Convergence rate toward shock wave under periodic perturbation for
generalized Korteweg-de Vries-Burgers equation

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ABSTRACT. In this paper, a viscous shock wave under space-periodic perturbation
of generalized Korteweg-de Vries-Burgers equation is investigated. It is shown that
if the initial periodic perturbation around the viscous shock wave is small, then the
solution time asymptotically tends to a viscous shock wave with a shift partially
determined by the periodic oscillations. Moreover the exponential time decay rate
toward the viscous shock wave is also obtained for some certain perturbations.

AMS subject classifications. 35Q53; 76L05.

1. INTRODUCTION

We consider generalized Korteweg-de Vries-Burgers (KdV-Burgers) equation:
\[ u_t + f(u)_x + \mu u_{xxx} - \gamma u_{xx}, \quad x \in \mathbb{R}, \quad t > 0, \]
where the flux \( f(u) \in C^3 \) is strictly convex, \( \mu > 0 \) the dispersive coefficient, \( \gamma > 0 \)
the viscosity. When \( \mu = 0 \), the equation (1.1) becomes the famous Burgers equation,
which admits viscous shock wave solution \( \phi(x - st) \) with shock propagation speed \( s \). The stability of viscous shock for the Burgers equation has been extensively
studied, see [9, 10, 19]. When \( \mu > 0 \), the KdV-Burgers equation (1.1) still admits
the viscous shock wave solution as \( \gamma \gg 1 \), that is the viscosity plays the main role,
see [1, 6]. Later, Bona-Rajopadhye-Schonbek [2] showed that this shock wave is
asymptotically stable in the case of \( f = \frac{1}{2} u^2 \) provided that the perturbation is small.
Moreover, the time decay rate was obtained in [14–16]. The exponential time decay
rate was further obtained in Yin-Zhao-Zhou [22] provided that the initial values
converge to some constants exponentially at the far field. We refer to [3, 12, 17, 21]
for more interesting works on KdV equation.

Note that all stability works above are based on \( L^2 \) integrability perturbation.
Namely the initial perturbation is \( L^2 \) integrable. If the initial perturbation is space
periodic, what about the stability of viscous shock? When \( \mu = \gamma = 0 \), the KdV-Burgers
equation becomes hyperbolic conservation law and the periodic perturbation problem
is interesting and challenging since the solution oscillates at the far field and
resonance may happen [13]. Indeed, Lax [11] and Dafermos [5] proved that the solution
time asymptotically tends to the periodic average. The asymptotic stability

Date: September 8, 2022.

Key words and phrases. periodic perturbations; asymptotic behavior Korteweg-de Vries-Burgers
equation; time decay rate; viscous shock wave.
Periodic perturbations for the KdV-Burgers equation

of shocks in both inviscid and viscous cases with periodic perturbation was obtained in [10,20,23]. We refer to [7,8] for more interesting works. When $\gamma=0$, the KdV-Burgers equation becomes KdV equation, and the periodic solutions were studied in [18]. Motivated by [7], we wonder whether the viscous shock wave constructed in [1,22] for the KdV-Burgers equation (1.1) is time asymptotically stable with periodic perturbations.

In this paper, we consider a Cauchy problem of (1.1) with the initial data satisfying

$$u_0(x) := u(x, 0) \rightarrow \left\{ \begin{array}{ll}
\bar{u}_l + w_{0l}(x), & x \rightarrow -\infty, \\
\bar{u}_r + w_{0r}(x), & x \rightarrow \infty,
\end{array} \right. \quad (1.2)$$

where $\bar{u}_l, \bar{u}_r$ are constants satisfying $\bar{u}_l > \bar{u}_r$. Function $w_{0i}(x) \in L^\infty(\mathbb{R})$ is a periodic function with period $p_i > 0, (i=r,l)$ satisfying

$$\frac{1}{p_i} \int_0^{p_i} w_{0i}(x) dx = 0. \quad (1.3)$$

We aim to prove that the shock wave is stable for the Cauchy problem (1.1)-(1.2). Roughly speaking, the solution not only exists globally but also tends to a viscous shock wave as time goes to infinity. Moreover, the exponential time decay rate toward the viscous shock wave is also obtained for some certain periodic perturbations. The precise statements of the main results are given in Theorem 2.1 and Theorem 2.2 in Section 2.

We outline the strategy as follows. We apply the anti-derivative method to study the stability of the traveling wave solution $\phi$, in which the anti-derivative of the perturbation $u-\phi$, namely, $\Psi(x,t) = \int_0^x (u-\phi)(y,t) dy$, “should” belong to some Sobolev spaces like $H^2(\mathbb{R})$. However, the method above can not be applicable directly in this paper since $u-\phi$ oscillates at the far field and hence does not belong to any $L^p$ space for $p \geq 1$. Motivated by [20], we introduce a suitable ansatz $U(x,t)$, which has the same oscillations as the solution $u(x,t)$ at the far field, so that $\int_0^{x=\infty} u(x,t) - U(x,t) dx$ belongs to some Sobolev spaces and the anti-derivative method is still available.

The ansatz is defined as $U = u_l(x,t) g_\eta(x-s t - \eta(t)) + u_r(x,t) [1 - g(x-s t - \eta(t))]$, where $s$ is the shock speed, $u_l$ is a periodic solution of (1.1) with the initial data $\bar{u}_l + w_{0l}(x)$ in (1.2) and is expected to have the same oscillation as $u$ near $x=-\infty$. Similarly $u_r$ is expected to have the same oscillation as $u$ near $x=\infty$. Thus $u(x,t) - U(x,t)$ is expected to be integrable. The shift $\eta(t)$ in the function $g_\eta$, related to the traveling wave $\phi$, is determined through an ODE equation (2.12) and partially depends on the initial periodic perturbations $w_{0l}$ and $w_{0r}$. The shift $\eta$ is used to guarantee the integral $\int_0^{x=\infty} (u-U)(y,t) dy = 0$ so that $\Psi(\pm\infty, t) = 0$ and thus $\Psi$ could belong to $L^2$. Moreover, we obtain the exponential time convergence rate toward the viscous shock wave by using a weighted energy estimate provided that the initial perturbation is located in a suitable weighted function space.
The rest of the paper will be arranged as follows. In Section 2, a suitable ansatz is constructed and the main results are stated. In Section 3, the stability problem is reformulated to a perturbation equation around the ansatz. In Section 4, some weighted a priori estimates are established. In Section 5, the main results are proved. In Section 6, some complementary proofs are provided.

Notation. The functional \( \| \cdot \|_{L^p(\Omega)} \) is defined by \( \| f \|_{L^p(\Omega)} = \left( \int_{\Omega} |f|^p(\xi)d\xi \right)^{1/p} \). The symbol \( \Omega \) is often omitted when \( \Omega = (-\infty, \infty) \). We denote for simplicity \( \| f \| = \left( \int_{-\infty}^{\infty} f^2(\xi)d\xi \right)^{1/2} \) as \( p = 2 \). In addition, \( H^m \) denotes the \( m \)-th order Sobolev space of functions defined by
\[
\| f \|_m = \left( \sum_{k=0}^{m} \| \partial^k_x f \|^2 \right)^{1/2}.
\]

2. Preliminaries and Main Results

2.1. Suitable Ansatz. A viscous shock profile \( \phi(x - st) \) is a traveling wave solution of (1.1). It satisfies
\[
\begin{align*}
\mu \phi''' - \gamma \phi'' - s \phi' + f'(\phi)\phi' &= 0, \\
\lim_{\xi \to -\infty} \phi(\xi) &= \bar{u}_l, \\
\lim_{\xi \to +\infty} \phi(\xi) &= \bar{u}_r,
\end{align*}
\]
where \( \xi = x - st, \quad s = \frac{d}{dt}, \) \( s \) is the shock speed defined by Rankine–Hugoniot condition
\[
s(\bar{u}_l - \bar{u}_r) = f(\bar{u}_l) - f(\bar{u}_r). \tag{2.2}
\]

Lemma 2.1. [22, Lemma 2.1] Assume that the Rankine–Hugoniot condition (2.2) and \( \gamma^2 + 4\mu(s - f'(\bar{u}_l)) \geq 0 \) holds. Then the equation (2.1) has a unique solution \( \phi(\xi) \), up to a shift. Moreover, it satisfies \( \phi'(\xi) < 0 \) and \( \bar{u}_r < \phi(\xi) < \bar{u}_l \) for all \( \xi \in \mathbb{R} \).

Let \( g(\xi) : = \frac{\phi(\xi) - \bar{u}_r}{\bar{u}_l - \bar{u}_r} \), it holds that
\[
\lim_{\xi \to -\infty} g(\xi) = 1, \quad \lim_{\xi \to +\infty} g(\xi) = 0. \tag{2.3}
\]

Then we have

Lemma 2.2. There exists a positive constant \( \sigma_0 \) such that
\[
\begin{align*}
0 < g(\xi) &\leq Ce^{-\sigma_0\xi}, & \xi > 0, \\
0 < 1 - g(\xi) &\leq Ce^{\sigma_0\xi}, & \xi < 0, \\
g'(\xi) &< 0, \\
|g^{(m)}(\xi)| &\leq Ce^{-\sigma_0|\xi|}, & \xi \in \mathbb{R}, m \in \mathbb{N}^+.
\end{align*}
\]
Here \( C > 0 \) depends on \( \bar{u}_l \) and \( \bar{u}_r \).

