Running coupling constants of the Luttinger liquid

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We compute the one-loop expressions of two running coupling constants of the Luttinger model. The obtained expressions have a nontrivial momentum dependence with Landau poles. The reason for the discrepancy between our results and those of other studies, which find that the scaling laws are trivial, is explained.

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INTRODUCTION

The Luttinger liquid, namely the one-dimensional system of interacting massless fermions with independent conservation laws for the left- and right-moving species, has been and still is the subject of many theoretical studies in Condensed Matter Physics (see, e.g., [1, 2, 3]). This is due to the fact that it has several characteristic properties which are very different from those of a Fermi liquid. Indeed, its momentum distribution is continuous at the Fermi energy even at zero temperature. Moreover, there are no fermionic quasi-particles in the system because its single-particle density of states vanishes as a power law when one approaches the Fermi energy. In addition, the Luttinger liquid has collective bosonic charge- and spin-density modes. The equations of motion of the system are solved with the help of the method of bosonization [4]. This method allows to replace the complicated fermionic dynamics by another one which involves noninteracting bosonic degrees of freedom only. The beta functions associated with the fermionic dynamics are believed to be equal to zero [5, 6, 7], a result which is commonly regarded as reflecting the triviality of the bosonic dynamics.

The dynamics of the Luttinger liquid may actually be less trivial than expected. It may indeed well be that the particles in the system interact also by means of long range forces, as suggested by the following line of argument. It is known that the one-dimensional propagator of a free massless fermion has a large anomalous dimension [8, 9]. This large anomalous dimension modifies the singularity structure in momentum space in such a way that the scattering amplitude for isolated particles vanishes together with the residue of the propagator on the mass shell [8, 9]. One may therefore say that a given particle moves slower or faster because of the nature, repulsive or attractive, of the interaction rather than because of the relative remoteness or closeness of other particles in the system. Such a behaviour clearly suggests that the interactions between massless fermions are able to act over large distances in one dimension. The scale parameter controlling this behaviour can only originate from a well-defined cutoff because the Luttinger model is known to be classically scale invariant [10].

This line of argument, however, doesn’t provide a proof for the existence of long range forces in the Luttinger liquid. Our aim in the present paper is to establish in a more reliable way whether such forces exist in the system or not. This is done with the help of a renormalization group study of two effective coupling constants of the model. The computations are done in the regularized version of the model with a sharp cutoff. We find that the one-loop expressions of the running coupling constants have Landau poles which line up in a scale invariant manner around the Fermi points and at definite momentum scales resulting from the finiteness of the cutoff. The fact that the momentum dependence of these quantities is singular around the Fermi points allows us to conclude that long range forces do exist in the Luttinger liquid and so that the dynamics of this system is strongly coupled.

VERTEX FUNCTIONS

The regularized Luttinger model with a sharp cutoff of value Λ is defined by the action $S_{\text{Reg}}(\Lambda) = S_0 + S_\ell(\Lambda)$, where

$$S_0 = \int_k [\bar{\psi}_-^*(k)k_+\psi_-^*(k) + \psi_+^*(k)k_-\psi_+^*(k)]$$

(1)

describes the free propagation of massless fermions and

$$S_\ell(\Lambda) = -g\int_{k, k', q} \bar{\psi}_-^*(k + q)\psi_+^*(k')\psi_-^*(k')\psi_+^*(k')\rho_\ell(\Lambda, k + q)\rho_\ell(\Lambda, k)$$

