A NOVEL FAST FEEDFORWARD NEURAL NETWORKS TRAINING ALGORITHM

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Abstract

In this paper1 a new neural networks training algorithm is presented. The algorithm originates from the Recursive Least Squares (RLS) method commonly used in adaptive filtering. It uses the QR decomposition in conjunction with the Givens rotations for solving a normal equation - resulting from minimization of the loss function. An important parameter in neural networks is training time. Many commonly used algorithms require a big number of iterations in order to achieve a satisfactory outcome while other algorithms are effective only for small neural networks. The proposed solution is characterized by a very short convergence time compared to the well-known backpropagation method and its variants. The paper contains a complete mathematical derivation of the proposed algorithm. There are presented extensive simulation results using various benchmarks including function approximation, classification, encoder, and parity problems. Obtained results show the advantages of the featured algorithm which outperforms commonly used recent state-of-the-art neural networks training algorithms, including the Adam optimizer and the Nesterov’s accelerated gradient.

Keywords: neural network training algorithm, QR decomposition, Givens rotations, approximation, classification.

1 Introduction

Artificial neural networks (ANNs) are one of the most common elements in the artificial intelligence. They are mathematical models of biological neurons. Their main properties include learning based on provided samples and gener-

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alizing real-life problems without knowing an exact formula. In recent years neural networks have been the subject of many research projects e.g. [1–7] and are used in the industry [8–10], medicine [11–13], finance sector [14–16], and many others [17–19].

When the ANN is to be used best to its advantage, it needs to be trained for a specific task. To achieve that, the training algorithm and a sufficient training set should be applied. Unfortunately, not every algorithm fits well each network and each training set. The objective of the ANN learning process is to find the global minimum of the error function. One of the biggest challenges in training algorithms is to avoid oscillations around the local minima while maintaining a reasonable convergence speed. This can be achieved by applying parameters to the training process. Most algorithms introduce at least one parameter - the learning rate (\( \eta \)) usually chosen experimentally from the range \((0, 1]\). It provides information on how big weight correction in a single step can be.

The backpropagation algorithm (BP) derived by J. Werbos in 1974 [1] completely revolutionized the world of artificial intelligence. The BP algorithm for the first time provided a good method for calculating errors and expected values for neurons in hidden layers. The BP algorithm is still one of the most popular training methods for artificial neural networks [20–22]. The basic idea is to calculate a gradient for each weight, scale it by learning rate (\( \eta \)) and apply it as a correction to its respective weight during each iteration. Unfortunately, the pure BP variant is burdened with many inconveniences. The BP algorithm is likely to becoming stuck in the local minima or overstepping the solution as the learning rate is mismatched. Due to that the BP algorithm requires many more epochs in order to converge below the given error threshold. Despite the aforementioned disadvantages the BP algorithm is commonly used for learning and acts as an indicator in most benchmarks.

Over the years many researchers have developed a lot of improvements for the BP algorithm. Since the BP method uses only first order derivatives, it is assumed to be the first order training algorithm. Some of the BP improvements were realised simply by applying additional parameters to the BP, which also makes them first order methods. Such approach is used, for example, in momentum variants of the BP. Other improvements introduce second order derivatives. This makes them closer to the Newton’s method. Such methods are called second order training algorithms. There are also learning methods that only approximate second order derivatives. Such approach was introduced in the Levenberg-Marquardt algorithm [23].

To overcome the problem of the slow training progress, the momentum variant of the BP algorithm (MBP) was introduced. The main idea was formulated by Polyak in [24] and it applies the second predefined training factor, so-called momentum (\( \alpha \)). This parameter is selected experimentally for each specific case but usually takes the value of 0.9. The momentum factor brings a degree of inertia to the training process which makes it less vulnerable to large gradient changes. It speeds up the training on the ‘flat spots’ of the error function by accumulating small gradient changes. It also helps to avoid local minima by maintaining the momentum from previous iterations. This suppresses sudden direction changes of the training process minimizing the risk of oscillation in a narrow ravine of the error function. Similar to the classic Back Propagation algorithm, the MBP uses predefined and training-time fixed values of its parameters (\( \eta \) and \( \alpha \)). This makes the Momentum Back Propagation algorithm not flexible enough and puts it in need of further improvements.

One of the well known improvements made to the MBP algorithm is a Nesterov’s Accelerated Gradient (NAG). The method was initially derived by Nesterov in 1983 [25] and was the subject of many research projects [26]. The NAG algorithm is classified as a first order method which requires only the local parameters of a neuron in order to calculate the weight update. Similar to the classical momentum, the NAG uses step (\( \eta \)) and momentum (\( \alpha \)) as the training parameters. The novelty here is an in-
termediate step in order to access the gradient’s direction and then apply the weight correction based on that information. This approach helps to mitigate inaccurate training steps and boost convergence for valid directions.

In order to achieve even better results, the intuition suggests the need to use variable values of learning factors during training. Such idea was used in the Quickprop (QProp) algorithm published by S. E. Fahlman in 1988[27]. The QProp is inspired by the Newton’s method, which makes it a second order training algorithm. In order to calculate the weight update, the QProp approximates the total loss function with a quadratic polynomial function. While many researchers report high performance of the QProp algorithm, it is still burdened with several drawbacks. The method is highly sensitive to initial weights values, which can result in a lack of convergence. Also, training done by the QProp algorithm can prove unstable in a case where the error function contains many local minima.

Most BP-based algorithms use an error function gradient value to determine the weight correction. An example of the exception from this rule is the Resilient Propagation (RProp) algorithm developed by M. Riedmiller and H. Braun in 1993[28]. The RProp is a self-tuned algorithm based on the steepest descent method. The only training factors here are the gain on the ‘positive’ and the ‘negative’ scenario. The weight update happens once per epoch and uses only local information stored in each neuron. The RProp algorithm is based on the idea that the magnitude of the gradient can occasionally be a poor factor in defining the next training step. This can happen in the case when the activation function becomes saturated. Then, the resulting gradient value is very small. To overcome this drawback the RProp algorithm uses only a sign of the gradient and reacts accordingly to its change through the training.

While neural networks gain popularity, the stochastic gradient descent based algorithms become in the center of interest. In 2014 D. P. Kingma and J. Ba proposed a very flexible method for stochastic optimization called Adam[29]. The algorithm is easy in implementation, has low memory requirements, and is proven to be very flexible. It utilizes several training parameters which in most applications retain fixed values. The Adam is a first order training algorithm that is derived from the SGD methods and is based on the adaptive estimates of the training momentum.

Over the couple of decades many algorithms were competing with the classic BP method using variable parameters, derivatives of higher orders, temporal updates, etc. Most of them proved successful but the training time still remains disappointing for various applications. The literature is missing a well performing, flexible and affordable in its implementation method for neural networks training. For that reason we made an attempt to establish a training method which is not burdened with the aforementioned shortcomings of the well known algorithms. The ultimate goal of our method is to reduce the number of epochs and the training time that are required for a valid neural network training.

