EXTENDED WEYL GROUPS, HURWITZ TRANSITIVITY AND WEIGHTED PROJECTIVE LINES II: THE WILD CASE

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Abstract. In this paper, which is a continuation of [5], we study extended Weyl groups of domestic and wild type. We start with an extended Coxeter-Dynkin diagram, attach to it a set of roots (vectors) in an \( R \)-space and define the extended Weyl groups as groups that are generated by the reflections related to these roots, the so-called simple reflections. We relate to such a group \( W \) a generalized root system, and determine the structure of \( W \). In particular we present a normal form for the elements of \( W \). Further we define Coxeter transformations in \( W \), and show the transitivity of the Hurwitz action on the set of reduced reflection factorizations of a Coxeter transformation where the reflections are the conjugates of the simple reflections in \( W \) (see Theorem 1.1).

We give two applications of Theorem 1.1 and of the results in [5]. In the context of representation theory of algebras, [5] Theorems 1.1 and 1.4 and Theorem 1.1 of this paper are the last steps that establish an isomorphism between the poset of thick subcategories that are generated by exceptional sequences of a hereditary connected ext-finite abelian \( k \)-category with a tilting object and the poset of elements in the extended Weyl group that are below a Coxeter transformation with respect to the absolute order (see Theorem 1.4). The second application concerns the theory of unimodal singularities (see Theorem 1.5). In particular, we provide an answer to a question of Brieskorn [10, Question 4] for the classical monodromy operator in the case of hyperbolic singularities.

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1. Introduction

Affine Coxeter systems are well studied, for instance group theoretical, topological or combinatorial properties of them are known (see for instance \[8, 11, 16, 24, 31\]). In \[5\] and in this paper we propose the systematic study of a generalization of simply laced affine Coxeter systems, more precisely the study of extended Weyl groups and extended Weyl systems. An extended Weyl group is a group that is generated by a certain set of reflections of a finite dimensional \(\mathbb{R}\)-space, which is equipped with a bilinear form (see Section 2). There are three different types of extended Weyl groups, those of domestic, tubular and of wild type corresponding to the three possible types of the related bilinear form. The extended Weyl groups of domestic type are precisely the simply laced affine Coxeter groups (see Remark 2.5(c)), and those of tubular type are elliptic Weyl groups (see \[5, 20, 28\]).

Apart from the domestic case, extended Weyl groups are not Coxeter groups. Nevertheless, they share several properties with Coxeter groups. For instance, every extended Weyl group \(W\) has a linear faithful representation, and is related to a generalized root system, which contains a “simple system” of roots, and the set \(S\) of reflections with respect to the roots in the simple system generates \(W\).

Therefore, each extended Weyl group is defined uniquely by a diagram, the so-called extended Coxeter Dynkin diagram, and contains special elements, the Coxeter transformations, which are the analogues of Coxeter elements in a Coxeter group (see Section 2).

Extended Weyl systems have already been partially studied in \[20, 22, 29\] as well as in \[30, 5\]. The origin of Kluitmanns \[20\] motivation to study these groups was his interest in simple elliptic singularities of types \(E_6^{(1,1)}, E_7^{(1,1)}\) and \(E_8^{(1,1)}\), as their monodromy groups are extended Weyl groups. In order to determine the distinguished bases of the corresponding Milnor lattices he considered an action of the braid group, the Hurwitz action (see Section 2), on the set of reduced reflection factorizations of a Coxeter transformation.

The motivation of the other authors was in the representation theory of algebras, more precisely the interest in the categories of coherent sheaves over a weighted projective line \(\text{coh}(X)\) in the sense of Geigle and Lenzing \[13\], or respectively in the derived equivalent categories of finite dimensional modules over canonical algebras (see the definition of canonical algebras in \[25\]). One can attach to such a category a generalized root system as well as a reflection group, which is an extended Weyl group (see \[5\] Section 2)). Also in the understanding of the category \(\text{coh}(X)\) the Hurwitz action of the braid group on the set of reduced factorizations of a Coxeter transformation into a product of reflections plays an important role, as we will explain in this paper.

We study the elliptic Weyl groups in \[5\]. In particular, extended Weyl groups of tubular type are elliptic Weyl groups. In this paper, which is a continuation of \[5\], we focus on the extended Weyl groups of wild as well as of domestic type. We uniformly describe their group structure (Subsection 3.2) and introduce a normal form for its elements (Subsection 3.3). Further we extend our results on the Hurwitz transitivity for Coxeter elements in Coxeter
groups [3] and Coxeter transformations in elliptic Weyl groups to Coxeter transformations in extended Weyl groups of domestic and wild type. More precisely, we show the following, where \( T = \bigcup_{w \in W} wSw^{-1} \) is the set of reflections in \( W \).

**Theorem 1.1.** Let \((W,S)\) be an extended Weyl system of domestic or wild type with simple system \(S\) of size \(n\), set of reflections \(T\) and Coxeter transformation \(c\). Then the Hurwitz action is transitive on the set of reduced reflection factorizations of \(c\), namely on the set \(\text{Red}_T(c) = \{(t_1, \ldots, t_n) \in T^n \mid t_1 \cdots t_n = c\}\).

Notice that our proof is uniform without a case distinction. Let \(\text{Red}^\text{gen}(c)\) be the set of reduced reflection factorizations of a Coxeter transformation \(c\) that generate \(W\). In an elliptic Weyl group the Hurwitz action is transitive on the set \(\text{Red}^\text{gen}(c)\) (see [3]). Furthermore, the authors proved that \(\text{Red}^\text{gen}(c) \neq \text{Red}_T(c)\) if \((W,S)\) is tubular. In the non-tubular case \(\text{Red}^\text{gen}(c) = \text{Red}_T(c)\) holds as a consequence of Theorem 1.1. This shows that the tubular and the non-tubular types behave differently. This distinction is based on the fact that for \(W\) of tubular type the rank of the radical \(R\) of the related bilinear form is two while for \(W\) of non-tubular type this rank is one. Therefore, we discuss the tubular and the non-tubular types in two different papers.

In the last section we apply Theorem 1.1 to the representation theory of hereditary categories. Every category of coherent sheaves over a weighted projective line is a hereditary category with a tilting object. The Grothendieck group of an ext-finite hereditary abelian \(k\)-category for an algebraically closed field \(k\) that has a tilting object is naturally equipped with a generalized root system and a reflection group. In particular, this is the case for the category of finitely generated \(A\)-modules for a finite dimensional hereditary \(k\)-algebra \(A\).

The reflection group attached to the category \(\text{mod}(A)\), also called Weyl group, is a Coxeter group \(W\) with set of simple reflections \(S\), which is obtained from a complete exceptional sequence. Such a sequence can be chosen such that it contains a complete set of representatives of the set of isomorphism classes of the finite dimensional simple \(A\)-modules (see [26]). As usual the product of the elements of \(S\) in arbitrary order is a Coxeter element of \((W,S)\).

According to Happel, the two just mentioned types of categories, that is those of finitely generated modules of a finite dimensional hereditary \(k\)-algebra and those related to weighted projective lines, are precisely up to derived equivalence the two types of hereditary connected ext-finite abelian \(k\)-categories with a tilting object ([14, Theorem 3.1]).

Ingalls and Thomas [18], and Igusa, Schiffler and Thomas [17] as well as Krause [21] and Hubery and Krause [15] obtained a combinatorial description of the poset of thick subcategories generated by exceptional sequences of \(\text{mod}(A)\) by providing an isomorphism (of posets) between this poset and a combinatorial object, the poset of non-crossing partitions in \(W\). The latter poset consists of all the elements of \(W\) that are in the intervall \([1,c]\) with respect to some order relation on \(W\), the so called absolute order (see [2]).

Beside our interest in the extended Weyl groups by themselves our main goal of this and the previous paper [5] is to transfer the results for the module categories \(\text{mod}(A)\) to the category of coherent sheaves over a weighted projective line, the other type of hereditary connected ext-finite abelian \(k\)-categories with a tilting object.

As a consequence of Theorem 1.1 and [3] Theorem 1.1 we obtain the desired isomorphism between the poset of thick subcategories of the category of coherent sheaves over a weighted projective line that are generated by an exceptional sequence, whose order relation is inclusion, and a subposet of the set of generalized non-crossing partitions \([id,c]\). We denote by \([id,c]\) the subposet of the interval poset \([id,c]\) that consists of all the elements \(w \in [id,c]\) which
possess a reduced factorization that can be extended to a reduced factorization of $c$ which generates $W$.