Proof. This lemma is a corollary of [22, Lemma 2.1] and the proof is omitted. \( \square \)
Periodic perturbations for the KdV-Burgers equation

We assume that the function \(u_i(x,t)\) is the solution of (1.1) with the initial data \((i=l,r)\):
\[
u_i_0(x) = \bar{u}_i + w_0(x).
\]
We introduce \(g_{\eta}(x,t)\) by
\[
g_{\eta}(x,t) = g(\xi - \eta(t)),
\]
satisfying
\[
\lim_{x \to -\infty} g_{\eta}(x,t) = 1, \quad \lim_{x \to +\infty} g_{\eta}(x,t) = 0,
\]
where \(\eta(t)\) is the shift of the shock profile, the exact expression of \(\eta(t)\) can be found in (2.12). Motivated by [20], we construct an ansatz below
\[
U(x,t) = u_l(x,t) g_{\eta}(x,t) + u_r(x,t) [1 - g_{\eta}(x,t)].
\]
Note that \(u_i, i=l, r\) is a periodic solution of (1.1) with the initial data \(\bar{u}_i + w_0(x)\) and is expected to have the same oscillation as \(u\) near \(x = \pm \infty\), respectively. Thus \(U(x,t)\) is expected to have the same oscillation as \(u(x,t)\) at the far fields.

2.2. Location of the Shift \(\eta(t)\). At the beginning of this subsection, we list a useful lemma, which will be used later. Equation (1.1) can be rewritten in the new coordinate \((\xi, t)\) as
\[
u_t - s u_\xi + f(u)_{\xi} + \mu u_{\xi \xi \xi} = \gamma u_{\xi \xi}, \quad \xi \in \mathbb{R}, \ t > 0.
\]

**Lemma 2.3.** Assume that \(u_0 \in H^{k+1}(0,p)\) is a periodic function with period \(p > 0\) for any integer \(k \geq 0\). Then the periodic solution \(u(\xi,t)\) of (2.7) satisfies
\[
\|\partial^k_{\xi} (u - \bar{u})\|_{L^\infty(\mathbb{R})} \leq C \|u_0 - \bar{u}\|_{H^{k+1}(0,p)} e^{-\theta t}, \quad t \geq 0,
\]
where \(\bar{u} = \frac{1}{p} \int_0^p u_0(\xi) d\xi\) and the positive constants \(C, \theta\) are independent of time \(t\).

The proof of Lemma 2.3 is left in Appendix.

Now we begin to study the property of the shift. Since \(U\) is not the solution of the KdV-Burgers equation (1.1), the error term is
\[
h := U_t - s U_\xi + f(U)_{\xi} + \mu U_{\xi \xi \xi} - \gamma U_{\xi \xi}.
\]
By (2.6), we have
\[
h = \{f(U) - f(u_l)_{\xi} g_{\eta} - f(u_r) (1 - g_{\eta})_\xi + \mu (u_l - u_r)_{\xi} g'_{\eta} - 2 \gamma (u_l - u_r) g''_{\eta}\}_{\xi}
+ (f(u_l) - f(u_r)) g'_{\eta} + \gamma (u_l - u_r) g''_{\eta} + \mu (u_l - u_r) g'''_{\eta}
- (u_l - u_r) \cdot g'_{\eta} \cdot (s + \eta'(t)).
\]

Subtracting (2.8) from (2.7) and integrating the resulting system with respect to \(\xi\) over \((-\infty, \infty)\), one has
\[
\frac{d}{dt} \int_{-\infty}^{\infty} (u - \bar{U})(\xi,t) d\xi = - \int_{-\infty}^{\infty} h(\xi,t) d\xi.
\]
To apply the anti-derivative method which is often used to study the stability of viscous shock, introduced in [10], we expect

$$\int_{-\infty}^{\infty} (u - U)(\xi, t) d\xi = 0, \quad (2.10)$$

holds for any $t \geq 0$. Then we compute from (2.10) that

$$0 = \int_{-\infty}^{\infty} (f(u_l) - f(u_r)) g'_0 d\xi - \int_{-\infty}^{\infty} (u_l - u_r) g'_0 (\eta(t) + s) d\xi$$

$$+ \gamma \int_{-\infty}^{\infty} (u_l - u_r) g''_0 d\xi + \mu \int_{-\infty}^{\infty} (u_l - u_r) g'''_0 d\xi. \quad (2.11)$$

Thus we obtain the following ODE for $\eta(t)$,

$$\begin{cases} 
\eta'(t) = \frac{\gamma \int_{-\infty}^{\infty} (u_l - u_r) g'_0 d\xi + \mu \int_{-\infty}^{\infty} (u_l - u_r) g''_0 d\xi + \int_{-\infty}^{\infty} (f(u_l) - f(u_r)) g'_0 d\xi}{\int_{-\infty}^{\infty} (u_l - u_r) g'_0 d\xi} - s, \\
\eta(0) = \eta_0. \quad (2.12)
\end{cases}$$

And the initial data $\eta_0$ of the equation (2.12) should satisfy $\int_{-\infty}^{\infty} (u - U)(\xi, 0) d\xi = 0$, i.e.,

$$\int_{-\infty}^{\infty} (u_0 - u_0)(\xi) (\xi - \eta_0) + (u_0 - u_0)(\xi) [1 - g(\xi - \eta_0)] d\xi = 0. \quad (2.13)$$

**Lemma 2.4.** There exists a small constant $\delta_0$, such that if $\delta := \max \|w_i\|_{H^1(\Omega)} < \delta_0; i = l, r$, the ODE problem (2.12) has a unique smooth solution $\eta(t) : [0, +\infty) \to \mathbb{R}$. Moreover, the shift $\eta(t)$ satisfies

$$|\eta'(t)| + |\eta(t) - \eta_\infty| \leq C \delta e^{-\theta t}, \quad t \geq 0, \quad (2.14)$$

where $C$ and $\theta$ are positive constants independent of time $t$, and

$$\eta_\infty = \frac{1}{u_l - u_r} (\eta_{\infty, 1} + \eta_{\infty, 2}),$$

$$\eta_{\infty, 1} = \int_{-\infty}^{0} (u_0 - \phi - w_{0l})(\xi) d\xi + \int_{0}^{+\infty} (u_0 - \phi - w_{0r})(\xi) d\xi,$$

$$\eta_{\infty, 2} = \int_{0}^{+\infty} \frac{1}{p_l} \int_{0}^{p_l} [f(u_l) - f(\bar{u}_l) - s u_l + s \bar{u}_l] (\xi, t) d\xi dt - \frac{1}{p_l} \int_{0}^{p_l} \int_{0}^{\xi} w_{0l}(y) dy d\xi$$

$$- \int_{0}^{+\infty} \frac{1}{p_r} \int_{0}^{p_r} [f(u_r) - f(\bar{u}_r) - s u_r + s \bar{u}_r] (\xi, t) d\xi dt + \frac{1}{p_r} \int_{0}^{p_r} \int_{0}^{\xi} w_{0r}(y) dy d\xi.$$
2.3. **Main Theorems.** Based on Lemma 2.4, we know (2.10) holds for any \( t \geq 0 \) provided that the solution of the equation (1.1) globally exists. Then we can define the anti-derivative of the perturbation \( \psi(\xi, t) : = u - U(\xi, t) \) by

\[
\Psi(\xi, t) := \int_{\xi}^{\xi_n} \psi(y, t) dy,
\]

so that \( \Psi \) belongs to some Sobolev space. We assume that the initial data satisfies

\[
\Psi_0(\xi) := \Psi(\xi, 0) \in H^3(\mathbb{R}).
\]

The first result is

**Theorem 2.1.** If (2.1), (2.2) and (2.16) hold, there exists a positive constant \( \epsilon_0 \), such that if

\[
\|\Psi_0\|_2^2 + \delta_0 \leq \epsilon_0^2,
\]

then there exists a unique global solution of (1.1), (1.2) satisfying

\[
\sup_{\xi} |(u - \phi_{\eta_\infty})(\xi, t)| \to 0 \ \forall t \to \infty,
\]

where \( \phi_{\eta_\infty} = \phi(\xi - \eta_\infty) \).

In order to obtain time decay rate of the solution, we further assume that

\[
\exp\left(\frac{\alpha}{2} |\xi - \xi^*_s| \right) \frac{\partial^i}{\partial \xi^i} \Psi_0(\xi) \in L^2(\mathbb{R}) (i = 0, 1, 2, 3),
\]

where \( \xi^*_s = \xi^* + \eta_\infty \) and \( f'(\phi(\xi^*)) = s, \ 0 < \alpha < \min\{\frac{2}{3\mu}, \sigma_0\} \).

The second result is

**Theorem 2.2.** If (2.1), (2.2), (2.16), (2.18) and (2.17) hold, then there exists a unique global solution of (1.1), (1.2) satisfying

\[
\sup_{\xi} |(u - \phi_{\eta_\infty})(\xi, t)| \leq Ce^{-\beta t}, \ \forall t > 0.
\]

Positive constants \( \beta \) satisfy

\[
\left\{ \begin{array}{l}
0 < \beta < \min\{C_0\alpha, \theta\}, \\
(C_0 - \frac{\beta}{\alpha})(3 \mu - \frac{2\alpha}{\alpha}) + (\alpha \mu + \gamma)^2 < 0,
\end{array} \right.
\]

Here

\[
\beta = \min\{C_0\alpha, \theta, C_0 - \frac{\alpha(\mu + \gamma)^2}{2\gamma - 3\alpha}\},
\]

where \( C_0 := G \cdot \left\{ \frac{B}{\gamma}, \min \{ |\phi(\xi^* - 1) - \phi(\xi^*)|, |\phi(\xi^* + 1) - \phi(\xi^*)| \} \right\} \), with \( G := \min_{u \in [a, b]} |f''(u)|, \ B := \min_{\xi \in [\xi^* - 1, \xi^* + 1]} |\phi'(\xi)|. \)
3. Reformulation of the Problem

Subtracting (2.8) from (2.7) and integrating the resulting system with respect to $\xi$, we have that
\[
\Psi_t - s \Psi_\xi + [f(u) - f(U)] + \mu \Psi_{\xi\xi\xi} - \gamma \Psi_{\xi\xi} = H,
\]
where $H = \int_{-\infty}^{\xi} -h(\xi, t)d\xi$. We show the following decay properties of the error term of $H$.

