Artinian bimodule with quasi-Frobenius bimodule of translations

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Abstract

For bimodule $A_M B$ we introduce notion of bimodule of translations $C M Z$, where $C$ is a quotient ring of a Schneider ring of $A_M B$ and $Z$ is its center.

We investigate mutual relationship of two cases:

(a) bimodule $A_M B$ is a Morita’s Artinian duality context and

(b) bimodule $C M Z$ is a Morita’s Artinian duality context.

Let’s note that sometimes bimodule of translations is called canonical bimodule and $C$ is called multiplication ring.

1 Introduction

Main results of this article previously were presented at Workshop [24]. Area of this investigation had arose in 1990-s when A.A.Nechaev has had attempts to generalize the notion of linear recurrent sequence over commutative ring to the cases of linear recurrences over non-commutative ring, module, bimodule [19, 20, 21, 14, 22].

Let’s note immediately that the necessity for the ring, module, bimodule to be quasi-Frobenius [2] was established promptly. Further one of the possible ways to determine linear recurrences over non-commutative ring, module or bimodule was this one [23].

Let $A_M B$ be an arbitrary bimodule. By the left translation [23] generated by the element $a \in A$ is called a natural map $\hat{a} : M \to M$ defined by equality: $\forall m \in M$.

*This article is dedicated to the memory of A.A.Nechaev
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\( \hat{a}(m) = am \). Analogously the right translation \( \hat{b} \) generated by the element \( b \in B \) is a natural map \( \hat{b} \) defined by the equality: \( \forall m \in M \ \hat{b}(m) = mb \). It is evidently that \( \hat{a} \in \text{End}(M) = \text{End}(M_\mathbb{Z}) \) and \( \hat{b} \in \text{End}(M) \). Subring \( \hat{A} = \{ \hat{a} \mid a \in A \} \) of the ring \( \text{End}(M) \) is called ring of left translations. Respectively ring \( \hat{B} = \{ \hat{b} \mid b \in B \} \) is called ring of right translations.

Let’s remember that rings \( (K,+,* \rangle \) and \( (R,\oplus,\otimes \rangle \) are called inversely isomorphic or anti-isomorphic \([2]\), if ring’s isomorphism takes place: \( K^{op} \cong R \) where \( K^{op} = (K,+,\tilde{*}) \) and operation \( \tilde{*} : K \times K \to K \) is defined according to the rule: \( k_1 \tilde{*} k_2 = k_2 * k_1, \ k_1, k_2 \in K \).

It is easy to see that if \( _A M \) is a faithful module then rings \( A \) and \( \hat{A} \) are isomorphic and if \( M_B \) is a faithful module then rings \( B \) and \( \hat{B} \) are inversely isomorphic .

Lemma 1. For arbitrary bimodule \( _A M_B \) this equality takes place: \( \hat{A} \cap \hat{B} = Z(\hat{A}) \cap Z(\hat{B}) \). Here \( Z(\hat{A}) \) and \( Z(\hat{B}) \) denote centers of corresponding rings.

Proof. Because of the associativity \( (am)b = a(mb) \) elements of the rings \( \hat{A} \) and \( \hat{B} \) are pair-wise commutative. Hence arbitrary element \( z \in \hat{A} \cap \hat{B} \) lies in \( Z(\hat{A}) \) and in \( Z(\hat{B}) \) concurrently, i.e. \( \hat{A} \cap \hat{B} \subset Z(\hat{A}) \cap Z(\hat{B}) \). Inverse inclusion is evident. \( \Box \)

Commutative ring \( Z = \hat{A} \cap \hat{B} \) (Lemma 1) is called \([23]\) common center of rings \( A \) and \( B \) in relation to bimodule \( M \). Let’s denote by \( \phi : \hat{A} \to A \) isomorphism constructed according to the rule: \( \hat{a} \overset{\phi}{\mapsto} a \) and by \( \psi : \hat{B} \to B \) inverse isomorphism of rings \( \hat{B} \) and \( B \) constructed by the rule: \( \hat{b} \overset{\psi}{\mapsto} b \). Then \( \phi(Z) < Z(A), \psi(Z) < Z(B) \).

Besides that to the set \( M \) may be correctly assigned structure of \( (Z,Z) \)-bimodule in such way that these identities hold: \( z \cdot m = \phi(z)m, \ m \cdot z = m\psi(z), \ zm = mz \).

\( (Z,Z) \)-bimodule \( \hat{A} \otimes Z \hat{B} \) has an associative multiplication operator \([25]\) Proposition 9.2(a)] defined by the rule: \( (\hat{a}_1 \otimes \hat{b}_1)(\hat{a}_2 \otimes \hat{b}_2) = \hat{a}_1 \hat{a}_2 \otimes \hat{b}_1 \hat{b}_2 \). Herewith if \( A \) and \( B \) are rings with identity then \( 1_{\hat{A} \otimes Z \hat{B}} = 1_{\hat{A}} \otimes 1_{\hat{B}} \). Now \( \hat{A} \otimes Z \hat{B} \) is a ring and \( M \) is a \([25]\) Proposition 10.1(a)] left \( \hat{A} \otimes Z \hat{B} \)-module with multiplication according to the rule: \( (\hat{a} \otimes \hat{b})m = \phi(\hat{a})m\psi(\hat{b}) \). If modules \( _A M \) and \( M_B \) are unitary then \( 1_{\hat{A} \otimes Z \hat{B}} = \epsilon \) is an identity mapping of \( M \).

Further we will need this generalization of \([25]\) §10.1, Proposition 1:
Proposition 2. Let $A, B$ are $Z$-algebras. Any $(A, B)$-bimodule $M$ is a left $A \otimes Z B^{op}$-module with scalar multiplication defined according to the rule: \((x \otimes y)u = (xu)y = x(uy)\) for $x \in A, y \in B, u \in M$.

Conversely, any left $A \otimes B^{op}$-module is an $(A, B)$-bimodule with operations: \(xu = (x \otimes 1_B)u, \quad uy = (1_A \otimes y)u\). If $M$ and $N$ are $(A, B)$-bimodules then $\text{Hom}_{A-B}(M, N) = \text{Hom}_{A \otimes B^{op}}(A \otimes B^{op} M, A \otimes B^{op} N)$.

Ring $C$ is called ring of translations of bimodule $A M_B$ [23] if it is generated in the ring $\text{End}(M)$ by the union of rings $\hat{A}$ and $\hat{B}$.

It takes place

Statement 3. Mapping $\lambda : \hat{A} \otimes \hat{B} \to C$ satisfying to the rule: $\lambda \left( \sum_{(i)} \hat{a}_i \otimes \hat{b}_i \right) = \sum_{(i)} \hat{a}_i \hat{b}_i$ is an epimorphism of rings and $\ker \lambda = \{ \delta \in \hat{A} \otimes \hat{B} \mid \delta M = \theta \}$ where $\theta$ is an identity element of $M$.

Proof. The first statement follows from element-wise commutativity of rings $\hat{A}$ and $\hat{B}$.
The second statement follows from the fact that $C M$ is a faithful module. 

Previously [23] bimodule $C M_Z$ was called canonical bimodule of bimodule $A M_B$.

Further we instead or denomination canonical bimodule will use denomination bimodule of translations.

Now linear recurrent sequence over bimodule $A M_B$ may be defined in such way [23]: it is a mapping $\mu : \mathbb{N}_0 \to M$ for which there exists a unitary polynomial $c(x) = x^m - \sum_{j=0}^{m-1} c_j x^j \in C[x]$ with property: $\forall i \in \mathbb{N}_0 \mu(i + m) = \sum_{j=0}^{m-1} c_j (\mu(i + j))$.

Herewith to generalize classical properties of linear recurrences onto newly arisen case it is necessary for bimodule $C M_Z$ to be quasi-Frobenius.

Let’s remember that according to definition [2] left-faithful and right-faithful bimodule $M$ over rings $A$ and $B$ with identities is called quasi-Frobenius if for every maximal left ideal $I$ of ring $A$ right annihilator $r_M(I) = \{ m \in M \mid Im = \theta \}$ is either irreducible $B$-module either equal to neutral element of $M$ and for every maximal right ideal $J$ of ring $B$ left annihilator $l_M(J) = \{ m \in M \mid mJ = \theta \}$ is either irreducible $A$-module either equal to $\theta$. Here and after $\theta$ is a neutral element of the group $(M, +)$. 
However it is a problem that there were not known any facts about relations between quasifrobeniusness of bimodule $CM_Z$ and quasifrobeniusness of bimodule $AM_B$. Initially was formulated

Hypothesis 4 ([23]). If bimodule $AM_B$ is a Morita’s Artinian duality context then bimodule of translations $CM_Z$ is also a Morita’s Artinian duality context.

However during multiyear process of attempts to prove this Hypothesis it become clear that it’s in general case wrong (Theorem 6) and moreover under additional conditions inverse implication holds (Theorem 5).

Resulting in these facts approach to definition of linear recurrences over bimodules suggested by V.L.Kurakin was adopted [9]. Theoretically this approach was based on results of [11].

Despite this results of this article are not negligible and were continued: notion of matrix linear recurrences is based on the Theorem 5, 10, 27 and the notion of skew linear recurrences is based on Theorem 6, 9.

Let’s now get to outline and prove main results of this article. In order of independence and completeness we will remember necessary definitions [7]. Module $AM$ is called injective if for every monomorphism $\alpha : AK \rightarrow AL$ and every homomorphism $\phi : AK \rightarrow AM$ there exists homomorphism $\psi : AL \rightarrow AM$ with property $\phi = \psi \alpha$. It is equivalent [7, Theorem 5.3.1(a)] to the fact that for every monomorphism $\xi : AM \rightarrow AN$ module $\xi(\mathcal{M})$ allocated in $N$ as a direct summand. According to [7, Theorem 5.6.4] for every $A$-module $L$ there exists unique up to isomorphism injective hull $\mathcal{I}(L)$ i.e. minimal according to order of inclusion injective module containing $L$. Module $AU > AL$ is called essential extension of module $AL$ if for every submodule $V$ of module $AU$ from condition $L \cap V = \theta$ follows that $V = \theta$. It is known [7, Teopema 5.6.6] that injective hull $\mathcal{I}(L)$ of module $L$ is a maximal essential extension of $L$.