**Theorem 1.2.** Let $X$ be a weighted projective line over an algebraically closed field $k$ of characteristic zero, $\text{coh}(X)$ the category of coherent sheaves over $X$, $\Phi$ the associated generalized root system, $W$ the extended Weyl group and $c \in W$ a Coxeter transformation. There exists an isomorphism of posets between

- the poset of thick subcategories of $\text{coh}(X)$ that are generated by an exceptional sequence and that are ordered by inclusion; and
- the subposet $[\text{id}, c]^{\text{gen}}$ of the interval poset $[\text{id}, c]$.

**Remark 1.3.** It is a consequence of Theorem 1.1 that $[\text{id}, c]^{\text{gen}} = [\text{id}, c]$ if $X$ is of wild or domestic type. Consequently, the assumptions (a) and (b) of [5, Theorem 1.1] are fulfilled for $X$ of wild or domestic type.

Theorem 1.2 together with [15, Theorem 1.2] and [14, Theorem 3.1] yields the following result (see Section 7):

**Theorem 1.4.** Let $C$ be a hereditary connected ext-finite abelian $k$−category with a tilting object over an algebraically closed field $k$ of characteristic zero. Let $\Phi$ be the associated generalized root system, $W$ the extended Weyl group and $c \in W$ a Coxeter transformation. Then there exists an isomorphism of posets between

- the poset of thick subcategories of $C$ that are generated by an exceptional sequence and that are ordered by inclusion; and
- the subposet $[\text{id}, c]^{\text{gen}}$ of the interval poset $[\text{id}, c]$.

As another application of Theorem 1.1 and [5, Theorem 1.3] we obtain a characterization of the set of all distinguished bases of vanishing cycles of a simple elliptic and hyperbolic singularity (see Section 7.2) and hereby answering a question of Brieskorn [10, Question 4] in these cases.

**Theorem 1.5.** Let $f$ be a simple elliptic or hyperbolic singularity, $\Lambda^*$ the set of vanishing cycles of $f$, $M$ the corresponding Milnor lattice and $h_\ast$ the monodromy operator of $f$. Then the set of all distinguished bases of vanishing cycles of $f$ is given by the set

$$\{ (\delta_1, \ldots, \delta_n) \in (\Lambda^*)^n \mid \text{span}_\mathbb{Z}(\delta_1, \ldots, \delta_n) = M, \ s_{\delta_1}\cdots s_{\delta_n} = h_\ast \}.$$ 

On the way to prove Theorem 1.1 we show some results on Coxeter groups, which are interesting by themselves. For instance the following. In a finite Coxeter group of rank $n$ with parabolic subgroup $P$ of rank $n-1$ all the reflections $t \in T$ which fulfill the property that they generate together with $P$ the whole Coxeter group are conjugate by the elements in $P$ (see [4, Theorem 1.5]). Here we show this result for the Coxeter groups whose Coxeter diagram is a star and its parabolic subgroup obtained by removing the unique vertex of degree at least three from the Coxeter diagram (see Corollary 5.3).

**Structure of the paper.** In Section 2 we recall the necessary framework, which is the extended Weyl group, the corresponding root system and the Coxeter transformation, from [5]. Then in Section 3 the structure of the extended Weyl groups of wild and domestic type is determined. In Section 4 we will show that all the Coxeter transformations are conjugate and will recollect the reflection length of a Coxeter transformation from [5].

The results of Sections 4 and 5 are used in Section 6 to prove the Main Theorem 1.1. The last section is dedicated to the applications of the main theorem.
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2. Notation, terminology and basic facts

In this section we introduce most of the notation that we use throughout this paper. In particular we recall the definition of a generalized root system and define the extended Weyl groups. These definitions, which can be partially found in [29] or [28], will be used throughout the paper.

2.1. The extended Coxeter-Dynkin diagram. We start by introducing a diagram, the so-called extended Coxeter-Dynkin diagram. For a non-negative integer \( r \in \mathbb{N}_0 \) and numbers \( p_i \in \mathbb{N} \) where \( 1 \leq i \leq r \) it is defined as follows:

\[
\begin{align*}
(1, p_1) & \quad (1, p_1 - 1) & \cdots & \quad (1, 2) & \quad (1, 1) & \quad (r, 1) & \quad (r, 2) & \quad (r, p_r - 1) & \quad (r, p_r) \\
(2, 1) & \quad (r - 1, 1) \\
(2, p_2) & \cdots & \quad (2, p_2 - 1) & \cdots & \quad (r - 1, p_{r-1} - 1) & \quad (r - 1, p_{r-1})
\end{align*}
\]

Figure 1. Extended Coxeter-Dynkin diagram

2.2. The extended space. We attach analogously to the Tits representation for Coxeter groups (see [16, Section 5.3]) a geometric datum to the extended Coxeter-Dynkin diagram. This will lead to the definition of the extended Weyl group and of a Coxeter transformation.

Let \( Q \) be the vertex set of the diagram in Figure 1. Define \( V \) to be the real vector space with basis \( B := \{ \alpha_\nu \mid \nu \in Q \} \). We further define a symmetric bilinear form \( (- | -) \) on \( V \) by setting

\[
(\alpha_\nu | \alpha_\omega) = \\
\begin{cases}
2 & \nu, \omega \in Q \text{ are connected by a dotted double bound or } \nu = \omega, \\
0 & \nu, \omega \in Q \text{ are disconnected}, \\
-1 & \nu, \omega \in Q \text{ are connected by a single edge},
\end{cases}
\]

and extending bilinearly to \( V \). We call \( (V, B, (- | -)) \) an extended space.

The following definition is a generalization of the notation of a root system in the theory of finite Coxeter systems and can be found in [25] Section 1.2. As usual, we define for \( \alpha \in V \) non-isotropic, that is \( (\alpha | \alpha) \neq 0 \), the reflection

\[
s_\alpha(v) := v - \frac{2(\alpha | v)}{(\alpha | \alpha)} \alpha \text{ for all } v \in V.
\]

Definition 2.1. A non-empty subset \( \Phi \subseteq V \) of non-isotropic vectors is called generalized root system if the following properties are satisfied
 Proposition and Definition 2.4. Let \((V, B, (- | -))\) be an extended space.

(a) The group \(W := \langle s_\alpha \mid \alpha \in B \rangle\) is called extended Weyl group. We call it wild, tubular or domestic if the signature of \((- | -)\) is \((|B| - 2, 1, 1)\), \((|B| - 2, 0, 2)\) or \((|B| - 1, 0, 1)\), respectively. Further we denote by \(R\) the radical of the form \((- | -)\). It is easy to see that the signatures that occur are exactly the following:

- tubular: \((m, 0, 2)\) for \(m = 4, 6, 7, 8\)
- domestic: \((m, 0, 1)\) for \(m \in \mathbb{N}\)
- wild: \((m, 1, 1)\) for \(m \geq 6\)

(b) If \(W\) is of domestic or wild type, then \(\dim R = 1\) and \(R = \text{span}_R (a)\) where \(a := \alpha_1 - \alpha_1\). If \(W\) is of tubular type, then \(\text{span}_R (a) \subset R\) and \(\dim_R (R) = 2\).

(c) The set \(S := \{ s_\alpha \mid \alpha \in B \}\) is called simple system and its elements simple reflections. We call \((W, S)\) an extended Weyl system. The set \(T := \bigcup_{w \in W} wS^{-1}\) is the set of reflections for \((W, S)\).

(d) Let \(\Phi \subseteq V\) be the minimal set that contains \(B\) and is closed under the action of \(W\), which is \(w(\beta) \in \Phi\) for all \(w \in W\) and \(\beta \in \Phi\). Then \(\Phi\) is a reduced and irreducible generalized root system (as the diagram restricted to \(\overline{B}\) is a tree) and \(\Phi = W(B)\). We call \(\Phi\) extended root system and the elements of \(B\) are called simple roots.
(e) Let $\Phi \subseteq V$ be the minimal set that contains $\overline{B} := B \setminus \{\alpha_1\}$ and is closed under the action of $\overline{W} = \langle s_0 \mid \alpha \in \overline{B} \rangle$. Then $\Phi$ is a simply laced generalized root system, which is irreducible and reduced, and it holds $\Phi = \overline{W}(\overline{B})$. We call the set $\Phi$ the projected root system. By construction, $\overline{(W, S)}$ is a Coxeter system, where $S = S \setminus \{s_{1^*}\}$. We call it the projected Coxeter system.