(2)
their interactions. This action is the simplest one of all those considered in the context of the so-called g-ology models \[1\]. The left- and right-moving spinless particles are represented by the fields \((\psi_-, \psi_+^*)\) and \((\psi_+, \psi_-^*)\) respectively. We use the conventions \(\int_k = \int dkd\mathbf{k}/(2\pi)^2\) and \(k = (k_0, \mathbf{k})\), where \(k_0\) and \(k\) are the frequency and momentum variables respectively. The momenta \(k_\pm\) in Eq. \[1\] are defined by the relation \(k_\pm = \pm kv_F - \mu - ik_0\), where \(v_F\) is the Fermi velocity and \(\mu = k_Fv_F\) the Fermi energy \((k_F\) is the Fermi momentum\). The cutoff functions \(\rho_{\pm}\) in Eq. \[2\] are Heaviside functions \(\Theta\); more precisely, \(\rho_{\pm}(\Lambda, k) = \Theta(\Lambda - |k + k_F|)\). We assume in this paper that the absolute value of the bare coupling constant \(g\) is small. It is important to bear in mind the fact that the expression of \(S_0\), since it contains the linearized form of the dispersion relation, is valid for small values of \(\Lambda\) only; the limit \(\Lambda \to \infty\) is therefore a purely formal limit in the context of the Luttinger model. We use here the method of the equation of motion to compute the expressions of the vertex functions needed in the renormalization group study. Although these expressions could be obtained by a straightforward application of the Feynman rules, we prefer to follow the more compact computational scheme offered by the chosen method because it shows in a more transparent manner the effect of the regularization on the dynamics of the model.

Let us first recall that the generating functional \(W[\eta_{\pm}, \eta_{\pm}]\) for the connected 2n-point Green functions \(G^{(2n)}(n = 1, 2, \ldots)\) of the model is defined by

\[
e^{-W} = \int D[\psi_-^*]D[\psi_-^*]D[\psi_-^*]e^{-S_{\text{int}}(\Lambda)-S_{\text{src}}},
\]

with

\[
S_{\text{src}} = \int_{\mathbf{k}_{\alpha} = \pm} [\eta_{\alpha}^*(\mathbf{k})\psi_{\alpha}(\mathbf{k}) + \psi_{\alpha}^*(\mathbf{k})\eta_{\alpha}(\mathbf{k})].
\]

One has then \((\alpha_i = \pm, i = 1, \ldots, n)\)

\[
G^{(2n)}_{\alpha_1,\ldots,\alpha_n}(p_1, \ldots, p_n, p_1', \ldots, p_n') = (2\pi)^{4n-2}\frac{\delta^{2n}W}{\delta \eta_{\alpha_1}^*(p_1)\cdots \delta \eta_{\alpha_n}^*(p_n)\delta \eta_{\alpha_1}(p_1')\cdots \delta \eta_{\alpha_n}(p_n')}.
\]

Each Green function is regularized by the product of the cutoff functions corresponding to the external and internal lines. However, since all external momenta belong necessarily to the intervals \([\pm k_F - \Lambda, \pm k_F + \Lambda]\), the cutoff functions corresponding to the external lines can be safely ignored. Let us also recall that the Legendre transform \(\Gamma[\psi_{\pm}, \psi_{\pm}]\) of \(W[\eta_{\pm}, \eta_{\pm}]\) is the generating functional for the 2n-point vertex functions \(\Gamma^{(2n)}(n = 1, 2, \ldots)\) of the model. Indeed, one has \((\alpha_i = \pm, i = 1, \ldots, n)\)

\[
\Gamma^{(2n)}_{\alpha_1,\ldots,\alpha_n}(p_1, \ldots, p_n, p_1', \ldots, p_n') = (2\pi)^{4n-2}\frac{\delta^{2n}\Gamma}{\delta \psi_{\alpha_1}^*(p_1)\cdots \delta \psi_{\alpha_n}^*(p_n)\delta \psi_{\alpha_1}(p_1')\cdots \delta \psi_{\alpha_n}(p_n')}.
\]

The two effective coupling constants considered in this paper are those related to the four-point vertex functions \(\Gamma^{(4)}_{++}\) and \(\Gamma^{(4)}_{++}\), whose expressions are computed below in the one-loop approximation.