Also we have to remember the notion of Brauer group of the field $F$ [25, §12.5].

Let $F$ be a field. Let’s denote by $\mathfrak{B}(F)$ class of all finite-dimensional simple $F$-algebras $A$ with property $Z(A) = F$ i.e. class of all finite-dimensional central simple $F$-algebras.

If algebras $A$ and $B$ contains in $\mathfrak{B}(F)$ then [25, §12.4, Proposition b(i)] $F$-algebra $A \otimes_F B$ contains in $\mathfrak{B}(F)$. 
Besides that \[25\] §12.5, Lemma | following conditions are equivalent:

(i) basic algebras of \( A \) and \( B \) are isomorphic;

(ii) there exists an algebra with division (body) \( D \in \mathfrak{B}(F) \) and naturals \( m \) and \( n \) such that \( A \cong D_{n,n} \) and \( B \cong D_{m,m} \);

(iii) there exists naturals \( r \) and \( s \) such that \( A \otimes F_{r,r} \cong B \otimes F_{s,s} \).

Algebras \( A, B \in \mathfrak{B}(F) \), underlying this conditions are called equivalent. Equivalence class of algebra \( A \) is denoted as \([A]\).

The set \( \mathfrak{B}(F) = \{[A] \mid A \in \mathfrak{B}(F)\} \) is \[25\] §12.5, Proposition a] an Abel group with operator \([A][B] = [A \otimes B]\), neutral element \([F]\) and the operation of taking inverse element \([A]^{-1} = [A^{op}]\). The group \( \mathfrak{B}(F) \) is called Brauer group of the field \( F \). Every class in group \( \mathfrak{B}(F) \) is represented by \[25\] §12.5, Proposition b(ii)] algebra with division which is unique up to isomorphism. If Brauer group of field \( F \) is trivial then every central simple \( F \)-algebra is a full matrix algebra over \( F \). If the field \( F \) is algebraically closed then its Brauer group is trivial \[25\] §12.5, Corollary]. Besides that Brauer group of the finite field is trivial too.

By socle \( \mathcal{S}(A) \) of the left \( A \)-module \( M \) is called \[6\] Chapter IV] the sum of all irreducible submodules of the \( A \)-module \( M \).

The main result of this work consists in following:

Theorem 5. Let bimodule \( A_M B \) be a faithful as a left \( A \)-module and as a right \( B \)-module simultaneously, bimodule of translations \( C_M Z \) of bimodule \( A_M B \) is quasi-Frobenius, \( Z \) is a local Artinian ring, \( Q = I(\bar{Z}) \) is an injective hull of the unique irreducible \( Z \)-module \( \bar{Z} \).

Then:

I. (a) Ring’s isomorphism takes place: \( C \cong Z_{n,n} \).

(b) Bimodule’s isomorphism takes place: \( C_M Z \cong z_{n,n} Q^{(n)}_Z \).

(c) Rings \( A, B \) are primary Artinian left-side and right-side simultaneously rings with identity and factor-rings \( \bar{A} = A/J(A), \bar{B} = B/J(B) \) of rings \( A \) and \( B \) respectively when activated modulo Jacobson’s radical are equivalent elements of the set \( \mathfrak{B}(\bar{Z}) \).

(d) Bimodule \( A\mathcal{S}(C) \bar{B} \) is quasi-Frobenius.

II. Bimodule \( A_M B \) is quasi-Frobenius if and only if the left socle \( \mathcal{S}(A_M) \) and the right
socle $\mathcal{S}(M_B)$ of bimodule $AM_B$ are equal to $\mathcal{S}(C_M)$.

III. Under additional condition of the form: Brauer group $B(\bar{Z})$ of the field $\bar{Z}$ is trivial bimodule $AM_B$ is quasi-Frobenius and moreover for some naturals $r, s \in \mathbb{N}$ such that $n = r \cdot s$ isomorphisms of rings and bimodules respectively takes place:

(a) $A \cong Z_{r,r}, B \cong Z_{s,s},$

(b) $AM_B \cong z_{r,r}Q_{r,s}z_{s,s}.$

Here we denote by $z_{r,r}Q_{r,s}z_{s,s}$ the set of orthogonal matrices of dimensions $r \times s$ filled by elements of the set $Q$ and usual matrix multiplication upon elements of $Z_{r,r}$ and $Z_{s,s}$ respectively. It is evident that $z_{r,r}Q_{r,s}z_{s,s}$ is a bimodule.

Conversion of the point III of Theorem 5 is wrong even in the finite case. Appropriate example may be constructed in class of so called GEO-rings \cite{18}. By definition the ring $S$ is called Galois-Eisenstein-Ore ring (or GEO-ring) if it is finite completely primary ring of principal ideals i.e. the ring $S$ contains unique (one-side) maximal ideal $p(S)$ and every one-side ideal of ring $S$ is principal. Commutative GEO-ring is called Galois-Eisenstein ring or $GE$-ring \cite{18}.

In arbitrary GEO-ring $S$ of characteristic $p^d$ contains Galois subring $R = GR(p^d, r)$ \cite{13} \cite{26}, $q = p^r$. All such rings are conjugated in $S$. Ring $R$ is called coefficients ring of $S$.

**Theorem 6.** (a) For arbitrary GEO-ring $S$ bimodule $S_S$ is a quasi-Frobenius.

(b) Let $S$ be a GEO-ring with a coefficients ring $R = GR(p^d, r)$ and in addition $d > 1$ and $(r, p) = 1$. Then bimodule of translations $C_SZ$ of quasi-Frobenius $S$-bimodule $S$ is a quasi-Frobenius bimodule if and only if $S$ is a GE-ring.

### 2 Preliminaries

To prove Theorems above it is necessary to remember a lot of definitions and facts connected firstly with different characterizations of quasi-Frobenius bimodules. Most deeply and fully quasi-Frobenius bimodules are investigated in the case when rings $A$ and $B$ are Artinian left-side and right-side respectively, i.e. it satisfy descending chain conditions of left (respectively, right) ideals. Such quasi-Frobenius bimodules in the monography
are called (Morita’s) Artinian duality context. In this case \[2\] Theorem 6(iv),(v)]
modules \(AM\) and \(MB\) are finitely generated.

Let’s remember other well-known results about quasi-Frobenius bimodules.

Let \(A\) be a left Artinian ring with unit \(e\), \(J(A)\) be a Jacobson radical of ring \(A\). Then \(J(A)\) (\[6\], Corollary III.1.1] is a maximal nil-potent two-sided ideal of ring \(A\) containing every one-side nil-ideal of ring \(A\). It is known \[6\], Chapter III\] that \(\bar{A} = A/J(A)\) is a direct sum of pair-wise orthogonal simple rings \(\bar{A}_1, \ldots, \bar{A}_t\) with units \(\bar{e}^{(1)}, \ldots, \bar{e}^{(t)}\) respectively.

Idempotent \(\bar{e}_l\), \(l \in \overline{1, t}\) of ring \(\bar{A}\) is called \[6\] primitive if left ideal \(\bar{A}\bar{e}_l\) of ring \(\bar{A}\) is irreducible as a left \(\bar{A}\)-module. Then \(\Delta_l = \text{End}(\bar{A}\bar{e}_l)\) is a body isomorphic to the ring \(\bar{A}\bar{e}_l\bar{A}\bar{e}_l = \bar{e}_l A \bar{e}_l\) and ring \(\bar{A}\) is a full matrix algebra of finite degree \(k_l\) over body \(\Delta_l\). Besides that modules \(\bar{A}\bar{e}_l, l = \overline{1, t}\) are all, up to isomorphism, irreducible \(A\)-modules.

Analogously idempotent \(e_l, l \in \overline{1, t}\) of ring \(A\) is called \[6\] primitive if left ideal \(\bar{A}\bar{e}_l\) of ring \(\bar{A}\) is irreducible left \(\bar{A}\)-module.

Idempotents \(\bar{e}_j, \bar{e}_l \in \bar{A}\) are called isomorphic if isomorphic left \(\bar{A}\)-modules \(\bar{A}\bar{e}_j\) and \(\bar{A}\bar{e}_l\). Analogously idempotents \(e_j, e_l \in A\) are called isomorphic if isomorphic left \(A\)-modules \(Ae_j\) and \(Ae_l\).

Let \(\bar{e}_l, l = \overline{1, t}\), are all non-isomorphic idempotents of ring \(\bar{A}\).

For every \(l = \overline{1, t}\) there exists idempotent \(e_l \in \bar{e}_l\), such that elements \(e_l, l = \overline{1, t}\), are all non-isomorphic primitive idempotents of ring \(A\), and there takes place \[6\] Proposition III.8.5\] the expansion unit \(e\) of ring \(A\) into the sum of primitive orthogonal idempotents

\[
e = \sum_{l=1}^{t} \sum_{i=1}^{k_l} e_{l,i},
\]

where every idempotent \(e_{l,i}\) is isomorphic to idempotent \(e_l\) (i.e. by definition \(A Ae_{l,i} \cong A Ae_l\)).

