{op}}$. Moreover, $f_p^\gamma$ is the unique symmetric solution.

2. If $(r^2, h_p(r))$ does not lie on the convex minorant of $x \mapsto h_p(x^{1/2})$, then $t_r^\gamma$ is not in the symmetric regime for $W_0$ and $\|\cdot\|_{\text{op}}$.

**Remark 6.** It is not hard to see that for $\tau = t(H, \cdot)$ and $\tau = \|\cdot\|_{\text{op}}$, the function $r \mapsto \tau(f_p^\gamma)$ is a continuous and non-decreasing function on $[p, 1]$ and that $\tau(f_p^\gamma) = \max_{f \in \mathcal{W}_{Ht}} \tau(f) = t_{\text{max}}$. Thus Theorems 7 and 8 cover the full range $[\tau(W_0), t_{\text{max}}]$. 

10
To establish Theorems 7 and 8, we follow the general approach introduced in [33]. Lemma [33] implies that (8) holds for \( \tau(g) = t(H,g) \) where \( H \) is a \( d \)-regular graph or \( \tau(g) = ||g||_{op} \), meaning that we can again reason about symmetry through the function \( I_{W_0} \) rather than \( J_{W_0} \). For \( r \in (0,1] \), let \( f_r^* \) be the bipartite graphon with value \( r \), and \( t_r^* = t(H, f_r^*) \) be the corresponding homomorphism density of a \( d \)-regular graph \( H \). We apply a generalized Hölder inequality to show that whenever \( f \in \mathcal{W}_0 \) satisfies \( t(H,f) \geq t(H,f_r^*) \), it holds that \( ||f||_d^d \geq 2\gamma(1-\gamma)r^d \) (Lemma [11]). Finally, we show that if \( (r^d,h_p(r)) \) lies on the convex minorant of \( x \mapsto h_p(x^{1/d}) \) and \( ||f||_d^d \geq 2\gamma(1-\gamma)r^d \), then \( I_{W_0}(f) \geq I_{W_0}(f_r^*) \), with equality occurring if and only if \( f = f_r^* \) (Lemma [10]). To establish the non-symmetric regime, we show that whenever \( (r^d,h_p(r)) \) is not on the convex minorant, we can construct a graphon \( g \) with \( t(H,g) > t(H,f_r^*) \) and \( I_{W_0}(g) < I_{W_0}(f_r^*) \) (Lemma [39]). This construction is more complicated than the one in [33], due to the bipartite nature of the underlying graph (see Figure 8 for the construction). The proof for the spectral norm \( \tau(g) = ||g||_{op} \) follows using similar arguments.

1.7 History and related work

The upper tail large deviation problem for subgraphs of \( G(n,p) \) has attracted considerable attention in Probability and Combinatorics. Chatterjee and Varadhan found the precise constant in the large deviation probability in the dense case by applying the theory of graph limits [14]. This approach does not work in the sparse regime where \( p \to 0 \), as graphon theory only applies to dense graphs.

The challenge of deriving an LDP for sparse graphs has attracted considerable attention in recent years. In the sparse regime, even determining the right order of this probability on the exponential scale proved to be considerably challenging. Following partial advances [26, 27, 28, 29, 35], this was finally resolved for \( H = K_3 \) in [11, 16]. Subsequently, [17] identified the right order of this probability for \( H = K_r, r \geq 4 \), and formulated a conjecture regarding the correct order for general subgraphs. See [34] for a recent counterexample to this general conjecture. Recently, the development of general theory [12, 22, 2] and problem-specific ideas [15, 1, 24, 3] have contributed to rapid progress on large deviations in the sparse setting. These results relate the large deviation probability to an entropic variational problem. In turn, some of these variational problems have also been solved [5, 5, 4], leading to deep insights regarding the structure of the random graph, conditioned on the rare event.

We emphasize that these remarkable results are mostly applicable for sparse random graphs or hypergraphs [30], and do not shed any direct insight on the problem considered in this paper. Instead, our work is the first step towards a full generalization of the work of [13] and [35] to block models. As [11] did for Erdős–Rényi graphs, we establish an LDP for block models and demonstrate the existence of a reentrant phase transition for the upper tail of \( d \)-regular subgraph counts. While we exhibit a reentrant phase transition for a limited class of block models, we show the existence of a symmetric regime for arbitrary block models. Our methods are inspired by the work of [33], which completely characterizes the symmetric and non-symmetric regimes for Erdős–Rényi graphs. Moreover, analogous to [33], we fully characterize the symmetric and non-symmetric regimes for bipartite Erdős–Rényi graphs. As discussed in the introduction, our work fits into the broader theme of large deviations for dense random graphs with inhomogeneities or constraints, and provides the first rigorous analysis of the large deviations problem for dense block models.

Outline: The rest of the paper is structured as follows. We establish our main LDP results, Theorem 1 and Theorem 2 in Section 2. In Section 3 we derive some analytic properties of \( \phi_r \) which are crucial in the analysis of the variational problem. Section 4 establishes the existence of a symmetric regime in the upper tail, while Section 5 establishes the existence of a non-symmetric regime in specific examples. Finally, we characterize the symmetric regime in Erdős–Rényi bipartite models in Section 6. We finish with some open problems in Section 7.

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2 Large deviation principle

In this section we establish the LDP. Since the space \((\mathcal{W}, \delta_{\infty})\) is compact, it will be enough to prove the bounds in Theorem 1 for balls in the metric \(\delta_{\infty}\); the precise statement is given in the following lemma.

**Lemma 2.** Since the space \((\mathcal{W}, \delta_{\infty})\) is compact, the bounds in Theorem 1 are equivalent to

1. For all \(\epsilon > 0\) and \(h \in \mathcal{W}\), \(\liminf_{kn \to \infty} \frac{1}{kn} \log \tilde{P}_{kn, W_0} (S(h, \epsilon)) \geq -J_{W_0}(h)\);

2. For all \(\tilde{g} \in \mathcal{W}\), \(\limsup_{kn \to \infty} \frac{1}{kn} \log \tilde{P}_{kn, W_0} (S(\tilde{g}, \epsilon)) \leq -J_{W_0}(\tilde{g})\).

The proof is standard (see e.g., [19, Theorems 4.1.11, 4.1.18]), and is thus omitted.

In Section 2.1, we begin by establishing several useful facts about the rate function and the space \(W_0\). We establish the LDP lower and upper bounds in Sections 2.2 and 2.3 respectively. Finally, in Section 2.4, we prove Theorem 2, which establishes upper tail large deviations for continuous graph parameters.

2.1 Preliminaries

In this section, we establish several useful analytical properties, most notably that \(\mathcal{W}_0\) is closed and that \(J_{W_0}\) is lower semi-continuous on \((\mathcal{W}, \delta_{\infty})\).

**Lemma 3.** For all \(W_0 \in B^r\), the set \(\mathcal{W}_0\) is closed in \(\mathcal{W}\) with respect to the cut metric topology \((\mathcal{W}, \delta_{\infty})\).

**Lemma 4.** For all \(W_0 \in B^r\), the function \(J_{W_0}(\cdot)\) is lower semi-continuous on \((\mathcal{W}, \delta_{\infty})\).

We start by stating some elementary properties of the relative entropy \(h_p(\cdot)\).

**Lemma 5.** Let \(\beta \in (0, 1/2)\) and let \(G_p(a, q) = aq - \log(p e^a + 1 - p)\). Then the following holds

(i) For all \(p \in [\beta, 1 - \beta]\), \(\|h_p\|_{\infty} \leq \log(2/\beta)\).

(ii) The family of functions \((h_p)_{p \in [\beta, 1 - \beta]}\) is equicontinuous on \([0, 1]\).

(iii) For all \(p \notin (0, 1), h_p(q) = \sup_{a \in G} G_p(a, q)\).

(iv) For \(q \notin (0, 1), \) the sup in (iii) is achieved by \(a = \log \left(\frac{\sqrt[q]{p} - \sqrt[q]{\frac{1-p}{p}}}\right)\).

**Proof.** (i) Follows by observing that \(|x \log x + (1-x) \log(1-x)| \leq \log 2\) and \(|x \log p| + |(1-x) \log p| \leq x |\log \beta + x| \log \beta| = -\log \beta|.

(ii) Follows from uniform continuity of the function \(x \mapsto x \log x + (1-x) \log(1-x)\).

(iii) and (iv) Is an elementary exercise and left to the reader.

The function \(I_{W_0}\) also comes up naturally in \([20]\) —however, in \([20]\) it is assumed that the base graphon \(W_0\) is bounded away from zero and one, and thus the function \(I_{W_0}\) is necessarily finite. This is not the case in our context. We use \(Im(W_0)\) to denote the image of \(W_0\) in \([0, 1]\).

**Proposition 1.** Let \(W_0 \in W\). If \(f \notin W_0\), then \(I_{W_0}(f) = \infty\). If \(f \in W_0\) and \(W_0\) obeys the assumption

\[ \beta = \inf \{w > 0 : w \in Im(W_0) \text{ or } 1 - w \notin Im(W_0)\} > 0, \]

then \(I_{W_0}(f) \leq \frac{1}{\beta} \log(2/\beta)\).

**Proof.** Since \(h_0(x) = \infty\) for \(x \neq 0\) and \(h_1(x) = \infty\) for \(x \neq 1\), it follows that \(I_{W_0}(f) = \infty\) when \(f \notin W_0\). To bound \(I_{W_0}(f)\) for \(f \in W_0\), observe that

\[ I_{W_0}(f) = \frac{1}{2} \int_{\Omega} h_{W_0(x,y)}(f(x,y)) \, dx \, dy, \]

since \(f\) and \(W_0\) agree on \(\Omega^r\). The proof is completed by invoking Lemma 5 (i).

**Proposition 2.** Let \(\epsilon > 0\) and assume that \(W_0 \in W\) obeys the condition \(10\). Then there exists \(\eta > 0\) such that if \(f, g \in W_0\) and \(\|f - g\|_{\infty} \leq \eta\), then \(|I_{W_0}(f) - I_{W_0}(g)| \leq \epsilon\).

**Proof.** Fix \(\epsilon > 0\). By Lemma 5 (ii) there exists an \(\eta > 0\) such that \(|h_p(u) - h_p(v)| \leq \epsilon\) whenever \(|u - v| \leq \eta\) and \(p \in [\beta, 1 - \beta]\). Inserted into \(11\) this, completes the proof.

We derive a variational representation for \(I_{W_0}\) using convex duality.

**Proposition 3.** Let \(W_0 \in W\) and let \(S\) be the set of all symmetric functions in \(L_2([0,1]^2)\). For \(a \in S\) and \(f \in W\), define

\[ K_{W_0}(f, a) = \int_{[0,1]^2} [a(x,y)f(x,y) - \log \left(W_0(x,y)e^{a(x,y)} + 1 - W_0(x,y)\right)] \, dx \, dy. \]

Then \(I_{W_0}(f) = \frac{1}{\beta} \sup_{a \in S} K_{W_0}(f, a)\).
Proof. First, we consider the case \( f \not\in \mathcal{W}_0 \) (in which case \( I_{\mathcal{W}_0}(f) = \infty \) by Proposition [1]). Then there exists \( \Gamma \subseteq [0,1]^2 \) with positive measure such that \( W_0(x,y) \in \{0,1\} \) and \( W_0(x,y) \neq f(x,y) \) for \((x,y) \in \Gamma\). Choosing

\[
a_M(x,y) = \begin{cases} 0 & (x,y) \not\in \Gamma \\ M & W_0(x,y) = 0 \\ -M & W_0(x,y) = 1. \end{cases}
\]

and taking \( M \to \infty \), we see that \( \sup_{a \in S} K_{\mathcal{W}_0}(f,a) = \infty \) in this case.

Next we consider the case \( f \in \mathcal{W}_0 \). Recalling the definition of \( G_p \) from Lemma [5] and noting that the integrand in [12] is zero if \( W_0(x,y) = f(x,y) \in \{0,1\} \), we then have \( K_{\mathcal{W}_0}(f,a) = \int_{\Omega} G_{\mathcal{W}_0(x,y)}(a(x,y),f(x,y))dxdy. \)

Combined with ([11] and Lemma [5](iii), this shows that

\[
I_{\mathcal{W}_0}(f) = \frac{1}{2} \int_{\Omega} h_{\mathcal{W}_0(x,y)}(f(x,y))dxdy \geq \frac{1}{2} \sup_{a \in S} \int_{\Omega} G_{\mathcal{W}_0(x,y)}(a(x,y),f(x,y))dxdy = \frac{1}{2} \sup_{a \in S} K_{\mathcal{W}_0}(f,a).
\]

To prove equality, we may w.l.o.g. assume that the right hand side is finite. We may further restrict the integrals on both sides to the subset \( \Omega \subseteq \Omega \) where \( f(x,y) \not\in \{0,1\} \), since the contributions of both sides to the complement can easily be seen to be equal. Finally, on \( \Omega \), we may use Lemma [5](iv) to conclude that

\[
\int_{\Omega} h_{\mathcal{W}_0(x,y)}(f(x,y))dxdy = \int_{\Omega} G_{\mathcal{W}_0(x,y)}(a_0(x,y),f(x,y))dxdy
\]

where \( a_0(x,y) = \log \left( \frac{1 - f(x,y)}{\mathcal{W}_0(x,y)} \right) \). If \( a_0 \in L^2(\Omega) \), the right hand side is bounded by the sup over all \( a \in S \), giving the desired upper bound. If it is not, we replace \( \Omega \) on both sides by its intersection with the set of points for which \( |a_0(x,y)| \leq M \) before bounding the right hand side by a sup over all square integrable \( a \). The proof is concluded by using the monotone convergence theorem.

The following propositions will be used to establish the lower semi-continuity of \( I_{\mathcal{W}_0} \). We recall that for any pseudo-metric space \( (S,d) \), the pseudo-metric induces a topology on \( S \), namely the topology generated by the open balls. With a slight abuse of notation, we denote this topological space by \( \mathcal{C} \). Finally, as in the case of a topology generated by a metric, a set \( C \subset \mathcal{W} \) is closed if and only if the limit of every convergent sequence with elements in \( \mathcal{C} \) lies in \( \mathcal{C} \) as well.

**Proposition 4.** For \( \mathcal{W}_0 \in \mathcal{B}^\ast \), the set \( \mathcal{W}_0 \) is a closed subset of \( \mathcal{W} \) with respect to the topology \((\mathcal{W},d_\mathcal{C})\).

**Proof.** Let \( \{f_n : n \geq 1\} \subseteq \mathcal{W}_0 \) be a convergent sequence of graphons satisfying \( d_\mathcal{C}(f_n,f) \to 0 \) for some \( f \in \mathcal{W} \). Then on each block \( I \times J \) of \( \mathcal{W}_0 \) that takes value 0 or 1, \( f_n = W_0 \) for all \( n \geq 1 \). Since \( |\int_{I \times J} f_n - f| \leq d_\mathcal{C}(f_n,f) \) and \( d_\mathcal{C}(f_n,f) \to 0 \), it follows that \( f = W_0 \) on \( I \times J \). Thus \( f \in \mathcal{W}_0 \).

**Proposition 5.** Let \( (S,d) \) be a pseudo metric space, and \( F \subseteq S \) be a closed subset of \( S \). Let \( f : F \to \mathbb{R} \) be a lower semi-continuous function on \( (F,d) \). The extension \( f^* : S \to \mathbb{R} \cup \{\infty\} \) where

\[
f^*(x) = \begin{cases} f(x) & x \in F \\ \infty & x \in S \setminus F \end{cases}
\]

is lower semi-continuous on \( (S,d) \).

**Proof.** We show that \( f^* \) is lower semi-continuous on \( S \) by demonstrating that for all \( \alpha \in \mathbb{R} \), the set \( \{x \in S | f^*(x) > \alpha \} \) is open. Observe

\[
\{x \in S | f^*(x) > \alpha \} = \{x \in F | f(x) > \alpha \} \cup (S \setminus F).
\]

By lower semi-continuity of \( f \) on \( F \), \( A = \{x \in F | f(x) > \alpha \} \) is open in \( F \), and so \( A^c \) is relatively closed with respect to \( F \) and therefore closed in \( S \) (since \( F \) is closed). It follows that

\[
\{x \in S | f^*(x) > \alpha \}^c = F \cap A^c
\]

is closed, and so we conclude that \( \{x \in S | f^*(x) > \alpha \} \) is open.

**Lemma 6.** For \( \mathcal{W}_0 \in \mathcal{B}^\ast \), the function \( I_{\mathcal{W}_0}(\cdot) \) is lower semi-continuous on \( \mathcal{W},d_\mathcal{C} \).
Proof. First, note that by Propositions 4 and 5 and the observation that $I_{W_0}(f) = \infty$ for all $f \in \mathcal{W} \setminus \mathcal{W}_1$, it is enough to establish that $I_{W_0}$ is lower semi-continuous on $(\mathcal{W}_1, d_\mathcal{W})$. Second, by Proposition 8, $I_{W_0}$ can be written as supremum over the functions $K_{W_0}(\cdot, a)$, so it will be enough to show that for all $a \in S$, the function $K_{W_0}(\cdot, a)$ is continuous on $(\mathcal{W}_1, d_\mathcal{W})$.

Consider two functions $f, g \in \mathcal{W}_2$, and observe that every $a \in S$ can be approximated by stepfunctions in $L^2$. Given $\varepsilon$ and $a$ we can therefore find $k < \infty$ and a $k$-step function $a_k$ such that $\|a - a_k\|_2 \leq \frac{\varepsilon}{2}$. As a consequence

$$\left| K_{W_0}(f, a) - K_{W_0}(g, a) \right| = \left| \int (f - g) \right| \leq \|f - g\|_1 \leq \|a_k(f - g)\|_1 + \frac{\varepsilon}{2} \leq \|a_k\|_\infty k^2 \|d_\mathcal{W}(f, g) + \frac{\varepsilon}{2}. $$

This is smaller than $\varepsilon$ if $d_\mathcal{W}(f, g) < \varepsilon/(\|a_k\|_\infty k^2)$, which proves that $K_{W_0}(\cdot, a)$ is continuous on $(\mathcal{W}_1, d_\mathcal{W})$, as required.

The following proposition will be used to prove that for any continuous graph parameter $\tau$, the function $\phi_\tau(\mathcal{W}_0, \cdot)$ is strictly positive on $(\tau(\mathcal{W}_0), t_{\text{max}}]$.

**Proposition 6.** For $W_0 \in \mathcal{W}$ and $f \in \mathcal{W}_1$, it holds that

$$J_{W_0}(f) = 0 \text{ if and only if } \delta_\mathcal{W}(f, W_0) = 0.$$

**Proof.** Noting that our definition of $J_{W_0}$ agrees with the definition of $J_{W_0}$ given in [20] whenever $f \in \mathcal{W}_1$, and that $\delta_\mathcal{W}(f, W_0) > 0$ if $f \notin \mathcal{W}_1$, the proposition follows from the analogous statement in [20] Lemma 2.2.

Our next result establishes that $\mathcal{W}_0$ is closed in the cut metric. For a partition $\mathcal{P}$ of $[0, 1]$, we define $W_\mathcal{P}$ as the step function graphon that is obtained by averaging over all blocks induced by the partition classes. Setting $\Gamma(x) \subseteq [0, 1]$ to be the partition class in $\mathcal{P}$ that contains $x$, we obtain

$$W_\mathcal{P}(x, y) = \frac{1}{|\Gamma(x)| \cdot |\Gamma(y)|} \int_{\Gamma(x) \times \Gamma(y)} W(u, v) \, du \, dv.$$ 

We call $\mathcal{P}$ an equipartition if all classes have the same measure, and use $|\mathcal{P}|$ to denote the number of classes in $\mathcal{P}$. Note that up to sets of measure zero, there is just one equipartition of $[0, 1]$ into $n$ intervals; for definiteness and consistency with our previous conventions, we use the partition $\{(0, 1/n], (1/n, 2/n], \ldots, ((n - 1)/n, 1]\}$.

**Lemma 7** (Corollary 3.4 of [8]). Let $f \in \mathcal{W}$ and $s$ be a positive integer. For every equipartition $\mathcal{Q}$ of $[0, 1]$, there is an equipartition $\mathcal{P}$ with $s|Q|$ classes such that $\mathcal{P}$ refines $\mathcal{Q}$ and

$$d_\mathcal{W}(f, f_\mathcal{P}) \leq \sqrt{\frac{20}{\log_2 s}}.$$ 

The next lemma follows from Lemma 7.

**Lemma 8.** Let $W_0 \in \mathcal{W}_1$. Then there exists a sequence of refining partitions $\mathcal{P}_k$ of $[0, 1]$ into equal length intervals such that for all $f \in \mathcal{W}_1$ there exists a sequence of step functions $f_k \in \mathcal{W}_1$ with steps in $\mathcal{P}_k$ such that (i) $(f_{k+1})_{\mathcal{P}_k} = f_k$ and (ii) $\delta_\mathcal{W}(f, f_k) \leq 1/k$ for all $k \geq 1$.

**Proof.** Let $s_k$ be such that $\sqrt{20/\log_2 s_k} \leq 1/k$, let $q_1$ be such that the lengths of the intervals described by $\gamma$ are integer multiples of $1/q_1$, define $q_k$ inductively by $q_k = s_k q_{k-1}$, and let $\mathcal{P}_k$ be the partition of $[0, 1]$ into intervals of length $1/q_k$. We will define $f_k$ as $f_k = (g_k)_{\mathcal{P}_k}$ where $g_k \in \mathcal{W}_1$ will be inductively be defined in such a way that (a) $(g_k)_{\mathcal{P}_{k-1}} = (g_{k-1})_{\mathcal{P}_{k-1}}$ for all $k \geq 2$, (b) $d_\mathcal{W}(f, g_k) = 0$ for all $k \geq 1$ and (c) $d_\mathcal{W}(g_k, (g_k)_{\mathcal{P}_k}) \leq 1/k$ for all $k \geq 1$. This clearly implies the statement of the lemma, since $g_k \in \mathcal{W}_1$ implies $g_k \in \mathcal{W}_0$ by the fact that $\mathcal{P}_k$ is a refinement of $\mathcal{P}_1$, 

$$(f_{k+1})_{\mathcal{P}_k} = (g_{k+1})_{\mathcal{P}_{k+1}} = (g_{k+1})_{\mathcal{P}_k} = (g_k)_{\mathcal{P}_k} = f_k$$

by (a) and the fact that $\mathcal{P}_{k+1}$ is a refinement of $\mathcal{P}_k$, and the two statements (b) and (c) imply (ii).

We start our inductive construction by setting $q_1 = f$. Noting that $d_\mathcal{W}(h, h') \leq 1$ for all $h, h' \in \mathcal{W}$, this shows that $q_1$ satisfies the inductive assumptions.

Let $k \geq 2$ and assume that $g_{k-1}$ satisfies the inductive assumption. By Lemma 8, we can find an equipartition $\mathcal{Q}_k$ of $[0, 1]$ into $q_k = s_k |\mathcal{P}_{k-1}|$ classes such that $\mathcal{Q}_k$ refines $\mathcal{P}_{k-1}$ and $d_\mathcal{W}(g_{k-1}, (g_{k-1})_{\mathcal{Q}_k}) \leq 1/k$. We now define a measure preserving bijection $\phi : [0, 1] \to [0, 1]$ as follows: Let $I$ be an interval.
in $P_{k-1}$, and let $Y_1, \ldots, Y_{s_k}$ those elements of $Q_k$ subdividing $I$. By Theorem A.7 in [25], we can find a measure preserving bijection from $I$ to itself such that the image of $Y_1, \ldots, Y_{s_k}$ are the $s_k$ intervals in $P_k$ that subdivide $I$. Doing this for all intervals in $Q_k$ we obtain a measure preserving bijection $\phi$ such that the images of the partition classes of $Q_k$ are the partition classes of $P_k$, and such that $\phi$ maps each interval in $P_{k-1}$ onto itself. Applying this bijection to $g_{k-1}$ gives a graphon $g_k \in W_{Q_k}$ such that $d_\square(g_k, g_{k-1}) = 0$ and $d_\square(g_k, (g_k)_{P_k}) = d_\square(g_{k-1}, (g_{k-1})_{Q_k}) \leq 1/k$. By the inductive assumption (b) we have that $d_\square(g_k, f) \geq 0$, and by the fact that $\phi$ maps each interval in $P_{k-1}$ onto itself we have that $(g_k)_{P_{k-1}} = (g_{k-1})_{P_{k-1}}$. This completes the inductive proof.

We show that $\tilde{W}_\Omega$ is closed, using ideas from the proof that $(\tilde{W}, \delta_\square)$ is compact [32, Theorem 5.1].

**Proof of Lemma 8** We establish the lemma by showing that $\tilde{W}_\Omega$ contains its limit points. Let $(\tilde{W}_n)_{n \geq 0}$ be a sequence of graphons in $\tilde{W}_\Omega$ that converges to $\tilde{W} \in \tilde{W}$. Since $\tilde{W}_n \in \tilde{W}_\Omega$, we may choose a sequence $W_n \in W_\Omega$ such that $d_\square(W_n, \tilde{W}) \to 0$. We claim that $\tilde{W} \in W_\Omega$.

By Lemma 8 we can find a sequence of refining partitions $P_k$ of $[0, 1]$ into intervals of length $1/|P_k|$ and sequences $W_n, k \in W_\Omega$ such that

(i) $d_\square(W_n, W_n, k) \leq 1/k$

(ii) $(W_{n, k+1})_{P_k} = W_n, k$.

Next we claim that it is possible to replace $(W_n)$ with a subsequence such that for all $k$, $W_{n, k}$ converges almost everywhere to a step function $U_k$ with steps made out of the intervals in $P_k$. Indeed, select a subsequence of $(W_n)$ such that the value of $W_n$ converges on the product of all intervals $I, J \in P_k$. We obtain $W_{n, 1} \to U_1$ almost everywhere for $U_1$ a step function on $s_k$ intervals of $[0, 1]$. Taking further subsequences for $k = 2, 3, \ldots$, we obtain a subsequence of $(W_n)$ such that $W_{n, k} \to U_k$ almost everywhere for all $k$. By the Dominated Convergence Theorem, $\|W_{n, k} - U_k\|_1 \to 0$, and so $d_\square(W_{n, k}, U_k) \to 0$. Each $U_k$ is a step function on $s_k$ intervals of $[0, 1]$. Note that since $W_{n, k} \in W_\Omega$ and $d_\square(W_{n, k}, U_k) \to 0$, Proposition 4 implies that $U_k \in W_\Omega$. For the remainder of this proof, we replace $(W_n)$ with this subsequence; doing so does not change the limit of the corresponding sequence in $\tilde{W}$.

Next we claim that the sequence $(U_k)_{k \geq 1}$ has a limit $U$ in $W_\Omega$. It follows from (ii) that $U_k = (U_k)_{P_k}$ for all $k$. Let $(x, y)$ be a uniform random point in $[0, 1]$. Since $U_k = (U_k)_{P_k}$, the sequence $(U_k(x, y), U_k(x, y), \ldots)$ is a martingale with respect to the canonical filtration. The random variables $U_k(x, y)$ are bounded, and so the Martingale Convergence Theorem [21, Theorem 4.2.11] implies that the sequence $(U_k(x, y), U_k(x, y), \ldots)$ converges with probability one. Thus there exists $U \in \tilde{W}$ such that $U_k \to U$ almost everywhere. By the Dominated Convergence Theorem, $\|U_k - U\|_1 \to 0$ and therefore $d_\square(U_k, U) \to 0$. Since $U_k \in W_\Omega$ for all $k$, and $W_\Omega$ is closed, it follows that $U \in W_\Omega$. Moreover $U \in \tilde{W}_\Omega$.

It remains to show that $d_\square(U, \tilde{W}) \to 0$ (as this implies that $d_\square(\tilde{W}, U) \to 0$, which establishes that the limit of the sequence is in $W_\Omega$). Let $\varepsilon > 0$. Choose $k > 3\varepsilon$ sufficiently large such that $\|U - U_k\|_1 < \varepsilon/3$. For this fixed $k$, there exists $n_0$ such that $\|U_k - W_{n, k}\|_1 \leq \varepsilon/3$ for all $n \geq n_0$. Observe that

$$
\delta_\square(U, W_n) \leq d_\square(U, U_k) + d_\square(U_k, W_{n, k}) + \delta_\square(W_{n, k}, W_n)
\leq \|U - U_k\|_1 + \|U_k - W_{n, k}\|_1 + \delta_\square(W_{n, k}, W_n) \leq \varepsilon.
$$

**Proof of Lemma 9** We modify the proof of [20, Lemma 2.1] to allow for $W_\Omega$ with values in $[0, 1]$. For $f \in \tilde{W}$, let

$$
H(f) = \inf_{g \in W : \delta_\square(g, f) = 0} I_{W_\Omega}(g).
$$

If $f \notin \tilde{W}_\Omega$, then $g \notin W_\Omega$ for all $g$ contributing to the infimum and by Proposition 1 $H(f) = \infty$. On the other hand, if $f \in \tilde{W}_\Omega$, there exists a $g \in W_\Omega$ contributing to the infimum, so with the help of Proposition 1 we conclude that $H(\cdot)$ is bounded on $\tilde{W}_\Omega$. Combined with the fact that $h \in B(\tilde{f}, \delta) \iff \delta_\square(\tilde{g}, h) = 0$ for some $\tilde{g} \in B(\tilde{f}, \delta)$,

we obtain that for $f \in \tilde{W}_\Omega$

$$
J_{W_\Omega}(f) = \sup_{\delta > 0} \inf_{h \in B(\tilde{f}, \delta)} I_{W_\Omega}(h) = \sup_{\delta > 0} \inf_{\tilde{g} \in B(\tilde{f}, \delta)} H(\tilde{g}) = \sup_{\delta > 0} \inf_{\tilde{g} \in B(\tilde{f}, \delta) \cap W_\Omega} H(\tilde{g}) = \lim_{h \to f} H(h).
$$

we obtain that for $f \in \tilde{W}_\Omega$
Therefore, $J_{W_0}(\cdot)$ is the pointwise limit in of a bounded function. This implies that $J_{W_0}(\cdot)$ is lower semi-continuous on $\tilde{W}_\Omega$.

Note that $J_{W_0}(f) = \infty$ for all $f \in \tilde{W} \setminus \tilde{W}_\Omega$. Therefore the lower semi-continuity of $J_{W_0}(\cdot)$ on $\tilde{W}$ follows by Proposition 5 and Lemma 8.

We close this preliminary section with a proposition and a lemma which will be used in the proofs of Theorems 3, 4 and 6.

**Proposition 7.** Let $\tau$ be a continuous graph parameter. Then the following holds:

(i) The set $\{g \in \tilde{B}^\tau : \tau(g) \geq t\}$ is a compact set in $(\tilde{W}, d_{c\tau})$.

(ii) The set $\{g \in \tilde{B}^\tau : \tau(g) > t\}$ is a closed set in $(\tilde{W}, d_{c\tau})$.

**Proof.** Since $\tilde{W}$ is compact, it suffices to show that $\{g \in \tilde{B}^\tau : \tau(g) \geq t\}$ is closed. Let $f_n \in \tilde{B}^\tau$ be such that $\tau(f_n) \geq t$ and $f_n$ converges to some graphon $\tilde{f}$. Since $\tau$ is continuous, $\lim_{n \to \infty} \tau(f_n) = \tau(\tilde{f}) \geq t$.

It remains to show that $f \in \tilde{B}^\tau$. Without loss of generality, we may assume $f_n \in B^\tau$, and write $f_n = (\alpha_{ij}^n)_{i,j \in [m]}$, where each $\alpha_{ij}^n \in [0, 1]$. By the compactness of $[0, 1]^{m^2}$, there exists a subsequence such that

$$\alpha_{ij}^{n_k} \to \beta_{ij} \text{ for all } i,j \in [m].$$

Let $g = (\beta_{ij})_{i,j \in [m]} \in B^\tau$. Since $f_{n_k} \to g$ pointwise and $d_{\square}(f_{n_k}, g) \leq \|f_{n_k} - g\|_1$, the Dominated Convergence Theorem implies that $d_{\square}(f_{n_k}, g) \to 0$. Since $d_{\square}(f_n, \tilde{f}) \to 0$, we have $d_{\square}(g, \tilde{f}) = 0$ and thus $f \in \tilde{B}^\tau$.

The proof of the second statement is identical, except that it starts from a sequence $f_n \in B^\tau$ such that $\tau(f_n) \geq t$ and $f_n$ converges to some $f \in W$ in the metric $d_{c\tau}$.

**Lemma 9.** Suppose $f \in W$ is of the form $f = \sum_{i,j \in [k]} \beta_{ij}1_{Y_i}1_{Y_j}$ with $\beta_{ij} = \beta_{ji} \in [0, 1]$ and $Y_1, \ldots, Y_k$ form a partition of $[0, 1]$ into measurable sets. Let $g \in \tilde{W}$. Then $\delta_{\square}(f, g) = 0$ if and only if there exists a partition of $[0, 1]$ into measurable subsets $\bar{Y}_1, \ldots, \bar{Y}_k$ such that $g = \sum_{i,j \in [k]} \beta_{ij}1_{\bar{Y}_i}1_{\bar{Y}_j}$ and $\lambda(Y_i) = \lambda(Y'_i)$ almost everywhere.

