Motivic $L$-Function Identities from CFT and Arithmetic Mirror Symmetry

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Abstract.

Exactly solvable mirror pairs of Calabi-Yau threefolds of hypersurface type exist in the class of Gepner models that include nondiagonal affine invariants. Motivated by the string theoretic automorphy established previously for models in this class it is natural to ask whether the arithmetic structure of mirror pairs varieties reflects the fact that as conformal field theories they are isomorphic. Mirror symmetry in particular predicts that the $L$-functions of the $\Omega$-motive of such pairs are identical. In the present paper this prediction is confirmed by showing that the $\Omega$-motives of exactly solvable mirror pairs are isomorphic. This follows as a corollary of the proof of a more general result establishing an isomorphism between nondiagonally and diagonally induced motives in this class of varieties. The motivic approach formulated here circumvents the difficulty that no mirror construction of the Hasse-Weil zeta function is known.

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1 Introduction

The purpose of this paper is to establish an isomorphism between motives associated to certain types of hypersurfaces that arise in the context of exactly solvable string compactifications. The motivation for this result comes from two independent, but related, programs. The first is to understand the relation between the conformal field theory $T_{\Sigma}$ on the string worldsheet $\Sigma$ and the structure of Calabi-Yau varieties $X$ by relating automorphic forms associated to motives $M(X)$ to automorphic forms derived from Kac-Moody algebras. The second aims at an understanding of mirror symmetry by using methods from arithmetic geometry.

Arithmetic mirror symmetry is an old issue that is made difficult by the fact that the mirror transformation exchanges even and odd cohomology of the variety. This implies that the basic arithmetic object associated to the variety $X$, given by the local zeta functions $Z(X/\mathbb{F}_p, t)$
defined for the finite fields $\mathbb{F}_p$ for any prime $p \in \mathbb{N}$, does not respect mirror symmetry, a fact that follows from the factorization of the zeta function into cohomological pieces, envisioned by Weil [1] and proven by Grothendieck [2]. Hence the arithmetic approach appears to suffer from the usual problems of understanding mirror symmetry in the geometric setting (see e.g. [3] for a more detailed discussion of this issue).

Guidance in this problem is provided by results which establish that the $L$-functions of certain varieties are identical to $L$-functions determined by modular forms on the string worldsheet (see e.g. [4] and references therein). Such an identity implies that the mirror $L$-function must be identical to the original one because the underlying conformal field theories of the varieties are isomorphic, hence the modular forms on the worldsheet are identical. This prediction of mirror symmetry was confirmed in the context of models with central charge $c = 3$ in ref. [5] and for some rigid mirror pairs in [6]. The advantage of elliptic curves is that their $L$-functions can be obtained directly by counting the points of $X/\mathbb{F}_p$ on the variety, while the cases of rigid mirrors only involved weighted Fermat hypersurfaces. In the more general case the extension of this test is made more difficult because it is necessary to have a definition of motives for nondiagonal varieties. Such a construction has recently been given in [7] and this result will be applied here in the context of exactly solvable string models to establish an isomorphism of their $\Omega$-motives, and to discuss its implications, in particular in the context of mirror symmetry. This isomorphism in particular implies the identity of the $L$-functions of the $\Omega$-motives of mirror pairs even though the Hasse-Weil zeta function is not invariant under mirror symmetry.

The outline of this article is as follows. Section 2 introduces the map from diagonal to nondiagonal varieties of $D$-type. Section 3 defines the relevant motives and proves the motivic isomorphism. Section 4 considers applications of the isomorphism in the context of the automorphic spacetime program and mirror symmetry, while the final section briefly discusses a second map from diagonal to nondiagonal manifolds.
2 The $AD$-space of hypersurfaces

The focus in the following is on certain types of hypersurfaces $X_n$ of dimension $n$ embedded in weighted projective spaces $\mathbb{P}(w_0, \ldots, w_{n+1})$ with weights $w_i \in \mathbb{N}$. The specific class of interest is generated by monomial and nondiagonal binomial building blocks given by

$$p_A^{(i)}(z_i) = z_i^{d_i},$$
$$p_D^{(i)}(z_i, z_{i+1}) = z_i^{d_i} + z_i z_{i+1}^{d_{i+1}}. \quad (1)$$

