THE DIFFERENCE $\lambda$-CALCULUS:
A LANGUAGE FOR DIFFERENCE CATEGORIES

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ABSTRACT. Cartesian difference categories are a recent generalisation of Cartesian differential categories which introduce a notion of “infinitesimal” arrows satisfying an analogue of the Kock-Lawvere axiom, with the axioms of a Cartesian differential category being satisfied only “up to an infinitesimal perturbation”. In this work, we construct a simply-typed calculus in the spirit of the differential $\lambda$-calculus equipped with syntactic “infinitesimals” and show how its models correspond to difference $\lambda$-categories, a family of Cartesian difference categories equipped with suitably well-behaved exponentials.

1. Introduction

A recent series of works introduced the concept of change actions and differential maps between them [5, 4] in order to study settings equipped with derivative-like operations. Although the motivating example was the eminently practical field of incremental computation, similar structures appear in more abstract settings such as the calculus of finite differences and Cartesian differential categories.

Of particular interest are Cartesian difference categories [2], a well-behaved class of change action models [4] that are much closer to the strong axioms of a Cartesian differential category [6] while remaining general enough for interpreting discrete calculus. A Cartesian difference category is a left additive category equipped with an “infinitesimal extension”, an operation that sends an arrow $f$ to an arrow $\varepsilon(f)$ which should be understood as $f$ being multiplied by an “infinitesimal” element – infinitesimal in the sense that it verifies the Kock-Lawvere axiom from synthetic differential geometry (we refer the reader to [12] for an introduction to SDG).

The interest of Cartesian differential categories is in part motivated by the fact that they provide models for the differential $\lambda$-calculus [11, 9], which extends the $\lambda$-calculus with

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linear combinations of terms and an operator that differentiates arbitrary λ-abstractions. The claim that differentiation in the differential λ-calculus corresponds to the standard, “analytic” notion is then justified by its interpretation in (a well-behaved class of) Cartesian differential categories \[7, 14\].

It is reasonable to ask, then, whether there is a similar calculus that captures the behavior of derivatives in difference categories – especially since, as it has been shown, these subsume differential categories. The issue is far from trivial, as many of the properties of the differential λ-calculus crucially hinge on derivatives being linear. Through this work we provide an affirmative answer to this question by defining untyped and simply-typed variants for a simple calculus which extends the differential λ-calculus with a notion of derivative more suitable to the Cartesian difference setting.

2. Cartesian Difference Categories

The theory of Cartesian difference categories is developed and discussed at length in \[2, 3\], but we present here the main definitions and results which we will use throughout the paper, referring the reader to \[3\] for the proofs.

**Definition 2.1.** A Cartesian left additive category (\[6, Definition 1.1.1\]) \(\mathcal{C}\) is a Cartesian category where every hom-set \(\mathcal{C}[A, B]\) is endowed with the structure of a commutative monoid \((\mathcal{C}[A, B], +, 0)\) such that \(0 \circ f = 0, (f + g) \circ h = (f \circ h) + (g \circ h)\) and \((f_1 + f_2) + (g_1, g_2) = (f_1 + g_1, f_2 + g_2)\).

An infinitesimal extension (\[2, Definition 8\]) in a Cartesian left additive category \(\mathcal{C}\) is a choice of a monoid homomorphism \(\varepsilon : \mathcal{C}[A, B] \to \mathcal{C}[A, B]\) for every hom-set in \(\mathcal{C}\). That is, \(\varepsilon(f + g) = \varepsilon(f) + \varepsilon(g)\) and \(\varepsilon(0) = 0\). Furthermore, we require that \(\varepsilon\) be compatible with the Cartesian structure, in the sense that \(\varepsilon((f, g)) = (\varepsilon(f), \varepsilon(g))\).

**Definition 2.2.** A Cartesian difference category (\[2, Definition 9\]) is a Cartesian left additive category with an infinitesimal extension \(\varepsilon\) which is equipped with a difference combinator \(\partial[\cdot]\) of the form:

\[
f : A \to B \\
\partial[f] : A \times A \to B
\]
satisfying the following coherence conditions (writing \(\partial^2[f]\) for \(\partial[\partial[f]]\)):

- **[C∂.0]** \(\partial \circ (x + \varepsilon(u)) = \partial x + \varepsilon(\partial f \circ \langle x, u \rangle)\)
- **[C∂.1]** \(\partial [f + g] = \partial [f] + \partial [g], \partial [0] = 0\) and \(\partial [\varepsilon(f)] = \varepsilon(\partial [f])\)
- **[C∂.2]** \(\partial \circ \langle x, u + v \rangle = \partial \circ \langle x, u \rangle + \partial \circ \langle x + \varepsilon(u), v \rangle\) and \(\partial \circ \langle x, 0 \rangle = 0\)
- **[C∂.3]** \(\partial \circ [\text{id}_A] = \pi_2\) and \(\partial \circ [\pi_1] = \pi_1 \circ \pi_2\) and \(\partial \circ [\pi_2] = \pi_2 \circ \pi_2\)
- **[C∂.4]** \(\partial \circ [\langle f, g \rangle] = \langle \partial \circ f, \partial \circ g \rangle\) and \(\partial \circ [\text{id}_A] = !_{A \times A}\)
- **[C∂.5]** \(\partial \circ [\text{id}_A] = \text{id} \circ [\langle f \circ \pi_1, \partial \circ f \rangle]\)
- **[C∂.6]** \(\partial^2 \circ [\langle x, u \rangle, \langle 0, v \rangle] = \partial \circ [\langle x + \varepsilon(u), v \rangle]\)
- **[C∂.7]** \(\partial \circ [\langle x, u \rangle, \langle v, 0 \rangle] = \partial^2 \circ [\langle x, v \rangle, \langle u, 0 \rangle]\)

As noted in \[2\], the axioms in a Cartesian differential category (see e.g. \([C∂.1–7]\) in \[6\]) correspond to the analogous axioms of the Cartesian difference operator, modulo certain “infinitesimal” terms, i.e. terms of the form \(\varepsilon(f)\). We state here the following two properties, whose proofs can be found in \[3\].

**Lemma 2.3.** Given any map \(f : A \to B\) in a Cartesian difference category \(\mathcal{C}\), its derivative \(\partial[f]\) satisfies the following equations:
i. \( \partial [f] \circ \langle x, \varepsilon(u) \rangle = \varepsilon(\partial [f]) \circ \langle x, u \rangle \)

ii. \( \varepsilon(\partial^2 [f]) \circ \langle \langle x, u \rangle, \langle v, 0 \rangle \rangle = \varepsilon^2(\partial^2 [f]) \circ \langle \langle x, u \rangle, \langle v, 0 \rangle \rangle \)

3. Difference \( \lambda \)-Categories

In order to give a semantics for the differential \( \lambda \)-calculus, it does not suffice to ask for a Cartesian differential category equipped with exponentials – the exponential structure has to be compatible with both the additive and the differential structure, in the sense of [7, Definition 4.4]. For difference categories we will require an identical equation, together with a condition requiring higher-order functions to respect the infinitesimal extension.

**Definition 3.1.** We remind the reader that a Cartesian left additive category is *Cartesian closed left additive* ([7, Definition 4.2]) whenever it is Cartesian closed and satisfies \( \Lambda(f+g) = \Lambda(f) + \Lambda(g), \Lambda(0) = 0 \).

A Cartesian difference category \( C \) is a *difference \( \lambda \)-category* if it Cartesian closed left additive and satisfies the following additional axioms:

[\( \partial \lambda.1 \)] \( \partial [\Lambda(f)] = \Lambda(\partial [f]) \circ ((\pi_1 \times \text{id}), (\pi_2 \times 0)) \)

[\( \partial \lambda.2 \)] \( \Lambda(\varepsilon(f)) = \varepsilon(\Lambda(f)) \)

Equivalently, let \( \text{sw} \) denote the map \( \langle \langle \pi_{11}, \pi_2 \rangle, \pi_{21} \rangle : (A \times B) \times C \to (A \times C) \times B \). Then the condition [\( \partial \lambda.1 \)] can be written in terms of \( \text{sw} \) as:

\[
\partial [\Lambda(f)] := \Lambda(\partial [f]) \circ (\text{id} \times (\text{id}, 0)) \circ \text{sw}
\]

Axiom [\( \partial \lambda.1 \)] is identical to its differential analogue [7, Definition 4.4], and it follows the same broad intuition. Given a map \( f : A \times B \to C \), we usually understand the composite \( \partial [f] \circ (\text{id}_{A \times B} \times (\text{id}_{A} \times 0_B)) : (A \times B) \times A \to C \) as a partial derivative of \( f \) with respect to its first argument. Hence, just as it was with differential \( \lambda \)-categories, axiom [\( \partial \lambda.1 \)] states that the derivative of a curried function is precisely the derivative of the uncurried function with respect to its first argument.

**Example 3.2.** Let \( C \) be a differential \( \lambda \)-category. Then the trivial Cartesian difference category obtained by setting \( \varepsilon(f) = 0 \) (as in [2, Proposition 1]) is a difference \( \lambda \)-category. Furthermore, the Kleisli category \( C_T \) induced by its tangent bundle monad (as in [2, Proposition 6]) is also a difference \( \lambda \)-category.

**Example 3.3.** The category \( \overline{\text{Ab}} \) ([2, Section 5.2]), which has Abelian groups as objects and arbitrary functions between their carrier sets as morphisms, is a difference \( \lambda \)-category with infinitesimal extension \( \varepsilon(f) = f \) and difference combinator \( \partial [f] (x, u) = f(x + u) - f(x) \). Given groups \( G, H \), the exponential \( G \Rightarrow H \) is the set of (set-theoretic) functions from \( G \) into \( H \), endowed with the group structure of \( H \) lifted pointwise (that is, \( (f+g)(x) = f(x) + g(x) \)). Evidently the exponential respects the monoidal structure and the infinitesimal extension. We check that it also verifies axiom [\( \partial \lambda.1 \)]:

\[
\partial [\Lambda(f)] (x, u)(y) = \Lambda(f)(x + u)(y) - \Lambda(f)(x)(y) \\
= f(x + u, y) - f(x, y) \\
= \Lambda(\partial [f] \circ (\text{id} \times (\text{id}, 0)) \circ \text{sw})(x, u)(y)
\]

A central property of differential \( \lambda \)-categories is a deep correspondence between differentiation and the evaluation map. As one would expect, the partial derivative of the evaluation map gives one a first-class derivative operator (see, for example, [7, Lemma 4.5],
which provides an interpretation for the differential substitution operator in the differential λ-calculus. This property still holds in difference categories, although its formulation is somewhat more involved.

**Lemma 3.4.** For any C-morphisms \( \Lambda(f) : A \to (B \Rightarrow C), e : A \to B \), the following identities hold:

i. \( \partial [\text{ev} \circ (\Lambda(f), e)] = \text{ev} \circ (\partial [\Lambda(f)], e \circ \pi_1) + \partial [f] \circ (\langle \pi_1 + \varepsilon(\pi_2), e \circ \pi_1 \rangle, \langle 0, \partial [e] \rangle) \)

ii. \( \partial [\text{ev} \circ (\Lambda(f), e)] = \text{ev} \circ (\partial [\Lambda(f)], e \circ \pi_1 + \varepsilon(\partial [e])) + \partial [f] \circ (\langle \pi_1, e \circ \pi_1 \rangle, \langle 0, \partial [e] \rangle) \)

As is the case in differential λ-categories, we can define a “differential substitution” operator on the semantic side. This operator is akin to post-composition with a partial derivative, and can be defined as follows.

**Definition 3.5.** Given morphisms \( s : A \times B \to C, u : A \to B \), we define their differential composition \( s \star u : A \times B \to C \) by:

\[
s \star u := \partial [s] \circ (\text{id}_{A \times B}, \langle 0_A, u \circ \pi_1 \rangle)
\]

We should understand the morphism \( s \star u \) as the partial derivative of \( s \) in its second argument, pre-composed with the morphism \( u \). This operator verifies the following useful properties (proofs of which can be found in [1, Chapter 6.4]).

**Lemma 3.6.** Let \( f : A \times B \times C \to D, g : A \to B, g' : A \times B \to B, e : A \times B \to C \) be arbitrary C-morphisms. Then the following identities hold:

i. \( (\text{ev} \circ (\Lambda(f), e)) \star g = \text{ev} \circ (\Lambda(f \star (e \star g)), e) + \text{ev} \circ (\Lambda(f) \star g, e \circ (\pi_1, \pi_2 + \varepsilon(g) \circ \pi_1)) \)

ii. \( \Lambda(f \star e) \star g = \Lambda[\Lambda^{-1}(\Lambda(f) \star g) \star e \circ (\text{id} + (0, \varepsilon(g))) + \epsilon(f \star e) \star (e \star g) + (f \star (e \star g))] \)

iii. \( \Lambda(f \star e) \circ (\pi_1, g') = \Lambda[\Lambda^{-1}(\Lambda(f) \circ (\pi_1, g')) \star (e \circ (\pi_1, g'))] \)

4. An Untyped Calculus of Differences

We proceed in a manner similar to Vaux [16] in his treatment of the algebraic λ-calculus; that is, we will first define a set of “unrestricted” terms \( \Lambda_\varepsilon \) which we will later consider up to an equivalence relation arising from the theory of difference categories.

**Definition 4.1.** The set \( \Lambda_\varepsilon \) of unrestricted terms of the \( \lambda_\varepsilon \)-calculus is given by the following inductive definition (assuming, as is usual, a countably infinite set of distinct variables \( x, y, z, \ldots \)):

Terms: \( s, t, e := x \mid \lambda x.t \mid (s \ t) \mid D(s) \cdot t \mid \varepsilon t \mid s + t \mid 0 \)

Since the only binder in the \( \lambda_\varepsilon \)-calculus is the usual \( \lambda \)-abstraction, the definition of free and bound variables is straightforward.

**Definition 4.2.** The set of free variables \( \text{FV}(t) \) of a term \( t \in \Lambda_\varepsilon \) is defined by induction on the structure of \( t \) as follows:

\[
\begin{align*}
\text{FV}(x) & := \{x\} \\
\text{FV}(\lambda x.t) & := \text{FV}(t) \setminus \{x\} \\
\text{FV}(s \ t) & := \text{FV}(s) \cup \text{FV}(t) \\
\text{FV}(D(s) \cdot t) & := \text{FV}(s) \cup \text{FV}(t) \\
\text{FV}(\varepsilon t) & := \text{FV}(t) \\
\text{FV}(s + t) & := \text{FV}(s) \cup \text{FV}(t) \\
\text{FV}(0) & := \emptyset
\end{align*}
\]
As usual, a variable \( x \) is free in \( t \) whenever \( x \in \text{FV}(t) \). An occurrence of a variable \( x \) in some term \( t \) is said to be bound whenever it appears in some subterm \( t' \) of \( t \) with \( x \notin \text{FV}(t) \). Two terms are said to be \( \alpha \)-equivalent if they are identical up to a renaming of all their bound variables.

In what follows, we will speak of terms only up to \( \alpha \)-equivalence. That is, we consider the terms \( \lambda x.x \) and \( \lambda y.y \) to be identical for all intents and purposes. Since this means we can rename bound variables freely, we will assume by convention that all bound variables appearing in any term \( t \in \Lambda \) are different from its free variables.

### 4.1. Differential Equivalence

Further to \( \alpha \)-equivalence, we introduce here the notion of differential equivalence of terms. The role of this relation is, as in [15], to enforce that the elementary algebraic properties of sums and actions are preserved. For example, we wish to treat the terms \( \lambda x.(0 + \varepsilon(s + t)) \) and \((\lambda x.\varepsilon t) + (\lambda x.\varepsilon s)\) as if they were equivalent (as it will be the case in the models). This equivalence relation also has the role of ensuring that the axioms of a Cartesian difference category are verified, especially regularity of derivatives.

**Definition 4.3.** A binary relation \( \sim \subseteq \Lambda \times \Lambda \) is contextual whenever it satisfies the conditions in Figure 1 below.

\[
\begin{align*}
t \sim t' & \implies \lambda x.t \sim \lambda x.t' \\
t \sim t' & \implies \varepsilon t \sim \varepsilon t' \\
s \sim s' \land t \sim t' & \implies s t \sim s' t' \\
s \sim s' \land t \sim t' & \implies D(s) \cdot t \sim D(s') \cdot t' \\
s \sim s' \land t \sim t' & \implies s + t \sim s' + t'
\end{align*}
\]

**Figure 1.** Contextuality on unrestricted \( \Lambda \)-terms

**Lemma 4.4.** Whenever \( \sim \) is contextual, if \( t \) is a subterm of \( s \) and \( t \sim t' \) then \( s \sim s' \), where \( s' \) is the term resulting from substituting the occurrence of \( t \) in \( s \) for \( t' \).

**Definition 4.5.** Differential equivalence \( \sim_{\varepsilon} \subseteq \Lambda \times \Lambda \) is the least equivalence relation which is contextual and contains the relation \( \sim_{1\varepsilon} \) below Figure 2 below.