(f) As $(\overline{W}, \overline{S})$ is a Coxeter system whose Coxeter diagram is a tree and whose simple roots are all of the same length, the action of $\overline{W} \subseteq W$ is transitive on $\overline{\Phi}$. This also implies that all the roots in $\overline{\Phi}$, and therefore also in $\Phi$, are of the same length.

(g) The elements $c := \left( \prod_{\alpha \in B \setminus \{\alpha_1, \alpha_1^*\}} s_\alpha \right) s_{\alpha_1} s_{\alpha_1^*}$ are called Coxeter transformations where we take the first $|B| - 2$ factors in an arbitrary order.

Remark 2.5.

(a) Extended Weyl groups of tubular type are also called tubular elliptic (see [5]).

(b) By definition the root system $\overline{\Phi}$ is a root subsystem of $\Phi$, and $\overline{W}$ is a subgroup of $W$. Let $p$ be the natural projection of $V$ onto $\overline{V} := V/R$. Then $p(\Phi)$ is isomorphic to $\overline{\Phi}$, and $p$ restricted to $\overline{\Phi}$ is injective. If it is convenient we will abbreviate $p(v)$ by $\overline{v}$ for $v \in V$. In particular, we have $\alpha = \overline{\alpha}$ for $\alpha \in \overline{\Phi}$ in our setting.

(c) Notice that the extended Weyl groups $W$ of domestic type are precisely the affine simply laced irreducible Coxeter groups. We claim: if $W$ is an extended Weyl group of domestic type then there is $Q \subseteq T$ such that $(\overline{W}, Q)$ is an affine Coxeter system of type $\overline{X}$ where $X$ is the type of the Coxeter system $(\overline{W}, \overline{S})$. Further the set of reflections in the Coxeter system $(\overline{W}, Q)$ coincides with $T$.

This can be seen as follows. If $W$ is of domestic type then $(- | -)$ restricted to $\text{span}_R(\overline{B})$ is positive definite, let us say $(\overline{W}, \overline{S})$ is of type $X$. Let $\overline{\alpha}$ be the highest root in the positive subsystem of type $X$ containing $\overline{B}$. By [2.4] (f) there is $w \in \overline{W}$ that maps $\alpha_1$ onto $-\overline{\alpha}$. This shows that $s_{-\overline{\alpha}+a} \in W$. Then, as $\overline{S}$ generates $\overline{W}$, the group $W$ is generated by $Q := S \cup \{s_{-\overline{\alpha}+a}\} \subseteq T$. As the diagram for $Q$ is an affine Coxeter diagram, $(\overline{W}, Q)$ is an affine Coxeter group with simple system $Q$ (see [116], Section 6.5). Let $\overline{T}$ be the set of reflections in $(\overline{W}, Q)$. Then $\overline{T} = \cup_{w \in W} wQw^{-1} \subseteq T$. As $w^{-1}(\overline{\alpha}) = \alpha_1$ it is $s_{\alpha_1^*}$ contained in $\overline{T}$ and therefore we also have $T \subseteq \overline{T}$.

On the other hand, if $(\overline{W}, Q)$ is an affine simply laced irreducible Coxeter system where $Q$ is chosen as in the last paragraph, then we obtain an extended Weyl system $(\overline{W}, \overline{S})$ by a similar procedure as just described, where we choose $\alpha_1$ as the root related to the unique vertex of degree three, if $W$ is not of type $\overline{A}_n$, and else $\alpha_1$ can be chosen as any vertex of the Coxeter diagram of $(\overline{W}, Q)$.

2.4. The Hurwitz action.

Definition 2.6. Let $G$ be an arbitrary group and $T \subseteq G$ a subset which is closed under conjugation. The braid group on $n$ strands, that is, the group

\[ B_n := \langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i - j| > 1, \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ for } 1 \leq i \leq n-2 \rangle \]

acts on $T^n$ as follows:

\[
\sigma_i(g_1, \ldots, g_n) = (g_1, \ldots, g_{i-1}, g_{i+1}, g_i, g_i g_{i+1}, g_{i+2}, \ldots, g_n),
\]

\[
\sigma_i^{-1}(g_1, \ldots, g_n) = (g_1, \ldots, g_{i-1}, g_i g_{i+1} g_i^{-1}, g_i, g_{i+2}, \ldots, g_n).
\]

This action is called Hurwitz action.
In this paper, we consider the Hurwitz action on $T^{[S]}$, where $W$ is an extended Weyl group, $S$ its simple system and $T$ its set of reflections.

3. Normal form and translation vector in extended Weyl groups

The structure of the extended Weyl groups of tubular type, that is of the elliptic Weyl groups, has been determined in [5]. In this section we study the extended Weyl systems $(W,S)$ of wild and domestic type. First we determine the related generalized root system $Φ$. Then we describe the elements of $W$ in terms of the root lattice of $Φ$ by using the so called Eichler-Siegel transformation (see [31,28]). This enables us to investigate the structure of the group $W$ and to introduce a normal form for the elements of $W$. Some of these results can also be found in [29].

Recall that for extended Weyl groups of domestic or wild type we have $\dim(R) = 1$ and $R = \text{span}_\mathbb{R}(a)$, where $a = \alpha_1^* - \alpha_1$.

3.1. The extended root system and its reflections.

Lemma 3.1. The following holds:

(a) The smallest generalized root system that contains $\{\alpha_1, \alpha_1^*\}$ is $Φ' := \{±\alpha_1 + ka \mid k \in \mathbb{Z}\}$.

(b) The generalized root system attached to $W$ is $Φ = \{±\alpha + ka \mid \alpha \in \Phi, k \in \mathbb{Z}\}$.

Proof. The group $\langle s_{\alpha_1}, s_{\alpha_1^*}\rangle$ is the dihedral group of type $A_1$, thus the corresponding root system is $Φ' = \{±\alpha_1 + ka \mid k \in \mathbb{Z}\}$ by [19] Proposition 6.3. This shows (a).

Set $Λ = \{±\alpha + ka \mid \alpha \in Φ, k \in \mathbb{Z}\}$. Next we prove that $Φ = Λ$. Therefore observe that $B \subseteq Λ$. Further it is easy to check that $W(Λ) \subseteq Λ$. Therefore [24] (d) implies that $Φ = W(B) \subseteq W(Λ) \subseteq Λ$. By (a) we have $\{±\alpha_1 + ka \mid k \in \mathbb{Z}\} \subseteq Φ$. The fact that $W(Λ) \subseteq Λ$ and $w_{α_1}w_{α_2}^{-1} = s_{w(α)}$ for all $w \in W$.

Lemma 3.2. Let $α \in Φ$. Then the following holds:

(a) The reflection $s_α$ is uniquely determined by $\text{Mov}(s_α) := (s_α - \text{id})(V)$, and it holds $\text{Mov}(s_α) = \text{span}_\mathbb{R}(α)$.

(b) Let $g \in O(V,(-,-)) = \{g \in \text{GL}(V) \mid (g(u) \mid g(v)) = (u \mid v) \text{ for all } u,v \in V\}$. Then $gs_αg^{-1} = s_{g(α)}$. In particular, $ws_αw^{-1} = s_{w(α)}$ for all $w \in W$.

Proof. As $α \in R$ assertion (a) follows directly from the definition of $s_α$. To prove (b) let $v \in V$ and $u = g^{-1}(v)$, and calculate $gs_αg^{-1}(v) - v = g(s_α(u) - u) \in g(\text{Mov}(α)) = \text{span}_\mathbb{R}(g(α))$, which implies the assertion by (a).

The following nice fact is a well-known property of reflections. For completeness we present a proof.

Lemma 3.3. Let $β_1, \ldots, β_m \in Φ$ be linearly independent and $v \in V$, then $s_{β_1} \cdots s_{β_m}(v) = v$ if and only if $s_{β_i}(v) = v$ for all $1 \leq i \leq m$.

