GRAVITATIONAL FIELD OF THE EARLY UNIVERSE: I. NON-LINEAR SCALAR FIELD AS THE SOURCE

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In this review article we consider three most important sources of the gravitational field of the Early Universe: self-interacting scalar field, chiral field and gauge field. The correspondence between all of them are pointed out. More attention is payed to nonlinear scalar field source of gravity. The progress in finding the exact solutions in inflationary universe is reviewed. The basic idea of ‘fine turning of the potential’ method is discussed and computational background is presented in details. A set of new exact solutions for standard inflationary model and conformally-flat space-times are obtained. Special attention payed to relations between ‘fine turning of the potential’ and Barrow’s approaches. As the example of a synthesis of both methods new exact solution is obtained.

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1. Introduction

One of the most promising recent achievement in the physics of the Early Universe is Inflationary Cosmology. The inflationary Universe became now an integral part of the hot Big Bang scenario and solved its long standing problems such as horizon, flatness, homogeneity [1]-[5].

To describe the gravitational field of the very Early Universe, in general case, we can consider the model with the action

\[ S = \int \sqrt{-g} d^4 x \left\{ L_G + L_M + L_{int} \right\}, \] (1)

where \( L_G = \frac{R + 2\Lambda}{2\kappa} \), \( L_{int} \) describes the interaction of a gravitational field to matter, \( L_M \) being the lagrangian density of matter. Henceforth we put \( L_{int} = 0 \).

It was realized [1],[3] that during inflationary stage the source of the gravitational field was gauge and Higgs fields described by the Grand Unified Theory (GUT). This model can be described by the the lagrangian

\[ L_M = -\frac{1}{\gamma} \left\{ \frac{1}{2} G_{\mu\nu}^a G^{\mu\nu a} + D_\mu \phi^a D^\mu \phi^a + V(|\phi|) \right\} \] (2)

In the relation (2), \( G_{\mu\nu}^a = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + \epsilon f_{bc}^a W^{b}_\mu W^{c}_\nu \), \( f_{bc}^a \) being a structure’s constants of a gauge group for the Yang-Mills field \( W^a_\mu \), \( D_\mu \phi^a = \partial_\mu \phi^a + \epsilon f_{bc}^a W^{b}_\mu \phi^c \) being the covariant derivative, \( \phi^a \) being the Higgs field. The potential can be taken in the form \( V(|\phi|) = \frac{1}{2} (\phi^a \phi^a - F^2)^2 \), where \( F = \langle |\phi^a| \rangle \neq 0 \) being the vacuum average of the field.

The action (1)-(2) leads to the system of Einstein-Yang-Mills-Higgs equations. To solve this self-consistent system is the real problem even under assumption about the symmetry. Therefore in inflationary scenarios it was used an effective model of a scalar field \( \phi \) with the potential of self-interaction \( V(\phi) \) for the sake of simplicity [3]. The action for 'standard inflationary model' reads

\[ S = \int \sqrt{-g} d^4 x \left\{ L_G + \frac{1}{2} \phi^a \phi^a g^{ik} - V(\phi) \right\}. \] (3)

Recently it was proposed so-called chiral inflationary model based on a nonlinear sigma model (NSM) with the self-interacting potential [4]. The action for this model can be presented as

\[ S = \int_{\mathcal{M}} \sqrt{-g} d^4 x \left\{ L_G + \frac{1}{2} h_{AB}(\phi) \phi^A_\mu \phi^B_\mu g^{ik} - W(\phi) \right\}. \] (4)

