Abstract. The a posteriori error estimate and greedy algorithms play central roles in reduced basis method (RBM). A classic RBM uses residual error estimator. The error is estimated in some parameter independent genetic norm and an estimation of the stability constant is also needed. This type of error estimate is not robust with respect to the parameter-dependent energy norm, and it measures only the difference between the RB solution and an unrealistic "truth" finite element solution. The choice of the RB error tolerance is often heuristic and will cause under or over-computing. The finite element solutions for RB snapshots are often computed on a non-optimal mesh, or only computed on a mesh which is good for some parameters in the RB parameter set but not for others.

With the parametric reaction-diffusion and diffusion equations as examples, for a class of parametric convex minimization variational problems, we develop primal-dual reduced basis methods (PD-RBM) with robust true error certifications and adaptive greedy algorithms to overcome the shortcomings of the classic a posteriori error estimator and greedy algorithms in RBM.

For convex minimization variational problems, the duality gap between the primal and dual functionals is a perfect a posteriori error estimator. This primal dual error estimator is robust with respect to the parameters of the problem, and it can be used for both the mesh refinements of finite element methods and the true RB error certification.

We develop three versions of adaptive greedy algorithms to balance the finite element error, the exact reduced basis error, and the adaptive mesh refinements. Extensive numerical tests are performed to test the PD-RBM for three greedy algorithms.

Key words. reduced basis method, robust a posteriori error estimate, greedy algorithm, primal-dual variational problems, true error certification

1. Introduction. The reduced basis method (RBM) is a very accurate and efficient method for solving parameterized problems many times for different parameter values within a given parametric domain, see review articles and books [18, 23, 22, 21, 15, 19, 13, 3].

As pointed out in [22], the RB methodology has several essential components: RB Galerkin projections; greedy sampling procedures; an offline-online computational strategy, and a rigorous a posteriori error estimation used for both the basis selection and the certification of the solution. In this paper, for a class of variational problems based on convex minimizations, we plan to work on two of these essential ingredients: a posteriori error estimations and greedy algorithms, and to develop primal-dual reduced basis methods (PD-RBM) with robust true error certification and adaptive greedy algorithms to overcome the shortcomings of the classic a posteriori error estimator and greedy algorithms in RBM.

Traditionally, the residual type of error estimator, which contains the discrete dual norm of the residual and an estimation of the stability constant, is used in...
RBM. As we will point out, the residual type of error estimator for RBM has several major shortcomings. On the other hand, for the adaptive finite element method (FEM), a posteriori error estimations also play the central role. For adaptive FEMs, beside the residual type error estimator, many other types error estimators with good properties are studied, \cite{1, 26}. For a class of variational problems based on convex minimizations, the primal-dual variational framework can be used to construct a perfect a posteriori error estimator\cite{7, 10, 14}. That is, the duality gap of the primal and dual functionals is a guaranteed upper bound of both approximation errors of primal and dual problems in the corresponding energy norms. With the primal-dual a posteriori error estimation, for such convex minimization variational problems, we can overcome the shortcomings of the residual type of error estimators.

For the greedy procedures, it seems that all current algorithms do not seek the balance of the FE approximation error, and the refinements of the FE mesh (DOFs), and the RB approximation error. An unbalanced algorithm will cause over-computations, thus will waste computational resources. Also, a non-optimal FE mesh for the RB snapshots will reduce the accuracy of the RBM. Thus, we need to have a greedy algorithm with an adaptive RB tolerance and adaptive finite element mesh refinements.

In this introduction, we will first list two problems, the reaction-diffusion problem and the diffusion problem, which will be used as examples in this paper. The shortcomings of the residual type of error estimator and greedy algorithms of the classic RBM will then be discussed. Then we give an overview of the objective, methods, and the layout of the paper.

1.1. Two model problems. We list two model problems that will be used as the context for our discussion of the primal-dual reduced basis method (PD-RBM) for convex minimization variational problems.

Let $\Omega$ be a bounded, polygonal domain in $\mathbb{R}^d$, $d = 2, 3$. For simplicity, we assume the computational domain $\Omega$ is not parametric dependent, while the coefficients and data (the right hand side and boundary conditions) can be parametric dependent. At some places, we will omit the dependence on $\mu$ if no confusion caused.

1.1.1. The reaction-diffusion equation. Consider the parametric reaction-diffusion equation:

\[
\begin{aligned}
-\nabla \cdot (\alpha(\mu)\nabla u(\mu)) + \kappa(\mu)u(\mu) &= f(\mu), \quad \text{in } \Omega, \\
\quad u(\mu) &= g_D(\mu), \quad \text{on } \Gamma_D, \\
-\alpha(\mu)\nabla u(\mu) \cdot n &= g_N(\mu), \quad \text{on } \Gamma_N.
\end{aligned}
\]

where $\Gamma_D$ and $\Gamma_N$ are Dirichlet and Neumann boundaries, separately. We assume $\Gamma_D$ is connected, $\overline{\Gamma_D} \cup \overline{\Gamma_N} = \partial \Omega$ and $\Gamma_D \cap \Gamma_N = \emptyset$. Let $n$ be the unit outward vector normal to the boundary. We assume $f(\mu) \in L^2(\Omega)$, $\alpha(\mu)$ is a symmetric, positive definite piecewise constant function, and $\kappa(\mu) > 0$ is a piecewise constant function. Let $H^1_{D,g}(\Omega) := \{v \in H^1(\Omega) : v = g_D \text{ on } \Gamma_D\}$ and $H^1_D(\Omega) := \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}$.

The weak formulation of (1.1) is: find $u(\mu) \in H^1_{D,g}(\Omega)$ such that

\[
a_{rd}(u(\mu), v; \mu) = F_{rd}(v; \mu), \quad \forall v \in H^1_D(\Omega),
\]
where the bilinear form $a_{rd}$ and the linear form $F_{rd}$ defined as:

$$a_{rd}(w; v; \mu) = (\alpha(\mu)\nabla w, \nabla v) + (\kappa(\mu)^2 w, v), \quad \forall w, v \in H^1(\Omega),$$

$$F_{rd}(v; \mu) = (f(\mu), v) - (g_N(\mu), v)_{\Gamma_N}, \quad \forall v \in H^1(\Omega).$$

Define the the potential energy functional:

$$J(v) = \frac{1}{2} \|\alpha(\mu)^{1/2} \nabla v\|^2_0 + \frac{1}{2} \|\kappa(\mu)v\|^2_0 - (f(\mu), v) + (g_N(\mu), v)_{\Gamma_N}.$$

Then the weak problem (1.2) can also be viewed as a minimization of $J_{rd}(\cdot; \mu)$:

$$J_{rd}(u(\mu); \mu) = \min_{v \in H^1_0(\Omega)} J_{rd}(v; \mu).$$

1.1.2. The diffusion equation. If taking $\kappa = 0$ in the reaction-diffusion equation, we have the diffusion equation. The lack of $\kappa$ actually will make the dual problem of the diffusion problem requiring a dual feasibility condition, see Sect. 6, which is quite different from that of the reaction-diffusion equation. This needs some completely different treatments in the context of RBM, as we will see later in see Sect. 7.

Here for simplicity, we assume that $\Gamma_D \neq \emptyset$ for the diffusion problem. (If $\Gamma_D = \emptyset$, to ensure the uniqueness and existence of the solution, we need to add a uniqueness condition $\int_{\Omega} u \, dx = 0$ and a compatibility condition $\int_{\Omega} g_N(\mu) \, dx = \int_{\Omega} f(\mu) \, dx$.)

We will discuss the variational (minimization problem) and weak formulations of the diffusion equation in details in Section 6.

1.2. Shortcomings of classic residual type of error estimates and greedy algorithms for RBM. First, we give a short review of the standard residual type a posteriori error estimator used in RBM.

Let $D$ be a compact parametric set in $\mathbb{R}^M$. Consider the linear, coercive parametric variational problem: for a $\mu \in D$, find $u(\mu) \in V$, such that

$$a(u(\mu); v; \mu) = f(v), \quad \forall v \in V,$$

where $V$ is an Hilbert space equipped with a parametric-independent norm $\| \cdot \|_V$, for example, the standard $H^1$ or $L^2$ norm. The corresponding inner product is defined as $(\cdot, \cdot)_V$. Let $V_{fe}$ be a finite element subspace of the abstract space $V$ and let $u_h(\mu) \in V_{fe}$ be the so called "truth" FE solution. The number of degrees of freedom (DOFs) of the FE space is usually very large.

For a set of $N$ carefully selected parameters, $S_N = \{\mu_1, \ldots, \mu_N\}$, which we call RB parameter set, the RB space is $V^N_{rb} = \text{span}\{u_h(\mu_i)\}_{i=1}^N$, and we have $V^N_{rb} \subset V_{fe}$. The reduced basis method is: find $u^N_{rb}(\mu) \in V^N_{rb}$, such that

$$a(u^N_{rb}(\mu); v; \mu) = f(v), \quad v \in V^N_{rb}.$$  

Define the residual $r(\cdot; \mu) \in V_{fe}^*$ (the dual space of $V_{fe}$) as:

$$r(v_h; \mu) := f(v_h) - a(u^N_{rb}(\mu); v_h; \mu), \quad \forall v_h \in V_{fe}.$$  

Define the discrete coercivity and continuity constants of the bilinear form by

$$C_{coer,h}(\mu) = \inf_{v_h \in V_{fe}} \frac{a(v_h, v_h; \mu)}{\|v_h\|^2_V} \quad \text{and} \quad C_{cont,h} = \sup_{w_h \in V_{fe}} \sup_{v_h \in V_{fe}} \frac{a(w_h, v_h; \mu)}{\|v_h\|_V \|w_h\|_V}.$$
Let $C_{\text{coer}, h}^{\text{LB}}(\mu)$ be a computable lower bound of the discrete coercivity constant $C_{\text{coer}, h}(\mu)$. We have $C_{\text{coer}, h}^{\text{LB}}(\mu) \leq C_{\text{coer}, h}(\mu)$.

Define the following $V$-norm a posteriori error estimator:

$$\Delta_V(\mu) := \frac{1}{C_{\text{coer}, h}^{\text{LB}}} \sup_{v_h \in V_{\text{fe}}} \frac{r(v_h; \mu)}{\|v_h\|_V},$$

where the dual norm is computed with the help of the Reisz representation. Then the following bound is true:

$$\|u_h(\mu) - u_{\text{rb}}^N(\mu)\|_V \leq \Delta_V(\mu) \leq \frac{C_{\text{cont}, h}}{C_{\text{coer}, h}^{\text{LB}}} \|u_h(\mu) - u_{\text{rb}}^N(\mu)\|_V.$$

We observe several major shortcomings of the above classic residual type of error estimates and the related greedy algorithms for RBM.

1. The error is not measured in the parametric-dependent intrinsic energy norm, and is not robust, thus with a possible very big gap between the estimated error and actual error. For many parametric dependent PDEs, it is often very important to get so-called robust error estimates, which means there should be no unknown genetic constant depending on parameters appeared in a priori or a posteriori error estimates, see [4, 24, 9, 8]. A non-robust result will under or over estimate the error. For parametric dependent problems, such under or over estimations are often very large and can be disastrous.

Take the reaction-diffusion problem with $u = 0$ on $\partial \Omega$ as an example. If we choose the $V$-norm to be the standard $H^1$-norm: $\|v\|_V = (\|\nabla v\|_0^2 + \|v\|_0^2)^{1/2}$. It is clear that the coercivity constant on the continuous level is $\min\{\alpha(\mu), \kappa(\mu)^2\}$ and the continuity constant on the continuous level is $\max\{\alpha(\mu), \kappa(\mu)^2\}$. Choosing $V_{\text{fe}}$ to be the standard linear conforming FE space. The discrete versions of these two constants are the same as long as the mesh is reasonably fine. Thus, the possible gap in (1.7) is

$$\frac{\max\{\alpha(\mu), \kappa(\mu)^2\}}{\min\{\alpha(\mu), \kappa(\mu)^2\}}.$$

This ratio can be extremely pessimistic.

On the other hand, if we choose the intrinsic energy norm:

$$\|v\|_{rd} = (\|\alpha^{1/2} \nabla v\|_0^2 + \|\kappa v\|_0^2)^{1/2}.$$

The continuity and coercivity constants of the bilinear form $a_{rd}(w, v)$ are both one with respect to $\|v\|_{rd}$. If the RB error is estimated in the intrinsic energy norm, we can have robust estimates with ratio in (1.7) to be one.

For residual error estimate for RBM, note that to compute the discrete dual norm or the Riesz representation, an inverse of an $N_{\text{fe}} \times N_{\text{fe}}$ matrix $M_h$, whose $(M_h)_{i,j} = (\lambda_i, \lambda_j)_V$, and $\lambda_i$ is a basis function of $V_{\text{fe}}$ and $N_{\text{fe}}$ is the dimension of $V_{\text{fe}}$, needed to be computed, see page 53 of [15]. To ensure that the online stage computation is independent of $N_{\text{fe}}$, the norm $\|\cdot\|_V$ and the corresponding inner product $(\cdot, \cdot)_V$ have to be parametric independent.

2. The computation of the lower bound of coercivity constant is often complicated, while its role in the error estimator is often limited. Most of the time, we need to use the Successive Constraint Method (SCM) [17] to estimate the discrete coercivity constant. But as shown in the reaction-diffusion equation example, even the estimation of the
constant is exact, it is still only one constant. From the linear algebra viewpoint, the
discrete coercivity constant is the smallest generalized singular value of the matrix
associated with bilinear form. While the ratio in (1.7) is the ratio of the biggest and
the smallest generalized singular values. On the other hand, the parameter energy
norm uses the all the information of the bilinear form and its associated matrix.

Thus, by our point (1), even we pay some non-trivial expense (a big eigenvalue
problem at offline stage and a small linear optimization problem at online stage) to
compute the discrete coercivity constant accurately, it still can largely over estimate
the error if we measure the error in parameter independent norms.

If we can measure the error in the energy norm and can avoid the computation
of the discrete coercivity constant, it will be a great advantage.

(3) The RB error and the FE error tolerances are often unbalanced. By the triangle
inequality, the RB error is bounded by the FE error and the RB-FE error.

\[ \| u(\mu) - u_{N}^{N}(\mu) \| \leq \| u(\mu) - u_{h}(\mu) \| + \| u_{h}(\mu) - u_{N}^{N}(\mu) \|. \]

Of course, what we are really interested in is the reduction and certification of the
true RB error.

We often assume that \( V_{fe} \) is extreme fine so that the FE error \( \| u(\mu) - u_{h}(\mu) \| \) is
small compared to the RB-FE error \( \| u_{h}(\mu) - u_{N}^{N}(\mu) \| \). We usually construct the RB
space with a large enough dimension to make the RB-FE error very small, often \( 10^{-6} \)
or even \( 10^{-10} \). In practice, the assumption that the FE error is very small is often
questionable. Especially for many challenging problems, a non-optimal mesh, even
with quite large degrees of freedom \( N_{fe} \), often will make the FE error to be quite large
for some specific \( \mu \), possibly only 0.1 or 0.5, which makes the very small certification
the RB-FE error quite meaningless and a big waste of computational resources. Thus,
when designing greedy algorithms, we need to balance the RB error with respect to
the FE approximation error.

(4) The RB snapshots are often computed on non-optimal meshes. In RBM, in order
to compute the inner products of different RB snapshots, we need to make sure that
all RB snapshots are computed in the same mesh. (Although in theory, we can com-
pute them in different meshes the use some grid transfer operators to compute their
inner products. But these grid transfer operators are highly non-trivial.) For many
challenging problems, different parameters \( \mu \) need different locally refined meshes.
Still taking the reaction-diffusion problem as example, for discontinuous \( \alpha \), the singu-
laritly often appears at the intersecting points of different \( \alpha(\mu) \), see [9]. For \( \kappa^{2} \) being
much bigger than the scale of \( \alpha \), a fine mesh is required near the boundary. See our
numerical experiment in Fig. 11 for an example. Thus, a naively chosen fixed mesh,
or a mesh only optimal for one specific choice of parameter \( \mu \) is not a good choice to
ensure the good quality of all RB snapshots.