The above conditions can be separated in a number of conceptually distinct groups corresponding to their purpose. These are as follows:

- The first block of equations simply states that \(+, 0\) define a commutative monoid and that \( \varepsilon \) defines a monoid homomorphism.
- The second block of equations amounts to stating that the monoid and infinitesimal extension structure on functions is pointwise.
- The third block of equations implies (and is equivalent to) stating that addition and infinitesimal extension are “linear”, in the sense that they are equal to their own derivatives (that is, \( \partial [+] = + \circ \pi_2 \) and \( \partial [\varepsilon] = \varepsilon \)).
- The fourth block of equations states structural properties of the derivative, such as the derivative conditions and the commutativity of second derivatives. Similar equations are also present in the differential \( \lambda \)-calculus, where they merely state that the derivative is additive (as opposed to regular).
Defined as the quotient set

\[ \text{Definition 4.6.} \]

\[ \epsilon \partial \]

In the particular setting of Cartesian difference categories, the last term boils down to the "duplication of infinitesimals" in \[ \epsilon(D(s) \cdot t) \cdot e = \epsilon^2(D(s) \cdot t) \cdot e. \] To understand the logic of the first equation, consider that the term \( D(\lambda x.s) \cdot \epsilon t \) should roughly correspond to \( \partial[[s]](x, \epsilon[t]) \) which expands to:

\[
\begin{align*}
\partial[[s]](x, \epsilon[t]) &= \partial[[s]](x, 0 + \epsilon[t]) \\
&= \partial[[s]](x, 0) + \epsilon \partial^2[[s]](x, 0, 0, [t]) \\
&= \epsilon \partial^2[[s]](x, 0, 0, [t])
\end{align*}
\]

In the particular setting of Cartesian difference categories, the last term boils down to \( \epsilon \partial[[s]](x, [t]) \) by axiom [C\partial6], hence the equality \( D(s) \cdot \epsilon t \sim_\epsilon \epsilon(D(s) \cdot t) \) is justified, allowing infinitesimal extensions to be "pulled out" of differential application. The duplication of infinitesimals is simply a syntactic version of Lemma 2.3.ii.

**Definition 4.6.** The set \( \lambda_\epsilon \) of **well-formed terms**, or simply **terms**, of the \( \lambda_\epsilon \)-calculus is defined as the quotient set \( \lambda_\epsilon := \text{\textup{\( \Lambda_\epsilon \)}}/\sim_\epsilon \). Whenever \( t \) is an unrestricted term, we write \( \hat{t} \) to refer to the well-formed term represented by \( t \), that is to say, the \( \sim_\epsilon \)-equivalence class of \( t \).

The notion of differential equivalence allows us to ensure that our calculus reflects the laws of the underlying models, but has the unintended consequence that our \( \lambda_\epsilon \)-terms are equivalence classes, rather than purely syntactic objects. We will proceed by defining a notion of canonical form of a term and a canonicalization algorithm which explicitly constructs the canonical form of any given term, thus proving that \( \sim_\epsilon \) is decidable.

\[
\begin{align*}
(s + t) + e &\sim_\epsilon (s + (t + e)) \\
\lambda x.0 &\sim_\epsilon (\lambda x.t) \\
\lambda x.(s + t) &\sim_\epsilon (\lambda x.s + (\lambda x.t)) \\
s + t &\sim_\epsilon t + s \\
0 &\sim_\epsilon 0 \\
\epsilon \cdot s &\sim_\epsilon \epsilon \cdot t \\
\epsilon(s + t) &\sim_\epsilon \epsilon s + \epsilon t
\end{align*}
\]

\[
\begin{align*}
D(0) \cdot e &\sim_\epsilon 0 \\
D(s + t) \cdot e &\sim_\epsilon (D(s) \cdot e) + (D(t) \cdot e) \\
D(\epsilon t) \cdot e &\sim_\epsilon \epsilon(D(t) \cdot e) \\
\epsilon(D(s) \cdot t) &\sim_\epsilon D(s) \cdot (\epsilon t) \\
\epsilon^2(D(s) \cdot t) \cdot e &\sim_\epsilon \epsilon(D(s) \cdot (\epsilon t)) \\
\epsilon \cdot (D(s) \cdot t) &\sim_\epsilon (s \cdot t) + \epsilon((D(s) \cdot t) \cdot e)
\end{align*}
\]

**Figure 2.** Differential equivalence on unrestricted \( \Lambda_\epsilon \)-terms
Definition 4.7. We define the sets $B_\varepsilon \subset B_\varepsilon^+ \subset B_\varepsilon^+ \subset C_\varepsilon^+ \subset C_\varepsilon^+ (\subset \Lambda_\varepsilon)$ of basic, positive, additive, positive canonical and canonical terms according to the following grammar:

- **Basic terms:** $s_b, t_b, e_b \in B_\varepsilon := x | \lambda x.t_b | (s_b t^*) | D(s_b) \cdot t_b$
- **Positive terms:** $s^+, t^+, e^+ \in B_\varepsilon^+ := s_b | s_b + (t^+)$
- **Additive terms:** $s^*, t^*, e^* \in B_\varepsilon^* := 0 | s^+$
- **Positive canonical terms:** $S^+, T^+ \in C_\varepsilon^+ := \varepsilon^k s^b | \varepsilon^k s^b + (S^+)$
- **Canonical terms:** $S, T \in C_\varepsilon := 0 | S^+$

We will sometimes abuse the notation and write $t^*$ or $t^b$ to denote well-formed terms whose canonical form is an additive or basic term respectively.

The above definition is somewhat technical, so a more informal description of canonical forms should be helpful. We observe that all the rules of differential equivalence can be oriented from left to right to obtain a rewrite system - intuitively, this rewrite system operates by pulling all the instances of addition and infinitesimal extension to the outermost layers of a term. Since every syntactic construct is additive except for application, basic terms may only contain additive terms as the arguments to a function application.

As infinitesimal extensions are themselves additive, we also want to disallow terms such as $\varepsilon(s + t)$, instead factoring out the extension into $\varepsilon s + \varepsilon t)$. A general canonical term $T \in C_\varepsilon$ then has the form:

$$T = \varepsilon^{k_1} t^b_1 + (\varepsilon^{k_2} t_2 + (\ldots + \varepsilon^{k_n} t^b_n) \ldots)$$

That is to say, a canonical term is similar to a polynomial with coefficients in the set of basic terms and a variable $\varepsilon$ (but note that canonical terms are always written in their “fully distributed” form, that is, we write $\varepsilon s + (\varepsilon t + \varepsilon^2 e)$ rather than $\varepsilon((s + t) + \varepsilon e))$.

We will freely abuse the notation and write $\sum_{i=1}^n \varepsilon^{k_i} t^b_i$ to denote a general canonical term, as this form is easier to manipulate in many cases. In particular, the canonical term 0 is precisely the sum of zero terms.

**Definition 4.8.** Given unrestricted terms $s, t$, we define their **canonical sum** $s \boxplus t$ by induction as follows:

$$0 \boxplus t := t \quad s \boxplus 0 := s \quad (s + s') \boxplus t := s + (s' \boxplus t)$$

**Lemma 4.9.** The canonical sum $S \boxplus T$ of any two canonical terms is a canonical term. Furthermore $\boxplus$ is associative and has 0 as an identity element. When $S_i$ are canonical terms, we will write $\boxplus_{i=1}^n S_i$ for the canonical term $S_1 \boxplus (S_2 \boxplus \ldots S_n) \ldots$.

**Definition 4.10.** Given an unrestricted $\lambda_\varepsilon$-term $t \in \Lambda_\varepsilon$, we define its **canonical form** $\text{can}(t)$ by structural induction on $t$ as follows:

- $\text{can}(0) := 0$
- $\text{can}(x) := x$
- $\text{can}(s + t) := \text{can}(s) \boxplus \text{can}(t)$
\( \text{can}(\varepsilon t) := \varepsilon^* \text{can}(t), \text{ where} \\
\varepsilon^* T := \begin{cases} \\
\sum_{i=1}^{n} \varepsilon^* \varepsilon^{k_i} t_i^b & \text{if } T = \sum_{i=1}^{n} \varepsilon^{k_i} t_i^b \\
T & \text{if } T = \varepsilon D(D(\varepsilon) \cdot u) \cdot v \\
\varepsilon T & \text{otherwise} \\
\end{cases} \)

- If \( \text{can}(t) = \sum_{i=1}^{n} \varepsilon^{k_i} t_i^b \) then \( \text{can}(\lambda x. t) := \sum_{i=1}^{n} \varepsilon^{k_i} (\lambda x. t_i^b) \)

- If \( \text{can}(s) = \sum_{i=1}^{n} \varepsilon^{k_i} s_i^b \) and \( \text{can}(t) = T \) then \( \text{can}(D(s) \cdot t) := \bigoplus_{i=1}^{n} (\varepsilon^*)^{k_i} \text{reg} \left( s_i^b, T \right) \)

where the regularization \( \text{reg}(s, T) \) is defined by structural induction on \( T \):
\[
\text{reg}(s, 0) := 0 \\
\text{reg} \left( s, \varepsilon^k t^b + T' \right) := \left( (\varepsilon^*)^k D(s) \cdot t^b \right) \bigoplus \left[ \text{reg}(s, T') \right] \\
\bigoplus \left[ (\varepsilon^*)^{k+1} D^* \left( \text{reg}(s, T') \right) \cdot t^b \right]
\]

and \( D^* \) denotes the extension of \( D \) by additivity in its first argument, that is to say:
\[
D^* \left( \sum_{i=1}^{n} \varepsilon^{k_i} s_i^b \right) \cdot t^b := \sum_{i=1}^{n} \varepsilon^{k_i} \left( s_i^b \cdot t^b \right)
\]

Observe that, whenever \( S \) is canonical and \( t^b \) is basic, the term \( D^*(S) \cdot t^b \) is also canonical. Therefore, by induction, the regularization \( \text{reg}(s^b, T) \) is indeed a canonical term, since canonicity is preserved by \( \varepsilon^*, \bigoplus \).

- If \( \text{can}(s) = \sum_{i=1}^{n} \varepsilon^{k_i} s_i^b \) and \( \text{can}(t) = T \), then
\[
\text{can}(s \cdot t) := \left[ \sum_{i=1}^{n} \varepsilon^{k_i} \left( s_i^b \cdot \text{pri}(T) \right) \right] \bigoplus \left[ \varepsilon^* \left( \sum_{i=1}^{n} \text{ap} \left( \text{reg}(s_i^b, \tan(T)), \text{pri}(T) \right) \right) \right]
\]

where the primal \( \text{pri} \) and tangent \( \tan \) components of a canonical term \( T \) correspond respectively to the basic terms with zero and non-zero \( \varepsilon \) coefficients, and \( \text{ap} \) is the additive extension of application.

\[
\begin{align*}
\text{pri}(0) & := 0 & \text{tan}(0) & := 0 \\
\text{pri}(\varepsilon^{k+1} t^b + T') & := \text{pri}(T') & \text{tan}(\varepsilon^{k+1} t^b + T') & := \varepsilon^{k+1} t^b + \tan(T') \\
\text{pri}(\varepsilon^0 t^b + T') & := t^b + \text{pri}(T') & \text{tan}(\varepsilon^0 t^b + T') & := \tan(T') \\
\text{ap} \left( \sum_{i=1}^{n} \varepsilon^{k_i} s_i^b, t^b \right) & := \sum_{i=1}^{n} \varepsilon^{k_i} (s_i^b \cdot t^b)
\end{align*}
\]

**Example 4.11.** The canonicalization algorithm is mostly straightforward, with only the cases for differential and standard application being of any interest. Since this is the part of the system where we diverge most from the differential \( \lambda \)-calculus, it is worth examining an example of regularization. Consider the following unrestricted term on free variables \( u, x, y, z \):
\[
t := D(u) \cdot (x + y + \varepsilon z)
\]
Applying the algorithm above, we compute its canonical form to be:

\[ \text{can} \left( D(u) \cdot (x + y + \varepsilon z) \right) \]

\[ = \text{reg} \left( u, x + y + \varepsilon z \right) \]

\[ = (D(u) \cdot x) \boxplus \text{reg} \left( u, y + \varepsilon z \right) \boxplus (\varepsilon^* D^* (\text{reg} \left( u, y + \varepsilon z \right)) \cdot x) \]

\[ = (D(u) \cdot x) \boxplus (D(u) \cdot y) \boxplus \text{reg} \left( u, \varepsilon z \right) \boxplus \varepsilon^* (D^* (\text{reg} \left( u, \varepsilon z \right)) \cdot y) \]

\[ + \text{reg} \left( \varepsilon^* D^* (\text{reg} \left( u, y + \varepsilon z \right)) \cdot x \right) \]

\[ = (D(u) \cdot x) \boxplus \left[ (D(u) \cdot y) \boxplus (\varepsilon D(u) \cdot z) \boxplus \varepsilon^* (\varepsilon D(u) \cdot z) \cdot y) \right] \]

\[ + \text{reg} \left( \varepsilon^* D^* (\text{reg} \left( u, y + \varepsilon z \right)) \cdot x \right) \]

\[ = (D(u) \cdot x) \boxplus \left[ D(u) \cdot y + \varepsilon(D(u) \cdot z) + \varepsilon(D(D(u) \cdot z) \cdot y) \right] \]

\[ + \text{reg} \left( \varepsilon^* D^* (\text{reg} \left( u, y + \varepsilon z \right)) \cdot x \right) \]

\[ = D(u) \cdot y + D(u) \cdot x \]

\[ + \varepsilon \left[ D(u) \cdot z + D(u) \cdot y \cdot x \right] \]

\[ + \varepsilon(D(D(u) \cdot z) \cdot y + D(D(u) \cdot z) \cdot x + \varepsilon(D(D(u) \cdot z) \cdot y) \cdot x)) \]

This is precisely the result we would expect from fully unfolding the expression \( \partial [u] (a, x+y+ \varepsilon(z)) \) in a Cartesian difference category and repeatedly applying regularity of the derivative!

**Theorem 4.12.** Every unrestricted \( \lambda_\varepsilon \)-term is differentially equivalent to its canonical form. That is to say, for all \( t \in \Lambda_\varepsilon \), we have \( t \sim_\varepsilon \text{can} \left( t \right) \).

**Proof.** The proof proceeds by straightforward induction on \( t \). We explicitly prove the case for the canonicalization of differential and standard application, as these are the only nontrivial cases. For this we will make use of the following results:

**Lemma 4.13.** Given canonical terms \( S, T \), we have:

\[ \varepsilon^* S \sim_\varepsilon \varepsilon S \]

\[ S \boxplus T \sim_\varepsilon S + T \]

\[ D^*(S) \cdot T \sim_\varepsilon D(S) \cdot T \]

\[ \text{ap} \left( S, T \right) \sim_\varepsilon S \cdot T \]

**Proof.** All four results follow by straightforward structural induction on \( S \). \( \square \)

**Lemma 4.14.** Given a basic term \( s^b \) and a canonical term \( T \) the differential application \( D(s^b) \cdot T \) is differentially equivalent to its regularized version \( \text{reg} \left( s^b, T \right) \).

**Proof.** The proof follows by induction on the structure of \( T \).

- When \( T = 0 \) we have \( \text{reg} \left( s^b, 0 \right) = 0 \sim_\varepsilon D(s^*) \cdot 0 \)
• When \( T = \varepsilon^{k} t^{b} + T' \) we have:

\[
\text{reg} \left( s^{b}, \varepsilon^{k} t^{b} + T' \right) = \left[ (\varepsilon^{*})^{k} D(s^{b}) \cdot t^{b} \right] \boxplus \left[ \text{reg} \left( s^{b}, T' \right) \right] \\
\boxplus \left[ (\varepsilon^{*})^{k+1} D^{*}(\text{reg} \left( s^{b}, T' \right)) \cdot t^{b} \right] \\
\sim_{\varepsilon} \varepsilon^{k} D(s^{b}) \cdot t^{b} + D(s^{b}) \cdot T' + \varepsilon^{k+1} D(D(s^{b}) \cdot T') \cdot t^{b}
\]

Going back to the proof of Theorem 4.12, the case for differential application is obtained as a straightforward corollary of Lemma 4.14. For conventional application, consider terms \( s, t \), and note that if \( \text{can} (t) = T \) then \( t \sim_{\varepsilon} \text{pri}(T) + \varepsilon \text{tan}(T) \). Then, for any basic term \( s^{b} \), we obtain:

\[
\text{can} \left( s^{b} t \right) = \left( s^{b} \text{pri}(T) \right) \boxplus \left[ \varepsilon^{*} \text{ap} \left( \text{reg} \left( s^{b}, \text{tan}(T) \right), \text{pri}(T) \right) \right] \\
\sim_{\varepsilon} s^{b} \text{pri}(T) + \varepsilon \left[ D(s^{b}) \cdot \text{tan}(T) \right] \text{pri}(T) \\
\sim_{\varepsilon} s^{b} \left[ \text{pri}(T) + \varepsilon \text{tan}(T) \right] \\
\sim_{\varepsilon} s^{b} t
\]

Our canonicalization algorithm is a result of orienting the equations in Figure 2. Note, however, that while most of these equivalences have a “natural” orientation to them, two of them are entirely symmetrical: those being commutativity of the sum and the derivative. Barring the imposition of some arbitrary total ordering on terms which would allow us to prefer the term \( x + y \) over \( y + x \) (or vice versa), we settle for our canonical forms to be unique “up to” these commutativity conditions.

**Definition 4.15. Permutative equivalence \( \sim_{+} \subseteq \Lambda_{\varepsilon} \times \Lambda_{\varepsilon} \) is the least equivalence relation which is contextual and satisfies the properties in Figure 3 below.**

\[
\begin{align*}
\text{s} + (\text{t} + \varepsilon) & \sim_{+} (\text{s} + \text{t}) + \varepsilon \\
\text{s} + \text{t} & \sim_{+} \text{t} + \text{s} \\
\text{D}(\text{D}(\text{s}) \cdot \text{t}) \cdot \varepsilon & \sim_{+} \text{D}(\text{D}(\text{s}) \cdot \varepsilon) \cdot \text{t}
\end{align*}
\]

**Figure 3. Permutative equivalence on unrestricted \( \Lambda_{\varepsilon} \)-terms**

We need to include associativity in the definition of permutative equality, as otherwise the canonical term \( x + (y + z) \) would not be permutatively equivalent to \( y + (x + z) \).