Here \((\mathcal{M}, g_{ik}(x))\) being a space-time, \((\mathcal{N}, h_{AB}(\phi))\) being a target space, \( \varphi = (\varphi^1, \ldots, \varphi^n) \), \( \varphi_A^A = \partial_k \varphi^A = \varphi^A_k \). The model (4) can be considered as generalisation of standard inflationary model [3] and in special case can be reduced to it. Namely, let us introduce new scalar field \( \Phi \) and a potential \( \tilde{V}(\Phi) \) which satisfy the relations

\[ \Phi_A^A, \Phi_\mu = h_{AB} \varphi^A_\mu \phi^B_\mu \] (5)

\[ \tilde{V}(\Phi) = W(\varphi^C) \] (6)
To realize the reduction above we have to restrict the metric of the chiral space \( h_{AB} \). Differentiating by coordinates \( x^i \) and \( x^j \) the relation (4) and then using (5) one can obtain the restriction on \( h_{AB} \):

\[
h_{AB} = \frac{\partial W}{\partial \phi^A} \frac{\partial W}{\partial \phi^B} \left( \frac{\partial V}{\partial \phi} \right)^{-2} \frac{\partial V}{\partial \phi} \neq 0. \quad (7)
\]

Thus the chiral inflationary model (4) can be reduced to the standard inflationary one (3) by virtue of the relations (6) and (3).

We can obtain as well the relation between GUTs (1)-(2) and chiral inflationary model (4). The model (4) can be obtained from (1)-(3), if we put \( W_{\mu} = 0 \) and \( h_{AB} = \text{diag}[1,1,\ldots,1] \). The last expression means that the target space are postulated to be an euclidian one. Thus the chiral inflationary model does not only simple reduction from GUT but it is the generalisation of a Higgs sector by virtue of the change the euclidian space by riemannian target space. It worth one more interesting feature of the chiral inflationary model (4). Namely the field equations

\[
\frac{1}{\sqrt{|g|}} \partial_i \left( \sqrt{|g|} \phi_A^i \right) - \Gamma_{C,AB} \phi^B_{,i} \phi^C_{,k} g^{ik} - \frac{\partial W}{\partial \phi^A} = 0 \quad (8)
\]

contain an additional term (the second one) which came from the intrinsic geometry of the target space. In standard inflationary models (3) based on self-interacting scalar field theory a similar term have been inserted by fenomenological way from quantum properties (1). Moreover it is well known the direct relation between the NSM with special symmetry and gauge theories (see, for example, (3)).

The exponential expansion of the Universe has been found by G.Ivanov (11) for a nonlinear scalar field with the potential \( V(\phi) = \frac{1}{2} \phi^2 - \frac{1}{2} \phi^4 \) in the framework of spatially-flat Friedmann-Robertson-Walker (FRW) spaces and has been interpreted as the Universe emerged from a quasi-vacuum state of matter. Ivanov’s solution has been obtained before the work by A.Guth (4) where the inflationary solution has been obtained with \( V(\phi) = \text{const} \).

Exact solutions of the power law inflationary type have been obtained for Liouville non-linearity \( V(\phi) = m e^{-\lambda \phi} \) (see also (11) and references quoted therein). The Liouville-type and some other non-linearities have been investigated and some exact and asymptotic solutions are presented in (12). Classical de Sitter type solutions was obtained in (13). New classes of exact solutions have been found in (14) by taking the scalar field as the function of time \( \phi = \phi(t) \) and then determining the evolution of the expansion scale factor \( K(t) \) and the potential \( V(\phi(t)) \) from it. This approach was applied in works (15), (16) as well.

Nevertheless approximations and numerical investigations are used in a large number of inflationary scenarios because of the difficulties in obtaining the exact solutions for inflationary models. The slow-roll approximation is the most common in use (17).

In (17), (18) the exact solutions have been obtained by taking, first, the scalar factor as the function of time \( K = K(t) \) and then determining the evolution of the potential \( V = V(t) \) and the evolution of a scalar field \( \phi = \phi(t) \), the dependence between \( V \) and \( \phi \) being, in general, parametric one. This approach called as the method of ‘fine turning of the potential’ (18).

Another point have been presented in (19), where exact general solutions were found for a single scalar field interacting through an exponential potential in the framework of background field equations for the Arnowitt-Deser-Misner (ADM) formalism. Approximate analytic solutions for slowly evolving multiple scalar fields are obtained also. The Hamilton-Jacobi theory for long-wavelength inhomogeneous universes are investigated in (20) in the framework of ADM formalism. Exact inhomogeneous solutions for Yang-Mills field minimally coupled to gravity have been obtained in (21).