1.3. Objective, methods, and layout of the paper. With the parametric
reaction-diffusion and diffusion equations as examples, for a class of parametric convex
minimization variational problems, in this paper, we develop primal-dual reduced
basis methods (PD-RBM) with robust true error certification and adaptive greedy
algorithms to overcome the shortcomings of the classic a posteriori error estimator
and greedy algorithms in RBM.

Many mathematical and physical important problems can be viewed as minimiza-
tions of a primal energy functional, like the reaction-diffusion equation and diffusion
equation. By introducing a Lagrangian, a corresponding dual problem, which is a
maximization of a dual energy functional, can be constructed. The pair, which con-
tains the minimizer of the primal problem and maximizer of the dual problem, is then the saddle point of the Lagrangian. For approximations of the primal and dual problems, the duality gap between the primal and dual functionals at their respective approximations is a guaranteed bound. Thus, we propose primal dual (PD) error estimators based on the duality gap. The PD error estimator can be both used on the FE and RB contexts. In the FE context, the primal dual error estimator is reliable, efficient, and robust with respect to the parameters, and can be used to drive the adaptive mesh refinements. When applied to the RBM, unlike the residual error estimator, the primal dual error estimator measures the true error in the robust parameter dependent energy norm, not some "truth" FE-RB error in non-robust parameter independent genetic norm, and there is no need to compute the coercivity constants. Thus, the PD a posteriori error estimator can overcome the shortcomings of the standard residual type error estimator for this class of convex minimization variational problems.

Even though the idea of using the PD error estimator is from the finite element community, we need to point out some important difference when applying it to RBM.

For the a posteriori error estimation for the finite elements by the primal-dual framework, where the goal is to estimate the error for the primal problem, an approximation of the dual solution is often only solved locally to reduce the computational cost, see [7, 10]. Sometimes, the dual variable is even constructed purely explicitly to ensure the construction is computationally cheap, see [7]. For parametric dependent problem, simple explicit construction may leads to the bound to be non-robust, that is, although it is a guaranteed bound, but the bound can be too big, see the example presented in [25]. Thus, even for local construction of the dual variables for FE a posteriori error estimates, we pointed out that a local optimization problem is necessary to ensure the error estimator is robust w.r.t. the parameters, see [10].

For the case of RBM, the scenario is quite different. A RB dual solution is also going to be used to compute the PD error estimator. Thus, when computing the RB snapshots of the dual problems, instead of computing locally, the better strategy is to solve the dual problems globally in full. This way, we can guarantee the primal and the dual FE approximations are of the same order of accuracy. Thus, we can ensure the duality gap and the error estimator of the RB snapshots computed by finite elements is tight and efficient. The good quality of the RB basis will then guarantee a good RB approximations and a tight error bound.

For the RB formulation of the dual problem, if a dual feasibility condition is required, using the technique similar to the treatment of the non-homogenous essential condition, we transform the dual RB problem into a standard Galerkin projection problem into the space with a homogenous dual feasibility condition.

We develop three versions of adaptive greedy algorithms to balance the finite element error, the exact reduced basis error, and the adaptive mesh refinements. Extensive numerical tests are performed to test the PD-RBM for three greedy algorithms. Earlier ideas of using adaptive numerical solutions in RB context can be found in [2, 28, 29], but the RB tolerance are all fixed in these papers, and the FE mesh is not optimal for all parameters in $S_N$. Compared to these results, the algorithms in this paper want to adaptively modify the RB tolerance, the FE tolerance, the FE mesh to balance the computational cost.

The idea of applying the primal and dual variational formulations to RBM was first proposed in [27]. Compared to [27], our treatment is more systematic and has several significant improvements. In [27], the dual problems are solved by local post-processing instead of global optimal finite element method (Galerkin projection into
the FE space). Thus our error estimator is tighter. We also correct identify the dual problem of the reaction-diffusion problem, where no dual feasibility conditions are needed. For the dual problem of the diffusion problem, instead of solving a saddle point problem, we transform the problem into a Galerkin projection problem in the space with a homogeneous dual feasibility condition.

The layout of the paper is as follows. We present in section 2 a short account of primal dual variational principle and its duality gap in the general abstract setting. Several abstract spaces and finite element spaces are given in section 3. Sections 4 and 5 apply the primal and dual framework to the reaction-diffusion equation and its corresponding primal dual RBM with PD error estimator. The PD-RBM for the diffusion is developed in sections 6 and 7. In section 8, we present three adaptive greedy algorithms, one with an adaptive RB tolerance and a fixed mesh, one with an adaptive mesh and a fixed RB tolerance, and one with double adaptivities. Extensive numerical experiments are performed to test the PD-RBM and the three greedy algorithms in section 8. In section 9, we make some concluding remarks and discuss some future plans.

2. Primal-Dual Variational Principles for a Class of Convex Problems.

For strongly convex minimization problems like the reaction-diffusion equation and the diffusion equation, the ideal framework for a posteriori error estimation is the primal and dual variational formulations [12, 14].

We first discuss an abstract framework: Let $Y$ and $Z$ be any two sets, and let $\mathcal{L} : Y \times Z \mapsto \mathbb{R}$ be a function. A point $(u, \sigma) \in Y \times Z$ is said to be a saddle point, if

$$\sup_{\tau \in Z} \mathcal{L}(u, \tau) = \mathcal{L}(u, \sigma) = \inf_{v \in Y} (v, \sigma).$$

By Theorem 7.16-1 of [11], if $(u, \sigma) \in Y \times Z$ is a saddle point of $\mathcal{L}$, then

$$\inf_{v \in Y} \sup_{\tau \in Z} \mathcal{L}(v, \tau) = \sup_{\tau \in Z} \mathcal{L}(u, \tau) = \mathcal{L}(u, \sigma) = \inf_{v \in Y} \sup_{\tau \in Z} \mathcal{L}(v, \tau).$$

Define the primal problem as:

$$J(u) = \inf_{v \in Y} J(v),$$

with the primal functional $J(v) := \sup_{\tau \in Z} \mathcal{L}(v, \tau)$,

and define the dual problem as:

$$J^*(\sigma) = \sup_{\tau \in Z} J^*(\tau),$$

with the dual functional $J^*(\tau) = \inf_{v \in Y} \mathcal{L}(v, \tau)$.

Since $(u, \sigma)$ is a saddle point, thus

$$J(u) = \mathcal{L}(u, \sigma) = J^*(\sigma).$$

Let $\tilde{u} \in Y$ be an approximation of $u$ and $\tilde{\sigma} \in Z$ be an approximation of $\sigma$, then

$$J(u) \leq J(\tilde{u}), \quad \forall \tilde{u} \in Y, \quad \text{and} \quad J^*(\tilde{\sigma}) \leq J^*(\sigma), \quad \forall \tilde{\sigma} \in Z.$$

So

$$0 \leq J(\tilde{u}) - J(u) = J(\tilde{u}) - J^*(\sigma) \leq J(\tilde{u}) - J^*(\tilde{\sigma}).$$

(2.1)
So the duality gap $J(\tilde{u}) - J^*(\tilde{\sigma})$ is a computable guaranteed bound for both the errors of $\tilde{u} - u$ and $\tilde{\sigma} - \sigma$. Since both the primal and dual problems are optimization problems, both approximation errors are often measured in some intrinsic energy norms.

To make sure the error bound is tight, we need to make sure that both the $\tilde{u}$ and $\tilde{\sigma}$ are good approximations of respective solutions and they should have the same approximation order.

Given a convex variational minimization problem (the primal problem), many concrete constructions of its dual problem can be found in [12]. The construction in [12] is based on an abstract operator framework and with homogeneous boundary conditions. In the following sections, we use the reaction-diffusion equation and the diffusion equation as example to show how to derive the dual problems with non-homogeneous boundary conditions elementarily.

3. Several families of abstract spaces and finite element spaces. First, define the standard $H(div; \Omega)$ space:

$$H(div; \Omega) = \{ \tau \in L^2(\Omega)^d : \nabla \cdot \tau \in L^2(\Omega) \}.$$  

Then we define

$$H_{N,g}(div; \Omega) = \{ \tau \in H(div; \Omega) : \tau \cdot n = g_N \text{ on } \Gamma_N \},$$

and

$$H_N(div; \Omega) = \{ \tau \in H(div; \Omega) : \tau \cdot n = 0 \text{ on } \Gamma_N \}.$$  

Let $T = \{K\}$ be a triangulation of $\Omega$ using simplicial elements. The mesh $T$ is assumed to be regular. For an element $K \in T$ and integer $k \geq 0$, let $P_k(K)$ the space of polynomials with degrees less than or equal to $k$. Define the discontinuous piecewise constant space $D_h$ as

$$D_h := \{ v \in L^2(\Omega) : v|_K \in P_0(K), K \in T \}.$$  

Define linear $H_1$-conforming finite element spaces $V_h$ as follows:

$$V_h := \{ v \in H^1(\Omega) : v|_K \in P_1(K), K \in T \}.$$  

And define $H(div)$-conforming Raviart-Thomas finite element spaces $\Sigma_h$ as follows:

$$\Sigma_h := \{ \tau \in H(div; \Omega) : \tau|_K \in P_0(K)^d + xP_0(K), \forall K \in T \}.$$  

For simplicity, we assume that $g_D$ defined on $\Gamma_D$ can be exactly approximated by the trace of $V_h$ on $\Gamma_D$, and that $g_N$ defined on $\Gamma_N$ can be exactly approximated by the normal trace of $\Sigma_h$ on $\Gamma_N$.

Define the finite element spaces with boundary conditions as follows:

$$V_{h,g} = \{ v \in V_h : v = g_D \text{ on } \Gamma_D \} \quad \text{and} \quad V_{h,0} = \{ v \in V_h : v = 0 \text{ on } \Gamma_D \},$$

$$\Sigma_{h,g} = \{ \tau \in \Sigma_h : \tau \cdot n = g_N \text{ on } \Gamma_N \} \quad \text{and} \quad \Sigma_{h,0} = \{ \tau \in \Sigma_h : \tau \cdot n = 0 \text{ on } \Gamma_N \}.$$  

In this paper, we use the lowest order spaces for simplicity, but the theory and methods developed in the paper can be applied to higher order finite element approximations.

4. The reaction-diffusion problem. We discuss the application of the primal-dual framework to the reaction-diffusion equation as our first example.
4.1. Primal and dual variational problems. As discussed in Section 1.1.1, the weak formulation in (1.2) may be rewritten as a functional minimization problem (the primal problem).

The primal problem Find \( u \in H^1_{D,g}(\Omega) \) such that

\[
J_{rd}(u) = \inf_{v \in H^1_{D,g}(\Omega)} J_{rd}(v),
\]

where the energy (primal) functional \( J_{rd} \) is defined as:

\[
J_{rd}(v) = \frac{1}{2} \left\{ (\alpha \nabla v, \nabla v) + (\kappa^2 v, v) \right\} - (f, v) + (g_N, v)_{\Gamma_N}.
\]

To find its dual problem, consider the following Lagrangian:

\[
L_{rd}(v, \tau) = J_{rd}(v) - \frac{1}{2} \|\alpha^{-1/2} \tau + \alpha^{1/2} \nabla v\|_0^2, \quad \forall v \in H^1_{D,g}(\Omega), \tau \in H_{N,g}(\text{div}; \Omega).
\]

It is clear that

\[
\sup_{\tau \in H_{N,g}(\text{div};\Omega)} L_{rd}(v, \tau) = L_{rd}(v, -\alpha \nabla v) = J_{rd}(v).
\]

By integration by parts, we have

\[
L_{rd}(v, \tau) = -\frac{1}{2} (\|\alpha^{-1/2} \tau\|_0^2 + \|\kappa^{-1}(\nabla \cdot \tau - f)\|_0^2 + \|\kappa v + \kappa^{-1}(\nabla \cdot \tau - f)\|_0^2)
\]

\[
- \langle \tau \cdot n, g_D \rangle_{\Gamma_D},
\]

which implies

\[
\inf_{v \in H^1_{D,g}(\Omega)} L_{rd}(v, \tau) = L_{rd}(\kappa^{-2}(f - \nabla \cdot \tau), \tau) = J^*_r(\tau),
\]

where the complimentary (dual) functional is given by

\[
J^*_r(\tau) = -\frac{1}{2} \|\alpha^{-1/2} \tau\|_0^2 - \frac{1}{2} \|\kappa^{-1}(\nabla \cdot \tau - f)\|_0^2 - \langle \tau \cdot n, g_D \rangle_{\Gamma_D}.
\]

We then can define the dual problem.

The dual problem. Find \( \sigma \in H_{N,g}(\text{div}; \Omega) \) such that

\[
J^*_r(\sigma) = \sup_{\tau \in H_{N,g}(\text{div};\Omega)} J^*_r(\tau).
\]

The corresponding weak formulation of (4.7) is: Find \( \sigma \in H_{N,g}(\text{div}; \Omega) \), such that

\[
b_{rd}(\sigma, \tau) = G_{rd}(\tau; \mu), \quad \forall \tau \in H_N(\text{div}; \Omega).
\]

The bilinear from \( b_{rd}(\cdot, \cdot) \) is defined as:

\[
b_{rd}(\tau, \rho) := (\alpha^{-1} \tau, \rho) + (\kappa^{-2} \nabla \cdot \tau, \nabla \cdot \rho), \quad \forall \tau, \rho \in H(\text{div}; \Omega),
\]

and the linear form is defined as:

\[
G_{rd}(\tau; \mu) := (\kappa^{-2} f, \nabla \cdot \tau) - \langle \tau \cdot n, g_D \rangle_{\Gamma_D}, \quad \forall \tau \in H(\text{div}; \Omega).
\]
Remark 4.1. In [27], the dual problem of the reaction-diffusion equation (see Section 3.3 of [27]) is unnecessarily complicated. Essentially, the formulation of the dual problem in Section 3.3 of [27] is the mixed formulation of the reaction-diffusion equation where an extra variable and an extra dual feasibility condition are needed. From our derivation, the dual problem (4.8) is very simple and straightforward.

The saddle point. A simple calculation with integration by parts gives

\[
2(J_{rd}(v) - J^*_{rd}(\tau)) = \| \alpha^{1/2} \nabla v + \alpha^{-1/2} \tau \|_0^2 + \| \kappa^{-1}(\nabla \cdot \tau + \kappa^2 v - f) \|_0^2
\]

for all \( v \in H_{D,g}^1(\Omega) \) and all \( \tau \in H_{N,g}(\text{div}; \Omega) \). This implies that \((\sigma, u)\) is a saddle point of \( L_{rd}(\tau, v) \) satisfying

\[
\inf_{v \in H_{D,g}^1(\Omega)} J_{rd}(v) = J_{rd}(u) = L_{rd}(\sigma, u) = J^*_{rd}(\sigma) = \sup_{\tau \in H_{N,g}(\text{div}; \Omega)} J^*(\tau).
\]

Moreover, we have

\[
\sigma = -\alpha \nabla u \quad \text{and} \quad \nabla \cdot \sigma + \kappa^2 u = f \quad \text{in} \ \Omega.
\]

4.2. Error relations. In this subsection, we discuss various error relations of the primal and dual problems of the reaction-diffusion equation. We assume that \( u \in H_{D,g}^1(\Omega) \) and \( \sigma \in H_{N,g}(\text{div}; \Omega) \) be the exact solution of the primal and dual problems respectively.