Decomposition (2.1) generates decomposition of the ring \(A\) into the direct sum of undecomposable left ideals:

\[
A = \sum_{l=1}^{t} \oplus \sum_{i=1}^{k_l} \oplus Ae_{l,i}
\]

Let’s denote \(e^{(l)} = \sum_{i=1}^{k_l} e_{l,i}, l = \overline{1, t}\). It is known \[6\] Theorem III.9.2\] that additive group
of the ring $A$ is represented as a direct sum of Abel groups

$$A = \sum_{l=1}^{t} \oplus A_l \oplus U,$$

(2.3)

where $A_l = e^{(l)}Ae^{(l)}$, $l = 1, t$, $U = \sum_{l \neq \nu} e^{(l)}Ae^{(\nu)}$. Herewith $A_l = S_{k_i,k_l}$, $l = 0, t$, are primarily rings with identities $e^{(l)}$ i.e. every of it is a full matrix ring over local left Artinian ring $S_l$ wherein it’s a preimage of ring $\bar{A}_l$ when activating modulo $J(A_l)$. Moreover herewith

$$J(A) = \sum_{l=1}^{t} J(A_l) \oplus U.$$

(2.4)

Further rings $A_l$, $l = 1, t$, we will call primarily components of ring $A$. If $t = 1$ then [6] Theorem III.8.1] $A = S_{k,k}$ is a primary ring.

Let’s remember that socle $\mathfrak{S}(AM)$ of the left $A$-module $M$ [6] Chapter IV] is a completely reducible left $A$-module satisfying condition: $\mathfrak{S}(AM) = r_M(J(A))$. Sum $\mathfrak{S}^{(V)}(AM) = \sum_{i \in I} \tilde{V}_i$ of all irreducible submodules $\tilde{V}_i, i \in I$, of module $AM$ isomorphic to module $AV$ is called homogeneous component of socle $\mathfrak{S}(AM)$ belonging to irreducible left $A$-module $V$.

If $AM_B$ is some bimodule and $V$ is a some left $A$-module then $B$-module $V_B^* = \text{Hom}_A(V,M)$ is called a right dual to $V$ with respect to bimodule $M$ [6] Chapter IV]. Relatively by left dual to the right $B$-module $W$ with respect to $AM_B$ is called $A$-module $AW^* = \text{Hom}_B(W,M)$.

If $AM_B$ is a quasi-Frobenius bimodule, $AV$ and $W_B$ are irreducible modules then $V_B^* \ni AW^*$ are also irreducible (either are non-zero). Moreover [2] Theorem 1] these equalities take place: $\mathfrak{S}^{(V)}(AM) = \mathfrak{S}^{(V^*)}(M_B), \mathfrak{S}^{(W)}(M_B) = \mathfrak{S}^{(W^*)}(AM)$, where $V_B^* = \text{Hom}_A(V,M)$, $AW^* = \text{Hom}_B(W,M)$. Hence $\mathfrak{S}(AM) = \mathfrak{S}(M_B)$.

Left $A$-module $M$ is called distinguished [2] if it contains an isomorphic image of every irreducible $A$-module or, what’s equivalent, for every irreducible left $A$-module $V \mathfrak{S}^{(V)}(AM) \neq \theta$. Left $A$-module $U$ is called minimal (in $M$) [2] if every submodule of $AM$ contains $U$. If minimal submodule exists it is a unique irreducible submodule of $AM$.

Now we can provide necessary characterizations of quasi-Frobenius bimodules:

Theorem 7. Following conditions are equivalent:
(1) [2, Theorem 6(iii)] Bimodule $A_M B$ is a quasi-Frobenius, rings $A$ and $B$ are left Artinian and right Artinian respectively.

(2) [2, Theorem 6(i)] Ring $A$ is a left Artinian, left $A$-module $M$ is an injective, finitely generated and distinguished, ring $\hat{B}$ is an endomorphism ring of module $A M$.

(3) [4, Theorem 23.25(6)] Every left ideal $I$ of ring $A$, every right $B$-submodule $W$ of module $M$ are satisfies to annihilator correspondences:

$$ I = l_A(r_M(I)), \quad W = r_M(l_A(W)), $$

(2.5)

eyery right ideal $J$ of ring $B$, every left $A$-submodule of $M$ satisfy to annihilator correspondences:

$$ J = r_B(l_M(I)), \quad V = l_M(r_B(V)). $$

(2.6)

(4) [2, Theorem 12] Bimodule $A_M B$ is a faithful, $A$ is a left Artinian ring, $B$ is a right Artinian ring, $A$ and $B$ have the same number $t$ of non-isomorphic primitive idempotents, namely: $e_1,\ldots,e_t, f_1,\ldots,f_t$ respectively, which under necessary re-enumeration satisfy conditions:

I. For every $l \in \overline{1,t}$ left $A$-module $Mf_l$ contains minimal $A$-submodule isomorphic to $\bar{A}e_l$, an image of $Ae_l$ activating modulo $J(A)e_l$;

II. For every $l \in \overline{1,t}$ right $B$-module $e_l M$ contains minimal $B$-submodule isomorphic to $\bar{f}_l B$, an image of $f_l B$ activating modulo $f_l J(B)$;

(5) [2, Theorem 6(ii)] Ring $B$ is right Artinian, right $B$-module $M$ is injective, finitely generated and distinguished, ring $\hat{A}$ is an endomorphism ring of module $M_B$.

If $A_M B$ is a quasi-Frobenius bimodule satisfying to conditions of Theorem [7] then [2, Theorem 7] there exist one-to-one Galois correspondences between the set of two-sided ideals $I$ of ring $A$, the set of $(A,B)$-subbimodules $N$ of bimodule $M$, and the set of two-sided ideals $J$ of ring $B$, which are established by equalities:

$$ N = r_M(I), \quad J = r_B(N); $$

$$ N = l_M(J), \quad I = l_A(N). $$

Herewith $N$ is a quasi-Frobenius $(A/I,B/J)$-bimodule. In particular, $\mathcal{S}(M)$ is a quasi-Frobenius $(A/J(A),B/J(B))$-bimodule.
Further we will use this Statement 8 (A.A.Nechaev, 1996г.). Left-faithful and right-faithful bimodule $A_M B$ over left Artinian and right Artinian correspondingly rings with units is quasi-Frobenius if and only if left socle $\mathcal{S}(A M)$ and right socle $\mathcal{S}(M B)$ of $M$ coincides and $(\bar{A}, \bar{B})$-bimodule

$$\mathcal{S}(M) = \mathcal{S}(A M) = \mathcal{S}(M B)$$

(2.7)
is quasi-Frobenius.

Proof. For quasi-Frobenius bimodule $A_M B$ we have equality: $\mathcal{S}(A M) = \mathcal{S}(M B)$ [2, Theorem 1]. Because $A_M B$ is satisfying to conditions of Theorem 7 we have that \[2\,\text{Theorem 7}\] $\mathcal{S}(M)$ is a quasi-Frobenius $(\bar{A}, \bar{B})$-bimodule.

Conversely let equalities (2.7) take place and bimodule $\bar{A}\mathcal{S}(M)\bar{B}$ is a quasi-Frobenius. Then bimodule $\bar{A}\mathcal{S}(M)\bar{B}$ satisfies conditions of point 4 of Theorem 7 and according to it notations these equalities take place: $\mathcal{S}(e_l M_B) = \mathcal{S}(\bar{e}_l \mathcal{S}(M_B))$ and $\mathcal{S}(A M_f l) = \mathcal{S}(\bar{A} \mathcal{S}(M) \bar{f}_l)$. Hence if conditions of point 4 of Theorem 7 take place relatively to bimodule $\bar{A}\mathcal{S}(M)\bar{B}$ then it implies satisfying of the same conditions in relation to bimodule $A_M B$. \[\square\]

Lemma 9. For Artinian duality context $A_M B$ this equality takes place: $\mathcal{Z}(\bar{A}) = \mathcal{Z}(\bar{B})$.

Proof. According to Theorem 7(2), $\bar{A} = \text{End}(M_B)$. Because an arbitrary element $z \in \mathcal{Z}(\bar{B})$ contains in $\text{End}(M_B)$ then $\mathcal{Z}(\bar{B}) \subset \bar{A}$ what follows $\mathcal{Z}(\bar{B}) \subset \mathcal{Z}(\bar{A})$. Converse inclusion may be stated by the same method. \[\square\]

It is also true

Statement 10. If $M$ is a bimodule over primary left-Artinian and right-Artinian rings with identities respectively $A = S_{k,k}$ and $B = T_{l,l}$ of the form:

$$A_M B = s_{k,k} W_{k,l} T_{l,l},$$

then bimodule $A_M B$ is a quasi-Frobenius if and only if bimodule $s W_T$ is a quasi-Frobenius.

Proof. Let suppose that bimodule $A_M B$ is a quasi-Frobenius. Let $e$ be a primitive idempotent of ring $A$. Then according to point (4) of Theorem 7 right $B = T_{l,l}$-module
eM = W^{(l)} contains a minimal submodule. Let suppose now that right T-module W does not contain minimal submodule. It means that module W_T contains at least two different irreducible submodules U and V. Then module W_{T,l}^{(l)} contains at least two different irreducible submodules U_{T,l}^{(l)} and V_{T,l}^{(l)}. It is a contradiction to existence in W_{T,l}^{(l)} of minimal submodule. Hence right module W_T also contains minimal submodule.

Analogously may be shown that existence of minimal submodule in S_{k,k} W^{(k)} implies existence of minimal submodule in _sW.

Hence according to point (4) of Theorem 4 bimodule _sW_T is a quasi-Frobenius.

The converse implication may be proven by the same arguments in reverse order. □

3 The Proof of Theorem 5

For convenience of reading let us duplicate the text of Theorem 5.

Theorem 11. Let bimodule _A_M_B be faithful as a left A-module and as a right B-module together, bimodule of translations C_M_Z of _A_M_B is a quasi-Frobenius, Z is local Artinian ring, Q = I(_Z) is an injective hull of unique irreducible Z-module _Z_Z.

Then:

I. (a) Isomorphism of rings takes place: C ≅ Z_{n,n}.

(b) Isomorphism of bimodules takes place: C_M_Z ≅ z_{n,n} Q^{(n)}_Z.

(c) Rings A, B are primary left-Artinian and right-Artinian rings with identities, herewith factor-rings _A = A/J(A), _B = B/J(B) of A and B respectively activated modulo Jacobson radicals are equivalent elements of the set _B(_Z).

(d) Bimodule _A M_B is a quasi-Frobenius.