Proof. Applying $s_{β_1}$ to the equality $s_{β_1} \cdots s_{β_m}(v) = v$ yields $s_{β_2} \cdots s_{β_m}(v) = s_{β_1}(v) = v - (v \mid β_1)β_1$. Therefore, $-(v \mid β_1)β_1 = s_{β_2} \cdots s_{β_m}(v) - v \in \text{span}_\mathbb{R}(β_2, \ldots, β_m)$, and the linear independence of $\{β_1, \ldots, β_m\}$ implies $(β_1 \mid v) = 0$. The latter is equivalent to $s_{β_1}(v) = v$. By induction we get that $s_{β_i}(v) = v$ for all $2 \leq i \leq m$. □
3.2. The structure of $W$. The next definition due to Saito is helpful for the understanding of $W$ (see also [5]).

**Definition 3.4** ([28] (1.14) Definition 1]). It is

$$E: V \otimes \mathbb{R} \to \text{End}(V), \quad \sum_{i} u_i \otimes \overline{v}_i \mapsto \left[ x \mapsto x - \sum_{i} (u_i | x) u_i \right],$$

the so called Eichler-Siegel map.

We define a binary operation $\circ$ on $V \otimes \mathbb{R} \overline{V}$ by setting for $x_1, x_2 \in V \otimes \mathbb{R} \overline{V}$:

$$x_1 \circ x_2 = x_1 + x_2 - I(x_1, x_2)$$

where

$$I: (V \otimes \mathbb{R} \overline{V}) \times (V \otimes \mathbb{R} \overline{V}) \to V \otimes \mathbb{R} \overline{V}, \quad (x_1, x_2) \mapsto I(x_1, x_2)$$

with

$$I(x_1, x_2) := \sum_{i_1, i_2} u_{1i_1} \otimes (v_{1i_1} | u_{2i_2}) \overline{v}_{2i_2}$$

for

$$x_j = \sum_{i_j} u_{ji_j} \otimes \overline{v}_{ji_j} \in V \otimes \mathbb{R} \overline{V}, \quad (j = 1, 2).$$

The operation $\circ$ yields a semi-group structure on $V \otimes \mathbb{R} \overline{V}$. We obtain directly:

**Proposition 3.5** ([28] 1.14]). Let $x_1, x_2 \in V \otimes \mathbb{R} \overline{V}$.

(a) The map $E$ is injective. It is bijective if and only if $R = 0$.

(b) The map $E$ is a homomorphism of semi-groups, that is $E(x_1 \circ x_2) = E(x_1)E(x_2)$.

(c) For a non-isotropic $v \in V$, the reflection $s_v$ is given by $s_v = E(v \otimes \overline{v})$.

(d) The inverse of the Eichler-Siegel map

$$E^{-1}: W \to V \otimes \mathbb{R} \overline{V}$$

on $W$ is well defined.

(e) The subspace $R \otimes \mathbb{R} \overline{V}$ is closed under $\circ$, and $\circ$ coincides on $(V \otimes \mathbb{R} \overline{V}) \times (R \otimes \mathbb{R} \overline{V})$ with the additive structure of $V \otimes \mathbb{R} \overline{V}$.

(f) Let $r \in \mathbb{R}$ and $v \in V$ non-isotropic. Then

$$E((v + ra) \otimes \overline{v}) = E(v \otimes \overline{v})E(ra \otimes \overline{v}) = s_vE(ra \otimes \overline{v}).$$

**Definition 3.6.** Let $L_a := \text{span}_\mathbb{Z}(B_a)$ where $B_a := \{a \otimes \overline{a} | a \in \overline{B}\}$, and set $L := \text{span}_\mathbb{Z}(B)$ and $\overline{L} := \text{span}_\mathbb{Z}(\overline{B})$.

We get as an immediate consequence of Proposition 3.5(b) and (c) the following.

**Lemma 3.7.**

(a) The three lattices $L/(L \cap R)$, $\overline{L}$ and $L_a$ are isomorphic;

(b) $W \subseteq E(L \otimes \mathbb{Z} \overline{L}) \subseteq E(V \otimes \mathbb{R} \overline{V})$.

**Lemma 3.8.** Let $\alpha \in \Phi$ and $k \in \mathbb{Z}$. Then the following holds.

(a) $s_\alpha s_{\alpha + ka} = E(ka \otimes \overline{\alpha})$;

(b) $wE(ka \otimes \overline{\alpha})w^{-1} = E(ka \otimes w(\alpha))$ for all $w \in W$;

(c) $E(L_a)$ is an abelian group, which is normalized by $W$. 
Lemma 3.9. The natural projection \( p : V \to \overline{V} \) induces an epimorphism \( \rho : W \to \overline{W} \).

Proof. As \( R \) is the radical of the bilinear form \((- \mid -\)) on \( V \), the group \( W \) acts trivially on \( R \). Further it also acts on \( \overline{V} \). This action is given by the map \( \rho : s_\alpha \mapsto s_\alpha \) for \( \alpha \in \Phi \) where \( s_\alpha \) is the reflection of \( \overline{V} \) with \( \text{Mov}(s_\alpha) = \text{span}(\overline{\alpha}) \).

As \( \overline{W} \) acts faithfully on the complement \( \text{span}_R(\{b \mid b \in \overline{B}\}) \) to \( R \) in \( V \), it also acts faithfully on \( \overline{V} \) and therefore \( \ker(\rho) \cap \overline{W} = \{1\} \). Since \( W = \langle s_\alpha \mid \alpha \in B \rangle \) and since \( E(a \otimes \overline{\alpha}_1) = s_{\alpha_1}s_{\alpha_1,*} \in \ker(\rho) \), it follows by the Dedekind identity that \( \ker(\rho) \) equals \( \langle E(a \otimes \overline{\alpha}_1) \rangle \), the normal closure \( M \) of \( E(a \otimes \overline{\alpha}_1) \) in \( W \), and that \( W = \overline{W} \ltimes M \). This yields that we may identify \( \rho(W) \) and \( \overline{W} \) via the isomorphism \( \rho|_{\overline{W}} \), and thereby get the assertion. \( \square \)

Proposition 3.10 ([28] 1.15, [29] Theorem 3.5). The short exact sequence

\[
0 \to E^{-1}(W) \cap (R \otimes_R (\overline{V})) \xrightarrow{E} W \xrightarrow{\rho} \overline{W} \to 1
\]

splits. Further \( E^{-1}(W) \cap (R \otimes_R (\overline{V})) = L_a \cong \overline{L} \) and

(a) \( W = \overline{W} \ltimes E(L_a) \);
(b) the action of \( \overline{W} \) on \( E(L_a) \) is given in Lemma 3.8 (b).

Proof. Let \( T_R := E^{-1}(W) \cap (R \otimes_R (\overline{V})) \). The following diagram is commutative

\[
\begin{array}{ccc}
0 & \to & T_R \\
\downarrow & & \downarrow \downarrow \\
0 & \to & R \otimes_R (\overline{V}) \\
& \to & \quad V \otimes_R (\overline{V}) \quad (V/R) \otimes_R (\overline{V}) \quad 0,
\end{array}
\]

and the second row is exact, which implies the exactness of the first row. In particular, this also implies that \( E(T_R) = \ker(\rho) \). The first sequence splits because of Lemma 3.9. By the same lemma, \( \ker(\rho) \) is \( \langle E(a \otimes \overline{\alpha}_1) \rangle \), the normal closure of \( E(a \otimes \overline{\alpha}_1) \) in \( W \). Since \( \overline{W} \) acts transitively on \( \overline{\Phi} \) we get that \( E(a \otimes \overline{\alpha}) \in \langle E(a \otimes \overline{\alpha}_1) \rangle \) for every \( \alpha \in \overline{\Phi} \). Therefore \( E(L_a) \subseteq \ker(\rho) \), and \( \ker(\rho) = E(L_a) \) follows with Lemma 3.8 (c). \( \square \)

3.3. The normal form for the elements in \( W \). Proposition 3.10 yields a normal form for the elements in \( W \). Observe if \( w \in W \), then \( w = \overline{w}P(w) \) where \( \overline{w} = \rho(w) \) and where \( P \) is the projection of \( W \) onto \( \ker(\rho) \). It is \( P(w) = E(a \otimes \overline{\beta}) \) for some \( \beta \in \overline{L} \) by Proposition 3.10.