The proposed algorithm originates from the Recursive Least Squares (RLS) method[30]. In adaptive filters, the RLS algorithm is used for adjusting filter weights. The main goal of the adaptive methods is to minimize a given error measure. Due to the similarity of an adaptive filter and a single neuron, similar algorithms (with some modifications) can be applied to neural networks as they are to the adaptive filters. The core of the proposed algorithm is based on the QR decomposition achieved by the Givens rotations (Givens rotations in QR decomposition - GQR), whose implementation is not highly complex and maintains high scalability. As presented in the following Sections, in the GQR algorithm each neuron is trained separately from the others, which opens the possibility for parallel training of all neurons in the network.

The summary of the article novelty and originality formulates as follows:

1. A new, original feedforward (FF) neural networks training algorithm has been proposed. It utilizes the Givens rotations to perform
2 The selected feedforward neural networks overview

There are many types of artificial neural networks. Ones of very useful ANNs are feed forward (FF) networks. They can be successfully applied in many demanding tasks. FF networks can consist of multiple layers, with a various number of neurons in each one. The last layer is called the output layer. All layers before it are the hidden layers. The Network’s input is an input to the first layer while the network’s output is the output from the last layer. Such network, with at least one hidden layer is called a multilayer perceptron (MLP). An example structure of an MLP is shown in Figure 1. A special case of the MLP networks are the fully connected MLP networks (FCMLP). Their layers are connected with all previous layers and the network’s input. An example of a simple FCMLP network is shown in Figure 2. Another variant of FF networks are the fully connected cascade (FCC) networks. They consist of ordered neurons. Each neuron in an FCC network is connected with all the preceding neurons and the network’s input creates the so-called “cascade”. An example structure of an FCC network is shown in Figure 3.
The NN recall phase is defined by formulas

\[
\begin{align*}
   s_i^{(l)} &= \sum_{j=0}^{N_l-1} w_{ij}^{(l)} x_j^{(l)}, \\
   y_i^{(l)}(t) &= f(s_i^{(l)}(t)),
\end{align*}
\]

(1)

where \( w_{ij}^{(l)} \) is j-th weight of i-th neuron in l-th layer, \( x_j^{(l)} \) is j-th input value in l-th layer, \( s_i^{(l)} \) is the linear output of i-th neuron in l-th layer, \( y_i^{(l)} \) is the nonlinear output of i-th neuron in l-th layer, \( f \) is an activation function and \( N_l \) is a number of neuron in l-th layer.

3 The Givens rotation

The Givens rotation \([31]\) is an orthogonal transformation method originated in n-dimensional linear algebra. In most cases rotations are limited to a single plain, which is stretched between two unit vectors: \( \text{span}\{e_p,e_q\} \) (\( 1 \leq p < q \leq n \)). The rotation itself is obtained by special matrix \( G_{pq} \), so called a rotation matrix or simply a rotation as shown in the following equation

\[
G_{pq} = \begin{bmatrix}
   1 & & & & & & 0 \\
   \vdots & \ddots & & & & & \vdots \\
   & & c & \cdots & s \\
   & & \vdots & \ddots & \vdots & \ddots & \vdots \\
   & & & & & -s & \cdots & c \\
   0 & \cdots & & & & & 1 \\
   & & & & & & p & q
\end{bmatrix}
\]

(2)

\[ g_{pp} = g_{qq} = c, \]

\[ g_{pq} = -g_{qp} = s, \]

(3)

where

\[ c^2 + s^2 = 1. \]

(4)

From equation (4) we know that \( G_{pq}^T G_{pq} = I \), which is the proof that matrix \( G_{pq} \) is in fact an orthogonal matrix. The rotation is performed by the following transformation

\[ x \rightarrow y = G_{pq}x, \]

(5)

which implies the given equalities

\[
\begin{align*}
   y_p &= cx_p + sx_q \\
   y_q &= -sx_p + cx_q \\
   y_i &= x_i \quad \text{for} \ i \neq p, q; i = 1, \ldots, n.
\end{align*}
\]

(6)

Consider a single rotation of vector \( a \in \mathbb{R}^n \). According to equations (5) and (6) a single rotation affects only two elements of vector \( a \). Those are \( a_p \) and \( a_q \). This provides an opportunity to set parameters \( c \) and \( s \), so that element \( a_q \) is to be superseded by the value of 0 as follows

\[ a_q = -sa_p + ca_q = 0. \]

(7)

In order to achieve that, parameters \( c \) and \( s \) of rotation matrix \( G_{pq} \) are calculated according to

\[ c = \frac{a_p}{\rho}, \quad s = \frac{a_q}{\rho}. \]

(8)

To take care of the numerical stability, \( \rho \) is depicted as

\[
\rho = \begin{cases}
   a_p \sqrt{1 + (a_q/a_p)^2} & \text{for} \ |a_p| \geq |a_q| \\
   a_q \sqrt{1 + (a_p/a_q)^2} & \text{for} \ |a_p| < |a_q|
\end{cases}
\]

(9)

4 QR decomposition based on rotations

The QR decomposition \([32, 33]\) is an iterative algorithm for transforming any non-singular matrix \( A \in \mathbb{R}^{m,n} \) to the product of orthogonal matrix \( Q \) and upper-triangle matrix \( R \) as follows

\[ A = QR, \]

(10)

where \( Q^T Q = I \), \( Q^T = Q^{-1} \) and \( r_{ij} = 0 \) for \( i > j \). Let \( a \in \mathbb{R}^m \) be a single column vector of matrix \( A^{m,n} \). Due to equations (5-9) it is possible to calculate a sequence of rotations \( G_{12}, G_{13}, \ldots, G_{1m} \) which can be given as a product

\[ G_1 = G_{12} \ldots G_{1,m-1} G_{1m}. \]

(11)
Matrix $G_1$ is also a rotation matrix. It is able to perform multiple rotations of vector $a$ at once, in order to eliminate $m - 1$ elements as shown in the following equation

$$\bar{a} = G_1 a = e_1 \rho = [\rho, 0, \ldots, 0]^T, \rho = \pm \|a\|_2. \quad (12)$$

While the elimination of the elements in the first column is taking place, all columns of matrix

$$A = A_1 = M_1 = \left[ \begin{array}{c} a_1 \\ B_1 \end{array} \right] \quad (13)$$

are affected by each rotation. This is the result of expanding equation (6) with an additional equation as follows

$$A_2 = G_1 A_1 = \bar{G}_1 M_1 = \left[ \bar{a}_1 \\ \bar{B}_1 \right] = \left[ \begin{array}{c} \rho_1 \\ 0 \end{array} \right]$$

At this stage, the first column of matrix $A$ is eliminated as shown in equation (12). The first row of matrix $A$ is also rotated as desired. It will not be affected by the oncoming rotations. In the next steps new rotations are performed as shown in the following equation

$$G_k = G_{k,k+1} \cdots G_{k,m-1} G_{km}$$

for $(k = 1, \ldots, m - 1)$. In each iteration of the algorithm, matrix $M_k$ is the remaining sub-matrix that still needs to be rotated due to the following equation

$$A_{k+1} = \tilde{G}_k M_k = \left[ \bar{a}_k \\ \bar{B}_k \right] = \left[ \begin{array}{c} \rho_k \\ 0 \end{array} \right]$$

