Exact solution of new integrable nineteen-vertex models and quantum spin-1 chains

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New exactly solvable nineteen vertex models and related quantum spin-1 chains are solved. Partition functions, excitation energies, correlation lengths, and critical exponents are calculated. It is argued that one of the non-critical Hamiltonians is a realization of an integrable Haldane system. The finite-size spectra of the critical Hamiltonians deviate in their structure from standard predictions by conformal invariance.

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1 Introduction

In a recent paper [1] a complete list of exactly solvable cases was presented for the three-state vertex model with ice rule and certain symmetries. As a criterion for integrability the “additive” Yang-Baxter equation was used and solved exhaustively for the Boltzmann weights in the considered class of models. The list in [1] comprises several well-known models, but also four non-trivial new ones. The solution to these models was not given in [1]. It is the purpose of the present paper to present the exact solution of the four models, namely the calculation of the thermodynamic properties, i.e. partition functions, correlation lengths, critical exponents, etc. At the same time we solve the associated quantum spin-1 chains.

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The vertex models considered in this paper are defined on a square lattice where spin variables are placed on the bonds. Each spin may take three values, say 0 or ±1, and there are interaction energies associated with each vertex depending on the local spin configuration. The corresponding local Boltzmann weight is denoted by $R_{\nu \beta}^{\mu \alpha}$ for a spin configuration $\mu, \nu, \alpha,$ and $\beta$ on the left, right, lower, and upper bond of the vertex, respectively.

The partition function for a lattice of size $N \times L$ with periodic boundary conditions is given by

$$Z = \text{Tr} \ T^L,$$

where the transfer-matrix $T$ is the product of the Boltzmann weights in a row:

$$T_{\beta_1, \ldots, \beta_N}^{\alpha_1, \ldots, \alpha_N} = \sum_{\mu_1, \ldots, \mu_N} R_{\mu_2 \beta_1}^{\mu_1 \alpha_1} R_{\mu_3 \beta_2}^{\mu_2 \alpha_2} \ldots R_{\mu_1 \beta_N}^{\mu_N \alpha_N}.$$

As in [1] we impose the ice-rule

$$\alpha + \mu = \beta + \nu$$

which leads to nineteen allowed, i.e. non-zero, Boltzmann weights, while the further symmetries

$$R_{\nu \beta}^{\mu \alpha} = R_{-\nu -\beta}^{-\mu -\alpha} = R_{\beta \nu}^{\alpha \mu} = R_{\mu \alpha}^{\nu \beta}.$$

reduce them to only 7 independent weights:

$$a = R_{11}^{11}, \ b = R_{-11}^{-11}, \ c = R_{1-1}^{-11}, \ e = R_{10}^{10}, \ g = R_{01}^{10}, \ p = R_{00}^{00}, \ d = R_{00}^{00}.$$

In the following we shall consider the four new models which in [1] have been derived as the solutions #8, #9, #2, and #3 of the Yang-Baxter equation and which we label as I to IV. The four models are defined by their respective (not normalized) Boltzmann weights which all depend on the spectral parameter $u$. Models I and II have no further variable interaction parameter and are parametrized (with slight changes as compared to [1]) as follows. With $U = \exp u$ we have the weights:

Model I:

$$a = \frac{4 - U^4}{3U^2}, \ b = \frac{2(U^4 - 1)}{3U^2}, \ c = \frac{U^4 + 2}{3U^2},$$

$$e = \frac{a}{U}, \ g = \frac{U^{1/2}b}{U}, \ p = 0, \ d = 1.$$

Model II:

$$a = \frac{\alpha(\alpha^2 - U^4)}{U^2(U^4 + \alpha)}, \ b = \frac{\alpha U^2(U^4 - 1)}{U^4 + \alpha}, \ c = \frac{\alpha^2 U^2}{U^4 + \alpha},$$
\[ e = Ua, \quad g = \frac{\sqrt{\alpha}}{U}b, \quad p = 0, \quad d = 1, \quad (7) \]

where \( \alpha = \frac{\sqrt{5}+1}{2} \).

