Propagation of shock wave of nitrogen gas in Titan stratosphere

H. G. Abdelwahed a, b and Abeer A. Mahmoud b

a College of Science and Humanitarian Studies, Physics Department, Prince Sattam Bin Abdul Aziz University, Alkharj, Kingdom of Saudi Arabia; b Theoretical Physics Research Group, Physics Department, Faculty of Science, Mansoura University, Mansoura, Egypt

ABSTRACT

Nitrogen gas is considered one of the significant standard gases in stratosphere of Titan, Saturn’s largest moon, which represents almost 98.4% of Titan atmosphere chemical composites. Hydrodynamics fluid model has been constructed to describe the Nitrogen gas flows in the inner part of the Titan stratosphere. Using Fourier analysis, the linear dispersion relation of the fluid equations has been derived. Exploration of the resultant dispersion relation elucidated that two independent phenomena occurred simultaneously. Moreover, the reductive perturbation method has been utilized, yielding Burgers equation. A shock wave enveloped solution has been obtained using the computational F-expansion method. The different effects of thermal conductivity and kinematic viscosity on propagating wave have been studied. They have found to cause dissipation of envelope wave but don’t affect the amplitude of envelope wave. Our investigations showed that the dissipation in the enveloped waves caused by thermal conductivity and kinematic viscosity.

1. Introduction

Fluid dynamics hypotheses have been successfully employed to investigate various processes in nature such as acoustics, aerodynamics, cryosphere science, fluid power, plasma physics and medicine, where the flows are preserved with heat and mass transfer. Modern experimental and numerical methods are obtainable to study velocity fields and share their features to exacting boundary conditions [1–6].

Titan is Saturn’s largest moon and one of the most interesting objects in our solar system. Titan gained notability because it is the only luminary in our system where it has proven liquid on its surface and has a planet-like atmosphere. Its atmosphere in the stratosphere region comprises approximately 98.4% nitrogen, 1.4% methane, and 0.1–0.2 hydrogen, charged with interaction with a light out of the atmosphere. Titan atmosphere is considered the densest nitrogen-rich atmosphere in the solar system. This large amount of nitrogen in the Titan atmosphere hasn’t come from the Saturn materials but from the Oort cloud associated with the comets. Although the surface pressure of the Titan atmosphere is about 1.45 times higher than that for Earth’s atmosphere, space missions measurements showed that its gravity is lower than that for the Earth’s [7,8]. Linear and nonlinear propagation waves analysis in fluid dynamics have been studied theoretically by many authors during the last three decades [9–20].

In this work, we study the linear and nonlinear properties for the major public gas in the Titan stratosphere, nitrogen gas, where we use the fluid model to describe the motion of the gas. This paper is organized as follows. In the next section, we state the fluid equations for nitrogen gas. Then we used the Fourier analysis to drive the dispersion relation, which gives all information about the linear propagation of the nitrogen gas waves in the Titan stratosphere in Section 3. The reductive perturbation method is used to derive a nonlinear equation in order to study the envelope wave’s nonlinear properties via Section 4. Finally, conclusions are given in Section 5.

2. The equations of fluid

The flow of nitrogen gas in one dimension can be described by using the Navier–Stokes equations [20]

\[
\frac{\partial (\rho u)}{\partial t} + \frac{\partial (\rho u^2)}{\partial x} = 0 \tag{1}
\]

\[
\rho \frac{\partial (uT)}{\partial t} + \rho u \frac{\partial (uT)}{\partial x} + \frac{\partial (\rho u^2 T)}{\partial x} = c_pT \frac{\partial h_0}{\partial x} + \frac{\partial (\rho u^2 T)}{\partial x} - \frac{\partial (\rho u^2)}{\partial x} = 0 \tag{2}
\]

\[
\rho \frac{\partial T}{\partial t} + \frac{h_0}{c_p} \frac{\partial h_0}{\partial x} + \frac{\partial (\rho u T)}{\partial x} = 0 \tag{3}
\]

where \( h_0 = \frac{u_0^2}{2} + c_pT \) is the total enthalpy per unit mass of the fluid. The thermal conductivity coefficient