Definition 3.11. Define a map \( tv : W \to \overline{L} \) by setting for \( w \in W \)

\[tv(w) = \beta \text{ if } \beta \in \overline{L} \text{ is such that } P(w) = E(a \otimes \overline{\beta}).\]

We call the pair \((\overline{w}, tv(w))\) the normal form of \( w \) and \( tv(w) \) the translation vector of \( w \).

Lemma 3.12. The following holds.
(a) Let \( w \in W \) such that \( \text{tv}(w) = \sum_{\beta \in \Phi} m_{\beta} \cdot \beta \) where \( m_{\beta} \in \mathbb{Z} \). Then
\[
w = \prod_{\beta \in \Phi} (s_{\beta} s_{\beta + \alpha})^{m_{\beta}}.
\]

(b) Let \( \alpha = \overline{\alpha} + k\alpha \in \Phi \) where \( \overline{\alpha} \in \overline{\Phi} \) and \( k \in \mathbb{Z} \). Then \( \text{tv}(s_{\alpha}) = k\overline{\alpha} \).

(c) For \( \gamma \in \overline{B} \), \( m_{\gamma} \in \mathbb{Z} \) and \( y \in \overline{W} \) we have
\[
\text{tv}(y^{-1} \prod_{\gamma \in \overline{B}} (s_{\gamma} s_{\gamma + \alpha})^{m_{\gamma}} y) = \sum_{\gamma \in \overline{B}} m_{\gamma} \cdot y^{-1}(\gamma) = y^{-1}\left( \text{tv}\left( \prod_{\gamma \in \overline{B}} (s_{\gamma} s_{\gamma + \alpha})^{m_{\gamma}} \right) \right).
\]

**Proof.** Lemma 3.8 (a) and Proposition 3.3 (e) yield (a), while Proposition 3.3 (f) yields assertion (b). The third assertion is a direct consequence of Lemma 3.8.

**Lemma 3.13.** The translation vector satisfies the following properties:

(a) \( \text{tv}(s_{1^*}) = \alpha_1 \);

(b) \( \text{tv}(s) = 0 \) for all \( s \in S \setminus \{ s_{1^*} \} \);

(c) \( \text{tv}(xy) = y^{-1}\text{tv}(x) + \text{tv}(y) \) for all \( x, y \in W \).

In particular, the translation vector of an element of \( W \) is uniquely determined by the properties (a) – (c).

**Proof.** Lemma 3.12 (b) yields \( \text{tv}(s_{1^*}) = \text{tv}(s_{\alpha_1 + \alpha}) = \alpha_1 \), which is (a). Assertion (b) follows from Proposition 3.5 (c) and (f). Next we prove (c). Let \( x, y \in W \) and \( \beta_x = \text{tv}(x) \), \( \beta_y = \text{tv}(y) \). Then by Lemma 3.8 (b)
\[
xy = \overline{x}E(a \otimes \beta_x)\overline{y}(a \otimes \beta_y) = \overline{xy}^{-1}E(a \otimes \beta_x)\overline{y}E(a \otimes \beta_y) = \overline{xy}E(a \otimes (\overline{y}^{-1}(\beta_x)) + \beta_y),
\]
which yields (c).

**Corollary 3.14.** The elements in \( T \) have normal form \( (s_{\alpha}, k\alpha) \) where \( k \) is any element in \( \mathbb{Z} \) and \( \alpha \) any root in \( \overline{\Phi} \).

**Proof.** If \( t \in T \), then by Lemma 3.1 (b) there is a root \( \alpha \in \overline{\Phi} \) and \( k \in \mathbb{Z} \) such that \( t = s_{\alpha + k\alpha} \). By Lemma 3.12 the normal form of \( t \) is \( (s_{\alpha}, k\alpha) \).

On the other hand Lemma 3.1 (b) implies that for every \( k \in \mathbb{Z} \) and \( \alpha \in \overline{\Phi} \) the tuple \( (s_{\alpha}, k\alpha) \) is the normal form of some reflection.

The following generalizes the respective property for affine simply laced Coxeter groups (see [31] Lemma 2.11). Our proof is a generalization of the one given in [31]. Therefore we leave out the details.

**Lemma 3.15.** Let \( \beta_1, \ldots, \beta_n \in \Phi \) such that \( \overline{\beta_1}, \ldots, \overline{\beta_n} \in \overline{\Phi} \) are linearly independent. Then \( \text{tv}(s_{\beta_1} \cdots s_{\beta_n}) = 0 \) if and only if \( \text{tv}(s_{\beta_1}) = \cdots = \text{tv}(s_{\beta_n}) = 0 \).

**Proof.** It is easy to prove that (see [31] Lemma 2.11)
\[
\text{tv}(s_{\beta_1} \cdots s_{\beta_n}) = \sum_{i=0}^{n-2} s_{\beta_{i+1}} \cdots s_{\beta_n} \text{tv}\left( s_{\beta_{i+1}} \right) + \text{tv}\left( s_{\beta_n} \right).
\]
Using the previous formula in an easy induction on the number of factors, the assertion follows.
4. The Coxeter Transformations

In this section we prove two properties for Coxeter transformations in extended Weyl systems that also hold for Coxeter elements in (finite) Coxeter systems. First we show that, as in finite Coxeter systems, all the Coxeter transformations are conjugate in $W$. Then we will recall that for a Coxeter transformation $c \in W$ the length of $c$ with respect to the set $T$ of all reflections is $|S|$.

We use the notation as introduced in Figure 1 and abbreviate the simple reflection with respect to the root $\alpha_{(i,j)}$ by $s_{ij}$. As defined in [2.4] (g), the element

$$c = \left( \prod_{i=1}^{p_i} s_{ij} \right) s_1 s_1^* \in W$$

is a Coxeter transformation of $(W, S)$, and every two Coxeter transformations only differ by the ordering of the reflections $s_{ij}$, where $1 \leq i \leq r$ and $1 \leq j \leq p_i$.

It is a consequence of [6] Chapter V, § 6, Lemma 1] that in the definition of the Coxeter transformation the ordering of the reflections $s_{ij}$, where $1 \leq i \leq r$ and $1 \leq j \leq p_i$, does not matter up to conjugacy. For the convenience of the reader we give a direct proof of this fact.

**Lemma 4.1.** Let $(W, S)$ be an extended Weyl system. Then in $W$ all the Coxeter transformations are conjugated.

**Proof.** Let $d$ be another Coxeter transformation of $(W, S)$. Because of the shape of the diagram and the definition of a Coxeter transformation, we have

$$d = \left( \prod_{i=1}^{p_i} s_{(i, j)} \right) s_1 s_1^*,$$

where $\pi_i$ is in $\text{Sym}(p_i)$ for $1 \leq i \leq r$. Thus $c_i := \prod_{j=1}^{p_i} s_{ij}$ and $d_i := \prod_{j=1}^{p_i} s_{(i, j)}$ are Coxeter elements in a Coxeter system $\Lambda_i$ of type $A_{p_i}$ with set of simple reflections $S_i := \{s_{ij} \mid 1 \leq j \leq p_i\}$ for every $1 \leq i \leq r$.

We claim that $P_i := \langle s_{i2}, \ldots, s_{ip_i} \rangle$ acts transitively on the set of Coxeter elements in $W_i := \langle S_i \rangle$, more precisely that there is $x_i \in P_i$ such that $c_i^{x_i} = d_i$. As $(W_i, S_i)$ is a Coxeter system of type $A_{p_i}$, the Coxeter elements $c_i$ and $d_i$ are $(p_i + 1)$-cycles in $W_i \cong \text{Sym}(p_i + 1)$. Without loss of generality we can assume that $c_i = (1 \ 2 \ \ldots \ \ p_i + 1)$ and $d_i = (1 \ k_2 \ \ldots \ k_{p_i + 1})$. Then $x_i \in P_i$ defined by $x_i(j) := k_j$ for $2 \leq j \leq p_i + 1$ maps $c_i$ onto $d_i$.

Now it follows, as every element of $P_i$ commutes with $s_1$, $s_1^*$ as well as with every element in $S_k$ for $k \neq i$, that $x := x_1 \cdots x_r$ conjugates $c$ onto $d$. □

**Remark 4.2.** Let $(W, S)$ be an extended Weyl system of domestic type. Direct calculations show that the Coxeter transformation $c$ defined above is conjugate to a Coxeter element in the affine Coxeter system $(W, Q)$ (see Remark 2.5). This yields that every Coxeter transformation of the extended Weyl system $(W, S)$ is also a Coxeter element of the affine Coxeter system $(W, Q)$, but not vice versa.