Infinitesimal translations \(\psi_\alpha \to \psi_\alpha + \epsilon (\alpha = \pm)\) of the integration variables \(\psi_\alpha\) in Eq. \[3\] lead to the two regularized equations of motion of the model, one for each species of particles. These two equations may be combined into a single one, namely,

\[
0 = \left\{ (2\pi)^2p_\alpha \frac{\delta}{\delta \eta_{\alpha}(\mathbf{p})} + \eta_{\alpha}^*(\mathbf{p}) \right\} + g(2\pi)^6\int_{\mathbf{k},\mathbf{p},\mathbf{q}} \rho_+(\Lambda, \mathbf{k} + \mathbf{q})\rho_-(\Lambda, \mathbf{k})\frac{\delta^3}{\delta \eta_{\alpha}^*(\mathbf{p})\delta \eta_{\alpha}^*(\mathbf{k})\delta \eta_{\alpha}(\mathbf{p} + \mathbf{q})\delta \eta_{\alpha}^*(\mathbf{k})} \right\} e^{-W}.
\]

If we let the functional derivative \(\delta/\delta \eta_{\alpha}^*(\mathbf{p}')\) act on the regularized equation of motion corresponding to the case \(\alpha = +\), we obtain the following regularized Dyson-Schwinger equation:

\[
0 = p_+G^{(2)}_{++}(\mathbf{p}', \mathbf{p}) + \delta(\mathbf{p} - \mathbf{p}') + g\int_{\mathbf{k},\mathbf{q}} \left[ G^{(4)}_{++}(\mathbf{p}', \mathbf{k}, \mathbf{p} + \mathbf{q}, \mathbf{k} - \mathbf{q}) - (2\pi)^2G^{(2)}(\mathbf{k}, \mathbf{k} - \mathbf{q})G^{(2)}_{++}(\mathbf{p}', \mathbf{p} + \mathbf{q}) \right] \rho_+(\Lambda, \mathbf{p} + \mathbf{q})\rho_-(\Lambda, \mathbf{k} - \mathbf{q})\rho_-(\Lambda, \mathbf{k}).
\]
The action of well-chosen third-order functional derivatives on the regularized equation of motion corresponding to the case $\alpha = +$, Eq. 7, leads to the identities needed to compute the one-loop expressions of the vertex functions $\Gamma^{(4)}_{++}$ and $\Gamma^{(4)}_{+-}$. If we let the functional derivative $\delta^3/\delta \eta^*_+(p') \delta \eta_-(r) \delta \eta^*_+(r')$ act on this equation of motion and then use the regularized Dyson-Schwinger equation, Eq. 5, we obtain the following identity:

$$0 = p_+ G^{(4)}_{+-}(r', p', r, p) - g \int_q \left[ G^{(6)}_{+-}(r', k, p', r, k - q, p + q) + (2\pi)^2 \left( G^{(2)}_{+-}(p', p + q) G^{(4)}_{+-}(r', k, r, k - q) - G^{(2)}_{+-}(k, k - q) G^{(4)}_{+-}(r', p', r, q + p) \right) \right] \rho_+(\Lambda, p + q) \rho_-(\Lambda, k - q) \rho_-(\Lambda, k).$$

The one-loop expression of $\Gamma^{(4)}_{+-}$ is obtained from this identity in the following way. First, we express each connected four- or six-point Green function in Eq. 9 in terms of its associated vertex function. Next, we replace each vertex function $\Gamma^{(4)}_{+-}(p, r, p', r')$ that has appeared in the integrand of the equation by its expression at leading order in $g$, namely,

$$\Gamma^{(4)}_{+-}(p, r, p', r') = g \delta(p + r - p' - r') (\delta_{\alpha_1, \alpha_2} - \delta_{\alpha_1, \alpha_2, -} + \delta_{\alpha_1, - \alpha_2, +} + \mathcal{O}(g^2))$$

(10)

The one-loop expression of $\Gamma^{(4)}_{+-}$ is obtained by means of the relation $\Gamma^{(4)}_{+-}(p, r, p', r') = \delta(p + r - p' - r') \Gamma^{(4)}_{+-}(p, r, p')$. If we let the functional derivative $\delta^3/\delta \eta^*_+(r') \delta \eta_+(r) \delta \eta^*_+(p')$ act on the regularized equation of motion corresponding to the case $\alpha = +$, Eq. 7, and then use the regularized Dyson-Schwinger equation, Eq. 5, we obtain another identity which is the following:

$$0 = p_+ G^{(4)}_{+-}(r', p', r, p) - g \int_q \left[ G^{(6)}_{+-}(r', k, p', r, k - q, p + q) + (2\pi)^2 \left( G^{(2)}_{+-}(p', p + q) G^{(4)}_{+-}(r', k, r, k - q) - G^{(2)}_{+-}(k, k - q) G^{(4)}_{+-}(r', p', r, q + p) \right) \right] \rho_+(\Lambda, p + q) \rho_-(\Lambda, k - q) \rho_-(\Lambda, k).$$