**Proof.** To prove the lemma, we will want to use Theorem 8.6 (vi) from [25]. This will require us to turn $f$ into what is called a twin-free graphon, defined as a graphon $W$ such that there exists no pair $(x, x') \in [0, 1]$ such that $W(x, \cdot) = W(x', \cdot)$ almost everywhere. Unfortunately, by its very definition, step functions are not twin-free. To remedy this, we introduce graphons over a general probability space $(\Omega, \mathcal{F}, \mu)$, defined as measurable functions $W : \Omega^2 \to [0, 1]$ such that $W(x, y) = W(y, x)$ for all $x, y \in \Omega$. We also need to define a cut distance between graphons $W_i$ on (potentially) different probability spaces $(\Omega_i, \mathcal{F}_i, \mu_i)$, $i = 1, 2$. It is defined as

$$\delta_{\square}(W_1, W_2) = \inf_{\mu} \sup_{S,T \in \mathcal{F}_1 \times \mathcal{F}_2} \left| \int (W_1(x, x') - W_2(y, y'))d\mu(x, y)d\mu(x', y') \right|$$

where the inf goes over all couplings of $\mu_1$ and $\mu_2$. It is easy to see that for graphons defined on $[0, 1]$, this definition agrees with the previous one (see, e.g., Lemma 3.5 in [8]). With these new definitions, we define two graphons $W_1, W_2$ over two possibly different probability spaces to be equivalent if $\delta_{\square}(W_1, W_2) = 0$.

With this definition, the graphon $f$ is equivalent to the “discrete” graphon $f'_\mu = \beta_{ij}$ where $i$ and $j$ lie in the probability space $([k], 2^{|k|}, \mu)$ with $\mu(i) = \lambda(Y_i)$ (Hint: use the coupling which pairs $i \in [k]$ with the uniform measure on $Y_i$). It is also easy to turn $f'$ into a twin free graph as follows: if $i$ and $j'$ are twins, i.e., if the $i^{th}$ and $j'$th row of $\beta$ are identical, just merge the sets $Y_i$ and $Y_j$ into a new set of measure $\lambda(Y'_i) = \lambda(Y_j)$, reducing $k$. Note that this does not change the function $f$, just the representation of the form $f = \sum_{i,j \in [k]} \beta_{ij}1_{Y_i}1_{Y_j}$. Iterating this procedure, we eventually obtain a twin free graphon $f'$ which has cut-distance zero from $f$, $\delta_{\square}(f', f) = 0$. Doing the same merger for the function $g$, we see that we may without loss of generality assume that $f'$ is twin free, i.e., that the rows of $\beta$ are pairwise distinct.

At this point, we use Theorem 8.6 (vi) from [25] which says that $\delta_{\square}(f', g) = 0$ if and only if there exists a measure preserving map $\phi : [0, 1] \to [k]$ such that $g(x, y) = f'_\phi(x, \phi(y))$ almost everywhere. Defining $Y'_i = \phi^{-1}(\{i\})$ proves the lemma.

### 2.2 Lower bound

In order to prove Statement (1) of Theorem 1, we closely follow the proof of Theorem 2.3 in [14].
Proof of Theorem 2. Statement (1). We will prove the bound in the form given in Lemma 2. Let $f_{G_{kn}}$ be the empirical graphon of a graph on $kn$ vertices drawn according to $P_{kn,W_0}$. First, we claim that if

$$\liminf_{n \to \infty} \frac{1}{(kn)^2} \log P_{kn,W_0} \left( d_G(f_{G_{kn}}, g) \leq \epsilon \right) \geq -I_{W_0}(g)$$  \hspace{1cm} (13)$$

holds for all $g \in W$ and $\epsilon > 0$, then the theorem follows.

To see this, we first observe that for $h \in W$, $0 < \eta \leq \epsilon/2$ and $g \in B(h, \eta)$

$$\tilde{P}_{kn,W_0}(S(h, \epsilon)) = P_{kn,W_0}(\delta_G(f_{G_{kn}}, h) \leq \epsilon) \geq P_{kn,W_0}(d_G(f_{G_{kn}}, g) \leq \epsilon/2)$$

where the identity follows from the definition of $P_{kn,W_0}$ and $S(h, \epsilon)$, and the lower bound follows upon noting that $\delta_G(f_{G_{kn}}, h) \leq d_G(f_{G_{kn}}, g) + \delta_G(g, h)$. Therefore, assuming (13) yields

$$\liminf_{n \to \infty} \frac{1}{(kn)^2} \log \tilde{P}_{kn,W_0}(S(h, \epsilon)) \geq -I_{W_0}(g)$$

for all $0 < \eta \leq \epsilon/2$ and all $g \in B(h, \epsilon/2)$. It follows that

$$\liminf_{n \to \infty} \frac{1}{(kn)^2} \log \tilde{P}_{kn,W_0}(S(h, \epsilon)) \geq - \sup_{\eta \in (0, \epsilon/2]} \inf_{g \in B(h, \eta)} I_{W_0}(g) = - \sup_{\eta > 0} \inf_{g \in B(h, \eta)} I_{W_0}(g) = -J_{W_0}(h)$$

as required.

We have shown that (13) implies the theorem; we now turn to its proof. By Proposition 8, it holds that $P_{kn,W_0}(S(h, \epsilon)) \geq P_{kn,W_0}(S(h, \epsilon/2))$. We may assume $g \in W_0$. Let $\epsilon > 0$. We define $(g_n)_{n \geq 1}$, a sequence of $kn \times kn$ block graphs that approximate $g$. Recall (1). For $i, j \in [kn]$,

$$p_{ij}^{(n)} = (kn)^2 \int \int_{[0,1]^{kn} \times [0,1]^{kn}} g(x,y) dx dy$$

and $g_n(x,y) = p_{kn,n}^{(n)}(x,y)$.

Since $\|g_n - g\|_1 \to 0$, in order to prove (13) it suffices to show that

$$\liminf_{n \to \infty} \frac{1}{(kn)^2} \log P_{kn,W_0}(B_{\epsilon,n}) \geq -I_{W_0}(g),$$

where $B_{\epsilon,n} = \{ f : d_G(f, g_n) \leq \epsilon/2 \}$. We will apply the following proposition, which is proved as part of Theorem 2.3 in [14]. For completion, we include a proof sketch in the Appendix (Section 8.2).

**Proposition 8.** Let $f_n$ be a graphon drawn from the measure $P_{kn,g_n}$, with $g_n$ as defined above. For any $\epsilon > 0$, it holds that

$$\lim_{n \to \infty} P_{kn,g_n}(d_G(f_n, g_n) \geq \epsilon) = 0.$$
By construction if $W_0(i/(kn), j/(kn)) = 0$, then $p(n)_{ij} = 0$ and if $W_0(i/(kn), j/(kn)) = 1$, then $p(n)_{ij} = 1$.

Using the convention that $0 \log(0/0) = 0$, we obtain

$$
\limsup_{n \to \infty} \frac{1}{(kn)^2} \int \log \left( \frac{dP_{kn,g_n}}{dP_{kn,W_0}} \right) dP_{kn,g_n}
= \limsup_{n \to \infty} \frac{1}{(kn)^2} \sum_{G \in \mathcal{G}(kn)} \mathbb{P}_{kn,g_n}(\{G\}) \log \left( \frac{\mathbb{P}_{kn,g_n}(\{G\})}{\mathbb{P}_{kn,W_0}(\{G\})} \right)
= \limsup_{n \to \infty} \frac{1}{(kn)^2} \sum_{G \in \mathcal{G}(kn)} \prod_{i,j \in [kn]} \left( p(n)_{ij} \mathbb{1}_{(i,j) \in E(G)} + (1 - p(n)_{ij}) \mathbb{1}_{(i,j) \notin E(G)} \right) \times
\left( \sum_{i,j \in [kn]} \frac{p(n)_{ij}}{W_0(\frac{i}{kn}, \frac{j}{kn})} \mathbb{1}_{(i,j) \in E(G)} + \frac{1 - p(n)_{ij}}{1 - W_0(\frac{i}{kn}, \frac{j}{kn})} \mathbb{1}_{(i,j) \notin E(G)} \right)
= \limsup_{n \to \infty} \frac{1}{(kn)^2} \sum_{i,j \in [kn]} p(n)_{ij} \log \frac{p(n)_{ij}}{W_0(\frac{i}{kn}, \frac{j}{kn})} + (1 - p(n)_{ij}) \log \frac{1 - p(n)_{ij}}{1 - W_0(\frac{i}{kn}, \frac{j}{kn})}
= \limsup_{n \to \infty} I_{W_0}(g_n) = I_{W_0}(g)
$$

where we used Proposition 2 in the last step. 

2.3 Upper bound

In this section we prove the upper bound of Theorem 1. The proof requires two key lemmas. The first one establishes that as long as we look at balls around block constant graphs, we can restrict to a finite number of measure preserving bijections, and the second one gives an upper bound on the probability that sampling a graphon and applying an invertible transformation yields a graphon in a particular $d_{\square}$ ball. This is formalized in the following two lemmas.

**Lemma 10.** Let $s_h, s_w \in \mathbb{Z}^+$ and $\eta > 0$. Given $\mu \in \mathbb{R}^{s_h}$, $\gamma \in \mathbb{R}^{s_w}$, $h \in \mathcal{B}^n$ and $W_0 \in \mathcal{B}^1$, there exists a finite set of invertible measure preserving transformations $T \subseteq \mathcal{M}$ with $|T| = N(\eta, s_w, s_h)$ such that the following holds for all $n > 12s_h^2 s_w/(kn)$: For all $\sigma \in \mathcal{M}_{kn}$ there exists $\tau \in T$ such that for

$$
\mathbb{P}_{kn,W_0}(d_{\square}(f^{\sigma}, h) \leq \varepsilon) \leq \mathbb{P}_{kn,W_0}(d_{\square}(f^{\tau}, h) \leq \varepsilon + \eta)
$$

where $f = f^{G_{kn}}$ is the empirical graphon obtained by sampling $W_0$ according to $\mathbb{P}_{kn,W_0}$.

**Lemma 11.** Let $\varepsilon > 0$ and $h \in \mathcal{W}$. Let $G_{kn}$ be the graph drawn from $\mathbb{P}_{kn,W_0}$, and let $f^{G_{kn}}$ denote the corresponding empirical graphon. For all invertible $\tau \in \mathcal{M}$, it holds that

$$
\limsup_{n \to \infty} \frac{1}{(kn)^2} \log \mathbb{P}_{kn,W_0} \left( d_{\square} \left( f^{G_{kn}}, h \right) \leq \varepsilon \right) \leq - \inf_{f^{\delta_{\square}(f,h) \leq \varepsilon}} I_{W_0}(f).
$$

To prove the second lemma, we use the following LDP upper bound with respect to the weak topology. Recall that the weak topology on $\mathcal{W}$ is the smallest topology under which the maps $f \mapsto \int_{[0,1]^2} f(x,y)g(x,y)dxdy$ are continuous for every $g \in L^2([0,1]^2)$.

**Theorem 9.** For every weakly closed set $F \subseteq \mathcal{W}$,

$$
\limsup_{n \to \infty} \frac{1}{(kn)^2} \log \mathbb{P}_{kn,W_0}(F) \leq - \inf_{f \in F} I_{W_0}(f).
$$

The proof of Theorem 9 is a straightforward generalization of Theorem 5.1 of [11]. For completeness we include the proof in the Appendix. We delay the proofs of Lemmas 10 and 11 to the end of the section and proceed to the proof of the upper bound in Theorem 9.

In the proof of the upper bound in Theorem 9 we also use the following version of the Weak Regularity Lemma, which follows directly from Theorem 3.1 in [11].

**Lemma 12** (Weak Regularity Lemma). Given $\varepsilon > 0$ there exists a finite set $H(\varepsilon) \subseteq \mathcal{W}$ of block-graphons, such that if $f$ is a uniform $n$-block graphon there exists $\sigma \in \mathcal{M}_n$ and an $h \in H(\varepsilon)$ such that

$$
d_{\square}(f^\sigma, h) < \varepsilon.
$$
Proof of Theorem 1 Statement 2. We prove the bound in the form given in Remark 2. Let \( H(\varepsilon/2) \) be an \( \varepsilon/2 \) net as given in Lemma 12 and let \( T^* \) be the union of the sets of invertible transformations \( T \) given by Lemma 10 for each \( h \in H(\varepsilon/2) \). We index the finite set as \( T^* = \{ \tau_1, \tau_2, \ldots, T(\eta, W_0, \varepsilon) \} \).

\[
\bar{\mathbb{P}}_{kn, W_0}(S(\tilde{g}, \alpha)) = \mathbb{P}_{kn, W_0} \left( \delta_{\Box}(f_{G_{kn}}, g) \leq \alpha \right) \\
\leq \sum_{h \in H(\varepsilon/2)} \mathbb{P}_{kn, W_0} \left( \bigcup_{\sigma \in M_{kn}} \left\{ f : \delta_{\Box}(f, g) \leq \alpha \right\} \cap \left\{ f : d_{\Box}(f, h) \leq \varepsilon/2 \right\} \right) \\
\leq \sum_{h \in H(\varepsilon/2)} \sum_{\sigma \in M_{kn}} \mathbb{P}_{kn, W_0} \left( \left\{ f : \delta_{\Box}(f, g) \leq \alpha \right\} \cap \left\{ f : d_{\Box}(f, h) \leq \varepsilon/2 \right\} \right) \\
\leq \sum_{h \in H(\varepsilon/2)} \sum_{\delta_{\Box}(g, h) \leq \alpha + \varepsilon/2} \mathbb{P}_{kn, W_0} \left( d_{\Box}(f_{G_{kn}}, h) \leq \varepsilon/2 \right) \\
\leq (kn)!|H(\varepsilon/2)| \max_{h \in H(\varepsilon/2)} \mathbb{P}_{kn, W_0} \left( d_{\Box}(f_{G_{kn}}, h) \leq \varepsilon/2 + \eta \right),
\]

where the last inequality follows from Lemma 10. Lemma 11 implies that

\[
\limsup_{n \to \infty} \frac{1}{(kn)^2} \log \bar{\mathbb{P}}_{kn, W_0}(S(\tilde{g}, \alpha)) \\
\leq \limsup_{n \to \infty} \frac{1}{(kn)^2} \log \left( (kn)!|H(\varepsilon/2)| \max_{h \in H(\varepsilon/2)} \mathbb{P}_{kn, W_0} \left( d_{\Box}(f_{G_{kn}}, h) \leq \varepsilon/2 + \eta \right) \right) \\
= \limsup_{n \to \infty} \frac{1}{(kn)^2} \log \left( \max_{h \in H(\varepsilon/2)} \mathbb{P}_{kn, W_0} \left( d_{\Box}(f_{G_{kn}}, h) \leq \varepsilon/2 + \eta \right) \right) \\
\leq \inf_{f : \delta_{\Box}(f, \tilde{g}) \leq \alpha + \varepsilon + \eta} I_{W_0}(f).
\]

Since \( \varepsilon, \eta \) were arbitrary, we can choose \( \varepsilon, \eta < \alpha/2 \) and obtain

\[
\limsup_{n \to \infty} \frac{1}{(kn)^2} \log \bar{\mathbb{P}}_{kn, W_0}(S(\tilde{g}, \alpha)) \leq - \inf_{f : \delta_{\Box}(f, \tilde{g}) \leq 2\alpha} I_{W_0}(f).
\]

Since \( - \inf_{f : \delta_{\Box}(f, \tilde{g}) \leq 2\alpha} I_{W_0}(f) \) is a non-increasing function as \( \alpha \to 0 \),

\[
\lim_{\alpha \to 0} \sup_{n \to \infty} \frac{1}{(kn)^2} \log \bar{\mathbb{P}}_{kn, W_0}(S(\tilde{g}, \alpha)) \leq \inf_{\alpha > 0} \left( - \inf_{f : \delta_{\Box}(f, \tilde{g}) \leq 2\alpha} I_{W_0}(f) \right) \\
= \sup_{\alpha > 0} \inf_{f : \delta_{\Box}(f, \tilde{g}) \leq 2\alpha} I_{W_0}(f) \\
= -J_{W_0}(g),
\]

as required.

We will use the following definition and proposition in the proof of Lemma 11.
Definition 14. Let \( \phi \in \mathcal{M} \). Let \( I_1, \ldots, I_k \) be a partition of \([0,1]\). We say that \( \phi \) respects the interval structure of \( I_1, \ldots, I_k \) if for all \( j \) and \( X \subseteq I_j \), \( \phi(X) \subseteq I_j \).

Proposition 9. Let \( h \) be a graphon that is constant on each block \( I_i \times I_j \) for \( i, j \in [k] \). If \( \phi \) is invertible and respects the interval structure of \( I_1 \ldots I_k \), then for all \( g \in \mathcal{W} \)
\[
d_{\square}(g^\phi, h) = d_{\square}(g, h).
\]

Proof. Note that since \( h \) is constant on each block \( I_i \times I_j \) and \( \phi \) respects the interval structure,
\[
h^\phi(x, y) = h(\phi(x), \phi(y)) = h(x, y).
\]
It follows that
\[
d_{\square}(g, h) = d_{\square}(g^\phi, h^\phi) = d_{\square}(g^\phi, h).
\]

Proof of Lemma 10. Let \( \{H_i : i \in [s_h]\} \) and \( \{W_j : j \in [s_w]\} \) denote the intervals of the block structure governed by \( \mu \) and \( \gamma \) respectively; formally, \( H_i = [0, \mu_i], W_i = [0, \gamma_i] \),
\[
H_i = \left( \sum_{j=1}^{i-1} \mu_j, \sum_{j=1}^{i-1} \mu_j \right) \quad \text{for } 2 \leq i \leq s_h,
\quad \text{and} \quad
W_i = \left( \sum_{j=1}^{i-1} \gamma_j, \sum_{j=1}^{i-1} \gamma_j \right) \quad \text{for } 2 \leq i \leq s_w.
\]
\( \{H_i\} \) are the intervals corresponding to the blocks of \( h \) and \( \{W_i\} \) are the intervals corresponding to the blocks of \( W_0 \).

We begin with a proof outline. First we construct a finite set of measure preserving transformations \( T \subseteq \mathcal{M} \) such that each \( \sigma \in \mathcal{M}_{kn} \) is “close” to a transformation \( \tau \in T \). Then, in order to compare \( f^\sigma \) and \( f^\tau \), we define \( \alpha, \beta \) invertible transformations that respect the intervals \( H_1, \ldots, H_{s_h} \). Proposition 8 implies that
\[
d_{\square}(f^\sigma, h) = d_{\square}((f^\sigma)^\alpha, h) \quad \text{and} \quad
d_{\square}(f^\tau, h) = d_{\square}((f^\tau)^\beta, h).
\]
It follows that
\[
\mathbb{P}_{kn, W_0}(d_{\square}(f^\sigma, h) \leq \varepsilon) = \mathbb{P}_{kn, W_0}(d_{\square}((f^\sigma)^\alpha, h) \leq \varepsilon), \quad \text{(14)}
\]
\[
\mathbb{P}_{kn, W_0}(d_{\square}(f^\tau, h) \leq \varepsilon + \eta) = \mathbb{P}_{kn, W_0}\left(d_{\square}((f^\tau)^\beta, h) \leq \varepsilon + \eta\right), \quad \text{(15)}
\]
where \( f = f^{G_{kn}} \) is the empirical graphon obtained by sampling \( W_0 \) according to \( \mathbb{P}_{kn, W_0} \). Finally, we will describe a coupling of \( f \) and \( g \), each with marginal distribution \( \mathbb{P}_{kn, W_0} \), that guarantees
\[
d_{\square}\left((f^\sigma)^\alpha, (g^\tau)^\beta\right) \leq \eta \quad \text{(16)}
\]
with probability one. The triangle inequality implies
\[
\mathbb{P}_{kn, W_0}(d_{\square}((f^\sigma)^\alpha, h) \leq \varepsilon) \leq \mathbb{P}_{kn, W_0}\left(d_{\square}((f^\tau)^\beta, h) \leq \varepsilon + \eta\right), \quad \text{(17)}
\]
The statement follows directly from (14), (15), and (17).

To complete the proof according to this outline, we must complete the following tasks:
(a) Define a finite net of measure preserving transformations \( T \subseteq \mathcal{M} \) with \( |T| = N(\eta, s_h, s_w) \).
(b) For each transformation \( \sigma \in \mathcal{M}_{kn} \), define a “close” transformation \( \tau \in T \). Informally, we will say that two transformations are close if they map approximately the same amount of mass from \( W_i \) to \( H_j \) for all \( i \in [s_w] \) and \( j \in [s_h] \).
(c) Define \( \alpha \) and \( \beta \) invertible transformations that respect the intervals \( H_1, \ldots, H_{s_h} \).
(d) Exhibit a coupling of \( f \) and \( g \) each sampled according to \( \mathbb{P}_{kn, W_0} \) that guarantees (10).

We begin with (a). For convenience we index vectors \( v \in \mathbb{R}^{s_w \times s_h} \) by pairs \( (i, j) \in [s_w] \times [s_h] \). Let
\[
V = \left\{ v \in \mathbb{R}^{s_w \times s_h} : \sum_{i=1}^{s_w} \sum_{j=1}^{s_h} v_{ij} = 1, \sum_{i=1}^{s_w} v_{ij} = \mu_j, \sum_{j=1}^{s_h} v_{ij} = \gamma_i \right\}.
\]
Recall that \( \{H_i\} \) are the intervals corresponding to the blocks of \( h \) and \( \{W_i\} \) are the intervals corresponding to the blocks of \( W_0 \). For each \( v \in V \) we associate an invertible measure preserving transformation

20
\(\tau_v \in \mathcal{M}\) that maps an interval of length \(v_{ij}\) contained in \(W_i\) to an interval that is contained completely in \(H_j\) for each \((i, j)\). To this end, let

\[
I_{11} = [0, v_{11}] \quad \text{and} \quad I_{ij} = \left(\sum_{a=1}^{i-1} \gamma_a + \sum_{b=1}^{j-1} v_{ab}, \sum_{a=1}^{i-1} \gamma_a + \sum_{b=1}^{j} v_{ab}\right), (i, j) \neq (1, 1).
\]

Note that the intervals \(I_{ij}\) are sorted first by the first index, then by the second index. Define \(\tau : [0, 1] \to [0, 1]\) to be the transformation that translates the intervals so they are first sorted by the second index, then by the first index,

\[
\tau(I_{11}) = [0, v_{11}] \quad \text{and} \quad \tau(I_{ij}) = \left(\sum_{a=1}^{i-1} \mu_b + \sum_{a=1}^{i} v_{aj}, \sum_{a=1}^{i-1} \mu_b + \sum_{a=1}^{i} v_{aj}\right), (i, j) \neq (1, 1).
\]

For an illustration of this transformation, see Figure 3.

![Figure 3: Illustration of the transformation \(I_{45} \mapsto \tau(I_{45})\).](image)

Observe that \(I_{ij} \subseteq W_i\) and \(\tau(I_{ij}) \subseteq H_j\). Note that \(V\) is a compact set, and thus we can construct a finite net \(V^* \subseteq V\) such that for all \(v \in V\), there exists \(v' \in V^*\) such that \(\|v - v'\|_{\infty} \leq \eta/(8s_w s_h)\). Let \(T = \{\tau_v | v \in V^*\} \cup \{\tau^{-1}_v | v \in V^*\}\).

Next we address (b). Let \(M = \{m_1, \ldots, m_n\}\) be the set of intervals corresponding to vertices in an empirical graphon with \(kn\) vertices, where \(m_1 = [0, \frac{1}{kn}]\) and

\[
m_i = \left(\frac{i-1}{kn}, \frac{i}{kn}\right) \quad \text{for} \ 2 \leq i \leq kn.
\]

We call the intervals in \(M\) “vertex intervals”.

For each transformation in \(M_{kn}\), we find a transformation in \(T\) that moves roughly the same amount of mass between intervals \(W_i\) and \(H_j\) for all \(i \in [s_w]\) and \(j \in [s_h]\). We construct an element in \(T\) that is close to \(\sigma^{-1}\) rather than \(\sigma\) to make the construction in the next section more convenient. (Note that both \(\sigma\) and \(\sigma^{-1}\) are in \(M_{kn}\)).

Let \(N_{ij}\) be the set of vertex intervals that are mapped from \(W_i\) to \(H_j\) under \(\sigma^{-1}\),

\[
N_{ij} = \{m_\ell \in M | m_\ell \subseteq W_i \text{ and } \sigma(m_\ell) \subseteq H_j\},
\]

and let \(n_{ij} = |N_{ij}|\).

Each vertex interval is contained in some \(W_i\) by construction. Since \(\sigma^{-1}\) maps at most \(s_h - 1\) vertex intervals to the boundary between intervals of \(h\), \(\sum_{i=1}^{s_w} \sum_{j=1}^{s_h} n_{ij} \geq kn - s_h + 1\). Define \(v \in \mathbb{R}^{s_w \times s_h}\), \(v_{ij} = n_{ij}/(kn)\). We claim that there exists \(v' \in V\) with \(v'_{ij} \geq v_{ij}\) for all \(i, j\) such that

\[
\|v - v'\|_1 \leq \frac{s_h - 1}{kn} < \frac{\eta}{8s_w s_h}.
\]

To see this, let \(\tilde{\mu}_j = \sum_{i=1}^{s_w} v_{ij}\) and \(\tilde{\gamma}_i = \sum_{j=1}^{s_h} v_{ij}\). Then \(\tilde{\mu}_j \leq \mu_j\) and \(\tilde{\gamma}_i \leq \gamma_i\) and

\[
\sum_i (\gamma_i - \tilde{\gamma}_i) = \sum_j (\mu_j - \tilde{\mu}_j) = 1 - \sum_{i=1}^{s_w} \sum_{j=1}^{s_h} v_{ij} =: \Delta, \quad \Delta \leq \frac{s_h - 1}{kn}.
\]

Taking a coupling \(p_{ij}\) of the probability distributions \((\frac{1}{h}(\gamma_i - \tilde{\gamma}_i))_{i \in [s_w]}\) and \((\frac{1}{h}(\mu_j - \tilde{\mu}_j))_{j \in [s_h]}\) and setting \(v'_{ij} = v_{ij} + p_{ij}\Delta\) proves the claim. (Observe that \(n \geq 8s_h s_w/(kn)\) by the assumption of the lemma.)
Choosing \( v^* \in V^* \) such that \( \|v' - v^*\|_{\infty} \leq \eta/(8s_w s_h) \), we then have

\[
\|v - v^*\|_{\infty} < \frac{\eta}{4s_w s_h}.
\]

We associate \( \sigma^{-1} \) with \( \tau_\sigma \in T \). Let \( \tau = \tau_\sigma^{-1} \), and note \( \tau \in T \).

Next we address (c), defining the transformations \( \alpha \) and \( \beta \). We must define \( \alpha \) and \( \beta \) in a way that, conveniently facilitates a coupling satisfying (16) in step (d). In particular, we will define a coupling so that \( (f^* \circ \tau) \) and \( (g^* \circ \tau) \) are identical on many sets of the form \( m \times m' \) where \( m \) and \( m' \) are vertex intervals.

We can exactly couple the values \( f(\alpha(x)), \sigma(\alpha(y)) \) and \( g(\beta(x)), \tau(\beta(y)) \) on \( m \times m' \), provided \( \sigma(\alpha(x)) \) and \( \tau(\beta(x)) \) are in the same interval \( W_i \) and \( \sigma(\alpha(y)) \) and \( \tau(\beta(y)) \) are in the same interval \( W_j \). In this case, both \( f(\alpha(x)), \sigma(\alpha(y)), g(\beta(x)), \tau(\beta(y)) \) are contained in \( W_i \times W_j \) and \( W_i \times W_j \) for some \( i, j \in [s_w] \). We construct \( \alpha \) and \( \beta \) so that at least a \((1 - \eta/2)\) fraction of the vertex intervals \( m \in M \) are synchronized. In step (d), we will couple the behavior of the vertices corresponding to \( \sigma(\alpha(m)) \) and \( \tau(\beta(m)) \) for each synchronized vertex interval \( m \).

Let \( n_{ij}, I_{ij}, v^* \) be as defined in part (b). Let \( k_{ij} = \min\{n_{ij}, [v^*_ij kn] - 1\} \). We will construct \( \alpha \) and \( \beta \) so that there are \( k_{ij} \) synchronized vertex intervals contained in \( I_{ij} \) whose images under \( \sigma \circ \alpha \) and \( \tau \circ \beta \) are contained in \( W_i \).

The transformations \( \sigma^{-1} \) and \( \tau^{-1} \) map approximately the same amount of mass from \( W_i \) to \( H_j \) for all \( i \in [s_w] \) and \( j \in [s_h] \), but the intersection of the image of \( W_i \) and \( H_j \) may be be very different under the two maps. We design \( \alpha^{-1} \) and \( \beta^{-1} \) so that \( \alpha^{-1} \circ \sigma^{-1} \) and \( \beta^{-1} \circ \tau^{-1} \) both map mass from \( W_i \) to the same subinterval of \( H_j \). Working with the inverse functions allows us to think of \( \alpha^{-1} \) and \( \beta^{-1} \) as functions that reorganize the images of \( W_i \) under \( \sigma^{-1} \) and \( \tau^{-1} \) (respectively) within each interval \( H_j \).

We now formally construct \( \alpha \) and \( \beta \) by constructing their inverses.

First we construct \( \alpha^{-1} \), as illustrated in Figure 3. There are \( n_{ij} \) vertex intervals contained in \( W_i \) that are mapped to vertex intervals in \( H_j \) under \( \sigma^{-1} \). Informally, \( \alpha^{-1} \) will rearrange the images of these vertex intervals within \( H_j \) by sorting them by their origin interval \( W_i \). Under \( \alpha^{-1} \), the image of vertex intervals originating in \( W_i \) are mapped to the leftmost vertex intervals contained completely in \( H_j \).

Formally, let \( a_{i1}^j, a_{i2}^j \ldots a_{ki}^j \) enumerate \( k_{ij} \) of the \( n_{ij} \) vertex intervals contained in \( W_i \) that are mapped to vertex intervals in \( H_j \) under \( \sigma^{-1} \). Let \( m_{ij}(i) \) be the \( i^{th} \) interval of \( M \) that is entirely contained in \( H_j \). Define \( \alpha \in M \) so that \( \alpha^{-1} \) translates the interval \( \sigma^{-1}(a_{ij}^k) \) to the vertex interval specified as follows:

\[
\alpha^{-1}(\sigma^{-1}(a_{ij}^k)) = m_j \left( \sum_{s=1}^{i-1} k_{s j} + \ell \right),
\]

and \( H_j \setminus \left( \bigcup_{s=1}^{i-1} \bigcup_{\ell=1}^{k_{s j}} \sigma^{-1}(a_{ij}^k) \right) \) maps to \( H_j \setminus \left( \bigcup_{s=1}^{i-1} \bigcup_{\ell=1}^{k_{s j}} \sigma^{-1}(a_{ij}^k) \right) \) under \( \alpha^{-1} \) in any invertible manner. Since \( \alpha^{-1}(H_i) = H_i \), \( \alpha \) and \( \alpha^{-1} \) respect the intervals \( H_1 \ldots H_{s_h} \).

Next we construct the map \( \beta^{-1} \), as illustrated in Figure 4. Recall the definition of \( I_{ij} \) described in the construction of \( \tau_\sigma \). Each interval \( I_{ij} \) is contained in \( W_i \) and \( \tau^{-1}(I_{ij}), \tau^{-1}(I_{k2}), \ldots \tau^{-1}(I_{k_{s_h}}) \) are consecutive intervals (in that order) whose union is \( H_j \). Unlike \( \sigma^{-1}, \tau^{-1} \) may not be in \( M_{k_{s_h}} \), and so the image of vertex intervals under \( \tau^{-1} \) are not necessarily vertex intervals. Informally, \( \beta^{-1} \) will map the images of vertex intervals under \( \tau^{-1} \) to vertex intervals in a way that maintains their relative order in \( H_j \).