We recall the reflection length of a Coxeter transformation $c$ of the extended Weyl group $W$, that is, we recall the number

$$\ell_T(c) := \min \{ n \in \mathbb{N} \mid t_1 \cdots t_n = c, \ t_i \in T \text{ for } 1 \leq i \leq n \}.$$

It will be an ingredient of the proof of Theorem 1.1.

**Proposition 4.3** ([5] Proposition 1.1). Let $W$ be an extended Weyl group of rank $n$. Then $\ell_T(c) = n$ for every Coxeter transformation $c \in W$. 
5. A remark on Coxeter groups whose diagram is a star

In this section we present some information on the root system $\Phi$ for a Coxeter system $(\overline{W}, \overline{S})$ whose diagram is a simply laced star. We use the notation as introduced in Definition 2.1(c). In particular, $\overline{B}$ is a simple system for $\Phi$, and we use the numbering of the roots in $\overline{B}$ as given in Figure 1. The reflection with respect to the root $\alpha_{ij} \in \overline{B}$ will again be abbreviated by $s_{ij}$. The results obtained in this section will be used in the proof of Theorem 1.1. We start with a little observation.

**Lemma 5.1.** Let $\beta_1, \ldots, \beta_n \in B$ such that $\overline{W} = \langle s_{\beta_1}, \ldots, s_{\beta_n} \rangle$. Then $L(\Phi) = L(\{\beta_1, \ldots, \beta_n\})$.

**Proof.** As the Coxeter diagram is a tree, all the reflections are conjugated in $\overline{W}$. In particular, for every $\beta \in \Phi$ there exists $w \in \overline{W} = \langle s_{\beta_1}, \ldots, s_{\beta_n} \rangle$ such that $s_w(\beta_1) = ws_{\beta_1}w^{-1} = s_\beta$. It follows that $\beta = \pm w(\beta_1) \in \text{span}_Z(\beta_1, \ldots, \beta_n)$. Therefore $L(\Phi) \subseteq \text{span}_Z(\beta_1, \ldots, \beta_n)$. □

**Lemma 5.2.** Let $\alpha \in B$ such that $\alpha = \alpha_1 + \sum_{i=1}^{r} \sum_{j=1}^{p_i} \lambda_{ij} \alpha_{ij}$, where $\lambda_{ij} \in \mathbb{N}_0$. Then for each $i \in \{1, \ldots, r\}$ either

(a) $\lambda_{ij} = 0$ for $1 \leq j \leq p_i$, or
(b) there is $m_i \in \{1, \ldots, p_i\}$ such that $\lambda_{ij} = 1$ for $1 \leq j \leq m_i$ and $\lambda_{ij} = 0$ for $m_i < j \leq p_i$.

**Proof.** First suppose that $\lambda_{11} = 0$ for some $\ell \in \{1, \ldots, r\}$. Furthermore, assume that $\lambda_{\ell m} \neq 0$ for some $m \in \{p_2, \ldots, p_r\}$. Without loss of generality we may assume that $m$ is maximal with this property, that is, $\lambda_{\ell j} = 0$ for all $j > m$. We calculate

$$s_{\ell 2} \cdots s_{\ell m}(\alpha) = \alpha_1 - \lambda_{\ell m} \alpha_{(\ell, 1)} + \sum_{j=2}^{m} (\lambda_{\ell, j-1} - \lambda_{\ell m}) \alpha_{(\ell, j)} + \sum_{i=1}^{r} \sum_{j=1}^{p_i} \lambda_{ij} \alpha_{(i, j)}$$

Since $s_{\ell 2} \cdots s_{\ell m}(\alpha)$ is a positive root it follows $\lambda_{\ell m} = 0$, contrary to our assumption. This shows (a).

Now assume $\lambda_{\ell 1} \neq 0$. Further let $m \in \{2, \ldots, p_\ell\}$ such that $\lambda_{\ell m} \neq 0$, but $\lambda_{\ell j} = 0$ for all $j > m$. Then

$$s_{\ell 1} \cdots s_{\ell m}(\alpha) = \alpha_1 + (1 - \lambda_{\ell m}) \alpha_{(\ell, 1)} + \sum_{j=2}^{m} (\lambda_{\ell, j-1} - \lambda_{\ell m}) \alpha_{(\ell, j)} + \sum_{i=1}^{r} \sum_{j=1}^{p_i} \lambda_{ij} \alpha_{(i, j)}.$$

Since $s_{\ell 2} \cdots s_{\ell m}(\alpha)$ is a positive root and $\lambda_{\ell m} \geq 1$ we obtain $\lambda_{\ell m} = 1$, hence $(1 - \lambda_{\ell m}) = 0$. As in the first part of this proof, this yields $\lambda_{\ell, j-1} - \lambda_{\ell m} = 0$ for $j \in \{2, \ldots, m\}$, which shows (b). □

**Corollary 5.3.** The parabolic subgroup $P := \{s_{ij} \mid 1 \leq i \leq r, 1 \leq j \leq p_i\}$ of $\overline{W}$ acts transitively on the set of roots $(\alpha_1 + \Lambda) \cap \Phi$, where $\Lambda := \text{span}_Z(\alpha_{(1,1)}, \ldots, \alpha_{(r, p_r)})$. In particular, $P$ acts by conjugation transitively on the set of reflections $\{t \in T \mid W = \langle P, t \rangle\}$.

**Proof.** Let $\beta = \sum_{i=1}^{r} \sum_{j=1}^{p_i} \lambda_{ij} \alpha_{(i, j)} \in \Lambda$ and $\alpha = \alpha_1 + \beta$. If $\lambda_{11} \neq 0$ for some $1 \leq i \leq r$, then there is $m_i \in \{1, \ldots, p_i\}$ such that $\lambda_{ij} = 1$ for $1 \leq j \leq m_i$ and $\lambda_{ij} = 0$ for $m_i < j \leq p_i$ by Lemma 5.2. If we set $w := s_{\ell 1} \cdots s_{\ell m}$ then we get, as in the proof of Lemma 5.2, that $w(\alpha)$ is a linear combination of simple roots that do not contain a root $\alpha_{(i, j)}$ for some $1 \leq j \leq p_i$. Now the assertion follows by induction on $r$.

We obtain the last assertion as the set of reflections with respect to the set of roots $(\alpha_1 + \Lambda) \cap \Phi$ equals $\{t \in T \mid W = \langle P, t \rangle\}$ by Lemma 5.1. □
6. Hurwitz transitivity for Coxeter transformations of extended Weyl groups of domestic or wild type

The aim of this section is to prove the main result of this paper, that is the Hurwitz transitivity on the set of reduced factorizations of a Coxeter transformation in the case of an extended Weyl group of domestic or wild type. First we collect some results on the Hurwitz action in Coxeter groups that we will need in the proof.

6.1. Hurwitz action in Coxeter groups. Let \((W, S)\) be an arbitrary Coxeter system of finite rank. Let as usual be the length of \(w \in W\) with respect to \(S\):
\[
\ell_S(w) := \min\{k \in \mathbb{N}_0 \mid w = s_1 \cdots s_k, \ s_1, \ldots, s_k \in S\}.
\]
Further let \(T := \bigcup_{w \in W} ws^{-1}\) be the set of reflections in \((W, S)\), and \(\ell_T\) the length function on \(W\) with respect to \(T\). A parabolic Coxeter element is an element in \(W\) that is a Coxeter element in a parabolic subgroup of \(W\) (see [3]). Let \(c_0\) be a parabolic Coxeter element of \(W\) of length \(\ell_T(c_0) = n\). Then the set of reduced reflection factorizations of \(c_0\) in \(W\) is
\[
\text{Red}_T(c_0) := \{(t_1, \ldots, t_n) \in T^n \mid c_0 = t_1 \cdots t_n\}.
\]
Next we quote three results on the Hurwitz action in Coxeter systems, whose proofs can be found at the given references.

Theorem 6.1 ([3] Theorem 1.3]). Let \((W, S)\) be a Coxeter system and \(c_0\) a parabolic Coxeter element in \(W\). Then the Hurwitz action of \(B_n\) is transitive on \(\text{Red}_T(c_0)\).