II.Bimodule _A_M_B is a quasi-Frobenius if and only if the left socle _B(_A_M_B) and the right socle _B(M_B) of _A_M_B coincide with _B(C_M_B).

III.If additional condition takes place: Brauer group _B(_Z) of the field _Z is trivial, then bimodule _A_M_B is a quasi-Frobenius and moreover for some r, s ∈ N such that n = r · s isomorphisms of rings and bimodules respectively take place:

(a) A ≅ Z_{r,r}, B ≅ Z_{s,s},

(b) _A_M_B ≅ Z_{r,s} Q_{r,s} Z_{s,s}.
3  THE PROOF OF THEOREM 5

Proof. I(a). Let $CM_Z$ be an Artinian duality context. Then according to Theorem 7(5) the right module $M_Z$ is an injective and hence it is a direct sum of some $n \in \mathbb{N}$ right modules $Q_Z$:

$$M_Z \cong Q_Z^{(n)},$$

where $Q = I(\tilde{Z}_Z)$ is an injective hull of unique irreducible $Z$-module $\tilde{Z} = Z/J(Z)$.

By the same Theorem the equality take place: $\hat{C} = C = \text{End}(M_Z)$. The existence of isomorphism (3.1) implies isomorphism:

$$C \cong D_{n,n},$$

where $D = \text{End}(Q_Z)$. Because of [4, Proposition 23.33], [15], equality takes place: $D = Z$ and isomorphism (3.2) takes a form:

$$C \cong Z_{n,n}.$$  

I(b). From (3.1) and (3.3) follows that $C_{M_Z} \cong z_{n,n}Q_Z^{(n)}$.

I(c). From (3.3) follows that $C$ is finitely generated left $Z$-module and finitely generated right $Z$-module simultaneously. Because $Z$ is a commutative Artinian ring then $C$ is also Artinian ring ( and not only left-Artinian how the Theorem 7(1) states ).

Under conditions of this Theorem left module $AM$ is a faithful. Hence rings $A$ and $\hat{A}$ are isomorphic. Because ring $\hat{A}$ is a subring of $C$ and hence contains ring $Z$, so module $\hat{A}_Z$ is a submodule of the finitely generated module $C_Z$ over commutative Artinian ring $Z$. Hence $\hat{A}$ is a two-sided Artinian ring and finitely generated $Z$-module. Because of the isomorphism, ring $A$ is also two-sided Artinian.

Analogously may be established that $\hat{B}$ is a two-sided Artinian ring and finitely generated $Z$-module and that ring $B$ is also two-sided Artinian.

Let’s establish now the relation between Jacobson radical $J(C)$ of ring $C$ and Jacobson radicals $J(\hat{A}), J(\hat{B})$ of rings $\hat{A}, \hat{B}$ respectively.

Let $\hat{a} \in J(\hat{A})$. Then right ideal $\hat{a}C$ generated by element $\hat{a}$ in ring $C$ is a nil-ideal. Indeed for arbitrary element $c = \sum_{(i)} \hat{a}_i \hat{b}_i \in C$ because of the element-wise commutativity...
of rings \( \hat{A} \) and \( \check{B} \) the equality takes place:

\[
(\hat{a}c)^{n(\hat{A})} = \sum_{(i_1, \ldots, i_n(\hat{A}))} \hat{a}i_{i_1} \cdots \hat{a}i_{i_n(\hat{A})} \cdot \check{b}_{i_1} \cdots \check{b}_{i_n(\hat{A})},
\]

where \( n(\hat{A}) \) is a nil-potent index of ideal \( J(\hat{A}) \). Because for arbitrary set \( (i_1, \ldots, i_n(\hat{A})) \) the inclusion takes place:

\[
\hat{a}i_{i_1} \cdots \hat{a}i_{i_n(\hat{A})} \in J(\hat{A})^{n(\hat{A})} = \hat{0},
\]

then \( (\hat{a}c)^{n(\hat{A})} = \hat{0} \).

Because every one-side nil-ideal of Artinian ring contains in Jacobson radical of it ring we have: \( \hat{a}C < J(C) \). Besides that, \( \hat{a} \in \hat{a}C \). So we have proved inclusion: \( J(\hat{A}) \subset J(C) \).

From other side the set \( J(C) \cap \hat{A} \) is a two-sided nil-potent ideal of ring \( \hat{A} \). Hence, \( J(C) \cap \hat{A} \triangleleft J(\hat{A}) \).

So we have a sequence of inclusions: \( J(\hat{A}) \subset J(C) \cap J(\hat{A}) \subset J(C) \cap \hat{A} \triangleleft J(\hat{A}) \) which means that all inclusions are equalities:

\[
J(\hat{A}) = J(C) \cap J(\hat{A}) = J(C) \cap \hat{A}
\]

Analogously may be established that

\[
J(\check{B}) = J(C) \cap J(\check{B}) = J(C) \cap \check{B}.
\]

Let’s denote by \( \mathcal{S}(M) \) the socle of the left module \( \mathcal{C}M \) which is equal to \([2, \text{Theorem 1}]\) the socle of the right module \( M_Z \). Because of the \([2, \text{Theorem 7}]\) bimodule \( \mathcal{C}\mathcal{S}(M)_Z \) is a quasi-Frobenius. An isomorphism

\[
\mathcal{C}\mathcal{S}(M)_Z \cong Z_{n,n}^{(n)}.
\]

may be established by the same arguments as an isomorphism \([3.4] \). From \( (3.8) \) follows that \( Z(\bar{C}) = \bar{Z} \).

Let’s show that rings \( \hat{A} \) and \( \check{B} \) are primary. Let \( \hat{A}_1, \ldots, \hat{A}_t \) and \( \check{B}_1, \ldots, \check{B}_t \) are all primary components of rings \( \hat{A} \) and \( \check{B} \) respectively.

Let \( \omega : C \to C/J(C) \) be a canonical epimorphism. Then

\[
\omega(C) = [\omega(\hat{A}), \omega(\check{B})],
\]

\( (3.9) \).
whereas
\[ \omega(\hat{A}) \cong \hat{A}/(\hat{A} \cap J(C)) = \hat{A}/J(\hat{A}) = \tilde{A} \cong \tilde{A} \quad (3.10) \]

and
\[ \omega(\check{B}) \cong \check{B}/(\check{B} \cap J(C)) = \check{B}/J(\check{B}) = \check{B} \cong \check{B}^{op}. \quad (3.11) \]

Hence because of the [6, Theorem III.9.2]
\[ \bar{C} = [\omega(\hat{A}), \omega(\check{B})] = \sum_{i=1}^{t} \oplus [\omega(\hat{A}_i), \omega(\check{B}_j)]. \quad (3.12) \]

Since \( \bar{C} \cong \tilde{Z}_{n,n} \) is a simple ring then between pair-wise orthogonal rings \([\omega(\hat{A}_i), \omega(\check{B}_j)]\), \( i = 1, t, j = 1, l \), there is only one non-zero. Let’s denote it \([\omega(\hat{A}_1), \omega(\check{B}_1)]\), and for all \( i, j \geq 2 \) inclusions take place: \( \hat{A}_i, \check{B}_j < J(C) \). Hence every element of every of rings \( \hat{A}_i, \check{B}_j, i, j \geq 2 \), is a nil-potent what is impossible because every of those rings contains an idempotent. It is a contradiction which shows that \( t = l = 1 \) and rings \( \hat{A}, \check{B} \) are primary.

Let’s note now that because of the pare-wise commutativity of rings \( \omega(\hat{A}) \) and \( \omega(\check{B}) \) inclusions take place:

\[ \tilde{Z} = Z(\bar{C}) < Z(\omega(\hat{A})), Z(\omega(\check{B})) < Z(\bar{C}) = \tilde{Z} \quad (3.13) \]

Hence \( \tilde{Z} = Z(\omega(\hat{A})) = Z(\omega(\check{B})) \). It means that rings \( \omega(\hat{A}) \) and \( \omega(\check{B}) \) are central \( \tilde{Z} \)-algebras.

Because rings \( \hat{A}, \check{B} \) are primary then isomorphisms take place: \( \omega(\hat{A}) \cong \overline{A}, \omega(\check{B}) \cong \overline{B} \), and rings \( \omega(\hat{A}), \omega(\check{B}) \) are simple algebras.

Since modules \( Z\hat{A}, Z\check{B} \) are finitely generated then \( \tilde{Z} \)-algebras \( \omega(\hat{A}), \omega(\check{B}) \) are of finite dimension.

So \( \omega(\hat{A}), \omega(\check{B}) \in \mathfrak{B}(\tilde{Z}) \). According to Structure Theorem of Wedderburn–Artin there exist algebras with division (body) \( \Delta, \nabla \) and naturals \( r, s \) such that \( \omega(\hat{A}) = \Delta_{r,r}, \omega(\check{B}) = \nabla_{s,s} \). Herewith \( Z(\Delta) = Z(\nabla) = \tilde{Z} \).

Because \( \bar{C} = [\omega(\hat{A}), \omega(\check{B})] \) and rings \( \omega(\hat{A}), \omega(\check{B}) \) are pair-wise commutative then the socle \( \mathfrak{S}(C M) \) is \( (\omega(\hat{A}), \omega(\check{B})^{op}) \)-bimodule. Consequently according to Statement \( \Box \) there exists an epimorphism \( \lambda : \omega(\hat{A}) \otimes \omega(\check{B}) \to \bar{C} \). Herewith \( \ker \lambda = l_{\omega(\hat{A}) \otimes \omega(\check{B})}(\mathfrak{S}(M)) \) is a left annihilator of module \( \mathfrak{S}(M) \) in ring \( \omega(\hat{A}) \otimes \omega(\check{B}) \). Because according to [25]
§12.4, Proposition b] $\omega(\hat{A}) \otimes \omega(\tilde{B}) \cong (\Delta \otimes \nabla)_{rs,rs}$ is a simple algebra and left module $\omega(\hat{A}) \otimes \omega(\tilde{B}) \mathcal{S}(CM)$ is non-zero then $\ker \lambda = \hat{0}$. Thus

$$\tilde{C} \cong (\Delta \otimes \nabla)_{rs,rs}. \quad (3.14)$$

Since $\tilde{C} \cong \tilde{Z}_{n,n}$ then Brauer classes of equivalence of $[\Delta \otimes \nabla]$ and $[\tilde{Z}]$ coincide i.e. classes $[\Delta]$ and $[\nabla]$ are \cite{25} §12.5, Proposition a] mutually inverse elements of group $\text{B}(\tilde{Z})$ and this equalities take place: $[\Delta]^{-1} = [\Delta^{op}] = [\nabla]$. Thereby \cite{25} §12.5, Proposition b(ii)] $\Delta^{op} \cong \nabla$.