The space of polynomials generated by $p_A$ and $p_D$ over a field $K$ then consists of varieties of the type

$$X_n = \left\{ \sum_i \alpha_i p_A^{(i)} + \sum_j \beta_j p_D^{(j)} = 0 \mid \alpha_i, \beta_j \in K \right\} \subset \mathbb{P}(w_0, \ldots, w_{n+1}), \quad (2)$$

with $n = n(i, j)$ depending on the range of $i$ and $j$, and is denoted by

$$H_K = \langle p_A^{(i)}, p_D^{(i)} \rangle_K. \quad (3)$$

On the space $H_K$ one can define a map that increases the number of links by replacing monomials by nondiagonal binomials. More precisely, consider any hypersurface $X$ defined by a polynomial containing at least one purely monomial term and a "trivial factor", i.e. it is of type

$$p_A(z_i, z_{i+1}) := z_i^a + z_i^{2}, \quad a \in 2\mathbb{N}, \quad (4)$$

and denote by $X_D$ the variety obtain by replacing $p_A$ by the nondiagonal factor

$$p_D(z_i, z_{i+1}) := z_i^{a/2} + z_i z_{i+1}^2. \quad (5)$$

The replacement $p_A(z_i, z_{i+1}) \mapsto p_D(z_i, z_{i+1})$ induces a map on the space of all CY hypersurfaces

$$s_D : H_K \longrightarrow H_K, \quad (6)$$

where $s_D$ is extended to the whole space $H_K$ by defining it to be the identity on those polynomials that do not contain a summand of the appropriate form.
3  \( A \)- and \( D \)-motives

The image \( s_D(X) \) of a variety \( X \) is in general topologically different from \( X \), indicated e.g. by their different Hodge numbers. The fact that the \( L \)-function of the \( \Omega \)-motive of diagonal varieties has been shown in several instances to be determined by the modular forms defined by the conformal field theory \( T_\Sigma \) on the worldsheet \( \Sigma \) raises the question whether the \( \Omega \)-motive of both varieties is the same even though the manifolds are topologically distinct. The purpose of this section is to establish such an isomorphism.

3.1 Diagonal and nondiagonal \( \Omega \)-motives

The notion of the \( \Omega \)-motive is universal, applicable to all Calabi-Yau varieties and Fano varieties of special type [4]. For the hypersurfaces considered in the present paper this general construction can be made explicit and computable. For diagonal varieties this has been described in the above reference. For nondiagonal varieties the construction of the \( \Omega \)-motives needs to be generalized. This is briefly reviewed below and more details can be found in [7].

Denote by \( X^d_n \) any diagonal hypersurface of type \( A \) or \( D \) of complex dimension \( n \) and degree \( d \) embedded in a weighted projective space. The polynomial defining the hypersurface contains monomials of the variables \( z_i \) occurring with degrees \( d_i \). For hypersurfaces of tadpole type defined by polynomials of the type \( p = \sum_{i=0}^{n} z_i^{d_i} + z_n z_n^{d_n + 1} \) define the integer \( v = \text{lcm}\{d_i\}_{i \neq n} \in \mathbb{N} \) and consider the \( v \)th root of unity \( \xi_v = e^{2\pi i/v} \) and the cyclic group \( \mu_v = \langle \xi_v \rangle \) it generates.

Associate to \( X = X^d_n \) the abelian number field \( K_X = \mathbb{Q}(\mu_v) \) and denote its Galois group by \( \text{Gal}(K_X/\mathbb{Q}) \). The cohomological realization \( H(M_\Omega) \) of the \( \Omega \) motive \( M_\Omega \) now is defined as the Galois orbit defined by the action of \( \text{Gal}(K_X/\mathbb{Q}) \) on the holomorphic \( n \)-form [7]. For diagonal varieties the integer \( v \) is given by the degree of the hypersurface.

For varieties of the type considered here the definition just given can be made more explicit because the cohomology class \( \Omega \) can be represented as a unit vector \( u_\Omega = (1, ..., 1) \in \mathbb{Z}^{n+2} \) and the action of the Galois group on \( \Omega \) can be expressed as the action of \( \text{Gal}(K_X/\mathbb{Q}) \cong (\mathbb{Z}/v\mathbb{Z})^\times \) on this vector. The actions differ for the diagonal and the nondiagonal cases. For
diagonal hypersurfaces the action of $\sigma_r \in \text{Gal}(K_d/\mathbb{Q})$ is given by $\sigma_r(u^i_\Omega) \equiv ru^i_\Omega \pmod{d_i}$ for all $i = 0, \ldots, n + 1$, while in the nondiagonal case the action of $\sigma_r \in \text{Gal}(K_v/\mathbb{Q})$ is given by a combination of an analogous modding relation in combination with the fact that the resulting images $u_r$ of $u_\Omega$ under $\sigma_r$ have to satisfy the constraint $d|\sum_i u^i_r w_i$ \cite{7}. With these definitions the cohomological realization of the $\Omega$-motive $M_\Omega$ of a hypersurface can be written as