Orthogonal matrix $Q$ is not needed to be explicitly calculated. During the process only rotation parameters $c$ and $s$ are directly required. Matrix $Q$ could be obtained by an inversion of the rotations according to the following equation

$$Q = G_1^T \cdots G_{m-1}^T =$$

$$= G_{1m}^T G_{2m}^T \cdots G_{m-1,m}^T. \quad (19)$$

5 Network training

The GQR algorithm can be applied to any multi-layered artificial neural network along with any differentiable activation function. The ultimate goal of a training process is to minimize the error measure which is given by the following equation

$$J(n) = \sum_{t=1}^{n} \lambda^{n-t} \sum_{j=1}^{N_t} \varepsilon_j^{(L)^2}(t) =$$

$$= \sum_{t=1}^{n} \lambda^{n-t} \sum_{j=1}^{N_t} \left[ d_j^{(L)}(t) - f \left( x_j^{(L)}(t) w_j^{(L)}(n) \right) \right]^2, \quad (20)$$

where $\lambda \in (0, 1)$ is a forgetting factor. This parameter defines how much the previous updates influence the current iteration of the algorithm.

The error minimization is based on the gradient method. In order to obtain an entry point to the GQR weight update algorithm, the error measure gradient needs to be calculated and equalled to 0 as follows

$$\frac{\partial J(n)}{\partial w_j^{(L)}(n)} = 2 \sum_{t=1}^{N_t} \lambda^{n-t} \sum_{j=1}^{N_t} \frac{\partial f_j^{(L)}(t)}{\partial w_j^{(L)}(n)} \varepsilon_j^{(L)}(t) =$$

$$= -2 \sum_{t=1}^{N_t} \lambda^{n-t} \sum_{j=1}^{N_t} \frac{\partial f_j^{(L)}(t)}{\partial w_j^{(L)}(n)} \varepsilon_j^{(L)}(t) = 0. \quad (21)$$

In the next step equation (21) is taken into further considerations.
will not be affected by the oncoming rotations. A
eliminated as shown in equation (12). The first
transformations of matrix
After

\[ G = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \]

\[ \bar{B} \]
is also rotated as desired. It
for
\[ k, k \]

\[ \hat{G}_{i,n} = 1 \]

At this stage, the first column of matrix

\[ G_{1} \rho \]

\[ \bar{B} \]

\[ \pm \| \]

(17)

Finally, the obtained form of an error measure
denotes the error value in each
\[ \frac{\partial y_i^{(l)}}{\partial s_j^{(l-1)}} \]

\[ w_{jp}^{(l)} \]

\[ \varepsilon_j^{(L)}(t) = 0, \]

\[ \varepsilon_p^{(l)}(t) \]

\[ \frac{\partial y_j^{(l+1)}}{\partial \bar{B}_{j,m}^{(l)}} u_{jp}^{(l+1)}(n) \varepsilon_j^{(l+1)}(t) \]

(23)

Then, the additional transformations of equation (22) are applied

\[ \sum_{t=1}^{n} \lambda^{n-t} \sum_{q=1}^{N_l} \frac{\partial \bar{y}_i^{(l)}}{\partial \bar{w}_i^{(l)}} \varepsilon_q^{(l)}(t) = \]

\[ = \sum_{t=1}^{n} \lambda^{n-t} \sum_{q=1}^{N_l} \frac{\partial y_j^{(l)}}{\partial \bar{w}_i^{(l)}} \frac{\partial \bar{w}_i^{(l)}}{\partial s_j^{(l-1)}} \varepsilon_j^{(l)}(t) = \]

\[ = \sum_{t=1}^{n} \lambda^{n-t} \sum_{q=1}^{N_l} \frac{\partial y_j^{(l)}}{\partial \bar{w}_i^{(l)}} w_{jp}^{(l+1)}(n) \varepsilon_j^{(l+1)}(t) = \]

\[ = \sum_{t=1}^{n} \lambda^{n-t} \sum_{q=1}^{N_l} \frac{\partial y_j^{(l)}}{\partial \bar{w}_i^{(l)}} \bar{B}_{j,m}^{(l)} \varepsilon_j^{(l)}(t) \]

(24)

Finally, the obtained form of an error measure given by equation (24) is linearised to the following equation

\[ f(\hat{b}_i^{(l)}(t)) \approx \]

\[ \approx f(s_i^{(l)}(t)) + f'(s_i^{(l)}(t))(\hat{b}_i^{(l)}(t) - s_i^{(l)}(t)). \]

(25)

Then, the normal form of equation (24) along with a linearisation step from equation (25) is given as follows

\[ \sum_{t=1}^{n} \lambda^{n-t} f^2(s_i^{(l)}(t)). \]

\[ \left[ \hat{b}_i^{(l)}(t) - x_i^{(l)}T(t) w_i^{(l)}(n) \right] x_i^{(l)}T(t) = 0. \]

(26)

The GQR algorithm entry point is obtained from equation (26) by presenting it in a vector form as shown in the following equation

\[ A_i^{(l)}(n) w_i^{(l)}(n) = h_i^{(l)}(n), \]

(27)

where

\[ A_i^{(l)}(n) = \sum_{t=1}^{n} \lambda^{n-t} f^2(s_i^{(l)}(t)) x_i^{(l)}(t) x_i^{(l)T}(t), \]

(28)

\[ h_i^{(l)}(n) = \sum_{t=1}^{n} \lambda^{n-t} f^2(s_i^{(l)}(t)) b_i^{(l)}(t) x_i^{(l)}(t). \]

(29)

To improve equation readability the following substitution is performed

\[ z_i^{(l)}(t) = f'(s_i^{(l)}(t)) x_i^{(l)}(t). \]

(30)

This results in a new form of equations 28 and 29 as follows

\[ A_i^{(l)}(n) = \sum_{t=1}^{n} \lambda^{n-t} z_i^{(l)}(t) z_i^{(l)T}(t), \]

(31)

\[ h_i^{(l)}(n) = \sum_{t=1}^{n} \lambda^{n-t} f'(s_i^{(l)}(t)) b_i^{(l)}(t) z_i^{(l)}(t), \]

(32)

where \( b_i^{(l)}(n) \) is a linear expected value depicted as

\[ b_i^{(l)}(n) = \begin{cases} f^{-1}(d_i^{(l)}(n)) & \text{for } l = L \\ s_i^{(l)}(n) + e_i^{(l)}(n) & \text{for } l = 1 \ldots L - 1. \end{cases} \]

(33)

The error \( e_i^{(k)}(n) \) is obtained in backpropagation process by

\[ e_i^{(k)}(n) = \sum_{j=1}^{N_{k+1}} f'(s_i^{(k)}(n)) w_{ji}^{(k+1)}(n) e_j^{(k+1)}(n) \]

for \( k = 1 \ldots L - 1. \)

(34)