We now formally describe \( \beta^{-1} \). Since \( I_{ij} \) has length \( v'_{ij} \), there are at least \([v'_{ij} kn] - 1 \) vertex intervals contained in \( I_{ij} \subseteq W_i \), all of which are mapped to \( H_j \) under \( \tau^{-1} \). Let \( b_{1ij}^j, b_{2ij}^j \ldots b_{k_{ij}}^j \) enumerate \( k_{ij} \) of these vertex intervals contained in \( I_{ij} \). Define \( \beta \in M \) so that \( \beta^{-1} \) translates the interval (which is not necessarily a vertex interval) \( \tau^{-1}(b_{ij}^k) \) to the vertex interval specified as follows:

\[
\beta^{-1}(\tau^{-1}(b_{ij}^k)) = m_j \left( \sum_{s=1}^{i-1} k_{s j} + \ell \right),
\]

and \( H_j \setminus \left( \bigcup_{s=1}^{i-1} \bigcup_{\ell=1}^{k_{s j}} \tau^{-1}(b_{ij}^k) \right) \) maps to \( H_j \setminus \left( \bigcup_{s=1}^{i-1} \bigcup_{\ell=1}^{k_{s j}} \beta^{-1}(b_{ij}^k) \right) \) under \( \beta^{-1} \) in any invertible manner. Since \( \beta(H_i) = H_i \), \( \beta \) and \( \beta^{-1} \) respect the intervals \( H_1 \ldots H_{s_h} \).
Figure 4: The construction of $\alpha^{-1}$. The tall solid vertical lines represent the divisions between the intervals $W_1,\ldots,W_3$, and the tall dashed vertical lines represent the divisions between the intervals $H_1,\ldots,H_4$. All arrows indicate that the respective transformations map the specified vertex intervals to vertex intervals by translation.

Figure 5: The construction of $\beta^{-1}$. The tall solid vertical lines represent the divisions between the intervals $w_1,\ldots,w_3$, and the tall dashed vertical lines represent the divisions between the intervals $h_1,\ldots,h_4$. The arrows corresponding to $\tau^{-1}$ illustrate that $\tau^{-1}$ map intervals to intervals by translation. The arrows depicting $\beta^{-1}$ show that $\beta^{-1}$ maps adjacent intervals of the form $\tau^{-1}(b_i^j)$ (shown by horizontal line segments) to vertex intervals by translation.
Next we construct $K$, a set of synchronized vertex intervals. Note that for all triples $i, j, \ell$ with $i \in [s_u], j \in [s_h]$, and $\ell \in [k_{ij}]$, $m_j \left( \sum_{i=1}^{j-1} k_{ij} + \ell \right)$ is a synchronized vertex interval since $a_i^j$ and $b_i^j$ are vertex intervals contained in $W_i$. Let

$$K = \left\{ m_j \left( \sum_{i=1}^{j-1} k_{ij} + \ell \right) \mid i \in [s_u], j \in [s_h], \text{ and } \ell \in [k_{ij}] \right\}.$$ 

Finally, we bound the size of $K$. Recall that by construction $v_{ij} = n_{ij}/(kn)$ and $\|v - v^*\|_\infty \leq \eta/(4s_ws_h)$. It follows that

$$|n_{ij} - v^*_j kn| \leq \frac{\eta kn}{4s_ws_h}.$$ 

Since $|v^*_j kn| - 1 - v^*_j kn| \leq 2$, it follows that

$$k_{ij} = \min(n_{ij}, |v^*_j kn| - 1) \geq n_{ij} - 2 - \frac{\eta kn}{4s_ws_h}.$$ 

We use this to lower bound the total number of synchronized intervals in $K$,

$$|K| = \sum_{i=1}^{s_u} \sum_{j=1}^{s_h} k_{ij} \geq \left( \sum_{i=1}^{s_u} \sum_{j=1}^{s_h} n_{ij} \right) - 2s_ws_h - \frac{\eta kn}{4} \geq kn - s_h + 1 - \frac{\eta kn}{4},$$

since $n \geq 12s^2 ws/(kn) \geq 12s_h s_w/(kn)$ by the assumption of the lemma.

Finally, we address (d). We construct a coupling of $f$ and $g$ so that $(f^\alpha)^\beta$ and $(g^\beta)^\beta$ agree on sets of the form $m_1 \times m_2$ where $m_1, m_2 \in K$ are synchronized intervals. Let $v_1$ and $v_2$ be the vertices in $f$ corresponding to the vertex intervals that are mapped to $m_1$ and $m_2$ respectively under $\sigma \circ \alpha$. Let $u_1$ and $u_2$ be the vertices in $g$ corresponding to the vertex intervals that are mapped to $m_1$ and $m_2$ respectively under $\tau \circ \beta$. By construction, $v_1$ and $u_1$ correspond to vertex intervals contained in the same interval $W_i$, and likewise $v_2$ and $u_2$ correspond to vertex intervals contained in the same interval $W_f$. Let $X_{ah}$ and $Y_{ah}$ be the indicator random variables for the events that there is an edge between vertices $a$ and $b$ in $f$ and $g$ respectively. Since $X_{ah} \sim \text{Bern}(W_0(i/(kn), j/(kn)))$ and $Y_{ah} \sim \text{Bern}(W_0(i/(kn), j/(kn)))$, we can couple $X_{ah}$ and $Y_{ah}$ exactly, which then guarantees that $(f^\alpha)^\beta$ and $(g^\beta)^\beta$ agree on the set $m_1 \times m_2$.

Since $(f^\alpha)^\beta$ and $(g^\beta)^\beta$ agree on the synchronized vertex intervals $(\bigcup_{m \in K} m)^2$ and $|\bigcup_{m \in K} m| \geq 1 - \eta/2$, it follows that $d_{\square}(f^\alpha, g^\beta) \leq \eta$, as desired. \qed

**Proof of Lemma 7.** Fix $\tau \in \mathcal{M}$. Note that $\tau$ is invertible and $d_{\square}(f^\tau, g) = d_{\square}(f, g^{-1})$. It follows that $\mathbb{P}_{kn,W_0}\left( d_{\square}(f^{G_{kn}}, h) \leq \varepsilon \right) = \mathbb{P}_{kn,W_0}\left( d_{\square}(f^{G_{kn}}, h^{-1}) \leq \varepsilon \right) = \mathbb{P}_{kn,W_0}\left( f^{G_{kn}} \in \left\{ g : d_{\square}(g, h^{-1}) \leq \varepsilon \right\} \right).$

Note [11] Lemma 5.4 implies that the set $\{ g : d_{\square}(g, h^{-1}) \leq \varepsilon \}$ is closed in the weak topology. Applying Theorem 9 we obtain

$$\limsup_{n \to \infty} \frac{1}{(kn)^2} \log \mathbb{P}_{kn,W_0}\left( d_{\square}(f^{G_{kn}}, h) \leq \varepsilon \right) = \limsup_{n \to \infty} \frac{1}{(kn)^2} \log \mathbb{P}_{kn,W_0}\left( f^{G_{kn}} \in \left\{ g : d_{\square}(g, h^{-1}) \leq \varepsilon \right\} \right) \leq - \inf_{f \in \mathcal{W} : d_{\square}(f, h^{-1}) \leq \varepsilon} I_{W_0}(f) \leq - \inf_{f \in \mathcal{H} : d_{\square}(f, \tilde{h}) \leq \varepsilon} I_{W_0}(f),$$

where the last line follows from the observation that $\delta_{\square}(g, h) \leq d_{\square}(g, h^{-1})$. \qed