**Lemma 3.1.** The error term $H$ satisfies:
\[
\left| \frac{\partial^j H}{\partial \xi^j} \right| \leq C \delta e^{-\theta t} e^{-\sigma_0 |\xi - \eta(t)|}, \quad j=0, 1, 2, 3.
\]

The proof is left in Section 5. We will seek the solution in the functional space $X_\epsilon(0, T)$ for any $0 \leq T < +\infty$,
\[
X_\epsilon(0, T) := \left\{ \Psi \in C([0, T]; H^3) \left| \Psi_\xi \in L^2(0, T; H^3), \sup_{0 \leq t \leq T} \|\Psi\|_3(t) \leq \epsilon \right. \right\}.
\]

**Proposition 3.1.** (a priori estimate) Suppose that $\Psi \in X_\epsilon(0, T)$ is the solution of (3.1), (2.16) for some time $T > 0$. There exists a positive constant $\epsilon_0$ independent of $T$, such that if
\[
\sup_{0 \leq t \leq T} \|\Psi(t)\|_3 \leq \epsilon \leq \epsilon_0
\]
for $t \in [0, T]$, then
\[
\|\Psi(t)\|_3^2 + \int_0^t \|\Psi_\xi(\tau)\|_3^2 d\tau \leq C_1 \left( \|\Psi_0\|_3^2 + \delta \right),
\]
for any $t \geq 0$. Here $C_1$ is independent of $T$.

4. Weighted Estimates

Throughout this section, we assume that the problem (3.1), (2.16) has a solution $\Psi \in X_\epsilon(0, T)$. At the begin of this section, we give a lemma and some equalities which will be use later.

**Lemma 4.1.** Under the same condition of Lemma 2.4, one gets
\[
\left| \frac{\partial^i}{\partial \xi^i} (U - \phi_{\eta_\infty}) \right| \leq C \delta e^{-\theta t}, \quad i=0, 1.
\]

**Proof.** By directly calculate, one gets
\[
U - \phi_{\eta_\infty} = [\phi_\eta(\xi) - \phi_{\eta_\infty}(\xi)] + \left\{ (u_l - \bar{u}_l) \frac{\phi_\eta(\xi) - \bar{u}_r}{u_l - \bar{u}_r} - (u_r - \bar{u}_r) \frac{\phi_\eta(\xi) - \bar{u}_l}{u_l - \bar{u}_r} \right\}
\]
\[
:= W_1 + W_2.
\]
With the aid of Lemma 2.4, we have
\[ |W_1|, |W_{1\xi}| \leq C|\eta(t) - \eta_\infty| \leq C\delta e^{-\beta t}. \]

By Lemma 2.3, it follows that \(|W_2|, |W_{2\xi}| \leq C\delta e^{-\beta t}\). Thus the proof of Lemma 4.1 is obtained.

Lagrange mean value theorem gives that
\[
2\Psi [f (U + \Psi t) - f (U)] = \{f'(U)\Psi^2\}_{\xi} - f''(U)U_{\xi}\Psi^2 + f''(\xi_1)\Psi\Psi_{\xi},
\]
\[
2\Psi_t [f (U + \Psi t) - f (U)]_{\xi} = 2\Psi [f' (\xi_2) \Psi \Psi_{\xi}] = \{2 f' (\xi_2) \Psi^2\}_{\xi} - 2 f' (\xi_2) \Psi\Psi_{\xi},
\]
\[
2\Psi_{\xi\xi} [f (U + \Psi t) - f (U)]_{\xi\xi} = \left\{2 \Psi_{\xi\xi} [f (U + \Psi t) - f (U)]_{\xi}\right\}_{\xi}
- 2 \Psi_{\xi\xi\xi} [f'' (\xi_3) U_{\xi} \Psi + f' (U + \Psi \xi) \Psi_{\xi}],
\]
\[
2\Psi_{\xi\xi\xi} [f (U + \Psi t) - f (U)]_{\xi\xi\xi} = \left\{2 \Psi_{\xi\xi\xi} [f (U + \Psi t) - f (U)]_{\xi}\right\}_{\xi}
- 2 \Psi_{\xi\xi\xi\xi} [f'' (\xi_3) U_{\xi} \Psi + f' (U + \Psi \xi) \Psi_{\xi} + f'' (U + \Psi \xi) \Psi_{\xi}^2]
- 2 \Psi_{\xi\xi\xi\xi} [2 f'' (U + \Psi \xi) \Psi_{\xi} U_{\xi} + f''' (\xi_4) \Psi_{\xi} U_{\xi}^2],
\]
where \(\xi_i\) between \(U\) and \(U + \Psi \xi_i, i=1, 2, 3, 4\).

**Lemma 4.2.** Under the same assumptions in Proposition 3.7, if \(\alpha, \beta\) satisfy (2.20) or \(\alpha=\beta=0\), we have the following inequality
\[
e^{\beta t} \int_{-\infty}^{+\infty} e^{\alpha |\xi - \xi_\infty|} |\Psi^2(t, \xi)| d\xi + \int_0^t e^{\beta \tau} \int_{-\infty}^{+\infty} e^{\alpha |\xi - \xi_\infty|} |\Psi^2(\tau, \xi)| d\xi d\tau
\leq C \int_{-\infty}^{+\infty} e^{\alpha |\xi - \xi_\infty|} |\Psi_0(\xi)| d\xi + C\delta. \tag{4.2}
\]

**Proof.** We multiply equation (3.3) by \(2e^{\beta t}e^{\alpha |\xi - \xi_\infty|} \Psi(\xi, t)\), and integrate result with respect to \(t\) and \(\xi\) over \([0, t] \times \mathbb{R}\), we have
\[
e^{\beta t} \int_{-\infty}^{+\infty} e^{\alpha |\xi - \xi_\infty|} |\Psi^2(t, \xi)| d\xi
+ 2 \gamma \int_0^t e^{\beta \tau} \int_{-\infty}^{+\infty} e^{\alpha |\xi - \xi_\infty|} |\Psi^2(\tau, \xi)| d\xi d\tau
+ \int_0^t e^{\beta \tau} \int_{-\infty}^{+\infty} e^{\alpha |\xi - \xi_\infty|} A_\alpha(\xi) |\Psi^2(\tau, \xi)| d\xi d\tau \tag{4.3}
\]
\[
:= \int_{-\infty}^{+\infty} e^{\alpha |\xi - \xi_\infty|} |\Psi_0^2(\xi)| d\xi + \sum_{i=1}^{8} \int_0^t e^{\beta \tau} \int_{-\infty}^{+\infty} e^{\alpha |\xi - \xi_\infty|} a_i d\xi d\tau,
\]
where
\[
A_\alpha(\xi) = - \text{sign} [(\xi - \eta_\infty) - (\xi_* - \eta_\infty)] \left(f' (\phi_{\eta_\infty}) - s\right) + f'' (\phi_{\eta_\infty}) |\phi_{\eta_\infty}|. \]
\[a_1 = \beta \Psi^2, \quad a_2 = -3\alpha \mu \text{sign}(\xi - \xi_*) \Psi^2, \quad a_3 = -2\alpha^2 \mu \Psi \Psi_\xi, \quad a_4 = -2\alpha \gamma \text{sign}(\xi - \xi_*) \Psi \Psi_\xi, \quad a_5 = -f''(\xi) \Psi_\xi^2, \quad a_6 = 2H \Psi, \]

\[a_7 = -\alpha \text{sign}(\xi - \xi_*) \left( f'(\phi_{\eta,\xi}) - f'(U) \right) \Psi_\xi^2, \quad a_8 = \left( f''(\phi_{\eta,\xi}) \left| \phi_{\eta,\xi,\xi} \right| - f''(U) \left| U_\xi \right| \right) \Psi^2. \]

One gets \( A_\alpha(\xi) \geq C_0 \alpha (\xref{22} Lemma 3.1). \) We rewrite \([2\gamma \Psi_\xi^2 + C_0 \alpha \Psi^2] - \Sigma_{j=1}^4 a_j \) as \((\Psi \Psi_\xi) \mathbf{M}(\Psi \Psi_\xi)^T\)

where the matrix \( \mathbf{M} : = \begin{pmatrix} C_0 \alpha - \beta & \mu \alpha^2 + \gamma \alpha \text{sign}(\xi - \xi_*) \\ \mu \alpha^2 + \gamma \alpha \text{sign}(\xi - \xi_*) & 2\gamma - 3\alpha \mu \text{sign}(\xi - \xi_*) \end{pmatrix} \).