The primal error relation. For any \( v \in H_{D,g}^1(\Omega) \), we have

\[
0 \leq 2(J_{rd}(v) - J_{rd}(u))
= \| \alpha^{1/2} \nabla v \|_0^2 + \| \kappa \|_0^2 - \| \alpha^{1/2} \nabla u \|_0^2 - \| \kappa \|_0^2 - 2(f, v - u) + 2(g_N, v - u) \gamma_N
= (\alpha \nabla (u + v), \nabla (v - u)) + (\kappa^2 (u + v), v - u) - 2(f, v - u) + 2(g_N, v - u) \gamma_N.
\]

Since \((\alpha \nabla u, \nabla (v - u)) + (\kappa^2 u, v - u) = (f, v - u) + 2(g_N, v - u) \gamma_N\), for all \( v \in H_{D,g}^1(\Omega) \), then

\[
2(J_{rd}(v) - J_{rd}(u)) = \| v - u \|_{rd}^2, \quad v \in H_{D,g}^1(\Omega),
\]

where the reaction-diffusion energy norm for the primal variable is defined as:

\[
\| v \|_{rd} = \left( \| \alpha^{1/2} \nabla v \|_0^2 + \| \kappa \|_0^2 \right)^{1/2}, \quad \forall v \in H^1(\Omega).
\]

The dual error relation. For any \( \tau \in H_{N,g}(\text{div}; \Omega) \), by a similar argument as the primal problem, we have

\[
2(J^*_{rd}(\sigma) - J^*_{rd}(\tau)) = \| \sigma - \tau \|_{rd}^2.
\]

where the reaction-diffusion energy norm for the dual variable is defined as:

\[
\| \tau \|_{rd} = \left( \| \alpha^{-1/2} \tau \|_0^2 + \| \kappa^{-1} \nabla \cdot \tau \|_0^2 \right)^{1/2}, \quad \forall \tau \in H(\text{div}; \Omega).
\]

Duality gap as guaranteed error upper bound. We have the following guaranteed error upper bounds for all \( v \in H_{D,g}(\Omega) \) and \( \tau \in H_{N,g}(\text{div}; \Omega) \),

\[
\| u - v \|_{rd}^2 = 2(J_{rd}(v) - J_{rd}(u)) = 2(J_{rd}(v) - J^*_{rd}(\sigma)) \leq 2(J_{rd}(v) - J^*_{rd}(\tau)),
\]
Define the combined energy norm as follows:

\[(v, \tau)_{rd} = \sqrt{\|v\|^2_{rd} + \|\tau\|^2_{rd}}.\]

For all \(v \in H_{D,g}(\Omega)\) and for all \(\tau \in H_{N,g}(\text{div}; \Omega)\), by (4.9), the following combined error relation holds:

\[\|(u - v, \sigma - \tau)\|_{rd}^2 = 2(J_{rd}(v) - J_{rd}^*(\tau)) = \alpha^{1/2}\|\nabla v + \alpha^{-1/2}\tau\|_0^2 + \|\kappa^{-1}(\nabla \cdot \tau + \kappa^2 v - f)\|_0^2.\]

4.3. FEMs for the primal and dual problems. The finite element approximation of the primal problem is: find \(u_h \in V_{h,g}\), such that

\[J_{rd}(u_h) = \min_{v \in V_{h,g}} J_{rd}(v),\]

or equivalently: find \(u_h \in V_{h,g}\), such that

\[a_{rd}(u_h, v) = F_{rd}(v),\quad \forall v \in V_{h,0}.\]

The FE approximation of the dual problem is: find \(\sigma_h \in \Sigma_{h,g}\), such that

\[J_{rd}^*(\sigma_h) = \min_{\tau \in \Sigma_{h,g}} J_{rd}(\tau),\]

or equivalently: find \(\sigma_h \in \Sigma_{h,g}\), such that

\[b_{rd}(\sigma_h, \tau) = G_{rd}(\tau),\quad \forall \tau \in \Sigma_{h,0}.\]

4.4. Robust and optimal a priori error estimates of FE approximations.

By the definition of energy norms, both bilinear forms of the primal and dual problems are coercive and continuous with constant one; it is very easy to prove the following robust and optimal by the standard FE analysis.

For the primal and dual FE problems, we have the best approximation results:

\[(u - u_h)_{rd} \leq \inf_{v_h \in V_{h,g}} \|u - v_h\|_{rd} \quad \text{and} \quad \|\sigma - \sigma_h\|_{rd} \leq \inf_{\tau_h \in \Sigma_{h,g}} \|\sigma - \tau_h\|_{rd}.\]

For the combined error, by (4.18), we have:

\[\|(u - u_h, \sigma - \sigma_h)\|_{rd} \leq \inf_{(v, \tau) \in V_{h,g} \times \Sigma_{h,g}} \|(u - v, \sigma - \tau)\|_{rd}.\]

Remark 4.2. Since we treat the primal and dual problems as a combined problem, we need to make sure that they are of the same order of approximation. Thus, for the continuous \(P_k, k \geq 1\) approximation of the primal variable, we should use the corresponding \(RT_{k-1}\) approximation of the dual variable.

4.5. PD-FE a posteriori error estimators and indicators. Choosing \(v = u_h\) and \(\tau = \sigma_h\) in the combined error relation (4.17), we have

\[\|(u - u_h, \sigma - \sigma_h)\|_{rd}^2 = 2(J_{rd}(u_h) - J_{rd}^*(\sigma_h))\]

\[= \|\alpha^{1/2}\nabla u_h + \alpha^{-1/2}\sigma_h\|_0^2 + \|\kappa^{-1}(\nabla \cdot \sigma_h + \kappa^2 u_h - f)\|_0^2.\]
The first identity (4.20) (duality gap) is perfect for error control for both adaptive FEM and online RBM basis selection. The second identity (4.21) can be used to drive the adaptive mesh refinement.

We define the local primal-dual (PD) error indicator to drive the mesh refinement:

$$\eta_{rd,h,K} = \left( \| \alpha^{1/2} \nabla u_h + \alpha^{-1/2} \sigma_h \|_{0,K}^2 + \| \kappa^{-1/2} (\nabla \cdot \sigma_h + \kappa^2 u_h - f) \|_{0,K}^2 \right)^{1/2}, \forall K \in T,$$

and define the global PD error estimator to control the error:

$$\eta_{rd,h} = \left( \sum_{K \in T} \eta_{rd,h,K}^2 \right)^{1/2} = 2(J_{rd}(u_h) - J_{rd}^*(\sigma_h)).$$

For the combined error, the PD error estimator $\eta_{rd,h}$ is both reliable and efficient with constant one:

$$\eta_{rd,h} = \| (u - u_h) - (\sigma - \sigma_h) \|_{rd}.$$

Locally, since $\sigma = -\alpha \nabla u$ and $f = \nabla \cdot \sigma + \kappa^2 u$, we have

$$\eta_{rd,h,K}^2 = \| \alpha^{1/2} \nabla u_h + \alpha^{-1/2} \sigma_h \|_{0,K}^2 + \| \kappa^{-1/2} (\nabla \cdot \sigma_h + \kappa^2 u_h - f) \|_{0,K}^2$$

$$= \| \alpha^{1/2} \nabla (u_h - u) + \alpha^{-1/2} (\sigma_h - \sigma) \|_{0,K}^2 + \| \kappa^{-1/2} \nabla \cdot (\sigma_h - \sigma) + \kappa (u_h - u) \|_{0,K}^2.$$

Then it is easy to derive that the following local efficiency holds:

$$\eta_{rd,h,K} \leq \| (u - u_h) - (\sigma - \sigma_h) \|_{rd,K}.$$

Thus, if we treat the primal and dual FE problems as a unified combined problem, the local indicator $\eta_{rd,h,K}$ and the global error estimator $\eta_{rd,h}$ are perfect for both mesh refinements and global error control. All error estimates hold without any unknown constant.

**Remark 4.3.** For the individual primal and dual finite element problems, $\eta_{rd,h}$ is of course a robust reliable error estimator with the reliability constant one.

$$\| u - u_h \|_{rd,h} \leq \eta_{rd,h} \text{ and } \| \sigma - \sigma_h \|_{rd} \leq \eta_{rd,h}.$$  

It is an ongoing research work to show that $\eta_{rd,h}$ is also robust efficient for the individual primal and dual finite element problems:

$$C \eta_{rd,h} \leq \| u - u_h \|_{rd} \text{ and } \eta_{rd,h} \leq C \| \sigma - \sigma_h \|_{rd},$$

where $C > 0$ is a constant independent of $\alpha$, $\kappa$, and the mesh size $h$.

### 5. RBMs for the primal and dual problems of the reaction-diffusion equation.

In this section, we use $\mu$-dependent forms, functionals, and norms to emphasize that $\alpha$, $\kappa$, and $f$ are $\mu$-dependent. The meanings of these notations are self-explicit.

For non-homogeneous essential boundary conditions (Dirichlet for the primal problem and Neumann for the dual problem), the treatment is standard. We often first find a lifting function satisfying the essential boundary condition and then change the problem into a problem with homogeneous essential boundary condition, see page 165 of [21].

Since the existence of non-homogeneous boundary conditions puts no essential difficulty on RBM but makes the presentation cumbersome, here, we assume that all boundary conditions are zero for simplicity:

$$g_D = 0 \text{ on } \Gamma_D \text{ and } g_N = 0 \text{ on } \Gamma_N.$$
5.1. RBM for the primal problem. Suppose that a set of \( N \) important parameters \( S_N = \{ \mu_1, \cdots, \mu_N \} \subset D \) is chosen, then the RB space for the primal problem is the span of the corresponding FE solutions.

\[
V_{rb}^N = \text{span}\{u_h(\mu_1), \cdots, u_h(\mu_N)\} \subset V_{h,0}.
\]

where \( u_h(\mu_i) \in V_{h,0} \), such that

\[
a_{rd}(u_h(\mu_i), v; \mu_i) = F_{rd}(v, \mu_i), \quad \forall v \in V_{h,0}.
\]

Then the RB approximation is: for any given \( \mu \in D \), find \( v_{rb}^N(\mu) \in V_{rb}^N \) such that

\[
a_{rd}(u_h^N(\mu), v_{rb}; \mu) = F_{rd}(v_{rb}; \mu), \quad \forall v_{rb} \in V_{rb}^N.
\]

With respect to the \( \mu \)-dependent energy norm, it is easy to show that the following best approximation estimate with constant one for any given \( \mu \in D \) holds:

\[
\|u(\mu) - u_{rb}^N(\mu)\|_{rd, \mu} \leq \inf_{v_{rb} \in V_{rb}^N} \|u(\mu) - v_{rb}\|_{rd, \mu}.
\]

5.2. RBM for the dual problem. With \( S_N = \{ \mu_1, \cdots, \mu_N \} \in D \) given, then the RB space for the dual problem is the span of the corresponding FE solutions of the dual problem:

\[
\Sigma_{rb}^N = \text{span}\{\sigma_h(\mu_1), \cdots, \sigma_h(\mu_N)\} \subset \Sigma_{h,0},
\]

where \( \sigma_h(\mu_i) \in \Sigma_{h,0} \), such that

\[
b_{rd}(\sigma_h(\mu_i), \tau; \mu_i) = G_{rd}(\tau; \mu_i), \quad \forall \tau \in \Sigma_{h,0}.
\]

Then the reduced basis approximation is: for any given \( \mu \in D \), find \( \sigma_{rb}^N(\mu) \in \Sigma_{rb}^N \) such that

\[
b_{rd}(\sigma_{rb}^N(\mu), \tau_{rb}; \mu) = G_{rd}(\tau_{rb}; \mu), \quad \forall \tau_{rb} \in \Sigma_{rb}^N.
\]

With respect to the \( \mu \)-dependent energy norm, we have the following best approximation estimate with constant one for any given \( \mu \in D \):

\[
\|\sigma(\mu) - \sigma_{rb}^N(\mu)\|_{rd, \mu} \leq \inf_{\tau_{rb} \in \Sigma_{rb}^N} \|\sigma(\mu) - \tau_{rb}^N(\mu)\|_{rd, \mu}.
\]

5.3. A priori combined error estimates for RBM. For the combined error, by (5.22) and (5.24), we have the following robust and optimal a priori error estimate:

\[
\|(u(\mu) - u_h(\mu), \sigma(\mu) - \sigma_h(\mu))\|_{rd, \mu} \leq \inf_{(v, \tau) \in V_{h,0} \times \Sigma_{h,0}} \|(u(\mu) - v, \sigma(\mu) - \tau)\|_{rd, \mu}.
\]

5.4. PD-RB a posteriori error estimator. Let \( v = u_{rb}^N(\mu) \) and \( \tau = \sigma_{rb}^N(\mu) \) in the combined error relation (4.17), we have:

\[
\|(u - u_h, \sigma - \sigma_h)\|_{rd, \mu}^2 = 2(J_{rd}(u_{rb}^N(\mu); \mu) - J_{rd}^*(\sigma_{rb}^N(\mu); \mu)).
\]

We define the primal dual reduced basis (PD-RB) error estimator:

\[
\eta_{rd,rb}(\mu) = \sqrt{2(J_{rd}(u_{rb}^N(\mu); \mu) - J_{rd}^*(\sigma_{rb}^N(\mu); \mu))}.
\]

It is a perfect error estimator for the combined RB energy error since \( \eta_{rd,rb}(\mu) = \|(u(\mu) - u_h(\mu), \sigma(\mu) - \sigma_h(\mu))\|_{rd, \mu} \).
5.5. RBMs in matrix forms. In this subsection, we present the primal and dual RBMs for the reaction-diffusion equation in matrix forms. With the affine decomposition assumption, the online computations of the RB solutions and the PD-RB error estimator are efficient and only depend on the dimension of RB spaces.

**Assumption 5.1.** (Affine decomposition) The bilinear forms \( a_{rd} \) and \( b_{rd} \) and the linear forms \( F_{rd} \) and \( G_{rd} \) can be decomposed into a finite sum of parameter-dependent functions multiplied by parameter-independent forms, i.e.,

\[
\begin{align*}
\ a_{rd}(w, v; \mu) &= \sum_{q=1}^{Q_{a,rd}} \theta_{q}^{a_{rd}}(\mu) a_{q}^{rd}(w, v), \\
\ F_{rd}(v; \mu) &= \sum_{q=1}^{Q_{F,rd}} \theta_{q}^{F,rd}(\mu) F_{q}^{rd}(v), \\
\ b_{rd}(\rho, \tau; \mu) &= \sum_{q=1}^{Q_{b,rd}} \theta_{q}^{b,rd}(\mu) b_{q}^{rd}(\rho, \tau), \\
\ G_{rd}(\tau; \mu) &= \sum_{q=1}^{Q_{G,rd}} \theta_{q}^{G,rd}(\mu) G_{q}^{rd}(\tau).
\end{align*}
\]

with the bilinear forms \( a_{q}^{rd} \) and \( b_{q}^{rd} \) and the linear forms \( F_{q}^{rd} \) and \( G_{q}^{rd} \) are independent of the parameter \( \mu \) and all \( \theta \) functions are scalar quantities which are independent of \( w, v, \) or \( \tau, \rho \).