(13)

The one-loop expression of $\Gamma^{(4)}_{++}$ is obtained from this identity by following the same procedure as for $\Gamma^{(4)}_{+-}$. We find that

$$\Gamma^{(4)}_{++}(p, r, p') = g^2 \int_q \left[ \frac{\rho_-(\Lambda, q) \rho_-(\Lambda, q + r - p')}{q_- (q + r - p')_-} - \frac{\rho_-(\Lambda, q) \rho_+(\Lambda, q + p - p')}{q_+ (q + p - p')_+} \right] + \mathcal{O}(g^3),$$

(14)

where $\Gamma^{(4)}_{++}(p, r, p')$ is introduced by means of the relation $\Gamma^{(4)}_{++}(p, r, p', r') = \delta(p + r - p' - r') \Gamma^{(4)}_{++}(p, r, p')$.

**RUNNING COUPLING CONSTANTS**

For later convenience, the variables of the vertex functions $\Gamma^{(4)}_{+\sigma}(\sigma = \pm)$ are parametrized as

$$k = (-iv_F(k + \Delta_k), k_F + k), \quad l = (-iv_F(l + \Delta_l), \sigma(k_F + l)), \quad k' = (-iv_F(k' + \Delta_{k'}), k_F + k'), \quad l' = (-iv_F(l' + \Delta_{l'}), \sigma(k_F + l')),$$

(15)
where the momentum variables $\Delta_k$, $\Delta_l$, $\Delta_{k'}$, and $\Delta_{l'}$ measure the distance from the mass shell. Conservation of energy imposes the condition $k + \Delta_k + l + \Delta_l = k' + \Delta_{k'} + l' + \Delta_{l'}$ and conservation of momentum the condition $k + \sigma l = k' + \sigma l'$. The combination of these two conditions leads to the additional constraint $\Delta_k + \Delta_l = \Delta_{k'} + \Delta_{l'}$ in the case of the vertex function $\Gamma_{++}^{(4)}$. The mass shell expression of $\Gamma_{++}^{(4)}$ depends on two momenta only (say, $k$ and $l$) because the combination of the two conservation conditions with the condition $\Delta_k = \Delta_l = \Delta_{k'} = \Delta_{l'} = 0$ gives the additional constraints $k = k'$ and $l = l'$ in the case of this vertex function. Performing the integrations in Eqs. (12) and (14), we find that

$$
\Gamma_{++}^{(4)}(k, l, k') = g + \frac{g^2}{4\pi^2 F^2} f_{+-}^{(4)}(k, l, k') + \mathcal{O}(g^3),
$$

$$
\Gamma_{++}^{(4)}(k, l, k') = \frac{g^2}{4\pi^2 F^2} f_{++}^{(4)}(k, l, k') + \mathcal{O}(g^3),
$$

(16)

where

$$
f_{+-}^{(4)}(k, l, k') = \ln \left[ \frac{(k' + l)^2 - (k' - l + \Delta_{k'} - \Delta_l)^2}{(k - l)^2 - (k + l + \Delta_k + \Delta_l)^2} \right] + \ln \left[ \frac{(2\Lambda - |k - l|)^2 - (k + l + \Delta_k + \Delta_l)^2}{(2\Lambda - |k' - l|)^2 - (k' - l + \Delta_{k'} - \Delta_l)^2} \right],
$$

$$
f_{++}^{(4)}(k, l, k') = \frac{k - k'}{k - k' + \frac{1}{2} (\Delta_k - \Delta_{k'})} - \frac{k' - l}{k' - l + \frac{1}{2} (\Delta_{k'} - \Delta_l)}. \tag{17}
$$

The renormalization group method allows us, starting from the truncated expressions of the vertex functions, Eqs. (10), (11), to compute the expressions of the running coupling constants we are interested in. This method is usually implemented in three steps which, in the present context, are the following.