Because of the (3.10) there exists an isomorphism $\mu : \omega(\hat{A}) \to \tilde{A}$. Since $\mathbf{Z}(\omega(\hat{A})) = \tilde{Z}$ then $\mathbf{Z}(\tilde{A}) = \mu(\tilde{Z}) \cong \tilde{Z}$. Consequently over an algebra $\tilde{A}$ may be established the structure of $\tilde{Z}$-module by have a put for all $\tilde{a} \in \tilde{A}, \tilde{z} \in \tilde{Z}$ $\tilde{z} \cdot \tilde{a} = \mu(\tilde{z}\mu^{-1}(\tilde{a}))$. Then $\tilde{A}$ is finite dimension central simple $\tilde{Z}$-algebra because it inherits properties of $\tilde{Z}$-algebra $\omega(\hat{A})$. Herewith basic algebras of $\tilde{A}$ and $\omega(\hat{A})$ are isomorphic to algebra with division $\Delta$.

Analogously because of (3.11) there exists an isomorphism $\nu : \omega(\tilde{B})^{op} \to \tilde{B}$. As above $\mathbf{Z}(\omega(\tilde{B})) \cong \mathbf{Z}(\omega(\tilde{B})^{op}) \cong \tilde{Z}$ what mean $\mathbf{Z}(\tilde{B}) = \nu(\tilde{Z}) \cong \tilde{Z}$. Structure of $\tilde{Z}$-module over $\tilde{B}$ is given by: for all $\tilde{b} \in \tilde{B}, \tilde{z} \in \tilde{Z}$ let’s put $\tilde{z} \cdot \tilde{b} = \nu(\tilde{z}\nu^{-1}(\tilde{b}))$. Now $\tilde{B}$ inherits properties of $\tilde{Z}$-algebra $\omega(\tilde{B})^{op}$ and becomes because of that finite dimensional central simple $\tilde{Z}$-algebra. Because basic algebras of $\tilde{B}$ and $\omega(\tilde{B})^{op}$ are isomorphic to algebra with division $\nabla^{op}$ then we have only to note that since $\Delta \cong \nabla^{op}$ basic algebras of $\tilde{Z}$-algebras $\tilde{A}$ and $\tilde{B}$ are isomorphic too.

I(d). Because of \cite{25} §12.4, Proposition b(iv)] an isomorphism takes place:

$$\Delta \otimes \nabla \cong \Delta \otimes \Delta^{op} \cong \tilde{Z}_{u,u}, \quad (3.15)$$

where $u = \dim_{\mathbf{Z}} \Delta$.

Hence according to (3.14) the equality takes place:

$$\dim_{\mathbf{Z}} \mathcal{S}(M) = r \cdot s \cdot \dim_{\mathbf{Z}} \Delta. \quad (3.16)$$

Now let’s calculate dimension of the vector space $\mathcal{S}(M)_{\mathbf{Z}}$ in other way.

Let $e_{i,j}, i, j = \overline{1,r}$, $f_{u,v}, u, v = \overline{1,s}$ be a full systems of matrix units of rings $\omega(\hat{A})$ and $\omega(\tilde{B})$ respectively. Since $\mathcal{S}(M)$ is $(\omega(\hat{A}), \omega(\tilde{B})^{op})$-bimodule then two-sided Pierce
decomposition of module $\mathfrak{S}(M)$ into direct sum of Abelian groups:

$$\mathfrak{S}(M) = \sum_{i=1}^{r} \oplus e_{i,i} \mathfrak{S}(M) f_{u,u}. \quad (3.17)$$

Every summand $e_{i,i} \mathfrak{S}(M) f_{u,u}, i = 1, r, u = 1, s,$ is a $(e_{i,i} \omega(\hat{A}) e_{i,i}, f_{u,u} \omega(\check{B})^{op} f_{u,u})$-bimodule. All these bimodules are pair-wise isomorphic and because

$$e_{i,i} \omega(\hat{A}) e_{i,i} \cong \Delta, \quad \omega(\hat{A}) \cong \Delta_{r,r},$$

$$f_{u,u} \omega(\check{B})^{op} f_{u,u} \cong \nabla^{op} \cong \Delta, \quad \omega(\check{B})^{op} \cong \Delta_{s,s},$$

the isomorphism takes place:

$$\omega(\hat{A}) \mathfrak{S}(M) \omega(\check{B})^{op} \cong \Delta_{r,s} \mathfrak{S}_{r,s} \Delta_{s,s}, \quad (3.18)$$

where $\mathfrak{S} = e_{1,1} \mathfrak{S}(M) f_{1,1}$.

From relation (3.18) equalities follow:

$$\dim \check{Z} \mathfrak{S}(M) = r \cdot s \cdot \dim \check{Z} \Delta \cdot \dim \mathfrak{S} = r \cdot s \cdot \dim \check{Z} \Delta \cdot \dim \mathfrak{S}_{\Delta}, \quad (3.19)$$

where $\dim \mathfrak{S}$ and $\dim \mathfrak{S}_{\Delta}$ are dimensions of left $\Delta \mathfrak{S}$ and right $\mathfrak{S}_{\Delta}$ vector spaces over the body $\Delta$.

Comparing (3.16) and (3.19) we find that

$$\dim \mathfrak{S} = \dim \mathfrak{S}_{\Delta} = 1. \quad (3.20)$$

Consequently

$$\Delta \mathfrak{S}_{\Delta} \cong \Delta \Delta_{\Delta} \quad (3.21)$$

is a quasi-Frobenius bimodule. Besides that from (3.21) follows an existence of isomorphism

$$\Delta_{r,s} \mathfrak{S}_{r,s} \Delta_{s,s} \cong \Delta_{r,s} \Delta_{r,s} \Delta_{s,s}, \quad (3.22)$$

which implies an existence of isomorphism

$$\omega(\hat{A}) \mathfrak{S}(M) \omega(\check{B})^{op} \cong \Delta_{r,s} \Delta_{r,s} \Delta_{s,s}. \quad (3.23)$$
Since \( \omega(\hat{A}) \cong \tilde{A}, \omega(\tilde{B})^{op} \cong (\tilde{B}^{op})^{op} = \tilde{B} \) then from existence of isomorphism \((3.23)\) follows the existence of isomorphism

\[
\tilde{A}\mathcal{S}(C\mathcal{M})_\tilde{B} \cong \Delta_{r,s},
\]

(3.24)

Statement \([10]\) shows that bimodule \(\Delta_{r,s} \Delta_{r,s} \Delta_{s,s}\) is a quasi-Frobenius. Consequently because of \((3.24)\) bimodule \(\tilde{A}\mathcal{S}(C\mathcal{M})_\tilde{B}\) is also quasi-Frobenius.

So point I of this Theorem is proven completely.

II. Now we can prove point II of this Theorem. Let \(\mathcal{S}(\hat{A}M) = \mathcal{S}(M_B) = \mathcal{S}(C\mathcal{M})\). As we already have established bimodule \(\tilde{A}\mathcal{S}(C\mathcal{M})_\tilde{B}\) is quasi-Frobenius. Hence because of the Statement \([8]\) bimodule \(\hat{A}M_B\) is also quasi-Frobenius.

Conversely if bimodule \(\hat{A}M_B\) is quasi-Frobenius then according to [2, Teorema 1] \(\mathcal{S}(\hat{A}M) = \mathcal{S}(M_B)\). From other side

\[
\mathcal{S}(\hat{A}M) = \mathcal{S}(\hat{A}M) = r_M(J(\hat{A})),
\]

\[
\mathcal{S}(M_B) = \mathcal{S}(\hat{B}M) = r_M(J(\hat{B})).
\]

Since \(J(\hat{A}), J(\hat{B}) \triangleleft J(C)\) then

\[
\mathcal{S}(M) = \mathcal{S}(\hat{A}M) = \mathcal{S}(M_B) > \mathcal{S}(C\mathcal{M}).
\]

Consequently quasi-Frobenius bimodule \(\tilde{A}\mathcal{S}(C\mathcal{M})_\tilde{B}\) is subbimodule of quasi-Frobenius bimodule \(\tilde{A}\mathcal{S}(C\mathcal{M})_\tilde{B}\). This inclusion with necessity have to be an equality i.e.

\[
\mathcal{S}(\hat{A}M) = \mathcal{S}(M_B) = \mathcal{S}(C\mathcal{M}).
\]

III. Since Brauer group \(B(\tilde{Z})\) of the field \(\tilde{Z}\) is trivial then \(\Delta \cong \nabla \cong \tilde{Z}\). Consequently \(\tilde{A}\mathcal{S}\Delta \cong \tilde{Z}\tilde{Z}\). Because of above and \((3.24)\) an isomorphism takes place:

\[
\tilde{A}\mathcal{S}(C\mathcal{M})_\tilde{B} \cong \tilde{Z}_{r,s} \tilde{Z}_{s,s}.
\]

(3.25)

Besides that from \((3.16)\) the equality follows: \(n = r \cdot s\).