$$H(M_\Omega) = \langle \text{Gal}(K_v(X)/\mathbb{Q}), \Omega \rangle$$

where $\Omega$ is represented by $u_\Omega$.

### 3.2 The motivic AD-isomorphism

Let $X_D$ be a variety generated by the action of $s_D$ on a variety $X$. Then the $\Omega$-motive $M_\Omega(X)$ of $X$ is isomorphic to the $\Omega$-motive $M_\Omega(X_D)$ of $X_D = s_D(X)$. As a consequence of this motivic AD-isomorphism the $L$-function of the $\Omega$-motive of $X_D$ is the same as the $L$-function of the $\Omega$-motive of $X$

$$L_\Omega(X_D, s) = L_\Omega(X, s).$$

This result will be derived in the remainder of the section from the structure of the underlying $N = 2$ superconformal field theory. Applications to motives of Calabi-Yau varieties and to mirror symmetry will be indicated in the following section.

### 3.3 Ingredients of the proof

The basic idea is to argue conformal field theoretically. The proof given below uses the fact that the cohomology of certain types of Calabi-Yau hypersurfaces is determined by the massless spectrum of tensor products of $N = 2$ superconformal minimal models endowed with a projection that ensure integral U(1) charges \cite{8}.
3.3.1 Exactly solvable tensor products

Minimal models with two supersymmetries are rational conformal field theories constructed from the affine algebra $A^{(1)}_1$ at conformal levels $k$ with central charge

$$c = \frac{3k}{k+2}, \quad k \in \mathbb{N}. \quad (9)$$

Such models are denoted by $k \left( A^{(1)}_1 \right)$, with notation $k_A$ and $K_D$ if the affine invariant needs to be indicated. In order for an $N = 2$ superconformal theory to describe a variety $X$ the central charge has to satisfy the dimension constraint

$$c = 3\text{dim}_\mathbb{C}X, \quad (10)$$

hence minimal models need to be tensored in order to saturate the central charge constraint

$$\bigotimes_{i=0}^{n+1} k_i \left( A^{(1)}_1 \right) : \quad c = \sum_{i=0}^{n+1} \frac{3k_i}{k_i + 2}. \quad (11)$$

There are two ingredients in the proof of the $L$-function relation for varieties associated to a tensor product $\bigotimes_i k_i \left( A^{(1)}_1 \right)$. In order to formulate them it is necessary to review briefly the relevant structures of such theories. In tensor products these fields are obtained from the fields of the individual factors $\varphi = \prod_i \varphi_i$, where $\varphi_i \in \text{Spec}(k_i \left( A^{(1)}_1 \right))$ are fields in the individual factors. Since $N = 2$ superconformal minimal models are rational there are a finite number of fields and the parametrization can be obtained explicitly in terms of quantum numbers given by $(\ell, q, s)$, denoted here by $\varphi^k_{\ell,q,s} \in \text{Spec}(k \left( A^{(1)}_1 \right))$, where the ranges of the quantum numbers $\ell, q, s$ are determined by the level $k$. The conformal weight and U(1) charges of these fields are given by

$$\Delta^k_{\ell,q,s} = \frac{(\ell + 2) - 2q^2}{4(k + 2)} + \frac{s^2}{8},$$

$$Q^k_{\ell,q,s} = -\frac{q}{k + 2} + \frac{s}{2}. \quad (12)$$

with $\ell \in \{0, 1, ..., k\}$, $\ell + q + s \in 2\mathbb{Z}$, and $|q - s| \leq \ell$ [8] and the identification $(\ell, q, s) \cong (k - \ell, q + k + 2, s + 2)$. The fields which correspond to the cohomology of the variety are
determined by the subset of chiral primary fields of the theories, characterized by the condition that their anomalous dimensions $\Delta$ and their $U(1)$ charge $Q$ are related as $Q(\phi) = 2\Delta(\phi)$. Combining these building blocks into marginal operators with specific integrality constraints for the total charge accounts for the cohomology of the critical theory.