Note, that equation (27) is a normal equation and it needs to be solved for all neurons of a network since each neuron computes its own linear response \( (s_i^{(l)}) \). Equation (27) can be solved using Matrix Inversion Lemma [34] but this method has a huge computational load. In the GQR algorithm the equation (27) is solved by the QR decomposition with the Givens rotation as shown in the previous Sections. This approach allows to solve the equation (27) easily.
and quickly with limited computational load. During the process \( Q_i^{(l)T}(n) \) matrix is implicitly calculated as shown in the following equations
\[
Q_i^{(l)T}(n) A_i^{(l)}(n) w_i^{(l)}(n) = Q_i^{(l)T}(n) h_i^{(l)}(n),
\]
\[
Q_i^{(l)}(n) Q_i^{(l)}(n) R_i^{(l)}(n) w_i^{(l)}(n) = Q_i^{(l)T}(n) h_i^{(l)}(n),
\]
\[
R_i^{(l)}(n) w_i^{(l)}(n) = Q_i^{(l)T}(n) h_i^{(l)}(n)
\]
where \( R_i^{(l)}(n) \) are the upper-triangle matrices (see Section 4). Vectors \( h_i^{(l)}(n) \) are rotated along with matrices \( A_i^{(l)}(n) \). The decomposition process transformed matrices \( A_i^{(l)}(n) \) into upper-triangle matrices \( R_i^{(l)}(n) \), whose inversion is not very complex. The approximated weights vectors are calculated according to the following equation
\[
\hat{w}_i^{(l)}(n) = R_i^{(l)-1}(n) Q_i^{(l)T}(n) h_i^{(l)}(n).
\]
At the final stage the correction vector \( \hat{w}_i^{(l)}(n) - w_i^{(l)}(n-1) \) is multiplied and applied to the network as follows
\[
w_i^{(l)}(n) = w_i^{(l)}(n-1) + \eta \left( \hat{w}_i^{(l)}(n) - w_i^{(l)}(n-1) \right) = (1-\eta) w_i^{(l)}(n-1) + \eta \hat{w}_i^{(l)}(n),
\]
where \( \eta \) is a learning rate.

The Algorithm 1 presents the full procedure of the GQR training algorithm for feedforward neural networks.

\section{Simulation results}

This Section contains a detailed analysis of the experimental results. To verify the performance and stability of the GQR algorithm extensive tests have been carried out. The benchmark contains nine training problems, which can be categorized by their nature and complexity level. The first group is function approximations, where the Logistic curve, Hang, Sinc, and Concrete problems belong. The second one is the classification which contains the Two Spirals, Abalone, and Iris datasets. Finally, the special case benchmarks are the encoder and parity problems. The Concrete, Abalone, and Iris datasets are taken from the UCI - Machine Learning Repository. The data have been normalized to the values in range \([-1, 1]\).

\begin{algorithm}
\caption{The GQR algorithm}
\begin{algorithmic}
\For {each sample \( n \)}
\For {each neuron \( i \)}
Compute equations (30), (31), (32)
\EndFor
\EndFor
\EndWhile
\end{algorithmic}
\end{algorithm}

Each benchmark also covers a wide range of networks. In order to simplify the nomenclature, the authors refer to FCC-n as a fully connected cascade network with \( n \) neurons. Similarly, by MLP-[\( n_L \) L] the authors refer to the multilayered perceptron with \( L \) layers and \( n_L \) neurons in each one. Additionally, prefix FC stands for a fully connected network.

Every scenario was trained by the GQR algorithm and most popular variants of the Back Propagation derivatives. This includes the Momentum variant (MBP), Nesterov’s Accelerated Gradient (NAG), Quick Propagation (QProp), Resilient Propagation (RProp) and the Adam algorithm. Every trial was run 100 times with
an identical setup but a newly generated initial network state.

### 6.1 Experiment methodology

The bench setup assumes several common parameters for each trial. The most significant ones are shown in Table 1. All experiments were retried 100 times in order to gather valuable statistics data. The training limit for all approximation and classification benchmarks was set to 1000 epochs. After reaching this value, the training is assumed to have failed. The target error threshold and its criterion was treated as a problem specific. Each sample of a training set was presented in a random order in each consecutive epoch.

#### Table 1. Common experiment setup

| Epoch limit | 1000 |
|-------------|------|
| Experiment retry count | 100 |
| Sequence type | Random |

The experiment was concluded with a great number of results. In order to gather the most valuable data, performance factor $\xi$ given by equation (38) was established. All data presented in the subsequent Sections were gathered with respect to the highest performance factor $\xi$.

\[
\xi_{\text{algorithm}} = \frac{\text{SuccessRatio}}{\text{EpochAverage}} \quad (38)
\]

The tables presented in the next Section contain common definitions for all columns. $\eta$, $\lambda$, $\alpha$, inc and dec are the training parameters of the respective algorithm. It needs to be noted that not all parameters are used by all algorithms. "SR" is a success ratio expressed as % of successful training trials. The column "Ep." stands for the average epoch count required to achieve a predefined error threshold. The "T" column shows an average convergence time in milliseconds.

### 6.2 Logistic curve approximation

The logistic curve is a non-linear single argument function given by the following equation

\[
f(x) = 4x(1-x) \quad x \in [0,1]. \quad (39)
\]

The training set consists of 11 samples that covers the argument range of $x \in [0,1]$. The experiment setup is summarized in Table 2.

#### Table 2. Setup for the logistic curve approximation

| Target error          | 0.001 |
|-----------------------|-------|
| Criterion             | Epoch average |
| Activation in hidden layers | Hyperbolic tangent |
| Teaching sequence size | 11    |

The logistic curve approximation was trained using FCC networks starting with 2 up to 7 neurons. Also a single FCMLP network with one hidden layer and the total of 6 neurons was used. The GQR algorithm performance in each scenario is shown in Table 3.

#### Table 3. The results of the training logistic curve problem by the GQR algorithm

| Network    | $\eta$ | $\lambda$ | SR | Ep. | T   |
|------------|--------|-----------|----|-----|-----|
| FCC-2      | 0.1    | 0.87      | 92 | 56.7 | 1.2 |
| FCC-3      | 0.05   | 0.8       | 86 | 38.09 | 1.00 |
| FCC-4      | 0.007  | 0.62      | 82 | 26.63 | 0.76 |
| FCC-5      | 0.01   | 0.63      | 66 | 18.52 | 0.64 |
| FCC-6      | 0.003  | 0.74      | 82 | 24.54 | 1.03 |
| FCC-7      | 0.007  | 0.73      | 53 | 18.96 | 0.99 |
| FCMLP-5-1  | 0.007  | 0.7       | 100| 18.33 | 0.58 |

The best 100% ratio was achieved using an FCMLP network with $\eta = 0.007$ and $\lambda = 0.7$. The lowest value of epoch average (18.33) emerges also for the FCMLP network. It can be observed that by increasing the number of neurons in FCC networks the success ratio drops significantly - from 92% to 53%.