A directly calculation gives if \((2.20)\) holds, the matrix \( \mathbf{M} \) is positive. Thus we can find a positive constant \( \sigma_1 \) such that

\[
\left( \begin{array}{c} \Psi \\ \Psi_\xi \end{array} \right) \mathbf{M} \left( \begin{array}{c} \Psi \\ \Psi_\xi \end{array} \right) \left\{ \begin{array}{l} > \sigma_1 [\Psi^2 + \Psi_\xi^2], \\
\alpha > 0, \\
= -2\gamma \Psi_\xi^2, \\
\alpha = 0. \end{array} \right. \tag{4.4} \]

Now we estimate the last four terms \( A_i \) on the right-hand side of \((4.3)\), where \( A_i = \int_0^t e^{\beta \tau} \int_{-\infty}^{+\infty} a_i d\xi d\tau. \) With the aid of Sobolev inequality, one gets

\[
A_5 \leq C \sup_{\tau \in [0,t]} \| \Psi(\tau) \|_{L^\infty} \int_0^t e^{\beta \tau} \int_{-\infty}^{+\infty} e^{\alpha |\xi - \xi_\tau|} \Psi_\xi^2(\tau, \xi) d\xi d\tau \leq C \epsilon \int_0^t e^{\beta \tau} \int_{-\infty}^{+\infty} e^{\alpha |\xi - \xi_\tau|} \Psi_\xi^2(\tau, \xi) d\xi d\tau. \tag{4.5} \}

By \((2.14)\), one gets that \( \eta \) is bounded. Thus, we can find a sufficient big positive constant \( N \), such that \( N \geq \max\{\xi_\tau, \pm \eta(t)\} \), if \( \beta < \theta, \alpha < \sigma_0 \), we have

\[
A_6 \leq C \sup_{\tau \in [0,t]} \| \Psi(\tau) \|_{L^\infty} \int_0^t e^{\beta \tau} \int_{-\infty}^{+\infty} e^{\alpha |\xi - \xi_\tau|} \| H \| d\xi d\tau \leq C \epsilon \delta \int_0^t e^{\beta \tau} e^{-\theta \tau} \int_{-\infty}^{+\infty} e^{\alpha |\xi - \xi_\tau|} e^{-\sigma_0 |\xi - \eta(t)|} (\tau, \xi) d\xi d\tau = C \epsilon \delta \int_0^t e^{(\beta - \theta) \tau} \left( \int_{-N}^{-N} + \int_{-N}^{N} + \int_{N}^{+\infty} \right) e^{\alpha |\xi - \xi_\tau|} e^{-\sigma_0 |\xi - \eta(t)|} d\xi d\tau \leq C \epsilon \cdot e^{(\sigma_0 + \alpha)N} \cdot \delta \int_0^t e^{(\beta - \sigma_0) \tau} \int_{-N}^{-N} e^{(\sigma_0 - \alpha) \xi} d\xi d\tau + C \epsilon \delta \int_0^t e^{(\beta - \theta) \tau} \int_{-N}^{-N} e^{\alpha |\xi - \xi_\tau|} e^{-\sigma_0 |\xi - \eta(t)|} d\xi d\tau + C \epsilon \cdot e^{(\sigma_0 + \alpha)N} \cdot \delta \int_0^t e^{(\beta - \theta) \tau} \int_{-N}^{N} e^{(\alpha - \sigma_0) \xi} d\xi d\tau \leq C \delta. \tag{4.6} \]
With the aid of Lemma 4.1 if \( \alpha \neq 0 \), we have
\[
A_7 \leq \alpha \int_0^t e^{\beta \tau} \int_{-\infty}^{+\infty} e^{\alpha |\xi - \xi_\tau|} \left| f'(\phi_{\eta_\tau}) - f'(U) \right| \Psi^2(\tau, \xi) d\xi d\tau
\]
\[
\leq C \int_0^t e^{\beta \tau} \int_{-\infty}^{+\infty} e^{\alpha |\xi - \xi_\tau|} |\phi_{\eta_\tau} - U| \Psi^2(\tau, \xi) d\xi d\tau
\]
\[
\leq C\delta \int_0^t e^{\beta \tau} \int_{-\infty}^{+\infty} e^{\alpha |\xi - \xi_\tau|} e^{-\theta \tau} \Psi^2(\tau, \xi) d\xi d\tau
\]
\[
\leq C\delta \int_0^t e^{\beta \tau} \int_{-\infty}^{+\infty} e^{\alpha |\xi - \xi_\tau|} \Psi^2(\tau, \xi) d\xi d\tau, \quad (4.7)
\]
and
\[
A_8 = \int_0^t e^{\beta \tau} \int_{-\infty}^{+\infty} e^{\alpha |\xi - \xi_\tau|} \left( f''(\phi_{\eta_\tau}) \phi_{\eta_\tau, \xi} - f''(\phi_{\eta_\tau}) |U_\xi| \right) \Psi^2(\tau, \xi) d\xi d\tau
\]
\[
+ \int_0^t e^{\beta \tau} \int_{-\infty}^{+\infty} e^{\alpha |\xi - \xi_\tau|} \left( f''(\phi_{\eta_\tau}) |U_\xi| - f''(U) |U_\xi| \right) \Psi^2(\tau, \xi) d\xi d\tau
\]
\[
\leq C \int_0^t e^{\beta \tau} \int_{-\infty}^{+\infty} e^{\alpha |\xi - \xi_\tau|} |\phi_{\eta_\tau, \xi} - U_\xi| \Psi^2(\tau, \xi) d\xi d\tau + C \int_0^t e^{\beta \tau} \int_{-\infty}^{+\infty} e^{\alpha |\xi - \xi_\tau|} |U_\xi| \phi_{\eta_\tau} - U \Psi^2(\tau, \xi) d\xi d\tau
\]
\[
\leq C\delta \int_0^t e^{\beta \tau} \int_{-\infty}^{+\infty} e^{\alpha |\xi - \xi_\tau|} \Psi^2(\tau, \xi) d\xi d\tau. \quad (4.8)
\]
On the other hand, for \( \alpha = \beta = 0 \), we have
\[
A_7 = 0,
\]
\[
A_8 \leq C \int_0^t \max\{|\phi_{\eta_\tau} - U|_{L^\infty}, |\phi_{\eta_\tau, \xi} - U_\xi|_{L^\infty}\} \|\Psi\|^2(\tau, \xi) d\tau \leq C\sup_{\tau \in [0,t]} \|\Psi\|^2 \int_0^t \delta e^{-\theta \tau} d\tau \leq C\delta, \quad (4.9)
\]
where we have used Lemma 4.1. Substituting (4.4)-(4.8) into (4.3), for \( \alpha \neq 0 \), one has
\[
e^{\beta t} \int_{-\infty}^{+\infty} e^{\alpha |\xi - \xi_\tau|} \Psi^2(\tau, \xi) d\xi + (\sigma_1 - C\delta) \int_0^t e^{\beta \tau} \int_{-\infty}^{+\infty} e^{\alpha |\xi - \xi_\tau|} \Psi^2(\tau, \xi) d\xi d\tau
\]
\[
+ (\sigma_1 - C\delta) \int_0^t e^{\beta \tau} \int_{-\infty}^{+\infty} e^{\alpha |\xi - \xi_\tau|} \Psi^2(\tau, \xi) d\xi d\tau \quad (4.10)
\]
\[
\leq \int_{-\infty}^{+\infty} e^{\alpha |\xi - \xi_\tau|} \Psi^2_0(\xi) d\xi + C\delta.
\]
Substituting (4.11)–(4.17) and (4.19) into (4.13), for $\alpha=\beta=0$, one has
\[
\int_{-\infty}^{+\infty} \Psi^2(t, \xi) d\xi + (2\gamma - C\epsilon) \int_{0}^{t} \int_{-\infty}^{+\infty} \Psi^2_\xi(\tau, \xi) d\xi d\tau \leq \int_{-\infty}^{+\infty} \Psi_0^2(\xi) d\xi + C\delta. \tag{4.11}
\]

Combining (4.11), (4.11), if $\alpha, \beta$ satisfy (2.20), we have complete the proof of Lemma 4.2.

**Lemma 4.3.** Under the same assumptions in Lemma 4.2, we have the following inequality
\[
e^{\beta t} \int_{-\infty}^{+\infty} e^{\alpha|\xi-\xi'|} \Psi^2_\xi(t, \xi) d\xi + \int_{0}^{t} e^{\beta \tau} \int_{-\infty}^{+\infty} e^{\alpha|\xi-\xi'|} \Psi^2_\xi(\tau, \xi) d\xi d\tau \leq C \int_{-\infty}^{+\infty} e^{\alpha|\xi-\xi'|} (\Psi_0^2 + \Psi_0^2) (\xi) d\xi + C\delta. \tag{4.12}
\]

**Proof.** Differentiating (3.1) with respect to $t$, multiplying the result by $2e^{\beta t} e^{\alpha|\xi-\xi'|} \Psi_\xi(\xi, t)$, integrating the resulting equation with respect to $t$, over $[0, t] \times \mathbb{R}$, one has
\[
e^{\beta t} \int_{-\infty}^{+\infty} e^{\alpha|\xi-\xi'|} \Psi^2_\xi(t, \xi) d\xi + 2\gamma \int_{0}^{t} e^{\beta \tau} \int_{-\infty}^{+\infty} e^{\alpha|\xi-\xi'|} \Psi^2_\xi(\tau, \xi) d\xi d\tau \leq \int_{-\infty}^{+\infty} e^{\alpha|\xi-\xi'|} \Psi^2_0(\xi) d\xi + \sum_{i=1}^{4} \int_{0}^{t} e^{\beta \tau} \int_{-\infty}^{+\infty} e^{\alpha|\xi-\xi'|} b_i d\xi d\tau, \tag{4.13}
\]
where
\[
b_1 = (\beta + \alpha | 2 f'(\xi_2) - s | \Psi^2_\xi, \quad b_2 = 3\alpha \mu \Psi^2_\xi, \quad b_3 = 2(\alpha^2 \mu + \alpha \gamma + | f'(\xi_2) |) | \Psi_\xi \Psi_\xi |, \quad b_4 = 2 | \Psi_\xi H |. \tag{4.14}
\]