Let’s show now that \(A \cong Z_{r,r}, B \cong Z_{s,s}\). Indeed rings \(A\) and \(B\) contain as subrings \(\phi(Z_{r,r}) \cong Z_{r,r}\) and \(\psi(Z_{s,s}) \cong Z_{s,s}\) respectively. Let’s denote by \(\tilde{\phi}(Z_{s,s})\) and \(\tilde{\psi}(Z_{s,s})\) images
of those subrings under converse mapping into $C$. Let’s denote by $\tilde{C}$ ring generated by $\hat{\phi}(Z_{r,r})$ and $\hat{\psi}(Z_{s,s})$ in $C$. According to Statement 3 ring $\tilde{C}$ is an epimorphic image of ring

$$\hat{\phi}(Z_{r,r}) \otimes Z_{s,s} \cong Z_{r,r} \otimes Z_{s,s},$$

i.e.

$$\tilde{C} \cong \hat{\phi}(Z_{r,r}) \otimes Z_{s,s} / I_{\hat{\phi}(Z_{r,r}) \otimes \hat{\psi}(Z_{s,s})}(M).$$

(3.26)

Since $C \cong Z_{n,n}$ and $n = rs$ then an isomorphism takes place:

$$C \cong Z_{r,r} \otimes Z_{s,s},$$

which implies isomorphisms:

$$CM \cong Z_{r,r} \otimes Z_{s,s} M \cong \hat{\phi}(Z_{r,r}) \otimes \hat{\psi}(Z_{s,s}) M.$$  

(3.27)

Because left module $CM$ is faithful and according to (3.27) left module

$$\hat{\phi}(Z_{r,r}) \otimes \hat{\psi}(Z_{s,s}) M$$

is also faithful. In view of isomorphism (3.26) and isomorphisms $\hat{\phi}(Z_{r,r}) \cong Z_{r,r}$, $\hat{\psi}(Z_{s,s}) \cong Z_{s,s}$ we conclude that $\tilde{C} \cong C$. Since $\tilde{C} = [\hat{\phi}(Z_{r,r}), \hat{\psi}(Z_{s,s})] < [\hat{A}, \hat{B}] = C$ then ring $\tilde{C}$ is embedded identically into ring $C$ and previous isomorphism is only an equality. Consequently $\hat{\phi}(Z_{r,r}) = \hat{A}$, $\hat{\psi}(Z_{s,s}) = \hat{B}$, where from we get isomorphisms of point III(a) of this Theorem.

Let’s now trace a chain of isomorphisms:

$$CM \cong A \otimes B^{op} M \cong AM_B.$$ 

From other side

$$CM \cong Z_{r,r} \otimes_{Z} Q^{(rs)} \cong Z_{r,r} \otimes Z_{s,s} Q^{(rs)} \cong Z_{r,r} \otimes_{Z} Q_{r,s} Z_{s,s}.$$ 

Here we have used a generalization of Proposition from §10.1 of [25]. So we have established an isomorphism of point III(b) of this Theorem.

Bimodule $Z_{r,r} \otimes_{Z} Q_{r,s} Z_{s,s}$ is quasi-Frobenius because of the Statement 10. Bimodule $AM_B$ is quasi-Frobenius because of the isomorphism of point III(b) of this Theorem.
4 Proof of the Theorem 6

For convenience let’s remember that the ring $S$ is called Galois-Eisenstein-Ore ring (or $GEO$-ring) if it is finite completely primary ring of principal ideals i.e. the ring $S$ contains unique (one-side) maximal ideal $\mathfrak{p}(S)$ and every one-side ideal of ring $S$ is principal. Commutative $GEO$-ring is called Galois-Eisenstein ring or $GE$-ring [18].

In arbitrary $GEO$-ring $S$ of characteristic $p^d$ contains Galois subring $R = GR(p^d, r)$ [13, 26], $q = p^r$. All such rings are conjugated in $S$. Ring $R$ is called coefficients ring of $S$.

And also let’s repeat conditions of Theorem 6:

Theorem 12. (a) For arbitrary $GEO$-ring $S$ bimodule $sSs$ is a quasi-Frobenius.
(b) Let $S$ be a $GEO$-ring with a coefficients ring $R = GR(p^d, r)$ and in addition $d > 1$ and $(r, p) = 1$. Then bimodule of translations $cS_Z$ of quasi-Frobenius $S$-bimodule $S$ is a quasi-Frobenius bimodule if and only if $S$ is a $GE$-ring.

Let’s denote by $n(S)$ the nilpotency index of ideal $\mathfrak{p}(S)$. The field $S/\mathfrak{p}(S)$ we denote by $\bar{S}$ and an image of $s \in S$ in $\bar{S}$ by $\bar{s}$. Let $\bar{S} = GF(q)$.

For every finite completely primary ring $S$ following statements are equivalent [18] Theorem 1.1, points I, II, VI, VII, VIII respectively [):

(a) $S$ is a $GEO$-ring;
(b) $|S| = q^{n(S)}$;
(c) Every one-side ideal of $S$ is a degree of $\mathfrak{p}(S)$;
(d) For every $\pi_t \in \mathfrak{p}(S)^t \setminus \mathfrak{p}(S)^{t+1}$, $t = 0, n(S) - 1$, this equalities take place: $\mathfrak{p}(S)^t = \pi_t S = S \pi_t$ (if $t = 0$ then we denote $\mathfrak{p}(S)^0 = S$).
(e) If $\gamma : \bar{S} \to S$ is a mapping with property

\[ \gamma(\bar{s}) = \bar{s} \quad \text{for all} \quad s \in S, \quad (4.1) \]

and $\pi_t \in \mathfrak{p}(S)^t \setminus \mathfrak{p}(S)^{t+1}$, $t = 0, n(S) - 1$, then an arbitrary element $s \in S$ is uniquely represented in the form: $s = \sum_{t=0}^{n(S)-1} s_t \pi_t$, where $s_t \in \gamma(\bar{S})$, $t = 0, n(S) - 1$, and also in the form: $s = \sum_{t=0}^{n(S)-1} \pi_t s'_t$, where $s'_t \in \gamma(\bar{S})$, $t = 0, n(S) - 1$. 

Let’s view $S$ as left unitary $R$-module. System of elements $s_1, \ldots, s_t \in RS$ is called [18, §2] free if there is no exists a non-trivial linear relation between these elements over $R$ and is called irreducible if no one of these elements is a linear combination over $R$ of others. Irreducible generating system of $RS$ is called basis of $RS$. By dimension $\dim RS$ is called a number of elements in basis of $RS$ which is equal to dimension of vector space $S/\mathfrak{p}(R)S$ over field $\bar{R}$.

Let $T(RS) = \{ s \in S \mid \mathfrak{p}(R)^{n(R)-1}s = 0 \}$ be a set of elements of $RS$ annihilated by some non-zero element from $R$. Then [18, Proposition 2.1] the number of elements in maximal free subsystem of $RS$ satisfies to equality: $\text{rang} RS = \dim \bar{R}(S/T(RS))$.

It is known [18, Corollary 2.3] that the equality takes place: $\dim RS = \dim SR$, $\text{rang} RS = \text{rang} SR$, and besides that an arbitrary basis of $RS$ is also a basis of $SR$ and conversely. Because of that we can speak simply about rank, dimension and basis of $S$ over $R$.

For some natural $e = e(S|R)$ [18, Theorem 1.1] these equalities take place:

$$\mathfrak{p}(R) \cdot S = S \cdot \mathfrak{p}(R) = S \cdot \mathfrak{p}(R)$$

and

$$\mathfrak{p}(R) \cdot S = \mathfrak{p}(S)^e.$$  

Besides that $n(S) = (n(R) - 1)e + \rho$ where $\rho = \rho(S|R)$ satisfies to inequalities: $1 \leq \rho \leq e$. Herewith [18, Proposition 5.1] $S$ is a $R$-bimodule of the rank $\rho = \rho(S|R)$ and dimension $e = e(S|R)$ whereas $T(S) = \mathfrak{p}(S)^\rho$.

Let $\sigma \in \text{Aut}(R)$. By the ring of Ore polynomials $R[x, \sigma]$ is called ring of polynomials with usual addition and multiplication established by the rule: $x \cdot r = r^\sigma \cdot x$, $r \in R$. If $t = t(\sigma)$ is an order of $\sigma$, then center $\mathbf{Z}(R[x, \sigma])$ of ring $R[x, \sigma]$ contains of and only of polynomials of the form $c_0 + c_1x^t + \cdots + c_tm^x$ where $c_0, \ldots, c_m \in R_\sigma = \{ r \in R \mid r^\sigma = r \}$.

Polynomial $x^t + c_{t-1}x^{t-1} + \cdots + c_0 \in R[x, \sigma]$ is called Eisenstein polynomial if $c_i \in \mathfrak{p}(R)$ for $i = 0, \ldots, t-1$ and $c_0 \not\in \mathfrak{p}(R)^2$ if $n(R) > 1$. Eisenstein polynomial of the form

$$c(x) = x^{tm} + c_{t(m-1)}x^{(m-1)} + \cdots + c_0$$ (4.2)

is called special if either

$$c(x) \in \mathbf{Z}(R[x, \sigma]),$$ (4.3)
either for some $a < m$

$$c(x) - c_{ta}x^{ta} \in \mathbb{Z}(R[x, \sigma]), \quad c_{ta}^\sigma - c_{ta} \in p(R)^{n(R)-1} \setminus \{0\}. \quad (4.4)$$

It is known [18, Theorem 5.2 II] that if $R$ is a ring of coefficients of $GEO$-ring $S$, $n(R) > 1$, $e = e(S|R)$, $\rho = \rho(S|R)$, then there exist an automorphism $\sigma \in \text{Aut}(R)$ such that $t = t(\sigma)$ divides $e$, $e = t \cdot m$, and special Eisenstein polynomial of the form (4.2) with the property:

$$S \cong R[x, \sigma]/I, \quad (4.5)$$

where $I = c(x)R[x, \sigma] + x^\rho p(R)^{n(R)-1}[x, \sigma]$. Herewith (4.4) is fulfilled only if $\rho = ta + 1$.

