Collective dipole oscillations in a mixture of Bose and Fermi superfluids in the BCS–BEC crossover

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Abstract

We present a study for the collective dipole oscillations of a mixture of Bose and Fermi superfluids in the crossover from Bardeen–Cooper–Schrieffer (BCS) superfluid to a molecular Bose–Einstein condensate (BEC) in the ENS experimental setup (Ferrier-Barbut et al 2014 Science 345 1035). The dynamics of the double superfluidity is described by coupled time-dependent order-parameter equations, which are Gross–Pitaevskii equations for the Bose superfluid, coupled to the order-parameter equation including the equation of state fitting from the experimental data for the Fermi superfluid in the BCS–BEC crossover. The numerical simulations show that due to the boson–fermion interaction, the frequencies of the dipole oscillations of the Bose and Fermi superfluids are both downshifted and the frequency shifts increase monotonically from the BCS side to BEC side, which are in agreement with the experiment. We further study the dependencies of the frequency shifts on the oscillation amplitudes and the nonlinear effects, in which the Bose and Fermi superfluids show different features. The frequency shifts of the Bose superfluid increase linearly as the oscillation amplitudes increase, while the frequency shifts of the Fermi superfluid show a quadratical increase. It is found that the intrinsic dipole oscillations of the Bose (Fermi) superfluid can be triggered by only boosting the Fermi (Bose) superfluid.

1. Introduction

The recent experimental realization of mixtures of Bose and Fermi superfluids in ultra-cold atoms advances an essential step to explore double superfluidity belonging to different statistics [1, 2], which is a long-sought goal in liquid helium, and opens a new scenario for the physics of superfluid mixtures [3–6]. Motivated by the breakthrough experiments [1–6], many theoretical studies have been done, such as Chandrasekhar–Clogston critical polarization [7], atom–dimer scattering properties [8, 9], quasiparticle excitation [10, 11], counterflow instability [12], dark-bright soliton [13], and vortex [14–16].

Collective excitations [17–21] play an important role in understanding the dynamics of quantum many-body systems [22–27]. Since the realization of a mixture of the Bose and Fermi superfluids in the BCS–BEC crossover [1], the experiments first concentrate on the studies of the simplest collective excitation, i.e. dipole mode [1, 2, 5, 6], which are quite useful to obtain deep insight in physical systems [28–30]. For a single component, the Kohn theorem [31] states that the center-of-mass of the entire cloud oscillates back and forth in a harmonic trap with the natural frequency of the trap independent of oscillation amplitude and interatomic interaction. However, in coupled systems inter-species interaction modifies the dipole oscillations and gives rise to a rich set of new phenomena [1, 2, 5, 6, 28–30, 32–34]. A long-time dipole oscillation is first observed in a mixture of Bose and Fermi superfluids. In the ENS experiment of 6Li–4Li superfluid mixture [1], the frequency of the Bose superfluid oscillating along the axial direction is found to be downshifted, and the value of the downshift increases monotonically from the BCS side to BEC side. Subsequently, the USTC group studied both
axial and radial dipole oscillations in the $^{41}$K–$^6$Li superfluid mixture which has a large mass imbalance [6]. The frequency shifts of the Bose and Fermi superfluids are both measured in high precision, and the frequency shift of the Fermi superfluid is showed to be much smaller than the bosonic counterpart. It is also observed that the frequency of the Bose superfluid is downshifted (upshifted) in the axial (radial) direction, while the frequencies of the Fermi superfluid are downshifted in the both directions. The variation of the frequency shift shows a non-monotonic behavior around the BCS side [35], differently from a monotonic increase in the $^6$Li–$^6$Li superfluid mixture.

From the theoretical point of view, it is a tough work to describe the dynamics of the superfluid Bose–Fermi mixtures, since they are complicated and macroscopic, which involve two different elements and three different internal states belonging to different statistics, huge particle numbers, intra-species and inter-species interactions, three-dimensional external potential and a very long-lived dynamics. In order to interpret the experimental results [1], the ENS group presents a perturbative calculation. The bosons are treated as a mesoscopic impurity evolving in an effective potential, which is composed of the harmonic trapping potential $V_{bo}$ of the bosons and the mean-field interaction term $\delta_{bf} n_f^0$, where $\delta_{bf}$ is the boson–fermion interaction and $n_f^0$ is the equilibrium density profile of the fermions in the absence of the bosons. The frequency of the dipole oscillation of the Bose superfluid is then written as

$$\tilde{\omega}_b = \omega_{he} \left[ 1 - \frac{1}{2} \delta_{bf} \left( \frac{dn_f^0}{\partial \mu_f} \right)_{r=0} \right],$$

with $\omega_{he}$ the axial frequency of the harmonic trap for the bosons and $\mu_f$ the chemical potential of the Fermi superfluids. Equation (1) is referred as ‘mean-field model’, for the reason that the Bose–Fermi interaction is treated at mean-field level, which can be also derived by the sum-rule method in the supplementary material of [1]. In a subsequent work based on the coupled hydrodynamic equations [36], a scaling method is used to analytically study the eigenfrequencies of the dipole modes of Bose–Fermi superfluid mixtures. If the boson–fermion interaction is ignored in the equilibrium density profiles, the theoretical results reduce to the mean-field model. However, if the boson–fermion interaction is included in the density profiles [34, 36], the signs of the frequency shifts of the Bose superfluid are affected by the ratio of the boson–fermion interaction to the boson interaction and change in the BCS–BEC crossover, which is inconsistent with the experimental observations.