In this review article we give the basic idea of a new insight on the potential of self-interaction. Using further development of ‘fine turning of the potential’ method (18) we obtain new exact solutions in the case of the conformally-flat spaces. The difficulties of obtaining the exact solutions for self-gravitating massive scalar field are pointed out. The set of generalised Barrow’s solutions (14) is obtained.

2. The effective self-interaction potential

The original inflationary scenario (3) as well as its first modifications (4), (5) are connected with the effective theory of self-interacting scalar field \( \phi \), minimally coupled to gravity, with the action (3).

As a rule the standard inflationary model (3) is analysed in the framework of the Friedmann-Robertson-Walker metric

\[
dS^2 = dt^2 - K^2(t) \left( \frac{dr^2}{1 - \epsilon r^2} + r^2(d\theta^2 + \sin^2 \theta d\varphi^2) \right). \quad (9)
\]

Here \( \epsilon = -1, 0, +1 \) corresponds to open, spatially-flat and closed Universe respectively.

A very important role in inflationary scenarios belongs to the effective potential of self-interaction \( V(\phi) \). The form of the \( V(\phi) \) reflects the physical phenomenon of the very early Universe: cosmological phase transitions and the symmetry restoration at high temperatures \( T \) in GUTs. It is well known that the form of a potential is changed while phase transitions occur and a temperature increase (22). The potential depends on the temperature and this dependence is due to quantum one-loop corrections in finite temperature field theory.
Thus the form of the effective scalar potential depends on the type of a field theory, we put into a physical basis, and tends to change when physical phenomenon occur in the development of a cosmological time $t$.

One more restriction on the form of the potential $V(\phi, T)$ came from the fine turning procedure [24]. Let us consider as an example the situation with the Coleman-Weinberg potential [25]

$$V(\phi) = A\phi^4\{\ln\left(\frac{\phi^2}{M^2} + 1\right)\}, \quad (10)$$
predicted by the $SU(5)$ GUT, and the fine turning of the potential (10). Here $M$ is an arbitrary renormalization mass and is a constant of order unity. It is happened that the potential (10) is unsuitable for matching the $SU(5)$-invariant GUT (with the $A \propto e^t > 10^{-2}$) with the observed rate of the density perturbations $\delta\rho$ (which needs $A < 10^{-12}$). To improve the situation Shafl and Vilenkin [21] took into consideration the three-level potential

$$V(\phi, \xi, T) = \frac{1}{2}a(\text{Tr}\Phi^2)^2 + \frac{1}{2}b^2\text{Tr}\Phi^4 + (H_5^2 H_5)\text{Tr}\Phi^2 + \frac{1}{4}\lambda_5(H_5^2 H_5)^2 + \frac{1}{4}\lambda_5\phi^4 + \frac{1}{2}\lambda_2\phi^2\text{Tr}\Phi^2 + \lambda_3\phi^2 H_5^2 H_5, \quad (11)$$

containing the adjoint and fundamental Higgs fields $\Phi$ and $H_5$ in addition to the inflationary field $\phi$. It should be mentioned here, that in [23] the kinetic terms from $\Phi$ and $H_5$ did not include in the model. This shortage has been avoided in the multiple [26] and double-field inflationary models [27]. Let us also mention the concrete high energy physics phenomenon which use double-field model with the potential, containing cross-interaction [28] $\frac{1}{2}\nu_\phi^2\phi^2$.

$$V(\phi, \xi) = \frac{1}{2}m_\phi^2\phi^2 + \frac{1}{4}\lambda_\phi\phi^4 - \frac{1}{2}m_\xi^2\xi^2 + \frac{1}{4}\lambda_\xi\xi^4 + \frac{1}{2}\nu_\phi^2\phi^2. \quad (12)$$

Thus we can see a tendency of changing the shape of the potential $V(\phi, T)$ if one needs to correct the inflationary scenario.