CONTACT Abeer A. Mahmoud salmafractal@gmail.com Theoretical Physics Research Group, Physics Department, Faculty of Science, Mansoura University, Mansoura 35516, Egypt

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is denoted by $\kappa$, the density by $\rho$, the time by $t$, the
temperature of the gas by $T$, the gas constant $R$, and $\gamma = C_p/C_v$. Also, $C_p(C_v)$ represent the specific heat capacity
for fluid at constant pressure (volume). Note that the shear stress $\tau_{xy}$ is related to viscosity $\mu$ and velocity $u$ by
Stokes relations $\tau_{xy} = \frac{4\mu u}{\kappa}$. All physical quantities exist in Equations (1)–(3)
normalized by their values at equilibrium as

$$\bar{\rho} = \frac{\rho}{\rho_0} - u = \frac{u}{u_0} - t = \frac{u_0}{\lambda}, \, \bar{k} = \frac{k}{\lambda},$$

$$\bar{\mu} = \frac{\mu}{\mu_0}, \, \bar{T} = \frac{T}{T_0}, \, \omega_0 = \sqrt{\gamma RT_0} \quad (4)$$

Then, the normalized governing equations in one
dimensional flow become as

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial (\bar{\rho} \bar{u})}{\partial x} = 0 \quad (5)$$

$$\bar{\rho} \frac{\partial \bar{u}}{\partial t} + \rho \bar{u} \frac{\partial \bar{u}}{\partial x} + \gamma \frac{\partial \bar{T}}{\partial x} \frac{4 \bar{\mu} \gamma^{3/2}}{3 R_e} = 0 \quad (6)$$

$$\bar{\rho} \left( \frac{\partial \bar{T}}{\partial t} + \bar{u} \frac{\partial \bar{T}}{\partial x} + \frac{4 \gamma (\gamma - 1) \bar{\mu}}{3 R_e} (\frac{\partial \bar{u}}{\partial x})^2 \right) = 0 \quad (7)$$

where $\rho_e = u_0 \rho_0 C_p/\kappa$ and $R_e = u_0 \rho_0 \lambda/\mu_0$ are called the Peclet number and Reynolds number, respectively. This set of equations is considered as coupled nonlinear partial differential equations, which is very difficult to deal with it as it is. There are different methods used to reduce these complicated nonlinear equations. The reductive perturbation method reduces unknown functions and equations to only one equation in one unknown function. This method used to study nonlinear small amplitude waves [21], whilst the Sagadeev Pseudo potential used to describe the propagation of nonlinear large-amplitude waves [22]. The information about the propagation of given fluid waves (as phase velocity, group velocity, propagation region, and so on) can be studied using the dispersion relation via linear analysis for the fluid system.

### 3. Linear analysis and dispersion relation

In this section, the Fourier analysis will be employed to investigate the dispersion relation for the Nitrogen fluid (5–7). First, we expand all the dynamical variables $\Psi = [\rho \, u \, T]$ around their equilibrium values

$$\Psi = \Psi_0 + \Psi_1 \quad (8)$$

where $\Psi_0 = [1 \quad 0 \quad 1]$, and $\Psi_1$ is a small perturbed term. Next, considering all the perturbed quantities to be proportional to $e^{i(kx - \omega t)}$, where $k$ and $\omega$ are representing the wavenumber and the angular frequency, respectively, i.e.

$$\Psi_1 = \bar{\Psi} \exp(i(kx - \omega t)) \quad (9)$$

Substituting from Equations (8) and (9) into Equations (5)–(7). The relation between wavenumber $k$ and angular
frequency $\omega$ obtained via the following dispersion relation,

$$\left( \frac{\gamma^2 k^2 \omega^2}{\bar{P}_e} + \frac{4 \mu R_e^{-1} \gamma k^2 \omega^2}{3 \bar{P}_e} - \frac{\gamma k^4}{\bar{P}_e} \right)
+ i \left( \frac{4 \mu}{3 \bar{P}_e} R_e^{-1} \gamma k^2 \omega + \gamma k^2 \omega - \gamma \omega^3 \right) = 0 \quad (10)$$