Automorphism $\sigma$ in (4.5) is uniquely determined by the ring $S$ and do not depends on the choice of coefficient ring $R$ [18, Proposition 5.5].

Let’s remember that $GEO$-ring $S$ is a Galois–Eisenstein ring ($GE$-ring) iff $t = t(\sigma) = 1$ i.e. Galois–Eisenstein ring is nothing but commutative Galois–Eisenstein–Ore ring [17].

Besides that [18, Proposition 5.7] if $\pi$ is a $\sigma$-element from $p(S) \setminus p(S)^2$ (i.e. for every $r \in R$ this equality satisfies: $\pi r = r^\sigma \pi$) then:

I) The centralizer $C(R)$ of ring $R$ in $S$ is a $GE$-ring [17]

$$C(R) = R[\pi^t] = R + R\pi^t + \cdots + R^{t(m-1)}, \quad (4.6)$$

whereas $e(C(R)|R) = m$, $\rho(C(R)|R) = \rho'$, $\rho' = \left\lceil \frac{\ell}{t} \right\rceil$, and

$$C(R) \cong R[y]/z(y)R[y] + y^{\rho'} p(R)^{n(R)-1}[y], \quad (4.7)$$

whereas $c(x) = z(x^t)$.

II) The center $\mathbb{Z}(S)$ is equal to

$$C(R)_\sigma = R_\sigma + R_\sigma \pi^t + \cdots + R_\sigma \pi^{t(m-1)}, \quad (4.8)$$

whereas $\rho \not\equiv 1(\text{mod } t)$ and is equal to

$$C(R)_\sigma + p(S)^{n(S)-1}$$

in other case.

It is known [7, §13] that arbitrary left-side and right-side together Artinian ring $T$ with identity is quasi-Frobenius if and only if $T$-bimodule $T$ is a quasi-Frobenius.
Proof of the Theorem

(a) Let’s use the Statement 8. These equalities take place:

\[ \mathcal{G}(S) = r_s(p(S)) = p(S)^{n(S)-1} = (\pi S)^{(n(R)-1)e^2 + \rho - 1} = \pi^{\rho - 1} \cdot (\pi S)^{(n(R)-1)e} = \pi^{\rho - 1} \cdot p(R)^{n(R)-1}, \]

whereas \( \pi \in p(S) \setminus p(S)^2 \). Analogously

\[ \mathcal{G}(S) = l_s(p(S)) = p(S)^{(n(R)-1)e^2 + \rho - 1} = \pi^{\rho - 1} \cdot p(R)^{n(R)-1}. \]

This way \( \mathcal{G}(S) = \mathcal{G}(S) = \mathcal{G}(S) \). Besides that \( \dim \mathcal{G}(S) = \dim_s \mathcal{G}(S) = 1 \). Hence according to point (4) of Theorem 7 bimodule \( \mathcal{G}(S) \) is quasi-Frobenius. Thus bimodule \( S ) \) and ring \( S \) is also quasi-Frobenius.

(b) Let’s note that \( n(R) = d \). Because \( d > 1 \) the ring \( S \) has a form:

\[ S \cong R[x, \sigma]/I, \]

where \( \rho = \rho(S|R), e = e(S|R), t = t(\sigma), \sigma \in \text{Aut}(R), I = c(x)R[x, \sigma] + x^np(R)^{n(R)-1}[x, \sigma], c(x) = x^m + c_{t(m-1)}x^{t(m-1)} + \cdots + c_0 \) is a special Eisenstein polynomial, whereas condition

\[ c(x) - c_{ta}x^{ta} \in \mathbb{Z}(R[x, \sigma]), \quad c_{ta}^n - c_{ta} \in p(R)^{n(R)-1} \setminus \{0\} \]

is satisfied only if \( \rho = ta + 1 \).

Besides that by \( r = \log_p q \) we have \( t|r \). Since under condition \( (r, p) = 1 \) we have then \( (t, p) = 1 \).

Let \( \mathcal{R} \) be a ring of translations of \( R \)-bimodule \( S \), \( C \) be a ring of translations of \( S \)-bimodule \( S \).

Let’s prove that ring \( \mathcal{R} \) is isomorphic to foreign direct sum of \( \min\{t, \rho\} \) copies of ring \( R = GR(p^d, r) \) and \( \max\{0, t - \rho\} \) copies of ring \( \hat{R} = R/p(R)^{n(R)-1} = GR(p^{d-1}, r) \).

Let’s denote by \( \epsilon_l \in \mathcal{R}, l = \frac{0, t-1}{t-1} \), endomorphisms of \( R \)-bimodule \( S \) of the form:

\[ \epsilon_l : S \to S^{(l)} = \sum_{j=0}^{e-1} \delta_{t,j} \quad (\text{mod } t) R \pi^j, \]

operating according to the rule:

\[ \epsilon_l \left( \sum_{j=0}^{e-1} r_j \pi^j \right) = \sum_{j=0}^{e-1} r_j \pi^j, \]
where $\delta_{\lambda,\mu}$ is a Kronecker symbol.

Since equalities take place: $S = \sum_{j=0}^{e-1} R\pi^j$ and $c(\pi) = 0$ where $\pi$ is a $\sigma$-element from $p(S) \setminus p(S)^2$ we have:

$$\hat{S} = \sum_{j=0}^{e-1} \hat{R}\pi^j, \quad \tilde{S} = \sum_{j=0}^{e-1} \tilde{R}\pi^j. \quad (4.14)$$

Hence

$$C = [\hat{S}, \tilde{S}] = \left[ \sum_{i=0}^{e-1} \hat{R}\pi^i, \sum_{j=0}^{e-1} \tilde{R}\pi^j \right] = \sum_{i,j=0, e-1} \left( \sum_{r_i, r_j \in R} \hat{r}_i\tilde{r}_j \right) \pi^i\pi^j. \quad (4.15)$$

It is evident that ring of translations $\mathcal{R}$ of $R$-bimodule $S$ may be represented in the form:

$$\mathcal{R} = \sum_{r_1, r_2 \in R} \hat{r}_1\tilde{r}_2. \quad (4.16)$$

Let’s operate the structure of this set. To do this let’s describe previously the ring $\hat{R}$.

Let $r \in R$ and $s = \sum_{l=0, e-1} r_\lambda \pi^\lambda \in S$. Then

$$\hat{r}(s) = s \cdot r = \sum_{l=0, e-1} r_\lambda r_\lambda \pi^\lambda = \sum_{l=0, e-1} (r_\lambda \pi^\lambda) = \sum_{l=0, e-1} \hat{r}_l r_\lambda \pi^\lambda \equiv 0, t - 1. \quad (4.17)$$

Define mapping $\phi_l : R \rightarrow \text{End}(S)$, $l = 0, t - 1$, by the rule: for every $r \in R$ put

$$\phi_l(r) \left( \sum_{j=0, t-1} r_j \pi^j \right) = \sum_{j=0, t-1} \delta_{l,j(\text{mod } t)} \cdot \left( r_\lambda \pi^\lambda \right) \left( r_j \pi^j \right), \quad (4.18)$$

i.e.

$$\phi_l(r)(S) = \left( \hat{r}_\lambda \pi^\lambda \right)(S^{(l)}). \quad (4.19)$$

Thus

$$\hat{r}(s) = \sum_{l=0, t-1} \phi_l(r)(s). \quad (4.20)$$

Let’s show that the family of set $\phi_l(R)$, $l = 0, t - 1$ are pair-wise orthogonal subrings of the ring $\text{End}(S)$. To do this, it suffices to prove that in fact there are elements $\epsilon_l \in \mathcal{R}$, $l = 0, t - 1$, with pointed in equality $\mathcal{R}(l)$ property which it is convenient to represent in the form

$$\epsilon_l(S^{(l)}) = \delta_{l,j} S^{(l)}, \quad l, j = 0, t - 1. \quad (4.21)$$

It is evident that an arbitrary mapping $\beta \in \mathcal{R}$ may be represented in the following form:

$$\beta(S) = \sum_{l=0}^{t-1} \hat{\beta}_l(S^{(l)}). \quad (4.22)$$
We pose the mapping $\beta$ into compliance a vector $\vec{\beta} = (\beta_0, \ldots, \beta_{t-1})$, $\beta_l \in R, l = 0, t - 1$, and determine the action of this vector on the set $S$ in this way:

$$\vec{\beta}(S) = (\beta_0, \ldots, \beta_{t-1}) \cdot \begin{pmatrix} S^{(0)} \\ \vdots \\ S^{(t-1)} \end{pmatrix} = \sum_{l=0}^{t-1} \beta_l(S^{(l)}) = \beta(S). \quad (4.22)$$

Let $\beta, \gamma \in R$, $r \in R$. Then mapping $\zeta = \hat{r} \cdot \beta \in R$ is represented by vector $\vec{\zeta} = r \cdot \vec{\beta}$, and mapping $\xi = \beta + \hat{r} \cdot \gamma \in R$ is represented by vector $\vec{\xi} = \vec{\beta} + r \cdot \vec{\gamma}$. Converse is also true.

Let’s remember that for $r \in R$

$$\vec{\zeta} = (r, r^\sigma, \ldots, r^{\sigma t-1}). \quad (4.23)$$

Let’s denote this vector as $\vec{\beta}(r)$ and consider the matrix

$$C(r) = \begin{pmatrix} \vec{\beta}(r) \\ \vec{\beta}(r^\sigma) \\ \vdots \\ \vec{\beta}(r^{\sigma t-1}) \end{pmatrix} = \begin{pmatrix} r, & r^\sigma, & \ldots, & r^{\sigma t-1} \\ r^\sigma, & r^{2 \sigma}, & \ldots, & r \\ \vdots & \vdots & \ddots & \vdots \\ r^{(t-1) \sigma}, & r, & \ldots, & r^{(t-2) \sigma} \end{pmatrix}. \quad (4.24)$$

Since $(t, p) = 1$ over field $\bar{R} = GF(q)$ there are exactly $t$ different roots from unit of degree $t$ namely: $\aleph_1, \ldots, \aleph_t$. Let $f_r(x) = \bar{r} + \bar{r}^\sigma \cdot x + \cdots + \bar{r}^{\sigma t-1} \cdot x^{t-1} \in \bar{R}[x]$. Then there is an equality:

$$\det C(r) = (-1)^{\frac{(t-1)(t-2)}{2}} \cdot \prod_{j=1}^{t} f_r(\aleph_j).$$

When $r \in R^* \setminus R_\sigma$ the right-hand side of previous equality differs from zero. Since with $r \in R^* \setminus R_\sigma$ and $(t, p) = 1$ the matrix $C(r)$ is invertible. Hence by equivalent transformations of rows matrix $C(r)$ is possible to lead to the identity matrix $E_{t \times t}$. It remains to remark that for every $l \in 0, t - 1$ row $E_{l+1}$ is a vector of desired mapping $\epsilon_l \in R$.