Summarising above we can come to the conclusion, that the form of the effective potential $V(\phi)$ does not fix and is subject to change with the evolution of the Universe. Put into basis possible variations of the $V(\phi(t))$ we can present a new insight on the problem of obtaining the exact solutions for inflationary models. Namely, what kind of the $V(\phi(t))$ admits the exact solutions with an exponential or power law expansion of the FRW Universe? The answer to this question will give us an explicit form of the potential which leads to the given rate of the expansion of the Universe. The next step should be to find an appropriate theory of matter which will be more close to the obtained form of the potential. We will not discuss the last problem here and will pay attention to the way of construction the $V(\phi)$ in the next section.

3. The method

The system of Einstein’s and nonlinear scalar field equations, corresponding to the model [3] in the FRW spaces [11], reads

$$\frac{K_{44}}{K} + 2\frac{K_4^2}{K^2} + 2\kappa = -\Lambda + \kappa V(\phi), \quad (13)$$
$$-3\frac{K_{44}}{K} = \Lambda + \kappa(\phi^2 - V(\phi)), \quad (14)$$
$$\phi_{44} + \frac{3K_4}{K} \phi_4 + \frac{dV(\phi)}{d\phi} = 0. \quad (15)$$

These equations present the standard inflationary model.

To obtain new class of exact solutions we will use a freedom in the choice of a potential’s form.

Considering the equations (13)-(15) one can find that the last equation (15) is the differential consequences of (13) and (14). Really, to prove this fact, one can differentiate (13) by $t$ to obtain the following equation

$$\frac{K_{44}}{K} + 3\frac{K_4 K_{44}}{K^2} - 4\frac{K_4^2}{K^3} - \frac{4K_4 \kappa}{K} - \frac{dV}{dt} = 0. \quad (16)$$

This equation can be rewritten as

$$\frac{K_{444}}{K} - 3\frac{K_4 K_{44}}{K^2} + 2\frac{K_4^2}{K^3} + 2\frac{K_4 \kappa}{K^3} - 6\frac{K_4}{K} \left(-\frac{K_{44}}{K} + \frac{K_4^2}{K^2} + \frac{\epsilon}{K^2}\right) - \frac{dV}{dt} = 0. \quad (17)$$

Now one can use two consequences of the Einstein’s equations (13) and (14):

- the sum of Einstein’s equations (13) and (14), firstly

$$\kappa\phi_4^2 = \frac{2}{K^2} \{-K_{44} + K_4^2 + \epsilon\}, \quad (19)$$

- and, secondly, the derivative by time $t$ from the relation (19)

$$\kappa\phi_{44}\phi_4 = \partial_4 \{-\frac{K_{44}}{K} + \frac{K_4^2}{K^2} + \frac{\epsilon}{K^2}\} \quad (20)$$

Inserting left hand side of (19) and (24) into (17) and using $\frac{d\phi_4}{dt} = \frac{d\phi_4}{d\phi} \phi_4$, we can divide the equations (17) by $\phi_4 \neq 0$. As the result one obtains the equation for the scalar field (13).

The following analysis will be used just for Einstein’s equations (13) and (14), which can be reduced to the form where the functions $V(t) \equiv V(\phi(t))$ and
4.1. Power law inflation

\[ V(t) = \frac{1}{k} \left( \Lambda + \frac{K_{14}}{K} + \frac{2K_4^2}{K^2} + \frac{2\epsilon}{K^2} \right), \quad (21) \]

\[ \phi(t) = \pm \sqrt{\frac{2}{k}} \int \left( \sqrt{-\frac{d^2\ln K}{dt^2} + \frac{\epsilon}{K^2}} \right) dt. \quad (22) \]

By giving the rate of the expansion as the function for a scale factor on time \( K = K(t) \), we can find the functions \( \phi(t) \) \( V(t) \) which are necessary for chosen type of the Universe’s evolution. It is obvious, that the pair of the function \( (21) \) and \( (22) \) gives the parametric dependence \( V = V(\phi) \). In some cases, after calculation of the right hand sides in \( (21), (22) \), it is possible to find the explicit dependence \( V = V(\phi) \) by eliminating \( t \).

4. The inflationary solutions

The exact solutions for exponential and power law type of inflation in the framework of ‘fine turning of the potential’ (FTP) method have been obtained in \[17, 22\].