**Theorem 4.16.** Given unrestricted terms \( s, t \in \Lambda_{\varepsilon} \), they are differentially equivalent if and only if their canonical forms are permutatively equivalent. More succinctly, \( s \sim_{\varepsilon} t \) if and only if \( \text{can} (s) \sim_{+} \text{can} (t) \)

**Proof.** As an immediate consequence of Theorem 4.12, we know that \( s \sim_{\varepsilon} t \) if and only if \( \text{can} (s) \sim_{\varepsilon} \text{can} (t) \). The desired result will then follow from the fact that differential equivalence is precisely equivalent to permutative equivalence on canonical terms.

Before proving this, we remark that \( \sim_{\text{can}} \) is reflexive, transitive and symmetric, which follows immediately from its definition and reflexivity, transitivity and symmetry of \( \sim_{+} \). We also prove the following two helpful results:
Lemma 4.17. For any unrestricted terms \( s, t, e \), the following equalities hold:

\[
\begin{align*}
\text{can}(\text{can}(s) + \text{can}(t)) &= \text{can}(s + t) \\
\text{can}(\varepsilon \text{can}(x)) &= \text{can}(\varepsilon(x)) \\
\text{can}(\text{can}(s) \text{ can}(t)) &= \text{can}(st) \\
\text{can}(D(\text{can}(s)) \cdot \text{can}(t)) &= \text{can}(D(s) \cdot t)
\end{align*}
\]

As a consequence, the relation \(~_{\text{can}}\) is contextual.

Proof. Follows immediately from the definition of \( \text{can}(\cdot) \) and observing that it only depends on the canonicalization of the subterms of the outermost syntactic form.

Lemma 4.18. Whenever \( S, T \) are canonical terms, \( S \sim_{\varepsilon} T \) if and only if \( S \sim_{+} T \).

Proof. Since \( \sim_{\varepsilon} \) is the least reflexive, transitive, symmetric and contextual relation that verifies the conditions in Figure 2, it follows that whenever \( S \sim_{\varepsilon} T \) it must be the case that \( S = S_1 \sim_{\varepsilon} S_2 \sim_{\varepsilon} \ldots \sim_{\varepsilon} S_n = T \), where \( \sim_{\varepsilon} \) denotes the contextual, symmetric (but not transitive) closure of \( \sim_{\varepsilon} \).

But for each such condition, we have \( \text{can}(\text{LHS}) = \text{can}(\text{RHS}) \), except for the two commutativity conditions, and in all cases we have \( \text{can}(\text{LHS}) \sim_{+} \text{can}(\text{RHS}) \). Since \( \sim_{\text{can}} \) is contextual, we have that \( S = S_1 \sim_{\text{can}} S_2 \sim_{\text{can}} \ldots \sim_{\text{can}} S_n = T \), and thus by transitivity we obtain \( S \sim_{\text{can}} T \).

Theorem 4.16 is then an immediate consequence of Lemma 4.18 and Theorem 4.12.

Corollary 4.19. Differential equivalence of \( \Lambda_{\varepsilon} \)-terms is decidable.

Proof. Permutative equivalence of two terms is decidable, since the set of \( \sim_{+} \)-equivalence classes of any term is finite and can be enumerated easily. Canonicalization is also decidable, since it is defined as a clearly well-founded recursion.

Corollary 4.20. The set \( \lambda_{\varepsilon} \) of well-formed terms corresponds precisely to the set of canonical terms up to permutative equivalence \( \mathcal{C}_{\varepsilon} / \sim_{+} \).

4.2. Substitution. As is usual, our calculus features two different kinds of application: standard function application, represented as \( (s \ t) \); and differential application, represented as \( D(s) \cdot t \). These two give rise to two different notions of substitution. The first is, of course, the usual capture-avoiding substitution. The second, differential substitution, is similar to the equivalent notion in the differential \( \lambda \)-calculus, as it arises from the same chain rule that is verified in both Cartesian differential categories and change action models.

Definition 4.21. Given terms \( s, t \in \Lambda_{\varepsilon} \) and a variable \( x \), the capture-avoiding substitution of \( s \) for \( x \) in \( t \) (which we write as \( t[s/x] \)) is defined by induction on the structure of \( t \) as in Figure 4 below.

Proposition 4.22. Capture-avoiding substitution respects differential equivalence. That is to say, whenever \( s \sim_{\varepsilon} s' \) and \( t \sim_{\varepsilon} t' \), it is the case that \( t[s/x] \sim_{\varepsilon} t'[s'/x] \).

Proof. It suffices to show that \( t[s/x] \sim_{\varepsilon} t'[s'/x] \) (the full result will then follow from transitivity and contextuality of \( \sim_{\varepsilon} \)), which can be proven by straightforward structural induction on \( t \).
\[
\begin{align*}
\text{Figure 4. Capture-avoiding substitution in } &\lambda\varepsilon \\
\frac{\partial x}{\partial x}(s) &:= s \\
\frac{\partial y}{\partial x}(s) &:= 0 \quad \text{if } x \neq y \\
\frac{\partial (\lambda y.t)}{\partial x}(s) &:= \lambda y.(t[s/x]) \quad \text{if } y \notin \text{FV}(t) \\
\frac{\partial (te)}{\partial x}(s) &:= (te[s/x]) \\
\frac{\partial (\varepsilon t)}{\partial x}(s) &:= \varepsilon (t[s/x]) \\
\frac{\partial (t+e)}{\partial x}(s) &:= (t[s/x]) + (e[s/x]) \\
0[s/x] &:= 0
\end{align*}
\]

**Definition 4.23.** Given terms \(s, t \in \Lambda_\varepsilon\) and a variable \(x\) which is not free in \(s\)\(^1\), the differential substitution of \(s\) for \(x\) in \(t\), which we write as \(\frac{\partial t}{\partial x}(s)\), is defined by induction on the structure of \(t\) as in Figure 5. We write

\[
\frac{\partial^k t}{\partial (x_1, \ldots, x_k)}(u_1, \ldots, u_k)
\]

to denote the sequence of nested differential substitutions

\[
(\partial((\partial t/\partial x_1)(u_1)) \ldots)/\partial x_k)(u_k)
\]

Most of the cases of differential substitution are identical to those in the differential \(\lambda\)-calculus (compare the above with the rules in e.g. [10]). There are, however, a number of notable differences which stem from our more general setting. First, we must point out that this definition in fact coincides exactly with the original notion of differential substitution in e.g [10], provided that one assumes the identity \(\varepsilon t = 0\) for all terms. This reflects the

\(^1\)One should emphasise this constraint. Differential substitution appears in the reduction of \(D(\lambda x.t) \cdot s\) into \(\lambda x. \frac{\partial t}{\partial x}(s)\), and so if \(x\) were free in \(s\) it would become bound by the enclosing \(\lambda\)-abstraction.
fact that every Cartesian differential category is in fact a Cartesian difference category with trivial infinitesimal extension.

All the differences in this definition stem from the failure of derivatives to be additive in the setting of Cartesian difference categories. Consider the case for
\[ \frac{\partial D(t(x))}{\partial x}(y) \]
and remember that the “essence” of a derivative in our setting lies in the derivative condition, that is to say, if \( t(x) \) is a term with a free variable \( x \), we seek our notion of differential substitution to satisfy a condition akin to Taylor’s formula:
\[
t(x + \varepsilon y) \sim \varepsilon t(x) + \varepsilon \frac{\partial t}{\partial x}(y)
\]

When the term \( t \) is a differential application, and assuming the above “Taylor’s formula” holds for all of its subterms (which we will show later), this leads us to the following informal argument:
\[
D(t(x + \varepsilon y)) \cdot (s(x + \varepsilon y)) \sim \varepsilon D\left(t(x) + \varepsilon \frac{\partial t}{\partial x}(y)\right) \cdot (s(x + \varepsilon y))
\]
\[
\sim \varepsilon D(t(x)) \cdot (s(x)) + \varepsilon D\left(\frac{\partial t}{\partial x}(y)\right) \cdot (s(x + \varepsilon y))
\]
\[
+ \varepsilon \frac{\partial t}{\partial x}(y) \cdot (s(x + \varepsilon y))
\]
\[
+ \varepsilon^2 D(D(t(x)) \cdot (s(x))) \cdot \left(\frac{\partial s}{\partial x}(y)\right)
\]

From this calculation, the differential substitution for this case arises naturally as it results from factoring out the \( \varepsilon \) and noticing that the resulting expression has precisely the correct shape to be Taylor’s formula for the case of differential application. The case for standard application can be derived similarly, although the involved terms are simpler. Differential substitution verifies some useful properties, which we state below (mechanised proofs are available in [1, Appendix A], although the details are more cumbersome than enlightening).

**Proposition 4.24.** Differential substitution respects differential equivalence. That is to say, whenever \( s \sim_{\varepsilon} s' \) and \( t \sim_{\varepsilon} t' \), it is the case that \( \frac{\partial t}{\partial x}(s) \sim_{\varepsilon} \frac{\partial t'}{\partial x}(s') \).

*Proof.* See [1, Lemma dsubst_diff] \( \square \)

**Proposition 4.25.** Whenever \( x \) is not free in \( t \), then \( \frac{\partial t}{\partial x}(u) \sim_{\varepsilon} 0 \).

*Proof.* See [1, Lemma dsubst_empty] \( \square \)

**Proposition 4.26.** Whenever \( x \) is not free in \( u, v \), then:
\[
\frac{\partial^2 t}{\partial x^2}(u, v) \sim_{\varepsilon} \frac{\partial^2 t}{\partial x^2}(v, u)
\]

*Proof.* See [1, Lemma dsubst_commute] \( \square \)
As we have previously mentioned, the rationale behind our specific definition of differential substitution is that it should verify some sort of “Taylor’s formula” (or rather, Kock-Lawvere formula), in the following sense:

**Theorem 4.27.** For any unrestricted terms \( s, t, e \) and any variable \( x \) which does not appear free in \( e \), we have

\[
s [t + \varepsilon e/x] \sim \varepsilon \left[ s[t/x] + \varepsilon \left( \frac{\partial s}{\partial x} (e) \right) [t/x] \right]
\]

We will often refer to the right-hand side of the above equivalence as the **Taylor expansion** of the corresponding term in the left-hand side.

**Proof.** The proof follows by induction on \( s \) and some involved calculations. A mechanised version can be found in [1, Theorem Taylor].

- When \( s = x \) we have \( \frac{\partial x}{\partial x} (e) = e \) and so:
  \[
x [t + \varepsilon e/x] = t + \varepsilon e = x [t/x] + \varepsilon \left( \frac{\partial x}{\partial x} (e) \right) [t/x]
  \]

- When \( s = y \neq x \) we have \( \frac{\partial y}{\partial x} (e) = 0 \) and so:
  \[
y [t + \varepsilon e/x] = y \sim \varepsilon y + 0 = y [y/x] + \varepsilon \left( \frac{\partial y}{\partial x} (e) \right) [t/x]
  \]

- When \( s = 0 \) we have LHS \( \sim \varepsilon 0 \sim \varepsilon \) RHS.

- When \( s = s' + s'' \) we have:
  \[
  (s' + s'') [t + \varepsilon e/x] = s' [t + \varepsilon e/x] + s'' [t + \varepsilon e/x]
  \]
  \[
  \sim \varepsilon \left( s'[t/x] + \varepsilon \left( \frac{\partial s'}{\partial x} (e) \right) [t/x] \right) + \left( s'' [t/x] + \varepsilon \left( \frac{\partial s''}{\partial x} (e) \right) [t/x] \right)
  \]
  \[
  \sim \varepsilon \left( s'[t/x] + s'' [t/x] \right) + \left( \frac{\partial s'}{\partial x} (e) \right) [t/x] + \left( \frac{\partial s''}{\partial x} (e) \right) [t/x]
  \]
  \[
  = (s' + s'') [t/x] + \left( \frac{\partial (s' + s'')} {\partial x} (e) \right) [t/x]
  \]

- When \( s = \varepsilon s' \) we have:
  \[
  (\varepsilon s') [t + \varepsilon e/x] = \varepsilon s' [t + \varepsilon e/x]
  \]
  \[
  \sim \varepsilon \left( s'[t/x] + \varepsilon \left( \frac{\partial s'}{\partial x} (e) \right) [t/x] \right)
  \]
  \[
  \sim \varepsilon \left( s'[t/x] + \varepsilon \left( \frac{\partial s'}{\partial x} (e) \right) [t/x] \right)
  \]
  \[
  \sim \varepsilon \left( \varepsilon s' \right) [t/x] + \varepsilon \left( \frac{\partial (\varepsilon s')}{\partial x} (e) \right) [t/x]
  \]

- When \( s = s' s'' \) we have:
  \[
  (s' s'') [t + \varepsilon e/x] = (s'[t + \varepsilon e/x]) (s'' [t + \varepsilon e/x])
  \]
  \[
  \sim \varepsilon \left[ s'[t/x] + \varepsilon \left( \frac{\partial s'}{\partial x} (e) \right) [t/x] \right] \left( s'' [t/x] + \varepsilon \left( \frac{\partial s''}{\partial x} (e) \right) [t/x] \right)
  \]
\[ \sim_{\epsilon} \left[ s'[t/x] \left( s''[t/x] + \epsilon \left( \left( \frac{\partial s''}{\partial x} (e) \right)[t/x] \right) \right) \right] \\
+ \epsilon \left[ \left( \left( \frac{\partial s'}{\partial x} (e) \right)[t/x] \right) \left( s''[t/x] + \epsilon \left( \left( \frac{\partial s''}{\partial x} (e) \right)[t/x] \right) \right) \right] \\
\sim_{\epsilon} (s'[t/x] \left( s''[t/x] \right)) \\
+ \epsilon \left[ \left( D(s'[t/x]) \cdot \left( \left( \frac{\partial s''}{\partial x} (e) \right)[t/x] \right) \right) \left( s''[t/x] \right) \right] \\
+ \epsilon \left( \left( \frac{\partial s'}{\partial x} (e) \right)[t/x] \right) \left( s''[t + \epsilon e/x] \right) [t/x] \\
\sim_{\epsilon} (s' s'')[t/x] + \epsilon \left( \left( \frac{\partial (s' + s'')}{\partial x} (e) \right)[t/x] \right) \\
\]

- When \( s = D(s') \cdot s'' \) we have:

\[ (D(s') \cdot s'')[t + \epsilon e/x] = D(s'[t + \epsilon e/x]) \cdot (s''[t + \epsilon e/x]) \]

\[ \sim_{\epsilon} D \left( s'[t/x] + \epsilon \left( \left( \frac{\partial s'}{\partial x} (e) \right)[t/x] \right) \right) \cdot (s''[t + \epsilon e/x]) \]

\[ \sim_{\epsilon} \partial \left( \left( s'[t/x] \right) \cdot (s''[t + \epsilon e/x]) \right) + \epsilon \left( D \left( \left( \frac{\partial s'}{\partial x} (e) \right)[t/x] \right) \cdot \left( s''[t + \epsilon e/x] \right) \right) \]

\[ \sim_{\epsilon} \partial \left( \left( s'[t/x] \right) \cdot \left( s''[t/x] + \epsilon \left( \left( \frac{\partial s''}{\partial x} (e) \right)[t/x] \right) \right) \right) \\
+ \epsilon \left( D \left( \left( \frac{\partial s'}{\partial x} (e) \right)[t/x] \right) \cdot \left( s''[t + \epsilon e/x] \right) \right) \]

\[ \sim_{\epsilon} \partial \left( \left( s'[t/x] \right) \cdot \left( s''[t/x] \right) \right) + \epsilon \left( \partial \left( \left( s'[t/x] \right) \right) \cdot \left( \left( \frac{\partial s''}{\partial x} (e) \right)[t/x] \right) \right) \\
+ \epsilon \left( \left( \frac{\partial s'}{\partial x} (e) \right)[t/x] \right) \cdot (s''[t + \epsilon e/x]) \]

\[ \sim_{\epsilon} \left( D(s') \cdot s'' \right)[t/x] + \epsilon \left( \partial \left( \left( s'[t/x] \right) \cdot \left( \frac{\partial s''}{\partial x} (e) \right)[t/x] \right) \right) \]

\[ + \epsilon \left( \left( \frac{\partial s'}{\partial x} (e) \right)[t/x] \right) \cdot \left( s''[t + \epsilon e/x] \right) \]
We can generalise this procedure to arbitrary sequences of differential substitutions, al-
t Theorem 4.27 and Lemma 4.28, is equivalent to
Alternatively, we can simply evaluate (λx.t)
where x
are computed by adding “perturbations” to the program input.
The process is remarkably similar to forward-mode automatic differentiation, where derivatives
can then be extracted from this result by extracting the term under the ε. This process is remarkably similar to forward-mode automatic differentiation, where derivatives are computed by adding “perturbations” to the program input.

Theorem 4.27 also allows us to unpack all of the substitutions in the definition of differential substitution. For example, the term \( \frac{\partial(t \cdot e)}{\partial x} (u) \) can be expanded to:
\[
\frac{\partial(t \cdot e)}{\partial x} (u) = D(t) \cdot \left( \frac{\partial e}{\partial x} (u) \right) e + \left[ \frac{\partial t}{\partial x} (u) \left( e \left[ \left( x + \varepsilon u \right)/x \right] \right) \right]
\]

We can generalise this procedure to arbitrary sequences of differential substitutions, although the terms involved are too complex to give a simple account.