Since in practice, some kind of orthonormalization are often performed to improve the condition number of the discrete RB problem, we assume that

\[
\begin{align*}
V_{rb}^{N} &= \text{span}\{u_{h}(\mu_{i})\}_{i=1}^{N} = \text{span}\{\xi_{i}\}_{i=1}^{N}, \\
\Sigma_{rb}^{N} &= \text{span}\{\sigma_{h}(\mu_{i})\}_{i=1}^{N} = \text{span}\{\phi_{i}\}_{i=1}^{N}.
\end{align*}
\]

We then pre-assemble in an offline procedure the \( Q \) corresponding matrices \( A_{rb}^{rd}, B_{rb}^{rd} \in \mathbb{R}^{N \times N} \) defined by

\[
\begin{align*}
(\ A_{rd}^{q})_{ij} &= a_{q}^{rd}(\xi_{j}, \xi_{i}), \quad 1 \leq i, j \leq N, 1 \leq q \leq Q_{a,rd}, \\
(\ B_{rd}^{q})_{ij} &= b_{q}^{rd}(\phi_{j}, \phi_{i}), \quad 1 \leq i, j \leq N, 1 \leq q \leq Q_{b,rd}, \\
(\ F_{rd}^{q})_{ij} &= F_{q}^{rd}(\xi_{j}), \quad 1 \leq j \leq N, 1 \leq q \leq Q_{F,rd}, \\
(\ G_{rd}^{q})_{ij} &= G_{q}^{rd}(\phi_{j}), \quad 1 \leq j \leq N, 1 \leq q \leq Q_{G,rd}.
\end{align*}
\]

For a given \( \mu \in \mathcal{D} \), at online stage, we assemble the matrices \( A_{rb}^{rd}(\mu), B_{rb}^{rd}(\mu) \in \mathbb{R}^{N \times N} \) and vectors \( F_{rb}^{rd}(\mu), G_{rb}^{rd}(\mu) \in \mathbb{R}^{N} \) as

\[
\begin{align*}
A_{rb}^{rd}(\mu) &= \sum_{q=1}^{Q_{a,rd}} \theta_{q}^{a,rd}(\mu) A_{rd}^{q}, \quad \text{and} \quad F_{rb}^{rd}(\mu) = \sum_{q=1}^{Q_{F,rd}} \theta_{q}^{F,rd}(\mu) F_{rd}^{q}, \\
B_{rb}^{rd}(\mu) &= \sum_{q=1}^{Q_{b,rd}} \theta_{q}^{b,rd}(\mu) B_{rd}^{q}, \quad \text{and} \quad G_{rb}^{rd}(\mu) = \sum_{q=1}^{Q_{G,rd}} \theta_{q}^{G,rd}(\mu) G_{rd}^{q}.
\end{align*}
\]

Then the primal RB problem (5.22) in the matrix form is: find \( u_{rb}^{N}(\mu) \in \mathbb{R}^{N} \), such that

\[
A_{rb}^{rd}(\mu) u_{rb}^{N}(\mu) = F_{rb}^{rd}(\mu).
\]

The dual RB problem (5.24) in the matrix form is: find \( \sigma_{rb}^{N}(\mu) \in \mathbb{R}^{N} \), such that

\[
B_{rb}^{rd}(\mu) \sigma_{rb}^{N}(\mu) = G_{rb}^{rd}(\mu).
\]
For the functional values, we have:

\[ J_{rd}(u_{rb}^N(\mu); \mu) = \frac{1}{2}(u_{rb}^N(\mu))^T A_{rb}^d(\mu) u_{rb}^N(\mu) - (u_{rb}^N(\mu))^T E_{rb}^d(\mu), \]

and \( J_{rd}^*(\sigma_{rb}^N(\mu); \mu) = -\frac{1}{2}(\sigma_{rb}^N(\mu))^T B_{rb}^d(\mu) \sigma_{rb}^N(\mu) + (\sigma_{rb}^N(\mu))^T G_{rb}^d(\mu). \)

Thus the PD-RB posteriori error estimator \( \eta_{rd,rb}(\mu) \) (5.28) can be effectively computed.

6. The Diffusion Equation. In this section, we discuss the PD-RBM for the diffusion equation. Different from the reaction-diffusion equation, where the primal and dual solutions are in condition free spaces, the diffusion equation is different. A dual feasibility condition is needed in the dual variational problem.

We assume that \( f|_K \in P_0(k) \), for each \( K \in T \).

6.1. The primal and dual problems for the diffusion equation. We drive the dual problem for the diffusion equation with non-homogenous boundary conditions.

The primal problem. For the pure diffusion problem, the primal problem is: find \( u \in H^1_{D,g}(\Omega) \), such that

\[
J_d(u) = \inf_{v \in H^1_{D,g}(\Omega)} J_d(v),
\]

where the primal functional is:

\[
J_d(v) := \frac{1}{2} \| \alpha^{1/2} \nabla v \|^2_0 - (f, v) + \langle g_N(\mu), v \rangle_{\Gamma_N}, \quad \forall v \in H^1_{D,g}(\Omega).
\]

Its equivalent Euler-Lagrange weak problem is: find \( u \in H^1_{D,g}(\Omega) \), such that

\[
a_d(u, v) = F_d(v), \quad v \in H^1_{D}(\Omega).
\]

where the bilinear form and the linear form are

\[
a_d(v, w) = (\alpha \nabla v, \nabla w) \quad \text{and} \quad F_d(v) = (f, v) - \langle g_N(\mu), v \rangle_{\Gamma_N}.
\]

The energy norm of the primal formulation of the diffusion problem is:

\[
\| v \|_d = \| \alpha^{1/2} \nabla v \|^2_0, \quad \forall v \in H^1_{D}(\Omega).
\]

To find its corresponding dual problem, consider the following Lagrangian:

\[
\mathcal{L}_d(v, \tau) = J_d(v) - \frac{1}{2} \| \alpha^{-1/2} \tau + \alpha^{1/2} \nabla v \|^2_0, \quad \forall v \in H^1_{D,g}(\Omega), \tau \in H_{N,g}(\text{div}; \Omega).
\]

Clearly, we have

\[
\sup_{\tau \in H_{N,g}(\text{div}; \Omega)} \mathcal{L}_d(v, \tau) = \mathcal{L}_d(v, -\alpha \nabla v) = J_d(v).
\]

Integrating by parts and completing the square give

\[
\mathcal{L}_d(\tau, v) = -\frac{1}{2} (\alpha^{-1} \tau, \tau) + (\nabla \cdot \tau - f, v) + \langle \tau \cdot n, g_D \rangle_{\Gamma_D},
\]
which implies
\[
\inf_{v \in H^1_{D,g}(\Omega)} \mathcal{L}_d(v, \tau) = \begin{cases} 
-\infty, & \text{if } \nabla \cdot \tau - f \neq 0, \\
-\frac{1}{2} (\alpha^{-1} \tau, \tau) - \langle \tau \cdot n, g_D \rangle_{\Gamma_D}, & \text{if } \nabla \cdot \tau - f = 0.
\end{cases}
\]

So the complimentary (dual) functional is:
\[
J_d^*(\tau) = -\frac{1}{2} (\alpha^{-1} \tau, \tau) - \langle \tau \cdot n, g_D \rangle_{\Gamma_D}, \quad \tau \in H_{N,g}(\text{div}; \Omega; f),
\]

where
\[
H_{N,g}(\text{div}; \Omega; f) := \{ \tau \in H(\text{div}; \Omega) : \nabla \cdot \tau = f, \tau \cdot n = g_N \text{ on } \Gamma_N \}.
\]

We then can define the dual problem.
**The dual problem.** The dual problem is: Find \( \sigma \in H_{N,g}(\text{div}; \Omega; f) \)
\[
J_d^*(\sigma) = \sup_{\tau \in H_{N,g}(\text{div}; \Omega; f)} J_d^*(\tau).
\]

Note that the dual problem can also be viewed as a constraint minimization problem (a mixed problem or a saddle point problem): find \((\sigma, w)\in H_{N,g}(\text{div}; \Omega) \times L^2(\Omega) \) such that
\[
\begin{cases}
(\alpha^{-1} \sigma, \tau) - (\nabla \cdot \tau, w) = -\langle \tau \cdot n, g_D \rangle_{\Gamma_D} \quad &\forall \tau \in H_N(\text{div}; \Omega), \\
(\nabla \cdot \sigma, v) = (f, v) \quad &\forall v \in L^2(\Omega).
\end{cases}
\]

The solution \( \sigma \) of the mixed problem (6.36) and the solution of the dual problem (6.35) are identical.

The energy norm of the dual formulation of the diffusion problem is
\[
[\tau]_d = \| \alpha^{-1/2} \tau \|_0^2, \quad \forall \tau \in H(\text{div}; \Omega).
\]

**The saddle point.** A simple calculation with integration by parts gives
\[
2(J_d(v) - J_d^*(\tau)) = \| \alpha^{1/2} \nabla v + \alpha^{-1/2} \tau \|_0^2, \quad \forall v \in H^1_{D,g}(\Omega), \tau \in H_{N,g}(\text{div}; \Omega; f).
\]

This implies that \((u, \sigma)\) is a saddle point of \( \mathcal{L}_d(v, \tau) \) satisfying
\[
\inf_{v \in H^1_{D,g}(\Omega)} J_d(v) = J_d(u) = \mathcal{L}_d(u, \sigma) = J_d^*(\sigma) = \sup_{\tau \in H_{N,g}(\text{div}; \Omega; f)} J_d^*(\tau).
\]

Moreover, for the primal and dual solutions, from (6.37) and (6.38), we have
\[
\sigma = -\alpha \nabla u \quad \text{in } \Omega.
\]

**6.2. Error relations.** In this subsection, we discuss various error relations of the primal and dual problems of the diffusion equation. We assume that \( u \in H^1_{D,g}(\Omega) \) and \( \sigma \in H_{N,g}(\text{div}; \Omega; f) \) be the exact solution of the primal and dual problems, respectively.

**The primal error relation.** Similar to the reaction-diffusion equation, for any \( v \in H^1_{D,g}(\Omega) \), we have
\[
\| v - u \|_0^2 = 2(J_d(v) - J_d(u)).
\]
The dual error relation. For the dual problem, similar to the reaction-diffusion equation, for any $\tau \in H_{N,g}(\text{div}; \Omega; f)$, we have

$$ (6.41) \quad \|\sigma - \tau\|_d^2 = 2(J_d(\sigma) - J_d(\tau)). $$

Duality gap as a guaranteed error upper bound. For all $v \in H_{D,g}(\Omega)$ and $\tau \in H(\text{div}; \Omega; f)$, the following bounds are true:

$$ (6.42) \quad \|u - v\|_d^2 = 2(J_d(v) - J_d(u)) = 2(J_d(v) - J_d^*(\sigma)) \leq 2(J_d(v) - J_d^*(\tau)), $$

$$ (6.43) \quad \|\sigma - \tau\|_d^2 = 2(J_d^*(\sigma) - J_d^*(\tau)) = 2(J_d(u) - J_d^*(\tau)) \leq 2(J_d(v) - J_d^*(\tau)). $$

Combined error relation. Define the combined energy norm for the diffusion equation as follows:

$$ (6.44) \quad \|(v, \tau)\| = \sqrt{\|v\|_d^2 + \|\tau\|_d^2}. $$

By (6.40), (6.41), (6.38), and (6.37), the following combined error relation holds: for all $v \in H_{D,g}(\Omega)$ and $\tau \in H_{N,g}(\text{div}; \Omega; f)$,

$$ (6.45) \quad \|(u - v, \sigma - \tau)\|_d = 2(J_d(v) - J_d^*(\tau)) = \|\alpha^{1/2}\nabla v + \alpha^{-1/2}\tau\|_0^2. $$

In literature, the identity

$$ (6.46) \quad \|u - v\|_d^2 + \|\sigma - \tau\|_d^2 = \|\alpha^{1/2}\nabla v + \alpha^{-1/2}\tau\|_0^2, \quad \forall v \in H_{D}^0(\Omega), \tau \in H(\text{div}; \Omega; f), $$

is the well known Prager-Syngue identity ([20], see, also, [7, 10]).

6.3. FEMs for the primal and dual problems. The finite element approximation of the primal problem is standard: seek $u_h \in V_{h,g}$, such that

$$ J_d(u_h) = \min_{v \in V_{h,g}} J_d(v), $$

or equivalently: find $u_h \in V_{h,g}$, such that

$$ a_d(u_h, v) = F_d(v), \quad \forall v \in V_{h,0}. $$

For the FE approximation of the dual problem, the mixed formulation is used. Find $(\sigma_h, w_h) \in \Sigma_{h,g} \times D_h$, such that

$$ (6.47) \quad \begin{cases} \quad (\alpha^{-1}\sigma_h, \tau) - (\nabla \cdot \tau, w_h) = -\langle \tau \cdot n, g_D \rangle_{\Gamma_D} \quad \forall \tau \in \Sigma_{h,0}, \\ \quad (\nabla \cdot \sigma_h, v) = (f, v) \quad \forall v \in D_h. \end{cases} $$

By the second equation of (6.47) and our assumption that the data $f$ is piecewise constant on $\mathcal{T}$, we have $\nabla \cdot \sigma_h = f$, thus

$$ \sigma_h \in \Sigma_{h,g} \cap H(\text{div}; \Omega; f) \subset H_{N,g}(\text{div}; \Omega; f). $$

6.4. Robust and optimal a priori error estimates of FE approximations. For both the primal and dual FE problems, we have the best approximation results:

$$ (6.48) \quad \|u - u_h\|_d \leq \inf_{u_h \in V_{h,g}} \|u - u_h\|_d \quad \text{and} \quad \|\sigma - \sigma_h\|_d \leq \inf_{\sigma_h \in \Sigma_{h,g}} \|\sigma - \sigma_h\|_d. $$

For the combined error, we have:

$$ (6.49) \quad \|(u - u_h, \sigma - \sigma_h)\|_d \leq \inf_{(u, \tau) \in V_{h,g} \times \Sigma_{h,g}} \|(u - v, \sigma - \tau)\|_d. $$
6.5. Primal-Dual FE a posteriori error estimators and indicators. Choosing $v = u_h$ and $\tau = \sigma_h$ in the combined error relation, we have

$$\|(u - u_h, \sigma - \sigma_h)\|^2_d = 2(J_d(u_h) - J^*_d(\sigma_h)) = \|a^{1/2}\nabla u_h + \alpha^{-1/2}\sigma_h\|_0^2. \tag{6.50}$$

We define the local PD-FE error indicator to drive the mesh refinements:

$$\eta_{d,h,K} = \|a^{1/2}\nabla u_h + \alpha^{-1/2}\sigma_h\|_{0,K}, \quad \forall K \in \mathcal{T},$$

and define the global PD-FE error estimator to control the error:

$$\eta_{d,h} = \left( \sum_{K \in \mathcal{T}} \eta_{d,h,K}^2 \right)^{1/2} = \sqrt{2(J_d(u_h) - J^*_d(\sigma_h))}.$$

For the combined error, $\eta_d$ is both reliable and efficient with constant one:

$$\eta_d = \|(u - u_h, \sigma - \sigma_h)\|_d.$$

Locally, by the triangle inequality and the fact $\sigma = -\alpha \nabla u$, the following local efficiency holds:

$$\eta_{d,h,K} = \|a^{1/2}\nabla u_h + \alpha^{-1/2}\sigma_h\|_{0,K} \leq \|(u - u_h, \sigma - \sigma_h)\|_{d,K}.$$

**Remark 6.1.** Using the result we obtained in [10], we can show that $\eta_{d,h}$ is also robust and efficient for the primal FE problem:

$$C \eta_{d,h} \leq \|u - u_h\|_d,$$

where $C > 0$ is a constant independent of $\alpha$ and the mesh size $h$. The proof of the robust efficiency for the dual (mixed) FE problem individually is an ongoing research work.

7. RBMs for the primal and dual problems of the diffusion equation. In this section, we assume $\alpha, f$ are $\mu$-dependent. Thus, we use $\mu$-dependent forms, functional, and norms, the meanings of these notations are self-explicit.

7.1. RBM for the primal problem. Suppose that a set of $N$ important parameters $S_N = \{\mu_i\}_{i=1}^N$ are chosen, then the RB space for the primal problem is the span of the corresponding FE solutions.

$$V_{\text{rb}}^N = \text{span}\{u_h(\mu_i)\}_{i=1}^N \subset V_{h,0}.$$  

Then the RB approximation is: for any given $\mu \in \mathcal{D}$, find $u_{\text{rb}}^N(\mu) \in V_{\text{rb}}^N$ such that

$$a_d(u_{\text{rb}}^N(\mu), v_{\text{rb}}; \mu) = F_d(v_{\text{rb}}; \mu), \quad \forall v_{\text{rb}} \in V_{\text{rb}}^N. \tag{7.51}$$

With respect to the $\mu$-dependent energy norm, we have the following best approximation estimates with constant one for any given $\mu \in \mathcal{D}$:

$$\|u(\mu) - u_{\text{rb}}^N(\mu)\|_{d,\mu} \leq \inf_{v_{\text{rb}} \in V_{\text{rb}}^N} \|u(\mu) - v_{\text{rb}}\|_{d,\mu}. \tag{7.52}$$
7.2. RBM for the dual problem. For the RBM of the dual problem, we have a dual feasibility condition $\nabla \cdot \tau = f$ needed to be satisfied. Handling this condition is essentially the same as the treatment of non-homogeneous essential boundary condition in the standard RBM for elliptic equations. We modify the problem to ensure that the RB space is a parameter free homogeneous space.