4.1. Power law inflation

To obtain the power law inflation let us start from the suggestion that

\[ K(t) = K_0 t^m, \quad (23) \]

where \( m > 1 \).

The integral in the right hand side of \( (22) \) can be calculated explicitly. Therefore we can find \( m \neq 1 \)

\[ V = \frac{m}{Kt^2} \left( \frac{At^2}{m} - (3m - 1 - 2\alpha t^{-2m+2}) \right) \]
\[ \phi(t) = \pm \sqrt{\frac{1}{2\kappa t^2}} \left\{ 2\sqrt{1 + \alpha t^{-2m+2}} + \right. \]
\[ + \ln \left( \frac{\sqrt{1 + \alpha t^{-2m+2}} + 1}{\sqrt{1 + \alpha t^{-2m+2}} - 1} \right) \} + \phi_0, \quad (25) \]

where \( \alpha = \epsilon K_0^{-2}/m \).

In the case of the spatially-flat Universe \( (\epsilon = 0) \) the solution for arbitrary \( m \) has the form

\[ \phi = \pm \sqrt{2m/K} \ln t + \phi_0, \quad (26) \]
\[ V(t) = \kappa^{-1} (\Lambda + (m + 3m^2)/t^2). \quad (27) \]

Eliminating \( t \) we find an exponential dependence \( V \) on \( \phi \)

\[ V(\phi) = \kappa^{-1} \{ \Lambda + (m + 3m^2)e^{-\sqrt{2m/K}(\phi - \phi_0)} \}, \quad (28) \]

which is usually the definition of the power law inflation \[8\].

In the case of open and closed Universe \( (\epsilon \neq 0) \) it is possible to find an explicit dependence \( V \) on \( \phi \) just for some values of \( m \). For example, if \( m = 1 \) (in this case the formulas \( (24) \) and \( (25) \) do not work)

\[ V(\phi) = \Lambda/K + \kappa^{-1} \exp\left\{ -\frac{2(\phi - \phi_0)}{\pm 2K_0^{-2}} \right\}. \quad (29) \]

4.2. Exponential inflation

The case when the scale factor \( K(t) \) of the Universe grows up very fast by the exponential type law have been analysed in \[14, 18\]. Let us mention about the differences in Barrow’s and ‘fine turning of the potential’ approaches. To find new exact solutions, in Barrow’s method \[14\], one needs to take the scalar field as the function of time \( \phi = \phi(t) \) and then to determine the evolution of the expansion scale factor \( K(t) \) and the potential \( V(\phi(t)) \) from it. This approach was applied in works \[15,16\] as well. In ‘fine turning of the potential’ method \[17,18\] the exact solutions have been obtained by taking, first, the scalar factor as the function of time \( K = K(t) \) and then determining the evolution of the potential \( V = V(t) \) and the evolution of a scalar field \( \phi = \phi(t) \), the dependence between \( V \) and \( \phi \) being, in general, parametric one. Thus we can combine both of the methods, if we will put the metric obtained by Barrow’s way as the seed solution in ‘fine turning of the potential’. As an example of a synthesis of both methods let us consider one of the exact solutions obtained in \[14\].

We can choose the scalar factor of the Universe in the form of the first class of exact solutions \[14\]:

\[ K(t) = K_0 \cdot \sinh^{A^2/2} \{2\lambda t\} \]
\[ (30) \]

where \( \lambda \)-constant.

This scalar factor \( (31) \) have been obtained for the spatially flat FRW Universe in \[14\]. For this solution, using \( (21)-(22) \), one can find

\[ V(t) = \frac{2\epsilon}{K_0^2} \sinh (2\lambda t)\]}
\[ \{ A + \lambda^2 A^2 \left[ (3A^2 - 2) \coth^2 2\lambda t + 2 \right] \}, \quad (31) \]
\[ \phi(t) = \]
\[ \int \sqrt{2A^2 \lambda^2 - \frac{\epsilon}{K_0^2} \sinh(2\lambda t)^2 - A^2} \pm \sqrt{\frac{2\epsilon dt}{\kappa}} \sinh(2\lambda t) \]
\[ (32) \]