Now we estimate the last two terms $B_i (i=3, 4)$ on the right-hand side of (4.13), where $B_4 = \int_{0}^{t} e^{\beta \tau} \int_{-\infty}^{+\infty} e^{\alpha|\xi-\xi'|} b_i d\xi d\tau$. With the aid of Cauhly inequality, one gets
\[
B_3 \leq \varepsilon_1 \int_{0}^{t} e^{\beta \tau} \int_{-\infty}^{+\infty} e^{\alpha|\xi-\xi'|} \Psi^2_\xi (\tau, \xi) d\xi d\tau + C\varepsilon_1 \int_{0}^{t} e^{\beta \tau} \int_{-\infty}^{+\infty} e^{\alpha|\xi-\xi'|} \Psi^2_\xi (\tau, \xi) d\xi d\tau. \tag{4.15}
\]

Using Lemma 3.1 similar to (4.6), we have
\[
B_4 \leq \varepsilon_1 \int_{0}^{t} e^{\beta \tau} \int_{-\infty}^{+\infty} e^{\alpha|\xi-\xi'|} \Psi^2_\xi (\tau, \xi) d\xi d\tau + C\varepsilon_1 \int_{0}^{t} e^{\beta \tau} \int_{-\infty}^{+\infty} e^{\alpha|\xi-\xi'|} H^2 (\tau, \xi) d\xi d\tau \leq \varepsilon_1 \int_{0}^{t} e^{\beta \tau} \int_{-\infty}^{+\infty} e^{\alpha|\xi-\xi'|} \Psi^2_\xi (\tau, \xi) d\xi d\tau + C\varepsilon_1 \delta. \tag{4.16}
\]
Substituting (4.13)-(4.16) into (4.11), choosing a sufficiently small constant $\varepsilon_1$, with the aid of Lemma 4.2, we obtain Lemma 4.3. □

**Lemma 4.4.** Under the same assumptions in Lemma 4.2, we have the following inequality

$$
\epsilon^\beta \int_{-\infty}^{+\infty} e^{\alpha|\xi| \xi} (\Psi_{\xi}^2(t, \xi)) d\xi + \int_{0}^{t} \epsilon^\beta \int_{-\infty}^{+\infty} e^{\alpha|\xi| \xi} (\Psi_{\xi\xi}(\tau, \xi)) d\xi d\tau \\
\leq C \int_{-\infty}^{+\infty} e^{\alpha|\xi| \xi} (\Psi_{\xi}^2(\xi) + \Psi_{\xi\xi}^2(\xi)) (\xi) d\xi + C\delta.
$$

(4.17) **s4.18?**

Here $C$ is a positive constant.

**Proof.** Differentiating of (3.1) with respect to $\xi$ twice, multiplying the result by $2e^{\beta t}e^{\alpha|\xi| \xi} \Psi_{\xi\xi}$ and integrating the result with respect to $t, \xi$ over $[0, t] \times \mathbb{R}$, one gets that

$$
\epsilon^\beta \int_{-\infty}^{+\infty} e^{\alpha|\xi| \xi} (\Psi_{\xi}^2(t, \xi)) d\xi + 2\gamma \int_{0}^{t} \epsilon^\beta \int_{-\infty}^{+\infty} e^{\alpha|\xi| \xi} (\Psi_{\xi\xi}^2(\tau, \xi)) d\xi d\tau \\
\leq \int_{-\infty}^{+\infty} e^{\alpha|\xi| \xi} (\Psi_{\xi\xi}^2(\xi)) d\xi + \sum_{i=1}^{6} \int_{0}^{t} \epsilon^\beta \int_{-\infty}^{+\infty} e^{\alpha|\xi| \xi} m_i(\tau, \xi)) d\xi d\tau,
$$

(4.18) **s4.19?**

where

$$
m_1=(\beta + \alpha |2f' (U + \Psi_{\xi}) - s|)e^{\alpha|\xi| \xi} \Psi_{\xi\xi}^2, \quad m_2=3\alpha \mu \Psi_{\xi\xi\xi},
$$

$$m_3=2\alpha |f'' (\xi_3) U_\xi \Psi_{\xi\xi}|, \quad m_4=2\alpha (\gamma + \alpha \mu) |\Psi_{\xi\xi} \Psi_{\xi\xi\xi}|, \quad m_5=2 \{f'' (\xi_3) U_\xi \Psi_{\xi} | + |f' (U + \Psi_{\xi}) \Psi_{\xi\xi}| \} |\Psi_{\xi\xi\xi}|, \quad m_6=2 |\Psi_{\xi\xi} H_{\xi\xi}|.
$$

Now we estimate the last four terms $M_i (i=3, 4, 5, 6)$ on the right-hand side of (4.18), where $M_i= \int_{0}^{t} \epsilon^\beta \int_{-\infty}^{+\infty} e^{\alpha|\xi| \xi} m_i(\xi) d\xi d\tau$. With the aid of Cauchy inequality, we have

$$
M_i \leq \epsilon^\beta \int_{0}^{t} \epsilon^\beta \int_{-\infty}^{+\infty} \xi \Psi_{\xi\xi}(\tau, \xi) d\xi d\tau \\
+ C \epsilon^\beta \int_{0}^{t} \epsilon^\beta \int_{-\infty}^{+\infty} \xi \Psi_{\xi\xi}^2(\tau, \xi)) (\tau, \xi) d\xi d\tau, \quad i=3, 4, 5.
$$

(4.19) **s4.20?**

Similar to (4.6), we have

$$
M_6 \leq C \int_{0}^{t} \epsilon^\beta \int_{-\infty}^{+\infty} \xi \Psi_{\xi\xi}(\tau, \xi) d\xi d\tau \\
+ C \int_{0}^{t} \epsilon^\beta \int_{-\infty}^{+\infty} \xi \Psi_{\xi\xi}^2(\tau, \xi)) d\xi d\tau \\
\leq C \int_{0}^{t} \epsilon^\beta \int_{-\infty}^{+\infty} \xi \Psi_{\xi\xi}(\tau, \xi) d\xi d\tau + C\delta.
$$

(4.20) **s4.23?**
using Lemma 3.1. Substituting (4.19)-(4.20) into (4.18), choosing a sufficiently small constant \( \varepsilon_2 \) with the aid of Lemma 3.2 and Lemma 4.3, we obtain the proof of Lemma 4.4.

**Lemma 4.5.** Under the same assumptions in Lemma 4.2, we have the following inequality

\[
e^{\beta t} \int_{-\infty}^{+\infty} e^{\alpha|\xi-x|} \psi_0^2 e^i \psi_{\xi d\xi}^2 e^{i\xi} + \int_0^t e^{\beta t} \int_{-\infty}^{+\infty} e^{\alpha|\xi-x|} \psi_0^2 e^i \psi_{\xi d\xi}^2 e^{i\xi} d\xi d\tau
\]

\[
\leq C \int_{-\infty}^{+\infty} e^{\alpha|\xi-x|} \left( \psi_0^2 + \psi_0^2 + \psi_0^2 + \psi_0^2 \right) (\xi) d\xi + C\delta.
\]

**Proof.** Differentiating of (3.1) with respect to \( \xi \) three times, multiplying the result by \( 2e^{\beta t}e^{\alpha|\xi-x|} \psi_{\xi d\xi}^2 e^{i\xi} \) and integrating the result with respect to \( t, \xi \) over \( [0, t] \times \mathbb{R} \), we have

\[
e^{\beta t} \int_{-\infty}^{+\infty} e^{\alpha|\xi-x|} \psi_0^2 e^i \psi_{\xi d\xi}^2 e^{i\xi} d\xi + 2\gamma \int_0^t e^{\beta t} \int_{-\infty}^{+\infty} e^{\alpha|\xi-x|} \psi_0^2 e^i \psi_{\xi d\xi}^2 e^{i\xi} d\xi d\tau
\]

\[
\leq \int_{-\infty}^{+\infty} e^{\alpha|\xi-x|} \psi_0^2 e^i \psi_{\xi d\xi}^2 e^{i\xi} d\xi + \int_0^t e^{\beta t} \int_{-\infty}^{+\infty} \sum_{i=1}^7 e^{\alpha|\xi-x|} n_i d\xi d\tau,
\]

where

\[
n_1 = (\beta + sa) \psi_{\xi d\xi}^2, \quad n_2 = 3\alpha \mu \psi_{\xi d\xi}^2,
\]

\[
n_3 = 2(\alpha^2 \mu + a \gamma + |f'(U + \psi)|) |\psi_{\xi d\xi}^2 \psi_{\xi d\xi}|, \quad n_4 = 2 |f''(\xi) U_{\xi d\xi} \psi_{\xi d\xi}^2|,
\]

\[
n_5 = 4 |f''(U + \psi) U_{\xi d\xi} \psi_{\xi d\xi}^2|, \quad n_6 = 2 |f''(\xi) U_{\xi d\xi}^2 \psi_{\xi d\xi}^2|,
\]

\[
n_7 = 2 |f''(U + \psi) \psi_{\xi d\xi}^2 \psi_{\xi d\xi}^2|, \quad n_8 = 2 |\psi_{\xi d\xi}^2 H_{\xi d\xi}|.
\]