The GQR algorithm was compared with some of the most common variants of the BP derivatives. Table 4 covers the results of the best trials of each algorithm along with all the relevant training parameters for the FCMLP-5-1 network.
Table 4. The logistic curve training summary for FCMLP-5-1 network.

| Alg.  | η   | α   | inc | dec | λ  | SR | Ep. | T     |
|-------|-----|-----|-----|-----|----|----|-----|-------|
| Adam  | 0.01| -   | -   | -   | -  | 100| 594.37 | 20.66 |
| BP    | 0.1 | -   | -   | -   | -  | 63 | 573.32 | 14.17 |
| MBP   | 0.01| 0.95| -   | -   | -  | 88 | 318.16 | 6.93  |
| NAG   | 0.07| 0.8 | -   | -   | -  | 100| 84.80  | 2.32  |
| QProp | 0.55| -   | -   | 1.05| -  | 91 | 255.29 | 7.63  |
| RProp | -   | 1.05| 0.7 | -   | -  | 99 | 207.67 | 4.45  |
| GQR   | 0.007| -  | -   | 0.7 | -  | 100| 18.33  | 0.58  |

For the logistic function approximation the most stable training process was held by the GQR and NAG algorithms. In both cases every training trial ended with a success. However, the GQR algorithm required only 18.33 epochs on average to establish the given training goal. The exemplary convergence process of the logistic curve approximation closest to the average is shown in Figure 4.

Figure 4. Convergence for the logistic curve problem using FCMLP-5-1 network

6.3 Hang approximation

Hang is a non-linear two argument function given as

\[ f(x_1, x_2) = \left(1 + x_1^{-2} + \sqrt{x_2^{-3}}\right)^2 \quad x_1, x_2 \in [1, 5]. \]

(40)

A training set consists of 50 samples that cover the argument range in \( x_1, x_2 \in [1, 5] \). Hang experiment setup is summarized in Table 5.

In the Hang function approximation benchmark, FCC networks with 8 up to 18 neurons and a single FCMLP network with one hidden layer with 16 neurons in total was trained. Table 6 summarizes the GQR algorithm performance in each scenario.

The success rate varies between 85 and 100%. The number of neurons in FCC networks clearly affects the average epoch count. The more neurons there are in a network, the faster the convergence is. The best performance with the highest success ratio (100%) is observed for the FCMLP network where barely 25 epochs (12.18 ms) on average were required for a successful training.

During the Hang approximation training the GQR algorithm and several variants of the BP derivatives were tested. Table 7 presents the best results with respect to the performance factor 38 achieved by all of the tested training methods using the FCMLP-15-1 network.

The GQR algorithm achieved the best 100% success ratio. It also manifests a very short convergence time (25.02 epochs and 12.18 ms on
average) comparing to all of the tested BP variants. Figure 5 illustrates the exemplary convergence process of the Hang function approximation that is the closest to the average.

![Figure 5. Convergence for the Hang approximation problem using FCMLP-15-1 network](image)

### 6.4 Sinc approximation

Sinc is a non-linear two argument function defined by the following formula

$$f(x_1, x_2) = \begin{cases} 
1 & x_1 = x_2 = 0 \\
\frac{\sin x_2}{x_2} & x_1 = 0 \land x_2 \neq 0 \\
\frac{\sin x_1}{x_1} \frac{\sin x_2}{x_2} & x_2 = 0 \land x_1 \neq 0 \\
\text{for other cases} & 
\end{cases}$$

(41)

The training set contains 121 samples, which corresponds to the unnormalized Sinc function with arguments in range $x_1, x_2 \in [-10, 10]$. The Sinc experiment setup is summarized in Table 8.

![Table 8. Setup for the Sinc approximation](image)

| Target error | 0.005 |
|--------------|-------|
| Criterion    | Epoch average |
| Activation in hidden layers | Hyperbolic tangent |
| Teaching sequence size | 121 |

During the Sinc experiment the same networks as in the Hang benchmark were used. This means the FCC with 8 to 18 neurons and FCMLP-15-1 networks were used in the experiment. The GQR algorithm performance across all scenarios is shown in Table 9.

![Table 9. The results of the Sinc training problem by the GQR algorithm](image)

| Network | $\eta$ | $\lambda$ | SR | Ep. | T |
|---------|--------|-----------|----|-----|---|
| FCC-8   | 0.03   | 0.94      | 100 | 10.39 | 9.58 |
| FCC-10  | 0.03   | 0.97      | 100 | 6.88  | 7.48 |
| FCC-12  | 0.03   | 0.975     | 100 | 6.03  | 11.92 |
| FCC-14  | 0.01   | 0.93      | 100 | 5.03  | 14.52 |
| FCC-16  | 0.01   | 0.96      | 100 | 4.51  | 21.47 |
| FCC-18  | 0.01   | 0.97      | 100 | 4.24  | 27.97 |
| FCMLP-15-1 | 0.009 | 0.99    | 100 | 108.36 | 94.29 |

The Sinc approximation problem showed a great stability of the GQR training resulting in 100% success ratio for all the scenarios. With the use of the FCC networks most trials required less than 11 epochs to achieve a defined error threshold. Also by increasing the neuron count in the FCC networks there is a visible trend of an average of fewer required epochs. The FCC network seems to be capable of handling the Sinc problem much better than the classic FCMLP network where the training took 108.36 epochs (94.29 ms) on average.

In order to maintain a consistent approach to the presented data across all benchmarks the best results for the FCMLP-15-1 network with respect to the performance factor 38 are shown in Table 10. In the Sinc scenario the same algorithms were used in comparison to the Hang training.

![Table 10. The Sinc training summary for the FCC-18 network](image)

| Alg. | $\eta$ | $\alpha$ | inc | dec | $\lambda$ | SR | Ep. | T |
|------|--------|---------|-----|-----|-----------|----|-----|---|
| Adam | 0.001  | -       | -   | -   | -         | 100| 34.14 | 43.92 |
| BP   | 0.007  | -       | -   | -   | -         | 100| 70.73 | 55.74 |
| MBP  | 0.0005 | 0.95    | -   | -   | -         | 100| 54.12 | 43.20 |
| NAG  | 0.001  | 0.95    | -   | -   | -         | 100| 64.06 | 134.99 |
| QProp| 0.68   | -       | -   | -   | -         | 58 | 308.36 | 282.13 |
| RProp| -      | 1.1     | 0.6 | -   | -         | 98 | 493.45 | 320.93 |
| GQR  | 0.01   | -       | -   | -   | 0.97      | 100| 4.24  | 27.97 |

The highest performance out of all tested algorithms is observed during the GQR training with 100% success ratio and 4.24 epochs (27.97 ms) on average. In Figure 6 presented is the exemplary convergence processes closest to the average of the Sinc function approximation training.
Figure 6. Convergence for the Sinc approximation problem using the FCC-18 network.