(i) In a first step, we introduce two effective coupling constants $\tilde{g}_{+-}$ and $\tilde{g}_{++}$ by choosing a subtraction point $(\tilde{k}, \tilde{l}, \tilde{k'})$ in the six-dimensional energy-momentum space and setting

$$
\tilde{g}_{+-} = \text{Re} \Gamma_{++}^{(4)}(\tilde{k}, \tilde{l}, \tilde{k'}),
$$

$$
\tilde{g}_{++} = \Gamma_{++}^{(4)}(\tilde{k}, \tilde{l}, \tilde{k'}). \tag{18}
$$

The subtraction point is chosen in such a manner that the absolute value of each effective coupling constant is small. It has to be noticed that if $\tilde{g}_{+-}$ is already present in the bare action, $\tilde{g}_{++}$ is fully generated by the dynamics.

(ii) Next, we choose a scheme, a trajectory in the energy-momentum space which can be parametrized by a variable, denoted by $\lambda$ in the sequel. The trajectory crosses the subtraction point for $\lambda = \tilde{\lambda}$. The trajectory usually chosen is the one associated with an homogeneous rescaling of the variables $k$, $\Delta_k$, etc., in which case $\lambda$ controls the scale dependence.

(iii) The values of the effective coupling constants $g_{\sigma}(\lambda)$ ($\sigma = \pm$) corresponding to an arbitrary point of the trajectory, that is, to a given value of $\lambda$, are then computed by integrating the beta functions along this trajectory from the subtraction point up to the considered point. The beta functions $\beta_{\sigma}(\lambda)$ give the rate of variation of the effective coupling constants as $\lambda$ increases, namely,

$$
\beta_{\sigma}(\lambda) = \lambda \frac{dg_{\sigma}(\lambda)}{d\lambda} = \frac{g_{\sigma}^2(\lambda)}{4\pi^2 F^2} \lambda \frac{d^2g_{\sigma}(\lambda)}{d\lambda^2} + \mathcal{O}(g_{\sigma}^3). \tag{19}
$$

The integration of these quantities leads to the following expression of the running coupling constants:

$$
g_{\sigma}(\lambda) = \frac{\tilde{g}_{\sigma}(\lambda)}{1 + \frac{\tilde{g}_{\sigma}(\lambda)}{4\pi^2 F^2} [f_{\sigma}(\lambda) - f_{\sigma}(\lambda)]} \tag{20}
$$

with $\tilde{g}_{\sigma} = g_{\sigma}(\tilde{\lambda})$. This expression, which gives at once the values of the effective coupling constants at any point of the trajectory, may be viewed as resulting from an optimized partial resummation of the full perturbation series of the vertex functions.

Notice that the particular trajectory which defines the scheme does not appear in the expression of Eq. (20). This circumstance allows us to generalize this expression to an arbitrary scheme, namely,

$$
g_{\sigma}(k, l, k') = \frac{\tilde{g}_{\sigma}(\lambda)}{1 + \frac{\tilde{g}_{\sigma}(\lambda)}{4\pi^2 F^2} [f_{\sigma}(k, l, k') - f_{\sigma}(k, l, k')]}. \tag{21}
$$
This expression offers the advantage that the effective coupling constants depend only on the coordinates \((k, l, k')\) of the point at which they are computed. Its derivation is based upon the fact that the rate of change of the renormalized coupling constants caused by an infinitesimal step in the energy-momentum space has a component along each direction of this space. The expression of such a component is

\[
\beta_{\eta, \sigma} (\lambda) = g \frac{\partial g_{\eta, \sigma} (q)}{\partial q} = \frac{g^2_{\eta, \sigma} (q)}{4 \pi v_F} \frac{\partial f_{\eta, \sigma} (q)}{\partial q} + \mathcal{O} \left( g^3_{\eta, \sigma} \right),
\]

(22)

where the symbol \(q\) represents a component of \((k, l, k', l')\). The integration of these “directional” beta functions along the chosen trajectory leads to the expression of Eq. (21), which, interestingly enough, may also be viewed as originating from an optimized partial resummation of the Bethe-Salpeter equation.