In the physical region all Boltzmann weights must be positive which leads to

\[ 0 \leq \text{Re } u \leq \lambda, \quad \lambda = \left\{ \begin{array}{ll}
\ln \sqrt{2}, & \text{for model I,} \\
\ln \sqrt{\alpha}, & \text{for model II.} 
\end{array} \right. \quad (8) \]

The models III and IV depend on an additional interaction parameter besides the spectral parameter \( u \) and are parametrized differently in different regions.

Model III: We have \( b = g = 0 \) and \( a = c = d \). With

\[ \Delta = \frac{a^2 + p^2 - e^2}{2ap}, \quad (-\infty < \Delta < \infty) \quad (9) \]

the parametrization depends on the value of \( \Delta \).

- \( \Delta > 1 \):
  \[
a = \frac{\sinh(\lambda + u)}{\sinh \lambda}, \quad p = \frac{\sinh u}{\sinh \lambda}, \quad e = 1, \quad \Delta = \cosh \lambda, \quad (10a)
\]

- \( \Delta < -1 \):
  \[
a = \frac{\sinh(\lambda - u)}{\sinh \lambda}, \quad p = \frac{\sinh u}{\sinh \lambda}, \quad e = 1, \quad \Delta = -\cosh \lambda, \quad (10b)
\]

- \(-1 < \Delta < 1 \):
  \[
a = \frac{\sin(\lambda - u)}{\sin \lambda}, \quad p = \frac{\sin u}{\sin \lambda}, \quad e = 1, \quad \Delta = -\cos \lambda, \quad (10c)
\]

with \( \lambda \geq 0 \).

Model IV: Again we have \( b = g = 0 \) and \( a = c \). However, now we require

\[ ad + p^2 - e^2 = 0, \quad (11) \]

which will prove to be a free fermion condition. Otherwise we shall use two different parametrizations

- a)
  \[
a = \cosh u \pm \cosh \lambda \sinh u, \quad d = \cosh u \mp \cosh \lambda \sinh u, \\
p = \sinh \lambda \sinh u, \quad e = 1, \quad (12a)
\]

- b)
  \[
a = \cos u \pm \sinh \lambda \sin u, \quad d = \cos u \mp \sinh \lambda \sin u, \\
p = \cosh \lambda \sin u, \quad e = 1, \quad (12b)\]
where \( \lambda \geq 0 \).

This completes the parametrization of the four models. We emphasize that all parametrizations satisfy the “standard initial condition”

\[ R_{\nu,\beta}^{\mu,\alpha}(u = 0) = \delta_{\mu,\beta}\delta_{\nu,\alpha}. \]  

(13)

One of the consequences of the Yang-Baxter equation is the existence of a family of commuting transfer matrices \( T(u) \) generated by the spectral parameter \( u \)

\[ [T(u), T(v)] = 0. \]

As a result of this commutativity the eigenvalue functions \( \Lambda(u) \) of \( T(u) \) possess the same analytic properties as the Boltzmann weights (i.e. analyticity up to poles imposed by the parametrization of the weights). It is well-known that a one-dimensional quantum spin-1 chain is associated with each 3-state two-dimensional vertex model. Its Hamiltonian \( H \) and momentum operator \( P \) are defined by

\[ \tau H = -\frac{d}{du} \ln T|_{u=0}, \quad P = -i \ln T(0), \]  

(14)

where \( \tau \) is an arbitrary positive scale factor. It turns out that because of the standard initial condition (13) the Hamiltonians are sums of local terms which are given explicitly in the following sections.

The plan of this paper is as follows. In section 2 we solve the models I and II. We calculate the partition function and the correlation lengths by employing analyticity of the eigenvalues \( \Lambda(u) \) and an important inversion identity. The models will turn out to be non-critical, independently of the spectral parameter \( u \) as all \( T(u) \) commute for different \( u \). In subsection 2.3 we derive the associated quantum spin-1 Hamiltonian and determine its spectrum from the spectrum of \( T(u) \).

In Section 3 the models III and IV are analyzed. They are mapped to the symmetric six-vertex model and a free fermion model where the interaction parameter \( \lambda \) plays the role of a crossing parameter or the chemical potential, respectively. Depending on \( \lambda \) the models show different behaviour: there are non-critical regimes with ferro- and antiferromagnetic order, and a critical antiferromagnetic regime. We also determine the spectrum of the associated one-dimensional Hamiltonians.

Section 4 contains a discussion.