6.5 Concrete dataset

The Concrete dataset corresponds to the highly nonlinear function which outputs the concrete compressive strength based on the used ingredients and age of the compound. The training set utilized in this benchmark contains 1030 samples with 8 inputs in each. Table 11 shows the initial training parameters.

| Table 11. The initial setup for the Concrete training |
|-----------------------------------------------------|
| Target error | 0.001 |
| Criterion | Epoch average |
| Activation in hidden layers | Hyperbolic tangent |
| Teaching sequence size | 1030 |

Due to high complexity of the benchmark function the GQR experiment covers wide range of tested networks. Table 12 presents the GQR algorithm performance for each attempted network.

| Table 12. The results of the Concrete training by the GQR algorithm. |
|---------------------------------------------------------------|
| Network | $\eta$ | $\lambda$ | $SR$ | $Ep.$ | $T$ |
| FCC-7 | 0.01 | 0.98 | 99 | 38.26 | 598.04 |
| FCC-8 | 0.005 | 0.98 | 100 | 27.13 | 507.89 |
| FCC-9 | 0.009 | 0.99 | 99 | 22.56 | 675.34 |
| FCMLP-4-4-1 | 0.009 | 0.98 | 100 | 25.82 | 428.77 |
| FCMLP-6-6-1 | 0.009 | 0.99 | 100 | 17.26 | 575.92 |
| FCMLP-8-8-1 | 0.009 | 0.99 | 100 | 13.88 | 864.35 |
| MLP-4-4-1 | 0.009 | 0.97 | 57 | 118.04 | 838.82 |
| MLP-6-6-1 | 0.01 | 0.98 | 100 | 32.46 | 432.07 |
| MLP-8-8-1 | 0.01 | 0.98 | 100 | 20.33 | 408.92 |

In order to establish the overall GQR algorithm performance it has been compared to other commonly used algorithms. Table 13 summarizes the achieved performance of the tested training methods for the MLP-6-6-1 network.

| Table 13. The summary of the Concrete training utilizing MLP-6-6-1 network. |
|---------------------------------------------------------------|
| Alg. | $\eta$ | $\alpha$ | $inc$ | $dec$ | $\lambda$ | $SR$ | $Ep.$ | $T$ |
| Adam | 0.001 | - | - | - | - | 100 | 165.30 | 615.37 |
| BP | 0.03 | - | - | - | - | 100 | 219.19 | 547.49 |
| MBP | 0.007 | 0.8 | - | - | - | 100 | 201.66 | 594.46 |
| NAG | 0.005 | 0.9 | - | - | - | 92 | 406.24 | 1937.19 |
| QProp | 0.05 | - | - | - | - | 17 | 782 | 1755.06 |
| RProp | - | - | 1.15 | 0.6 | - | 41 | 664.17 | 1350.04 |
| GQR | 0.01 | - | - | 0.98 | 100 | 32.46 | 432.07 |

The Concrete Compressive Strength benchmark manifests the great GQR performance. Figure 7 shows exemplary close to average convergence process of the tested methods.

Figure 7. Example convergence for the Concrete problem using the MLP-6-6-1 network.

6.6 Two spirals classification

The two spirals is a well known classification problem which acts as a universal benchmark for neural network training algorithms. The used training set contains 96 samples, which corresponds to points in a 3-dimensional space. Each point "belongs" to either bottom or top spirals. The two spirals experiment setup is summarized in Table 14.
Table 14. Setup for the two spirals classification

| Target error | Criterion | Activation in hidden layers | Teaching sequence size |
|--------------|-----------|-----------------------------|------------------------|
| 0.05         | Epoch average | Hyperbolic tangent         | 96                     |

Similar to the previous experiments the two spirals problem involved the FCC with 8 to 18 neurons and a fully connected FCMLP network containing 3 hidden layers with a total of 16 neurons. Table 15 summarizes the GQR algorithm performance in each scenario with respect to performance factor 38.

Table 15. The results of training the Two Spirals problem by the GQR algorithm.

| Network        | η   | λ   | SR  | Ep. | T   |
|----------------|-----|-----|-----|-----|-----|
| FCC-8          | 0.03| 0.995| 89  | 57.31 | 36.28 |
| FCC-10         | 0.005 | 0.975 | 82  | 34.74 | 38.46 |
| FCC-12         | 0.009 | 0.99 | 92  | 27.51 | 40.98 |
| FCC-14         | 0.01 | 0.995 | 97  | 25.61 | 55.94 |
| FCC-16         | 0.009 | 0.995 | 96  | 23.49 | 71.98 |
| FCC-18         | 0.003 | 0.97 | 82  | 20.32 | 81.42 |
| FCMLP-5-5-5-1  | 0.009 | 0.99 | 99  | 25.46 | 54.70 |

In all cases the GQR algorithm achieved a rather satisfying success ratio ($SR \geq 82\%$) with fewer than 57 epochs on average. The FCMLP network seems to be the most stable having 99% of successful trials with an average of 25.46 required epochs. It can be observed that the success ratio is reduced to 82% in the FCC-18 scenario.

The two spirals classification benchmark involved the same set of BP derivatives as for the previous trials in comparison to the GQR algorithm. Table 16 presents the best results in respect to performance factor (38) achieved in the FCMLP-5-5-5-1 network training across all the tested training methods.

Table 16. The Two Spirals training summary for the FCMLP-5-5-5-1 network.

| Alg.   | η   | α   | inc | dec | SR  | Ep. | T   |
|--------|-----|-----|-----|-----|-----|-----|-----|
| Adam   | 0.001 | -   | -   | -   | 94  | 552.65 | 294.15 |
| BP     | 0.007 | -   | -   | -   | 81  | 629.73 | 224.24 |
| MBP    | 0.0005 | 0.95 | -   | -   | 84  | 591.38 | 214.16 |
| NAG    | 0.005 | 0.75 | -   | -   | 41  | 583.49 | 443.39 |
| QProp  | 0.007 | -   | -   | -   | 29  | 578.69 | 209.36 |
| RProp  | -    | 1.05 | 0.75 | -   | 47  | 594.43 | 224.25 |
| GQR    | 0.009 | -   | -   | -   | 0.99 | 25.46 | 54.70 |

The success ratio diversity proved to be very big. The highest value of 99% was achieved by the GQR algorithm with only 25.46 epochs (54.70 ms) on average. The worst performing algorithms in the Two Spirals benchmark turns out to be the QProp, NAG and RProp (success ratio below 50%). The experiment confirms that the classic BP and its momentum variant is able to achieve a rather high success ratio for very low values of training step ($\eta$). However, this results in a big training effort in terms of average amount of time and epoch number. The example of closest to the average convergence processes for the best trials of the Two Spirals classification training is shown in Figure 8.

Figure 8. Convergence for the Two Spirals problem using the FCMLP-5-5-5-1 network

6.7 Abalone dataset

The Abalone dataset is a universal classification benchmark that found many use cases in the Machine Learning domain. The training sequence contains 4177 samples with 8 attributes. Each corresponds to the physical measurement of the abalone. Based on this information, a well trained neural network can estimate the age of the abalone. Table 17 contains experiment setup.