### 2.4 Proof of Theorem 2

We begin with the following theorem, which is a direct adaptation of [14] Theorem 3.1] to general $k$-block graphons $W_0$. As usual, for $\tilde{f} \in \mathcal{W}$ and $H \subseteq \mathcal{W}$, define $\delta_{\square}(\tilde{f}, \tilde{H}) \triangleq \inf_{\tilde{h} \in \tilde{H}} \delta_{\square}(\tilde{f}, \tilde{h})$. 

```
Theorem 10. Let $\bar{F}$ be a closed subset of $\overline{W}$, and let $\bar{F}^0$ be its interior. Suppose

$$\inf_{h \in \bar{F}} J_{W_0}(h) = \inf_{h \in \bar{F}} J_{W_0}(\bar{h}).$$

(18)

Let $\bar{F}^*$ be the subset of $\bar{F}$ where $J_{W_0}$ is minimized. Then $\bar{F}^*$ is non-empty and compact, and

$$\min_{h \in \bar{F}} J_{W_0}(h) = - \lim_{n \to \infty} \frac{1}{(kn)^2} \log \tilde{P}_{k_n,W_0}(\bar{F}).$$

(19)

If $\min_{h \in \bar{F}} J_{W_0}(h) < \infty$, then for all sufficiently large $n$ and all $\varepsilon > 0$, $\tilde{P}_{k_n,W_0}(\bar{f}_{G_kn} \in \bar{F}) > 0$ and

$$\tilde{P}_{k_n,W_0} \left( \bar{C}(\bar{f}_{G_kn}, \bar{F}^*) \geq \varepsilon | \bar{f}_{G_kn} \in \bar{F} \right) \leq e^{-C(\varepsilon, \bar{F}) (kn)^2},$$

where $C(\varepsilon, \bar{F})$ is a positive constant depending only on $\varepsilon$ and $\bar{F}$. In particular, if $\bar{F}^*$ contains only one element $h^*$ (and $J_{W_0}(h^*) < \infty$), then the conditional distribution of $\bar{f}_{G_kn}$ given $\bar{f}_{G_kn} \in \bar{F}$ converges to the point mass at $h^*$ as $n \to \infty$.

Proof. Since $\overline{W}$ is compact and $\bar{F}$ is closed, $\bar{F}^*$ is also compact. By Lemma 4 the function $J_{W_0}$ is lower semi-continuous on $\bar{F}$. Since $\bar{F}$ is compact, $J_{W_0}$ must attain its minimum on $\bar{F}$. Therefore, $\bar{F}^*$ is non-empty. Moreover, by the lower semi-continuity of $J_{W_0}$, $\bar{F}^*$ is closed, and hence compact. Finally, by Theorem 14

$$- \inf_{h \in \bar{F}^0} J_{W_0}(h) \leq \inf_{n \to \infty} \frac{1}{(kn)^2} \log \tilde{P}_{k_n,W_0}(\bar{F}^0) \leq \lim_{n \to \infty} \frac{1}{(kn)^2} \log \tilde{P}_{k_n,W_0}(\bar{F}) \leq - \inf_{h \in \bar{F}^0} J_{W_0}(h) = - \min_{h \in \bar{F}^0} J_{W_0}(h).$$

Combined with (18) this proves (19).

Next, assume that the inf in (18) is finite. This is only compatible with (19) if $\tilde{P}_{k_n,W_0}(\bar{F}^0) > 0$ for all $n$ larger than some $n_0$. Fix $\varepsilon > 0$ and let

$$\bar{F}_\varepsilon \triangleq \left\{ \bar{h} \in \bar{F} : \delta_{\overline{C}}(\bar{h}, \bar{F}^*) \geq \varepsilon \right\},$$

which is also a closed subset. Observe that $\bar{F}_\varepsilon \cap \bar{F}^* = \emptyset$. Then

$$\tilde{P}_{k_n,W_0} \left( \delta_{\overline{C}}(\bar{f}_{G_kn}, \bar{F}^*) \geq \varepsilon | \bar{f}_{G_kn} \in \bar{F} \right) \leq \frac{\tilde{P}_{k_n,W_0} \left( \bar{f}_{G_kn} \in \bar{F}_\varepsilon \right)}{\tilde{P}_{k_n,W_0} \left( \bar{f}_{G_kn} \in \bar{F}^0 \right)},$$

Using Theorem 14 again, this shows that,

$$\lim_{n \to \infty} \frac{1}{(kn)^2} \log \tilde{P}_{k_n,W_0} \left( \delta_{\overline{C}}(\bar{f}_{G_kn}, \bar{F}^*) \geq \varepsilon | \bar{f}_{G_kn} \in \bar{F} \right) \leq \lim_{n \to \infty} \frac{1}{(kn)^2} \log \tilde{P}_{k_n,W_0} \left( \bar{f}_{G_kn} \in \bar{F}_\varepsilon \right) - \lim_{n \to \infty} \frac{1}{(kn)^2} \log \tilde{P}_{k_n,W_0} \left( \bar{f}_{G_kn} \in \bar{F}^0 \right) \leq \inf_{h \in \bar{F}^0} J_{W_0}(h) - \inf_{h \in \bar{F}_\varepsilon} J_{W_0}(h).$$

It now suffices to show that $\inf_{h \in \bar{F}} J_{W_0}(h) < \inf_{h \in \bar{F}^0} J_{W_0}(h)$. Clearly, $\inf_{h \in \bar{F}} J_{W_0}(h) \leq \inf_{h \in \bar{F}_\varepsilon} J_{W_0}(h)$. Suppose that equality holds. The compactness of $\bar{F}_\varepsilon$ and the lower semi-continuity of $J_{W_0}$ (Lemma 4) imply that there exists $\bar{g} \in \bar{F}_\varepsilon$ that attains the infimum. It follows that $J_{W_0}(\bar{g}) = \inf_{h \in \bar{F}_\varepsilon} J_{W_0}(h) = \inf_{h \in \bar{F}^0} J_{W_0}(h)$. But then $\bar{g} \in \bar{F}^*$, and so $\bar{F}_\varepsilon \cap \bar{F}^* \neq \emptyset$, which is a contradiction. 

Proof of Theorem 3 We will prove the theorem by establishing condition (18) in Theorem 10. We first note that the continuity of $\phi_t(W_0, \cdot)$ at $t$ excludes the trivial case $\tau(W) = \{t\}$, since then $\tau(W) = \{t\}$ as well, which shows that $\phi_t(W_0, \cdot)$ jumps from a finite constant to $\infty$ at $t$. 25
Next we recall that \( \phi_\tau(W_0,t) = \min\{J_{W_0}(f) : f \in \tilde{F}\} \), where \( \tilde{F} = \{\tilde{f} \in \tilde{W} : \tau(\tilde{f}) \geq t\} \). By the continuity of \( \tau \), \( \tilde{F} \) is closed. Also by the continuity of \( \tau \), the interior of \( F \) is the set \( \tilde{F}^0 = \{\tilde{f} \in \tilde{W} : \tau(\tilde{f}) > t\} \), which for all \( \varepsilon > 0 \) is a subset of \( \tilde{F}_\varepsilon = \{\tilde{f} \in \tilde{W} : \tau(\tilde{f}) \geq t + \varepsilon\} \). As a consequence,

\[
\phi_\tau(W_0,t) = \inf_{h \in \tilde{F}} J_{W_0}(h) \leq \inf_{h \in \tilde{F}^0} J_{W_0}(h) \leq \inf_{h \in \tilde{F}_\varepsilon} J_{W_0}(h) = \phi_\tau(W_0,t + \varepsilon).
\]

Sending \( \varepsilon \to 0 \) and using the continuity of \( \phi_\tau(W_0,\cdot) \) at \( t \), we see that the first inequality is saturated, proving (18).

The proof is completed by noting that \( \phi_\tau(W_0,t) < \infty \) if and only if \( t \leq t_{\max} \). \( \square \)

3 \( \phi_\tau \): Monotonicity, continuity, and examples

In this section, we establish some analytical properties of the function \( \phi_\tau \), which will be critical for our discussion of symmetry/symmetry-breaking in the subsequent sections. Section 3.1 collects some preliminary properties of homomorphism densities and the cut distance. In Section 3.2, we introduce the “sufficient increase property”, which guarantees the continuity of \( \phi_\tau \). Further, we establish that homomorphism densities satisfy this property, and the operator norm satisfies this property under additional assumptions. Finally, Section 3.3 establishes an alternative variational representation of \( \phi_\tau \) at points of continuity. Using this representation, we identify a class of parameters \( \tau \) such that \( \phi_\tau \) is strictly increasing.

3.1 Preliminaries

In subsequent sections, it will be necessary to express the homomorphism density as the sum of interval labeled homomorphisms and identify interval labeled homomorphisms that are always zero on \( W_0 \). Given \( W_0 \in B^r \), let \( I_1, I_2, \ldots, I_m \) be the intervals of \( \gamma \), i.e. \( I_1 = [0, \gamma_1] \) and \( I_j = (\sum_{i=1}^{j-1} \gamma_i, \sum_{i=1}^{j} \gamma_i] \) for \( j \geq 2 \). In the previous section we used \( k \) to denote the total number of intervals and assumed that all intervals have the same length. Here the intervals need not be the same length— to emphasize this we now use \( m \) to denote the total number of intervals. The blocks of \( W_0 \) have the form \( I_i \times I_j \) for \( i, j \in [m] \). Let \( v = |v(H)| \), and let \( Y \in [m]^v \) be a vector of vertex interval indices. Define the interval-labeled homomorphism density as

\[
t(H,g,Y) = \int_{x_1 \in I_{Y_1}} \int_{x_2 \in I_{Y_2}} \cdots \int_{x_v \in I_{Y_v}} \prod_{(i,j) \in E(H)} g(x_i, x_j) \, dx_v \cdots dx_2 dx_1.
\]

In other words, \( t(H,g,Y) \) accounts for the homomorphisms in which the \( j^{th} \) vertex is in \( I_{Y_j} \) for all \( j \in [v] \) and so

\[
t(H,g) = \sum_{Y \in [m]^v} t(H,g,Y).
\]

Next, we define relevant blocks to be the blocks whose values may affect the homomorphism density of a graphon in \( W_{00} \). Increasing the value of a graphon \( g \) in \( W_{00} \) on a relevant block has the potential to increase \( t(H,g) \). Figure 6 gives an example of a block that is not relevant.
Definition 15. Fix a finite graph $H$ and $W_0$. We say an interval labeling vector $Y$ is irrelevant with respect to $W_0$ if there exists $(i, j) \in \epsilon(H)$ such that the block $I_Y \times I_Y$ takes value zero on $W_0$. Equivalently, $Y$ is irrelevant if $\phi(H, g, Y) = 0$ for all $g \in W_0$. We say $Y$ is relevant otherwise.

We say a block $I_a \times I_b$ contributes to the interval-labeled homomorphism density $\phi(H, g, Y)$ if $Y_i = a$ and $Y_j = b$ for some $(i, j) \in E(H)$. We say the block $I_a \times I_b$ is relevant if $I_a \times I_b$ contributes to some $\phi(H, g, Y)$ with $Y$ relevant. Let $R \subseteq [0, 1]^2$ be the union of all relevant blocks.

Note that $I_a \times I_b$ is relevant if and only if $p_{ab} > 0$ and $\phi(H, W_0)$ strictly decreases when $p_{ab}$ is lowered.

Our next result establishes that if the cut distance between two graphons is at least a constant, one can find a region where the values on the graphons differ by at least a constant. This result will be crucially used to establish the "sufficient increase property" in this section. In our subsequent discussion, we will use this result to establish the existence of nearby graphons with lower entropy.

Lemma 13. Let $f, g \in W$. Let $S^+_\beta = \{(x, y) \in [0, 1]^2 : f(x, y) - g(x, y) \geq \beta\}$ and $S^-_\beta = \{(x, y) \in [0, 1]^2 : g(x, y) - f(x, y) \geq \beta\}$.

1. If $f \geq g$ pointwise and $d_{\Omega}(f, g) \geq \epsilon$, then $|S^+_{\epsilon/2}| \geq \epsilon/2$.

2. If $d_{\Omega}(f, g) \geq \epsilon$, then $|S^+_{\epsilon/4}| \geq \epsilon/4$ or $|S^-_{\epsilon/4}| \geq \epsilon/4$.

Proof. Suppose $f \geq g$ pointwise and $d_{\Omega}(f, g) \geq \epsilon$. Since $f \geq g$ pointwise, $d_{\Omega}(f, g) = \|f - g\|_1$. It follows that

$$\epsilon \leq d_{\Omega}(f, g) = \|f - g\|_1 = \int_{[0, 1]^2} f - g \leq |S^+_{\epsilon/2}| + \frac{\epsilon}{2}(1 - |S^+_{\epsilon/2}|),$$

and so $|S^+_{\epsilon/2}| \geq \epsilon/(2 - \epsilon) > \epsilon/2$.

Next suppose $d_{\Omega}(f, g) \geq \epsilon$ with no additional assumptions on $f, g \in W$. Let $S^+ = \{(x, y) \in [0, 1]^2 : f(x, y) \geq g(x, y)\}$ and $S^- = \{(x, y) \in [0, 1]^2 : f(x, y) < g(x, y)\}$. Since $d_{\Omega}(f, g) \geq \epsilon$, there exists $A, B \subseteq [0, 1]$ such that

$$\int_{A \times B} f - g \geq \epsilon.$$

Observe

$$\left| \int_{A \times B} f - g \right| \leq \int_{A \times B} |f - g| = \int_{(A \times B) \cap S^+} f - g + \int_{(A \times B) \cap S^-} g - f.$$

It follows that $\int_{(A \times B) \cap S^+} f - g \geq \epsilon/2$ or $\int_{(A \times B) \cap S^-} g - f \geq \epsilon/2$. Suppose $\int_{(A \times B) \cap S^+} f - g \geq \epsilon/2$. Then

$$\frac{\epsilon}{2} \leq \int_{(A \times B) \cap S^+} f - g \leq \int_{(A \times B) \cap S^+_{\epsilon/4}} 1 + \int_{(A \times B) \cap (S^+ \setminus S^+_{\epsilon/4})} \frac{\epsilon}{4} \leq |S^+_{\epsilon/4}| + \frac{\epsilon}{4}(1 - |S^+_{\epsilon/4}|).$$

It follows that $|S^+_{\epsilon/4}| \geq \epsilon/4$. A similar argument shows that $|S^-_{\epsilon/4}| \geq \epsilon/4$ when $\int_{(A \times B) \cap S^-} g - f \geq \epsilon/2$. \qed

3.2 Establishing the continuity of $\phi_{\tau}$

In this subsection we establish that $\phi_{\tau}$ is continuous for certain graph parameters $\tau$. Throughout this section, we consider $W_0$ to be fixed. When $\tau$ is clear from context, we let $\phi(t) = \phi_{\tau}(W_0, t)$.

Lemma 14. Let $H$ be a finite graph, let $\tau = \tau(H, \cdot)$, let $W_0 \in B^\tau$, and set $t_{\max} = \tau_{\max}(W_0)$. Then $\phi$ is continuous on $R \setminus \{t_{\max}\}$.

Lemma 15. Let $\tau(g) = \|g\|_{op}$, let $W_0$ be a two-block bipartite graphon with $W_0 \in B^{(\gamma, 1-\gamma)}$, and set $t_{\max} = \tau_{\max}(W_0)$. Then $\phi$ is continuous on $R \setminus \{t_{\max}\}$.

In order to establish the above lemmas, we describe the sufficient increase property of $\phi$ and show that this property guarantees that $\phi_{\tau}$ is continuous. Then we show that homomorphism densities have this property (with any block constant base graphon $W_0$), and the operator norm has this property when $W_0$ is a two block bipartite graphon.

Definition 16. We say that $\tau$ has the sufficient increase property on $W_0$ if the following is true. Let $t_{\max} = \max_{g \in W_0} \tau(g)$ and $t_{\min} = \min_{g \in W_0} \tau(g)$. Fix any $\eta > 0$ and $t_{\min} \leq t < t_{\max}$. Then there exist $\alpha = \alpha(t, t_{\max}, \eta), \beta = \beta(t, t_{\max}, \eta) > 0$, such that the following holds for all $g \in W_0$. If $\tau(g) \geq t - \alpha$, then there exists $g^* \in \{f : \|f - g\|_{op} \leq \eta\} \cap W_0$ such that $\tau(g^*) \geq t + \beta$.

Lemma 16. Let $W_0 \in B^\tau$, and let $\tau$ be a continuous graph parameter that has the sufficient increase property on $W_0$, and let $t_{\max} = \tau_{\max}(W_0)$ Then $\phi$ is continuous on $R \setminus \{t_{\max}\}$. \qed
In [33], Lubetzky and Zhao studied the variational problem [9] when $W_0$ is a constant graphon. They defined a “nice graph parameter” as a graph parameter $\tau$ that is (i) continuous with respect to $\delta_{\Box}$ and (ii) has the property that every local extremum of $\tau$ with respect to $L_\infty$ is necessarily a global extremum. They show that for any nice graph parameter $\tau$, $\phi_\tau$ is continuous in the setting where the base graphon $W_0$ is constant. Their proof technique cannot be directly adapted to the setting where $W_0$ is a block constant graphon with a zero or one block. When $W_0$ take values zero or one, the entropy function $I_{W_0}$ can be infinite, creating a technical hurdle. In particular, it is not clear how to establish right continuity of $\phi_\tau$ for arbitrary nice graph parameters. We instead use Definition [10] as a sufficient condition for the right continuity of $\phi_\tau$.

Before proving Lemma 16, we establish the left continuity of $\phi_\tau$ without any assumptions on the base graphon $W_0$ or the continuous graph parameter $\tau$.

**Lemma 17.** Let $\tau$ be a continuous graph parameter and let $W_0 \in B^\gamma$. Then $\phi$ is left-continuous.

**Proof.** We first note that we may assume that $t \leq \tau_{\max}(\tilde{W})$, since $\phi = \infty$ and hence constant above $\tau_{\max}(\tilde{W})$. Let $t_n \nearrow t$. Since $\phi$ is non-decreasing in $t$, the sequence $\phi(t_n)$ has a limit, and $\lim_{n \to \infty} \phi(t_n) \leq \phi(t)$. To prove an upper bound on $\phi(t)$, recall the definition (8) of $\phi(t)$ as a minimum. For each $k \geq 1$, there exists $\tilde{g}_k$ such that $\tau(\tilde{g}_k) \geq t_k$ and $I_{W_0}(\tilde{g}_k) = \phi(t_k)$. By the compactness of $\tilde{W}$, there exists a convergent subsequence $\tilde{g}_{k_j}$ such that $\delta_{\Box}(\tilde{g}_{k_j}, \tilde{g}) \to 0$. Since $\tau(\tilde{g}_{k_j}) \geq t_{k_j}$ and $t_{k_j} \nearrow t$, it follows that $\tau(\tilde{g}) \geq t$, and thus $\phi(t) \leq I_{W_0}(\tilde{g})$. Combined with the lower semi-continuity of $I_{W_0}$ (Lemma 4), we get

$$
\phi(t) \leq I_{W_0}(\tilde{g}) \leq \liminf_{j \to \infty} I_{W_0}(\tilde{g}_{k_j}) = \liminf_{j \to \infty} \phi(t_{k_j}) = \lim_{n \to \infty} \phi(t_n).
$$

**Remark 7.** Since $\phi$ is left-continuous (Lemma 17) and non-decreasing, $\phi$ can have at most countably many points of discontinuity.

We now prove Lemma 16 which establishes the continuity of $\phi_\tau$ when $\tau$ has the sufficient increase property.

**Proof of Lemma 16.** By Lemma 17 it suffices to establish the right-continuity of $\phi$ at $t$. By assumption, $t \neq t_{\max}$. Since $\phi$ is constant on $(-\infty, t_{\min}]$ and $(t_{\max}, \infty)$ (where it is 0 and $\infty$, respectively), we may assume that $t_{\min} \leq t < t_{\max}$. Consider a sequence $t_n \uparrow t$, and an arbitrary $\varepsilon > 0$. We need to show that there exists $n$ sufficiently large such that $\phi(t_n) \leq \phi(t) + \varepsilon$.

Let $\eta > 0$ be such that if $f, g \in \mathcal{W}_0$ and $\|f - g\|_\infty \leq \eta$, then $|I_{W_0}(f) - I_{W_0}(g)| < \varepsilon$; Proposition 2 guarantees the existence of such an $\eta$. Let $\tilde{g} \in \mathcal{W}_0$ be such that $\tau(\tilde{g}) \geq t$ and $\phi(t) = I_{W_0}(\tilde{g})$. By definition of $I_{W_0}$ there exists a sequence $f_k \in \mathcal{W}_0$ such that

$$
I_{W_0}(f_k) \to I_{W_0}(\tilde{g}) \quad \text{and} \quad \delta_{\Box}(f_k, \tilde{g}) \to 0.
$$

Since $\tau$ has the sufficient increase property on $\mathcal{W}_0$, there exist $\alpha, \beta > 0$ such that

$$
\tau(g) \geq t - \alpha \implies \exists g^* \text{ with } \|g^* - g\|_\infty \leq \eta \text{ and } \tau(g^*) \geq t + \beta.
$$

Since $\tau$ is continuous in $(\tilde{W}_0, \delta_{\Box})$ and $\tau(g) \geq t$, there exists $k_0$ sufficiently large such that for all $k \geq k_0$, $\tau(f_k) \geq t - \alpha$. Thus, for all $k \geq k_0$, there exists $f_k'$ such that $\tau(f_k' ) \geq t + \beta$ and $\|f_k' - f_k\|_\infty \leq \eta$.

The choice of $\eta$ implies

$$
|I_{W_0}(f_k') - I_{W_0}(f_k)| \leq \varepsilon.
$$

By compactness of $(\tilde{W}_0, \delta_{\Box})$, there exists a convergent subsequence such that $f_{k_j}' \to \tilde{h}$ for some $\tilde{h} \in \tilde{W}_0$.

Since $\tau$ is continuous with respect to $\delta_{\Box}$, $\tau(f_{k_j}') \to \tau(\tilde{h})$, and so $\tau(\tilde{h}) \geq t + \beta$. It follows that

$$
\phi(t + \beta) \leq I_{W_0}(\tilde{h}) \leq \liminf_{j \to \infty} I_{W_0}(f_{k_j}') \leq \liminf_{j \to \infty} I_{W_0}(f_{k_j}) + \varepsilon = I_{W_0}(\tilde{g}) + \varepsilon = \phi(t) + \varepsilon.
$$

Taking $n$ sufficiently large such that $t_n \leq t + \beta$ and noting $\phi(t_n) \leq \phi(t + \beta)$ yields the desired statement. □
Next we establish that homomorphism densities have the sufficient increase property. To this end, we introduce the following graphon $g^{+\eta}$,

$$
g^{+\eta}(x, y) = \begin{cases} 
g(x, y) & (x, y) \not\in \Omega \\
\min\{g(x, y) + \eta, 1\} & \text{otherwise}
\end{cases}
$$

and note the following fact.

**Fact 1.** Let $\ell, u, w, z \in \mathbb{R}$ with $u \geq 0$. Suppose that $\ell \geq uz$ and $\ell \geq w - z$. Then $\ell \geq \frac{uw}{u+w}$.

**Proof.** Note that for all $z \in \mathbb{R}$, $uz \geq \frac{uw}{u+w}$ or $w - z \geq \frac{uw}{u+w}$. The fact follows directly. \qed

**Lemma 18.** Let $\tau = t(H, \gamma)$ where $H$ is a finite graph, let $\gamma \in \Delta_m$ and $W_0 \in B^\gamma$, and let $t_{\text{max}} = \tau_{\text{max}}(W_0)$. Fix $\eta > 0$. Then there exists $c = c(\gamma, H, \eta) \in (0, 1]$ such that $\tau(g^{+\eta}) \geq \tau(g) + c(t_{\text{max}} - \tau(g))^2$ for all $g \in W_0$.

**Proof of Lemma 18** Define $g^{\text{max}}$ as follows

$$
g^{\text{max}}(x, y) = \begin{cases} 
g(x, y) & (x, y) \in \{(0, 1]^2 \setminus R\} \cup \Omega^c \\
1 & \text{otherwise},
\end{cases}
$$

where $R$ is the union of relevant blocks, as defined above. Note that since $g = W_0 \in [0, 1]^2 \setminus \Omega$, $g^{\text{max}} \in W_\Omega$. Also note that $\tau(g^{\text{max}}) = \max_{f \in W_\Omega} \tau(f) = t_{\text{max}}$.

Let $\epsilon_H = |E(H)|$, $v = |V(H)|$, and let

$$
d = \frac{\tau(H, g^{\text{max}}) - \tau(H, g)}{\epsilon_H} = \frac{t_{\text{max}} - \tau(g)}{\epsilon_H}.
$$

Since the statement of the lemma is trivial if $\tau(g) = t_{\text{max}}$ we may assume w.l.o.g. that $d > 0$. The Counting Lemma \cite{22} Lemma 10.23 implies that $\delta_{\triangle}(g^{\text{max}}, g) \geq d$, and so

$$
d_\triangle(g, g^{\text{max}}) \geq \delta_{\triangle}(g, g^{\text{max}}) \geq d.
$$

Let $S = \{(x, y) \in [0, 1]^2 : g^{\text{max}} - g \geq d/2\}$. Since $g^{\text{max}} \geq g$ pointwise, Lemma \cite{13} implies that $|S| \geq d/2$. Let $\eta' \triangleq \min\{\eta, d/2\}$. It follows that $g^{+\eta} - g \geq \eta'$ on $S$. By construction, $S \subseteq R$. Recall that $m$ denotes the number of blocks in $W_0$. Therefore, there are at most $m^2$ relevant blocks of $[0, 1]^2$ of the form $I_i \times I_j$ for $i, j \in [m]$. Thus, there exists $a, b \in [m]$ such that $I_a \times I_b$ is relevant and $|(I_a \times I_b) \cap S| \geq d/(2m^2)$.

It suffices to show that increasing $g$ to $g^{+\eta}$ on $(I_a \times I_b) \cap S$ yields a constant increase in the homomorphism density. Since $I_a \times I_b$ is a relevant block, there exists a relevant $Y \in [m]^v$ such that $Y_p = a$, $Y_q = b$ for some $\{p, q\} \in E(H)$.

Define

$$
Z_{\mathcal{F}}(H, g, Y) = \int_{x_1 \in I_{I_1}} \cdots \int_{x_v \in I_{I_v}} \prod_{(i, j) \in E(H)} g(x_i, x_j) 1\{(x_p, x_q) \not\in S\} dx_v \ldots dx_1.
$$

$$
Z_{S}(H, g, Y) = \int_{x_1 \in I_{I_1}} \cdots \int_{x_v \in I_{I_v}} \prod_{(i, j) \in E(H)} g(x_i, x_j) 1\{(x_p, x_q) \in S\} dx_v \ldots dx_1.
$$

In other words, $Z_{S}(H, g, Y)$ accounts for the homomorphisms in which the $\{p, q\}$ edge lies in $S$, and $Z_{\mathcal{F}}(H, g, Y)$ accounts for the homomorphisms in which the $\{p, q\}$ edge does not lie in $S$. Therefore,

$$
t(H, g, Y) = Z_{\mathcal{F}}(H, g, Y) + Z_{S}(H, g, Y).
$$

Since $g^{+\eta} \geq g$ pointwise, $Z_{\mathcal{F}}(H, g^{+\eta}, Y) \geq Z_{\mathcal{F}}(H, g, Y)$. Next we derive two lower bounds on $Z_{S}(H, g^{+\eta}, Y)$.

First observe

$$
Z_{S}(H, g^{+\eta}, Y) = \int_{x_1 \in I_{I_1}} \cdots \int_{x_v \in I_{I_v}} \prod_{(i, j) \in E(H)} g^{+\eta}(x_i, x_j) 1\{(x_p, x_q) \in S\} dx_v \ldots dx_1
$$

$$
= \int_{x_1 \in I_{I_1}} \cdots \int_{x_v \in I_{I_v}} g^{+\eta}(x_p, x_q) \prod_{(i, j) \in E(H)} g^{+\eta}(x_i, x_j) 1\{(x_p, x_q) \in S\} dx_v \ldots dx_1
$$

$$
\geq \int_{x_1 \in I_{I_1}} \cdots \int_{x_v \in I_{I_v}} (g(x_p, x_q) + \eta') \prod_{(i, j) \in E(H)} g(x_i, x_j) 1\{(x_p, x_q) \in S\} dx_v \ldots dx_1
$$

29
\[ Z_\text{S}(H,g^{+\eta},Y) \geq Z_\text{S}(H,g,Y)(1 + \eta'). \]

The final inequality follows from noting that \( g + \eta' \geq (1 + \eta')g \). Our goal is to lower bound the difference \( t(H,g^{+\eta'}) - t(H,g) \) by a constant. The above computation implies that \( t(H,g^{+\eta'}) - t(H,g) \geq \eta' Z_\text{S}(H,g,Y) \). This lower bound may not be sufficient if \( Z_\text{S}(H,g,Y) \) is too small. We derive another lower bound for this case.

Let \( \beta = \min_{\in\{0,1\}}|I_j| \). Observe that for all \( \{i,j\} \in E(H) \), \( g^{+\eta}(x_i,x_j) \geq \eta \) when \( x_i \in I_Y \) and \( x_j \in I_Y \), as \( Y \) is relevant. It follows that

\[ Z_\text{S}(H,g^{+\eta},Y) \geq \eta^{\|S \cap (I_a \times I_b)\|} \prod_{j \in \{m\} \setminus \{p,q\}} |I_{Y_j}| \geq \eta^{\|S \cap (I_a \times I_b)\|} \beta^{\nu - 2}. \]

We have shown that \( Z_\text{S}(H,g^{+\eta},Y) - Z_\text{S}(H,g,Y) \geq \eta' Z_\text{S}(H,g,Y) \) and

\[ Z_\text{S}(H,g^{+\eta},Y) - Z_\text{S}(H,g,Y) \geq \eta^{\|S \cap (I_a \times I_b)\|} \beta^{\nu - 2} - Z_\text{S}(H,g,Y). \]

Applying Fact 1 with \( u = \eta', w = \eta^{\|S \cap (I_a \times I_b)\|} \beta^{\nu - 2} \), \( z = Z_\text{S}(H,g,Y) \), and \( \ell = Z_\text{S}(H,g^{+\eta},Y) - Z_\text{S}(H,g,Y) \), we obtain

\[ Z_\text{S}(H,g^{+\eta},Y) - Z_\text{S}(H,g,Y) \geq \frac{\eta' \eta^{\|S \cap (I_a \times I_b)\|} \beta^{\nu - 2}}{\eta + 1}. \]

We now simplify our lower bound using the facts that \( |S \cap (I_a \times I_b)| \geq d/(2m^2) \), \( \eta'/(1 + \eta') \geq \eta'/2 \), \( \eta' \geq \eta d/2 \), and \( d = (t_{\max} - \tau(g))/\epsilon_H \), obtaining

\[ \frac{\eta' \eta^{\|S \cap (I_a \times I_b)\|} \beta^{\nu - 2}}{\eta + 1} \geq \frac{\eta^{\|S \cap (I_a \times I_b)\|} \beta^{\nu - 2}}{8m^2} \geq \frac{\eta^{\|S \cap (I_a \times I_b)\|} \beta^{\nu - 2}}{8e_4 m^2}. \]

Set \( c = \frac{\eta^{\|S \cap (I_a \times I_b)\|} \beta^{\nu - 2}}{8e_4 m^2} \). Note that \( c \) is a function only of \( H \), \( \gamma \), and \( \eta \). It follows that

\[ t(H,g^{+\eta},Y) = Z_\text{S}(H,g^{+\eta},Y) + Z_\text{S}(H,g^{+\eta},Y) \]

\[ \geq Z_\text{S}(H,g,Y) + Z_\text{S}(H,g,Y) + c(t_{\max} - \tau(g))^2 \]

\[ = t(H,g,Y) + c(t_{\max} - \tau(g))^2. \]

Thus, \( t(H,g^{+\eta}) \geq t(H,g) + c(t_{\max} - \tau(g))^2 \).

Next we establish that \( \phi_\tau \) is continuous when \( \tau \) is a homomorphism density by using Lemma 19 to show that homomorphism densities have the sufficient increase property.

**Proof of Lemma 19.** By Lemma 10 it suffices to show that \( \tau \) has the sufficient increase property. Fix \( \eta \), \( t \) and \( t_{\max} \). If \( \tau(g) > t + \frac{1}{8}(t_{\max} - t) \), we can choose \( g' = g \) (and any \( \beta \leq \frac{1}{8}(t_{\max} - t) \)). Therefore, w.l.o.g., \( \tau(g) \leq t + \frac{1}{8}(t_{\max} - t) \). Then \( t_{\max} - \tau(g) \geq \frac{1}{8}(t_{\max} - t) \) and by Lemma 18 \( g^{+\eta_n} \geq \tau(g) + \frac{1}{8}(t_{\max} - t)^2 \).

Choosing \( \alpha = \frac{1}{8}(t_{\max} - t)^2 \), the assumption \( \tau(g) > t - \alpha \) then implies

\[ \tau(g^{+\eta_n}) \geq t + \frac{c}{8}(t_{\max} - t)^2. \]

Setting \( \beta = \alpha \) and noting that \( \beta \leq \frac{1}{8}(t_{\max} - t)^2 \leq \frac{1}{8}(t_{\max} - t) \) completes the proof.

Next we show that the operator norm has the sufficient increase property. We begin with the following lemma.

**Lemma 19.** Let \( \tau(g) = \|g\|_{\text{op}} \) and let \( W_0 \in B(g^{1-\gamma}) \) be a bipartite graphon. Let \( t_{\max} = \tau_{\max}(W_0) \), and let \( g^{+\eta} \) be as defined in (20). Fix any \( \eta > 0 \) and \( \tau(g) < t_{\max} \). Then

\[ \tau(g^{+\eta}) \geq \max \left\{ \eta^{\frac{1}{2} t_{\max}} \left( 1 + \frac{\eta^4}{2\eta^3} (t_{\max} - \tau(g))^2 \right) \right\}. \]

**Proof.** Recalling the definition of a bipartite graphon \( f^t_p \) from Section 1.6, we note that except for the trivial case \( p \in \{0,1\} \) (in which case \( W_0 = \{W_0\} \) and \( \tau(g) = t_{\max} \) for all \( g \in W_0 \)), the set \( \Omega \) is \([0,\gamma] \times (\gamma,1] \cup (\gamma,1] \times [0,\gamma] \). Let \( h^{\text{max}} = f^t_p \) be the graphon that takes value 1 on \( \Omega \) and agrees with \( W_0 \) on \( \Omega^c \) (where both are 0). Note that if \( f \leq g \) pointwise, then \( \|f\|_{\text{op}} \leq \|g\|_{\text{op}} \). It follows that \( \tau(h^{\text{max}}) = t_{\max} \). The graphon \( h^{\text{max}} \) satisfies \( f \leq h^{\text{max}} \) for every \( f \in W_0 \).
To prove the lower bound \( \eta_{\text{max}} \), we note that \( g^{+\eta} \geq f_0^{+\eta} \) pointwise, implying that \( \|g^{+\eta}\|_{\text{op}} \geq \|f_0^{+\eta}\|_{\text{op}} = \eta \|f_0\|_{\text{op}} = \eta \|\eta\|_{\text{op}} \).

To prove the second lower bound, we start by applying \cite{33} Lemma 3.6, which states that for \( f, g \in \mathcal{W} \)
\[
\frac{\|f\|_{\text{op}} - \|g\|_{\text{op}}^2}{4} \leq \delta_{\mathcal{C}}(f, g).
\]
Let \( d = \left( \frac{\|f\|_{\text{op}} - \|g\|_{\text{op}}^2}{4} \right)^{\frac{1}{2}} \). It follows that
\[
d \leq \delta_{\mathcal{C}}(h_{\text{max}}, g) \leq d_{\mathcal{C}}(h_{\text{max}}, g).
\]
Next, let \( \eta' = \min\{\eta, d/2\} \) and \( S = \{(x, y) \in \Omega : h_{\text{max}}(x, y) - g(x, y) \geq \eta' \} \). Since \( h_{\text{max}} = g \) on \([0, 1]^2 \setminus \Omega \) and \( h_{\text{max}} \geq g \) pointwise, Lemma \cite{33} implies that \( |S| \geq d/2 \).