The following lemma is a result about the Hurwitz orbits of non-reduced reflection factorizations.

Lemma 6.2 ([32] Lemma 2.3]). Let \((W, S)\) be a Coxeter system, \(w \in W\) and \(w = t_1 \cdots t_{n+2k}\) where \(n = \ell_S(w)\), \(k \in \mathbb{N}\) and \(t_i \in T\) for \(1 \leq i \leq n + 2k\). Then there exists a braid \(\sigma \in B_{n+2k}\) and reflections \(r_1, \ldots, r_n, r_i, r_{i_1}, \ldots, r_{i_k}\) such that
\[
\sigma(t_1, \ldots, t_{n+2k}) = (r_1, \ldots, r_n, r_{i_1}, r_{i_1}, \ldots, r_{i_k}, r_{i_k}).
\]
The proof of the next lemma also appears implicitly in the proof of [23] Theorem 1.1.

Lemma 6.3 ([32] Lemma 2.6]). Let \((W, S)\) be a Coxeter system and let \(t_1, \ldots, t_n, t \in T\). Then \((t_1, \ldots, t_n, t, t)\) and \((t_1, \ldots, t_n, t^w, t^w)\) lie in the same Hurwitz orbit for every \(w \in \langle t_1, \ldots, t_n \rangle\).

6.2. The Proof of Theorem 1.1] We use again the notation introduced in Section 2. So, let \((W, S)\) be an extended Weyl group of rank \(m\) of wild or domestic type with simple system \(S\), set of reflections \(T\) and a Coxeter transformation \(c \in W\). By definition \(c\) admits a factorization into pairwise different simple reflections such that \(s_1 s_1^*\) is a suffix of this factorization.

In this subsection we prove the Hurwitz transitivity on the set of reduced reflection factorizations of \(c\), namely the transitivity of the braid group action on the set
\[
\text{Red}_T(c) := \{(t_1, \ldots, t_m) \in T^m \mid c = t_1 \cdots t_m\}.
\]
Notice that by Proposition 4.3 the elements of \(\text{Red}_T(c)\) are the shortest possible reflection factorizations of \(c\). First we consider a special situation.

Lemma 6.4. Let \(s_{\beta_1} s_{\beta_2} = s_1 s_1^*\) with \(\beta_1, \beta_2 \in \Phi\). Then \((s_{\beta_1}, s_{\beta_2})\) and \((s_1, s_1^*)\) lie in the same Hurwitz orbit, that is, \(B_2(s_1, s_1^*) = \{(t_1, t_2) \in T^2 \mid t_1 t_2 = s_1 s_1^*\}\).
Proof. By considering the projection we get $s_{\alpha_1} s_{\alpha_2} = s_{\alpha_2} s_{\alpha_1} = s_1 s_1^\ast = 1 \in \overline{W}$, that is we can assume that $\beta := \frac{s_1}{s_2} \in \overline{\Phi} = \text{span}_{\mathbb{Q}}(\{B\}) \cap \overline{\Phi}$. We have $tv(s_{\beta_i}) = \lambda_i \beta, \lambda_i \in \mathbb{Z}$ for $i = 1, 2$ (see Lemma 3.12 (b)). Therefore we obtain by Lemma 3.13 (c)

$$\alpha_1 = tv(s_1 s_1^\ast) = tv(s_{\alpha_1} s_{\alpha_2}) = s_{\beta} (tv(s_{\beta_1})) + tv(s_{\beta_2}) = s_{\beta}(\lambda_1 \beta + \lambda_2 \beta = (\lambda_2 - \lambda_1) \beta.$$ 

As the projected root system $\overline{\Phi}$ is reduced we get $\beta = \pm \alpha_1$. By Lemma 3.1 $s_{\beta_1}$ and $s_{\beta_2}$ are reflections of the infinite dihedral group $\overline{W} := \langle s_1, s_1^\ast \rangle$ with simple system $S' := \{s_1, s_1^\ast\}$. Therefore $s_{\beta_1} s_{\beta_2}$ is a reduced reflection factorization of the Coxeter element $s_1 s_1^\ast$ in $\overline{W}$. By [3 Theorem 1.3] the factorizations $(s_{\beta_1}, s_{\beta_2})$ and $(s_1, s_1^\ast)$ lie in the same Hurwitz orbit. 

Proof of Theorem 1.1 Let $c = s_1' \cdots s_m' s_1 s_1^\ast$ be a factorization of a Coxeter transformation $c$ in $(\overline{W}, S)$ in the pairwise different simple reflections of the extended Weyl group $(\overline{W}, S)$, that is $S = \{s_1', \ldots, s_m', s_1, s_1^\ast\}$. By Proposition 4.3 this factorization is $T$-reduced. Let $\alpha'_i \in \overline{\Phi}$ such that $s_{\alpha'_i} = s_i'$ for $1 \leq i \leq m - 2$, and $s_{\alpha_1} = s_1, s_{\alpha_1^\ast} = s_1^\ast$. Further fix a reduced factorization $(t_1, \ldots, t_m) \in \text{Red}_T(c)$. We prove the theorem by showing that there exists a braid $\tau \in \mathcal{B}_m$ such that

$$\tau(t_1, \ldots, t_m) = (s_1', \ldots, s_m', s_1, s_1^\ast).$$

Consider the factorization $(\overline{t}_1, \ldots, \overline{t}_m) \in \overline{W}_m$ where $\overline{t}_i \in \overline{T}$ for $1 \leq i \leq m$ that is induced by $(t_1, \ldots, t_m) \in \overline{W}_m$. In $\overline{W}$ we calculate

$$\overline{t}_1 \cdots \overline{t}_m = \overline{c} = s_1' \cdots s_{m-2} s_1 s_1^\ast = s_1' \cdots s_{m-2},$$

where $s_i' = s_i'$ for $1 \leq i \leq m$ and $s_1^\ast = s_1$. Since $\overline{t}_i(\overline{c}) = m - 2$, Lemma 6.2 yields the existence of a braid $\tau_1 \in \mathcal{B}_{m+2}$ and a reflection $t \in T$ such that

$$\tau_1(\overline{t}_1, \ldots, \overline{t}_m) = (\overline{t}_1, \ldots, \overline{t}_{m-2}, t, t).$$

Thus we have

$$\overline{t}_1' \cdots \overline{t}_{m-2}' = \overline{t}_1' \cdots \overline{t}_{m-2}' tt = \overline{c} = s_1' \cdots s_{m-2}'.$$

Therefore $\overline{t}_1' \cdots \overline{t}_{m-2}'$ is a reduced reflection factorization of the parabolic Coxeter element $\overline{c}$ of the Coxeter system $(\overline{W}, S)$. By Theorem 6.1 there exists a braid $\tau_2 \in \mathcal{B}_m$ such that

$$\tau_2(\overline{t}_1', \ldots, \overline{t}_{m-2}', t, t) = (s_1', \ldots, s_m', t, t).$$

Applying the braid $\tau_2 \tau_1$ to the initial factorization and using Lemma 3.9 we obtain

$$\tau_2 \tau_1(t_1, \ldots, t_m) = (t_1', \ldots, t_m', t_a, t_b)$$

where $t_1', \ldots, t_m', t_a, t_b \in T$ such that $\overline{t}_i = s_i'$ for $1 \leq i \leq m - 2$ and $\overline{t}_a = \overline{t}_b = t$. Let $\alpha \in \overline{\Phi}$ such that $t = \overline{\alpha}$. Then $t_a t_b$ has normal form $(1, \lambda \alpha)$ for some $\lambda \in \mathbb{Z}$ by Lemma 4.8 (a). Therefore we obtain using Lemma 3.13 (c), and setting $\beta := tv(t_1' \cdots t_{m-2}')$:

$$\alpha_1 = tv(c) = tv(t_1' \cdots t_{m-2}' t_a t_b) = \beta + tv(t_a t_b) = \beta + \lambda \alpha.$$ 

The fact that $\overline{t}_i = s_i'$ for $1 \leq i \leq m - 2$ yields $\overline{t}_1' \cdots \overline{t}_{m-2}' \in P := \langle s_1', \ldots, s_m' \rangle$ and $\beta \in \text{span}_\mathbb{Z}(\alpha_1', \ldots, \alpha_1')$. As $\{\alpha_1, \alpha_1', \ldots, \alpha_1'\}$ is linear independent, it follows $\lambda \neq 0$ and therefore

$$\alpha = \frac{1}{\lambda} \alpha_1 - \frac{1}{\lambda} \beta \in \overline{\Phi}.$$ 

The latter implies, as $\alpha \in \overline{\Phi}^+$, that $\lambda = 1$. Hence $\alpha = \alpha_1 - \beta$. 
By Corollary 3.3 there exists $x \in P$ such that $t^x = s^x_\alpha = s_1$ and therefore we get $t^x_a = t^x_b = s_1$ by Lemma 3.9. Since $t^i = s^i_i$ for $1 \leq i \leq m - 2$, Lemma 6.3 yields the existence of a braid $\tau_3 \in B_m$ such that 

$$\tau_3(t''_1, \ldots, t''_{m-2}, t'_a, t'_b) = (t'_1, \ldots, t''_{m-1}, t'_a, t'_b).$$

Set $t'_a = t^x_a$ and $t'_b = t^x_b$, and observe that $tv(t'_a t'_b) = \lambda' \alpha_1$ for some $\lambda' \in \mathbb{Z}$.