A mass shell expression of \(g_{\eta, \sigma}^{p-p}\) which is valid in the particle-particle \((p-p)\) channel is obtained by using Eq. (20) in the case of the linear trajectory \(k (\lambda) = l (\lambda) = \lambda \Lambda \ (|\lambda| < 1)\). We find that

\[
g_{\eta, \sigma}^{p-p} (\lambda) = \frac{g_{\eta, \sigma}^{p-p} (\tilde{\lambda})}{1 + \frac{g_{\eta, \sigma}^{p-p} (\lambda)}{4 \pi v_F} \left[ \ln \left( \frac{1 + |\lambda|}{1 - |\lambda|} \right) - \ln \left( \frac{1 + |\lambda|}{1 - |\lambda|} \right) \right]}.
\]

(23)

This equation clearly diverges and the Landau pole is found at

\[
\lambda_{L}^{p-p} = \frac{1 + \tilde{\lambda} - (1 - \tilde{\lambda}) e^{- \frac{4 \pi v_F}{g_{\eta, \sigma}^{p-p} (\lambda)}}}{1 + \tilde{\lambda} + (1 - \tilde{\lambda}) e^{- \frac{4 \pi v_F}{g_{\eta, \sigma}^{p-p} (\lambda)}}}.
\]

(24)

The effective interaction between the particles has obviously not the same sign in the particle-hole channel as in the particle-particle channel. It is easy to verify that the function \(f_{\eta, \sigma}^{p-h}\), the one used to define the effective coupling constant \(\tilde{g}_{\eta, \sigma}\), Eq. (18), changes sign if the momentum \(l\) is transformed into \(-l\); a mass shell expression of \(g_{\sigma, \eta}^{p-h}\) which is valid in the particle-hole \((p-h)\) channel can therefore be obtained by using Eq. (20) in the case of the linear trajectory \(k (\lambda) = -l (\lambda) = \lambda \Lambda \ (|\lambda| < 1)\). The result is

\[
g_{\eta, \sigma}^{p-h} (\lambda) = \frac{g_{\eta, \sigma}^{p-h} (\tilde{\lambda})}{1 - \frac{g_{\eta, \sigma}^{p-h} (\lambda)}{4 \pi v_F} \left[ \ln \left( \frac{1 - (1 + |\lambda| - \lambda)}{1 - (1 + |\lambda| - \lambda)} \right) - \ln \left( \frac{1 + (1 + |\lambda| - \lambda)}{1 - (1 + |\lambda| - \lambda)} \right) \right]}.
\]

(25)

This equation diverges too and the Landau pole is at

\[
\lambda_{L}^{p-h} = \frac{1 + \tilde{\lambda} - (1 - \tilde{\lambda}) e^{\frac{4 \pi v_F}{g_{\eta, \sigma}^{p-h} (\lambda)}}}{1 + \tilde{\lambda} + (1 - \tilde{\lambda}) e^{\frac{4 \pi v_F}{g_{\eta, \sigma}^{p-h} (\lambda)}}}.
\]

(26)

We use now Eq. (21) to obtain \(g_{\sigma, \eta}^{p-h}\) on the mass shell,

\[
g_{\sigma, \eta}^{p-h} (k, l) = \frac{\tilde{g}_{\sigma, \eta}}{1 + \frac{\tilde{g}_{\sigma, \eta}}{4 \pi v_F} \left[ \ln \left( \frac{(2 \Lambda - |k - l|)^2 - (k + l)^2}{(2 \Lambda - |k + l|)^2 - (k - l)^2} \right) - \ln \left( \frac{(2 \Lambda - |k - l|)^2 - (k + l)^2}{(2 \Lambda - |k + l|)^2 - (k - l)^2} \right) \right]}.
\]

(27)