Since \( T_\eta \) is a self-adjoint compact nonnegative linear operator, there exists \( u \in L_2([0, 1]) \) such that \( u(x) \geq 0 \), \( \|u\|_2 = 1 \) and \( T_\eta u(x) = \|g\|_{\text{op}} u(x) \) for all \( x \in [0, 1] \) \cite{11} Proposition 2.11. We will derive a lower bound on \( \tau(g^{+\eta})^2 \) by showing that for some \( c > 0 \),
\[
T_{\eta^+} u(x) \geq T_\eta u(x) + c,
\]
for \( x \) in some subset of \([0, 1]\). The construction of this subset will depend on \( S \) and \( u \). For ease of notation, we let \( \varepsilon = \eta/2 \). Define
\[
A_\varepsilon = \{x \in [0, 1] : u(x) \geq \varepsilon\} \quad \text{and} \quad A_\varepsilon^c = [0, 1] \setminus A_\varepsilon.
\]
We will consider two cases that depend on the size of \( A_\varepsilon^c \). In each case we find a subset of \([0, 1]\) and \( c > 0 \) satisfying \eqref{21}.

Before proceeding to the cases, we establish a useful property of \( u \). Define
\[
u_1 = \int_0^\gamma u(x) \, dx \quad \text{and} \quad \nu_2 = \int_0^1 u(x) \, dx.
\]
Since \( \|u\|_2 = 1 \), there exists \( x \in [0, 1] \) such that \( u(x) \geq 1 \). We assume that \( x \in [0, \gamma] \). After completing the proof under this assumption, we will discuss how a similar argument applies when \( x \in (\gamma, 1] \). Since \( x \in [0, \gamma] \), \( g \) is zero on \( \{x \} \times [0, \gamma] \) and so
\[
\|g\|_{\text{op}} \leq \|g\|_{\text{op}} u(x) = T_\eta u(x) = \int_0^1 g(x, y) u(y) \, dy = \int_0^1 g(x, y) u(y) \, dy \leq \nu_2.
\]

**Case 1:** \( |A_\varepsilon^c \cap [0, \gamma]| \geq d/16 \).
Let \( x \in A_\varepsilon^c \cap [0, \gamma] \). Observe
\[
T_\eta u(x) = \|g\|_{\text{op}} u(x) \leq \|g\|_{\text{op}} \varepsilon = \frac{\eta \|g\|_{\text{op}}}{2}.
\]
Note that \( g^{+\eta}(x, y) \geq \eta \) for all \( y \in (\gamma, 1] \) and \( g^{+\eta}(x, y) = 0 \) for all \( y \in [0, \gamma] \) by construction. It follows that
\[
T_{\eta^+} u(x) = \int_\gamma^1 g^{+\eta}(x, y) u(y) \, dy \geq \eta \nu_2 \geq \eta \|g\|_{\text{op}}.
\]
Thus for all \( x \in A_\varepsilon^c \cap [0, \gamma] \)
\[
T_{\eta^+} u(x) - T_\eta u(x) \geq \frac{\eta \|g\|_{\text{op}}}{2}.
\]
Observe that
\[
\tau(g^{+\eta})^2 = \|g^{+\eta}\|_{\text{op}}^2 \geq \|T_{\eta^+} u\|_{\text{op}}^2 = \int_0^1 (T_{\eta^+} u(x))^2 \, dx
\]
\[
\geq \int_{A_\varepsilon^c \cap [0, \gamma]} \left( T_\eta u(x) + \frac{\eta \|g\|_{\text{op}}}{2} \right)^2 \, dx + \int_{(A_\varepsilon^c \cap [0, \gamma])^c} T_\eta u(x)^2 \, dx
\]
\[
\geq \|T_\eta u\|_{\text{op}}^2 + |A_\varepsilon^c \cap [0, \gamma]| \frac{\eta^2 \|g\|_{\text{op}}^2}{4} \geq \tau(g)^2 + \frac{d \eta^2 \|g\|_{\text{op}}^2}{64}.
\]

31
Case 2: $|A_c^e \cap [0, \gamma]| < d/16$.
Recall $|S| \geq d/2$. For each $x \in [0, 1]$, define $S_x = \{ y \in [0, 1] : (x, y) \in S \}$, and let $X = \{ x \in (\gamma, 1) : |S_x| > d/8 \}$. Since $g$ is symmetric, $(x, y) \in S$ if and only if $(y, x) \in S$. It follows that

$$
\frac{d}{4} \leq \frac{|S|}{2} = \int_0^1 |S_x| \leq |X| + \frac{d}{8}(1 - \gamma - |X|) \leq |X| + \frac{d}{8}(1 - |X|).
$$

Therefore $|X| \geq d/8$. Note that for all $x \in X$, $|S_x \cap A_c| > d/16$ because $|S_x| > d/8$, $S_x \subseteq [0, \gamma]$ and $|A_c^e \cap [0, \gamma]| < d/16$. Recall that for $y \in S_x$, $g^{+\eta}(x, y) \geq g(x, y) + \eta'$ and that for $y \in A_c$, $u(y) \geq \varepsilon$. Therefore, for all $x \in X$,

$$
T_y^\eta u(x) = \int_0^1 g^{+\eta}(x, y)u(y)dy \geq \int_{S_x \cap A_c} (g(x, y) + \eta')u(y)dy + \int_{(S_x \cap A_c)^c} g(x, y)u(y)dy
$$

$$
\geq T_y u(x) + \varepsilon \eta' |S_x \cap A_c| \geq T_y u(x) + \frac{d \varepsilon \eta'}{16}.
$$

Finally, observe that

$$
\tau(g^{+\eta})^2 \geq \int_0^1 (T_y^\eta u(x))^2dx \geq \int_X \left( T_y u(x) + \frac{d \varepsilon \eta'}{16} \right)^2dx + \int_{X^c} T_y u(x)^2dx
$$

$$
\geq \| T_y u \|^2 + |X| \left( \frac{d \varepsilon \eta'}{16} \right)^2 \geq \tau(g)^2 + \frac{d^2 \varepsilon^2 \eta'^2}{211}.
$$

Recalling that $\varepsilon = \eta/2$, $\eta' \geq \eta d/2$, and $d = (t_{\max} - \tau(g))^4/4$, we see that in the first case, we have

$$
\tau(g^{+\eta})^2 - \tau(g)^2 \geq \frac{d \eta^2 \| g \|^2_{op}}{64} = \frac{\eta^2 (t_{\max} - \tau(g))^4 \tau(g)^2}{2^{16}}
$$

and in the second case,

$$
\tau(g^{+\eta})^2 - \tau(g)^2 \geq \frac{d^2 \varepsilon^2 \eta'^2}{211} \geq \frac{d^4 \eta^2}{2^{15}} \left( \frac{\eta d}{2} \right)^2 = \frac{\eta^4 d^6}{2^{15}} = \frac{\eta^4 (t_{\max} - \tau(g))^{20}}{2^{25}}.
$$

Therefore, in both cases

$$
\tau(g^{+\eta})^2 - \tau(g)^2 \geq \frac{\eta^4 (t_{\max} - \tau(g))^{20}}{2^{25}} \tau(g)^2.
$$

Finally we revisit our assumption prior to the case work that the value $x$ such that $u(x) \geq 1$ is in $[0, \gamma]$. Suppose instead that $x \in (\gamma, 1]$. The equation analogous to (22) implies that $u_1 \geq \|g\|_{op}$. Now switching the roles of $[0, \gamma]$ and $(\gamma, 1]$ in Case 1 gives the same lower bounds on $\tau(g^{+\eta})^2$.

Next we establish the continuity of $\phi_\tau$ when $\tau$ is the operator norm and $W_0$ is a bipartite graphon with $W_0 \in B^{(\gamma, 1-\gamma)}$. We use Lemma 19 to establish the sufficient increase property.

Proof of Lemma 19. By Lemma 19, it suffices to show that $\tau$ has the sufficient increase property. Fix $\eta$, $t$ and $t_{\max}$. If $t = 0$, the choice $\beta = \eta t_{\max}$ and the first bound in Lemma 19 implies that $\tau(g^{+\eta}) \geq \beta$.

If $t > 0$, the proof proceeds along the same lines as the proof of Lemma 13. Indeed, as before, the required bound is easy if $\tau(g) > t + \frac{1}{2}(t_{\max} - t)$. In this case, we again choose $g^* = g$ (and any $\beta \leq \frac{1}{2}(t_{\max} - t)$). Therefore, w.l.o.g., $\tau(g) \leq t + \frac{1}{2}(t_{\max} - t)$. But then $t_{\max} - \tau(g) \geq \frac{1}{2}(t_{\max} - t)$ and by the second bound in Lemma 19,

$$
\tau(g^{+\eta}) \geq (1 + c) \tau(g) \quad \text{where} \quad c = \sqrt{1 + \frac{\eta^4}{2^{25}} (t_{\max} - t)^{20} - 1}.
$$

Choosing $\alpha = \min\{t/2, ct/4\}$ and $\beta = \min\{ct/4, (t_{\max} - t)/2\}$, the assumption $\tau(g) \geq t - \alpha$ then implies

$$
\tau(g^{+\eta}) \geq (t - \alpha) + (t - \alpha)c \geq t - ct/4 + ct/2 \geq t + ct/2 \geq t + \beta.
$$

\qed
3.3 Properties of $φ_τ$ at points of continuity

Lemma 20 states that at points of continuity, the function $φ_τ(W_0, t)$ can be alternatively expressed as the minimum of $I_{W_0}$ over a subset of $W$. We use this characterization of $φ_τ$ to establish that $φ_τ$ is strictly increasing when $τ$ is an increasing graph parameter and $φ_τ$ is continuous (Lemma 21).

**Lemma 20.** Fix $W_0 ∈ B^τ$ and let $t ∈ R$. If $φ_τ(W_0, ·)$ is continuous at $t$, then

\[
φ_τ(W_0, t) = \inf\{I_{W_0}(f) : f ∈ W, τ(f) ≥ t\}.
\]

**Lemma 21.** Let $τ$ be a graph parameter that is uniformly continuous (with respect to $δ_Ω$) and increasing, meaning that if $f ≥ g$ pointwise then $τ(f) ≥ τ(g)$. Suppose that $φ_τ(W_0, ·)$ is continuous on the open interval $(τ(W_0), τ_{\text{max}}(W_0))$. Then $φ_τ(W_0, t)$ is strictly increasing on $[τ(W_0), τ_{\text{max}}(W_0))$.

**Remark 8.** Note that the Counting Lemma [32, Lemma 10.23] and [33, Lemma 3.6] imply that homomorphism densities and the operator norm are each uniformly continuous with respect to $δ_Δ$. Lemma 21 directly implies that if $τ$ is a uniformly continuous increasing graph parameter, $φ_τ(W_0, ·)$ is continuous on $(τ(W_0), τ_{\text{max}}(W_0))$, and $f$ is a minimizer of the variational problem (24), then $τ(f) = t$ for all $t ∈ [τ(W_0), τ_{\text{max}}(W_0))$.

**Proof of Lemma 20.** Let $h(t) = \inf\{I_{W_0}(f) : f ∈ W, τ(f) ≥ t\}$. It is clear from the definition that $φ ≤ h$. We will show that the right continuity of $φ$ at $t$ implies that $φ(t) ≥ h(t)$.

To this end, we claim that for all $a ∈ R$,

\[
\inf\{I_{W_0}(f) : τ(f) > a\} = \inf\{I_{W_0}(f) : τ(f) > a\}.
\]

Indeed, it is clear the left hand side is at most the right hand side. Since both sides are infinite if the set $\{f ∈ W_0 : τ(f) > a\}$ is empty, we may assume that this set contains at least one $f ∈ W_0$ such that $τ(f) > a$. By the definition of $I_{W_0}$, there exists $g_0$ such that $δ_Ω(g_0, f) → 0$ and $I_{W_0}(g_0) → I_{W_0}(f)$. By continuity of $τ$, there exists $n_0$ sufficiently large such that for all $a > n_0$, $τ(g_0) > a$. Thus

\[
I_{W_0}(f) ≥ \inf\{I_{W_0}(h) : τ(f) > a\},
\]

and so (25) follows.

We now turn to showing that if $φ$ is right continuous at $t$, then $φ(t) ≥ h(t)$. Applying (25), we obtain

\[
φ(t + ε) ≥ \inf\{I_{W_0}(f) : τ(f) > t + ε/2\}
= \inf\{I_{W_0}(f) : τ(f) > t + ε/2\}
≥ \inf\{I_{W_0}(f) : τ(f) > t\}
= h(t)
\]

for any $ε > 0$. It follows by right continuity of $φ$ at $t$ that

\[
φ(t) = \lim_{ε → 0} φ(t + ε) ≥ h(t).
\]

**Lemma 22.** Fix $W_0 ∈ W$ and let $f, g ∈ W_0$. There exists $η = η(ε, W_0) > 0$ such that if $f ≥ g ≥ W_0$ pointwise and $d_Ω(f, g) ≥ ε$, then

\[
I_{W_0}(g) ≤ I_{W_0}(f) − η.
\]

**Proof.** We may assume that $ε < 2(1 − p)$ for $p ∈ Im(W_0) \setminus \{0, 1\}$. Recall the definition

\[
S_{ε/2}^+ = \{(x, y) ∈ [0, 1]^2 : f − g ≥ ε/2\}.
\]

Lemma 13 implies that $|S_{ε/2}^+| > ε/2$. Let

\[
η' = \min_{p ∈ Im(W_0) \setminus \{0, 1\}} \min_{ε ∈ [p, 1 − ε/2]} [h_p(x + ε/2) − h_p(x)] > 0.
\]

Since $f ≥ g ≥ W_0$ and $h_ν(·)$ is increasing on $[p, 1]$, we have that

\[
h_{W_0}(f(x, y)) ≥ h_{W_0}(g(x, y) + ε/2) ≥ h_{W_0}(g(x, y)) + η'.
\]
for all $(x, y) \in S^+_e/2$. As a consequence,

\[ I_{W_0}(f) = \int_{[0, 1]^2} h_{W_0(x,y)}(f(x,y)) \, dx \, dy \]

\[ \geq \int_{[0, 1]^2 \setminus S^+_e/2} h_{W_0(x,y)}(g(x,y)) \, dx \, dy + \int_{S^+_e/2} (h_{W_0(x,y)}(g(x,y) + \eta')) \, dx \, dy \]

\[ \geq I_{W_0}(g) + \eta' |S^+_e/2| \geq I_{W_0}(g) + \frac{\varepsilon \eta'}{2}. \]

Taking $\eta = \eta' \varepsilon/2$ completes the proof. \(\square\)

Proof of Lemma 21. The lemma is trivial if $\tau(W_0) = \tau_{\text{max}}(W_1)$ so we may assume that $\tau(W_0) < \tau_{\text{max}}(W_1)$. Furthermore, since $\phi_r(W_{\alpha, \cdot})$ is non-decreasing, it is enough to prove that it is strictly increasing on the open interval $(\tau(W_0), \tau_{\text{max}}(W_0))$. Let $\tau(W_0) < t_1 < t_2 < \tau_{\text{max}}(W_1)$. We will prove that $\phi_r(W_0, t_1) < \phi_r(W_0, t_2)$ by applying Lemma 20 and showing that

\[ \inf \{ I_{W_0}(f) : \tau(f) \geq t_1 \} < \inf \{ I_{W_0}(f) : \tau(f) \geq t_2 \}. \]

To establish the above statement, it suffices to show that there exists $\eta = \eta(t_1, t_2) > 0$ such that the following is true. If $f \in W_0$ is such that $\tau(f) \geq t_2$, then there exists $g' \in W_0$ such that $\tau(g') \geq t_1$ and $I_{W_0}(g') \leq I_{W_0}(f) - \eta$.

Given $f \in W_0$ with $\tau(f) \geq t_2$, define $g \in W_0$ such that $g(x, y) = \max\{W_0(x,y), f(x,y)\}$. Since $g \geq f$ pointwise and $\tau$ is increasing, $\tau(g) \geq t_2$. Moreover $g \geq W_0$ pointwise. Next define $g_\alpha \in W_0$ where

\[ g_\alpha(x, y) = \begin{cases} W_0(x, y) + \alpha (g(x, y) - W_0(x, y)) & (x, y) \in \Omega \\ g(x, y) & (x, y) \in \Omega^c. \end{cases} \]

By construction $g_0 = g$ and $g_0 = W_0$, and so $\tau(g_0) \geq t_2$, and $\tau(g_0) \leq t_1$. Since $\tau(g_\alpha)$ decreases continuously as $\alpha \to 0$, there exists some $\alpha_0$ such that $\tau(g_{\alpha_0}) = t_1$. Let $g' = g_{\alpha_0}$. Since $\tau(g) - \tau(g') \geq t_2 - t_1$, the uniform continuity of $\tau$ implies that $\Delta_{\varepsilon/2}(g, g') \geq \beta$ for some positive $\beta = \beta(t_2 - t_1)$. It follows that $\Delta_{\varepsilon/2}(g, g') \geq \beta$. Note also that $g, g' \in W_0$ and $g \geq g'$ pointwise. Therefore Lemma 22 implies that there exists $\eta = \eta(\beta)$ such that $I_{W_0}(g') \leq I_{W_0}(g) - \eta$. By construction, $I_{W_0}(g) \leq I_{W_0}(f)$ and so $I_{W_0}(g') \leq I_{W_0}(f) - \eta$. \(\square\)

4 The symmetric regime in general block models

In this section we prove Theorems 3 and 4 which establish that for any $d$-regular graph $H$ and graphons $W_0 \in B^*$, there exists a symmetric regime for $W_0$ and $t(H, \cdot)$.

Notation. Throughout this section we fix a particular $d$-regular subgraph $H$ with $d \geq 2$. Since we consider $d$ to be fixed, we suppress the dependence on $d$ in our notation.

Definition 17. Let $p \in (0, 1)$ and $d \geq 2$. We define $\psi_p : (0, 1) \to \mathbb{R}$ as

\[ \psi_p(x) = h_p(x^{1/d}), \]

and let $\hat{\psi}_p(x)$ denote the convex minorant of $\psi_p(x)$.

Proposition 13 in the Appendix collects some useful properties of $\hat{\psi}_p$. Next, we introduce notation that allows us to reason about individual blocks within a graphon. Let $m \in \mathbb{N}$ and let $\gamma \in \Delta^m$ be a vector of interval widths that determines block membership. Let $I_j$ for $j \in [m]$ be as given in Definition 6. For $x \in [0, 1]$, recall that $k(x)$ denotes the membership of $x$, so that $x \in I_{k(x)}$.

Let $f \in W$. For each $i, j \in [m]$, we define a function $f_{ij} : [0, 1]^2 \to [0, 1]$ that describes $f$ restricted to the block $I_i \times I_j$ by

\[ f_{ij}(x, y) = f \left( \sum_{k=0}^{i-1} \gamma_k + x \gamma_i, \sum_{k=0}^{j-1} \gamma_k + y \gamma_j \right) \]  \[ (26) \]

2 Strictly speaking, the function $f_{ij}$ contains a little more information than is contained in $f$ restricted to $I_i \times I_j$, namely, it represents the function $f$ restricted to the closure of $I_i \times I_j$. But this difference only appears on a set of measure zero, and is thus inconsequential; furthermore, the relation \([24]\) holds for all $(x, y) \in [0, 1]^2$. 

34
with $\gamma_0 = 0$. We write $f = (f_{ij})_{i,j \in [m]}$ to indicate that
\[ f(x, y) = f_{k(x), k(y)}(r(x), r(y)), \tag{27} \]
where
\[ r(x) = \frac{x - \sum_{i=0}^{k(x)-1} \gamma_i}{\gamma_{k(x)}} \tag{28} \]
(see Figure 7). By an abuse of notation, when a graphon $f$ takes constant values on the blocks defined by $\gamma$ (as in Definition 12), we write $f = (f_{ij})_{i,j \in [m]}$ where each $f_{ij} \in \mathbb{R}$ is a constant rather than a constant function.

In this section, we will utilize the restricted functions $f_{ij}$. Note that in contrast to the original graphon $f \in W$, the $f_{ij}$ functions are not necessarily symmetric. However, we will continue to use the cut distance $d_{\square}$ on these functions. In particular, we will crucially use Lemma 13—we note that the proof of this result does not utilize the symmetry of the functions, and thus continues to hold in this extended setting.

![Figure 7: Illustration of the graphon $f = (f_{ij})_{i,j \in [m]}$ for $m = 3$. The indicated point in the graphon $f$ is in the $(1, 2)$ block. The point is mapped to a point in the function $f_{12}$, with a scaled position.](image)

**Definition 18.** Let $d > 0$. For a graphon $f = (f_{ij})_{i,j \in [m]}$, we define the corresponding $d$-averaged block constant graphon
\[ f^* = (\|f_{ij}\|_d)_{i,j \in [m]}, \]
where the $d$-norm is defined as $\|g\|_d = \left( \int_{[0,1]^2} g(x, y)^d \, dx \, dy \right)^{1/d}$.

**Definition 19.** We say that $x \in [0, 1]$ has a strictly convex $\varepsilon$-neighborhood with respect to $\psi_p$ if for all $w \in \{ y \in (0, 1) : |x - y| \leq \varepsilon \}$, it holds that $\psi_p''(w) > 0$.

Note that as a consequence of Proposition 13 if $x \in [0, 1]$ has a strictly convex $\varepsilon$-neighborhood with respect to $\psi_p$, then $\psi_p(y) = \hat{\psi}_p(y)$ in the $\varepsilon$-neighborhood of $x$.

**Definition 20.** Let $W_0 \in B^*$ with $W_0 = (p_{ij})_{i,j \in [m]}$. We say that the graphon $f = (f_{ij})_{i,j \in [m]}$ satisfies the $\varepsilon$-convex minorant condition with respect to $W_0$ if for all $(i, j)$ such that $p_{ij} \in (0, 1)$, the value $\|f_{ij}\|_d^*$ has a strictly convex $\varepsilon$-neighborhood with respect to $\psi_{p_{ij}}$.

**Remark 9.** Strictly speaking, the above definitions don’t just depend on $W_0$ but on the representation of $W_0$ as a step-function in $B^*$. Explicitly, for the same $W_0$ represented as a function in $B^*$ and $B^{*'}$, the above definitions have a different meaning. We trust that this will not cause any confusion. Note also that this ambiguity does not arise in the context of Definition 12 and Theorems 3 and 4, where we assume that $W_0 \in B^{*'}$.
4.1 A key lemma for establishing the symmetric regime

We will establish the symmetric regimes for δ small and δ large (Theorems 4 and 4 respectively) by applying the following lemma.

**Lemma 23.** Let $W_0 ∈ B^*$ and $ε > 0$. Suppose $\tilde{f}$ is a minimizer of the variational problem (6) and there exists a sequence of graphons $f_n ∈ W_0$ such that each $f_n$ satisfies the $ε$-convex minorant condition, $d_Ω(f_n, \tilde{f}) → 0$, and $I_{W_0}(f_n) → J_{W_0}(\tilde{f})$. Then $f_0 ∈ B^*$ and $J_{W_0}(f) = I_{W_0}(g)$ for some $g ∈ B^*$ with $\tilde{g} = \tilde{f}$.

The proof relies on the following two lemmas. We delay their proofs until the end of the subsection.

**Lemma 24.** Let $H$ be a $d$-regular graph. Let $f = (f_{ij})_{i,j∈[m]}$ and $f^* = (∥f_{ij}∥d)_{i,j∈[m]}$. Then $t(H, f) ≤ t(H, f^*)$.

**Lemma 25.** Let $H$ be a $d$-regular graph and let $W_0 ∈ B^*$ with $W_0 = (p_{ij})_{i,j∈[m]}$. Let $f = (f_{ij})_{i,j∈[m]} ∈ W_0$, and set $f^* = (∥f_{ij}∥d)_{i,j∈[m]}$ to be the $d$-averaged graphon. Assume that $f ∈ W_0$ satisfies the $ε$-convex minorant condition with respect to $W_0$ for some $ε > 0$. If $d_Ω(f, f^*) ≥ α > 0$, there exists $η = η(ε, d, W_0, α) > 0$, such that $I_{W_0}(f) ≥ I_{W_0}(f^*) + η$.

Consequently, $I_{W_0}(f) = I_{W_0}(f^*)$ if and only if $f ∈ B^*$.

**Proof of Lemma 23.** Let $\tilde{f}$ be a minimizer of the variational problem (6). Let $f_n ∈ W_0$ be such that each $f_n$ satisfies the $ε$-convex minorant condition, $d_Ω(f_n, \tilde{f}) → 0$, and $I_{W_0}(f_n) → J_{W_0}(\tilde{f})$. For each $f_n = (f_{ij})_{i,j∈[m]}$, define $f^*_n = (∥f_{ij}∥d)_{i,j∈[m]}$ to be the corresponding $d$-averaged graphon. We claim that $d_Ω(f_n, f^*_n) → 0$.

Indeed, suppose for the sake of contradiction that there exists $β > 0$ and a subsequence $\{f_{n_k}\}_{k≥1}$ such that $d_Ω(f^*_n, f^*_n) ≥ β$ for all $k$. By Lemma 24 there exists some $η > 0$ such that $I_{W_0}(f_{n_k}) ≥ I_{W_0}(f^*_n) + η$. Consider the sequence $\{f^*_n\}_{k≥1}$. By the compactness of $W_0$, there exists a convergent subsequence $f^*_{n_k} → f^*$ for some $f^* ∈ W_0$. By Lemma 24 $t(H, f^*_{n_k}) ≥ t(H, f^*_n)$ and so $t(H, f^*) ≥ t(H, f)$ follows. It follows that $J_{W_0}(f^*) ≤ \lim \inf J_{W_0}(f_{n_k}) + \eta ≥ J_{W_0}(f^*_n) + η ≥ \min \{J_{W_0}(\tilde{g}) : τ(g) ≥ t\} + η$, and thus we have reached a contradiction.

We have shown that $d_Ω(f^*_n, f^*_n) → 0$. It follows that $d_Ω(f^*_n, \tilde{f}) → 0$. Since each $f^*_n ∈ B^*$, we can write $f^*_n = (α^n)_{i,j∈[m]}$ where each $α^n_{ij} ∈ [0, 1]$. By the compactness of $[0, 1]^2$, there exists a subsequence such that $α_{ij}^n → β_{ij}$ for all $i, j ∈ [m]$. Let $g = (β_{ij})_{i,j∈[m]}, g ∈ B^*$. Since $f^*_{n_k} → g$ pointwise and $d_Ω(f^*_{n_k}, g) ≤ ||f^*_{n_k} - g||_1$, the Dominated Convergence Theorem implies that $d_Ω(f^*_{n_k}, g) → 0$. Since $d_Ω(f^*_{n_k}, \tilde{f}) → 0$, we have $d_Ω(\tilde{g}, \tilde{f}) = 0$. Thus $\tilde{f} ∈ B^*$. Note $τ(g) ≥ t$.

Next we show that $I_{W_0}(g) = J_{W_0}(\tilde{f})$. Since $d_Ω(\tilde{f}, \tilde{f}) → 0$, $\lim inf_{n→∞} I_{W_0}(f^*_n) ≥ J_{W_0}(\tilde{f})$. Further, since each $f_n$ satisfies the $ε$-convex minorant condition, then by Lemma 24 $\lim inf_{n→∞} I_{W_0}(f^*_n) ≤ \lim_{n→∞} I_{W_0}(f_n) = J_{W_0}(\tilde{f})$. Thus, $\lim_{n→∞} I_{W_0}(f^*_n) = J_{W_0}(\tilde{f})$. Since $f^*_{n_k} → g$ pointwise and $f^*_{n_k}, g ∈ W_0$, the continuity of $h_p$ for any fixed $p ∈ (0, 1)$ implies that $I_{W_0}(g) = \lim_{k→∞} I_{W_0}(f^*_{n_k}) = J_{W_0}(\tilde{f})$.

4.1.1 Proofs of supporting lemmas

We turn to the proofs of Lemmas 24 and 25.

**Proof of Lemma 24.** The Lemma is a direct consequence of the generalized Hölder inequality of 23 (stated as Theorem 11 in the Appendix). Let $v = |V(H)|$. Recall the definition of $t(H, f)$:

$$t(H, f) = \int_{[0,1]^v} \prod_{(i,j)∈E(H)} f(x_i, x_j) dx_1 ⋯ dx_v.$$
We break up the integration over the blocks specified by the vector $\gamma$. Recall the definition of $I_j$ (Definition 6). We have

$$t(H, f) = \sum_{i_1 = 1}^{m_1} \cdots \sum_{i_v = 1}^{m_v} \int_{x_{i_1} \in I_{i_1}} \cdots \int_{x_{i_v} \in I_{i_v}} \prod_{(a, b) \in E(H)} f(x_a, x_b) dx_1 \cdots dx_v.$$ 

Recall that $f(x, y) = f_k(x, k(y)) (r(x), r(y))$ as stated in (28) and (27). It follows that

$$f(x_a, x_b) = f_k(x_a, k(x_a)) (r(x_a), r(x_b)) = f_{i_a, i_b} (r(x_a), r(x_b)).$$

Substituting and applying a change of variables, we obtain

$$t(H, f) = \sum_{i_1 = 1}^{m_1} \cdots \sum_{i_v = 1}^{m_v} \left( \prod_{j=1}^{v} \gamma_{i_j} \right) \int_{x \in [0,1]^v} \prod_{(a, b) \in E(H)} f_{i_a, i_b} (x_a, x_b) dx_1 \cdots dx_v.$$ 

By the generalized Hölder inequality (Theorem 11),

$$t(H, f) \leq \sum_{i_1 = 1}^{m_1} \cdots \sum_{i_v = 1}^{m_v} \left( \prod_{j=1}^{v} \gamma_{i_j} \right) \prod_{(a, b) \in E(H)} \| f_{i_a, i_b} \|_d = t(H, f^*).$$ 

□

We will apply Lemmas 26 to 28 in the proof of Lemma 25.

**Lemma 26.** Fix $\gamma \in \mathbb{R}^m$, and let $f, g \in C^0$ be two graphons with $f = \{f_{i,j}\}_{(i,j) \in [m]^2}$ and $g = \{g_{i,j}\}_{(i,j) \in [m]^2}$. Then $d_{\Box}(f, g) \leq \max_{i,j} d_{\Box}(f_{i,j}, g_{i,j})$.

**Proof.** Recalling the definition of $d_{\Box}$, we will need to bound $\sup_{S,T} | \int_{S \times T} (f - g) |$ over all measurable subsets $S, T \subset [0,1]$. Fix two such subsets, and let $I_i$ be the $i$th block. By the triangle inequality,

$$\left| \int_{S \times T} (f(x, y) - g(x, y)) \, dx \, dy \right| = \left| \sum_{(i,j) \in [m]^2} \int_{(S \times T) \cap (I_i \times I_j)} (f(x, y) - g(x, y)) \, dx \, dy \right| \leq \sum_{(i,j) \in [m]^2} \int_{(S \times T) \cap (I_i \times I_j)} |f(x, y) - g(x, y)| \, dx \, dy.$$ 

Setting $T_i = r(T \cap I_i)$ and $S_i = r(T \cap I_i)$, where $r$ is as described in (28) and (27), we write the right hand side as

$$\sum_{(i,j) \in [m]^2} \left| \eta_i \eta_j \int_{(S_i \times T_i)} (f_{i,j}(x, y) - g_{i,j}(x, y)) \, dx \, dy \right| \leq \sum_{i,j} \eta_i \eta_j \max_{k,\ell} d_{\Box}(f_{k,\ell}, g_{k,\ell}) = \max_{k,\ell} d_{\Box}(f_{k,\ell}, g_{k,\ell}).$$

Since $S, T \subset [0,1]$ were arbitrary, this completes the proof. □

**Definition 21.** Given $f : [0,1]^2 \to [0,1]$ measurable, $c \in \mathbb{R}$, and $\varepsilon \in [0,1]$, define the sets

$$A^+_\varepsilon(f, c) = \{(x, y) \in [0,1]^2 : f(x,y) - c \geq \varepsilon \}$$

$$A^-_\varepsilon(f, c) = \{(x, y) \in [0,1]^2 : c - f(x,y) \geq \varepsilon \}.$$ 

**Lemma 27.** Let $f : [0,1]^2 \to [0,1]$ be measurable, and let $g$ be the constant graphon that takes value $\|f\|_d$. There exists $\beta = \beta(d, \varepsilon)$ such that if $d_{\Box}(f, g) \geq \varepsilon$, then

$$|A^+_\beta(f^d, ||f^d||_d^2)| \geq \beta \quad \text{and} \quad |A^-_\beta(f^d, ||f^d||_d^2)| \geq \beta$$ 

where $f^d$ denotes the function $f^d(x, y) = f(x, y)^d$. 

37
Proof. Without loss of generality, we may assume that \( \varepsilon \) is small enough such that \( 1 \geq \varepsilon / 4 \geq d(\varepsilon / s)^4 \).

We begin by observing that for \( u, v \geq 0 \) satisfying \( u - v \geq \varepsilon / 4 \) and \( a(x) = x^d \),

\[
a(u) - a(v) = \int_v^u a'(x) \, dx \geq \int_{v+\varepsilon/8}^{u+\varepsilon/4} a'(x) \, dx \geq \frac{\varepsilon}{8} a' \left( \frac{\varepsilon}{8} \right) = d \left( \frac{\varepsilon}{8} \right)^4. \tag{29}
\]

Let \( f : [0, 1]^2 \to [0, 1] \) satisfy the hypotheses of the lemma. Lemma 13 implies that \( |A_{\varepsilon/4}^\pm(f, \|f\|d)| \geq \varepsilon / 4 \) or \( |A_{\varepsilon/4}^\pm(f, \|f\|d)| \geq \varepsilon / 4 \). We will establish the result in the case that \( |A_{\varepsilon/4}^\pm(f, \|f\|d)| \geq \varepsilon / 4 \). The other case follows by an analogous argument. Note that if \( f - \|f\|d \geq \varepsilon / 4 \), then (29) implies

\[
f^d - \|f\|d ^2 \geq d \left( \frac{\varepsilon}{8} \right)^d.
\]

Let \( c = d(\varepsilon / 8)^d \). Since \( |A_{\varepsilon/4}^\pm(f, \|f\|d)| \geq \varepsilon / 4 \) and \( A_{\varepsilon/4}^\pm(f, \|f\|d) \subseteq A_\varepsilon^\pm(f^d, \|f\|d^2) \), it follows that \( |A_\varepsilon^\pm(f^d, \|f\|d^2)| \geq \varepsilon / 4 \). For ease of notation, let \( A^+ = A_\varepsilon^\pm(f^d, \|f\|d^2) \) and \( A^- = A_\varepsilon^\pm(f^d, \|f\|d^2) \).

Observe

\[
c^2 \leq \int_{A^\pm(f^d, \|f\|d^2)} f^d - \|f\|d^2 \leq \int_{A^+} f^d - \|f\|d^2 = \int_{A^-} \|f\|d^2 - f^d \leq |A_{\varepsilon/2}^\pm(f^d, \|f\|d^2)| + \frac{c^2}{2}.
\]

It follows that \( |A_{\varepsilon/2}^\pm(f^d, \|f\|d^2)| \geq c^2/2 \). Since \( |A_{\varepsilon/2}^\pm(f^d, \|f\|d^2)| \geq |A_{\varepsilon}^\pm(f, \|f\|d)| \geq c \geq c^2/2 \), taking \( \beta = c^2/2 \) completes the proof. \( \Box \)

Lemma 28. Let \( p \in (0, 1) \), and \( f : [0, 1]^2 \to [0, 1] \) measurable. There exists \( \eta = \eta(p, \varepsilon, \beta) > 0 \) such that if \( |A_\varepsilon^\pm(f,\|f\|_1)|, |A_\varepsilon^\pm(f,\|f\|_1)| \geq \beta \) and the value \( \|f\|_1 \) has a convex \( \varepsilon \)-neighborhood with respect to \( \psi_p(x) = h_p(x^{1/d}) \), then

\[
\int_{[0,1]^2} \hat{\psi}_p(f) \geq \hat{\psi}_p(\|f\|_1) - \eta.
\]

Proof. For ease of notation let \( z = \|f\|_1 \). Since \( z \) has a strictly convex \( \varepsilon \)-neighborhood with respect to \( \psi_p \), \( \psi_p = \hat{\psi}_p \) in a neighborhood around \( z \). Since \( \hat{\psi}_p \) is differentiable, it follows that \( \psi_p \) is differentiable at \( z \), and so \( \psi'_p(z) = \psi'_p(z) \) is a subdifferential of \( \psi_p \) at \( z \). Let

\[
g(w) = \psi_p(z) + \psi'_p(z)(w - z).
\]

Moreover, since \( \psi'_p(z) \) is a subdifferential of \( \hat{\psi}_p \) at \( z \), \( \hat{\psi}_p(w) \geq g(w) \) for all \( w \in [0, 1] \). Since \( g \) is a linear function,

\[
\int_{[0,1]^2} g(f(x,y)) = g \left( \int_{[0,1]^2} f(x,y) \right) = g(\|f\|_1) = \psi_p(\|f\|_1) = \hat{\psi}_p(\|f\|_1). \tag{30}
\]

Define

\[
d(w) = \psi_p(w) - g(w),
\]

and note that \( d'(w) = \psi'_p(w) - \psi'_p(z) \). Applying the Fundamental Theorem of Calculus twice, we obtain

\[
d(w) - d(z) = \int_z^w d'(a) \, da = \int_z^w \int_z^a \psi'_p(b) \, db \, da \geq \frac{(w - z)^2}{2} \min\{\psi'_p(x) : x \in [z, w] \cup [z, w]\},
\]

provided the minimum on the right is non-negative. The above computation covers both the cases \( z < w \) and \( z \geq w \). For this reason the minimum in the final expression is over the union of the intervals \( [z, w] \) and \([w, z]\), one of which will always be empty or a singleton.

Next, we construct a set \( S \subseteq [0, 1]^2 \) and choose \( \eta' > 0 \) such that for all \( (x, y) \in S \),

(i) \( \psi_p(f(x,y)) = \hat{\psi}_p(f(x,y)) \),

(ii) \( |f(x,y) - z| \geq \beta \),

(iii) \( \psi'_p(b) \geq \eta' \) for all \( b \in [f(x,y), z] \cup [z, f(x,y)] \), and

(iv) \( |S| \geq \beta \).

Our construction of \( S \) depends on \( \psi_p \) and \( z \). Let \( p_0 \) be as given in Proposition 13. There are three cases concerning \( p_0 \). In each case, \( \eta' \) is well-defined because it is the minimum of a continuous function over a compact set.