In this work, we simulate the ENS experiment for the dipole oscillations of the Bose–Fermi superfluid mixtures by carrying out an extensive numerical simulation for coupled cylindrically-symmetric order-parameter equations. The coupled time-dependent order-parameter equations [36, 37] are formulated with Gross–Pitaevskii equation [38, 39] for Bose superfluids and the order-parameter equation [40–42] with the equation of state fitting from the experimental data [43] for Fermi superfluids in the BCS–BEC crossover.

We first study the dipole oscillations of Bose–Fermi superfluid mixtures in small amplitudes, and compare the frequency shifts of the dipole oscillations of the Bose and Fermi superfluids in the BCS–BEC crossover. We find that the dipole oscillations of the Bose superfluid have one intrinsic mode and one fermion-forced mode, in contrast to the oscillations of the Fermi superfluid having only one intrinsic mode. These features are different from previous studies [33] on degenerate Bose–Fermi mixtures, where the boson oscillations have only one mode, while the fermion oscillations include three modes. The numerics show that the value of the frequency shift of the Bose superfluid increases monotonically from the BCS side to BEC side, in consistent with the experiment. The monotonic increase of the frequency shift of the Fermi superfluid from the BCS side to BEC side is also predicted. By calculating the time evolutions of Bose–Fermi interaction energy, the increase of the frequency shift is found to be resulted from the increase of the exchange of the Bose–Fermi interaction energy between other energy. Subsequently, we study the frequency shifts of the dipole oscillations of the Bose and Fermi superfluids as a function of the oscillation amplitude. As the oscillation amplitude increases, the frequency shift of the Bose superfluid increases linearly, while the frequency shift of the Fermi superfluid shows a quadratical increase. By plotting time evolutions of the radial and axial density profiles of the Bose and Fermi superfluids, we find that as the oscillation amplitude increases, the nonlinear effect first results in the distortions of the axial density profiles of the Fermi superfluids, then the axial density profiles of the Bose superfluids, finally even the radial excitations of the Fermi superfluids. In order to mimic dipole oscillations of the Bose superfluid in the effective potential provided by the Fermi superfluid, we also study dipole oscillation of the Bose (Fermi) superfluid in an initially static Fermi (Bose) superfluid. We find that the frequency shifts of the dipole oscillations of the Bose superfluid are downshifted, irrespective of the effective potential that is repulsive or attractive. The frequency shifts of the Bose and Fermi superfluids in the BCS–BEC crossover are found to be the same as those in the previous cases, indicating that the dipole oscillations triggered are intrinsic modes.

The rest of the paper is organized as follows. In section 2, we introduce the theoretical model and the numerical method. In section 3, we present the results for the collective dipole oscillations of the Bose–Fermi
superfluid mixtures in the BCS–BEC crossover, and compare to the experiment. The conclusion is given in section 4.

2. Coupled order-parameter equations

We consider a Bose–Fermi mixture in which both species are superfluid. The Bose superfluidity is a condensation of bosons, while the Fermi superfluidity is originated from pairing of fermions in two different spin states and changes in the crossover from a BEC of molecules to a BCS superfluid of fermion pairs. Superfluid is quantum phase of matter in which the atoms condensate into a single macroscopic quantum state, indicating that it can be described by a complex macroscopic wave function (order parameter). In terms of two complex-value order parameters $\Psi_b$ for bosons [38, 39] and $\Psi_f$ for fermion pairs [44, 45], respectively, the coupled time-dependent order-parameter equations for dynamics of a Bose–Fermi superfluid mixture are given by [36, 37]

$$i\hbar \frac{\partial \Psi_b(r, t)}{\partial t} = \left[ -\frac{\hbar^2 \nabla^2}{2m_b} + V_{bo}(r) + g_b|\Psi_b|^2 + 2g_{bf}|\Psi_f|^2 \right] \Psi_b(r, t), \quad (2a)$$

$$i\hbar \frac{\partial \Psi_f(r, t)}{\partial t} = \left[ -\frac{\hbar^2 \nabla^2}{2m_p} + V^o_f(r) + \mu_p(n_f) + 2g_{bf}|\Psi_b|^2 \right] \Psi_f(r, t), \quad (2b)$$

$$\mu_p(n_f) = 2\mu(n_f) = 2\mu\left( \frac{n_f}{n_o} \right)^\gamma, \quad \mu^0 = \epsilon_f [\sigma(\eta) - \eta \frac{\partial \sigma(\eta)}{\partial \eta}],$$

$$\gamma = \gamma(\eta) = \frac{n_f}{n_o} \frac{\partial \mu}{\partial n_f} = \frac{2\sigma(\eta) - 2\mu^0(\eta) + \eta^2(\eta) + \eta^2(\eta)}{\sigma(\eta)} - \frac{\sigma(\eta)}{\gamma(\eta)}, \quad (2c)$$