2 Solution of models I and II

To solve models I and II we shall apply the analytic method developed in [2] which uses functional equations and avoids the more cumbersome Bethe ansatz. We also refer to [2] for more detailed proofs and derivations of corresponding results. A

*In the usual terminology of the six-vertex model one uses “ferro- (antiferro-) electric order” rather than “ferro- (antiferro-) magnetic order”.

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well known consequence of the Yang-Baxter equation for $R$-matrices satisfying the standard initial condition (13) is the unitarity property

$$
\sum_{\gamma,\delta} R_{\gamma\delta}^{\alpha\alpha}(u) R^{\gamma\delta}_{\nu\beta}(-u) = \varphi(u) \delta_{\alpha\nu} \delta_{\mu\beta},
$$

(15)

where

$$
\varphi(u) = a(u)a(-u).
$$

(16)

Explicitly we obtain

model I: $\varphi(u) = \frac{1}{9}(4 - U^4)(4 - \frac{1}{U^4})$

model II: $\varphi(u) = -\alpha^3 \frac{(U^4 - \alpha^2)(U^4 - \alpha^{-2})}{(U^4 + \alpha)(U^4 + \alpha^{-1})}$.

(17)

Furthermore we have the additional crossing symmetry

$$
R_{\nu\beta}^{\mu\alpha}(u) = R_{-\alpha\nu}^{-\beta\mu}(\lambda - u)
$$

(18)

where the crossing parameter $\lambda$ is given in (8).

As shown in [2] the local relations (15) and (18) imply the global inversion relation for the transfer matrices

$$
T(u)T(u + \lambda) = \varphi(u)^N \left[ I_N + O(e^{-N}) \right],
$$

(19)

where $I_N$ is the identity matrix and $O(\exp(-N))$ is a correction which is exponentially small in the thermodynamic limit ($N \to \infty$). (The correction term is identically zero for $u = 0$ since the transfer matrices $T(0)$ and $T(\lambda)$ reduce to right and left shift operators.)

From the local crossing symmetry (18) and (4) we also obtain

$$
T^+(u) = T(\lambda - u^*).
$$

(20)

Relations (19) and (20) directly imply functional equations for the eigenvalues $\Lambda(u)$

$$
\Lambda(u)\Lambda(u + \lambda) = \varphi(u)^N \left[ 1 + O(e^{-N}) \right],
$$

(21)

$$
\Lambda^*(u) = \Lambda(\lambda - u^*),
$$

which will be solved in the next subsections for the largest and next-largest eigenvalues subject to some obvious analytical properties (and $N$ even). We note that models III and IV also satisfy (15), but relation (18) does not hold. Therefore the method of solution for these models is different (see section 3).
2.1 Largest eigenvalue and partition function

Here the eigenvalue $\Lambda_0(u)$ which is the largest in the physical region is determined. It is convenient to define

$$\Psi(u) = \lim_{N \to \infty} \frac{\Lambda_0^{1/N}(u)}{N}$$

for which we have

- (i) analyticity in the physical region $0 \leq \text{Re} u \leq \lambda$, no zeros therein,
- (ii) periodicity or antiperiodicity under $u \to u + \frac{1}{2} \pi i$,
- (iii) inversion identity:

$$\Psi(u)\Psi(u + \lambda) = \varphi(u). \quad (23)$$

For deriving the unique solution we closely follow [2].

Model I: The ansatz

$$\Psi(u) = \frac{4}{3} F(U) F \left( \sqrt{\frac{2}{U}} \right) \quad (24)$$

satisfies the crossing symmetry $\Psi(u) = \Psi(\lambda - u)$ for real $\Psi$. Inserting (24) into (23) yields a functional equation which is satisfied if only

$$F(U)F(\sqrt{2}U) = 1 - \frac{1}{4U^4}. \quad (25)$$

Solving this equation we obtain

$$F(U) = \prod_{n=0}^{\infty} \frac{1 - \frac{1}{U^{\frac{1}{2}4^{n+2}}}}{1 - \frac{1}{U^{\frac{1}{2}4^{n+2}}}}. \quad (26)$$

The function defined by (24) and (26) satisfies (i) and is periodic in the sense of (ii). Therefore it is identical to the partition function per site (22). As explained in [2] the solution is unique.