First, remember we assume $g_N = 0$, and we need the following space

$$H_N(\text{div}; \Omega; f) = \{ \tau \in H_N(\text{div}; \Omega) : \nabla \cdot \tau = f \}.$$

The notation $H_N(\text{div}; \Omega; f; \mu)$ is used when we need to emphasize the dependency of $f$ on $\mu$.

We want to solve the following abstract optimization problem (we already assume $g_D = 0$ for simplicity):

$$\sup_{\tau \in H_N(\text{div}; \Omega; f; \mu)} -\|\alpha^{-1/2}(\mu) \tau\|_0^2. \tag{7.53}$$

The key task is to ensure the dual compatibility condition: $\nabla \cdot \tau = f(\mu)$. 

**Assumption 7.1. (Affine decomposition of $f$)** We assume that $f(\mu)$ allows the affine decomposition

$$f(\mu) = \sum_{q=1}^{Q_f} \theta_q^f(\mu) f_q, \tag{7.54}$$

where $f_q$ are piecewise constant functions on the mesh $T$.

First, we find a set of parameter independent $\sigma_q^f \in \Sigma_{h,0}$, such that

$$\nabla \cdot \sigma_q^f = f_q, \quad q = 1, \ldots, Q_f,$$

This can be done by finding a least-squares solution of $\sigma_q^f \in \Sigma_{h,0}$, such that

$$\langle \nabla \cdot \sigma_q^f, v \rangle = (f_q, v) \quad \forall v \in D_h, \tag{7.55}$$

in the sense that if the matrix problem for (7.55) is $D \sigma_q^f = b$, we solve $D^T D \sigma_q^f = D^T b$ to get a $\sigma_q^f$. Then we can construct

$$\sigma^f(\mu) = \sum_{q=1}^{Q_f} \theta_q^f(\mu) \sigma_q^f, \tag{7.56}$$

satisfying $\nabla \cdot \sigma^f(\mu) = f(\mu)$. With $\sigma^f(\mu)$ available,

$$H_N(\text{div}; \Omega; f; \mu) = \sigma^f(\mu) + H_N(\text{div}; \Omega; 0),$$

where $H_N(\text{div}; \Omega; 0) = \{ \tau \in H_N(\text{div}; \Omega) : \nabla \cdot \tau = 0 \}$ is a parameter free space.

Then the optimization problem (7.53) is the same as: Find $\sigma_0(\mu) \in H_N(\text{div}; \Omega; 0)$, such that

$$\|\alpha^{-1/2}(\mu)(\sigma_0(\mu) + \sigma^f(\mu))\|_0 = \inf_{\tau \in H_N(\text{div}; \Omega; 0)} \|\alpha^{-1/2}(\mu)(\tau + \sigma^f(\mu))\|_0, \tag{7.57}$$

which is equivalent to: Find $\sigma^0(\mu) \in H_N(\text{div}; \Omega; 0)$, such that

$$\langle \alpha^{-1}(\mu) \sigma^0(\mu), \tau \rangle = -\langle \alpha^{-1}(\mu) \sigma^f(\mu), \tau \rangle, \quad \forall \tau \in H_N(\text{div}; \Omega; 0). \tag{7.58}$$
We then construct our RBM based on (7.58).

Suppose that $S_N = \{\mu_i\}_{i=1}^N$ is known, and the corresponding mixed FE approximations $\sigma_h(\mu_i)$ of (6.47) are computed. Then, for each $\sigma_h(\mu_i)$, let $\sigma^0_h(\mu_i) = \sigma_h(\mu_i) - \sigma_f(\mu_i)$. Then $\sigma^0_h(\mu_i) \in \Sigma_{h,0} \cap H_N(\text{div}; \Omega; 0)$. The divergence-free RB space for the dual variable is

$$\Sigma_{rb}^N = \text{span}(\sigma^0_h(\mu_i))_{i=1}^N.$$  

**RBM for the dual problem.** For any given $\mu \in \mathcal{D}$, find $\sigma_{rb}^N(\mu) = \sigma_f(\mu) + \sigma_{rb}^N(\mu)$, with $\sigma_{rb}^N(\mu) \in \Sigma_{rb}^0$ such that

$$b_d(\sigma_{rb}^N(\mu), \tau; \mu) = -b_d(\sigma_f(\mu), \tau; \mu), \quad \forall \tau \in \Sigma_{rb}^N,$$

with the bilinear form:

$$b_d(\rho, \tau; \mu) = (\alpha^{-1}(\mu) \rho, \tau), \quad \forall \rho, \tau \in H(\text{div}; \Omega).$$

For a $\mu$-independent $f$, $\sigma_f(\mu) = \sigma_f$ is also $\mu$-independent, we only need to compute one $\sigma_f$.

From problems (7.58) and (7.59), it is easy to get the following best approximation property:

$$\left\| \sigma^0(\mu) - \sigma^0_{rb}(\mu) \right\|_{d,\mu} \leq \inf_{\tau^N_{rb} \in \Sigma_{rb}^N} \left\| \sigma^0(\mu) - \tau^N_{rb} \right\|_{d,\mu}.$$

Note that

$$\sigma(\mu) - \sigma^0_{rb}(\mu) = \sigma^0(\mu) + \sigma_f(\mu) - (\sigma^0_{rb}(\mu) + \sigma_f(\mu)) = \sigma^0(\mu) - \sigma^0_{rb}(\mu).$$

Thus, with respect to the $\mu$-dependent energy norm, for the dual reduced basis approximation of the diffusion problem, we have the following best approximation a priori estimate with constant one for any given $\mu \in \mathcal{D}$:

$$\left\| \sigma(\mu) - \sigma^0_{rb}(\mu) \right\|_{d,\mu} \leq \inf_{\tau^N_{rb} \in \Sigma_{rb}^N} \left\| \sigma(\mu) - \sigma_f(\mu) - \tau^N_{rb} \right\|_{d,\mu}.$$

**Remark 7.2.** In this remark, we discuss the non-homogeneous Neumann boundary case, which is non-trivial for this problem. We first assume $\Gamma_D$ is not empty. The optimization problem is (we still assume $g_D = 0$ for simplicity)

$$\sup_{\tau \in H_{N,g}(\text{div}; \Omega; f; \mu)} -\|\alpha^{-1/2}(\mu)\tau\|_0.$$

By the same technique, we need to find a $\sigma_f^g(\mu) \in H_{N,g}(\text{div}; \Omega; f; \mu)$, then the maximization problem is the same as: Find $\sigma_0(\mu) \in H_N(\text{div}; \Omega; 0)$, such that

$$\|\alpha^{-1/2}(\mu)\sigma_0(\mu) + \sigma_f^g(\mu)\|_0 = \inf_{\tau \in H_N(\text{div}; \Omega; 0)} \|\alpha^{-1/2}(\mu)\tau + \sigma_f^g(\mu)\|_0.$$

The strategy is still divide and conquer. We wish to find a $\sigma_f^g(\mu) \in H_N(\text{div}; \Omega; f; \mu)$ with the decomposition:

$$\sigma_f^g(\mu) = \sigma_f^0(\mu) + \sigma_0^g(\mu),$$

where $\sigma_f^0(\mu)$ and $\sigma_0^g(\mu) \in H(\text{div}; \Omega)$, and

$$\nabla \cdot \sigma_f^0(\mu) = f(\mu), \quad \sigma_f^0(\mu) \cdot n = 0 \text{ on } \Gamma_N \quad \nabla \cdot \sigma_0^g(\mu) = 0, \quad \sigma_0^g(\mu) \cdot n = g_N(\mu) \text{ on } \Gamma_N.$$
The construction of $\sigma^{f_0}$ is identical to the construction in (7.56) with an affine decomposition assumption. For $\sigma^{g_0}(\mu)$, the construction is similar with a proper affine decomposition assumption on $g_N$. Once an affine decomposed $\sigma^{f_g}(\mu)$ is available, the rest is identical to the $g_N = 0$ case.

For the case that $\Gamma_N = \partial \Omega$, $f(\mu)$ and $g_N(\mu)$ are not independent. The compatibility condition

$$\int_{\Omega} f(\mu)dx = \int_{\partial \Omega} g_N(\mu)dx$$

is required. Thus, we need to assume that the following affine decomposition is true:

$$f(\mu) = \sum_{q=1}^{Q_f} \theta^f_q(\mu)f_q, \quad g_N(\mu) = \sum_{q=1}^{Q_f} \theta^g_q(\mu)g_q, \quad \int_{\Omega} f_qdx = \int_{\partial \Omega} g_qdx, q = 1, \ldots, Q_f.$$

With this assumption, we wish to find a $\sigma^{f_g}(\mu) \in H_N(\text{div}; \Omega; f; \mu)$ with the decomposition:

$$\sigma^{f_g}(\mu) = \sum_{q=1}^{Q_f} \theta^f_1(\mu)\sigma^f_q,$$

where $\sigma^f_q \in \Sigma_{h,g}$ is $\mu$-independent, and

$$\nabla \cdot \sigma^f_q = f_q \text{ in } \Omega \quad \text{and} \quad \sigma^f_q \cdot n = g_q \text{ on } \partial \Omega,$$

which is solvable due to the compatibility condition $\int_{\Omega} f_qdx = \int_{\partial \Omega} g_qdx$ is true.

### 7.3. RB combined a priori and a posteriori error estimations

For any $\mu \in \mathcal{D}$, we have the following robust and optimal a priori error estimate for the combined RB problem:

$$\| (u(\mu) - u^N_{rb}(\mu), \sigma(\mu) - \sigma^N_{rb}(\mu)) \|_{d,\mu} \leq \inf_{(v_{rb}, \tau_{rb}) \in V_{rb}^N \times \Sigma_{rb}} \| (u(\mu) - v_{rb}, \sigma(\mu) - \sigma^{f_g}(\mu) - \tau_{rb}) \|_{d,\mu}.$$

We define the PD-RB a posteriori error estimator:

$$\eta_{d,rb}(u^N_{rb}, \sigma^N_{rb}; \mu) = \sqrt{2(J_d(u^N_{rb}(\mu); \mu) - J^N_{rb}(\sigma^N_{rb}(\mu); \mu))}.$$

It is a perfect error estimator for the combined reduced basis energy error since $\eta_{d,rb}(\mu) = \| (u(\mu) - u^N_{rb}(\mu), \sigma(\mu) - \sigma^N_{rb}(\mu)) \|_{d,\mu}$.

### 7.4. RBM for the diffusion equation in matrix forms

In this subsection, we discuss the RBM for the diffusion equation in matrix forms. Since the RBM for the primal problem has no essential difference from the reaction-diffusion case, here we only discuss the RBM for the dual problem in matrix forms.

**Assumption 7.3.** (**Affine decomposition**) The bilinear form $b_d$ can be decomposed into a finite sum of parameter-dependent functions multiplied by parameter-independent forms, i.e.,

$$b_d(\rho, \tau; \mu) = \sum_{q=1}^{Q_{b,d}} \theta^b_q(\mu)b^d_q(\rho, \tau),$$

with the bilinear form $b^d_q$ is independent of the parameter $\mu$ and all $\theta$ functions are scalar quantities which are independent of $\tau$ and $\rho.$
We also assume that the Assumption 7.1 (Affine Assumption on \( f \)) is true. Suppose
\[
\Sigma_{rb}^N = \text{span}\{\sigma_h^N(\mu_i)\}_{i=1}^N = \text{span}\{\psi_i\}_{i=1}^N.
\]
We then pre-assemble in an offline procedure the \( Q \) corresponding matrices \( \mathcal{B}_{rb}^d \in \mathbb{R}^{N \times N}, \mathcal{C}_{q}^d \in \mathbb{R}^{Q \times Q}, \) and vectors \( \mathcal{G}_{p,q}^d \in \mathbb{R}^N, \) defined by
\[
(\mathcal{B}_{rb}^d)_{ij} = \theta_{q}^{b,d}((\mu)\mathcal{B}_{rb}^d), \quad (\mathcal{C}_{q}^d)_{ij} = \theta_{q}^{f}((\mu)\mathcal{C}_{q}^d), \quad (\mathcal{G}_{p,q}^d)_{ij} = \theta_{p}^{f}((\mu)\mathcal{G}_{p,q}^d),
\]

With those pre-assembled matrices and vectors, all the rest steps can be done in an effective online fashion with the computational costs not depending on the size of finite element problems.

For a given \( \mu \in \mathcal{D} \), at online stage, we assemble the matrices \( \mathcal{B}_{rb}^d(\mu) \in \mathbb{R}^{N \times N}, \mathcal{C}_{q}^d(\mu) \in \mathbb{R}^{Q \times Q}, \) and vector \( \mathcal{G}_{rb}^d(\mu) \in \mathbb{R}^N \) as
\[
\mathcal{B}_{rb}^d(\mu) = \sum_{q=1}^{Q_{b,d}} \theta_{q}^{b,d}(\mu)\mathcal{B}_{rb}^d, \quad \mathcal{C}_{rb}^d(\mu) = -\sum_{q=1}^{Q_{b,d}} \theta_{q}^{f}(\mu)\theta_{p}^{b,d}(\mu)\mathcal{G}_{p,q}^d,
\]
and
\[
\mathcal{C}_{rb}^d(\mu) = \sum_{q=1}^{Q_{b,d}} \theta_{q}^{b,d}(\mu)\mathcal{C}_{q}^d.
\]

The dual RB problem (7.59) in the matrix form is: Find \( \sigma_{rb}^N(\mu) \in \mathbb{R}^N \), such that
\[
\mathcal{B}_{rb}^d(\mu)\sigma_{rb}^N(\mu) = \mathcal{G}_{rb}^d(\mu).
\]

Also, define vector \( \sigma_f(\mu) \in \mathbb{R}^{Q_f} \) as \( (\theta_{1}^{f}(\mu), \cdots, \theta_{Q_f}^{f}(\mu))^T \).

We can compute the dual functional at \( \sigma_{rb}^N(\mu) \) as:
\[
J_d^*(\sigma_{rb}^N(\mu);\mu) = -\frac{1}{2}(\sigma_{rb}^N(\mu))^T\mathcal{B}_{rb}^d(\mu)\sigma_{rb}^N(\mu) + (\sigma_{rb}^N(\mu))^T\mathcal{C}_{rb}^d(\mu) - \frac{1}{2}(\sigma_f(\mu))^T\mathcal{C}_{rb}^d(\mu)\sigma_f(\mu),
\]
and thus the RB posteriori error estimator \( \eta_{d,rb}(u_{rb}^N, \sigma_{rb}^N;\mu) \) can be effectively computed at the online stage.

Note that if we choose a fixed training set \( \Xi_{\text{train}} \subset \mathcal{D} \), the value of \( (\sigma_f(\mu))^T\mathcal{C}_{rb}^d(\mu)\sigma_f(\mu) \) which corresponds to \( (\alpha^{-1}(\mu)\sigma_f(\mu), \sigma_f(\mu)) \), only needs to be computed once for each \( \mu \in \Xi_{\text{train}} \).

**Remark 7.4.** For the simple case that \( f \) is \( \mu \)-independent, we have \( \sigma_f(\mu) = 1 \), and the matrix \( \mathcal{C}_{rb}^d(\mu) \) is a just scalar:
\[
\mathcal{C}_{rb}^d(\mu) = \sum_{q=1}^{Q_{a,d}} \theta_{q}^{a,d}(\mu)\mathcal{C}_{q}^d \quad \text{with} \quad \mathcal{C}_{q}^d = \sigma_{f}^d(\sigma_f^f, \sigma_f^f), \quad 1 \leq q \leq Q_{a,d}.
\]

Furthermore, vectors with the name \( G \) can be simplified as:
\[
\mathcal{G}_{rb}^d(\mu) = -\sum_{p=1}^{Q_{b,d}} \theta_{p}^{b,d}(\mu)\mathcal{G}_{p}^d \quad \text{with} \quad (\mathcal{G}_{p}^d)_{j} = \theta_{p}^{b}(\sigma_f^f, \psi_j), \quad 1 \leq j \leq N, 1 \leq p \leq Q_{b,d}.
\]
Remark 7.5. In [27], the dual solution is not solved as a global finite element problem (i.e., a global optimization problem in the corresponding FE space), but is recovered through a local post-processing procedure. In our algorithm, we suggest to solve the dual FE approximation directly. This is due to two reasons: The first reason is that the quality of the a posteriori error estimator is based on how tight the duality gap is. To make the duality gap tight, we should solve the dual variational problems globally for the FEM and RBM. The second reason is that even for the a posteriori error estimation in FEM, one often chooses to use local problems, see [7, 10], we should do the global optimization in our context. Since for the standard FE a posteriori error estimators, we do not want the error estimator to be expensive as the original problem, while in the RB context, all these steps are offline steps. A good quality of RB set is more important than several solutions of offline problems. Plus, our greedy algorithms developed in the next sections will balance the RB tolerance and FE approximation errors to make sure we do not over-compute too many basis functions.