(1) If $p > p_0$, then $\psi''_p$ is positive on $[0, 1]$. Let $S = A_{\beta}^+(f, z)$ and $\eta' = \min\{\psi''_p(x) : x \in [0, 1]\}$.

(2) If $p < p_0$, then the function $\psi_p$ has two inflection points $r_1$ and $r_2$, and $\psi''_p(x) > 0$ on $[0, r_1)$ and $(r_2, 1]$. Note that $z \notin (r_1 - \varepsilon, r_2 + \varepsilon)$, since $z$ has a strictly convex $\varepsilon$-neighborhood with respect to $\psi_p$.

- If $z \leq r_1 - \varepsilon$, let $\eta' = \min\{\psi''_p(x) : x \in [0, r_1 - \varepsilon]\}$. Note $\eta' > 0$ since $\psi''_p$ is positive on $[0, r_1 - \varepsilon]$.
- Let $S = A_{\beta}^+(f, z)$.

- If $z \geq r_2 + \varepsilon$, let $\eta' = \min\{\psi''_p(x) : x \in [r_2 + \varepsilon, 1]\}$. Note $\eta' > 0$ since $\psi''_p$ is positive on $[r_2 + \varepsilon, 1]$.
- Let $S = A_{\beta}^+(f, z)$.

(3) If $p = p_0$, then the function $\psi_p$ has one point $r$ such that $\psi''_p(r) = 0$ and $\psi''_p(x) > 0$ on $[0, r)$ and $(r, 1]$. Construct $\eta'$ and $S$ as in the above case with $r = r_1 = r_2$. Namely, if $z \leq r - \varepsilon$, let $\eta' = \min\{\psi''_p(x) : x \in [0, r - \varepsilon]\}$ and $S = A_{\beta}^+(f, z)$. If $z \geq r + \varepsilon$, let $\eta' = \min\{\psi''_p(x) : x \in [r + \varepsilon, 1]\}$ and $S = A_{\beta}^+(f, z)$.

It is easy to check in each case that $S$ satisfies properties (i)-(iv). Note that if $(x, y) \in S$ and $f = f(x, y)$, then properties (i), (ii), and (iii) imply that

$$\hat{\psi}_p(f) - g(f) = \psi_p(f) - g(f) = d(f) \geq \frac{(f - z)^2}{2} \min\{\psi''_p(b) : b \in [z, f] \cup [z, f]\} \geq \eta' \frac{\beta^2}{2}. \quad (31)$$

Recall that $\hat{\psi}_p(w) \geq g(w)$ for all $w \in [0, 1]$. It follows by (30) and (31)

$$\int_{[0, 1]^2} \hat{\psi}_p(f(x, y)) \, dxdy - \hat{\psi}_p(\|f\|_1) = \int_{[0, 1]^2} \psi_p(f(x, y)) - g(f(x, y)) \, dxdy \geq \int_S \hat{\psi}_p(f(x, y)) - g(f(x, y)) \, dxdy \geq |S| \frac{\eta' \beta^2}{2} \geq \eta' \frac{\beta^3}{2}. \quad (32)$$

Taking $\eta = \eta' \beta^3/2$ completes the proof. \hfill \square

Proof of Lemma 24 Suppose $f$ is a graphon that satisfies the $\varepsilon$-convex minorant condition, and has the property that $d_{\varepsilon}(f, f^*) \geq \alpha$. Since $d_{\varepsilon}(f, f^*) \geq \alpha$, Lemma 24 implies that there exists some $a, b \in [\varepsilon]$ such that $d_{\varepsilon}(f_{ab}, \|f_{ab}\|_d) \geq \alpha$ (where by an abuse of notation, $\|f_{ab}\|_d$ denotes the constant function that takes that value $\|f_{ab}\|_d$). Since $f \in W_{0, \alpha}$, $f_{ij}$ is constant whenever $p_{ij} \in (0, 1)$ and the values $a, b$ are such that $p_{ab} \in (0, 1)$. Lemma 24 implies that there exists $\beta = \beta(d, \alpha)$ such that $|A_{\beta}^+(f_{ab}, \|f_{ab}\|_d)| \geq \beta$ and $|A_{\beta}^+(f_{ab}, \|f_{ab}\|_d)| = \beta$.

For each $(i, j)$ such that $p_{ij} \in (0, 1)$, Lemma 24 implies that there exists $\eta_{ij} = \eta_{ij}(p_{ij}, \beta)$ such that if $g : [0, 1]^2 \to [0, 1]$, $|A_{\beta}^+(g, \|g\|_1)| \geq \beta$ and the value $\|g\|_1$ has a convex $\varepsilon$-neighborhood with respect to $\psi_{p_{ij}}(x) = h_{p_{ij}}(x^{1/4})$, then

$$\int_{[0, 1]^2} \hat{\psi}_{p_{ij}}(g) \geq \hat{\psi}_{p_{ij}}(\|g\|_1) + \eta_{ij}. \quad (33)$$

Let $\eta' = \min_{i,j} \{\eta_{ij} : p_{ij} \in (0, 1)\}$.

Since $f$ satisfies the $\varepsilon$-convex minorant condition, the value $\|f_{ab}\|_d = \|f_{ab}\|_1$ has a strictly convex $\varepsilon$-neighborhood with respect to $\psi_{p_{ij}}$. Applying (32) to $f_{ab}$, we obtain

$$\int_{[0, 1]^2} h_{p_{ab}}(f_{ab}(x, y)) \, dxdy \geq \int_{[0, 1]^2} \psi_{p_{ab}}(f_{ab}(x, y)) \, dxdy \geq \int_{[0, 1]^2} \hat{\psi}_{p_{ab}}(f_{ab}(x, y)) \, dxdy \geq \hat{\psi}_{p_{ab}}(\|f_{ab}\|_d) + \eta' = h_{p_{ab}}(\|f_{ab}\|_d) + \eta'. \quad (34)$$

For all $i, j$ such that $p_{ij} \in (0, 1)$, the point $(\|f_{ij}\|_d, h_{p_{ij}}(\|f_{ij}\|_d))$ lies on the convex minorant of $\psi_{p_{ij}}(x) = h_{p_{ij}}(x^{1/4})$, and so applying Jensen’s Inequality we obtain

$$\int_{[0, 1]^2} h_{p_{ij}}(f_{ij}(x, y)) \, dxdy = \int_{[0, 1]^2} \psi_{p_{ij}}(f_{ij}(x, y)) \, dxdy \geq \int_{[0, 1]^2} \hat{\psi}_{p_{ij}}(f_{ij}(x, y)) \, dxdy \geq \hat{\psi}_{p_{ij}}(\|f_{ij}\|_d) = \psi_{p_{ij}}(\|f_{ij}\|_d) = h_{p_{ij}}(\|f_{ij}\|_d). \quad (35)$$
Since \( f \in \mathcal{W}_0 \), it holds that \( f(x,y) = W_0(x,y) \) for all \((x,y)\) such that \( W_0(x,y) \in \{0,1\} \). Let \( c_{ij} \) be the indicator that \( p_{ij} \in (0,1) \). Observe

\[
I_{W_n}(f) = \sum_{i=1}^{m} \sum_{j=1}^{m} \gamma_i c_{ij} \int_{[0,1]^2} h_{p_{ij}}(f_{ij}(x,y)) dx dy \\
\geq \eta' \gamma_a \gamma_b + \sum_{i=1}^{m} \sum_{j=1}^{m} \gamma_i c_{ij} h_{p_{ij}}(\|f_{ij}\|_d) \\
= \eta' \gamma_a \gamma_b + I_{W_n}(f^*) .
\]

Taking \( \eta = \min_{(i,j)} \{ \gamma_i c_{ij} : p_{ij} \in (0,1) \} \) yields the desired result. \( \square \)

### 4.2 Symmetry with a unique minimizer for small \( \delta \)

We now prove Theorem \ref{thm:symmetry}, which establishes the existence of a symmetric regime with a unique minimizer when \( \delta \) is small. We state the key lemmas used in the proof and defer the proofs of these lemmas to the end of the subsection.

**Lemma 29.** Given \( W_0 \in \mathcal{B}^n \) and a finite \( d \)-regular graph \( H \), there exist \( \delta_0 = \delta_0(H,W_0) \) and \( \varepsilon = \varepsilon(W_0) > 0 \) such that the following is true. For all \( 0 < \delta < \delta_0 \), if \( f^* \) is a minimizer of the variational problem \ref{eq:variational_problem} with \( \delta = (1 + \delta) \varepsilon \), then there exists a sequence of graphons \( f_n \in \mathcal{W}_0 \) such that \( \delta \varepsilon(f_n,f^*) \to 0 \), \( I_{W_n}(f_n) \to I_{W_0}(f^*) \) and each \( f_n \) satisfies the \( \varepsilon \)-convex minorant condition.

**Lemma 30.** Given \( W_0 \in \mathcal{B}^n \), a finite graph \( H \), and \( t \leq t_{\text{max}} \), there is a unique minimizer of the minimization problem

\[
\min \{I_{W_0}(g) : g \in \mathcal{B}^n, t(H,g) \geq t \}. \tag{33}
\]

We now use Lemmas 29 and 30 to prove Theorem 5.

**Proof of Theorem 5.** Let \( W_0 \in \mathcal{B}^{n^*} \), and let \( H \) be a finite \( d \)-regular graph. Let \( \delta_0 > 0 \) and \( \varepsilon > 0 \) be as in Lemma 29 and assume that \( \delta < \delta_0 \). Suppose that \( f^* \) is a minimizer of the variational problem \ref{eq:variational_problem} with \( \delta = (1 + \delta) \varepsilon \). By Lemma 29 there exists a sequence of graphons \( f_n \in \mathcal{W}_0 \) such that each \( f_n \) satisfies the \( \varepsilon \)-convex minorant condition, \( \delta \varepsilon(f_n,f^*) \to 0 \), and \( I_{W_n}(f_n) \to I_{W_0}(f^*) \). Lemma 29 implies that \( \tilde{f} \in \mathcal{B}^n \) and \( J_{W_0}(\tilde{f}) = I_{W_0}(g) \) for some \( g \in \mathcal{B}^n \) with \( \tilde{g} = \tilde{f} \). Thus, the problem is in the symmetric regime.

Next we establish that a unique element of \( \mathcal{W}_0 \) minimizes \ref{eq:variational_problem}. Let \( h \) be a minimizer of the variational problem \ref{eq:variational_problem} with \( \delta = (1 + \delta) \varepsilon \). By the above argument we may assume that \( h \in \mathcal{B}^n \) and \( I_{W_0}(h) = J_{W_0}(h) \). Note that

\[
\min \{I_{W_0}(g) : g \in \mathcal{B}^n, t(H,g) \geq (1 + \delta) \varepsilon \} \leq I_{W_0}(h) = J_{W_0}(h) \]

and

\[
\min \{I_{W_0}(g) : g \in \mathcal{B}^n, t(H,g) \geq (1 + \delta) \varepsilon \} \geq \min \{J_{W_0}(\tilde{g}) : \tilde{g} \in \mathcal{B}^n, t(H,\tilde{g}) \geq (1 + \delta) \varepsilon \}
\]

\[
= J_{W_0}(\tilde{h}) = I_{W_0}(h).
\]

It follows that

\[
J_{W_0}(\tilde{h}) = I_{W_0}(h) = \min \{I_{W_0}(g) : g \in \mathcal{B}^n, t(H,g) \geq (1 + \delta) \varepsilon \}.
\]

By Lemma 30 there is a unique minimizer of the rightmost expression; \( h \) must be that unique minimizer, and so the solution to the variational problem is unique. \( \square \)

### 4.2.1 Proofs of supporting lemmas

We now prove Lemmas 29 and 30. We first show that graphons with homomorphism density close to \( t(H,W_0) \) must be close to \( W_0 \) pointwise except possibly on a small set.

**Lemma 31.** Let \( W_0 \in \mathcal{B}^n \) such that \( \gamma \in \mathbb{R}^n \), and let \( H \) be a finite \( d \)-regular graph. Suppose \( f \geq W_0 \) pointwise and \( t(H,f) \leq (1 + \delta) t(H,W_0) \) for \( \delta > 0 \). Let \( Y \) be a relevant interval labeling vector with respect to \( W_0 \). Suppose \( I_a \times I_b \) contributes to \( Y \), i.e. \( Y_i = a \) and \( Y_j = b \) for some \( i, j \in E(H) \). (See Definition \ref{def:relevant}) Let \( S_{ij} = \{(x,y) \in I_a \times I_b : f(x,y) \geq W_0(x,y) + \varepsilon \} \). Then

\[
|S_{ij}| \leq \frac{\delta t(H,W_0) p_{ab} \gamma_a \gamma_b}{t(H,W_0,Y)},
\]

where \( p_{ab} \) is the value of \( W_0 \) on \( I_a \times I_b \), \( \gamma_a = |I_a| \), and \( \gamma_b = |I_b| \).
Proof. Let \( v = |V(H)| \). Observe
\[
t(H, f, Y) = \int_{x_1 \in I_1} \cdots \int_{x_v \in I_v} \prod_{\{u, w\} \in E(H)} f(x_u, x_w) \, dx_v \cdots dx_1
\]
\[
\geq \left( \prod_{u \in [v] \setminus \{i, j\}} |I_{Y_u}| \prod_{\{u, w\} \in E(H) \setminus \{i, j\}} p_{Y_u} \right) \left( \int_{x_i \in I_a} \int_{x_j \in I_b} f(x_i, x_j) \, dx_i \, dx_j \right)
\]
\[
\geq \left( \prod_{u \in [v] \setminus \{i, j\}} |I_{Y_u}| \prod_{\{u, w\} \in E(H) \setminus \{i, j\}} p_{Y_u} \right) \left( \int_{(x_i, x_j) \in I_a \times I_b \setminus S_e} p_{ab} + \int_{(x_i, x_j) \in S_e} (\varepsilon + p_{ab}) \right)
\]
\[
= \left( \prod_{u \in [v] \setminus \{i, j\}} |I_{Y_u}| \prod_{\{u, w\} \in E(H) \setminus \{i, j\}} p_{Y_u} \right) (|I_a||I_b|p_{ab} + \varepsilon|S_e|)
\]
\[
= t(H, W_0, Y) + \varepsilon \frac{|S_e|}{p_{ab} \gamma_a \gamma_b} t(H, W_0, Y).
\]

We use this to lower bound the homomorphism density of \( f \), and obtain
\[
(1 + \delta)t(H, W_0) \geq t(H, f) = \sum_{Z \in [m]^v} t(H, f, Z)
\]
\[
\geq \varepsilon \frac{|S_e|}{p_{ab} \gamma_a \gamma_b} t(H, W_0, Y) + \sum_{Z \in [m]^v} t(H, W_0, Z)
\]
\[
= \varepsilon \frac{|S_e|}{p_{ab} \gamma_a \gamma_b} t(H, W_0, Y) + t(H, W_0).
\]

The statement of the lemma follows directly.

\( \square \)

**Lemma 32.** Fix \( W_0 \in B^d \) and \( H \) a finite \( d \)-regular graph. There exist \( \delta = \delta(H, W_0), \varepsilon = \varepsilon(W_0) > 0 \) sufficiently small such that the following is true. If \( f \in W_0 \), \( f \geq W_0 \) pointwise, \( f = W_0 \) on irrelevant blocks (Definition 15), and \( t(H, f) \leq (1 + \delta) t(H, W_0) \), then \( f \) satisfies the \( \varepsilon \)-convex minorant condition.

**Proof.** Let \( W_0 = (p_{ij})_{i,j \in [m]} \). By Proposition 13 for \( p_{ij} \in (0, 1) \) the function \( \psi_{ij} : (0, 1) \to \mathbb{R} \) is either convex with \( \psi_{ij}'' > 0 \), is convex with \( \psi_{ij}'' = 0 \) at exactly one value greater than \( p_{ij} \), or has two inflection points greater than \( p_{ij} \). If \( \psi_{ij}'' \) is convex with \( \psi_{ij}'' > 0 \), then the convex minorant criterion is trivially satisfied for the \( (i, j) \) block. Otherwise, let \( q_{ij} \) be such that \( q_{ij}'' \) is the smallest value greater than \( p_{ij}'' \) such that \( \psi_{ij}''(q_{ij}''(x)) = 0 \). Let \( P = \{(i, j) : \psi_{ij}''(x) = 0 \text{ for some } x \in [p_{ij}'', 1]\} \) and define
\[
\varepsilon = \min_{(i,j) \in P} (q_{ij}'' - p_{ij}'')/2.
\]

Let \( f \in W_0 \), \( f \geq W_0 \) pointwise, \( f = W_0 \) on irrelevant blocks (Definition 15), and \( t(H, f) \leq (1 + \delta) t(H, W_0) \). When \( I_a \times I_b \) is irrelevant, \( \|f_{ab}\|^2 = p_{ab} \). It follows from Proposition 13 that the \( \varepsilon \)-convex minorant condition holds. It suffices to show that for \( \delta \) sufficiently small, the value \( \|f_{ab}\|^2 \) has a convex \( \varepsilon \)-neighborhood with respect to \( \psi_{p_{ab}} \) for all \((a, b) \in P \) such that \( I_a \times I_b \) is relevant. Fix such a pair \((a, b)\).

Let \( S_\eta = \{(x, y) \in I_a \times I_b : f(x, y) \geq W_0(x, y) + \eta\} \). Let \( S'_\eta = (I_a \times I_b) \setminus S_\eta \). Observe
\[
\|f_{ab}\|^2 = \frac{1}{\gamma_a \gamma_b} \left( \int_{S_\eta} f_{ab}(r(x), r(y))'' \, dx \, dy + \int_{S'_\eta} f_{ab}(r(x), r(y))'' \, dx \, dy \right)
\]
\[
\leq \frac{1}{\gamma_a \gamma_b} \left( |S_\eta|(p_{ab} + \eta)^d + |S_\eta| \right)
\]
\[
\leq (p_{ab} + \eta)^d + \frac{1}{\gamma_a \gamma_b} |S_\eta|.
\]

Let \( \eta(\delta) = \sqrt{\frac{\delta t(H, W_0)p_{ab} \gamma_a \gamma_b}{t(H, W_0, Y)}} \). By Lemma 33
\[
\eta(\delta)|S_\eta(\delta)| \leq \frac{\delta t(H, W_0)p_{ab} \gamma_a \gamma_b}{t(H, W_0, Y)} \implies |S_\eta(\delta)| \leq \eta(\delta).
\]
Then there exists a sequence \( \{b_i\}_{i \geq 1} \) such that \( f_n \in W_0 \), \( f_n \geq W_0 \) pointwise, \( f_n = W_0 \) on irrelevant blocks, \( I_{W_0}(f_n) \to J_{W_0}(\tilde{f}) \) and \( \delta_{\infty}(f_n, \tilde{f}) \to 0 \).

**Lemma 33.** Let \( f \) be a minimizer of the variational problem \((1)\) with \( t = (1 + \delta)t(H, W_0) \) for \( \delta > 0 \). Then there exists a sequence \( \{f_n\}_{n \geq 1} \) such that \( f_n \in W_1 \), \( I_{W_0}(f_n) \to J_{W_0}(f) \) and \( \delta_{\infty}(f_n, f) \to 0 \). Let \( R \subseteq [0, 1]^2 \) be the union of the relevant blocks (Definition 19). Define \( f_n' \in W_1 \) such that

\[
f_n'(x, y) = \begin{cases} 
\max\{f_n(x, y), W_0(x, y)\} & (x, y) \in R \\
W_0(x, y) & (x, y) \notin R
\end{cases}
\]

Note that this ensures that \( f_n' \geq W_0 \) pointwise, and note further that \( f_n' = W_0 \) whenever \( f_n' \neq f_n \), which in turn implies that \( h_{W_0}(x, y)(f_n'(x, y)) \leq h_{W_0}(x, y)(f_n(x, y)) \) for all \( (x, y) \in [0, 1]^2 \) and hence \( I_{W_0}(f_n') \leq I_{W_0}(f_n) \).

We claim that the lemma follows from showing that \( \delta_{\infty}(f_n', f_n) \to 0 \). Indeed, if \( \delta_{\infty}(f_n', f_n) \to 0 \), then \( \liminf I_{W_0}(f_n') \geq J_{W_0}(f) \). Since \( I_{W_0}(f_n') \leq I_{W_0}(f_n) \) for each \( n \), \( \limsup I_{W_0}(f_n') \leq \liminf I_{W_0}(f_n) = J_{W_0}(f) \). Thus, \( I_{W_0}(f_n') \to J_{W_0}(f) \) as \( n \to \infty \). Note that by construction \( f_n' \geq W_0 \) pointwise and \( f_n' = W_0 \) on irrelevant blocks. This completes the proof.

It remains to prove that \( \delta_{\infty}(f_n', f_n) \to 0 \). Suppose for contradiction that there exists \( \epsilon > 0 \) and a subsequence such that \( \delta_{\infty}(f_{n_i}', f_{n_i}) \geq \epsilon \) for all \( i \geq 1 \).

Let

\[
\eta = \min_{p \in I_m(W_0) \setminus \{0\}} \min\{h_p(p + \epsilon/4), h_p(p - \epsilon/4)\}.
\]

We will show that \( I_{W_0}(f_{n_i}') \leq I_{W_0}(f_{n_i}) - \epsilon\eta/4 \), and then use this to derive a contradiction.

Indeed, let \( S^+_{\epsilon/4} = \{(x, y) \in [0, 1]^2 : f_{n_i}'(x, y) - f_{n_i}(x, y) \geq \epsilon/4\} \) and \( S^-_{\epsilon/4} = \{(x, y) \in [0, 1]^2 : f_{n_i}(x, y) - f_{n_i}'(x, y) \geq \epsilon/4\} \) and \( S_{\epsilon/4} = S^+_{\epsilon/4} \cup S^-_{\epsilon/4} \). Lemma \((13)\) implies that \( |S_{\epsilon/4}| \geq \epsilon/4 \).

If \((x, y) \in S^+_{\epsilon/4} \cap R\), then \( \max\{f_{n_i}(x, y), W_0(x, y)\} - f_{n_i}(x, y) \geq \epsilon/4 \). It follows that \( f_{n_i}'(x, y) = W_0(x, y) \). If \((x, y) \in S^-_{\epsilon/4} \setminus R\), then \( W_0(x, y) - f_{n_i}(x, y) \geq \epsilon/4 \). In both cases \( f_{n_i}(x, y) \leq W_0(x, y) - \epsilon/4 \), and so \( h_{W_0}(x, y)(f_{n_i}(x, y)) \geq \eta \). Therefore

\[
h_{W_0}(x, y)(f_{n_i}(x, y)) = 0 \leq h_{W_0}(x, y)(f_{n_i}(x, y)) - \eta.
\]

If \((x, y) \in S^-_{\epsilon/4} \setminus R\), then \( f_{n_i}'(x, y) \notin R \) because \( f_n' \geq f_n \) on \( R \). It follows that \( f_{n_i}'(x, y) = W_0(x, y) \), and so \( f_{n_i}(x, y) \geq W_0(x, y) + \epsilon/4 \). Thus \( h_{W_0}(x, y)(f_{n_i}(x, y)) \geq \eta \), which implies

\[
h_{W_0}(x, y)(f_{n_i}(x, y)) = 0 \leq h_{W_0}(x, y)(f_{n_i}(x, y)) - \eta.
\]

Recall that \( h_{W_0}(x, y)(f_{n_i}(x, y)) \leq h_{W_0}(x, y)(f_{n_i}(x, y)) \) for all \((x, y) \in [0, 1]^2 \) and that \( |S| \geq \epsilon/4 \). Therefore

\[
I_{W_0}(f_{n_i}') \leq \int_{[0, 1]^2 \setminus S} h_{W_0}(x, y)(f_{n_i}(x, y)) + \int_S \left[h_{W_0}(x, y)(f_{n_i}(x, y)) - \eta\right] \leq I_{W_0}(f_{n_i}) - \frac{\epsilon\eta}{4}.
\]

Next, consider the sequence \( \{\tilde{f}_{n_i}\}_{i \geq 1} \). By the compactness of \( W_1 \), there exists a convergent subsequence \( \tilde{f}_{n_i} \to \tilde{h} \) for some \( \tilde{h} \in W_\tilde{I} \). Since \( f_{n_i}' \geq f_{n_i} \) on all relevant blocks, \( t(H, f_{n_i}) \geq t(H, f_{n_i}) \), and so \( t(H, f_{n_i}) \geq t(H, f) \geq (1 + \delta)t(H, W_0) \). It follows that \( J_{W_0}(\tilde{h}) \geq \min\{J_{W_0}(\tilde{g}) : \tau(y) \geq (1 + \delta)t(H, W_0)\} \) which in turn implies

\[
J_{W_0}(\tilde{f}) = \liminf J_{W_0}(f_{n_i}) \geq \liminf J_{W_0}(f_{n_i}') + \frac{\epsilon\eta}{4} \geq J_{W_0}(\tilde{h}) + \frac{\epsilon\eta}{4} \geq \min\{J_{W_0}(\tilde{g}) : \tau(y) \geq (1 + \delta)t(H, W_0)\} + \frac{\epsilon\eta}{4} = J_{W_0}(\tilde{f}) + \frac{\epsilon\eta}{4}.
\]

We thus have reached a contradiction. □
Proof of Lemma 29. By Lemma 32 there exists $\delta_0, \varepsilon$ such that if $f \in \mathcal{W}_0$, $f \geq W_0$ pointwise, $f = W_0$ on irrelevant blocks (Definition 14), and $(H, f) \leq (1 + \delta_0) t(H, W_0)$, then $f$ satisfies the $\varepsilon$-convex minorant condition. Let $0 < \delta < \delta_0$, and let $\tilde{f}$ be a minimizer of the variational problem (30) with $t = (1 + \delta) t(H, W_0)$.

Remark 8 implies that $t(H, f) = (1 + \delta) t(H, W_0)$. By Lemma 33 there exists a sequence $(f_n)_{n \geq 1}$ such that $f_n \in \mathcal{W}_0$, $f_n \geq W_0$ pointwise, $f_n = W_0$ on irrelevant blocks, $I_{W_0}(f_n) \to I_{W_0}(f)$, and $\delta_0(\tilde{f}, f_n) \to 0$. Since $\delta_0(\tilde{f}, f_n) \to 0$, $t(H, f_n) \to (1 + \delta) t(H, W_0)$. Since $\delta < \delta_0$, there exists $n_0$ such that for all $n \geq n_0$, $t(H, f_n) \leq (1 + \delta_0) t(H, W_0)$. It follows by the assumption on $\delta_0$ that for all $n \geq n_0$, $f_n$ satisfies the $\varepsilon$-convex minorant condition.

The next proposition will be used in the proof of Lemma 30.

**Proposition 10.** Let $t(H, W_0) \leq t \leq t_{\max}$. If $f$ is a minimizer of

$$\min \{ I_{W_0}(g) : g \in \mathcal{B}_t, t(H, g) \geq t \},$$

then $t(H, f) = t$ and $W_0 \leq f$ pointwise, with equality on the irrelevant set.

Proof. Let $W_0 = (p_{ij})_{i,j \in [m]}$ and let $f = (\alpha_{ij})_{i,j \in [m]}$ be a minimizer. First, if $\alpha_{ij} < p_{ij}$, we can decrease $I_{W_0}$ while maintaining the constraint $t(H, g) \geq t$ by increasing $\alpha_{ij}$. Next observe that if $I_x \not\in I_x$ is irrelevant, then $t(H, f)$ does not depend on $f_{ij}$, since $h_p(\beta)$ has a unique minimum at $\beta = p$, this implies $f_{ij} = p_{ij}$.

To prove the last statement, suppose for contradiction that $t(H, f) > t$. There exists a relevant block $I_a \times I_b$ such that $\alpha_{ab} > p_{ab}$. Let $R$ be the union of the relevant blocks. Let $f_\beta$ be the following graphon

$$f_\beta(x, y) = \begin{cases} \beta & (x, y) \in I_a \times I_b, \\ f(x, y) & (x, y) \in R \setminus I_a \times I_b, \\ W_0(x, y) & \text{otherwise}. \end{cases}$$

Since $h_p(\beta)$ is strictly increasing for $\beta \in [p, 1]$, $I_{W_0}(f_\beta)$ is strictly increasing for $\beta \geq p_{ab}$. Combined with the continuity of $t(H, f_\beta)$ as a function of $\beta$, we conclude that there exists $\beta \in [p_{ab}, \alpha_{ab})$ such that $t(H, f_\beta) > t$ and $I_{W_0}(f) > I_{W_0}(f_\beta)$. This is a contradiction.

Proof of Lemma 30. We need to show that there exists a unique vector $\alpha$ such that $f_\alpha = (\alpha_{ij})_{i,j \in [m]}$ minimizes $I_{W_0}(f)$ subject to $t(H, f) \geq t$. Existence is guaranteed by Proposition 4 and the continuity of $t(H, \cdot)$, so all we need to show is uniqueness. By Proposition 11 it suffices to consider $f_\alpha$ with $f \geq W_0$ pointwise with equality on the irrelevant set. Keeping this and the fact that $\alpha_{ij} = \alpha_{ji}$ in mind, it will be enough to label $f$ by the set of $\alpha_{ij}$ such that $ij \in R = \{(i, j) : (i, j) \text{ is relevant}, i \leq j\}$.

Assume there were two distinct minimizers $f_\alpha$ and $f_{\alpha'}$, let $kl$, $k \geq l$ be such that $\alpha_{kl} \neq \alpha'_{kl}$, and let $R' = \{ij \in R : ij \neq kl\}$. By Proposition 11 we know that $t(H, f_\alpha) = t(H, f_{\alpha'}) = t$, and by exchanging the roles of $\alpha$ and $\alpha'$ if needed, we may assume that $p_{kl} \leq \alpha_{kl} < \alpha'_{kl} \leq 1$.

Consider the constraint function

$$F(\alpha_{kl}, \alpha) = t(H, f_\alpha), \quad \alpha = (\alpha_{ij})_{i,j \in R'}$$

and the implicit equation

$$F(x, \alpha + \lambda(\alpha - \alpha)) = t \quad \text{where} \quad \lambda \in (0, 1). \quad (34)$$

Assume for a moment that there exists $\lambda \in (0, 1)$ such that $F(\alpha_{kl}, \alpha + \lambda(\alpha - \alpha)) \geq t$. By the monotonicity of $t(H, f_\alpha)$ and the fact that $\alpha'_{kl} > \alpha_{kl}$, we then have that $t(H, f_\alpha + \lambda(f_{\alpha'} - f_\alpha)) > t$. Since $h_p(\beta)$ is strictly convex, $I_{W_0}(f_\alpha + \lambda(f_{\alpha'} - f_\alpha)) < I_{W_0}(f_{\alpha'}) = I_{W_0}(f_\alpha)$, which is a contradiction.

We thus may assume that $F(\alpha_{kl}, \alpha + \lambda(\alpha - \alpha)) < F(\alpha_{kl}, \alpha)$ for all $\lambda \in (0, 1)$. Since $kl$ is relevant, which implies that $\frac{\partial}{\partial \lambda} F > c$ for some constant $c > 0$, and since $F$ is a finite polynomial, the above equation has a unique solution $x^\lambda \in (\alpha_{kl}, \alpha'_{kl})$ as long as $\lambda$ is sufficiently small. But this means the vector $\beta = (x^\lambda, \alpha + \lambda(\alpha - \alpha))$ fulfills the constraint. Since all its components are convex combinations of the components of $\alpha$ and $\alpha'$, and at least one of them (the first one) is a non-trivial convex combination, we again get the contradiction that $I_{W_0}(f_{\beta}) < I_{W_0}(f_\alpha)$ by the strict convexity of $h_p(\beta)$.

43
4.3 A symmetric regime for larger $\delta$

We now establish the existence of a symmetric regime for larger $\delta$ (Theorem 4). As in the proof of Theorem 3 we will apply Lemma 23 and Lemma 34.

**Lemma 34.** Fix $W_0 \in B^\gamma$ and $H$ a finite $d$-regular graph. There exists $\delta = \delta(W_0, H), \varepsilon = \varepsilon(W_0) > 0$ for which $(1 + \delta)t(H, W_0) < \max_{g \in W_0} t(H, g)$ such that the following is true: If $f \in W_1$, $f \geq W_0$ pointwise, $f = W_0$ on irrelevant blocks, and $t(H, f) \geq (1 + \delta)t(H, W_0)$, then $f$ satisfies the $\varepsilon$-convex minorant condition.

**Proof.** By Proposition 13 for $p_{ij} \in (0, 1)$ the function $\psi_{p_{ij}} : (0, 1) \rightarrow \mathbb{R}$ is either convex with $\psi_{p_{ij}}'' > 0$, or is convex with $\psi_{p_{ij}}'' < 0$ at exactly one value greater than $p_{ij}$, or has two inflection points greater than $p_{ij}$. If $p_{ij}$ is such that $\psi_{p_{ij}}'' > 0$, then the convex minorant criterion is trivially satisfied for the $(i, j)$ block. Otherwise, let $q_{ij}$ be such that $q_{ij}^d$ is the largest value between $p_{ij}$ and 1 such that $\psi_{p_{ij}}''(q_{ij}) = 0$. Let $P = \{(i, j) : \psi_{p_{ij}}''(x) = 0$ for some $x \in [p_{ij}, 1]\}$ and define

$$
\varepsilon = \min_{(i, j) \in P} (1 - q_{ij})/2.
$$

Let $g_{ij}$ be the graphon in $W_1$ that takes value $q_{ij} + \varepsilon^{1/4}$ on the block $I_i \times I_j$ and takes value 1 on the rest of $\Omega$. Let $\delta$ be such that $(1 + \delta)t(H, W_0) = \max_{(i, j) \in P} t(H, g_{ij})$.

Suppose $f \in W_1$, $f \geq W_0$ pointwise, $f = W_0$ on irrelevant blocks, and $t(H, f) \geq (1 + \delta)t(H, W_0) \geq t(H, g_{ij})$. Lemma 23 implies that $t(H, g_{ij}) \leq t(H, f) \leq t(H, f^*)$ where $f^* = (\|f_{ij}\|_d, i, j \in [n])$ is the $d$-averaged graphon. When $I_a \times I_b$ is irrelevant, $\|f_{ab}\|_d = p_{ab}$. It follows from Proposition 13 that the $\varepsilon$-convex minorant condition holds on irrelevant blocks. Next consider a relevant block $I_i \times I_j$. Since $t(H, f^*) \geq t(H, g_{ij})$ and $g_{ij} \geq f_{ij}^*$ on $[0, 1]^2 \setminus (I_i \times I_j)$, $f^*$ must be greater than $g_{ij}$ on $I_i \times I_j$. It follows that $\|f_{ij}\|_d \geq q_{ij} + \varepsilon^{1/4}$, and so $\|f_{ij}\|_d^2 \geq q_{ij}^2 + \varepsilon$. By Proposition 13 $\psi_{p_{ij}}'' > 0$ on $(q_{ij}, 1)$, and so $\|f_{ij}\|_d$ has a strictly convex $\varepsilon$-neighborhood with respect to $\psi_{p_{ij}}''$. \qed

**Proof of Theorem 4.** Let $W_0 \in B^{\gamma, *}$. By Lemma 34 there exist $\delta, \varepsilon > 0$ such that if $f \in W_1$, $f \geq W_0$ pointwise, $f = W_0$ on irrelevant blocks, and $t(H, f) \geq (1 + \delta)t(H, W_0)$, then $f$ satisfies the $\varepsilon$-convex minorant condition. Let $\eta$ be such that $(1 + \eta)t(H, W_0) = (1 - \eta)t_{\max}$. Since $(1 + \delta)t(H, W_0) < \max_{g \in W_0} t(H, g)$, it holds that $\eta > 0$. Let $t \in ((1 - \eta)t_{\max}, t_{\max}]

Let $g$ be a minimizer of the variational problem (9) with this value of $t$. Lemma 33 implies that there exists a sequence $g_n \in W_0$ such that $I_{W_0}(g_n) \rightarrow I_{W_0}(\tilde{g})$, each $g_n \geq W_0$ pointwise, and $\delta_{\max}(g_n, \tilde{g}) \rightarrow 0$. It follows that $t(H, g_n) \rightarrow t(H, \tilde{g}) \geq t$. Since $t > (1 + \delta)t(H, W_0)$, there exists some $n_0$ such that for all $n \geq n_0$, $t(H, g_n) \geq (1 + \delta)t(H, W_0)$. It follows by Lemma 34 that for all $n \geq n_0$, $g_n$ satisfies the $\varepsilon$-convex minorant condition. Lemma 23 implies that $\tilde{g} \in B^{\gamma, *}$, which establishes symmetry in this case.

Uniqueness of the symmetric optimizer follows from Lemma 30 using exactly the same argument as in the proof of Theorem 6. \qed

5 Symmetry breaking in special cases

Recall the definition

$$
f_{p, q, \gamma}(x, y) = \begin{cases} 
p & \text{if } (x, y) \in [0, \gamma]^2 \\
r & \text{if } (x, y) \in (\gamma, 1]^2 \\
q & \text{otherwise.}
\end{cases}
$$

We prove Theorem 6 which establishes the existence of a non-symmetric regime for graphons of the form $f_{0, p, p}^* f_{1, p, p}$ and $f_{1, p, 0}^*$ when $p$ is sufficiently small. In previous sections we let $\gamma \in \Delta_0$ and defined $B^\gamma$ as the set of block graphons in which the interval structure is given by the vector $\gamma$. In this section, we let $\gamma \in [0, 1]$ and let $B^{(\gamma, 1-\gamma)}$ denote the set of block graphons with two intervals, the first of length $\gamma$ and the second of length $1 - \gamma$.

**Lemma 35.** Let $W_0$ be a graphon with $Im(W_0) \in \{0, p, 1\}$. If $g \in W_1$, then $I_{W_0}(g) = I_{W_0}(\tilde{g})$. Moreover, if $\tau$ is a continuous graph parameter and $W_0$ is a graphon of the form $f_{\gamma, p, p}$ with $\gamma \in \{0, 1\}$, or a graphon the form $f_{\gamma, p, x, z}$ with $z_1 = z_2 \in \{0, 1\}$, then

$$