Model II: The ansatz

$$\Psi(u) = \frac{U^2}{\alpha + U^4} F(U) F \left( \sqrt{\frac{\alpha}{U}} \right), \quad (27)$$

leads to

$$F(U)F(\sqrt{\alpha}U) = 1 - \frac{1}{\alpha^2 U^4}, \quad (28)$$

which is solved by

$$F(U) = \prod_{n=0}^{\infty} \frac{1 - \frac{1}{U^{\frac{1}{2}4^{n+2}}}}{1 - \frac{1}{U^{\frac{1}{2}4^{n+2}}}}. \quad (29)$$

Again the function defined by (27) and (29) satisfies (i) and is periodic in the sense of (ii).

These are the final results for the partition function per site. In Section 2.3 the ground state energy of the related quantum spin chain is calculated from $\Psi(u)$. 

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2.2 Next-largest eigenvalues and correlation length

We now consider all eigenvalues of the transfer matrix for which

\[ l(u) := \lim_{N \to \infty} \frac{\Lambda(u)}{\Lambda_0(u)} \tag{30} \]

is finite. The properties of \( l(u) \) are:

- (i) analyticity in the physical region \( 0 \leq \text{Re} \ u \leq \lambda \), zeros are allowed,
- (ii) periodicity (antiperiodicity) under \( u \to u + \frac{1}{2} \pi i \) for even (odd) magnetization of the considered state,
- (iii) inversion relation \( l(u)l(u + \lambda) = 1 \), \tag{31}
- (iv) crossing symmetry \( l^*(u) = l(\lambda - u^*) \).

The properties (iii) and (iv) follow from (21). The magnetization for a state \( (\alpha_1, \ldots, \alpha_N) \) of vertical spins \( (\alpha_i = 0, \pm 1) \) is defined as \( M = \sum_{i=1}^{N} \alpha_i \) and is an even or odd integer. This quantity is conserved by the transfer matrix \( T \). Property (ii) follows by inspecting the weights (6), (7) in the product row of \( T \).

Applying (iii) twice we obtain \( l(u + 2\lambda) = l(u) \). Therefore, \( l(u) \) has two periods \( 2\lambda \) and \( \frac{1}{2} \pi i \). A doubly periodic, meromorphic function is an elliptic function and is determined by the location of its zeros \( \Theta_j \) and poles \( \Theta_j + \lambda \). We obtain

\[ l(u) = \prod_{j=1}^{\nu} \sqrt{k} \text{snh} \left[ \frac{4K}{\pi}(u - \Theta_j) \right], \tag{32} \]

where \( \text{snh} \) is the elliptic function of modulus \( k \in (0, 1) \) (see for instance [3, 4]) which is defined by requiring that the corresponding periods \( K(k), K'(k) \) satisfy

\[ \frac{K'}{K} = \frac{4\lambda}{\pi}. \tag{33} \]

As \( \text{snh}(...) \) is antiperiodic under \( u \to u + \frac{1}{2} \pi i \), it follows from (ii) that \( \nu \) is even (odd) for even (odd) magnetization of the corresponding eigenstate. The zeros \( \Theta_j \) are free parameters restricted only by

\[ \text{Re} \ \Theta_j = \frac{\lambda}{2}, \tag{34} \]

which follows from (iv).

From the band of next-largest eigenvalues \( (\nu = 1) \) it is possible to derive [5, 6] the correlation length as

\[ \xi = -\frac{2}{\ln k}. \tag{35} \]

From (33) we find numerically \( \xi_I = 308.93145 \) and \( \xi_{II} = 7105.70704 \), for models I and II, respectively, which are surprisingly large values.
2.3 Quantum spin chains

According to (14) there are quantum spin-1 chains associated with models I and II. After some calculation (see [2]) we obtain

\[ H = \sum_{j=1}^{N} H_{j,j+1}, \]  

(36)

with local interactions:

Model I:

\[ H_{j,j+1} = -A_j^2 - 2B_j^2 - \frac{1}{\sqrt{2}}(A_jB_j + B_jA_j) + \frac{5}{8}((S_j^x)^2 + (S_{j+1}^z)^2) + 2I, \]  

(37)

Model II:

\[ H_{j,j+1} = 2B_j - \frac{4}{\alpha}A_j^2 - 6B_j^2 - \frac{4}{\sqrt{\alpha}}(A_jB_j + B_jA_j) + (9 - 4\alpha)((S_j^z)^2 + (S_{j+1}^z)^2) - (8 - 8\alpha)I, \]  

(38)

where we have used the definition

\[ A_j = S_j^x S_{j+1}^x + S_j^y S_{j+1}^y, \quad B_j = S_j^z S_{j+1}^z, \]  

(39)

in terms of the standard spin operators \( S^{x,y,z} \) of spin one, \( I \) is the unit operator, and we have adjusted a scale factor \( \tau = 8/3 \) for model I.