### 3.3.2 Modular invariants

The detailed spectrum of the theory depends on the modular invariants $N_{\ell,\overline{\ell}}, M_{q,\overline{q}}, R_{s,\overline{s}}$ that appear in the partition function

$$Z^k = \sum N_{\ell,\overline{\ell}} M_{q,\overline{q}} R_{s,\overline{s}} \chi^k_{\ell,q,s} \chi^k_{\overline{\ell},\overline{q},\overline{s}},$$

where $\chi^k_{\ell,q,s}$ denote the basic characters. Most important in the present context is the modular invariant $N_{\ell,\overline{\ell}}$ associated to the affine part of the $N = 2$ superconformal theory. These invariants have been classified and follow an ADE pattern [9, 10]. It is known in particular that for each even conformal level $k \in 2\mathbb{N}$ the partition function is modular invariant with both the diagonal $A$-invariant as well as the $D$-invariant.

The choice of the modular invariants therefore determines how the left- and right-moving sectors are tensored together. In the following these tensor product fields will be denoted by $\phi_{\ell,q,s}$; neglecting the level index $k$. Here the $(\ell, q, s)$ denotes the left-moving sector, while $(\overline{\ell}, \overline{q}, \overline{s})$ denotes the right-moving sector.

### 3.4 The proof

The basic observation in the proof of the motivic $AD$-isomorphism is that the map $s_D$ is the geometric counterpart of the replacement of the diagonal $A$-invariant in the partition function of a level $k$–minimal model by an affine $D$-invariant

$$A_i^{(1)} : k_A \rightarrow k_D,$$

where the affine invariants of the remaining factors of the tensor product $\otimes_i k_i$ are left unchanged. There are two steps in the proof that the motives are isomorphic, with a resulting
$L$-function identity. These two steps reflect the definition of the $\Omega$-motive outlined above:

1. The first ingredient is that the field $\varphi_\Omega$ corresponding to the holomorphic form $\Omega$ is unchanged in the transition $k_A \mapsto k_D$.

2. The second ingredient is that not only is $\varphi_\Omega$ remain unchanged, but that the Galois group also remains the same.

For nondiagonal hypersurfaces the motive is determined by the orbit of the Galois group of $\mathbb{Q}(\mu_v)$, where $v = \text{lcm}\{d_i\}_{i \neq n}$, while for $A$-motives the rank of the motives is determined by the degree of the variety $d = \text{deg}(X)$.

### 3.4.1 Part I: invariance of the holomorphic form $\Omega$

The basic reason for the invariance of $\varphi_\Omega$ under the map $s_D$ is because in the transition all factors except one remain the same. It follows that the factor $\varphi^{k_A}_{\ell_A, m_A, s_A}$ that is changed must be replaced by a chiral primary field of exactly the same weight and the same charge, which will imply that the field must be invariant under the exchange of the affine invariant.

It is useful to first recall that the $D$-theory is the resolution of the $\mathbb{Z}_2$-quotient of the diagonal model at level $k$

$$k_D = \text{res}(k_A/\mathbb{Z}_2),$$

where $\mathbb{Z}_2$ is short-hand notation for $\mathbb{Z}/2\mathbb{Z}$. This quotient isomorphism has been discussed in detail in [11] in Landau-Ginzburg framework of [12, 13, 14, 15]. The resolution of the quotient theory introduces a single twist field, denoted by $\tilde{\Psi}$, and the two models can described by the superpotentials

$$W_A = \Phi^{k+2} + \Psi^2$$
$$W_D = \Phi^{\frac{k+2}{2}} + \Phi \tilde{\Psi}^2,$$

with a map

$$\Phi = \tilde{\Phi}^{1/2}, \quad \Psi = \tilde{\Phi}^{1/2} \tilde{\Psi},$$

(17)
involving fractional exponents, hence called fractional transform in [11, 16]. The ideal of the 
$D$-theory is generated by
\[ \mathcal{I}_D = \langle \tilde{\Phi}^k, \tilde{\Psi}^2, \tilde{\Phi} \tilde{\Psi} \rangle \] (18)
leads to the spectrum
\[ \text{Spec}(k_D) = \{1, \tilde{\Phi}, ..., \tilde{\Phi}^k, \tilde{\Psi}\}. \] (19)
The weights of the fields in the $D$-model from the superpotential as
\[ \text{wt}(\tilde{\Phi}) = \frac{2}{k+2}, \quad \text{wt}(\tilde{\Psi}) = \frac{k}{2(k+2)}; \] (20)
consistent with the structure of the ideal.