\min\{I_{W_0}(\tilde{f}) : \tilde{f} \in B^{(\gamma, 1-\gamma)}, \tau(\tilde{f}) \geq t\} = \min\{I_{W_0}(f) : f \in B^{(\gamma, 1-\gamma)} \cup B^{(1-\gamma, \gamma)}, \tau(f) \geq t\},
$$

44
provided \( t \in \mathbb{R} \) is such that the above minima are finite. If \( W_0 \) is a graphon of the form \( f_{z_1,p,z_2} \) with 
\( z_1, z_2 \in \{0, 1\} \) and \( z_1 \neq z_2 \), then
\[
\min\{J_{W_0}(\tilde{f}) : \tilde{f} \in \bar{B}^{(\gamma,1-\gamma)}, \tau(\tilde{f}) \geq t\} = \min\{I_{W_0}(f) : f \in B^{(\gamma,1-\gamma)}, \tau(f) \geq t\},
\]
again provided \( t \in \mathbb{R} \) is such that these minima are finite.

Proposition 11 establishes that \( B^{(\gamma,1-\gamma)} \) is compact under the \( \delta_\square \) topology and that \( B^{(\gamma,1-\gamma)} \) and \( B^{(1-\gamma,\gamma)} \) are closed sets under the \( \delta_\square \). Therefore all the above minima are well-defined.

**Lemma 36.** Let \( 0 < \gamma < 1 \), and let \( H \) be a d-regular graph with \( v \) vertices, and assume that

1. \( W_\gamma = f_{0,p,p}^\gamma, \mathcal{C}(\gamma) = B^{(\gamma,1-\gamma)} \cup B^{(1-\gamma,\gamma)}, \) and \( 0 < t < t(H, f_0^0,0,1) \) or
2. \( W_\gamma = f_{1,p,p}^\gamma, \mathcal{C}(\gamma) = B^{(\gamma,1-\gamma)} \cup B^{(1-\gamma,\gamma)}, \) and \( t(H, f_1^0,0,0) < t < 1 \) or
3. \( W_\gamma = f_{1,p,0}^\gamma, \mathcal{C}(\gamma) = B^{(\gamma,1-\gamma)}, \) and \( t(H, f_1^0,0,0) < t < t(H, f_1^1,1,0) \).

Then there exists \( p_0 > 0 \) such that if \( p < p_0 \),
\[
\inf\{I_{W_\gamma}(f) : t(H,f) \geq t\} < \min\{I_{W_\gamma}(f) : f \in \mathcal{C}(\gamma), t(H,f) \geq t\}.
\]

Note that in the second and third case, we cover all \( t \in (t_{\min}, t_{\max}) \), while we don’t do that in the first case; specifically, we do not consider \( t \in (t(H, f_0^0,0,1), t(H, f_0^1,1,1)) \).

**Proof of Theorem 30** We apply Lemma 36 to conclude that there exists \( p_0 > 0 \) such that if \( p < p_0 \),
\[
\inf\{I_{W_\gamma}(f) : t(H,f) \geq t\} < \min\{I_{W_\gamma}(f) : f \in \mathcal{C}(\gamma), t(H,f) \geq t\}.
\]

Lemmas 29 and 35 imply that
\[
\min\{J_{W_\gamma}(\tilde{g}) : t(H,g) \geq t\} < \min\{J_{W_\gamma}(\tilde{f}) : \tilde{f} \in \bar{B}^{(\gamma,1-\gamma)}, t(H,f) \geq t\}.
\]

Therefore if \( \tilde{g} \) is a minimizer of \( \mathcal{B}(\gamma,1-\gamma) \), then \( \tilde{g} \notin \bar{B}^{(\gamma,1-\gamma)} \), meaning \( t \) is not in the symmetric regime. \( \square \)

### 5.1 Proof of Lemma 35

**Proposition 11.** Let \( W_0 \) be a graphon such that \( \text{Im}(W_0) \subseteq \{0,p,1\} \). Let \( c_p(f) = \int_{[0,1]^2} h_p(f(x,y))dxdy \),

and let \( \Omega_q = \{(x,y) \in [0,1]^2 : W_0(x,y) = q\} \) for \( q \in \{0,p,1\} \). For all \( f \in W_\Omega \),
\[
I_{W_0}(f) = I_p(f) - |\Omega_0|h_p(0) - |\Omega_1|h_p(1).
\]

**Proof.** Suppose \( f \in W_\Omega \). Then for \( q \in \{0,1\}, f = q \) almost everywhere on \( \Omega_q \). It follows that
\[
I_{W_0}(f) = \int_{\Omega_p} h_p(f(x,y))dxdy + \int_{\Omega_0} h_0(f(x,y))dxdy + \int_{\Omega_1} h_1(f(x,y))dxdy
\]
\[
= \int_{\Omega_p} h_p(f(x,y))dxdy
\]
\[
= \int_{[0,1]^2} h_p(f(x,y))dxdy - |\Omega_0|h_p(0) - |\Omega_1|h_p(1)
\]
\[
= I_p(f) - |\Omega_0|h_p(0) - |\Omega_1|h_p(1).
\]

\( \square \)

**Lemma 37.** Let \( \bar{W}_\Omega \) be defined with respect to the graphon \( W_0 \).

(i) Suppose \( W_0 = f_{z_1,p,z_2}^\gamma \) and \( z \in \{0,1\} \). If \( \tilde{f} \in B^{(\gamma,1-\gamma)} \cap \bar{W}_\Omega \), there exists \( g \) such that \( \delta_\square(\tilde{f},g) = 0 \) and \( g \in (B^{(\gamma,1-\gamma)} \cup B^{(1-\gamma,\gamma)}) \cap \Omega_\gamma \).

(ii) Suppose \( W_0 = f_{z_1,p,z_2}^\gamma \) and \( z_1, z_2 \in \{0,1\} \). If \( \tilde{f} \in B^{(\gamma,1-\gamma)} \cap \bar{W}_\Omega \) and \( z_1 = z_2 \), there exists \( g \) such that \( \delta_\square(\tilde{f},g) = 0 \) and \( g \in (B^{(\gamma,1-\gamma)} \cup B^{(1-\gamma,\gamma)}) \cap \Omega_\gamma \).

**Proof.** Since \( \tilde{f} \in B^{(\gamma,1-\gamma)} \cap \bar{W}_\Omega \), there exists some \( g \in B^{(\gamma,1-\gamma)} \) and \( h \in W_\Omega \) such that \( \delta_\square(\tilde{f},h) = 0 \) and \( \delta_\square(\tilde{f},g) = 0 \). Thus \( \delta_\square(h,\tilde{g}) = 0 \). Since \( g \) is of the form \( g = \sum_{\{ij\}} \alpha_{ij}1_{Y_1} \times 1_{Y_2} \) with \( Y_1 = [0,\gamma] \) and \( Y_2 = (\gamma,1] \), we can use Lemma 33 to conclude that \( h \) must be of the same form with appropriate set \( Y'_i \) of sizes \( \gamma \) and \( 1-\gamma \).

First suppose that \( W_0 \) has the form \( f_{z_1,p}^\gamma \) for \( z \in \{0,1\} \). We consider cases:
(i) If \( |Y'_1 \cap [0, \gamma]| > 0 \), then \( \alpha_{11} = z \) and \( g \in \mathcal{W}_{\Omega} \), which completes the proof.

(ii) If \( |Y'_1 \cap [0, \gamma]| = 0 \), it must be the case that \( \gamma \leq 1/2 \) and \( |Y'_2 \cap [0, \gamma]| > 0 \), implying that \( \alpha_{22} = z \). Note that we can always re-define \( g \) on the measure zero set \( \{ \} \) so that \( g^0 \in \mathcal{B}^{(1-\gamma, \gamma)} \) for \( \delta(x) = 1 - x \).

Since \( \gamma \leq 1/2 \), \( g^0 = z \) on \([0, \gamma] \times [0, \gamma]\) meaning \( g^0 \in \mathcal{W}_0 \). By construction, \( g^0 \in \mathcal{B}^{(1-\gamma, \gamma)} \) and \( \delta_2(g^0, f) = 0 \).

Next, suppose that \( W_0 \) has the form \( f_i^2, z_i, z_2 \) for \( z_1, z_2 \in \{ 0, 1 \} \). As \( h \in \mathcal{W}_n \), \( h \) takes value \( z_1 \) on \([0, \gamma]^2\) and value \( z_2 \) on \((\gamma, 1]^2\).

(i) If \( |Y'_1 \cap [0, \gamma]| > 0 \) and \( |Y'_2 \cap (\gamma, 1]| > 0 \), then \( \alpha_{11} = z_1 \), \( \alpha_{22} = z_2 \) and again \( g \in \mathcal{W}_n \).

(ii) If \( |Y'_1 \cap [0, \gamma]| = 0 \), then \( \gamma \leq 1/2 \), \( |Y'_2 \cap [0, \gamma]| > 0 \) and \( |Y'_2 \cap (\gamma, 1]| > 0 \). It follows that \( \alpha_{11} = z_2 \) and \( \alpha_{22} = z_1 \).

- Suppose \( \gamma = 1/2 \). As before, by re-defining \( g \) on the boundary if necessary, we note that for \( \phi(x) = 1 - x \), \( g^0 \) takes value \( z_1 \) on \([0, \gamma] \times [0, \gamma]\), and value \( z_2 \) on \((\gamma, 1] \times (\gamma, 1]\). Thus \( g^0 \in \mathcal{W}_n \).

- By construction \( g^0 \in \mathcal{B}^{(\gamma, 1-\gamma)} \) and \( \delta_2(g^0, f) = 0 \).

- Suppose \( \gamma < 1/2 \), then \( |Y'_2| = 1 - \gamma = |(\gamma, 1]| \leq 1/2 \). Thus \( |Y'_2 \cap (\gamma, 1]| > 0 \), and \( \alpha_{22} = z_2 \). Thus \( z_2 = z_1 \). This implies that \( g \in \mathcal{W}_0 \).

(iii) The case that \( |Y'_2 \cap (\gamma, 1]| = 0 \) follows analogously to the above case.

Note that if \( \gamma \neq 1/2 \), cases (ii) and (iii) only occur when \( z_1 = z_2 \). Therefore when \( z_1 \neq z_2 \), \( g \in \mathcal{B}^{(\gamma, 1-\gamma)} \cap \mathcal{W}_0 \).

\[ \square \]

Proof of Lemma 36: First we show that if \( g \in \mathcal{W}_0 \), then \( J_{W_0}(\tilde{g}) = I_{W_0}(g) \). Let \( g \in \mathcal{W}_0 \). There exists a sequence of graphons \( \{g_n\}_{n \geq 1} \) with each \( g_n \in \mathcal{W}_0 \) such that \( I_{W_0}(g_n) \to J_{W_0}(g) \) and \( \delta_2(g_n, g) \to 0 \). It follows that there exists a sequence \( \phi_n \in \mathcal{M} \) such that \( \delta_2(\phi_n, g) \to 0 \). Note that \( I_p(g_n^\alpha) = I_p(g_n) \). Let \( c = |(\Omega_0, \mathcal{H}_p, 0) + |(\Omega_1, \mathcal{H}_p, 1) \). By Proposition 11, \( I_{\tilde{W}_0}(g) = I_p(g) - c \) and \( I_{\tilde{W}_0}(g_n) = I_p(g_n) - c \) since \( g_n \in \mathcal{W}_0 \). Leveraging the lower semi-continuity of \( I_p \) with respect to \( \delta_2 \) (Lemma 36), we obtain

\[ J_{W_0}(\tilde{g}) = \lim_{n \to \infty} \inf I_{W_0}(g_n) = \lim_{n \to \infty} \inf I_p(g_n) - c = \lim_{n \to \infty} \inf I_p(g_n^\alpha) - c = I_p(g) - c = I_{W_0}(g). \]

Since the definition of \( J_{W_0} \) implies that \( J_{W_0}(\tilde{g}) \leq I_{W_0}(g) \), it follows that \( J_{W_0}(\tilde{g}) = I_{W_0}(g) \).

Next suppose \( W_0 \) is of the form \( f^2, z, z_2 \) where \( z_1, z_2 \in \{ 0, 1 \} \) and \( z_1 = z_2 \). Clearly \( \min\{I_{W_0}(f) : f \in \mathcal{B}^{(\gamma, 1-\gamma)} \cup \mathcal{B}^{(1-\gamma, \gamma)} \cap \mathcal{M} \} = \min\{I_{W_0}(\tilde{g}) : \tilde{g} \in \mathcal{B}^{(\gamma, 1-\gamma)} \cap \mathcal{M} \} \). Let \( h \) be such that \( J_{W_0}(h) = \min\{I_{W_0}(\tilde{g}) : \tilde{g} \in \mathcal{B}^{(\gamma, 1-\gamma)} \cap \mathcal{M} \} \). Lemma 37 implies that we may assume \( h \in (\mathcal{B}^{(\gamma, 1-\gamma)} \cup \mathcal{B}^{(1-\gamma, \gamma)}) \cap \mathcal{W}_0 \). Observe

\[ \min\{I_{W_0}(f) : f \in \mathcal{B}^{(\gamma, 1-\gamma)} \cap \mathcal{M} \} = J_{W_0}(h) = I_{W_0}(h) \]

\[ \geq \min\{I_{W_0}(f) : f \in \mathcal{B}^{(\gamma, 1-\gamma)} \cup \mathcal{B}^{(1-\gamma, \gamma)} \cap \mathcal{M} \} \]

This establishes the claim in these cases. The proof for \( W_0 = f^2, z_1, z_2 \), \( z_1, z_2 \in \{ 0, 1 \} \), \( z_1 \neq z_2 \) is analogous.

5.2 Proof of Lemma 36

Our construction of a non-symmetric graphon with lower entropy than any symmetric graphon is different for each of the three cases. Each proof uses the following proposition.

**Proposition 12.** Let \( p \in (0, 1) \) and let \( W_0 \in \mathcal{W} \) be a graphon that takes values in \([0, p, 1]\). For \( i \in \{0, p, 1\} \), let \( \Omega_i = \{(x, y) \in [0, 1] : W_0(x, y) = i\} \). Let \( E \) be the graph with two vertices and one edge. If \( f \in \mathcal{W}_n \), then

\[ \lim_{p \to 0} \frac{I_{W_n}(f)}{\log 1/p} = \frac{1}{2} \left( (\log f(x, f) - |\Omega_i|) \right). \]

**Proof.** First observe

\[ \lim_{p \to 0} \frac{h_p(\alpha)}{\log 1/p} = \lim_{p \to 0} \frac{\alpha \log \frac{p}{\alpha} + (1 - \alpha) \log \frac{1-p}{1-\alpha}}{\log \frac{1}{p}} = \alpha. \]  \hspace{1cm} (35)

Since \( \frac{h_p(f)}{\log 1/p} \) is bounded as \( p \to 0 \) and \( f \in \mathcal{W}_n \), the Dominated Convergence Theorem implies that

\[ \lim_{p \to 0} \frac{I_{W_n}(f)}{\log 1/p} = \lim_{p \to 0} \frac{1}{2} \int_{\Omega_p} \frac{h_p(f)}{\log 1/p} = \frac{1}{2} \int_{\Omega_p} \lim_{p \to 0} \frac{h_p(f)}{\log 1/p} = \frac{1}{2} \int_{\Omega_p} f = \frac{1}{2} \left( (\log f(x, f) - |\Omega_i|) \right). \]

\[ \square \]
5.2.1 Erdős–Rényi graphs with a planted independent set

In this subsection, we prove the existence of a non-symmetric regime for \(d\)-regular subgraph counts in graphs of the form \(f_{p,p}^{1}\) when \(p\) is sufficiently small. We will do this by showing that the union of isolated vertices with a clique will have lower relative entropy than the minimum in \(B^{(\gamma,1-\gamma)} \cup B^{(1-\gamma,\gamma)}\).

**Proof of Lemma 36 Statement 1** Let \(W_0^\gamma = f_{p,p}^{1}\). First note that

\[
\min\{I_{W_0}(f) : f \in B^{(\gamma,1-\gamma)} \cup B^{(1-\gamma,\gamma)}, t(H, f) \geq t\} = \min\{I_{W_0}(f_{\eta,\alpha,\beta}^{\gamma}) : z \in \{\gamma, 1 - \gamma\}, \alpha, \beta \in [0, 1], t(H, f_{\eta,\alpha,\beta}^{\gamma}) \geq t\}
\]

(36)

since \(I_{W_0}(f_{\eta,\alpha,\beta}^{\gamma}) = \infty\) when \(\eta \neq 0\).

Define the non-symmetric graphon \(\chi_t\) as follows.

\[
\chi_t(x, y) = \begin{cases} 
1 & (x, y) \in [1 - t^{\frac{1}{\gamma}}, 1]^2 \\
0 & \text{otherwise.}
\end{cases}
\]

In other words, the graphon \(\chi_t\) is the union of a clique and isolated vertices that has the required subgraph density (by the fact that \(t(H, \chi_t) \leq t\)). Note that the assumption that \(t \leq t(H, f_{0,0,1}^{\gamma})\) implies that \(t^{1/\gamma} \leq 1 - \gamma\) and so \(\chi_t \in W_0\). By (36), it suffices to show

\[
\lim_{p \to 0} \frac{I_{W_0}(\chi_t)}{\log \frac{1}{p}} < \lim_{p \to 0} \frac{I_{W_0}(f_{0,0,1}^{\gamma})}{\log \frac{1}{p}}
\]

(37)

for each triple \(z \in \{\gamma, 1 - \gamma\}, \alpha, \beta \in [0, 1]\) such that \(t(H, f_{0,0,1}^{\gamma}) \geq t\).

Let \(E\) be the graph with two vertices and one edge. By Proposition 12 for \(z \in \{\gamma, 1 - \gamma\}\)

\[
\lim_{p \to 0} \frac{I_{W_0}(f_{0,0,1}^{\gamma})}{\log \frac{1}{p}} = \frac{1}{2} t(E, f_{0,0,1}^{\gamma}) \quad \text{and} \quad \lim_{p \to 0} \frac{I_{W_0}(\chi_t)}{\log \frac{1}{p}} = \frac{1}{2} t(E, \chi_t) = \frac{1}{2} t^{\frac{1}{\gamma}}.
\]

Therefore to establish (37), it suffices to show that

\[
t^{\frac{1}{\gamma}} \leq t(H, f_{0,0,1}^{\gamma})^{\frac{1}{\gamma}} < t(E, f_{0,0,1}^{\gamma}).
\]

By the generalized Hölder inequality (Theorem 11) and the facts that \(e(H) = dv/2\) and \(g^d \leq g\),

\[
t(H, g) \leq \left( \int_{[0,1]^2} (g(x,y))^d \, dx \, dy \right)^{\frac{1}{d}} \leq \left( \int_{[0,1]^2} g(x,y) \, dx \, dy \right)^{\frac{1}{d}} = t(E, g)^{\frac{1}{d}}
\]

(38)

The second inequality is strict if \(g\) is not the constant graphon that takes value one. Taking \(g = f_{0,0,1}^{\gamma}\) and rearranging establishes (37).

5.2.2 Erdős–Rényi graphs with a planted clique

In this subsection, we prove the existence of a non-symmetric regime for \(d\)-regular subgraph counts in graphs of the form \(f_{p,p}^{1}\) when \(p\) is sufficiently small. Again it will be the union of a clique and isolated vertices which has lower relative entropy than the minimizer in \(B^{(\gamma,1-\gamma)} \cup B^{(1-\gamma,\gamma)}\).

**Proof of Lemma 36 Statement 2** Let \(W_0^\gamma = f_{p,p}^{1}\). First note that

\[
\min\{I_{W_0}(f) : f \in B^{(\gamma,1-\gamma)} \cup B^{(1-\gamma,\gamma)}, t(H, f) \geq t\} = \min\{I_{W_0}(f_{\eta,\alpha,\beta}^{\gamma}) : z \in \{\gamma, 1 - \gamma\}, \alpha, \beta \in [0, 1], t(H, f_{\eta,\alpha,\beta}^{\gamma}) \geq t\}
\]

(39)

since \(I_{W_0}(f_{\eta,\alpha,\beta}^{\gamma}) = \infty\) when \(\eta \neq 1\). Define the non-symmetric graphon \(\chi_t\) as follows.

\[
\chi_t(x, y) = \begin{cases} 
1 & (x, y) \in [0, t^{\frac{1}{\gamma}}]^2 \\
0 & \text{otherwise.}
\end{cases}
\]

In other words, the graphon \(\chi_t\) is the union of a clique and isolated vertices that has the required subgraph density. Note that since \(t \geq t(H, f_{1,0,0}^{\gamma}), t^{1/\gamma} \geq \gamma\) and so \(\chi_t \in W_0\). By (39), it suffices to show that

\[
\lim_{p \to 0} \frac{I_{W_0}(\chi_t)}{\log \frac{1}{p}} < \lim_{p \to 0} \frac{I_{W_0}(f_{1,0,0}^{\gamma})}{\log \frac{1}{p}}
\]
for each triple $z \in \{\gamma, 1 - \gamma\}$, $\alpha, \beta \in [0, 1]$ such that $t(H, f_{1,0,1}) \geq t$.

Let $E$ be the graph with two vertices and one edge. By Proposition [12] for $z \in \{\gamma, 1 - \gamma\}$
\[
\lim_{p \to 0} \frac{I_{W_0}(f_{1,0,1})}{\log \frac{1}{p}} = \frac{t(E, f_{1,0,1}) - \gamma^2}{2} \quad \text{and} \quad \lim_{p \to 0} \frac{I_{W_0}(\chi_1)}{\log \frac{1}{p}} = \frac{t(E, \chi_1) - \gamma^2}{2} = \frac{\gamma^2 - \gamma^2}{2}.
\]

Therefore, it suffices to show that
\[
\frac{1}{2} \left( t^2 - \gamma^2 \right) < \frac{1}{2} \left( t(E, f_{1,0,1}) - \gamma^2 \right) \iff t^2 \leq t(H, f_{1,0,1})^2 < t(E, f_{1,0,1}).
\]

This holds by [35].

### 5.2.3 Erdős–Rényi graphs with a planted clique and independent set

In this subsection, we prove the existence of a non-symmetric regime for $d$-regular subgraph counts in graphons of the form $f_{\eta,0,0}$ when $p$ is sufficiently small. This time, it will be the union of a clique, a bipartite complete graph and independent vertices which has lower relative entropy than the minimizer in $B^{(\gamma,1-\gamma)}$.

**Proof of Lemma 36, Statement 3.** Let $W_0 = f_{\eta,0,0}$ and $t(H, f_{\eta,0,0}) < t < t(H, f_{1,1,0})$. First note that
\[
\min\{I_{W_0}(f) : f \in B^{(\gamma,1-\gamma)}, t(H, f) \geq t\} = \min\{I_{W_0}(f_{1,0,1}) : \alpha \in [0, 1], t(H, f_{1,0,1}) \geq t\} = \min\{I_{W_0}(f_{1,0,1}) : \alpha \in [0, 1], t(H, f_{1,0,1}) = t\}.
\]

The first equality follows because $I_{W_0}(f_{\eta,0,1}) = \infty$ when $\eta \neq 1$ or $\beta \neq 0$. The second equality follows by Proposition [10].

We construct a non-symmetric graphon $\chi_\alpha$ such that $t(H, \chi_\alpha) = t(H, f_{1,0,1})$. Let
\[
\chi_\alpha(x,y) = \begin{cases} 1 & (x,y) \in [0, \gamma + (1 - \gamma)\alpha^d]^2 \setminus (\gamma, 1]^2 \\ 0 & \text{otherwise}. \end{cases}
\]

Let $s_k$ be the number of labeled independent sets of size $k$ in $H$, and let $v = |V(H)|$. In any homomorphism of $H$ in $f_{1,0,1}$, the vertices of $H$ mapped to the interval $(\gamma, 1]$ must form an independent set. Counting homomorphisms by the number of vertices that map to $(\gamma, 1]$, we obtain
\[
t(H, f_{1,0,1}) = \sum_{k=0}^v s_k \gamma^{v-k}(1 - \gamma)^k \alpha^d = \sum_{k=0}^v s_k \gamma^{v-k}(1 - \gamma)^{\alpha^d} 1^{dk} = t(H, \chi_\alpha).
\]

By [11], it suffices to show that
\[
\lim_{p \to 0} \frac{I_{W_0}(\chi_\alpha)}{\log \frac{1}{p}} < \lim_{p \to 0} \frac{I_{W_0}(f_{1,0,1})}{\log \frac{1}{p}} \tag{41}
\]

for all $\alpha$ such that $t(H, f_{1,0,1}) = t$. Since $t(H, f_{1,0,1}) < t < t(H, f_{1,1,0})$, $t(H, f_{1,0,1}) \neq t$ when $\alpha \in [0, 1)$. Thus, it suffices to establish [11] when $\alpha \in (0, 1)$.

Observe using [35] that
\[
\lim_{p \to 0} \frac{I_{W_0}(\chi_\alpha)}{\log \frac{1}{p}} = \lim_{p \to 0} \frac{(1 - \gamma)\alpha^d \gamma h_p(1) + (1 - \gamma - (1 - \gamma)\alpha^d) \gamma h_p(0)}{\log \frac{1}{p}} = (1 - \gamma)\gamma \alpha^d
\]

and
\[
\lim_{p \to 0} \frac{I_{W_0}(f_{1,0,1})}{\log \frac{1}{p}} = \lim_{p \to 0} \frac{(1 - \gamma)\gamma h_p(\alpha)}{\log \frac{1}{p}} = (1 - \gamma)\gamma \alpha.
\]

Noting that $0 < \alpha < 1$ establishes [11], and completes the proof. \qed
6 Bipartite Erdős–Rényi graphs

In this section, we prove Theorems \( \Theta \) and \( \delta \) which precisely identify the symmetric and non-symmetric regimes for \( d \)-regular subgraph counts and the operator norm in bipartite Erdős–Rényi graphs respectively. Throughout this section, we fix \( p, \gamma \in (0, 1) \). We let \( \psi(x) = h_p(x^{1/d}) \), and let \( \psi \) denote the convex minorant of \( \psi \). We use the notation \( f_\gamma \) to denote the bipartite graphon with density \( p \) and blocks of size \( \gamma \) and \( 1 - \gamma \), as illustrated in Figure 2.

6.1 Density of \( d \)-regular subgraphs

We will apply Lemmas \( \beta \) and \( \delta \) to identify the symmetric and non-symmetric regimes respectively.

**Lemma 38.** Let \( H \) be a \( d \)-regular graph with \( d \geq 1 \). Let \( 0 < p \leq r \leq 1 \) be such that \((r^d, h_p(r)) \) is on the convex minorant of \( \psi \). If \( f \in W_\Omega \) and \( t(H, f) \geq t(H, f_\gamma) \), then \( I_{W_\gamma}(f) \geq I_{W_\gamma}(f_\gamma) \) with equality if and only if \( f = f_\gamma \) almost everywhere.

**Lemma 39.** Let \( H \) be a \( d \)-regular graph with \( d \geq 1 \). Let \( 0 < p < r < 1 \) be such that \((r^d, h_p(r)) \) is not on the convex minorant of \( \psi \). Then there exists \( g \in W \) such that \( t(H, g) > t(H, f_\gamma) \) and \( I_{W_\gamma}(g) < I_{W_\gamma}(f_\gamma) \).

We now prove Theorem \( \Theta \) which completely characterizes the symmetric and non-symmetric regimes for \( d \)-regular homomorphism densities in bipartite Erdős–Rényi graphons.

**Proof of Theorem \( \Theta \)** Suppose that the point \((r^d, h_p(r)) \) lies on the convex minorant of \( \psi \). We will show that \( t_\gamma = t(H, f_\gamma) \) is in the symmetric regime for \( t(H, \cdot) \). Let \( g \in W_\Omega \) be such that \( J_{W_\gamma}(g) = \min \{ J_{W_\gamma}(f) : t(H, f) \geq t_\gamma \} \). We may assume that \( g \in W_\Omega \). By Lemma 38, \( J_{W_\gamma}(g) = I_{W_\gamma}(g) \). Since \( t(H, g) \geq t_\gamma \) and \((r^d, h_p(r)) \) lies on the convex minorant of \( \psi \), Lemma 38 implies that \( I_{W_\gamma}(g) \geq I_{W_\gamma}(f_\gamma) \). Since \( I_{W_\gamma}(g) = J_{W_\gamma}(g) = \min \{ J_{W_\gamma}(f) : t(H, f) \geq t_\gamma \} \geq I_{W_\gamma}(f_\gamma) \), it follows that \( I_{W_\gamma}(f_\gamma) = I_{W_\gamma}(g) \). Lemma 38 implies that \( g = f_\gamma \), meaning that \( f_\gamma \) is the unique symmetric solution.

Next, suppose that the point \((r^d, h_p(r)) \) does not lie on the convex minorant of \( \psi \). We will show that \( t_\gamma = t(H, f_\gamma) \) is not in the symmetric regime for \( t(H, \cdot) \). Lemma 38 implies that there exists \( g \in W_\Omega \) such that \( t(H, g) > t(H, f_\gamma) \) and \( I_{W_\gamma}(g) < I_{W_\gamma}(f_\gamma) \). By Lemma 38, \( J_{W_\gamma}(g) = I_{W_\gamma}(g) \). We apply Lemma 38 and obtain

\[
\min \{ J_{W_\gamma}(f) : t(H, g) \geq t_\gamma, f \in \mathcal{B}^{(1,1-\gamma)} \} = \min \{ I_{W_\gamma}(f) : t(H, f) \geq t_\gamma \} \geq I_{W_\gamma}(f_\gamma)
\]

The second equality follows by noting that if \( \gamma \neq 1/2 \) and \( g \in \mathcal{B}^{(1,1-\gamma)} \), then \( I_{W_\gamma}(g) = \infty \) or \( g \) is the zero graphon. The third equality follows by noting that \( I_{W_\gamma}(f_\gamma) \) and \( t(H, f_\gamma) \) are increasing functions of \( \gamma \).

6.1.1 Proof for the symmetric regime

The following lemma describes a norm condition on \( f \) that implies that the graphon \( f_\gamma \) has lower entropy.

**Lemma 40.** Suppose that \( d \geq 1 \) and \( p \leq r \leq 1 \) are such that the point \((r^d, h_p(r)) \) lies on the convex minorant of \( \psi \) and

\[
\| f \|_{\text{d}}^2 \geq 2\gamma(1-\gamma)r^d.
\]

Then \( I_{W_\gamma}(f) \geq I_{W_\gamma}(f_\gamma) \), with equality occurring if and only if \( f = f_\gamma \) almost everywhere.

**Proof.** The statement is trivial if \( f \not\in W_\Omega \). For \( f \in W_\Omega \),

\[
I_{W_\gamma}(f) = \int_0^\gamma \int_0^1 h_p(f(x,y)) \, dx \, dy = \int_0^\gamma \int_0^1 \psi(f^d(x,y)) \, dx \, dy
\]

\[
\geq \int_0^\gamma \int_0^1 \hat{\psi}(f^d(x,y)) \, dx \, dy
\]

\[
\geq (1-\gamma)\hat{\psi}\left(\frac{1}{\gamma(1-\gamma)}\int_0^\gamma \int_0^1 f^d(x,y) \, dx \, dy\right)
\]

\[
(42)
\]

\[
\geq (1-\gamma)\hat{\psi}(r^d)
\]

\[
(43)
\]

\[
= (1-\gamma)\psi(r^d) = (1-\gamma)h_p(r) = I_{W_\gamma}(f_\gamma).
\]
Note that \((12)\) is an application of Jensen’s inequality, and \((43)\) is due to \(\hat{\psi}\) being an increasing function on \([r^d, 1]\). If \(f \neq f^\gamma\), then the step using Jensen’s inequality is a strict inequality. Therefore, \(I_{W_0}(f) \geq I_{W_0}(f^\gamma)\), with equality occurring if and only if \(f = f^\gamma\) almost everywhere.

The following lemma establishes a norm condition on graphons that satisfy the subgraph density requirement.

**Lemma 41.** Let \(H\) be a \(d\)-regular graph with \(d \geq 1\). Let \(f \in W_1\) be such that \(t(H, f) \geq t(H, f^\gamma)\). Then \(\|f\|_d^d \geq 2(1 - \gamma)r^d\).

**Proof.** We may assume \(H\) is bipartite. Since \(H\) is \(d\)-regular, \(H\) must have \(m\) vertices in each partition class and \(d m\) edges for some \(m \in \mathbb{Z}^+\). Let \(c\) be the number of connected components of \(H\). Note that \(t(H, f^\gamma) = 2^c(\gamma(1 - \gamma))m r^{dm}\). Let \(f\) be any graphon such that \(I_{W_0}(f) < \infty\) and \(t(H, f) \geq t(H, f^\gamma)\). Then \(t(H, f^\gamma) \leq t(H, f)\) implies

\[
2^c \gamma^m (1 - \gamma)^m r^{dm} \leq \int_{[0,1]^{2m}} \prod_{(i,j) \in E(H)} f(x_i, x_j) dx_1, \ldots, dx_{2m} \tag{43}
\]

\[
= 2^c \int_0^\gamma \int_0^1 \cdots \int_0^1 \prod_{(i,j) \in E(H)} f(x_i, x_j) dx_1, \ldots, dx_{2m} \tag{44}
\]

\[
\leq 2^c \gamma^m (1 - \gamma)^m \prod_{(i,j) \in E(H)} \left( \frac{1}{\gamma(1 - \gamma)} \int_{x_i=0}^\gamma \int_{x_j=\gamma}^1 f(x, x)^d dx_j dx_i \right) \tag{45}
\]

\[
= 2^c \gamma^m (1 - \gamma)^m \left( \frac{1}{\gamma(1 - \gamma)} \int_{x_i=0}^\gamma \int_{y=\gamma}^1 f(x, y)^d dy dx \right) \tag{46}
\]

\[
= 2^c \left( \int_{x_i=0}^\gamma \int_{y=\gamma}^1 f(x, y)^d dy dx \right)^m \geq 2^c \left( \frac{1}{2} \|f\|_d^d \right)^m .
\]

In \((46)\) we rewrite the density by ordering the vertices so that they alternate between the sides of the bipartition. The factor \(2^c\) accounts for the fact that within each component, a partition class of vertices can map to either \([0, \gamma]\) or \((\gamma, 1]\), and the other partition class will map to the other interval. The generalized Hölder inequality from Theorem \(12\) implies \((45)\). In the application of Theorem \(12\) we set \(p_i = d\) for every \(i \in \{2m\}\). The measures are given by

\[
\mu_{2k+1}(x) = \begin{cases} 0 & 0 \leq x \leq \gamma \\ \frac{1}{1 - \gamma} & \gamma < x \leq 1 \end{cases} \quad \text{and} \quad \mu_{2k}(x) = \begin{cases} 0 & 0 \leq x \leq \gamma \\ \frac{1}{1 - \gamma} & \gamma < x \leq 1 \end{cases}
\]

for \(k \in \{0, 1, \ldots, m\}\), and \(A_1, \ldots, A_{c(H)}\) correspond to the set \(E(H)\). We conclude that \(\|f\|_d^d \geq 2(1 - \gamma)r^d\).

**Proof of Lemma \(39\)** Let \(f \in W_1\) be such that \(t(H, f) \geq t(H, f^\gamma)\). Lemma \(12\) implies that \(\|f\|_d^d \geq 2(1 - \gamma)r^d\). It follows by Lemma \(10\) that \(I_{W_0}(f) \geq I_{W_0}(f^\gamma)\) with equality if and only if \(f = f^\gamma\).

**6.1.2 Proof for the non-symmetric regime**

**Proof of Lemma \(39\)** Since \((r^d, h_p(r))\) is not on the convex minorant of \(\psi(x) = h_p(r^{1/d})\), we may use Proposition \(10\) to conclude there exist \(r_1, r_2, r\) such that \(p < r_1 < r < r_2 \leq 1\) and \((r^d, h_p(r))\) lies strictly above the line segment joining \((r^d_1, h_p(r_1))\) and \((r^d_2, h_p(r_2))\). Let \(s\) be such that

\[
r^d = sr^d_1 + (1 - s)r^d_2,
\]

and thus

\[
sh_p(r_1) + (1 - s)h_p(r_2) < h_p(r). \tag{46}
\]

We use the values \(r_1, r_2, s\) to define a family of graphons \((g^\varepsilon)_{\varepsilon > 0}\). We will prove that for \(\varepsilon > 0\) sufficiently small (i) \(t(H, g^\varepsilon) > t(H, f)\) and (ii) \(I_{W_0}(g^\varepsilon) < I_{W_0}(f^\gamma)\). Define

\[
\begin{align*}
\alpha_1 &= \gamma s \varepsilon^2 & I_1 &= [0, \alpha_1] \\
\alpha_2 &= (1 - \gamma) s \varepsilon^2 & I_2 &= (\gamma, \gamma + \alpha_2] \\
\alpha_3 &= (1 - \gamma) ((1 - s) s^2 + \varepsilon^3) & I_3 &= (1 - \alpha_3, 1] \\
\alpha_4 &= \gamma ((1 - s) s^2 + \varepsilon^3) & I_4 &= (\gamma - \alpha_4, \gamma].
\end{align*}
\]

(47)
Let
\[ I_{c4} = [0, \gamma] \setminus (I_1 \cup I_4) \quad \text{and} \quad I_{c3} = (\gamma, 1] \setminus (I_2 \cup I_3). \]

Define
\[
g^\varepsilon(x, y) = \begin{cases} 
0 & (x, y) \in ([0, \gamma] \times [0, \gamma]) \cup ((\gamma, 1] \times (\gamma, 1]) \\
r_1 & (x, y) \in (I_1 \times I_{c3}) \cup (I_{c3} \times I_1) \cup (I_2 \times I_{c4}) \cup (I_{c4} \times I_2) \\
r_2 & (x, y) \in (I_1 \times I_{c4}) \cup (I_{c4} \times I_1) \cup (I_2 \times I_{c3}) \cup (I_{c3} \times I_2) \\
r & \text{otherwise}.
\end{cases}
\]

Figure 8 illustrates the construction of the graphon \( g^\varepsilon \).

\[
\begin{array}{c|c|c}
\alpha_2 & \alpha_3 & \alpha_1 \\
\hline
\gamma & r & r \\
& r_1 & r \\
\gamma & r & r \\
0 & r_1 & r \\
0 & r_2 & r \\
1 - \gamma & r_1 & r \\
1 - \gamma & r_2 & r \\
\end{array}
\]

Figure 8: Construction of \( g^\varepsilon \).