For these Hamiltonians the ground state energy per site \( e_0 = \lim E_0/N \) is easily calculated from \( \Psi(u) \) (22)

\[ \tau e_0 = -(\ln \Psi)'(0), \]  

(40)

and numerically we have \( (e_0)_I = -0.338201, \quad (e_0)_II = -1.923384. \)

More interesting are the low-lying energy-momentum excitations (see [2,4])

\[ E - E_0 = -\frac{1}{\tau}(\ln l)'(0) = \sum_{j=1}^{\nu} \varepsilon(p_j), \]  

\[ P - P_0 = -i \ln l(0) = \sum_{j=1}^{\nu} p_j \]  

(41)

with energy-momentum dispersion

\[ \varepsilon(p) = \frac{4K}{\tau \pi} \sqrt{(1-k)^2 + 4k \sin^2 p}. \]  

(42)

Obviously, the Hamiltonian has a gap \( \Delta = \varepsilon(0) (\nu = 1) \)

\[ \Delta = \frac{4K}{\tau \pi} (1-k) > 0 \]  

(43)

with numerical values \( \Delta_I = 0.0110033 \) and \( \Delta_{II} = 0.0018375 \) for models I and II, respectively. The decay of correlations in the groundstate of the Hamiltonian is described by the correlation length (33).
3 Solution of models III and IV

The crossing symmetry (18) does not hold in the case of models III and IV. Therefore, the solution in these cases cannot be obtained by the methods of the previous chapter. However, it is straightforward to see that models III and IV can be mapped to the six-vertex model, i.e. a two-state vertex model with six allowed local spin configurations.

The general idea of the mapping is to ignore the signs of all non-zero spin values in the allowed configurations of the three-state model. This is a unique prescription as \( b = 0 \) and \( a = c \) for models III and IV, see (3), (10) and (12). The remaining non-zero vertex configurations and corresponding weights are shown in Fig. 1. Other configurations than those shown in Fig. 1 are not possible because \( g = 0 \) for models III and IV. As a spin 1 line could equally well be a spin \(-1\) line, to such a line corresponds an “internal” degeneracy 2.

\[
\begin{array}{ccccccc}
1 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 1 & 0 \\
& a & d & e & e & p & p \\
\end{array}
\]

Fig. 1 The allowed vertices and the Boltzmann weights of the six-vertex model of interacting spins \( \sigma = 0,1 \). Note that \( a \) and \( d \) may be different (model IV).

It is clear that the partition functions of the three-state and the related two-state model are identical in the thermodynamic limit. For finite systems the equivalence can be made correct by introducing a “seam” of modified weights in the Nth column of the six-vertex model

\[
R_{\alpha}^{\mu \beta} \to e^{i\beta \varphi} R_{\alpha}^{\mu \beta},
\]

where \( \varphi \) is a “twist” which practically takes any real value.

The equivalence of the three-state models with periodic boundary conditions and the six-vertex models with seam are to be understood in terms of the spectra of the corresponding transfer matrices acting on finite chains of length \( N \). The action of the transfer matrices can be thought of as a rearrangement of zero-spins in a background of non-zero spins without changing the sequence of non-zero spins. [Note that there are no intersections of lines of non-zero spins among the allowed vertex configurations.] Each time a non-zero spin is moved from column \( N \) to column 1 the background configuration suffers a shift by one lattice constant which amounts to multiplying the initial configuration by \( \exp (i \varphi) \), where \( \varphi \) is the background momentum. In the reduced description by the equivalent six-vertex model there is no such change of the background configuration, but the factor \( \exp (i \varphi) \) is imposed by the modified boundary condition (44). If we denote by \( N_{\pm}, N_{0} \) the numbers of \( \pm 1 \)
and 0 spins within a row, which are conserved by the action of the transfer matrices of models III and IV, we have

\[
\begin{align*}
N_0 + N_+ + N_- &= N, \\
N_+ - N_- &= \text{magnetization } M, \\
\varphi &= \text{multiple of } \frac{2\pi}{N_+ + N_-}.
\end{align*}
\] (45)

In the following subsections we treat models III and IV separately as they correspond to qualitatively different symmetric and asymmetric six-vertex models, respectively.