Table 17. The initial setup for the Abalone training

| Target error | 0.012 |
| Criterion    | Epoch average |
| Activation in hidden layers | Hyperbolic tangent |
| Teaching sequence size | 4177 |
To establish the best training parameters and a suitable network topology a detailed experiment has been conducted. Several networks and a wide range of training parameters have been examined. Table 18 presents the GQR algorithm performance along with the best training parameters.

Table 18. The results of the Abalone training by the GQR algorithm.

| Network     | η    | λ  | SR | Ep. | T    |
|-------------|------|----|----|-----|------|
| FCC-3       | 0.003| 0.99| 100| 15.30| 310.59|
| FCC-4       | 0.003| 0.99| 100| 6.90 | 201.57|
| FCC-5       | 0.0007| 0.99| 100| 5.70 | 228.87|
| FCMLP-2-2-1 | 0.005| 0.99| 100| 5.30 | 157.60|
| FCMLP-4-4-1 | 0.0007| 0.99| 100| 3.20 | 288.70|
| FCMLP-6-6-1 | 0.005| 0.98| 100| 2.80 | 461.72|
| MLP-2-2-1   | 0.009| 0.99| 100| 4.30 | 64.65 |
| MLP-4-4-1   | 0.007| 0.99| 100| 2.90 | 99.70 |
| MLP-6-6-1   | 0.003| 0.96| 100| 2.90 | 135.35|

The GQR algorithm has also been compared with several well-known training methods. The experiment results show that the proposed algorithm performance is superior to the classic methods of feedforward neural networks training. Table 19 contains the benchmark summary of all attempted algorithms.

Table 19. The summary of the Abalone training utilizing MLP-2-2-1 network.

| Alg.     | η    | α  | inc | dec | λ | SR | Ep. | T    |
|----------|------|----|-----|-----|---|----|-----|------|
| Adam     | 0.001| -  | -   | -   | - | 100| 20.80| 101.25|
| BP       | 0.05 | -  | -   | -   | - | 100| 29.90| 95.89 |
| MBP      | 0.01 | 0.7| -   | -   | - | 100| 25.40| 92.72 |
| NAG      | 0.007| 0.9| -   | -   | - | 100| 61.90| 480.41|
| QProp    | 0.51 | -  | -   | 1.1 | 0.45| 100| 289  | 812.73|
| RProp    | -    | -  | 1.1 | 0.45| -  | 100| 212.30| 566.06|
| GQR      | 0.009| -  | -   | -   | 0.99| 100| 4.30 | 64.65 |

The great performance of the GQR algorithm in the Abalone benchmark can also be observed in Figure 9, which shows exemplary, close to the average convergence process for the MLP-2-2-1 network.

Figure 9. Example convergence for the Abalone problem using the MLP-2-2-1 network

6.8 Iris dataset

The Iris benchmark utilizes the well known Iris plant dataset. It contains a total of 150 samples with 4 attributes which correspond to the one out of three iris plant classes. Each class is represented by 50 samples. Table 20 contains the Iris benchmark’s initial setup.

Table 20. The initial setup for the Iris training

| Target error | 0.05 |
|---------------|------|
| Criterion     | Epoch average |
| Activation in hidden layers | Hyperbolic tangent |
| Teaching sequence size | 150 |

The performance of the GQR algorithm during Iris dataset training has been tested on several feedforward neural network topologies. The results have been combined and presented in Table 21. It can be observed that the MLP networks handle Iris training better than their fully connected counterparts.

Table 21. The results of the Iris training by the GQR algorithm.

| Network     | η    | λ  | SR | Ep. | T    |
|-------------|------|----|----|-----|------|
| FCMLP-2-2-3 | 0.1  | 0.99| 95 | 26.31| 23.34|
| FCMLP-4-4-3 | 0.03 | 0.98| 96 | 19.34| 44.80|
| FCMLP-6-6-3 | 0.05 | 0.98| 97 | 16.43| 67.33|
| MLP-2-2-3   | 0.03 | 0.99| 89 | 20.65| 6.25 |
| MLP-4-4-3   | 0.01 | 0.98| 98 | 15.48| 10.93|
| MLP-6-6-3   | 0.03 | 0.98| 100| 12.83| 22.74|
The Iris dataset has also been tested with a set of BP derived training methods and compared to the proposed GQR algorithm. Table 22 contains results of the iris plant classification problem.

**Table 22.** The summary of the Iris training utilizing MLP-2-2-3 network.

| Alg. | ω | α | inc | dec | λ | SR | Ep. | T |
|------|---------|------|-----|-----|---|----|-----|---|
| Adam | 0.01    | -    | -   | -   | - | 100 | 38.63 | 9.43 |
| BP   | 0.05    | -    | -   | -   | - | 100 | 53.06 | 6.65 |
| MBP  | 0.003   | 0.95 | -   | -   | - | 100 | 47.24 | 7.31 |
| NAG  | 0.009   | 0.9  | -   | -   | - | 100 | 68.82 | 18.90 |
| QProp| 0.7     | -    | -   | -   | - | 79  | 180.97 | 24.19 |
| RProp| -       | 1.5  | 0.6 | -   | - | 86  | 179.98 | 26.30 |
| GQR  | 0.03    | -    | -   | -   | 0.99 | 89  | **20.65** | **6.25** |

Figure 10 shows exemplary, closest to the average training convergence process of selected algorithms.

**Figure 10.** Example convergence for the Iris problem using the MLP-2-2-3 network

### 6.9 Encoder

The encoder/decoder problems, also simply called “encoders”, are special cases of classification tasks in which the network has to find a way to simulate one-hot code encoder. The biggest challenge for this benchmark is to duplicate the presented input pattern in the output layer having only a few hidden neurons available. To handle this problem correctly the MLP-\(N-M-N\) networks are used. This structure corresponds to \(N\) network inputs/outputs and \(M\) neurons in a single hidden layer. The training set also contains \(N\) samples. In order to prevent a network from “memorizing” the patterns rather than extract unique features, the “tight” encoders are used. In this case the number of hidden neurons is calculated as \(M = \log_2 N\).

In this paper the tight encoder MLP-2-4 was used. The training goal was set to 0.1 as an error average. The sigmoid function was used as an activation in the hidden neurons. The setup summary is shown in Table 23.

**Table 23.** Setup for the MLP-2-4 encoder problem

| Target error | 0.1 |
| Criterion | Epoch average |
| Activation in hidden layers | Sigmoid |
| Teaching sequence size | 4 |

The MLP-2-4 encoder was trained by the GQR and selected BP derived algorithms. During the process, the experiment reveals the best individual parameters for every method. It results with a high success ratio of 100% across all the tested methods. The detailed summary of the selected parameters and individual methods performance are shown in Table 24.