Lemma 4.30. For any basic terms \( s^b, t^b \), variables \( x_1, \ldots, x_n \) and basic terms \( u_1^b, \ldots, u_n^b \) such that none of the \( x_i \) appear free in the \( u_i \), the differential substitution
\[
\frac{\partial^k(D(s^b) \cdot t^b)}{\partial(x_1, \ldots, x_n)} (u_1^b, \ldots, u_n^b)
\]
is differentially equivalent to a sum of terms of the form
\[
\varepsilon^2 D^l(v) \cdot (w_1, \ldots, w_l)
\]
where \( v \) is of the form
\[
\frac{\partial^p t^b}{\partial(x_1^{(t)}, \ldots, x_p^{(t)})} (u_1^{(t)}, \ldots, u_p^{(t)})
\]
and every \( w_j \) is of the form
\[
\frac{\partial^{q_j} e^b}{\partial \left( x_1^{(w_j)}, \ldots, x_{q_j}^{(w_j)} \right)} (u_1^{(w_j)}, \ldots, u_{q_j}^{(w_j)})
\]
where \( 1 \leq l \leq 2^n \) and each pair of sequences \( x_i^{(t)}, u_i^{(t)} \) corresponds to a reordering of some subsequence of the \( x_i, u_i^b \).

**Proof.** Straightforward induction on \( k \) and applying Theorem 4.27.

**Lemma 4.31.** For any basic term \( s^b \) and additive term \( t^* \), variables \( x_1, \ldots, x_n \) and basic terms \( u_1^b, \ldots, u_n^b \) such that none of the \( x_i \) appear free in the \( u_i^b \), the differential substitution
\[
\frac{\partial^k (s^b t^*)}{\partial (x_1, \ldots, x_n)} (u_1^b, \ldots, u_n^b)
\]
is differentially equivalent to a sum of terms of the form
\[
\varepsilon^s D^l(v) \cdot (w_1, \ldots, w_l)
\]
where \( v \) is of the form
\[
\frac{\partial^p t^b}{\partial (x_1^{(t)}, \ldots, x_p^{(t)})} (u_1^{(t)}, \ldots, u_p^{(t)})
\]
and every \( w_j \) is of the form
\[
\frac{\partial^{q_j} e^b}{\partial \left( x_1^{(w_j)}, \ldots, x_{q_j}^{(w_j)} \right)} (u_1^{(w_j)}, \ldots, u_{q_j}^{(w_j)})
\]
where \( 1 \leq l \leq 2^n \) and each pair of sequences \( x_i^{(t)}, u_i^{(t)} \) corresponds to a reordering of some subsequence of the \( x_i, u_i^b \).

The above results may seem overly weak and arcane, but at its core they make a very simple statement: if one applies any number of differential substitutions to the term \( D(s) \cdot t \) (or \( s \cdot t \)) and “cranks the lever”, pushing the substitutions as far down the term as possible, then all the differential substitutions in the resulting term are applied to either \( s \) or \( t \), and their arguments are a reordering of some subsequence of the arguments to the initial differential substitution.

**Theorem 4.32.** Differential substitution is regular, that is, for any unrestricted terms \( s, u, v \) where \( x \) does not appear free in either \( u \) or \( v \), we have:
\[
\frac{\partial s}{\partial x} (0) \sim \varepsilon 0
\]
\[
\frac{\partial s}{\partial x} (u + v) \sim \varepsilon \frac{\partial s}{\partial x} (u) + \left( \frac{\partial s}{\partial x} (v) \right) [x + \varepsilon u/x]
\]

**Proof.** Both properties follow by induction on \( s \). As the proof involves immense amounts of tedious calculations, we refer the reader to [1, Theorem Regularity].
4.3. The Operational Semantics of $\lambda_\varepsilon$. With the substitution operations we have introduced so far, we can now proceed to give a small-step operational semantics as a reduction system.

**Definition 4.33.** The one-step reduction relation $\rightsquigarrow \subseteq \Lambda_\varepsilon \times \Lambda_\varepsilon$ is the least contextual relation satisfying the reduction rules in Figure 6 below.

\[
\begin{align*}
(\lambda x.t) \ s & \rightsquigarrow_{\beta} [t/s][x] \\
D(\lambda x.t) \cdot s & \rightsquigarrow_{\partial} \lambda x. (\frac{\partial t}{\partial x}(s))
\end{align*}
\]

**Figure 6.** One-step reduction rules for $\lambda_\varepsilon$

We write $\rightsquigarrow^+$ to denote the transitive closure of $\rightsquigarrow$, and $\rightsquigarrow^*$ to denote its transitive, reflexive closure.

While the one-step reduction rules for $\lambda_\varepsilon$ may seem identical to those in the differential $\lambda$-calculus (see [10]), they are in fact not equivalent, as our notion of differential substitution diverges substantially.

The above one-step reduction is defined as a relation from unrestricted terms to unrestricted terms, but it is not compatible with differential equivalence. That is to say, there may be differentially equivalent terms $t \sim_\varepsilon t'$ such that $t'$ can be reduced but $t$ cannot. For example, consider the term $(\lambda x. x + 0) 0$, which contains no $\beta$-redexes that can be reduced. This term is, however, equivalent to $(\lambda x. x) 0$, which clearly reduces to 0.

We could lift the one-step reduction relation to well-formed terms by setting $t \rightsquigarrow t'$ whenever there exist $s, s'$ such that $t \sim_\varepsilon s, t' \sim_\varepsilon s'$ and $s \rightsquigarrow s'$. This is not very satisfactory, however, as it would make one-step reduction undecidable. Indeed, in order to check whether $s \sim_\varepsilon s'$ it would be necessary to check whether $s \rightsquigarrow s'$ for all their (infinitely many) representatives $s, s'$!

Another problem with this definition lies in the fact that the term $0$ (ostensibly a value which should not reduce) can also be written as $0 \ t$ for any term $t$. Whenever $t$ reduces to $t'$ in one step, then according to the previous definition so does $0 \ t$ reduce to $0 \ t'$, which is equivalent to $0$. Hence zero reduces to itself, rather than being a normal form!

Fortunately the canonical form of a term $t$ gives us a representative of $t$ which is "maximally reducible", that is to say, whenever any representative of $t$ can be reduced, then so can $\text{can}(t)$, possibly in zero steps.

**Theorem 4.34.** Reduction is compatible with canonicalization. That is to say, if $s \rightsquigarrow s'$, then $\text{can}(s) \rightsquigarrow^* s''$ for some $s'' \sim_\varepsilon s'$.

**Proof.** We prove the theorem by induction on the depth at which reduction happens and the number of non-canonical $\beta$-redexes in the term (that is, the number of redexes of the form $(\lambda x.s) (t + \varepsilon u)$). The most important cases are the ones where it happens at the outermost level, but we explicitly show some of the other cases. Before we do so, however, we state the following auxiliary properties:

**Lemma 4.35.** Whenever $T \sim_\varepsilon \lambda x.t$ where $T$ is a canonical term and $t$ is an unrestricted term, then $T$ is of the form $\sum_{i=1}^n \varepsilon^{k_i} (\lambda x.t_i^b)$, and additionally $t \sim_\varepsilon \sum_{i=1}^n \varepsilon^{k_i} t_i^b$. 
Lemma 4.36. Whenever \( s + t \sim^* e \) then \( e = s' + t' \) with \( s \sim^* s' \) and \( t \sim^* t' \). Whenever \( \varepsilon s \sim^* e \) then \( e = \varepsilon s' \) with \( s \sim^* s' \). In particular, whenever \( \text{can} \left( t \right) \sim^* t' \) then \( t' = \sum_{i=1}^{n} \varepsilon^{k_i} t_i'^{b} \), where \( \text{can} \left( t \right) = \sum_{i=1}^{n} \varepsilon^{k_i} t_i^{b} \) and \( t_i^{b} \sim^* t_i'^{b} \). Note that the \( t_i'^{b} \) may not be basic terms and thus \( t' \) may not be canonical.

Lemma 4.37. Whenever \( s \sim^* s' \) and \( t \sim^* t' \), then \( s \sqcup t \sim^* s' \sqcup t' \).

Lemma 4.38. Whenever \( s \sim^* s' \) and \( \text{can} \left( t \right) \sim^* t' \), then \( \text{reg} \left( s, \text{can} \left( t \right) \right) \sim^* \text{reg} \left( s', t' \right) \).

We proceed now to prove one of the cases where reduction happens in a subterm of \( s \), which illustrates the ideas for the other cases:

- Let \( s = D(t) \cdot u \) and \( s' = D(t') \cdot u \), with \( t \sim t' \). By Lemma 4.17, we can write \( \text{can} \left( s \right) \) as \( \text{can} \left( D(\text{can} \left( t \right)) \cdot \text{can} \left( u \right) \right) \). Let \( \text{can} \left( t \right) = \sum_{i=1}^{n} \varepsilon^{k_i} t_i^{b} \). By induction and Lemma 4.36, we have \( t_i^{b} \sim^* t_i''^{b} \) and \( t' \sim_{\varepsilon} \sum_{i=1}^{n} \varepsilon^{k_i} t_i''^{b} \). Applying the previous auxiliary lemmas, we obtain:

\[
\text{can} \left( s \right) = \text{can} \left( D(\text{can} \left( t \right)) \cdot u \right) \\
= \biguplus_{i=1}^{n} (\varepsilon^* k_i) \text{reg} \left( t_i^{b}, \text{can} \left( u \right) \right) \\
\sim^* \biguplus_{i=1}^{n} (\varepsilon^* k_i) \text{reg} \left( t_i''^{b}, \text{can} \left( u \right) \right) \\
\sim_{\varepsilon} D \left( \sum_{i=1}^{n} \varepsilon^{k_i} t_i''^{b} \right) \cdot u \\
\sim_{\varepsilon} D(t') \cdot u
\]

- Let \( s = (t \ e) \) and \( s' = (t' \ e') \), with \( e \sim e' \). The result follows from the previous auxiliary lemmas and the fact that the primal \( \text{pri} \) and tangent \( \text{tan} \) components commute with reduction.

- Every other case is either immediate or follows from similar arguments.

The more involved cases are those when reduction happens at the outermost level of \( s \). For brevity we will focus on the non-trivial cases where the underlying \( \lambda \)-abstraction involves only basic terms, as the more general cases follow by unfolding the \( \lambda \)-abstraction into a canonical sum and applying the primitive cases below to each summand separately.

- Let \( s = D(\lambda x.s^{b}) \cdot t \) and \( s' = \lambda x.\frac{\partial s^{b}}{\partial x} \left( t \right) \), and write \( T \) for \( \text{can} \left( t \right) \). The proof proceeds then by induction on the number of summands of \( T \).
  - When \( T = 0 \) we have \( \text{can} \left( s \right) = 0 \). On the other hand:
    \[
s' = \lambda x.\frac{\partial s^{b}}{\partial x} \left( t \right) \sim_{\varepsilon} \lambda x.\frac{\partial s^{b}}{\partial x} \left( 0 \right) \sim_{\varepsilon} 0
    \]
  - When \( T = \varepsilon^{k} t^{b} + T' \), we apply the induction hypothesis and Lemma 4.35 to obtain

\[
\text{can} \left( D(\lambda x.s^{b}) \cdot T' \right) \sim^* \sum_{i=1}^{n} \varepsilon^{k_i} (\lambda x.w_i) \sim_{\varepsilon} \lambda x.\frac{\partial s^{b}}{\partial x} \left( T' \right)
\]

Then the canonical form \( \text{can} \left( s \right) \) reduces as follows:

\[
\text{can} \left( s \right) \\
= (\varepsilon^* k) D(\lambda x.s^{b}) \cdot t^{b} \\
+ \left[ \text{reg} \left( D(\lambda x.s^{b}) \cdot T' \right) \sqcup \varepsilon^* D^* \left( \text{reg} \left( D(\lambda x.s^{b}) \cdot T' \right) \right) \cdot t^{b} \right]
\]
\[\begin{align*}
&= (\varepsilon^* k^D) (\lambda x. s^b) \cdot t^b \\
&\quad + \left[ \text{can} \left( D(\lambda x. s^b) \cdot T' \right) \right] \varepsilon^* D^* \left( \text{can} \left( D(\lambda x. s^b) \cdot T' \right) \right) \cdot t^b \\
\quad \leadsto^* (\varepsilon^* k^D) \left( \lambda x. \frac{\partial s^b}{\partial x} (t^b) \right) \\
&\quad + \left[ \left( \sum_{i=1}^n \varepsilon^{k_i} (\lambda x. w_i) \right) \varepsilon^* D^* \left( \sum_{i=1}^n \varepsilon^{k_i} (\lambda x. w_i) \right) \cdot t^b \right] \\
&= (\varepsilon^* k^D) \left( \lambda x. \frac{\partial s^b}{\partial x} (t^b) \right) \\
&\quad + \left[ \left( \sum_{i=1}^n \varepsilon^{k_i} (\lambda x. w_i) \right) \varepsilon^* \left( \sum_{i=1}^n \varepsilon^{k_i} D(\lambda x. w_i) \cdot t^b \right) \right] \\
\quad \leadsto^* (\varepsilon^* k^D) \left( \lambda x. \frac{\partial s^b}{\partial x} (t^b) \right) \\
&\quad + \left[ \left( \sum_{i=1}^n \varepsilon^{k_i} (\lambda x. w_i) \right) \varepsilon^* \left( \sum_{i=1}^n \varepsilon^{k_i} \lambda x. \frac{\partial w_i}{\partial x} (t^b) \right) \right] \\
&\quad \leadsto^e \lambda x. \left( \frac{\partial s^b}{\partial x} (\varepsilon^k t^b) \right) + \lambda x. \left( \frac{\partial s^b}{\partial x} (T') \right) + \varepsilon \left( \lambda x. \frac{\partial}{\partial x} \left( \sum_{i=1}^n \varepsilon^{k_i} w_i \right) \right) \left( t^b \right) \\
&\quad \leadsto^e \lambda x. \left( \frac{\partial s^b}{\partial x} (\varepsilon^k t^b) \right) + \lambda x. \left( \frac{\partial s^b}{\partial x} (T') \right) + \varepsilon \left( \lambda x. \frac{\partial}{\partial x} \left( T' \right) \right) \left( t^b \right) \\
&\quad \leadsto^e \lambda x. \left( \frac{\partial s^b}{\partial x} (\varepsilon^k t^b) \right) + \left( \frac{\partial s^b}{\partial x} (T') \right) \left[ x + \varepsilon t^b / x \right]
\end{align*}\]

By Theorem 4.32, this last term is precisely the term \( \lambda x. \frac{\partial s^b}{\partial x} (\varepsilon^k t^b + T') \), which is equivalent to \( s' \) and thus the proof is concluded.

- Let \( s = (\lambda x. s^b) \cdot t \) and \( s' = s^b \cdot [t/x] \), and suppose \( \text{can'}(t) = t^* + \varepsilon^* T \) (that is, \( \text{pri}(\text{can'}(t)) = t^* \) and \( \tan(\text{can'}(t)) = T \), with \( t^* \) additive and \( T \) canonical).

\[ \text{can'}(s) = (\lambda x. s^b) \cdot t^* \varepsilon^* \text{ap} \left( \text{reg} \left( \lambda x. s^b, T \right), t^* \right) \]

Since the term \( (\lambda x. s^b) \cdot T \) contains one less non-canonical \( \beta \)-redex than the term \( s \), we apply our induction hypothesis to obtain

\[ \text{reg} \left( \lambda x. s^b, T \right) = \text{can'} \left( (\lambda x. s^b) \cdot T \right) \leadsto^* \sum_{i=1}^n \varepsilon^{k_i} (\lambda x. w_i) \leadsto^e \lambda x. \frac{\partial s^b}{\partial x} (T) \]

and therefore \( \frac{\partial s^b}{\partial x} (T) \leadsto^e \sum_{i=1}^n \varepsilon^{k_i} w_i \). With this in mind we continue to reduce the previous equation:

\[ (\lambda x. s^b) \cdot t^* \varepsilon^* \text{ap} \left( \text{reg} \left( \lambda x. s^b, T \right), t^* \right) \]

\[ \leadsto^* s^b \cdot [t^*/x] \varepsilon^* \text{ap} \left( \sum_{i=1}^n \varepsilon^{k_i} (\lambda x. w_i), t^* \right) \]
\[ s^b \left[ t^* / x \right] \varepsilon^* \left( \sum_{i=1}^{n} \varepsilon^j_i (\lambda x. w_i) \ t^* \right) \]

\[ \sim^{*} s^b \left[ t^* / x \right] \varepsilon^* \left( \sum_{i=1}^{n} \varepsilon^j_i (w_i \left[ t^* / x \right]) \right) \]

\[ = s^b \left[ t^* / x \right] \varepsilon^* \left( \sum_{i=1}^{n} \varepsilon^j_i \left( \left[ t^* / x \right] \right) \right) \]

\[ \sim_{\varepsilon} s^b \left[ t^* / x \right] + \varepsilon \left( \frac{\partial s^b}{\partial x} (T) \left[ t^* / x \right] \right) \]

\[ \sim_{\varepsilon} s^b \left[ t^* + \varepsilon T / x \right] \]

\[ \sim_{\varepsilon} s^b \left[ t^* / x \right] \]

The above result then legitimises our proposed “existential” definition of reduction of well-formed terms, as it shows that, in order to reduce a given term, it suffices to reduce its canonical form. It also gets rid of the “reducing zero” problem, as canonical forms do not contain “spurious” representations of zero.

**Definition 4.39.** Given well-formed terms \( s, s' \), we say that \( s \) **reduces to** \( s' \) **in one step**, and write \( s \sim t \), whenever \( \text{can}(s) \sim s'' \) and \( s'' \sim_{\varepsilon} s' \), for some canonical form \( \text{can}(s) \) of \( s \).