Now we estimate the last six terms \( N_i (i = 3, 4, 5, 6, 7, 8) \) on the right-hand side of (4.22), where \( N_i = \int_0^t e^{\beta t} \int_{-\infty}^{+\infty} e^{\alpha|\xi-x|} n_i d\xi d\tau. \) With the aid of Cauchy inequality, for \( i = 3, 4, 5, 6, \) we have

\[
N_i \leq \varepsilon_3 \int_0^t e^{\beta t} \int_{-\infty}^{+\infty} e^{\alpha|\xi-x|} \psi_{\xi d\xi}^2 e^{i\xi} (\tau, \xi) d\xi d\tau
\]

\[
+C \varepsilon_3 \int_0^t e^{\beta t} \int_{-\infty}^{+\infty} e^{\alpha|\xi-x|} (\psi_{\xi d\xi} + \psi_{\xi d\xi} + \psi_{\xi d\xi})^2 (\tau, \xi) d\xi d\tau.
\]

For \( N_7, \) with the help of Sobolev inequality, we have

\[
N_7 \leq C \int_0^t e^{\beta t} \|\psi\|_3 \int_{-\infty}^{+\infty} e^{\alpha|\xi-x|} |\psi_{\xi d\xi}^2 \psi_{\xi d\xi}| (\tau, \xi) d\xi d\tau
\]

\[
\leq \varepsilon_3 \int_0^t e^{\beta t} \int_{-\infty}^{+\infty} e^{\alpha|\xi-x|} \psi_{\xi d\xi}^2 e^{i\xi} (\tau, \xi) d\xi d\tau
\]

\[
+C \varepsilon_3 \int_0^t e^{\beta t} \int_{-\infty}^{+\infty} e^{\alpha|\xi-x|} \psi_{\xi d\xi}^2 (\tau, \xi) d\xi d\tau.
\]
Using Lemma 3.1 similar to (4.6), we have
\[
N_8 \leq C \int_0^t e^{\beta \tau} \int_{-\infty}^{+\infty} e^{\alpha |\xi - \xi_*|} \Psi_{\xi \xi \xi \xi}^2 (\tau, \xi) d\xi d\tau \\
+ C \int_0^t e^{\beta \tau} \int_{-\infty}^{+\infty} e^{\alpha |\xi - \xi_*|} H_{\xi \xi \xi \xi}^2 (\tau, \xi) d\xi d\tau
\]
\[
\leq C \int_0^t e^{\beta \tau} \int_{-\infty}^{+\infty} e^{\alpha |\xi - \xi_*|} \Psi_{\xi \xi \xi \xi}^2 (\tau, \xi) d\xi d\tau + C \delta. \tag{4.25}
\]
Substituting (4.23)-(4.25) into (4.22), choosing a sufficiently small constant \(\varepsilon_3\), with the aid of Lemma 4.2-Lemma 4.4, we obtain the proof of Lemma 4.5. □

5. PROOF OF THE MAIN RESULTS

5.1. Proof of Theorem 2.1 Taking \(\alpha = \beta = 0\) in Lemma 4.2-Lemma 4.5, one gets Proposition 3.1. Making full use of Proposition 3.1, we can extend the unique local solution \(\Psi\) to \(T = +\infty\) by the standard continuation process. As long as Proposition 3.1 is proved, we can extend the unique local solution \(\Psi\) to \(T = +\infty\) by the standard continuation process. We have the following lemma.

Lemma 5.1. Suppose \(\Psi_0 \in H^3\) there exists a positive constant \(\varepsilon_1 = \frac{\varepsilon_0}{\sqrt{C_1}}\), such that if
\[
\|\Psi_0\|_3^2 + \delta \leq \varepsilon_1^2,
\]
then the initial problem (3.1), (2.16) has a unique global solution
\[
\sup_{t \geq 0} \|\Psi(t)\|_3^2 + \int_0^{+\infty} \|\Psi_\xi(\tau)\|_3^2 d\tau \leq C_1 \left(\|\Psi_0\|_3^2 + \delta\right). \tag{5.1}
\]

Combining Lemma 4.1 and Lemma 5.1 we complete the proof of Theorem 2.1

5.2. Proof of Theorem 2.2 Taking \(\alpha > 0, \beta > 0\), in Lemma 4.2-Lemma 4.5 we have

Lemma 5.2. Suppose that \(\Psi(t, \xi)\) is a global smooth solution to the Cauchy problem (3.1), (2.16). If (2.18) holds, there exists a positive constant \(\varepsilon_2 < \varepsilon_1\), such that if \(\|\Psi_0\|_3^2 + \delta \leq \varepsilon_2^2\), we have
\[
\|\Psi(t)\|_H^3 \leq C_2 e^{-\beta t} \left\{\int_{-\infty}^{+\infty} e^{\alpha |\xi - \xi_*|} \left(\Psi_0^2_{\xi} + \Psi_0^2_{\xi \xi} + \Psi_0^2_{\xi \xi \xi} + \Psi_0^2_{\xi \xi \xi \xi}\right) (\xi) d\xi + \delta\right\}.
\]
Here \(C_2 > 0\) is a constant, \(\alpha\) and \(\beta\) are two positive constants, satisfying (2.20).

Theorem 2.1 gives that \(\Psi(t, \xi)\) is a global smooth solution. Combining \(\theta > \beta\), Lemma 4.1 and Lemma 5.2 we complete the proof of Theorem 2.2

6. PROOF OF LEMMAS 2.4, 3.1

For convenience, we define
\[
w_l(\xi, t) : = v_l(\xi, t) - \bar{v}_l, \quad w_r(\xi, t) : = u_r(\xi, t) - \bar{v}_r. \tag{6.1}
\]
6.1. Proof of Lemma 2.4

Proof. Using the similar method in [20], we obtain that there exists a unique $\eta_0$, such that the initial data satisfies (2.13). With the aid of Rankine-Hugoniot condition (2.2) and Lemma 2.3, one can easily prove that

$$ |\eta'(t)| \leq C\delta e^{-\theta t}. \quad (6.2) $$

Once (6.2) is proved, one gets that there exists a constant $\eta_{\infty}$, such that

$$ \eta(t) - \eta_{\infty} = - \int_{t}^{\infty} \eta'(t)dt. \quad (6.3) $$

However, the exact expression of $\eta_{\infty}$ is not easy to obtain. Motivated by [19], now we find this constant. For $y \in (0, 1), N \in \mathbb{N}^*$, we define the domain

$$ \Omega_y^N : = \{(\xi, \tau) : \eta(\tau) + (-N + y)p_l \leq \xi \leq \eta(\tau) + (N + y)p_r, 0 \leq \tau \leq t \}. $$

We define $G(z) := f(z) - sz$. With the aid of the equations of $u_l$ and $u_r$, we have

$$ \iint_{\Omega_y^N} \mathcal{E}(\xi, \tau)d\xi d\tau = 0, $$

where

$$ \mathcal{E}(\xi, \tau) = \left( \partial_t u_l + \partial_\xi G(u_l) + \mu \partial^3_\xi u_l - \gamma \partial^2_\xi u_l \right) g_\eta $$

$$ + \left( \partial_t u_r + \partial_\xi G(u_r) + \mu \partial^3_\xi u_r - \gamma \partial^2_\xi u_r \right) (1 - g_\eta) \quad (6.4) $$

Then integrating by parts, and using Green formula, we have

$$ \iint_{\Omega_y^N} \mathcal{D}(\xi, \tau)g_\eta' d\xi d\tau $$

$$ = A^N(y, t) - A^N(y, 0) - B^N_l(y, t) + B^N_r(y, t), \quad (6.5) $$

where

$$ \mathcal{D}(\xi, \tau) : = - \eta'(t) (u_l - u_r) + (G(u_l) - \gamma \partial_\xi u_l + \mu \partial^2_\xi u_l) - (G(u_r) - \gamma \partial_\xi u_r + \mu \partial^2_\xi u_r), $$
Thus, we have

\[ A^N(y, t) = \int_{\eta(t)+(-N+y)p_t}^{\eta(t)+(N+y)p_r} \left[ u_t(\xi, t) g_\eta(\xi) + u_r(\xi, t) (1 - g_\eta(\xi)) \right] d\xi, \]

\[ A^N(y, 0) = \int_{\eta_0+(N+y)p_r}^{\eta_0+(N+y)p_t} \left[ u_t(\xi, 0) g_{\eta_0}(\xi) + u_r(\xi, 0) (1 - g_{\eta_0}(\xi)) \right] d\xi, \]

\[ B^N_t(y, t) = \int_0^t \left\{ (G(u_t) - \gamma \partial_\xi u_t + \mu \partial_2^2 u_t) g_\eta + (G(u_r) - \gamma \partial_\xi u_r + \mu \partial_2^2 u_r) (1 - g_\eta) \right. \]

\[ - \left. \eta'(\tau) [u_1 g_\eta + u_r (1 - g_\eta)] \right\} (\eta(\tau) + (-N + y)p_t, \tau) d\tau, \]

\[ B^N_r(y, t) = \int_0^t \left\{ (G(u_t) - \gamma \partial_\xi u_t + \mu \partial_2^2 u_t) g_\eta + (G(u_r) - \gamma \partial_\xi u_r + \mu \partial_2^2 u_r) (1 - g_\eta) \right. \]

\[ - \left. \eta'(\tau) [u_1 g_\eta + u_r (1 - g_\eta)] \right\} (\eta(\tau) + (-N + y)p_r, \tau) d\tau. \]