**Table 24.** The training summary for the MLP-2-4 encoder.

| Alg. | ω | α | inc | dec | λ | SR | Ep. | T |
|------|---------|------|-----|-----|---|----|-----|---|
| Adam | 0.01    | -    | -   | -   | - | 100 | 136.02 | 3.24 |
| BP   | 0.9     | -    | -   | -   | - | 100 | 49.28 | 0.88 |
| MBP  | 0.3     | 0.85 | -   | -   | - | 100 | 26.78 | 0.71 |
| NAG  | 0.1     | 0.9  | -   | -   | - | 100 | 36.35 | 1.04 |
| QProp| 0.1     | -    | -   | -   | - | 100 | 31.90 | 0.64 |
| RProp| -       | 1.3  | 0.7 | -   | - | 100 | 32.02 | 0.58 |
| GQR  | 0.07    | -    | -   | -   | **0.62** | 100 | **7.87** | **0.20** |

The best performance was achieved by the GQR algorithm which requires only 7.87 epochs (0.20 ms) on average to establish the accepted error criterion. The classic BP method took fewer than 50 epochs. It is worth noting that despite the GQR’s bigger complexity comparing to BP derived variants, it is still faster due to a significant reduction of required epochs. The exemplary training process that is the closest to the average trial is shown in Figure 11.
Figure 11. Example convergence for the encoder problem using the MLP-2-4 network.

6.10 Parity detection

The $n$-bit parity detection problem originates from the electronic circuits domain. This technique is commonly used for detecting transmission errors. In the parity benchmark a neural network needs to simulate the logical $n$-input XNOR gate in order to detect the even number of high states in the input vector. The training set contains $n^2$ samples.

An effort was made to train the set of networks to simulate a 4-bit parity detection circuit. The training sequence consists of 16 samples. The single output of the network should be either of low or high state (logical 0 or 1) depending on the number of high bits in the presented sample. The error criterion was set to be not greater than the maximum of 0.1 in an epoch. The sigmoid was used as an activation in the hidden layer. Table 25 contains detailed information regarding the parity benchmark setup.

| Target error | 0.1 |
| Criterion | Epoch max |
| Activation in hidden layers | Sigmoid |
| Teaching sequence size | 25 |

The proposed GQR algorithm manifests superior performance for each trained network. Table 26 contains the results achieved by the GQR training.

The initial approach to the parity problem was to find the best parameters for each of the tested training methods. All of them achieved a high (100%) value of the success ratio. Table 27 contains the best parameters and statistics summary.

Table 25. Setup for the MLP-2-4 encoder problem

Table 26. The results of training the 4-bit parity detection problem by the GQR algorithm.

| Network  | $\eta$ | $\lambda$ | SR  | Ep. | $T$ |
|----------|--------|-----------|-----|-----|-----|
| FCC-2    | 0.09   | 0.99      | 100 | 4.64| 0.15|
| FCC-3    | 0.1    | 0.985     | 100 | 4.51| 0.26|
| FCC-4    | 0.1    | 0.985     | 100 | 4.40| 0.25|
| MLP-10-1 | 0.3    | 0.985     | 100 | 2.29| 0.22|
| MLP-2-2-1| 0.3    | 0.985     | 100 | 2.43| 0.09|
| MLP-3-3-1| 0.3    | 0.985     | 100 | 2.41| 0.13|
| MLP-4-4-1| 0.3    | 0.985     | 100 | 2.35| 0.17|

Table 27. The 4-bit parity detection training summary for the MLP-2-2-1 network.

| Alg. | $\eta$ | $\alpha$ | inc | dec | $\lambda$ | SR | Ep. | $T$ |
|------|--------|----------|-----|-----|-----------|----|-----|-----|
| Adam | 0.01   | -        | -   | -   | -         | 100| 4.02| 0.11|
| BP   | 0.03   | -        | -   | -   | -         | 100| 6.83| 0.17|
| MBP  | 0.007  | 0.85     | -   | -   | -         | 100| 3.65| 0.10|
| NAG  | 0.009  | 0.95     | -   | -   | -         | 96 | 71.98| 2.45|
| QProp| 0.03   | -        | -   | -   | -         | 93 | 15.61| 0.38|
| RProp| -      | 1.45     | 0.25| -   | -         | 93 | 16.41| 0.39|
| GQR  | 0.3    | -        | -   | 0.995| 100       | 2.43| 0.09|

The parity benchmark strongly manifests the great performance of the GQR algorithm. Despite the strict error criterion our training method requires no more than 2.43 epochs (0.09 ms) on average to satisfy the rough training requirements. The second best performing method in this scenario turns out to be the MBP algorithm with an average of 3.65 required epochs (0.10 ms). Figure 12 presents an exemplary, closest to the average, convergence process for the 4-bit parity detection problem. It is also worth noting how, the GQR algorithm is able to drop the error directly into the desired threshold in only a single epoch.
number of high states in the input vector. The input XNOR gate in order to detect the even
parity network needs to simulate the logical mission errors. In the parity benchmark a neuro-
technique is commonly used for detecting trans-

6.10 Parity detection

tates from the electronic circuits domain. This
work to simulate a 4-bit parity detection cir-

Table 26 contains the results achieved by the presented sample. The error criterion was set
FQR training.

to be not greater than the maximum of

Table 25 contains detailed information regarding the parity bench-
tation in the hidden layer. Table 25 contains de-

The parity benchmark strongly manifests the
initial approach to the parity problem

Table 27 contains the best parameters and statis-
ty and reproducibility of results across all the
tested scenarios while maintaining a very high
success ratio.

Figure 12. Example convergence for the 4-bit
parity problem using the MLP-4-2-2-1
network. Note that chart series are presenting
the average error value while the training
target is maximum epoch error.

7 Conclusions

The GQR algorithm originates from the Re-
cursive Least Squares method, which is widely
used in the adaptive filters area. As shown in
this paper the presented method significantly
differs from the well known Back Propagation
algorithm and its derived variants. It also sig-
nificantly speeds up the training in terms of
average epoch count and time duration. The
GQR algorithm is characterized by high stabil-
ity and reproducibility of results across all the
tested scenarios while maintaining a very high
success ratio.

Based on the convergence curves presented
in Section 6, the typical behaviour of the tested
methods can be observed. The curves of the BP
algorithm and its momentum variants (MBP
and NAG) are rugged while slowly progressing
towards the optimal solution. Also, the com-
mon behaviour of the QProp and RProp algo-
rithms can be observed. The training processes
carried out by both methods are rather smooth
and only occasionally burdened with some dis-
turbances. The GQR training process manifests
the shortest convergence time by a sudden drop
of the global error value and then progresses to
the fine tuning if it is still needed.

The presented algorithm performs weight
update after each sample presentation. It can
be noted that despite a bigger complexity when
compared to the BP based algorithms, the GQR
algorithm gives a much better performance in
terms of epochs count and training time. The
training time of the presented method can be re-
duced by parallel implementation. It is caused
by high scalability of the GQR algorithm be-
cause an update of each neuron requires only
its local data that are available just after the
error backpropagation.

The most important advantages of the GQR
algorithm can be summarized:

– the number of training epochs is significantly
reduced,
– training time is much shorter,
– overall success ratio is very high,
– the method can be parallelized.

Our further research will focus on a paral-
lel variant of the GQR algorithm similar to the
methods presented in [35–38]. Also, the pre-

tased method is going to be attempted in the
Convolutional Neural Networks (CNN) train-

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