In order to make the CFT theoretic realization of the holomorphic $\Omega$ form explicit in the 
$D$-theory it is useful to recall the structure of the corresponding chiral primary field in the 
underlying conformal field theory and its corresponding Landau-Ginzburg model. It was 
pointed out by Boucher-Friedan-Kent [17] that the chiral primary field $\varphi_\Omega$ corresponding to 
the holomorphic form $\Omega$ in an $N = 2$ superconformal field theory is characterized by its 
anomalous dimension $\Delta$ and U(1)-charge $Q$
\[ (\Delta, Q)(\varphi_\Omega) = \left(\frac{c}{6}, \frac{c}{3}\right), \quad c = 3\dim \mathbb{C} X. \] (21)
This translates into the same condition for the Landau-Ginzburg field $\Phi_\Omega$ corresponding to 
$\varphi_\Omega$. It follows from this and the central charge relation of the tensor product (11) that in the 
diagonal tensor model the field $\varphi_\Omega$ corresponding to the holomorphic $\Omega$-form is given by
\[ \varphi_\Omega = \prod_{i=0}^{n+1} \varphi_{k_i, -k_i, 0}. \] (22)
In the corresponding diagonal Landau-Ginzburg theory, determined by the superpotential
\[ W(\Phi_i) = \sum_i \Phi_i^{k_i+2}, \] (23)
the U(1) charges of the fields in the spectrum are given by
\[ Q(\Phi_i^{k_i}) = \frac{k_i}{k_i + 2}. \] (24)
It follows from this that the field $\Phi_\Omega$ must be constructed from the highest powers of the fields $\Phi_i$ that is consistent with the ideal generated by the Landau-Ginzburg potential, i.e.

$$\Phi_\Omega = \prod_i \Phi_i^{k_i},$$  \hspace{1cm} (25)$$

leading to

$$Q(\Phi_\Omega) = \sum_i \frac{k_i}{k_i + 2} = \frac{c}{3}. $$ \hspace{1cm} (26)$$

The transition $k_A \mapsto k_D$ is only possible for even $k$, therefore the highest power $\Phi^k$ is invariant under the $\mathbb{Z}_2$ that defines the $D$-invariant, hence it survives the quotient.

### 3.4.2 Part II: invariance of the Galois group

It remains to show that while the Galois group generating the motivic orbit is different in general for diagonal and nondiagonal varieties it is unchanged in the exchange of the affine invariants $k_A \mapsto k_D$. This follows by noting that $v_A = \text{lcm}\{a, 2\} = a$ since $a$ must be even: for $k_A$ one has the superpotential $W_A = \Phi^a + \Psi^2$, but the replacement $k_A \mapsto k_D$ can be applied to even $k$. The $D$-model superpotential $W_D = \tilde{\Phi}^{a/2} + \tilde{\Phi} \tilde{\Psi}^2$ contributes to the Galois field $K_v v_D = \text{lcm}\{a/2, 2\} = a$ since in general $a/2$ does not have to be even. Combining these two facts shows that the motive-generating group is in fact the same for both models.

### 4 Applications

#### 4.1 Automorphic motives from Kac-Moody algebras

One of the consequences of string theory is that it suggests a relation between two-dimensional conformal field theory and the geometry of Calabi-Yau varieties. In the arithmetic framework this translates into relations between $L$-functions arise in the context of a string theoretic application of the arithmetic Langlands program in combination with Grothendieck’s theory of motives. The precise formulation of this framework involves the interpretation of automorphic
forms derived from pure or mixed motives that arise in Calabi-Yau varieties to modular forms that come from Kac-Moody algebras. In this context one encounters results like the following.

Let $X_2^{\text{ND}}$ be a K3 surface of $D$-type in the weighted projective space $\mathbb{P}_{(1,1,2,2)}$ and $X_2^{\text{BP}}$ the diagonal K3 surface in the space $\mathbb{P}_{(1,1,1,3)}$. The motivic $AD$-isomorphism then implies that the $L$-functions of the $\Omega$-motives of these two K3 surfaces are identical. Because the latter hypersurface is known to be modular with an $\Omega$-motivic modular form of weight 3 and level 27 [18] it follows that the $\Omega$-motive of the nondiagonal surface is modular as well

$$f_\Omega(X_2^{\text{B,ND}}, q) = f_\Omega(X_2^{\text{A,BP}}, q) = f_{3,27}(q) \in S_3(\Gamma_0(27), e).$$

(27)

More details can be found in [7].