### 3.1 Model III

Model III is defined by the requirement \(a = c = d\), and \(b = g = 0\), otherwise any values for \(a\), \(p\) and \(e\) are allowed. The importance of the parameter \(\Delta\) (9) is known from the 6-vertex model [3, 7], and the different physical regions have different parametrizations (10). The associated quantum spin-1 Hamiltonian (14) is easily calculated. It is again of the form (36) with

\[
H_{j,j+1} = -[A_j + A_j B_j + B_j A_j] + \Delta \left[ (S^z_j)^2 + (S^z_{j+1})^2 - 2B^2_j - 1 \right],
\] (46)

in the whole range of \(\Delta\) with the abbreviations (39). In deriving (46) from (10) we have adjusted different scale factors \(\tau\) in the three different regions of parametrization.

For \(\Delta > 1\) the model possesses simple (frozen in) ferromagnetic order, the intervals \(-1 < \Delta < 1\) and \(\Delta < -1\) correspond to the critical and non-critical antiferromagnetic regimes, respectively.

For \(\Delta < -1\), where \(\Delta = -\cosh \lambda\) according to (10b), correlation functions decay with length [5,6]

\[
\xi = -\frac{1}{\ln k},
\] (47)

where \(k\) is the elliptic modulus defined by the requirement

\[
\frac{K'}{K} = \frac{2\lambda}{\pi},
\] (48)

For \(-1 < \Delta < 1\), where \(\Delta = -\cos \lambda\), the 6-vertex model is critical and the correlation functions decay algebraically. For critical systems the energy levels of the associated Hamiltonian (14) on a finite chain of length \(N\) and periodic boundary conditions are expected to scale like

\[
E_0 = N e_0 - \frac{\pi v}{6N(c)},
\] (49)
for the groundstate energy and like
\[ E - E_0 = \frac{2\pi}{N} v x, \]
\[ P = \frac{2\pi}{N} s, \] (50)
for the excited states where \( v \) is the velocity of the elementary excitations. Systems which are conformally invariant \([8, 9]\) have a unique groundstate and the finite-size amplitudes \( c, x, \) and \( s \) are identical to the central charge of the underlying field theory, the scaling dimensions and spins of primary fields \([10, 11]\). For instance the asymptotics of two-point functions are given by
\[ C_r \simeq \frac{1}{r^{2x}}. \] (51)

From the correspondence of model III with a six-vertex model with seam \([12, 13]\) we indeed find (49) and (50) to hold with \( c = 1 \) and
\[ x = \frac{1 - \lambda/\pi}{2} n^2 + \frac{1}{2(1 - \lambda/\pi)} \left( m - \frac{\varphi}{2\pi} \right)^2 + I + \bar{I}, \]
\[ s = n \left( m - \frac{\varphi}{2\pi} \right) + N \frac{\varphi}{4\pi} + I - \bar{I}. \] (52)
The groundstate lies in the sector with \( N_0 = N_+ + N_- = N/2 \) and twist \( \varphi = 0 \). For the excited states \( n = N_0 - N/2, m \) may take any integer value and \( I, \bar{I} \) are non-negative integers.

However, the numbers \( x \) and \( s \) in general do not correspond to scaling dimensions and spins. The problem can be traced back to the degeneracy of the groundstate of the system. There are about \( \simeq 2^{N/2} \) background spin configurations for which \( N_+ + N_- = N/2 \) with momentum zero. (This implies a residual entropy of \( \frac{1}{2} \ln 2 \).) So, most of the correlation functions are ill-defined if they probe the background spin configuration, for instance \( \langle S^z_i S^z_j \rangle \) or simply \( \langle S^z \rangle \), as any magnetization between \(-N/2\) and \(+N/2\) may be realized by one of the \( \simeq 2^{N/2} \) groundstates. Other correlation functions are independent of the particular groundstate like \( \langle (S^z_i)^2 (S^z_j)^2 \rangle \) and are given by the corresponding correlation function of the six-vertex model (without twist).