Clearly, the linear dispersion relation (10) has two parts: real and imaginary; this means that there are two independendent phenomena that occurred in this system. To determine the damping or growth rate, consider angular frequency $\omega$ has two parts as

$$\omega = \omega_\text{r} + i \omega_\text{i} \quad (11)$$

where $\omega_\text{r}$ represents the angular frequency of the propagating wave, and $\omega_\text{i}$ represents the damping or growth rate. By substituting from Equation (11) into Equation (10), yield the following two equations:

$$\frac{\gamma^2 k^2 \omega_\text{r}^2 \omega_\text{i}^2}{\bar{P}_e} - \frac{\gamma^2 k^2}{\bar{P}_e} \omega_\text{i}^2 + \frac{4 \mu R_e^{-1} \gamma k^2 \omega_\text{i}^2}{3 \bar{P}_e}$$

$$- \frac{4 \mu}{3 \bar{P}_e} R_e^{-1} \gamma k^4 \omega_\text{r}^2 - \frac{\gamma k^4}{\bar{P}_e} - \gamma k^2 \omega_\text{i} - \frac{\gamma k^4}{\bar{P}_e} R_e^{-1} \gamma k^4 \omega_\text{i}$$

$$+ 3 \gamma \omega_\text{r} \omega_\text{i}^2 - \gamma \omega_\text{i}^3 = 0 \quad (12a)$$

$$2 \frac{\gamma^2 k^2}{\bar{P}_e} \omega_\text{r} \omega_\text{i} + \frac{8 \mu R_e^{-1} \gamma k^2 \omega_\text{r} \omega_\text{i} + \gamma k^2 \omega_\text{i}}{3 \bar{P}_e}$$

$$+ \frac{4 \mu}{3 \bar{P}_e} R_e^{-1} \gamma k^4 \omega_\text{r}$$

$$+ 3 \gamma \omega_\text{r} \omega_\text{i}^2 - \gamma \omega_\text{i}^3 = 0 \quad (12b)$$

Maple programming is used to solve these two equations, and study the effect of wave number $k$, and the viscosity of nitrogen gas on both $\omega_\text{r}$ and $\omega_\text{i}$. Where at heat capacity at constant pressure $C_p = 1.45 * 10^7$ (kg.K)$^{-1}$, $T_0 = 155$ K, $\rho_0 = 3 * 10^{-14}$ kg/m$^3$, $\gamma = \frac{g}{mg} = \frac{27.21*10^{-5}T^0_8}{T_0 T^0_p}$ and at altitude 150km as taken from observation [23], as appeared from Figure (1)a which shows the variation of absolute value of real part of angular frequency $\omega_\text{r}$ with wave number $k$ at different values of normalized fluid viscosity ($\mu = 1, \mu = 1.5, \mu = 2$). This figure show how the real part of angular frequency affected by fluid viscosity, where at $\mu = 2$ (dashed curve) $\omega_\text{r}$ increased with increasing the angular frequency until $k \approx 6$ the behaviour inverted. At $\mu = 2$ and $k > 7.5$ the value of $\omega_\text{r}$ inverted to imaginary. Decreasing the value of fluid viscosity $\mu = 1.5$ (dotted curve), the behaviour of real part of angular frequency with wave number changed at $k \approx 8$ and
The relation between the (a) – real part of angular frequency \( \omega_r \) and wave number \( k \) at different values of kinematic viscosity \( \mu_1 \) (b) – imaginary part of angular frequency \( \omega_i \) and wave number \( k \) at different values of kinematic viscosity \( \mu_1 \) (c) – real part of angular frequency \( \omega_r \) and kinematic viscosity \( \mu_1 \) at different values of wave number \( k \) (d) – imaginary part of angular frequency \( \omega_i \) and kinematic viscosity \( \mu_1 \) at different values of wave number \( k \).