It is easy to see from \( (31) \) that \( 2\lambda^2 A^2 \) plays role an effective (positive) cosmological constant and when \( t \to \infty, V(t) \) will approach to \( V(t)|_{t=0} \) for open \( (\epsilon = -1) \) or closed \( (\epsilon = 1) \) universes. The scalar field can be calculated exactly when, for example, \( |A| = \sqrt{2} \).

\[ \phi(t) = \pm \sqrt{\frac{2}{\kappa}} \sqrt{\frac{A^2}{2} - \frac{\epsilon}{4\lambda^2 K_0^2} \ln \tanh(\lambda t) +}
\[ + \phi_0 = \tilde{\lambda} \ln \tanh(\lambda t) + \phi_0. \quad (33) \]
The dependence $V = V(\phi)$ can be presented as direct one for open universe

$$V(\phi) = A + 2\lambda^2 + \cosh^2(\frac{\phi}{A}) \left[ 2\lambda^2 - \frac{2}{K_0^2} \right].$$  \hfill (34)

5. Conformally-flat solutions for nonlinear scalar fields

Let us consider the case of the conformally-flat metric

$$dS^2 = A\{-d\xi^2 - (dx^2)^2 -(dx^3)^2 + (dx^4)^2\}$$  \hfill (35)

as an application of the method above. In (35) $A = A(x^3,x^4)$. The analogy of the formulas (21) and (22) can be presented in the form

$$V(\phi(x^3,x^4)) = \kappa^{-1}(\frac{A_{33} - A_{44}}{2A^2}),$$  \hfill (36)

$$\phi_3^2 + \phi_4^2 = \kappa^{-1}\left\{ -\frac{A_{33} + A_{44}}{A} + \frac{3(A_3^2 + A_4^2)}{2A^2} \right\}. \hfill (37)

In special cases it is possible to reduce solutions (36) and (37) to those ones obtained earlier. The construction of new solutions is also possible.

5.1. Cosmological and static solutions

Few examples will show the possibilities of the method.

- Let $A = A_0 e^{H_0z}$, $\eta = x^4$. The solution is

$$V(\phi) = -\frac{H_0^2}{\kappa A_0} e^{\pm \sqrt{2}\phi},$$  \hfill (38)

$$\phi = \pm \frac{H_0}{\sqrt{2}\kappa} \eta.$$  \hfill (39)

- Let $A = A_0 e^{H_0z}$, $z = x^3$. Then the solution is

$$V(\phi(\eta)) = -\frac{H_0^2}{\kappa A_0} e^{\pm \sqrt{2}\phi},$$  \hfill (40)

$$\phi = \pm \frac{H_0}{\sqrt{2}\kappa} \eta.$$  \hfill (41)

Both cases (38) and (40) correspond to Liouville-type nonlinearity for the $V(\phi)$. These solutions have been obtained in the work [3] by suggesting the Liouville form of the potential $V(\phi)$. It is clear that the cosmological solution (38)-(39) is the same as (29) for $m = 1$, but calculated in the conformal time $\eta$.

- Let $A = e^\beta$, where $\beta = \frac{1}{2}c_1 \eta^2 + c_2 \eta + c_3$. Then, integrating (36)-(37), we can find

$$V(\phi(\eta)) = -\kappa^{-1}e^{-\beta}(c_1 + (c_1 \eta + c_2)^2),$$  \hfill (42)

$$\phi(\eta) = \pm \frac{\sqrt{2}}{\sqrt{\kappa c_1}} \left( \frac{1}{2} \eta \sqrt{\eta^2} - c_1 - \frac{1}{2} c_1 \ln |\eta + \sqrt{\eta^2} - c_1| \right),$$  \hfill (43)

$$\tilde{\eta} = \frac{c_1 \eta + c_2}{\sqrt{2}}.$$  \hfill (44)

- Let $A = e^\beta$, where $\beta = \frac{a_1}{2} \eta^2 + a_3$. Then the potential can be presented in the form

$$V(\phi(\eta)) = -\kappa^{-1}e^{-\beta}\{a_1 \eta^2 + \frac{a_2}{9} \eta^6\}.$$  \hfill (45)