With the aid of (2.12), we have

\[ \int_{\Omega_N^t} D(\xi, \tau) g_\eta' d\xi d\tau = 0 \quad \text{as} \, N \to +\infty. \]

(i) The integrals on \( \{ \tau = 0 \} \) and \( \{ \tau = t \} \). With the help of Lemma 2.3, we have \( \| w_t \|_{L^\infty} + \| w_r \|_{L^\infty} \leq C\delta e^{-\alpha t} \). With the aid of (1.3), one gets

\[ J^N(y, t) := A^N(y, t) - A^N(y, 0) \]

\[ = \int_{(-N+y)p_t}^{(N+y)p_r} \left[ w_1(\xi, \eta(t), t) g(\xi) + w_r(\xi, \eta(t), t) (1 - g(\xi)) \right] d\xi \]

\[ - \int_{\eta_0+(-N+y)p_r}^{\eta_0+(N+y)p_t} \left[ w_{1r}(\xi) g(\xi) (\xi - \eta_0) + w_{0r}(\xi) (1 - g(\xi - \eta_0)) \right] d\xi \]

\[ \leq C\delta \left( e^{-\alpha t} \right) + \int_{\eta_0+(-N+y)p_r}^{\eta_0+y(p_t)} (w_{1r} - w_{0r})(\xi) (1 - g(\xi - \eta_0)) d\xi \]

\[ - \int_{\eta_0+y(p_t)}^{\eta_0+(N+y)p_t} \left[ w_{1r}(\xi) g(\xi - \eta_0) + w_{0r}(\xi) (1 - g(\xi - \eta_0)) \right] d\xi \]

\[ - \int_{\eta_0+(N+y)p_r}^{\eta_0+y(p_r)} (w_{1r} - w_{0r})(\xi) g(\xi - \eta_0) d\xi. \]

Thus, we have

\[ J(y, t) \leq C\delta \left( e^{-\alpha t} \right) + \int_{-\infty}^{\eta_0+y(p_t)} (w_{1r} - w_{0r})(\xi) (1 - g(\xi - \eta_0)) d\xi \]

\[ - \int_{\eta_0+y(p_t)}^{\eta_0+(N+y)p_t} \left[ w_{1r}(\xi) g(\xi - \eta_0) + w_{0r}(\xi) (1 - g(\xi - \eta_0)) \right] d\xi \]

\[ - \int_{\eta_0+(N+y)p_r}^{+\infty} (w_{1r} - w_{0r})(\xi) g(\xi - \eta_0) d\xi. \]
where \( J(y, t) = \lim_{N \to \infty} J^N(y, t) \). With the aid of (2.10), it follows that

\[
0 = - \int_{-\infty}^{\infty} [u_0(\xi) - \phi(\xi - \eta_0) - w_{0\ell}(\xi)g(\xi - \eta_0) - w_{0r}(\xi)(1 - g(\xi - \eta_0))] \, d\xi
\]

\[
= - \int_{-\infty}^{0} (u_0 - \phi - w_{0\ell})(\xi) \, d\xi - \int_{0}^{+\infty} (u_0 - \phi - w_{0r})(\xi) \, d\xi \\
+ (\bar{u}_l - \bar{u}_r) \eta_0 - \int_{-\infty}^{0} (w_{0\ell} - w_{0r})(\xi)(1 - g(\xi - \eta_0)) \, d\xi \\
+ \int_{0}^{+\infty} (w_{0\ell} - w_{0r})(\xi)g(\xi - \eta_0) \, d\xi.
\]

(6.8) \( \{? \}

Together with (6.7), we have

\[
J(y, t) \leq C\delta \left( e^{-\alpha t} \right) + (\bar{u}_l - \bar{u}_r) \eta_0 \\
- \int_{-\infty}^{0} (u_0 - \phi - w_{0\ell})(\xi) \, d\xi - \int_{0}^{+\infty} (u_0 - \phi - w_{0r})(\xi) \, d\xi \\
+ \int_{0}^{\eta_0 + y\rho_i} w_{0\ell}(\xi) \, d\xi - \int_{0}^{\eta_0 + y\rho_r} w_{0r}(\xi) \, d\xi.
\]

(6.9) \( \{? \}

Since \( \int_{0}^{p_i} w_{0i}(\xi) \, d\xi = 0 \) for \( i = l, r \), \( \int_{0}^{y} w_{0i}(\xi) \, d\xi \) is periodic with respective to \( y \) with period \( p_i \). Therefore

\[
\int_{0}^{1} \int_{0}^{\eta_0 + y\rho_i} w_{0i}(\xi) \, d\xi \, dy = \frac{1}{p_i} \int_{0}^{p_i} \int_{0}^{\eta_0 + y} w_{0i}(\xi) \, d\xi \, dy \\
= \frac{1}{p_i} \int_{0}^{p_i} \int_{0}^{y} w_{0i}(\xi) \, d\xi \, dy.
\]

(6.10) \( \{? \}

So, we have

\[
\int_{0}^{1} J(y, t) \, dy \leq C\delta \left( e^{-\alpha t} \right) + (\bar{u}_l - \bar{u}_r) \eta_0 \\
- \int_{-\infty}^{0} (u_0 - \phi - w_{0\ell})(\xi) \, d\xi - \int_{0}^{+\infty} (u_0 - \phi - w_{0r})(\xi) \, d\xi \\
+ \frac{1}{p_i} \int_{0}^{p_i} \int_{0}^{y} w_{0\ell}(\xi) \, d\xi \, dy - \frac{1}{p_r} \int_{0}^{p_r} \int_{0}^{y} w_{0r}(\xi) \, d\xi \, dy.
\]

(6.11) \( \{? \}

(ii) The integrals on two sides. Since \( u_l \) is periodic, it holds that

\[
B_i^N(y, t) = \int_{0}^{1} \left\{ (G(u_l) - \gamma \partial_{\xi} u_l + \mu \partial_{\xi}^2 u_l)(\eta(\tau) + yp_l, \tau)g((-N + y)p_i) \\
- \eta'(\tau)u_l(\eta(\tau) + yp_l, \tau) \\
+ [\cdots] (1 - g_{\eta})(\eta(\tau) + (-N + y)p_i, \tau) \right\} \, d\tau.
\]

(6.12) \( \{? \5.40 \}
where \([\cdots]\) denotes the remaining terms which are bounded. Then by taking the limit \(N \to +\infty\) in (6.12) and using Lemma 2.2, one can get
\[
\lim_{N \to +\infty} \int_0^1 B_1^N (y, t) \, dy = \int_0^t \frac{1}{p_l} \int_{0}^{p_l} G(u_l)(\xi, \tau) \, d\xi \, d\tau - \bar{u}_l(\eta(t) - \eta_0). \tag{6.13}
\]
Similarly, it holds that
\[
\lim_{N \to +\infty} \int_0^1 B_r^N (y, t) \, dy = \int_0^t \frac{1}{p_r} \int_{0}^{p_r} G(u_r)(\xi, \tau) \, d\xi \, d\tau - \bar{u}_r(\eta(t) - \eta_0). \tag{6.14}
\]
Now, with the calculations in (i) and (ii), one can integrate the equation (6.5) with respect to \(y\) over \((0, 1)\), and then let \(N \to +\infty\), for any \(t > 0\)
\[
\int_0^1 J(y, t) \, dy + (\bar{u}_l - \bar{u}_r)(\eta(t) - \eta_0)
= \int_0^t \left[ \frac{1}{p_l} \int_{0}^{p_l} G(u_l)(\xi, \tau) \, d\xi - \frac{1}{p_r} \int_{0}^{p_r} G(u_r)(\xi, \tau) \, d\xi \right] \, d\tau. \tag{6.15}
\]
Note also that for \(i = l, r\),
\[
\int_0^t \frac{1}{p_i} \int_{0}^{p_i} G(u_i) \, dy \, d\tau = \int_0^t \frac{1}{p_i} \int_{0}^{p_i} \left( G(u_i) - G(\bar{u}_i) \right) \, dy \, d\tau + G(\bar{u}_i) t
\leq \int_0^t \frac{1}{p_i} \int_{0}^{p_i} \left( G(u_i) - G(\bar{u}_i) \right) \, dy \, d\tau + C \delta e^{-\alpha t} + G(\bar{u}_i) t. \tag{6.16}
\]
Thus we have the proof of Lemma 2.4.