Notice that the first term on the right-hand side of the first equation in Eq. (17) is vanishing on the mass shell. As a result, the scale dependence, i.e. the deviation from a scale invariant result, arises from the presence of the cutoff \(\Lambda\) only. The Landau poles corresponding to the divergences of this expression belong to the line of the plane \((k, l)\),

\[
\frac{4 \pi v_F}{\tilde{g}_{\sigma, \eta}} = \ln \left( \frac{(2 \Lambda - |k - l|)^2 - (k + l)^2}{(2 \Lambda - |k + l|)^2 - (k - l)^2} \right) - \ln \left( \frac{(2 \Lambda - |k - l|)^2 - (k + l)^2}{(2 \Lambda - |k + l|)^2 - (k - l)^2} \right). \tag{28}
\]

This line stays away from the Fermi points for any generic choice of the bare coupling constant, that is, in the absence of fine tuning. The behaviour of \(g_{\sigma, \eta}^{p-h}\) that is valid slightly away from the mass shell is similar to the one of \(g_{\eta, \sigma}^{p-h}\) on the mass shell and the corresponding equations can be obtained from Eqs. (28) by adding terms of the order \(\mathcal{O} (\Delta)\) on the right-hand sides.
The coupling strength \( g_{++} \) is always vanishing at the order of one loop on mass shell. Using Eq. (21) again, we obtain
\[
g_{++} = \frac{\tilde{g}_{++}}{1 + \frac{g_{++}}{4\pi^2} \left[ \frac{k-k'}{k-k'+\frac{1}{2}(\Delta_k-\Delta_{k'})} - \frac{k'-l}{k'-l+\frac{1}{2}(\Delta_{k'}-\Delta_l)} - \frac{k-\tilde{k}'}{k-\tilde{k}'+\frac{1}{2}(\Delta_k-\Delta_{k'})} + \frac{k'-l}{k'-l+\frac{1}{2}(\Delta_{k'}-\Delta_l)} \right]} \tag{29}
\]
off-shell. The Landau poles are now spread over a five-dimensional hypersurface whose implicit equation is
\[
\frac{4\pi^2}{\tilde{g}_{++}} = \frac{k-k'}{k-k'+\frac{1}{2}(\Delta_k-\Delta_{k'})} - \frac{k'-l}{k'-l+\frac{1}{2}(\Delta_{k'}-\Delta_l)} - \frac{k-\tilde{k}'}{k-\tilde{k}'+\frac{1}{2}(\Delta_k-\Delta_{k'})} + \frac{k'-l}{k'-l+\frac{1}{2}(\Delta_{k'}-\Delta_l)}. \tag{30}
\]
This hypersurface approaches the Fermi point of the right-moving particles in the limit \( \Delta_k, \Delta_l, \) and \( \Delta_{k'} \to 0 \) for any value of \( \tilde{g}_{++} \).

Several comments are in order at this point. We start with the scale dependence of the coupling constants \( g_{+-} \) and \( g_{++} \). The latter is scale invariant according to Eq. (29) and its beta function is vanishing in the conventional schemes where the variables \( k, \Delta_k, \) etc. are scaled homogeneously. The scale invariance of \( g_{+-} \) is broken by the cutoff, as shown by Eq. (21). The limit \( \Lambda \to \infty \) carried out in a given finite order of the perturbation expansion suppresses this breakdown and one would naively expect that the renormalized theory recovers the classical scale invariance. But the partial infinite resummation of the perturbation expansion, carried out by the generalized renormalization group scheme, indicates that the interactions become strong within certain kinematical regions because of the accumulation of large contributions of higher orders in the perturbation expansion. These regions are characterized by the scale of the cutoff. This prevents us from removing the cutoff and restoring the naive, classical scale invariance at the same time.

Another remark is about the nature of the singularities of the effective coupling strengths. The divergence in Eqs. (21) and (29) does not necessarily mean that the theory becomes strongly coupled. In fact, the effective interaction strengths should be defined by scattering amplitudes or other physical observables which reflect the interactions among elementary or dressed excitations. Such a construction always involves a finite resolution in the energy-momentum space which can be imagined as coming from a smearing of some Green functions in the kinematical space. As long as the singularities are integrable, as is the case of the dimensionless running coupling strength \( \frac{1}{\tilde{g}_{++}} \) in Eq. (21), such a smeared coupling strength remains finite despite the diverging numerical values the effective coupling constant may take in the vicinity of the Landau poles. But the non-integrable singularity of \( g_{++} \) appearing in Eq. (29) produces a diverging coupling strength for an arbitrary localisation of the constituents in the energy-momentum space.