**Proposition 4.40.** Whenever \( s \sim t \) then for any term \( t \) we have \( s + t \sim s' + t \).

Furthermore, when \( t = t^b \) is a basic term (in particular \( t^b \) is not differentially equivalent to zero), we also have \( D(s) \cdot t \sim^{+} D(s') \cdot t \).

Conversely, whenever \( s \) is not differentially equivalent to zero and \( t \sim t' \), then \( s \cdot t \sim^{+} s \cdot t' \) and \( D(s) \cdot t \sim^{+} D(s) \cdot t' \).

The wording of the above definition specifies that a well-formed term reduces to another whenever any of its canonical forms reduces. As we have shown before, canonical forms are in fact only unique up to commutativity of addition and derivatives. Addition is not problematic, since it respects reduction; that is to say, if a sum \( s + t \) reduces to \( s' + t \), then its permutation \( t + s \) also reduces to \( t' + s' \). Symmetry of derivatives raises a more significant issue: consider the following diagram, which does not commute:

\[ D(D(\lambda x. t) \cdot u) \cdot v \sim_{\varepsilon} D(D(\lambda x. t) \cdot v) \cdot u \]

\[ D(\lambda x. \frac{\partial}{\partial x} (u)) \cdot v \not\sim_{\varepsilon} D(\lambda x. \frac{\partial}{\partial x} (v)) \cdot u \]

One-step reduction is still computable, since the set of canonical forms of any given term is finite, but we will have to keep this behaviour in mind when showing confluence.

A proof of confluence for \( \lambda \varepsilon \) will proceed by the standard Tait/Martin-Löf method by introducing a notion of parallel reduction on terms.

**Definition 4.41.** The **parallel reduction** relation between (unrestricted) terms is defined according to the deduction rules in Figure 7.
The parallel reduction relation can be extended to well-formed terms by setting $t \cong t'$ whenever $\textsf{can}(t) \cong t''$ with $t'' \sim \varepsilon t'$ for some canonical form of $t$.

**Remark 4.42.** Our definition of parallel reduction differs slightly from the usual in the rule $(\cong_{\beta})$, which allows reducing a newly-formed $\lambda$-abstraction. This is necessary because our calculus contains terms of the shape $(D(\lambda x.s) \cdot u) t$, which we need to parallel reduce in a single step to $(\partial s/\partial x(u))[t/x]$. The original presentation of the differential $\lambda$-calculus opted instead for adding an extra parallel reduction rule to allow for the case of reducing an abstraction under a differential application. Similarly, our rule $(\cong_{\partial})$ allows reducing terms of the form $D(D(\lambda x.s) \cdot u) \cdot v$, which can be entirely reduced in a single parallel reduction step.

One convenient property of the parallel reduction relation lies in its relation to canonical forms. As we saw in Theorem 4.34, canonical forms are “maximally reducible”, but don’t respect the number of reduction steps. This is no longer the case for parallel reduction: the process of canonicalization only duplicates regexes “in parallel” (that is, by copying them onto multiple separate summands) or in a “parallelizable series” (i.e. a differential application may be regularized into a term of the form $D(D(\ldots) \cdot u) \cdot v$, which can be entirely reduced in a single parallel reduction step).

**Theorem 4.43.** Whenever $s \cong s'$, then $\textsf{can}(s) \cong s''$ for some $s'' \sim \varepsilon s'$.

**Proof.** It suffices to inspect the proof of Theorem 4.34 and convince oneself that all of the reductions introduced by the proof can be lifted into a single instance of parallel reduction.

We also state the following standard properties of parallel reduction, all of which can be proven by straightforward induction on the term.

**Lemma 4.44.** Parallel reduction sits between one-step and many-step reduction. That is to say: $\Rightarrow \subseteq \cong \subseteq \Rightarrow^*$, and furthermore $\Rightarrow \subseteq \cong \subseteq \Rightarrow^*$.

**Lemma 4.45.** The parallel reduction relation is contextual. In particular, every term parallel-reduces to itself.

**Lemma 4.46.** Parallel reduction cannot introduce free variables. That is to say: whenever $t \cong t'$, we have $\text{FV}(t') \subseteq \text{FV}(t)$.

**Lemma 4.47.** Whenever $\lambda x.t \cong u$, it must be the case that $u = \lambda x.t'$ and $t \cong t'$.

**Lemma 4.48.** Whenever $s \cong s'$ and $t \cong t'$ then $s[t/x] \cong s'[t'/x]$, and furthermore there is some $w$ with $\partial s/\partial x(t) \cong w \sim \varepsilon \partial s'/\partial x(t')$. 

---

**Figure 7.** Parallel reduction rules for $\varepsilon$
The corresponding cases for differential substitution are slightly more technically involved. In the way, as differential substitution may "unfold" an application into a differential application followed by a one-step reduction, the result of parallel-reducing a canonical term need not be canonical.

When the last rule applied is \((\equiv_{\beta})\), that is: \(s = e \mathbin{u}, s' = e' \mathbin{[u'/y]}\), with \(e \equiv \lambda y.e', u \equiv u'\).

By the induction hypothesis, we have \(e [t/x] \equiv \lambda y.e' [t'/x]\) and \(u [t/x] \equiv u' [t'/x]\), hence:
\[
s [t/x] = e [t/x] \mathbin{u [t/x]} \equiv (e' [t'/x]) \mathbin{[(u' [t'/x]) / y]} = (e' [u'/y]) [t'/x]
\]

When the last rule applied is \((\equiv_{\theta})\), that is: \(s = D(e) \mathbin{u}, s' = \lambda y.\partial e \mathbin{u'}\), with \(e \equiv \lambda y.e', u \equiv u'\).

As before, we apply the induction hypothesis and obtain:
\[
s [t/x] = D(e [t/x]) \mathbin{u [t/x]} \equiv \lambda y.\left(\frac{\partial (e' [t'/x])}{\partial y}\right) (u' [t'/x]) = \left(\lambda y.\frac{\partial e'}{\partial y} (u')\right) [t'/x]
\]

The corresponding cases for differential substitution are slightly more technically involved.

When the last rule applied is \((\equiv_{\beta})\), that is: \(s = (e u), s' = e' [u'/y]\), with \(e \equiv \lambda y.e', u \equiv u'\).

By the induction hypothesis and applying the definition of differential substitution, we have \(\frac{\partial s}{\partial x} (t) \equiv \lambda y.\frac{\partial e'}{\partial x} (t')\) and \(\frac{\partial u}{\partial y} (t) \equiv \frac{\partial u'}{\partial y} (t')\). By applying the previous proof we also obtain \(u [x + \varepsilon t/x] \equiv u' [x + \varepsilon t'/x]\) hence:
\[
\frac{\partial s}{\partial x} (t) = \left(\left(\frac{\partial (e' [t'/x])}{\partial y}\right) \mathbin{u [t'/x]}\right) \left(\frac{\partial e}{\partial x} (t) \mathbin{u [x + \varepsilon t/x]}\right)
\]
\[
\equiv \left[\frac{\partial e'}{\partial y} (\frac{\partial u'}{\partial x} (t')) [u'/y]\right] + \left(\frac{\partial e'}{\partial x} (t')\right) [(u' [x + \varepsilon t'/x]) / y]
\]

On the other hand, since \(y\) is not free in either \(u'\) or \(t'\), applying Lemma 4.29, we obtain:
\[
\frac{\partial s'}{\partial x} (t') = \frac{\partial (e' [u'/y])}{\partial x} (t')
\]
\[
\equiv_{\varepsilon} \left(\frac{\partial e'}{\partial x} (t')\right) [(u' [x + \varepsilon t'/x]) / y] + \left(\frac{\partial e'}{\partial y} (\frac{\partial u'}{\partial x} (t'))\right) [u'/y]
\]

When the last rule applied is \((\equiv_{\theta})\), that is: \(s = D(\lambda y.e) \mathbin{u}, s' = \lambda y.\frac{\partial e'}{\partial x} (u')\), with \(e \equiv e', u \equiv u'\). As before, we apply the induction hypothesis and obtain:
\[
s [t/x] = D(\lambda y.e [t/x]) \mathbin{u [t/x]} \equiv \lambda y.\left(\frac{\partial (e' [t'/x])}{\partial y}\right) (u' [t'/x]) = \left(\lambda y.\frac{\partial e'}{\partial y} (u')\right) [t'/x]
\]

We first prove that parallel reduction has the diamond property when applied to canonical terms, taking care that it holds up to differential equivalence (note that, much like one-step reduction, the result of parallel-reducing a canonical term need not be canonical). For this, we introduce the usual notion of a full parallel reduct of a term.

---

2Observe that the reasoning here would not hold if we had opted to define parallel reduction in the “standard” way, as differential substitution may “unfold” an application into a differential application followed by a standard one.
Theorem 4.51. For any unrestricted term \( t \), the number of outermost abstractions is maximised precisely whenever \( t \) is of the form \( \lambda x.w \), for some term \( w \).

Proof. The proof follows by inspection of the parallel reduction rules. Consider a derivation of \( s \leadsto s' \), which will be of the form \( \lambda x_1.x_2.\ldots.x_n.t \) (with \( n \) possibly equal to 0). In general, the amount of abstractions at the outermost level of the term depends on our choice of a derivation for \( s \leadsto s' \). Suppose then that we pick a derivation for which \( s \leadsto s' \) is maximal. If this derivation does not use the rules \( \leadsto_{\text{ap}} \) or \( \leadsto_{\text{D}} \), then it is already the case that \( s' = s_\downarrow \). On the other hand, if it contains either rule, it is straightforward to see that replacing the last application of \( \leadsto_{\text{ap}} \) or \( \leadsto_{\text{D}} \) by \( \leadsto_\beta \) or \( \leadsto_{\beta^\prime} \) respectively the resulting term has at least as many \( \lambda \)-abstractions at the outermost level as the previous one. Iterating this process, we obtain that the number of outermost abstractions is maximised precisely whenever \( s' = s_\downarrow \).

Theorem 4.52. For any unrestricted terms \( s, s' \) such that \( s \leadsto s' \), there is an unrestricted term \( w \) such that \( s' \leadsto w \) and \( w \leadsto_\varepsilon s_\downarrow \).

Proof. The proof follows by induction on the derivation of \( s \leadsto s' \). Most cases are straightforward, and the rest follow as a corollary of Lemma 4.48, as we now show.

- The last applied rule is \( \leadsto_\beta^\prime \), that is, \( s = \langle t \cdot e \rangle \), \( s' = t' [e'/x] \), with \( t \leadsto \lambda x.t', e \leadsto e' \).
  
  By the induction hypothesis we have \( e' \leadsto w_e \leadsto e_\downarrow \) and \( \langle \lambda x.t' \rangle \leadsto \langle \lambda x.w_t \rangle \leadsto t_\downarrow \).
  
  By Lemma 4.50, it follows that \( t_\downarrow \) has the form \( \lambda x.v_t \), and since \( \langle \lambda x.w_t \rangle \leadsto_\varepsilon \langle \lambda x.v_t \rangle \) we also have \( w_t \leadsto_\varepsilon v_t \).
  
  Then:
  
  \[
  s' = t' [e'/x] \leadsto w_t [w_e/x] \leadsto_\varepsilon v_t [e_\downarrow/x] = (t \varepsilon_\downarrow)
  \]

- The last applied rule is \( \leadsto_\beta \), that is, \( s = \langle D(t) \cdot u \rangle \), \( s' = \lambda x \frac{\partial u}{\partial x} (u') \), with \( t \leadsto \lambda x.t', u \leadsto u' \).
  
  By the induction hypothesis we have \( u' \leadsto w_u \leadsto u_\downarrow \) and \( \lambda x.t' \leadsto \lambda x.w_t \leadsto t_\downarrow \).
  
  Again we must have \( t_\downarrow = \lambda x.v_t \) with \( w_t \leadsto_\varepsilon v_t \).
  
  Then:
  
  \[
  s' = \frac{\partial t'}{\partial x} (u') \leadsto \frac{\partial w_t}{\partial x} (w_u) \leadsto_\varepsilon \frac{\partial v_t}{\partial x} (v_u) = (D(t) \cdot u) \varepsilon_\downarrow
  \]

Corollary 4.52. Parallel reduction has the diamond property up to differential equivalence. That is to say, for any unrestricted term \( t \) and terms \( t_1, t_2 \) such that \( t \leadsto t_1 \) and \( t \leadsto t_2 \), there are terms \( u, v \) making the following diagram commute:
Theorem 4.54. The reduction relation $\Rightarrow$ has the diamond property. That is, whenever $s \Rightarrow u$ and $s \Rightarrow v$ there is a term $c$ such that $u \Rightarrow c$ and $v \Rightarrow c$.

Proof. Consider a well-formed term $s$, and suppose that $s \Rightarrow u$ and $s \Rightarrow v$. In particular, this means there are two canonical forms $\text{can}(s)_1$, $\text{can}(s)_2$ of $s$ such that $\text{can}(s)_1 \Rightarrow u$ and $\text{can}(s)_2 \Rightarrow v$. These canonical forms $\text{can}(s)_1$, $\text{can}(s)_2$ are equivalent up to permutative equivalence, and so their full parallel reducts are differentially equivalent as per Lemma 4.53. Denote their $\sim_{\varepsilon}$-equivalence class by $c$. Therefore since $\text{can}(s)_1 \sim_{\varepsilon} \text{can}(s)_1 \sim_{\varepsilon} c$ and $\text{can}(s)_2 \sim_{\varepsilon} \text{can}(s)_2 \sim_{\varepsilon} c$ it follows that $u \sim_{\varepsilon} c$ and $v \sim_{\varepsilon} c$. \hfill $\Box$

Corollary 4.55. The reduction relation $\rightsquigarrow$ is confluent.

4.4. Encoding the Differential $\lambda$-Calculus. It is immediately clear, from simply inspecting the operational semantics for $\lambda_{\varepsilon}$, that it is closely related to the differential $\lambda$-calculus – indeed, every Cartesian differential category is a Cartesian difference category, and this connection should also be reflected in the syntax.

As it turns out, there is a clean translation that embeds $\lambda_{\varepsilon}$ into the differential $\lambda$-calculus, which proceeds by deleting every term that contains an $\varepsilon$. The intuition behind this scheme should be apparent: every single differential substitution rule in $\lambda_{\varepsilon}$ is identical to the corresponding case for the differential $\lambda$-calculus, once all the $\varepsilon$ terms are cancelled out.

Definition 4.56. Given an unrestricted $\lambda_{\varepsilon}$ term $t$, its $\varepsilon$-erasure is the differential $\lambda$-term $\lceil t \rceil$ defined inductively according to the rules in Figure 8 below.

Proposition 4.57. The erasure $\lceil t \rceil$ is invariant under differential equivalence. That is to say, whenever $t \sim_{\varepsilon} t'$, it is the case that $\lceil t \rceil = \lceil t' \rceil$.
\[
\begin{align*}
\lceil x \rceil & := x \\
\lceil 0 \rceil & := 0 \\
\lceil s + t \rceil & := \lceil s \rceil + \lceil t \rceil & (rcl) \\
\lceil \varepsilon t \rceil & := 0 \\
\lceil s t \rceil & := \lceil s \rceil \lceil t \rceil \\
\lceil D(s) \cdot t \rceil & := D(\lceil s \rceil) \cdot \lceil t \rceil
\end{align*}
\]

**Figure 8.** \(\varepsilon\)-erasure of a term \(t\)

**Proof.** Follows immediately from inspecting the differential equivalence rules in Figure 2 and noticing that the erasure of both sides coincides.

Note that the standard presentation of the differential \(\lambda\)-calculus does not distinguish between equivalent terms, so terms like \(s + t\) and \(t + s\) are not merely equivalent but in fact identical.

**Proposition 4.58.** Erasure is compatible with standard and differential substitution. That is to say, for any terms \(s, t\) and a variable \(x\), we have:

\[
\lceil s \lceil t / x \rceil \rceil = \lceil s \rceil \lceil [t] / x \rceil \frac{\partial s}{\partial x} (\lceil t \rceil) = \frac{\partial \lceil s \rceil}{\partial x} (\lceil t \rceil)
\]

**Corollary 4.59.** Whenever \(s \sim s'\), then \(\lceil s \rceil \sim^* \lceil s' \rceil\).

These results form the syntactic obverse to the purely semantic fact that every Cartesian differential category is (trivially) a Cartesian difference category (see [3, Section 4.3] for a proof): the former exhibits the differential \(\lambda\)-calculus as an instance of \(\lambda_\varepsilon\) where the \(\varepsilon\) operator is “crossed out”, whereas the later shows that every Cartesian differential category can be understood as a “degenerate” Cartesian difference category, in the same sense that the corresponding infinitesimal extension is just the zero map.

5. **Simple Types for \(\lambda_\varepsilon\)**

Much like the differential \(\lambda\)-calculus, \(\lambda_\varepsilon\) can be endowed with a system of simple types, built from a set of basic types using the usual function type constructor.

**Definition 5.1.** The set of **types** and **contexts** of the \(\lambda_\varepsilon\)-calculus is given by the following inductive definition:

- Types: \(\sigma, \tau := t \mid \sigma \Rightarrow \tau\)
- Contexts: \(\Gamma := \emptyset \mid \Gamma, x : \tau\)

assuming a countably infinite set of basic types \(t, s, \ldots\) is given.

The typing rules for the \(\lambda_\varepsilon\)-calculus are given in Figure 9 below, and should not be in the least surprising, as they are identical to the typing rules for the differential \(\lambda\)-calculus, with the addition of a typing rule for the infinitesimal extension of a term. As one would expect, our type system enjoys all the “usual” structural properties and their proofs follow by straightforward induction on the typing derivation. Note, however, that all of these typing rules operate on unrestricted terms, rather than on well-formed terms, for reasons that we will clarify later.