The $\Omega$-motivic modular form $f_{3,27}(q)$ has physical significance because it admits an interpretation in terms of Hecke indefinite modular forms that are associated to Kac-Peterson string functions, which are modular forms of half-integral weight [19]. Such relations therefore link the spacetime geometry to the worldsheet theory $T_{\Sigma}$, providing a direct method to pass from the geometry of the extra dimensions to the worldsheet and vice versa. In this way the automorphic approach provides a concrete realization of an emergent spacetime program in string theory.

### 4.2 Arithmetic mirror symmetry

One of the discoveries triggered by string theory is mirror symmetry [20, 21]. As mentioned in the introduction, from the zeta function point of view mirror symmetry has been puzzling since its rational form treats the even and odd cohomology groups differently. It has thus been of some interest to search for a quantum analog of the zeta function of varieties that would be invariant under mirror symmetry. In the present discussion the focus instead is on the universal motivic $L$-function that exists for any Calabi-Yau variety. In the class of Gepner models the construction of Greene and Plesser describes the mirror theory as a quotient of the Gepner model with respect to a discrete symmetry group. In some cases this group is $\mathbb{Z}_2$ and the fractional transform (17) is a 1-1 transformation that maps the quotient model to hypersurface
Thus there exist pairs of mirror manifolds that are obtained by replacing the diagonal affine invariant in the $A$-model by the affine $D$-invariant, leading to a weighted projective hypersurface of $D$-type. The motivic $AD$-isomorphism $H(M_\Omega(X)) \cong H(M_\Omega(s_D(X)))$ implies that the motivic $L$-functions of such mirror pairs are identical $L_\Omega(X,s) = L_\Omega(s_D(X),s)$ independent of any input about the automorphic structure of the motives and their $L$-functions.

The motives that arise from exactly solvable mirror pairs among the Gepner models have quite high rank because the degrees of the hypersurfaces are large. This makes an explicit analysis of the automorphic structure of these models quite challenging. The lowest rank motive is obtained from mirror pairs of varieties of degree 264. This arises from the diagonal model $(1_A \otimes 6_A \otimes 31_A \otimes 86_A)_{GSO}$, associated to the Calabi-Yau threefold embedded in the weighted projective space $\mathbb{P}(3,8,33,88,132)$. The fractional transform of the mirror quotient leads to the $D$–type hypersurface

$$\mathbb{P}(3,8,33,88,132)[264]^{(57,81)}/\mathbb{Z}_2 \cong \mathbb{P}(3,8,66,88,99)[264]^{(81,57)},$$

leading to a pure motive of rank 80. The other exactly solvable mirror pairs constructed in [22] lead to even higher rank motives.

The virtue of the isomorphism established above is that even without a proof of automorphy it follows that the high-rank motives of exactly solvable mirror pairs in this class of theories have the same as the $L$-functions. This is exactly as one would expect if the motives were automorphic and confirms this prediction of mirror symmetry, extending the elliptic curve results of [5] to higher dimensions.

5 A generalized map

It is natural to ask whether there are other identities between $L$-functions that generalize the $AD$-isomorphism above, independent of whether an exactly solvable model is known or not.
It turns out that this is the case. Define the map

$$s_{\text{ND}} : \quad p_A = x^a + y^b \longrightarrow x^{\frac{a+1}{2}} + xy^b, \quad \text{for } a \text{ odd.}$$

The central charge constraint applied to the map $s_{\text{ND}}$

$$\frac{1}{a} + \frac{1}{b} = \frac{2}{a+1} + \frac{1}{b} \left( 1 - \frac{2}{a+1} \right)$$

leads to the constraint

$$a = \frac{b}{b-2}.$$  

Since $a$ is an integer the only solutions are $b = 3, 4$, leading to

$$(a, b) = (3, 3), \quad (a, b) = (2, 4).$$

The assumption that $a$ is odd then leaves just a single solution for the central charge constraint:

$$p_A = x^3 + y^3 \longrightarrow x^2 + xy^3.$$  

This map is of interest because it explains $L$-function identities between nondiagonal and diagonal motives that are not based on exactly solvable models.

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