We close this Section by pointing out the way the apparent contradiction between our results and the well-known vanishing of the usual beta function can be resolved.

(i) The beta function computed in Refs. [3, 8] is based upon a particular choice of the subtraction point, namely \( p_0 = r_0 = \tilde{p}_0 = \tilde{r}_0 = 0, (\Delta_k = -k, \Delta_l = -l, \Delta_{k'} = -k', \Delta_{l'} = -l') \) and \( k = -l = -k' = -l'/3 \). The special feature of this point is that the radiative corrections \( f_{+-} \) and \( f_{++} \), given by Eqs. (17), are vanishing. This subtraction point together with the usual, homogeneous scaling implies vanishing beta functions. But the generalized scheme used in this work picks up the radiative corrections arising from an arbitrary displacement in the kinematical space and reveals the non-naturalness of this cancellation.

(ii) Another argument, the one used in Ref. [7], rests on the study of the behaviour of the bare coupling constant in the blocking procedure. The bare coupling constant was defined in that work at the Fermi point and the possible energy-momentum dependence was treated perturbatively. The point is that the leading order result, the \( O(g^2) \) coupling constant at the Fermi point, agrees with Eq. (12). The singular energy-momentum dependence generated by the partial resummation in Eqs. (21) and (29) suggests that the dependence of the coupling constant on the energy-momentum can not be taken into account by expanding around the Fermi point and the simplicity of the scenario spelled out in Ref. [7] appears misleading.

(iii) A more formal argument has also been used to justify the triviality of the scaling laws. This argument is based upon the (supposed) equivalence between the behaviour of the running coupling constant in two different limits, the one where the cutoff is removed and the one where each energy-momentum variable \( k \) tends towards zero. Indeed, since these variables and \( \Lambda \) are the only scales (\( k_F \) is absent from the beta functions), the dimensionless running coupling constants should not depend on them separately but rather on the ratios \( k/\Lambda \). Consequently, the limit \( k \to 0 \) is supposed to coincide with the limit \( \Lambda \to \infty \) where the scale dependence drops, as one can see by inspecting Eqs. (17). The fallacy of this argument is usually shown by recalling the phenomenon of dimension transmutation [12]. This phenomenon is found in asymptotically free, classically scale invariant models where the cutoff generates a scale which remains finite even in a renormalized theory where the cutoff is already removed. The Luttinger model offers a simpler mechanism to break the scale invariance by its Landau poles, as pointed out after Eq. (30) above.
SUMMARY AND CONCLUSION

Our aim in this paper was to establish the existence of long range forces in the Luttinger liquid in a reliable manner. This has been done with the help of a renormalization group study of two effective coupling constants of the model. The computations have been done in the one-loop order. We have shown that the running coupling constants have Landau poles in the vicinity of the Fermi points for arbitrarily weak bare interactions. The singular momentum dependence of the effective coupling constants in the vicinity of the Fermi points shows that the interactions between the fermionic excitations are of long range.

The problem of the breakdown of the perturbation expansion in the vicinity of the Fermi points is difficult to solve. It is known that the correspondence between the fermionic and bosonic versions of the model can be established at any finite order of the perturbation expansion in powers of the particle mass or interaction strength. However, since the strong interactions close to the Fermi points modify the single particle dispersion relation in a fundamental manner, the perturbative approach may not be sufficient to prove this correspondence. We believe that a more systematic study of the scaling laws established in both versions of the theory is needed to clarify this question.

The partial resummation, performed by the integration of the beta functions, remains consistent as long as the effective coupling strengths are weak. When the one-loop beta functions predict large values for the coupling constants, the computation is no longer reliable and one usually assumes that the true dynamics is indeed strongly coupled in terms of the degrees of freedom used. A similar problem, namely the one of the soft gluons, blocks the way towards the analytical understanding of Hadronic Physics. We believe that any progress towards more definite conclusions about the dynamics of the Luttinger model requires some non-perturbative method in order to treat the strong, long range interactions, just as in the case of QCD.

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