Figure 1. The relation between the (a) – real part of angular frequency \( \omega_r \) and wave number \( k \) at different values of kinematic viscosity \( \mu_1 \) (b) – imaginary part of angular frequency \( \omega_i \) and wave number \( k \) at different values of kinematic viscosity \( \mu_1 \) (c) – real part of angular frequency \( \omega_r \) and kinematic viscosity \( \mu_1 \) at different values of wave number \( k \) (d) – imaginary part of angular frequency \( \omega_i \) and kinematic viscosity \( \mu_1 \) at different values of wave number \( k \).

inverted to imaginary at \( k > 10 \). Again decreasing the value of fluid viscosity \( \mu = 1 \) (lined curve), the transfer point of behaviour of \( \omega_r \) at \( k \approx 10 \).

The relation between the absolute value for the imaginary part of angular frequency \( \omega_i \) and wave number \( k \) at different values of normalized fluid viscosity (\( \mu = 1 \), \( \mu = 1.5 \), \( \mu = 2 \)) shows graphically (Figure 1(b)) that, at \( \mu = 2 \) (dashed curve) \( \omega_i \) slightly increased with increased wave number until \( k \approx 6 \) the imaginary part of angular frequency quickly increased with wave number \( k \geq 6 \). At \( \mu = 1.5 \) (dotted curve), \( \omega_i \) quickly increased with wave number \( k \geq 8 \), but at \( \mu = 1 \) (lined curve) the quickly increased of \( \omega_i \) with wave number \( k \geq 10 \). In the next section, the reductive perturbation method will be used to study the effects of both viscosity and thermal conductivity on the shape of the propagated wave.
4. Reductive perturbation method and shock wave

In this paper to study the propagation of nonlinear waves, the following stretched space–time coordinates introduced [24,25]

\[
\xi = \varepsilon^{1/2}(x - V_p t), \tau = \varepsilon^{3/2} t, \mu = \varepsilon^{1/2} \mu
\]

where \(V_p\) is the phase velocity of the nitrogen gas and \(\varepsilon\) is a parameter which measures the size of the perturbation amplitude. Furthermore, all unknown dynamical variables in Equations (5–7) are expanded asymptotically as power series in \(\varepsilon\) about their equilibrium values as

\[
\begin{bmatrix}
\rho \\
u \\
T
\end{bmatrix} =
\begin{bmatrix}
1 \\
u_1 \\
T_1
\end{bmatrix} + \varepsilon \begin{bmatrix}
\rho_1 \\
u_2 \\
T_2
\end{bmatrix} + \varepsilon^2 \begin{bmatrix}
\rho_2 \\
u_3 \\
T_3
\end{bmatrix} + \ldots
\]

(14)

Substituting from Equations (13) and (14) into the basic set of Equations (5–7), the first order in \(\varepsilon\) gives

\[
\rho_1 = \frac{1}{V_p} u_1,
\]

\[
T_1 = \frac{\gamma V_p^2 - 1}{V_p} u_1 \text{ and } \gamma V_p^3 + \gamma V_p = 0
\]

(15)

Eliminating the second-order perturbed quantities from the set of equations which yielded from the second order of \(\varepsilon\) and making use of the first-order results, we obtain the Burgers’ equation

\[
\frac{\partial \rho_1}{\partial \tau} + A \rho_1 \frac{\partial \rho_1}{\partial \xi} + B \frac{\partial^2 \rho_1}{\partial \xi^2} = 0
\]

(16)

where the coefficient of a nonlinear term is given by

\[
A = \frac{\gamma^2 V_p^3 + 3 \gamma V_p^3 - 2 \gamma V_p}{3 \gamma V_p^2 - \gamma}
\]

(17a)

and the dissipative coefficient is

\[
B = \frac{-5 \gamma V_p^2 + \sigma (1 - \gamma V_p^2)}{3 \gamma V_p^2 - \gamma}, \quad \sigma = \frac{4 \mu}{3 \Re}, \quad \sigma = \frac{\gamma}{P_e}
\]

(17b)