where the two isotopes feel the same harmonic potential $V_{bo} = m_b(\omega_{b,1}^2 r^2 + \omega_{b,2}^2 z^2)/2 = m_f(\omega_{f,1}^2 r^2 + \omega_{f,2}^2 z^2)/2$ in the ENS experiment, with $r^2 = x^2 + y^2$ and $m_b(m_f)$ is the mass of a bosonic (fermionic) atom. The atomic density of Bose superfluid is $n_b = |\Psi_b|^2$ and the total atomic number is determined by the normalization $N_b = \int n_b(r, t) \, dr$. The strength of the boson–boson interaction $g_{bb} = 4\pi \hbar^2 a_{bb}/m_b$ is related to the scattering length $a_{bb}$. Differently from the bosonic part, $\Psi_f$ is the order parameter for condensed fermion pairs, thus in equation (2b) $n_f = 2n_p$ is the mass of a fermion pair, and the harmonic potential $V^o_{bo} = 2V_{bo}$, as well as the equation of state $\mu_p(n_f) = 2\mu(n_f)$ are both for fermion pairs. The present experiments suggest that almost all of fermions can be paired condensed at available low temperature even in the BCS side [1]. The atomic density $n_f = 2n_p$ is twice fermion pair density $n_p = |\Psi_f|^2$, and the total atomic number is determined by the normalization $N_f = 2n_p(r, t) \, dr$. The two-spin fermionic interaction in different superfluid regimes is characterized by the equation of state equation (2c), which is made by a polytropic approximation [46], where the reference chemical potential $\mu^0$ is proportional to the Fermi energy $\epsilon_f = (\hbar k_f)^2/(2m_f) = \hbar (6n_p \omega_{f,1}^2 \omega_{f,2}^2)^{1/3}$ defined in a cylindrically symmetric trap and reference atomic number density is $n_o = (2m_f \epsilon_f)^{1/2}/(3\pi^2 \hbar^4)$. The effective polytropic index $\gamma$ and reference chemical potential $\mu^0$ are determined by $\sigma(\eta)$ as a function of the dimensionless interaction $\eta = 1/k_f a_f/k_E$ [36]. In order to be close to experimental observations, $\sigma(\eta)$ is based on the explicit expressions of fitting functions from the experimental data [43].

The order-parameter equation (2b) for condensed fermion pairs without the boson–fermion interaction is derived from the density function theory [40, 41], or superfluid hydrodynamic equations by incorporating proper quantum pressure term [42]. The order-parameter equation depending only on a single wave function for the condensed state has the computational advantage. Very recently, the order-parameter equation is used to interpret the experiments on Josephson effect between two fermionic superfluids [47, 48]. The frequencies of Josephson oscillations calculated from the order-parameter equation is shown to be in agreement with the experimental data in the regimes from the BCS to the unitarity limit for small initial imbalance and barrier height [47]. For large initial imbalance, the phase slippage induced by vortex nucleation is found to be the dominant mechanism for emergent dissipations, which is also confirmed by the numerical simulations for the order-parameter equation. However, the order-parameter equation is essentially a zero-temperature theory describing condensed fermion pairs without the fermionic degrees of freedom. It does not contain the dissipative terms describing dynamical coupling of the condensate to the normal components and transferring energy to single particle excitations. Thus, it fails for the BCS side of the crossover and large excitations [49, 50], where the quantum depletion of condensate, breaking of fermionion pairs and finite-temperature effect play a vital role. To describe the interaction between Bose and Fermi superfluids, we have introduced the boson–fermion interaction $g_{bf} = 2\pi \hbar^2 a_{bf}/m_{bf}$ in terms of boson–fermion scattering length $a_{bf}$ and reduced mass...
m_{bf} = m_b m_f / (m_b + m_f) \[8, 9\]. It is worth noting that when \(a_b\) is comparable to \(a_f\) in the BEC limit \(1/k_f a_f \gg 1\), the interaction should be replaced by the boson–dimer interaction \[51\].

In the ground state we can set \(\Psi_b(r, t) = \Psi_{b0}(r)e^{-i\omega t/\hbar}\) with chemical potential \(\mu_b\) for Bose superfluids, and \(\Psi_f(r, t) = \Psi_{f0}(r)e^{-i\omega t/\hbar}\) with chemical potential \(\mu_f\) for Fermi superfluids. The ground states \(\Psi_{b0}(r)\) and \(\Psi_{f0}(r)\) can be obtained by solving the coupled order-parameter equations \(2a\) and \(2b\) using the split-time-step Crank–Nicolson method in a imaginary–time propagation \[52, 53\]. The program algorithms can be found in a Fortran language version \[52\] and a C version \[53\]. Subsequently, we add axial momentums to the superfluids by boosting the order parameters in the following way \[33\] to study the dipole oscillations

\[
\begin{align*}
\Psi_b(r, t = 0) &= e^{i\lambda_b z}\Psi_{b0}(r), \\
\Psi_f(r, t = 0) &= e^{i\lambda_f z}\Psi_{f0}(r),
\end{align*}
\]

(3a)

with the boost parameters \(\lambda_b\) and \(\lambda_f\). The dynamics of the Bose–Fermi superfluids in the harmonic trap are obtained by solving the coupled order-parameter equations in a real-time propagation. We define center-of-mass positions on z coordinate as \(z_b(t) = \frac{1}{N_b} \int dr \, z|\Psi_b(r, t)|^2\) for bosons and \(z_f(t) = \frac{1}{N_f} \int dr \, |\Psi_f(r, t)|^2\) for fermions, respectively. In the absence of the boson–fermion interaction \(g_{bf} = 0\), it is known that the Bose and Fermi superfluids oscillate sinuosoidally and independently \[31\]. The time dependencies of \(z_b\) and \(z_f\) are given by \(z_b = A_b \sin(\omega_{b2} t)\) and \(z_f = A_f \sin(\omega_{f2} t)\), respectively, in which the initial amplitudes \(A_b\) and \(A_f\) are related to the boost parameters by \(A_b = \lambda_b m_b / (m_b + m_f)\) and \(A_f = \lambda_f m_f / (2m_f + m_f)\), respectively.