We summarize that only for \( \varphi = 0 \) the set of exponents of algebraically decaying correlation functions is given by (50). This is a little unfortunate as for continuous dependence of \( x \) on \( \varphi \) we could have expected logarithmic behaviour of the correlations \( C_r \simeq \int d\varphi / r^{2x(\varphi)} \simeq 1 / (r^{2x(0)} \ln r) \).

### 3.2 Model IV

In this case we are dealing with an asymmetric six-vertex model as \( a \neq d \) (see Fig.1). The associated quantum spin-1 Hamiltonian is again of the form (50) and
uniformly given by

\[ H_{j,j+1} = -[A_j + A_j B_j + B_j A_j] + \alpha_0 \left[ 1 - (S_j^z)^2 - (S_{j+1}^z)^2 \right], \tag{53} \]

where the coupling \( \alpha_0 \) takes all real values, \(-\infty < \alpha_0 < \infty\). The range \( 1 \leq |\alpha_0| \) follows from the parametrization (12a) with \( \alpha_0 = \pm \coth \lambda \), while the range \( |\alpha_0| \leq 1 \) follows from (12b) with \( \alpha_0 = \pm \tanh \lambda \), and we have adjusted the scale factors \( \tau = \sinh \lambda \) and \( \cosh \lambda \) in these two cases, respectively.

To solve model IV we do not use the parametrization (12), but a different one which is more appropriate to the asymmetric six-vertex model (and does not satisfy (13)). The asymmetric model can be thought of as a symmetric model in an external magnetic field \( B \) acting on horizontal and vertical bond spins, with new weights \( \tilde{R} \) such that

\[ R^\alpha_\beta = e^{B(\alpha+\beta+\mu+\nu)} \tilde{R}^\alpha_\beta, \tag{54} \]

where \((a/d)^{1/4} = e^B\).

To satisfy (11) with \( e = 1 \) the \( \tilde{R} \) can be parametrized by

\[ \tilde{R}_{11}^{00} = \tilde{R}_{00}^{10} = \rho \cos v, \quad \tilde{R}_{01}^{10} = \tilde{R}_{10}^{01} = \rho, \quad \tilde{R}_{01}^{01} = \tilde{R}_{10}^{10} = \rho \sin v \tag{55} \]

with fixed scale \( \rho = e^{-2B} \) and an appropriate parameter \( v \) which satisfies \( 0 \leq \Re v \leq \frac{\pi}{2} \) in the physical region. The eigenvalues of the transfer matrix of the six-vertex model in external fields (plus “twisted boundary conditions”) are given by

\[ \Lambda(v) = \rho^N \left(-e^{2B}\right)^{N-N_0} \cdot \frac{e^{2NB+i\varphi} \cos^N v q(v + \frac{\pi}{2}) + \sin^N v q(v - \frac{\pi}{2})}{q(v)}, \tag{56} \]

where

\[ q(v) = \prod_{j=1}^{N_0} \sin(v - v_j) \tag{57} \]

is a function of \( N_0 \) Bethe ansatz numbers \( v_j \). Equation (56) is easily derived (see for instance [3, 14]) as the vertical magnetic field contributes the overall factor \( e^{2B(N-N_0)} \) and the horizontal field and the twist amount to the prefactor \( e^{2BN} e^{i\varphi} \) of the first term on the right hand side of (54).

Inserting (57) simplifies (56):

\[ \Lambda(v) = \left[e^{2BN+i\varphi} \cos^N v + (-1)^{N_0} \sin^N v \right] \prod_{j=1}^{N_0} \left(-e^{-2B \cos(v - v_j)} \sin(v - v_j) \right). \tag{58} \]

The factoring is in fact due to the “free fermion” condition (11). As \( \Lambda \) must be analytic at the zeros \( v_j \) of \( q(v) \), the first factor must vanish which determines the \( v_j \)

\[ \tan^N v_j = (-1)^{N_0+1} e^{2BN+i\varphi}, \tag{59} \]
which means that the Bethe ansatz equations have decoupled. Inserting (59) into (58) we obtain the final solution in terms of the original weights \( a \) and \( p \)

\[
\Lambda = \left[e^{i\varphi}a^N + (-1)^{N_0}p^N\right] \prod_{j=1}^{N_0} \frac{(1 - p^2)^{\Theta_j} + ap}{a(a - p\Theta_j)},
\]