Next we claim that \( t(H, g^\varepsilon) > t(H, f_\varepsilon^g) \) for sufficiently small \( \varepsilon \). Let \( m \) be such that \( H \) has \( 2m \) vertices and \( dm \) edges. Let \( c \) be the number of connected components of \( H \). Note that the only embeddings of \( H \) into \( g^\varepsilon \) which contribute a value other than \( r^e(H) \) to the integral in \( t(H, g^\varepsilon) \) are such that at least one vertex of \( H \) is mapped to \( \bigcup_{i=1}^c I_j \). Since each \( \alpha_i \) is of order \( \varepsilon^2 \), in order to compute \( t(H, g^\varepsilon) - t(H, f_\varepsilon^g) \) up to error \( O(\varepsilon^4) \) it suffices to consider embeddings in which only one vertex is mapped to \( \bigcup_{i=1}^c I_j \). Observe

\[
t(H, g^\varepsilon) - t(H, f_\varepsilon^g) = 2^c \int_0^1 \int_0^\gamma \cdots \int_0^\gamma \left( \prod_{(i,j) \in E(H)} g^\varepsilon(x_i, y_j) - r^e(H) \right) dx_1 \cdots dx_{2m}
\]

\[
= 2^c m \left[ \alpha_1 \gamma^{-m-1} (1 - \gamma)^m (r_1^d - r^d) r^{e(H) - d} + \alpha_2 (1 - \gamma)^m (r_1^d - r^d) r^{e(H) - d}
+ \alpha_3 (1 - \gamma)^m (r_2^d - r^d) r^{e(H) - d} + \alpha_4 (1 - \gamma)^m (r_2^d - r^d) r^{e(H) - d} \right] + O(\varepsilon^4)
\]

\[
= 2^c m \gamma^m (1 - \gamma)^m r^{e(H) - d} \left( s \varepsilon^2 (r_1^d - r^d) + (1 - s) \varepsilon^2 + \varepsilon^3 \right) \left( r_2^d - r^d \right) + O(\varepsilon^4)
\]

Since \( r_2 > r \), the above computation implies that \( t(H, g^\varepsilon) - t(H, f_\varepsilon^g) > 0 \) for \( \varepsilon \) sufficiently small.

Next we show that \( I_{W_0}(g^\varepsilon) < I_{W_0}(f_\varepsilon^g) \) for sufficiently small \( \varepsilon \). Observe

\[
I_{W_0}(g^\varepsilon) - I_{W_0}(f_\varepsilon^g) = (\alpha_1 (1 - \gamma - \alpha_2 - \alpha_3) + \alpha_2 (\gamma - \alpha_1 - \alpha_4)) (h_p(r_1) - h_p(r))
+ (\alpha_3 (\gamma - \alpha_1 - \alpha_4) + \alpha_4 (1 - \gamma - \alpha_2 - \alpha_3)) (h_p(r_2) - h_p(r))
\]

\[
= 2 \gamma (1 - \gamma) \left( 1 - \varepsilon^2 - \varepsilon^3 \right) \left[ s \varepsilon^2 (h_p(r_1) - h_p(r)) + (1 - s) \varepsilon^2 + \varepsilon^3 \right] \left[(h_p(r_2) - h_p(r))\right]
\]

\[
= 2 \gamma (1 - \gamma) \varepsilon^2 \left[(h_p(r_1) + (1 - s) h_p(r_2) - h_p(r) + \varepsilon (h_p(r_2) - h_p(r)))\right].
\]

Using the condition (46), we conclude that there exists \( \varepsilon \) sufficiently small such that \( I_{W_0}(g^\varepsilon) - I_{W_0}(f_\varepsilon^g) < 0 \), as desired.
6.2 Largest eigenvalue

In this subsection we prove Theorem 8, which characterizes the symmetric and non-symmetric regimes for the largest eigenvalue of the adjacency matrix of a bipartite graph. Recall that $\| \cdot \|_{op}$ is a continuous extension of the normalized graph spectral norm (Lemma 1). We will use the following two lemmas to prove Theorem 8.

**Lemma 42.** Let $\gamma \in (0,1)$ and let $W_0 = f^γ$. For every $f$ such that $f \in W_1$, we have $\|f\|_1 \leq \|f\|_{op} \leq \sqrt{\text{II}_2} \|f\|_2$.

**Proof of Lemma 42.** Let $\psi(x) = h_p(\sqrt{F})$. Suppose that the point $(r^2, h_p(r))$ lies on the convex minorant of $\psi$. We will show that $t^2 = \|f\|_{op}$ is in the symmetric regime for $t(H,.)$. Let $g \in B^r$ be such that $J_{W_0}(\tilde{g}) = \{g \in W_0 : \|g\|_{op} \geq t^2\}$. We may assume that $g \in W_0$. By Lemma 43, $J_{W_0}(\tilde{g}) = I_{W_0}(g)$. Since $\|g\|_{op} \geq t^2 = r \sqrt{\text{II}_2}$, Lemma 12 implies that $\|g\|_2 \geq r \sqrt{\text{II}_2}$. Next, by Lemma 40, we have $J_{W_0}(g) = I_{W_0}(f^γ)$ with equality if and only if $g = f^γ$. Since $I_{W_0}(g) = I_{W_0}(f^γ)$, it follows that $I_{W_0}(f^γ) = I_{W_0}(g)$. By Lemma 40, we conclude that $g = f^γ$, meaning that $f^γ$ is the unique symmetric solution.

Next, suppose that the point $(r^2, h_p(r))$ does not lie on the convex minorant of $\psi$. We will show that $t^2 = \|f\|_{op}$ is not in the symmetric regime for $t(H,.)$. Lemma 43 implies that there exists $g \in W_2$ such that $\|g\|_{op} > r \sqrt{\text{II}_2}$ and $I_{W_0}(g) < I_{W_0}(f^γ)$. Let $\gamma, \gamma'$ be as defined in the proof of Lemma 39. We have already shown that $I_{W_0}(g') < I_{W_0}(f^γ)$ for small enough $\varepsilon > 0$. It remains to show that $\|g'\|_{op} > r \sqrt{\text{II}_2}$. For this claim, it suffices to exhibit a function $u \in L^2([0,1])$ such that $(T_{g'}u)(x) > r \sqrt{\text{II}_2} u(x)$ for all $x \in [0,1]$. Recall the definitions given in (47). Let

$$u(x) = \begin{cases} \frac{\sqrt{\text{II}_2}}{\gamma - \alpha_1 - \alpha_4} r_1 & x \in I_1 \\ r_2 \sqrt{1 - \gamma} & x \in I_{1/4} \\ \frac{\sqrt{\text{II}_2}}{\gamma - \alpha_1 - \alpha_4} r_2 & x \in I_4 \\ \frac{\gamma}{\gamma - (1 - \gamma - \alpha_2 - \alpha_3)} r_1 & x \in I_2 \\ r_2 \sqrt{\gamma} & x \in I_{1/23} \\ \frac{\sqrt{\text{II}_2}}{\gamma - (1 - \gamma - \alpha_2 - \alpha_3)} r_2 & x \in I_3. \end{cases}$$

To derive the upper bound, we use the Cauchy–Schwarz inequality. Observe that for any $u : [0,1] \to \mathbb{R}$,

$$\|T_{g'}u\|^2 = \int_0^1 \left( \int_0^1 f(x,y) u(y) dy \right)^2 dx$$

$$= \int_0^r \left( \int_0^r f(x,y) u(y) dy \right)^2 dx + \int_r^\gamma \left( \int_0^r f(x,y) u(y) dy \right)^2 dx$$

$$\leq \int_0^r u(y)^2 dy \left( \int_0^r \int_0^r f(x,y)^2 dy dx \right) + \int_r^\gamma u(y)^2 dy \left( \int_0^r \int_0^r f(x,y)^2 dy dx \right)$$

$$= \frac{1}{2} \|u\|^2_2 \|f\|^2_2.$$

It follows that $\|T_{g'}u\|_2 \leq \frac{1}{\sqrt{2}} \|f\|_2 \|u\|_2$ for all $u$, and thus $\|f\|_{op} \leq \frac{1}{\sqrt{2}} \|f\|_2$.

**Proof of Lemma 43.** As stated in [33], the left inequality follows from the observation that

$$\|f\|_1 = \|T_{f_1}1\|_1 \leq \|T_{f_2}1\|_2 \leq \|f\|_{op}.$$
Recall that
\[ 1 - \varepsilon^2 - \varepsilon^3 = \frac{1}{1 - \gamma}(1 - \gamma - \alpha_2 - \alpha_3) = \frac{1}{1 - \gamma}(1 - \gamma - \alpha_2 - \alpha_3). \]

We consider six cases, and assume \( \varepsilon \) is sufficiently small in each. For \( x \in I_1, \)
\[
T_{g^*} u(x) = \int_0^1 g^*(x, y) u(y) dy
= \alpha_2 r \cdot \frac{\sqrt{\gamma}}{1 - \gamma} (1 - \gamma - \alpha_2 - \alpha_3)r_1 + (1 - \gamma - \alpha_2 - \alpha_3)r_1 \cdot \sqrt{\gamma}r + \alpha_3 r \cdot \frac{\sqrt{\gamma}}{1 - \gamma} (1 - \gamma - \alpha_2 - \alpha_3)r_2
\]
\[ > (1 - \gamma - \alpha_2 - \alpha_3)r_1 \sqrt{\gamma}r = \frac{1}{\gamma} (\gamma - \alpha_1 - \alpha_4) \sqrt{\gamma}r_1 r
\]
\[ = r \sqrt{\gamma} (1 - \gamma) \frac{1 - \gamma}{\gamma} (\gamma - \alpha_1 - \alpha_4)r_1 = r \sqrt{\gamma} (1 - \gamma) u(x). \]

For \( x \in I_2, \)
\[
T_{g^*} u(x) > (1 - \gamma - \alpha_2 - \alpha_3)r_2 \sqrt{\gamma}r = \frac{1}{\gamma} (\gamma - \alpha_1 - \alpha_4) r_2 \sqrt{\gamma}r
\]
\[ = r \sqrt{\gamma} (1 - \gamma) \frac{1 - \gamma}{\gamma} (\gamma - \alpha_1 - \alpha_4)r_2 = r \sqrt{\gamma} (1 - \gamma) u(x). \]

For \( x \in I_3, \)
\[
T_{g^*} u(x) > (\gamma - \alpha_1 - \alpha_4)r_1 \sqrt{1 - \gamma}r = \frac{\gamma}{1 - \gamma} (1 - \gamma - \alpha_2 - \alpha_3)r_1 \sqrt{1 - \gamma}r
\]
\[ = r \sqrt{\gamma} (1 - \gamma) \frac{\sqrt{\gamma}}{1 - \gamma} (1 - \gamma - \alpha_2 - \alpha_3)r_1 = r \sqrt{\gamma} (1 - \gamma) u(x). \]

For \( x \in I_4, \)
\[
T_{g^*} u(x) > (\gamma - \alpha_1 - \alpha_4)r_2 \sqrt{1 - \gamma}r = \frac{\gamma}{1 - \gamma} (1 - \gamma - \alpha_2 - \alpha_3)r_2 \sqrt{1 - \gamma}r
\]
\[ = r \sqrt{\gamma} (1 - \gamma) \frac{\sqrt{\gamma}}{1 - \gamma} (1 - \gamma - \alpha_2 - \alpha_3)r_2 = r \sqrt{\gamma} (1 - \gamma) u(x). \]

Next consider \( x \in I_{43}. \) Using the fact that \( r^2 = sr_1^2 + (1 - s)r_2^2, \) along with \( \alpha_2 = (1 - \gamma) \varepsilon \) and \( \alpha_3 = (1 - \gamma) (1 - s) \varepsilon^2 + \varepsilon^3, \) we obtain
\[
T_{g^*} u(x) = (1 - \gamma - \alpha_2 - \alpha_3) \left( \alpha_2 \frac{\sqrt{\gamma}}{1 - \gamma} r_1^2 + \sqrt{\gamma}r^2 + \alpha_3 \frac{\sqrt{\gamma}}{1 - \gamma} r_2^2 \right)
\]
\[ = (1 - \gamma)(1 - \varepsilon^2 - \varepsilon^3) \left( \frac{\sqrt{\gamma}}{1 - \gamma} (\alpha_2 r_1^2 + \alpha_3 r_2^2) + \sqrt{\gamma}r^2 \right)
\]
\[ = (1 - \gamma)(1 - \varepsilon^2 - \varepsilon^3) \left( \frac{\sqrt{\gamma}}{1 - \gamma} ((1 - \gamma) s \varepsilon^2 r_1^2 + (1 - \gamma) (1 - s) \varepsilon^2 + \varepsilon^3) r_2^2 + \sqrt{\gamma}r^2 \right)
\]
\[ = \sqrt{\gamma} (1 - \gamma)(1 - \varepsilon^2 - \varepsilon^3) \left( \varepsilon r^2 + \varepsilon^3 r_2^2 + r^2 \right) = \sqrt{\gamma} (1 - \gamma) (r^2 + (r_2 - r) \varepsilon^3 + O(\varepsilon^4))
\]
\[ > \sqrt{\gamma} (1 - \gamma) r^2 = r \sqrt{\gamma} (1 - \gamma) \sqrt{1 - \gamma} r = r \sqrt{\gamma} (1 - \gamma) u(x), \]

where the inequality holds for sufficiently small \( \varepsilon > 0 \) since \( r_2 > r. \)

Similarly, for \( x \in I_{34}, \)
\[
T_{g^*} u(x) = (\gamma - \alpha_1 - \alpha_4) \left( \alpha_1 \frac{\sqrt{\gamma}}{1 - \gamma} r_1^2 + \sqrt{1 - \gamma}r^2 + \alpha_4 \frac{\sqrt{1 - \gamma}}{\gamma} r_2^2 \right)
\]
\[ = \gamma(1 - \varepsilon^2 - \varepsilon^3) \left( \frac{\sqrt{1 - \gamma}}{\gamma} (\alpha_1 r_1^2 + \alpha_4 r_2^2) + \sqrt{1 - \gamma}r^2 \right)
\]
\[ = \gamma(1 - \varepsilon^2 - \varepsilon^3) \left( \gamma s \varepsilon^2 r_1^2 + \gamma ((1 - s) \varepsilon^2 + \varepsilon^3) r_2^2 + \sqrt{1 - \gamma}r^2 \right)
\]
\[ = \gamma \sqrt{1 - \gamma}(1 - \varepsilon^2 - \varepsilon^3) \left( s \varepsilon^2 r_1^2 + ((1 - s) \varepsilon^2 + \varepsilon^3) r_2^2 + r^2 \right) \]
We collect here some questions arising naturally from our investigations.

To prove the upper bound LDP in the weak topology, Theorem 9, we will use a general LDP upper bound $\varepsilon > 0$ where again the inequality holds for sufficiently small $\varepsilon$ (Theorem 4.1 of [11]).

7 Open questions

We collect here some questions arising naturally from our investigations.

1. Our results establish a “reentrant phase transition” in upper tail large deviations for homomorphism densities in specific block model random graphs. Note that our results on the symmetric regime are quite general, and applicable for arbitrary block graphons. In contrast, our proof for the existence of a symmetry breaking regime is case-specific, and does not generalize directly. It is natural to believe that this reentrant phase transition phenomenon should hold for a much wider family of block graphons, and it would be interesting to investigate this further.

2. A natural follow up question concerns the precise boundary between the symmetric and non-symmetric regimes. So far, this boundary has been identified for very homogeneous graphs—the Erdős-Rényi random graph in [33] and the Erdős-Rényi bipartite graph in this article. The rate function in these specific examples simplifies considerably, and is expressed in terms of an appropriate relative entropy functional. We expect the general case to be significantly more challenging, due to the intractable nature of the rate function $J_{\Phi_0}$. Progress in this direction will likely require considerably new ideas, and is beyond the scope of this paper.

3. Another natural direction of inquiry concerns the behavior of the optimizer(s) in the symmetry breaking regime. In fact, we do not even know whether the upper tail variational problem (6) has a unique optimizer in the symmetry-breaking phase. Any tangible progress on this uniqueness question would be a promising start in this direction. Moreover, it would be of interest to identify the structure of the optimizer(s) in the non-symmetric regime. These questions remain open even for Erdős–Rényi graphs, and were already raised in [14] and [33].

4. Finally, we note that our analysis of the upper tail variational problem (6) is restricted to regular subgraphs. Non-trivial extensions to non-regular graphs will likely require new ideas, and will provide new insights on the upper tail problem.

8 Appendix

8.1 Weak topology LDP upper bound

To prove the upper bound LDP in the weak topology, Theorem 9 we will use a general LDP upper bound given in [11] Section 4.3 which we restate as Lemma 44 below.

We need some notation. Let $\mathcal{H}$ be a real topological vector space whose topology satisfies the Hausdorff property. Let $\mathcal{H}^*$ denote the dual space of continuous linear functionals on $\mathcal{H}$. Let $\mathcal{B}$ denote the Borel sigma-algebra of $\mathcal{H}$ and let $\{\mu_n\}_{n \geq 1}$ be a sequence of probability measures on $(\mathcal{H}, \mathcal{B})$. Define the logarithmic moment generating function $\Lambda_n : \mathcal{H}^* \to (-\infty, \infty]$ of $\mu_n$ as

$$\Lambda_n(\lambda) = \log \int \exp(\lambda(x)) \, d\mu_n(x).$$

Given a “rate” $\{\varepsilon_n\}_{n \geq 1}$, i.e., a sequence of positive real numbers $\varepsilon_n$ tending to 0, we define $\bar{\Lambda} : \mathcal{H}^* \to [-\infty, \infty]$ and its Fenchel-Legendre transform $\bar{\Lambda}^* : \mathcal{H} \to [-\infty, \infty]$ as

$$\bar{\Lambda}(\lambda) = \limsup_{n \to \infty} \varepsilon_n \Lambda_n(\lambda/\varepsilon_n)$$

$$\bar{\Lambda}^*(x) = \sup_{\lambda \in \mathcal{H}^*} (\lambda(x) - \bar{\Lambda}(\lambda)).$$

Lemma 44 (Theorem 4.1 of [11]). For any compact set $\Gamma \subseteq \mathcal{H}$,

$$\limsup_{n \to \infty} \varepsilon_n \log \mu_n(\Gamma) \leq - \inf_{x \in \Gamma} \bar{\Lambda}^*(x).$$
Proof of Theorem 3. We closely follow the proof of Theorem 5.1 in [11]. Let $\mathcal{H}$ be the vector space $L^2([0,1]^2)$ with the weak topology. For each $a, f \in \mathcal{H}$, define $\lambda_n(f)$ in the dual space $\mathcal{H}^*$ as

$$\lambda_n(f) = \int_{[0,1]^2} a(x, y) f(x, y) dx dy.$$ 

Setting $\mu_n = \mathbb{P}_{kn, W_0}$, define $\Lambda_n : \mathcal{H}^* \to \mathbb{R}$ as

$$\Lambda_n(\lambda) = \log \int_{\mathcal{H}} \exp (\lambda(f)) \, d\mathbb{P}_{kn, W_0}(f) = \log \left( \mathbb{E}_{f \sim \mathbb{P}_{kn, W_0}} [\exp (\lambda(f))] \right).$$

Set $\varepsilon_n = \frac{2}{(kn)^2}$, and let

$$\bar{\Lambda}(\lambda) = \limsup_{n \to \infty} \frac{2\Lambda_n((kn)^2 \lambda/2)}{(kn)^2}.$$ 

Let $f_G^G$ be the empirical graphon on $kn$ vertices drawn from $W_0$. For $i, j \in [kn]$, let $X_{ij}$ be the indicator for the event that $\{i, j\}$ is an edge in $G$. Since $G$ is a simple undirected graph $X_{ij} = X_{ji}$ and $X_{ii} = 0$. For ease of notation let $W_{ij}^G = W_0(i/(kn), j/(kn))$. Note $X_{ij} \sim \text{Bern}(W_{ij}^G)$. Let $I_{kn, i} = [i - 1]/kn, i/kn]$ for $i = 2, \ldots, kn$ and let $B_{i,j,n}$ be the square $I_{kn, i} \times I_{kn, j}$. Let $S$ be the set of symmetric $L^2$ functions. For $a \in S$, let $\hat{a}_n$ denote the level $kn$ approximant, i.e.

$$\hat{a}_n(x, y) = (kn)^2 \int_B a(w, z) \, dw \, dz$$

where $B = B_{i,j,n}$ is such that $(x, y) \in B_{i,j,n}$.

Observe that for an empirical graphon $f_G^G$

$$\lambda_n(f_G^G) = \sum_{1 \leq i, j \leq kn, i \neq j} X_{ij} \int_{B(i,j,n)} a(x, y) \, dx dy = \sum_{1 \leq i, j \leq kn} X_{ij} \int_{B(i,j,n) \cup B(j,i,n)} a(x, y) \, dx dy.$$ 

Recall $X_{ij} \sim \text{Bern}(W_{ij}^G)$, and so for any $\theta$

$$\mathbb{E} [\exp (\theta X_{ij})] = W_{ij}^G \exp (\theta) + 1 - W_{ij}^G.$$ 

Since the events $\{X_{ij}\}_{i<j}$ are independent, it follows that for any $a \in S$

$$\Lambda_n((kn)^2 \lambda_n/2) = \log \left( \mathbb{E}_{f \sim \mathbb{P}_{kn, W_0}} [\exp ((kn)^2 \lambda_n(f)/2)] \right)$$

$$= \log \left( \mathbb{E}_{f \sim \mathbb{P}_{kn, W_0}} \left[ \exp \left( \frac{(kn)^2}{2} \sum_{1 \leq i, j \leq kn} X_{ij} \int_{B(i,j,n) \cup B(j,i,n)} a(x, y) \, dx dy \right) \right] \right)$$

$$= \log \prod_{1 \leq i, j \leq kn} \left( W_{ij}^G \exp \left( \frac{(kn)^2}{2} \int_{B(i,j,n) \cup B(j,i,n)} a(x, y) \, dx dy \right) + 1 - W_{ij}^G \right)$$

$$= \sum_{1 \leq i, j \leq kn} \log \left( W_{ij}^G \exp \left( \frac{(kn)^2}{2} \int_{B(i,j,n) \cup B(j,i,n)} a(x, y) \, dx dy \right) + 1 - W_{ij}^G \right)$$

$$= (kn)^2 \int_{[0,1]^2 \setminus B_n} \log \left( W_0(x, y) \exp (\hat{a}_n(x, y)) + 1 - W_0(x, y) \right) \, dx dy$$

$$= (kn)^2 \int_{[0,1]^2 \setminus B_n} \log (W_0(x, y) \exp (\hat{a}_n(x, y)) + 1 - W_0(x, y)) \, dx dy$$

(49)

where $B_n = \bigcup_{i=1}^{kn} B(i, i, n)$. Next we consider $u_p(x) \triangleq \log (pe^x + 1 - p)$, in order to reason about the limit of the above integral as $n \to \infty$. For $p \in [0, 1]$, $u'_p(x) = pe^x/(pe^x + 1 - p)$, and so $|u'_p(x)| \leq 1$ everywhere. Thus $|u_p(x) - u_p(y)| \leq |x - y|$ for all $x, y \in \mathbb{R}$. It follows that

$$| \log (W_0(x, y) \exp (\hat{a}_n(x, y)) + 1 - W_0(x, y)) - \log (W_0(x, y) \exp (a_n(x, y)) + 1 - W_0(x, y)) |$$

$$\leq | \hat{a}_n(x, y) - a(x, y) |.$$ 

By Proposition 2.6 of [11], $\hat{a}_n \to a$ in $L^2$, and therefore using the above inequality and the Cauchy–Schwarz inequality, we obtain

$$\int_{[0,1]^2} \log (W_0(x, y) \exp (\hat{a}_n(x, y)) + 1 - W_0(x, y)) - (\log (W_0(x, y) \exp (a(x, y)) + 1 - W_0(x, y))) \, dx dy$$

55
\[ \leq \int_{[0,1]^2} |\hat{a}_n(x, y) - a(x, y)| dxdy \]

\[ \leq \left( \int_{[0,1]^2} (\hat{a}_n(x, y) - a(x, y))^2 dxdy \right)^{1/2} \rightarrow 0. \]

Therefore,

\[ \lim_{n \to \infty} \int_{[0,1]^2} \log \left( W_0(x, y) \exp \left( \hat{a}_n(x, y) \right) + 1 - W_0(x, y) \right) dx \: dy = \int_{[0,1]^2} \log \left( W_0(x, y) \exp (a(x, y)) + 1 - W_0(x, y) \right) dx \: dy. \quad (50) \]

Next we consider the limit of the above integral over the set \( B_n \). Since \(|u_p(x) - u_p(y)| \leq |x - y|\), taking \( y = 0 \), we obtain \(|u_p(x)| \leq |x|\). The Cauchy–Schwarz inequality then implies that

\[ \left| \int_{B_n} \log \left( W_0(x, y) \exp \left( \hat{a}_n(x, y) \right) + 1 - W_0(x, y) \right) dx \: dy \right| \leq \left( \frac{|B_n|}{2} \right)^{1/2} \left( \int_{B_n} (\hat{a}_n(x, y))^2 dx \: dy \right)^{1/2} \leq \frac{\|\hat{a}_n\|_2}{\sqrt{kn}}. \]

Since \( \lim_{n \to \infty} \hat{a}_n(x, y) = a(x, y) \) for all \((x, y) \in [0, 1]^2\),

\[ \lim_{n \to \infty} \int_{B_n} \log \left( W_0(x, y) \exp \left( \hat{a}_n(x, y) \right) + 1 - W_0(x, y) \right) dx \: dy = 0. \quad (51) \]

Finally we use (49), (51), and (50) to compute for \( a \in S \)

\[ \bar{\Lambda}(\lambda_n) = \limsup_{n \to \infty} \frac{2 \Lambda_n \left( (kn)^2 \lambda_n / 2 \right)}{(kn)^2} \]

\[ = \limsup_{n \to \infty} \int_{[0,1]^2 \setminus B_n} \log \left( W_0(x, y) \exp \left( \hat{a}_n(x, y) \right) + 1 - W_0(x, y) \right) dx \: dy \]

\[ = \limsup_{n \to \infty} \int_{[0,1]^2} \log \left( W_0(x, y) \exp \left( \hat{a}_n(x, y) \right) + 1 - W_0(x, y) \right) dx \: dy \]

\[ = \int_{[0,1]^2} \log \left( W_0(x, y) \exp (a(x, y)) + 1 - W_0(x, y) \right) dx \: dy. \]

For \( f \in H \), let

\[ \bar{\Lambda}^*(x) \triangleq \sup_{\lambda \in \Lambda^+} (\lambda(f) - \bar{\Lambda}(\lambda)). \]

By Proposition 3

\[ \bar{\Lambda}^*(f) \geq \sup_{\lambda \in \Lambda} (\lambda_0(f) - \bar{\Lambda}(\lambda_0)) = 2I_{W_0}(f). \]

Combined with the compactness of the weak topology [11 Proposition 2.8], [11 Theorem 4.1], stated here as Lemma 24 implies that

\[ \limsup_{n \to \infty} \frac{2}{(kn)^2} \log \mathbb{P}_{kn,W_0}(F) \leq \inf_{f \in P} 2I_{W_0}(f). \]

\[ \square \]

### 8.2 Other useful results

The following theorem was proven in [23]. We include the form as given by [33].

**Theorem 11.** Let \( \mu_1, \ldots, \mu_n \) be probability measures on \( \Omega_1, \ldots, \Omega_n \), respectively, and let \( \mu = \prod_{i=1}^n \mu_i \) be the product measure on \( \Omega = \prod_{i=1}^n \Omega_i \). Let \( A_1, \ldots, A_n \) be nonempty subsets of \([n] = \{1, \ldots, n\}\) and write \( \Omega_A = \prod_{i \in A} \Omega_i \) and \( \mu_A = \prod_{i \in A} \mu_i \). Let \( f_i \in L^p(\Omega_{A_i}, \mu_{A_i}) \) with \( p_i \geq 1 \) for each \( i \in [m] \) and suppose in addition that \( \sum_{i,l \in A_i} \frac{1}{p_i} \leq 1 \) for each \( l \in [n] \). Then

\[ \int \prod_{i=1}^m |f_i| d\mu \leq \left( \int |f_i|^{p_i} d\mu_{A_i} \right)^{1 \over m}. \]
In particular, when \( p_i = d \) for every \( i \in [m] \) we have
\[
\int \prod_{i=1}^{m} |f_i| d\mu \leq \prod_{i=1}^{m} \left( \int |f_i|^d d\mu_A \right)^{\frac{1}{d}}.
\]

**Proof of Proposition** Let \( \epsilon > 0 \) be arbitrary. The proof is very similar to that of Theorem 2.3 \cite{14}, and we only sketch it here. Since both \( f_n \) and \( g_n \) are block graphs, the distance \( d_\square(f_n, g_n) \) can be written as a maximum over \( 4^k_n \) pairs of sets \( S, T \subseteq [0, 1] \), that are unions of a subset of the intervals used in the definition of \( g_n \). But given \( S \) and \( T \), the expectation of \( \int_{S \times T} f_n \) is \( \int_{S \times T} g_n \). Azuma’s inequality then shows that the probability that the difference is larger than \( \epsilon \) is bounded by \( e^{-c\epsilon^2 kn^2} \) for some universal constant \( c > 0 \). The union bound now implies the proposition. \( \square \)

**Proposition 13** (Lemma A.1 of \cite{33}). Let \( d \geq 1 \) and \( p \in (0, 1) \). Consider \( \psi_p(x) = h_p(x^{1/d}) \) with domain \([0, 1]\).
1. The function \( \psi_p(x) \) is decreasing on \([0, p^d]\) and increasing on \([p^d, 1]\).
2. Let \( p_0 = \frac{d - 1}{d - 1 + c_p^2} \).
   (a) If \( p > p_0 \), then \( \psi_p(x) \) is convex and \( \psi''_p > 0 \) on \([0, 1]\).
   (b) If \( p = p_0 \), then \( \psi_p(x) \) is convex and \( \psi''_p = 0 \) at exactly one point in \((p^d, 1)\).
   (c) For \( p < p_0 \), the function \( \psi_p(x) \) has exactly two inflection points \( r_1^d \) and \( r_2^d \) with \( p < r_1 < r_2 < 1 \).

8.3 Behavior at \( t = t_{\text{max}} \).

8.3.1 Proof of Theorem

In this subsection we prove Theorem 5.

**Proof of Theorem** First, observe \( t(H, \hat{f}_{\text{max}}) = t_{\text{max}} \). Consider the optimization problem
\[
\min \{ J_{W_0}(\hat{f}) : t(H, \hat{f}) \geq t_{\text{max}} \}. \tag{52}
\]
Let \( \hat{f}_* \) be an optimizer of (52). Lemma 33 states that there exists a sequence \( f_n \in W_0 \) such that \( \delta_{\square}(f_n, \hat{f}_*) \to 0, t(H, f_n) \to t_{\text{max}}, f_n \geq W_0 \) pointwise, and \( f_n = W_0 \) on irrelevant blocks. For \( n \geq n_0 \), the \( f_n \) satisfy the \( \epsilon \)-convex minorant condition (this follows from Lemma 33). Lemma 23 implies that \( \hat{f}_* \in B^7 \). and \( J_{W_0}(\hat{f}_*) = I_{W_0}(g) < \infty \) for some \( g \in B^7 \). Therefore
\[
I_{W_0}(g) = \min \{ J_{W_0}(\hat{f}) : t(H, \hat{f}) \geq t_{\text{max}} \} \leq I_{W_0}(f_{\text{max}}) < \infty.
\]
and thus \( g \in B^7 \cap W_0 \). Further, \( \delta_{\square}(f_*, \hat{f}_*) = 0 \) implies \( t(H, g) = t_{\text{max}} \). This implies \( g = 1 \) on the relevant blocks. Moreover, \( I_{W_0}(g) \leq I_{W_0}(f_{\text{max}}) \) which is possible if \( g = W_0 \) on the irrelevant blocks. We therefore conclude that \( g = f_{\text{max}} \). We then have \( f_*=f_{\text{max}} \) and \( J_{W_0}(f_*) = J_{W_0}(f_{\text{max}}) = I_{W_0}(f_{\text{max}}) \). We have thus established that \( f_{\text{max}} \) is the unique optimizer to the upper tail variational problem for \( t = t_{\text{max}} \).

Next, we have,
\[
P_{kn,W_0} \left( \delta_{\square}(f^{G_{kn}}, \hat{f}_{\text{max}}) \geq \delta, t(H, G_{kn}) \geq t_{\text{max}} \right) = \frac{P_{kn,W_0} \left( \delta_{\square}(f^{G_{kn}}, \hat{f}_{\text{max}}) \geq \delta, t(H, G_{kn}) \geq t_{\text{max}} \right)}{P_{kn,W_0} \left( t(H, G_{kn}) \geq t_{\text{max}} \right)} \tag{53}
\]
Note that the set \( \{ \hat{f} : \delta_{\square}(\hat{f}, \hat{f}_{\text{max}}) \geq \delta, t(H, \hat{f}) \geq t_{\text{max}} \} \) is closed, and thus Theorem 11 implies
\[
\lim_{n \to \infty} \frac{1}{(kn)^{2}} \log P_{kn,W_0} \left( \delta_{\square}(f^{G_{kn}}, \hat{f}_{\text{max}}) \geq \delta, t(H, G_{kn}) \geq t_{\text{max}} \right) \leq - \inf \{ J_{W_0}(\hat{f}) : \delta_{\square}(\hat{f}, \hat{f}_{\text{max}}) \geq \delta, t(H, \hat{f}) \geq t_{\text{max}} \} =: -C'.
\]

57
We conclude that for every $\varepsilon > 0$, there exists $N(\varepsilon)$ such that if $n \geq N(\varepsilon)$, then

$$
\frac{1}{(kn)^2} \log \mathbb{P}_{kn,W_0} \left( \delta_\square(f_{G_{kn}}, \hat{f}_{\max}) \geq \delta, t(H, G_{kn}) \geq t_{\max} \right) \leq -C' + \varepsilon,
$$

or equivalently

$$
\mathbb{P}_{kn,W_0} \left( \delta_\square(f_{G_{kn}}, \hat{f}_{\max}) \geq \delta, t(H, G_{kn}) \geq t_{\max} \right) \leq \exp \left( -(kn)^2 (C' - \varepsilon) \right) .
$$

(54)

Next, we turn to the denominator. Let $(I_i \times I_j)_{i,j \in [k]}$ be the blocks of $W_0$, and let $R \subset [0,1]^2$ denote the union of the relevant blocks. Recall the definition of $k(\cdot)$ from (2), let $A = (A_{ij})_{i,j \in [kn]}$ denote the adjacency matrix of $G_{kn}$ and let $S$ be the set of relevant edges:

$$
S = \left\{ (i,j) \in [kn]^2 : I_k\left(\frac{i}{k}\right) \times I_k\left(\frac{j}{k}\right) \subset R \right\}.
$$

Observe that

$$
\mathbb{P}_{kn,W_0}(t(H, G_{kn}) = t_{\max}) = \mathbb{P} \left( \cap_{(i,j) \in S \{A_{ij} = 1\}} \right)
$$

$$
= \prod_{a \cdot I_a \times I_a \subset R} p_a(2) \prod_{a < b \cdot I_a \times I_b \subset R} p_{ab}^{-2}
$$

$$
= \exp \left( \frac{n}{2} \sum_{a \cdot I_a \times I_a \subset R} \log(p_a) + n^2 \sum_{a < b \cdot I_a \times I_b \subset R} \log(p_{ab}) \right)
$$

$$
= \exp \left( \frac{1}{2} \sum_{a < b \cdot I_a \times I_b \subset R} \log(p_{ab}) - \frac{n}{2} \sum_{a \cdot I_a \times I_a \subset R} \log(p_a) \right)
$$

$$
= \exp \left( -\frac{1}{2} (kn)^2 \sum_{a < b \cdot I_a \times I_b \subset R} \frac{1}{k^2} \log\left(\frac{1}{p_{ab}}\right) - \frac{n}{2} \sum_{a \cdot I_a \times I_a \subset R} \log(p_a) \right)
$$

$$
= \exp \left( -(kn)^2 I_{W_0}(f_{\max})(1 + o(1)) \right).
$$

Recall that $I_{W_0}(f_{\max}) = J_{W_0}(\hat{f}_{\max})$, so that

$$
\mathbb{P}_{kn,W_0}(t(H, G_{kn}) \geq t_{\max}) = \exp \left( -(kn)^2 J_{W_0}(\hat{f}_{\max})(1 + o(1)) \right) .
$$

(55)

Applying (54) and (55) to (53), we obtain for $n \geq N(\varepsilon)$

$$
\mathbb{P}_{kn,W_0} \left( \delta_\square(f_{G_{kn}}, \hat{f}_{\max}) \geq \delta, t(H, G_{kn}) \geq t_{\max} \right) \leq \exp \left( -(kn)^2 (C' - \varepsilon) + (kn)^2 J_{W_0}(\hat{f}_{\max})(1 + o(1)) \right)
$$

$$
= \exp \left( -(kn)^2 \left( C' - \varepsilon - J_{W_0}(\hat{f}_{\max})(1 + o(1)) \right) \right) .
$$

Recalling that $\hat{f}_{\max}$ is the unique optimizer of (52), the proof is complete by observing that

$$
C' = \inf \left\{ J_{W_0}(\tilde{f}) : \delta_\square(\tilde{f}, \hat{f}_{\max}) \geq \delta, t(H, \tilde{f}) \geq t_{\max} \right\}
$$

$$
> \inf \left\{ J_{W_0}(\tilde{f}) : t(H, \tilde{f}) \geq t_{\max} \right\} = J_{W_0}(\hat{f}_{\max}).
$$

\qed

8.3.2 Elaboration on Remark 4

Let $W_0$ be a uniform $k$ block graphon, and $\tau = t(H, \cdot)$, where $H$ is a finite $d$-regular graph. By Lemma 21 $t(H, \tilde{f}) \leq t(H, f^*) \leq t_{\max}$, with equality if and only if $f^* = 1_{W_0>0}$, as all non-trivial blocks of $W_0$ are relevant. Thus $f^* = 1_{W_0>0}$ is the unique solution to $t(H, f) = t_{\max}$ in this setting.

Note that if $W_0 = f_0^2$, and $\tau(\tilde{f}) = \|\tilde{f}\|_{op}$, $t_{\max} = \sqrt{\gamma(1 - \gamma)}$. Further, using Lemma 24 we conclude that $\|\tilde{f}_{G}\|_{op} \geq t_{\max}$ implies that $\|\tilde{f}_{G}\|_2 \geq 2\gamma(1 - \gamma)$. This is possible if and only if $\tilde{f}_{G} = f_0^\gamma$. This establishes the desired claim.
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