Our dual RB problem is also different from the dual problem for the same diffusion equation in [27] where a saddle point problem is considered. We believe our approach here is more appropriate since our version is based on a standard weighted projection and can be used to handle the non-homogenous Neumann conditions more naturally. Our approach here is also easier to prove the best approximation properties of the RB space, see (7.60).

8. RB greedy algorithms and numerical tests. In this section, we propose three greedy algorithms for our Primal Dual Reduced Basis Methods (PD-RBM).

For both the reaction-diffusion and the diffusion equations, we have robust and efficient primal-dual a posteriori error for the combined problems for both the FE and RB approximations. We will omit the subscripts \(rd\) or \(d\) on all our solutions, estimators, etc, in this section.

For the combined primal-dual problem, we use \(\eta_h(u_h, \sigma_h; \mu)\) as the notation for the global PD-FE a posteriori estimator and \(\eta_{h,K}(u_h, \sigma_h; \mu)\) as the notation for the local PD-FE a posteriori error indicator on an element \(K \in \mathcal{T}\). The notation \(\eta_{rb}(u_{rb}^N, \sigma_{rb}^N; \mu)\) is used as PD-RB a posteriori error estimator.

8.1. Greedy algorithm on a fixed mesh with an adaptive RB tolerance. In practice, many model reduction problems are computed on a fixed mesh. It is important to re-examine the greedy algorithm to see with an exact PD energy error estimator, what is the best stopping criteria and greedy strategy.

First, we propose a greedy algorithm on a fixed mesh with an adaptive RB tolerance.

In the classic RB greedy algorithm, we choose a fixed RB error tolerance \(\varepsilon_{rb}\), and stop the greedy algorithm when all error estimators for \(\mu\) in the training set \(\Xi_{\text{train}}\) are smaller than \(\varepsilon_{rb}\). In the classic algorithm, the estimator estimates the difference between the RB solution and the ”truth” FE solution, in the extreme case, when all points in \(\Xi_{\text{train}}\) are used to computed RB snapshots, the estimator is zero. So the greedy algorithm will stop no matter how small the RB tolerance is. On the other hand, if the PD-RB error estimator is used, it is a measure of the true energy error, which will never be zero or converges to zero on a fixed mesh (except for some trivial cases). So we need to adaptively choose the RB error tolerance \(\varepsilon_{rb}\) to make the algorithm stoppable.

The notation \(\varepsilon_h\) is used for the FE error tolerance. Suppose the RB parameter
set $S_N = \{\mu_i\}_{i=1}^N$ is chosen, define:

$$
\varepsilon_h^0 = 0 \quad \text{and} \quad \varepsilon_h^n := \max\{\eta_h(u_h, \sigma_h; \mu_1), \ldots, \eta_h(u_h, \sigma_h; \mu_n)\}, \quad 1 \leq n \leq N.
$$

It is easy to see that

$$
\varepsilon_h^n = \max(\varepsilon_h^{n-1}, \eta_h(u_h, \sigma_h; \mu_n)).
$$

So $\varepsilon_h^n \geq \varepsilon_h^{n-1}$, $\varepsilon_h^n$ is monotonically increasing with respect to the RB dimension $n$.

We use the notation $\varepsilon_{rb}$ for the RB error tolerance and we want to determine a reasonable $\varepsilon_{rb}^N$, such that when $\max_{\mu \in \Xi_{\text{train}}} \eta_{rb}(u_{rb}^N, \sigma_{rb}^N; \mu) \leq \varepsilon_{rb}^N$, we can terminate the greedy algorithm.

It is reasonable to assume that for those $\eta_i \in S_N$, their FE error and RB error are the same, thus we should define $\varepsilon_{rb}^n$ such that

$$
\varepsilon_{rb}^n \geq \varepsilon_h^n.
$$

On the other hand, choosing $\varepsilon_{rb}^n$ too close to $\varepsilon_h^n$ will result many RB basis functions needed to be computed. From the triangle inequality, for a fixed $\mu$, and a combined norm $\|\cdot\|$ for both the primal and dual variables, we have

$$
\|\mu(u, \sigma) - (u_{rb}^N, \sigma_{rb}^N)\| \leq \|\mu(u, \sigma) - (u_h, \sigma_h)\| + \|\mu(u_h, \sigma_h) - (u_{rb}^N, \sigma_{rb}^N)\|. \tag{8.63}
$$

Thus, to keep the balance of the FE tolerance $\varepsilon_h$ and RB tolerance $\varepsilon_{rb}$, one simple and reasonable choice is

$$
\varepsilon_{rb}^n := \max\{r_{rb, fe}\varepsilon_h^n, \varepsilon_{rb}^{n-1}\}, \quad 1 \leq n \leq N,
$$

where

$$
r_{rb, fe} > 1
$$

is a fixed number, which is the ratio between the $\varepsilon_{rb}^n$ and $\varepsilon_h^n$ (in the case that $\varepsilon_{rb}^0$ is small). In this paper, we test two values of $r_{rb, fe}$, 2 and 1.1.

From (8.63), the definition (8.64) means that for a parameter $\mu \in \Xi_{\text{train}} \setminus S_N$, we keep the difference between the RB solutions and the FE solutions roughly the same ($r_{rb, fe} = 2$) or a fraction ($r_{rb, fe} = 1.1$) of the FE error.

If $\varepsilon_{rb}^0$ is reasonably small, in the end, $\varepsilon_{rb}^N$ is $r_{rb, fe}\varepsilon_h^N$, and they are the same magnitude. Note that, this choice includes the case that $\varepsilon_{rb}^0$ is big compared to the FE approximation errors, and $\varepsilon_{rb}^n$ will remain big.

Algorithm 1 is an exact energy error certified PD-RB greedy algorithm on a fixed mesh.

Since the computation of RB primal and dual solutions are based on optimizations, we then have, for a fixed $\mu$,

$$
\eta_{rb}(u_{rb}^N, \sigma_{rb}^N; \mu) \leq \eta_{rb}(u_{rb}^{N-1}, \sigma_{rb}^{N-1}; \mu).
$$

Which means, the more basis functions, the smaller the energy error. Thus we can use the saturation trick we developed in [16] to avoid going through all $\mu \in \Xi_{\text{train}}$ is Step 4 of Algorithm 1. In this saturation trick, when we loop for the all $\mu \in \Xi_{\text{train}}$, we can skip those $\mu$s whose error estimators in the previous stages (with smaller RB dimensions) are already smaller than the current temporary maximum error. Algorithm 2 is the Algorithm for the saturation trick, and can be used to replace the Step 4 of Algorithm 1:

$$
\mu_{N+1} = \arg\max_{\mu \in \Xi_{\text{train}}} \eta_{rb}(u_{rb}^N, \sigma_{rb}^N; \mu).
$$
for all \( \mu \) in the tables and figures: maxerror

In our numerical experiments, we use the notation an adaptive RB tolerance.

Output: updated Vector Input:

Saturation trick to choose Algorithm 2

\textbf{Algorithm 2} Saturation trick to choose \( \mu_{\text{max}} \) without computing error estimators for all \( \mu \in \Xi_{\text{train}} \).

\textbf{Input:} Vector \( \eta_{\text{saved}} \), whose length is the cardinality of \( \Xi_{\text{train}} \). (if \( N = 1 \), then \( \eta_{\text{saved}} = \infty \) for all \( \mu \in \Xi_{\text{train}} \). If \( N > 1 \), it is a vector of the saved values of old error estimations of each \( \mu \in \Xi_{\text{train}} \).

\textbf{Output:} updated \( \eta_{\text{saved}}, \mu_{N+1} \)

1: error\(_{\text{tmpmax}} = 0.
2: for all \( \mu \in \Xi_{\text{train}} \) do
3: \hspace{1em} if \( \eta_{\text{saved}}(\mu) > \text{error}_{\text{tmpmax}} \) then
4: \hspace{2em} Compute \( \eta_{\text{rb}}(u_{\text{rb}}^N, \sigma_{\text{rb}}^N; \mu) \), and let \( \eta_{\text{saved}}(\mu) = \eta_{\text{rb}}(u_{\text{rb}}^N, \sigma_{\text{rb}}^N; \mu) \);
5: \hspace{2em} if \( \eta_{\text{rb}}(u_{\text{rb}}^N, \sigma_{\text{rb}}^N; \mu) > \text{error}_{\text{tmpmax}} \) then
6: \hspace{3em} error\(_{\text{tmpmax}} = \eta_{\text{rb}}(u_{\text{rb}}^N, \sigma_{\text{rb}}^N; \mu), \mu_{\text{max}} = \mu; \)
7: \hspace{2em} end if
8: \hspace{1em} end if
9: \hspace{1em} end for
10: Choose \( \mu_{N+1} = \mu_{\text{max}} \);

\subsection*{8.2. Numerical experiments for greedy algorithm on a fixed mesh and an adaptive RB tolerance.}

In our numerical experiments, we use the notation \texttt{maxerror} in the tables and figures:

\[ \text{maxerror} = \max_{\mu \in \Xi_{\text{train}}} \eta_{\text{rb}}(u_{\text{rb}}^N, \sigma_{\text{rb}}^N; \mu). \]

\textbf{8.2.1. The reaction-diffusion equation.} Our first example is a two-parameter reaction-diffusion equation on an L-shape domain \( \Omega = (-1,1)^2 \setminus (-1,0)^2 \) with a two dimensional parameter \( \mu = (\mu_1, \mu_2) \):

\[ -\nabla \cdot (\alpha(\mu) \nabla u) + u = 1 \quad \text{in} \ O, \quad u = 0 \text{ on } \partial O, \]

where

\[ \alpha(\mu) = \begin{cases} 10^{\mu_1} & \text{if } x y > 0, \\ 10^{\mu_2} & \text{if } x y \leq 0. \end{cases} \]

where \( \mu \in \mathcal{D} = [-2,2]^2 \). A uniform mesh with 197633 nodes, 393216 elements, and 590848 edges is given. Thus the number of degrees of freedom (DOFs) for the primal problems is 197633 (to be more accurate, we should remove the boundary nodes), and
the number of DOFs for the dual problem is roughly 984064. We choose a fixed Ξ_{\text{\text{train}}} to be a random uniform discretization of $[-1,1]^2$ of $10^5$ points. We use MATLAB code:

```matlab
rng(5000); train = 4*rand(100000)-2;
```

to generate the train set Ξ_{\text{\text{train}}}, where `rng(5000)` is used to produce a predictable sequence of random numbers for repeatable computations.

The original $\varepsilon_{rb}^0$ is chosen to be $10^{-3}$, which is a too strong requirement for the current mesh. We choose $\mu_1 = [0,0]$. The max RB dimension $N_{\text{max}}$ is chosen to be 20, which is more than enough for all our tests in the paper.

### Table 1

| RB Dim | 1     | 2     | 3     | 4     | 5     | 6     |
|--------|-------|-------|-------|-------|-------|-------|
| $\mu_{[1]}$ | 0.0000 | -1.9996 | 1.8099 | -1.9936 | -0.9741 | -1.9980 |
| $\mu_{[2]}$ | 0.0000 | 1.9808 | -1.9997 | -1.9882 | 1.9858 | -0.6277 |
| $\varepsilon_h$ | 0.0070 | 0.0220 | 0.0220 | 0.0220 | 0.0250 | 0.1250 |
| $\varepsilon_{rb}$ | 0.0141 | 0.0440 | 0.0440 | 0.0440 | 0.0499 | 0.2500 |
| maxerror | 1.5634 | 1.3010 | 0.6635 | 0.4611 | 0.2483 | 0.2406 |
| skipped | 0 | 90913 | 63232 | 83083 | 56558 | 97789 |
| test error | 0.2406 | 0.2406 | 0.2406 | 0.2406 | 0.2406 | 0.2406 |

**Fig. 1.** Selection of RB points $S_N$ (left) and convergence history of RB maxerror (right) for the reaction-diffusion equation with the adaptive RB tolerance greedy algorithm on a fixed mesh, $r_{rb,fe} = 2$

First, we test the case with $r_{rb,fe} = 2$.

On the left of Fig. 1, we depict the RB parameter set $S_N$. The bigger the size, the earlier the point is selected. We also show the numerical values of the 6 $\mu_i$ on the second and third rows of Table 1. On the 4th and 5th rows of Table 1, $\varepsilon_h$ and $\varepsilon_{rb}$ are shown. It is clear that both of them are monotonically non-decreasing. The final $\varepsilon_{rb}$ is of reasonable size, which is quite big compared to the common practice of the standard residual type RB a posteriori error estimator which often needs $\varepsilon_{rb}$ to be $10^{-6}$ or even smaller.

On the right of Fig. 1, we show the decay of maxerror and the increasing of $\varepsilon_h$ and $\varepsilon_{rb}$ with respect to RB space dimensions. We can see a clear spectral convergence of maxerror and the adaptivity of $\varepsilon_h$ and $\varepsilon_{rb}$ from the figure. The algorithm stops
when $\text{maxerror} \leq \varepsilon_{rb}$. The values of $\text{maxerror}$, $\varepsilon_h$ and $\varepsilon_{rb}$ are also shown on Table 1.

On the 7th row of Table 1, we show the efficiency of the saturation trick (Algorithm 2). It is found that the trick saves a large amount of computations of the loop in $\Xi_{\text{train}}$.

We also test the quality of the resulting RB set by running 10000 random online test cases, the max error of these 10000 test cases is 0.2406, which is very similar to the final $\text{maxerror}$ 0.2406, and is less than our $\varepsilon_{rb} = 0.2500$. For the later numerical tests for the both equations, we test the online RB problems with the same set of random parameters, and we list the maximum test combined RB energy error on the last row of the numerical results table.

![Image](image-url)

**Fig. 2.** Selection of RB points $S_N$ (left) and convergence history of RB maxerror (right) for the reaction-diffusion equation with an adaptive RB tolerance greedy algorithm on a fixed mesh, $r_{rb,fe} = 1.1$.

On Fig. 2, we showed the numerical results with $r_{rb,fe} = 1.1$. Note that 3 more basis functions are added. We skip the numerical results table for the $r_{rb,fe} = 1.1$ case.

### 8.3. The diffusion equation

Our second example is a two-parameter pure-diffusion equation on the same L-shape domain.

$$-\nabla \cdot (\alpha(\mu) \nabla u) = 1 \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial \Omega.$$  

We use the same setting as the reaction-diffusion example in $\alpha(\mu)$, $D$, the mesh, the original $\varepsilon_{rb}^0$, $\Xi_{\text{train}}$, $N_{\text{max}}$, and $\mu_1$.

| Table 2 |
| --- |
| **Convergence results for the PD-RBM adaptive RB tolerance greedy algorithm for diffusion equation on a fixed mesh, $r_{rb,fe} = 2$** |
| RB Dim | 1 | 2 | 3 | 4 |
| $\mu_{[1]}$ | 0.0000 | -1.9996 | 1.9936 | -1.9970 |
| $\mu_{[2]}$ | 0.0000 | 1.9808 | -1.9999 | -1.0199 |
| $\varepsilon_h$ | 0.0077 | 0.0354 | 0.0354 | 0.3261 |
| $\varepsilon_{rb}$ | 0.0154 | 0.0708 | 0.0708 | 0.6522 |
| maxerror | 3.6569 | 2.3279 | 0.6066 | 0.3506 |
| skipped | 0 | 90693 | 55251 | 83013 |
| test error | 0.3461 |
The numerical test results can be seen in Table 2 and Fig. 3 for $r_{rb,fe} = 2$. On Fig. 4, we showed the numerical results with $r_{rb,fe} = 1.1$. Note that for this case, the same number of RB basis function are computed as the $r_{rb,fe} = 2$ case. In both experiments for both equations, we clearly find that a non-optimal uniform mesh, even with quite a large number of DOFs, the final $\varepsilon_h$ and $\varepsilon_{rb}$ should be quite big, and the RB dimension should be reasonably small.

\begin{equation}
\varepsilon^N_h \leq \text{maxerror} \leq \varepsilon^N_{rb},
\end{equation}

where $N$ is the final RB space dimension.