According to the above rules, typing derivations are **invertible**, that is to say, whenever \(\Gamma \vdash t : \tau\) and \(t\) is of the form \(s + \varepsilon\), then it must be the case that \(\Gamma \vdash s : \tau\) and \(\Gamma \vdash \varepsilon : \tau\), and
so on. One property that fails to hold is uniqueness of typings: indeed the term 0 admits any type, as do terms such as 0 + 0 or (λx.0) y.

The following “standard” properties also hold, and can be proven by straightforward induction on the relevant typing derivation.

**Proposition 5.2 (Weakening).** Whenever $\Gamma \vdash t : \tau$, then for any context $\Sigma$ which is disjoint with $\Gamma$ it is also the case that $\Gamma, \Sigma \vdash t : \tau$.

**Proposition 5.3 (Substitution).** Whenever $\Gamma, x : \tau \vdash s : \sigma$ and $\Gamma \vdash t : \tau$, we have:

(i) $\Gamma \vdash s \left[ t/x \right] : \sigma$

(ii) $\Gamma, x : \tau \vdash \frac{\partial s}{\partial x} (t) : \sigma$

**Theorem 5.4 (Subject reduction).** Whenever $\Gamma \vdash t : \tau$ and $t \rightsquigarrow t'$ then $\Gamma \vdash t' : \tau$.

Since we have defined well-formed terms as equivalence classes of unrestricted terms, we might ask if typing is compatible with this equivalence relation. The answer is unfortunately no, that is to say, there are ill-typed terms that are differentially equivalent to well-typed terms. In particular, the term $(0 t)$ is differentially equivalent to the term 0, but while the later is trivially well-typed, the former will not be typable for many choices of $t$ (for example, whenever $t = (x x)$). A weaker version of this property does hold, however, that makes use of canonicity.

**Proposition 5.5.** Whenever $\Gamma \vdash t : \tau$, then $\Gamma \vdash \text{can}(t) : \tau$, and furthermore whenever $\Gamma \vdash \text{can}(t) : \tau$ then every canonical form of $t$ admits the same type.

*Proof.* The proof proceeds by induction on the typing derivation, by noting that every operation involved in canonicalization respects the typing rules.

**Remark 5.6.** The above issue could have been entirely avoided by circumventing the untyped calculus altogether and instead defining and operating on well-typed (unrestricted) terms directly. We have preferred to work out the untyped case first for two reasons: first, to mimic the development of the differential $\lambda$-calculus. Second, since differentiation of control and fixpoint operators is suspect (in that there is not an “obvious” choice of a derivative for them), we hope that working in an untyped calculus featuring Church encodings and a $Y$ combinator can illustrate what the “natural” choice for their derivatives should be.

Before stating a progress theorem for $\lambda_\varepsilon$, we must point out one small subtlety, as the definition of reduction of unrestricted terms depends on the particular representation chosen for the term. For example, the terms $((\lambda x.x) + 0) 0$ and $((\lambda x.x) + 0)$ are equivalent, but the first one contains no $\beta$-redexes, whereas the second one reduces to 0 in one step. We can prove that progress holds for canonical terms, however, as those are “maximally reducible”.

---

**Figure 9.** Simple types for $\lambda_\varepsilon$
Definition 5.7. A canonical term $T$ is a **canonical value** whenever it is of the form

$$T = \sum_{i=1}^{i} \varepsilon_{i}(\lambda x_{i}.t_{i})$$

Theorem 5.8 (Progress). Whenever a canonical term $T$ admits a typing derivation $\vdash T : \tau$, then either $T$ is a canonical value or there is some term $t'$ with $T \vDash t'$.

Proof. The proof proceeds by induction on the structure of $T$.

- When $T = 0$ then $T$ is trivially a canonical value.
- When $T = \varepsilon_{k}s^{b}$, then $s^{b}$ has the form $\lambda x.e^{b}$ or $D(e^{b}) \cdot u^{b}$. In the first case $T$ is already a canonical value. In the second case, note that the term $e^{b}$ is itself a canonical term and a strict subterm of $T$. By inversion of the typing rules, we have that $\vdash e^{b} : \tau$ (note that the type of a differential application is the same as the type of its body, unlike the case of standard application). Hence either $e^{b}$ reduces (and therefore so does $T$), or it is a canonical value, i.e. $e^{b}$ is of the form $\lambda x.w^{b}$. But in the last case then $T = \varepsilon_{k}D(\lambda x.w^{b}) \cdot s^{b}$ and therefore $T \vDash \varepsilon_{k}\lambda x.\frac{\partial w^{b}}{\partial x}(s^{b})$.
- When $T = T_{1} + T_{2}$, then either both $T_{1}, T_{2}$ are canonical values, and then so is $T$, or one of $T_{1}, T_{2}$ reduces, in which case so does $T$. \qed

Definition 5.9. We extend typing judgements to well-formed terms by setting $\Gamma \vdash t : \tau$ whenever $\Gamma \vdash \text{can}(t) : \tau$.

Corollary 5.10 (Subject reduction for well-formed terms). Whenever $\Gamma \vdash t : \tau$ and $t \vDash t'$, then $\Gamma \vdash t' : \tau$.

Proof. Since $t \vDash t'$, there is some canonical form $T = \text{can}(t)$ such that $T \vDash t'' \vDash t'$, and furthermore $\Gamma \vdash T : \tau$. By definition of Theorem 5.4, we have that $\Gamma \vdash t'' : \tau$ and therefore by Proposition 5.5 $\Gamma \vdash \text{can}(t'') : \tau$, from which it follows that $\Gamma \vdash t' : \tau$. \qed

Corollary 5.11 (Progress for well-formed terms). Whenever $\Gamma \vdash t : \tau$ then either $t \vDash t'$ or every canonical form $\text{can}(t)$ is a canonical value.

5.1. **Strong Normalisation.** With our typing rules in place, we set out to show that $\lambda z$ is strongly normalising. Our proof follows the structure of Ehrhard and Regnier’s [10] and Vaux’s[15], which use an adaptation of the well-known argument by reducibility candidates. Our proof will be somewhat simpler, however, due to two main reasons: first, we are not concerning ourselves with terms with coefficients on some general rig; and second, we have defined unrestricted and canonical terms as inductive types, and so we can freely use induction on the syntax of our terms. We will need some auxiliary results, which we prove now.

Lemma 5.12. Given an unrestricted term $t$, there are only finitely many terms $t'$ such that $t \vDash t'$.

Proof. Since our reduction relation is defined by simple induction on the syntax of $t$, it suffices to observe that any term $t$ will only contain a finite number of applications where reduction may take place. \qed

Lemma 5.13. Given a well-formed term $t$, there are only finitely many canonical terms $T$ such that $T \vDash t$. 

**Definition 5.9.** We extend typing judgements to well-formed terms by setting $\Gamma \vdash t : \tau$ whenever $\Gamma \vdash \text{can}(t) : \tau$. 

**Corollary 5.10 (Subject reduction for well-formed terms).** Whenever $\Gamma \vdash t : \tau$ and $t \vDash t'$, then $\Gamma \vdash t' : \tau$. 

**Proof.** Since $t \vDash t'$, there is some canonical form $T = \text{can}(t)$ such that $T \vDash t'' \vDash t'$, and furthermore $\Gamma \vdash T : \tau$. By definition of Theorem 5.4, we have that $\Gamma \vdash t'' : \tau$ and therefore by Proposition 5.5 $\Gamma \vdash \text{can}(t'') : \tau$, from which it follows that $\Gamma \vdash t' : \tau$. \qed

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**Corollary 5.10 (Subject reduction for well-formed terms).** Whenever $\Gamma \vdash t : \tau$ and $t \vDash t'$, then $\Gamma \vdash t' : \tau$. 

**Proof.** Since $t \vDash t'$, there is some canonical form $T = \text{can}(t)$ such that $T \vDash t'' \vDash t'$, and furthermore $\Gamma \vdash T : \tau$. By definition of Theorem 5.4, we have that $\Gamma \vdash t'' : \tau$ and therefore by Proposition 5.5 $\Gamma \vdash \text{can}(t'') : \tau$, from which it follows that $\Gamma \vdash t' : \tau$. \qed

**Corollary 5.11 (Progress for well-formed terms).** Whenever $\Gamma \vdash t : \tau$ then either $t \vDash t'$ or every canonical form $\text{can}(t)$ is a canonical value.
Proof. By Theorem 4.16, we know that any two canonical forms for \( t \) must be permutatively equivalent. But any term has a finite number of permutative equivalence classes, hence a term \( t \) only has finitely many canonical forms. \( \square \)

As a corollary of the two previous results, whenever a well-formed term \( t \) is strongly normalising, by König's lemma there is a longest sequence of well-formed terms \( t = t_1, t_2, \ldots, t_n \) such that \( \text{can} (t_i) \sim t_i' \sim \varepsilon t_{i+1} \). We write \(|t|\) to indicate the length of this sequence. The following result is then immediate.

Lemma 5.14. Whenever \( t \) is strongly normalising and \( t \sim t' \), we have \(|t| > |t'|\).

Lemma 5.15. A term \( s + t \) is strongly normalising if and only if \( s, t \) are strongly normalising.

Proof. The proof in the first direction proceeds by induction on \(|s + t|\).

Suppose \( s + t \) is strongly normalising. and suppose \( S, T \) are canonical forms for \( s, t \) respectively. Then \( S + T \) is a canonical form for \( s + t \) (up to associativity of addition), and any sequence of reductions

If \(|s + t| = 0\) it follows that its canonical form \( S + T \) does not reduce, and hence neither do the separate \( S, T \), thus \( s, t \) are normal forms.

On the other hand, suppose \( S \sim s', T \sim t' \) then \( s + t \sim s' + t' \) which is therefore strongly normalising, with \(|s' + t'| < |t|\). Hence, by induction, \( s' \) and \( t' \) are strongly normalising for any choices of canonical forms \( S, T \) and reducts \( s', t' \).

In the opposite direction, the proof follows similarly by induction on \(|s| + |t|\). The base case is equally trivial. For the inductive step, since any canonical form for \( s + t \) is (up to commutativity of addition) of the form \( S + T \), with \( S, T \) being canonical forms for \( s, t \) respectively, it follows that if \( s + t \sim e \). Without loss of generality, we assume that \( S \sim S' \) with \( \text{can}(e) = S' + T \). Then \( s \sim S' \) and \( e = S' + t \). Now \(|S'| < |s|\), and so we apply the induction hypothesis and obtain that \( e \) must be strongly normalising, and therefore so is \( s + t \).

Lemma 5.16. A term \( s + t \) is strongly normalising if and only if \( s \) is.

Definition 5.17. For every type \( \tau \) we introduce a set \( R_\tau \) of well-formed terms of type \( \tau \).

We do so by induction on \( \tau \).

- Whenever \( \tau = \tau \) is a primitive type, \( s \in R_\tau \) if and only if \( s \) is strongly normalising.
- Whenever \( \tau = \sigma_1 \Rightarrow \sigma_2 \), \( s \in R_{\sigma_1 \Rightarrow \sigma_2} \) if and only if for any additive term \( t^* \in R_{\sigma_1} \) and for any sequence \( v^b_1, \ldots, v^b_n \) of basic terms \( v^b \in R_{\sigma_1} \) of length \( n \geq 0 \) we have

\[
(D^n(s) \cdot (v^b_1, \ldots, v^b_n)) t^* \in R_{\sigma_2}.
\]

If \( \ell \in R_\tau \) we will often just say that \( \ell \) is reducible if the choice of \( \tau \) is clear from the context.

Lemma 5.18. Whenever \( \ell \in R_\tau \), then for any two distinct variables \( x, y \) the renaming \( \ell | y/x \) is also in \( R_\tau \).

Proof. Straightforward induction on \( \tau \).

Lemma 5.19. Whenever \( \ell \in R_\tau \), then \( \ell \) is strongly normalising.

Proof. By induction on \( \tau \). When \( \tau \) is a primitive type, the result follows trivially. Let \( \tau = \sigma_1 \Rightarrow \sigma_2 \) and \( \ell \in R_\tau \). By the induction hypothesis we know that for all \( u \in R_{\sigma_2} \) the application \( \ell u \) is strongly normalising.
Now suppose some canonical form of $\bar{t}$ reduces, that is, $T \rightsquigarrow \bar{t}'$. Since $\text{can}(t\ u^*) = \text{ap}(\text{can}(t), \text{pri}(u^*)) = \text{ap}(T, \text{can}(u^*))$, it follows that $\text{can}(t\ u) \rightsquigarrow^+ \text{ap}(\bar{t}', \text{pri}(u))$. Hence if there were any infinite sequence of reductions starting from $\bar{t}$, so would there be an infinite sequence of reductions starting from $t\ u$. Since $t\ u$ is strongly normalising, this must be impossible and so $\bar{t}$ must be strongly normalising as well.

**Lemma 5.20.** Whenever $s, t \in \mathcal{R}_\tau$, then both $s + t, s_e$ are in $\mathcal{R}_\tau$. Conversely, whenever $s + t$ is in $\mathcal{R}_\tau$ then so are $s, t$.

**Proof.** When $\tau$ is a primitive type the proof is a straightforward corollary of Lemmas 5.15 and 5.16.

When $\tau = \sigma_1 \Rightarrow \sigma_2$, consider an additive term $u^* \in \mathcal{R}_{\sigma_1}$. We ask whether the application $(s + t)\ u^*$ is in $\mathcal{R}_{\sigma_2}$. But note that $\text{can}((s + t)\ u^*)$ is equal to $\text{can}(s\ u^*) + \text{can}(t\ u^*)$ (modulo commutativity of the sum) and therefore $(s + t)\ u^* = s\ u^* + t\ u^*$. Since $s\ u^*$ and $t\ u^*$ are both in $\mathcal{R}_{\sigma_2}$, it follows by the induction hypothesis that so is $(s + t)\ u^*$. The same reasoning shows that $(D(s + t) \cdot v^b)\ u^*$ is in $\mathcal{R}_{\sigma_2}$.

The proof for $\varepsilon$ follows by a simpler but otherwise identical procedure.

On the opposite direction, consider a sum $s + t \in \mathcal{R}_\tau$. When $\tau$ is a primitive type then $s + t$ is strongly normalising and therefore so are $s, t$, hence they are regular. On the other hand, if $\tau = \sigma_1 \Rightarrow \sigma_2$, we have that for any $e^* \in \mathcal{R}_{\sigma_1}$ the reducible $(s + t)\ e^*$ is equal to $(s\ e^*) + (t\ e^*)$. By the induction hypothesis, both $(s\ e^*)$ and $(t\ e^*)$ are regular. A similar argument proves that differential applications of $s$ and $t$ are also regular, thus $s, t \in \mathcal{R}_{\sigma_1 \Rightarrow \sigma_2}$.

**Lemma 5.21.** Whenever $s \in \mathcal{R}_{\sigma_1 \Rightarrow \tau}$ and $t \in \mathcal{R}_\sigma$ then $D(s) \cdot t \in \mathcal{R}_{\sigma_1 \Rightarrow \tau}$.

**Proof.** Pick a canonical form $T$ of $t$. The proof proceeds by induction on the number of summands of $T$. If $T = 0$ or $T = \varepsilon^k t^b$ the result follows directly by definition of $\mathcal{R}_{\sigma_1 \Rightarrow \tau}$.

Now suppose $T = \varepsilon^k t^b + T'$. Then

\[
D(s) \cdot t \rightsquigarrow_{\varepsilon} D(s) \cdot (t^b + T')
\]\n
\[
\rightsquigarrow_{\varepsilon} D(s) \cdot t^b + D(s) \cdot T' + \varepsilon D(D(s) \cdot T') \cdot t^b
\]

Now $D(s) \cdot t^b$ is evidently reducible (as $s$ is reducible by hypothesis and, by Lemma 5.20, $t^b$ is also reducible), and by the induction hypothesis so is $D(s) \cdot T'$, from which also follows that $D(D(s) \cdot T') \cdot t^b$ is reducible as well. By Lemma 5.20 it follows that $\bar{t}$ is reducible.

**Corollary 5.22.** A well-formed term $\bar{t}$ is in $\mathcal{R}_\tau$ if and only if some canonical form $T = \text{can}(\bar{t})$ is of the form $\sum_{i=1}^{n} \varepsilon^k t^b_i$ with $t^b_i \in \mathcal{R}_\tau$ for each $1 \leq i \leq n$.

**Lemma 5.23.** Whenever $s \in \mathcal{R}_\tau$, $t \rightsquigarrow^+ \bar{t}'$, then $\bar{t}' \in \mathcal{R}_\tau$.

**Proof.** We proceed by induction on $\tau$. When $\tau$ is a primitive type, we have that $\bar{t}$ is strongly normalising and therefore so is $\bar{t}'$, hence $\bar{t}' \in \mathcal{R}_\tau$.

When $\tau = \sigma_1 \Rightarrow \sigma_2$, we pick some additive reducible term $e^*$ and a sequence of reducible basic terms $u_{1,1}^b, \ldots, u_{k,1}^b$, and establish that $D^k(t') \cdot (u_{1,1}^b, \ldots, u_{k,1}^b)\ e^*$ is reducible. But this is immediate: since $t$ reduces to $\bar{t'}$, then so does $D^k(t) \cdot (u_{1,1}^b, \ldots, u_{k,1}^b)\ e^*$ reduce to $D^k(t') \cdot (u_{1,1}^b, \ldots, u_{k,1}^b)\ e^*$ and, by induction, $D^k(t') \cdot (u_{1,1}^b, \ldots, u_{k,1}^b)\ e^*$ is reducible.