The ENS experimental parameters are used as follows \[1\]: the total number of \(^7\)Li bosons and \(^6\)Li fermions are given by \(N_b = 4 \times 10^4\) and \(N_f = 3.5 \times 10^5\), respectively. The radial and axial frequencies of the harmonic trap for bosons (fermions) are given by \(\omega_{b2} = 2\pi \times 550\ \text{Hz} (\omega_{f2} = 2\pi \times 595\ \text{Hz})\) and \(\omega_{f2} = 2\pi \times 15.27\ \text{Hz}\) (\(\omega_{b2} = 2\pi \times 16.8\ \text{Hz}\)), respectively. By a magnetic field induced Feshbach resonance, the scattering length \(a_f\) of two-spin fermionic atoms is tunable across the BCS–BEC crossover, and the scattering length \(a_b\) of bosonic atoms is correspondingly tuned in a controlled way. The scattering lengths \(a_f = -1582(1 + 262.3/(B - 832.18))\) and \(a_b = -18.24(1 + 237.8/(B - 893.95))(1 - 4.518/(B - 845.54))\) in units of Bohr radius \(a_0\) as a function of magnetic field \(B\) are provided in the supplementary material of \[1\]. In contrast, the scattering length \(a_{bf} = 40.8a_0\) is independent on the magnetic field and the Bose–fermion interaction remains constant throughout the crossover. In the presence of the boson–fermion interaction, the density profiles of the Bose superfluids at equilibrium are slightly changed, while the density profiles of the Fermi superfluids, especially in the central regions where the Bose superfluids occupy, are significantly modified. For \(g_{bf} > 1\), the Fermi density shows a flat distribution in the center region, and for \(g_{bf} < 1\) \(g_{bf} > 1\), it shows a upcurved (downcurved) distribution \[36\].

3. Numerical results

3.1. Frequency shift across the BCS–BEC crossover

We present the numerical results for the time dependencies of the center-of-mass positions \(z_b\) of the Bose superfluid and \(z_f\) of the Fermi superfluid for two different initial amplitudes, i.e. \(A_{bf} = 20\ \mu m\) and \(A_{bf} = 60\ \mu m\), in figure 1(a) for the BEC side \(1/k_f a_f = 0.68\), figure 1(b) the unitary limit \(1/k_f a_f = 0\) and figure 1(c) the BCS side \(1/k_f a_f = -0.2\). The oscillations of Bose–Fermi mixtures are induced by introducing the global phases \(\theta_b = \lambda_b \varphi\) in equation \(3a\) and \(\theta_f = \lambda_f \varphi\) in equation \(3b\), which correspond to giving initial velocities \(v_{b2} = (\hbar / m_b) \nabla \theta_b = \hbar \lambda_b / m_b\) for the Bose superfluid and \(v_{f2} = (\hbar / m_f) \nabla \theta_f = \hbar \lambda_f / m_f\) for the Fermi superfluid by the phase-velocity relations. It is seen that the sinusoidal oscillations of the Bose superfluids are significantly modulated by interacting with the Fermi superfluids, while the Fermi oscillations are slightly changed. The initial oscillation amplitudes also affect the dipole oscillations. In the following, we first discuss the cases of the small oscillation amplitude \(A_{bf} = 20\ \mu m\), and compare the frequency shifts of the dipole oscillations of the Bose and Fermi superfluid in the BCS–BEC crossover.

Due to the boson–fermion interaction, the oscillations are no longer simple and include various modes. In order to separate these modes, we look at the strength functions \(S_{bf}(\omega)\), which are defined as the Fourier transformation of \(z_{bf}\)

\[S_{bf}(\omega) = \int_{t_1}^{t_f} dt \, z_{bf} (t) \sin(\omega t),\]

where we have set \(t_1 = 0\) and \(t_f = 1447\ \text{ms}\). In figure 2 we plot the strength functions of the dipole oscillations of the Bose (solid lines) and Fermi (dashed lines) superfluids with the small initial amplitudes \(A_{bf} = 20\ \mu m\) in the BCS–BEC crossover. Figure 2(a) corresponds to the case in the BEC side, figures 2(b) and (c) are the cases in the unitary regime, and figure 2(d) for the BCS side. These figures show that the boson strength functions \(S_b\) have two sharp peaks: one peak \(\tilde{\omega}_b\) locates at the left side of the frequency \(\omega_{b2}\) (denoted by the left dotted line), which corresponds to the intrinsic frequencies of the boson oscillations. Another small one is around the frequency \(\omega_{f2}\)
Figure 1. Time evolutions of the center-of-mass positions $z_b (z_f)$ of the Bose (Fermi) superfluid for the oscillation amplitudes $A_b = 20 \mu m$ and $A_f = 60 \mu m$, respectively: (a) the BEC side ($k_f a_f = 0.68, a_b = 20a_b$), (b) the unitary limit ($k_f a_f = 0, a_b = 69a_b$) and (c) the BCS side ($k_f a_f = -0.2, a_b = 45a_b$). The Bose–Fermi superfluid mixture contains $N_b = 4 \times 10^4$ Li bosons and $N_f = 3.5 \times 10^5$ Li fermions in a cylindrically symmetric trap with radial frequency $\omega_{lb} = 2\pi \times 550$ Hz ($\omega_{lf} = 2\pi \times 595$ Hz) and axial frequency $\omega_{lz} = 2\pi \times 15.27$ Hz ($\omega_{fz} = 2\pi \times 16.8$ Hz) for bosons (fermions). The boson–fermion scattering length is fixed by $a_{bf} = 40.8a_b$.