Burger’s Equation (16) can be solved analytically using the following travelling wave transform

\[
X = \xi - M \tau
\]

(18)

According to the F-expansion method [26,27], Equation (19) has a general solution with the form

\[
\rho(X) = \sum_{r=0}^{n} h_r(H)^r + \sum_{r=1}^{n} s_r(H)^{-r}
\]

(20)

where \(M\) is the travelling wave velocity, substituting from this transformation into Equation (16), the following ordinary differential equation of the second kind with constants coefficients obtained

\[
\frac{d^2 \rho}{dX^2} + \frac{A}{B} \frac{d \rho}{dX} - \frac{M d \rho}{B} = 0
\]

(19)

Figure 2. The 3-dimensional density for the major public gas in Titan’s stratosphere with velocity of envelope wave \(M = 0.4\) and at \(a - V_p = 1\), and \(b - V_p = -1\).
by Mathematica Software yields

\[ h_0 = \frac{M}{A}, h_1 = \sqrt{\frac{1 - 2M}{B}}, s_1 = \frac{M^2}{2A^2h_1} \]

\[ H = H(X) \text{ satisfies the following ordinary differential equation (Riccati Equation)} \]

\[ \frac{dH}{dX} + aH^2 + bH + c_1 = 0, \quad (21) \]

where \( a, b, c_1 \) are arbitrary constants, Equation (22) has the following exact solution

\[ H(X) = \frac{M}{A} - \frac{M}{A} \tanh \left( \frac{M}{2B} (X) \right) \quad (22) \]

This solution gives a shock shape wave, as shown in Figures (2) and (3), where two 3D envelope shock waves appear, as shown in Figure (2). The appearance of these two envelope waves depends on the values of acoustic velocity \( V_p \), where from the analytical analysis of both dispersion relation (10) and complementary Equation (15), we find that \( V_p = \pm 1 \). The formula of the envelope wave (22) indicates that the probability of propagating of positive or negative shock waves depends mainly on the values of nonlinear coefficient \( A \) which are directly proportional to \( V_p \). As demonstrated from Equation (17b), the dissipation in propagating waves depends mainly on fluid viscosity and thermal conductivity. If a sudden change in temperature of nitrogen gas happens [28], the thermal conductivity of nitrogen may change, so the dependence of the shape of envelope wave on thermal conductivity and viscosity is graphically studied. As shown in Figure (3), the steepness of shock wave decreases as both thermal conductivity and kinematic viscosity increase.

5. Conclusion

The Navier–Stokes equations had been used to describe the motion of nitrogen gas which is considered as the major public component in the Titan stratosphere. The linear analysis was made to study the linear properties of propagating waves via Fourier analysis. A complex dispersion relation is obtained, which means that there are two independent phenomena (absorption, scattering, transmission, and so on) for the propagating wave made simultaneously. By testing the real and imaginary parts, we find that the envelope nitrogen wave is a damped wave, and the damping rate is directly proportional to fluid viscosity, at \( \mu = 2 \) (dashed curve), the real and imaginary parts of angular frequency are slightly increased with increased wave number until \( k \geq 6 \) the real part decreased and imaginary part of angular frequency quickly increased with wave number. By decreasing the kinematic viscosity the damping rate will decrease and the envelope wave will take a propagation longer time as shown in Figure 1(c). To study the nonlinear properties of propagating waves, Burger’s equation in first perturbed density was derived by using the reductive perturbation method. Shock wave enveloped from the solution of the resultant equation. All thermodynamics properties of nitrogen gas can be obtained from Equation (15), and from the relations between these properties and temperature and velocity. From studying the effect of thermal conductivity and kinematic viscosity, we think that the dissipation in envelope wave depended mainly on thermal conductivity and kinematic viscosity, but don’t affect the amplitude of the envelope wave.

An actual explosion in Titan’s atmosphere would generate spherical waves in 3D, where wave energy density would decay with radius \( r \) as \( 1/r^2 \), so in future work, we will study the spherical waves by investigating the propagation of nitrogen fluid in spherical coordinate.

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