It is easy to see that there are equalities:

$$\phi_l(r) = \hat{r} \cdot \epsilon_l = \hat{r}^{\sigma l} \cdot \epsilon_l, \quad l = 0, t - 1, \quad (4.25)$$

from which it follows that $\phi_l(R), l = 0, t - 1$, are pair-wise orthogonal subrings of ring $R$.  


Besides that now the equality is evident:

\[ R = \sum_{l=0}^{t-1} \phi_l(R). \] (4.26)

Let’s describe now rings \( \phi_l(R) \), \( l = 0, t - 1 \). To do this let’s note that for arbitrary \( r \in R \) \( \phi_l(r) = 0 \) if and only if \( r^{t-1} \cdot \pi^l = 0 \), i.e. \( r^{t-1} \cdot \pi^l \in \mathfrak{p}(S)^{n(\mathfrak{p})} \), whereas \( n(S) = e(n(R) - 1) + \rho, 0 \leq \rho \leq e - 1 \). Let \( r \in \mathfrak{p}(R)^\kappa \). Then \( r^{t-1} \cdot \pi^l \in \mathfrak{p}(S)^{\kappa + l} \) and equality \( r^{t-1} \cdot \pi^l = 0 \) is achieved if and only if \( e\kappa + l \geq e(n(R) - 1) + \rho, \) i.e. \( l \in \overline{\rho, e - 1} \), \( \kappa = n(R) - 1 \). Let’s note now that \( t \leq e \). Hence these isomorphisms take place:

\[ \phi_l(R) \cong R, l = 0, \min\{t, \rho\} - 1, \] \[ \phi_l(R) \cong R/\mathfrak{p}(R)^{n(R)-1} = GR(q^{d-1}, p^{d-1}), l = \overline{\rho, t - 1}. \] (4.27) (4.28)

Let’s describe now the ring \( \bar{C} = C/J(C) \). Let’s note that \( C = \mathcal{R}[\hat{\pi}, \check{\pi}] \). Since \( \hat{\pi} \) and \( \check{\pi} \) are nil-potent elements of ring \( C \) then \( \hat{\pi}, \check{\pi} \in J(C) \). Consequently \( \bar{C} = C/J(C) \cong \sum_{l=0}^{t-1} \overline{\phi_l(R)} \) is isomorphic to foreign direct sum of \( t \) copies of the field \( \bar{R} = GF(q) \).

Immediately from the definitions it follows that for arbitrary GEO-ring \( S \) common center \( Z \) of rings \( \hat{S} \) and \( \check{S} \) is equal to \( Z(S) \).

Let’s suppose now that bimodule \( cSZ \) is quasi-Frobenius. Then since \( Z \) is local Artinian ring we can apply results of Theorem 5. In particular according to point I(a) of Theorem 5 the ring \( C \) is isomorphic to \( Z_{n,n} \) whereas \( n = \dim_Z \mathfrak{S}(S) \). Let’s remark that \( \bar{Z} = \overline{R_\sigma} \) and \( \dim_Z \mathfrak{S}(S) = t = \text{ord} \sigma \).

Thus ring \( \bar{C} = C/J(C) \cong \sum_{l=0}^{t-1} \overline{\phi_l(R)} \) is isomorphic to foreign direct sum of \( t \) copies of the field \( \bar{R} = GF(q) \) and has to be isomorphic to ring of \( t \times t \)-matrices over ring \( R_\sigma \).

But the ring \( \bar{C} \) is isomorphic to matrix ring only if \( t = 1 \) i.e. if \( S \) is a GE-ring.

Converse implication is evident because firstly an arbitrary GE-ring is quasi-Frobenius and secondly for arbitrary commutative ring \( S \) ring of translations of \( S \)-bimodule \( S \) and common center of \( \hat{S} \) and \( \check{S} \) relatively to \( S \) coincide with \( S \). \( \square \)

References

[1] Atiyah, M. F.; Macdonald, I. G. , Introduction to commutative algebra, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., pp. ix+128 (1969)
[2] Azumaya G. A duality theory for injective modules (Theory of quasi-Frobenius modules) // Amer. J. Math.—1959—v.81—N1— p.249-278

[3] Eisenbud D. Subrings of Artinian and Noetherian rings // Math.Ann.—1970 —v.185— p.247

[4] Faith, C., Algebra: Rings, Modules, and Categories, I, XXII +565 pgs., Springer-Verlag, Grundlehren Math. Wiss Bd 190, Berlin, 1973.

[5] Goltvanitsa M.A., Zaitsev S.N., Nechaev A.A. Skew linear recurring sequences of maximal period over galois rings // в журнале Journal of Mathematical Sciences, издательство Plenum Publishers (United States), том 187, № 2, с. 115-128 (2012)

[6] Jacobson N., Structure of Rings, AMS (1956)

[7] Kasch, F., Moduln und ringe, B. G. TEUBNER GmbH, Stuttgart (1978)

[8] Kuzmin A.S., Kurakin V.L., Mikhailov A.V., Nechaev A.A. Linear recurrences over rings and modules. // J. Math. Science (Contemporary Math. and its Appl., Thematic surveys)—1995—v.76—N6—p.2793-2915

[9] Kurakin V.L., Mikhailov A.V., Nechaev A.A. Polylinear recurring sequences over a bimodule // в сборнике Krob, Daniel (ed.) et al., Formal power series and algebraic combinatorics. Proceedings of the 12th international conference, FPSAC’00, Moscow, Russia, June 26-30, 2000

[10] Kurakin V.L., Mikhailov A.V., Nechaev A.A., Tyupyschev V.N. Linear and polylinear recurring sequences over an abelian groups and modules // J.Math.Sci.(Contemporary Math. and its Appl., Thematic surveys)— vol.102—6—2000—P.4598–4626

[11] V. L. Kurakin , A. A. Nechaev Quasi-Frobenius bimodules of functions on a semigroup // Communications in Algebra, Volume 29, Issue 9, pages 4079-4094 (2001)

[12] Lambek,Joachim, Lectures on rings and modules, Blaisdell Pub. Co., 183pp. (1966)

[13] McDonald C. Finite rings with identity // New York: Marcel Dekker—1974—495p.
[14] Mikhalev A.V., Nechaev A.A. Linear recurring sequences over modules // Acta Applicandae Mathematicae, Kluwer Academic Publishers (Netherlands), v. 42, 2, pp. 161-202 (1996)

[15] Muller B.J. Linear compactness and Morita duality // J. Algebra—16(1970)—p.464-467

[16] Nagata M. Local rings // Int. Tracts in Pure and Appl. Math.—13—1962—New York

[17] Nechaev A.A. On the structure of finite commutative rings with an identity, Mathematical Notes, Consultants Bureau (United States), v. 10, pp. 840-845 (1971)

[18] Nechaev A.A., Finite principal ideal rings, Math. USSR, Sb. 20(1973), v. 20, pp. 364-382 (1974)

[19] Nechaev A.A. Linear recurrent sequences over quasi-Frobenius modules // Russian Mathematical Surveys, Turpion - Moscow Ltd. (United Kingdom), v. 48, 3, pp. 209-210 (1993)

[20] Nechaev A.A. Linear codes and multilinear recurrences over finite rings and quasi-Frobenius modules // Doklady Mathematics, Maik Nauka/Interperiodica Publishing (Russian Federation), v. 52, 3, pp. 404-407 (1995)

[21] Nechaev A.A. Finite quasi-Frobenius modules. Applications to codes and linear recurrences. (in Russian) // Fundamental’naya i Prikladnaya Matematika, v. 1, 1, pp. 229-254 (1995)

[22] Nechaev A. A Polynomial recurring sequences over modules and quasi-Frobenius modules. (English) // [CA] Fong, Y. (ed.) et al., First international Tainan-Moscow algebra workshop. Proceedings of the international conference, Tainan, Taiwan, July 23–August 22, 1994, Berlin: de Gruyter, pp. 283-298 (1996)

[23] Nechaev A.A., Tsypyschev V.N., Polylinear recurrences over bimodule, Math. Method and Appl., Proceedings Third Mathematical Readings of Moscow State Social Univ., Jan. 23–29, 1995, pp. 95-100
[24] Nechaev A.A., Tsypyshev V.N. Artinian bimodule with quasi-Frobenius canonical bimodule // Proc. Int. Workshop devoted to 70-th anniversary of scientific algebraic workshop of Moscow State University founded by O.J. Smidt in 1930. Mech-math. dpt. of Moscow St. Univ., November 13-16, 2000, Proceedings, pp.39-40 (2000)

[25] Pierce, Richard S., Associative algebras, Graduate Texts in Mathematics 88, New York: Springer-Verlag, pp. xii+436, ISBN 0-387-90693-2 (1982)

[26] Radghavendran R. A class of finite rings // Compositio Math.—1970— v.22—N1— p.49-57

[27] Tsypyshev V. N. Matrix linear congruent generator over a Galois ring of odd characteristic // Proceedings of the 5th Int. Conf. "Algebra and number theory: modern problems and applications", Tula State Pedagogic Univ., Tula, 2003, p 233-237 (In Russian) MathSciNet: 2035586