Figure 2. Strength functions (in unit of $10^{-3}$) of the dipole oscillations of the Bose (solid line) and Fermi (dashed line) superfluids with the oscillation amplitudes $A_b = 20 \mu m$ for the cases of figure 1 in (a) the BEC side, (b) and (c) the unitary regime, and (d) the BCS side. The two vertical dotted lines denote the frequencies $\omega_{lb}$ and $\omega_{lf}$, respectively.
denoted by the right dotted line) and its amplitude monotonously decreases from the BEC side to BCS side, which are fermion-forced oscillations. In contrast, the fermion strengths have only one shark peak locating left and very close to the frequency \( \omega_f \).

The relative frequency shifts \( \delta \omega_b / (\omega_b k_f a_b f) \equiv (\omega_b - \tilde{\omega}_b) / (\omega_b k_f a_b f) \) of the dipole oscillations of the Bose superfluids as a function of the dimensionless interaction \( 1 / k_f a_f \) are shown by the squares in figure 3(a). It is seen that the frequency of the Bose superfluid is downshifted in the BCS–BEC crossover, and the value of the frequency shift monotonically increases from the BCS to the BEC side, which is in consistent with the experiment and the mean-field model (1). It is noting that the discrepancy between the experiment and the numerical results becomes significantly obvious in the BCS side \( 1 / k_f a_f = -0.29 \), where the order-parameter equation for Fermi superfluids is unreliable and the fermionic degree of freedom is important. It would be interesting to study the pair-breaking effects on the BCS side by resorting to a microscopic theory [49, 50]. In figure 3(b) we predict the relative frequency shifts \( \delta \omega_f / (\omega_f k_f a_f f) \equiv (\omega_f - \tilde{\omega}_f) / (\omega_f k_f a_f f) \) of the dipole oscillations of the Fermi superfluids. One can see that the frequencies of the dipole oscillations of the Fermi superfluid in the BCS–BEC crossover are also downshifted, and the value of the frequency shifts monotonically increase from the BCS side to BEC side. The fact that the frequency shift of the Fermi superfluid with larger particle number is one order smaller than that of the bosonic counterpart is also observed in \(^{41}\text{K}\)–\(^{6}\text{Li}\) mixture by the high-precision measurements [6].

The total energy of the Bose–Fermi superfluid mixture \( E[\Psi_0, \Psi_S] = E_b[\Psi_0] + E_f[\Psi_S] + E_{bf}[\Psi_0, \Psi_S] \) is the sum of the energy of the Bose superfluid \( E_b \), the energy of the Fermi superfluid \( E_f \) and the Bose–Fermi interaction energy \( E_{bf} \). The Bose superfluid energy \( E_b = E_{bh} + E_{pt} + E_{bh} \) of the kinetic energy \( E_{bh} \), the trapping potential energy \( E_{pt} \), and the interaction energy \( E_{bh} \), where the three contributions are given by:

\[
E_b[\Psi_0] = \int \mathrm{d}r \left[ \frac{\hbar^2}{2m_b} \left| \nabla \Psi_0 \right|^2 + V_{\text{trapping}}[\Psi_0]^2 + \frac{1}{2} g_0 \left| \Psi_0 \right|^4 \right]
\]  

(5)

The Fermi superfluid energy \( E_f = E_{fh} + E_{ht} + E_{fi} \) is also the sum of the kinetic energy \( E_{fh} \), the trapping potential energy \( E_{ht} \), and the interaction energy \( E_{fi} \). We compare the energy of the Bose superfluid in the BCS–BEC crossover at the initial time \( t = 0 \) for \( A_{\lambda f} = 20 \mu m \) in figure 4(a), and correspondingly the energy of the Fermi superfluid in figure 4(b). The Bose–Fermi interaction energy is defined as:

\[
E_{bf}[\Psi_0, \Psi_S] = \int \mathrm{d}r 2 g_{bf} \left| \Psi_0 \right|^2 \left| \Psi_S \right|^2,
\]

(7)
which is related to the overlap integral of the Bose and Fermi density distributions. From bottom to top in figure 4(c), despite the Bose–Fermi interaction $g_{bf}$ remains constant along the BCS–BEC crossover, the Bose–Fermi interaction energy increases monotonically from the BCS side to BEC side. This is because as we pass from the BCS side to BEC side, the decreases of the bosonic (figure 4(a)) and fermionic (figure 4(b)) interactions result in narrowing the Bose and Fermi density distributions, respectively, and increasing the corresponding number densities. Thus the Bose and Fermi densities overlap integral equation (7) increases from the BCS to BEC side. Furthermore, we find in figure 4(c) that the range of the oscillation of the Bose–Fermi interaction energy and its exchange between other energy increases monotonically from the BCS to BEC side, which implies a more significant effect on the oscillations and leads to a larger frequency shift in the BEC side. By comparing to the energy of the Bose (figure 4(a)) and Fermi superfluid (figure 4(b)), one can find that the Bose–Fermi interaction energy is relatively small. From this perspective, the chosen experimental parameters realize a weak coupling between the Bose and Fermi superfluids. Such small Bose–Fermi coupling cannot result in the distortions of density profiles or other excitations during the evolutions, and allows one to study long-time dipole oscillations without damping.