8.4. Greedy algorithm with an adaptive mesh and a fixed RB tolerance.

In this subsection, we discuss a greedy algorithm with a spatial adaptivity for the field variable and a fixed RB tolerance.

For parameter dependent PDEs, it is obvious that a uniform mesh is not best for approximations due to possible singularity, sharp layers, and other solution features.
So it is reasonable to use an adaptively refined mesh to compute RB snapshots. On the other hand, when assembling the RB offline matrices, we need to compute the inner products of different RB snapshots. It is essential to make sure that all RB snapshots to be computed on a same mesh. Although in principle, we do can use different meshes for different RB snapshots \( u_h(\mu_i) \), but this will need the some non-trivial grid transfer operators to compute the inner products. In this paper, we try to construct a common mesh for all \( \mu \in \Sigma_N \) to compute snapshots.

In this algorithm, we assume that the user chooses a fixed and reasonable small RB error tolerance \( \varepsilon_{rb} \). The FE tolerance \( \varepsilon_{h} = \varepsilon_{rb}/r_{rb} \), is also fixed.

Standard adaptive FEM algorithm can be written as loops of the form

\[
\text{SOLVE} \rightarrow \text{ESTIMATE} \rightarrow \text{MARK} \rightarrow \text{REFINE}.
\]

The algorithm starts from an initial mesh \( T_0 \). To get \( T_{k+1} \) from \( T_k \), we solve the primal and dual FE problems on \( T_k \). On the estimate step, we compute the FE error indicators \( \eta_{h,K} \) and a global estimator \( \eta_h \). Then we mark those elements whose error indicators are big. We use the Dörfler bulk marking strategy: we mark a set of elements \( M_k \subset T_k \) such that

\[
\sum_{K \in M_k} \eta_{h,K}^2 \geq \theta \eta_h^2, \quad \text{for some } \theta \in (0,1).
\]

In our numerical experiments in the paper, \( \theta = 0.3 \) is used. Then those marked elements and possible more neighboring elements are refined to get a new mesh. The refine step ensures the new mesh is still shape regular and conforming. We use the newest vertex bisection in our algorithm. The solve-estimate-mark-refine loop continues until the global error estimator \( \eta_h \) is smaller than the prescribed FE error tolerance \( \varepsilon_h \).

Starting from \( \mu_1 \) and an initial mesh \( T_0 \), we can adaptively refine the FE mesh until the error estimator \( \eta_h(\mu_1) \) is smaller than \( \varepsilon_h \). The resulting mesh is \( T_1 \). For the next selected \( \mu_2 \in \Sigma_N \), if the current mesh \( T_1 \) (which is optimal for \( \mu_1 \)) is good enough to guarantee \( \eta_h(\mu_2) \leq \varepsilon_h \), no mesh refinement is needed. Otherwise, we use \( T_1 \) as an initial mesh, use \( \eta_{h,K}(\mu_2) \) as error indicators, adaptively refine the FE mesh until the error estimator \( \eta_h(\mu_2) \) is smaller \( \varepsilon_h \), the resulting mesh is \( T_2 \). If \( T_2 \neq T_1 \), then we need to recompute \( u_h(\mu_1) \) and \( \sigma_h(\mu_1) \) on \( T_2 \) to ensure that all RB snapshots are on a same mesh. In this algorithm, we assume we have reasonable large computational resources and do not restrict the dimensions of FE approximations. We list the detailed algorithm in Algorithm 3.

Since \( V_{rb}^N = \text{span}\{u_h(\mu_i)\}_{i=1}^N \) is computed on \( T_N \) and \( T_N \supset T_{N-1} \), we should have confidence that \( V_{rb}^N \) has a better quality than \( V_{rb}^{N-1} \). Similarly, we should have confidence that \( \Sigma_{rb}^N \) is better than \( \Sigma_{rb}^{N-1} \). Thus, for a fixed \( \mu \), the following result is still true:

\[
\eta_{rb}(u_{rb}^N, \sigma_{rb}^N, \mu) \leq \eta_{rb}(u_{rb}^{N-1}, \sigma_{rb}^{N-1}, \mu).
\]

Thus, in the step 4 of Algorithm 3, we still can use the saturation trick (Algorithm 2) to avoid computations of all \( \mu \in \Xi_{\text{train}} \).

8.5. Numerical experiments for the greedy algorithm with an adaptive mesh and a fixed RB tolerance. In this subsection, we test the Algorithm 3. Unless specified, we use the same problem settings as the previous section.
Algorithm 3 A primal-dual exact energy error certified PD-RB greedy algorithm with an adaptive mesh and a fixed RB tolerance (an adaptive mesh refinement greedy algorithm).

Input: Training set $\Xi_{\text{train}} \subset D$, FE tolerance $\varepsilon_h$, RB tolerance $\varepsilon_{rb} = r_{rb, fe} \varepsilon_h$, maximal RB dimension $N_{\text{max}}$, initial coarse FE mesh $T_0$, $N = 1$, $\mu_1$, flag $\text{refined}=0$.

Output: $N$, $V_N^{rb}$, $\Sigma_N^{rb}$.

1: Starting from mesh $T_{N-1}$, adaptively compute FE solutions $u_N(\mu_N)$ and $\sigma_N(\mu_N)$ with $\eta_{h,K}(u_N, \sigma_N; \mu)$ as the FE error indicator, compute until $\eta_{h,K}(u_N, \sigma_N; \mu) \leq \varepsilon_h$. The final mesh is $T_N$. Modify the flag $\text{refined} = 1$, if $T_N \neq T_{N-1}$.

2: If $\text{refined} = 1$, recompute $u_i(\mu_i)$ and $\sigma_i(\mu_i)$, $i=1, \ldots, N-1$ on the current mesh $T_N$ and reset the flag $\text{refined} = 0$.

3: Update $V_N^{rb}$ and $\Sigma_N^{rb}$.

4: Choose

$$\mu_{N+1} = \arg\max_{\mu \in \Xi_{\text{train}}} \eta_{rb}(u_N^{rb}, \sigma_N^{rb}; \mu).$$

5: If $\eta_{rb}(u_N^{rb}, \sigma_N^{rb}; \mu) > \varepsilon_{rb}$ and $N < N_{\text{max}}$, then set $N:=N+1$, goto 1. Otherwise, terminate.

---

8.5.1. The reaction-diffusion equation. We first choose $\varepsilon_h = 0.08$ and $\varepsilon_{rb} = 0.16$ ($r_{rb, fe} = 2$). The initial mesh $T_0$ is shown as the first mesh on Fig. 5.

The numerical test results can be found in Table 3 and Fig. 6 for $r_{rb, fe} = 2$. We need 7 RB snapshots to reduce $\text{maxerror}$ to be less than $\varepsilon_{rb} = 0.16$. From the 8th row of table, we see that the mesh was refined for 5 times (including the mesh generated for $\mu_1$). We depicted those meshes on Fig. 5. The latter meshes are the refinements of previous ones. And we can clearly see that refinements on different areas of the domain due to the features of solutions corresponding to later $\mu_i \in S_N$.

On the rows 5 and 6 of Table 3, we list the number of FE DOFs for the primal and dual problems. All the numbers are increasing due to mesh refinements. But compared with Table 1, we can clearly see that number of DOFs used with adaptive mesh refinements are much smaller than a fixed mesh for a similar (even smaller) $\varepsilon_{rb}$.
Table 3
Convergence results for adaptive PD-RBM for the reaction-diffusion equation, $r_{rb,fe} = 2$

| RB Dim | 1    | 2    | 3    | 4    | 5    | 6    | 7    |
|--------|------|------|------|------|------|------|------|
| $\mu_{[1]}$ | 0.0000 | 1.9996 | 1.9936 | -1.9925 | -0.9900 | -1.9980 | 1.9950 |
| $\mu_{[2]}$ | 0.0000 | 1.9808 | -1.9999 | -1.9928 | 1.9862 | -0.6277 | -0.9530 |
| maxerror | 1.5659 | 1.3014 | 0.6667 | 0.4630 | 0.2594 | 0.2349 | 0.1405 |
| $N_{fe,p}$ | 286 | 1671 | 1874 | 2375 | 2375 | 3441 | 3441 |
| $N_{fe,d}$ | 1265 | 7890 | 8877 | 11298 | 11298 | 16556 | 16556 |
| skipped | 0 | 90681 | 62268 | 82660 | 51212 | 90430 | 47504 |
| refined | 1 | 1 | 1 | 1 | 0 | 1 | 0 |
| test error | | | | | | | 0.1406 |

Fig. 6. Selection of RB points $S_N$ (left) and convergence history of RB error (right) for the reaction-diffusion equation on an adaptive mesh, $r_{rb,fe} = 2$

Also, we find that for the adaptive mesh greedy algorithm, the saturation trick also saves a good amount of computation in the loop for all $\mu \in \Xi_{train}$.

On Fig. 7, we show the result with $\varepsilon_h = 0.8$ and $\varepsilon_{rb} = 0.88$ ($r_{rb,fe} = 1.1$). We need 10 RB snapshots and the final $N_{fe,p} = 5645$ and $N_{fe,d} = 27476$.

Fig. 7. Selection of RB points $S_N$ (left) and convergence history of RB error (right) for the reaction-diffusion equation with the adaptive mesh refinement greedy algorithm, $r_{rb,fe} = 1.1$

8.5.2. The diffusion equation. We first choose $\varepsilon_h = 0.08$ and $\varepsilon_{rb} = 0.16$ ($r_{rb,fe} = 2$). The initial mesh $T_0$ is the same as the reaction-diffusion example.
Different from the reaction-diffusion example, due to the compatibility condition, we need to recompute $\sigma_f$ whence the mesh is refined.

![Adaptively refined meshes for RB snapshots for the diffusion equation, $r_{rb,fe} = 2$](image)

**Table 4**

| RB Dim | 1   | 2   | 3   | 4   | 5   |
|--------|-----|-----|-----|-----|-----|
| $\mu [1]$ | 0   | -1.9996 | 1.9936 | -1.9970 | -1.9976 |
| $\mu [2]$ | 0   | 1.9808 | -1.9999 | -1.0199 | -0.2083 |
| maxerror | 3.7450 | 2.3489 | 0.6195 | 0.1696 | 0.0995 |
| $N_{fe,p}$ | 351 | 9651 | 11148 | 21552 | 26249 |
| $N_{fe,d}$ | 1582 | 47290 | 54643 | 106319 | 129600 |
| skipped | 0   | 90695 | 54872 | 57756 | 76380 |
| refined | 1   | 1   | 1   | 1   | 1   |
| test error | 0.0993 |

The numerical test results can be found in Table 4 and Fig. 9. We need 5 RB snapshots to reduce maxerror to be less than $\varepsilon_{rb} = 0.16$. From the 8th row of table, we see that the mesh was refined for 5 times. We depicted the meshes on Fig. 8.

The numerical test results for $\varepsilon_h = 0.08$ and $\varepsilon_{rb} = 0.88$ can be found in Fig. 10. We need 6 RB snapshots to reduce maxerror to be less than $\varepsilon_{rb} = 0.88$.

**8.6. A double-adaptive greedy algorithm with an optimal mesh for the whole RB parameter set.** In this greedy algorithm, we plan to do both the spatial and RB tolerance adaptivities and to generate a mesh optimal for the whole RB parameter set $S_N$.

It is very clear that different parameters $\mu$ may require very different FE meshes. In Fig. 11, we show three meshes and corresponding solutions with parameter points
Fig. 9. Selection of RB points $S_N$ (left) and convergence history of RB error (right) for diffusion equation with the adaptive mesh refinement greedy algorithm, $r_{rb, fe} = 2$

Fig. 10. Selection of RB points $S_N$ (left) and convergence history of RB error (right) for diffusion equation with the adaptive mesh refinement greedy algorithm, $r_{rb, fe} = 1.1$

$(-2, 2), (2, -2),$ and $(-2, -2)$ generated by the PD-FE error estimator $\eta_{rd, h}$ until the $\eta_{rd, h} \leq 0.03$ for the reaction-diffusion equation. In Fig. 12, we show three meshes and corresponding solutions with parameter points $(0, 0), (-2, 2),$ and $(2, -2)$ generated by the PD-FE error estimator $\eta_{d, h}$ until the $\eta_{d, h} \leq 0.1$ for the diffusion equation.

In practical computations, the computational resource is not unlimited. Suppose the maximum number of DOFs that the FE solver can handle is $N_{fe, max}$. When adaptive mesh refinement method is used, we cannot use more DOFs than $N_{fe, max}$. One naive method is that we just stop refining the mesh when $N_{fe, max}$ is reached and treat it as a fixed mesh algorithm as Algorithm 1. This treatment has a problem due to that different parameters have their own optimal meshes. If the mesh with DOFs $\approx N_{fe, max}$ is optimal for $\mu_1 = (2, -2)$ and no further mesh refinements can be done. Then for parameters $(-2, 2)$ or $(-2, -2)$, the mesh is not good. Thus, we need an adaptive mesh refinement algorithm which is in some sense optimal for the whole parameter set $S_N = \{\mu_i\}_{i=1}^N$ with maximum number of DOFs $N_{fe, max}$.

It is important to recall that the underline principle of adaptive mesh refinement is the so-called "equi-distribution" of the error, i.e., each element has a similar size of the error.
Fig. 11. Different adaptively refined meshes for different parameters for the reaction-diffusion equation. The 1st column, \( \mu = (-2, 2) \), the 2nd column, \( \mu = (2, -2) \), the 3rd column, \( \mu = (-2, -2) \)

Fig. 12. Different adaptively refined meshes for different parameters for the diffusion equation. The 1st column, \( \mu = (0, 0) \), the 2nd column, \( \mu = (-2, 2) \), the 3rd column, \( \mu = (2, -2) \)

Note that we have the following PD sum energy error relation:

\[
\sum_{i=1}^{N} \| (u(\mu_i) - u_h(\mu_i), \sigma(\mu_i) - \sigma_h(\mu_i)) \|^2_{\mu_i} \\
= \sum_{i=1}^{N} J(u_h(\mu_i); \mu_i) - J^*(\sigma_h(\mu_i); \mu_i) = \sum_{i=1}^{N} \eta_h(u_h, \sigma_h; \mu_i)^2, \tag{8.67}
\]

where all \( \mu_i \in S_N \). Thus, if our goal is minimizing the sum energy error

\[
\left( \sum_{i=1}^{N} \| (u(\mu_i) - u_h(\mu_i), \sigma(\mu_i) - \sigma_h(\mu_i)) \|^2_{\mu_i} \right)^{1/2},
\]
we define the PD-sum error indicator:

\[ \eta_{h,K}(u_h, \sigma_h; S_N) = \left( \sum_{i=1}^{N} \eta_{h,K}(u_h, \sigma_h; \mu_i) \right)^{1/2}. \]

With this error indicator, the mesh that it generates equally distributes the PD-sum energy error, thus it is optimal for the reduction of sum energy error for the whole \( S_N \):

Now, we explain our spatial and RB tolerance double adaptive greedy algorithm with an optimal mesh for the whole \( S_N \) under the condition that number of DOFs of the FE mesh can not exceeds \( N_{fe,max} \).