Similar as above we can use Lemma 3.13(c) to obtain

$$\alpha_1 = tv(c) = tv(t'_1 \ldots t''_{m-2} t'_a t'_b) = \beta + tv(t'_a t'_b) = \beta + \lambda' \alpha_1.$$ 

This yields, as $\alpha_1$ and $\beta$ are linear independent, that $\beta = 0$ and $\lambda' = 1$.

Since the roots related to the reflections $t^i$ where $1 \leq i \leq m - 2$ are linearly independent, Lemma 3.15 yields that

$$tv(t''_1) = \ldots = tv(t''_{m-2}) = 0,$$

and therefore $t''_i = s'_i$ for $1 \leq i \leq m - 2$. From $t''_1 = t''_2 = s_1$ and $\lambda' = 1$ follows $t'_a t'_b = s_1 s_1^\ast$.

Therefore by Lemma 6.4 there exists a braid $\tau_4 \in B_m$ such that

$$\tau_4(s'_1, \ldots, s'_{m-2}, t'_a, t'_b) = (s'_1, \ldots, s'_{m-2}, s_1, s_1^\ast).$$

Altogether, setting $\tau := \tau_4 \tau_3 \tau_2 \tau_1 \in B_m$, we obtain

$$\tau(t_1, \ldots, t_m) = (s'_1, \ldots, s'_{m-2}, s_1, s_1^\ast),$$

the assertion. $\square$

7. Two applications of the main theorem

7.1. The category of coherent sheaves over a weighted projective line. In the representation theory of finite dimensional algebras hereditary categories are an important tool, since they serve as prototypes for many phenomena appearing there. An important class of these objects are the hereditary ext-finite abelian $k$–categories with tilting object where $k$ is an algebraically closed field. They are classified up to derived equivalence by Happel in [13]. Up to derived equivalence they are the categories of coherent sheaves over a weighted projective line in the sense of Geigle and Lenzing [13] and the categories of finite dimensional modules over a hereditary finite dimensional $k$–algebra.

One way to study their structure is to investigate the lattice of thick subcategories.

**Definition 7.1.** Let $A$ be an abelian category and $H$ a full subcategory of $A$. The category $H$ is called thick if it is abelian and closed under extensions.

If $H$ is a collection of objects in $A$, we denote the smallest thick subcategory that contains $H$ by $\text{Thick}(H)$ and call it the thick subcategory generated by $H$.

In [5] the authors show that one can attach to the category of coherent sheaves over a weighted projective line an extended Weyl group by proving the existence of a complete exceptional sequence that induces an extended Coxeter-Dynkin diagram as given in Figure 4. This information can be used to 'classify' to some extend all the thick subcategories of an hereditary ext-finite abelian category with tilting object that are generated by an exceptional sequence, as we will describe next. First we include the necessary definitions.

**Definition 7.2.** In an abelian category $A$ an object $E$ is called exceptional if $\text{End}_A(E) = k$ and $\text{Ext}^i_A(E,E) = 0$ for $i > 0$. A pair $(E,F)$ of exceptional objects in $A$ is called exceptional pair provided it holds in addition $\text{Ext}^i_A(F,E) = 0$ for $i > 0$. A sequence $E = (E_1, \ldots, E_r)$ of exceptional objects in $A$ is called exceptional sequence of length $r$ if $\text{Ext}^i_A(E_j, E_i) = 0$ for $i < j$.
and all $s > 0$. An exceptional sequence in $\mathcal{A}$ is called **complete** if the smallest thick subcategory which contain the sequence is $\mathcal{A}$.

In order to formulate our 'classification' of the thick subcategories that are generated by an exceptional sequence we need some further notation in the extended Weyl group.

**Definition 7.3.** Let $W$ be an extended Weyl group, $T$ its set of reflections and $c$ a Coxeter transformation. Define a partial order on $W$ by

$$x \leq y \text{ if and only if } \ell_T(x) = \ell_T(x) + \ell_T(x^{-1}y).$$

We let $[id, c] = \{x \in W \mid id \leq x \leq c\}$, and call the poset $([id, c], \leq)$ interval poset. The poset $[id, c]$ consists of the prefixes of the reduced reflection factorizations of $c$. We call the factorization $(t_1, \ldots, t_n)$ of $c$ generating, if $W = \langle t_1, \ldots, t_n \rangle$. Then the subposet $[id, c]^{\text{gen}}$ of $[id, c]$ is by definition the set of prefixes of reduced generating factorizations of $c$ ordered by $\leq$.

An easy consequence of our main result Theorem 1.1 is that $[id, c] = [id, c]^{\text{gen}}$ for a Coxeter transformation $c \in W$ and $W$ of domestic or wild type. For instance for $W$ of tubular type $E_6^{(1,1)}$ the authors show in [5] that $[id, c] \neq [id, c]^{\text{gen}}$ for each Coxeter transformation. By [14] Theorem 3.1, the combination of [21] Theorem 6.10 with [7] Theorem 5.1 and [5] Theorem 1.1, together with the main result of this paper (Theorem 1.1) yields the following theorem.

**Theorem 1.4.** Let $\mathcal{C}$ be a hereditary connected ext-finite abelian $k$–category with a tilting object. Let $\Phi$ be the associated generalized root system, $W$ the extended Weyl group and $c \in W$ a Coxeter transformation. Then there exists an order preserving bijection between

- the poset of thick subcategories of $\mathcal{C}$ that are generated by an exceptional sequence and ordered by inclusion; and
- the subposet $[id, c]^{\text{gen}}$ of the interval poset $[id, c]$.

### 7.2. Singularity theory

For details and definitions we refer to [10] and the references therein.

Let $f : (\mathbb{C}^{k+1}, 0) \to (\mathbb{C}, 0)$ be a holomorphic function germ with an isolated singularity at the origin and let $k$ be even. Further assume that $f$ defines a **simple elliptic singularity** in the sense of Saito [27] or a **hyperbolic singularity**, that is, a singularity (up to stable equivalence) of the series

$$T_{p,q,r} : x^p + y^q + z^r + a xyz,$$

where $a \neq 0$, $2 \leq p \leq q \leq r$ and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$ (see also [2]). These singularities are unimodal and by [12] a Coxeter-Dynkin diagram $\Gamma$ with respect to a distinguished base is given by a diagram as in Figure 2.

The monodromy group of this singularity is given by the extended Weyl group $W$ attached to the diagram $\Gamma$. Denote by $\Lambda^*$ the set of vanishing cycles of $f$ and by $M$ the corresponding Milnor lattice. They are given by the generalized root system attached to $\Gamma$ and the corresponding root lattice. Denote by $h_\ast : M \to M$ the (classical) monodromy operator of $f$. We can identify the latter one with a Coxeter transformation in $W$. As a consequence of Theorems 1.1 and [5] Theorem 1.3 we obtain:

**Theorem 1.5** The set of all distinguished bases of vanishing cycles of $f$ is given by the set

$$\{(\delta_1, \ldots, \delta_n) \in (\Lambda^*)^n \mid \text{span}_\mathbb{Z}(\delta_1, \ldots, \delta_n) = M, \ s_{\delta_1}\cdots s_{\delta_n} = h_\ast\}.$$
Figure 2. The diagram $T_{p,q,r}$

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