3.2. Dependence of frequency shift on oscillation amplitude

In this subsection, we study the dipole oscillations of the Bose–Fermi superfluid mixtures for different oscillation amplitudes, and the nonlinear effects on the density profiles of the Bose and Fermi superfluids. The dependence of the frequency shift of the Bose superfluid on the oscillation amplitude is shown in figure 5(a), and those of the fermionic counterpart shown in figure 5(b). It is seen that as the oscillation amplitude increases, the frequency shifts of the Bose and Fermi superfluid both increase. Compared to the Bose superfluids, the oscillation amplitudes take a more significant effect on the frequency shifts on the Fermi superfluids, which can be also seen in figure 1. By fitting the numerical results (the dashed lines), we find the frequency shift of the Bose superfluid is a linear function of the oscillation amplitude, while the frequency shift of the Fermi superfluid shows a quadratical increase. We extract the frequency shifts for the Bose and Fermi superfluids in the zero amplitude limit ($A_b = A_f = 0$), and plot them in the figures 3(a) and (b) by ($\times$), respectively.

As the oscillation amplitude increases, the kinetic collective energy increases and is transformed into other excitations. In order to study the nonlinear effects on the dipole oscillations, in figure 6 we plot the time evolutions of the radial and axial density profiles of the Bose and Fermi superfluids, respectively, at the unitarity $1/k_f a_f = 0$ for the different oscillation amplitudes. For the small oscillation amplitude $A_{bf} = 20 \mu m$ in figure 6(a), no difference is found in the radial density profiles of the Bose and Fermi superfluids. In the axial direction, the Bose and Fermi superfluids do the dipole oscillations as a whole due to the weak coupling, and the modification of the density profile of the Fermi superfluid changes as the Bose superfluid moving inside it. As the oscillation amplitude increases $A_{bf} = 60 \mu m$ in figure 6(b), the radial densities at $z = 0$ of the Bose and Fermi superfluids oscillate due to the large central displace in the axial direction, and no radial excitation is found. After
a long time of the dynamics, one can find that the kinetic collective energy first results in the density fluctuations of the Fermi superfluid in the axial direction in figure 6(b4). For the relatively large oscillation amplitude $A_{bf} = 80\ \mu m$ in figure 6(c), the axial density profiles of the Fermi superfluid are significantly distorted in figure 6(c3) and then the axial density of the Bose superfluid in figure 6(c4). We find that the larger axial dipole oscillations even couple to the radial modes, and result in the distortions of the radial density profiles of the Fermi superfluid in figure 6(c2). The distortions of the density distributions of the Fermi superfluid are seem to be easier and more obvious as the oscillation amplitudes increases, which can be considered as the reason that the frequency shift of the Fermi superfluid increases more significantly than the bosonic counterpart.
In the mean-field model, the frequency of the dipole oscillation of the Bose superfluid is calculated by assuming the Bose superfluid moving in the harmonic trap combined with an effective potential provided by the Fermi superfluid. If one ignores the effect of the Bose superfluid on the density profile of the Fermi superfluid, the effective potential is repulsive in the BCS—BEC crossover in which the oscillation frequency of the Bose superfluid is downshifted [54]. However, the density profile curvature of the Fermi superfluid is inverted for the case of $g_{bf}/g_b > 1$ by taking into account the effect of the Bose superfluid [36], and the effective potential becomes attractive [36] where the frequency of the Bose superfluid is upshifted. Therefore, it is interesting to study the dipole oscillations of the Bose (Fermi) superfluid in an initially static Fermi (Bose) superfluid, on the other hand, by analogy with a superfluid pasting an obstacle [54, 55].