(60)

where

\[
\Theta_j = e^{i k_j}, \quad k_j = \frac{2\pi}{N}I_j - \frac{\varphi}{N}.
\]

(61)

The \( I_j \) are arbitrary, however distinct integers or half-integers for odd or even number \( N_0 \), respectively. Obviously, the influence of the twist angle \( \varphi \) is small, vanishing for \( N \to \infty \).

The physical properties are most easily discussed for the associated Hamiltonian (53). Its eigenvalues follow from (14)

\[
E = 2 \sum_{j=1}^{N_0} [\alpha_0 - \cos k_j] - N\alpha_0,
\]

\[
P = \sum_{j=1}^{N_0} k_j + \varphi,
\]

(62)

for both cases of parametrization (12).

The properties depend on the coupling \( \alpha_0 \). If \( \alpha_0 > 1 \) the groundstate is the \( N_0 \)-vacuum, i.e. the state with \( N_0 = 0 \). With a view to (53) this means that all \( S_z = 1 \) or \(-1\) along the chain, the degeneracy is therefore \( 2^N \). For \( \alpha_0 < -1 \) the groundstate is the vacuum of non-zero spins. All \( S_z \) along the chain are zero, i.e. \( N_0 = N \) and all \( k_j \)-modes in (62) are occupied. The state is unique. In both cases we have \( E_0 = -N|\alpha_0| \) and the gap to the excitations is \( \Delta = 2(|\alpha_0| - 1), N \to \infty \).

For \( |\alpha_0| \leq 1 \) all \( k_j \)-modes of negative energy in (62) are occupied in the ground state. Its energy is

\[
E_0 = -2\frac{\cos \frac{\varphi}{N}}{\sin \frac{\varphi}{N}} \sin \frac{N_0}{N}\pi - (N - 2N_0)\alpha_0.
\]

(63)

The model is critical and the underlying conformal field theory has central charge \( c = 1 \). The “scaling dimensions” and “spins” (see also subsection 3.1) are given by

\[
x = \frac{(\Delta n)^2}{4} + \left(m - \frac{\varphi}{2\pi}\right)^2 + I + \bar{I}
\]

\[
s = \Delta n \left(m - \frac{\varphi}{2\pi}\right) + (N - N_0)\frac{\varphi}{2\pi} + I - \bar{I}.
\]

(64)

where \( \Delta n \) is the change in the particle number as compared to the ground state, the integer \( m \) describes an asymmetry in the momentum distribution \( k_j \), and \( I, \bar{I} \) are non-negative integers arising from particle-hole excitations.
4 Conclusion

We have studied four exactly solvable three-state vertex models and corresponding integrable spin-1 chains.

For models I and II the groundstate energy of their associated Hamiltonians, the elementary excitations and the correlation lengths could be calculated. These systems turned out to be non-critical. Unfortunately, the methods of section 2 did not allow for the calculation of any order parameter or the multiplicity of the groundstate. However, the patterns of the elementary excitations indicate that model I is an integrable realization of a Haldane system [15, 16] and model II has Néel order. On chains with an even number $N$ of sites the number $\nu$ of elementary excitations may be even or odd, $\nu = 1, 2, 3, \ldots$. This excludes dimerization which would imply $\nu$ even. On chains with an odd number of sites a similar analysis to that of section 2 shows that the groundstate of model I is still separated by a gap from the first band of one-particle excitations, whereas model II has a groundstate which corresponds to the lowest edge of a one-particle band. If not conclusive, these findings are at least consistent with a Haldane system [17, 18] for model I and Néel order for model II.

Models III and IV comprise different physical phases depending on the value of their coupling parameters $\Delta$ and $\alpha_0$, respectively. The physical properties were derived by a mapping to the six-vertex model. We have seen that these three-state models contain spurious degrees of freedom which essentially do not participate in the dynamics. This is the reason for the high degeneracy of the ground states for models I and II (with residual entropy). It also is the reason why a simple application of conformal invariance relating finite-size corrections of energy levels to scaling dimensions fails.

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