6.2. Proof of Lemma 3.1

Proof. The proof is motivate by [20]. With the aid of (2.9), when \(\xi < \eta(t)\), one has
\[
H = -f(U) + f(u_l)g_\eta + f(u_r)(1 - g_\eta) - 3\mu(u_l - u_r)g_\eta' + 2\gamma(u_l - u_r)g_\eta'
- \int_{-\infty}^{\xi} \mathfrak{R}(\xi, t) \, d\xi : = H_1 + H_2,
\]
where
\[
\mathfrak{R}(\xi, t) := (f(u_l) - f(u_r))g_\eta' + \gamma(u_l - u_r)g_\eta'' + \mu(u_l - u_r)g_\eta''' - (u_l - u_r) \cdot g_\eta' \cdot (s + \eta'(t)).
\]
Using Lemma 2.3–Lemma 2.4 one gets
\[
H_1 = -f(U) + f(u_l) + \gamma(\bar{u}_l - \bar{u}_r)g_\eta' - \mu(\bar{u}_l - \bar{u}_r)g_\eta'' + \int_{-\infty}^{\xi} (\bar{u}_l - \bar{u}_r) \cdot g_\eta' \cdot s \, d\xi
\leq C \delta e^{-\alpha t} e^{\sigma_0(\xi - \eta(t))},
\]
and
and
\[
H_2 = - \int_{-\infty}^{\xi} (\bar{u}_r - \bar{u}_l) \cdot g_\eta(t) d\xi + [f(u_r) - f(\bar{u}_r) - f(u_l) + f(\bar{u}_l)](1 - g_\eta) \\
+ 2\gamma(w_l - w_r)g'_\eta - 3\mu(w_l - w_r)\xi g''_\eta \\
+ \int_{-\infty}^{\xi} (w_l - w_r) \cdot g'_\eta \cdot (s + \eta'(t)) - [f(u_l) - f(\bar{u}_l) + f(u_r) - f(\bar{u}_r)]g'_\eta d\xi \\
- \int_{-\infty}^{\xi} \gamma(w_l - w_r) \cdot g''_\eta + \mu(w_l - w_r) \cdot g''_\eta d\xi \\
\leq C\delta e^{-\beta t} \{ \int_{-\infty}^{\xi} (|g''_\eta| + |g''_\eta| + |g'_\eta|) d\xi + (1 - g_\eta) \} \\
\leq C\delta e^{-\beta t} e^{\sigma_0(\xi - \eta(t))}.
\]

As to \( \xi > \eta(t) \), using the same method, we have \( H(\xi, t) \leq C\delta e^{-\beta t} e^{\sigma_0(\xi - \eta(t))} \). Similar, it follows that \( |H_\xi|, |H_{\xi\xi}|, |H_{\xi\xi\xi}| \leq C\delta e^{-\beta t} e^{\sigma_0(\xi - \eta(t))} \). \( \Box \)

7. Appendix

7.1. Proof of Lemma 2.3 In this section, we write \( \| \cdot \|_{H^{k+1}(\Omega)} \) as \( \| \cdot \|_{k+1} \), for convenience.

Claim 1. For any integer \( k \geq 0 \),
\[
\| \partial^k_\xi (u - \bar{u}) \|^2 + \int_0^t \| \partial^k_\xi u \|^2 d\tau \leq C \| (u_0 - \bar{u}) \|^2_k \quad \forall t > 0,
\]
(7.1) \( \Box \)

where \( C \) is a constant depends only on \( u_0, p \) and \( f \). Then we prove Claim 1 by the induction method.

step 1: We will prove (7.1) is true when \( k=0 \). Multiplying \( u - \bar{u} \) on each side of (2.7) integrating with respect to \( \xi \) over \([0, p] \), one gets
\[
\frac{d}{dt} (\| u - \bar{u} \|^2) + 2\gamma \| \partial_\xi u \|^2 = 0, \quad \forall t > 0.
\]
(7.2) \( \Box \)

Integrating (7.2) with respect to \( t \) over \([0, t] \), we have
\[
\| u - \bar{u} \|^2 + 2\gamma \int_0^t \| \partial_\xi u \|^2 d\tau = \| u_0 - \bar{u} \|^2, \quad \forall t > 0.
\]
(7.3) \( \Box \)

step 2: We assume (7.1) is true when \( k=2, \ldots, m-1 \). We will prove (7.1) is true when \( k=m \). Taking the derivative \( \partial^m_\xi \) in (2.7), multiplying \( \partial^m_\xi u \) on each side, integrating the result over \([0, p] \), with the aid of Cauchy inequality, one has
\[
\frac{d}{dt} (\| \partial^m_\xi u \|^2) + 2\gamma \| \partial^m_\xi u \|^2 \leq \gamma \| \partial^{m+1}_\xi u \|^2 + C\gamma \sum_{k=1}^{m} \| \partial^k_\xi u \|^2.
\]
Thus
\[
\frac{d}{dt} \left( \| \partial_t^m u \|^2 \right) + \gamma \| \partial_t^{m+1} u \|^2 \leq C \sum_{k=1}^{m} \| \partial_t^k u \|^2.
\] (7.4) \text{ ??}

Integrating (7.4) over \([0, t]\). Thus for \forall t > 0
\[
\| \partial_t^m u \|^2 + \gamma \int_0^t \| \partial_t^{m+1} u \|^2 \, dt \leq C \sum_{k=1}^{m} \int_0^t \| \partial_t^k u \|^2 \, dt + \| \partial_t^m u_0 \|^2
\]
\[\leq C \|(u_0 - \bar{u})\|_m^2.
\]
Thus Claim 1 is true.

Claim 2. For each \(k \geq 0, C > 0\), it follows that
\[
\| \partial_t^k (u - \bar{u})(\xi, t) \|^2 \leq C \|u_0 - \bar{u}\|^2 e^{-\theta t} \quad \forall t \leq 0.
\] (7.5) \text{ ??}

\textbf{step 1}: We will prove (7.5) is true when \(k=0\). With the aid of Poincare inequality on \([0, p]\), there exists a constant \(\theta > 0\), which depends only on \(p\), such that
\[
\int_0^p (\partial_t^1 u)^2 (\xi, \tau) \, d\xi \geq \frac{\theta}{2^2} \int_0^p (u - \bar{u})^2(\xi, t) \, d\xi.
\] (7.6) \text{ ??}

Combining (7.2) and (7.6), we have
\[
\| (u - \bar{u})(\xi, t) \|^2 \leq C \|(u_0 - \bar{u})\|^2 e^{-\theta t} \quad \forall t \leq 0.
\]

\textbf{step 2}: We assume (7.5) is true when \(k=1, 2, \ldots, m-1\). We will prove (7.5) is true when \(k=m\). For any \(j, 0 \leq j \leq m\), combining (7.2) and (7.4), we have
\[
\frac{d}{dt} \left( \| \partial_t^j (u - \bar{u}) \|^2 \right) + \gamma \| \partial_t^{j+1} u \|^2 \leq C \sum_{k=1}^{j} \| \partial_t^k u \|^2.
\] (7.7) \text{ ??}

Letting \(j=m\) in (7.7), we have
\[
\frac{d}{dt} \| \partial_t^m u \|^2 \leq C_m \sum_{k=1}^{m} \| \partial_t^k u \|^2 \leq C \| u_0 - \bar{u} \|^2_{m-1} e^{-\theta t} + C_m \| \partial_t^m u \|^2,
\] (7.8) \text{ ??}

where \(C_m > \theta\). We have used Claim 1 and Sobolev inequality. Letting \(j=m-1\) in (7.7), one gets that \(\forall t \geq 0\)
\[
\frac{d}{dt} \| \partial_t^{m-1} (u - \bar{u}) \|^2 + \gamma \| \partial_t^{m} u \|^2 \leq C \| u_0 - \bar{u} \|^2_{m-1} e^{-\theta t}.
\] (7.9) \text{ ??}

Then multiply \(\frac{2C_m}{\gamma}\) on (7.9). Add (7.8) to the result. One gets \(\forall t \geq 0\)
\[
\frac{d}{dt} \left( \frac{2C_m}{\gamma} \| \partial_t^{m-1} (u - \bar{u}) \|^2 + \| \partial_t^{m} u \|^2 \right) + C_m \| \partial_t^m u \|^2 \leq C e^{-\theta t} \| u_0 - \bar{u} \|^2_{m-1}.
\] (7.10) \text{ ??}

Denote
\[
E_m(t) := \frac{2C_m}{\gamma} \| \partial_t^{m-1} (u - \bar{u}) \|^2 + \| \partial_t^m u \|^2.
\]
Taking $k = m - 1$ in (7.5), with the aid of (7.10), for $\forall t \geq 0$, we have

$$E'_m(t) + C_m E_m(t) \leq \tilde{C}_m \|u_0 - \bar{u}\|_{m-1}^2 e^{-\theta t}. \quad (7.11)$$

Here $\tilde{C}_m$ is a new constant depends on $m$. Multiplying (7.11) by $e^{C_m t}$, we have

$$\frac{d}{dt}[e^{C_m t} E_m(t)] \leq \tilde{C}_m \|u_0 - \bar{u}\|_{m-1}^2 e^{(C_m - \theta)t}.$$ 

Thus

$$e^{C_m t} E_m(t) \leq E_m(0) + \frac{\tilde{C}_m}{C_m - \theta} \|u_0 - \bar{u}\|_{m-1}^2 [e^{(C_m - \theta)t} - 1].$$

With the aid of $C_m > \theta$, one gets

$$E_m(t) \leq E_m(0) e^{-C_m t} + \frac{\tilde{C}_m}{C_m - \theta} \|u_0 - \bar{u}\|_{m-1}^2 e^{-\theta t} \leq C \|u_0 - \bar{u}\|_m^2 e^{-\theta t}.$$ 

The proof of Claim 2 is accomplished. Then by Sobolev inequality and Claim 2, for any integer $k \leq 0$ and $t \leq 0$, we have

$$\left\|\partial^k \xi(u - \bar{u})\right\|_{L^\infty(\mathbb{R})} = \left\|\partial^k \xi(u - \bar{u})\right\|_{L^\infty(\Omega)} \leq C \left\|u_0 - \bar{u}\right\|_{H^{k+1}(\Omega)} e^{-\theta t}.$$ 

Thus, we finish the proof of Lemma 2.3.

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