We show the time dependencies of the center-of-mass positions of the Bose ($z_b$) and Fermi ($z_f$) superfluids at unitarity ($1/k_f a_f = 0, g_{bf}/g_b = 0.6$) for the initial oscillation amplitudes $A_b = 40 \mu m, A_f = 0$ in figure 7(a) and $A_b = 0, A_f = 40 \mu m$ in figure 7(b). The oscillation of the Bose (Fermi) superfluid in figures 7(a), (b) are shown to be sinusoidal and independent, whose strength function has only one positive peak in figures 7(a3), (b3), and the frequency of the intrinsic mode $\tilde{\omega}_b(\tilde{\omega}_f)$ is found to be downshifted. In contrast, the oscillation of the Fermi (Bose) superfluid that is initially static is a forced vibration with a beat. The strength function includes two modes: the one is the forced Fermi (Bose) mode $\tilde{\omega}_f(\tilde{\omega}_b)$ caused by the Bose (Fermi) oscillation, and the other one is the intrinsic mode of the Bose (Fermi) oscillation. We find that the frequencies of the two modes are both downshifted, but interestingly the signs of the two peaks are the same for the two different cases, i.e. one corresponding to the Bose superfluid is negative, while another one corresponding to the fermionic counterpart is positive. It is reasonable because the two relative motions are essentially the same. We also study the cases in the BEC ($1/k_f a_f = 0.68, g_{bf}/g_b = 2.2$) and BCS regimes ($1/k_f a_f = -0.2, g_{bf}/g_b = 1.9$) which have the same characteristics. It is inconsistent with the analytical analysis that in the case of $g_{bf}/g_b > 1$ the frequency of the Bose oscillation is upshifted. We present the frequency shifts $\omega_{bz} - \tilde{\omega}_b$ of the dipole oscillations of the Bose superfluids in figure 3(a) and $\omega_{zf} - \tilde{\omega}_f$ of the Fermi superfluids in figure 3(b) by ($\triangle$) for the case of $A_b = 40 \mu m, A_f = 0$, and ($\bigtriangledown$) for $A_b = 0, A_f = 40 \mu m$. It is deduced that the characteristics of the frequency downshifts may be determined by the intrinsic properties of the Bose–Fermi superfluid mixture, independent on the initial oscillation amplitude and interaction, however, a confirmed conclusion should be based on clear analytical expressions in the future.

Finally, we show the time evolutions of the axial density profiles at $r = 0$ of the Bose and Fermi superfluids for the case of $1/k_f a_f = 0$ in figure 8(a) and the case of $1/k_f a_f = 0.68$ in figure 8(b). One can find that the Fermi density profiles alter as the bosons move and the bosons feel the average effects of the modulating effective potentials. This is different from optical potentials that keep stationary and unchanged when they interact with superfluids [54, 55]. Furthermore, it is pointed that the mean-field model is based on a self-consistently
perturbative method by ignoring the Bose–Fermi coupling, and it should be not appropriate to further include the coupling in the effective potentials in such self-consistent theory.

4. Conclusion

In conclusion, we have carried out a numerical simulation for the coupled time-dependent order-parameter equations to perform a detailed study of the dipole oscillations of the mixtures of the Bose and Fermi superfluids in the ENS experimental setup. The frequency shifts of the Bose superfluid in the BCS–BEC crossover are in agreement with the mean-field model which is based on a perturbative analysis and the experimental data. The smaller frequency shifts of the Fermi superfluid are also predicted. We then study the frequency shifts of the Bose and Fermi superfluids for different oscillation amplitudes. As the oscillation amplitude increases, the frequency shifts of the Bose superfluid increases linearly, while the Fermi superfluid shows a quadratical increase. The characteristics of the frequency downshifts along the BCS–BEC crossover are found to be independent on initial oscillation amplitudes. However, the origin of the downshift is currently unclear. We plan to investigate this issue by means of an analytical approach to gain critical factors on the signs of frequency shifts. The coupled time-dependent order-parameter equations, which only depend on two macroscopic wavefunctions, but includes tunable experiment parameters may provide us a phenomenological way to understand the diverse superfluid dynamics in the future experiments for Bose–Fermi superfluid mixtures.

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References

[1] Ferrier-Barbut I, Delehaye M, Laurent S, Grier A T, Pierce M, Rem B S, Chevy F and Salomon C 2014 Science 345 1035
[2] Delehaye M, Laurent S, Ferrier-Barbut I, Jin S, Chevy F and Salomon C 2015 Phys. Rev. Lett. 115 265303
[3] Yao X-C, Chen H-Z, Wu Y-P, Liu X-P, Wang X-Q, Jiang X, Deng Y, Chen Y-A and Pan J-W 2016 Phys. Rev. Lett. 117 145301
[4] Ikemachi T, Ito A, Aratake Y, Chen Y, Koashi M, Kuwata-Gonokami M and Horikoshi M 2017 J. Phys. B: At. Mol. Opt. Phys. 50 01LT01
[5] Roy R, Green A, Bowler R and Gupta S 2017 Phys. Rev. Lett. 118 055301