**Definition 5.24.** A basic term $t^b$ is **neutral** whenever it is not a $\lambda$-abstraction. In other words, a basic term is neutral whenever it is of the form $x, (s\ t)$ or $D(s) \cdot u$. 

A canonical term \( T \) is neutral whenever it is of the form \( \sum_{i=1}^{n} e^{k_i} s^b_i \), where each of the \( s^b_i \) are neutral. In particular, 0 is a neutral term.

A well-formed term \( \ell \) is neutral whenever some canonical form (and therefore all of its canonical forms) is neutral.

**Lemma 5.25.** Whenever \( \ell \) is neutral and every \( \ell' \) such that \( \ell \xrightarrow{1} \ell' \) is in \( R_\tau \), then so is \( \ell \).

**Proof.** When \( \tau \) is a primitive type the proof is immediate, as \( \ell' \in R_\tau \) implies \( \ell' \) is strongly normalising and therefore so is \( \ell \).

When \( \tau = \sigma_1 \Rightarrow \sigma_2 \), we show the reasoning for standard application first. We select arbitrary \( e^{k_1}, u^{b_1}_1, \ldots, u^{b_k}_k \in R_\sigma_1 \) and show that whenever \( (D^k(t) \cdot (u^{b_1}_1, \ldots, u^{b_k}_k)) e^* \) reduces then it reduces to a reducible term (hence our desired result will follow by induction on \( \tau \)).

We prove this property by induction on \( Q := |e^*| + |u^{b_1}| + \ldots + |u^{b_k}| \) which is well-defined since, by hypothesis, all of the involved terms are strongly normalising.

When \( Q = 0 \) then all of our chosen terms are normal, and so, since \( t \) is neutral, if \( (D^k(t) \cdot (u^{b_1}_1, \ldots, u^{b_k}_k)) e^* \) reduces it must be that \( \ell \xrightarrow{1} \ell' \). By hypothesis, \( \ell' \in R_{\sigma_1 \Rightarrow \sigma_2} \) and therefore \( (D^k(t') \cdot (u^{b_1}_1, \ldots, u^{b_k}_k)) e^* \) is reducible.

When \( Q > 0 \), then a reduction may occur in \( t \), in which case we apply the previous reasoning, or in one of the applied terms, in which case we apply the induction hypothesis on \( Q \). induction hypothesis.

**Lemma 5.26.** If, for all \( t^* \in R_\sigma_1 \) where \( x \) does not appear free, the term \( s[t^*/x] \) is in \( R_\sigma_2 \) and, for all \( u^b \) where \( x \) does not appear free, the term \( (\partial_u (u^b)) [t^*/x] \) is in \( R_\sigma_2 \), then the term \( \lambda x. s \) is in \( R_{\sigma_1 \Rightarrow \sigma_2} \).

**Proof.** As a corollary of Lemma 5.20, it suffices to check the case when \( s \) is some basic term \( s = s^b \).

Pick any variable \( y \neq z \). Since the variable \( y \) itself is an additive term in \( R_{\sigma_1} \), by hypothesis we have \( s[y/x] \in R_{\sigma_2} \). But since reducible terms are closed under renaming as per Lemma 5.18, this means that so is \( s \).

Now consider an arbitrary \( e^* \in R_{\sigma_1} \). We show that the application \((\lambda x. s^b) e^* \) is in \( R_{\sigma_2} \).

As it is a neutral term, by Lemma 5.25 it suffices to prove that every one-step reduct of \((\lambda x. s^b) e^* \) is in \( R_{\sigma_2} \). We do so by induction on \( |s^b| + |e^*| \). The term \((\lambda x. s^b) e^* \) reduces to one of:

- \( s^b[e^*/x] \), which by hypothesis is a representative of a term in \( R_{\sigma_2} \).
- \((\lambda x. s') e^* \) with \( s^b \xrightarrow{1} s' \). Then \( s' \in R_{\sigma_2} \) and since \( |s'| < |s^b| \) we apply our induction on \( |s^b| \) to obtain that \((\lambda x. s') e^* \in R_{\sigma_2} \).
- \((\lambda x. s^b) e' \) with \( e^* \xrightarrow{1} e' \). Then \( e' \in R_{\sigma_1} \) and since \( |e'| < |e^*| \) we apply our induction on \( |e^*| \) to obtain that \((\lambda x. s^b) e' \in R_{\sigma_2} \).

By a similar argument we can show that \((D^k(\lambda x. s^b) \cdot (u^{b_1}_1, \ldots, u^{b_k}_k)) e^* \) is reducible, applying Lemma 5.25 and using induction in \(|e^*| + |u^{b_1}| + \ldots + |u^{b_k}| \).

**Theorem 5.27.** Consider a well-formed term \( \ell \) which admits a typing of the form \( x_1 : \sigma_1, \ldots, x_n : \sigma_n \vdash \ell : \tau \) and assume given the following data:

- A sequence of basic terms \( u^{b_1}_1, \ldots, u^{b_k}_k \in R_{\sigma_1} \).
- An arbitrary sequence of indices \( i_1, \ldots, i_k \in \{1, \ldots, n\} \) (possibly with repetitions).
- A sequence of additive terms \( s^b_i \in R_{\sigma_{i_1}}, \ldots, s^b_i \in R_{\sigma_{i_k}} \).
such that none of the variables \(x_1, \ldots, x_i\) appear free in the \(d^b_1, s^*_i\). Then the term

\[
\frac{\partial^k t}{\partial(x_{i_1}, \ldots, x_{i_k})} (d^b_{i_1}, \ldots, d^b_{i_k}) [s^*_1, \ldots, s^*_n/x_1, \ldots, x_n]
\]

is in \(R_\tau\).

**Proof.** Throughout the proof we will write \(x^i, s^*_i, d^b_i\) as a shorthand for the corresponding sequences \(x_1, \ldots, x_n\), etc.

By definition of \(\tau\), we know that there is some canonical form \(T\) (in fact any canonical form) of \(t\) such that \(x_1 : \sigma_1, \ldots, x_n : \sigma_n \vdash T : \tau\). We prove our property holds by induction on this typing derivation. Furthermore, by Lemma 5.20, it suffices to consider the case when \(T\) is in fact some basic term \(t^b\). We proceed now by case analysis on the last rule of the typing derivation:

- \(t^b = x_i\) (and therefore \(\tau = \sigma_i\))
  
  If the sequence of indices \(i_1, \ldots, i_k\) is empty then the substitution \(\tau'\) is exactly equal to \(s^*_i\) and therefore \(\tau' \in R_{\sigma_i}\).
  
  If the sequence of indices \(i_1, \ldots, i_k\) is exactly the sequence containing only \(i\) then since \(x_i\) does not appear free in the substituted term \(d^b\) then \(t'\) is differentially equivalent to \(\frac{\partial s^*_i}{\partial x_i}(d^b_i)\) and therefore \(\tau' = d^b \in R_{\sigma_i}\).
  
  If the list of indices contains two or more indices, or does not contain \(i\), then \(\tau' \sim \varepsilon 0\) and therefore \(\tau'\) is trivially in \(\varepsilon \in R_{\sigma_i}\) (either the derivative)

- \(t^b = D(s^b) \cdot e^b\)

Applying Lemma 4.30, we know that the term

\[
\left(\left(\frac{\partial^k t}{\partial x_{i_1} \ldots \partial x_{i_k}}\right) (d^b_{i_1}, \ldots, d^b_{i_k})\right) [s^*_1, \ldots, s^*_n/x_1, \ldots, x_n]
\]

is equivalent to a sum of terms of the form

\[
(\varepsilon^2 D^l (v \cdot x^i_{s_i^*/x_i}) \cdot (w_1 \cdot x^i_{s_i^*/x_i}), \ldots, w_l \cdot x^i_{s_i^*/x_i})
\]

Again by Lemmas 5.20 and 5.21, it suffices to show that each of the \(v \cdot x^i_{s_i^*/x_i}, w_j \cdot x^i_{s_i^*/x_i}\) are reducible. But by Lemma 4.31, we know that \(v\) has the form

\[
\frac{\partial s^b}{\partial(x_{j_1}, \ldots, x_{j_m})} (d^b_{j_1}, \ldots, d^b_{j_m})
\]

Since \(s^b\) is a subterm of \(t^b\) its typing derivation is therefore a sub-derivation of the one for \(t^b\). We apply the induction hypothesis, obtaining that \(v \cdot x^i_{s_i^*/x_i}\) is reducible (as each of the \(d^b_j\) are reducible). By a similar argument, each of the \(w_j \cdot x^i_{s_i^*/x_i}\) are reducible as well, and therefore so is \(t^b\).

- \(t^b = (s^b \cdot e^*\)

Applying Lemma 4.30, we know that the term

\[
\left(\left(\frac{\partial^k t}{\partial x_{i_1} \ldots \partial x_{i_k}}\right) (d^b_{i_1}, \ldots, d^b_{i_k})\right) [s^*_1, \ldots, s^*_n/x_1, \ldots, x_n]
\]

is equivalent to a sum of terms of the form

\[
(\varepsilon^2 D^l (v \cdot x^i_{s_i^*/x_i}) \cdot (w_1 \cdot x^i_{s_i^*/x_i}), \ldots, w_l \cdot x^i_{s_i^*/x_i}) (e^* \cdot x^i_{s_i^*/x_i})
\]
Again by Lemma 5.20 it suffices to show that every such term is reducible. First, by Lemma 4.31, we know that \( v \) has the form

\[
\frac{\partial^p s^b}{\partial (x_{j_1}, \ldots, x_{j_m})} (d_{j_1}^b, \ldots, d_{j_m}^b)
\]

Since \( s^b \) is a subterm of \( t^b \) its typing derivation is therefore a sub-derivation of the one for \( t^b \). We apply the induction hypothesis, obtaining that \( v \frac{s^T_i}{x_i} \) is reducible (as each of the \( d_i^b \) are reducible). By a similar argument, each of the \( w_i \frac{s^T_i}{x_i} \) are reducible as well, as is \( e^* \frac{s^T_i}{x_i} \). Thus, by Lemma 5.21, the entire differential application is reducible.

But since \( t^b \) is an application of the reducible term

\[
e^\varepsilon D^\varepsilon \left( v \frac{s^T_i}{x_i} \right) \cdot \left( w_1 \frac{s^T_i}{x_i}, \ldots, w_l \frac{s^T_i}{x_i}, s^n_i/x_1, \ldots, x_n \right)
\]

to the reducible term \( e^* \frac{s^T_i}{x_i} \), it follows then that \( t^b \) is itself reducible.

\( \bullet \) \( t^b = \lambda y . s^b \)

Pick fresh variables \( z_1, \ldots, z_n \). By the induction hypothesis, we know that both of the following substitutions are reducible:

\[
\left( \frac{t^b}{z_1, \ldots, z_n, e^*/x_1, \ldots, x_n} \right)
\]

\[
\left( \frac{\partial t^b}{\partial y} \left( \frac{d^b}{x_1, \ldots, x_n, e^*/x_1, \ldots, x_n} \right) \right)
\]

But since reducible terms are closed under renaming, then the terms \( t^b [e^*/y] \) and \( \left( \frac{\partial t^b}{\partial y} \left( \frac{d^b}{e^*/y} \right) \right) \) are also reducible. Hence, by Lemma 5.26, the term \( \lambda y . s^b \) is reducible. \( \Box \)

**Corollary 5.28** (Strong normalisation). Whenever a closed well-formed term is typable with type \( \underline{\cdot} : \tau \), it is strongly normalising.

**Proof.** By the previous result, we know that any such term satisfies \( \underline{\cdot} \in \mathcal{R}_\tau \), and therefore \( \underline{\cdot} \) is strongly normalising. \( \Box \)

### 6. Semantics

It is a well-known result that the differential \( \lambda \)-calculus can be soundly interpreted in any differential \( \lambda \)-category, that is to say, any Cartesian differential category where differentiation “commutes with” abstraction (in the sense of [7, Definition 4.4]).

The exact same result holds for the difference \( \lambda \)-calculus and difference \( \lambda \)-categories. In what follows we will consider a fixed difference \( \lambda \)-category \( C \), and proceed to define interpretations for the types, contexts and terms of the simply-typed \( \lambda \)-calculus.

**Definition 6.1.** Given a \( t \)-indexed family of objects \( O_t \), we define the interpretation \( [\tau] \) of a type \( \tau \) by induction on its structure by setting \( [t] := O_t, [\sigma \Rightarrow \tau] := [\sigma] \Rightarrow [\tau] \). We lift the interpretation of types to contexts in the usual way. Or, more formally, we have: \( [\cdot] := 1, [\Gamma, x : \tau] := [\Gamma] \times [\tau] \).

**Definition 6.2.** Given a well-typed unrestricted \( \lambda \)-term \( \Gamma \vdash t : \tau \), we define its interpretation \( \llbracket t \rrbracket : [\Gamma] \rightarrow [\tau] \) inductively as in Figure 10 below. When \( \Gamma \) and \( \tau \) are irrelevant or can be inferred from the context, we will simply write \( \llbracket t \rrbracket \).
Theorem 6.5. Whenever \( \sim \), which will simplify the task considerably.

Precisely, to stating that 

\[
\frac{\text{Lemma 6.3.}}{\partial [\Lambda^-(\cdot)] \circ (\text{id}, \langle 0, g \circ \pi_1 \rangle) = \partial [\text{ev}] \circ \langle \_ \rangle \circ}
\]

Lemma 6.4. Define the relation \( \sim_{\|} \subseteq \Lambda_{\varepsilon} \) by letting \( s \sim_{\|} t \) whenever there exist \( \Gamma, \tau \) such that \( [\Gamma \vdash s : \tau] = [\Gamma \vdash t : \tau] \). Then the relation \( \sim_{\|} \) is contextual.

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\[
\begin{align*}
(x_i : \pi_i)_{i=1}^n \vdash x_k : \pi_k & := \pi_2 \circ \pi_1^{n-k} : \prod_{i=1}^n [\pi_i] \rightarrow [\pi_k] \\
[\Gamma \vdash 0 : \tau] & := 0 : [\Gamma] \rightarrow [\tau] \\
[\Gamma \vdash s + t : \tau] & := [s] + [t] : [\Gamma] \rightarrow [\tau] \\
[\Gamma \vdash \varepsilon t : \tau] & := \varepsilon [t] : [\Gamma] \rightarrow [\tau] \\
[\Gamma \vdash \lambda x.t : \sigma \Rightarrow \tau] & := \Lambda[t] : [\Gamma] \rightarrow [\sigma \Rightarrow \tau] \\
[\Gamma \vdash D(s) \cdot t : \sigma \Rightarrow \tau] & := \Lambda(\partial [\Lambda^-(\cdot)] \circ (\text{id}, \langle 0, [t] \circ \pi_1 \rangle)) : [\Gamma] \rightarrow [\tau]
\end{align*}
\]

\[\text{Figure 10. Interpreting } \lambda_{\varepsilon} \text{ in } C\]

First, we show a general equivalence between syntactic and semantic second derivatives which will simplify the task considerably.

\[\Lambda^-(\llbracket D(s) \cdot u \cdot v \rrbracket)\]

\[= \partial [\Lambda^-(\llbracket D(s) \cdot u \rrbracket)] \circ (\text{id}, \langle 0, [v] \circ \pi_1 \rangle)\]

\[= \partial^2 [\Lambda^-(\llbracket s \rrbracket)] \circ (\text{id}, \langle 0, [u] \circ \pi_1 \rangle) \circ (\text{id}, \langle 0, [v] \circ \pi_1 \rangle)\]

\[= \partial^2 [\Lambda^-(\llbracket s \rrbracket)] \circ (\text{id}, \langle 0, [u] \circ \pi_1 \rangle) \circ \partial (\text{id}, \langle 0, [u] \circ \pi_1 \rangle) \circ (\text{id}, \langle 0, [v] \circ \pi_1 \rangle)\]

\[= \partial^2 [\Lambda^-(\llbracket s \rrbracket)] \circ (\text{id}, \langle 0, [u] \circ \pi_1 \rangle) \circ (\text{id}, \langle 0, [v] \circ \pi_1 \rangle) \circ (\text{id}, \langle 0, [v] \circ \pi_1 \rangle)\]

\[= \partial^2 [\Lambda^-(\llbracket s \rrbracket)] \circ (\text{id}, \langle 0, [u] \circ \pi_1 \rangle) \circ (\text{id}, \langle 0, [v] \circ \pi_1 \rangle) \circ (0, [v] \circ \pi_1, 0)\]

With the above, most of the conditions become trivial. For example, we can prove that regularity of the syntactic derivative follows from semantic regularity (that is to say, \([C\partial_2]\)) with the following calculation:
• $D(s) \cdot (u + v) \sim_{\varepsilon}^1 D(s) \cdot u + D(s) \cdot v + \varepsilon(D(D(s) \cdot u) \cdot v)$

$$[D(s) \cdot (u + v)] = \Lambda(\partial[f] \circ (\text{id}, (0, [u] \circ \pi_1 + [v] \circ \pi_1)))$$

$$= \Lambda(\partial[f] \circ (\text{id}, (0, [u] \circ \pi_1) + [v] \circ \pi_1)))$$

$$= \Lambda[\partial[f] \circ (\text{id}, [u] \circ \pi_1))]$$

$$+ (\varepsilon(\partial[f] \circ (\text{id}, (0, [u] \circ \pi_1)), (0, [v] \circ \pi_1))))$$

$$= \Lambda[\partial[f] \circ (\text{id}, (0, [u] \circ \pi_1)))]$$

$$+ \Lambda[\varepsilon(\partial[f] \circ (\text{id}, (0, [v] \circ \pi_1), (0, [u] \circ \pi_1)))]$$

$$= \Lambda[\partial[f] \circ (\text{id}, (0, [u] \circ \pi_1)))]$$

$$+ \varepsilon(\partial[f] \circ (\text{id}, (0, [v] \circ \pi_1), (0, [u] \circ \pi_1)))]$$

$$= [D(s) \cdot u] + [D(s) \cdot v] + [\varepsilon(D(D(s) \cdot u) \cdot v)]$$

Most of the other conditions follow from similar arguments. For example, commutativity of the syntactic second derivative follows from commutativity of the semantic second derivative. The third and fifth conditions, dealing with infinitesimal extensions, may seem harder to prove, but they are both corollaries of axiom [C0.6], as we showed in Lemma 2.3. It remains to show that the syntactic derivative condition holds; this is not hard, but we do it explicitly as the derivative condition is such a central notion.