On each loop of the greedy algorithm, we start from a FE mesh from the previous iteration \( T_{N-1} \), which is already good for \( S_{N-1} \) and a FE tolerance \( \varepsilon_{h}^{N-1} \). Now we adaptively compute FE solutions \( u_h(\mu_N) \) and \( \sigma_h(\mu_N) \) with \( \eta_{h,K}(u_h, \sigma_h; \mu_N) \) as the FE error indicator, compute the best possible solutions \( \eta_{h,K}(u_h, \sigma_h; \mu) \leq \varepsilon_h^{N-1} \) and the number of FE DOFs \( \leq N_{fe,max} \). The best possible is in the sense that if \( N_{fe,max} \) is big enough to ensure that \( \eta_{h,K}(u_h, \sigma_h; \mu_N) \leq \varepsilon_h^{N-1} \), we will use an algorithm similar to Algorithm 3 to generate a mesh \( T_N \) (it is also possible not refinements from \( T_{N-1} \) is needed). If \( N_{fe,max} \) is not big enough to ensure \( \eta_{h,K}(u_h, \sigma_h; \mu_N) \leq \varepsilon_h^{N-1} \), we use as many DOFs as we can to generate a new mesh with the number of FE DOFs \( \leq N_{fe,max} \).

Now we have two situations. If the \( N_{fe,max} \) is enough to ensure all RB snapshots for \( \mu_i \in S_N \) can be computed with the previous level FE tolerance \( \varepsilon_h^{N-1} \), we just need to let \( \varepsilon_h^N = \varepsilon_h^{N-1} \) and \( \varepsilon_{rb}^N = \varepsilon_{rb}^{N-1} \) and add \( u_h(\mu_N) \) and \( \sigma_h(\mu_N) \) to the corresponding RB sets.

On the other hand, if the \( N_{fe,max} \) is not enough to ensure all RB snapshots for \( \mu_i \in S_N \) can be computed with the previous level FE tolerance \( \varepsilon_h^{N-1} \), we need to regenerate a new mesh which is good for all \( S_N \). Starting from an initial mesh \( T_0 \), we adaptively compute FE solutions \( u_h(\mu_i) \) and \( \sigma_h(\mu_i) \), \( i = 1, \cdots, N \), with the sum error estimator \( \eta_{h,K}(u_h, \sigma_h; S_N) \) as the FE error indicator, compute best possible solution with \( \varepsilon_h^{N-1} \) and \( N_{fe,max} \). The final mesh is \( T_N \). Note that for each \( \mu_i \), the FE problems are solved independently, thus, the number of DOFs of each individually problem is still limited.

We define the new FE and RB tolerances to be:

\[ \varepsilon_h^N := \max\{ \varepsilon_h^{N-1}, \eta(u_h, \sigma_h; \mu_i), i = 1 : N \} \quad \text{and} \quad \varepsilon_{rb}^N := \sum_{i=1}^{N} \eta_{rb}(u_{rb}^N, \sigma_{rb}^N; \mu). \]

That is, \( \varepsilon_h^N \) is the worst FE error of all \( \mu_i \in S_N \) using \( T_N \).

For the saturation trick, the relation

\[ \eta_{rb}(u_{rb}^N, \sigma_{rb}^N; \mu) \leq \eta_{rb}(u_{rb}^{N-1}, \sigma_{rb}^{N-1}; \mu). \]

is not always true since now the meshes \( T_N \) and \( T_{N-1} \) are different. But as we discussed in [16], as long as the assumption that the better quality of the RB space, the smaller the error estimator is true, we can use the saturation trick without problem. After all, we only need to find the almost worst one.

We list the detailed algorithm in Algorithm 4.

8.6.1. The reaction-diffusion equation. We first choose initial \( \varepsilon_h^0 = 0.01 \) and \( \varepsilon_{rb}^0 = 0.02 \) (\( r_{rb,fe} = 2 \)). The maximum number of FE DOFs for the dual problem is chosen to be \( N_{fe,max} = 2 \times 10^4 \). Since the primal FE problem uses less DOFs than
Algorithm 4 A primal-dual exact energy error certified RB greedy algorithm with double adaptivities and an optimal mesh for the RB parameter set \( S_N \) (a double adaptive greedy algorithm)

Input: Training set \( \Xi_{\text{train}} \subset \mathcal{D} \), initial FE accuracy tolerance \( \varepsilon_0^h > 0 \), initial RB tolerance \( \varepsilon_0^{rb} = r_{rb,fe}\varepsilon_0^h \), maximal RB dimension \( N_{\text{max}} \), maximum number of FE DOFs \( N_{\text{fe,max}} \), initial coarse mesh \( T_0 \) (whose number of DOFs \( \leq N_{\text{fe,max}} \)), \( \mu_1 \), \( N = 1 \), flag enough = 1, flag refined = 0.

Output: \( N, \varepsilon^N_{\text{rb}}, \varepsilon^N_{\text{fe}}, V^N_{\text{rb}}, \Sigma^N_{\text{rb}}, T_N \).

1: Starting from mesh \( T_{N-1} \), adaptively compute FE solutions \( u_h(\mu_N) \) and \( \sigma_h(\mu_N) \) with \( \eta_{h,K}(u_h, \sigma_h; \mu) \) as the FE error indicator, compute best possible solutions \( \eta_{h,K}(u_h, \sigma_h; \mu) \leq \varepsilon^{N-1}_h \) and the number of FE DOFs \( \leq N_{\text{fe,max}} \). The final mesh is \( T_N \). Mark refined = 1 if the mesh is refined, otherwise refined = 0; If \( N_{\text{fe,max}} \) is not enough to make \( \eta_{h,K}(u_h, \sigma_h; \mu) \leq \varepsilon^{N-1}_h \), enough = 0.

2: if enough = 1 then
3: \( \varepsilon^N_h = \varepsilon^{N-1}_h, \varepsilon^N_{rb} = r_{rb,fe}\varepsilon^h_h \).
4: If refined = 1, recompute \( u_h(\mu_i) \) and \( \sigma_h(\mu_i) \), i=1,⋯,N-1 on the current mesh \( T_N \).
5: Update \( V^N_{rb} \) and \( \Sigma^N_{rb} \).
6: else
7: Starting from mesh \( T_0 \), adaptively compute FE solutions \( u_h(\mu_i) \) and \( \sigma_h(\mu_i) \), i=1,⋯,N, with \( \eta_{h,K}(u_h, \sigma_h; S_N) \) as the FE error indicator, compute best possible solution with \( \varepsilon^{N-1}_h \) and \( N_{\text{fe,max}} \). The final mesh is \( T_N \). refined = 0.
8: Update \( V^N_{rb} \) and \( \Sigma^N_{rb} \).
9: Update \( \varepsilon^N_h := \max\{\varepsilon^{N-1}_h, \eta(u_h, \sigma_h; \mu_i), i = 1 : N\} \) and \( \varepsilon^N_{rb} := r_{rb,fe}\varepsilon^N_h \).
10: end if
11: Choose
\[ \mu_{N+1} = \arg\max_{\mu \in \Xi_{\text{train}}} \eta_{rb}(u^N_{rb}, \sigma^N_{rb}; \mu). \]
12: If \( \eta_{rb}(u^N_{rb}, \sigma^N_{rb}; \mu_{N+1}) > \varepsilon^N_{rb} \) and \( N < N_{\text{max}} \), then set \( N := N+1 \), goto 1. Otherwise, terminate.

the dual FE problem, we make no restriction on it. Here, we choose \( N_{\text{fe,max},d} \) smaller (in stead of a bigger number, for example \( 10^6 \)) in order to show the effectiveness of the algorithm.

The numerical test results can be found in Table 5 and Fig. 13 for \( (r_{rb,fe} = 2) \). We need 7 RB snapshots to balance the RB dimension, \( \varepsilon_{rb} \), and \( N_{\text{fe,max}} \).

From the 10th row of table, we see that the \( N_{\text{fe,max}} \) was not enough in 4 occasions. For a smaller RB dimension \( N \), \( \varepsilon^N_{\text{fe}} \) and \( \varepsilon^N_{rb} \) are smaller, so we need more DOFs for each \( \mu_i \) to satisfy the error requirement, but since the \( N \) is smaller, the situation that different \( \mu_i \) needs very different meshes are less severe. For a bigger RB dimension \( N \), the situation is different, more \( \mu_i \) in \( S_N \) but a bigger FE tolerance \( \varepsilon_h \). Thus it is a dynamic balance of the dimension \( N \) and the FE and RB tolerances.

We show the four refined meshes corresponding to the 4 non-enough cases on Fig. 14. It is easy to see that the last two meshes are more balanced for the whole \( S_N \), which is quite similar to the combination of all three meshes depicted in Fig. 11.

With everything else unchanged, the numerical test results can be found in Fig. 15 for \( r_{rb,fe} = 1.1 \), 10 RB snapshots are needed.
Table 5
Convergence results for the PD-RBM with the double-adaptive greedy algorithm for the reaction-diffusion equation, \( r_{rb,fe} = 2 \)

| RB Dim | 1    | 2    | 3    | 4    | 5    | 6    | 7    |
|--------|------|------|------|------|------|------|------|
| \( \mu_1 \) | 0.0000 | -1.9996 | 1.9936 | -1.9925 | -0.9900 | -1.9980 | 1.9950 |
| \( \mu_2 \) | 0.0000 | 1.9808 | -1.9999 | -1.9928 | 1.9862 | -0.6277 | -0.9530 |
| \( \varepsilon_h \) | 0.0195 | 0.0565 | 0.0565 | 0.0580 | 0.0580 | 0.0782 | 0.0782 |
| \( \varepsilon_{rb} \) | 0.0389 | 0.1130 | 0.1130 | 0.1159 | 0.1159 | 0.1565 | 0.1565 |
| maxerror | 1.5635 | 1.3017 | 0.6650 | 0.4621 | 0.2535 | 0.2346 | 0.1346 |
| \( N_{fe,p} \) | 4076 | 3444 | 3727 | 4123 | 4123 | 4099 | 4099 |
| \( N_{fe,d} \) | 19755 | 16639 | 18002 | 19970 | 19970 | 19814 | 19814 |
| skipped | 0 | 90892 | 62694 | 83021 | 50943 | 92645 | 46529 |
| enough | 0 | 0 | 1 | 0 | 1 | 0 | 1 |
| test error | &nbsp; &nbsp; | &nbsp; | &nbsp; | &nbsp; | &nbsp; | &nbsp; | 0.1345 |

Fig. 13. Selection of RB points \( S_N \) (left) and convergence history of RB error (right) for the reaction-diffusion equation on with the double-adaptive greedy algorithm, \( r_{rb,fe} = 2 \)

8.6.2. The diffusion equation. We first choose initial \( \varepsilon_h^0 = 0.05 \) and \( \varepsilon_{rb}^0 = 0.1 \) (\( r_{rb,fe} = 2 \)) and \( N_{fe,\text{max},d} = 2 \times 10^4 \). The numerical test results can be found in Table 6 and Fig. 16 for \( r_{rb,fe} = 2 \). We need 4 RB snapshots to balance the RB dimension, \( \varepsilon_{rb} \), and \( N_{fe,\text{max}} \).

From the 10th row of table, we see that the \( N_{fe,\text{max}} \) was not enough in 3 occasions. We show the four refined meshes corresponding to the 1st adaptive mesh and the other 3 non-enough cases on Fig. 17. It is easy to see that the last two meshes are more balanced for all the \( \mu_i \in S_N \), which is quite similar to the combination of all three meshes depicted in Fig. 12.

The numerical test results can be found in Fig. 18 for \( r_{rb,fe} = 1.1 \). We need 5 RB snapshots to balance the RB dimension, \( \varepsilon_{rb} \), and \( N_{fe,\text{max}} \).

8.7. Remarks on the numerical tests. We note that although we use quite different meshes and use the saturation trick in the greedy algorithms, the selection of the set \( S_N \) and their orders are identical for all three algorithms and both equations.

The saturation trick also saves a large percent of offline loop in \( \mu \in \Xi_{\text{train}} \) for all three algorithms and both equations.

The 10000 online random tests for all algorithms and both equations also show that the RB sets are of good quality and the RB tolerances are well selected.

From our numerical experiments with \( \text{ratio}_{fe,rb} = \varepsilon_{rb}/\varepsilon_h = 2 \) or 1.1, in general
Fig. 14. Four refined meshes for the double-adaptive greedy algorithm for the reaction-diffusion equation.

Fig. 15. Selection of RB points $S_N$ (left) and convergence history of RB error (right) for the reaction-diffusion equation with the double-adaptive greedy algorithm, $\frac{r_{rb}}{r_{fe}} = 1.1$.

More RB dimensions are needed for a smaller $\frac{r_{rb}}{r_{fe}}$ for our adaptive RB tolerance Algorithms 1 and 4. For Algorithm 3 with a fixed RB tolerance and an adaptive FEM, a smaller $\frac{r_{rb}}{r_{fe}}$ requires more DOFs for the FEM. For more complicated problems, a smaller $\frac{r_{rb}}{r_{fe}}$ probably will require many more RB dimensions for our Algorithms 1 and 4 and many more FEM DOFs for Algorithms 2.

9. Concluding comments and future plans. In this paper, with the parametric reaction-diffusion and diffusion equations as examples, for a class of parametric convex minimization variational problems, we develop primal-dual reduced basis methods with robust true error certification and adaptive greedy algorithms to over-
The primal dual error estimator is robust with respect to the parameters of the problem, it measures the true error in the energy norm for RBM, and it can be used for both the mesh refinements of finite element methods and the true RB error certification and basis selection of RBM. We develop three versions of adaptive greedy algorithms to balance the finite element error, the exact reduced basis error, and the adaptive mesh refinements.

In the paper, we assume the data (the right hand side and boundary conditions)
Table 6
Convergence results for PD-RBM with the double-adaptive greedy algorithm for diffusion equation, $r_{rb, fe} = 2$

| RB Dim | 1     | 2     | 3     | 4     |
|--------|-------|-------|-------|-------|
| $\mu_1$ | 0.0000 | -1.9996 | 1.9936 | -1.9973 |
| $\mu_2$ | 0.0000 | 1.9808 | -1.9999 | -1.0361 |
| $\varepsilon_h$ | 0.0500 | 0.1242 | 0.1707 | 0.2003 |
| $\varepsilon_{rb}$ | 0.1000 | 0.2484 | 0.3415 | 0.4006 |
| maxerror | 3.7357 | 2.3461 | 0.6537 | 0.2432 |
| $N_{fe,p}$ | 814 | 3791 | 2675 | 3694 |
| $N_{fe,d}$ | 3793 | 18398 | 12844 | 17821 |
| skipped | 0 | 90624 | 56967 | 67516 |
| enough | 1 | 0 | 0 | 0 |
| test error | | | | 0.2424 |

Fig. 18. Selection of RB points $S_N$ (left) and convergence history of RB error (right) for the diffusion equation on with the double-adaptive greedy algorithm, $r_{rb, fe} = 1.1$.

are piecewise polynomials to simplify the presentation. It is easy to show that for problems with general non-polynomial data, a simple perturbation argument like that in the page of 181 of [6] can be used to estimate the error between the solutions with the general data and polynomial data. This error term can be controlled if we refine the mesh corresponding to the data.

For the convergence of the greedy algorithms presented in the paper, it is not hard to show that they will convergence in a genetic $H^1$ norm under suitable assumptions with the help of the $n$-Width theory. To analyze the convergence with respect to the parameter dependent energy norm is a much harder task and will be a future work.

Besides simple linear elliptic problems, we are plan to apply the PD-RBM to other more challenging problems in the primal and dual variational framework. For example, for the obstacle problem, which is a variational inequality problem, not only we have an inequality dual feasibility condition, the primal solution should also satisfy an inequality constraint. Some famous nonlinear problems like the $p$-Laplacian are also in this framework. These problems all pose different challenges to the construction, efficient implementation, and analysis of the PD-BRM.

For many other problems not fitted in the framework, for example, non-symmetric problems, the reasonable choice is using the artificial minimization principle: The Least-Squares principle [5]. Although some simple version of the method is already
applied to second order elliptic equations in [28], it is an ongoing work to construct the true residual certified least-squares variational RBM for a wide range of linear and nonlinear problems.

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