Figure 8. Time evolutions of the axial density profiles (in units of μm⁻³) of the Bose (n_b, narrow curves) and Fermi (n_f, broad curves) superfluids for the cases of figure 1 (a) at the unitarity and (b) in the BEC regime with the initial oscillation amplitudes (1) A_b = 40 μm, A_f = 0 and (2) A_b = 0, A_f = 40 μm.
[6] Wu Y-P, Yao X-C, Liu X-P, Wang X-Q, Wang Y-X, Chen H-Z, Deng Y, Chen Y-A and Pan J-W 2018 Phys. Rev. B 97 020506(R)
[7] Ozawa T, Recati A, Delehaye M, Chevy F and Stringari S 2014 Phys. Rev. A 90 043608
[8] Zhang R, Zhang W, Zhai H and Zhang P 2014 Phys. Rev. A 90 063614
[9] Cui X 2014 Phys. Rev. A 90 041603(R)
[10] Zheng W and Zhai H 2014 Phys. Rev. Lett. 113 265304
[11] Kinnunen J I and Bruun G M 2013 Phys. Rev. A 91 041605(R)
[12] Chevy F 2015 Phys. Rev. A 91 063606
Abad M, Recati A, Stringari S and Chevy F 2015 Eur. Phys. J. D 69 126
[13] Tylutki M, Recati A, Dalfovo F and Stringari S 2016 New J. Phys. 18 053014
[14] Jiang Y, Qi R, Shi Z-Y and Zhai H 2017 Phys. Rev. Lett. 118 080403
[15] Pan J-S, Zhang W, Yi W and Guo G-C 2017 Phys. Rev. A 95 063614
[16] Wen L and Li J 2014 Phys. Rev. A 90 053621
[17] Jin D S, Ensher J R, Matthews M R, Wieman C E and Cornell E A 1996 Phys. Rev. Lett. 77 420
[18] Mewes M O, Andrews M R, van Druten N J, Kurn D M, Durfee D S, Townsend C G and Ketterle W 1996 Phys. Rev. Lett. 77 988
[19] Modugno G, Modugno M, Riboli F, Roati G and Inguscio M 2002 Phys. Rev. Lett. 89 190404
[20] Kinast J, Hemmer S L, Gehm M E, Turlapov A and Thomas J F 2004 Phys. Rev. Lett. 92 150402
[21] Altmeyer A, Riedl S, Kohstall C, Wright M J, Geuresen R, Bartenstein M, Chin C, Denschlag J I and Grimm R 2007 Phys. Rev. Lett. 98 040401
Guajardo E S R, Tey M K, Sidorenkov I A and Grimm R 2013 Phys. Rev. A 87 063601
[22] Stringari S 1996 Phys. Rev. Lett. 77 2360
Astrakharchik G E, Combescot R, Leyronas X and Stringari S 2005 Phys. Rev. Lett. 95 030404
[24] Bulgac A and Bertsch G F 2005 Phys. Rev. Lett. 94 070401
[25] Adhikari S K 2010 J. Phys. B: At. Mol. Opt. Phys. 43 085304
[26] Tylutki M, Recati A, Dalfovo F and Stringari S 2016 New J. Phys. 18 053014
[27] Pu H and Bigelow N P 1998 Phys. Rev. Lett. 80 1134
[28] Pu H, Zhai W, Wilkens M and Meystre P 2002 Phys. Rev. Lett. 88 070408
[29] Maddaloni P, Modugno M, Fort C, Minardi F and Inguscio M 2000 Phys. Rev. Lett. 85 24113
[30] Ferlaino F, Brecha R I, Hannaford P, Riboli F, Roati G, Modugno G and Inguscio M 2003 J. Opt. B: Quantum Semiclass. Opt. 5 S3
[31] Zhang G E et al 2012 Phys. Rev. Lett. 109 115301
[32] Kohn W 1961 Phys. Rev. 123 1242
[33] Modugno G, Dalfovo F, Fort C, Maddaloni P and Minardi F 2000 Phys. Rev. A 62 053607
[34] Maruyama T and Bertsch G F 2008 Phys. Rev. A 77 063611
[35] Maruyama T and Bertsch G F 2006 Phys. Rev. A 73 013610
[36] Zhang R 2018 Chin. Phys. Lett. 35 046701
[37] Wen W, Chen B and Zhang X 2017 J. Phys. B: At. Mol. Opt. Phys. 50 035301
[38] Adhikari S K and Salasnich L 2008 Phys. Rev. A 78 043616
Adhikari S K and Malomed B A 2007 Phys. Rev. A 76 043626
Adhikari S K, Malomed B A, Salasnich L and Toigo F 2010 Phys. Rev. A 81 053630
[39] Dalfovo F, Giorgini S, Pitaevskii L and Stringari S 1999 Rev. Mod. Phys. 71 463
[40] Pethick C J and Smith H 2008 Bose–Einstein Condensation in Dilute Gases 2nd edn (Cambridge: Cambridge University Press)
[41] Salasnich L, Manini N and Toigo F 2008 Phys. Rev. A 77 043609
Salasnich L and Toigo F 2008 Phys. Rev. A 78 053626
[42] Adhikari S K 2008 Phys. Rev. A 77 045602
[43] Wen W, Zhou Y and Huang G 2008 Phys. Rev. A 77 033623
[44] Navon N, Nascimbène S, Chevy F and Salomon C 2010 Science 328 729
[45] Leggett A J Quantum Liquids: Bose Condensation and Cooper Pairing in Condensed-Matter Systems (Oxford: Oxford University Press)
[46] Zhang W, Pu H, Search C P, Myestre F and Wright E M 2003 Phys. Rev. A 67 021601(R)
[47] Manini N and Salasnich L 2005 Phys. Rev. A 71 033625
[48] Baltz J M and Salasnich L 2007 J. Phys. B: At. Mol. Phys. 40 095304
[49] Vlahos L 2015 Science 350 1505
[50] Bulgac A, Forbes M M, Kelley M M, Roche K J and Wlazloowski G 2012 Phys. Rev. Lett. 109 053626
[51] Bulgac A, Forbes M M and Wlazloowski G 2017 J. Phys.: Conf. Ser. 77 A 118 080403
[52] Petrov D S 2003 Phys. Rev. A 67 011601(R)
[53] Muruganandam P and Adhikari S K 2009 Comput. Phys. Commun. 180 1888
[54] Vukragovic D, Vidanovic I, Balaz A, Muruganandam P and Adhikari S K 2012 Comput. Phys. Commun. 183 2013
[55] Albert M, Paul T, Pavloff N and Leboeuf P 2008 Phys. Rev. Lett. 100 250405
[56] Engels P and Atherton C 2007 Phys. Rev. Lett. 99 160405