• $s \cdot (t + \varepsilon e) \sim_{\varepsilon}^1 (s \cdot t) + \varepsilon((D(s) \cdot e) \cdot t)$

$$[s \cdot (t + \varepsilon e)]$$

$$= \text{ev} \circ \langle [s], [t] + \varepsilon [e] \rangle$$

$$= \Lambda^{-}[s] \circ \langle \text{id}, [t] + \varepsilon [e] \rangle$$

$$= \langle \Lambda^{-}[s] \circ \langle \text{id}, [t] \rangle \rangle + \varepsilon(\partial[\Lambda^{-}[s]] \circ \langle \text{id}, [0, [e] \circ \pi_1] \rangle, (0, [0] \circ \pi_1))$$

$$= [s \cdot t] + \varepsilon([\partial[\Lambda^{-}[s]] \circ \langle \text{id}, [0, [e] \circ \pi_1] \rangle \rangle \circ \langle \text{id}, [t] \rangle)$$

$$= [s \cdot t] + \varepsilon[\text{ev} \circ \langle \Lambda^{-}[s] \circ \langle \text{id}, (0, [e] \circ \pi_1)) \rangle \circ \langle [t] \rangle)$$

$$= [s \cdot t] + \varepsilon[\text{ev} \circ \langle [D(s) \cdot e] \cdot [t] \rangle)$$

$$= [s \cdot t] + \varepsilon([D(s) \cdot e] \cdot [t])$$

**Lemma 6.6.** Let $t$ be some unrestricted $\lambda_e$-term. The following properties hold:

i. If $\Gamma \vdash t : \tau$ and $x$ does not appear in $\Gamma$ then $[\Gamma, x : \sigma \vdash t : \tau] = [\Gamma \vdash t : \pi_1 \circ \pi_1$]

ii. If $\Gamma, x : \sigma_1, y : \sigma_2 \vdash t : \tau$ then $[\Gamma, y : \sigma_2, x : \sigma_1 \vdash t : \tau] = [\Gamma, x : \sigma_1, y : \sigma_2 \vdash t : \tau] \circ \text{sw}$

The morphism $\text{sw}$ above is the obvious isomorphism between $(A \times B) \times C$ and $(A \times C) \times B$, which can be defined explicitly by:

$$\text{sw} := \langle \pi_{11}, \pi_{21} \rangle : (A \times B) \times C \to (A \times C) \times B$$

**Lemma 6.7.** Let $\Gamma, x : \tau \vdash s : \sigma$, with $s$ some unrestricted $\lambda_e$-term. Then:

i. Whenever $\Gamma, x : \tau \vdash t : \tau$, then $[s \cdot t/x]_{\Gamma} = [s]_{\Gamma, x : \tau} \circ [t]_{\Gamma, x : \tau}$

ii. Whenever $\Gamma \vdash t : \tau$, then $\frac{\partial s}{\partial x}(t)_{\Gamma, x : \tau} = \partial [s]_{\Gamma, x : \tau} \circ \langle \text{id}, (0, [t]_{\Gamma \circ \pi_1} \rangle$. Or, using the notation in Definition 3.5, $\frac{\partial s}{\partial x}(t) = [s] \cdot [t]$. 

\[\square\]
Proof. The proof follows roughly the structure of [7, Theorem 4.11], taking into account the differences in our notion of differential substitution. Note also that we prove substitution in the case that the variable $x$ is not free in $t$. This is because we require this (stronger) form of substitution to write $e[x + \varepsilon(t)/x]$ in some cases of differential substitution.

Both properties will follow by induction on the typing derivation of $s$. The only non-trivial case for the first one is differential application. For this, we must show that $[D(s[t/x]) \cdot (u[t/x])]$ is equal to $[D(s) \cdot u] \circ \langle \text{id}, [t] \rangle$. Expanding the term we obtain:

$$\Lambda \left( \Lambda^{-\langle [s], 0, [u] \rangle} \right) \circ \langle \pi_1, [t] \rangle$$

which concludes the proof.

We show now the cases for differential substitution.

- $s = x$ Then $[s] = \pi_2$ and
  $$\partial [\langle [s], 0, [t] \circ \pi_1 \rangle] = \pi_2 \circ \langle \text{id}, 0, [t] \circ \pi_1 \rangle = [t] \circ \pi_1 = [\Gamma, x : \tau \vdash t : \tau]$$

- $s = y \neq x$
  Then $[s] = \pi_2 \circ \pi_1^n \circ \pi_1$ and
  $$\partial [\langle [s], 0, [t] \circ \pi_1 \rangle] = \pi_2 \circ \pi_1^n \circ \pi_1 \circ \pi_2 \circ \langle \text{id}, 0, [t] \circ \pi_1 \rangle = 0 = [\Gamma, x : \tau \vdash 0 : \tau]$$

- $s = \varepsilon s_1$
  Then $[s] = \varepsilon [s_1]$ and
  $$\partial [\langle [s], 0, [t] \circ \pi_1 \rangle] = \varepsilon (\partial [\langle [s_1], 0, [t] \circ \pi_1 \rangle]) = \varepsilon \left[ \frac{\partial s_1}{\partial x} (t) \right] = \left[ \frac{\partial (\varepsilon s_1)}{\partial x} (t) \right]$$

- The case $s = \sum_{i=1}^n s_i$ follows by a similar argument as the previous one.

- $s = \lambda y.s_1 : \sigma_1 \Rightarrow \sigma_2$
  Then $\Gamma, x : \tau, y : \sigma_1 \vdash \varepsilon s_1 : \sigma_2$ and therefore, by the induction hypothesis, we know that:

$$\left[ \frac{\partial s_1}{\partial x} (t) \right]_{\Gamma, x : \tau, y : \sigma_1} = \left[ \frac{\partial s_1}{\partial x} (t) \right]_{\Gamma, y : \sigma_1, x : \tau} \circ \text{sw}$$

Obtaining the final result is just a matter of applying this identity and writing $\partial [\Lambda([s_1])]$ in terms of the swapping map $\text{sw}$ as remarked in Definition 3.1.

$$\partial [\langle [s], 0, [t] \circ \pi_1 \rangle] = \partial [\Lambda([s_1])] \circ \langle \text{id}, 0, [t] \circ \pi_1 \rangle$$
To simplify the calculations, we write \( s \) for \( \Lambda^-(\llbracket s_1 \rrbracket) \). The result follows as a consequence of Lemma 3.6(iii).

\[
\left[ \frac{\partial (s_1 e)}{\partial x} (t) \right] = \left[ \frac{\partial s_1}{\partial x} (t) \right] e + \left[ \frac{\partial s_1}{\partial x} (t) \right] \left( e [x + \varepsilon t/x] \right) = \text{ev} \circ \langle \Lambda (s * (\llbracket e \rrbracket * [t])), [e] \rangle + \text{ev} \circ \langle \llbracket s_1 \rrbracket * [t], [e] \rangle \circ \langle \pi_1, \pi_2 + \varepsilon (\llbracket [t] \rrbracket \circ \pi_1) \rangle = (\text{ev} \circ \langle \llbracket s_1 \rrbracket, [e] \rangle) * [t] = \llbracket s_1 e \rrbracket * [t]
\]

\( s = s_1 e \)

We will again abbreviate \( \Lambda^-(\llbracket s_1 \rrbracket) \) as \( s \) to make the subsequent calculations more readable. The result then follows by applying of Lemma 3.6(ii).

\[
\left[ \frac{\partial (D(s_1) \cdot u)}{\partial x} (t) \right] = \left[ D(s_1) \cdot \frac{\partial u}{\partial x} (t) \right] + \left[ D \left( \frac{\partial s_1}{\partial x} (t) \right) \cdot (u [x + \varepsilon (t)/x]) \right] + \varepsilon \left[ \left( D(D(s_1) \cdot u) \cdot \left( \frac{\partial u}{\partial x} (t) \right) \right) \right] = \Lambda \left( s * \left[ \frac{\partial u}{\partial x} (t) \right] \right) + \Lambda \left( \Lambda^- \left[ \frac{\partial s_1}{\partial x} (t) \right] \right) \ast \llbracket u [x + \varepsilon (t)/x] \rrbracket + \varepsilon \Lambda \left( \left( \Lambda^- \left[ (D(s_1) \cdot u) \right] \right) \ast \left[ \frac{\partial u}{\partial x} (t) \right] \right) = \Lambda \left( s \ast \left[ \llbracket u \rrbracket * [t] \right] \right) + \Lambda \left( \left( \Lambda^- (\Lambda^- (\llbracket s_1 \rrbracket) * [t]) \right) \ast \llbracket u [x + \varepsilon (t)/x] \rrbracket \right) + \varepsilon \Lambda \left( \left( \Lambda^- (s \ast \left[ \llbracket u \rrbracket \right]) \right) \ast \left( \llbracket u \ast [t] \right) \right)
\[
\begin{align*}
= \Lambda (s \ast ([u] \ast [t])) + \Lambda \left((\Lambda^{-1}(\Lambda([s_1]) \ast [t])) \ast ([u] \circ (\pi_1, \pi_2 + \varepsilon([t]) \circ \pi_1))\right) \\
+ \varepsilon \Lambda \left((\Lambda^{-1}(s \ast [u])) \ast ([u] \ast [t])\right)
\end{align*}
\]

\[
= (\Lambda (s \ast [u])) \ast [t]
\]

\[
= (\Lambda (\Lambda^{-1}([s_1]) \ast [u])) \ast [t]
\]

\[
= ([D(s_1) \cdot u]) \ast [t]
\]

\[
\square
\]

**Definition 6.8.** Given well-formed terms \(s, s'\), we define the equivalence relation \(\sim_{\beta \eta}\) as the least contextual equivalence relation that contains the one-step reduction relation \(\triangleright\triangleright\).

**Corollary 6.9.** The interpretation \([\_]\) is sound, that is to say, whenever \(s \sim_{\beta \eta} s'\) then \([s] = [s']\), independently of the choice of representatives \(s, s'\).

**Definition 6.10.** Recall that a simply-typed theory is a collection of equational judgements of the form \(\Gamma \vdash s = t : \sigma\) where \(\Gamma \vdash s : \sigma\) and \(\Gamma \vdash t : \sigma\) are derivable. We say that a simply-typed theory is a difference \(\lambda\)-theory if it is closed under all rules in the system \(\lambda^x\beta\eta\partial\) (comprising the contextual rules for all the constructs of the \(\lambda\)-calculus augmented by products, \(\sim_\varepsilon\) equivalence, and the surjective pairing, \(\beta, \eta\) and \(\partial\) laws, and last being the equational version of \(\sim_\beta\)).

Given an interpretation \([\_]_\mathcal{M}\) of \(\lambda^x\) in \(\mathcal{C}\), we say that \(\mathcal{M} = [\_]_\mathcal{M}\) is a model of a difference \(\lambda\)-theory \(\mathcal{T}\) if for every typed equational judgement \(\Gamma \vdash s = t : \sigma\) in \(\mathcal{T}\), we have that \([\Gamma \vdash s : \sigma]_\mathcal{M}\) and \([\Gamma \vdash t : \sigma]_\mathcal{M}\) are the same morphism.

A model homomorphism \(h : \mathcal{M} \rightarrow \mathcal{N}\) is given by isomorphisms \(h_t : [t]_\mathcal{M} \rightarrow [t]_\mathcal{N}\) for each basic type \(t\), and \(h_\sigma \times h_\tau := h_\sigma \circ h_\tau\), and

\[
h_{s \Rightarrow \tau} := h_\sigma^{-1} \Rightarrow h_\tau := \Lambda(h_\tau \circ \text{ev} \circ (\text{id} \times h_\sigma^{-1})).
\]

We write \(\text{Mod}_{\text{Diff}}(\mathcal{T}, \mathcal{C})\) for the category whose objects are all models of difference \(\lambda\)-theory \(\mathcal{T}\) in a difference \(\lambda\)-category \(\mathcal{C}\), and whose morphisms are model homomorphisms.

**Definition 6.11.** Let \(\mathcal{C}\) and \(\mathcal{D}\) be difference \(\lambda\)-categories. We say that a functor \(F : \mathcal{C} \rightarrow \mathcal{D}\) is a difference \(\lambda\)-functor if \(F\) preserves the following:

- additive structure: \(F(f + g) = F(f) + F(g)\), and \(F(0) = 0\)
- infinitesimal extension: \(F(\varepsilon(f)) = \varepsilon(F(f))\)
- products via the isomorphism \(\Phi := (F(\pi_1), F(\pi_2))\)
- exponentials via the isomorphism \(\Psi := \Lambda(F(\text{ev}) \circ \Phi)\)
- difference combinator: \(F(\partial [f]) = \partial [F(f)] \circ \Phi\).

We write \(\text{Dif}\lambda\text{-Func}(\mathcal{C}, \mathcal{D})\) for the category of difference \(\lambda\)-functors \(\mathcal{C} \rightarrow \mathcal{D}\) and natural isomorphisms.

**Definition 6.12.** Given a difference \(\lambda\)-theory \(\mathcal{T}\), we say that a category, denoted \(\text{Cl}(\mathcal{T})\), is classifying if there is a model of the theory in \(\text{Cl}(\mathcal{T})\), and this model is “generic”, meaning that for every differential \(\lambda\)-category \(\mathcal{D}\), there is a natural equivalence

\[
\text{Dif}\lambda\text{-Func}(\text{Cl}(\mathcal{T}), \mathcal{D}) \simeq \text{Mod}_{\text{Diff}}(\mathcal{T}, \mathcal{D}).
\]

(6.1)

The classifying category (unique up to isomorphism) is the “smallest” in the sense that given a model of the theory \([\_]_\mathcal{D}\) in a difference \(\lambda\)-category \(\mathcal{D}\), there is a difference \(\lambda\)-functor \(F : \text{Cl}(\mathcal{T}) \rightarrow \mathcal{D}\) such that the interpretation \([\_]_\mathcal{D}\) can be factored through the canonical interpretation in the classifying category, i.e., \([\_]_\mathcal{D} = F \circ [\_]_{\text{Cl}(\mathcal{T})}\).
Conjecture 6.1 ComPLEtEness. Every difference λ-theory $\mathcal{F}$ has a classifying difference λ-category $\text{Cl}(\mathcal{F})$.

7. Conclusions and Future Work

We have defined here the difference λ-calculus, which generalises the differential λ-calculus in exactly the same manner as Cartesian difference categories generalise their differential counterpart. While this calculus is of theoretical interest, it lacks most practical features, such as iteration or conditionals, and it is not immediately obvious how to extend it with these. It is not clear, for example, precisely when iteration combinators are differentiable in the difference category sense.

The problem of iteration is closely related to integration, which is itself the focus of current work on the differential side [8, 13]. Indeed, consider a hypothetical extension of the difference λ-calculus equipped with a type of natural numbers (with the identity as its corresponding infinitesimal extension, that is to say, $\varepsilon_N = \text{id}_N$). How should an iteration operator \texttt{iter} be defined? The straightforward option would be to give it the usual behavior, that is to say:

\[
\text{iter } Z z s \mapsto z \\
\text{iter } (S\ n) z s \mapsto s \text{ (iter } n z s)
\]

These reduction rules entail that every object involved must be complete, that is to say, for every $s, t : A$, there is some $u : A$ with $s + \varepsilon(u) = t$ – such an element is given by the term $((D(\lambda n.\text{iter } n s (\lambda x.t)) \cdot (S\ Z))\ Z)$.

This would rule out a number of interesting models and so it seems unsatisfactory. An alternative is to define the iteration operator by:

\[
\text{iter } Z z s \mapsto z \\
\text{iter } (S\ n) z s \mapsto (\text{iter } n z s) + \varepsilon(s \text{ (iter } n z s))
\]

Fixed $z, s$, and defining the map $\mu(n) := \text{iter } n z s$, its derivative $D[\mu](n, S\ Z)$ is precisely $s(\mu(n))$. Or, in other words, the function $\mu : \mathbb{N} \to A$ is a "curve" which starts at $z$ and whose derivative at a given point $n$ is $s(\mu(n))$ – this boils down to stating that the curve $\mu$ is an integral curve for the vector field $s$ satisfying the initial condition $\mu(Z) = z$! Hence it may be possible to understand iteration as a discrete counterpart of the Picard-Lindelöf theorem, which states that such integral curves always exist (locally).

It would be of great interest to extend λε with an iteration operator and give its semantics in terms of differential (or difference) equations. Studying recurrence equations using the language of differential equations is a very useful tool in discrete analysis; for example, one can treat the recursive definition of the Fibonacci sequence as a discrete ODE and use differential equation methods to find a closed-form solution. We believe that in a language which frames iteration in such terms may be amenable to optimisation by similar analytic methods.

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