The expression for the scalar field reads

$$\phi(\eta) = \pm \frac{3}{2a_1 \sqrt{2\kappa}} \left\{ \frac{1}{2} \eta \sqrt{\eta^2} - c_1 - \frac{1}{2} c_1 \ln |\eta + \sqrt{\eta^2} - c_1| \right\},$$  \hfill (46)

$$\tilde{\eta} = \frac{a_1 \eta^2}{3}.$$  \hfill (47)

- Let $A = e^\beta$, where $\beta = \frac{a_1}{2} \eta^2 + a_3$. The static analogy can be obtained from the case above by substituting: $\eta \rightarrow z$, $V(\phi(\eta)) \rightarrow -V(\phi(z))$.

It should be noted here that the list of analytical solutions can be extended if one can solve (37) for given gravitational field $A(x^3,x^4)$.

5.2. Solitary wave solutions

Let $A = A(\theta)$, $\phi = \phi(\theta)$, where $\theta = z - u_0 \eta$. That is we are looking for solutions of a solitary wave type. Equations (36)-(37) are reduced to

$$V(\phi(\theta)) = \kappa^{-1}(1 - u_0^2)\frac{A_{\theta\theta}}{2A^2}$$  \hfill (48)

$$\phi_\theta^2 = \kappa^{-1}\left\{ -\frac{A_{\theta\theta}}{A} + \frac{3A_\theta^2}{2A^2} \right\}. \hfill (49)

Using the analogy of equations (48)-(49) to those ones for cosmological and static solutions above we can conclude that all solutions presented in this section can be obtained by virtue of the substitutions: $\eta \rightarrow \theta$, $V(\phi(\eta)) \rightarrow \pm (1 - u_0^2)^{-1}V(\phi(\theta))$. It is interesting to note that, when $u_0^2 = 1$, e.g. $u_0^2$ equal to the speed of light, the potential disappears.

5.3. The massive scalar field

The massive scalar field can be considered as the simplest chaotic inflationary model [3]. The analysis of the chaotic inflation scenario has been based on the asymptotic solution, when $m \ll 1$, $\phi \gg 1$. To find the exact solutions in the framework of general relativity with the massive scalar field is the long standing problem. To understand the reason of this fact we can insert the potential

$$V(\phi) = \frac{1}{2}m^2\phi^2$$  \hfill (50)
in the general formulas (18), (20). After simple manipulations with (20), (21), using (16), it is possible to find the test equation for exact solutions of self-gravitating massive scalar field in the conformally-flat spaces (16) in the form:

\[ \frac{1}{2m^2} \left( \frac{A_{44}}{2A^2} \right)^2 + \frac{A_{44}}{2A^2} \left( \frac{3A^2}{2A^2} \right) - \frac{A_{44}}{A} = 0. \]  

(51)

Unfortunately the solution of the form \( A = A_0 \eta^{-2} \) leads to \( \phi = \text{const} \) and can be identified with solutions for gravitational vacuum with \( V(\phi) = \text{const} = \Lambda \). Any other solution of test equation (51) by respect of \( A(\eta) \) solves the problem of exact solutions for massive scalar field in standard inflationary model.

The complications of the third order non-linear differential equation (51) are the reason of impossibility to find the exact solutions in standard inflationary model. Nevertheless it is possible to calculate by virtue of numerical study the divergence an asymptotic and numerical solutions from the exact one.

6. Conclusions

This review article are devoted to better understanding the gravitational field of the very early Universe. We have presented three most important sources of the gravitation: self-interacting scalar field, chiral field and gauge field. The correspondence between all of them are considered. In this contribution more attention have been payed to nonlinear scalar field source of gravity. The progress in finding the exact solutions in inflationary universe is reviewed. The basic idea of ‘fine turning of the potential’ method is discussed and computational background is presented in details. A set of new exact solutions for standard inflationary model and conformally-flat space-times are obtained. Special attention payed to relations between ‘fine turning of the potential’ and Barrow’s approaches. As the example of a synthesis of both methods new exact solution is obtained.

The analogous presentation of chiral and gauge fields will appear in future publications.

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